Stronger Second-law-like Inequalities Implying Advantage of Feedback Control over Memory Cost

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We report second-law-like inequalities (SLLIs) implying that feedback control increases maximum work more than compensates memory cost. Our SLLIs are stronger bounds than the second law (SLT) and the previous SLLIs that incorporate the correlations of a memory. The previous SLLIs claim that the memory’s correlation increases the maximum work, but it is canceled out by memory cost. In contrast, our SLLIs reveal that when the subsystems of a controlled system internally correlate, the maximum work by open-loop control can be less than the feedback control, and this advantage of the feedback control can grow linearly with the system size under keeping the memory cost constant.

Studies to exorcise Maxwell’s demon \[\text{[1]}\] shed light on the relationship between information theory and statistical physics \[\text{[2–7]}\]. Researchers reported models for the Maxwell demon, including the demon as a feedback controller \[\text{[8–28]}\], information reservoir such as a bit sequence \[\text{[29–41]}\], and the unifying approaches \[\text{[42–44]}\]. In those studies, researchers have shown second-law-like inequalities (SLLIs), establishing that the second law of thermodynamics (SLT)

\[
\Delta S(X_{\text{tot}}) + Q_{\text{tot}} \geq 0, \tag{1}
\]

is not violated by the demon in principle, where we denote by \(S(X_{\text{tot}})\) and \(Q_{\text{tot}}\) the entropy and the heat transfer of the whole system in contact with a heat bath. We assume that all processes are isothermal processes with a heat bath of temperature \(T\) in this Letter. We set \(k_B T = 1\).

The SLLIs appeared in those previous studies can be tentatively classified into two types: the first one, which was reported in the studies of the feedback control model, asserting that the entropy of the system can be reduced by the amount of information gathered by the controller (demon):

\[
\Delta S(X) + Q \geq \mathcal{I}, \tag{2}
\]

and the second one, which was reported in the studies of the information reservoir, asserting that work can be extracted in exchange for the entropy production of the information reservoir:

\[
W \leq \Delta S(X), \tag{3}
\]

where \(X\) denotes the state of the controlled system such as gases enclosed in a piston or a bit sequence; \(Q\) denotes the heat transfer from the controlled system; \(W\) denotes the work extracted from the controlled system; For \(\mathcal{I}\) in the inequality \(\text{[2]}\), several quantities such as transfer entropy have been employed. In the following, we specify it as the mutual information production: \(\mathcal{I} = I(X';Y) - I(X;Y)\), where \(Y\) is the memory’s state. Then, the SLLI \(\text{[2]}\) coincides with the one shown in Ref. \(\text{[18]}\).

Both types of the previous SLLIs do not imply that feedback control increases the net maximum work. Here, we call the upper limit of extracted work that accounts for the memory cost determined by the Landauer’s principle \(\text{[13, 16]}\) the net maximum work. In other words, the net maximum work is the upper limit of \(W\) minus the lower limit of the memory cost. According to the Landauer’s principle, the work required to exploit the memory is greater than \(\mathcal{I}\). Thus, the SLLI \(\text{[2]}\) does not imply that the net maximum work increases by measurements because \(\mathcal{I}\) is canceled out by the memory cost. Likewise, the SLLI \(\text{[4]}\) does not provide the difference of the net maximum work between feedback and open-loop control because all of the terms in the SLLI \(\text{[3]}\) do not vary depending on whether the control is feedback or open-loop.

Our first main result is that the sum of the entropy bounds for all subsystems leads to the first SLLI

\[
\Delta S(X_{\text{tot}}) + Q_{\text{tot}} \geq D \tag{4}
\]

implying that feedback control increases the net maximum work. Here, \(D\) is the positive quantity defined later. Due to the positivity of \(D\), this SLLI is a stronger bound than the SLT \(\text{[1]}\). In terms of extracting work, \(D\) coincides with how much the system’s entropy production is useless for a controller to extract the work from the system. Although \(D\) can grow in proportional to the system size, the memory cost can be constant with the system size when there are correlations within the system. Hence, the feedback control can vanish \(D\) with significantly less memory cost than \(D\). Thus, the feedback control can increase the net maximum work.

In addition, the sum of the entropy bounds of the controlled subsystems leads to the stronger inequality than SLLI \(\text{[2]}\):

\[
\Delta S(X) + Q \geq \mathcal{I} + D^{X|Y}, \tag{5}
\]

where \(D^{X|Y}\) denotes the positive quantity similar to \(D\). This inequality is our second main result and derived at the end of the main result part. Likewise, the stronger bound corresponding to SLLI \(\text{[3]}\) is immediately derived from SLLI \(\text{[3]}\) by regarding all subsystems as a controlled
system such as a bit sequence and ignoring the change in the internal energy:

$$W \leq \Delta S (X_{\text{tot}}) - D,$$

(6)

**Main result** – We study a classical system in contact with a heat bath of temperature $T$. We assume that the system and the heat bath constitute a closed system. The system consists of non-overlapping subsystems $1, 2, \ldots, N$. The evolution is Markovian with discretized time intervals, and we focus on a single step of this discrete Markov process. The initial and final states of the $k$-th subsystem are denoted by $X_k$ and $X'_k$, respectively. Let $\Gamma_{X_k}$ be the subset of $X_{\text{tot}} = \{X_1, \ldots, X_N\}$ that influences the evolution of $X_k$ other than $X_k$ itself. In other words, the state $X_k$ evolves depending only on $\{X_k\} \cup \Gamma_{X_k}$. We denote by $\Omega$ the number of the coarse-grained subsystems.

We premise the local detailed balance for each subsystem, which result in the entropy bound for a subsystem $[48]$:

$$\Omega_{X_k} := S(X_k | \Gamma_{X_k}) - S(X_k | \Gamma_{X_k}) + Q_{X_k} \geq 0,$$

(7)

where we denote by $\Omega_{X_k}$ the total entropy production accompanied with the evolution of $X_k$ by $Q_{X_k}$. The heat transfer from the $k$-th subsystem to the heat bath. The dependency among subsystems induces the graph $G^0 = (V_0, E_0)$ as follows:

$$V_0^p := \{\{X_1\}, \{X_2\}, \ldots, \{X_N\}\},$$

(8)

$$E_0^p := \{\{(j)\}\{k\} | X_j \in \Gamma_{X_k}, X_k \in X_{\text{tot}}\},$$

(9)

where we denote by $\{\{j\}\}{\{k\} }$ the directed edge from the vertex $\{j\}$ to the vertex $\{k\}$. The graph $G^p_0$ may contain cycles. To eliminate dependencies causing these cycles, we consider the coarse-grained subsystems where original subsystems are merged if they are members of the same cycle in $G^p_0$. Hereinafter, $\{X_j\}$ denotes the states of the $j$-th subsystem in the coarse-grained subsystems, and $N$ denotes the number of the coarse-grained subsystems. We denote by $G^p = (V^p, E^p)$ the graph induced from the coarse-grained subsystems by the same way shown in Eqs. (8) and (9). By its definition, $G^p$ is a directed acyclic graph (DAG).

Let $G^{cc} = (V^{cc}, E^{cc})$ be a connected component of $G^p$. The graph sequence $G^{cc}_1, G^{cc}_2, \ldots, G^{cc}_{\Gamma_{V^{cc}}}$ is constructed by the following procedure applying the edge contraction repeatedly to $G^{cc}$:

1. Initialization: let $j = 1$, $G^{seq}_{1} = G^{cc}$.
2. If the graph $G^{seq}_j$ has exactly one vertex, terminate. Otherwise, go to step 3.
3. Select a vertex couple $(p, c)$ from $G^{seq}_j$, such that every child of $p$ is a sink, and $c$ is a child of $p$.
4. Execute the edge contraction for $p$ and $c$, and let the result of it be $G^{seq}_{j+1}$.
5. Increment $j$, and go back to step 2.

![FIG. 1. (Color online). The schematic diagram of the sets defined as Eqs. (15) and $K_{p,c}$, $L_{p,c}$, and $M_{p,c}$.](image)

This procedure is feasible because a DAG with more than two vertices always has a vertex such that has only sink as its children. The essence of this procedure is that any $G^cc$ is a DAG, and the adjacency of $G^p$ preserves $[50]$. For simplicity, we introduce a convention regarding collections of subsystems $V$. Suppose a function $f$ takes an arbitrary number of subsystems as arguments. Then, we just write $f(V)$ in the sense of $f(X_{V_1}, X_{V_2}, \ldots)$, where $X_{V_j}$ is a $j$-th member of $V$. Likewise, we denote by $V'$ the collection of the subsystem’s final states in $V$. Furthermore, we denote by $\text{Flat}(V)$ the set of the subsystems created by flattening the nested structure of $V$. For example, if $V = \{(X_1, X_2), (X_3, X_4)\}$, then

$$P(V') = P(X_1, X_2, X_3, X_4),$$

(10)

$$V' = \{(X_{1}', X_{2}'), (X_{3}', X_{4}')\},$$

(11)

$$\text{Flat}(V) = \{X_1, X_2, X_3, X_4\}.$$ (12)

We note by $Q_V := \sum_{X \in \text{Flat}(V)} Q_X$ the heat transfer from all the subsystems composing $V$. We denote by $\Gamma_V := \bigcup_{X \in \text{Flat}(V)} \Gamma_X$ the sets of subsystems influencing the evolution of the members of $V$. We use another notation to ignore the empty set as the conditioning event:

$$P(V | \emptyset) := P(V), \quad S(V | \emptyset) := S(V),$$

(13)

$$I(V; W | \emptyset) := I(V; W).$$

(14)

In addition, we prepare notations for sets of subsystems regarding the dependency of $p$ and $c$:

$$\Gamma_c := \Gamma_c \cap p, \quad \Gamma_c^- := \Gamma_c \backslash p,$$

$$\Gamma_{p,c} := \Gamma_p \cap \Gamma_c, \quad \Gamma_{p,c}^- := \Gamma_{p,c} \backslash p,$$

$$\Gamma_{c,p} := \Gamma_c \backslash \Gamma_p, \quad \Gamma_{c,p}^- := \Gamma_{c,p} \backslash p.$$ (15)

As illustrated in Fig. 1, these sets have the following meaning: $\Gamma_c^p$ are the subsystems that are members of $p$ and influence to $c$; $\Gamma_{p,c}$ are the subsystems that influence $p$ but not $c$; $\Gamma_{c,p}$ are the subsystems that influence both $p$ and $c$; $\Gamma_{c,p}^-$ are the subsystems that influences $c$ but not $p$. We also note that $\Gamma_c^- = \Gamma_{c,p}^- \cup \Gamma_{c,p}^-$ and...
The minus symbol “−” put at the superscript of \( \Gamma \) represents the elimination of \( p \).

The key quantity \( D \) is the sum of two modules, each caused by the independency of the evolution within or among connected components in \( G^\Gamma \). The first module \( \hat{D}_{G^\Gamma} \), which is caused by the independency within a connected component, is provided by

\[
\hat{D}_{G^\Gamma} := - \sum_{(p, c) \in V_{G^\Gamma}} (J_{p, c} + K_{p, c} + L_{p, c} + M_{p, c}),
\]

where \( V_{G^\Gamma} \) is the set of all \((p, c)\) selected through constructing the graph sequence \( G_{1}^{eq}, \ldots, G_{|V_{G^\Gamma}|}^{eq} \), and each term of \( \hat{D}_{G^\Gamma} \) is mutual information production as follows:

\[
J_{p, c} := I(p'; c' \mid \Gamma_{p, c}^-) - I(p; c' \mid \Gamma_{p, c}^-),
\]
\[
K_{p, c} := I(p'; \Gamma_{c p, c}^- \mid \Gamma_p) - I(p; \Gamma_{c p, c}^- \mid \Gamma_p),
\]
\[
L_{p, c} := I(c'; p, \Gamma_c) - I(c; p, \Gamma_c),
\]
\[
M_{p, c} := I(c'; p \mid \Gamma_c) - I(c; p \mid \Gamma_c),
\]

where we used the convention introduced in Eqs. (10) and (11). The independencies of the evolutions induced from the definitions (17) lead to the negativity of \( J_{p, c}, K_{p, c}, L_{p, c}, \) and \( M_{p, c} \). For example, \( J_{p, c} \) is the mutual information production through the independent process of \( p \) from \( c' \) conditioned on \( \Gamma_{p, c}^- \). This fact leads to the negativity of \( J_{p, c} \). Consequently, \( \hat{D}_{G^\Gamma} \) is positive. Likewise, the following quantity is caused by the independency among the connected components in \( G^\Gamma \):

\[
\overline{D} := \sum_{j=2}^{N_{cc}} \left[ I(V_{j}^{cc}; V_{j-1}^{cc}) - I(V_{j}^{cem}; V_{j-1}^{cem}) \right],
\]

where we denote by \( N_{cc} \) the number of connected components in \( G^\Gamma \) and by \( V_{j}^{cc} \) all vertices in the \( j \)-th connected component of \( G^\Gamma \). We use the colons to represent the union of indexed sets such as \( V_{cc}^{1, j-1} := \bigcup_{k=1}^{j-1} V_{k}^{cc} \). Since subsystems contained in different connected components evolve mutually independently, the mutual information production \( I(V_{j}^{cem}; V_{j-1}^{cem}) - I(V_{j}^{cc}; V_{j-1}^{cc}) \) is negative, which leads to the positivity of \( \overline{D} \). We define \( D \) as the sum of the two contributions:

\[
D := \sum_{G^\Gamma \in G^\Gamma} \hat{D}_{G^\Gamma} + \overline{D}.
\]

The first result of this Letter is the SLLI (11) with \( D \) provided by Eq. (22). For a rigorous proof of this result, see Theorem 1 in Section S3. where we demonstrate that the sum \( \sum_{k=1}^{N_{cc}} \Omega_{X_k} \) results in the SLLI (11). Moreover, since all of its modules are positive, \( D \) takes a positive value. The positivity is formally proved in Theorem 2 in Section S3. Thus, by the positivity of \( D \), the SLLI (11) is a stronger bound than the SLT (1). In particular, if all subsystems influence each other, then \( D \) vanishes, and SLLI coincides with the SLT (1).
Then, although the SLT \(1\) and the SLLIs \(2\) claim the same net maximum work \( (N - 2) \ln 2 \) for all four cases, our SLLI \(4\) asserts a strict limit such that no work can be extracted without the feedback control:

\[
W \leq \begin{cases} 
(N - 1) \ln 2 & \text{(feedback control)} \\
0 & \text{(OLM or OL)}
\end{cases} 
\]

(23)

because we have \(5\):

\[
D = \begin{cases} 
(N - 1) \ln 2 & \text{(feedback control)} \\
(N - 2) \ln 2. & \text{(OL)}
\end{cases} 
\]

(24)

Accordingly, only the feedback control enjoys the positive net maximum work \( (N - 2) \ln 2 \) despite \(- \ln 2 \) or zero for the OLM or the OL case. In addition, this advantage of the feedback control increases with the number of particles, which means the feedback control can be significantly beneficial more at a macroscopic level.

Example 2 – To observe how \(J_{p,c}\) appears, we consider a two-body system with the following dependency:

\[
\Gamma_{X_1} = \emptyset, \quad \Gamma_{X_2} = \{X_1\} 
\]

(25)

Then, \(N_{cc} = 1\) and \((p,c) = (X_1, X_2)\). Suppose that all subsystems take binary states, and the entropy of each subsystem takes the maximum amount: \(S(X_j) = S(\{X_j\}) = \ln 2\) for all \(j\). We assume the following correlations:

\[
\begin{align*}
I(X_1; X_2) &= I(X_1; X_2') = I(X_2; X_2') = \ln 2, \\
I(X_1; X_1') &= I(X_2; X_1') = I(X_2'; X_1') = 0.
\end{align*} 
\]

(26) (27)

Then, we have \(\Delta S(X_{tot}) = 2 \ln 2\). Since \(J_{p,c} = \ln 2\) and \(K_{p,c} = L_{p,c} = M_{p,c} = 0\), we obtain \(D = 2 \ln 2\). Assuming that the internal energy is constant, our inequality \(6\) implies \(W \leq \ln 2\) although the SLT \(1\) and the SLLIs \(2\) assert that the net maximum work is \(2 \ln 2\).

Example 3 – To observe how \(K_{p,c}, L_{p,c}\) and \(M_{p,c}\) appear, we consider a system consisting of five subsystems with the following dependency:

\[
\begin{align*}
\Gamma_{X_1} &= \{X_3\}, \quad \Gamma_{X_2} = \{X_3, X_5\}, \quad \Gamma_{X_3} = \{X_4\}, \\
\Gamma_{X_4} &= \{X_5\}, \quad \Gamma_{X_5} = \emptyset
\end{align*} 
\]

(28)

Then, \(N_{cc} = 1\) and the sequence of \((p,c)\) can be \((X_3, X_1), \{(X_1, X_3), X_2\}, \{(X_2, X_1, X_3)\}, \{X_5, \{X_1, X_2, X_3, X_4\}\}\). We suppose all subsystems take binary states except \(X_1\), which takes four states. We denote by \(X_{1H}\) and \(X_{1V}\) the first and second bit of \(X_1\). The entropy of each subsystem takes the maximum amount: \(S(X_j) = S(\{X_j\}) = \ln 2\) for all \(j\) except \(S(X_1) = S(\{X_1\}) = 2 \ln 2\). There are no correlations among the subsystems other than the following:

\[
\begin{align*}
I(X_{1H}; X_2) &= I(X_2; X_2') = I(X_{1V}; X_3) = I(X_3; X_5) = \ln 2.
\end{align*} 
\]

(29)

Then, we have \(\Delta S(X_{tot}) = 4 \ln 2\). Since \(J_{p,c} = K_{p,c} = L_{p,c} = M_{p,c} = 0\) other than \(L_{X_4, X_1} = K_{(X_1, X_1), X_2} = M_{(X_1, X_1), X_2} = \ln 2\), we obtain \(D = 3 \ln 2\). Assuming that the internal energy is constant, our inequality \(6\) implies \(W \leq \ln 2\), although the SLT \(1\) and the SLLIs \(2\) assert that the maximum work is \(4 \ln 2\). Indeed, when the third subsystem performs feedback control, the maximum work \(2 \ln 2\) is achieved. Meanwhile, if the dependency is provided by \(\Gamma_{X_1} = \emptyset, \Gamma_{X_2} = \Gamma_{X_3} = \Gamma_{X_4} = \Gamma_{X_5} = \{X_1\}\), then \(D\) is zero, and our SLLI claims that the same upper limit \(W = 4 \ln 2\) as the SLT and the SLLIs \(2\).

Discussion – By the definition, \(D\) is the decrease in mutual informations between subsystems evolving independently. Precisely, \(\hat{D}_{CC}\) is the negative sum of \(J_{p,c}, K_{p,c}, L_{p,c}\), and \(M_{p,c}\) that are the mutual information productions through the independent processes conditioned on the directly influencing subsystems. For example, \(K_{p,c}\) is the mutual information production between \(p\) and \(\Gamma_c^\ast\), where \(p\) evolves independently from \(\Gamma_c^\ast\) conditioned on \(\Gamma_{pc}^\ast\). We can refer to Fig.4 where the arrows represent dependencies, to understand the independencies regarding \(J_{p,c}, K_{p,c}, L_{p,c}\), and \(M_{p,c}\). Likewise, \(\hat{D}\) is the decrease in the mutual informations through the mutually independent processes, as mentioned earlier. Hence, \(D\) is varied by changing the dependency and the conditioning event.

In Example 1, we can interpret that the feedback control reduces \(D\) by turning the independent process not conditioned on other states into the independent process conditioned on the correlated memory’s state. When the controller neither performs the measurement nor influences the controlled system’s evolution, \(D\) becomes a smaller amount through the independent processes that are not conditioned on another state. In contrast, in the feedback control, the memory state influences the controlled system’s evolution, and then \(D\) becomes the maximum amount decrease through the independent process conditioned on the correlated memory state. Since the measurement provides the correlation: \(I(X_1; X_j)\), this conditioning by the memory state causes less \(D\) in the feedback control.

At last, we compare the present study with the exorcise of the Maxwell demon. The latter study recovers the SLT from a situation in which the feedback control appeared to extract more work than the upper limit by the second law. In contrast, this study illustrates that the feedback control improves the maximum work, which is lower than what the SLT claims in the open-loop control. Moreover, our SLLIs imply that the feedback control does not violate the SLT because our SLLIs are always tighter than the SLT.

Conclusion – We reported the first SLLIs \(1\) – \(6\) implying that the feedback control can increase the net maximum work. Although both the SLLIs \(2\) and \(3\) do not imply that the feedback control increases the net maximum work, our SLLIs imply it because the open-loop control decreases \(D\), which leads to less net maximum work than what the SLT claims. By the positivity of \(D\), out
SLLIs are stronger bounds than the SLT and the previous SLLIs. Since all components of $D$ are the decreases of the mutual informations among the subsystems, the advantage of the feedback control suggested in this Letter comes from the internal correlations of the controlled system. Our work might serve as a theoretical basis for quantifying the usefulness of information processing in physical systems.

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[50] See Lemma 1 in Section S2.

[51] For the formal definition of the independent evolution, see Definition 2 in Section S3. In addition, Lemma 3 in this section establishes the negativity of the mutual information productions such as $F_{p,c}$, $K_{p,c}$, $L_{p,c}$ and $M_{p,c}$.

[52] See Corollary 1 in Section S3.

[53] See Section S5.

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Supplemental Material to “Stronger Second-law-like Inequalities Implying Advantage of Feedback Control over Memory Cost”

S1. THE ENTROPY BOUND FOR A SUBSYSTEM

In this section, we provide the derivation of Eq. (7) of the main text. Let $X_k$ be a subsystem that evolves conditioned on $\Gamma_{X_k}$. We premise the local detailed balances for the subsystem $k$:

$$Q_{X_k} = \left\langle \ln \frac{P(X_k' \mid X_k, \Gamma_{X_k})}{P(X_k \mid X_k', \Gamma_{X_k})} \right\rangle$$  \hfill (S1)

In a broad class of nonequilibrium dynamics including Langevin dynamics, the relation (S1) holds [27]. It follows that $Q_{\text{tot}} = \sum_{k=1}^{N} Q_{X_k}$. Now by the chain rule, we obtain

$$\frac{P(X_{k'} \mid X_k, \Gamma_{X_k})}{P(X_k \mid X_k', \Gamma_{X_k})} = \frac{P(X_{k'} \mid \Gamma_{X_k}) P(X_k \mid X_k', \Gamma_{X_k})}{P(X_k' \mid \Gamma_{X_k}) P(X_k \mid \Gamma_{X_k})},$$  \hfill (S2)

and

$$P(X_{\text{tot}}, X_{\text{tot}}') = P(X_{\text{sys}}, X_{\text{sys}}', \Gamma_{X_k} \mid X_k, X_k', \Gamma_{X_k}) P(X_k, X_k', \Gamma_{X_k}),$$  \hfill (S3)

where we denote that $X_{\text{sys}} := X_{\text{tot}} \setminus X_k \setminus \Gamma_{X_k}$ and $X_{\text{sys}}' := X_{\text{tot}}' \setminus X_k' \setminus \Gamma_{X_k}$. Then, we infer

$$\left\langle \frac{P(X_{k'}, X_k, \Gamma_{X_k})}{P(X_k, X_k', \Gamma_{X_k})} \right\rangle = \int P(X_{\text{tot}}, X_{\text{tot}}') \frac{P(X_{k'}, X_k, \Gamma_{X_k})}{P(X_k, X_k', \Gamma_{X_k})} dX_{\text{tot}} dX_{\text{tot}}' \hfill (S4)$$

$$\equiv \int P(X_{\text{sys}}, X_{\text{sys}}', \Gamma_{X_k} \mid X_k, X_k', \Gamma_{X_k}) dX_{\text{sys}} dX_{\text{sys}}' d\Gamma_{X_k}$$

$$\times \int \frac{P(X_{k'}, X_k, \Gamma_{X_k})}{P(X_k, X_k', \Gamma_{X_k})} dX_k dX_k' d\Gamma_{X_k} = 1. \hfill (S5)$$

Meanwhile, we infer

$$\left\langle \frac{P(X_{k'}, X_k, \Gamma_{X_k})}{P(X_k, X_k', \Gamma_{X_k})} \right\rangle = \left\langle \frac{P(X_{k'}, \Gamma_{X_k}) P(X_k \mid X_k', \Gamma_{X_k})}{P(X_k, \Gamma_{X_k}) P(X_k' \mid X_k, \Gamma_{X_k})} \right\rangle \hfill (S7)$$

$$= \left\langle \frac{P(X_{k'}, \Gamma_{X_k}) P(X_k \mid X_k', \Gamma_{X_k})}{P(X_k, \Gamma_{X_k}) P(X_k' \mid X_k, \Gamma_{X_k})} \right\rangle \hfill (S8)$$

$$= \left\langle \exp \left\{ - \left[ \ln P(X_{k'} \mid \Gamma_{X_k}) - \ln P(X_k \mid \Gamma_{X_k}) + \ln P(X_k' \mid X_k, \Gamma_{X_k}) \right]^+ \right\} \right\rangle \hfill (S9)$$

$$= 1. \hfill (S10)$$

Let us apply the Jensen’s inequality for the equality between the RHS of (S9) and (S10). Then, by the definition of Shannon entropy and the local detailed balance condition (S1), it yields the inequality (7).

S2. PROPERTIES OF THE GRAPH SEQUENCE

We denote by Parent$(v, G)$, Child$(v, G)$, Ancest$(v, G)$, and Decend$(v, G)$ the set of the parents, children, ancestors and descendants of $v$ in the graph $G$, respectively. Likewise, let Sink$(G)$ be the set of the sink in $G$.

The following Lemma shows two essential properties of the graph sequence $G_{1\text{seq}}, \ldots, G_{|V_c|\text{seq}}$.

Lemma 1. Suppose that $G^\Gamma$ is the graph induced from the dependency of subsystems and $G_{1\text{seq}}, \ldots, G_{|V_c|\text{seq}}$ is the graph sequence induced from a connected component of $G^\Gamma$. Then, the following properties hold:

a) The graph $G_{j\text{seq}}$ is a DAG for all $j \in \{1, \cdots, |V_c|\}$.

b) The adjacency of the graph $G_{j\text{seq}}$ comprehends the dependency of the subsystems. Precisely, let $v_k$ and $v_l$ be vertices of $G_{j\text{seq}}$; If $X_k \in v_k$, $X_l \in v_l$, and $X_l \in \Gamma_{X_k}$, then $v_l \in \text{Parent}(v_k, G_{j\text{seq}})$. 

Proof. Proof of a] Since $G^c$ is a DAG, the conclusion is established by verifying the statement that if $G^c_{j-1}$ is a DAG, then $G^c_{j-1}$ is a DAG as well. Let $c$ and $p$ be the vertices of $G^c_{j-1}$ replaced with $v_{new}$ by the edge contraction. Let $V_{p,c} = V_{j-1} \setminus \{c, p\}$. Since the edge contraction does not change the adjacency between vertices in $V_{p,c}$, it does not create a cycle in which all vertices are members of $V_{p,c}$. Thus, the only condition we need to verify is that there is no cycle containing $v_{new}$ in $G^c_{j-1}$.

Let us verify this condition. Since $c \in \text{Sink}(G^c_{j-1})$ and $\text{Child}(p, G^c_{j-1}) \subseteq \text{Sink}(G^c_{j-1})$, $\text{Descend}(v_{new}, G^c_{j-1}) \subseteq \text{Sink}(G^c_{j-1})$. Let $v$ be a vertex in $G^c_{j-1}$. Since a sink does not have descendants, $\text{Sink}(G^c_{j-1}) \cap \text{Ancest}(v, G^c_{j-1}) = \emptyset$ for all $v \in G^c_{j-1}$. Thus, $\text{Descend}(v_{new}, G^c_{j-1}) \cap \text{Ancest}(v, G^c_{j-1}) = \emptyset$. By setting $v = v_{new}$, we have $\text{Descend}(v_{new}, G^c_{j-1}) \cap \text{Ancest}(v_{new}, G^c_{j-1}) = \emptyset$, which implies that there is not a cycle that contains $v_{new}$. This completes the proof.

Proof of b] Let $G^c_{j-1} = (V_{j-1}^c, E_{j-1}^c)$ and $G^c_{j} = (V_{j}^c, E_{j}^c)$. The edge contraction builds $G^c_{j}$ from $G^c_{j-1}$ so that all edges of $(p, c)$ are inherited to the new vertex by changing the terminals from the replaced vertex to the new vertex. Thus, the graph $G^c_{j}$ reflects all adjacency of $G^c_{j-1}$. Precisely, if $X_j \in \mathcal{X}_{j-1}$, $X_k \in \mathcal{X}_{k-1}$, $v_k \in \text{Parent}(v_j, G^c_{j-1})$, $X_j \in \mathcal{X}_j$, $X_k \in \mathcal{X}_k$, then $\tilde{v}_k \in \text{Parent}(\tilde{v}_j, G^c_j)$.

Since the dependency of the subsystems determines the adjacency in $G^c$, we establish the statement [b] by applying the property [S11] recursively to the sequence $G^c_{1}, \ldots, G^c_{\mathcal{V}^c_{\mathcal{E}}}$. Note that $G^c_{1}$ is a connected component of $G^c$.

S3. DERIVATION OF OUR SECOND-LAW-LIKE INEQUALITY

Here, we see the proof of the inequality [4]. To this end, since $\Omega_{X_k} \geq 0$, it is sufficient to verify that $\sum_{k=1}^{N} \Omega_{X_k} = \Delta S(X_{tot}) + Q_{tot} - D$. Extending the definition of $\Omega_{X_k}$ given in Eq. (7), let us recursively define $\Omega_v$ for all $v$ that is a vertex of $G^c_{1}, \ldots, G^c_{\mathcal{V}^c_{\mathcal{E}}}$ as follows:

$$\Omega_v = \sum_{X_j \in \text{Flat}(v)} \Omega_{X_j}$$  \hspace{1cm} (S12)

We denote by $\hat{v}_{G^c}$ the vertex of the last graph $G^c_{\mathcal{V}^c_{\mathcal{E}}}$ of the graph sequence. Since $\hat{v}_{G^c}$ is generated by the edge contractions for all vertices in $G^c$,

$$\Omega_{G^c} := \Omega_{\hat{v}_{G^c}} = \sum_{v \in \mathcal{V}^c_{\mathcal{E}}} \Omega_{v}$$  \hspace{1cm} (S13)

where we simply denote $\Omega_{\hat{v}_{G^c}}$ by $\Omega_{G^c}$. Let $V^v$ be the set containing all of $(p, c)$ selected in the process of generating a specific vertex $v$. Because $\hat{v}_{G^c}$ is the only single vertex of the last graph $G^c_{\mathcal{V}^c_{\mathcal{E}}}$, $\Omega_{\hat{v}_{G^c}}$ is identical to $\hat{v}_{G^c}$ by the definition of $\Omega_{G^c}$. The following Lemma claims that $\Omega_{G^c}$ is decomposed into the entropy production inside a connected component and $\hat{D}_{G^c}$.

Lemma 2. Let $G^c$ be the DAG induced from the dependencies of the subsystems, and $G^c = (V^c, E^c)$ be a connected component of $G^c$. Then,

$$\Omega_{G^c} = S(V^c) - S(V^c) + Q_{\mathcal{V}^c_{\mathcal{E}}} - \hat{D}_{G^c}.$$  \hspace{1cm} (S14)

Proof. Suppose that $|V^c| = 1$. Then, $V^c = \{\hat{v}_{G^c}\}$. Thus, $\Omega_{G^c} = \Omega_{\hat{v}_{G^c}}$. Likewise,

$$S(V^c) = S(\hat{v}_{G^c}), \quad S(V^c) = S(\hat{v}_{G^c}), \quad Q_{\mathcal{V}^c_{\mathcal{E}}} = Q_{\hat{v}_{G^c}}, \quad \hat{D}_{G^c} = 0.$$  \hspace{1cm} (S15)

Hence, the RHS of Eq. (S14) is identical to $S(\hat{v}_{G^c}) - S(\hat{v}_{G^c}) + Q_{\hat{v}_{G^c}}$. Since $\hat{v}_{G^c}$ contains exactly one subsystem due to the condition $|V^c| = 1$, by the definition of $\Omega_{\hat{v}_{G^c}}$ shown in (S12), $\Omega_{\hat{v}_{G^c}} = S(\hat{v}_{G^c}) - S(\hat{v}_{G^c}) + Q_{\hat{v}_{G^c}}$. By these observations, Eq. (S14) holds if $|V^c| = 1$.

Suppose that $|V^c| \geq 2$ and $(p, c) \in V^c_{\mathcal{E}}$. Let $D_{p,c}$ be the quantity provided by

$$D_{p,c} := - J_{p,c} - K_{p,c} - L_{p,c} - M_{p,c}$$  \hspace{1cm} (S16)

Let us observe that the conclusion [S14] follows if the following equation holds:

$$\Omega_p + \Omega_c = S(p’, c’ | \Gamma_{p,c}) - S(p, c | \Gamma_{p,c}) + Q_{p,c} - D_{p,c} - \sum_{(\bar{p}, \bar{c}) \in V^p} D_{\bar{p}, \bar{c}} - \sum_{(\bar{p}, \bar{c}) \in V^c} D_{\bar{p}, \bar{c}}$$  \hspace{1cm} (S17)
For a connected component $G^{cc} = (V^{cc}, E^{cc})$, let $(p_{\text{fin}}, c_{\text{fin}})$ be the vertex couple selected from $\mathcal{G}_{V^{cc}}$ as the target of the edge contraction. Since $v_{G^{cc}} = p_{\text{fin}} \cup c_{\text{fin}},$

\[
\text{Flat}(V^{cc}) = \text{Flat}(p_{\text{fin}} \cup c_{\text{fin}}),
\]
\[
\Omega_{G^{cc}} = \Omega_{p_{\text{fin}}} + \Omega_{c_{\text{fin}}}. 
\]  
(S18)  
(S19)

Since $G^{seq}_{V^{cc}}$ has only two vertices, $p_{\text{fin}}$ and $c_{\text{fin}},$

\[
\Gamma_{p_{\text{fin}} \cup c_{\text{fin}}} = \emptyset. 
\]  
(S20)

By the convention shown in Eq. (14) and the observations (S18) and (S20),

\[
S(p_{\text{fin}}, c_{\text{fin}} | \Gamma_{p_{\text{fin}} \cup c_{\text{fin}}}^-) = S(V^{cc}),
\]
\[
S(p_{\text{fin}}, c_{\text{fin}} | \Gamma_{p_{\text{fin}} \cup c_{\text{fin}}}^+) = S(V^{cc'}). 
\]  
(S21)  
(S22)

From Eq. (18),

\[
Q_{p_{\text{fin}} \cup c_{\text{fin}}} = Q_{V^{cc}}. 
\]  
(S23)

Since $V_{G^{cc}} = (p_{\text{fin}}, c_{\text{fin}}) \cup V^{p} \cup V^{c},$ by the definition of $\hat{D}_{G^{cc}}$ in Eq. (16),

\[
\hat{D}_{G^{cc}} = D_{p_{\text{fin}}, c_{\text{fin}}} + \sum_{(\bar{p}, \bar{c}) \in V^{p_{\text{fin}}}} D_{\bar{p}, \bar{c}} + \sum_{(\bar{p}, \bar{c}) \in V^{c_{\text{fin}}}} D_{\bar{p}, \bar{c}}. 
\]  
(S24)

Now, we use Eq. (17) as a hypothesis. Then,

\[
\Omega_{G^{cc}} \overset{\text{(S19)}}{=} \Omega_{p_{\text{fin}}} + \Omega_{c_{\text{fin}}}
\]
\[
\overset{\text{(S13)}}{=} S(p_{\text{fin}}, c_{\text{fin}} | \Gamma_{p_{\text{fin}} \cup c_{\text{fin}}}^-) - S(p_{\text{fin}}, c_{\text{fin}} | \Gamma_{p_{\text{fin}} \cup c_{\text{fin}}}^+) + Q_{p_{\text{fin}}} - D_{p_{\text{fin}}} - \sum_{(\bar{p}, \bar{c}) \in V^{p}} D_{\bar{p}, \bar{c}} - \sum_{(\bar{p}, \bar{c}) \in V^{c}} D_{\bar{p}, \bar{c}}. 
\]  
(S25)  
(S26)

By Eqs. (S21) - (S24), the right-hand side of Eq. (S26) coincides with the right-hand side of Eq. (S14). Accordingly, it suffices to verify Eq. (S17) to prove the conclusion (S14).

To demonstrate Eq. (S17), we prepare two relations that hold for any $(p, c) \in V_{G^{cc}}$. Firstly, suppose that $v_{\text{new}} = p \cup c$. Since $V^{\text{new}} = (p, c) \cup V^{p} \cup V^{c},$ we have the first relation

\[
\sum_{(\bar{p}, \bar{c}) \in V^{\text{new}}} D_{\bar{p}, \bar{c}} = D_{p_{\text{fin}}} + \sum_{(\bar{p}, \bar{c}) \in V^{p}} D_{\bar{p}, \bar{c}} + \sum_{(\bar{p}, \bar{c}) \in V^{c}} D_{\bar{p}, \bar{c}}. 
\]  
(S27)

Secondly, by Eq. (S91),

\[
S(p' | p, \Gamma_{p_{\text{fin}}}) = S(p' | c', p, \Gamma_{p_{\text{fin}}}). 
\]  
(S28)

By Eq. (S95),

\[
I(p'; c' | p, \Gamma_{p_{\text{fin}}}) = S(p' | p, \Gamma_{p_{\text{fin}}}) - S(p' | c', p, \Gamma_{p_{\text{fin}}}). 
\]  
(S29)

By Eq. (S28), the right-hand side of the above vanishes. Thus,

\[
I(p'; c' | p, \Gamma_{p_{\text{fin}}}) = 0. 
\]  
(S30)

By Eqs. (S94) and (S96),

\[
S(p | \Gamma_p) + S(c | \Gamma_c) \overset{\text{(S90)}}{=} S(p | \Gamma_p) + S(c | p, \Gamma_c^-) + I(p; c | \Gamma_c)
\]
\[
\overset{\text{(S90)}}{=} S(p | \Gamma_{p_{\text{fin}}}) + S(c | p, \Gamma_{p_{\text{fin}}}) + I(p; \Gamma_{p_{\text{fin}}}^- | \Gamma_p) + I(c; \Gamma_{p_{\text{fin}}}^- | p, \Gamma_c^-) + I(p; c | \Gamma_c)
\]
\[
\overset{\text{(S33)}}{=} S(p, c | \Gamma_{p_{\text{fin}}}) + I(p; \Gamma_{p_{\text{fin}}}^- | \Gamma_p) + I(c; \Gamma_{p_{\text{fin}}}^- | p, \Gamma_c^-) + I(c; p | \Gamma_c). 
\]  
(S31)  
(S32)  
(S33)
By Eqs. (S30) and (S33),

\[ S\left(c' \mid p, \Gamma_{p,c}\right) = S\left(c' \mid p', p, \Gamma_{p,c}\right) + I\left(p'; c' \mid p, \Gamma_{p,c}\right) \]  
(S34)

\[ S\left(c' \mid p', \Gamma_{p,c}\right) - I\left(p; c' \mid p', \Gamma_{p,c}\right) + I\left(p'; c' \mid p, \Gamma_{p,c}\right) \]  
(S35)

\[ S\left(c' \mid p', \Gamma_{p,c}\right) + J_{p,c}. \]  
(S36)

With the similar way of the derivation of Eq. (S33),

\[ S\left(p' \mid \Gamma_p\right) + S\left(c' \mid \Gamma_c\right) = S\left(p' \mid \Gamma_p\right) + S\left(c' \mid p, \Gamma_c\right) + I\left(p; c' \mid \Gamma_c\right) \]  
(S37)

\[ S\left(p' \mid \Gamma_{p,c}\right) + S\left(c' \mid p, \Gamma_{p,c}\right) + I\left(p'; \Gamma_{c,p} \mid \Gamma_p\right) + I\left(c'; p \mid \Gamma_c\right) \]  
(S38)

\[ S\left(p' \mid \Gamma_{p,c}\right) + S\left(c' \mid p', \Gamma_{p,c}\right) + J_{p,c} + I\left(p'; \Gamma_{c,p} \mid \Gamma_p\right) + I\left(c'; p \mid \Gamma_c\right). \]  
(S39)

Hence, by (S39) - (S33), we have a second relation

\[ S\left(p' \mid \Gamma_p\right) - S\left(p \mid \Gamma_p\right) + S\left(c' \mid \Gamma_c\right) - S\left(c \mid \Gamma_c\right) = S\left(p', c' \mid \Gamma_{p,c}\right) - S\left(p, c \mid \Gamma_{p,c}\right) - D_{p,c}. \]  
(S40)

Now, we demonstrate Eq. (S17) by mathematical induction for the sequences of \((p, c)\) selected in the graph sequence \(G_1^{\text{seq}}, \ldots, G_{\text{tot}}^{\text{seq}}\). As the base case, we verify that Eq. (S17) holds under the hypothesis that vertex couple \((p, c)\) is selected from \(G_1^{\text{seq}}\). Since any vertex of \(G_1^{\text{seq}}\) contains exactly one subsystem, by the definition of \(\Omega_{X_k}\) shown in Eq. (7),

\[ \Omega_p = S\left(p' \mid \Gamma_p\right) - S\left(p \mid \Gamma_p\right) + Q_p, \]  
(S41)

\[ \Omega_c = S\left(c' \mid \Gamma_c\right) - S\left(c \mid \Gamma_c\right) + Q_c. \]  
(S42)

Thus, by Eq. (S40),

\[ \Omega_p + \Omega_c = S\left(p', c' \mid \Gamma_{p,c}\right) - S\left(p, c \mid \Gamma_{p,c}\right) + Q_{p,c} - D_{p,c}. \]  
(S43)

Since \((p, c)\) is the first vertex couple due to the hypothesis of the base case, both \(V^p\) and \(V^c\) are the empty sets. Then, the last two terms in Eq. (S17) vanish. Consequently, Eq. (S43) coincides with Eq. (S17). This completes the base case of the induction.

As the induction step, we demonstrate Eq. (S17) for \((p, c)\) selected in \(G_{j+1}^{\text{seq}}\) under the hypothesis that Eq. (S17) holds for \((p, c)\) selected from any of \(G_1^{\text{seq}}, \ldots, G_j^{\text{seq}}\). Let \((p_+, c_+)\) be the vertex couple selected from \(G_{j+1}^{\text{seq}}\). Suppose that \(v \in G_{j+1}^{\text{seq}}\). Then, because any vertex in \(G_{j+1}^{\text{seq}}\) is either a vertex of \(G_\Gamma\) or a newly generated vertex through the edge contraction and because we hypothesized that Eq. (S17) holds for \((p, c)\) selected from any of \(G_1^{\text{seq}}, \ldots, G_j^{\text{seq}}\),

\[ \Omega_v = S\left(v' \mid \Gamma_v\right) - S\left(v \mid \Gamma_v\right) + Q_v - \sum_{(\bar{p}, \bar{c}) \in V_v} D_{\bar{p}, \bar{c}}, \]  
(S44)

where we use Eq. (S27) to obtain the last term of the right-hand side. Since both \(p_+\) and \(c_+\) are vertices of \(G_{j+1}^{\text{seq}}\), we can assign either \(p_+\) or \(c_+\) to \(v\) in Eq. (S44):

\[ \Omega_{p_+} = S\left(p_+ \mid \Gamma_p\right) - S\left(p \mid \Gamma_p\right) + Q_{p_+} - \sum_{(\bar{p}, \bar{c}) \in V_p} D_{\bar{p}, \bar{c}}, \]  
(S45)

\[ \Omega_{c_+} = S\left(c_+ \mid \Gamma_c\right) - S\left(c \mid \Gamma_c\right) + Q_{c_+} - \sum_{(\bar{p}, \bar{c}) \in V_c} D_{\bar{p}, \bar{c}}. \]  
(S46)

By Eq. (S40), the sum of (S45) and (S46) coincides with Eq. (S17). This completes the induction to verify Eq. (S17). Accordingly, the conclusion Eq. (S14) follows.

**Theorem 1.** Let \(X_1, \ldots, X_N\) be the state of the subsystems of the whole system of which the state is denoted by \(X_{\text{tot}}\). Suppose that the graph \(G_\Gamma\) induced from the dependency among \(X_1, \ldots, X_N\) does not contain a cycle. Then,

\[ \sum_{k=1}^{N} \Omega_{X_k} = \Delta S\left(X_{\text{tot}}\right) + Q_{\text{tot}} - D. \]  
(S47)
Proof. Let $G^{cc} = (V^{cc}, E^{cc})$ be a connected component of $G^\Gamma$. We write the indexed set of all connected components of $G^\Gamma$ as follows:

$$
\{G_j^{cc}\}_{j=1,2,\ldots,N^\Gamma} = \{(V_j^{cc}, E_j^{cc})\}_{j=1,2,\ldots,N^\Gamma},
$$

(S48)

where $N^\Gamma$ is the number of the connected component in $G^\Gamma = (V^\Gamma, E^\Gamma)$. Since $\bigcup_{j=1}^{N^\Gamma} G_j^{cc} = G^\Gamma$, we have

$$
V^\Gamma = V^{cc}_{1:N^\Gamma},
$$

(S49)

where we denote by using a colon that $V^{cc}_{1:j} := \bigcup_{k=1}^{j} V_k^{cc}$. Since $S(V^\Gamma) = S(X_{tot})$, we have $S(V^{cc}_{1:N^\Gamma}) = S(X_{tot})$ by (S49). Consequently,

$$
\sum_{G^{cc} \in G^\Gamma} S(V^{cc}) = \sum_{j=1}^{N^\Gamma} S(V_j^{cc})
$$

(S50)

By a similar calculation,

$$
\sum_{G^{cc} \in G^\Gamma} S(V^{cc}) = S(X_{tot}') + \sum_{j=2}^{N^\Gamma} I(V_j^{cc'}, V_{1:j-1}^{cc'}).
$$

(S53)

By the definition of $\overline{D}$ shown in Eq. (21), the difference of Eq. (S53) and Eq. (S52) leads to

$$
\sum_{G^{cc} \in G^\Gamma} [S(V^{cc'}) - S(V^{cc})] = \Delta S(X_{tot}) - \overline{D}.
$$

(S54)

Meanwhile, by taking the sum for $G^{cc}$ on both sides of Eq. (S14),

$$
\sum_{k=1}^{N} \Omega_{X_k} = \sum_{G^{cc} \in G^\Gamma} \left[ S(V^{cc'}) - S(V^{cc}) \right] + \sum_{G^{cc} \in G^\Gamma} \left[ Q_{V^{cc}} - \hat{D}_{G^{cc}} \right],
$$

(S55)

where we use the following relation:

$$
\sum_{k=1}^{N} \Omega_{X_k} \overset{S19}{=} \sum_{G^{cc} \in G^\Gamma} \Omega_{G^{cc}}.
$$

(S56)

By the definition of $Q_{tot}$ and $D$,

$$
\sum_{G^{cc} \in G^\Gamma} \left[ Q_{V^{cc}} - \hat{D}_{G^{cc}} \right] = Q_{tot} - D + \overline{D}.
$$

(S57)

Thus, by substituting Eqs. (S54) and (S57) for Eq. (S55), we have Eq. (S17). This completes the proof.

We rewrite $X_1$ and $X_2:N$ as $Y$ and $X$. Let $D^X$ be $D$ determined by the definition (22) under the setting where the subsystems $2,3,\ldots,N$ constitute the whole system, and the original dependency among them remains. Let $\hat{\Gamma}_{X_k} := \Gamma_{X_k} \setminus Y$. Then, by Theorem I, we have

$$
D^X = S(X') - S(X) - \sum_{k=2}^{N} \left[ S(X_k' | \hat{\Gamma}_{X_k}) - S(X_k | \hat{\Gamma}_{X_k}) \right],
$$

(S58)
Likewise, let $D^{X|Y}$ be the quantity provided by adding $Y$ as the conditioning event in $\mathcal{J}_{p,c}$ to $\mathcal{M}_{p,c}$ when calculating $D^X$. Then by Eq. (S58),

$$
D^{X|Y} = S(X' | Y) - S(X | Y) - \sum_{k=2}^{N} \left[ S \left( X_k' \mid \hat{\Gamma}_{X_k}, Y \right) - S \left( X_k \mid \hat{\Gamma}_{X_k}, Y \right) \right] 
$$

(S59)

$$
= S(X' | Y) - S(X | Y) - \sum_{k=2}^{N} \left[ S(X_k' \mid \Gamma_{X_k}) - S(X_k \mid \Gamma_{X_k}) \right] 
$$

(S60)

The following corollary establishes the SLLI corresponding to Eq. (S6).

**Corollary 1.** We assume the same hypothesis in Theorem [2]. Then,

$$
\sum_{k=2}^{N} \Omega_{X_k} = \Delta S(X) + Q - I(X'; Y) + I(X; Y) - D^{X|Y} 
$$

(S61)

**Proof.** We infer

$$
\sum_{k=2}^{N} \Omega_{X_k} \equiv \sum_{k=2}^{N} \left[ S(X_k' \mid \Gamma_{X_k}) - S(X_k \mid \Gamma_{X_k}) + Q_{X_k} \right] 
$$

(S62)

$$
\equiv S(X' | Y) - S(X | Y) + Q - D^{X|Y} 
$$

(S63)

$$
= S(X') - S(X) - I(X'; Y) + I(X; Y) + Q - D^{X|Y} 
$$

(S64)

This completes the proof. \[\square\]

**S4. POSITIVITY OF $D$**

In preparation for establishing the positivity of $D$, we formally define the independence of random variables in Definition (5.4). We see only the definition of conditional independence because it contains a not-conditional one as a special case. The origin of the positivity of $D$ is that the specific types of mutual information production are non-increasing through independent processes. Using the results of Lemmas (5.5 and 5.6) Lemmas (5.5 and 5.6) establish that these types of mutual information production can be rewritten by negative mutual information, which immediately result in the monotonicity of the specific types of mutual information.

**Definition 1** (Independence (5.4)). Let $A, B,$ and $Z$ be random variables. If

$$
P(A, B, Z)P(Z) = P(A, Z)P(B, Z), 
$$

(S65)

then $A$ is independent of $B$ conditioned on $Z$, denoted by $A \perp B \mid Z$.

When $P(Z) \neq 0$, by dividing Eq. (S65) by $P(Z)$ twice, we obtain

$$
P(A, B \mid Z) = P(A \mid Z)P(B \mid Z). 
$$

(S66)

Suppose that $A \perp B \mid Z$. Then, by Eq. (S66),

$$
P(A \mid Z) = \frac{P(A, B \mid Z)}{P(B \mid Z)} = P(A \mid B, Z). 
$$

(S67)

We note that $X_k \perp X_j \mid \Gamma_{X_k}$ if $X_j \notin \Gamma_{X_k}$.

We denote by $Z \leadsto Z'$ the time step from $Z$ to $Z'$. The independency of processes is defined as follows.

**Definition 2** (Independent process). Let $A, A', B, B'$, and $Z$ be random variables.

- If $A' \perp B \mid (A, Z)$ holds, the time step $A \leadsto A'$ is called an independent process of $A$ from $B$ conditioned on $Z$. We also say that the random variable $A$ evolves independently from $B$ conditioned on $Z$.

- If both $A' \perp B \mid (A, Z)$ and $B' \perp A \mid (B, Z)$ hold, the time step $(A, B) \leadsto (A', B')$ is called a mutually independent process conditioned on $Z$. We also say that the random variable $A$ and $B$ evolves mutually independently conditioned on $Z$. 
Lemma 3. Let $A, A', B, B'$, and $Z$ be random variables.

- Assume that the time step $(A, B) \rightarrow (A', B')$ is an mutually independent process conditioned on $Z$. Then,

$$I (A'; B' \mid A, B, Z) = 0. \quad (S68)$$

- Assume that $A$ evolves independently from $B$ conditioned on $Z$. Then,

$$I (A'; B \mid A, Z) = 0. \quad (S69)$$

Proof. Proof of Eq. (S68): By the chain rule,

$$P(A', B' \mid A, B, Z) = P(A' \mid B', A, B, Z)P(B' \mid A, B, Z). \quad (S70)$$

By the chain rule and Eq. (S67), the hypothesis of the dependency leads to

$$P(A' \mid B', A, B, Z) = P(A' \mid A, B, Z). \quad (S71)$$

By substituting Eq. (S71) for Eq. (S70),

$$P(A', B' \mid A, B, Z) = P(A' \mid A, B, Z)P(B' \mid A, B, Z). \quad (S72)$$

Therefore,

$$I (A'; B' \mid A, B, Z) \overset{\text{def.}}{=} \langle \ln \frac{P(A', B' \mid A, B, Z)}{P(A' \mid A, B, Z)P(B' \mid A, B, Z)} \rangle = 0. \quad (S73)$$

Proof of Eq. (S69): By the chain rule and Eq. (S67),

$$P(A', B \mid A, Z) = P(A' \mid A, Z)P(B \mid A', A, Z) = P(A' \mid A, Z)P(B \mid A, Z). \quad (S74)$$

As with the proof of Eq. (S68) above, this observation completes the proof.

Lemma 4. Let $A, A', B$, and $B'$ be random variables. Assume that $(A, B) \rightarrow (A', B')$ is a non-conditional mutually independent process. Then,

$$I (A'; B') = I (A'; B'; A; B). \quad (S75)$$

Proof. By Eq. (S68),

$$I (A'; B' \mid A, B) = 0. \quad (S76)$$

Likewise,

$$I (A'; B'; A \mid B) \overset{(S74)}{=} I (B'; A \mid B) - I (B'; A \mid A', B) \overset{(S69)}{=} 0. \quad (S77)$$

See Section S6 for the definition of the interaction information, so to speak, multivariate mutual information. By a similar computation,

$$I (A'; B'; B \mid A) = 0. \quad (S78)$$

From these observations,

$$I (A'; B') \overset{(S74)}{=} I (A'; B'; A; B) + I (A'; B' \mid A, B) + I (A'; B'; A \mid B) + I (A'; B'; B \mid A) \quad (S79)$$

$$= I (A'; B'; A; B). \quad (S80)$$

Lemma 5. Let $A, A', B, W,$ and $Z$ be random variables. Assume that $A$ evolves independently from $B$ conditioned on $Z$. Then,

$$I (A'; B \mid W, Z) - I (A; B \mid W, Z) = -I (A; B \mid A', W, Z) \quad (S81)$$
Proof.

\[ I(\mathbf{A}'; B \mid W, Z) - I(\mathbf{A}; B \mid W, Z) = I(\mathbf{A}'; B \mid A, W, Z) + I(\mathbf{A}'; B \mid A, W, Z) - I(\mathbf{A}; B \mid W, Z) \quad (\text{S82}) \]

\[ \overset{\text{S81}}{=} -I(\mathbf{A}; B \mid \mathbf{A}', W, Z) \quad (\text{S83}) \]

Lemma 6. Let \( \mathbf{A}', \mathbf{B}, \) and \( \mathbf{B}' \) be random variables. Assume that \( \mathbf{A}, \mathbf{B} \sim \mathbf{A}', \mathbf{B}' \) is a non-conditional mutually independent process. Then,

\[ I(\mathbf{A}'; \mathbf{B}') - I(\mathbf{A}; \mathbf{B}) = -I(\mathbf{A}; \mathbf{B} \mid \mathbf{A}', \mathbf{B}') \quad (\text{S84}) \]

Proof.

\[ I(\mathbf{A}'; \mathbf{B}') - I(\mathbf{A}; \mathbf{B}) \overset{\text{S81}}{=} I(\mathbf{A}'; \mathbf{B}'; \mathbf{B}) - I(\mathbf{A}; \mathbf{B}) \overset{\text{S81}}{=} -I(\mathbf{A}; \mathbf{B} \mid \mathbf{A}', \mathbf{B}') \quad (\text{S85}) \]

The following theorem establishes the positivity of \( D \).

Theorem 2. Suppose that \( G^F \) is the graph induced by the dependency of the subsystems. Then, \( D \), defined as in Eq. (\text{S24}), is positive: \( D \geq 0 \).

Proof. Let \( G^cc \) be a connected component of \( G^F \), and \( (p, c) \) be a vertex couple that is the target of the edge contraction. By Eq. (\text{S81}), the dependency provided by the definitions (\text{I5}) leads to

\[ J_{p,c} = -I(p; c' \mid p', \Gamma_{p,c}), \quad K_{p,c} = -I(p; \Gamma_{p,c}^{-} \mid p', \Gamma_p), \quad L_{p,c} = -I(c; \Gamma_{p,c}^{-} \mid c', p, \Gamma_c), \quad M_{p,c} = -I(c; p \mid c', \Gamma_c) \quad (\text{S86}) \]

Thus, all \( J_{p,c}, K_{p,c}, L_{p,c}, \) and \( M_{p,c} \) are non-conditional independent processes. Hence, by Eq. (\text{S84}),

\[ (V^cc_{j}; V^cc_{1:j-1}) - I(V^cc_{j}; V^cc_{1:j-1}) = I(V^cc_{j}; V^cc_{1:j-1} \mid V^cc'_{j}; V^cc'_{1:j-1}) \geq 0. \quad (\text{S87}) \]

Since \( D \) is identical to the sum of the left-hand side of the inequality (\text{S87}), it follows that \( D \geq 0 \). Lastly, \( D \) is defined as the sum of the \( D_{G^cc} \) and \( D_{cc} \). Hence, positivity of \( D_{G^cc} \) and \( D \), \( D \) is positive.

S5. CALCULATION OF \( D \) IN EXAMPLE 1

Here, we calculate \( D \) appeared in Example 1.

A. Feedback control to the ideal gases

The feedback control from the controller \( X_1 \) to the ideal gases \( X_{2:N} \) has the dependency as follows:

\[ \Gamma_{X_k} = \begin{cases} \emptyset & (k = 1) \\ \{X_1\} & (k = 2, 3, \ldots, N), \end{cases} \quad (\text{S88}) \]

Then, we can choose the sequence of the vertex couple \( (p, c) \) as follows:

\[ (X_1, X_2), (X_{1:2}, X_3), \ldots, (X_{1:j-1}, X_j), \ldots, (X_{1:N-1}, X_N). \quad (\text{S89}) \]

We calculate \( J_{p,c} \) through \( M_{p,c} \) by Eqs. (\text{S80}) and the assumption regarding the initial and final states. Since \( I(X_{1:j-1}; X_j' \mid X_{1:j-1}') = 0 \) and \( \Gamma_{p,c} = \emptyset \), we have \( J_{p,c} = 0 \). Since \( \Gamma_{p,c} = \Gamma_{p,c}^{-} = \emptyset \), we have \( K_{p,c} = L_{p,c} = 0 \). Since \( I(X_j; X_{1:j-1} \mid X_j', X_1) = 0 \), we have \( M_{p,c} = 0 \). To obtain \( M_{p,c} \), we use \( \Gamma_c = X_1 \) and \( S(X_j \mid X_1) = 0 \) for \( j \geq 2 \). Hence, we have \( D_{G^cc} = 0 \). Since \( G^F \) has exactly one connected component, we have \( D = 0 \). Accordingly, \( D = 0 \).
B. Open-loop control to the ideal gases

As described in the main text, we can consider the cases with and without the measurement (OLM and OL) for the open-loop control. In addition, there are two variations for the OL case in terms of the influence of the memory state on the controlled system.

With the measurement, namely the OLM case, the memory’s state does not influence the controlled system’s evolution. Then, the dependency is provided by \( \Gamma_{X_k} = \emptyset \) for all \( k \in \{1, 2, \ldots, N\} \). The graph \( G^D \) has \( N \) connected components: \( N^{cc} = N \). Since \( |V^{cc}| = 1 \), we have \( V_{G^{cc}} = \emptyset \). Hence, we obtain \( \hat{D}_{G^D} = 0 \) by the definition of \( \hat{D}_{G^D} \). Hence, we have \( D = \overline{D} \). Since all vertices contain exactly one subsystem, we have

\[
\overline{D} = \sum_{j=2}^N I(X_j; X_{1:j-1}) - I(X'_j; X'_{1:j-1}) \tag{S90}
\]

By the correlations assumed in the main text, we obtain \( \overline{D} = (N-1)\ln 2 \). Consequently, \( D = (N-1)\ln 2 \).

Without measurement, namely the OL case, if the memory’s state does not influence the controlled system, the dependency is the same as the OLM case. Thus, \( D \) is provided by the right-hand side of \( [S90] \). Since the measurement is not performed, we have

\[
I(X_1; X_j) = 0 \quad (j \geq 2). \tag{S91}
\]

Accordingly, we obtain \( D = (N-2)\ln 2 \).

Furthermore, when the memory’s state influences the controlled system in the OL case, the dependency and the sequence of \( (p, c) \) are the same as those in the feedback control shown in Eqs. \( [S88] \) and \( [S89] \). The only difference from the feedback control case is that the memory state \( X_1 \) does not correlate to the controlled system, as shown in Eq. \( [S91] \). Then, we still have \( \mathcal{J}_{p,c} = \mathcal{K}_{p,c} = \mathcal{L}_{p,c} = 0 \) by the same calculation in the feedback control case. Meanwhile, we have \( \mathcal{M}_{p,c} = 0 \) when \( (p, c) = (X_1, X_2) \), and otherwise we have \( \mathcal{M}_{p,c} = \ln 2 \) as described follows. The memory is not correlated to the controlled system: \( I(X_j; X_1) = 0 \) for \( j \geq 2 \). In the initial state, all subsystems of the controlled system are mutually correlated: \( I(X_j; X_k) = \ln 2 \) for \( j, k \geq 2 \). Moreover, all correlations of the controlled system vanish in the final state: \( I(X'_j; X_k) = 0 \) for \( j, k \geq 2 \). Thus, we obtain

\[
\mathcal{M}_{p,c} \mathcal{S}S88 \mathcal{S} - I(X_j; X_{1:j-1} \mid X'_j, X_1) = \begin{cases} 0 & \text{if } (p, c) = (X_1, X_2) \\ \ln 2 & \text{otherwise}. \end{cases} \tag{S92}
\]

Hence, we have \( \hat{D}_{G^{cc}} = (N-2)\ln 2 \) because the sequence \([S90]\) consists of \( (N-1) \) pairs of \( p \) and \( c \). In addition, we have \( \overline{D} = 0 \) because the number of the connected components is 1. Consequently, we obtain \( D = (N-2)\ln 2 \).

S6. Formulas of Shannon information

In this section, we summarize formulas of Shannon information that we have used previously. The conditioning variables can be \( \emptyset \), and then Eq. \( [114] \) are applied.

Let \( A_{1:n} = \{A_1, A_2, \ldots, A_n\} \) be random variables. Conditional interaction information, so to speak, multivariate mutual information is recursively defined as follows \([53, 54]\):

\[
I(A_1; A_2 \ldots; A_{j+1} \mid Z) := I(A; A_2; \ldots; A_j \mid Z) - I(A_1; \ldots; A_j \mid A_{j+1}, Z). \tag{S93}
\]

If \( Z = \emptyset \), this becomes non-conditional one. By taking \( n = 2 \), \( A_1 = A \), \( A_2 = B \) and \( A_3 = C \) in Eq. \( [S93] \), we have

\[
I(A; B \mid Z) = I(A; B; C \mid Z) + I(A; B \mid C, Z). \tag{S94}
\]

Let \( A, B, \) and \( Z \) be random variables. Then, the following relations are well known \([57]\):

\[
S(A; B \mid Z) = S(A \mid Z) + S(A \mid B, Z), \tag{S95}
\]

\[
S(A \mid B, Z) = S(A \mid Z) + I(A; B \mid Z). \tag{S96}
\]

Proposition 1. Let \( A_{1:n} = \{A_1, A_2, \ldots, A_n\} \), and \( Z \) be random variables. Then,

\[
S(A_{1:n} \mid Z) = \sum_{j=1}^n S(A_j \mid Z) - \sum_{j=2}^n I(A_j; A_{1:j-1} \mid Z). \tag{S97}
\]
Proof. By the chain rule of entropy,

\[ S(A_{1:n} \mid Z) = S(A_1 \mid Z) + \sum_{j=2}^{n} S(A_j \mid A_{1:j-1}, Z). \]  

(S98)

By Eq. [S96],

\[ S(A_j \mid A_{1:j-1}, Z) = S(A_j \mid Z) - I(A_j; A_{1:j-1} \mid Z). \]  

(S99)

Substituting Eq. [S99] for Eq. [S98], we establish Eq. [S97].

By replacing the order of the indices 1, 2, ..., n with 2, 3, ..., n, 1, Eq. [S97] can be rewritten as follows:

\[ S(A_{1:n} \mid Z) = \sum_{j=1}^{n} S(A_j \mid Z) - \sum_{j=3}^{n} I(A_j; A_{2:j-1} \mid Z) + I(A_1; A_{2:n} \mid Z). \]  

(S100)

**Proposition 2.** Let A, B, C, D, and Z be random variables. Then the following properties hold:

\[ I(A; B \mid Z) - I(A; B; C \mid D, Z) = I(A; B \mid C, Z) + I(A; B; C \mid D, Z), \]  

(S101)

\[ I(A; B) = I(A; B; C; D) + I(A; B \mid C, D) + I(A; B; C \mid D) + I(A; B; D \mid C). \]  

(S102)

**Proof. Proof of Eq. (S101):** Using Eq. [S93] twice,

\[ I(A; B \mid Z) - I(A; B; C \mid D, Z) = I(A; B \mid C, Z) + I(A; B; C \mid Z) - I(A; B; C; D \mid Z) \]  

(S103)

\[ = I(A; B \mid C, Z) + I(A; B; C \mid D, Z). \]  

(S104)

It completes the proof.

**Proof of Eq. (S102):** By Eq. [S101],

\[ I(A; B) = I(A; B; C; D) + I(A; B \mid C) + I(A; B; C \mid D). \]  

(S105)

By Eq. [S93],

\[ I(A; B \mid C) = I(A; B \mid C, D) + I(A; B; D \mid C). \]  

(S106)

Substituting Eq. [S106] for Eq. [S105], Eq. [S102] holds.