Smooth weight structures and birationality filtrations on motivic categories

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Abstract

We study various triangulated motivic categories and introduce a vast family of aisles (these are certain classes of objects) in them. These aisles are defined in terms of the corresponding "motives" (or motivic spectra) of smooth varieties in them; we relate them to the corresponding homotopy $t-$structures. We describe our aisles in terms of stalks at function fields and prove that they widely generalize the ones corresponding to slice filtrations. Further, the filtrations on the "homotopy hearts" $H_{\text{eff}}^{\text{hom}}$ of the corresponding effective subcategories that are induced by these aisles can be described in terms of (Nisnevich) sheaf cohomology as well as in terms of the Voevodsky contractions $-1$. Respectively, we express the condition for an object of $H_{\text{eff}}^{\text{hom}}$ to be weakly birational (i.e., that its $n+1$th contraction is trivial or, equivalently, the Nisnevich cohomology vanishes in degrees $> n$ for some $n \geq 0$) in terms of these aisles; this statement generalizes well-known results of Kahn and Sujatha. Next, these classes define weight structures $w_{\text{Smooth}}^s$ (where $s = (s_j)$ are non-decreasing sequences parameterizing our aisles) that vastly generalize the Chow weight structures $w_{\text{Chow}}$ defined earlier. Using general abstract nonsense we also construct the corresponding adjacent $t-$structures $t_{\text{Smooth}}^s$ and prove that they give the birationality filtrations on $H_{\text{eff}}^{\text{hom}}$.

Moreover, some of these weight structures induce weight structures on the corresponding $n-$birational motivic categories (these are the localizations by the levels of the slice filtrations). Our results also yield some new unramified cohomology calculations.

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1 Introduction

In [BoK18] the authors constructed and studied in detail the Chow weight structure $w_{\text{Chow}}$ on the category $DM^{eff}_R(k)$ of Voevodsky’s motives (and on its ”stable” version $DM_R(k)$), where $k$ is a perfect field and $R$ is the coefficient ring. The main advantage of this definition (in contrast to the earlier one in [Bon11]) was that it did not depend on any resolution of singularities results, and the characteristic $p$ of $k$ was not required to be invertible in $R$ (if $p > 0$).

In the current text we consider similar (but much more general) definitions in various motivic categories, and relate the corresponding classes of objects (that are aisles) to the corresponding (well-known) homotopy $t$—structures $t_{\text{hom}}$. The filtrations given by these aisles generalize the slice and (Chow) weight ones (the latter are considered in several papers of the first author). We also consider the effective versions of our motivic categories.
and obtain several conditions for an object from the heart of the homotopy $t$–structure $H_{\text{eff}}^\text{hom}$ to be weakly birational (see Proposition 3.2.1 and Remark 3.2.2(1)). Further, since our weight structures $w^s_{\text{smooth}}$ (generated by $\mathcal{M}(\text{SmVar})(j)[s_j]$, where $s = s_j$ is a non-decreasing sequence) are smashing, there exist $t$-structures $t^s_{\text{smooth}}$ (right) adjacent to $w^s_{\text{smooth}}$. This allows us to prove interesting properties of our filtration on $H_{\text{eff}}^\text{hom}$: the levels $F^n$ of this filtration consist of $i$–birational objects, and they give right adjoint functors to the embeddings $H^i_{\text{eff}} \hookrightarrow H_{\text{eff}}^\text{hom}$. Thus we generalize the corresponding results of [KaS17]. We also obtain a curious statement about unramified cohomology (see Proposition 4.2.8).

Let us now describe the contents of the paper. Some more information of this sort can be found at the beginnings of sections. By $\mathcal{D}^\text{eff} \subset \mathcal{D}$ we denote the motivic categories we consider.

In §2 we recall several definitions and results on triangulated categories, $t$–structures, and recall the motivic categories relevant to us (these are $\text{SH}^S_1(k)$, $\text{SH}^\text{eff}(k) \subset \text{SH}(k)$, $\text{DM}^\text{eff}(k) \subset \text{DM}(k)$, and $D^\text{eff}_{A^1}(k) \subset D_{A^1}(k)$) along with their basic properties. Further we discuss homotopy $t$–structures on them in a rather axiomatic manner. Some of the details are postponed until §5.

In §3 we define "smoothly generated" aisles corresponding to non-decreasing sequences $s = (s_j)$, in our motivic categories, and obtain several comparison theorems. Next, we describe these aisles in terms of stalks corresponding to function fields. Further, we consider the corresponding smooth filtrations on $H_{\text{eff}}^\text{hom}$ and prove some of their properties. Finally, using our results we prove that the homotopy $t$–structure restricts to $\mathcal{D}^{n-\text{bir}}$.

In §4 we recall basic definitions and properties of weight structures. Next we define the main weight structures on our categories — the $A_s$-smooth ones, and relate them to unramified cohomology. Using general existence results, we also define $A_s$-smooth $t$-structures (that are right adjacent to our $w^s_{\text{smooth}}$), and relate them to the birationality filtration on $H_{\text{eff}}^\text{hom}$. Next we relate our weight structures to the corresponding $n$–birational motivic categories. In particular, we study the weight-exactness of the functors $-\langle n \rangle$ and of the localizations $p_n : \mathcal{D}^\text{eff} \rightarrow \mathcal{D}^{n-\text{bir}}$.

In §5 we discuss (in more detail) some definitions and results for motivic categories different from $\text{SH}^\text{eff}(k) \subset \text{SH}(k)$. We explain that localizing coefficients for these categories yields new examples. This gives $R$-linear versions of the categories above, where $R \subset \mathbb{Q}$; it follows that the category $\text{SH}(k)^+$ can also be added to the examples mentioned above.
2 Preliminaries

In §2.1 we give some definitions and conventions related to (mostly) triangulated categories.
In §2.2 we recall basic definitions and properties of t-structures.
In §2.3 we recall some basics on various motivic categories.
In §2.4 we introduce and discuss homotopy $t$-structures on these categories.
In §2.5 we discuss our axioms mainly in the case of the categories $\text{SH}^{\text{eff}}(k) \subset \text{SH}(k)$.

2.1 Categorical definitions and notation

- Given a category $\mathcal{B}$ and $M, N \in \text{Obj } \mathcal{B}$, we say that $M$ is a retract of $N$ if $\text{id}_M$ can be factored through $N$ (recall that if $\mathcal{B}$ is triangulated then $M$ is a retract of $N$ if and only if $M$ is its direct summand).

- A subcategory $\mathcal{D}$ of $\mathcal{B}$ is said to be retraction-closed in $\mathcal{B}$ if it contains all $\mathcal{B}$-retracts of its objects.

- The full subcategory $\text{Kar}_\mathcal{B}(\mathcal{D})$ of $\mathcal{B}$ whose objects are all $\mathcal{B}$-retracts of objects of $\mathcal{D}$ will be called the retraction-closure of $\mathcal{D}$ in $\mathcal{B}$. It is easily seen that $\text{Kar}_\mathcal{B}(\mathcal{D})$ is retraction-closed in $\mathcal{B}$: if $\mathcal{B}$ and $\mathcal{D}$ are additive then $\text{Kar}_\mathcal{B}(\mathcal{D})$ is additive as well.

- We say that an additive category $\mathcal{D}$ is Karoubian if any its idempotent endomorphism is isomorphic to the composition of a retract and a coretraction of the type $M \oplus N \to M \to M \oplus N$.

- For any $A, B, C \in \text{Obj } \mathcal{C}$ we will say that $C$ is an extension of $B$ by $A$ if there exists a distinguished triangle $A \to C \to B \to A[1]$. A class $\mathcal{P} \subset \text{Obj } \mathcal{C}$ is said to be extension-closed if it is closed with respect to extensions and contains $0$.

- For $X, Y \in \text{Obj } \mathcal{C}$ we will write $X \perp Y$ if $\mathcal{C}(X, Y) = \{0\}$. For $D, E \subset \text{Obj } \mathcal{C}$ we write $D \perp E$ if $X \perp Y$ for all $X \in D$, $Y \in E$. Given $D \subset \text{Obj } \mathcal{C}$ we will write $D^\perp$ for the class

$$\{Y \in \text{Obj } \mathcal{C} : X \perp Y \forall X \in D\}.$$  

Dually, $^\perp D$ is the class $\{Y \in \text{Obj } \mathcal{C} : Y \perp X \forall X \in D\}$.

- Given $f \in \mathcal{C}(X, Y)$ we will call the third vertex of (any) distinguished triangle $X \xrightarrow{f} Y \to Z$ a cone of $f$.

\footnote{Recall that different choices of cones are connected by non-unique isomorphisms.}
• All coproducts in this paper will be small.

• Assume that $C$ is smashing, that is, closed with respect to coproducts. For $D \subseteq C$ ($D$ is a triangulated category that may be equal to $C$) one says that $\mathcal{P} \subseteq \text{Obj } C$ generates $D$ as a localizing subcategory of $C$ if $D$ is the smallest full strict triangulated subcategory of $C$ that contains $\mathcal{P}$ and is closed with respect to $C$-coproducts.

• $M \in \text{Obj } C$ is said to be compact if the functor $C(M, -) : C \to \text{Ab}$ respects coproducts.

• $C$ is said to be compactly generated if it is generated by a set of compact objects as its own localizing subcategory.

2.2 Basics on $t$-structures

Let us recall some notations and properties on $t$–structures. In contrast to the original definitions in [BBD82], our convention for $t$–structures will be homological.$^2$

Definition 2.2.1. A pair of strict subcategories $C_{t \geq 0}, C_{t < 0} \subseteq \text{Obj } C$ will be said to define a $t$-structure $t$ on a triangulated category $C$ if they satisfy the following conditions.

(i) $C_{t \geq 0}[1] \subseteq C_{t \geq 0}$ and $C_{t < 0}[-1] \subseteq C_{t < 0}$.

(ii) $C_{t \geq 0} \perp C_{t < 0}$.

(iii) For any $M \in \text{Obj } C$ there exists a distinguished triangle

$$M_{t \geq 0} \to M \to M_{t < 0} \to M_{t \geq 0}[1]$$

such that $M_{t \geq 0} \in C_{t \geq 0}$, $M_{t < 0} \in C_{t < 0}$.

We also need the following auxiliary definitions.

Definition 2.2.2. 1. $C_{t \geq n} := C_{t \geq 0}[n]$ (resp. $C_{t \leq n+1} := C_{t < n} := C_{t < 0}[n]$) for any integer $n \in \mathbb{Z}$.

2. The heart of $t$ is the category $\mathcal{H}t = C_{t \geq 0} \cap C_{t \leq 0} \subseteq C$: recall that it is an abelian category.

3. We will say that $t$ is left (resp. right) non-degenerate if $\cap_{i \in \mathbb{Z}} C_{t \leq i} = \{0\}$ (resp. $\cap_{i \in \mathbb{Z}} C_{t \geq i} = \{0\}$). We say that $t$ is non-degenerate if it is both left and right non-degenerate.

$^2$Here we follow [Mor]. The homological and cohomological convention are related in the usual way: $C_{t \leq n} = C_{t \geq -n}$ and $C_{t \geq n} = C_{t \leq -n}$. 

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Remark 2.2.3. 1. The triangle in axiom (iii) is essentially functorial in $M$. Thus, we get a well-defined functor $\tau_{\geq 0} : C \to C_{\geq 0}$ (resp. $\tau_{< 0} : C \to C_{< 0}$) which is right (resp. left) adjoint to the inclusion functor $C_{\geq 0} \to C$ (resp. $C_{< 0} \hookrightarrow C$). Also we put $\tau_{\geq n}(M) := \tau_{\geq 0}(M[-n])[n]$ (resp. $\tau_{\leq n}(M) := \tau_{< 0}(M[n + 1])[-n - 1]$).

2. The functor $H^0_t := \tau_{\geq 0} \circ \tau_{< 0}$ sends $C$ into $H^t_t$; it is homological (i.e., converts distinguished triangles into long exact sequences). Moreover, we will write $H^n_t$ for $H^0_t \circ [-n]$.

3. One can easily check that $t$ is non-degenerate if and only if the family of functors $(H^n_t)_{n \in \mathbb{Z}}$ is conservative.

4. Consider two categories $C$ and $D$ endowed with $t$–structures. One says that a functor $F : C \to D$ is left (resp. right) $t$–exact if it respects the $t$-negativity (resp. $t$-positivity) of objects. We will say that $F$ is $t$–exact if it is both left and right $t$–exact.

Proposition 2.2.4. Let $P \subset \text{Obj } C$ be a set of compact objects. Then there exists a unique $t$–structure $t$ on $C$ such that $C_{\geq 0}$ is the smallest subclass of $\text{Obj } C$ that contains $P$ and is stable with respect to extensions, the suspension [1], and coproducts.

Proof. This is precisely Theorem A.1 of [AJS03].

Definition 2.2.5. We will call the $t$–structure as in the previous proposition the $t$–structure generated by $P$.

Remark 2.2.6. 1. For a $t$–structure given by Proposition 2.2.4, the functors $\tau_{\geq 0}, \tau_{< 0}$, and $H^0_t$ respect coproducts (see Proposition A.2 of ibid.).

2. The $t$–structure as in the proposition is left non-degenerate if and only if $P$ generates $C$ as its own localizing subcategory (obvious; see Lemma 1.2.9 of [BoD17]).

2.3 On various motivic categories

Now we recall some basics on motivic categories. Below $k$ will always be a perfect field of characteristic $p$, and $\mathcal{D}(k)$ is one of the motivic categories listed below; $\mathcal{M} = \mathcal{M}_{\mathcal{D}} : \text{SmVar} \to \mathcal{D}(k)$ will denote the corresponding "$\mathcal{D}$–motive" functor from the category $\text{SmVar}$ of smooth $k$-varieties. We will always assume that $\mathcal{D}$ is triangulated monoidal with the tensor unit given by $1_{\mathcal{D}} = \mathcal{M}_{\mathcal{D}}(\text{Spec}(k))$. Moreover, we assume that $\mathcal{M}_{\mathcal{D}}$ sends products of varieties into tensor products.

$\mathcal{M}_{\mathcal{D}}$ will satisfy the homotopy invariance property, that is, $\mathcal{M}_{\mathcal{D}}(\mathbb{A}^1) \cong 1_{\mathcal{D}}$. 

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Remark 2.3.1. 1. Now let us introduce some notation related to "Tate-type twists".
Firstly, we set $T = \text{Cone}(\mathcal{M}_D(\mathbb{G}_m) \to \mathcal{M}_D(\mathbb{A}^1))$. Below we will always assume that $T$ is $\otimes$-invertible in $\mathcal{D}$ (yet cf. Remark 2.3.2(4) below). For $C \in \text{Obj} \mathcal{D}$ and $n \in \mathbb{Z}$ we set $C(n) = C \otimes T^\otimes n$ and $C\{n\} = C(n)[-n]$.

2. These twists are clearly "coherent with respect to functors that commute with $\mathcal{M}$" in various motivic categories below.
Moreover, homotopy invariance implies $\mathcal{M}_D(\mathbb{G}_m) \cong 1_\mathcal{D} \bigoplus T[-1]$. Furthermore, it is well-known (and follows from the so-called Mayer-Vietoris property) for the categories we consider that $\mathcal{M}_D(\mathbb{P}^1) \cong 1_\mathcal{D} \bigoplus T$.
These splittings also exist in the motivic categories of the type $\mathcal{D}^{eff}$ that we discuss in Remark 2.3.2(2-3) below.

- We will write $SH(k)$ for the $\mathbb{P}^1$–stable motivic homotopy category, and $\mathcal{M}_{SH} : \text{SmVar} \to SH(k)$ for the corresponding infinite suspension spectrum functor (see [Mor], §5.1).
- We will write $DM(k)$ for the category of Voevodsky’s motives, and $\mathcal{M}_{DM} : \text{SmVar} \to DM(k)$ for the corresponding ("usual") $DM$-motive functor (see §5.1 of [Deg11] for the detail).
- We will write $D_{\mathbb{A}^1}(k)$ for the $\mathbb{A}^1$–derived category as defined in §5.3.20 of [CiD19], and $\mathcal{M}_{\mathbb{A}^1} : \text{SmVar} \to D_{\mathbb{A}^1}(k)$ for the corresponding functor (see also §6.2 of [Mor]).
- We will write $\text{MGl} – \text{Mod}(k)$ for the category of $\text{MGl}$-modules in $SH(k)$; see Propositions 7.2.14, 7.2.18 of [CiD19], Example 1.3.1(3) of [BoD17], or §2.2 of [Deg13].

Remark 2.3.2. 1. All the categories discussed above are well known to be smashing and compactly generated by the objects $\mathcal{M}_D(X)\{i\}$ for $X \in \text{SmVar}, i \in \mathbb{Z}$.

Furthermore, the functors $\mathcal{M}_D$ factor through the corresponding subcategories of compact objects.

2. For each of these $\mathcal{D}$ we will write $\mathcal{D}^{eff}(k)$ for the localising subcategory of $\mathcal{D}(k)$ generated by objects of the form $\mathcal{M}(X)$ for $X \in \text{SmVar}$. Thus $\mathcal{D}^{eff}(k)$ is also compactly generated by these objects.

3. We will write $SH^{S^1}(k)$ for the $S^1$–stable motivic homotopy category, and $\mathcal{M}_{SH^{S^1}} : \text{SmVar} \to SH^{S^1}(k)$ for the corresponding infinite suspension spectrum functor (see [Mor], §4.1). The category $SH^{S^1}(k)$ is compactly generated by the objects $\mathcal{M}(X)$ for $X \in \text{SmVar}$. Note that there exists an adjunction $\sigma : SH^{S^1}(k) \rightleftarrows SH(k) : \omega$ (see Remark 5.1.11 of [Mor]).
The category $SHS^1(k)$ is not equivalent to the effective category $D^{eff}$ for any "motivic" $D$. Nonetheless, by abuse of notation we will take $SHS^1(k)$ as one of the possibilities for the category $D^{eff}(k)$ in all the formulations in this paper except the ones in §4.3.

4. In categories of the type $D^{eff}(k)$ the twists of the types $-\langle n \rangle$ and $-\{ n \}$ are defined for $n \geq 0$ only. Note also that these functors are not fully faithful on $SHS^1(k)$ (in contrast to other $D^{eff}(k)$).

2.4 Homotopy $t$-structures: recollection

Now we introduce some definitions, which will be very important further in this paper. We will write $D$ for some of the motivic categories from section 2.3 and $D^{eff}$ for its effective version.

**Definition 2.4.1.** 1. Denote by $t_{hom}^D$ the $t$–structure on $D$ generated by $M(X)\{i\}$ for $X \in \text{SmVar}$, $i \in \mathbb{Z}$ (see Definition 2.2.5).

2. Denote by $t_{hom}^{D^{eff}}$ the $t$–structure on $D^{eff}$ generated by $M(X)$ for $X \in \text{SmVar}$.

3. For $i \in \mathbb{Z}$, $C \in D$ and $j \in \mathbb{Z}$ (resp. $C \in D^{eff}$ and $j \geq 0$) we define the functor $C_{j}(-) : \text{SmVar}^{op} \rightarrow \text{Ab}$ as $X \mapsto D(M(X)\{j\}, C[i])$ (resp. $X \mapsto D^{eff}(M(X)\{j\}, C[i])$).

More generally, for any cohomological functor from $D(k)$ to $\text{Ab}$ and $X \in \text{SmVar}$, $i, j \in \mathbb{Z}$ we define $H^i_j(M(X))$ as $H(M(X)\{j\}[-i])$. For $i, j$ as above we will write $D^{eff}(M(K)\{j\}[i], -)$ (resp. $D(M(K)\{j\}[i], -)$) for the following functor from $D^{eff}$: $C \mapsto \lim_{X, k(X) = K} \text{C}_{j}^{-i}(X)$ (resp. for the functor on $D$: $C \mapsto \lim_{X, k(X) = K} C_{j}^{-i}(X)$).

5. Denote by $-\langle 1 \rangle$ the right adjoint to $-\{1\}$; this is the so-called Voevodsky contraction\footnote{Essentially following Definition 4.3.10 of [Mor]; note that the latter is equivalent to the original definition from [Voe00b]. This adjoint exists since $D^{eff}$ is compactly generated (see Remark 2.3.2(2,3)), and $-\{1\}$ is exact and respects coproducts.}. For $i > 0$ the $i$-th iteration of $-\langle 1 \rangle$ will be denoted by $-\langle i \rangle$.

We introduce some "axioms" characterizing homotopy $t$–structures on motivic categories that we consider in this paper.

(A1) Let $H$ be a cohomological functor from $D(k)$ to $\text{Ab}$ and $X \in \text{SmVar}$. Then there exists a convergent (coniveau) spectral sequence as follows:

$$E_1^{p,q} = \prod_{x \in X^{(p)}} H^q_p(x) \Rightarrow H^{p+q}_0(M(X)),$$
where \( X^p \) is the set of points of \( X \) of codimension \( p \geq 0 \), and for a presentation of \( x \in X^p \) as \( \lim_{\mathcal{I}} X_i \) for \( X_i \in \text{SmVar} \) we define \( H^q_{\mathcal{I}}(x) \) as \( \varprojlim \ H(\mathcal{M}(X_i)\{p\}[-q]) \).

\[
(A2) \quad \mathcal{D}(k)_{\text{hom} \geq 0} = \{ C \in \mathcal{D}(k) | \mathcal{D}(\mathcal{M}(K)\{j\}, C[i]) = \{0\} \text{ for all function fields } K/k, j \in \mathbb{Z}, i > 0 \}.
\]

In the effective setting one should take \( j \geq 0 \) in Axiom (A2) instead.

**Corollary 2.4.2.**

1. The endofunctor \((-)_{-1}\) is \( t_{\text{hom}} \)-exact.

2. The functor \( \Phi : H^D_{\mathcal{I}} \to Psh(Pts, \text{Ab}) \), \( \Phi(F) = F^0_*(-) \) (see the related definitions above) is conservative, exact, and commutes with coproducts. Here we write \( Pts \) for the set of function fields (that is, finitely generated extensions) over the base field \( k \).

Moreover, the functor \( \Phi_0 : F \mapsto F^0_*(-) \) is conservative on \( H^D_{\mathcal{I}}^{\text{eff}} \).

3. \( t^D_{\text{hom}} \) and \( t^{\text{eff}}_{\text{hom}} \) are non-degenerate (see Definition [2.2.2](3))

**Proof.**

1. The right \( t_{\text{hom}} \)-exactness follows from \( (A2) \). Next we note that \(-\{1\}\) is right \( t_{\text{hom}} \)-exact, thus \((-)_{-1}\) is also left \( t_{\text{hom}} \)-exact (as a right adjoint).

2. The description of \( \mathcal{D}(k)_{\text{hom} \geq 0} \) provided by \( (A2) \) immediately yields that \( \Phi(F) \neq 0 \) if \( F \in \mathcal{D}(k)_{\text{hom} \geq 0} \setminus \mathcal{D}(k)_{\text{hom} \geq 1} \). Next, \( \Phi \) commutes with coproducts, since it is defined in terms of functors corepresented by compact objects.

To prove exactness, we apply the functor \( \Phi \) to the exact sequence \( 0 \to F' \to F \to F'' \to 0 \) in \( H^D_{\mathcal{I}} \). Our definitions immediately imply \( \Phi(F''[-1]) = 0 \), whereas \( \Phi(F'[1]) = 0 \) by \( (A2) \).

We should also prove that if \( F \in \text{Obj} H^D_{\mathcal{I}}^{\text{eff}} \) and \( \Phi_0(F) = 0 \) then \( F^0_*(-) = \{0\} \) for \( i > 0 \). Firstly, \( F^*_*(\mathcal{M}((\mathbb{G}_m)^{\times i})) \) equals \( \varinjlim_j F^*_*(\mathcal{M}((\mathbb{G}_m)^{\times i}(X_j))) \) for a presentation \( \text{Spec}(K) = \varprojlim X_j \). We should prove that \( F^0_*((\mathcal{M}((\mathbb{G}_m)^{\times i})(X_j))) = \{0\} \) for \( p + q = 0 \). We take the spectral sequence as in \( (A1) \) for the variety \( (\mathbb{G}_m)^{\times i}(X_j) \) for each \( j \). By our assumption and \( (A2) \), the corresponding \( E^{p,q}_2 \) vanish if \( q > 0 \) and \( p = q = 0 \). Further, for \( q < 0 \) (i.e., in the case \( p = -q \)) these groups vanish by the property (ii) for \( t \)-structures and the right \( t \)-exactness of \(-\{p\}\). Thus \( F^0_*((\mathcal{M}((\mathbb{G}_m)^{\times i}(X_j))) = 0 \) for any \( j \), and passing to the direct limit we obtain our assertion.

3. Can be easily obtained by combining a spectral sequence argument (Axiom \( (A1) \)) with the corresponding compact generation properties; see Remarks [2.2.3](3) and [2.3.2](1–2). \( \square \)
It will be shown below that axioms (A1–A2) hold for all aforementioned motivic categories (and for some other ones).

2.5 The case of $SH(k)$ and $SH^{eff}(k)$

In this section, we check our axioms and discuss the related definitions for the categories $SH^{eff}(k) \subset SH(k)$ that appear to give the most interesting examples.

**Definition 2.5.1.** We will write $\pi_i(E)_j$ for the Nisnevich sheaf on SmVar that is associated to the presheaf $E^j_i$.

**Lemma 2.5.2.**
1. There exist convergent spectral sequences as in (A1) for these cases.
2. Axiom (A2) is fulfilled.

**Proof.**
1. See Proposition 4.3.1(I.3) and Remark 4.3.2(2) of [Bon18b] (note that the construction there was inspired by the corresponding results from [CTHK]).
2. See Proposition 5.1.1(5) of [Bon18b].

**Remark 2.5.3.** Thus, our axioms (A1) and (A2) above are fulfilled for $SH(k)$ and $SH^{eff}(k)$. Therefore, Corollary 2.4.2 can be applied in this case as well.

**Proposition 2.5.4.**
1. $SH(k)_{\geq 0} = \{ E \in SH(k) | \pi_i(E)_j = 0 \text{ for } i > 0, j \in \mathbb{Z} \}$.
2. $SH(k)_{\leq 0} = \{ E \in SH(k) | \pi_i(E)_j = 0 \text{ for } i < 0, j \in \mathbb{Z} \}$.
3. There is an adjunction $i^{SH} : SH^{eff}(k) \rightleftarrows SH(k) : \omega^{SH}$. Moreover, $i^{SH}$ is right $t_{hom}$–exact, and $\omega^{SH}$ is $t_{hom}$–exact.
4. The functor $E \mapsto \pi_0(E)_0$ induces an equivalence of categories $\mathcal{Ht}^{SH}$ and $HL_n(k)$, where $HL_n(k)$ is the category of homotopy modules (see Definition 5.2.4 of [Mor]). Next, the functor $E \mapsto \pi_0(E)_0$ induces an equivalence $\mathcal{Ht}^{SH^{eff}} \cong H^{fr}(k)$, where $H^{fr}(k)$ is the category of homotopy invariant stable Nisnevich sheaves with framed transfers (see §1 of [GP] for this definitions).
5. For $E \in \mathcal{Ht}$, $X \in SmVar$, and $n \in \mathbb{Z}$ we have $H^n_{Nis}(X, \pi_0(E)_0) \cong E^n_0(X)$. 

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Proof. 1.2. This has been proved in Theorem 2.3 of [Hoy15].
3. See Corollary 3.3.7(2) of [BoD17] or Proposition 2.2.5(1) of [Bon20a].
4. See Theorem 5.2.6 of [Mor] and Theorem 5.14 of [BaY20].
5. This fact is well-known; see Proposition 5.1.1(8) of [Bon18b].

3 "Smooth" aisles and birationality filtrations
in motivic categories

This section contains the central technical results of the paper.
In §3.1 we define certain aisles in terms of motives of smooth varieties and
describe them in terms of the stalks corresponding to function fields. Next
we prove that $t_{\text{hom}}$—homology respect these classes.
In §3.2 we introduce the $n$–birational categories $\mathcal{D}^{n-bir}$. We use our aisles
to study the birationality filtration on $H_{\text{eff}}^{t_{\text{hom}}}$, and prove that the homotopy
$t$–structure restricts to $\mathcal{D}^{n-bir} \subset \mathcal{D}^{\text{eff}}$.

3.1 Smoothly generated aisles: definition and main properties

Definition 3.1.1. If $s = (s_j)$ is a non-decreasing sequence in $\mathbb{Z} \cup \{\pm \infty\}$, $j \in \mathbb{Z}$,	hen we define an aisle (see Remark 3.1.4 below) $\mathcal{A}_s$ as $(\{\mathcal{M}_R(\text{SmVar}) \langle j \rangle [t_j]\})^{t \leftarrow}$,
where we take all $t_j < s_j$, $t_j \in \mathbb{Z}$.

We will also use the following modification of this definition in the effective setting: we take $j \geq 0$ and take the orthogonal in $\mathcal{D}^{\text{eff}}$.

Lemma 3.1.2. 1. We have the following equality: $\mathcal{A}_s = \{C \in \mathcal{D}(\mathcal{M}(K) \langle j \rangle [t_j], C) = 0\}$ for all function fields $K/k$, $j \in \mathbb{Z}$, and $t_j < s_j$ (see Definition 2.4.1(4) for
the corresponding notation).
2. Similar assertion holds in the effective case if one takes $j \geq 0$.

Proof. 1. We should prove that $\mathcal{A}_s = (\{\cup_{j \in \mathbb{Z}} \mathcal{M}_R(K)(j)[t_j]\})^{t \leftarrow}$. Obviously,
the first of these classes is contained in the second one; cf. Definition 2.4.1(4).
Conversely, if $N$ belongs to the second class then the convergent coniveau
spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in X(p)} H_j^{q-t_j}(x) \Rightarrow H_j^{p+q-t_j}(\mathcal{M}(X))$$

from (A1) yields that it belongs to $\mathcal{A}_s$; here $X^p$ is the set of points of $X$ of
codimension $p$, and $H_j^{q-\cdots}$ are the following cohomology theories (on SmVar):
$H_j^{q}(X) := \mathcal{D}(\mathcal{M}_R(X)(p)[-p-q], N)$ (see Definition 2.4.1(3)).
2. In the effective case the proof is completely similar. \[\square\]

**Proposition 3.1.3.** The following conditions for an object \( C \in \mathfrak{D} \) are equivalent:

1. \( C \in A_s \).
2. \( H^r_{\text{hom}}(C)[r] \in A_s \) for all \( r \in \mathbb{Z} \).

The obvious effective version of this statement is valid as well.

**Proof.** By the previous lemma it suffices to prove that \( \mathfrak{D}(\mathcal{M}(K)\{j\}[t], C) \cong \mathfrak{D}(\mathcal{M}(K)\{j\}, H^r_{\text{hom}}(C)) \) for all function fields \( K/k \) and \( j \in \mathbb{Z} \). This isomorphism is well known. By the definition of \( t^\mathfrak{D}_{\text{hom}} \) (see Proposition 2.5.4(1,2)) and the adjunction from Remark 2.2.3 it suffices to verify it for \( C \in \mathfrak{D}_{t^\mathfrak{D}_{\text{hom}}} \), and in the latter case it follows from the well-known Corollary 2.4 of [Hoy15].

In the effective case, the proof is literally the same. \[\square\]

**Remark 3.1.4.** 1. It is noteworthy that our \( A_s \) are aisles (see §1 of [AJS03] for the definition and discussions on this notion). Indeed, the pair \((A_s, A_s^\perp)\) gives a \( t^-\)structure for each \( A_s \); see §4.2 below.

2. Note also that the assumption that the sequence \( s \) in Definition 3.1.1 is non-decreasing is not restrictive. Indeed, if we take \( s'_j = \sup_{i \leq j} s_i \) (for the corresponding values of \( j \)) then our definition easily implies the equality \( A_{s'} = A_s \). To obtain the non-obvious inclusion here it suffices to recall that \( \mathcal{M}(X) \langle n \rangle \) is a retract of \( \mathcal{M}(\mathbb{P}^n_X) \) for any \( X \in \text{SmVar} \).

For this reason, we only consider non-decreasing sequences; this also simplifies some of the formulations (see Propositions 4.2.2 and 4.3.1 below).

### 3.2 Weakly birational categories in terms of smooth aisles

In this section we apply our results to shift-stable aisles.

- Let \( n \geq 0 \). Let us recall some properties of the \( n^-\)birational motivic category \( \mathfrak{D}^{n^-\text{bir}} := \mathfrak{D}^{\text{eff}}\{n+1\}^\perp \). Firstly, we have an adjunction \( p_n : \mathfrak{D}^{\text{eff}} \rightleftarrows \mathfrak{D}^{\text{eff}}/\mathfrak{D}^{\text{eff}}\{n+1\} : i^{(n)} \); here \( p_n \) is the corresponding localization, the functor \( i^{(n)} \) is fully faithful and induces an equivalence \( \mathfrak{D}^{\text{eff}}/\mathfrak{D}^{\text{eff}}\{n+1\} \cong \mathfrak{D}^{n^-\text{bir}} \), and there exists a right adjoint \( R_{nr,n} \) to \( i^{(n)} \). These statements follow from well-known abstract nonsense; see Proposition 3.6 and Theorem 1.4(2) of [Pel13] or Theorem A.2.6 and Lemma 4.5.4 of [KaS17].

\(^4\)The motivation for our terminology is that the localization functor \( \mathcal{M}^{n^-\text{bir}}_R := p_n \circ \mathcal{M}_R \) sends all open immersions \( U \to V \) with \( \text{codim}_V(V \setminus U) \geq n+1 \) into isomorphisms.
• Also we recall very briefly the notion of so-called slices; see §1 of [PE13] and §4.2 of [KaS17] for details. We have an adjunction \( i_n : \mathcal{D}^{\text{eff}} \{ n \} \rightleftarrows \mathcal{D}^{\text{eff}} : r_n \), which defines a functor \( \nu^\geq n : \mathcal{D}^{\text{eff}} \to \mathcal{D}^{\text{eff}} \) as the composition \( i_n \circ r_n \). Then we have the following slice filtration triangle for any \( C \in \mathcal{D}^{\text{eff}} \):

\[
\nu^\geq n(C) \rightarrow C \rightarrow i^{(n-1)}p_{n-1}(C) \rightarrow \nu^\geq n(C)[1]
\]

(3.2.1)

Next we study the properties of objects from the heart of \( t^{\text{eff}}_{\text{hom}} \) with respect to the \( A_{s} - \)filtration. Recall that objects of \( H_{t}^{\text{eff}}_{\text{hom}} \) for all categories considered in this article are "sheaf-like" (see Propositions 2.5.3(4), 5.1.3(2), 5.1.1(3), 5.1.2(3) for the corresponding descriptions).

**Proposition 3.2.1.** Assume that \( r \geq -1 \) is fixed, and define a sequence \( s^{r-\text{bir}} \) as follows:

\[ s^{r-\text{bir}}_j = -\infty \text{ for } 0 \leq j \leq r \text{ and } s^{r-\text{bir}}_j = +\infty \text{ for } j \geq r + 1. \]

Then for \( S \in \text{Obj} \ H_{t}^{\text{eff}}_{\text{hom}} \subset \mathcal{D}^{\text{eff}} \) the following conditions are equivalent:

1. \( S \in A_s \).
2. The Nisnevich cohomology of \( S \) vanishes in degrees \( > r \).
3. \( S(K\{m\}) = \{0\} \) for all function fields \( K/k \) and \( m > r \), where \( S(K\{m\}) = \mathcal{D}^{\text{eff}}(\mathcal{M}(K)\{m\}, S) \).
4. \( S(K\{r + 1\}) = \{0\} \) for all function fields \( K/k \).
5. \( S_{r-1} = 0 \) (see Definition 2.4.1(5) for this notation).

**Proof.** Axiom (A2) along with Proposition 3.1.2 immediately give the equivalence of conditions (1) and (3). Recalling the spectral sequence

\[ E_1^{p,q} = \coprod_{x \in X^p} S^q_p(x) \Rightarrow S_0^{p+q}(\mathcal{M}(X)) \]

given by (A1) (here \( X^p \) is the set of points of \( X \) of codimension \( p \)). Now we note that \( \mathcal{D}^{\text{eff}}(\mathcal{M}(X), S[p + q]) \cong H_{t_{\text{Nis}}}^{p+q}(X, S) \) (see Proposition 2.5.3(5) and Remark 5.1.5). It easily follows that (3) \( \Rightarrow \) (2).

Next, recall that \( S(K\{m\}) = \lim_{X,k(\mathcal{M}(\mathcal{D}(X) \otimes T^m[-m], S).} \) Since \( \mathcal{D}(\mathcal{M}(X) \otimes T^m, S) \) is a retract of \( \mathcal{D}(\mathcal{M}(\mathcal{D}((\mathbb{P}^1)^m(X)), S[m]) \) (see Remark 2.3.1(2)), we obtain (2) \( \Rightarrow \) (3).

Moreover, condition (3) obviously implies (4). Next, if (4) is fulfilled then \( S_{r-1} = 0 \) since we can apply Corollary 2.4.2(2) to \( S_{r-1} \). Lastly, if (5) is valid then \( S_{m} = 0 \) for all \( m > r \); hence \( S_{m}(K) = \{0\} \) for all function fields \( K/k \) by Axiom (A2). 

**Remark 3.2.2.** 1. It is easily seen that \( A_{s-\text{bir}} = \mathcal{D}^{\text{eff}}\{r + 1\} \); see Remarks 2.3.2(2,3).
The objects of $\mathfrak{D}^{\text{eff}}\{r + 1\}^+$ (that also satisfy the equivalent conditions of the proposition above) will be called $r$-birational. We will also say that they are weakly birational. Note also that our Proposition 3.2.1 generalizes Proposition 2.5.2 of [KaS17] (which follows from it if we put $r = 0$).

2. Clearly, conditions 1, 2, and 3 of the proposition are equivalent in the ("degenerate") case $r = +\infty$ as well.

**Proposition 3.2.3.** $t^\text{eff}_{\text{hom}}$ restricts to a (homotopy) $t$-structure $t^\text{n-bir}_{\text{hom}}$ on the category $\mathfrak{D}^\text{n-bir}$. Respectively, the functor $p_n$ is right $t$-exact, whereas $i^n(n)$ is $t$-exact and $R_{nr,n}$ is left $t$-exact.

**Proof.** It suffices to prove that the $t_{\text{hom}}$—truncations preserve the class $\mathfrak{D}^\text{n-bir} := \mathfrak{D}^{\text{eff}}\{n + 1\}^+$, and the latter fact follows from Proposition 3.1.3 immediately.

**Remark 3.2.4.** Our Proposition 3.2.3 widely generalizes Theorem 4.4.1 of [KaS17] (only case of $\mathcal{DM}^{\text{eff}}(k)$ and $n = 0$ was considered there). Note also that the arguments used in ibid. cannot be applied for $n > 0$.

## 4 Smooth weight and $t$-structures and their applications

In §4.1 we recall the basics of the theory of weight structures. In §4.2 we give the definition of smooth weight structures $w^{\text{eff}}_{\text{Smooth}}$ and $w^{\text{eff},s}_{\text{Smooth}}$ corresponding to certain sequences $s_j$, and prove their main properties. We also express the conditions of weak birationality in terms of our $w^{\text{eff},s}_{\text{Smooth}}$. Next we define a filtration on the $H^{\text{eff}}_{\text{hom}}$ using the adjacent $t$—structure $t^{\text{eff}}_{\text{Smooth}}$, and prove some of its interesting properties. We also apply our results to the study of unramified cohomology.

In §4.3 we prove some weak weight-exactness property of the functor $-\langle n \rangle$ and so we obtain that $w^{\text{eff}}_{\text{Smooth}}$ induces a weight structure on the $n$—birational category $\mathfrak{D}^\text{n-bir}$ (actually, this is also true for $w^{\text{eff},s}_{\text{Smooth}}$ under certain assumptions on $s = (s_j)$).

### 4.1 Weight structures: a short recollection

Let us recall the definitions and some properties of weight structures.
Definition 4.1.1. A pair of subclasses $C_{\leq 0}, C_{\geq 0} \subset \text{Obj } C$ will be said to define a weight structure $w$ for a triangulated category $C$ if they satisfy the following conditions:

(i) $C_{\geq 0}, C_{\leq 0}$ are retraction-closed in $C$ (i.e., contain all $C$-retracts of their objects).

(ii) Semi-invariance with respect to translations. $C_{\leq 0} \subset C_{\leq 0}[1], C_{\geq 0}[1] \subset C_{\geq 0}$.

(iii) Orthogonality. $C_{\leq 0} \perp C_{\geq 0}[1]$.

(iv) Weight decompositions. For any $M \in \text{Obj } C$ there exists a distinguished triangle $X \to M \to Y \to X[1]$ such that $X \in C_{\leq 0}$, $Y \in C_{\geq 0}[1]$.

We will also need the following definitions.

Definition 4.1.2. Let $i, j \in \mathbb{Z}$; assume that a triangulated category $C$ is endowed with a weight structure $w$.

1. The full category $H_w \subset C$ whose object class is $C_{= 0} = C_{\geq 0} \cap C_{\leq 0}$ is called the heart of the weight structure $w$.

2. $C_{\geq i}$ (resp. $C_{\leq i}$, resp. $C_{= i}$) will denote $C_{\geq 0}[i]$ (resp. $C_{\leq 0}[i]$, resp. $C_{= 0}[i]$).

3. We will say that a weight structure $w$ is generated by a class $P \subset \text{Obj } C$ if $C_{\geq 0} = (\cup_{i > 0} P[-i])^\perp$.

4. We will call $\cup_{i \in \mathbb{Z}} C_{w \geq i}$ the class of $w$-bounded below objects of $C$; it will be denoted by $C_+$.

5. We will say that $w$ is smashing if both the category $C$ and the class $C_{w \geq 0}$ is closed with respect to (small) $C$-coproducts (cf. Proposition 4.1.4(2)).

6. Let $C'$ be a triangulated category endowed with a weight structures $w'$; let $F : C \to C'$ be an exact functor.

$F$ is said to be weight-exact (with respect to $w, w'$) if it maps $C_{w \leq 0}$ into $C'_{w' \leq 0}$ and sends $C_{w \geq 0}$ into $C'_{w' \geq 0}$.

---

In the current paper we use the so-called homological convention for weight structures; whereas in [Bon10a] the cohomological convention was used. In the latter convention the roles of $C_{w \leq 0}$ and $C_{w \geq 0}$ are interchanged, i.e., one considers $C_{w \leq 0} = C_{w \geq 0}$ and $C_{w \geq 0} = C_{w \leq 0}$.
7. Let $D$ be a full triangulated subcategory of $C$.

We will say that $w$ restricts to $D$ whenever the couple $(C_{w \leq 0} \cap \text{Obj } D, C_{w \geq 0} \cap \text{Obj } D)$ is a weight structure on $D$.

Remark 4.1.3. A weight decomposition (of any $M \in \text{ Obj } C$) is (almost) never canonical.

Still for any $m \in \mathbb{Z}$ axiom (iv) gives the existence of distinguished triangle

$$w_{\leq m} M \rightarrow M \rightarrow w_{\geq m+1} M$$

with some $w_{\geq m+1} M \in C_{w \geq m+1}$ and $w_{\leq m} M \in C_{w \leq m}$; we will call it an $m$-weight decomposition of $M$.

We will often use this notation below (even though $w_{\geq m+1} M$ and $w_{\leq m} M$ are not canonically determined by $M$); we will call any possible choice either of $w_{\geq m+1} M$ or of $w_{\leq m} M$ (for any $m \in \mathbb{Z}$) a weight truncation of $M$. Moreover, when we will write arrows of the type $w_{\leq m} M \rightarrow M$ or $M \rightarrow w_{\geq m+1} M$ we will always assume that they come from some $m$-weight decomposition of $M$.

Proposition 4.1.4. Let $C$ be a triangulated category, $n \geq 0$; we will assume that $w$ is a fixed weight structure on $C$.

1. The axiomatics of weight structures is self-dual, i.e., for $D = C^{op}$ (so $\text{Obj } D = \text{Obj } C)$ there exists the (opposite) weight structure $w'$ for which $D_{w' \leq 0} = C_{w \geq 0}$ and $D_{w' \geq 0} = C_{w \leq 0}$.

2. $C_{w \leq 0}$ is closed with respect to $C$-coproducts.

3. $C_{w \geq 0} = (C_{w \leq -1})^\perp$ and $C_{w \leq 0} = \perp C_{w \geq 1}$. Thus if $w$ is generated by a class $P$ then $P \subset C_{w \leq 0}$.

4. Let $m \leq l \in \mathbb{Z}$, $X, X' \in \text{ Obj } C$; fix certain weight decompositions of $X[-m]$ and $X'[-l]$. Then any morphism $g : X \rightarrow X'$ can be extended to a commutative diagram of the corresponding distinguished triangles (see Remark 4.1.3(2)):

$$\begin{array}{ccc}
w_{\leq m} X & \rightarrow & X \rightarrow w_{\geq m+1} X \\
\downarrow & & \downarrow g \\
w_{\leq l} X' & \rightarrow & X' \rightarrow w_{\geq l+1} X'
\end{array}$$

Moreover, if $m < l$ then this extension is unique (provided that the rows are fixed).
Proof. See Remark 1.1.2(1), Proposition 1.3.3(1,2,5) and Lemma 1.5.1(1,2) of [Bon10a].

Proposition 4.1.5. Assume that $\mathcal{C}$ is a compactly generated category.

1. Let $\mathcal{P}$ be a set of compact objects of $\mathcal{C}$. Then there exists (a unique) weight structure on $\mathcal{C}$ that is generated by $\mathcal{P}$, and $\mathcal{P} \subset \mathcal{C}_{w \leq 0}$.

2. Assume that an exact functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ respects coproducts, $w$ is a weight structure on $\mathcal{C}$ that is generated by some class $\mathcal{P} \subset \text{Obj} \mathcal{C}$, and $w'$ is a weight structure on $\mathcal{C}'$. Then $F$ is left weight-exact if and only if $F(\mathcal{P}) \subset \mathcal{C}_{w \leq 0}$.

Proof. See Theorem 5 of [Pau12] (cf. also Proposition 1.2.3(II) of [BoK18]).

4.2 Smooth weight and $t$–structures, unramified cohomology, and the weakly birational filtration

Definition 4.2.1. For any category of the type $\mathcal{D}^{eff}$ or $\mathcal{D}$ we will write $w^{\text{eff},s}_{\text{Smooth}}$ (resp. $w^s_{\text{Smooth}}$) for the weight structure defined by $\mathcal{A}_s$ on $\mathcal{D}^{eff}$ (resp. on $\mathcal{D}$); that is, we assume that the class of weight-non-negative objects equals $\mathcal{A}_s$.

We will write just $w^{\text{eff}}_{\text{Smooth}}$ and $w^\text{Smooth}$ in the case where all $s_j$ are zero.

Proposition 4.2.2. Assume that $S \in \text{Obj} \mathcal{H}_{\text{non}}^{eff} \subset \mathcal{D}^{eff}$. Then the following conditions are equivalent:

1. $S \in \mathcal{D}^{eff}_{w^{\text{eff},s}_{\text{Smooth}} \geq 0}$.

2. Condition 2 of Proposition 3.2.1 is fulfilled if one takes $r$ to be the minimal integer $m \geq -1$ such that $s_{m+1} \geq -m - 1$ if $m$ of this sort exists and $r = +\infty$ otherwise (see Remark 3.2.2(2)).

Proof. 1 $\Rightarrow$ 2. This implication follows from the definition of $w^{\text{eff},s}_{\text{Smooth}}$, Proposition 2.3.4(5) and Remark 5.1.3.

2 $\Rightarrow$ 1. Condition (2) implies that $\mathcal{D}^{eff}(\mathcal{M}(X), S[i]) = \{0\}$ for $i > r$. This implies that $S$ belongs to the class $\mathcal{A}_s$, where we choose the sequence $s$ as in the formulation. Namely, we set $s_{r+1} = -r - 1$ if the first of the alternatives in (2) holds, and $(s_j) = -\infty$ otherwise (recall that we assume all our sequences to be non-decreasing).

Remark 4.2.3. 1. Clearly, in the case $r < +\infty$ conditions 3–5 of Proposition 3.2.1 are equivalent to our ones as well.

2. Actually, our principal examples are $s = s^{r-bir}$ (see Proposition 3.2.1) and $s = 0$ (that corresponds to the smooth weight structure $w^{\text{eff}}_{\text{Smooth}}$).
Now we define some new $t$-structures.

**Definition 4.2.4.** 1. We will say that a $t$-structure $t$ is right adjacent to $w$ whenever $C_{w,0} = C_{t,0}$.

2. If $H : C \to A$ is contravariant functor, $A$ is an abelian category, then we define $W^j(H)(X) := \text{Im}(H(w_j(X)) \to H(X))$ (see Remark 4.1.3). The map thus defined gives a canonical subfunctor of $H$ (in particular, this image does not depend on the choice of a weight decomposition of $X[j]$); see Proposition 2.1.2 of [Bon10a].

We will omit the adjective "right" in this definition and simply write "adjacent", since further we will need this case of the definition only. Recall also that $\mathcal{D}^{\text{eff}}$ satisfies the Brown representability property since it is compactly generated (see Ch. 8 of [Nee01]).

**Construction-Definition 4.2.5.** 1. Since all the weight structures $w^{\text{eff},s,\text{Smooth}}$ are smashing (see Definition 4.1.2(iii)), there exists a $t$-structure $t^{\text{eff},s,\text{Smooth}}$ adjacent to it; see Theorem 3.2.3(I) of [Bon19]. Thus we have a canonical map $\tau^{t^{\text{eff},s,\text{Smooth}}}_{\geq -j}(E) \to E$ for any $E \in \mathcal{D}^{\text{eff}}$; see Definition 2.2.1(iii) and Remark 2.2.3(1). Further, we can apply $H^{t^{\text{eff}}}_{\text{hom}}$ to this map, and get a canonical morphism $H^{t^{\text{eff}}}_{\text{hom}}(\tau^{t^{\text{eff},s,\text{Smooth}}}_{\geq -j}(E)) \to H^{t^{\text{eff}}}_{\text{hom}}(E)$. As above, we will write just $t^{\text{eff}}_{\text{Smooth}}$ in the case $s_j = 0$ (for $j \geq 0$).

2. If we apply the previous construction to $E \in \text{Obj } H^{t^{\text{eff}}}_{\text{hom}}$ we get maps from $H^{t^{\text{eff}}}_{\text{hom}} \cap \mathcal{D}^{\text{eff}}_{\text{Smooth} \geq -j}$ to $H^{t^{\text{eff}}}_{\text{hom}}$. They give rise to the following filtration: $F^{j,s,}(E) = \text{Im}_{H^{t^{\text{eff}}}_{\text{hom}}(\tau^{t^{\text{eff},s,\text{Smooth}}}_{\geq -j}(E))) \to H^{t^{\text{eff}}}_{\text{hom}}(E) = E$.

Now we will study in more detail our filtration for the $t$-structure, generated by $A_s$.

**Theorem 4.2.6.** Let $-1 \leq i \leq +\infty$, $j \in \mathbb{Z}$.

1. The category $H^{i-\text{bir},s}$ of $i$-birational objects is a Serre abelian subcategory of $H^{t^{\text{eff}}}_{\text{hom}}$, and we denote by $j_i$ this inclusion.

   Fix $E \in \text{Obj } H^{t^{\text{eff}}}_{\text{hom}}$.

2. The object $F^{j,s,}(E)$ is the maximal $r$-birational subobject of $E$ in $H^{t^{\text{eff}}}_{\text{hom}}$, where we take $r$ to be the minimal $m \geq -1$ such that $s_{m+1} \geq -j$ if $m$ of this sort exists and $r = +\infty$ otherwise (see Remark 3.2.2(2)).

   Thus, $F^{j,s,}$ is right adjoint to the inclusion $j_i$. 

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3. The higher unramified part functor $R_{nr,j}$ (see §3.2, also §7.1 of [KaS17]) gives the $t$–truncation $\tau_{\geq 0}^{\text{eff}, \text{r–bir, smooth}}$.

4. Similarly, $w_{\text{Smooth}}^{\text{eff}, \text{r–bir}}$–truncations give the corresponding slices (see the beginning of §3.2).

5. $E$ belongs to $H^\text{eff}_{\text{Smooth}}$ if and only if it is birational (see Remark 3.2.2(1)).

Proof. 1. Suppose that $0 \to S' \to S \to S'' \to 0$ is an exact sequence in $H^\text{eff}_{\text{hom}}$. Since the stalks as in condition (4) of Proposition 3.2.1 give exact functors $H^\text{eff}_{\text{hom}} \to \text{Ab}$ (see our Corollary 2.4.2(2)), $S$ belongs to $H^{i-\text{bir,s}}_{\text{Smooth}}$ if and only if $S'$ and $S''$ do.

2. The sheaf $F^{j,s}(E)$ is $r$–birational by Propositions 4.2.2, 3.1.3 and assertion (1) above. Let $\tilde{E} \xrightarrow{j} E$ be some $r$–birational subobject of $E$. Since $\tilde{E} \in D^\text{eff}_{\text{Smooth} \geq -j}$, the adjunction corresponding to $\tau_{\geq -j}^{\text{eff,s} \text{smooth}}$ (see Remark 2.2.3(1)) yields that $f$ factors through $\tau_{\geq -j}^{\text{eff,s} \text{smooth}}(E)$. Applying $H^\text{hom}_0$ to the corresponding commutative triangle we obtain that $f$ also factors through $F^{j,s}(E)$.

3. Our assertion follows from the adjunctions $D^\text{eff}_{\text{Smooth} \geq 0} \rightleftarrows D^\text{eff}_{\text{Smooth} \leq 0}$ and $j(i) \dashv R_{nr,j}$.  

4. Note that our weight structure is also a $t$–structure (by definition). Thus, our statement is an obvious consequence of the definition of slices and corresponding adjunctions (cf. assertion (3) above and §3.2).

5. Obvious; note that $D^\text{eff}_{\text{hom} \leq 0} \subset D^\text{eff}_{\text{Smooth} \leq 0}$. \hfill $\square$

Remark 4.2.7. 1. In particular, for $E \in \text{Obj } H^\text{eff}_{\text{hom}}$ we have $F^j(E) = E$ if and only if $E$ is $j$–birational.

2. It would be interesting to relate our filtration with the filtration from Theorem 15 of [Bac18] (in the case $D^\text{eff} = SH(k)^{\text{eff}}$).

3. For the weight structure as in assertion (4) above $W^*(D^\text{eff}(-, R(n)[m]))$ gives the filtration from Definition 5.2.1 and Corollary 5.3.3 of [Pel17]. Note that for $m = 2n$ and $D^\text{eff} = DM^\text{eff}$ this is a candidate for a Bloch-Beilinson-Murre filtration (see Theorem 6.1.4, Conjecture 6.1.7, Proposition 6.1.8 and Rem. 6.1.9 of ibid.).

Next we apply our results to the study of unramified cohomology.

Corollary 4.2.8. 1. If $S \in \text{Obj } H^\text{eff}_{\text{hom}}$ then the $H^\text{eff}_{\text{hom}}$–monomorphism $c(S) : F^0(S) \to S$ from Construction-Definition 4.2.5 gives the unramified part of $D^\text{eff}(-, S)$ in the sense of [KaS17, Definition 7.2.1].
2. Let $M \in H_{\text{w}_\text{Smooth}}^{\text{eff}}$ and for some $X \in \text{SmVar}$ and $f : \mathcal{M}_R(X) \to M$ assume that $p_0(f)$ is an isomorphism. Then $\mathcal{D}^\text{eff}_R(f, S)$ is monomorphic, and its image yields the unramified part of $\mathcal{D}^\text{eff}_R(M_R(X), S)$.

**Proof.**

1. See Proposition 2.6.3 and Theorem 7.3.1 of [KaS17], and [Aso, Lemma 4.2] (note that the proofs given there are applicable to any of the motivic categories from §2.3).

2. We argue similarly to [Bon20b, Theorem 2.2.3]. Consider the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{D}^\text{eff}_R(M, F^0(S)) & \xrightarrow{\cong} & \mathcal{D}^\text{eff}_R(M_R(X), F^0(S)) \\
\downarrow^{c(S)_*} & & \downarrow \\
\mathcal{D}^\text{eff}_R(M, S) & \xrightarrow{\cong} & \mathcal{D}^\text{eff}_R(M_R(X), S)
\end{array}
$$

Firstly, by adjunctions and Theorem 4.2.6 we have $\mathcal{D}^\text{eff}_R(i(0)p_0(M), S) \cong \mathcal{D}^\text{eff}_R(M, i(0)R_{nr, 0}(S)) \cong \mathcal{D}^\text{eff}_R(M, F^0(S))$. Now, let us look at the following exact sequence coming from the slice filtration triangle (3.2.1):

$$
0 \to \mathcal{D}^\text{eff}_R(M, F^0(S)) \to \mathcal{D}^\text{eff}_R(M, S) \to \mathcal{D}^\text{eff}_R(\nu \geq 1(M), S)
$$

It remains to prove that the map $c(S)_*$ is epimorphic. This fact follows from the sequence above along with the orthogonality axiom for $t$–structures, since $\nu \geq 1(M) \in \mathcal{D}^\text{eff}_{\text{hom} \geq 1}$ (see Lemma 2.2.4(2) of [BoK18] and [BaY20, Lemma 6.1(2)]).

**Proposition 4.2.9.**

1. Let $E, X \in \mathcal{D}^\text{eff}$. Then $W^i(\mathcal{D}^\text{eff}(-, E))(X) \cong \text{Im}(\mathcal{D}^\text{eff}(X, \tau^\text{eff}_{\text{Smooth}}^\geq -i)(E) \to \mathcal{D}^\text{eff}(X, E))$.

2. For all $i, j$ and $X, Y \in \mathcal{D}^\text{eff}$ we have a functorial isomorphism

$$
\mathcal{D}^\text{eff}(X, \tau^\text{eff}_{\text{Smooth}}^\geq -j)(Y)[j + i] \cong \text{Im}(\mathcal{D}^\text{eff}(w^\text{eff}_{\text{Smooth} \geq j}(X), Y[i]) \to \mathcal{D}^\text{eff}(w^\text{eff}_{\text{Smooth} \geq j-1}(X), Y[i + 1])).
$$

3. $X \in \mathcal{D}^\text{eff}_{w^\text{Smooth}=i} \implies \forall Y \in \mathcal{D}$ we have $\mathcal{D}(X, Y) \cong \mathcal{D}(X, H^i_{\tau^\text{Smooth}}(Y))$.

4. $\mathcal{D}^\text{eff}_{w^\text{Smooth} \geq 0} = \text{Obj}(\mathcal{D}^+ \cap (\sqcup_{i < 0} \mathcal{D}^\text{eff}_{\text{Smooth}=i}))$ (see Definition 4.1.2(4) for the first piece of this notation).

5. If $C \in \mathcal{D}^\text{eff}_{\text{hom} \leq i}$, $U \in \text{SmVar}$, $i \geq 0$ then

$$
\mathcal{D}(M(U), \tau^\text{eff}_{\geq -i-1}(C[-i - 1])) \cong \mathcal{D}(w^\text{eff}_{\text{Smooth} \geq -i}(M(U), C).
$$
Proof. 1, 2. See Theorem 4.4.2(6,7) of [Bon10a].

3. See Theorem 2.6.1(4) of [Bon10b].

4. Obviously, the first of these classes is contained in the second one. Let $X$ be an object of $\mathcal{D}^+$. To check the inverse inclusion, it suffices to prove that $X \perp Y$ for $X \perp (\bigcup_{i<0} \mathcal{D}^t_{\text{Smooth}}=i)$, and for all $Y \in \mathcal{D}^t_{\text{Smooth}} \leq -1$. Since $X$ is $w^\text{Smooth}$-bounded below, we have $X \perp \tau^t_{\text{Smooth}} \leq -1$ for some $Y \in \mathcal{D}^t_{\text{Smooth}} \leq -1$. Thus, considering appropriate $t-$decompositions for $Y$ and its truncations, we obtain the statement in question.

5. Take the weight decompositions as in Proposition 4.4.2(4) for $m = -i-2, l = -i-1, g = \text{id}_{M(U)}$. Applying the functor $\mathcal{D}(-, C)$ one can check that the map $\mathcal{D}(w^\text{Smooth} > -1, M(U), C) \to \mathcal{D}(w^\text{Smooth} > -1, M(U), C)$ is injective. Now the claim follows from Theorem 4.4.2(7) of [Bon10a] immediately. 

4.3 More on $n$-birational categories and weight-exact localisations

In this section we prove some weak weight-exactness properties of the functor $-\langle n \rangle$. As a consequence, we get the $w^\text{eff}$ and $w^\text{Smooth}$-exactness of this functor, as well as of the localization functor $p_n$. Note that in this paragraph, we exclude the case of $SH^S_1(k)$ (cf. Remark 2.3.2(3,4)).

Proposition 4.3.1. Assume that $k, n \geq 0$ are fixed and $s_{j+n} - s_j \leq k$ for $j \geq 0$; take $C \in \mathcal{D}^t_{\text{eff}, s} \geq 0$ for $s = (s_j)$. Then $C\langle n \rangle \in \mathcal{D}^t_{\text{eff}, s} \geq -k$.

Proof. Let $C \in \mathcal{D}^t_{\text{eff}, s} \geq 0$; and consider the following functors from the category $\text{SmVar}$ of smooth $k$-varieties: $H^q_p(X) := \mathcal{D}^t_{\text{eff}}(\mathcal{M}(X) \{p\}[-q], C\langle n \rangle[-k])$ for each $p \geq 0, q \in \mathbb{Z}, X \in \text{SmVar}$. We should verify that $H^q_{p+q}(X) = \{0\}$ for all $q > 0, X \in \text{SmVar}$. Let us write the coniveau spectral sequence as in (A1):

$$E^{p,q}_1 = \prod_{x \in X(p)} H^q_{j+p}(x) \Rightarrow H^p_{j+q}(\mathcal{M}(X)).$$

Then $E^{0,q}_1 = \{0\}$ for $q \geq 0$ by (A2). Further, for a function field $K/k$ and $p > 0$ it is easily seen that $H^q_{p+j}(K) = \mathcal{D}^t_{\text{eff}}(\mathcal{M}(K) \{j+p-n\}[k-j-p-q], C) = \{0\}$ (see Definition 2.4.1(4) for this notation; cf. also the proof of Proposition 3.1.2). It obviously follows that $H^p_{j+q}(X) = \{0\}$ if $p + q > 0$, and that is what we need.

Remark 4.3.2. Clearly, a similar property holds in the non-effective case (cf. Definition 3.1.1).
Corollary 4.3.3. 1. The endofunctors $-\langle n \rangle$ are $w_{\text{Smooth}}^{\text{eff}}$-exact (resp. $w_{\text{Sm}}^{\text{eff}}$-exact) for $n \geq 0$ (resp. $n \in \mathbb{Z}$).

2. The embedding $i : D_{\text{eff}} \hookrightarrow D$ is weight-exact with respect to $w_{\text{Smooth}}^{\text{eff}}$ and $w_{\text{Sm}}^{\text{eff}}$.

3. Take a sequence $(s_j)_{j \geq 0}$ such that $s_{j+n} - s_j \leq 1$ for all $j \geq 0$. Then the functor $p_n$ is weight-exact with respect to the weight structures $w_{\text{Smooth}}^{\text{eff},s}$ and $w_{\text{Sm}}^{\text{eff},s}$, where the weight structure $w_{\text{Sm}}^{\text{eff},s}$ is defined via $\mathcal{M}_{R}^{\text{eff},s}$ similarly to Definitions 4.2.1 and 3.1.1.

4. $D_{\text{eff}}^{w_{\text{Smooth}}^{\text{eff}}} \subset D_{\text{eff}}^{w_{\text{Sm}}^{\text{eff}}}$.

Proof. 1. Take $s_j = 0$ for $j \geq 0$ (resp. $j \in \mathbb{Z}$) in the previous proposition.

2. From Proposition 4.1.5(2) we obtain that $i$ is left weight-exact. To prove the right weight-exactness, we take $C \in D_{w_{\text{Smooth}}^{\text{eff}}}^{\geq 0}$, and consider the group $D(\mathcal{M}(R)(X)\langle n \rangle[s], C)$, where $X \in \text{SmVar}$, $n \in \mathbb{Z}$, $s < 0$. Then in the case of $n \geq 0$ this group vanishes by the previous assertion. If $n < 0$, then $D(\mathcal{M}(R)(X)\langle n \rangle[s], C) \cong D^{\text{eff}}(\mathcal{M}(R)(X)[s], C\langle -n \rangle)$, and $C\langle -n \rangle \in D_{w_{\text{Smooth}}^{\text{eff}}}^{\geq 0}$, and this group vanishes also. Thus, we are done.

3. Take a weight decomposition $X \to C \to Y \to X[1]$ with respect to $w_{\text{Smooth}}^{\text{eff},s}$, and twist it by $-\langle n \rangle$; then we have $Y\langle n \rangle \in D_{w_{\text{Smooth}}^{\text{eff}}}^{\leq 0} \cap D^{\text{eff}}\langle n \rangle$ by Proposition 4.3.1 and $X\langle n \rangle \in D_{w_{\text{Smooth}}^{\text{eff}}}^{\leq 0} \cap D^{\text{eff}}\langle n \rangle$ by definition. Thus our statement follows from Theorem 3.1.3(2) of [BoS19].

4. Immediately from definition and Proposition 2.5.4(1) (see also Lemma 2.2.4(3) of [BoK18]).

5 Our constructions in motivic categories distinct from $SH$

For the convenience of the reader, now we discuss our main constructions and results in the case where $D$ is distinct from $SH$. As noted earlier, these results are quite similar to those obtained in Proposition 2.5.4.

5.1 Our notions and assumptions in various motivic categories

In this section, we check our axioms and discuss the related definitions for the motivic categories distinct from $SH^{s_1}(k) \subset SH(k)$.

Firstly, recall that there exists a canonical exact monoidal connecting functor from $SH^{s_1}(k)$ into each our motivic category $D(k)$; it send $\mathcal{M}_{SH}^{s_1}(X)$
into the corresponding $\mathcal{M}_D(X)$. This gives the isomorphism $\mathcal{M}_D(\mathbb{P}^1) \cong 1_D \oplus T$ mentioned in the beginning of §2.3. Thus the spectral sequences of axiom (A1) in all of these $\mathcal{D}(k)$ can be obtained from the one for $SH^{S^1}(k)$.

Now let us check axiom (A2) for each of our categories separately; cf. Lemma 2.5.2 above.

The case of $SH^{S^1}(k)$

We will write $\pi^{A^1}_n(E)$ for the Nisnevich sheaf associated to the presheaf $X \mapsto E_{-n}(X) = SH^{S^1}(\mathcal{M}(X)[n], E), X \in \text{SmVar}, n \in \mathbb{Z}$.

**Proposition 5.1.1.**

1. $E \in SH^{S^1}(k)_{\geq 0}$ if and only if $\pi^{A^1}_n(E) = 0$ for $n < 0$.
2. $E \in SH^{S^1}(k)_{\leq 0}$ if and only if $\pi^{A^1}_n(E) = 0$ for $n > 0$.
3. The functor $\pi^{A^1}_n(E)$ gives an equivalence of $H^*_{\text{hom}}$ and $SHI(k)$, where $SHI(k)$ is the category of strictly homotopy invariant sheaves (see Definition 4.3.5 of [Mor]).
4. (A2) is fulfilled.

**Proof.** 1,2. Similarly to Theorem 2.3 of [Hoy15].
3. See Lemma 4.3.7(2) of [Mor].
4. See Lemma 6.1.6 of [Mor1].

The case of $D_{A^1}(k)$ and $D^{eff}_{A^1}(k)$

**Proposition 5.1.2.**

1. $E \in D_{A^1}(k)_{\geq 0}$ if and only if $H^{A^1}_{m,n}(E) = 0$ for $(m, n) \in \mathbb{Z} \times \mathbb{Z}^-$.
2. $E \in D_{A^1}(k)_{\leq 0}$ if and only if $H^{A^1}_{m,n}(E) = 0$ for $(m, n) \in \mathbb{Z} \times \mathbb{Z}^+$.
3. There exist equivalences $H^{A^1}_{*,0} : H^*_{\text{hom}} \to HI_*(k)$ and $H^{A^1}_{0,*} : H^0_{\text{hom}} \to HI_*^{eff}(k)$.
4. (A2) is fulfilled.

**Proof.** 1, 2. See subsection 16.2.4 of [CiD19] or Corollary 2.1.72 of [Ayo07].
3. See Example 4.1.2 of [BoD17], Theorem 8.12 and Corollary 8.14 of [AN].
4. Similarly to the previous case; see Remark 8 of [Mor1].

The case of $DM(k)$ and $DM^{eff}(k)$
**Proposition 5.1.3.** 1. There exist equivalences $H^0_t : H_{DM} \to H_{tr}^r(k)$ and $H^0_t : H_{DM}^{eff} \to H_I^r(k)$, where $H_I^r(k)$ and $H_I(k)$ are the categories of homotopy modules with transfers and homotopy invariant sheaves with transfers, respectively (see [Voe00, Definition 3.1.9] and [Deg11, Definition 1.3.2]).

2. (A2) is fulfilled.

**Proof.** 1. See [Voe00] Proposition 3.1.12 and [Deg11] Corollary 5.2 and Theorem 5.11.
2. This is well known; see §5 of [Deg11] (cf. also Lemma 2.2.4 of [BoK18]). □

**The case of MGl − Mod(k)**

**Proposition 5.1.4.** 1. There exists an equivalence $O|_H : H^MGl \simeq H_{tr}^r(k)$.

2. (A2) is fulfilled.

**Proof.** 1. See Remark 4.3.3 of [BoD17].
2. Easy from the corresponding result for $SH(k)$, $t_{hom}$—exactness and conservativity of the "forgetful" functor $O$, which is right adjoint to the natural connecting functor $L : SH(k) \to MGl − Mod(k)$ (for which $L(M_{SH}(X)\{i\}[n]) = MGl(X)\{i\}[n]$); see Example 2.3.3 of [BoD17] and §2.2.5 of [Deg13]. □

**Remark 5.1.5.** Recall also that for each of the categories $\mathfrak{D}_{eff}(k)$ there is a "forgetful" $t_{hom}^{eff}$—exact functor $\Psi_{S}$ to $SH^{S^1}(k)$, which is right adjoint to the natural functor $SH^{S^1}(k) \to \mathfrak{D}_{eff}$ (easily follows from our description of $t_{hom}^{eff}$ in terms of homotopy sheaves, see also Lemma 6.2 (2) of [BaY20]).

Let $E \in H^0_{t_{hom}^{eff}}$; then $E^0_n(X) \cong H^k_{Nis}(X, \pi_0^N(\Psi_S(E)))$ for all $X \in SmVar, n \in \mathbb{Z}$. Indeed, in the case $\mathfrak{D}_{eff} = SH^{S^1}(k)$ this statement is well known and is given by Proposition 1.4.6(6) of [Bou18b]. The general case follows from this one via the adjunction isomorphism $E^0_n(X) \cong \Psi_S(E)_n^0(X)$; cf. Proposition 2.5.4(5).

### 5.2 On localizations of coefficients in motivic categories

Now we recall some basics on localizations of coefficient rings (actually, we "start from" the case where the coefficient ring is just $\mathbb{Z}$) in compactly generated triangulated categories.

Below $S \subset \mathbb{Z}$ will be a set of prime numbers; denote the ring $\mathbb{Z}[S^{-1}]$ by $R$. 24
Proposition 5.2.1. Assume that $C$ is compactly generated by a small subcategory $C'$. Denote by $C_{S \text{-- tors}}$ the localizing subcategory of $C$ (compactly) generated by $\text{Cone}(c' \xrightarrow{s} c')$ for $c' \in \text{Obj}C'$, $s \in S$. Then the following statements are valid.

1. $C_{S \text{-- tors}}$ contains the cones of $c \xrightarrow{s} c$ for all $c \in \text{Obj}C$, $s \in S$.

2. The Verdier quotient $C_R = C / C_{S \text{-- tors}}$ exists (i.e., morphisms form sets); the localization functor $l : C \to C_R$ respects coproducts and converts compact objects into compact ones. Moreover, $C_R$ is generated by $l(\text{Obj}C')$ as a localizing subcategory.

3. For any $c \in \text{Obj}C$, $c' \in \text{Obj}C'$, we have $C_R(l(c'), l(c)) \cong C(c', c) \otimes Z_R$.

4. $l$ possesses a right adjoint $G$ that is a full embedding functor. The essential image of $G$ consists of those $M \in \text{Obj}C$, such that $sId_M$ is an automorphism for any $s \in S$.

5. Assume char($k$) = $p$, and $p \in S$ if $p > 0$. Then the category $\mathcal{D}_R^p(k)$ is rigid. Moreover, $\mathcal{D}_R^p(k)$ is the smallest thick subcategory of $\mathcal{D}_R(k)$ containing all $\mathcal{M}_R(P)\{i\}$ for $P \in \text{SmPrVar}$, $i \in \mathbb{Z}$. Thus, the set $\mathcal{M}_R(\text{SmPrVar})$ compactly generates $\mathcal{D}_R(k)$.

Surely, the corresponding statements are fulfilled for the effective case.

Proof. 1, 2, 3, 4. See Proposition 1.2.5 of [Bon20a].

5. See [Bon20a, Proposition 2.2.3(9)] (cf. also Corollary 2.4.8 of [BoD17] and [Bon11, Lemma 2.3.1]).

Remark 5.2.2. 1. Since Chow$_R$ is Karoubian and connective in $DM_R$ (i.e., Chow$_R \perp \text{Chow}_{R[i]}$ for all $i > 0$; see Remark 3.1.2 of [BoK18]), Corollary 2.1.2 of [BoS18b] gives a unique weight structure $w_{\text{Chow, gm}}$ on the smallest full retraction-closed triangulated subcategory of $DM_R$ containing Chow$_R$ such that $Hw_{\text{Chow, gm}} = \text{Chow}_R$.

An important observation is that if char($k$) = 0 or char($k$) = $p \in S$, then $w_{\text{smooth, eff}}$ and $w_{\text{smooth}}$ is also generated by the motives of smooth projective varieties (see Theorem 2.1.2(2) of [BoK18] and the proposition above). Moreover, in these cases $DM_R(k)$ and MGI $-$ Mod$_R(k)$ differ from other motivic categories in that we can describe the hearts of $w_{\text{smooth}}$ ”geometrically”; these are just the corresponding ”big” categories of Chow motives. Respectively, in this case Proposition 4.2.8(2) gives a quite explicit calculation of unramified cohomology. Moreover,
in these case \( w_{\text{Smooth}} \) and \( w_{\text{Smooth}} \) restrict to the corresponding subcategories \( DM_\text{eff, gm, R}(k) \subset DM_\text{gm, R}(k) \) of compact objects (which equal the smallest strict triangulated subcategories of \( DM_\text{R}(k) \subset DM_R(k) \) that contain the corresponding Chow motives), and \( Hw_{\text{Chow, gm}} \) is equivalent to the corresponding additive category of Chow motives \( CH_\text{R} \). These types of Chow weight structures originate from §6.5 of [Bon10a] and [Bon11] (cf. also Remark 3.1.4 and Proposition 3.2.6 of [Bon21]).

Note also that one of the obstructions for inclusion \( CH_\text{R} \subset Hw_{\text{Chow}} \) is the non-triviality of the Hopf map \( \eta \) (see §6.2 of [Mor]). It is closely related to orientability, cf. [BKWX20].

2. Take \( S = \{2\} \), and recall the decomposition \( SH(k)[1/2] = SH(k)^+ \times SH(k)^- \) (induced by the symmetry involution of \( \mathbb{P}^1 \wedge \mathbb{P}^1 \); see §6 of [Lev]). It easily follows that we can also add \( SH(k)^+ \) to the examples listed in §5.1.

3. Summarizing the above, we obtain that all the results of the previous sections can be applied directly to every motivic category mentioned in §2.3.

### 5.3 Supplementary remarks and questions

Possibly the matters mentioned below will be studied in consequent papers.

**Remark 5.3.1.** 1. It would be interesting to generalize our weight and \( t \)-structure definitions and results to relative motives over general base schemes. We also plan to study the weight-exactness for various functors between motivic categories.

2. In [BKWX20] a \( t \)-structure generated by objects of the form \( Th_X(\xi) \), where \( X \in \text{SmProj}/k \) and \( \xi \in K(X) \) was considered. Probably, this \( t \)-structure is dual (in some sense) to our \( t_{\text{Smooth}} \).

3. In the papers [BoS14] and [BoK20] certain Chow-weight homology functors from the categories \( DM_\text{gm, eff}(k) \) and \( DM_\text{eff}(k) \) were studied in detail. Using the results of this article (especially §4.3), one can generalize this theory to any motivic category. The most interesting case is \( SH(S) \) (for a "reasonable" base scheme \( S \)).

4. For which objects \( S \in H_\text{Chow} \) is our filtration finite (i.e., there exists \( N \) with \( F^N(S) = S \))? Is it exhaustive?
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