SEMISTABILITY AND \(CAT(0)\) GEOMETRY

ROSS GEGHEGAN

Abstract. We explain why semistability of a one-ended proper \(CAT(0)\) space can be determined by the geodesic rays. This is applied to boundaries of \(CAT(0)\) groups.

1. Geometry

Let \(X\) be a proper \(CAT(0)\) space having one end, and let \(\partial X\) denote its compactifying boundary. One says that \(X\) has semistable fundamental group at infinity (or has one strong end) if any two proper rays in \(X\) are properly homotopic. A point of \(\partial X\) is, by definition, an equivalence class of geodesic rays in \(X\) any two of which are boundedly close. (See [BH99] for \(CAT(0)\) matters.) Since geodesic rays are proper one may ask if semistability can be determined by the geodesic rays alone.

Theorem 1.1. The one-ended proper \(CAT(0)\) space \(X\) has semistable fundamental group at infinity if and only if any two geodesic rays in \(X\) are properly homotopic.

Remark. If it is the case that any two geodesic rays in \(X\) are properly homotopic through geodesic rays (i.e. every level of the homotopy a geodesic ray), then \(\partial X\) is path connected. Theorem 1.1 describes something weaker than path connectedness.

An inverse sequence of groups \(\{H_n, f^m_n\}\) is said to be semistable (or Mittag-Leffler) if, for each \(n\), the images of the bonding homomorphisms \(f^m_n : H_m \to H_n\) are the same for all but finitely many values of \(m > n\). The relevance is this: Let \(\{K_n\}\) be an exhausting sequence of compact subsets of \(X\) such that, for all \(n\), \(K_n\) is a subset of the interior of \(K_{n+1}\). Choose a suitably parametrized proper base ray \(\omega\) in \(X\) and consider the inverse sequence of groups \(\{\pi_1(X - K_n, \omega), (\text{inclusion})\#\}\). This sequence of groups is semistable if and only if any two proper rays in \(X\) are properly homotopic; see [Geo08], Section 16.1 for details. This explains the terminology above.

Remark. It has long been known ([Kra77]) that if a metrizable compact connected space (e.g. \(\partial X\)) is locally connected\(^1\) then it has a shape theoretic property called “pointed 1-movable”. Our \(\partial X\) has this property if and only if \(X\) has semistable fundamental group at infinity. So Theorem 1.1 is mainly of interest when \(\partial X\) is not known to be locally connected.

Remark. That theorem of Krasinkiewicz [Kra77] says more: A metrizable compact connected space is pointed 1-movable if and only if it is shape equivalent to a locally connected compact connected metrizable space. This applies to our \(\partial X\).

\(^1\)Equivalently, if it is a Peano space, i.e. the continuous image of a closed interval.

Date: March 16, 2017.

2010 Mathematics Subject Classification. Primary 20F65; Secondary 57M07.

Key words and phrases. CAT(0) space, semistable, geodesic ray, boundary.
Remark. Any compact metrizable space can be the boundary of a proper CAT(0) space so there are many proper CAT(0) spaces $X$ which do not have semistable fundamental group at infinity. For example, a dyadic solenoid can be such a boundary.

**Open Question 1**: Is it true that when the full isometry group of $X$ acts cocompactly on $X$ then $X$ has semistable fundamental group at infinity?

(Compare this with the better-known group theoretical Open Question 2 posed in Section 2.)

We note that a necessary condition for the full isometry group of $X$ to act cocompactly is that the Lebesgue covering dimension and the cohomological dimension of $\partial X$ be the same [GO07].

We write $\hat{X}$ for the compact absolute retract (AR) $X \cup \partial X$. It is easy to see that the set $\partial X$ is a $Z$-set in $\hat{X}$. Recall that this means that for any open set $U$ in $\hat{X}$ the inclusion map $U \cap X \to U$ is a homotopy equivalence\(^3\). So shape theory can be directly applied to the metric compactum $\partial X$ as it sits naturally in the AR $\hat{X}$.

We need some shape theoretic terminology.

1. A strong shape component of $\partial X$ is a proper homotopy class of proper rays in $X$.
2. The proper ray $c : [0, \infty) \to X$ ends at the point $p \in \partial X$ if the map $c$ extends continuously to $\hat{c} : [0, \infty] \to \hat{X}$ by mapping the point $\infty$ to $p$.
3. Two points $p$ and $q$ of $\partial X$ lie in the same component of joinability if there are proper rays in $X$ ending at $p$ and at $q$ which are properly homotopic.

Remark. For strong shape theory see, for example, Sect. 17.7 of Geo08. Components of joinability were introduced in KM79.

**Lemma 1.2.** If two proper rays end at the same point of $\partial X$ then they are properly homotopic.

*Proof.* Let the proper rays $c$ and $c'$ end at the point $p$. AR’s are locally simply connected, so we can choose a basic system of neighborhoods of $p$ in $\hat{X}$

$$\cdots \subseteq W_n \subseteq V_n \subseteq U_n = W_{n-1} \subseteq V_{n-1} \subseteq U_{n-1} \subseteq \cdots$$

such that

(i) $[c(n), \infty) \cup [c'(n), \infty) \subseteq W_n$;
(ii) any two points in $W_n$ can be joined by a path in $V_n$;
(iii) any loop in $V_n$ is homotopically trivial in $U_n$.

For each $n$ choose a path $\omega_n$ in $V_n$ joining $c(n)$ to $c'(n)$. Choose a trivializing homotopy in $U_{n-1}$ of the loop formed by the segments $[c'(n-1), c'(n)]$, $[c(n-1), c(n)]$, $\omega_{n-1}$ and $\omega_n$. Because $\partial X$ is a $Z$-set all these homotopies can be pulled off $\partial X$ with the required amount of control. Thus they can be fitted together to give a proper homotopy between $c$ and $c'$. □

\(^2\)For a sketched proof of this observation of Gromov see the Appendix to GO07.

\(^3\)Less formally: for any positive $\epsilon$, any map of a polyhedron $P$ into $\hat{X}$ can be $\epsilon$-homotoped off $\partial X$, holding fixed an arbitrarily large closed subset of $P$ lying in the pre-image of $X$. 
We denote by \([c]\) the strong shape component of \(\partial X\) defined by the proper ray \(c\). If some \(c'\) in \([c]\) ends at a point of \(\partial X\) we say that the strong shape component \([c]\) of \(\partial X\) is non-empty. In general there may also be empty strong shape components, meaning examples of \([c]\) containing no such \(c'\). The existence of these is explored in [GK91] where, in particular the following is proved ([GK91] Corollary 3.6 and Theorem 5.1):

**Theorem 1.3.** If \(\partial X\) has an empty strong shape component then it has uncountably many such, and also uncountably many components of joinability.

**Remark.** The proof in [GK91] is for compact metrizable spaces in general, not specifically for boundaries of \(CAT(0)\) spaces.

**Proof of Theorem 1.1:** The “only if” direction is trivial. Assume that any two geodesic rays in \(X\) are properly homotopic. Then \(\partial X\) has only one component of joinability. By Theorem 1.3, \(\partial X\) has no empty strong shape components. So any strong shape component \([c]\) contains a proper ray \(c'\) which ends at a point \(p\) of \(\partial X\). There is also a geodesic ray ending at \(p\), and, by Lemma 1.2, it is properly homotopic to \(c'\). It follows that \(\partial X\) has only one strong shape component. In other words, \(X\) has semistable fundamental group at infinity.

\(\square\)

2. **Group Theory**

Let the group \(G\) act geometrically (i.e. properly discontinuously and cocompactly) on the one-ended proper \(CAT(0)\) space \(X\). Then the discrete group \(G\) is quasi-isometric to \(X\), so \(G\) has semistable fundamental group at infinity (in the usual sense) if and only if the same is true of \(X\). (See, for example, Section 18.2 of [Geo08] where quasi-isometry is implicitly proved.) The relevance to group theory is that in order to show \(G\) has semistable fundamental group at infinity, one need only check the condition on geodesic rays given in Theorem 1.1. And since semistable fundamental group at infinity is a quasi-isometry invariant, this can be checked on any proper \(CAT(0)\) space on which \(G\) acts geometrically.

This is of interest because of the following long-studied problem:

**Open Question 2:** Is it true that every (one-ended) \(CAT(0)\) group has semistable fundamental group at infinity?

The theorem of Krasinkiewicz [Kra77] mentioned in Section 1 implies that if there exists a \(CAT(0)\) group whose boundary does not have semistable fundamental group at infinity then that boundary is not shape equivalent to a (connected) locally connected metrizable compact space.

As a corollary to Theorem 1.1 we have:

**Corollary 2.1.** If \(\partial X\) is path connected then \(G\) has semistable fundamental group at infinity.

Many \(CAT(0)\) groups are known to have semistable fundamental group at infinity, and for some of these groups spaces \(X\) exist having path connected boundaries. All one-ended word-hyperbolic groups have connected and locally connected boundary in the sense of Gromov; such a boundary is path connected. So when a one-ended word-hyperbolic group acts geometrically on a proper \(CAT(0)\) space \(X\) then \(\partial X\) is path connected. All Artin groups and Coxeter groups have semistable fundamental group at infinity, but it has been conjectured
that right angled Artin groups which do not split as direct products of infinite groups cannot have path connected boundaries.

A well-known example due to Croke and Kleiner [CK00] gives a $CAT(0)$ group $G$, actually a right angled Artin group, which has the following properties:
1. $G$ has semistable fundamental group at infinity;
2. $G$ acts geometrically on two proper $CAT(0)$ spaces whose boundaries are not homeomorphic;
3. Those boundaries are not path connected.

M. Mihalik has asked:

**Open Question 3:** If $G$ acts geometrically on $CAT(0)$ spaces $X$ and $X'$ and if $\partial X$ is path connected must $\partial X'$ be path connected?

The matter of when boundaries associated with $CAT(0)$ groups are path connected seems to be not fully understood.

**Acknowledgment:** I thank Chris Hruska, Mike Mihalik and Kim Ruane for helpful comments on an earlier draft of this note.

**References**

[BH99] Martin R. Bridson and André Haefliger, Metric spaces of non-positive curvature, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR 1744486 (2000k:53038)

[CK00] Christopher B. Croke and Bruce Kleiner, Spaces with nonpositive curvature and their ideal boundaries, Topology 39 (2000), no. 3, 549–556. MR 1746908

[Geo08] Ross Geoghegan, Topological methods in group theory, Graduate Texts in Mathematics, vol. 243, Springer, New York, 2008. MR 2365352

[GK91] Ross Geoghegan and Józef Krasinkiewicz, Empty components in strong shape theory, Topology Appl. 41 (1991), no. 3, 213–233. MR 1135099

[GO07] Ross Geoghegan and Pedro Ontaneda, Boundaries of cocompact proper $CAT(0)$ spaces, Topology 46 (2007), no. 2, 129–137. MR 2313968 (2008c:57004)

[KM79] Józef Krasinkiewicz and Piotr Minc, Generalized paths and pointed 1-movability, Fund. Math. 104 (1979), no. 2, 141–153. MR 551664

[Kra77] Józef Krasinkiewicz, Local connectedness and pointed 1-movability, Bull. Acad. Polon. Sci. S Sci. Math. Astronom. Phys. 25 (1977), no. 12, 1265–1269. MR 500986

Ross Geoghegan, Department of Mathematical Sciences, Binghamton University (SUNY), Binghamton, NY 13902-6000, USA

E-mail address: ross@math.binghamton.edu