Laboratory experiments and numerical simulations of inertial wave-interactions in a rotating spherical shell

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Abstract. In homogeneous rotating fluids inertial waves can occur. These waves are an essential element for the motion in the liquid outer core of the Earth, the atmosphere and the oceans. We investigate such waves for the rotating spherical shell geometry. For this geometry, a theoretical understanding is hampered due to the occurrence of singularities and a complicated Ekman layer structure. In the experiment, the inner sphere's rotation is modulated in form of a sinus curve. The waves propagate in the fluid with a fixed angle respect to the rotation axis. The angle depends on the wave frequency. Due to multiple reflections at the spherical boundaries, wave energy can be focused on certain orbits.

Preliminary results show the generation of higher harmonics in the equatorial region of the shell. We also find a vertical layer, touching the inner sphere’s equator. The layer is reminiscent of a Stewartson layer but shows a different spatial structure. Finally, the laboratory experiment is compared with numerical simulations. Good agreement can be found.

1. Introduction

In geophysical fluids, like the atmosphere, the oceans or the liquid core of the Earth, periodic flows can be found on all scales. For planetary scales, such flows are caused e.g. by tidal forces. Waves are spatially and temporally dependent phenomena and are characterized by periodic motions. Particles deflect from their rest position and restoring forces drive them back to their initial positions at which they overshoot. Such particle oscillation can be organized in the form of waves. The restoring forces define the kind of waves. This yields a wave driven and slowly varying flow, called mean flow. Waves transport momentum and can deliver it locally.

In rotating systems, inertial waves play a outstanding role and are the result from a subtle interplay between inertial and Coriolis forces. A similar class of waves can be also find in stratified fluids (Maas & Lam (1995); Maas (2001)), where the role of the rotation is played by stratification. The propagation of internal waves in enclosed systems causes repeated reflections at the boundaries and generates complicated structures. In case of multiple reflections, e.g. on the curved boundaries of a spherical shell, the wave energy can be focused on certain orbits, called wave attractors (Maas (2001); Rieutord et al. (2001); Harlander & Maas (2007)). Generally wave attractors point to internal boundary layers that are detached from the boundaries. Inertial waves also generate mean flows (Tilgner (2007)).

Wave attractors related internal boundary layers have been studied experimentally for about
10 years. Several authors (Hollerbach (1995); Dintrans et al. (1999); Rieutord et al. (2001)) have shown numerically that waves and wave-attractors exists in spherical shells. Tilgner showed in his work (Tilgner (1999, 2007)) that in fast rotating spherical shells attractors exist and also that two attractors can coexist for certain frequencies. Further he showed that decreasing the radius of the inner sphere, the number of attractors is also decreasing. Later, he showed that a zonal flow can be excited by an interaction of inertial waves, comparable to rectified currents in a rotating box (Maas (2001)), or a rotating cylindrical gap (Swart et al. (2009)). We experimentally investigate such waves for the spherical shell geometry, for which a theoretical understanding is hampered not only due to the occurrence of singularities, but also due to a complicated Ekman layer structure. The waves propagate in the fluid with a fixed angle respect to the rotation axis. The study will reveal, whether low Ekman numbers imply internal boundary layers and whether resulting wave attractors can be observed.

We will first introduce the governing equations, followed by the experimental setup and experimental results. Further we show the comparison between experimental data and numerical simulations and finally we give a conclusion and an outlook.

2. The governing equations

The gap of the spherical shell with width $d$ is filled with a homogeneous incompressible fluid with a kinematic viscosity $\nu$ and rotates about the $\hat{z}$ axis at the angular velocity $\Omega$. The dimensionless equation of motion for the velocity $\vec{v}(r, t)$ is

$$\frac{\partial \vec{v}}{\partial t} = -Ro \vec{v} \cdot \nabla \vec{v} - 2\hat{z} \times \vec{v} - \nabla p + Ek \nabla^2 \vec{v}, \quad (1)$$

$$\nabla \cdot \vec{v} = 0, \quad (2)$$

when we scaled $(\vec{v}, \vec{x}, t, p)$ by $(U, d, \Omega^{-1}, 2d\rho\Omega U)$. The unit vector in the $z$-direction is denoted by $\hat{z}$.

The experiment is governed by the Rossby number

$$Ro = \frac{U}{\Omega_0 d} = \varepsilon, \quad (3)$$

when $U = d\Omega_0 \varepsilon$ is the typical dimensional velocity and the Ekman number

$$Ek = \frac{\nu}{\Omega_0 d^2}. \quad (4)$$

The rotation of the inner sphere varies in time in form of a sinus curve forcing the particles to be deflected away from their rest position. Coriolis forces drives particles back to their initial position where they overshoot due to inertia. This mechanism gives the rise to propagating inertial waves. The outer sphere rotates with $\Omega_0$, the inner sphere with

$$\Omega(t) = \Omega_0(1 + \varepsilon \cdot \sin[\omega t]), \quad \text{when } 0 < \omega < 2\Omega_0, \ 0 < \varepsilon < 1. \quad (5)$$

3. Experimental setup

The spherical shell experiment is relevant for geophysical flows of planetary scale. Fig. 1 shows the setup of the experiment.

The radius ratio between the outer acrylic glass shell ($r_o = 120 \ mm$) and the black anodized aluminum inner sphere ($r_i = 60 \ mm$) is $\eta = 0.5$. The gap ($d = 60 \ mm$) is the area of observation and is filled with an incompressible low viscosity oil (Wacker Ak 0.65) with the kinematic viscosity of $\nu = 0.65 \ mm^2/s$ (at $T = 25^\circ C$). Tab. 1 specifies the geometric parameter of the experiment.
Figure 1. Setup of the experiment (right) with the laser sheet components (left) for optical visualization.

Table 1. Geometric parameter of the experiment.

| parameter             | notation | value  |
|-----------------------|----------|--------|
| radius inner sphere   | \( r_i \) | 60 mm  |
| radius outer sphere   | \( r_o \) | 120 mm |
| gap width             | \( d \)  | 60 mm  |
| radius ratio          | \( \eta = r_i/r_o \) | 0.5    |

To investigate the wave driven flow in the gap, the rotation is varied as described in the Eq. 5. The frequency of the inner sphere modulates with \( \omega = \omega' \cdot \Omega \). We used \( 0.2 < \omega' < 1 \), by increments of 0.1.

The Ekman number of \( 4 \cdot 10^{-5} \) results from fluid properties of the chosen oil, the geometry and the rotation speed of the outer sphere \( \Omega_0 = 0.7183 \) Hz. The range for the Rossby number is \( 0.2 < Ro < 1.0 \), we changed this number by increments of 0.1.

Experiments with different modulation frequencies \( \omega' \) and amplitudes \( \varepsilon \) have been performed. Fig. 2 shows experimental images for different Rossby numbers (\( \varepsilon \)) with \( Ek = 4 \cdot 10^{-5} \) and a frequency of \( \omega' = 0.4 \).

The experiments show the generation of higher harmonics in the equatorial region of the shell for different excitation frequencies. The increasing amplitude leads to a larger turbulent part of
Figure 2. Experimental results for $Ek = 4 \cdot 10^{-5}$ and frequency of $\omega' = 0.4$. The waves are generated for different Rossby numbers, a) with $\varepsilon = 0.3$, b) with $\varepsilon = 0.5$ and c) with $\varepsilon = 0.7$. The higher the amplitude $\varepsilon$, the higher is the turbulent part of the wave formation. Nevertheless, the dominant angles of the waves can also be seen for large $\varepsilon$.

4. Comparison of experimental and numerical data

The velocity, the kinetic energy and the vorticity are characteristic properties to identify wave attractors. To visualize the flow structure and also the estimate wave propagation, we installed a laser sheet illumination. To visualize the flow, aluminium flakes are injected into the fluid. The flat flakes orient in the direction of the shear and reflect the laser light in the expand meridional illumination plane shown in Fig. 2. With this visualization method we can qualitatively determine the inertial wave frequencies but due to the small velocities it is hard to find the numerically predicted wave attractors (see Fig. 3).

Wave attractors are linear features that are dominant as long as the wave amplitude is small. It appears that with the applied visualization technique, such structures are difficult to observe.
Figure 3. Snapshots of the shear layers from numerical simulations for the meridional flow for $Ek = 10^{-5}$ of $\omega' = 0.5$ (1), $\omega' = 0.6$ (2), $\omega' = 1.0$ (3), and $\omega' = 1.4$ (4). It shows the occurrences of the wave attractors at different frequencies for $\eta = 0.5$.

It seems that a more elaborate measurement technique is needed to evidence the wave attractors. However, for larger amplitudes $\varepsilon$, the experimental results are in good agreement with numerical simulations. Fig. 4 shows the comparison between the data. In both, experiment and numerical simulation, the Rossby number is fixed at $Ro = \varepsilon = 0.5$. The Ekman number is $Ek = 4 \cdot 10^{-5}$ in the experiment and $Ek = 1 \cdot 10^{-5}$ in the numerical simulation. For those $\varepsilon$ we observe the generation of higher harmonics in the equatorial region of the shells. This reflects the transition to the nonlinear regime, when the modulation amplitude is large. Then apart from the forcing frequency, additional frequencies exist (see Tab. 2). In the case of $\omega' = 0.8$ we found only one frequency in the experiment, but the numerical simulation reveal also three frequencies. This might again be due to the limited ability of the visualization technique to detect weaker structures in the flow field.

Figure 4. Comparison of experimental data ($Ek = 4 \cdot 10^{-5}$, $Ro = 0.5$) with numerical simulations ($Ek = 10^{-5}$, $Ro = 0.5$). Fig. a) and b) left show the experimental images obtained by applying the meridional laser sheet. In the center we show the azimuthal velocity and on the right side the meridional stream function of the numerical simulation. The lines in the center Figures qualitatively show the dominant frequencies that are excited (see also Tab. 2).

5. Conclusion

Theoretical and numerical investigations predict that nonlinear reflection of internal waves from sloping walls in stratified fluids generates higher harmonics (Tabaei et al. (2005); Gerkema (2006)). Also experiments of internal waves that are reflected at sloping walls (Peacock & Tabaei (2005)) confirm these numerical findings. We suggest that the harmonics, we observe
Table 2. Frequencies corresponding to the lines in Fig. 4 for the experimental and the numerical data, according to dispersion relation of inertial waves \( \omega = \pm 2\Omega_0 \cdot \cos \theta \), where \( \Omega_0 \) is the mean angular velocity and \( \theta \) is the angle between the orientation of the phase line and the horizontal direction. The error could be reduced by using more elaborate optical measurement techniques.

| \( \omega \) | Exp. error | Num. error | Exp. error |
|------------|-------------|-------------|-------------|
| \( \omega_1 \) | 0.52 \( \pm 0.06 \) | 0.62 \( \pm 0.0 \) | 1.0 \( \pm 0.0 \) |
| \( \omega_2 \) | 1.0 \( \pm 0.09 \) | 1.06 \( \pm 0.12 \) | – \( \pm – \) | 1.62 \( \pm 0.12 \) |
| \( \omega_3 \) | 1.41 \( \pm 0.07 \) | 1.58 \( \pm 0.04 \) | – \( \pm – \) | 1.96 \( \pm 0.04 \) |

in the Fig. 2 and 4, exist due to the same wave interaction mechanism but for inertial waves. These waves are excited when the wave beams (see Fig. 2) touch the inner sphere. It is known that the Ekman layer becomes singular there (Stewartson (1966)). The existence of the vertical stationary shear layer is less clear. Experiments with frequencies outside the internal wave band (\( \omega' > 2 \)) should reveal, whether this sharp vertical layer is related to an interaction of inertial waves or due to instabilities in the Ekman layer itself.

6. Outlook

In the future, measurements with particle image velocimetry (PIV) will complete our analysis of the wave field in the spherical shell. For the attractor regime, Rieutord et al. (2001) have shown that the wave beams are nested layers. They show that the thickness of the thinnest and most internal layer scales with \( Ek^{1/3} \). By varying the Ekman number we will investigate scaling laws for the thickness of the wave beams in the nonlinear regime.

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