The $c_2$ invariant is invariant

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Abstract

In this article we continue the work started by F. Brown, O. Schnetz and K. Yeats in [7] and prove that for a big subset of log-divergent graphs the $c_2$ invariants in all 4 different representations of the Feynman period (parametric and dual parametric representations, position and momentum spaces) coincide.

Introduction

For a connected graph $G$ with $N_G$ edges, $n_G + 1$ vertexes, and $h_G := N_G - n_G$ cycles, the graph polynomial and the dual graph polynomial are defined by

$$
\Psi_G = \sum_T \prod_{e \notin T} \alpha_e \quad \text{and} \quad \varphi_G = \sum_T \prod_{e \in T} \alpha_e \in \mathbb{Z}[\alpha_1, \ldots, \alpha_{N_G}],
$$

with $\alpha_i$s the Schwinger parameters (edge variables) and $T$ running over all spanning trees of $G$. Recall that a graph $G$ is said to be log-divergent if $N_G = 2h_G$, and a log-divergent graph $G$ is primitive log divergent if for any proper subgraph $\gamma \subset G$ the following inequality holds: $2h_\gamma < N_\gamma$. It the case $G$ is log-divergent, one has the associated Feynman period $I_G$ defined by an integral of a differential form with double poles along $\Psi_G = 0$. Similarly, the other form of the Feynman period is the integral $I_G^{\text{dual}}$ with poles along $\varphi_G$ with inverted variables. The more natural representation for physicists is the one in momentum space ($I_G^{\text{mom}}$)(see [12]), while the position space ($I_G^{\text{pos}}$) is where some good techniques effectively help in the computations, as Gegenbauer polynomials ([9]), etc. The connection of these different approaches are shown on the following diagram:

\[ \text{parametric space} \quad \leftrightarrow \quad \text{Schwinger trick} \quad \leftrightarrow \quad \text{momentum space} \]

\[ \text{Cremona transform.} \quad \leftrightarrow \quad \text{Fourier transform.} \]

\[ \text{dual parametric space} \quad \leftrightarrow \quad \text{Schwinger trick} \quad \leftrightarrow \quad \text{position space} \]

Figure 1

For a primitive log-divergent graph the 4 integrals defined in this spaces give the same value (up to multiplication by $\pi^2$). See [14], Section 2 for more explanation.
In practice, it’s quite complicated to compute the period $I_G$ analytically in any of these representations, and usually can be done only for small graphs. On the other hand, the values of $I_G$ for many known examples of graphs are lying in the $\mathbb{Q}$-algebra spanned by multiple zeta values (MZV), see [3], [14]. One knows the deep connection of MZV to algebraic geometry and to mixed Tate motives. This motivates the study of the arithmetic and algebraic nature of the poles of $I_G$, i.e. of the graph hypersurface $X_G$ defined by the vanishing of $\Psi_G = 0$ in affine (or projective) setting.

For the structure of $\Psi_G$ see [4], [8]. The Kontsevich conjecture on the number of rational points of $X_G$ was discussed in [1], [14], [11], [6]. The cohomological approach for study of $X_G$ and motivic point of view on the Feynman period can be found in [2], [10], [5].

Recall that for $G$ with $n_G \geq 2$ one has the congruence $\#X_G(\mathbb{F}_q) \equiv 0 \mod q^2$ counting $\mathbb{F}_q$-rational points for a fixed $q$ of (the base change to $\mathbb{F}_q$ of) $X_G$. One defines

$$c_2(G)_q := \#X_G(\mathbb{F}_q)/q^2 \mod q.$$  

Motivated by the known examples, one makes the following conjecture (see Conjecture 5 in [6]):

**Conjecture 1.** If $I_{G_1} = I_{G_2}$ for two primitive log-divergent graphs $G_1$ and $G_2$, then $c_2(G_1)_q = c_2(G_2)_q$.

In other words, $c_2$ invariant should play a role of a discrete analogue of the Feynman period. One can even define the $c_2(G)$ invariant in the Grothendick ring $K_0(Var_k)$ of varieties over a field, and can ask for the same question (this is partially done in [7] and in our article in dual setting). Since we have no Chevalley-Warning vanishing in $K_0(Var_k)$ (by the result of Huh in [13]), the question is more complicated.

It was natural to expect the existence and coincidence of the analogues of the $c_2(G)_q$ invariants in all 4 spaces in Figure 1, since the values of the integral representations coincide.

The relation on the level of the $c_2$ invariant in the upper row in Figure 1 was studied in [7]. There was defined the $c_2^{mom}(G)_q$ invariant for a graph with $N_G \leq 2h_G$, $h_G \geq 2$ in Proposition-Definition 17 in [7], and then there was proved the following theorem (see Theorem 18 loc. cit.):

**Theorem 2.** Let $G$ be a log-divergent graph (i.e. $N_G = 2h_G$) with $h_G \geq 3$. Then the $c_2$ invariants in parametric and momentum spaces coincide:

$$c_2^{mom}(G)_q = c_2(G)_q.$$  

In this article we discuss the analogues of $c_2$ invariant for the remaining two spaces: dual parametric and position spaces.

In section 1 we study the properties of the dual graph polynomial $\varphi_G$ and define $c_2(G)_{dual}$. The situation is very similar (but dual) to the case of $\Psi_G$. 

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Section 2 contains the computation of the classes of the dual graph hypersurface and of its singular locus in the Grothendiek ring, this is a translation of the results for Ψ$_G$ from [7] to our setting with minor modifications.

In section 3 we do the computations for point-counting functions in position space. We try to follow a similar strategy to the one was used in [7] for the case momentum space. We define $c_2^{pos}(G)_q$ out of the configuration of quadrics (in the vertex variables) in the denominator of the differential form of $I^{pos}_G$, and then prove

**Theorem 3.** For a log-divergent graph $G$ with $n_G \geq 3$, the $c_2$ invariants in dual parametric space and in position space coincide:

$$c_2^{dual}(G)_q = c_2^{pos}(G)_q.$$

Section 4 contains the proof of the coincidence of $c_2$ invariants in the left column of Figure 1. Hence, together with theorems above, one gets the coincidence of all 4 $c_2$ invariants. For the proof we need to restrict to the graphs we call duality admissible (see Definition 29). This class contains log-divergent graphs which are planar or have enough triangles.

**Theorem 4.** Let $G$ be a duality admissible graph. Then the $c_2$ invariants for parametric and for dual parametric representations coincide:

$$c_2^{dual}(G)_q = c_2(G)_q.$$  

This is the main theorem of the article. This part of the Figure 1 was assumed to be the hardest one, see the discussion at the end of Section 3 in [7].

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1 Dual graph polynomials

We start with a graph $G$ that consists of the set of vertexes $V(G)$ and the set of edges $E(G)$. We define $N_G := |E(G)|$ and $n_G = |V(G)| - 1$. The Euler formula then implies that $h_G := N_G - n_G$ is the number of cycles (independent loops). This $h_G$ can be also seen as the rank of the first homology group of $G$ ([2], Section 2). We use the index set $I_N := \{1, \ldots, N_G\}$ for labelling of the elements of the set $E(G)$, so $E(G) := \{e_i\}_{i \in I_N}$. To each edge $e_i$, we associate a variable (Schwinger parameter) $\alpha_i$.

For a connected graph $G$, one defines the first Symanzik polynomial, or simply the graph polynomial, denoted by $\Psi_G$ as in [11]. Equivalently, $\Psi_G$ is defined as the determinant of the matrix

$$M(G) = \begin{pmatrix} \Delta(\alpha) & E \\ -E^T & 0 \end{pmatrix} \in \text{Mat}_{N+n,N+n}(\mathbb{Z}_{\{\alpha_i\}_{i \in I_N}}),$$

(6)
where $\Delta(\alpha)$ is the diagonal matrix with entries $\alpha_1, \ldots, \alpha_N$, and $E \in \text{Mat}_{N,n}(\mathbb{Z})$ is the incidence matrix after deleting the last column, $N = N_G$, $n = n_G$ (see [4], Section 2.2). Out of this matrix, one can define the Dodson polynomials $\Psi_{I,J}^{G,K}$ by $\Psi_{I,J}^{G,K} := \det M(G)(I; J)_K$, where $M(G)(I; J)_K$ obtained from $M(G)$ after removing rows indexed by $I$ and columns indexed by $J$, and after putting $\alpha_t = 0$ for all $t \in K$. For simplicity, we write $\Psi_{I,K}^{G}$ for $\Psi_{I,I}^{G,K}$. These Dodson polynomials satisfy many identities like contraction-deletion formula, the first and second Dodson identities, etc. (see [4]).

In contrast to $\Psi_{G}^{I}$, one also has $\varphi_{G}^{I} = \sum_{T} \prod_{e \in T} \alpha_e \in \mathbb{Z}[\{\alpha_i\}_{i \in I_N}]$, \hspace{1cm} (7)

the dual graph polynomial. Let $\iota : \mathbb{Z}[\{\alpha_i\}_{i \in I_N}] \to \mathbb{Z}[\{\alpha_i\}_{i \in I_N}]$ be the following Cremona transformation: for a polynomial $P \in \mathbb{Z}[\{\alpha_i\}_{i \in I_N}]$ dependant on the variables indexed by $I$, $\iota(P)(\alpha_1, \ldots, \alpha_N) = P(\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_N}) \prod_{i \in I} \alpha_i$. We often call the application of this transformation simply the dualization. By the very definition, $\varphi_{G} = \iota(\Psi_{G}^{I})$. Define $\varphi_{I,J}^{G} := \iota(\Psi_{I,J}^{G})$. Starting with the contraction-deletion formula for a graph polynomial, $\Psi_{G}^{I} = \Psi_{G,\alpha_k}^{k} + \Psi_{G,k}^{G}$ (formula (11) in [7]), inverting the variables and multiplying with $\prod_{i \in I_N \setminus k} \alpha_i$, one gets the similar-looking contraction-deletion formula for the dual graph polynomial:

$$\varphi_{G}^{I} = \varphi_{G,\alpha_k}^{k} + \varphi_{G,k}^{G}$$ \hspace{1cm} (8)

for any $k \in I_N$. Moreover, $\varphi_{G,\alpha_k}^{k} = \varphi_{G/\alpha_k}^{G}$ and $\varphi_{G,k}^{G} = \varphi_{G/\alpha_k}^{G/\alpha_k}$ with $G/\alpha_k$ (resp. $G/\alpha_k$) denoting the graph $G$ after deletion (resp. contraction) of the edge $e_k$.

We can easily derive the formulas for special cases of $G$:

1). If an edge $e_1 \in E(G)$ forms a tadpole (self-loop), then

$$\varphi_{G} = \varphi_{G/1}. \hspace{1cm} (9)$$

2). If two edges $e_1, e_2 \in E(G)$ form a cycle of length 2 (double-edge), then

$$\varphi_{G} = \varphi_{G/1/2}(\alpha_1 + \alpha_2) + \varphi_{G/12}. \hspace{1cm} (10)$$

For $I \cap J = \emptyset$ and $|I| = |J|$ define $\varphi_{I,J}^{G} := \iota(\Psi_{I,J}^{G})$. Sometimes we fix $G$ and omit the subscript to make the formulas more readable. Fix two indexes $i \neq j$ and consider the special case of the (first) Dodson identity for $\Psi$:

$$\Psi^i \Psi^j + \Psi \Psi^{ij} = (\Psi^{i,j})^2. \hspace{1cm} (11)$$

This identity follows from the (studied by Dodgson) identities on the minors of a symmetric matrix, knowing that $\Psi_{G}^{I}$ is a determinant of the matrix $M(G)(I; J)_K$. \hspace{1cm}
becomes symmetric after getting rid of the minus sign in the case of an even number of vertexes. Dualizing, we get

\[ \varphi_i \varphi_j + \varphi_{ij} = (\varphi^{ij})^2 \alpha_i \alpha_j. \] (12)

Applying (8) twice and taking the coefficients of \( \alpha_i \alpha_j \), we obtain

\[ \varphi_j \varphi_i + \varphi_{ij} = (\varphi^{ij})^2. \] (13)

Using the expansions of \( \varphi, \varphi^i \) and \( \varphi^j \) in \( \alpha_i \) and \( \alpha_j \) (by (8)), one computes

\[ \varphi_j \varphi_i + \varphi_{ij} = (\varphi^{ij})^2. \] (14)

More generally, define the dual Dodgson polynomials by

\[ \varphi^{IS,JS}_{G,K} := \iota(\Psi_{IK,JK}^{G,S}) \] (15)

for any \( I, J, S, K \subset I_N \) pairwise non-overlapping with \( |I| = |J| \). One immediately gets \( \varphi^{IS,JS}_{G,K} := \iota(\Psi_{I,JG,K/S}^{I,J}) = \varphi^{I,J}_{G,K/S} \).

With this definition we get the non-natural \( \varphi^{I,J} = \varphi_j \) from the point of view of the graph polynomial, but the identities on dual Dodgson polynomials become looking very similar to the case of \( \Psi_G \). The dual Dodgson polynomials satisfy

\[ \varphi^{I,J}_{G,K} = \pm \varphi^{I_t,J_t}_{G,K_t} \pm \varphi^{I,J}_{G,K_t} \] (16)

for \( t \in I_N \setminus (I \cup J \cup K) \) and possibly overlapping \( I, J \). The signs in the formula can be explained by using spanning forest polynomials similar to the case of \( \Psi_G \), see [8], Section 2. Recall ([6], Section 2.2) that the first Dodgson identity is

\[ \Psi_{IKx,JKx}^{IS,JS} - \Psi_{IKx}^{IS,JSax,JSbx} = \pm \Psi_{IKx}^{IS,JS} \Psi_{IKx}^{ISax,JSbx} \] (17)

for \( I, J, S, K \subset I_N \) and non-overlapping, \( |I| = |J| \) and \( a, b, x \in I_N \setminus (I \cup J \cup S \cup K) \). The sign depends on the order of \( a, b \) and \( x \). Dualizing this, we get the (dual) Dodgson identity for dual Dodgson polynomials

\[ \varphi_{S}^{IKx,JKx} \varphi_{Sx}^{IKa,JKb} - \varphi_{Sx}^{IKx,JKx} \varphi_{S}^{IKa,JKb} = \pm \varphi_{S}^{IKx,JKb} \varphi_{S}^{IKa,JKx}. \] (18)

We can also derive the Dodgson identity of the second type for dual Dodgson polynomials by dualizing the one for Dodgson polynomials:

\[ \varphi_{S}^{IKx,JKx} \varphi_{Sx}^{IKb,JK} - \varphi_{Sx}^{IKx,JKx} \varphi_{S}^{IKb,JK} = \pm \varphi_{S}^{IKx,JK} \varphi_{S}^{IKa,JKx}. \] (19)

where \( I, J, S, K \subset I_N \) are non-overlapping, \( |I| = |J| + 1 \) and \( a, b, x \in I_N \setminus (I \cup J \cup S \cup K) \).

Define the resultant \([f, g]_k \) of two polynomials \( f = f^k \alpha_k + f_k \) and \( g = g^k \alpha_k + g_k \) linear in a variable \( \alpha_k \) by \([f, g]_k = f^k g - f_k g^k \). The next lemma is the analogue of Lemma 21 in [7] with the same proof.
Lemma 5. For any 3 distinct edges $i, j, k$ of $G$ the following identity holds
\[
[\varphi^i, \varphi^j]_k = \varphi^{ij,jk}\varphi^{i,k} - \varphi^{ij,jk}\varphi^{i,j}. \tag{20}
\]

Proof. The proof is completely the same as in [7] after replacing $\Psi_G$ with $\varphi_G$ because of the similarity of the Dodgson identities (18) and the contraction-deletion formulas (8) for $\Psi_G$ and $\varphi_G$. \hfill \Box

Corollary 6. Fix an element $k \in I_N$ and let $I$ be the ideal of $\mathbb{Q}[\{\alpha_i\}_{i \in I_N}]$ generated by $\varphi^k$ and $\varphi_k$ for some $k$. Then
\[
[\varphi^i, \varphi^j]_k \in \text{Rad}(I) \tag{21}
\]

Proof. Using (18), one computes
\[
(\varphi^i,k)^2 = [\varphi^i, \varphi^j]_k = [\varphi^i]_k = \varphi^k\varphi_i^k - \varphi^k\varphi_i^k \in I. \tag{22}
\]
Thus $\varphi^i,k \in I$ and similarly $\varphi^i,j \in I$. The lemma above implies the statement. \hfill \Box

Proposition 7. Let $G$ be a graph with edges $E(G)$ labelled with the set $I_N$ and $I = \{1, 2, \ldots, t\} \subset I_N$ a subset.

i. If the edges labelled with $I$ form a corolla (all the edges incident to one fixed vertex), then
\[
\varphi_{G,1} = \sum_{i \in I \setminus 1} \lambda_i \alpha_i \varphi^{1,i}_G, \quad \text{where } \lambda_i = \pm 1. \tag{23}
\]

ii. If the edges labelled with $I$ form a (topological) loop, then
\[
\varphi_{G}^1 = \sum_{i \in I \setminus 1} \lambda_i \varphi^{1,i}_G, \quad \text{where } \lambda_i = \pm 1. \tag{24}
\]

Proof. For part i, we start with the formula for the graph polynomial with given edges forming a corolla $\Psi^1_G = \sum_{i \in I} \lambda_i \varphi^{1,i}_G$ (see Lemma 31 in [4]). Dualization immediately gives (23). For the part ii, with edges forming a loop, we can just dualize the formula $\Psi_{G,1} = \sum_{i \in I} \lambda_i \varphi^{1,i}_G$ that was proved in [7], Proposition 24. \hfill \Box

Corollary 8. Let $G$ be a connected graph with more than 1 edge and let us fix any edge of $G$, say $e_1$. Then there exists a subset $I = \{1, \ldots, t\} \subset I_N$ such that $\varphi^1$ lies in the radical $\text{Rad}(I)$ of the ideal $I \subset \mathbb{Z}[\{\alpha_i\}_{i \in I_N \setminus 1}]$ spanned by $\varphi^1$, and $\varphi^{1,i}_G$ for all $i \in I \setminus 1$. 

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Proof. Since $G$ is connected, one of the endpoints of the edge $e_1$ has degree bigger than 1. For $I \subset I_N$ we take the set that indexes the edges of the corolla of that endpoint. Using the Dodgson identity \((15)\) or \((14)\), one computes

\[
(\varphi^1_i)^2 = [\varphi^1_i, \varphi^i]_1 = [\varphi^i + \varphi^i\alpha_i, \varphi^i]_1 = \varphi^1_i \varphi^1_i - \varphi^1_i \varphi^1_i \in \mathcal{I}.
\]  

(25)

Thus, $\varphi^1_i \in \text{Rad}(\mathcal{I})$ for each $i \in I \setminus 1$. Now Proposition 7, part i implies the statement. \hfill \qed

We return to the representation for $\Psi_G$ as a determinant of the matrix \((6)\). Working with blocks, we can modify the matrix as follows:

\[
\begin{pmatrix}
\Delta(\alpha) & E \\
-E^T & 0
\end{pmatrix}
\begin{pmatrix}
J_N & -\Delta(\alpha)^{-1}E \\
0 & J_n
\end{pmatrix} =
\begin{pmatrix}
\Delta(\alpha) & 0 \\
-E^T & E^T \Delta(\alpha) E
\end{pmatrix}
\]  

(26)

with $\Delta(\alpha)^{-1} := \Delta(\alpha)^{-1}$. Here $J_d$ denotes the $d \times d$ identity matrix for $d = n, N$. Taking determinants of both sides, we get

\[
\Psi(\alpha) \cdot 1 = \prod_{i \in I_N} \alpha_i \cdot \det(E^T \Delta(\alpha) E).
\]

(27)

Substituting $\alpha_i \mapsto 1/\alpha_i$, $i \in I_N$, one obtains

\[
\varphi(\alpha) = \prod_{i \in I_N} \alpha_i \cdot \Psi(1/\alpha) = \det(E^T \Delta(\alpha) E).
\]

(28)

It is convenient to define

\[
P_G(\alpha) := E^T \Delta(\alpha) E \in \text{Mat}_{n\times n}(\mathbb{Z}[\{\alpha_i\}_{i \in I_N}])
\]

(29)

with $\varphi_G = \det P_G(\alpha)$ as above. One easily sees that the matrix $P(\alpha)$ can be written as $P(\alpha) = \sum \alpha_i P_i$, where $P_i \in \text{Mat}_{n\times n}(\mathbb{Z})$ for an edge $e_i = (v_s, v_t)$ has entry 1 at $(s, s)$ and $(t, t)$, $-1$ at $(s, t)$ $(t, s)$, and 0 elsewhere (the special case is when one of the endpoints of the edge is the last variable that corresponds to the removed column of $E$, then the matrix has only one entry).

Now we are going to diagonalize $P(\alpha) = P_G(\alpha)$ with respect to certain $n$ variables (modulo the others).

**Proposition 9.** Let $G$ be log-divergent graph and let $T$ be a spanning tree of $G$. Then there exists a matrix $\tilde{P}(\alpha) \in \text{Mat}_{n\times n}(\mathbb{Z}[\alpha])$ obtained from $P(\alpha) = P_G(\alpha)$ by elementary row and column operations such that for any $i$, $1 \leq i \leq n_G$, there exists a variable appearing at the only entry $\tilde{P}_{i,i}$.

**Proof.** Assume for a moment that we have a Hamiltonian path in our graph, that is a path $T$ with consecutive edges $e_1, \ldots, e_n$ with no loops which contains all the
vertices. Then, permuting the edges we can write the matrix $P$ in the form

$$
\begin{pmatrix}
\ddots & : & : & : \\
\vdots & \alpha_{n-1} + \alpha_{n-2} + c & -\alpha_{n-1} & c_{n-2,n} \\
\vdots & -\alpha_{n-1} & \alpha + \alpha_{n-1} + b & -\alpha_n \\
\vdots & c_n & -\alpha_n & \alpha_n + a \\
\end{pmatrix}
$$

(30)

where $a, b, c$ and $c_{ij}$ do depend only on the other variables $\alpha_e$ for $e \in E(G) \setminus E(T)$ (each non-specified entry $(i,j)$ of the matrix is denoted by $c_{ij}$). The variable $\alpha_n$ is only in the named 4 entries on the matrix. In general, $\alpha_i, e_i \in E(T)$, is contained in 4 entries $(P)_{t,s}$, $i - 1 \leq t, s \leq i$. Our basic operation $op(i, j)$ is the following: add the $i$-th row to the $j$-th row, and then add the $i$-th column to the $j$-th one. Apply $op(n, n - 1)$ to the matrix above. Then the matrix takes the form

$$
\begin{pmatrix}
\ddots & : & : & : \\
\vdots & \alpha_{n-1} + \alpha_{n-2} + c & -\alpha_{n-1} & c_{n-2,n} \\
\vdots & a + \alpha_{n-1} + b & a & \alpha_n + a \\
\vdots & a & \alpha_n & \alpha_n + a \\
\end{pmatrix}
$$

(31)

The variable $\alpha_n$ sits only at bottom right corner. Similarly, doing the basic operations step by step $op(n - 1, n - 2), op(n - 2, n - 3), \ldots op(2, 1)$, we can bring all the variables $\alpha_e, e \in E(T)$ to the diagonal, i.e. $\alpha_e$ appears only at the entry $(e,e)$, as desired.

Unfortunately, the statement about the existence of the Hamiltonian path similar to that one of the Hamiltonian cycle is proved only for graphs with big enough degrees of the vertexes and seems to be wrong for primitive log-divergent graphs with big enough $h_G$. We try to modify the proof above.

Consider now $T$ a given spanning tree of $G$ with edges $E(T) = E' \subset E(G)$, $|E'| = n$ and prove that the matrix $P$ can be transformed into the matrix where the variables $\alpha_e, e \in E'$, appear only and only on the diagonal. To do this we can forget the other variables (put the variables $\alpha_e$ equal zero for $e \notin E'' = E(G) \setminus E'$). Let’s (re)number the edges of $T$ in the following way. Take a vertex which is not a branch point to be a (top) root of the tree, fixing some planar embedding, and number to edges going down and left to the right. More precisely, if we come to the branch point then we go down the left branch. When the left branch is numbered (had come do a leaf), we return to the last branch point and go on with the next (from left to right) branch.

One can get the intuition of the numeration algorithm by analysing the following example of a spanning tree $T$ of a graph with 7 vertices.
The root will be the vertex we throw away in the procedure of construction of the block \( E \) in the matrix for \( \Psi_G \). The matrix \( P(\alpha) \) for this example modulo the ideal \( I_T \subset \mathbb{Z}[\{\alpha_e\}_{e \in E(G)}] \) generated by \( \alpha_e, e \notin E' \), takes the form:

\[
\begin{pmatrix}
\alpha_1 + \alpha_2 + \alpha_5 & -\alpha_2 & 0 & 0 & -\alpha_5 & 0 \\
-\alpha_2 & \alpha_2 + \alpha_3 + \alpha_4 & -\alpha_3 & -\alpha_4 & 0 & 0 \\
0 & -\alpha_3 & \alpha_3 & 0 & 0 & 0 \\
0 & -\alpha_4 & 0 & \alpha_4 & 0 & 0 \\
-\alpha_5 & 0 & 0 & 0 & \alpha_5 + \alpha_6 & -\alpha_6 \\
0 & 0 & 0 & 0 & -\alpha_6 & \alpha_6
\end{pmatrix}
\]

Here \( \circ \) denotes an entry congruent to 0 modulo \( I_T \). Doing the basic operations \( op(i, j) \) for the pairs of rows and columns \((i, j)\) equal \((4, 2), (3, 2), (3, 2), (6, 5), (5, 1), (2, 1)\), one gets the diagonal matrix with entries \( \alpha_1, \ldots, \alpha_6 \).

Now consider the case of a general log divergent graph \( G \) with a spanning tree \( T \). We diagonalize the matrix \( P(\alpha) \) by induction on the number \( m \) of branch points of \( T \). For \( m = 0 \) this is the case of a Hamiltonian path described above. Assume that for smaller \( m \) and all graphs the desired matrix is build. Consider the branch point \( R \) with the biggest depth in the rooted tree (the lowest on the picture similar to the example above) or the leftmost one of such points (if several). According to the numeration of edges, the left branch consists of edges \( e_s, \ldots, e_{s+p} \) for some \( s, p \geq 1 \). Since the leftmost branch of \( R \) has no more branch points, we can diagonalize this block as in case of \( m = 0 \) by applying \( p \) basic operations. This corresponds to \( op(3, 2) \) in the example. The variables \( \alpha_s, \ldots, \alpha_{s+p} \) are brought to the diagonal. After forgetting these \( p \) rows and columns with indeces from \( s + 1 \) to \( s + p \), the diagonalization of the remaining part follows from the induction hypothesis for the tree \( T/\{e_{s+1} \ldots e_{s+p+q}\} \). The matrix \( \tilde{P}(\alpha) \) is constructed.

**Definition 10.** Let \( k \) be a field. The dual graph hypersurface \( Z_G \) of a connected graph \( G \) is defined by the vanishing of \( \varphi_G : Z_G := V(\varphi_G) \subset \mathbb{A}^{N_G}_k \).

**Definition 11.** Define the singular locus of the dual graph hypersurface \( Z_G \) by

\[
\text{Sing}(Z_G) := \{ \alpha \in \mathbb{A}^{N_G}_k | \varphi_G(\alpha) = \frac{\partial}{\partial \alpha_i} \varphi_G(\alpha) = 0, \forall i \leq N_G \}.
\]

(32)
Proposition 12. Assume that the first $n_G$ edges of $G$ form a spanning tree.
Then the ideal of $\text{Sing}(Z_G)$ in $k[\{\alpha\}_{i \in I_N}]$ is

$$I(\text{Sing}(Z_G)) = k[\{\alpha\}_{i \in I_N}] \left\langle \varphi_G, \frac{\partial}{\partial \alpha_i} \varphi_G \mid i \leq n_G \right\rangle,$$  \hfill (33)

(is generated by the derivatives for the only $n_G$ edges).

Proof. The inclusion of the right hand side of (33) into the left one is clear. So we are going to prove the opposite inclusion, that is: $\varphi_G^i \in \mathcal{I}'$ for all $i \in I_N$, where $\mathcal{I}' := \langle \varphi_G, \varphi_G^i \mid i \leq n_G \rangle$. Denote by $T$ the tree formed by the edges $e_1, \ldots, e_{n_G}$. Recall that in Proposition 9 we have constructed the matrix $\tilde{P}(\alpha)$ that is a "diagonalization" of $P(\alpha)$ with respect to $n_G$ variables corresponding to the edges of a given spanning tree $T$. We can denote $\tilde{P}(\alpha)$ again by $P(\alpha)$. After reordering of the variables we can assume that $\alpha_i$ is only in (a linear summand of) $P_{i, i}$ for $i = 1, \ldots, n_G$. Here $P_{I, J} = P_{I, J}(t)$, $I, J \subset I_N$ denotes the matrix that we get from $P(\alpha)$ after deleting $I$ rows and $J$ columns. Thus $\varphi_G^{i}(\alpha) = P_{i, i}(\alpha)$ for any $i = 1, \ldots, n_G$. Consider any edge $e_{j}$, $j > n_G$ with endpoints $v_{s}$ and $v_{t}$. Since $T$ is a spanning tree, there exist a path from $v_{s}$ to $v_{t}$ that lies in $T$, say $e_{j_1}, \ldots, e_{j_r}$, $1 \leq j_i \leq n_G$, for $i = r$. These edges together with $e_{j}$ form a loop. By Proposition 7 ii, one now gets

$$\varphi_G^{j} = \sum_{i} \lambda_i \varphi_G^{i, j} \text{ with } \lambda = \pm 1.$$  \hfill (34)

The Dodgson identities (14) for the symmetric matrix $P = P(\alpha)$

$$\det P_{i, i} \det P_{j, j} - \det P \det P_{i, j} = (\det P_{i, j})^2$$  \hfill (35)

imply $\varphi_G^{i, j} \in \mathcal{I}'$ for any $1 \leq i \leq h_G$, $1 \leq j \leq N_G$. By formula (34) above, one now gets $\varphi_G^{j} \in \mathcal{I}'$ for $1 \leq j \leq N_G$. \hfill \Box

Lemma 13. In terms of the matrix $P_G(\alpha)$, the singular locus $\text{Sing}(Z_G)$ is given by

$$\text{Sing}(Z_G) = \left\{ \alpha \in \mathbb{A}^{N_G} \mid \text{rank } P_G(\alpha) < n_G - 1 \right\}.$$  \hfill (36)

Proof. Since the rank of a matrix is stable under the elementary row and column operations, Proposition 9 yields that it is enough to prove the statement for $P(\alpha) := \tilde{P}(\alpha)$ with variables ordered in the way $T$ being a spanning tree formed by $e_1, \ldots, e_{n_G}$. Consider $t \in \text{Sing}(Z_G)$. It follows that $\det P_{i, i}(t) = \partial_{\alpha_i} \varphi_G(t) = 0$ for $i = 1, \ldots, n_G$ and $\varphi_G(t) = 0$. The Dodgson identity (35) now implies $\det P_{i, j}(t) = 0$ for $i, j = 1, \ldots, h_G$. Hence rank $P(t) < h_G - 1$.

For the opposite inclusion in (36), consider a point $t$ on the right hand side. Since rank $P(t) < n - 1$, we get $\varphi_G(t) = \det P(t) = 0$ and $\varphi_G^{i}(t) = \det \tilde{P}^{i, i}(t) = 0$ for $i = 1, \ldots, n$. Thus Proposition 12 yields $t \in \text{Sing}(Z_G)$. \hfill \Box
2 [Z_G] and [Sing(Z_G)] in K_0(Var_k)

The main theorems of this article concern the relations between the number of \( \mathbb{F}_q \)-rational points of certain varieties. Nevertheless, the part of the computations are valid for \( K_0(Var_k) \) that is more likely from the geometric point of view.

Fix a field \( k \). The Grothendieck ring of varieties \( K_0(Var_k) \) is defined as a free \( \mathbb{Z} \) module generated by the isomorphism classes \([X]\) of separated schemes \( X \) of finite type over \( k \) modulo the following relation: \([X] = [Y] + [X \setminus Y]\) for closed subschemes \( Y \subseteq X \). The ring structure is given by the product \([X] \cdot [X'] = [(X \times Y)_{red}]\). The unit is \([1] = [\text{Spec} k]\) and the Lefschetz motive is \( \mathbb{L} := [A_k^1] \).

In our situation, the polynomials \( \Psi \) and \( \varphi \) are linear with respect to each of the variables. Recall the standard tool for computing the class in the Grothendieck ring using linearity (see [6], Lemma 16):

**Lemma 14.** Let \( f^1, f_1, g^1, g_1 \in k[\alpha_2, \ldots, \alpha_N] \). Then

1. \( [f^1 \alpha + f_1] = [f^1, f_1] \mathbb{L} + \mathbb{L}^{N-1} - [f^1] \).
2. \( [f^1 \alpha + f_1, g^1 \alpha + g_1] = [f^1, f_1, g^1, g_1] \mathbb{L} + [f^1 g_1 - g^1 g_1] - [f^1, g^1] \).

**Proposition 15.** Let \( G \) be a graph with \( h_G \geq 2 \). Then in \( K_0(Var_k) \)

\[
[\varphi_G] \equiv 0 \mod \mathbb{L}^2. \tag{37}
\]

**Proof.** By Euler’s formula, the condition \( h_G \geq 2 \) is equivalent to \( n_G + 2 \leq N_G \), and \( n = n_G \) is the degree of \( \varphi_G \). If \( G \) is disconnected, then \( \varphi_G = 0 \) and there is nothing to prove. Assume \( G \) is a connected graph. By induction we prove that for \( f \in \mathbb{Z}[\alpha_1, \ldots, \alpha_r] \) of degree \( \leq r \), and for any \( G \) with at least 2 loops and any edge of \( G \), say \( e_1 \), there exist elements \( a(f), b(G, 1), c(G) \in K_0(Var_k) \) such that

1. \( [f] = a(f) \mathbb{L} \mod \mathbb{L}^2 \).
2. \( [\varphi_{G, 1}, \varphi_{G}^1] = b(G, 1) \mathbb{L} \mod \mathbb{L}^2 \).
3. \( [\varphi_G] = c(G) \mathbb{L}^2 \mod \mathbb{L}^3 \).

1) For \( r = 1 \) the statement is obvious. By Lemma 14(i) for \( f = f^1 \alpha_1 + f_1 \), one computes \( [f] = \mathbb{L}^{r-1} - [f^1] + [f^1, f_1] \mathbb{L} \). Since the degree of \( f^1 \) is also less then the number of variables, we can construct \( a(f) \) inductively:

\[
a(f) := [f^1, f_1] - a(f^1). \tag{38}
\]

2) Fix any other edge \( e_2 \). By contraction-deletion ([8]) for \( G \setminus 1 \) and \( G / 1 \), \( \varphi_G = \varphi_{G, G}^2 \alpha_2 + \varphi_{G,G,1} \) and \( \varphi_{G,1} = \varphi_{G,1}^2 \alpha_2 + \varphi_{G,G,12} \). The Dodgson identity ([15]) reads \( \varphi_G = \varphi_{G,2} \varphi_{G,1} - \varphi_{G,G,1}^2 \varphi_{G,ed} = (\varphi_{G,G,1}^2)^2 \). The Lemma 14 implies

\[
[\varphi_G, \varphi_{G,1}] = \mathbb{L}[\varphi_{G,2}, \varphi_{G,1}, \varphi_{G,G,12}, \varphi_{G,G,12}] + [\varphi_{G,G,12}] - [\varphi_{G,G,12}, \varphi_{G,G,1}] \tag{39}
\]

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Note that $\deg \varphi_G^{1,2} = n - 1 \leq N_G - 3$, thus $\varphi_G^{1,2}$ satisfies the conditions in for part 1). For a non-connected graph $G$ define $b(G, 1) := 0$. Otherwise, inductively

$$b(G, 1) := [\varphi_{G,2}^{1}, \varphi_{G,1}^{2}, \varphi_{G,12}^{1,2}, \varphi_{G,12}^{1} + a(\varphi_G^{1,2}) - b(G/2, 1),$$  

(40)

We can choose $e_2$ on each step in the way to decrease $n$, then the base of the induction is a graph with 1 vertex and $N_G - n \geq 2$ self-loops that is trivial.

3) Since $\varphi_G$ is linear in $\alpha_1$, Lemma 14(i) implies

$$[\varphi_1] = [\varphi_{G,1}, \varphi_{G,1}^1]\mathbb{L} + \mathbb{L}^{N_G - 1} - [\varphi_G^1].$$  

(41)

If $G$ less than 2 vertexes, then $G$ should be formed by 1 vertex and $\ell \geq 2$ self-loops. One computes $c(G) = 1$ for $\ell = 2$, and $c(G) = 0$ otherwise. When $n_G \geq 1$, define $c(G)$ inductively by

$$c(G) := b(G, 1) - c(G/1).$$  

(42)

$$\Box$$

**Definition 16.** For a graph $G$ define the invariant $c_2^{\text{dual}}(G)$ to be the element $c(G)$ from the proof above. In other words,

$$c_2^{\text{dual}}(G) := [\varphi_G]/\mathbb{L}^2 \mod \mathbb{L}. $$  

(43)

If one of the loops of $G$ is of length 2, using (10) one can easily prove that $c_2^{\text{dual}}(G) = 0$ since we can get rid of one of the variables and get a fibration with each fibre isomorphic to $\mathbb{A}^1$.

In the case $G$ has a loop of length 3, we are able to give a concrete description of the $c_2^{\text{dual}}(G)$ invariant.

**Proposition 17.** Let $G$ be a graph with 3 edges (say $e_1, e_2, e_3$) forming a triangle and $h_G \geq 3$. Then

$$c_2^{\text{dual}}(G) \equiv [\varphi_G^{1,2}, \varphi_G^{13,23}] \mod \mathbb{L}. $$  

(44)

**Proof.** Recall that the proof of the corresponding statement for the graph polynomial uses the special structure of $\Psi_G$ in the case of the existence of a 3-valent vertex, see Lemma 24 in [10]. There is also a nice structure of $\Psi_G$ in the case of the existence of a triangle in $G$, it can be found in Example 33, [4]:

$$\Psi_G = f^{123} \alpha_1 \alpha_2 \alpha_3 + (f^1 + f^2) \alpha_1 \alpha_2 + (f^1 + f^3) \alpha_1 \alpha_3 + (f^2 + f^3) \alpha_2 \alpha_3$$

$$+ f^0 (\alpha_1 + \alpha_2 + \alpha_3),$$  

(45)

together with $f^0 f^{123} = f^1 f^2 + f^2 f^3 + f^1 f^3$, where $f^{123} = \Psi_{123}^{13}, f^0 = \Psi_{ijk}^{13}$, $f^i = \Psi_{ij,k}^{13}$ for $\{i, j, k\} = \{1, 2, 3\}$. We dualize this using (15) to get a convenient formula for $\varphi_G$:

$$\varphi_G = g_0 (\alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_1 \alpha_3) + (g_1 + g_2) \alpha_1 \beta_3 + (g_1 + g_3) \alpha_2$$

$$+ (g_2 + g_3) \alpha_1 + g_{123},$$  

(46)
with the only identity
\[ g_0 g_{123} = g_1 g_2 + g_2 g_3 + g_1 g_3. \] (47)

Here \( g_{123} = \varphi_{123} \), \( g_0 = \varphi_i^j \), \( g_i = \varphi_i^{j,k} \), \( g_i + g_j = \varphi_i^{j,k} \), \( \{i, j, k\} = \{1, 2, 3\} \). The formula looks identical to that for \( \psi_i \) in the case \( G \) has a 3-valent vertex (see [4], Example 32), so can use the same strategy as in the proof of Proposition 23 in [6] to prove that
\[ [\varphi_G] = \mathbb{L}^{N-1} + \mathbb{L}^3[g_0, g_1, g_2, g_3; g_{123}] - \mathbb{L}^2[g_0, g_1, g_2, g_3]. \] (48)

Thus \( c_2^{dual}(G) \equiv [g_0, g_1, g_2, g_3] \mod \mathbb{L} \). The next part of the proof goes similar as the proof of Lemma 24 in [6]. By (47), the inclusion-exclusion formula yields
\[ [g_0, g_3] = [g_0, g_1, g_2, g_3] = [g_0, g_1, g_3] + [g_0, g_2, g_3] - [g_0, g_1, g_2, g_3], \] (49)
and \( [g_0, g_1 + g_3] = [g_0, g_1 + g_3, g_3] = [g_0, g_1, g_3] \). By contraction-deletion [8], \( [g_0, g_1 + g_3] = [\varphi_1^{12}, \varphi_2^{13}] = [\varphi_G, \varphi_{G',1}] \) for \( G' = G \setminus 3/2 \). Since \( G \) has at least 3 loops, the graph \( G' \) has \( h_{G'} \geq 2 \). We can use Proposition [13] statement 2) and get \( \mathbb{L}[\varphi_G, \varphi_{G',1}] \). By symmetry, we can also get the divisibility \( \mathbb{L}[g_0, g_2, g_3] \). Now (48) and (49) imply
\[ [\varphi_G] \equiv \mathbb{L}^2[g_0, g_1, g_2, g_3] \equiv \mathbb{L}^2[g_0, g_3] \equiv \mathbb{L}^2[\varphi_3^{1,2}, \varphi_3^{13,23}] \mod \mathbb{L}^3. \] (50)

The statement follows from the definition of \( c_2^{dual}(G) \).

We are going to use Proposition 29 from [7]. This is the simultaneous elimination of one variable from the ideal on the Grothendick ring whose generators are all linear in that variable.

**Proposition 18.** Let \( f_1, \ldots, f_n \) are linear in \( \alpha \), say \( f_i = f_i^\alpha + f_{1,\alpha}, 1 \leq i \leq n. \) Then

\[
[f_1, \ldots, f_n] = [f_1^\alpha, f_1, \alpha, \ldots, f_n^\alpha, f_n, \alpha] \mathbb{L}^+ + \\
[[f_1, f_2], \alpha, \ldots, [f_1, f_n], \alpha] - [f_1^\alpha, \ldots, f_n^\alpha] \\
\sum_{k=1}^{n-2} ([f_1^\alpha, f_1, \alpha, \ldots, f_k^\alpha, f_k, \alpha, [f_{k+1}, f_{k+2}], \alpha, \ldots, [f_{k+1}, f_n], \alpha]) \\
- [f_1^\alpha, f_1, \alpha, \ldots, f_k^\alpha, f_k, \alpha] \] (51)

Now we will again use the singular locus of the dual graph hypersurface \( \text{Sing}(Z_G) \), see Definition [11]. In the Grothendick ring one immediately gets
\[ [\text{Sing}(Z_G)] = [\varphi_G, \varphi_{G',1}, \ldots, \varphi_G^{N_G}] \in K_0(\text{Var}_k). \] (52)
Proposition 19. Let $G$ be a connected graph with $N = N_G$ edges and with $h_G \geq 2$. Then in $K_0(Var_k)$ one has

$$[\text{Sing}(Z_G)] + [\text{Sing}(Z_{G/1})] = L[\varphi^1, \varphi_1, \{\varphi^1_i, \varphi^1_i\}_{i=2..N}] + [\varphi^1, \varphi_1]$$

for some edge $e_1$.

Proof. The proof is very similar to the proof of Lemma 30 in [7]. For $e_1$ we can choose any edge which deletion does not disconnect $G$. We write $[\text{Sing}(Z_G)] = [\varphi, \varphi^1, \ldots, \varphi^N]$ and apply Proposition (18) to the set of polynomials $\varphi, \varphi^1, \ldots, \varphi^N$ linear in the variable $\alpha = \alpha_1$. Each summand of the sum on the right hand side in (51) is of the form

$$[\varphi^1, \varphi_1, \ldots, \varphi^1t, \varphi^1_i, [\varphi^{t+1}, \varphi^{t+2}]_1, \ldots [\varphi^{t+1}, \varphi^n]_1] - [\varphi^1, \varphi_1, \ldots, \varphi^1t, \varphi^1_i].$$

By Corollary 6 for any $a \neq b \in I_N \setminus 1$, the resultant $[\varphi^a, \varphi^b]_1$ is in the radical of the ideal spanned by $\varphi^1, \varphi_1$. In the Grothendick ring we see only the reduced scheme structure (ideal is undistinguishable from its radical). It follows that the two classes above sum to 0 for every $t$. Hence (51) reduces to

$$[\text{Sing}(Z_G)] = [\varphi^1, \varphi_1, \ldots, \{\varphi^1t, \varphi^1_i\}_{i=1}^t] - [\varphi^1, \{\varphi, \varphi^i\}_{i=1}^t] - [\varphi^1, \{\varphi^1t\}_{i=1}^t]$$

with $t$ ranging from 2 to $N$ in each of the three expressions on the right hand side. Since $[\varphi^1, \varphi^1_i] = \varphi^1 \varphi^1_i - \varphi^1 \varphi^1t$, the middle summand on the right hand side simplifies as $[\varphi^1, \{\varphi, \varphi^i\}_{i=1}^t] = [\varphi^1, \{\varphi \varphi^1t\}_{i=1}^t]$. Distinguishing the cases $\varphi_1 = 0$ and $\varphi_1 \neq 0$, one computes

$$[\varphi^1, \{\varphi_1 \varphi^1t\}_{i=1}^t] = [V(\varphi^1, \{\varphi_1 \varphi^1t\}_{i=1}^t) \setminus V(\varphi^1, \varphi^1_i), \{\varphi_1 \varphi^1t\}_{i=1}^t] + [\varphi_1, \varphi^1_i, \{\varphi \varphi^1t\}_{i=1}^t] = [V(\varphi^1, \varphi^1_i), \{\varphi_1 \varphi^1t\}_{i=1}^t] + [\varphi_1, \varphi^1_i] - [\varphi^1, \varphi_1, \{\varphi^1t\}_{i=1}^t].$$

Now we can consider a corollary in $G$ which contains the edge $e_1$ and we apply Corollary 8. It follows that $\varphi_1 \in \text{Rad}(I)$ for the ideal $I \subseteq \mathbb{Z}[\{\alpha_i\}_{i \notin I}]$ generated by $\varphi^1$, and $\{\varphi^{1r}\} \in I_{1} \setminus I$ for some $I \subseteq I_N$. Thus the second and the third summand on the last expression in (56) cancel each other. The last term on the right in (55) defines the singular locus of the dual graph hypersurface for the graph $G/1$. \(\square\)

Theorem 20. Let $G$ be a graph with at least 2 loops. Then for the singular locus of the dual graph hypersurface of $G$, the following congruence holds:

$$[\text{Sing}(Z_G)] \equiv 0 \mod L.$$

Proof. If $G$ is disconnected then $\varphi_G = 0$ and there is nothing to proof. If $G$ has a self-loop, say formed by an edge $e_1$, then by (5) all the $\varphi^1_i$ for $G$ are independent of $\alpha_1$. It follows that we can project down to the situation for $G \setminus 1$ with fibres $\mathbb{A}^1$, the statement follows.
If $G$ has a loop of length 2, then by (10) one can write
\[
\varphi_G = \varphi_{G\setminus 1/2} (\alpha_1 + \alpha_2) + \varphi_{G\setminus 12}.
\]
After the changing of the variables $\alpha_2 := \alpha_1 + \alpha_2$, we can again project to the situation for $G\setminus 1$ with fibres $\mathbb{A}^1$ and (57) holds.

So we can assume that the graph $G$ is connected with no self-loops or double edges. The proof goes by the induction on the number of edges $N_G$. The assumptions on $G$ implies $N_G \geq 5$. Since $h_G \geq 2$ is equivalent to $n_G + 2 \leq N_G$ by Euler’s formula, we are able to use statement 2) in the proof of Proposition 15 and we get $[\varphi_G^1, \varphi_{G,1}] \equiv 0 \mod L$. Hence, (53) implies
\[
[Sing(Z_G)] \equiv -[Sing(Z_{G/1})] \mod L. \tag{58}
\]
If the graph $G/1$ still has no double edges, then it has again at least 2 loops. In any case, by the induction hypothesis, $[Sing(Z_{G/1})] \equiv 0 \mod L$. \qed

3 The $c_2$ invariant in position space

Fix some field $k$ ($k$ can be $\mathbb{F}_q$, or (the usual for physicists) $\mathbb{R}$). For the convenience of the computation, we work not with Euclidian metric, but with the metric defined as
\[
|x|^2 = x^1 x^2 + x^3 x^4, \quad \text{for} \quad x = (x^1, x^2, x^3, x^4) \in k^4. \tag{59}
\]
Consider a log-divergent graph $G$ with $N$ edges $\{e_i\}_{i \in I_N}$ and $n_G + 1$ vertexes. To each vertex we associate a variable $x_j$, $j = 1, \ldots, n + 1$, $n = n_G$. The propagator attached to an edge $e_i$ with endpoints with variables $x_s$ and $x_t$ is of the form
\[
\frac{1}{q_i(x)} = \frac{1}{||x_s - x_t||^2} \in \text{Frac}(\mathbb{Z}[\{x_i^j\}_{i,j}]), \tag{60}
\]
with $1 \leq i \leq N$, $1 \leq j \leq 4$. For a primitive log-divergent graph $G$ the Feynman period representation in the position space is defined to be the value
\[
I^\text{pos}_G := \int_{\mathbb{P}^{4n-1}} \frac{\Omega(x)}{q_1 \ldots q_{N_G}}, \tag{61}
\]
where we first put one vertex (variable $x_{n+1}$) equal zero, and $\Omega(x)$ is a standard differential form in projective space. We are be interested in the configuration of quadrics $q_i$ in $\mathbb{A}^{4n}$. One can easily translate the results from projective space to affine one and vice versa, for counting points we prefer the affine setting.

Consider the universal quadric
\[
Q(\alpha, x) = \sum_{i=1}^{N_G} \alpha_i q_i(x) \in \mathbb{Z} [\{\alpha_i\}_{i \in I_N}, \{x_j\}_{j=1, \ldots, n}] \tag{62}
\]
depending on the edge variables $\alpha_i$ and the vertex variables $x_j$. This is the key tool of the Schwinger trick, see Figure 1.
We return to (60) and consider two adjacent vertices with associated variables \( a \) and \( b \). The denominator can be written as

\[
|a - b|^2 = (a^2a^4b^2b^4)
\]

\[
\begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
a^1 \\
a^3 \\
b^1 \\
b^3
\end{pmatrix}
\]

(63)

It follows that the universal quadric (52) can be written as coming from a matrix consisting of blocks of the shape (63) multiplied by \( \alpha_i \)s. Permuting the rows and columns, one gets

\[
Q(\alpha, x) = \begin{pmatrix}
x^2 \\
x^4
\end{pmatrix}^T \begin{pmatrix}
P(\alpha) & 0 \\
0 & P(\alpha)
\end{pmatrix} \begin{pmatrix}
x^1 \\
x^3
\end{pmatrix},
\]

(64)

where \( x_j \) is a vector build up of consecutive coordinates \( x_1^j, \ldots, x_n^j \), \( 1 \leq j \leq 4 \), and \( P(\alpha) \in \text{Mat}_{n,n}(\mathbb{Z}[\{\alpha_i\}_{i \in I_N}]) \) is the matrix in (29).

Recall that in Proposition 9 we have constructed the matrix \( \tilde{P}(\alpha) \) out of \( P_G(\alpha) \) by the diagonalization with respect to the edges of any fixed spanning tree \( T \) of \( G \). We need 2 more propositions.

**Proposition 21.** For a graph \( G \) with \(|E(G)| = N \) and for a subset of edges \( I \subset I_N \), define by \( P_I \) the matrix \( \tilde{P}(\alpha)|_{\alpha_i = 0, i \notin I} \) that is obtained from \( \tilde{P}(\alpha) \) by setting to zero all the variables with indexes in \( I_N \setminus I \). Then

\[
\mathbb{L}^{|I|}[\{q_i(x)\}_{i \in I}] = \mathbb{L}^{2n}[P_I \cdot x^2, P_I \cdot x^4],
\]

(65)

where \( \{q_i(x)\}_{i \in I} \) denotes the class of the vanishing of all the \( q_i \)s (with \( i \) in the given set).

**Proof.** We compute the number of points of the quadric \( Q_I(\alpha, x) = \sum_{i \in I} \alpha_i q_i(x) \) in two ways projecting to the space of edge variables \( \alpha \) or vertex variables \( x \). First consider a projection of \( Q \) to \( \mathbb{A}^{|I|}(x_j^i) \), \( 1 \leq j \leq 4 \), \( i \in I \). Since \( Q_I \) is linear in all \( \alpha_i \), the general fiber is isomorphic to \( \mathbb{A}^{|I|-1} \). In the case of the intersection of all the quadrics \( q_i \) (writing \( \{q_i(x)\}_{i \in I} \) for the class in the Grothendick ring in this situation), the fibre is isomorphic to \( \mathbb{A}^{|I|} \). We get

\[
[Q_I] = \mathbb{L}^{|I|-1}\left(\mathbb{L}^{4n} - \{q_i(x)\}_{i \in I}\right) + \mathbb{L}^{|I|}[\{q_i(x)\}_{i \in I}].
\]

(66)

On the other hand, comparing to (64), \( Q_I(\alpha, x) \) can be rewritten in the form

\[
Q_I(\alpha, x) = \begin{pmatrix}
x^2 \\
x^4
\end{pmatrix}^T \begin{pmatrix}
P_I(\alpha) & 0 \\
0 & P_I(\alpha)
\end{pmatrix} \begin{pmatrix}
x^1 \\
x^3
\end{pmatrix},
\]

(67)

and thus defines a fibration to \( \mathbb{A}^{|I|+2n}(\alpha_i, x_j^1, x_j^3) \), \( i \in I \), \( 1 \leq j \leq n \) with fibres linear subspaces in the variables \( x_j^2 \) and \( x_j^4 \). One computes

\[
[Q_I(\alpha, x)] = \mathbb{L}^{2n-1}(\mathbb{L}^{4n} - [P_I \cdot x^2, P_I \cdot x^4]) + \mathbb{L}^{2n}[P_I \cdot x^2, P_I \cdot x^4].
\]

(68)

Together with (66) this yields the statement. \(\square\)
Proposition 22. Define $\varphi_{G,I} := \det P_I(\alpha) = \varphi_G|_{\alpha_k=0,k \notin I}$ for $I \subset I_N$. Then

$$[P_I \cdot x^2, P_I \cdot x^4] \equiv \mathbb{L}|^I| + (\mathbb{L}^2 - 1)[\varphi_{G,I}] - \mathbb{L}^2[\text{rank } P_I < n - 1] \mod \mathbb{L}^4. \quad (69)$$

Proof. The equation $P_I \cdot x^2 = 0$ is a system of $n$ linear equations in the variables $x^2$, thus the vanishing locus of this system is isomorphic to $\mathbb{A}^r$ for $r = \text{corank } P_I$. The equation $P_I \cdot x^4 = 0$ gives the same system but in the variables $x^4$. It follows that

$$[P_I \cdot x^2, P_I \cdot x^4] \equiv \mathbb{L}|^I| + \mathbb{L}^2[\text{corank } P_I = 1] \mod \mathbb{L}^4$$

$$\equiv \mathbb{L}|^I| - \mathbb{L}^2([\text{corank } P_I > 0] - [\text{corank } P_I > 1]) \mod \mathbb{L}^4. \quad (70)$$

Since $(\varphi_G = 0) \iff (\text{corank } P_I > 0)$, one obtains (69).

From now we need to reduce to the computation of number of rational points over finite fields.

Consider $F_1, \ldots, F_r \subset \mathbb{Z}[a_1, \ldots, a_N]$ and fix $q = p^s$ a prime power. Denote by $ar{F}_i$ the reduction of $F_i$ modulo $q$. Define $[F_1, \ldots, F_r]_q \in \mathbb{N}_0$ the number of $\bar{F}_q$-rational points of the variety $V(\bar{F}_1, \ldots, \bar{F}_r)$.

Similarly to what happens in momentum space, our object of interest is the count points function for the product $[q_1 \ldots q_N]_q$ that is a denominator of the differential form in the representation of a period in position space. We are going to use Chevaley-Warning theorem. The possible analogue of this result in the Grothendick ring of varieties is called the geometric Chevaley-Warning question has a negative answer (see [13]). This means that the results for the counting points functions over $\mathbb{F}_q$ below cannot be easily lifted to the Grothendick ring.

The counting points functor factors through the Grothendick ring of varieties mapping $1$ to $1$ and $\mathbb{L}$ to $q$, so the results of the previous 2 propositions and the results of Section 2 imply the corresponding congruences for the number of points. For instance, the following definition corresponds to Definition 16 and will be used later in the section.

Definition 23. For a graph $G$ define the invariant $c_2^{\text{dual}}(G)_q$ by

$$c_2^{\text{dual}}(G)_q := [\varphi_G]_q/q^2 \mod q. \quad (71)$$

Theorem 24. (Chevalley-Warning) Let $F_1, \ldots, F_r \in \mathbb{Z}[a_1, \ldots, a_N]$ be polynomials with $\sum_i \deg F_i < N$. Then for any prime power $q$

$$[F_1, \ldots, F_r]_q \equiv 0 \mod q. \quad (72)$$

Proposition 25. For any graph $G$ with $2n_G \geq N_G$, one has

$$[q_1 \ldots q_N]_q \equiv (-q)^{2n-N_G}([\varphi_G]_q + q^2[\text{Sing}(\mathbb{Z}_G)]_q$$

$$- q \sum_{i \in I_N} [\varphi_{G,i}]_q + q^2 \sum_{i,j \in I_N} [\varphi_{G,i,j}]_q \mod q^3 \quad (73)$$
**Proof.** First we apply the inclusion-exclusion formula

\[
[q_{1} \ldots q_{N_{G}}]_{q} = \sum_{I \subseteq I_{N}} (-1)^{|I|} [\{q_{i}\}_{i \in I}]_{q}.
\]  

(74)

Proposition 21 implies \([\{q_{i}\}_{i \in I}]_{q} = q^{2n-|I|}[P_{I} \cdot x^{2}, P_{I} \cdot x^{4}]_{q}\). We also use Proposition 15 under assumption \(2n \geq N\). Since \(\varphi_{G,I} = \varphi_{G,I}(I_{N}\setminus I)\) (the dual graph polynomial for the graph obtained from \(G\) by contracting a set of edges complementary to \(I\) in \(I_{N}\)), Lemma 8 implies that \([\varphi_{G,I}]_{q} \equiv 0 \mod q^{2}\) for \(2n \geq N\) until \(I = I_{N}\), and \([\varphi_{G,I_{N}}]_{q} = [\text{Sing}(Z_{G})]\) by Proposition 12. Since \(\varphi_{G,I} = \text{det} P_{\bar{I}}(\alpha)\), one gets \([\text{rank} P_{\bar{I}} < n-1]_q \equiv 0 \mod q^{2}\) until \(I = I_{N}\). Thus

\[
[\{q_{i}\}_{i \in I}]_{q} \equiv
\begin{cases}
-q^{2n-N}(-[\varphi_{G}]_{q} + q^{2}[\text{Sing}(Z_{G})]) \mod q^{3}, & I = I_{N}, \\
n-q^{2n-|I|}(-[\varphi_{G,I}]_{q}) \mod q^{3}, & |I| = N-1, N-2,
0 \mod q^{3}, & |I| < N-3.
\end{cases}
\]  

(75)

Summing everything together using (74), one gets (73). \(\square\)

**Corollary 26.** For \(G\) a graph with \(N_{G} \leq 2n_{G}\), \(n_{G} \geq 2\) one has

\[
[q_{1} \ldots q_{N_{G}}]_{q} \equiv 0 \mod q^{2}.
\]  

(76)

**Proof.** Proposition 25 trivially implies the statement for \(2n > N + 1\), so we need to take care of the cases \(2n = N + 1\) and \(2n = N\), \(n \geq 2\). By Proposition 15 \(q^{2}|[\varphi_{G}]|\) for \(2n = N_{G}\) and \(n \geq 2\), and \(q|[\varphi_{G}]|\) for \(2n = N_{G} + 1\) and \(n \geq 2\). For the middle summand of the right hand side of (73) we have \(q|[\varphi_{G'}]|\) for \(2n = N_{G}\), where \(G' = G//e\) for any \(e \in E(G)\). Now (76) follows. \(\square\)

Using this corollary, we can give the following definition.

**Definition 27.** Let \(G\) be a graph with \(N_{G} \leq 2n_{G}\) and \(n_{G} \geq 2\). We define the \(c_{2}\) invariant of \(G\) in position space as follows:

\[
c_{2}^{\text{pos}}(G)_{q} := [q_{1} \ldots q_{N_{G}}]_{q}/q^{2} \mod q^{3}.
\]  

(77)

Now we are able to prove the coincidence of \(c_{2}\) invariants in the dual parametric space (Definition 23) and in position space.

**Theorem 28.** Let \(G\) be a graph with \(n_{G} \geq 3\).

1. If \(N_{G} > 2n_{G}\), then \(c_{2}^{\text{pos}}(G)_{q} = 0\).

2. If \(N_{G} = 2n_{G}\) (i.e. \(G\) is log-divergent), then

\[
c_{2}^{\text{dual}}(G)_{q} = c_{2}^{\text{pos}}(G)_{q}.
\]  

(78)
Proof. 1) We are going to use the formula (73). In the case $2n_G > N_G + 2$ the statement holds for trivial reasons.

If $2n_G = N_G + 2$, then $q|\varphi_G$ by Proposition 15 for $N_G \geq n_G + 2$ and directly for $n_G = 3$ and $N_G = 4$.

If $2n_G = N_G + 1$, then $N_G \geq n_G + 2$, thus again $q^2|\varphi_G$. We also have $G\parallel e$ disconnected or $N_{G\parallel e} \geq n_{G\parallel e} + 1$, hence $q||\varphi_{G\parallel e}$. The statement follows.

2) For the part 2) we have $N_G = 2n_G$, so either $G\parallel e$ is disconnected or $N_{G\parallel e} \geq n_{G\parallel e} + 1$, hence $q^2|\varphi_{G\parallel e}$. Similarly, either $G/\langle e_1e_2 \rangle$ is disconnected or $N_{G/\langle e_1e_2 \rangle} \geq n_{G/\langle e_1e_2 \rangle} + 1$, hence $q||\varphi_{G/\langle e_1e_2 \rangle}$. Thus, formula (73) reduces to

$$[q_1 \ldots q_N] \equiv ([\varphi_G]_q + q^2[\text{Sing}(Z_G)]_q) \mod q^3 \quad (79)$$

The statement follows from Theorem 20 and the definitions of $c^\text{pos}_2(G)_q$ and $c^\text{dual}_2(G)_q$. \qed

4 The $c_2$ invariant respects dualization

In this section we will prove the coincidence of $c_2(G)_q$ and $c^\text{dual}_2(G)_q$ for a subset of log-divergent graphs $G$ which we call duality admissible.

We cannot use this proof for $c_2$ in the Grothendieck ring $K_0(Var_k)$ since we will intensively apply Chevalley-Warning vanishing. Nevertheless, starting from now, we omit the index $q$ and write $[Y]$ for the number of rational points of an affine scheme $Y$ over $\mathbb{F}_q$ for a fixed prime power $q$. This will make the formulas more readable. We also define $[Y]'$ to be $[Y \cap (\mathbb{G}_m)^N]$ for a fixed embedding of $(\mathbb{G}_m)^N \hookrightarrow \mathbb{A}_k^N$, where $Y \subset \mathbb{A}_k^N$ is an affine scheme. For instance, the function $f \mapsto [f]'$ counts the number of solutions of $f$ with non-zero coordinates.

For example, since $\varphi_f' := \iota(\Psi_f')$, one has a bijection between non-zero solutions of $\Psi_f = 0$ and non-zero solutions of $\varphi_f = 0$, thus

$$[\varphi_f'] = [\Psi_f'].$$

Assume for a moment that $\Psi \in \mathbb{Z}[\alpha_1, \ldots, \alpha_N]$ is any polynomial of degree $n$ linear with respect to each of the variable (not necessarily a graph polynomial).

Grouping the summands by the number of the variables $\alpha_i$ which are zero, we get

$$[\Psi] = [\Psi]' + \sum_i [\Psi_i]' + \sum_{i,j} [\Psi_{ij}]' + \sum_{i,j,k} [\Psi_{ijk}]' + \ldots = [\Psi]' + \sum_{t=1}^N \sum_{|I|=t} [\Psi_I]' \quad (81)$$

On the other hand, computing affinely, in the solutions for a summand $[\Psi_I]$ the variables $\alpha_j$, $j \notin I$ are allowed to vanish. By inclusion-exclusion, one has

$$[\Psi] = [\Psi]' + \sum_i [\Psi_i] - \sum_{i,j} [\Psi_{ij}] + \sum_{i,j,k} [\Psi_{ijk}] - \ldots = [\Psi]' + \sum_{t=1}^N (-1)^{t+1} \sum_{|I|=t} [\Psi_I] \quad (82)$$

We should restrict our attention to the following type of graphs.
Definition 29. We say that a log-divergent graph $G$ with $h_G, n_G \geq 2$ is called duality admissible if for any subsets of edges of $G$ indexed by $I, J \subset I_N$, $|I| < |J|$, for the subquotient graph $G \backslash I / J$ the following holds: $G \backslash I / J$ is disconnected, or is planar, of has a loop of length at most 3.

The essential part of the definition is the existence of a 3-loop in any subquotient graph to be able to get the good divisibility conditions for $[\varphi^I_j]$ by Proposition 17. The corresponding divisibility for the dual situation, i.e. for $[\Psi^I_j]$, is easier to be satisfied since a log-divergent graph always has a 3-valent vertex. An example of a log-divergent graph that has no 3-loops can be found in [14] on Figure 1,d) (after deletion of one of the vertexes).

Proposition 30. Let $G$ be a graph with $N = N_G$ edges.

1. If $G$ is log-divergent ($N = 2n$), then

$$[\Psi^I_j] \equiv 0 \mod q^3$$

for any $I, J \subset I_N$ with $|I| > |J| \geq 0$, $|J| \leq n - 3$.

2. If $G$ is, moreover, duality admissible, then

$$[\varphi^I_j] \equiv 0 \mod q^3$$

for any $I, J \subset I_N$ with $|J| > |I| \geq 0$, $|I| \leq n - 3$.

Proof. 1). We can assume $G$ is connected, otherwise the divisibility is clear. Since $G$ in log-divergent, $G$ has $(N_G, h_G, n_G) = (2n, n, n)$. We know $\Psi^I_{G,J} = \Psi_{G'}$ for the graph $G' \equiv G \backslash I / J$. Again, assume $G'$ is connected. Each deletion of an edge of $G$ decreases $h_G$, and each contraction of an edge decreases $n_G$. Thus $G'$ has $(N_{G'}, h_{G'}, n_{G'}) = (2n - |I| - |J|, n - |I|, n - |J|)$. If $G'$ has a vertex of degree 1 with an incident edge $e_1$, then $\Psi'_G$ is independent of $\alpha_1$ and one computes $[\Psi_{G'}] = q[\Psi_{G''}]$ for $G'' \equiv G / 1$. The divisibility $[\Psi_{G''}]$ is standard, follows from the analogue of Proposition 15, see [6], Lemma 16. Now one gets $q^3[\Psi_{G''}]$. If $G'$ has a 2-valent vertex with incident edges $e_1$ and $e_2$, then, after the change of the variables $\alpha_2 := \alpha_1 + \alpha_2$ one gets rid of $\alpha_1$ and obtains $[\Psi_{G'}] = q[\Psi_{G''}]$ for $G'' \equiv G / 1$ (see [6], Lemma 17, 1)). Thus $q^3[\Psi_{G''}]$ in this case.

Consider now the case when all the vertexes of $G'$ are of degrees $\geq 3$. Since $G$ is log-divergent, there should exist a vertex of $G'$ of degree 3. Indeed, $N_G = 2n$ and $|I| > |J|$ imply $N_{G'} < 2n_{G'}$. But on the other hand, each vertex has $\geq 4$ edges and each edge is counted twice, so $2(n_{G'} + 1) \leq N_{G'}$, a contradiction.

If now $G'$ has a $J \leq n - 3$, then $n_{G'} \geq 3$ and the Lemma 24 in [6] gives us $[\Psi_{G'}] \equiv q^2[\Psi_{G',3}^{1,2}, \Psi_{G',3}^{13,23}] \mod q^3$. Since $2h_{G'} < N_{G'}$, we apply Chevalley-Warning (Theorem 24) to the polynomials in the last square brackets and get $[\Psi_{G''}] \equiv 0 \mod q^3$.

2) Let $G' \equiv G \backslash I / J$ again in the way that $\varphi^I_{G,J} = \varphi_{G'}$. Instead of the vertices of small degree, we look at loops of small length. Similarly to part 1), we consider
the cases of the existence of a self-loop or a doubled edge (2-loop) and use (9), (10), and Proposition 15 to get \( q^3 | [\varphi_G] \).

Now consider the case when all the loops of a \( G' \) are of length at least 3. Assume \( G' \) is planar. There is a notion of the planar dual graph \( G' \) of a planar graph \( G' \). (see (2.2) in [12]). Its vertexes (resp. cycles) correspond to cycles (resp. vertexes) of the original graph, \( h_{G'} = n_G \) and \( n_{G'} = h_{G'} \). The important identity is \( \tilde{\varphi}_{G'} = \Psi_{G'}. \) Thus, one can use the results of part 1) and derive \( [\varphi_{G'}] = 0 \ mod \ q^3 \).

The last case to consider is \( G' \) has no self-loops or 2-loops and is not planar. By the very definition, since \( G \) is duality admissible, \( G' \) should have a loop of length 3 (say, formed by edges \( e_1, e_2 \) and \( e_3 \)). Thus, by Proposition 17 one gets \( [\varphi_{G'}] \equiv [\varphi_{G',3}, \varphi_{G}] \ mod \ q^3 \). We are again able to apply Chevaley-Warning (Theorem 21) for the two polynomials \( \varphi_{G',3}, \varphi_{G} \) and get \( [\varphi_{G'}] \equiv 0 \ mod \ q^3 \).

Now we prove the main theorem of this section.

**Theorem 31.** Let \( G \) be an duality admissible graph with \( h_G, n_G \geq 2 \). Then

\[ c_2(G)q = c_2^{\text{dual}}(G)q. \]  \hspace{1cm} (85)

**Proof.** Define \( n := n_G = h_G, N := N_G = 2n \). Let \( \Psi = \Psi_G \) be the graph polynomial and \( \varphi = \varphi_G \) the dual one. Denote by \( P \) the \( \mathbb{Q} \)-algebra generated by the sums of the point-counting functions. It is spanned by the functions \( q \mapsto \#(F_q) \) from the set of prime powers to integers, \( Y \in \text{Var}_\mathbb{Q} \). Consider the elements \( S_t := \sum_{I,J}[\varphi_I J]' \), where the sum goes over all \( I, J \subset I_N \) with \( |I| = |J| = t \), \( t = 1, \ldots, n \). Identity (80) shows that \( S_t \) respects Cremona transformation, i.e. symmetric under \( (\Psi \leftrightarrow \varphi) \). By (82), \( S_t \) is in \( P \) for any \( t \). One also has \( q^3 := [\mathbb{A}^3] \in P \).

Let \( I \subset P \) be the ideal generated by \( q^3 \) and \( S_t, 1 \leq t \leq n - 1 \).

We start with \( \Psi \) and apply formula (81):

\[ [\Psi] = [\Psi]' + \sum_{t=1}^{N} \sum_{|I|=t} [\Psi_I]' . \]  \hspace{1cm} (86)

Using the duality \( [\varphi_I]' = [\varphi_I']' \) for all \( I, J \subset I_N \), one gets

\[ [\Psi] = [\varphi]' + \sum_{t=1}^{N} \sum_{|I|=t} [\varphi_I]' . \]  \hspace{1cm} (87)

We always assume \( \Psi_I' = 0 \) and \( \varphi_I' = 0 \) for \( I \cap J \neq \emptyset \). For each \( [\varphi_I]' \) we substitute the expression from (82) applied to \( \Psi := \varphi_I' \) and get

\[ [\Psi] = [\varphi]' + \sum_{t=1}^{N} \sum_{|I|=t} \left( [\varphi_I] + \sum_{s=1}^{N-t} (-1)^s \sum_{|J|=s} [\varphi_J] \right) . \]  \hspace{1cm} (88)
We know that $[\varphi_j'] = [\varphi_{G^r}] = 0 = [\Psi]$ with $G' = G\backslash J/\mathcal{I}$ for $|I| > n$ or $|J| > n$, so we can reduce the upper bound of the summation signs from $N$ to $n$. Since $[\varphi_j'] \equiv 0 \mod q^3$ by Proposition 30 part 2 for all $I, J \subset I_N$ with $|I| > |J|$ and $|J| \leq n - 3$, we forget these summands shifting to the computations modulo $q^3$. There are also summands $[\varphi_j']$ with $|I| > |J| \geq n - 2$. Assume such term is non-zero. Then it is of the form $[\varphi_{G^r}]$ for $G' = G\backslash J/\mathcal{I}$ with $n_{G'} \leq h_{G^r} \leq 2$. We call such graph the small graph. We have $[\varphi_{G^r}] = q^2$ for 3 different non-isomorphic possibilities of such graphs with $h_{G^r} = 2$ and $[\varphi_{G^r}] = q$ for 1 case with $h_{G^r} = 1$.

The numbers of such subquotient graphs for different possibilities is a part of the local information of $G$. We will collect all terms $[\varphi_{G^r}]$ we get for small graphs (together with the dual objects of the next steps) to the sums denoted by $A_r$'s.

Now, the summands $[\varphi_j']$ of the last brackets of (88) with $|I| = |J| = t$ do not need to be 0, but they sum up to the element $S_t \in \mathcal{I}$. Thus one gets

$$[\Psi] \equiv [\varphi] + \sum_{t=1}^{n} \sum_{|I|=t} \sum_{s=t+1}^{n-t} (-1)^s \sum_{|J|=s} [\varphi_j'] + A_1 \mod \mathcal{I}. \quad (89)$$

Using induction of $r$, $1 \leq r \leq N$, we now prove the following statement:

$$[\Psi] = \begin{cases} 
[\varphi] + \sum_{t=r}^{n} \sum_{|I|=t} \sum_{s=t+1}^{n-t} d_{t,s}^{(r)} \sum_{|J|=s} [\varphi_j'] + A_r \mod \mathcal{I}, & r \text{ odd}, \\
[\Psi] + \sum_{t=r}^{n} \sum_{|I|=t} \sum_{s=t+1}^{n-t} d_{t,s}^{(r)} \sum_{|J|=s} [\varphi_j'] + A_r \mod \mathcal{I}, & r \text{ even}. 
\end{cases} \quad (90)$$

Here $A_r$ again is again a sum of terms $[\varphi_{G^r}]$ for small graphs and the duals. Formula (89) is the base of the induction, $r = 1$ and $d_{t,s}^{(1)} = (-1)^s$. For general $r$, start first with an odd $r$ and the congruence

$$[\Psi] \equiv [\varphi] + \sum_{t=r}^{n} \sum_{|I|=t} \sum_{s=t+1}^{n-t} d_{t,s}^{(r)} \sum_{|J|=s} [\varphi_j'] + A_r \mod \mathcal{I}. \quad (91)$$

First apply (82) for each $\Psi = \varphi_j'$:

$$[\Psi] \equiv [\varphi] + \sum_{t=r}^{n} \sum_{|I|=t} \sum_{s=t+1}^{n-t} d_{t,s}^{(r)} \sum_{|J|=s} ([\varphi_j'] + \sum_{p=1}^{n-t-s} \sum_{|K|=p} [\varphi_j']^p) + A_r \mod \mathcal{I} \quad (92)$$

with the rightmost summation going over all $K \subset I_N \backslash (I + J)$. Collecting the summands by the cardinality of indexes, we get

$$[\Psi] \equiv [\varphi] + \sum_{t=r}^{n} \sum_{|I|=t} \sum_{s=t+1}^{n-t} b_{t,s}^{(r)} \sum_{|J|=s} [\varphi_j'] + A_r \mod \mathcal{I}. \quad (93)$$
The coefficients \( b_{t,s}^{(r)} \) depend only on \( d_{i,j}^{(r)} \), \( i = |I| \leq t \), \( j = |J| \leq s \), but not on \( I \) and \( J \) itself. Using the duality, we rewrite

\[
\[\Psi\] \equiv [\Psi] + \sum_{t=r}^{n} \sum_{|I|=t}^{n-t} \sum_{|J|=s}^{n-t} b_{t,s}^{(r)} [\Psi_I^J] + A_r \mod \mathcal{I},
\]

(94)

Now, using (81) for each \( \Psi = \Psi_I^J \), one can rewrite the formula above as

\[
\[\Psi\] \equiv [\Psi] + \sum_{t=r}^{n} \sum_{|I|=t}^{n-t} \sum_{|J|=s}^{n-t} b_{t,s}^{(r)} \left( [\Psi_I^J] + (-1)^p \sum_{|K|=p}^{n-t-s} [\Psi_I^K] \right) + A_r \mod \mathcal{I}.
\]

(95)

By Proposition 30, part 1, we can get rid of all the summands \( [\Psi_I^J] \) for \( |J'| \geq |I'| \), \( |I'| \leq n - 3 \), while the sums \( \sum_{|I'|=|I|} [\Psi_I^{J'}] \), \( |I'| = |J'| \) contribute to 0 \( \mod \mathcal{I} \). We also sum up all the terms for small graph (here \( G' \) with \( h_{G'} < n_{G'} \leq 2 \)) adding \( A_r \) denote the result by \( A_{r+1} \).

Collecting the remaining summands by the cardinality of indexes, one gets

\[
\[\Psi\] \equiv [\Psi] + \sum_{s=r+1}^{n} \sum_{|I|=s}^{n-s} \sum_{|J|=s}^{n-s} \sum_{|I'|=t}^{n-t} d_{s,t}^{(r+1)} [\Psi_I^J] + A_{r+1} \mod \mathcal{I}
\]

(96)

for some integer coefficients \( d_{s,t}^{(r+1)} \) (linearly) depending on \( b_{i,j}^{(r)} \), \( i \leq t \), \( j \leq s \).

On the Figure 3 on the left there are indicated the pairs \((I, J)\) for which the summands \( [\Psi_I^J] \) appear in formula (94). The middle picture shows the pairs \((J, I)\) and \((J, IK)\) such that \( \Psi_I^J \) appear in formula (95). The right picture shows what summands \( [\Psi_I^J] \) survive in (97). Reflecting the right picture, we see that we have decreased the number of the (fat) points (terms surviving in the sum) by 1 level.
So, interchanging \( s \) and \( t \), as well as \( I \) and \( J \) in (97), one obtains the statement for \( r+1 \) in (90):

\[
[\Psi] \equiv [\Psi] + \sum_{t=r+1}^{n} \sum_{|J|=t} \sum_{s=t+1}^{n} a_{s,t}^{(r+1)}[\Psi] + A_{r+1} \mod I.
\] (97)

The conditions (duality and vanishing lemmas) we used above are symmetric under \( \Psi \leftrightarrow \phi \) in the right hand side of equations (91) - (97). This implies the proof for the case \( r \) is even starting with formula (91) after substituting \( \phi = \Psi \). This finishes our inductive proof of (90).

The polynomials \( \Psi \) and \( \phi \) are of degree \( n \) of \( N = 2n \) variables. On the \( r = n-3 \)-th step we get rid of all the summands in the big sums on the right of (90). Indeed, consider the case \( r \) is odd. On that step we derive (97) with terms with \( |J| > |I| \geq n-2 \) (corresponding to small graphs). But these terms are considered to be in \( A_{r+1} \) already. The same holds in the case \( r \) is even.

So we get \( [\Psi] \equiv [\phi] + A_{n-2} \equiv [\Psi] + A_{n-2} \mod I \). In other words,

\[
[\Psi] = [\Psi] + a(\Psi) + \sum_{i=1}^{n} u_i(\Psi)S_i + v(\Psi)q^3
\] (98)

for \( a(\Psi) := A_{r-2}, v(\Psi), u_i(\Psi) \in \mathcal{P}, 1 \leq i \leq N \).

Now we want to do the similar computation for \([\phi]\). One can again use the symmetry between \( \Psi \) and \( \phi \) in the applied conditions (Proposition 30) and duality. Starting with formula (86), we do the same swapping \( \Psi \) with \( \phi \) both on the left and on the right hand side of each formula till we finally derive

\[
[\phi] = [\phi] + a(\phi) + \sum_{i=1}^{n} u_i(\phi)S_i + v(\phi)q^3
\] (99)

for \( a(\phi), v(\phi), u_i(\phi) \in \mathcal{P}, 1 \leq i \leq N \). We do not have control on the relation between \( v(\Psi) \) and \( v(\phi) \), but the coincidence of the coefficients \( a_{s,t}^{(r)}, b_{s,t}^{(r)} \) in (86)- (98) for \( \Psi \) and \( \phi \), implies \( u_i(\Psi) = u_i(\phi) \) for any \( 1 \leq i \leq n \). Now (98) and (99) imply

\[
[\Psi] - [\phi] = \sum_{i=1}^{n} (u_i(\Psi) - u_i(\phi))S_i + v(\Psi)q^3 - v(\phi)q^3
= (v(\Psi) - v(\phi))q^3 + (a(\Psi) - a(\phi)).
\] (100)

Lets show that \( a(\Psi) = a(\phi) \), i.e. \( a \) is stable under duality. By the discussion before (97) for \( \Psi \) in the case \( r \) is odd, \( A_{r+1}(\Psi) \backslash A_r(\Psi) \) is a sum of terms of the form \([\Psi_\gamma] \), where \( \gamma = G \backslash J / I \) with \( h_\gamma < n_\gamma \leq 2 \). Let’s look at the case \( n_\gamma = 2 \) and \( h_\gamma = 1 \). To obtain \( \gamma \) one can first delete and contract \( n - 2 \) edges and then delete one more edge. Consider any subquotient graph \( \tilde{\gamma} \) of \( G \) with \( \tilde{\gamma} = \tilde{G} \backslash J / I' \)
with $I' \subset I$, $|I| = |J| = n - 2$, $I \setminus I' = \{i_1\}$ and $\gamma = \tilde{\gamma} \setminus i_1$. Then one can rewrite $[\Psi_{\tilde{\gamma}}]$ by (81) in terms of $[\Psi_{\gamma'}]$, summands $[\Psi_{\gamma}]$ (for $\gamma$ as above a subgraph of $\tilde{\gamma}$), and terms of smaller graphs. Now we can do the opposite: rewrite $\varphi_{\tilde{\gamma}}$ by (81) is terms of $[\varphi_{\gamma'}]$, the terms $[\varphi_{\gamma}]$ for $\gamma'$ a quotient graph of $\tilde{\gamma}$ with $n_{\gamma'} = 1$, and terms for smaller graphs. One can check that the coefficients for in this two formulas coincide for terms that give $[\Psi] = q^2$. Similar for smaller graphs, that means for case $[\varphi] = q$. It follows now that $a$ as a the sum of such formulas, is also stable for interchange $\Psi$ and $\varphi$. The other possibility to see the symmetry of of the formulas in $\Psi$ and $\varphi$ for small graphs is to use the planar duality (the small graphs a planar). Thus $A_{r+1}(\Psi) \setminus A_r(\Psi) = A_{r+1}(\varphi) \setminus A_r(\varphi)$ for an odd $r$. The case of even $r$ works similar. It follows that $a(\Psi) = A_{n-2}(\Psi) = A_{n-2}(\varphi) = a(\varphi)$.

Since $[\Psi] \equiv q^2 \cdot c_2(G) \mod q^3$ and $[\varphi] \equiv q^2 \cdot c^\text{dual}_2(G) \mod q^3$, formula (100) finally yields

$$c^\text{dual}_2(G) \equiv c_2(G) \mod q^3.$$ (101)

References

[1] Belkale,P. Brosnan,P. Matroids, motives and a conjecture of Kontsevich Duke Math. Journal, Vol. 116 (2003), 147-188.

[2] Bloch,S. Esnault,H. Kreimer,D. On motives associated to graph polynomials Comm. Math. Phys. 267 (2006), no. 1, 181-225.

[3] Broadhurst,D. Kreimer,D. Knots and Numbers in $\phi^4$ Theory to 7 Loops and Beyond, Int. J. Mod. Phys. C6, (1995) 519-524.

[4] Brown,F. On the periods of some Feynman integrals arXiv:0910.0114v2

[5] Brown,F. Doryn,D. Framings of for graph hypersurfaces, arXiv:1301.3056

[6] Brown,F. Schnetz,O. A $K3$ in $\phi^4$, Duke Math. Journal, Vol. 161, No. 10 (2012), 1817-1862

[7] Brown,F. Schnetz,O. Yeats,K. Properties of $c_2$ invariants of Feynman graphs arXiv:1203.0188

[8] Brown,F Yeats,K. Spanning forest polynomials and the transcendental weight of Feynman graphs, Comm. Math. Phys. 301:357-382, (2011)

[9] Chertykin,K. Kataev,A. Tkachev,F. The Gegenbauer polynomial x-space technique Nucl. Ph. B174(1980) 345-477

[10] Doryn,D. On the cohomology of graph hypersurfaces associated to certain Feynman graphs, Comm. Num. Th. Phys. 4 (2010), 365-415.
[11] Doryn, D. *On one example and one counterexample in counting rational points on graph hypersurfaces*, Let. Math. Phys., Vol. 97 (2011), Is. 3, 303-315

[12] Itzykson, J. Zuber, J. *Quantum Field Theory*. Mc-Graw-Hill, (1980).

[13] Huh, J. *A counterexample to the geometric Chevalley-Warning conjecture* [arXiv:1307.7765v3]

[14] Schnetz, O. *Quantum field theory over $F_q$*, The Electronic Jour. of Combin. 18, #P102 (2011).

[15] Schnetz, O *Quantum periods: A census of $\phi^4$ transcendentals*, Comm. Num. Th. Phys. 4, no. 1 (2010), 1-48.