THE RIEMANN SURFACE OF A STATIC DISPERSION MODEL
AND REGGE TRAJECTORIES

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Abstract

The S-matrix in the static limit of a dispersion relation is a matrix of a
finite order \( N \) of meromorphic functions of energy \( \omega \) in the plane with cuts
\((-\infty, -1], [+1, +\infty)).\) In the elastic case it reduces to \( N \) functions
\( S_i(\omega) \)
connected by the crossing symmetry matrix \( A.\) The scattering of a neutral
pseudoscalar meson with an arbitrary angular momentum \( l \) at a source
with spin \( 1/2 \) is considered (\( N=2 \)). The Regge trajectories of this model
are explicitly found.

The analytic structure of physical amplitudes in gauge theories with confinement
was investigated in ref.[1]. It was shown that the analytic structure of hadron
physical amplitudes established in old proofs of dispersion relations remains valid
in QCD. It is well known[2] that the static limit of a dispersion relation is equivalent
to the system of nonlinear integral equations[3]. Below, we will study this type
of equations reducing them to a nonlinear boundary value problem [1]. It consists
of the following series of conditions on \( S_i, S^-\)matrix elements:

\( a)\) \( S_i(z) \) — meromorphic functions in the complex plane \( z \)
with cuts \((-\infty, -1], [+1, +\infty)),\)

\( b)\) \( S^*_i(z) = S_i(z^*)\),

\( c)\) \[ | S_i(\omega + i0) |^2 = 1 \text{ at } \omega \geq 1 \]
\[ S_i(\omega + i0) = \lim_{\epsilon \to +0} S_i(\omega + i\epsilon), \]

\( d)\) \( S_i(-z) = \sum_{j=1}^{N} A_{ij} S_j(z)\).
Real values of the variable $z$ represent the total energy $\omega$ of a relativistic particle scattered at a fixed center. The requirement for functions $S_i(z)$ being meromorphic results from the static limit of the scattering problem [5]. The elastic condition of unitarity (1c) is valid only on the right cut of the plane $z$. On the left cut, functions $S_i(z)$ are given by the conditions of crossing symmetry (1d). The matrix of crossing symmetry $A$ is defined by the group under which the $S$-matrix is invariant; see, for instance [4]. Let us write conditions (1) in the matrix form. To this end, we introduce the column $S^{(0)}(z) = [S_1(z), S_2(z), \ldots, S_N(z)]$, where the upper index denotes the physical sheet of the Riemann surface of the $S$-matrix. Conditions (1a,b,d) refer to the physical sheet, while the unitarity condition (1c) can be extended to complex values of $\omega$, being of a component-wise form, $S_i^{(0)}(z)S_i^{(1)}(z) = 1$. The matrix form of the unitarity condition (1c) is derived by the nonlinear operation of inversion $I$ according to the formula $IS(z) = [1/S_1(z), 1/S_2(z), \ldots, 1/S_N(z)]$. As a result, conditions (1a,b,c,d) assume the form

\begin{align}
& a) \quad S^{(0)}(z) - a \text{ column of meromorphic functions in the complex plane } z \\
& \text{ with cuts } (-\infty, -1], [+1, +\infty) \\
& b) \quad S^{(0)*}(z) = S^{(0)}(z^*) \\
& c) \quad S^{(1)}(z) = IS^{(0)}(z) \\
& d) \quad S^{(0)}(-z) = AS^{(0)}(z)
\end{align}

(2)

Analytic continuation onto unphysical sheets will be defined as follows:

\begin{align}
S^{(p)}(z) = (IA)^p S^{(0)}(z(-1)^p).
\end{align}

(3)

By using the definition (3), we can easily continue the unitarity condition (2c) and crossing symmetry (2d) on to unphysical sheets

\begin{align}
& IS^{(p)}(z) = S^{(1-p)}(z), AS^{(p)}(z) = S^{(-p)}(-z)
\end{align}

(4)
and we arrive at the formula

\[(IA)^q S^{(p)}(z) = S^{(q+p)}(z(-1)^q). \tag{5}\]

For example, the scattering of a neutral pseudoscalar pions at a fixed nucleon with spin 1/2 is defined by the condition (1) and the two-row matrix

\[A = \frac{1}{2l+1} \begin{pmatrix} -1 & 2l + 2 \\ -2l & 1 \end{pmatrix}, \quad l \in N. \tag{6}\]

Let us introduce the function \(X = S_1/S_2\) and consider it for \(z = 0\). Then the continuation of \(X\) on to the first unphysical sheet is determined by the rule

\[X^{(1)} = \frac{2lX^{(0)} + 1}{-X^{(0)} + (2l + 2)}\]

and together with the crossing symmetry condition (4) gives the following expression for \(X^{(n)}\)

\[X^{(n)} = \frac{n - (l + 1)}{n + l}, \quad X^{(0)} = -(1 + 1/l). \tag{7}\]

Thus, on any unphysical sheet \(n\) the ratio \(S_1/S_2\) is defined at \(z = 0\) and for construction of \(S_1\) and \(S_2\) it is sufficient to find any of them. Let us denote \(S_2\) by \(\varphi = S_2\). This function is determined by the system of functional equations

\[\varphi^{(n)}\varphi^{(1-n)} = 1, \tag{8}\]

\[\varphi^{(n)}\varphi^{(-n)} = \frac{n + l}{n - l}, \tag{9}\]

which follows from the unitarity and the crossing symmetry conditions (4) on the unphysical sheets. Here only those equalities are used from (4), which were not used for derivation of eq. (7). Equation (8) has an obvious solution in the ring of meromorphic functions

\[\varphi^{(n)} = \frac{G(n)}{G(1 - n)}, \tag{10}\]

where \(G(n)\) is an entire function. Solution (10) can be represented in another form \(\ln \varphi^{(n)} = g(n - 1/2)\), where \(g(n - 1/2)\) is any odd function of its argument.
That form of $\ln \varphi^{(n)}$ is convenient for the solution to eq.(9) which is now of the form

$$g(n + 1) + g(n) = \ln \frac{n + 1/2 + l}{n + 1/2 - l}.$$ 

A partial solution of this nonhomogeneous difference equation can be found by subsequent substitutions of a unknown functions according to the formulae

$$g_{m}(n) = g_{m+1}(n) + \ln \frac{n + (-1)^m \alpha_{m+1}}{n - (-1)^m \alpha_{m+1}},$$

where $\alpha_k = 1/2 + l - k$ and $g_0(n) = g(n)$. The function $g_k$ obeys the equation

$$g_k(n + 1) + g_k(n) = \ln \frac{n + 1/2 + (-1)^k (l - k)}{n + 1/2 - (-1)^k (l - k)}.$$ 

It is clear that

$$g_l(n + 1) + g_l(n) = 0 \quad (11)$$

and a general solution to this equation gives a trivial solution of the problem (1) which does not depend on $l$. Therefore, one gets

$$\varphi^{(n)} = \prod_{m=1}^{l} \frac{n - 1/2 - (-1)^m (1/2 + l - m)}{n - 1/2 + (-1)^m (1/2 + l - m)}. \quad (12)$$

One has an infinite product in formulae (12) for noninteger $l \in \mathbb{R}$. Now eq.(11) is of the form

$$g(n + 1) + g(n) = \ln(-1) \quad (13)$$

In this case one has instead of eq.(12)

$$\varphi^{(n)} = \psi(n) \frac{\Gamma[-\frac{n+l}{2} + 1] \Gamma[\frac{n-l}{2}]}{\Gamma[-\frac{n-1-l}{2} + 1] \Gamma[-\frac{n-1+l}{2}]} \quad (14)$$

where $\psi(n)$ is a general solution of eq.(13) with properties

$$\psi(n + 1)\psi(n) = 1, \quad \psi(n)\psi(-n) = 1 \quad (15)$$

Till now one of the unitarity conditions (1c) was not used and it diverges the following result

$$n(z) = 1/\pi \arcsin z + i\sqrt{z^2 - 1}\beta(z), \quad (16)$$
where $\beta(z) = -\beta(-z)$ is a meromorphic function. Equation (16) shows that the Riemann surface of the model has an algebraic branch points at $z = \pm 1$ and a logarithmic one at infinity. Now formulae (7,14,15,16) give the general solution to the problem (1) for matrix (6). The function $\psi$ can be determined from the requirement that eq.(14) turns to eq.(12) for integer $l$. This gives $\psi(n) = -\cot(n)$ for $l$ even and $\psi(n) = -\tan(n)$ for $l$ odd.

Let us remind that in eq.(14) $l \in R$, but it is clear that this relation can be continue to $l \in C$ and allows explicit determination of the Regge trajectories with definite signature $l_k^\pm(z)$. The common part of the set of Regge trajectories set for $J_\pm = l \pm 1/2$ is of the form $l_k^\pm(z) = \{2 - n(z) + 2k, n(z) + 2k \mid k = 0, 1, 2 \cdots\}$. The Regge trajectories for $J_- = l - 1/2$ contained one additional trajectory $l_{J_-}^\pm(z) = -n(z)$.

All the Regge trajectories of the model depends on one function $\beta(z)$.

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