Local and nonlocal advected invariants and helicities in magnetohydrodynamics and gas dynamics I: Lie dragging approach

G M Webb¹, B Dasgupta¹, J F McKenzie¹,², Q Hu¹,³ and G P Zank¹,³

¹ CSPAR, The University of Alabama in Huntsville, Huntsville, AL 35805, USA
² Department of Mathematics and Statistics, Durban University of Technology, Steve Biko Campus, Durban South Africa and School of Mathematical Sciences, University of Kwa-Zulu, Natal, Durban, South Africa
³ Department of Space Science, The University of Alabama in Huntsville, Huntsville, AL 35899, USA

E-mail: gmw0002@uah.edu

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Abstract
In this paper advected invariants and conservation laws in ideal magnetohydrodynamics (MHD) and gas dynamics are obtained using Lie dragging techniques. There are different classes of invariants that are advected or Lie dragged with the flow. Simple examples are the advection of the entropy S (a 0-form), and the conservation of magnetic flux (an invariant 2-form advected with the flow). The magnetic flux conservation law is equivalent to Faraday’s equation. The gauge condition for the magnetic helicity to be advected with the flow is determined. Different variants of the helicity in ideal fluid dynamics and MHD including: fluid helicity, cross helicity and magnetic helicity are investigated. The fluid helicity conservation law and the cross-helicity conservation law in MHD are derived for the case of a barotropic gas. If the magnetic field lies in the constant entropy surface, then the gas pressure can depend on both the entropy and the density. In these cases the conservation laws are local conservation laws. For non-barotropic gases, we obtain nonlocal conservation laws for fluid helicity and cross helicity by using Clebsch variables. These nonlocal conservation laws are the main new results of the paper. Ertel’s theorem and potential vorticity, the Hollman invariant, and the Godbillon–Vey invariant for special flows for which the magnetic helicity is zero are also discussed.

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1. Introduction

Advected invariants and conservation laws in ideal magnetohydrodynamics (MHD) and gas dynamics have wide applications in space plasma physics, fusion and laboratory plasmas, and fluid dynamics. In space plasma physics and solar physics, magnetic helicity is of major interest in describing the topology and linkage of magnetic fields (e.g. Berger and Field 1984, Moffatt 1969, Moffatt and Ricca 1992, Woltjer 1958, Berger 1999, Berger and Ruzmaikin 2000, Bieber et al 1987, Yahalom and Lynden Bell 2008, Yahalom 2013, Webb et al 2010a, 2010b), Ruzmaikin and Akhmetiev (1994), Akhmetiev and Ruzmaikin (1995) and Berger (1990) investigated higher order knot invariants in MHD, known as Sato–Levine invariants. These invariants are associated with two-sided surfaces known as Seifert surfaces in which the knot is imbedded. They illustrated the theory using the Whitehead link and Borromean rings. Kuznetsov and Ruban (1998, 2000) and Kuznetsov (2006) have developed the Hamiltonian dynamics of vortex and magnetic field lines in hydrodynamic type systems. They use a mixed Eulerian and Lagrangian description (the so-called vortex line representation (VLR)). The VLR mapping describes the compressibility of the vortex lines even in incompressible flows, and can be used to describe the merging and collapse of the vortex lines. This work is clearly important in the development of topological fluid dynamics and Hamiltonian fluid dynamics, but is not explicitly addressed in the present paper.

The present paper gives a synopsis of advected invariants in MHD and gas dynamics. A short account of this work is given by Webb et al (2013) in a preliminary conference paper. The discussion is based in part on the paper of Tur and Yanovsky (1993), who use the ideas of Lie dragging of vectors, \( n \)-forms \((n = 1, 2, 3)\), scalars and tensors (i.e. conserved physical quantities), and the algebra of exterior differential forms to determine the advected invariants.

The concept of Lie dragging of advected invariants in MHD and gas dynamics was investigated by Tur and Yanovsky (1993) who extended previous work by Moiseev et al (1982). These ideas also appear in recent work on advected invariants and conservation laws in MHD and hydrodynamical models by Cotter and Holm (2013). Cotter and Holm (2013) use an approach based on the Eulerian, Euler Poincaré formulation of ideal hydrodynamical models. Cotter and Holm (2013) derive conservation laws associated with Noether’s second theorem, due to fluid relabeling symmetries. Their analysis does not require the introduction of Lagrangian variables. However, to obtain some of the conservation laws it is necessary to include Lagrange multipliers to take into account constraints in their variational principle. Hydon and Mansfield (2011) discuss and extend Noether’s second theorem by using Lagrange multipliers which explicitly shows that for variational problems involving free functions and infinite dimensional Lie algebraic structures, that there are differential relations between the different Euler operators occurring in Noether’s second theorem that must be taken into account (in Padhye and Morrison (1996a, 1996b) these relations are referred to as generalized Bianchi identities). The work of Cotter and Holm (2013) extends previous techniques used to derive conservation laws due to fluid relabeling symmetries. Earlier work by Salmon (1982, 1988) and Padhye and Morrison (1996a, 1996b) used a Lagrangian fluid dynamics approach.

Section 2 outlines the model equations.

Section 3 gives an overview of helicity in ideal fluid mechanics and MHD. The local helicity conservation law in ideal fluid mechanics is given for the case of an isobaric equation of state for the gas (i.e. the pressure \( p = p(\rho) \) is the equation of state for the gas). Integral forms of the helicity conservation equation and Ertel’s theorem in ideal fluid mechanics are discussed. Conservation laws for magnetic helicity and cross helicity in MHD are described. The concept of relative helicity in MHD is also described.
Section 4 outlines a theory for advected invariants in ideal fluid mechanics and MHD, based on the Lie dragging of invariant geometrical quantities with the fluid (e.g. vector fields and \( p \)-forms where \( p = 0, 1, 2, 3 \) in 3D MHD). The discussion is based on the work of Tur and Yanovsky (1993) on advected invariants in fluid and MHD systems of equations. We discuss the concept of topological charge for invariant advected differential forms in fluid dynamics and MHD. We also derive and discuss the Godbillon–Vey topological invariant. The Godbillon–Vey invariant in an MHD flow, arises for example, if \( \mathbf{A} \cdot \nabla \times \mathbf{A} = 0 \) where \( \mathbf{A} \) is the magnetic vector potential and \( \mathbf{B} = \nabla \times \mathbf{A} \) is the magnetic induction. In such a flow, the magnetic helicity is zero. However, there is a higher order topological invariant (the Godbillon–Vey invariant), which in general is non-zero. Thus a zero magnetic helicity field, can still have a non-trivial topology. The Godbillon–Vey invariant also occurs in ideal fluid dynamics for flows in which the fluid helicity \( \mathbf{u} \cdot \nabla \times \mathbf{u} = 0 \).

Section 5 gives an overview of the use of Clebsch variables in Lagrangian and Hamiltonian fluid mechanics. We also discuss the canonical and non-canonical Poisson bracket for MHD (e.g. Morrison and Greene 1980, 1982, Holm and Kupershmidt 1983a, 1983b) and Weber transformations.

Section 6 uses a Clebsch variable formulation of ideal fluid mechanics to derive a nonlocal helicity conservation law (6.1) for a fluid with a non-barotropic equation of state (i.e. \( p = p(\rho, S) \)). Clebsch variables are also used to derive the nonlocal cross-helicity conservation law (6.29) in MHD. These nonlocal conservation laws for helicity and cross helicity are two new results obtained in the present paper.

Section 7 concludes with a summary and discussion.

2. The model

The magnetohydrodynamic equations can be written in the form:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{2.1}
\]

\[
\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot \left[ \rho \mathbf{u} \mathbf{u} + \left( p + \frac{B^2}{2\mu} \right) \mathbf{I} - \frac{\mathbf{B} \mathbf{B}^T}{\mu} \right] = 0, \tag{2.2}
\]

\[
\frac{\partial S}{\partial t} + \mathbf{u} \cdot \nabla S = 0, \tag{2.3}
\]

\[
\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) + \mathbf{u} \nabla \cdot \mathbf{B} = 0, \tag{2.4}
\]

where \( \rho, \mathbf{u}, p, S \) and \( \mathbf{B} \) correspond to the gas density, fluid velocity, pressure, specific entropy, and magnetic induction \( \mathbf{B} \) respectively, and \( \mathbf{I} \) is the unit \( 3 \times 3 \) dyadic. The gas pressure \( p = p(\rho, S) \) is a function of the density \( \rho \) and entropy \( S \), and \( \mu \) is the magnetic permeability. Equations (2.1)–(2.2) are the mass and momentum conservation laws, (2.3) is the entropy advection equation and (2.4) is Faraday’s equation in the MHD limit.

In classical MHD, (2.1)–(2.4) are supplemented by Gauss’ law:

\[
\nabla \cdot \mathbf{B} = 0 \tag{2.5}
\]

which implies the non-existence of magnetic monopoles.

It is useful to keep in mind the first law of thermodynamics:

\[
T \, dS = dQ = dU + p \, dV \quad \text{where} \quad V = \frac{1}{\rho}, \tag{2.6}
\]

where \( U \) is the internal energy per unit mass and \( V = 1/\rho \) is the specific volume. To be more exact, the first law of thermodynamics should be \( \delta Q = dU + \delta W \), in order to emphasize that
in general, the internal energy density is a perfect differential, whereas $\delta Q$ and $\delta W$ are not necessarily perfect differentials (e.g. for the case of heat conduction and dissipation $\delta Q$ is not a perfect differential). However, for ideal gases, with no dissipation $\delta Q = T \, dS$ where $T$ is the temperature and $dS$ is a perfect differential. The integrability of Pfaffian differential forms of the form $X \, dx + Y \, dy + Z \, dz = 0$ is only assured iff $X \cdot \nabla \times X = 0$ where $X = (X, Y, Z)$. This result plays an important role in Caratheodory thermodynamics (e.g. Sneddon 1957). Using the internal energy per unit volume $\varepsilon = \rho U$ instead of $U$, (2.6) may be written as:

$$T \, dS = \frac{1}{\rho} \left( \delta \varepsilon - h \, d\rho \right) \quad \text{where} \quad h = \frac{\varepsilon + p}{\rho}, \quad (2.7)$$

is the enthalpy of the gas. Assuming $\varepsilon = \varepsilon(\rho, S)$ (2.7) gives the formulae:

$$\rho T = \varepsilon S, \quad h = \varepsilon p, \quad p = \rho \varepsilon - \varepsilon, \quad (2.8)$$

relating the temperature $T$, enthalpy $h$ and pressure $p$ to the internal energy density $\varepsilon(\rho, S)$. From (2.7) we obtain:

$$T \, dS = dh - \frac{1}{\rho} \, dp \quad \text{and} \quad - \frac{1}{\rho} \nabla p = T \nabla S - \nabla h, \quad (2.9)$$

which is useful in the further analysis of the momentum equation for the system.

It is worth noting that the different variants of helicity (fluid helicity, cross helicity and magnetic helicity) are pseudo-scalars as they reverse sign under space reversal $x \rightarrow -x$ (e.g. $u \cdot \nabla \times u$ has this property). This is an important property of helicity as it measures parity symmetry breaking. A parity invariant flow has zero helicity.

3. Helicity in fluids and MHD

In this section we give a brief overview of helicity and vorticity conservation laws in ideal fluid dynamics and MHD. For ideal barotropic fluids, with no magnetic field we discuss the helicity conservation law involving the helicity density $h_f = u \cdot \omega$, where $\omega = \nabla \times u$ is the fluid vorticity. The integral $H_f = \int_{V_m} h_f \, d^3x$ over a volume $V_m$ moving with the fluid, is known as the fluid helicity. It plays a key role in topological fluid dynamics in the description of the linkage of the vorticity streamlines (e.g. Moffatt 1969, Arnold and Khesin 1998). The integral $H_m = \int_{V_m} A \cdot B \, d^3x$ in MHD is known as the magnetic helicity, where $B = \nabla \times A$ is the magnetic induction and $A$ is the magnetic vector potential. It describes the linkage of the magnetic field lines (Woltjer 1958, Berger and Field 1984). A further quantity of interest in MHD is the cross helicity $H_c = \int_{V_m} u \cdot B \, d^3x$ which describes the topology of the magnetic field and fluid velocity streamlines. One of the main aims of the present paper is to show how these fluid and MHD invariants are obtained by Lie dragging invariant differential forms and scalars with the flow (Tur and Yanovsky 1993). We also describe helicity and cross-helicity conservation laws in MHD and gas dynamics.

3.1. Helicity in fluid dynamics

In a barotropic, ideal fluid, in which the pressure $p = p(\rho)$, is independent of the entropy $S$, the helicity density

$$h_f = u \cdot \omega \quad \text{where} \quad \omega = \nabla \times u, \quad (3.1)$$

satisfies the helicity conservation law:

$$\frac{\partial h_f}{\partial t} + \nabla \cdot \left( u h_f + \left( h - \frac{1}{2} |u|^2 \right) \omega \right) = 0. \quad (3.2)$$
The total helicity for a fluid volume $V_m$ moving with the fluid is conserved following the flow (e.g. Moffatt 1969). Thus, for a barotropic fluid,

$$\frac{dH_f}{dt} = 0 \quad \text{where} \quad H_f = \int_{V_m} \mathbf{u} \cdot \nabla \times \mathbf{u} \, d^3x,$$

(3.3)

where $H_f$ is the total helicity of the fluid in the volume $V_m$. For the conservation law (3.3) to apply, it is required that that the component of the vorticity $\omega_n$ normal to the boundary $\partial V_m$ vanish on $\partial V_m$, i.e. $\omega_n = \mathbf{\omega} \cdot \mathbf{n} = 0$ on $\partial V_m$. Here $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ is the total Lagrangian time derivative following the flow.

To derive (3.2), note that for an ideal gas, the momentum equation for the fluid:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p,$$

(3.4)

can be written in the form:

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \mathbf{\omega} = T \nabla S - \nabla \left( h + \frac{1}{2} |\mathbf{u}|^2 \right).$$

(3.5)

For the case of a barotropic equation of state, there is no $T \nabla S$ term in (3.5).

Taking the curl of the momentum equation (3.5) gives the vorticity equation:

$$\frac{\partial \mathbf{\omega}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{\omega}) = \nabla \times (T \nabla S).$$

(3.6)

Taking the scalar product of $\mathbf{\omega}$ with the momentum equation (3.5) and adding the scalar product of $\mathbf{u}$ with the vorticity equation (3.6) gives the equation

$$\frac{d}{dt} \left( \frac{\mathbf{u} \cdot \mathbf{\omega}}{\rho} \right) + \nabla \cdot \left[ \left( \mathbf{u} \cdot \mathbf{\omega} \right) \mathbf{u} + \left( h - \frac{1}{2} |\mathbf{u}|^2 \right) \mathbf{\omega} \right] = \mathbf{\omega} \cdot (T \nabla S) + \mathbf{u} \cdot \nabla \times \nabla S.$$

(3.7)

For a barotropic fluid there are no entropy gradients i.e. $\nabla S = 0$, and in that case (3.7) reduces to the helicity conservation law (3.2). The integral conservation law (3.3) can be derived by using the identity:

$$\frac{d}{dt} \left( \frac{\mathbf{u} \cdot \mathbf{\omega}}{\rho} \right) = -\frac{\mathbf{\omega}}{\rho} \cdot \nabla \left( h - \frac{1}{2} |\mathbf{u}|^2 \right).$$

(3.8)

The detailed proof of $dH_f/dt = 0$ is given by Moffatt (1969).

There are other conservation laws for ideal fluids. For example Kelvin’s theorem implies that the circulation $\Gamma = \int_C \mathbf{u} \cdot d\mathbf{x}$ is conserved following the flow, for an ideal, barotropic fluid, where $C$ is a closed path moving with the fluid, i.e. $d\Gamma/dt = 0$, where $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ is the Lagrangian time derivative moving with the flow (this result also holds if there is a conservative, external gravitational field present). However, the circulation is not conserved if there are entropy gradients in the flow, in which case $d\Gamma/dt = \int_A (\nabla T \times \nabla S) \cdot \mathbf{n} \, dA$, where $A$ is the area enclosing $C$ with normal $\mathbf{n}$.

**Theorem 3.1.** Ertel’s theorem. Ertel’s theorem in ideal fluid mechanics states that the potential vorticity $q = \mathbf{\omega} \cdot \nabla S/\rho$ is a scalar invariant advected with the flow, i.e.,

$$\frac{d}{dt} \left( \frac{\mathbf{\omega} \cdot \nabla S}{\rho} \right) = 0,$$

(3.9)

where $\mathbf{\omega} = \nabla \times \mathbf{u}$ is the fluid vorticity.

**Proof.** The vorticity equation (3.6) may be written in the form:

$$\frac{d\mathbf{\omega}}{dt} + \mathbf{\omega} \nabla \cdot \mathbf{u} - \mathbf{\omega} \cdot \nabla \mathbf{u} = \nabla \times (T \nabla S).$$

(3.10)
Using the mass continuity equation (2.1) in the form $\nabla \cdot \mathbf{u} = -(\partial \rho / \partial t)/\rho$ in (3.10) gives the equation:

$$\frac{d}{dt} \left( \frac{\omega}{\rho} \right) - \frac{\omega}{\rho} \nabla \mathbf{u} = - \nabla T \times \nabla S / \rho.$$  \hspace{1cm} (3.11)

Also using the entropy advection equation $dS/dt = 0$, we obtain:

$$\frac{d}{dt} \nabla S = \nabla \left( \frac{dS}{dt} \right) - (\nabla \mathbf{u})^T \cdot \nabla S \equiv - (\nabla \mathbf{u})^T \cdot \nabla S.$$  \hspace{1cm} (3.12)

Taking the scalar product of (3.11) with $\nabla S$ and the scalar product of (3.12) with $\omega / \rho$ and adding the resultant equations gives Ertel’s theorem (3.9). This completes the proof. \hfill \Box

### 3.2. Helicity in MHD

Magnetic helicity in space and fusion plasmas has been investigated as a key quantity describing the topology of magnetic fields (e.g. Moffatt 1969, 1978, Moffatt and Ricca 1992, Berger and Field 1984, Finn and Antonsen 1985, 1988, Rosner et al 1989, Low 2006, Longcope and Malanushenko 2008). The magnetic helicity $H$ is defined as:

$$H = \int_V \omega_1 \wedge d\omega_1 = \int_V d^3x \mathbf{A} \cdot \mathbf{B},$$  \hspace{1cm} (3.13)

where $\omega_1 = \mathbf{A} \cdot d\mathbf{x}$ is the magnetic vector potential 1-form, $d\omega_1 = \mathbf{B} \cdot d\mathbf{s}$ is the magnetic field 2-form; $\mathbf{B} = \nabla \times \mathbf{A}$ is the magnetic induction, $\mathbf{A}$ is the magnetic vector potential and $V$ is the isolated volume in which the magnetic field configuration of interest is located. The magnetic helicity is an invariant of MHD (Elsässer 1956, Woltjer 1958, Moffat 1969, 1978). In (3.13) it is assumed that the normal magnetic field $\mathbf{B}_n = \mathbf{B} \cdot \mathbf{n}$ vanishes on the boundary $\partial V$ of the volume $V$. The magnetic helicity (3.13) when expressed as an integral of $\omega_1 \wedge d\omega_1$ is known as the Hopf invariant.

For open-ended magnetic field configurations, a gauge independent definition of relative helicity for a magnetic field configuration in a volume $V$ (Finn and Antonsen 1985, 1988) is

$$H_r = \int_V d^3x (\mathbf{A}_1 + \mathbf{A}_2) \cdot (\mathbf{B}_1 - \mathbf{B}_2),$$  \hspace{1cm} (3.14)

(see also Berger and Field (1984) for an equivalent definition) where $\mathbf{B}_1 = \nabla \times \mathbf{A}_1$ describes the magnetic field of interest and $\mathbf{B}_2 = \nabla \times \mathbf{A}_2$ is a reference magnetic field with the same normal flux as $\mathbf{B}_1$. In many applications the reference magnetic field is a potential magnetic field, i.e. $\nabla \times \mathbf{B}_2 = 0$. Relative helicity is now commonly used in the modeling of solar magnetic structures (Longcope and Malanushenko 2008, Low 2006). Bieber et al (1987) and Webb et al (2010b) investigated the relative helicity of the Parker interplanetary spiral magnetic field. Berger and Ruzmaikin (2000) investigated the injection of magnetic helicity into the solar wind from the photospheric base based on observational data and the differential rotation of the Sun. Webb et al (2010a) obtained the relative helicity of shear and torsional Alfvén waves.

#### 3.2.1. Magnetic helicity conservation equation

For ideal MHD, $h_m = \mathbf{A} \cdot \mathbf{B}$ satisfies the conservation law:

$$\frac{\partial h_m}{\partial t} + \nabla \cdot [\mathbf{u} h_m + \mathbf{B} (\phi_E - \mathbf{A} \cdot \mathbf{u})] = 0,$$  \hspace{1cm} (3.15)

where

$$\mathbf{E} = - \nabla \phi_E - \frac{\partial \mathbf{A}}{\partial t} = - \mathbf{u} \times \mathbf{B}, \quad \mathbf{B} = \nabla \times \mathbf{A}.$$  \hspace{1cm} (3.16)

In (3.16) $\mathbf{E} = - \mathbf{u} \times \mathbf{B}$ is Ohm’s law in the infinite conductivity limit.
To derive (3.15) use Faraday’s law \( \mathbf{B}_t + \nabla \times \mathbf{E} = 0 \) and \( \mathbf{B} = \nabla \times \mathbf{A} \) to obtain the equations:

\[
\mathbf{B}_t + \nabla \times \mathbf{E} = 0, \tag{3.17}
\]

\[
\mathbf{A}_t + \mathbf{E} + \nabla \phi_E = 0, \tag{3.18}
\]

where \( \phi_E \) is the electric field potential. Note that the curl of (3.18) gives Faraday’s law (3.17). Combining (3.17)–(3.18) gives the equation

\[
\frac{\partial}{\partial t} (\mathbf{A} \cdot \mathbf{B}) + \nabla \cdot (\mathbf{E} \times \mathbf{A} + \phi_E \mathbf{B}) = -2\mathbf{E} \cdot \mathbf{B}. \tag{3.19}
\]

Noting \( \mathbf{E} = -\mathbf{u} \times \mathbf{B} \) and \( \mathbf{E} \cdot \mathbf{B} = 0 \) in (3.19) gives helicity conservation equation (3.15) for ideal MHD.

The total magnetic helicity \( H_m = \int_{V_m} \mathbf{A} \cdot \mathbf{B} \, d^3x \) moving with the flow is invariant, i.e. \( dH_m/dt = 0 \) provided \( \mathbf{B} \cdot \mathbf{n} = 0 \) on the boundary surface \( \partial V_m \) of the volume \( V_m \). The proof is similar to the proof that the helicity in ideal fluid mechanics: \( H_f = \int_{V_f} \mathbf{u} \cdot \nabla \times \mathbf{u} \, d^3x \) is conserved following the flow (see (3.3) and also Moffatt (1969, 1978)).

Consider the choice of the gauge for \( \mathbf{A} \). By setting \( \mathbf{B} = \nabla \times \mathbf{A}, \) (3.16) may be written in the form

\[
\frac{d\tilde{\mathbf{A}}}{dt} = \nabla (\mathbf{A} \cdot \mathbf{u} - \phi_E) - (\nabla \mathbf{u})^\top \cdot \mathbf{A}, \tag{3.20}
\]

where \( d/dt = \partial / \partial t + \mathbf{u} \cdot \nabla \) is the Lagrangian time derivative. Using the gauge transformation:

\[
\tilde{\mathbf{A}} = \mathbf{A} + \nabla \Lambda \quad \text{where} \quad \frac{d\Lambda}{dt} + \mathbf{A} \cdot \mathbf{u} - \phi_E = 0, \tag{3.21}
\]

in (3.20), Faraday’s equation, for \( \tilde{\mathbf{A}} \) reduces to:

\[
\frac{d\tilde{\mathbf{A}}}{dt} + (\nabla \mathbf{u})^\top \cdot \tilde{\mathbf{A}} = 0, \tag{3.22}
\]

Equation (3.22) can also be written in the form:

\[
\frac{\partial \tilde{\mathbf{A}}}{\partial t} - \mathbf{u} \times (\nabla \times \tilde{\mathbf{A}}) + \nabla (\mathbf{u} \cdot \tilde{\mathbf{A}}) = 0. \tag{3.23}
\]

The latter equation is equivalent to (3.16) for the electric field \( \mathbf{E} = -\mathbf{u} \times \mathbf{B} \) in the form:

\[
\mathbf{E} = -\nabla (\mathbf{u} \cdot \tilde{\mathbf{A}}) - \frac{\partial \tilde{\mathbf{A}}}{\partial t}, \tag{3.24}
\]

which shows that the electric potential in the new gauge is \( \hat{\phi}_E = \mathbf{u} \cdot \tilde{\mathbf{A}} \). The evolution equation (3.23) is equivalent to the equation \( d/dt(\hat{\mathbf{A}} \cdot d\mathbf{x}) = 0 \) (see section 4), which shows that the 1-form \( \alpha = \hat{\mathbf{A}} \cdot d\mathbf{x} \) is Lie dragged by the flow. Combining (3.22) with Faraday’s equation for \( \mathbf{B} \) gives the helicity transport equation:

\[
\frac{\partial \hat{h}}{\partial t} + \nabla \cdot (\hat{h} \mathbf{u}) = 0, \tag{3.25}
\]

where \( \hat{h} = \hat{\mathbf{A}} \cdot \mathbf{B} \).

The gauge choice (3.21) appears to be the best choice of the gauge potential \( \Lambda \) in the formulation of magnetic helicity and magnetic helicity related conservation laws (section 4), since it fits in with the idea that \( \hat{\mathbf{A}} \cdot d\mathbf{x} \) is an invariant, Lie dragged 1-form, and gives the simplest continuity equation for the helicity conservation law (3.25).

A question that naturally arises is what happens to the advection equation (3.23) in the limit as \( |\mathbf{u}| \to 0 \). Assuming \( \partial \hat{\mathbf{A}} / \partial t \sim \epsilon \hat{\mathbf{g}}(\mathbf{x}) \) and \( |\mathbf{u}| = \epsilon \), then in the limit as \( \epsilon \to 0 \), (3.23) reduces to the equation:

\[
\epsilon [\hat{\mathbf{g}} - \hat{\mathbf{u}} \times \mathbf{B} + \nabla A_\parallel] = 0, \quad \text{where} \quad A_\parallel = \mathbf{A} \cdot \hat{\mathbf{u}}, \tag{3.26}
\]
constancy of the fluid helicity of interest. It is straightforward to adapt the argument (3 as a rugged invariant of MHD (Matthaeus MHD invariant, it is implicitly assumed that but will not be pursued further in the present paper. consistent to use this gauge in this limit? This issue needs to be investigated in further detail, but will not be pursued further in the present paper.

3.2.2. Cross Helicity in MHD. The cross helicity (for \(p = p(\rho)\)) is defined as the integral:

\[
C[u, B] = \int_B d^3 \mathbf{x} \cdot \mathbf{B}.
\]

It is a Casimir of barotropic MHD (\(p = p(\rho)\)), i.e. \(\{F, C\} = 0\), for any functional \(F\) where \(\{\ldots\}\) is the MHD Poisson bracket (Padhye and Morrison 1996a, 1996b). It is also referred to as a rugged invariant of MHD (Matthaeus et al 1982). In order for the cross helicity to be an MHD invariant, it is implicitly assumed that \(\mathbf{B} \cdot \mathbf{n} = 0\) on the boundary \(\partial V\) of the volume \(V\) of interest. It is straightforward to adapt the argument (3.3) used to show the invariance or constancy of the fluid helicity \(H_f\) to show that \(\partial H_c/\partial t = 0\) where \(H_c = \int_{V_m} \mathbf{u} \cdot \mathbf{B} \, d^3 x\) is the cross helicity for a volume \(V_m\) moving with the fluid.

The cross-helicity density conservation law (for \(p = p(\rho)\)) is:

\[
\frac{\partial h_c}{\partial t} + \nabla \cdot \left[ u h_c + B \left( h - \frac{1}{2} |u|^2 \right) \right] = 0 \quad \text{where} \quad h_c = u \cdot B, \quad (3.30)
\]

and \(h = (p + \varepsilon)/\rho\) is the gas enthalpy. Equation (3.30) also holds if \(p = p(\rho, S)\) and \(\mathbf{B} \cdot \nabla S = 0\). Conservation law (3.30) is due to fluid relabeling symmetries. To derive (3.30) we use the Faraday and momentum equations:

\[
\frac{d\mathbf{B}}{dt} - \nabla \times (\mathbf{u} \times \mathbf{B}) = 0, \quad \frac{d\mathbf{u}}{dt} = - \frac{1}{\rho} \nabla p + \frac{\mathbf{J} \times \mathbf{B}}{\rho}, \quad (3.31)
\]

where \(\mathbf{J} = \nabla \times \mathbf{B}/\mu_0\) and \(d\mathbf{u}/dt = (\partial + \mathbf{u} \cdot \nabla) \mathbf{u}\) and we assume \(\nabla \cdot \mathbf{B} = 0\). Using the first law of thermodynamics (2.9) the momentum equation for the MHD fluid can be written in the form:

\[
\mathbf{u}_\tau - \mathbf{u} + \frac{1}{2} |\mathbf{u}|^2 \nabla \left( h - \frac{1}{2} |\mathbf{u}|^2 \right) - \frac{\mathbf{J} \times \mathbf{B}}{\rho} - T \nabla S = 0, \quad (3.32)
\]

where \(\mathbf{u}_\tau = \partial \mathbf{u}/\partial t\). Taking the scalar product of Faraday’s equation with \(\mathbf{u}\) and the scalar product of the momentum equation (3.32) with \(\mathbf{B}\) gives the cross-helicity equation:

\[
\frac{\partial}{\partial t} \left( \mathbf{u} \cdot \mathbf{B} \right) + \nabla \cdot [\mathbf{E} \times \mathbf{u} + (h + (1/2) |\mathbf{u}|^2) \mathbf{B}] = \nabla \cdot \mathbf{B} \cdot \nabla S. \quad (3.33)
\]

If \(\nabla S = 0\), equation (3.33) reduces to the cross-helicity conservation law (3.30).
The helicity conservation equation (3.2) holds for a barotropic gas, in which there are no entropy gradients. Similarly, the cross-helicity conservation law (3.30) holds provided \( B \cdot \nabla S = 0 \).

4. Advected invariants

Tur and Yanovsky (1993) developed a formalism for geometrical objects \( G \) (tensors, \( p \)-forms and vectors) that are advected with the flow in ideal gas dynamics and MHD. The basic requirement for \( G \) to be advected or Lie dragged with the flow \( u \) is that
\[
\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) G = \left( \frac{\partial}{\partial t} + L_u \right) G = 0, \tag{4.1}
\]
where \( L_u \) denotes the Lie derivative with respect to the vector field \( u \). As in the Calculus of exterior differential forms and in differential geometry (e.g. Harrison and Estabrook 1971, Misner et al 1973, Fecko 2006), vector fields \( V \) and 1-forms \( \omega = A_i \text{d}x^i \equiv A \cdot \text{d}x \) are dual.

4.1. Exterior differential forms and vector fields

Useful discussions of the algebra of exterior differential forms may be found for example in the books by Frankel (1997), Bott and Tu (1982), Misner et al (1973), Marsden and Ratiu (1994), Holm (2008a, 2008b), Flanders (1963). A useful short summary is given in the paper by Harrison and Estabrook (1971), who develop a geometric approach to invariance groups and solutions of partial differential systems using Cartan’s geometric formulation of partial differential equations in the language of exterior differential forms and vector fields. This formalism was used by Tur and Yanovsky (1993) in their work on advected invariants in fluids and MHD plasmas.

The vector field \( V \) in 3D Cartesian geometry is thought of as a directional derivative operator:
\[
V = V^i \frac{\partial}{\partial x^i} + V^j \frac{\partial}{\partial y^j} + V^z \frac{\partial}{\partial z^z}, \tag{4.2}
\]
and the 1-form \( A \cdot \text{d}x \) has the form:
\[
\omega = A \cdot \text{d}x = A_x \, \text{d}x + A_y \, \text{d}y + A_z \, \text{d}z. \tag{4.3}
\]
The inner product of vector \( u \) and 1-form \( \omega \) is the scalar or dot product:
\[
\langle u, \omega \rangle = \left( u^i \frac{\partial}{\partial x^i} + u^j \frac{\partial}{\partial y^j} + u^z \frac{\partial}{\partial z^z}, A_x \, \text{d}x + A_y \, \text{d}y + A_z \, \text{d}z \right) = u^i A_i + u^j A_j + u^z A_z \equiv u \cdot A. \tag{4.4}
\]
Equivalent notations for the inner product are:
\[
\langle u, \omega \rangle \equiv u \cdot \omega \equiv i_u \omega, \tag{4.5}
\]
where \( i_u \) denotes inner product of contravariant field \( u \) with a covariant field \( \omega \):
\[
\left( \frac{\partial}{\partial x^i}, \text{d}x^j \right) = \delta_{ij}, \quad \text{e.g.} \quad \left( \frac{\partial}{\partial x}, \text{d}x \right) = 1, \quad \left( \frac{\partial}{\partial x}, \text{d}y \right) = 0. \tag{4.6}
\]
In a differentiable manifold of dimension \( n \), a \( p \)-form \( \omega \) can be thought of as a completely antisymmetric covariant \( p \)-rank tensor, described by its anti-symmetric components \( \omega_{\mu_1 \cdots \mu_p} \). The \( p \)-form in general can be expressed in the form:
\[
\omega = \omega_{\mu_1 \cdots \mu_p} \, \text{d}x^{\mu_1} \wedge \text{d}x^{\mu_2} \wedge \cdots \wedge \text{d}x^{\mu_p}. \tag{4.7}
\]
1-forms can be thought of as elements of the cotangent space at a point of a manifold, involving the mapping of the vector fields in the tangent space onto the reals. In (4.7) \( \wedge \) denotes a non-commutative anti-symmetrized multiplication. At each point of the manifold, the forms have a Grassmann algebra defined by the properties of the wedge product operator \( \wedge \). Over the manifold we may use the three operations of exterior differentiation, \( d \), of contraction with a vector field \( V \) (a contravariant vector \( V^\mu \)), \( V \cdot \omega \), and of Lie derivative with respect to \( V \), \( \mathcal{L}_V \omega \). These operations give forms of rank \( p + 1 \), \( p - 1 \) and \( p \) respectively.

Misner et al (1973) and Schutz (1980) and other texts emphasize the geometrical picture of the commutator of two vector fields in terms of the closure of the quadrilateral associated with the two vector fields. In Misner et al (1973), the Faraday 2-form is visualized in terms of an egg-crate like structure. For example, the magnetic field 2-form \( B \cdot dS \), has a geometric structure associated with the oriented surface element \( dS \) and the vector field \( B \), which describes magnetic flux tubes. The lie dragging of vectors, differential forms and tensors involves the concept of parallel transport, in which the change in the geometric quantity of interest at a point along a Lie orbit or trajectory, must be pulled back to the initial point involved in the derivative to make sense. It is also useful in some applications to use the dual of vector fields and forms using the hodge star formalism (e.g. Flanders 1963, Frankel 1997, Fecko 2006).

Some of the basic properties of the wedge product, \( \wedge \), of the exterior derivative \( d \) and the Lie derivative \( \mathcal{L}_V \) are given below.

Let \( \omega \) be a \( p \)-form, \( \sigma \) a \( q \)-form, \( f \) a 0-form, \( c \) a constant, \( V \) and \( W \) be vector fields, then:

\[
\omega \wedge \sigma = (-1)^{pq} \sigma \wedge \omega,
\]

\[
d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^p \omega \wedge d\sigma,
\]

\[
\begin{align*}
d\omega &= 0, \quad dc = 0, \\
(V + W) \cdot \omega &= V \cdot \omega + W \cdot \omega, \quad (fV) \cdot \omega = f(V \cdot \omega), \\
V \cdot (\omega \wedge \sigma) &= (V \cdot \omega) \wedge \sigma + (-1)^p \omega \wedge (V \cdot \sigma).
\end{align*}
\]

(4.8)

Cartan’s magic formula for the Lie derivative of the \( p \)-form \( \omega \):

\[
\mathcal{L}_V \omega = V \cdot d\omega + d(V \cdot \omega),
\]

is a particularly useful formula in applications. Other Lie derivative formulae are:

\[
\begin{align*}
\mathcal{L}_V f &= V \cdot df, \quad \mathcal{L}_V d\omega = d(\mathcal{L}_V \omega), \\
\mathcal{L}_V (\omega \wedge \sigma) &= (\mathcal{L}_V \omega) \wedge \sigma + \omega \wedge (\mathcal{L}_V \sigma), \\
\mathcal{L}_V (W \cdot \omega) &= [V, W] \cdot \omega + W \cdot (\mathcal{L}_V \omega).
\end{align*}
\]

(4.9)

**Exterior derivative formula relations (vector notation)**

\[
\begin{align*}
df &= \nabla f \cdot dx, \\
d(V \cdot dx) &= (\nabla \times V) \cdot dS \quad \text{(Stokes thm)}, \\
d(A \cdot dS) &= (\nabla \cdot A) dV \quad \text{(Gauss thm)}, \\
d^2 f &= d(\nabla f \cdot dx) = (\nabla \times \nabla f) \cdot dS = 0 \quad \text{(Poincaré lemma)}, \\
d^2(V \cdot dx) &= d[(\nabla \times V) \cdot dS] = \nabla \cdot (\nabla \times V) dV = 0 \quad \text{(Poincaré lemma)} \\
X \cdot (V \cdot dx) &= V \cdot X, \\
X \cdot (B \cdot dS) &= -(X \times B) \cdot dx, \\
X \cdot dV &= X \cdot dS, \\
(dX \cdot dV) &= d(X \cdot dS) = (\nabla \cdot X) dV.
\end{align*}
\]

(4.10)
useful in describing advected invariants. 1-forms, 2-forms, 3-forms and vector fields are given below. These formulae are particularly Fecko (2006).

For vector fields

\[ \mathcal{L}_X f = X \cdot df = X \cdot \nabla f, \]
\[ \mathcal{L}_X (V \cdot dx) = (-X \times (\nabla \times V) + \nabla (X \cdot V)) \cdot dx, \]
\[ \mathcal{L}_X (B \cdot dS) = (-\nabla \times (X \times B) + X (\nabla \cdot B)) \cdot dS, \]
\[ \mathcal{L}_X (f dV) = \nabla \cdot (X f) dV. \] (4.13)

For vector fields \( X \) and \( Y \)

\[ \mathcal{L}_X Y = [X, Y] = (X \cdot Y - Y \cdot X) \cdot \nabla \equiv \text{ad}_X(Y). \] (4.14)

Here \([X, Y]\) is the left Lie Bracket.

For a 1-form density \( m = m \cdot dx \otimes dV \):

\[ \mathcal{L}_X m = (\nabla \cdot (X \otimes m) + (\nabla X)^T \cdot m) \cdot dx \otimes dV =: \text{ad}_X^* m. \] (4.15)

The pairing between the 1-form density \( m \) and the vector field \( u \) is defined by the inner product:

\[ \langle m, u \rangle = \int \langle u, m \rangle dV. \] (4.16)

Vector fields can be either left or right invariant vector fields. Thus, associated with the group transformation \( x = g x_0 \), the right invariant vector field \( u = g x_0 = g g^{-1} x \) defines the right invariant vector field \( u = g g^{-1} \). The corresponding left invariant version of the same vector field is \( v = g^{-1} g \). The right and left Lie brackets are related by \([U, V]_R = -[U, V]_L\). The left Lie bracket is used in (4.14). The right Lie bracket used in (4.15) is given by:

\[ \text{ad}_U(V) = [U, V]^R = (V \cdot \nabla U - U \cdot \nabla V) \cdot \nabla. \] (4.17)

A more detailed discussion of the difference between right and left vector fields of a Lie algebra are given by Marsden and Ratiu (1994), Holm et al (1998), Holm (2008a, 2008b) and Fecko (2006).

**Lie dragging of forms and vector fields.** Useful formulas for the Lie dragging of 0-forms, 1-forms, 2-forms, 3-forms and vector fields are given below. These formulae are particularly useful in describing advected invariants.

For 0-forms or functions \( I \)

\[ \frac{dI}{dt} = \frac{\partial I}{\partial t} + u \cdot \nabla I = 0. \] (4.18)

For 1-forms \( S \cdot dx \)

\[ \frac{d}{dt} (S \cdot dx) = \left( \frac{\partial S}{\partial t} - u \times (\nabla \times S) + \nabla (u \cdot S) \right) \cdot dx = 0. \] (4.19)

For 2-forms \( B \cdot dS \)

\[ \frac{d}{dt} (B \cdot dS) = \left( \frac{\partial B}{\partial t} - \nabla \times (u \times B) + u (\nabla \cdot B) \right) \cdot dS = 0. \] (4.20)

For 3-forms \( \rho \, dx \wedge dy \wedge dz \)

\[ \frac{d}{dt} (\rho \, dx \wedge dy \wedge dz) = \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) \right) \, dx \wedge dy \wedge dz = 0 \] (4.21)

For vector fields (the dual of 1-forms): \( J = J^i \nabla_i \)

\[ \frac{dJ}{dt} = \frac{\partial J}{\partial t} + [u, J] = 0, \quad \text{where} \quad [u, J] = (u \cdot \nabla J^i - J \cdot \nabla u^i) \nabla_i, \] (4.22)

is the Lie bracket of \( u \) and \( J \). There are many invariants which are advected with the flow involving \( A, B, S, \) and \( \rho \) (e.g. Tur and Yanovsky 1993).
4.2. Applications

Consider the quantities

\[ S' = \nabla S(x, t), \quad \Gamma' = \frac{A \cdot B}{\rho}, \quad \rho' = A \cdot B. \]  

One can show that

\[ \frac{d}{dt} \Gamma' = 0, \quad \frac{d}{dt} (S' \cdot dx) = 0, \quad \frac{d}{dt} (B \cdot dS) = 0, \quad \frac{d}{dt} (\rho' dx \wedge dy \wedge dz) = 0, \]  

where

\[ \frac{d}{dt} = \frac{\partial}{\partial t} + u \cdot \nabla \equiv \frac{\partial}{\partial t} + \mathcal{L}_u, \]  

is the Lagrangian or advective time derivative following the flow, and \( \mathcal{L}_u \) denotes the Lie derivative with respect to the vector field \( u \). Here \( \Gamma' \) is a scalar or 0-form, \( \nabla S \cdot dx \) is a 1-form, \( B \cdot dS \) is a 2-form and \( \rho' dx \wedge dy \wedge dz \) is a 3-form, which are advected invariants that are Lie dragged by the flow (i.e. these quantities remain invariant moving with the flow). The advection invariance of the Faraday 2-form \( B \cdot dS \) is equivalent to Faraday’s equation (2.4). There are many other invariants. Some integral invariants are:

\[ \Gamma_1 = \int_{\gamma(t)} \Phi A \cdot dl, \quad \Gamma_1 = \int_{\Sigma(t)} \Phi B \cdot dS, \quad \Gamma_2 = \int_{\Omega(t)} \Phi (A \cdot B) \, d^3x, \]

\[ I_4 = \int_{\Omega(t)} \Phi \left[ \nabla S \times \nabla \left( \frac{A \cdot B}{\rho} \right) \right] \, d^3x, \]

\[ \Phi = \Phi\left( \frac{A \cdot B}{\rho}, S, \frac{B}{A \cdot B} \cdot \nabla \left( \frac{A \cdot B}{\rho} \right), \frac{B}{\rho} \cdot \nabla \left( \frac{B \cdot S}{\rho} \right), \cdots \right). \]

where \( \Phi \) is an arbitrary function of its arguments.

4.3. Lie dragging

**Example 1.** Consider the results of Lie dragging the Faraday 2-form:

\[ \beta = B_1 \, dy \wedge dz + B_2 \, dz \wedge dx + B_3 \, dx \wedge dy \equiv B \cdot dS, \]  

where \( u = u^i \partial_i + u^i \partial_j + u^i \partial_z = u \cdot \nabla \) is the vector field representing the fluid velocity (here we use the notation of vector fields and 1-forms used in modern differential geometry (e.g. Misner et al. 1973), in which the base vectors for contravariant vector fields are written as \( \partial_i \equiv e^i \) and the base vectors for 1-forms or covariant vector fields are written as \( dx^i \equiv e^i \)). We use Cartan’s magic formula:

\[ \mathcal{L}_u (\beta) = u \lrcorner \, d\beta + d(u \lrcorner \beta). \]  

Calculating \( d\beta \) and \( u \lrcorner \, d\beta \) and \( d(u \lrcorner \beta) \) gives:

\[ d\beta = \nabla \cdot B \, dx \wedge dy \wedge dz, \quad u \lrcorner \, d\beta = \nabla \cdot B \, (u \cdot dS), \]

\[ u \lrcorner \beta = -(u \times B) \cdot dx, \quad d(u \lrcorner \beta) = -\nabla \times (u \times B) \cdot dS. \]

Using the results (4.29) in Cartan’s formula (4.28) gives

\[ \left( \frac{\partial}{\partial t} + \mathcal{L}_u \right) \beta = \left( \frac{\partial B}{\partial t} - \nabla \times (u \times B) + u \nabla \cdot B \right) \cdot dS = 0, \]

which implies Faraday’s equation (note \( \nabla \cdot B \) is advected with the flow if \( \nabla \cdot B \neq 0 \) (e.g. in numerical MHD)).
4.3.1. Entropy and mass advection.

Example 2. Consider the effect of Lie dragging the 1-form:
\[ \alpha = A_x \, dx + A_y \, dy + A_z \, dz \equiv A \cdot d\mathbf{x}. \] (4.31)

Using Cartan’s magic formula:
\[ \mathcal{L}_u (\alpha) = u \cdot d\alpha + d(u \cdot \alpha), \] (4.32)

and the results
\[ d\alpha = (\nabla \times A) \cdot d\mathbf{S}, \quad u \cdot d\alpha = -[u \times (\nabla \times A)] \cdot d\mathbf{x}, \]
we obtain
\[ u \cdot \alpha = (u \cdot A), \quad d(u \cdot \alpha) = \nabla (u \cdot A) \cdot d\mathbf{x}, \] (4.33)

If \((\partial_t + \mathcal{L}_u)\alpha = 0\), then \(\alpha\) is Lie dragged with the flow. Comparing (4.34) with (3.23)–(3.24) it follows that \(\mathbf{A} \cdot d\mathbf{x}\) is Lie dragged by the flow if \(\phi_E = u \cdot A\). In this special gauge \(\mathbf{A} \cdot \mathbf{B}/\rho\) is an advected invariant (see (4.49)).

Example 3. Faraday’s equation (2.4) combined with the mass continuity equation (2.1) implies:
\[ \frac{\partial \mathbf{b}}{\partial t} + [\mathbf{u}, \mathbf{b}] = \left( \frac{\partial \mathbf{b}}{\partial t} + \mathcal{L}_u \right) \mathbf{b} = 0 \quad \text{where} \quad \mathbf{b} = \frac{\mathbf{B}}{\rho}, \] (4.35)

and \([\mathbf{u}, \mathbf{b}]\) is the Lie bracket of the vector fields \(\mathbf{u}\) and \(\mathbf{b}\), i.e.
\[ \mathcal{L}_u (\mathbf{b}) = [\mathbf{u}, \mathbf{b}] = [\mathbf{u}, \mathbf{b}] \nabla \mathcal{I} = (u^i \nabla_j b^i - b^i \nabla_j u^i) \nabla_j. \] (4.36)

The vector field \(\mathbf{b}\) is Lie dragged with the fluid, and hence
\[ b_i \frac{\partial}{\partial x^i} = b_i^0 \frac{\partial}{\partial x_0^i} \quad \text{or} \quad b^i = x_j b^j, \quad x_j = \frac{\partial x^j}{\partial x_0^i}, \] (4.37)

where \(\mathbf{x} = \mathbf{x}(x_0, t)\) is the Lagrangian map. From (4.35) and (4.37) we obtain:
\[ B^i = \frac{x_j B_j^i(x_0)}{J} \quad \text{where} \quad J = \det(x_{ij}), \] (4.38)

which is Cauchy’s solution for \(\mathbf{B}\) (e.g. Newcomb 1962, Parker 1979).

4.3.1. Entropy and mass advection. The entropy \(S = S(x_0)\), is a 0-form (i.e. a function) which is Lie dragged with the fluid, i.e.
\[ \left( \frac{\partial}{\partial t} + \mathcal{L}_u \right) S = \frac{\partial S}{\partial t} + \mathbf{u} \cdot \nabla S = 0, \] (4.39)

which is (4.18) for the advection of a 0-form \(I\), but with \(I \rightarrow S\). The integral of (4.39) is \(S = S(x_0)\), where \(x_0\) is the Lagrange fluid label for which \(\mathbf{x} = x_0\) at time \(t = 0\).

Consider the mass 3-form:
\[ \beta = \rho \, dx \wedge dy \wedge dz. \] (4.40)

Using Cartan’s formula (4.28) we find \(d\beta = 0\) as \(\beta\) is a 3-form in 3D \(xyz\)-space, and
\[ u \cdot \beta = \rho u \cdot dS, \] which implies:
\[ \mathcal{L}_u (\beta) = 0 + d(u \cdot \beta) = \nabla \cdot (\rho u) \, dx \wedge dy \wedge dz \] (4.41)

and
\[ \left( \frac{\partial}{\partial t} + \mathcal{L}_u \right) \beta = \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) \right) \, d^3x = 0. \] (4.42)

Equation (4.42) is the same as (4.21) for an advected 3-form \(\rho \, dx \wedge dy \wedge dz\). The integral of (4.42) is:
\[ \rho \, dx_0 = \rho_0 \, dx_0, \quad \text{where} \quad \rho = \rho_0(x_0)/J, \quad J = \det(x_{ij}). \] (4.43)

Thus the mass continuity, entropy advection and Faraday’s equation can all be expressed in terms of the Lie dragging of forms by the vector field \(\mathbf{u}\).
4.4. Theorems for advected invariants

**Theorem 4.1.** If $\omega^p$ is an invariant, then $\omega^{p+1} = d\omega^p$ is an invariant $(p+1)$-form.

**Proof.** $\omega^p$ is invariant implies:

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_u\right)\omega^p = 0.$$  \hfill (4.44)

Take $d$ of (4.44). Use $d\delta = \delta d$, and $d\mathcal{L}_u = \mathcal{L}_u d$ gives (4.44) but with $\omega^p \to \omega^{p+1}$. \hfill \Box

**Example.** The entropy $S$ is a scalar invariant implies $\alpha = dS = \nabla S \cdot dx$ is a conserved advected 1-form.

**Theorem 4.2.** Let $\omega^k_1$ and $\omega^l_2$ be advected $k$ and $l$-form invariants, then $\omega^{k+l} = \omega^k_1 \wedge \omega^l_2$ is an advected $(k+l)$-form invariant.

**Proof.** Use

$$\frac{\partial}{\partial t} (\omega_1 \wedge \omega_2) = \frac{\partial \omega_1}{\partial t} \wedge \omega_2 + \omega_1 \wedge \frac{\partial \omega_2}{\partial t}$$

$$\mathcal{L}_u (\omega_1 \wedge \omega_2) = \mathcal{L}_u (\omega_1) \wedge \omega_2 + \omega_1 \wedge \mathcal{L}_u (\omega_2).$$ \hfill (4.45)

to get

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_u\right) (\omega_1 \wedge \omega_2) = 0.$$ \hfill (4.46)

**Example.** $\omega_1 = S_1 \cdot dx$ and $\omega_2 = S_2 \cdot dx$ are advected 1-forms, then

$$\omega_1 \wedge \omega_2 = (S_1 \times S_2) \cdot dS,$$ \hfill (4.47)

is an advected 2-form, and $(S_1 \times S_2)/\rho$ is an advected invariant vector field.

There are further theorems on the formation of advected invariants from known advected invariants described by Tur and Yanovsky (1993). Some of these theorems are listed below, without proof. Cartan’s magic formula is useful in many of the proofs.

**Theorem 4.3.** If $\omega$ is a conserved $p$-form, and $J$ is a conserved vector, then $\omega^{(p-1)} = J \cdot \omega$ is a conserved $(p-1)$-form.

**Theorem 4.4.** If $\omega$ is an invariant $p$-form, and $J$ is an invariant vector field, then $\omega' = \mathcal{L}_J \omega$ is an invariant $p$-form.

**Theorem 4.5.** If $J_1$ and $J_2$ are invariant vector fields then so is $[J_1, J_2]$ iff $[J_1, J_2, u]$ are elements of a Lie algebra.

**Comment.** The question of Lie algebraic structures for fluid relabeling symmetries has been addressed by Volkov et al (1995). Their work shows that there is a hidden supersymmetry in hydrodynamical systems (i.e. ideal MHD and hydrodynamics), with respect to the odd Buttin bracket.

4.5. Magnetic helicity

$\alpha = A \cdot dx$ is advected 1-form for the magnetic vector potential, provided the gauge for $A$ is chosen so that $\phi_E = u \cdot A$. The Lie dragging condition for $\alpha = A \cdot dx$ implies

$$\frac{\partial A}{\partial t} - u \times (\nabla \times A) + \nabla (u \cdot A) = 0.$$ \hfill (4.48)

This equation can be written as $dA/dt + (\nabla u)^T \cdot A = 0$. The magnetic field $A = A \cdot dS$ and the vector field $B = B / \rho$ are Lie dragged with the flow. Thus, $B \cdot (A \cdot dx) \equiv A \cdot B / \rho$ is a Lie dragged scalar invariant. Thus, we obtain the magnetic helicity conservation law:

$$\frac{d}{dt} \left(\frac{A \cdot B}{\rho}\right) = 0 \quad \text{or} \quad \frac{\partial h_m}{\partial t} + \nabla \cdot (h_m u) = 0,$$ \hfill (4.49)

where $h_m = A \cdot B$ is the magnetic helicity in the gauge $\phi_E = u \cdot A$. 

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4.6. The Ertel invariant and related invariants

In this section we discuss Ertel’s theorem in gas dynamics, and the generalization of Ertel’s equation to MHD (e.g. Kats 2003). The MHD generalization of Ertel’s theorem uses the Clebsch variable representation of the fluid velocity, that arises from using Lagrangian constraints in the variational principle for MHD discussed by Zakharov and Kuznetsov (1997). We also discuss the Hollmann (1964) invariant, which is related to the Ertel invariant (e.g. Tur and Yanovsky 1993). The Ertel invariant is:

\[ I_e = \frac{\omega \cdot \nabla S}{\rho} \]

where \( \omega = \nabla \times \mathbf{u} \).

(4.50)

To derive the Ertel invariant we use the Clebsch representation for \( \mathbf{u} \):

\[ \mathbf{u} = \nabla \phi - r \nabla S - \lambda \nabla \mu, \]

\[ \phi = \int_0^t \left( \frac{1}{2} |\mathbf{u}|^2 - h \right)(x_0, t') \, dt', \quad r = -\int_0^t T_0(x_0, t') \, dt', \]

(4.51)

where \( h = (p + \epsilon)/\rho \) is the enthalpy, \( S \) is the entropy, \( \phi \) is the velocity potential, and \( T_0(x_0, t) = T(x, t) \) is the temperature. \( \lambda \) and \( \mu \) are related to the Lin constraints associated with vorticity in a Lagrangian variational principle with constraints (e.g. Zakharov and Kuznetsov 1997). The Clebsch variable representation for \( \mathbf{u} \) is related to Weber transformations.

Let

\[ \mathbf{w} = \mathbf{u} - \nabla \phi + r \nabla S \equiv -\lambda \nabla \mu, \]

(4.52)

\( \nabla \times \mathbf{w} \) represents the component of the vorticity of the fluid that is not generated by entropy gradients, i.e. it does not depend on \( \nabla S \). The 1-form \( \alpha = \mathbf{w} \cdot d\mathbf{x} \) is Lie dragged with the fluid. Thus \( \mathbf{w} \) satisfies the equation (4.19):

\[ \frac{\partial \mathbf{w}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{w}) + \nabla (\mathbf{u} \cdot \mathbf{w}) = 0. \]

(4.53)

It follows that \( \mathbf{b} = (\nabla \times \mathbf{w})/\rho \) is a Lie dragged vector field and \( \nabla S \cdot d\mathbf{x} \) is a conserved 1-form (Tur and Yanovsky 1993). Thus, \( \mathbf{b} \cdot (\nabla S \cdot d\mathbf{x}) = \mathbf{b} \cdot \nabla S \) is a conserved scalar. Inspection of \( \mathbf{b} \cdot \nabla S \) reveals that:

\[ I_e \equiv \mathbf{b} \cdot \nabla S = \frac{\nabla \times (\mathbf{u} + r \nabla S - \nabla \phi)}{\rho} \cdot \nabla S = \frac{\nabla \times \mathbf{u}}{\rho} \cdot \nabla S, \]

(4.54)

is the Ertel invariant.

Theorem 4.6. The generalization for the Ertel invariant in MHD is (Kats 2003):

\[ I_e^{(m)} = \frac{\nabla \times (\mathbf{u} - \mathbf{u}_M)}{\rho} \cdot \nabla S, \]

(4.55)

where

\[ \mathbf{u}_M = - \frac{(\nabla \times \Gamma) \times \mathbf{B}}{\rho} - \frac{\Gamma (\nabla \cdot \mathbf{B})}{\rho}, \]

(4.56)

\[ \frac{\partial \Gamma}{\partial t} - \mathbf{u} \times (\nabla \times \Gamma) + \nabla (\Gamma \cdot \mathbf{u}) = -\frac{\mathbf{B}}{\mu_0}, \]

(4.57)

and \( \mu_0 \) is the magnetic permeability. We can also write (4.57) as:

\[ \frac{d}{dt} (\Gamma \cdot d\mathbf{x}) = -\frac{\mathbf{B} \cdot d\mathbf{x}}{\mu_0}. \]

(4.58)
Proof. Use the Clebsch representation for $u$:

$$u = \nabla \phi - r \nabla S - \lambda \nabla \mu + u_M. \quad (4.59)$$

Inspection shows that $w$ satisfies the equation:

$$w = u - \nabla \phi + r \nabla S - u_M \equiv -\lambda \nabla \mu, \quad (4.60)$$

and hence $\alpha = w \cdot dx$ is an invariant 1-form. It follows that $b = \nabla \times w / \rho$ is a Lie advected vector field. $dS = \nabla S \cdot dx$ is an invariant advected 1-form. Thus, $I^m_e = b \cdot \nabla S$ is an invariant scalar given by:

$$I^m_e = \nabla \times \left( u - \nabla \phi + r \nabla S - u_M \right) \cdot \nabla S / \rho \equiv \nabla \times \left( u - u_M \right) \cdot \nabla S / \rho. \quad (4.61)$$

The quantity $I^m_e$ is the MHD analogue of the Ertel invariant. It reduces to the Ertel invariant in the case where $u_M$ is zero. $\square$

Theorem 4.7. The Hollmann invariant is:

$$I_h = (\nabla \times u) \cdot \frac{\nabla S \times \nabla I_e}{\rho} \quad \text{where} \quad I_e = \frac{(\nabla \times u) \cdot \nabla S}{\rho}, \quad (4.62)$$

is the Ertel invariant. Here $\phi$ is the Clebsch potential in (4.51) associated with potential flow. The Hollmann invariant $I_h$ is Lie dragged with the flow.

Proof. $\omega_1 = \nabla S \cdot dx$ and $\omega_2 = \nabla I_e \cdot dx$ are conserved 1-forms. Thus, $\omega = \omega_1 \wedge \omega_2 = (\nabla S \times \nabla I_e) \cdot dS$ is a conserved 2-form, and

$$b = \nabla S \times \nabla I_e / \rho, \quad (4.63)$$

is a conserved vector. $\alpha = w \cdot dx$ is a conserved 1-form, where

$$w = u - \nabla \phi + r \nabla S, \quad (4.64)$$

and $w$ satisfies the equation:

$$\frac{\partial w}{\partial t} - u \times (\nabla \times w) + \nabla (u \cdot w) = 0. \quad (4.65)$$

Using (4.63) and (4.64) it follows that

$$I_h = w \cdot b \equiv \left( u - \nabla \phi \right) \cdot \frac{\nabla S \times \nabla I_e}{\rho}, \quad (4.66)$$

is a scalar invariant (i.e. the Hollmann invariant). $\square$

Similarly, the MHD version of the Hollmann invariant is:

$$I^m_h = w_m \cdot b_m = \left( u - u_M - \nabla \phi \right) \cdot \left( \frac{\nabla S \times \nabla I^m_e}{\rho} \right), \quad (4.67)$$

where

$$w_m = u - \nabla \phi + r \nabla S - u_M, \quad b_m = \frac{\nabla S \times \nabla I^m_e}{\rho}. \quad (4.68)$$

4.7. Topological invariants

Topological invariants and integrals of differential forms over a volume $V$ that are non-zero are sometimes referred to as topological charges. A more complete discussion is given by Tur and Yanovsky (1993). First we recall the definitions of closed and exact differential forms.
Definition 1. A p-form \( \omega^p \) is closed if its exterior derivative \( d\omega^p = 0 \).

Definition 2. A p-form \( \omega^p \) is exact if it can be expressed as the exterior derivative of a \((p-1)\)-form \( \omega^{p-1} \), i.e., \( \omega^p = d\omega^{p-1} \). It is assumed that \( \omega^p \) and \( \omega^{p-1} \) are sufficiently smooth and differentiable on a star-shaped region of the manifold on which the forms are defined.

Lemma 4.8. (Poincaré). The Poincaré lemma states that if \( \omega \) is a contractible open set of \( \mathbb{R}^n \), then any closed p-form defined on \( \omega \) is exact, for any integer \( 0 < p \leq n \).

Definition 3. Contractibility means that there is a homotopy \( F_t : X \times [0,1] \rightarrow X \) that continuously deforms \( X \) to a point. Thus every cycle \( c \) in \( X \) is the boundary of some cone. One can take the cone to be the image of \( c \) under the homotopy. A dual version of this result gives the Poincaré lemma.

From the above definitions, it follows that an exact \( p \)-form is closed, but a closed \( p \)-form is not necessarily exact. To verify these statements, note that if \( \omega^p \) is exact, then \( \omega^p = d\omega^{p-1} \) for some \( p-1 \) form \( \omega^{p-1} \). By the Poincaré lemma, \( d\omega^p = d\omega^{p-1} = 0 \) (i.e. the Poincaré lemma states that \( d\omega = 0 \) for a differential form \( \alpha \), where \( \alpha \) is sufficiently differentiable, i.e. at least twice differentiable on the star shaped region of the manifold \( M \) on which the form is defined). However, a closed form \( \omega^p \) with \( d\omega^p = 0 \) is not necessarily exact, i.e. there might not exist a \((p-1)\) form such that \( \omega^p = d\omega^{p-1} \). The word exact is synonymous with the notion of global integrability.

An invariant integral of the form (e.g. magnetic helicity):

\[
I = \int_V \omega \wedge d\omega, \tag{4.69}
\]

where \( \omega = \mathbf{A} \cdot d\mathbf{x} \) is an advected invariant 1-form, and with \( d\omega = \nabla \times \mathbf{A} \cdot d\mathbf{S} \) for which the integral (4.69) is non-zero, defines a non-zero topological charge known as the Hopf invariant. A classical example of an MHD solution with non-zero topological charge is the MHD topological soliton (e.g. Kamchatnov 1982) and related topological MHD solutions (Semenov et al 2002).

If \( \beta = \omega \cdot d\mathbf{S} \) is an advected invariant 2-form, then \( \mathbf{J} = \omega/\rho \equiv (\omega/\rho)\partial/\partial x^i \) is an invariant advected vector field, and \( d\beta = \nabla \cdot \omega \, d^3x = \nabla \cdot (\rho \mathbf{J}) \, d^3x = 0 \) if \( \nabla \cdot (\rho \mathbf{J}) = 0 \). If \( \nabla \cdot (\rho \mathbf{J}) \neq 0 \), the integral \( P = \int d\beta \) has non-zero topological charge. Examples of 2-forms with non-zero topological charge can be constructed from the wedge product of two invariant 1-forms. For example, if

\[
\omega_{S_1}^1 = \mathbf{S}_1 \cdot d\mathbf{x}, \quad \omega_{S_2}^1 = \mathbf{S}_2 \cdot d\mathbf{x},
\]

are invariant 1-forms, then

\[
\omega^2 = \omega_{S_1}^1 \wedge \omega_{S_2}^1 = \mathbf{S}_1 \cdot d\mathbf{x} \wedge \mathbf{S}_2 \cdot d\mathbf{x} = (\mathbf{S}_1 \times \mathbf{S}_2) \cdot d\mathbf{S},
\]

is an invariant 2-form. Taking the exterior derivative of \( \omega^2 \) gives

\[
d\omega^2 = \nabla \cdot (\mathbf{S}_1 \times \mathbf{S}_2) \, d^3x.
\]

In general \( \nabla \cdot (\mathbf{S}_1 \times \mathbf{S}_2) \neq 0 \), and hence the 3-form \( d\omega^2 \) has non-zero topological charge. More precisely, the topological charge for a volume \( V = D_3(t) \) is given by the equivalent expressions:

\[
P^\prime = \int_{D_3(t)} \omega^2 = \int_{\partial D_3(t)} \omega^2 = \int_{\partial D_3(t)} (\mathbf{S}_1 \times \mathbf{S}_2) \cdot d\mathbf{S}. \tag{4.73}
\]

Thus \( P^\prime \) is zero if the normal component of \( \mathbf{S}_1 \times \mathbf{S}_2 \) is zero on the boundary \( \partial D_3(t) \) of the volume \( D_3(t) \) of the region of interest.
Example 1. For compressible ideal fluid flows:

\[ \omega^1 = \nabla \cdot \mathbf{d}x, \quad \omega^2 = (\mathbf{u} - \nabla \phi) \cdot \mathbf{d}S, \quad \omega^3 = \mathbf{w} \cdot \mathbf{d}x, \quad (4.74) \]

are invariant 1-forms advected with the flow. \( \mathbf{w} \cdot \mathbf{d}x \) is an invariant advected 1-form, where \( \mathbf{u} = \nabla \phi - \nabla S - \lambda \nabla \mu \) is the Clebsch representation for the fluid velocity \( \mathbf{u} \). The 2-form \( \omega^2 \) with properties:

\[ \omega^2 = \omega^1 \wedge \omega^1 = \nabla S \times (\mathbf{u} - \nabla \phi) \cdot \mathbf{d}S, \quad \text{do}^2 = \nabla \cdot [\nabla S \times (\mathbf{u} - \nabla \phi)] \mathbf{d}^3x, \quad (4.75) \]

is an advected invariant 2-form. Using the identity

\[ \nabla \cdot (\mathbf{E} \times \mathbf{A}) = \mathbf{A} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{A}, \quad (4.76) \]

with \( \mathbf{E} = \nabla S \) and \( \mathbf{A} = \mathbf{u} - \nabla \phi \) in (4.75) we obtain:

\[ \text{do}^2 = - \nabla S \cdot \nabla \times \mathbf{u} \mathbf{d}^3x = - \rho \mathbf{I}_e \mathbf{d}^3x, \quad (4.77) \]

where \( \mathbf{I}_e \) is the Ertel invariant. In this case, in general \( \text{do}^2 = \nabla \cdot (\mathbf{\rho} \mathbf{J}) \mathbf{d}^3x \neq 0 \) where \( \mathbf{\rho} = \nabla \times (\mathbf{u} - \nabla \phi) \). This example shows that if \( \rho \mathbf{I}_e \neq 0 \), the Ertel invariant can give rise to topological charge in ideal fluid mechanics.

Example 2. For ideal MHD,

\[ \omega^1 = \tilde{\mathbf{A}} \cdot \mathbf{d}x, \quad \omega^2 = \nabla S \cdot \mathbf{d}x, \quad (4.78) \]

are invariant advected 1-forms. The 2-form:

\[ \omega^2 = \omega^1 \wedge \omega^1 = (\tilde{\mathbf{A}} \times \nabla S) \cdot \mathbf{d}S, \quad (4.79) \]

is an advected invariant 2-form, with exterior derivative:

\[ \text{do}^2 = \nabla \cdot (\tilde{\mathbf{A}} \times \nabla S) \mathbf{d}^3x = [\nabla \times \tilde{\mathbf{A}} \cdot \nabla S - \tilde{\mathbf{A}} \cdot (\nabla \times \nabla S)] \mathbf{d}^3x \]

\[ \equiv (\mathbf{B} \cdot \nabla S) \mathbf{d}^3x = \mathbf{\rho} \mathbf{I}_b \mathbf{d}^3x, \quad (4.80) \]

where \( \mathbf{I}_b = \mathbf{B} \cdot \nabla S / \mathbf{\rho} \) is an invariant, advected scalar. In this case \( \text{do}^2 = \nabla \cdot (\mathbf{\rho} \mathbf{J}) \mathbf{d}^3x \) where \( \mathbf{\rho} \mathbf{J} = \tilde{\mathbf{A}} \times \nabla S \). If the integral \( I^2 = \int \text{do}^2 \) is non-zero then it gives a non-zero topological charge associated with the scalar \( \mathbf{I}_b = \mathbf{B} \cdot \nabla S / \mathbf{\rho} \).

4.8. The Godbillon–Vey invariant

In an MHD flow, in which \( \tilde{\mathbf{A}} \cdot \nabla \times \tilde{\mathbf{A}} = 0 \) the magnetic helicity \( \tilde{\mathbf{A}} \cdot \mathbf{B} = 0 \). The question arises of whether the magnetic field in this case has a non-trivial topology. It turns out that the field can still have a non-trivial topology if the higher order topological invariant, the Godbillon–Vey invariant is non-zero. The same question also arises in ordinary fluid dynamics for flows in which \( \mathbf{u} \cdot \nabla \times \mathbf{u} = 0 \). A discussion and derivation of the Godbillon–Vey invariant is given below (see also Tur and Yanovsky (1993)).

Consider the Pfaffian differential form (1-form) \( \tilde{\mathbf{\omega}}^1 = \tilde{\mathbf{A}} \cdot \mathbf{d}x \), for which

\[ \text{d} \tilde{\mathbf{\omega}}^1 = (\nabla \times \tilde{\mathbf{A}}) \cdot \mathbf{d}S \]

and

\[ \tilde{\mathbf{\omega}}^1 \wedge \text{d} \tilde{\mathbf{\omega}}^1 = \mathbf{A} \cdot \mathbf{d}x \times (\nabla \times \tilde{\mathbf{A}}) \cdot \mathbf{d}S = (\tilde{\mathbf{A}} \cdot \nabla \times \tilde{\mathbf{A}}) \mathbf{d}^3x. \quad (4.81) \]

The Pfaffian differential equation:

\[ \text{d} \tilde{\mathbf{\omega}}^1 = \mathbf{A} \cdot \mathbf{d}x = 0, \quad (4.82) \]

determines planes perpendicular to the vector field \( \tilde{\mathbf{A}} \) at each point. For these planes to exist, i.e. for the Pfaffian equation (4.82) to have a solution requires that the integrability conditions

\[ \text{d} \tilde{\mathbf{\omega}}^1 \wedge \text{d} \tilde{\mathbf{\omega}}^1 = (\tilde{\mathbf{A}} \cdot \nabla \times \tilde{\mathbf{A}}) \mathbf{d}^3x = 0 \quad (4.83) \]
are satisfied. If
\[ \tilde{A} \cdot \nabla \times \tilde{A} = 0, \] (4.84)
the Pfaffian equation (4.82) is integrable (e.g. Sneddon 1957).

Tur and Yanovsky (1993) discuss the geometric obstruction to integrability when \( \tilde{A} \cdot \nabla \times \tilde{A} \neq 0 \) in terms of non-closure of the integral paths. Note that the helicity or Hopf invariant
\[ \tilde{I}' = \int_V \tilde{\omega}_A^1 \wedge \tilde{\omega}_A^1 = \int_V \tilde{A} \cdot \nabla \times \tilde{A} \, d^3 x, \] (4.85)
is non-zero only if \( \tilde{A} \cdot \nabla \times \tilde{A} \neq 0 \) in some region in the volume \( V \) (i.e. \( \tilde{A} \cdot \nabla \times \tilde{A} = 0 \) throughout the whole of \( V \) is not possible). Thus \( I' \neq 0 \) implies \( \alpha = \tilde{A} \cdot \, dx \) is non-integrable in sub-regions of \( V \) where \( \alpha \) does not change sign.

A natural question (e.g. Tur and Yanovsky 1993), is: given that the differential form \( \tilde{\omega}^1 = \tilde{A} \cdot \, dx \) is integrable, and satisfies the integrability condition (4.83), are there then higher order topological invariants that have non-zero topological charge? The answer to this question is yes, there is a higher order topological quantity that can be non-zero in this case called the Godbillon–Vey invariant. It is defined by the equation
\[ I^F = \int_{D^3(t)} \eta \cdot \nabla \times \eta \, d^3 x \quad \text{where} \quad \eta = \frac{\tilde{A} \times \tilde{B}}{|	ilde{A}|^2}. \] (4.86)
where \( B = \nabla \times \tilde{A} \), and \( B \cdot n = 0 \) on the boundary \( \partial D^3(t) \) of the region \( D^3(t) \) with outward normal \( n \). \( I^F \) is a topological invariant that is advected with the flow, i.e.,
\[ \frac{dI^F}{dt} = 0, \] (4.87)
where \( d/dt = \partial / \partial t + u \cdot \nabla \) is the Lagrangian time derivative moving with the flow. It is important to note that the Godbillon–Vey invariant (4.86) only applies to zero helicity flows for which \( \tilde{A} \cdot \nabla \times \tilde{A} = 0 \).

In (4.86) \( \eta \) is defined by the integrability equation:
\[ d\tilde{\omega}^1_A = \omega^1_\eta \wedge \tilde{\omega}^1_A, \] (4.88)
where
\[ \tilde{\omega}^1_A = \tilde{\alpha} \cdot \, dx, \quad \text{and} \quad \omega^1_\eta = \eta \cdot \, dx, \] (4.89)
are 1-forms. Taking the exterior derivative of \( \tilde{\omega}^1_A \) and using it in (4.88) we obtain the equivalent flux equation:
\[ (\nabla \times \tilde{A}) \cdot dS = (\eta \times \tilde{A}) \cdot dS \quad \text{or} \quad \nabla \times \tilde{A} = \eta \times \tilde{A}. \] (4.90)
From (4.90) we obtain:
\[ \tilde{A} \times (\nabla \times \tilde{A}) = \tilde{A} \times (\eta \times \tilde{A}) = (\tilde{A} \cdot \tilde{A}) \eta - (\tilde{A} \cdot \eta) \tilde{A}. \] (4.91)
The general solution of (4.91) for \( \eta \) is:
\[ \eta = \frac{1}{|	ilde{A}|^2} (\tilde{A} \times B + \eta \cdot \tilde{A} \tilde{A}). \] (4.92)
By dropping the arbitrary component of \( \eta \) parallel to \( \tilde{A} \) we obtain the solution (4.86) for \( \eta \).

A derivation of the Godbillon–Vey invariant (4.86) and the invariance equation (4.87) for \( I^F \) (see also Tur and Yanovsky (1993)) is outlined below.

**Proof of Godbillon–Vey formula (4.87).** The Frobenius integrability condition (4.83) is satisfied if there exists a 1-form \( \omega^1_\eta \) such that
\[ d\tilde{\omega}^1_A = \omega^1_\eta \wedge \tilde{\omega}^1_A. \] (4.93)
Note that
\[ \tilde{\omega}_A^1 \wedge d\tilde{\omega}_A^1 = \tilde{\omega}_A^1 \wedge (\omega_\eta^1 \wedge \tilde{\omega}_A^1) = -\tilde{\omega}_A^1 \wedge \omega_\eta^1 \wedge \omega_\eta^1 = 0, \]
where we used the associative and anti-symmetry properties of the \( \wedge \) operation. Equation (4.93) ensures \( d\tilde{\omega}_A^1 = 0 \) whenever \( \tilde{\omega}_A^1 = 0 \). The condition \( d\tilde{\omega}_A^1 = 0 \) implies by the Poincaré lemma that there exist a 0-form \( \Phi \) such that \( \tilde{\omega}_A^1 = d\Phi \). The Pfaffian equation \( \tilde{\omega}_A^1 = \tilde{A} \cdot dx = 0 \) is then satisfied by \( \Phi(x, y, z) = \text{const.} \) Equation (4.93) implies that the set of forms \( \{\tilde{\omega}_A^1, d\tilde{\omega}_A^1\} \) is a closed ideal of differential forms which are in involution according to Cartan’s theory of differential equations (e.g. Harrison and Estabrook 1971), i.e. the equations \( \tilde{\omega}_A^1 = 0 \) are integrable and satisfy the integrability conditions (4.83)). Equations (4.93) are similar to the Maurer Cartan equations, which are differential conditions in differential geometry.

We require that \( d\tilde{\omega}_A^1 \) is advected with the flow, i.e.
\[ \left( \frac{\partial}{\partial t} + L_u \right) d\tilde{\omega}_A^1 \equiv \left( \frac{\partial}{\partial t} + L_u \right) (\omega_\eta^1 \wedge \tilde{\omega}_A^1) = 0. \] (4.95)

Expanding (4.95) using the properties of the Lie derivative \( L_u \) gives:
\[ \left[ \left( \frac{\partial}{\partial t} + L_u \right) \tilde{\omega}_n \right] \wedge \tilde{\omega}_A^1 + \omega_\eta^1 \wedge \left[ \left( \frac{\partial}{\partial t} + L_u \right) \tilde{\omega}_A^1 \right] = 0. \] (4.96)

Using (4.96) and the condition that \( \tilde{\omega}_A^1 \) is Lie dragged with the flow (4.96) simplifies to:
\[ \left[ \left( \frac{\partial}{\partial t} + L_u \right) \omega_\eta^1 \right] \wedge \tilde{\omega}_A^1 = 0. \] (4.97)

Equation (4.97) is satisfied if
\[ \left( \frac{\partial}{\partial t} + L_u \right) \omega_\eta^1 = \alpha \tilde{\omega}_A^1. \] (4.98)

Equation (4.98) can also be written in the form:
\[ \frac{\partial \eta}{\partial t} - u \times (\nabla \times \eta) + \nabla (u \cdot \eta) = \alpha \tilde{A}. \] (4.99)

Taking the scalar product of (4.99) with \( \tilde{A} \) gives:
\[ \alpha |\tilde{A}|^2 = \tilde{A} \cdot \left[ \frac{\partial \eta}{\partial t} - u \times (\nabla \times \eta) + \nabla (u \cdot \eta) \right]. \] (4.100)

An alternative expression for \( \alpha \) can be obtained by noting that \( \tilde{A} \cdot dx \) is Lie dragged with the flow. Thus, \( \tilde{A} \) satisfies (3.23), and hence:
\[ 0 = \eta \cdot \left[ \frac{\partial \tilde{A}}{\partial t} - u \times (\nabla \times \tilde{A}) + \nabla (u \cdot \tilde{A}) \right]. \] (4.101)

Noting that \( \tilde{A} \cdot \eta = \tilde{A} \cdot (\tilde{A} \times B/|\tilde{A}|^2) = 0 \) and adding (4.100) and (4.101) we obtain:
\[ \alpha = \frac{1}{|\tilde{A}|^2} \{ \tilde{A} : [u \cdot \nabla \eta + (\nabla u)^T \cdot \eta] + \eta \cdot [u \cdot \nabla \tilde{A} + (\nabla u)^T \cdot \tilde{A}] \}. \] (4.102)

Next we investigate if the 3-form:
\[ \omega_\eta^3 = \omega_\eta^1 \wedge d\omega_\eta^1, \] (4.103)
is an advected (Lie dragged) 3-form. We find:
\[ \left( \frac{\partial}{\partial t} + L_u \right) \omega_\eta^3 = -d \left( \alpha d\tilde{\omega}_A^1 \right). \] (4.104)
To derive (4.104) first note that
\[
\left( \frac{\partial}{\partial t} + \mathcal{L}_u \right) \omega^3_t = \left( \frac{\partial}{\partial t} + \mathcal{L}_u \right) \omega^3_t + d \omega^3_t + \omega^3_t \wedge \left( \frac{\partial}{\partial t} + \mathcal{L}_u \right) \omega^3_t.
\]
\[
= \alpha \omega^3_t \wedge d \omega^3_t + \omega^3_t \wedge \mathcal{L}_u \omega^3_t.
\]
(4.105)

Next we use the fact that \(d \omega^3_t \wedge \tilde{\omega}_t = 0\) which follows by noting
\[
d(d \omega^3_t) = 0 \iff d(\omega^3_t \wedge \tilde{\omega}_t) = d \omega^3_t \wedge \tilde{\omega}_t - \omega^3_t \wedge d \tilde{\omega}_t.
\]
(4.106)

and that \(\omega^3_t \wedge d \tilde{\omega}_t = 0\) by (4.93). Thus,
\[
\left( \frac{\partial}{\partial t} + \mathcal{L}_u \right) \omega^3_t = \omega^3_t \wedge \left\{ d(\alpha \omega^3_t) = -d(\omega^3_t \wedge \alpha \omega^3_t) \right\}.
\]
(4.107)

which reduces to (4.104).

Next we consider the Godbillon–Vey integral:
\[
I^t = \int_{\partial D^3(t)} \omega^3_t^{\partial} \wedge d \omega^3_t \equiv \int_{\partial D^3(t)} \eta \cdot \nabla \times \eta \, d^3 x.
\]
(4.108)

Using (4.104) gives:
\[
\frac{\partial I^t}{\partial t} = \int_{\partial D^3(t)} \frac{\partial}{\partial t} \omega^3_t \wedge d \omega^3_t = \int_{\partial D^3(t)} \left[ - \mathcal{L}_u(\omega^3_t) - d(\alpha \omega^3_t) \right].
\]
(4.109)

However, using Cartan’s magic formula gives
\[
\mathcal{L}_u(\omega^3_t) = d(\eta \cdot \omega^3_t) + \eta \cdot d \omega^3_t = d(\eta \cdot \omega^3_t).
\]
(4.110)

(note \(\omega^3_t\) is a 3-form and hence \(d \omega^3_t = 0\)). From (4.110) and (4.109) we obtain:
\[
\frac{\partial I^t}{\partial t} = \int_{\partial D^3(t)} - d(\eta \cdot \omega^3_t + \alpha \omega^3_t) = -\int_{\partial D^3(t)} (\eta \cdot \omega^3_t + \alpha \omega^3_t).
\]
(4.111)

Writing
\[
\psi = \eta \cdot \nabla \times \eta.
\]
(4.112)

(4.111) can be written in the form:
\[
\int_{D^3(t)} \frac{\partial \psi}{\partial t} 
\]
\[
\frac{\partial I^t}{\partial t} = \int_{D^3(t)} - \left\{ \psi \cdot ( \eta \cdot d \omega^3_t + \alpha (\tilde{\mathbf{A}} \cdot d \mathbf{x}) \right\}
\]
\[
= -\int \left\{ \psi \cdot [ \eta \cdot d \mathbf{x} ] \right\} \cdot ( \nabla \times \eta ) \cdot d \mathbf{S} + \alpha ( \nabla \times \tilde{\mathbf{A}} ) \cdot d \mathbf{S}
\]
\[
= -\int \left\{ \psi \cdot ( \eta \cdot d \mathbf{x} ) \right\} \cdot d \mathbf{S} + \alpha \mathbf{B} \cdot d \mathbf{S}
\]
\[
= -\int \left\{ \nabla \cdot ( \psi \mathbf{u} + \alpha \mathbf{B} ) \right\} \cdot d^3 x.
\]
(4.113)

Equation (4.113) implies the conservation law:
\[
\frac{\partial \psi}{\partial t} + \nabla \cdot ( \mathbf{u} \psi + \alpha \mathbf{B} ) = 0,
\]
(4.114)

where \(\alpha\) is given in (4.102).

Integrating the continuity equation (4.114) for \(\psi\) over the volume \(D^3(t)\), and using the results
\[
\psi \, d^3 x = \psi(x_0) \, d^3 x_0, \quad d^3 x = J \, d^3 x_0, \quad \psi J = \psi_0(x_0), \quad \frac{d \ln J}{d t} = \nabla \cdot \mathbf{u},
\]
(4.115)
from Lagrangian fluid mechanics where $J = \det(x_{ij})$ is the Jacobian determinant of $x_{ij} = \partial x^i / \partial x^j_0$ of the Lagrangian map relating the Eulerian position coordinate $x$ and the Lagrangian label $x_0$ where $x = x_0$ at $t = 0$, we obtain

$$0 = \int_{D'(t)} \left[ \frac{\partial \psi}{\partial t} + \nabla \cdot (u \psi + \alpha B) \right] d^3 x$$

$$= \int_{D'(t)} \left[ \frac{\partial \psi}{\partial t} + \left( \psi \frac{d \ln J}{d t} + u \cdot \nabla \psi \right) \right] J d^3 x_0$$

$$= \int_{D'(t)} \left[ \frac{d \psi}{d t} \psi d^3 x + \psi \frac{d}{d t} (d^3 x) \right].$$

In the second line in (4.116) there is no contribution from the $\alpha B$ term, because if we apply Gauss’s theorem $\nabla \cdot (\alpha B) d^3 x \rightarrow \alpha B \cdot d S = \alpha B \cdot \tilde{A} d S / |\tilde{A}| = 0$ and because $B \cdot \tilde{A} = 0$ is the integrability condition for $\tilde{A} \cdot d x = 0$. The last integral in (4.116) can be recognized as $d I^g / d t$. Thus, (4.116) implies the Lagrangian conservation law:

$$\frac{d I^g}{d t} = 0.$$  (4.117)

Thus $I^g$ is a constant moving with the flow. This completes the proof of (4.87).

\section{5. Hamiltonian approach}

In this section we discuss the Hamiltonian approach to MHD and gas dynamics. In section 5.1 we give a brief description of a constrained variational principle for MHD using Lagrange multipliers to enforce the constraints of mass conservation; the entropy advection equation; Faraday’s equation and the so-called Lin constraint describing in part, the vorticity of the flow (i.e. Kelvin’s theorem). This leads to Hamilton’s canonical equations in terms of Clebsch potentials. A basic reference is the paper by Zakharov and Kuznetsov (1997). The Lagrange multipliers define the Clebsch variables, which give a representation for the fluid velocity $u$. In section 5.2 we transform the canonical Poisson bracket obtained from the Clebsch variable approach to a non-canonical Poisson bracket written in terms of Eulerian physical variables (see e.g. Morrison and Greene 1980, 1982, Morrison 1982, and Holm and Kupershmidt 1983a, 1983b for more details). In section 5.3 we discuss the connection between the Clebsch variable approach and Weber transformations. Our main aim is to obtain the Clebsch variable evolution equations that follow from the variational principle. We use these evolution equations and Clebsch variables later to obtain nonlocal fluid helicity and cross-helicity conservation laws in the next section.

\subsection*{5.1. Clebsch variables and Hamilton’s equations}

Consider the MHD action (modified by constraints):

$$J = \int d^3 x \, dt L,$$  (5.1)

where

$$L = \left[ \frac{1}{2} \rho u^2 - \epsilon (\rho, S) - \frac{B^2}{2 \mu_0} \right] + \phi \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) \right) + \beta \left( \frac{\partial S}{\partial t} + u \cdot \nabla S \right)$$

$$+ \lambda \left( \frac{\partial \mu}{\partial t} + u \cdot \nabla \mu \right) + \Gamma \cdot \left( \frac{\partial B}{\partial t} - \nabla \times (u \times B) + u (\nabla \cdot B) \right).$$  (5.2)
The Lagrangian in curly brackets equals the kinetic minus the potential energy (internal thermodynamic energy plus magnetic energy). The Lagrange multipliers $\phi$, $\beta$, $\lambda$, and $\Gamma$ ensure that the mass, entropy, Lin constraint, Faraday equations are satisfied. We do not enforce $\nabla \cdot B = 0$, since we are interested in the effect of $\nabla \cdot B \neq 0$ (which is useful for numerical MHD where $\nabla \cdot B \neq 0$). It is straightforward to impose $\nabla \cdot B = 0$ if desired, although some care is required in the formulation of the Poisson bracket, to ensure that the Jacobi identity is satisfied (e.g. Morrison and Greene 1982, Morrison 1982).

Stationary point conditions for the action are $\delta J = 0$, $\delta J/\delta u = 0$ gives the Clebsch representation for $u$:

$$u = \nabla \phi - \frac{\beta}{\rho} \nabla S - \frac{\lambda}{\rho} \nabla \mu + u_M$$

where

$$u_M = \frac{- (\nabla \times \Gamma) \times B}{\rho} - \frac{\Gamma}{\rho} \nabla \cdot B$$

is magnetic contribution to $u$. Setting $\delta J/\delta \phi$, $\delta J/\delta \beta$, $\delta J/\delta \lambda$, $\delta J/\delta \Gamma$ consecutively equal to zero gives the mass, entropy advection, Lin constraint, and Faraday (magnetic flux conservation) constraint equations:

$$\rho_t + \nabla \cdot (\rho u) = 0,$$
$$S_t + u \cdot \nabla S = 0,$$
$$\mu_t + u \cdot \nabla \mu = 0,$$
$$B_t - \nabla \times (u \times B) + u (\nabla \cdot B) = 0.$$  \tag{5.3}

Setting $\delta J/\delta \rho$, $\delta J/\delta S$, $\delta J/\delta \mu$, $\delta J/\delta B$ equal to zero gives evolution equations for the Clebsch potentials $\phi$, $\beta$, $\lambda$, and $\Gamma$ as:

$$- \left( \frac{\partial \phi}{\partial t} + u \cdot \nabla \phi \right) + \frac{1}{2} u^2 - h = 0,$$
$$\frac{\partial \beta}{\partial t} + \nabla \cdot (\beta u) + \rho T = 0,$$
$$\frac{\partial \lambda}{\partial t} + \nabla \cdot (\lambda u) = 0,$$
$$\frac{\partial \Gamma}{\partial t} - u \times (\nabla \times \Gamma) + \nabla (\Gamma \cdot u) + \frac{B}{\mu_0} = 0.$$  \tag{5.4}

Equation (5.6) is related to Bernoulli’s equation for potential flow. The $\nabla (\Gamma \cdot u)$ term in (5.9) is associated with $\nabla \cdot B \neq 0$. Taking the curl of (5.9) gives:

$$\frac{\partial \tilde{\Gamma}}{\partial t} - \nabla \times (u \times \tilde{\Gamma}) = - \frac{\nabla \times B}{\mu_0}$$

where $\tilde{\Gamma} = \nabla \times \Gamma$.  \tag{5.5}

Equations (5.6)–(5.10) can be written in the form:

$$\frac{d}{dt} \phi = \frac{1}{2} u^2 - h, \quad \frac{d}{dt} \left( \frac{\beta}{\rho} \right) = -T,$$
$$\frac{d}{dt} \lambda = 0 \quad \text{or} \quad \frac{d}{dt} \left( \frac{\lambda}{\rho} \right) = 0,$$
$$\frac{d}{dt} (\Gamma \cdot dx) = \frac{B}{\mu_0} \cdot dx$$

where $d/dt = \partial/\partial t + u \cdot \nabla$, is the Lagrangian time derivative following the flow and $J = \nabla \times B/\mu_0$ is the current.
Introduce the Hamiltonian functional
\[ H = \int H \, d^3x \quad \text{where} \quad H = \frac{1}{2} u^2 + \epsilon(\rho, S) + \frac{B^2}{2\mu_0}. \]  \hfill (5.12)

Substitute the Clebsch expansion (5.3)–(5.4) for $u$ in (5.12). Evaluating the variational derivatives of $H$ gives Hamilton’s equations:
\[
\begin{align*}
\frac{\partial \rho}{\partial t} &= \frac{\delta H}{\delta \rho}, \quad \frac{\partial \phi}{\partial t} = -\frac{\delta H}{\delta \phi}, \quad \frac{\partial S}{\partial t} = \frac{\delta H}{\delta S}, \\
\frac{\partial \mu}{\partial t} &= \frac{\delta H}{\delta \mu}, \quad \frac{\partial \lambda}{\partial t} = \frac{\delta H}{\delta \lambda}. 
\end{align*}
\]  \hfill (5.13)

Here $\{\rho, \phi\}, \{S, \beta\}, \{\mu, \lambda\}, \{B, \Gamma\}$ are canonically conjugate variables.

The canonical Poisson bracket is:
\[
\{F, G\} = \int d^3x \left( \frac{\delta F}{\delta \rho} \frac{\delta G}{\delta \phi} - \frac{\delta F}{\delta \phi} \frac{\delta G}{\delta \rho} + \frac{\delta F}{\delta \rho} \frac{\delta G}{\delta \mu} \frac{\delta \mu}{\delta \lambda} \frac{\delta \lambda}{\delta \rho} \right). \]  \hfill (5.14)

It is straightforward to verify that the canonical Poisson bracket (5.14) satisfies the linearity, skew symmetry and Jacobi identity necessary for a Hamiltonian system (i.e. the Poisson bracket defines a Lie algebra).

### 5.2. Non-canonical poisson brackets

Morrison and Greene (1980, 1982) introduced non-canonical Poisson brackets for MHD for the case $\nabla \cdot B = 0$. Morrison and Greene (1982) and Morrison (1982) discuss the form of the Poisson bracket if $\nabla \cdot B \neq 0$. Morrison (1982) discusses the proof of the Jacobi identity. Holm and Kupershmidt (1983) point out that the Poisson bracket has the form expected for a semi-direct product Lie algebra, for which the Jacobi identity is automatically satisfied. Chandre et al (2013) discuss the $\nabla \cdot B = 0$ constraint using Dirac’s method of constraints and the Dirac bracket.

Introduce the new variables:
\[ M = \rho u, \quad \sigma = \rho S. \]  \hfill (5.15)

The formulae for the transformation of variational derivatives in the old variables $(\rho, \phi, S, \beta, B, \Gamma)$ in terms of the new variables $(\rho, \sigma, B, M)$ are:
\[
\begin{align*}
\frac{\delta F}{\delta \rho} &= \frac{\delta F}{\delta \rho} + \frac{\delta F}{\delta \sigma} + \frac{\delta F}{\delta M} \cdot \nabla \phi, \quad \frac{\delta F}{\delta \phi} = -\nabla \cdot \left( \frac{\delta F}{\delta M} \right), \\
\frac{\delta F}{\delta S} &= \frac{\delta F}{\delta \sigma} + \nabla \cdot \left( \frac{\delta F}{\delta B} \right), \quad \frac{\delta F}{\delta \beta} = -\frac{\delta F}{\delta M} \cdot \nabla S, \\
\frac{\delta F}{\delta B} &= \left[ \frac{\delta F}{\delta M} \right] \left( \frac{\delta F}{\delta M} \cdot \nabla \Gamma \right) \frac{\delta F}{\delta \beta} + \nabla \left( \frac{\delta F}{\delta M} \cdot \nabla \Gamma \right) \frac{\delta F}{\delta \beta} + \nabla \left( \frac{\delta F}{\delta M} \cdot \nabla \Gamma \right) \frac{\delta F}{\delta \beta} \\
\frac{\delta F}{\delta \mu} &= \nabla \cdot \left( \frac{\delta F}{\delta M} \right), \quad \frac{\delta F}{\delta \lambda} = -\frac{\delta F}{\delta M} \cdot \nabla \mu. 
\end{align*}
\]  \hfill (5.16)

Note that
\[ M = \rho u = \rho \nabla \phi - \beta \nabla S + B \cdot \left( \nabla \Gamma \right)^T - B \cdot \nabla \Gamma - \Gamma \left( \nabla \cdot B \right). \]  \hfill (5.17)
Using the transformations \((5.16)\) in the canonical Poisson bracket \((5.14)\) we obtain the Morrison and Greene (1982) non-canonical Poisson bracket:

\[
\{F, G\} = - \int d^3x \left[ \rho \frac{\delta F}{\delta \rho} \nabla \cdot \left( \frac{\delta G}{\delta \rho} \right) - \frac{\delta G}{\delta \rho} \nabla \cdot \left( \frac{\delta F}{\delta \rho} \right) \right] + \sigma \left[ \frac{\delta F}{\delta \sigma} \nabla \cdot \left( \frac{\delta G}{\delta \sigma} \right) - \frac{\delta G}{\delta \sigma} \nabla \cdot \left( \frac{\delta F}{\delta \sigma} \right) \right] + M \cdot \left[ \left( \frac{\delta F}{\delta M} \right) \nabla \cdot \left( \frac{\delta G}{\delta M} \right) - \frac{\delta G}{\delta M} \nabla \cdot \left( \frac{\delta F}{\delta M} \right) \right] + B \cdot \left[ \left( \frac{\delta F}{\delta B} \right) \nabla \cdot \left( \frac{\delta G}{\delta B} \right) - \frac{\delta G}{\delta B} \nabla \cdot \left( \frac{\delta F}{\delta B} \right) \right] + B \cdot \left[ \left( \frac{\nabla F}{\delta B} \right) \cdot \left( \frac{\delta G}{\delta B} \right) - \frac{\delta G}{\delta B} \cdot \left( \frac{\nabla F}{\delta B} \right) \right].
\]

(5.18)

The bracket \((5.18)\) has the Lie–Poisson form and satisfies the Jacobi identity for all functionals \(F\) and \(G\) of the physical variables, and in general applies both for \(\nabla \cdot B \neq 0\) and \(\nabla \cdot B = 0\).

### 5.3. Weber transformations

The classical Weber transformation uses the Lagrangian map: \(x = x(x_0, t)\) to integrate the Eulerian momentum equation to get the Clebsch representation for \(u\). The Eulerian momentum conservation equation can be written as:

\[
\frac{\partial}{\partial t} (\rho u) + \nabla \cdot \left( \rho u \otimes u + \rho l + \left( \frac{B^2}{2\mu_0} - \frac{B \otimes B}{\mu_0} \right) \right) = 0,
\]

or as:

\[
\frac{du}{dt} = T \nabla S - \nabla h + \frac{J \times B}{\rho} + B \frac{\nabla \cdot B}{\mu_0 \rho}.
\]

(5.19) \hspace{1cm} (5.20)

Use:

\[
\frac{du}{dt} = \frac{\partial u}{\partial t} + \omega \times u + \nabla \left( \frac{1}{2} |u|^2 \right)
\]

where \(\omega = \nabla \times u\),

\[
\frac{d}{dt} \left( u \cdot \nabla \right) = \left[ \frac{\partial u}{\partial t} + \omega \times u + \nabla \left( |u|^2 \right) \right] \cdot \nabla,
\]

(5.21)

to get

\[
\frac{d}{dt} (u \cdot \nabla) = \left[ T \nabla S + \nabla \left( \frac{1}{2} |u|^2 - h \right) + \frac{J \times B}{\rho} + B \frac{\nabla \cdot B}{\mu_0 \rho} \right] \cdot \nabla.
\]

(5.22)

On the right-hand side (RHS) of \((5.22)\) for the magnetic terms we use:

\[
\frac{d}{dt} \left[ \left( \frac{\nabla \times \Gamma}{\rho} \right) \times B \right] \cdot \nabla = \left( \frac{\nabla B}{\rho} \right) \cdot \nabla.
\]

(5.23)

On the RHS of \((5.22)\) for the gas terms we use:

\[
\frac{d}{dt} \left( \frac{\nabla \phi}{\rho} \right) \cdot \nabla = \left( \frac{1}{2} |u|^2 - h \right) \cdot \nabla,
\]

\[
\frac{d}{dt} \left( r \nabla S \cdot \nabla \right) = -T \nabla S \cdot \nabla, \quad \frac{d}{dt} \left( \frac{\lambda}{\rho} \right) = 0,
\]

(5.24) \hspace{1cm} (5.25)

to obtain the Clebsch representation \(u = u_h + u_M\) in \((5.3)-(5.4)\).
Proposition 5.1. Equations (5.22)–(5.25) imply the Clebsch representation \( u = u_h + u_M \) in (5.3)–(5.4).

Proof. Using (5.23)–(5.25) in (5.22) gives:
\[
\frac{d}{dt}(w \cdot d\mathbf{x}) = 0, \tag{5.26}
\]
where
\[
w = u - \left( \nabla \phi - r \nabla S - \frac{\nabla \times \mathbf{B}}{\rho} \right) \times \left( \frac{\nabla \mathbf{B}}{\rho} \right). \tag{5.27}
\]
Integration of (5.26) gives
\[
w \cdot d\mathbf{x} = f_0(\mathbf{x}_0)^k d\mathbf{x}_0^k \quad \text{or} \quad w^j = f_0(\mathbf{x}_0)^k \partial x_0^k / \partial x^j. \tag{5.28}
\]
Using the initial data \( w^j = f_0(\mathbf{x}_0)^j = f_{g00}(\mathbf{x}_0) \partial g_{00} / \partial x_0^j \) at \( t = 0 \) gives
\[
w = -\tilde{\lambda} \nabla \mu, \tag{5.29}
\]
where \( \tilde{\lambda} = -f_{g00} \) and \( \mu = g_{00} \). Equations (5.27)–(5.28) then give:
\[
u = \nabla \phi - \tilde{\lambda} \nabla \mu - r \nabla S - \frac{\nabla \times \mathbf{B}}{\rho} \times \left( \frac{\nabla \mathbf{B}}{\rho} \right), \tag{5.30}
\]
which is the Clebsch representation (5.3)–(5.4) for \( u \), where \( \tilde{\lambda} = \lambda / \rho \).

The proof of (5.23) is sketched below. Note that \( \mathbf{b} = \mathbf{B} / \rho \) is an advected vector field. The 1-form on the LHS of (5.23) can be written as
\[
\alpha = \mathbf{b} \cdot ((\tilde{\Gamma} \cdot d\mathbf{S}) = (\tilde{\Gamma} \times \mathbf{b}) \cdot d\mathbf{x} \equiv [(\nabla \times \tilde{\Gamma}) \times \mathbf{B} / \rho] \cdot d\mathbf{x}, \tag{5.31}
\]
where \( \tilde{\Gamma} = \nabla \times \Gamma \). The RHS of (5.23) is:
\[
\begin{align*}
\frac{d\alpha}{dt} &= \frac{d\mathbf{b}}{dt} \cdot (\tilde{\Gamma} \cdot d\mathbf{S}) + \mathbf{b} \cdot \frac{d}{dt} (\tilde{\Gamma} \cdot d\mathbf{S}) \\
&= 0 - \mathbf{b} \cdot (\tilde{\mathbf{J}} \cdot d\mathbf{S}) \equiv -\frac{\mathbf{J} \times \mathbf{B}}{\rho} \cdot d\mathbf{x}. \tag{5.32}
\end{align*}
\]
This establishes (5.23). There are similar proofs for (5.24) and (5.25). \( \square \)

6. Nonlocal helicity conservation laws

In this section we look again at the helicity conservation law (3.2) and the cross-helicity conservation law (3.30). The helicity conservation law (3.2) requires that the gas or fluid be barotropic (i.e. \( p = p(\rho) \)) is independent of the entropy \( S \) in order for this conservation law to apply. Similarly, the cross-helicity conservation equation (3.30) only applies, if either (i) the gas is barotropic with \( p = p(\rho) \) or if (ii) \( \mathbf{B} \cdot \nabla S = 0 \), which implies that the magnetic field lies in the constant entropy surface. Using Clebsch variables allows one to obtain analogous nonlocal conservation laws corresponding to the helicity in ordinary fluid dynamics, and the cross-helicity conservation law in MHD.

Proposition 6.1. The generalized helicity conservation law in ideal fluid mechanics can be written in the form:
\[
\begin{align*}
\frac{\partial}{\partial t} [\Omega \cdot (u + r \nabla S)] + \nabla \cdot \left\{ u [\Omega \cdot (u + r \nabla S)] + \Omega \left( \frac{1}{2} |u|^2 \right) \right\} &= 0. \tag{6.1}
\end{align*}
\]
The nonlocal conservation law (6.1) depends on the Clebsch variable formulation of ideal fluid mechanics in which the fluid velocity \( \mathbf{u} \) is given by the equation:

\[
\mathbf{u} = \nabla \phi - r \nabla S - \tilde{\lambda} \nabla \mu, \tag{6.2}
\]

where \( \phi, r, S, \tilde{\lambda}, \) and \( \mu \) satisfy the equations:

\[
\frac{d\phi}{dt} = \frac{1}{2} |\mathbf{u}|^2 - h, \quad \frac{dr}{dt} = -T, \quad \frac{dS}{dt} = \frac{d\tilde{\lambda}}{dt} = \frac{d\mu}{dt} = 0, \tag{6.3}
\]

and \( d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla \) is the Lagrangian time derivative following the flow. In (6.1) the generalized vorticity \( \Omega \) is defined by the equations:

\[
\mathbf{w} = \mathbf{u} - \nabla \phi + r \nabla S \equiv -\tilde{\lambda} \nabla \mu, \tag{6.4}
\]

\[
\Omega = \nabla \times \mathbf{w} = \omega + \nabla r \times \nabla S, \tag{6.5}
\]

where \( \omega = \nabla \times \mathbf{u} \) is the fluid vorticity. The 1-form \( \alpha = \mathbf{w} \cdot d\mathbf{x} \) and the 2-form \( \beta = d\alpha = \Omega \cdot dS \) are advected invariants, i.e.

\[
\frac{d\alpha}{dt} = \left( \frac{\partial}{\partial t} + \mathcal{L}_u \right) \alpha = \left[ \frac{\partial \mathbf{w}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{w}) + \nabla (\mathbf{u} \cdot \mathbf{w}) \right] \cdot d\mathbf{x} = 0, \tag{6.6}
\]

\[
\frac{d\beta}{dt} = \left( \frac{\partial}{\partial t} + \mathcal{L}_u \right) \beta = \left[ \frac{\partial \Omega}{\partial t} - \nabla \times (\mathbf{u} \times \Omega) \right] \cdot dS = 0. \tag{6.7}
\]

An alternative form of the conservation law (6.1) is:

\[
\frac{\partial}{\partial t} (\mathbf{u} \cdot \Omega + \beta I_e) + \nabla \cdot \left[ (\mathbf{u} \cdot \Omega) \mathbf{u} + (\beta I_e) \mathbf{u} + \Omega \left( h - \frac{1}{2} |\mathbf{u}|^2 \right) \right] = 0, \tag{6.8}
\]

where

\[
\beta = r \rho, \quad I_e = \frac{\omega \cdot \nabla S}{\rho} \equiv \frac{\Omega \cdot \nabla S}{\rho} \tag{6.9}
\]

in which \( I_e \) is the Ertel invariant and the Clebsch variable \( \beta \) satisfies the evolution equation:

\[
\frac{\partial \beta}{\partial t} + \nabla \cdot (\beta \mathbf{u}) = -\rho T. \tag{6.10}
\]

**Proof.** Equation (6.6) states that the 1-form \( \alpha = \mathbf{w} \cdot d\mathbf{x} \) is Lie dragged by the flow. This is proved, by calculating the evolution of the forms \( \mathbf{u} \cdot d\mathbf{x}, -\nabla \phi \cdot d\mathbf{x} \) and \( r \nabla S \cdot d\mathbf{x} \) moving with the flow (see (5.31) et seq. for a proof the \( d/dt (\mathbf{w} \cdot d\mathbf{x}) = 0 \)). Alternatively one can prove (6.6) by noting \( \mathbf{w} = -\tilde{\lambda} \nabla \mu \) and that \( \tilde{\lambda} \) and \( \mu \) are advected with the flow. It is straightforward to show:

\[
\frac{\partial \mathbf{w}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{w}) + \nabla (\mathbf{u} \cdot \mathbf{w}) = -\nabla \mu \left( \frac{d\tilde{\lambda}}{dt} \right) - \tilde{\lambda} \nabla \left( \frac{d\mu}{dt} \right) = 0, \tag{6.11}
\]

which verifies (6.6).

Because \( \alpha = \mathbf{w} \cdot d\mathbf{x} \) is an advected invariant 1-form, then \( \beta = d\alpha = \Omega \cdot dS \) is an advected invariant 2-form and hence by (4.20) \( \Omega \) satisfies the equation:

\[
\frac{\partial \Omega}{\partial t} - \nabla \times (\mathbf{u} \times \Omega) + (\nabla \cdot \Omega) \mathbf{u} = 0, \tag{6.12}
\]

where \( \nabla \cdot \Omega = 0. \)
From (3.5), the momentum equation for the system may be written in the form:

\[
\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \omega + \nabla \left( \frac{1}{2} |\mathbf{u}|^2 \right) = TS\nabla S - \nabla h. \tag{6.13}
\]

Combining (6.12) and (6.13) gives the equation:

\[
\mathbf{I}e \cdot \left[ \frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \omega + \nabla \left( \frac{1}{2} |\mathbf{u}|^2 \right) - TS\nabla S \right] + \mathbf{u} \cdot \left[ \frac{\partial \mathbf{I}e}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{I}e) \right] = 0. \tag{6.14}
\]

Equation (6.14) reduces to:

\[
\frac{\partial}{\partial t} \left( \mathbf{u} \cdot \mathbf{I}e \right) + \nabla \cdot \left[ \mathbf{u} \times (\mathbf{u} \times \mathbf{I}e) \right] + \nabla \cdot \left[ \left( \frac{1}{2} |\mathbf{u}|^2 \right) \mathbf{I}e \right] = T \mathbf{I}e \cdot \nabla S. \tag{6.15}
\]

Since \( \mathbf{I}e = \omega + \nabla r \times \nabla S \), the RHS of (6.15) may be written as:

\[
T \mathbf{I}e \cdot \nabla S = T \omega \cdot \nabla S = \rho TL, \tag{6.16}
\]

where \( L = \omega \cdot \nabla S / \rho \) is the Ertel invariant. Using (6.10) in (6.16) we obtain:

\[
T \mathbf{I}e \cdot \nabla S = - \left[ \frac{\partial \beta}{\partial t} + \nabla \cdot (\beta \mathbf{u}) \right] L = - \left[ \frac{\partial (\beta L)}{\partial t} + \nabla \cdot (\beta L \mathbf{u}) \right] + \beta \frac{dL}{dt}. \tag{6.17}
\]

However \( dL / dt = 0 \). Thus, using (6.17) in (6.15) we obtain the conservation law:

\[
\frac{\partial}{\partial t} \left( \mathbf{u} \cdot \mathbf{I}e + \beta L \right) + \nabla \cdot \left[ \left( \mathbf{u} \cdot \mathbf{I}e \right) \mathbf{u} + \left( \beta L \right) \mathbf{u} + \mathbf{I}e \left( h - \frac{1}{2} |\mathbf{u}|^2 \right) \right] = 0, \tag{6.18}
\]

which is (6.8). By noting:

\[
\beta L = r \rho (\omega \cdot \nabla S) / \rho = r \omega \cdot \nabla S = r \mathbf{I}e \cdot \nabla S, \tag{6.19}
\]

(6.18) reduces to (6.1). This completes the proof \( \square \)

**Remark 1.** Since \( \alpha \) and \( \beta \) are advected invariants, then the 3-form

\[
\gamma = \alpha \wedge \beta = (\mathbf{w} \cdot \mathbf{I}e) \, d^3x, \tag{6.20}
\]

is also an advected invariant. However

\[
\mathbf{w} \cdot \mathbf{I}e = (-\check{\lambda} \nabla \mu) \cdot (-\nabla \check{\lambda} \times \nabla \mu) = 0. \tag{6.21}
\]

Taking into account (6.21) the conservation law (6.1) can also be written in the form:

\[
\frac{\partial}{\partial t} \left( \mathbf{I}e \cdot \nabla \phi \right) + \nabla \cdot \left[ \left( \mathbf{u} \cdot \mathbf{I}e \cdot \nabla \phi \right) + \mathbf{I}e \left( h - \frac{1}{2} |\mathbf{u}|^2 \right) \right] = 0. \tag{6.22}
\]

**Remark 2.** The conservation laws (6.1), or equivalently (6.22) is a nonlocal conservation law that involves the nonlocal potentials:

\[
r(x, t) = - \int_0^t T_0(x_0, t') \, dt' + r_0(x_0), \tag{6.23}
\]

\[
\phi(x, t) = \int_0^t \left( \frac{1}{2} |\mathbf{u}|^2 - h \right) (x_0, t') \, dt' + \phi_0(x_0), \tag{6.24}
\]

where \( x = f(x_0, t) \) and \( x_0 = f^{-1}(x, t) \) are the Lagrangian map and the inverse Lagrangian map. The temperature \( T(x, t) = T_0(x_0, t) \) and \( r_0(x_0) \) and \( \phi_0(x_0) \) are ‘integration constants’. 
Remark 3. The conservation law (6.1) can also be written in the form:

\[ \frac{\partial D}{\partial t} + \nabla \cdot F = 0, \]  

(6.25)

where

\[ D = \omega \cdot u + u \cdot \nabla r \times \nabla S, \]  

(6.26)

\[ F = u(\omega \cdot u) + \omega (h - \frac{1}{2}|u|^2) + u (u \cdot \nabla r \times \nabla S) + (\nabla r \times \nabla S) (h - \frac{1}{2}|u|^2). \]  

(6.27)

For barotropic or constant entropy flows \( \nabla S = 0 \) and the conservation law (6.1) reduces to the usual fluid helicity conservation form:

\[ \frac{\partial}{\partial t} (\omega \cdot u) + \nabla \cdot \left[ u(\omega \cdot u) + \omega \left( h - \frac{1}{2}|u|^2 \right) \right] = 0 \]  

(6.28)

where \( h = (\varepsilon + p)/\rho \) is the entropy of the gas.

Proposition 6.2. The generalized cross-helicity conservation law in MHD can be written in the form:

\[ \frac{\partial}{\partial t} (B \cdot (u + r \nabla S)) + \nabla \cdot \left[ u [B \cdot (u + r \nabla S)] + \left( h - \frac{1}{2}|u|^2 \right) B \right] = 0, \]  

(6.29)

where

\[ u = \nabla \psi - r \nabla S - \tilde{\lambda} \nabla \mu - \frac{(\nabla \times \Gamma) \times B}{\rho} - \Gamma \frac{\nabla \cdot B}{\rho}, \]  

(6.30)

is the Clebsch variable representation for the fluid velocity \( u \), and \( r(x,t) \) is the Lagrangian temperature integral (6.23) moving with the flow.

In the special cases of either (i) \( B \cdot \nabla S = 0 \) or (ii) the case of a barotropic gas with \( p = p(\rho) \), the conservation law (6.29) reduces to the usual cross-helicity conservation law:

\[ \frac{\partial}{\partial t} (u \cdot B) + \nabla \cdot \left[ u (u \cdot B) + \left( h - \frac{1}{2}|u|^2 \right) B \right] = 0, \]  

(6.31)

In general the cross-helicity conservation equation (6.29) is a nonlocal conservation law, in which the variable \( r(x,t) \) is a nonlocal potential given by (6.23).

Proof. The simplest approach is to start from the cross-helicity equation (3.33) with source term \( 7B \cdot \nabla S \), i.e.,

\[ \frac{\partial}{\partial t} (u \cdot B) + \nabla \cdot \left[ (u \cdot B)u + \left( h - \frac{1}{2}|u|^2 \right) B \right] = 7B \cdot \nabla S, \]  

(6.32)

and then show that the source term can be written as a pure space and time divergence term. First note that

\[ 7B \cdot \nabla S = \rho T \frac{B \cdot \nabla S}{\rho} = \rho T \psi \quad \text{where} \quad \psi = \frac{B \cdot \nabla S}{\rho}. \]  

(6.33)

Note that \( d\psi/dt = 0 \) as \( \psi \) is an advected invariant. Using the Eulerian mass continuity equation and (6.3) we obtain:

\[ \frac{\partial (\rho r)}{\partial t} + \nabla \cdot (u \rho r) = \rho \frac{dr}{dt} = -\rho T. \]  

(6.34)

It follows from (6.33) and (6.34) that \( 7B \cdot \nabla S = \rho T \psi \) and hence:

\[ 7B \cdot \nabla S = -\psi \left( \frac{\partial}{\partial t} \rho \right) + \nabla \cdot (u \rho r) \equiv -\frac{\partial}{\partial t} (\rho r \psi) + \nabla \cdot (u \rho r \psi) \equiv -\alpha \left( r \cdot B \cdot \nabla S + \nabla \cdot [u (r B \cdot \nabla S)] \right). \]  

(6.35)

Use of (6.35) in (6.32) then gives the conservation law (6.29). This completes the proof. \( \square \)
7. Concluding remarks

The main aim of the present paper is to provide an overview of the Lie dragging and conservation laws associated with fluid relabeling symmetries in MHD and fluid dynamics. Two notable new results in the paper are the generalization of the helicity conservation equation in ideal fluid mechanics and the generalization of the cross-helicity conservation law in ideal MHD. In most derivations of these conservation laws it is assumed either that (i) the gas is isentropic and the gas pressure is isobaric or (ii) in the case of cross-helicity conservation law in MHD \( p = p(\rho, S) \) and \( \mathbf{B} \cdot \nabla S = 0 \), meaning that the magnetic field lies in the \( S = \text{const} \) surfaces; in MHD the assumption \( p = p(\rho) \) also leads to the cross-helicity conservation law. The assumptions (i) and (ii) ensure that source terms dependent on \( \mathbf{F} \) flux vanish. The resultant conservation laws are local, meaning that the conserved density (the usual Clebsch variables used in the variational principle are \( \phi, \beta = \lambda \rho \) (e.g. Zakharov and Kuznetsov 1997)). The resultant helicity conservation equation (6.1) applies for a general non-isobaric equation of state for the gas (i.e. conservation law). The assumptions (i) and (ii) ensure that source terms dependent on \( \mathbf{F} \) flux vanish. The resultant conservation laws are local, meaning that the conserved density \( D \) and flux \( \mathbf{F} \) in the conservation law depend only on the local variables \( (\rho, \mathbf{u}, \mathbf{B}, S, T) \). A preliminary account of advected invariants in MHD using the ideas of Lie dragging is given in Webb et al (2013). It turns out that one can obtain a nonlocal version of the fluid helicity conservation equation by using Clebsch variables to describe the fluid. In this formulation the fluid velocity is represented in terms of the Clebsch potentials by the formula:

\[
\mathbf{u} = \nabla \phi - r \nabla S - \lambda \nabla \mu,
\]

(7.1)

(the usual Clebsch variables used in the variational principle are \( \phi, \beta = \lambda \rho \) and \( \lambda = \lambda_t \rho \) (e.g. Zakharov and Kuznetsov 1997)). The resultant helicity conservation equation (6.1) applies for a general non-isobaric equation of state for the gas (i.e. \( p = p(\rho, S) \), but also involves the nonlocal Clebsch potentials

\[
\begin{align*}
    r(x, t) &= -\int_0^t T_0(x_0, t') \, dt' + r_0(x_0), \\
    \phi(x, t) &= \int_0^t \left( \frac{1}{2} |\mathbf{u}|^2 - h \right)(x_0, t') \, dt' + \phi_0(x_0)
\end{align*}
\]

(7.2)

where \( T_0(x_0, t) = T(x, t) \) is the temperature of the gas and \( x = x(x_0, t) \) is the solution of the differential equation system \( \frac{dx}{dt} = \mathbf{u}(x, t) \) with \( x = x_0 \) at time \( t = 0 \) (this is technically referred to as the Lagrangian map). Here \( r_0(x_0) \) and \( \phi_0(x_0) \) are integration ‘constants’. A similar non-local conservation law (6.29) for the cross helicity is obtained by using the Clebsch variables appropriate for MHD.

An alternative account of MHD conservation laws, Lie symmetries and variational methods is to use the Euler–Poincaré approach to Noether’s theorems adopted by Cotter and Holm (2013) (see also Holm et al (1998)). The Euler–Poincaré variational approach takes into account known symmetries of the Lagrangian and uses Eulerian variations of the action. In the case of Lagrangian fluid mechanics the Lagrangian map \( x = x(x_0, t) \equiv g(x_0) \) can be thought of as a group of diffeomorphisms that map the Lagrange labels \( x_0 \) onto the Eulerian position of the fluid element \( x \). Note that the group element \( g \) has inverse element \( g^{-1} \) where \( x_0 = g^{-1} x \), provided the Jacobian of the map is non-zero and bounded, and that the identity element \( e \) corresponds to the transformation \( x = e x_0 = x_0 \). The use of Lie symmetries for differential equations and Noether’s theorems are described in standard texts (e.g. Olver 1993).

The relationship between the helicity and cross helicity conservation laws for barotropic and non-barotropic equations of state for the gas, will be investigated in a companion paper (paper II) using Noether’s theorems, fluid relabeling symmetries and gauge transformations. The relationship between the fluid relabeling symmetries and the Casimir invariants (e.g. Padhye and Morrison 1996a, 1996b, Padhye 1998) will also be investigated in paper II.

Other approaches to conservation laws and Noether’s theorems may be useful in future analyses. Anco and Bluman (2002a, 2002b) have developed a method to determine
conservation laws of a system of partial differential equations that does not invoke Noether’s theorem and a variational formulation of the equations (see also Bluman et al (2010)). Noether’s theorems and conservation laws using the method of moving frames has been developed by Goncalves and Mansfield (2012). This approach investigates the mathematical structure behind the Euler–Lagrange equations. They give examples of variational problems that are invariant under semi-simple Lie algebras. The method of moving frames and its relation to Lie pseudo algebras was developed by Fels and Olver (1998).

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