q-ANALOGUE OF THE DUNKL TRANSFORM ON THE REAL LINE

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Abstract. In this paper, we consider a q-analogue of the Dunkl operator on \( \mathbb{R} \), we define and study its associated Fourier transform which is a q-analogue of the Dunkl transform. In addition to several properties, we establish an inversion formula and prove a Plancherel theorem for this q-Dunkl transform. Next, we study the q-Dunkl intertwining operator and its dual via the q-analogues of the Riemann-Liouville and Weyl transforms. Using this dual intertwining operator, we provide a relation between the q-Dunkl transform and the \( q^2 \)-analogue Fourier transform introduced and studied in [17, 18].

Key Words: q-Dunkl operator, q-Dunkl transform, q-Dunkl intertwining operator.

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1. Introduction

The Dunkl operator on \( \mathbb{R} \) of index \( \left( \alpha + \frac{1}{2} \right) \) associated with the reflection group \( \mathbb{Z}_2 \) is the differential-difference operator \( \Lambda_\alpha \) introduced by C. F. Dunkl in [3] by

\[
\Lambda_\alpha(f)(x) = \frac{df(x)}{dx} + \left( \alpha + \frac{1}{2} \right) \frac{f(x) - f(-x)}{x}, \quad \alpha \geq -\frac{1}{2}.
\]

These operators are very important in pure mathematics and physics. They provide a useful tool in the study of special functions with root systems [4, 2] and they are closely related to certain representations of degenerate affine Hecke algebras [1, 16], moreover the commutative algebra generated by these operators has been used in the study of certain exactly solvable models of quantum mechanics, namely the Calogero-Sutherland-Moser models, which deal with systems of identical particles in a one dimensional space [15, 13].

In [5], C. F. Dunkl has introduced and studied a Fourier transform associated with the operator \( \Lambda_\alpha \), called Dunkl transform, but the basic results such as inversion formula and Plancherel theorem were established later by M. F. E. de Jeu in [10, 11].

C. F. Dunkl has proved in [4] that there exists a linear isomorphism \( V_\alpha \), called the Dunkl intertwining operator, from the space of polynomials on \( \mathbb{R} \) of degree \( n \) onto itself, satisfying the transmutation relation

\[
\Lambda_\alpha V_\alpha = V_\alpha \frac{d}{dx}, \quad V_\alpha(1) = 1.
\]

Next, K. Trimèche has proved in [19] that the operator \( V_\alpha \) can be extended to a topological isomorphism from \( \mathcal{E}(\mathbb{R}) \), the space of \( C^\infty \)-functions on \( \mathbb{R} \), onto itself satisfying the relation (2).

The goal of this paper is to provide a similar construction for a q-analogue context. The analogue transform we employ to make our construction is based on some q-Bessel functions and orthogonality results from [14], which have important applications to q-deformed mechanics. The q-analogue of the Bessel operator and the Dunkl operator are defined in
terms of the \(q^2\)-analogue differential operator, \(\partial_q\), introduced in [18].

This paper is organized as follows: In Section 2, we present some preliminaries results and notations that will be useful in the sequel. In Section 3, we establish some results associated with the \(q\)-Bessel transform and study the \(q\)-Riemann-Liouville and the \(q\)-Weyl operators. In Section 4, we introduce and study a \(q\)-analogue of the Dunkl operator (1) and we deal with its eigenfunctions by giving some of their properties and providing for them a \(q\)-integral representations of Mehler type as well as an orthogonality relation. In section 5, we define and study the \(q\)-Dunkl intertwining operator and its dual via the \(q\)-Riemann-Liouville and the \(q\)-Weyl transforms. Finally, in Section 6, we study the Fourier transform associated with the \(q\)-Dunkl operator (\(q\)-Dunkl transform), we establish an inversion formula, prove a Plancherel theorem and we provide a relation between the \(q\)-Dunkl transform and the \(q^2\)-analogue Fourier transform (see [17] [18]).

2. Notations and preliminaries

For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper. We refer the reader to the general references [8] and [12], for the definitions, notations and properties of the \(q\)-shifted factorials and the \(q\)-hypergeometric functions. Throughout this paper, we assume \(q\in[0, 1]\) and we denote \(R_q = \{\pm q^n : n \in \mathbb{Z}\}\), \(R_{q,+} = \{q^n : n \in \mathbb{Z}\}\).

2.1. Basic symbols. For \(x \in \mathbb{C}\), the \(q\)-shifted factorials are defined by

\[
(x; q)_0 = 1; \quad (x; q)_n = \prod_{k=0}^{n-1} (1 - qx^k), \quad n = 1, 2, \ldots; \quad (x; q)_\infty = \prod_{k=0}^{\infty} (1 - qx^k).
\]

We also denote

\[
[x]_q = \frac{1 - q^x}{1 - q}, \quad x \in \mathbb{C} \quad \text{and} \quad [n]_q! = \frac{(q; q)_n}{(1 - q)_n}, \quad n \in \mathbb{N}.
\]

2.2. Operators and elementary special functions.

The \(q\)-Gamma function is given by (see [9])

\[
\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x}, \quad x \neq 0, -1, -2, \ldots
\]

It satisfies the following relations

\[
\Gamma_q(x + 1) = [x]_q \Gamma_q(x), \quad \Gamma_q(1) = 1 \quad \text{and} \quad \lim_{q \to 1^-} \Gamma_q(x) = \Gamma(x), \quad \Re(x) > 0.
\]

The \(q\)-trigonometric functions \(q\)-cosine and \(q\)-sine are defined by (see [17] [18])

\[
\cos(x; q^2) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{x^{2n}}{[2n]_q!}, \quad \sin(x; q^2) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{x^{2n+1}}{[2n + 1]_q!},
\]

The \(q\)-analogue exponential function is given by (see [17] [18])

\[
e(z; q^2) = \cos(-iz; q^2) + i \sin(-iz; q^2).
\]
These three functions are absolutely convergent for all \( z \) in the plane and when \( q \) tends to 1 they tend to the corresponding classical ones pointwise and uniformly on compacts. Note that we have for all \( x \in \mathbb{R}_q \) (see [17])

\[
|\cos(x; q^2)| \leq \frac{1}{(q; q)_\infty}, \quad |\sin(x; q^2)| \leq \frac{1}{(q; q)_\infty},
\]

and

\[
|e(ix; q^2)| \leq \frac{2}{(q; q)_\infty}.
\]

The \( q^2 \)-analogue differential operator is (see [17] [18])

\[
\partial_q(f)(z) = \begin{cases} 
\frac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1-q)z} & \text{if } z \neq 0 \\
\lim_{x \to 0} \partial_q(f)(x) & \text{(in } \mathbb{R}_q) \\
\end{cases}
\]

Remark that if \( f \) is differentiable at \( z \), then \( \lim_{q \to 1} \partial_q(f)(z) = f'(z) \).

A repeated application of the \( q^2 \)-analogue differential operator \( n \) times is denoted by:

\[
\partial_q^n f = f, \quad \partial_q^{n+1} f = \partial_q(\partial_q^n f).
\]

The following lemma lists some useful computational properties of \( \partial_q \), and reflects the sensitivity of this operator to parity of its argument. The proof is straightforward.

**Lemma 1.**

1) \( \partial_q \sin(x; q^2) = \cos(x; q^2), \ \partial_q \cos(x; q^2) = -\sin(x; q^2) \) and \( \partial_q e(x; q^2) = e(x; q^2) \).

2) For all function \( f \) on \( \mathbb{R}_q \), \( \partial_q f(z) = \frac{f_e(q^{-1}z) - f_o(z)}{(1-q)z} + \frac{f_o(z) - f_o(qz)}{(1-q)z} \).

3) For two functions \( f \) and \( g \) on \( \mathbb{R}_q \), we have
   * if \( f \) even and \( g \) odd
     \[
     \partial_q(fg)(z) = q\partial_q(f)(qz)g(z) + f(qz)\partial_q(g)(z) = \partial_q(g)(z)f(z) + qg(qz)\partial_q(f)(qz);
     \]
   * if \( f \) and \( g \) are even
     \[
     \partial_q(fg)(z) = \partial_q(f)(z)g(q^{-1}z) + f(z)\partial_q(g)(z) .
     \]

Here, for a function \( f \) defined on \( \mathbb{R}_q \), \( f_e \) and \( f_o \) are its even and odd parts respectively. The \( q \)-Jackson integrals are defined by (see [9])

\[
\int_0^a f(x) d_q x = (1-q) a \sum_{n=0}^{\infty} q^n f(a q^n), \quad \int_a^b f(x) d_q x = (1-q) \sum_{n=0}^{\infty} q^n \left( b f(b q^n) - a f(a q^n) \right),
\]

\[
\int_0^\infty f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} q^n f(q^n), \quad \int_{-\infty}^\infty f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} q^n \left[ f(q^n) + f(-q^n) \right],
\]

provided the sums converge absolutely. In particular, for \( a \in \mathbb{R}_{q, +} \),

\[
\int_a^\infty f(x) d_q x = (1-q) a \sum_{n=-\infty}^{-1} q^n f(a q^n),
\]
The following simple result, giving \( q \)-analogues of the integration by parts theorem, can be verified by direct calculation.

**Lemma 2.**
1) For \( a > 0 \), if \( \int_{-a}^{a} (\partial_q f)(x) g(x) d_q x \) exists, then

\[
\int_{-a}^{a} (\partial_q f)(x) g(x) d_q x = 2 \left[ f_o(q^{-1}a)g_o(a) + f_o(a)g_e(q^{-1}a) \right] - \int_{-a}^{a} f(x)(\partial_q g)(x) d_q x.
\]

2) If \( \int_{-\infty}^{\infty} (\partial_q f)(x) g(x) d_q x \) exists,

\[
\int_{-\infty}^{\infty} (\partial_q f)(x) g(x) d_q x = -\int_{-\infty}^{\infty} f(x)(\partial_q g)(x) d_q x.
\]

**2.3. Sets and spaces.**

By the use of the \( q^2 \)-analogue differential operator \( \partial_q \), we note:

- \( E_q(\mathbb{R}_q) \) the space of functions \( f \) defined on \( \mathbb{R}_q \), satisfying

\[
\forall n \in \mathbb{N}, \ a \geq 0, \ P_{n,a}(f) = \sup \left\{ |\partial_q^k f(x)|; 0 \leq k \leq n; x \in [-a,a] \cap \mathbb{R}_q \right\} < \infty
\]

and

\[
\lim_{x \to 0} \partial_q^n f(x) \quad (\text{in } \mathbb{R}_q) \quad \text{exists}.
\]

We provide it with the topology defined by the semi norms \( P_{n,a} \).

- \( E_{s,q}(\mathbb{R}_q) \) the subspace of \( E_q(\mathbb{R}_q) \) constituted of even functions.

- \( S_q(\mathbb{R}_q) \) the space of functions \( f \) defined on \( \mathbb{R}_q \) satisfying

\[
\forall n, m \in \mathbb{N}, \ P_{n,m,q}(f) = \sup_{x \in \mathbb{R}_q} |x^m \partial_q^n f(x)| < +\infty
\]

and

\[
\lim_{x \to 0} \partial_q^n f(x) \quad (\text{in } \mathbb{R}_q) \quad \text{exists}.
\]

- \( S_{s,q}(\mathbb{R}_q) \) the subspace of \( S_q(\mathbb{R}_q) \) constituted of even functions.

- \( D_q(\mathbb{R}_q) \) the space of functions defined on \( \mathbb{R}_q \) with compact supports.

- \( D_{s,q}(\mathbb{R}_q) \) the subspace of \( D_q(\mathbb{R}_q) \) constituted of even functions.

Using the \( q \)-Jackson integrals, we note for \( p > 0 \) and \( \alpha \in \mathbb{R} \),

- \( L_q^p(\mathbb{R}_q) = \left\{ f : \|f\|_{p,q} = \left( \int_{-\infty}^{\infty} |f(x)|^p d_q x \right)^{\frac{1}{p}} < \infty \right\} \),

- \( L_q^p(\mathbb{R}_q+,+) = \left\{ f : \|f\|_{p,q} = \left( \int_{0}^{\infty} |f(x)|^p d_q x \right)^{\frac{1}{p}} < \infty \right\} \),

- \( L_{a,q}^p(\mathbb{R}_q) = \left\{ f : \|f\|_{p,a,q} = \left( \int_{-\infty}^{\infty} |f(x)|^p |x|^{2\alpha+1} d_q x \right)^{\frac{1}{p}} < \infty \right\} \),

- \( L_{a,q}^p(\mathbb{R}_q+,+) = \left\{ f : \|f\|_{p,a,q} = \left( \int_{0}^{\infty} |f(x)|^p x^{2\alpha+1} d_q x \right)^{\frac{1}{p}} < \infty \right\} \),

- \( L_q^\infty(\mathbb{R}_q) = \left\{ f : \|f\|_{\infty,q} = \sup_{x \in \mathbb{R}_q} |f(x)| < \infty \right\} \),
2.4. $q^2$-Analogous Fourier transform. R. L. Rubin defined in [18] the $q^2$-analogous Fourier transform as

$$\hat{f}(x; q^2) = K \int_{-\infty}^{\infty} f(t)e(-itx; q^2)dt,$$

where $K = \frac{(1 + q)^{\frac{1}{2}}}{2\Gamma(q^2) \left( \frac{q^2}{2} \right)}$.

Letting $q \uparrow 1$ subject to the condition

$$\log(1 - q) \log(q) \in 2\mathbb{Z},$$

gives, at least formally, the classical Fourier transform. In the remainder of this paper, we assume that the condition (16) holds.

It was shown in [18] that $\hat{f}(., q^2)$ verifies the following properties:

1) If $f(u), uf(u) \in L^1_q(\mathbb{R}_q)$, then $\partial_q \left( \hat{f}(x; q^2) \right) = (-iu f(u))(x; q^2)$.

2) If $f, \partial_q f \in L^1_q(\mathbb{R}_q)$, then $(\partial_q f)\sim(x; q^2) = ixf(x; q^2)$.

3) $\hat{f}(., q^2)$ is an isomorphism from $L^2_q(\mathbb{R}_q)$ onto itself. For $f \in L^2_q(\mathbb{R}_q)$, we have $\forall x \in \mathbb{R}_q, \left( \hat{f}\right)^{-1}(x; q^2) = \hat{f}(-x; q^2)$ and $\|\hat{f}(., q^2)\|_{2,q} = \|f\|_{2,q}$.

3. $q$-Bessel Fourier Transform

The normalized $q$-Bessel function is defined by

$$j_\alpha(x; q^2) = \sum_{n=0}^{+\infty} (-1)^n \frac{\Gamma_q(\alpha+n+1)q^{n(n+1)}}{\Gamma_q(\alpha+n+1)\Gamma_q(n+1)} \left( \frac{x}{1+q} \right)^{2n}.$$

Note that we have

$$j_\alpha(x; q^2) = (1 - q^2)^\alpha \Gamma_q(\alpha+1)((1-q)x)^{-\alpha} J_\alpha((1-q)x; q^2),$$

where

$$J_\alpha(x; q^2) = \frac{x^\alpha(q^{2\alpha+2}; q^2)_\infty}{(q^{2\alpha}; q^2)_\infty} \varphi_1(0; q^{2\alpha+2}; q^2, q^2x^2)$$

is the Jackson’s third $q$-Bessel function.

Using the relations (17) and (6), we obtain

$$j_{-\frac{1}{2}}(x; q^2) = \cos(x; q^2),$$

$$j_{\frac{1}{2}}(x; q^2) = \frac{\sin(x; q^2)}{x},$$

and

$$\partial_q j_\alpha(x; q^2) = -\frac{x}{[2\alpha + 2]q} j_{\alpha+1}(x; q^2).$$

In [6], the authors proved the following estimation.
Lemma 3. For \( \alpha \geq -\frac{1}{2} \) and \( x \in \mathbb{R}_q \),

- \(|j_\alpha(x; q^2)| \leq \begin{cases} 
\frac{(-q^2; q^2)_\infty(-q^{2\alpha+1}; q^2)_\infty}{(q^{2\alpha+1}; q^2)_\infty} \left( \begin{array}{c} 1, \\
q^{-\frac{\log(1-q)|x|}{\log q}} \end{array} \right)^2, & \text{if } |x| \leq \frac{1}{1-q} \\
q^{2\alpha+1}; q^2)_\infty & \text{if } |x| \geq \frac{1}{1-q}
\end{cases} \)

- for all \( v \in \mathbb{R} \), \( j_\alpha(x; q^2) = o(x^{-v}) \) as \(|x| \to +\infty \) (in \( \mathbb{R}_q \)).

As a consequence of the previous lemma and the relation (22), we have for \( \alpha \geq -\frac{1}{2} \),

\[ j_\alpha(x; q^2) \in S_{*q}(\mathbb{R}_q) \]

With the same technique used in [7], we can prove that for \( \alpha > -\frac{1}{2} \), \( j_\alpha(\cdot; q^2) \) has the following \( q \)-integral representation of Mehler type

\[ j_\alpha(x; q^2) = C(\alpha; q^2) \int_0^1 W_\alpha(t; q^2) \cos(xt; q^2) dt, \]

where

\[ C(\alpha; q^2) = (1 + q) \frac{\Gamma_q^2(\alpha + 1)}{\Gamma_q(\frac{1}{2}) \Gamma_q^2(\alpha + \frac{1}{2})} \]

and

\[ W_\alpha(t; q^2) = \frac{(t^2 q^2; q^2)_\infty}{(t^2 q^{2\alpha+1}; q^2)_\infty}. \]

Remark: Since the functions \( W_\alpha(\cdot; q^2) \) and \( \cos(\cdot; q^2) \) are even and \( \sin(\cdot; q^2) \) is odd, we can write for \( \alpha > -\frac{1}{2} \),

\[ j_\alpha(x; q^2) = \frac{1}{2} C(\alpha; q^2) \int_{-1}^1 W_\alpha(t; q^2) e(-ixt; q^2) dt. \]

In particular, using the inequality (8), we obtain

\[ |j_\alpha(x; q^2)| \leq \frac{2}{(q; q)_\infty}, \forall x \in \mathbb{R}_q. \]

Proposition 1. For \( x, y \in \mathbb{R}_{q,+} \), we have

\[ (xy)^{\alpha+1} \int_0^{+\infty} j_\alpha(xt; q^2) j_\alpha(yt; q^2) t^{2\alpha+1} dt = \frac{(1 + q)\Gamma_q^2(\alpha + 1)}{(1 - q)} \delta_{x,y} \]

Proof. The result follows from the relation (18) and the orthogonality relation of the Jackson’s third \( q \)-Bessel function \( J_\alpha(\cdot; q^2) \) proved in [14].

Using the same technique as in [7], one can prove the following result.

Proposition 2. For \( \lambda \in \mathbb{C} \), the function \( j_\alpha(\lambda x; q^2) \) is the unique even solution of the problem

\[ \left\{ \begin{array}{l}
\Delta_{a,q} f(x) = -\lambda^2 f(x), \\
f(0) = 1,
\end{array} \right. \]
where \( \Delta_{\alpha,q} f(x) = \frac{1}{|x|^{2\alpha+1}} \partial_q [|x|^{2\alpha+1} \partial_q f(x)]. \)

**Definition 1.** The \( q \)-Bessel Fourier transform is defined for \( f \in L^1_{\alpha,q}(\mathbb{R}_{q,+}) \), by

\[
F_{\alpha,q}(f)(\lambda) = c_{\alpha,q} \int_0^\infty f(x)j_\alpha(\lambda x; q^2) x^{2\alpha+1}d_q x
\]

where

\[
c_{\alpha,q} = \frac{(1+q)^{-\alpha}}{\Gamma_q(\alpha+1)}. \]

Letting \( q \uparrow 1 \) subject to the condition \([16]\), gives, at least formally, the classical Bessel-Fourier transform. Some properties of the \( q \)-Bessel Fourier transform are given in the following result.

**Proposition 3.** 1) For \( f \in L^1_{\alpha,q}(\mathbb{R}_{q,+}) \), we have \( F_{\alpha,q}(f) \in L^\infty_q(\mathbb{R}_{q,+}) \) and

\[
\| F_{\alpha,q}(f) \|_{\infty,q} \leq \frac{2c_{\alpha,q}}{(q;q)_\infty} \| f \|_{1,q}.
\]

2) For \( f, g \in L^1_{\alpha,q}(\mathbb{R}_{q,+}) \), we have

\[
\int_0^\infty \int_0^\infty f(x)F_{\alpha,q}(g)(x)x^{2\alpha+1}d_q x = \int_0^\infty \int_0^\infty F_{\alpha,q}(f)(\lambda)g(\lambda)\lambda^{2\alpha+1}d_q \lambda.
\]

3) If \( f \) and \( \Delta_{\alpha,q} f \) are in \( L^1_{\alpha,q}(\mathbb{R}_{q,+}) \), then

\[
F_{\alpha,q}(\Delta_{\alpha,q} f)(\lambda) = -\lambda^2 F_{\alpha,q}(f)(\lambda).
\]

4) If \( f \) and \( x^2 f \) are in \( L^1_{\alpha,q}(\mathbb{R}_{q,+}) \), then

\[
\Delta_{\alpha,q}(F_{\alpha,q}(f)) = -F_{\alpha,q}(x^2 f).
\]

**Proof.** 1) follows from the definition of \( F_{\alpha,q} \) and the relation \([27]\).

2) Let \( f, g \in L^1_{\alpha,q}(\mathbb{R}_{q,+}) \).

Since for all \( \lambda, x \in \mathbb{R}_{q,+} \), we have \( |j_\alpha(\lambda x; q^2)| \leq \frac{2}{(q;q)_\infty} \), then

\[
\int_0^+ \int_0^+ | f(x)g(\lambda)j_\alpha(\lambda x; q^2)|x^{2\alpha+1}\lambda^{2\alpha+1}d_q x d_q \lambda \leq \frac{2}{(q;q)_\infty} \| f \|_{1,a,q} \| g \|_{1,a,q} < \infty.
\]

So, by the Fubini’s theorem, we can exchange the order of the \( q \)-integrals and obtain,

\[
\int_0^\infty f(x)F_{\alpha,q}(g)(x)x^{2\alpha+1}d_q x = \int_0^\infty \int_0^\infty f(x)g(\lambda)j_\alpha(\lambda x; q^2)x^{2\alpha+1}\lambda^{2\alpha+1}d_q \lambda d_q x
\]

\[
= \int_0^\infty g(\lambda) \left( \int_0^\infty f(x)j_\alpha(\lambda x; q^2)x^{2\alpha+1}d_q x \right) \lambda^{2\alpha+1}d_q \lambda
\]

\[
= \int_0^\infty F_{\alpha,q}(f)(\lambda)g(\lambda)\lambda^{2\alpha+1}d_q \lambda.
\]
3) For \( f \in L^1_{\alpha,q}(\mathbb{R}_{q,+}) \) such that \( \triangle_{\alpha,q}f \in L^1_{\alpha,q}(\mathbb{R}_{q,+}) \), let \( \tilde{f} \) be the even function defined on \( \mathbb{R}_q \) whose \( f \) is its restriction on \( \mathbb{R}_{q,+} \). We have \( \triangle_{\alpha,q}f = \triangle_{\alpha,q}\tilde{f} \) and

\[
\mathcal{F}_{\alpha,q}(\triangle_{\alpha,q}f)(\lambda) = c_{\alpha,q} \int_0^\infty (\triangle_{\alpha,q}f)(x)j_\alpha(x \lambda; q^2)x^{2\alpha+1}d_qx 
\]

(33)

\[
= \frac{c_{\alpha,q}}{2} \int_{-\infty}^\infty (\triangle_{\alpha,q}\tilde{f})(x)j_\alpha(x \lambda; q^2)|x|^{2\alpha+1}d_qx. 
\]

(34)

So, Proposition 2 and two \( q \)-integrations by parts give the result.
4) The result follows from Proposition 2.

**Proposition 4.** If \( f \in L^1_{\alpha,q}(\mathbb{R}_{q,+}) \), then

\[
\forall x \in \mathbb{R}_{q,+}, \quad f(x) = c_{\alpha,q} \int_0^\infty \mathcal{F}_{\alpha,q}(f)(\lambda)j_\alpha(\lambda x; q^2)\lambda^{2\alpha+1}d_q\lambda. 
\]

**Proof.** The result follows from the relation (27), Proposition 1 and the Fubini’s theorem.

**Theorem 1.** 1) **Plancherel formula**

For all \( f \in \mathcal{D}_{*q}(\mathbb{R}_q) \), we have

\[
\|\mathcal{F}_{\alpha,q}(f)\|_{2,\alpha,q} = \|f\|_{2,\alpha,q}. 
\]

(35)

2) **Plancherel theorem**

The \( q \)-Bessel transform can be uniquely extended to an isometric isomorphism on \( L^2_{\alpha,q}(\mathbb{R}_{q,+}) \) with \( \mathcal{F}^{-1}_{\alpha,q} = \mathcal{F}_{\alpha,q} \).

**Proof.** 1) Let \( f \in \mathcal{D}_{*q}(\mathbb{R}_q) \), it is easy to show that \( \mathcal{F}_{\alpha,q}(f) \) is in \( L^1_{\alpha,q}(\mathbb{R}_{q,+}) \). From Proposition 1 we have \( f = \mathcal{F}_{\alpha,q}(\mathcal{F}_{\alpha,q}(f)) \), so using the relation (33), we obtain

\[
\|f\|_{2,\alpha,q}^2 = \int_0^\infty f(x)\tilde{f}(x)x^{2\alpha+1}d_qx = \int_0^\infty \mathcal{F}_{\alpha,q}(\mathcal{F}_{\alpha,q}f)(x)\tilde{f}(x)x^{2\alpha+1}d_qx 
\]

\[
= \int_0^\infty \mathcal{F}_{\alpha,q}(f)(x)\mathcal{F}_{\alpha,q}(\tilde{f})(x)x^{2\alpha+1}d_qx = \|\mathcal{F}_{\alpha,q}(f)\|_{2,\alpha,q}^2. 
\]

2) The result follows from 1), Proposition 2 and the density of \( \mathcal{D}_{*q}(\mathbb{R}_q) \) in \( L^2_{\alpha,q}(\mathbb{R}_{q,+}) \).

**Definition 2.** For \( \alpha > -\frac{1}{2} \), the \( q \)-Riemann-Liouville operator \( R_{\alpha,q} \) is defined for \( f \in \mathcal{E}_{*q}(\mathbb{R}_q) \) by

\[
R_{\alpha,q}(f)(x) = \frac{1}{2}C(\alpha; q^2) \int_{-1}^1 W_\alpha(t; q^2)f(xt)d_qt. 
\]

(36)

The \( q \)-Weyl operator is defined for \( f \in \mathcal{D}_{*q}(\mathbb{R}_q) \) by

\[
t^\alpha R_{\alpha,q}(f)(t) = \frac{(1+q)^{-\alpha+\frac{1}{2}}}{\Gamma_q(\alpha+\frac{1}{2})} \int_{-q^1[t]}^{q^1[t]} W_\alpha \left( \frac{t}{x}; q^2 \right) f(x)x^{2\alpha}d_qx. 
\]

(37)
In the end of this section, we shall give some useful properties of these two operators. First, by simple calculus, one can easily prove that for $f \in \mathcal{E}_{\ast,q}(\mathbb{R}_q)$ and $g \in \mathcal{D}_{\ast,q}(\mathbb{R}_q)$, we have

$$
\frac{c_{\alpha,q}}{2} \int_{-\infty}^{\infty} R_{\alpha,q}(f)(x)g(x)|x|^{2\alpha+1}d_qx = K \int_{-\infty}^{\infty} f(t)^{\ast}R_{\alpha,q}(g)(t)d_qt.
$$

Next, using the relation (20), we obtain

$$
\alpha(\gamma^2) = R_{\alpha,q}(e(-i;i^2q^2)).
$$

**Lemma 4.** The operator $R_{\alpha,q}$ is continuous from $\mathcal{E}_{\ast,q}(\mathbb{R}_q)$ into itself.

**Proof.** Let $f$ be in $\mathcal{E}_{\ast,q}(\mathbb{R}_q)$. The function $x \rightarrow R_{\alpha,q}(f)(x)$ is an even function on $\mathbb{R}_q$. By $q$-derivation under the $q$-integral sign, we deduce that for all $n \in \mathbb{N}$,

$$
\partial_q^n R_{\alpha,q}(f)(x) = \frac{1}{2}C(\alpha;q^2) \int_{-1}^{1} W_{\alpha}(t;q^2)t^n(\partial_{q}^n f)(xt)d_qt.
$$

Then,

$$
\forall \alpha \geq 0, \forall n \in \mathbb{N}, ||P_{\alpha,n}(R_{\alpha,q}(f))|| \leq P_{\alpha,n}(f) < \infty.
$$

This relation together with the Lebesgue theorem proves that $R_{\alpha,q}(f)$ belongs to $\mathcal{E}_{\ast,q}(\mathbb{R}_q)$ and it shows that the operator $R_{\alpha,q}$ is continuous from $\mathcal{E}_{\ast,q}(\mathbb{R}_q)$ into itself. ■

Using the previous lemma and making a proof as in Theorems 3 and 4 of [7], we obtain the following result.

**Theorem 2.** The $q$-Riemann-Liouville operator $R_{\alpha,q}$ is a topological isomorphism from $\mathcal{E}_{\ast,q}(\mathbb{R}_q)$ onto itself and it transmutes the operators $\Delta_{\alpha,q}$ and $\partial_q^2$ in the following sense

$$
\Delta_{\alpha,q} R_{\alpha,q} = R_{\alpha,q} \partial_q^2.
$$

**Theorem 3.** The $q$-Weyl operator $^{\ast}R_{\alpha,q}$ is an isomorphism from $\mathcal{D}_{\ast,q}(\mathbb{R}_q)$ onto itself, it transmutes the operators $\Delta_{\alpha,q}$ and $\partial_q^2$ in the following sense

$$
^{\ast}R_{\alpha,q} \Delta_{\alpha,q} = \partial_q^2(\ast R_{\alpha,q})
$$

and for $f \in \mathcal{D}_{\ast,q}(\mathbb{R}_q)$, we have

$$
\mathcal{F}_{\alpha,q}(f) = (^{\ast}R_{\alpha,q}(f)) \gamma(:,q^2).
$$

**Proof.** The first part of the result can be proved as Proposition 3 of [7] page 158. The relation (12) is a consequence of the relations (38) and (39). Let us now, prove the relation (11). Let $g \in \mathcal{D}_{\ast,q}(\mathbb{R}_q)$. For all $f \in \mathcal{D}_{\ast,q}(\mathbb{R}_q)$, we have, using the $q$-integration by parts theorem, the relations (38) and (40),

$$
K \int_{-\infty}^{\infty} \partial_q^2(\ast R_{\alpha,q}g)(x)f(x)d_qx = K \int_{-\infty}^{\infty} (\ast R_{\alpha,q}g)(x)\partial_q^2 f(x)d_qx
$$

$$
= \frac{c_{\alpha,q}}{2} \int_{-\infty}^{\infty} g(x)R_{\alpha,q}\partial_q^2 f(x)|x|^{2\alpha+1}d_qx = \frac{c_{\alpha,q}}{2} \int_{-\infty}^{\infty} g(x)\Delta_{\alpha,q} R_{\alpha,q} f(x)|x|^{2\alpha+1}d_qx
$$

$$
= -\frac{c_{\alpha,q}}{2} \int_{-\infty}^{\infty} \partial_q g(x) \partial_q (R_{\alpha,q} f)(x)|x|^{2\alpha+1}d_qx = \frac{c_{\alpha,q}}{2} \int_{-\infty}^{\infty} \Delta_{\alpha,q} g(x) R_{\alpha,q} f(x)|x|^{2\alpha+1}d_qx
$$

$$
= K \int_{-\infty}^{\infty} \ast R_{\alpha,q}(\Delta_{\alpha,q} g)(x)f(x)d_qx.
$$
4. The $q$-Dunkl operator and its eigenfunctions

For $\alpha \geq -\frac{1}{2}$, consider the operators:

\begin{equation}
H_{\alpha,q} : f = f_e + f_o \mapsto f_e + q^{2\alpha+1}f_o
\end{equation}

and

\begin{equation}
\Lambda_{\alpha,q}(f)(x) = \partial_q [H_{\alpha,q}(f)](x) + [2\alpha + 1]_q \frac{f(x) - f(-x)}{2x}.
\end{equation}

It is easy to see that for a differentiable function $f$, the $q$-Dunkl operator $\Lambda_{\alpha,q}(f)$ tends, as $q$ tends to 1, to the classical Dunkl operator $\Lambda_{\alpha}(f)$ given by (1).

In the case $\alpha = -\frac{1}{2}$, $\Lambda_{\alpha,q}$ reduces to the $q^2$-analogue differential operator $\partial_q$.

Some properties of the $q$-Dunkl operator $\Lambda_{\alpha,q}$ are given in the following proposition.

**Proposition 5.**

i) If $f$ is odd then $\Lambda_{\alpha,q}(f)(x) = q^{2\alpha+1}\partial_q f(x) + [2\alpha + 1]_q \frac{f(x)}{x}$ and if $f$ is even then $\Lambda_{\alpha,q}(f)(x) = \partial_q f(x)$.

ii) If $f$ and $g$ are of the same parity, then

\begin{equation}
\int_{-\infty}^{+\infty} \Lambda_{\alpha,q}(f)(x)g(x)|x|^{2\alpha+1}d_qx = 0.
\end{equation}

iii) For all $f$ and $g$ such that \( \int_{-\infty}^{+\infty} \Lambda_{\alpha,q}(f)(x)g(x)|x|^{2\alpha+1}d_qx \) exists, we have

\begin{equation}
\int_{-\infty}^{+\infty} \Lambda_{\alpha,q}(f)(x)g(x)|x|^{2\alpha+1}d_qx = -\int_{-\infty}^{+\infty} \Lambda_{\alpha,q}(g)(x)f(x)|x|^{2\alpha+1}d_qx.
\end{equation}

iv) The operator $\Lambda_{\alpha,q}$ lives $E_q(\mathbb{R}_q)$, $S_q(\mathbb{R}_q)$ and $D_q(\mathbb{R}_q)$ invariant.

**Proof.** i) is a direct consequence of the definition of $\Lambda_{\alpha,q}$.

ii) follows from the properties of the $q$-integrals and the fact that $\Lambda_{\alpha,q}$ change the parity of functions.

iii) From ii) we have the result when $f$ and $g$ are of the same parity.

Now, suppose that $f$ is even and $g$ is odd. Using Lemma \[2\] the property i) of $\Lambda_{\alpha,q}$ and the properties of the $q^2$-analogue differential operator $\partial_q$ we obtain

\begin{align*}
\int_{-\infty}^{+\infty} \Lambda_{\alpha,q}(f)(x)g(x)|x|^{2\alpha+1}d_qx &= \int_{-\infty}^{+\infty} \partial_q(f)(x)g(x)|x|^{2\alpha+1}d_qx \\
&= -\int_{-\infty}^{+\infty} f(x)\partial_q [g(x)|x|^{2\alpha+1}]d_qx \\
&= -\int_{-\infty}^{+\infty} f(x)\left[q^{2\alpha+1}\partial_q g(x) + [2\alpha + 1]_q \frac{g(x)}{x}\right]|x|^{2\alpha+1}d_qx \\
&= -\int_{-\infty}^{+\infty} f(x)\Lambda_{\alpha,q}(g)(x)|x|^{2\alpha+1}d_qx.
\end{align*}
iv) follows from the facts that for $f \in E_q(\mathbb{R})$,

$$
\Lambda_{\alpha,q}(f)(x) = \partial_q [H_{\alpha,q}(f)](x) + \frac{[2\alpha + 1]_q}{2} \int_{-1}^{1} \partial_q(f)(xt) dt
$$

and for $f \in S_q(\mathbb{R})$,

$$
\Lambda_{\alpha,q}(f)(x) = \partial_q [H_{\alpha,q}(f)](x) + [2\alpha + 1]_q \int_{0}^{1} \partial_q(f_o)(xt) dt - \partial_q [H_{\alpha,q}(f)](x) - [2\alpha + 1]_q \int_{1}^{\infty} \partial_q(f_o)(xt) dt.
$$

Let us now introduce the eigenfunctions of the $q$-Dunkl operator.

**Theorem 4.** For $\lambda \in \mathbb{C}$, the $q$-differential-difference equation:

$$
\begin{cases}
\Lambda_{\alpha,q}(f) = i\lambda f \\
f(0) = 1
\end{cases}
$$

has as unique solution, the function

$$
\psi_{\lambda}^{\alpha,q} : x \mapsto j_\alpha(\lambda x; q^2) + \frac{i\lambda x}{[2\alpha + 2]_q} j_{\alpha+1}(\lambda x; q^2).
$$

**Proof.** Let $f = f_e + f_o$. The problem (46) is equivalent to the system

$$
\begin{cases}
\partial_q f_e(x) + q^{2\alpha+1} \partial_q f_o(x) + [2\alpha + 1]_q \frac{f_o(x)}{x} = i\lambda f_e(x) + i\lambda f_o(x) \\
f_e(0) = 1,
\end{cases}
$$

which is equivalent to

$$
\begin{cases}
\partial_q f_e(x) = i\lambda f_o(x) \\
q^{2\alpha+1} \partial^2_q f_e(x) + [2\alpha + 1]_q \frac{\partial_q f_e(x)}{x} = -\lambda^2 f_e(x) \\
f_e(0) = 1.
\end{cases}
$$

Now, using Proposition 2 and the relation (22), we obtain

$$
f_e(x) = j_\alpha(\lambda x; q^2) \\
f_o(x) = \frac{1}{i\lambda} \partial_q(j_\alpha(\lambda x; q^2)) = \frac{i\lambda x}{[2\alpha + 2]_q} j_{\alpha+1}(\lambda x; q^2).
$$

Finally, for $\lambda \in \mathbb{C}$,

$$
\psi_{\lambda}^{\alpha,q}(x) = f(x) = j_\alpha(\lambda x; q^2) + \frac{i\lambda x}{[2\alpha + 2]_q} j_{\alpha+1}(\lambda x; q^2).
$$

The function $\psi_{\lambda}^{\alpha,q}(x)$, called $q$-Dunkl kernel has an unique extention to $\mathbb{C} \times \mathbb{C}$ and verifies the following properties.

**Proposition 6.** 1) $\Lambda_{\alpha,q}\psi_{\lambda}^{\alpha,q} = i\lambda \psi_{\lambda}^{\alpha,q}$.

2) $\psi_{\lambda}^{\alpha,q}(x) = \psi_{\lambda}^{\alpha,q}(\lambda)$, $\psi_{a\lambda}^{\alpha,q}(x) = \psi_{\lambda}^{\alpha,q}(ax)$ and $\psi_{\lambda}^{\alpha,q}(x) = \psi_{-\lambda}^{\alpha,q}(x)$, for $\lambda, x \in \mathbb{R}$ and $a \in \mathbb{C}$. 

\[\square\]
If \( \alpha = -\frac{1}{2} \), then \( \psi_\alpha^{\alpha,q}(x) = e(i\lambda x; q^2) \).

For \( \alpha > -\frac{1}{2} \), \( \psi_\alpha^{\alpha,q} \) has the following \( q \)-integral representation of Mehler type

\[
\psi_\alpha^{\alpha,q}(x) = \frac{1}{2} C(\alpha; q^2) \int_{-1}^{1} W_\alpha(t; q^2)(1 + t) e(i\lambda xt; q^2) dt,
\]

where \( C(\alpha; q^2) \) and \( W_\alpha(t; q^2) \) are given respectively by (24) and (25).

4) For all \( n \in \mathbb{N} \) we have

\[
|\partial_q^n \psi_\alpha^{\alpha,q}(x)| \leq \frac{4}{(q; q)_\infty} |\lambda|^n, \quad \forall \lambda, x \in \mathbb{R}_q.
\]

In particular for all \( \lambda \in \mathbb{R}_q \), \( \psi_\alpha^{\alpha,q} \) is bounded on \( \mathbb{R}_q \) and we have

\[
|\psi_\alpha^{\alpha,q}(x)| \leq \frac{4}{(q; q)_\infty}, \quad \forall x \in \mathbb{R}_q.
\]

5) For all \( \lambda \in \mathbb{R}_q \), \( \psi_\alpha^{\alpha,q} \in \mathcal{S}_q(\mathbb{R}_q) \).

Proof. 1) and 2) are immediate consequences of the definition of \( \psi_\alpha^{\alpha,q} \).
3) If \( \alpha = -\frac{1}{2} \) then the relations (20), (21) and (7) give the result.

If \( \alpha > -\frac{1}{2} \), using the definition of \( \psi_\alpha^{\alpha,q} \), the parity of the function \( j_\alpha(\lambda x; q^2) \) and the relations (26) and (22), we obtain

\[
\psi_\alpha^{\alpha,q}(x) = j_\alpha(\lambda x; q^2) + \frac{1}{i\lambda} \partial_q(j_\alpha(\lambda x; q^2))
\]

\[
= \frac{C(\alpha; q^2)}{2} \int_{-1}^{1} W_\alpha(t; q^2)e(i\lambda xt; q^2) dt + \frac{1}{i} \frac{C(\alpha; q^2)}{2} \int_{-1}^{1} W_\alpha(t; q^2)ite(i\lambda xt; q^2) dt,
\]

which achieves the proof.

4) By induction on \( n \) we prove that

\[
\partial_q^n \psi_\alpha^{\alpha,q}(x) = \frac{C(\alpha; q^2)}{2} (i\lambda)^n \int_{-1}^{1} W_\alpha(t; q^2)(1 + t)t^n e(i\lambda xt; q^2) dt.
\]

So, the fact that \( |e(ix; q^2)| \leq \frac{2}{(q; q)_\infty} \) gives the result.

5) The result follows from Lemma 3, the relation (22) and the properties of \( \partial_q \). \( \blacksquare \)

The function \( \psi_\alpha^{\alpha,q} \) verifies the following orthogonality relation.

**Proposition 7.** For all \( x, y \in \mathbb{R}_q \), we have

\[
\int_{-\infty}^{+\infty} \psi_\alpha^{\alpha,q}(x) \overline{\psi_\alpha^{\alpha,q}(y)} |\lambda|^{2\alpha+1} dq = \frac{4(1 + q)^{2\alpha} \Gamma^2_q (\alpha + 1) \delta_{x,y}}{(1 - q)|xy|^{\alpha+1}}.
\]
Proof. Let $x, y \in \mathbb{R}_q$, the use of the relation (58) and the properties of the $q$-Jackson’s integral lead to

$$\int_{-\infty}^{+\infty} \psi_\Lambda^{\alpha,q}(x) \psi_\Lambda^{\alpha,q}(y) |\lambda|^{2\alpha+1} d\lambda$$

$$= \int_{-\infty}^{+\infty} j_\alpha(\lambda x; q^2) j_\alpha(\lambda y; q^2) |\lambda|^{2\alpha+1} d\lambda + \frac{xy}{2\alpha + 2} \int_{-\infty}^{+\infty} j_{\alpha+1}(\lambda x; q^2) j_{\alpha+1}(\lambda y; q^2) |\lambda|^{2\alpha+3} d\lambda$$

$$= \frac{2(1+q)^{2\alpha} \Gamma^2_q(\alpha+1) \delta_{|x|,|y|}}{(1-q)|xy|^\alpha+1} + \frac{2xy(1+q)^{2\alpha+2} \Gamma^2_q(\alpha+2) \delta_{|x|,|y|}}{[2\alpha + 2]^2 q(1-q)|xy|^\alpha+2}$$

$$= \frac{2(1+q)^{2\alpha} \Gamma^2_q(\alpha+1) \delta_{|x|,|y|}}{(1-q)|xy|^\alpha+1}(1 + \text{sgn}(xy)) = \frac{4(1+q)^{2\alpha} \Gamma^2_q(\alpha+1) \delta_{x,y}}{(1-q)|xy|^\alpha+1}.$$  

\[ \Box \]

5. \textit{q–Dunkl intertwining operator}

**Definition 3.** We define the \textit{q–Dunkl intertwining operator} $V_\alpha$ on $E_q(\mathbb{R}_q)$ by

$$\forall x \in \mathbb{R}_q, V_{\alpha,q}(f)(x) = \frac{C(\alpha; q^2)}{2} \int_{-1}^{1} W_\alpha(t; q^2)(1 + t) f(x t) d_q t,$$

where $C(\alpha; q^2)$ and $W_\alpha(t; q^2)$ are given by (54) and (55) respectively.

**Theorem 5.** We have

i) $V_{\alpha,q}(e(-i\lambda x; q^2)) = \psi_\Lambda^{\alpha,q}(x)$, $\lambda, x \in \mathbb{R}_q$.

ii) $V_{\alpha,q}$ verifies the following transmutation relation

$$\Lambda_{\alpha,q} V_{\alpha,q}(f) = V_{\alpha,q}(\partial_q f), \quad V_{\alpha,q}(f)(0) = f(0).$$

**Proof.** i) follows from the relation (58).

ii) Let $f = f_o + f_e \in E_q(\mathbb{R}_q)$, we have on the one hand

$$V_{\alpha,q}(\partial_q f)(x) = \frac{C(\alpha; q^2)}{2} \int_{-1}^{1} W_\alpha(t; q^2) \partial_q f_o(x t) d_q t + \frac{C(\alpha; q^2)}{2} \int_{-1}^{1} W_\alpha(t; q^2) t \partial_q f_e(x t) d_q t.$$ 

On the other hand, we have

$$\Lambda_{\alpha,q} V_{\alpha,q}(f)(x) = \frac{C(\alpha; q^2)}{2} \int_{-1}^{1} W_\alpha(t; q^2) t \partial_q f_e(x t) d_q t + \frac{q^{2\alpha+1} C(\alpha; q^2)}{2} \int_{-1}^{1} W_\alpha(t; q^2) t^2 \partial_q f_o(x t) d_q t$$

$$+ \frac{[2\alpha + 1] q C(\alpha; q^2)}{2x} \int_{-1}^{1} W_\alpha(t; q^2) t f_o(x t) d_q t.$$ 

Now, using a $q$-integration by parts and the facts that

$$\partial_q [(1 - q^2 t^2) W_\alpha(q t; q^2)] = -[2\alpha + 1] q t W_\alpha(t; q^2)$$

and

$$(1 - q^2 t^2) W_\alpha(q t; q^2) = (1 - t^2 q^{2\alpha+1}) W_\alpha(t; q^2),$$
we get
\[ [2\alpha + 1]q^{-\frac{C(\alpha; q^2)}{2x}} \int_{-1}^{1} W_\alpha(t; q^2) t f_0(xt) dt = \frac{C(\alpha; q^2)}{2} \int_{-1}^{1} (1 - q^2 t^2) W_\alpha(qt; q^2) \partial_q f_0(xt) dt = \frac{C(\alpha; q^2)}{2} \int_{-1}^{1} (1 - t^2 q^{2\alpha+1}) W_\alpha(t; q^2) \partial_q f_0(xt) dt, \]
which completes the proof.

**Theorem 6.** For all \( f \in \mathcal{E}_q(\mathbb{R}_q) \), we have
\[
\forall x \in \mathbb{R}_q, V_{\alpha,q}(f)(x) = R_{\alpha,q}(f_\varepsilon)(x) + \partial_q R_{\alpha,q}I_q(f_0)(x),
\]
where \( R_{\alpha,q} \) is given by (36) and \( I_q \) is the operator given by
\[
\forall x \in \mathbb{R}_q, I_q(f_0)(x) = \int_{0}^{[qx]} f_0(t) dt.
\]

**Proof.** From the definitions of the \( q \)-Dunkl intertwining and the \( q \)-Riemann-Liouville operators, we have
\[
V_{\alpha,q}(f)(x) = \frac{C(\alpha; q^2)}{2} \int_{-1}^{1} W_\alpha(t; q^2) (1 + t)(f_0(xt) + f_\varepsilon(xt)) dt
= \frac{C(\alpha; q^2)}{2} \int_{-1}^{1} W_\alpha(t; q^2) f_\varepsilon(xt) dt + \frac{C(\alpha; q^2)}{2} \int_{-1}^{1} W_\alpha(t; q^2) t f_0(xt) dt.
\]

On the other hand, by \( q \)-derivation under the \( q \)-integral sign and the fact that
\[ \partial_q(I_qf_\varepsilon) = f_\varepsilon, \]
we obtain
\[
\partial_q[R_{\alpha,q}I_q(f_0)](x) = \frac{C(\alpha; q^2)}{2} \int_{-1}^{1} W_\alpha(t; q^2) t \partial_q(I_qf_\varepsilon)(xt) dt = \frac{C(\alpha; q^2)}{2} \int_{-1}^{1} W_\alpha(t; q^2) t f_0(xt) dt.
\]
This gives the result. □

**Theorem 7.** The transform \( V_{\alpha,q} \) is an isomorphism from \( \mathcal{E}_q(\mathbb{R}_q) \) onto itself, its inverse transform is given by
\[
\forall x \in \mathbb{R}_q, V_{\alpha,q}^{-1}(f)(x) = R_{\alpha,q}^{-1}(f_\varepsilon)(x) + \partial_q (R_{\alpha,q}^{-1}I_q(f_0))(x),
\]
where \( R_{\alpha,q}^{-1} \) is the inverse transform of \( R_{\alpha,q} \).

**Proof.** Let \( H \) be the operator defined on \( \mathcal{E}_q(\mathbb{R}_q) \) by
\[
H(f) = R_{\alpha,q}^{-1}(f_\varepsilon) + \partial_q (R_{\alpha,q}^{-1}I_q(f_0)).
\]
We have \( V_{\alpha,q}(f) = R_{\alpha,q}(f_\varepsilon) + \partial_q (R_{\alpha,q}I_q(f_0)) \), \( R_{\alpha,q}(f_\varepsilon) \) is even and \( \partial_q(R_{\alpha,q}I_q(f_0)) \) is odd, then
\[
HV_{\alpha,q}(f) = R_{\alpha,q}^{-1}R_{\alpha,q}f_\varepsilon + \partial_q R_{\alpha,q}^{-1}I_q(\partial_q R_{\alpha,q}I_q(f_0))
= f_\varepsilon + \partial_q R_{\alpha,q}^{-1}I_q(\partial_q R_{\alpha,q}I_q(f_0)).
\]
Using the fact that for $\varphi \in \mathcal{E}_q(\mathbb{R}_q)$, $I_q(\partial_q \varphi)(x) = \varphi(x) - \lim_{t \to 0} \varphi(t)$, we obtain

$$I_q(\partial_q R_{\alpha,q} I_q(f_o)) = R_{\alpha,q} I_q(f_o).$$

So,

$$R_{\alpha,q}^{-1} I_q(\partial_q R_{\alpha,q} I_q(f_o)) = I_q(f_o)$$

and

$$\partial_q R_{\alpha,q}^{-1} I_q(\partial_q R_{\alpha,q} I_q(f_o)) = \partial_q I_q(f_o) = f_0.$$ 

Thus,

$$HV_{\alpha,q}(f) = f_e + f_o = f.$$

With the same technique, we prove that $V_{\alpha,q}H(f) = f$.

**Definition 4.** For $f \in \mathcal{D}_q(\mathbb{R}_q)$ and $\alpha > -\frac{1}{2}$, we define the $q$-transpose of $V_{\alpha,q}$ by

$$V_q^\alpha(f)(t) = M_{\alpha,q} \int_{|x| \geq |t|} W_{\alpha} \left( \frac{t}{x} : q^2 \right) \left( 1 + \frac{t}{x} \right) f(x) \frac{|x|^{2\alpha+1}}{x} dq x,$$

where $W_{\alpha}(\cdot; q^2)$ is given by (22) and

$$M_{\alpha,q} = \frac{(1 + q)^{-\alpha + \frac{1}{2}}}{2 \Gamma_{q^2}(\alpha + \frac{1}{2}).}$$

Note that by simple computation, we obtain for $f \in \mathcal{E}_q(\mathbb{R}_q)$ and $g \in \mathcal{D}_q(\mathbb{R}_q)$

$$\frac{c_{\alpha,q}}{2} \int_{-\infty}^{+\infty} V_{\alpha,q}(f)(x) g(x) |x|^{2\alpha+1} dq x = K \int_{-\infty}^{+\infty} f(t) (V_q^\alpha g)(t) dt.$$ 

**Proposition 8.** For $f \in \mathcal{D}_q(\mathbb{R}_q)$, we have

$$\partial_q (V_q^\alpha f) = (V_q^\alpha \Lambda_{\alpha,q})(f).$$

**Proof.** Using a $q$-integration by parts and the relations (58), (53) and (45), we get for all $f \in \mathcal{D}_q(\mathbb{R}_q)$ and $g \in \mathcal{E}_q(\mathbb{R}_q)$,

$$K \int_{-\infty}^{+\infty} g(x) \partial_q (V_q^\alpha f)(x) dx = -K \int_{-\infty}^{+\infty} \partial_q g(x) (V_q^\alpha f)(x) dx$$

$$= -\frac{c_{\alpha,q}}{2} \int_{-\infty}^{+\infty} V_{\alpha,q}(\partial_q g)(x) f(x) |x|^{2\alpha+1} dq x$$

$$= -\frac{c_{\alpha,q}}{2} \int_{-\infty}^{+\infty} \Lambda_{\alpha,q}(V_q^\alpha g)(x) f(x) |x|^{2\alpha+1} dq x$$

$$= \frac{c_{\alpha,q}}{2} \int_{-\infty}^{+\infty} V_{\alpha,q}(g)(x) \Lambda_{\alpha,q} f(x) |x|^{2\alpha+1} dq x$$

$$= K \int_{-\infty}^{+\infty} g(x) (V_q^\alpha \Lambda_{\alpha,q} f)(x) dq x.$$ 

As $g$ is arbitrary in $\mathcal{E}_q(\mathbb{R}_q)$, we obtain the result.
Theorem 8. For \( f \in \mathcal{D}_q(\mathbb{R}) \), we have

\[
\forall x \in \mathbb{R}, (t^{V_{\alpha,q}})(f)(x) = (t^{R_{\alpha,q}})(f_e)(x) + \partial_q \left[t^{R_{\alpha,q}}J_q(f_o)\right](x),
\]

where \( t^{R_{\alpha,q}} \) is given by (37) and \( J_q \) is the operator defined by

\[
J_q(f_o)(x) = \int_{-\infty}^{x} f_o(x) d_q x.
\]

Proof. Let \( f, g \in \mathcal{D}_q(\mathbb{R}) \), using Theorem 6 the relation (38) and a \( q \)-integration by parts, we obtain

\[
\frac{c_{\alpha,q}}{2} \int_{-\infty}^{+\infty} V_{\alpha,q}(g)(x)f(x)|x|^{2\alpha+1} d_q x = \frac{c_{\alpha,q}}{2} \int_{-\infty}^{+\infty} [R_{\alpha,q}(g_o)(x) + \partial_q R_{\alpha,q} \Delta_{\alpha,q} f_o(x)] f(x)|x|^{2\alpha+1} d_q x
\]

\[
= \frac{c_{\alpha,q}}{2} \int_{-\infty}^{+\infty} R_{\alpha,q}(g_o)(x) \cdot f_e(x).|x|^{2\alpha+1} d_q x + \frac{c_{\alpha,q}}{2} \int_{-\infty}^{+\infty} \partial_q R_{\alpha,q} \Delta_{\alpha,q} f_o(x).f_o(x).|x|^{2\alpha+1} d_q x
\]

\[
= K \int_{-\infty}^{+\infty} (t^{R_{\alpha,q}})(f_e)(x)\cdot g_o(x) d_q x - \frac{c_{\alpha,q}}{2} \int_{-\infty}^{+\infty} R_{\alpha,q} \Delta_{\alpha,q} f_o(x).\partial_q f_o(x).|x|^{2\alpha+1} d_q x.
\]

It is easily seen that the map \( J_q \) is bijective from \( \mathcal{D}_q^*(\mathbb{R}) \) onto \( \mathcal{D}_q^*(\mathbb{R}) \) and \( J_q^{-1} = \partial_q \), where \( \mathcal{D}_q^*(\mathbb{R}) \) is the subspace of \( \mathcal{D}_q(\mathbb{R}) \) constituted of odd functions. Hence, by writing \( f_o = \partial_q J_q f_o \) and by making use of (40) and (38) we get

\[
\frac{c_{\alpha,q}}{2} \int_{-\infty}^{+\infty} R_{\alpha,q} \Delta_{\alpha,q} f_o(x).\partial_q f_o(x).|x|^{2\alpha+1} d_q x
\]

\[
= \frac{c_{\alpha,q}}{2} \int_{-\infty}^{+\infty} R_{\alpha,q} \Delta_{\alpha,q} J_q f_o(x) d_q x = K \int_{-\infty}^{+\infty} I_q(g_o)(x) \cdot (t^{R_{\alpha,q}})(f_e)(x) d_q x = -K \int_{-\infty}^{+\infty} \partial_q I_q(g_o)(x) \cdot \partial_q (t^{R_{\alpha,q}}) J_q f_o(x) d_q x.
\]

Since \( \partial_q I_q(g_o)(x) = g_o(x) \), then

\[
\frac{c_{\alpha,q}}{2} \int_{-\infty}^{+\infty} V_{\alpha,q}(g)(x)f(x)|x|^{2\alpha+1} d_q x = K \int_{-\infty}^{+\infty} g(x) \left[(t^{R_{\alpha,q}}) f_e(x) + \partial_q (t^{R_{\alpha,q}}) J_q f_o(x)\right] d_q x.
\]

As \( g \) is arbitrary in \( \mathcal{D}_q(\mathbb{R}) \), this relation when combined with (38) gives the result. \[\Box\]

Theorem 9. The transform \( (t^{V_{\alpha,q}}) \) is an isomorphism from \( \mathcal{D}_q(\mathbb{R}) \) onto itself, its inverse transform is given by

\[
\forall x \in \mathbb{R}, (t^{V_{\alpha,q}})^{-1}(f)(x) = (t^{R_{\alpha,q}})^{-1}(f_e)(x) + \partial_q \left[(t^{R_{\alpha,q}})^{-1} J_q(f_o)\right](x),
\]

where \( (t^{R_{\alpha,q}})^{-1} \) is the inverse transform of \( t^{R_{\alpha,q}} \).

Proof. Taking account of the relation \( J_q \partial_q f(x) = f(x) \) for all \( f \in \mathcal{D}_q(\mathbb{R}) \) and proceeding as in Theorem 7 we obtain the result. \[\Box\]
6. \(q\)-Dunkl Transform

**Definition 5.** Define the \(q\)-Dunkl transform for \(f \in L^1_{\alpha,q}(\mathbb{R}_q)\) by

\[
F^\alpha,q_D(f)(\lambda) = \frac{c_{\alpha,q}}{2} \int_{-\infty}^{+\infty} f(x)\psi^{\alpha,q}_{-\lambda}(x)|x|^{2\alpha+1}d_qx,
\]

where \(c_{\alpha,q}\) is given by (31).

**Remarks:**
1) It is easy to see that in the even case \(F^\alpha,q_D\) reduces to the \(q\)-Bessel Fourier transform given by (30) and in the case \(\alpha = -\frac{1}{2}\), it reduces to the \(q^2\)-analogue Fourier transform given by (15).
2) Letting \(q \uparrow 1\) subject to the condition (16), gives, at least formally, the classical Bessel-Dunkl transform.

Some properties of the \(q\)-Dunkl transform are given in the following proposition.

**Proposition 9.** i) If \(f \in L^1_{\alpha,q}(\mathbb{R}_q)\) then \(F^\alpha,q_D(f) \in L^\infty_q(\mathbb{R}_q)\),

\[
\|F^\alpha,q_D(f)\|_{\infty,q} \leq \frac{2c_{\alpha,q}}{(q; q)_\infty} \|f\|_{1,\alpha,q}
\]

and

\[
\lim_{\lambda \to \infty} F^\alpha,q_D(f)(\lambda) = 0.
\]

ii) For \(f \in L^1_{\alpha,q}(\mathbb{R}_q)\),

\[
F^\alpha,q_D(\Lambda_{\alpha,q}f)(\lambda) = i\lambda F^\alpha,q_D(f)(\lambda).
\]

iii) For \(f, g \in L^1_{\alpha,q}(\mathbb{R}_q)\),

\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F^\alpha,q_D(f)(\lambda)g(\lambda)|\lambda|^{2\alpha+1}d_q\lambda = \int_{-\infty}^{+\infty} f(x)F^\alpha,q_D(g)(x)|x|^{2\alpha+1}d_qx.
\]

**Proof.** i) Follows from the definition of \(F^\alpha,q_D(f)\), the Lebesgue theorem and the fact that 
\(|\psi^{\alpha,q}_{-\lambda}(x)| \leq \frac{4}{(q; q)_\infty}\), for all \(\lambda, x \in \mathbb{R}_q\).

ii) Using the relation (45) and Proposition 6 we obtain the result.

iii) Let \(f, g \in L^1_{\alpha,q}(\mathbb{R}_q)\).

Since for all \(\lambda, x \in \mathbb{R}_q\), we have 
\(|\psi^{\alpha,q}_{\lambda}(x)| \leq \frac{4}{(q; q)_\infty}\), then

\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(x)g(\lambda)|\psi^{\alpha,q}_{\lambda}(x)||x|^{2\alpha+1}d_q\lambda \leq \frac{4}{(q; q)_\infty} \|f\|_{1,\alpha,q} \|g\|_{1,\alpha,q}.
\]

So, by the Fubini’s theorem, we can exchange the order of the \(q\)-integrals, which gives the result. \(\blacksquare\)
Theorem 10. For all $f \in L^1_{\alpha,q}(\mathbb{R}_q)$, we have
\begin{equation}
\forall x \in \mathbb{R}_q, \quad f(x) = \frac{c_{\alpha,q}}{2} \int_{-\infty}^{+\infty} F_D^{\alpha,q}(f)(\lambda)\psi^\alpha_q(x) |\lambda|^{2\alpha+1} d\lambda
= F_D^{\alpha,q}(F_D^{\alpha,q}(f))(x).
\end{equation}

Proof. Let $f \in L^1_{\alpha,q}(\mathbb{R}_q)$ and $x \in \mathbb{R}_q$. Since for all $\lambda, t \in \mathbb{R}_q$, we have
\[ |\psi^\alpha_q(t)| \leq \frac{4}{(q; q)_\infty}, \quad \text{and } \lambda \mapsto \psi^\alpha_q(x) \text{ is in } S_q(\mathbb{R}_q), \text{ then}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(t)\psi^\alpha_q(t)\psi^\alpha_q(x)||t\lambda|^{2\alpha+1}dtd\lambda \leq \frac{4}{(q; q)_\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(t)||\psi^\alpha_q(x)||t\lambda|^{2\alpha+1}dtd\lambda
= \frac{4}{(q; q)_\infty} \|f\|_{1,\alpha,q}\|\psi^\alpha_q(\cdot)\|_{1,\alpha,q}.
\]

Hence, by the Fubini’s theorem, we can exchange the order of the $q$-integrals and by Proposition 7, we obtain
\[
\frac{c_{\alpha,q}}{2} \int_{-\infty}^{\infty} F_D^{\alpha,q}(f)(\lambda)\psi^\alpha_q(x) |\lambda|^{2\alpha+1} d\lambda
= \left( \frac{c_{\alpha,q}}{2} \right)^2 \int_{-\infty}^{\infty} f(t) \left( \int_{-\infty}^{\infty} \psi^\alpha_q(t)\psi^\alpha_q(x)|\lambda|^{2\alpha+1}d\lambda \right) |t|^{2\alpha+1}dt = f(x).
\]
The second equality is a direct consequence of the definition of the $q$-Dunkl transform, Proposition 6 and the definition of the $q$-Jackson integral.

Theorem 11. i) Plancherel formula
For $\alpha \geq -1/2$, the $q$-Dunkl transform $F_D^{\alpha,q}$ is an isomorphism from $S_q(\mathbb{R}_q)$ onto itself. Moreover, for all $f \in S_q(\mathbb{R}_q)$, we have
\begin{equation}
\|F_D^{\alpha,q}(f)\|_{2,\alpha,q} = \|f\|_{2,\alpha,q}.
\end{equation}

ii) Plancherel theorem
The $q$-Dunkl transform can be uniquely extended to an isometric isomorphism on $L^2_{\alpha,q}(\mathbb{R}_q)$. Its inverse transform $(F_D^{\alpha,q})^{-1}$ is given by:
\begin{equation}
(F_D^{\alpha,q})^{-1}(f)(x) = \frac{c_{\alpha,q}}{2} \int_{-\infty}^{+\infty} f(\lambda)\psi^\alpha_q(x) |\lambda|^{2\alpha+1}d\lambda = F_D^{\alpha,q}(f)(-x).
\end{equation}

Proof. i) From Theorem 10, to prove the first part of i) it suffices to prove that $F_D^{\alpha,q}$ lives $S_q(\mathbb{R}_q)$ invariant. Moreover, from the definition of $S_q(\mathbb{R}_q)$ and the properties of the operator $\partial_q$ (Lemma 1), one can easily see that $S_q(\mathbb{R}_q)$ is also the set of all function defined on $\mathbb{R}_q$, such that for all $k, l \in \mathbb{N}$, we have
\[
\sup_{x \in \mathbb{R}_q} |\partial_q^k \left( x^l f(x) \right) | < \infty \text{ and } \lim_{x \to 0} \partial_q^k f(x) \text{ exists.}
\]
Now, let $f \in S_q(\mathbb{R}_q)$ and $k, l \in \mathbb{N}$. On the one hand, using the notation $\Lambda^{0}_{\alpha,q}f = f$ and $\Lambda^{n+1}_{\alpha,q}f = \Lambda_{\alpha,q}(\Lambda^{n}_{\alpha,q}f)$, $n \in \mathbb{N}$, we obtain from the properties of the operator $\Lambda_{\alpha,q}$ that for
Proof. Using the relation (58) and Theorem 5, we obtain for
\begin{equation}
∀ \alpha, \beta \geq -1/2.
\end{equation}
On the other hand, from the relation (49), we have
\begin{equation}
\lambda^l F^{t α,q}_D(f)(λ) = (-i)^l F^{t α,q}_D(λ^{l α,q} f)(λ).
\end{equation}
By Theorem 10, we deduce that
\begin{equation}
F^{t α,q}_D(q^{−α/2} f)(x) = F^{t α,q}_D(f)(−x), \quad x ∈ R_q.
\end{equation}
Finally, the Plancherel formula (67) is a direct consequence of the second equality in Theorem 10 and the relation (65).
ii) The result follows from i), Theorem 10 and the density of $S_2(\mathbb{R}_q)$ in $L^2_\alpha,q(\mathbb{R}_q)$.

\textbf{Theorem 12.} The $q$-Dunkl transform and the $q^2$-analogue Fourier transform are linked by
\begin{equation}
∀ f ∈ D_q(\mathbb{R}_q), \quad F^{α,q}_D(f) = [t V_{α,q}(f)](−1; q^2).
\end{equation}

\textbf{Proof.} Using the relation (58) and Theorem 5, we obtain for $f ∈ D_q(\mathbb{R}_q)$,
\begin{align*}
[t V_{α,q}(f)](−1) & = K \int_{−∞}^{+∞} (t V_{α,q}(f)(t)e(−iλt; q^2))d_q t \\
& = \frac{c_{α,q}}{2} \int_{−∞}^{+∞} V_{α,q}(e(−iλx; q^2)) f(x)|x|^{2α+1}d_q x \\
& = \frac{c_{α,q}}{2} \int_{−∞}^{+∞} f(x)ψ^{α,q}_−(x)|x|^{2α+1}d_q x \\
& = F^{α,q}_D(f)(λ).
\end{align*}

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