Poincaré and Sobolev Type Inequalities for Intrisic Rectifiable Varifolds

Julio Cesar Correa Hoyos

Abstract. We prove a Poincaré, and a general Sobolev type inequalities for functions with compact support defined on a $k$-rectifiable varifold $V$ defined on a complete Riemannian manifold with positive injectivity radius and sectional curvature bounded above. Our techniques allow us to consider Riemannian manifolds $(M^n, g)$ with $g$ of class $C^2$ or more regular, avoiding the use of Nash’s isometric embedding theorem. Our analysis permits to do some quite important fragments of geometric measure theory also for those Riemannian manifolds carrying a $C^2$ metric $g$, that is not $C^{k+\alpha}$ with $k+\alpha > 2$. The class of varifolds we consider are those which first variation $\delta V$ lies in an appropriate Lebesgue space $L^p$ with respect to its weight measure $\|V\|$ with the exponent $p \in \mathbb{R}$ satisfying $p > k$.

1. Introduction

The ordinary Sobolev inequality has been studied for many years in the Euclidean case, as well as in the Riemannian case, and its prominent role in the theory of partial differential equations is well known. In [Mir67] Miranda obtained a Sobolev inequality for functions defined on minimal graphs, The Miranda’s proof follows from the Isoperimetric inequality proved in [FF60] for integer currents and a procedure introduced by De Giorgi in [BDGM69a]. Bombieri used a refined version of this new inequality, De Giorgi and Miranda (employing the isoperimetric inequality of [FF60]), to derive gradient bounds for solutions to the minimal surface equation (see [BDGM69b]).

In [MS73], Michael and Simon prove a general Sobolev type inequality, which proof follows an argument which is, in some aspects, evocative of the potential theory. That inequality is obtained on what might be termed a generalized manifold and in particular, the classical Sobolev inequality, a Sobolev inequality on graphs of weak solutions to the mean curvature equation, and a Sobolev inequality on arbitrary $C^2$ submanifolds of $\mathbb{R}^n$ (of arbitrary co-dimension) are derived.

On the other hand, Allard in a pioneering work [All72] proved a Sobolev type inequality for non-negative, compact supported functions defined on a varifold $V$, whose first variation $\delta V$ lies in an appropriate Lebesgue space endowed with the measure $\|\delta V\|$, by generalization of the Isoperimetric inequality for integer currents in [FF60]. The proof of this inequality given by Allards follows from the monotonicity formula derived from the computation of the first variation of a varifold.

Date: January 24, 2020.

Key words and phrases. Metric geometry, calculus of variations, geometric measure theory, analysis on manifolds, first variation of a varifold, Michael-Simon inequality.
for a suitable perturbation of a radial vector field.

Following the ideas of Michael and Simon, Hoffman and Spruk proved in [HS74] a general Sobolev inequality for submanifolds $N$ of a Riemannian manifold $M$, satisfying geometric restrictions involving the volume of $M$, the sectional curvatures and the injectivity radius of $M$. Their proof is inspired in the Michael and Simon work, therefore, is an extrinsic perspective.

Since this general Sobolev inequality has been largely studied in different contexts from an extrinsic point of view, a natural question is whether or not this kind of inequality remains valid from an intrinsic point of view. For functions with compact support on a varifold $V$ whose first variation $\delta V$ lies in an appropriate Lebesgue space with respect to $\|\delta V\|$.

In this paper, we show an intrinsic Riemannian analog to the Allard result, considering a $k$-dimensional varifold $V$ defined in an $n$-dimensional Riemannian manifold $(M^n, g)$ (with $1 \leq k \leq n$) defined intrinsically. We achieve this goal by recovering a monotonicity inequality (instead of monotonicity equality) in this intrinsic Riemannian context, which takes into account the bounds on the geometry of $M$. Then we follow the ideas of Simon and Michael in [MS73] and [Sim83], to get a local version of the desired inequality. Finally, The Sobolev type inequality is then obtained by a standard covering argument.

1.1. A quick overview. We present a natural extension of Allard’s work. In fact, instead of define a general varifold on a Riemannian manifold via an isometric embedding (i.e., as a Radon measure on $G_k(i(M))$ where $i: M^n \to U \subset \mathbb{R}^N$ is an isometric embedding), is defined as a nonnegative, real extended valued, Radon measure on $G_k(M^n)$, the Grassmannian manifold whose underlying set is the union of the sets of $k$-dimensional subspaces of $T_x M$ as $x$ varies on $M^n$. This point of view has a consequence: more freedom on the regularity of the metric (i.e., our theory holds even for the case of only $C^2$ metrics, when the Nash embedding does not exists). In fact, there is a gap in the theory of isometric embeddings in $\mathbb{R}^N$, precisely in the case when the metric is $C^2$ but not $C^{2+\alpha}$ with $\alpha > 0$ it is not known whether an isometric immersion into Euclidean space exists. On the other hand, whenever the metric is $C^{k,\alpha}$ with $k + \alpha > 2$ there exists isometric embeddings of class $C^{k+\alpha}$. If the metric is $C^{k,\alpha}$ with $k + \alpha < 2$ then there are isometric embeddings $C^{1+\frac{\alpha}{2}}$. The first theorem is proved with the aid of the ”hard implicit function theorem”, la Nash-Moser. The second is proved using the technics of the first paper by John Nash about isometric embeddings, compare [Nas54]. This freedom constitutes, as shown in [Nar18] and [NOA18], a powerful tool to tackle problems in noncompact Riemannian manifolds of bounded geometry, or compact Riemannian manifolds with variable metric and also, possibly noncompact Riemannian manifolds of bounded geometry and variable metrics.

Our main goal is to reproduce Allard’s Theorem 7.3 in this new context along the lines of the proofs of Theorems 18.5 and 18.6 of [Sim83]. To do so, as in [HS74], our ambient manifold must satisfy some geometric conditions, namely positive injectivity radius and sectional curvature bounded above, which from now on we refer as bounded geometry, see Definition 3.4. On the other hand, since we are interested
in varifolds satisfying the conditions of the celebrated Theorem 8.1 [All72] (from now on called Allard Conditions (AC), see Definition 2.10) is necessary to study the properties of varifolds satisfying these conditions. Roughly speaking, we are interested in varifolds whose first variation has no boundary term, having generalized mean curvature belonging to $L^p$ with respect to $\|\delta V\|$ for some $p > k$, and whose density ratio is close to that of an Euclidean disk in a small ball, and density bounded below far from 0, $\|V\|$-a.e.

The main contributions of this paper are:

(i) to define rectifiable sets, exploiting the local structure of the ambient manifold, and so, define rectifiable varifolds ans in Chapter 4 Definition...

(ii) to give an intrinsic $L^p$-monotonicity formula valid for manifolds with bounded geometry and not only in $\mathbb{R}^N$, without using Nash’s isometric embedding Theorem. See Theorem 1.

(iii) to prove, intrinsically, Poincaré and Sobolev type inequalities for $C^1(S)$ non-negative functions defined over a $\mathcal{H}^k$-rectifiable set $S \subset M$. Compare Theorems 2 and 3.

(iv) to prove that the best constant $C > 0$ in the extrinsic Sobolev inequality (1.4) depends only on the dimension $k$ of $V$, which is a remarkable fact, since one a priori expects that $C$ could depend also on the bounds of the geometry of the ambient manifold $n, k, inj_M, b$. For the meaning of the former constants see Definition 3.4.

The rest of this introduction will describe in more detail the contributions of this paper.

1.1.1. Monotonicity Inequality. Given a general varifold $V$, the classic way to deduce a “monotonicity formula” is to compute the first variation of $V$ at a perturbations, by smooth functions, of (Euclidean) radial vector fields, then seems natural to consider vector fields of the form $\gamma(t) (u \nabla u)(x)$, where $\gamma(t)$ is a smooth real valued function which vanishes for large values of the variable $t$, and $u(x) := dist_{(M,g)}(x, \xi)$ for some $\xi \in M$.

The classic calculation uses the fact that the divergence of a radial vector field over a $k$ dimensional plane is exactly $k$, which is not true in general Riemannian manifolds, so it is here where the bounded geometry plays a central role, since the Rauch comparison Theorem gives bounds on such quantity. The price to pay in making our monotonicity formula using this comparison geometry argument is that there is no equality anymore, instead we have the following inequality.

**Theorem 1.** Let $(M^n, g)$ be a complete Riemannian manifold with $g$ of class at least $C^2$ having bounded geometry and Levi-Civita connection $\nabla_g$, such that $r_0 \cot \theta(r_0) > 0$, and let $V \in \mathcal{V}_k(M^n)$ satisfying (AC), then for any $0 < s < r_0$, and $h \in C^1(M)$ non-negative. There exists a constant $c = c(s,b) \in [0,1]$ such that, if we set $u(x) = r_\xi(x) = dist_{(M,g)}(x, \xi)$ we have for all $0 < s < r_0$
\begin{equation}
\frac{d}{ds} \left( \frac{1}{s^k} \int_{B_s(\xi,s)} h(y) d\|V\|(y) \right) \geq \frac{d}{ds} \int_{G_k(B_s(\xi,s))} h(y) \frac{|\nabla^S u|^2}{r^k} dV(y, T) \\
+ \frac{1}{s^{k+1}} \left( \int_{G_k(B_s(\xi,s))} \langle \nabla h(y), (u\nabla u)(y) \rangle_g dV(y, T) \right) \\
+ \frac{(c(s) - 1)k}{s} \int_{B_s(\xi,s)} h(y) d\|V\|(y) \\
+ \frac{1}{s^{k+1}} \int_{B_s(\xi,s)} (H_g, h(y)(u\nabla u)(y))_g d\|V\|(y).
\end{equation}

Here $H_g$ is as in Proposition 2.1.

1.1.2. Rectifiable varifolds. In this subsection we define the main object of our investigation, namely rectifiable varifolds. We follow here the approach of Chapter 4 of [Sim83] without treating the rectifiability of general varifolds under conditions of the first variation and the density of their weights. Before to give the following definition it is worth to recall here the classical Rademacher Theorem.

**Theorem 1.1** (Rademacher). If $f$ is Lipschitz on $\mathbb{R}^n$, then $f$ is differentiable $\mathcal{L}^n$-almost everywhere; that is, the gradient $\nabla f(x)$ exists and
\[
\lim_{y \to x} \frac{f(y) - f(x) - \nabla f(x) \cdot (y - x)}{|y - x|} = 0,
\]
for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$.

**Definition 1.2** (cf. [AK00], Definition 5.3, pg. 536). Let $(M^n, g)$ be a complete Riemannian manifold. We say that a Borel set $S \subset M$ is countable $H^k_g$-rectifiable if there exists countable many Lipschitz functions $f_j : \mathbb{R}^k \to M^n$ such that
\[
H^k_g \left( S \setminus \bigcup_{j \in \mathbb{N}} f_j(\mathbb{R}^k) \right) = 0.
\]

For any $x \in \bigcup_{j \in \mathbb{N}} f_j(\mathbb{R}^k)$ such that $f_j$ is differentiable at $y = f_j^{-1}(x)$, we define the approximate tangent space $\text{Tan}^k (S, x)$ as $d f_y(\mathbb{R}^k) \leq T_y M^n$.

**Remark 1.3.** Observe that the preceding definition is well posed, since, if $x \in f_j(\mathbb{R}^k) \cap f_i(\mathbb{R}^k)$ and $y_l \in f^{-1}_j(x)$ for $l \in \{i, j\}$, then $d f_{y_l}(\mathbb{R}^k) = d f_{y_j}(\mathbb{R}^k)$. The reader could consult [AK00], Definition 5.5, pg. 536.

**Definition 1.4** (cf. [Sim83] Chapter 4, pg. 77). Let $S$ be a countably $H^k_g$-rectifiable, subset of $(M^n, g)$ and let $\theta \in \mathcal{L}^1_{loc}(S, H^k_g)$. Corresponding to such a pair $(S, \theta)$ we define the rectifiable $k$-varifold $\nu(S, \theta)$ to be, the equivalence class of all pairs $(S, \tilde{\theta})$, where $\tilde{S}$ is countably $H^k_g$-rectifiable with $H^k_g((S \Delta \tilde{S})) = 0$ and where $\tilde{\theta} = \theta, H^k_g$-a.e. on $S \cap \tilde{S}$.

So, we can naturally induce a (general) varifold from a rectifiable one as follows.
**Definition 1.5.** Given an $\mathcal{H}^k$-rectifiable varifold $v(S, \theta)$ on $M^n$ there is a corresponding (general) $k$-varifold $V \in \mathcal{V}_k(M^n)$ (also denoted by $v(S, \theta)$), defined by

$$V(A) := v(S, \theta)(A) = \mathcal{H}^k_A, \theta(\pi(\{(x, T_xM) : x \in S \} \cap A)), \quad A \subset G_k(M^n),$$

where $S_*$ is the set of $x \in S$ such that $S$ has an approximate tangent space $T_xS$ with respect to $\theta$ at $x$. Evidently $v(S, \theta)$, so defined, has weight measure $\|v(S, \theta)\| = \mathcal{H}^k_{g^*} \theta$.

1.1.3. **Poincaré and Sobolev inequalities.** In the special case in which, $V = v(S, \theta)$ is a rectifiable varifold such that the mean vector field belongs to a certain Lebesgue and Sobolev inequalities.

\[ h(\xi) \leq e^{(\Lambda + c^* k)\rho} \left( \frac{1}{\omega_k} \rho^k \int_{B_\rho(\xi, \rho)} h d\|V\| + \int_{B_\rho(\xi, \rho)} |\nabla^S h|_{\theta} d\|V\| \right), \]

for all $\xi \in \text{spt } \|V\|$ and for all $0 < \rho < r_0$. From this, together with an approximation argument and Fubini’s Theorem we deduce the following Poincaré inequality.

**Theorem 2.** Let $(M^n, g)$ be a complete Riemannian manifold with bounded geometry, let $V := v(S, \theta)$ a rectifiable varifold satisfying (AC). Suppose: $\ h \in C^1(M)$, $h \geq 0$, $B_g(\xi, 2\rho) \subset B_g(\xi, r_0)$ for $\xi \in S$ fixed, $|H_g|_g \leq \Lambda$ for some $\Lambda > 0$, $\theta > 1$ $\|V\|$-a.e. in $B_g(\xi, r_0)$ and for some $0 < \alpha < 1$

\[ \|V\| \left( \{x \in B_g(\xi, \rho) : h(x) > 0 \} \right) \leq \omega_k (1 - \alpha) \rho^k \quad \text{and} \quad e^{(\Lambda + c^* k)\rho} \leq 1 + \alpha. \]

Suppose also that, for some constant $\Gamma > 0$

\[ \|V\| \left( B_g(\xi, 2\rho) \right) \leq \Gamma \rho^k. \]

Then there are constants $\beta := \beta(k, \alpha, r_0, b) \in]0, \frac{1}{2}[$ and $C := C(k, \alpha, r_0, b) > 0$ such that

\[ \int_{B_\rho(\xi, 2\rho)} h d\|V\| \leq C \rho \int_{B_\rho(\xi, \rho)} |\nabla^S h|_g d\|V\|. \]

Here $H_g$ is as in Proposition 2.1.

Finally, the Sobolev inequality follows from Theorem (1) again in the special case in which $V = v(S, \theta)$ is a rectifiable varifold such that the mean vector field belongs to a certain Lebesgue space and a standard covering argument (c.f. [Sim83] Theorem 3.3 pg. 11).

**Theorem 3.** Let $(M^n, g)$ be a complete Riemannian manifold with bounded geometry, let $V = v(S, \theta)$ a $k$-rectifiable varifold satisfying (AC). Suppose $\ h \in C^1_0(M)$ non negative, and $\theta \geq 1$ $\|V\|$-a.e. in $S$. Then there exists $C := C(k) > 0$ such that

\[ \left( \int_S h^{\frac{k}{k-1}} d\|V\| \right)^{\frac{k-1}{k}} \leq C \int_S \left( |\nabla^S h|_g + h \left( |H_g|_g - c^* k \right) \right) d\|V\|. \]

Here $H_g$ is as in Proposition 2.1.
1.2. Structure of the paper. In Section 2 the definition of general varifold of
[All72] is extended to a complete Riemannian manifold as well as the first variation
of them. In Section 3 Theorem 1 is proved, by testing the first variation of a
general varifold with a suitable radial deformation field. Here the bounded geometry
assumptions of the ambient manifold, play an essential role. In Section 3 are also
proved several results concerning the monotonicity of the density ratio under
$L^p$-type assumptions on the weak mean curvature field $H_g$ of the varifold. Finally, in
Section 4 Theorem 2 and Theorem 3 are proved, as a consequence of Theorem 1
and a standard covering argument.

2. INTRINSIC RIEMANNIAN THEORY OF GENERAL VARIFOLDS

2.1. General Varifolds. Now we introduce the notations and concepts relative to
varifolds that we need to make a Riemannian intrinsic theory of varifolds. In this
respect we closely follow [All72], [Sim83], [Lei12], [NOA18]. In what follows $V$
will always denote a varifold and $dv_g$ the Riemannian measure of $(M^n,g)$.

Definition 2.1. For any $k, n \in \mathbb{N}$, $n \geq 2$, $1 \leq k \leq n - 1$, let $M^n$ a $n$-dimensional
manifold. We say that $V$ is a $k$-dimensional varifold in $M$, if $V$ is a nonnegative, real
extended valued, Radon measure on $G_k(M^n)$ the Grassmannian manifold whose
underlying set is the union of the sets $Gr(k, T_x M)$, where $Gr(k, T_x M)$ denotes the
set of $k$-dimensional subspaces of $T_x M^n$, as $x$ varies on $M^n$ (compare with section
2.6 of [All72]). For every $k \in \{1, ..., n-1\}$, we define $V_k(M^n)$ to be the space
of all $k$-dimensional varifolds on $M^n$ endowed with the weak topology induced
by $C^0(G_k(M^n))$, say the space of continuous compactly supported functions on
$G_k(M^n)$ endowed with the compact open topology.

Definition 2.2. Let $V \in V_k(M^n)$, $g$ is a Riemannian metric on $M^n$, we say
that the nonnegative Radon measure on $M^n$, $\|V\|$ is the weight of $V$, if $\|V\| = \pi_\#(V)$, here $\pi$ indicates the natural fiber bundle projection $\pi : G_k(M^n) \to M^n,$
$\pi : (x, S) \mapsto x,$ for every $(x, S) \in G_k(M^n)$, $x \in M^n$, $S \in Gr(k, T_x M),$
$\|V\|(A) := V(\pi^{-1}(A)).$

Remark 2.3. Recall that $\pi$ is a proper map because the fibers of the fiber bundle
$G_k(M^n) \xrightarrow{\pi} M^n$ are compact.

As the reader has noticed, an abstract varifold can be a quite strange object,
because it is hard to work with Borel sets on $G_k(M^n)$ in an operative way (here
operative way has to be understated in a sense to be specified later in this section);
but in the sequel, we also define the weight of $V$ which is a Radon measure on $\Sigma$
obtained from $V$ by ignoring the fiber variable. The next theorem illustrates how
to "simplify" a varifold in an operative way. This result is a direct application
of a well known disintegration Theorem, which can be found in [AFP00] Theorem
2.28. However the following is an adaptation of it into the context of our intrinsic
varifolds.

Theorem 2.4 (Disintegration Theorem for Varifolds). Let $(M^n, g)$ a $n$-dimensional
Riemannian manifold, let $V \in V_k(M^n)$, and $\pi : G_k(M^n) \to M$ be the canonical
projection onto $M^n$. Then there exists a family of Radon measures $\{\pi_x\}_{x \in M^n}$ such
that, the map \( x \mapsto \pi_x \) is \( ||V|| \)-measurable and the following relations are satisfied

\[(2.1) \quad \pi_x(B) := \lim_{r \downarrow 0} \frac{V(B_g(x,r) \times B)}{||V||||B(x,r)||}, \quad \text{for all } B \in \mathcal{B}(\text{Gr}(k,T_x M^n)), \]

\[(2.2) \quad \pi_x(\text{Gr}(k,T_x M^n) \setminus \{ S : S \subset \text{Tan}(M^n,x) \}) = 0, \quad \text{and } \pi_x(\text{Gr}(k,T_x M^n)) = 1, \]

\[(2.3) \quad f(x,\cdot) \in L^1(\text{Gr}(k,T_x M^n),\pi_x), \quad \text{for } ||V|| - \text{a.e. } x \in M^n, \]

\[(2.4) \quad x \mapsto \int_{\text{Gr}(k,T_x M^n)} f(x,S)d\pi_x(S) \in L^1(M^n,||V||), \quad \text{is } ||V|| - \text{measurable}, \]

\[(2.5) \quad \int_{G_k(M^n)} f(x,S)d\nu(x,T) = \int_{M^n} \left( \int_{\text{Gr}(k,T_x M^n)} f(x,S)d\pi_x(S) \right) d||V||(x), \]

for any \( f \in L^1(G_k(M^n),V) \). Moreover, if \( \pi_x' \) is any other \( ||V|| \)-measurable map satisfying (2.4) and (2.5) for every bounded Borel function with compact support and such that \( x \mapsto \pi_x'(\text{Gr}(k,T_x M^n)) \in L^1(M^n,||V||) \), then \( \pi_x = \pi_x' \) for \( ||V|| \)-a.e. \( x \in M^n \).

**Corollary 2.5.** Let \( V \in \mathcal{V}_k(M^n) \), with the same notation of the Theorem 2.4, the equality

\[ V = ||V|| \otimes \pi_x, \]

holds.

### 2.2. The First Variation of a Varifold

According to [Al72], we associate to each varifold \( V \) a vector valued distribution, which depends on the metric \( g \) of the ambient space, called the **first variation of \( V \)**. If this first variation is (as Radon measure) absolute continuous with respect to weight measure \( ||V|| \) in analogy with the smooth case we define the generalized mean curvature vector \( H_g \).

**Definition 2.6.** Let \((M^n, g)\) a \( n \)-dimensional Riemannian manifold with Levi-Civita connection \( \nabla \), \( \mathcal{X}_k^1(M) \) the set of differentiable vector fields on \( M \) and \( V \in \mathcal{V}_k(M) \) a \( k \)-dimensional varifold \((k \leq n)\). We define the first variation of \( V \) along the vector field \( X \in \mathcal{X}_k^1(M) \) as

\[ \delta V(X) := \int_{G_k(M)} \langle \nabla X(x) \circ p_T, p_T \rangle_g dV(x,T), \]

where the inner product in the integrand is the one defined in \( \text{Hom}(T_x M,T_x M) \), and \( \nabla X : \mathcal{X}(M) \to \mathcal{X}(M) \) such that \( \nabla X(Y) := \nabla Y X \).

Let \( \{ \tau_1, \ldots, \tau_k \} \) an orthonormal basis of \( T \in \text{Gr}(k,T_x M) \) for \( x \) given, and \( \{ \tau_1, \ldots, \tau_k, \tau_{k+1}, \ldots, \tau_n \} \) the completion to an orthonormal basis for \( T_x M \), then for given \( X \in \mathcal{X}_k^1(M) \),

\[ \delta V(X) = \int_{G_k(M)} \text{trace} \left( (\nabla X(x) \circ p_T) \circ p_T \right) dV(x,T) \]

\[ = \int_{G_k(M)} \sum_{i=1}^k \langle \tau_i, \nabla \tau_i X(x) \rangle_g dV(x,T), \]

so, we have that, for all \( X \in \mathcal{X}_k^1(M) \), the definition (2.6) is equivalent to say,

\[(2.6) \quad \delta V(X) = \int_{G_k(M)} \text{div}_T X(x) dV(x,T), \]
where, for given \( x \), and \( \{\tau_1, \ldots, \tau_k\} \) an orthonormal basis of a fixed \( T \in \text{Gr}(k, T_x M) \),

\[
\text{div}_T X(x) = \sum_{i=1}^{k} \langle \tau_i, \nabla_{\tau_i} X(x) \rangle_g .
\]

Then (2.6) give us formula with more geometric meaning than the merely definition, on the other hand, by analogy with the smooth case, we desire to related in some way the first variation of a varifold with a "mean curvature" vector, to do so, let us first analyze the total variation of the first variation.

**Definition 2.7.** Let \( V \in \mathbf{V}_k(M) \) with \((M, g)\) a \( n \)-dimensional Riemannian manifold, we say that \( V \) has locally bounded first variation in \( U \subset M \) open set, if for all \( W \subset U \) open set, there exists a constant \( C := C(W) \) such that

\[
|\delta V(X)| \leq C|X|_{L^\infty(W, V)} ,
\]

for all \( X \in \mathcal{X}_c^0(W) \).

**Remark 2.8.** Notice that the definition of bounded first variation is valid for each \( X \in \mathcal{X}_c^0(M) \) while the first variation, \( \delta V \), is defined on \( \mathcal{X}_c^1(M) \). However, defining an extension \( \tilde{\delta V} : \mathcal{X}_c^0(M) \to \mathbb{R} \) of \( \delta V \) as

\[
\tilde{\delta V}(X) := \lim_{\varepsilon \downarrow 0} \delta V(X_\varepsilon),
\]

where \( (X^1, \ldots, X^n) = X \in \mathcal{X}_c^0(M) \), and \( X_\varepsilon \in \mathcal{X}_c^1(M) \) is a "\( C^1 \) approximation" of \( X \) defined as follows:

Let \( \{(\Phi_i, U_i)\}_{i \in \Lambda} \) an atlas of \( M \), since we are interested in compactly vector fields we can choose \( \Lambda = \{1, \ldots, N\} \) such that \( \{U_i\}_{i=1}^N \) is an open covering for \( \text{spt} \, X \), and consider \( \{\psi_i\}_{i=1}^N \) a partition of unity subordinate to \( \{U_i\}_{i=1}^N \), then

\[
X_\varepsilon(p) := \sum_{i=1}^{\varepsilon} \left( \sum_{j=1}^{N} \left( \psi_j \left. X^i \right|_{U_i} \ast \varphi^\varepsilon \right) (p) \right) \frac{\partial}{\partial x_i},
\]

where \( \varphi^\varepsilon \) is an standard approximation of the identity in \( \mathbb{R}^n \), and \( \ast \) denotes the usual convolution.

Clearly \( X_\varepsilon \) is independent of the choice of charts and defined on \( \text{spt} \, X \). Furthermore by standard theory of convolutions, we know that,

\[
\psi_j \left. X^i \right|_{U_i} \ast \varphi^\varepsilon \to \psi_j \left. X^i \right|_{U_j}
\]

uniformly in compacts, when \( \varepsilon \to 0 \). Hence

\[
\|X - X_\varepsilon\|_{L^\infty} \to 0, \quad \text{when} \quad \varepsilon \to 0,
\]

and \( X_\varepsilon \in \mathcal{X}_c^1(M) \).

On the other hand, given \( X^1_\varepsilon \in \mathcal{X}_c^0(M) \) as above such that \( X^1_\varepsilon \to X \) and \( X^1 \to X \), assume that \( V \) has bounded first variation on \( M \), then

\[
|\delta V(X^1_\varepsilon - X^1)| \leq C\|X^1_\varepsilon - X^1\|_{L^\infty} \to 0 \quad \text{when} \quad \varepsilon \to 0,
\]

which implies that

\[
\tilde{\delta V} : \mathcal{X}_c^0(M) \to \mathbb{R}
\]

is well defined and \( \tilde{\delta V} \equiv \delta V \) on \( \mathcal{X}_c^1(M) \). By abuse of notation we will identify \( \tilde{\delta V} \) with \( \delta V \).
Proposition 2.1. Let $V \in \text{V}_k(M^n)$ with locally bounded first variation in $(M^n, g)$. Then the total variation $\|\delta V\|$ is a Radon measure. Furthermore, there exist a $\|V\|$-measurable function $H_g: M \to TM$, and $Z \subset M$ with $\|V\|(Z) = 0$, such that

$$\delta V(X) = -\int_{M^n} \langle X, H_g \rangle_g d\|V\| + \int_{M^n} \langle X, \nu \rangle_g d\|\delta V\|_{\text{sing}},$$

where $\nu$ is a $\|\delta V\|$-measurable function with $|\nu(x)|_g = 1$, and $\|\delta V\|_{\text{sing}} = \|\delta V\|_Z$. We call $H_g$ the generalized mean curvature vector of $V$.

Proof. Since by hypothesis $V$ has locally bounded first variation, for all open set $W \subset \subset M$

$$\|\delta V\|(W) = \sup \{ |\delta V(X)| : X \in \mathcal{X}_V(W), |X| \leq 1 \} \leq C < \infty.$$

Then by the Riesz Representation Theorem for $\mathcal{X}_V(M)$ (where we are identifying $\overline{\delta V}$ with $\delta V$) there exists $\nu: M \to TM \|\delta V\|$-measurable function with $|\nu(x)|_g = 1$ $\|\delta V\|$-a.e. and for all $X \in \mathcal{X}_V(M)$

$$\delta V(X) = \int_M \langle X, \nu \rangle_g d\|\delta V\|,$$

and $\|\delta V\|$ is a Radon measure. This last assertion prove the first part of the theorem as in the Euclidean case. To prove the second part of the theorem we need to make use of the fact that the Vitali symmetry property holds for every metric balls of $(M^n, g)$. This guarantees that we can apply the Lebesgue Differentiation Theorem to space $B_g(p, r)$. Thus, by the Lebesgue Differentiation Theorem we get

$$D_{\|V\|} \|\delta V\| = \lim_{\mu \to 0} \frac{\|\delta V\|(B_g(x, \rho))}{\|V\|(B_g(x, \rho))}$$

exists $\|V\|$-a.e and (writing $-H_g(x) := D_{\|V\|} \|\delta V\|(x)\nu(x)$) we are lead to the following formula

$$\delta V(X) = -\int_M \langle X, H_g(x) \rangle_g d\|V\|(x) + \int_M \langle X(x), \nu(x) \rangle_g d\|\delta V\|_{\text{sing}},$$

where $\|\delta V\|_{\text{sing}} = \|\delta V\|_\{x : D_{\|V\|} \|\delta V\| = +\infty\}$. □

Remark 2.9. Let $(M^n, g)$ be an $n$-dimensional Riemannian manifold, let $V \in \text{V}_k(M)$ such that for some $C > 0$, $p > k$, $0 < \rho < \text{inj}_\xi(M, g)$ for fixed $\xi \in M$

$$\delta V(X) \leq C \left( \int_{B_g(\xi, \rho)} |X|^{\frac{n}{n-k}} d\|V\| \right)^{\frac{n-1}{q}},$$

Then, by Hölder inequality we have that

$$|\delta V(X)| \leq C_1 \|X\|_{L^\infty(\|V\|, \text{spt} X \cap B_g(\xi, \rho))},$$

where $C_1(\xi, \rho) := C (\|V\| (\text{spt} X \cap B_g(\xi, \rho)))^{\frac{n-1}{q}}$. Then $V$ has locally bounded first variation, furthermore, $\|\delta V\|$ is a Radon measure on $M$. Roughly speaking, if we assume that $\text{spt} X \subset A$, for $A$ a given $\|V\|$-negligible set, and take the supremum over all such $X$’s with $\|X\|_{L^\infty(\|V\|)} = 1$ then by (2.7) we get that

$$\|\delta V\|(A) \leq C (\|V\|(A))^{\frac{n-1}{q}} = 0,$$
which implies that $\|\delta V\| \ll \|V\|$. In fact, technically to prove the preceding equation we need to be more careful, i.e., we need to take a sequence of open sets $A \subset U_j \subset B_g(\xi, \rho), \forall j$ such that $U_j \to A$, then using (2.7) we have that

$$
\delta V(X) \leq C \left( \int_{U_j} |X| \frac{1}{p} d\|V\| \right)^{\frac{q}{q-1}}, \forall X \in \mathcal{X}^1(\mathcal{U}_j).
$$

Then taking limits, it easy to check that

$$
\|\delta V\|(A) = \lim_{j \to \infty} \|\delta V\|(U_j) \leq \lim_{j \to \infty} C (\|V\|(U_j))^{\frac{q-1}{q}} = 0.
$$

Hence, the Radon-Nikodym Theorem gives the existence of a $L^1(M^n, \|V\|)$ function $H_g$ such that

\begin{equation}
(2.8) \quad \delta V(X) = \int_{G_{\mathcal{U}}(M)} \text{div}_S X(y) dV(y, T) = - \int_{M} \langle H_g(y), X(y) \rangle d\|V\|(y).
\end{equation}

On the other hand, (2.7) means that $\delta V(X)$ is a bounded linear functional on the space $L^1_{\text{loc}}(\Gamma(TM), \|V\|)$ of the $L^1_{\text{loc}}$ vector fields on $M$ with respect to $\|V\|$. So by well known theorems of measure theory (compare Theorem 6.16 of [Rud87]) stating that the topological dual of $L^q$ is $L^p$, i.e., $(L^q)^* = L^p$ we have that there exists a function $f \in L^p$ such that $\delta V(X) = \int_{B_q(\xi, \rho)} \langle f, X \rangle_g d\|V\|(y)$ and the best constant $C > 0$ in (2.7) is given by $\|f\|_{L^p}$. Combining (2.7) and (2.8) we get that $f = H_g$ and so that $H_g \in L^1_{\text{loc}}(\Gamma(TM), \|V\|)$. Finally, notice that, for all $X \in \mathcal{X}^1(M)$ (again using the Hölder inequality)

$$
\delta V(X) \leq \|H_g\|_{L^p(\|V\|, M)} \|X\|_{L^\frac{p}{p-1}(\|V\|, M)}.
$$

At this stage we have proved that (2.7) implies that $V$ has locally bounded first variation, no singular part and the generalized mean curvature vector field $H_g$, satisfies an $L^p$ condition.

**Definition 2.10** (Allard’s Conditions). Given $V$ a $k$-dimensional varifold, we say that $V$ satisfies an *Allard’s type condition* for the generalized mean curvature, if $V$ satisfies (2.7). From now on, denoted as $(AC)$.

This kind of conditions will play a key role in the section (3.1) when we study the monotonicity behaviour of the density ratio.

### 3. Monotonicity and Consequences

The aim of this section is to obtain information about $V$ from their first variation $\delta V(X)$, for an appropriate choice of $X \in \mathcal{X}^1_0(U)$. This kind of result is known as “Monotonicity Formula”, and is the step zero in any interior regularity theory.

#### 3.1. Weighted Monotonicity Formulae for Abstract Varifolds

Let $(M, g)$ a Riemannian manifold with Levi-Civita connection $\nabla$ and $V \in \mathcal{V}_k(M)$ satisfying $(AC)$, fix $\xi \in M$ and assume $inj_\xi(M, g) > 0$. Choose $r_0 > 0$ such that $r_0 < inj_\xi(M, g)$, and, for fixed $\varepsilon > 0$ let $\gamma_\varepsilon \in C^1((-\infty, 1])$, such that

$$
\gamma_\varepsilon(y) := \begin{cases} 
1 & \text{if } y \leq \varepsilon, \\
0 & \text{if } y > 1,
\end{cases} \quad \text{and} \quad \gamma_\varepsilon'(y) < 0 \text{ if } \varepsilon < y < 1,
$$

Then

\begin{equation}
\delta V(X) = \int_{G_{\mathcal{U}}(M)} \text{div}_S X(y) dV(y, T) = - \int_{M} \langle H_g(y), X(y) \rangle d\|V\|(y).
\end{equation}

On the other hand, (2.7) means that $\delta V(X)$ is a bounded linear functional on the space $L^1_{\text{loc}}(\Gamma(TM), \|V\|)$ of the $L^1_{\text{loc}}$ vector fields on $M$ with respect to $\|V\|$. So by well known theorems of measure theory (compare Theorem 6.16 of [Rud87]) stating that the topological dual of $L^q$ is $L^p$, i.e., $(L^q)^* = L^p$ we have that there exists a function $f \in L^p$ such that $\delta V(X) = \int_{B_q(\xi, \rho)} \langle f, X \rangle_g d\|V\|(y)$ and the best constant $C > 0$ in (2.7) is given by $\|f\|_{L^p}$. Combining (2.7) and (2.8) we get that $f = H_g$ and so that $H_g \in L^1_{\text{loc}}(\Gamma(TM), \|V\|)$. Finally, notice that, for all $X \in \mathcal{X}^1(M)$ (again using the Hölder inequality)

$$
\delta V(X) \leq \|H_g\|_{L^p(\|V\|, M)} \|X\|_{L^\frac{p}{p-1}(\|V\|, M)}.
$$

At this stage we have proved that (2.7) implies that $V$ has locally bounded first variation, no singular part and the generalized mean curvature vector field $H_g$, satisfies an $L^p$ condition.

**Definition 2.10** (Allard’s Conditions). Given $V$ a $k$-dimensional varifold, we say that $V$ satisfies an *Allard’s type condition* for the generalized mean curvature, if $V$ satisfies (2.7). From now on, denoted as $(AC)$.

This kind of conditions will play a key role in the section (3.1) when we study the monotonicity behaviour of the density ratio.

### 3. Monotonicity and Consequences

The aim of this section is to obtain information about $V$ from their first variation $\delta V(X)$, for an appropriate choice of $X \in \mathcal{X}^1_0(U)$. This kind of result is known as “Monotonicity Formula”, and is the step zero in any interior regularity theory.
Then we can consider the radial perturbation
\[
\tilde{X}_{s,ε}(x) = \left( γ_ε \left( \frac{u(x)}{s} \right) (u∇u) \right)(x), \text{ for } 0 < |s| < r_0,
\]
where \( u(x) = r_ξ(x) = \text{dist}(M, g)(x, ξ) \). Now, let \( y \in M \) given, and \( T \in \text{Gr}(k, T_y M) \). Let \( \{e_1^T, \ldots, e_k^T\} \) an orthonormal basis for \( S \), then
\[
div T \tilde{X}_{s,ε} = \sum_{i=1}^{k} \left< e_i^T, ∇_{e_i^T γ_ε} \left( \frac{u}{s} \right) (u∇u) \right>_g
\]
\[
= γ_ε \left( \frac{u}{s} \right) \frac{u}{s} \sum_{i=1}^{k} (e_i^T, ∇u)^2 + γ_ε \left( \frac{u}{s} \right) \sum_{i=1}^{k} \left< e_i^T, ∇_{e_i^T} (u∇u) \right>_g
\]
\[
= γ_ε \left( \frac{u}{s} \right) \text{div}_T (u∇u) + γ_ε' \left( \frac{u}{s} \right) \frac{u}{s} |∇^T u|^2_g,
\]
where \( ∇^T u \) is the orthogonal projection of \( ∇u \) onto \( S \). Let \( ∇^{T^\perp} u \) denote the orthogonal projection of \( ∇u \) onto \( T^\perp \), then,
\[
|∇^T u|^2_g + |∇^{T^\perp} u|^2_g = |∇u|^2_g = 1,
\]
since \( u \) is a distance function. Therefore
\[
div T \tilde{X}_{s,ε} = γ_ε \left( \frac{u}{s} \right) \text{div}_T (u∇u) + γ_ε' \left( \frac{u}{s} \right) \frac{u}{s} |∇^T u|^2_g.
\]
Although this choice is enough to get many useful information, let us consider a general case, which will be used in the sequel. Let \( h \in C^1(U) \) a non-negative function, and consider
\[
(3.1) \quad X_{s,ε}(y) := h(y)\tilde{X}_{s,ε} = h(y) \left( γ_ε \left( \frac{u(y)}{s} \right) (u∇u) \right)(y), \text{ for } 0 < |s| < r_0.
\]
Then, if \( \{e_1^S, \ldots, e_k^T\} \) is as above,
\[
div T X_{s,ε}(y) = \sum_{i=1}^{k} \left< e_i^T, ∇_{e_i^T h(y)}\tilde{X}_{s,ε}(y) \right>_g
\]
\[
= h(y) \sum_{i=1}^{k} \left< e_i^T, ∇_{e_i^T} \tilde{X}_{s,ε}(y) \right>_g + \sum_{i=1}^{k} \left( \left< ∇_{e_i^T} h(y) \right> e_i^T, \tilde{X}_{s,ε}(y) \right>_g
\]
\[
= h(y) \text{div}_T \tilde{X}_{s,ε} + \left< ∇^T h(y), \tilde{X}_{s,ε} (y) \right>_g.
\]
By the definition of the first variation, we know that
\[
\delta V(X_{s,ε}) := \int_{G_0(U)} \text{div}_T X_{s,ε}(y)dV(y, T)
\]
\[
= \int_{G_0(U)} h(y) \text{div}_T \tilde{X}_{s,ε}(y)dV(y, T) + \int_{G_0(U)} \left< ∇^T h(y), \tilde{X}_{s,ε}(y) \right>_g dV(y, T).
\]
Then, replacing (3.1) in (3.2) and the information above, we have
\[ \delta V(X, \varepsilon) = \int_{G_k(U)} h(y) \gamma \left( \frac{u(y)}{s} \right) \text{div}_T(u \nabla u)(y) dV(y, T) \]
\[ + \int_{G_k(U)} \frac{h(y)u(y)}{s} \gamma \left( \frac{u(y)}{s} \right) dV(y, T) \]
\[ - \int_{G_k(U)} \frac{h(y)u(y)}{s} \gamma \left( \frac{u(y)}{s} \right) \left| \nabla^T\!_{\!g} u \right|_{g}^2 dV(y, T) \]
\[ + \int_{G_k(U)} \gamma \left( \frac{u(y)}{s} \right) \left\langle (\nabla^T h(y), (u \nabla u)(y)) \right\rangle_{g} dV(y, T) \]
\[ (3.3) \]

If we compare (3.3) with the Euclidean case (see Cf. [Sim83] (4.14) and (4.24)) we note the lack of a dimensional term, then, in order to have an intrinsic result, we need to compare it in some way with the Euclidean case. To do this we use the Rauch’s comparison theorem, applied as in Lemma 3.6 of [HS74], which states:

**Lemma 3.1.** Let \((M, g)\) be a complete Riemannian manifold, with Levi-Civita connection \(\nabla\), let \(b \in \mathbb{R}\) such that \(\text{Sec}_g \leq b\), assume \(br_0 < \pi\). Then
\[ \text{div}_T(u \nabla u)(x) \geq ku(x) \cot_b(u(x)), \]
for all \(x \in B_g(\xi, r_0)\).

**Remark 3.2.** Before to continue, let us give some comments on this Lemma. First to all, recall that \(\cot_b(s)\) is defined as
\[ \cot_b(s) = \frac{cs_b(s)}{sn_b(s)}, \]
where \(sn_b(s)\) is the unique solution to
\[
\begin{align*}
x''(s) + bx(s) &= 0 \\
x(0) &= 0 \\
x'(0) &= 1,
\end{align*}
\]
and \(cs_b(s) = sn'_b(s)\).

Then, we have three cases

Case 1: assume \(b = 0\), then
\[ s \cot_b(s) = 1. \]

Case 2: Assume \(b > 0\), then
\[ s \cot_b(s) = s \sqrt{b} \cdot \frac{\cos(\sqrt{bs})}{\sin(\sqrt{bs})}. \]
An asymptotic analysis when \(s \to 0\) gives
\[ s \cot_b(s) \sim 1 - O(s^2) \]
\[ (3.4) \]
Case 3: Assume $b < 0$, then

$$s \cot_b(s) = \sqrt{|b|s} \cdot \frac{\cosh(\sqrt{|b|s})}{\sinh(\sqrt{|b|s})}.$$  

As in the previous case, an asymptotic analysis when $s \to 0$ gives

$$(3.5) \quad s \cot_b(s) \sim 1 + O(s^2).$$

Remark 3.3. Notice that under the conditions of Lemma 3.1 we can compare the divergence of the radial field $u\nabla u$ in the ambient manifold $M^n$ with the same quantity calculated in a space-form, so, is natural to guess that this is the framework to get some "monotonicity" behaviour (inspired in the results from the euclidean case). Furthermore, in view of Remark 3.2 above, we expect to recovery in some sense the euclidean case, this is why from now we will be in the setting of the Lemma 3.1, i.e.

**Definition 3.4 (Bounded Geometry).** We say that a complete Riemannian manifold $(M^n, g)$ has bounded geometry (or satisfy $(BG)$), if $\text{Sec}_g < b$ for some constant $b \in \mathbb{R}$ and for every $\xi \in M$ there is $r_0$ such that $0 < r_0 < \text{inj}_\xi(M, g)$ and $r_0b < \pi$.

In view of the Remarks (3.3) and (3.2) we can define $c(s) := c(s, b)$ as:

$$c(s) = \begin{cases} 
    s\sqrt{b} \cot(\sqrt{bs}) & b > 0 \\
    0 & b \leq 0,
\end{cases}$$

where $b \in \mathbb{R}$. Then, by the Lemma (3.1),

$$\text{div}_T((u\nabla u)(x)) \geq kc(u(x), b),$$

for all $x \in B_g(\xi, r_0)$, furthermore we can assume $r_0 \cot_b(r_0) > 0$, since we are interested in small geodesic balls around $\xi$.

Then, substituting in (3.3) we have

$$\delta V(X_{s,\varepsilon}) \geq \int_{G_b(M)} k \gamma \left( \frac{u(y)}{s} \right) dV(y, T)$$

$$\geq \int_{G_b(M)} \gamma \left( \frac{u(y)}{s} \right) \langle \nabla^T h(y), (u\nabla u)(y) \rangle_g dV(y, T)$$

$$+ \int_{G_b(M)} \gamma \left( \frac{u(y)}{s} \right) \langle \nabla^T h(y), (u\nabla u)(y) \rangle_g dV(y, T)$$

$$+ \int_{G_b(M)} h(y)u(y) \gamma \left( \frac{u(y)}{s} \right) dV(y, T)$$

$$- \int_{G_b(M)} h(y)u(y) \gamma \left( \frac{u(y)}{s} \right) \left| \nabla^T u \right|_g^2 dV(y, T).$$

On the other hand, since $c(s)$ is decreasing, $c(u(x)) \geq c(s) \geq c(r_0)$ whenever $0 \leq u(x) \leq s \leq r_0 = r_0(b)$. Hence,
\[- \int_{G_k(M)} \left[ h k \gamma_\varepsilon \left( \frac{u}{s} \right) + \frac{h u}{s} \gamma_\varepsilon' \left( \frac{u}{s} \right) \right] dV \geq \int_{G_k(M)} \left[ -\frac{h u}{s} \gamma_\varepsilon' \left( \frac{u}{s} \right) \| \nabla T u \|^2_g dV \right.
\]
\[+ \int_{G_k(M)} \gamma_\varepsilon \left( \frac{u}{s} \right) \langle \nabla T h, (u \nabla u)_g \rangle dV \]
\[+ (c(s) - 1) \int_{G_k(M)} h k \gamma_\varepsilon \left( \frac{u}{s} \right) dV - \delta V(X_{s,\varepsilon}), \]
now, dividing by \(s^{k+1}\),
\[- \int_{G_k(M)} h(y)I(s)dV(y,T) \geq \int_{G_k(M)} \frac{h(y)}{s^k} \frac{\partial}{\partial s} \left( \gamma_\varepsilon \left( \frac{u(y)}{s} \right) \right) \left\| \nabla T^\perp u \right\|_g^2 dV(y,T) \]
\[+ \frac{1}{s^{k+1}} \left( \int_{G_k(M)} \gamma_\varepsilon \left( \frac{u(y)}{s} \right) \langle \nabla T h(y), (u \nabla u)_g \rangle dV(y,T) \right) \]
\[+ \frac{(c(s) - 1) k}{s} \int_{G_k(M)} h(y) \gamma_\varepsilon \left( \frac{u(y)}{s} \right) dV(y,T) - \frac{\delta V(X_{s,\varepsilon})}{s^{k+1}}, \]
where
\[I(s) := \frac{k}{s^{k+1}} \gamma_\varepsilon \left( \frac{u(y)}{s} \right) + \frac{u(y)}{s^{k+2}} \gamma_\varepsilon' \left( \frac{u(y)}{s} \right) \]
\[= - \frac{d}{ds} \left( \frac{1}{s^k} \right) \gamma_\varepsilon \left( \frac{u(y)}{s} \right) - \frac{1}{s^k} \frac{d}{ds} \left( \gamma_\varepsilon' \left( \frac{u(y)}{s} \right) \right) \]
\[= - \frac{d}{ds} \left( \frac{1}{s^k} \gamma_\varepsilon \left( \frac{u(y)}{s} \right) \right). \]
Differentiating under the sign of integral we have
\[\frac{d}{ds} \left( \frac{1}{s^k} \int_{G_k(M)} h \gamma_\varepsilon \left( \frac{u}{s} \right) dV \right) \geq \int_{G_k(M)} \frac{h}{s^k} \frac{\partial}{\partial s} \left( \gamma_\varepsilon \left( \frac{u}{s} \right) \right) \left\| \nabla T^\perp u \right\|_g^2 dV \]
\[+ \frac{1}{s^{k+1}} \left( \int_{G_k(M)} \gamma_\varepsilon \left( \frac{u}{s} \right) \langle \nabla T h, (u \nabla u)_g \rangle dV \right) \]
\[+ \frac{(c(s) - 1) k}{s} \int_{G_k(M)} h \gamma_\varepsilon \left( \frac{u}{s} \right) dV - \frac{\delta V(X_{s,\varepsilon})}{s^{k+1}}. \]
Now, by Theorem (2.4),
\[\int_{G_k(M)} h(y) \gamma_\varepsilon \left( \frac{u(y)}{s} \right) dV(y,T) = \int_{M} h(y) \gamma_\varepsilon \left( \frac{u(y)}{s} \right) d\|V\|(y) \]
and
\[\int_{G_k(M)} h(y) k \gamma_\varepsilon \left( \frac{u(y)}{s} \right) dV(y,T) = \int_{M} h(y) k \gamma_\varepsilon \left( \frac{u(y)}{s} \right) d\|V\|(y). \]
Thus,
\[
\frac{d}{ds} \left( \frac{1}{s^k} \int_M h \gamma \left( \frac{u}{s} \right) d\|V\| \right) \geq \int_{G_k(M)} \frac{h}{s^k} \frac{\partial}{\partial s} \left( \gamma \left( \frac{u}{s} \right) \right) \left| \nabla^{T_{\perp}} u \right|^2_g dV \\
+ \frac{1}{s^{k+1}} \left( \int_{G_k(M)} \gamma \left( \frac{u}{s} \right) \langle \nabla^T h, (u \nabla u) \rangle_g dV \right) \\
+ \frac{(c(s) - 1)}{s} \frac{k}{s^k} \int_{G_k(M)} h \gamma \left( \frac{u}{s} \right) d\|V\| \\
- \frac{\delta V(X_{s,\varepsilon})}{s^{k+1}}.
\]

(3.6)

Remark 3.5. Notice that, by the Remark (3.2) the behavior of \( c(s) := c(s, b) \), we have that
\[
\frac{(c(s) - 1)}{s} = -O(s) = O(s), \text{ as } s \to 0^+
\]
furthermore, this is decreasing, and
\[
\frac{(c(s) - 1)}{s} \geq \frac{(c(r_0) - 1)}{r_0} := c_1^s
\]

Theorem 3.6 (Fundamental Weighted Monotonicity Inequality). Let \((M^n, g)\) be a complete Riemannian Manifold with Levi-Civita connection \(\nabla\), satisfying (BG) such that \(r_0 \cot(b(r_0)) > 0\), and let \(V \in \mathbf{V}_k(M^n)\) satisfying (AC), then for any \(0 < s < r_0\). There exists a constant \(c = c(s, b) \in \]0, 1\] such that, if we set \(u(x) = r_\xi(x) = \text{dist}_{(M, g)}(x, \xi)\) we have for all \(0 < s < r_0\)
\[
\frac{d}{ds} \left( \frac{1}{s^k} \int_{B_g(\xi,s)} h d\|V\| \right) \geq \frac{1}{s^{k+1}} \left( \int_{G_k(B_g(\xi,s))} \langle \nabla^T h, (u \nabla u) \rangle_g dV \right) \\
+ \frac{(c(s) - 1)}{s} \frac{k}{s^k} \int_{B_g(\xi,s)} h d\|V\| \\
+ \frac{1}{s^{k+1}} \int_{B_g(\xi,s)} \langle H, h(u \nabla u) \rangle_g d\|V\| \\
+ \frac{d}{ds} J(s),
\]

(3.7)

where
\[
J(s) := \int_{G_k(B_g(\xi,s))} h(y) \left| \nabla^{T_{\perp}} u \right|^2_g dV(y, T).
\]

Proof. Let \(\{\varepsilon_j\}_{j \in \mathbb{N}}\) a sequence, such that \(\varepsilon_j \uparrow 1\) when \(j \to \infty\) and \(\{\gamma_{\varepsilon_j}\} \subset C_0^1([-\infty, 1])\) a sequence of mollifiers, such that \(\gamma_{\varepsilon_j} \to \chi_{[-\infty, 1]}\) pointwise from below. Considering \(X_{s,\varepsilon_j}\) as in (3.1) by the previous reasoning (see (3.6)) we have for
By the Lebesgue Dominated Donvergence Theorem, we have
\[ \gamma \leq \frac{1}{s^{k+1}} \left( \int_{G_k(M)} \gamma_{\epsilon_j} \left( \frac{u}{s} \right) d\|V\| \right) \geq \int_{G_k(M)} \frac{h}{s^k} \frac{\partial}{\partial s} \left( \gamma_{\epsilon_j} \left( \frac{u}{s} \right) \right) \| \nabla^T u \|_g^2 dV \\
+ \frac{1}{s^{k+1}} \left( \int_{G_k(M)} \gamma_{\epsilon_j} \left( \frac{u}{s} \right) \left( \nabla h, (u\nabla u)_g \right) dV \right) \\
+ \frac{(c(s) - 1)}{s^k} \int_{G_k(M)} h\gamma_{\epsilon_j} \left( \frac{u}{s} \right) d\|V\|| \\
- \delta V(X_{s,\epsilon_j})_{s^{k+1}}. \]

Letting \( j \to \infty \), in virtue of Lebesgue dominated convergence theorem, we have for all \( 0 < s < r_0 \),
\[
\lim_{j \to \infty} \int_M h(y)\gamma_{\epsilon_j} \left( \frac{u(y)}{s} \right) d\|V\||(y) = \int_M h(y)\chi_{B_s(y)}(y)d\|V\||(x) \\
= \int_{B_s(y)} h(y)d\|V\||(y). \tag{3.8}
\]

On the other hand, since \( |\gamma'_{\epsilon_j} \left( \frac{u(x)}{s} \right) | < 0 \), for all \( j \in \mathbb{N} \) and for all \( x \in M \) such that \( \epsilon_j \leq u(x)/s \leq 1 \), and \( 0 < s < r_0 \). Then
\[
\frac{\partial}{\partial s} \left( \gamma_{\epsilon_j} \left( \frac{u(x)}{s} \right) \right) = \gamma'_{\epsilon_j} \left( \frac{u(x)}{s} \right) \left( \frac{-u(x)}{s^2} \right) > 0.
\]

Therefore, since \( h \geq 0 \)
\[
h(y)\frac{\epsilon^k_s}{s^k} \frac{\partial}{\partial s} \left( \gamma_{\epsilon_j} \left( \frac{u(y)}{s} \right) \right) \| \nabla^T u(y) \|_g^2 \leq \int_{G_k(M)} \frac{h(y)}{s^k} \frac{\partial}{\partial s} \left( \gamma_{\epsilon_j} \left( \frac{u(y)}{s} \right) \right) \| \nabla^T u(y) \|_g^2 \\
\leq \int_{G_k(M)} \frac{h(y)}{s^k} \frac{\partial}{\partial s} \left( \gamma_{\epsilon_j} \left( \frac{u(y)}{s} \right) \right) \| \nabla^T u(x) \|_g^2 dV,
\]

for all \( n \in \mathbb{N} \). Integrating over \( G_k(M) \),
\[
\int_{G_k(M)} h(y)\frac{\epsilon^k_s}{s^k} \frac{\partial}{\partial s} \left( \gamma_{\epsilon_j} \left( \frac{u(y)}{s} \right) \right) \| \nabla^T u \|_g^2 dV \leq \int_{G_k(M)} \frac{h(y)}{s^k} \frac{\partial}{\partial s} \left( \gamma_{\epsilon_j} \left( \frac{u}{s} \right) \right) \| \nabla^T u(x) \|_g^2 dV \\
\leq \int_{G_k(M)} \frac{h(y)}{s^k} \frac{\partial}{\partial s} \left( \gamma_{\epsilon_j} \left( \frac{u}{s} \right) \right) \| \nabla^T u \|_g^2 dV,
\]

and making \( j \to \infty \), we have in distributional sense
\[
\lim_{j \to \infty} \int_M h(y)\frac{1}{s^k} \frac{\partial}{\partial s} \left( \gamma_{\epsilon_j} \left( \frac{u(y)}{s} \right) \right) \| \nabla^T u \|_g^2 dV(y, T) = \frac{d}{ds} J(s). \tag{3.9}
\]

Finally, since \( V \) satisfies (AC), we have (see Remark 2.9)
\[
\delta V(X_{s,\epsilon_j}) = - \int_M h(y)\gamma_{\epsilon_j} \left( \frac{u(y)}{s} \right) \left( H_g(y), (u\nabla u)_g \right) d\|V\||(y),
\]
then, taking \( X_s(y) = h(y)\chi_{B_s}(y)(u\nabla u)(y) \),
\[
|\delta V(X_{s,\epsilon_j} - X_s)| \leq \int_M |h(y)| \left| \gamma_{\epsilon_j} \left( \frac{u(y)}{s} \right) - \chi_{s^{-1},1} \left( \frac{u(y)}{s} \right) \right| |H_g|_g d\|V\||(y).
\]

By the Lebesgue Dominated Donvergence Theorem, we have \( \gamma_{\epsilon_j} \to \chi_{s^{-1},1} \) in \( L^1 \) sense, then, letting \( 1 \leq p' < +\infty \) be the conjugate exponent to \( 1 < p \leq +\infty \), by
Hölder inequality we have that
\[ \| \gamma_{e_j} - \chi \|_{L^{p'}(\|V\|, B_g(\xi, s))} \to 0, \]
for all \( s \in ]0, r_0[ \) and \( 1 \leq p' \leq +\infty \). Therefore,
\[ \int_M \| H_g \| |h| \left( \gamma_{e_j} - \chi \right)_{-1,1} \left( \frac{u}{s} \right) d\|V\| \leq \|h\|_{L^\infty(\|V\|)} \| H_g \|_{L^{p'}} \| \gamma_{e_j} - \chi \|_{L^{p'}}. \]
Hence,
\[ \lim_{j \to \infty} \delta V(X_{s, e_j}) = \frac{1}{\sigma^k} \int_{B_g(\xi, \sigma)} \left\langle H, h \left( u \nabla u \right) \right\rangle g \, d\|V\|(y), \]
and then the result follows from (3.8), (3.9), and (3.10) \( \square \)

**Corollary 3.7 (Weighted Monotonicity Formula).** Under the hypothesis of the theorem above, if \( 0 < \sigma < \rho < r_0 \). Then
\[ \frac{1}{\rho^k} \int_{B_g(\xi, \rho)} hd\|V\| - \frac{1}{\sigma^k} \int_{B_g(\xi, \sigma)} hd\|V\| \geq - \int_{\sigma}^{\rho} \frac{1}{s^{k+1}} \left( \int_{B_g(\xi, s)} |\nabla T h| dV \right) \]
\[ - \int_{\sigma}^{\rho} \frac{1}{s^k} \left( \int_{B_g(\xi, s)} h(|H_g| - kc_*) d\|V\| \right) ds \]
\[ + \left( J(\rho) - J(\sigma) \right), \]
where \( J(s) \) is defined as in Theorem 3.6.

**Proof.** Integrating over \([\sigma, \rho] \subset [0, r_0]\), from Theorem 3.6 we have,
\[ \int_{\sigma}^{\rho} \left( \frac{1}{s^k} \int_{B_g(\xi, s)} \, d\|V\| \right) ds \geq \int_{\sigma}^{\rho} \left( \frac{1}{s^{k+1}} \left( \int_{G_b(\xi, s)} \left\langle \nabla T h, (u \nabla u) \right\rangle g \, dV \right) \right) ds \]
\[ + \int_{\sigma}^{\rho} \frac{(c(s) - 1) k}{s} \int_{B_g(\xi, s)} \, d\|V\| \right) ds \]
\[ + \int_{\sigma}^{\rho} \left( \frac{1}{s^{k+1}} \int_{B_g(\xi, s)} \left\langle H_g, h(u \nabla u) \right\rangle g \, d\|V\| \right) ds \]
\[ + \int_{\sigma}^{\rho} \frac{d}{ds} J(s) ds. \]
Applying the Fundamental Theorem of Calculus to the left-hand term, we have
\[ \int_{\sigma}^{\rho} \left( \frac{1}{s^k} \int_{B_g(\xi, s)} \, d\|V\| \right) ds = \frac{1}{\rho^k} \int_{B_g(\xi, \rho)} \, d\|V\| - \frac{1}{\sigma^k} \int_{B_g(\xi, \sigma)} \, d\|V\|. \]
To estimate the right-hand term, we precede in four steps. For the last term, again from the Fundamental Theorem of Calculus we have,
\[ \int_{\sigma}^{\rho} \frac{d}{ds} J(s) ds = J(\rho) - J(\sigma). \]
For the first term, we have
\begin{equation}
\int_{\sigma}^{\rho} \left( \frac{1}{s^{k+1}} \left( \int_{G_{k}(B_{g}(\xi,s))} \langle \nabla^{T} h, (u \nabla u) \rangle_{g} dV \right) \right) ds \geq - \int_{\sigma}^{\rho} \left( \int_{G_{k}(B_{g}(\xi,s))} |\nabla^{T} h|_{g} dV \right) ds
\end{equation}

Now, estimating the third and fourth term, (see Remark 3.5) we have

\begin{equation}
\int_{\sigma}^{\rho} \frac{1}{s^{k+1}} \left( \int_{B_{g}(\xi,s)} (H_{g}, h(u \nabla u))_{g} d\|V\| \right) ds + \int_{\sigma}^{\rho} \frac{e(s) - 1}{s^{k+1}} \left( \int_{B_{g}(\xi,s)} h d\|V\| \right) ds
\end{equation}

\begin{equation}
\geq - \int_{\sigma}^{\rho} \frac{1}{s^{k}} \left( \int_{B_{g}(\xi,s)} |H_{g}|_{g} h d\|V\| \right) ds + \int_{\sigma}^{\rho} \frac{kc^{*}_{1}}{s^{k}} \left( \int_{B_{g}(\xi,s)} h d\|V\| \right) ds
\end{equation}

\begin{equation}
\geq - \int_{\sigma}^{\rho} \frac{1}{s^{k}} \left( \int_{B_{g}(\xi,s)} h(H_{g}(y) - kc^{*}_{1}) d\|V\| \right) ds.
\end{equation}

Hence, the result follows from (3.11), (3.12), (3.13), and (3.14)

**Remark 3.8.** Let \( h : \text{spt} \|V\| \subset M \to \mathbb{R} \) such that, \( h \equiv 1 \), and \( J(s) \) as in Theorem (3.6) then, under the hypothesis of Theorem 3.6 as a particular case we have that there exists \( c \in [0,1] \) such that, for all \( 0 < s < r_{0} \)

\begin{equation}
\frac{d}{ds} \left( \frac{\|V\| (B_{g}(\xi,s))}{s^{k}} \right) \geq \frac{d}{ds} J(s) + \frac{(c(s) - 1) k}{s^{k}} \int_{B_{g}(\xi,s)} d\|V\| \right)
\end{equation}

Now, assume that \( V \) has locally bounded first variation in \( B_{g}(\xi,2\rho_{0}) \) for some \( \rho_{0} < r_{0} \), and let \( \Lambda \geq 0 \) such that

\( \|\delta V\|(B_{g}(\xi,\rho)) \leq \Lambda \|V\|(B_{g}(\xi,\rho)), \) for \( 0 < \rho < \rho_{0} \),

notice that the hypothesis above are much stronger than \( (AC) \), and those implies that for all \( X \in \mathfrak{X}^{c}(M) \) with \( \text{spt} X \subset B_{g}(\xi,s_{\rho_{0}}) \),

\( \delta V(X) = \int_{G_{k}(M)} \text{div}_{S}X(y)dV(y,T) = - \int_{M} (H_{g}(y), X(y))_{g} d\|V\|(y). \)

Then it is not so hard to see that (3.15) remains valid and \( |H_{g}(y)|_{g} \leq \Lambda \) for \( \|V\| \)-a.e. \( y \in B_{g}(\xi,\rho_{0}) \). Then, setting \( c^{*}_{1} \) as in the Remark 3.5,

\begin{equation}
\frac{d}{ds} \left( \frac{\|V\| (B_{g}(\xi,s))}{s^{k}} \right) \geq \frac{d}{ds} J(s) - \frac{1}{s^{k}} \int_{B_{g}(\xi,s)} \frac{|u \nabla u|_{g}}{s} |H_{g}|_{g} d\|V\| \right) \geq \frac{d}{ds} J(s) - \frac{1}{s^{k}} \int_{B_{g}(\xi,s)} (|H_{g}(y)|_{g} - kc^{*}_{1}) d\|V\|(y)
\end{equation}

\begin{equation}
\geq \frac{d}{ds} J(s) - (\Lambda - kc^{*}_{1}) \frac{\|V\| (B_{g}(\xi,s))}{s^{k}}.
\end{equation}

Hence,

\begin{equation}
\frac{d}{ds} \left( \frac{\|V\| (B_{g}(\xi,s))}{s^{k}} \right) + (\Lambda - kc^{*}_{1}) \frac{\|V\| (B_{g}(\xi,s))}{s^{k}} \geq \frac{d}{ds} J(s) > 0.
\end{equation}

Finally, multiplying by \( e^{(\Lambda-kc^{*}_{1})s} \) we have
\[ e^{(\Lambda - kc)^s} \frac{d}{ds} \left( \frac{\|V\| (B_g(\xi, s))}{s^k} \right) + (\Lambda - kc)^s \frac{\|V\| (B_g(\xi, s))}{s^k} \geq 0, \]

then

\[
\frac{d}{ds} \left( e^{(\Lambda - kc)^s} \frac{\|V\| (B_g(\xi, s))}{s^k} \right) \geq 0.
\]

Therefore, the function

\[
f : [0, \rho_0 [ \rightarrow [0, +\infty[ \quad s \mapsto e^{(\Lambda - kc)^s} \frac{\|V\| (B_g(\xi, s))}{s^k}
\]

in non-decreasing.

In the sequel we exploit the \( L^p \) condition over the generalized mean curvature \( H \) to get similar results, concerning to the monotonicity of this ratio.

### 3.2. \( L^p \) Monotonicity Formula

In this section we prove the monotonicity behaviour of the density ratio, as in Remark (3.8), but instead of the hypothesis of such we assume \( L^p \) boundedness for \( H \), i.e. under the hypothesis \((AC)\).

First to all, from now on we assume \((M, g)\) to be an \( n \)-dimensional complete Riemannian manifold satisfying \((BG)\), and let

\[
c^* = c^*(b, r_0) = \frac{c(b, r_0) - 1}{r_0},
\]

where \( c(b, s) \) is as in Remark (3.2).

Notice that, if \( V \in \mathcal{V}_k(M^n) \) satisfy \((AC)\), and considering the special case on the Remark (3.8) we have in distributional sense that,

\[
\frac{d}{ds} \left( \frac{\|V\| (B_g(\xi, s))}{s^k} \right) \geq \frac{d}{ds} J(s) + \frac{1}{s^{k+1}} \int_{B_g(\xi, s)} \left( kc^* s + \langle H_g, u \nabla u \rangle \right) d\|V\|,
\]

where \( J(s) \) is as in Theorem 3.6. The equation above is the Intrinsic Riemannian version of the fundamental monotonicity identity of [Sim83], and from now on we refer to this as fundamental monotonicity inequality, also notice that, since

\[
J(s) := \int_{G_k(B_g(\xi, s))} \frac{\|\nabla T_{-u} u\|^2}{r_\xi^k} dV \quad \text{and} \quad \|V\| (B_g(\xi, s)),
\]

are increasing in \( s \), the inequality (3.17) holds also in the classical sense. Furthermore, from the analysis of this inequality, naturally follows a monotonic behaviour of the density ratio.

**Theorem 3.9.** Given \( \xi \in M \), let \( 0 < \alpha \leq 1 \) and \( \Lambda \) be a positive constant. Let \( V \in \mathcal{V}_k(M) \) satisfying \((AC)\) and for all \( 0 < s < r_0 \)

\[
\frac{1}{\alpha} \int_{B_g(\xi, s)} |H_g|_g d\|V\|(x) \leq \Lambda \left( \frac{s}{r_0} \right)^{\alpha - 1} \|V\| (B_g(\xi, s)).
\]
Then the function
\[ f(s) := e^{\lambda(s)} \frac{\|V\|(B_g(\xi,s))}{s^k}, \]
is a non-decreasing function, where
\[ \lambda(s) := \left( \Lambda \left( \frac{s}{\tau_0} \right)^{\alpha-1} - c^*k \right)s, \]
and in fact, for \(0 < \sigma < \rho \leq r_0\),
\[ e^{\lambda(\rho)} \frac{\|V\|(B_g(\xi,\rho))}{\rho^k} - e^{\lambda(\sigma)} \frac{\|V\|(B_g(\xi,\sigma))}{\sigma^k} \geq \int J(\rho) - J(\sigma). \tag{3.18} \]

\textbf{Proof.} Notice that it is enough to prove (3.18) to guarantee that \(f\) is non-decreasing. To prove (3.18), we multiply the fundamental monotonicity inequality (3.17) by \(e^{\lambda(s)} \geq 1\), we have
\[
e^{\lambda(s)} \frac{d}{ds} \left( \frac{\|V\|(B_g(\xi,s))}{s^k} \right) \geq e^{\lambda(s)} \frac{d}{ds} J(s) + e^{\lambda(s)} \int_{B_g(\xi,s)} \left( kc^*s + \langle H_g, u \nabla u \rangle \right) d\|V\| \]
\[
\geq \frac{d}{ds} J(s) + e^{\lambda(s)} \int_{B_g(\xi,s)} \left( |H_g| \left| u \nabla u \right| - kc^*s \right) d\|V\| \]
\[
\geq \frac{d}{ds} J(s) - e^{\lambda(s)} \int_{B_g(\xi,s)} \left( |H_g| \left| u \nabla u \right| - kc^* \right) d\|V\|. \]

On the other hand, by hypothesis
\[- \int_{B_g(\xi,s)} \left( |H_g| - kc^* \right) d\|V\| \geq - \left( \alpha \Lambda \left( \frac{s}{\tau_0} \right)^{\alpha-1} - kc^* \right) \|V\|(B_g(\xi,s)), \]
therefore
\[
e^{\lambda(s)} \frac{d}{ds} \left( \frac{\|V\|(B_g(\xi,s))}{s^k} \right) + \left( \alpha \Lambda \left( \frac{s}{\tau_0} \right)^{\alpha-1} - kc^* \right) e^{\lambda(s)} \frac{\|V\|(B_g(\xi,s))}{s^k} \geq \frac{d}{ds} J(s). \]

Finally, since
\[ \frac{d}{ds} \lambda(s) = \alpha \Lambda \left( \frac{s}{\tau_0} \right)^{\alpha-1} - kc^*, \]
hence
\[ \frac{d}{ds} \left( e^{\lambda(s)} \frac{\|V\|(B_g(\xi,s))}{s^k} \right) \geq J(s) \geq 0, \]
and (3.18) follows readily by integrating over the interval \([\sigma, \rho]\).  

\textbf{Theorem 3.10.} Given \(\xi \in M\), let \(p > k \geq 1\) and \(\Gamma\) be a positive constant. Let \(V \in V_k(M)\) satisfying \((AC)\) such that
\[ \left( \int_{B_g(\xi,r_0)} |H_g|^p d\|V\| \right)^{1/p} \leq \Gamma. \]
Then there exists a positive constant \( c_0^* = c_0^*(p, k, b, r_0) \) such that

\[
(3.19) \quad \left( \frac{\|V\|(B_\rho(\xi, \sigma))}{\rho^k} \right)^{\frac{1}{p'}} - \left( \frac{\|V\|(B_\rho(\xi, \rho))}{\rho^k} \right)^{\frac{1}{p'}} \leq \frac{\Gamma + c_0^*}{p - k} \left( \rho^{1-k/p} - \sigma^{1-k/p} \right),
\]

for every \( 0 < \sigma < \rho \leq r_0 \).

Furthermore, the functions \( f : [0, R[ \rightarrow [0, +\infty) \) defined as

\[
f(s) := \left( \frac{\|V\|(B_s(\xi, s))}{s^k} \right)^{\frac{1}{p'}} + \frac{\Gamma + c_0^*}{p - k} s^{1-k/p},
\]

is monotone non-decreasing.

**Proof.** From the Hölder inequality in the fundamental monotonicity inequality (3.17) we have for all \( 0 < s < r_0 \) that

\[
\frac{d}{ds} \left( \frac{\|V\|(B_s(\xi, s))}{s^k} \right) \geq \frac{d}{ds} J(s) + \frac{1}{s^{k+1}} \int_{B_s(\xi, s)} \left( k c^* + \langle H_g, u \nabla u \rangle_g \right) d\|V\|
\]

\[
\geq \frac{d}{ds} J(s) - \frac{1}{s^k} \int_{B_s(\xi, s)} \left| H_g \right|_g d\|V\|
\]

\[
\geq \frac{d}{ds} J(s) - \frac{1}{s^k} \int_{B_s(\xi, s)} \left| H_g \right|_g d\|V\| + k c^* \frac{\|V\|(B_s(\xi, s))}{s^k}
\]

\[
\geq \frac{d}{ds} J(s) - \frac{1}{s^k} \left( \frac{\|V\|(B_s(\xi, s))}{s^k} \right)^{\frac{1}{p'}} \|H_g\|_{L^p(B_s(\xi, s))} + k c^* \frac{\|V\|(B_s(\xi, s))}{s^k},
\]

where \( p' \) is the conjugate exponent to \( p \). Now, by hypothesis we know that

\[
\|H_g\|_{L^p(\|V\|, B_s(\xi, s))} \leq \|H_g\|_{L^p(\|V\|, B_s(\xi, r_0))} \leq \Gamma,
\]

and, we also have that \( J(s) \geq 0 \) for all \( 0 < s < r_0 \), then

\[
\frac{d}{ds} \left( \frac{\|V\|(B_s(\xi, s))}{s^k} \right) \geq \left( \frac{\|V\|(B_s(\xi, s))}{s^k} \right)^{\frac{1}{p'}} \left( \Gamma - k c^* \left( \frac{\|V\|(B_s(\xi, s))}{s^k} \right)^{\frac{1}{p'}} \right)
\]

Now, by Remarks 3.5 and 3.2, \( c^* \leq 0 \), then for all \( 0 < s < r_0 \)

\[
\Gamma - k c^* \left( \frac{\|V\|(B_s(\xi, s))}{s^k} \right)^{\frac{1}{p'}} \leq \Gamma - k c^* \left( \frac{\|V\|(B_s(\xi, r_0))}{s^k} \right)^{\frac{1}{p'}}.
\]

Hence

\[
\frac{d}{ds} \left( \frac{\|V\|(B_s(\xi, s))}{s^k} \right) \geq -\left( \frac{\|V\|(B_s(\xi, s))}{s^k} \right)^{\frac{1}{p'}} \left( \Gamma + c_2^* \right),
\]

where \( c_2^* = c_2^*(k, p, b, r_0) \) is a positive constant such that

\[
-k c^* \left( \frac{\|V\|(B_s(\xi, r_0))}{s^k} \right)^{\frac{1}{p'}} \leq c_2^*,
\]

such constant always exists because \( \|V\| \) is a Radon measure.

On the other hand, since

\[
\frac{d}{ds} \left( \frac{\|V\|(B_s(\xi, s))}{s^k} \right)^{\frac{1}{p'}} = \frac{1}{p} \left( \frac{\|V\|(B_s(\xi, s))}{s^k} \right)^{\frac{1}{p'}} \frac{d}{ds} \left( \frac{\|V\|(B_s(\xi, s))}{s^k} \right),
\]

we have that

\[
\frac{d}{ds} \left( \frac{\|V\|(B_s(\xi, s))}{s^k} \right)^{\frac{1}{p'}} \geq \frac{\Gamma + c_2^*}{ps^{\frac{1}{p}}},
\]
and the result follows integrating the inequality above over \([\sigma, \rho] \subset [0, r_0]\).

**Corollary 3.11.** If \( H \in L^p_{loc}(M, TM, \mathcal{H}^k_0) \) for some \( p > k \), then the density

\[
\Theta^k(\|V\|, x) := \lim_{\rho \downarrow 0} \frac{\|V\|((B_{\rho}(x)))}{\omega_k \rho^k},
\]

does exists at every \( x \in B_{\rho}(\xi, r_0) \). Furthermore, \( \Theta^k(\|V\|, \cdot) \) is an upper-semi-continuous function in \( M^n \), i.e.

\[
\Theta^k(\|V\|, x) \geq \limsup_{y \to x} \Theta^k(\|V\|, x) \quad \forall x \in B_{\rho}(\xi, r_0).
\]

**Proof.** Let

\[
f(s) := \left( \frac{\|V\|((B_{\rho}(\xi, s)))}{s^k} \right)^{1/p} + \frac{\Gamma + c_1^*}{p - k} s^{1-1/p}.
\]

By the Theorem 3.10, for \( 0 < \sigma < \rho \leq r_0 \)

\[
f(\rho) - f(\sigma) = -\left( \left( \frac{\|V\|((B_{\rho}(\xi, \sigma)))}{\sigma^k} \right)^{1/p} - \left( \frac{\|V\|((B_{\rho}(\xi, \rho)))}{\rho^k} \right)^{1/p} \right) + \frac{\Gamma + c_1^*}{p - k} (\rho^{1-1/p} - \sigma^{1-1/p}) \geq 0.
\]

Then \( f \) is a non-decreasing function, therefore, the limit \( f(s) \to l \) when \( s \downarrow 0^+ \), exists with \( l \in [0, +\infty[. \) Hence the limit

\[
\lim_{s \downarrow 0} \frac{\|V\|((B_{\rho}(\xi, s)))}{s^k}
\]

also exists, furthermore, is equal to

\[
\lim_{s \downarrow 0} \frac{\|V\|((B_{\rho}(\xi, s)))}{s^k}.
\]

On the other hand, let \( \varepsilon > 0 \) fixed, such that \( 0 < \sigma < \rho, B_{\rho}(x, \rho + \varepsilon) \subset B_{\rho}(\xi, r_0) \) and \( d_{\rho}(x, y) < \varepsilon \), then, we also deduce from Theorem 3.10 and the monotonicity of \( \|V\| \) that

\[
\left( \frac{\|V\|((B_{\rho}(\xi, s)))}{\sigma^k} \right)^{1/p} \leq \left( \frac{\|V\|((B_{\rho}(\xi, \rho)))}{\rho^k} \right)^{1/p} + \frac{\Gamma + c_1^*}{p - k} \rho^{1-1/p}
\]

\[
\leq \left( \frac{\|V\|((B_{\rho}(\xi, \rho + \varepsilon)))}{\rho^k} \right)^{1/p} + \frac{\Gamma + c_1^*}{p - k} \rho^{1-1/p}.
\]

Letting \( \sigma \downarrow 0 \), we thus have

\[
(\Theta^k(\|V\|, y))^{1/p} \leq \left( \frac{\|V\|((B_{\rho}(\xi, \rho + \varepsilon)))}{\rho^k} \right)^{1/p} + \frac{\Gamma + c_1^*}{\omega_k (\rho + \varepsilon)^k/p} \rho^{1-1/p}.
\]

Now, let \( \delta > 0 \) be given and choose \( \varepsilon \ll \rho < \delta \) so that

\[
\left( \frac{\|V\|((B_{\rho}(\xi, \rho + \varepsilon)))}{\omega_k (\rho + \varepsilon)^k/p} \right) \leq (\Theta^k(\|V\|, x))^{1/p} + \delta.
\]

The inequality above gives

\[
(\Theta^k(\|V\|, y))^{1/p} \leq (\Theta^k(\|V\|, x))^{1/p} + \frac{\Gamma + c_1^*}{\omega_k (\rho + \varepsilon)^k/p} \rho^{1-1/p},
\]
provided \( d_g(y, x) < \varepsilon \) and the desired upper-semi-continuity follows straightforward.

\[ \square \]

**Proposition 3.1.** Let \( V \in \mathcal{V}_k(M) \), \( \xi \in M \) fixed, and assume that
\[
\| V \|(B_\gamma(\xi, \sigma)) \leq \beta \sigma^n \quad \text{for } 0 < \sigma < r_0,
\]
then
\[
\int_{B_\gamma(\xi, \rho)} d_g(x, \xi)^{\alpha-k} \| V \|(x) \leq \frac{k \beta \rho^\alpha}{\alpha},
\]
for any \( \rho \in ]0, r_0[ \) and \( 0 < \alpha < k \).

The proof of this is based in the following technical lemma.

**Lemma 3.12.** Let \((X, \mu)\) a measure space, \( \gamma > 0 \), \( f \in L^1(\mu) \), \( f \geq 0 \), then
\[
\int_0^\infty t^{\gamma-1} \mu(E_t)dt = \frac{1}{\gamma} \int E_0 f^\gamma(x)d\mu(x),
\]
where \( E_t := \{x : f(x) > t\} \). More generally
\[
\int_{t_0}^\infty t^{\gamma-1} \mu(E_t)dt = \frac{1}{\gamma} \int_{E_{t_0}} (f^\gamma(x) - t_0^\gamma)d\mu(x),
\]
for each \( t_0 \geq 0 \).

This is a classical Lemma from measure theory, and its proof can be found in [Mat95] pg. 15.

**Proof of Proposition 3.1.** Let in the previous Lemma, \( f(x) = |x - \xi|^{-\gamma} \), \( \gamma = k - \alpha \) and \( t_0 = 1/\rho \), then
\[ (3.20) \]
\[
\int_{E_{1/\rho}} (|x - \xi_g|^{\alpha-k} - \rho^{\alpha-k}) d\| V \|(x) = (\alpha - k) \int_{1/\rho}^\infty t^{(k-\alpha)-1} \| V \|(E_t)d\| V \|(x).
\]
Notice that
\[
E_t := \{x : f(x) > t\} = \left\{ x : \frac{1}{|x - \xi_g|} > t \right\} = \{x : |x - \xi_g| < 1/t\} := B_\gamma(\xi, 1/t).
\]
On the other hand, by hypothesis, we have
\[
\int_{1/\rho}^\infty t^{(k-\alpha)-1} \| V \|(B_\gamma(\xi, 1/t))d\mu(x) \leq \int_{1/\rho}^\infty \frac{t^{(k-\alpha)-1} \beta}{t^k} d\| V \||(x) = \frac{\beta \rho^\alpha}{\alpha},
\]
for any \( 0 < 1/t < r_0 \). Putting together the information above in (3.20), we have
\[
\int_{B_\gamma(\xi, \rho)} (|x - \xi_g|^{\alpha-k} - \rho^{\alpha-k}) d\| V \| = \int_{B_\gamma(\xi, \rho)} (|x - \xi_g|^{\alpha-k}) d\| V \| - \int_{B_\gamma(\xi, \rho)} \rho^{\alpha-k} d\| V \|
\]
\[
= \int_{B_\gamma(\xi, \rho)} (|x - \xi_g|^{\alpha-k}) d\| V \| - \rho^{\alpha-k} \| V \|(B_\gamma(\xi, \rho))
\]
\[
\leq \frac{(k-\alpha) \beta \rho^\alpha}{\alpha},
\]
then,
\[
\int_{B_g(\xi, \rho)} \left( |x - \xi|^{\alpha - k} \right) d\|V\| \leq \frac{(k - \alpha)\beta \rho^\alpha}{\alpha} + \rho^{\alpha - k} \|V\|(B_g(\xi, \rho)) \\
\leq \frac{(k - \alpha)\beta \rho^\alpha}{\alpha} + \beta \rho^k \\
= \frac{k \beta \rho^\alpha}{\alpha},
\]
provided \( \rho \in ]0, r_0[ \).

\[\square\]

4. Poincaré and Sobolev Type Inequalities for Intrinsic Varifolds

In this final section, a Poincaré and Sobolev-type inequalities are proved for non-negative functions defined on \( S \subset M^n, \mathcal{H}^k_{g}\)-rectifiable sets as a consequence of the Fundamental Weighted Monotonicity Inequality 3.6 in the particular case in which \( V := v(S, \theta) \), with \( \theta \in L^1_{loc}(\mathcal{H}^k_{g} \cup S) \) and mean curvature vector \( H_g \in L^1_{loc}(\|V\|) \).

Therefore along this section we are under the assumptions of Theorem 3.6, i.e.

\((BG)\): Let \((M^n, g)\) be a complete Riemannian Manifold with Levi-Civita connection \( \nabla \). We say that \((M^n, g)\) satisfies \((BG)\), if \( \text{Sec}_g \leq b \) for some constant \( b \in \mathbb{R} \), in case \( b > 0 \) we assume furthermore \( br_0 < \pi \).

\((AC)\): We say that \( V \in \mathcal{V}_k(M) \) satisfies the \((AC)\) condition, whenever

\[
\delta V(X) \leq C(V) \left( \int_{B_g(\xi, r_0)} |X|^2 \frac{d\|V\|}{\|V\|} \right)^{\frac{p-1}{p}}, \forall X \in X^1_c(B_g(\xi, r_0)),
\]

for some \( p > k \).

Furthermore, we need to explain some notation: in what follows \( \xi \in M \) is a fixed point and \( r_0 < \text{inj}_\xi(M, g) \) such that \( r_0 \cot_\xi(r_0) > 0 \), according to Definition 3.4.

4.1. Poincaré type inequality. The Poincaré type inequality which we proof is a consequence of the following lemma, which is a direct application of the formula (3.6).

**Lemma 4.1.** Assume \((M^n, g)\) satisfying \((BG)\) and \( V \in \mathcal{V}_k(M^n) \) satisfying \((AC)\). Let \( \Lambda > 0 \) such that \( |H|g \leq \Lambda \), and \( h \in C^1(M) \), then for \( \|V\|\text{-}a.e. \ \xi \in \text{spt} \|V\| \), and for all \( 0 < \rho < r_0 \)

\[
(4.1) \quad h(\xi) \leq e^{(\Lambda + c^*)\rho} \left( \frac{1}{\omega_k} \rho^k \int_{B_g(\xi, \rho)} h d\|V\| + \int_{B_g(\xi, \rho)} \frac{|\nabla^g_{\xi} h|^2}{\rho^{\lambda - 1}} d\|V\| \right),
\]

where \( c^* \) is as in Remark 3.5.

**Proof.** By Hypothesis \((BG)\) and \((AC)\) we are in the conditions of Theorem 3.6, then, for \( \xi \in \text{spt} \|V\| \) and for all \( 0 < s < r_0 \) we have
Then

\[
\frac{d}{ds}\left(\frac{1}{s^k} \int_{B_g(\xi,s)} h\gamma_{\varepsilon} \left(\frac{u}{s}\right) d\|V\|\right) \geq \int_{B_g(\xi,s)} \frac{h}{s^k} \frac{\partial}{\partial s} \left(\gamma_{\varepsilon} \left(\frac{u}{s}\right)\right) \left|\nabla_g^s u\right|^2 d\|V\| \\
+ \frac{1}{s^k} \int_{B_g(\xi,s)} \gamma_{\varepsilon} \left(\frac{u}{s}\right) \left\langle \nabla_g^s h, u\nabla_g u\right\rangle_g d\|V\| \\
+ \frac{k \varepsilon^*}{s^k} \int_{B_g(\xi,s)} h\gamma_{\varepsilon} \left(\frac{u}{s}\right) d\|V\| \\
+ \frac{1}{s^{k+1}} \int_{B_g(\xi,s)} \gamma_{\varepsilon} \left(\frac{u}{s}\right) \left(h H_g, u\nabla_g u\right)_g d\|V\|,
\]

where \(\gamma_{\varepsilon} \in C^1(\mathbb{R})\) is, such that, given \(0 < \varepsilon < 1\),

\[
\gamma_{\varepsilon}(s) := \begin{cases} 
1, & \text{if } s < \varepsilon \\
0, & \text{if } s > 1,
\end{cases}
\]

and

\[
\gamma_{\varepsilon}'(s) < 0 \quad \text{if } \varepsilon < s < 1.
\]

Then

\[
\frac{d}{ds}\left(\frac{1}{s^k} \int_{B_g(\xi,s)} h\gamma_{\varepsilon} \left(\frac{u}{s}\right) d\|V\|\right) \geq \frac{1}{s^{k+1}} \int_{B_g(\xi,s)} \gamma_{\varepsilon} \left(\frac{u}{s}\right) \left\langle H_g h + \nabla_g^s h, u\nabla_g u\right\rangle_g d\|V\| \\
+ \frac{\varepsilon^* k}{s^k} \int_{B_g(\xi,s)} h\gamma_{\varepsilon} \left(\frac{u}{s}\right) d\|V\| \\
\geq -\frac{1}{s^{k+1}} \gamma_{\varepsilon} \left(\frac{u}{s}\right) \left|h H_g + \nabla_g^s h\right| u d\|V\| \\
+ \frac{\varepsilon^* k}{s^k} \int_{B_g(\xi,s)} h\gamma_{\varepsilon} \left(\frac{u}{s}\right) d\|V\|,
\]

therefore

\[
\frac{d}{ds}\left(\frac{1}{s^k} \int_{B_g(\xi,s)} \gamma_{\varepsilon} \left(\frac{u}{s}\right) h d\|V\|\right) \geq -\frac{1}{s^{k+1}} \int_{B_g(\xi,s)} \gamma_{\varepsilon} \left(\frac{u}{s}\right) u \left|\nabla_g^s h\right|_g d\|V\| \\
- \frac{1}{s^k} \int_{B_g(\xi,s)} \gamma_{\varepsilon} \left(\frac{u}{s}\right) h \left(H_g \left| - \varepsilon^* k\right\rangle d\|V\|.
\]

Since, by hypothesis we have that \(|H_g|_g \leq \Lambda\), we have

\[
\frac{d}{ds}\left(\frac{1}{s^k} \int_{B_g(\xi,s)} \gamma_{\varepsilon} \left(\frac{u}{s}\right) h d\|V\|\right) \geq -\frac{1}{s^{k+1}} \int_{B_g(\xi,s)} \gamma_{\varepsilon} \left(\frac{u}{s}\right) u \left|\nabla_g^s h\right|_g d\|V\| \\
- \frac{\Lambda + \varepsilon^* k}{s^k} \int_{B_g(\xi,s)} \gamma_{\varepsilon} \left(\frac{u}{s}\right) h d\|V\|.
\]

Now, let \(\{\varepsilon_j\}_{j \in \mathbb{N}}\) a sequence of real numbers such that \(\varepsilon \uparrow 1\) as \(j \to \infty\), and \(\{\gamma_{\varepsilon_j}\}_{j \in \mathbb{N}} \subset C^1(\mathbb{R})\) such that

\[
\gamma_{\varepsilon_j}(s) := \begin{cases} 
1, & \text{if } s < \varepsilon_j \\
0, & \text{if } s > 1,
\end{cases}
\]
and

\[ \gamma_j'(s) < 0, \quad \text{if} \quad \varepsilon_j < s < 1, \]

and \( \varepsilon_j \to \chi_{-\infty, 1}\) pointwise from below. Then, reasoning as above we have (as in the proof of Theorem 3.6) that for all \( j \in \mathbb{N} \), and for all \( 0 < s < r_0 \)

\[
\frac{d}{ds} \left( \frac{1}{s^k} \int_{B_g(\xi, s)} \gamma_{\varepsilon_j} \left( \frac{u}{s} \right) h d\|V\| \right) \geq -\frac{1}{s^{k+1}} \int_{B_g(\xi, s)} \gamma_{\varepsilon_j} \left( \frac{u}{s} \right) u \left| \nabla_g h \right|_g d\|V\| \]

\[- \frac{\Lambda + c^* k}{s^k} \int_{B_g(\xi, s)} \gamma_{\varepsilon_j} \left( \frac{u}{s} \right) h d\|V\|.\]

Multiply the above inequality by \( e^{\tilde{\lambda}(s)} \) and letting \( j \to \infty \) we have in distributional sense in \( s \),

\[
e^{\tilde{\lambda}(s)} \frac{d}{ds} \left( \frac{1}{s^k} \int_{B_g(\xi, s)} h d\|V\| \right) + e^{\tilde{\lambda}(s)} \frac{\Lambda + c^* k}{s^k} \int_{B_g(\xi, s)} h d\|V\| \]

\[
\geq -e^{\tilde{\lambda}(s)} \frac{1}{s^{k+1}} \int_{B_g(\xi, s)} u \left| \nabla_g h \right|_g d\|V\|,\]

where

\[ \tilde{\lambda}(s) := (\Lambda + c^* k) s, \]

therefore, since \( \tilde{\lambda}'(s) = \Lambda + c^* k \), we have

\[
\frac{d}{ds} \left( \frac{e^{\tilde{\lambda}(s)} \int_{B_g(\xi, s)} h d\|V\|}{s^k} \right) \geq -\frac{e^{\tilde{\lambda}(s)} \left( \Lambda + c^* k \right)}{s^{k+1}} \int_{B_g(\xi, s)} u \left| \nabla_g h \right|_g d\|V\|.
\]

Hence, integrating over \( [\sigma, \rho] \subset ]0, r_0[, \)

\[
e^{\tilde{\lambda}(\rho)} \frac{e^{\tilde{\lambda}(\rho)} \int_{B_g(\xi, \rho)} h d\|V\|}{\rho^k} \geq -\int_{\sigma}^{\rho} \frac{e^{\tilde{\lambda}(s)} \left( \Lambda + c^* k \right)}{s^{k+1}} \int_{B_g(\xi, s)} u \left| \nabla_g h \right|_g d\|V\| ds
\]

\[
\geq -e^{\tilde{\lambda}(\rho)} \int_{\sigma}^{\rho} \frac{1}{s^{k+1}} \left( \int_{B_g(\xi, s)} u \left| \nabla_g h \right|_g d\|V\| \right) ds,
\]

then, rearranging terms,

\[
\frac{1}{\omega_k \sigma^k} \int_{B_g(\xi, \sigma)} h d\|V\| \leq e^{\tilde{\lambda}(\rho)} \left( \frac{1}{\omega_k \rho^k} \int_{B_g(\xi, \rho)} h d\|V\| + \frac{1}{\omega_k} \int_{\sigma}^{\rho} \frac{1}{s^{k+1}} \left( \int_{B_g(\xi, s)} u \left| \nabla_g h \right|_g d\|V\| \right) ds \right).
\]

Now, letting \( \tau = 1/s \) in the inequality above, we have that

\[
\int_{\sigma}^{\rho} \left( \frac{1}{s} \right)^{k+1} \left( \int_{B_g(\xi, s)} u \left| \nabla_g h \right|_g d\|V\| \right) ds = -\int_{\frac{1}{\rho}}^{-\frac{1}{\sigma}} \tau^{k-1} \int_{B_g(\xi, \tau)} u \left| \nabla_g h \right|_g d\|V\|,
\]

then, if we let for \( A \subset M \)

\[ \nu(A) := \int_A u \left| \nabla_g h \right|_g d\|V\|, \]
since \( u \) is an increasing function, \( \nu \) defines a measure on \((M, g)\), so by Lemma 3.12

\[
\int_0^\rho \frac{1}{s^{k+1}} \left( \int_{B_s(\xi, \rho)} u \| \nabla_g^S h \|_g d\|V\| \right) ds = -\int_1^{\frac{1}{\rho}} \tau^{-k-1} \nu \left( B_g \left( \xi, \frac{1}{\tau} \right) \right) d\tau
\]

\[
= \int_1^{\frac{1}{\rho}} \tau^{-k-1} \nu \left( \left\{ x \in M : \frac{1}{u(x)} > \tau \right\} \right) d\tau
\]

\[
= \frac{1}{k} \int \left\{ \frac{1}{\rho^k} < \frac{1}{u(x)} < \frac{1}{\rho} \right\} \left( \frac{1}{\sigma^k} - \frac{1}{\rho^k} \right) d\nu
\]

\[
= \int_{B_g(\xi, \rho) \setminus B_g(\xi, \sigma)} \frac{1}{u} - \frac{1}{\rho^k} d\nu
\]

\[
\leq \frac{1}{k} \int_{B_g(\xi, \rho)} \frac{1}{u} d\nu
\]

\[
= \frac{1}{k} \int_{B_g(\xi, \rho)} \frac{u |\nabla_g^S h|_g d\|V\|}{u^{k-1}}
\]

\[
= \frac{1}{k} \int_{B_g(\xi, \rho)} |\nabla_g^S h|_g d\|V\|.
\]

Then

\[
\frac{1}{\omega_k \sigma^k} \int_{B_g(\xi, \sigma)} h d\|V\| \leq e^{\lambda(\rho)} \left( \frac{1}{\omega_k \rho^k} \int_{B_g(\xi, \rho)} h d\|V\| + \frac{1}{k \omega_k} \int_{B_g(\xi, \rho)} |\nabla_g^S h|_g d\|V\| \right).
\]

Finally, let \( \sigma \downarrow 0 \), and the result follows, since, \( \xi \in \text{spt} \|V\| \) implies \( \Theta^k(\|V\|, \xi) = 1 \) for \( \|V\| \)-a.e. \( \xi \), therefore \( \xi \) is a Lebesgue point \( \|V\| \)-a.e. \( \square \)

**Theorem 4.2** (Poincaré-Type Inequality for Intrinsic Varifolds). Let \((M^n, g)\) be a complete Riemannian manifold satisfying \((BG)\), let \( V := \nu(S, \theta) \) a rectifiable varifold satisfying \((AC)\). Suppose: \( h \in C^1(M), \ h \geq 0, \ B_g(\xi, 2\rho) \subset B_g(\xi, r_0) \) for \( \xi \in S \) fixed, \( |H|_g \leq \Lambda \) for some \( \Lambda > 0, \theta > 1 \) \( \|V\| \)-a.e. in \( B_g(\xi, r_0) \) and for some \( 0 < \alpha < 1 \)

\[
\|V\| \left( \{ x \in B_g(\xi, \rho) : h(x) > 0 \} \right) \leq \omega_k (1 - \alpha) \rho^k \quad \text{and,}
\]

\[
e^{(\Lambda + k\epsilon^*)\rho} \leq 1 + \alpha.
\]

Suppose also that, for some constant \( \Gamma > 0 \)

\[
(4.5)
\]

\[
(4.6)
\]

Then there are constants \( \beta := \beta(k, \alpha, r_0, b) \in [0, \frac{1}{2}] \) and \( C := C(k, \alpha, r_0, b) > 0 \) such that

\[
\int_{B_g(\xi, 2\rho)} h d\|V\| \leq C \rho \int_{B_g(\xi, \rho)} |\nabla_g^S h|_g d\|V\|.
\]

**Proof.** Take \( \beta \in [0, 1/2] \) an arbitrary parameter, to be specified later and \( \lambda(s) \) as in (4.3). Applying the previous Lemma with \( \eta \in B_g(\xi, \beta \rho) \cap \text{spt} \|V\|, \)
in place of $\xi$, hence, for all $0 < \rho < r_0$

$$h(\rho) \leq e^{\hat{\lambda}(1-\beta)\rho} \left( \frac{1}{\omega_k(1-\beta)^k} \int_{B_\rho(\eta, (1-\beta)\rho)} V \|dV\| + \frac{1}{k\omega_k} \int_{B_\rho(\eta, (1-\beta)\rho)} \|\nabla_\omega h\|_g \|dV\| \right).$$

Since $\rho/2 < (1-\beta)\rho < \rho$, we have $B_\rho(\eta, (1-\beta)\rho) \subset B_\rho(\xi, \rho)$ and

$$h(\rho) \leq e^{\hat{\lambda}(\rho)} \left( \frac{1}{\omega_k(1-\beta)^k} \int_{B_\rho(\xi, \rho)} V \|dV\| + \frac{1}{k\omega_k} \int_{B_\rho(\xi, \rho)} \|\nabla_\omega h\|_g \|dV\| \right). \tag{4.7}$$

Now, let $t_0 \geq 0$ fixed, and $\gamma \in C^1(\mathbb{R})$ a fixed non-decreasing function with

$$\gamma(t) = 0, \text{ if } t \leq 0 \quad \text{and} \quad \gamma(t) \leq 1 \text{ if } t > 0,$$

and apply (4.7) with $f(x) := \gamma(h(x) - t_0)$ in place of $h$, then

$$f(\rho) \leq e^{\hat{\lambda}(\rho)} \left( \frac{1}{\omega_k(1-\beta)^k} \int_{B_\rho(\xi, \rho)} f \|dV\| + \frac{1}{k\omega_k} \int_{B_\rho(\xi, \rho)} \|\nabla_\omega h\|_g \|dV\| \right).$$

By hypothesis (4.5) we know that:

$$e^{\hat{\lambda}(\rho)} \leq 1 + \alpha$$

and that

$$\|V\| \{(x \in B_\rho(\xi, \rho) : h(x) > 0)\} \leq \omega_k(1-\alpha)^k,$$

hence,

$$f(\rho) \leq (1 + \alpha) \left( \frac{1}{\omega_k(1-\beta)^k} \int_{B_\rho(\xi, \rho)} f \|dV\| + \frac{1}{k\omega_k} \int_{B_\rho(\xi, \rho)} \gamma' \left( h(x) - t_0 \right) \|\nabla_\omega h\|_g \|dV\| \right)$$

$$\leq (1 + \alpha) \left( \|V\| \{(h > 0) \cap B_\rho(\xi, \rho)\} \right. + \frac{1}{k\omega_k} \int_{B_\rho(\xi, \rho)} \gamma' \left( h(x) - t_0 \right) \|\nabla_\omega h\|_g \|dV\| \right)$$

$$\leq \frac{1 - \alpha^2}{(1-\beta)^k} + \frac{1 + \alpha}{k\omega_k} \int_{B_\rho(\xi, \rho)} \gamma' \left( h(x) - t_0 \right) \|\nabla_\omega h\|_g \|dV\|.$$
To calculate the left term, let \( \|V\| (B_g(\xi, \beta \rho) \cap E_{t_0 + \varepsilon}) \leq \frac{2(1 + \alpha)}{\alpha^2 \omega_k} \int_{B_g(\xi, \beta \rho) \cap E_{t_0 + \varepsilon}} \left( \int_{B_g(\xi, \beta \rho)} \frac{\gamma'(h(x) - t_0)}{r_{\eta}^{k-1}} \right) d\|V\|(x) \) d\|V\|(\eta)

\[
\leq \frac{2(1 + \alpha)}{\alpha^2 \omega_k} \int_{B_g(\xi, \beta \rho)} \gamma'(h(x) - t_0) \left| \nabla^S g h(x) \right|_g \left( \int_{B_g(\xi, \beta \rho) \cap E_{t_0 + \varepsilon}} \frac{1}{r_{\eta}^{k-1}} d\|V\|(\eta) \right) d\|V\|(x).
\]

Finally, letting \( \varepsilon \downarrow 0 \), the result follows.
4.2. Sobolev type Inequality.

Lemma 4.3. Assume \((M^n, g)\) satisfying \((BG)\) and \(V := v(S, \theta)\), with \(S \subset M\) \(H^{\delta}_{\theta}\)-rectifiable, and \(\theta \in L^1_{\text{loc}}(|V|)\), such that \((AC)\) is satisfied. Let \(h \in C^1(M)\) non-negative, then for all \(\xi \in \text{spt} \|V\|\) and for all \(0 < \rho < r_0\)

\[
\frac{1}{\omega_{k+1}} \int_{B_k(\xi, \sigma)} h|d\|V\| \leq \frac{1}{\omega_k} \int_{B_1(\xi, \rho)} h|d\||V| + \int_{\rho}^{r_0} \frac{1}{s} \left( \int_{B_s(\xi, s)} \left| \nabla_{\vartheta}^0 h \right|_g + h \left( |H_g|_g - c^*k \right) \right) ds
\]

where \(c^*\) is as in Remark 3.5.

Proof. By Hypothesis \((BG)\) and \((AC)\) we are in the conditions of Theorem 3.6, then, for \(\xi \in \text{spt} \|V\|\) and for all \(0 < s < r_0\)

\[
\frac{d}{ds} \left( \frac{1}{s} \int_{B_s(\xi, s)} h\gamma_{\epsilon} \left( \frac{u}{s} \right) d||V|| \right) \geq \int_{B_1(\xi, s)} \frac{h}{s} \frac{\partial}{\partial s} \left( \gamma_{\epsilon} \left( \frac{u}{s} \right) \right) \left| \nabla_{\vartheta}^0 u \right|^2 \|V\|
\]

\[
+ \frac{1}{s} \int_{B_1(\xi, s)} \gamma_{\epsilon} \left( \frac{u}{s} \right) \left( \nabla_{\vartheta}^0 h, u \nabla_{\vartheta}^0 u \right)_g d\|V\|
\]

\[
+ \frac{k c^*}{s} \int_{B_1(\xi, s)} h\gamma_{\epsilon} \left( \frac{u}{s} \right) d\|V\|
\]

\[
+ \frac{1}{s^{k+1}} \int_{B_1(\xi, s)} \gamma_{\epsilon} \left( \frac{u}{s} \right) \left( H_g h, u \nabla_{\vartheta}^0 u \right)_g d\|V\|
\]

where \(\gamma_{\epsilon} \in C^1(\mathbb{R})\) is, such that, given \(0 < \epsilon < 1\),

\[
\gamma_{\epsilon}(s) := \begin{cases} 
1, & \text{if } s < \epsilon \\
0, & \text{if } s > 1,
\end{cases}
\]

and

\[
\gamma'_{\epsilon}(s) < 0 \text{ if } \epsilon < s < 1.
\]

Then

\[
\frac{d}{ds} \left( \frac{1}{s^{k+1}} \int_{B_1(\xi, s)} h\gamma_{\epsilon} \left( \frac{u}{s} \right) d\|V\| \right) \geq \int_{B_1(\xi, s)} \gamma_{\epsilon} \left( \frac{u}{s} \right) \left( H_g h + \nabla_{\vartheta}^0 h, u \nabla_{\vartheta}^0 u \right)_g \|V\|
\]

\[
+ \frac{c^* k}{s} \int_{B_1(\xi, s)} h\gamma_{\epsilon} \left( \frac{u}{s} \right) d\|V\|
\]

\[
\geq - \frac{1}{s^{k+1}} \int_{B_1(\xi, s)} \gamma_{\epsilon} \left( \frac{u}{s} \right) \left| H_g h + \nabla_{\vartheta}^0 h \right|_g^2 d\|V\|
\]

\[
+ \frac{c^* k}{s} \int_{B_1(\xi, s)} h\gamma_{\epsilon} \left( \frac{u}{s} \right) d\|V\|,
\]

therefore

\[
\frac{d}{ds} \left( \frac{1}{s} \int_{B_1(\xi, s)} \gamma_{\epsilon} \left( \frac{u}{s} \right) h|d\||V\| \right) \geq - \frac{1}{s} \int_{B_1(\xi, s)} \gamma_{\epsilon} \left( \frac{u}{s} \right) \left( \left| \nabla_{\vartheta}^0 h \right|_g + h \left( |H_g|_g - c^*k \right) \right) d\|V\|.
\]
Now, let \( \{\varepsilon_j\}_{j \in \mathbb{N}} \) a sequence of real numbers such that \( \varepsilon \uparrow 1 \) as \( j \to \infty \), and \( \{\gamma_{\varepsilon_j}\}_{j \in \mathbb{N}} \subset C^1(\mathbb{R}) \) such that
\[
\gamma_{\varepsilon_j}(s) := \begin{cases} 1, & \text{if } s < \varepsilon_j \\ 0, & \text{if } s > 1, \end{cases}
\]
and
\[
\gamma'_{\varepsilon_j}(s) < 0 \quad \text{if } \varepsilon_j < s < 1,
\]
and \( \varepsilon_j \to \chi_{]-\infty,1]} \) pointwise from below. Then, reasoning as in the proof of Theorem (3.6), for all \( j \in \mathbb{N} \), and for all \( 0 < s < r \)
\[
\frac{d}{ds} \left( \frac{1}{s^k} \int_{B_s(\xi,s)} \gamma_{\varepsilon_j} \left( \frac{u}{\lambda_s} \right) h \|V\| \right) \geq -\frac{1}{s^k} \int_{B_s(\xi,s)} \gamma_{\varepsilon_j} \left( \frac{u}{\lambda_s} \right) \left( |\nabla g h|_g + h \left( |H_g|_g - c^* k \right) \right) d\|V\|.
\]
Letting \( j \to \infty \) we have in distributional sense in \( s \),
\[
\frac{d}{ds} \left( \frac{1}{\lambda_s^k} \int_{B_{\lambda_s}(\xi,s)} \gamma_{\varepsilon_j} \left( \frac{u}{\lambda_s} \right) h \|V\| \right) \geq -\frac{1}{\lambda_s^k} \int_{B_{\lambda_s}(\xi,s)} \left( |\nabla g h|_g + h \left( |H_g|_g - c^* k \right) \right) d\|V\|.
\]
Finally, integrating over \( [\sigma,\rho[ \subset ]0,\rho_0[ \) we have
\[
\int_{\sigma}^{\rho} \frac{d}{ds} \left( \frac{1}{s^k} \int_{B_s(\xi,s)} h \|V\| \right) = \frac{1}{\rho^k} \int_{B_{\rho}(\xi,\rho)} h \|V\| - \frac{1}{\sigma^k} \int_{B_{\sigma}(\xi,\sigma)} h \|V\|
\geq -\int_{\sigma}^{\rho} \left( \frac{1}{s^k} \int_{B_s(\xi,s)} \left( |\nabla g h|_g + h \left( |H_g|_g - c^* k \right) \right) d\|V\| \right) ds.
\]
Therefore,
\[
\frac{1}{\omega_k \sigma^k} \int_{B_{\sigma}(\xi,\sigma)} h \|V\| \leq \frac{1}{\omega_k \rho^k} \int_{B_{\rho}(\xi,\rho)} h \|V\| + \int_{\sigma}^{\rho} \frac{1}{s^k} \left( \int_{B_s(\xi,s)} \left|\nabla g h\right|_g + h \left( |H_g|_g - c^* k \right) \right) d\|V\| ds.
\]

\[\square\]

**Lemma 4.4.** Suppose \( f \) and \( g \) are bounded non-decreasing functions on \( ]0,\infty[ \), and
\[
1 \leq \frac{1}{\sigma^k} f(\sigma) \leq \frac{1}{\rho^k} f(\rho) + \int_{0}^{\rho} \frac{1}{s^k} g(s) ds,
\]
where \( 0 < \sigma < \rho < \infty \). Then, there exists \( \rho \in ]0,\rho_0[ \) such that
\[
f(\rho) \leq \frac{1}{2} 5^k \rho_0 g(\rho),
\]
where
\[
\rho_0 = 2 \left( \lim_{\rho \to \infty} f(\rho) \right)^{\frac{1}{k}}.
\]

**Proof.** We argue by contradiction. Assume that for every \( \rho \in ]0,\rho_0[ \)
\[
f(5\rho) > \frac{1}{2} 5^k \rho_0 g(\rho),
\]
then, by (4.11),

(4.12)

\[
1 \leq \sup_{0 < \sigma < \rho_0} \frac{1}{\sigma^k} f(\sigma) \leq \frac{1}{\rho_0^k} f(\rho_0) + \int_{0}^{\rho_0} \frac{1}{s^k} g(s) ds
\]

\[
< \frac{1}{\rho_0^k} f(\rho_0) + \frac{2}{5^k \rho_0} \int_{0}^{\rho_0} \frac{1}{s^k} f(5s) ds
\]

\[
= \frac{1}{\rho_0^k} f(\rho_0) + \frac{2}{5^k \rho_0} \int_{0}^{5\rho_0} \frac{5^k}{s^k} f(s) \frac{ds}{5}
\]

\[
= \frac{1}{\rho_0^k} f(\rho_0) + \frac{2}{5^k \rho_0} \left( \int_{0}^{\rho_0} \frac{1}{s^k} f(s) ds + \int_{\rho_0}^{5\rho_0} \frac{1}{s^k} f(s) ds \right)
\]

\[
\leq \frac{1}{\rho_0^k} f(\rho_0) + \frac{2}{5} \sup_{0 < \sigma < \rho_0} \frac{1}{\sigma^k} f(\rho_0) + \frac{2}{5} \lim_{s \to \infty} f(s) \int_{\rho_0}^{5\rho_0} \frac{1}{s^k} ds
\]

\[
\leq \frac{1}{\rho_0^k} \lim_{s \to \infty} f(s) + \frac{2}{5} \sup_{0 < \sigma < \rho_0} \frac{1}{\sigma^k} f(\rho_0) + \frac{2}{5} \lim_{s \to \infty} f(s) \int_{\rho_0}^{5\rho_0} \frac{1}{s^k} ds
\]

\[
\leq \left(1 + \frac{2}{5}\right) \frac{1}{\rho_0^k} \lim_{s \to \infty} f(s) + \frac{1}{2} \sup_{0 < \sigma < \rho_0} \frac{1}{\sigma^k} f(\rho_0)
\]

\[
\leq \left(1 + \frac{1}{2}\right) \frac{1}{\rho_0^k} \lim_{s \to \infty} f(s) + \frac{1}{2} \sup_{0 < \sigma < \rho_0} \frac{1}{\sigma^k} f(\rho_0),
\]

then,

\[
\sup_{0 < \sigma < \rho_0} \frac{1}{\sigma^k} f(\sigma) < \frac{3}{\rho_0^k} \lim_{s \to \infty} f(s).
\]

Hence

\[
\sup_{0 < \sigma < \rho_0} \frac{1}{\sigma^k} f(\sigma) < \frac{1}{2} \left( \sup_{0 < \sigma < \rho_0} \frac{1}{\sigma^k} f(\sigma) + 3 \frac{1}{\rho_0^k} \lim_{s \to \infty} f(s) \right) < 3 \frac{1}{\rho_0^k} \lim_{s \to \infty} f(s) < 4 \frac{1}{\rho_0^k} \lim_{s \to \infty} f(s),
\]

and then, by (4.14) we have,

\[
1 \leq \sup_{0 < \sigma < \rho_0} \frac{1}{\sigma^k} f(\sigma) < \frac{1}{2} \left( \sup_{0 < \sigma < \rho_0} \frac{1}{\sigma^k} f(\sigma) + 3 \frac{1}{\rho_0^k} \lim_{s \to \infty} f(s) \right) < \frac{1}{\rho_0^k} \lim_{s \to \infty} f(s).
\]

Finally, since

\[
\rho_0^k = 2^k \lim_{s \to \infty} f(s),
\]

and dividing by 2, we have

\[
\frac{1}{2} \leq \frac{1}{2} \sup_{0 < \sigma < \rho_0} f(\sigma) < \frac{1}{2^{k-1}},
\]

which clearly is a contradiction since \(k \geq 2\).

\[
\Box
\]

**Theorem 4.5** (Sobolev-Type Inequality for Intrinsic Varifolds). Let \((M^n, g)\) be a complete Riemannian manifold satisfying (BG), let \(V = v(S, \theta)\) a rectifiable
varifold satisfying (AC). Suppose \( h \in C^1_0(S) \) non negative, and \( \theta \geq 1 \) \( \|V\| \)-a.e. in \( S \). Then there exists \( C := C(k) > 0 \) such that

\[
\left( \int_S h^{\frac{k}{k-1}} d\|V\| \right)^{\frac{k-1}{k}} \leq C \int_S \left( |\nabla g h|_g + h \left( |H_g|_g - c^* k \right) \right) d\|V\|.
\]

**Proof.** Notice that, for Lemma 4.3, for all \( 0 < \sigma < \rho < r_0 \)

\[
\frac{1}{\omega_k g^k} \int_{B_g(\xi, \sigma)} hd\|V\| \leq \frac{1}{\omega_k \rho^k} \int_{B_g(\xi, \rho)} hd\|V\| + \int_\sigma^\rho \frac{1}{s^{k-1}} \int_{B_g(\xi, s)} \left( |\nabla g h|_g + h \left( |H_g|_g - c^* k \right) \right) d\|V\| \, ds,
\]

but, since by hypothesis \( h \in C^1_0(B_g(\xi, r_0)) \) it remains valid for all \( 0 < \sigma < \rho < \infty \), hence we can apply the Lemma 4.4 with the choices

\[
f(\rho) := \frac{1}{\omega_k} \int_{B_g(\xi, \rho)} hd\|V\| \]

\[
g(\rho) := \frac{1}{\omega_k} \int_{B_g(\xi, \rho)} \left( |\nabla g h|_g + h \left( |H_g|_g - c^* k \right) \right) d\|V\|
\]

provided that \( \xi \in \text{spt} \|V\| \) and \( h(\xi) \geq 1 \).

Thus for \( \|V\|\)-a.e \( \xi \in \{ x \in \text{spt} \|V\| : h(x) \geq 1 \} \) we have

\[
\rho < 2 \left( \frac{1}{\omega_k} \int_M hd\|V\| \right)^\frac{1}{k},
\]

and

\[
\int_{B_g(\xi, 5\rho)} hd\|V\| \leq 5^k \left( \frac{1}{\omega_k} \int_M hd\|V\| \right)^\frac{1}{k} \int_{B_g(\xi, \rho)} \left( |\nabla g h|_g + h \left( |H_g|_g - c^* k \right) \right) d\|V\|.
\]

Now, since \( \{ x \in \text{spt} \|V\| : h(x) \geq 1 \} \subset \text{spt} \|V\| \cap \text{spt} h \) which is compact, in virtue of Vitali’s five Lemma (cf. [Sim83] Theorem 3.3 pg. 11) we can select a collection of balls

\[
\{ B_g(\xi_i, \rho_i) \}_{i \in \Lambda},
\]

such that, for all \( i \in \Lambda, \xi_i \subset \{ x \in \text{spt} \|V\| : h(x) \geq 1 \} \) and, there exists \( \Lambda' \subset \Lambda \) such that for every \( i \neq j, i, j \in \Lambda' \) it holds \( B_g(\xi_i, \rho_i) \cap B_g(\xi_j, \rho_j) = \emptyset \) and

\[
\{ x \in \text{spt} \|V\| : h(x) \geq 1 \} \subset \bigcup_{j \in \Lambda'} B_g(\xi_j, 5\rho_j) = \bigcup_{i \in \Lambda} B_g(\xi_i, \rho_i).
\]

Then, by (4.14)

\[
\int_{\{ h \geq 1 \} \cap \text{spt} \|V\|} hd\|V\| \leq \sum_{j \in \Lambda'} \int_{B_g(\xi_j, 5\rho_j)} hd\|V\|
\]

\[
\leq \sum_{j \in \Lambda'} \left( 5^k \left( \frac{1}{\omega_k} \int_M hd\|V\| \right)^\frac{1}{k} \int_{B_g(\xi_j, \rho_j)} \left( |\nabla g h|_g + h \left( |H_g|_g - c^* k \right) \right) d\|V\| \right)
\]

\[
\leq 5^k \left( \frac{1}{\omega_k} \int_M hd\|V\| \right)^\frac{1}{k} \sum_{j \in \Lambda'} \int_{B_g(\xi_j, \rho_j)} \left( |\nabla g h|_g + h \left( |H_g|_g - c^* k \right) \right) d\|V\|
\]

\[
\leq 5^k \left( \frac{1}{\omega_k} \int_M hd\|V\| \right)^\frac{1}{k} \int_M \left( |\nabla g h|_g + h \left( |H_g|_g - c^* k \right) \right) d\|V\|.
\]
Let \( \gamma_\varepsilon \in C^1(\mathbb{R}) \) a non-decreasing function such that, for given \( \varepsilon > 0 \),

\[
\gamma_\varepsilon(t) := \begin{cases} 
1, & \text{if } 0 < \varepsilon < t \\
0, & \text{if } t \leq 0,
\end{cases}
\]

and consider, for \( t_0 \geq 0 \) given,

\[
f(x) := \gamma_\varepsilon(h(x) - t_0),
\]

then, applying (4.15) with \( f \) instead of \( h \), and setting \( S_\alpha := \{ x \in S : h(x) > \alpha \} \), we get,

\[
\|V\| (S_{t_0+\varepsilon}) = \int_{\{x \in M : h(x) - t_0 > \varepsilon\}} h d\|V\| \\
= \int_{\{x \in S : f(x) \geq 1\}} f d\|V\| \\
\leq 5^k \left( \frac{1}{\omega_k} \int_S f d\|V\| \right)^{\frac{1}{k}} \int_S (\gamma'_\varepsilon(h - t_0) |\nabla^g h|_g + f (|H_g|_g - c^* k)) d\|V\| \\
\leq \frac{5^k}{\omega_k} (\|V\|(S_{t_0}))^{\frac{1}{k}} \int_S (\gamma'_\varepsilon(h - t_0) |\nabla^g h|_g + f (|H_g|_g - c^* k)) d\|V\|
\]

now, multiplying by \((t_0 + \varepsilon)^{\frac{1}{k}}\) we have,

\[
(t_0 + \varepsilon)^{\frac{1}{k}} \|V\| (S_{t_0+\varepsilon}) \leq \frac{5^k}{\omega_k} \left( \int_{S_{t_0}} (t_0 + \varepsilon)^{\frac{1}{k}} d\|V\| \right)^{\frac{1}{k}} \int_S (\gamma'_\varepsilon(h - t_0) |\nabla^g h|_g + f (|H_g|_g - c^* k)) d\|V\| \\
= \frac{5^k}{\omega_k} \left( \int_{S_{t_0}} (t_0 + \varepsilon)^{\frac{1}{k}} d\|V\| \right)^{\frac{1}{k}} \int_S (\gamma'_\varepsilon(h - t_0) |\nabla^g h|_g + f (|H_g|_g - c^* k)) d\|V\| \\
\leq \frac{5^k}{\omega_k} \left( \int_{S_{t_0}} (h + \varepsilon)^{\frac{1}{k}} d\|V\| \right)^{\frac{1}{k}} \int_S \left( \int_S \frac{\partial}{\partial t_0} \gamma'_\varepsilon(h - t_0) |\nabla^g h|_g d\|V\| + \int_S f (|H_g|_g - c^* k) d\|V\| \right) \\
\leq \frac{5^k}{\omega_k} \left( \int_S (h + \varepsilon)^{\frac{1}{k}} d\|V\| \right)^{\frac{1}{k}} \int_S \left( \int_S \frac{\partial}{\partial t_0} \gamma'_\varepsilon(h - t_0) |\nabla^g h|_g d\|V\| + \int_S f (|H_g|_g - c^* k) d\|V\| \right).
\]

Integrating the above inequality on \( t_0 \) in the interval \([0, \infty]\), we have

\[
\int_0^\infty (t_0 + \varepsilon)^{\frac{1}{k}} \|V\| (S_{t_0+\varepsilon}) dt_0 \leq \\
\frac{5^k}{\omega_k} \left( \int_S (h + \varepsilon)^{\frac{1}{k}} d\|V\| \right)^{\frac{1}{k}} \int_0^\infty \left( \int_S \frac{\partial}{\partial t_0} \gamma'_\varepsilon(h - t_0) |\nabla^g h|_g d\|V\| + \int_S f (|H_g|_g - c^* k) d\|V\| \right) dt_0.
\]

First, notice that the left hand side, by the Lemma 3.12, is equal to,

\[
\int_0^\infty (t_0 + \varepsilon)^{\frac{1}{k}} \|V\| (S_{t_0+\varepsilon}) dt_0 = \frac{k}{k} \int_{S_0} (h - \varepsilon)^{\frac{1}{k}} d\|V\| = \frac{k}{k} \int_{\{x \in S : h(x) > \varepsilon\}} (h - \varepsilon)^{\frac{1}{k}} d\|V\|.
\]
on the other hand, we can estimate the right hand side as follows, first by Fubini’s Theorem
\[
\int_0^\infty \left( \int_S \left( -\frac{\partial}{\partial t_0} \gamma_\varepsilon (h(x) - t_0) \right) \left| \nabla^S_{\gamma} h \right| g \, d\|V\| \right) \, dt_0 = \int_S \left| \nabla^S_{\gamma} h \right| g \left( -\frac{d}{dt} \int_0^\infty \gamma_\varepsilon (h(x) - t_0) \, dt_0 \right) \, d\|V\| \\
= \int_S \left| \nabla^S_{\gamma} h \right| g (\gamma_\varepsilon (h(x))) \, d\|V\| \\
\leq \int_S \left| \nabla^S_{\gamma} h \right| (x) \, d\|V\|.
\]

And, again by Fubini’s Theorem,
\[
\int_0^\infty \left( \int_S f(x) \left( |H| g - c^* k \right) \, d\|V\| (x) \right) \, dt = \int_S \left( |H| g - c^* k \right) \left( \int_0^\infty \gamma_\varepsilon (h(x) - t_0) \, dt_0 \right) \, d\|V\| \\
= \int_S \left( |H| g - c^* k \right) \left( \int_{h(x)}^{-\infty} -\gamma_\varepsilon (w) \, dw \right) \, d\|V\| \\
\leq \int_S h \left( |H| g - c^* k \right) \, d\|V\|.
\]

Therefore
\[
\frac{k - 1}{k} \int_{M_\varepsilon} (h - \varepsilon)^{\frac{k}{k-1}} \, d\|V\| \leq \frac{5^k}{\omega_k^\frac{1}{k}} \left( \int_M (h + \varepsilon)^{\frac{k}{k+1}} \, d\|V\| \right)^{\frac{k}{k-1}} \int_M \left( \left| \nabla^S_{\gamma} h \right| g + h \left( |H| g - c^* k \right) \right) \, d\|V\|,
\]
then, letting \( \varepsilon \to 0 \) we obtain
\[
\left( \int_S h^{\frac{k}{k+1}} \, d\|V\| \right)^{\frac{k-1}{k}} \leq C \int_S \left( \left| \nabla^S_{\gamma} h \right| g + h \left( |H| g - c^* k \right) \right) \, d\|V\|,
\]
where
\[
C := C(k) := \frac{k \cdot 5^k}{(k - 1) \omega_k^\frac{1}{k}}.
\]
\[\square\]
References

[AFP00] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. Functions of bounded variation and free discontinuity problems. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000. 6

[AK00] Luigi Ambrosio and Bernd Kirchheim. Rectifiable sets in metric and Banach spaces. Math. Ann., 318(3):527–555, 2000. 4

[All72] William K. Allard. On the first variation of a varifold. Ann. of Math. (2), 95:417–491, 1972. 1, 3, 6, 7

[BDGM69a] E. Bombieri, E. De Giorgi, and M. Miranda. Una maggiorazione a priori relativa alle ipersuperfici minimali non parametriche. Arch. Rational Mech. Anal., 32:255–267, 1969. 1

[BDGM69b] E. Bombieri, E. De Giorgi, and M. Miranda. Una maggiorazione a priori relativa alle ipersuperfici minimali non parametriche. Arch. Rational Mech. Anal., 32:255–267, 1969. 1

[FF60] Herbert Federer and Wendell H. Fleming. Normal and integral currents. Ann. of Math. (3), 72:458–520, 1960. 1

[HS74] David Hoffman and Joel Spruck. Sobolev and isoperimetric inequalities for Riemannian submanifolds. Comm. Pure Appl. Math., 27:715–727, 1974. 2, 12

[LeL12] Camillo De Lellis. Allard’s interior regularity theorem: an invitation to stationary varifolds. http://www.math.uzh.ch/fileadmin/user/delellis/publikation/allard35.pdf, 2012. 6

[Mat95] Pertti Mattila. Geometry of sets and measures in Euclidean spaces, volume 44 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability. 23

[Mir67] Mario Miranda. Diseguaglianze di Sobolev sulle ipersuperfici minimali. Rend. Sem. Mat. Univ. Padova, 38:69–79, 1967. 1

[MS73] J. H. Michael and L. M. Simon. Sobolev and mean-value inequalities on generalized submanifolds of $\mathbb{R}^n$. Comm. Pure Appl. Math., 26:361–379, 1973. 1, 2

[Nar18] Stefano Nardulli. Regularity of isoperimetric regions that are close to a smooth manifold. Bull. Braz. Math. Soc. (N.S.), 49(2):199–260, 2018. 2

[Nas54] John Nash. $C^1$ isometric imbeddings. Ann. of Math. (2), 60:383–396, 1954. 2

[NOA18] Stefano Nardulli and Luis Eduardo Osorio Acevedo. Sharp Isoperimetric Inequalities for Small Volumes in Complete Noncompact Riemannian Manifolds of Bounded Geometry Involving the Scalar Curvature. International Mathematics Research Notices, 06 2018. rny131. 2, 6

[Rud87] Walter Rudin. Real and complex analysis. McGraw-Hill Book Co., New York, third edition, 1987. 10

[Sim83] Leon Simon. Lectures on geometric measure theory, volume 3 of Proceedings of the Centre for Mathematical Analysis, Australian National University. Australian National University, Centre for Mathematical Analysis, Canberra, 1983. 2, 3, 4, 5, 6, 12, 19, 33

Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo
E-mail address: jccorrea@ime.usp.br