Effective electrostatic attraction between electrons due to quantum interference

Marina F. B. Cenni, Raul Corrêa, and Pablo L. Saldanha
Departamento de Física, Universidade Federal de Minas Gerais,
Caixa Postal 701, 30161-970, Belo Horizonte, MG, Brazil
(Dated: August 30, 2018)

PACS numbers: 03.65.Ta, 03.75.-b

Electric charges of equal sign repel each other, while charges of opposite sign suffer an attraction. This sentence represents a fundamental principle not only for the scientific community, but is well known by the general public as well. It is hard to imagine that such principle could be violated. But here we show that this principle may be contradicted by a quantum interference phenomenon with post-selection.

Quantum interference may result in many non-intuitive phenomena such as interferometry with massive particles [1,2], quantum delayed choice experiments [3,4], quantum erasers [7–10], “interaction-free” measurements [11–13] or the Hong-Ou-Mandel effect [14–16]. Here we discuss a novel example of the counter-intuitive characteristics of quantum interference in a gedanken experiment inspired on a recent work by Aharonov et al. [17] and following the discussions of an earlier paper from our group [18]. In Ref. [17] the authors discuss the classical limit of the radiation pressure and the difference in interpretation arising from classical and quantum descriptions, by treating one of the mirrors of an interferometer quantum mechanically. The authors have shown how it is possible for the quantum combination of two possibilities, one in which light pushes a mirror outwards and other that leaves it still, to result in an inward pull in the mirror. Our previous paper generalizes this result by considering anomalous shifts in momentum associated with general quantum objects, and proposes feasible ways of testing the effect in the laboratory outside the weak interaction regime [18]. With this we have introduced the concept of the ‘quantum interference of force’ effect and its close relation to the wave-particle duality.

Here we describe a gedanken experiment based on the quantum interference of force phenomenon [18], by considering two electrons that propagate parallel to each other in a Mach-Zehnder interferometer and post-selecting the interferometer port where each electron exits. We show that the quantum superposition of the situations where the electrons propagate in the same interferometer arm, repelling each other, with the situations where they propagate in opposite arms, with no interaction, may result in an effective attraction between them. This effective electrostatic attraction between the electrons is manifested on the momentum distribution of each electron, that change its mean value in the direction of the other electron with the propagation through the interferometer.

In the proposed gedanken experiment we consider a two-paths Mach-Zehnder interferometer with two electrons $e_1$ and $e_2$ sent at the same time through the apparatus, as depicted in Fig. 1. The electrons can be distinguished from one another by the $z$ component of their position, with their separation $d$ being much larger than the width of their wave functions. Apart from this displacement, the states of the electrons are essentially the same. Both paths are considered to be free from any external influence and isolated from each other so that only the electromagnetic interactions between electrons taking the same path are allowed to take place. We associate the orthogonal state vectors $|A_i\rangle$ and $|B_i\rangle$ with the distinguishable paths of propagation possible for the electrons during their travel through the system, and the vectors $|C_i\rangle$ and $|D_i\rangle$ with the possible exit ports of the Mach-Zehnder interferometer, matching the labeling given by Fig. 1. The reflection and transmission coefficients for each beam-splitter BS$_1$ and BS$_2$ are the same, denoted by $ir$, and $t = \sqrt{1-r^2}$, with $r$ and $t$ real. The initial quantum state of the $z$ component of the electrons momentum will be denoted by $|\Phi_i\rangle$, with $i = \{1,2\}$.

We consider a post-selection of the totality of events where electron $e_1$ exits the interferometer by $D_1$ and $e_2$ exits by $C_2$, as indicated in Fig. 1. By considering this post-selection choice, the final joint state of the system that consists of the two electrons will be a coherent sum over the amplitudes associated with all the possible ways for this system to have evolved in time towards this final state. There are in total four possibilities of evolution for the described system: two where the electrons take different paths inside the interferometer and therefore do not interact, and two where they do travel by the same path and an electric interaction between them exists during some time interval. In the first two cases where the elec-
To closely analyze these results, we shall specify the initial wave functions for the $z$ component of the electrons momentum $\Phi_z(p) = \langle p|\Phi_z\rangle$ as Gaussian distributions with width $W$ centered in zero:

$$\Phi_z(p) = \frac{\pi^{-\frac{1}{4}}}{\sqrt{W}} \exp\left[-\frac{1}{2}\left(\frac{p}{W}\right)^2\right], \quad (6)$$

where the origin of the $z$ axis for each electron was defined at the corresponding center of its position wave function. If the electrons separation is much larger than the width of their wave functions and if this width does not change appreciably during the electrons time travel along the interferometer, the electrons interaction results in shifts $\delta$ on their momentum wave functions without altering their Gaussian forms [19]. The exact magnitude of $\delta$ will depend on the electrons separation $d$ and on the interaction time. In this case, the wave functions for the $z$ component of the electrons momentum altered by the interaction become

$$\Phi_z^+ (p) \equiv \langle p|\Phi_z^+\rangle = \Phi_z (p + \delta), \quad (7)$$

$$\Phi_z^- (p) \equiv \langle p|\Phi_z^-\rangle = \Phi_z (p - \delta), \quad (8)$$

which correspond to momentum shifts of $\mp \delta$ in the wave functions. We note that electron $e_1$ gains a negative momentum while electron $e_2$ gains a positive momentum of the same amplitude.

It is possible to analyze the quantum states associated to each of the electrons separately by taking the partial traces over the post-selected state of Eq. (5). In this way, the state $\rho_1$ associated to electron $e_1$ is

$$\rho_1 = \text{Tr}^2(\langle \Phi_{ps}|\Phi_{ps}\rangle) =$$

$$= \langle \Phi_1| \Phi_1 | + | e^{-i\alpha} \cos(\phi)|\Phi_1 | +$$

$$+ I e^{i\alpha} \cos(\phi)|\Phi_1^- | + | e^{i\alpha} \sin(\phi)|\Phi_1^+ |, \quad (9)$$

apart from a normalization factor, with

$$I = \int \Phi_2(p)\Phi_2(p - \delta)dp = \exp\left(-\frac{\delta^2}{4W^2}\right). \quad (10)$$

In the same manner, the state $\rho_2$ associated to the electron $e_2$ is

$$\rho_2 = | \Phi_2\rangle \langle \Phi_2 | + I e^{-i\alpha} \cos(\phi)|\Phi_2 | +$$

$$+ I e^{i\alpha} \cos(\phi)|\Phi_2^- | + | e^{i\alpha} \sin(\phi)|\Phi_2^+ |, \quad (11)$$

apart from a normalization factor.

Both states $\rho_1$, from Eq. (9), and $\rho_2$, from Eq. (11), which were derived from the entangled pure state of Eq. (5), represent mixed states for the electrons $e_1$ and $e_2$ individually. We are able to obtain the probability distributions for the electrons momenta as $P_1(p) = \text{Tr}(\rho_1|p\rangle\langle p|)$ and $P_2(p) = \text{Tr}(\rho_2|p\rangle\langle p|)$, obtaining

$$P_1(p) = \Phi_1^2 (p) + \cos^2(\phi)|\Phi_1^- | +$$

$$+ 2I \cos(\phi) \cos(\alpha)|\Phi_1 (p + \delta), \quad (12)$$

$$P_2(p) = \Phi_2^2 (p) + \cos^2(\phi)|\Phi_2^- | +$$

$$+ 2I \cos(\phi) \cos(\alpha)|\Phi_2 (p - \delta), \quad (13)$$

$$\text{FIG. 1}. \text{Two electrons propagate parallel to each other in a Mach-Zehnder interferometer, entering by the indicated ports. Beam splitter BS}_1 \text{ splits the incident wave functions and the mirrors M redirect the electrons to interfere at the second beam-splitter BS}_2. \text{The lines represent the center of the wave functions of the electrons } e_1 \text{ (blue) and } e_2 \text{ (red) while propagating in the interferometer. The distance } d \text{ between the electrons paths is considered to be much larger than the width of their wave function, such that the electrons can be labeled as } e_1 \text{ and } e_2 \text{ due to their spatial distinguishability. If the electrons propagate in the same arm they repel each other, while if they propagate in opposite arms their interaction is negligible. We will post-select the events where electron } e_1 \text{ exits by } D_1 \text{ and electron } e_2 \text{ exits by } C_2. \text{ Electrons don’t interact, the state of the system will evolve as:}$$

- $e_1$ goes through path $A_1$ and $e_2$ goes through $B_2$:

$$- r^2t^2 e^{i\phi}|\Phi_1 (D_1)|\Phi_2 (C_2), \quad (1)$$

- $e_1$ goes through path $B_1$ and $e_2$ goes through $A_2$:

$$- t^2 r^2 e^{i\phi}|\Phi_1 (D_1)|\Phi_2 (C_2), \quad (2)$$

where the vector states associated to each electron individually are labeled accordingly, and $\phi$ represents an extra phase for an electron propagation through path $A_i$ in relation to a propagation through path $B_i$.

In turn, considering that the interaction between the electrons will change their momentum states, the quantum state associated with the last two possibilities of evolution where the electrons take the same path and therefore interact will evolve as:

- $e_1$ goes through path $A_1$ and $e_2$ goes through $A_2$:

$$- r^2 t^2 e^{i(2\phi + \alpha)}|\Phi_1^- (D_1)|\Phi_2^+ (C_2), \quad (3)$$

- $e_1$ goes through path $B_1$ and $e_2$ goes through $B_2$:

$$- r^2 t^2 e^{i\alpha}|\Phi_1^- (D_1)|\Phi_2^+ (C_2), \quad (4)$$

where we have taken the vectors $|\Phi_i\rangle$ to represent the electrons momentum states that were disturbed by their electromagnetic interaction, and $\alpha$ represents a phase gained due to the interaction. Considering the combination of these four probability amplitudes, the post-selected electrons momentum state is

$$|\Phi_{ps}\rangle \propto |\Phi_1\rangle|\Phi_2\rangle + e^{i\alpha} \cos(\phi)|\Phi_1^- |+\Phi_2^+ \rangle. \quad (5)$$
apart from normalization factors. Both probability distributions have the same form except for a sign change in $\delta$.

Fig. 2 displays the counter-intuitive result that we want to emphasize in our paper. Fig. 2(a) shows the initial distributions of the $z$ component of the electrons momenta, given by the modulus squared of the momentum wave functions of Eq. (6). Fig. 2(b) shows the momentum distributions for the situations where the electrons propagate through the same path in the interferometer, given by Eqs. (7) and (8) with the parameters $\delta = 0.3W$, $\phi = 3\pi/4$ and $e^{i\alpha} = 1$. We see that the quantum superposition of an electrostatic repulsion between the electrons with no interaction may result in an effective attraction between them.

The expectation value of the momentum of the electron $1$ leaving the interferometer at the post-selection condition is

$$
\langle p_1 \rangle_{ps} = \frac{\int_{-\infty}^{\infty} dp_1 P_1(p)p}{\int_{-\infty}^{\infty} dp_1 P_1(p)} = \frac{-\delta}{1 + \cos^2(\phi) + 2 \cos(\phi) \exp(-\frac{-\delta^2}{2W^2})},
$$

with $P_1(p)$ given by Eq. (12) with $e^{i\alpha} = 1$. It is straightforward to show that $(p_2)_{ps} = -(p_1)_{ps}$. The anomalous behavior of an effective attraction between the electrons depicted in Fig. 2 happens for many values of the interferometer parameters. In Fig. 3 we plot the value of $\langle p_1 \rangle_{ps}/W$ as a function of the parameters $\delta/W$ and $\phi$.
means that if the post-selection of electron \( e_1 \) normalized by the width of the distribution, \( \langle p_1 \rangle_{ps}/W \), as a function of \( \delta/W \) and \( \phi \). Anomalous positive values, associated to an effective electrostatic attraction between the electrons, are evident.

for \( e^{i\alpha} = 1 \). We note that anomalous positive values for \( \langle p_1 \rangle_{ps}/W \) occur in a large range of parameters. Note also that for \( \delta > 2W \) the effective interaction between the electrons is always repulsive, since the displaced wave function of Eq. (6) becomes almost orthogonal to the original wave function of Eq. (6), and the interference disappears.

It is important to mention that, independently of the parameters used in the interferometer, the average interaction between the electrons is always repulsive when we consider all possible events, without post-selection. This means that if the post-selection of electron \( e_1 \) exiting by \( D_1 \) and electron \( e_2 \) exiting by \( D_2 \) results in an effective attraction between them, as in the situation depicted in Fig. 2, the average interaction in the other situations (electron \( e_1 \) by \( D_1 \) and electron \( e_2 \) by \( D_2 \), electron \( e_1 \) by \( C_1 \) and electron \( e_2 \) by \( C_2 \), electron \( e_1 \) by \( C_1 \) and electron \( e_2 \) by \( D_2 \) ) is necessarily repulsive, such that the average total interaction is repulsive. We demonstrate this behavior in the Appendix, showing the validity of the Ehrenfest theorem in this situation, which is a way to say that the average momentum is conserved when one does not post-select the results.

To conclude, we have shown that the quantum superposition of the electrostatic repulsion between two electrons (when they propagate in the same arm of an interferometer) with an absence of interaction (when they propagate in opposite arms) may result in an effective electrostatic attraction between them, given the appropriate post-selection. So, in this scenario, the common sense that two charges of equal sign always repel each other is violated due to a quantum interference effect.

This work was supported by the Brazilian agencies CNPq, CAPES and FAPEMIG.

**Appendix: Validity of the Ehrenfest theorem**

As we have mentioned, although we are able to observe the anomalous effect of attraction between two electrons in the system when the appropriate post-selection of exit ports is made, it is necessary that the average interaction between the electrons be repulsive overall. This expectation is derived from the Ehrenfest theorem, which states that the behavior of the averages of quantum observables should agree with those expected classically. Here we show how the Ehrenfest theorem applies to our interferometer.

First we note that there are in total 4 possibilities of paths for the two particles inside the apparatus, and 4 possible ways that they can leave the system at the end of the experiment, making up for a total of 16 evolution possibilities for the system. This means that a priori we have a 16 term superposition for our complete two-electron state after they leave the apparatus. The four term superposition for the joint state just before the action of BS\(_2\) can be written as:

\[
irte^{i\phi}e^{i\alpha}|\Phi_1^{-},A_1\rangle|\Phi_2^{+},A_2\rangle + irte^{i\alpha}|\Phi_1^{-},B_1\rangle|\Phi_2^{+},B_2\rangle + r^2e^{i\phi}|\Phi_1,A_1\rangle|\Phi_2,B_2\rangle - r^2e^{i\phi}|\Phi_1,B_1\rangle|\Phi_2,A_2\rangle, \tag{A.1}
\]

where we have taken into account the existence or not of an interaction between the electrons and the appropriate phase gains due to the evolution of the system as done in our previous discussion.

The effect of BS\(_2\) over the different terms of this state can be written as:

\[
|A_1\rangle|A_2\rangle \Rightarrow (r^2|C_1\rangle|C_2\rangle - r^2|D_1\rangle|D_2\rangle + irt(|C_1\rangle|D_2\rangle + |D_1\rangle|C_2\rangle)),
|B_1\rangle|B_2\rangle \Rightarrow -r^2|C_1\rangle|C_2\rangle + r^2|D_1\rangle|D_2\rangle + irt(|C_1\rangle|D_2\rangle + |D_1\rangle|C_2\rangle),
|A_1\rangle|B_2\rangle \Rightarrow irt(|C_1\rangle|C_2\rangle + |D_1\rangle|D_2\rangle + r^2|C_1\rangle|D_2\rangle - r^2|D_1\rangle|C_2\rangle),
|B_1\rangle|A_2\rangle \Rightarrow irt(|C_1\rangle|C_2\rangle + |D_1\rangle|D_2\rangle) - r^2|C_1\rangle|D_2\rangle + r^2|D_1\rangle|C_2\rangle. \tag{A.2}
\]

By plugging Eqs. (A.2) into Eq. (A.1), we reach the final joint state superposition for the electrons leaving the
where $P$ is what happens to electron $e_1$ sive when no post-selection is made, we can focus on should be. wave function of Eq. (5) used to derive our results, as it must be always null or negative, namely $\langle p \rangle = \langle p \rangle_{CC} P_{CC} + \langle p \rangle_{DD} P_{DD} + \langle p \rangle_{CD} P_{CD} + (p)_{DC} P_{DC}, \quad (A.4)$ where $P_{jk}$ is the probability of detecting electron $e_1$ at exit $j$ and $e_2$ at $k$, and $\langle p_{1j} \rangle$ is the respective average momentum for this detection. This quantity can be derived by repeating the process done in Eqs. (9), (12) and (14) for each of the four exit ports possibilities, $|C_1 \rangle |C_2 \rangle$, $|D_1 \rangle |D_2 \rangle$, $|C_1 \rangle |D_2 \rangle$, and $|D_1 \rangle |C_2 \rangle$. Some straightforward algebra shows us that the total average momentum gained by electron $e_1$ is

$$\langle p_1 \rangle = (t^4 + r^4) |\Phi_1 \rangle |p_1 \rangle |\Phi_1 \rangle + 2t^2 r^2 |\Phi_1 \rangle |p_1 \rangle |\Phi_1 \rangle = -2t^2 r^2 \delta. \quad (A.5)$$

This perfectly agrees with our classical intuition, as the first term incorporates the probability that the electrons are either both transmitted or both reflected by BS$_1$ (they do not interact), and the second term considers the probability that one of the electrons is transmitted and the other is reflected at BS$_1$ (they do interact). So the final average momentum is simply the momentum gained when they do interact $|\Phi_1 \rangle |p_1 \rangle |\Phi_1 \rangle = -\delta$, which comes from a repulsive interaction, times the probability of interacting $2t^2 r^2$. Therefore the average interaction is repulsive, in agreement with momentum conservation and the Ehrenfest theorem.

[1] R. P. Feynman, R. B. Leighton, and M. Sands, The Feynman Lectures on Physics, Vol. III (Basic Books, New York, 2010).
[2] A. Tonomura, J. Endo, T. Matsuda, T. Kawasaki, and H. Ezawa, Am. J. Phys. 57, 117 (1989).
[3] R. Bach, D. Pope, S.-H. Liou, and H. Batelaan, New J. Phys. 15, 033018 (2013).
[4] R. Ionicioiu and D. R. Terno, Phys. Rev. Lett. 107, 230406 (2011).
[5] A. Peruzzo, P. Shadbolt, N. Brunner, S. Popescu, and J. L. O’Brien, Science 338, 634 (2012).
[6] F. Kaiser, T. Coudreau, P. Milman, D. B. Ostrowsky, and S. Tannzilli, Science 338, 637 (2012).
[7] M. O. Scully, B.-G. Englert, and H. Walther, Nature 351, 111 (1991).
[8] T. J. Herzog, P. G. Kwiat, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. 75, 3034 (1995).
[9] S. Durr, T. Nonn, and G. Rempe, Phys. Rev. Lett. 81, 5705 (1998).
[10] S. P. Walborn, M. O. Terra Cunha, S. Pádua, and C. H. Monken, Phys. Rev. A 65, 033818 (2002).
[11] A. C. Elitzur and L. Vaidman, Found. Phys. 23, 987 (1993).
[12] P. Kwiat, H. Weinfurter, T. Herzog, A. Zeilinger, and M. A. Kasevich, Phys. Rev. Lett. 74, 4763 (1995).
[13] J. Peise, B. Lucke, L. Pezé, F. Deuretzbacher, W. Ertmer, J. Arlt, A. Smerzi, L. Santos, and C. Klempt, Nat. Commun. 6, 6811 (2015).
[14] C. K. Hong, Z. Y. Ou, and L. Mandel, Phys. Rev. Lett. 59, 2044 (1987).
[15] E. Bocquillon, V. Freulon, J.-M. Berroir, P. Degiovanni, B. Plaçais, A. Cavanna, Y. Jin, and G. Fève, Science 339, 1054 (2013).
[16] S. Mahle, P. Oppel, R. Wiegner, and J. von Zanthier, J. Mod. Opt. 64, 921 (2016).
[17] Y. Aharonov, A. Bozser, S. Nussinov, S. Popescu, J. Tollaksen, and L. Vaidman, New J. Phys. 15, 093006 (2013).
[18] R. Corrêa, M. F. B. Cenni, and P. L. Sal Danha, arXiv:1802.10370.
[19] R. P. Feynman and A. R. Hibbs, Quantum mechanics and path integrals, Vol. I (New York, 2010).