REMARKS ON THE CONVERGENCE OF AN ALGORITHM FOR CURVATURE-DEPENDENT MOTIONS OF HYPERSURFACES

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Abstract. We consider a threshold-type algorithm for curvature-dependent motions (CDM for short) of hypersurfaces. This algorithm was numerically studied by Kimura - Notsu [13], Esedoḡlu - Ruuth - Tsai [7] and Mohammad - Švadlenka [16], where they used the signed distance function as the level set function for CDM. The convergence of this algorithm and its optimal rate have been considered in Ishii - Kimura [12]. In this paper we give different approaches to the optimal rate of convergence to the smooth and compact CDM from [12]. As for the optimality, we give a more precise estimate than that in [12].

1. Introduction. In this paper we study the convergence of a threshold-type algorithm for curvature-dependent motions (CDM for short) of hypersurfaces. This algorithm is introduced and numerically studied by Kimura - Notsu [13].

Let \( \{\Gamma(t)\}_{0 \leq t < T_0} \) be a family of compact hypersurfaces in \( \mathbb{R}^N \). We say this family is a CDM if \( \Gamma(t) \) moves by the following equation:

\[
V = \kappa + \langle \mathbf{b}, \mathbf{n} \rangle + g \quad \text{on} \quad \Gamma(t), \ t \in (0, T_0).
\]

Here \( T_0 > 0, \mathbf{n} = \mathbf{n}(t, x) \) is the inner unit normal vector field on \( \Gamma(t), V = V(t, x) \) is the velocity of \( \Gamma(t) \) in the direction of \( \mathbf{n}, \kappa = \kappa(t, x) \) is the \((N - 1)\)-times mean curvature of \( \Gamma(t) \), \( \mathbf{b} = \mathbf{b}(t, x) \) denotes a given vector field in \( \mathbb{R}^N \), \( g = g(t, x) \) is a forcing term and \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( \mathbb{R}^N \). As is well known, in the case of \( \mathbf{b} \equiv 0 \) and \( g \equiv 0 \), \( \{\Gamma(t)\}_{t \in [0, T_0]} \) is called a mean curvature flow (MCF for short). The CDM and MCF are mathematical models of the motion of an interface by its surface tension, a transport term and an external force and they arise in various fields such as two-phase flow problems, phase transitions, image processing and so on.

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1103
From the viewpoints of these applications, many people have studied numerical methods for CDM. Among them, we treat the following algorithm: Let $C_0$ be a compact set in $\mathbb{R}^N$ and fix a time step $h > 0$. For $k = 0, 1, 2, \ldots$, set $b_k(t, x) := b(t + kh, x)$ and $g_k(t, x) := g(t + kh, x)$. Let $w_0 = w_0(t, x)$ be a solution of the initial value problem for the linear parabolic equation with $k = 0$:

$$L_k[w] + g_k = 0 \text{ in } (0, h) \times \mathbb{R}^N, \quad L_k[w] := w_t - \Delta w + \langle b_k, Dw \rangle,$$

(2)

$$w(0, x) = d(x, C_k) \quad \text{for } x \in \mathbb{R}^N,$$

(3)

where $d(x, E)$ is the signed distance function to $\partial E$ defined by

$$d(x, E) := \begin{cases} \operatorname{dist}(x, \partial E) & \text{for } x \in E, \\ -\operatorname{dist}(x, \partial E) & \text{for } x \notin E, \end{cases}$$

(4)

for each closed subset $E(\neq \emptyset)$ of $\mathbb{R}^N$. We then set

$$C_1 := \{w_0(h, \cdot) \geq 0\}(:= \{x \in \mathbb{R}^N \mid w_0(h, x) \geq 0\}).$$

(5)

Let $w_1$ be a unique solution of (2) - (3) with $k = 1$. Again we define $C_2$ as the set in (5) with $w_1$ replacing $w_0$. Repeating this process, we have a sequence $\{C_k\}_{k=0}^{+\infty}$ of compact subsets of $\mathbb{R}^N$. We set

$$C^h(t) := C_k \quad \text{for } t \in [kh, (k + 1)h), \quad k = 0, 1, 2, \ldots$$

(6)

Letting $h \to 0$, we formally obtain a limit flow $\{C(t)\}_{t \geq 0}$ of compact sets in $\mathbb{R}^N$ and observe that $\partial C(t)$ moves by (1) starting from $\partial C(0) = \partial C_0$.

The above algorithm was numerically studied by Kimura - Notsu [13]. In their paper Kimura and Notsu proposed a fully discrete finite element scheme based on the level set method of the signed distance function. In [13, Section 4] they gave some numerical examples for MCF with a forcing term. Esedoḡlu - Ruuth - Tsai [7] considered various geometric motions, including CDM, MCF with triple junctions and motion by surface diffusion. Mohammad - Švadlenka [16] gave an extension of these approaches to the vector-valued one for numerical computation of multiphase problems. On the other hand, our algorithm is regarded as a variant of the Bence - Merriman - Osher (BMO for short) algorithm and that by Chambolle (cf. [2] and [3]). Many people studied their algorithms and generalizations (e.g., [9], [1], [17], [15], [4], [5], [8], etc.). Among them Vivier [17] and Leoni [15] generalized the BMO algorithm using linear/semilinear parabolic equations and proved the convergence of their scheme to the anisotropic CDMs associated with these equations. Chambolle - Novaga [4], [5] considered an extension of [3] for the anisotropic CDM and showed the convergence to the regular flow. Although our algorithm is quite similar to theirs, the differences between their algorithm and ours are the choice of the initial data and the fact that (2) has a transport term. In the (generalized) BMO algorithm they choose the initial data

$$w(0, x) = \begin{cases} 1 & \text{for } x \in C_k, \\ -1 & \text{for } x \notin C_k, \end{cases} \quad (= \operatorname{sgn}^*(d(x, C_k)))$$

instead of (3), where $\operatorname{sgn}^*(r) := 1$ for $r \geq 0$, $:= -1$ for $r < 0$.

As for the convergence of our algorithm, Ishii - Kimura [12] proved the convergence to the generalized CDM under the nonfattening assumption. (cf. [1, Corollary 1.3], [12, (2.16) and Theorem 5.6], etc.). In addition they obtained the rate of convergence to the smooth and compact CDM and showed its optimality with respect to the order of $h$ in the case of a circle evolving by curvature. However, the arguments in [12] are complicated since we divide the argument into two parts to get the rate
of convergence. First we showed the convergence itself and then we obtained its rate with using this convergence, because we needed some uniform local estimates and Taylor expansion of the solution \( w_k \) of (2) to derive the rate of convergence. Also, the proofs of the convergence and its rate are based on cumbersome calculations.

The main purpose of this paper is to give different approaches to the optimal rate of convergence to the smooth and compact CDM. As a result, we are able to simultaneously prove the convergence and its rate of the algorithm and to obtain the asymptotic behavior of the radius of a circle evolving by the algorithm as \( h \rightarrow 0 \) in terms of that of a circle by evolving curvature. The approach in this paper is simpler and more straightforward than that in [12] and our optimality estimate is more precise than that in [12].

In deriving the rate of the convergence of the algorithm, the basic idea is to construct sub- and super-solutions of (2) in each time interval \([kh, (k+1)h)\) \((k = 0, 1, 2, \ldots)\) which approximate the solution \( w_k \) of (2) - (3) near \( \Gamma(t) \). For this purpose we employ the signed distance function to the CDM. To show the optimality of our estimate we use an explicit solution of the level set equation to (1) with \( b \equiv 0 \) and \( g \equiv 0 \). Let \( \Gamma(t) := \{ | \cdot | = \sqrt{1 - 2(N - 1)t} \} \) \((= \{ x \in \mathbb{R}^N \mid |x| = \sqrt{1 - 2(N - 1)t} \})\). Then \( \Gamma(t) \) evolves by mean curvature starting from the unit sphere. Set \( \rho(t, r) := 1 - \sqrt{r^2 + 2(N - 1)t} \). Then we see that \( \Gamma(t) = \{ \rho(t, |\cdot|) = 0 \} \) and that \( \bar{\rho}(t, x) = \rho(t, |x|) \) satisfies

\[
\bar{\rho}_t - \Delta \bar{\rho} + \frac{\langle D^2 \bar{\rho} \bar{\rho}_t, D\bar{\rho} \rangle}{|D\bar{\rho}|^2} = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^N, \quad (7)
\]

\[
\bar{\rho}_t - \Delta \bar{\rho} \geq 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^N. \quad (8)
\]

Note that (7) holds in the sense that

\[
\bar{\rho}_t + F^*(D\bar{\rho}, D^2\bar{\rho}) = \bar{\rho}_t + F_*(D\bar{\rho}, D^2\bar{\rho}) = 0 \quad \text{for each } (t, x) \in (0, +\infty) \times \mathbb{R}^N.
\]

Here \( F^* \) and \( F_* \) are defined by

\[
F^*(p, X) := \limsup_{(q,Y) \to (p,X)} F(q, Y), \quad F_*(p, X) := -(F^*)^*(p, X),
\]

\[
F(p, X) := -\text{tr} X + \frac{\langle Xp, p \rangle}{|p|^2} \quad \text{for } (p, X) \in (\mathbb{R}^N \setminus \{0\}) \times \mathbb{S}^N.
\]

The inequality (8) is equivalent to

\[
\rho_t - \rho_{rr} - \frac{N - 1}{r} \rho_r \geq 0 \quad \text{in } (0, +\infty) \times (0, +\infty), \quad \rho_r(t, 0) = 0 \quad \text{for } t \in (0, +\infty). \quad (9)
\]

We use (9) to construct sub- and super-solutions of (2) with \( b \equiv 0 \) and \( g \equiv 0 \) in order to prove the optimality of our estimate. We have mentioned in the above paragraph that our approach is simpler and more straightforward than that in [12]. As a by-product, we have an approximation of the solution \( w_k \) of (2) - (3) (resp., (33) - (34) - (35) in section 4 below) by means of the signed distance function to \( \Gamma(t) \) (resp., the function \( \rho \) defined as above).

This paper is organized as follows. In section 2 we state our assumptions and briefly recall the smooth and compact CDM and the solution \( w_k \) of (2) - (3). In section 3 we derive the rate of convergence of the algorithm to the smooth and compact CDM. Section 4 is devoted to the optimality of our estimate in the case of a circle evolving by curvature.
We use the following notation in this paper: For \( u : \mathbb{R}^N \rightarrow \mathbb{R} \), \( v : [0, T) \times \mathbb{R}^N \rightarrow \mathbb{R} \) and \( \mu \in \mathbb{R} \),
\[
\{ u \geq \mu \} := \{ x \in \mathbb{R}^N \mid u(x) \geq \mu \},
\{ v \geq \mu \} := \{(t, x) \in [0, T) \times \mathbb{R}^N \mid v(t, x) \geq \mu \},
\{ v(t, \cdot) \geq \mu \} := \{ x \in \mathbb{R}^N \mid v(t, x) \geq \mu \},
\end{equation}

\( (p, q) := \) the inner product between \( p, q \in \mathbb{R}^N \),
\( [r] := \) Gauss symbol for \( r \in \mathbb{R} \),
\( d_H(A, B) := \max \left\{ \sup_{x \in A} \text{dist} \ (x, B), \sup_{x \in B} \text{dist} \ (x, A) \right\} \) for \( A, B \subset \mathbb{R}^N \),
\( \| \cdot \|_{\infty} := \| \cdot \|_{L^{\infty}(0,T) \times \mathbb{R}^N)} \).

For \( f \) (resp., \( f = (f_1, \cdots, f_N) \)) : \([0, T) \times \mathbb{R}^N \rightarrow \mathbb{R} \) (resp., \( \mathbb{R}^N \)),
\[
Df := \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N} \right) \quad \text{(resp., } Df := \left( \frac{\partial f_i}{\partial x_j} \right)_{1 \leq i, j \leq N} \text{)}
\end{equation}

In addition, \( K_i \) and \( M_j \) (\( i, j \in \mathbb{N} \)) denote positive constants independent of small \( \delta > 0 \) and \( k = 0, 1, 2, \ldots \).

2. Preliminaries.

2.1. Assumptions and solutions of (1) and (2). We make the following assumptions:

(A0) \( \Gamma_0 \) is a compact hypersurface in \( \mathbb{R}^N \) and is of class \( C^4 \).

(A1) \( b, g \in C^{1,2}([0, +\infty) \times \mathbb{R}^N) \cap W^{1,2,\infty}((0, +\infty) \times \mathbb{R}^N) \).

Then we see from [10], [12, Appendix] etc. that for some \( T_0 > 0 \), there uniquely exists a smooth and compact CDM \( \{ \Gamma(t) \}_{0 \leq t < T_0} \) with the initial data \( \Gamma(0) = \Gamma_0 \).

2.2. The signed distance function to the CDM. Let \( \{ \Gamma(t) \}_{0 \leq t < T_0} \) be the smooth and compact CDM. For each \( t \in [0, T_0) \) let \( C(t) \subset \mathbb{R}^N \) be the compact set satisfying \( \partial C(t) = \Gamma(t) \). Define \( d = d(t, x) \) by (4) with \( E = C(t) \). Then for any \( \varepsilon \in (0, T_0) \), there exists a \( \delta > 0 \) such that \( d \in C^{2,4}(\mathcal{N}_{\varepsilon,30\delta}) \), where \( \mathcal{N}_{\varepsilon,30\delta} := \bigcup_{t \in [0, T_0-\varepsilon]} \{ \{ x \mid \| d(t, x) \| \leq 30\delta \} \} \) (cf. [10, Section 2] and [12, Section 8], etc.). As \( d_t = -V, \quad Dd = n \) and \( \Delta d = -\kappa, \) (1) turns to
\[
d_t - \Delta d + (b, Dd) + g = 0 \quad \text{on } \Gamma(t), \quad t \in (0, T_0).
\]

We derive an inequality which \( d \) satisfies in \( \mathcal{N}_{\varepsilon,30\delta} \). For any \( (t, x) \in \mathcal{N}_{\varepsilon,30\delta} \), set
\[
\kappa^i(t, x) := \text{the principal curvatures at } x \in \Gamma(t),
\]
\[
\kappa(t, x) := \sum_{i=1}^{N-1} \kappa^i(t, x) \quad \text{the sum of the principal curvatures at } x \in \Gamma(t),
\]
\[
z(t, x) := x - d(t, x)Dd(t, x) \quad \text{the point on } \Gamma(t) \text{ satisfying } |d(t, x)| = |x - z(t, x)|.
\]
Then it follows from (11) that for \((t, x) \in \mathcal{N}_{\varepsilon, 30\delta}\)
\[
d_t(z(t, x)) - \Delta d(t, z(t, x)) + \langle b(t, z(t, x)), Dd(t, z(t, x)) \rangle + g(t, z(t, x)) = 0.
\]

On the other hand, we see by [11, Section 14.6] that
\[
\kappa(t, x) = \sum_{i=1}^{N-1} \frac{\kappa^i(t, z(t, x))}{1 - \kappa^i(t, z(t, x))} d(t, x) \quad \text{for} \quad (t, x) \in \mathcal{N}_{\varepsilon, 30\delta}.
\]

In view of \(\kappa = -\Delta d \text{ on } \Gamma(t), \ t \in [0, T_0]\), we get for all \((t, x) \in \mathcal{N}_{\varepsilon, 30\delta}\),
\[
|\Delta d(t, z(t, x)) - \{\Delta d(t, x) + \kappa^s(t, x) d(t, x)\}| \leq M_1(d(t, x))^2,
\]
where \(\kappa^s = \kappa^s(t, x) := \sum_{i=1}^{N-1} \{\kappa^i(t, z(t, x))\}^2\). Besides by (A1) and Taylor’s theorem we get for \((t, x) \in \mathcal{N}_{\varepsilon, 30\delta}\)
\[
d_t(t, x) = d_t(t, z(t, x)), \ Dd(t, z(t, x)) = Dd(t, x),
\]
\[
|b(t, z(t, x)) - \{b(t, x) - d(t, x) Db(t, x) Dd(t, x)\}| \leq M_2(d(t, x))^2,
\]
\[
|g(t, z(t, x)) - \{g(t, x) - d(t, x) Dg(t, x), Dd(t, x)\}| \leq M_2(d(t, x))^2.
\]
Combining (13) - (16) with (12), we obtain
\[
|d_t - \Delta d - \omega d + \langle D(d Db), Dd \rangle + g| \leq M_3 d^2 \quad \text{on } \mathcal{N}_{\varepsilon, 30\delta}.
\]

Here \(\omega := \kappa^s + \langle D(Db Dd), Dd \rangle + \langle Dg, Dd \rangle\).

2.3. Auxiliary functions. For any fixed \(m \in \mathbb{N} \text{ and } y \in \mathbb{R}^N\), we define \(v_m = v_m(t, x; y) := \sqrt{|x - y|^2 + 2mt}\) for \((t, x) \in [0, +\infty) \times \mathbb{R}^N\). It is easy to see that for \((t, x) \in (0, +\infty) \times \mathbb{R}^N\),
\[
v_m(t, x) - \Delta v_m = \frac{(m - N + 1)|x - y|^2 + 2m(m - N)t}{(|x - y|^2 + 2mt)^{3/2}} \begin{cases} \leq 0 & \text{if } m \leq N - 1, \\ \geq 0 & \text{if } m \geq N. \end{cases}
\]

We remark that \(v_N\) is used in section 3 to construct sub- and super-solutions of (2). Note that for each \(R > 0\), the set \(\{v_{N-1}(t, \cdot : 0) = R\}\) is a sphere evolving by mean curvature and shrinks to a point at \(t = R^2/(N - 1)\).

3. Rate of convergence to CDM. In this section we give an alternative proof of Theorem 3.1 below, which has already been obtained in [12, Theorem 6.1]. For any given smooth and compact hypersurface \(\Gamma_0 \subset \mathbb{R}^N\), there is a unique, smooth and compact CDM \(\{\Gamma(t)\}_{0 \leq t < T_0}\) locally in time satisfying \(\Gamma(0) = \Gamma_0\), \(\text{cf.} \ [10] \text{ and} \ [12, \text{Appendix}]. \) Let \(\{C(t)\}_{0 \leq t < T_0}\) be a family of compact sets in \(\mathbb{R}^N\) such that \(\partial C(t) = \Gamma(t)\). For each \(h > 0\) and \(k = 0, 1, \ldots, \lfloor(T_0 - \varepsilon)/h\rfloor - 1\), let \(w^k\) be the solution of (2) - (3). Throughout this section we set \(C^h(t) := \{w^k(t - kh, \cdot) \geq 0\}\) instead of (6).

**Theorem 3.1.** Assume (A0) and (A1). For any \(\varepsilon > 0\) there are constants \(L_0, h_0 > 0\) such that
\[
d_H(C^h(t), C(t)) \leq L_0 h \quad \text{for all } t \in [0, T_0 - \varepsilon] \text{ and } h \in (0, h_0).
\]

This theorem is derived from the following proposition.
Proposition 3.2. (cf. [12, Lemma 6.2]) Assume (A0) and (A1). For any \( \varepsilon > 0 \), there exist \( K_1 > 0 \) and \( t_1 > 0 \) such that for any \( h \in (0, t_1) \) and \( k = 0, 1, \ldots, \floor{(T_0 - \varepsilon)/h} - 1 \), if \( d_H(C^{\alpha}(kh), C(kh)) \leq \alpha_k \) for some \( \alpha_k \in [0, 2\delta] \), then

\[
d_H(C^{\alpha}(t + kh), C(t + kh)) \leq \frac{K_1 t^2 + \alpha_k}{1 - K_1 t} \quad \text{for } t \in [0, h].
\]

The proof of this proposition in [12] is complicated because it is based on some uniform estimates of solutions of (2) in \( \mathcal{N}_{c, 30\delta} \) and because we need to show the convergence itself of our algorithm in advance to apply them. In this section we adopt a different approach from [12] to show Proposition 3.2. Note that the following arguments derive the convergence and its rate simultaneously.

The basic idea to prove the above proposition is a suitable construction of a sub-and a super-solution of (2) which approximate \( w_k \) near \( \Gamma(t + kh) \) for \( t \in [0, h] \) and \( k = 0, 1, \ldots, \floor{(T_0 - \varepsilon)/h} - 1 \). This is the main difference from [12]. For this purpose, we first do this construction in \( \mathcal{M}_k := \{(t, x) \mid 0 \leq t < h, \ |d_k(0, x)| \leq 15\delta \} \). Note that by the smoothness of the CDM, we are able to take \( t_2 > 0 \) such that for any \( h \in (0, t_2) \) and \( k = 0, 1, \ldots, \floor{(T_0 - \varepsilon)/h} - 1, \mathcal{M}_k \subset \mathcal{N}_{c, 30\delta} \) and \( d_k \in C^{2,1}(\mathcal{M}_k) \). We also set

\[
f_k = f_k(t, x) = f(t + kh, x) \quad \text{for } f = b, g, d \text{ and } \kappa^a.
\]

We define \( d_k = d_k(t, x) \) and \( \omega_k = \omega_k(t, x) \) by

\[
\begin{align*}
\overline{d}_k &:= (1 - t\omega_k)d_k - M_3 t d_k^2 - K_2 t^2 - \alpha_k, \\
\underline{d}_k &:= (1 - t\omega_k)d_k + M_3 t d_k^2 + K_2 t^2 + \alpha_k, \\
\omega_k &:= \kappa^a + \langle Db_k Dd_k, Dd_k \rangle + \langle Dg_k, Dd_k \rangle.
\end{align*}
\]

Here \( \alpha_k \) is nonnegative and selected later. Lemma 3.3 below shows that \( \overline{d}_k \) and \( \underline{d}_k \) are, respectively, a classical subsolution and a classical supersolution of (2) in \( \mathcal{M}_k \). In Lemmas 3.4 and 3.5 below we extend \( d_k \) and \( \omega_k \) in \([0, h) \times \mathbb{R}^N\) as a viscosity subsolution and a viscosity supersolution of (2), respectively. We apply the comparison principle for viscosity solutions to estimate \( w_k \) near \( \Gamma(t + kh) \) for \( t \in [0, h] \). We then use this estimate to show the assertion of Proposition 3.2.

Now we construct a classical subsolution of (2).

**Lemma 3.3.** For large \( K_2 > 0, d_k \) and \( \omega_k \) are, respectively, a classical subsolution and a classical supersolution of (2) in \( \mathcal{M}_k \) for all \( k = 0, 1, \ldots, \floor{(T_0 - \varepsilon)/h} - 1 \) and \( h \in (0, t_2) \). In addition, if \( d_H(C^{\alpha}(kh), C(kh)) \leq \alpha_k \) for some \( \alpha_k \in [0, 2\delta] \), then \( d_k(0, \cdot) \leq \overline{d}(\cdot, C^{\alpha}(kh)) \leq \underline{d}_k(0, \cdot) \) in \( \mathbb{R}^N \). Here \( t_2 > 0 \) is chosen so that \( \mathcal{M}_k \subset \mathcal{N}_{c, 30\delta} \) and \( d_k \in C^{2,1}(\mathcal{M}_k) \).

**Proof.** We prove only the subsolution case because the other one is similarly proved.

It is seen by (17) that

\[
L_k[d_k] + g_k = L_k[d_k] - \omega_k d_k + g_k - M_3 d_k^2 - t(L_k[\omega_k d_k + M_3 d_k^2] + 2K_2) \\
\leq t(\langle L[\omega d + M_3 d^2] \rangle_\infty - 2K_2).
\]

where \( L[w] := w_t - \Delta w + \langle b, Dw \rangle \) and \( \langle \cdot \rangle_\infty := \| \cdot \|_{L_\infty(\mathcal{N}_{c, 30\delta})} \). Hence taking \( K_2 > 0 \) sufficiently large, we see that \( d_k \) is a classical subsolution of (2) in \( \mathcal{M}_k \).

If \( d_H(C^{\alpha}(kh), C(kh)) \leq \alpha_k \) for some \( \alpha_k \in [0, 2\delta] \), then it is seen that

\[
d_k(0, \cdot) = d_k(0, \cdot) - \alpha_k \leq d(\cdot, C^{\alpha}(kh)) \leq d_k(0, \cdot) + \alpha_k = \underline{d}_k(0, \cdot) \quad \text{in } \mathbb{R}^N.
\]

Thus the proof is completed. \( \square \)
We extend $d_k$ and $\overline{d}_k$ to the domain $[kh,(k+1)h) \times \mathbb{R}^N$ as, respectively a viscosity subsolution and a viscosity supersolution of (2). See [6] and [14] for the definitions and the theory of viscosity solutions.

Let $\eta_{k,i} = \eta_{k,i}(x)$ ($i = 1, 2, 3$) be smooth cut-off functions satisfying

$$\eta_{k,1} = \begin{cases} 1 & \text{on } \{6\delta \leq d_k(0,. \leq 8\delta]\}, \\ 0 & \text{on } \mathbb{R}^N \setminus \{5\delta \leq d_k(0,. \leq 9\delta]\}, \end{cases}$$

$$\eta_{k,2} = \begin{cases} 1 & \text{on } \{-8\delta \leq d_k(0,. \leq -5\delta]\}, \\ 0 & \text{on } \mathbb{R}^N \setminus \{-9\delta \leq d_k(0,. \leq -4\delta]\}, \end{cases}$$

$$\eta_{k,3} = \begin{cases} 1 & \text{on } \{2\delta \leq d_k(0,. \leq 5\delta]\}, \\ 0 & \text{on } \mathbb{R}^N \setminus \{\delta \leq d_k(0,. \leq 5\delta]\}, \end{cases}$$

$$0 \leq \eta_{k,i} \leq 1 \text{ for } \mathbb{R}^N, \delta ||D\eta_{k,i}||_{L^\infty(\mathbb{R}^N)} + \delta^2 ||D^2\eta_{k,i}||_{L^\infty(\mathbb{R}^N)} \leq M_4.$$ 

For $\alpha_k \in (0, 2\delta)$, $y \in \{d_k(0,. \leq \pm 10\delta]\}$ and $(t, x) \in \mathcal{M}_k$, define

$$\rho_k = \rho_k(t, x) := d_k(t, x) - K_3 t \{\eta_{k,1}(x) + \eta_{k,2}(x)\} - K_4 t^2$$

$$w^+_k(t, x, y) = 10\delta - v_N(t, x, y) - K_5 \sqrt{t} [\eta_{k,3}(x) - K_6 t - K_7 t^{3/2} - \alpha_k]$$

if $y \in \{d_k(0,. \geq 10\delta]\}$,

$$w^-_k(t, x, y) = -6\delta - v_N(t, x, y) - K_6 t - K_7 t^{3/2} - \alpha_k$$

if $y \in \{d_k(0,. \leq -10\delta]\}$,

where $v_N$ is given in subsection 2.3.

**Lemma 3.4.** There exist large $K_j > 0$ ($j = 3, 4, 5, 6, 7$) and $t_3 \in (0, t_2)$ such that for $h \in (0, t_3)$ and $k = 0, 1, \ldots, ([T_0 - \varepsilon]/h) - 1$,

$$L_k[\rho_k] + g_k \leq 0 \text{ in } \mathcal{M}_k, L_k[w^+_k] + g_k \leq 0 \text{ in } (0, h) \times \mathbb{R}^N, \quad (21)$$

$$\rho_k(t, x) \leq w^+_k(t, x; x + 2\delta Dd_k(0, x)) \quad \text{for } t \in [0, h], \ x \in \{d_k(0,. \leq 8\delta]\} \quad (22)$$

$$w^+_k(t, x, y) \leq \rho_k(t, x) \quad \text{for } t \in [0, h], \ x \in \{d_k(0,. \leq 3\delta]\} \quad (23)$$

and $y \in \{d_k(0,. \geq 10\delta]\}$,

$$w^-_k(t, x, y) \leq \rho_k(t, x) \quad \text{for } t \in [0, h], \ x \in \{d_k(0,. \leq -3\delta]\} \quad (24)$$

and $y \in \{d_k(0,. \geq -10\delta]\}$,

$$\rho_k(t, x) \leq w^-_k(t, x; x - 2\delta Dd_k(0, x)) \quad \text{for } t \in [0, h] \quad (25)$$

and $x \in \{d_k(0,. \leq -8\delta]\}$.

**Proof.** We observe by Lemma 3.3 and (18) with $m = N$ that

$$L_k[\rho\_k] + g_k \leq t(K_3 \langle L_k[\eta_{k,1} + \eta_{k,2}]\rangle_\infty - 2K_4) \text{ in } \mathcal{M}_k,$$

$$L_k[w^+_k] + g_k \leq ||b||_\infty + ||g||_\infty - K_6 + \sqrt{t} \left( K_5 \langle L_k[\eta_{k,3}]\rangle_\infty - \frac{3K_7}{2} \right)$$

in $\{d_k(0,. \leq 8\delta]\}$,

$$L_k[w^-_k] + g_k \leq ||b||_\infty + ||g||_\infty - K_6 \text{ in } (0, h) \times \mathbb{R}^N.$$

Since it readily follows that

$$d_k(t, x) = d_k(0, x) + t d_k, t(x, \theta) \quad \text{for } (t, x) \in \mathcal{M}_k \text{ and } \theta = \theta(t, x) \in (0, 1),$$

we see that
\[
\rho_k(t,x) \leq 8\delta + M_5 t - K_3 t - \alpha_k \text{ for } t \in [0,h], \ x \in \{d_k(0,\cdot) = 8\delta\},
\]
\[
\rho_k(t,x) \geq 3\delta - M_6 t - (K_2 + K_4) t^2 - \alpha_k \text{ for } t \in [0,h], \ x \in \{d_k(0,\cdot) = 3\delta\},
\]
\[
\rho_k(t,x) \geq -3\delta - M_6 t - (K_2 + K_4) t^2 - \alpha_k \text{ for } t \in [0,h], \ x \in \{d_k(0,\cdot) = -3\delta\},
\]
\[
\rho_k(t,x) \leq -8\delta + M_5 t - K_3 t - \alpha_k \text{ for } t \in [0,h], \ x \in \{d_k(0,\cdot) = -8\delta\},
\]
where \( M_5 := 8\delta \langle \omega \rangle_\infty + \langle d_1 \rangle_\infty (1 + \langle \omega \rangle_\infty) \) and \( M_6 := M_2 + M_3 (8\delta + \langle d_1 \rangle_\infty)^2 \). Furthermore, choosing \( t_{3,1} \in (0,t_2) \) small, we observe that for all \( h \in (0,t_{3,1}) \),
\[
\begin{align*}
\omega^+_k(t,x;x+2\delta Dd_k(0,x)) &\geq 8\delta - \frac{Nt}{2\delta} - K_6 t - K_7 t^{3/2} - \alpha_k \\
&\text{for } t \in [0,h], \ x \in \{d_k(0,\cdot) = 8\delta\} \\
\omega^+_{\pm}(t,x;y) &\leq 10\delta - \sqrt{(7\delta)^2 + 2Nt} - K_5 \sqrt{t} - \alpha_k \\
&\leq 3\delta - \frac{Nt}{7\delta} + \frac{N^2t^2}{(7\delta)^2} - K_5 \sqrt{t} - \alpha_k \\
&\text{for } t \in [0,h], \ x \in \{d_k(0,\cdot) = 3\delta\} \text{ and } y \in \{d_k(0,\cdot) = 10\delta\}, \\
\omega^-_{\pm}(t,x;y) &\leq -10\delta - \alpha_k \\
&\text{for } t \in [0,h], \ x \in \{d_k(0,\cdot) = -3\delta\} \text{ and } y \in \{d_k(0,\cdot) = -10\delta\}, \\
\omega^-_k(t,x;x-2\delta Dd_k(0,x)) &\geq -8\delta - \frac{Nt}{2\delta} - K_6 t - K_7 t^{3/2} - \alpha_k \\
&\text{for } t \in [0,h] \text{ and } x \in \{d_k(0,\cdot) = -8\delta\}.
\end{align*}
\]
Here we have used the inequality:
\[
1 + \frac{s}{2} - \frac{s^2}{4} \leq \sqrt{1+s} \leq 1 + \frac{s}{2} \text{ for small } |s| < 1.
\]
Choose
\[
K_3 \geq M_5 + \frac{N}{2\delta} + K_6 + 1, \ K_4 \geq K_3 (\langle L_k | \eta_{k,1} + \eta_{k,2} \rangle)_{\infty},
\]
\[
K_5 \geq M_6 + \frac{N^2}{(7\delta)^2} + K_2 + K_4, \ K_6 \geq \|b\|_{\infty} + \|g\|_{\infty}, \ K_7 \geq K_5 (\langle L_k | \eta_{k,3} \rangle)_{\infty}.
\]
We get (21) by the choices of \( K_4, K_6 \) and \( K_7 \) and the differential inequalities at the beginning of this proof. From the choices of \( K_3 \) and \( K_5 \) and the above inequalities for \( \rho_k \) and \( \omega^\pm_k \), for each \( h \in (0,t_{3,1}) \) we have (23), (24) and
\[
\rho_k(t,x) \leq \omega^+_k(t,x;x+2\delta Dd_k(0,x)) + t(K_7 \sqrt{t} - 1) \\
&\text{for } t \in [0,h] \text{ and } x \in \{d_k(0,\cdot) = 8\delta\} \\
\rho_k(t,x) \leq \omega^-_k(t,x;x-2\delta Dd_k(0,x)) + t(K_7 \sqrt{t} - 1) \\
&\text{for } t \in [0,h] \text{ and } x \in \{d_k(0,\cdot) = -8\delta\}.
\]
Furthermore taking \( t_3 := \min\{t_{3,1}, K_7^{-2}\} \langle t_2 \rangle \), we have (22) and (25) for all \( h \in (0,t_3) \) and \( k = 0, 1, \ldots, [(T_0 - \varepsilon)/h] - 1 \). Therefore the proof of Lemma 3.4 is completed. \( \square \)
For any $h \in (0, t_3)$, define a function $w_k = w_k(t, x)$ by

$$w_k(t, x) := \begin{cases} \sup_{y \in \{d_k(0, \cdot) \geq 10\delta\}} w_k^+(t, x; y) & \text{for } (t, x) \in [0, h] \times \{d_k(0, \cdot) \geq 9\delta\}, \\ \max \left\{ \rho_k(t, x), \sup_{y \in \{d_k(0, \cdot) = 10\delta\}} w_k^+(t, x; y) \right\} & \text{for } (t, x) \in [0, h] \times \{3\delta \leq d_k(0, \cdot) < 9\delta\}, \\ \max \left\{ \rho_k(t, x), \sup_{y \in \{d_k(0, \cdot) = -10\delta\}} w_k^-(t, x; y) \right\} & \text{for } (t, x) \in [0, h] \times \{-12\delta \leq d_k(0, \cdot) \leq -3\delta\}, \\ \sup_{y \in \{d_k(0, \cdot) = -10\delta\}} w_k^-(t, x; y) & \text{for } (t, x) \in [0, h] \times \{d_k(0, \cdot) < -12\delta\}. \end{cases}$$

Then the following lemma asserts that $d_k$ can be extended as a viscosity subsolution of (1.2) in $[0, h) \times \mathbb{R}^N$.

**Lemma 3.5.** The function $w_k$ is a viscosity subsolution of (2). Moreover, if $d_H(C^h(kh), C(kh)) \leq \alpha_k$ for some $\alpha_k \in [0, 2\delta)$, then $w_k(0, \cdot) \leq d(\cdot, C^h(kh))$ in $\mathbb{R}^N$.

**Proof.** Applying the stability of viscosity solutions (cf. [6], [14], etc.), we see that $w_k$ is a viscosity subsolution of (2). Assume that $d_H(C^h(kh), C(kh)) \leq \alpha_k$ for some $\alpha_k \in [0, 2\delta)$. Then we have

$$w_k^+(0, x; y) = 10\delta - |x - y| - \alpha_k \leq d_k(0, x) - \alpha_k \quad \text{for } x \in \{d_k(0, \cdot) \geq 3\delta\} \text{ and } y \in \{d_k(0, \cdot) = 10\delta\}$$

$$w_k^-(0, x; y) = -10\delta - |x - y| - \alpha_k \leq d_k(0, x) - \alpha_k \quad \text{for } x \in \{d_k(0, \cdot) \leq -3\delta\} \text{ and } y \in \{d_k(0, \cdot) = -10\delta\}.$$  

Therefore we conclude that $w_k(0, \cdot) \leq d(\cdot, C^h(kh))$ in $\mathbb{R}^N$ and this is our desired extension of $d_k$ in $[0, h) \times \mathbb{R}^N$. \hfill $\square$

Recall that $\overline{d}_k$ is given by (20) and that $\eta_{k,i}$ ($i = 1, 2, 3$) are defined after the proof of Lemma 3.3. As for a supersolution of (2), define

$$\overline{\rho}_k = \overline{w}_k(t, x) = \overline{d}_k(t, x) + K_3 t \{\eta_{k,1}(x) + \eta_{k,2}(x)\} + K_4 t^2$$

$$\overline{w}_k^+ = \overline{w}_k^+(t, x; y) := 10\delta + v_N(t, x; y) + K_5 \sqrt{t} \eta_{k,3}(x) + K_6 t + K_7 t^{3/2} + \alpha_k \quad \text{if } y \in \{d_k(0, \cdot) = 10\delta\},$$

$$\overline{w}_k^- = \overline{w}^-(t, x; y) := -6\delta + v_N(t, x; y) + K_6 t + K_7 t^{3/2} + \alpha_k \quad \text{if } y \in \{d_k(0, \cdot) \leq -10\delta\},$$

$$\overline{w}_k(t, x) := \begin{cases} \sup_{y \in \{d_k(0, \cdot) \geq 10\delta\}} \overline{w}_k^+(t, x; y) & \text{for } (t, x) \in [0, h] \times \{d_k(0, \cdot) \geq 9\delta\}, \\ \max \left\{ \overline{\rho}_k(t, x), \sup_{y \in \{d_k(0, \cdot) = 10\delta\}} \overline{w}_k^+(t, x; y) \right\} & \text{for } (t, x) \in [0, h] \times \{3\delta \leq d_k(0, \cdot) < 9\delta\}, \\ \max \left\{ \overline{\rho}_k(t, x), \sup_{y \in \{d_k(0, \cdot) = -10\delta\}} \overline{w}_k^-(t, x; y) \right\} & \text{for } (t, x) \in [0, h] \times \{-12\delta \leq d_k(0, \cdot) \leq -3\delta\}, \\ \sup_{y \in \{d_k(0, \cdot) = -10\delta\}} \overline{w}_k^-(t, x; y) & \text{for } (t, x) \in [0, h] \times \{d_k(0, \cdot) < -12\delta\}. \end{cases}$$
where \( v_N \) is given in subsection 2.3 and set

\[
\begin{align*}
\inf_{y \in \{d_k(0, \cdot) = 10\delta\}} w^+_k(t, x; y) & \quad \text{for } (t, x) \in [0, h] \times \{d_k(0, \cdot) \geq 9\delta\}, \\
\min \left\{ \bar{p}_k(t, x), \inf_{y \in \{d_k(0, \cdot) = 10\delta\}} w^+_k(t, x; y) \right\} & \quad \text{for } (t, x) \in [0, h] \times \{3\delta < d_k(0, \cdot) < 9\delta\}, \\
\min \left\{ \bar{p}_k(t, x), \inf_{y \in \{d_k(0, \cdot) = -10\delta\}} \underline{w}_k(t, x; y) \right\} & \quad \text{for } (t, x) \in [0, h] \times \{-12\delta \leq d_k(0, \cdot) \leq -3\delta\}, \\
\inf_{y \in \{d_k(0, \cdot) \leq -10\delta\}} \underline{w}_k(t, x; y) & \quad \text{for } (t, x) \in [0, h] \times \{d_k(0, \cdot) < -12\delta\}.
\end{align*}
\]

Then we are able to prove by the same way as Lemmas 3.4 and 3.5 that \( \underline{w}_k \) is a viscosity supersolution of (2) satisfying \( d(\cdot, C(kh)) \leq \underline{w}(\cdot, \cdot) \) in \( \mathbb{R}^N \) for all \( h \in (0, t_3) \) and \( k = 0, 1, \ldots, [T_0 - \varepsilon]/h - 1 \). Here \( t_3 \) is given in Lemma 3.4. Figure 1 shows the outline of the graphs of \( w_k \) and \( \underline{w}_k \). Note that \( w_k \) and \( \underline{w}_k \) grow at most linearly as \( |x| \to +\infty \) since so does \( v_N \).

\[
\text{Figure 1. Graphs of } w_k, \underline{w}_k \text{ and } \overline{w}_k \text{ (thick curves)}
\]

It follows from (10) and the constructions of \( w_k \) and \( \underline{w}_k \) that for all \( h \in (0, t_3) \) and \( k = 0, 1, \ldots, [T_0 - \varepsilon]/h - 1 \)

\[
|w_k(t, x)|, |\underline{w}_k(t, x)|, |\overline{w}_k(t, x)| \leq M_7(1 + |x|) \quad \text{for } (t, x) \in [0, h] \times \mathbb{R}^N,
\]

\[
\underline{w}_k(0, \cdot) \leq w_k(0, \cdot) = d(\cdot, C_k) \leq \overline{w}_k(0, \cdot) \quad \text{in } \mathbb{R}^N.
\]

Thus we use the comparison principle for viscosity solutions and the continuity of these functions to obtain \( w_k \leq \underline{w}_k \leq w_k \) in \( [0, h] \times \mathbb{R}^N \) for all \( k = 0, 1, \ldots, [T_0 - \varepsilon]/h - 1 \) and \( h \in (0, t_3) \).

We are now in a position to give

The proof of Proposition 3.2. Suppose that \( d_H(C^h(kh), C(kh)) \leq \alpha_k \) for some \( \alpha_k \in [0, 2\delta] \). Set \( \overline{C}^h(t + kh) := \{ \overline{w}_k(t, \cdot) \geq 0 \} \) and \( \underline{C}^h(t + kh) := \{ \underline{w}_k(t, \cdot) \geq 0 \} \).
Step 1. We prove that for some $t_{1,1} > 0$ and all $h \in (0, t_{1,1})$, $k = 0, 1, \ldots, [T_0 - \varepsilon]/h - 1$ and $t \in [0, h)$,
\begin{align}
& C^h(t + kh) \subset C^h(t + kh) \subset C^h(t + kh), \\
& C^h(t + kh) \subset C^h(t + kh) \subset C^h(t + kh).
\end{align}
(29)\hspace{1cm}(30)

Since $w_k \leq w_k \leq \overline{w_k}$ in $[0, h] \times \mathbb{R}^N$ for each $k = 0, 1, \ldots, [T_0 - \varepsilon]/h - 1$ and $h \in (0, t_3)$, we get on $[0, h] \times \{|d(kh, \cdot)| \leq 3\delta\}$
\begin{align}
(1 - t\omega_k) d_k - M_3 t d_k^2 - (K_2 + K_4) t^2 - \alpha_k \leq w_k \leq (1 - t\omega_k) d_k + M_3 t d_k^2 + (K_2 + K_4) t^2 + \alpha_k.
\end{align}
(31)

We choose $t_{1,1} \in (0, t_3)$ such that for all $h \in (0, t_{1,1})$ and $k = 0, 1, \ldots, [T_0 - \varepsilon]/h - 1$,
\begin{align}
|p_k| & \geq |d_k| - t(\omega d)_\infty - M_3 t (d^2)_\infty - K_3 t - (K_2 + K_4) t^2 - \alpha_k \geq \frac{\delta}{2}
\quad \text{on } [0, h] \times \{|d_k(0, \cdot)| \leq 15\delta\}, \\
\overline{w_k}^+ & \geq 3\delta - \frac{2N}{\dot{\gamma}(t\omega_k + K_2 + K_4 + K_7) - \delta} \geq \frac{\delta}{2}
\quad \text{on } [0, h] \times \{|d_k(0, \cdot)| \leq 9\delta\}, \\
\underline{w_k}^- & \leq -6\delta \quad \text{on } [0, h] \times \{|d_k(0, \cdot)| \leq -3\delta\}.
\end{align}

Recall that $\omega = \kappa^* + \langle Db Dd, Dd \rangle + \langle Dg, Dd \rangle$. It is easy to see that the following holds for all $h \in (0, t_{1,1})$ and $k = 0, 1, \ldots, [T_0 - \varepsilon]/h - 1$:
\begin{align}
\sup_{y \in \{d(t, \cdot) \geq 10\delta\}} w_k^+(t, x; y) = 10\delta - \sqrt{\delta^2 + 2N t - \sqrt{t(K_2 + K_6 + K_7) - \alpha_k}} \geq 4\delta
\quad \text{on } [0, h] \times \{|d_k(0, \cdot)| \geq 9\delta\}.
\end{align}

Therefore we get for all $h \in (0, t_{1,1})$ and $k = 0, 1, \ldots, [T_0 - \varepsilon]/h - 1$,
\begin{align}
|w_k| \geq \frac{\delta}{2} \quad \text{on } [0, h] \times \{|d_k(0, \cdot)| \geq 3\delta\}.
\end{align}

Here we have used the assumption $\alpha_k \in (0, 2\delta)$. Similarly, we are able to show that
\begin{align}
|p_k| \geq \frac{\delta}{2} \quad \text{on } [0, h] \times \{|d_k(0, \cdot)| \geq 3\delta\}.
\end{align}

Taking $t_{1,2} > 0$ small, we observe that the following holds for all $h \in (0, t_{1,2})$ and $k = 0, 1, \ldots, [T_0 - \varepsilon]/h - 1$
\begin{align}
|Dw_k|, |D\overline{w}_k| \geq 1 - M_3 t > 0 \quad \text{on } [0, h] \times \{|d_k(0, \cdot)| \leq 3\delta\}.
\end{align}

Hence it follows that for all $h \in (0, t_{1,2})$, $k = 0, 1, \ldots, [T_0 - \varepsilon]/h - 1$ and $t \in [0, h]$, $\partial C^h(t + kh)$ and $\partial C^h(t + kh)$ are hypersurfaces and satisfy
\begin{align}
\partial C^h(t + kh), \partial C^h(t + kh) \subset \{|d_k(0, \cdot)| \leq 3\delta\}.
\end{align}
(32)

Recalling $C^h(t + kh) = \{w_k(t, \cdot) \geq 0\}$, from $w_k \leq w_k \leq \overline{w_k}$ in $[0, h] \times \mathbb{R}^N$ we get
(29) for $t \in [0, h)$, $k = 0, 1, \ldots, [T_0 - \varepsilon]/h - 1$ and $h \in (0, t_{1,2})$. Moreover, since $w_k(t, \cdot) < \overline{w_k}(t, \cdot)$ on $\{d(t, \cdot) = 0\} = \partial C(t + kh)$, we have (30).

Step 2. We show the assertion of Proposition 3.2.

Define $t_1 := \min\{t_{1,1}, t_{1,2}\}$. Fix $h \in (0, t_1)$, $k = 0, 1, \ldots, [T_0 - \varepsilon]/h - 1$, $t \in [0, h]$ and $x \in C^h(t + kh)$. We assume that $x \notin C(t + kh)$ because, if otherwise, dist $(x, C(t + kh)) = 0$. Then it follows from (29), (32) and the definition of $w_k$ that
\begin{align}
(1 - t\omega_k(t, x)) d_k(t, x) - M_3 t (d_k(t, x))^2 - (K_2 + K_4) t^2 - \alpha_k \leq 0 \quad \text{and } |d_k(0, x)| \leq 3\delta.
\end{align}
Thus we get
\[ d_k(t, x) \leq \frac{1 - t\omega_k(t, x) - \sqrt{(1 - t\omega_k(t, x))^2 - 4M_3t((K_2 + K_4)^2 + \alpha_k)}}{2M_3t} \]
\[ = \frac{1 - t\omega_k(t, x)}{2M_3t} \left( 1 - \sqrt{1 - \frac{4M_3t((K_2 + K_4)^2 + \alpha_k)}{(1 - t\omega_k(t, x))^2}} \right) \]
\[ \leq \frac{(K_2 + K_4)^2 + \alpha_k}{-t\omega_k(t, x)} \left( 1 - \frac{4M_3((K_2 + K_4)^2 + \alpha_k)}{1 - t\omega_k(t, x)} \right)^{-1} \]
\[ = \frac{(K_2 + K_4)^2 + \alpha_k}{1 - t\omega_k(t, x) + 4M_3(K_2 + K_4)^2 + 2\delta}. \]

Here we have used the fact \( 1 - \sqrt{1-s} \leq s/2(1-s) \) for small \( s > 0 \) and \( \alpha_k \in (0, 2\delta) \). Similarly, using \((1 - t\omega_k(t, x))d_k(t, x) + M_3t((d_k(t, x))^2 + (K_2 + K_4)^2 + \alpha_k \geq 0 \) we have
\[ d_k(t, x) \geq \frac{-((K_2 + K_4)^2 + \alpha_k)}{1 - t\omega(t, x) + 4M_3(K_2 + K_4)^2 + 2\delta}. \]

Thus we obtain
\[ \sup_{x \in C^h(t + kh)} \text{dist}(x, C(t + kh)) \leq \frac{(K_2 + K_4)^2 + \alpha_k}{1 - t\omega(t, x) + 4M_3(K_2 + K_4)^2 + 2\delta}. \]

for \( t \in [0, h] \).

Take any \( x \in C(t + kh) \). Then we may consider \( x \notin C^h(t + kh) \) because of the same reason as above. Choose \( r \in \mathbb{R} \) such that \( x + rDd_k(t, x) \in \partial C^h(t + kh) \). Since \( Dd_k(t, x + rDd_k(t, x)) = Dd_k(t, x) \) for \( r \in (-3\delta, 3\delta) \) in view of (30), we calculate that
\[ 0 = w_k(t, x + rDd_k(t, x)) \]
\[ = w_k(t, x) + r(Dw_k(t, x + \theta rDd_k(t, x)), Dd_k(t, x)) \]
\[ = -(K_2 + K_4)^2 - \alpha_k + r(Dd_k(t, x), Dd_k(t, x)) \]
\[ -rt(D(\omega_kd_k + M_3d_k^2)(t, x + \theta rDd_k(t, x)), Dd_k(t, x)). \]

Setting \( M_9 := \|D(\omega_kd_k + M_3d_k^2)\|_\infty \), we get
\[ |r| \leq \frac{(K_2 + K_4)^2 + \alpha_k}{1 - M_9t} \quad \text{for} \quad t \in [0, h] \quad \text{and} \quad k = 0, 1, \ldots, \lceil(T_0 - \varepsilon)/h\rceil - 1. \]

Consequently we have for all \( h \in (0, t_1) \), \( k = 0, 1, \ldots, \lceil(T_0 - \varepsilon)/h\rceil - 1, \)
\[ \sup_{x \in C(t + kh)} \text{dist}(x, C^h(t + kh)) \leq \frac{(K_2 + K_4)^2 + \alpha_k}{1 - M_9t} \quad \text{for} \quad t \in [0, h]. \]

Setting \( K_1 := \max\{K_2 + K_4, \omega, \infty, 4M_3(K_2 + K_4) + 2\delta, M_9\} \), we obtain the desired estimate. \( \square \)

**Remark 3.6.** (1) In [12] we used some local uniform estimates of the solution to (2) in \( N_{\varepsilon, 30\delta} \). However, thanks to the explicit forms of \( \underbar{w}_k \) and \( \overbar{w}_k \) on \([0, h] \times \{dk(0, \cdot)\} \leq 3\delta \), we do not need such estimates.

(2) Since \( dk(0, \cdot) \) is Lipschitz continuous in \( \mathbb{R}^N \) and (A1) is assumed, we see that \( w_k = dk + O(\sqrt{h}) \) in \([0, h] \times \mathbb{R}^N \). On the other hand, we obtain \( w_k = dk + O(h) \) on \( M_k \) for \( k = 0, 1, \ldots, \lceil(T_0 - \varepsilon)/h\rceil - 1 \) and \( h \in (0, t_3) \) from (31). This estimate is better than the above one. Here \( O(h) \) and \( O(\sqrt{h}) \) are uniform for \( k = 0, 1, \ldots, \lceil(T_0 - \varepsilon)/h\rceil - 1. \)
Proof of Theorem 3.1. Let $L_0 := (e^{K_1 T_0} - 1)$ and $h_0 := \min\{1/K_1, \delta, \delta/L_0, t_1\}$. We use Proposition 3.2 with $\alpha_0 = 0$ to have

$$d_H(C^h(t), C(t)) \leq \frac{K_1 t^2}{1 - K_1 t}$$

for all $t \in [0, h]$ and $h \in (0, h_0)$.

Then it is seen that for all $t \in [0, h],

$$\frac{K_1 t^2}{1 - K_1 t} \leq \frac{K_1 h^2}{1 - K_1 h} = \left(\frac{1}{1 - K_1 h} - 1\right) h \leq \left\{\left(\frac{1}{1 - K_1 h}\right)^{[T_0/h]} - 1\right\} h \leq L_0 h(< 2\delta).$$

Hence we get $\alpha_1 := K_1 h^2/(1 - K_1 h) \in (0, 2\delta)$ for all $h \in (0, h_0)$. Applying Proposition 3.2 again, we have

$$d_H(C^h(t + h), C(t + h)) \leq \frac{K_1 t^2 + \alpha_1}{1 - K_1 t}$$

for all $t \in [0, h]$ and $h \in (0, h_0)$.

We calculate that for all $t \in [0, h],

$$\frac{K_1 t^2 + \alpha_1}{1 - K_1 t} \leq \left\{\left(\frac{1}{1 - K_1 h}\right)^2 - 1\right\} h \leq \left\{\left(\frac{1}{1 - K_1 h}\right)^{[T_0/h]} - 1\right\} h \leq L_0 h(< 2\delta).$$

Inductively, we set $\alpha_k := \frac{K_1 h^2 + \alpha_{k-1}}{1 - K_1 h}$ and observe that for any $h \in [0, h_0]$, if $\alpha_k \in (0, 2\delta)$, then it follows from Proposition 3.2 that

$$d_H(C^h(t + kh), C(t + kh)) \leq \frac{K_1 t^2 + \alpha_k}{1 - K_1 t}$$

for all $t \in [0, h]$ and $h \in (0, h_0)$.

Since we compute that

$$\frac{K_1 t^2 + \alpha_k}{1 - K_1 t} \leq \frac{K_1 h^2 + \alpha_k}{1 - K_1 h} = \sum_{l=1}^{k+1} \frac{K_1 h^2}{(1 - K_1 h)^l} \leq \left\{\left(\frac{1}{1 - K_1 h}\right)^{[T_0/h]} - 1\right\} h \leq (e^{K_1 T_0} - 1)h \leq L_0 h(< 2\delta),$$

we obtain $d_H(C^h(t + kh), C(t + kh)) \leq L_0 h$ and $\alpha_{k+1} < 2\delta$ for all $t \in [0, h]$, $k = 0, 1, \ldots, \left[(T - \epsilon_0)/h\right] - 1$ and $h \in (0, h_0)$. Thus we obtain the desired result.

Remark 3.7. The arguments of this section cannot be applied to the case where $\{\Gamma(t)\}_{t \in [0, T_0]}$ is a generalized CDM because the smoothness of $\Gamma(t)$ is essentially used. We have a partial result in [12] which gives the rate of convergence to the regular portion of the generalized CDM. This results is valid even if $\Gamma(t)$ has some singular points. However, to our best knowledge, there is no result such as Theorem 3.1 around singular points.

4. Optimality. This section is devoted to the optimality of the estimate in Theorem 3.1 with respect to the order $h$. For this purpose we consider the radial case. For simplicity, we set $N = 2$, $R(t) := \sqrt{1 - 2t}$, $T_0 := 1/2$, $C(t) := \{x \in \mathbb{R}^2 \mid |x| \leq R(t)\}$, $b \equiv 0$ and $g \equiv 0$. Since it suffices to consider the radial solution, we treat the following problem:
\( \mathbb{L}[w_k] := w_{k,t} - w_{k,rr} - \frac{w_k}{r} = 0, \ w_k = w_k(t, r) \) in \((0, h) \times (0, +\infty)\), \hspace{1cm} (33)

\( w_k, r(t, 0) = 0 \) for \( t > 0 \), \hspace{1cm} (34)

\( w_k(0, r) = R_k - r \) for \( r \in [0, +\infty) \), \hspace{1cm} (35)

\( C_k := \{ w_k(h, | \cdot |) \geq 0 \}, \ C_0 := \text{cl} B(0, 1), \)

\( R_k := \text{radius of } C_k, \ R_0 := 1. \)

For \( h > 0, k = 0, 1, \ldots, [(T_0 - \varepsilon)/h] - 1 \) and \( t \in [kh, (k + 1)h) \), set

\[ C^h(t) := \{ w_k(t - kh, |x|) \geq 0 \}, \ R^h(t) := \text{radius of } C^h(t). \]

The following proposition says that for each \( h > 0 \), \( C^h(t) \) evolves faster than \( C(t) \).

**Proposition 4.1.** ([12, Proposition 6.4]) \( C^h(t) \subset C(t) \) for all \( t \in [0, T_0] \) and \( h > 0 \).

Combining Theorem 3.1 with this proposition, we see that for any \( \varepsilon \in (0, 1/100) \), there exists \( h_0 > 0 \) such that

\[ R(t) \geq \sqrt{2\varepsilon}, \ R(t) \geq R^h(t) \geq R(t) - L_0h \geq \sqrt{\varepsilon}. \] \hspace{1cm} (36)

for all \( t \in [0, T_0 - \varepsilon] \) and \( h \in (0, h_0) \).

Under these settings, we are able to prove the following estimate.

**Theorem 4.2.** For any \( \varepsilon \in (0, 1/100) \), there exist \( L_1 > 0 \) and \( h_1 > 0 \) such that for all \( h \in (0, h_1) \),

\[ |R^h(t) - \{ R(t) - t^2 \}| \leq L_1 t^{5/2} \] \hspace{1cm} for \( t \in [0, h] \),

\[ \left| R^h(t) - \left\{ R(t) + \frac{h \log R(t)}{R(t)} \right\} \right| \leq \frac{L_1 h^{3/2}}{R(t)} \] \hspace{1cm} for \( t \in [h, T_0 - \varepsilon] \).

This implies the optimality of the estimate in Theorem 3.1 with respect to the order \( h \) and is more precise than the estimate in [12].

As mentioned in the Introduction, the idea to prove this theorem is to employ an explicit solution of (7). For \( h > 0 \) and \( k = 0, 1, 2, \ldots, [(T_0 - \varepsilon)/h] - 1 \), let \( \rho_k(t, r) := R^h(kh) - \sqrt{r^2 + 2t} \ (R^h(0) = 1) \). Then as seen in the Introduction, \( \rho_k \) satisfies (7) and (9) with \( N = 2 \). More precisely,

\[ \mathbb{L}[\rho_k] = \frac{2t}{(r^2 + 2t)^{3/2}} \geq 0 \] \hspace{1cm} in \((0, h) \times (0, +\infty)\).

Note that for \( t \leq \varepsilon^2 \)

\[ \mathbb{L}[\rho_k] \leq \frac{1}{\sqrt{2t}} \] \hspace{1cm} for \( r \geq 0 \), \hspace{1cm} \mathbb{L}[\rho_k] \leq \frac{2t}{\varepsilon^3} \] \hspace{1cm} for \( r \geq \varepsilon \). \hspace{1cm} (37)

We modify \( \rho_k \) to construct a classical subsolution of (33). Define

\[ \rho^*_k = \rho^*_k(t, r) := \rho_k(t, r) - \frac{t^2}{(r^2 + 2t)^{3/2}}. \]

Then we have the following.
Lemma 4.3. For any \( \varepsilon \in (0, 1/100) \), \( h \in (0, T_0/2) \) and \( k = 0, 1, \ldots, \lfloor (T_0 - \varepsilon)/h \rfloor - 1 \), \( \rho_k \) satisfies
\[
-\frac{3}{4\sqrt{2t}} \leq L[\rho_k] \leq 0 \quad \text{for} \ t \in (0, \varepsilon^2) \text{ and } r \in [0, \sqrt{t}/2], \tag{38}
\]
\[
0 \leq L[\rho_k] \leq \frac{3}{4\sqrt{2t}} \quad \text{for} \ t \in (0, \varepsilon^2) \text{ and } r \in [\sqrt{t}/2, \varepsilon), \tag{39}
\]
\[
0 \leq L[\rho_k] \leq \frac{12t^2}{r^b} \quad \text{for} \ t \in (0, \varepsilon^2) \text{ and } r \in [\varepsilon, +\infty). \tag{40}
\]

Proof. Some calculations yield that
\[
L[\rho_k] = \frac{6t^2(2r^2 - t)}{(r^2 + 2t)^{7/2}}.
\]
Then (38) and (40) are valid. As \( 2r^2 - t \geq 0 \) and \( r^2 + 2t \geq 2\sqrt{2t}r \) for \( r \geq (t/2)^{1/2} \), we have for \( (t, r) \in (0, \varepsilon^2) \times (\varepsilon, +\infty) \),
\[
L[\rho_k] \leq \frac{3t^{1/4}(2r^2 - t)}{16 \cdot 2^{1/4} r^{7/2}} \leq \frac{3t^{1/4}}{8 \cdot 2^{1/4} r^{3/2}} \leq \frac{3}{4\sqrt{2t}}.
\]
Hence we obtain (39) and the proof is completed. \( \square \)

We modify \( \rho_k \) to construct a viscosity subsolution of (33). Let \( \eta_i = \eta_i(r) \ (i = 1, 2) \) be smooth cut-off functions satisfying
\[
\eta_1 = \begin{cases} 
1 & \text{on } [\varepsilon/2, 3\varepsilon/2], \\
0 & \text{on } \mathbb{R} \setminus [\varepsilon/3, 5\varepsilon/3], \\
\varepsilon\|\eta_i'\|_{L^\infty(\mathbb{R})} + \varepsilon^2\|\eta_i''\|_{L^\infty(\mathbb{R})} & \leq M_{10}, \\
\eta_2(r) := \eta_1(r - 2\varepsilon).
\end{cases}
\]

We set
\[
\psi_{k,1} = \psi_{k,1}(t, r) := \rho_k(t, r) - 2\sqrt{t},
\]
\[
\psi_{k,2} = \psi_{k,2}(t, r) := \rho_k(t, r) - 2\sqrt{t} \eta_1(r) - K_8^t^{3/2},
\]
\[
\psi_{k,3} = \psi_{k,3}(t, r) := \rho_k(t, r) - K_8^t^{3/2} \eta_2(r) - K_8^t^{3/2}.
\]
Then we see from (37) that \( \psi_{k,1} \) is a classical subsolution of (33) satisfying (34) with \( w_k = v_{k,1} \) and \( \psi_{k,1}(0, r) = \rho_k(0, r) \) for all \( r \in [0, +\infty) \). As for \( \psi_{k,2} \) and \( \psi_{k,3} \) we have the following lemma.

Lemma 4.4. Let \( \varepsilon \in (0, 1/100) \). There exist \( t_4 \in (0, \varepsilon^4) \) and large \( K_8, K_9 > 0 \) such that for all \( h \in (0, t_4) \), \( v_{k, i} \ (i = 2, 3) \) satisfies, respectively,
\[
L[\psi_{k,2}] \leq 0 \quad \text{in } (0, t_4) \times (\varepsilon/2, +\infty), \quad L[\psi_{k,3}] \leq 0 \quad \text{in } (0, t_4) \times (5\varepsilon/2, +\infty),
\]
\[
\psi_{k,2}(t, \varepsilon) < \psi_{k,1}(t, \varepsilon), \quad \psi_{k,2}(t, 2\varepsilon) > \psi_{k,1}(t, 2\varepsilon) \quad \text{for all } t \in [0, t_4],
\]
\[
\psi_{k,3}(t, 3\varepsilon) < \psi_{k,2}(t, 3\varepsilon), \quad \psi_{k,3}(t, 4\varepsilon) > \psi_{k,2}(t, 4\varepsilon) \quad \text{for all } t \in [0, t_4].
\]

Proof. As for \( \psi_{k,2} \), we calculate that
\[
L[\psi_{k,2}] \leq L[\rho_k] - \frac{\eta_1}{\sqrt{t}} + \sqrt{t} \left\{ \frac{6M_7}{\varepsilon^2} - \frac{3}{2} K_8 \right\} \quad \text{in } (0, \varepsilon^2) \times (\varepsilon/2, +\infty),
\]
\[
\psi_{k,2}(0, r) = \rho_k(0, r) \quad \text{for } r \geq \varepsilon,
\]
\[ v_{k,1}(t, \varepsilon) < \rho_k(t, \varepsilon) - 2 \sqrt{t} = v_{k,1}(t, \varepsilon) \] for \( t \in (0, \varepsilon^2) \),

\[ v_{k,2}(t, 2\varepsilon) \geq v_{k,1}(t, 2\varepsilon) + \sqrt{t} \left( \frac{3}{2} - K_9 t \right) \] for \( t \in (0, \varepsilon^2) \).

It follows from Lemma 4.3 that

\[
L_v[\underline{\rho}_k] := \begin{cases} 
0 & \text{for } r \in (\varepsilon/2, \sqrt{t}/2) \text{ if } t \in (\varepsilon^2/2, \varepsilon^2), \\
\frac{1}{\sqrt{t}} & \text{for } t \in (0, \varepsilon^2) \text{ and } r \in (\max\{\varepsilon/2, \sqrt{t}/2\}, \varepsilon), \\
\frac{12r^2}{\varepsilon^2} \leq \frac{12\sqrt{t}}{\varepsilon^2} & \text{for } t \in (0, \varepsilon^2) \text{ and } r \in (\varepsilon, +\infty).
\end{cases}
\]

Taking \( K_8 \geq 6\varepsilon^{-2}(M_7 + 2) \) and \( t_{4,1} = 3/2K_8(<\varepsilon^2) \), we get

\[ L_v[\underline{\rho}_k] \leq 0 \] for \( (t, r) \in (0, t_{4,1}) \times (\varepsilon/2, +\infty) \),

\[ v_{k,2}(t, 2\varepsilon) > v_{k,1}(t, 2\varepsilon) \] for all \( t \in (0, t_{4,1}) \).

On the other hand, we see that

\[ L_v[\underline{\rho}_{k,3}] \leq L_v[\underline{\rho}_k] + t^{3/2} \left( \frac{2K_8M_{10}}{\varepsilon^2} - \frac{5}{2}K_9 \right) \] in \( (0, \varepsilon^2) \times (5\varepsilon^2/2, +\infty) \),

\[ \underline{\rho}_{k,3}(0, r) = \rho_k(0, r) \] for \( r \geq 2\varepsilon \),

\[ \underline{\rho}_{k,3}(t, 3\varepsilon) < \rho_k(t, 3\varepsilon) - K_9 t^{3/2} = \underline{\rho}_{k,2}(t, 3\varepsilon) \] for \( t \in (0, \varepsilon^2) \),

\[ \underline{\rho}_{k,3}(t, 4\varepsilon) = \underline{\rho}_{k,2}(t, 4\varepsilon) + t^{3/2}(K_8 - K_9) \] for \( t \in (0, \varepsilon^2) \).

Since \( L_v[\underline{\rho}_k] \leq t^{3/2}/\varepsilon^4 \) for \( (t, r) \in (0, \varepsilon^2) \times (5\varepsilon^2/2, +\infty) \) by Lemma 4.3, if we choose \( K_9 \geq \varepsilon^{-2}(K_8M_{10} + \varepsilon^{-2}) \) and \( t_{4,2} := K_8/(K_9 + 1)(<\varepsilon^4) \), we get

\[ L_v[\underline{\rho}_{k,3}] \leq 0 \] for \( (t, r) \in (0, t_{4,2}) \times (5\varepsilon^2/2, +\infty) \),

\[ \underline{\rho}_{k,3}(t, 4\varepsilon) > \underline{\rho}_{k,2}(t, 4\varepsilon) \] for all \( t \in (0, t_{4,2}) \).

Hence, setting \( t_4 := \min\{t_{4,1}, t_{4,2}\}(<\varepsilon^4) \), we complete the proof. \( \square \)

Consequently, we define for \( h \in (0, t_4) \)

\[ w_k := w_k(t, r) := \begin{cases} 
v_{k,1}(t, r) & \text{for } (t, r) \in [0, h) \times [0, \varepsilon/2), \\
\max\{v_{k,1}(t, r), v_{k,2}(t, r)\} & \text{for } (t, r) \in [0, h) \times [\varepsilon/2, 2\varepsilon), \\
v_{k,2}(t, r) & \text{for } (t, r) \in [0, h) \times [2\varepsilon, 5\varepsilon/2), \\
\max\{v_{k,2}(t, r), v_{k,3}(t, r)\} & \text{for } (t, r) \in [0, h) \times [5\varepsilon/2, 4\varepsilon), \\
v_{k,3}(t, r) & \text{for } (t, r) \in [0, h) \times [4\varepsilon, +\infty). 
\end{cases}
\]

Applying the theory of viscosity solutions we have the following proposition.

**Proposition 4.5.** For each \( \varepsilon \in (0, 1/100) \), let \( t_4 > 0 \) be given as in Lemma 4.4. For \( h \in (0, t_4) \) and \( k = 0, 1, 2, \ldots, [(T_0 - \varepsilon)/h] - 1 \), \( w_k \) is a viscosity subsolution of (33) in \( (0, h) \times (0, +\infty) \) satisfying (34) and (35).

We construct a supersolution of (33). Let \( \eta_i \) (\( i = 1, 2 \)) and \( K_j \) (\( j = 8, 9 \)) be the same as above. Define

\[ \overline{v}_{k,1} = \overline{v}_{k,1}(t, r) := \rho_k(t, r) + 2\sqrt{t}, \]

\[ \overline{v}_{k,2} = \overline{v}_{k,2}(t, r) := \rho_k(t, r) + 2\sqrt{M_1(r)} + K_8 t^{3/2}, \]

\[ \overline{v}_{k,3} = \overline{v}_{k,3}(t, r) := \rho_k(t, r) + 2K_8 t^{3/2} \eta_2(r) + K_9 t^{5/2}. \]
For $h \in (0, t_4)$, set
\[
\bar{w}_k = \bar{w}_k(t, r) := \begin{cases}
\varpi_k,1(t, r) & \text{for } (t, r) \in [0, h] \times [0, \varepsilon/2], \\
\min\{\varpi_k,1(t, r), \varpi_k,2(t, r)\} & \text{for } (t, r) \in [0, h] \times [\varepsilon/2, 2\varepsilon], \\
\varpi_k,2(t, r) & \text{for } (t, r) \in [0, h] \times [2\varepsilon, 5\varepsilon/2], \\
\min\{\varpi_k,2(t, r), \varpi_k,3(t, r)\} & \text{for } (t, r) \in [0, h] \times [5\varepsilon/2, 4\varepsilon], \\
\varpi_k,3(t, r) & \text{for } (t, r) \in [0, h] \times [4\varepsilon, +\infty].
\end{cases}
\]

Then we observe that $\bar{w}_k$ is a viscosity supersolution of (33) satisfying (34) and (35) for all $k = 0, 1, 2, \ldots, [(T_0 - \varepsilon)/h] - 1$ and $h \in (0, t_4)$. See Figure 2 for the outline of the graphs of $w_k, \bar{w}_k$ and $\bar{w}_k$.

**Remark 4.6.** The point of the constructions of $w_k$ and $\bar{w}_k$ is the modification of $\rho_k$ near $r = 0$. Then it is necessary that $R(t) > 4\varepsilon$ because if otherwise, we cannot estimate $R(t) - R^h(t)$ precisely. The choice of $\varepsilon \in (0, 1/100)$ and the estimate (36) assure this fact. In the following arguments we always assume $\varepsilon \in (0, 1/100)$.

Since it is clear that
\[
|w_k(t, r)|, |\bar{w}_k(t, r)|, |\bar{w}_k(t, r)| \leq M_{11}(1 + r) \quad \text{for all } (t, x) \in [0, h] \times \mathbb{R},
\]
\[
w_{k,r}(t, 0) = w_{k,r}(t, 0) = \varpi_{k,r}(t, 0) = 0 \quad \text{for all } t \in (0, h],
\]
\[
w_k(0, r) = w_k(0, r) = \bar{w}_k(0, r) \quad \text{for } r \in [0, +\infty),
\]
applying the comparison principle for viscosity solutions and the continuity of these functions, we have $w_k \leq \varpi_k \leq \bar{w}_k$ on $[0, h] \times [0, +\infty)$. In particular, we have
\[
\left|w_k(t, r) - \left\{\rho_k(t, r) - \frac{t^2}{(r^2 + t)^{3/2}}\right\}\right| \leq M_{12}h^{5/2} \quad \text{for all } (t, r) \in [0, h] \times [4\varepsilon, +\infty).
\]

(41)

**Figure 2.** Graphs of $w^k, \bar{w}^k$ and $\bar{w}^k$ (thick curves)

This gives the second order approximation of $w_k$ with respect to small $t \in [0, h]$ and the $M_{12}$ is uniform for $k = 0, 1, \ldots, [(T_0 - \varepsilon)/h] - 1$ and $h > 0$. In addition we observe that for $h \in (0, t_4), k = 0, 1, \ldots, [(T_0 - \varepsilon)/h] - 1$ and $t \in [0, h),$
\[
C^h(t + kh) \subset C^h(kh + t) \subset \overline{C^h}(t + kh),
\]
where
\[ C^h(t + kh) := \{w_k(t, | \cdot |) \geq 0\}, \quad \overline{C}^h(t + kh) := \{\overline{w}_k(t, | \cdot |) \geq 0\}, \]
\[ C^h(t + kh) := \{w_k(t, | \cdot |) \geq 0\}. \]

We estimate the difference \( R(t + kh) - R^h(t + kh) \) for \( t \in [0, h], k = 0, 1, \ldots, [(T_0 - \varepsilon)/h] - 1 \) and \( h \in (0, t_4) \). Set
\[ R_k(t) := R(t + kh), \quad R_k^h(t) := R^h(t + kh), \quad R_k := R(kh), \quad R_k^h := R^h(kh), \]
and define \( \alpha_k = \alpha_k(t) := \langle R_k^h - \sqrt{(R_k^h(t))^2 + 2t} \rangle / t^2 \). Since \( R_k(t) = \sqrt{R_k^2 - 2t} \), we see that
\[ R_k(t) - R_k^h(t) = \sqrt{(R_k^h)^2 - 2t} - \sqrt{(R_k^h)^2 - 2t + \sqrt{(R_k^h)^2 - 2t} - R_k^h(t)} \]
\[ = \frac{(R_k^h)^2 - (R_k^h)^2}{\sqrt{(R_k^h)^2 - 2t} + \sqrt{(R_k^h)^2 - 2t} - \sqrt{(R_k^h)^2 - 2t} + R_k^h(t)} \]
\[ =: I_{k,1} + I_{k,2} \]
Note that the first term of the right-hand side of this formula is nonnegative because of Proposition 4.1.

We estimate \( \alpha_k \). From (41) and \( w_k(t, R_k^h(t)) = 0 \) we see that
\[ \left| R_k^h - \sqrt{(R_k^h(t))^2 + 2t} - \frac{t^2}{(R_k^h(t))^2 + 2t} \right| \leq M_{12} \sqrt{t}. \]
Thus
\[ \left| \alpha_k - \frac{1}{(R_k^h - \alpha_k t^2)^3} \right| \leq M_{12} \sqrt{t}. \quad (43) \]
If \( \alpha_k t^2 \ll \sqrt{\varepsilon} \leq R_k^h(kh) = R_k^h, \) then we formally get \( |\alpha_k - (R_k^h)^{-3}| \approx M_{12} \sqrt{t}\). Indeed, we have the following estimate.

Lemma 4.7. Let \( t_4 > 0 \) be given as in Lemma 4.4. There are \( K_{10} > 0 \) and \( t_5 \in (0, t_4) \) such that for \( h \in (0, t_5) \) and \( k = 0, 1, \ldots, [(T_0 - \varepsilon)/h] - 1 \),
\[ \frac{1}{(R_k^h)^3} - K_{10} \sqrt{t} \leq \alpha_k(t) \leq \frac{1}{(R_k^h)^3} + K_{10} (t + \sqrt{t}) \quad \text{for all } t \in [0, h]. \]

Proof. The (43) turns to
\[ (R_k^h - \alpha_k t^2)^3(\alpha_k - M_{12} \sqrt{t}) \leq 1, \quad (R_k^h - \alpha_k t^2)^3(\alpha_k + M_{12} \sqrt{t}) \geq 1. \]
Define \( f_1(\alpha) := (R_k^h - \alpha_k t^2)^3(\alpha_k - M_{12} \sqrt{t}) \) and \( f_2(\alpha) := (R_k^h - \alpha_k t^2)^3(\alpha_k + M_{12} \sqrt{t}) \). Besides taking (36) into account we take \( t_5 \in (0, t_4) \) such that \( \sqrt{\varepsilon}/2 \geq M_{12} \sqrt{t} \) for each \( t \in (0, t_5) \). We then observe that \( f_1 \) and \( f_2 \) satisfy
\[ f_1 \left( \frac{1}{(R_k^h)^3} - M_{12} \sqrt{t} \right) \leq 1, \quad f_2 \left( \frac{1}{(R_k^h)^3} - M_{12} \sqrt{t} \right) \leq 1, \]
\[ f_1 \left( \frac{1 + 2t^2/(R_k^h)^3}{(R_k^h)^3} \right) + M_{12} \sqrt{t} \geq 1, \quad f_2 \left( \frac{1 + 2t^2/(R_k^h)^3}{(R_k^h)^3} \right) + M_{12} \sqrt{t} \geq 1. \]
Since we have chosen \( t_4 < \varepsilon^4 \) at the end of the proof of Lemma 4.4, it is obvious that \( t_5 < \varepsilon^4 \). Thus from (36) and this fact, we have for all \( t \in (0, t_5) \)
\[ 0 \leq \frac{1}{(R_k^h)^3} - M_{12} \sqrt{t} \leq \varepsilon^{-3/2}, \quad 0 \leq \frac{(1 + 2t^2/(R_k^h)^3)}{(R_k^h)^3} + M_{12} \sqrt{t} \leq 3 \varepsilon^{-3/2}. \]
Furthermore, we are able to verify that \( f_1 \) and \( f_2 \) are increasing for all \( \alpha \in [0, 3e^{-3/2}] \) and \( t \in (0, t_3) \), we get for each \( h \in (0, t_3) \) and \( t \in [0, h] \),

\[
\frac{1}{(R_h^k)^3} - M_{12}\sqrt{t} \leq \alpha_k(t) \leq \frac{(1 + 2t^2/(R_h^k)^4)^3}{(R_h^k)^3} + M_{12}\sqrt{t}.
\]

Thus the proof is completed by taking \( K_{10} := M_{12} \).

We estimate \( I_{k,1} \) and \( I_{k,2} \) of (42).

**Lemma 4.8.** Let \( t_5 > 0 \) be the same constant as in Lemma 4.7. There exists \( K_{11} > 0 \) such that for any \( h \in (0, t_5) \), \( k = 0, 1, \ldots, [T_0 - \varepsilon]/h - 1 \) and \( t \in [0, h] \),

\[
(R_h^k - R_h^k) \left( 1 + \frac{t}{R_h^k} \right) \leq I_{k,1} \leq (R_h^k - R_h^k) \left( 1 + \frac{t}{R_h^k} + K_{11}h^2 \right),
\]

\[
\alpha_k t^2 \left( 1 + \frac{t}{R_h^k} \right) \leq I_{k,2} \leq \alpha_k t^2 \left( 1 + \frac{t}{R_h^k} + K_{11}h^2 \right).
\]

**Proof.** By use of (26) we get for small \( h > 0 \) and all \( t \in [0, h] \),

\[
r - \frac{t}{r} - \frac{t^2}{r^3} \leq \sqrt{r^2 - 2t} \leq r - \frac{t}{r} \quad \text{for } r = R_k, R_h^k.
\]

Thus there exists \( M_{13} > 0 \) such that for all \( t \in [0, h] \)

\[
(R_h^k + R_h^k) \left( 1 - \frac{t}{R_h^k R_h^k} - M_{13}t^2 \right) \leq \sqrt{R_h^k^2 - 2t} + \sqrt{(R_h^k)^2 - 2t} \leq (R_h^k + R_h^k) \left( 1 - \frac{t}{R_h^k R_h^k} \right).
\]

Similarly, we use \( R_h^k(t) = \sqrt{(R_h^k - \alpha_k t^2)^2 - 2t} \) to have

\[
(2R_h^k - \alpha_k t^2) \left( 1 - \frac{t}{R_h^k (R_h^k - \alpha_k t^2)} - M_{14}t^2 \right) \leq \sqrt{(R_h^k)^2 - 2t} + R_h^k(t) \leq (2R_h^k - \alpha_k t^2) \left( 1 - \frac{t}{R_h^k (R_h^k - \alpha_k t^2)} \right),
\]

for some \( M_{14} > 0 \). Combining the above two inequalities with (42), we obtain

\[
(R_h^k - R_h^k) \left( 1 - \frac{t}{R_h^k R_h^k} \right)^{-1} \leq I_{k,1} \leq (R_h^k - R_h^k) \left( 1 - \frac{t}{R_h^k R_h^k} - M_{13}t^2 \right)^{-1},
\]

\[
\alpha_k t^2 \left( 1 - \frac{t}{R_h^k (R_h^k - \alpha_k t^2)} \right)^{-1} \leq I_{k,2} \leq \alpha_k t^2 \left( 1 - \frac{t}{R_h^k (R_h^k - \alpha_k t^2)} - M_{14}t^2 \right)^{-1},
\]

for all \( t \in [0, h] \). We use (36) to obtain

\[
\left( 1 - \frac{t}{R_h^k R_h^k} \right)^{-1} \geq 1 + \frac{t}{R_h^k R_h^k} \geq 1 + \frac{t}{R_h^k},
\]

\[
\left( 1 - \frac{t}{R_h^k (R_h^k - \alpha_k t^2)} \right)^{-1} \geq 1 + \frac{t}{R_h^k (R_h^k - \alpha_k t^2)} \geq 1 + \frac{t}{R_h^k}.
\]
Proof. Let \( \alpha_k \geq K_{11} \geq L_0/(\sqrt{2\epsilon})^3 + \max\{M_{15} + M_{16}, M_{17}\} \) sufficiently large, we obtain the result. \( \square \)

**Proposition 4.9.** Let \( t_5 > 0 \) be given as in Lemma 4.7. For any \( h \in (0, t_5), \) \( k = 0, 1, \ldots, \lfloor (T_0 - \epsilon)/h \rfloor - 1 \) and \( t \in [0, h], \) we have

\[
I_{k, 4} \leq R_h(t) - R^h_0(t) \leq I_{k, 3},
\]

\[
I_{k, 3} := \sum_{l=0}^{k-1} \alpha_l(h) t^{2l} \prod_{i=1}^{k-1} \left(1 + \frac{h}{R^l_i} + K_{11} h^2\right) \cdot \left(1 + \frac{t}{R^l_i} + K_{11} h^2\right)
\]

\[
+ \alpha_k(t) t^2 \left(1 + \frac{t}{R^l_k} + K_{11} h^2\right),
\]

\[
I_{k, 4} := \sum_{l=0}^{k-1} \alpha_l(h) t^{2l} \prod_{i=1}^{k-1} \left(1 + \frac{h}{R^l_i} + K_{11} h^2\right) \cdot \left(1 + \frac{t}{R^l_i} + K_{11} h^2\right) + \alpha_k(t) t^2 \left(1 + \frac{t}{R^l_k} + K_{11} h^2\right).
\]

Proof. Note that \( R_k = R_k(0) = R(kh) \) and \( R^h_k = R^h_k(0) = R^h(kh) \) for \( k = 0, 1, \ldots, \lfloor (T_0 - \epsilon)/h \rfloor - 1, \) and \( h > 0. \) For \( k = 0, \) since \( I_{0, 1} = 0, \) we have

\[
\alpha_0(t) t^2 \left(1 + \frac{t}{R^l_0} + K_{11} h^2\right) \leq R_0(t) - R^h_0(t) \leq \alpha_0 t^2 \left(1 + \frac{t}{R^l_0} + K_{11} h^2\right). \tag{44}
\]

In the case \( k = 1, \) combining (42), Lemma 4.8 with \( k = 1 \) and this estimate, we get

\[
\alpha_0(h) t^2 \left(1 + \frac{h}{R^l_0} + K_{11} h^2\right) \left(1 + \frac{t}{R^l_1} + K_{11} h^2\right) + \alpha_1(t) t^2 \left(1 + \frac{t}{R^l_1} + K_{11} h^2\right) \leq R_1(t) - R^h_1(t)
\]

\[
\leq \alpha_0 h^2 \left(1 + \frac{h}{R^l_0} + K_{11} h^2\right) \left(1 + \frac{t}{R^l_1} + K_{11} h^2\right) + \alpha_1 t^2 \left(1 + \frac{t}{R^l_1} + K_{11} h^2\right).
\]

The desired estimate can be obtained by induction. \( \square \)

We are now in a position to give

the proof of Theorem 4.2. As \( R_0 = R^h_0 = 0, \) we see by (42) and (44) that for all \( t \in [0, h], \)

\[
|R^h(t) - \{R(t) - t^2\}| \leq K_{12} t^{5/2}.
\]

For \( t \in [h, T_0 - \epsilon], \) we estimate \( I_{k, 3} \) and \( I_{k, 4} \) to have the desired result. Recall \( R_i = R(ih) = \sqrt{1 - 2ih} \) for \( i = 0, 1, \ldots, \lfloor (T_0 - \epsilon)/h \rfloor - 1. \) Fix \( k \geq 1 \) and \( 0 \leq t \leq k. \)
**Step 1.** We estimate $I_{k,3}$.

Set $h \in (0,t_5)$. We see by the fact that $\log(1 + s) \leq s$ for $s > -1$ and the convexity of the function $(1 - 2s)^{-1}$ for $s \in [0,1/2]$ that

$$
\log \left( \prod_{i=l}^{k} \left( 1 + \frac{h}{R_i^2} + K_{11} h^2 \right) \cdot \left( 1 + \frac{t}{R_k^2} + K_{11} h^2 \right) \right)
$$

$$
= \sum_{i=l}^{k} \log \left( 1 + \frac{h}{R_i^2} + K_{11} h^2 \right) + \log \left( 1 + \frac{t}{R_k^2} + K_{11} h^2 \right)
$$

$$
\leq \sum_{i=l}^{k-1} \left( \frac{h}{R_i^2} + K_{11} h^2 \right) + \frac{t}{R_k^2} + K_{11} h^2
$$

$$
\leq \int_{t}^{k h + t} \frac{1}{1 - 2s} ds + K_{11} (k - l + 1) h^2
$$

$$
= \log \frac{R_l}{R_k(t)} + K_{11} (k - l + 1) h^2.
$$

Similarly

$$
\log \left( 1 + \frac{t}{(R_k)^2} + K_{11} h^2 \right) \leq \log \frac{R_k}{R_k(t)} + K_{11} h^2.
$$

Hence we have from Taylor’s theorem, Lemma 4.7 and the fact $kh \leq 1/2$

$$
I_{k,3} \leq \sum_{l=0}^{k-1} \alpha_l(h) h^2 \exp \left( \log \frac{R_l}{R_k(t)} + K_{11} (k - l + 1) h^2 \right)
$$

$$
+ \alpha_k(t) t^2 \exp \left( \log \frac{R_k}{R_k(t)} + K_{11} h^2 \right)
$$

$$
\leq \left\{ \sum_{l=0}^{k-1} \frac{\alpha_l(h) h^2 R_l}{R_k(t)} + \frac{\alpha_k(t) t^2 R_k}{R_k(t)} \right\} (1 + K_{11} h)
$$

$$
\leq \left[ \frac{h^2}{R_k(t)} \sum_{l=0}^{k-1} \left\{ \frac{1}{(R_l)^2} + K_{10} \sqrt{h} \right\} + \frac{t^2}{R_k(t)} \left\{ \frac{1}{(R_k)^2} + K_{10} \sqrt{h} \right\} \right] \cdot (1 + K_{11} h)
$$

$$
\leq \frac{h}{R_k(t)} \left\{ \int_{0}^{kh + t} \frac{1}{(R(s))^2} ds + \frac{K_{10} \sqrt{h}}{2} \right\} (1 + K_{11} h)
$$

$$
\leq -h \log \frac{R_k(t)}{R_k(t)} + \frac{M_{18} h^{3/2}}{R_k(t)}
$$

for any $h \in [0,t_6]$ and some small $t_6 \in (0,t_5)$. Here we have used the inequality $e^s \leq 1 + 2s$ for small $s > 0$.

**Step 2.** We estimate $I_{k,4}$.

We use $\log(1 + s) \geq s - s^2/2$ for $s \geq 0$ to get

$$
\log \left[ \prod_{i=1}^{k-1} \left\{ 1 + \frac{h}{(R_i)^2} \right\} \cdot \left\{ 1 + \frac{t}{(R_k)^2} \right\} \right]
$$

$$
= \sum_{i=l}^{k-1} \log \left\{ 1 + \frac{h}{(R_i)^2} \right\} + \log \left\{ 1 + \frac{t}{(R_k)^2} \right\}
$$
\[ \geq \sum_{i=1}^{k-1} \left\{ \frac{h}{(R_i)^2} - \frac{h^2}{2(R_i)^4} \right\} + \frac{t}{(R_k)^2} - \frac{t^2}{2(R_k)^4} = \ast. \]

Since it is observed by the first estimate of (36) and the convexity of \(1/(1 - 2s)\) for \(s \in [0, 1/2]\) that for \(i = 0, 1, \ldots, [(T_0 - \varepsilon) / h] - 1\)
\[ h \frac{1}{(R_i)^2} \geq \int_{t_{i+1}}^{t} \frac{1}{(R(s))^2} ds - \frac{h^2}{2\varepsilon^2}, \quad (45) \]
we get by \(kh \leq 1/2\)
\[ \ast \geq \int_{t_{i+1}}^{t} \frac{1}{(R(s))^2} ds - \frac{(k - l + 1)h^2}{\varepsilon^2} - \frac{(k - l + 1)h^2}{2\varepsilon^2} \geq \log \frac{R_t}{R_k(t)} - \frac{3h}{4\varepsilon^2}. \]

Consequently we get
\[ I_{k,4} \geq \sum_{l=0}^{k-1} \alpha_l(h)h^2 \exp \left( \log \frac{R_t}{R_k(t)} - \frac{3h}{4\varepsilon^2} \right) + \alpha_k(t)h^2 \exp \left( \log \frac{R_k}{R_k(t)} - \frac{3h}{4\varepsilon^2} \right) \]
\[ \geq \left( \sum_{l=0}^{k-1} \alpha_l(h)h^2 \frac{R_t}{R_k(t)} + \alpha_k(t)h^2 \frac{R_k}{R_k(t)} \right) \left( 1 - \frac{3h}{4\varepsilon^2} \right) \]
\[ \geq \left\{ \frac{h^2}{R_k(t)} \sum_{l=0}^{k-1} \left( \frac{1}{(R_i)^2} - K_{10} \sqrt{h} \right) + \frac{t^2}{R_k(t)} \left( \frac{1}{(R_k)^4} - K_{10} \varepsilon \right) \right\} \]
\[ \cdot \left( 1 - \frac{3h}{4\varepsilon^2} \right) \]
\[ = \ast \ast. \]

We use (45) again to have
\[ \ast \ast \geq \frac{h}{R_k(t)} \left\{ \int_{t_0}^{t} \frac{1}{(R(s))^2} ds \left( 1 - \frac{3h}{4\varepsilon^2} \right) - \frac{K_{10}}{2} \varepsilon \right\} \]
\[ \geq \frac{-h \log R_k(t)}{R_k(t)} - \frac{M_{19}h^{3/2}}{R_k(t)}. \]

Letting \( L_1 := \max\{K_{12}, M_{18}, M_{19}\} \) and \( h_1 := t_6 \), we have obtained the desired result. \( \square \)

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