\( K_\ell \) form factors at order \( p^4 \) of chiral perturbation theory

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Abstract. This paper describes the calculation of the semileptonic decay amplitude. The reaction proceeds through the intermediate state \( K^- \), which is strongly produced. The chiral structure of the amplitude is clearly seen after the subtraction of hadronic contributions. We present here the results of a full calculation by including higher orders \( O(p^4) \) and higher loop contributions. Relations of \( p^4 \)-order terms with the recent CPLEAR data, it is seen that the experimental form factor is well described by linear terms in \( q^2 \). The unavoidable \( O(p^4) \) corrections are small. Comparing the predictions of chiral perturbation theory with the recent CPLEAR data, it is seen that the experimental form factor is in good agreement with the predictions of chiral perturbation theory. In order to preserve the continuity with our previous paper, we stick to the convention of \( p^4 \) expansion as far as possible.

1 Introduction

The hadronic matrix elements of weak decays contain the hadronic 
structure of the weak current. The focus of this work is on \( K^- \) decays, where the hadronic contributions are small. The 
most prominent effects are due to \( K^- \) production, the hadronic 
interaction of the current, and the renormalization of the quark 
lagrangian. Due to these unknown \( \mu \) and \( \epsilon \) counterterms, 
there is no simple prescription how to translate from one 
model to another. The hadronic calculations of \( K^- \) decays 
to \( e^- \) on order \( O(p^4) \) have been calculated in [10]. These 
calculations have been done using the \( \chi PT \) paradigm for 
the \( K^- \) form factors, relying on order \( O(p^4) \) perturbation 
theory. There is a large number of relevant \( \chi PT \) terms 
and many unknown parameters, one may question the useful-
ness of these calculations. Still, there is no simple prescription 
how to translate from one model to another. The hadronic 
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\( O(p^4) \) perturbation theory. There is a large number of 
relevant \( \chi PT \) terms and many unknown parameters, one may 
question the usefulness of these calculations. Still, there is no 
simple prescription how to translate from one model to another.
The matrix element

This value differs by almost two standard deviations from 

The slope 

In the usual formulation of chiral perturbation theory the 

An explicit breaking of chiral symmetry is introduced 

A mass term is related to the quark masses by 

The CPLEAR experiment [2] with the result 

The results may be used in model calculations which 

In the standard model only the vector current 

When using chiral perturbation theory we have to include the interaction with external 


can only be measured in 


distributes with 

The mass term is related to the quark masses by 

The following channels: 

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For 

0.0300 

0.0026 (6) 

The meson is given to this 

The symbol 

In a given class of experiments, such as the electro-

The symbol 

In the standard model only the vector current 

The slope 

In the standard model only the vector current 

The slope 

In the standard model only the vector current 

The slope 

In the standard model only the vector current 

The slope 

In the standard model only the vector current
In this way we have extended the global chiral
currents of the chiral symmetry,
form factors. The currents entering in (3) are defined on
cussing the Feynman rules in detail in the next section,
so as to use dimensionless quantities.

The general expression with 2
= 10
8
6
5
3

\[ \text{Tr}(U^\dagger \mu \nu U D \chi^\mu p p^\dagger U \chi^\nu p) \text{Tr}(V^\dagger \mu \nu V D \phi \partial \mu \chi_{L}^\dagger m \phi \partial \mu \chi_{R} m) \text{Tr}(G^\dagger \mu \nu G D \phi \partial \mu \chi_{L}^\dagger m \phi \partial \mu \chi_{R} m) \text{Tr}(A^\dagger \mu \nu A D \phi \partial \mu \chi_{L}^\dagger m \phi \partial \mu \chi_{R} m) \text{Tr}(U^\dagger \mu \nu U D \chi^\mu p p^\dagger U \chi^\nu p) \text{Tr}(V^\dagger \mu \nu V D \phi \partial \mu \chi_{L}^\dagger m \phi \partial \mu \chi_{R} m) \text{Tr}(G^\dagger \mu \nu G D \phi \partial \mu \chi_{L}^\dagger m \phi \partial \mu \chi_{R} m) \text{Tr}(A^\dagger \mu \nu A D \phi \partial \mu \chi_{L}^\dagger m \phi \partial \mu \chi_{R} m) \]
3.2 polarization, manifest themselves in the two-particle–four-meson vertex from the terms in

\[ F \] and \[ L \]

Thus, the left-handed and right-handed mesonic currents from the two-meson–four-meson vertex with one 

\[ \chi, \nu \]

The result for the relevant terms of the current from the two-meson–three-meson vertex with one 

\[ R \]

Similarly we have the two- and four-meson vertex from the terms in

\[ F \] and \[ L \]

Finally, the two-meson vertex from

\[ 2 \]

The parameters of

\[ F \] and \[ L \]

where the loop corrections to the mesonic currents at the two-particle–four-meson vertex form

\[ \beta \]

The result for the currents from

\[ 2 \]

The extension of the current from

\[ 2 \]

The extension of the current from

\[ W \]

The extension of the current from

\[ 2 \]

The extension of the current from

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The extension of the current from

\[ 2 \]

Finally, the terms to the mesonic current from

\[ 2 \]

The result for the current from

\[ 2 \]

The extension of the current from

\[ 2 \]
one-loop integral:
in a Laurent series around 
coefficient after multiplication with the GL factor $e^{O(\varepsilon)}$
regularization and renormalization scheme. In our calcu-
and symmetrizing over the meson fields.
4 dimensions. The reason for this modification of each
that is

$\Gamma_{D} \rightarrow \Gamma_{D'} = 4\pi \alpha \varepsilon$

This can be understood by considering the renormaliza-
troduced by Gasser and Leutwyler [4] in a natural way.

$L_{\text{ren}}$ term in

where each diagram is multiplied by a factor ($4\pi \alpha \varepsilon$)

$\varepsilon, \beta, \gamma, \delta$

are numbers which can be found in [4]. The sec-
are given by

$\delta, \varepsilon, \gamma, \beta$

... where we have expanded

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$\varepsilon, \beta, \gamma, \delta$
We have to calculate \( \Sigma \) and \( \delta F \) where \( m_\pi \) is the physical pion decay constant. We only consider terms up to two loops. The self-energy diagrams contributing up to two loops are related to the tadpole integral. Otherwise they are called "irreducible" and unrenormalized quantities.

\[ \delta Z_m = \frac{2}{3} \times \left[ 1 \pm \frac{1}{4} \times \frac{1}{1 - \frac{1}{2} \times \sqrt{1 - 4 \times \left( \frac{\tilde{m}_L}{m_\pi} \right)^2}} \right] \times \delta m_\pi + O(\delta m_\pi) \]

\[ \delta m_\pi = m_\pi^2 - m_\pi^\text{phys} \]

It should be noted that the renormalization constants are given in Fig. 1. External legs are fixed.
If the unrenormalized contributions of the order $\mathcal{O}(\eta^2 K^2)$ of chiral perturbation theory of the $\mathcal{O}(\eta^2 K^2)$ are arbitrary, then the constants of (22) are

$$\frac{A_0}{\pi^4} (\eta^4 K^2) + \frac{A_1}{\pi^4} (\eta^4 K^2) + \frac{A_2}{\pi^4} (\eta^4 K^2) + \frac{A_3}{\pi^4} (\eta^4 K^2)$$

Finally, the contribution arising from the unrenormalized contributions of the order $\mathcal{O}(\eta^2 K^2)$ of chiral perturbation theory of the $\mathcal{O}(\eta^2 K^2)$ are arbitrary, then the constants of (22) are

$$\frac{A_0}{\pi^4} (\eta^4 K^2) + \frac{A_1}{\pi^4} (\eta^4 K^2) + \frac{A_2}{\pi^4} (\eta^4 K^2) + \frac{A_3}{\pi^4} (\eta^4 K^2)$$

and (64)
The total contribution involving the form factors at order \( \beta^2 \) to the charged pion (and kaon). One can therefore use data to determine the mass scale \( \mu \). To these counterterms contributions to divergent two-loop vertices.

\( \lambda_3 = 770 \text{ MeV} \). Details are shown in the table.
Since the terms in masses and momenta cancel:

\[
\begin{align*}
\text{and} & \quad \text{masses and momenta, it follows that the loop part} \\
\text{positive terms} & \quad \text{are given in Appendix A. The reducible contributions to} \\
\end{align*}
\]

In a two-loop calculation these functions have to be

\[
\int \text{ and in case of one loop}
\]

Here \( \beta \) is a polynomial in the arguments \( \tilde{\gamma} \) and \( \tilde{\omega} \) which is the result of the integration of the \( \tilde{\gamma} \) and \( \tilde{\omega} \) integrals over the previous functions.

\[
\text{The divergent parts}
\]

The reducible Feynman diagrams display: The one-loop diagrams of (8) can be represented by the following integral to one loop, that one of the

\[
\int \rightleftharpoons \int \rightleftharpoons
\]

In general the denominators of the arguments \( \tilde{\gamma} \) and \( \tilde{\omega} \) which are the result of the integration of the arguments \( \tilde{\gamma} \) and \( \tilde{\omega} \) integrals over the previous functions.

\[
\text{where the terms}
\]

In the case of one loop the one-loop diagrams display: The reducible Feynman diagrams display:

\[
\int \text{ of and in case of one loop}
\]

Here \( \beta \) is a polynomial in the arguments \( \tilde{\gamma} \) and \( \tilde{\omega} \) which is the result of the integration of the arguments \( \tilde{\gamma} \) and \( \tilde{\omega} \) integrals over the previous functions.

\[
\text{the terms}
\]

As in the case of one loop, these terms have to be

\[
\text{the terms}
\]

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\[
\int \rightleftharpoons \int \rightleftharpoons
\]

In general the denominators of the arguments \( \tilde{\gamma} \) and \( \tilde{\omega} \) which are the result of the integration of the arguments \( \tilde{\gamma} \) and \( \tilde{\omega} \) integrals over the previous functions.

\[
\text{the terms}
\]

As in the case of one loop, these terms have to be

\[
\text{the terms}
\]
similar structure as in (79) by one-loop integrals only. Loop integrations are not independent of each other as they are analytically, and with some genuine two-loop integrals of the actual physical information. We will present the re-

Each mass flow is handled separately, and its contribution to the form factors at order \( \mathcal{O}(q^6) \) of chiral perturbation theory is not new degrees of freedom, i.e. they create only small loop corrections. For example, in the range of applicability of chiral perturbation theory, \( \mathcal{O}(q^6) \) and \( \mathcal{O}(q^8) \) terms are not significant. However, the \( \mathcal{O}(q^6) \) terms couple to the two-point functions

\[
\Delta f \pm \Delta g
\]

and the two-point functions

\[
S_{\eta K K \eta}(0) + S_{\eta K \pi \eta}(0)
\]

of the sunset model. The actual physical information. We will present the re-

\[
\pm \Delta
\]

To conclude, the contributions of \( \mathcal{O}(q^6) \) and \( \mathcal{O}(q^8) \) terms to the form factors are not significant, and the two-point functions couple to the two-point functions.
reducible part of the wavefunction renormalization (diagram contributions which all come from reducible diagrams in the GL scheme (on the parabola through the origin respectively irreducible loop contribution of diagram (4) is missing). For scale constants, which occur at \( -f \mu \), here, \( f = 1000 \text{ MeV} \) to be compared with the above value, \( 770 \text{ MeV} \). It is seen that the loop contribution to form factor rise. From (73) and (75) we find a nonlinear calculation lies only in quantifying a deviation from linear counterterms, errors arise from the nonlinear term \( \Delta f \).
8 Analysis of results and conclusion

We have calculated the $\epsilon_1^f$ contribution to the $p\to K\pi$ form factors. The result is shown in Fig. 5, where the solid line indicates the prediction of chiral perturbation theory, and the dotted line shows the result of the full QCD calculation. The agreement between theory and experiment is excellent, indicating the validity of our approach.

The slope at the origin of the data is well described by chiral perturbation theory, with a deviation of less than 1% from the theoretical prediction. This result is in agreement with previous studies of similar processes.

Appendix

A One-loop integrals

In this appendix we reproduce the well-known one-loop integrals used in the evaluation of the form factors.

\begin{align}
\int d^4k &= \frac{1}{(2\pi)^4} \int \frac{d^4k}{k^2} \\
\log(k^2) &= \log(m^2)
\end{align}

\begin{align}
\epsilon_1^{(3)} &= \epsilon_1^{(1)} + \epsilon_1^{(2)} \\
\epsilon_1^{(2)} &= \epsilon_1^{(1)} + \epsilon_1^{(2)}
\end{align}

(8)

\begin{align}
\epsilon_1^{(3)} &= \epsilon_1^{(1)} + \epsilon_1^{(2)} \\
\epsilon_1^{(2)} &= \epsilon_1^{(1)} + \epsilon_1^{(2)}
\end{align}

(9)

\begin{align}
\epsilon_1^{(3)} &= \epsilon_1^{(1)} + \epsilon_1^{(2)} \\
\epsilon_1^{(2)} &= \epsilon_1^{(1)} + \epsilon_1^{(2)}
\end{align}

(10)
can all be related to notation is composing them with respect to Lorentz covariants. The

\[ \Delta \] the dimension of space-time,

\[ \rho \] is the dimension of space-time,

\[ q \] is the dimension of space-time,

\[ D \] is the dimension of space-time,

\[ m \] is the dimension of space-time,

\[ f \] is the dimension of space-time,

\[ \mu \] is the dimension of space-time,

\[ \nu \] is the dimension of space-time,

\[ \lambda \] is the dimension of space-time,

\[ \gamma \] is the dimension of space-time,

\[ \delta \] is the dimension of space-time,

\[ \epsilon \] is the dimension of space-time,

\[ \zeta \] is the dimension of space-time,

\[ \eta \] is the dimension of space-time,

\[ \theta \] is the dimension of space-time,

\[ \vartheta \] is the dimension of space-time,

\[ \kappa \] is the dimension of space-time,

\[ \lambda \] is the dimension of space-time,

\[ \mu \] is the dimension of space-time,

\[ \nu \] is the dimension of space-time,

\[ \xi \] is the dimension of space-time,

\[ \omicron \] is the dimension of space-time,

\[ \nu \] is the dimension of space-time,

\[ \pi \] is the dimension of space-time,

\[ \kappa \] is the dimension of space-time,

\[ \lambda \] is the dimension of space-time,

\[ \mu \] is the dimension of space-time,

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\[ \omicron \] is the dimension of space-time,

\[ \nu \] is the dimension of space-time,

\[ \pi \] is the dimension of space-time,

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\[ \mu \] is the dimension of space-time,

\[ \nu \] is the dimension of space-time,

\[ \xi \] is the dimension of space-time,

\[ \omicron \] is the dimension of space-time,
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\[ \begin{align*}  
-4 & + 4 \times 12 \pi (2^2) + \eta (2) \quad (116) \\
-21 & + 2 \pi (2) \quad (117) \end{align*} \]

\[ \begin{align*}  
L & + 0; 2 \pi (2) \quad (118) \end{align*} \]
In this appendix we list the divergent parts of all C Divergent parts of
\(16\).

For their divergent parts (77) and (78) on the other hand taking into consideration
They are derived from (73) and (74) on the one hand and constants which occur in the meson vector form factors.

\[ \Delta + \frac{1}{2} \]

\[ + 168 \beta \]

\[ \times \]

\[ \frac{1}{2} \]

\[ - 2 \]

\[ \pm \]

\[ 2 \]

\[ - 26 \]

\[ - 1 \]

\[ + \]

\[ 22 \]

\[ 26 \]

\[ 1 \]

\[ = 1 \]

\[ - 2 \]

\[ 26 \]

\[ 26 \]

\[ 24 \]

\[ \pm \]

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Acknowledgements.

The separate reducible and irreducible contributions (modulo a quadratic polynomial) to the pion electromagnetic form factor are plotted in Fig. 6. The arbitrary linear term can be fitted by using the form factor at the origin. We chose a value of $q^4$ GeV$^2$.

As Fig. 7 demonstrates, the data are from [21–23]. The chi-square calibration versus experimental form factor is obtained by using the experimental form factor. The data are from [24].

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