LAGUERRE CHARACTERIZATION OF SOME HYPERSURFACES

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Abstract. Let $x : M^{n-1} \to \mathbb{R}^n$ $(n \geq 4)$ be an umbilical free hypersurface with non-zero principal curvatures. Then $x$ is associated with a Laguerre metric $g$, a Laguerre tensor $L$, a Laguerre form $C$, and a Laguerre second fundamental form $B$, which are invariants of $x$ under Laguerre transformation group. We denote the Laguerre scalar curvature by $R$ and the trace-free Laguerre tensor by $\tilde{L} := L - \frac{1}{n}tr(L)g$. In this paper, we prove a local classification result under the assumption of parallel Laguerre form and an inequality of the type 

$$\|\tilde{L}\| \leq cR,$$

where $c = \frac{1}{(n-1)\sqrt{(n-2)(n-1)}}$ is appropriate real constant, depending on the dimension.

1. Introduction

Let $x : M^{n-1} \to \mathbb{R}^n$ be an umbilical free hypersurface with non-zero principal curvatures. Let $\xi : M \to S^{n-1}$ be its unit normal. Let $\{e_1, e_2, \ldots, e_{n-1}\}$ be the orthonormal basis for $TM$ with respect to $dx \cdot dx$, consisting of unit principal vectors. Let $r_i = \frac{1}{k_i}$, $r = \frac{r_1 + r_2 + \cdots + r_{n-1}}{n-1}$ be the curvature radius and mean curvature radius of $x$ respectively, where $k_i \neq 0$ is the principal curvature corresponding to $e_i$. We define $\rho = \sqrt{\sum_i (r_i - r)^2}$, $\bar{E}_i = r_i e_i$, $1 \leq i \leq n-1$. Then $g = \rho^2 d\xi \cdot d\xi$ is a Laguerre invariant metric, $\{\bar{E}_1, \bar{E}_2, \ldots, \bar{E}_{n-1}\}$ is an orthonormal basis for $III = d\xi \cdot d\xi$. The normalized scalar curvature of Laguerre metric $g$ will be denoted by $R$ and is called the normalized Laguerre scalar curvature. Two basic Laguerre invariants of $x$, the Laguerre form $C = \sum_i C_i \omega_i$ and the Laguerre tensor $L = \sum_{ij} L_{ij} \omega_i \otimes \omega_j$, are defined by

$$C_i = -\rho^{-2} \left( \bar{E}_i(r) - \bar{E}_i(\log \rho)(r_i - r) \right),$$

Received May 22, 2015.
2010 Mathematics Subject Classification. Primary 53A40, 53B25.
Key words and phrases. Laguerre geometry, hypersurfaces.

Supported by Scientific Research Fund of Yunnan Provincial Education Department (Grant No. 2014Y445 and 2015Y101).

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(2) \[ L_{ij} = \rho^{-2} \left( \text{Hess}_{ij}(\log \rho) - \bar{E}_i(\log \rho)\bar{E}_j(\log \rho) + \frac{1}{2} \left( \|\nabla \log \rho\|^2 - 1 \right) \delta_{ij} \right), \]

where \((\text{Hess}_{ij})\) and \(\nabla\) are the Hessian matrix and the gradient operator with respect to the third fundamental form \(III = d\xi \cdot d\xi\).

Laguerre geometry of surfaces in \(\mathbb{R}^3\) has been developed by Blaschke and his school (see [1]). Recently, there has been some renewed interest for the surface of \(\mathbb{R}^3\) in Laguerre geometry (see [2, 3, 4, 9]).

In [7], Li and Wang studied Laguerre differential geometry of oriented hypersurfaces in \(\mathbb{R}^n\). For any umbilical-free hypersurface \(x: M \rightarrow \mathbb{R}^n\) with non-zero principal curvatures, Li and Wang defined a Laguerre invariant metric \(g\), a Laguerre second fundamental form \(B\), a Laguerre form \(C\) and a Laguerre tensor \(L\) on \(M\), and showed that \(\{g, B\}\) is a complete Laguerre invariant system for hypersurfaces in \(\mathbb{R}^n\) with \(n \geq 4\). In the case \(n = 3\), a complete Laguerre invariant system for surfaces in \(\mathbb{R}^3\) is given by \(\{g, B, L\}\).

In [8], authors classified hypersurfaces with parallel Laguerre second fundamental form. Laguerre tensor is a codazzi tensor, which is another Laguerre invariant. An eigenvalue of Laguerre tensor \(L\) of \(x\) is called a Laguerre eigenvalue of \(x\). If Laguerre eigenvalues of \(x\) are equal, i.e., \(L = \sum_{i,j} \lambda_i \omega_i \otimes \omega_j\), and Laguerre form is vanishing, then \(x\) is called Laguerre isotropic hypersurface.

We define the trace-free Laguerre tensor \(\tilde{L} := L - \frac{1}{n-1} \text{tr}(L)g\). Authors classified hypersurfaces with vanishing Laguerre form \(C\) and vanishing trace-free Laguerre tensor \(\tilde{L}\) in [6].

In this paper, we prove the following local result:

**Theorem 1.1.** Let \(x: M^{n-1} \rightarrow \mathbb{R}^n\) \((n \geq 4)\) be an umbilical free hypersurface with non-zero principal curvatures. If its Laguerre form \(C\) is parallel and \(\|\tilde{L}\| \leq R (n-3)\sqrt{(n-2)(n-1)}\),

then \(R\) is constant, we have equality \(\|\tilde{L}\| = \frac{R}{(n-3)\sqrt{(n-2)(n-1)}}\) and \(M^{n-1}\) is Laguerre equivalent to an open subset of one of the following hypersurfaces in \(\mathbb{R}^n\):

(i) the images of \(\tau\) of the hypersurface \(\tilde{x}\) in \(\mathbb{R}^n_0\) with mean curvature radius \(r = 0\) and \(\rho = \text{constant}\), where for the definition of \(\tau\), please refer to [6].

(ii) the hypersurface \(\tilde{x}: H^1 \times S^{n-2} \rightarrow \mathbb{R}^n\) by

\[ \tilde{x}(w, v, u) = \sqrt{\frac{(n-1)(n-3)}{R}} \left( \frac{v}{u}, \frac{u}{w}(1 + w) \right), \]

where \(u: S^{n-2} \rightarrow \mathbb{R}^{n-1}\) and \((w, v): H^1 \rightarrow \mathbb{R}_1^2\) are the canonical embeddings.

We organize the paper as follows. In Section 2 we give Laguerre invariants for hypersurfaces in \(\mathbb{R}^n\). In Section 3, we make calculations for the example
2. Laguerre geometry of hypersurfaces in $\mathbb{R}^n$

In this section we review the Laguerre invariants and structure equations for hypersurfaces in $\mathbb{R}^n$. For the detail we refer to [7].

Let $\mathbb{R}_2^{n+3}$ be the space $\mathbb{R}^{n+3}$ equipped with the inner product

$$\langle X, Y \rangle = -x_1y_1 + x_2y_2 + \cdots + x_{n+2}y_{n+2} - x_{n+3}y_{n+3}.$$ 

Let $C_{n+2}$ be the light-cone in $\mathbb{R}_2^{n+3}$ given by $C_{n+2} = \{ X \in \mathbb{R}_2^{n+3} | \langle X, X \rangle = 0 \}$.

Let $LG$ be the subgroup of orthogonal group $O(n+1,2)$ on $\mathbb{R}_2^{n+3}$ given by

$$LG = \{ T \in O(n+1,2) \mid \varsigma T = \varsigma \},$$

where $\varsigma = (1, -1, \vec{0}, 0)$, where $\vec{0} \in \mathbb{R}^n$, is a light-like vector in $\mathbb{R}_2^{n+3}$.

Let $x : M \to \mathbb{R}^n$ be an umbilical free hypersurface with non-zero principal curvatures. Let $\xi : M \to S^{n-1}$ be its unit normal. Let $\{ e_1, e_2, \ldots, e_{n-1} \}$ be the orthonormal basis for $TM$ with respect to $dx \cdot dx$, consisting of unit principal vectors. We write the structure equations of $x : M \to \mathbb{R}^n$ by

$$e_j(e_i(x)) = \sum_k \Gamma^k_{ij}e_k(x) + k_i \delta_{ij} \xi; e_i(\xi) = -k_i e_i(x), \quad 1 \leq i, j, k \leq n-1,$$

where $k_i \neq 0$ is the principal curvature corresponding to $e_i$. Let

$$r_i = \frac{1}{k_i}, \quad r = \frac{r_1 + r_2 + \cdots + r_{n-1}}{n-1},$$

be the curvature radius and mean curvature radius of $x$, respectively. We define Laguerre position vector of $x$ by

$$Y = \rho(x \cdot \xi, -x \cdot \xi, \xi, 1) : M \to C_{n+2} \subset \mathbb{R}_2^{n+3},$$

where $\rho = \sqrt{\sum (r_i - r)^2} > 0$.

**Theorem 2.1.** Let $x, \tilde{x} : M \to \mathbb{R}^n$ be two umbilical oriented hypersurfaces with non-zero principal curvatures. Then $x$ and $\tilde{x}$ are Laguerre equivalent if and only if there exists $T \in LG$ such that $\tilde{Y} = YT$.

From the theorem we know that

$$g = \langle dY, dY \rangle = \rho^2 \delta \xi \cdot d\xi = \rho^2 III$$

is a Laguerre invariant metric, where $III$ is the third fundamental form of $x$, we call $g$ the Laguerre metric of $x$. Let $\Delta$ be the Laplacian operator of $g$, then we have

$$N = \frac{1}{n-1} \Delta Y + \frac{1}{2(n-1)^2} \langle \Delta Y, \Delta Y \rangle Y,$$

and

$$\eta = \left( \frac{1}{2} (1 + |x|^2) \right) \frac{1}{2} (1 - |x|^2, x, 0) + r(x \cdot \xi, -x \cdot \xi, \xi, 1)$$
From (3) we get
\[\langle Y, Y \rangle = \langle N, N \rangle = 0, \langle N, \eta \rangle = 0, \langle \eta, \varsigma \rangle = -1.\]

Let \(\{E_1, E_2, \ldots, E_{n-1}\}\) be an orthonormal basis for \(\mathfrak{g} = \langle dY, dY \rangle\) with dual basis \(\{\omega_1, \omega_2, \ldots, \omega_{n-1}\}\) and write \(Y_i = E_i(Y), 1 \leq i \leq n-1\). Then we have the following orthogonal decomposition,
\[R^{n+3}_2 = \text{Span}\{Y, N\} \oplus \text{Span}\{Y_1, Y_2, \ldots, Y_{n-1}\} \oplus \text{Span}\{\eta, \varsigma\}.\]

We call \(\{Y, N, Y_1, \ldots, Y_{n-1}, \eta, \varsigma\}\) a Laguerre moving frame in \(\mathbb{R}^{n+3}_2\) of \(x\). By taking derivatives of this frame we obtain the following structure equations:

(4) \(E_i(N) = \sum_j L_{ij} Y_j + C_i \varsigma,\)

(5) \(E_j(Y_i) = L_{ij} Y + \delta_{ij} N + \sum_k \Gamma_{kij} Y_k + B_{ij} \varsigma,\)

(6) \(E_i(\eta) = -C_i Y + \sum_j B_{ij} Y_j.\)

From these equations we obtain the following basic Laguerre invariants:

(i) The Laguerre metric \(g = \langle dY, dY \rangle\);

(ii) The Laguerre second fundamental form \(B = \sum_{ij} B_{ij} \omega_i \otimes \omega_j;\)

(iii) The Laguerre tensor \(L = \sum_{ij} L_{ij} \omega_i \otimes \omega_j;\)

(iv) The Laguerre form \(C = \sum_i C_i \omega_i,\) where \(L_{ij} = L_{ji}, B_{ij} = B_{ji}.\)

By taking further derivatives of (4)-(6), we get the following relations between these invariants:

(7) \(L_{ij,k} = L_{ik,j};\)

(8) \(C_{i,j} - C_{j,i} = \sum_k (B_{ik} L_{kj} - B_{kj} L_{ki});\)

(9) \(B_{ij,k} - B_{ik,j} = C_j \delta_{ik} - C_k \delta_{ij};\)

(10) \(R_{ijkl} = L_{jk} \delta_{il} + L_{il} \delta_{jk} - L_{ik} \delta_{jl} - L_{jl} \delta_{ik},\)

where \(\{L_{ij,k}\}, \{C_{ij}\}\) and \(\{B_{ij,k}\}\) are covariant derivatives of the tensors \(\{L_{ij}, C_i, B_{ij}\}\) with respect to the Laguerre metric \(\mathfrak{g}\), respectively, and \(R_{ijkl}\) is the curvature tensor of \(\mathfrak{g}\). Moreover, we have the following identities (see [7]):

(11) \(\sum_{i,j} (B_{ij})^2 = 1, \sum_i B_{ii} = 0, \sum_{i,j,i} B_{ij,i} = (n - 2)C_j,\)

(12) \(\sum_i L_{ii} = -\frac{1}{2(n-1)} \langle \Delta Y, \Delta Y \rangle Y,\)

(13) \(R_{ik} = -(n-3)L_{ik} - \left(\sum_i L_{ii}\right) \delta_{ik},\)
(14) \[ R = -2(n-2) \sum_i L_{ii} = \frac{n-2}{(n-1)} \langle \Delta Y, \Delta Y \rangle Y \]

is the normalized scalar curvature.

In the case \( n \geq 4 \), we know from (11) and (14) that \( C_i \) and \( L_{ij} \) are completely determined by the Laguerre invariants \( \{g, B\} \), thus we get:

**Theorem 2.2.** Two umbilical free oriented hypersurfaces in \( \mathbb{R}^n \) \((n > 3)\) with non-zero principal curvatures are Laguerre equivalent if and only if they have the same Laguerre metric \( g \) and Laguerre second fundamental form \( B \).

In the case \( n = 3 \), a complete Laguerre invariant system for surfaces in \( \mathbb{R}^3 \) is given by \( \{g, B, L\} \).

We define \( \tilde{E}_i = r_i e_i, 1 \leq i \leq n-1 \). Then \( \{\tilde{E}_1, \tilde{E}_2, \ldots, \tilde{E}_{n-1}\} \) is an orthonormal basis for \( III = d\xi \cdot d\xi \). Then \( \{E_i = \rho^{-1} \tilde{E}_i \mid 1 \leq i \leq n-1\} \) is an orthonormal basis for the Laguerre metric \( g \). By direct calculations, we obtain the following local expressions:

(15) \[ g = \sum_i (r_i - r)^2 III = \rho^2 III, \ B_{ij} = \rho^{-1}(r - r_i)\delta_{ij}. \]

3. Typical examples

In this section, for the purpose of proving Theorem 1.1, we will consider a umbilic-free hypersurface \( M \) in \( \mathbb{R}^n \), and then calculate the Laguerre invariants for \( x : H^1 \times S^{n-2} \rightarrow \mathbb{R}^n \).

**Example 3.1.** We denote by \( H^1 = \{(w, v) \in \mathbb{R}_+^2 \mid -w^2 + v^2 = -1, w > 0\} \) the hyperbolic space embedded in the Minkowski space \( \mathbb{R}^2_1 \). We define \( x : H^1 \times S^{n-2} \rightarrow \mathbb{R}^n \) by

(16) \[ x(w, v, u) = \left( \frac{v}{w}, \frac{u}{w}, (1 + w) \right), \]

then \( x \) satisfies

(17) \[ \mathbf{C} \equiv 0, \ \nabla B = 0, \]

(18) \[ R = (n-1)(n-3) = \text{const}, \]

(19) \[ ||\mathbf{L}|| = \sqrt{\frac{n-1}{n-2}}. \]

In fact: clearly \( x \) is a hypersurface with the unit normal field \( \xi = (\frac{\hat{w}}{w}, \frac{\hat{u}}{w}) \), and the first and the second fundamental forms of \( x \) are given by

\[ I = dx \cdot dx = \frac{1}{w^2} \{ -dw \cdot dw + dv \cdot dv + (1+w)^2 du \cdot du \}, \]

\[ II = -dx \cdot d\xi = -\frac{1}{w^2} \{ -dw \cdot dw + dv \cdot dv + (1+w)du \cdot du \}, \]
respectively. Therefore $x$ has two principal curvature

(20) \[ k_1 = -1, \quad k_2 = \cdots = k_{n-1} = -\frac{1}{w+1}. \]

From (20) we see that

\[
\begin{align*}
r &= \frac{r_1 + r_2 + \cdots + r_{n-1}}{n-1} = -\frac{(n-2)w + (n-1)}{n-1}, \\
\rho^2 &= \sum_i (r_i - r)^2 = \frac{n-2}{n-1}w^2.
\end{align*}
\]

From (15) we get the Laguerre metric

\[ g = \frac{n-2}{n-1}(-dw^2 + dv \cdot dv + du \cdot du) \]

Therefore, $g = g_1 + g_2$, where $g_1, g_2$ have constant sectional curvature $\frac{n-1}{n-2}, -\frac{n-1}{n-2}$ respectively. And the Laguerre second fundamental form is given by using (15),

\[ B_{ij} = b_i \delta_{ij}, \]

\[ b_1 = -\sqrt{\frac{n-2}{n-1}}, \quad b_2 = \cdots = b_{n-1} = \sqrt{\frac{1}{(n-1)(n-2)}}. \]

From (1) we get $C_i = 0, 1 \leq i \leq n-1$, that is (17).

Let $L_{ij} = a_i \delta_{ij}$, from (10) we get

\[ a_1 = -\frac{n-1}{2(n-2)}, \quad a_2 = \cdots = a_{n-1} = -\frac{n-1}{2(n-2)}. \]

Thus we have

\[ tr L = \sum_{i=1}^{n-1} a_i = -\frac{(n-1)(n-3)}{2(n-2)} \]

and $\tilde{L}_{ij} = L_{ij} - \frac{tr L}{n-1} \delta_{ij} = \tilde{a}_i \delta_{ij}$ with

\[ \tilde{a}_1 = 1, \quad \tilde{a}_2 = \cdots = \tilde{a}_{n-1} = -\frac{1}{n-2}. \]

This gives

\[ ||L||^2 = \sum_{i=1}^{n-1} \tilde{a}_i^2 = \frac{n-1}{n-2}. \]

On the other hand, from (14), we have

\[ R = (n-1)(n-3). \]
4. The proof of the main theorem

We are going to calculate the Laplacian of the length of the Laguerre second fundamental form. By definition and (11) we have

\[ 0 = \frac{1}{2} \Delta \left( \sum (B_{ij})^2 \right) = \sum (B_{ij,k})^2 + \sum B_{ij}B_{ij,kk}. \]  

On the other hand, using (9) and Ricci identities, noting that the Laguerre form \( C \) is parallel, we obtain

\[ B_{ij,kk} = B_{kk,ij} + B_{lk}R_{lijk} + B_{il}R_{lkjk}. \]

Form (10), (11) and the above equation, we easily obtain

\[ B_{ij}B_{ij,kk} = -B_{ij}^2L_{kk} - (n-1)B_{ij}B_{il}L_{lj}. \]

Inserting (21) into (22), we get the following lemma:

**Lemma 4.1.** Let \( x : M^{n-1} \to \mathbb{R}^n \) be an umbilical free hypersurface with non-zero principal curvatures. If the Laguerre form \( C \) of \( x \) is parallel, then

\[ \|\nabla B\|^2 - (n-1)\text{tr}(B^2L) - \text{tr}(L) = 0. \]

We state the following lemma which is needed in the proof of main theorem.

**Lemma 4.2 (cf. [3]).** Let \( a_1, \ldots, a_{n-1} \) and \( b_1, \ldots, b_{n-1} \) be \( 2(n-1) \) real numbers satisfying \( \sum a_i = 0 \), \( \sum b_i = 0 \). Then

\[ \left| \sum a_i b_i^2 \right| \leq \frac{n-3}{\sqrt{(n-1)(n-2)}} \sqrt{\sum a_i^2 \sum b_i^2}. \]

Moreover, if \( \sum a_i^2 \neq 0 \) and \( \sum b_i^2 \neq 0 \), then equality holds if and only if there are \( (n-2) \) pairs of numbers \( (a_i, b_i) \) take the same value \( (a, b) \).

The proof of the main theorem. Define the free-trace tensor

\[ \tilde{L} := L - \frac{1}{n-1} \text{tr}(L)g. \]

Since the Laguerre form \( C \) of \( x \) is parallel, from (8) we have \( BL = LB \). Hence, we can choose \( \{E_i\} \) such that both, \( B \) and \( \tilde{L} \), are simultaneously diagonal, and therefore we can apply Lemma 4.2:

\[ \text{tr}(\tilde{L}B^2) \leq \frac{n-3}{\sqrt{(n-1)(n-2)}} \|\tilde{L}\|\|B\|^2. \]

Since the quantities on both side of (25) are invariant under orthogonal transformations, inequality (25) is independent of the choice of \( \{E_i\} \). From (23) and (25) we have

\[ 0 \geq \|\nabla B\|^2 - 2\text{tr}(L) - \frac{(n-1)(n-3)}{\sqrt{(n-1)(n-2)}} \|\tilde{L}\|\|B\|^2. \]
Putting (11) and (13) into (26), we have

\[(27) \quad 0 \geq \| \nabla B \|^2 + \frac{1}{n-2} \left( R - (n-3)\sqrt{(n-2)(n-1)} \| \tilde{L} \| \right) .\]

The assumption of the theorem

\[\| \tilde{L} \| \leq \frac{1}{(n-3)\sqrt{(n-2)(n-1)}} R,\]

and (26) imply

\[(28) \quad \nabla B = 0, \quad \| \tilde{L} \| = \frac{1}{(n-3)\sqrt{(n-2)(n-1)}} R,\]

and we have equality in the inequality of (28). We consider the two cases:

**Case(I):** \( \tilde{L} = 0. \)

If \( \tilde{L} = 0, \) then from (28) and (14), we have

\[(29) \quad R = 0, \quad L = \frac{1}{n-1} \text{tr}(L) g = 0.\]

This together with Theorem 1.1 in [6] implies that \( M \) is Laguerre equivalent to the images of \( \tau \) of hypersurface \( \tilde{x} \) in \( R^0_n \) with mean curvature radius \( r = 0 \) and \( \rho \) = constant.

**Case(II):** \( \tilde{L} \neq 0. \)

Now we assume that \( \tilde{L} \neq 0. \) Since the Laguerre form is parallel, we can choose \( \{ E_i \} \) such that both, \( B \) and \( L, \) are simultaneously diagonal. Let \( \mu_1, \ldots, \mu_n-1 \) and \( \lambda_1, \ldots, \lambda_n-1 \) are the eigenvalues of inequality holds, Lemma 4.2 gives

\[(30) \quad \mu_2, \ldots, \mu_n-1 =: \mu, \quad \lambda_2, \ldots, \lambda_n-1 =: \lambda.\]

The relations \( \text{tr}(B) = 0 \) and \( \| B \|^2 = 1 \) imply

\[(31) \quad \mu_1 = -\sqrt{\frac{n-2}{n-1}}, \quad \mu = \sqrt{\frac{1}{(n-1)(n-2)}}.\]

We use the following convention on the ranges of indices:

\[(32) \quad 1 \leq i, j, k, \ldots \leq (n-1), \quad 2 \leq \alpha, \beta, \gamma, \ldots \leq (n-1).\]

Since \( \nabla B = 0, \) we have

\[(33) \quad 0 = B_{1i,k} \omega_k = dB_{1\alpha} + B_{1k} \omega_{k\alpha} + B_{k\alpha} \omega_k = (\mu_1 - \mu) \omega_{1\alpha}.\]

As \( x \) is umbilic-free, we have

\[(34) \quad \omega_{1\alpha} = 0,\]

this gives

\[(35) \quad -\frac{1}{2} R_{1\alpha i j} \omega_i \wedge \omega_j = d \omega_{1\alpha} - \omega_{1i} \wedge \omega_{i\alpha} = 0.\]
Form the Gauss equation (2.8) we have
(36) \[ 0 = R_{1\alpha\alpha} = -\lambda_1 - \lambda. \]
That is
(37) \[ \lambda_1 = -\lambda. \]
We are going to show that both \( \lambda_1 \) and \( \lambda \) are constant. In fact, noting that \( L_{ij} = 0 \) for \( i \neq j \), from (34) we get
(38) \[ L_{1\alpha,k} = dL_{1\alpha} + L_{1k}\omega_\alpha + L_{k\alpha}\omega_k = 0. \]
In particular, we have
(39) \[ L_{\alpha\alpha,1} = L_{\alpha1,\alpha} = L_{1\alpha,\alpha} = 0, \quad L_{11,\alpha} = L_{1\alpha,1}. \]
(38) and (39) give
(40) \[ L_{11,1} = -L_{\alpha\alpha,1}. \]
Hence
(41) \[ d\lambda = 0. \]
From this and (37) we see that both \( \lambda_1 \) and \( \lambda \) are constant. From (34), it follows that the two distributions, defined by \( \omega_1 = 0 \) and \( \omega_2 = \cdots = \omega_{n-1} = 0 \), are both integrable and thus give a local decomposition of \( M \). Then every point of \( M \) has a neighborhood \( U \) which is a Riemannian product \( V_1 \times V_2 \), where \( V_1 \) and \( V_2 \) are simply connected, with \( \dim V_1 = 1 \) and \( \dim V_2 = n - 2 \). Since \( n \geq 4 \), the sectional curvature of \( V_2 \) is given by
(42) \[ R_{\alpha\beta\alpha\beta} = -2\lambda. \]
\( V_2 \) is a manifold with constant curvature. From (28) and (37) we see that
(43) \[ \lambda = -\frac{R}{2(n - 2)(n - 3)}. \]
Hence
(44) \[ B_{11} = -\sqrt{\frac{n - 2}{n - 1}}, \quad B_{\alpha\alpha} = \sqrt{\frac{1}{(n - 1)(n - 2)}}. \]
(45) \[ L_{11} = \frac{R}{2(n - 2)(n - 3)}, \quad L_{\alpha\alpha} = \frac{R}{2(n - 2)(n - 3)}. \]
Now we compare with Example 3.1 and then consider the following example: the hypersurface \( \tilde{x} : H^1 \times S^{n-2} \to \mathbb{R}^n \) by
\[ \tilde{x}(w, v, u) = \sqrt{\frac{(n - 1)(n - 3)}{R}} \left( \frac{v}{\sqrt{w}} \frac{u}{1 + w} \right), \]
where \( u : S^{n-2} \to \mathbb{R}^{n-1} \) and \((w, v) : H^1 \to \mathbb{R}_1^2 \) are the canonical embeddings.
We get that the laguerre metric \( \tilde{g} \) of \( \tilde{x} \)
\[ \tilde{g} = \left( \frac{(n - 2)(n - 3)}{R} \right) \left( -dw^2 + dv \cdot dv + du \cdot du \right) = \tilde{g}_1 + \tilde{g}_2, \]
where \( \tilde{g}_1 = \frac{(n-2)(n-3)}{R} \left( -dw^2 + dv \cdot dv \right) \) and \( \tilde{g}_2 = \frac{(n-2)(n-3)}{R} \left( du \cdot du \right) \).

We know that \( x : M^{n-1} \to \mathbb{R}^n \) and \( \tilde{x} : H^1 \times S^{n-2} \to \mathbb{R}^n \) have the same Laguerre invariants. Thus from Theorem 2.2 \( x \) and \( \tilde{x} \) are locally Laguerre equivalent. \( \square \)

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