THE 1-LOOP SELF-ENERGY OF AN ELECTRON
IN A STRONG EXTERNAL MAGNETIC FIELD REVISITED

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Abstract: I calculate the 1-loop self-energy of the lowest Landau level of an electron of mass $m$ in a strong, constant and uniform external magnetic field $B$, beyond its always used truncation at $(\ln L)^2$, $L = \frac{|e|B}{m^2}$. This is achieved by evaluating the integral deduced in 1953 by Demeur and incompletely calculated in 1969 by Jancovici, which I recover from Schwinger’s techniques of calculation. It yields

$$\delta m \simeq \frac{\alpha m}{4\pi} \left[ (\ln L - \gamma_E - \frac{3}{2})^2 - \frac{9}{4} + \frac{\pi}{\beta - 1} + \frac{\pi^2}{6} + \frac{\pi \Gamma[1 - \beta]}{L^{\beta - 1}} + \right.$$

$$\frac{1}{L} \left( \frac{\pi}{2 - \beta} - 5 \right) + O\left( \frac{1}{L^2} \right)$$

with $\beta \simeq 1.175$ for $75 \leq L \leq 10000$. The $(\ln L)^2$ truncation exceeds the precise estimate by 45% at $L = 100$ and by more at lower values of $L$, due to neglecting, among others, the single logarithmic contribution. This is doubly unjustified because it is large and because it is needed to fulfill appropriate renormalization conditions. Technically challenging improvements look therefore necessary, for example when resumming higher loops and incorporating the effects of large $B$ on the photonic vacuum polarization, like investigated in recent years.

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1 Generalities

We shall be concerned in this short note, with the self-energy of an electron at 1-loop in the presence of a strong, constant and uniform external magnetic field $B$. The electron propagator is described by the sum of the 2 diagrams of Fig. 1.

![Fig. 1: 1-loop radiative correction to the mass of an electron.](image)

in which the double horizontal lines, external as well as internal, stand for an electron of mass $m$ in an external $B$. The electron mass is defined as the pole of its propagator, which is the only gauge invariant definition. Renormalization conditions are set accordingly. The self-energy that we shall calculate is the second diagram. For the sake of simplicity, we shall restrict external electrons to lie in the lowest Landau level. This does not apply to the internal electron propagator, which includes a summation on all Landau levels.

1.1 Motivation

The uses of the self-energy of an electron in a strong external $B$ generally rely on its “leading” double logarithmic term proportional to $\left(\ln \frac{|e|B}{m^2}\right)^2$ and its eventual transmutation into a single logarithmic behavior on accounting for the accompanying modifications of the photonic vacuum polarization. The double logarithmic term was first extracted in 1969 by Jancovici from a general formula deduced by Demeur in 1953. At the very end of Jancovici mentions the presence of potentially large, single logarithmic and constant corrections, but the constant $A$ could not be determined at that time. The asymptotic double logarithmic behavior at $B \to \infty$ was also obtained by Loskutov and Skobelev in 1977 in the 2-dimensional limit of QED which is a suitable approximation at this limit. However, only the kinematical domains of integrations leading to the double logs were accounted for and the eventual presence of large but non double-logarithmic corrections was not investigated. The next step is a resummation of the same double-logarithmic terms in rainbow-type diagrams argued to be dominant. In Loskutov and Skobelev have shown in 1981 that the result exponentiates. A slightly different result, non-exponential, was obtained later in 1999 by Gusynin and Smilga in which were still only concerned by resuming double logarithmic terms. An important modification to be brought to these results had already been shown earlier in 1983 in again by Loskutov and Skobelev, then studied more extensively in 2002 by Kuznetsov, Mikheev and Osipov: accounting for the effective photon mass induced by asymptotically strong magnetic fields shrinks the double logarithm down to a single logarithmic behavior. While in an exponentiation still occurs, a different result is obtained in which higher Landau levels for the virtual electron are also included.

In all these calculations, only double logarithmic terms were considered at the start, which would then be eventually resummed and corrected by an effective photon mass. This makes that the corresponding results never incorporate the starting large single logarithm and constant which, as we shall show, strongly damp the $\ln^2$ truncation of the 1-loop electron self-energy. This neglect is all the more unfortunate as the large single logarithm is tightly connected to renormalization conditions and to the corresponding counterterms. Getting meaningful results requires indeed that the appropriate renormalization conditions should be fulfilled at each order of the resummation process, and that the same

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5 This is why I do not pay a fair enough tribute to the many authors that contributed to this subject, and I apologize for this. I will instead insist on very small details, generally not mentioned, that can help the reader.

6 As far as I could see, Demeur’s calculations, performed with techniques which are unfamiliar today, have not been reproduced. They have been critically examined and completed by Newton at small values of $\frac{|e|B}{m^2}$, but this path seems to have then been abandoned.

7 see also.
care be due when including the corrections to the photon vacuum polarization by the external $B$. Calculations as they have been done up to now, that gave birth to many developments, for example in condensed matter physics\textsuperscript{8}, do not seem to worry about these criteria, which can jeopardize their conclusions and predictions.

1.2 The procedure

I will re-calculate the Demeur-Jancovici integral and explicitly display the large corrections that strongly damp the double logarithmic behavior of the electron self-energy in a strong external $B$. Special attention will be paid to the counterterm that ensures suitable renormalization conditions for the electron self-energy.

To cast this on solid grounds, I will first show, in section 2, how this integral can be recovered by using the formalism developed by Schwinger in the late 1940’s \textsuperscript{11}. The corresponding calculations are explained in details in the book by Dittrich and Reuter \textsuperscript{12} in 1985 (which includes a long list of references). One finds in there, in particular, the expression for the renormalized 1-loop mass operator $\Sigma(\pi)$, where $\pi_{\mu} = p_{\mu} - eA_{\mu}$, for an electron in an external $B$, as deduced in 1974 by Tsai \textsuperscript{13}.\textsuperscript{9} It will be the starting point of the original calculations.

I will then make then use of Demeur’s technique \textsuperscript{2} to sandwich the mass operator $\Sigma(\pi)$ between two “privileged” electron states $|\psi\rangle$ (to reproduce the terminology of Demeur and previous authors, in particular Luttinger \textsuperscript{14}), on mass-shell. This restricts, but greatly simplifies the calculations. This matrix element corresponds to $\delta m$ of the electron at 1-loop in the presence of $B$. The privileged state, that always exists in the presence of $B$, is the one with energy $m$.

In our present terminology, it corresponds to the Lowest Landau Level (LLL) and, on mass shell, it satisfies the Dirac equation $(\frac{\hbar}{m} + m)|\psi\rangle = 0\textsuperscript{10}$. Then, I will show how changes of variables cast $\delta m$ in the form deduced by Demeur \textsuperscript{2} and used by Jancovici \textsuperscript{1}. It is a convergent double integral that only depends on $\frac{|e|Bm^2}{m^2}$Its rigorous exact analytical evaluation lies beyond my ability. However, a trick due to M.I. Vysotsky in his study of the screening of the Coulomb potential in an external magnetic field \textsuperscript{15} comes to the rescue: the part of the integrand that resists analytical integration can be nearly perfectly fitted inside the range of integration by a simpler function that can be analytically integrated.

2 The 1-loop self-energy $\Sigma$ for the lowest Landau level of an electron in external $B$; equivalence between the calculations by Schwinger and Demeur

2.1 The general formula for the electron self-energy operator at 1-loop

For this work to be self-contained, I recall the main steps in the determination of the operatorial expression of the self-energy of an electron in an external $B$ deduced by Tsai \textsuperscript{13}. I closely follow the book by Dittrich and Reuter \textsuperscript{12}, more precisely the paragraphs 2 and 3, that I summarize here. It means that the present subsection does not include anything original and owes everything to \textsuperscript{12}.

The self-energy $\Sigma(x',x'')$ includes 2 internal propagators:

* the free photon propagator, that we shall take in the Feynman gauge, arguing of the gauge independence of Schwinger’s techniques of calculation \textsuperscript{11}\textsuperscript{16}

\begin{equation}
\begin{split}
  i\Delta_{\mu\nu}(x' - x'') &= -i\eta_{\mu\nu}D(x' - x''), \\
  D(x' - x'') &= \int \frac{d^4k}{(2\pi)^4} e^{ik(x' - x'')} \frac{1}{k^2 - i\epsilon}.
\end{split}
\end{equation}

\textsuperscript{8}see for example the review \textsuperscript{10}.

\textsuperscript{9} At the end of his paper, Tsai just states that his calculation, which uses the techniques and results of Schwinger, yields, when projected on the ground state of the electron, “...the known result of Demeur” (this correspondence is the subject of subsection \textsuperscript{2.4}.

\textsuperscript{10}I use Schwinger’s metric $(-, +, +, +)$.
* the electron propagator in the presence of an external magnetic field as determined by Schwinger [16] and then re-expressed and used by Tsai [13] to make the link with the calculations by Demeur [2]

\[ G(x',x'',B) \equiv i < 0 | T \psi(x') \bar{\psi}(x'') | 0 >= \Phi(x',x'') \int \frac{d^4p}{(2\pi)^4} e^{ip(x'-x'')} G(p,B), \] (2)

in which the phase \( \Phi(x',x'') \), which ensures gauge invariance, is given by

\[ \Phi(x',x'') = \exp \left[ ie \int_{x''}^{x'} dx_\mu \left( A^\mu(x) + \frac{1}{2} F^{\mu\nu}(x'_\nu - x''_\nu) \right) \right], \] (3)

and (this is eq.(2.47b) of [12])

\[ G(p,B) = i \int_0^\infty ds \ e^{-is1 \left( m^2 - i\epsilon + p_1^2 + \frac{1}{4} \gamma_3 p_1^2 \right)} e^{iz\gamma_3} \left( m - \not{p}_1 - e^{-iz\gamma_3} \not{p}_1 \right), \] with \( z = eBs_1 \). (4)

e stands everywhere in this work for the charge of the electron \( e = -|e| < 0 \). The metric that is used is \((-1,1,1,1)\) and the notations are the following

\[ \sigma^3 = \sigma^{12} = \frac{i}{2}[\gamma_1,\gamma_2] = \text{diag}(1,-1,1,-1), \]

\[ p_\parallel = (p_0,0,0,p_3), \quad p_\perp^2 = -p_0 + p_3^2, \quad p_\perp = (0,p_1,p_2,0), \quad p_\parallel^2 = p_1^2 + p_2^2, \] (5)

The propagator (4) includes all Landau levels of the electron.

The constant external \( B \) is chosen in the \( z \)-direction such that \( F_{12} = -F_{21} = B \) (therefore the notations “\( \parallel \)” and “\( \perp \)” have a natural meaning).

The phase (3) is independent of the choice of the path of integration because the curl of the integrand vanishes. Choosing a straight line of integration \( x(t) = x'' + t(x' - x''), t \in [0,1] \) leads to the familiar expression

\[ \Phi(x',x'') = e^{ie \int_{x''}^{x'} dx_\mu A^\mu(x)}. \] (6)

### 2.1.1 The unrenormalized self-energy

- In terms of the quantities above, the 1-loop self-energy writes (“c.t.” stands for “counterterms”)

\[ \Sigma(x',x'',B) = ie^2 \gamma^\mu G(x',x'',B) D(x' - x'') \gamma_\mu + \text{c.t.} \] (7)

that is

\[ \Sigma(x',x'',B) = \Phi(x',x'') \int \frac{d^4p}{(2\pi)^4} e^{ip(x'-x'')} \Sigma(p,B), \]

\[ \Sigma(p,B) = ie^2 \gamma^\mu \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - i\epsilon} G(p-k,B) \gamma_\mu + \text{c.t.} \] (8)

One introduces a second Schwinger parameter \( s_2 \) for the photon propagator

\[ \frac{1}{k^2 - i\epsilon} = i \int_0^\infty ds_2 e^{-is_2(k^2 - i\epsilon)}, \] (9)

and get eq. (3.11) of [12]

\[ \Sigma(p,B) = -ie^2 \int_0^\infty ds_1 \int_0^\infty ds_2 \int \frac{d^4k}{(2\pi)^4} e^{-is_2(k^2 - i\epsilon)} \frac{1}{k^2 - i\epsilon} \left( m^2 + (p-k)^2 + \frac{1}{4} \gamma_3 (p-k)^2 \right) \gamma^\mu \frac{e^{iz\gamma_3}}{\cos z} \left( m - (\not{p}_\parallel - \not{k}_\parallel) - \frac{e^{-iz\gamma_3}}{\cos z} (\not{p}_\perp - \not{k}_\perp) \right) \gamma_\mu + \text{c.t.} \] (10)
• The next step is to change variables: one goes from $s_1$ and $s_2$ to $s$ and $u$ such that
\[ s_1 = su, \quad s_2 = s(1-u) \Rightarrow \int_0^\infty ds_1 \int_0^\infty ds_2 = \int_0^\infty ds \int_0^1 du, \quad Y = eBsu. \tag{11} \]

$\Sigma(p, B)$ can then be cast in the form
\[
\Sigma(p, B) = -ie^2 \int_0^\infty ds \int_0^1 du Y \left\{ \int \frac{d^4 k}{(2\pi)^4} e^{-i\chi} \gamma^\mu e^{iY\sigma^3} \left[ m - (1-u) \frac{p_\|}{\cos Y} + \frac{e^{-iY\sigma^3}}{1-u+u \tan Y} \frac{\hat{p}_\perp}{\cos Y} \right] \gamma_\mu + c.t. \right\}.
\]

\[
\chi = um^2 + \phi + (k_\| - up_\|)^2 + \left( 1-u + u \frac{\tan Y}{Y} \right) \left( k_\perp - \frac{u \tan Y}{1-u+u \tan Y} p_\perp \right)^2,
\]

\[
\phi = u(1-u) p_\| + \frac{u}{Y} (1-u) \sin Y \frac{1}{(1-u) \cos Y + u \sin Y \frac{p_\perp^2}{Y}}.
\tag{12}\]

The $k$ integration, which only occurs inside the curly bracket in [12] can now be performed by shifting the integration variables inside $\chi$ and by using the standard integral $\int_{-\infty}^{+\infty} dx e^{\pm iAx^2} = e^{\pm \frac{x^2}{A}} A^{1/2}, A > 0$, which yields eq. (3.27) of [12] ($\alpha = \frac{e^2}{4\pi}$)
\[
\Sigma(p, B) = \frac{\alpha m}{2\pi} \int_0^\infty ds \int_0^1 du \frac{e^{-i\chi(um^2+\phi)}}{(1-u) \cos Y + u \sin Y \frac{Y}{Y}} e^{iY\sigma^3} \left[ 1 + e^{-2iY\sigma^3} \right.
\]
\[
\left. + (1-u) e^{-2iY\sigma^3} \frac{p_\|}{m} + (1-u) \frac{e^{-iY\sigma^3}}{(1-u) \cos Y + u \sin Y \frac{Y}{Y}} \frac{\hat{p}_\perp}{m} \right] + c.t. \tag{13}\]

At this stage, the integrations on $s$ and $u$ cannot be done explicitly.

• The last and crucial step to get the self-mass $\delta m$ of a given state $| \psi >$ on mass-shell $((\not{m}+m) | \psi > = 0)$ is to go to the so-called “space representation” and $\Sigma(\pi)$ defined by
\[
\Sigma(x', x'', B) = \Phi(x', x'') \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x' - x'')} \Sigma(p, B) = < x' | \Sigma(\pi) | x'' >. \tag{14}\]

Note that the phase $\Phi(x', x'')$ gets now “included” in $\Sigma(\pi)$.

For this, one has to go through the manipulations of pages 47-50 of [12]. We shall only write here the intermediate formulæ (eventually correcting for some misprints). One is led to introduce
\[
\Delta = (1-u)^2 + 2u(1-u) \cos Y \frac{\sin Y}{Y} + u^2 \left( \frac{\sin Y}{Y} \right)^2, \tag{15}\]

the angle $\beta$ such that
\[
\cos \beta = \frac{(1-u) \cos Y + u \frac{\sin Y}{Y}}{\Delta^{1/2}}, \quad \sin \beta = \frac{(1-u) \sin Y}{\Delta^{1/2}}, \tag{16}\]

and
\[
\Phi = u(1-u) \left( m^2 - \frac{\beta^2}{\Delta} \right) + \frac{u}{Y} (\beta - (1-u)Y) \frac{\pi^2}{\Delta} - u^2 \frac{e}{2} \pi_{\mu \nu} F^{\mu \nu}. \tag{17}\]

One gets then
\[
\Sigma(\pi) = \frac{\alpha m}{2\pi} \int_0^\infty ds \int_0^1 du e^{-i\chi s^2 m^2} \left[ \frac{e^{-i\Phi}}{\Delta^{1/2}} \left[ 1 + e^{-2iY\sigma^3} + (1-u) e^{-2iY\sigma^3} \frac{\hat{\pi}}{m} \right] \right.
\]
\[
\left. + (1-u) \left( \frac{1-u}{\Delta} + u \frac{\sin Y}{Y} e^{-iY\sigma^3} \frac{\hat{\pi}}{m} - e^{-2iY\sigma^3} \frac{\hat{\pi}}{m} \right) \right] + c.t. \tag{18}\]

\[11\]In eq. (3.31) of [12], the first 2 expressions for $\cos \beta$ should be replaced by their inverse.

\[12\]There is a sign misprint in the definition (3.38b) of $\Phi$ in [12], which has been corrected here. The correct sign is the one in eq. (3.35) of [12].

\[13\]This $\Phi$ should not be confused with the phase $\Phi(x', x'')$ of [3].
2.1.2 The renormalization conditions and the counterterms

Since we are working in a gauge invariant formalism, consistency requires that the mass of the electron be also defined in a gauge invariant way, that is as the pole of its propagator.

* At \( B = 0 \), the electron propagator writes

\[
G(p) = \frac{1}{\not{p} + m_0 + \Sigma(p)},
\]

in which \( \Sigma(p) \) is the bare quantity, and the renormalized electron mass is accordingly defined by

\[
m = m_0 + \delta m, \quad \delta m = \Sigma(p)_{\not{p} + m = 0}.
\]

* At \( B \neq 0 \), the electron propagator is

\[
G = \frac{1}{\not{\varphi} + m_0 + \Sigma(\pi)}.
\]

We define, in analogy with eq. (20), the mass of the electron as the pole of its propagator by

\[
m = m_0 + \Sigma(\pi)_{\not{\varphi} + m = 0} \Leftrightarrow \delta m = \Sigma(\pi)_{\not{\varphi} + m = 0}.
\]

\( \delta m \) depends on the external field. Note that, on mass-shell, \( \varphi^2 \equiv -\pi^2 + \frac{\alpha}{2} \sigma_{\mu\nu}F^{\mu\nu} = m^2 \).

The counterterms are determined by the two equations (3.39) and (3.40) of \[12\] (we restore the superscript “\( \text{ren} \)” to make clear that one deals now with the renormalized quantities)\[^{14}\]

\[
\lim_{\not{\varphi} + m = 0} \lim_{B \to 0} \Sigma_{\text{ren}}(\pi) = 0, \quad \lim_{\not{\varphi} + m = 0} \lim_{B \to 0} \frac{\partial \Sigma_{\text{ren}}(\pi)}{\partial \not{\varphi}} = 0.
\]

They ensure that, after turning off \( B, (\not{\varphi} \to \not{\varphi}) \), the renormalization conditions \( \Sigma_{\text{ren}}(p)_{\not{\varphi} + m = 0} = 0 \) and \( \frac{\partial \Sigma_{\text{ren}}(p)}{\partial \not{p}} \bigg|_{\not{\varphi} + m = 0} = 0 \) are fulfilled (compare with (20)).

Since the renormalization conditions are expressed at \( B = 0 \), one needs the following limits at \( Y \equiv eBu \to 0 \)

\[
\begin{align*}
\beta & \to 0 \quad (1 - u)Y + O(Y^2), \\
\Phi & \to 0 \quad u(1 - u)(m^2 - \varphi^2) = u(1 - u)(m^2 - \not{\varphi}^2), \\
\Delta & \to 0 \quad 1.
\end{align*}
\]

At \( B = 0, \pi = p, \) and one gets

\[
\Sigma(p) = \frac{\alpha m}{2\pi} \int_0^\infty \frac{ds}{s} \int_0^1 du \ e^{-isu^2m^2} \left[ e^{-isu(1-u)} (m^2 - \varphi^2) (2 + (1-u)\frac{\not{p}}{m}) + \text{c.t.} \right],
\]

and, at \( \not{p} + m = 0, \) point at which the renormalization conditions are expressed

\[
\Sigma(p)_{\not{p} + m = 0} = \frac{\alpha m}{2\pi} \int_0^\infty \frac{ds}{s} \int_0^1 du \ e^{-isu^2m^2} \left[ (1 + u) + \text{c.t.} \right].
\]

To fulfill the first renormalization condition, we must therefore introduce a first counterterm

\[
\text{c.t.}_1 = -(1 + u).
\]

To implement the second renormalization condition, one calculates

\[
\frac{\partial \Sigma(p)}{\partial \not{p}} = \frac{\alpha m}{2\pi} \int_0^\infty \frac{ds}{s} \int_0^1 du \ e^{-isu^2m^2} \left[ (-isu)(1-u)(-2\varphi)e^{-isu(1-u)(m^2 - \varphi^2)} \left( 2 + (1-u)\frac{\not{p}}{m} \right) + \frac{1-u}{m} e^{-isu(1-u)(m^2 - \varphi^2)} \right],
\]

\[^{14}\]These renormalization conditions are carefully explained in p. 38-41 of [12]. Their importance is also emphasized by Ritus in [14].
such that, at $\dot{\pi} + m = 0$ one gets

$$\frac{\partial \Sigma(p)}{\partial \pi} \big|_{\dot{\pi} + m = 0} = \frac{\alpha m}{2\pi} \int_0^1 \frac{ds}{s} \int_0^1 du e^{-isu^2m^2} \left[ -2ismu(1-u^2) + \frac{1-u}{m} \right].$$  \hspace{1cm} (29)$$

This leads to the second counterterm

$$c.t. = - (\pi / m) \left( \frac{1-u}{m} - 2imu(1-u^2)s \right),$$  \hspace{1cm} (30)$$
in which the factor $(\pi / m)$ ensures that the first renormalization condition keeps satisfied.

### 2.1.3 The renormalized 1-loop self-energy in the presence of $B$

Collecting all terms yields

$$\Sigma(\pi) = \frac{\alpha m}{2\pi} \int_0^1 \frac{ds}{s} \int_0^1 du e^{-isu^2m^2} \left\{ e^{-is\Phi} \left[ \frac{1+e^{-2iY\sigma^3} + (1-u)e^{-2iY\sigma^3}}{\Delta} \right] \frac{\Phi}{m} \right\}$$

$$+ (1-u) \left( \frac{1-u}{\Delta} + \frac{u}{Y} \sin Y \frac{e^{-iY\sigma^3} - e^{-2iY\sigma^3}}{m} \right)$$

$$- (1+u) - (\pi / m) \left\{ \frac{1-u}{m} - 2imu(1-u^2)s \right\} \right\}.$$  \hspace{1cm} (31)$$

It is eq. (3.44) of [12], which coincides with the operatorial expression of the self-energy of an electron in an external field deduced by Tsai [13].

### 2.2 Projecting $\Sigma(\pi)$ on the “privileged state”: $\delta m$ for the lowest Landau level

The spectrum of a Dirac electron in a pure magnetic field directed along $z$ is [13]

$$\epsilon_n^2 = m^2 + p_z^2 + (2n + 1 + \sigma z) |e|B,$$  \hspace{1cm} (32)$$
in which $\sigma_z = \pm 1$ is $2 \times$ the spin projection of the electron on the $z$ axis. So, at $n = 0, \sigma_z = -1, p_z = 0, \epsilon_n = m$: this so-called “privileged state” is nothing more than the lowest Landau level.

We can consider $A_\mu = \begin{pmatrix} A_0 = 0 \\ A_x = 0 \\ A_y = xB \\ A_z = 0 \end{pmatrix}$ such that $F_{12} = B$ is the only non-vanishing component of the classical external field $F_{\mu\nu}$. Then, the wave function of the privileged state of energy $m$ writes [14] [19]

$$\psi_{n=0,\sigma_z=-1, p_z=0} = \frac{1}{\sqrt{N}} \left( \frac{|e|B}{\pi} \right)^{1/4} e^{-|e|B \frac{y}{\pi}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, N \equiv \frac{L_y}{L_z}.$$

Following [22], in order to determine $\delta m$ for the (on mass-shell) LLL, we shall sandwich the general self-energy operator $\Sigma(\pi)$ between two states $|\psi >$ defined in (33) and satisfying $(\pi / m) |\psi > = 0$.

The expression (31) involves $\Phi$ that we shall replace by $-m$, $\Delta$ that needs not be transformed, and $\Phi$ which involves $m^2 - \pi^2, \pi^2_1$ and $\sigma_{\mu\nu} F_{\mu\nu}$. The only non-vanishing component of $F_{\mu\nu}$ being $F^{12} = B, \sigma_{\mu\nu} F_{\mu\nu} = \sigma_{12} F^{12} + \sigma_{21} F^{21} = 2 \sigma_{12} F^{12} \equiv 2 \sigma_3 B$. Since the electron is an eigenstate of the Dirac equation in the presence of $B$, $m^2 - \pi^2$ can be taken
Therefore, on mass-shell, \( \pi^2_\perp \equiv \pi^2_1 + \pi^2_2 \) is also identical, since the privileged state has \( p_z = 0 \) and we work in a gauge with \( A_z = 0 \), to \( \bar{\pi}^2 \equiv \pi^2_1 + \pi^2_2 \). One has \( \frac{\bar{\pi}^2}{(\pi/h)^2} = -\pi^2 + (e/2) \sigma_{\mu\nu} F^{\mu\nu} \) such that \( \pi^2_\perp = -\bar{\pi}^2 + \pi^2_0 + \sigma_3 eB \). Since our gauge for the external \( B \) has \( A_0 = 0 \), \( \sigma^2_0 = p^2_0 \), which is the energy squared of the electron, identical to \( m^2 \) for the privileged state. Therefore, on mass-shell, \( \pi^2_\perp = \sigma_3 eB \). When sandwiched between privileged states,

\[
\langle \psi | \sigma^3 | \psi \rangle = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \text{diag}(1, -1, 1) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = -1 \text{ such that } \sigma^3 \text{ can be replaced by } -1
\]

and \( \Phi \) shrinks to \( u eB \left( 1 - \frac{\beta}{Y} \right) \). \( \sigma^3 \) can also be replaced by \(-1\) in the exponentials of (31).

\( \Sigma(\pi) \) in (31) also involves a term proportional to \( \frac{\bar{\pi}}{Y} \). Since the privileged state has \( p_z = 0 \) and we work at \( A_z = 0 \), this is also equal to \( \gamma^9 \pi_\mu \gamma^9 \pi_0 = \bar{\pi} + \gamma^0 p^0 \). \(< \psi | \bar{\pi} | \psi \rangle = -m \) such that \(< \psi | \bar{\pi} | \psi \rangle = \langle \psi | -m + \gamma^0 p^0 | \psi \rangle \). Since \( \gamma^0 = \text{diag}(1, 1, -1, -1) \), eq. (33) yields \( \langle \psi | \bar{\pi} | \psi \rangle = -m + p^0 \).

The energy \( p^0 \) of the privileged state \( | \psi \rangle \) being equal to \( m \), this term vanishes.

Gathering all information and simplifications leads finally to

\[
\delta m_{LLL} \equiv \Sigma(\pi)_{\beta + m = 0} = \frac{\alpha m}{2\pi} \int_0^\infty \frac{ds}{s} \int_0^1 du \, e^{-isu^2m^2 \left[ e^{-i\Phi(Y)} \right] \Delta(u, Y) (1 + u e^{2iY}) - (1 + u)_{\text{c.t.}}}.
\]

(34)

with \( Y = eBs u \). \( \Delta(u, Y) \) is the same as in (15), \( \sin \beta \) and \( \cos \beta \) the same as in (16). \( \Phi \) in (17) has shrunk to

\[
\Phi(u, Y) = u eB \left( 1 - \frac{\beta(u, Y)}{Y} \right) = u eB - \frac{\beta(u, Y)}{s}.
\]

(35)

Equivalently

\[
\delta m_{LLL} \equiv \Sigma(\pi)_{\beta + m = 0} = \frac{\alpha m}{2\pi} \int_0^\infty \frac{ds}{s} \int_0^1 du \, e^{-isu^2m^2 \left[ e^{i[sueB + \beta(u, Y)]} + u e^{i[sueB + \beta(u, Y)]} \right] \Delta(u, Y) - (1 + u)_{\text{c.t.}}},
\]

(36)

which is the expression that we have to evaluate.

### 2.3 A few remarks

* At \( B \to 0 \), \( Y \to 0 \), \( \beta \sim (1 - u)Y + \mathcal{O}(Y^2) \) yields \( \Phi_B = 0 \). One also has \( \Delta_{B=0} = 1 \) such that \( \Sigma(\pi)_{B=0} = \frac{\alpha m}{2\pi} \int_0^\infty \frac{ds}{s} \int_0^1 du \, e^{-isu^2m^2} [(1 + u) - (1 + u)] = 0 \). This agrees with the renormalization condition (23).

* \( \Delta(u, Y) \), which occurs by its square root, is a seemingly naughty denominator. Its zeroes \( u_\pm \) can be written \( u_+ = u_- = \frac{1 - \sin(Y/Y)}{\xi(Y)} e^{iY} \), with \( \xi(Y) = 1 - 2 \left[ \frac{\sin(Y/2Y)}{Y} + \left( \frac{\sin(Y)}{Y} \right)^2 \right] \). The real zeroes \( u_+ = 1 = u_- \) are degenerate and are located at \( Y = n\pi, n \neq 0 \), values at which \( \beta = 0 \).

* The renormalized \( \delta m \) given by (34) is finite. The contribution \( (1 + u) \) from the counterterm is tailored for this.

* The (infinite) contribution to \( \delta m_{LLL} \) coming from this counterterm corresponds to its value at \( B = 0 \). It does not depend on \( B \). It has been evaluated in pp. 53-56 of (12). \( \delta m_{B=0} = \lim_{s \to 0} \frac{3\alpha m}{4\pi} \left( -\gamma E + \ln \frac{1}{i m^2 s_0} + \frac{5}{6} \right) \), where \( s_0 \) is the lower limit of integration for the Schwinger parameter \( s_1 \) attached to the electron propagator. It coincides with the result given by Ritus in (17).

### 2.4 Changing variables; the Demen-Jancovici integral [2][1]

We first perform the change of variables

\[
(u, s) \to (u, Y = eBs u) \Rightarrow \frac{du \, ds}{s} = \frac{du \, dY}{Y}.
\]

(37)
In Dittrich-Reuter \cite{12}, \(e\) stands for the (negative) charge of the electron\(^1\). Therefore, \(Y < 0\), too, and \(\int_0^\infty \frac{ds}{s} = \int_0^- dY \). Then, \(\delta m\) in (36) becomes
\[
\delta m_{LLL} = \frac{\alpha m}{2\pi} \int_0^- dY \int_0^1 du \, e^{-iuY} \frac{\Delta(u,Y)}{\sqrt{\Delta(u,Y)}} - (1 + u)
\]

which is seen to only depend on \(\frac{eB}{m^2}\). The divergence of \(\delta m\) occurs now at \(Y \to 0\). The change (37) introduces a dependence of the counterterm on \(\eta\).

It is interesting to expand the sole \(e^{i3\beta}\) into \(\cos \beta + i \sin \beta\), to use the expressions (34) of \(\cos \beta\) and \(\sin \beta\), to cast \(\delta m\) in the form
\[
\delta m_{LLL} = \frac{\alpha m}{2\pi} \int_0^- dY \int_0^1 du \, e^{-iuY} \frac{\Delta(u,Y)}{\sqrt{\Delta(u,Y)}} - (1 + u)
\]

and to notice that \(\Delta(u,Y) = (1 - u + u \sin Y \, e^{-iY})(1 - u + u \sin Y \, e^{-iY})\) to simplify the previous expression into
\[
\delta m_{LLL} = \frac{\alpha m}{2\pi} \int_0^- dY \int_0^1 du \, e^{-iuY} \frac{\Delta(u,Y)}{\sqrt{\Delta(u,Y)}} - (1 + u)
\]

Expressing \(\sin Y\) in the denominator in terms of complex exponentials gives
\[
\delta m_{LLL} = \frac{\alpha m}{2\pi} \int_0^- dY \int_0^1 du \, e^{-iuY} \frac{\Delta(u,Y)}{\sqrt{\Delta(u,Y)}} - (1 + u)
\]

Going to \(t = -iY\) yields
\[
\delta m_{LLL} = \frac{\alpha m}{2\pi} \int_0^+ dt \int_0^1 du \, e^{it} \frac{\Delta(u,Y)}{\sqrt{\Delta(u,Y)}} - (1 + u)
\]

Last, we change to \(z = ut \Rightarrow du \, dt = \frac{du \, dz}{t}\) and get
\[
\delta m_{LLL} = \frac{\alpha m}{2\pi} \int_0^+ dz \int_0^1 du \, e^{-iz} \frac{\Delta(u,Y)}{\sqrt{\Delta(u,Y)}} - (1 + u)
\]

It still differs from eq. 3 of Jancovici \cite{11} by the 2 following points:
* one has \(e^{+iz\frac{m^2}{e}}\) instead of \(e^{-iz\frac{m^2}{e}}\) in \cite{11}; this is due to \(e > 0\) in there, while, here, \(e < 0\);
* one has \(\int_0^+ dt \) instead of \(\int_0^+ dt\); a Wick rotation is needed: \(\int_0^+ dt \) to \(\int_0^\infty dt\) in infinite circle + \(\int_0^0 = 2\pi \sum \text{residues}\).

Because of \(e^{-iz\frac{m^2}{e}}\) the contribution on the infinite 1/4 circle is vanishing. That the residue at \(z = 0\) vanishes is trivial as long as \(u\) is not vanishing. The expansion of the terms between square brackets in (43) at \(z \to 0\) writes indeed
\[
u = 1 + (-\frac{3}{3} + \frac{1}{2} + u)z + (-\frac{3}{3} - \frac{1}{2} + \frac{7}{3} + u) z^2 + \mathcal{O}(z^3),
\]
which seemingly displays poles at \(u = 0\). However, without expanding, it also writes, then \(\frac{2}{3} z - \frac{1}{2} = 0\), which shows that the poles at \(u = 0\) in the expansion at \(z \to 0\) are fake and

\(^{14}\) Unlike in \cite{13}, in which, like in Schwinger, both \(q\) and \(e\) are introduced. In there, \(e\) has the meaning of the elementary charge \(e > 0\).

\(^{16}\) To summarize in a symbolic (and dirty) way, this change of variables amounts to rewriting \(\delta m_{LLL} = \left(\infty + \eta(\frac{|eB|}{m^2})\right) - \infty\) as \(\delta m = \left(\infty + \eta(\frac{|eB|}{m^2})\right) - \left(\infty + \eta(\frac{|eB|}{m^2})\right)\). \(\xi\) is the dependence on \(\frac{|eB|}{m^2}\) generated by the change of variables. We shall then regularize the canceling infinities to get rid of them and calculate separately \(\eta\) and \(-\xi\) which give respectively the \(\left(\ln \frac{|eB|}{m^2}\right)^2\) and \(\ln \frac{|eB|}{m^2}\) terms.
that the residue at \( z = 0 \) always vanishes. Other poles (we now consider eq. (42)) can only occur when the denominator of the first term inside brackets vanishes. That the corresponding \( u^{\text{pole}} = \frac{2t + e^{-2t} - 1}{2t + e^{-2t} - 1} \) should be real constrains them to occur at \( t \rightarrow i\pi, n \in \mathbb{N} > 0 \) and \( u \rightarrow 1 \). In general, they satisfy \( 2t(1 - u) + u(1 - e^{-2t}) = 0 \) which, setting \( t = t_1 + it_2, t_1, t_2 \in \mathbb{R} \), yields the 2 equations \( e^{-2t_1} \cos 2t_2 = 1 + 2\eta t_1, e^{-2t_2} \sin 2t_2 = -2\eta t_2, \eta = \frac{1 - u}{u} \geq 0 \). Since \( t_1 \rightarrow 0 \), one may expand the first relation at this limit, which yields \( \cos 2t_2 - 1 = 2t_1(\eta + \cos 2t_2) \). As \( t_2 \rightarrow n\pi \), \( \cos 2t_2 > 0 \) and \( \cos 2t_2 - 1 < 0 \), which, since \( \eta > 0 \), constrains \( t_1 \) to stay negative. Therefore, the potentially troublesome poles lie in reality on the left of the imaginary \( t \) axis along which the integration is done and should not be accounted for when doing a Wick rotation. It gives

\[
\delta m_{LLL} = \frac{\alpha m}{4\pi} 2 \int_{0}^{\infty} dz \int_{0}^{1} du \ e^{-z} \frac{2}{\pi^2} \left[ \frac{2 (1 + u e^{-2z/u})}{2z(1 - u) + u^2 (1 - e^{-2z/u})} - 1 + u \frac{\eta}{z} \right] \text{ from c.t.} \tag{44}
\]

(44) is now the same as Jancovici’s eq. 3 (1) (see eqs. (46,47) below). This proves in particular that the latter (and therefore Demeur’s calculation (2)) satisfy the same “on mass shell” renormalization conditions (23), which was not clear in (2).

3 Calculating Jancovici’s integral [1]

3.1 Generalities and definition

Along with Jancovici (1), let us write the rest energy of the electron

\[
E_0 = m(c^2) \left( 1 + \frac{\alpha}{4\pi} I(L) \right), \quad L = \frac{(\hbar c) B}{(e^3) m^2} \tag{45}
\]

in which, at all orders in \( B \)

\[
I(L) = 2 \int_{0}^{\infty} dz \ e^{-z/L} \int_{0}^{1} dv \ \frac{2 (1 + v e^{-2z/v})}{2z(1 - v) + v^2 (1 - e^{-2z/v})} = 2 \int_{0}^{\infty} dz \ e^{-z/L} \int_{0}^{1} dv \ f(v, z), \tag{46}
\]

\[
f(v, z) = \frac{2 (1 + v e^{-2z/v})}{2z(1 - v) + v^2 (1 - e^{-2z/v})} - 1 + v \frac{\eta}{z}.
\]

Jancovici (1) defines accordingly (we set hereafter \( \hbar = 1 = c \))

\[
\delta m = \frac{\alpha m}{4\pi} I(L) \tag{47}
\]

such that the \( I(L) \) in (46) coincides with the one in (44). Since the same external electron states are concerned in the 2 calculations, we have proved that they are equivalent.

\( I(L) \) has been obtained from Demeur’s original integral (2)\(^{18,19}\)

\[
D(L) = \int_{0}^{1} dv \ (1 + v) \int \frac{dw}{w} \ w \ e^{ivw} \frac{2iLw(ve^{2Lw} + 1)}{(1 + v)(ve^{2Lw} + 2iLw(1 - v) - v)} \tag{48}
\]

by subtracting its value at \( B = 0 \leftrightarrow L = 0 \) and after the change of variables \( z = -iLv\). Therefore, (47) corresponds to the magnetic radiative corrections to the electron mass, after subtracting the self-energy of the “free” (i.e. at \( B = 0 \)) electron \(^{20}\). The latter corresponds to the term \( \propto \frac{1 + iv}{v} \) in the integrand of (46). Accordingly, (47) satisfies \( \delta m B \rightarrow 0 \).

---

\(^{17}\)The 2nd relation then tells us that \( \sin 2\pi < 0 \), which means that the poles correspond to \( t_2 = n\pi - \epsilon, \epsilon > 0 \).

\(^{18}\)It is eq. (21) of §8: “La self-énergie de l’électron”, p. 78 of (2).

\(^{19}\)It has been manifestly obtained with an internal photon in the Feynman gauge (see eq. (1) p. 56 of (2)).

\(^{20}\)See Demeur (2) chapitre III “Les corrections radiatives magnétiques”, §1 “La self-énergie”, p.55
3.2 First steps: a simple convergent approximation for \( L \equiv eB/m^2 > 75 \)

We want an analytical expression for \( I(L) \) valid for large values of the magnetic field, say \( |e|B/m^2 > 75 \). That \( I(L) \) can easily be integrated numerically makes checks easy.

The two integrals making up \([40]\) both diverge at \( z \to 0 \). The cancellation of the divergences is ensured by the first renormalization condition \([23]\), but its practical implementation needs a regularization.

Following Jancovici \([1]\), one splits \( I(L) \) into \( \int_0^\infty dz = \int_0^a dz + \int_a^\infty dz \), with \( a \) large enough such that \( e^{-2z/v} \ll 1 \) can be neglected inside \( f(v,z) \). Since \( v \in [0,1] \), this requires at least \( a \geq 1 \), that we check numerically. \( I(L) \) can then be approximated by

\[
I(L) \approx 2 \int_0^a dz \ e^{-z/L} \int_0^1 dv \ f(v,z) + 2 \int_a^\infty dz \ e^{-z/L} \int_0^1 dv \ \left( \frac{2}{v^2 + 2z(1-v)} - \frac{1+v}{z} \right),
\]

in which the second integral is manifestly convergent. We focus on the first one, which includes the two canceling divergences. It turns out, as in \([1]\), that, for \( L \) large enough, for example \( L > 75 \), its numerical value decreases with \( a \) and that one can go very safely down to \( a = 1 \) at which it is totally negligible with respect to the value of the full \( I(L) \).

We thus approximate, for \( L \geq 75 \)

\[
I(L) \approx \frac{2}{L} \int_0^\infty dz \ e^{-z/L} \int_0^1 dv \ \frac{2}{v^2 + 2z(1-v)} - 3 \Gamma(0,1/L) = 2J(L) - 3 \Gamma(0,1/L)
\]

The result of the change of variables done in subsection \([2,4]\) associated with the regularization-approximation just performed is a sum of two finite integrals. The most peculiar and also the most important for our purposes is the second one which originates from the counter-term and includes the large divergence. It turns out, as in \([1]\), that, for \( L \) large enough such that \( \epsilon |e|B/m^2 > 75 \), the integral \( \int_0^1 dv \ \frac{2}{v^2 + 2z(1-v)} \) can be easily performed analytically, too, leading to

\[
I(L) \approx \frac{2}{L} \int_1^\infty dz \ e^{-z/L} \ln \left( \frac{z - 1 + \sqrt{z(z-2)}}{\sqrt{z(z-2)}} \right) - 3 \Gamma(0,1/L).\]

The result of the change of variables done in subsection \([2,4]\) associated with the regularization-approximation just performed is a sum of two finite integrals. The most peculiar and also the most important for our purposes is the second one which originates from the counter-term and includes the large \( \ln \frac{|e|B}{m^2} \) generally ignored. Its occurrence is non-trivial and only appears through the change of variables \([37]\).

3.3 Further evaluation

\[
J(L) \equiv \int_1^\infty dz \ e^{-z/L} \ g(z), \ g(z) = \frac{\ln \left( z - 1 + \sqrt{z(z-2)} \right)}{\sqrt{z(z-2)}}
\]

cannot be integrated exactly but, following \([15]\), one can find an accurate approximation for the integrand

\[
g_{\text{app}}(z) \approx \frac{\ln z}{z} + \frac{\pi}{2} \frac{1}{z^\beta}, \quad \beta = 1.175
\]

as shown on Fig. 2 below where the 2 curves for the exact \( g (\text{blue}) \) and the approximate \( g_{\text{app}} (\text{yellow}) \) are practically indistinguishable.

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21 We proceed as follows. Though \( f(0,z) = 0, f(v,z) \) cannot be integrated \( \int_0^1 dv \) at small \( z \) because, as already mentioned in subsection \([2,4]\), its expansion has (fake) poles at \( v = 0 \) and numerical integration becomes itself hazardous. To achieve it safely, we regularize the first integral in \([49]\) by introducing a small parameter \( \epsilon \), replace \( \int_0^1 dv \ f(v,z) \) with \( \int_1^\infty dv \ f(v,z) \), then decrease \( \epsilon = 10^{-3}, 10^{-6}, 10^{-9} \ldots \) while checking stability.

22 This term was neglected in eq. (4) of \([1]\), where only \( \ln^2 \) are focused on.
Without the term $\frac{\pi}{2} \frac{1}{z^2}$, $g$ would go to 0 instead of $\frac{\pi}{2}$ at $z = 1$. This term yields in particular the term $\frac{1}{L^{\beta-1}}$ in the expansion of $J_{\text{app}}$ at $L \to \infty$. The integration can now be done analytically, leading to

$$J_{\text{app}}(L) = \int_{1}^{\infty} dz \, e^{-z/L} \left( \frac{\ln z}{z} + \frac{\pi}{2} \frac{1}{z^2} \right) = \frac{\pi}{2} \text{Ei}(\beta, \frac{1}{L}) + \text{MeijerG} \left[ \left\{ \frac{1}{2}, \frac{1}{2} \right\}, \left\{ 0 \right\}, \left\{ 0 \right\}, \left\{ 0 \right\}, \frac{1}{L} \right].$$

We compare in Fig. 3 the integrals $J(L)$ (blue) and $J_{\text{app}}(L)$ (yellow), which prove extremely close to each other.

### 3.4 Final result

The final result is obtained by expanding $J_{\text{app}}(L)$ and $\Gamma(0, 1/L)$ at large $L$

$$J_{\text{app}} \overset{L \to \infty}{\sim} \frac{\pi}{L^\beta} \left( \frac{\pi}{2} L \Gamma[1 - \beta] + O\left( \frac{1}{L^2} \right) \right) + \frac{\gamma_E}{2} + \frac{\pi^2}{12} \left( \frac{6}{\beta - 1} + \pi \right) - \frac{1}{2} \ln L (2\gamma_E - \ln L) + \frac{-1 + \frac{\pi}{4} - \ln L}{L} + O\left( \frac{1}{L^2} \right),$$

$$\Gamma(0, 1/L) \overset{L \to \infty}{\sim} -\gamma_E + \ln L + \frac{1}{L} + O\left( \frac{1}{L^2} \right) \quad \text{(comes from the counterterm)}$$

which yields for $I(L)$ written in (52)

$$I_{\text{app}}(L, \beta) \overset{L \geq 75}{\approx} \gamma_E \underbrace{\ln L - \gamma_E - \frac{3}{2}}_{\text{from c.t.}} + \frac{9}{4} \frac{\pi}{\beta - 1} + \frac{\pi^2}{6} + \frac{\pi \Gamma[1 - \beta]}{L^{\beta-1}} - \ln L \left( 2\gamma_E + \frac{3}{\text{from c.t.}} \right) + O\left( \frac{1}{L^2} \right)$$

$$+ \frac{1}{L} \left( \frac{\pi}{2 - \beta} - \frac{2 - \beta}{\text{from c.t.}} \right) + O\left( \frac{1}{L^2} \right).$$

The terms under-braced “from c.t.” result from the subtraction of the electron self-energy at $B = 0$; they include a large $-3(\ln L - \gamma_E)$, which therefore originates from the counterterm (together with part of the constant term in $\delta m$).
At $L \geq 75$ the term $\propto 1/L$ can be very safely neglected and one can approximate

$$I_{app}(L, \beta) \approx \left( \ln L - \gamma_E - \frac{3}{2} \right)^2 - \frac{9}{4} + \frac{\pi}{\beta - 1} + \frac{\pi^2}{6} + \frac{\Gamma(1-\beta)}{L^{\beta-1}} + O\left(\frac{1}{L}\right), \quad \beta \approx 1.175 \quad (57)$$

which is very different, as we shall see, from the brutal approximation $I_{app} \approx (\ln L)^2$ that has been systematically used in the following years. At $\beta = 1.175$, one gets explicitly

$$I_{app}(L, \beta = 1.175) \approx \ln L \left( \ln L - 4.15443 \right) - \frac{20.4164}{L^{0.175}} + 21.6617 + O\left(\frac{1}{L}\right). \quad (58)$$

We plot in Fig. 4 the different contributions to the Demeur-Jancovici integral: the green curve is the constant term, the yellow one is the inverse power, the brown one the $\ln$ contribution, the red one the $(\ln)^2$, and the blue curve is the global result. The comparison between the red and blue curve is that between the systematically used $(\ln)^2$ approximation and our accurate evaluation (57). A large cancellation between $(\ln)^2$ and $\ln$ terms makes in particular the role of the large constant important.

![Fig. 4: contributions to the Demeur-Jancovici integral; constant term (green), inverse power (yellow), $\ln$ (brown), $(\ln)^2$ (red), sum of all (blue).](image)

The $(\ln L)^2$ exceeds by 45% the precise estimate at $L = 100$ and still by 32% at $L = 10000$. These values of $L$ correspond to already gigantic magnetic fields that cannot be produced on earth (hundred times the Schwinger “critical” $B_c$). The absolute difference increases with $L$ while the relative difference decreases very slowly. One needs $L > 2 \times 10^{17}$ for the relative error to be smaller than 1/10, which is a totally unrealistic value of $B$.

Jancovici mentioned at the end of his work [1] a refined estimate $I(L) \approx (\ln 2L - \gamma_E - \frac{3}{2})^2 + A$ with $-6 \leq A \leq +7$. Actually, the value $A = 3.5$ yields a good agreement with our calculation in the range $75 \leq L \leq 100000$, as shown in Figs. 5. It corresponds to $I(L)_{Jancovici} \approx (\ln L)^2 - 1.768 \ln L + 5.416$.

![Fig. 5: comparison between the present calculation of $I(L)$ (blue) and Jancovici’s final refined estimate with $A = 3.5$, $I(L) \approx (\ln 2L - \gamma_E - \frac{3}{2})^2 + 3.5$ (yellow).](image)

A comparison is due between the present calculation (58) and Jancovici’s, in particular because the former involves $(\ln L + \ldots)^2$ as seen in (56) while the latter involves $(\ln 2L + \ldots)^2$. The result is that, though being very close

$\text{23They exactly cancel at } \ln L \approx 4.15443 \leftrightarrow B \approx 63 B_0, \text{ where } B_0 = \frac{m^2}{|e|} \text{ is the “Schwinger critical field”}$. 


numerically, the former includes, in addition to the $\ln^2$, large canceling ($\ln$, constant and inverse power) contributions, while the latter includes smaller log, constant and no inverse power. This could raise questions about which evaluation is closer to the exact result. However, the accuracy of the “analytical approximation” to $J(L)$ that we performed in subsection 3.3 and the fact that it is hard to know how Jancovici got his “tedious but straightforward” estimate tend to support our calculation and the presence, in particular, of a large single logarithm.

4 Concluding remarks and challenges

In view of these results, it appears illegitimate to approximate the integral of Demeur-Jancovici (and the corresponding $\delta m$ of the electron at 1-loop) by the sole term proportional to $\left( \ln \frac{|eB|}{m^2} \right)^2$. Still, all formal manipulations that have been made until recently, like resummations at a higher number of loops of a certain class of diagrams, have only concerned the double log contribution and its eventual later shrinking to a single log by the modification of the photonic vacuum polarization. The stakes for improvements are twofold: include large corrections and fulfill suitable renormalization conditions. They are obviously technically challenging, since the manipulations mentioned above should be generalized beyond the “leading (double-)log approximation”. Achieving a reliable resummation at a large number of loops is all the more non-trivial as it furthermore needs to satisfy at each order the appropriate renormalization conditions, that, as we have seen, control in particular the large single logarithm. This however becomes necessary when going to very large values of $\ln \frac{|eB|}{m^2}$ or when considering theories more strongly coupled than standard QED.

A second obvious challenge is to extend the present calculation to higher Landau levels of the external electrons. Though it is premature to make any prospect, the sharp damping of $\delta m$ with respect to previous approximations that we have found at 1-loop cannot but suggest that physical consequences could go along the same way. This is left for later investigations.

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24 The screening of the Coulomb potential and effective mass for the photon are obtained by resumming the geometric series of vacuum polarizations at 1-loop (see for example [15]). Consistency forbids therefore that a massive photon be inserted into the 1-loop self-energy of the electron.
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