CARLSSON’S RANK CONJECTURE AND A CONJECTURE ON SQUARE-ZERO UPPER TRIANGULAR MATRICES

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Abstract. Let $k$ be an algebraically closed field of characteristic 2 and $A$ the polynomial algebra in $r$ variables with coefficients in $k$. Carlsson [8] conjectured that for any DG-$A$-module $M$ of dimension $N$ as a free $A$-module, if the homology of $M$ is nontrivial and finite dimensional as a $k$-vector space, then $2^r \leq N$. Here we state a stronger conjecture about varieties of square-zero upper-triangular $N \times N$ matrices with entries in $A$. Using stratifications of these varieties via Borel orbits, we show that the stronger conjecture holds when $N < 8$ or $r < 3$. As a consequence, we obtain a new proof for many of the known cases of Carlsson’s conjecture and yields new results when $N > 4$ and $r = 2$.

1. Introduction

A well-known conjecture in algebraic topology states that if $(\mathbb{Z}/2\mathbb{Z})^r$ acts freely and cellularly on a finite CW-complex homotopy equivalent to $S^{n_1} \times \ldots \times S^{n_m}$, then $r$ is less than or equal to $m$. This conjecture is known to be true in several cases: In the equidimensional case, i.e., $n := n_1 = \ldots = n_m$, Carlsson [7], Browder [5], and Benson-Carlson [4] proved the conjecture under the assumption that the induced action on homology is trivial. Without the homology assumption, Conner [10], Adem-Browder [1], and Yalçın [18] showed that the equidimensional conjecture holds except for $n = 3$ or $n = 7$. In the non-equidimensional case, the conjecture is proved by Smith [17] for $r = 1$, Heller [11] for $r = 2$, Carlsson [9] for $r = 3$, Refai [15] for $r = 4$, and Okutan-Yalçın [14] for products in which the average of the dimensions is sufficiently large compared to the differences between them. The general case $r \geq 5$ is still open.

Let $G = (\mathbb{Z}/2\mathbb{Z})^r$ and $k$ be an algebraically closed field of characteristic 2. Assume that $G$ acts freely and cellularly on a finite CW-complex $X$ homotopy equivalent to a product of $m$ spheres. One can consider the cellular chain complex $C_* (X; k)$ as a finite chain complex of free $kG$-modules with homology $H_* (C_* ; k)$ that has dimension $2^m$ as a $k$-vector space. Hence, a stronger and purely algebraic conjecture can be stated as follows: If $C_*$ is a finite chain complex of free $kG$-modules with nonzero homology then $\dim_k H_* (C_*) \geq 2^r$.

Let $A$ be the polynomial algebra in $r$ variables with coefficients in $k$. Using a functor from the category of chain complexes of $kG$-modules to the category of differential graded $A$-modules, Carlsson in [6], [8] showed that the above algebraic conjecture is equivalent to the following conjecture:

Conjecture 1. (Conjecture II.8 in [8]) Let $k$ be an algebraically closed field of characteristic 2, $A$ the polynomial algebra in $r$ variables with coefficients in $k$ and $N$ a positive
integer. If \((M, \partial)\) is a free DG-A-module of dimension \(N\) whose homology is nonzero and finite dimensional as a \(k\)-vector space, then \(N \geq 2^r\).

Carlsson [9], and Avramov, Buchweitz and Iyengar [3] proved Conjecture 1 for \(N \leq 6\) while Refai [15] dealt with \(N \leq 8\). In this paper we consider the conjecture from the viewpoint of algebraic geometry. Carlsson [9] showed that the above conjecture is implied by the following:

**Conjecture 2.** Let \(k\) be an algebraically closed field of characteristic 2, \(r\) a positive integer, and \(N = 2n\) an even positive integer. Assume that there exists a nonconstant morphism \(\psi\) from the projective variety \(\mathbb{P}^{r-1}_k\) to the weighted quasi-projective variety of rank \(n\) square-zero upper triangular \(N \times N\) matrices \((x_{ij})\) with \(\deg(x_{ij}) = d_i - d_j + 1\) for some \(N\)-tuple of nonincreasing integers \((d_1, d_2, \ldots, d_N)\). Then \(N \geq 2^r\).

We will give a more precise statement for Conjecture 2 in Section 2 after discussing necessary definitions and notation. We propose the following which is stronger than Conjecture 2:

**Conjecture 3.** Let \(k, r, N, n\) and \(\psi\) be as in Conjecture 2. Assume \(1 \leq \mathcal{R}, \mathcal{C} \leq N\) and the value of \(x_{ij}\) at every point in the image of \(\psi\) is 0 whenever \(i \geq N - \mathcal{R} + 1\) or \(j \leq \mathcal{C}\). Then \(N \geq 2^r - 1(\mathcal{R} + \mathcal{C})\).

Note that in Conjecture 3 we have \(2^r - 1(\mathcal{R} + \mathcal{C}) \geq 2^r\) because \(1 \leq \mathcal{R}\) and \(1 \leq \mathcal{C}\). The main result of the paper is a proof of Conjecture 3 when \(N < 8\) or \(r < 3\), see Theorem 1 and Theorem 2. As Conjecture 3 is the strongest conjecture mentioned above, we obtain proofs of all the conjectures in this introduction under the same conditions, including the main result of Carlsson in [8]. Also note that for \(r = 2\), taking \(N > 4\) gives novel results not covered in the literature. The main tool we use is the stratification of certain varieties obtained by considering the action of a Borel group, as described in Section 2. Finally, Section 3 concludes with examples and problems.

## 2. Varieties of square-zero matrices

Throughout this paper we assume that \(k\) is an algebraically closed field of characteristic 2, \(n\) is a positive integer, \(N = 2n\), and \(d = (d_1, d_2, \ldots, d_N)\) is an \(N\)-tuple of nonincreasing integers.

### 2A. Statements of conjectures.

We give here the notation for the affine and projective varieties used to prove the conjectures discussed above. First we fix an affine variety \(U_N\), a ring \(R(U_N)\), and a subvariety \(V_N\) as follows:

- \(U_N\) is the affine variety of strictly upper triangular \(N \times N\) matrices over \(k\).
- \(R(U_N) = k[ x_{ij} \mid 1 \leq i < j \leq N ]\) is the coordinate ring of \(U_N\).
- \(V_N\) is the subvariety of square zero matrices in \(U_N\).

Define an action of the unit group \(k^*\) on \(U_N\) by \(\lambda \cdot (x_{ij}) = (\lambda^{d_i - d_j + 1}x_{ij})\) for \(\lambda \in k^*\). Using this action we set two more notation:

- \(U(d)\) is the weighted projective space given by the quotient of \(U_N\) by the action of \(k^*\) defined above.
- \( R(U(d)) \) is the homogeneous coordinate ring of \( U(d) \). In other words, \( R(U(d)) \) is \( R(U_N) \) considered as a graded ring with \( \deg(x_{ij}) = d_i - d_j + 1 \).

Note that the polynomial \( p_{ij} = \sum_{m=i+1}^{j-1} x_{im} x_{mj} \) in \( R(U(d)) \) is a homogeneous polynomial of degree \( d_i - d_j + 2 \) whenever \( 1 \leq i < j \leq N \). Similarly, the \( n \times n \)-minors of \((x_{ij})\) are homogeneous polynomials in \( R(U(d)) \). Hence, we define two subvarieties of \( U(d) \) as follows:

- \( V(d) \) is the projective variety of square zero matrices in \( U(d) \).
- \( L(d) \) is the subvariety of matrices of rank less than \( n \) in \( V(d) \).

So we can restate Conjecture 2:

**Conjecture 4.** Let \( k \) be an algebraically closed field of characteristic 2, \( r \) a positive integer, and \( d \) an \( N \)-tuple of nonincreasing integers. If there exists a nonconstant morphism \( \psi \) from the projective variety \( \mathbb{P}_k^{r-1} \) to the quasi-projective variety \( V(d) - L(d) \), then \( N \geq 2^r \).

Let \( U \) be an open subset of \( V(d) \). We say \( \psi : \mathbb{P}_k^{r-1} \to U \) is a nonconstant morphism if \( \psi \) is represented by \((\psi_{ij})\) so that the following conditions are satisfied:

1. (I) for every \( i, j \) the component \( \psi_{ij} \) is a homogeneous polynomial,
2. (II) for every \( \gamma \in \mathbb{P}_k^{r-1} \) there exists \( i, j \) such that \( \psi_{ij}(\gamma) \neq 0 \),
3. (III) there exists \( i, j \) such that \( \psi_{ij} \) is nonconstant.

This is not the standard definition of being nonconstant but its use allows us to treat the case \( r = 1 \) with the same methods as \( r > 1 \). In particular, if \( \psi : \mathbb{P}_k^{r-1} \to U \) is a nonconstant morphism, \( \psi \) can be considered as a function from \( \mathbb{P}_k^{r-1} \) to \( U \) represented by a nonconstant polynomial map \( \widetilde{\psi} \) from \( \mathbb{A}_k^{r-1} \) to the cone over \( U \) such that \( \widetilde{\psi}(\mathbb{A}_k^{r-1} - \{0\}) \) does not contain the zero matrix in \( V(d) \). Each indeterminate \( x_{ij} \) can be viewed as homogeneous polynomial in \( R(U(d)) \). Hence for \( 1 \leq \mathcal{R}, \mathcal{C} \leq N \) we define an important subvariety of \( V(d) \):

- \( V(d)_{\mathcal{R}\mathcal{C}} \) is the subvariety of \( V(d) \) given by the equations \( x_{ij} = 0 \) for \( i \geq N - \mathcal{R} + 1 \) or \( j \leq \mathcal{C} \).

Now we restate the Conjecture 3 as follows:

**Conjecture 5.** Let \( k \) be an algebraically closed field of characteristic 2, \( r \) a positive integer, and \( d \) an \( N \)-tuple of nonincreasing integers. If there exists a nonconstant morphism \( \psi \) from the projective variety \( \mathbb{P}_k^{r-1} \) to the quasi-projective variety \( V(d)_{\mathcal{R}\mathcal{C}} - L(d) \), then \( N \geq 2^{r-1}(\mathcal{R} + \mathcal{C}) \).

Hence, the varieties defined above are the main interest in this paper.

**2B. Action of a Borel subgroup on \( V_N \).** Here we introduce an action of a Borel subgroup in the group of invertible \( N \times N \) matrices on the varieties discussed in the previous section. First we set a notation for the Borel subgroup.

- \( B_N \) is the group of invertible upper triangular \( N \times N \) matrices with coefficients in \( k \).

The group \( B_N \) acts on \( V_N \) by conjugation.
• $V_N/B_N$ denotes the set of orbits of the action of $B_N$ on $V_N$.
• $B_N$ denotes the $B_N$ orbit that contains $X \in V_N$.

A partial permutation matrix is a matrix having at most one nonzero entry in each row and column, which is 1. Using a result of Rothbach (Theorem 1 in [16]), one can see that each $B_N$ orbit of $V_N$ contains a unique partial permutation matrix. Hence we introduce the following notation.

• $\text{PM}(N)$ denotes the set of nonzero $N \times N$ strictly upper triangular square-zero partial permutation matrices.

There is a one-to-one correspondence between $\text{PM}(N)$ and $V_N/B_N$ sending $P$ to $B_P$. We can identify these partial permutation matrices with a subset of permutations in the symmetric group $\text{Sym}(N)$:

• $\text{P}(N)$ is the set of involutions in $\text{Sym}(N)$; i.e., the set of non-identity permutations whose square is the identity $()$.

For $P \in \text{PM}(N)$ and $\sigma \in \text{P}(N),$

• $\sigma_P$ denotes the permutation in $\text{P}(N)$ that sends $i$ to $j$ if $P_{ij} = 1$;
• $P_\sigma$ denotes the partial permutation matrix in $\text{PM}(N)$ that satisfies $(P_\sigma)_{ij} = 1$ if and only if $\sigma(i) = j$ and $i < j$.

Clearly the assignments $P \mapsto \sigma_P$ and $\sigma \mapsto P_\sigma$ are mutual inverses and so define a one-to-one correspondence between $\text{P}(N)$ and $\text{PM}(N)$.

2C. A partial order on the set of orbits. There are important partial orders on $V_N/B_N$, $\text{P}(N)$, $\text{PM}(N)$, all of which are equivalent under the one-to-one correspondences mentioned above (cf. [16]). We begin with $V_N/B_N$. For Borel orbits $B, B' \in V_N/B_N$,

• $B' \leq B$ means the closure of $B$, considered as a subspace of $V(d)$, contains $B'$.

Second, we define a partial order on $\text{PM}(N)$. To do this, we consider ranks of certain minors of partial permutation matrices. In general, for an $N \times N$ matrix $X$,

• $r_{ij}(X)$ denotes the rank of the lower left $((N - i + 1) \times j)$ submatrix of $X$, where $1 \leq i < j \leq N$.

For partial permutation matrices $P', P \in \text{PM}(N),$

• $P' \leq P$ means $r_{ij}(P') \leq r_{ij}(P)$ for all $i, j$.

Third, we define a partial order on $\text{P}(N)$. For positive integers $i < j$, let $\sigma(i, j)$ denote the product of the permutations $\sigma$ and $(i, j)$ and $\sigma'^{(i,j)}$ the conjugate of $\sigma$ by $(i, j)$. For $\sigma, \sigma' \in \text{P}(N),$

• $\sigma' \leq \sigma$ if $\sigma'$ can be obtained from $\sigma$ by a sequence of moves of the following form:
  - Type I replaces $\sigma$ with $\sigma(i,j)$ if $\sigma(i) = j$ and $i \neq j$.
  - Type II replaces $\sigma$ with $\sigma'^{(i,j)}$ if $\sigma(i) = i' < \sigma(i')$.
  - Type III replaces $\sigma$ with $\sigma'^{(j,j')} if \sigma(j) < \sigma(j') < j' < j$.
  - Type IV replaces $\sigma$ with $\sigma'^{(j,j')} if \sigma(j') < j' = \sigma(j)$.
  - Type V replaces $\sigma$ with $\sigma'^{(i,j)} if i < \sigma(i) < \sigma(j) < j$.
The idea of describing order by moves comes from [12]. Although we use different names for moves, the set of possible moves are same. We represent a permutation $(i_1, j_1)(i_2, j_2) \ldots (i_s, j_s)$ in $\mathbf{P}(N)$ by the matrix
\[
\begin{pmatrix}
i_1 & i_2 & \ldots & i_s \\
j_1 & j_2 & \ldots & j_s
\end{pmatrix}.
\]
For example, we draw the Hasse diagram of $\mathbf{P}(4)$ in which each edge is labelled by the type of the move it represents.

When $N \geq 6$, the Hasse diagram for $\mathbf{P}(N)$ is too large to draw here. We are actually only interested in a small part of this diagram, which we discuss in Section 2F. One can consider the above Hasse diagram as a stratification of $V_4$. In the next section, we use the stratification of $V_N$ to stratify $V(d)$.

2D. Stratification of $V(d)$. For $d = (d_1, d_2, \ldots, d_N)$ an $N$-tuple of nonincreasing integers, $\lambda \in k^*$, and $X = (x_{ij}) \in V_N$, we have
\[
\lambda \cdot X = \lambda \cdot (x_{ij}) = (\lambda^{d_i - d_j + 1}x_{ij}) = D_\lambda I_\lambda X D_\lambda^{-1}
\]
where $D_\lambda$ denotes the diagonal matrix with entries $\lambda^{d_1}, \lambda^{d_2}, \ldots, \lambda^{d_N}$ and $I_\lambda$ is the scalar matrix with all diagonal entries $\lambda$. Let $P_X \in \mathbf{PM}(N)$ be the unique partial permutation matrix in the Borel orbit of $X$. Consider $b$ in $B_N$ such that
\[
P_X = b^{-1}Xb.
Let $I_{\lambda,X}$ be the diagonal matrix whose $j$th entry is $\lambda$ if $(P_X)_{ij} = 1$ for some $i$ and 1 otherwise. Then we have
\[ I_{\lambda} P_X = I_{\lambda,X}^{-1} P_X I_{\lambda,X}. \]
Hence, we get
\[ \lambda \cdot X = D_{\lambda} b I_{\lambda,X}^{-1} b^{-1} X b I_{\lambda,X} b^{-1} D_{\lambda}^{-1} = Z^{-1} XZ \]
where $Z = b I_{\lambda,X} b^{-1} D_{\lambda}^{-1}$ is in $B_N$. Thus, for any $X \in V(d)$ there exists a well-defined Borel orbit in $V_N/B_N$ that contains a representative of $X$ in $V_N$. Hence we can set the following notation. For $X \in V(d)$,
- $B_X$ denotes the Borel orbit in $V_N/B_N$ that contains a representative of $X$ in $V_N$.
- Let $\psi : \mathbb{P}^{r-1} \to V(d) - L(d)$ be a nonconstant morphism. There is a lift of this morphism to a morphism from $\mathbb{A}_k^r - \{0\}$ to the cone over $V(d) - L(d)$ that can be extended to a morphism $\tilde{\psi} : \mathbb{A}_k^s \to V_N$. Since $\mathbb{A}_k^s$ is an irreducible affine variety, there exist a unique maximal Borel orbit among the Borel orbits that intersects the image of $\tilde{\psi}$ nontrivially. Note that this maximal Borel orbit is independent of the lift and extension we selected because it is also maximal in the set $\{B_X \mid X \in V(d)\}$. Hence we associate a permutation to the nonconstant morphism $\psi$ as follows:
  - $\sigma_\psi$ is the permutation that corresponds to the unique maximal Borel orbit $B_X$ where $X$ is in the image of $\psi$.

Note that every point in the image of a morphism $\psi$ as above must have rank $n$. Hence $\sigma_\psi$ must be a product of $n$ distinct transpositions. In Section 2E, we restrict our attention to such permutations.

2E. Operations on polynomial maps from $\mathbb{A}_k^s$ to $V_N$. Another way to see that $B_X$ is well-defined for $X \in V(d)$ is to consider the fact that a minor of a representative of $X$ is zero if and only if the corresponding minor of another representative is zero. We use this fact several times to prove our main result. Hence we introduce the following notation. For $X \in V_N$,
- $m_{i_1 \ldots i_k}^{j_1 \ldots j_k} (X)$ denotes the determinant of the $k \times k$ submatrix obtained by taking the $i_1^{th}$, $i_2^{th}$, ..., $i_k^{th}$ rows and $j_1^{th}$, $j_2^{th}$, ..., $j_k^{th}$ columns of $X$.

First note that $m_{i_1 \ldots i_k}^{j_1 \ldots j_k}$ can be considered as a morphism from $V_N$ to $k$, and hence can be composed with the morphism $\tilde{\psi}$ mentioned in the previous section. Here we discuss several other morphisms that we can compose with such morphisms. For $u \in k$,
- $R_{i,j}(u)$ is the function that takes a square matrix $M$ and multiplies the $i^{th}$ row of $M$ by $u$ and adds it to the $j^{th}$ row of $M$ while multiplying the $j^{th}$ column of $M$ by $u$ and adding it to the $i^{th}$ column of $M$.

Note that $R_{i,j}(u)(M)$ is a conjugate of $M$. In fact, they are in the same Borel orbit when $M$ is in $V_N$ and $i > j$. Hence, for $i > j$, we can consider $R_{i,j}(u)$ as an operation that takes a morphism from $\mathbb{A}_k^s$ to $V_N$ and transforms it to a morphism from $\mathbb{A}_k^{s+1}$ to $V_N$ by considering $u$ as a new indeterminate and applying $R_{i,j}(u)$ to the morphism. For $v \in k^*$,
- $D_i(v)$ denotes the function that takes a square matrix $M$ and multiplies the $i^{th}$ row of $M$ by $v$ and the $i^{th}$ column of $M$ by $1/v$. 

Let \( q \) be a polynomial on \( s \) indeterminates. We view \( D_i(q) \) as an operation that takes a rational map from the quasi affine variety \( \mathbb{A}_k^s - Z \) to \( V_N \) and transforms into a rational map from \( \mathbb{A}_k^s - Z \cup V(q) \) to \( V_N \) by applying \( D_i(q) \), using the following notation:

- \( V(q_1, q_2, \ldots, q_k) \) is the variety determined by the equations \( q_1 = q_2 = \ldots = q_k = 0 \).

We use the above notation also for varieties in projective spaces determined by homogeneous polynomials \( q_1, q_2, \ldots, q_k \).

**2F. Rank of orbits and proof of first main result.** Each \( \sigma \in P(N) \) is a product of disjoint transpositions. Hence for \( \sigma \in P(N) \), we define the number of transpositions in \( \sigma \) to be the rank of \( \sigma \). Note that under the one-to-one correspondence between \( P(N) \) and \( PM(N) \), the rank of a permutation is equal to the rank of the corresponding partial permutation matrix.

- \( RP(N) \) denotes the permutations in \( P(N) \) of rank \( n \).

Note that all moves other than type I preserve the rank of \( \sigma \). Indeed, the only way of obtaining \( \sigma \) of smaller rank by applying these moves is by deleting a transposition, which is the effect of move of type I. Also note that it is impossible to have move of type II and move of type IV between two permutations in \( RP(N) \). For example, we draw the Hasse diagram for \( RP(6) \) where each dotted line denotes a move of type III and solid line denotes a move of type V:

![Hasse diagram of \( RP(6) \)](image)

By considering such Hasse diagrams and especially the maximal elements we prove our first main result.

**Theorem 1.** Conjecture 5 holds for \( N < 8 \).

**Proof.** Assume \( N < 8 \), \( d = (d_1, d_2, \ldots, d_N) \) is an \( N \)-tuple of nonincreasing integers, and \( \psi : \mathbb{P}_k^{r-1} \to V(d) - L(d) \) is a nonconstant morphism. Then \( \sigma = \sigma_\psi \) is in \( RP(N) \). By considering the above Hasse diagrams we note that there exists a unique maximal \( \sigma' \in RP(N) \) such that \( \sigma \) can be obtained from \( \sigma' \) by a sequence of moves of type III.
Since moves of type III do not change the number of leading zero rows and ending zero columns, the Borel orbit corresponding to $\sigma$ is contained in $V(d)_{RC}$ if and only if the Borel orbit corresponding to $\sigma'$ is contained in $V(d)_{RC}$ for all $R, C$. Hence it is enough to consider the cases where $\sigma$ is less than or equal to a maximal element in $\mathbb{RP}(N)$ for $N = 2, 4, 6$. Therefore, we prove the following eight statements:

(i) If $\sigma = (1, 2)$ then $r < 2$. Assume to the contrary that $\sigma = (1, 2)$ and $r \geq 2$. If we restrict the morphism $\psi$ to $\mathbb{P}_k^1$ considered as a subspace of $\mathbb{P}_k^{r-1}$, which we also denote $\psi$, we get a map of the form

$$
\psi(x, y) = \begin{bmatrix}
0 & p_{12} \\
0 & 0
\end{bmatrix}
$$

where $p_{12}$ is a homogeneous polynomial in $k[x, y]$. Since $k$ is algebraically closed, there exists $\gamma \in \mathbb{P}_k^1$ such that $p_{12}(\gamma) = 0$. This means $\psi(\gamma)$ is in $L(d)$, which is a contradiction.

(ii) If $\sigma \leq (1, 2)(3, 4)$ then $r < 3$. Suppose to the contrary that $\sigma \leq (1, 2)(3, 4)$ and $r \geq 3$. Here we restrict the morphism $\psi$ to $\mathbb{P}_k^2$, we get a map of the form

$$
\psi(x, y, z) = \begin{bmatrix}
0 & p_{12} & p_{13} & p_{14} \\
0 & 0 & 0 & p_{24} \\
0 & 0 & 0 & p_{34} \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

Note that there exists $\gamma$ in $\mathbb{P}_k^2$ such that $p_{12}(\gamma) = 0$ and $p_{13}(\gamma) = 0$.

Again this means $\psi(\gamma) \in L(d)$. Hence this case is proved by contradiction as well.

(iii) If $\sigma \leq (1, 4)(2, 3)$ then $r < 2$. To prove this suppose we have

$$
\psi(x, y) = \begin{bmatrix}
0 & 0 & p_{13} & p_{14} \\
0 & 0 & p_{23} & p_{24} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

Let $m_{j_1, j_2, \ldots, j_k}^{i_1, i_2, \ldots, i_k}$ be as in Section 2E and use the same notation to denote its composition with $\psi$. Then there exists $\gamma$ in $\mathbb{P}_k^1$ such that

$$
m_{34}^{12}(\gamma) = (p_{13}p_{24} - p_{23}p_{14})(\gamma) = 0.
$$

This again gives a contradiction.

(iv) If $\sigma \leq (1, 2)(3, 4)(5, 6)$ then $r < 3$. Suppose otherwise. We have

$$
\psi(x, y, z) = \begin{bmatrix}
0 & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\
0 & 0 & 0 & p_{24} & p_{25} & p_{26} \\
0 & 0 & 0 & p_{34} & p_{35} & p_{36} \\
0 & 0 & 0 & 0 & 0 & p_{46} \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$
If $p_{12}$ and $p_{13}$ are not relatively prime homogeneous polynomials then there exists $\gamma \in \mathbb{P}^2_k$ such that
\[
p_{12}(\gamma) = 0, \quad p_{13}(\gamma) = 0, \quad \text{and} \quad m_{456}^{123}(\gamma) = 0
\]
which leads to a contradiction. Hence we can assume $p_{12}$ and $p_{13}$ are relatively prime. Since $\psi^2 = 0$ and $k$ has characteristic 2, we have $p_{12}p_{24} = p_{13}p_{34}$ and $p_{12}p_{25} = p_{13}p_{35}$. This means that $p_{12}$ divides $p_{34}$ and $p_{35}$, and similarly $p_{13}$ divides $p_{24}$ and $p_{25}$. Then there exists $\gamma$ in $\mathbb{P}^2_k$ such that
\[
p_{12}(\gamma) = 0 \quad \text{and} \quad p_{13}(\gamma) = 0.
\]
This means $p_{12}, p_{13}, p_{24}, p_{24}, p_{24},$ and $p_{24}$ all vanish at $\gamma$. Hence $\psi(\gamma) \in L(d)$, which is a contradiction.

(v) If $\sigma \leq (1, 2)(3, 6)(4, 5)$ then $r < 3$, and (vi) If $\sigma \leq (1, 4)(2, 3)(5, 6)$ then $r < 3$.

These cases are symmetric, so it is enough to prove (v). Consider
\[
\psi(x, y, z) = \begin{bmatrix}
0 & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\
0 & 0 & 0 & 0 & p_{25} & p_{26} \\
0 & 0 & 0 & 0 & p_{35} & p_{36} \\
0 & 0 & 0 & 0 & p_{45} & p_{46} \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

We modify $\psi$ by operations in Section 2E. First apply $R_{6,5}(u)$ to $\psi$. If $p_{46} + up_{45} \neq 0$, then we can apply $D_5(1/p_{46} + up_{45})$ and then $R_{5,6}(p_{45})$, giving us a matrix of the form
\[
\begin{bmatrix}
0 & p_{12} & p_{13} & p_{14} & \ast & \ast \\
0 & 0 & 0 & 0 & m_{56}^{24} & p_{26} + up_{25} \\
0 & 0 & 0 & 0 & m_{56}^{34} & p_{36} + up_{35} \\
0 & 0 & 0 & 0 & 0 & p_{46} + up_{45} \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Hence by selecting a correct value for $u$ we would be done if $V(m_{56}^{24}, m_{56}^{34}) \not\subseteq V(p_{45})$. This means we can assume
\[
V(m_{56}^{24}, m_{56}^{34}) \subseteq V(p_{45}).
\]

Similarly, we can also assume
\[
V(m_{56}^{23}, m_{56}^{34}) \subseteq V(p_{35}) \quad \text{and} \quad V(m_{56}^{23}, m_{56}^{24}) \subseteq V(p_{25}).
\]

Therefore,
\[
V(m_{56}^{23}, m_{56}^{34}, m_{56}^{24}) \subseteq V(p_{25}, p_{35}, p_{45}) = \emptyset.
\]

Thus, $\{m_{56}^{23}, m_{56}^{34}, m_{56}^{24}\}$ is a regular sequence in $k[x, y, z]$. Note that we can assume $p_{45}$ and $p_{46}$ are relatively prime or we could find $\gamma$ such that $m_{56}^{23}(\gamma) = m_{56}^{24}(\gamma) = p_{45}(\gamma) = p_{46}(\gamma) = 0$. Hence this is a contradiction as we have
\[
p_{45}m_{56}^{23} + p_{25}m_{56}^{34} + p_{35}m_{56}^{24} = 0.
\]
(vii) If $\sigma \leq (1,6)(2,3)(4,5)$ then $r < 2$.

To prove this case consider

$$\psi(x, y) = \begin{bmatrix}
0 & 0 & p_{13} & p_{14} & p_{15} & p_{16} \\
0 & 0 & p_{23} & p_{24} & p_{25} & p_{26} \\
0 & 0 & 0 & 0 & p_{35} & p_{36} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$ 

Again by applying $R_{i,j}(u)$ and $D_i(v)$ for some $u, v$ we can assume that

$$V(m_{356}^{124}) \subseteq V(p_{13}, p_{23}) \quad \text{and} \quad V(m_{456}^{123}) \subseteq V(p_{14}, p_{24}).$$

Hence $\{m_{356}^{124}, m_{456}^{123}\}$ must be a regular sequence in $k[x, y]$. However this is impossible because the determinant of

$$\begin{bmatrix}
gcd(p_{13}, p_{14}) & p_{15} & p_{16} \\
gcd(p_{13}, p_{14}) & p_{25} & p_{26} \\
0 & \gcd(p_{35}, p_{45}) & \gcd(p_{36}, p_{46})
\end{bmatrix}$$

divides both $m_{356}^{124}$ and $m_{456}^{123}$.

(viii) If $\sigma \leq (1,6)(2,5)(3,4)$ then $r < 2$.

It is enough to consider a root of $m_{456}^{123}$ to prove this case. \[\square\]

Note that in the above proof the last two cases prove Conjecture $\ref{conj:main}$ when $N \leq 6$ and $r \leq 2$. In the rest of the paper we will generalize these ideas to prove the conjecture for $r \leq 2$. To do this we examine the dimensions of these varieties.

2G. Orbit dimensions and proof of second main result. We now introduce notation for dimensions of these varieties. In this section, for $\sigma \in \mathbb{P}(N)$, if the rank of $\sigma$ is $s$, then we obtain two sequences of numbers $i_1, \ldots, i_s$ and $j_1, \ldots, j_s$ satisfying the following:

$$\sigma = (i_1, j_1)(i_2, j_2)\ldots(i_s, j_s)$$

with $i_1 < i_2 < \cdots < i_s$ and $i_a < j_a$ for all $1 \leq a \leq s$. In \cite{Melnikov}, Melnikov gives a formula for the dimension of a Borel orbit $B_\sigma$ for $\sigma$ in $\mathbb{P}(N)$ as follows:

- $f_t(\sigma) := \#\{j_p \mid p < t, j_p < j_t\} + \#\{j_p \mid p < t, j_p < i_t\}$ for $2 \leq t \leq s$.
- $\dim(\sigma) = Ns + \sum_{t=1}^{s} (i_t - j_t) - \sum_{t=2}^{s} f_t(\sigma)$

We define a new subset of $R\mathbb{P}(N)$ as below:

- $DP(N)$ is the set of all $\sigma$ in $R\mathbb{P}(N)$ such that $\dim(\sigma') = \dim(\sigma) - 1$ whenever $\sigma'$ is a permutation obtained from applying a single move of type I to $\sigma$.

For instance, the following is the Hasse diagram for $DP(8)$. 
Figure 3. Hasse diagram of DP(8)

Note that in the Hasse diagram of DP(8) all moves are of type V. Now we prove this in general.

**Lemma 1.** If the transposition (1, N) appears in σ, then σ /∈ DP(N).

**Proof.** If σ ∈ DP(N), we have 3 ≤ σ(2) ≤ N − 1 since σ has rank n. If we apply a move of type I that sends σ to σ′ = σ(2, σ(2)), then dim(σ′) = dim(σ) − 3, which contradicts the definition of DP(N).

**Proposition 1.** If σ ∈ DP(N), then j_p < j_t for all p < t, which means we can not apply a move of type III to σ.

**Proof.** By Lemma 1 we have j_t − i_t ≤ N − 2 where 2 ≤ t ≤ n. If we apply a move of type I to σ, then the difference of Borel dimension must be −1. Thus f_t(σ) is nonzero for all t, hence j_p < j_t.

**Lemma 2.** For every X in V(d) − L(d) we have

\[ r_{ij}(X) ≥ j - i + 1 - n. \]

**Proof.** The rank of X is n, so r_{1N}(X) = n. The result follows from the inequality

\[ r_{ij}(X) + (i - 1) + (N - j) ≥ r_{1N}(X). \]

We now define our last set of permutations:
• $\text{MP}(r, N)$ is the set of minimal permutations in $\text{P}(N)$ that appear as a permutation in the form $\sigma_\psi$ for some nonconstant morphism $\psi : \mathbb{P}_k^{r-1} \to V(d) - L(d)$ for some $d$.

We now state and prove our second main result.

**Theorem 2.** Conjecture 4 holds for $r \leq 2$.

*Proof.* We have $\text{MP}(1, N) = \{(1, n + 1)(2, n + 2)\ldots(n, N)\}$ (see Example 1). This means Conjecture 4 holds for $r = 1$, because $N - n + (n + 1) - 1 = N \leq N$. Hence it is enough to prove Conjecture 4 for $r = 2$. Suppose that Conjecture 4 does not hold for $r = 2$. Then there exists an $N$-tuple of nonincreasing integers $d = (d_1, d_2, \ldots, d_N)$, two positive integers $R, C$, and a nonconstant morphism $\psi : \mathbb{P}_k^1 \to V(d)_{R \cap - L(d)}$ such that $N < 2(R + C)$, equivalently $n < R + C$. Write $\sigma_\psi = (i_1, j_1)(i_2, j_2)\ldots(i_n, j_n)$ with $1 = i_1 < i_2 < \cdots < i_n$ and $i_a < j_a$ for all $1 \leq a \leq n$.

First assume that $\sigma$ is in $\text{DP}(N)$. By Proposition 1, we have $j_1 < j_2 < \cdots < j_n = N$. Therefore, $C = j_1 - 1$ and $R = N - i_n$. Moreover, for every $a$ we have $j_a > C$ and $i_a < N - R + 1$. Set $I := \{i_1, \ldots, i_n\}$. Note that $\{1, \ldots, C\} \subseteq I$ since $\forall a, j_a > C$. So, $i_a = a$ if $a \leq C$. Similarly, $j_a - n = a$ if $a \geq N - R + 1$. Set

$$\hat{i} := n - R + 1 \quad \text{and} \quad \hat{j} := C$$

and

$$\bar{i} := N - R + 1 \quad \text{and} \quad \bar{j} := C + n.$$

We have

$$\bar{j} - \bar{i} = R + C - n - 1 = \bar{j} - \hat{j}.$$  

In particular, $\hat{i} \leq \bar{i}$ and $\bar{i} \leq \bar{j}$, since we suppose that $n < R + C$.

For $i \leq i'$ and $j \leq j'$, let $A_{i,j}^{i',j'}$ denote the submatrix of the partial permutation matrix $P_\sigma$ obtained by considering the rows from $j$ to $j'$ and columns from $i$ to $i'$. Note that $A_{1,N}^{1,7}$ has $C$-many 1’s and $A_{2,N}^{1,7}$ has $R$-many 1’s. Hence $A_{2,N}^{1,7}$ must have $(R + C - n)$-many 1’s.

However, there is no 1 in $A_{2,N}^{1,7}$; otherwise there would exist $a$ such that $i_a < \hat{i} = n - R + 1$ and $j_a \geq \bar{j}$, which leads to a contradiction by considering the number of 1’s in the region determined by the union of $A_{1,n}^{1,7}$ and $A_{1,0}^{1,7}$. Similarly, there is no 1 in $A_{j+1,N}^{1,7}$. Hence the $(R + C - n) \times (R + C - n)$-submatrix $A_{2,j}^{2,7}$ contains $(R + C - n)$-many 1’s. Thus, $A_{2,j}^{2,7}$ is the identity matrix of dimension $R + C - n$. This in particular means that for every element $X$ in the image of $\psi$ we have

$$r_{\hat{i} \hat{j} - 1}(X) = r_{n-R+1, N-R}(X) = 0$$

and

$$r_{\hat{i}+1 \hat{j}}(X) = r_{C+1, C+n}(X) = 0.$$  

Since $k$ is algebraically closed, there exists a root of the minor $m_{\hat{i} \hat{j}+1 \ldots \hat{j}}$. Thus, there exists an element $X$ in the image of $\psi$ such that

$$r_{\hat{i} \hat{j}}(X) \leq \bar{j} - \hat{i}.$$
we define
\[ r_{n-R+1,c+n}(X) \leq R + C - n - 1. \]
Lemma 2 implies that for every element \( X \) in the image of \( \psi \) we have
\[ r_{n-R+1,c+n}(X) \geq C + n - (n - R + 1) + 1 - n = R + C - n. \]
This is a contradiction.

Now assume that \( \sigma \) is not in \( \text{DP}(N) \). We recursively define perturbations of \( \psi \) so that we can again use the square submatrix \( A_{\frac{R}{2}} \) to get a contradiction similar to that of the previous case. Set \( \psi^0 = \psi \), \( n_0 = 0 \), \( Z_0 = \emptyset \). We have a rational map \( \psi_0 : A^{2+n_0}_k - Z_0 \rightarrow V_N \).

Now given
\[ \psi^s : A^{2+n_s}_k - Z_s \rightarrow V_N \]
we define \( \psi^{s+1} : A^{2+n_{s+1}}_k - Z_{s+1} \rightarrow V_N \) when \( \sigma_s = \sigma_{\psi^s} \) is not in \( \text{DP}(N) \).

Assume \( \sigma_s \) is not in \( \text{DP}(N) \) there exists a move of type III that we can apply on \( \sigma_s \). Hence, we can define
\[ l'_s := \min\{ l' \mid l' < l < \sigma_s(l) < \sigma_s(l') \} \]
and
\[ l_s := \sigma_s(\min\{ \sigma_s(l) \mid l_s < l < \sigma_s(l) < \sigma_s(l'_s) \}) \].
In case \( l'_s < l_s \) we define
\[ n_{s+1} := n_s + l_s - l'_s + 1. \]
Note that \( l_s > l'_s \) and so \( n_{s+1} \geq n_s + 2 \). Hence the affine variety \( Z_s \) can be considered as a subvariety of \( A^{2+n_{s+1}}_k \) by considering \( A^{2+n_s}_k \subset A^{2+n_{s+1}}_k \). Here we write \((x, y, u_1, u_2, \ldots u_{n_{s+1}})\) to denote a point in \( A^{2+n_{s+1}}_k \). Hence \( A^{2+n_s}_k \) corresponds to the points where \( u_{n_{s+1}} = \ldots = u_{n_{s+1}} = 0 \).

Represent \( \psi^s \) by a matrix \( (\psi^s_{ij}) \) with \((i, j)\)-entry \( \psi^s_{ij} \in k(x, y, u_1, u_2, \ldots u_{n_s}) \). Let \( p_i \) denote the entry \( \psi^s_{i,\sigma_i(u_i)} \). Let \( \bar{p} \) denote the greatest monic polynomial in \( k[x, y] \) that divides all the entries \( p_{l'_s}, p_{l'_s+1}, \ldots, p_{l_s} \). Set \( p'_m := p_m/\bar{p} \) for \( m \in \{l'_s, l'_s+1, \ldots, l_s\} \). We define
\[ Z_{s+1} = Z_s \cup V\left( u_{n_{s+1}} : \sum_{i=l'_s}^{l_s} u_{n_{s+1}+i-l'_s+p'_i} \right) \]
We obtain \( \psi^{s+1} \) from \( \psi^s \) by first applying \( D_{l'_s}(u_{n_{s+1}}) \), then \( R_{l'_s}(u_i) \) where \( l'_s < i \leq l_s \), then \( D_i(\sum_{i=l'_s}^{l_s} u_{n_{s+1}+i-l'_s+p'_i}) \) for \( l'_s < i \leq l_s \), and finally applying \( R_{l'_s}(p'_i) \) for \( l'_s < i \leq l_s \).

We can repeat this process until it is no longer possible to find a move of type III with \( l'_s < l_s \). Then we can continue with the symmetric (with respect to the diagonal of the matrix running from the lower left entry to the upper right entry) operations. To continue with symmetric operations we can define
\[ l'_s := \sigma_s(\max\{ \sigma_s(l') \mid l' < l < \sigma_s(l) < \sigma_s(l') \}) \]
and
\[ l_s := (\max\{ \sigma_s(l) \mid l' < l < \sigma_s(l) < \sigma_s(l'_s) \}) \].
We repeat the symmetric operations as long as we have \( \sigma_s(l'_s) > j \).
At the end of this process we obtain a rational map $\psi^t$ from the quasi affine variety $U = A_k^{2+n_t} - Z_t$ to $V_N$ where we have $r_{\frac{k-1}{t}}(X) = r_{n-\frac{R}{t}}(X) = 0$ and $r_{\frac{C+1}{t}}(X) = r_{C+n}(X) = 0$ for every $X$ in the image of this rational map. Denote the composition of $m_{i\ldots j}$ with $\psi^t$ by $m$. Notice that $m$ is a polynomial in $k[\{x,y,u_1,\ldots,u_n\}]$. Define another polynomial $p$ in the same polynomial algebra as follows:

$$p = \prod_{s=1}^{t} u_{n_s+1} \sum_{i=I'_s}^{l_s} u_{n_s+i-I'_s+1}p_{i}'.$$

Since in each sum $p_{i}'$s have no common factor in $k[x,y]$, there exists a solution to the equations $p = 1$ and $m = 0$. Thus, there exists $\gamma \in U$ such that $m_{i\ldots j}(\gamma) = 0$, which is again a contradiction by Lemma 2.

3. Examples and Problems

One might ask how strict the upper bound for $r$ in Conjecture 5 is. We claim that

$$r \leq \left\lfloor \log_2 \left( \frac{N}{R+C} \right) \right\rfloor + 1.$$

In all the following examples, we define $\psi$ from $\mathbb{P}_k^{r-1}$ to $V(d)_{R+C} - L(d)$ for different values of $r$, $N$ and $R+C$ where $r = \left\lfloor \log_2 \left( \frac{N}{R+C} \right) \right\rfloor + 1$. It follows that we do not have a better upper bound for $r$ in these cases.

**Example 1.** For $r = 1$, $N = 2n$, $d = (0,\ldots,0)$ and $R+C = N$, define

$$\psi(x) = \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} \quad \text{where} \quad M = \begin{bmatrix} x & 0 \\ \ddots & \ddots \\ 0 & x \end{bmatrix}.$$ 

Note that $\sigma_\psi = (1,n+1)(2,n+2)\ldots(n,N)$. This example shows that $\text{MP}(1,N) = \{(1,n+1)(2,n+2)\ldots(n,N)\}$.

**Example 2.** For $r = 2$ and $N = 4$, $d = (0,0,0,0)$ and $R+C = 2$, define

$$\psi(x,y) = \begin{bmatrix} 0 & x & y & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

In this example, $\sigma_\psi = (1,2)(3,4)$. Hence, $\text{MP}(2,4) = \{(1,2)(3,4)\}$. 

Example 3. For $r = 2$, $N = 6$, $d = (0, -1, -1, -1, -1, -1)$, and $\mathcal{R} + \mathcal{C} = 3$, set:

$$
\psi(x, y) = \begin{bmatrix}
0 & x^2 & xy & y^2 & 0 & 0 \\
0 & 0 & 0 & 0 & y & 0 \\
0 & 0 & 0 & 0 & x & y \\
0 & 0 & 0 & 0 & 0 & x \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

Here, we have $\sigma_{\psi} = (1, 2)(3, 5)(4, 6)$. Considering the Hasse diagram for $\mathbb{R}\mathbb{P}(6)$ in Figure 2 and symmetry it is clear that

$$
\mathbb{M}\mathbb{P}(2, 6) = \{(1, 2)(3, 5)(4, 6), (1, 3)(2, 4)(5, 6)\}.
$$

The above example can be generalized as follows:

Example 4. For $r = 2$, $N = 2n$, $d = (0, -n + 2, \ldots, -n + 2)$, and $\mathcal{R} + \mathcal{C} = n$, set:

$$
\psi(x, y) = \begin{bmatrix}
0 & x^{n-1} & x^{n-2}y & \ldots & y^{n-1} & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & y & : \\
0 & 0 & \ldots & x & y & & & \\
& & \ddots & \ddots & \ddots & y & & \\
& & & \ddots & \ddots & \ddots & \ddots & \\
& & & & \ddots & \ddots & \ddots & \ddots & \\
& & & & & \ddots & \ddots & \ddots & \ddots & \\
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& & & & & & & & & & & & & & & & & \ddots & \ddots & \ddots & \ddots & \\
& & & & & & & & & & & & & & & & & & \ddots & \ddots & \ddots & \ddots &
\end{bmatrix}.
$$

We can use the above examples to obtain new ones by the chess board construction given as follows

Construction 1. Let $(l_1, l_2, \ldots, l_m)$ be an $m$-tuple of positive integers and $V(d)(l_1, l_2, \ldots, l_m)$ the subvariety of $V(d)$ such that $x_{ij} = 0$ when $l_1 + l_2 + \ldots + l_{(s-1)} + 1 \leq i < j \leq l_1 + l_2 + \ldots + l_s$ for some $1 \leq s \leq m$. For example, the following matrix $\psi(x, y)$ is in $V(d)(1, 3, 2)$ where $d$ is 6-tuple of nonincreasing integers;

$$
\psi(x, y) = \begin{bmatrix}
0 & x^2 & xy & y^2 & 0 & 0 \\
0 & 0 & 0 & 0 & y & 0 \\
0 & 0 & 0 & 0 & x & y \\
0 & 0 & 0 & 0 & 0 & x \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

Take $\psi_1 \in V(d_1)(l_{11}^1, l_{12}^1, \ldots, l_{1n}^1)$ where $d_1$ is $N_1$-tuple nonpositive integers and $\psi_2 \in V(d_2)(l_{21}^2, l_{22}^2, \ldots, l_{2n}^2)$ where $d_2$ is $N_2$-tuple of nonincreasing integers. We arrange a $(N_1 + N_2) \times (N_1 + N_2)$ matrix in a $2m \times 2m$-chessboard as follows: The $ij$-square contains a $\ell_{[x+1]}^i \times \ell_{[x+1]}^j$ matrix such
that \( \varepsilon_k = 1 \) if \( k \) is odd or \( \varepsilon_k = 2 \) if \( k \) is even integer. Now we color the \( ij \) square black if \( \varepsilon_i \neq \varepsilon_j \) and white if \( \varepsilon_i = \varepsilon_j \). Fill in the \( ij \) square with zeros if it is a black square and otherwise fill in with \((x_{i'j'})\) where \( i \leq i' \leq i + 1 \) and \( j \leq j' \leq j + 1 \) part of \( \psi_{\varepsilon_i} \) where

\[
\tilde{s} = \left\lfloor \frac{s+1}{2} \right\rfloor - 1 \quad \text{and} \quad \tilde{s} = \sum_{m=1}^{l_{\varepsilon_i}} f_{\varepsilon_i} + 1 \quad \text{and} \quad \tilde{s} = \sum_{m=1}^{l_{\varepsilon_i}} f_{\varepsilon_i}.
\]

For instance, using chessboard construction we can obtain an example as follows:

**Example 5.** For \( r = 2, N = 4 + 6 = 10, d = (0, 0, -1, -1, -1, -1, -1, -1, -1, -1) \), and \( R + C = 2 + 3 = 5 \) we obtain an example by applying the chess board construction on the morphisms in Examples 2 and 3.

We also have other well-known constructions like the Koszul complex construction giving us examples as below.

**Example 6.** For \( r = 3, N = 8 \) and \( R + C = 2 \), define

\[
\psi(x, y, z) = \begin{bmatrix}
0 & x & y & z & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & y & z & 0 & 0 & 0 \\
0 & 0 & 0 & x & 0 & z & 0 & 0 \\
0 & 0 & 0 & 0 & x & y & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

In this example, \( \sigma_{\psi} = (1, 2)(3, 5)(4, 6)(7, 8) \).

We end with a few questions for future research. Notice that all examples discussed above are in \( \mathbf{DP}(N) \). Hence one can ask:

**Question 1.** Is \( \mathbf{MP}(r, N) \subseteq \mathbf{DP}(N) \)?
For all these examples $\frac{N}{2r-1}$ is an integer. For instance, we can find an example see Example 7 for $r = 3$, $N = 12$ but we do not know the answer of the following question:

**Question 2.** Is there any example for $r = 3$ and $N = 10$? More precisely, can we say that $MP(3,10)$ is nonempty?

Note that the following example can not be obtained by the constructions we mentioned above.

**Example 7.** For $r = 3$, $N = 12$, $d = (0,0,-1,\ldots,-1)$ and $R + C = 3$, consider

\[
\begin{bmatrix}
0 & x & y^2 & yz & z^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & y^2 & yz & 0 & z^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & x & 0 & z & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & x & y & 0 & z & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x & y & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & z & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Note that we can obtain an example for $r = 3$ and $N = 4s$ for every $s \geq 2$ by using the examples for $r = 3$, $N = 8$ and the example for $r = 3$, $N = 12$ and applying the chessboard construction as many times as necessary. If the answer to question 2 is negative, then one can ask the following question:

**Question 3.** Do there exist any periodicity results about nonemptiness of $MP(r,N)$?

Another observation we make about these examples is that there always exists a sequence of permutations $\sigma_1 < \sigma_2 < \cdots < \sigma_r$ such that the image of the morphism contains a point from each Borel orbit corresponding to the these $\sigma_i$’s and each pair of consecutive $\sigma_i$’s consist of distinct transpositions. For example, putting $x = 1$ and $y = 0$ to $\psi$ in Example 2 we get a point in the Borel orbit corresponding to permutation $\sigma_2 = (1,2)(3,4)$. and putting $x = 0$ and $y = 1$, we get $\sigma_1 = (1,3)(2,4)$. Hence one could ask the following question:

**Question 4.** Given $\sigma$ in $MP(r,N)$ does there exist a morphism $\psi: \mathbb{P}^{r-1} \to V(d) - L(d)$ with a sequence permutations $\sigma_1 < \sigma_2 < \cdots < \sigma_r$ and points $X_1, X_2, \ldots, X_r$ in the image of $\psi$ such that $\sigma_\psi = \sigma$ and $X_i$ is in the Borel orbit of $\sigma_i$ for all $i$ and $\sigma_i$ and $\sigma_{i+1}$ has no common transpositions?

If the answer is affirmative to this question then one can say that the inequalities

\[
\frac{n(n+1)}{2} \leq \dim(\sigma_i) \leq n^2
\]

and

\[
\dim(\sigma_i) + \left\lceil \frac{n}{2} \right\rceil \leq \dim(\sigma_{i+1})
\]
hold and they give the inequality \( N \geq 2r \).

Note that Allday and Puppe \([2]\) have related results: If \( k, A, r, N, \) and \( M \) are as in Conjecture \([1]\) then they prove \( N \geq 2r \). Moreover, Avramov, Buchweitz and Iyengar \([3]\) verified that \( N \geq 2r \) in a more general case.

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