Keywords: Large set, Steiner triple system, Kirkman triple system
Large sets of Kirkman triple systems with order $q^n + 2$

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Abstract

The existence of Large sets of Kirkman Triple Systems (LKTS) is an old problem in combinatorics. Known results are very limited, and a lot of them are based on the works of Denniston [2, 3, 5, 4]. The only known recursive constructions are an tripling construction by Denniston [5] and a product construction by Lei [9], both constructs an LKTS$(uv)$ on the basis of an LKTS$(v)$.

In this paper, we describe an construction of LKTS$(q^n + 2)$ from LKTS$(q + 2)$, where $q$ is a prime power of the form $6t + 1$. We could construct previous unknown LKTS$(v)$ by this result, the smallest among them have $v = 171, 345, 363$.

1. Introduction

A Steiner triple system of order $v$, denoted by STS$(v)$, is a pair $(X, B)$, where $X$ is a $v$-set and $B$ is a collection of triples (called blocks) of $X$ such that every pair of $X$ is contained in a unique block of $B$. Let $(X, B)$ be an STS$(v)$. If there exists a partition $A = \{P_1, P_2, \ldots, P_{(v-1)/2}\}$ of $B$ such that each part $P_i$ forms a parallel class, i.e., a partition of $X$, then the STS$(v)$ is called resolvable and $A$ is called a resolution. A resolvable STS$(v)$ is usually called a Kirkman triple system of order $v$ (briefly KTS$(v)$). It is well known that a KTS$(v)$ exists if and only if $v \equiv 3 \pmod{6}$.

Two STS$(v)$s on the same set $X$ are said to be disjoint if they have no triples in common. By a simple counting argument, there can be at most $v - 2$ mutually disjoint STS$(v)$s on the same set $X$; such a set of $v - 2$ disjoint systems must contain every possible triple in $X$, and is called a large set of
STS($v$)s and denoted by LSTS($v$). A large set of Kirkman triple systems of order $v$, denoted by LKTS($v$), is an LSTS($v$) where each STS($v$) is resolvable.

The existence problem of LKTS is posed by Sylvester in 1850 as an extension of Kirkman’s schoolgirl problem. The existence of LSTS has been completely solved by Lu, Teirlinck, and Teirlinck, but existence of LKTS is much harder to establish, and only limited results are known. Kirkman showed the existence of an LKTS(9) in 1850, and Denniston constructed an LKTS(15) more than one century later. Denniston, Chang and Ge, and Zhou and Chang gave further direct constructions for LKTS. The first recursive construction of LKTS was a tripling construction by Denniston, which was later generalized by Lei to a product construction. Zhang and Zhu improved the above mentioned tripling and product constructions, removing the need of transitive KTS. The known result for existence of LKTS are summarized below:

**Theorem 1.1.**

1. There exists an LKTS($3^{s}5^{b}7^{m}r\prod_{i=1}^{s}(2\cdot13^{n_{i}}+1)\prod_{j=1}^{b}(2\cdot7^{m_{j}}+1)$) for integers $r \in \{1, 7, 11, 13, 17, 35, 43, 67, 91, 123\}\cup\{2^{2p+1}5^{2q}+1\mid p, q \geq 1\}, a, n_{i}, m_{j} \geq 1, b, s, t \geq 0$ and $a + s + t \geq 2$ when $b \geq 1$ and $r \neq 1$.

2. There exists an LKTS($3\prod_{i=1}^{s}(2q_{i}^{n_{i}}+1)\prod_{j=1}^{t}(4^{m_{j}}-1)$), where $s + t \geq 1, n_{i}, m_{j} \geq 1, q_{i} \equiv 7 \pmod{12}$ and $q_{i}$ is a prime power.

To this day, known recursive constructions of LKTS are all product constructions; that is, they construct an LKTS($pq$) from an LKTS($p$) and some auxiliary design on $q$ points. In this paper, we provide a construction of LKTS($q^{n}+2$) from known LKTS($q+2$), which serves as a good complement to the known product constructions.

**2. Main results**

The main result we will prove in this paper is

**Theorem 2.1.** If there exists an LKTS($q+2$) for $q \equiv 1 \pmod{6}$ a prime power, then there exists an LKTS($q^{n}+2$) for every integer $n \geq 2$.

LKTS($q+2$) is known for $q = 7, 13, 19, 25, 31, 37, 43, 49, 61$ and many more values. By Theorem 2.1 we can deduce that:

**Corollary 2.2.** There exists an LKTS($q^{n}+2$) for $n \geq 1$ and $q \in \{7, 13, 19, 25, 31, 37, 43, 49, 61\}$.
Taken into account the product constructions in [16, 7], we have the following:

**Corollary 2.3.** There exists an LKTS$(3a^b(q^n + 2)\prod_{i=1}^{s}(2\cdot 13^{n_i} + 1)\prod_{j=1}^{t}(2\cdot 7^{m_j} + 1))$ for integers $q \in \{7, 13, 19, 25, 31, 37, 43, 49, 61\}$, $n_i, m_j \geq 1$, $a, b, n, s, t \geq 0$.

This result enables us to construct an infinite family of previous unknown LKTS, the smallest of which have order $v = 171, 345, 363, 363, 513, 627, 855, 963, 1035, 1089$.

We will construct an LKTS on the point set $W \bigcup \{∞_1, ∞_2\}$, where $W$ is a vector space of dimension $n$ over $\mathbb{F}_q$, and $∞_i$ are two additional points. The details of the construction are exhibited in the next section.

3. The Construction

Let $q = 6t + 1$ be a prime power, $(Y = \mathbb{F}_q \bigcup \{∞_1, ∞_2\}, D_i|_{i \in \mathbb{F}_q})$ be a large set of KTS$(q + 2)$, where we take $D_i$ to be the design that contains the triple $\{∞_1, ∞_2, i\}$, and $W = \mathbb{F}^n$ be a linear space over $\mathbb{F}_q$. Without loss of generality, let $Q_{i,j}(j = 0, 1, \ldots, (q - 1)/2)$ be the parallel classes of $D_i$, where $\{∞_1, ∞_2, i\} \in Q_{i,0}$. We will construct an LKTS$(q^n + 2)$ on the point set $X = W \bigcup \{∞_1, ∞_2\}$.

For each triple $A \subset Y$ and point $x \in W$, we denote by $Ax$ the set $\{ax|a \in A\}$, where $∞_i x = ∞_i, i = 1, 2$.

The main ingredient of our construction is a large set of ”frames” over $W$, resolvable into partial parallel classes, and invariant under translations in $W$. The triples in the ”base” frame are exactly those of the form $\{x, y, -x - y\}$ where $x$ and $y$ are independent. The ”holes” within each frame are 1-dimensional affine subspaces of $W$ (or geometrically, lines in $W$) that passes a given point; We make use of the triples from the original large set $Y$ to fill in the holes.

We will only describe the ”base” frame, which contains all the zero-sum triples that are non-collinear. It is easy to show that the total number of such triples are $(q^n - q)(q^n - 1)^2/6$. We will describe a partition of these triples into $(q^n - 1)/2$ partial parallel classes, where each class contains $(q^n - q)/3$ triples.

We fix a family of skew-symmetric bilinear forms $f_L : L \times L \rightarrow \mathbb{F}_q$ for each 2-dimensional subspace $L$ of $W$. 
Let $g$ be a primitive root of $q$, and $\omega = g^{2t}$ be a third root of unity in $\mathbb{F}_q$. We define, for each ordered linearly independent pair of points $(u, v) \in W^2$, the triple
\[
T(u, v) = \{u + v, \omega u + \omega^2 v, \omega^2 u + \omega v\}. \tag{1}
\]
It is easy to see that $T(u, v) = T(\omega u, \omega^2 v) = T(\omega^2 u, \omega v)$. The following lemma gives the basis of our construction.

**Lemma 3.1.** Let $K$ be a 1-dimensional subspace of $W$, and $L \supseteq K$ is an 2-dimensional space containing $K$. For any $u \in K$ and $c \in \mathbb{F}_q^*$, the $q(q-1)/3$ triples $\{\pm T(u, v) | v \in L, f_L(u, v) = g^m c, 0 \leq m < t\}$ is a partition of $L \setminus K$. We will denote this set by $P_{u, L, c}$.

**Proof.** As $L \setminus K$ have $q(q-1)$ elements, it suffices to prove that every element in $L \setminus K$ belongs to one of such triple.

Let $v_0 \in L$ such that $f_L(u, v_0) = 1$. For a given $c \in \mathbb{F}_q^*$, the set of vectors $v$ such that $f_L(u, v) = c$ are exactly $cv_0 + bu$ for $b \in \mathbb{F}_q$.

Thus we have $T(u, cv_0 + bu) = \{(1 + b)u + cv_0, (\omega + b\omega^2)u + \omega^2 cv_0, (\omega^2 + b\omega)u + \omega cv_0\}$, thus $\bigcup_{f_L(u, v) = c} T(u, v) = \bigcup_{b \in \mathbb{F}_q} T(u, cv_0 + bu) = \{bu + \omega^a cv_0 | a = 0, 1, 2, b \in \mathbb{F}_q\}$

Therefore $\bigcup_{f_L(u, v) = g^m c} (\pm T(u, v)) = \{\pm bu \pm \omega^a g^m cv_0 | a = 0, 1, 2, b \in \mathbb{F}_q, 0 \leq m < (q-1)/6\} = \{bu + av_0 | b \in \mathbb{F}_q, a \in \mathbb{F}_q^*\} = L \setminus K$. \hfill $\Box$

**Lemma 3.2.** Let $L_K$ be the set of all 2-dimensional subspaces of $W$ containing $K$ (so that $|L_K| = \frac{2^{n-1}-1}{q-1}$), then the sets $\{L \setminus K | L \in L_K\}$ give a partition of $W \setminus K$.

**Corollary 3.3.** The set $P_{u,c} = \bigcup_{L \in U_{K_i}} P_{u, L, c}$ gives a partition of $W \setminus K_i$.

We will now construct the frame from the partial parallel classes defined above.

Let $U = \{K_i | 1 \leq i \leq (q^n - 1)/(q - 1)\}$ be the set of 1-dimensional subspaces of $W$. We choose a generator $u_i \in K_i$ in each $K_i$, and denote $V = \{u_i | 1 \leq i \leq (q^n - 1)/(q - 1)\}$.

**Lemma 3.4.** The union of partial parallel classes $P_{g^au_i, \omega^b}$ where $0 \leq a < t$, $b = 0, 1, 2$ and $1 \leq i \leq q+1$ contains every zero-sum non-collinear triple exactly once.
Proof. The total number of triples in these partial parallel classes is \( \frac{q(q-1)}{3} \cdot \frac{q^n-1}{q-1} \cdot 3 \cdot \frac{q^n-1}{q-1} = (q^n - q)(q^n - 1)/6 \), which agrees with the total number of zero-sum non-collinear triples. We only need to prove that every such triple \( \{x, y, -x - y\} \) belongs to at least one of these classes.

Let \( u_0 = (1 - \omega^2)(\omega x - y)/3 \) and \( v_0 = (1 - \omega^2)(\omega y - x)/3 \) so that we have \( \{x, y, -x - y\} = T(u_0, v_0) \). Let \( L \) be the subspace spanned by \( x \) and \( y \), we have \( f_L(u_0, v_0) = (\omega^2 - \omega) f_L(x, y)/3 \). Since \( f_L(u_0, v_0) = -f_L(v_0, u_0) \), without loss of generality, we can assume that \( f_L(u_0, v_0) = g^m \omega^3 \) with \( 0 \leq m < (q - 1)/6 \).

Assume now that \( u_0 = g^d u_i \) with \( u_i \in V \) and \( 0 \leq d < q - 1 = 2t \). We write \( d = ct + a \) where \( c = 0, 1, 2, 3, 4 \) and \( 0 \leq a < t \). We can now conclude that \( \{x, y, -x - y\} = T(u_0, v_0) = (-1)^c T(\omega^{-c} g^a u_i, v_0) = (-1)^c T(g^a u_i, \omega^{-c} v_0) \) belongs to \( P_{g^a u_i, \omega^3} \) and therefore \( P_{g^d u_i, \omega^3} \).

\( \square \)

It is an immediate conclusion from the above lemma that the union of all translations of the partial parallel classes contains every non-collinear triple in \( W \).

We now proceed to fill the holes in the frame. For each \( K_i \in U \), we construct a system of representatives \( R_i \) of the cosets of \( K_i \), and an associated function \( p_i : W \to \mathbb{F}_q \) such that \( x - p_i(x) u_i \in R_i \) for all \( x \in W \).

Recall that \( Q_{i,j} \) are the parallel classes of \( D_i \) and that \( \{\infty_1, \infty_2, i\} \in Q_{i,0} \).

**Theorem 3.5.** For each \( w \in W \), the following parallel classes

\[
P_{w, u_i, a+b} = (w + P_{g^a u_i, \omega^3}) \bigcup \{w - p_i(w) u_i + A u_i | A \in Q_{p_i(w), a+b+1}\} \tag{2}
\]

for \( u_i \in V, \ 0 \leq a < t \) and \( b = 0, 1, 2 \) and

\[
P_{w, *} = \bigcup_{u_i \in V} \{w - p_i(w) u_i + A u_i | A \in Q_{p_i(w), 0}\} \tag{3}
\]

constitutes a KTS\( (q^n + 2) \), which we will denote by \( B_w \).

**Proof.** By Lemma 3.1, \( w + P_{g^a u_i, \omega^3} \) is a partition of \( W \setminus (w + K_i) \), while \( \{w - p_i(w) x_i + A u_i | A \in Q_{p_i(w), a+b+1}\} \) is a partition of \( \{\infty_1, \infty_2\} \bigcup (w + K_i) \). Therefore \( P_{w, u_i, a+b+1} \) is indeed a parallel class. On the other hand, for each \( u_i \in V, \{w - p_i(w) x_i + A u_i | A \in Q_{p_i(w), 0}\} \) contains the triple \( \{w, \infty_1, \infty_2\} \) as well as other \( (q - 1)/3 \) triples that gives a partition of \( w + (K_i \setminus 0) \), so the union of these is also a parallel class.
We now prove that $B_w$ is an STS, and therefore a KTS by the argument above. The number of parallel classes in $B_w$ equals $(q^n + 1)/2$, which agrees with the expected number of a KTS$(q^n + 2)$, so it suffices to find a triple $B$ containing $\{x, y\}$ in one of these classes for all $x, y \in X$.

We distinguish the following cases.

1. $\{x, y\} = \{\infty_1, \infty_2\}$. We know that $\{w, \infty_1, \infty_2\} \in P_{w,*}$, so we can take $B = \{w, \infty_1, \infty_2\}$.
2. $y \in \{\infty_1, \infty_2\}, x \in W$. If $x = w$ we take $B = \{w, \infty_1, \infty_2\}$ as above. Otherwise, let $x = au_i + w$ where $a \in \mathbb{F}_q$, then there is a unique block $B_0$ containing $\{a - p_i(w), y\}$ in $D_{p_i(w)}$. Take $B = w - p_i(w)u_i + B_0u_i$.
3. $\{x, y\} \subset W$, and $x - w, y - w$ are linear dependent. Similarly, let $x = au_i + w, y = bu_i + w$ where $a, b \in \mathbb{F}_q$, there is a unique block $B_0$ containing $\{a - p_i(w), b - p_i(w)\}$ in $D_{p_i(w)}$. Take $B = w - p_i(w)u_i + B_0u_i$.
4. $\{x, y\} \subset W$, and $x - w, y - w$ are linear independent. By Lemma 3.3, the triple $\{x - w, y - w, 2w - x - y\}$ belongs to one of the partial parallel classes $P_{g^a u_i, \omega^b}$. Take $B = \{x, y, 3w - x - y\} \in w + P_{g^a u_i, \omega^b} \subset P_{w, u_i, a + b + 1}$.

\[\square\]

**Theorem 3.6.** The set $\{B_w\}_{w \in W}$ is an LKTS$(q^n + 2)$.

**Proof.** We only need to find, for each triple $\{x, y, z\} \subset X$, an element $w \in W$ such that $\{x, y, z\} \in B_w$.

Like the proof of the previous lemma, we distinguish four cases.

1. $\{y, z\} = \{\infty_1, \infty_2\}$. Take $w = x$.
2. $\{x, y\} \subset W, z \in \{\infty_1, \infty_2\}$. Let $x = r + ax_i, y = r + bx_i$ where $a, b \in \mathbb{F}_q$ and $r \in R_i$, then there is a unique $d \in \mathbb{F}_q$ such that $\{a, b, z\} \in D_d$. Take $w = r + dx_i$.
3. $\{x, y, z\} \subset W$, and are collinear. Similarly, let $x = r + ax_i, y = r + bx_i, z = r + cx_i$ where $a, b, c \in \mathbb{F}_q$ and $r \in R_i$, there is a unique $d \in \mathbb{F}_q$ such that $\{a, b, c\} \in D_d$. Take $w = r + dx_i$.
4. $\{x, y, z\} \subset W$, and are not collinear. Take $w = (x + y + z)/3$.

\[\square\]

**Remark 3.7.** Suppose the original large set $\{D_i\}_{i \in \mathbb{F}_q}$ is invariant under translations in $\mathbb{F}_q$, i.e. $D_i = D_0 + i$ and $Q_{i,j} = Q_{0,j} + i$ holds for all $i$ and $j$. For all $w \in W$ and $A \in Q_{p_i(w), j}$, we define $A_0 = A - p_i(w) + p_i(0)$.
so that $A_0 \in Q_{p_i(0),j}$, and therefore $w - p_i(w)u_i + Au_i = w - p_i(0) + A_0u_i$. By (2) and (3), we can infer that $B_w = B_0 + w$, thus the large set $\{B_w\}_{w \in W}$ is also invariant under translations in $W$. We can also prove that different choice of $R_i$ gives the same design in this case.

4. Example: construction of an LTKS(171)

In this section, we present an example to illustrate our construction, applied to the case $q = 13$ and $n = 2$. The resulting large set is of order $13^2 + 2 = 171$, and is previously unknown. The basis of this construction is the LKTS(15) by Denniston [3], as described below:

$$
\begin{align*}
Q_{0,0} & = \{\{\infty, \infty, 0\}, \{1, 4, 5\}, \{2, 6, 11\}, \{3, 7, 10\}, \{8, 9, 12\}\} \\
Q_{0,1} & = \{\{\infty, 1, 6\}, \{\infty, 2, 8\}, \{0, 10, 12\}, \{3, 5, 9\}, \{4, 7, 11\}\} \\
Q_{0,2} & = \{\{\infty, 1, 2, 5\}, \{\infty, 2, 4, 9\}, \{0, 3, 11\}, \{1, 7, 12\}, \{6, 8, 10\}\} \\
Q_{0,3} & = \{\{\infty, 1, 3, 12\}, \{\infty, 2, 5, 7\}, \{0, 4, 6\}, \{1, 8, 11\}, \{2, 9, 10\}\} \\
Q_{0,4} & = \{\{\infty, 1, 4, 10\}, \{\infty, 2, 11, 12\}, \{0, 5, 8\}, \{1, 2, 3\}, \{6, 7, 9\}\} \\
Q_{0,5} & = \{\{\infty, 1, 7, 8\}, \{\infty, 2, 3, 6\}, \{0, 1, 9\}, \{2, 4, 12\}, \{5, 10, 11\}\} \\
Q_{0,6} & = \{\{\infty, 1, 9, 11\}, \{\infty, 2, 1, 10\}, \{0, 2, 7\}, \{3, 4, 8\}, \{5, 6, 12\}\}
\end{align*}
$$

and $Q_{i,j} = Q_{0,j} + i$ for $i \in \mathbb{F}_{13}$. We denote the point $(a, b) \in \mathbb{F}_{13}^2$ by $ab$, and choose $g = 2, \omega = g^4 = 3, V = \{0_1, 0_1, 1_1, 1_2, 1_3, 1_4, 1_5, 1_6, 1_7, 1_8, 1_9, 1_{10}, 1_{11}, 1_{12}\}$, and the skew-symmetric bilinear form $f(a_b, c_d) = ad - bc$. By Remark 3.7, the 169 designs can be obtained as translations of a base design $B_{0_0}$ on the set $X = \{\infty_1, \infty_2\} \cup \mathbb{F}_{13}^2$.

By (3), we obtain the first parallel class of $B_{0_0}$:
\[ P_{0,0} = \{ \{\infty, \infty, 0, 0\}, \{10, 40, 50\}, \{30, 70, 100\}, \{20, 60, 110\}, \{80, 90, 120\}, \{01, 04, 05\}, \{05, 07, 100\}, \{02, 06, 011\}, \{08, 09, 012\}, \{11, 44, 55\}, \{33, 77, 1010\}, \{22, 66, 1111\}, \{88, 99, 1212\}, \{21, 84, 105\}, \{63, 17, 710\}, \{42, 126, 911\}, \{38, 59, 1112\}, \{31, 124, 25\}, \{93, 87, 410\}, \{62, 56, 711\}, \{118, 19, 1012\}, \{41, 34, 75\}, \{123, 27, 1110\}, \{82, 116, 511\}, \{68, 109, 912\}, \{51, 74, 125\}, \{23, 97, 1110\}, \{102, 46, 311\}, \{18, 69, 812\}, \{61, 114, 45\}, \{53, 37, 810\}, \{122, 106, 1111\}, \{98, 29, 712\}, \{71, 24, 95\}, \{83, 107, 510\}, \{12, 36, 1211\}, \{48, 119, 612\}, \{81, 64, 15\}, \{113, 47, 210\}, \{32, 96, 1011\}, \{128, 79, 512\}, \{91, 104, 65\}, \{13, 117, 1210\}, \{52, 26, 811\}, \{78, 39, 412\}, \{101, 14, 115\}, \{43, 57, 910\}, \{72, 86, 611\}, \{28, 129, 312\}, \{111, 54, 35\}, \{73, 127, 610\}, \{92, 16, 411\}, \{108, 89, 212\}, \{12, 94, 85\}, \{103, 67, 310\}, \{112, 76, 211\}, \{58, 49, 112\}\} \]

We describe one of the other 84 classes, namely \( P_{0, u_1 = 10, 1} \). By (2), this class is given by \( P_{0, u_1} = P_{u_1, 1} \cup \{-p_1(0) u_1 + A u_1 | A \in Q_{p_1(0), 1}\} = P_{u_1, 1} \cup \{A u_1 | A \in Q_{0, 1}\} \). The complete description of the class is given below:
\[ P_{0,0,1,1} = \{ \{\infty, 1, 0, 6\}, \{\infty, 2, 0, 8\}, \{0, 10, 12\}, \{3, 0, 9\}, \{4, 7, 11\}, \{1, 3, 9, 5\}, \{12, 10, 4, 10\}, \{12, 5, 9\}, \{12, 11, 10, 4\}, \{2, 12, 9\}, \{11, 1, 8, 17\}, \{3, 8, 2\}, \{10, 12, 5, 11\}, \{3, 8, 5\}, \{10, 11, 5, 11\}, \{4, 4, 5\}, \{9, 12, 9, 8\}, \{42, 4, 5, 6\}, \{9, 11, 9, 8\}, \{5, 0, 8\}, \{8, 12, 0, 5\}, \{5, 2, 0, 5\}, \{8, 11, 0, 8, 5\}, \{6, 3, 9, 11\}, \{7, 12, 4, 21\}, \{6, 9, 5, 11\}, \{7, 11, 4, 27\}, \{7, 1, 5, 9, 13\}, \{6, 12, 8, 12\}, \{7, 2, 5, 10\}, \{6, 11, 8, 12\}, \{8, 1, 9, 4\}, \{5, 12, 12, 9\}, \{8, 1, 5, 4\}, \{5, 11, 12, 9\}, \{9, 1, 10, 7\}, \{4, 12, 34, 61\}, \{9, 2, 10, 7\}, \{4, 11, 3, 6\}, \{10, 1, 6, 9, 10\}, \{3, 12, 7, 3\}, \{10, 2, 6, 5\}, \{3, 11, 7, 3\}, \{11, 2, 9, 0\}, \{2, 12, 11, 4, 10\}, \{11, 2, 5, 0\}, \{2, 11, 11, 0\}, \{12, 1, 11, 3\}, \{11, 24, 10\}, \{12, 11, 3\}, \{11, 28, 10\}, \{0, 1, 7, 6\}, \{0, 12, 6, 7\}, \{0, 2, 7, 5, 6\}, \{0, 11, 6, 8, 7\} \}

For a complete list of the 85 parallel classes of \( B_{0,0} \), see the appendix.

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Appendix

Here we give a complete list of the 85 parallel classes of the base design $\mathcal{B}_0$, constructed in Section 4. The point $a_b$ is represented as $ab$, A,B,C denotes 10, 11 and 12 respectively, and $\infty_1$ are denoted by $XX$ and $YY$. 