ON THE CAUCHY PROBLEM FOR HIGHER DIMENSIONAL BENJAMIN-ONO AND ZAKHAROV-KUZNETSOV EQUATIONS

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Abstract. A family of dispersive equations is considered, which links a higher-dimensional Benjamin-Ono equation and the Zakharov-Kuznetsov equation. For these fractional Zakharov-Kuznetsov equations new well-posedness results are proved using transversality and time localization to small frequency dependent time intervals.

1. Introduction. In this note well-posedness of the higher-dimensional fractional Benjamin-Ono equations

\[
\begin{align*}
\partial_t u + \partial_{x_1} (-\Delta)^{a/2} u &= u \partial_{x_1} u, \quad (t, x) \in \mathbb{R} \times \mathbb{K}^n, \quad 1 \leq a \leq 2 \\
u(0) &= u_0 \in H^s(\mathbb{K}^n),
\end{align*}
\]

(1)

is discussed, where \(n \geq 2\) and \(\mathbb{K} \in \{\mathbb{R}, \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})\}\).

In the one-dimensional case, (1) becomes the Benjamin-Ono equation (cf. [1], see e.g. [28] for a recent survey) for \(a = 1\) and the Korteweg-De Vries equation (see [15] for the sharp global well-posedness result) for \(a = 2\). In the one-dimensional case the equations are best understood and extensively studied. In higher dimensions (1) yields a generalization of the Benjamin-Ono equation for \(a = 1\) (cf. [12, 22, 25]) and for \(a = 2\) the Zakharov-Kuznetsov equation (cf. [32]) is recovered.

By local well-posedness we mean that for any real-valued \(u_0 \in H^s\) there is \(T = T(||u_0||_{H^s})\) such that \(S_T^* : H^s \to C([0,T], H^s)\) extends uniquely to a continuous mapping \(S_T^* : H^s \to C([0,T], H^s)\).

The energy method [2] yields well-posedness for \(s > \frac{a+2}{2}\), but neglects the dispersive properties. These are clearly stronger in Euclidean space than in the fully periodic case. We discuss solutions in Euclidean space first, for which we can show stronger well-posedness results consequently.

Already in the one-dimensional case it is well-known that the data-to-solution mapping for dispersion coefficients \(1 \leq a < 2\) is not uniformly continuous (cf. [11, 20, 23]).

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Also, in two-dimensions it was proved for $a = 1$ in [12] that the data-to-solution mapping is not $C^2$. Local well-posedness was proved for $a = 1$ provided that $s > 5/3$ in [12] using short-time linear Strichartz estimates (see also [19]).

Here, we improve the local well-posedness for $n = 2$ and interpolate between $a = 1$ and $a = 2$ to recover in the limiting case the local well-posedness for the Zakharov-Kuznetsov equation $s > 1/2$ (cf. [5, 24]) in two dimensions and $s > 1$ in three dimensions [24, 26]. The results in higher dimensions seem to be new for $1 < a \leq 2$.

In the present work, we use transversality and localization of time to small frequency dependent time intervals (cf. [13, 29]) to prove the following theorem:

**Theorem 1.1.** Let $n \geq 2$, $K = \mathbb{R}$, $1 \leq a < 2$ and $s > \frac{n+3}{2} - a$. Then (1) is locally well-posed.

**Remark 1.** Following [29], the method of proof extends to equations with polynomial nonlinearities

$$\partial_t u + \partial_{x_1} D^a u = \partial_{x_1} (u^k), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n$$

for $k \geq 2$, which we do not cover explicitly.

In case $a = 2$, the below arguments point out that (1) is semilinear, and no frequency dependent time localization is required. Thus, in the Euclidean case $a = 2$ will not be considered. In two dimensions, the optimal $C^2$ local well-posedness $s > -1/4$ was proved by Kinoshita in [16] after the first submission of this article. This result was proved by refined transversality considerations making use of the nonlinear Loomis-Whitney inequality. This argument might very well lead to further improvements of the results from this article.

We sketch the argument. Let $N \in 2^{N_0}$ ($N_0 = \mathbb{N} \cup \{0\}$) denote a dyadic number and $P_N$ the inhomogeneous Littlewood-Paley projector, i.e.,

$$(P_N f)(\xi) = \begin{cases} \Phi(\xi) \hat{f}(\xi), & N = 1, \\ \chi_N(\xi) \hat{f}(\xi), & \text{else}, \end{cases}$$

where $\Phi, \chi_N \in C^\infty_c$, $\text{supp}(\Phi) \subseteq B(0, 2)$, $\text{supp}(\chi_N) \subseteq B(0, 2N) \backslash B(0, N/2)$ and $\Phi + \sum_N \chi_N \equiv 1$.

Further, let $S_a(t)$ denote the linear propagator of (1), that is

$$\widehat{S_a(t)u_0}(\xi) = e^{-it|\xi_1||\xi|^a} \hat{u}_0(\xi).$$

The most problematic interaction happens in case a low frequency interacts with a high frequency because the derivative nonlinearity

$$\partial_{x_1} (P_N u P_K u) \quad (K \ll N)$$

possibly requires one to recover a whole derivative. The derivative loss is partially ameliorated by the following bilinear Strichartz estimate:

**Proposition 1.** Let $n \geq 2$, $K, N \in 2^{N_0}$, $K \ll N$. Then, we find the following estimate to hold:

$$\|P_N S_a(t)u_0 P_K S_a(t)v_0\|_{L^2_t L^2_x(\mathbb{R} \times \mathbb{R}^n)} \lesssim \left(\frac{K^{n-1}}{N^n} \right)^{1/2} \|P_N u_0\|_{L^2} \|P_K v_0\|_{L^2}. \quad (2)$$

This proposition is a consequence of general transversality considerations (cf. [3]). Apparently, this is still insufficient to recover the derivative loss for $1 \leq a < 2$. 
To overcome the gap, we additionally localize time in a frequency dependent way (cf. [13]).

In the following we motivate for which frequency dependent time localization we can treat the most problematic $\text{High} \times \text{Low} \to \text{High}$-interaction utilizing (2). For $K \ll N$, one finds

$$
\| \partial_x (P_N S_a(t) u_0 P_K S_a(t) v_0) \|_{L^1([0,T];L^2(\mathbb{R}^n))} \\
\lesssim N \| T \|^{1/2} \| P_N S_a(t) u_0 P_K S_a(t) v_0 \|_{L^2([0,T];L^2_2(\mathbb{R}^n))} \\
\lesssim |T|^{1/2} N \left( \frac{K^{n-1}}{N^a} \right)^{1/2} \| P_N u_0 \|_{L^2(\mathbb{R}^n)} \| P_K v_0 \|_{L^2(\mathbb{R}^n)}. \tag{3}
$$

This suggests that for $T(N) = N^{a-2}$ this peculiar interaction can be estimated for $s > (n-1)/2$, which will be carried out in Section 4. In the one-dimensional case this had been done for dispersion generalized Benjamin-Ono equations (cf. [7]).

This argument will be sufficient to handle $\text{High} \times \text{Low} \to \text{High}$-interactions and $\text{High} \times \text{High} \to \text{High}$-interactions, $n \geq 3$, we prove a weaker transversality estimate in Proposition 5, but one can as well utilize linear Strichartz estimates (cf. [12]):

**Proposition 2.** Let $n \geq 3$, $1 \leq a \leq 2$ and $2 \leq p, q \leq \infty$, $p \neq \infty$. Then, we find the following estimate to hold

$$
\| S_a(t) f \|_{L_t^1(\mathbb{R};L_x^2(\mathbb{R}^n))} \lesssim \| f \|_{H^s(\mathbb{R}^n)} \\
\| S_a(t) f \|_{L_t^1([0,T];L_x^2(\mathbb{R}^n))} \lesssim_T \| f \|_{H^s(\mathbb{R}^n)}. \tag{4}
$$

provided that $\frac{2}{q} + \frac{2}{p} = 1$ and $s = n \left( \frac{1}{2} - \frac{1}{p} \right) - \frac{a+1}{q}$.

Since localization in time erases the dependence on the initial data, one still has to carry out energy estimates, which will give a worse regularity threshold to close the argument, namely $s > \frac{n+1}{2} - a$. This will be done in Section 5.

We will use a variant of the function spaces from [13, 4] to prove a priori estimates in the first step. Next, $L^2$-Lipschitz dependence for initial data of higher regularity is discussed. Finally, continuous dependence is proved by the Bona-Smith argument (cf. [2]).

The strategy of the proof closely follows the arguments from [29]. Therein the argument was applied to periodic solutions, where dispersive effects are weaker.

Another Benjamin-Ono-Zakharov-Kuznetsov equation was considered in [27]:

$$
\begin{align*}
\partial_t u - \partial_x D_x^a u + \partial_x \partial_{x2} u &= u \partial_x u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \\
&= u_0 \in H^s(\mathbb{R}^n).
\end{align*}
$$

Here, only dispersion in the $x_1$-component was decreased. Local and global well-posedness results for this model were also proved via frequency dependent time localization.

Lastly, we remark that the local well-posedness result from Theorem 1.1 gives global well-posedness in the energy space $H^{a/2}(\mathbb{R}^2)$ for sufficiently large $a$ in the two-dimensional case due to conservation of energy

$$
E(u) = \int_{\mathbb{R}^2} |D^{a/2} u|^2 - \frac{1}{3} u^3(t,x) dx.
$$

Another conserved quantity is the mass

$$
M(u) = \int_{\mathbb{R}^2} u^2(t,x) dx,
$$

where the second term is the mass of the solution.
but a well-posedness result in $L^2$ seems to be far beyond the methods of this paper.

Iteration of Theorem 1.1 for $s = a/2$ yields:

**Corollary 1.** Let $n = 2, K = \mathbb{R}$ and $a > 5/3$. Then, (1) is globally well-posed for $s = a/2$.

We turn to a discussion of the fully periodic case, in which case the following theorem is shown:

**Theorem 1.2.** Let $K = T, n \geq 2, 1 < a \leq 2$ and $s > (n + 1)/2$. Then, (1) is locally well-posed in $H^s(T^n)$. For $a = 1$, (1) is locally well-posed in $H^s(T^n)$ for small initial data.

In case $n = 2$ this improves the results from [21] where local well-posedness was proved in $H^s(T^2)$ for $a = 2$ provided that $s > 5/3$. In [21] short-time linear Strichartz estimates were used. In the present work this result is modestly improved by transversality considerations and corresponding results are proven in higher dimensions. The results in higher dimensions appear to be the first results below the energy threshold. However, the covered regularities are still far from the energy space. To make further progress, one presumably needs a better comprehension of the resonance set which appears to be more delicate than for the Kadomtsev--Petviashvili-equations (cf. [9]). This might be possible by the considerations from [16].

The strategy of proof is the same as for solutions on Euclidean space: In suitable function spaces we will prove a priori estimates for solutions, Lipschitz continuous dependence for differences of solutions in $L^2$ for higher regular initial data and finally continuous dependence in $H^s$ by the Bona-Smith approximation. The conclusion of the proof is similar to the Euclidean case.

The key ingredient will be bilinear convolution estimates for the space-time Fourier transform of functions, which are localized in frequency and modulation. These will be derived in Subsection 7.2. Here, the transversality considerations from Euclidean space will again come into play. However, we always have to localize time at least reciprocally to the highest involved frequency so that transversality becomes observable. Therefore, we can not lower the regularity, for which our method of proof yields local well-posedness as the dispersion coefficients increase compared to the Euclidean case.

After deriving these bilinear convolution estimates, the argument follows the real line case. The smallness assumption in case $a = 1$ is merely technical, and in Subsection 7.5 we outline how it can be removed.

**2. Bilinear Strichartz estimates.** The purpose of this section is to prove bilinear Strichartz estimates as stated in Proposition 1. The below estimates can hold globally in time only in Euclidean space, however, in Subsection 7.2 we shall see how bilinear estimates can partially be recovered in the fully periodic case after frequency dependent time localization. Linear Strichartz estimates in the Euclidean space are discussed in the appendix.

Whereas the proof is straight-forward in case of separated frequencies, it requires more care to treat the $High \times High \times High$-interaction

\[
\int_{\mathbb{R}^2 \times [0, T]} P_{N_1} S_a(t) u_0 P_{N_2} S_a(t) v_0 P_{N_3} S_a(t) w_0 \, dx \, dy \, dt, \quad N \sim N_1 \sim N_2 \sim N_3, \quad (5)
\]

where we shall see that it is still amenable to a bilinear Strichartz estimate.
Proposition 4. Let $U_i$ be open sets in $\mathbb{R}^n$, $\varphi_i \in C^1(U_i, \mathbb{R})$ and let $u_i$ have Fourier support in balls of radius $r$, which are contained in $U_i$ for $i = 1, 2$. Moreover, suppose that $|\nabla \varphi(\xi_1) - \nabla \varphi(\xi_2)| \geq M > 0$, whenever $\xi_1 \in U_1$, $\xi_2 \in U_2$.

Then, we find the following estimate to hold:

$$\|e^{it\varphi(\nabla)}u_1 e^{it\varphi(\nabla)}u_2\|_{L^2_x(\mathbb{R} \times \mathbb{R}^n)} \lesssim_{n} \frac{r^{n-1}}{M^{1/2}} \|u_1\|_{L^2(\mathbb{R}^n)} \|u_2\|_{L^2(\mathbb{R}^n)}. \quad (6)$$

The proof is omitted. In the context of $X^{s,b}$-spaces, we revisit the argument in Lemma 7.4.

In order to apply Proposition 3, we have to analyze the group velocity $v_a(\xi) = -\nabla \varphi_a(\xi)$, where $\varphi_a(\xi) = \xi_1 |\xi|^a$.

We have

$$\partial_1 \varphi_a(\xi) = |\xi|^a + a \xi_1^2 |\xi|^{a-2}, \quad \partial_2 \varphi_a(\xi) = a \xi_1 \xi_2 |\xi|^{a-2}. \quad (7)$$

Proof of Proposition 1. First, divide $B_{2N} \setminus B_{N/2}$ into finitely overlapping balls of radius $K$, which we denote by the family $(R_L)$. Then, from almost orthogonality

$$\|P_N S(t)u_0 P_K S(t)v_0\|_{L^2}^2 \lesssim \sum_L \|R_L S(t)u_0 P_K S(t)v_0\|_{L^2}^2. \quad (8)$$

To estimate the terms from the sum, we use Proposition 3. From (7) we find $|\partial_1 \varphi(\xi)| \geq (N/2)^a$ for $|\xi| \geq N/2$ and $|\partial_2 \varphi_a(\xi)| \leq (1 + a)(2K)^a$ and (6) implies

$$(8) \lesssim \sum_L \left( \frac{K^{n-1}}{N^a} \right) \|R_L u_0\|_{L^2}^2 \|P_K v_0\|_{L^2}^2 = \left( \frac{K^{n-1}}{N^a} \right)^{1/2} \|P_N u_0\|_{L^2}^2 \|P_K v_0\|_{L^2}^2,$$

which completes the proof. \qed

Next, we turn to the case of three comparable frequencies in the plane as depicted in (5). We prove the following proposition:

Proposition 4. Let $N \gg 1$ and suppose that $\xi_1 \in \mathbb{R}^2$, $N/8 \leq |\xi_i| \leq 8N$ for $i = 1, 2, 3$ and $\xi_1 + \xi_2 + \xi_3 = 0$. Then, there are $i, j \in \{1, 2, 3\}$ with $|v_a(\xi_i) - v_a(\xi_j)| \gtrsim N^a$.

Proof. A key observation is that for $|\lambda_2| \leq c|\lambda|$ or $|\lambda_1| \leq c|\lambda|$, where $c$ is a small constant, a Taylor expansion of $|\lambda|$ around the large component reveals

$$\partial_1 \varphi_a(\lambda) = (1 + a)|\lambda_1|^a + O(\lambda_2^2 |\lambda_1|^{a-2})$$

$$= (1 + a)|\lambda_1|^a + O(c^2 |\lambda_1|^{a}) \quad (|\lambda_2| \leq c|\lambda|),$$

$$\partial_1 \varphi_a(\lambda) = |\lambda_2|^a + O(c^2 |\lambda_2|^{a}) \quad (|\lambda_1| \leq c|\lambda|).$$

This means that as soon as one component dominates the other one, the propagation into $x_1$-direction is essentially governed by the group velocity associated to a (fractional) one-dimensional Benjamin-Ono equation. In the one-dimensional case the claim is immediate from the mean-value theorem. Momentarily, we shall be more precise about the choice of $i$ and $j$.

Let $\xi_1 + \xi_2 + \xi_3 = 0$, $\xi_i \in \mathbb{R}$, $|\xi_i| \in [N/16, 16N]$. We partition $[N/16, 16N]$ and $[-16N, -N/16]$ into smaller intervals $(I_i)$ of size $cN$, $c \ll 1$. With the intervals having length $cN$, there is no loss summing up the intervals at last. Let $\xi_i \in I_i$ for $i = 1, 2, 3$. Note that $I_1$ and $-I_2$ must be separated as this would otherwise
violate the convolution constraint. In case $I_1$ and $I_2$ are separated, the estimate $|\xi_1^\alpha - |\xi_2^\alpha| \gtrsim N^\alpha$ is immediate. If there is no separation between $I_1$ and $I_2$, the intervals are neighbouring and $\pm I_2$ will be separated from $I_1$ and $I_2$ due to otherwise impossible frequency interaction.

We come back to the higher-dimensional problem.

To deal with different sizes of the components for $\lambda \in \mathbb{R}^2$, we introduce the notations $\lambda \in (A, B)$, where $A, B \in \{\text{Low, Medium, High}\}$ and $\lambda \in (X, Y)$, where $X, Y \in \{+,-\}$ to indicate $\lambda_1 \geq 0, \lambda_2 \leq 0$. E.g. $\lambda \in (\text{High}(+), \text{Medium}(-))$ means $|\lambda_1| \geq c|\lambda|$, $|\lambda_2| \in [c^3|\lambda|, \frac{c|\lambda|}{2}]$, $\lambda_1 \geq 0, \lambda_2 \leq 0$ or $\lambda \in (\text{Low}, \text{High}(-))$ means $|\lambda_1| \leq c^3|\lambda|$, $|\lambda_2| \geq \frac{c|\lambda|}{2}$, $\lambda_2 \leq 0$.

Here, $c$ is a small dimensional constant chosen so that the error terms in the above Taylor expansion can be neglected in the following considerations.

We sort the frequencies according to the above system.

Suppose that the components of any frequency are all at least of medium size, so that no component of the three frequencies is low.

Then, by (7), $|\partial_2 \varphi_{\alpha}(\lambda)| \geq c^3|\lambda|^\alpha$ for $i = 1, 2, 3$. Next, observe that for $\xi_i \in (+,+)$ or $\xi_i \in (-,-)$, we have $\partial_2 \varphi(\xi_i) \geq c^3|\xi|^\alpha$, and in case of mixed signs $\xi_i \in (+,-)$ or $\xi_i \in (-,+)$, we have $\partial_2 \varphi_{\alpha}(\xi_i) \leq -c^3|\xi|^\alpha$, and the estimate $|\partial_2 \varphi_{\alpha}(\xi_i) - \partial_2 \varphi_{\alpha}(\xi_j)| \geq c^3|\xi|^\alpha$ is immediate.

Next, we turn to the case where all components have size greater than $c^3|\xi|$ and all frequencies are of equal signs (the case of mixed signs will be analogous).

Say

$$
\xi_1 \in (\text{High}(+), \text{Medium}(+)), \quad \xi_2 \in (\text{High}(+), \text{High}(+)),
$$

$$
\xi_3 \in (\text{High}(-), \text{High}(-)).
$$

Write $\xi_{21} = \alpha \xi_{11}$, $\xi_{22} = \beta \xi_{12}$, where $\alpha, \beta \in [c^5, c^{-5}]$, and it follows

$$
|\partial_2 \varphi_{\alpha}(\xi_1) - \partial_2 \varphi_{\alpha}(\xi_3)|
= \left|\frac{a \xi_{11} \xi_{12}}{\xi_{11}^2 + \xi_{12}^2} - \frac{a(1 + \alpha) \xi_{11}(1 + \beta) \xi_{12}}{((1 + \alpha)^2 \xi_{11}^2 + (1 + \beta)^2 \xi_{12}^2)^{\frac{3}{2}}}\right|
$$

$$
\geq c^5 a \frac{|\xi_{11} \xi_{12}|}{\xi_{11}^2 + \xi_{12}^2} \gtrsim |\xi|^\alpha.
$$

Next, we suppose that there is one low component involved, say $\xi_1 \in (\text{Low}, \text{High})$. Suppose that there is a frequency $\xi_j \in (\text{High}, \text{High})$. Then, we find $|\partial_2 \varphi_{\alpha}(\xi_1)| = O(c^3|\xi|^\alpha)$ and $|\partial_2 \varphi_{\alpha}(\xi_j)| \gtrsim c^2|\xi|^\alpha$, hence $|\partial_2 \varphi_{\alpha}(\xi_1) - \partial_2 \varphi_{\alpha}(\xi_j)| \gtrsim c^3|\xi|^\alpha$, which yields the desired transversality.

With $|\xi_{12}| \sim |\xi|$ there is another frequency, say $\xi_2$ with $|\xi_{22}| \sim |\xi|$ and by the above consideration suppose next that $\xi_2 \in (\text{Low}, \text{High})$ or $\xi_2 \in (\text{Medium, High})$.

Either way, $|\xi_{11}| \leq |\xi_{11}| + |\xi_{12}| \leq c|\xi_{11}|$, and we can expand $\partial_1 \varphi(\xi_1)$ in the second component of the frequencies to find that the analysis reduces to the one-dimensional fractional Benjamin-Ono equation, and hence, there are $\xi_i$ and $\xi_j$ with

$$
|\partial_1 \varphi_{\alpha}(\xi_i) - \partial_1 \varphi_{\alpha}(\xi_j)| \gtrsim |\xi|^\alpha.
$$

The same argument applies in case $\xi_1 \in (\text{High}, \text{Low})$. In case there is $\xi_j \in (\text{High}, \text{High})$ the difference satisfies $|\partial_2 \varphi_{\alpha}(\xi_1) - \partial_2 \varphi_{\alpha}(\xi_j)| \gtrsim c^2|\xi|^\alpha$ and in case there is no $\xi_j \in (\text{High}, \text{High})$, we can expand in the first frequency component to reduce the analysis to the one-dimensional fractional Benjamin-Ono equation according to which there are $\xi_i, \xi_j$ such that $|\partial_1 \varphi_{\alpha}(\xi_i) - \partial_1 \varphi_{\alpha}(\xi_j)| \gtrsim |\xi|^\alpha$. 


Remark 2. Proposition 4 proves the existence of essentially disjoint regions $(U^j_i)_{i=1,2,3}$, in Fourier space such that for the decomposition

$$\int_{[0,T] \times \mathbb{R}^2} P_{N_1} S_a(t) u_1 P_{N_2} S_a(t) u_2 P_{N_3} S_a(t) u_3 dt dx$$

$$= \sum_{i_m,j_n} \int_{[0,T] \times \mathbb{R}^2} \prod_{k=1}^{3} P_{N_k} S_a(t) u_k^{i_k,j_k}$$

we have for any non-trivial product, i.e.,

$$\int_{[0,T] \times \mathbb{R}^2} \prod_{k=1}^{3} P_{N_k} S_a(t) u_k^{i_k,j_k} \neq 0$$

the existence of $m, n \in \{1, 2, 3\}$ with

$$|v_a(\xi_m) - v_a(\xi_n)| \gtrsim N^a$$

provided that $\xi_m \in \text{supp}(u_m^{i_m,j_m})$ and $\xi_n \in \text{supp}(u_n^{i_n,j_n})$.

This means that the product

$$P_{N_m} S_a(t) u_m^{i_m,j_m} P_{N_n} S_a(t) u_n^{i_n,j_n}$$

is amenable to a bilinear Strichartz estimate from Proposition 3.

These transversality considerations for comparable frequencies do not appear to remain true in higher dimensions. Instead, we revisit the proof of Proposition 4 to prove the following weaker result in higher dimensions, which is still sufficient for our purposes:

**Proposition 5.** Let $1 \leq a \leq 2$, $n \geq 3$ and $\xi_i \in \mathbb{R}^n$, $i = 1, 2, 3$ with $\xi_1 + \xi_2 + \xi_3 = 0$ and $|\xi_i| \sim 2^k$. Further, suppose that $\max_{i=1,2,3} |\xi_i| \sim 2^l$.

Then, there are $\xi_i$ and $\xi_j$ such that

$$|v_a(\xi_i) - v_a(\xi_j)| \gtrsim 2^l 2^{k(a-1)}.$$

**Proof.** First, we deal with the case $n = 3$. In order to lighten the notation further, we use the less precise notation $\lesssim$, compared to the more carefully defined regions above. Again, the below argument can be made more precise borrowing notations from the proof of Proposition 4.

By symmetry and convolution constraint, we can suppose that $|\xi_{11}| \sim 2^l$, $|\xi_{21}| \sim 2^l$. If $|\xi_{31}| \ll 2^l$, then there will be another component of $\xi_3$ having size $2^k$, by symmetry say $|\xi_{32}| \sim 2^k$.

By convolution constraint, there is $i \in \{1; 2\}$ such that $|\xi_{i2}| \sim 2^k$. Then we find

$$|\partial_2\varphi_a(\xi_i) - \partial_2\varphi_a(\xi_3)| \gtrsim 2^l 2^{k(a-1)}$$

Thus, we suppose in the following that $|\xi_{11}| \sim |\xi_{21}| \sim |\xi_{31}| \sim 2^l$.

If there is no other component among $\xi_{ji}$, $j = 1, 2, 3$, $i = 2, \ldots, n$ that is comparable to $2^k$, then the analysis reduces to the one-dimensional fractional Benjamin-Ono equation after expansion of $\partial_1\varphi_a$.

Thus, we suppose in the following that there is a component say $|\xi_{12}| \sim 2^k$. By convolution constraint we can suppose further that $|\xi_{22}| \sim 2^k$.

If $|\xi_{32}| \ll 2^k$, then it follows $|\partial_2\varphi_a(\xi_1) - \partial_2\varphi_a(\xi_3)| \gtrsim 2^l 2^{k(a-1)}$.

Thus, we suppose in the following that $|\xi_{12}| \sim |\xi_{22}| \sim |\xi_{32}| \sim 2^k$. The proof is complete. □
Next, we take the third component into account: If $|\xi_{13}| \ll 2^k$ for $i = 1, 2, 3$, then the third component can be neglected, and the claim follows from the two-dimensional argument.

If $|\xi_{13}| \sim 2^k$, there exists a third component of another frequency of comparable size by convolution constraint, say $|\xi_{23}| \sim 2^k$. If $|\xi_{33}| \ll 2^k$, then we find

$$|\partial_3 \varphi_a(\xi_1) - \partial_3 \varphi_a(\xi_3)| \gtrsim 212^{(a-1)k}.$$  

Thus, we can suppose that $|\xi_{33}| \sim 2^k$ for $i = 1, 2, 3$.

In the next step we take the signs into account. If the product of the signs of the first and second or first and third component differs, then the claim follows from the observation that the respective partial derivatives are of opposite signs.

Thus, we suppose that

$$\text{sgn}(\xi_{11}\xi_{12}) = \text{sgn}(\xi_{11}\xi_{12}), \quad i, j \in \{1, 2, 3\},$$

$$\text{sgn}(\xi_{11}\xi_{13}) = \text{sgn}(\xi_{j1}\xi_{j3}). \quad (10)$$

There will be two frequencies, where the first component has the same sign, say $\xi_1$ and $\xi_2$, and one frequency where the first component has a different sign, that is $\xi_3$ in the current setting.

By (10), the signs of the other components must also be equal at $\xi_1$ and $\xi_2$ and different for $\xi_3$. Write $\xi_{21} = \alpha \xi_{11}$, $\xi_{22} = \beta \xi_{12}$, $\xi_{23} = \gamma \xi_{13}$ and suppose that $\beta \geq \gamma$, where $\alpha \sim \beta \sim \gamma \sim 1$. Then we compute along the lines of (9)

$$\frac{(1 + \alpha)(1 + \beta)}{(1 + \alpha)^2 \xi_{11}^2 + (1 + \beta)^2 \xi_{12}^2 + (1 + \gamma)^2 \xi_{13}^2 \frac{2}{2+\gamma}} \geq \frac{1}{1 + (\alpha \wedge \beta)} \frac{1}{(\xi_{11}^2 + \xi_{12}^2 + \xi_{13}^2 \frac{2}{2+\gamma}).$$

Consequently,

$$|\partial_2 \varphi_a(\xi_1) - \partial_2 \varphi_a(\xi_3)|$$

$$= |a \xi_{11} \xi_{12}| \frac{(1 + \alpha)(1 + \beta)}{(1 + \alpha)^2 \xi_{11}^2 + (1 + \beta)^2 \xi_{12}^2 + (1 + \gamma^2) \xi_{13}^2 \frac{2}{2+\gamma}} - \frac{1}{(\xi_{11}^2 + \xi_{12}^2 + \xi_{13}^2 \frac{2}{2+\gamma})} \geq \frac{1}{(\xi_{11}^2 + \xi_{12}^2 + \xi_{13}^2 \frac{2}{2+\gamma})} \geq 212^{(a-1)n}.$$  

This proves the claim for $n = 3$. The above arguments allow us to deal with higher dimensions inductively.

3. Function spaces. In this section we discuss the short-time function spaces, which are used to prove the local well-posedness results in Euclidean space. The iteration scheme is the same for solutions in Euclidean space and for fully periodic solutions. However, in Euclidean space we do not have to use the Fourier transform in time, which allows for a simplification of the construction compared to the periodic case.

Short-time $L^2$-valued $U^{p_r}/V^{p_r}$-spaces will be utilized like in [4, 29]. Here, we will be very brief and instead refer to these works for a presentation of the basic function space properties. The notation will be the same as in the aforementioned works. For a careful exposition, see [9, 10]. The $V^{p_r}$-spaces are the usual function spaces
containing functions of bounded $p$-variation and the $U^p$-spaces are atomic spaces which are the respective predual spaces. Roughly speaking, $U^2$ serves as a substitute for $H^{1/2}$, which does not embed into $L^\infty$, but any $U^p$-function is bounded.

The $U^p/V^p$-spaces are adapted to free solutions in the usual way:

$$
\|u\|_{U^p(T;L^2)} = \|S_a(-t)u(t)\|_{U^p(T;L^2)},
$$
$$
\|v\|_{V^p(T;L^2)} = \|S_a(-t)v(t)\|_{V^p(T;L^2)},
$$
$$
\|w\|_{DU^p(T;L^2)} = \|S_a(-t)w(t)\|_{DU^p(T;L^2)}.
$$

Motivated by (3), we choose $T(N) = N^{a-2}$ as frequency dependent time localization.

Below we shall only deal with the case $1 \leq a < 2$, since for $a = 2$ the localization to small frequency dependent time intervals is no longer necessary in Euclidean space, and the analysis comes down to the Fourier restriction analysis without localization in time from [5]. This is divergent from the fully periodic case, where it is known that even for $a = 2$ (1) cannot be solved via Picard iteration (cf. [21]).

Letting $\chi_I$ denote a sharp cut-off to a time interval $I$, the short-time $U^2$-space, into which the solution to (1) will be placed, is given by

$$
\|u\|^2_{F^2_a(T)} = \sum_{N \geq 1} N^{2s} \sup_{|I| = N^{a-2}, \, t \in [0,T]} \|P_N \chi_I u\|^2_{U^2_a(I;L^2)}.
$$

The corresponding space for the nonlinearity is defined by

$$
\|f\|^2_{F^2_a(T)} = \sum_{N \geq 1} N^{2s} \sup_{|I| = N^{a-2}, \, t \in [0,T]} \|P_N \chi_I f\|^2_{U^2_a(I;L^2)},
$$

and the energy space is

$$
\|u\|^2_{E^2(T)} = \sum_{N \geq 1} N^{2s} \sup_{t \in [0,T]} \|P_N u(t)\|^2_{L^2}.
$$

The short-time norm of a smooth solution to (2) is propagated as follows:

$$
\|u\|_{F^2_a(T)} \lesssim \|u\|_{E^2(T)} + \|\partial_{x_1}(u^2)\|_{N^2_a(T)}
$$

(cf. [4]).

Moreover, since $U^p_a$-atoms are piecewise free solutions estimates for free solutions extend to $U^p_a$-functions.

**Proposition 6.** Let $n \geq 3$, $1 \leq a \leq 2$, $N \in 2^{\mathbb{N}_0}$ and $I$ be an interval. Suppose that

$$
2/q + 2/p = 1, \, 2 \leq q, p < \infty.
$$

Then, we find the following estimate to hold:

$$
\|P_N u(t)\|_{L^q(I;L^p(\mathbb{R}^n))} \lesssim N^s \|P_N u_0\|_{U^2_a(I;L^2)},
$$

where $s = n \left( \frac{1}{2} - \frac{1}{p} \right) - \frac{a+1}{q}$.

This also remains valid for bilinear estimates.

**Proposition 7.** Let $1 \leq a < 2$, $N_1 \gg N_2$ and $I$ be an interval with $|I| = N_1^{a-2}$. Then, we find the following estimates to hold:

$$
\|P_{N_1} u_1 P_{N_2} u_2\|_{L^2_a(I \times \mathbb{R}^n)} \lesssim \left( \frac{N_2^{a-1}}{N_1^{a-2}} \right)^{1/2} \|P_{N_1} u_1\|_{U^2_a(I)} \|P_{N_2} u_2\|_{U^2_a(I)},
$$

(11)
\[ \|P_{N_1}u_1 P_{N_2}u_2\|_{L^2_t L^2_x(I \times \mathbb{R}^n)} \lesssim \left( \frac{N_2^{n-1}}{N_1^{a}} \right)^{1/2} \log(N_1)^2 \|P_{N_1}u_1\|_{V^2_2(I)} \|P_{N_2}u_2\|_{V^2_2(I)}. \]

(12)

**Proof.** (11) is immediate from atomic decompositions (cf. [9, Proposition 2.19]), and (12) follows from an interpolation argument (cf. [9, Proposition 2.20]). \(\square\)

4. **Nonlinear estimates.** This section is devoted to the propagation of the nonlinearity in the short-time function spaces.

**Proposition 8.** Let \( \mathbb{K} = \mathbb{R} \), \( 1 \leq a < 2 \), \( n \geq 2 \), \( s > (n - 1)/2 \). Then, we find the following estimates to hold:

\[ \|\partial_x(uv)\|_{N^a(T)} \lesssim \|u\|_{F^a_2(T)} \|v\|_{F^a_2(T)}, \]

(13)

\[ \|\partial_x(uv)\|_{N^a_2(T)} \lesssim \|u\|_{F^a_2(T)} \|v\|_{F^a_2(T)}, \]

(14)

**Proof.** After using Littlewood-Paley theory, we are reduced to the analysis of \( \text{High} \times \text{Low} \rightarrow \text{High} \), \( \text{High} \times \text{High} \rightarrow \text{High} \) - and \( \text{High} \times \text{High} \rightarrow \text{Low} \)-interaction. Carrying out the summation in the short-time function spaces gives (13) and (14).

Suppose that \( N_3 \sim N_1 \gg N_2 \) (\( \text{High} \times \text{Low} \rightarrow \text{High} \)-interaction), and let \( I \) be an interval with \( |I| \lesssim N_3^{-a/2} \). Then, we compute

\[ \|P_{N_3} \partial_x(P_{N_1} u P_{N_2} v)\|_{DU^2_2(I)} \lesssim N_1 \|P_{N_1} u P_{N_2} v\|_{L^2_{t,x}} \]

\[ \lesssim N_1 N_3^{-a/2} \|P_{N_1} u P_{N_2} v\|_{L^2_{t,x}} \]

\[ \lesssim N_2^{n-1/2} \|P_{N_1} u\|_{U^2_2(I)} \|P_{N_2} v\|_{U^2_2(I)}. \]

Suppose that \( N_1 \sim N_2 \sim N_3 \) (\( \text{High} \times \text{High} \rightarrow \text{High} \)-interaction) and \( n = 2 \). Using duality, we have

\[ \|P_{N_1} \partial_x(P_{N_2} u P_{N_3} v)\|_{DU^2_2(I)} = \sup_{\|w\|_{V^2_2} = 1} \int \int P_{N_1} w \partial_x(P_{N_2} u P_{N_3} v) dx dt. \]

(15)

Now we use Proposition 4 to apply a bilinear Strichartz estimate on two factors, say \( w \) and \( u \), to find

\[ (15) \lesssim N_1 \sup_w \|P_{N_1} w P_{N_2} u\|_{L^2_{t,x}} \|P_{N_3} v\|_{L^2_{t,x}} \]

\[ \lesssim N_1 N_2^{-a/4} \log(N_2) \sup_w \|P_{N_1} w\|_{V^2_2(I)} \|P_{N_2} u\|_{V^2_2(I)} N_3^{a/2-1} \|P_{N_3} v\|_{U^2_2(I)}, \]

which is sufficient.

For \( n \geq 3 \) we use two \( L^4_{t,x} \)-Strichartz estimates instead:

\[ \|P_{N_3} \partial_x(P_{N_1} u P_{N_2} v)\|_{DU^2_2(I)} \lesssim N_3 \|P_{N_1} u P_{N_2} v\|_{L^4_{t,x}} \]

\[ \lesssim N_3 N_3^{a/2} N_3^{n-4} \|P_{N_1} u\|_{U^2_2(I)} \|P_{N_2} v\|_{U^2_2(I)} \]

\[ \lesssim N_3^{a/2} \|P_{N_1} u\|_{U^2_2(I)} \|P_{N_2} v\|_{U^2_2(I)}, \]

which is again sufficient.

At last, suppose that \( N_3 \ll N_1 \sim N_2 \) (\( \text{High} \times \text{High} \rightarrow \text{Low} \)-interaction). Here, we have to add localization in time which amounts to a factor \( (N_1/N_3)^{2-a} \). More
concretely, we have to decompose $I = \bigcup_i J_i$ with $|J_i| \lesssim N_1^{2-a}$. We use duality to write
\[
\|P_{N_3} \partial_x (P_{N_1} u P_{N_2} v)\|_{L^2_{L^2}(t)} \\
\lesssim N_3 \sum_i \sup_w \int_{J_i} \int P_{N_3} w P_{N_1} u P_{N_2} v dx dt \\
\lesssim N_3 \sum_i \sup_w \|P_{N_3} w P_{N_1} u\|_{L^2_{L^2}(J_i \times \mathbb{R}^n)} \|P_{N_2} v\|_{L^2_{L^2}(J_i \times \mathbb{R}^n)} \\
\lesssim N_3 \left( \frac{N_1}{N_3} \right)^{2-a} N_1^{\frac{2-a}{2}} \left( \frac{N_2}{N_1} \right)^{1/2} \\
\times \log^2 (N_1) \|P_{N_1} u\|_{F^a_2} \|P_{N_2} v\|_{F^a_2} \\
\lesssim (N_1/N_3)^{1-a} N_3^{2-a} \log^2 (N_1) \|P_{N_1} u\|_{F^a_2} \|P_{N_2} v\|_{F^a_2}.
\]

Carrying out the summation is straightforward for $s > (n-1)/2$. 

5. Energy estimates. In this section, energy estimates are carried out in Euclidean space. First, we turn to the energy estimate, which will yield a priori estimates provided that $s > s_a := \frac{n+3}{2} - a$.

**Proposition 9.** Let $\mathcal{K} = \mathbb{R}$, $n \geq 2$, $1 \leq a < 2$ and let $u$ be a smooth solution to (1). Then, we find the following estimate to hold
\[
\|u\|^2_{E^s(T)} \lesssim \|u_0\|^2_{H^s} + T \|u\|^3_{E^s(T)}
\]
provided that $s > s_a$.

**Proof.** The fundamental theorem of calculus yields
\[
\|P_N u(t)\|_{L^2}^2 = \|P_N u_0\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^n} P_N u P_N \partial_x (u^2) dx dt.
\]
The time integral we treat with Littlewood-Paley decompositions and analyze the possible interactions separately.

Suppose that $N_1 \sim N_3 \gg N_2$. Then integration by parts and a commutator estimate yields after localization in time to intervals of size $N_1^{2-a}$
\[
\left| \int \int_{\mathbb{R}^n} P_{N_3} u \partial_x (P_{N_2} u P_{N_3} u) dx dt \right| \lesssim N_2 T N_1^{2-a} \|P_{N_1} u P_{N_2} u\|_{L^2_{L^2}} \|P_{N_3} u\|_{L^2_{L^2}} \\
\lesssim T N_2 N_1^{2-a} \left( \frac{N_2}{N_1} \right)^{1/2} N_1^{2-a} \prod_i \|P_{N_i} u\|_{F^a_{n_i}} \\
\lesssim T N_2^{s_a} \left( \frac{N_2}{N_1} \right)^{a-1} \prod_i \|P_{N_i} u\|_{F^a_{n_i}}
\]
In case $N_1 \lesssim N_2 \sim N_3$ there is no point to integrate by parts, but apart from that the estimate is concluded along the lines of the above argument. 

Next, we prove the energy estimates which will yield Lipschitz continuity in $L^2$ for initial data in $H^s$, $s > s_a$ and continuity of the data-to-solution mapping after invoking the Bona-Smith approximation.
Proposition 10. Let $\mathcal{K} = \mathbb{R}$, $n \geq 2$, $1 \leq a < 2$ and $u_1, u_2$ be two smooth solutions to (1). Set $v = u_1 - u_2$. Then, we find the following estimates to hold:

\[
\begin{align*}
\|v\|^2_{E^0(T)} &\lesssim \|v(0)\|^2_{E^0} + T \|v\|^2_{E^0(T)} (\|u_1\|_{F^s(T)} + \|u_2\|_{F^s(T)}), \\
\|v\|^2_{E^s(T)} &\lesssim \|v(0)\|^2_{H^s} + T \|v\|^2_{E^s(T)} \\
&+ T (\|v\|^2_{F^s(T)} \|u_2\|_{F^s(T)} + \|v\|_{F^s(T)} \|v\|_{F^s(T)} \|u_2\|_{F^s(T)})
\end{align*}
\]

provided that $s > s_a$.

Proof. After the same reductions as above, we have to estimate

\[
\left| \int \int P_{N_i} v \partial x_1 (P_{N_i} u P_{N_i} v) dx dt \right|
\]

for $N_1 \sim N_3 \gg N_2$, $N_1 \lesssim N_2 \sim N_3$ and $N_3 \lesssim N_1 \sim N_2$.

The first case can be dealt with like in the corresponding estimate for solutions because we can still integrate by parts.

The second case does not require integration by parts and thus can be estimated like above.

In the case $N_3 \ll N_1 \sim N_2$ we estimate

\[
(18) \lesssim N_1 T N_1^{2-a} \|P_{N_1} v P_{N_3} v\|_{L^2_{x,t}} \|P_{N_2} u\|_{L^2_{x,t}} \lesssim T N_1^{2-a} N_3^{\frac{a-1}{2}} \|P_{N_1} v\|_{F^0} \|P_{N_2} u\|_{F^0} \|P_{N_3} v\|_{F^0}.
\]

This yields (16) after summation. To prove (17), one writes

\[
\partial_t v + \partial x_1 D^s v = v \partial x_1 v + \partial x_1 (u_2 v).
\]

The first term has the same symmetries like the term we encountered when proving a priori estimates for solutions. For the second term, the only new estimate one has to carry out (due to impossibility to integrate by parts) is

\[
\sum_{1 \leq K \leq N} N^{2s} \int \int P_{N} v \partial x_1 (P_{N} u_2 P_{K} v) dx dt \lesssim T \|v\|_{F^0(T)} \|v\|_{F^s(T)} \|u_2\|_{F^s(T)},
\]

which follows by the above means. 

6. Proof of Theorem 1.1.

Proof of Theorem 1.1. We shall be brief because the concluding arguments are already standard (cf. [13]). Below we fix $s > s_a$.

By rescaling we are reduced to consider sufficiently small initial data. Firstly, we only consider initial data $u_0 \in H^\infty(\mathbb{R}^n) = \cap_{k \geq 0} H^k(\mathbb{R}^n)$. The energy method yields existence of solutions in $C([0, T^*], H^s(\mathbb{R}^n))$ for $s > n/2+1$, where $\lim_{t \to T^*} \|u(t)\|_{H^s} = \infty$.

In a first step, we prove a priori estimates from

\[
\begin{align*}
\|u\|_{F^s(T)} &\lesssim \|u\|_{F^s(T)} + \|\partial x_1 (u^2)\|_{N^s(T)} \\
\|u\partial x_1 u\|_{N^s(T)} &\lesssim \|u\|^2_{F^s(T)} \\
\|u\|_{E^s(T)}^2 &\lesssim \|u_0\|^2_{H^s} + T \|u\|^3_{F^s(T)}
\end{align*}
\]

for solutions to (1) by a bootstrap argument.

The above set of estimates yields

\[
\|u\|^2_{F^s(T)} \lesssim \|u_0\|^2_{H^s} + \|u\|^4_{F^s(T)} + T \|u\|^3_{F^s(T)}.
\]
Next, we invoke continuity of $E^s(T)$ and
\[
\lim_{T \to 0} \|u\|_{E^s(T)} \lesssim \|u_0\|_{H^s}, \quad \lim_{T \to 0} \|\partial_{x_1}(u^2)\|_{N^2_0(T)} = 0.
\]
For details, see e.g. [18].

Consequently, the above set of estimates yields
\[
\|u\|_{F^2_s(T)} \lesssim \|u_0\|_{H^s}(19)
\]
provided that $\|u_0\|_{H^s}$ is chosen sufficiently small.

For $s' > s$ we have
\[
\left\{ \begin{aligned}
\|u\|_{F^{s'}_s(T)} &\lesssim \|u\|_{E^{s'}(T)} + \|\partial_{x_1}(u^2)\|_{N^{s'}_s(T)} \\
\|\partial_{x_1}(vu_1)\|_{N^0_0(T)} &\lesssim \|v\|_{F^0_s(T)} + \|u\|_{F^2_s(T)} \\
\|v\|_{E^0_s(T)} &\lesssim \|v(0)\|_{L^2} + T\|v\|_{F^0_s(T)}(\|u_1\|_{F^2_s(T)} + \|u_2\|_{F^2_s(T)})
\end{aligned} \right.
\]
yield an a priori estimate for $v$ in $L^2$ in dependence of $\|u_1\|_{H^s}$ for $s > s_0$.

Lastly, the set of estimates
\[
\left\{ \begin{aligned}
\|v\|_{F^2_s(T)} &\lesssim \|v\|_{E^s(T)} + \|\partial_{x_1}(v(u_1 + u_2))\|_{N^0_0(T)} \\
\|\partial_{x_1}(vu_1)\|_{N^2_0(T)} &\lesssim \|v\|_{F^2_s(T)} + \|u\|_{F^2_s(T)} \\
\|v\|_{E^0_s(T)} &\lesssim \|v(0)\|_{L^2} + T\|v\|_{F^0_s(T)}(\|u_1\|_{F^2_s(T)} + \|u_2\|_{F^2_s(T)})
\end{aligned} \right.
\]
allows us to conclude continuous dependence on the initial data by the classical Bona-Smith approximation (cf. [2, 13]).

For this purpose, let $u_2$ be the solution associated to $P_{\leq N} u_0$ and $u_1$ be the solution associated to $u_0$.

Due to the difference of initial data consisting only of high frequencies, the gain from estimating $\|v\|_{E^0}$ compensates the loss from
\[
\|u_2\|_{F^2_s(T)} \lesssim \|P_{\leq N} u_0\|_{H^2} \lesssim N^s \|P_{\leq N} u_0\|_{H^s}.
\]
The data-to-solution mapping $H^s \to C([0, T], H^s) \cap F^s(T)$, which can also be constructed by the above means, is continuous, but not uniformly continuous because the approximation depends on the distribution of the Sobolev energy along the high frequencies, i.e., $\|P_{\leq N} u_0\|_{H^s}$. The proof is complete.

7. Periodic solutions to fractional Zakharov-Kuznetsov equations. Below, the above considerations regarding short-time nonlinear and energy estimates are extended to the fully periodic case. Firstly, the function spaces are introduced.
7.1. Function spaces in the periodic case. We will be brief because the function spaces are defined completely analogous to [8] with the basic function space properties remaining valid.

For $k \in \mathbb{N}_0$ let

$$\mathbb{R}^n \supseteq A_k = \begin{cases} 2^{k-1} < |\xi| \leq 2^k, & k \geq 1, \\ |\xi| \leq 1, & k = 0 \end{cases}$$

denote dyadic ranges and by $P_k$ the corresponding frequency projectors, i.e.,

$$\hat{P}_k u(\xi) = 1_{A_k}(\xi) \hat{u}(\xi).$$

Most of the time it will be fine to work with sharp cutoffs though in Subsection 7.4 we adapt to smooth cutoffs which will be denoted by $\tilde{\eta}$.

We write $\eta \in C^1(\mathbb{R}^n, \mathbb{R})$ the following regions in Fourier space:

$$D^a_{k,j} = \{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n | \xi \in A_k, 2^j - 1 \leq |\tau - \varphi_a(\xi)| \leq 2^j \},$$

$$D^a_{k,j} = \{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n | \xi \in A_k, |\tau - \varphi_a(\xi)| \leq 2^{j+1} \}. \quad (20)$$

Let $\eta_0 : \mathbb{R} \times [0, 1]$ denote an even, smooth function $\text{supp}(\eta_0) \subseteq [-5/4, 5/4]$. For $k \in \mathbb{N}$ we set

$$\eta_k(\tau) = \eta_0(\tau/2^k) - \eta_0(\tau/2^{k-1}).$$

We write $\eta \leq m = \sum_{j=0}^m \eta_j$ for $m \in \mathbb{N}$.

For $1 \leq a \leq 2$, $k \in \mathbb{N}_0$ define the $X^{s,b}$-type normed spaces

$$X_{a,k} = X_{a,k}(\mathbb{R} \times \mathbb{Z}^n) = \{ f \in L^2(\mathbb{R} \times \mathbb{Z}^n) \mid \text{f is supported in } \mathbb{R} \times A_k \}$$

$$\| f_k \|_{X_{a,k}} = \sum_{j=0}^{\infty} 2^{j/2} \| \eta_j(\tau - \varphi_a(\xi)) f_k \|_{L_x^2 L_t^2} < \infty.$$

Partitioning the modulation variable through a sum over $\eta_j$ yields the estimate:

$$\| \int_\mathbb{R} |f_k(\xi, \tau)| d\tau \|_{L_x^2} \lesssim \| f_k \|_{X_{a,k}}.$$  

Also, we record the estimate

$$\sum_{j=l+1}^{\infty} 2^{j/2} \| \eta_j(\tau - \varphi_a(\xi)) \| \int_{\mathbb{R}} |f_k(\xi, \tau') 2^{-l} (1 + 2^{-l} |\tau - \tau'|)^{-4} d\tau' \|_{L_t^2}$$

$$+ 2^{l/2} \| \eta_l(\tau - \varphi_a(\xi)) \| \int_{\mathbb{R}} |f_k(\xi, \tau') 2^{-l} (1 + 2^{-l} |\tau - \tau'|)^{-4} d\tau' \|_{L_t^2} \lesssim \| f_k \|_{X_{a,k}}, \quad (21)$$

which is [8, Equation (3.5)].

In particular, we find for a Schwartz-function $\gamma \in \mathcal{S}(\mathbb{R})$, $k, l \in \mathbb{N}$, $t_0 \in \mathbb{R}$, $f \in X_{a,k}$ the estimate

$$\| \mathcal{F}_{t,x} [\gamma (2^lt - t_0) \mathcal{F}_{t,x}^{-1}(f)] \|_{X_{a,k}} \lesssim \| f \|_{X_{a,k}}.$$  

The $X_{a,k}$-spaces relate to the space-time Fourier transform of the original functions after frequency localization. Let

$$E_k = \{ \hat{\phi} : T^n \rightarrow \mathbb{R} \mid \hat{\phi} \text{ supported in } A_k, \| \hat{\phi} \|_{E_k} = \| \hat{\phi} \|_{L^2} < \infty \}.$$
Next, define

\[ F_{a,k,\delta} = \{ u_k \in C(\mathbb{R}; E_k) \mid \| u_k \|_{F_{a,k,\delta}} = \sup_{t_k \in \mathbb{R}} \| F[u_k \eta_0 (2^{1+\delta})^k (t - t_k)] \|_{X_{a,k}} < \infty \}, \]

\[ N_{a,k,\delta} = \{ u_k \in C(\mathbb{R}; E_k) \mid \| u_k \|_{N_{a,k,\delta}} = \sup_{t_k \in \mathbb{R}} \| (\tau - \varphi_\alpha(\xi) + i2^{1+\delta})^k \|_{X_{a,k}} < \infty \}. \]

For \( T \in (0, 1] \), let

\[ F_{a,k,\delta}(T) = \{ u_k \in C([-T,T]; E_k) \mid \| u_k \|_{F_{a,k,\delta}(T)} = \inf_{\tilde{u}_k \in \mathbb{R}} \| \tilde{u}_k \|_{F_{a,k,\delta}} < \infty \}, \]

\[ N_{a,k,\delta}(T) = \{ u_k \in C([-T,T]; E_k) \mid \| u_k \|_{N_{a,k,\delta}(T)} = \inf_{\tilde{u}_k \in \mathbb{R}} \| \tilde{u}_k \|_{N_{a,k,\delta}} < \infty \}. \]

The spaces \( F_{a,\delta}^s(T), N_{a,\delta}^s(T), E^s(T) \) are assembled by Littlewood-Paley theory. Let \( C = C([-T,T], H^\infty(\mathbb{T}^n)) \) and define

\[ F_{a,\delta}^s(T) = \{ u \in C \mid \| u \|_{F_{a,\delta}^s(T)} = \sum_{k \in \mathbb{N}_0} 2^{2sk} \| P_k u \|_{F_{a,k,\delta}(T)}^2 < \infty \}, \]

\[ N_{a,\delta}^s(T) = \{ u \in C \mid \| u \|_{N_{a,\delta}^s(T)} = \sum_{k \in \mathbb{N}_0} 2^{2sk} \| P_k u \|_{N_{a,k,\delta}(T)}^2 < \infty \}, \]

\[ E^s(T) = \{ u \in C \mid \| u \|_{E^s(T)} = \sum_{k \geq 0} \sup_{t_k \in [-T,T]} 2^{2sk} \| P_k u(t_k) \|_{F_{a,k,\delta}}^2 < \infty \}. \]

The multiplier properties (cf. [13, Eq. (2.21), p. 273]) hold independently of the dispersion relation. We record basic properties of the short-time function spaces introduced above. The next lemma establishes the embedding \( F_{a,\delta}^s(T) \hookrightarrow C([0,T], H^s) \).

**Lemma 7.1.** (i) We find the estimate

\[ \| u \|_{L^\infty_t L^2_x} \lesssim \| u \|_{F_{a,\delta}^s}, \]

which holds for any \( u \in F_{a,\delta}^s \) with implicit constant independent of \( k \).

(ii) Suppose that \( s \in \mathbb{R}, T > 0 \) and \( u \in F_{a,\delta}^s(T) \). Then, we find the estimate

\[ \| u \|_{C([0,T], H^s)} \lesssim \| u \|_{F_{a,\delta}^s(T)} \]

to hold.

**Proof.** See [8, Lemma 3.2, 3.3].

We state the linear energy estimate for the above short-time \( X^{s,\delta} \)-spaces. The proof, which is carried out on the real line in [13, Proposition 3.2., p. 274] and in the periodic case in [8, Proposition 4.1.], is omitted.

**Proposition 11.** Let \( T \in (0,1], 1 \leq a \leq 2, \delta \geq 0 \) and \( u, v \in C([-T,T], H^\infty) \) satisfy the equation

\[ i\partial_t u + \varphi(\nabla/i)u = v \text{ in } (-T,T) \times \mathbb{T}^n. \]

Then, we find the following estimate to hold for any \( s \in \mathbb{R} \):

\[ \| u \|_{F_{a,\delta}^s(T)} \lesssim \| u \|_{E^s(T)} + \| v \|_{N_{a,\delta}^s(T)}. \]
For the large data theory we have to define the following generalizations in terms of regularity in the modulation variable to the $X_k$-spaces:

$$X_{a,k,\delta} = \{ f : \mathbb{R} \times \mathbb{Z}^n \to \mathbb{C} \mid \text{supp}(f) \subseteq \mathbb{R} \times A_k, \| f \|_{X_{a,k,\delta}} = \sum_{j=0}^{\infty} 2^{bj}\| \eta_j(\tau - \varphi(\xi))f(\tau,\xi) \|_{L^2_xL^2_t} < \infty \},$$

where $b \in \mathbb{R}$. The short-time spaces $F_{a,k,\delta}^b$, $F_{a,\delta}^b(T)$ and $X_{a,k,\delta}^b$, $X_{a,\delta}^b(T)$ are defined following along the above lines with $X_{a,k}$ replaced by $X_{a,k}^b$.

Indeed, in a similar spirit to the treatment of $X_{T}^{s,b}$-spaces, we can trade regularity in the modulation variable for a power of $T$:

**Lemma 7.2.** [8, Lemma 3.4] Let $T > 0$, $\delta \geq 0$ and $b < 1/2$. Then, we find the following estimate to hold:

$$\| P_k u \|_{F_{a,k,\delta}^b} \lesssim T^{(1/2-b)-} \| P_k u \|_{F_{a,k,\delta}^b}$$

for any function $u$ with temporal support in $[-T,T]$ and implicit constant independent of $k$.

Below we have to consider the action of sharp time cutoffs in the $X_k$-spaces. Recall from the usual $X^{s,b}$-space-theory that multiplication with a sharp cutoff in time is not bounded. However, we find the following estimate to hold:

**Lemma 7.3.** [8, Lemma 3.5] Let $k \in \mathbb{Z}$. Then, for any interval $I = [t_1, t_2] \subseteq \mathbb{R}$, we find the following estimate to hold:

$$\sup_{j \geq 0} 2^{j/2}\| \eta_j(\tau - \varphi(n))F_{t,x}[1_I(t)P_k u]\|_{L^2_xL^2_t} \lesssim \| F_{t,x}(P_k u) \|_{X_k}$$

with implicit constant independent of $k$ and $I$.

### 7.2. Bilinear estimates

Next, we derive bilinear convolution estimates for space-time Fourier transforms of functions localized in frequency and modulation.

Due to the lack of dispersion, Strichartz estimates like in Euclidean space are not available. Still the supports of the involved functions can be estimated by the group velocity.

**Lemma 7.4.** Let $1 \leq a \leq 2$, $k_i, j_i \in \mathbb{N}$, $f_i : \mathbb{R} \times \mathbb{Z}^n \to \mathbb{R}_+$, $f_i \in L^2$, $\text{supp}(f_i) \subseteq D_{k_i,\leq j_i}$, for $i \in \{1, 2, 3\}$. Set $\delta = (a - 1)/2$.

(a) Suppose that $k_2 \leq k_1 - 10$. Then, we find the following estimate to hold:

$$\int_{\mathbb{R} \times \mathbb{Z}^n} (f_1 * f_2) f_3 \lesssim \| f_1 \|_{L^2} 2^{j_2/2} 2^{(a-1)} k_2/2 \| f_2 \|_{L^2} (1 + 2^{j_2 - k_j}) \| f_3 \|_{L^2}. \tag{22}$$

(b) Suppose that $|k_1 - k_2| \leq 10, |k_2 - k_3| \leq 10$ and $j_i \geq (1 + \delta) k_i$. Further, suppose that $f_i(\tau, \xi) = 0$ provided that $|\xi| \notin [2^{j_i}, 2^{j_i+1})$, where $i = 1, 2, 3$ and $l = \max_{i=1,2,3} l_i$. Then, we find the following estimate to hold:

$$\int_{\mathbb{R} \times \mathbb{Z}^n} (f_1 * f_2) f_3 \lesssim 2^{-l/2} 2^{-(1+\delta) k_1} 2^{k_1/2} \prod_{i=1}^{3} 2^{j_i/2} \| f_i \|_{L^2}. \tag{23}$$

(c) The estimate

$$\int_{\mathbb{R} \times \mathbb{Z}^n} (f_1 * f_2) f_3 \lesssim 2^{n_{k_{\min}}/2} 2^{j_{\min}/2} \prod_{i=1}^{3} \| f_i \|_{L^2} \tag{24}$$

holds true.
Proof. (a),(b): Set \( f_i^\#(\tau, \xi) = f_i(\tau + \varphi_a(\xi), \xi) \) for \( i \in \{1, 2, 3\} \) such that \( \|f_i^\#\|_{L^2_\tau L^2_\xi}^2 = \|f_i\|_{L^2_\tau L^2_\xi}^2 \) and

\[
\int_{\mathbb{R}^2} (f_1 * f_2) f_3 = \int_{\mathbb{R}^2} \sum_{\xi_1, \xi_2} f_1^\#(\tau_1, \xi_1) f_2^\#(\tau_2, \xi_2) f_3^\#(\tau_0u_1 + \tau_2 + \Omega, \xi_1 + \xi_2) d\tau_1 d\tau_2,
\]

where

\[
\Omega_a(\xi_1, \xi_2) = \varphi_a(\xi_1 + \xi_2) - \varphi_a(\xi_1) - \varphi_a(\xi_2).
\]

The considerations from Section 2 can be utilized in the following way:

Consider \( u_i \in L^2(\mathbb{R} \times \mathbb{T}^n) \), real-valued, with \( f_i = \mathcal{F}_{t,x}[u_i] \), \( \text{supp}(f_i) \subseteq D^a_{k_i, \leq j_i} \), and moreover,

\[
|\nabla \varphi_a(\xi_2) - \nabla \varphi_a(\xi_3)| \gtrsim V \quad \xi_i \in \text{supp}(f_i).
\]

Suppose that \( |\partial_1 \varphi_a(\xi_1) - \partial_1 \varphi_a(\xi_2)| \gtrsim V \). In case another partial derivative dominates the conclusion follows \textit{mutatis mutandis}.

Next, we compute by applying the Cauchy-Schwarz inequality in \( \xi_2 \)

\[
\int_{\mathbb{R}^2} \sum_{\xi_1} f_1^\#(\tau_1, \xi_1) \sum_{\xi_2} f_2^\#(\tau_2, \xi_2) \tilde{f}_3^\#(\tau_1 + \tau_2 + \Omega_a(\xi_1, \xi_2), \xi_1 + \xi_2) d\tau_1 d\tau_2 \\
\lesssim \int d\tau_1 d\tau_2 \sum_{\xi_1} f_1^\#(\xi_1, \tau_1) 2^{(n-1)k_2} (1 + 2^{3j} V)^{1/2} \\
\times \left( \sum_{\xi_2} |f_2^\#(\tau_2, \xi_2)|^2 \tilde{f}_3^\#(\tau_1 + \tau_2 + \Omega_a, \xi_1 + \xi_2)^2 \right)^{1/2}.
\]

The estimate is concluded by applications of the Cauchy-Schwarz inequality in the remaining variables:

\[
\lesssim 2^{(n-1)k_2} (1 + 2^{3j} V)^{1/2} \int d\tau_2 \sum_{\xi_1} \left( \int d\tau_1 |f_1^\#(\tau_1, \xi_1)|^2 \right)^{1/2} \\
\times \left( \sum_{\xi_2} |f_2^\#(\tau_2, \xi_2)|^2 \tilde{f}_3(\cdot, \xi_1 + \xi_2) \right)^{1/2} \\
\lesssim 2^{(n-1)k_2} (1 + 2^{3j} V)^{1/2} \|f_1\|_{L^2_\tau L^2_\xi} \|f_2\|_{L^2_\tau L^2_\xi} \int d\tau_2 \left( \sum_{\xi_2} |f_2^\#(\tau_2, \xi_2)|^2 \right)^{1/2} \\
\lesssim 2^{2j/2} 2^{(n-1) k_2/2} (1 + 2^{3j} V)^{1/2} \prod_{i=1}^3 \|f_i\|_{L^2}.
\]

Above \( \tilde{f}(\tau, \xi) = f(\tau - \xi) \). The argument together with Propositions 4 and 5 yields (a) and (b). (c): This follows from two applications of the Cauchy-Schwarz inequality without using the resonance function. \( \square \)

7.3. Nonlinear estimates.

Proposition 12. Let \( n \geq 2, 1 \leq a \leq 2 \) and \( T \in (0, 1] \). Set \( \delta = (a - 1)/2 \). We find the following estimates to hold:

\[
\|\partial_x(uv)\|_{N^a_{\delta}(T)} \lesssim T^{(\delta)} \|u\|_{F^a_{\delta}(T)} \|v\|_{F^a_{\delta}(T)}.
\]  

(25)
\[ \| \partial_{x_1} (uv) \|_{N_{a,k}^0(T)} \lesssim T^{\varepsilon(a)} \| u \|_{p^a_{N_{a,k}^0}(T)} \| v \|_{p^a_{N_{a,k}^0}(T)}. \] (26)

with \( \varepsilon(1) = 0 \) and some \( \varepsilon(a) > 0 \) for \( 1 < a \leq 2 \) provided that \( n/2 < s \leq s' \).

**Remark 3.** The argument below yields nonlinear estimates up to \( s > (n - 1)/2 \). The regularity threshold \( s > (n + 1)/2 \) comes from carrying out energy estimates.

**Proof.** Choose \( \tilde{u}, \tilde{v} \in C(\mathbb{R}, H^{n+2}) \) such that
\[ \| P_k \tilde{u} \|_{F_{a,k,\delta}} \leq 2 \| P_k u \|_{F_{a,k,\delta}(T)} \text{ and } \| P_k \tilde{v} \|_{F_{a,k,\delta}} \leq 2 \| P_k v \|_{F_{a,k,\delta}(T)} \]
for \( k \in \mathbb{N} \). Set \( u_k = P_k \tilde{u} \) and \( v_k = P_k \tilde{v} \). Due to (25), we can suppose that \( \tilde{u} \) and \( \tilde{v} \) are supported in \( [-2T, 2T] \). Then it suffices to consider the interactions: \( High \times Low \rightarrow High \):
\[ \| P_k \partial_{x_1} (u_k v_k) \|_{N_{a,k,\delta}} \lesssim T^{\varepsilon(a)} 2^{(n-1)k/2} \| u_k \|_{F_{a,k_1,\delta}} \| v_k \|_{F_{a,k_2,\delta}} \quad (k_2 \leq k - 10), \]
\[ (27) \]

\( High \times High \rightarrow High \):
\[ \| P_k \partial_{x_1} (u_k v_k) \|_{N_{a,k,\delta}} \lesssim T^{\varepsilon(a)} 2^{(n-1)k/2} \| u_k \|_{F_{a,k_1,\delta}} \| v_k \|_{F_{a,k_2,\delta}} \quad (|k_1 - k| \leq 10), \]
\[ (28) \]

\( High \times High \rightarrow Low \):
\[ \| P_k \partial_{x_1} (u_k v_k) \|_{N_{a,k,\delta}} \lesssim T^{\varepsilon(a)} 2^{(n-1)k_1/2} \| u_k \|_{F_{a,k_1,\delta}} \| v_k \|_{F_{a,k_2,\delta}} \quad (k \leq k_1 - 10). \]
\[ (29) \]

The above estimates yield the claim by the definition of the function spaces after summing over the frequencies. The extra factor \( T^{\varepsilon(a)} \) stems from not making use of the full range of modulation regularity from \(-1/2\) to \( 1/2 \), but using some room in the modulation regularity together with Lemma 7.2.

We start with \( High \times Low \rightarrow High \)-interaction. By the definition of \( N_{a,k,\delta} \) and \( F_{a,k,\delta}\)-spaces, it suffices to show the estimate
\[ 2^k \sum_{j \geq 2} 2^{-j/2} \| 1_{D_{k,j}^0} (f_{k_1,j_1} * f_{k_2,j_2}) \|_{L^2} \lesssim 2^{(n-1)k/2} \prod_{i=1}^2 2^{(1/2) - j} \| f_{k_i,j_i} \|_{L^2}. \] (30)

It will be enough to have one modulation regularity less than \( 1/2 \). Here,
\[ f_{k_i,j_i} (\tau, \xi) = \begin{cases} \eta_{j_i} (\tau - \varphi_a (\xi)) \mathcal{F}_{t,x}[u_i], & j_i > (1 + \delta)k \\ \eta_{\leq j_i} (\tau - \varphi_a (\xi)) \mathcal{F}_{t,x}[u_i], & j_i = (1 + \delta)k. \end{cases} \]

To prove (30), we use duality and apply estimate (22) to find
\[ \| 1_{D_{k,j}^0} (f_{k_1,j_1} * f_{k_2,j_2}) \|_{L^2} = \sup_{\| g_{a,j} \|_{L^\infty} = 1} \int_{\mathbb{R}^n} g_{k,j} (f_{k_1,j_1} * f_{k_2,j_2}) \]
\[ \lesssim 2^{(n-1)k_2/2} \| \eta_{\leq j} \|_{L^\infty} \| f_{k_1,j_1} \|_{L^2}. \]

For the \( High \times High \rightarrow High \)-interaction, we split the sum over the output modulation variable into \((1 + \delta)k \leq j \leq 2k\) and \( j \geq 2k\) and further, introduce an additional
frequency localization in the $x_1$-variable, to find for frequency localized pieces

$$2^{1/2} \sum_{(1+\delta)k \leq j \leq 2k} 2^{-j/2} \|1_{D^{\delta}_{k,i,j}}(f^{l_1}_{k_1,j_1} \ast f^{l_2}_{k_2,j_2})\|_{L^2}$$

$$\lesssim 2^k \sum_{(1+\delta)k \leq j \leq 2k} 2^{-j/2} 2j/2 2^{(n-2)k/2} 2^{j/2} \|f^{l_1}_{k_1,j_1}\|_{L^2} L^2_j$$

$$\lesssim 2^{n(k/2)+1} \prod_{i=1}^2 2^{j_i/2} \|f^{l_i}_{k_i,j_i}\|_{L^2}.$$

after applying duality and estimate (23), where $(f^{l_i}_{k_i,j_i})(\tau, \xi) = 0$ if $|\xi_1| \notin [2^k, 2^{k+1}]$ and $t^* = \max(l, t_1, t_2)$. Summation over $l$ and $l_i$ only gives a logarithmic factor, not changing the estimate effectively.

For the high modulation output apply duality and estimate (24) to find

$$2^k \sum_{j \geq 2k} 2^{-j/2} \|1_{D^{\delta}_{k,i,j}}(f^{l_1}_{k_{1,j_1}} \ast f^{l_2}_{k_{2,j_2}})\|_{L^2}$$

$$\lesssim 2^k \sum_{j \geq 2k} 2^{-j/2} 2^{(n-1)k/2} \prod_{i=1}^2 2^{j_i/2} \|f^{l_i}_{k_i,j_i}\|_{L^2}$$

$$\lesssim 2^{(n-1)k/2} \prod_{i=1}^2 2^{j_i/2} \|f^{l_i}_{k_i,j_i}\|_{L^2}.$$

Interpolation with trivial estimates from Bernstein’s inequality allows us to lower $j_i/2 \to (1/2-)j_i$ at the cost of $2^{(0+)k}$.

For $High \times High \rightarrow Low$-interaction, we argue similarly: Taking into account the additional time localization, it suffices to prove

$$2^{(1+\delta)k_1-\delta k} \sum_{j \geq (1+\delta)k} 2^{-j/2} \|1_{D^{\delta}_{k_1,i,j}}(f^{l_1}_{k_1,j_1} \ast f^{l_2}_{k_2,j_2})\|_{L^2} \lesssim 2^{k_1/2} \prod_{i=1}^2 2^{j_i/2} \|f^{l_i}_{k_i,j_i}\|_{L^2},$$

where $\text{supp}(f^{l_i}_{k_i,j_i}) \subseteq D^{\delta}_{k_1,i,j_i}, j_i \geq (1+\delta)k_1$ for $i = 1, 2$.

We split the sum over $j$ into $(1+\delta)k \leq j \leq 2k_1, j \geq 2k_1$.

In the first case, we use duality and apply (22) to find

$$2^{k_1} 2^{(k_1-k)} \sum_{j \geq (1+\delta)k} 2^{-j/2} 2^{j/2} 2^{(n-1)k/2} 2^{(1+\delta)k_i/2} \prod_{i=1}^2 2^{j_i/2} \|f^{l_i}_{k_i,j_i}\|_{L^2}$$

$$\lesssim (2k_1 - k) 2^{(n-1)k/2} \prod_{i=1}^2 2^{j_i/2} \|f^{l_i}_{k_i,j_i}\|_{L^2}.$$

In the second case, estimate (24) yields

$$2^{k_1} 2^{(k_1-k)} \sum_{j \geq 2k_1} 2^{-j/2} \|1_{D^{\delta}_{k_1,i,j}}(f^{l_1} \ast f^{l_2})\|_{L^2}$$

$$\lesssim 2^{k_1} \sum_{j \geq 2k_1} 2^{-j/2} 2^{nk/2} 2^{-(1+\delta)k_1/2} \prod_{i=1}^2 2^{j_i/2} \|f^{l_i}_{k_i,j_i}\|_{L^2}$$

$$\lesssim 2^{nk/2} \prod_{i=1}^2 2^{j_i/2} \|f^{l_i}_{k_i,j_i}\|_{L^2}.$$
Like in \( \text{High} \times \text{High} \rightarrow \text{High} \)-interaction, we can lower the modulation regularity by interpolation with trivial estimates. The proof is complete. \( \square \)

### 7.4. Energy estimates.

The purpose of this section is to propagate the energy norm of solutions and differences of solutions in terms of short-time norms. We prove the following proposition:

**Proposition 13.** Let \( n \geq 2, \ 1 \leq a \leq 2, \ T \in (0,1), \ s > (n+1)/2 \) and \( u \in C([-T,T], H^\infty(\mathbb{T}^n)) \) be a smooth solution to (1). Set \( \delta = (a-1)/2 \). Then, there is \( \theta > 0 \) such that we find the following estimate to hold:

\[
\|u\|^2_{L^2(T)} \lesssim \|u_0\|^2_{H^s} + T^\theta \|u\|^3_{F_{a,\delta}^s(T)}. \tag{31}
\]

For two solutions \( u_i \in C([-T,T], H^\infty) \) to (1) the function \( v = u_1 - u_2 \) satisfies the estimate

\[
\|v\|^2_{L^2(T)} \lesssim \|v(0)\|^2_{H^s} + T^\theta \|v\|^2_{F_{a,\delta}^s(T)}(\|u_1\|_{F_{a,\delta}^s(T)} + \|u_2\|_{F_{a,\delta}^s(T)}) \tag{32}
\]

and

\[
\|v\|^2_{L^2(T)} \lesssim \|v(0)\|^2_{H^s} + T^\theta \|v\|^2_{F_{a,\delta}^s(T)} + T^\theta \|v\|_{F_{a,\delta}^s(T)} \|v\|_{F_{a,\delta}^s(T)} \|u_2\|_{F_{a,\delta}^s(T)} \tag{33}
\]

for some \( \theta > 0 \).

Like in the Euclidean case we find for the evolution of the \( L^2 \)-norm of the frequencies

\[
\|\hat{\hat{P}}_k u(t_k)\|^2_{L^2} = \|\hat{\hat{P}}_k u(0)\|^2_{L^2} + 2 \int_{[0,t_k] \times \mathbb{T}^n} \hat{\hat{P}}_k u \hat{\hat{P}}_k v dx dt
\]

where \( u \) solves the following forced equation:

\[
\partial_t u + \partial_{x_1} D^\alpha u = \partial_{x_1} (uv).
\]

The key estimates are carried out in the following lemma:

**Lemma 7.5.** Let \( T > 0, \ u_i \in F_{a,k_i,\delta}(T), \ i = 1, 2, 3 \). We find the following estimate to hold:

\[
\left| \int_{[0,T] \times \mathbb{T}^n} u_1 u_2 u_3 dx dt \right| \lesssim T^{\theta/2} 2^{(n-1)k_2/2} \prod_{i=1}^3 \|u_i\|_{F_{a,k_i}(T)} \tag{34}
\]

provided that \( k_2 < k_1 - 10 \) for some \( \theta > 0 \).

Suppose that \( |k_1 - k_2| \leq 10, \ |k_1 - k_3| \leq 10, \ u_i \in F_{a,k_i,\delta}(T), \ i = 1, 2, 3 \) and \( F_x(u_i)(\xi) = 0 \) whenever \( |\xi| \notin [2^{i}, 2^{i+1}] \). Set \( l^* = \max_{i=1,2,3} l_i \). Then, there is \( \theta > 0 \), such that the following estimate holds:

\[
\left| \int_{[0,T] \times \mathbb{T}^n} u_1 u_2 u_3 dx dt \right| \lesssim T^{\theta/2} 2^{-l^*/2} 2^n k_2/2 \prod_{i=1}^3 \|u_i\|_{F_{a,k_i,\delta}(T)}. \tag{35}
\]

Suppose that \( k_1 < k - 10 \). Then, we find the following estimate to hold:

\[
\left| \int_{[0,T] \times \mathbb{T}^n} \hat{\hat{P}}_k u \partial_{x_1} \hat{\hat{P}}_k (u \hat{\hat{P}}_{k_1} v) dx dt \right| \lesssim T^{\theta/2} 2^{(n+1)k_1/2} \sum_{|m_k - k_1| \leq 5} \|v\|_{F_{a,k_1,\delta}(T)} \times \sum_{|k - k'| \leq 10} \|P_{k'} u\|^2_{F_{a,k',\delta}(T)}. \tag{36}
\]

Furthermore, estimates (34) and (36) hold true after replacing \( \hat{\hat{P}}_k \).
Proof. By symmetry we can assume that \( k_1 \leq k_2 \leq k_3 \). Let \( \tilde{u}_i \in F_{a,k_i,\delta} \) with \( \|\tilde{u}_i\|_{F_{a,k_i,\delta}} \leq 2\|n_i\|_{F_{a,k_i,\delta}(T)}, \; i = 1, 2, 3 \) from the definitions.

The \( \tilde{u}_i \) will be denoted by \( u_i \) to lighten the notation. In order to estimate the functions in the short-time function spaces, time has to be localized according to the highest frequency. Let \( \gamma : \mathbb{R} \to [0, 1] \) be a smooth function supported in \([-1, 1]\) with

\[
\sum_{n \in \mathbb{Z}} \gamma^3(x - n) \equiv 1 \; \forall x \in \mathbb{R}.
\]

The left-hand side of (34) is dominated by

\[
\sum_{|n| \leq CT2^{(1+\delta)k_3}} \sum_{j_i \geq (1+\delta)k_i} \int_{\mathbb{R} \times \mathbb{Z}^n} \eta_{j_1}(\tau - \varphi_a(\xi)) F_{t,x}(u_1 \gamma(2^{(1+\delta)k_3} t - n)1_{[0,T]}(t))
\]

\[
(\eta_{j_2}(\tau - \varphi_a(\xi)) F[u_2 \gamma(2^{(1+\delta)k_3} t - n)]) \ast (\eta_{j_3} F[u_3 \gamma(2^{(1+\delta)k_3} t - n)]) d\xi dt
\]

\[
= \sum_{n \in A} (\ldots) + \sum_{n \in B} (\ldots),
\]

(37)

where

\[
A = \{ n \in \mathbb{Z} | \gamma(2^{(1+\delta)k_3} \cdot - n)1_{[0,T]} \neq \gamma(2^{(1+\delta)k_3} \cdot - n) \},
\]

\[
B = \{ n \in \mathbb{Z} | \gamma(2^{(1+\delta)k_3} \cdot - n)1_{[0,T]} = \gamma(2^{(1+\delta)k_3} \cdot - n) \}.
\]

In (37) read \( \eta_{j_i} = \eta_{\leq j_i} \); it is sufficient to derive bounds for this modulation variable decomposition according to (21).

 Apparently, \(|A| \leq 10, |B| \leq C_0 T 2^{(1+\delta)k_3} \) for \( T \geq 2^{-(1+\delta)k_3} \). For \( T \ll 2^{-(1+\delta)k_3} \) we can suppose that \(|A| = 1 \) and \( B = \emptyset \). We shall only give details for \( T \geq 2^{-(1+\delta)k_3} \).

The main contribution of \( B \) is handled first.

Denote

\[
f_{k_{1,i},j_1} = \eta_{j_1}(\tau - \varphi_a(\xi)) F_{t,x}(u_1 \gamma(2^{(1+\delta)k_3} t - n)1_{[0,T]}(t)), \; i = 1, 2, 3.
\]

We do not distinguish between different values of \( n \) because the following estimates are independent of \( n \).

In case \( k_1 \leq k_2 - 10 \) an application of (22) yields

\[
\sum_{n \in B} (\ldots) \lesssim T^{2(1+\delta)k_3} \sum_{j_1 \geq (1+\delta)k_1} 2^{j_1/2} 2^{(n-1)k_1/2} (1 + 2^{2-\varepsilon k_2}) \prod_{i=1}^3 \|f_{k_{i,j_i}}\|_{L^2}
\]

\[
\lesssim T^{2(n-1)k_1/2} \sum_{i=1}^3 2^{j_i/2} \|f_{k_{i,j_i}}\|_{L^2}
\]

because \( |k_2 - k_3| \leq 5 \) and \( j_3 \geq (1+\delta)k_3 \).

In case \( |k_1 - k_2| \leq 10, |k_2 - k_3| \leq 10 \) an application of (23) gives

\[
\sum_{n \in B} (\ldots) \lesssim T^{n k_3/2 - l^*} \sum_{i=1}^3 2^{j_i/2} \|f_{k_{i,j_i}}\|_{L^2}
\]

\[
\lesssim T^{n k_3/2 - l^*} \sum_{i=1}^3 2^{j_i/2} \|f_{k_{i,j_i}}\|_{L^2}.
\]
For the boundary terms note that sharp cutoffs in time are almost bounded in $X_{a,k}$, that is for an interval $I \subseteq \mathbb{R}$, $k \in \mathbb{N}_0$, $f_k \in X_{a,k}$ and $f_k' = \mathcal{F}(1_I(t)\mathcal{F}^{-1}(f_k))$ (cf. [13, p. 291])

$$\sup_{j \in \mathbb{N}} 2^{j/2}\|\eta_j(\tau - \varphi_a(\xi))f_k'\|_{L^2} \lesssim \|f_k\|_{X_{a,k}}.$$  

An application of Cauchy-Schwarz yields

$$\sum_{n \in \mathbb{N}} (\ldots) \lesssim \sum_{j_i \geq (1+\delta)k_i} 2^{j_i/2}2^{nk_i/2} \prod_{i=1}^3 \|f_i\|_{L^2} \lesssim 2^{k_1}2^{-k_3/2} \prod_{i=1}^2 \sum_{j_i \geq (1+\delta)k_i} 2^{j_i/2}\|f_{k_i,j_i}\|_{L^2} \sup_{j \in \mathbb{N}} 2^{j/2}\|f_j\|_{L^2}.$$  

Interpolation with trivial estimates allows us to lower the modulation regularity and gives the additional factor of $T^6$ by Lemma 7.2.

For the proof of (36) we integrate by parts (cf. [13]) to find

$$\int_{[0,T] \times \mathbb{T}^n} \hat{P}_k u \hat{P}_k(\partial_{x_1} u \hat{P}_k, v) dxdt$$

\begin{equation}
\leq \int_{[0,T] \times \mathbb{T}^n} \hat{P}_k u \hat{P}_k(\partial_{x_1} u \hat{P}_k, v) dxdt + C \sum_{i=1}^n \int_{\mathbb{R}^{d+1}} \hat{F}(\hat{P}_k u)(\tau, \xi) \times \int_{\mathbb{R} \times \mathbb{Z}^n} \hat{F}(\hat{P}_k \partial_{x_1} v)(\tau_1, \xi_1) \hat{F} v(\tau - \tau_1, \xi - \xi_1) \psi_i(\xi, \xi_1) d\xi_1 d\tau_1 d\xi d\tau,
\end{equation}

where $\psi_i, i = 1, \ldots, n$ are bounded and regular multipliers. The resulting expressions can be handled by (22). \hfill \Box

We are ready to prove Proposition 13.

**Proof of Proposition 13.** Following the remark after Proposition 13, we find for a solution to (1)

$$\|\hat{P}_k u(t_k)\|^2_{L^2} = \|\hat{P}_k u(0)\|^2_{L^2} + 2 \int_0^T ds \int_{\mathbb{T}^n} dx \hat{P}_k u \hat{P}_k(\partial_{x_1} u^2).$$

For the integral we consider the following interactions: $High \times Low \rightarrow High$:

$$\int_0^T \int_{\mathbb{T}^n} \hat{P}_k u \hat{P}_k(\partial_{x_1} u \hat{P}_k, v) dxdt \quad (k_1 \leq k - 10), \quad (38)$$

$High \times High \rightarrow High$:

$$\int_0^T \int_{\mathbb{T}^n} \hat{P}_k u \hat{P}_k(\partial_{x_1} u \hat{P}_k, u) dxdt \quad (|k - k_1| \leq 10), \quad (39)$$

$High \times High \rightarrow Low$:

$$\int_0^T \int_{\mathbb{T}^n} \hat{P}_k u \partial_{x_1}(\hat{P}_k, u \hat{P}_k, u) dxdt \quad (k \leq k_1 - 10). \quad (40)$$

$High \times Low \rightarrow High$-interaction is estimated by (36) to

$$\begin{equation}
(38) \lesssim T^{\theta d^2(\alpha+1)^2} \sum_{|m-k| \leq 5} \|P_m u\|^2_{F_{a,m},\delta(T)} \sum_{|m_1-k_1| \leq 5} \|P_{m_1} u\|_{F_{a,m_1},\delta(T)}
\end{equation}$$

and summing over $k_1 \leq k - 10$ and square summing over $k$ gives (31).
In case of $High \times High \rightarrow High$-interaction the functions are additionally partitioned in the first frequency. Estimate (35) is used to obtain

$$\sum_{|m-k|\leq 5} \|P_m u_1\|_{F_{a,m,\delta}(T)}^2 \sum_{|m_1-k_1|\leq 10} \|P_{m_1} u\|_{F_{a,m_1,\delta}(T)}$$

and square summing over $k$ gives (31).

For $High \times High \rightarrow Low$-interaction we do not integrate by parts, and the argument for $High \times Low \rightarrow High$-interaction is used.

To prove (36), we write

$$\|\hat{P}_k v(t_k)\|_{L^2}^2 = \|\hat{P}_k v(0)\|_{L^2}^2 + 2 \int_0^T \int_{\mathbb{T}^n} \hat{P}_k \hat{P}_k \partial_{x_1}(v(u_1 + u_2)) dx dt$$

and estimate $High \times High \rightarrow High$-interaction and $High \times High \rightarrow Low$-interaction like above to obtain (36). In case of $High \times Low \rightarrow High$-interaction one finds two different terms:

$$\int_0^T \int_{\mathbb{T}^n} \hat{P}_k \hat{P}_k \partial_{x_1}(v(u_1 + u_2)) dt dx \quad (k_1 \leq k - 10)$$

and

$$\int_0^T \int_{\mathbb{T}^n} \hat{P}_k \hat{P}_k \partial_{x_1}(u_1 + u_2) \hat{P}_k \hat{P}_k v dx dt \quad (k_1 \leq k - 10).$$

(41) is estimated along the above lines because we can integrate by parts to arrange the derivative on the smallest frequency.

For (42) we use estimate (34) instead to find

$$\sum_{|m-k|\leq 5} \|P_m v\|_{F_{a,m,\delta}(T)} \sum_{|m_1-k_1|\leq 10} \|P_{m_1} v\|_{F_{a,m_1,\delta}(T)}$$

and square summing in $k$ and summing over $k_1 \leq k - 10$ gives (32).

To prove (33), the solution to the difference equation is rewritten as

$$\partial_t v + \partial_{x_1} D^a v = \partial_{x_1}(v^2) + \partial_{x_1}(vu_2).$$

When estimating $\|v\|_{F_s(T)}$ for $s > (n + 1)/2$ the contribution of $\partial_{x_1}(v^2)$ can be handled as in the proof of (31), which gives

$$\sum_k 2^{2ks} \int_0^T \int_{\mathbb{T}^n} \hat{P}_k \hat{P}_k \partial_{x_1}(v^2) dx dt \lesssim T\|v\|_{F_s(T)}^2.$$

The contribution of $\partial_{x_1}(vu_2)$ can be treated like in the proof of (31) except for the interaction

$$\int_0^T \int_{\mathbb{T}^n} \hat{P}_k \hat{P}_k \partial_{x_1}(u_2 \hat{P}_k \hat{P}_k v) dx dt \quad (k_1 \leq k - 10)$$

because here we can not integrate by parts like above. Instead estimate (34) and square summing in $k$ and summation in $k_1 \leq k - 10$ gives

$$\sum_{k,k_1\leq k-10} 2^{2ks} \int_0^T \int_{\mathbb{T}^n} \hat{P}_k \hat{P}_k \partial_{x_1}(u_2 \hat{P}_k \hat{P}_k v) dx dt \lesssim T\|v\|_{F_{a,\delta}(T)} \|u_2\|_{F_{a,\delta}(T)} \|v\|_{F_{a,\delta}(T)}.$$
Appendix: Linear Strichartz estimates.

Remark 4. A remedy to the smallness restriction for $a = 1$ is to rescale the torus and introduce a low frequency weight such that the resulting norm is subcritical. These ideas are implemented in [6, 30]. For the sake of a concise presentation, this is not carried out in the present article.

Appendix: Linear Strichartz estimates. In this section linear Strichartz estimates for solutions to (1) posed on $\mathbb{R}^n$ are discussed. We start with a dispersive estimate which was proved for $a = 1$ in [12].

Proposition 14. Let $a \geq 1$, $n \geq 3$ and $\psi : \mathbb{R}^n \to \mathbb{R}$ be a smooth radial function supported in $B_n(0,2) \setminus B_n(0,1/2)$. Then, we find the following estimate to hold

$$
\int \psi(|\xi|) e^{i(\xi_1(|\xi|^n + x \cdot \xi))} d\xi \leq C|t|^{-1}
$$

with $C$ only depending on $n$, $\psi$ and $a$.

The modifications for $a \geq 1$ are straight-forward; we give the details for the sake of completeness.

Proof. We rewrite the integral in spherical coordinates to find

$$
I(x,t) = \int_0^\infty dr r^{n-1} \psi(r) \int_{S^{n-1}} d\sigma(\omega) e^{it(r^{a+1}\omega_1 + x_1 r + \cdots + x_n r)}
$$

$$
= \int_0^\infty \rho(r) \hat{\psi}(y_{x,t}(r)) dr,
$$

where $y_{x,t}(r) = (tr^{a+1} + x_1 r, x_2 r, \ldots, x_n r)$.

Recall the decay (cf. [31, Theorem 1.2.1, p.52])

$$
|\hat{\psi}(y)| \lesssim (1 + |y|)^{-\frac{n+2}{2}}.
$$

This is already enough to prove the claim for $n \geq 4$.

Indeed, partition supp$(f) = E_1 \cup E_2$, where $E_1 = \{ r \in \text{supp}(\rho) \mid |tr^{a+1} + x_1 r| \leq 1 \}$ and $|E_1| \lesssim |t|^{-1}$. To see this, note that $|tr^{a+1} + x_1 r| \leq 1$ implies $|tr^a + x_1| \leq 2$ and, by change of variables,

$$
\int_{1/2}^{2} 1_{\{|tr^a + x_1| \leq 2\}}(r) \rho(r) dr = \int_{r^a \sim 1} 1_{\{|tr^a + x_1| \leq 2\}}(r') dr' \leq C|t|^{-1}
$$

where $C$ depends on $\psi$, $n$ and $a$. 

7.5. Proof of Theorem 1.2.

Proof of Theorem 1.2. Fix $s > (n+1)/2$. We only prove a priori estimates for smooth initial values because the modifications for $L^2$-Lipschitz continuity and continuous dependence follow like in Section 6. From Propositions 11, 12 and 13 we find

$$
\begin{align*}
\|u\|_{F^s_{x,t}(T)} &\lesssim \|u\|_{E^s(T)} + \|\partial_{x_1}(u^2)\|_{N_{x,t}^s(T)} \\
\|u\partial_{x_1} u\|_{N_{x,t}^s(T)} &\lesssim T^s \|u\|_{F^s_{x,t}(T)}^2 \\
\|u\|_{F^s_{x,t}(T)}^3 &\lesssim \|u_0\|_{H^s}^2 + T^\theta \|u\|_{F^s_{x,t}(T)}^3.
\end{align*}
$$

For $1 < a \leq 2$ we have $\varepsilon(a) > 0$. These estimates allow us to conclude the a priori bound by bootstrapping the $F^s_{x,t}(T)$-norm, even for large initial data. For $a = 1$, since $\varepsilon(a = 1) = 0$, the derived set of estimates only give a priori bounds for small initial data. Further details are omitted to avoid repetition. □
Similarly, $E_2 \subseteq \{ r \in \text{supp}(\rho) | |tr^a + x_1| \geq 2 \}$, and consequently,
\[
\int_{E_2} \rho(r)|\hat{\sigma}(y_{x,t}(r))| dr \leq \int_{|tr^a + x_1| \geq 2} \rho(r)|tr^{a+1} + x_1r|^{-\frac{n-1}{2}} dr
\]
\[
\leq C \int_{|tr + x_1| \geq 2} |tr + x_1|^{-\frac{n-1}{2}} dr
\]
\[
= C|t|^{-\frac{n-1}{2}} \int_{|r + x_1/t| \geq 2/|t|} |r + x_1/t|^{-\frac{n-1}{2}} dr.
\]
After a linear change of variables, we estimate the final expression by $C|t|^{-1}$.

We turn to $n = 3$. Here, we make use of the asymptotic expansion
\[
\hat{\sigma}(y) = e^{i\|y\|} + e^{-i\|y\|} + \mathcal{E}_{x,t}(y),
\]
where $|\mathcal{E}_{x,t}(y)| \lesssim \|y\|^{-2}$ ($\|y\| \gg 1$).

Set $\phi(r) = \sqrt{f(r)}$, where $f(r) = (tr^{a+1} + x_1r)^2 + r^2\|x'\|^2$ and
\[
F_1 = \{ r \in \text{supp}(\rho) | |tr^{a+1} + x_1r| \leq 1 \} \cap \{ r \in \text{supp}(\rho) | |f'(r)| \leq |t| \} \supseteq E^1,
\]
\[
F_2 = \{ r \in \text{supp}(\rho) | |tr^{a+1} + x_1r| \geq 1, \ |f'(r)| \geq |t| \} \subseteq E^2.
\]
Below, we see that $|F_1| \lesssim |t|^{-1}$, which means that this contribution is controlled by $|\hat{\sigma}| \lesssim 1$.

Moreover, the contribution of $\mathcal{E}_{x,t}$ when integrating over $F_2$ is controlled by the higher dimensional argument due to $F_2 \subseteq E^2$ and sufficient decay to run the above argument.

A computation yields
\[
f'(r) = 2t^2(a + 1)r(r^a - r_-)(r^a - r_+),
\]
\[
r_{\pm} = \frac{(a + 2)x_1}{2(a + 1)t} \pm \sqrt{\left(\frac{a + 2}{a + 1}\right)^2 \left(\frac{x_1}{t}\right)^2 - \frac{x_1^2}{(a + 1)t^2} - \frac{\|x'\|^2}{(a + 1)t^2}}.
\]
We can suppose that $\frac{x_1}{t} \sim 1$ and $\frac{\|x'\|^2}{t^2} \ll 1$, since otherwise $|f(r)| \gtrsim |t|$, so that the roots are real and separated.

In fact, $|r_{\pm}| \sim 1$ and $|r_r - r_-| \sim 1$. Moreover, whenever $f'$ vanishes, then $|f''|$ is still bounded away from zero and thus, $|F_1| \lesssim |t|^{-1}$.

For the contribution of $e^{i\|y\|/\|y\|}$ over $F_2$ note that we can write
\[
\int e^{i\phi(r)} \rho(r) dr \sim \int \frac{d}{dr}[e^{i\phi(r)}] \rho(r) \frac{dr}{f'(r)} dr
\]
Next, the domain of integration is divided into a finite union of intervals, where $\rho/f'$ is monotone. On each such interval integration by parts yields the desired result. \qed

**Remark 5.** The dispersive estimate follows also from [17, Proposition 4.7].

From the dispersive estimate Strichartz estimates are derived by standard arguments.

**Proof of Proposition 2.** For $n \geq 3$ the dispersive estimate and conservation of mass give by interpolation
\[
\|S_a(t)P_1f\|_{L^p(\mathbb{R}^n)} \lesssim \|\hat{P}_1f\|_{L^{p'}(\mathbb{R}^n)} \quad (2 \leq p \lesssim \infty)
\]
and combination with the $TT^*$-argument (cf. [14]) proves Strichartz estimates

$$\|S_a(t)P_1f\|_{L^r_t(L^p_x(\mathbb{R}^n))} \lesssim \|\tilde{P}_Nf\|_{L^2(\mathbb{R}^n)}$$

provided that $\frac{2}{q} + \frac{2}{p} = 1$, $p \neq \infty$. A scaling argument gives for $p, q$ like above

$$\|S_a(t)P_Nf\|_{L^r_t(L^p_x(\mathbb{R}^n))} \lesssim N^s\|\tilde{P}_Nf\|_{L^2(\mathbb{R}^n)}, \quad s = n\left(1 - \frac{1}{p}\right) - \frac{a + 1}{q},$$

and (4) follows from Littlewood-Paley theory.
L. Molinet and D. Pilod, Bilinear Strichartz estimates for the Zakharov-Kuznetsov equation and applications, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **32** (2015), 347–371.

D. E. Pelinovsky and V. I. Shrira, Collapse transformation for self-focusing solitary waves in boundary-layer type shear flows, *Physics Letters A*, **206** (1995), 195–202.

F. Ribaud and S. Vento, Well-posedness results for the three-dimensional Zakharov-Kuznetsov equation, *SIAM J. Math. Anal.*, **44** (2012), 2289–2304.

F. Ribaud and S. Vento, Local and global well-posedness results for the Benjamin-Ono-Zakharov-Kuznetsov equation, *Discrete Contin. Dyn. Syst.*, **37** (2017), 449–483.

J.-C. Saut, Benjamin-Ono and intermediate long wave equations: Modeling, IST and PDE, *Nonlinear Dispersive Partial Differential Equations and Inverse Scattering*, Fields Inst. Commun., Springer, New York, **83** (2019), 95–160.

R. Schippa, On shorttime bilinear Strichartz estimates and applications to the Shrira equation, *Nonlinear Anal.*, **198** (2020), 111910.

R. Schippa, On a priori estimates and existence of periodic solutions to the modified Benjamin-Ono equation below $H^{1/2}(T)$, e-prints, arXiv:1704.07174.

C. D. Sogge, *Fourier Integrals in Classical Analysis*, Second edition, Cambridge Tracts in Mathematics, 210, Cambridge University Press, Cambridge, 2017.

V. Zakharov and E. Kuznetsov, On three dimensional solitons, *J. Exp. Theor. Phys.*, **39** (1974), 285–286.

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