CALCULATING WITH TOPOLOGICAL ANDRÉ-QUILLEN THEORY, I: HOMOTOPICAL PROPERTIES OF UNIVERSAL DERIVATIONS AND FREE COMMUTATIVE S-ALGEBRAS

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Abstract. We adopt the viewpoint that topological André-Quillen theory for commutative $S$-algebras should provide usable (co)homology theories for doing calculations in the sense traditional within Algebraic Topology. Our main emphasis is on homotopical properties of universal derivations, especially their behaviour in multiplicative homology theories. There are algebraic derivation properties, but also deeper properties arising from the homotopical structure of the free algebra functor $\mathbb{P}_R$ and its relationship with extended powers of spectra. In the connective case in ordinary mod$p$ homology, this leads to useful formulae involving Dyer-Lashof operations in the homology of commutative $S$-algebras. Although many of our results could no doubt be obtained using stabilisation, our approach seems more direct. We also discuss a reduced free algebra functor $\tilde{\mathbb{P}}_R$.

Contents

Introduction 2
Notation, etc 2
1. Recollections on topological André-Quillen theory 2
2. Homotopical properties of universal derivations 3
3. The free commutative algebra functor 5
4. Power operations and the free functor 8
5. The reduced free commutative algebra functor 13
6. The ordinary homology of free commutative $S$-algebras 17
7. The free commutative $S$-algebra functor and $\Omega^\infty \Sigma^\infty \mathbb{Z}$ 22
8. Some calculations 23
Appendix A. A proof and a Lemma 29
Appendix B. Some formulae 31
References 32

Date: 07/08/2013 version 7

2010 Mathematics Subject Classification. Primary 55P43; Secondary 13D03, 55N35, 55P48.

Key words and phrases. $S$-module, $S$-algebra, cell algebra, topological André-Quillen (co)homology, power operations.

Part of this work was carried out in the period 2007–8 when the author was supported by a YFF Norwegian Research Council grant while at the University of Oslo, and later while receiving an EPSRC research grant. The author would like to thank Maria Basterra, Marcel Bökstedt, Bob Bruner, Helen Gilmour, Nick Kuhn, Tyler Lawson, Mike Mandell, Peter May, Birgit Richter, Constanze Roitzheim, Markus Szymik and John Rognes for helpful comments and encouragement over many years, and especially Philipp Reinhard for detailed comments and also supplying the proof in Appendix A.

This research was supported by funding from RCUK.
Introduction

Topological André-Quillen homology and cohomology theories for commutative $S$-algebras were introduced by Maria Basterra, building on ideas of Igor Kříž as well as algebraic André-Quillen theory. Subsequent work, both individually and jointly in various combinations, by Basterra, Gilmour, Goerss, Hopkins, Kuhn, Lazarev, Mandell, McCarthy, Minasian, Reinhard, Richter, Robinson, Whitehouse as well as the present author, has laid out the basic structure and provided key relationships with other areas.

In this work we continue to adopt the viewpoint of [4], regarding TAQ as providing usable (co)homology theories for doing calculations in the sense traditional within Algebraic Topology.

Our main emphasis is on homotopical properties of universal derivations, especially their behaviour in multiplicative homology theories. As the name suggests, there are algebraic derivation properties, but also deeper properties arising out of the homotopical structure of the free algebra functor and its relationship with extended powers of spectra. In the connective case and in ordinary mod $p$ homology, this leads to useful formulae involving Dyer-Lashof operations in the homology of commutative $S$-algebras. It seems likely that many of these results are obtainable using stabilisation, but our approach seems more direct. We remark that work of Mike Mandell [21] suggests that it might be more natural to replace commutative $S$-algebras by algebras in $M$ over his operad $G$ and work with those. Related results on the homology of the free commutative $S$-algebra functor also appear in work of Nick Kuhn & Jason McCarthy [17,18].

We also discuss a reduced free algebra functor $\tilde{P}_R$ which we learnt of from Tyler Lawson. This takes as input $R$-modules under a fixed cofibrant replacement for the $R$-module $R$ and gives rise to a Quillen adjunction.

$$C_R \xrightarrow{\tilde{P}_R} S^0_R/\mathcal{M}_R \xleftarrow{U} S_R/\mathcal{M}_R$$

We will use this in a sequel to study spectral sequences related to those studied by Maria Basterra [7, section 5] and Haynes Miller [26].

We give some sample calculations, but our main concern is with laying the groundwork for future applications.

In two brief appendices we supply a proof of a basic result, an adjunction result, and some formulae for calculating Dyer-Lashof operations.

Notation, etc. When working over a fixed commutative ground ring $k$ such as $\mathbb{F}_p$, we often write $\otimes$ for $\otimes_k$, Hom for $\text{Hom}_k$, etc.

1. Recollections on topological André-Quillen theory

We will assume the reader is familiar with Basterra’s foundational paper [7] and the further development of its ideas in [4]. All of this is founded on the notions of $S$-modules and commutative $S$-algebras of [12]. We briefly spell out some of the main ingredients.

If $R$ is a commutative $S$-algebra, then its category of (left) $R$-modules $\mathcal{M}_R$ is a model category and the category of commutative $R$-algebras $C_R$ consists of the commutative monoids in $\mathcal{M}_R$ with monoidal morphisms. There is a free $R$-algebra functor $F_R: \mathcal{M}_R \rightarrow C_R$ left adjoint to the forgetful functor $U: C_R \rightarrow \mathcal{M}_R$, and this pair gives a Quillen adjunction. We denote the derived (or homotopy) categories by $\bar{\mathcal{M}}_R, \bar{\mathcal{C}}_R$. 


For a pair of commutative $R$-algebras $A \rightarrow B$ there is a $B$-module $\Omega_A(B)$ which is well defined up to isomorphism in $\mathcal{M}_B$. This comes with a canonical morphism in $\mathcal{M}_A$, the universal derivation

$$\delta_{(B,A)}: B \rightarrow \Omega_A(B),$$

caraketerised by a natural isomorphism

$$\mathcal{T}_{\mathcal{M}_A}/B(B,B \vee X) \cong \mathcal{T}_{\mathcal{M}_B}(\Omega_A(B), X),$$

where $X \in \mathcal{M}_B$ and $B \vee X$ denotes the square zero extension of $B$ by $X$ viewed as a $B$-algebra over $B$.

Topological André-Quillen homology and cohomology with coefficients in a $B$-module $M$ are defined by

$$TAQ_\ast(B, A; M) = \pi_\ast(M \wedge_B \Omega_A(B)),$$

$$TAQ^\ast(B, A; M) = \pi_{-\ast}(\mathcal{F}_B(\Omega_A(B), M)) = \mathcal{T}_{\mathcal{M}_B}(\Omega_A(B), M)^\ast,$$

where

$$\mathcal{T}_{\mathcal{M}_B}(\Omega_A(B), X)^n = \mathcal{T}_{\mathcal{M}_B}(\Omega_A(B), \Sigma^n X).$$

When $E$ is a (unital) $B$ ring spectrum, the composition

$$\theta$$

(1.1) $\pi_\ast(B) \xrightarrow{E_\ast(B)} \xrightarrow{E_\ast(\Omega_A(B))} E^B_\ast(\Omega_A(B)) \xrightarrow{\phi} \mathcal{T}AQ_\ast(B, A; E)$

is the TAQ-Hurewicz homomorphism.

In [4] we showed how this could be interpreted as a cellular theory for cellular commutative $R$-algebras. A key ingredient was the basic observation that for an $R$-module $Z$,

$$\Omega_R[\mathbb{P}RZ] \cong \mathbb{P}RZ \wedge_R Z.$$

For completeness we give a proof of this in Appendix A.

In [4] we developed the theory of connective $p$-local commutative $S$-algebras along the lines of [5] for spectra, making crucial use of TAQ with coefficients in $HF_p$. In both of those works, one important outcome was the ability to detect minimal atomic objects using the vanishing of the appropriate Hurewicz homomorphism in positive degrees.

2. Homotopical properties of universal derivations

Let $A$ be a commutative $S$-algebra. As pointed out in [19], for a commutative $A$-algebra $B$, the universal derivation $\delta_{(B,A)}: B \rightarrow \Omega_A(B)$ is a homotopy derivation in the sense of the following discussion.

Let $R$ be a commutative $S$-algebra, let $E$ be an $R$ ring spectrum and let $M$ be a left $E$-module in the sense that there is a morphism $\mu: E \wedge_R M \rightarrow M$ in $\mathcal{T}_{\mathcal{M}_R}$ satisfying appropriate associativity and unital conditions. We denote the product on $E$ by $\phi: E \wedge_R E \rightarrow E$. 

3
**Definition 2.1.** A morphism $\partial: E \to M$ in $\mathcal{M}_R$ is a homotopy derivation if the following diagram in $\mathcal{M}_R$ commutes.

\[
\begin{array}{ccc}
E \wedge_R E & \xrightarrow{\varphi} & E \\
\downarrow & & \downarrow \partial \\
E \wedge_R M \vee M \wedge_R E & \xrightarrow{\mu} & M \\
\end{array}
\]

Now let $A$ be a commutative $S$-algebra and let $B$ be a commutative $A$-algebra. Following the remarks at end of [19, section 3], we recall that the universal derivation $\delta(B,A): B \to \Omega_A(B)$ is a morphism in the derived category of $A$-modules $\mathcal{M}_A$ which is also a homotopy derivation in the sense that the following diagram commutes in $\mathcal{M}_A$.

\[
\begin{array}{ccc}
B \wedge_A B & \xrightarrow{\prod} & B \\
\downarrow & & \downarrow \delta(B,A) \\
B \wedge_A \Omega_A(B) & \xrightarrow{\text{mult}} & \Omega_A(B) \\
\end{array}
\]

where elements of $\mathcal{M}_A(X,Y)$ are added in the usual way.

Now suppose that $E$ is a commutative $B$ ring spectrum; this implies that $E$ is a $B$-module and there is a unit morphism of $E$ ring spectra $B \to E$ in $\mathcal{M}_B$. Then on smashing with copies of $E$, (2.2) gives another commutative diagram

\[
\begin{array}{ccc}
E \wedge_A B \wedge_A E \wedge_A B & \xrightarrow{\text{switch}} & E \wedge_A B \\
\downarrow & & \downarrow \delta(B,A) \\
E \wedge_A B \wedge_A E \wedge_A B \wedge_A \Omega_A(B) & \xrightarrow{\text{prod} \times \text{mult}} & E \wedge_A \Omega_A(B) \\
\end{array}
\]

which shows that the commutative $E_\ast$-algebra $E_\ast^A B = \pi_\ast(E \wedge_A B)$ admits the $E_\ast$-module homomorphism

\[ (\delta(B,A))_\ast: E_\ast^A B \to E_\ast^A \Omega_A(B). \]

Of course $E_\ast^A \Omega_A(B)$ is also a left $E_\ast^B$-module since $\Omega_A(B)$ is a left $B$-module. Composing $(\delta(B,A))_\ast$ with the natural homomorphism $E_\ast^A \Omega_A(B) \to E_\ast^B \Omega_A(B)$, we obtain an $E_\ast$-module homomorphism

\[ \Delta(B,A): E_\ast^A B \to E_\ast^A \Omega_A(B) \to E_\ast^B \Omega_A(B). \]

We also have an augmentation $\varepsilon: E_\ast^A B \to E_\ast$ induced by applying $\pi_\ast(\cdot)$ to the evident composition

\[ E \wedge_A B \to E \wedge_A E \to E. \]

Clearly $\varepsilon$ is a morphism of $E_\ast$-algebras.
Lemma 2.2. $(\delta_{(B,A)})_*$ and $\Delta_{(B,A)}$ are $E_*$-derivations, so for $u,v \in E_* B$,

$$(\delta_{(B,A)})_*(uv) = u\delta_{(B,A)}(v) \pm v\delta_{(B,A)}(u),$$

$$\Delta_{(B,A)}(uv) = \varepsilon(u)\Delta_{(B,A)}(v) \pm \varepsilon(v)\Delta_{(B,A)}(u),$$

where the signs are determined from the degrees of $u,v$ with the usual sign convention. In particular, if $u,v \in \ker \varepsilon : E_* B \to E_*$, then

$$\Delta_{(B,A)}(uv) = 0,$$

so $\Delta_{(B,A)}$ annihilates non-trivial products.

Proof. This involves diagram chasing using the definitions. \qed

We will often write $\delta$ and $\Delta$ for $\delta_{(B,A)}$ and $\Delta_{(B,A)}$ when $(B,A)$ is clear from the context.

3. The free commutative algebra functor

For a $R$-module $X$ there is a free commutative $R$-algebra

$$\mathbb{P}_R X = \bigvee_{j \geq 0} X^{(j)}/\Sigma_j.$$ 

When $R = S$ or a localisation of $S$, we will set $\mathbb{P} = \mathbb{P}_S$.

If $X$ is cofibrant as an $R$-module then $\mathbb{P}_R X$ is cofibrant as a commutative $R$-algebra. The functor $\mathbb{P}_R$ is left adjoint to the forgetful functor $U : \mathcal{C}_R \to \mathcal{M}_R$, so for $A \in \mathcal{C}_R$,

$$\mathcal{C}_R(\mathbb{P}_R(-), A) \cong \mathcal{M}_R(-, A),$$

where $A = U A$ is regarded as an $R$-module. In fact,

$$(3.1) \quad \mathcal{C}_R \xrightarrow{\mathbb{P}_R} \mathcal{M}_R \xleftarrow{U} \mathcal{C}_R$$

is a Quillen adjunction [12].

As it is a left adjoint, $\mathbb{P}_R$ preserves colimits, including pushouts. As cell and CW $R$-modules are defined as iterated pushouts, applying $\mathbb{P}_R$ to the skeleta leads to cell or CW skeleta. To make this explicit, suppose that $X$ is an $R$-module with CW skeleta $X^{[n]}$, and attaching maps

$$j_n : \bigvee_i S^n_{i,R} \to X^{[n]},$$

where $S^n_{i,R} = \mathbb{P}_R S^n$ is the cofibrant model for the sphere spectrum in $\mathcal{M}_R$. Setting $D^n_{i,R} = \mathbb{P}_R D^n$, the $(n+1)$-skeleton $X^{[n+1]}$ is defined by the pushout diagram

$$\bigvee_i S^n_{i,R} \xrightarrow{\gamma} X^{[n]} \xleftarrow{\gamma} \bigvee_i D^n_{i,R} \to X^{[n+1]}$$

which induces the pushout diagram

$$\mathbb{P}_R\bigvee_i S^n_{i,R} \xrightarrow{\gamma} \mathbb{P}_R X^{[n]} \xleftarrow{\gamma} \mathbb{P}_R\bigvee_i D^n_{i,R} \to \mathbb{P}_R X^{[n+1]}.$$
in \( \mathcal{C}_R \). So we obtain a CW filtration on \( \mathbb{P}_R X \) with \( n \)-skeleton
\[
\mathbb{P}_R^{(n)}X = (\mathbb{P}_R X)^{(n)} = \mathbb{P}_R(X^n).
\]

By \cite{4} proposition 1.6, in the homotopy category of \( \mathbb{P}_R X \)-modules \( \mathcal{T}_R \mathcal{M}_{\mathbb{P}_R X} \),
\[
\Omega_R(\mathbb{P}_R X) \cong \mathbb{P}_R X \wedge_R X.
\]

The universal derivation
\[
\delta_{(\mathbb{P}_R X,R)} \in \mathcal{T}_R \mathcal{M}_R(\mathbb{P}_R X, \Omega_R(\mathbb{P}_R X)) \cong \mathcal{T}_R \mathcal{M}_R(\mathbb{P}_R X, \mathbb{P}_R X \wedge_R X)
\]
has the homotopy derivation property shown in the homotopy commutative diagram (2.1). Furthermore, \( \delta_{(\mathbb{P}_R X,R)} \) corresponds to the inclusion \( X \hookrightarrow \mathbb{P}_R X \wedge_R X \) under the sequence of isomorphisms
\[
\mathcal{T}_R \mathcal{C}_R/\mathbb{P}_R X(\mathbb{P}_R X, \mathbb{P}_R X \vee \Omega_R(\mathbb{P}_R X)) \cong \mathcal{T}_R \mathcal{M}_{\mathbb{P}_R X}(\Omega_R(\mathbb{P}_R X), \Omega_R(\mathbb{P}_R X))
\]
\[
\cong \mathcal{T}_R \mathcal{M}_{\mathbb{P}_R X}(\mathbb{P}_R X \wedge_R X, \mathbb{P}_R X \wedge_R X)
\]
\[
\cong \mathcal{T}_R \mathcal{M}_R(\mathbb{P}_R X, \mathbb{P}_R X \wedge_R X).
\]

(3.2)

We will describe \( \delta_{(\mathbb{P}_R X,R)} \) as a morphism in the homotopy category \( \mathcal{T}_R \mathcal{M}_R \) using this identification.

Nick Kuhn has pointed out that \cite{17} gives a closely related analysis of extended powers, and proves far more about their coproduct structure induced by the pinch map.

Suppose that \( X' \) is a second copy of \( X \). A representative for the homotopy class of the pinch map \( \operatorname{pi} : X \rightarrow X \vee X' \) induces morphisms of commutative \( R \)-algebras

\[
\begin{array}{ccc}
\mathbb{P}_R(X \vee X') & \xrightarrow{\mathbb{P}_R \operatorname{pi}} & \mathbb{P}_R X \vee \mathbb{P}_R X \wedge_R X'\\
\cong & & \cong
\end{array}
\]

\[
\mathbb{P}_R X \wedge_R \mathbb{P}_R X' \quad \mathbb{P}_R X \wedge_R (R \vee X')
\]

where \( R \vee X' \) and \( \mathbb{P}_R X \vee \mathbb{P}_R X \wedge_R X' \) are square zero extensions of \( R \) and \( \mathbb{P}_R X \) respectively, and the horizontal morphism kills the wedge summands \( (X')^r/\Sigma_r \) with \( r \geq 2 \). Restricting to the summand \( X \) in \( \mathbb{P}_R X \) we obtain the pinch map \( \operatorname{pi} \), and then on applying the isomorphism of (3.2) we find that the resulting composition

\[
\begin{array}{ccc}
\mathbb{P}_R X & \xrightarrow{\delta} & \mathbb{P}_R X \vee \mathbb{P}_R X \wedge_R X'\\
& & \mathbb{P}_R X \wedge_R (R \vee X')
\end{array}
\]

agrees with the universal derivation \( \delta_{(\mathbb{P}_R X,R)} \).

In the homotopy category of \( S \)-modules, \( \delta_{(\mathbb{P}_R X,R)} \) is equivalent to a coproduct of maps

\[
\delta_{(\mathbb{P}_R X,R),n} : ES\Sigma_n \wedge \Sigma_n X(n) \longrightarrow (ES\Sigma_{n-1} \wedge \Sigma_{n-1} X^{(n-1)}) \wedge X,
\]

where we have of course identified \( X' \) with \( X \). In fact these are the transfer maps \( \tau_{n-1,1} \) of \cite{11} definition II.1.4, i.e.,

\[
\delta_{(\mathbb{P}_R X,R),n} = \tau_{n-1,1} : ES\Sigma_n \wedge \Sigma_n X(n) \longrightarrow (ES\Sigma_{n-1} \wedge \Sigma_{n-1} X^{(n-1)}) \wedge X.
\]

(3.3)
The derivation property of $\delta_{[P_X,S]}$ is just a consequence of the commutativity of the diagram (3.4) below. We will give a brief explanation of this.

For detailed accounts of the stable homotopy theory involved, see [11, 20, 24]. We remark that in [11, chapter II, p. 24], the pinch map is referred to as the ‘diagonal’ since in the stable category finite products and coproducts coincide. To ease notation and exposition, we take $R = S$ and set $P = P_S$; however the general case is similar.

Let $E\Sigma_{m+n}$ be a free contractible $\Sigma_{m+n}$-space, and let $Y$ be a $\Sigma_{m+n}$-spectrum for $m, n \geq 1$; we are interested in the case where $Y = X^{(m+n)}$, the $(m + n)$-th smash power of $X$. The equivariant half smash product $E\Sigma_{m+n} \wedge Y$ is a free $\Sigma_{m+n}$-spectrum, and the evident inclusions of subgroups

\[
\Sigma_m \times \Sigma_n \xrightarrow{\iota} \Sigma_{m+n} \xrightarrow{i} \Sigma_{m+n-1}
\]

induce morphisms of spectra

\[
E\Sigma_{m+n} \wedge \Sigma_{m+n} Y \xrightarrow{\tau_{m,n}} E\Sigma_{m+n} \wedge \Sigma_{m+n} Y \xleftarrow{E\Sigma_{m+n} \wedge \Sigma_{m+n-1} Y}
\]
on orbit spectra. There are also transfer maps

\[
E\Sigma_{m+n} \wedge \Sigma_{m+n} Y \xrightarrow{\tau_{m,n}} E\Sigma_{m+n} \wedge \Sigma_{m+n} Y
\]

\[
E\Sigma_{m+n} \wedge \Sigma_{m+n} Y \xrightarrow{\tau_{m+n-1,1}} E\Sigma_{m+n} \wedge \Sigma_{m+n-1} Y
\]

associated with these inclusions of subgroups. We will use the double coset formula of [20, §IV.6]. We are in the situation of [20, theorem IV.6.3], and our first task is to identify representatives for the double cosets in

\[
\Sigma_m \times \Sigma_n \backslash \Sigma_{m+n}/\Sigma_{m+n-1}.
\]

An elementary exercise with cycle notation shows that the following are true:

- the elements of $\Sigma_{m+n}/\Sigma_{m+n-1}$ are the distinct left cosets $(r, m + n)\Sigma_{m+n-1}$ where $1 \leq r \leq m + n - 1$, together with $\Sigma_{m+n-1}$;
- by definition, the elements of $\Sigma_m \times \Sigma_n \backslash \Sigma_{m+n} / \Sigma_{m+n-1}$ are the $\Sigma_m \times \Sigma_n$-orbits in $\Sigma_{m+n} / \Sigma_{m+n-1}$ and these are represented by $(m, m + n)$ and id. In fact the orbit of $(m, m + n)$ contains all the $(r, m + n)$ with $1 \leq r \leq m$, and the orbit of the identity $I$ contains all of the transpositions $(m + r, m + n)$ with $1 \leq r \leq n$.

It is straightforward to verify the identities

\[
\Sigma_m \times \Sigma_{n-1} = \Sigma_m \times \Sigma_n \cap \Sigma_{m+n-1},
\]

\[
\Sigma_{m-1} \times \Sigma_n = \Sigma_m \times \Sigma_n \cap (m, m + n)\Sigma_{m+n-1}(m, m + n).
\]
Now the double coset formula tells us that in the homotopy category, there is a commutative
diagram having the following form.

(3.4)

\[
\begin{array}{ccc}
E \Sigma_{m+n} \times \Sigma_{m} \times \Sigma_{n} Y & \xrightarrow{\tau_{n-1,1}} & E \Sigma_{m+n} \times \Sigma_{m} \times \Sigma_{n} Y \\
E \Sigma_{m+n} \times \Sigma_{m} \times \Sigma_{n-1} Y \vee E \Sigma_{m+n} \times \Sigma_{m-1} \times \Sigma_{n} Y & \xrightarrow{\text{fold}} & E \Sigma_{m+n} \times \Sigma_{m+n-1} Y \\
\end{array}
\]

4. Power operations and the free functor

We will describe another result on the effect of certain transfer maps in homology that sheds
light on the calculation of universal derivations. We begin by recalling some standard facts
about the homology of extended powers.

Let \( p \) be a prime and let \( V = V_\ast \) be a graded \( \mathbb{F}_p \)-vector space. The inclusion \( C_p \leq \Sigma_p \) of
the subgroup of cyclic permutations \( C_p = \langle \gamma \rangle \) with \( \gamma = (1, 2, \ldots, p) \) has index \((p-1)!\), so the
associated transfer homomorphism provides a splitting for the induced homomorphism in group
homology with coefficients in the \( p \)-fold tensor power \( V^\otimes p \) with the obvious action.

\[
\begin{array}{c}
H_*(C_p; V^\otimes p) \\
\xrightarrow{\text{Tr}_{C_p}} \\
H_*(\Sigma_p; V^\otimes p)
\end{array}
\]

Furthermore, the homology of the subgroup \( \Sigma_{p-1} \leq \Sigma_p \) is trivial in positive degrees, \( i.e., \)
\[
H_*(\Sigma_{p-1}; V^\otimes p) = H_0(\Sigma_{p-1}; V^\otimes p) = (V^\otimes p)_{\Sigma_{p-1}}.
\]

Hence the associated transfer homomorphism is also zero in positive degrees, \( i.e., \) for \( k > 0 \),

(4.1) \[
0 = \text{Tr}_{\Sigma_{p-1}}: H_k(\Sigma_{p}, V^\otimes p) \rightarrow H_k(\Sigma_{p-1}, V^\otimes p).
\]

In fact the diagram of subgroup inclusions

\[
\begin{array}{ccc}
1 & \xleftarrow{C_p} & \Sigma_p \\
\downarrow & & \downarrow \\
\Sigma_{p-1} & \xrightarrow{\text{fold}} & \Sigma_p
\end{array}
\]

induces a commutative diagram of split epimorphisms.

\[
\begin{array}{ccc}
V^\otimes p & \xrightarrow{\text{Tr}_{\Sigma_{p-1}}} & H_*(C_p; V^\otimes p) \\
\downarrow & & \downarrow \\
(V^\otimes p)_{\Sigma_{p-1}} & \xrightarrow{\text{Tr}_{C_p}} & H_*(\Sigma_p; V^\otimes p)
\end{array}
\]

This can be generalised to \( \Sigma_{p^m} \) where \( m \geq 2 \). Then the \( p \)-order of \( \vert \Sigma_{p^m} \vert \) is
\[
\text{ord}_p \vert \Sigma_{p^m} \vert = \frac{(p^m - 1)}{(p-1)} = p^{m-1} + p^{m-2} + \cdots + p + 1.
\]
Writing

\[ \Sigma_p^{k} = \Sigma_p \times \cdots \times \Sigma_p, \]

the wreath product

\[ \Sigma_p \wr \Sigma_p^{m-1} = \Sigma_p \rtimes \Sigma_p^{m-1} \]

has \( p \)-order

\[ \text{ord}_p | \Sigma_p | \Sigma_p^{m-1} | = 1 + p \frac{(p^m - 1) - 1}{(p - 1)} = \frac{p^m - 1}{p - 1}, \]

so an argument using transfer shows that the inclusion induces a split epimorphism.

\[ H_*(\Sigma_p \wr \Sigma_p^{m-1}; \mathbb{F}_p) \to H_*(\Sigma_p^m; \mathbb{F}_p) \]

Another calculation shows that

\[ \text{ord}_p | \Sigma_p^m - 1 | = \frac{(p^m - 1)}{(p - 1)} - m = (p^{m-1} + p^{m-2} + \cdots + p + 1) - m \]

and

\[ \text{ord}_p | \Sigma_p^{(p-1)} \times \Sigma_p^{m-1} - 1 | = (p - 1) \frac{(p^{m-1} - 1)}{(p - 1)} + \frac{(p^{m-1} - 1)}{(p - 1)} - (m - 1) \]

\[ = (p^{m-1} + p^{m-2} + \cdots + p + 1) - m \]

\[ = \text{ord}_p | \Sigma_p^m - 1 |. \]

Therefore

\[ \Sigma_p^{(p-1)} \times \Sigma_p^{m-1} - 1 \leq \Sigma_p^m - 1 \]

and these subgroups of \( \Sigma_p^m \) have the same \( p \)-order, hence the inclusion induces an isomorphism

\[ H_*(\Sigma_p^{(p-1)} \times \Sigma_p^{m-1} - 1; \mathbb{F}_p) \cong H_*(\Sigma_p^m - 1; \mathbb{F}_p). \]

Consider the commuting diagram of subgroup inclusions

\[ \begin{array}{ccc} 
\Sigma_p & \xrightarrow{1} & \Sigma_p^m \\
| & & | \\
\Sigma_p \wr \Sigma_p^{m-1} & \xrightarrow{p^m} & \Sigma_p^m \\
| & & | \\
\Sigma_p^{(p-1)} \times \Sigma_p^{m-1} - 1 & \xrightarrow{1} & \Sigma_p^{(p-1)} \times \Sigma_p^{m-1} \\
\end{array} \]

in which the arrows are decorated with the \( p \)-power factors of the indices, i.e., if \( H \leq G \) then the number would be \( p^{\text{ord}_p [G:H]} \). Applying homology with coefficients in \( V^\otimes p^m \) with the evident
action of \(\Sigma p^m\), \(H_s(-; V^\otimes p^m)\), we obtain a commutative diagram of induced homomorphisms (solid arrows) and transfer homomorphisms (dashed arrows).

\[
\begin{array}{c}
H_s(\Sigma_p \otimes \Sigma_{p-1}; V^\otimes p^m) \\
\downarrow \\
H_s(\Sigma_p \otimes \Sigma_{p-1} \times \Sigma_{p-1}; V^\otimes p^m)
\end{array}
\quad \text{and} \quad
\begin{array}{c}
H_s(\Sigma_{p-1}; V^\otimes p^m) \\
\downarrow \\
H_s(\Sigma_{p-1} \times \Sigma_{p-1}; V^\otimes p^m)
\end{array}
\]

As the transfer is contravariantly functorial with respect to homomorphisms induced from inclusions, it is enough to show that \(\text{Tr}_{\Sigma_{p-1}} \otimes \Sigma_{p-1} \times \Sigma_{p-1} \times \Sigma_{p-1}\) is zero in positive degrees to deduce that the same holds for \(\text{Tr}_{\Sigma_{p-1}}\). But this follows since

\[
H_s((\Sigma_p \otimes \Sigma_{p-1}; V^\otimes p^m) \cong H_s((\Sigma_p; H_s((\Sigma_{p-1}; V^\otimes p^{m-1})^\otimes p))
\]

and by (4.1) we already know the result for all transfer homomorphisms of the form

\[
\text{Tr}_{\Sigma_{p-1}} : H_s((\Sigma_p; W^\otimes p) \longrightarrow H_s((\Sigma_{p-1}; W^\otimes p)
\]

for some \(\mathbb{F}_p\)-vector space \(W\). To summarise, we have verified

**Lemma 4.1.** For \(m \geq 1\), the transfer

\[
\text{Tr}_{\Sigma_{p-1}} : H_s((\Sigma_p; V^\otimes p^m) \longrightarrow H_s((\Sigma_{p-1}; V^\otimes p^m)
\]

is zero in positive degrees.

Recall the standard 2-periodic projective \(\mathbb{F}_p[C_p]\)-resolution of \(\mathbb{F}_p\),

\[
(4.2) \quad \mathbb{F}_p \quad \mathbb{F}_p[C_p] \quad \mathbb{F}_p[C_p] \quad \mathbb{F}_p[C_p] \quad \mathbb{F}_p[C_p] \quad \cdots
\]

where \(C_p = \langle \gamma \rangle\) generated by the \(p\)-cycle \(\gamma = (1, 2, \ldots, p)\). Tensoring over \(\mathbb{F}_p[C_p]\) gives a complex

\[
0 \quad \mathbb{F}_p e_0 \otimes V^\otimes p \quad \mathbb{F}_p e_1 \otimes V^\otimes p \quad \mathbb{F}_p e_2 \otimes V^\otimes p \quad \cdots
\]

whose homology is \(H_s(C_p; V^\otimes p)\).

Let \(X\) be a connective cofibrant \(S\)-module, and let \(x \in H_n(X; \mathbb{F}_p)\) be a non-zero element. Recall the algebraic results of [22, lemma 1.4]. If \(p\) is a prime, then there are elements

\[
e_r \otimes x^\otimes p = e_r \otimes x \otimes \cdots \otimes x
\]

which survive to non-zero homology classes in \(H_s(C_p; H_s(X; \mathbb{F}_p)^\otimes p)\) for \(r \geq 0\), and where if \(p\) is odd,
• $n$ is even and $r = 2s(p - 1)$ or $r = 2(s + 1)(p - 1) - 1$ for $0 \leq s \in \mathbb{Z}$,
• $n$ is odd, $r = (2s + 1)(p - 1)$ or $r = (2s + 1)(p - 1) - 1$ for $0 \leq s \in \mathbb{Z}$.

These map to non-zero elements
\[ \tilde{Q}_r x, \beta \tilde{Q}_r x \in H_*(\Sigma_p; \mathbb{H}_*(X; \mathbb{F}_p)^{\otimes p}) \]
depending on the parity of $r$. There is a canonical isomorphism
\[ H_* (\Sigma_p; \mathbb{H}_* (X; \mathbb{F}_p)^{\otimes p}) \xrightarrow{\cong} H_* (\mathbb{E} \Sigma_p \ltimes \Sigma_p X^{(p)}) \]
and we also denote the images of $\tilde{Q}_r x, \beta \tilde{Q}_r x$ by the same symbols. The natural weak equivalence
\[ \mathbb{E} \Sigma_p \ltimes \Sigma_p X^{(p)} \xrightarrow{\sim} X^{(p)}/\Sigma_p \]
induces an isomorphism
\[ \cong \]
\[ H_* (\Sigma_p; \mathbb{H}_* (X; \mathbb{F}_p)^{\otimes p}) \cong H_* (\mathbb{E} \Sigma_p \ltimes \Sigma_p X^{(p)}; \mathbb{F}_p) \cong H_* (X^{(p)}/\Sigma_p; \mathbb{F}_p) \]
sending $\tilde{Q}_r x$ to the element which we will denote by $\overline{Q}_r x \in H_* (X^{(p)}/\Sigma_p; \mathbb{F}_p)$. When $p$ is odd, whenever $2r \geq n$ we will set
\[ \overline{Q}^r x = (-1)^r \nu(n) \overline{Q}_{2r-n} (p-1) x, \]
\[ \beta \overline{Q}^r x = (-1)^r \nu(n) \overline{Q}_{2r-n} (p-1) x, \]
in keeping with upper indexing for Dyer-Lashof operations, where
\[ \nu(n) = (-1)^{n(n-1)(p-1)/4} \left( \left( (p-1)/2 \right)! \right)^n, \]
which does not depend on $r$. When $p = 2$, whenever $r \geq n$ we set
\[ \overline{Q} x = \overline{Q}_{r-n} x. \]

The action of the Dyer-Lashof operations $Q^r, \beta Q^r$ on $H_* (\mathbb{F} X; \mathbb{F}_p)$ described by Steinberger in [II, chapter III] is consistent with this notation; we will write $Q^r \cdot x, \beta Q^r \cdot x$ when applying such an operation to an element $x \in H_* (\mathbb{F} X; \mathbb{F}_p) \subseteq H_* (\mathbb{F} X; \mathbb{F}_p)$ to avoid potential confusion when $X$ is itself a commutative $S$-algebra.

**Lemma 4.2.** For a connective cofibrant $S$-module $X$, and an element $x \in H_* (\mathbb{F} X; \mathbb{F}_p)$, under the natural map
\[ \mathbb{E} \Sigma_p \ltimes \Sigma_p X^{(p)} \xrightarrow{\rho} X^{(p)}/\Sigma_p \]
in $H_* (-; \mathbb{F}_p)$ we have
\[ \rho_*(\overline{Q}^r x) = Q^r \cdot x, \quad \rho_*(\beta \overline{Q}^r x) = \beta Q^r \cdot x. \]

**Proof.** The basic observation is that a commutative $S$-algebra is an algebra over the monad $\mathbb{P} \circ \mathbb{U}$ where the two model categories $\mathcal{M}_S$ and $\mathcal{E}_S$ are related by the Quillen adjunction of (3.1) with $R = S$.

\[ \mathcal{E}_S \xrightarrow{\mathbb{P}} \mathcal{M}_S \]

The definition of the Dyer-Lashof operations for a commutative $S$-algebra $A$ involves the composition
Theorem 4.3. Let $X$ be a connective cofibrant $S$-module. Then
\[(\delta_{(\mathbb{P}X,S)})_* : H_* (\mathbb{P}X; \mathbb{F}_p) \rightarrow H_* (\Omega_S (\mathbb{P}X); \mathbb{F}_p)\]
annihilates every element of the form $Q^I x$, where $\text{length}(I) > 0$ and $x \in H_* (X; \mathbb{F}_p)$.

Proof. This follows from the observation (3.3) together with Lemma 4.1. \hfill \Box

This generalises to give

Theorem 4.4. Let $A$ be a connective commutative $S$-algebra. Then
\[(\delta_{(A,S)})_* : H_* (A; \mathbb{F}_p) \rightarrow H_* (\Omega_S (A); \mathbb{F}_p)\]
annihilates every element of the form $Q^I a$ with $\text{length}(I) > 0$ and $a \in H_* (A; \mathbb{F}_p)$. Hence the homomorphism
\[\Delta_{(A,S)} : H_* (A; \mathbb{F}_p) \rightarrow H_*^A (\Omega_S (A); \mathbb{F}_p) = \text{TAQ}_* (A, S; H \mathbb{F}_p)\]
also annihilates all such elements.

Proof. Using the observation in the Proof of Lemma 4.2 we know there is a morphism of commutative $S$-algebras $P A \to A$ extending the multiplication. Choose a cofibrant replacement $A^c \to A$ for the underlying $S$-module of $A$. By naturality there is a commutative diagram in the homotopy category of $S$-modules

$$
\begin{array}{ccc}
PA^c & \to & PA \\
\delta_{(PA^c, S)} & \downarrow & \delta_{(PA, S)} \\
PA^c \land A^c & \to & \Omega_S(PA) \\
\end{array}
$$

and on applying $H^*(-) = H^*(-; \mathbb{F}_p)$ we obtain an algebraic commutative diagram.

$$
\begin{array}{ccc}
H_*(PA^c) & \to & H_*(PA) \\
(\delta_{(PA^c, S)})_* & \downarrow & (\delta_{(PA, S)})_* \\
H_*(PA^c \land A^c) & \to & \Omega_S(PA) \\
\end{array}
$$

Since an element of the form $Q^I a$ lifts back to an element $Q^I a' \in H_*(PA^c)$ as explained above, the result follows. The result about $\theta'$ is immediate from the definition (1.1). $\square$

5. The reduced free commutative algebra functor

Throughout, we fix a cofibrant commutative $S$-algebra $R$. The two model categories $\mathcal{M}_R$ and $\mathcal{C}_R$ are related by the Quillen adjunction

$$
\begin{array}{ccc}
\mathcal{C}_R & \xrightarrow{\mathcal{P}_R} & \mathcal{M}_R \\
\mathcal{U} & \xleftarrow{\mathcal{F}_R} & \\
\end{array}
$$

where the right adjoint $\mathcal{U}$ is the forgetful functor. For a cofibrant $R$-module $Z$, inclusion of the basepoint $* \to X$ induces a cofibration of commutative $R$-algebras $R = \mathcal{P}_R * \to \mathcal{P}_R Z$, so $\mathcal{P}_R Z$ is cofibrant in the model category $\mathcal{C}_R$. More generally a (acyclic) cofibration $f : X \to Y$ in $\mathcal{M}_R$ induces a (acyclic) cofibration $\mathcal{P}_R f : \mathcal{P}_R X \to \mathcal{P}_R Y$ in $\mathcal{C}_R$.

In $\mathcal{M}_R$, $R$ is not cofibrant and we denote its functorial cofibrant replacement by $S^0_R = \mathcal{F}_R S^0$ and a weak equivalence induced by a map of spectra $S^0_R \to R$ which represents the unit.

$$
* \to S^0_R \\
\sim \\
\to R$$

There is a unique induced morphism $\mathcal{P}_R S^0_R \to R$ in $\mathcal{C}_R$, but this need not be a cofibration. Using the functorial factorisation in $\mathcal{C}_R$ we obtain a commutative diagram in $\mathcal{C}_R$

$$
\begin{array}{ccc}
\mathcal{P}_R S^0_R & \to & \mathcal{P}_R S^0_R \\
\sim & \downarrow & \sim \\
\mathcal{R} & \xrightarrow{\sim} & R \\
\end{array}
$$

which we use to fix the left hand arrow.
We will make use of the comma category $S^0_R/\mathcal{M}_R$ of $R$-modules under $S^0_R$, whose objects are the morphisms $S^0_R \to X$ and whose morphisms are the commuting diagrams

$$
\begin{array}{ccc}
S^0_R & \rightarrow & X \\
\downarrow & & \downarrow \\
Y & \rightarrow & X
\end{array}
$$

with initial object $\text{Id}_{S^0_R}$ and terminal object $S^0_R \to \ast$. This inherits a model structure from $\mathcal{M}_R$. Given $i: S^0_R \to X$ in $S^0_R/\mathcal{M}_R$, we obtain the induced morphism $P_Ri: P_RS^0_R \to P_RX$ in $\mathcal{C}_R$. If $i$ is a cofibration, then $i$ is cofibrant in $S^0_R/\mathcal{M}_R$ and $P_Ri: P_RS^0_R \to P_RX$ is a cofibration in $\mathcal{C}_R$: we will then write $X/S^0_R$ for the cofibre of $i$. We obtain a pushout diagram of commutative $R$-algebras

$$
\begin{array}{ccc}
P_RS^0_R & \rightarrow & P_RX \\
\downarrow & \nearrow \gamma & \downarrow \\
\tilde{R} & \rightarrow & \tilde{R} \wedge P_RS^0_R \ P_RX
\end{array}
$$

and we set

$$
\widetilde{P}_RX = \tilde{R} \wedge P_RS^0_R \ P_RX.
$$

If $i^c: S^0_R \to X^c$ is the cofibrant replacement of $i$ in the comma category, then the pushout diagram of solid arrows in

$$
\begin{array}{ccc}
P_RS^0_R & \rightarrow & \ P_RX^c \\
\downarrow & \nearrow \gamma & \downarrow \\
\tilde{R} & \rightarrow & \tilde{R} \wedge P_RS^0_R \ P_RX^c
\end{array}
$$

defines the homotopy pushout of the first diagram,

$$
\widetilde{P}_RX^h = \tilde{R} \wedge P_RS^0_R \ P_RX^c.
$$

Of course $\widetilde{P}_RX^h$ is well-defined in the homotopy category $\mathcal{C}_R$, and we have in effect defined it by making functorial choices.

The model categories $S^0_R/\mathcal{M}_R$ and $\mathcal{C}_R$ are related by the Quillen adjunction

$$
\begin{array}{ccc}
\mathcal{C}_R & \xrightarrow{P_R} & S^0_R/\mathcal{M}_R \\
\tilde{U} & \text{adj} & \end{array}
$$

where the right adjoint $\tilde{U}$ is the forgetful functor sending $A$ to the composition

$$
\begin{array}{ccc}
S^0_R & \xrightarrow{R} & A
\end{array}
$$
in \( \mathcal{M}_R \). The total left derived functor of \( \widetilde{\mathcal{P}}_R \) is \( \widetilde{\mathcal{P}}^h_R \) and
\[
\begin{array}{c}
\widetilde{\mathcal{P}}^h_R \\
\downarrow \\
\mathcal{M}_R/\mathcal{R}
\end{array}
\]

is a derived adjunction on homotopy categories.

**Remark 5.1.** In \( S^0_R/\mathcal{M}_R \), pushouts are defined using pushouts in \( \mathcal{M}_R \), and we use the symbol \( \lor \) to indicate such a pushout.

\[
\begin{array}{c}
S^0_R \\
\downarrow \\
X
\end{array} \lor \begin{array}{c}
Y \\
\downarrow \\
S^0_R
\end{array}
\]

Since the reduced free algebra functor \( \widetilde{\mathcal{P}}_R : \mathcal{M}_R \rightarrow \mathcal{C}_R \) is a left adjoint it preserves pushouts, so for two \( R \)-modules \( X, Y \) under \( S^0_R \):
\[
(5.1) \quad \widetilde{\mathcal{P}}_R(X \lor Y) \cong \widetilde{\mathcal{P}}_R X \lor_R \widetilde{\mathcal{P}}_R Y.
\]

In particular, if we have a connective CW \( R \)-module with distinguished bottom cell \( S^0_R \rightarrow X \), then the \( n \)-skeleton \( X^{[n]} \) gives rise to the \( n \)-skeleton of the CW commutative \( R \)-algebra \( \widetilde{\mathcal{P}}_R X \),
\[
(5.2) \quad \widetilde{\mathcal{P}}^{(n)}_R X = (\widetilde{\mathcal{P}}_R X)^{(n)} = \widetilde{\mathcal{P}}_R (X^{[n]}).
\]

We already know that for cofibrant \( X \), in the homotopy category \( \mathcal{M}_{\widetilde{\mathcal{P}}_R X} \),
\[
\Omega_R(\mathcal{P}_R X) \cong \mathcal{P}_R X \lor_R X.
\]

**Proposition 5.2.** Let \( X \in S^0_R/\mathcal{M}_R \) be cofibrant. Then in the homotopy category \( \mathcal{M}_{\widetilde{\mathcal{P}}_R X} \),
\[
\Omega_R(\mathcal{P}_R X) \cong \mathcal{P}_R X \lor_R X/S^0_R.
\]

**Proof.** First we recall some observations appearing in [4]. For a pushout diagram of cofibrations of commutative \( R \)-algebras
\[
\begin{array}{c}
A \\
\downarrow \\
C
\end{array} \lor \begin{array}{c}
B \\
\downarrow \\
B \land_A C
\end{array}
\]

we have
\[
(5.3) \quad \Omega_B(B \land_A C) \sim B \land_A \Omega_A(C)
\]
by [7, proposition 4.6].

Now assume that \( i : S^0_R \rightarrow X \) is a cofibration, and consider
\[
\widetilde{\mathcal{P}}_R X = \widetilde{\mathcal{P}} \land_{\mathcal{P}_R S^0_R} \mathcal{P}_R X.
\]

Notice that by \( (5.3) \),
\[
\Omega_{\mathcal{P}_R X} (\widetilde{\mathcal{P}}_R X) \sim \mathcal{P}_R X \land_{\mathcal{P}_R S^0_R} \Omega_{\mathcal{P}_R S^0_R} (\widetilde{R}),
\]
and since there is a weak equivalence of \( R \)-algebras \( \widetilde{R} \rightarrow_R R \), the sequence
\[
R \rightarrow \mathcal{P}_R S^0_R \rightarrow \widetilde{R}
\]
has an associated cofibre sequence of the form
\[
\tilde{R} \wedge_{P_R S^0_R} \Omega_R(P_R S^0_R) \longrightarrow \Omega_R(\tilde{R}) \longrightarrow \Omega_{\tilde{P}_R S^0_R}(\tilde{R}) \longrightarrow \cdots
\]
where
\[
\Omega_R(\tilde{R}) \sim \Omega_R(R) \sim *. 
\]
Therefore
\[
\Omega_{\tilde{P}_R S^0_R}(\tilde{R}) \sim \tilde{R} \wedge_{\tilde{P}_R S^0_R} \sum \Omega_R(\tilde{P}_R S^0_R) \sim \tilde{R} \wedge_{\tilde{P}_R S^0_R} \tilde{P}_R S^0_R \wedge \Sigma S^0_R \sim \tilde{R} \wedge \Sigma S^0_R ,
\]
and so
\[
\Omega_{\tilde{P}_R X}(\tilde{P}_R X) \sim \tilde{P}_R X \wedge R \Sigma S^0_R. 
\]
Similarly, from the sequence
\[
R \longrightarrow P_R X \longrightarrow \tilde{R} \wedge_{\tilde{P}_R S^0_R} P_R X
\]
we obtain a cofibre sequence of \(\tilde{P}_R X\)-modules
\[
\tilde{P}_R X \wedge X \longrightarrow \Omega_R(\tilde{P}_R X) \longrightarrow \tilde{P}_R X \wedge \Sigma S^0_R \longrightarrow \tilde{P}_R X \wedge X \longrightarrow \cdots
\]
which is equivalent to
\[
\Omega_R(\tilde{P}_R X) \sim \tilde{P}_R X \wedge X / S^0_R ,
\]
as required. \(\square\)

Now let \(A\) be a commutative \(R\)-algebra. On composing the unit \(R \longrightarrow A\) with the weak equivalence \(S^0_R \longrightarrow R\) we obtain the object
\[
S^0_R \sim \longrightarrow R \longrightarrow A
\]
in \(S^0_R / M_R\). Using functorial factorisation we obtain a cofibrant replacement
\[
A^c \sim \longrightarrow A
\]
and so
\[
\tilde{P}^h_R A = \tilde{R} \wedge_{\tilde{P}_R S^0_R} P_R A^c.
\]

**Remark 5.3.** The multiplication on a commutative \(R\)-algebra \(A\) extends to a morphism of commutative \(R\)-algebras \(\tilde{P}_R A \longrightarrow A\). This follows from the evident commutative diagram of solid arrows
\[
\begin{array}{ccc}
P_R S^0_R & \longrightarrow & P_R A \\
\| & \downarrow & \| \\
\tilde{R} & \sim \longrightarrow & \tilde{P}_R A \\
\| & \downarrow & \| \\
R & \sim \longrightarrow & A \\
\end{array}
\]
where the curved arrows come from the unit and extension of the product respectively.

6. The ordinary homology of free commutative $S$-algebras

For a commutative ring $k$, and a graded $k$-module $V_*$, we will write $k(V_*)$ for the free commutative graded $k$-algebra on $V_*$. If $V_*$ is connective and $V_0$ is a cyclic $k$-module, we set $V_* = V_*/V_0$.

Let $X$ be a cofibrant connective spectrum.

**Theorem 6.1.** The rational homology of $\mathbb{P}X$ is given by

$$H_*(\mathbb{P}X; \mathbb{Q}) = \mathbb{Q}(H_*(X; \mathbb{Q})).$$

In positive characteristic, the next result is fundamental. We use the standard convention for Dyer-Lashof monomials so that in an indexing sequence

$$I = (\varepsilon_1, i_1, \varepsilon_2, i_2, \ldots, \varepsilon_\ell, i_\ell),$$

each $i_r$ is positive, when $p = 2$ all the $\varepsilon_i$ are zero, while for odd $p$, $\varepsilon_i = 0, 1$. The length of $Q^I$ is $\text{length}(Q^I) = \ell$.

**Theorem 6.2.** For $p$ a prime, $H_*(\mathbb{P}X; \mathbb{F}_p)$ is the free commutative graded $\mathbb{F}_p$-algebra generated by elements $\overline{Q}^I x_j$, where $x_j$ for $j \in J$ gives a basis for $H_*(X; \mathbb{F}_p)$ and $I = (\varepsilon_1, i_1, \varepsilon_2, i_2, \ldots, \varepsilon_\ell, i_\ell)$ is admissible and satisfies $\text{excess}(I) + \varepsilon_1 > |x_j|$.

So for $p = 2$, this gives the polynomial ring

$$H_*(\mathbb{P}X; \mathbb{F}_2) = \mathbb{F}_2[\overline{Q}^I x_j : j \in J, \text{excess}(I) + \varepsilon_1 > |x_j|].$$

Of course these results are very similar to those for the homology of $\Omega^\infty \Sigma^\infty Z$ for a space $Z$; for a convenient overview of the latter, see [23].

**Sketch of why Theorems 6.1 and 6.2 hold.** We learnt some of the following from Mike Mandell, see also [7, section 6].

Let $R$ be a commutative $S$-algebra. The free functor $\mathbb{P}R$ for, sends $R$-modules to commutative $R$-algebras. As it is a left adjoint it preserves pushouts, so for $R$-modules $X, Y$,

$$\mathbb{P}R(X \vee Y) \cong \mathbb{P}R(X) \wedge_R \mathbb{P}R(Y).$$

For any commutative $R$-algebra $A$, base change gives

$$A \wedge_R \mathbb{P}R(-) = \mathbb{P}A(A \wedge_R (-)).$$

If $R = S$ and $H = Hk$ for a field $k$, then

$$H \wedge \mathbb{P}(-) = \mathbb{P}H(H \wedge (-)).$$

Applying $\pi_*(-)$ gives a functor sending spectra to commutative graded $k$-algebras,

$$X \mapsto H_*(\mathbb{P}(X)) = \pi_*(\mathbb{P}H(H \wedge (-))),$$

which preserves pushouts, in particular it sends wedges to tensor products. For any spectrum $X$, as an $H$-module, $H \wedge X$ is equivalent to a wedge of suspensions of $H$, so the calculation of $H_*(\mathbb{P}(X)) = H_*(\mathbb{P}(X); k)$ reduces to that for spheres. For $k = \mathbb{Q}$ this gives the rational result.

When $k = \mathbb{F}_p$, for a sphere $S^n$ the answer is the free commutative $k$-algebra on admissible Dyer-Lashof monomials $Q^I$ applied to an element $s_n$, i.e., elements of the form $Q^I \cdot s_n = \overline{Q}^I s_n$, where the indexing sequences $I$ are admissible and satisfy $\text{excess}(I) > |s_n| = n$. Although this
makes sense for \( n \in \mathbb{Z} \) (for the general case see \[11\] chapter III), we only require the case where \( n \geq 0 \).

Using the ideas of Section \[11\] we know that for \( x \in H_*(X) \), the element \( \overline{Q}^j x \) is the image under the canonical homomorphism

\[
H_*(E \ltimes \Sigma_0, X^{(p^j)}) \cong H_*(X^{(p^j)}/\Sigma_p) \xrightarrow{\varepsilon} H_*(\mathbb{P}X)
\]

of an element obtained by forming iterated wreath powers of \( x \) in

\[ H_*(E \Sigma_p \ltimes \Sigma_0, X^{(p^j)}) \]

Of course we can view \( \overline{Q}^j \) as obtained from \( x \) by applying the Dyer-Lashof monomial

\[ Q^j = \beta^{e_1} Q^{j_1} \cdots \beta^{e_t} Q^{j_t} \]

which exists as an operation on the homology of any commutative \( S \)-algebra. \( \square \)

We will describe the analogous results for \( \mathbb{P}X \). Our main computational tool is the Künneth spectral sequence, and we will use its multiplicative properties and compatibility with the action of the Dyer-Lashof operations, see \[9\] for some related results on this.

We begin by stating the rational result whose proof we leave to the reader.

**Theorem 6.3.** The rational homology of \( \mathbb{P}X \) is given by

\[ H_*(\mathbb{P}X; \mathbb{Q}) = \mathbb{Q}(\tilde{H}_*(X; \mathbb{Q})). \]

The positive characteristic case is of course more interesting.

**Theorem 6.4.** Let \( p \) be a prime. If \( X \) is a \( p \)-local Hurewicz spectrum, then \( H_*(\mathbb{P}X; \mathbb{F}_p) \) is the free commutative graded \( \mathbb{F}_p \)-algebra generated by elements \( \overline{Q} x_j \), where \( x_j \) for \( j \in J \) gives a basis for \( \tilde{H}_*(X; \mathbb{F}_p) = H_*(X/S^0; \mathbb{F}_p) \) and \( I = (e_1, i_1, e_2, \ldots, e_t, i_t) \) is admissible and satisfies \( \text{excess}(I) + e_1 > |x_j| \).

**Proof.** We set \( H_*(-) = H_*(-; \mathbb{F}_p) \).

Since the pushout agrees with the smash product over \( \mathbb{P}S^0 \), there is a first quadrant Künneth spectral sequence with

\[ E^2_{s,t} = \text{Tor}^{H_*(\mathbb{P}S^0)}_{s,t}(H_*(\mathbb{P}X), \mathbb{F}_p) \Rightarrow H_{s+t}(\mathbb{P}X). \]

Here \( i_0' : H_*(\mathbb{P}S^0) \rightarrow H_*(\mathbb{P}X) \) embeds the domain as a subalgebra since \( x_0 = i_*(s_0) \) generates \( H_0(X) = \mathbb{F}_p \) and

\[ i_0'(s_0) = x_0 - 1, \]

\[ i_0'(\overline{Q} x_0) = \overline{Q} x_0 - \overline{Q}(1) = \overline{Q}(x_0) \quad \text{if length}(I) > 0. \]

By the freeness of \( H_*(\mathbb{P}X) \), it is a free \( i_0' H_*(\mathbb{P}S^0) \)-module, hence

\[
E^2_{s,s} = \text{Tor}^{H_*(\mathbb{P}S^0)}_{s,s}(H_*(\mathbb{P}X), \mathbb{F}_p) = H_*(\mathbb{P}X) \otimes_{i_0' H_*(\mathbb{P}S^0)} \mathbb{F}_p = H_*(\mathbb{P}X)/(x_0 - 1, \overline{Q} x_0 : \text{length}(I) > 0).
\]
Thus the spectral sequence collapses at the $E^2$-term and the result follows. 

Armed with Theorems 4.3, we have

**Theorem 6.5.** Let $p$ be a prime and let $X$ be a $p$-local connective cofibrant $S$-module. Then the universal derivation $\delta(\tilde{P}_X,S)$ induces the derivation

$$
\Delta(\tilde{P}_X,S) : H_*(\tilde{P}_X;F_p) \rightarrow \text{TAQ}_*(\tilde{P}_X, S; H_F) \cong H_*(X/S^0; F_p)
$$

which acts on the $Q^j x$ with $x \in H_*(X; F_p)$ by the rule

$$
\Delta(Q^j x) = \begin{cases} 
  x & \text{if length}(I) = 0, \\
  0 & \text{if length}(I) > 0.
\end{cases}
$$

**Example 6.6.** Consider the suspension spectrum $\Sigma^{-2}_\infty CP^\infty = \Sigma^{-2}_\infty CP^\infty$. A complex orientation for a ring spectrum $E$ is the homotopy class of a map $\Sigma^{-2}_\infty CP^\infty \rightarrow E$ such that the restriction

$$
S = S^0 = \Sigma^{-2}_\infty CP^1 = \Sigma^{-2}_\infty S^2 \rightarrow E
$$

is homotopic to the unit of $E$. When $E$ is a commutative $S$-algebra, such a map $\Sigma^{-2}_\infty CP^\infty \rightarrow E$ induces a unique morphism of commutative $S$-algebras

$$
\mathbb{P}\Sigma^{-2}_\infty CP^\infty \rightarrow E.
$$

Because of the condition involving the bottom cell, there is a commutative diagram of solid arrows

![Commutative Diagram](https://via.placeholder.com/150)

and hence a unique dotted arrow making the whole diagram commute. This shows that $\mathbb{P}\Sigma^{-2}_\infty CP^\infty$ is universal for maps $S^0 \rightarrow E$ which give complex orientations. Of course the inclusion map $\Sigma^{-2}_\infty CP^\infty \rightarrow \mathbb{P}\Sigma^{-2}_\infty CP^\infty$ itself provides a complex orientation.

**Lemma 6.7.** The universal complex orientation

$$
\Sigma^{-2}_\infty CP^\infty = \Sigma^{-2}_\infty MU(1) \rightarrow MU
$$

induces a rational equivalence of commutative $S$-algebras $\sigma : \mathbb{P}\Sigma^{-2}_\infty CP^\infty \rightarrow MU$. Furthermore, the inclusion $\Sigma^{-2}_\infty CP^\infty \rightarrow \mathbb{P}\Sigma^{-2}_\infty CP^\infty$ induces a morphism of ring spectra

$$
MU \rightarrow \mathbb{P}\Sigma^{-2}_\infty CP^\infty
$$

which provides a splitting of $\sigma$ in the homotopy category $\text{h}_MU$.

**Proof.** The rational result is straightforward since an argument using the Künneth spectral sequence gives

$$
H_*(\mathbb{P}\Sigma^{-2}_\infty CP^\infty; Q) = Q[\beta_r : r \geq 1],
$$
where \( \tilde{\beta}_r \) is the image of the canonical generator \( \beta_r \in H_{2r}(\mathbb{CP}^\infty) \). Then the morphism \( \tilde{\Sigma}^{\infty-2}\mathbb{CP}^\infty \to MU \) induces an isomorphism of rings

\[
H_*(\tilde{\Sigma}^{\infty-2}\mathbb{CP}^\infty; \mathbb{Q}) \to H_*(MU; \mathbb{Q}).
\]

It is easy to see that

\[
H_*(\tilde{\Sigma}^{\infty-2}\mathbb{CP}^\infty; \mathbb{Z}) \to H_*(MU; \mathbb{Z})
\]

is epic.

The composition

\[
\Sigma^{\infty-2}\mathbb{CP}^\infty \to MU \to \tilde{\Sigma}^{\infty-2}\mathbb{CP}^\infty \xrightarrow{\sigma} MU
\]

is homotopic to the canonical orientation, so the composition

\[
MU \to \tilde{\Sigma}^{\infty-2}\mathbb{CP}^\infty \xrightarrow{\sigma} MU
\]

is homotopic to the identity by the classical universality of the commutative ring spectrum \( MU \) described by Adams [1]. □

The morphism \( \tilde{\Sigma}^{\infty-2}\mathbb{CP}^\infty \to MU \) can be converted into a fibration (in either of the two model categories \( \mathcal{M}_S \) or \( \mathcal{C}_S \)), giving a commutative diagram

\[
\begin{array}{ccc}
\tilde{\Sigma}^{\infty-2}\mathbb{CP}^\infty & \sim & (\tilde{\Sigma}^{\infty-2}\mathbb{CP}^\infty)'\\
\downarrow & & \downarrow \\
MU & & MU
\end{array}
\]

where \( (\tilde{\Sigma}^{\infty-2}\mathbb{CP}^\infty)' \) is cofibrant in \( \mathcal{C}_S \). The map \( (\tilde{\Sigma}^{\infty-2}\mathbb{CP}^\infty)' \to MU \) is a morphism in the subcategory \( \mathcal{M}_{(\tilde{\Sigma}^{\infty-2}\mathbb{CP}^\infty)'} \) of \( \mathcal{M}_S \).

**Corollary 6.8.** The fibre of \( (\tilde{\Sigma}^{\infty-2}\mathbb{CP}^\infty)' \to MU \) is rationally trivial.

A version of the next result appears in [6].

**Proposition 6.9.** For a prime \( p \), there can be no morphism of commutative \( S_\mathbb{F}_p \)-algebras \( \theta: MU \to (\tilde{\Sigma}^{\infty-2}\mathbb{CP}^\infty)_{(p)} \) for which \( \sigma \circ \theta \) is homotopic to the identity. Hence there can be no morphism of commutative \( S \)-algebras \( \theta: MU \to \tilde{\Sigma}^{\infty-2}\mathbb{CP}^\infty \) for which \( \sigma \circ \theta \) is homotopic to the identity.

**Proof.** It suffices to prove the result for a prime \( p \), and we will assume all spectra are localised at \( p \). Assume such a morphism \( \theta \) existed. Then by naturality of \( \Omega_S \), there are (derived) morphisms of \( MU \)-modules and a commutative diagram

\[
\begin{array}{ccc}
\Omega_S(MU) & \xrightarrow{id} & \Omega_S(MU) \\
\xrightarrow{\sigma_*} & & \xrightarrow{\sigma_*} \\
\Omega_S(\tilde{\Sigma}^{\infty-2}\mathbb{CP}^\infty) & \xrightarrow{\theta_*} & \Omega_S(\tilde{\Sigma}^{\infty-2}\mathbb{CP}^\infty)
\end{array}
\]

which induces a commutative diagram in \( \text{TAQ}_*(\mathbb{Q}^\infty/\mathbb{Q}) \)

\[
\begin{array}{ccc}
H_*(\Sigma^2\mathbb{F}_p) & \xrightarrow{\theta_*} & H_*(\Sigma^{\infty-2}\mathbb{CP}^\infty_2; \mathbb{F}_p) \\
\xrightarrow{\sigma_*} & & \xrightarrow{\sigma_*} \\
H_*(\Sigma^2\mathbb{F}_p) & \xrightarrow{\theta_*} & H_*(\Sigma^{\infty-2}\mathbb{CP}^\infty_2; \mathbb{F}_p)
\end{array}
\]

where \( \mathbb{CP}^\infty_2 = \mathbb{CP}^\infty/\mathbb{CP}^1 \).
It is standard that

\[ H_n(\Sigma^{\infty-2}\mathbb{C}P_2^\infty; \mathbb{F}_p) = \begin{cases} 
\mathbb{F}_p & \text{if } n \geq 2 \text{ and is even,} \\
0 & \text{otherwise.}
\end{cases} \]

On the other hand, when \( p = 2 \),

\[ H_*(ku; \mathbb{F}_2) = \mathbb{F}_2[\zeta_1^2, \zeta_2^2, \zeta_3, \ldots] \subseteq A(2)* \]

with \( |\zeta_s| = 2^s - 1 \), while when \( p \) is odd, \( \Sigma^2 ku \sim \bigvee_{1 \leq r \leq p-1} \Sigma^{2r}\ell \) with

\[ H_*(\ell; \mathbb{F}_2) = \mathbb{F}_p[\zeta_1, \zeta_2, \zeta_3, \ldots] \otimes \Lambda(\tau_r : r \geq 2) \]

where \( |\zeta_s| = 2p^s - 2 \) and \( |\tau_s| = 2p^s - 1 \). Clearly this means that no such \( \theta \) can exist. \( \square \)

At the prime 2, \( \Sigma^{\infty-2}\mathbb{C}P^\infty \) is known to be minimal atomic \([5, \text{proposition 5.9}]\). The next result shows that the functor \( \tilde{P} \) need not preserve this property; see Proposition 6.12 for a converse to this.

**Proposition 6.10.** The 2-local commutative \( S \)-algebra \( \tilde{P}\Sigma^{\infty-2}\mathbb{C}P^\infty(2) \) is not minimal atomic.

**Proof.** If \( \tilde{P}\Sigma^{\infty-2}\mathbb{C}P^\infty(2) \) were minimal atomic then by \([4, \text{theorem 3.3}]\), the TAQ Hurewicz homomorphism (induced from the universal derivation)

\[ \theta: \pi_n(\tilde{P}\Sigma^{\infty-2}\mathbb{C}P^\infty(2)) \longrightarrow \text{TAQ}_n(\tilde{P}\Sigma^{\infty-2}\mathbb{C}P^\infty(2), S; H\mathbb{F}_2)) \]

would be trivial for \( n > 0 \).

By naturality, there is a commutative diagram

\[
\begin{array}{cccc}
\pi_*(\tilde{P}\Sigma^{\infty-2}\mathbb{C}P^\infty(2)) & \xrightarrow{\sigma_*} & \pi_*(MU(2)) \\
\text{TAQ}_n(\tilde{P}\Sigma^{\infty-2}\mathbb{C}P^\infty(2), S; H\mathbb{F}_2)) & \xrightarrow{\sigma_*} & \text{TAQ}_n(MU(2), S; H\mathbb{F}_2)) \\
\cong & & \cong \\
H_*(\mathbb{C}P_2^\infty; \mathbb{F}_2) & \xrightarrow{\sigma_*} & H_*(\Sigma^2 ku; \mathbb{F}_2) \\
\end{array}
\]

in which the surjectivity of the top row follows from Lemma 6.7. The 2-primary calculations of \([4, \text{section 5}]\) show that the right hand Hurewicz homomorphism \( \theta \) is non-zero in positive degrees, hence so is the left hand one. Therefore \( \tilde{P}\Sigma^{\infty-2}\mathbb{C}P^\infty(2) \) cannot be minimal atomic. \( \square \)

We leave the interested reader to formulate and verify analogues for an odd prime \( p \) based on desuspensions of the \( p \)-local summands of \( \Sigma^\infty\mathbb{C}P^\infty(p) \).

**Remark 6.11.** We point out that \( \sigma_*: H_*(\mathbb{C}P_2^\infty; \mathbb{F}_2) \longrightarrow H_*(\Sigma^2 ku; \mathbb{F}_2) \) is different from the homomorphism induced by any map of spectra \( \Sigma^{\infty-2}\mathbb{C}P^\infty_2 \longrightarrow \Sigma^2 ku \) which is an equivalence on the bottom cell. For such a map composed with the natural map \( \Sigma^2 ku \longrightarrow H\mathbb{F}_2 \) induces the homomorphism in homology given by

\[ \Sigma^{-2}\beta_n \mapsto \begin{cases} 
\xi_n^4 & \text{if } n = 2^s, \\
0 & \text{otherwise.}
\end{cases} \]
But also $\Sigma^{-2}\beta_n \mapsto b_{n-1}$ in $H_*(MU; \mathbb{F}_2)$ and under the TAQ-Hurewicz homomorphism,

$$b_{n-1} \mapsto \begin{cases} \xi_n^2 & \text{if } n = 2^s + 1, \\ 0 & \text{otherwise.} \end{cases}$$

As promised above, here is a positive result relating the additive and multiplicative notions of minimal atomic. We regard $p$-local spectra as equivalent to $S$-modules.

**Proposition 6.12.** Let $p$ be a prime and let $S$ be the $p$-local sphere spectrum. Suppose that $X$ is a connective Hurewicz $S$-module with chosen bottom cell $S^0 \to X$. If $\tilde{P}X$ is minimal atomic as a commutative $S$-algebra, then $X$ is minimal atomic as an $S$-module.

**Proof.** Working in $S^0./M$ we can replace $X$ by a CW spectrum which is weakly equivalent to it so we will assume that this has been done. Using observations in Remark 5.1, we can relate the two $n$-skeleta. The $(n+1)$-skeleton $X^{[n+1]}$ is constructed using a map of $S$-modules

$$i^n: \bigvee S^n \to X^{[n]}$$

for which

$$\ker[i^n_*: \pi_n(\bigvee S^n) \to \pi_n(X^{[n]})] \subseteq p \pi_n(\bigvee S^n).$$

Similarly, we form the $(n+1)$-skeleton $\tilde{P}^{(n+1)}X$ is constructed from $\tilde{P}^{(n)}X$ using a morphism of $S$-modules

$$j^n: \bigvee S^n \to \tilde{P}^{(n)}X$$

for which

$$\ker[j^n_*: \pi_n(\bigvee S^n) \to \pi_n(\tilde{P}^{(n)}X)] \subseteq p \pi_n(\bigvee S^n).$$

In $S^0./M$ there is a commutative diagram

$$\begin{array}{ccc} & \bigvee S^n & \\\\ X^{[n]} & \xrightarrow{\text{incl}} & \tilde{P}^{(n)}X \\\\ \xleftarrow{i^n} & & \xrightarrow{j^n} \\\\ \end{array}$$

in which $i^n$ provides the attaching maps for the $(n+1)$-cells of $X$. Clearly

$$\ker[j^n_*: \pi_n(\bigvee S^n) \to \pi_n(X^{[n]})] \subseteq \ker[j^n_*: \pi_n(\bigvee S^n) \to \pi_n(\tilde{P}^{(n)}X)] \subseteq p \pi_n(\bigvee S^n),$$

and it follows that $X$ is nuclear. \hfill $\square$

7. THE FREE COMMUTATIVE $S$-ALGEBRA Functor and $\Omega^\infty\Sigma^\infty Z$

Let $Z$ be a connected based space. The infinite loop space $\Omega^\infty\Sigma^\infty Z$ gives rise to a commutative $S$-algebra $\Omega^\infty\Sigma^\infty Z$, i.e., the suspension spectrum of the based space $\Omega^\infty\Sigma^\infty Z$ with a disjoint basepoint.

The natural (based) map $Z \to \Omega^\infty\Sigma^\infty Z$ viewed as an unbased map induces a based map

$$\Sigma^\infty Z \to \Sigma^\infty \Omega^\infty \Sigma^\infty Z$$

which extends uniquely to a morphism of ring spectra

$$\mathbb{F}\Sigma^\infty Z \to \Sigma^\infty \Omega^\infty \Sigma^\infty Z.$$
The base points in $Z$ and $\Omega^\infty \Sigma^\infty Z$ pick out maps from the sphere $S$ and there is a commutative diagram of commutative $S$-algebras

(7.1)

where the acyclic fibration $\widetilde{S} \to S$ is that of Section 5.

**Proposition 7.1.** For a connected based space $Z$, the morphism

$$\widetilde{\Sigma}^\infty Z \xrightarrow{\sim} \Sigma^\infty \Omega^\infty \Sigma^\infty Z$$

of (7.1) is a weak equivalence.

**Proof.** If $k = \mathbb{Q}$ and $k = \mathbb{F}_p$ for $p$ a prime, comparison of the known answers for $H_*(\widetilde{\Sigma}^\infty Z; k)$ and $H_*(\Omega^\infty \Sigma^\infty Z; k)$ shows that this morphism induces an isomorphism $H_*(-; k)$. It follows that it induces an isomorphism on $H_*(-; \mathbb{Z})$, hence it is a weak equivalence. $\square$

8. Some calculations

Armed with our earlier results, we revisit some of the calculations of [4, section 5].

First we consider the TAQ-Hurewicz homomorphism for the $E_\infty$ Thom spectrum $\text{MU}$. By work of Basterra and Mandell, in the homotopy category $\text{hM}_{\text{MU}}$,

$$\Omega_S(\text{MU}) \cong \text{MU} \wedge \Sigma^2 ku.$$

At the prime $p = 2$, the Hurewicz homomorphism factors through $H_*(\text{MU}; \mathbb{F}_2)$

$$\pi_*(\text{MU}) \xrightarrow{\theta} H_*(\text{MU}; \mathbb{F}_2) \xrightarrow{\theta'} \text{TAQ}_4(\text{MU}, S; \text{HPP}_2) \cong H_*(\Sigma^2 ku; \mathbb{F}_2)$$

where

$$H_*(\text{MU}; \mathbb{F}_2) \xrightarrow{\theta'} H_{*-2}(\Sigma^2 ku; \mathbb{F}_2)$$

is a derivation and

$$\theta'(b_r) = \begin{cases} \Sigma^2 \xi_s^2 & \text{if } r = 2^s, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\zeta_s = \xi(\xi_s)$ is the conjugate of the Milnor generator in $\mathbb{F}_2[\zeta_1^2, \zeta_2^2, \zeta_3, \ldots] \subseteq A(2)_s$. This tells us that the elements $b_2^r$ are not decomposable in terms of the Dyer-Lashof action, and recovers part of Kochman’s result Theorem B.2.
For an odd prime $p$, the TAQ-Hurewicz homomorphism decomposes into $(p - 1)$ pieces corresponding to the Adams splitting of $p$-local connective $K$-theory, giving
\[
\Sigma^{2ku(p)} \sim \bigvee_{1 \leq r \leq p-1} \Sigma^{2r}\ell.
\]
This gives an equivalent homomorphism
\[
\theta' : H_\ast(MU; \mathbb{F}_p) \longrightarrow \bigoplus_{1 \leq r \leq p-1} H_\ast(\Sigma^{2r}\ell; \mathbb{F}_p).
\]
Here
\[
H_\ast(\ell; \mathbb{F}_p) = \mathbb{F}_p[\zeta_i : i \geq 1] \otimes \lambda_\ell(\tau_j : j \geq 2) \subseteq A(p)_\ast,
\]
and the component corresponding to $\Sigma^{2r}\ell$ is determined in terms of generating functions by
\[
\sum_{i \geq 0} b_i t^i \mapsto t^r \left( \sum_{j \geq 0} \xi_j t^{p^j-1} \right)^r.
\]
It follows that $b_k \mapsto 0$ unless $k \equiv r \mod (p - 1)$. Write $k = np^e$ with $p \nmid n$, so $n \equiv r \mod (p - 1)$. Now set
\[
n - r = (p - 1)(s_0 + s_1 p + \cdots + s_d p^d)
\]
with $0 \leq s_i \leq p - 1$ and $s_d \neq 0$. Then we obtain
\[
b_{(s(p-1)+r)p^e} \mapsto \left( r - s_0, s_0 - s_1, \ldots, s_{d-1} - s_d, s_d \right) \xi_0^{r-s_0} \xi_{s_0-s_1} \xi_{e+1} \cdots \xi_{e+d-1}^{s_d-s_{d-1}} \xi_{e+d}^{s_{d-1}-s_d} \xi_{e+d+1}.
\]
Notice that this can only give a non-zero answer if the following inequalities are satisfied:
\[
1 \leq s_d \leq s_{d-2} \leq \cdots \leq s_1 \leq s_0 \leq r.
\]
In these cases $b_{(s(p-1)+r)p^e}$ must be Dyer-Lashof indecomposable, and so we again recover Kochman's odd primary result of Theorem B.2.

Here is another example, the reader is invited to compare it with that of $MU(2)$ and $MSp(2)$ in [3, proposition 5.1].

**Proposition 8.1.** The 2-local commutative $S$-algebra $MSU(2)$ is not minimal atomic.

**Proof.** We recall that $H_\ast(MSU; \mathbb{F}_2)$ is a polynomial algebra with a generator in each even degree greater than 2. There are many explicit generating families known, for example see [2,3]. In fact, $H_\ast(MSU; \mathbb{F}_2)$ can be identified as a subalgebra of $H_\ast(MU; \mathbb{F}_2)$, and then there are polynomial generators $a_n \in H_{2n}(MU; \mathbb{F}_2)$ so that
\[
H_\ast(MSU; \mathbb{F}_2) = \mathbb{F}_2[a_{2s}^2 : s \geq 0] \otimes \mathbb{F}_2[a_{2k} : s > 0, k > 1 \text{ odd}] \subseteq H_\ast(MU; \mathbb{F}_2).
\]
We will write $a'_n$ the generator in degree $2n$ where $n \geq 2$, for our purposes it is not important which choice we make here.

By [3, theorem 19(a)], the Dyer-Lashof indecomposables in $H_\ast(MSU; \mathbb{F}_2)$ are the algebra generators appearing in degrees of the form $2^m + 2^n$ where $m, n \geq 0$; this includes the case $2^s = 2^{s-1} + 2^{s-1}$ where $s \geq 1$.

Since there is a weak equivalence of infinite loop spaces $BSU \sim \Omega^{\infty} \Sigma^4 ku$, by [3],
\[
\Omega S(MSU) \cong MSU \wedge \Sigma^4 ku.
\]
Therefore the TAQ-Hurewicz homomorphism factors as

\[ \pi_\ast(MSU) \xrightarrow{\theta} H_\ast(MSU; \mathbb{F}_2) \xrightarrow{\theta'} \text{TAQ}_\ast(MSU, S; H\mathbb{F}_2) \cong H_\ast(\Sigma^4 ku; \mathbb{F}_2) \]

and in fact using the geometrically defined generators described in [2] it can be shown that

\[ \text{im } \theta' = \mathbb{F}_2 \{ \sum^4 \xi_m^2 \xi_n^2 : m, n \geq 1 \} . \]

Here \( \theta' \) has the effect

\[ a'_{2n+1} \mapsto \sum^4 \xi_n^4 \quad (n \geq 0), \quad a'_{2m+2n} \mapsto \sum^4 \xi_m^2 \xi_n^2 \quad (n > m \geq 0) . \]

However, this alone does not give us the result. We would like to use [4, theorems 3.2,3.4], so we must show that

\[ \theta : \pi_n(MSU) \rightarrow \text{TAQ}_n(MSU, S; H\mathbb{F}_2) \]

is non-trivial for some \( n > 0 \). For this we will use work of Pengelley [27] on the Adams spectral sequence for \( \pi_n(MSU_{(2)}) \). In [27, theorem 2.6] it is shown that there are polynomial generators \( y_{sk} \in H_{sk}(MSU; \mathbb{F}_2) \) for which the Adams differential \( d_2 \) satisfies

\[ d_2y_{sk} = \begin{cases} hq'_{k-1} \neq 0 & \text{if } k = 2^n, \\ 0 & \text{if } k \text{ is not a power of } 2, \end{cases} \]

and furthermore all trivial higher differentials in the spectral sequence are trivial. For our purposes what matters here is that each generator \( y'_{2m+3+2n+3} \) where \( n > m \geq 0 \) is in the image of the classical Hurewicz homomorphism and under the TAQ-homomorphism it maps to \( \Sigma^4 \xi_m^2 \xi_n^2 \neq 0 \). This means that \( MSU_{(2)} \) cannot be minimal atomic. \( \square \)

If the summand BoP of \( MSU_{(2)} \) were to have a commutative \( S \)-algebra structure, then Pengelley’s results would imply that the mod 2 TAQ-Hurewicz homomorphism was trivial, hence BoP would be minimal atomic. However, this depends on the observation that the classical mod 2 Hurewicz homomorphism is trivial so we already know it is minimal atomic as a spectrum [5] and hence it would be as a commutative \( S \)-algebra. So the use of TAQ would not be really necessary.

Here are some more examples.

**Example 8.2.** Let \( p \) be a prime and set \( H = H\mathbb{F}_p, H_\ast(\ast) = H_\ast(\ast; \mathbb{F}_p) \). Then TAQ-Hurewicz homomorphism

\[ \theta' : H_\ast(H) \rightarrow \text{TAQ}_\ast(H, S; H) \]

has the following effect on

\[ H_\ast(H) = A(p)_\ast = \begin{cases} \mathbb{F}_p[\xi_i : i \geq 1] \otimes \Lambda(\tau_j : j \geq 0) & \text{if } p \text{ is odd,} \\ \mathbb{F}_2[\xi_i : i \geq 1] & \text{if } p = 2. \end{cases} \]

When \( p \) is odd,

\[ \theta'(\tau_0) \neq 0, \quad \theta'(\tau_i) = \theta'(\xi_i) = 0 \quad (i \geq 1) . \]

When \( p = 2 \),

\[ \theta'(\xi_1) \neq 0, \quad \theta'(\xi_i) = 0 \quad (i \geq 2) . \]
The vanishing results follow from Steinberger’s calculations of Dyer-Lashof operations in [11, chapter III, theorem 2.3]. The non-triviality results use the fact that the unit \( S \to H \) is 0-connected, hence by Basterra [7, lemma 8.2], \( \Omega_S(H) \) is 0-connected, see also [4, corollary 1.3].

Next we will consider the case of \( MO \). The infinite loop space \( BO \) has Thom spectrum \( MO \) which admits the structure of an \( E_\infty \) ring spectrum or equivalently of a commutative \( S \)-algebra. By Thom’s theorem, this is known to split as a wedge of suspensions of \( H = HF_2 \) even as a ring spectrum

\[
MO \sim \bigvee_\alpha \Sigma^n H.
\]

But as we will see, no such splitting can happen in \( \Omega \mathcal{C}_S \) because of obstructions lying in TAQ. Here the underlying infinite loop space is \( BO \) and the associated spectrum is \( ko \langle 1 \rangle \), the 0-connected cover of \( ko \). In the above splitting, the generalized Eilenberg-Mac Lane ring spectrum on the right hand side realises the graded polynomial ring

\[
(8.1) \quad \pi_\ast(MO) = F_2[z_n : n \geq 1 \text{ is not of the form } 2^s - 1],
\]

where \( z_n \) has degree \( n \). For more on such ring spectra, see [10]. Let \( h : \pi_n(MO) \to H_n(MO) \) denote the usual mod 2 homology Hurewicz homomorphism. By Thom’s theorem, \( h \) is a monomorphism and for the polynomial generators \( z_n \) of \( (8.1) \), the Hurewicz images \( h(z_n) \) form part of a set of polynomial generators for \( H_\ast(MO) \) which has one generator in each positive degree.

By a result of Basterra and Mandell [8],

\[
\Omega_S(MO) = MO \wedge ko \langle 1 \rangle,
\]

where \( ko \langle 1 \rangle \) is the 0-connected cover of \( ko \), defined by the cofibre sequence of \( ko \)-modules

\[
ko \langle 1 \rangle \longrightarrow ko \longrightarrow HZ \longrightarrow \Sigma ko \langle 1 \rangle.
\]

On applying mod 2 homology \( H_\ast(\Sigma^{-1}ko) \) we obtain a short exact sequence

\[
0 \to H_\ast(\Sigma^{-1}H) \to H_\ast(\Sigma^{-1}HZ) \to H_\ast(ko \langle 1 \rangle) \to 0
\]

from which we deduce that as an \( H_\ast ko \)-module,

\[
(8.2) \quad H_\ast(ko \langle 1 \rangle) = H_\ast(ko)\{\Sigma^{-1}\zeta_1^2, \Sigma^{-1}\zeta_2, \Sigma^{-1}\zeta_1^2\zeta_2\},
\]

i.e., the free \( H_\ast(ko)\)-module on the generators \( \Sigma^{-1}\zeta_1^2, \Sigma^{-1}\zeta_2, \Sigma^{-1}\zeta_1^2\zeta_2 \) which have degrees 1, 2, 4 respectively.

We will make use of the TAQ-Hurewicz homomorphism

\[
\theta : \pi_n(MO) \to TAQ_n(MO, S; F_2) = H_n(ko \langle 1 \rangle),
\]

and so we need to understand the mod 2 homology \( H_\ast(ko \langle 1 \rangle) \). In the dual Steenrod algebra

\[
A(2)_\ast = H_\ast(H) = F_2[\xi_r : r \geq 1] = F_2[\zeta_r : r \geq 1],
\]

each generator \( \xi_r \in A(2)_{2r-1} \) is in the image of the natural map

\[
H_{2r}(\mathbb{R}P^\infty) \longrightarrow H_{2r}(\Sigma H) = A(2)_{2r-1},
\]

and \( \zeta_r = \chi(\xi_r) \), the Hopf-algebra conjugate of \( \xi_r \).
Now since $\pi_1(ko \langle 1 \rangle) = F_2$, there is a canonical non-trivial homotopy class $\psi: ko \langle 1 \rangle \to \Sigma H$ inducing an isomorphism on $\pi_1(-)$. The horizontal composition in the diagram

\[
\begin{array}{ccc}
HZ & \xrightarrow{\psi} & \Sigma ko \langle 1 \rangle \\
\downarrow \text{reduction mod 2} & & \downarrow \\
\text{Sq}^2 & & H
\end{array}
\]

factors as shown. In order to calculate the effect of the $H_s(ko)$-module homomorphism $\psi_s: H_s(ko \langle 1 \rangle) \to H_{s-1}(H)$, we first note that for $r = 1, 2$, the composition

$ko \to H \xrightarrow{\text{Sq}^r} \Sigma^2 H$

is trivial, hence it induces the trivial map on $H_s(ko)$. Using the Cartan formula for $\text{Sq}^2$, for any element $w \in H_s(ko)$ we obtain

\[(8.3) \quad \psi_s(w\Sigma^{-1}\zeta_2^2) = w, \quad \psi_s(w\Sigma^{-1}\zeta_2) = w\xi_1, \quad \psi_s(w\Sigma^{-1}\zeta_1^2\zeta_2) = w(\zeta_2 + \zeta_2^3) = w\xi_2.\]

In particular it follows that $\psi_s: H_s(ko \langle 1 \rangle) \to H_{s-1}(H)$ is a monomorphism. We also note that the factorisation of $\eta: \Sigma ko \to ko$ through a $ko$-module map $\tilde{\eta}: \Sigma ko \to ko \langle 1 \rangle$ induces

$\tilde{\eta}_s: H_s(\Sigma ko) \to H_s(ko \langle 1 \rangle); \quad \tilde{\eta}_s(w) = w\Sigma^{-1}\zeta_1^2$.

**Proposition 8.3.** For any choice of generators $z_n$ in (8.1), the TAQ-Hurewicz homomorphism $\theta: \pi_* (MO) \to H_*(ko \langle 1 \rangle)$ satisfies

$\theta(z_n) = \begin{dcases} 
0 & \text{if } n \neq 2^s, \\
\Sigma^{-1}\zeta_2 & \text{if } n = 2, \\
\Sigma^{-1}\zeta_1^2\zeta_2 & \text{if } n = 4, \\
\Sigma^{-1}\zeta_1^2\zeta_s & \text{if } n = 2^s \text{ with } s \geq 3.
\end{dcases}$

Hence $MO$ is not a minimal atomic 2-complete commutative $S$-algebra.

**Proof.** Choose polynomial generators $a_n \in H_n(MO)$ so that when $n + 1$ is not a power of 2, $h(z_n) = a_n$.

Note that Kochman’s results in [15] give the action of the Dyer-Lashof operations on $H_*(BO)$ and the Dyer-Lashof indecomposables are spanned by the polynomial generators $a_{2^s}$ for $s \geq 0$. Thus we should only expect $h(z_n)$ to be non-zero when $n = 2^s$ for some $s \geq 0$.

The calculation of $\psi_s \circ \theta$ require similar methods to those used for $MU$ in [4] section 3]. The crucial point is the determination of the homomorphism

$H_*(\mathbb{RP}^\infty) \to H_*(BO) \to H_*(ko \langle 1 \rangle)$

induced by the natural inclusion $\mathbb{RP}^\infty \to BO$ and the evaluation

$\Sigma^\infty BO = \Sigma^\infty \Omega^\infty \Sigma^\infty ko \to ko$.

Composing with $\psi$ and applying homology we obtain

$H_*(\mathbb{RP}^\infty) \to H_*(ko \langle 1 \rangle) \xrightarrow{\psi_s} H_*(\Sigma H) = A(2)_{s-1},$
where $\psi$ is monic. Since $H^1(\mathbb{R}P^\infty) = \mathbb{F}_2$, our composition is the standard one which maps the generator $\gamma_n \in H_n(\mathbb{R}P^\infty)$ according to the rule

$$\gamma_n \mapsto \begin{cases} 
\xi_s & \text{if } n = 2^s - 1, \\
0 & \text{otherwise}.
\end{cases}$$

Using (8.3) we see that $\theta'(a_{2^s})$ has the form claimed.

The statement about $MO$ not being minimal atomic follows from Thom’s result since by definition, for each $s \geq 1$, $a_{2^s}$ is the Hurewicz image of a homotopy element. □

For completeness, we mention the following result which appeared in the unpublished preprint of Krč [16], unpublished work of Basterra and Mandell, and Lazarev [19].

**Theorem 8.4.** There is an isomorphism

$$\text{TAQ}^*(HF_2, S; HF_2) \cong \mathbb{F}_2[\Sigma SQ^I : I = (i_1, \ldots, i_t) \text{ admissible, } i_t \geq 4].$$

Here the symbols $SQ^I$ behave like the analogous symbols $Sq^I$ in the Steenrod algebra $A(2)^*$, and we regard the empty sequence as admissible. However, the right hand side should not be regarded as a module over the Steenrod algebra $A(2)^*$, and this is merely an isomorphism of vector spaces. Here the suspension $\Sigma(-)$ indicates a degree shift of $+1$. There is a duality between $\text{TAQ}^*(HF_2, S; HF_2)$ and $\text{TAQ}_n(HF_2, S; HF_2)$, i.e.,

$$\text{TAQ}^n(HF_2, S; HF_2) \cong \text{Hom}_F(\text{TAQ}_n(HF_2, S; HF_2), \mathbb{F}_2).$$

We end with a result that is essentially a generalisation of [14, proposition 2.11]; several examples of this kind were given in Helen Gilmour’s thesis [13]. In the planned part II of this work, we will describe computations in the setting of a Miller-type spectral sequence for computing $\text{TAQ}$ which also lead to such results.

**Proposition 8.5.** There is no morphism of commutative $S$-algebras $HF_2 \rightarrow MO$.

**Proof.** Once again we set $H = HF_2$.

If such a morphism $H \rightarrow MO$ existed, the generator $\zeta_1 \in H_1H = A(2)_1$ would map to the algebra generator $a_1 \in H_1MO$. Using the Dyer-Lashof action calculated by Kochman [15], see Theorem [B.1], we have

$$Q^4 a_1 \equiv a_5 \mod \text{decomposables}.$$  

As in the proof of [14, proposition 2.11], this leads to a contradiction since there is no degree 5 indecomposable in $A(2)_*$.

We will not give the details here, but it seems worth mentioning that the Thom spectrum $MU/O$ associated to the infinite loop space $U/O$ which is the fibre in the sequence

$$U/O \rightarrow BO \rightarrow BU,$$

is a core for $MO$. It turns out that $H_*(MU/O; \mathbb{F}_2)$ embeds into $H_*(MO; \mathbb{F}_2)$ as a polynomial subalgebra on odd degree generators the only Dyer-Lashof indecomposable has degree 1. In fact

$$\Omega_5(MU/O) \cong MU/O \wedge \Sigma ko$$

and so

$$\text{TAQ}_n(MU/O, S; HF_2) = H_*(\Sigma ko; \mathbb{F}_2).$$
and under the TAQ-Hurewicz homomorphism the Dyer-Lashof indecomposable generator is sent to $\Sigma 1$.

**APPENDIX A. A proof and a Lemma**

For completeness we outline a proof of [4, proposition 1.6], due to Philipp Reinhard; unfortunately this was only produced after that paper was published. Our approach is similar to that of McCarthy and Minasian in [25, theorem 6.1], however this appears to be incorrect as stated (at one stage they seem to assume that $M$ is an algebra).

**Proposition A.1.** Let $R$ be a commutative $S$-algebra and let $X$ be a cofibrant $R$-module. Then there is a weak equivalence of $\mathbb{P}_R X$-modules

$$\Omega_R(\mathbb{P}_R X) \sim \mathbb{P}_R X \wedge_R X.$$  

**Proof.** For every $M \in \mathcal{M}_{\mathbb{P}_R X}$ there is an adjunction

$$\mathcal{C}_R/\mathbb{P}_R X(\mathbb{P}_R X, \mathbb{P}_R X \vee M) \cong \mathcal{M}_R/\mathbb{P}_R X(X, \mathbb{P}_R X \vee M),$$

where $\mathcal{M}_R/\mathbb{P}_R X$ denotes the category of $R$-modules over $\mathbb{P}_R X$. Because the forgetful functor $\mathcal{C}_R/\mathbb{P}_R X \rightarrow \mathcal{M}_R/\mathbb{P}_R X$ respects fibrations and acyclic fibrations, the adjunction passes to homotopy categories, giving

$$\overline{\mathcal{C}}_R/\mathbb{P}_R X(\mathbb{P}_R X, \mathbb{P}_R X \vee M) \cong \overline{\mathcal{M}}_R/\mathbb{P}_R X(X, \mathbb{P}_R X \vee M).$$

Now we have

$$\mathcal{M}_R/\mathbb{P}_R X(\mathbb{P}_R X, M) \cong \mathcal{M}_R/X(X, X \vee M)$$

and the adjunction again passes to homotopy categories and gives

$$\overline{\mathcal{M}}_R/\mathbb{P}_R X(\mathbb{P}_R X, M) \cong \overline{\mathcal{M}}_R/X(X, X \vee M).$$

Since in the homotopy category $X \vee M$ is the product of $X$ and $M$, we have

$$\overline{\mathcal{M}}_R/X(X, X \vee M) \cong \overline{\mathcal{M}}_R(X, M).$$

By using the free functor from $R$-modules to $\mathbb{P}_R X$-modules, we obtain

$$\overline{\mathcal{M}}_{\mathbb{P}_R X}(X, X \vee M) \cong \overline{\mathcal{M}}_{\mathbb{P}_R X}(\mathbb{P}_R X \wedge_R X, M).$$

Thus we have shown that

$$\overline{\mathcal{M}}_{\mathbb{P}_R X}(\Omega_R(\mathbb{P}_R X), M) \cong \overline{\mathcal{M}}_{\mathbb{P}_R X}(\mathbb{P}_R X, \mathbb{P}_R X \vee M) \cong \overline{\mathcal{M}}_{\mathbb{P}_R X}(\mathbb{P}_R X \wedge_R X, M).$$

Using Yoneda’s lemma, we obtain the desired equivalence

$$\Omega_R(\mathbb{P}_R X) \sim \mathbb{P}_R X \wedge_R X.$$

We also give a useful result on the adjunction for a commutative $R$-algebra. Let $A$ be a cofibrant commutative $R$-algebra and let

$$A^c \xrightarrow{\sim} A$$

be its functorial cofibrant replacement in the model category of $R$-modules $\mathcal{M}_R$. Let

$$\mu: \mathbb{P}_R A^c \rightarrow \mathbb{P}_R A \rightarrow A$$
be the extension of the multiplication. We have

$$\Omega_R(\mathbb{P}_R A^c) \cong \mathbb{P}_R A^c \land_R A^c,$$

and also the $A$-module $\Omega_R(A)$ becomes a $\mathbb{P}_R A^c$-module via pullback along $\bar{\mu}$. Writing $\delta$ (without decorations) for universal derivations, we obtain a commutative diagram in $\mathcal{M}_R$ (with the pentagon commuting in $\mathcal{M}_{\mathbb{P}_R A^c}$).

\[
\begin{array}{ccc}
R \land_R A^c & \cong & \mathbb{P}_R A^c \\
\downarrow \sim & \delta & \downarrow \bar{\mu} \land \delta \\
A^c & \to & \mathbb{P}_R A^c \land_R A^c \\
\parallel & \downarrow \omega(\bar{\mu}) & \downarrow \omega(\bar{\mu}) \\
A & \to & \Omega_R(A) \\
\parallel & \downarrow \mult & \downarrow \mult \\
& & \Omega_R(A)
\end{array}
\]

(A.1)

Here $\omega(\bar{\mu})$ denotes the induced ‘derivative’ morphism $\Omega_R(\mathbb{P}_R A^c) \to \Omega_R(A)$.

**Lemma A.2.** Suppose that $M$ is an $A$-module and therefore an $\mathbb{P}_R A^c$-module. Then the induced morphism on $\text{TAQ}_*(-)$, $\bar{\mu}_*$, is given by the following commutative diagram.

\[
\begin{array}{ccc}
\text{T AQ}_* (\mathbb{P}_R A^c, R; M) & \xrightarrow{\bar{\mu}_*} & \text{T AQ}_* (A, R; M) \\
\parallel & \parallel & \parallel \\
\pi_*(M \land_R A^c) & \xrightarrow{(I \land \delta(A,R))^*} & \pi_*(M \land_A \Omega_R(A)) \\
\parallel & \parallel & \parallel \\
\pi_*(M \land_R \Omega_R(A)) & \xrightarrow{(I \land \delta(A,R))^*} & \pi_*(M \land_A \Omega_R(A))
\end{array}
\]

**Proof.** This is obtained by applying $\pi_*(M \land_R -)$ to (A.1). \qed

Since the universal derivation restricts trivially to $R$, there is an induced map

$$\bar{\delta}_{(A,R)} : A^c / S_R^0 \to \Omega_R(A).$$

So for the reduced free algebra there is a similar commutative diagram.

\[
\begin{array}{ccc}
\text{T AQ}_* (\mathbb{P}_R A^c, R; M) & \xrightarrow{\bar{\mu}_*} & \text{T AQ}_* (A, R; M) \\
\parallel & \parallel & \parallel \\
\pi_*(M \land_R A^c / S_R^0) & \xrightarrow{(I \land \delta(A,R))^*} & \pi_*(M \land_A \Omega_R(A)) \\
\parallel & \parallel & \parallel \\
\pi_*(M \land_R \Omega_R(A)) & \xrightarrow{(I \land \delta(A,R))^*} & \pi_*(M \land_A \Omega_R(A))
\end{array}
\]
APPENDIX B. SOME FORMULAE

We begin by recalling formula due to Kochman [15]. Actually his results are for the infinite loop spaces such as $BU$, but the Thom isomorphism commutes with the Dyer-Lashof operations so we will interpret them in the homology of the Thom spectrum $MU$ with its $E_{\infty}$ structure inherited from that of $BU$.

Let $p$ be a prime. We will write $H_*(\mathbb{Z}) = H_*(-; \mathbb{F}_p)$. Let $b_r \in H_{2r}(MU)$ be the generator obtained as the image of $\beta_{r+1} \in H_{2r+2}(MU(1)) \cong H_{2r+2}(\mathbb{C}P^\infty)$ under the homomorphism induced by the canonical map $MU(1) \to \Sigma^2 MU$ as in [1]. We will use the notation $x \equiv y$ as shorthand for $x \equiv y \mod \text{decomposables}$. We also interchangeably use the notations

$$(a, b) = (b, a) = \left(\begin{array}{c} a + b \\ b \end{array}\right) = \left(\begin{array}{c} a + b \\ a \end{array}\right)$$

for binomial coefficients, where this is taken to be zero if $a < 0$ or $b < 0$. We will use the well-known congruence

$$(B.1) \quad \left(\begin{array}{c} n_0 + n_1 p + \cdots + n_k p^k \\ m_0 + m_1 p + \cdots + m_k p^k \end{array}\right) \equiv \left(\begin{array}{c} n_0 \\ m_0 \end{array}\right) \cdots \left(\begin{array}{c} n_k \\ m_k \end{array}\right) \mod p$$

when $0 \leq m_i, n_i \leq p - 1$.

**Theorem B.1.** In $H_*(MU)$ we have

- if $p$ is odd, $Q^r b_n \approx (-1)^{r+n+1}(n, r - n - 1)b_{n+r(p-1)}$,

- if $p = 2$, $Q^{2r} b_n \approx (n, r - n - 1)b_{n+r}$.

Note that in the $p = 2$ case there are analogous results for $H_*(MO; \mathbb{F}_2)$.

The Dyer-Lashof operations annihilate 1 and the Cartan formula implies that they act on the indecomposable quotient. In [15, theorem 10], Kochman determined the indecomposable generators which are not in the image of any Dyer-Lashof operations of positive degree. We set

$$Q_{DL}H_*(MU) = QH_*(MU)/\{Q^s x : s \geq 1, x \in QH_*(MU)\}.$$

**Theorem B.2.** The indecomposables $Q_{DL}H_*(MU)$ have the following elements as a basis:

- if $p$ is odd, $b_{np^r}$ where $p \nmid n$ and $n = (\sum_{i=0}^{k} s_i p^{i})(p - 1) + r$ with $r = 1, 2, \ldots, p - 1$ and if $\sum_{i=0}^{k} s_i p^{i} \neq 0$, $0 \leq s_i \leq (p - 1)$ and $1 \leq s_k \leq s_{k-1} \leq \cdots \leq s_0 \leq r$,

- if $p = 2$, $b_{2^t}$ where $t \geq 0$.

The Dyer-Lashof indecomposability of the stated generators can be deduced from our results on the TAQ-Hurewicz homomorphism. As an exercise in computing with binomial coefficients modulo a prime, we have

**Proposition B.3.** Suppose that $p$ is a prime and $n$ has $p$-adic expansion

$$n = n_sp^s + \cdots n_{s+t}p^{s+t}$$

where $n_s \neq 0 \neq n_{s+t}$ and $t > 0$.

If $p$ is odd then

$$Q^{n-n_sp^s} b_{n_sp^s} \approx \pm \left(\begin{array}{c} n - n_sp^s - 1 \\ n_sp^s \end{array}\right) b_n \neq 0.$$
If $p = 2$ then

$$Q^{2n-2s+1} b_{2s} \approx \left( n - 2s - 1 \right) b_n \not\approx 0.$$ 

Proof. In each case working modulo $p$ we have

$$\left( n - n_s p^s - 1 \right) \equiv \left( n - n_s p^s - p^k \right) + \left( p^k - 1 \right) \equiv \left( n - n_s p^s - p^k \right) \left( p^k - 1 \right) \not\equiv 0,$$

where $p^k$ is the highest power of $p$ dividing $(n - n_s p^s)$, and we use the fact that

$$p^k - 1 = (p - 1)p^{k-1} + \cdots + (p - 1)p^s + \cdots (p - 1)p + (p - 1)$$

with $n_s \leq p - 1$. \qed

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