STABILITY PROPERTIES AND GAP THEOREM FOR COMPLETE $f$-MINIMAL HYPERSURFACES

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Abstract. In this paper, we study complete oriented $f$-minimal hypersurfaces properly immersed in a cylinder shrinking soliton $(\mathbb{S}^n \times \mathbb{R}, \bar{g}, f)$. We prove that such hypersurface with $L_f$-index one must be either $\mathbb{S}^n \times \{0\}$ or $\mathbb{S}^{n-1} \times \mathbb{R}$, where $\mathbb{S}^{n-1}$ denotes the sphere in $\mathbb{S}^n$ of the same radius. Also we prove a pinching theorem for them.

1. Introduction

The $f$-minimal hypersurfaces in smooth metric measure spaces are the generalization of self-shrinkers in the Euclidean space and have been studied recently. See, for instance, [1], [5], [6], [7], [12], [14], [17], [20]. Recall a hypersurface $(\Sigma, g)$ isometrically immersed in a Riemannian manifold $(M, \bar{g})$ is called an $f$-minimal hypersurface if its mean curvature $H$ satisfies that,

$$H = \langle \nabla f, \nu \rangle,$$

where $\nu$ is the unit normal to $\Sigma$, $f$ is a smooth function on $M$, and $\nabla f$ denotes the gradient of $f$ on $M$. It is known that an $f$-minimal hypersurface can be viewed in two basic ways: 1) it is a critical point of the weighted volume functional $\int_{\Sigma} e^{-f} d\sigma$ of $\Sigma$, where $d\sigma$ denote the volume element of $(\Sigma, g)$; 2) it is a minimal hypersurface in $(M, \tilde{g})$, where the new metric $\tilde{g} = e^{-\frac{2}{n}f} \bar{g}$ of $M$ is conformal to $\bar{g}$. Besides, $f$-minimal hypersurfaces appear in the study of mean curvature flow of a hypersurface in an ambient manifold evolving by Ricci flow. Recently Lott [21] and Magni-Mantegazza-Tsats [22] proved that Huisken’s monotonicity formula holds when the ambient is a gradient Ricci soliton solution to the Ricci flow. Lott [21] introduced the concept of mean curvature soliton for the mean curvature flow of a hypersurface in a gradient Ricci soliton solution. By its definition, a mean

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curvature soliton is just an $f$-minimal hypersurface, where $f$ is the potential function of the ambient gradient Ricci soliton. Some compactness theorem of $f$-minimal surfaces in three-dimensional manifolds were proved by Cheng-Mejia-Zhou [5], [6].

Motivated by the above facts, Cheng-Mejia-Zhou [7] studied $f$-minimal hypersurfaces in a cylinder shrinking soliton of type $(\mathbb{S}^n \times \mathbb{R}, \mathcal{g}, f)$ with the potential function $f$. In [7], the authors studied the drifted Laplacian $\Delta f = \Delta f - \langle \nabla f, \nabla \cdot \rangle$ of geometric quantities for $f$-minimal hypersurfaces and derived a Simons’ type equation and some other equations. These equations involve the Barky-Émery Ricci curvature $\text{Ric}_f := \text{Ric} + \nabla^2 f$ of the ambient manifold and have simpler expressions when the ambient manifold is a gradient Ricci soliton. Furthermore, the stability properties and gap phenomenon of closed, i.e., compact and without boundary, $f$-minimal hypersurfaces immersed in a cylinder shrinking soliton of type $(\mathbb{S}^n \times \mathbb{R}, \mathcal{g}, f)$ were studied. Especially, the classification of the closed immersed $f$-minimal hypersurfaces with $L_f$-index one and a pinching theorem of the norm $|A|^2$ of the second fundamental form were obtained (see the definitions of $L_f$ operator and $L_f$-index in Section 2). In this paper, we will discuss the complete noncompact case. First, we prove the following

**Theorem 1.** Let $\Sigma$ be a complete oriented $f$-minimal hypersurface properly immersed in the cylinder shrinking soliton $(\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}, \mathcal{g}, f)$. Then $L_f$-index of $\Sigma$ satisfies that $L_f\text{-ind}(\Sigma) \geq 1$.

Moreover $L_f$-index of $\Sigma$ is one if and only if $\Sigma$ is either $\mathbb{S}^n(\sqrt{2(n-1)}) \times \{0\}$ or $\mathbb{S}^{n-1}(\sqrt{2(n-1)}) \times \mathbb{R}$, where $\mathbb{S}^{n-1}(\sqrt{2(n-1)})$ denotes the totally geodesic sphere of $\mathbb{S}^n(\sqrt{2(n-1)})$.

Here $\mathbb{S}^n(\sqrt{2(n-1)})$ denote the round sphere in the Euclidean space $\mathbb{R}^{n+1}$ of radius $\sqrt{2(n-1)}$ centered at the origin. The cylinder shrinking soliton $(\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}, \mathcal{g}, f)$ is a triple satisfying that it has the product metric $\mathcal{g}$ of $(\mathbb{S}^n(\sqrt{2(n-1)})$ and $\mathbb{R}$, $f(x, h) = \frac{h^2}{4}, (x, h) \in (\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}$ (see more details in Section 2.

Secondly, we prove a pinching theorem as follows:
Theorem 2. Let $\Sigma$ be a complete oriented $f$-minimal hypersurface properly immersed in the cylinder shrinking soliton $(\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}, \bar{g}, f)$, $n \geq 3$. Assume that the norm $|A|$ of the second fundamental form of $\Sigma$ satisfies

\[
|A|^2 - \frac{1}{4} \leq \frac{1}{4} \left( 1 - \frac{8}{n-1} \alpha^2 (1 - \alpha^2) \right),
\]

where $\alpha = \langle \frac{\partial}{\partial h}, \nu \rangle$, $h$ denotes the coordinate of the second factor $\mathbb{R}$ in $(\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R})$. Then $\Sigma$ must be one of the following three cases:

- $\mathbb{S}^n(\sqrt{2(n-1)}) \times \{0\}$,
- $\mathbb{S}^{n-1}(\sqrt{2(n-1)}) \times \mathbb{R}$, and
- $T \times \mathbb{R}$, where $T$ denotes the minimal Clifford torus $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{S}^l(\sqrt{2l}) \subset \mathbb{S}^n(\sqrt{2(n-1)})$ in $\mathbb{S}^n(\sqrt{2(n-1)})$, $k + l = n - 1$.

Theorem 2 implies that

Corollary 1. There is no any complete oriented $f$-minimal hypersurface properly immersed in the cylinder shrinking soliton $(\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}, \bar{g}, f)$, $n \geq 3$, so that its norm $|A|$ of second fundamental form satisfies

\[
\frac{1}{4} \left( 1 - \frac{2}{n-1} \right) \leq |A|^2 \leq \frac{1}{4} \left( 1 + \sqrt{1 - \frac{2}{n-1}} \right).
\]

Remark 1. The cases of closed $f$-minimal hypersurfaces in Theorems 1 and 2 and Corollary 1 were proved in [7] (see [7] Th.1, Th.2, and Corollary 1).

For an immersed $f$-minimal hypersurface in a shrinking gradient Ricci soliton with certain condition on $f$, the properness of its immersion, its polynomial volume growth, and its finite weighted volume are equivalent each other ([6]). This guarantees integrability of some weighted integrals. Furthermore if there are some integrability conditions on $|A|^2$, one may obtain the integral identities, which are derived from Simons’ type equation and other identities (Propositions 7 and 8). As an immediate application of the integral identities, we prove the rigidity of complete totally geodesic $f$-minimal hypersurfaces in $(\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}, \bar{g}, f)$ as follows:

Theorem 3. Let $\Sigma$ be a complete oriented $f$-minimal hypersurface properly immersed in $(\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}, \bar{g}, f)$. Then $\Sigma$ is totally geodesic if and only if it is either $\mathbb{S}^n(\sqrt{2(n-1)}) \times \{0\}$ or $\mathbb{S}^{n-1}(\sqrt{2(n-1)}) \times \mathbb{R}$.
When \( f \)-minimal hypersurfaces has \( L_f \)-index one, the results on the spectrum of the operator \( L_f \) imply the integrability of \( |A|^2 \) (Lemma 1). Hence the adapted integral identity can be used in order to make classification (Theorem 1). Also the pinching condition of \( |A|^2 \) make the use of integral identities possible in the proof of Theorem 2.

We mentioned that for self-shrinkers, the stability properties were studied by Colding-Minicozzi [11] and Hussey [16], and gap phenomenon was discussed, for instance, by Le-Sesum [19] and Cao-Li [3].

The rest of this paper is organized as follows: In Section 2 we give some notations and conventions as preliminaries; In Section 3 we give some spectrum properties of the operator \( \Delta_f + q \) which will be used in Section 5; In Section 4 we prove some integral identities for \( f \)-minimal hypersurfaces and Theorem 3; In Section 5 we study \( L_f \)-index and prove Theorem 1; In Section 6 the proof of Theorem 2 is given.

2. Notations and conventions

Throughout this paper, we will use the same conventions and notations as in [4], unless otherwise specified. For instance, \((M^{n+1}, \overline{g}, e^{-f} d\mu)\) denotes a smooth metric measure space, where \( f \) is a smooth function on \( M \) and \( d\mu \) is the volume element of \( M \) induced by the metric \( \overline{g} \). \((\Sigma, g)\) denotes a hypersurface isometrically immersed in \((M, \overline{g})\) respectively. We denote by a bar all quantities on \((M, \overline{g})\), for instance by \( \overline{\nabla}, \overline{\text{Ric}}, \) and \( \overline{\text{Ric}}_f \) the Levi-Civita connection, the Ricci curvature tensor, and the Bakry-Émery Ricci curvature of \((M, \overline{g})\) respectively. We denote still by \( f \) the restriction of \( f \) on \( \Sigma \). Also we denote for instance by \( \nabla, \text{Ric}, \Delta, \) and \( d\sigma \), the Levi-Civita connection, the Ricci curvature tensor, the Laplacian, and the volume element of \((\Sigma, g)\) respectively. The second fundamental form \( A \) of \( \Sigma \) is defined by

\[
A : T_p \Sigma \times T_p \Sigma \to \mathbb{R}, \quad A(X, Y) = \langle \overline{\nabla}_X \nu, Y \rangle,
\]

where \( p \in \Sigma, X, Y \in T_p \Sigma, \) \( \nu \) is a unit normal vector at \( p \). In a local orthonormal system \( \{e_i\}, i = 1, \ldots, n \) of \( \Sigma \), the components of \( A \) are denoted by \( a_{ij} = A(e_i, e_j) = \langle \overline{\nabla}_{e_i} \nu, e_j \rangle \). The mean curvature \( H \) and the weighted
mean curvature of $\Sigma$ is defined, respectively, by

$$H = \text{tr} A = \sum_{i=1}^{n} a_{ii},$$
$$H_f = H - \langle \nabla f, \nu \rangle.$$

$(\Sigma, g)$ is called $f$-minimal if its mean curvature $H$ satisfies that

$$H = \langle \nabla f, \nu \rangle,$$

for any $p \in \Sigma$, where $\nu$ is the unit normal to $\Sigma$.

The drifted Laplacian $\Delta_f$ and $L_f$ operator on $\Sigma$ are defined by

$$\Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle,$$
$$L_f = \Delta_f + |A|^2 + \text{Ric}_f(\nu, \nu).$$

It holds that for $u \in C^1_0(\Sigma)$, $w \in C^2(\Sigma)$,

$$\int_{\Sigma} u \Delta_f we^{-f} dv = - \int_{\Sigma} \langle \nabla u, \nabla w \rangle e^{-f} dv.$$

Since $\Delta_f$ is self-adjoint in the weighted space $L^2(e^{-f} d\sigma)$, we may define a symmetric bilinear form on the space $C^\infty_0(\Sigma)$ of compactly supported smooth functions on $\Sigma$ by, for $\phi, \psi \in C^\infty_0(\Sigma)$,

$$B_f(\phi, \psi) := - \int_{\Sigma} \phi L_f \psi e^{-f} d\sigma\tag{2}$$
$$= \int_{\Sigma} \left[ \langle \nabla \phi, \nabla \psi \rangle - (|A|^2 + \text{Ric}_f(\nu, \nu)) \phi \psi \right] e^{-f} d\sigma.$$

Now suppose that $(\Sigma, g)$ is an $f$-minimal hypersurface. $\Sigma$ is called $L_f$-stable if $B_f(\phi, \phi) \geq 0$ for all $\phi \in C^\infty_0(\Sigma)$. It is known that $L_f$-stability $\Sigma$ means that the weighted volume $\int_{\Sigma} e^{-f} d\sigma$ of $\Sigma$ is locally minimizing. Furthermore, we may define the $L_f$-index of $\Sigma$ analogous to defining the (Morse) index of the operator $\Delta + q$ (cf [13]). Consider the Dirichlet eigenvalue problems of $L_f$ on a compact domain $\Omega \subset \Sigma$:

$$L_f u + \lambda u = 0, \quad u \in \Omega; \quad u|_{\partial \Omega} = 0.$$

It is known that the $L^2(e^{-f} d\sigma)$-spectrum of the operator $L_f$ on $\Omega$ is discrete and the set of all (Dirichlet) eigenvalues, counted with multiplicity, is an increasing sequence

$$\lambda_1(\Omega) < \lambda_2(\Omega) \leq \cdots$$

with $\lambda_i(\Omega) \to \infty$ as $i \to \infty$. 

The $L_f$-index of $\Omega$, denoted by $L_f\text{-ind}(\Omega)$, is defined as the number of negative Dirichlet eigenvalues of $L_f$ on $\Omega$ counted with multiplicity. The variational characterization of eigenvalues implies that if $\Omega_1 \subset \Omega_2$, then $L_f\text{-ind}(\Omega_1) \leq L_f\text{-ind}(\Omega_2)$. Hence one may define the $L_f$-index of $\Sigma$ as follows.

**Definition 1.** The $L_f$-index of $\Sigma$, denoted by $L_f\text{-ind}(\Sigma)$, is defined to be the supremum over compact domains of $\Sigma$ of $L_f$-index of compact domain, that is, $L_f\text{-ind}(\Sigma) = \sup_{\Omega \subset \subset \Sigma} L_f\text{-ind}(\Omega)$.

By Definition 1 $L_f\text{-ind}(\Sigma) = 0$ if and only if $\Sigma$ is $L_f$-stable.

If $\Sigma$ is an $n$-dimensional hypersurface in a complete manifold $M^{n+1}$, $\Sigma$ is said to have polynomial volume growth if, for a $p \in M$ fixed, there exist constants $C$ and $d$ so that for all $r \geq 1$,

\begin{equation}
\text{Vol}(B_r^M(p) \cap \Sigma) \leq Cr^d,
\end{equation}

where $B_r^M(p)$ is the extrinsic ball of radius $r$ centered at $p$, $\text{Vol}(B_r^M(p) \cap \Sigma)$ denotes the volume of $B_r^M(p) \cap \Sigma$. When $d = n$ in (3), $\Sigma$ is said to be of Euclidean volume growth.

Now we recall the definition and some related facts of gradient Ricci solitons (cf Chapter 4 of [4], or [2]). A quadruple $(M, \overline{g}, f, \rho)$ is called a gradient Ricci soliton if it satisfies that

\[ \overline{\text{Ric}} + \nabla^2 f = \frac{\rho}{2} \overline{g}, \]

where $\rho$ is a constant. If $\rho = 0$, $\rho > 0$, or $\rho < 0$, the gradient Ricci soliton $(M, \overline{g}, f, \rho)$ is called steady, shrinking or expanding respectively.

It is known that a gradient Ricci soliton $(M, \overline{g}, f, \rho)$ has an associated time-dependent version, i.e., a gradient Ricci soliton solution to the Ricci flow. By abuse of notations, a gradient Ricci soliton solution to the Ricci flow is often called gradient Ricci soliton if there is no confusion.

Cylinder shrinking solitons of type $\mathbb{S}^n \times \mathbb{R}$, $n \geq 2$: Let $\mathbb{S}^n(\sqrt{2(n-1)})$ denote the round sphere in the Euclidean space $\mathbb{R}^{n+1}$ of radius $\sqrt{2(n-1)}$ centered at the origin. Consider the triple $(\mathbb{S}^n(\sqrt{2(n-1)}), \mathbb{R}, g(t), f(t))$, $t \in (-\infty, 0)$, with the metric

\[ \overline{g}(t) = 2(n-1)|t|ds^2_n + dh^2, \]
and
\[
f(t)(x,h) = \frac{h^2}{4|t|}, \quad (x,h) \in S^n(\sqrt{2(n-1)}) \times \mathbb{R}.
\]
where \(ds_n^2\) and \(dh^2\) denote the canonical metrics on the unit round sphere in \(\mathbb{R}^{n+1}\) and \(\mathbb{R}\) respectively. One may verify that
\[
\overline{Ric}_f(t) = \overline{Ric} + \nabla^2(f(t)) = \frac{1}{2|t|}\overline{g}(t).
\]
Hence for each \(t\), \((S^n(\sqrt{2(n-1)}) \times \mathbb{R}, \overline{g}(t), f(t))\) is a gradient shrinking Ricci soliton and they are the examples of cylinder shrinking solitons of type \(S^n \times \mathbb{R}\). Moreover, it is known that \((S^n(\sqrt{2(n-1)}) \times \mathbb{R}, \overline{g}(t), f(t))\), \(t \in (-\infty, 0)\), is the gradient shrinking Ricci soliton solution to the Ricci flow \(\frac{\partial}{\partial t}g = -2\overline{Ric}\).

In this paper, we denote by \((S^n(\sqrt{2(n-1)}) \times \mathbb{R}, \overline{g}, f)\) the cylinder shrinking soliton \((S^n(\sqrt{2(n-1)}) \times \mathbb{R}, \overline{g}(-1), f(-1))\). To study the \(f\)-minimal hypersurfaces in a cylinder shrinking soliton of type \(S^n \times \mathbb{R}\), it suffices to consider the ambient is \((S^n(\sqrt{2(n-1)}) \times \mathbb{R}, \overline{g}, f)\).

We also need the following result:

**Proposition 1.** ([6], Corollary 1) Let \((M^m, \overline{g}, f)\) be a complete shrinking gradient Ricci soliton with \(\overline{Ric}_f = \frac{1}{2}\overline{g}\). Assume that \(f\) is a convex function, i.e., \(\nabla^2 f \geq 0\), and satisfies \(|\nabla f|^2 \leq f\). If \(\Sigma\) is an immersed complete \(f\)-minimal submanifold in \(M\), then for \(\Sigma\) the properness of immersion, polynomial volume growth, and finite weighted volume are equivalent.

Throughout this paper, we assume that \(f\)-minimal hypersurfaces are orientable.

### 3. \(L_f\)-index of a weighted manifold

In this section, suppose that \((\Sigma, g, e^{-f}d\sigma)\) is a complete smooth metric measure space. Here \(\Sigma\) is not necessarily a hypersurface. Let \(q(x)\) be a continuous function on \(\Sigma\) and consider the operator \(L_f = \Delta_f + q(x)\) on \(\Sigma\). Recall that the bottom of the spectrum of the operator \(L_f\), denoted by
$\mu_1(L_f)$, satisfies that

$$\mu_1(L_f) = \inf_{u \neq 0, u \in C_0^\infty(\Sigma)} \frac{\int_\Sigma (|\nabla u|^2 - qu^2)e^{-f} d\sigma}{\int_\Sigma u^2 e^{-f} d\sigma}.$$ 

Just substitute $q(x)$ for $|A|^2 + \text{Ric}_f(\nu, \nu)$ in Section 2, one can similarly define the $L_f$-index of $\Sigma$, denoted by $L_f\text{-ind}(\Sigma)$, to be the supremum over compact domains of $\Sigma$ of the number of negative (Dirichlet) eigenvalues of $L_f$. In [15], Fischer-Colbrie showed that if a complete minimal hypersurface has finite index, then it is stable outside of a compact set and there is a positive function $u$ on it so that $Ju = (\Delta + |A|^2 + \text{Ric}(\nu, \nu))u = 0$ outside of the compact set. Moreover, she proved an equivalent statement of finite index of a complete minimal hypersurface ([15], Proposition 2) and also proved that if the index of a complete minimal hypersurface is finite, then the bottom of the spectrum of stability operator $J$ is finite (see (4) in [15]).

An analogous argument gives the following weighted version for $L_f$:

**Proposition 2.** The following are equivalent:

(i) $\Sigma$ has finite $L_f$-index;

(ii) There exists a finite dimensional subspace $W$ of the weighted space $L^2(e^{-f} d\sigma)$ having an orthonormal basis $\psi_1, \ldots, \psi_k$ consisting of eigenfunctions with eigenvalues $\lambda_1, \ldots, \lambda_k$ respectively. Each $\lambda_i$ is negative and for $\phi \in C_0^\infty(\Sigma) \cap W^\perp$, $Q(\phi, \phi) \geq 0$.

Moreover if the $L_f\text{-ind}(\Sigma) < \infty$, then $L_f\text{-ind}(\Sigma) = \dim W$ and the bottom $\mu_1$ of the spectrum of $L_f$ is finite. Furthermore if $1 \leq L_f\text{-ind}(\Sigma) < \infty$, $\mu_1$ is the least negative $L^2(e^{-f} d\sigma)$ eigenvalue.

Denote by $W^{1,2}(e^{-f} d\sigma)$ the weighted Sobolev space which is the set of the functions $u$ on $\Sigma$ satisfying that $\int_\Sigma (u^2 + |\nabla u|^2)e^{-f} d\sigma < \infty$. $W^{1,2}(e^{-f} d\sigma)$ has the norm:

$$\|u\|_{W^{1,2}(e^{-f} d\sigma)} := \left(\int_\Sigma (u^2 + |\nabla u|^2)e^{-f} d\sigma\right)^{\frac{1}{2}}.$$ 

Through a similar proof, we can extend Lemmas 9.15 and 9.25 in [11] for the operator $L = \Delta - \frac{1}{2}(x, \nabla \cdot) + |A|^2 + \frac{1}{2}$ on hypersurfaces in $\mathbb{R}^{n+1}$ to the operator $L_f$. 

Proposition 3. If $\mu_1(L_f) \neq -\infty$, then there exists a positive $C^2$ function $u$ on $\Sigma$ with $L_fu = -\mu_1(L_f)u$. Moreover, if $w$ is in the weighted $W^{1,2}(e^{-f}d\sigma)$ space and $L_fw = -\mu_1(L_f)w$, then $w = Cu$ for some $C \in \mathbb{R}$.

Proposition 4. Suppose that $h$ is a $C^2$ function with $L_fh = -\mu h$, $\mu \in \mathbb{R}$. If $h > 0$ and $\phi$ is in $W^{1,2}(e^{-f}d\sigma)$, then

$$\int_{\Sigma} \phi^2 (2q + |\log h|^2) e^{-f}d\sigma \leq \int_{\Sigma} (4|\nabla \phi|^2 - 2\mu \phi^2) e^{-f}d\sigma.$$  

4. Integral identities of complete $f$-minimal hypersurfaces

From this section, we start to consider complete $f$-minimal hypersurfaces immersed in the cylinder shrinking soliton $(\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}, \overline{g}, f)$, $n \geq 2$ with $f(x, h) = \frac{h^2}{4}$, $(x, h) \in \mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}$ and the metric

$$\overline{g} = g_{\mathbb{S}^n(\sqrt{2(n-1)})} + d\xi^2.$$  

$(\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}, \overline{g}, f)$ associates to a smooth metric measure space $(\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}, \overline{g}, e^{-f}d\mu)$. It is known (cf [2]) that

$$\nabla f = \frac{h}{2} \frac{\partial}{\partial h}, \quad \nabla^2 f = \frac{1}{2} \frac{\partial}{\partial h} \otimes \frac{\partial}{\partial h}, \quad \text{Ric} = \frac{1}{2} \overline{g} - \frac{1}{2} \frac{\partial}{\partial h} \otimes \frac{\partial}{\partial h}, \quad \text{Ric}_f = \frac{1}{2} \overline{g}.$$  

(5)

For an $f$-minimal hypersurface $\Sigma$ immersed in $\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}$,

$$0 = H_f = H - \frac{h}{2} \left( \frac{\partial}{\partial h}, \nu \right) = H - \frac{h}{2} \alpha,$$

where $\alpha = \left( \frac{\partial}{\partial h}, \nu \right)$. Hence $\Sigma$ satisfies

$$H = \frac{h\alpha}{2}.$$  

(6)

Under a local orthonormal frame $\{e_i\}_{i=1}^n$ on $\Sigma$,

$$\nabla_{e_i} \alpha = \left( \nabla_{e_i} \nu, \frac{\partial}{\partial h} \right) = \sum_{j=1}^n a_{ij} \langle e_j, \frac{\partial}{\partial h} \rangle.$$

(7) $|\nabla \alpha|^2 = \sum_{i=1}^n |\nabla_{e_i} \alpha|^2 \leq \sum_{i=1}^n \left( \sum_{j=1}^n a_{ij}^2 \right) \left( \sum_{j=1}^n \langle e_j, \frac{\partial}{\partial h} \rangle^2 \right) \leq |A|^2.$
The operator $L_f$ on $\Sigma$ is equal to
\begin{equation}
L_f = \Delta - \frac{h}{2} \left( \frac{\partial}{\partial h} \right)^T, \nabla \right) + |A|^2 + \frac{1}{2}.
\end{equation}
Now we give some examples of $f$-minimal hypersurfaces in $(\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}, g, f)$.

**Example 1.** (4) Lemma 1) The slice $\mathbb{S}^n(\sqrt{2(n-1)}) \times \{0\}$ is $f$-minimal and totally geodesic. Furthermore a complete immersed $f$-minimal hypersurface is in a horizontal slice $\mathbb{S}^n(\sqrt{2(n-1)}) \times \{h\}, h \in \mathbb{R}$ fixed, if and only if it is $\mathbb{S}^n(\sqrt{2(n-1)}) \times \{0\}$.

**Example 2.** Assume that $\Sigma_1$ is an immersed hypersurface in $\mathbb{S}^n(\sqrt{2(n-1)})$. Then the product $\Sigma = \Sigma_1 \times \mathbb{R}$ is $f$-minimal in $(\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}, \bar{g}, f)$ if and only if $\Sigma_1$ is minimal in $\mathbb{S}^n(\sqrt{2(n-1)})$. Particularly, the totally geodesic hypersurface $\mathbb{S}^{n-1}(\sqrt{2(n-1)}) \times \mathbb{R}$ is $f$-minimal in $\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}$, where the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}(\sqrt{2(n-1)}) \subset \mathbb{S}^n(\sqrt{2(n-1)})$.

The reason is that: the unit normal $\nu$ of $\Sigma$ is also the unit normal of $\Sigma_1$ and $\nu \in TS^n(\sqrt{2(n-1)})$. Then $\langle \nabla f, \nu \rangle = 0$ and $\nabla_{\partial \nu} \nu = 0$. Hence $H_f(x, h) = H(x, h) = H_{\Sigma_1}(x), (x, h) \in \Sigma_1 \times \mathbb{R}$. This implies that $H_f = 0$ on $\Sigma$ if and only if $\Sigma_1$ is minimal in $\mathbb{S}^n(\sqrt{2(n-1)})$.

Note that $\nabla^2 f \geq 0$ and $|\nabla f|^2 = f = \frac{h^2}{4}$. Applying Proposition 4 we get that

**Proposition 5.** For any complete immersed $f$-minimal hypersurface $\Sigma$ in $(\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}, \bar{g}, f)$, the properness of immersion, polynomial volume growth, and finite weighted volume are equivalent.

**Remark 2.** Proposition 5 will be used frequently in the proofs of this paper without mentioned. For self-shrinkers in $\mathbb{R}^{n+1}$, the statement that properly immersed self-shrinkers in $\mathbb{R}^{n+1}$ implies the Euclidean volume growth and finite weighted volume was proved in [13]. The equivalence of properness of immersion, finite weighted volume, polynomial volume growth was proved in [8].

A Simons’ type equation and the other identities for $f$-minimal hypersurfaces immersed in $(\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}, \bar{g}, f)$ were derived in [7] as follows:
Proposition 6. Let $\Sigma$ be an $f$-minimal hypersurface immersed in $(\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}, g, f)$. Then

\begin{align*}
(9) \quad L_f \alpha &= \frac{1}{2} \alpha, \\
(10) \quad \frac{1}{2} \Delta_f \alpha^2 &= |\nabla \alpha|^2 - |A|^2 \alpha^2, \\
(11) \quad \frac{1}{2} \Delta_f H^2 &= |\nabla H|^2 - (|A|^2 + \frac{1}{2}) H^2 + \frac{1}{2} \langle \nabla \alpha^2, \nabla f \rangle, \\
(12) \quad \frac{1}{2} \Delta_f |A|^2 &= |\nabla A|^2 + |A|^2 (\frac{1}{2} - |A|^2) - \frac{1}{n-1} (|\nabla \alpha|^2 - \alpha^2 |A|^2) \\
&\quad - \frac{1}{n-1} (\alpha^2 f - \langle \nabla \alpha^2, \nabla f \rangle).
\end{align*}

Now we prove the identities in Proposition 6 imply some integral identities for complete $f$-minimal hypersurfaces (the case of closed $f$-minimal hypersurfaces was proved in [7]). We denote by $p \in \Sigma$ and $B_j$ a fixed point and the geodesic sphere of $\Sigma$ of radius $j$ centered at $p$, respectively. Let $\varphi_j$ be the nonnegative cut-off functions satisfying that $\varphi_j$ is 1 on $B_j$, $|\nabla \varphi_j| \leq 2$ on $B_{j+1} \setminus B_j$, and $\varphi_j = 0$ on $\Sigma \setminus B_{j+1}$.

Proposition 7. Let $\Sigma$ be a complete oriented $f$-minimal hypersurface properly immersed in $(\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}, g, f)$. Assume that $\int_\Sigma |A|^2 e^{-f} < \infty$. Then

\begin{align*}
(13) \quad \int_\Sigma |A|^2 e^{-f} d\sigma &= 0, \\
(14) \quad \int_\Sigma |\nabla \alpha|^2 e^{-f} d\sigma - \int_\Sigma \alpha^2 |A|^2 e^{-f} d\sigma &= 0.
\end{align*}

Proof. Note $L_f = \Delta_f + |A|^2 + \frac{1}{2}$. (9) implies that

\begin{align*}
(15) \quad \Delta_f \alpha + |A|^2 \alpha &= 0.
\end{align*}

If $\Sigma$ is closed, we get (13) and (14) by integrating (15) and (10), and then using the Stokes’ formula respectively. Consider the case of non-compact $\Sigma$. Since $|\alpha| \leq 1$ and $|\nabla \alpha|^2 \leq |A|^2$, we have

\begin{align*}
\int_\Sigma |\nabla \alpha|^2 e^{-f} < \infty, \quad \int_\Sigma \alpha^2 |A|^2 e^{-f} < \infty.
\end{align*}
Here and thereafter, for simplicity of notations, we omit $d\sigma$ in the computation of integrals. For any positive integer $k$,

$$\left| \int_{\Sigma} \varphi_j(\Delta_f \alpha^k)e^{-f} \right| = k \int_{\Sigma} \alpha^{k-1} \langle \nabla \varphi_j, \nabla \alpha \rangle e^{-f} \leq k \left( \int_{\Sigma} |\varphi_j|^2 e^{-f} \right)^{\frac{1}{2}} \left( \int_{\Sigma} |\nabla \alpha|^2 e^{-f} \right)^{\frac{1}{2}} \leq 2k \left( \int_{B_{j+1} \setminus B_j} e^{-f} \right)^{ \frac{1}{2}} \left( \int_{\Sigma} |\nabla \alpha|^2 e^{-f} \right)^{\frac{1}{2}}$$

(16)

Since the right-hand side of (16) tends to zero as $j \to \infty$,

$$\lim_{j \to \infty} \int_{\Sigma} \varphi_j(\Delta_f \alpha^k)e^{-f} = 0.$$  

Multiplying (15) by the cut-off functions $\varphi_j$, we have

$$\int_{\Sigma} \varphi_j(\Delta_f \alpha)e^{-f} + \int_{\Sigma} \varphi_j |A|^2 \alpha e^{-f} = 0.$$  

(18)

Letting $j \to \infty$ in (18), by (17) for $k = 1$ and the monotone convergence theorem, we get the identity

$$\int_{\Sigma} |A|^2 \alpha e^{-f} = 0,$$

that is (13).

Now we prove (14). Multiplying (10) by $\varphi_j$ and integrating, we have

$$\frac{1}{2} \int_{\Sigma} \varphi_j(\Delta_f \alpha^2)e^{-f} = \int_{\Sigma} \varphi_j |\nabla \alpha|^2 - \int_{\Sigma} \varphi_j |A|^2 \alpha^2 e^{-f}.$$  

(19)

Letting $j \to \infty$ on both sides of (19), by (17) for $k = 2$ and the monotone convergence theorem, we get (14).

□

**Proposition 8.** Let $\Sigma$ be a complete oriented $f$-minimal hypersurface properly immersed in $(S^n(\sqrt{2(n-1)}) \times \mathbb{R}, \bar{g}, f)$. Assume that $\int_{\Sigma} |A|^4 e^{-f} < \infty$. Then

$$- \int_{\Sigma} |\nabla H|^2 e^{-f} + \int_{\Sigma} H^2 |A|^2 e^{-f} + \frac{1}{4} \int_{\Sigma} \alpha^2 (1 - \alpha^2) e^{-f} = 0,$$

(20)

$$\int_{\Sigma} |\nabla A|^2 e^{-f} + \int_{\Sigma} |A|^2 (\frac{1}{2} - |A|^2) e^{-f} - \frac{1}{2(n-1)} \int_{\Sigma} \alpha^2 (1 - \alpha^2) e^{-f} = 0.$$  

(21)

**Proof.** Suppose that $\Sigma$ is complete noncompact (the compact case was considered in [7]). First we prove (20). By $|\alpha| \leq 1$ and finite weighted volume of $\Sigma$,

$$\int_{\Sigma} \alpha^2 (1 - \alpha^2) e^{-f} < \infty.$$
Note that
\[
\int_{\Sigma} |A|^2 e^{-f} \leq \left( \int_{\Sigma} |A|^4 e^{-f} \right)^{\frac{1}{2}} \left( \int_{\Sigma} e^{-f} \right)^{\frac{1}{2}} < \infty.
\]
By \( H^2 \leq n|A|^2 \) and the assumptions of the proposition,
\[
\int_{\Sigma} H^2 e^{-f} < \infty, \quad \int_{\Sigma} |A|^2 H^2 e^{-f} < \infty.
\]
Under a local orthonormal frame \( \{e_i\}_{i=1}^n \) on \( \Sigma \),
\[
\nabla_{e_i} H = \langle \nabla_{e_i} \nabla f, \nu \rangle + \langle \nabla f, a_{ij} e_j \rangle
\]
\[
= \frac{1}{2} \left( e_i, \frac{\partial}{\partial h} \right) \langle \nu, \frac{\partial}{\partial h} \rangle + \frac{a_{ij} h^2}{2} \langle e_j, \frac{\partial}{\partial h} \rangle.
\]
Then
\[
|\nabla H|^2 = \sum_{i=1}^n |\nabla_{e_i} H|^2
\]
\[
\leq \frac{1}{2} \sum_{i=1}^n \left( e_i, \frac{\partial}{\partial h} \right)^2 \langle \nu, \frac{\partial}{\partial h} \rangle^2 + \sum_{i,j=1}^n \frac{a_{ij}^2 h^2}{2} \langle \nabla_{e_i} H, e_j \rangle^2
\]
\[
\leq \frac{1}{2} \alpha^2 (1 - \alpha^2) + \frac{h^2}{2} |A|^2
\]
\[
\leq \frac{1}{2} \alpha^2 (1 - \alpha^2) + \frac{1}{4} (h^4 + |A|^4).
\]
Since \( \Sigma \) has polynomial volume growth, for any positive integer \( k \),
\[
\int_{\Sigma} h^k e^{-f} = \int_{\Sigma \cap B_{n_0}^M (p)} h^k e^{-\frac{k^2}{4}} + \sum_{i=0}^{\infty} \int_{\Sigma \cap (B_{r_0 + i+1}^M(p) \setminus B_{r_0 + i}^M(p))} h^k e^{-\frac{k^2}{4}}
\]
\[
\leq C_1 \text{Vol}(\Sigma \cap B_{r_0}^M (p)) + C \sum_{i=0}^{\infty} (r_0 + i + 1)^k e^{-\frac{k^2}{4}} (r_0 + i + 1)^{d+k} \text{Vol}(\Sigma \cap B_{r_0 + i+1}^M(p))
\]
\[
\leq C \left[ r_0^d + \sum_{i=0}^{\infty} e^{-\frac{k^2}{4}} (r_0 + i + 1)^{d+k} \right] < \infty.
\]
\[\eqref{eq1}, \eqref{eq2}\] and the assumption of the proposition imply that
\[
\int_{\Sigma} |\nabla H|^2 e^{-f} < \infty.
\]
Multiplying (11) by $\varphi_j^2$, we have
\[
\frac{1}{2} \int_{\Sigma} \varphi_j^2 (\Delta_f H^2) e^{-f} = \int_{\Sigma} \varphi_j^2 \nabla H^2 e^{-f} - \int_{\Sigma} \varphi_j^2 |A|^2 H^2 e^{-f} - \frac{1}{2} \int_{\Sigma} \varphi_j^2 H^2 e^{-f} + \frac{1}{2} \int_{\Sigma} \varphi_j^2 (\nabla \alpha^2, \nabla f) e^{-f}.
\]
(24)

Note
\[
\left| \int_{\Sigma} \varphi_j^2 (\Delta_f H^2) e^{-f} \right| = 4 \left| \int_{\Sigma} H \varphi_j (\nabla \varphi_j, \nabla H) e^{-f} \right|
\]
\[
\leq 4 \left( \int_{\Sigma} |\nabla \varphi_j|^2 H^2 e^{-f} \right)^{\frac{1}{2}} \left( \int_{\Sigma} \varphi_j^2 |\nabla H|^2 e^{-f} \right)^{\frac{1}{2}}
\]
\[
\leq 4 \left( \int_{B_{j+1} \setminus B_j} H^2 e^{-f} \right)^{\frac{1}{2}} \left( \int_{\Sigma} |\nabla H|^2 e^{-f} \right)^{\frac{1}{2}}.
\]
(25)

Since the right-hand side of (25) tends to zero as $j \to \infty$,
\[
\lim_{j \to \infty} \int_{\Sigma} \varphi_j^2 (\Delta_f H^2) e^{-f} = 0.
\]
(26)

Observe that
\[
\Delta_f f = \sum_{i=1}^n ((\nabla^2 f)_{ii} - a_{ii} f) - (\nabla f, \nabla f)
\]
\[
= \frac{1}{2} \sum_{i=1}^n (e_i, \frac{\partial}{\partial h})^2 - H f - |\nabla f|^2
\]
\[
= \frac{1}{2} (1 - \alpha^2) - |\nabla f|^2
\]
\[
= \frac{1}{2} (1 - \alpha^2) - f.
\]
(27)

By Stokes’ formula, (27) and $H^2 = \left( \frac{\hbar \alpha}{2} \right)^2 = \alpha^2 f$, we have
\[
\frac{1}{2} \int_{\Sigma} \varphi_j^2 (\nabla \alpha^2, \nabla f) e^{-f}
\]
\[
= -\frac{1}{2} \int_{\Sigma} \varphi_j^2 \alpha^2 (\Delta_f f) e^{-f} - \frac{1}{2} \int_{\Sigma} \alpha^2 (\nabla \varphi_j^2, \nabla f) e^{-f}
\]
\[
= -\frac{1}{4} \int_{\Sigma} \varphi_j^2 \alpha^2 (1 - \alpha^2) e^{-f} + \frac{1}{2} \int_{\Sigma} \varphi_j^2 H^2 e^{-f}
\]
\[
- \int_{\Sigma} \varphi_j \alpha^2 (\nabla \varphi_j, \nabla f) e^{-f}.
\]
(28)
By $|\nabla f| \leq |\nabla f| \leq \frac{h}{2}$,
\[
\left( \int_{\Sigma} \varphi_j \alpha^2 \langle \nabla \varphi_j, \nabla f \rangle e^{-f} \right)^2 \leq \left( \int_{\Sigma} |\nabla \varphi_j|^2 e^{-f} \right) \left( \int_{\Sigma} \varphi_j^2 |\nabla f|^2 e^{-f} \right) \\
\leq \left( \int_{B_{j+1} \setminus B_j} \right) \left( \int_{\Sigma} h^2 e^{-f} \right).
\]

This implies that
\[
\lim_{j \to \infty} \int_{\Sigma} \varphi_j \alpha^2 \langle \nabla \varphi_j, \nabla f \rangle e^{-f} = 0.
\]

Letting $j \to \infty$ on the both side of (28), by (29) and the monotone convergence theorem, we have
\[
\lim_{j \to \infty} \frac{1}{2} \int_{\Sigma} \varphi_j^2 \langle \nabla \alpha^2, \nabla f \rangle e^{-f} = -\frac{1}{4} \int_{\Sigma} \alpha^2 (1 - \alpha^2) e^{-f} + \frac{1}{2} \int_{\Sigma} H^2 e^{-f}.
\]

Letting $j \to \infty$ on the both side of (24) and using (26), (30) and the monotone convergence theorem, we get (20), that is
\[
- \int_{\Sigma} |\nabla H|^2 e^{-f} + \int_{\Sigma} H^2 |A|^2 e^{-f} + \frac{1}{4} \int_{\Sigma} \alpha^2 (1 - \alpha^2) e^{-f} = 0.
\]

Now we prove (21). Multiplying (12) by $\varphi_j^2$, we have
\[
\frac{1}{2} \int_{\Sigma} \varphi_j^2 \Delta f |A|^2 e^{-f} = \int_{\Sigma} \varphi_j^2 |\nabla A|^2 e^{-f} + \frac{1}{2} \int_{\Sigma} \varphi_j^2 |A|^2 e^{-f} - \int_{\Sigma} \varphi_j^2 |A|^4 e^{-f} - \frac{1}{n - 1} \int_{\Sigma} \varphi_j^2 |\nabla \alpha|^2 e^{-f} + \frac{1}{n - 1} \int_{\Sigma} \varphi_j^2 \alpha^2 |A|^2 e^{-f} - \frac{1}{n - 1} \int_{\Sigma} \varphi_j^2 (\nabla \alpha^2, \nabla f) e^{-f}.
\]

Using $|\nabla A| \leq |\nabla A|$,
\[
\frac{1}{2} \int_{\Sigma} \varphi_j^2 \Delta f |A|^2 e^{-f} = -2 \int_{\Sigma} \langle \varphi_j \nabla \varphi_j, |A| \nabla |A| \rangle e^{-f} \\
\leq \epsilon \int_{\Sigma} \varphi_j^2 |\nabla A|^2 e^{-f} + \frac{1}{\epsilon} \int_{\Sigma} |A|^2 |\nabla \varphi_j|^2 e^{-f} \\
\leq \epsilon \int_{\Sigma} \varphi_j^2 |\nabla A|^2 e^{-f} + \frac{1}{\epsilon} \int_{\Sigma} |A|^2 |\nabla \varphi_j|^2 e^{-f}.
\]
Substitute (32) into (31). We have

\[
(1 - \epsilon) \int_{\Sigma} \varphi_j^2 |\nabla A|^2 e^{-f} \\
\leq \int_{\Sigma} \varphi_j^2 |A|^4 e^{-f} + \frac{1}{\epsilon} \int_{\Sigma} |A|^2 |\nabla \varphi_j|^2 e^{-f} \\
+ \frac{1}{n-1} \int_{\Sigma} \varphi_j^2 |\nabla \alpha|^2 e^{-f} - \frac{1}{n-1} \int_{\Sigma} \varphi_j^2 \alpha^2 |A|^2 e^{-f} \\
+ \frac{1}{n-1} \int_{\Sigma} \varphi_j^2 \alpha^2 f e^{-f} - \frac{1}{n-1} \int_{\Sigma} \varphi_j^2 (\nabla \alpha^2, \nabla f) e^{-f}.
\]  

(33)

Observe that \( \int_{\Sigma} \alpha^2 f e^{-f} = \int_{\Sigma} H^2 e^{-f} < \infty \) and all terms on the right-hand side of (33) converge as \( j \to \infty \). By the monotone convergence theorem,

\[
\int_{\Sigma} |\nabla A|^2 = \lim_{j \to \infty} \int_{\Sigma} \varphi_j^2 |\nabla A|^2 < \infty.
\]

Furthermore,

\[
\left| \frac{1}{2} \int_{\Sigma} \varphi_j^2 (\Delta f |A|^2) e^{-f} \right| = 2 \left| \int_{\Sigma} |A| \varphi_j (\nabla \varphi_j, \nabla |A|) e^{-f} \right| \\
\leq 2 \left( \int_{\Sigma} |\nabla \varphi_j|^2 |A|^2 e^{-f} \right)^{\frac{1}{2}} \left( \int_{\Sigma} \varphi_j^2 |\nabla |A||^2 e^{-f} \right)^{\frac{1}{2}} \\
\leq 4 \left( \int_{B_{j+1} \setminus B_j} |A|^2 e^{-f} \right)^{\frac{1}{2}} \left( \int_{\Sigma} |\nabla A|^2 e^{-f} \right)^{\frac{1}{2}}.
\]  

(34)

Since the right-hand side of (34) tends to zero as \( j \to \infty \),

\[
\lim_{j \to \infty} \int_{\Sigma} \varphi_j^2 (\Delta f |A|^2) e^{-f} = 0.
\]  

(35)

Letting \( j \to \infty \) on both sides of (31) and using the monotone convergence theorem, (35), (14), (30) and \( H^2 = \alpha^2 f \), we get (21).

\[\square\]

The integral identities in Proposition 8 can be used to classify complete totally geodesic \( f \)-minimal hypersurfaces.

**Proof of Theorem 3**. Examples 4 and 2 say that \( f \)-minimal hypersurfaces \( S^n(\sqrt{2(n-1)}) \times \{0\} \) and \( S^{n-1}(\sqrt{2(n-1)}) \times \mathbb{R} \) are totally geodesic. Now we prove the inverse. Since \( A \equiv 0 \) on \( \Sigma \), by (20), we have

\[
\int_{\Sigma} \alpha^2 (1 - \alpha^2) e^{-f} = 0.
\]

Note \( |\alpha| \leq 1 \). \( \alpha^2 (1 - \alpha^2) \equiv 0 \). Hence either \( \alpha \equiv 0 \) or \( \alpha^2 \equiv 1 \) on \( \Sigma \).
(i) the case of $\alpha^2 \equiv 1$. Without lost of generality, suppose $\alpha \equiv 1$. This means that $\nu = \frac{\partial}{\partial h}$ on $\Sigma$. Hence $\Sigma$ must be in a horizontal slice $S^n(\sqrt{2(n-1)}) \times \{h\}$. By Example 1, $\Sigma$ must be $S^n(\sqrt{2(n-1)}) \times \{0\}$.

(ii) the case of $\alpha \equiv 0$. In this case, $\frac{\partial}{\partial h} \in T_p \Sigma$ for any $p \in \Sigma$. This implies that any vertical line $\{x\} \times \mathbb{R}$ passing through $\Sigma$ must be a curve in $\Sigma$. Thus $\Sigma = \Sigma_1 \times \mathbb{R}$, where $\Sigma_1 \subset S^n(\sqrt{2(n-1)})$. Let $\pi : S^n(\sqrt{2(n-1)}) \times \mathbb{R} \to S^n(\sqrt{2(n-1)})$ denote the projection onto the first factor $S^n(\sqrt{2(n-1)})$. Since the rank of the differential $d\pi|_{\Sigma}$ is $n-1$, $\Sigma_1$ is an $(n-1)$-dimensional hypersurface in $S^n(\sqrt{2(n-1)})$. Like Example 2, the unit normal $\nu$ of $\Sigma$ is the unit normal of $\Sigma_1$. $\Sigma$ is $f$-minimal in $S^n(\sqrt{2(n-1)}) \times \mathbb{R}$ if and only if $\Sigma_1$ is minimal in $S^n(\sqrt{2(n-1)})$. In addition, $\Sigma$ is properly immersed if and only if $\Sigma_1$ is closed. Furthermore, since $\nabla f \nu = 0$, $A(x,h) = A_{\Sigma_1}(x), (x,h) \in \Sigma_1 \times \mathbb{R}$. Hence $\Sigma_1$ is totally geodesic in $S^n(\sqrt{2(n-1)})$. Therefore $\Sigma_1$ is $S^{n-1}(\sqrt{2(n-1)})$ and thus $\Sigma$ is $S^{n-1}(\sqrt{2(n-1)}) \times \mathbb{R}$.

□

5. $L_f$-index of complete $f$-minimal hypersurfaces

Our purpose of this section is to prove Theorem 1. First we prove that $S^{n-1}(\sqrt{2(n-1)}) \times \mathbb{R} \subset (S^n(\sqrt{2(n-1)}) \times \mathbb{R}, \bar{g}, f)$ has $L_f$-index one.

**Proposition 9.** The $f$-minimal hypersurface $S^{n-1}(\sqrt{2(n-1)}) \times \mathbb{R}$ in the cylinder shrinking soliton $(S^n(\sqrt{2(n-1)}) \times \mathbb{R}, \bar{g}, f)$ satisfies that $L_f$-ind $(S^{n-1}(\sqrt{2(n-1)}) \times \mathbb{R}) = 1$.

**Proof.** For $\Sigma = S^{n-1}(\sqrt{2(n-1)}) \times \mathbb{R}$, the normal $\nu \in TS^n(\sqrt{2(n-1)})$, $\nabla f = (\nabla f)^\top = \frac{h}{2} \frac{\partial}{\partial h}$, and $|A|^2 = 0$. Hence,

$$L_f = \Delta - \frac{h}{2} \left( \frac{\partial}{\partial h}, \nabla \cdot \right) + \frac{1}{2}.$$  

(36)

Let $\psi(x), x \in S^{n-1}(\sqrt{2(n-1)})$ and $\rho(h), h \in \mathbb{R}$ satisfy the following eigenvalue problems respectively:

$$\Delta_{S^{n-1}(\sqrt{2(n-1)})} \psi(x) = -\lambda \psi(x),$$  

(37)

$$\frac{d^2 \rho}{dh^2}(h) - \frac{h}{2 \frac{dp}{dh}}(h) = -\eta \rho(h).$$  

(38)
Let \( \{ e_i \}, i = 1, \cdots, n-1 \), denotes the orthonormal frame on \( \mathbb{S}^{n-1}(\sqrt{2(n-1)}) \).

Then

\[
L_f \psi(x) \rho(h) = \sum_{i=1}^{n-1} \nabla_i \nabla_i (\psi(x) \rho(h)) + \nabla \frac{\partial}{\partial h} \nabla \frac{\partial}{\partial h} (\psi(x) \rho(h)) \\
- \frac{h}{2} \frac{\partial}{\partial h} (\nabla(\psi(x) \rho(h))) + \frac{1}{2} \psi(x) \rho(h) \\
= \left( \Delta_{\mathbb{S}^{n-1}(\sqrt{2(n-1)})} \psi(x) \right) \rho(h) + \frac{1}{2} \psi(x) \rho(h) \\
+ \psi(x) \left( \frac{d^2 \rho}{dh^2}(h) - \frac{h d \rho}{2 dh}(h) \right) \\
= (-\lambda - \eta + \frac{1}{2}) \psi(x) \rho(h). 
\]

(39)

It is known that the eigenvalues of the Laplacian \( \Delta_{\mathbb{S}^{n-1}(\sqrt{2(n-1)})} \), counted with multiplicity, are

\[
\{ \lambda_k : k = 0, 1, 2, \ldots \} = \{ 0, \frac{1}{2}, \frac{3}{2}, \frac{n}{n-1}, \ldots \}
\]

and there exists a complete orthonormal system for space \( L^2(d\sigma_{\mathbb{S}^{n-1}(\sqrt{2(n-1)})}) \) consisting of eigenfunctions \( \psi_k \) of \( \Delta_{\mathbb{S}^{n-1}(\sqrt{2(n-1)})} \) associated to \( \lambda_k \). On the other hand, for the operator \( \frac{d^2}{dh^2} - \frac{h}{2} \frac{d}{dh}, h \in \mathbb{R} \), it is known that its spectrum on \( L^2(\mathbb{R}, e^{-h^2/4}dh) \) is discrete, the eigenvalues of the operator \( \frac{d^2}{dh^2} - \frac{h}{2} \frac{d}{dh}, h \in \mathbb{R} \), counted by multiplicity, are

\[
\{ \eta_l : l = 0, 1, 2, \ldots \} = \{ 0, \frac{1}{2}, 1, \ldots \},
\]

and the so-called Hermite polynomials \( \rho_l(h) \) are orthonormal eigenfunctions associated to \( \eta_l \), which form a complete orthonormal system for the weighted space \( L^2(\mathbb{R}, e^{-h^2/4}dh) \). Through a standard argument, one can verify that \( \{ \psi_k(x) \rho_l(h) \} \) is a complete orthonormal system for space \( L^2(\Sigma, e^{-f}d\sigma) \), where \( d\sigma = d\sigma_{\mathbb{S}^{n-1}(\sqrt{2(n-1)})}dh \), and \( \psi_k(x) \rho_l(h) \) are the eigenfunctions associated to the eigenvalues counted with multiplicity:

\[
\mu_{k,l} = \lambda_k + \eta_l - \frac{1}{2}.
\]
Note that
\[ \mu_{0,0} = -\frac{1}{2}, \]
\[ \mu_{k,l} \geq 0 \quad \text{for all other} \quad k, l, \]
\[ \mu_{k,l} \to +\infty. \]

Hence the spectrum of the operator \( L_f \) is discrete. \( L_f \) has only a negative eigenvalue \( \mu_{0,0} \) with multiplicity one and the associated eigenfunction \( \psi_0 \rho_0 \equiv 2\sqrt{\pi} \). By the variational characterization of eigenvalues, we have that for any \( \varphi \in \{\psi_0 \rho_0\}^{-1} \cap C_0^\infty(\Sigma) \), \( Q(\varphi, \varphi) \geq 0 \). Hence by Proposition 2, the \( L_f \)-index of \( S_n^{-1}(\sqrt{2(n-1)}) \times \mathbb{R} \) is 1.

**Remark 3.** In the proof of Proposition 9, the discreteness of \( L_f \) can also be obtained by Corollary 3 in [9], since \( S_n^{-1}(\sqrt{2(n-1)}) \times \mathbb{R} \) is totally geodesic and \( L_f = \Delta_f + \frac{1}{2} \). The discreteness of the spectrum of \( \Delta_f \) implies that the spectrum of \( \Delta_f + \frac{1}{2} \) is also discrete.

**Lemma 1.** Let \( \Sigma \) be a complete oriented properly immersed \( f \)-minimal hypersurface in \((S^n(\sqrt{2(n-1)}) \times \mathbb{R}, \mathcal{G}, f)\). If \( \mu_1(L_f) \neq -\infty \), then
\[
\int_{\Sigma} |A|^2 e^{-f} < \infty.
\]

**Proof.** Let \( \varphi_j \) be the cut-off functions as before. So \( \varphi_j \in W^{1,2}(e^{-f} d\sigma) \). Since \( \mu_1(L_f) \neq -\infty \), by Proposition 3 there is a \( C^2 \) positive function \( h \) on \( \Sigma \) satisfying \( L_f h = -\mu_1(L_f) h \). By Proposition 4 we have
\[
\int_{\Sigma} \varphi_j^2 |A|^2 e^{-f} \leq \int_{\Sigma} (4|\nabla \varphi_j|^2 - 2\mu_1(L_f) \varphi_j^2) e^{-f}.
\]

So
\[
\int_{B_j} |A|^2 e^{-f} \leq \int_{\Sigma} (4 + 2|\mu_1(L_f)|) e^{-f} < \infty.
\]

Letting \( j \to \infty \), we obtain the conclusion. \( \square \)

Now we are ready to prove Theorem 1 which classifies the complete noncompact \( f \)-minimal hypersurfaces in \((S^n(\sqrt{2(n-1)}) \times \mathbb{R}, \mathcal{G}, f)\) of \( L_f \)-index one.
Proof of Theorem 1. It was proved in [6] that there is no complete stable \( f \)-minimal hypersurface with finite weighted volume, immersed in \( M^{n+1} \) with \( \text{Ric}_f \geq \frac{1}{2} \). Hence, Proposition [5] implies that \( L_f\text{-ind}(\Sigma) \geq 1 \). By Lemma 2 in [7] and Proposition [9], \( S^n(\sqrt{2(n-1)}) \times \{0\} \) and \( S^{n-1}(\sqrt{2(n-1)}) \times \mathbb{R} \) have \( L_f \)-index one. Now we prove the converse. Suppose that \( L_f\text{-ind}(\Sigma) = 1 \).

In the case of \( \alpha \not\equiv 0 \) on \( \Sigma \). By (9),

\[
L_f \alpha = \frac{1}{2} \alpha.
\]

Note \( \int_{\Sigma} \alpha^2 e^{-f} \leq \int_{\Sigma} e^{-f} < \infty \). Hence \( \alpha \) is an \( L^2(e^{-f}d\sigma) \) eigenfunction with negative eigenvalue \( -\frac{1}{2} \). Since \( L_f \)-index of \( \Sigma \) is one, by Proposition [2] the bottom \( \mu_1(L_f) \) of the spectrum of \( L_f \) satisfies that \( \mu_1(L_f) = -\frac{1}{2} \). By \( |\nabla \alpha|^2 \leq |A|^2 \) and Lemma [11],

\[
\int_{\Sigma} |\nabla \alpha|^2 e^{-f} \leq \int_{\Sigma} |A|^2 e^{-f} < \infty.
\]

Hence \( \alpha \) is in \( W^{1,2}(e^{-f}d\sigma) \). By Proposition [3] \( \alpha > 0 \) on \( \Sigma \) without lost of generality. Note the integrability of \( |A|^2 \) implies that \( (13) \) holds on \( \Sigma \), that is, \( \int_{\Sigma} |A|^2 \alpha e^{-f} = 0 \). Thus \( |A| \equiv 0 \) on \( \Sigma \). Note \( S^{n-1}(\sqrt{2(n-1)}) \times \mathbb{R} \) has \( \alpha \equiv 0 \). By Theorem [3] \( \Sigma \) must be \( S^n(\sqrt{2(n-1)}) \times \{0\} \).

In the case of \( \alpha \equiv 0 \) on \( \Sigma \). By the proof of Theorem [3] and Example [2] \( \Sigma \) is \( \Sigma_1 \times \mathbb{R} \), where \( \Sigma_1 \) is a closed minimal hypersurface in \( S^n(\sqrt{2(n-1)}) \). Note

\[
L_f = \Delta_f + |A|^2 + \frac{1}{2}
\]

\[
= \Delta + \langle \nabla f, \nabla \cdot \rangle + |A_{\Sigma_1}|^2 + \frac{1}{2}
\]

\[
= \left( \Delta_{\Sigma_1} + |A_{\Sigma_1}|^2 + \text{Ric}_{S^n(\sqrt{2(n-1)})}(\nu, \nu) \right) + \left( \frac{\partial^2}{\partial h^2} + \frac{1}{2} \frac{\partial}{\partial h} \right).
\]

Observe that the first part of the right-hand side of (42) is just the Jacobi operator \( J_{\Sigma_1} \) for minimal hypersurface \( \Sigma_1 \). If the index of \( \Sigma_1 \) is more than 1, then there exist at least two linearly independent eigenfunctions \( \psi_i(x), i = 1, 2 \) of \( J_{\Sigma_1} \) corresponding to negative eigenvalues. Obviously \( \int_{\Sigma} \psi_i(x)e^{-\frac{\nu^2}{4}}d\sigma < \infty \). Thus \( \psi_1(x) \) and \( \psi_2(x) \) are the two linearly independent \( L^2(e^{-f}d\sigma) \) eigenfunctions of \( L_f \) associated to negative eigenvalues. By Proposition [2] \( L_f\text{-ind}(\Sigma) \geq 2 \), which contradicts the assumption. Hence the index of \( \Sigma_1 \) must less than 2. It is known that in round
sphere $S^n(\sqrt{2(n-1)})$, there is no closed stable minimal hypersurface and a 
closed index one minimal hypersurface must be the totally geodesic spheres $S^{n-1}(\sqrt{2(n-1)})$ ([23]). Therefore $\Sigma_1 = S^{n-1}(\sqrt{2(n-1)})$ and so $\Sigma = S^{n-1}(\sqrt{2(n-1)}) \times \mathbb{R}$.

□

Remark 4. There are weaker conditions on the non-existence of $L_f$-stable $f$-
minimal hypersurfaces in a manifold with $\overline{Ric}_f \geq \frac{1}{2}$. In fact, when $f$-
minimal hypersurface $\Sigma^n$ has the weighted volume growth satisfying $\lim_{r \to \infty} \frac{\log V^\Sigma_f(r)}{r} = \beta, \beta < \sqrt{2}$, using an estimate of the bottom $\mu_1(\Delta_f)$ of the spectrum of $\Delta_f$ by [1], [23], one can get that $L_f$-ind$(\Sigma) \geq 1$. This result was proved by Impera and Rimoldi in [17] (Theorem 3.9 [17]). Since we only studied properly im-
mersed $f$-minimal hypersurfaces, we prefer to let the first part of Theorem [1] as stated.

6. Pinching theorem

Recall the gap phenomenon for the compact minimal hypersurfaces in 
sphere was first studied in [25] by using the integral identity derived from Simons’ equation. In this section, we will prove a pinching theorem and sub-
sequently obtain a gap phenomenon for the norm of the second fundamen-
tal form of complete $f$-minimal hypersurfaces in $(S^n(\sqrt{2(n-1)}) \times \mathbb{R}, \tilde{g}, f)$.

Proof of Theorem [3]. We only need to consider the case when $\Sigma$ is non-
compact. [11] says that $|A|$ is bounded. So $\int_{\Sigma} |A|^2 e^{-f} < \infty$ and $\int_{\Sigma} |A|^4 e^{-f} < \infty$. Observe that the assumption

$$\left| |A|^2 - \frac{1}{4} \right| \leq \frac{1}{4} \left( \sqrt{1 - \frac{8}{n-1} \alpha^2 (1 - \alpha^2)} \right)$$

is equivalent to

$$|A|^2 \left( \frac{1}{2} - |A|^2 \right) - \frac{1}{2(n-1)} \alpha^2 (1 - \alpha^2)$$

$$\leq - (|A|^2 - \frac{1}{4})^2 + \left( \frac{1}{4} \right)^2 \left[ 1 - \frac{8}{n-1} \alpha^2 (1 - \alpha^2) \right] \geq 0.$$ (43)

[13] and [21] imply that for $n \geq 3$, on $\Sigma$

$$\nabla A \equiv 0,$$
and
\begin{equation}
|A|^2 \left( \frac{1}{2} - |A|^2 \right) - \frac{1}{2(n-1)} \alpha^2 (1 - \alpha^2) = 0.
\end{equation}

Hence, $|A|^2$ and $H$ are constants. By (20), we have
\[ \int_{\Sigma} |A|^2 H^2 e^{-f} + \frac{1}{4} \int_{\Sigma} \alpha^2 (1 - \alpha^2) e^{-f} = 0. \]

This implies that
\[ \alpha^2 (1 - \alpha^2) \equiv 0 \quad \text{and} \quad H \equiv 0. \]

Then on $\Sigma$,
\[ \alpha \equiv 0 \quad \text{or} \quad \alpha^2 \equiv 1. \]

If $\alpha^2 \equiv 1$, we may assume that $\nu \equiv \frac{\partial}{\partial h}$. By Example 1, $\Sigma$ must be $\mathbb{S}^n(\sqrt{2(n-1)}) \times \{0\}$.

If $\alpha^2 \equiv 0$, as we have shown in the proof of Theorem 1, $\Sigma$ is $\Sigma_1 \times \mathbb{R}$, where $\Sigma_1$ be a closed minimal hypersurface in $\mathbb{S}^n(\sqrt{2(n-1)})$. Moreover $\nu$ is the normal of $\Sigma_1$. Since $|A|^2$ is constant on $\Sigma$, by (44), $|A| \equiv 0$ or $|A|^2 \equiv \frac{1}{2}$ on $\Sigma$. If $A \equiv 0$, by Theorem 3, $\Sigma$ must be $\mathbb{S}^{n-1}(\sqrt{2(n-1)}) \times \mathbb{R}$. If $|A|^2 \equiv \frac{1}{2}$, since $\frac{\partial}{\partial h}$ is parallel, the second fundamental form $A_{\Sigma_1}$ of $\Sigma_1$ satisfies that $|A_{\Sigma_1}|^2 \equiv \frac{1}{2}$. By the result on minimal hypersurfaces in sphere by Chern-do Carmo-Kobayashi [10] and Lawson [18], $\Sigma_1$ must be minimal clifford torus $T = \mathbb{S}^k(\sqrt{2k}) \times \mathbb{S}^l(\sqrt{2l})$, where $k + l = n - 1$. Hence $\Sigma$ is $T \times \mathbb{R}$.

\[ \square \]

Corollary 1 is a straightforward consequence of Theorem 2 since
\[ 4\alpha^2 (1 - \alpha^2) \leq 1. \]

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