Free Lévy Matrices and Financial Correlations

Zdzislaw Burda\textsuperscript{a,b}, Jerzy Jurkiewicz\textsuperscript{a}\textsuperscript{†}, Maciej A. Nowak\textsuperscript{a,†}, Gabor Papp\textsuperscript{c,d} and Ismail Zahed\textsuperscript{**}

\textsuperscript{a} M. Smoluchowski Institute of Physics, Jagellonian University, Cracow, Poland
\textsuperscript{b} Fakultät für Physik, Universität Bielefeld P.O.Box 100131, D-33501 Bielefeld, Germany
\textsuperscript{c} Department of Physics and Astronomy, SUNY-Stony- Brook, NY 11794 U. S. A.
\textsuperscript{d} HAS Research Group for Theoretical Physics, Eötvös University, Budapest, H-1518 Hungary

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We consider a covariance matrix composed of asymmetric and free random Lévy matrices. We use the results of free random variables to derive an algebraic equation for the resolvent and solve it to extract the spectral density. For an appropriate choice of asymmetry and Lévy index (a/2 = 3/4) the free eigenvalue spectrum is in remarkable agreement with the one obtained from the covariance matrix of the SP500 financial market. Our results are of interest to a number of stochastic systems with power law noise.

1. The basic concept of independence of commuting random variables has been generalized by Voiculescu to noncommuting ones such as random matrices, using the powerful theory of free random variables \cite{1}. The extension covers formally all stable distributions such as the Lévy ones. Free random matrices with gaussian fixed asymmetry have been applied to many physical problems \cite{2,3}. Recently we have extended the concept of free random Lévy variables to matrices \cite{4}, and suggested that the results may be relevant for addressing the issue of noise in stochastic systems with power law distributions. The latters are encountered in physics (e. g. solar wind data), biophysics (e. g. heartbeat data) and finances (e. g. financial time series) \cite{5,6}.

A quantitative way of addressing the issue of noise in gaussian stochastic systems is through the use of covariance matrices. In nongaussian systems, this concept is usually substituted by the one of covariance \cite{7} or tail covariance \cite{8}. In this letter, we will show that the concept of freeness as extended to matrices \cite{9}, allows for a simple analytical understanding of the eigenvalue structure of the covariance matrix, where the underlying noise is inherently power law distributed.

In section 2 we recall some results for free random matrices, which will be useful in the further part of the paper, with an emphasis on the Coulomb gas construction. In section 3, we define the covariance for random asymmetric Lévy matrices and derive a closed algebraic equation for the resolvent in the large size limit and for fixed asymmetry. We solve it to derive pertinent spectral distributions for Lévy covariances. In section 4, we show that our results may be relevant to the understanding of financial spectral correlations, a point of recent interest \cite{10}. Our conclusions are in section 5.

2. The resolvent for an $N \times N$ random matrix $M$ is generically given by

$$G(z) = \frac{1}{N} \left\langle \text{Tr} \left( \frac{1}{z-M} \right) \right\rangle. \quad (1)$$

In the following we consider the unitary ensemble of Hermitian matrices (other ensembles can be studied similarly) in the large-$N$ limit. The averaging is carried with the measure $e^{-N \text{Tr} V(M)} dM$. For standard random matrices the potential $V(M)$ is usually an even power series in $M$ bounded from below. For free random Lévy matrices, the potential is not analytic in $M$. It is known explicitly in two cases \cite{2}: $V(M) = \ln(M^2 + b^2)$ (Cauchy) and $V(M) = (1/M + \ln M^2)$ (Lévy-Smirnov), and implicitly as we now discuss.

The measure given above depends only on eigenvalues of the $M$ matrix. The rotational degrees of freedom can be integrated out yielding the Coulomb-gas representation

$$\rho(\lambda_1, \ldots, \lambda_N) \prod_i d\lambda_i = \prod_i d\lambda_i e^{-NV(\lambda_i)} \prod_{i<j} (\lambda_i - \lambda_j)^2, \quad (2)$$

where

$$V'(\lambda) = 2 \text{Re } G(\lambda + i0). \quad (3)$$

For free Lévy matrices $G(z)$ satisfies an algebraic equation in the $N \to \infty$ limit \cite{11} ($\alpha \neq 1$)

$$bG^\alpha(z) - (z-a)G(z) + 1 = 0, \quad (4)$$

in the upper half-plane, and follows by Cauchy reflection in the lower half-plane. The parameter $b$ is related to the Lévy index $\alpha$, asymmetry $\beta$, and range or scale $\gamma$ \cite{12}. In particular, for symmetric distributions ($\beta = 0$), it reads

$$b = -\gamma e^{i\alpha\pi/2}. \quad (5)$$
The marginal case $\alpha = 1$ is discussed in [4]. Equation (6) can be solved analytically in many cases [3], and numerically by starting at large $z$ and moving inward, using continuity and the asymptotic value of the physical branch

$$G(z) \to 1/z + b/z^{1+\alpha}. \tag{6}$$

The spectral density is $\rho(\lambda) = -\text{Im}G(\lambda + i0)/\pi$. For finite $N$ we expect corrections proportional to $1/N^p$ with some positive $p$, depending on the ensemble. In the following we shall neglect these corrections.

To show that this is legitimate and that (3-4) define correctly the free random Lévy ensemble, we have numerically calculated the spectral density using a Monte-Carlo simulation of the Coulomb gas in the Cauchy and Lévy-Smirnov case for large, but finite $N$. In Fig. 1a we show the results for the Cauchy case and in Fig. 1b we show the results for the Lévy-Smirnov case. For large $N$, the results asymptote $(b/\pi)/(\lambda^2 + b^2)$ (Cauchy) and $\sqrt{4\lambda - 1}/(2\pi\lambda^2)$ (Lévy-Smirnov) shown in solid lines. Other cases can be obtained by solving (3-4) to generate the potential, and then Monte-Carlo sampling to generate any n-point density. Note that the Cauchy and Lévy-Smirnov cases are directly accessible by random matrix sampling since the measure is explicitly known in these two cases (and only these two cases).

3. One way of quantifying the fluctuations in a stochastic system is through the use of a covariance matrix $C$,

$$C_{ij} = \frac{1}{T^{2/\alpha}} \sum_{t=1}^{T} M_{ti} M_{tj}, \tag{7}$$

where $M_{ti}$ is $T \times N$ (asymmetric). The normalization in (7) follows from the fact that the “variance” for Lévy matrices grows like $T^{2/\alpha}$ where $\alpha = 2$ is the expected diffusive limit (Brownian). Free Lévy ensembles are super-diffusive for $0 < \alpha < 1$ and sub-diffusive for $1 < \alpha < 2$ with $\alpha = 1$ the critical divide.

The spectrum of $C$ contains important information about the character of the correlations. It follows from the resolvent $W(z)$ of $C$. Since (7) involves a product $MM^T$, the resolvent of $C$ can be related to the resolvent of $M$ using Voiculescu’s powerful machinery of $R$ and $S$ transforms for free random variables [4]. Following the procedure described in [3], in our case we find that $W(z)$ satisfies the transcendental equation

$$\gamma^{2/\alpha} W(z\gamma^{2/\alpha}) = W_{\gamma=1}(z) \equiv \frac{1 + w(z)}{z}, \tag{8}$$

where $w(z)$ satisfies the multi-valued equation

$$-e^{\frac{2\pi m}{\alpha}} w^{2/\alpha} \cdot z = (1 + w)(w + m), \tag{9}$$

for fixed asymmetry $m = T/N$, and Lévy asymmetry parameter $\beta = 0$. The range of the Lévy distribution enters the formula only through the prefactor of $W(z)$, and the normalization of $z$. Thus, using the scaling on the left-hand side of (8), one can easily rescale the $\gamma = 1$ result to any other value of $\gamma$. The phase factor on the left hand side of (8) can be deduced from (9).

The distribution of eigenvalues of the covariance matrix follows from the discontinuity of $w(z)$,

$$\rho(\lambda) = \frac{1}{\pi\lambda} \text{Im} w(\lambda + i0). \tag{10}$$

The density of eigenvalues of index $\alpha$ and range $\gamma$ satisfies the self-affine property $\rho_{\gamma}(\lambda) = \rho(\lambda/\gamma^{2/\alpha})/\gamma^{2/\alpha}$. Its tail-behavior is given by

$$\rho(\lambda) \approx \frac{1}{\pi} \frac{m^{\alpha/2} \sin \alpha \pi}{\lambda^{1+\alpha/2}}, \tag{11}$$

for $\lambda \gg \lambda_{\text{min}} > 0$ and symmetric ($\beta = 0$) entries. The distribution is unimodal and unbounded from above. Since the covariance is a square, the tail distribution is characterized by an index $\alpha/2$ since the entries $M_{ti}$, are Lévy distributed with index $\alpha$.

![FIG. 1. Right: Metropolis Monte Carlo simulation of the Coulomb gas with $\alpha = 1/2$ and $\beta = 1$ (crosses) compared to the FRV result (line). Left: the same for $\alpha = 1$ and $\beta = 0$.](image1)

![FIG. 2. Left: Spectral density of FRV with $\alpha = 1/2$ and different asymmetry parameters $m$. Right: spectral density for several indices $\alpha$, at $m = 3.22$.](image2)
For a symmetric Cauchy distribution ($\alpha = 1$), the above analysis yields for the resolvent

$$W_{\gamma=1}(z) = \frac{1}{z} \left( 1 - \frac{(m+1) \mp \sqrt{(m-1)^2 - 4mw}}{2(1 + z)} \right),$$

from which the density of eigenvalues follows

$$\rho(\lambda) = \frac{1}{\pi} \frac{\sqrt{m}}{\lambda(\lambda + 1)} (\lambda - \lambda_{\min})^{1/2},$$

with $\lambda_{\min} = (m - 1)^2/4m$. The spectrum starts at the minimum eigenvalue $\lambda_{\min}$ and is unbounded from above. In general the minimal eigenvalue is determined by

$$\lambda_{\min} = \frac{(1 + w_\ast)(m + w_\ast)}{(-w_\ast)^{2/\alpha}},$$

where $w_\ast$ is the negative solution of the second order equation

$$(\alpha - 1)w_\ast^2 - (1 - \frac{\alpha}{2})(m + 1)w_\ast - m = 0.$$  

Some typical spectra are shown in Fig. 3a. We now illustrate the relevance of these distributions for the covariance spectrum in a financial market, a point of current interest [12].

4. Recently, it was pointed out that financial covariances are permeated by Gaussian noise in the low-lying eigenvalue region with consequences on risk assessment [11]. Here we show that the financial covariance spectrum is throughout permeated by free Lévy noise. For that consider the empirical matrix of relative returns

$$C_{ij} = \frac{1}{T} \sum_t M_{it} M_{jt} \quad \text{with} \quad M_{it} = \frac{x_{it+1} - x_{it}}{x_{it} - \langle M \rangle},$$

where $x_{it}$ are the raw returns made of the price $x_{i0}$ of stock $i$ at time $t$. Several other definitions are also possible, see [13]. The returns are normalized to the initial price $x_{i0}$ to insure scale invariance. They carry zero mean after subtracting the average relative return $\langle M \rangle$. For the raw prices we will use the daily quotations of $N = 406$ stocks from the SP500 market over the period of $T + 1 = 1309$ days from 01.01.1991 till 06.03.1996 (ignoring dividends). For these data the matrix asymmetry is $m = 1308/406 \approx 3.22$.

In Fig. 3 we compare the analytical results following from the free Lévy covariance to the ones following from the SP500 market setting the scale of the free covariance to match the one of the market. Fig. 3 left shows the distribution of eigenvalues for free Lévy matrices with index $\alpha/2 = 3/4$ and asymmetry $m = 2$, versus the raw SP500 data. Fig. 3 right shows the same distribution of eigenvalues for free Lévy matrices versus the reshuffled SP500 data. The reshuffled data follow from a random reshuffling of the price series for a fixed stock, destroying all inter-stock correlations. Our optimal fit preserves the index of the cumulative distribution, albeit for a smaller asymmetry ($m = 2 < 3.22$). It is remarkable that our free Lévy fit to the spectrum of the covariance suggests that the time series of returns is power law distributed with index $\alpha = 3/2$ which is consistent with detailed studies of the SP500 financial time series [7].

Finally, we would like to mention that there exists an alternative construction for random matrices with spectral Lévy disorder, as discussed in [11], where the matrix elements are populated from independent, one-dimensional stable distributions. As a result, these Random Lévy Matrix (RLM) ensembles are not rotational invariant, contrary to the Free Lévy Matrix (FLM) ensembles discussed here. We note that RLM ensembles exhibit interesting correlations between the eigenvectors, which are absent (or rather trivial) in the case of the rotation invariant ensembles such as the GUE, GOE, GSpE or FLM ones. For detailed studies of covariances based on RLM ensembles we refer to [12].

5. We have shown that the resolvent of covariances constructed from free random Lévy matrices, obeys an algebraic equation for all $1 \leq \alpha \leq 2$. The ensuing distribution of eigenvalues was constructed for a range of $\alpha$’s. We have argued that this distribution may be relevant to a large number of problems where power law fluctuations are expected, an example being the SP500 market. Indeed, we have shown that the spectral density of the SP500 financial covariance is in overall agreement with a free Lévy distribution of index $\alpha/2 = 3/4$. Free Lévy noise may be dominant in financial covariances, a point of relevance to the central issue of assessing risk in finances.
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