Gravitational Collapse of Perfect Fluid

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Abstract

The spherical gravitational collapse of a compact packet consisting of perfect fluid is studied. The spacetime outside the fluid packet is described by the out-going Vaidya radiation fluid. It is found that when the collapse has continuous self-similarity the formation of black holes always starts with zero mass, and when the collapse has no self-similarity, the formation of black holes always starts with a finite non-zero mass. The packet is usually accompanied by a thin matter shell. The effects of the shell on the collapse are also studied.

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I. INTRODUCTION

Critical phenomena in gravitational collapse have attracted much attention since the pioneering work of Choptuik. From the known results the following emerges: In general critical collapse of a fixed matter field can be divided into three different classes according to the self-similarities that the critical solution possesses. If the critical solution has no self-similarity, continuous or discrete, the formation of black holes always starts with a mass gap (Type I collapse), otherwise it will start with zero mass (Type II collapse), and the mass of black holes takes the scaling form $M_{BH} \propto (P - P^*)^\gamma$, where $P$ characterizes the strength of the initial data. In the latter case, the collapse can be further divided into two subclasses according to whether the critical solution has continuous self-similarity (CSS) or discrete self-similarity (DSS). Because of this difference, the exponent $\gamma$ is usually also different. Whether the critical solution is CSS, DSS, or none of them, depending on both the matter field and the regions of the initial data space. The co-existence of Type I and Type II collapse was first found in the SU(2) Einstein-Yang-Mills case, and later extended to both the Einstein-scalar case and the Einstein-Skyrme case, while the co-existence of CSS and DSS critical solutions was found in the Brans-Dicke theory. The uniqueness of the exponent in Type II collapse is well understood in terms of perturbations, and is closely related to the fact that the critical solution has only one unstable mode. This property now is considered as the main criterion for a solution to be critical.

While the uniqueness of the exponent $\gamma$ crucially depends on the numbers of the unstable modes of the critical solution, that whether or not the formation of black holes starts with a mass gap seemingly only depends on whether the spacetime has self-similarity or not. Thus, even the collapse is not critical, if a spacetime has CSS or DSS, the formation of black holes may still turn on with zero mass. If this speculation is correct, it may have profound physical implications. For example, if Nature forbids the formation of zero-mass black holes, which are essentially naked singularities, it means that Nature forbids solutions with self-similarity. To study this problem in its generality term, it is found difficult.
will be referred as Paper I, gravitational collapse of massless scalar field and radiation fluid is studied, and it was found that when solutions have CSS, the formation of black holes indeed starts with zero-mass, while when solutions have no self-similarity it starts with a mass gap.

In this Letter, we shall generalize the studies given in Paper I to the case of perfect fluid with the equation of state \( p = k\rho \), where \( \rho \) is the energy density of the fluid, \( p \) the pressure, and \( k \) an arbitrary constant, subjected to \( 0 \leq k \leq 1 \). We shall show that the emerging results are consistent with the ones obtained in Paper I. Specifically, we shall present two classes of exact solutions to the Einstein field equations that represent spherical gravitational collapse of perfect fluid, one has CSS, and the other has neither CSS nor DSS. It is found that such formed black holes usually do not have finite masses. To remedy this shortage, we shall cut the spacetime along a time-like hypersurface, say, \( r = r_0(t) \), and then join the internal region \( r \leq r_0(t) \) with an asymptotically flat out-going Vaidya radiation fluid, using Israel’s method [11]. It turns out that in general such a junction is possible only when a thin matter shell is present on the joining hypersurface [12]. Thus, the finally resulting spacetimes will represent the collapse of a compact packet of perfect fluid plus a thin matter shell. The effects of the thin shell on the collapse are also studied. It should be noted that by properly choosing the solution in the region \( r \geq r_0(t) \), in principle one can make the thin shell disappear, although in this Letter we shall not consider such possibilities. The notations will closely follow the ones used in Paper I.

II. EXACT SOLUTIONS REPRESENTING GRAVITATIONAL COLLAPSE OF PERFECT FLUID

In this section, we shall present two classes of solutions to the Einstein field equations,

\[
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = (\rho + p)u_\mu u_\nu - pg_{\mu\nu},
\]

where \( u_\mu \) is the four-velocity of the perfect fluid considered. The general metric of spherically symmetric spacetimes that are conformally flat is given by [10].
\[
\frac{ds^2}{\dd{t}^2 - \dd{r}^2} = G(t, r) \left[ dt^2 - h^2(t, r) \left( dr^2 + r^2 d\Omega^2 \right) \right],
\]

where \( d\Omega^2 \equiv d\theta^2 + \sin^2 \theta d\varphi^2 \), \( \{x^\mu\} \equiv \{t, r, \theta, \varphi\} \) (\( \mu = 0, 1, 2, 3 \)) are the usual spherical coordinates, and

\[
h(t, r) = \begin{cases} 
1, \\
(f_1(t) + r^2)^{-1},
\end{cases}
\]

with \( f_1(t) \neq 0 \). The Friedmann-Robertson-Walker (FRW) metric corresponds to \( G(t, r) = G(t) \) and \( f_1(t) = \text{Const.} \). The corresponding Einstein field equations are given by Eqs. (2.20) - (2.23) in Paper I. Integrating those equations, we find two classes of solutions. In the following, we shall present them separately.

\textbf{α) Class A Solutions:} The first class of the solutions is given by

\[
G(t, r) = (1 - Pt)^{2\xi}, \quad h(r) = 1, \quad p = k\rho = 3k\xi^2 P^2 (1 - Pt)^{-2(\xi + 1)}, \quad u_\mu = (1 - Pt)^{-\xi} \delta_\mu^t,
\]

where \( P \) is a constant and characterizes the strength of the solutions (See the discussions given below), and \( \xi \equiv 2/(1 + 3k) \). This class of solutions is actually the FRW solutions and has CSS symmetry [13]. However, in this Letter we shall study them in the context of gravitational collapse.

To study the physical properties of these solutions, following Paper I we consider the following physical quantities,

\[
m^f(t, r) \equiv \frac{R}{2}(1 + R_{,\alpha}R_{,\beta}g^{\alpha\beta}) = \frac{\xi^2 P^2 \rho^3}{2(1 - Pt)^{2-\xi}},
\]

\[
\mathcal{R} \equiv R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{18\xi^2(1 + \xi^2)P^4}{(1 - Pt)^{4(1+\xi)}},
\]

where \( R \) is the physical radius of the two sphere \( t, r = \text{Const.} \), and \( m^f(t, r) \) is the local mass function [14]. From Eq.(4) we can see that the spacetime is singular on the space-like hypersurface \( t = P^{-1} \). The nature of the singularity depends on the signature of the parameter \( P \). When \( P < 0 \), it is naked, and the corresponding solutions represent white holes [10]. When \( P = 0 \), the singularity disappears and the corresponding spacetime is
Minkowski. When $P > 0$, the singularity hides behind the apparent horizon, which locates on the hypersurface,

$$r = r_{AH} \equiv \frac{1 - Pt}{\xi P}, \quad (5)$$

with $r_{AH}$ being a solution of the equation $R_{,\alpha} R_{,\beta} g^{\alpha \beta} = 0$. Thus, in the latter case the solutions represent the formation of black holes due to the gravitational collapse of the fluid. The corresponding Penrose diagram is similar to that given by Fig.1(a) in Paper I. Note that, although the spacetime singularity is always space-like, the nature of the apparent horizon depends on the choice of the parameter $k$. In fact, when $1/3 < k \leq 1$, it is space-like; when $k = 1/3$, it is null; and when $0 \leq k < 1/3$, it is time-like. Substituting Eq.(5) into Eq.(4), we find that $m^f_{AH}(t, r_{AH}) = (P \xi)^{r_{AH}^{1+\xi}}$. Thus, as $r_{AH} \to +\infty$, we have $m^f_{AH} \to +\infty$. That is, the total mass of the black hole is infinitely large. To get a physically reasonable model, one way is to cut the spacetime along a time-like hypersurface, say, $r = r_0(t)$, and then join the part $r \leq r_0(t)$ with one that is asymptotically flat [3]. We shall consider such junctions in the next section.

\(\beta\) Class B Solutions: The second class of solutions are given by

$$G(t, r) = \sinh^{2\xi} \left[2\alpha\xi^{-1}(t_0 - \epsilon t)\right], \quad h(r) = (r^2 - \alpha^2)^{-1}, \quad (6)$$

where $\epsilon = \pm 1$, $\xi$ is defined as in Eq.(3), $t_0$ and $\alpha(\equiv \sqrt{-f_1})$ are constants. Introducing a new radial coordinate $\bar{r}$ by $d\bar{r} = h(r)dr$, the corresponding metric can be written in the form

$$ds^2 = \sinh^{2\xi}[2\xi^{-1}(t_0 - \epsilon t)] \left\{dt^2 - dr^2 - \frac{\sinh^2(2r)}{4}d^2\Omega\right\}. \quad (7)$$

Note that in writing the above equation we had, without loss of generality, chosen $\alpha = 1$, and dropped the bar from $\bar{r}$. The energy density and four-velocity of the fluid are given, respectively, by

$$p = k\rho = 12k \sin^{-2(\xi+1)}[2(t_0 - \epsilon t)/\xi], \quad u_\mu = \sin^{-\xi}[2(t_0 - \epsilon t)/\xi]\delta^\mu_\tau, \quad (8)$$

while the relevant physical quantities are given by
\[ m^f(r,t) = \frac{1}{4} \sinh^3(2r) \sinh^{\xi-2} \left[ 2\xi^{-1}(t_0 - \epsilon t) \right], \]

\[ \mathcal{R} = 288 \left( 1 + \xi^2 \right) \xi^{-2} \sinh^{-4(\xi+1)} \left[ 2\xi^{-1}(t_0 - \epsilon t) \right]. \tag{9} \]

The apparent horizon now is located at

\[ r = r_{AH} \equiv \xi^{-1}(t_0 - \epsilon t). \tag{10} \]

From Eq. (9) we can see that the solutions are singular on the hypersurface \( t = \epsilon t_0 \). When \( \epsilon = -1 \) it can be shown that the corresponding solutions represent cosmological models with a naked singularity at the initial time \( t = -t_0 \), while when \( \epsilon = +1 \) the singularity is hidden behind the apparent horizon given by Eq. (10), and the solutions represent the formation of black holes due to the collapse of the fluid. In the latter case the total mass of black holes is also infinitely large. To remedy this shortage, in the next section we shall make “surgery” to this spacetime, so that the resulting black holes have finite masses.

### III. MATCHING THE SOLUTIONS WITH OUTGOING VAIKYA SOLUTION

In order to have the black hole mass finite, we shall first cut the spacetimes represented by the solutions given by Eqs. (3), and (7) along a time-like hypersurface, and then join the internal part with the out-going Vaidya radiation fluid. In the present two cases since the perfect fluid is comoving, the hypersurface can be chosen as \( r = r_0 = \text{const} \). Thus, the metric in the whole spacetime can be written in the form

\[ ds^2 = \begin{cases} A(t,r)^2 dt^2 - B(t,r)^2 dr^2 - C(t,r)^2 d\Omega^2, & (r \leq r_0), \\ \left( 1 - \frac{2m(v)}{R} \right) dv^2 + 2dvR^2 - R^2 d\Omega^2, & (r \geq r_0), \end{cases} \tag{11} \]

where the functions \( A(t,r) \), \( B(t,r) \) and \( C(t,r) \) can be read off from Eqs. (1), (3) and (7). On the hypersurface \( r = r_0 \) the metric reduces to

\[ ds^2 |_{r=r_0} = g_{ab} d\xi^a d\xi^b = d\tau^2 - R(\tau)^2 d\Omega^2, \tag{12} \]

where \( \xi^a = \{\tau, \theta, \varphi\} \) are the intrinsic coordinates of the surface, and \( \tau \) is defined by
\[ d\tau^2 = A^2(t, r_0)dt^2 = \left(1 - \frac{2M(\tau)}{R}\right)dv^2 + 2dv dR, \]  

(13)

where \(v\) and \(R\) are functions of \(\tau\) on the surface, and \(R(\tau) \equiv C(t, r_0), \ M(\tau) \equiv m(v(\tau)).\)

The extrinsic curvature on the two sides of the surface defined by

\[ K^\pm_{ab} = -n^\pm_\alpha \left[ \frac{\partial^2 x^\alpha}{\partial \xi^a \partial \xi^b} - \Gamma^\pm_{\alpha \beta \delta} \frac{\partial x^\beta}{\partial \xi^a} \frac{\partial x^\delta}{\partial \xi^b} \right], \]

(14)

has the following non-vanishing components \[13\]

\[ K^-_{\tau\tau} = -\frac{i^2 A_a A}{B}, \quad K^-_{\theta\theta} = \sin^{-2} \theta K^-_{\phi\phi} = \frac{C_a C}{B}, \]

\[ K^+_{\tau\tau} = \frac{\dot{v}}{\bar{v}} - \frac{\dot{v}M(\tau)}{R^2}, \quad K^+_{\theta\theta} = \sin^{-2} \theta K^+_{\phi\phi} = R \left\{ \dot{v} \left(1 - \frac{2M(\tau)}{R}\right) + \dot{R} \right\}, \]

(15)

where \(i \equiv dt/d\tau, (\ )_\mu \equiv \partial(\ )/\partial x^\mu\) and \(n^\pm_\alpha\) are the normal vectors defined in the two faces of the surface. Using the expression \[11\]

\[ [K_{ab}]^- - g_{ab} [K]^- = -8\pi \tau_{ab} \]

(16)

we can calculate the surface energy-momentum tensor \(\tau_{ab}\), where \([K_{ab}]^- = K^+_{ab} - K^-_{ab}, \ [K]^- = g^{ab} [K_{ab}]^-\), and \(g_{ab}\) can be read off from Eq.(12). Inserting Eq.(15) into the above equation, we find that \(\tau_{ab}\) can be written in the form

\[ \tau_{ab} = \sigma w_a w_b + \eta \left( \theta_a \theta_b + \phi_a \phi_b \right), \]

(17)

where \(w_a, \ \theta_a\) and \(\phi_a\) are unit vectors defined on the surface, given respectively by \(w_a = \delta^\tau_a, \ \theta_a = R\delta^\theta_a, \ \phi_a = R \sin \theta \delta^\phi_a\), and \(\sigma\) can be interpreted as the surface energy density, \(\eta\) the tangential pressure, provided that they satisfy certain energy conditions \[17\]. In the present case \(\sigma\) and \(\eta\) are given by

\[ \sigma = \frac{1}{4\pi R} \left\{ \dot{R} - \frac{1}{\bar{v}} + J'(r_0) \right\}, \]

\[ \eta = \frac{1}{16\pi R \bar{v}} \left\{ \bar{v}^2 - 2\bar{v}R - 2\bar{v}J'(r_0) + 1 \right\}, \]

(18)

where \(J(r) = r\) for Class A solutions, and \(J(r) = \sinh(2r)/2\) for Class B solutions, and a prime denotes the ordinary differentiation with respect to the indicated argument. Note
that in writing Eq. (18) we had used Eq. (13), from which it is found that the total mass of the collapsing ball, which includes the contribution from both the fluid and the shell, is given by

\[ M(\tau) = \frac{R}{2\dot{\theta}^2} \left( \ddot{\theta}^2 + 2\dot{\theta}\dot{R} - 1 \right). \]  

(19)

To fix the spacetime outside the shell we need to give the equation of state of the shell. In order to minimize the effects of the shell on the collapse, in the following we shall consider the case \( \eta = 0 \), which reads

\[ \dot{\theta}^2 - 2\ddot{\theta}R - 2J'(r_0)\dot{\theta} + 1 = 0. \]  

(20)

To solve the above equation, let us consider the two classes of solutions separately.

**A. Class A Solutions**

In this case, it can be shown that Eq. (20) has the first integral,

\[ \dot{\theta}(\tau) = \frac{x - 2(v_0 - 1)R_0}{x - 2v_0R_0}, \]  

(21)

where \( R(\tau) \equiv R_0x^\xi \), \( R_0 \equiv r_0P^{\frac{1}{\xi + 1}} \), \( x \equiv [(\xi + 1)(\tau_0 - \tau)]^{\frac{1}{\xi + 1}} \), and \( v_0 \) and \( \tau_0 \) are integration constants. Substituting the above expressions into Eq. (19), we find that

\[ M(x) = \frac{R_0^2x^{\xi - 1}}{[x - 2(v_0 - 1)R_0]^2} \left\{ (2 - \xi)x^2 + 2(\xi - 1)(2v_0 - 1)R_0x + 4\xi v_0(1 - v_0)R_0^2 \right\}. \]  

(22)

At the moment \( \tau = \tau_{AH} \) (or \( x = x_{AH} = \xi R_0 \)), the shell collapses inside the apparent horizon. Consequently, the total mass of the formed black hole is given by

\[ M_{BH} \equiv M(x_{AH}) = \frac{\xi^{\xi + 1}P^\xi}{[\xi - 2(v_0 - 1)]^2} \left\{ \xi(2 - \xi) + 2(\xi - 1)(2v_0 - 1) + 4v_0(1 - v_0) \right\}, \]  

(23)

which is finite and can be positive by properly choosing the parameter \( v_0 \) for any given \( \xi \). The contribution of the fluid and the thin shell to the black hole mass is given, respectively,
From the above equations we can see that all the three masses are proportional to $P$, the parameter that characterizes the strength of the initial data of the collapsing ball. Thus, when the initial data is very weak ($P\to 0$), the mass of the formed black hole is very small ($M_{BH}\to 0$). In principle, by properly tuning the parameter $P$ we can make it as small as wanted. Recall that now the solutions have CSS. It should be noted that due to the gravitational interaction between the collapsing fluid and the thin shell, we have $M_{BH} \neq m^f_{BH} + m^{shell}_{BH}$, unless $\xi = 2$, which corresponds to null dust. In the latter case, it can be shown that by choosing $v_0 = 1$ we can make the thin shell disappear, and the collapse is purely due to the null fluid. Like the cases with thin shell, by properly tuning the parameter $P$ we can make black holes with infinitesimal mass. When $\xi = 1/2$ or 1, which corresponds, respectively, to the massless scalar field or to radiation fluid, the solutions reduce to the ones considered in [10].

Note that although the mass of black holes takes a scaling form in terms of $P$, the exponent $\gamma$ is not uniquely defined. This is because in the present case the solution with $P = P^* = 0$ separates black holes from white holes, and the latter is not the result of gravitational collapse. Thus, the solutions considered here do not really represent the critical collapse. As a result, we can replace $P$ by any function $P(\bar{P})$, and for each of such replacements, we will have a different $\gamma$ [15]. However, such replacements do not change the fact that by properly tuning the parameter we can make black holes with masses as small as wanted.

1 While a unique definition of the total mass of a thin shell is still absent, here we simply define it as $m^{shell}_{BH} \equiv 4\pi R^2 \sigma$. Certainly we can equally use other definitions, such as $m^{shell}_{BH} \equiv M_{BH} - m^f_{BH}$, but our final conclusions will not depend on them.
B. Class B Solutions

In this case, the first integral of Eq.(20) yields

\[ \dot{v} = \cosh(2r_0) - \sinh(2r_0) \tanh(t + t_1), \]

(25)

where \( t_1 \) is an integration constant. At the moment \( t = t_{AH} \) the whole ball collapses inside the apparent horizon, and the contribution of the fluid and the shell to the total mass of the just formed black holes are given, respectively, by

\[
\begin{align*}
    m_{BH}^f &\equiv m_{AH}^f(\tau_{AH}) = \frac{1}{4} \sinh^{\xi+1}(2r_0), \\
    m_{BH}^{shell} &\equiv 4\pi R^2(\tau_{AH})\sigma(\tau_{AH}) = -\frac{1}{2} \sinh^{\xi+1}(2r_0) \frac{\cosh[t_1 - t_0 + \xi r_0]}{\cosh[t_1 - t_0 - (2 - \xi) r_0]}.
\end{align*}
\]

(26)

From the above expressions we can see that for any given \( r_0 \), \( m_{BH}^f \) and \( m_{BH}^{shell} \) are always finite and non-zero. Thus, in the present case black holes start to form with a mass gap. It should be noted that although \( m_{BH}^f \) is positive, \( m_{BH}^{shell} \) is negative. This undesirable feature makes the model very unphysical. One may look for other possible junctions. However, since the fluid is co-moving, one can always make such junctions across the hypersurface \( r = r_0 = Const. \). Then, the contribution of the collapsing fluid to the total mass of black holes will be still given by Eq.(26), and the total mass of the formed black hole then can be written in the form,

\[ M_{BH} = \frac{1}{4} \sinh^{\xi+2}(2r_0) + M_{rest}^{BH}, \]

where \( M_{BH}^{rest} \) denotes the mass contribution of the rest part of the spacetime, which is non-negative for any physically reasonable junction. Therefore, in this case for any physical junction, the total mass of black holes will be finite and non-zero.

IV. CONCLUSION

In this Letter, we have studied two classes of solutions to the Einstein field equations, which represent the spherical gravitational collapse of a packet of perfect fluid, accompanied
usually by a thin matter shell. The first class of solutions has CSS, and black holes always start to form with a zero-mass, while the second class has neither CSS nor DSS, and the formation of black holes always starts with a mass gap. The existence of the matter shell does not affect our main conclusions. These solutions provide further evidences to support our speculation that the formation of black holes always starts with zero-mass for the collapse with self-similarities, CSS or DSS.

It should be noted that none of these two classes of solutions given above represent critical collapse. Thus, whether the formation of black holes starts with zero mass or not is closely related to the symmetries of the collapse (CSS or DSS), rather than to whether the collapse is critical or not.

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