Survival of the quantum anomalous Hall effect in orbital magnetic fields as a consequence of the parity anomaly

Jan Böttcher, Christian Tutschku, Laurens W. Molenkamp, and E. M. Hankiewicz

1 Institut für Theoretische Physik und Astrophysik, Universität Würzburg, Würzburg, Germany.
2 Physikalisches Institut, ETP, Universität Würzburg, Würzburg, Germany.

Recent experimental progress in condensed matter physics gives the prospect to observe the parity anomaly in two-dimensional Dirac-like materials. Using effective field theories and analyzing band structures in external out-of-plane magnetic fields (orbital fields), we show that topological properties of quantum anomalous Hall (QAH) insulators are related to the parity anomaly. We demonstrate that the QAH phase survives in orbital fields, violates Onsager relations, and can be therefore distinguished from a quantum Hall (QH) phase. As a fingerprint of the QAH phase in increasing orbital fields, we predict a transition from a quantized Hall plateau with $\sigma_{xy} = -e^2/h$ to a not perfectly quantized plateau, caused by scattering processes between counterpropagating QH and QAH edge states. This transition could be especially important in paramagnetic QAH insulators, such as Hg$_{1-y}$Mn$_y$Te/CdTe quantum wells, where exchange interaction and orbital fields compete.

**Introduction.** Condensed matter analogs of the Dirac equation have opened new directions to study quantum anomalies in the solid-state laboratory. An anomaly occurs, when a symmetry of a classical theory cannot be maintained in the associated quantum theory. For instance, in massless, (2+1)D quantum electrodynamics, parity symmetry is broken during renormalization if one insists on gauge invariance (parity anomaly) [12–20]. As a consequence of this process, a Chern-Simons (CS) term with a non-zero Chern number is induced [16].

In condensed matter physics, an analogous system is a Chern/quantum anomalous Hall (QAH) insulator which describes a single Dirac fermion with a momentum dependent mass or, equivalently, half of the Bernevig-Hughes-Zhang (BHZ) model [21, 22]. In this paper, we examine hallmarks of the parity anomaly in two-dimensional QAH insulators subjected to an external out-of-plane magnetic field (orbital field). In particular, we demonstrate that the parity anomaly enables us to distinguish the QAH from a quantum Hall (QH) phase. This is due to the fact, that although both phases are described by the same topological invariant, the Chern number [23], their origin is very different: QH phases are induced by the parity anomaly [12–20]. As a signature, we demonstrate that the survival of the QAH phase is characterized by a unique charge pumping, as long as the Dirac point is in the bulk gap: Increasing $H$ generates a net charge flow from the QAH edge states (charge depletion) into the full valence band (charge accumulation). With a further increase of the orbital field the Dirac point is pushed into the bulk valence bands, resulting in counterpropagating QH and QAH edge states, which are not protected from backscattering if disorder is present. We predict, that these two effects give rise to a system size dependent transition from $\sigma_{xy} = -e^2/h$ to a not perfectly quantized Hall plateau. The average value of this plateau depends on details of the scattering mechanisms. Such a transition could be observed in Hg$_{1-y}$Mn$_y$Te [26] or in Bi-based QAH insulators [27–30].

**Model.** We start with a Chern/QAH insulator described by a single, non-trivial block of the BHZ model:

$$\mathcal{H}(k) = (M - Bk^2) \sigma_x - Dk^2 \sigma_0 + A (k_x \sigma_x - k_y \sigma_y),$$

(2)

where $k = k_x + k_y \sigma_z$, $\sigma_i$ are the Pauli matrices, $A$ mixes both (pseudo)spin-components, $D$ introduces a particle-hole asymmetry, and $B$, as well as $M$ were defined before. The spectrum is obtained numerically by mapping the Hamiltonian on a stripe geometry with finite length $L_x$ in the $e_x$-direction (hard wall boundary conditions) and periodic boundary conditions along the $e_y$-direction [31]. In Fig. [1a], the band structure with $\mathcal{C}_{\text{QAH}} = -1$ is displayed, with chiral QAH edge states traversing the Dirac mass gap. Since $D \neq 0$, the Dirac point is close to the conduction band edge and the QAH edge states are strongly (pseudo)spin polarized [32].

Next, we implement an orbital field $H = H e_z$ in the Landau gauge $A = -y H e_x$ [Figs. [1b]-(c)]. This field has two main effects: Firstly, bulk subbands evolve into Landau levels (LLs) for $|y| \ll L_x$. All LLs with $n \neq 0$ come...
in pairs of energy $E_n^\pm$, except for a single $n=0$ LL with energy $E_0$. This causes an asymmetry in the spectrum further discussed in Ref. [35]. Secondly, the orbital field lowers the energy of the Dirac point successively, until it enters the valence band at a critical field $H_{\text{scat}}$ and later becomes buried deep in the valence band [31] [32]. The evolution of the Dirac point is determined by $E_D(H) \approx E_D(0) - g_{\text{eff}} \mu_B H$, where $g_{\text{eff}} = m_0 v_x L_x / \hbar$ [32]. Here, $v_x$ is the edge state velocity, $\mu_B$ is the Bohr magneton, $m_0$ is the bare electron mass and $E_D(0)$ is the Dirac point energy at $H = 0$. Note that the QAH edge states survive (up to finite size gaps) even in large orbital fields [Figs. 1(a,c)] since they are protected from hybridization with bulk states by wave function localization and spin polarization.

**Effective Action.** To understand the survival of the QAH edge states better, we derive the corresponding low energy effective bulk Lagrangian $\mathcal{L}_{\text{eff}}^{\text{bulk}}$ by computing the particle number in the continuum/bulk model [12] [16],

$$\langle N \rangle_{\mu} = \frac{1}{\hbar} \int dx \sum_\alpha \langle [\psi_\alpha(x), \psi_\alpha(x)] \rangle_{\mu} = \langle N_0 \rangle_{\mu} - \frac{\eta_\mu}{2}.$$  

Here, $\langle \ldots \rangle_{\mu}$ denotes the expectation value with respect to an arbitrary chemical potential $\mu$, $\psi(x)$ is a field operator (two-spinor), destroying a particle at $x$, $N_0$ is the normal ordered fermion number operator and $\eta_\mu$ is the spectral asymmetry, quantifying the difference in the amount of positive and negative eigenenergy solutions. From the requirement of Lorentz covariance one can now determine the induced three current $j_{\text{ind}}^{\mu} = \sigma_{xy} e^{\mu\nu} \partial_\nu a_\mu$ arising as a response to a small perturbing field $a_\mu$, applied on top of the underlying orbital field $H$. Here, $\sigma_{xy}$ is the Hall conductivity, $j_{\text{ind}}^0$ is the induced bulk charge density, and $j_{\text{ind}}^{1,2}$ is the induced bulk current density in $x$- and $y$-direction, respectively. Since $j_{\text{ind}}^0 = \delta S_{\text{eff}}^{\text{bulk}} / \partial \mu$, with $S_{\text{eff}}^{\text{bulk}} = \int dx \mathcal{L}_{\text{eff}}^{\text{bulk}}$, we can compute the corresponding effective bulk Lagrangian which is one of the main results of our paper [35]:

$$\mathcal{L}_{\text{eff}}^{\text{bulk}}(\mu, H) = \frac{\sigma_{xy}(\mu, H)}{2} e^{\mu\nu} a_\mu \partial_\nu a_\rho,$$  

where $e^{\mu\nu}$ is the Levi-Civita symbol. This is a topological CS term [36] with quantized Hall conductivity

$$\sigma_{xy} = \kappa_{\text{QAH}} - \kappa_{\text{QH}}^0 \Theta(|\mu| - |E_0|) - \sum_{n=1}^{\infty} s \kappa_{\text{QH}}^n \Theta[|\mu - E_n^0|].$$

According to their physical origin, we have separated the Hall conductivity into three parts:

$$\kappa_{\text{QAH}} = \frac{2 e^2}{\hbar} \left[ \text{sgn}(M - B/\hbar^2) + \text{sgn}(B) \right],$$  

$$(4a)$$

$$\kappa_{\text{QH}}^0 = \frac{2 e^2}{\hbar} \left[ \text{sgn}(M - B/\hbar^2) - \text{sgn}(\mu H) \text{sgn}(B) \right],$$  

$$(4b)$$

$$\kappa_{\text{QH}} = \frac{e^2}{\hbar} \text{sgn}(eH).$$  

$$\kappa_{\text{QH}} = \frac{e^2}{\hbar} \Theta\left(|\mu| - |E_0|\right).$$  

CS terms only arise if parity and time-reversal symmetry are broken [10] [36]. Since the parameters $M$ and $B$ [35] as well as the orbital field $H$ violate these symmetries, they give rise to CS terms [17] [19] [37]. Here, we distinguish two types of CS terms: The first type, Eq. (4a), is defined by its exclusive relation to the Dirac ($M$) and the effective ($B$) mass, resulting in a violation of the Onsager relation. It is hence related to the inverted band structure and thus to the parity anomaly. Equation (4a) does not arise from a single LL, but is rather connected to the asymmetry of the entire spectrum, encoded in the spectral asymmetry $\eta_{\mu} = 2 n_0 \text{sgn}(eH) \kappa_{\text{QAH}}$, where $n_0$ is the LL degeneracy [35] [38]. Hence, it does not depend on a Heaviside function. The second type of CS terms [Eqs. (4b), (4c)] describes conventional QH physics, generated by an orbital field, as indicated by the $\text{sgn}(eH)$-dependence [35]. In contrast to the first type, the origin of each of these CS terms is related to a single LL, reflected by the Heaviside functions. They can only contribute to the Hall conductivity if $|\mu| > |E_0|$. In order to derive the corresponding edge theories, we have to add a new degree of freedom to $\mathcal{L}_{\text{eff}}^{\text{bulk}}$. This can be inferred from the fact that any CS term changes by a total derivative under a local gauge transformation, $\mathcal{L}_{\text{eff}}^{\text{bulk}} \rightarrow \mathcal{L}_{\text{eff}}^{\text{bulk}} + \delta \mathcal{L}_{\text{eff}}^{\text{bulk}}$, causing a violation of charge conservation, $\partial_\mu j_{\text{ind}}^\mu \neq 0 \big|_{\partial \Omega}$, at the boundary $\partial \Omega$ [36] [39]. To cancel this U(1)-anomaly, we must enlarge our description by an effective edge Lagrangian $\mathcal{L}_{\text{eff}}^{\text{eff}}$, which restores gauge invariance via anomaly cancellation between edge and
where $j_{\text{ind}}^\mu$ symbolize induced currents at the left/right edge of the stripe geometry. This procedure is the field-theoretical analog to the bulk-boundary correspondence \[13\]. Equation \[5\] implies that an increasing orbital field induces charge accumulation in the bulk which is compensated by a charge depletion at the edges (no charge is conserved by single LLs and charge flow appears only between an edge state and its associated LL. Note that $\rho = 0$ for $j_{\text{ind}}^\mu = 0$).\[4\] \[4\] The amount of induced bulk charge is given by $j_{\text{ind}}^0$.\

\[
\frac{\partial_j j_{\text{tot}}^\mu}{\partial_j j_{\text{tot}}^\mu} = \sum_{i=1}^{\infty} \frac{\chi_i}{\chi_i} \left( \delta_{\Psi_{\text{QH}}}(E_n) - \delta_{\Psi_{\text{QAH}}}(E_n) \right) \epsilon_{\Psi_{\text{QH}}} \partial_{\Psi_{\text{QAH}}} \frac{\partial_j j_{\text{tot}}^\mu}{\partial_j j_{\text{tot}}^\mu} \frac{\partial_j j_{\text{tot}}}{\partial_j j_{\text{tot}}^\mu} - \partial_j j_{\text{ind}}^\mu, \tag{5}
\]

where $j_{\text{ind}}^\mu$ are the induced currents at the left/right edge of the stripe geometry. This procedure is the field-theoretical analog to the bulk-boundary correspondence \[13\]. Equation \[5\] implies that an increasing orbital field induces charge accumulation in the bulk which is compensated by a charge depletion at the edges (no charge is conserved by single LLs and charge flow appears only between an edge state and its associated LL). Note that $\rho = 0$ for $j_{\text{ind}}^\mu = 0$. The amount of induced bulk charge is given by $j_{\text{ind}}^0$.\[4\] \[4\] The amount of induced bulk charge is given by $j_{\text{ind}}^0$.\[4\]

\[
L_{\text{eff}}^{\mu} = L_{\text{QH}}^{\mu} \delta(y - y_{\text{QH}}) + L_{\text{QAH}}^{\mu} \delta(y - y_{\text{QAH}}),
\]

\[
L_{\text{eff}}^{\mu} = \chi_i \left( \partial_{\Psi_{\text{QH}}} + \hbar \epsilon_{\Psi_{\text{QAH}}}/e \right) \partial_{\Psi_{\text{QAH}}} \delta_{\Psi_{\text{QH}}}(E_n) \delta_{\Psi_{\text{QAH}}}(E_n) \xi_0 \Theta(|\mu| - |E_0|) \tag{6a}
\]

\[
+ \sum_{n \neq \pm \infty} \xi_i \left( \partial_{\Psi_{\text{QH}}} + \hbar \epsilon_{\Psi_{\text{QAH}}}/e \right) \partial_{\Psi_{\text{QAH}}} \delta_{\Psi_{\text{QH}}}(E_n) \delta_{\Psi_{\text{QAH}}}(E_n) \xi_i \Theta(|\mu| - |E_n|), \tag{6b}
\]

where $\chi$ and $\xi$ define QAH (QH) edge states. In this case, $D_{xy} = \hbar^2 c_0^2 \kappa_0^2$ determines the Hamiltonian. Note that the chirality of the QAH edge states is unaltered for $j_{\text{ind}}^\mu = 0$.\[4\] \[4\] The amount of induced bulk charge is given by $j_{\text{ind}}^0$.\[4\]

\[
H_{\text{crit}} = \text{sgn}(eH) \frac{\hbar M}{e B}. \tag{7}
\]

In contrast, Eqs. \[6b\] and \[6c\] are related to Eqs. \[4b\] and \[4c\] and define QH edge states. These states are bound by single LLs and charge flow appears only between an edge state and its associated LL. Note that only the chirality of the QAH edge states is unaltered for $H \rightarrow -H$ (at constant chemical potential).

**Charge pumping.** To highlight the differences in the charge pumping between QAH and QH phases, we consider here an impurity-free system and comment on (in)elastic scattering effects in the next section. We simulate the evolution of the charge distribution as a function of the orbital field within the time-dependent Schrödinger equation. As in typical experiments, the carrier density (not chemical potential) will be kept constant.\[48\] \[49\] \[50\] \[51\] In particular, we consider a vector potential $\mathbf{A}(t) = \mathbf{A}(t_i) + \mathbf{a}(t)$ with $t \in [t_i, 0, t_i]$, where $\mathbf{A}(t_i)$ is a time-independent background field and $\mathbf{a}(t) = -yH(t)\mathbf{e}_x$ is a time-dependent perturbation.

At initial time $t_i$, the system is described by the solutions of the Schrödinger equation $|\psi_{j,k}(t_i)|$, where $j$ labels the j-th subband. For $t > t_i$, the perturbation is switched on and each initially occupied state, with $j \leq j_{\text{max}}$ and $k \leq k_{\text{max}}$, evolves under unitary time evolution to $|\psi_{j,k}(t)|$.

The quantities $j_{\text{max}}$ and $k_{\text{max}}$ are determined by the initial chemical potential $\mu$. Linearly increasing the orbital field with time, we trace the occupation of states in each instantaneous spectrum, defined by the time-independent Schrödinger equation $H(t)|\psi_{j,k}(t)| = E_{j,k}(t)|\psi_{j,k}(t)|$ (here $t$ only parametrizes the Hamiltonian). Their occupation probabilities are determined by $P_{j,k}(t) = \sum_{j=0}^{j_{\text{max}}} \langle |\psi_{j,k}(t)| |\psi_{j,k}(t)| \rangle^2$.\[52\] At $t_i$, the ground state for $H$ is the QAH phase and is determined by $\mathbf{A}(t_i) = \mathbf{0}$ with $\mu$ located at the Dirac point [Fig. 1(a)], whereas for $H > H_{\text{crit}}$ the QH phase a finite background field $\mathbf{A}(t_i) = -yH_0\mathbf{e}_x$ has to be applied and $\mu$ is placed above the first LL [Fig. 1(d)]. The numerical results, presented in Figs. 1 and 2, are independent of the time scale in which $H(t)$ is ramped up, provided that $t_{\text{min}}^0 \ll t_i \ll t_{\text{max}}^0$. The lower bound prevents excitations across bulk gaps $E_g$ and is therefore determined by $t_{\text{min}}^0 = h/E_g \sim 10^{-13}$ s. For $H > H_{\text{crit}}$, the upper bound comes from the necessity to overcome hybridization gaps forming between the QAH edge states and bulk LLs. As long as these hybridization gaps are finite size gaps, exponentially suppressed by the system size, $t_{\text{max}}^0$ tends to infinity.

Let us now discuss the numerical results, starting with the QAH phase under initial condition (I). Increasing $H(t)$ with time, the occupation of the eigenstates and the induced charge carrier density $n_{\text{ind}}(x,t) = -e n_{\text{ind}}(x,t)$ evolve as shown in Figs. 1(a)-(c) and Fig. 2 where $n_{\text{ind}}(x,t) = \sum_i k_i P_{i,k_i}(t)|\phi_{i,k_i}(x,t)|^2 - n_{\text{back}}$, where $n_{\text{back}}$ ensures that $n_{\text{ind}}(x,t_i) = 0$. Starting from a flat
(zero) charge density distribution, an increase of $H(t)$ causes a net charge flow from the QAH edge states (charge depletion) into all valence band LLs (charge accumulation) via charge polarization (redistribution of weights of wave functions). During this process, all valence band LLs, including the $n = 0$ LL, remain filled. As illustrated in the inset of Fig. 2, this causes a linear increase of the bulk charge with $j_{\text{ind}} = \sigma_{xy} \nabla \times a = \kappa_{\text{QAH}} H(t)$. Since this type of pumping is bound to the existence of the QAH edge states, it can only exist for $H < H_{\text{crit}}$ [Eq. (7)]. Hence, these results are consistent with our conclusions based on the Callan-Harvey mechanism following from Eq. (5).

In contrast, our results for the QH phase under initial condition ($H$) are shown in Fig. 1(d) and in the inset of Fig. 2. In agreement with our field-theoretical approach, we find that the bulk charge originates purely from the associated QH edge states, implying a saturation of the charge accumulation already for small orbital fields. This is therefore further evidence that the QAH edge states are related to a distinct CS term, which is connected to the spectral asymmetry $\eta_{\mu}$ and not to a single LL.

**Experimental signatures.** We have so far considered an impurity-free system. What are consequences of taking disorder and, therefore, (in)elastic scattering into account? As long as the Dirac point is above the $n = 0$ LL, i.e., for $H < H_{\text{scat}}$, the system is in its ground state. Scattering cannot cause relaxation of the induced bulk charge and, hence, disorder cannot affect the results of Figs. 1(b) and 2. The hallmark of the QAH effect is a quantized Hall plateau with $\sigma_{xy} = \kappa_{\text{QAH}}$ whose length scales with $H_{\text{scat}} \sim L_y^{-1}$. This is depicted by region I in Fig. 3 and follows from $g_{\text{eff}} \sim L_y$ [55]. For $H > H_{\text{scat}}$, the system is driven into a state with no common chemical potential, whose signature is a selective population of states (charge inversion), shown in Fig. 1(c). This charge inversion is protected by momentum conservation, since direct relaxation processes, such as spontaneous emission, are exponentially suppressed by the spatial localization of the wave functions. However, since realistic systems are rather imperfect, (in)elastic scattering between occupied QH and unoccupied QAH edge states facilitate momentum, as well as energy relaxation as indicated by region II in Fig. 3. As a result, the charge inversion relaxes eventually, until a common chemical potential has set in. In this new ground state, counterpropagating QAH and QH edge states coexist at a single boundary. For instance in the inset of region II, at the right boundary, the QAH edge state has a positive velocity ($v_x = \hbar^{-1} \partial E / \partial k_x > 0$), while the QH edge state has a negative velocity ($v_x < 0$). Similarly to Ref. [49], which uses the Landauer-Büttiker formalism, we might expect deviations from a perfect quantization of the Hall conductivity $\sigma_{xy}$ arising from scattering between QH and QAH edge states. When the transmission probability $T_{i,j}$ between contact $i$ and contact $j$ on a typical Hall bar is approximately symmet-ric, meaning $T_{i,i+1} \approx T_{i+1,i}$, we expect a noisy $\sigma_{xy} = 0$ plateau. While for $T_{i,i+1} \neq T_{i+1,i}$, the average value of $\sigma_{xy}$ can significantly deviate from zero (wiggly line in Fig. 3). Finally for $H > H_{\text{crit}}$, the Dirac mass gap is closed and $\sigma_{xy} = 0$, indicated by region III in Fig. 3.

**Realization.** Typical materials in which this crossover should be observed include Hg$_{1-x}$Mn$_x$Te/CdTe quantum wells, described by the BHZ model [21, 22, 26, 50]. In the discussion above, we assumed that the spin-down block of the BHZ model is trivial and, hence, does not qualitatively affect the discussed physics. Nevertheless, analogous equations for the spin-down block can be derived replacing $(M, B) \rightarrow (-M, -B)$, Zeeman ($g_z$) and exchange ($G_{\text{ex}}$) terms can be incorporated, replacing $M \rightarrow M \pm g(H)$, where $g(H) \equiv g_z H \pm G_{\text{ex}}(H)$ [22] and $+(-)$ applies to the spin-up(down) block. In the full BHZ model, $g \neq 0$ breaks time-reversal symmetry and drives the system into the QAH phase if $(M + g - B/l_H^2)(M - g - B/l_H^2) < 0$, extending the definition of QAH insulators to orbital fields [22]. Since the exchange interaction in Hg$_{1-x}$Mn$_x$Te is paramagnetic [51], a finite orbital field is needed to drive the system into the QAH phase. The second spin-block causes therefore an additional transition from the quantum spin Hall to the charge pumping region I. In Bi-based QAH insulators, one should in general be able to observe similar transitions as shown in Fig. 3 given that signatures of both the QH and the QAH effect are observed at relatively small orbital fields [27, 30].

**Conclusions.** Studying QAH insulators opens new possibilities to understand consequences of the parity anomaly in solid state systems. We have shown that the parity anomaly in orbital fields is linked to the survival of the QAH edge states, a violation of the Onsager relations and a unique charge pumping from the QAH edge states to the full LL spectrum. As a fingerprint of this survival,
we predict a transition from $\sigma_{xy} = -e^2/h$ (QAH effect) to a noisy Hall plateau (scattering between counterpropagating QAH and QH edge states) with increasing orbital fields. The experimental verification of our theoretical predictions in Hg$_{1-x}$Mn$_x$Te quantum wells is underway [20]. In the future, it would be interesting to study signatures of quantum anomalies in models beyond the BHZ model and analyze transport signatures of counterpropagating QH and QAH edge states.

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* These two authors contributed equally to this work.

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A. Remnant of QAH effect in orbital fields

In this section, we are going to show that the band inversion of a non-trivial Chern insulator survives up to orbital fields of

\[ H_{\text{crit}} = \text{sgn}(eH) \frac{M \phi_0}{B 2\pi}, \tag{8} \]

where \( \phi_0 = h/e \). In Section A.1, this relation is first proven from a pure bulk approach, while in Section A.2, we supply further evidence using an edge approach. A more rigorous argument will be presented in App. B.

Bulk Approach

In this appendix, we take a closer look on the features of an inverted band structure surviving in orbital fields. In particular, we study a 2D Chern insulator, which is denoted by [21]

\[ \mathcal{H} = (M - Bk^2) \sigma_x - Dk^2 \sigma_0 + A (k_x \sigma_x - k_y \sigma_y), \tag{9} \]

where all parameters have been discussed in the main text. For simplicity, let us adopt this Hamiltonian in the pseudospin basis of the spin-up block of Hg\(_{1-x}\)Mn\(_x\)Te, i.e. \{\(E_1, \uparrow\), \(H1, \uparrow\)\}. The energy spectrum of Eq. (9) is determined by solving the corresponding Schrödinger equation:

\[ E^{\pm}(k_x, k_y) = -Dk^2 \pm \sqrt{A^2k^2 + (M - Bk^2)^2}. \tag{10} \]

The system has a non-trivial topology if \(M/B > 0\). In the following, we focus without loss of generality on \(M, B < 0\) [21]. A consequence of this condition is that the band structure is inverted, meaning that the band edge energy of the \(E1\)-band lies below the band edge energy of the \(H1\)-band. An exemplary case is depicted in Fig. 4(a), where the color code displays the pseudospin character of the wave functions highlighting the inverted band structure. In particular, Fig. 4(a) depicts the band structure for a stripe geometry and only the inset shows the continuous bulk band structure as determined by Eq. (10). In momentum space, we observe that the band structure is inverted close to the \(\Gamma\)-point, characterized by \(M - Bk^2 < 0\), while it is normally ordered for momenta satisfying \(M - Bk^2 > 0\). This property results from a momentum dependent renormalization of the Dirac mass \(M\) by the effective mass \(Bk^2\). The band ordering is solely determined by the diagonal terms of the Hamiltonian. However, closing of the Dirac mass gap at finite momentum cannot be observed in band structure plots, since the off-diagonal terms cause hybridization between bulk bands. Nevertheless, the bulk band inversion is removed at \(k_{\text{crit}}^2 = M/B\), where \(M - Bk^2 = 0\) implying that the renormalized Dirac mass changes its sign.

Now, we raise the question, what happens to the band inversion in the presence of an orbital field? As it is well known, an orbital field induces a parabolic confinement and increases (decreases) the energy of \(E_1 (H1)\) - states independently on the orbital field direction. It should therefore counteract the band inversion. Since the band inversion is protected by the Dirac mass gap, a finite critical orbital field is needed to remove the inversion completely. In Fig 4(b), the band structure is shown for \(H = 1.5\) T and, on the first glance, the color code seems to reveal that the band inversion is still present. In the following, we give a proof for Eq. (8) based on a simple analysis of the bulk LL spectrum. Let us start by replacing the gauge-independent momentum operators, \(\pi_i = k_i + eA_i/h\), by ladder operators

\[ \pi_+ \rightarrow \sqrt{\frac{2}{l_H}} \begin{cases} a^\dagger & \text{for } s > 0 \\ -a & \text{for } s < 0 \end{cases}, \]

\[ \pi_- \rightarrow \sqrt{\frac{2}{l_H}} \begin{cases} a & \text{for } s > 0 \\ -a^\dagger & \text{for } s < 0 \end{cases}, \]

where \(s = \text{sgn}(eH)\) and \(l_H = \sqrt{\hbar/|eH|}\). An appropriate ansatz to solve the Schrödinger equation is given by

\[ \psi_{n,k_x}^{\pm}(y) \propto \begin{cases} (M - \beta n - \frac{s\phi}{2} \pm c_n) \langle y|n, k_x \rangle \sqrt{n} & \text{for } s > 0 \\ (M - \beta n - \frac{s\phi}{2} \pm c_n) \langle y|n - 1, k_x \rangle \sqrt{n} & \text{for } s < 0 \end{cases} \tag{11} \]

FIG. 4. Band structure of a Chern insulator on a stripe geometry with parameters chosen as in Fig. 1 of main text. The color code indicates that the respective wave functions consist of a mixture of \(E1\) (red) and \(H1\) (cyan) components. In (a) the band structure is depicted at \(H = 0\) T. One clearly observes the band inversion close to the \(\Gamma\)-point. The inset shows the continuous bulk band structure as determined by Eq. (10). In (b) the spectrum is shown for \(H = 1.5\) T.
and

\[
\psi_{0,k_x}(y) \propto \begin{cases} 
(y|0,k_x) & s > 0 \\
0 & s < 0
\end{cases}
\]

where we neglected normalization constants for simplicity, \( \alpha = \sqrt{2A/l_H} \), \( \beta = 2B/l_H^2 \), and \( \delta = 2D/l_H^2 \). The corresponding LL energies are [33]

\[
E_{n\neq 0}^\pm = -s\beta/2 - n\delta \pm \epsilon_n, \quad (13)
\]

\[
E_0 = s(M - \beta/2) - \delta/2, \quad (14)
\]

where \( \epsilon_n = \sqrt{\alpha^2 n^2 + (M - n\beta - s\delta/2)^2} \). A hallmark of a Dirac-like Hamiltonian is the relativistic structure of the LL spinors, following from the off-diagonal structure of the Hamiltonian. The \( n = 0 \) LL is completely decoupled and, therefore, pseudospin polarized, while all other LLs (for \( n > 0 \)) are formed from hybridization between the \( n \)-th \( E1 \) and \( (n-1) \)-th \( H1 \) LLs [33]. This asymmetric coupling in orbital fields causes an asymmetry in the LL spectrum further discussed in App. B and in Ref. [12].

A sketch of this hybridization process is shown in Fig. 5(a). Notice that without hybridization, the system would include two decoupled \( n = 0 \) LLs.

Above, we have shown for \( H = 0 \), that the diagonal elements in Eq. (9) characterize the band ordering in momentum space. These elements will also define the band ordering of the Chern insulator in orbital fields. To understand this in detail, let us first study the evolution of the \( E1 \) and \( H1 \) LLs without hybridization. In such a case, \( A \) is effectively zero and the system is driven by its diagonal elements. This is illustrated by the solid lines in Fig. 5(b), where it is found that each pair of LLs with opposite pseudospin, but with the same LL indices. Blue dotted lines show the evolution of LLs with hybridization for \( A = 45 \) meVnm.

FIG. 5. Hybridization and gap closing of the bulk LL spectrum in orbital fields. Except for \( A \), all parameters are taken from the caption of Fig. 1. The hybridization between \( E1 \) and \( H1 \) bands in orbital fields is schematically demonstrated in (a), where red (cyan) indicates \( E1 \) (\( H1 \)) - pseudospin character. In (b), the LL energies without coupling \( (A = 0) \) are depicted by solid lines and are explicitly numerated by LL indices. Blue dotted lines show the evolution of LLs with hybridization for \( A = 45 \) meVnm.

at which the band structure becomes normally ordered. We therefore deduce that \( H_{\text{crit}} \) defines the critical orbital field at which the band inversion is completely removed.

To avoid any misunderstanding, we want to emphasize that the argumentation above has been given for a single block of the BHZ model. The discussed crossing should not be confused with the one, occurring in the full BHZ model. In this model, one observes a crossing between two pseudospin polarized \( n = 0 \) LLs [33], where each level belongs to a distinct spin-block.

**Edge approach**

The upper proof is based on a pure bulk calculation. Here, we show that the result for \( H_{\text{crit}} \) can be reproduced focusing on the evolution of the QAH edge states in orbital fields. As described in the main text, the QAH edge states are successively lowered in energy as we increase the orbital field (for \( \text{sgn}(eH) > 0 \)). For simplicity, Fig. 6 sketches this scenario by only taking into account the flat bulk \( n = 0 \) LL, as well as the QAH edge states. With this simplified model, we are aiming to determine an upper
limit until which the QAH edge states can survive in orbital fields. By survival of the QAH edge states, we mean that even for $H \neq 0$ the QAH edge states and bulk LLs remain decoupled up to finite size gaps, exponentially vanishing in the limit $L_y \rightarrow \infty$.

First, let us take a closer look on the properties of the $n = 0$ LL, whose energy is given by Eq. (14). The degeneracy of this level and therefore its width in momentum space increases linearly with $H$:

$$k_{\text{max}} = \frac{eH L_y}{2\hbar},$$

where $2k_{\text{max}}$ is the full width of the LL [Fig. 6]. The associated wave functions $\psi_{0,k_y}(y)$ of the zeroth LL are each centered at $y(k_x) = \frac{L_y}{2} k_x$ and their spatial width decreases linearly as we increase the orbital field.

Now, let us analyze the evolution of the QAH edge states in orbital fields for which an analytic expression was derived by Zhou et al. [32]:

$$E_{\text{edge}} \pm (k_x, H) = E_D(0) - \mu_B g_{\text{eff}}(L_y) H \pm \hbar v_x k_x,$$  \hspace{1cm} (15)

where $v_x$ is the edge state velocity, $\mu_B$ is the Bohr magneton and $E_D(0)$ is the energy of the Dirac point at $H = 0$. The effective g-factor reads

$$g_{\text{eff}}(L_y) = m_0 v_x h^{-1} \left[ L_y - \lambda_1^{-1} - \lambda_2^{-1} - 2(\lambda_1 + \lambda_2)^{-1} \right],$$

where $\lambda_{1,2}$ are the decay length scales of the edge state and $m_0$ is the bare electron mass. For $L_y \gg \lambda_{1,2}^{-1}$, we further simplify $g_{\text{eff}}(L_y) \approx m_0 v_x h^{-1} L_y$.

The crossing between the zeroth LL and the QAH edge states in momentum space, as shown in Fig. 6(a), is denoted by $k_{\text{cross}}$. It moves to larger momentum values for increasing orbital fields. This is due to the fact that the QAH edge states are pushed down in energy while the increasing orbital fields. This is due to the fact that the noted by $k_{\text{cross}}$, the wave functions of the bulk LL (solid red line) and of the QAH edge state (solid blue line) have an exponentially small overlap. Only if the crossing of the QAH edge state and the bulk LL happens close to $k_{\text{max}}$ (wave function indicated by red dashed line), they would start to hybridize causing a finite gap in the spectrum.

![FIG. 6. Energy gaps between a QAH edge state (blue) and the $n = 0$ LL (red). (a) The sketch shows that the QAH edge states was shifted down in energy by the orbital field and crosses the flat bulk LL at $k_{\text{cross}}$. (b) If $k_{\text{cross}} \ll k_{\text{max}}$, the wave functions of the bulk LL (solid red line) and of the QAH edge state (solid blue line) have an exponentially small overlap. Only if the crossing of the QAH edge state and the bulk LL happens close to $k_{\text{max}}$ (wave function indicated by red dashed line), they would start to hybridize causing a finite gap in the spectrum.](image)

**B. Charge counting and effective field theory of Chern insulators**

In this appendix, we are going to derive the bulk charge carrier density and the effective bulk Lagrangian of a gapped Chern insulator in orbital fields at arbitrary chemical potentials. We take again Eq. (9) as our starting point. For simplicity, we focus on the particle-hole symmetric case, $D = 0$. In literature, similar calculations have been so far performed solely for massive $(2+1)$D quantum electrodynamics [12][13][15][17][19], i.e. a Chern insulator without the $Bk^2$-term. Our new results are consistent with these calculations in the limit $B \rightarrow 0$.

Like Dirac systems, a Chern insulator (continuum model) has an infinite Dirac sea, which causes infinities in many physical observables. To obtain finite results, those infinities have to be carefully subtracted (renormalized). For $H = 0$, this can be achieved by the physical requirement that the fermion number needs to vanish, if the chemical potential is located at the charge neutrality point $E_Z$. For a particle-hole symmetric Chern insulator,
the charge neutrality point should be defined in the middle of the bulk gap, i.e. at $E_z = 0$. Physically, we satisfy this constrain by choosing antisymmetrization as the appropriate operator ordering for the bulk fermion number operator \cite{15}:

$$ N = \frac{1}{2} \int d\mathbf{x} \sum_{\alpha} \left[ \psi_{\alpha}^\dagger (\mathbf{x}), \psi_{\alpha} (\mathbf{x}) \right], \quad (16) $$

where $\psi(\mathbf{x})$ is a field operator (two component spinor).

To calculate the bulk fermion number with respect to $E_z$, the renormalization scheme, as given by Eq. \((16)\), must be maintained for $H \neq 0$. In this case, the field operators can be expanded in terms of the LL spinors of the conduction band $u_{n,k}(\mathbf{x}) = e^{ik\mathbf{x}\cdot \mathbf{r}}v_{n,k}(\mathbf{y})$, as well as the valence band $v_{n,k}(\mathbf{x}) = e^{ik\mathbf{x}\cdot \mathbf{r}}\overline{v}_{n,k}(\mathbf{y})$, which have been already defined in Eqs. \((11)\) and \((12)\):

$$ \psi (\mathbf{x}) = \sum_{k_x,n} b_{n,k_x} u_{n,k_x}(\mathbf{x}) + \sum_{k_x,n} d_{n,k_x}^\dagger v_{n,k_x}(\mathbf{x}). \quad (17) $$

Here $b_{n,k_x}$ destroys an electron in the $n$-th conduction band LL with momentum $k_x$, and $d_{n,k_x}^\dagger$ creates a hole in the $n$-th valence band LL with momentum $k_x$. The LL energies are determined by Eqs. \((13)\) and \((14)\). The zeroth LL is unique since it is either part of the valence or conduction band. As a result, for $E_{n=0} > E_z$ the first sum in Eq. \((17)\) runs from $n = 0 \ldots \infty$ and the second sum from $n = 1 \ldots \infty$. The situation is reversed, if the zeroth LL is located at an energy $E_{n=0} < E_z$.

All fermionic operators fulfill conventional anti-commutation relations

$$ \{ b_{n,k_x}, b_{m,q_x}^\dagger \} = \delta_{n,m} \delta_{k_x,q_x}, \quad \{ d_{n,k_x}, d_{m,q_x}^\dagger \} = \delta_{n,m} \delta_{k_x,q_x}. \quad (18) $$

Inserting now Eq. \((17)\) into Eq. \((16)\), we obtain

$$ N = \frac{1}{2} \left( \sum_{k_x,n} [b_{n,k_x}, b_{m,q_x}^\dagger] + \sum_{k_x,n} [d_{n,k_x}^\dagger, d_{m,q_x}] \right) 
\quad = N_0 - \eta_H / 2, \quad (19) $$

where we made use of Eq. \((18)\).

Here, the normal ordered fermion number operator $N_0$ and the, so called, spectral asymmetry $\eta_H$ are given by \cite{15}:

$$ \eta_H = \sum_{E > E_z} \sum_{E < E_z} \frac{1}{E - E_z} = \sum_{n} \text{sgn} (E - E_z). \quad (20) $$

The spectral asymmetry $\eta_H$ quantifies the asymmetry of the entire eigenvalue spectrum. This means that it counts the difference in the amount of states between valence and conduction band. If $\eta_H$ becomes different from zero by applying an external field, eigenstates of a Hamiltonian must have gone from the conduction into the valence band (or vice versa) \cite{18}. It is a topological quantity since it is invariant under small, local perturbations \cite{13}.

At $H = 0$, the spectral asymmetry $\eta_H$ vanishes due to the underlying particle-hole symmetry. This argumentation is however not valid in an orbital field where this symmetry is violated. Here every summand in Eq. \((21)\) contributes to $\eta_H$, since there is no symmetry argument which allows us to cancel summands from the first with the second sum. Due to the fact that $\eta_H$ consists of two infinite sums, which are separately divergent, a regularization scheme has to be introduced. Here, we make use of a heat-kernel regularization \cite{11}:

$$ n \geq 0: \quad 1 \rightarrow e^{-\kappa|E_n|}, $$

where $\kappa > 0$ ensures the absolute convergence of both sums. After regularization, we obtain

$$ \eta_H (\kappa) = \frac{1}{n_0} \left( \sum_{k_x,n=1} e^{-\kappa E_n} - \sum_{k_x,n=1} e^{\kappa E_n} + \sum_{k_x} c e^{-\kappa E_0} \right) \cdot (21) $$

where $c = \text{sgn} (eH) \text{sgn} (M - \beta/2)$. Here, the last term marks the contribution of the zeroth LL and, in the second equality, we made use of the momentum independence of the eigenvalue spectrum to show that the LL degeneracy $n_0 = \sum_{k_x} = S/(2\pi l_H^2)$, where $S$ is the area of the system. Using simple algebraic manipulations and the geometric series, we find that

$$ \eta_H (\kappa) / n_0 \approx 2 s e^{-\kappa \text{sgn}(\beta)} \left( \frac{2\pi \kappa}{M} \right)^2 \text{sinh} (\kappa \beta/2) \times \left[ \frac{1}{1 - e^{-\kappa |\beta|}} - 1 \right] + c e^{-\kappa |E_0|}. $$

The spectral asymmetry is defined as the analytic continuation for $\kappa \rightarrow 0^+$, this means $\eta_H = \lim_{\kappa \rightarrow 0^+} \eta_H (\kappa)$, resulting in

$$ \eta_H = n_0 \text{sgn} (eH) \text{sgn} (M - \beta/2) + \text{sgn} (B). \quad (22) $$

With Eqs. \((20)\) and \((22)\), we are finally in the position to calculate the functional dependence of the bulk charge density $j_{\text{bulk}}^0 (\mu, H)$. Let us first focus on the case $\mu = E_z$ characterizing the ground state. Here, the bulk charge density is determined by $j_{\text{bulk}}^0 (\mu = E_z, H) = -e(\text{vac}|N|\text{vac})/S$, where $|\text{vac}\rangle = \Pi_{n,k_x} d_{n,k_x}|0\rangle$. Since the operators are normally ordered with respect to $E_g, N_0(|\text{vac}\rangle = 0$. Thus in the ground state the bulk charge carrier density at constant chemical potential $\mu = E_z$ is purely defined by the spectral asymmetry:

$$ j_{\text{bulk}}^0 (\mu = E_z, H) = \kappa_{\text{QAH}} H = \frac{e}{2\hbar} \eta_H, \quad \kappa_{\text{QAH}} = \frac{e^2}{2\hbar} \left[ \text{sgn} (M - \beta/2) + \text{sgn} (B) \right]. $$
This demonstrates that the Hall conductivity in the ground state is solely determined by \( \eta_H \). In contrast to the half-quantized Hall conductivity, obtained for a massive two-dimensional Dirac operator \([12, 19]\), we find that the effective mass parameter \( Bk^2 \) takes the role of a regulator at high energies, resulting in the required integer quantization of the Hall conductivity \([3]\). The asymmetry of the entire spectrum acts as if effectively a partner of the zeroth LL exists at large energies. In the limit of zero orbital field, the Chern number reproduces the zero field result obtained in Ref. \([25]\). This demonstrates that the spectral asymmetry arises due to the inverted band structure. Furthermore, Eq. (22) reveals that the spectral asymmetry vanishes when the \( n = 0 \) LL spectrum loses all information on the band inversion. We therefore conclude that \( \eta_H \) contains information about the band inversion and, therefore, about the QAH phase in orbital fields. Let us emphasize that a conventional QH-effect is always characterized by an \( \eta_H = 0 \).

Now, it is straightforward to determine the bulk charge carrier density for an arbitrary (constant) chemical potential \( \mu \) [cf. Eq. (19)]:

\[
\begin{align*}
  j_\text{bulk}^0(\mu, H) &= -\frac{e}{S} \langle \Phi(\mu)|N|\Phi(\mu) \rangle \\
  &= -\frac{e}{S} \langle \Phi(\mu)|N_0|\Phi(\mu) \rangle + j_\text{bulk}^0(\mu = E_0, H),
\end{align*}
\]

where \( |\Phi(\mu)\rangle \) denotes a many-particle state for which all states are filled up to the chemical potential. According to Eq. (20), the term proportional to \( N_0 \) is entirely determined by the LL energies. Hence, we obtain

\[
-\frac{e}{S} \langle \Phi(\mu)|N_0|\Phi(\mu) \rangle = \left\{ -\kappa_{QH}^0 \Theta(|\mu| - |E_0|) \\
-\sum_{n=1}^{\infty} \kappa_{QH} \Theta(n(\mu - E_n^0)) \right\} H,
\]

where \( \kappa_{QH}^0 \) and \( \kappa_{QH} \) are defined in Eq. (4) of the main text.

Our results can be easily extended to include a Zeeman Hamiltonian \( h_0 = \sigma_z g_s H \), or an additional exchange Hamiltonian \( h_{\text{ex}} = \sigma_z g_s E_{\text{ex}}(H) \) \([22]\). We only need to apply the replacement \( M \rightarrow M + g_s H + G_{\text{ex}}(H) \). Since the extension to broken particle-hole symmetry is more tedious, further details will be given in Ref. \([38]\), where it will be shown that our results are in general unaltered by the \( D \)-parameter.

Finally, let us comment on the effective bulk Lagrangian \( L_{\text{eff}}^0(\mu, H) \) characterizing the response of our system to a small perturbing field \( a_\mu \) on top of an underlying background field \( H \). According to Eq. (23), this small perturbation induces an additional bulk charge carrier density \( j_{\text{ind}}^0 = \sigma_{xy} \nabla \times a \) on top of \( j_\text{bulk}^0 \). One can then deduce the missing two spatial components of the induced three current \( j_{\text{ind}}^0 \) by the requirement of Lorentz covariance \([12]\):

\[
  j_{\text{ind}}^0(\mu) = \sigma_{xy}(\mu, H) e^{\mu\rho} \partial_\rho a_\rho.
\]

Finally, the effective bulk Lagrangian follows from the fact that \( j_{\text{ind}}^0(\mu) = \delta S_{\text{eff}}^0(\mu, H)/\delta a_\mu \), where the induced effective action is given by \( S_{\text{eff}}^0 = \int d^3x L_{\text{eff}}^0 \).

**QAH vs. QH response**

In the following, we elaborate further on the interpretation of Eq. (4) of the main text (equivalently, Eq. (23)). For simplicity, we assume that \( D = 0 \) and that the electron charge \( e > 0 \).

Let us first focus on the trivial case \( M/B < 0 \), implying that only \( \kappa_{QH}^0 \) and \( \kappa_{QH} \) contribute to the total Hall conductivity \( \sigma_{xy} \) since \( \kappa_{QAH} = 0 \). In this case, the evolution of the LL energies as a function of the orbital field, given by Eqs. (13) and (14), is shown in Figs. 7(a) and (b), for positive and negative orbital fields, respectively. For \( n \geq 1 \), all LLs come in pairs \( E_n^\pm \) indicating that every

![FIG. 7. Evolution of bulk LL energies is depicted as function of orbital field (LL fan). For (a) and (c) \( H > 0 \), while for (b) and (d) \( H < 0 \). Black numbers indicate the total Hall conductivity \( \sigma_{xy} \) [e²/h] in the respective region. The Hall conductivity changes only if a LL (blue line) is crossed. For (a) and (b) we use \( M = 0 \) and \( B = -685 \text{ meVnm}^2 \), or \( D = 0 \) and \( A = 365 \text{ meVnm}^2 \). For (c) and (d) the same parameters as in (a) and (b) are used except for \( M = -10 \text{ meV} \), resulting in \( \kappa_{QAH} = 0 \). Below this critical field the Chern number is placed such that \( |\mu| < |E_0| \) (depicted by green shaded area), the Hall conductivity remains invariant for \( H \rightarrow -H \). For clarity, the \( n = 0 \) LL and the first pair of LLs with \( n = 1 \) is explicitly marked in all plots.](image-url)
conduction band LL has a partner in the valence band. According to Eq. (4c), each of these valence (conduction) band LL contributes $\sigma_{xy} = +e^2/h$ to the total Hall conductivity for $H > 0$. For $H < 0$, they contribute with the opposite sign. We identified this sgn(eH) dependence in the main text as the characteristic feature of a conventional QH / LL response. The sgn(eH) dependence also holds for the $n = 0$ LL, however with the peculiar difference that it lacks a partner [3].

Thus, it follows that the building blocks of the Chern Hamiltonian transform under parity via

$$\psi^\dagger(r) M \sigma_x \psi(r) \xrightarrow{P} -\psi^\dagger(r) M \sigma_x \psi(r),$$

$$\psi^\dagger(r) B k^2 \sigma_x \psi(r) \xrightarrow{P} -\psi^\dagger(r) B k^2 \sigma_x \psi(r),$$

$$\psi^\dagger(r) D k^2 \sigma_0 \psi(r) \xrightarrow{P} \psi^\dagger(r) D k^2 \sigma_0 \psi(r),$$

$$\psi^\dagger(r) A k_x \sigma_y \psi(r) \xrightarrow{P} \psi^\dagger(r) A k_x \sigma_y \psi(r).$$

Since the Dirac mass operator $\psi^\dagger(r) M \sigma_x \psi(r)$, as well as the effective mass operator $\psi^\dagger(r) B k^2 \sigma_x \psi(r)$ change sign under $P$, they break parity symmetry explicitly.

In Eq. (3) of the main text, we constructed an effective action for a Chern insulator in terms of a small perturbing vector potential $a_\mu$. This was done by effectively integrating out the fermionic sector. Using this procedure, a Chern-Simons term of odd parity was induced [36]:

$$e^{\mu \nu \rho} a_\mu \partial_\nu a_\rho \xrightarrow{P} -e^{\mu \nu \rho} a_\mu \partial_\nu a_\rho.$$  

For zero orbital field, this parity-breaking Chern-Simons term arose from the parity breaking mass terms $M$ and $B$, as discussed above. Hence for $H = 0$, we expect that the Chern-Simons level is exclusively a function of these parameters. In particular, we find that $C_{\text{QAH}} = [\text{sgn}(M) + \text{sgn}(B)]/2$, which is in agreement with Ref. [24].

### D. Numerical approach

This appendix gives more details on the numerical approach, which was employed to study QAH insulators in time-dependent orbital fields $H(t)$. In particular, we show how we were able to visualize the charge pumping shown in Figs. 1 and 2 of the main text. We apply the Peierls substitution in the Landau gauge to introduce the orbital field $H(t)$. This implies that $k_x$ is a good quantum number and, therefore, enables us to write the Hamiltonian and its corresponding Hilbert space as a direct sum:

$$H(t) = \bigoplus_{k_x} H_{k_x}(t).$$

The numerical simulations can be therefore carried out on each Hilbert subspace $H_{k_x}$ separately.

At the initial time $t_i$, the eigenstates of Eq. (9) are solutions of the time-independent Schrödinger equation:

$$H(t_i) |\psi_{j, k_x}(t_i)\rangle = E_{j, k_x}(t_i) |\psi_{j, k_x}(t_i)\rangle.$$  

The orbital field is now increased as a function of time and the evolution of all eigenstates is traced via the time-dependent Schrödinger equation,

$$i\hbar \partial_t |\psi_{j, k_x}(t)\rangle = H(t) |\psi_{j, k_x}(t)\rangle.$$  

C. Parity symmetry

In this appendix, we systematically analyze in which way the building blocks of the Chern Hamiltonian in Eq. (9) change under parity transformation. In $(2+1)$D, a parity transformation $P$ is defined via [36]

$$r = (t, x, y) \xrightarrow{P} (t, -x, y) = \tilde{r},$$

$$k = (\omega, k_x, k_y) \xrightarrow{P} (\omega, -k_x, k_y) = \tilde{k}.$$  

From this, we can deduce that the fermionic spinor operators transform under parity as [36]

$$\psi(r) \xrightarrow{P} \sigma_y \psi(\tilde{r}).$$
We compute the time-evolution of eigenstates numerically, using an iterative procedure:

\[ |\psi_{j,k_{z}}(t+\Delta t)\rangle = e^{-i\mathcal{H}_{t} \Delta t / \hbar} |\psi_{j,k_{z}}(t)\rangle \]

where \( U(t_{1}, t_{2}) \) denotes the unitary time evolution of each state from \( t_{1} \rightarrow t_{2} \) and \( \Delta t \) has to be chosen small enough to ensure convergence. After the time \( t \), we obtain

\[ |\psi_{j,k_{z}}(t)\rangle = U(t, t_{1}) |\psi_{j,k_{z}}(t_{1})\rangle . \tag{24} \]

We apply this iterative procedure to analyze the evolution of the following non-interacting, many-particle state in orbital fields, where all states are filled up to the chemical potential \( \mu \):

\[ |\Phi(\mu, t = t_{1})\rangle = \prod_{j \leq j_{\text{max}} \atop k_{z} \leq k_{\text{max}}} |\psi_{j,k_{z}}(t_{1})\rangle , \]

where \( j_{\text{max}} \) and \( k_{z_{\text{max}}} \) are determined by a given \( \mu \).

Now, tracing these initially filled states via Eq. (24) enables us to determine two characteristic, time-dependent quantities. Firstly, we can compute the induced charge density distribution, which was used in Fig. 2 of the main text, to study charge flow in QAH insulators:

\[ j_{\text{ind}}^{0}(\mathbf{x}, t) = -e \sum_{j \leq j_{\text{max}} \atop k_{z} \leq k_{\text{max}}} \psi_{j,k_{z}}^{\dagger}(\mathbf{x}, t) \psi_{j,k_{z}}(\mathbf{x}, t) - j_{\text{back}}^{0} , \tag{25} \]

where \( j_{\text{back}}^{0} \) ensures \( j_{\text{ind}}^{0}(\mathbf{x}, t) = 0 \). Secondly, we can identify the states which are responsible for this charge flow. Therefore, we trace the filling probabilities of each instantaneous eigenstate at time \( t \), in the time-independent Schrödinger equation \( \mathcal{H}(t) |\phi_{j,k_{z}}(t)\rangle = E_{j,k_{z}}(t) |\phi_{j,k_{z}}(t)\rangle \).

Note that here \( t \) is not a dynamical variable, defining the time evolution of states as in Eq. (24), but rather parametrizes the eigensystem of the Hamiltonian at time \( t \). In particular, the occupation probability of an eigenstate \( |\phi_{j,k_{z}}(t)\rangle \), depicted in Fig. 1 of the main text, is given by:

\[ P_{j,k_{z}}(t) = \sum_{j \leq j_{\text{max}}} |\langle \psi_{j,k_{z}}(t) |\phi_{j,k_{z}}(t)\rangle|^{2} . \tag{26} \]

This quantity can be used to recast Eq. (25) to the form, shown in the main text:

\[ j_{\text{ind}}^{0}(\mathbf{x}, t) = -e \sum_{i,k_{z}} P_{i,k_{z}}(t) |\phi_{i,k_{z}}(\mathbf{x}, t)\rangle^{2} - j_{\text{back}}^{0} . \]

If our results are supposed to be experimentally accessible, the results shown in Fig. 1 and 2 of the main text should not depend on how fast we increase the orbital field. In the numerical approach, we raise \( H(t) \) within the time interval \([t_{1} = 0, t_{2}]\) corresponding to a ramping speed

\[ v_{\text{ramp}} = \frac{H_{\text{max}}}{t_{f}} \quad \text{with} \quad H_{\text{max}} = H(t_{f}) . \tag{27} \]

For a fixed \( H_{\text{max}} \), different ramping speeds can be therefore simulated by varying \( t_{f} \). Our results in Fig. 1 and 2 of the main text fulfill the following requirements: the ramping time \( t_{f} \) and, therefore, the ramping speed \( v_{\text{ramp}} \) have to be chosen such that

\[ t_{f_{\text{min}}}^{(i)} \ll t_{f} \ll t_{f_{\text{max}}}^{(ii)} , \tag{28} \]

where (ii) only needs to be fulfilled for \( H > H_{\text{scat}} \) [Fig.1 (c) of the main text].

(i) The lower (upper) bound on \( t_{f} (v_{\text{ramp}}) \) results from the fact that particles should not be excited between bulk bands. Therefore, \( H(t) \) has to be increased on a time scale which is adiabatic with respect to any bulk energy gap \( E_{g} \). In particular, this implies that \( t_{f_{\text{min}}} \ll t_{f} \) with

\[ t_{f_{\text{min}}} = \frac{\hbar}{E_{g}} . \tag{29} \]

In order to overcome \( E_{g} \), which can be on the order of a few tenth of meV, we would need to ramp up \( H_{\text{max}} \) (a few Tesla) on a very small time scale \( t_{f} \ll 10^{-13} \text{s} \).

(ii) The upper (lower) bound on \( t_{f} (v_{\text{ramp}}) \) is caused by the fact that, for \( H > H_{\text{scat}} \), unoccupied QAH edge states and occupied bulk LLs form finite hybridization gaps \( \Delta_{\text{Hyb}} \) as the QAH edge states are lowered in energy with increasing \( H(t) \) [cf. Fig. 1(c) of the main text]. Our goal is to ensure that the QAH edge states and all bulk LLs separately maintain their initial filling probabilities throughout this process. As a result, for \( H > H_{\text{scat}} \), we need to choose \( t_{f} \) such that we diabatically overcome \( \Delta_{\text{Hyb}} \). Diabatically means that neither the filling probabilities nor the local densities of the QAH edge states

\[ \text{FIG. 8. Finite size gap } \Delta_{\text{Hyb}}(L_{y}) \text{ for } k_{x} = 0 \text{ and } H_{\text{scat}} < H < H_{\text{crit}}, \text{ forming between QAH edge states and the } n = 0 \text{ LL, as a function of the system length } L_{y} \text{ (depicted in Log-Log plot). Parameters are taken from the caption of Fig. 1 in the main text. The linear behavior shows that } \Delta_{\text{Hyb}}(L_{y}) \text{ drops exponentially with } L_{y}. \text{ Here, we find that } \Delta_{0} = 100 \text{ meV, and } \lambda = 0.05 \text{ nm}^{-1} \text{ [cf. Eq. (30)]. We obtain analogous results for } k_{x} \neq 0. \]
and the bulk LL wave functions change, if they pass each other with increasing orbital field. Analogous to Eq. (29), this implies that $t_f \ll t_f^{\text{max}}$ with

$$t_f^{\text{max}} = \frac{\hbar}{\Delta_{\text{Hyb}}}.$$

If $\Delta_{\text{Hyb}}$ are finite size gaps [cf. App. A.2], satisfying

$$\Delta_{\text{Hyb}}(L_y) = \Delta_0 \ e^{-\lambda L_y} \quad \text{with} \quad \lambda > 0,$$  \hspace{1cm} (30)

time-scales, which are experimentally possible to reach, become accessible since $t_f^{\text{max}}$ increases exponentially. As shown in Fig. 8 we find for $H_{\text{scat}} < H < H_{\text{crit}}$ and $k_x = 0$ (same holds for $k_x \neq 0$) that energy gaps forming between QAH edge states and the $n = 0$ LL are exponentially small. In a typical macroscopic Hall bar [26], the system length can be on the order of $L_y \approx 10 \mu m$ [26], which implies that $t_f^{\text{max}}$ can be approximately infinite compared to all other experimental time scales. It is therefore plausible that even for $H > H_{\text{scat}}$ the QAH charge pumping could be experimentally observable in macroscopically large systems provided that scattering between the QH and the QAH edge states can be strongly suppressed. However, in a conventional device elastic and inelastic scattering events between the QAH and the QH edge states cause relaxation of the charge inversion which ultimately leads to a transition to region II indicated in Fig. 3 of the main text.