Exact correspondence between Renyi entropy flows and physical flows

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(Dated: March 2, 2015)

We present a universal relation between the flow of a Renyi entropy and the full counting statistics of energy transfers. We prove the exact relation for a flow to a system in thermal equilibrium that is weakly coupled to an arbitrary time-dependent and non-equilibrium system. The exact correspondence, given by this relation, provides a simple protocol to quantify the flows of Shannon and Renyi entropies from the measurements of energy transfer statistics.

Exact correspondences between seemingly different concepts play important role in all fields of physics. An example is the fluctuation-dissipation theorem, which states that the linear response of a system to externally applied forces corresponds to the system fluctuations \[ \langle \xi E \rangle = \lim_{M \to \infty} M^{-1} \int_{-T}^{T} \partial S_M/\partial M \text{ d}t. \] The theorem sets constraints on phenomenological theories whereby important developments achieved, for instance in quantum transport and quantum computation. \cite{1, 2}. The theorem can be extended to nonlinear responses \cite{3} and to full counting statistics (FCS) \cite{4}, giving more extended sets of such relations similar to Crooks’ formula \cite{5}. In this paper we present a relation similar to the fluctuation-dissipation theorem that provides an exact correspondence between the flows of Renyi entropy and FCS of energy transfers.

In transport theory, stationary flow of a physical quantity can take place from a system into an infinitely large system. In the case a quantity is locally conserved in each system, its flow is determined only by the interaction between the two systems \cite{6}. The traditional examples include electric current, which is the flow of charge, and energy flow. There are also conserved quantities which are not physical in strict sense. An example is the generalization of entropy by Alfred Renyi into Shannon entropy by defining \[ \frac{\partial S}{\partial M} \approx \frac{1}{T} \int_{-T}^{T} \frac{E}{\beta} \text{ d}E, \] = \sum_n p_n^M \text{ with } p_n \text{ being the probability to be in state n and arbitrary } M > 0. \cite{7}. Quantum generalization of the Renyi entropy is obviously conserved in a system under Hamiltonian evolution, with the Hamiltonian involving only the degrees of freedom of this system \cite{8}. For a system in thermal equilibrium at temperature \( T \) this entropy corresponds to the difference of free energies, i.e. \( \ln S_M = F(T) - F(T/M) \). The Renyi entropies have been studied in strongly interacting systems \cite{9, 10, 11}, in particular spin chains \cite{12, 13, 14}.

The Renyi entropies in quantum physics are considered unphysical, or non-observable, due to their non-linear dependence on density matrix. So is the Shannon entropy, which is derived from the Renyi entropy \[ S = \lim_{M \to \infty} \partial S_M/\partial M, \] \cite{9}. Such quantities cannot be determined from immediate measurements; Instead their quantification seems to be equivalent to determining the density matrix. This requires reinitialization of the density matrix between many successive measurements \cite{14}.

Therefore the flows of Renyi entropy between systems \( F_M \equiv -d \ln S_M/dt \) are conserved measures of non-physical quantities. The same pertains to Shannon entropy flow \cite{15}. An interesting and non-trivial question is: Is there any relation between the flows of Renyi entropy and the physical flows? An idea of such relation was first put forward by Levitov and Klich in \cite{15}, where they proposed that the Shannon entropy flow can be quantified from the measurement of Full Counting Statistics (FCS) of charge transfers. The validity of this relation is restricted to vanishing temperature and obviously to the systems where interaction occurs by means of charge transfer. In this paper we present a relation which is similar in spirit. It gives a correspondence between the flows of Renyi and Shannon entropies and the FCS of energy transfer in the limit of weak coupling.

I. DEFINITIONS AND RESULT

We consider two quantum systems \( X \) and \( Y \). We assume that the system \( X \) is infinitely large and is kept in thermal equilibrium at temperature \( T \). The system \( Y \) is arbitrary: it can encompass several degrees of freedom as well as infinitely many of those. It does not have to be in thermal equilibrium and in general is subject to time-dependent forces. It is convenient to assume that these forces are periodic with period \( \tau \). However this period does not enter explicitly in formulation of our result, which is also valid for aperiodic forces. The only requirement is that there is a stationary limit of the flows of physical quantities to the system \( X \). The stationary limit is defined by averaging the instantaneous flow over the period \( \tau \). For aperiodic forces it is determined by averaging over sufficiently long time interval.

The energy transfer is statistical. The FCS of energy transfers concentrates on the probability \( P(E_{tr}, T) \) to have energy transfer of \( E_{tr} \) during time interval \( T \), \cite{16, 17}. In the low frequency limit of long \( T \) all statistical cumulants of the energy transfer are proportional to \( T \) and are determined from the generating function \( F(\xi) \) = \( \int dE_{tr} P(E_{tr}, T) \exp(i\xi E_{tr}) \approx \exp(-T \bar{f}(\xi)) \). The parameter \( \xi \) is a characteristic parameter and cumulants are given by expansion of \( \bar{f}(\xi) \) in \( \xi \) at \( \xi = 0 \).

For quantification of the Renyi entropy flow we need to define an auxiliary FCS of energy transfer. The most general interaction Hamiltonian is \( \hat{H} = \sum_n \hat{X}_n \hat{Y}_n \) with
\[ \hat{X}_n \text{ being operators in the space of the system in thermal equilibrium, and } \hat{Y}_n \text{ being those in the space of the arbitrary system. Let us replace } \hat{Y}_n \text{ with average values } \bar{Y}_n \rightarrow (\bar{Y}_n). \text{ The result in Hamiltonian is that of the equilibrium system subject to time dependent external forces. Those induce energy transfers to the system to be characterized by a FCS. We discuss below possible physical realization of the scheme.} \]

We have two FCSs. We denote their generating functions with \( f_i(\xi) \) (incoherent) and \( f_c(\xi) \) (coherent).

Our main result is the following exact correspondence:

\[
\bar{F}_M^{(\beta)}/M = \bar{f}_i^{(M\beta)}(\xi^*) - \bar{f}_c^{(M\beta)}(\xi^*), \quad \xi^* = i\beta(M-1) \tag{1}
\]

which indicates that the Renyi entropy flow of the order \( M \) to the system kept at temperature \( T = 1/k_B \beta \) is exactly equal to the difference of FCS of incoherent and coherent energy transfers to the system kept at temperature \( T/M \) at the fixed characteristic parameter \( \xi^* \). This relation is valid in the limit of weak coupling, where the interaction between the systems can be treated perturbatively.

There is an obvious classical limit of the arbitrary system: all operators \( \hat{Y}_n \) are just numbers corresponding to classical forces acting on the system in thermal equilibrium. In this case the dynamics of the system is governed by the Hamiltonian in degrees of freedom of the system and therefore will be unitary. In this case, the trace of any power of density matrix, as is used in the definition of Renyi entropy, will not change in time: There will be no entropy flow. This result can also be understood from the correspondence \( \xi^i = \xi \xi^c \).

\section{II. DERIVATION OF THE RESULT}

Here we discuss the proof of the exact correspondence in Eq. (1) in the weak coupling regime where we can restrict ourselves to the first non-vanishing order of perturbation theory. To start with, we determine the FCS generating function of energy transfers using a diagrammatic representation of a pseudo-density matrix. Then we obtain the Renyi entropy flow from a multi-contour technique, and we demonstrate the correspondence of the two. The general formalism is illustrated by applications to two particular types of systems: the simplest quantum heat engine and a harmonic oscillator system coupled to environments.

\subsection{A. Full Counting Statistics}

Interactions between two systems influences the statistics of conserved quantities such as current and energy flows. In our consideration of FCS of energy transfer, we follow the lines of reference [16, 17]. We specify it to our situation where the interaction Hamiltonian between system \( X \) in thermal equilibrium and an arbitrary system \( Y \) is given by \( \hat{H} = \sum_n \hat{X}_n \hat{Y}_n \). The FCS of energy transfer in system \( X \) during the time interval \([0, T]\) can be determined from the following generating function:

\[
F_T(\xi) \equiv \text{Tr}_X \hat{\rho}_X (T), \tag{2}
\]

using the dynamics of the pseudo-density matrix \( \hat{\rho} \):

\[
\hat{\rho}_X (T) = \text{Tr}_X \left\{ \mathcal{F} e^{-i \sum_m J^m dt_1 \hat{X}_m (t_1 + \frac{\xi}{2}) \hat{Y}_m (t_1)} \times \right.
\]

\[
\left. \rho(0) \left( \mathcal{F} e^{-i \sum_m J^m dt_2 \hat{X}_m (t_2 - \frac{\xi}{2}) \hat{Y}_m (t_2)} \right) \right\} \tag{3}
\]

where \( \mathcal{F} \) (\( \mathcal{F} \)) denotes (anti-) time order operator. This quantity can be rewritten as a Keldysh partition function with integral taken over a Keldysh contour.

Fig. (1) shows that there are four possible diagrams for the evolution of \( \hat{\rho} \) in the second order. Time moves forward from left to right. Each diagram contains a double-contour, the outer (inner) represents the evolution of system \( Y \) (\( X \)). Determining FCS for system \( X \) requires to shift \( \hat{X} \) operators in time with \( \pm \xi/2 \), with \( \xi \) being the characteristic parameter. The value of the shift is opposite for forward and backward contours corresponding to time evolution of bra and ket states.

Let us consider second order perturbation for \( d\hat{\rho}_X/dt \). An element of this diagram is the average \( \langle X(t), \hat{X}(t') \rangle \). This average is performed over the states of thermal equilibrium. We define

\[
S_{mn}^{(\beta)}(t-t') \equiv \langle \hat{X}_n(t') \hat{X}_m(t) \rangle. \tag{4}
\]

The spectral density is \( S_{mn}^{(\beta)}(\omega) = \int dt (t-t') \exp(i\omega(t-t'))S_{mn}^{(\beta)}(t-t') \). Since \( t-t' \) is large in Eq. (3) we can shift the lower bound \( 0 \rightarrow -\infty \). Due to Markov approximation we can replace \( \rho_X (0) \) with \( \rho_X (T) \).

This allows us to compute the mean value of generating function \( \bar{f}(\xi) = -\text{Tr}(d\hat{\rho}_X(t)/dt)/\text{Tr}\hat{\rho}_X(t) \) averaged over the period \( \tau \). This can be explicitly obtained from Eqs. (2) and (3) and the diagrams in Fig. (1):

\[
f(\xi) = \frac{1}{\tau} \int_0^\tau \int_{-\infty}^t dt' dt' \sum_{mn}
\]

\[
\left\{ \left[ \langle \hat{X}_n \left( t + \frac{\xi}{2} \right) \hat{X}_m \left( t' - \frac{\xi}{2} \right) \rangle - \langle \hat{X}_m \left( t' - \frac{\xi}{2} \right) \hat{X}_n \left( t - \frac{\xi}{2} \right) \rangle \right] \times \langle \hat{Y}_m(t') \hat{Y}_n(t) \rangle \right\}
\]

\[
+ \left[ \langle \hat{X}_n \left( t + \frac{\xi}{2} \right) \hat{X}_m \left( t - \frac{\xi}{2} \right) \rangle - \langle \hat{X}_m \left( t - \frac{\xi}{2} \right) \hat{X}_n \left( t + \frac{\xi}{2} \right) \rangle \right] \times \langle \hat{X}_n(t) \hat{Y}_m(t') \rangle \right\} \tag{5}
\]
After some easy steps, the generating function of the correlator defined in Eq. (4) helps to simplify Eq. (5).

FCS of incoherent energy transfer: The incoherent energy transfer takes place between the system $X$ and $Y$ using the interaction Hamiltonian $H = \sum_n X_n Y_n$. The correlator defined in Eq. (4) helps to simplify Eq. (5).

FIG. 1: Diagrammatic illustration of dynamics of $\bar{\rho}$ using multi-contour evolution. Arrows indicate the direction of following Keldysh contour. In each diagram two interactions take place at vertices $m$ and $n$ and their correlator link the vertices by a solid line. Auxiliary times $\pm \xi/2$ shift the times when interactions act on the contour of $X$.

\[
\tilde{f}_i^{(\beta)} (\xi) = -\sum_{mn} \int \frac{d\omega}{2\pi} \left( e^{-i\omega\xi} - 1 \right) S_{mn}^{(\beta)} (\omega) Y_{mn} (\omega)
\]

FCS of coherent energy transfer: Since a driving force is externally applied, another type of energy exchange is possible to take place between the driving force and the system $X$. In this sense, two opposite energy transfers occur between the external force and system $X$, one at $t$ and the other at $t'$. The two transfers are correlated from within the system $X$. One way to consider this energy transfer is to replace $Y_n$ with the driving energy: $\hat{Y}_n \rightarrow \langle \hat{Y}_n \rangle$. The interaction Hamiltonian is $\hat{H} = \sum_n X_n \langle \hat{Y}_n \rangle + h.c.$ The full counting statistics of coherent energy transfers can be described in a similar way as that of incoherent energy transfer discussed above:

\[
\tilde{f}_c^{(\beta)} (\xi) = -\sum_{mn} \int \frac{d\omega}{2\pi} \left( e^{-i\omega\xi} - 1 \right) S_{mn}^{(\beta)} (\omega) Y_{mn} (\omega),
\]

\[
Y_{mn} (\omega) = \frac{1}{\tau} \int_0^\tau dt \int_{-\infty}^t dt' \left\{ \langle \hat{Y}_m (t') \rangle \langle \hat{Y}_n (t) \rangle e^{-i\omega(t-t')} + \langle \hat{Y}_m (t) \rangle \langle \hat{Y}_n (t') \rangle e^{i\omega(t-t')} \right\}
\]

with $Y_{mn}$ being spectral density of the forces acting on the system $X$.

Using the relation between $S$ and response function $\tilde{X}_{mn}$ (see Appendix [2]) one can rewrite the coherent FCS in more comprehensive way:

\[
\bar{f}_c^{(\beta)} (\xi) = -\int_0^\infty \frac{d\omega}{2\pi} \sum_{mn} \tilde{X}_{mn}^{(\beta)} (\omega) Y_{mn} (\omega) \times \left[ (e^{-i\omega\xi} - 1) \tilde{n} (\omega/T) + (e^{i\omega\xi} - 1) (\tilde{n} (\omega/T) + 1) \right]
\]

Statistical cumulants $C_n$ can be determined from the FCS generating functions from $C_n = i^n d^n \bar{f}/d\xi^n$ at $\xi = 0$.

B. Renyi entropy flow

The fluctuation relations are traditionally formulated in terms of entropy production that is computed using classical states [13]. When it comes to quantum, the Shanon entropy is known to be non-linear in density matrix and its change is not necessarily related to the expectation value of any operator. This problem raises a careful consideration of entropy, specially that with current technology developments entropy production in small scale systems is revealing the rich physics yet to be fully probed [19, 20].

A generalization of Shanon entropy is the Renyi entropies. To evaluate the flow of Renyi entropy (R-flow) we need to use the perturbation theory for the $M$-th power of its density matrix. [9] To this end, we use a multi-contour Keldysh technique. We consider $M$ copies of an isolated world. The contour for the degrees of freedom of $X$ encompasses all of the worlds and closes. This imposes the trace over the matrix multiplication of $\rho_X$. For other degrees of freedom in $Y$, the bra and ket parts of the contours are closed within each world providing the partial trace over these degrees of freedom: that yields $\rho_X = Tr_Y \rho$ for each world. The relevant diagrams are pairwise-grouped.

The average flow of Renyi entropy during a period $\tau$ is simply determined from $\bar{F}_\rho = (1/\tau) \int_0^\tau \bar{F}_\rho(t)dt$. In the second order we expect two interactions of the form indicated above Eq. (2). The two interactions can be either in the same world, or in different ones. The same-world diagrams have been considered in [9]. The different-world diagrams contain contributions from quantum coherence terms and are present when driving force is applied. We studied the contribution of the quantum coherence on the Renyi entropy flows in [22].
Salized correlators of single-world interactions

correlator can be easily shown to be related to the generator

governs the dynamics associated to multiple-world interactions. In this case the energy is exchanged between different worlds. Summing over all possible diagrams and using the generalized KMS relation to simplify the result the flow of Renyi entropy from different-world interactions can be found:

\[
\mathcal{F}_M \big|_{1w} = -\frac{M}{\tau} \int_0^T dt \int_{-\infty}^{\infty} dt' \sum_{mn} \text{Tr}_X \left\{ \hat{X}_n(t') \rho_X \hat{X}_m(t) \rho_X^{M-1} - \hat{X}_m(t) \hat{X}_n(t') \rho_X^{M} \right\}
\]

The generalised correlators in system X are defined as

\[
S^{N,M}_{mn}(t-t') = \frac{\text{Tr}_X \left\{ \hat{X}_n(t') \rho_X^{N} \hat{X}_m(t) \rho_X^{M-N} \right\}}{\text{Tr}_X \left\{ \rho_X^{M} \right\}}.
\]  

We generalized the Kubo-Martin-Schwinger (KMS) relation [21] to M-worlds in [22]. The Fourier transforming of generalized correlator thermal in a thermal equilibrium with temperature-independent dynamical susceptibility \(\tilde{\chi}_{mn}(\omega)\) can be determined from the relation (see Appendix [3]):

\[
S^{N,M}_{mn}(\omega) = \exp(\beta N \omega) \bar{n}(M\omega/T) \tilde{\chi}_{mn}(\omega).
\]

Note that since the time-difference \(t-t'\) in the correlator is large we can shift the lower bound of time integral over \(t'\) from 0 to \(-\infty\).

Note that in a system with temperature-dependent \(\tilde{\chi}_{mn}(\omega)\) requires rescaling its temperature to \(T/M\). This correlator can be easily shown to be related to the generalized correlators of single-world interactions \(S^{0,M}_{mn}\) in the following form: \(S^{N,M}_{mn}(\omega) = \exp(\beta N \omega)S^{0,M}_{mn}(\omega)\), where \(S^{0,M}_{mn}(\omega) = S^{(\beta^{*})}_{mn}(\omega)\) which is the standard spectral density in an environment of rescaled temperature from \(T \rightarrow T^{*} = 1/k_B\beta^{*} = 1/M\beta\).

Using these definitions Eq. (11) is simplified,

\[
\mathcal{F}_M \big|_{1w} = -\frac{M}{\tau} \int_0^T dt \int_{-\infty}^{\infty} dt' \sum_{mn} \frac{d\omega}{2\pi} \left( e^{\beta(M-1)\omega} - 1 \right) S^{(\beta^{*})}_{mn}(\omega)
\]

\[
	imes \left( \hat{Y}_m(t') \hat{Y}_n(t) e^{-i\omega(t-t')} + \hat{Y}_m(t) \hat{Y}_n(t') e^{i\omega(t-t')} \right)
\]

\[(14)\]

Multi-world R-flow: Similarly one can calculate the dynamics associated to multiple-world interactions. In this case the energy is exchanged between different worlds. Summing over all possible diagrams and using the generalized KMS relation to simplify the result the flow of Renyi entropy from different-world interactions can be found:

\[
\mathcal{F}_M \big|_{mw} = -\frac{M}{\tau} \int_0^T dt \int_{-\infty}^{\infty} dt' \int \frac{d\omega}{2\pi} \sum_{mn} \left( e^{\beta(M-1)\omega} - 1 \right) S^{0,M}_{mn}(\omega)
\]

\[
\times \left( \hat{Y}_m(\omega) e^{i\omega(t-t')} \langle \hat{Y}_m(t) \rho_Y \rangle \langle \hat{Y}_n(t') \rho_Y \rangle \right) (e^{\beta\omega} - 1)
\]

\[(15)\]

Details of this calculation can be found in Appendix [C].

The total flow of Renyi entropy can be obtained by summing over Eq. (14) and (15). By factorizing terms evolving with the same frequency (i.e. \(e^{\pm i\omega(t-t')}\)) the final result is

\[
\mathcal{F}_M = -M \sum_{m,n} \int \frac{d\omega}{2\pi} \left( e^{\beta(M-1)\omega} - 1 \right) S^{0,M}_{mn}(\omega)
\]

\[
\times \left( \langle \hat{Y}_mn(\omega) - \hat{Y}_mn(\omega) \rangle \right)
\]

\[(16)\]

with \(Y_{mn}\) and \(Y_{mn}\) defined in Eqs. (7) and (9), respectively.

Correspondence: Comparing eq. (16) with (6) and (8) one can conclude the exact correspondence mentioned in Eq. (1).

The case of Shannon entropy flow: Let us discuss here how the correspondence look like for the Shannon entropy flow. The Shannon entropy \(S = \text{Tr}_p \ln \rho\) can be genuinely defined from the Renyi entropy in the following form: \(S = -\lim_{M \to 1} \partial S_M / \partial M\).

Using the correspondence (1) the flow of Shannon entropy is

\[
\mathcal{F}_S = (i\beta) \lim_{\xi \to 0} \partial (f_1 - f_c) / \partial \xi.
\]

The flow of Shannon entropy exactly corresponds to

\[
\mathcal{F}_S^{(\beta)} = \frac{Q^{(\beta^{*})}_{1/c} - Q^{(\beta^{*})}_c}{T}
\]

\[(17)\]

with \(Q^{(\beta^{*})}_{1/c}\) the incoherent and coherent dissipated energy in a system of temperature \(T^{*} = 1/k_B\beta^{*}\) and \(\beta^{*} \equiv M\beta.\)
\section{III. Example 1: The Simplest Quantum Heat Engine}

A quantum heat engine (QHE) is a system of several discrete quantum states connected to several environments at different temperatures. The motivation for research in QHE comes from studying models of photocells and photosynthesis [22]. It has been demonstrated that quantum effects can dramatically change the thermodynamics of QHEs [24] and their fluctuations [25] manifesting the role of quantum coherences.

The simplest QHE of our interest is made of a probe environment weakly coupled to a two level system (TLS) whose states are $|0\rangle$ and $|1\rangle$. The TLS itself is also coupled to other heat baths at different temperatures as well as a coherent driving force with frequency $\Omega$ matching the two level energy difference. The specifics and the simplicity of the situation is that all energy exchanges take place by quanta $\hbar\Omega$. The interaction between the two level system and the probe is governed by the interaction Hamiltonian: $H_{int} = X_{01}(t)|0\rangle\langle1|e^{i\omega t} + X_{10}(t)|1\rangle\langle0|e^{-i\omega t}$.

![FIG. 3: Schematics of a QHE. A quantum system with two sets of states separated by energy $E_1 - E_0$ driven by external field at matching frequency. The system interacts with a number of environments that induce transitions between the states. We study the R-flows to a weakly coupled probe environment.](image)

In Ref. [22] we explicitly derived the Renyi entropy flow for the probe environment of this system using perturbative expansion of the probe dynamics. Here we make an attempt to determine the R-flow using the full counting statistics method and the correspondence of Eq. (1) manifesting the role of quantum coherence.

In the TLS, transition from upper level to lower one takes place by the operator $\hat{Y}_{01}(t) = |1\rangle\langle0|e^{-i\omega t}$ and the opposite one by $\hat{Y}_{10}(t) = |0\rangle\langle1|e^{i\omega t}$. Moreover: $\langle\hat{Y}_{10}(t)\hat{Y}_{01}(t')\rho_s\rangle = \rho_{01}e^{i\omega(t-t')}$, and $\langle\hat{Y}_{01}(t)\hat{Y}_{10}(t')\rho_s\rangle = \rho_{10}e^{i\omega(t-t')}$. Also $\langle\hat{Y}_{01}\rho_s(t)\rangle = \rho_{01}e^{i\omega t}$ and $\langle\hat{Y}_{10}\rho_s(t)\rangle = \rho_{10}e^{i\omega t}$.

Using the Kubo-Martin-Schwinger (KMS) relation [21] in Eq. (13) we can introduce the excitation transition rate $\Gamma_\uparrow = \bar{n}(\Omega/T)\chi_{01,10}$, and the emission rate $\Gamma_\downarrow = e^{\beta\Gamma_\uparrow}$ with the Bose function $\bar{n}(\Omega/T) = 1/(e^{\beta\Omega} - 1)$.

Using Eq. (10) the full counting statistics of heat dissipation in the incoherent energy transfer is:

$$\tilde{f}_i^{(\beta)}(\xi^*) = \left( e^{-i\xi^*\Omega} - 1 \right) \frac{\bar{n}(\Omega/T)}{\bar{n}(\Omega/T)} \left[ \Gamma_\downarrow p_1 - \Gamma_\uparrow p_0 \right]$$

Similarly, using Eq. (9), the full counting statistics of energy transfer through quantum coherence flow becomes

$$\tilde{f}_e^{(\beta)}(\xi^*) = \left( e^{-i\xi^*\Omega} - 1 \right) \frac{\bar{n}(\Omega/T)}{\bar{n}(\Omega/T)} \left( \Gamma_\downarrow - \Gamma_\uparrow \right) \rho_{01}\rho_{10}$$

where we used $S_{mn,pq}(\omega) = e^{\beta\omega} S_{pq,mn}(\omega)$.

Notice that Eqs. (18) and (19) are the FCSs associated to Poissonian probabilities. The reason is that the probe environment is weakly coupled to the quantum heat engine. Since we consider the dynamics to be Markovian, the time lag between two successive emissions in equilibrium environments at fixed temperatures is long. The events of transmissions of energy are uncorrelated. Moreover due to the weak coupling the energy transfers take place at low transmission probability. Such a process can be described by Poissonian probability $p_k = e^{-\bar{n}\hbar\Omega/k!}$ for exchanging $k$ quanta of energy $\hbar\Omega$, where $\bar{n}$ is the average number of quanta transmitted during time $[0, \tau]$. In the case the coupling of interaction between the two systems is not weak enough, or the emissions take place in short time intervals such that they become correlated, the Poissonian probability for emissions and absorptions are no longer valid.

From the correspondence Eq. (1) the R-flow can be obtained from subtracting the two FCSs at $\xi^*$ and $\beta^*$:

$$\mathcal{F}_M^{\beta} = \frac{M\bar{n}(\Omega/T)}{\bar{n}(\Omega/T) - \bar{n}(\Omega/T)\left( \rho_1\Gamma_\downarrow - \rho_0\Gamma_\uparrow + (\Gamma_\downarrow - \Gamma_\uparrow)\rho_{01}\rho_{10} \right)}$$

In the second line of Eq. (20) the first two terms are the dissipation of heat and the third term is the energy transfer due to quantum coherence flow. In conclusion, what we calculated above matches the R-flow result we obtained earlier in Ref. [22].

\section{IV. Example 2: A Driven Harmonic Oscillator Coupled to Heat Baths}

Let us consider a single harmonic oscillator of frequency $\omega_0$ with Hamiltonian $\hat{H} = \omega_0(\hat{a}^\dagger\hat{a} + 1/2)$ is coupled to a number of environments at different temperatures with different coupling strength. We concentrate on a probe environment which is weakly coupled to the oscillator. In addition the oscillator is driven by external force at frequency $\Omega$.

We calculate the Renyi entropy flow to the probe environment. The coupling Hamiltonian between the harmonic oscillator and the probe reservoir is $\hat{H}(t) = \hat{X}(t)\hat{a}^\dagger(t) + h.c.$ with $\hat{X}$ being the probe reservoir operator. The Fourier transform of the correlator is:
\[ S_{\text{in}}^{(\beta)}(\omega) = \int \exp(-i\omega t) S_{\text{in}}^{(\beta)}(t) d\omega/2\pi. \] Due to conservation of energy the energy exchange occurs either with quantum \(\hbar \Omega\) or with quantum \(\hbar \omega_0\).

We note that the time dependence of the average of two operators can be written as \(\langle \hat{a}(t)\hat{a}(t') \rangle = \langle \hat{a}^\dagger \hat{a} \rangle\) \(e^{i\omega_0 (t-t')} + \langle \hat{a}(t)\hat{a}(t') \rangle\), where the \(\langle \hat{a}(t) \rangle\) is due to the oscillator is driven by external force. This corresponds to the fact that the oscillator can oscillate both at its own frequency and at the frequency of external force.

Obtaining the FCS of energy transfers is straightforward from the diagrams of Eq. 1. The incoherent and coherent flows are:

\[
\begin{align*}
-f_i^{(\beta)}(\xi) &= S^{(\beta)}(\omega_0)(\langle aa^\dagger \rangle)(e^{-i\omega_0 \xi} - 1) \\
&+ S^{(\beta)}(-\omega_0)(\langle aa^\dagger \rangle)(e^{i\omega_0 \xi} - 1) \\
&+ S^{(\beta)}(\Omega)(\langle a_+^\dagger a_+ \rangle)(e^{-i\Omega \xi} - 1) \\
&+ S^{(\beta)}(-\Omega)(\langle a_+^\dagger a_+ \rangle)(e^{i\Omega \xi} - 1) \\
-f_c^{(\beta)}(\xi) &= S^{(\beta)}(\Omega)(\langle a_+ \rangle)(e^{-i\Omega \xi} - 1) \\
&+ S^{(\beta)}(-\Omega)(\langle a_+ \rangle)(e^{i\Omega \xi} - 1)
\end{align*}
\]

Substituting these FCSs in the correspondences of Eq. 1 using the values of \(\xi^*\) and \(\beta^*\), the flow of Renyi entropy after using the relation using \(S^{(\beta)}(-\omega) = \exp(\beta \omega)S^{(\beta)}(\omega)\) is determined to:

\[
\tilde{F}_M^{(\beta)} = M(e^{\beta(M-1)\omega_0} - 1) S^{(M\beta)}(\omega_0) \times \left\{ \langle (aa^\dagger) \rangle e^{\beta \omega_0} - \langle (aa^\dagger) \rangle \right\}.
\] (21)

Given \(T'\) to be the effective temperature of the harmonic oscillator \(\langle (aa^\dagger) \rangle = \bar{n}(\omega_0/T') + 1\) and \(\langle (aa^\dagger) \rangle = \bar{n}(\omega_0/T)\). The KMS relation of Eq. 13 helps to describe the correlator in the thermal bath in terms of its dynamical susceptibility, i.e.

\[
\bar{n}(M\omega/\hbar T) \chi^{(M\beta)}(\omega).
\] These help to simplify Eq. 21 into:

\[
\tilde{F}_M^{(\beta)} = \frac{M \bar{n}(M\omega_0/\hbar T) \chi^{(M\beta)}(\omega)}{\bar{n}(M\omega_0/\hbar T) - \bar{n}(\omega_0/\hbar T)} \left\{ \bar{n}(\omega_0/T') - \bar{n}(\omega_0/T) \right\}.
\] (22)

The entropy flow is robust in the sense that it only depends on the probe and harmonic oscillator temperatures and completely insensitive to external driving force. The entropy flow changes sign at temperature \(T = T'\).

V. DISCUSSION

In this paper we prove an exact correspondence between the flow of Renyi (as well as Shannon) entropy and the full counting statistics of energy transfers. This correspondence is valid for the flow to the system in thermal equilibrium that is weakly coupled to an arbitrary system out of equilibrium subject to arbitrary time-dependent forces.

In the case of time-dependent external forces we need to introduce an auxiliary full counting statistics of energy transfers. This is FCS for the case when the quantum forces acting on the system in thermal equilibrium \(Y\)'s are replaced by their averages. The usual FCS can be in principle measured directly. The same applies to the auxiliary FCS although the measurement protocol is more involved. Let us describe this protocol.

Let us notice that the forces correspond to operators \(\hat{Y}_n\) and therefore can be in principle measured directly as an expectation value of this observable. The output of this measurement is a function \(\langle \hat{Y}_n(t) \rangle\) which is periodic with period \(\tau\). From this point one can proceed in two ways. First way in to build an artificial system that interacts with system \(X\) classically and program it to exert classical forces on system \(X\) with values that are given by the results of the first measurement. One then collects the statistics of energy transfers to obtain the auxiliary FCS. The second way is more practical. One notices that the response of \(\hat{X}_n\) on the forces is linear one in the limit of weak coupling, so instead of measuring the statistics of energy transfer one can measure the matrix of response functions \(\chi_{mn}(\omega)\). Then the auxiliary FCS can be evaluated with the aid of Eq. 10.

This R-flow/FCS correspondence permits quantification of R-flows which are not accessible by direct measurement being non-linear function of density matrix.

This has many advantages; for instance a complete understanding of the flow of entropies can help to identify the sources of fidelity loss in quantum communications.

Our derivation was restricted to the second order perturbative dynamics. There are indications the theorem formulated is not valid in higher orders of perturbation theory. It is interesting to find a similar correspondence that is valid in all order of interaction coupling.
Acknowledgments

The research leading to these results has received funding from the European Union Seventh Framework Programme (FP7/2007-2013) under grant agreement n 308850 (INFEHONs).

Appendix A: A relation

One can easily prove that in general for any multi-argument function \( f(\omega, \cdots) \) the following relation holds:

\[
\int d\omega \left( e^{\beta(M-1)\omega} - 1 \right) e^{\beta\omega} S_{mn}^{0,M}(\omega) f(\omega, \cdots) = -\int d\omega \left( e^{\beta(M-1)\omega} - 1 \right) S_{mn}^{0,M}(\omega) f(-\omega, \cdots)
\]

(A1)

This can be easily proven by changing variable \( \omega \rightarrow -\omega \) and using the relation between spectral density function of negative and positive frequencies and simplifying using easy algebra.

Appendix B: Generalized KMS

The generalized correlator of two operators \( A \) and \( B \) is defined (see eq. (4)):

\[
S_{AB}^{N,M}(\omega) = \int d\tau e^{i\nu\tau} \text{Tr} \{ \hat{A}(0) \rho_b^N \hat{B}(\tau) \rho_b^{M-N} \} / \text{Tr} \rho_b^M
\]

This correlator in the energy eigenbasis can be rewritten in matrix form

\[
S_{nm,nn}^{N,M} = \int d\tau e^{i\nu\tau} A_{nm} e^{-\beta E_n} \sum_m Z(\beta)^N \times B_{mn} e^{(E_m - E_n)\tau} e^{-\beta E_n(M-N)} \frac{Z(\beta)^M}{Z(\beta)^{M-N}} \frac{Z(\beta)^M}{Z(\beta)^{M-N}}
\]

(B1)

\[
= 2\pi \delta (E_m - E_n + \nu) \frac{A_{nm} B_{mn} e^{-\beta E_n}}{Z(\beta)^M} e^{\beta N\nu}
\]

where \( Z(\beta) \) is the partition function defined as \( Z(\beta) = \sum e^{-\beta E_i} \). The standard correlator is \( S_{AB}(\omega) = \int d\tau e^{i\nu\tau} \text{Tr} \{ A(0) B(\tau) \rho_b \} / \text{Tr} \rho_b \) becomes equal to

\[
S_{AB}(\omega) = 2\pi \delta (E_m - E_n + \nu) A_{nm} B_{mn} e^{-\beta E_n} / Z(\beta)
\]

where KMS relation links this to dynamical susceptibility: \( S_{AB}(\nu) = \bar{\chi}_{AB}(\nu) \tilde{\nu}(\nu/T) \). Substituting this in (B1) a generalized KMS relation is obtained:

\[
S_{AB}^{N,M}(\omega) = \bar{\nu}(M\omega/T) e^{\beta\omega} \hat{\chi}_{AB}(\omega)
\]

(B2)

Appendix C: Multiple-world dynamics

Typical diagrams corresponding to the multi-world terms are listed in Fig. (5).

FIG. 5: Typical multicontour diagrams for the R-flow with interaction legs at nodes \( m \) and \( n \) in two different worlds.

Similar to the analysis for one world, we can write detailed diagrammatic values for the evolution of R-flow in the case two interactions occurs in two different worlds. After Fourier transformation the flow from the diagrams (e-l) of Fig. (5) becomes:

\[
\tilde{F}_M|_{(e-l)} = -\frac{1}{T} \int_0^T dt \int_{-\infty}^t dt' \int d\omega \sum_{mn} e^{-i\omega(t-t')} \left\{ e^{i\omega(t-t')} \left( S_{mn}^{N-2,M}(\omega) - 2S_{mn}^{N-1,M}(\omega) + S_{mn}^{N,M}(\omega) \right) \right. \\
\left. + e^{-i\omega(t-t')} \left( S_{mn}^{N-2,M}(\omega) - 2S_{mn}^{N-1,M}(\omega) + S_{mn}^{N,M}(\omega) \right) \right\}
\]

This must be summed over all possibilities. When the first interaction is at the topmost world the second one can run between \( n = 2 \) to \( M \). However, when we put the first interaction at the second topmost world the second interaction can have maximally \( M-1 \) world distance with it, therefore \( n = 2 \) to \( M - 1 \). Note that we already consider the both positive and negative energy exchanges in the summation of diagrams (e-l). Extending this discussion one can find the following total summation for all multi-world diagrams:
\[ \mathcal{F}_M|_{m_w} = -\frac{1}{\tau} \int_0^\tau dt \int_{-\infty}^t dt' \int \frac{d\omega}{2\pi} \sum_{mn} \left\{ e^{i\omega(t-t')} \left( S_{mn}^{N-2,M}(\omega) - 2S_{mn}^{N-1,M}(\omega) + S_{mn}^{N,M}(\omega) \right) + e^{-i\omega(t-t')} \left( S_{mn}^{N-2,M}(\omega) - 2S_{mn}^{N-1,M}(\omega) + S_{mn}^{N,M}(\omega) \right) \right\} \]

Changing \( \omega \rightarrow -\omega \) in terms with indices \( S_{nm} \) and using the relation \( S_{AB}(-\omega) = S_{BA}^{-N,M}(\omega) \) that can be easily concluded from the Fourier transforming eq. (12) and simplifying the summation using the KMS relation, all multi-world diagrams sum into

\[ \mathcal{F}_M|_{m_w} = -\frac{M}{\tau} \int_0^\tau dt \int_{-\infty}^t dt' \int \frac{d\omega}{2\pi} \sum_{mn} \left( e^{\beta(M-1)\omega} - 1 \right) \times S_{mn}^{0,M}(\omega) \mathcal{Y}_{m,w}(t',\omega) \mathcal{Y}_n(t,\omega) \left( e^{\beta\omega} - 1 \right) \]

(C1)

Simplifying the integration using a relation that comes in Eq. (A1) we can further simplify this relation into

\[ \mathcal{F}_M|_{m_w} = M \int \frac{d\omega}{2\pi} \sum_{mn} \left( e^{\beta(M-1)\omega} - 1 \right) S_{mn}^{0,M}(\omega) \mathcal{Y}_{m,w}(t,\omega) \]

(C2)