Semiclassical superoscillations: interference, evanescence, post-WKB

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Abstract

The concept of superoscillations is extended beyond bandlimited functions, to include monochromatic waves in space-varying media, such as wavefunctions representing quantum particles in non-constant potentials. ‘Semiclassical superoscillations’ are defined as places where the phase gradient (local wavevector, or weak momentum) exceeds the classically allowed momentum. The phenomenon is essentially quantum (or, in optics, essentially beyond geometrical optics), and can arise in three different ways: from interference, evanescence (e.g. barrier penetration), and WKB corrections. Illustrations are given, including superpositions of few and many states in a linear potential, angular-momentum circular harmonic-oscillator states, eigenstates of degenerate harmonic-oscillator modes, and above-barrier scattering.

1. Introduction

As currently studied, superoscillations [1, 2] occur in bandlimited functions that in some regions vary faster than their fastest Fourier component. There have been many mathematical investigations [3, 4], and many applications [2]. It is well understood that this mathematical phenomenon occurs where functions possess very small values, as a result of near-perfect destructive interference between their Fourier components. If the functions represent waves, as we consider here, this physical interpretation indicates that superoscillations are regions that are nonclassical, if the waves are quantum, or, if the waves are electromagnetic, places where geometrical optics fails qualitatively. This is the aspect we will explore here, usually employing quantum terminology for brevity.

Superoscillations join a number of known signals of quantumness, for example negative values of Wigner functions and negative probabilities [5–7], barrier penetration, and the discreteness of energy levels in bound systems. The quantumness of superoscillations is obvious from one of the origins of the concept: the phenomenon constitutes a special case of weak measurement theory [8, 9], in which operators acting on a preselected state, that is then restricted by postselection with a different state, can yield values outside the spectrum of the operator: superweak values. For superoscillations, the preselected state represents a wavefunction, the operator is momentum and the postselected state is position [10, 11]. Then the weak value of the momentum operator is (up to a factor $\hbar$) the local wavevector, i.e. the phase gradient of the wave, and superweak momenta correspond to superoscillations. The advantage of displaying superoscillations visually, as we do here, is that the nonclassical regions are made vividly apparent.

This conventional definition of superoscillations, in terms of bandlimitness, seriously restricts the applicability of the concept, because it applies only to quantum particles in regions where the potential is constant, or light where the refractive index is constant. Such waves can be represented as superpositions of plane waves whose wavevectors all have the same length, and superoscillations occur where the phase gradient exceeds this length. When the potential or refractive index varies, causing classically trajectories or optical rays to curve, waves are usually not bandlimited and the conventional definition cannot be applied. Here we explore a simple generalisation that applies to such cases while preserving the essence of the
concept of superoscillations: we consider a wave as superoscillatory at a given spatial location if the length of its phase gradient vector exceeds the classically allowed momentum there. We refer to this phenomenon as *semiclassical superoscillations*.

The paper is structured as follows. Section 2 gives the general theory, defining semiclassical superoscillations and describing their occurrence in three situations: where there is interference between two or more classical paths, in classically forbidden regions, i.e. evanescent waves; and by an extension of WKB theory where a single classical path contributes. Section 3 illustrates semiclassical superoscillations with a number of examples. Section 3.1: superpositions of states in a linear potential; when many such states are included, these superpositions represent 'Airy–Gauss random fields'; section 3.2: angular-momentum eigenstates of isotropic two-dimensional harmonic oscillators; section 3.3: entangled degenerate harmonic-oscillator states; section 3.4: scattering above a potential barrier. The concluding section 4 lists some possible directions for further investigation. The Appendix gives details of the calculation of the superoscillation probability for Airy–Gauss random fields.

To avoid confusion, we note that here we consider stationary states, that is monochromatic waves, not time-dependent ones. For evolving waves, it is always possible to choose initial states that are bandlimited and superoscillatory, and explore whether and how the superoscillations persist. This question has been addressed already [12–14], and we do not pursue it further here.

2. General theory

We consider waves that are complex functions $\psi(r)$ of position $r$ in any number of dimensions $d$; usually we will consider $d = 2$. The quantity to be studied is the local wavevector $k(r)$, defined as the phase gradient of the wave, or, equivalently [11], the weak value of the momentum operator with position postselected:

$$k = \frac{\p}{\hbar} = -i\nabla, \quad k(r) = \nabla \text{arg} \psi(r) = \text{Im} \nabla \log \psi(r) = \text{Re} \frac{\langle r | \hat{k} | \psi \rangle}{\langle r | \psi \rangle}.$$  \hfill (2.1)

The waves to be studied are solutions of the time-independent Schrödinger equation for a particle of mass $m$ moving in a potential $U(r)$, equivalent to the Helmholtz equation with varying refractive index, represented by a position-dependent wavenumber $k_0(r)$:

$$\nabla^2 \psi(r) + k_0^2(r) \psi(r) = 0, \quad k_0(r) = \frac{1}{\hbar} \sqrt{2m (E - U(r))}.$$  \hfill (2.2)

We will often simplify notation by choosing units such that $\hbar^2/2m = 1$.

Our criterion for $r$ to lie in a region of semiclassical superoscillations of $\psi(r)$ is

$$|k(r)|^2 > k_0^2(r).$$  \hfill (2.3)

This requires some explanation. If the potential or refractive index is constant, say $U_0$, $k_0$ is the free-space wavenumber, i.e. $2\pi/(\text{wavelength}$,$\lambda$), and when $\psi(r)$ is a superposition of propagating plane waves it is a bandlimited function, with $k_0^2(r) = 2m (E - U_0)/\hbar^2$, so the criterion is equivalent to the familiar one. In classically forbidden regions $E < U(r)$, $k_0^2(r) < 0$, so the criterion (2.3) implies that such regions of evanescent waves are always superoscillatory, consistent with their interpretation as nonclassical. We note that the criterion for boundaries of the superoscillatory regions, i.e. $|k(r)| = k_0(r)$, were recently identified as zero contours of the Madelung–de Broglie–Bohm quantum potential [15]; semiclassical superoscillations occur where the potential is negative.

Semiclassically, solutions of (2.2) in classically allowed regions are superpositions of WKB waves associated with classical paths, and when there are several such paths their superoscillations are strongest near the places where destructive interference produces intensity zeros, corresponding to phase singularities (= wave vortices = wave dislocations) [10, 16, 17], at which $|k(r)|$ is infinite. When there is only one such classical path, there are no phase vortices, and the question of superoscillations is more delicate. We analyse this in one dimension using WKB theory.

This familiar technique [18, 19] starts by writing the wave—the solution of (2.2)—exactly, in the form

$$\psi(x) = a(x) \exp \left( \frac{i}{\hbar} \int_0^x dx' p(x') \right), \quad a^2(x) = \frac{1}{p(x)},$$  \hfill (2.4)

in which $p(x)$ and $a(x)$ are real functions, also dependent on $\hbar$. The phase gradient $p(x)$ satisfies a (nonlinear) Riccati equation, whose systematic solution generates the $\hbar$ expansion. We need only the
leading correction [20], for which the local wavenumber is

\[ k(x) = \frac{p(x)}{\hbar} = k_0(x) \left(1 + \frac{\hbar^2}{64m} \left(\frac{4U''(x)}{(E - U(x))^2} + \frac{5U'(x)^2}{(E - U(x))^3}\right) + O(\hbar^4)\right). \] (2.5)

With (2.3), this leads to a simple explicit approximate criterion for semiclassical superoscillations associated with a single classical path:

\[ U''(x) > \frac{5}{4} \frac{U'(x)^2}{(E - U(x))^3}. \] (2.6)

3. Illustrative examples

3.1. ‘Falling waves’ in a linear potential

We consider two dimensions, with the following time-independent Schrödinger equation, written after scaling away irrelevant constants:

\[ \left(-\partial_x^2 - \partial_y^2 + y\right) \psi(r) = 0, \quad r = (x, y). \] (3.1)

This is written for energy \( E = 0 \); different energies simply correspond to shifting the wave up or down in \( y \). This equation can represent electrons in a constant electric field, or neutrons falling in a constant gravitational field: ‘falling waves’. Exact solutions involve the Airy function [21, 22], and are superpositions of elementary waves with different \( x \) wavenumbers \( Q \):

\[ \psi_Q(r) = \text{Ai} \left(y + Q^2\right) \exp(iQx). \] (3.2)

The classical counterpart of this zero-energy wave, for given \( Q \), is a family of parabolic trajectories, starting at different horizontal positions \( x_0 \) and reaching a maximum height \(-Q^2\):

\[ x(t) = x_0 + 2Qt, \quad y(t) = -Q^2 - t^2. \] (3.3)

Semiclassically, deep in the classically allowed region \( y < -Q^2 \), asymptotics of \( \text{Ai} \) enables \( \psi_Q \) to be separated into two locally plane waves travelling upwards and downwards:

\[ \psi_Q(r) \approx \psi_{Q+}(r) + \psi_{Q-}(r), \quad \psi_{Q\pm}(r) = \frac{\exp\left(i \left(Qx \pm \frac{3}{2} (-y - Q^2)^{3/2}\right)\right)}{2\sqrt{\pi}(-y - Q^2)^{1/4}}. \] (3.4)

For each such wave, the two local momenta are

\[ k_{Q\pm}(r) = \nabla \arg \psi_{Q\pm}(r) = Qe_\pm \pm \sqrt{-y - Q^2} e_y \Rightarrow |k_{Q\pm}(r)|^2 = -y. \] (3.5)

The length of these vectors is independent of \( Q \), and indeed gives the classically allowed boundary for any superposition, and the criterion (2.3) indicates that superoscillations occur in the classically forbidden region \( y > -Q^2 \). This is for each individual wave (3.2). But for superpositions of such waves, the exact local wavevector, defined by (2.1), is different from \(|k_{Q\pm}(r)|\), and the criterion (2.3) for semiclassical superoscillations is nontrivial:

\[ |k(r)|^2 > -y. \] (3.6)

A rich variety of superpositions of the waves (3.2) can be constructed. One of the simplest is a superposition of three waves:

\[ \psi(r) = \psi_0(r) + \frac{1}{2} i (\psi_{-1}(r) + \psi_1(r)), \]

\[ = \text{Ai} \left(y\right) + i \text{Ai} \left(y + 1\right) \cos x. \] (3.7)

Figure 1(a) depicts this wave using a convention we will follow for subsequent pictures: the region of semiclassical superoscillations is represented by the wave’s phase, colour-coded by hue, and the region that is not superoscillatory is shaded black. Immediately evident are the phase singularities where all colours meet, and \( \psi(r) = 0 \); these points are always superoscillatory because \(|k(r)| \) is infinite there. For (3.7) the phase singularities occur where \( y \) is a zero of the Airy function and \( x = (n + 1/2)\pi \) (\( n \) integer). These superoscillations in the classically allowed region are associated with near-destructive interference between the three waves in this superposition. Note that the superoscillatory regions get closer together deeper into the classically allowed region, i.e. as \( y \) decreases. This is because for large negative \( y \) the classically allowed wavelength, proportional to \( 1/|k_{\text{osc}}| = 1/\sqrt{(-y)} \), which sets the vertical scale of the oscillations, gets smaller.

Figure 1(b) shows the wave amplitude in the same region, with a notation we will use for the other pictures: the region of semiclassical superoscillations is shaded grey.

Figure 1(c) shows the wave amplitude in the same region, with a notation we will use for the other pictures: the region of semiclassical superoscillations is shaded grey.
The contrasting class of superpositions involves many waves, with random phases and an $x$ spectrum spanning a range of values of $Q$:

$$\psi(\mathbf{r}) = \sum_{n=1}^{N} \psi_{Q_n}(\mathbf{r}) \exp(i(Q_n x + \phi_n)), \quad N \gg 1. \tag{3.8}$$

By the central limit theorem, this defines a class of Gaussian random functions, constituting an ‘Airy–Gauss’ ensemble. Superpositions of this type have been studied before, as a model for the nodal structure of random waves influenced by boundaries ([23], and section 4 of [24]).

Figure 1(b) shows a wave chosen from this ensemble. Again the classically forbidden region $y > 0$ is entirely superoscillatory, and in the classically allowed region there are superoscillatory domains centred on the phase singularities. Examinations of figure 1(b) and similar pictures (not shown) that we have calculated for larger regions of the $x, y$ plane reveals a feature of the topology of these semiclassical superoscillatory regions, shared with conventional superoscillatory regions in similar superpositions of plane waves: the regions consist of largely connected thin filaments, with relatively few isolated superoscillations.

It is known that for Gaussian random superpositions of plane waves in two dimensions, the ‘superoscillation probability’, that a randomly chosen position lies in a region of superoscillations, is exactly 1/3 [25] (with similar values in other dimensions [26]). A natural question is: does this result survive the generalisation to varying potentials. The answer, from a detailed calculation in the appendix, is that for Airy–Gauss ensembles the semiclassical superoscillation probability tends to 1/3 deep in the classically allowed region $y \ll 0$, but rises to unity at the classical boundary $y = 0$, as illustrated in figure 2. This result probably holds for more general ensembles of waves in two dimensions with smooth classical boundary curves; in higher dimensions, the details might be slightly different, but the general behaviour is likely to be similar: deep in the classically allowed region, the previously understood value for superpositions of plane waves (close to 1/3), rising to unity at the classical boundary and into the forbidden region.
3.2. 2D isotropic harmonic oscillator: angular-momentum eigenstates

An example where the semiclassical superoscillations are entirely associated with evanescent waves is the discrete set of angular-momentum eigenstates in a two-dimensional harmonic oscillator. The Schrödinger equation is

$$(-\nabla^2 + r^2) \psi(r) = E \psi(r), \quad r = (r \cos \theta, r \sin \theta), \quad (3.9)$$

with wavefunctions, involving the generalised Laguerre polynomials \([21]\), and energies, given by

$$\psi_{l,n}(r) = r^{|l|} \exp \left(-\frac{1}{2} r^2\right) L_{|l|}^{\frac{|l|}{2}} (r^2) \exp (i l \theta), \quad E_{l,n} = 2 (2n + l + 1). \quad (3.10)$$

The local wavevector is directed azimuthally: from (2.1),

$$k(r) = \frac{l}{r} \mathbf{e}_\theta. \quad (3.11)$$

Thus the criterion (2.3) for semiclassical superoscillation is

$$\frac{l^2}{r^2} > E - r^2. \quad (3.12)$$

This shows that superoscillations fill two classically forbidden regions, bounded by circles:

$$r < r_m = \sqrt{\frac{1}{2} (E - \sqrt{E^2 - 4l})} \quad \& \quad r > r_p = \sqrt{\frac{1}{2} (E + \sqrt{E^2 - 4l})}. \quad (3.13)$$

The classical orbits corresponding to the state \(\psi_{l,n}\) constitute a family of closed ellipses with different orientations \(\phi\):

$$x(t) + i y(t) = \frac{1}{2} \exp (i \phi) \left[ (r_p + r_m) \exp (2it) + (r_p - r_m) \exp (-2it) \right]. \quad (3.14)$$

Figure 3 illustrates one of these states, and the corresponding family of classical orbits. In this case, there are no semiclassical superoscillations in the classically allowed region, even though there are radial oscillations (from the Laguerre polynomials), because these are perpendicular to the local wavevector (3.11) and there is no azimuthal interference that would generate phase singularities.

3.3. 2D isotropic harmonic oscillator: entangled 1D states

The 2D isotropic harmonic oscillator is also separable in Cartesian coordinates. Bound states of the Hamiltonian

$$H = \hat{k}_x^2 + \hat{k}_y^2 + \hat{x}^2 + \hat{y}^2, \quad (3.15)$$

are superpositions of the one-dimensional Gaussian-modulated Hermite polynomials \([21]\)

$$\psi_{l,m}(x) = \frac{\exp \left(-\frac{x^2}{2}\right)}{\pi^{1/4} \sqrt{2^m m!}} H_m(x). \quad (3.16)$$

Figure 3. The wave (3.10), for \(l = 20, n = 2\), represented by its phase in the region of semiclassical superoscillations and black elsewhere, including the corresponding family of elliptical classical orbits (white).
The phase gradient, and the semiclassical superoscillation criterion (2.3), are

$$|k_{m,n}(r)|^2 = |\text{Im} \nabla \log |\psi_{m,n}(r)|^2 > E_{m,n} - x^2 - y^2.$$  

The classically allowed region is the disk with radius $r = \sqrt{2(m+n+1)}$.

Figure 4 depicts two examples. Each illustrates two kinds of superoscillations: the classically forbidden (evanescent) region outside the black circle, and destructive interference in the classically allowed region within the circle. As anticipated, the number of phase singularities increases with the quantum numbers, i.e. semiclassically.

3.4. Above-barrier scattering
Waves above a parabolic barrier provide a convenient example to illustrate the post-WKB criterion (2.6) for one-dimensional waves where there is only one classical path, and also to create entangled superpositions in two dimensions. The 2D Hamiltonian, for which all $x$ is classically allowed (for $E > 0$), is

$$\hat{H} = \hat{k}_x^2 + \hat{k}_y^2 - \hat{x}^2 - \hat{y}^2.$$  

Degenerate eigenstates in the continuum are combinations of the one-dimensional eigensolutions, involving parabolic cylinder functions [21]:

$$\psi_1(x,E,c_1,c_2) = c_1D_{-\frac{1}{2}i}((1-i)x) + c_2D_{-\frac{1}{2}i}((1+i)x).$$  

The contribution with coefficient $c_1$ represents a complex-valued wave with energy $E$ incident from $x = -\infty$, with a transmitted wave travelling towards $x = +\infty$; for $x < 0$ there is a reflected wave interfering with the incident wave and travelling back towards $x = -\infty$. The coefficient $c_2$ represents the parity-reversed wave, i.e. incident from $x = +\infty$.

For $x > 0$, there is a single transmitted wave travelling in the $+x$ direction, so the post-WKB expression (2.5), can be applied to this potential $U(x) = -x^2$, for which the classical wavenumber is

$$k_0(x) = \sqrt{E + x^2}.$$  

The resulting phase gradient is

$$k(x) = k_0(x) \left(1 + \frac{1}{32(E + x^2)^3} \left(12x^2 - 8E\right) + \cdots\right).$$  

The criterion (2.3) for semiclassical superoscillations is therefore $x > \sqrt{2E/3}$.

This one-dimensional wave is illustrated in figure 5. In figure 5(a) the oscillations of $|\psi_1(x)|$ for $x < 0$ represent interference between the incident and reflected waves; for $x > 0$ there is only the transmitted wave, and therefore no interference oscillations in the intensity. In figure 5(b), the same phenomena are illustrated by the local wavenumber $k(x)$. For $x < 0$ the interference oscillations above the classical curve indicate semiclassical superoscillations, and for $x > 0$ the quantum $k(x)$ closely approaches the classical momentum $k_0(x)$. In figure 5(c) the magnification for $x > 0$ shows how the post-WKB approximation (3.22) for the quantum wavenumber gets better as $x$ increases. Superoscillations correspond to the curves.

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**Figure 4.** Superoscillatory regions for the isotropic harmonic oscillator entangled states (3.11), for (a) $(m,n) = (2,4)$; (b) $(m,n) = (7,4)$. 

It will suffice to consider the simplest complex entangled superposition of separable product states, namely

$$\psi_{m,n}(r) = \psi_{1,m}(x)\psi_{1,n}(y) + i\psi_{1,m}(x)\psi_{1,n}(y), \quad \text{energy } E_{m,n} = 2(m+n+1).$$  

The resulting phase gradient is

$$\psi_{1,n}(x) = \psi_{1,m}(x)\psi_{1,n}(y), \quad \text{energy } E_{m,n} = 2(m+n+1).$$  

The phase gradient, and the semiclassical superoscillation criterion (2.3), are

$$|k_{m,n}(r)|^2 = |\text{Im} \nabla \log |\psi_{m,n}(r)|^2 > E_{m,n} - x^2 - y^2.$$  

The classically allowed region is the disk with radius $r = \sqrt{2(m+n+1)}$. 

Figure 4 depicts two examples. Each illustrates two kinds of superoscillations: the classically forbidden (evanescent) region outside the black circle, and destructive interference in the classically allowed region within the circle. As anticipated, the number of phase singularities increases with the quantum numbers, i.e. semiclassically.
Figure 5. The one-dimensional wave (3.20) with \( c_1 = 1, c_2 = 0 \) and energy \( E = 1 \). (a) Re \( \psi_1(x) \) (red curve); Im \( \psi_1(x) \) (full black curve); \( |\psi_1(x)| \) (dashed black curve). (b) Local wavenumber \( k(x) \) (phase gradient) (red), and the classical momenta \( \pm k_0(x) \) (black curves). (c) Quantum/classical ratio \( k(x)/k_0(x) \), red: exact, black: the ratio with \( k(x) \) replaced by the approximation (3.22). (In (b), the lower black curve corresponds to the solution (3.20) with \( c_1 = 0, c_2 = 1 \).)

lying above 1; the fact that the approximation (black curve) overestimates the onset of superoscillations by approximately a factor 2 is unsurprising because the first-order WKB correction is inadequate so close to the potential barrier at \( x = 0 \).

In order to extend this function to two dimensions, we first create the separable product state with energy \( E \):

\[
\psi_{\text{sep}}(x, y, E, 1, 1, 0, 0) = \psi_1(x, E, 1, 0, 0) \psi_1(y, E - 1, 0, 0). \tag{3.23}
\]

Figure 6(a) illustrates a particular such product, of a wave incident from \( x = -\infty \) and one from \( y = -\infty \), i.e. an incident (plus reflected) wave in the third quadrant:

\[
\psi_{\text{sep}}(x, y) = \psi_1\left(x, 1, \frac{1}{2}, 0\right) \psi_1\left(y, 1, \frac{1}{2}, 0\right). \tag{3.24}
\]

Interferences are too weak to generate phase singularities, but there are still superoscillations. Separability is indicated by the rectilinear structure of the superoscillatory region. Nonseparable states, i.e. bipartite entangled states, are general superpositions of these products, with the general form

\[
A\psi_{\text{sep}}(x, y, E, 1, 1, 0, 0, 0, 0) + B\psi_{\text{sep}}(x, y, E, 2, 0, 0, 0, 0). \tag{3.25}
\]

Figure 6(b) shows the particular combination

\[
\psi_{\text{nonsep}}(x, y) = \psi_1\left(x, 1, \frac{1}{2}, 0\right) \psi_1\left(y, 1, \frac{1}{2}, 0\right) - \frac{3}{4} \psi_1\left(x, 1, \frac{1}{4}, 0\right) \psi_1\left(y, 1, \frac{3}{4}, 0\right). \tag{3.26}
\]
Figure 6. Superoscillations for waves above the potential barrier $U(r) = -x^2 - y^2$; (a) the separable state (3.24); (b) the nonseparable (entangled) state (3.26).

Now there are phase singularities in the interference region $x < 0, y < 0$ (third quadrant), but none in the transmitted wave region $x > 0, y > 0$ (first quadrant).

4. Concluding remarks

We have explored a generalisation of conventional superoscillations, applicable for bandlimited functions, to functions representing waves that need not be bandlimited, in which ‘semiclassical superoscillations’ occur where the phase gradient exceeds the local classically allowed momentum rather than the bandlimit. Highlighting the regions of superoscillation gives a vivid representation of the nonclassicality of such waves. Three different kinds of superoscillation can occur, associated with destructive interference between waves corresponding to classical paths, evanescent waves in classically forbidden regions, and semiclassical corrections in classically allowed regions where a single path contributes.

The theoretical ideas studied here could be pursued in a number of directions. Most important is to find ways of experimentally observing the superoscillatory regions in quantum or optical waves. Other possible directions are:

- Investigating different kinds of random waves, such as those with statistics that are anisotropic, or non-Gaussian;
- Understanding the largely connected (percolation) topology of the superoscillatory regions, indicated in figure 1(b) and similar calculations;
- Further extending the concept of superoscillations to include magnetic fields, where the Schrödinger equation replacing (2.2) includes a vector potential $A(r)$, and the semiclassical superoscillations are naturally determined by the weak values of the kinetic momentum $\hat{p} - A(\hat{r})$ rather than the canonical momentum $\hat{p}$ (for a preliminary discussion for the Aharonov–Bohm effect, see [15]).
- Investigating ways of rendering pictorially the superoscillatory regions for waves in dimensions $d > 2$;
- Developing existing ideas [27] that superoscillations are much weaker for vector waves, by investigating their superoscillatory regions;
- Investigating generalisations of the phase gradient as a descriptor of superoscillations, to describe the oscillations of waves represented by real functions of position. An earlier proposal [28] was to use the weak value of the square of one of the momentum components, e.g. $-\partial^2_x \psi(r) / \psi(r)$, but a definitive and more symmetrical generalisation, applicable in any dimension, is lacking.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).
Appendix. Superoscillation probability for Airy–Gauss ensemble of falling waves

Let \( P_k(\mathbf{k}, y) \) denote the probability that at a randomly chosen \( x \) and given \( y \) the local wavevector \( \mathbf{k}(r) \), defined by (2.1), has the value \( \mathbf{k} \). Then, using the criterion (3.6), the probability that the wave at the chosen point is superoscillatory, conveniently written as 1 minus the probability that it is not, is, with \( k = k \{ \cos \theta, \sin \theta \} \),

\[
P_{\text{super}}(y) = 1 - \Theta (-\gamma) \int_0^\infty dk \int_0^{2\pi} d\theta \, P_k(\mathbf{k}, y) . \tag{A.1}
\]

Therefore it is necessary to calculate

\[
P_k(\mathbf{k}, y) = \langle \delta (\mathbf{k} - \mathbf{k}(r)) \rangle = \frac{1}{(2\pi)^2} \int d\mathbf{s} \exp \left( -is \cdot \mathbf{k} \right) \exp \left( is \cdot \mathbf{k}(r) \right) , \tag{A.2}
\]

in which \( \langle \ldots \rangle \) denotes ensemble averaging. It is convenient to separate the wave into its real and imaginary parts, i.e.

\[
\psi(r) = u(r) + iv(r) . \tag{A.3}
\]

Then \( u \) and \( v \) are real Gaussian random functions, and the local wavevector, whose distribution we need to calculate, is

\[
\mathbf{k}(r) = \frac{u(r) \nabla v(r) - v(r) \nabla u(r)}{u(r)^2 + v(r)^2} . \tag{A.4}
\]

Therefore we need averages involving the six quantities \( u, v \) and their \( x \) and \( y \) derivatives.

The statistics of Gaussian random functions are determined by quadratic averages and correlations. These are easily calculated from (3.8) by averaging over the phases \( \phi_q \). We make the natural stipulation that the spectrum of \( x \) wavenumbers \( Q_x \) is even in \( Q_x \) and consider the simplest case in which this uniform along the \( Q \) axis. Then the only nonzero quadratic averages are

\[
\langle u(r)^2 \rangle = \langle v(r)^2 \rangle = \int_0^\infty dQ A_i^2(y + Q^2) \equiv I_1(y) ,
\]

\[
\langle u_3(r)^2 \rangle = \langle v_3(r)^2 \rangle = \int_0^\infty dQ A_i^2(y + Q^2) \equiv I_2(y) ,
\]

\[
\langle u_2(r)^2 \rangle = \langle v_2(r)^2 \rangle = \int_0^\infty dQ A_i^2(y + Q^2) \equiv I_3(y) ,
\]

\[
\langle u(r) u_3(r) \rangle = \langle v(r) v_3(r) \rangle = \int_0^\infty dQ A_i(y + Q^2) A_i(y + Q^2) \equiv I_4(y) .
\]

The probability distribution of the six-vector

\[
\mathbf{V} = \{ u, v, u_3, u_2, v_3, v_2 \} . \tag{A.6}
\]

can be calculated by inverting the \( 6 \times 6 \) matrix of correlations specified using (A.5). The result, with derivatives denoted by subscripts, is

\[
P_V(V) = \frac{\exp \left( -\frac{1}{2} \mathbf{V} \cdot \mathbf{M}^{-1} \cdot \mathbf{V} \right)}{(2\pi)^3 \det \mathbf{M}} = \frac{\exp \left[ -\frac{1}{2} \left( \frac{\langle x^2 \rangle}{I_2} + \frac{I_2 (\langle u^2 + v^2 \rangle - 2I_3 (\langle u_3 \rangle + \langle v_3 \rangle) + I_3 (\langle u_3 \rangle + \langle v_3 \rangle)^2)}{(I_2 - I_1 \langle I_3 \rangle)} \right) \right]}{(2\pi)^3 I_2 (I_1 I_3 - I_2^2)} . \tag{A.7}
\]

To calculate the exponential average in (A.2), it is convenient to integrate first over the four derivatives \( u_3, u_2, v_3, v_2 \), and then \( u \) and \( v \). The result is

\[
\langle \exp (s \cdot \mathbf{k}(r)) \rangle_V = \frac{1}{2\pi I_1} \int du \int d\gamma \exp \left( -\frac{u^2 + \gamma^2}{2I_1} - \frac{I_1 I_2 s_k^2 + (I_1 I_3 - I_2^2) s_s^2}{2I_1 (u^2 + \gamma^2)} \right) . \tag{A.8}
\]

\[
= A(\theta_s, y) sK_1(A(\theta_s, y) s) ,
\]

\[
A(\theta_s, y) = \frac{1}{\pi} \int_0^{2\pi} d\theta \, \exp \left( -\frac{1}{2I_1} \left( \cos \theta - \frac{s}{s_s} \right)^2 - \frac{I_1 I_3 - I_2^2}{2I_1 (u^2 + \gamma^2)} \right) .
\]
in which
\[ A(\theta_k, y) \equiv \frac{1}{I_1} \sqrt{\frac{\alpha(y)}{\cos^2 \theta_k + \beta(y)}} \left( 1 - \gamma(y) \right) \sin^2 \theta_k, \]
(A.9)

involving the following ratios of the integrals (A.5):
\[ \alpha(y) \equiv \frac{I_1(y)}{I_2(y)}, \quad \beta(y) \equiv \frac{I_1(y)}{I_5(y)} \quad \gamma(y) \equiv \frac{I_1(y)^2}{I_1(y)I_5(y)}. \]
(A.10)

To evaluate the Fourier transform and \( k \) integral in (A.2), it is convenient to integrate first over the directions \( \theta_k \) of \( k \), then the length of the vector \( s \):
\[ \frac{1}{(2\pi)^2} \int ds \int s \cdot s \exp(-i k \cdot s) \exp(i k (r) \cdot s) \cdot \]
\[ = \frac{A(\theta_k, y)}{2\pi} \int ds s^2 K_1(A \theta_k, y) s \int_{0}^{\infty} = \frac{A(\theta_k, y)^2}{\pi(A(\theta_k, y)^2 + k^2)}. \]
\[ \text{Then integrating over the length } k \text{ gives, finally, the superoscillatory probability:} \]
\[ P_{\text{super}}(y) \approx 1 - \frac{\Theta(-y)|y|}{\left(1 + \alpha(y)|y|\right)\left(1 - \gamma(y) + \beta(y)|y|\right)}. \]
(A.12)

This function is plotted in figure 2.

To understand the limiting behaviour for \( y \to -\infty \), we need the asymptotics of the Airy function in the integrals (A.5). The lowest approximation suffices, namely [21]
\[ \text{Ai}(y + Q^2) \approx \frac{\Theta(y - Q^2)}{2\pi \sqrt{|y - Q^2|}} (y \ll -1). \]
(A.13)

Using these results in \( I_1, I_2 \) and \( I_5 \), and an analogous procedure for \( I_4 \), leads to
\[ \alpha(y) \approx \beta(y) \approx \frac{2}{|y|} \quad \gamma(y) \approx \frac{\text{oscillatory}}{\pi|y|^{3/2}}, \quad (y \ll -1). \]
(A.14)

and then (A.12) gives
\[ P_{\text{super}}(y) \xrightarrow{y \to -\infty} \frac{1}{3}. \]
(A.15)

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