Study on mechanical effect of adjacent vertical prestressed tendons

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Abstract. The web cracking of continuous rigid frame bridge is still a common phenomenon. A part of the reason is that the vertical prestress does not work well. At present, the mechanical research of vertical prestressed reinforcement is not comprehensive, the mechanical effect between two adjacent vertical prestressed tendons is not thorough, theoretical analysis cannot be carried out through formula derivation, and finite element modeling is relatively dependent on, so the work is cumbersome and the application is not convenient. In view of this problem, this paper uses elastic mechanics theory to deduce the stress effect formula of two adjacent vertical prestressed tendons, and verifies the correctness of the formula through finite element modeling. In addition, this paper also uses the finite element theory to carry on the modeling calculation, obtains the distribution law of the web stress caused by the vertical prestress along the height of the web and the longitudinal direction of the box girder, providing guidance for the vertical prestressed design of rigid frame bridge.

1. General instructions
When the main tensile stress or shear stress in the web is greater than the tensile strength of concrete, cracks will appear on the web. In order to suppress the aggravation of cracks, the vertical prestressed steel bundles are usually installed in the prestressed concrete box girder to reduce the shear stress and main tensile stress in the web of the box girder.

In recent years, experts and scholars at home and abroad have done a lot of research on the effect of vertical prestress, especially on the stress field of box girder web under the action of a single vertical prestressed steel bundle. Based on the finite element and mathematical methods, Zhong Xingu, Li Feng and others analyzed the stress field of box girder web under the action of vertical prestress, and found that the compressive stress storage near the neutral axis of the web is insufficient when applying the vertical prestress value designed according to the current vertical prestress design method. Shen Mingyan and others used a narrow thin plate to simulate the web, derived the analytical formula of the web preload field under the action of a single vertical prestressed steel bundle, modified the coefficient of 0.6 given in the standard formula, and introduced the correction coefficient K. Zhao Baojun and others simulated the web with a narrow thin plate, simulated the vertical prestress with concentrated force, derived the analytical and numerical solutions under the action of single vertical prestress, studied the vertical prestress loss of box girder webs systematically combined with the field measured data, and established the calculation formula of long-term loss rate of vertical prestress.

However, the existing research on vertical prestressing effect mainly focuses on the single prestress, considers the prestress diffusion effect and applies numerical fitting, superposition principle and other analysis methods. These studies failed to carry out theoretical analysis through formula derivation and...
relied on finite element modeling, which was cumbersome and inconvenient to apply. It can be seen that the research on vertical prestressing effect is not comprehensive. Based on the investigation and statistics of the webs of several continuous rigid frame bridges in Shaanxi Province, it is found that the cracking of webs at 1/4 span is still common, which means that the vertical prestressing effect is not ideal. It is necessary to make further research on the effect and spacing arrangement, so as to play the role of vertical prestressing more efficiently.

2. Calculation formula derivation of vertical prestress effect

Because the stress difference between thin plate model and segment model is small when the size, position and constraint conditions of vertical prestress are similar, the thin plate model is used for simplified analysis, the following assumptions are made: (1) Under the action of vertical prestress, the plane section assumption is satisfied, and the section is still in the elastic stage; (2) The stress distribution of vertical prestressed reinforcement in the direction of web thickness is uniform; (3) The effect of other steel tendons or ordinary steel bars and the influence of prestressed pipe deviation are not considered.

2.1 Calculation formula of slab internal stress caused by single vertical prestress

Assuming that the length of the web is $2l$, the thickness of the web is $t$, and the height of the web is $2c$, the formula of stress distribution in web caused by single vertical prestress is obtained:

$$\sigma_y |_{y=0} = -\frac{4P}{lt} \sum_{n=1}^{\infty} \frac{1}{\sinh \frac{2n\pi c}{l} + \cosh \frac{2n\pi c}{l}} \left(1 + \frac{n\pi}{l} \cdot \cosh \frac{n\pi c}{l}\right) \cos \frac{n\pi x}{l} \sin \frac{n\pi c}{l}$$

The formula simulates the web internal force caused by a single prestressed tendon, but for the web internal force caused by more than two prestressed tendons, the current research is limited to numerical simulation and superposition, and the obtained results are only numerical solutions rather than analytical solutions, which makes the calculation results more error.

2.2 Calculation formula of internal stress of slab caused by two vertical prestress

Two or more vertical prestressed loads are discontinuous. If polynomial function is used to calculate, the boundary conditions cannot be satisfied. Therefore, Fourier series should be used for expansion, and then the stress function in the form of trigonometric series should be used for calculation.

Taking two vertical prestressing forces as examples, the concentrated force is replaced by the distributed force $P$ in a certain range, and the distribution function $p(x)$ is expanded by Fourier series.

Suppose the range of distributed force $2\delta$ is ($\delta$ is infinitesimal), the length of web is $2l$, the thickness of web is $t$, the height of web is $2c$, and the spacing of vertical prestressed reinforcement is $2a$. The specific stress diagram is shown in Figure 1 below.

![Figure 1](image_url)

Figure 1. Calculation diagram of internal stress caused by two vertical prestressing.

Then the concentration of distributed force is expressed by concentrated force as follows:

$$p(x) = \frac{P}{2\delta \cdot t}$$ (1)
The load distribution function \( p(x) \) on the plate can be expanded into Fourier series in the interval \((-l \leq x \leq l)\)

\[
p(x) = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}
\]  

(2)

In the formula: \( n=1,2,3… \)

The three coefficients are as follows:

\[
A_0 = \frac{1}{l} \int_{-l}^{l} p(x) \, dx
\]
\[
A_n = \frac{1}{l} \int_{-l}^{l} p(x) \cos \frac{n\pi x}{l} \, dx
\]
\[
B_n = \frac{1}{l} \int_{-l}^{l} p(x) \sin \frac{n\pi x}{l} \, dx
\]

(3)

Since the load \( p(x) \) is symmetrical with respect to the y-axis, the coefficient \( B_n \) in equation (2) must be equal to zero, and the other two coefficients shall be calculated according to formula (3), and the coefficients are calculated as follows:

\[
A_0 = \frac{1}{l} \int_{-l}^{l} p(x) \, dx = \frac{2}{l} \int_{0}^{a+\delta} p \, dx = \frac{2}{l} \cdot 2p \delta = \frac{4p\delta}{l}
\]
\[
A_n = \frac{1}{l} \int_{-l}^{l} p(x) \cos \frac{n\pi x}{l} \, dx = \frac{2}{l} \int_{0}^{a+\delta} \cos \frac{n\pi x}{l} \, dx
\]
\[
= \frac{2}{l} \cdot \frac{l}{n\pi} \left[ \sin \frac{n\pi}{l} (a + \delta) - \sin \frac{n\pi}{l} (a - \delta) \right] = \frac{2p}{n\pi} \cdot 2 \cos \frac{n\pi a}{l} \sin \frac{n\pi \delta}{l}
\]

(4)

The load on the slab can be expressed by the following continuous function:

\[
p(x) = \frac{2p\delta}{l} + \sum_{n=1}^{\infty} \left( \frac{4p}{n\pi} \sin \frac{n\pi \delta}{l} \cos \frac{n\pi a}{l} \right) \cos \frac{n\pi x}{l}
\]

(5)

The stress effect of vertical prestressing is transformed into the stress effect of formula (5) on the web, which can be divided into two parts: one is \( \frac{2p\delta}{l} \), representing the average value of the load \( p \) over the full length of the web, which corresponds to the uniform compression stress in the Y direction of the web:

\[
\sigma_y = -\frac{2p\delta}{l}, \sigma_x = \tau_{xy} = 0
\]

the other one is:

\[
\sum_{n=1}^{\infty} \left( \frac{4p}{n\pi} \sin \frac{n\pi \delta}{l} \cos \frac{n\pi a}{l} \right) \cos \frac{n\pi x}{l},
\]

which is the stress corresponding to load distributed in cosine series.

Now we are solving the stress component corresponding to the load distributed by cosine series in formula (5).

Selecting the stress function:

\[
\varphi(x, y) = \sum_{n=1}^{\infty} f_n(y) \cos \frac{n\pi x}{l}
\]

(6)

Let:

\[
k_n = \frac{n\pi}{l},
\]

Substituting the stress function \( \varphi(x, y) \) into the biharmonic equation is as follows:

\[
\nabla^4 \varphi = \sum_{n=1}^{\infty} f_n(y) k_n^4 \cos k_n x - 2 \sum_{n=1}^{\infty} f_n'(y) k_n^2 \cos k_n x + \sum_{n=1}^{\infty} f_n^{(4)}(y) \cos k_n x = 0
\]

(7)

The results are as follows:

\[
\sum_{n=1}^{\infty} \left[ k_n^4 f_n(y) - 2k_n^2 f_n'(y) + f_n^{(4)}(y) \right] \cos k_n x = 0
\]

(8)

Then the fourth order linear ordinary differential equation
must be established, and the general solution of which is as follows:

\[ f_n(y) = (C_n' + D_n'y)e^{kn'y} + (E_n' + F_n'y)e^{-kn'y} \]  

(9)

It is rewritten as hyperbolic function in the following form:

\[ f_n(y) = C_n \cosh kn'y + D_n \sinh kn'y + E_n y \cdot \cosh kn'y + F_n y \cdot \sinh kn'y \]  

(10)

When \( f_n(y) \) is in the form of equation (10), the stress function can satisfy the compatibility equation. In this case, the stress function \( \varphi(x,y) \) is an equation with undetermined coefficient \( C_n, D_n, E_n, F_n \). The stress component is deduced from the stress function \( \varphi(x,y) \) as follows:

\[
\sigma_x = \frac{\partial^2 \varphi}{\partial y^2} = \sum_{n=1}^{\infty} k_n^2 \left( C_n \cosh kn'y + D_n \sinh kn'y + \frac{2}{k_n} E_n y \cdot \cosh kn'y + E_n y \cdot \cosh kn'y \right) \cos kn'x 
\]

\[
\sigma_y = \frac{\partial^2 \varphi}{\partial x^2} = -\sum_{n=1}^{\infty} k_n^2 \left( C_n \sinh kn'y + D_n \cosh kn'y + \frac{E_n}{k_n} \cosh kn'y + E_n y \cdot \cosh kn'y \right) \cos kn'x 
\]

\[
\tau_{xy} = \frac{\partial^2 \varphi}{\partial x \partial y} = \sum_{n=1}^{\infty} k_n^2 \left( C_n \sinh kn'y + D_n \cosh kn'y + \frac{E_n}{k_n} \cosh kn'y + E_n y \cdot \cosh kn'y \right) \sin kn'x 
\]

(11)

For this model, the boundary conditions are as follows:

When \( y = \pm c \),

Then

\[
\sigma_y = -\sum_{n=1}^{\infty} \left( \frac{4p}{k_n l} \sin k_n \delta \cos k_n a \right) \cos k_n x, \quad \tau_{xy} = 0. \]  

(12)

Because the load is distributed in the upper and lower edges, \( \sigma_y \) is axisymmetric about the x-axis, Therefore, the coefficients of the antisymmetric functions \( \text{shkn'y} \) and \( \text{ychkn'y} \) in the expression \( \sigma_y \) in equation (10) are 0, in other words: \( D_n=0, E_n=0 \).

Then the boundary condition equation (12) is substituted into the stress component expression (11) derived from the stress function, and let \( D_n=0, E_n=0 \), then:

\[
\tau_{xy}|_{y=\pm c} = \sum_{n=1}^{\infty} k_n^2 \left( C_n \sinh kn'y + F_n y \cdot \cosh kn'y \right) \sin kn'x \bigg|_{y=\pm c} = 0 
\]

\[
\sigma_y|_{y=\pm c} = -\sum_{n=1}^{\infty} k_n^2 \left( C_n \cosh kn'y + F_n y \cdot \sinh kn'y \right) \cos kn'x \bigg|_{y=\pm c} = -\sum_{n=1}^{\infty} \frac{4p}{k_n l} \sin k_n \delta \cos k_n a \cos k_n x 
\]

Then:

\[
C_n = \frac{8p}{k_n l} \cdot \frac{\text{shkn'c}+k_n \cosh kn'c}{2k_n \cosh kn'c+\text{shkn'c}} \cos k_n a \sin k_n \delta 
\]

\[
F_n = -\frac{8p}{k_n l} \cdot \frac{\text{shkn'c}}{2k_n \cosh kn'c+\text{shkn'c}} \cos k_n a \sin k_n \delta 
\]

(13)

The coefficient \( C_n, D_n, E_n, F_n \) is introduced into equation (11) and simplified, then:

\[
\sigma_y = -\frac{8p}{l} \sum_{n=1}^{\infty} \frac{\text{shkn'c} \sin k_n \delta \cos k_n a}{\sin 2k_n \cosh 2k_n c+2k_n c} \left( \frac{1}{k_n} \text{chkn'y} + c \cdot \text{chkn'c} \cdot \text{chkn'y} - y \text{shkn'y} \right) \cos k_n x 
\]

(14)

By substituting formula (1) and

\[
k_n = \frac{n\pi}{l}
\]
into equation (14), the vertical stress corresponding to the load distributed in cosine series is obtained:

$$\sigma_y = - \frac{4P}{\delta l t} \sum_{n=1}^{\infty} \frac{\sin n \pi \delta / l}{n \pi \delta / l} \left( c h \frac{n \pi y}{l} + \frac{n \pi c}{l} \cdot c t h \frac{n \pi c}{l} \right) \cdot \left( c h \frac{n \pi y}{l} - \frac{n \pi y}{l} \cdot s h \frac{n \pi y}{l} \right) \cos \frac{n \pi x}{l}$$

By adding the above formula with the vertical stress ($\sigma_y = - \frac{2P}{l t}$) caused by uniform compression, the final expression of vertical stress in slab caused by two vertical prestressed reinforcement is obtained:

Since the length $l$ of the plate is assumed to be much larger than that of other dimensions, i.e. the plate is assumed to be a long strip, so $\delta \rightarrow 0$, and the limit of the above formula is taken as follows:

$$\lim_{\delta \rightarrow 0} \sigma_y = 0 - \frac{4P}{l t} \lim_{\delta \rightarrow 0} \sum_{n=1}^{\infty} \frac{\sin n \pi \delta / l}{n \pi \delta / l} \cdot \frac{\sinh n \pi c / l}{n \pi c / l} \cdot \left( c h \frac{n \pi y}{l} + \frac{n \pi c}{l} \cdot c t h \frac{n \pi c}{l} \cdot c h \frac{n \pi y}{l} - \frac{n \pi y}{l} \cdot s h \frac{n \pi y}{l} \right) \cos \frac{n \pi x}{l}$$

Then:

$$\sigma_y = - \frac{4P}{l t} \sum_{n=1}^{\infty} \frac{\sinh n \pi c / l}{\sinh n \pi c / l} \cdot \left( c h \frac{n \pi y}{l} + \frac{n \pi c}{l} \cdot c t h \frac{n \pi c}{l} \right) \cdot \left( c h \frac{n \pi y}{l} - \frac{n \pi y}{l} \cdot s h \frac{n \pi y}{l} \right) \cos \frac{n \pi x}{l}$$

Let formula (15) take $y = 0$ to obtain the change of $\sigma_y$ along the x-axis direction of the center line of beam height:

$$\sigma_y |_{y=0} = - \frac{4P}{l t} \sum_{n=1}^{\infty} \frac{\sinh n \pi c / l}{\sinh n \pi c / l} \cdot \left( c h \frac{n \pi y}{l} + \frac{n \pi c}{l} \cdot c t h \frac{n \pi c}{l} \right) \cos \frac{n \pi x}{l}$$

3. Calculation example analysis of vertical prestress calculation formula

The finite element model of web is established by ANSYS. Solid65 element is used for concrete and link8 element is used for finish rolling rebar, take the vertical prestress as $P=1kN$, take the plate length as $2l=40m$, take the thickness of the plate as $t=0.5m$, take the height of the plate as $2c=3m$, the vertical prestressed spacing is $2a=3m$, the model of web and vertical prestressed reinforcement is established as shown in the following Figure.2:

Figure 2. Finite element results of plate internal stress caused by two vertical prestressing forces.

Output the vertical stress value of the web along the center line of 5 ~ 15m plate length, import the stress value into origin, and draw the corresponding stress diagram.

Then, according to formula (16) and using the same model parameters as the finite element method, the variation rule of $\sigma_y$ along the x-axis direction of the infinite strip on the center line ($y = 0$) of the plate can be drawn by Origin software.

The results of the above two calculation methods are drawn in a graph with the same coordinates to facilitate the comparison of the results of finite element calculation and formula calculation, as shown in the following Figure.3:
Figure 3. Comparison between finite element analysis and formula calculation results of two vertical prestressing forces in slab.

It can be seen that the calculation results of formula (16) are basically consistent with the results of finite element model analysis, which shows that formula (16) can be applied to the calculation of web stress caused by multiple vertical prestressed tendons. In the setting of vertical prestress, the effective prestress after deducting the long-term loss is less than the standard requirements, and the stress calculated by the formula is generally lower than the analysis result of the finite element model, so the calculation result of the formula is more conservative and the design is safer.

4. The change law of internal stress caused by two vertical prestressing forces

4.1 Distribution law of stress in slab along the height direction caused by vertical prestress

For formula (15), the vertical prestress is taken as \( P=1KN \), the length of the slab is \( l=40m \), the thickness of the slab is \( t=0.5m \), the height of the slab is \( 2c=3m \), and the vertical prestressed spacing is \( 2a=3m \). Taking different values of \( y \) in the formula, we can get the stress distribution at different heights of the web plate under the action of vertical prestressed steel bars, and thus obtain the distribution law of the stress in the slab along the height direction caused by vertical prestressing.

Figure 4. Calculation results of internal stress formula at \( y = 1.5m \) height.

Figure 5. Calculation results of stress formula in plate at \( y = 1m \) height.
Compared with the results in Figure 4, Figure 5 and Figure 6, the stress distribution law of vertical prestress along the height direction of web can be obtained: (1) the stress at the position of vertical prestressed reinforcement is larger, and its value gradually increases from the center line of the beam to the top (bottom) of the beam; (2) the stress in the middle of the two prestressed tendons is small, and its value gradually decreases from the center of the beam to the top (bottom) of the beam. That is, at the position of two prestressed bars, the section stress closer to the upper (lower) edge of the beam is greater, and in the middle of the two vertical prestressed tendons, the section stress closer to the upper (lower) edge of the beam is smaller. The closer to the edge of the beam, the more uneven the stress distribution is, which is caused by the diffusion effect of prestress.

4.2 Distribution of stress in slab caused by vertical prestress along longitudinal direction of slab
For formula (15), take the vertical prestress as $P=1KN$, plate length $2l=40m$, plate thickness $t=0.5m$, slab height $2c=3m$, $y=0$ (That is, take the center position of web height for analysis). Taking different vertical prestressed spacing $2a$ and comparing different prestressed tendon spacing, the distribution law of stress in slab caused by vertical prestress along the length direction of slab is obtained.

Figure 7. Calculation results of internal stress formula at the center line of 0.5m prestressed space.

Figure 8. Calculation results of internal stress formula at the center line of 1 m prestressed space.
According to Figure 7, Figure 8, and Figure 9, the distribution law of longitudinal stress under the action of vertical prestressed reinforcement can be summarized: (1) The stress of vertical prestressed reinforcement is larger near the position of prestressed reinforcement, and the farther away from the reinforcement, the smaller the stress is; (2) With the increase of vertical prestressed reinforcement spacing, it tends to be the effect of two independent vertical prestressed reinforcement, that is, the vertical prestress in the middle of two vertical prestressed reinforcement is small, and with the increase of vertical prestressed reinforcement spacing, the stress in the middle of two vertical prestressed reinforcement decreases continuously; (3) When the spacing of vertical prestressed steel bars is small, the effect is similar to that of a vertical prestressed steel bar, that is, the maximum stress in the middle is gradually reduced to both sides. The stress in the middle of two vertical prestressed steel bars is not significantly reduced, but even slightly higher than that of Prestressed steel bars.

5. Conclusion
Based on the data of Shaanxi continuous rigid frame bridge, this paper studies the stress effect of two adjacent vertical prestressing forces by means of theoretical formula derivation and finite element calculation. The conclusion is as follows:

(1) Based on the theory of elastic mechanics, this paper derives the calculation formula of stress effect of box girder web under the action of two vertical prestressed steel bars, and the correctness of the formula is verified by finite element modeling;

(2) The web stress is calculated by using two formulas for calculating the action effect of vertical prestressed steel bundles, and then we obtain the stress distribution of vertical prestress along the height of web. The stress value at the position of the vertical prestressed reinforcement increases gradually from the center line of the beam to the top (bottom) of the beam; The stress value of the middle position of the two prestressed tendons gradually decreases from the center of the beam to the top (bottom) of the beam, that is, the section stress closer to the upper (lower) edge of the beam is greater at the position of the two prestressed tendons, and the section stress closer to the upper (lower) edge of the beam is smaller at the middle position of the two vertical prestressed tendons.

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