Convergence Properties of Spline-Like Cardinal Interpolation Operators Acting on $\ell^p$ Data

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Abstract In this article, we provide examples of interpolation and approximation methods for $\ell^p$ data. We also show that the resulting interpolants share convergence properties similar to those enjoyed by splines.

Keywords Cardinal interpolation · Splines · Multiquadrics · Gaussians

Mathematics Subject Classification 41A05 · 41A30 · 42C20

1 Introduction

This paper continues the study of convergence properties of cardinal interpolation operators, which was popularized by the spline methods developed by Schoenberg and others, see [12]. We start with the Whittaker–Kotel’nikov–Shannon sampling theorem, which allows one to identify a Paley–Wiener (or band-limited) function $f$ with its samples on the integers $\{f(j) : j \in \mathbb{Z}\}$ via the reconstruction

$$f = \sum_{j \in \mathbb{Z}} f(j) \text{sinc}(\cdot - j).$$

A modern account of this result is given by Unser in [13]. Examining the reconstruction formula, we see that if we started with discrete data, then we could generate
a continuous signal. However, the sinc kernel decays slowly, which means that the convergence of the sum above will depend heavily on the data we are using. Cardinal splines, so called because they interpolate data on the integers, may be used to interpolate $\ell^2$ data, then by letting the degree increase to infinity, the corresponding Paley–Wiener function is recovered pointwise. One way understand this result is to view an $n$-th order spline $\beta^n$ as a filtering process. The spline filters tend to the sinc filter as the degree increases to infinity, see for instance [3]. That is, if we use $\beta^n$ to interpolate the data $\{y_j: j \in \mathbb{Z}\} \in \ell^2$, we have

$$\sum_{j \in \mathbb{Z}} y_j \beta^n(\cdot - j) \to \sum_{j \in \mathbb{Z}} y_j \text{sinc}(\cdot - j).$$

This connection has been generalized by several authors, for instance, by de Boor, Höllig, and Riemenschneider in [5], and Aldroubi and Unser in [2]. Sufficient conditions on a family of functions to possess the limiting property above were studied by the author in [8], in the context of $L^2$ cardinal interpolation.

In [9], Marsden, Richards, and Riemenschneider showed that splines may also be used to recover functions from the space

$$\mathcal{A} = \left\{ f \in L^p(\mathbb{R}) : f(x) = (2\pi)^{-1/2} \int_{-\pi}^{\pi} e^{ix\xi} d\beta(\xi), \beta \in C([-\pi, \pi]) \right\}. \quad (1)$$

We wish to extend this result to a general class of interpolants, those that are built from what we call spline-like cardinal interpolators.

The remainder of this paper is organized into three sections, the first dealing with definitions and preliminary facts concerning spline-like cardinal interpolators, the second proving the convergence result, and finally in the third section we show that this class of interpolants includes odd-degree cardinal splines, the Gaussian which was the subject of [11], Poisson kernels, and two families of multiquadrics. Of these examples, the convergence property for the Poisson kernels appears to be new, as does the result concerning the family of multiquadrics $\{((\cdot)^2 + c^2)^{\alpha-1/2} : \alpha \in \mathbb{N}\}$.

## 2 Definitions and Basic Facts

We begin with a convention for the Fourier transform.

**Definition 1** The **Fourier transform** of a function $f \in L^1(\mathbb{R})$, is defined to be the function

$$\hat{f}(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} f(x) e^{-ix\xi} dx.$$

We make the usual extension to the class of tempered distributions using this convention. Our work will also make use of the mixed Hilbert transform which is described below, more details may be found in [9]. For $x \in \mathbb{R}$, let $m_x$ be the integer such that
If \( f = \{ f(j) \} \in \ell^p \) and \( 1 < p < \infty \), we define the mixed Hilbert transform, denoted \( \mathcal{H}[f] \), to be

\[
\mathcal{H}[f](x) = \sum_{j \neq m_x} \frac{f(j)}{x-j}.
\]

Proposition 1.3 in [9] provides a constant depending only on \( p \) such that

\[
\| \mathcal{H}[f] \|_{L^p(\mathbb{R})} \leq C_p \| f \|_{\ell^p}.
\]

We need a definition similar to the one found in [8]. For our purposes, we make the following definition.

**Definition 2** We say that a function \( \phi \) satisfies the interpolating conditions if it satisfies all of the following:

(A1) \( \phi \) is a real valued slowly increasing function on \( \mathbb{R} \),

(A2) \( \hat{\phi}(\xi) \geq 0 \) and \( \hat{\phi}(\xi) \geq m > 0 \) in \([-\pi, \pi]\),

(A3) \( \phi \in C^1(\mathbb{R} \setminus \{0\}) \), and

(A4) there exists \( \epsilon > 0 \) such that for \( j = 0, 1 \) we have

\[
\hat{\phi}^{(j)}(\xi) = O(|\xi|^{-(1+\epsilon)}) \quad \text{as} \quad |\xi| \to \infty.
\]

These conditions, assure us that the fundamental function \( L_\phi \) defined by its Fourier transform via the formula

\[
\hat{L}_\phi(\xi) = (2\pi)^{-1/2} \frac{\hat{\phi}(\xi)}{\sum_{j \in \mathbb{Z}} \hat{\phi}(\xi + 2\pi j),}
\]

is continuous and solves the following interpolation problem.

**Problem** Find a function \( L \in L^2(\mathbb{R}) \) that satisfies \( L(j) = \delta_{0,j} \) for \( j \in \mathbb{Z} \).

Functions defined by (3) are often called fundamental functions. By interpolator, we mean the function \( \phi \) from which a fundamental function is built. A fundamental function allows us to solve an \( \ell^2 \) interpolation problem using interpolants of the form

\[
\mathcal{I}_\phi[f](x) = \sum_{j \in \mathbb{Z}} f(j) L_\phi(x-j),
\]

where \( L_\phi \) is defined in (3) and \( \{ f(j) \} \in \ell^2 \). Henceforth, we focus on interpolants of this form.

Since we are interested in matters of convergence, we need to introduce a parameter. To this end we let \( A \subset (0, \infty) \) be an unbounded index set which could be discrete or continuous depending on the example. We wish to make sure that our parameter interacts well with the limit.
Definition 3 A collection of functions \( \{ \phi_\alpha : \alpha \in A \} \) will be called a spline-like family of cardinal interpolators if the following conditions are satisfied:

(B1) for all \( \alpha \in A \), \( \phi_\alpha \) satisfies the interpolating conditions,

(B2) \[ \sum_{m \in \mathbb{Z}, n \neq m} \left\| \hat{\phi}_\alpha' \cdot + 2\pi m \hat{\phi}_\alpha \cdot + 2\pi n \right\|^2_{L^1([-\pi, \pi])} \leq C, \text{ independent of } \alpha, \]

(B3) for \( j \neq 0 \), \( \lim_{\alpha \to \infty} \hat{\phi}_\alpha(\xi + 2\pi j)/\hat{\phi}(\xi) = 0 \) for almost every \( |\xi| \leq \pi \).

This definition seems quite restrictive, nevertheless many popular choices of interpolator satisfy these conditions. The condition (B2) is motivated by the quotient rule, as can be seen below, and may be improved for certain choices of interpolators. The main idea is to try to encapsulate both local and global regularity information in a single condition. The reader interested in examples may skip ahead to the final section.

We turn now to preliminary results which will aid in the proof of the main theorem. Most of these are straightforward consequences of the conditions listed above. For the remainder of the paper, we fix a particular spline-like family of cardinal interpolators \( \{ \phi_\alpha : \alpha \in A \} \), and note that the particular value of the constant \( C \) depends on its occurrence and may change from one line to the next.

Lemma 1 If \( f = \{ f(j) \} \in \ell^2 \), \( L_{\phi_\alpha} \) is defined by (3), and \( \mathcal{S}_{\phi_\alpha}[f] \) is defined by (4), then for each \( \alpha \in A \), \( L_{\phi_\alpha} \in L^2(\mathbb{R}) \cap C(\mathbb{R}) \) and \( \mathcal{S}_{\phi_\alpha}[f] \in L^2(\mathbb{R}) \cap C(\mathbb{R}) \).

Proof To see that \( L_{\phi_\alpha} \in L^2(\mathbb{R}) \cap C(\mathbb{R}) \), we need only consider the Fourier transform.

\[
\int \mathbb{R} |\hat{L}_{\phi_\alpha}(\xi)|^2 d\xi = \sum_{j \in \mathbb{Z}} \left| \hat{L}_{\phi_\alpha}(\xi - 2\pi j) \right|^2 d\xi = \int_{-\pi}^{\pi} \sum_{j \in \mathbb{Z}} |\hat{L}_{\phi_\alpha}(\xi - 2\pi j)|^2 d\xi = (2\pi)^{1/2}
\]

Thus \( L_{\phi_\alpha} \in C(\mathbb{R}) \). Since \( \phi_\alpha \) satisfies (A2), the calculation for \( L_{\phi_\alpha} \in L^2(\mathbb{R}) \) is similar. To see that \( \mathcal{S}_{\phi_\alpha}[f] \in L^2(\mathbb{R}) \), we again use the Fourier transform,

\[
\int \mathbb{R} |\hat{\mathcal{S}}_{\phi_\alpha}[f](\xi)|^2 d\xi = \int \mathbb{R} \left| \hat{L}_{\phi_\alpha}(\xi) \sum_{k \in \mathbb{Z}} f(k)e^{ik\xi} \right|^2 d\xi
\]

\[
= \sum_{j \in \mathbb{Z}} \int_{-\pi}^{\pi} \left| \hat{L}_{\phi_\alpha}(\xi - 2\pi j) \sum_{k \in \mathbb{Z}} f(k)e^{ik\xi} \right|^2 d\xi
\]

\[
\leq \int_{-\pi}^{\pi} \left| \sum_{j \in \mathbb{Z}} f(j)e^{ij\xi} \right|^2 d\xi = \| \{ f(j) \} \|_{\ell^2}^2.
\]

The last equality coming from Parseval’s formula. A similar argument shows that \( \mathcal{S}_{\phi_\alpha}[f] \in L^1(\mathbb{R}) \), hence \( \mathcal{S}_{\phi_\alpha}[f] \in C(\mathbb{R}) \). \( \square \)
Lemma 2 For all $\alpha \in A$, $L_{\phi_{\alpha}}$ satisfies $L_{\phi_{\alpha}}(j) = \delta_{0,j}$ for $j \in \mathbb{Z}$.

Proof Using the inversion formula for the Fourier transform, we get for $j \in \mathbb{Z}$:

$$
L_{\phi_{\alpha}}(j) = (2\pi)^{-1/2} \int_{\mathbb{R}} \hat{L}_{\phi_{\alpha}}(\xi) e^{ij\xi} d\xi
$$

$$
= (2\pi)^{-1/2} \sum_{k \in \mathbb{Z}} \int_{-\pi}^{\pi} \hat{L}_{\phi_{\alpha}}(\xi - 2\pi k) e^{ij\xi} d\xi
$$

$$
= (2\pi)^{-1} \int_{-\pi}^{\pi} e^{ij\xi} d\xi = \delta_{0,j}.
$$

We introduce an operator which plays an important role throughout the remainder of the paper, the Whittaker map $\mathcal{W}: \ell^p \to L^p(\mathbb{R})$, given by

$$
\mathcal{W}([\{c_j\}](x)) = \sum_{j \in \mathbb{Z}} c_j \frac{\sin(\pi(x-j))}{\pi(x-j)}.
$$

We mention a few results concerning $\mathcal{W}$. In [9], it is shown (in Lemma 1.4) that this a continuous mapping for $1 < p < \infty$, i.e.

$$
\|\mathcal{W}([\{c_j\}])\|_{L^p(\mathbb{R})} \leq C_p \|\{c_j\}\|_{\ell^p},
$$

where $C_p$ depends only on $p$. Furthermore, it is shown (in Theorem 3.4) that the space

$$
\mathcal{B} = \{f : f = \mathcal{W}([\{c_j\}]), \{c_j\} \in \ell^p\}
$$

is equivalent to the space $\mathcal{A}$ defined in (1).

3 Main Result

Our goal is to prove a generalization of Theorem 3.4 in [9] (see also [10]), which gives another equivalent formulation of the space $\mathcal{A}$. Our argument closely resembles the one given there. We begin by finding the $L^p$-norm of $L_{\phi_{\alpha}}$.

Lemma 3 If $\{\phi_{\alpha} : \alpha \in A\}$ is a spline-like family of cardinal interpolants, then there exists a constant $C$ independent of $\alpha$ such that

$$
\|D\hat{L}_{\phi_{\alpha}}\|_{L^1(\mathbb{R})} \leq C
$$

Proof The quotient rule gives us

$$
D\hat{L}_{\phi_{\alpha}}(\xi) = (2\pi)^{-1/2} \frac{\hat{\phi}'_{\alpha}(\xi) P_{\alpha}(\xi) - \hat{\phi}_{\alpha}(\xi) P'_{\alpha}(\xi)}{P^2_{\alpha}(\xi)},
$$
where \( P_\alpha(\xi) = \sum_{j \in \mathbb{Z}} \hat{\phi}_\alpha(\xi - 2\pi j) \). We have

\[
\int_{\mathbb{R}} \left| \frac{\hat{\phi}'_\alpha(\xi) P_\alpha(\xi) - \hat{\phi}_\alpha(\xi) P'(\xi)}{P^2_\alpha(\xi)} \right| d\xi \\
= \sum_{m \in \mathbb{Z}} \int_{(2m-1)\pi}^{(2m+1)\pi} \left| \frac{\hat{\phi}'_\alpha(\xi) P_\alpha(\xi) - \hat{\phi}_\alpha(\xi) P'(\xi)}{P^2_\alpha(\xi)} \right| d\xi \\
= \sum_{m \in \mathbb{Z}} \int_{-\pi}^{\pi} \left| \frac{\hat{\phi}'_\alpha(\xi + 2\pi m) P_\alpha(\xi) - \hat{\phi}_\alpha(\xi + 2\pi m) P'(\xi)}{P^2_\alpha(\xi)} \right| d\xi.
\]

The integrand may be simplified by letting \( u_\alpha(\xi) = \sum_{j \neq 0} \hat{\phi}_\alpha(\xi - 2\pi j) \) for \(|\xi| \leq \pi\). Then (A3) and (A4) imply that we can differentiate term by term, so we get

\[
\sum_{m \in \mathbb{Z}} \int_{-\pi}^{\pi} \left| \frac{\hat{\phi}'_\alpha(\xi + 2\pi m) P_\alpha(\xi) - \hat{\phi}_\alpha(\xi + 2\pi m) P'(\xi)}{P^2_\alpha(\xi)} \right| d\xi \\
\leq 2 \int_{-\pi}^{\pi} \sum_{m \in \mathbb{Z}} \sum_{n \neq m} \left| \frac{\hat{\phi}'_\alpha(\xi + 2\pi m) \hat{\phi}_\alpha(\xi + 2\pi n) \hat{\phi}^2_\alpha(\xi)}{P^2_\alpha(\xi)} \right| d\xi \\
\leq 2C.
\]

The interchange of the integral and the sum is justified by Tonelli’s theorem. We have also used the triangle inequality and reindexed the second double sum to combine terms. The final bound comes from (B2) and is independent of \( \alpha \).

Remark. We could replace (B2) with (6) with little to no change in the proof below. In practice, if we can simplify \( D \hat{L}_{\phi_\alpha} \) we may use (6) or some variant of (B2) such as

\[
(B2)' \sum_{m \in \mathbb{Z}, n \neq m} \left\| \frac{\hat{\phi}'_\alpha(\cdot + 2\pi m) \hat{\phi}_\alpha(\cdot + 2\pi n)}{\hat{\phi}^2_\alpha(\cdot)} \right\|_{L^\infty([-\pi, \pi])} \leq C, \text{ independent of } \alpha.
\]

If need be, we can actually check that the \( L^1 \)-norm is bounded for some terms while the \( L^\infty \)-norm is bounded for others.

We have the following two straightforward corollaries.

**Corollary 1** If the conditions of the above lemma are met, then we have the pointwise bound

\[
|L_{\phi_\alpha}(x)| \leq C (1 + |x|)^{-1},
\]

where \( C \) is independent of \( \alpha \).

**Proof** This follows from taking the Fourier transform of \((1 + x) L_{\phi_\alpha}(x)\).

**Corollary 2** For \( 1 < p < \infty \), \( \| L_{\phi_\alpha} \|_{L^p(\mathbb{R})} \leq C_p \), where \( C_p \) is independent of \( \alpha \).
Proof. This follows immediately from the above corollary.

We may now establish a bound for \( \| J_{\phi_{\alpha}}\|_{L^p} \to L^p \).

**Proposition 1** Let \( \{ \phi_{\alpha} : \alpha \in A \} \) be a spline-like family of cardinal interpolators, \( f = \{ f(j) \} \in \ell^p \), and \( 1 < p < \infty \), then

\[
\| J_{\phi_{\alpha}}[f] \|_{L^p(\mathbb{R})} \leq C_p \| f \|_{\ell^p},
\]

(8)

where \( J_{\phi_{\alpha}} \) is defined as in (4) and \( C_p \) is independent of both \( f \) and \( \alpha \).

**Proof** The method of proof is similar to Theorem 3.1 in [9], we encourage the reader to consult that proof as well. We will use the converse of Hölder’s inequality. To this end, we suppose that \( 1 < p < \infty \) and \( f = \{ f(j) \} \in \ell^p \) satisfies \( f(j) = 0 \) for \( |j| > N \). Letting \( q \) be the conjugate exponent to \( p \), i.e. \( 1/p + 1/q = 1 \), we suppose that \( g \in L^q(\mathbb{R}) \) is supported in \([-R, R]\). Recalling that \( m_x \) is the integer which satisfies \( m_x - 1/2 \leq x < m_x + 1/2 \), we have

\[
\left| \int_{\mathbb{R}} J_{\phi_{\alpha}}(x)g(x)dx \right| = \left| \int_{-R}^{R} \sum_{|j| \leq N} f(j) L_{\phi_{\alpha}}(x-j)g(x)dx \right|
\]

\[
\leq \left| \int_{-R}^{R} |f(m_x)| L_{\phi_{\alpha}}(x-m_x)g(x)dx \right| + \left| \int_{-R}^{R} \sum_{j \neq m_x} f(j) L_{\phi_{\alpha}}(x-j)g(x)dx \right|
\]

\[
\leq C_p \| f \|_{\ell^p} \| g \|_{L^q(\mathbb{R})} + \left| \int_{-R}^{R} \sum_{j \neq m_x} f(j) L_{\phi_{\alpha}}(x-j)g(x)dx \right|
\]

Here we have used Corollary 2 along with Hölder’s inequality to estimate the first term, we note that the constant \( C_p \) is independent of \( \alpha \). It remains to bound the second term.

\[
\left| \int_{-R}^{R} \sum_{j \neq m_x} f(j) L_{\phi_{\alpha}}(x-j)g(x)dx \right|
\]

\[
= (2\pi)^{-1/2} \left| \int_{-R}^{R} \sum_{j \neq m_x} f(j) \int_{\mathbb{R}} \hat{L}_{\phi_{\alpha}}(\xi) e^{i(x-j)\xi} d\xi g(x)dx \right|
\]

\[
= (2\pi)^{-1/2} \left| \int_{-R}^{R} \sum_{j \neq m_x} \frac{f(j)}{x-j} \int_{\mathbb{R}} e^{i\xi} D\hat{L}_{\phi_{\alpha}}(\xi) d\xi g(x)dx \right|
\]

\[
\leq C \int_{-R}^{R} |\mathcal{H}[f](x)g(x)| dx \leq C^*_p \| f \|_{\ell^p} \| g \|_{L^q(\mathbb{R})}
\]

Here we have used the Fourier representation of \( L_{\phi_{\alpha}} \) and integrated by parts, then used the bound (6) along with Hölder’s inequality and (2). The constant \( C^*_p \) is independent of \( \alpha \), so combining the two estimates and taking \( N, R \to \infty \) completes the proof. \( \square \)
Proposition 2 Let \( \{ \phi_\alpha : \alpha \in A \} \) be a spline-like family of cardinal interpolators, \( f = \{ f(j) \} \in \ell^p \), and \( 1 < p < \infty \), then we have

\[
\lim_{\alpha \to \infty} \| \mathcal{I}_{\phi_\alpha}[f] - \mathcal{W}[f] \|_{L^p(\mathbb{R})} = 0.
\]  

(9)

Proof From Corollary 2, we may apply the uniform boundedness theorem. Thus it is enough to check the result on \( y_j = \{ \delta_{j,k} : k \in \mathbb{Z} \} \). We have

\[
\| \mathcal{I}_{\phi_\alpha}[y_j] - \mathcal{W}[y_j] \|_{L^p(\mathbb{R})} = \left\| L_{\phi_\alpha} \left( \cdot - j \right) - \frac{\sin(\pi(\cdot - j))}{\pi(\cdot - j)} \right\|_{L^p(\mathbb{R})}.
\]

Corollary 1 combined with the dominated convergence theorem will finish the proof, provided that we show the pointwise convergence. We will use the Fourier transform. We have

\[
\left| L_{\phi_\alpha}(x) - \frac{\sin(\pi x)}{\pi x} \right| = (2\pi)^{-1/2} \left| \int_{\mathbb{R}} \hat{L}_{\phi_\alpha}(\xi) e^{ix\xi} \, d\xi - \int_{-\pi}^{\pi} e^{ix\xi} \, d\xi \right| 
\leq (2\pi)^{-1/2} \left( \int_{-\pi}^{\pi} \left| \hat{L}_{\phi_\alpha}(\xi) - 1 \right| d\xi + \sum_{j \neq 0} \int_{-\pi}^{\pi} \left| \hat{L}_{\phi_\alpha}(\xi - 2\pi j) \right| d\xi \right).
\]

We will estimate the terms in parentheses separately, for the first term we have

\[
\int_{-\pi}^{\pi} \left| \hat{L}_{\phi_\alpha}(\xi) - 1 \right| d\xi \leq \int_{-\pi}^{\pi} \left| \frac{u_\alpha(\xi)}{\hat{\phi}_\alpha(\xi) + u_\alpha(\xi)} \right| d\xi
\leq \int_{-\pi}^{\pi} \left| \frac{\hat{\phi}_\alpha(\xi - 2\pi j)}{\hat{\phi}_\alpha(\xi)} \right| d\xi.
\]

The function \( u_\alpha \) is the same one introduced in the proof of Lemma 3. Since (B2) and (B3) hold, the dominated convergence theorem shows that this term tends to 0. The calculation for the second term is similar:

\[
\sum_{j \neq 0} \int_{-\pi}^{\pi} \left| \hat{L}_{\phi_\alpha}(\xi - 2\pi j) \right| d\xi \leq \int_{-\pi}^{\pi} \left| \frac{\hat{\phi}_\alpha(\xi - 2\pi j)}{\hat{\phi}_\alpha(\xi)} \right| d\xi.
\]

Again, the terms tend to 0 by the dominated convergence theorem. Hence we have shown that

\[
\lim_{\alpha \to \infty} \left| L_{\phi_\alpha}(x) - \frac{\sin(\pi x)}{\pi x} \right| = 0,
\]

which completes the proof. \( \square \)
These propositions lead us to our main result, stated in the theorem below. We use the shorthand $L^p - \lim$ to mean that the convergence takes place with respect to the $L^p$-norm.

**Theorem 1** Let $1 < p < \infty$ and $\{\phi_\alpha : \alpha \in A\}$ be a spline-like family of cardinal interpolators, then the following spaces are equivalent.

\[
\mathcal{A} = \{ f \in L^p(\mathbb{R}) : f(x) = \int_{-\pi}^{\pi} e^{ix\xi} d\beta(\xi), \beta \in C([-\pi, \pi]) \}
\]

\[
\mathcal{B} = \{ f : f = \mathcal{W}[\{c_j\}], \{c_j\} \in \ell^p \}
\]

\[
\mathcal{C} = \{ f : f = L^p - \lim_{\alpha \to \infty} \mathcal{I}_{\phi_\alpha}[\{c_j\}], \{c(j)\} \in \ell^p \}
\]

**Proof** That $\mathcal{A}$ and $\mathcal{B}$ are equivalent is shown Theorem 3.4 in [10]. The previous proposition shows that $\mathcal{B}$ and $\mathcal{C}$ are equivalent. \qed

We have the following corollary concerning the recovery of functions from the space $\mathcal{A}$.

**Corollary 3** If $f \in \mathcal{A}$ and $\{\phi_\alpha : \alpha \in A\}$ is a spline-like family of cardinal interpolators, then

\[
\lim_{\alpha \to \infty} \left| f(x) - \mathcal{I}_{\phi_\alpha}[\{f(j)\}](x) \right| = 0,
\]

and the convergence is uniform.

**Proof** Theorem 1 implies that $f(x) = \mathcal{W}[\{f(j)\}](x)$, hence the pointwise estimate from Proposition 2 provides the limit, as well as uniformity. \qed

### 4 Examples

In this section we provide examples of spline-like families of cardinal interpolators. Some of these results appear in the literature. Specifically, the spline analog of the $L^p$ result that is obtained here was worked out by Marsden, Richards, and Riemenschneider in [9] and the Gaussian analog was worked out by Riemenschneider and Sivakumar in [11]. Several of the examples below are exponential in the frequency domain, these examples fit into the framework described by de Boor, Höllig, and Riemenschneider in [5] for $L^2$. Additionally, in the language of Aldroubi and Unser, all of these examples are more or less interpolating generating functions for $L^2([-\pi, \pi])$, see Section 4.3 in [2] for more details. The work done in [8] for the space $L^2$ suggested that an appropriate generalization to $L^p$ should exist.

The interpolating conditions (A1)–(A4), are generally straightforward exercises which we leave to the reader. We will focus on checking (B2) and (B3). The remark following Lemma 3 suggests that we check (B2) by breaking up the double sum over $\{(m, n) : m \in \mathbb{Z}, n \neq m\}$ into three pieces, $\Lambda_1 = \{(m, 0) : m \neq 0\}$, $\Lambda_2 = \{(0, n) : n \neq 0\}$, and $\Lambda_3 = \{(m, n) : m, n \neq 0\}$. The basic strategy in all of our examples will be to integrate the derivative term after making preliminary estimates.
Remark Since the conditions (A1)–(A4) and (B1)–(B3) are only sufficient, it is likely that one need not be so strict. For instance, condition (A2) requires $\hat{\phi}_\alpha(\xi) \geq 0$. This condition is used in [8] to prove an interpolation result. However, one could start with a family of functions which satisfies an $L^2$ interpolating condition, then we could drop this requirement. Similarly, (B2) could be replaced with three separate growth conditions, corresponding to the categories $\Lambda_1, \Lambda_2$, and $\Lambda_3$ above.

One could also start with a single generating function and develop sufficient conditions for the main result here to hold for the family of convolution powers of the generating function. This procedure was examined in [2] and [5]. This provides a discrete exponential parameter in the frequency domain as illustrated in Sect. 4.2 for the Gaussian.

4.1 Odd Degree Splines

We will examine the spline interpolator whose definition may be found in [9]. Owing to the positivity requirement in (A2), we will only consider odd degree splines, i.e. those which reduce to an odd degree polynomial off of the integers, whose Fourier transforms are given by $\hat{\phi}_\alpha(\xi) = |\xi|^{-2\alpha}$ for $\alpha \in \mathbb{N}$. The reader is encouraged to examine [9] for more details. We verify (B2) presently. If $(m, n) \in \Lambda_1$, we have

$$\left\| \frac{\hat{\phi}'_\alpha(\cdot + 2\pi m)}{\hat{\phi}_\alpha} \right\|_{L^1([-\pi, \pi])} = 2\alpha \int_{-\pi}^{\pi} |\xi + 2\pi m|^{-2\alpha-1} |\xi|^{2\alpha} d\xi \leq 2\alpha \pi 2\alpha \int_{-\pi}^{\pi} |\xi + 2\pi m|^{-2\alpha-1} d\xi \leq 2(2|m| - 1)^{-2\alpha}.$$

The estimate for $(m, n) \in \Lambda_2$ is similar, we have

$$\left\| \frac{\hat{\phi}'_\alpha \hat{\phi}_\alpha(\cdot + 2\pi n)}{\hat{\phi}_\alpha^2} \right\|_{L^1([-\pi, \pi])} = 2\alpha \int_{-\pi}^{\pi} |\xi|^{2\alpha-1} |\xi + 2\pi n|^{-2\alpha} d\xi \leq 2\alpha \pi^{-2\alpha} (2|n| - 1)^{-2\alpha} \int_{-\pi}^{\pi} |\xi|^{2\alpha-1} d\xi = 2(2|n| - 1)^{-2\alpha}.$$

We turn our attention now to $(m, n) \in \Lambda_3$; here we have,

$$\left\| \frac{\hat{\phi}'_\alpha(\cdot + 2\pi m)\hat{\phi}_\alpha(\cdot + 2\pi n)}{\hat{\phi}_\alpha^2} \right\|_{L^1([-\pi, \pi])} \leq 2\alpha \pi^{-2\alpha} (2|m| - 1)^{-2\alpha} \int_{-\pi}^{\pi} |\xi + 2\pi m|^{-2\alpha-1} d\xi \leq 2(2|m| - 1)^{-2\alpha} (2|m| - 1)^{-2\alpha}.$$
Combining these estimates show that (B2) is satisfied. Condition (B3) is checked by noting that for \(-\pi < \xi < \pi\) and \(j \neq 0\),

\[
\lim_{\alpha \to \infty} \frac{\hat{\phi}_\alpha(\xi + 2\pi j)}{\hat{\phi}_\alpha(\xi)} = \lim_{\alpha \to \infty} \frac{|\xi|^{2\alpha}}{|\xi + 2\pi j|^{2\alpha}} = 0.
\]

### 4.2 Gaussian Kernels

We now consider \(\{\exp(-x^2/(4\alpha)) : \alpha \geq 1\}\), which is the focus of [11], although we have changed the parameter to suit our purposes. We have \(\hat{\phi}_\alpha(\xi) = \exp(-\alpha\xi^2)\), so that (B1) is clear. We begin by checking (B2). For \((m, n)\) \(\in \Lambda_1\), we have,

\[
\left\| \hat{\phi}_\alpha(\cdot + 2\pi m)/\hat{\phi}_\alpha \right\|_{L_1([\pi, \pi])} = 2\alpha \int_{-\pi}^\pi |\xi + 2\pi| \exp\left(\alpha(\xi^2 - (\xi + 2\pi m)^2)\right) d\xi \\
\leq 2\alpha \exp(\alpha\pi^2) \int_{-\pi}^\pi |\xi + 2\pi m| \exp\left(-\alpha(\xi + 2\pi m)^2\right) d\xi \\
\leq 2 \exp\left(-4\alpha\pi m(|m| - 1)\right).
\]

For \((m, n)\) \(\in \Lambda_2\), we see that

\[
\left\| \hat{\phi}_\alpha(\cdot + 2\pi n)/\hat{\phi}_\alpha^2 \right\|_{L_1([\pi, \pi])} = 2\alpha \int_{-\pi}^\pi |\xi| \exp\left(\alpha(\xi^2 - (\xi + 2\pi n)^2)\right) d\xi \\
\leq 2\alpha \exp\left(-\alpha\pi^2(2|n| - 1)^2\right) \int_{-\pi}^\pi |\xi| \exp\left(\alpha\xi^2\right) d\xi \\
= 4\alpha \exp\left(-\alpha\pi^2(2|n| - 1)^2\right) \int_0^\pi \xi \exp\left(\alpha\xi^2\right) d\xi \\
\leq 4 \exp\left(-4\alpha\pi^2|n|(|n| - 1)\right).
\]

Finally, we suppose \((m, n)\) \(\in \Lambda_3\), then we have

\[
\left\| \hat{\phi}_\alpha(\cdot + 2\pi m)\hat{\phi}_\alpha(\cdot + 2\pi n)/\hat{\phi}_\alpha^2 \right\|_{L_1([\pi, \pi])} \\
= 2\alpha \int_{-\pi}^\pi |\xi + 2\pi m| \exp\left(\alpha(2\xi^2 - (\xi + 2\pi m)^2 - (\xi + 2\pi n)^2)\right) d\xi \\
\leq 2\alpha \exp\left(-\alpha\pi^2(4|n|^2 - 4|n| - 1)\right) \int_{-\pi}^\pi |\xi + 2\pi m| \exp\left(-\alpha(\xi + 2\pi m)^2\right) d\xi \\
\leq 2 \exp\left(-4\alpha\pi^2|m|(|m| - 1)\right) \exp\left(-4\alpha\pi^2|n|(|n| - 1)\right)
\]

Thus we can see that the double sum will be bounded independent of \(\alpha\) so that (B2) is satisfied.
Checking condition (B3) is straightforward; for $|\xi| < \pi$ and $j \neq 0$, we have

$$\lim_{\alpha \to \infty} \frac{\hat{\phi}_\alpha(\xi + 2\pi j)}{\hat{\phi}_\alpha(\xi)} = \lim_{\alpha \to \infty} \exp(-4\pi \alpha j (\xi + \pi j)) = 0.$$ 

Thus $\{\exp(-x^2/(4\alpha)) : \alpha \geq 1\}$ is a spline-like family of cardinal interpolators. Notice that we could have also started with $\phi_1(x) = \exp(-x^2/4)$ and defined $\phi_2 = \phi_1 \ast \phi_1$ and so on, so that $\hat{\phi}_\alpha(\xi) = \exp(-\alpha \xi^2)$, where $\alpha \in \mathbb{N}$.

4.3 Poisson Kernels

We now consider $\{(x^2 + \alpha^2)^{-1/2} : \alpha \geq 1\}$, the family of Poisson kernels, whose Fourier transforms are given by $\hat{\phi}_\alpha(\xi) = \exp(-\alpha |\xi|)$. Since (B1) is clear, we check (B2). Suppose $(m, n) \in \Lambda_1$, we have

$$\left\| \frac{\hat{\phi}_\alpha' \cdot + 2\pi m)}{\hat{\phi}_\alpha(\cdot + 2\pi n)} \right\|_{L^1([-\pi, \pi])} = \alpha \int_{-\pi}^{\pi} \exp(\alpha(|\xi| - |\xi + 2\pi|)) d\xi \leq \alpha \exp(-\alpha \pi (2|m| - 1)) \int_{-\pi}^{\pi} \exp(\alpha |\xi|) d\xi = 2 \exp(-\alpha \pi (2|m| - 1)) \int_{-\pi}^{\pi} \exp(2\alpha \xi) d\xi \leq 4 \exp(-2\alpha \pi (|m| - 1)).$$

For $(m, n) \in \Lambda_2$, we have

$$\left\| \frac{\hat{\phi}_\alpha \hat{\phi}_\alpha' \cdot + 2\pi n)}{\hat{\phi}_\alpha^2(\cdot + 2\pi n)} \right\|_{L^1([-\pi, \pi])} = \alpha \int_{-\pi}^{\pi} \exp(\alpha(|\xi| - |\xi + 2\pi|)) d\xi \leq \alpha \exp(-\alpha \pi (2|n| - 1)) \int_{-\pi}^{\pi} \exp(\alpha |\xi|) d\xi \leq 4 \exp(-2\alpha \pi (|n| - 1)).$$

Examining $(m, n) \in \Lambda_3$, we see that

$$\left\| \frac{\hat{\phi}_\alpha' \cdot + 2\pi m)}{\hat{\phi}_\alpha(\cdot + 2\pi n)} \right\|_{L^1([-\pi, \pi])} = \alpha \int_{-\pi}^{\pi} \exp(\alpha(2|\xi| - |\xi + 2\pi m| - |\xi + 2\pi n|)) d\xi \leq 2 \alpha \exp(-2\alpha \pi (|m| + |n| - 1)) \int_{0}^{\pi} \exp(2\alpha \xi) d\xi \leq 2 \exp(-2\alpha \pi (|m| - 1)) \exp(-2\alpha \pi (|n| - 1)).$$
Thus we can see (B2) holds. To see that condition (B3) holds note that for any $|\xi| < \pi$ and $j \neq 0$,

$$\lim_{\alpha \to \infty} \exp(\alpha(|\xi| - |\xi + 2\pi j|)) = 0,$$

because the exponent is negative.

### 4.4 Multiquadrics I

We consider the family of multiquadrics $\{((\cdot)^2 + c^2)^{\alpha-1/2} : \alpha \in \mathbb{N}\}$ for a fixed $c > 0$. These kernels correspond to ‘smoothed out’ odd degree splines and the convergence result for these interpolation operators appears to be new. We need the Fourier transforms which may be found in [7]; neglecting constants we have $\hat{\phi}_\alpha(\xi) = |\xi|^{-\alpha} K_\alpha(c|\xi|)$, where

$$K_\alpha(u) = \int_0^\infty \exp(-u \cosh(t)) \cosh(at) dt, \quad u > 0$$

is the Macdonald function, also known as the modified Bessel function of the second kind. We will make use of the following estimates, which we state as a lemma.

**Lemma 4** For $|a| \geq 1/2$ and $u > 0$, we have the inequalities

$$(\pi/2)^{1/2}u^{-1/2} \exp(-u) \leq K_\alpha(u) \leq (2\pi)^{1/2}u^{-1/2} \exp(-u) \exp(a^2/(2u)). \quad (10)$$

**Proof** This is just a combination of Corollary 5.12 and Lemma 5.13 in [14]. These bounds also appeared in [1] as asymptotic relations. $\square$

The interpolatory conditions follow easily from the definition of the Macdonald function, also of interest is that $\hat{\phi}_\alpha$ decreases on $(0, \infty)$. As usual, we start with (B2); however, the cases $\alpha = 1$ and $\alpha > 1$ require different arguments. We start with $\alpha = 1$ and show that the double sum in (B2) is bounded. We suppose that $(m, n) \in \Lambda_1$ and see that

$$\left\| \hat{\phi}'(\cdot + 2\pi m)/\hat{\phi}_1 \right\|_{L^1((-\pi, \pi))} = c \int_{-\pi}^{\pi} \frac{|\xi + 2\pi m|^{-1} K_2(c|\xi + 2\pi m|)}{|\xi|^{-1} K_1(c|\xi|)} d\xi$$

$$\leq \frac{c\pi}{K_1(c\pi)} \int_{-\pi}^{\pi} |\xi + 2\pi m|^{-1} K_2(c|\xi + 2\pi|) d\xi$$

$$\leq \frac{2K_1(c\pi(2|m| - 1))}{K_1(c\pi)} (2|m| - 1)^{-1}.$$
For \((m, n) \in \Lambda_2\), we have
\[
\left\| \frac{\phi'_1(\cdot + 2\pi n)\hat{\phi}_1}{\phi_1^2} \right\|_{L^1([\pi, \pi])} \\
= c \int_{-\pi}^{\pi} \frac{\xi^{-1} K_2(c|\xi|) (\xi + 2\pi n)^{-1} K_1(c|\xi + 2\pi n|)}{|\xi|^{-1} K_1(c|\xi|)} d\xi \\
\leq 2c|\pi|^{-1} |K_1(c|\pi|^{-1} K_1(c|\pi + 2\pi n|) d\xi \\
= \frac{2K_1(c|\pi(2|n| - 1)| K_1(c|\pi(2|n| - 1)| (2|n| - 1)^{-1}.}
\]

Now suppose that \((m, n) \in \Lambda_3\), we have
\[
\left\| \frac{\phi'_1(\cdot + 2\pi m)\hat{\phi}_1(\cdot + 2\pi n)\hat{\phi}_1}{\phi_1^2} \right\|_{L^1([\pi, \pi])} \\
= c \int_{-\pi}^{\pi} \frac{\xi + 2\pi m|^{-1} K_2(c|\xi + 2\pi m|) (\xi + 2\pi n)^{-1} K_1(c|\xi + 2\pi n|)}{|\xi|^{-2} K_1^2(c|\xi|)} d\xi \\
\leq \frac{c|\pi|^{-1} |K_1(c|\pi(2|n| - 1)| (2|n| - 1)^{-1}}{K_1^2(c|\pi|) (2|m| - 1)^{-1}}.
\]

Now we may use (10) to see that
\[
\frac{K_1(c|\pi|)}{K_1(c|\pi|)} \leq 2 \exp \left( (1/(2c|\pi|)) \right) (2|n| - 1)^{-1/2} \exp \left( -2c|\pi(2|n| - 1)| \right),
\]
thus the double sum is bounded. We move on to the \(\alpha > 1\) case. Following the above estimates with \(\alpha\) in the place of 1 and using the fact that \(K_\alpha\) is decreasing, we get the following bounds:
\[
\Lambda_1 : 2(2|n| - 1)^{-\alpha},
\Lambda_2 : 2(2|n| - 1)^{-\alpha}, \text{ and }
\Lambda_3 : 2(2|n| - 1)^{-\alpha}(2|n| - 1)^{-\alpha}.
\]

Since \(\alpha > 1\), we no longer need to use (10) to ensure convergence of the series.
To check (B3), note that if \(|\xi| < \pi\) and \(j \neq 0\), we have
\[
\left| \frac{\phi_\alpha(\xi + 2\pi j)}{\phi_\alpha(\xi)} \right| \leq \left( \frac{|\xi|}{|\xi + 2\pi j|} \right)^\alpha,
\]
which tends to 0 as \(\alpha \to \infty\).
If \( \{ \alpha_j : j \in \mathbb{N} \} \subseteq [1/2, \infty) \setminus \mathbb{N} \) satisfies \( \text{dist}(\{ \alpha_j \}, \mathbb{N}) > 0 \), then a slight modification of the argument given above will show that for \( c > 0 \) fixed, \( \{ (c^2 + c^2)^{\alpha_j} : j \in \mathbb{N} \} \) is a spline-like family of cardinal interpolators as well.

4.5 Multiquadrics II

We will now consider the family of multiquadrics \( \{ (c^2 + c^2)^{\alpha} : c \geq 1 \} \), where \( a \in \mathbb{R} \) is fixed. The case that \( a = 1/2 \) is the subject of [4], while the case \( a = 1 \) is the Poisson kernel above. More recently, these interpolators were studied in [6]. This time the Fourier transforms are given by constant multiples of \( \hat{\phi}_c(\xi) = |\xi|^{-(a+1/2)} K_{a+1/2}(c|\xi|) \) where \( a \neq \tilde{N} = \mathbb{N} \cup \{0\} \cup \{-k - 1/2 : k \in \mathbb{N}\} \). In order to use (10), we continue with \( a \in (\mathbb{R} \setminus \tilde{N}) \cap \{ a \in \mathbb{R} : |a + 1/2| \geq 1/2 \} \). Condition (B1) is straightforward and checked in [8]. We may recycle most of the estimates above to check (B2), except we must use (10). We have the following termwise estimates

\[
\begin{align*}
\Lambda_1: & 4 \exp \left( a^2/(2\pi) \right) \frac{\exp(-2c\pi(|m| - 1))}{(2|m| - 1)^{a+1/2}}, \\
\Lambda_2: & 4 \exp \left( a^2/(2\pi) \right) \frac{\exp(-2c\pi(|n| - 1))}{(2|n| - 1)^{a+1/2}}, \text{ and} \\
\Lambda_3: & 8 \exp \left( a^2/\pi \right) \frac{\exp(-2c\pi(|m| - 1)) \exp(-2c\pi(|n| - 1))}{(2|m| - 1)^{a+1/2} (2|n| - 1)^{a+1/2}}.
\end{align*}
\]

Combining these estimates shows that (B2) is satisfied. Checking (B3), we have for \( |\xi| < \pi \) and \( j \neq 0 \)

\[
\left| \frac{\hat{\phi}_c(\xi + 2\pi j)}{\hat{\phi}_c(\xi)} \right| \leq 2 \exp \left( a^2/(2\pi) \right) \left( \frac{|\xi|}{|\xi + 2\pi j|} \right)^{1/2} \exp \left( c(|\xi| - |\xi + 2\pi j|) \right) \\
\leq 2 \exp \left( a^2/(2\pi) \right) \exp \left( c(|\xi| - |\xi + 2\pi j|) \right).
\]

These terms tend to 0 as \( c \to \infty \), which means that \( \{ (c^2 + c^2)^{\alpha} : c \geq 1 \} \) is a spline-like family of cardinal interpolators.

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