ESSENTIAL NORMS OF WEIGHTED COMPOSITION OPERATORS BETWEEN HARDY SPACES IN THE UNIT BALL

ZHONG-SHAN FANG AND ZE-HUA ZHOU *

Abstract. Let \( \varphi(z) = (\varphi_1(z), \cdots, \varphi_n(z)) \) be a holomorphic self-map of \( B_n \) and \( \psi(z) \) a holomorphic function on \( B_n \), and \( H(B_n) \) the class of all holomorphic functions on \( B_n \), where \( B_n \) is the unit ball of \( C^n \), the weight composition operator \( W_{\psi,\varphi} \) is defined by \( W_{\psi,\varphi} = \psi f(\varphi) \) for \( f \in H(B_n) \). In this paper we estimate the essential norm for the weighted composition operator \( W_{\psi,\varphi} \) acting from the Hardy space \( H^p \) to \( H^q \) (\( 0 < p, q \leq \infty \)). When \( p = \infty \) and \( q = 2 \), we give an exact formula for the essential norm. As their applications, we also obtain some sufficient and necessary conditions for the bounded weighted composition operator to be compact from \( H^p \) to \( H^q \).

1. Introduction

Let \( B_n \) be the unit ball of \( C^n \) with boundary \( \partial B_n \), \( \sigma \) the normalized rotation invariant measure on \( \partial B_n \). The class of all holomorphic functions on domain \( B_n \) will be denoted by \( H(B_n) \). Let \( \varphi(z) = (\varphi_1(z), \cdots, \varphi_n(z)) \) be a holomorphic self-map of \( B_n \) and \( \psi(z) \) is in \( H(B_n) \). Multiplication operator, Composition operator and weighted composition operator are defined as follows:

\[
M_{\psi}(f)(z) = \psi(z) \cdot f(z);
\]

\[
C_{\varphi}(f)(z) = f(\varphi(z));
\]

\[
W_{\psi,\varphi}(f)(z) = \psi(z) \cdot f(\varphi(z))
\]

for any \( f \in H(B_n) \) and \( z \in B_n \).

If let \( \psi \equiv 1 \), then \( W_{\psi,\varphi} = C_{\varphi} \); if let \( \varphi = Id \), then \( W_{\psi,\varphi} = M_{\psi} \). So we can regard weighted composition operator as a generalization of a multiplication operator and a composition operator. It is easy to show that \( C_{\varphi} \) and \( W_{\psi,\varphi} \) take \( H(B_n) \) into itself.

Shapiro’s monograph \cite{Shap1} gives an interesting account of these developments. See also Cowen and MacCluer’s book \cite{CowMac} for a comprehensive treatment of these and other related problems with composition operators.

In the recent years, boundedness and compactness of composition operators between several spaces of holomorphic functions have been studied by many authors: by Smith \cite{Smi1} between Bergman and Hardy spaces, by Jarchow and Ried \cite{JarR} between generalized Bloch-type spaces and Hardy spaces, between Bloch spaces and Besov spaces and BMOA and VMOA in Tian’s thesis \cite{JarR}, on BMOA by Simth \cite{Smi2}, and by Simth and Zhao \cite{SmiZ} from Bergman and Hardy spaces and

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Blanch space into $Q_p$ spaces. All of papers above focus on studying the composition operators in function spaces for 1-dimensional case.

More recently, there have been many papers focused on studying the same problems for $n$-dimensional case: by Luo and Shi [LS1] between Hardy spaces on the unit ball. [LS2] weighted Bergman spaces on bounded symmetric domains, by Zhou and Shi [ZS1] [ZS2] [ZS3] on the Bloch space in polydisk or classical symmetric domains, Gorkin and MacCluer [GorM] between hardy spaces in the unit ball, and Lipschitz space in polydisc by Zhou [Zho]. In all these works the main goal is to relate function theoretic properties of $L_1$ spaces to boundedness and compactness of $C_\phi$.

The essential norm of an operator $T$ is by definition its distance to the compact operators; that is

$$||T||_e := \inf \{|T - K| : K \text{ compact}\}.$$  

Notice that $||T||_e = 0$ if and only if $T$ is compact, so that estimates on $||T||_e$ lead to the conditions for $T$ to be compact.

In general, there is no easy way to determine the essential norms of composition operator or weighted composition operator.

Let $f$ be in $H(B_n)$. For $0 < p < \infty$, $f$ is said to be in the Hardy space $H^p(B_n)$ provided that

$$||f||_p = \sup_{0<r<1} \int_{\partial B_n} |f(r\xi)|^p d\sigma(\xi) < \infty.$$  

The Banach space of bounded holomorphic functions on $B_n$ in the sup norm is denoted by $H^\infty$.

When $f \in H^p$, then $f$ has radial limits at almost every ([d$\sigma$]) point of $\partial B_n$, and its $H^p$ norm is also given by the $L^p(d\sigma)$ norm of its radial limit function $f^*$. That is

$$||f||_p = \int_{\partial B_n} |f^*(\xi)|^p d\sigma(\xi).$$

Typically we continue to write $f(\xi)$ for the radial limit; occasionally for clarity we use the special notation $f^*(\xi)$ for $\lim_{r \to 1} f(r\xi)$. In the whole of paper, $E = \{\xi \in \partial B_n : |f^*(\xi)| = 1\}$, which we call it the extreme set of $\varphi$.

It is well known that $C_\varphi$ is always bounded on $H^p(D)$ for $0 < p \leq \infty$, this is a consequence of a theorem of J. Littlewood, see [CowMac], where $D = B_1$ is an unit disk. In 1987, J. Shapiro [Shap2] determined precisely when $C_\varphi$ acts compactly on $H^p(D)$, for $p < \infty$, and gave a formula for the essential norm of $C_\varphi$ acting on $H^2(D)$ in terms of the Nevanlinna counting function for $\varphi$. In 2002, L. Zheng [Zhe] proved the essential norm of $C_\varphi$ acting on $H^\infty(D)$ is 1 whenever $C_\varphi$ is not compact on $H^\infty(D)$ (equivalently, whenever $||\varphi||_\infty = 1$); it is also true when $D$ is replaced by the unit ball [GorMS]. For $0 \leq p < q > 0$, $C_\varphi$ acting from $H^p(D)$ to $H^q(D)$ will of course be bounded. H. Jarchow [JarN] and T. Goebeler [Goe] shew independently that $C_\varphi$ is compact if and only if $|E| = 0$.

It seems reasonable to expect the essential norm to be given by a formula that involves $|E|$. In fact, P.Gorkin and B.MacCluer [GorM] pointed out the essential norm of $C_\varphi$ acting from $H^\infty(D)$ to $H^2(D)$ is precisely $|E|^2$, and they have obtained the same results in the setting of Hardy spaces $H^p(B_n)$ (we write it $H^p$ in the following) and also gave some simple estimates for the essential norm of a composition operator acting from $H^\infty$ to $H^q$ for $q \neq 2$ and for $0 < p < \infty$, from $H^p$ to $H^q$ under a natural additional condition. Here the additional condition is that there exists
0 < p < ∞ such that $C_\varphi : H^p \to H^p$ is bounded, which is naturally satisfied in the case $n = 1$. This assumption has two properties of interest to us:

1. No set of positive measure in $\partial B_n$ is mapped by $\varphi^*$ to a set of measure 0 in $\partial B_n$ (see Corollary 3.38 of [GorM]);
2. If $f \in H^p(B_n)$, then for a.e. $\sigma_\xi \in \partial B_n$, $(f \circ \varphi)^*(\xi) = f^*(\varphi^*(\xi))$ (see Lemma 1.6 in [Mac]).

In our paper, in addition to extend corresponding cases in [GorM] to the weighted composition operator, we also get the lower estimates for the essential norm of a weighted composition operator from $H^p$ to $H^q$ for $1 < p \leq q \leq \infty$.

The remainder of the present paper is assembled as follows: In section 2, we refer the reader some Lemmas which needs in next sections. In section 3, we will show that the essential norm of the bounded weighted composition operator $W_{\psi, \varphi}$ is precisely $\left( \mu_{\psi, \varphi, 2}(\varphi(E)) \right)^{1/2}$ for the case $p = \infty, q = 2$ (Theorem 3.1), and give an estimate for the case $p = \infty, q \neq 2$ (Theorem 3.2). In section 4, we give the upper estimate for the case $1 < p < \infty$ (Theorem 4.1) and lower estimate for the case $1 < q < p < \infty$ (Theorem 4.2). The fundamental ideas of the proof are those used by Gorkin and MacCluer in [GorM], but some new techniques are still used in this section because of the citation of the new measure induced by $\psi$ and $\varphi$ and the difference between weighted composition operator and composition operator. If $\psi = 1$, $W_{\psi, \varphi} = C_\varphi$, we can completely the corresponding results in [GorM].

In sections 5 and 6 (not be considered in Gorkin and MacCluerin’s paper), using different methods, we obtain some estimates for the essential norms of the weighted operator acting from $H^p$ to $H^\infty$ for $p > 1$ (Theorem 5.2) and from $H^p$ to $H^q$ for $1 < p \leq q < \infty$ (Theorem 6.2). All of them are done under the same additional condition. As their applications, we also obtained some sufficient and necessary conditions for the weighted composition operator to be compact from $H^p$ to $H^q$ for the above cases. For convenience, we always abbreviate $H^p(B_n)$ to $H^p$.

2. Some Lemmas

**Lemma 2.1.** Let $\varphi$ is holomorphic self-map of $B_n$ and $\psi \in H^p$, where $0 < p < \infty$. For any measurable subset $E$ of $\partial B_n$, denote $\mu_{\psi, \varphi, p}(E) = \int_{\varphi^{-1}(E) \cap \partial B_n} |\psi|^p d\sigma$. Then

$$\int_{\partial B_n} g d\mu_{\psi, \varphi, p} = \int_{\partial B_n} |\psi|^p(g \circ \varphi) d\sigma,$$

where $g$ is an arbitrary measurable positive function in $\overline{B}_n$.

**Proof** If $g$ is a measurable simple function defined on $\overline{B}_n$ given by $g = \sum_{i=1}^n \alpha_i \chi_{E_i}$, then

$$\int_{\overline{B}_n} g d\mu_{\psi, \varphi, p} = \sum_{i=1}^n \alpha_i \mu_{\psi, \varphi, p}(E_i) = \sum_{i=1}^n \alpha_i \int_{\varphi^{-1}(E_i) \cap \partial B_n} |\psi|^p d\sigma = \int_{\partial B_n} |\psi|^p(\sum_{i=1}^n \alpha_i \chi_{\varphi^{-1}(E_i) \cap \partial B_n}) d\sigma = \int_{\partial B_n} |\psi|^p(g \circ \varphi) d\sigma.$$
Now, if $g$ is a measurable positive function in $\overline{B}_n$, then we can take an increasing sequence $\{g_m\}$ of positive and simple functions such that $g_m(z) \to g(z)$ for all $z \in \overline{B}_n$, it follows that

$$\int_{\overline{B}_n} g_m d\mu_{\psi, \varphi, p} \to \int_{\overline{B}_n} g d\mu_{\psi, \varphi, p}.$$  

On the other hand, $|\psi|^p(g_m \circ \varphi)$ is an increasing sequence such that

$$|\psi(z)|^p(g_m(\varphi(z))) \to |\psi(z)|^p(g(\varphi(z)))$$

for all $z \in \overline{B}_n$, so

$$\int_{\overline{B}_n} g_m d\mu_{\psi, \varphi, p} = \int_{\partial B_n} |\psi|^p(g_m \circ \varphi)d\sigma \to \int_{\partial B_n} |\psi|^p(g \circ \varphi)d\sigma.$$  

And the conclusion follows by the uniqueness of the limit.

**Lemma 2.2.** (See p116 in [Zhu2]) Suppose $0 < p < \infty$ and $f \in H^p$. Then $|f(z)| \leq \frac{|f|_p}{(1-|z|^p)_p}$ for all $z \in B_n$.

**Lemma 2.3.** Let $\Omega$ be a domain in $C^n$, $f \in H(\Omega)$. If a compact set $K$ and its neighborhood $G$ satisfy $K \subset G \subset \subset \Omega$ and $\rho = \text{dist}(K, \partial G) > 0$, then

$$\sup_{z \in K} \frac{\partial f}{\partial z_j}(z) \leq \frac{\sqrt{n}}{\rho} \sup_{z \in G} |f(z)|.$$  

**Proof** Since $\rho = \text{dist}(K, \partial G) > 0$, for any $a \in K$, the polydisc

$$P_a = \{(z_1, \cdots, z_n) \in C^n : |z_j - a_j| < \frac{\rho}{\sqrt{n}}, j = 1, \cdots, n\}$$

is contained in $G$. Using Cauchy inequality, we have

$$\left|\frac{\partial f}{\partial z_j}(a)\right| \leq \frac{\sqrt{n}}{\rho} \sup_{z \in \partial P_a} |f(z)| \leq \frac{\sqrt{n}}{\rho} \sup_{z \in G} |f(z)|.$$  

So the Lemma follows.

**Lemma 2.4.** For fixed $0 < \delta < 1$, let $G = \{z \in B_n : |z| \leq 1 - \delta\}$. Then

$$\limsup_{r \to 1} \sup_{z \in G} |f(z) - f(rz)| = 0$$

for any $f \in H^p(B_n)$.

**Proof**

$$\sup_{z \in G} |f(z) - f(rz)| = \sup_{z \in G} \left|\sum_{j=1}^n (f(rz_1, rz_2, \cdots, rz_{j-1}, z_j, \cdots, z_n) - f(rz_1, rz_2, \cdots, rz_j, z_{j+1}, \cdots, z_n))\right|$$

$$\leq \sup_{z \in G} \sum_{j=1}^n \left|\int_0^1 |z_j - t z_j| \frac{\partial f}{\partial z_j}(rz_1, rz_2, \cdots, rz_{j-1}, tz_j, z_{j+1}, \cdots, z_n) dt\right|$$

$$\leq (1 - r)n \sup_{z \in G} \left|\frac{\partial f}{\partial z_j}(z)\right|.$$  

Define $G_1 = \{z \in B_n : |z| \leq 1 - \frac{\delta}{2}\}$, then $G \subset G_1$ and $\text{dist}(G, \partial G_1) = \frac{\delta}{2}$. 

It follows from Lemma 2.3 that
\[
\sup_{z \in G} \left| \frac{\partial f}{\partial z_j}(z) \right| \leq \frac{2\sqrt{n}}{\delta} \sup_{z \in G_1} |f(z)|.
\]
If \( p = \infty \), then
\[
\sup_{z \in G} |f(z) - f(rz)| \leq \frac{2(1-r)n\sqrt{n}}{\delta} ||f||_\infty.
\]
For \( 0 < p < \infty \), it follows from Lemma 2.2 that
\[
\sup_{z \in G} |f(z) - f(rz)| \leq \frac{2(1-r)n\sqrt{n}}{\delta} \sup_{z \in G_1} ||f||_p (1 - |z|^n)^{n/p} \leq \frac{2(1-r)n\sqrt{n}}{\delta} ||f||_p \left( \frac{2}{3} \right)^{n/p}.
\]
Let \( r \to 1 \), the conclusion follows.

**Lemma 2.5.** (See corollary 1.3 in [CowMac] ) A sequence in a reflexive functional Banach space converges weakly if and only if it is bounded and converges point-wise.

**Lemma 2.6.** Assume \{\( f_m \)\} is a bounded sequence in \( H^p(B_n)(p > 1) \), and \{\( f_m \)\} converges weakly to 0, then for any compact operator \( K \) from \( H^p(B_n) \) to \( Y \) (\( Y \) is a normalized linear space), we have \( \|Kf_m\|_Y \to 0 \).

**Proof** This is easily followed by Lemma 2.5 and the property of compact operator.

3. FROM \( H^\infty \) TO \( H^q \)

**Case 1.** \( p = \infty, q = 2 \)

It is well known that for any \( f \in H(B_n) \), \( f \) has homogeneous expansion \( f(z) = \sum_{s=0}^\infty F_s(z) \), where \( F_s(z) \) is the homogeneous polynomial \( \sum_{|\alpha| = s} c(\alpha)z^\alpha \), \( z^\alpha = z^{\alpha_1} \cdots z^{\alpha_n} \), \( \alpha = (\alpha_1, \cdots, \alpha_n) \), and \( |\alpha| = \alpha_1 + \cdots \alpha_n \).

If \( f \in H^2(B_n) \), then
\[
||f||_2^2 = \sum_{\alpha} |c(\alpha)|^2 ||z^\alpha||_2^2,
\]
where
\[
||z^\alpha||_2^2 = \frac{(n-1)!|\alpha|}{(n-1 + |\alpha|)!},
\]
where \( \{\frac{z^\alpha}{||z^\alpha||_2} \} \) is an orthonormal basis for \( H^2(B_n) \), and \( c(\alpha) = D^\alpha f(0)/\alpha! \) with \( \alpha! = \alpha_1! \cdots \alpha_n! \). If necessary, we refer the reader to see [Rud].

For \( m \) a positive integer, define the operators from \( H^2(B_n) \) to itself:
\[
R_m(\sum_{s=0}^\infty F_s) = \sum_{s=m+1}^\infty F_s
\]
and
\[
Q_m = I - R_m.
\]
It is easy to show that \( R_m \) is compact and \( \|R_m\| = 1 \).
Lemma 3.1. $W_{\psi, \varphi} : H^\infty \to H^q$, $0 < q < \infty$ is bounded if and only if $\psi \in H^q$.

Proof If $W_{\psi, \varphi}$ is bounded, let $f = 1$, then $W_{\psi, \varphi}f = \psi f(\varphi) = \psi \in H^q$. Conversely, apparently we have $||W_{\psi, \varphi}f||_q \leq ||\psi||_q ||f||_\infty$ for any $f \in H^\infty$, that is, $||W_{\psi, \varphi}|| \leq ||\psi||_q^q$.

Using the same methods as that of Gorkin-MacCluer in [GorM], with minor modifications, we can obtain the following Lemmas 3.2 and 3.3. But for the reader’s convenience, we give still the detail proof for the results.

Lemma 3.2. If $W_{\psi, \varphi} : H^\infty \to H^2$ and $\psi \in H^2$, then

$$||W_{\psi, \varphi}||_e = \lim_{m \to \infty} ||R_m W_{\psi, \varphi}||.$$ 

Proof On one hand, by hypothesis and Lemma 3.1, we know $W_{\psi, \varphi}$ is bounded, so the compactness of $Q_m$ implies that $Q_m W_{\psi, \varphi}$ is also compact,

$$||W_{\psi, \varphi}||_e = ||(R_m + Q_m)W_{\psi, \varphi}||_e = ||R_m W_{\psi, \varphi}||_e \leq ||R_m W_{\psi, \varphi}||,$$ 

it follows that

$$||W_{\psi, \varphi}||_e \leq \lim inf_{n \to \infty} ||R_m W_{\psi, \varphi}||.$$

On the other hand, let $K : H^\infty \to H^2$ be compact. Since $||R_m|| = 1$,

$$||W_{\psi, \varphi} - K|| \geq ||R_m (W_{\psi, \varphi} - K)|| = ||R_m W_{\psi, \varphi} - R_m K|| \geq ||R_m W_{\psi, \varphi}|| - ||R_m K||.$$

Note that $K$ is compact, the image of the unit ball in $H^\infty$ under $K$ has compact closure in $H^2$. Since $||R_m|| = 1$ and $R_m K$ tends to 0 point-wise in $H^2$, $R_m K$ tends to 0 uniformly on the unit ball of $H^\infty$, that is $||R_m K|| \to 0$ as $n \to \infty$. It follows that

$$||W_{\psi, \varphi}||_e \geq \lim sup_{m \to \infty} ||R_m W_{\psi, \varphi}||,$$

this completes the proof.

Lemma 3.3. For $W_{\psi, \varphi} : H^\infty \to H^2$ and $\psi \in H^2$, if $k$ is fixed positive integer and $g$ is any non-constant holomorphic function on $B_n$ with $||g||_\infty \leq 1$, then $||Q_k W_{\psi, \varphi}(g^m)||_2 \to 0$ as $m \to \infty$.

Proof If $\alpha$ is a multi-index with $|\alpha| \leq k$, then

$$||z^\alpha||^2 = \frac{(n-1)!n!}{(n-1+|\alpha|)!} \leq (k!)^n \equiv c(n, k).$$

Since $D^n(0, \frac{1}{2n}) \subseteq B_n$ and Cauchy’s estimates, for any holomorphic function $F$ in $B_n$, we have

$$\frac{D^n F(0)}{\alpha!} \leq (2n)^{|\alpha|} ||F||_{\infty, D^n(0, \frac{1}{2n})}$$

where $||F||_{\infty, D^n(0, \frac{1}{2n})}$ denotes the maximum modulus of $F$ on the polydisc $D^n(0, \frac{1}{2n})$.

Since the series coefficients for $F$ are $c(\alpha) = \frac{D^n F(0)}{\alpha!}$, we get the series coefficients for $\psi \cdot g^m \circ \varphi$ are bounded above by

$$(2n)^{|\alpha|} ||\psi \cdot g^m \circ \varphi||_{\infty, D^n(0, \frac{1}{2n})}.$$ 

Let $c = \max |\psi|$ and $s = \max |g \circ \varphi|$ on $D^n(0, \frac{1}{2n})$, then $s < 1$ by hypothesis. This implies that $||\psi \cdot g^m \circ \varphi||_{\infty, D^n(0, \frac{1}{2n})} \leq cs^m$, which tends to 0 as $m \to \infty$. For
fixed $k$, $\|Q_k W_{\psi,\varphi}(g^m)\|^2 = \sum_{|\alpha| \leq k} |c(\alpha)||z^\alpha|^2$, where $c(\alpha)$ is the coefficients of $z^\alpha$ in the expansion of $\psi \cdot (g \circ \varphi)^m$. By the above estimate, we have

$$\|Q_k W_{\psi,\varphi}(g^m)\|^2 \leq \sum_{|\alpha| \leq k} ((2n)^k c s^m)^2 c(n,k) \leq c'(n,k)s^{2m}.$$ 

For fixed $k$, the last expression tends to 0 as $m \to \infty$.

**Lemma 3.4.** Let $\epsilon > 0$, set $E_\epsilon = \{\xi \in \partial B_n : |\varphi(\xi)| \geq 1 - \epsilon\}$ and let $E^c_\epsilon$ denote its complement in $\partial B_n$, $\psi \in H^2$. Define an operator $K : H^\infty \to H^2$ by $K(f) = P(\chi_{E^c_\epsilon} \psi \cdot (f \circ \varphi))$, where $P$ is the orthogonal projection of $L^2$ onto $H^2$ (where we identify a function in $H^2$ with its radial limit function). Then $K$ is compact from $H^p$ to $H^2$, for any $2 < p \leq \infty$.

**Proof** Let $\{f_m\}$ be a sequence from the unit ball of $H^p$. By Lemma 2.4, $\{f_m\}$ is a normal family when $2 < p < \infty$, and this is obviously true for $p = \infty$. So there is a subsequence which converges uniformly on compact subset of $B_n$, to say $f$. For simplicity we still denote this subsequence as $\{f_m\}$. Clearly $f \in H^p$. So

$$\|Kf_m - Kf\|^2 \leq ||P||^2 \|\chi_{E^c_\epsilon} \psi \cdot (f_m \circ \varphi) - \chi_{E^c_\epsilon} \psi \cdot (f \circ \varphi)\|^2 \leq \int_{\partial B_n} |\chi_{E^c_\epsilon} \psi \cdot (f_m \circ \varphi) - \chi_{E^c_\epsilon} \psi \cdot (f \circ \varphi)|^2 d\sigma = \int_{E^c_\epsilon} |\psi \cdot (f_m \circ \varphi) - \psi \cdot (f \circ \varphi)|^2 d\sigma.$$ 

Since $\{f_m\}$ are uniformly bounded on $E^c_\epsilon$ and $\psi \in H^2$, the above expression tends to 0 as $n \to \infty$ by Lebesgue’s dominated convergence theorem. This verifies the compactness of $K$.

**Theorem 3.1.** For $W_{\psi,\varphi} : H^\infty \to H^2$ and $\psi \in H^2$, then $\|W_{\psi,\varphi}\|_e = (\mu_{\psi,\varphi,2}(\varphi(E)))^{1/2}$, where $E = \{\xi \in \partial B_n : |\varphi^*(\xi)| = 1\}$.

**Proof** we consider the lower estimate first.

Let $g$ be a non-constant inner function on $B_n$ and set $h = g^m$ for a positive integer $m$, then

$$\|W_{\psi,\varphi}(g^m)\|^2 = \int_{\partial B_n} |\psi \cdot (h^* \circ \varphi^*)|^2 d\sigma = \int_{E^c_\epsilon} |h^*| d\mu_{\psi,\varphi,2} \geq \int_{\varphi(E)} |h^*| d\mu_{\psi,\varphi,2} \geq \mu_{\psi,\varphi,2}(\varphi(E))$$

where the last inequality follows by the fact that $|h^*| = 1$ a.e $[d\mu]$ on $\varphi(E)$, this is true that $h$ is inner and the restriction of $\mu_{\psi,\varphi,2}$ to $\partial B_n$ is absolutely continuous with respect to $\sigma$.

In fact, for any measurable subset $E$ of $\partial B_n$,

$$\mu_{\psi,\varphi,2}(E) = \int_{\varphi^{-1}(E) \cap \partial B_n} |\psi|^2 d\sigma,$$

by hypothesis of $C_{\varphi}$, if $\sigma(E) = 0$, then $\sigma(\varphi^{-1}(E)) = 0$, and $\mu_{\psi,\varphi,2}(E) = 0$ follows. So

$$\|R_k W_{\psi,\varphi}\| \geq \|R_k W_{\psi,\varphi}(g^m)\| \geq \|W_{\psi,\varphi}(g^m)\| - \|Q_k W_{\psi,\varphi}(g^m)\| \geq \mu_{\psi,\varphi,2}(\varphi(E)) - \|Q_k W_{\psi,\varphi}(g^m)\|.$$
for all $m$.

Fix $k$ and let $m \to \infty$ and apply Lemma 3.2 we obtain

$$
||R_k W_{\psi,\varphi}|| \geq (\mu_{\psi,\varphi,2}(\varphi(E)))^{1/2}
$$

for any $k$. Now let $k \to \infty$, by Lemma 3.1 we have the desired lower estimate on

$$
||W_{\psi,\varphi}||_e.
$$

Now we turn to the upper estimate.

Take $K$ as in Lemma 3.3, for any $g \in H^\infty$ with $||g||_\infty = 1$, we have

$$
||W_{\psi,\varphi}(g) - K(g)||_2 = ||\varphi \cdot g \circ \varphi - P(\chi_{E}\varphi \cdot (f \circ \varphi))||_2
$$

$$
= ||P(\chi_{E}\varphi \cdot (f \circ \varphi))||_2 \leq ||\chi_{E}\varphi \cdot (f \circ \varphi)||_2
$$

$$
= (\int_{E_n} |\varphi \cdot g \circ \varphi|^2 d\sigma)^{1/2} \leq ||g \circ \varphi||_\infty (\int_{E_n} |\psi|^2 d\sigma)^{1/2}
$$

$$
\leq ||g \circ \varphi||_\infty (\int_{\varphi^{-1}(\varphi(E)_e)} |\psi|^2 d\sigma)^{1/2}
$$

$$
= ||g \circ \varphi||_\infty (\mu_{\psi,\varphi,2}(\varphi(E)_e))^{1/2}.
$$

Let $\epsilon_m \downarrow 0$ and $K_m$ the corresponding operator defined by

$$
K_m(f) = P(\chi_{E_m} \varphi \cdot (f \circ \varphi)).
$$

For $p = \infty$ we have

$$
||W_{\psi,\varphi}||_e \leq ||W_{\psi,\varphi} - K_m|| \leq (\mu_{\psi,\varphi,2}(\varphi(E_{\epsilon_m})))^{1/2}
$$

for all $m$, and let $m \to \infty$, as desired.

**Corollary 3.1.** $W_{\psi,\varphi} : H^\infty \to H^2$ is compact if and only if $\psi \in H^2$ and $\sigma(E) = 0$.

**Proof.** If $W_{\psi,\varphi}$ is compact, it is obviously bounded, it follows from Lemma 3.1 that $\psi \in H^2$. From Theorem 3.1, the compactness of $W_{\psi,\varphi}$ implies $\mu_{\psi,\varphi,2}(\varphi(E)) = 0$, so $\sigma(\varphi^{-1}(\varphi(E)) \cap \partial B_n) = 0$ (see 5.5.9 in [Rud]), therefore $0 \leq \sigma(E) \leq \sigma(\varphi^{-1}(\varphi(E)) \cap \partial B_n) = 0$, $\sigma(E) = 0$.

On the other hand, if $\psi \in H^2$, from the proof of theorem 3.1, it follows that

$$
||W_{\psi,\varphi}||_e \leq (\int_{E_n} |\psi|^2 d\sigma)^{1/2}
$$

when $\epsilon \to 0$ and since $\sigma(E) = 0$, we get $||W_{\psi,\varphi}||_e = 0$, so $W_{\psi,\varphi}$ is compact.

In the above proof, set $\psi = 1 \in H^2$, then $||W_{1,\varphi}||_e = ||C_{\varphi}||_e \leq \sigma(E)^{1/2}$. And if set $\psi = 1$ in theorem 3.1, then

$$
||C_{\varphi}||_e \geq (\mu_{1,\varphi,2}(E))^{1/2} = \sigma(\varphi^{-1}(\varphi(E)))^{1/2},
$$

so $\sigma(\varphi^{-1}(\varphi(E))) = \sigma(E)$, we have the following Corollary

**Corollary 3.2.** (Theorem 1 in [GorM]) $C_{\varphi} : H^\infty \to H^2$ is bounded and

$$
||C_{\varphi}||_e = \sigma(E)^{1/2}.
$$

**Case 2.** $p = \infty, q \neq 2$

**Theorem 3.2.** Suppose $W_{\psi,\varphi} : H^\infty \to H^q (q > 1)$, and $\psi \in H^q$, then

$$
\frac{1}{2}(\mu_{\psi,\varphi,q}(\varphi(E)))^{1/q} \leq ||W_{\psi,\varphi}||_e \leq 2(\mu_{\psi,\varphi,q}(\varphi(E)))^{1/q}.
$$
Thus, for that \( K \) is any compact operator. For any positive integer \( m \)

we consider upper estimate first. Obviously \( W_{\psi,r\varphi} \) is compact for any

fixed \( 0 < r < 1 \). Let \( E_\epsilon = \{ \xi \in \partial B_n : |\varphi(\xi)| \geq 1 - \epsilon \} \) and let \( E_\epsilon^c \) denote its

complement in \( \partial B_n \). So

\[
\| W_{\psi,\varphi} - W_{\psi,r\varphi} \| = \sup_{|f| = 1} \| (W_{\psi,\varphi} - W_{\psi,r\varphi})f \|_q
\]

\[
= \sup_{|f| = 1} \left( \int_{\partial B_n} |\psi(f \circ \varphi) - \psi(f \circ (r\varphi))|^q d\sigma \right)^{1/q}
\]

\[
= \sup_{|f| = 1} \left( \int_{E_\epsilon} |\psi(f \circ \varphi) - \psi(f \circ (r\varphi))|^q d\sigma \right)^{1/q}
\]

\[
+ \sup_{|f| = 1} \left( \int_{E_\epsilon^c} |\psi(f \circ \varphi) - \psi(f \circ (r\varphi))|^q d\sigma \right)^{1/q}.
\]

Apply Lemma 2.4, we can choose \( r \) sufficiently close to 1 to make the second term

less than \( \epsilon \| \varphi \|_q \). For the first term, the triangle inequality yields

\[
|f \circ \varphi(\xi) - f \circ (r\varphi)(\xi)| \leq 2
\]

So, the first term is less than

\[
2\left( \int_{E_\epsilon} |\psi|^q d\sigma \right)^{1/q} \leq 2\left( \int_{\varphi^{-1}(E_\epsilon) \cap \partial B_n} |\psi|^q d\sigma \right)^{1/q} = 2(\mu_{\psi,\varphi,q}(\varphi(E_\epsilon)))^{1/q}.
\]

Let \( m \downarrow 0 \), and \( E_{\epsilon_m} = \{ \xi \in \partial B_n : |\varphi(\xi)| \geq 1 - \epsilon_m \} \), then \( \mu_{\psi,\varphi,q}(\varphi(E_{\epsilon_m})) \to \mu_{\psi,\varphi,q}(\varphi(E)) \), the upper estimate follows.

Now we turn to lower estimate. Let \( f \) be a non-constant inner function in \( B_n \),

\( K \) is any compact operator. For any positive integer \( m \), the sequence \( \{ f^m \} \) are

in the unit ball of \( H^\infty \). So there exists a subsequence \( \{ f^{m_k} \} \) such that \( \{ K(f^{m_k}) \} \)

converges in norm. Therefore, given \( \epsilon > 0 \), there exists \( M \) such that \( \| K(f^{m_k}) - K(f^{m_l}) \|_q < \epsilon \) for any \( k, l > M \). Fix \( k > M \), there exists \( r \) with \( 0 < r < 1 \) such that \( (\psi(f \circ \varphi)^{m_k})(z) = \psi(rz)(f \circ \varphi(rz))^{m_k} \) satisfies

\[
\| (\psi(f \circ \varphi)^{m_k})_r \|_q \geq \| (\psi(f \circ \varphi)^{m_k}) \| - \epsilon.
\]

Thus, for \( m \geq M \)

\[
\| W_{\psi,\varphi} - K \| \geq \| (W_{\psi,\varphi} - K) \frac{f^{m_k} - f^{m_l}}{2} \|_q
\]

\[
\geq \frac{1}{2} \| (\psi(f \circ \varphi)^{m_k}) - (\psi(f \circ \varphi)^{m_l}) \|_q - \epsilon/2
\]

\[
\geq \frac{1}{2} \| (\psi(f \circ \varphi)^{m_k})_r \|_q - \| (\psi(f \circ \varphi)^{m_l})_r \|_q - \epsilon/2
\]

\[
\geq \frac{1}{2} \| (\psi(f \circ \varphi)^{m_k})_r \|_q - \| (\psi(f \circ \varphi)^{m_l})_r \|_q - \epsilon.
\]

letting \( l \to \infty \) and \( h = f^{m_k} \), we have

\[
\| W_{\psi,\varphi} - K \| \geq \frac{1}{2} \| (\psi(f \circ \varphi)^{m_k})_r \|_q - \epsilon
\]

\[
= \frac{1}{2} \left( \int_{\partial B_n} |\psi^* \cdot (h^* \circ \varphi^*)|^q d\sigma \right)^{1/q} - \epsilon
\]

\[
= \frac{1}{2} \left( \int_{B_n} |h^*|^q d\mu_{\psi,\varphi,q} \right)^{1/q} - \epsilon
\]

\[
\geq \frac{1}{2} \left( \int_{\varphi(E)} |h^*|^q d\mu_{\psi,\varphi,q} \right)^{1/q} - \epsilon
\]

\[
\geq \frac{1}{2} (\mu_{\psi,\varphi,q}(\varphi(E)))^{1/q} - \epsilon.
\]
Now letting $\epsilon \to 0$ yields the result.

**Corollary 3.3.** $W_{\psi, \varphi} : H^\infty \to H^q$ is compact if and only if $\psi \in H^q$ and $\sigma(E) = 0$.

**Proof** Combining Lemma 3.1 and Theorem 3.2, the corollary follows.

**Corollary 3.4.** (Theorems 2 and 3) $C_\varphi : H^\infty \to H^q$ is compact if and only if $\psi \in H^q$ and $\sigma(E) = 0$.

**Proof** Let $\psi = 1 \in H^q$, then $W_{\psi, \varphi} = C_\varphi$, the corollary follows by Theorem 3.2.

4. **From $H^p$ to $H^q$ for $1 < q < p < \infty$**

**Theorem 4.1.** Assume $W_{\psi, \varphi} : H^p \to H^q$ $(1 < p < \infty)$ is bounded, then

$$||W_{\psi, \varphi}||_e \geq (\mu_{\psi, \varphi, q}(\varphi(E)))^{1/q}.$$ 

**Proof** Let $g$ be a non-constant inner function on $B_n$ and set $h = g^m$ for a positive integer $m$. Then $||g^m||_p = 1$ for any $m$, and $g^m$ converges weakly to 0 as $m \to \infty$, thus $||Kf_w|| \to 0$ for any compact operator from $H^p$ to $H^q$ when $|w| \to 1$. Like in Theorem 3.1, we have

$$||W_{\psi, \varphi} - K|| \geq \limsup_{m \to \infty} ||(W_{\psi, \varphi} - K)(g^m)||_q \geq \limsup_{m \to \infty} ||W_{\psi, \varphi}(g^m)||_q - \limsup_{m \to \infty} ||K(g^m)||_q = \limsup_{m \to \infty} ||W_{\psi, \varphi}(g^m)||_q = \limsup_{m \to \infty} (\int_{\partial B_n} |\psi^* \cdot (h^* \circ \varphi^*)|^q d\sigma)^{1/q} = \limsup_{m \to \infty} (\int_{\varphi(E)} |h^*| d\mu_{\psi, \varphi, q})^{1/q} \geq \limsup_{m \to \infty} (\int_{\varphi(E)} |h^*| d\mu_{\psi, \varphi, q})^{1/q} \geq (\mu_{\psi, \varphi, 2}(\varphi(E)))^{1/q}.$$ 

This ends the proof.

**Corollary 4.1.** Assume $W_{\psi, \varphi} : H^p \to H^q$, $p > 1, 0 < q < \infty$ is compact, then $\sigma(E) = 0$.

**Remark 1.** We will show that when $0 < p < q < \infty$ and $W_{\psi, \varphi} : H^p \to H^q$ is bounded, then $\mu_{\psi, \varphi, q}(\varphi(E)) = 0$ (see Corollary 6.1), So the above estimate is useless.

**Theorem 4.2.** Suppose $1 < q < p < \infty$ and there exists $r > q$ such that $W_{\psi, \varphi} : H^p \to H^r$ $(1 < p < \infty)$ is bounded, then

$$||W_{\psi, \varphi}||_e \leq ||P|| \cdot ||W_{\psi, \varphi}||_{p,r} \cdot \sigma(E)^{\frac{r-q}{qr}}$$

where $P$ is the Szegő projection of $L^q(\sigma)$ onto $H^q$.

**Proof** We consider the operator $K : H^p \to H^q$ defined by $K(f) = P(\chi_{E^c} \psi \cdot (f \circ \varphi))$. 
where $P$ is the Szegő projection of $L^q(\sigma)$ onto $H^q$. Like in Lemma 3.3, $K$ is compact operator from $H^p$ to $H^q$. So for any $g \in H^p$ with $\|g\|_p = 1$, we have
\[
\|W_{\psi,\varphi}(g) - K(g)\|_q = \|\psi \cdot g \circ \varphi - P(\chi_{E_\epsilon} \psi \cdot (f \circ \varphi))\|_q
\]
\[
= \|P(\chi_{E_\epsilon} \psi \cdot (f \circ \varphi))\|_q
\]
\[
\leq \|P\| \cdot \|\chi_{E_\epsilon} \psi \cdot (f \circ \varphi)\|_q
\]
\[
= \|P\| \cdot \left( \int_{E_\epsilon} |\psi \cdot g \circ \varphi|^q d\sigma \right)^{\frac{1}{q}}
\]
\[
\leq \|P\| \cdot \left( \int_{\partial B_n} \chi_{E_\epsilon} |\psi \cdot g \circ \varphi|^q d\sigma \right)^{\frac{1}{q}}
\]
\[
\leq \|P\| \cdot \|W_{\psi,\varphi}(g)\|_r \sigma(E_\epsilon)^{\frac{q}{q'}}
\]
\[
\leq \|P\| \cdot \|W_{\psi,\varphi}\|_{p,r} \cdot \sigma(E_\epsilon)^{\frac{q}{q'}}.
\]

Letting $\epsilon \to 0$ yields the conclusion.

5. From $H^p$ to $H^\infty$

**Theorem 5.1.** For $W_{\psi,\varphi} : H^p \to H^\infty$, and $0 < p < \infty$, then $W_{\psi,\varphi}$ is bounded if and only if $\sup_{z \in B_n} \frac{\psi(z)}{(1 - |\varphi(z)|^2)^{n/p}} < \infty$.

**Proof** " $\Rightarrow$ " For any $w \in B_n$, define $f_w(z) = \frac{(1 - |w|^2)^{n/p}}{(1 - <z,w>)^2n/p}$, and it is easy to check $\|f_w\|_p = 1$. So
\[
C \geq \|W_{\psi,\varphi}\| = \sup_{\|f\|_p = 1} \|W_{\psi,\varphi}f\|_\infty \geq \sup_{z \in B_n} \|W_{\psi,\varphi}f_{w}\|_\infty
\]
\[
= \sup_{w \in B_n} \sup_{z \in B_n} |\psi(z)| |f_{w}(\varphi(z))|
\]
setting $w = \varphi(z)$, as desired.

" $\Leftarrow$ "
\[
\|W_{\psi,\varphi}\| = \sup_{\|f\|_p = 1} \|W_{\psi,\varphi}f\|_\infty = \sup_{\|f\|_p = 1} \sup_{z \in B_n} |\psi(z)| f(\varphi(z))
\]
\[
\leq \sup_{\|f\|_p = 1} \sup_{z \in B_n} |\psi(z)| \frac{\|f\|_p}{(1 - |\varphi(z)|^2)^{n/p}} = \sup_{z \in B_n} \frac{|\psi(z)|}{(1 - |\varphi(z)|^2)^{n/p}}
\]

**Theorem 5.2.** For $W_{\psi,\varphi} : H^p \to H^\infty$ ($p > 1$), and $W_{\psi,\varphi}$ is bounded, then
\[
\lim_{\delta \to 0} \sup_{\text{dist}(\varphi(z), \partial B_n) < \delta} \frac{|\psi(z)|}{(1 - |\varphi(z)|^2)^{n/p}} \leq \|W_{\psi,\varphi}\|_e
\]
\[
\leq 2 \lim_{\delta \to 0} \sup_{\text{dist}(\varphi(z), \partial B_n) < \delta} \frac{|\psi(z)|}{(1 - |\varphi(z)|^2)^{n/p}}.
\]

**Proof** We consider the upper estimate first.
For any fixed $0 < r < 1$, it is easy to check that $W_{\psi,\varphi}$ is compact. Thus
\[
\|W_{\psi,\varphi}\|_e \leq \|W_{\psi,\varphi} - W_{\psi,\varphi,r}\|.
\]
Now for any $0 < \delta < 1$
\[
\|W_{\psi, r} - W_{\psi, r}\| = \sup_{\|f\|_p = 1} \| (W_{\psi, r} - W_{\psi, r}) f \|_\infty \\
= \sup_{\|f\|_p = 1} \sup_{z \in B_n} |\psi(z)| \cdot |f(\varphi(z)) - f(r\varphi(z))| \\
\leq \|\psi\|_\infty \sup_{\|f\|_p = 1} \sup_{\text{dist}(\varphi(z), \partial B_n) < \delta} |f(\varphi(z)) - f(r\varphi(z))| \\
+ \sup_{\|f\|_p = 1} \sup_{\text{dist}(\varphi(z), \partial B_n) < \delta} |\psi(z)| \cdot |f(\varphi(z)) - f(r\varphi(z))|.
\]

From Lemma 2.4, we can choose $r$ sufficiently close to 1 such that the first term of the right hand side is less than any given $\epsilon$. And we denote the second term by $I$. Then,
\[
I \leq \sup_{\|f\|_p = 1} \sup_{\text{dist}(\varphi(z), \partial B_n) < \delta} |\psi(z)| \cdot |f(\varphi(z))| + \sup_{\|f\|_p = 1} \sup_{\text{dist}(\varphi(z), \partial B_n) < \delta} |f(\varphi(z)) - f(r\varphi(z))| \frac{\|f\|_p}{(1 - |\varphi(z)|^2)^{n/p}} + \frac{\|f\|_p}{(1 - |r\varphi(z)|^2)^{n/p}}
\]
\[
\leq 2 \sup_{\text{dist}(\varphi(z), \partial B_n) < \delta} |\psi(z)| \frac{\|f\|_p}{(1 - |\varphi(z)|^2)^{n/p}}.
\]

Now let $r \to 1$ first, then let $\delta \to 0$, we get the desired upper estimate.

We now turn to the lower estimate.

Let $K$ be any compact operator from $H^p$ to $H^\infty$. For any $w \in B_n$ define $f_w(z) = \frac{(1 - |w|^2)^{n/p}}{(1 - \langle z, w \rangle)^{n/p}}$, it is easy to check $\|f_w\|_p = 1$ and $f_w$ converge weakly to 0 as $|w| \to 1$, thus $\|Kf_w\| \to 0$ when $|w| \to 1$.

So for any $0 < \delta < 1$
\[
\|W_{\psi, r} - K\| \geq \limsup_{|w| \to 1} \| (W_{\psi, r} - K) f_w \|_\infty \\
\geq \limsup_{|w| \to 1} \|W_{\psi, r} f_w \|_\infty - \limsup_{|w| \to 1} \|K f_w\|_\infty \\
= \limsup_{|w| \to 1} \sup_{z \in B_n} |\psi(z)| \|f_w(\varphi(z))\| \\
\geq \limsup_{|w| \to 1} \sup_{\text{dist}(\varphi(z), \partial B_n) < \delta} |\psi(z)| \|f_w(\varphi(z))\|.
\]

Let $\delta \to 0$ then $|\varphi(z)| \to 1$ and set $w = \varphi(z)$, we obtain the lower estimate of $\|W_{\psi, r}\|_e$.

**Corollary 5.1.** Assume $W_{\psi, r} : H^p \to H^\infty$ is bounded, then it is compact if and only if
\[
\lim_{\delta \to 0} \sup_{\text{dist}(\varphi(z), \partial B_n) < \delta} \frac{|\psi(z)|}{(1 - |\varphi(z)|^2)^{n/p}} = 0.
\]

**Remark 2.** If $\|\varphi\|_\infty < 1$, then $E = \{ z \in \overline{B_n} \ | \varphi(z) = 1 \} = \emptyset$, without the loss of generality, we set
\[
\lim_{\delta \to 0} \sup_{\text{dist}(\varphi(z), \partial B_n) < \delta} \frac{|\psi(z)|}{(1 - |\varphi(z)|^2)^{n/p}} = 0.
\]
6. FROM $H^p$ TO $H^q$ FOR $1 < p \leq q < \infty$

**Definition** Let $\beta \geq 1$. A finite and positive measure $\mu$ on is called a $\beta$-Carleson measure. If there is a constant $M \leq \infty$ such that $\mu(S_h(\xi)) \geq M h^\beta$ for all $\xi \in \partial B_n$ and $0 < h < 2$, it is called vanishing $\beta$-Carleson measure if $\lim_{h \to 0} \sup_{\xi \in \partial B_n} \frac{\mu(S_h(\xi))}{h^\beta} = 0$.

**Lemma 6.1.** (see corollary 2 in [LS1]). Let $\mu$ be a finite and positive measure on $\overline{B}_n$, and $0 < p \leq q < \infty$, then the following statement are equivalent:

(i) $\mu$ is a bounded $\frac{1}{p} - \text{Carleson}$ measure.

(ii) There is a constant $C < \infty$ so that

\[
\int_{\overline{B}_n} |f|^p \, d\mu \leq C \|f\|_p^q
\]

for all $f$ in $\overline{B}_n$.

**Lemma 6.2.** (X) Suppose that $0 < p \leq q < \infty$ and $W_{\psi, \varphi} : H^p \to H^q$ is bounded, then the following conditions are equivalent:

(i) $W_{\psi, \varphi}$ is vanishing $\frac{1}{p} - \text{Carleson}$ measure

(ii) $W_{\psi, \varphi} : H^p \to H^q$ is compact operator.

**Theorem 6.1.** For $0 \leq p \leq q < \infty$, then the following statement are equivalent:

(i) $\mu_{\psi, \varphi, q}$ is a bounded $\frac{1}{p} - \text{Carleson}$ measure.

(ii) $W_{\psi, \varphi} : H^p \to H^q$ is bounded.

(iii)

\[
\sup_{z \in B_n} \int_{B_n} \frac{(1 - |z|^2)^{nq/p}}{|1 - <w, z>|^{2nq/p}} d\mu_{\psi, \varphi, q}(w) < \infty.
\]

**Proof** (i) $\Rightarrow$ (ii) From Lemma 3.4, if $\mu_{\psi, \varphi, q}$ is bounded $\frac{1}{p} - \text{Carleson}$ measure, then there exists constant $C$ such that

\[
\int_{\overline{B}_n} |f|^p d\mu_{\psi, \varphi, q} \leq C \|f\|_p^q
\]

for any $f \in H^p(B_n)$. Apply Lemma 2.1, and put $g = |f|^q$, we have

\[
\int_{B_n} |f|^q d\mu_{\psi, \varphi, q} = \int_{\partial B_n} |\psi|^q |f \circ \varphi|^q d\sigma = \|W_{\psi, \varphi} f\|_q.
\]

So

\[
\|W_{\psi, \varphi}(f)\|_q \leq C^{1/q} \|f\|_p
\]

for any $f \in H^p(B_n)$. That is, $W_{\psi, \varphi} : H^p \to H^q$ is bounded.

(ii) $\Rightarrow$ (iii) For any $z \in B_n$, set $f_z(w) = \frac{(1 - |z|^2)^{nq/p}}{|1 - <w, z>|^{2nq/p}}$, then $\|f_w\|_p = 1$

\[
C \geq \|W_{\psi, \varphi}\|^q = \sup_{\|f\|_p = 1} \|W_{\psi, \varphi} f\|_q \geq \sup_{z \in B_n} \|W_{\psi, \varphi} f_z\|_q
\]

\[
= \sup_{z \in B_n} \left( \int_{\partial B_n} |\psi|^p |f_z \circ \varphi|^q d\sigma \right) = \sup_{z \in B_n} \int_{B_n} |f_z|^q d\mu_{\psi, \varphi, q}
\]

\[
= \sup_{z \in B_n} \int_{B_n} \frac{(1 - |z|^2)^{nq/p}}{|1 - <w, z>|^{2nq/p}} d\mu_{\psi, \varphi, q}(w)
\]
Assume that

\[
M = \sup_{z \in B_n} \int_{B_n} \frac{(1 - |z|^2)^{nq/p}}{|1 - <w, z| |2nq/p} d\mu_{\psi, \varphi, q}(w) < \infty
\]

we show that \( \mu_{\psi, \varphi, q} \) is a bounded \( \frac{2}{p} \)-Carleson measure.

First let \( z = 0 \), then \( \mu_{\psi, \varphi, q}(B_n) \leq M \). Thus \( \mu_{\psi, \varphi, q} \) is finite and hence \( \mu_{\psi, \varphi, q}(S_h(\xi)) \leq M \leq 4M^{nq/p} \) for all \( \xi \in \partial B_n \) and \( h \geq (\frac{1}{4})^{\frac{1}{nq/p}} \). Suppose \( h \leq (\frac{1}{4})^{\frac{1}{nq/p}} \) and \( \xi \in \partial B_n \).

Let \( \xi_0 = (1 - \frac{1}{2} h) \xi \), then for any \( w \in S_h(\xi) \),

\[
|1 - <w, \xi_0>| = |1 - \frac{1}{2} h + \frac{1}{2} h| < w, \xi_0| = |(1 - \frac{1}{2}) (1 - <w, \xi> + h)| \leq |(1 - \frac{1}{2}) h| + \frac{h}{2} \leq \frac{3h}{2}
\]

and \( 1 - |\xi_0|^2 = (1 - |\xi_0|)(1 + |\xi_0|) \geq (1 - |\xi_0|) \), we have

\[
\frac{(1 - |\xi_0|^2)^{nq/p}}{1 - <w, \xi_0>} \geq \frac{(1 - |\xi_0|^{nq/p}}{\frac{3h}{2}} \geq \frac{c}{h^{nq/p}}.
\]

So

\[
M \geq \int_{B_n} \frac{(1 - |\xi_0|^2)^{nq/p}}{|1 - <w, \xi_0>} |2nq/p d\mu_{\psi, \varphi, q}(w)
\]

\[
\geq \int_{S_h(\xi)} \frac{c}{h^{nq/p}} d\mu_{\psi, \varphi, q} \geq \frac{c \mu_{\psi, \varphi, q}(S_h(\xi))}{h^{nq/p}}
\]

Therefore, \( \mu_{\psi, \varphi, q} \) is bounded \( \frac{2}{p} \)-Carleson measure.

Corollary 6.1. If \( 0 < p < q < \infty \) and \( W_{\psi, \varphi} : H^p \to H^q \) is bounded, then \( \mu_{\psi, \varphi, q}(\varphi(E)) = 0 \).

Proof Denote \( g \) the Radon – Nikodým derivative of \( \mu_{\psi, \varphi, q}| \partial B_n \) with respect to \( \sigma \), \( \mu_{\psi, \varphi, q} \) is absolutely continuous with respect to \( \sigma \) on \( \partial B_n \), so it follows that

\[
g(b) = \lim_{h \to 0} \frac{1}{\sigma(S_h(b))} \int_{S_h(b)} \mu_{\psi, \varphi, q}(S_h(b)) \geq \lim_{h \to 0} \frac{c \mu_{\psi, \varphi, q}(S_h(b))}{\sigma(S_h(b))} \geq 0
\]

almost everywhere in \( \partial B_n \). Where the penultimate inequality uses the fact that \( \sigma(S_h(b)) \) is roughly proportional to \( h^n \) (see P67 in [Rud]). Now we have \( \mu_{\psi, \varphi, q}| \partial B_n = 0 \), the corollary is proved.

Theorem 6.2. For fixed \( 1 < p \leq q < \infty \) and weighted composition operator \( W_{\psi, \varphi} : H^p \to H^q \) is bounded, then

\[
||W_{\psi, \varphi}||_e \geq \lim_{|w| \to 1} \int_{B_n} \frac{(1 - |w|^2)^{nq/p}}{|1 - <z, w>} |2nq/p d\mu_{\psi, \varphi, q}(z).
\]

Proof Let \( K \) be any compact operator from \( H^p \) to \( H^\infty \). For any \( w \in B_n \) define \( f_w(z) = \frac{(1 - |w|^2)^{nq/p}}{(1 - <z, w>)^{nq/p}} \), it is easy to check \( ||f_w||_p = 1 \) and \( f_w \) converge weakly to 0 as
\[ |w| \to 1, \text{ thus } \|Kf_w\| \to 0 \text{ when } |w| \to 1. \] So for any \( 0 < \delta < 1 \),

\[
\|W_{\psi, \varphi} - K\| \geq \limsup_{|w| \to 1} \| (W_{\psi, \varphi} - K)f_w \|_q
\]

\[
\geq \limsup_{|w| \to 1} \|W_{\psi, \varphi}f_w\|_q - \limsup_{|w| \to 1} \|Kf_w\|_q
\]

\[
= \limsup_{|w| \to 1} \int_{\partial B_n} \frac{|\psi(z)|^q}{|1 - \psi(z), w|^{2nq/p}} d\sigma(z)
\]

\[
\geq \limsup_{|w| \to 1} \int_{\mathcal{B}_n} \frac{(1 - |w|^2)^{nq/p}}{|1 - z, w|^{2nq/p}} d\mu_{\psi, \varphi, q}(z)
\]

The conclusion follows.

We cannot give the upper estimate in the above form, but we have the following theorem.

**Theorem 6.3.** Assume \( 1 < p \leq q < \infty \) and \( W_{\psi, \varphi} : H^p \to H^q \) is bounded, then \( W_{\psi, \varphi} : H^p \to H^q \) is compact if and only if

\[
\lim_{|w| \to 1} \int_{\mathcal{B}_n} \frac{(1 - |w|^2)^{nq/p}}{|1 - z, w|^{2nq/p}} d\mu_{\psi, \varphi, q}(z) = 0.
\]

**Proof** The necessary condition follows by theorem 6.2. We consider the sufficient condition. By Lemma 6.2, we only have to show \( \mu_{\psi, \varphi, q} \) is vanishing \( \frac{q}{p} - \text{Carleson measure} \). From the proof of (iii) \( \Rightarrow \) (i) in theorem 6.1, for any \( z \in \partial B_n \), set \( |z_0| = 1 - \frac{h}{2} \). Suppose

\[
\lim_{|w| \to 1} \int_{\mathcal{B}_n} \frac{(1 - |w|^2)^{nq/p}}{|1 - z, w|^{2nq/p}} d\mu_{\psi, \varphi, q}(z) = 0.
\]

That is, \( \forall \epsilon > 0, \exists 1 > r > 0 \), when \( |w| > r \) we have

\[
| \int_{\mathcal{B}_n} \frac{(1 - |w|^2)^{nq/p}}{|1 - z, w|^{2nq/p}} d\mu_{\psi, \varphi, q}(z) | < \epsilon.
\]

When \( h < 2(1 - r) \), for any \( z \in \partial B_n \), the corresponding \( |z_0| > r \), so

\[
\epsilon > \int_{\mathcal{B}_n} \frac{(1 - |z|^2)^{nq/p}}{|1 - z, z_0|^{2nq/p}} d\mu_{\psi, \varphi, q}(w)
\]

\[
\geq \int_{S_h(z)} \frac{c}{h^{nq/p}} d\mu_{\psi, \varphi, q}
\]

\[
\geq \frac{c\mu_{\psi, \varphi, q}(S_h(z))}{h^{nq/p}}.
\]

This is true for any \( z \in \partial B_n \). So \( \mu_{\psi, \varphi, q} \) is vanishing \( \frac{q}{p} - \text{Carleson measure} \).

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Department of Mathematics  
Tianjin Polytechnic University  
Tianjin 300160  
P.R. China.  
E-mail address: fangzhongshan@yahoo.com.cn

Department of Mathematics  
Tianjin University  
Tianjin 300072  
P.R. China.  
E-mail address: zehuazhou2003@yahoo.com.cn