Kinetic theory and thermalization of weakly interacting fermions

Jani Lukkarinen*

University of Helsinki, Department of Mathematics and Statistics
P.O. Box 68, FI-00014 Helsingin yliopisto, Finland

December 19, 2017

Abstract

Weakly interacting quantum fluids allow for a natural kinetic theory description which takes into account the fermionic or bosonic nature of the interacting particles. In the simplest cases, one arrives at the Boltzmann–Nordheim equations for the reduced density matrix of the fluid. We discuss here two related topics: the kinetic theory of the fermionic Hubbard model, in which conservation of total spin results in an additional Vlasov type term in the Boltzmann equation, and the relation between kinetic theory and thermalization.

1 Introduction

Kinetic theory describes motion which is transport dominated in the sense that typically the solutions to the kinetic equations correspond to constant velocity, i.e., ballistic, motion intercepted by collisions whose frequency is order one on the kinetic space-time scales. Weakly interacting quantum fluids provide one such example system, as discussed in detail in [10].

We focus here on one particular case of a weakly interacting quantum fluid, the case of weakly interacting fermions hopping on a lattice. Such a model would arise physically as a description of a fluid of electrons in a crystal background potential. For our purposes, this model has also other attractive properties, namely, it has interesting non-trivial kinetic theory with relatively few technical and mathematical difficulties. Much of the discussion below can be straightforwardly adapted to bosonic lattice systems, at least for initial data which exclude formation of Bose–Einstein condensate. For more details about such extensions, we refer to [10]; for instance, Remark 2.3 summarizes the changes and new properties which arise for fermions and bosons moving not on a lattice, but in the continuum $\mathbb{R}^3$.

The purpose of this contribution is not to provide a comprehensive review of literature on kinetic theory and properties of fermionic systems. Instead, we focus on building a bridge between mathematically rigorous results and the physics of fermionic systems. To this end, we begin with a fairly detailed Section 2 on the definition of finite system of fermions hopping on a periodic lattice with pair interactions between the particles, from the point of view of both a fixed particle number Hilbert space and the full antisymmetric Fock space. In Section 3 we recall how the probabilistic concepts of classical particle systems can be generalised into systems of fermions, namely, the definition of quasi-free states, reduced density matrices, and truncated correlation functions. The weak coupling limit of the first reduced density matrix of a translation invariant system and its approximation by the appropriate spatially homogeneous Boltzmann equation is reviewed in Section 4. As a conclusion, we discuss in Sections 5 and 6 the implications of the properties of the solutions to the Boltzmann equation on thermalization in the original fermionic lattice system. Acknowledgements and references can be found at the end of the text.

*E-mail: jani.lukkarinen@helsinki.fi
2 Dynamics of lattice fermions

We recall in this section the mathematical description of fermions, possibly with spin, which are hopping on a finite periodic lattice of length \( L > 1 \). The particles move on a lattice whose points are labelled by \( \Lambda := \mathbb{Z}^d / (L \mathbb{Z}^d) \) which we parametrize by a square centred at the origin. For instance, if \( L \) is even, we use the parametrisation

\[
\Lambda = \left\{ -\frac{L}{2} + 1, \ldots, \frac{L}{2} - 1, \frac{L}{2} \right\}^d.
\]

In particular, all arithmetic on \( \Lambda \) is performed “modulo \( L \)”: if \( x, y \) are in the above parametrisation of \( \Lambda \subset \mathbb{Z}^d \), then \( x + y \in \mathbb{Z}^d \) needs to be identified with its counterpart in the parametrisation. Explicitly, \( x + y \) is equal to \( x + y - Lm \in \Lambda \) where \( m \in \mathbb{Z}^d \) is the unique vector for which \( x + y - Lm \in \Lambda \).

To describe Fourier transforms, we employ the corresponding discrete dual lattice \( \Lambda^* := \Lambda / L = (L^{-1} \mathbb{Z}^d) / \mathbb{Z}^d \). If needed, we use the parametrisation implied by the above notation; for instance for even \( L \), we use \( \Lambda^* = \left\{ -\frac{1}{2} + \frac{1}{L}, \ldots, \frac{1}{2} - \frac{1}{L}, \frac{1}{L} \right\}^d \). The arithmetic on \( \Lambda^* \) is then performed modulo 1, i.e., using the arithmetic inherited from the \( d \)-torus \( T^d = \mathbb{R}^d / \mathbb{Z}^d \supset \Lambda^* \).

Such periodic arithmetic is particularly well adapted for use of discrete Fourier transforms. For a function \( f: \Lambda \to \mathbb{C} \) we take its Fourier transform to be the function \( \hat{f}: \Lambda^* \to \mathbb{C} \) defined by the formula

\[
\hat{f}(k) := \sum_{x \in \Lambda} e^{-i 2 \pi k \cdot x} f(x), \quad k \in \Lambda^*.
\]

The inverse transform of \( g: \Lambda^* \to \mathbb{C} \) is then given by \( \hat{g}: \Lambda \to \mathbb{C} \) defined by

\[
\hat{g}(x) := \int_{\Lambda^*} dk e^{i 2 \pi k \cdot x} g(k) = \frac{1}{|\Lambda|} \sum_{k \in \Lambda^*} e^{i 2 \pi k \cdot x} g(k), \quad x \in \Lambda.
\]

Here and in the following we use the shorthand notation

\[
\int_{\Lambda^*} dk \cdots = \frac{1}{|\Lambda|} \sum_{k \in \Lambda^*} \cdots.
\]

On a finite lattice, the discrete Fourier transform is always pointwise invertible, i.e., for all \( x \in \Lambda \), \( k \in \Lambda^* \), \( (\hat{f}(f_k) = f(x) \), \( (\hat{g}(g_k) = g(k) \).

We assume that the dominant free evolution is defined by giving the dispersion relation \( \omega: T^d \to \mathbb{R} \) corresponding to free evolution after a thermodynamic limit \( L \to \infty \) has been taken. More precisely, we let the periodic lattice hopping potential \( \alpha: \Lambda \to \mathbb{R} \) be defined by the inverse Fourier transform of the map \( \omega|_{\Lambda^*} \),

\[
\alpha(x; L) := \int_{\Lambda^*} dk e^{i 2 \pi k \cdot x} \omega(k), \quad x \in \Lambda.
\]  

The function \( \alpha \) determines the free \( n \)-particle Hamiltonian \( H_0^{(n)} \) by its action on \( n \)-particle wave vectors \( \psi: \Lambda^n \to \mathbb{C} \),

\[
H_0^{(n)} \psi(x_1, \ldots, x_n) = \sum_{j=1}^{n} \sum_{y \in \Lambda} \alpha(x_j - y) \psi(x_1, \ldots, y, \ldots, x_n).
\]

The above construction allows an \( L \)-independent diagonalisation of \( H_0^{(n)} \) by taking the discrete Fourier transform:

\[
(H_0^{(n)} \psi) (k_1, \ldots, k_n) = \sum_{j=1}^{n} \omega(k_j) \tilde{\psi}(k_1, \ldots, k_n).
\]

We assume that the dispersion relation is smooth and symmetric, \( \omega(-k) = \omega(k) \). Then \( \alpha(x; L) \) is always real, and denoting the inverse Fourier transform (i.e., the Fourier series) of \( \omega \) by \( \alpha \), we then have \( \alpha(x; L) \to \alpha(x) \) for each fixed \( x \in \mathbb{Z}^d \) as \( L \to \infty \). In adition, the range of \( \alpha \) is finite, in the sense that \( |\alpha(x)| \) decreases faster than any power as \( |x| \to \infty \).
An explicit often considered example case is nearest neighbour hopping. This corresponds to
\[
\omega(k) = c - \sum_{\nu=1}^{d} \cos(2\pi k_\nu),
\]
where \(c \in \mathbb{R}\) is any constant. For instance, choosing \(c = d\), one obtains the standard discrete Laplacian,
\[
\sum_{y \in \Lambda} \alpha(x - y)\psi(y) = \frac{1}{2} \sum_{\nu=1}^{d} (2\psi(x) - \psi(x - e_\nu) - \psi(x + e_\nu)),
\]
where \(e_\nu\) denotes the unit vector in direction \(\nu\). For vectors \(\psi\) which are obtained by taking values of a slowly varying function \(\psi : \mathbb{R}^{d} \to \mathbb{C}\), the right hand side can be approximated by \(-\frac{1}{2} \nabla^{2}\psi(x)\). Therefore, in this case one may also think of \(H_{0}^{(n)}\) as a discrete approximation of the standard free \(n\)-particle Hamiltonian, with particle mass normalized to one.

We construct a pair-interaction potential \(V(x; L)\) analogously, starting from its Fourier-transform \(\hat{V} : \mathbb{T}^{d} \to \mathbb{C}\) and defining
\[
V(x; L) := \int_{\Lambda^{*}} dk \, e^{i2\pi k \cdot x} \hat{V}(k), \quad x \in \Lambda.
\]
(2)

To make the potential real-valued and symmetric, we assume that \(\hat{V}\) is real-valued and symmetric. The \(n\)-particle pair-interaction potential \(V^{(n)}\) is then defined via the formula
\[
V^{(n)}(x_{1}, \ldots, x_{n}; L) := \frac{1}{2} \sum_{i \neq j}^{n} V(x_{i} - x_{j}; L).
\]

The potential function acts as a multiplication operator on wave-vectors, and we do not make any distinction in the notation between the function and the operator. Thus, if \(\psi : \Lambda^{n} \to \mathbb{C}\) is an \(n\)-particle wave vector, then
\[
V^{(n)}(x_{1}, \ldots, x_{n}) = V^{(n)}(x_{1}, \ldots, x_{n}; L)\psi(x_{1}, \ldots, x_{n}).
\]

Naturally, if \(n = 1\), we have \(V^{(n)} = 0\).

After these preliminaries, we define the full \(n\)-particle Hamiltonian by choosing an interaction strength \(\lambda \geq 0\), setting \(H_{0}^{(0)} = 0\), and for \(n \geq 1\) defining
\[
H^{(n)}_{\lambda} := H_{0}^{(n)} + \lambda V^{(n)}.
\]

The corresponding evolution equation for \(n\)-particle wave vectors \(\psi(t)\) is
\[
\partial_{t}\psi(t) = -iH^{(n)}_{\lambda}\psi(t).
\]

The \(n\)-particle Hilbert space is here finite-dimensional, \(H_{n} = (\mathbb{C}^{\Lambda})^* \otimes \mathbb{C}^{\Lambda^{n}}\). By construction, the evolution preserves particle number and each \(H^{(n)}_{\lambda}\) is a bounded self-adjoint operator on \(H_{n}\). Thus their direct sum \(H_{\lambda} := \bigoplus_{n=0}^{\infty} H^{(n)}_{\lambda}\) defines a self-adjoint operator on the full Fock space \(\mathcal{F} := \bigoplus_{n=0}^{\infty} H_{n}\). More precisely, the domain of the operator is
\[
D(H_{\lambda}) := \left\{ \Psi \in \mathcal{F} \mid \sum_{n=0}^{\infty} \|H^{(n)}_{\lambda}\Psi_{n}\|^{2} < \infty \right\},
\]
and the action of \(H_{\lambda}\) on \(\Psi = (\Psi_{0}, \Psi_{1}, \ldots) \in D(H_{\lambda})\) yields the vector \((H^{(n)}_{\lambda}\Psi_{n})_{n=0}^{\infty} \in \mathcal{F}\). (The proof of these properties can be found for instance in [16, Theorem 2.23].) An analogous construction holds for the potential terms \(V^{(n)}\) alone, and the corresponding full Fock space operator is denoted by \(V\); clearly, \(\lambda V = H_{\lambda} - H_{0}\) on the domain of \(H_{\lambda}\).

Each \(H^{(n)}_{0}\) and \(V^{(n)}\) clearly commutes with permutations of particle labels (i.e., with all of the operators \(Q_{\pi}\) defined by \((Q_{\pi}\psi)(x_{1}, \ldots, x_{n}) = \psi(x_{\pi(1)}, \ldots, x_{\pi(n)})\), there \(\pi\) is any permutation of \(\{1, 2, \ldots, n\}\)). Thus \(H_{\lambda}\) leaves invariant both the fermionic Fock space \(\mathcal{F}_{-},\).
containing those $\Psi \in F$ for which each $\Psi_n$ is antisymmetric under permutations of particle labels, and the bosonic Fock space $F_+$, containing only symmetric $\Psi_n$.

From now on, we focus on the corresponding fermionic lattice system which is defined by wave vectors $\Psi(t) \in \mathcal{F}_-$ and the semigroup generated by the restriction of $H_\Lambda$ to $\mathcal{F}_-$. Since wave vectors with only finitely many non-zero particle sectors belong to $D(H_\Lambda)$ and form a dense set in $\mathcal{F}_-$, we find that for any $\Psi(0) \in \mathcal{F}_-$, the $n$-particle sector of the time-evolved wave function can be obtained by solving the matrix evolution equation

$$\partial_t \Psi_n(t) = -iH_\Lambda^{(n)}\Psi_n(t),$$

with initial data $\Psi_n(0)$.

### 2.1 Dynamics in terms of creation and annihilation operators

Antisymmetry of wave vectors is one of the most important features of fermionic quantum systems, and it can alter the properties of time-evolution significantly. Controlling the effect of antisymmetry is difficult in the above formulation of the time-evolution. A better alternative is offered by representing the time-evolution as an evolution equation of the corresponding fermionic creation and annihilation operators. We summarize their main properties below and refer to [1] Section 5.2 for more mathematical details.

In the present finite lattice case, the Fock space has been constructed using a one-particle space $\mathfrak{h} := C^\Lambda$ and the corresponding (distinguishable) $n$-particle sectors $\mathcal{H}_n := \mathfrak{h}^\otimes n = C^{\Lambda n}$. Let $P^{(n)}_-$ denote the orthogonal projection onto the subspace of antisymmetric functions in $\mathcal{H}_n$; explicitly,

$$(P^{(n)}_- \psi)(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\pi \in S_n} (-1)^\pi \psi(x_{\pi(1)}, \ldots, x_{\pi(n)}),$$

where $S_n$ denotes the group of permutations of the set $\{1, 2, \ldots, n\}$ and $(-1)^\pi$ is the sign of the permutation $\pi \in S_n$. Since we consider a system of identical fermions, at any time, a wave vector $\Psi \in \mathcal{F}_-$ satisfies $P^{(n)}_- \Psi_n = \Psi_n$ for all $n$.

Given a one-particle wave vector $g \in \mathfrak{h}$, we define the corresponding annihilation operator $a(g)$ as the map which takes a vector $\Psi \in \mathcal{F}_-$ and removes the first particle from each of its sectors, with a weight proportional to the overlap with $g$. More precisely, for a fixed particle number $n \geq 1$, there is a unique bounded linear map $A_n(g) : \mathcal{H}_n \to \mathcal{H}_{n-1}$ such that for any collection of one-particle wave vectors $f_j \in \mathfrak{h}$,

$$A_n(g) \left( \bigotimes_{j=1}^n f_j \right) = \sqrt{n!} g \otimes f_1 \otimes \bigotimes_{j=2}^n f_j,$$

where $(g, f)$ is the one-particle scalar product, defined here conjugate linear in the first argument, i.e., $(g, f) = \sum_{x \in \Lambda} g(x)^* f(x)$. We then define the fermionic annihilation operator $a(g) : \mathcal{F}_- \to \mathcal{F}_-$ by the rule

$$(a(g) \Psi)_n = P^{(n)}_- A_{n+1}(g) P^{(n+1)}_- \Psi_{n+1} = P^{(n)}_- A_{n+1}(g) \Psi_{n+1}, \quad n \geq 0, \ \Psi \in \mathcal{F}_-.$$

In general, annihilation operators are unbounded on the appropriate Fock space, and one has to worry about the domain of the operator in its definition. However, it is a remarkable consequence of the antisymmetrisation that $a(g)$ is in fact a bounded operator on $\mathcal{F}_-$, and the normalisation $\sqrt{n}$ added above guarantees that its operator norm is the same as the norm of the wave vector $g$, i.e., we always have $\|a(g)\| = \|g\|_\mathfrak{h}$.

The adjoint of $a(g)$, which we denote here by $a^*(g)$, is called the creation operator at the vector $g \in \mathfrak{h}$. The creation operator can indeed be interpreted as creating a particle with wave vector $g$ at the first position (and hence shifting the labels of the existing particles by one). This interpretation is based on a more direct construction analogous to the one for $a(g)$ above. Namely, there is a unique bounded linear map $C_n(g) : \mathcal{H}_n \to \mathcal{H}_{n+1}$ such that for any collection of one-particle wave vectors $f_j \in \mathfrak{h}$,

$$C_n(g) \left( \bigotimes_{j=1}^n f_j \right) = \sqrt{n+1} g \otimes f_1 \otimes \cdots \otimes f_n.$$
We also set $C_0(g) 1 = g \in \mathcal{H}_1$. The fermionic creation operator is then given by $c(g) : \mathcal{F}_- \to \mathcal{F}_-$, and it satisfies $(c(g) \Psi)_0 = 0$, and

$$(c(g) \Psi)_n = P_{-}^{(n)}C_{n-1}(g)P_{-}^{(n-1)}\Psi_{n-1} = P_{-}^{(n)}C_{n-1}(g)\Psi_{n-1}, \quad n \geq 1, \quad \Psi \in \mathcal{F}_-.$$ 

One can check that then indeed $c(g) = a^*(g)$ which implies that also $\|c(g)\| = \|g\|_h$.

One important reason why working with the creation and annihilation operators simplifies the analysis of time-evolution is that they satisfy fairly simple algebraic rules for swapping the order of any two such operators. Namely, they satisfy the following canonical anticommutation relations: for any one-particle vectors $f, g \in \mathcal{H}$, we have

$$a(f)a(g) + a(g)a(f) = 0 = a(f)^*a(g)^* + a(g)^*a(f)^*,
$$
$$a(f)a(g)^* + a(g)^*a(f) = (f, g)_1,$$ 

where “1” denotes the identity operator on $\mathcal{F}_-$. In particular, $a(f)^2 = 0 = a^*(f)^2$, and if $(e_{\ell})$ is any orthonormal basis of $\mathcal{H}$, we have

$$a(e_{\ell})a(e_{\ell'})^* + a(e_{\ell'})^*a(e_{\ell}) = 1_{\{\ell = \ell'\}}1,$$ 

with $1_{\{P\}}$ denoting the general characteristic function of the condition $P$: we define $1_{\{P\}} = 1$, if $P$ is true, and $1_{\{P\}} = 0$, if $P$ is false.

Moreover, tensor products in $\mathcal{H}_n$ are conveniently expressed in terms of products of creation operators acting on the vacuum $\Omega = (1, 0, 0, \ldots) \in \mathcal{F}_-$. Namely, if $g_j \in \mathcal{H}$, $j = 1, 2, \ldots, n$, are given, then $\otimes g_j \in \mathcal{H}_n$ after antisymmetrisation defines a vector $\Psi \in \mathcal{F}_-$ by setting all other components to zero, i.e., setting $\Psi = P_{-}^{(n)}(\otimes g_j)$ and $\Psi_m = 0$, for $m \neq n$. This vector can also be obtained from

$$\Psi = \frac{1}{\sqrt{n!}}a^*(g_1) \cdots a^*(g_n)\Omega.$$  

The collection of creation and annihilation operators corresponding to the standard unit vector orthonormal basis $(e_{x})_{x \in \Lambda}$, where $(e_{x})_y = 1_{\{x = y\}}$ for all $x, y \in \Lambda$, is of particular interest to us. We employ the following standard shorthand notations:

$$a(x) := a(e_x), \quad a^*(x) := a^*(e_x) = a(x)^*, \quad x \in \Lambda.$$  

These operators can be thought of as annihilating or creating a particle at the site $x$. By (3), they satisfy the following simple anticommutation relations for any $x, y \in \Lambda$,

$$a(x)a(y) + a(y)a(x) = 0 = a(x)^*a(y)^* + a(y)^*a(x)^*,
$$
$$a(x)a(y)^* + a(y)^*a(x) = a(x)^*a(y) + a(y)^*a(x) = 1_{\{x = y\}}1.$$ 

We can also use the creation operators to generate an orthonormal basis for $\mathcal{F}_-$. For this, first define

$$e(x_1, \ldots, x_n) := a^*(x_1) \cdots a^*(x_n)\Omega, \quad x_i \in \Lambda, \quad i = 1, 2, \ldots, n.$$ 

The orthonormal basis may be constructed by collecting all non-repeating sequences of arbitrary length and then choosing one representative for each collection of sequences which differ by a permutation of particle labels. The actual choice does not does not play much role: if $(x_i) \in \Lambda^n$ and $\pi \in S_n$ is some permutation, then by the anticommutation relations

$$e(x_{\pi(1)}, \ldots, x_{\pi(n)}) = (-1)^\pi e(x_1, \ldots, x_n),$$

and hence the choice merely affects signs of the basis vectors.

After these preliminaries, it is straightforward to check that wave vectors and interaction potentials may also be represented using the creation and annihilation operators. Namely, if $\Psi \in \mathcal{F}_-$, $n \in \mathbb{N}$, and $x \in \Lambda^n$, we have

$$\Psi_n(x_1, \ldots, x_n) = (\otimes_{i=1}^n e_{x_i}, \Psi_n)_{\mathcal{H}_n} = (P_{-}^{(n)}(\otimes_{i=1}^n e_{x_i}), \Psi_n)_{\mathcal{H}_n},$$

and hence by (3),

$$\Psi_n(x_1, \ldots, x_n) = \frac{1}{\sqrt{n!}}(a^*(x_1) \cdots a^*(x_n)\Omega, \Psi)_{\mathcal{F}_-}.$$
Moreover, the anticommutation relations imply that if \( x, y \) and \( x_i \in \Lambda, i = 1, 2, \ldots, n \), then

\[
a^\ast(x)a(y)a^\ast(x_1)\cdots a^\ast(x_n)\Omega = \sum_{i=1}^{n} \mathbb{1}_{\{y=x_i\}}a^\ast(x_1)\cdots a^\ast(x_{i-1})a^\ast(x)a^\ast(x_{i+1})\cdots a^\ast(x_n)\Omega.
\]

Using these two properties it is now straightforward to check that the earlier defined operators \( H_0 \) and \( V \) on the fermionic Fock space have the following representations in terms of creation and annihilation operators,

\[
H_0 = \sum_{x,y \in \Lambda} \alpha(x-y;L)a(x)^\ast a(y),
\]

\[
V = \frac{1}{2} \sum_{x,y \in \Lambda} V(x-y;L)a(x)^\ast a(y)^\ast a(y)a(x).
\]

The above right hand sides are finite sums in the Banach space of bounded operators on \( F_- \), and thus \( H_0, V, \) and \( H_\lambda = H_0 + \lambda V \) are also bounded operators on the fermionic Fock space.

The time-evolution of any initial data \( \Psi(0) \in F_- \) under the semigroup \( U_t := e^{-itH_\lambda} \) can be solved if we can solve the time-evolution of the annihilation operators, i.e., it suffices to study

\[
a(x,t) := e^{itH_\lambda}a(x)e^{-itH_\lambda},
\]

and its adjoint

\[
a^\ast(x,t) := e^{itH_\lambda}a^\ast(x)e^{-itH_\lambda}.
\]

This follows from our definition that the Hamiltonian acts trivially on the vacuum sector, \((H_\lambda)\Omega = 0\), and thus

\[
a^\ast(x_1,t)\cdots a^\ast(x_n,t)\Omega = e^{itH_\lambda}a^\ast(x_1)\cdots a^\ast(x_n)\Omega,
\]

implying that

\[
\Psi_n(x_1,\ldots,x_n,t) = \frac{1}{\sqrt{n!}}(a^\ast(x_1)\cdots a^\ast(x_n)\Omega, e^{-iH_\lambda}\Psi(0))_{F_-}
\]

\[= \frac{1}{\sqrt{n!}}(a^\ast(x_1,t)\cdots a^\ast(x_n,t)\Omega, \Psi(0))_{F_-}.\]

Since the Hamiltonian is a bounded operator, we can directly differentiate the definition and obtain

\[
\partial_t a(x,t) = -i\omega(t)a(x), H_\lambda e^{-iH_\lambda}.
\]

The computation of the commutator is straightforward using the anticommutation relations, yielding

\[
[a(x), H_\lambda] = \sum_{y \in \Lambda} \alpha(x-y;L)a(y) + \lambda \sum_{y \in \Lambda} V(x-y;L)a(y)^\ast a(y)a(x).
\]

Therefore, we find that in order to solve the original (linear) evolution equation in the fermionic Fock space, it suffices to solve the following non-linear operator evolution equation on the space of bounded operators on \( F_- \),

\[
\partial_t a(x,t) = -i\sum_{y \in \Lambda} \alpha(x-y;L)a(y,t) - i\lambda \sum_{y \in \Lambda} V(x-y;L)a^\ast(y,t)a(y,t)a(x,t).
\]

In Fourier variables, after defining

\[
\hat{a}(k,t) := \sum_{x \in \Lambda} e^{-2\pi ik \cdot x} a(x,t),
\]

we obtain

\[
\partial_t \hat{a}(k,t) = -i\omega(k)\hat{a}(k,t)
\]

\[= \lambda \int \{k \in \Lambda^3 \}^d dk_1 dk_2 dk_3 \delta(k-k_1-k_2-k_3)\hat{V}(k_1+k_2)\hat{a^\ast}(k_1,t)\hat{a}(k_2,t)\hat{a}(k_3,t),
\]

where \( \delta_\lambda(k) := |\Lambda| \{ k \in \Lambda \} \) is a “discrete Dirac \( \delta \)-function” and \([\hat{a}(k,t)]^\ast = \hat{a^\ast}(-k,t)\).
2.2 Fermionic systems with spin interactions and the Hubbard model

Spin is an integral part of description of quantum mechanical particles. For instance, by the spin-statistics relation, all fermionic particles possess a half-integer spin. In particular, the spin cannot be zero, so the above fermionic description is not yet completely adequate for physical fermions.

Spin is a one-particle property, and hence affects the definition of the one-particle Hilbert space $\mathfrak{h}$ above. It is determined by a half-integer value $S \in \mathbb{N}_0/2$, resulting in $2S + 1$ new “internal” degrees of freedom which are labelled by values in $\sigma_S := \{ -S, -S + 1, \ldots, S \}$. There are several equivalent ways of defining the wave vector of a particle with a non-zero spin: one can either think that they are multicomponent wave-vectors, or that each lattice site is augmented with extra degrees of freedom, $\psi(x, \sigma) \in \mathbb{C}^{2S}$, or that each lattice site is augmented with $D$ extra degrees of freedom, $\psi(x, \sigma) \in \mathbb{C}$, $\sigma \in \sigma_S$. These descriptions are quantum mechanically equivalent since the identification

$$\psi(x)_\sigma = \phi(x, \sigma) = \langle e_x \otimes e_\sigma, \phi \rangle$$

provides a mapping $\psi \rightarrow \phi$ which turns out to be a Hilbert space isomorphism between $\otimes_{\sigma \in \sigma_S} L^2(\Lambda)$ and $L^2(\Lambda \times \sigma_S)$. The second equality above also yields an isomorphism, namely the standard one between $L^2(\Lambda \times \sigma_S)$ and $L^2(\Lambda) \otimes L^2(\sigma_S)$.

Hence, most of the discussion in the previous sections holds verbatim if we replace $\omega = \pi k$ and thus depending only on one dispersion relation function which is typically chosen to be nearest neighbour, $\omega(k) = -\sum_{\nu=1}^d \cos(2\pi k_\nu)$. The pair interactions in the Hubbard model are taken to be onsite only,

$$V = \frac{1}{2} \sum_{x \in \Lambda, \sigma, \sigma'} V_{\sigma, \sigma'}(x, \sigma)^* a(x, \sigma')^* a(x, \sigma),$$

and thus depending only on one dispersion relation function which is typically chosen to be nearest neighbour, $\omega(k) = -\sum_{\nu=1}^d \cos(2\pi k_\nu)$. The pair interactions in the Hubbard model are taken to be onsite only,
and, since \(a(x, \sigma)^2 = 0\) and \(V_{\sigma\sigma'} = V_{\sigma\sigma'}(0; L)\), \(\sigma, \sigma' \in \{\pm 1\}\), form a real symmetric \(2 \times 2\) matrix, without loss of generality, we may set \(V_{\sigma\sigma} = 0\) and use \(V_{\sigma+} = V_{\sigma-}\) as the sole real parameter. It is usually included in the definition of the coupling \(\lambda\), and thus the general fermionic spin-\(\frac{1}{2}\) onsite interactions are covered by the interaction
\[
V_{\text{Hubbard}} = \sum_{x \in \Lambda} a(x, +)^* a(x, -)^* a(x, -) a(x, +) . \tag{16}
\]

Let us point out that onsite potentials fall into the class of translation invariant potentials studied in the previous subsection. Namely, they correspond to choosing potentials whose Fourier transforms are constant, \(V_{\sigma\sigma'}(k) = V_{\sigma\sigma'}\), for all \(k \in \mathbb{T}^d\).

The main difficulties compared to deriving the evolution equations in the earlier discussed case are notational. We skip the parts which are similar to the earlier computations, and merely record the outcome in a form which is easy to use in computations involving products of creation and annihilation operators.

We label annihilation operators with an additional label \(\tau = -1\) and creation operators with \(\tau = +1\), and consider their dynamics after Fourier transform of the spatial degrees of freedom. Explicitly, we define
\[
a(k, \sigma, -1, t) := \hat{a}(k, \sigma, t) = \sum_{x \in \Lambda} e^{-i2\pi x \cdot k} a(x, \sigma, t) , \tag{17}
a(k, \sigma, +1, t) := \hat{a}^*(k, \sigma, t) = \sum_{x \in \Lambda} e^{-i2\pi x \cdot k} a^*(x, \sigma, t) . \tag{18}
\]

These operators are connected via operator adjoints, \(\{a(k, \sigma, \tau, t)\}^* = a(-k, \sigma, -\tau, t)\). Since now
\[
\partial_t a(x, \sigma, \tau, t) = -i \sum_{x' \in \Lambda} a_{\sigma\sigma'}(x - x'; L) a(x', \sigma', t)
\]
\[
- i \lambda \sum_{x' \in \Lambda} V_{\sigma\sigma'}(x - x'; L) a^*(x', \sigma', \tau, t) a(x', \sigma', t) a(x, \sigma) ,
\]

the above operators satisfy the following closed evolution equations
\[
\partial_t a(k, \sigma, \tau, t) = i \tau \sum_{\sigma' \in \mathbb{S}} \omega_{\sigma\sigma'}(k; \tau) a(k, \sigma', \tau, t)
\]
\[
+ i \tau \lambda \sum_{\sigma'_{1,2,3} \in \mathbb{S}} \int_{(\mathbb{T}^d)^3} dk_1 dk_2 dk_3 \delta_{\chi}(k - k_1 - k_2 - k_3)
\]
\[
\times \hat{V}_{\sigma, \sigma_{1,2,3}}(k_1, k_2, k_3; \tau) a(k_1, \sigma_{1,1,1}, 1, t) a(k_2, \sigma_{2,2,1}, \tau, t) a(k_3, \sigma_{3,3,1}, -1, t) , \tag{19}
\]

where \(\omega_{\sigma\sigma'}(k; -1) := \omega_{\sigma\sigma'}(k), \omega_{\sigma\sigma'}(k; +1) := \omega_{\sigma\sigma'}(k)\), and
\[
\hat{V}_{\sigma, \sigma_{1,2,3}}(k_1, k_2, k_3; -1) = \{\sigma_1 = \sigma_{2,3} = \sigma\} \hat{V}_{\sigma_{1,2,3}}(k_1 + k_2) ,
\]
\[
\hat{V}_{\sigma, \sigma_{1,2,3}}(k_1, k_2, k_3; +1) = \{\sigma_1 = \sigma_{2,3} = \sigma\} \hat{V}_{\sigma_{1,2,3}}(k_2 + k_3) .
\]

Here we need the above equations only in two special cases. First, if there is no spin, the equation reduces to
\[
\partial_t a(k, \tau, t) = i \tau \omega(k) a(k, \tau, t) + i \tau \lambda \int_{(\mathbb{T}^d)^3} dk_1 dk_2 dk_3 \delta_{\chi}(k - k_1 - k_2 - k_3)
\]
\[
\times \hat{V}(k_1, k_2, k_3; \tau) a(k_1, 1, t) a(k_2, \tau, t) a(k_3, -1, t) , \tag{20}
\]

with \(\hat{V}(k_1 + k_2, k_3; -1) = \hat{V}(k_1 + k_2)\) and \(\hat{V}(k_1, k_2, k_3; 1) = \hat{V}(k_2 + k_3)\). Secondly, for the Hubbard model, the equations can be simplified into
\[
\partial_t a(k, \sigma, \tau, t) = i \tau \omega(k) a(k, \sigma, \tau, t) + i \tau \lambda \int_{(\mathbb{T}^d)^3} dk_1 dk_2 dk_3 \delta_{\chi}(k - k_1 - k_2 - k_3)
\]
\[
\times a(k_1, \tau, 1, t) a(k_2, -\sigma, \tau, t) a(k_3, -\tau \sigma, -1, t) . \tag{21}
\]

The most standard notation for the Hubbard model uses the potential \(U \sum_x n(x, +) n(x, -)\) where \(n(x, \sigma) := a(x, \sigma)^* a(x, \sigma)\). This is seen to be equivalent to the present case after setting \(U = \lambda\) and using the anticommutation relations.
3 States, reduced density matrices, and truncated correlation functions

A state in classical mechanics is a probability measure describing the distribution of positions and velocities of the particles at some fixed time. Thus it can be used to compute the statistics of all observables, i.e., measurable functions of the positions and velocities at that time. In Hamiltonian mechanics, an initial state given at time \( t = 0 \) determines the state at all times \( t \in \mathbb{R} \). Often it is simpler to study the evolution of physical properties of the system by inspecting the evolution starting from some suitably chosen random initial state rather than from a deterministic state with fixed values for the initial positions and velocities of the particles.

A state at time \( t \) in quantum mechanics is defined as a map \( \rho_t \) which associates to each observable \( A \) a number \( \rho_t[A] \) which gives the limiting value for statistical averages of this observable measured in repeated experiments. This is analogous to the expectation value map under the probability measure which defines the state in the classical case. The more precise mathematical definition of a state takes two ingredients: the collection of observables \( \mathcal{A} \), which is assumed to be some subspace of bounded operators, closed under adjoint and containing the identity operator, and a positive linear functional \( \rho : \mathcal{A} \to \mathbb{C} \) of norm 1.

For instance, a Borel probability measure \( \mu \) of wave vectors \( \Psi \in \mathcal{H} \), \( \| \Psi \| = 1 \), generates a state by setting for any bounded operator \( A \) on \( \mathcal{H} \)\n\[
\rho[A] := \int \mu(d\psi) \langle \psi, A\psi \rangle.
\]

Most often a state is determined by giving a trace-class operator \( \rho \) on \( \mathcal{H} \) such that \( \rho \) is positive, \( \text{Tr} \rho = 1 \), and setting \( \rho[A] = \text{Tr}[\rho A] \) for all \( A \in \mathcal{A} \). Such an operator \( \rho \) is called the density matrix of the state (note that we do not make a distinction in the notation between the state and its density matrix). If the Hilbert space is separable, such as our Fock spaces are, then for instance all states given by the above Borel probability measures have a density matrix associated with them.

The \( n \)-th reduced density matrix \( \rho_n \) is an analogous quantity which is obtained from the full density matrix by taking a partial trace over the degrees of freedom which concern particle labels higher than \( n \). The general construction is discussed in [1] Section 6.3.3 and in [10] Section 3, but there is a more direct definition available for the present system of lattice fermions: Given a state \( \rho \) on the fermionic Fock space, we first define
\[
\rho_n(z_1, z_1', \ldots, z_n, z_n') := \rho[a^*(z_1) \cdots a^*(z_n) a(z_1) \cdots a(z_n)].
\]

Here each \( z_i \) and \( z_i' \) belongs to the one-particle label set, i.e., \( z_i \in \Lambda \) in the spinless case and \( z_i \in \Lambda \times \sigma_S \) for spin-\( S \) particles. The collection of these complex numbers defines the reduced density matrix \( \rho_n \), which is a positive operator on \( \mathbb{H}^\otimes n \), via the formula
\[
\langle \otimes_i z_i, \rho_n(\otimes_i z_i') \rangle = \rho_n(z_1, z_1', \ldots, z_n, z_n').
\]

In quantum mechanics, given an initial density matrix \( \rho(0) = \rho \), the expectation of a time-evolved observable \( A(t) = U_t^* A U_t \) satisfies
\[
\rho[A(t)] = \text{Tr}[\rho U_t^* A U_t] = \text{Tr}[U_t \rho U_t^* A],
\]
by cyclicity of trace. Hence, we define the time-evolved density matrix \( \rho(t) := U_t \rho U_t^* \) for which \( \rho(t)[A] = \rho[A(t)] \). The reduced time-evolved density matrices may thus be obtained as expectations of time-evolved creation and annihilation operators: by replacing each \( a(z) \) in \( \rho[A(t)] \) by \( a(z, t) = U_t^* a(z) U_t \), we obtain the reduced density matrix \( \rho(t)_n \).

Considering the earlier observation that time-evolved annihilation operators suffice to determine the time-evolution of wave vectors, it is not surprising that reduced density matrices play an important role in the physics of quantum fluids. For instance, the expectation of the hopping Hamiltonian \( H_0 \) may be computed from \( \rho(t) \), by the formula
\[
\rho(t)[H_0] = \sum_{x, y \in \Lambda} \sum_{\sigma, \sigma' \in \sigma} \alpha_{\sigma \sigma'}(x - y; L) \rho(t)_1((x, \sigma), (y, \sigma')).
\]
Indeed, for kinetic theory, the central goal is to describe the evolution of \( \rho(t) \), a positive operator on \( \mathfrak{h} \), in the limit of weak coupling.

In fact, there is a class of fermionic states, called \textit{quasifree states}, for which \( \rho \) uniquely determines all other reduced density matrices: if \( \rho \) is quasifree, then for all \( n \geq 1 \) the corresponding density matrix is given as a determinant of an \( n \times n \) matrix,

\[
\rho_n(z_1, z_1', \ldots, z_n, z_n') = \det(\rho(1(z, z'))),_{i,j=1,\ldots,n}.
\]

To simplify analysis of states which are not quasifree but close to such, one can introduce \textit{truncated correlation functions} \( \rho^T \) which are analogous to cumulants of random variables in classical probability theory. The construction below applies to a state \( \rho \) on a fermionic system which is \textit{even}: it is assumed that an expectation of any observable remains invariant if we change \( a(z) \) to \( -a(z) \) for all \( z \). As explained in more detail in [1] pp. 42–43, given an even state \( \rho \) to each even length sequence \( (a_1, a_2, \ldots, a_m) \) of creation and annihilation operators one may associate a truncated expectation \( \rho^T[a_1, a_2, \ldots, a_m] \) such that the expectation of any product of even length can be expressed as a sum over partitions. Explicitly,

\[
\rho[a^T] = \sum_{\Pi \in \mathcal{P}_2(I)} \varepsilon(\Pi) \prod_{S \in \Pi} \rho^T[a_S],
\]

(23)

where \( I = (1, 2, \ldots, n) \), \( a^T := a_1 \cdots a_n \), \( \mathcal{P}_2(I) \) denotes the collection of partitions of \( I \) into even length subsequences, \( \varepsilon(\Pi) \) is the sign of the permutation which takes \( I \) to \( \Pi = (S_1, \ldots, S_m) \), and for a subsequence \( S = (s_1, \ldots, s_m) \) of \( I \) we have used the shorthand notation \( a_S = (a_{s_1}, \ldots, a_{s_m}) \). Note that odd sequences for even states have always zero expectation, so this is the antisymmetrised analogue of the moments-to-cumulants formula of probability.

The above definition requires careful consideration of the signs of each term. The following identity can also serve as a basis for a recursive definition of the truncated expectations,

\[
\rho[a^T] = \sum_{m \in S \subseteq I} \varepsilon(S, I \setminus S) \rho^T[a_S] \rho[a^{T \setminus S}],
\]

(24)

where \( m \in I \) is any fixed label and \( \varepsilon(S, I \setminus S) \) is the sign of the permutation \( I \to (S, I \setminus S) \). (Note that all terms where \( S \) has an odd length are zero in the sum, since then also \( I \setminus S \) is odd, so we could have restricted the sum to even subsequences here.) For instance, \( \rho^T[a_1, a_2] = \rho[a_1, a_2] \), and for \( n = 4 \) we have

\[
\rho[a_1 a_2 a_3 a_4] = \rho^T[a_1, a_2, a_3, a_4] + \rho^T[a_1, a_2] \rho[a_3 a_4] - \rho^T[a_1, a_3] \rho[a_2 a_4] + \rho^T[a_1, a_4] \rho[a_2 a_3],
\]

and thus

\[
\rho^T[a_1, a_2, a_3, a_4] := \rho[a_1 a_2 a_3 a_4] - \rho[a_1 a_2] \rho[a_3 a_4] + \rho[a_1 a_3] \rho[a_2 a_4] - \rho[a_1 a_4] \rho[a_2 a_3],
\]

and, in accordance with (23), also

\[
\rho[a_1 a_2 a_3 a_4] = \rho^T[a_1, a_2, a_3, a_4] + \rho^T[a_1, a_2] \rho^T[a_3, a_4] - \rho^T[a_1, a_3] \rho^T[a_2, a_4] + \rho^T[a_1, a_4] \rho^T[a_2, a_3].
\]

(25)

The truncated correlation functions can be used to characterise quasifree states: an even state \( \rho \) is quasifree if and only if \( \rho^T[a_1, a_2, \ldots, a_n] = 0 \) for all \( n > 2 \). This is completely analogous with characterisation of Gaussian measures by vanishing of their higher order cumulants. Even for states which are not quasifree, the truncated correlation functions enjoy properties which are typically not valid for direct expectations:

1. If \( n > 2 \), then \( \rho^T[a_1, a_2, \ldots, a_n] \) is completely antisymmetric with respect to permutation of its arguments: if \( \pi \in S_n \), we have \( \rho^T[a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(n)}] = (-1)^{\pi(a_{1}, a_{2}, \ldots, a_{n})} \rho^T[a_1, a_2, \ldots, a_n] \).

   (For a proof, consider a basic odd permutation which swaps two neighbouring labels \( m \) and \( m' \), and then use (23) and the anticommutation relations.)
2. If \( \rho \) is an equilibrium Gibbs state at sufficiently small activity and corresponding to a short range interaction, all reduced density matrices are typically decaying summably in the separation of their spatial arguments. For a precise statement and assumptions under which this result holds, see [1] Theorem 6.3.21, and further discussion can be found in [12]. In particular, keeping one of the sites fixed, Fourier transforms of the reduced density matrices are typically uniformly bounded in the lattice size \( L \), unlike those of the corresponding expectations.

4 Weak coupling limit and quantum kinetic theory

For kinetic theory, we are interested in the evolution of the first truncated reduced density matrix \( \rho_1(x', \sigma', x, \sigma; t) = \rho^{\sigma'}[a^{\sigma'}(x', t), a(x, \sigma; t)] \). There is no difference between the truncated and direct reduced density matrices for the first reduced density matrix of an even state of fermions but for higher order density matrices there is a difference in their properties. Most notably, for systems which which are eventually well approximated by Gibbs states of the type discussed in item 2 at the end of Section 3, one would expect the truncated correlation functions to decay in the distance. Then, Fourier transforms in these variables are given by “nice” functions, for instance, uniformly bounded in the lattice size or with a uniformly bounded \( L^2(\mathbb{R}^d) \)-norm. In contrast, the Fourier transform of the corresponding moments would be a fairly complicated sum over “\( \delta_{\Lambda} \)-distributions”.

Here we consider only initial data which are both gauge invariant and translation invariant. The first condition means that the initial data does not contain correlations between different particle sectors, and this property is preserved by the present type of evolution. It simplifies the resulting analysis since for gauge invariant states all moments, which do not have the same number of creation and annihilation operators, are zero. For instance, then \( \rho[a(y, \sigma') t] a(x, \sigma; t)] = 0 = \rho[a^{\sigma'}(y, \sigma'; t) a^{\sigma}(x, \sigma; t)] \).

For translation invariance, we require that all moments are invariant under periodic spatial translations of the lattice \( \Lambda \). For the present translation invariant \( H_0 \) and \( V \) also this property is preserved by the time-evolution. As a consequence, any one of the spatial arguments of the correlation functions can be translated to the origin. In particular, there is a function \( F : \mathbb{C}^{2 \times 2} \rightarrow \mathbb{C}^{2 \times 2} \) for which
\[
\rho_1(x', \sigma', x, \sigma; t) = F_{\sigma'\sigma}(x' - x, t).
\]
The Wigner function is defined as the discrete Fourier transform of \( F \),
\[
W_{\sigma'\sigma}(k, t) := \sum_{y \in \Lambda} e^{-i2\pi y \cdot k} F_{\sigma'\sigma}(y, t) = \int_{\Lambda^*} dk' \rho[a(k, \sigma', 1, t) a(k', \sigma, -1, t)].
\] (26)

Using the properties of adjoints, it is straightforward to check that the so defined \( \sigma_S \times \sigma_S \) matrix \( W(k, t) \) is always Hermitian. In addition, translation invariance may be invoked to prove that
\[
\rho[a(k, \sigma', 1, t) a(k', \sigma, -1, t)] = W_{\sigma'\sigma}(k, t) \delta_{\chi}(k + k') .
\] (27)

We also introduce the related notation \( \tilde{W} \) for the corresponding expectation where the order of the operators has been swapped. More precisely, we define as matrices
\[
\tilde{W}(k, t) := 1 - W(k, t),
\] (28)
where \( 1 \) denotes the diagonal unit matrix. By the anticommutation relations, then
\[
\rho[a(k', \sigma, -1, t) a(k, \sigma', 1, t)] = \tilde{W}_{\sigma'\sigma}(k, t) \delta_{\chi}(k + k').
\] (29)

The quantum kinetic equation will concern the time-evolution of the above Hermitian matrix-valued Wigner functions. There are a number of differences in the computations depending on whether there are spin-interactions present or not, and we have split the discussion accordingly below.
4.1 Fermionic Boltzmann–Nordheim equation

We begin with a case in which the spin-degrees of freedom evolve independently. As mentioned above, this case can be handled ignoring the spin degrees of freedom and thus we can use the spinless results and notations. We adapt here the method introduced in [8] for derivation of a phonon Boltzmann equation for the weakly nonlinear discrete Schrödinger equation from the evolution hierarchy of truncated correlation functions. For comparison, a derivation of the Boltzmann–Nordheim equation using direct perturbation expansions of moments and their graph representations can be found in [10].

It should be stressed that neither method currently produces a mathematically rigorous derivation of fermionic kinetic theory. In particular, it is not yet known which precise assumptions are needed for the kinetic approximation to work nor are there any rigorous bounds for the accuracy of the approximation. From the point of view of the truncated correlation function hierarchy, the key missing ingredient is a control of the evolution of decay properties of correlation functions. Here we do not go into any detail about the role played by the terms ignored in the derivations below but more details about why their effects are in general expected to be lower order in the weak coupling limit \( \lambda \to 0 \) can be found in [8][10].

Let us also point out one case in which rigorous control has been possible: in [11], the kinetic scaling limit of time-correlations of equilibrium distributed fields with discrete nonlinear Schrödinger evolution are proven to follow the above scenario. In this case, the state itself is stationary and the good decay properties of the truncated correlation functions are provided by the initial data which can be studied with methods from equilibrium statistical mechanics.

Differentiating (20) and recalling the adjoint relations yields the following representation for the time derivative of the Wigner function of translation invariant states

\[
\begin{align*}
\partial_t W(k,t) &= 2 \text{Re} \left( \int_{\Lambda^s} dk' \rho(\partial a(k,1,t) a(k',-1,t)) \right).
\end{align*}
\]

We use (20) to compute the derivative, yielding

\[
\begin{align*}
\int_{\Lambda^s} dk' \rho(\partial a(k,1,t) a(k',-1,t)) &= \omega(k) \int_{\Lambda^s} dk' \rho[a(k,1,t) a(k',-1,t)] \\
&+ i\lambda \int_{(\Lambda^s)^4} dk_1 dk_2 dk_3 dk_4 \hat{V}(k_2+k_3) \delta_\lambda(k-k_1-k_2-k_3) \\
&\times \rho[a(k_1,1,t) a(k_2,1,t) a(k_3,-1,t) a(k_4,-1,t)].
\end{align*}
\]

The first term on the right is purely imaginary and does not contribute to the real part. In the second term, the expectation is antisymmetric with respect to the swap \( k_1 \leftrightarrow k_2 \), and thus we can conclude that

\[
\partial_t W(k,t) = \text{Re} \left[ i\lambda \int_{(\Lambda^s)^4} dk_1 dk_2 dk_3 dk_4 \left( \hat{V}(k_2+k_3) - \hat{V}(k_1+k_3) \right) \\
\times \delta_\lambda(k-k_1-k_2-k_3) \rho[a(k_1,1,t) a(k_2,1,t) a(k_3,-1,t) a(k_4,-1,t)] \right].
\]

We represent the remaining expectation in terms of truncated expectations using (20). Since \( \hat{V} \) is real, all terms involving second order truncated correlation functions produce terms which are purely imaginary and, hence, they do not contribute to the derivative of the Wigner function. Therefore,

\[
\begin{align*}
\partial_t W(k,t) &= \text{Re} \left[ i\lambda \int_{(\Lambda^s)^4} dk_1 dk_2 dk_3 dk_4 \left( \hat{V}(k_2+k_3) - \hat{V}(k_1+k_3) \right) \\
&\times \delta_\lambda(k-k_1-k_2-k_3) \rho^T[a(k_1,1,t), a(k_2,1,t), a(k_3,-1,t), a(k_4,-1,t)] \right].
\end{align*}
\]
Computation of derivatives of higher order truncated correlation functions would be simplified by introducing the associated Wick polynomials, as was observed in \cite{8} for commuting fields. However, it is still possible to work out the necessary combinatorics and cancellations by hand for the fourth order terms which are needed to compute the collision operator of kinetic theory. Namely, after a somewhat lengthy computation employing the symmetry of \( \lambda \) subleading in \( \lambda \) at the kinetic time scales \( t \propto \lambda^{-2} \), due to the “integrals” over the oscillatory phase factors. The remaining terms yield the approximation

\[
W(\hat{k}, t) = \frac{e^{-i(t(2\omega + 2\omega) + \omega) - \omega)}{p} + a(k_1, 1, t), a(k_2, 1, t), a(k_3, -1, t), a(k_4, -1, t)} = i\lambda e^{-i(t(2\omega + 2\omega) + \omega) - \omega)}{\delta(\lambda)(k_1 + k_2 + k_3 + k_4) (\hat{V}(k_2 + k_3) - \hat{V}(k_1 + k_3)}
\times \left[ \hat{W}(k_2)W(-k_3)W(-k_4) - W(k_1)W(-k_3)W(-k_4) + W(k_1)W(k_2)W(-k_4) - W(k_1)W(k_2)\hat{W}(-k_3) \right]
+ \text{(higher order truncated functions)},
\] (35)

where we have introduced the shorthand notations \( \omega_i := \omega(k_i), \hat{W} = 1 - W \), and each \( W \) and \( \hat{W} \) factor is evaluated at \( t \).

We then integrate the above time-derivatives from 0 to \( t \). The terms involving higher order truncated functions (4th and 6th in \( \delta \)), as well as the substitution term involving the 4th order truncated correlation at time 0, are expected to contribute only terms which are subleading in \( \lambda \) at the kinetic time scales \( t \propto \lambda^{-2} \). Hence, integration over \( k_4 \) is straightforward and swapping the sign of \( k_3 \), the order of time-integrals, and denoting \( W_i := W(k_i, s) \) and \( \hat{W} := 1 - W \), we arrive at the approximation

\[
W(k_0, t) - W(k_0, 0) \approx \lambda^2 \int_0^t ds \int \frac{d\lambda}{(\delta^2)} d\lambda d\lambda d\lambda d\lambda __{s} \omega_0 - \omega_1 - \omega_0 \)\)
\times \left( \hat{V}(k_2 - k_3) - \hat{V}(k_1 - k_3) \right)^2 \delta(\lambda_0 - k_1 - k_2 - k_3) \times \left[ -W_2 W_3 W_0 + W_1 W_3 W_0 - W_1 W_3 W_0 + W_1 W_2 W_3 \right],
\] (37)

The real part of the remaining oscillatory time-integral formally convergences to \( \pi \delta(\omega_0 - \omega_3 - \omega_1 - \omega_2) \) as \( t \to \infty \). In fact, the \( \delta \)-function approximation should only be used after the thermodynamic limit \( L \to \infty \) has been taken; for a finite lattice, also values for which \( \omega_1 + \omega_2 - \omega_3 - \omega_0 \) is not exactly zero but close enough to zero (e.g., \( o(L^{-2}) \)), will contribute to the collision term. Assuming that the thermodynamic limit of the function \( W \) exists and using the same notation for the limit, we obtain

\[
W(k_0, t) - W(k_0, 0) \approx \int_0^t ds \tilde{C}_{BN}[W(\ell, s)](k_0), \] (38)

where a relabelling \( k_1 \leftrightarrow k_3 \) yields the following more standard form of a fermionic Boltzmann–Nordheim collision operator

\[
\tilde{C}_{BN}[W](k_0) := \pi \lambda^2 \int_{(\gamma^2)} \frac{d\lambda d\lambda d\lambda d\lambda}{(\delta^2)} d\lambda d\lambda d\lambda d\lambda __{s} \omega_0 - \omega_1 - \omega_2 - \omega_3 \)
\times \left( \hat{V}(k_1 - k_2) - \hat{V}(k_1 - k_3) \right)^2 \delta(\lambda_0 - k_1 - k_2 - k_3) \times \left[ \hat{W}_1 W_2 W_3 - W_0 W_2 W_3 - W_0 W_3 W_0 + W_0 W_2 W_3 \right].
\] (39)
The kinetic equation obtained by replacing the approximation sign in (38) by an equals sign is called the (spatially homogeneous) fermionic Boltzmann–Nordheim equation. The term in square brackets in (39) is then usually written in a more symmetric form as
\[ W_0 W_1 W_2 W_3 - W_0 W_1 W_2 W_3. \]
However, it should be noted that, since the highest order terms indeed cancel, the collision operator has a nonlinearity of third order, not of fourth order.

The above lattice kinetic theories have two conserved quantities, \( \int dk \omega(k) W(k, t) \) related to energy and \( \int dk W(k, t) \) related to particle density. The mathematical properties of their solutions have mainly been studied in the continuum case for which instead of the lattice wave potential but includes spin-interactions. By (30),
\[ \Sigma \text{ is time-independent. Therefore, the dominant term in the time-derivative } (40), \]

It depends on the expectation
\[ \Sigma_{\sigma} : = \int_{X^*} dk' W_{-\sigma}(k') = \rho(a^*(0, \sigma) a(0, \sigma)] = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \rho(a^*(x, \sigma) a(x, \sigma)] , \]
i.e., on the spin correlation matrix. These expectations are conserved by the time evolution of the Hubbard model, and hence the matrix \( \Sigma_{\sigma} \) is time-independent. Therefore, the dominant term in the time-derivative (40) is given by
\[ i\lambda (W_{\sigma}(k) \Sigma_{-\sigma}, -\sigma; -\sigma) - W_{-\sigma}(k) \Sigma_{\sigma}, -\sigma; -\sigma - W_{\sigma}(k) \Sigma_{-\sigma}, -\sigma + W_{-\sigma}(k) \Sigma_{\sigma}, -\sigma) , \]
which is most conveniently written as the \( \langle \sigma', \sigma \rangle \)-component of the commutator
\[ -i\lambda [\Sigma, W(k, t)] . \]

New terms arise also in the computation of the second order term in \( \lambda \). The computations are in principle completely analogous to those in the previous subsection but one has to carefully consider the propagation of the spin variable. After taking the thermodynamic limit \( L \to \infty \) and neglecting terms which are expected to be higher order in \( \lambda \), new features compared to the spinless case arise. Most importantly, since one takes a Hermitian, not
is physically related to the Pauli exclusion principle and it can be checked to follow from
\[ \int_0^\infty d\epsilon e^{i\omega\epsilon} = \pi\delta(\omega) + i\text{P.V.} \frac{1}{\omega}, \]
where “P.V.” denotes a Cauchy principal value when integrating over the real variable \( \omega \). The terms arising from the imaginary part do not resemble usual collision integrals. Instead, they combine into conservative Vlasov-type terms, similarly to what occurred above for the lowest order contribution.

The final evolution equation is most conveniently written as an evolution equation for the Hermitian \( 2 \times 2 \)-matrix \( W(k,t) \), \( k \in \mathbb{T}^d \). It reads
\[
\partial_t W(k,t) = C_{\text{Hubb}}[W(\cdot,t)](k) - i \left[ H_{\text{eff}}[W(\cdot,t)](k), W(k,t) \right], \quad (45)
\]
where the collision operator may be written as
\[
C_{\text{Hubb}}(k) := \lambda^2 \int_{(\mathbb{T}^d)^3} dk_1 dk_2 dk_3 \delta(k_0 + k_1 - k_2 - k_3) \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3) \times (W_0 W_2 J[W_1 W_3] + J[W_0 W_1] W_2 W_3 - W_0 W_2 J[W_1 W_3] - J[W_0 W_1] W_2 W_3) \quad (46)
\]
using the matrix operation \( J[A] := 1 \text{Tr} \ A - A \in \mathbb{C}^{2 \times 2} \). The “effective Hamiltonian” in the matrix commutator term is given by
\[
H_{\text{eff}}[W](k_0) := \lambda \Sigma + \lambda^2 \text{P.V.} \int_{(\mathbb{T}^d)^3} dk_1 dk_2 dk_3 \delta(k_0 + k_1 - k_2 - k_3) \times \frac{1}{\omega_0 + \omega_1 - \omega_2 - \omega_3} \left( W_2 J[W_1 W_3] + W_2 J[W_1 W_3] \right). \quad (47)
\]

Also the Hubbard–Boltzmann equation (45) can be derived using direct perturbation expansions and their graph representations, as has been done in [3] for more general spin-interaction potentials and with a slightly different splitting between the terms in \( H_0 \) and \( V \) operators. Neither of these derivations provides rigorous estimates of how accurately the solutions to the Hubbard–Boltzmann equation describe the original fermionic reduced density matrices. The principal value integral, in particular, is somewhat troublesome from a mathematical point of view.

The precise mathematical meaning of the terms appearing in the Hubbard–Boltzmann equation (45), as well as the existence and uniqueness of its solutions for physically relevant initial data, have been studied in [10]. It is shown there that for the nearest neighbour Hubbard model with a sufficiently high dimension, \( d \geq 3 \), any Lebesgue measurable initial data \( W_0(k) \) satisfying the matrix constraint \( 0 \leq W_0(k) \leq 1 \) allows a global solution to (45) which is also unique among solutions satisfying the constraint \( 0 \leq W(k,t) \leq 1 \). (The constraint is physically related to the Pauli exclusion principle and it can be checked to follow from the earlier mentioned properties of the fermionic creation and annihilation operators.) This solution is also proven to conserve energy and total spin. More precisely, the real observable \( \int dk \omega(k) \text{Tr} W(k,t) \) and the matrix observable \( \int dk W(k,t) \) are constants along the solutions. Together these properties show that the approximations leading to the Hubbard–Boltzmann equation are consistent, and the resulting kinetic equation should have range of validity similar to the more standard kinetic theories such as the Boltzmann–Nordheim equation derived earlier.

5 Thermalization in spatially homogeneous kinetic theory

For ergodic systems, time averages of observables will converge to ensemble averages when the averaging period is taken to infinity. In fact, the ensembles covered by such limits could be identified with thermal equilibrium states of the system. However, for system with local conservation laws the approach to global equilibrium typically takes a very long time, often
diverging when the system size is increased: for instance, for systems with normal heat conductivity heat relaxation occurs diffusively and thus involves time-scales of order $L^2$ for systems of spatial diameter $L$.

For physical transport phenomena one is interested in the state of the system at mesoscopic timescales, i.e., times which are long in microscopic units but short on the macroscopic scale. If the system has only short range interactions, even though its state could not yet be well approximated by the global equilibrium state, often time-averages of observables local to a point in space can be ever better approximated by one of the equilibrium states. This allows describing the evolution of the state of the system by first parametrizing its equilibrium states and then inspecting the evolution of these parameters. A common example would be introduction of space-time dependent temperature function related to the temperature parameter of the canonical Gibbs state for those systems where total energy is conserved by the evolution.

Systems, which have the above local approximation property, are said to be in local thermal equilibrium, and thermalization refers to the approach to one of the local thermal equilibrium states from the given initial state. The thermalization time, i.e., the time it takes for local thermal equilibrium states to become good approximations, is typically mesoscopic, not macroscopic.

In fact, kinetic theory provides a method of estimating the thermalization process and times. We focus here on thermalization of spatially homogeneous states. This simplifies the analysis since the slow processes associated with spatial relaxation of the equilibrium parameters are then absent. As explained below, kinetic theory indicates that the Wigner function relaxes to stationary states labelled by a few parameters and hence one would expect local equilibrium or quasi-equilibrium to be reached already at kinetic timescales proportional to $\lambda^{-2}$. The key to these properties is finding an entropy functional satisfying an H-theorem for the appropriate kinetic evolution. The vanishing of entropy production restricts the functional form of stationary solutions and allows their explicit parametrisation.

5.1 Thermalization without spin-interactions

The entropy functional associated with the spatially homogeneous fermionic Boltzmann–Nordheim equation,

$$\partial_t W(k,t) = \mathcal{C}_{\text{BN}}[W(k,t)](k),$$

where the collision operator is defined in (39), is given by

$$S[W] := - \int_{\mathbb{R}^d} dk \left( W(k) \log W(k) + \tilde{W}(k) \log \tilde{W}(k) \right).$$

(48)

Computing the time-derivative, one obtains

$$\frac{d}{dt} S[W(t)] = \sigma[W(t)],$$

where the entropy production functional is

$$\sigma[W] = \pi \int_{(\mathbb{R}^d)^4} dk_1 dk_2 dk_3 dk_4 \delta(k_1 + k_2 - k_3 - k_4) \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)$$

$$\times \left( \tilde{V}(k_2 - k_3) - \tilde{V}(k_2 - k_4) \right)^2 G(W_1 W_2 W_3 W_4, W_1 W_2 \tilde{W}_3 \tilde{W}_4),$$

(49)

with $G(x, y) = (x - y) \ln(x/y)$. Since $\sigma[W] \geq 0$ for physical Wigner functions with $W, \tilde{W} \geq 0$, this proves that $S$ satisfies an analogue of the H-theorem of classical rarefied gas Boltzmann equation.

In particular, any stationary solution to the kinetic equation needs to satisfy $\sigma[W^{(eq)}] = 0$. For sufficiently non-degenerate $\tilde{V}$ and $\omega$, the only regular solutions to this equation are given by the two-parameter family

$$W^{(eq)}_{\beta, \mu}(k) = (e^{(\omega(\lambda^2 - k)) - \mu})^{-1},$$

(50)

where the values of the parameters $\beta, \mu \in \mathbb{R}$ could also be fixed by giving the values for the conserved energy and particle density observables. These Wigner functions can also
be obtained by considering the one-particle reduced density matrix of the standard grand canonical Fermi–Dirac states after setting λ = 0, cf. [1] Proposition 5.2.23. These states are gauge invariant and quasifree and thus the Wigner function determines all other reduced density matrices.

It is clear that \( \hat{V}(k) \) cannot be a constant since then \( \mathcal{C}_{\text{FSE}}[W] = 0 \), but otherwise the function \( \hat{V} \) can be fairly arbitrary for this result to hold; one merely needs that the difference \( \hat{V}(k_2 - k_3) - \hat{V}(k_2 - k_4) \) is nonzero almost everywhere on the manifold defined by the two \( \delta \)-constraints. The conditions on the dispersion relation \( \omega \) are more intricate but in two and higher dimensions quite generally the above solutions should be the only stationary ones, see [10, Appendix B.1] and [13] for detailed conditions and more discussion on the topic.

In case \( \hat{V} \) and \( \omega \) are such that the only stationary solutions are given by (eq), one expects that for any regular initial data the solution of the fermionic Boltzmann–Nordheim equation converges as \( t \to \infty \) to the unique function \( W_{\beta, \mu}^{(eq)} \) where \( \beta, \mu \in \mathbb{R} \) are determined by the initial energy and particle number. Unlike for the corresponding bosonic equation, the solutions cannot diverge since they satisfy \( 0 \leq W \leq 1 \) at all times. Thus the space of regular stationary solutions should suffice to cover all asymptotic limits of the solutions. The convergence to a regular stationary solution has been proven for certain continuum models and initial data in [7].

The above results suggest that thermalization timescale for weakly interacting spinless lattice fermions is in great generality given by the kinetic timescale, \( t \propto \lambda^{-2} \). It is also consistent with the hypotheses that, apart from special degenerate interactions, the only equilibrium parameters are related to the conservation of energy and particle number. More precisely, one can use \( \beta \) and \( \mu \) of the standard grand canonical Fermi–Dirac states on the fermionic Fock space as parameters.

5.2 Thermalization in the Hubbard model

The spin-structure of the Hubbard–Boltzmann equation (45) leads to some new phenomena compared to the above spinless Boltzmann–Nordheim case. The entropy functional needs to be generalised to

\[
S[W] := - \int dk \left( \text{Tr} \, W \ln W + \text{Tr} \, \hat{W} \ln \hat{W} \right),
\]

where \( W \) is a \( 2 \times 2 \) Hermitian matrix. Computing its derivative requires some effort, yielding

\[
\frac{d}{dt} S[W(t)] = \sigma[W(t)],
\]

where the entropy production functional is again positive, \( \sigma[W] \geq 0 \). To write down the entropy production, let us first diagonalize the matrices \( W(k) \), yielding an eigensystem \( (\lambda_a(k), \psi_a(k)) \), \( a = 1, 2 \), for each \( k \in \mathbb{T}^d \). Then

\[
\sigma[W](k) := \frac{\pi}{4} \int d^4k \delta(k_1 + k_2 - k_3 - k_4) \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \sum_{a \in \{1, 2\}} \left( \lambda_1 \lambda_2 \lambda_3 \lambda_4 - \lambda_1 \lambda_2 \lambda_3 \lambda_4 \right) \ln \frac{\lambda_1 \lambda_2 \lambda_3 \lambda_4}{\lambda_1 \lambda_2 \lambda_3 \lambda_4} |\langle \psi_1, \psi_3 \rangle \langle \psi_2, \psi_4 \rangle - \langle \psi_1, \psi_4 \rangle \langle \psi_2, \psi_3 \rangle|^2
\]

where \( \psi_i := \psi_{a_i}(k_i), \lambda_a := \lambda_a(k_i) \) and \( \lambda := 1 - \lambda \).

The solution of the condition \( \sigma[W] = 0 \) is no longer quite as straightforward as before, and one has to consider a few degenerate cases separately. However, if \( d \geq 2 \), the non-degeneracy conditions mentioned earlier are satisfied for the nearest neighbour interaction of the Hubbard model, and thus the analysis of the two \( \delta \)-constraints is simplified. As derived in [11], then one of the following possibilities needs to be realized by physical stationary solutions \( W^{(eq)}(k) \) which are Hermitian matrices satisfying \( 0 \leq W(k) \leq 1 \) for every \( k \in \mathbb{T}^d \). First, choose a spin-basis such that the total spin-correlation matrix \( \Sigma \) is diagonal. Then one of the following cases holds:
1. There are grand canonical parameters $\beta, \mu_+, \mu_-$, fixed by the diagonal matrix $\Sigma$ and the energy, such that

$$W^{(\infty)}(k) = \begin{pmatrix} g_+(k) & 0 \\ 0 & g_-(k) \end{pmatrix},$$

where $g_\pm(k) := (1 + e^{\beta(\omega(k) - \mu_\pm)})^{-1}$ are standard Fermi–Dirac distributions.

2. One of the bands is empty and the other is arbitrary: there is a function $f(k)$ with $0 \leq f(k) \leq 1$ and $\sigma \in \{\pm 1\}$ such that $W_{\sigma\sigma}(k) = f(k)$ and all other elements of $W(k)$ are zero.

3. One of the bands is full and the other is arbitrary: there is a function $f(k)$ with $0 \leq f(k) \leq 1$ and $\sigma \in \{\pm 1\}$ such that $W_{\sigma\sigma}(k) = f(k)$, $W_{-\sigma,-\sigma}(k) = 1$, and all off-diagonal elements of $W(k)$ are zero.

These solutions are expected to behave differently when occurring as asymptotic stationary states in the Hubbard model. If the initial data is such that both bands are partially filled, i.e., if one can find $\beta, \mu_+, \mu_-$ and a unitary matrix $U$ such that the function $W^{(\infty)}$ in (52) satisfies $\int dk U^\dagger W(k, t) U = \int dk W^{(\infty)}(k)$ and $\int dk \omega(k) Tr W(k, t) = \int dk \omega(k) Tr W^{(\infty)}(k)$ initially, and hence for all $t$, then one expects $W(k, t) \to UW^{(\infty)}(k)U^\dagger$ as $t \to \infty$.

However, if one of the bands is either empty or fully then, no thermalization can be expected. In fact, this property is not only an artefact of the kinetic theory but it can also be realised in the original Hubbard model. Consider an initial wave vector for which there are no particles with $-\sigma$-spin. Then the pair-interaction $V$ acting on the vector produces zero and, since the free Hamiltonian does not mix the two bands, one can check that Hubbard model evolution equations are satisfied by the solution of the free evolution generated by $H_0$.

The free semigroup leaves for instance all quasifree states invariant and one can choose the Wigner function of the $+\sigma$-component arbitrarily.

The above situation is radically changed if $d = 1$. This case is known to be integrable, see [4] for a review of the one-dimensional Hubbard model, and the large number of conserved quantities is reflected also in the kinetic evolution. As shown in [6], in this case one may take in the stationary solutions in item 1 above instead of the standard Fermi–Dirac distributions $g_\pm$ any functions which are of the form $(1 + e^{\beta(f(k) - \mu_\pm)})^{-1}$ for some real periodic function $f$ which satisfies the antisymmetry condition $f(\frac{1}{2} - k) = -f(k)$. Hence, one needs infinitely many parameters to describe the stationary solutions. The various scenarios for the convergence towards a steady state are explored numerically in [3]. There it is also observed that adding a next-to-nearest neighbour term to the free evolution appears to lift the degeneracy, leaving only the standard Fermi–Dirac distributions as possible limits, similarly to what was stated above for the cases with $d \geq 2$.

6 Concluding Remarks

Reliable study of large scale evolution of a system of weakly interacting fermions is a challenge both to numerical simulations and to theoretical analysis. We advocate here using kinetic equations not only to reproduce standard folklore results, such as convergence towards Fermi–Dirac distribution, but as a tool for systematic study of the approach to equilibrium and thermalization in these systems. Even lacking complete mathematical control over the accuracy and applicability of the kinetic approximation, analysis of kinetic equations can provide testable predictions and reveal possible sources of “anomalies” and other degeneracies. For instance, the role of the dispersion relation and dimensionality in the Hubbard model revealed in the above references encourages such studies in other models.

The almost unreasonable usefulness of kinetic theory begs for better understanding of its underpinnings, in particular, of what is the most accurate connection between the microscopic evolution and the kinetic theory and what are the most appropriate kinetic equations for this purpose. These questions lie in the realm of mathematically rigorous study of scaling limits producing observables which exactly follow some kinetic equation. However, ultimately the goal should be in also extracting practical information about the error in such approximations and how well the approximations extend beyond their apparent regions of applicability, as dictated by the convergence of the scaling limits.
For instance, finding answers to the following open questions could benefit from mathematically rigorous approaches:

1. For which initial data does the corresponding solution to the kinetic equation converge towards the stationary solution determined by the values of the conserved quantities? Could one estimate the rate of convergence?

2. How would the kinetic equations and their solutions change for general spin-interactions, including also interactions with external magnetic fields?

3. If the initial state of the system is not spatially homogeneous, when does its evolution follow an inhomogeneous Boltzmann equation? Are there ways of improving the accuracy of the model, for instance, by including a Vlasov-Poisson-type correction?

4. Could one improve the accuracy of the kinetic equation by “renormalizing” the microscopic observables? How much?

Acknowledgements

I am most grateful to Herbert Spohn for our collaboration and many discussions about validity and properties of kinetic theory. Most of the results here are based on his works and on our joint collaborations. The related research has been made possible by support from the Academy of Finland and also partially supported by the French Ministry of Education through the grant ANR (EDNHS).

References

[1] Bratteli, O., Robinson, D.W.: Operator Algebras and Quantum Statistical Mechanics II. Springer, New York (1981)

[2] Dolbeault, J.: Kinetic models and quantum effects: A modified Boltzmann equation for Fermi-Dirac particles. Arch. Ration. Mech. Anal. **127**, 101–131 (1994)

[3] Escobedo, M., Mischler, S., Valle, M.A.: Homogeneous Boltzmann equation in quantum relativistic kinetic theory. Electron. J. Diff. Eqns. Monograph **04**, 1–85 (2003)

[4] Essler, F.H.L., Frahm, H., Göhmann, F., Klümper, A., Korepin, V.E.: The One-Dimensional Hubbard Model. Cambridge University Press (2005)

[5] Fürst, M.L.R., Lukkarinen, J., Mei, P., Spohn, H.: Derivation of a matrix-valued Boltzmann equation for the Hubbard model. J. Phys. A: Math. Theor. **46**(48), 485,002 (2013)

[6] Fürst, M.L.R., Mendl, C.B., Spohn, H.: Matrix-valued Boltzmann equation for the Hubbard chain. Phys. Rev. E **86**, 031122 (2012)

[7] Lu, X., Wennberg, B.: On stability and strong convergence for the spatially homogeneous Boltzmann equation for Fermi–Dirac particles. Arch. Ration. Mech. Anal. **168**(1), 1–34 (2003)

[8] Lukkarinen, J., Marcozzi, M.: Wick polynomials and time-evolution of cumulants. J. Math. Phys. **57**(8), 083,301:1–27 (2016)

[9] Lukkarinen, J., Mei, P., Spohn, H.: Global well-posedness of the spatially homogeneous Hubbard-Boltzmann equation. Comm. Pure Appl. Math. **68**(5), 758–807 (2015)

[10] Lukkarinen, J., Spohn, H.: Not to normal order—Notes on the kinetic limit for weakly interacting quantum fluids. J. Stat. Phys. **134**(5), 1133–1172 (2009)

[11] Lukkarinen, J., Spohn, H.: Weakly nonlinear Schrödinger equation with random initial data. Invent. Math. **183**(1), 79–188 (2011)

[12] Salmhofer, M.: Clustering of fermionic truncated expectation values via functional integration. J. Stat. Phys. **134**(5), 941–952 (2009)

[13] Spohn, H.: Collisional invariants for the phonon Boltzmann equation. J. Stat. Phys. **124**, 1131–1135 (2006)

[14] Teschl, G.: Mathematical Methods in Quantum Mechanics: With Applications to Schrödinger Operators, first edn. American Mathematical Society (2009). Graduate Studies in Mathematics, Volume 99