Momentum-resolved spectroscopy of a 1D superfluid using a single atomic impurity

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We present a general and non-invasive probing scheme to perform full momentum-resolved spectroscopy of a cold atomic gas loaded into an optical lattice using a single quantum impurity. The protocol relies on weak collisional interactions and subsequent population measurements of the impurity. By tuning a few controllable external parameters the impurity-lattice interaction can be engineered, and using two sets of measurements, performed with the impurity in two different positions, the full dispersion relation of the superfluid phonons can be reliably extracted.

I. INTRODUCTION

Cold atoms in optical lattices allow to engineer and investigate non-trivial Hamiltonian models in a fully controllable way [1–3], with the possibility of tuning the interactions [4] by means of Feshbach resonances [5, 6]. In these setups typical condensed matter physics effects and models can be simulated [7–9] with the lack of lattice defects. In this context, the Bose-Hubbard model is perhaps the most celebrated example [10, 12]. This model has been extensively studied theoretically [13–18] and a great number of experimental verifications have been performed [19–22]. Furthermore, recent experiments in the context of quantum information and simulations using cold atoms in optical lattices also suggest that the Bose-Hubbard model can be of practical relevance for technological applications [23]. As for most systems in condensed matter physics, probing of cold atoms in optical lattices is usually performed via semi-classical methods that can be rather invasive or even destructive, depending on the specific technique or quantity to be measured. A prominent example are the superfluid excitations of a Bose-Hubbard gas. These have been resolved in energy using techniques such as magnetic gradients [19] and lattice depth modulation [24, 25]. A full momentum-resolved spectroscopy leading to the measurement of the superfluid dispersion relation has been performed using Bragg spectroscopy in [26, 27]. This method, however, relies on two-photon processes, thus implying the exchange of energy and momentum with the atoms in the lattice. Although all of these techniques have been successfully applied, they strongly interfere with the dynamics of the gas and one may wonder whether it could be possible to extract similar information without disturbing the system so much. Very recently, single and controllable quantum objects have been proposed as an alternative tool to investigate collective properties of large many-body systems. Successful examples in optical lattice systems range from transport properties [28] to temperature estimation [29] and measure of quantum correlations [30, 31]. Further instances include probing of cold free and trapped gases [32–36], spin chains [37], Fermi systems [38–42], Coulomb crystals [43, 44] and generically critical systems [45]. Recently, an interesting spectroscopic protocol was presented to study some energy-resolved spectral features of atomic gases in optical lattices [46]. Here, we take a step further and propose an experimentally feasible application of the protocol outlined in [47]. We show how to perform full momentum-resolved spectroscopy of a cold 1D superfluid gas in an optical lattice using a single atomic impurity. The impurity is harmonically trapped in an auxiliary potential well and brought into contact (and interaction) with the surrounding gas. By properly controlling the coupling strength and the position of the impurity, it is possible to engineer a two-stage spectroscopic protocol that allows for the reconstruction of the dispersion relation ω(k) of the quasi-particle excitations of the atomic gas. We call such an impurity a quantum probe.

II. PROTOCOL

Our scheme is depicted in Fig. 1 and it can be implemented by using a either Rb-K setup [48, 49] or a modification of the spin-dependent Cs scheme described in [50]. The protocol consists of two subsequent steps in which transitions between energy levels of the impurity are observed. The transition rates strongly depend on the position of the impurity within the lattice. By combining the outcomes of two sets of energy-resolved measurements corresponding to different impurity positions, it is possible to momentum-resolve both the excitation spectrum and spectral density of the Bose gas. The two key features of our protocol being 1) measurements are to be performed only on the probe, hence causing minimal disturbance to the atomic gas, and 2) no momentum exchange between the probe and the gas ever occurs. The dynamics of an ensemble of bosonic atoms trapped in a one-dimensional optical potential and cooled to its lowest energy band is governed by the Bose-Hubbard Hamiltonian [10–12]:

\[
\hat{H}_B = -J \sum_{\langle i,j \rangle} \hat{c}_i^\dagger \hat{c}_j + \frac{U}{2} \sum_i \hat{n}_i (\hat{n}_i - 1) - \mu \sum_i \hat{n}_i, \tag{1}
\]

Here, \( \hat{c}_i^\dagger, \hat{c}_i \) are local boson ladder operators labelled by the lattice site with \( \hat{n}_i \equiv \hat{c}_i^\dagger \hat{c}_i \). \( \langle \rangle \) in the first sum selects nearest
type $\Lambda$, with the usual assumption of contact potential, and coupling to the Bose gas is taken to be of the density-density of the trapping potential) and reads $\hat{H}_{int} = g \sum_{m,n,j} \int dxdydz \psi_n^*(x,y,z) \psi_m(x,y,z) \times$

\[
\omega_j(x) \delta_j(y,z) \delta_j(y,z) |m\rangle \otimes |\beta_j^\dagger \rangle, \tag{2}
\]

in which we have assumed a three dimensional spatially extended probe, although the lattice is effectively one dimensional. Here, $g$ is the impurity-gas coupling constant, $\psi_m(x,y,z) = \langle x,y,z|m \rangle$ is the $m$-th unperturbed impurity energy eigenfunction, while $\omega_j(x)$ is the Wannier eigenfunction corresponding to the $i$-th lattice site. The effective coupling between the impurity and the bosons at the $i$-th site of the lattice depends upon the overlap integrals $\phi_{nm} = \int dxdydz \psi_n^*(x,y,z) \psi_m(x,y,z) \omega_j(x)^2 \delta_j(y,z)$. In what follows we assume the probe to be spatially localized around one specific site that we label $0$. This allows us to drop the summation over the site index and simplify Eq. 2. Furthermore, by employing the Bogoliubov approximation and expressing the number of bosons at site $0$ in terms of the Bogoliubov modes [14], the interaction Hamiltonian can be rewritten as

\[
\hat{H}_{int} = g \sum_{n,m,j} \phi_{nm} |n\rangle \langle m| \otimes \left[ n_0 + \sum_k \beta_k (\hat{b}_k^\dagger + \hat{b}_k) \right], \tag{3}
\]

in which $\beta_k = \sqrt{\frac{\nu_k}{\lambda_k}} (u_k + v_k)$, with $u_k, v_k$ being the Bogoliubov coefficients, whose analytical expression can be found, e.g., in [14]. While in real space the impurity couples locally to one specific lattice site (that is $0$), in the momentum space it couples to all of the Bogoliubov modes. The interaction (3) describes transitions between different energy levels of the probe associated to phonon propagating through the lattice. In the following, we take the probe to be initialized in its unperturbed ground state $|0\rangle$, while the gas loaded into the lattice is in a thermal state $\rho_0 \propto \exp \left(-\beta \sum_k \omega_j(k) \hat{b}_k^\dagger \hat{b}_k \right)$. In this way, only ground-to-excited state transitions of the atomic probe have to be considered. The probability $\Gamma_{0-n}$ for such a transition to occur within time $t$ can be easily computed in the weak coupling limit

\[
\Gamma_{0-n}(t) = g^2 \phi_{00}^2 \left\{ \Gamma_0 + \sum_k \Gamma_k (\omega, t) + \Gamma_k^\dagger (\omega, t) \right\}, \tag{4}
\]

with $\Gamma_0 = \lambda_1 (\omega_n, t) n_0 \Gamma_0^\dagger, \Gamma_k (\omega, t) \equiv \beta_k^2 \lambda_1 (\omega + \omega(k), t) (1 + n(\omega(k)))$ and $\Gamma_k^\dagger (\omega, t) \equiv \beta_k^2 \lambda_2 (\omega - \omega(k), t) n(\omega(k))$. The latter three quantities are expressed in terms of the probe transition frequency $\omega_0 \equiv \nu_n - \nu_0$, the Bose-Einstein distribution at temperature $\beta^{-1}$, $n(\omega)$, and two auxiliary functions $\lambda_1 (\omega, t) = 2 [1 - \cos (\omega t)] / \omega^2, \lambda_2 (\omega - \omega(k), t) = \lambda_1 (\omega - \omega(k), t)$ if $\omega \neq \omega(k)$, $\lambda_2 (\omega - \omega(k), t) = t^2$ if $\omega = \omega(k)$.

To go further in the analysis, we consider a specific trapping potential for the probe and, as a simple and yet physically relevant example, we analyse the case of an harmonic trap. For the sake of clarity, we first discuss a one-dimensional impurity trap, in which the longitudinal spreading of the impurity wave function can be neglected. Later on we will move to a more realistic three dimensional trapping well.

Figure 1: Sketch of the two different steps of the protocol. In the upper panel, the trapped impurity is located at a minimum of the optical lattice and its ground state wave function overlaps with the Wannier state of that site only. The lower panel, instead, shows the impurity localized near a maximum of the lattice, with a ground state wave function large enough to couple with both of the adjacent sites.
A. Quantum Probe: Impurity atom in 1D harmonic trap

In the simple 1-D harmonic case, the probe eigenenergies are \( \nu_n = \nu(n + \frac{1}{2}) \), while the unperturbed eigenfunctions are given in terms of the Hermite polynomials \( H_n \) and read
\[
\psi_n^0(z) = \frac{(m\nu)^{1/4}}{2^{n/2}\Gamma(n+1/2)}H_n(\sqrt{m\nu}z)e^{-\frac{z^2}{2m\nu}} \text{, where } m \text{ is the impurity mass. Here the } z \text{ axis (along which the probe trapping well extends) is imagined to be orthogonal to the lattice axis; with this spatial arrangement the interaction Hamiltonian fully satisfies the localization assumption discussed above. As a side effect of the harmonic approximation, the parity of the probe eigenstates implies that transitions are only induced between even numbered levels, [52]. Assuming that the minimum of the harmonic trap coincides with a selected minimum of the optical lattice, the amplitude \( \varphi_{nm} \) entering the probabilities above, becomes \( \varphi_{nm} = \sqrt{n\nu^0_n}(0) \sqrt{\nu_n^{1/2}(-1)^{n+m}y_n^{1/2}y_m^{1/2}} \), in which \( \nu_n = \frac{\Gamma(n+1/2)}{\Gamma(n)} \text{ is the Euler Gamma function ratio. As a result, the transition probability from the ground to the } n \text{-th excited level reads}
\]
\[
\Gamma_{0\rightarrow n} = g_n^2 \nu \lambda_1(n\nu, t)n_0^2 + g_n^2 \nu \sum_k \Gamma_k(n\nu, t) + \Gamma_k(n\nu, t),
\]
in which \( g_n = \sqrt{\nu_n^0(0)} \nu - \sqrt{\nu_n y_0} \).

III. QUANTUM PROBE: IMPURITY ATOM IN 3D HARMONIC TRAP

In a realistic experimental situation, the three dimensional spatial extension of the probe wave function has to be taken into account. We consider a 3D harmonic trap and assume the trap frequency to be tailored (and controllable) along one direction orthogonal to the lattice. The confinement in the two other directions is kept fixed. The unperturbed probe wave functions are now given by three factors, one for each spatial coordinate, \( \psi_0(x) = \psi_n^0(x)\psi_m(y)\psi_k(z) \). As before, we are interested in measuring transition probabilities between impurity states along the \( z \) direction, and assume the \( x \) and \( y \) degrees of freedom to be frozen. We therefore need to evaluate \( \Gamma_{0\rightarrow(n,0,0)} \). As depicted in Fig. 1, the protocol develops in two subsequent steps with the probe brought onto I) a minimum, and II) a maximum of the optical potential. The information obtained by using these two steps allows for the reconstruction of the full dispersion relation \( \omega(k) \), that is momentum-resolved spectroscopy of the gas through measurements on the impurity. As sketched in Fig. 1, in configuration I) the impurity interacts with one lattice site only, while in configuration II), due to a suitable choice of the longitudinal confining frequency \( \nu_0 \), the impurity is coupled with two adjacent sites at the same time. The transition probabilities corresponding to the two positions can be computed as before. For case I), we obtain an expression which is identical to Eq. (4), but for the pre-factor:
\[
\Gamma^I_{0\rightarrow(n,0,0)} = g_{1n}^2 \nu \left\{ \lambda_1(n\nu, t)n_0^2 + \sum_k [\Gamma_\nu(n\nu, t) + \Gamma_k(n\nu, t)] \right\},
\]
where the pre-factor \( g_{1n}^2 = \nu \nu_0 Y_0 Z_0 \), is expressed in terms of the spatial overlap \( X_0 = \int dx \psi_{n0}(x)\omega_0(x) \), and of the constants \( Y_0 = \sqrt{m\nu^0_0} \nu_0 \) and \( Z_0 = (-1)^n \sqrt{m\nu^{1/2} y_0^{1/2}} \), related to the confinement in the transverse directions. For case II), assuming that the probe interacts with equal strength with the two adjacent sites, the transition probability reads
\[
\Gamma^II_{0\rightarrow(0,0,0)} = \nu \left\{ 2\lambda_1(n\nu, t)n_0^2 + \sum_k [(1 + \cos(ka))[\Gamma_\nu(n\nu, t) + \Gamma_k(n\nu, t)] \right\},
\]
where the new pre-factor, \( g_{1n}^2 = 2\nu X_0 Y_0 Z_0 \), has a similar expression to the one for case I) above, but with the contribution \( X_0 = \int dx \psi_{n0}(x - \frac{a}{2})\omega_0(x) + \omega_0(x)\omega_1(x) \), calculated using a shifted ground state wave function. These probabilities can be obtained experimentally by i) initialising the probe in its ground state, and ii) measuring the population of a selected excited state after a given time. To reconstruct the excitation spectrum of the atomic gas, this procedure should be repeated for different values of the energy difference between the two involved impurity levels. This can be done, in the harmonic case, by manipulating the frequency of the probe confinement trap. Indeed, the transition probability \( \Gamma_{0\rightarrow n} \) is a function of the energy difference between the probe levels as well as the overlap between the lattice Wannier states and the unperturbed eigenfunctions of the impurity. If the interaction time \( T_f \) is large enough, resonance peaks will emerge when scanning the probability \( \Gamma_{0\rightarrow n} \) for different trapping frequencies. Indeed, (for case I) we have that
\[
\Gamma^I_{0\rightarrow(0,0,0)} \approx 2 g_{1n}^2 \nu_0^2 (\omega(k)) T_f^2.
\]
This probability is displayed in Fig. 2 as a function of the impurity energy gap for a 65-site lattice at 1 nK. The peaks are located precisely at the frequen-
cies of the phononic excitations and their height is proportional to both the occupation of each Bogoliubov mode and the spectral density $\beta_k^2$; in particular, the progressive damping at higher frequencies is also due to the thermal character of the atomic gas.

In order to fully reconstruct the dispersion relation $\omega(k)$ and spectral density $\beta(k)$, their dependence on the wave number $k$ is also needed, which can be obtained via step II. When the probe is located at a maximum of the optical potential and under the assumption that its longitudinal confinement length $x_0$ is comparable to the lattice constant $a$, its wave function overlaps with the Wannier states of the two adjacent sites. This gives rise to the extra $\cos(ka)$ factor in Eq. (7), which is crucial in order to associate the wave number $k$ to each excitation frequency $\omega(k)$. Again for a sufficiently long interaction time, the transition probabilities simplifies

$$\Gamma^{\text{II}}_{(0,0,n)} \beta_k^2(n(\omega(k)))T_J^2(1 + \cos(ka)),$$

leading to the following ratio

$$\frac{\Gamma^{\text{II}}}{\Gamma^{\text{I}}} \frac{\beta_k^2}{\beta_k^2} = [1 + \cos(ka)]. \quad (8)$$

Therefore, by measuring both $\Gamma^{\text{I}}$ and $\Gamma^{\text{II}}$, it is possible to discriminate the wave number corresponding to each peak, thus probing the Bogoliubov dispersion relation, even if the exact values of the effective coupling constants are unknown. In other words, by performing twice energy-resolved measurements of the impurity for two different spatial configurations, it is possible to perform a full momentum-resolved spectroscopy of the gas. This feature is very important, since possible hopping between next nearest neighbours will generate spectral features resulting in a non monotonic dispersion law that can only be captured when one can discriminate energies in $k$. The reconstructed dispersion relation is displayed in Fig. 3, in comparison with the exact analytic values $\omega(k)$, (black line and dots). The reconstructed points are obtained from transition rates to whom a statistical noise of 1%, 2%, 5% and 10% is applied, and a discrete sampling of the transition frequency is also taken into account. All the reconstructed curves are able to capture the relevant features and behaviour of the the Bogoliubov spectrum. One can also extract the spectral density of Eq. 3. Indeed, Fig. 4 shows the comparison between the exact spectral density (black line and dots) and the reconstructed ones for the same optical lattice and sources of error considered in Fig. 2 and Fig. 3. The reconstructed relation is able to capture the main features of the exact spectral density. We notice that the low $k$ sector is more sensitive to noise, this is quite reasonable given the nature of relation Eq 8. Low $k$ quantities are still visible in Fig. 2, however is not possible to associate a proper wave vector. The number of excitations in a particular Bogoliubov mode strictly depends on the temperature of the atomic gas. At low temperatures, high energy excitations are mostly suppressed; therefore, a good probing requires a larger interaction time. In this case, the effective interaction strength appearing in the transition probability becomes proportional to the trap frequency, i.e. $g \sim \nu$. As a result, all of the relevant parameters must be chosen consistently with the perturbative approach. In particular, to avoid coupling of the probe with bosons on more than two sites, a crucial condition to fulfil is $mv_0 > 4/\nu$ (see appendices for details). Even after this restriction has been taken into account our probing scheme remains feasible with current technology [48, 50, 53, 54], considering specifically techniques to properly manipulate the impurity, checking its location and perform the energy measurements [55–58]. Furthermore, it can be also applied to the Mott phase although only energy differences in the Bogoliubov spectrum can be efficiently extracted in this case.

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**Figure 3:** Comparison between the analytic excitation spectrum $\omega(k)$ (black) and the frequencies extracted via the probing protocol using the local atomic probe in both configurations I and II described in the text. The lattice parameters are the same as in Fig. 2. Red, blue, green and orange markers correspond to a noise of 1%, 2%, 5% and 10% respectively.

**Figure 4:** Comparison between exact spectral function $\beta_k^2$ (black line) and values extracted from the rates with noise. The lattice parameters are the same as in Fig. 2. The legend is the same as in Fig. 3.
IV. CONCLUSIONS

Concluding, we have presented an experimentally feasible protocol to perform momentum-resolved spectroscopy of a 1D superfluid cold gas in an optical lattice via energy-resolved measurements on a single and controllable quantum system. Our proposal exemplifies the essence of the quantum probing approach, wherein some of the typical complexity of a many-body system is broken down by imprinting it onto the open dynamics of a smaller system, and therefore locally extracted. This process is not at all obvious a priori. Importantly, the protocol is potentially non-invasive as it acts on the gas as a small density perturbation, whose effects are rapidly suppressed after each measurement. Furthermore, it can be extended to investigate other lattice models, and generalised to a multi-probe schemes aimed at studying genuine many-body features, such as quantum correlations.

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Appendix A: Details on the derivation of the transition rates

The most general density-density interaction allowing for transitions among different Wannier states takes the following form

\[
\hat{H}_{\text{int}} = g \sum_{n,m,i,j} \int dx dy dz \psi_n^*(x,y,z)\psi_m(x,y,z)\omega_i(x)\omega_j(x) \\
\times \langle n | m \rangle \otimes \delta_i(y,z)\delta_j(y,z)\hat{c}_i^n \hat{c}_j^n, \tag{A1}
\]

To compute the transition rates in the weak coupling regime one has to calculate the quantity \(\langle 0, 0, n, \hat{H}_{\text{int}} | 0, 0, 0 \rangle\) up to the first order in \(g\). For the two cases considered in the manuscript we find

Case I

\[
\langle 0, 0, n, \hat{H}_{\text{int}} | 0, 0, 0 \rangle \simeq g\psi_{n,=0}(y = 0)^2\psi_{n,=0}(z = 0)\psi_{n,}(z = 0) \\
\times \int dx \psi_{n,=0}(x)^2\omega_0(x)\omega_0(x)\hat{c}_{0}^n \hat{c}_0^0 \tag{A2}
\]

Case II

\[
\langle 0, 0, n, \hat{H}_{\text{int}} | 0, 0, 0 \rangle \simeq g\psi_{n,=0}(y = 0)^2\psi_{n,=0}(z = 0)\psi_{n,}(z = 0) \\
\times \int dx \psi_{n,=0}(x - \frac{a^2}{2})^2 \sum_{i,j=0,1} \omega_i(x)\omega_j(x)\hat{c}_i^n \hat{c}_j^n \tag{A3}
\]

In case I, the site in which the impurity is immersed is labelled "0" and it is assumed that the interaction depends on the local boson number operator related to this site only. In the next section we are going to show that this local approximation is indeed good, calculating numerically the overlapping integrals.

In case II, instead, the impurity is embedded between sites "0" and "1", and the terms that dominate the dynamics are those containing the number operators of the two sites as well as the transition operator describing tunnelling between them. We then, need to expand the following combinations \(\hat{c}_1^n \hat{c}_0^0\), \(\hat{c}_0^n \hat{c}_1^n\), and \(\hat{c}_1^n \hat{c}_0^n\) in terms of the Bogoliubov operators of the condensate \(\hat{b}_k = \hat{u}_ke^{\hat{v}_k} - \hat{v}_ke^{\hat{u}_k}\), obtaining

\[
\hat{c}_1^n \hat{c}_0^0 = \frac{1}{\sqrt{N_z}} \sum_k e^{ikx_0} \hat{c}_0^n \hat{c}_0^0, \\
\hat{c}_0^n \hat{c}_1^n = \frac{1}{\sqrt{N_z}} \sum_k e^{-ikx_0} \hat{c}_0^n \hat{c}_0^0, \\
\hat{c}_1^n \hat{c}_0^n = \frac{1}{\sqrt{N_z}} \sum_k e^{ikx_0} \hat{c}_0^n \hat{c}_0^0 \\
\simeq n_0 + \frac{n_0}{N_z} \sum_{k \neq 0} (e^{ikx_0} \hat{c}_k + e^{-ikx_0} \hat{c}_k)
\]

\[
= n_0 + \frac{n_0}{N_z} \sum_{k \neq 0} \left[ (\hat{u}_ke^{\hat{v}_k} + \hat{v}_ke^{\hat{u}_k})\hat{b}_k + (\hat{u}_ke^{-\hat{v}_k} + \hat{v}_ke^{-\hat{u}_k})\hat{b}_k^\dagger \right], 	ag{A4}
\]

in which, due to the linear nature of the Bogoliubov approximation, we have neglected any process involving more than one non-zero momentum operator. Hence, Eqs. A2-A3 can be recast as follows

\[
\langle 0, 0, n, \hat{H}_{\text{int}} | 0, 0, 0 \rangle \simeq \phi Y_{00} Z_{n,0} \times \\
\left[ n_0 + \frac{n_0}{N_z} \sum_{k \neq 0} \beta_k \left( e^{ikx_0} \hat{b}_k + e^{-ikx_0} \hat{b}_k^\dagger \right) \right], \\
\langle 0, 0, n, \hat{H}_{\text{int}} | 0, 0, 0 \rangle_b \simeq Y_{00} Z_{n,0} \times \\
\left\{ 2(\varphi + \varphi')n_0 + (\varphi + \varphi') \sqrt{n_0} \sum_{k \neq 0} \beta_k \left[ e^{ikx_0} + e^{-ikx_0} \right] \right\} \hat{b}_k + h.c. \right\}
\]

in which we have defined the amplitude \(\beta_k = \sqrt{n_0} (u_k + v_k)\), \(\phi = \int dx \varphi_{n,=0}(x)\omega_0(x)\omega_0(x)\), \(\varphi = \int dx \psi_{n,=0}(x - a/2)^2\omega_0(1)(x)\omega_0(1)(x)\) and \(\varphi' = \int dx \psi_{n,=0}(x - a/2)^2\omega_0(1)(x)\omega_0(0)(x)\).

Appendix B: Physical Setting

The boson condensate is trapped in a one-dimensional optical potential \(V(x) = V_0 \sin^2\left(\frac{2\pi x}{L}\right)\) that can be generated using two laser beams, and having lattice parameter \(a = \frac{\lambda}{2}\). Different lattice potentials generate Hubbard models with different \(J, U\) and Wannier states. In Fig.1 we show the values of the overlapping integrals for different optical potentials as a function of the harmonic length of the trapped impurity.
Figure 5: Upper panel: overlapping integrals \( \phi \) (left) and \( \phi' = \int dx \psi_n(x) \psi_1(x) \) (right). Lower panel: \( \varphi \) (left) and \( \varphi' \) (right). The overlapping integrals are computed using Wannier states for different potential depth expressed in terms of the recoil energy \( E_R \). Wannier states for different realizations of optical lattices have been generated using the software package developed at University of Oxford by the group of Prof. Dieter Jaksch [60]. The algorithm implemented in the software package is described in [18]. For further details, see [61, 62].

Beside the expression given in the main text, the wave functions of the impurity can be expressed in terms of the length 

\[
x_0 = \sqrt{\frac{k}{\pi}}, \quad \psi_n(x) = \frac{x_0^{1/4} e^{-x_0/4}}{2^{1/4} \pi^{1/4}} H_n\left(\frac{x}{x_0}\right) e^{-\frac{x^2}{2x_0^2}}.
\]

For the case I we show \( \phi \) and \( \phi' \), they represent the integral containing the ground state function of the probe and the Wannier function of the same site and of its nearest neighbor respectively. We see how the second one is far smaller than the first one. This proves that the local approximation is good enough for a harmonically trapped probe, within the specified trapping-length range.
