SEMILINEAR SCHRÖDINGER EVOLUTION EQUATIONS WITH INVERSE-SQUARE AND HARMONIC POTENTIALS VIA PSEUDO-CONFORMAL SYMMETRY

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Abstract. We consider the Cauchy problems for Schrödinger equations with an inverse-square potential and a harmonic one. Since the Mehler type formulas are completed, the pseudo-conformal transforms can be constructed. Thus we can convert the problems into the nonautonomous Schrödinger equations without a harmonic oscillator.

1. Introduction. In this article, we consider the following Cauchy problems

\[
\begin{cases}
i \frac{\partial u}{\partial t} = (-\Delta + V + \omega^2|x|^2)u + g_0(u), \\
u(0) = u_0;
\end{cases}
\]

\(\text{(NLS)}^+\)

\[
\begin{cases}
i \frac{\partial u}{\partial t} = (-\Delta + V - \omega^2|x|^2)u + g_0(u), \\
u(0) = u_0,
\end{cases}
\]

\(\text{(NLS)}^-\)

where \(i = \sqrt{-1}, \omega > 0, \Delta = \partial^2_{x_1} + \partial^2_{x_2} + \cdots + \partial^2_{x_N}\) (Laplacian in \(\mathbb{R}^N\)), and \(|x| := (x_1^2 + x_2^2 + \cdots + x_N^2)^{1/2}\). \(V\) is a real-valued potential of homogeneous of degree \(-2\), that is,

\[
V(\mu x) = \mu^{-2}V(x) \quad \text{a.a.} \ x \in \mathbb{R}^N, \ \forall \ \mu > 0.
\]

The potentials of the inverse-square scales represent some physical phenomena with singular poles; see e.g., Suzuki [17, Section 1]. Especially, we consider the Calogero–Moser models:

\[
\sum_{j=1}^{N} \left( -\frac{\partial^2}{\partial x_j^2} + \frac{k}{x_j^2} + g x_j^2 \right) = -\Delta + \sum_{j=1}^{N} k_j x_j^{-2} + g |x|^2 =: -\Delta + V_{\text{CM}} + g |x|^2.
\]

The potentials \(V_{\text{CM}}\) are satisfies (1.1). This system is integrable (see Calogero [3, 4] and Moser [9]). Unfortunately, we need more considerations to solve the semilinear evolution equations (nonlinear perturbation to the Calogero–Moser Hamiltonian) since the poles arise in not only the origin but also the hyperplanes.

A usual harmonic oscillator \(-\Delta + \omega^2|x|^2\) is also integrable. The system is commonly applied to the many kinds of problems both in physics and mathematics.

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We omit to comment on the application. On the other hand, the Hamiltonian $-\Delta - \omega^2 |x|^2$ is employed with analyzing the Schwinger effect. The phenomena is the prediction that electron-positron pairs are generated in strong electric fields and hence the fields decay (see e.g., Aouda–Kanda–Naka–Toyoda [1]).

Here we remark that the system $-\Delta + V$ with super-quadratic growth of $-V$ is not essentially selfadjoint and hence $\exp(-it(-\Delta + V))$ cannot be defined; see e.g., Dunford–Schwartz [6, Chapter 13, Section 6, Corollary 22]. This implies that one of the thresholds of admissible negative potentials is $-\omega^2 |x|^2$. In general, $-\Delta + V$ is essentially selfadjoint in $C_0^\infty(\mathbb{R}^N)$ if $V \in L^p(\mathbb{R}^N)$ ($p \geq 3$ if $N \leq 3; p > 2$ if $N = 4$; $p \geq N/2$ if $N \geq 5$) and $W(x) \geq -C|x|^2 - D$; see e.g., [12, Theorem X.38 and Corollary]: the Faris–Lavine theorem. Here we remark that (1.1) implies that $V \notin L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for $\max\{2, N/2\} \leq p < \infty$. For example, $|x|^{-2} \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ ($p < N/2$) and $|x|^{-2} \notin L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ ($p \geq N/2$). Thus we need another approach to verifying the selfadjointness of $-\Delta + V - \omega^2 |x|^2$ for $V$ satisfying (1.1).

In another viewpoint of $(\text{NLS})^+$ and $(\text{NLS})^-$, $g_0(u)$ represents the nonlinear term so that the charge and the energy are conserved; for example, local nonlinearities as like $g_0(u) = \lambda |u|^{p-1}u$ ($\lambda \in \mathbb{R}$) and nonlocal nonlinearities as like $g_0(u) = \lambda u |x|^{-\alpha} * |u|^2$ ($\lambda \in \mathbb{R}$).

If $V \equiv 0$, then $(\text{NLS})^+$ is converted into

$$u(t) = S_0^{+\omega}(t)u_0 - i \int_0^t S_0^{+\omega}(t - \tau)g_0(u(\tau)) \, d\tau,$$

where $S_0^{+\omega}(t) := \exp(-it(-\Delta + \omega^2 |x|^2))$ (double-sign corresponds). The same formula is also completed for $(\text{NLS})^-$ (with alteration of $S_0^{+\omega}$ to $S_0^{-\omega}$). Here we have explicit representation for $S_0^{+\omega}(t)$ known as the Mehler formulas (see e.g., Feynman–Hibbs [7, Cahpter 8])

\[
S_0^{+\omega}(t)\varphi(x) = \int_{\mathbb{R}^N} \left( \frac{\omega}{2\pi i \sin(2\omega t)} \right)^{N/2} \exp\left( \frac{i\omega(|x|^2 + |y|^2)}{2\tan(2\omega t)} - \frac{i\omega x \cdot y}{\sin(2\omega t)} \right)\varphi(y) \, dy;
\]

\[
S_0^{-\omega}(t)\varphi(x) = \int_{\mathbb{R}^N} \left( \frac{\omega}{2\pi i \sinh(2\omega t)} \right)^{N/2} \exp\left( \frac{i\omega(|x|^2 + |y|^2)}{2\tanh(2\omega t)} - \frac{i\omega x \cdot y}{\sinh(2\omega t)} \right)\varphi(y) \, dy.
\]

Now we consider the relation among $S_0^{\pm\omega}(t)$ and

\[
S_0(t)\varphi(x) := \exp(it\Delta)\varphi(x) = \int_{\mathbb{R}^N} \frac{1}{(4\pi it)^{N/2}} \exp\left( \frac{i|x-y|^2}{4t} \right)\varphi(y) \, dy,
\]

\[
m(\sigma)\varphi(x) := \exp\left( \frac{i\sigma}{4} |x|^2 \right)\varphi(x).
\]

Here the valuables in the exponential functions are calculated

\[
\frac{i\omega(|x|^2 + |y|^2)}{2\tan(2\omega t)} - \frac{i\omega x \cdot y}{\sin(2\omega t)} = -\frac{i\omega (1 - \cos(2\omega t))(|x|^2 + |y|^2)}{2\sin(2\omega t)} + \frac{i\omega |x-y|^2}{2\sin(2\omega t)},
\]

\[
\frac{i\omega(|x|^2 + |y|^2)}{2\tanh(2\omega t)} - \frac{i\omega x \cdot y}{\sinh(2\omega t)} = \frac{i\omega (\cosh(2\omega t) - 1)(|x|^2 + |y|^2)}{2\sinh(2\omega t)} + \frac{i\omega |x-y|^2}{2\sinh(2\omega t)}.
\]
The solvability of \((\text{NLS})\) is not suitable since the admissible range of \(p\) is limited so that the potential free case is not followed (see e.g., Okazawa–Suzuki–Yokota \([10]\)). Note that the usual contraction method (so-called the Kato–Lions method) is not suitable since the admissible range of \(p\) is limited so that the potential free case is not followed (see e.g., Okazawa–Suzuki–Yokota \([10]\)).

The second aim is the solvability of \((\text{NLS})\) since the admissible range of \(p\) is limited so that the potential free case is not followed (see e.g., Okazawa–Suzuki–Yokota \([10]\)).

The first aim of this paper is verifying the Mehler type formula for \((\text{NLS})\) and \((\text{NLS})^+\) in the weighted energy spaces. Here \(\Delta_D\) is the Laplacian \(\partial_x^2\) in \((0, \infty)\) with the homogeneous Dirichlet boundary condition.

The first aim of this paper is verifying the Mehler type formula for \(S^{\pm\omega}_0(t) := \exp(-i(-\Delta + V \pm \omega^2 |x|^2))\). We can rewrite \(S^{\pm\omega}_0(t)\) by using \(S^{\sigma}_0(t) := \exp(-i(\Delta + V))\) and \(m(\sigma)\) as above. This implies the confirmation of the Strichartz estimates for \(S^{\pm\omega}_0(t)\).

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some nonautonomous problems without oscillation term. Moreover, the conversion allows to solve the scattering problems respect to \((NLS)^-\).

This article is divided into 4 sections. In Section 2 we consider \(\exp(-it(P \pm \omega^2|x|^2))\). We show the Mehler type formula. The pseudo-conformal transforms respect to \((NLS)^+\) and \((NLS)^-\) are defined in this section. In section 3 we solve \((NLS)^-\) via the energy methods. Also the scattering problems are also considered. Since \(P - \omega^2|x|^2\) is not bounded below, we need to convert into other problems. In Section 4 we conclude some remarks in this article.

1.1. Notations. First \(L^p(\Omega)\) is the usual Lebesgue space. \(p' := p/(p - 1)\) is the Hölder conjugate of \(p (1 \leq p \leq \infty)\). Note that \(1/p + 1/p' = 1\) and the dual space of \(L^p(\Omega) (1 \leq p < \infty, \Omega \subset \mathbb{R}^N)\) is isomorphic to \(L^{p'}(\Omega)\). The Lebesgue spaces are also considered as the vector-valued spaces \(L^p(I; L^q(\mathbb{R}^N))\), as known as the Bochner spaces. The norm of these spaces is defined

\[
\|u\|_{L^p(I; L^q)} := \|\|u(t)\|_{L^q(\mathbb{R}^N)}\|_{L^p(I)}.
\]

Let \(A : D(A) \rightarrow L^2(\mathbb{R}^N)\) be a closed operator. Here \(D(A)\) represents the domain of \(A\): \(D(A) = \{u; Au \in L^2(\mathbb{R}^N)\}\). Note that \(D(A)\) is not always a subspace of \(L^2(\mathbb{R}^N)\), conveniently. Especially, \(H^s(\mathbb{R}^N) := D((1 - \Delta)^{s/2}) (s \in \mathbb{R})\) is called the (fractional) Sobolev space.

Let \(P\) be a nonnegative and selfadjoint operator on \(L^2(\mathbb{R}^N)\). Then we set

\[
\mathcal{D} := D((1 + P)^{1/2}) \subset L^2(\mathbb{R}^N), \quad \Sigma := \mathcal{D} \cap D((1 + |x|^2)^{1/2}).
\]

Here we can see the triplet \(\mathcal{D} \subset L^2(\mathbb{R}^N) \subset \mathcal{D}^* (\mathcal{D}^*\ \text{is a dual of }\mathcal{D})\). Thus we have \(\mathcal{D}^* = D((1 + P)^{-1/2})\) and \(\Sigma^* = \mathcal{D}^* + D((1 + |x|^2)^{-1/2})\).

The (inverse) trigonometric functions and the (inverse) hyperbolic functions are effectively used in this paper. Especially, \(\arctan x\) and \(\tanh x\) are the inverse functions of \(\tan x\) and \(\tanh x\). Here we can see that

\[
\arctan x = \int_0^x \frac{1}{1 + t^2} dt, \quad \tanh x = \int_0^x \frac{1}{1 - t^2} dt = \frac{1}{2} \log \frac{1 + x}{1 - x}.
\]

Moreover, we have

\[
\tan \left(\frac{1}{2} \arctan x\right) = \frac{x}{1 + \sqrt{1 + x^2}}, \quad \tanh \left(\frac{1}{2} \tanh x\right) = \frac{x}{1 + \sqrt{1 - x^2}}.
\]

We use the two functions for simple notation for \(\omega > 0\).

\[
\xi_+(t) := \frac{\tan(2\omega t)}{2\omega}, \quad \xi_-(t) := \frac{\tanh(2\omega t)}{2\omega}.
\]

Note that the inverse functions are calculated as

\[
\xi_+^{-1}(t) = \frac{\arctan(2\omega t)}{2\omega}, \quad \xi_-^{-1}(t) = \frac{\tanh(2\omega t)}{2\omega}.
\]

Finally, we define the key operators in the considerations below:

\[
d(\beta)f(x) := \beta^{N/2} f(\beta x), \quad \beta > 0; \quad m(\sigma)f(x) := \exp \left(\frac{i \sigma |x|^2}{4}\right) f(x), \quad \sigma \in \mathbb{R}.
\]

For simple notation, we use

\[
\mathcal{S}_V(t) := \exp(-itP); \quad P = -\Delta + V.
\]

Throughout in this paper, we require that \(P\) is nonnegative and selfadjoint in \(L^2(\mathbb{R}^N)\) and satisfies the dilation invariance of domain:

\[
\{d(\beta)u; u \in D(1 + P)\} = D(1 + P) \quad \forall \beta > 0.
\]
Here the dilation invariance is not always satisfied. Metafune–Sobajima [8] showed that for any selfadjoint extensions of \(-\Delta + a|x|^{-2}\) \((a < -(N - 2)^2/4)\) the dilation invariance is only verified at most countable \(\beta\).

**Definition 1.1.** We impose the following setting for \(V\) throughout in this article. Let \(V\) be an inverse-square potential (1.1) which satisfies \(V_{|S^{N−1}} \in L^\infty(S^{N−1})\) and

\[
\delta_V := \inf \left\{ \int_{S^{N−1}} \left| \nabla_{S^{N−1}} u \right|^2 + V |u|^2 \, d\sigma; \|u\|_{L^2(S^{N−1})} = 1 \right\} + \frac{(N - 2)^2}{4} \geq 0. \tag{1.2}
\]

Note that \(P\) is nonnegative and selfadjoint in \(L^2(\mathbb{R}^N)\) with the dilation invariance.

Here much of the results in Section 2 are fulfilled for more general cases, for example, the Calogero–Moser Hamiltonian \(-\Delta + V_{CM}\).

2. **Pseudo-conformal symmetry.** In this section, we consider the propagators \(S^+_V(t) = \exp(-it(P \pm \omega^2|x|^2))\) via the representation by \(S_V(t)\), the dilation \(d(\beta)\), and the multiplier \(m(\sigma)\). Here the consequences in this section are trivial if \(V(x) \equiv 0\). Since we do not define a suitable unitary operator compatible with \(P\) (as like the Fourier transform), we need other techniques to deriving the pseudo-conformal symmetry.

2.1. **Relation among the three operators and representation for the propagators.**

**Lemma 2.1.** Let \(V\) satisfy (1.1) and (1.2). Then one has

\[
d(\beta) m(\sigma) = m(\beta^2 \sigma) d(\beta), \quad \beta > 0, \quad \sigma \in \mathbb{R}; \tag{2.1}
\]

\[
S_V(\tau) d(\beta) = d(\beta) S_V(\beta^2 \tau), \quad \beta > 0, \quad \tau \in \mathbb{R}; \tag{2.2}
\]

\[
S_V(\tau) m(\sigma) = m\left(\frac{\sigma}{1 + \sigma \tau}\right) d\left(\frac{1}{1 + \sigma \tau}\right) S_V\left(\frac{\tau}{1 + \sigma \tau}\right),
\]

\[
\tau \in \mathbb{R}, \quad \sigma \in \mathbb{R}, \quad 1 + \sigma \tau > 0. \tag{2.3}
\]

**Proof.** It is simple to check (2.1). Now we consider (2.2). To end this, let \(u(t, x) := S_V(t)\varphi(x)\) and \(v(t, x) := d(\beta) S_V(\beta^2 t)\varphi(x) = \beta^{N/2} u(\beta^2 t, \beta x)\). Note that \(i u_t - Pu = 0\). Here we see from \(Pd(\beta) = \beta^2 d(\beta) P\) that

\[
(i v_t - Pv)(t, x)
\]

\[
= i \beta^{N/2} \beta^2 u_t(\beta^2 t, \beta x) - \beta^{N/2} \beta^2 (Pu)(\beta^2 t, \beta x)
\]

\[
= \beta^{N/2+2} (i u_t - Pu)(\beta^2 t, \beta x) = 0
\]

and \(v(0, x) = \beta^{N/2} u(0, \beta x) = d(\beta) \varphi(x)\). Thus the selfadjoint of \(P\) implies \(v(t, x) = S_V(t) d(\beta) \varphi(x)\). This is the consequence of (2.2).

Next we show (2.3). To end this, let \(u(t, x) := S_V(t)\varphi(x)\) and

\[
v(t, x) := m\left(\frac{\sigma}{1 + \sigma \tau}\right) d\left(\frac{1}{1 + \sigma \tau}\right) S_V\left(\frac{t}{1 + \sigma \tau}\right) \varphi(x)
\]

\[
= (1 + \sigma \tau)^{-N/2} m\left(\frac{\sigma}{1 + \sigma \tau}\right) u\left(\frac{t}{1 + \sigma \tau}, \frac{x}{1 + \sigma \tau}\right)
\]

Note that \(i u_t - Pu = 0\). Here we see that

\[
v_t = (1 + \sigma \tau)^{-N/2} m\left(\frac{\sigma}{1 + \sigma \tau}\right)
\]

\[
\times \left[ \left(-\frac{N \sigma}{2(1 + \sigma \tau)} - \frac{i \sigma^2 |x|^2}{4(1 + \sigma \tau)^2} \right) (u) + \frac{(u_t)}{(1 + \sigma \tau)^2} - \frac{\sigma x \cdot (\nabla u)}{(1 + \sigma \tau)^2} \right].
\]
\[ \Delta v = (1 + \sigma t)^{-N/2} m \left( \frac{\sigma}{1 + \sigma t} \right) \times \left[ \left( \frac{i N \sigma}{2 (1 + \sigma t)} - \frac{\sigma^2 |x|^2}{4 (1 + \sigma t)^2} \right) (u) + \frac{i \sigma x}{(1 + \sigma t)^2} \cdot (\nabla u) + \frac{(\Delta u)}{(1 + \sigma t)^2} \right], \]

\[ Vv = (1 + \sigma t)^{-N/2} m \left( \frac{\sigma}{1 + \sigma t} \right) \left( Vu \right) \frac{(Vu)}{(1 + \sigma t)^2}, \]

where the valuables in \((u), (u_t), (\nabla u), (\Delta u), \) and \((Vu)\) are \((t/(1 + \sigma t), x/(1 + \sigma t))\). Thus we have

\[ (i v_t - P v)(t, x) = i v_t(t, x) + \Delta v(t, x) - Vv(t, x) = (1 + \sigma t)^{-N/2 - 2} m \left( \frac{\sigma}{1 + \sigma t} \right) \left[ i u_t + \Delta u - Vu \right] \left( \frac{t}{1 + \sigma t} - \frac{x}{1 + \sigma t} \right) = 0 \]

and \(v(0, x) = m(\sigma) u(0, x) = m(\sigma) \varphi(x).\) Thus the selfadjointness of \(P\) implies \(v(t, x) = S_v(t)m(\sigma)\varphi(x).\) This is the consequence of (2.3).

Define the two families of operators as

\[ A_+(t) := m(-2\omega t) S_v \left( \frac{t}{\omega(1 + t^2)} \right) m(-2\omega t), \]

\[ A_-(t) := m(2\omega t) S_v \left( \frac{t}{\omega(1 - t^2)} \right) m(2\omega t). \]

In a simple view of the definition, we see that \(A_\pm(t)^{-1} = A_\pm(-t)\) (double-sign corresponds). It follows from (2.3) that for \(|t| < 1\)

\[ A_+(t) = m \left( \frac{4\omega t}{1 + t^2} \right) d \left( \frac{1 + t^2}{1 - t^2} \right) S_v \left( \frac{t}{\omega(1 - t^2)} \right), \quad (2.4) \]

\[ A_-(t) = m \left( \frac{4\omega t}{1 + t^2} \right) d \left( \frac{1 - t^2}{1 + t^2} \right) S_v \left( \frac{t}{\omega(1 + t^2)} \right). \quad (2.5) \]

Especially,

\[ A_+ \left( \frac{s}{1 + \sqrt{1 - s^2}} \right) = m(-2\omega s) d(\sqrt{1 + s^2}) S_v \left( \frac{s}{2\omega} \right), \quad s \in \mathbb{R}; \]

\[ A_- \left( \frac{s}{1 + \sqrt{1 - s^2}} \right) = m(2\omega s) d(\sqrt{1 - s^2}) S_v \left( \frac{s}{2\omega} \right), \quad s \in (-1, 1). \quad (2.6) \]

Here we can see the addition formulas for \(A_+(t)\) and \(A_-(t).\)

**Proposition 1.** Assume that \(|ts| < 1.\) Then

\[ A_+(t) A_+(s) = A_+ \left( \frac{t + s}{1 - ts} \right), \quad (2.8) \]

\[ A_-(t) A_-(s) = A_- \left( \frac{t + s}{1 + ts} \right). \quad (2.9) \]

**Proof.** First we show (2.8). Simple calculation implies

\[ A_+(t) A_+(s) = m(-2\omega t + s) m(2\omega s(1 + t^2)) S_v \left( \frac{t}{\omega(1 + t^2)} \right) m(-2\omega(t + s)) \]

\[ \times S_v \left( \frac{s}{\omega(1 + s^2)} \right) m \left( 2\omega \frac{t(1 + s^2)}{1 - ts} \right) m \left( -2\omega \frac{t + s}{1 - ts} \right). \]
Thus we can apply (2.3) as follows:
\[
m \left(2 \omega \frac{s(1 + t^2)}{1 - ts}\right) S_V \left(\frac{t}{\omega(1 + t^2)}\right)
= S_V \left(\frac{t(1 - ts)}{\omega(1 + t^2)(1 + ts)}\right) d \left(\frac{1 + ts}{1 - ts}\right) \left(2 \omega s\frac{(1 + t^2)}{1 + ts}\right),
\]
\[
S \left(\frac{s}{\omega(1 + s^2)}\right) m \left(2 \omega \frac{t(1 + s^2)}{1 - ts}\right)
= m \left(2 \omega t(1 + s^2)\right) d \left(\frac{1 - ts}{1 + ts}\right) S_V \left(\frac{s(1 - ts)}{\omega(1 + s^2)(1 + ts)}\right).
\]
Hence we obtain
\[
A_+(t) A_+(s)
= m \left(-2 \omega \frac{t + s}{1 - ts}\right) S_V \left(\frac{t(1 - ts)}{\omega(1 + t^2)(1 + ts)} + \frac{s(1 - ts)}{\omega(1 + s^2)(1 + ts)}\right) m \left(-2 \omega \frac{t + s}{1 - ts}\right).
\]
Now the variables in \( S_V \) is calculated
\[
\frac{t(1 - ts)}{\omega(1 + t^2)(1 + ts)} + \frac{s(1 - ts)}{\omega(1 + s^2)(1 + ts)} = \frac{(t + s)(1 - ts)}{\omega(1 + t^2)(1 + s^2)}
= \frac{t + s}{1 - ts} \times \left[1 + \left(\frac{t + s}{1 - ts}\right)^2\right]^{-1}.
\]
Thus we can conclude (2.8). In a way similar to the above we can also prove (2.9).

Proposition 1 implies that \( S_V^{+\omega}(t) := A_+(\tan(\omega t)) \) and \( S_V^{-\omega}(t) := A_-(\tanh(\omega t)) \) satisfy the semigroup properties
\[
S_V^{+\omega}(t) S_V^{+\omega}(s) = S_V^{+\omega}(t + s) \quad (|\tan(\omega t) \tan(\omega s)| < 1);
S_V^{-\omega}(t) S_V^{-\omega}(s) = S_V^{-\omega}(t + s).
\]
Note that \(|\tanh(\omega t)| < 1 \) for all \( t \in \mathbb{R} \). Now we consider the generators of \( S_V^{+\omega}(t) \) and \( S_V^{-\omega}(t) \). Since we can also see that \( S_V^{+\omega}(t) \varphi \to \varphi \ (t \to 0) \) strongly in \( L^2(\mathbb{R}^N) \), the properties of \( C_0 \)-semigroups are fully verified, the generators can be considered.

Proposition 2. Suppose the assumption in Lemma 2.1.
(i) The infinitesimal generator of \( S_V^{+\omega}(t) = A_+(\tan(\omega t)) \) \((|t| < \pi/(4 \omega)) \) is \(-i(P + \omega^2|x|^2)\).
(ii) The infinitesimal generator of \( S_V^{-\omega}(t) = A_-(\tanh(\omega t)) \) is \(-i(P - \omega^2|x|^2)\).

Proof. We see from (2.4) and (2.5) that
\[
S_V^{+\omega}(t) = A_+(\tan(\omega t)) = m \left(-2 \omega \tan(2 \omega t)\right) d \left(\frac{1}{\cos(2 \omega t)}\right) S_V \left(\frac{\tan(2 \omega t)}{2 \omega}\right),
\]
\[
S_V^{-\omega}(t) = A_-(\tanh(\omega t)) = m \left(2 \omega \tanh(2 \omega t)\right) d \left(\frac{1}{\cosh(2 \omega t)}\right) S_V \left(\frac{\tanh(2 \omega t)}{2 \omega}\right).
\]
On the other hand, we see the following strong convergence in \( L^2(\mathbb{R}^N) \).
\[
\lim_{\sigma \to 0} \frac{m(\sigma)\varphi - \varphi}{\sigma} = \frac{i |x|^2}{4} \varphi,
\]
\[
\lim_{\beta \to 1} \frac{d(\beta)\varphi(x) - \varphi(x)}{\beta - 1} = \frac{N}{2} \varphi(x) + x \cdot \nabla \varphi(x),
\]
for \( \varphi \in D(1 + P) \cap D(1 + |x|^2) \). Now we show (i). Let \( \varphi \in D(1 + P) \cap D(1 + |x|^2) \). We see that

\[
\lim_{\tau \to 0} \frac{S_V(\tau) \varphi - \varphi}{\tau} = -iP\varphi
\]

where

\[
S_V^\pm(t) \varphi = \left(1 \mp \frac{2}{\omega} \tan(2\omega t) \right) \exp \left( \pm \frac{2}{\omega} \cos(2\omega t) \right) S_V(2\omega t) \varphi,
\]

\[
S_V(t) = \left(1 + \frac{2}{\omega} \tan(2\omega t) \right) \exp \left( \frac{2}{\omega} \cos(2\omega t) \right) S_V(2\omega t) \varphi.
\]

Proposition 2 implies that

\[
\exp(-it(P + \omega^2 |x|^2)) = S_V^+(t)
\]

\[
= m(-2\omega \tan(2\omega t)) S_V \left( \frac{\sin(2\omega t)}{2\omega} \right) m(-2\omega \tan(\omega t))
\]

\[
= m(-2\omega \tan(2\omega t)) d \left( \frac{1}{\cos(2\omega t)} \right) S_V \left( \frac{\tan(2\omega t)}{2\omega} \right), \quad |t| < \frac{\pi}{4\omega};
\]

\[
\exp(-it(P - \omega^2 |x|^2)) = S_V^-(t)
\]

\[
= m(2\omega \tanh(\omega t)) S_V \left( \frac{\sinh(2\omega t)}{2\omega} \right) m(2\omega \tanh(\omega t))
\]

\[
= m(2\omega \tanh(2\omega t)) d \left( \frac{1}{\cosh(2\omega t)} \right) S_V \left( \frac{\tanh(2\omega t)}{2\omega} \right),
\]

and \( D(P \pm \omega^2 |x|^2) \cap L^2(\mathbb{R}^N) \supset D(1 + P) \cap D(1 + |x|^2) \). We denote \( S_V^{\pm}(t) := \exp(-it(P \pm \omega^2 |x|^2)) \) (double-sign corresponds). In particular, we see that

\[
S_V^{\pm}(\xi^+_{-1}(t)) = m(-4\omega^2 t) d \left( \sqrt{1 + 4\omega^2 t^2} \right) S_V(t),
\]

\[
S_V^{\pm}(\xi^-_{-1}(t)) = m(4\omega^2 t) d \left( \sqrt{1 - 4\omega^2 t^2} \right) S_V(t).
\]

2.2. Definition and properties of new pseudo-conformal transforms. First, we consider the following inhomogeneous problems

\[
\begin{cases}
  i \frac{\partial u}{\partial t} = (P + \omega^2 |x|^2) u + f_+(t), \\
  u(0) = 0;
\end{cases}
\]

\[
\begin{cases}
  i \frac{\partial u}{\partial t} = Pu + f_0(t), \\
  u(0) = 0;
\end{cases}
\]

\[
\begin{cases}
  i \frac{\partial u}{\partial t} = Pu + f_0(t), \\
  u(0) = 0;
\end{cases}
\]
Proposition 3. Suppose the assumption in Lemma 2.1.

If \( u \) satisfies (2.14) then \( v = C_{-} \) satisfies (2.16).

Proof. We prove (i). The Duhamel principle and \( S_{\nu}^{+}(t) = A_{+}(\tan(\omega t)) \) yield

\[
u(t) = -iA_{+}(\tan(\omega t)) \int_{0}^{t} A_{+}(\tan(\omega s))^{-1} f_{+}(s) \, ds.
\]

Thus we see that

\[
u(t) = -iB(t) \int_{0}^{\xi_{+}^{-1}(t)} A_{+}(-\tan(\omega s)) f_{+}(s) \, ds,
\]

where

\[
B(t) := m \left( \frac{4\omega^{2}t}{1 + 4\omega^{2}t^{2}} \right) d\left( \frac{1}{\sqrt{1 + 4\omega^{2}t^{2}}} \right) A_{+}\left( \tan\left( \frac{\arctan(2\omega t)}{2} \right) \right).
\]

Applying (2.6) and (2.1), we see that \( B(t) = S_{\nu}(t) \). On the one hand, the integration can be substituted \( s = \xi_{+}^{-1}(\sigma) \). (2.6) implies

\[
\int_{0}^{\xi_{+}^{-1}(t)} A_{+}(\tan(\omega s))^{-1} f_{+}(s) \, ds = \int_{0}^{t} A_{+}\left( \tan\left( \frac{1}{2} \arctan(2\omega \sigma) \right) \right)^{-1} f_{+}(\xi_{+}^{-1}(\sigma)) \frac{d\sigma}{1 + 4\omega^{2}\sigma^{2}}
\]

\[
= \int_{0}^{t} S_{\nu}(-\sigma) d\left( \frac{1}{\sqrt{1 + 4\omega^{2}\sigma^{2}}} \right) m(4\omega^{2}\sigma) f_{+}(\xi_{+}^{-1}(\sigma)) \frac{d\sigma}{1 + 4\omega^{2}\sigma^{2}}
\]

\[
= \int_{0}^{t} S_{\nu}(-\sigma)(1 + 4\omega^{2}\sigma^{2})^{-1}(C_{+} f_{+})(\sigma) \, d\sigma.
\]
Thus we have  
\[ v(t) = -i \int_0^t S_V(t-s)(1 + 4\omega^2 s^2)^{-1}(C_+ f_+)(s) \, ds. \]
Hence \( v \) satisfies (2.15) with \( f_0(t) = (1 + 4\omega^2 t^2)^{-1}(C_+ f_+)(t) \). In a way similar to (i), we can prove (ii). \( \square \)

Proposition 3 implies that if \( u \) satisfies
\[
\begin{aligned}
& \begin{cases}
  i \frac{\partial u}{\partial t} = (P + \omega^2 |x|^2) u + f_+(t), \\
  u(0) = u_0,
\end{cases} \\
\end{aligned}
\]
then \( v = C_+ u \) satisfies
\[
\begin{aligned}
& \begin{cases}
  i \frac{\partial v}{\partial t} = P v + f_0(t), \\
  v(0) = v_0
\end{cases}
\tag{2.17}
\end{aligned}
\]
with \( v_0 := u_0 \) and \( f_0(t) := (1 + 4\omega^2 t^2)^{-1}(C_+ f_+)(t) \). Similarly, if \( u \) satisfies
\[
\begin{aligned}
& \begin{cases}
  i \frac{\partial u}{\partial t} = (P - \omega^2 |x|^2) u + f_-(t), \\
  u(0) = u_0,
\end{cases}
\end{aligned}
\]
then \( v = C_- u \) satisfies (2.17) with \( v_0 := u_0 \) and \( f_0(t) := (1 - 4\omega^2 t^2)^{-1}(C_- f_-)(t) \).

The pseudo-conformal transforms are also applicable to the scattering problems.

**Lemma 2.2.** Suppose the assumption in Lemma 2.1.

(i) Let \( v(t) = C_+ u(t) \). Then
\[
S_V(-t)v(t) = S_V(-t)(C_+ u)(t) = S_V^+(-\xi_+^{-1}(t)) u(\xi_+^{-1}(t));
\]

(ii) Let \( v(t) = C_- u(t) \). Then
\[
S_V(-t)v(t) = S_V(-t)(C_- u)(t) = S_V^-(-\xi_-^{-1}(t)) u(\xi_-^{-1}(t)).
\]

**Proof.** Applying (2.1) and (2.12), we see that
\[
S_V(-t)(C_+ u)(t)
= S_V(-t) d\left( \frac{1}{\sqrt{1 + 4\omega^2 t^2}} \right) m(4\omega^2 t) u(\xi_+^{-1}(t))
= S_V^+\left( \xi_+^{-1}(t) \right)^{-1} u(\xi_+^{-1}(t)).
\]
This is nothing but (i). (ii) is shown in a way similar to (i). \( \square \)

The transforms \( C_+ \) are applied to the semilinear Schrödinger evolution equations with the attractive/repulsive harmonic potential. Here we need to pay attention to the fact that the suitable space for the pseudo-conformal transform is not \( D \) (or \( C(I;D) \)) but \( \Sigma \) (or \( C(I;\Sigma) \)). In fact we see that \( C_+ \) is a bijective mapping from \( C(-\pi/(4\omega), \pi/(4\omega); \Sigma) \) to \( C(\mathbb{R}; \Sigma) \) and \( C_- \) is a bijective mapping from \( C(\mathbb{R}; \Sigma) \) to \( C(-1/(2\omega), 1/(2\omega); \Sigma) \). We consider the case for \( C_- \) to apply it to later arguments. Let \( v(s) = (C_- u)(s) \). Simple calculations imply
\[
x d(\beta) \varphi = \frac{1}{\beta} d(\beta)(x \varphi), \quad x m(\sigma) \varphi = m(\sigma)(x \varphi).
\]
Applying these we have \( x v(t) = \sqrt{1 - 4\omega^2 t^2} C_- (x u)(t) \) and
\[
\|x v(t)\|_{L^2(\mathbb{R}^N)}^2 = (1 - 4\omega^2 t^2) \|x u(\xi_-^{-1}(t))\|_{L^2(\mathbb{R}^N)}^2.
\tag{2.18} \]
Similarly, we see that
\[ \nabla d(\beta)\varphi = \beta d(\beta)(\nabla \varphi), \quad \nabla m(\sigma)\varphi = m(\sigma)\left(\nabla \varphi + \frac{i\sigma}{2} x \varphi\right). \]

We also obtain
\[ \nabla v(t) = \frac{1}{\sqrt{1 - 4\omega^2 t^2}} \left[ \mathcal{C}_-(\nabla u(t) - 2i\omega^2 t \mathcal{C}_-(xu(t)) \right]. \]

Thus we have
\[
\begin{align*}
\| P^{1/2} v(t) \|^2_{L^2(\mathbb{R}^N)} &= \frac{1}{1 - 4\omega^2 t^2} \left[ \| P^{1/2} u(\xi^{-1}(t)) \|^2_{L^2(\mathbb{R}^N)} + 4\omega^4 t^2 \| x u(\xi^{-1}(t)) \|^2_{L^2(\mathbb{R}^N)} \right. \\
&\quad \left. - 4\omega^2 t \text{Im} \int_{\mathbb{R}^N} x u(\xi^{-1}(t)) : \nabla u(\xi^{-1}(t)) \, dx \right]. \tag{2.19}
\end{align*}
\]

\[
\begin{align*}
\| P^{1/2} u(t) \|^2_{L^2(\mathbb{R}^N)} &= (1 - 4\omega^2 \xi_-(t)^2) \| P^{1/2} v(\xi_-(t)) \|^2_{L^2(\mathbb{R}^N)} \\
&\quad + \frac{4\omega^4 \xi_-(t)^2}{1 - 4\omega^2 \xi_-(t)^2} \| x v(\xi_-(t)) \|^2_{L^2(\mathbb{R}^N)} \\
&\quad + 4\omega^2 \xi_-(t) \text{Im} \int_{\mathbb{R}^N} x \bar{v}(\xi_-(t)) \cdot \nabla v(\xi_-(t)) \, dx. \tag{2.20}
\end{align*}
\]

Hence (2.19) implies that if \( u \in C(\mathbb{R}; \Sigma) \), then \( v = \mathcal{C}_- u \in C(-1/(2\omega), 1/(2\omega); \Sigma) \), and if \( v \in C(-1/(2\omega), 1/(2\omega); \Sigma) \), then \( u = \mathcal{C}_-^{-1} v \in C(\mathbb{R}; \Sigma) \). These conclusions are broken down if we replace \( \Sigma \) by \( \mathcal{D} \). Moreover, Lemma 2.2 implies that the following two equivalences are completed (double-sign corresponds):

\[
\begin{align*}
\mathcal{S}_V^+ \left( \pm \frac{\pi}{4\omega} \right) u(\pm \frac{\pi}{4\omega}) &= \lim_{t \to \pm \pi/(4\omega) - 0} \mathcal{S}_V^\pm(-t) u(t) = u_\pm \\
\xleftrightarrow{s \to \pm \infty} \mathcal{S}_V(-s)v(s) u_\pm = u_\pm, \tag{2.22}
\end{align*}
\]

\[
\begin{align*}
\lim_{t \to \pm \infty} \mathcal{S}_V^\pm(-t) u(t) &= u_\pm \\
\xleftrightarrow{s \to \pm \infty} \mathcal{S}_V \left( \pm \frac{1}{2\omega} \right) u(\pm \frac{1}{2\omega}) = \lim_{s \to \pm (1/(2\omega) - 0)} \mathcal{S}_V(-s)v(s) u_\pm = u_\pm, \tag{2.23}
\end{align*}
\]

strongly in \( L^2(\mathbb{R}^N) \) or \( \Sigma \) (not in \( \mathcal{D} \)). (2.22) seems to be applicable to the nonlinear scattering problems without the harmonic oscillation. We apply (2.23) in the later section.

In the end of this section, we give one of the applications of another representation for \( \exp(-it(P + \omega^2|x|^2)) \). Since we can specify the propagators \( \mathcal{S}_V^\pm(t) = \exp(-it(P + \omega^2|x|^2)) \), we consider the commutator for the linear term of \( (\text{NLS})^+ \) and \( (\text{NLS})^- \): \( i \partial_t - P + \omega^2|x|^2 \). Define \([A, B] := AB - BA\).
Let $S$ be a densely defined selfadjoint operator on the complex Hilbert space $X$, and $A$ be a closed operator on $X$. Then we see that
\[
\frac{d}{dt}[\exp(-itS)A\exp(itS)] = -i\exp(-itS)[S, A]\exp(+itS).
\]
Here we have\[
\begin{align*}
[P, |x|^2/4] &= \begin{bmatrix} -\Delta + V, |x|^2/4 \end{bmatrix} = \begin{bmatrix} -\Delta, |x|^2/4 \end{bmatrix} = -\frac{N}{2} - x \cdot \nabla, \\
[P, x \cdot \nabla] &= \begin{bmatrix} -\Delta + V, x \cdot \nabla \end{bmatrix} = -2\Delta + 2V = 2P.
\end{align*}
\]
Thus we see that\[
\begin{align*}
\frac{d}{dt}\exp(-itP)|x|^2/4\exp(+itP) &= (-i)\exp(-itP)\begin{bmatrix} P, |x|^2/4 \end{bmatrix}\exp(+itP) \\
&= i\exp(-itP)\begin{bmatrix} N/2 + x \cdot \nabla \end{bmatrix}\exp(+itP), \\
\frac{d^2}{dt^2}\exp(-itP)|x|^2/4\exp(+itP) &= (-i)i\exp(-itP)\begin{bmatrix} P, N/2 + x \cdot \nabla \end{bmatrix}\exp(+itP) \\
&= \exp(-itP)(2P)\exp(+itP) = 2P.
\end{align*}
\]
Thus we obtain\[
\begin{align*}
\mathcal{L}_0(t) := \exp(-itP)|x|^2/4\exp(+itP) &= \mathcal{L}_0(0) + \mathcal{L}_0'(0) t + \frac{1}{2}\mathcal{L}_0''(0) t^2 \\
&= \frac{|x|^2}{4} + it \begin{bmatrix} N/2 + x \cdot \nabla \end{bmatrix} + t^2 P = \left(\frac{x}{2} + it \nabla\right)^2 + t^2 V.
\end{align*}
\]
$\mathcal{L}_0(t)$ is commuted with $i\partial_t - P$. Moreover, $\mathcal{L}_0(t)$ is nonnegative operator and we see that
\[
\mathcal{L}_0(t) = t^2m(1/t)Pm(-1/t), \quad \mathcal{L}_0(t)^{1/2} = |t|m(1/t)P^{1/2}m(-1/t).
\]
Note that $\mathcal{L}_0(t)^{1/2} = \exp(-itP)(|x|/2)\exp(+itP)$. Moreover, we have\[
\begin{align*}
\mathcal{L}_0(t)\mathcal{S}_V(s) &= \mathcal{S}_V(s)\mathcal{L}_0(t-s), \quad \mathcal{L}_0(t)^{1/2}\mathcal{S}_V(s) = \mathcal{S}_V(s)\mathcal{L}_0(t-s)^{1/2}.
\end{align*}
\]
Especially, we can obtain\[
\begin{align*}
\|\mathcal{L}_0(t)^{1/2}\varphi\|_{L^2(\mathbb{R}^N)}^2 &= \left\| \frac{|x|^2}{2} \exp(itP)\varphi \right\|_{L^2(\mathbb{R}^N)}^2 \\
&= \int_{\mathbb{R}^N} \left[ \left(\frac{x}{2} + it \nabla\right)\varphi \right]^2 + t^2 V|\varphi|^2 \ dx \\
&= \frac{1}{4}\|x|\varphi\|_{L^2(\mathbb{R}^N)}^2 + t^2 \|P^{1/2}\varphi\|_{L^2(\mathbb{R}^N)}^2 - t\text{Im} \int_{\mathbb{R}^N} x\varphi \cdot \nabla \varphi \ dx.
\end{align*}
\]
Now we consider the commutator respect to $P \pm \omega^2|x|^2$. We see from (2.10) and (2.11) that\[
\begin{align*}
\mathcal{L}_0^+(t) := \exp(-it(P + \omega^2|x|^2))|x|^2/4 \exp(+it(P + \omega^2|x|^2)) \\
&= \mathcal{S}_V^+(t)^{-1}|x|^2/4 \mathcal{S}_V^+(t) = \mathcal{S}_V(\xi_+(t))|x|^2/4 \cos^2(2\omega t)\mathcal{S}_V(-\xi_+(t)) \\
&= \cos^2(2\omega t)\mathcal{L}_0(\xi_+(t)), \quad |t| < \frac{\pi}{4\omega} \\
\mathcal{L}_0^-(t) := \exp(-it(P - \omega^2|x|^2))|x|^2/4 \exp(+it(P - \omega^2|x|^2)) \\
&= \cosh^2(2\omega t)\mathcal{L}_0(\xi_-(t)).
\end{align*}
\]
Note that \( \mathcal{L}_\pm(t) \) is commuted with \( i \partial_t - P \mp \omega^2 |x|^2 \) (double-sign corresponds). In this article, we only apply the commutators \( \mathcal{L}_0(t) \). We can also consider the relations between \( \mathcal{C}_\pm \) (pseudo-conformal transform) and \( \mathcal{L}_\pm \) (commutator).

\[
\mathcal{C}_+(\mathcal{L}_+(t)u(t)) = \mathcal{L}_0(t)(\mathcal{C}_+u(t)),
\]
\[
\mathcal{C}_-(\mathcal{L}_-(t)u(t)) = \mathcal{L}_0(t)(\mathcal{C}_-u(t)).
\]

The right hand sides of (2.27) and (2.28) are computed as follows with applying (2.24), (2.1), and \( Pd(\beta) = \beta^2 d(\beta) P \).

\[
\mathcal{L}_0(t)(\mathcal{C}_\pm u(t))
= t^2 m(1/t)Pm(-1/t)m\left(\frac{\pm 4\omega^2 t}{1 \pm 4\omega^2 t^2}\right)d\left(\frac{1}{\sqrt{1 \pm 4\omega^2 t^2}}\right)u(\xi_\pm^{-1}(t))
= t^2 m(1/t)\frac{1}{1 \pm 4\omega^2 t^2}d\left(\frac{1}{\sqrt{1 \pm 4\omega^2 t^2}}\right)Pm(-1/t)u(\xi_\pm^{-1}(t))
= \frac{1}{1 \pm 4\omega^2 t^2}m\left(\frac{\pm 4\omega^2 t}{1 \pm 4\omega^2 t^2}\right)d\left(\frac{1}{\sqrt{1 \pm 4\omega^2 t^2}}\right)\mathcal{L}_0(t)u(\xi_\pm^{-1}(t)).
\]

On the other hand, the left hand side of (2.27) is calculated that

\[
\mathcal{C}_+(\mathcal{L}_+(t)u(t))
= m\left(\frac{4\omega^2 t}{1 \pm 4\omega^2 t^2}\right)d\left(\frac{1}{\sqrt{1 \pm 4\omega^2 t^2}}\right)\mathcal{C}_+(\xi_+^{-1}(t))u(\xi_+^{-1}(t))
= m\left(\frac{4\omega^2 t}{1 \pm 4\omega^2 t^2}\right)d\left(\frac{1}{\sqrt{1 \pm 4\omega^2 t^2}}\right)\cos^2(2\omega \xi_+^{-1}(t))\mathcal{L}_0(t)u(\xi_+^{-1}(t))
= \frac{1}{1 \pm 4\omega^2 t^2}m\left(\frac{4\omega^2 t}{1 \pm 4\omega^2 t^2}\right)d\left(\frac{1}{\sqrt{1 \pm 4\omega^2 t^2}}\right)\mathcal{L}_0(t)u(\xi_+^{-1}(t)).
\]

Here \( \mathcal{L}_+(t) = \cos^2(2\omega t)\mathcal{L}_0(\xi_+(t)) \) is applied. Similarly, it follows from \( \mathcal{C}_-(t) = \cosh^2(2\omega t)\mathcal{L}_0(\xi_-(t)) \) that

\[
\mathcal{C}_-(\mathcal{L}_-(t)u(t))
= m\left(\frac{-4\omega^2 t}{1 \pm 4\omega^2 t^2}\right)d\left(\frac{1}{\sqrt{1 \pm 4\omega^2 t^2}}\right)\mathcal{C}_-(\xi_-^{-1}(t))u(\xi_-^{-1}(t))
= \frac{1}{1 \pm 4\omega^2 t^2}m\left(\frac{-4\omega^2 t}{1 \pm 4\omega^2 t^2}\right)d\left(\frac{1}{\sqrt{1 \pm 4\omega^2 t^2}}\right)\mathcal{L}_0(t)u(\xi_-^{-1}(t)).
\]

Thus we conclude (2.27) and (2.28).

2.3. Applications the pseudo-conformal symmetry to the Strichartz estimates. The pair \((\tau, \rho)\) is said to be an admissible pair if

\[
\frac{2}{\tau} + \frac{N}{\rho} = \frac{N}{2}, \quad \tau \geq 2, \quad \rho \geq 2.
\]

Proposition 3 yields the Strichartz estimates for \( \exp(-it(P \pm \omega^2 |x|^2)) \). \( L^r(I; L^p) \) is an abbreviation of \( L^r(I; L^p(\mathbb{R}^N)) \) for simple notations.

**Theorem 2.3.** Suppose the assumption in Lemma 2.1.

(a) Assume that \( P \) satisfies the short-time homogeneous Strichartz estimate with the admissible pair \((\tau, \rho)\), that is,

\[
\| \exp(-itP)\varphi \|_{L^r(-T,T; L^p)} \leq C(\tau, T) \| \varphi \|_{L^2(\mathbb{R}^N)}.
\]

(2.29)
Then $P + \omega^2|x|^2$ satisfies the short-time homogeneous Strichartz estimate with the same pair $(\tau, \rho)$:

$$\|\exp(-it(P + \omega^2|x|^2))\varphi\|_{L^r_t(-\tau,\tau;L^q_x)} \leq C'(\tau, T) \|\varphi\|_{L^2(\mathbb{R}^N)}. \quad (2.30)$$

Moreover, $P - \omega^2|x|^2$ satisfies the global-in-time homogeneous Strichartz estimate with the same pair $(\tau, \rho)$:

$$\|\exp(-it(P - \omega^2|x|^2))\varphi\|_{L^r_t(\mathbb{R};L^q_x)} \leq C'(\tau) \|\varphi\|_{L^2(\mathbb{R}^N)}; \quad (2.31)$$

(b) Assume that $P$ satisfies the short-time inhomogeneous Strichartz estimate with the admissible pairs $(\tau_1, \rho_1)$ and $(\tau_2, \rho_2)$, that is,

$$\left\| \int_0^t \exp(-i(t-s)(P + \omega^2|x|^2))\Phi(s)\,ds \right\|_{L^r_t(-\tau,T;L^q_x)} \leq C'(\tau_1,\tau_2) \|\Phi\|_{L^r_t(-\tau,T;L^q_x)}, \quad (2.33)$$

Then $P + \omega^2|x|^2$ satisfies the short-time inhomogeneous Strichartz estimates with the same pairs $(\tau_1, \rho_1)$ and $(\tau_2, \rho_2)$:

$$\left\| \int_0^t \exp(-i(t-s)(P + \omega^2|x|^2))\Phi(s)\,ds \right\|_{L^r_t(-\tau,T;L^q_x)} \leq C'(\tau_1,\tau_2) \|\Phi\|_{L^r_t(-\tau,T;L^q_x)}, \quad (2.34)$$

Proof. First we remark the integrations. Simple calculation implies

$$\int_0^t \exp(-i(t-s)P)\Phi(s)\,ds \leq C(\tau_1,\tau_2,T) \|\Phi\|_{L^r_t(-\tau,T;L^q_x)}. \quad (2.32)$$

Moreover, if $\xi \in C^1(a,b;\mathbb{R})$ with $\xi'(t) > 0$, then we have

$$\|\xi'(t)^{1/q}F(g(t))\|_{L^q(a,b;X)} = \|F(t)\|_{L^q(\xi(\xi(t));X)}.$$
Now we solve tentials. Theorem 3.3 showed that for $0 < s < \frac{3}{2}$ Burq–Planchon–Stalker–Tahvildar-Zadeh [2] mentioned that

$$\int_0^t \left\| \exp(-i(t-s)(P + \omega^2 |x|^2)) \Phi(s) \right\|_{L^\infty(-T,T;L^\infty)_{\omega^2}} ds = \left\| (\cos(2\omega t))^{-N/2-N/\rho_2} \int_0^t S_V(\xi(t) - s) (C_+ \Phi)(s) \right\|_{L^\sigma(-T,T;L^\sigma)_{\omega^2}} $$

Thus we see that

$$\int_0^t \left\| \exp(-i(t-s)(P + \omega^2 |x|^2)) \Phi(s) \right\|_{L^\infty(-T,T;L^\infty)_{\omega^2}} ds = \left\| (\cos(2\omega t))^{-N/2-N/\rho_2} \int_0^t S_V(\xi(t) - s) (C_+ \Phi)(s) \right\|_{L^\sigma(-T,T;L^\sigma)_{\omega^2}} $$

$$= C(\tau_1, \tau_2, \xi(T)) \left\| m\left(\frac{4\omega^2 t}{1 + 4\omega^2 t^2} \right) \Phi(\xi^{-1}(t)) \right\|_{L^1(I_+(T);L^1)_{\omega^2}}$$

$$= C(\tau_1, \tau_2, \xi(T)) \left\| m\left(\frac{4\omega^2 t}{1 + 4\omega^2 t^2} \right) \Phi(\xi^{-1}(t)) \right\|_{L^1(I_+(T);L^1)_{\omega^2}}$$

where $I_+(T) := (-\xi(T), \xi(T))$. Extending the interval step by step, we obtain (2.33). In a way similar way to the above, (2.34) is confirmed by Proposition 3 (ii) and the representation of $C^{-1}$. \hfill $\Box$

### 3. Global existence and scattering for NLS with repulsive harmonic potentials

Now we solve (NLS). Assume that $V$ satisfies (1.1) and (1.2). Then Burq–Plancho–Stalker–Tahvildar-Zadeh [2] mentioned that

$$\delta V = \frac{\delta V}{\delta V + \|V\|_{L^\infty(S^{N-1})}} \| (1 - \Delta)^{1/2} u \|^2_{L^2(\mathbb{R}^N)}$$

$$\leq \left\| (1 + P)^{1/2} u \right\|^2_{L^2(\mathbb{R}^N)}$$

$$\leq \left(1 + \frac{\|V\|_{L^\infty(S^{N-1})}}{(N-2)^2} \right) (1 - \Delta)^{1/2} u \right\|^2_{L^2(\mathbb{R}^N)}.$$ (3.1)

This implies that $D = D((1 + P)^{1/2}) \subseteq H^1(\mathbb{R}^N)$ if $\delta V > 0$. Moreover, [2] proved that $P$ satisfies the global-time homogeneous Strichartz estimate (2.29) with $C(\tau, \infty) < +\infty$ and the global-time inhomogeneous estimate (2.32) with $C(\tau_1, \tau_2, \infty) < +\infty$. On the other hand, if $\delta V = 0$, then $D = D((1 + P)^{1/2}) \supseteq H^1(\mathbb{R}^N)$. Suzuki [17, Theorem 3.3] showed that for $0 < s < 1$

$$\left\| (-\Delta)^{s/2} u \right\|^2_{L^2(\mathbb{R}^N)} \leq C(V, s) \| P^{s/2} u \|^2_{L^2(\mathbb{R}^N)},$$ (3.2)

where

$$C(V, s) := \frac{\Gamma((1 + s + \|V\|_{L^\infty(S^{N-1})})/2)}{\Gamma((1 - s)/2)} \frac{\Gamma((1 - s)/2)}{\Gamma((1 + s)/2)} \Gamma((1 + s)/2).$$

and the global-time Strichartz estimates (2.29) and (2.32) without the endpoint ($\tau = 2$).

Let $g_0(u) = f(t, x, u)$ be a local nonlinearity with gauge invariant as like $\lambda |u|^{p-1} u$, that is, $f(t, x, u) \in C$ if $t \in \mathbb{R}$, $x \in \mathbb{R}^N$, and $u \in C$ with $f(t, x, e^{i\theta} u) = e^{i\theta} f(t, x, u)$ ($\theta \in \mathbb{R}$). Here we see that

$$d(\beta) f(t, x, u(x)) = \frac{\beta N}{2} f(t, \beta x, \beta^{-N/2} d(\beta) u(x)).$$
We consider the energy functional of \( f(t, x, u(x)) \):

\[
F(t, u) := \int_{\mathbb{R}^N} F(t, x, u(x)) \, dx,
\]
where \( F(t, x, u) := \int_0^{|u|} f(t, x, \sigma) \, d\sigma. \)

Then we have

\[
F(t, \partial_t u) = \int_{\mathbb{R}^N} F(t, x, \partial_t^N u(x)) \, dx = \beta^{-N} \int_{\mathbb{R}^N} F(t, \beta^{-1} y, \partial_t^N u(y)) \, dy.
\]

Especially, if \( f(t, x, u) \) does not depend on \( x \), then \( F(t, \partial_t u) = \beta^{-N} F(t, \partial_t^N u) \).

Thus we conclude that if \( v := C \_ u \), then

\[
F(t, v(t)) = (1 - 4\omega^2 t^2)^{N/2} F(t, (1 - 4\omega^2 t^2)^{-N/4} u(\xi^{-1}(t))).
\]

If \( u \) solves \((\text{NLS})^-\) with \( g(u) = f(t, x, u(t, x)) \) as above, then \( v = C \_ u \) solves

\[
\begin{cases}
i \frac{\partial v}{\partial t} = (-\Delta + V)v + \bar{g}_0(t, v), \\
v(0) = u_0,
\end{cases}
\]

where \( \bar{g}_0 \) is defined by

\[
\bar{g}_0(t, v) = (1 - 4\omega^2 t^2)^{-1-\gamma/4} f\left(\frac{\xi^{-1}(t)}{\sqrt{1 - 4\omega^2 t^2}}, \frac{x}{(1 - 4\omega^2 t^2)^{N/4}} v(t, x)\right).
\]

Especially, let \( g_0(u) = \lambda \text{sgn}(u) \). Then

\[
\bar{g}_0(t, v) = \lambda(1 - 4\omega^2 t^2)^{-1+\gamma/2+N(p-1)/4} |\text{sgn}(u)|^{p-1} v.
\]

Thus \((\text{NLS})^-\) is turned into a nonautonomous problem \((3.4)\).

### 3.1. Local existence and uniqueness.
Considering the converted problem \((3.4)\), we see that \((\text{NLS})^-\) can be solved. Since \( P - \omega^2 |x|^2 \) is not nonnegative, we cannot consider the energy space for \( P - \omega^2 |x|^2 \). Thus we cannot apply the energy methods directly to \((\text{NLS})^-\). This is the reason to the application for the transformation from \((\text{NLS})^-\) to the nonautonomous problems.

**Theorem 3.1.** Let \( g_0(u) \) be the local nonlinearity which is satisfied

1. **(L1)** \( g_0(0) = 0 \) and there exist \( C > 0 \) and \( 1 < p < 1 + 4/(N - 2) \) such that

\[
|g_0(z_2) - g_0(z_1)| \leq C(1 + |z_1|^{p-1} + |z_2|^{p-1}) |z_2 - z_1| \quad \forall z_1, z_2 \in \mathbb{C};
\]

2. **(L2)** \( g(x) \in \mathbb{R} \) for all \( x \in \mathbb{R} \) and \( g_0(e^{it} z) = e^{it} g_0(z) \) for all \( \theta \in \mathbb{R} \) and \( z \in \mathbb{C} \).

Then for every \( u_0 \in \Sigma \), there uniquely exists a local solution \( u \in C(\bar{T}; \Sigma) \cap C^1(\bar{T}; \mathcal{D}^*) \) to \((\text{NLS})^-\). Moreover, \( u \) conserves the charge and energy:

\[
\|u(t)\|_{L^2(\mathbb{R}^N)} = \|u_0\|_{L^2(\mathbb{R}^N)}, \quad E(u(t)) = E(u_0) := \frac{1}{2}\|P^{1/2} u_0\|_{L^2(\mathbb{R}^N)}^2 - \frac{\omega^2}{2} \|\text{sgn}(u_0)\|_{L^2(\mathbb{R}^N)}^2 + G_0(u_0).
\]

Furthermore, \( u \) satisfies the virial identity:

\[
\frac{d^2}{dt^2} \|x |u(t)| \|_{L^2(\mathbb{R}^N)}^2 = 8 \|P^{1/2} u(t)\|_{L^2(\mathbb{R}^N)}^2 + 8\omega^2 \|\text{sgn}(u(t))\|_{L^2(\mathbb{R}^N)}^2 - 8N G_0(u(t)) + 4N G_1(u(t)).
\]

Here we have set

\[
F_0(z) := \int_0^{|z|} g_0(s) \, ds, \quad F_1(z) := g_0(z) \, z,
\]
SEMILINEAR SCHRODINGER EQUATIONS WITH $|x|^{-2}$ AND $|x|^2$

We prepare for the energy methods for nonautonomous semilinear Schrödinger equations established by Suzuki [16, Theorem 2.1].

$$G_j(u) := \int_{\mathbb{R}^N} F_j(u(x))\,dx \quad (j = 0, 1).$$

where $S$ is nonnegative and selfadjoint in the (complex) Hilbert space $X$. $X := D((1 + S)^{1/2})$ and $X^* := D((1 + S)^{-1/2})$ is the dual space of $X$. Here we see the triplet $X_S := X = X^* \subset X^*_S$. We give assumption for the nonlinearity $g : [-T,T] \times X_S \rightarrow X^*_S$. For the simple notation, we use $B_M := \{u \in X_S; \|u\|_{X_S} \leq M\}$.

(A1) Existence of energy functional: there exists $G \in C([-T,T] \times X_S; \mathbb{R})$ whose real Fréchet differential $d_{X_S}G(t,u)$ is exactly $g(t,u)$, that is, given $u \in X_S$ and $t \in [-T,T]$ for every $\varepsilon > 0$ there exists $\delta = \delta(u,\varepsilon) > 0$ such that

$$\|G(t,u + \varepsilon) - G(t,u) - \text{Re} \langle g(t,u), v \rangle_{X^*_S, X_S} \| \leq \varepsilon \|v\|_{X_S} \quad \forall v \in B_M;$$

(A2) Lipschitz continuity of $g$ in $u$:

$$\|g(t,u) - g(t,v)\|_{X_S} \leq C(M)\|u - v\|_{X_S} \quad \forall t \in [-T,T], \forall u, v \in B_M;$$

(A3) Hölder-like continuity of $g$ in $t$: there exists $\varphi \in L^1(-T,T)$ with $\varphi(t) \geq 0$ such that

$$\|g(t,u) - g(s,u)\|_{X_S} \leq C(M) \int_s^t \varphi(\sigma)\,d\sigma \quad \forall t, s \in [-T,T], \forall u \in B_M;$$

(A4) Hölder-like continuity of $G$: given $M > 0$, for all $\delta > 0$ there exists a constant $C_{1,\delta}(M) > 0$ such that

$$\|G(t,u) - G(t,v)\| \leq \delta + C_{1,\delta}(M)\|u - v\|_X \quad \forall t \in [-T,T], \forall u, v \in B_M;$$

(A5) Hölder type continuity of $G$: $G(t,u)$ is partially differentiable in $t$ for every $u \in X_S$. Moreover, there exists $\varphi \in L^1(-T,T)$, and for any $M > 0$ and $\delta > 0$ there exists a constant $C_{2,\delta}(M) > 0$ such that

$$\|G(t,u) - G(t,v)\| \leq \varphi(t)\|u - v\|_X \quad \text{a.a. } t \in (-T,T), \forall u \in B_M;$$

(A6) Gauge condition:

$$\text{Re} \langle g(t,u), iu \rangle_{X^*_S, X_S} = 0 \quad \forall t \in [-T,T], \forall u \in X_S;$$

(A7) Closedness condition: let $I \subset (-T,T)$ be an open interval. Assume that $\{w_n\}_n$ is any bounded sequence in $L^\infty(-T,T; X_S)$ such that

$$\left\{ \begin{array}{l}
  w_n(t) \rightarrow w(t) \quad (n \rightarrow \infty) \quad \text{weakly in } X_S \quad \text{a.a. } t \in I, \\
  g(t, w_n) \rightarrow f \quad (n \rightarrow \infty) \quad \text{weakly}^* \quad \text{in } L^\infty(I; X^*_S).
\end{array} \right.$$ Then

$$\text{Re} \int_I \langle f(t), iw(t) \rangle_{X^*_S, X_S}\,dt = \lim_{n \rightarrow \infty} \text{Re} \int_I \langle g(t, w_n(t)), iw_n(t) \rangle_{X^*_S, X_S}\,dt.$$ Here $f(t) = g(t, w(t))$ is guaranteed if $w_n(t) \rightarrow w(t) \quad (n \rightarrow \infty)$ strongly in $X$ a.a. $t \in I$.

(A8) there exists $\varepsilon > 0$ such that

$$G(t,u) \geq -((1-\varepsilon)/2) \| (1 + S)^{1/2}u \|_X^2 - C(\|u\|_X) \quad \forall t \in [-T,T], \forall u \in X_S;$$
there exists \( \psi \in L^1(-T,T) \) with \( \psi(t) \geq 0 \) such that
\[
\text{sgn}(t) G_t(t,u) \leq \psi(t) \left[ ||(1+S)^{1/2}u||^2_X + C(||u||_X) \right] \quad \text{a.a. } t \in (-T,T), \forall u \in X_S.
\]

**Theorem 3.2.** (i) (see [16, Theorem 2.1 and Remark 2.8]) Assume that (A1)–(A7) and \( u_0 \in X_S \). Then there exist \( T_0 \in (0,T] \) (dependent on \( ||u_0||_{X_S} \)) and \( u \in C_w([-T_0,T_0];X_S) \cap W^{1,\infty}(-T_0,T_0;X_S^*) \) such that \( u \) is a solution to (ACP).

The solution satisfies the pseudo-conservation laws:
\[
E(t,u(t)) - E(0,u_0) \leq \int_0^t G_t(s,u(s)) \, ds \quad \forall t \in [-T_0,T_0],
\]
where \( E(t,\varphi) = (1/2)|| (1+S)^{1/2} \varphi||^2_X + G(t,\varphi) \). If the solution is unique, (3.8) is strictly equal and \( u \in C([-T_0,T_0];X_S) \cap C^1([-T_0,T_0];X_S^*) \).

(ii) (see [16, Theorem 2.7]) Assume (A1)–(A9). Then the local solution \( u \in C_w([-T_0,T_0];X_S) \cap W^{1,\infty}(-T_0,T_0;X_S^*) \) to (ACP) as in (i) can be extended globally in time \( t \in [-T,T] \).

We divide the proof of Theorem 3.1 into 3 stages as below. We have some remarks of the relation between \( u \) and \( v := C_-u \). The fact that \( u \in C([-T_1,T_2];\Sigma) \) solves (NLS)– is turned into that \( v \in C([-\xi_-(T_1),\xi_+(T_2)];\Sigma) \) solves (3.4). Also, if \( u \) is a global solution, that is, \( u \in C(\mathbb{R};\Sigma) \), then \( v \) belongs to \( C(-1/(2\omega),1/(2\omega);\Sigma) \), continuous on the open interval \((-1/(2\omega),1/(2\omega))\); the reverse is also true.

**Stage 1.** We solve (3.4) via the energy methods (Theorem 3.2 (i)). To end this we verify the conditions (A1)–(A7) with \( T = 1/(2\omega) - \varepsilon \) for sufficiently small \( \varepsilon > 0 \), and the uniqueness. In this paper, we only describe the check (A7) and the uniqueness.

**Stage 2.** We show the continuity of \( v(t) \) in \( \Sigma \) and the virial identity. Note that the virial identities are also verified even if the nonlinearities are non-autonomous (dependent on the time variable \( t \)). In this paper, we only comment the ideas for the proof of the virial identities and present the summary.

**Stage 3.** We recur to \( u \) of the solution to (NLS)–. Especially, we confirm the energy conservation (3.6) and the virial identity (3.7).

**Stage 1 for proof of Theorem 3.1.** If \( u \) satisfies (NLS)–, then \( v = C_-u \) satisfies (3.4) with \( v_0 := u_0 \) and
\[
g_0(t,v) := (1 - 4\omega^2t^2)^{-1-N/4} g_0((1 - 4\omega^2t^2)^{N/4} v).
\]

First we solve (3.4) by applying the energy methods (Theorem 3.2). To end this, we set \( S := P = -\Delta + V, \) and
\[
g(t,v) := g_0(t,v), \quad G(t,v) := (1 - 4\omega^2t^2)^{-1-N/2} G_0((1 - 4\omega^2t^2)^{N/4} v).
\]

We check (A1)–(A7) with \( T = 1/(2\omega) - \varepsilon \) for sufficiently small \( \varepsilon > 0 \). We omit to describe the verification (A1)–(A6), since the non-degenerateness of \( 1 - 4\omega^2t^2, \) the simple calculations, and applying the Sobolev type inequality (see (3.1), (3.2), and the continuous inclusion \( H^s \subset L^{2N/(N-2s)} \))
\[
||u||_{L^p(\mathbb{R}^N)} \leq C(p, P) ||u||^{N/(1-2/p)}_{L^2(\mathbb{R}^N)} ||u||^{N/(2-1/p)}_D \quad \forall u \in D
\]
for \( 2 \leq p \leq 2N/(N-2) \) if \( \delta_V > 0 \) and \( 2 \leq p < 2N/(N-2) \) if \( \delta_V = 0 \). Also, especially for (A5), we see that
\[
G_t(t,v) = -8\omega^2t(1 - 4\omega^2t^2)^{-2-N/2}
\]
Thus we only show (A7). Note that we can divide \(g_0\) into \(g_1 + g_2\) so that \(g(0) = 0\) for \(l = 1, 2\) and
\[
|g_1(z_2) - g_1(z_1)| \leq K_1 |z_2 - z_1| \quad \forall z_1, z_2 \in \mathbb{C},
\]
\[
|g_2(z_2) - g_2(z_1)| \leq K_2 (|z_1|^{p-1} + |z_2|^{p-1}) |z_2 - z_1| \quad \forall z_1, z_2 \in \mathbb{C}.
\]
Additionally, we define
\[
\tilde{g}_l(t, v) := (1 - \omega^2 t^2)^{-1-N/4} g_l((1 - \omega^2 t^2)^{N/4}) (l = 1, 2).
\]
Thus we obtain for any \(v_1, v_2 \in \mathbb{C}\) and \(t \in (-1/(2\omega), 1/(2\omega))\)
\[
|\tilde{g}_1(t, v_2) - \tilde{g}_1(t, v_1)| \leq K_1 (1 - \omega^2 t^2)^{-1} |v_2 - v_1|,
\]
\[
|\tilde{g}_2(t, v_2) - \tilde{g}_2(t, v_1)| \leq K_2 (1 - \omega^2 t^2)^{-1+N(p-1)/4} (|v_1|^{p-1} + |v_2|^{p-1})
\]
\[
\times |v_2 - v_1|.
\]
Now let \(\{w_n\}_n\) be a (uniformly) bounded subsequence in \(L^\infty(I; \mathcal{D})\), where \(I\) is an open interval with \(\overline{I} \subset (-1/(2\omega), 1/(2\omega))\). Assume that there exists \(\tilde{f}(t) \in L^\infty(I; \mathcal{D})\) such that
\[
\begin{aligned}
& w_n \to w (n \to \infty) \text{ weakly in } \mathcal{D} \ a.a. t \in I, \\
& \tilde{g}(t, w_n) \to \tilde{f}(t) (n \to \infty) \text{ weakly* in } L^\infty(I; \mathcal{D}^*).
\end{aligned}
\]
Then it suffices to show that \(\tilde{f}(t) = \tilde{g}(t, w)\). To prove this let \(\Omega \subset \mathbb{R}^N\) be an arbitrary bounded domain with \(C^1\)-boundary. It follows from the Rellich compactness theorem (see e.g. [14, Lemma 4.5]): the inclusion \(\mathcal{D} \subset L^p(\Omega)\) is compact if \(2 \leq p < 2N/(N-2)\) that
\[
\tilde{w}_n(t) \to \tilde{w}(t) (n \to \infty) \text{ strongly in } L^2(\Omega) \text{ and } L^{p+1}(\Omega) \ a.a. t \in I.
\]
Hence (3.10), (3.11), and Sobolev embeddings (3.9) yield that the strong convergence \(\tilde{g}_1(t, w_n(t)) \to \tilde{g}_1(t, w(t)) (n \to \infty)\) in \(L^2(\Omega)\) a.a. \(t \in I\), and \(\tilde{g}_2(t, w_n(t)) \to \tilde{g}_2(t, w(t)) (n \to \infty)\) in \(L^{1+1/p}(\Omega)\). Note that since \(I \subset (-/(2\omega), 1/(2\omega))\) and the uniform boundedness yields the limits are strong and uniform convergences. Applying the dominated convergence theorem, we can see that for any \(\chi(t) \in L^1(I; L^2(\Omega))\)
\[
\int_I \int_\Omega \tilde{g}_1(t, w_n(t)) \chi(t) \, dx \, dt \to \int_I \int_\Omega \tilde{g}_1(t, w(t)) \chi(t) \, dx \, dt \quad (n \to \infty),
\]
and that for any \(\chi(t) \in L^1(I; L^{p+1}(\Omega))\)
\[
\int_I \int_\Omega \tilde{g}_2(t, w_n(t)) \chi(t) \, dx \, dt \to \int_I \int_\Omega \tilde{g}_2(t, w(t)) \chi(t) \, dx \, dt \quad (n \to \infty).
\]
These yield the weak* convergences of \(\tilde{g}_1(t, w_n(t)) \to \tilde{g}_1(t, w(t)) (n \to \infty)\) in \(L^\infty(I; L^2(\Omega))\) and \(\tilde{g}_2(t, w_n(t)) \to \tilde{g}_2(t, w(t)) (n \to \infty)\) in \(L^\infty(I; L^{1+1/p}(\Omega))\).

On the other hand, the uniform boundedness of \(w_n(t)\) implies that there exist \(\tilde{f}_1 \in L^\infty(I; L^2(\mathbb{R}^N))\) \(\tilde{f}_2 \in L^\infty(I; L^{1+1/p}(\mathbb{R}^N))\), and a subsequence of \(\{w_n(t)\}_n\) such that
\[
\tilde{g}_1(t, w_n(t)) \to \tilde{f}_1(t) (n \to \infty) \text{ weakly* in } L^\infty(I; L^2(\mathbb{R}^N)),
\]
\[
\tilde{g}_2(t, w_n(t)) \to \tilde{f}_2(t) (n \to \infty) \text{ weakly* in } L^\infty(I; L^{1+1/p}(\mathbb{R}^N)),
\]
where we denote the subsequence of \(\{w_n(t)\}_n\) as \(\{w_n(t)\}_\hat{n}\). Obviously, \(\tilde{f}(t) = \tilde{f}_1(t) + \tilde{f}_2(t)\). Restricting the whole space \(\mathbb{R}^N\) to any bounded open subset \(\Omega\),
we see that \( \tilde{f}(t) = \tilde{g}(t, w(t)) \) on \( I \times \Omega \). Since \( \Omega \) is arbitrary, we conclude that \( \tilde{f}(t) = \tilde{g}(t, w(t)) \). Naturally, (A7) is verified.

Uniqueness of (3.4) is verified by the Strichartz estimates. To end this, let \( v_l \) \((l = 1, 2)\) be the solutions in \( L^\infty(-T_1, T_2; D) \) to (3.4) with the same initial values \( v_1(0) = u_0 = v_2(0) \in D \). We show \( v_1 = v_2 \). Here \( v_1 \) satisfies the following integral equations:

\[
v_1(t) = \exp(-itP)v_0 - i \int_0^t \exp(i(t-s)P)g_0(s, v_1(s)) \, ds.
\]

Apply (3.10), (3.11), and the Strichartz estimates (2.29) and (2.29) with the admissible pairs \((\infty, 2)\) and \((r, p + 1) \((r := 4(p + 1)/[N(p - 1)])\). We see that

\[
\|v_1(t) - v_2(t)\|_{L_t^\infty(I; L_x^p)} \leq C(\infty, \tau, \omega) K_1 \|(1 - 4\omega^2t^2)^{-1} [v_1(t) - v_2(t)]\|_{L_t^1(I; L_x^2)} + C(\tau, \omega) K_2 \|(1 - 4\omega^2t^2)^{-1+N(p-1)/4} \times [\|v_1(t)\|_{L_t^p} + \|v_2(t)\|_{L_t^p}] [\|v_1(t) - v_2(t)\|_{L_t^1(I; L_x^{1+r})}] \leq C(\infty, \tau, \omega) K_1 \|(1 - 4\omega^2t^2)^{-1} [L_t^1(I)\|v_1(t) - v_2(t)\|_{L_t^\infty(I; L_x^2)} + 2C(\tau, \omega) K_2 \|(1 - 4\omega^2t^2)^{-1+N(p-1)/4} L_t^r(I; L_x^{1+r}) \times [C(p + 1, V) M]^{p-1} [\|v_1(t) - v_2(t)\|_{L_t^1(I; L_x^{1+r})}],
\]

where \( M := \max\{\|v_1(t)\|_{L_t^\infty(I; D)}, \|v_2(t)\|_{L_t^\infty(I; D)}\} \) and we have applied (3.9). Put \( I = (-T_M, T_M) \) \((T_M > 0 \) is sufficiently small) and \((\tau, \rho) = (\infty, 2) \) or \((r, p + 1)\). Then we see that \( v_1(t) = v_2(t) \) on \((-T_1, T_2)\). Extending the interval step by step, we can conclude the full uniqueness \( v_1(t) = v_2(t) \) on \((-T_1, T_2)\).

**Conclusion of Stage 1.** Let \( \bar{g}_0(t, v) := (1 - 4\omega^2t^2)^{-1-N/4} \tilde{g}_0((1 - 4\omega^2t^2)^{N/4}v) \). Assume that (L1) and (L2). Then for any \( v_0 \in D \) there uniquely exists a local solution \( v \in C([-T_M, T_M]; D) \cap C^1([-T_M, T_M]; D^*) \) to (3.4). Moreover, \( v \) satisfies the following identities:

\[
\|v(t)\|_{L_t^2(\mathbb{R}^N)} = \|v_0\|_{L_t^2(\mathbb{R}^N)}, \quad E(t, v(t)) = E(0, v_0)
\]

\[
+ \int_0^t -8\omega^2s (1 - 4\omega^2s^2)^{-2-N/2} G_d((1 - 4\omega^2s^2)^{N/4}v(s)) \, ds,
\]

where

\[
E(t, \varphi) := \frac{1}{2} \|P_{1/2}^1 \varphi\|_{L_t^2(\mathbb{R}^N)}^2 + (1 - 4\omega^2t^2)^{-1-N/2} G_0((1 - 4\omega^2t^2)^{N/4} \varphi),
\]

\[
G_d(\varphi) := \frac{N}{4} G_1(\varphi) - \frac{N + 2}{2} G_0(\varphi).
\]

**Stage 2 for proof of Theorem 3.1.** First [17, Proposition 3.6] shows that if \( \delta_V \geq 0 \) as in (1.2) and \( \varphi \) is a real-valued and radially symmetric function, then one has

\[
\left|\operatorname{Im} \int_{\mathbb{R}^N} \varphi x \cdot \nabla u \, dx\right| \leq \|P_{1/2}^1 u\|_{L_t^2(\mathbb{R}^N)} \|x \varphi u\|_{L_t^2(\mathbb{R}^N)}.
\]
Applying this we can see the continuity of \( v(t) \) in \( \Sigma \). Put \( \eta(t) := \|x\|/(1 + \varepsilon |x|^2) v(t) \|_{L^2(\mathbb{R}^N)}^2 \). Then we see that

\[
\eta'(t) = 4 \text{Im} \int_{\mathbb{R}^N} \frac{1 - \varepsilon |x|^2}{(1 + \varepsilon |x|^2)^2} x v(t) \cdot \nabla v(t) \, dx.
\]

Applying (3.14) with \( \varphi(x) = (1 - \varepsilon |x|^2)/(1 + \varepsilon |x|^2)^3 \), we see that

\[
|\eta'(t)| \leq 4 \left\| \frac{1 - \varepsilon |x|^2}{(1 + \varepsilon |x|^2)^3} x v(t) \right\|_{L^2(\mathbb{R}^N)} \left\| P^{1/2} v(t) \right\|_{L^2(\mathbb{R}^N)}
\]

\[
\leq 4 \eta(t)^{1/2} \left\| P^{1/2} v(t) \right\|_{L^2(\mathbb{R}^N)}.
\]

Since \( v(t) \in C(\mathcal{T}; D) \), we see the uniform boundedness of \( \eta(t) \) respect to \( \varepsilon > 0 \) and \( t \in \mathcal{T} \). Thus letting \( \varepsilon \to +0 \), we see \( v(t) \in C(\mathcal{T}; \Sigma) \) and

\[
\frac{d}{dt} \|x v(t)\|_{L^2(\mathbb{R}^N)}^2 = 4 \text{Im} \int_{\mathbb{R}^N} x \overline{v(t)} \cdot \nabla v(t) \, dx.
\]

Next we consider the virial identities for (3.4). Here the verification is similar as in [13, Theorem 5.2]. To derive the virial identities we need to consider the approximated problems

\[
\begin{cases}
v_{\varepsilon, \delta} = -\Delta v_{\varepsilon, \delta} + \rho_\delta \ast V_a v_{\varepsilon, \delta} + \rho_\varepsilon \ast [\tilde{g}_0(t, \rho_\varepsilon \ast v_{\varepsilon, \delta})] \quad \text{in } \mathbb{R} \times \mathbb{R}^N, \\
v_{\varepsilon, \delta}(0, x) = u_0(x) \quad \text{in } \mathbb{R}^N,
\end{cases}
(3.15)
\]

where \( V_a := V + a|x|^{-2} \) and \( \rho_\delta \) is the Friedrichs mollifier \( (\rho_\delta \to \delta_0 \quad (b \to +0) \) in the sense of distribution; \( \delta_0 \) is the delta function). Here the parameters in (3.15) are \( \varepsilon > 0, \delta > 0, \) and \( a; \) \( a \) is set as 0 if \( \delta_\varepsilon > 0 \) or \( a \) if \( \delta_\varepsilon = 0 \). We see that there uniquely exists the global in time solution to (3.15), since \( \rho_\delta \ast [V + a|x|^{-2}] \in C^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \). After the confirmation of the virial identities for (3.15), letting \( \delta \to +0, a \to +0 \) (only the case \( \delta_\varepsilon = 0 \); see also [15, Stage 2 in Section 3]), and \( \varepsilon \to +0 \) in order. Here we need to apply the Strichartz estimates (2.32) to let \( \varepsilon \to +0 \) (see e.g., [13, Stage 4 in Section 3]).

**Conclusion of Stage 2.** The solution \( v(t) \) to (3.4) constructed in Stage 1 is continuous in \( \Sigma \). Moreover, \( v(t) \) satisfies

\[
\frac{d}{dt} \|x v(t)\|_{L^2(\mathbb{R}^N)}^2 = 4 \text{Im} \int_{\mathbb{R}^N} x \overline{v(t)} \cdot \nabla v(t) \, dx,
(3.16)
\]

\[
\frac{d^2}{dt^2} \|x v(t)\|_{L^2(\mathbb{R}^N)}^2 = 8 \|P^{1/2} v(t)\|_{L^2(\mathbb{R}^N)}^2 - 8N (1 - 4\omega^2 t^2)^{-1-N/2} G_0(1 - 4\omega^2 t^2)^{N/4} v(t)
\]

\[
+ 4N (1 - 4\omega^2 t^2)^{-1-N/2} G_1(1 - 4\omega^2 t^2)^{N/4} v(t)
\]

\[
= 16E(t, v(t)) + 16(1 - 4\omega^2 t^2)^{-1-N/2} D_\omega((1 - 4\omega^2 t^2)^{N/4} v(t)).
(3.17)
\]

**Stage 3 for proof of Theorem 3.1.** Unique existence of local solution to (NL-S)\(-\) is verified in Stages 1 and 2. Here we need to remark the regularity of solution. In Stage 2, we see that \( v \), the solution of (3.4), belongs to \( C([-T, T]; \Sigma) \cap C^1([-T, T]; D^*) \subset C([-T, T]; \Sigma) \cap C^1([-T, T]; \Sigma^*) \) if \( u_0 \in \Sigma \). Since the pseudo-conformal transforms preserve the continuity in \( \Sigma \) (also in \( \Sigma^* \) but not in \( D \) nor \( D^* \)), \( u = C^{-1} v \) is the solution to (NLS\(-\)), which belongs to \( C([-T', T']; \Sigma) \cap \)
It remains to show the identities (3.5), (3.6), and (3.7). It is simple for the charge identity (3.5). We see from (3.5) that
\[
\|u(t)\|_{L^2(\mathbb{R}^N)} = \|(C_1^{-1}u)(t)\|_{L^2(\mathbb{R}^N)} = \|v(\xi_-(t))\|_{L^2(\mathbb{R}^N)}
\]
Next we consider (3.7), the virial identities for \((\text{NLS})^\dagger\). We see from (2.18) that
\[
\|x|u(t)|^2\|_{L^2(\mathbb{R}^N)} = \cosh^2(2\omega t)\|x|v(s)|^2\|_{L^2(\mathbb{R}^N)} \quad (s := \xi_- (t)).
\]
Thus \(\|x|u(t)|^2\|_{L^2(\mathbb{R}^N)}\) is continuously differentiable twice. Applying (3.16), (2.20) and (2.18), we obtain
\[
\frac{d}{dt}\|x|u(t)|^2\|_{L^2(\mathbb{R}^N)} = 4\omega \cosh(2\omega t) \sinh(2\omega t) \|x|v(s)|^2\|_{L^2(\mathbb{R}^N)} + \frac{d}{ds}\left[\frac{2\omega \sinh(4\omega t)}{\cosh^2(2\omega t)} \frac{d}{ds}\|x|v(s)|^2\|_{L^2(\mathbb{R}^N)}\right]
\]
Next we differentiate again. Applying (2.21) and (3.16), we see that
\[
\frac{d^2}{dt^2}\|x|u(t)|^2\|_{L^2(\mathbb{R}^N)} = 8\omega^2 \cosh(4\omega t) \|x|v(s)|^2\|_{L^2(\mathbb{R}^N)} + \frac{2\omega \sinh(4\omega t)}{\cosh^2(2\omega t)} \frac{d}{ds}\left[\frac{2\omega \sinh(4\omega t)}{\cosh^2(2\omega t)} \frac{d}{ds}\|x|v(s)|^2\|_{L^2(\mathbb{R}^N)}\right]
\]
It follows from (3.17), (2.21), and (2.18) that
\[
\frac{d^2}{dt^2}\|x|u(t)|^2\|_{L^2(\mathbb{R}^N)} = 8\left[\frac{1}{\cosh^2(2\omega t)} \|P^{1/2}v(s)|^2\|_{L^2(\mathbb{R}^N)} + 2\omega \tanh(2\omega t) \text{Im} \int_{\mathbb{R}^N} x \overline{v(s)} \cdot \nabla v(s) \, dx\right]
\]
Finally, we verify (3.6). Applying (2.21) and (3.13), we see that
\[
E(u(t)) = \frac{1}{2} \frac{1}{\cosh^2(2\omega t)} \|P^{1/2}v(s)|^2\|_{L^2(\mathbb{R}^N)} + \omega \tanh(2\omega t) \text{Im} \int_{\mathbb{R}^N} x \overline{v(s)} \cdot \nabla v(s) \, dx
\]
ity of global extension of the local solutions to
We need more assumption for these studies. To end this, we define
Global existence and scattering problems.
This leads the energy conservation (3.6). Thus we have proved (3.5), (3.6), and
There exist (L5)
There exist (L4)
There exist (L3)
Note that (3.13) as follows:
\[ \Phi(z) := \frac{N}{4} F_1(z) - \frac{N + 2}{2} F_0(z). \]
Since \( \Phi(s) \) can be differentiated by \( s \), we can calculate with (3.16), (3.17), and
(3.13) as follows:
\[ \Phi'(s) = -8 \omega^2 s E(s, v(s)) + (1 - 4 \omega^2 s^2)[-8 \omega^2 s (1 - 4 \omega^2 s^2)^{-2 - N/2} G_d((1 - 4 \omega^2 s^2)^{N/4} v(s))] + \frac{\omega^2 s}{2} \partial_s \| |v(s)\|_{L^2(\mathbb{R}^N)}^2 \| |v(s)\|_{L^2(\mathbb{R}^N)}^2 = 0. \]
This leads the energy conservation (3.6). Thus we have proved (3.5), (3.6), and
(3.7). The proof of Theorem 3.1 is fully finished.

3.2. Global existence and scattering problems. Next we consider the possibility of global extension of the local solutions to (NLS)\(^-\) and the scattering problems.
We need more assumption for these studies. To end this, we define
\[ F_d(z) := \frac{N}{4} F_1(z) - \frac{N + 2}{2} F_0(z). \]
Note that
\[ G_d(u) = \int_{\mathbb{R}^N} F_d(u(x)) \, dx. \]

(L3) There exist \( 1 < q < 1 + 4/N \) and \( C_1, C_2 \geq 0 \) such that for any \( z \in \mathbb{C} \)
\[ F_0(z) \geq -C_1(|z|^2 + |z|^{q+1}), \quad F_d(z) \geq -C_2(|z|^2 + |z|^{q+1}); \]

(L4) There exist \( C_1 > 0 \) and \( 1 + 4/N \leq p_1 \leq p_2 < 1 + 4/(N - 2) \) such that for any \( z_1, z_2 \in \mathbb{C} \)
\[ |g_0(z_2) - g_0(z_1)| \leq C_3(|z_1|^{p_1 - 1} + |z_2|^{p_1 - 1} + |z_1|^{p_2 - 1} + |z_2|^{p_2 - 1})|z_2 - z_1|; \]

(L5) There exist \( C_2 > 0 \) and \( 1 + 4/N < p_1 \leq p_2 < 1 + 4/(N - 2) \) such that for any \( z_1, z_2 \in \mathbb{C} \)
\[ |F_d(z_2) - F_d(z_1)| \leq C_4(|z_1|^{p_1} + |z_2|^{p_1} + |z_1|^{p_2} + |z_2|^{p_2})|z_2 - z_1|. \]

Theorem 3.3 (Global existence). Assume that \( g_0 \) satisfies (L1), (L2), and (L3). Then for any \( u_0 \in \Sigma \) the local solution to (NLS)\(^-\) constructed in Theorem 3.1 can be extended globally in time \( t \in \mathbb{R} \).

Theorem 3.4 (Existence of scattering states). Assume that \( g_0 \) satisfies (L1), (L2), and (L3) with \( C_1 = 0 \) and \( C_2 = 0 \). Then for any \( u_0 \in \Sigma \) the global solution to (NLS)\(^-\) constructed in Theorem 3.1 scatters in free solution in the following sense:
\[ \lim_{t \to \pm \infty} \exp(it(P - \omega^2 |x|^2)) u(t) = u_\pm \] strongly in \( \Sigma \). (3.18)

Theorem 3.5 (Existence of wave operators). Assume that \( g_0 \) satisfies (L1)–(L5). Then for any \( u_+ \in \Sigma \) (resp. \( u_- \in \Sigma \)) there exists a global solution to (NLS)\(^-\) with (3.18).
It is similar to verify the certainty of global extension (Theorem 3.3) and the asymptotic free of solutions (Theorem 3.4). The difference between these is the range of \( t \) for the required conditions. If the solution \( v(t) = C - u(t) \) to (3.4) exists in \( t \in \left(-1/(2\omega) - \delta, 1/(2\omega) - \delta\right) \), \( u(t) \) is a global solution to \((\text{NLS})^-\). On the other hand, if \( v(t) \) exists in \([-1/(2\omega), 1/(2\omega)]\), especially, \( v(\pm 1/(2\omega)) \) makes sense, \( u(t) \) has a standard scattering states (3.18) (see (2.23)). We bring confirming \((A8)\) and \((A9)\) with \( T = 1/(2\omega) - \varepsilon \) (for any small \( \varepsilon > 0 \)) to the certainty of global extension (Theorem 3.3), while \((A8)\) and \((A9)\) with \( T = 1/(2\omega) \) makes for the asymptotic free of solutions (Theorem 3.4). Now we consider \((A8)\) and \((A9)\). It follows from (1.3) that

\[
G(t, v) \geq - C_1(1 - 4\omega^2 t^2)^{-1-N/2} \int_{\mathbb{R}^N} \left[(1 - 4\omega^2 t^2)^{N/4} v^2 + \| (1 - 4\omega^2 t^2)^{N/4} v \|^q + 1 \right] \, dx
\]

\[
\geq - C_1(1 - 4\omega^2 t^2)^{-1} \| v \|^2_{L^2(\mathbb{R}^N)} - C_1 C(q + 1, P) q + 1(1 - 4\omega^2 t^2)^{-1 + N(q-1)/4} \| v \|^{q + 1 - N(q-1)/2} \| v \|_{L^2(\mathbb{R}^N)}^{N(q-1)/2},
\]

(3.19)

\[
\text{sgn}(t) G_t(t, v) \leq 8\omega^2 |t| (1 - 4\omega^2 t^2)^{-2-N/2} C_2 \int_{\mathbb{R}^N} \left[(1 - 4\omega^2 t^2)^{N/4} v^2 + \| (1 - 4\omega^2 t^2)^{N/4} v \|^q\right] \, dx
\]

\[
\leq 8C_2 \omega^2 |t| (1 - 4\omega^2 t^2)^{-2} \left[ \| v \|^2_{L^2(\mathbb{R}^N)} + C(q + 1, P) q + 1(1 - 4\omega^2 t^2)^{N(q-1)/4} \| v \|^{q + 1 - N(q-1)/2} \| v \|_{L^2(\mathbb{R}^N)}^{N(q-1)/2} \right].
\]

(3.20)

It follows from \( N(q - 1)/2 < 2 \) and the Young inequality that for any \( \delta_0 > 0 \)

\[
\| v \|^{q + 1 - N(q-1)/2} \| v \|_{L^2(\mathbb{R}^N)}^{N(q-1)/2} \leq \delta_0 \| v \|^2_{L^2(\mathbb{R}^N)} + \frac{4 - N(q - 1)}{4} \left( \frac{N(q - 1)}{4\delta_0} \right) 4/[4 - N(q - 1)] \times \| v \|_{L^2(\mathbb{R}^N)}^{2[q(q+1)-N(q-1)]/[4-N(q-1)]}.
\]

(3.21)

**Proof of Theorem 3.3.** Fix \( \varepsilon > 0 \) (as in \((A8)\)) and \( \delta > 0 \) for sufficiently small. (3.19) and (3.21) (with appropriate choice of \( \delta_0 > 0 \)) yield that for any \( t \in [-1/(2\omega) + \delta, 1/(2\omega) - \delta] \) and \( v \in D \)

\[
G(t, v) \geq - C_1 \| v \|^2_{L^2(\mathbb{R}^N)} + \frac{1 - \varepsilon}{2} \| v \|_{L^2(\mathbb{R}^N)}^2 - C'_1 \| v \|^{2[q(q+1)-N(q-1)]/[4-N(q-1)]}_{L^2(\mathbb{R}^N)}
\]

This implies that \((A8)\). On the other hand, (3.20) and (3.21) yield that for any \( t \in [-1/(2\omega) + \delta, 1/(2\omega) - \delta] \) and \( v \in D \)

\[
\text{sgn}(t) G_t(t, v) \leq 8C_2 \omega^2 |t| (1 - 4\omega^2 t^2)^{-2} \left[ \| v \|^2_{L^2(\mathbb{R}^N)} + C' \left( \| v \|^2_{L^2(\mathbb{R}^N)} + \| v \|^{2[q(q+1)-N(q-1)]/[4-N(q-1)]}_{L^2(\mathbb{R}^N)} \right) \right].
\]

Since \( |t|(1 - 4\omega^2 t^2)^{-2} \in L^1(-1/(2\omega) + \delta, 1/(2\omega) - \delta) \), we can verify \((A9)\). Since \((A8)\) and \((A9)\) are verified in \([-1/(2\omega) + \delta, 1/(2\omega) - \delta] \) for any sufficiently small \( \delta > 0 \). Thus we see that the solution \( v(t) \) to (3.4) with \( v_0 \in \Sigma \) belongs to \( C([-1/(2\omega) + \delta, 1/(2\omega) - \delta]; \Sigma) \) for any \( \delta > 0 \). Thus we get the solution to (3.4) in the class \( C(-1/(2\omega), 1/(2\omega); \Sigma) \). Hence setting \( v_0 := u_0 \in \Sigma \) and \( u(t) := (C^{-1} v)(t) \), we obtain the global solution to \((\text{NLS})^-\) in \( C(\mathbb{R}; \Sigma) \). \( \square \)
Proof of Theorem 3.4. It follows from (3.19) and $C_1 = 0$ that $G(t, v) \geq 0$ for any $t \in (-1/(2\omega), 1/(2\omega))$ and $v \in D$. Moreover, (3.20) and $C_2 = 0$ yield that $\text{sgn}(t) G(t, v) \leq 0$ for any $t \in (-1/(2\omega), 1/(2\omega))$ and $v \in D$. These imply (A8) and (A9). Since (A8) and (A9) are verified in $[-1/(2\omega), 1/(2\omega)]$. Thus we have the solution to (3.4) with $v_0 \in \Sigma$ belonging to $C([1/(2\omega), 1/(2\omega)]; \Sigma)$. Especially, there exist $v_+ := v(1/(2\omega))$, $v_- := v(-1/(2\omega)) \in \Sigma$ such that

$$
\lim_{t \to 1/(2\omega) + 0} v(t) = v_-, \quad \lim_{t \to 1/(2\omega) - 0} v(t) = v_+,
$$
strongly in $\Sigma$. Since $\mathcal{S}_V(t)$ is continuous group on $\Sigma$, we see that

$$
\lim_{t \to 1/(2\omega) + 0} \mathcal{S}_V(-t)v(t) = \mathcal{S}_V\left(\frac{1}{2\omega}\right)v_-, \quad \lim_{t \to 1/(2\omega) - 0} \mathcal{S}_V(-t)v(t) = \mathcal{S}_V\left(-\frac{1}{2\omega}\right)v_+,
$$
strongly in $\Sigma$. It follows from (2.23) that $u(t) := (C^{-1}_0 v)(t)$, the solution to $\text{(NLS)}^-$ with $u_0 = v_0 \in \Sigma$, satisfies

$$
\lim_{t \to -\infty} \mathcal{S}_V(-t)u(t) = \mathcal{S}_V\left(\frac{1}{2\omega}\right)v_-, \quad \lim_{t \to +\infty} \mathcal{S}_V(-t)u(t) = \mathcal{S}_V\left(-\frac{1}{2\omega}\right)v_+,
$$
strongly in $\Sigma$. Putting $u_+ := \mathcal{S}_V(-1/(2\omega))v_+$ and $u_- := \mathcal{S}_V(1/(2\omega))v_-$, we can conclude (3.18).

Construction of the wave operators is followed by the existence of the local solution to (3.4) with initial value $v(\pm 1/(2\omega)) = v_\pm \in \Sigma$. To end this we check the conditions (A1)–(A7) again ($t$ is near $\pm 1/(2\omega)$). (L4) implies that $g(t, v)$ and $G(t, v)$ can be extended continuously to $t = \pm 1/(2\omega)$. Thus the verification is simple for (A1), (A4), (A2), and (A7) (former two are the conditions for $G(t, v)$, while latter two are for $g(t, v)$). (A6) is obviously true by (L2). Moreover, (A5) is followed by (L5) and the verification is similar to (A4). We need delicate calculation for (A3).

Since $g(t, u)$ is even function of $t$, we can set $0 \leq t < s < 1/(2\omega)$. For simple notation, we set $\tau := 1 - 4\omega^2t^2$ and $\sigma := 1 - 4\omega^2s^2$. Note that $0 < \sigma < \tau \leq 1$. We see that

$$
g(t, v) - g(s, v) = \tau^{-1}N/4|g_0(\tau^{N/4}v) - g_0(\sigma^{N/4}v)| + |\tau^{-1}N/4 - \sigma^{-1}N/4|g_0(\sigma^{N/4}v)
= I_1 + I_2.
$$

Simple calculations with (L4) imply that

$$
|I_1| \leq \frac{C_3}{\tau^{1+N/4}} \left(|\tau^{N/4}v|^{p_1-1} + |\sigma^{N/4}v|^{p_1-1} + |\tau^{N/4}v|^{p_2-1} + |\sigma^{N/4}v|^{p_2-1}\right)
\times \left|\tau^{N/4}v - \sigma^{N/4}v\right|
\leq 2C_3 |\tau^{N/4} - \sigma^{N/4}| \left[\tau^{N(p_1-2)/4-1}v|^{p_1} + \tau^{N(p_2-2)/4-1}v|^{p_2}\right],
$$

$$
|I_2| \leq (\tau \sigma)^{-1}N/4|\tau^{1+N/4} - \sigma^{1+N/4}| \left[C_3 \left(|\sigma^{N/4}v|^{p_1-1} + |\sigma^{N/4}v|^{p_2-1}\right)\right] \left|\tau^{N/4}v\right|
= C_3 |\tau^{1+N/4} - \sigma^{1+N/4}| \left|\tau^{-1-N/4}|\tau^{N(p_1-1)/4-1}v|^{p_1} + \sigma^{N(p_2-1)/4-1}v|^{p_2}\right|.
$$

Thus we see that

$$
|g(t, v) - g(s, v)| \leq 2C_3 \left[\tau^{N(p_1-2)/4-1}|v|^{p_1} + \tau^{N(p_2-2)/4-1}|v|^{p_2}\right] \int_\sigma^\tau \frac{N}{4} r^{N/4-1} dr
$$
+ C_3 \frac{1}{2}N^2 |\sigma|^N \eta \left[ |\sigma|^N \eta^N u \right] \int_\sigma^T \left[ \frac{N}{2} + \frac{N + 4}{4} \epsilon^{N(p_j - 1)/4 - 2} \right] dr.

Multiplying by \( w \in \mathcal{D} \) and integrating over \( \mathbb{R}^N \), we can calculate
\[
\| g(t, v) - g(s, v) \|_{\mathcal{D}^*} = \sup \left\{ \Re \int_{\mathbb{R}^N} [g(t, v) - g(s, v)] \overline{w} \, dx ; \|w\|_{\mathcal{D}} \leq 1 \right\}
\leq \sup \left\{ C_3 \int_{\mathbb{R}^N} |v|^{p_j} |w| \, dx \right\} \left( \int_\sigma^T \left[ \frac{N}{2} + \frac{N + 4}{4} \epsilon^{N(p_j - 1)/4 - 2} \right] dr \right)^{1/2} \leq \frac{2}{p_j} C_3 \left( \int_{\mathbb{R}^N} |v|^{p_j} \, dx \right)^{1/2}.\]

and the integrand of the right-hand-side belongs to \( L^1(-1/(2\omega), 1/(2\omega)) \). Hence we can conclude (A3).

**Proof of Theorem 3.5.** First we remark that the assumption of Theorem 3.5 implies the global existence of solution to \( (\text{NLS})^- \) (also the local existence of \( (\text{NLS}) \) in \( (-1/(2\omega), 1/(2\omega)) \)). We need to prove that there uniquely exists a local solution to \( (3.4) \) on \( [1/(2\omega) - \delta, 1/(2\omega)] \) (resp. \( [-1/(2\omega), -1/(2\omega) + \delta] \)) for sufficiently small \( \delta > 0 \) with the initial condition \( v(1/(2\omega)) = v_+ \in \Sigma \) (resp. \( v(-1/(2\omega)) = v_- \in \Sigma \)). Here the energy methods can be applicable since the conditions (A1)–(A7) are fulfilled. Thus we can get the solution to \( (3.4) \) with \( v(1/(2\omega)) = v_+ \in \Sigma \) (resp. \( v(-1/(2\omega)) = v_- \in \Sigma \)).

In a way similar to the proof of Theorem 3.4, \( u(t):=(C^{-1}v)(t) \) satisfies \( (\text{NLS})^- \) and the strong convergence in \( \Sigma \) such that
\[
\lim_{t \to +\infty} S_{V}^{-\omega}(-t)u(t) = S_{V}(-1/(2\omega))v_+ \quad (\text{resp. } \lim_{t \to -\infty} S_{V}^{-\omega}(-t)u(t) = S_{V}(1/(2\omega))v_-).
\]

Here for any \( u_+ \in \Sigma \) (resp. \( u_- \in \Sigma \)) put \( v_+ := S_{V}(1/(2\omega))u_+ \) (resp. \( v_- := S_{V}(-1/(2\omega))u_- \)). The above arguments imply that there exists the global solution to \( (\text{NLS})^- \) with the condition
\[
\lim_{t \to +\infty} S_{V}^{-\omega}(-t)u(t) = u_+ \quad (\text{resp. } \lim_{t \to -\infty} S_{V}^{-\omega}(-t)u(t) = u_-) \text{ strongly in } \Sigma.
\]

This concludes the proof. \( \Box \)

The presence of repulsive harmonic potential makes easy to scatter. If we can apply the Kato methods (contraction principle), the range of the power of the
nonlinearities can be extended, for example, we can admit that the conditions (L4) and (L5) are relaxed: $1 < p_1 < p_2 < 1 + 4/(N - 2)$.

To end this section, we see the transition for the solution. If the solutions of the nonlinear problems scatter, the effects of the nonlinearities may be disabled. We give the affirmative results.

**Proposition 4.** Assume that (L1)–(L5) with $p_1 > 1 + 4/N$ in (L4). Let $u$ be a global solution to (NLS) with $u(0) = u_0 \in \Sigma$ and

$$u_+ = \lim_{t \to \infty} \exp(it(P - \omega^2|x|^2))u(t) \quad \text{strongly in } \Sigma.$$  

Then the following identities are hold.

$$\|u_+\|_{L^2(\mathbb{R}^N)}^2 = \|u_0\|_{L^2(\mathbb{R}^N)}^2$$  

$$(3.22)$$

$$\frac{1}{2}\|(1 + P)^{1/2}u_+\|_{L^2(\mathbb{R}^N)}^2 - \frac{\omega^2}{2}\|\langle x \rangle u_+\|_{L^2(\mathbb{R}^N)}^2 = E(u_0)$$  

$$(3.23)$$

$$\|\langle x \rangle u_+\|_{L^2(\mathbb{R}^N)}^2 = \|\langle x \rangle u_0\|_{L^2(\mathbb{R}^N)}^2 + \int_0^\infty 4\sinh(4\omega s)G_d(u(s))\,ds.$$  

$$(3.24)$$

**Proof.** We divide the proof into 3 steps. Here we define $v := C_-u$.

**Step 1.** The charge identity (3.22) is followed by (3.5), (3.12), and (2.23):

$$\|u_0\|_{L^2(\mathbb{R}^N)}^2 = \|u(\xi_-)(t)\|_{L^2(\mathbb{R}^N)}^2 = \|(C_-u)(t)\|_{L^2(\mathbb{R}^N)}^2 = \|v(t)\|_{L^2(\mathbb{R}^N)}^2$$

$$= \|v(1/(2\omega))\|_{L^2(\mathbb{R}^N)}^2 = \|Sv(1/(2\omega))u_+\|_{L^2(\mathbb{R}^N)}^2 = \|u_+\|_{L^2(\mathbb{R}^N)}^2.$$  

**Step 2.** We consider the energy identity (3.23). We see from (2.23), (2.25) and (2.26) that

$$\|\langle x \rangle u_+\|_{L^2(\mathbb{R}^N)}^2 = 4\|Sv\left(-\frac{1}{2\omega}\right)L_0\left(\frac{1}{2\omega}\right)^{1/2}v\left(\frac{1}{2\omega}\right)\|_{L^2(\mathbb{R}^N)}^2$$

$$= \|\langle x \rangle v(1/(2\omega))\|_{L^2(\mathbb{R}^N)}^2 = \frac{1}{\omega^2}\|P^{1/2}v(1/(2\omega))\|_{L^2(\mathbb{R}^N)}^2$$

$$- \frac{2}{\omega}\text{Im}\int_{\mathbb{R}^N} \langle x \rangle v(1/(2\omega)) \cdot \nabla v(1/(2\omega)) \, dx.$$  

Here we define

$$\Phi(t) := \|\langle x \rangle v(t)\|_{L^2(\mathbb{R}^N)}^2 - 4t\text{Im}\int_{\mathbb{R}^N} \langle x \rangle v(t) \cdot \nabla v(t) \, dx.$$  

Then we see that $\Phi(0) = \|\langle x \rangle u_0\|_{L^2(\mathbb{R}^N)}^2$ and that

$$\Phi\left(\frac{1}{2\omega}\right) = \|\langle x \rangle v(1/(2\omega))\|_{L^2(\mathbb{R}^N)}^2 - \frac{2}{\omega}\text{Im}\int_{\mathbb{R}^N} \langle x \rangle v(1/(2\omega)) \cdot \nabla v(1/(2\omega)) \, dx$$

Since (3.16) and (3.17), we see that

$$\Phi'(t) = -4t\frac{d}{dt}\text{Im}\int_{\mathbb{R}^N} \langle x \rangle v(t) \cdot \nabla v(t) \, dx = -t\frac{d^2}{dt^2}\|\langle x \rangle v(t)\|_{L^2(\mathbb{R}^N)}^2$$

$$= -16tE(t, v(t)) - 16t(1 - 4\omega^2t^2)^{-1-N/2}G_d((1 - 4\omega^2t^2)^{-N/4}v(t)).$$  

$$= \frac{d}{dt}\left(\frac{2(1 - 4\omega^2t^2)}{\omega^2}E(t, v(t))\right).$$
Thus we have
\[ \|xu_+\|^2_{L^2(\mathbb{R}^N)} = \frac{1}{\omega^2} \left\| P^{1/2}u \left( \frac{1}{2\omega} \right) \right\|^2_{L^2(\mathbb{R}^N)} + \|xu_0\|^2_{L^2(\mathbb{R}^N)} - \frac{2}{\omega^2}E(0,u_0). \] (3.25)

Applying (2.23) and rearranging (3.25), we obtain (3.23).

**Step 3.** Finally, we consider (3.24). First we see from (2.23), (3.13), and (3.3) that
\[
\frac{1}{2}\|P^{1/2}u_+\|^2_{L^2(\mathbb{R}^N)} = E(1/(2\omega), v(1/(2\omega)))
= E(0,v(0)) + \int_0^{1/(2\omega)} 8\omega^2 s \left( 1 - 4\omega^2 s^2 \right)^{-2-N/2} G_d\left( (1 - 4\omega^2 s^2)^{N/4} v(s) \right) ds
= \frac{1}{2}\|P^{1/2}u_0\|^2_{L^2(\mathbb{R}^N)} + G(u_0) + \int_0^{1/(2\omega)} \frac{8\omega^2 s}{(1 - 4\omega^2 s^2)^2} G_d(u(\xi^{-1}(s))) ds.
\]

Thus we have
\[
\frac{1}{2}\|P^{1/2}u_+\|^2_{L^2(\mathbb{R}^N)} = \frac{1}{2}\|P^{1/2}u_0\|^2_{L^2(\mathbb{R}^N)} + G(u_0) + \int_0^{\infty} 2\omega \sinh(4\omega \sigma) G_d(u(\sigma)) d\sigma. \] (3.26)

Subtracting (3.26) from (3.23), we conclude (3.24). \(\Box\)

In the end of this section, we consider the special case \(g_0(u) := \lambda|u|^{p-1}u\). We have
\[ F_0(z) = \frac{\lambda}{p+1}|z|^{p+1}, \quad F_d(z) = \frac{\lambda N(p-1) - 4}{p+1}|z|^{p+1}. \]

It follows from Theorem 3.1 that if \(\lambda \in \mathbb{R}\) and \(1 < p < 1 + 4/(N-2)\), then for every \(u_0 \in \Sigma\) there uniquely exists a local solution \(u \in C([-T,T];\Sigma) \cap C^1([-T,T];\Sigma^*)\) to \((\text{NLS})^-\). Assume further that \(\lambda > 0\) or \(\lambda < 0\) with \(1 \leq p < 1 + 4/N\). Then Theorem 3.3 yields the local solution \(u\) to \((\text{NLS})^-\) can be extended globally in time \(t \in \mathbb{R}\).

The scattering problems are considered in \(\lambda > 0\) and \(1 + 4/N \leq p < 1 + 4/(N-2)\) in this article. Theorem 3.4 implies that the global solution to \((\text{NLS})^-\) scatters in the free solution \(\exp(-it(P - \omega^2|x|^2)) u_+\) as \(t \to +\infty\). On the one hand, Theorem 3.5 ensures that the wave operator related to \((\text{NLS})^-\) can be constructed. Finally, assume further \(1 + 4/N < p < 1 + 4/(N-2)\). Then we see from Proposition 4 that the relation between the quantities of the initial value \(u_0\) and ones of the final value \(u_+\).

4. Concluding remarks.

4.1. Solvability of \((\text{NLS})^+\). Since \(P + \omega^2|x|^2\) is nonnegative and selfadjoint in \(L^2(\mathbb{R}^N)\), we can apply Theorem 3.2 (see also Okazawa–Suzuki–Yokota [11]). Here \(D(1 + P + \omega^2|x|^2) = D(1 + P) \cap D(1 + |x|^2)\) and \(D((1 + P + \omega^2|x|^2)^{1/2}) = \Sigma\). Existence of solution is simpler since we need not convert into the nonautonomous problems. Uniqueness of solution is followed by the Strichartz estimates (2.33).

**Theorem 4.1.** Let \(g_0(u)\) be the local nonlinearity which is satisfied (L1) and (L2). Then for every \(u_0 \in \Sigma\) there uniquely exists a local solution \(u \in C(T;\Sigma) \cap C^1(T;\Sigma^*)\) to \((\text{NLS})^+\). Moreover, \(u\) conserves the charge and energy:
\[ \|u(t)\|^2_{L^2(\mathbb{R}^N)} = \|u_0\|^2_{L^2(\mathbb{R}^N)}. \]
Furthermore, $u$ satisfies the virial identity:

$$\frac{d^2}{dt^2}\|x|u(t)\|_{L^2(\mathbb{R}^N)}^2 = 16E(u(t)) - 16\omega^2 \|x|u(t)\|_{L^2(\mathbb{R}^N)}^2 + 16G_0(u(t)).$$

4.2. Representation of $\exp(-it(P + \omega^2|x|^2))$ for general $t \in \mathbb{R}$. We can generalize the addition formula (2.8) as follows:

$$A_+(t)A_+(s) = \begin{cases} A_+(\frac{t+s}{1-ts}) & 1-ts > 0, \\ A_+(1)^2A_+\left(\frac{ts-1}{t+s}\right) & 1-ts \leq 0, t > 0, s > 0, \\ A_+(1)^{-2}A_+\left(\frac{ts-1}{t+s}\right) & 1-ts \leq 0, t < 0, s < 0. \end{cases}$$

Thus we can conclude that

$$S^+_V(t) = \exp(-it(P + \omega^2|x|^2)) = A_+(1)^{2(2\omega t)/\pi}A_+(\tan(\omega t)).$$

Here $|s|$ is the floor function, the largest integer less than or equal to $s$.

If $V = 0$, then $A_+(1)\varphi(x)$ and $A_+(1)^2\varphi(x)$ are calculated as

$$A_+(1)\varphi(x) = m(-2\omega)S_0(1/(2\omega))m(-2\omega)\varphi(x)$$

$$= \frac{i^{-N/2}$$}$$2 \pi$$^{N/2}$$}{^N} \int_{\mathbb{R}^N} e^{-i\omega x \cdot y} \varphi(y) dy = i^{-N/2}d(\omega)F\varphi(x),$$

$$A_+(1)^2\varphi(x) = i^N d(\omega)F d(1/\omega)F \varphi(x) = i^N Fd(1/\omega)F \varphi(x)$$

$$= i^N F^2\varphi(x) = i^N \varphi(-x).$$

Here we see that $A_+(1)$ can be regarded as the Fourier transform. Now we define the Fourier type transform as $F_V := i^{N/2}d(1/\omega)m(-2\omega)S_0(1/(2\omega))m(-2\omega)$. Note that $F_V$ is independent of $\omega$. Moreover, we can see that

$$F_V^{-1}|x|^2F_V = P, \quad F_V^{-1}e^{-it|x|^2}F_V = S_V(t).$$

4.3. Blowing up in finite time of the solution to $(\text{NLS})^-$. In a view of Carles [5, Proposition 4.1], we also consider the blowing up in finite time. We apply the Glassy method.

**Theorem 4.2.** Let $g_0(u)$ be the local nonlinearity which is satisfied ($\text{L1}$), ($\text{L2}$), and $F_d(z) \leq 0$. Then for every $u_0 \in \Sigma$ with $E(u_0) + \omega^2\|x|u_0\|_{L^2(\mathbb{R}^N)}^2 < 0$ the local solution $u$ to $(\text{NLS})^-$ blows up in finite time, in the future or in the past. More precisely, if

$$J(u_0) := \text{Im} \int_{\mathbb{R}^N} x \bar{u}_0 \cdot \nabla u_0 \, dx \leq 0,$$

then $u(t)$ blows up in the future, that is, there exists $T_1 > 0$ such that $u$ belongs to $C([0,T_1); \Sigma)$ and

$$\lim_{t \to T_1^-} \|(1 + P)^{1/2}u(t)\|_{L^2(\mathbb{R}^N)}^2 = +\infty,$$
On the one hand, if \( J(u_0) \geq 0 \), then \( u(t) \) blows up in the past, that is, there exists \( T_1 > 0 \) such that \( u \in C((-T_1,0]; \Sigma) \) and

\[
\lim_{t \to -T_1^+} \|(1 + P)^{1/2} u(t)\|_{L^2(\mathbb{R}^N)}^2 = +\infty.
\]

Furthermore, if \( E(u_0) + \omega^2 \|x|u_0\|_{L^2(\mathbb{R}^N)}^2 + \omega |J(u_0)| < 0 \), then \( u(t) \) blows up in the past and in future.

Define \( \varphi(t) := \|x|u(t)\|_{L^2(\mathbb{R}^N)}^2 \). (3.7) implies that

\[
\varphi''(t) = 16E(u(t)) + 16\omega^2 \varphi(t) + 16G_d(u(t)).
\]

\( F_d(z) \leq 0 \) and the energy conservation (3.6) ensure that

\[
\varphi''(t) \leq 16E(u_0) + 16\omega^2 \varphi(t).
\]

The differential inequality can be solved as

\[
\varphi(t) \leq \varphi(0) \cosh(4\omega t) + \frac{\varphi'(0)}{4\omega} \sinh(4\omega t) + \int_0^t \frac{\sinh(4\omega(t-s))}{4\omega} E(u_0) \, ds
\]

\[
= \left\|x|u_0\right\|_{L^2(\mathbb{R}^N)}^2 \cosh(4\omega t) + \frac{J(u_0)}{\omega} \sinh(4\omega t) + \frac{\cosh(4\omega t) - 1}{\omega^2} E(u_0).
\]

The assumption of \( u_0 \) in Theorem 4.2 yields that there exists \( T_1 \) such that \( \varphi(T_1) < 0 \). This is not allowed for global in time solutions.

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