Scaling theory of Tomonaga-Luttinger liquid with $1/r^\beta$ type long-range interactions

Hitoshi Inoue

Department of Physics, Kyushu University 33, Hakozaki, Higashi-ku, Fukuoka 812-8581, Japan

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Abstract

We discuss effects of $1/r^\beta$ type long-range (LR) interactions in a tight-binding model by utilizing the bosonization technique, renormalization group and conformal field theory (CFT). We obtain the low energy action known for Kibble’s model which generates the mass gap in 3 dimension when $\beta = 1$, the Coulomb force case. In one dimension, the dispersion relations predict that the system remains gapless even for $\beta = 1$ and the existences of Tomonaga-Luttinger liquid (TLL) when $\beta > 1$. When $\beta = 1$, the LR interactions break TLL in the long wavelength limits, even if the strength is very small. We make the more precise arguments from the standpoint of the renormalization group and CFT. Finally we derive the accurate finite size scaling of energies and thermodynamics. Moreover we proceed to numerical calculations, considering the LR umklapp process terms. We conclude that the TLL phase become wider in the strength space of interactions as the power $\beta$ approaches to 1.

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I. INTRODUCTION

The electron systems have attracted our much attention in the low and high energy physics. As the dimension of the electron systems decrease, the charge screening effects become less important. In spite of these facts models with short range interactions have been adopted in many researches of one dimensional electron systems. The recent advanced technology make it possible to fabricate quasi-one-dimension systems. Actually in a low temperature the effect of Coulomb forces have been observed in GaAs quantum wires [1], quasi-one-dimensional conductors [2–4] and 1D Carbon nanotubes [5–7].

A role of a $1/r$ Coulomb repulsive force was investigated on long distance properties by bosonization techniques [8]. The charge correlation decays very slowly with distance suggesting that the ground state is the Wigner crystal rather than the Tomonaga-Luttinger liquid (TLL).

On one hand the insulator-metal transition caused by the long-range (LR) Coulomb interactions have been discussed [9]. For $1/r^2$ type weak interactions the ground state is TLL which is explained by the Gaussian CFT [10,11]. It was reported that the strong $1/r^2$ interactions make the system gapless to gapful through the generalized Kosterlitz-Thouless transition [12].

The aim of this paper is to find the precise finite size scaling of energies and estimate the range of TLL in the strength space of the $1/r^β(β ≥ 1)$ type LR interactions for various powers $β$. The strategy is as follows. In section II A and B we discuss the long wavelength behaviors of the system when we consider the LR forward scatterings by making use of bosonization techniques and the renormalization group (RG) method. In the section II C we find the corrections caused by such the $β > 1$ LR forwards scatterings in the finite size scaling of energies and the thermodynamics. In the section III, we analyze a tight-binding model with the $β > 1$ type LR interactions numerically by considering these corrections in the finite size scaling. We show numerical data of the drude weight and compressibility and determine the range of TLL in the interactions strength space. In section IV we argue the
case $\beta = 1$.

II. FIELD THEORETICAL APPROACHES

A. Low energy action

Schulz analyzed the effect of LR Coulomb interactions by bosonization technique [8]. We extend the action to general $1/r^\beta$ type interactions. Namely we write the action:

$$S = \int d\tau dx \frac{1}{2\pi K} (\nabla \phi)^2 + g \int d\tau dx dx' \partial_x \phi(x, \tau)V(|x - x'|)\partial_x' \phi(x', \tau),$$

(2.1)

where $V(x) = \frac{1}{|x|}$ and $K$ is the TL parameter. The second term in (2.1) comes from the forward scattering in the fermion picture. Precisely speaking, there are other oscillating terms in the effective action. Now we focus on what happens in considering only the forward scattering terms of the charge density freedom. We shall discuss the effects of the oscillating terms later in the section III. This action is known for Kibble’s model [13], the interactions of which induce the mass gap in three dimension when $\beta = 1$. The situations in one dimension are different from that in three dimension.

To discuss in the Fourier space we choose the form $V(x) = \frac{1}{(x^2 + \alpha^2)^{\beta/2}}$ which contains the cut-off $\alpha$ in order to remove the ultra-violet divergences. The expression in the wave number space of this action is

$$S = \int dq dw \left\{ \frac{2\pi}{K}(q^2 + w^2) + gq^2 V(q) \right\} |\phi(q, w)|^2,$$

(2.2)

where $V(q)$ is the Fourier transformation of $V(x)$ in one dimension. From this the dispersion relation is

$$w^2 = q^2 \left\{ 1 + \frac{gK}{2\pi} V(q) \right\}.$$  

(2.3)

We can derive the long wavelength behavior of $V(q)$:
\[ V(q) \sim -A + B \ln q \text{ for } \beta = 1 \]
\[ \sim C + Dq^{\beta - 1} \text{ for } \beta > 1 (\beta \neq \text{ odd integer}), \]
\[ \sim E + Fq^2 \ln q + Gq^2 + \cdots \text{ for } \beta = 3 \]
\[ \sim I + Jq^2 + Kq^2 \ln q + Lq^4 + \cdots \text{ for } \beta = 5 \]
\[ \cdots, \quad (2.4) \]
where \( A, \cdots, L \) are constant (See APPENDIX A). From this we see that the system is gapless when \( \beta \geq 1 \) and it is expected to be TLL when \( \beta > 1 \), that is, \( w \sim q \). When \( \beta = 1 \), the LR interactions drive the ground state from TLL to the Wigner crystal [8]. The slowest decaying part of the density correlation function is given by
\[ < \rho(x)\rho(0) > \sim \cos(2k_Fx)\exp(-c\sqrt{\log x}), \quad (2.5) \]
where \( c \) is a function of \( K \) [8]. This feature is not like that of TLL:
\[ < \rho(x)\rho(0) > \sim -K \frac{1}{x^2} + \text{const.} \frac{\cos(2k_Fx)}{|x|^K}. \quad (2.6) \]

\textbf{B. The treatments by the renormalization group}

For the moment we focus our argument on the case \( \beta > 1 \) and \( \beta \neq \text{ odd integer} \) which is expected to belongs to TLL from the dispersion relations (2.4). As the C term in similarities (2.4) give the linear dispersion, the C term is marginal. To see the essence of LR interactions we separate the interactions term \( g \) into two parts:
\[ V(q) = \int dx \frac{e^{iqx} - 1}{(x^2 + \alpha^2)^{\beta/2}} + \int dx \frac{1}{(x^2 + \alpha^2)^{\beta/2}} \]
\[ \equiv V_{\text{long}}(q) + V_{\text{short}}. \quad (2.7) \]
Equivalently the expression in real space is
\[ V(x) = V(x) - V_{\text{short}} \delta(x) + V_{\text{short}} \delta(x) \]
\[ \equiv V_{\text{long}}(x) + V_{\text{short}} \delta(x). \quad (2.8) \]
Hence the present action is rewritten to

\[
S = \int d\tau dx \frac{1}{2\pi K}(\nabla \phi)^2 + g \int d\tau dx dx' \partial_x \phi(x, \tau) V_{\text{long}}(x - x') \partial_{x'} \phi(x', \tau).
\]

\[
= \int dq dw \left\{ \frac{2\pi}{K'} (q^2 + w^2) + gq^2 V_{\text{long}}(q) \right\} |\phi(q, w)|^2.
\]

(2.9)

We derive the RG eq. of g:

\[
\frac{dg}{dl} = (1 - \beta) g
\]

(2.10)

(See APPENDIX B.). Just later we also find the consistency of this eq. by conformal field theory.

For the case \( \beta = 1 \) we can derive the RG eq. of \( g \) and the velocity(See APPENDIX B.).

C. The finite size scaling of energies

We know the energy size scaling in the Gaussian CFT [14–17]:

\[
\Delta E_n = \frac{2\pi vx_n}{L} E_g = e_g L - \frac{\pi vc}{6L}.
\]

(2.11)

Considering the LR interactions, we can extract the corrections from these energy size scalings precisely (See APPENDIX C.):

\[
\Delta E_n = \frac{2\pi vx_n}{L} (1 + g \frac{\text{const.}}{L^{\beta-1}} + g \frac{1}{L^2} + O(1/L^2))
\]

\[
E_g = e_g L - \frac{\pi vc}{6L} (1 + g \frac{\text{const.}}{L^{\beta-1}} + g \frac{1}{L^2} + O(1/L^2)),
\]

(2.12)

where \( \beta(>1) \) is not odd integer. For the \( \beta = \text{odd integer} \), the logarithmic corrections appear. The details are shown in APPENDIX C and the \( \beta = 1 \) case is discussed later. The \( O(1/L^2) \) terms come from the irrelevant field \( L_{-2} \bar{L}_{-2} \) and the LR g term. The first scaling of eqs. (2.12) imply that the LR forward scatterings

\[
g \int dx' \partial_x \phi(x, \tau) V(x - x') \partial_{x'} \phi(x', \tau)
\]

(2.13)
have the scaling dimension $x_g = \beta + 1$ effectively, which is consistent with the RG eq. of $g$ (2.10). The solution of eq. (2.10) is

$$g(l) = g(0)e^{(1-\beta)l} = g(0)e^{(1-\beta)\ln L} = g(0)\frac{1}{L^\beta - 1},$$

(2.14)

where we use $l = \ln L$. Finally we obtain the accurate finite size scaling:

$$\Delta E_n = \frac{2\pi v x_n}{L}(1 + \text{const.}\frac{1}{L^{2(\beta-1)}} + \frac{\text{const.}}{L^{\beta+1}} + O(1/L^2))$$

$$E_g = \epsilon_g L - \frac{\pi v c}{6L}(1 + \text{const.}\frac{1}{L^{2(\beta-1)}} + \frac{\text{const.}}{L^{\beta+1}} + O(1/L^2)).$$

(2.15)

Moreover we can discuss the thermodynamics properties. We replace $E_g \rightarrow f/T$ and $L \rightarrow v/T$ in the ground state energy of eq. (2.15), where $f$ is the free energy per temperature.

We obtain the low temperature behaviors of $f$:

$$f = -\frac{\pi v c}{6v^2}(1 + \text{const.}(\frac{T}{v})^{2(\beta-1)} + \frac{\text{const.}}{L^{2(\beta-1)}} + \frac{\text{const.}}{L^{\beta+1}} + \frac{T}{v}^2).$$

(2.16)

Thus the specific heat $C = -\frac{\partial^2 f}{\partial T^2}$ is

$$C = \frac{\pi c T}{3v}(1 + \text{const.}(\frac{T}{v})^{2(\beta-1)} + \frac{\text{const.}}{L^{2(\beta-1)}} + \frac{\text{const.}}{L^{\beta+1}} + \frac{T}{v}^2).$$

(2.17)

III. NUMERICAL CALCULATIONS

We have investigated the properties of the energy scaling and derived the corrections terms caused by the LR forward scatterings. Let us consider the following tight-binding Hamiltonian with LR interactions:

$$H = -i \sum_j (c^\dagger_{j+1}c_j + h.c) + \frac{V}{2} \sum_{i \neq j} (n_i - <n>)V(|i-j|)(n_j - <n>)V(|i-j|),$$

(3.1)

where $V(i-j) = \frac{1}{(2\pi \sin(\pi |i-j|)/L)^\beta}$ and $n_j = c^\dagger_j c_j$. And we impose the periodic boundary condition and $<n> = 1/2$. By the straight forward bosonization technique, the effective action of (3.1) for the arbitrary filling can be written by

$$S = \int d\tau dx \frac{1}{2\pi K}(\nabla \phi)^2 + g \int d\tau dx dx' \partial_{x'} \phi(x, \tau)V(x - x')\partial_{x'} \phi(x', \tau)$$

$$+ g \text{const.} \int d\tau dx dx' \cos(2k_F x + \sqrt{2} \phi(x, \tau))V(x - x') \cos(2k_F x' + \sqrt{2} \phi(x', \tau)).$$

(3.2)
The last oscillating term consists of the umklapp process term cos $2\sqrt{2}\phi$ and the longer range interactions. We have found the corrections in eqs. (2.15) caused by the irrelevant LR forward scatterings ($\beta > 1$ and $\beta \neq$ odd integer) in the previous section. However the oscillating terms may disturb the TLL and cause the mass gap. We numerically investigate how long range the TLL phase survive for the strength of the LR interactions. By making use of eqs. (2.15), we can calculate the compressibility $\chi = K/v$ and the drude weight $D = vK$ within the TLL framework.

The operators $\cos \sqrt{2}\phi$ and $e^{\pm i\sqrt{2}\theta}$ have the scaling dimensions $K/2$ and $1/2K$ in the TLL. The operators have the symmetries $q = \pi, S_z = 0$ and $q = \pi, S_z = 1$ respectively. The explicit excitations describing the compressibility and the drude weight are

$$\chi = K/v = 1/(2L\Delta E(S_z = 1, q = \pi))$$

$$D = vK = 2L\Delta E(q = \pi).$$

(3.3)

In Fig. 1 and 2 we plot the compressibility $\chi$ and the drude weight $D = vK$ versus the interactions strength $g$ for the various powers $\beta$. For $g < 0$ the $\chi$ exhibits the rapid increase which suggests the phase separation. In spin variables’ language of (3.1), that is, XXZ model, this phase separation is nothing but the ferromagnetic phase. Hence for the larger $\beta$ the point of the phase separation approaches to $-1$. For $g > 0$ we see the subtle tendency that the $\chi$ become smaller as the $\beta$ is smaller. For the drude weight of $g > 0$ we find that the $D$ become larger as the $\beta$ approach to the Coulomb interactions case $\beta = 1$.

In Fig. 3 we plot the velocity versus the interactions strength $g$ for the various powers $\beta$. The velocity is defined by

$$v = \frac{L}{2\pi} \Delta E(q = 2\pi/L).$$

(3.4)

Note that the velocities are finite values for $\beta > 1$ on the contrary of the Coulomb interactions case $\beta = 1$ (See APPENDIX B 2.). There are the points where the velocities are zero, implying the existences of the phase separation.

In Fig. 4 we plot the normalization $\frac{D}{\chi v}$ versus the interactions strength $g$ for the various powers $\beta$, where $D$, $\chi$ and $v$ are defined by eqs. (3.3) and eq. (3.4) respectively. If the
system belongs to TLL, this value is expected to be 1. We see that the TLL region become wider as the $\beta$ approaches to 1 for $g > 0$. Reversely for $g < 0$ the TLL region is smaller as $\beta$ goes to 1.

IV. DISCUSSIONS AND SUMMARY

We have investigated the range of the TLL theoretically and numerically by utilizing the scaling argument of renormalization group and CFT. We have found that the TLL region become wider and the drude weight become larger as the power $\beta$ approaches to 1 which is the Coulomb case. It was found experimentally \[18\] that the amplitude of the persistent current in the micron-size Au loops is larger than a predicted value \[19\]. Recently it has been found by the self harmonic approximation treatments that the localization length in the sine-Gordon model with randomness become larger when the Coulomb interactions are considered \[20\]. The our present results seem to give the additional confirmations to these features.

We have found that the TLL is broken for the stronger LR interactions. We expect that the system is gapful for stronger $g$ and the ground state become two fold degenerate independently of $\beta(> 1)$. Yamanaka et al derived the necessary conditions for the gap generations in the fermion systems of one dimension with nonperturbative arguments \[21\]:

$$n\rho = \text{integer},$$

(4.1)

where $n$ is the period of ground state and $\rho = N_F/L$ is the density of the fermion. Hence $n = 2$ is derived for $\rho = 1/2$ ($k_F = \pi/2$). This is reasonable, because the LR interactions include the short range umklapp process term $\cos 2\sqrt{2}\phi$ which cause the mass gap and 2 fold degeneracy of the system.

Let us argue the case $\beta = 1$. Assuming the $g$ term is the small perturbation, we obtain the energy size scaling (See APPENDIX B and APPENDIX C.):

$$\Delta E_n = \frac{2\pi v_p x_n}{L} (1 + O(g) + g(0) \text{ const.} \ln(1/L))$$
\[ E_g = e_g L - \frac{\pi v_0 c}{6L}(1 + O(g) + g(0) \text{ const. } \ln(1/L)), \tag{4.2} \]

where we assume \(g(l) \sim g(0) \sim \text{const.}\) (See APPENDIX B.). We should interpret that the velocity \(v_0\) is for no perturbations. According to these results, the \(g\) terms break the TLL behaviors \(L\Delta E = \text{const.}\) for \(L \to \infty\). This estimation is consistent with a numerical report that TLL is broken by the Coulomb interactions \[22\].

Through the replacements same as \(\beta > 1\) case, we obtain the temperature dependences of the free energy and specific heat:

\[ f = -\frac{\pi c}{6v_0} T^2(1 + O(g) + g \text{ const. } \ln(\frac{T}{v_0})) \tag{4.3} \]

and

\[ C = \left\{ \frac{\pi c}{3v_0} + O(g) \right\} T + g \text{ const. } \frac{T}{v_0} \ln\left(\frac{T}{v_0}\right), \tag{4.4} \]

where const. is the positive constant (See the negative C term of eqs. (C9) in APPENDIX C.). These suggest that the TLL is broken even for the slight \(g\) when \(T\) is near to 0 because \(|g\frac{T}{v_0} \ln\left(\frac{T}{v_0}\right)|\) become larger than \(T\). When \(T = 0\), even for small \(g\), the system belongs to other universality class in which the density correlation function is given by (2.5). However we can justify that the TLL holds if \(O(e^{-1/g}) << T < 1\) because the \(|g\frac{T}{v_0} \ln\left(\frac{T}{v_0}\right)|\) is smaller than \(T\). For this case it is convincing that the \(g\frac{T}{v_0} \ln\left(\frac{T}{v_0}\right)\) terms are the small perturbations. Therefore we can have the pictures for the case \(\beta = 1\) (See Fig. 8.).

In summary, by utilizing the bosonization technique, the renormalization group and the CFT we analyzed the TLL with LR forward scatterings. And we could obtain the accurate finite size scaling of the energies and thermodynamics properties. By making use of these scaling relations and the numerical calculations we found that the range of TLL and the drude weight increase as the interactions’ power \(\beta\) approaches to 1 which is the Coulomb

\[1\] Though the scaling (C5) is valid for finite size (finite temperature), we are treating the lower temperature behaviors now. Therefore we should write \(O(e^{-1/g}) << T < O(1)\).
interactions’ one. Furthermore we obtained the specific heat for $\beta = 1$, which deviates from the linear $T$ to $T + gT \ln T$, where the small $g$ takes the positive value if the LR interactions are repulsive. This implies the TLL holds when $O(e^{-1/g}) \ll T < 1$.

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**APPENDIX A: THE LONG WAVELENGTH BEHAVIOR OF $V(Q)$**

The Fourier form of the interaction $V(x)$ is given by

$$V(q) = 2\sqrt{\pi} \frac{\alpha^{\beta/2-1/2}}{\Gamma(\beta/2)^2} \frac{q^{\beta/2-1/2} K_{\beta/2-1/2}(\alpha q)}{q^{\beta/2-1/2} K_{\beta/2-1/2}(\alpha q)}, \quad (A1)$$

where $K_\nu(x)$ is the modified Bessel function of $\nu$th order and $\Gamma(x)$ is the gamma function. When $\beta = 1$, $V(q) = K_0(\alpha q) \sim -\ln q$ ($q \to 0$) which was already discussed by Schulz [8].

When $\beta > 1$ and $\beta \neq$ odd integer,

$$V(q) = \int_{-\infty}^{\infty} dx \frac{e^{iqx}}{(x^2 + \alpha^2)^{\beta/2}} = \int dx \frac{1}{(x^2 + \alpha^2)^{\beta/2}} + \int dx \frac{e^{iqx} - 1}{(x^2 + \alpha^2)^{\beta/2}}$$

$$= \text{const.} + \frac{1}{q^{1-\beta}} \int dx' \frac{e^{iqx' - 1}}{(x'^2 + q^2 \alpha^2)^{\beta/2}} \sim \text{const.} + Aq^{\beta-1}, \quad A: \text{const.} \quad (A2)$$

When $\beta$ is odd integer, the $V(q)$ is shown in the main part of the present paper.

**APPENDIX B: RENORMALIZATION GROUP EQUATION**

1. $\beta > 1$ and $\beta \neq$ odd integer

We derive the renormalization group equations heuristically. Let us start from the action (2.9):
\[ S = \sum_w \sum_{q=\Lambda}^{\Lambda/b} \frac{2\pi}{K} (q^2 + w^2) |\phi(q, w)|^2 + g \sum_w \sum_{q=\Lambda}^{\Lambda/b} q^2 (V(q) - V_{\text{short}}) |\phi(q, w)|^2 \]
\[ = \sum_w \{ \sum_{q=\Lambda/b}^{\Lambda/b} + \sum_{q=\Lambda}^{\Lambda} \} + g \sum_w \{ \sum_{q=-\Lambda/b}^{\Lambda/b} + \sum_{q=-\Lambda}^{\Lambda} \} \}. \tag{B1} \]

The partition function is
\[ Z = \int D\phi_{\text{slow}} D\phi_{\text{fast}} \exp(-S^0_{\text{slow}} - S^0_{\text{fast}} - S^g_{\text{slow}} - S^g_{\text{fast}}). \tag{B2} \]

Thus we can integrate out \( S_{\text{fast}} \) (\(|q| > \Lambda/b \) component) simply and obtain
\[ Z = \int D\phi_{\text{slow}} \exp(-S^0_{\text{slow}} - S^g_{\text{slow}}). \tag{B3} \]

The remaining procedure of the renormalization is the scale transformation
\[ q \to q/b, \ w \to w/b \ \text{and} \ \phi \to \phi b^2, \tag{B4} \]
where we choose the dynamical exponent 1. The results are
\[ S^0_{\text{slow}} \to S^0 \]
\[ S^g_{\text{slow}} \to g \sum_w \sum_{q=\Lambda}^{\Lambda/b} q^2 (V(q/b) - V_{\text{short}}) |\phi(q, w)|^2 \]
\[ \to gb^{1-\beta} \sum_w \sum_{q=-\Lambda}^{\Lambda} q^2 (V(q) - V_{\text{short}}) |\phi(q, w)|^2, \tag{B5} \]
where we use the behavior \( V(q) - V_{\text{short}} \sim q^{\beta-1} \) from (2.4). Hence we obtain the renormalization group eq.
\[ \frac{dg(b)}{db} = (1 - \beta) \frac{g(b)}{b}. \tag{B6} \]

We put \( l = \ln b \) and obtain eq. (2. 10).

2. \( \beta = 1 \)

The dispersion relations of the Coulomb interactions case include the marginal part \( w \sim q \) and \( w \sim q^{1/2} \ln q \) as well as the \( \beta > 1 \) case. However it is difficult to know the explicit
separated function like eqs. (2.7) and (2.8). Therefore we renormalize by using the bare 
\[ V(q) \sim -A \log q + B. \]

Integrating out the fast part, we obtain the effective action of the slow part

\[ S_{\text{slow}} = \sum_w \sum_{q=-\Lambda/b}^{\Lambda/b} \frac{2\pi}{K} (vq^2 + w^2/v)|\phi(q, w)|^2 + g \sum_w \sum_{q=-\Lambda/b}^{\Lambda/b} q^2V(q)|\phi(q, w)|^2, \]

(B7)

where we dare to leave the velocity in the Gaussian part. Note that we need not the renormalization of the velocity in the case \( \beta > 1 \). After the scale transformation (B 4), we obtain the eqs.

\[
\begin{align*}
\frac{dg}{dl} &= 0 \\
\frac{d}{dl} \left( \frac{v}{K} \right) &= \frac{gA}{2\pi} \\
\frac{d}{dl} \left( \frac{1}{vK} \right) &= 0.
\end{align*}
\]

(B8)

We see that the \( K \) and the velocity \( v \) is renormalized in stead of the no renormalization of \( g \). The velocity \( v \) can be written by

\[ v(b) = \sqrt{\text{const.} + g \text{ const.} \ln b} \sim \sqrt{\ln L}. \]

(B9)

The velocity shows the weak divergence for long distances, which is consistent with the estimations of \( v = \frac{dw}{dq} \) from the eqs. (2.3) and (2.4).

**APPENDIX C: THE FINITE SIZE SCALING OF ENERGY**

We write the Hamiltonian in the finite strip from the action (2.1):

\[ H = H_{\text{TLL}} + g \int_{-L/2}^{L/2} d\sigma_1 d\sigma_2 \partial_{\sigma_1} \phi(\sigma_1) \partial_{\sigma_2} \phi(\sigma_2) V(|\sigma_1 - \sigma_2|) \theta(|\sigma_1 - \sigma_2| - \alpha_0), \]

(C1)

where the \( H_{\text{TLL}} \) is the TLL part of the Hamiltonian. We introduce the step function \( \theta(x) \) to avoid the ultra violet divergences which come from \( V(x) \) and the operator product expansion of \( \partial_\sigma \phi(\sigma) \). For the small perturbation \( g \) the ground state energy \( E_g \) varies as
\[ E'_g - E_g = g \int_{-L/2}^{L/2} d\sigma_1 d\sigma_2 V(|\sigma_1 - \sigma_2|) < 0|\partial_{\sigma_1} \phi(\sigma_1) \partial_{\sigma_2} \phi(\sigma_2)|0 > \theta(|\sigma_1 - \sigma_2| - \alpha_0) \]
\[ = g \int_{-L/2}^{L/2} d\sigma_1 d\sigma_2 V(|\sigma_1 - \sigma_2|) \left[ < 0|\partial_{\sigma_1} \varphi(w_1) \partial_{\sigma_2} \varphi(w_2)|0 > + < 0|\partial_{\bar{\sigma}_1} \bar{\varphi}(\bar{w}_1) \partial_{\bar{\sigma}_2} \bar{\varphi}(\bar{w}_2)|0 > \right] \]
\[ = \frac{2\pi}{L} \left[ \frac{1}{\sin^2 \frac{\pi}{L}} \right] \]

where we introduce the coordination \( w = \tau + i\sigma \) \((-L/2 < \sigma < L/2, -\infty < \tau < \infty\)). From the characters of the Gaussian part (TLL part) we can separate as \( \phi(\sigma, \tau) = \varphi(w) + \bar{\varphi}(\bar{w}) \) and derive \( < 0|\partial_{\bar{\sigma}_1} \bar{\varphi}(\bar{w}_1) \partial_{\bar{\sigma}_2} \bar{\varphi}(\bar{w}_2)|0 > = 0 \). The content of the brackets is modified as follows:

\[ = \left[ \frac{2\pi}{L} \right] ^2 \frac{1}{2\sin^2 \frac{\pi}{L}} \]

where we transform the correlations \( < \partial_{z_1} \varphi(z_1) \partial_{z_2} \varphi(z_2) > = 1/(z_1 - z_2)^2 \) in \( \infty \times \infty \) \( z \) plane to the present strip \( w \) thorough \( z = \exp \frac{2\pi w}{L} \). At present case \( \partial_w \varphi(w) \) (\( \partial_{\bar{w}} \bar{\varphi} \)) have the spin \( s = 1(-1) \) and conformal dimension \( \Delta = 1(\bar{\Delta} = 1) \). Hence we obtain

\[ E'_g - E_g = \frac{g}{8} \left( \frac{2\pi}{L} \right) ^2 L \int_{-L}^{L} dxV(|x|) \frac{1}{\sin^2 \frac{\pi}{L}} \theta(|x| - \alpha_0) \]
\[ = \frac{g}{8} \left( \frac{2\pi}{L} \right) ^2 \int_{-1}^{1} dx' V(|Lx'|) \frac{1}{\sin^2 \frac{\pi}{L}} \theta(L|x'| - \alpha_0) \]
\[ = \frac{g}{8} \left( \frac{2\pi}{L} \right) ^2 \left( \frac{\pi}{L} \right) ^\beta \int_{-1}^{1} dx' \frac{1}{(\sin \pi |x'|)^\beta} \frac{1}{\sin^2 \frac{\pi}{L}} \theta(|x'| - \alpha_0) \]

where we impose the periodic boundary condition and use the interaction potential \( V(x) = 1/(L \sin \frac{2\pi x}{L})^\beta \). Putting \( \epsilon = \alpha_0/L \) for convenience, we give the differential of the integral part:

\[ \frac{\partial}{\partial \epsilon} \int_{-1}^{1} dx' \frac{1}{(\sin \pi |x'|)^\beta} \frac{1}{\sin^2 \frac{\pi}{L}} \theta(|x'| - \epsilon) \]
\[ = \frac{1}{(\sin \pi |\epsilon|)^\beta} \frac{1}{\sin^2 \pi \epsilon} \]

After integrating the Taylor expansion about \( \epsilon \) of this quantity, we obtain

\[ \int_{-1}^{1} dx' \frac{1}{(\sin \pi |x'|)^\beta} \frac{1}{\sin^2 \pi x'} \theta(|x'| - \epsilon) \]
\[
\frac{1}{\pi^2} \text{const.} - \frac{(\pi \epsilon)^{-\beta - 1}}{-\beta - 1} - \frac{\beta + 2}{3!(-\beta + 1)} (\pi \epsilon)^{-\beta + 1} \\
+ \frac{1}{-\beta + 3} \{ -\frac{1}{5!}(\beta + 2) + \frac{1}{72}(\beta + 1)(\beta + 2) \} (\pi \epsilon)^{-\beta + 3} + O((\pi \epsilon)^{-\beta + 5})].
\]

(C6)

where \( \beta \neq \text{odd integer} \). Therefore we can write the corrections in the form:

\[
E_g' - E_g = g[A/L^3 + BL + C/L + D/L^3 + O(1/L^5)].
\]

(C7)

where A, B, C and D are the finite constant values. The form of B and C terms is just same as the second term of eqs. (2.11). We assume that these terms should be renormalized to TLL, because the LR interactions inevitably contain the short range types of interactions which are reduced to \((\partial_x \phi)^2\). It is natural and reasonable that in the corrections of the ground state energy there are the same contributions as the TLL forms of (2.11). Thus we think that the intrinsic contributions of the LR interactions are

\[
E_g' - E_g = g[A/L^3 + B/L + C/L + D/L^3 + O(1/L^5)].
\]

(C8)

This result is consistent with the spectrum analysis and the RG results for \( \beta > 1 \). For the \( \beta = \text{odd number} \) case, there exist the logarithmic corrections instead of the eq. (C. 6). We write the results for the respective \( \beta \) specifically:

\[
E_g' - E_g = g[A/L^3 + B/L + C/L + D/L^3 + E/L^5 + O(1/L^7)] \text{ for } \beta = 1 \\
g[A/L^3 + B/L + C/L + D/L^3 + E/L^5 + O(1/L^7)] \text{ for } \beta = 3 \\
g[A/L^5 + B/L + C/L + D/L^3 + E/L^5 + O(1/L^7)] \text{ for } \beta = 5 \\
\ldots
\]

(C9)

Next we derive the corrections for the energy of the excited state:

2If the B and C term in eq. (C. 7) give the important contributions to the excitations, the TLL is broken by such the contributions. This is discrepant with the spectrum results and the RG arguments. Hence we assume that the B and C contributions in eq. (C. 7) are not intrinsic. Especially we should regard the B term nonuniversal bulk constants.
\[ E'_n - E_n = g \int_{-L/2}^{L/2} d\sigma_1 d\sigma_2 V(|\sigma_1 - \sigma_2|) < n|\partial_{\sigma_1} \phi(\sigma_1) \partial_{\sigma_2} \phi(\sigma_2)|n > \theta(|\sigma_1 - \sigma_2| - \alpha_0) \]

\[ = g \int_{-L/2}^{L/2} d\sigma_1 d\sigma_2 V(|\sigma_1 - \sigma_2|) \sum_\alpha < n|\partial_{\sigma_1} \phi(\sigma_1)|\alpha > < \alpha|\partial_{\sigma_2} \phi(\sigma_2)|n > \theta(|\sigma_1 - \sigma_2| - \alpha_0) \]

\[ = g \sum_\alpha C_{n\alpha} C_{\alpha n} (2\pi)^2 \int_{-L/2}^{L/2} d\sigma_1 d\sigma_2 V(|\sigma_1 - \sigma_2|) e^{2\pi i (s_n - s_\alpha) / L} \theta(|\sigma_1 - \sigma_2| - \alpha_0) \]

\[ = g \sum_\alpha C_{n\alpha} C_{\alpha n} \left( \frac{2\pi}{L} \right)^2 L \int_0^L dx V(|x|) \cos \frac{2\pi}{L} (s_n - s_\alpha) x \theta(|x| - \alpha_0) \]

\[ = g \sum_\alpha C_{n\alpha} C_{\alpha n} \left\{ \frac{(s_n - s_\alpha)}{L^3} + \frac{B}{-\beta + 1} + \frac{C(s_n - s_\alpha)}{-\beta + 3} \right\} \]

\[ + \frac{D(s_n - s_\alpha)}{-\beta + 5} + O\left(\frac{1}{L^7}\right) \]  \( \beta \neq \text{odd integer} \)

\[ = g \sum_\alpha C_{n\alpha} C_{\alpha n} \left\{ \frac{A(s_n - s_\alpha)}{L^3} + \frac{B}{L} + \frac{C(s_n - s_\alpha)}{L^3} \right\} \]

\[ + D(s_n - s_\alpha) \frac{1}{L^5} + O\left(\frac{1}{L^7}\right) \]  \( \beta = 1 \)

\[ = g \sum_\alpha C_{n\alpha} C_{\alpha n} \left\{ \frac{A(s_n - s_\alpha)}{L^3} + \frac{B}{L} + \frac{C(s_n - s_\alpha)}{L^3} \ln \frac{1}{L} \right\} \]

\[ + D(s_n - s_\alpha) \frac{1}{L^5} + O\left(\frac{1}{L^7}\right) \]  \( \beta = 3 \)

\[ = g \sum_\alpha C_{n\alpha} C_{\alpha n} \left\{ \frac{A(s_n - s_\alpha)}{L^3} + \frac{B}{L} + \frac{C(s_n - s_\alpha)}{L^3} \ln \frac{1}{L} \right\} \]

\[ + D(s_n - s_\alpha) \frac{1}{L^5} + O\left(\frac{1}{L^7}\right) \]  \( \beta = 5 \)

\[ \ldots, \]

(C10)

where we use the results by Cardy [17]:

\[ < n|\phi(\sigma)|\alpha > = C_{n\alpha} (\frac{2\pi}{L})^3 e^{2\pi i (s_n - s_\alpha) / L}. \]  \( \) (C11)

We find the dependences of $1/L$ which should be renormalized to TLL. We think that this is caused by the LR interactions which include the short range interactions. The short interactions is nothing but the part of TLL. Hence it is plausible for the $1/L$ dependences to appear. These situations are same as the ground state properties that we discussed just previously. Strictly speaking, in the eqs. (C10) there is an ambiguity whether the $\Sigma_\alpha$ commute with the integrals. However it is difficult to prove the communtations unexpectedly,
because we must know whether the sum of the operator product expansion coefficients
\[ \sum C_{\alpha}^2 \] is finite, or not. We would transfer this problem to future works.
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FIG. 1. The extrapolated compressibility $\chi = K/v$ is plotted versus the strength $g$. We use the size dependence $K/v(L) = K/v(\infty) + \frac{a}{L^2} + \frac{b}{L^3}$ for $\beta = 1.5$, where $a$ and $b$ are determined numerically (See eqs. (2.15).). We use $K/v(L) = K/v(\infty) + \frac{a}{L^2} + \frac{b}{L^3}$ for $\beta = 2, 2.5$ and $K/v(L) = K/v(\infty) + \frac{a}{L^2} + \frac{b}{L^4}$ for $\beta \geq 4$. 
FIG. 2. The extrapolated drude weight $D = vK$ is plotted versus the strength $g$. We use the same scaling as compressibility.
FIG. 3. The spin wave velocity $v$ is plotted versus the strength $g$. We use same scaling as compressibility.

FIG. 4. The normalization $D/\chi v^2$ is plotted versus the strength $g$.

Wigner crystal \hspace{1cm} \text{Tomonaga-Luttinger liquid}

$0 \hspace{1cm} e^{-1/g}$

$T$
FIG. 5. The schematic phase diagram versus temperature when $\beta = 1$. Though the TLL generally lies for the low temperature $T$, the TLL is broken as $T \to 0$ when $\beta = 1$. Instead the Wigner crystal phase emerges for the lower temperature side than $O(e^{-1/g})$. However the TLL holds when $O(e^{-1/g}) \ll T < 1$. The value $O(e^{-1/g})$ is not the transition point but the point where the perturbation theory breaks down.