Quintessence and phantom emerging from the split-complex field, split-quaternion field and split-complex DBI field

Changjun Gao

Key Laboratory for Computational Astrophysics, National Astronomical Observatories, Chinese Academy of Sciences, Beijing, 100012, China and
State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China

Xuelei Chen

Key Laboratory for Computational Astrophysics, National Astronomical Observatories, Chinese Academy of Sciences, Beijing, 100012, China and
Center of High Energy Physics, Peking University, Beijing 100871, China

You-Gen Shen

Shanghai Astronomical Observatory, Chinese Academy of Sciences, Shanghai 200030, China

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Motivated by the mathematic theory of split-complex numbers (or hyperbolic numbers, also perplex numbers) and the split-quaternion numbers (or coquaternion numbers), we define the notion of split-complex scalar field and the split-quaternion scalar field. Then we explore the cosmic evolution of these scalar fields in the background of spatially flat Friedmann-Robertson-Walker Universe. We find that both the quintessence field and the phantom field could naturally emerge in these scalar fields. Introducing the metric of field space, these theories fall into a subclass of the multi-field theories which have been extensively studied in inflationary cosmology. Using the brane world model, the split-complex Dirac-Born-Infeld Lagrangian is constructed and analyzed.

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I. INTRODUCTION

More and more accurate and convincing astronomical observations [1-3] indicate that dark energy dominates our current Universe. Although a host of observationally viable dark energy models have been proposed, the nature of dark energy is still undetermined. The Einstein cosmological constant is in many respects the most economical solution to the problem of dark energy. But it is confronted with two fundamental problems: the fine tuning problem and the coincidence problem (see e.g. [4]). So one turn to the study of other options for dark energy, for example, quintessence [5-7], quintom [8], k-essence [9], Chaplygin gas [10], holographic dark energy [11] and so on.

One find that in many models of quintessence field, there exist the so-called tracker solutions. In these solutions, the quintessence field always has an energy density closely tracks (but is smaller than) that of the radiation until the matter-radiation equality. By this way, the coincidence problem is solved [5-7]. Phantom energy is introduced into the study of cosmic evolution by Caldwell [12, 13]. The Lagrangian of phantom takes the form \( L_\phi = -\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi + V(\phi) \), with \( \phi \) the phantom field and \( V(\phi) \) the phantom potential. By changing the sign of kinetic term in quintessence field by hand, this form of Lagrangian could be obtained. One then find that the energy density of phantom actually increases with cosmic time. So the fate of the Universe is a Big Rip.

Now we want to ask: can we unify quintessence and phantom in a natural way? The answer is yes. We find that the quintessence and phantom can naturally emerge in the theory of split-complex scalar field, the split-quaternion scalar field and the split-octonion scalar field. This finding is motivated by the mathematic theory of split-complex numbers (or hyperbolic numbers, also perplex numbers), the split-quaternion numbers (or coquaternion numbers) and the split-octonion numbers [14] (for references, see also the Wikipedia [1]).

The paper is organized as follows. In section II, we define the notion of split-complex scalar field and show quintessence and phantom could emerge from this field. In section III, we investigate the cosmic evolution of the split-complex field and show that the detail of dynamics is closely related to the initial conditions on the quintessence and phantom. In section IV, we investigate the cosmic evolution of the split-quaternion field. In section V, in the framework of brane world, we analyze the split-complex Dirac-Born-Infeld field. In section VI, the linear perturbations of these fields in the background

\[ 1 \text{http://en.wikipedia.org/wiki/Split-complex number.} \]
of Friedmann-Robertson-Walker Universe are present. Conclusions and discussions are given in section VII. Throughout this paper, we adopt the system of units in which \( G = c = \hbar = 1 \) and the metric signature \((- , +, +, +)\).

II. SPLIT-COMPLEX SCALAR FIELD

A. What is split-complex scalar field

The mathematic theory of split-complex numbers (or hyperbolic numbers, also perplex numbers) can be found in Ref. [14] or the Wikipedia \(^2\). Motivated by the theory of these numbers, we define the split-complex scalar field \( \Phi \) as follows

\[
\Phi = \phi + j \psi ,
\]

(1)

where \( \phi \) and \( \psi \) are two real scalar fields. The quantity \( j \) is similar to the imaginary unit \( i \) except that \( j^2 = +1 \).

Choosing \( j^2 = -1 \) results in the conventional complex scalar field. It is this change of sign which distinguishes the split-complex scalar field from the ordinary complex one. The quantity \( j \) here is not a real number but an independent quantity. Namely, it is not equal to \( \pm 1 \).

Just as for ordinary complex field, one can define the notion of a split-complex conjugate as follows

\[
\Phi^* = \phi - j \psi .
\]

(2)

Then the modulus of a split-complex scalar field is given by the isotropic quadratic form

\[
\Phi \Phi^* = \phi^2 - \psi^2 .
\]

(3)

This quadratic form is split into positive and negative parts, in contrast to the positive definite form of the ordinary complex scalar field.

Similar to the ordinary complex field which can be written in the form of Euler’s formula

\[
\Phi = \phi e^{i\theta} = \phi \cos \theta + i\phi \sin \theta ,
\]

(5)

the split-complex field has the Euler’s formula as follows

\[
\Phi = \phi e^{j\theta} = \phi \cosh \theta + j\phi \sinh \theta .
\]

(6)

This can be derived from a power series expansion using the fact that \( \cosh \) has only even powers while that for \( \sinh \) has odd powers. It follows that \( \Phi \Phi^* = \phi^2 \).

B. Quintessence and phantom from split-complex field

We shall consider the theory of a massive, split-complex, self-interacting scalar field with the Lagrangian density as follows

\[
\mathcal{L} = \frac{1}{2} \nabla_{\mu} \Phi \nabla^{\mu} \Phi^* + \frac{1}{4} \lambda^2 (\Phi \Phi^* + m^2/\lambda^2)^2 , \quad (7)
\]

with \( m \) the mass of the scalar and \( \lambda \) a coupling constant. Substituting Eq. (1) into Eq. (7), we obtain

\[
\mathcal{L} = \frac{1}{2} \nabla_{\mu} \phi \nabla^{\mu} \phi - \frac{1}{2} \nabla_{\mu} \psi \nabla^{\mu} \psi + \frac{1}{4} \lambda^2 (\phi^2 - \psi^2 + m^2/\lambda^2)^2 . \quad (8)
\]

If we take the field \( \Phi \) in the Lagrangian Eq. (7) as the ordinary complex scalar field, \( \Phi = \phi + i\psi \), the Lagrangian takes the form

\[
\mathcal{L} = \frac{1}{2} \nabla_{\mu} \phi \nabla^{\mu} \phi + \frac{1}{2} \nabla_{\mu} \psi \nabla^{\mu} \psi + \frac{1}{4} \lambda^2 (\phi^2 + \psi^2 + m^2/\lambda^2)^2 . \quad (9)
\]

It is apparent there is sign of difference before the terms \( \frac{1}{2} \nabla_{\mu} \phi \nabla^{\mu} \psi \) and \( \psi^2 \). This is due to the fact that \( i^2 = -1 \) and \( j^2 = 1 \). One can recognize that Lagrangian, Eq. (8) is nothing but the Hessefield proposed by Wei et al in Ref. [15]. \( \phi \) and \( \psi \) plays the role of quintessence and phantom, respectively. The difference of Hessefield from the quintom field \[8\]

\[
\mathcal{L} = \frac{1}{2} \nabla_{\mu} \phi \nabla^{\mu} \phi - \frac{1}{2} \nabla_{\mu} \psi \nabla^{\mu} \psi + V (\phi, \psi) , \quad (10)
\]

is that the form of the scalar potential is greatly constrained by \( V(\phi^2 - \psi^2) \) in Hessefield. We point out that there is a remarkable difference in our motivation from the Hessefield. Ref. [15] propose the Lagrangian density of Hessefield as follows

\[
\mathcal{L} = \frac{1}{2} \nabla_{\mu} \Phi \nabla^{\mu} \Phi + \frac{1}{2} \nabla_{\mu} \Phi^* \nabla^{\mu} \Phi^* + V (\Phi^2 + \Phi^*^2) . \quad (11)
\]

with \( \Phi \) the conventional complex scalar field. However, our Lagrangian density Eq. (7) is different from Eq. (11) not only on the Lagrangian expression but also on the physical meaning of scalar field \( \Phi \). In Fig. 1 we plot the scalar potential \( V \propto (\Phi^* \Phi + M^2)^2 \) with the quintessence field \( \phi \) and the phantom field \( \psi \). The potential has the vanishing absolute vacuum energy on the hyperbola \( \phi^2 - \psi^2 + M^2 = 0 \).

C. symmetry

The theory of Lagrangian Eq. (7) has the symmetry that it is invariant after a hyperbolic rotation

\[
\Phi \rightarrow \Phi e^{j\alpha} , \quad (12)
\]

\(^2\) [http://en.wikipedia.org/wiki/Split–complex number].
to a conserved charge which is given by the formula

\[ \text{Then the Noether theorem tells us this symmetry leads to a conserved charge which is given by the formula} \]

\[ Q = \frac{1}{2} \int d^3x \left( \Phi^* \partial_0 \Phi - \Phi \partial_0 \Phi^* \right), \] (15)

in the background of four dimensional Minkowski space-time. Here \( \partial_0 \) represents the derivative with the time.

D. equation of motion and energy-momentum tensor

Introduce the metric tensor \( \eta_{IJ} \) in the field space \( \Phi_I = (\phi, \psi) \),

\[ \eta_{IJ} = \eta^{IJ} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \] (16)

where \( I, J = 1, 2 \). Eq. (15) can be expressed as

\[ \mathcal{L} = \frac{1}{2} \nabla_\mu \Phi_I \nabla^\mu \Phi_I + \frac{1}{4} \lambda^2 \left( \Phi_I \Phi_I^* + m^2 \lambda^2 \right)^2. \] (17)

With this form, the theory belongs to the general multiple scalar field theory considered in Ref. 16–20. The equation of motion and the energy momentum are give by

\[ \nabla_\mu \nabla^\mu \Phi_I - m^2 \Phi_I - \lambda^2 \Phi_I \Phi_I^* \Phi_I = 0, \] (18)

and

\[ T_{\mu\nu} = -\nabla_\mu \Phi_I \nabla^\mu \Phi_I + g_{\mu\nu} \mathcal{L}. \] (19)

From the equation of motion, we see both the quintessence field \( \phi \) and the phantom field \( \psi \), have the same positive mass \( m \). This is different from the usual phantom field with the Lagrangian

\[ \mathcal{L} = -\frac{1}{2} \nabla_\mu \psi \nabla^\mu \psi + V(\psi), \] (20)

and the equation of motion

\[ \nabla_\mu \nabla^\mu \psi + m^2 \psi = 0, \] (21)

where \( m^2 = V_{\psi\psi} \left( \frac{\psi}{\psi^*} \right) \). The value of \( \psi = \psi_0 \) corresponds to the global minimum of the potential. In this case, the phantom field acquires an imaginary mass which would lead to the classically and quantum instability. Then why is there such a difference in the mass? The reason are as follows. The usual phantom scalar potential has the form \( V = \frac{1}{2}m^2 \psi^2 \) at the global minimum. However, for the Lagrangian Eq. (13), the phantom potential takes the form \( V = -\frac{1}{2}m^2 \psi^2 \) at the global minimum. Thus it is very important that the potential of the split-complex scalar is constrained to be \( V(\phi^2 - \psi^2) \) in order to avoid the problem of instability.

E. conformal invariant split-complex field

Recently, Kallosh and Linde 21 proposed a simple conformally invariant two-field model of dS/AdS space. The model consists of two real scalar fields, \( \phi \) and \( \psi \), which has the \( SO(1, 1) \) symmetry:

\[ \mathcal{L}_{KL} = \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{2} \nabla_\mu \psi \nabla^\mu \psi + \frac{1}{12} (\phi^2 - \psi^2)^2 \frac{\lambda}{4} \] (22)

Here \( R \) is the Ricci scalar and \( \lambda \) a coupling constant. This theory is locally conformal invariant under the following transformations,

\[ \tilde{g}_{\mu\nu} = e^{-2\sigma(x)} g_{\mu\nu}, \quad \tilde{\phi} = e^{\sigma(x)} \phi, \quad \tilde{\psi} = e^{\sigma(x)} \psi. \] (23)

The global \( SO(1, 1) \) symmetry is a boost between these two fields. Using the concept of our split-complex scalar field \( \Phi_I \), the theory is equivalent to

\[ \mathcal{L} = \frac{1}{2} \nabla_\mu \Phi_I \nabla^\mu \Phi_I + \frac{1}{12} R \Phi_I \Phi_I - \frac{1}{4} \lambda (\Phi_I \Phi_I^*)^2. \] (24)

In other words, the two scalar fields considered in Ref. 21 is exactly the conformal invariant split-complex field.
III. DYNAMICS OF SPLIT-COMPLEX SCALAR FIELD

In this section, we investigate the cosmic evolution of the split-complex field in the background of spatially flat Friedmann-Robertson-Walker (FRW) Universe

\[ ds^2 = -dt^2 + a(t)^2 \left( dr^2 + r^2 d\Omega^2 \right) , \]  

(25)

where \( a(t) \) is the cosmic scale factor. We model all other matter sources present in the Universe as perfect fluids. These matter sources can be baryonic matter, relativistic matter, and dark energy. We assume there is no interaction between the split-complex scalar field and other matter fields, other than by gravity. Then the Einstein action between the split-complex scalar field and other matter fields, other than by gravity. Then the action can be written in the following autonomous form

\[
\begin{align*}
\frac{dX}{dN} &= -3X - \frac{3}{2} \lambda_\phi Z^2 - X \frac{\dot{H}}{H^2} , \\
\frac{dY}{dN} &= -3Y - \frac{3}{2} \lambda_\psi Z^2 - Y \frac{\dot{H}}{H^2} , \\
\frac{dZ}{dN} &= -\frac{3}{2} Z \left( \lambda_\phi X + \lambda_\psi Y \right) - Z \frac{\dot{H}}{H^2} , \\
\frac{dU}{dN} &= -\frac{3}{2} U - U \frac{\dot{H}}{H^2} , \\
\frac{d\lambda_\phi}{dN} &= \frac{3X \left( \lambda_\phi^2 - \lambda_\psi^2 \right)}{2 + 2 \sqrt{1 - s^2 \left( \lambda_\phi^2 - \lambda_\psi^2 \right)}} + \frac{3}{2} \lambda_\phi \left( X \lambda_\phi - Y \lambda_\psi \right) , \\
\frac{d\lambda_\psi}{dN} &= \frac{3Y \left( \lambda_\phi^2 - \lambda_\psi^2 \right)}{2 + 2 \sqrt{1 - s^2 \left( \lambda_\phi^2 - \lambda_\psi^2 \right)}} + \frac{3}{2} \lambda_\psi \left( X \lambda_\phi - Y \lambda_\psi \right) ,
\end{align*}
\]

(31)

together with a constraint equation

\[ X^2 - Y^2 + Z^2 + U^2 + \frac{\kappa^2 \rho_r}{3H^2} = 1 . \]

(32)

Here

\[ s = \frac{\sqrt{6} \mu \kappa}{4} \lambda , \]

(33)

and

\[ \frac{\dot{H}}{H^2} = -2 - X^2 + Y^2 + 2Z^2 + \frac{1}{2} U^2 . \]

(34)

The equation of state \( w \) and the fraction of the energy density for the split-complex scalar field are

\[ w \equiv \frac{X^2 - Y^2 - Z^2}{X^2 - Y^2 + Z^2} , \]

\[ \Omega_\Phi \equiv \frac{X^2 - Y^2 + Z^2}{X^2 - Y^2 + Z^2} , \]

(35)

As an example, we consider \( s = 1 \). Physically, the quintessence field \( \psi \) would roll down the potential and the phantom \( \psi \) roll up the potential. Therefore we can safely assume \( X > 0, Y > 0, \lambda_\phi > 0, \lambda_\psi < 0 \). We investigate the cosmology model with the following values of parameters: \( \Omega_{k0} = 0, \Omega_{m0} = 0.27, \Omega_{r0} = 8.1 \times 10^{-5}, \Omega_{\Lambda0} = 0.73 \), which are consistent with current observations \cite{22}.

In Fig. \[ 2 \] we plot the phase plane for the evolution of the split-complex scalar with a range of different initial conditions. The point \( (0, 0, 0) \) corresponds to the radiation or matter dominated epochs. The point \( (0, 0, 0) \) is unstable and the point \( (0, 0, 1) \) is stable and thus...
an attractor. These trajectories show that the Universe always ends at the split-complex scalar potential dominated epoch.

In Fig. 3, we plot the evolution of density fractions for radiation, dark matter and the split-complex scalar field. This shows that the split-complex scalar field can mimic the dark energy very well.

In Fig. 4, we plot the evolution of the equation of state for the split-complex scalar. It behaves as the stiff matter at higher redshifts and a cosmological constant for the lower redshifts. It can cross the phantom divide around the value of $N = -5$ (redshift 3000).

IV. DYNAMICS OF SPLIT-QUATERNION SCALAR FIELD

A. split-quaternion scalar field

In this subsection, we define the notion of split-quaternion scalar field and the split-octonion field. We find they actually consists of two quintessence fields and two phantom fields. The number of quintessence is exactly the same as that of the phantom. To this end, we start from the theory of split-quaternion number $q$ (or coquaternion) which is given by

$$q = u + ix + jy + k z ,$$  \hspace{1cm} (36)

where $u, x, y, z$ are real numbers. The quantity $i, j, k$ here are not real numbers but independent quantities. The

[http://en.wikipedia.org/wiki/Split−quaternion]
products of these elements are \[23\]

\[
\begin{align*}
ij &= k = -ji, \\
jk &= i = -kj, \\
ki &= -i = -jk, \\
i^2 &= j = -ik, \\
j^2 &= +1, \\
k^2 &= +1,
\end{align*}
\]

and hence \(ijk = 1\). A split-quaternion has a conjugate

\[
q^* = u - ix - jy - kz ,
\]

and multiplicative modulus

\[
qq^* = w^2 + x^2 - y^2 - z^2.
\]

This quadratic form is split into positive and negative parts, in contrast to the positive definite form on the algebra of quaternions.

We now define the split-quaternion scalar field \(\Phi\) as

\[
\Phi = \phi_1 + i\phi_2 + j\psi_1 + k\psi_2.
\]

Then the Lagrangian

\[
\mathcal{L} = \frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi^* + \frac{1}{4\lambda^2} \left( \Phi \Phi^* + \frac{6\Lambda}{\kappa^2} \right)^2,
\]

can be written as

\[
\mathcal{L} = \frac{1}{2} \nabla_\mu \phi_1 \nabla^\mu \phi_1 + \frac{1}{2} \nabla_\mu \phi_2 \nabla^\mu \phi_2
\]

\[
-\frac{1}{2} \nabla_\mu \psi_1 \nabla^\mu \psi_1 - \frac{1}{2} \nabla_\mu \psi_2 \nabla^\mu \psi_2
\]

\[
+ \frac{1}{4\lambda^2} \left( \phi_1^2 + \phi_2^2 - \psi_1^2 - \psi_2^2 + \frac{6\Lambda}{\kappa^2} \right)^2.
\]

Here \(\Lambda\) is a positive constant. It is apparent that the theory consists of two quintessence field \(\phi_1, \phi_2\) and two phantom field \(\psi_1, \psi_2\). Introduce the metric tensor \(\eta_{IJ}\) in the space of field \(\Phi_I = (\phi_1, \phi_2, \psi_1, \psi_2)\),

\[
\eta_{IJ} = \eta^{IJ} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\]

where \(I, J = 1, 2, 3, 4\). Eq. (42) can be expressed as

\[
\mathcal{L} = \frac{1}{2} \nabla_\mu \Phi_I \nabla^\mu \Phi^*_I + \frac{1}{4\lambda^2} \left( \Phi_I \Phi^*_I + \frac{6\Lambda}{\kappa^2} \right)^2.
\]

With this form, the theory also belongs to the general multiple scalar field theory considered in Ref. [16–20]. Furthermore, we could define the split-octonion field (motivated by the split-octonion)\(^4\) by introducing the metric tensor in the space of field \(\Phi_I = (\phi_1, \phi_2, \phi_3, \psi_1, \psi_2, \psi_3, \psi_4)\),

\[
\eta_{IJ} = \eta^{IJ} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix},
\]

where \(I, J = 1, 2, 3, 4, 5, 6, 7, 8\). The Lagrangian Eq. (44) can also describe a massive, self-interacting split-octonion scalar field.

### B. dynamics

In this section, we study the dynamics of the split-quaternion scalar field in the background of spatially flat FRW Universe. For simplicity, we only consider the cosmic evolution of the split-quaternion scalar field.

The Einstein equations and the equation of motion of the scalar fields take the form

\[
3H^2 = \kappa^2 \left( \frac{1}{2} \dot{\Phi}_I \ddot{\Phi}^*_I + V \right),
\]

\[
2\dot{H} + 3H^2 = -\kappa^2 \left( \frac{1}{2} \dot{\Phi}_I \ddot{\Phi}^*_I - V \right),
\]

and

\[
\ddot{\Phi}_I + 3H \dot{\Phi}_I + V_{\phi I} = 0,
\]

respectively.

Introduce the following dimensionless quantities

\[
X \equiv \frac{\kappa \dot{\phi}_1}{\sqrt{6} H}, \quad \lambda_{\phi_1} \equiv \frac{3}{\sqrt{6\kappa}} \frac{V_{\phi_1}}{V},
\]

\[
Y \equiv \frac{\kappa \dot{\phi}_2}{\sqrt{6} H}, \quad \lambda_{\phi_2} \equiv \frac{3}{\sqrt{6\kappa}} \frac{V_{\phi_2}}{V},
\]

\[
Z \equiv \frac{\kappa \dot{\psi}_1}{\sqrt{6} H}, \quad \lambda_{\psi_1} \equiv -\frac{3}{\sqrt{6\kappa}} \frac{V_{\psi_1}}{V},
\]

\[
U \equiv \frac{\kappa \dot{\psi}_2}{\sqrt{6} H}, \quad \lambda_{\psi_2} \equiv -\frac{3}{\sqrt{6\kappa}} \frac{V_{\psi_2}}{V},
\]

\[
N \equiv \ln a,
\]

then the equations of motion can be written in the fol-
lowing autonomous form
\[
\begin{align*}
\frac{dX}{dN} &= -3X - XB - \lambda_{\phi_1} A, \\
\frac{dY}{dN} &= -3Y - YB - \lambda_{\phi_2} A, \\
\frac{dZ}{dN} &= -3Z - ZB - \lambda_{\psi_1} A, \\
\frac{dU}{dN} &= -3U - UB - \lambda_{\psi_2} A, \\
\frac{d\lambda_{\phi_1}}{dN} &= -X\lambda_{\phi_1}^2 - Y\lambda_{\phi_1} \lambda_{\phi_2} - Z\lambda_{\phi_1} \lambda_{\psi_1} - U\lambda_{\phi_1} \lambda_{\psi_2} \\
&\quad+ \frac{XC}{1 + \sqrt{1 - AC}}, \\
\frac{d\lambda_{\phi_2}}{dN} &= -X\lambda_{\phi_1} \lambda_{\phi_2} - Y\lambda_{\phi_2}^2 - Z\lambda_{\phi_2} \lambda_{\psi_1} - U\lambda_{\phi_2} \lambda_{\psi_2} \\
&\quad+ \frac{YC}{1 + \sqrt{1 - AC}}, \\
\frac{d\lambda_{\psi_1}}{dN} &= X\lambda_{\phi_1} \lambda_{\psi_1} + Y\lambda_{\phi_2} \lambda_{\psi_1} + Z\lambda_{\psi_1}^2 + U\lambda_{\psi_2} \lambda_{\psi_1} \\
&\quad+ \frac{ZC}{1 + \sqrt{1 - AC}}, \\
\frac{d\lambda_{\psi_2}}{dN} &= X\lambda_{\phi_1} \lambda_{\psi_2} + Y\lambda_{\phi_2} \lambda_{\psi_2} + Z\lambda_{\psi_1} \lambda_{\psi_2} + U\lambda_{\psi_2}^2 \\
&\quad+ \frac{UC}{1 + \sqrt{1 - AC}},
\end{align*}
\]
\[
A \equiv 1 - X^2 - Y^2 + Z^2 + U^2,
\]
\[
B \equiv -3(X^2 + Y^2 - Z^2 - U^2),
\]
\[
C \equiv \lambda_{\phi_1}^2 + \lambda_{\phi_2}^2 - \lambda_{\psi_1}^2 - \lambda_{\psi_2}^2,
\]
(49)
together with a constraint equation
\[
X^2 + Y^2 - Z^2 - U^2 + \frac{k^2 V}{3H^2} = 1.
\]
(50)
The equation of state \(w\) of the split-quaternion scalar field is
\[
w \equiv -1 - \frac{2}{3} \frac{\dot{H}}{H^2} = -1 - \frac{2}{3} B.
\]
(51)

We note that the coupling constant \(\lambda\) is not present in above equations. So there is only one free parameter, \(\Lambda\). As an example, we consider \(\Lambda = 1\). Physically, the quintessence \(\phi_1, \phi_2\) would roll down the potential and the phantom \(\psi_1, \psi_2\) roll up the potential. Therefore we shall assume \(\lambda_{\phi_1} > 0, \lambda_{\phi_2} > 0, \lambda_{\psi_1} < 0, \lambda_{\psi_2} < 0, X < 0, Y < 0, Z > 0, U > 0\).

In Fig. 7 and Fig. 8, respectively, we plot the evolution of the scaled velocities \(X, Y, Z, U\) for the quintessence fields \(\phi_1, \phi_2\) and the phantom fields \(\psi_1, \psi_2\), respectively, with respect to the equation of state \(w\). They show that if the kinetic energy of quintessence dominates over that of phantom initially, \(X^2 + Y^2 > Z^2 + U^2\) \((X(0) = -1.22, Y(0) = -0.95, Z(0) = 0.99, U(0) = 0.70, \lambda_{\phi_1}(0) = 1/20, \lambda_{\phi_2}(0) = 1/20, \lambda_{\psi_1}(0) = -1/20, \lambda_{\psi_2}(0) = -1/20)\), the equation of state of the split-quaternion field would evolve from \(+1\) to \(-1\) with the point \((X, Y, Z, U, w) = (0, 0, 0, 0, -1)\) as the attractor (Fig. 7). On the other hand, if the kinetic energy of phantom dominates over that of quintessence initially, \(X^2 + Y^2 < Z^2 + U^2\) \((X(0) = -0.95, Y(0) = -0.78, Z(0) = 1.2, U(0) = 1.0, \lambda_{\phi_1}(0) = 1/20, \lambda_{\phi_2}(0) = -1/20, \lambda_{\psi_1}(0) = -1/20, \lambda_{\psi_2}(0) = -1/20)\), the equation of state of the split-quaternion field would evolve from minus infinity to \(-1\) with the point \((X, Y, Z, U, w) = (0, 0, 0, 0, -1)\) as the attractor (Fig. 8). Corresponding to Fig. 7 and Fig. 8, we plot the equation of state in Fig. 1 and Fig. 2 respectively.

In Fig. 9 we plot the evolution of the scaled velocities \(X, Y, Z, U\) for the quintessence fields \(\phi_1, \phi_2\), the phantom fields \(\psi_1, \psi_2\), respectively, with respect to the equation of state \(w\). The initial values are put by \((X(0) = -1, Y(0) = -1.1, Z(0) = 1, U(0) = 1.1, \lambda_{\phi_1}(0) = 1/20, \lambda_{\phi_2}(0) = 1/20, \lambda_{\psi_1}(0) = -1/27, \lambda_{\psi_2}(0) = -1/20)\). The figure shows that the equation of state of the split-quaternion field could cross the phantom divide with the point \((X, Y, Z, U, w) = (0, 0, 0, 0, -1)\) as the attractor. In Fig. 10 we plot evolution of the scaled velocities \(X, Y, Z, U\) for the quintessence fields \(\phi_1, \phi_2\), the phantom fields \(\psi_1, \psi_2\), respectively, with respect to the equation of state \(w\). The initial values are put by \((X(0) = -0.8, Y(0) = -0.99, Z(0) = 0.805, U(0) = 0.99, \lambda_{\phi_1}(0) = 1/15, \lambda_{\phi_2}(0) = 1/15, \lambda_{\psi_1}(0) = -1/20, \lambda_{\psi_2}(0) = -1/20)\). The figure shows that the equation of state of the split-quaternion field cold cross the phantom divide with the point \((X, Y, Z, U, w) = (0, 0, 0, 0, -1)\) as the attractor. In Fig. 11 and Fig. 12, we plot the evolution of their equation of state, respectively.

In all, the detail of dynamics of the split-quaternion scalar field is closely related to the initial conditions on the fields.

V. SPLIT-COMPLEX DBI FIELD

A. Multi-field theory

In the proceeding sections, we have studied the cosmic evolution of the split-complex field and the split-quaternion field with the Lagrangian as follows, \(\mathcal{L} = \frac{1}{2} \partial_\mu \Phi^I \partial^\mu \Phi_I + V\), Eq. (12) and (43). In this subsection, we extend the expression of Lagrangian and show that they actually belong to the multi-field theories studied in the inflationary cosmology [17, 20]. To this end, we define
\[
X^{IJ} \equiv -\frac{1}{2} \partial_\mu \Phi^I \partial^\mu \Phi^J.
\]
(52)

In the spirit of k-inflation [24], the very general Lagrangian is of the form
\[
\mathcal{L} = K \left( X^{IJ}, \Phi^I \right),
\]
(53)
where $I = 1, \cdots, N$ labels the multiple fields. Here $\Phi_I$ can be split-complex field ($N=2$), split-quaternion field ($N=4$), split-octonion field ($N=8$) and so on. The field indices $I, J$ are raised and lowered with the metric tensor $\eta_{IJ}$ and $\eta^{IJ}$ of the field space. If $\eta^{IJ}$ is replaced with the most general metric $\bar{g}_{IJ}(\Phi^K)$, it is just the extensively studied multi-field theory in inflationary cosmology [16, 20].

### B. split-complex DBI Lagrangian

In this subsection, we focus on the construction of a split-complex DBI Lagrangian which governs the dynamics of a D3-brane. To this end, we consider a D3-brane with tension $T_3$ evolving in a 6-dimensional spacetime described by the metric (following Ref. [23])

$$
\begin{align*}
\text{d}s_6^2 &= h^{-1/2}(y^K)\ g_{\mu\nu}\ dx^\mu dx^\nu + h^{1/2}(y^K)\ \eta_{IJ}\ dy^I dy^J \\
&= H_{AB}dY^A dY^B,
\end{align*}
$$

(54)

with coordinates $Y^A = (x^\mu, y^J)$. Here $\mu = 0, 1, 2, 3$ and $J = 1, 2, 3$. $h(y^K)$ is the warp factor, $g_{\mu\nu}$ is the metric for our four dimensional spacetime, $\eta_{IJ}$ is the metric tensor for the internal spacetime. We embed the mobile D3-brane in this 6-dimensional spacetime whose world-volume dynamics are described by the DBI action. The D3-brane is free to move on the internal compact Calabi-Yau manifold.

The kinetic part of the DBI action

$$
\mathcal{L}_{DBI} = -T_3\sqrt{-\det\gamma_{\mu\nu}},
$$

(55)

depends on the determinant of the induced metric on the 3-brane

$$
\gamma_{\mu\nu} = H_{AB}\partial_\mu Y^A_\ell \partial_\nu Y^B_\ell.
$$

(56)

The embedding of the brane is defined by $Y^A_\ell(x^\mu)$. Let $Y^A_\ell = (x^\mu, \phi^J(x^\mu))$, we find

$$
\gamma_{\mu\nu} = h^{-1/2}\left(g_{\mu\nu} + h\eta^{IJ}\partial_\mu \phi^I \partial_\nu \phi^J\right).
$$

(57)

Then the DBI Lagrangian Eq. (55) becomes

$$
\mathcal{L}_{DBI} = -T_3 h^{-1} \sqrt{-g} \sqrt{\det \left( \delta^\ell_{\mu} + h\eta^{IJ} \partial_\mu \phi^I \partial_\nu \phi^J \right)}.
$$

(58)

Define $f$ and $\Phi^I$ as follows
FIG. 7: The evolution of the equation of state for the split-quaternion scalar when the kinetic energy of quintessence dominates over that of phantom (with the same strength of fields). It behaves as the stiff matter at higher redshifts and a cosmological constant for the lower redshifts.

FIG. 8: The evolution of the equation of state for the split-quaternion scalar when the kinetic energy of phantom dominates over that of quintessence (with the same strength of fields). The equation of state is always smaller than unit one.

\[ f = \frac{\hbar}{T_3}, \quad \Phi^I = \sqrt{T_3}\phi^I, \]

we obtain the DBI Lagrangian

\[ \mathcal{L}_{DBI} = -f^{-1}\sqrt{-g}\left(\sqrt{\mathcal{D}} - 1\right) + \sqrt{-g}V(\Phi^I), \]

\[ \mathcal{D} \equiv \det \left(\delta^\mu_\nu + f\eta^{IJ}\partial^\nu\phi_I\partial_\nu\phi_J\right). \]

The potential term \( V(\Phi^I) \) is present in the above DBI Lagrangian which arises from the brane’s interactions with bulk fields or other branes. In the next, we will assume the potential is given by Eq. (17) or Eq. (44) for simplicity. Now we identify \( \Phi^I \) as the split-complex field and \( \eta^{IJ} \) as the metric defined by Eq. (16). Then Eq. (60) is the split-complex DBI Lagrangian.

We note that one time-like extra dimension appears in the internal geometry due to the fact that

\[ \eta^{IJ} = \eta^{IJ} = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right). \]

Extra timelike dimension usually leads to the presence of tachyon field. The tachyon field violates the causality, the conserved probability and makes massive bodies unstable. But the effect is not unacceptable experimentally if the scale of the extra dimension is bounded below a sufficiently small size \[\text{(26, 27)}\]. In fact, there is no priori reason why extra dimensions cannot be timelike, and the observational constraints on braneworld models with extra timelike dimensions have been greatly discussed in \[\text{(26, 27)}\]. In the past years, such models were under consideration in \[\text{(26, 27)}\]. It is found that a timelike extra dimension could lead to very interesting new features. For instance, a contracting braneworld generically bounces as it reaches a high density thereby leading to the absence of the “Big Crunch” and “Big Bang” singularities in General Relativity.
Attractor

\[ X(t) = -0.8, \quad Y(t) = -0.99, \quad Z(t) = 0.805, \quad U(t) = 0.99, \quad \lambda_{\phi_1}(0) = 1/15, \quad \lambda_{\phi_2}(0) = 1/15, \quad \lambda_{\psi_1}(0) = -1/20, \quad \lambda_{\psi_2}(0) = -1/20. \]

The figure shows that the equation of state of the split-quaternion field cold cross the phantom divide with the point \((X,Y,Z,U,w) = (0,0,0,0,-1)\) as the attractor.

One can express the DBI Lagrangian in Eq. (60) explicitly in terms of \(X_{IJ}\) defined in Eq. (52). As shown in [20], one have

\[ \mathcal{D} = 1 - 2f X^I_I + 4f^2 X^I_I X^J_J - 8f^3 X^I_I X^J_J X^K_K + 16f^4 X^I_I X^J_J X^K_K X^L_L, \]

where the brackets denote antisymmetrisation on the field indices. If there are three scalar fields, the last term disappears because of the antisymmetrization over the field indices. For two scalar fields, the last two terms disappear; and for one scalar field, only the first two terms remain. For more than four scalar fields, the truncation at order \(f^4\) is natural if one considers \(\mathcal{D}\) as the determinant of a 4 \(\times\) 4 matrix.

The split-complex field consists of two scalar fields. So the corresponding DBI Lagrangian takes the form

\[ \mathcal{L}_{SC} = \sqrt{-g} \left( \frac{1}{2} - \frac{1}{7} \sqrt{1 - 2f X^I_I + 4f^2 X^I_I X^J_J} \right) \]

\[-\sqrt{-g} V. \]

C. equations of motion

In the background of spatially flat FRW Universe, the scalar fields depend only on cosmic time, \(t\), and

\[ X^{IJ} = \frac{1}{2} \Phi^I \Phi^J. \]

So the term \(4f^2 X^I_I X^J_J\) in Eq. (63) vanishes because of the antisymmetrisation on field indices. However, in inhomogeneous universe, this term are present as higher order in gradients. Ref. [22] show that it drastically changes the behavior of cosmic perturbations. From this discussion we see this kind of DBI Lagrangian does not belong to the k-essence-like \(K(\eta^{IJ} X_{IJ}, \Phi^I)\) the-
ory because it does not uniquely depend on $X = \eta^{IJ}X_{IJ}$.

In the background of spatially flat FRW Universe, the Einstein equations and the equations of motion of $\Phi^I$ read

$$3H^2 = \kappa^2 \left[ \frac{1}{f} \left( \frac{1}{\sqrt{1 - 2f X^2}} - 1 \right) + V \right],$$

$$2\dot{H} + 3H^2 = -\kappa^2 \left[ \frac{1}{f} \left( 1 - \sqrt{1 - 2f X^2} \right) - V \right],$$

$$\frac{1}{a^3} \frac{d}{dt} \left( \frac{a^3 \phi}{\sqrt{1 - 2f X^2}} \right) + \frac{\partial V}{\partial \phi} = 0. \quad (64)$$

For simplicity, we assume the warp factor $f$ has the form

$$f = \frac{1}{V}, \quad (65)$$

and the scalar potential $V$ is given by

$$V = V_0 \left( \Phi^I \Phi_I + \frac{12 \Lambda^2}{\kappa^2} \right)^2. \quad (66)$$

Here $V_0$ and $\Lambda$ are two constants. Then Eqs. (64) become

$$3H^2 = \kappa^2 \left( \frac{a^3 SC}{T^2 L_{SC}} \right),$$

$$2\dot{H} + 3H^2 = \kappa^2 \left( \frac{L_{SC}}{a^3} \right),$$

$$\left[ (L_{SC})_{,\phi} \right] - (L_{SC})_{,\phi} = 0,$$

$$\left[ (L_{SC})_{,\psi} \right] - (L_{SC})_{,\psi} = 0,$$

$$L_{SC} \equiv \frac{a^3}{f} \sqrt{1 - f \phi^2 + f \psi^2}. \quad (67)$$

In the next subsection, we shall investigate the dynamics of above equations of motion and find that the split-complex DBI field could cross the phantom divide [34].

It is apparent if the kinetic energy of quintessence $\phi^2$ is larger than the kinetic energy of phantom $\psi^2$, the equation of state would be larger than $-1$. On the other hand, if the kinetic energy of quintessence $\phi^2$ is smaller than that of phantom $\psi^2$, the equation of state would be smaller than $-1$. Then the behavior of phantom divide crossing may be achieved. In the next, we shall show it is indeed the case.

In order to investigate the evolution of the equation of state numerically, we should rewrite Eqs. (67) in the form of autonomous system. To this end, we define

$$X \equiv \sqrt{f} \phi, \quad Y \equiv \sqrt{f} \psi,$$

$$\lambda_\phi \equiv \sqrt{\frac{3}{f}} f_\phi, \quad \lambda_\psi \equiv \frac{\sqrt{3}}{f} f_\psi,$$

$$N \equiv \ln a. \quad (70)$$

Then we obtain the following autonomous form for the equations of motion

$$\frac{dX}{dN} = -\frac{1}{2} X^2 \lambda_\phi A - \frac{1}{2} XY \lambda_\psi A - XB + 2\lambda_\phi A^5,$$

$$\frac{dY}{dN} = -\frac{1}{2} XY \lambda_\phi A - \frac{1}{2} Y^2 \lambda_\psi A - YB - 2\lambda_\psi A^5,$$

$$\frac{d\lambda_\phi}{dN} = -\frac{1}{2} XAC + \frac{1}{2} X^2 \lambda_\phi A + \frac{1}{2} Y \lambda_\phi A,$$

$$\frac{d\lambda_\psi}{dN} = -\frac{1}{2} YAC + \frac{1}{2} Y^2 \lambda_\psi A + \frac{1}{2} X \lambda_\phi A,$$

$$A \equiv (1 - X^2 + Y^2)^{1/4},$$

$$B \equiv 3 (1 - X^2 + Y^2),$$

$$C \equiv 1 - \sqrt{1 - A^2 \left( \lambda_\phi^2 - \lambda_\psi^2 \right)}, \quad (71)$$

together with a constraint equation

$$1 = \frac{\kappa^2}{3H^2 f}. \quad (72)$$

The equation of state $w$ of the split-complex scalar field is

$$w = - \left( 1 - X^2 + Y^2 \right). \quad (73)$$

We note that the constant $V_0$ in the scalar potential $V$ is not present in above equations. So there is only one free parameter, $\Lambda$. As an example, we consider $\Lambda = 1$. Physically, the quintessence $\phi$ would roll down the potential and the phantom $\psi$ roll up the potential. Therefore we shall assume $\lambda_\phi > 0, \lambda_\psi > 0, X < 0, Y > 0$.

In Fig. 13 and Fig. 14 we plot the evolution of the equation of state for the split-complex field with different initial conditions. Fig. 13 shows that if phantom dominates over quintessence initially, $(X(0) = -0.99, Y(0) = 0.999, \lambda_\phi(0) = 0.01, \lambda_\psi(0) = 0.9)$, the equation of state would evolve from minus infinity to $-1$ with the two times of phantom divide crossing. On the other hand, if quintessence dominates over phantom initially, $(X(0) =
−0.998, \( Y(0) = 0.001 \), \( \lambda_\eta(0) = 0.01 \), \( \lambda_\eta(0) = 0.99997 \), the equation of state would evolve from zero to \(-1\) with one time of phantom divide crossing. In all, the detail of dynamics of the split-complex scalar field is closely related to the initial conditions on the fields.

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VI. LINEAR PERTURBATIONS

Up to now, we have explored the cosmic evolution of split-complex field, split-quaternion field and the split-complex DBI field, respectively. In this section, we focus on the linear perturbations for the split-complex field. The extension of the result to split-quaternion and split-octonion field is straightforward.

We start from the action as follows (following the convention and notation in Ref. [16])

\[
S = \int d^4x \sqrt{-g} P(Y, \Phi^I),
\]

where

\[
Y = \eta_{IJ} X^I X^J + \frac{b(\Phi^K)}{2} (X^2 - X^I X^J),
\]

where \( \Phi^K \) is the split-complex field. \( \eta_{IJ} \) is defined by Eq. (16). When \( b = 0 \), the Lagrangian, Eq. (17) is included as a particular case of \( P(X) \). On the other hand, when \( b = -2f \), the split-complex DBI Lagrangian, Eq. (63) is included in it [16].

In order to study the evolution of linear perturbations in the background of FRW Universe, we expand the above action to second order, including both metric and scalar field perturbations. In the uniform curvature gauge, the perturbed split-complex field takes the form

\[
\Phi^I(x, t) = \Phi^I_0(t) + Q^I(x, t),
\]

where \( Q^I \) denotes the field perturbations. In the following, we will usually drop the subscript "0" on \( \Phi^I_0 \) and simply identify \( \Phi^I \) as the fields in FRW Universe unless otherwise stated.

The second order action can be expressed as

\[
S_{(2)} = \frac{1}{2} \int dtd^3x a^2 \left[ (\tilde{P} Y \eta_{IJ} + \tilde{P} Y \Phi^I \Phi^J) \dot{Q}^I \dot{Q}^J - \frac{1}{a^2} \tilde{P} Y [(1 + b X) \eta_{IJ} - b X_{IJ}] \partial_i Q^I \partial^i Q^J - \tilde{M}_{IJ} Q^I Q^J + 2 \tilde{P} Y \Phi^I Q^J \dot{Q}^I \right],
\]

with the effective squared mass matrix

\[
\tilde{M}_{IJ} = -\tilde{P}_{IJ} + \frac{X \tilde{P}_Y}{H} (\tilde{P} Y \Phi^I \Phi^J) + \frac{X \tilde{P}_Y}{2H^2} (1 - \frac{1}{c_{ad}^2}) \dot{\Phi}^I \dot{\Phi}^J - \frac{1}{a^2} \tilde{P} Y \left[ \frac{a^2}{2H^2} \tilde{P}_Y \left( 1 + \frac{1}{c_{ad}^2} \right) \dot{\Phi}^I \dot{\Phi}^J \right].
\]

Here \( \tilde{P}_Y \) and \( \tilde{P}_J \) denote the derivative of \( \tilde{P} \) with respect to \( Y \) and \( \Phi^I \). Dot denotes the derivative with respect to cosmic time \( t \). \( a \) and \( H \) are the scale factor and Hubble parameter of the Universe, respectively. \( c_{ad} \) is the sound speed for adiabatic perturbations defined by

\[
c_{ad}^2 = \frac{\tilde{P}_Y}{\tilde{P}_Y + 2X \tilde{P}_Y Y}. \tag{79}
\]
The sound speed squared of entropy perturbations is defined by

\[ c_{en}^2 = 1 + bX. \] (80)

For Lagrangian, Eq. (17), we have

\[ c_{ad} = c_{en} = 1. \] (81)

For the split-complex DBI Lagrangian, Eq. (63), we have

\[ c_{ad} = c_{en} = 1 + bX. \] (82)

\[ = 1 - 2fX \]

\[ = 1 - f\dot{\phi}^2 + f\ddot{\psi}^2. \] (83)

The perturbations can be decomposed into adiabatic and entropy part as follows, \( Q^I = Q_\sigma e^I_\sigma + Q_s e^I_s, \) where

\[ e^I_\sigma = e^I_1 = \frac{\dot{\Phi}^I}{\sqrt{P_{XJ}^I\Phi^J\Phi^K}}, \]

and \( e^I_s \) is the unit vector orthogonal to \( e^I_\sigma \). Use the conformal time \( \tau = \int dt/a(t) \) and define the canonically normalized fields

\[ v_\sigma = \frac{a}{c_{ad}}Q_\sigma, \quad v_s = \frac{a}{c_{en}}Q_s, \] (84)

one derive the equations of motion for \( v_\sigma \) and \( v_s \)

\[ v''_\sigma + \xi v'_\sigma + \left( c_{en}^2k^2 - \frac{\alpha''}{\alpha} + a^2\mu_s^2 \right) v_\sigma - \frac{z'}{z}\xi v_\sigma = 0, \]

\[ v''_s - \xi v'_s + \left( c_{ad}^2k^2 - \frac{z''}{z} \right) v_s - \left( \frac{z\xi'}{z} \right) v_s = 0. \] (85)

Here the prime denotes the derivative with respect to \( \tau \) and

\[ \xi \equiv \frac{a}{\dot{\sigma}P_{Y}c_{ad}} \left[ (1 + c_{ad}^2)\dot{P}_s - c_{ad}^2\dot{\sigma}^2\dot{P}_Y^s \right], \]

\[ \mu_s^2 \equiv \frac{-\dot{P}_{ss}}{P_{Y}} - \frac{1}{2c_{ad}^2k^2}P_{s}^2 - \frac{2\dot{P}_{Y}P_{s}}{P_{Y}^2} + \frac{2P_{Y}\dot{P}_s}{P_{Y}^2}, \]

\[ z \equiv \frac{a\dot{\sigma}}{c_{ad}\sqrt{\dot{P}_Y}}, \quad \alpha \equiv a\sqrt{\dot{P}_Y}, \] (86)

with

\[ \dot{\sigma} \equiv \sqrt{2\dot{X}}, \quad \dot{P}_s \equiv \dot{P}_{Y}e^I_s\sqrt{\dot{P}_Yc_{en}}, \]

\[ \dot{P}_Y \equiv \dot{P}_{Y}e^I_s\sqrt{\dot{P}_Yc_{en}}, \]

\[ \dot{P}_{ss} \equiv \dot{P}_{Y}e^I_s\sqrt{\dot{P}_Yc_{en}}. \] (87)

In the limit of weak coupling, \( \xi \approx 0, \) small effective mass, \( \mu_s \approx 0 \) and slow-roll, \( \dot{H} \approx 0, \) one have \( z''/z = 2/\tau^2 \) and \( \alpha''/\alpha = 2/\tau^2. \) So the equations of motion turn out to be

\[ v''_s + \left( c_{en}^2k^2 - \frac{2}{\tau^2} \right) v_s = 0, \]

\[ v''_s + \left( c_{ad}^2k^2 - \frac{2}{\tau^2} \right) v_s = 0. \] (88)

Refs. [33] and [36] have ever shown that above equations lead to exact scale invariant power spectrum.

VII. CONCLUSION AND DISCUSSION

In conclusion, motivated by the mathematic theory of split-complex numbers, the split-quaternion numbers and split-octonion numbers, we have proposed the notion of split-complex scalar field, split-quaternion scalar field and the split-octonion scalar field. We find that the well-known quintessence and phantom fields could naturally emerge in these fields. So in order to construct a phantom field, one need not resort to quintessence by changing the sign of its kinetic term by hand. We note that the conventional complex scalar field usually can not generate a phantom field by the decomposition, \( \Phi = \phi + i\psi \) except for a non-canonical form of Lagrangian [15].

We have explored the cosmic evolution of these scalar fields. We find the trajectories of the evolution are completely determined by the initial conditions on the strength of quintessence and phantom. Specifically, if the quintessence dominates over the phantom initially, the equation of state would evolves from +1 to −1. On the contrary, if the phantom dominates over quintessence initially, the equation of state would evolves from minus infinity to −1. In both cases, the crossing of phantom divide could be realized. Furthermore, it is always the de Sitter solution that plays the role of an attractor.

Introducing the metric of field space, these theories fall into a subclass of the multi-field theories which have been extensively studied in inflationary cosmology [16–20]. In view of the importance of DBI type Lagrangian, we construct the split-complex DBI Lagrangian in the framework of brane world model. Compared to the usual DBI Lagrangian, a significant difference is present in split-complex DBI Lagrangian. Namely, some timelike dimensions are present in its internal geometry. Extra timelike dimension usually leads to the presence of tachyon field. But the effect is not unacceptable experimentally if the scale of the extra dimension is bounded below a sufficiently small size [24, 25].

Acknowledgments

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