PARABOLIC PROJECTIVE FUNCTORS IN TYPE $A$

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Abstract. We classify projective functors on the regular block of Rocha-Caridi’s parabolic version of the BGG category $\mathcal{O}$ in type $A$. In fact, we show that, in type $A$, the restriction of an indecomposable projective functor from $\mathcal{O}$ to the parabolic category is either indecomposable or zero. As a consequence, we obtain that projective functors on the parabolic category $\mathcal{O}$ in type $A$ are completely determined, up to isomorphism, by the linear transformations they induce on the level of the Grothendieck group, which was conjectured by Stroppel in [St3].

1. Introduction and description of the results

Category $\mathcal{O}$ associated to a fixed triangular decomposition $\mathfrak{g} = n_- \oplus \mathfrak{h} \oplus n_+$ of a semi-simple complex finite dimensional Lie algebra $\mathfrak{g}$ was introduced in [BGG]. For each parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing $\mathfrak{h} \oplus n_+$, there is a parabolic version $\mathcal{O}^p$ of $\mathcal{O}$ introduced in [RC]. An important role in understanding the structure of both $\mathcal{O}$ and $\mathcal{O}^p$ is played by the so-called projective functors, that is endofunctors of these categories isomorphic to direct summands of tensoring with finite dimensional $\mathfrak{g}$-modules. Indecomposable projective functors on $\mathcal{O}$ were classified, in terms of the action of the Weyl group $W$ of $\mathfrak{g}$ on $\mathfrak{h}^*$, in [BG].

Let $\mathcal{O}_0$ denote the principal block of $\mathcal{O}$, that is the indecomposable direct summand of $\mathcal{O}$ containing the trivial $\mathfrak{g}$-module. Formulated in modern terms, the main result of [BG] asserts that the action of projective functors on $\mathcal{O}_0$ categorifies, using the Grothendieck group decategorification, the right regular representation of $W$, see [Ma4, Lecture 5] for details. In particular, isomorphism classes of indecomposable projective functors on $\mathcal{O}_0$ turn out to be in a natural bijection with elements in $W$ and have a nice combinatorial description in terms of Kazhdan-Lusztig combinatorics from [KL], see [Ma4, Lecture 7] for details.

In the case of a Lie algebra of type $A$, the action of projective functors on $\mathcal{O}$ and, especially, on $\mathcal{O}^p$ for a maximal parabolic subalgebra $\mathfrak{p}$ plays a crucial role in the category $\mathcal{O}$ reformulation, given in [St3], of Khovanov homology for oriented links, originally defined in [Kh]. In particular, the paper [St3] establishes the following two properties for projective functors on $\mathcal{O}_0$ for a maximal parabolic subalgebra $\mathfrak{p}$ in type $A$:

- The restriction of an indecomposable projective functor from $\mathcal{O}_0$ to $\mathcal{O}_0^p$ is either indecomposable or zero, see [St3, Theorem 5.1].
- A projective functor on $\mathcal{O}_0^p$ is completely determined, up to isomorphism, by the linear transformation it induces on the level of the Grothendieck group, see [St3, Theorem 5.7].
Moreover, it is conjectured in [St3, Conjecture 3.3] that the second property should hold in the general case (note that the first property fails outside type $A$, see, for example, [St3, Example 3.7(c)]). The action of projective functors on arbitrary parabolic categories in type $A$ is used for categorification of other quantum link invariants, see [MS3]. The aim of the present paper is to prove [St3, Conjecture 3.3] for any parabolic category $\mathcal{O}_0^p$, not necessarily a maximal one, in type $A$. Our main result is the following theorem.

**Theorem 1.** Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and $\mathfrak{p}$ be any parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{h} \oplus \mathfrak{n}_+$. Then we have the following:

(i) The restriction of an indecomposable projective functor from $\mathcal{O}_0$ to $\mathcal{O}_0^p$ is either indecomposable or zero.

(ii) A projective functor on $\mathcal{O}_0^p$ is completely determined, up to isomorphism, by the linear transformation it induces on the level of the Grothendieck group.

For an explicit description of which indecomposable projective functors survive restriction from $\mathcal{O}_0$ to $\mathcal{O}_0^p$, given in terms of Kazhdan-Lusztig combinatorics, we refer the reader to Formula (1).

The approach to prove [St3, Theorem 5.1] and [St3, Theorem 5.7] in [St3] heavily relies on the relation of indecomposable projective functors which survive restriction from $\mathcal{O}_0$ to $\mathcal{O}_0^p$, for maximal $\mathfrak{p}$, to braid avoiding permutations. This is, clearly, not available for the general case. In fact, our approach to prove Theorem 1 is completely different and is crucially based on several advances in the abstract 2-representation theory of finitary 2-categories which were made in the series [MM1, MM2, MM3, MM4, MM5, MM6] of papers by Vanessa Miemietz and the second author. For Theorem 1(ii) we also use a result of Steffen König and Changchang Xi from [KX2] which asserts that the Cartan determinant of a cellular algebra is non-zero.

The paper is organized as follows. In Section 2 we collect all necessary preliminaries from the Lie algebra side of the story and then in Section 3 we collect all necessary preliminaries from the 2-representation side. Theorem 1 is proved in Section 4. The final Section 5 contains various speculations related to one of the original approaches to prove Theorem 1 which did not work. This approach was based on an attempt to first prove the following:

**Conjecture 2.** For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, let $L \in \mathcal{O}_0$ be simple and $\theta$ be an indecomposable projective functor on $\mathcal{O}_0$. Then $\theta L$ is either an indecomposable module or zero.

In Section 5 we discuss the evidence we have for the validity of this conjecture and also approaches that could perhaps be used to prove it.

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2. Preliminaries from Lie theory

2.1. Generalities. We work over $\mathbb{C}$. For a Lie algebra $\mathfrak{a}$, we denote by $U(\mathfrak{a})$ the universal enveloping algebra of $\mathfrak{a}$.

2.2. Category $\mathcal{O}$. Let $\mathfrak{g}$ be a finite dimensional semi-simple complex Lie algebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and set $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}_+$. With this decomposition one associates the corresponding BGG category $\mathcal{O}$, as defined in [BGG], which is the full subcategory of the category of all $\mathfrak{g}$-modules consisting of all finitely generated modules on which the action of $\mathfrak{h}$ is diagonalizable and the action of $U(\mathfrak{n}_+)$ is locally finite.

For $\lambda \in \mathfrak{h}^*$, we denote by $L(\lambda)$ the simple highest weight $\mathfrak{g}$-module with highest weight $\lambda$. These exhaust simple objects in $\mathcal{O}$, up to isomorphism. The module $L(\lambda)$ is the unique simple quotient of the Verma module $\Delta(\lambda)$ and also of the indecomposable projective module $P(\lambda)$. There is a contravariant simple preserving duality on $\mathcal{O}$ denoted by $M \mapsto M^\ast$. We set $\nabla(\lambda) := \Delta(\lambda)^\ast$ and $I(\lambda) := P(\lambda)^\ast$. Then $I(\lambda)$ is the indecomposable injective envelope of $L(\lambda)$.

We refer the reader to [Hu] for more details on category $\mathcal{O}$.

2.3. The principal block $\mathcal{O}_0$. The Weyl group $W$ of $\mathfrak{g}$ acts on $\mathfrak{h}^*$ in the usual way. We also consider the dot-action given by $w \cdot \lambda = w(\lambda + \rho) - \rho$ where $\rho$ is the half-sum of all positive roots. We denote by $w_0$ the longest element in $W$.

The principal block $\mathcal{O}_0$ of $\mathcal{O}$ is the Serre subcategory of $\mathcal{O}$ generated by all $L(w \cdot 0)$ for $w \in W$. It is a direct summand of $\mathcal{O}$ consisting of all modules in $\mathcal{O}$ which have the same generalized central character as the trivial $\mathfrak{g}$-module $L(0)$. To simplify notation, for $w \in W$, we set $L(w) := L(w \cdot 0)$, $\Delta(w) := \Delta(w \cdot 0)$, $\nabla(w) := \nabla(w \cdot 0)$, $P(w) := P(\mathfrak{g} \cdot 0)$ and $I(w) := I(w \cdot 0)$.

We consider the finite dimensional associative algebra $\mathbb{A} = \text{End}_\mathcal{O} \left( \bigoplus_{w \in W} P(w) \right)^{\text{op}}$

and have the usual equivalence between $\mathcal{O}_0$ and the category $\mathbb{A}$-mod of finite dimensional left $\mathbb{A}$-modules. The algebra $\mathbb{A}$ is quasi-hereditary, with respect to the weight poset $W$ equipped with the usual Bruhat order $\preceq$. The duality $\ast$ on $\mathcal{O}$ restricts to a duality on $\mathcal{O}_0$ which, in turn, gives an involution on $\mathbb{A}$ that fixes pointwise a complete set of primitive orthogonal idempotents, in particular, $\mathbb{A} \cong \mathbb{A}^{\text{op}}$.

2.4. Parabolic subcategories. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$ and let $W^\mathfrak{p}$ be the corresponding parabolic subgroup of $W$. The parabolic category $\mathcal{O}_0^\mathfrak{p}$ is the full subcategory of $\mathcal{O}_0$ consisting of all modules on which the action of $U(\mathfrak{p})$ is locally finite. In can be alternatively described as the Serre subcategory of $\mathcal{O}_0$ generated by all $L(w)$ where $w \in W^\mathfrak{p}$, the set of shortest representatives of the cosets in $w_0 \backslash W$.

The exact inclusion of $\mathcal{O}_0^\mathfrak{p}$ into $\mathcal{O}_0$ admits a left adjoint, denoted $Z^\mathfrak{p}$ and called the Zuckerman functor, defined as the largest quotient contained in $\mathcal{O}_0^\mathfrak{p}$. For $w \in W^\mathfrak{p}$, we set $\Delta^\mathfrak{p}(w) := Z^\mathfrak{p} \Delta(w)$ and $P^\mathfrak{p}(w) := Z^\mathfrak{p} P(w)$. The duality $\ast$ restricts to $\mathcal{O}_0^\mathfrak{p}$ and we set $\nabla^\mathfrak{p}(w) := \Delta^\mathfrak{p}(w)^\ast$ and $P^\mathfrak{p}(w) := P^\mathfrak{p}(w)^\ast$. 

set the restriction of the Bruhat order to \( W \). We set
\[
\mathcal{H}^p = \text{End}_{\mathcal{O}}( \bigoplus_{w \in W^p} P^p(w))^\text{op},
\]
so \( \mathcal{O}_0^p \) is equivalent to \( \mathcal{H}^p\text{-mod} \). The algebra \( \mathcal{H}^p \) is quasi-hereditary with respect to the restriction of the Bruhat order to \( W^p \) and also \( \mathcal{H}^p \cong (\mathcal{H}^p)\text{op} \).

### 2.5. Hecke algebra and Kazhdan-Lusztig combinatorics

Denote by \( S \) the set of simple reflections in \( W \) corresponding to our fixed triangular decomposition of \( g \). Let \( l : W \to \mathbb{Z} \) denote the length function on \( W \) with respect to \( S \). Then, associated to the pair \((W, S)\), we have the Hecke algebra \( \mathcal{H} = \mathcal{H}(W, S) \), which is a free \( \mathbb{Z}[v, v^{-1}] \)-module on generators \( H_w \), where \( w \in W \), and multiplication is uniquely defined using the following formulae:
\[
H_x H_y = H_{xy} \quad \text{whenever} \quad l(xy) = l(x) + l(y)
\]
and
\[
H_s^2 = H_s + (v^{-1} - v)H_s, \quad \text{for} \quad s \in S.
\]

We note that we use the normalization of \([So2]\).

There is a unique involution \( \mathcal{H} \) which maps \( H_x \mapsto (H_x)^{-1} \) and \( v \mapsto v^{-1} \), see \([KL] \ [So2]\). We denote by \( H_w \), for \( w \in W \), the corresponding Kazhdan-Lusztig basis element, see \([So2]\). Let \( \leq_l, \leq_R \) and \( \leq_j \) denote the Kazhdan-Lusztig left, right and two-sided preorders, respectively. The corresponding equivalence relations will be denoted \( \sim_l, \sim_R \) and \( \sim_j \), respectively, the are called Kazhdan-Lusztig cells. For \( w \in W \), we denote by \( L_w, R_w \) and \( J_w \) the left, right and two-sided Kazhdan-Lusztig cell containing \( w \), respectively. In what follows we abbreviate, as usual, “Kazhdan-Lusztig” simply by “KL”.

In the special case of \( g = sl_n \), we have \( W = S_n \) where simple reflections are given by elementary transpositions. In this case there is a nice description of KL-cells in terms of the Robinson-Schensted correspondence (cf. \([Sa] \) Section 3.1), see for example \([Na] \) and references therein. In particular, this correspondence shows that different left (resp. right) KL-cells inside a two-sided KL-cell are not comparable with respect to the left (resp. right) preorders. The same correspondence also shows that the intersection of a right and a left KL-cells inside the same two-sided KL-cell consists of precisely one element. Finally, each left (or right) KL-cell contains a unique involution, called the Duflo involution of the KL-cell.

Lusztig’s a-function \( a : W \to \mathbb{Z}_{\geq 0} \), defined in \([Lu1] \) and \([Lu2] \), is associated to the KL-combinatorics. In type \( A \) it is uniquely determined by the properties that it is constant on two-sided KL-cells of \( W \) and that \( a(w) = l(w) \) whenever \( w \) is the longest element in \( W_p \) for some parabolic subalgebra \( p \) of \( g \) containing \( b \).

### 2.6. Subcategories associated to right KL-cells

For a right KL-cell \( R \) of \( W \), define \( \hat{R} := \{ x \in W \mid x \leq_R R \} \) and let \( \mathcal{O}_0^R \) denote the Serre subcategory of \( \mathcal{O}_0 \) generated by all \( L(x) \), where \( x \in \hat{R} \). These categories were introduced in \([MS2]\). If \( p \) is a parabolic subalgebra of \( g \) containing \( b \), and \( w_p \) is the longest element in \( W_p \), then \( \mathcal{O}_0^p = \mathcal{O}_0^{R_{w_p}} \), where \( w_p = w_{w_0} \), see \([MS2] \) Remark 14].

Similarly to \( \mathcal{O}_0^p \), the exact inclusion of \( \mathcal{O}_0^R \) into \( \mathcal{O}_0 \) admits a left adjoint, denoted by \( Z^R \). For \( x \in \hat{R} \), the indecomposable projective cover \( Z^R P(x) \) of \( L(x) \) in \( \mathcal{O}_0^R \) is...
denoted by $P^R(x)$. We also define

$$\mathbb{A}^R := \text{End}_O(\bigoplus_{x \in R} P^R(x))^{\text{op}},$$

so that $O_0^R$ is equivalent to $\mathbb{A}^R$-$\text{mod}$. The duality $\ast$ restricts to $O_0^R$ and hence gives an involution on $\mathbb{A}^R$ which stabilizes a fixed set of primitive orthogonal idempotents. We note that $\mathbb{A}^R$ is not quasi-hereditary in general, see [MS2, Section 5.3].

For $x \in \bar{R}$, the indecomposable projective module $P^R(x)$ in $O_0^R$ is injective if and only if $x \in R$, see [Ma3, Theorem 6].

### 2.7. Graded setup.

By graded, we will always mean $\mathbb{Z}$-graded.

A graded finite dimensional associative algebra $A = \bigoplus_{i \in \mathbb{Z}} A_i$ is called positively graded provided that $A_i = 0$ for $i < 0$ and $A_0$ is semi-simple. Consider the category $A$-$\text{mod}$ of finite dimensional graded $A$-modules with the usual endofunctor $(1)$ which shifts the grading. For a graded $A$-module $M$, we write

$$\max(M) = \max\{i \in \mathbb{Z} \mid M_i \neq 0\} \quad \text{and} \quad \min(M) = \min\{i \in \mathbb{Z} \mid M_i \neq 0\}.$$  

The graded length of $M$ is then $\text{grl}(M) := \max(M) - \min(M) + 1$.

If $A$ is positively graded and $M$ is a graded $A$-module, then $M$ has a filtration by $A$-submodules $0 = M^{m+1} \subseteq M^m \subseteq \cdots \subseteq M^{k+1} \subseteq M^k = M$ where we have $k = \min(M)$, $m = \max(M)$ and $M^j = \bigoplus_{i \geq j} M_i$. This filtration is called the grading filtration. All subquotients of this filtration are semi-simple.

### 2.8. Graded category $O$.

The algebra $\mathbb{A}$ can be positively graded. In fact, it is a Koszul algebra, see [So1, BGS, Ma2]. The algebras $\mathbb{A}^p$ and $\mathbb{A}^R$ are quotients of $\mathbb{A}$ and in this way they inherit from $\mathbb{A}$ a positive grading, see for example [Ma3, Section 2.3]. We denote by $\mathbb{A}^p_\mathbb{Z}$, $\mathbb{A}^p_\mathbb{R}$ and $\mathbb{A}^R_\mathbb{Z}$ the graded versions of these algebras, and by $\mathbb{Z}O_0^p$, $\mathbb{Z}O_0^p$ and $\mathbb{Z}O_0^R$ the corresponding categories of finite dimensional graded modules.

For each of the above categories, we then have a forgetful functor to the corresponding non-graded categories $O_0$, $O_0^p$ and $O_0^R$. We call a module $M$ in any of the non-graded categories gradable if it is isomorphic to an image of some graded module $\hat{M}$ under the corresponding forgetful functor. The module $\hat{M}$ is then called a graded lift of $M$. All the modules $L(x)$, $P(x)$, $P^p(x)$ and $P^R(x)$, for $x \in W$, are gradable, see [St1, Theorem 2.1] and [St3, Theorem 2.1]. By [St1, Lemma 1.5], a graded lift of any of the aforementioned modules will be unique up to isomorphism and shift of grading. If $M$ is one of those modules, we will denote by $\hat{M}$ the graded lift whose head is concentrated in degree zero. Similarly, we define graded lifts for Verma modules and their quotients. For dual Verma modules and injective modules, the standard graded lift is the one in which the socle is concentrated in degree zero.

### 2.9. Projective functors.

Following [BG], a projective functor on $O$ is a functor isomorphic to a direct summand of the functor of tensoring with a finite dimensional $\mathfrak{g}$-module. Projective functors on $O_0$ were classified in [BG]. It turns out that
indecomposable projective functors on $O_0$ are in bijection with elements in $W$. For $w \in W$, we denote the corresponding projective endofunctor of $O_0$ by $\theta_w$. The functor $\theta_w$ is the unique, up to isomorphism, projective functor with the property $\theta_w P(e) = P(w)$.

From [St1, Theorem 8.2] (see also [St3, Corollary 3.2]) it follows that each $\theta_w$ admits a graded lift to an endofunctor of $\mathbb{Z}O_0$ and this lift is unique up to isomorphism and shift of grading. We denote the corresponding functor by $\tilde{\theta}_w$ which we normalize by the condition $\tilde{\theta}_w \tilde{P}(e) = \tilde{P}(w)$.

### 2.10. Decategorification of the action of projective functors

Let $\tilde{P}$ denote the additive tensor category of graded projective functors. It acts on $\mathbb{Z}O_0$ in the natural way. The Grothendieck group $[\mathbb{Z}O_0]$ of $\mathbb{Z}O_0$ is identified with $H$ by sending $[\Delta(w)]$ to $H_w$ under the convention that multiplication by $v$ corresponds to the shift $\langle 1 \rangle$ of grading. The split Grothendieck group $[\tilde{P}]_{\oplus}$ of $\tilde{P}$ is similarly identified with $H$ by sending $[\tilde{\theta}_w]$ to $H_{w'}$. In this way the action of $\tilde{P}$ on $\mathbb{Z}O_0$ gives the right regular representation of $H$, see [Ma4, Theorem 7.11] for more details. The ungraded version $P$ of $\tilde{P}$ is defined similarly and categorifies the right regular $\mathbb{Z}[W]$-module, see also Subsection 3.5.

### 3. Preliminaries from 2-representation theory

#### 3.1. Finitary and fiat 2-categories

We denote by $\text{Cat}$ the category of small categories. A 2-category is a category enriched over $\text{Cat}$. Thus, a 2-category $\mathcal{C}$ consists of objects and morphism categories $\mathcal{C}(i,j)$, objects of which, in turn, are 1-morphisms of $\mathcal{C}$ and morphisms of which are 2-morphisms of $\mathcal{C}$. As usual, we denote by $\circ_0$ and $\circ_1$ the horizontal and vertical composition of 2-morphisms, respectively. We refer to [Le] for more details.

Following [MM1], we call a 2-category $\mathcal{C}$ finitary provided that

- $\mathcal{C}$ has finitely many objects;
- each morphism category $\mathcal{C}(i,j)$ is $\mathbb{C}$-linear, additive and idempotent split with finitely many isomorphism classes of indecomposable objects and finite dimensional spaces of 2-morphisms;
- all compositions are biadditive and $\mathbb{C}$-bilinear when applicable;
- all identity 1-morphisms are indecomposable.

Furthermore, we call a finitary 2-category $\mathcal{C}$ weakly fiat provided that

- $\mathcal{C}$ has a weak anti-automorphism $(-)^*$ which reverses direction of both 1-morphisms and 2-morphisms;
- $\mathcal{C}$ has adjunction 2-morphisms $\alpha : F \circ F^* \to 1_j$ and $\beta : 1_i \to F^* \circ F$ such that $\alpha_F \circ_1 F(\beta) = \text{id}_F$ and $F^*(\alpha) \circ_1 \beta_{F^*} = \text{id}_{F^*}$.

Here $1_j$ is the identity 1-morphism for the object $j$, further, $\text{id}_F$ is the identity 2-morphism for the 1-morphism $F$ and, finally, $F(\beta)$ stands for $\text{id}_F \circ_0 \beta$ and $\alpha_F$.
stands for $\alpha \circ \text{id}_F$. If $(-)^*$ is involutive, then $\mathcal{C}$ is called flat. For example, $\mathcal{P}$ is biequivalent to a fiat 2-category, see Subsection [5.5] for details.

A 2-representation of a 2-category $\mathcal{C}$ is a strict 2-functor from $\mathcal{C}$ to $\text{Cat}$. All 2-representations of $\mathcal{C}$ form a 2-category, denoted $\mathcal{C}$-mod, in which 1-morphisms are non-strict 2-natural transformations and 2-morphisms are modifications, see e.g. [MM3] for details.

A 2-representation of $\mathcal{C}$ is called additive if it is given by an additive $\mathbb{C}$-linear action of $\mathcal{C}$ on additive, idempotent split, $\mathbb{C}$-linear categories with finitely many isomorphism classes of indecomposable objects. The 2-category of additive 2-representations of $\mathcal{C}$ is denoted by $\mathcal{C}$-amod.

3.2. Combinatorics of finitary 2-categories. For a finitary 2-category $\mathcal{C}$ consider the set $\mathcal{S}_\mathcal{C}$ of isomorphism classes of indecomposable 1-morphisms in $\mathcal{C}$. The set $\mathcal{S}_\mathcal{C}$ has the natural structure of a multisemigroup given by

$$[F] \circ [G] = \{[H] \mid H \text{ is isomorphic to a direct summand of } F \circ G\},$$

see [MM2, Section 3] for details. The left preorder $\leq_L$ on $\mathcal{S}_\mathcal{C}$ is given by $F \leq_L G$ if and only if $[G] \in \mathcal{S}_\mathcal{C} \circ [F]$. An equivalence class of $\leq_L$ is called a left cell. The right preorder $\leq_R$ and the corresponding right cells are defined similarly using right multiplication. The two-sided preorder $\leq_J$ and the corresponding two-sided cells are defined similarly using two-sided multiplication.

3.3. Principal and cell 2-representations. For a finitary 2-category $\mathcal{C}$ and an object $i \in \mathcal{C}$, we denote by $\mathcal{P}_i$ the corresponding principal 2-representation $\mathcal{C}(\mathbb{1}, \_)$.

Let $\mathcal{L}$ be a left cell in $\mathcal{S}_\mathcal{C}$ and $i$ be the object in $\mathcal{C}$ which is the origin of all 1-morphisms in $\mathcal{L}$. Denote by $\mathcal{N}_\mathcal{L}$ the additive closure inside $\mathcal{P}_i$ of all 1-morphisms $F$ such that $F \geq_L \mathcal{L}$. Then $\mathcal{N}_\mathcal{L}$ is a 2-representation of $\mathcal{S}_\mathcal{C}$ by restriction. This 2-representation has a unique maximal $\mathcal{C}$-invariant ideal $\mathcal{I}_\mathcal{L}$ and the quotient $\mathcal{N}_\mathcal{L}/\mathcal{I}_\mathcal{L}$ is called the cell 2-representation of $\mathcal{C}$ corresponding to $\mathcal{L}$ and denoted by $\mathcal{C}_\mathcal{L}$, see [MM2, Section 6.5] for details.

A two-sided cell $\mathcal{J}$ is called regular provided that different left (resp. right) cells inside $\mathcal{J}$ are not comparable with respect to the left (resp. right) order. A regular two-sided cell $\mathcal{J}$ is called strongly regular provided that the intersection of any left and any right cell inside $\mathcal{J}$ consists of exactly one element. For example, all two-sided cells of the tensor category $\mathcal{P}$ for $g$ of type $A$ are strongly regular, see [MM1, Subsection 7.1] for details. We note that, due to the fact that the action of $\mathcal{P}$ on $\mathcal{O}_0$ is a right action, the Kazhdan-Lusztig left (resp. right) order as defined in Subsection [2.3] corresponds to the right (resp. left) order as defined in this subsection.

3.4. Transitive and simple transitive 2-representations. Let $\mathcal{C}$ be a finitary 2-category and $\mathcal{M} \in \mathcal{C}$-amod. The 2-representation $\mathcal{M}$ is called transitive provided that for any $i \in \mathcal{C}$ and any indecomposable object $X \in \mathcal{M}(i)$, the additive closure of all objects of the form $\mathcal{M}(F)X$, where $F$ is a 1-morphism in $\mathcal{C}$, equals $\mathcal{M}$. A transitive 2-representation $\mathcal{M}$ of $\mathcal{C}$ is called simple transitive if it does not have any proper $\mathcal{C}$-invariant ideals, see [MM5] for details. The arguments in this paper use
crucially the following statement proved in [MM1, Theorem 43], [MM5, Theorem 18] and [MM6, Theorem 31].

**Theorem 3.** Let $\mathcal{C}$ be a weakly fiat 2-category in which all two-sided cells are strongly regular.

(i) Each simple transitive 2-representation of $\mathcal{C}$ is equivalent to a cell 2-representation.

(ii) For any two left cells inside the same two-sided cell, the corresponding cell 2-representation of $\mathcal{C}$ are equivalent.

3.5. **The 2-category of projective functors.** We fix some small category $\mathcal{A}$ equivalent to $\mathcal{O}_0$ and consider the 2-category $\mathcal{P}$ of projective endofunctors of $\mathcal{A}$, see [MM1, Subsection 7.1]. This is a fiat 2-category with one object $\mathbf{1}$ (which should be thought of as $\mathcal{A}$) and a weak involution given by $\theta_w = \theta_{w^{-1}}$, for $w \in W$. Indecomposable 1-morphisms in this category are exactly $\theta_w$, for $w \in W$, up to isomorphism. Because of the right action, cell 2-representations of $\mathcal{P}$ are indexed by right KL-cells. For a right KL-cell $\mathbf{R}$, the corresponding cell 2-representation of $\mathcal{P}$ is equivalent to the action of $\mathcal{P}$ on the additive category of projective-injective modules in $\mathcal{O}_0^\mathbf{R}$, see [MS2] and [MM1] for details.

If $\mathfrak{g}$ is of type $\mathcal{A}$, then, as already mentioned in Subsection 2.5, all two-sided cells in $\mathcal{P}$ are strongly regular. In particular, Theorem 3 gives a complete description of all simple transitive 2-representations of $\mathcal{P}$ in this case, up to equivalence.

Recall (see for example [MM1] Lemma 12] or [Ma3] Proposition 1] or [Mat] (1.4)]

that, for $x, y \in W$, we have

(1) \[ \theta_x L(y) \neq 0 \quad \text{if and only if} \quad x^{-1} \leq_L y \quad \text{if and only if} \quad x \leq_R y^{-1}. \]

For any KL-right cell $\mathbf{R}$, we have $L(y) \in \mathcal{O}_0^{\mathbf{R}}$ if and only if $y \leq_R \mathbf{R}$. This, combined with Formula (1), completely determines which $\theta_x$ survive restriction to $\mathcal{O}_0^{\mathbf{R}}$.

Now, for a fixed parabolic $\mathfrak{p}$ in $\mathfrak{g}$ containing $\mathbf{b}$, we can consider the Serre subcategory $\mathcal{A}^\mathfrak{p}$ of $\mathcal{A}$ which corresponds to $\mathcal{O}_0^\mathfrak{p}$. The action of $\mathcal{P}$ preserves $\mathcal{O}_0^\mathfrak{p}$ and hence we can define the 2-category $\mathcal{P}^\mathfrak{p}$ as the 2-category given by the additive closure of the 2-action of $\mathcal{P}$ on $\mathcal{O}_0^\mathfrak{p}$, the so-called image completion of the 2-action in the sense of [MM2, Subsection 7.3]. In more detail:

- the 2-category $\mathcal{P}^\mathfrak{p}$ has the same objects as $\mathcal{P}$;
- 1-morphisms in $\mathcal{P}^\mathfrak{p}$ are all endofunctors of $\mathcal{O}_0^\mathfrak{p}$ which belong to the additive closure of endofunctors given by the action of 1-morphisms in $\mathcal{P}$ on $\mathcal{O}_0^\mathfrak{p}$;
- 2-morphisms in $\mathcal{P}^\mathfrak{p}$ are all natural transformations of endofunctors of $\mathcal{O}_0^\mathfrak{p}$.

We note that, while $\mathcal{P}$ is fiat, the 2-category $\mathcal{P}^\mathfrak{p}$ is, at the present stage, only weakly fiat. In fact, it is Theorem [14] which implies that $\mathcal{P}^\mathfrak{p}$ is also fiat.

We would like to mention once more that, due to the fact that the action of $\mathcal{P}$ on $\mathcal{O}_0^\mathfrak{p}$ is a right action, the Kazhdan-Lusztig left (resp. right) order as defined in Subsection 2.5 corresponds to the right (resp. left) order as defined in this subsection. In particular, for any simple reflection $s$ and any $w \in W$ such that $l(sw) > l(w)$, we have $\theta_s \theta_w = \theta_{ws} \oplus$ other terms, see [BG, St3, MS2] for details.
4. Proof of Theorem \[1\]

4.1. Proof of Theorem \[1\]. For the rest of this section we set \( g = \mathfrak{sl}_n \) and let \( p \) be a parabolic subalgebra of \( g \) containing \( b \). Recall that \( w'_0 \) denotes the longest element of the parabolic Weyl group \( W_p \) corresponding to \( p \). Set \( w_p = w'_0 w_0 \) and let \( R := R_{w_p} \). Then we have \( O_0^p = O_0^R \).

For a module \( M \), denote by \( \ell(M) \) the Loewy length of \( M \), i.e. the shortest length of a filtration of \( M \) with semi-simple quotients.

**Lemma 4.** Let \( M \in ZO_0^p \). Then \( \ell(M) \leq \text{grl}(M) \).

**Proof.** Since \( A Z \) is Koszul, it is positively graded. This property is inherited by the quotient \( A p Z \). Thus, the grading filtration of \( M \) is a filtration of length \( \text{grl}(M) \) with semi-simple subquotients, and thus \( \ell(M) \leq \text{grl}(M) \). \( \square \)

Let, from now on, \( x \in W \) be such that \( \theta_x \) is non-zero when restricted to \( O_0^p \).

**Lemma 5.** There is some \( y \in \hat{R} \) such that \( x \sim_J y \).

**Proof.** Since \( \theta_x \) is non-zero when restricted to \( O_0^R \), there is some \( z \leq_R w_p \) such that \( x \leq_R z^{-1} \), see Formula \([1]\). But then \( x \leq_R z^{-1} \sim_J z \leq_R w_p \) and thus \( x \leq_J w_p \). Therefore we have to show that for any two-sided KL-cell \( J \) such that \( J \leq_J R \) we have \( J \cap \hat{R} \neq \emptyset \).

To prove this we recall that the action of projective functors on \( O_0^p \) categorifies, after extending scalars to \( \mathbb{C} \), the induced sign \( \mathbb{C}[W] \)-module by \([MS2 \text{ Proposition 30}]\). This module is a direct sum of Specht modules, where the Specht module for a partition \( \lambda \) occurs at least once whenever \( \lambda \leq \mu \), where \( \mu \) is the partition corresponding to \( p \) and \( \leq \) denotes the dominance ordering, see \([Sa \text{ Corollary 2.4.7}]\). On the other hand, the Kazhdan-Lusztig cell \( \mathbb{C}[W] \)-module associated to a right KL-cell inside a two-sided KL-cell is exactly the Specht module for the partition corresponding to the two-sided KL-cell via the Robinson-Schensted correspondence, see \([KL \text{ Theorem 1.4}]\) and \([Na \text{ Theorem 4.1}]\). The 2-representation corresponding to the action of projective functors on \( O_0^p \) has a weak Jordan-Hölder series in the sense of \([MM5 \text{ Section 4.3}]\) corresponding to the right KL-cells in \( \hat{R} \) (the corresponding subquotients are unique in the sense of \([MM5 \text{ Theorem 8}]\)). In the Grothendieck group, this gives a Jordan-Hölder series for the induced sign \( \mathbb{C}[W] \)-module by KL-cell modules corresponding to the right KL-cells which appear in \( \hat{R} \). Hence, whenever a Specht module corresponding to a partition \( \lambda \) occurs in the induced sign \( \mathbb{C}[W] \)-module, there must be a two-sided KL-cell corresponding to \( \lambda \) which intersects \( \hat{R} \) non-trivially. Since the dominance order coincides with the two-sided order by \([Ge \text{ Theorem 5.1}]\), the claim of the lemma follows. \( \square \)

Let \( J \) be the two-sided cell containing \( x \) and \( R' \subseteq \hat{R} \) be a right cell such that \( R' \cap J \neq \emptyset \), which exists by Lemma \([5]\) (note that \( R' \) is not uniquely determined by these properties).

**Lemma 6.** There is a unique \( y \in R' \) such that \( \theta_x L(y) \neq 0 \) and with this choice of \( y \) the module \( \theta_x L(y) \) is indecomposable.
Proof. According to Formula (1), the inequality \( \theta_x L(y) \neq 0 \) is equivalent to the inequality \( x^{-1} \leq_L y \). Since \( x \sim_J y \) by assumptions, we have \( x^{-1} \sim_J y \). Together with \( x^{-1} \leq_L y \), we thus have \( x^{-1} \sim_L y \) by regularity of \( J \). Due to strong regularity of \( J \), we thus have that \( y \) is the unique element in \( \mathcal{L}_{x^{-1}} \cap R' \).

Now let \( y \) be given as above and let \( R'' = R_x \). By [MM1, Theorem 43] and [MM1, Subsection 7.1], the cell 2-representations of \( \mathcal{P} \) corresponding to \( R' \) and \( R'' \) are equivalent, so it suffices to prove that \( \theta_x L(y') \) is indecomposable if we take \( y' \in \mathcal{L}_{x^{-1}} \cap R' \). However, the unique element in this latter intersection is precisely the Duflo involution in \( R'' \), and then \( \theta_x L(y') = \mathcal{P} R''(x) \), see [Ma3, Theorem 6] or [MM1, Section 4.5], which is indecomposable. □

Apart from the above, we will also need the following lemma.

**Lemma 7.** Let \( \mathcal{C} \) be a finitary 2-category and \( \mathcal{C}' \) be an image completion of \( \mathcal{C} \). Let \( \Psi : \mathcal{C} \to \mathcal{C}' \) be the corresponding canonical 2-functor. Then the pullback, via \( \Psi \), of a transitive 2-representation of \( \mathcal{C}' \) is a transitive 2-representation of \( \mathcal{C} \).

Proof. This is clear from the definitions since, for any 1-morphism \( F \) in \( \mathcal{C}' \), there is a 1-morphism \( G \) in \( \mathcal{C} \) and a 1-morphism \( H \) in \( \mathcal{C} \) such that \( F \oplus G \) is isomorphic to \( \Psi(H) \). □

Consider \( x \in W \) such that the restriction of \( \theta_x \) to \( \mathcal{O}_x^p \) is non-zero. Assume that this restriction decomposes. Let \( \overline{\theta}_x \) denote the unique indecomposable direct summand of \( \theta_x \) such that \( \overline{\theta}_x L(y) \neq 0 \) where \( y \) is as in Lemma 6. Let \( F_x \) be such that \( \theta_x = \overline{\theta}_x \oplus F_x \).

Assume that \( F_x \) is non-zero and consider some \( z \in W \) such that \( F_x L(z) \neq 0 \). Choose \( z \) in a two-sided cell such that \( a(z) \) is minimal possible with the property \( F_x L(z) \neq 0 \). Because of Lemma 6 and also our choice of \( \overline{\theta}_x \), we cannot have \( x \sim_R z^{-1} \), so we have \( x <_R z^{-1} \) and hence \( x <_J z \). Thus, by [Ma3, Proposition 1], we have \( \max(\theta_x L(z)) < a(z) \) and \( \min(\theta_x L(z)) > -a(z) \), implying the inequality \( \ell(\theta_x L(z)) \leq \ell(\theta_x L(z)) < 2a(z) + 1 \) by Lemma 3 (see Subsection 2.7 for definitions of min and max).

Consider the defining 2-representation \( \mathcal{N} \) of \( \mathcal{P}^p \). Let \( \mathcal{M} \) be the induced additive 2-representation of \( \mathcal{P}^p \) on the additive closure of all objects of the form \( \theta_w L(z) \), where \( w \in W \). Let \( \mathcal{P}_p \) be the quotient of \( \mathcal{P} \) by the 2-ideal generated by all \( \theta_w \) which annihilate \( \mathcal{O}_x^p \). Then \( \mathcal{P}_p \) is flat and all two-sided cells in \( \mathcal{P}_p \) are strongly regular, by construction. Moreover, by Lemma 5 the indexing set for elements of each two-sided cell of \( \mathcal{P}_p \) intersects the set \( \mathcal{R} \).

Now, let \( \mathcal{M}' \) be the simple transitive subquotient of \( \mathcal{M} \) containing \( F_x L(z) \). So far, this is a 2-representation of \( \mathcal{P}_p \). We may consider \( \mathcal{M}' \) as a 2-representation of \( \mathcal{P}_p \) via the canonical 2-functor \( \mathcal{P}_p \hookrightarrow \mathcal{P}^p \). This is, by construction, a transitive 2-representation of \( \mathcal{P}_p \) and hence also of \( \mathcal{P}_p \), by Lemma 7. Let \( \mathcal{M}' \) be the simple transitive quotient of \( \mathcal{M}' \), now as a 2-representation of \( \mathcal{P}_p \). From the above estimates and construction, we have the inequalities \( 0 < \ell(\mathcal{F}_x L(z)) < 2a(z) + 1 \).

By Theorem 3(b), \( \mathcal{M}' \) is equivalent to a cell 2-representation of \( \mathcal{P}_p \) which corresponds to some right KL-cell, say \( \mathcal{R} \subset \mathcal{R} \). Since \( \ell(\mathcal{F}_x L(z)) < 2a(z) + 1 \), we cannot have \( a(\mathcal{R}) \geq a(z) \). Indeed, the cell 2-representation corresponding to \( \mathcal{R} \)
consists of objects of Loewy length $2a(R) + 1$ by [Ma3 Corollary 7] since this cell 2-representation is modelled on the category of projective-injective objects corresponding to $R$ and they all have Loewy length $2a(R) + 1$. Therefore this cell 2-representation cannot contain the object $F_zL(z)$ of strictly smaller Loewy length.

On the other hand, by construction, $M'$ does not annihilate $F_z$ and $z$ is chosen such that, for all $z' \in R$ with $a(z') < a(z)$, we have $F_zL(z') = 0$. Hence $a(R) < a(z)$ is not possible either. The obtained contradiction shows that $F_z = 0$ and completes the proof of Theorem 1(i).

4.2. Proof of Theorem 1(ii). Let $F$ and $G$ be two projective functors on $O_0^R$. By Theorem 1(i), we may write

$$F \cong \bigoplus_{w \leq_j w_p} a_w \theta_w \quad \text{and} \quad G \cong \bigoplus_{w \leq_j w_p} b_w \theta_w,$$

for some non-negative integers $a_w$ and $b_w$. We need to show that, if $F$ and $G$ induce the same linear operators on the Grothendieck group $[O_0^R]$ of $O_0^R$, then $a_w = b_w$ for all $w \in W$.

Using Formula 1 and induction on the two-sided order, it is sufficient to consider the case

$$F \cong \bigoplus_{w \in J} a_w \theta_w \quad \text{and} \quad G \cong \bigoplus_{w \in J} b_w \theta_w,$$

where $J$ is a fixed two-sided KL-cell such that $J \leq_j w_p$. Let $R'$ be a right KL-cell in $J \cap R$, which exists due to Lemma 5.

As the classes of simple modules $L(x)$, for $x \in R'$, are linearly independent in $[O_0^R]$, it suffices to show that, for $w \in J$, the matrices

$$M_w := ([\theta_wL(x) : L(y)])_{y,x \in R'}$$

are linearly independent. Since $J$ is strongly regular, using Formula 1 we see that it is enough to show that the matrices $M_w$ are linearly independent for $w$ in a fixed left KL-cell $L$ of $J$. Note that, by Formula 1, each $M_w$ has a unique non-zero column (our convention is that columns are indexed by $x$).

Consider the cell 2-representation $C_{R'}$ of $S$. Let $Q_{R'}$ be the opposite of the endomorphism algebra of the multiplicity free sum of all indecomposable projective-injective modules in $O_{0}^{R'}$. Then $C_{R'}(1) \cong Q_{R'}$-mod. By [MM1 Theorem 43], the functors $\theta_w$, for $w \in L$, act as projective functors on $Q_{R'}$-mod in the sense of [MM1 Section 7.3]. Hence, putting together the non-zero columns of the matrices $M_w$, for $w \in L$, produces the Cartan matrix of $Q_{R'}$. Therefore we only need to show that the Cartan matrix of $Q_{R'}$ is non-degenerate.

In fact, we claim that $Q_{R'}$ is a cellular algebra, in which case the fact that its Cartan matrix is non-degenerate follows from [KK2 Proposition 1.2]

To prove that $Q_{R'}$ is a cellular algebra, consider another right cell $R''$ in $J$ which we choose such that $R''$ contains $w_q$ for some parabolic subalgebra $q$ of $g$ containing $b$. This is possible because we are in type $A$. Consider $O_0^q$ and let $T$ be a multiplicity free direct sum of all indecomposable projective-injective objects in $O_0^q$. Then the opposite of the endomorphism algebra of $T$ is isomorphic to $Q_{R'}$ by [MS1 Theorem 5.4] (see also [MS2 Theorem 18] or [MM1 Theorem 43]). On
the other hand, the associative algebra $\mathcal{A}$ of $\mathcal{O}$ is quasi-hereditary with simple preserving duality. In particular, $\mathcal{A}$ is cellular by \cite[Corollary 4.2]{KX1}. As the duality fixes projective-injective modules, it fixes $T$ and hence the endomorphism algebra of $T$ is cellular by \cite[Proposition 4.3]{KX1}. This completes the proof of Theorem\[1\].

5. Action of projective functors on simple modules

5.1. Action of projective functors on simple modules. Using Formula (1), the statement of Conjecture 2 has a more precise reformulation.

**Conjecture 2'.** For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, let $x, y \in W$ be such that $x^{-1} \leq_L y$. Then $\theta_x L(y)$ is indecomposable.

We note that the conjectured statement is not extendable outside type $A$. For example, $\theta_{st} L(ts) \cong \theta t L(t) \oplus \theta t L(tst)$ is decomposable in type $B_2$.

5.2. $J$-comparable indices. Our first observation is that Conjecture 2' is true in the case $x \sim_J y$ by Lemma 2.

5.3. Translation through a wall. The following claim is fairly well-known to experts but we failed to find a proper reference.

**Proposition 8.** Conjecture 2' is true if $x$ is the longest element in some parabolic subgroup of $W$, moreover, the corresponding statement is true for $\mathfrak{g}$ of any type.

**Proof.** If $x$ is the longest element in some parabolic subgroup, then $\theta_x$ is the translation through the intersection of walls which correspond to all simple reflections for this parabolic subgroup. For simplicity, we will simply say “a wall” instead of “intersection of all walls”.

As translation to a wall sends simple modules to simple modules or zero (because of 1-dimensionality of the highest weight), by adjunction, translation from a wall (which is biadjoint to the translation to a wall) sends a simple module to a module with simple top, in particular, to an indecomposable module. Therefore translation through the wall, which is the composition of a translation to a wall and from a wall, sends a simple module to an indecomposable module (or zero). □

5.4. Projectives in $\mathcal{O}_0$. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$. The module $P^\mathfrak{p}(e)$ has simple socle by [MST1, Lemma 4.7]. We denote this socle by $L(d_{\mathfrak{p}})$. If $\mathcal{R}$ is the right KL-cell in $W$ such that $\mathcal{O}_0^\mathfrak{p} = \mathcal{O}_0^\mathcal{R}$, then $d_{\mathfrak{p}}$ is the Duflo involution in $\mathcal{R}$, see [Ma3, Corollary 3]. The following claim is fairly well-known to experts but we failed to find a proper reference.

**Proposition 9.** Conjecture 2' is true if $y = d_{\mathfrak{p}}$ for some $\mathfrak{p}$, moreover, the corresponding statement is true for $\mathfrak{g}$ of arbitrary type.

**Proof.** Let $P$ be a multiplicity free projective generator of $\mathcal{O}_0^\mathfrak{p}$ and $Q$ be the maximal injective summand of $P$. Let $\mathcal{Q}$ be the opposite of the endomorphism algebra of $Q$. By [St2, Theorem 10.1], the functor $\text{Hom}_{\mathcal{O}}(Q, \cdot)$ from $\mathcal{O}_0^\mathfrak{p}$ to $\mathcal{Q}$-mod is full and faithful on projective modules in $\mathcal{O}_0^\mathfrak{p}$.
Consider the trace $\text{Tr}_Q(P)$ of $Q$ in $P$, that is the submodule of $P$ generated by all images of $Q$ in $P$. Each endomorphism of $P$ restricts to $\text{Tr}_Q(P)$, moreover, the previous paragraph guarantees that this restriction map induces an isomorphism

$$\text{End}_Q(P) \cong \text{End}_Q(\text{Tr}_Q(P)).$$

Since projective functors are adjoint to projective functors and preserve projective-injective modules, we have $\text{Tr}_Q(\theta M) \cong \theta \text{Tr}_Q(M)$ for any projective functor $\theta$ and any $M \in \mathcal{O}_0^\theta$.

Put together, the above implies that the endomorphism algebra of the module

$$\theta_wL(d_p) = \theta_w\text{Tr}_Q(\theta_w P^p(e)) \cong \text{Tr}_Q(\theta_w P^p(e)) = \text{Tr}_Q(P^p(x))$$

and the endomorphism algebra of the module $P^p(x)$ are isomorphic. Therefore $\theta_wL(d_p)$ is indecomposable as $P^p(x)$ is.

5.5. ** Tilting modules in $\mathcal{O}_0^p$. ** Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$, $W_\mathfrak{p}$ the corresponding parabolic subgroup of $W$, $w_0^\mathfrak{p}$ the longest element in $W_\mathfrak{p}$ and $w_0 = w_0^\mathfrak{p}w_0$. The following claim is fairly well-known to experts but we failed to find a proper reference.

**Proposition 10.** Conjecture [3] is true if $y = w_{\mathfrak{p}}$ for some $\mathfrak{p}$, moreover, the corresponding statement is true for $\mathfrak{g}$ of arbitrary type.

**Proof.** Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$ such that $w_0^\mathfrak{q} = w_0 w_0^\mathfrak{p}w_0$. Note that $w_0^\mathfrak{p} = w_0 w_0^\mathfrak{q}$.

For a positive integer $i$, denote by $W_{\mathfrak{q}}^i$ the set of all elements in $W_{\mathfrak{q}}$ of length $i$ and write $m$ for the length of $w_0^\mathfrak{q}$. By [Lep] Section 4, the module $P^\mathfrak{q}(e) = \Delta^\mathfrak{q}(e)$ has a BGG type resolution of the following form:

$$0 \rightarrow \Delta(w_0^\mathfrak{q}) \rightarrow \bigoplus_{w \in W_{\mathfrak{q}}^{m-1}} \Delta(w) \rightarrow \bigoplus_{w \in W_{\mathfrak{q}}^{m-2}} \Delta(w) \rightarrow \cdots \rightarrow \Delta(e) \rightarrow \Delta^\mathfrak{q}(e) \rightarrow 0.$$ 

Consider the derived twisting functor $\mathcal{L}T_{w_0}$. By [AS] Theorem 2.2, the complex $\mathcal{L}T_{w_0} \Delta(w)$ is, in fact, a module, for all $w \in W$. For any reduced decomposition $w_0 = s_1 s_2 \cdots s_k$, we have

$$\mathcal{L}T_{w_0} \cong \mathcal{L}T_{s_1} \circ \mathcal{L}T_{s_2} \circ \cdots \circ \mathcal{L}T_{s_k},$$

see e.g. [MSt] Remark 4.3(4). In particular, we have $\mathcal{L}T_{w_0} \cong \mathcal{L}T_x \circ \mathcal{L}T_y$, for any $x, y \in W$ such that $xy = w_0$ and $l(x) + l(y) = l(w_0)$. Note that we can write

$$w_0 = (w_0^\mathfrak{q}w_0^{-1})(w_0w^{-1})$$

and $l(w_0) = l(w_0^\mathfrak{q}w_0^{-1}) + l(w_0w^{-1})$. Combining the above with [AS] Formula (2.3) and Theorem 2.3, we see that

$$\mathcal{L}T_{w_0} \Delta(w) \cong \mathcal{L}T_{w_0 w_0^{-1}} \circ \mathcal{L}T_{w_0 w^{-1}} \Delta(w) \cong \mathcal{L}T_{w_0 w_0^{-1}} \Delta(w_0) \cong \nabla(w_0 w).$$

Therefore, $\mathcal{L}T_{w_0}$ maps Verma modules to dual Verma modules. Thus, applying $\mathcal{L}T_{w_0}$ to the resolution in (2), produces a coresolution of $L(w_0^\mathfrak{q})$ by dual Verma modules (see, for example, the proof of [MSt] Proposition 4.4 for details).

Applying $\theta_w$ to $P^\mathfrak{q}(e)$, produces (if non-zero) an indecomposable projective module in $\mathcal{O}_0^\mathfrak{q}$. Since twisting commutes with projective functors, see [AS] Theorem 3.2,
and, being a self-equivalence of the derived category of $\mathcal{O}_0$ by [AS, Corollary 4.2], preserves indecomposability, the claim follows.

The modules of the form $\theta_x L(w_p)$ are exactly the indecomposable tilting modules in the category $\mathcal{O}^p_0$. We refer the reader to [MS1] and [CM] for more details on the techniques used in the above proof and further results in this direction.

5.6. General reduction to involutions.

**Proposition 11.** Conjecture 2' is true if and only if it is true for all $x$ and $y$ such that $y^2 = e$.

**Proof.** Let $x, w \in W$ with $x^{-1} \leq_R w$ and let $y \in W$ be the Duflo involution in the left KL-cell of $w$. By [MS2, Proposition 35], the cell 2-representations of $\mathcal{P}$ corresponding to the right KL-cells $R_w$ and $R_y$ are equivalent and this equivalence swaps $L(w)$ and $L(y)$. Thus, $\theta_x L(w)$ is indecomposable if and only if $\theta_x L(y)$ is. The claim follows. □

Another way to formulate Proposition 11 is to say that the property that Conjecture 2' is true for all $x$ is an invariant of left KL-cells with respect to $y$.

5.7. Connection to the double centralizer property. Let $R$ be a right KL-cell and $d \in R$ the corresponding Duflo involution. Let $P$ be a multiplicity free projective generator in $\mathcal{O}^\hat{R}_0$ and $Q$ be the maximal injective summand of $P$. Let $\mathcal{Q}$ be the opposite of the endomorphism algebra of $Q$. The proof of Proposition 9 implies that, if the functor $\text{Hom}_{\mathcal{O}}(Q, -)$ from $\mathcal{O}^\hat{R}_0$ to $Q$-mod is full and faithful on projective modules in $\mathcal{O}^\hat{R}_0$ (this property is equivalent to a certain double centralizer property, see for example [Ma3, Section 3.4]), then Conjecture 2' is true for $y = d$ and for any $x$.

By [Ma3, Theorem 11], the condition that $\text{Hom}_{\mathcal{O}}(Q, -)$ is full and faithful on projective modules in $\mathcal{O}^\hat{R}_0$ is equivalent to the condition that Kostant’s problem has the positive solution for $L(d)$ (see [KM] for more details on Kostant’s problem). We refer the reader to [KM, Ma3, Ka] for many examples of elements for which Kostant’s problem has the positive solution. However, as it is shown in [KM], Kostant’s problem can have the negative solution for some $d$, even in type $A$ (the smallest example exists for the algebra $\mathfrak{sl}_4$).

5.8. Further speculations. Assume that we are in the situation as in the previous subsection, but such that there is no double centralizer property for our choice of $d$. Then the restriction map

$$\text{End}_{\mathcal{O}}(P) \to \text{End}_{\mathcal{O}}(\text{Tr}_Q(P))$$

is still injective but no longer surjective. By construction, the algebra $\text{End}_{\mathcal{O}}(P)$ is positively graded, moreover, $\text{Tr}_Q(P)$ is a graded submodule of $P$. Therefore the algebra $\text{End}_{\mathcal{O}}(\text{Tr}_Q(P))$ is a graded algebra.

To prove Conjecture 2', it would be sufficient to show that all homogeneous components of the graded quotient space

$$\text{End}_{\mathcal{O}}(\text{Tr}_Q(P))/\text{End}_{\mathcal{O}}(P)$$

are graded.
have strictly positive degrees. Indeed, in such a case these new components would only contribute to the Jacobson radical of End\(_{O}(\text{Tr}_{Q}(P))\) and hence no essentially new idempotents can be created.

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