Testing volume independence of large N gauge theories on the lattice

A. González-Arroyo* and Masanori Okawa

Lattice 2014

* Instituto de Física Teórica UAM/CSIC
Departamento de Física Teórica, UAM

June 24, 2014
Large N and the lattice

- Two powerful non-perturbative methods which can be used in combination.
- The concept of volume independence or reduction emerged from this confluence.
- large N SU(N) gauge theories are interesting in their own right.

In this work we restrict ourselves to pure gauge on a symmetric box $L^4$ and Wilson action

$$S_W = -bN \sum_P \text{Tr}(U(P))$$

where $b = \beta/(2N^2) = 1/\lambda_L$

We also restrict the observables $O$ to small Wilson loops $W(R, T)$ (simple and precise).
The problem

A lattice expectation value: $O(b, N, L)$
The large $N$ quantity is given by

$$O_\infty(b) = \lim_{N \to \infty} \lim_{L \to \infty} O(b, N, L)$$

$$\leftarrow O(b, N) \rightarrow$$
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O_\infty(b) = \lim_{N \to \infty} \lim_{L \to \infty} O(b, N, L) \quad \leftarrow \quad O(b, N) \quad \rightarrow
\]

Do the limits commute?
The problem

A lattice expectation value: \( O(b, N, L) \)

The large N quantity is given by

\[
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\]

\[\longleftrightarrow O(b, N) \rightarrow \]

- Do the limits commute?
- Are there phase transition points on the x or y axis?
A lattice expectation value: $O(b, N, L)$

The large $N$ quantity is given by

$$O_\infty(b) = \lim_{N \to \infty} \lim_{L \to \infty} O(b, N, L)$$

$$\leftarrow O(b, N) \rightarrow$$

- Do the limits commute?
- Are there phase transition points on the $x$ or $y$ axis?

$$\lim_{N \to \infty} O(b, N, L) = O_\infty(b) \quad \text{VOLUME INDEPENDENCE}$$
Important ingredients/results

- **The statement can be** \( b \)-**dependent**
  
  True at strong coupling, wrong at weak coupling (BHN)?

- **The statement can depend on the boundary conditions**
  
  Twisted Boundary conditions (GAO), discrete \( Z_N \) fluxes through each plane \( n_{\mu\nu} \):

  \[
  S_W = -bN \sum_P Z(P) \text{Tr}(U(P))
  \]

  All that matters is \( Z(\text{Plane}) = \exp\{2\pi i \frac{n(\text{Plane})}{N}\} \)

- **There might be phase transitions at fixed** \( L_c(b) \)
  
  Partial reduction (NN)

- **Might depend on the lattice action (UY)-...**
Weak coupling results (large $b$)

The theory can be studied by standard weak-coupling/perturbative methods.

$$O(b, N, L) = \hat{O}_0 - \sum_{j=1} \hat{O}_j(N, L) \frac{1}{b^j}.$$ 

- Leading order dominated by flat connections: $\hat{O}_0 = \hat{W}_0 = 1$

For periodic B.C.: TORONS

For T.B.C: orthogonal twists

+ isotropy $\Rightarrow N = \hat{L}^2$ and $n(\text{Plane}) = k\hat{L}$

**SYMMETRIC TWIST** $k$ coprime with $\hat{L}$

- $\hat{W}^{PBC}_1(N, L) = \hat{W}_1(N, \infty) - \frac{R^2 T^2}{24 L^4} (1 - \frac{1}{N^2}) + \ldots$ 

- $\hat{W}^{TBC}_1(N, L) = \hat{W}^{PBC}_1(\infty, L\hat{L}) - \frac{1}{N^2} \hat{W}^{PBC}_1(\infty, L) \sim \hat{W}_1(N, \infty) + \mathcal{O}(\frac{1}{L^6 N^2})$

- General arguments suggest this is true to all orders (GAO)

(now estimating size of $\hat{W}^{TBC}_2(N, L, k)$ with M. Garcia Perez)
Results in Non-perturbative weak-coupling region

Can we recover volume independence by simply setting $k \neq 0$? Results by several authors (IO-TV) show one cannot keep $k$ fixed

A surviving candidate (GAO 2010):

$$\lim_{\hat{L} \to \infty} O(b, N = \hat{L}^2, L, k_{\hat{L}}) = O_\infty(b)$$

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**THIS WORK**

Make a **DIRECT** test of the proposal by a **QUANTITATIVELY PRECISE** measurement of $W(R, T; b, N, L, k)$

Our study includes ordinary LGT with PBC ($k = 0$), the EK model ($L = 1 \; k = 0$) and TEK model ($L = 1 \; k \neq 0$)

Volume independence $\Leftrightarrow W(R, T; b, \infty, L, k) = W_{N=\infty}(R, T; b)$
We studied the plaquette \((R = T = 1)\) at \(b = 0.36\) at various \(L\) and \(N\). This is a very precise quantity (errors \(10^{-5}\)).
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$N$-dependence of plaquette e.v.

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For periodic boundary conditions ($k = 0$)

![Graph showing the dependence of plaquette on $1/N^2$.](image)
We studied the plaquette \((R = T = 1)\) at \(b = 0.36\) at various \(L\) and \(N\). This is a very precise quantity (errors \(10^{-5}\)). And twisted boundary conditions \(k \neq 0\) (TEK \(L = 1\)).

![Graph showing the dependency of plaquette on \(1/N^2\)]
$N$-dependence of TEK

We tried various values of $N = \hat{L}^2$
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\(N\)-dependence of TEK

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\[\begin{array}{ccccccc}
\text{N} & 1369 & 841 & 529 & 289 & 121 & 81 \\
\text{E}_{\text{infty}}(0.36) & 0.55795 & 0.558 & 0.55805 & 0.5581 & 0.55815 & 0.5582 & 0.55825
\end{array}\]
Other values of $b$ and $R$

The result extends to other values of $b$:
Example $b = 0.37$: The $1/N^2$ is approximately universal

![Graph showing $E$ vs $1/N^2$ for $b=0.37$.]
Other values of $b$ and $R$

The same is true about other Wilson loops $R = 2, 3, 4$
Global description for all $N$

All the data points obtained over the last 3 years can be fitted to a unified picture:

$$\bar{E}(b, N) = \frac{1 + \sum_{n=1}^{3} \sum_{m=0}^{\min(2,n)} a_{nm} x^n y^m}{1 + \sum_{n=1}^{3} \sum_{m=0}^{\min(2,n)} a'_{nm} x^n y^m}$$

with $x = 1/b_I = 1/(bE(b))$ and $y = 1/N^2$.

$a'_{nm}$ adjusted to match perturbative expansion to 3 loops.
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$a'_{nm}$ adjusted to match perturbative expansion to 3 loops

- The formula is invertible
- matches the data within $\mathcal{O}(10^{-5})$ errors
- The $N = \infty$ parameters are determined with TEK only
- Similar results can be obtained for $W(R, R)$
A final view

Plaquette for all $N$

- $N=3$
- $N=4$
- $N=6$
- $N=8$
- $N=16$

Perturbative to order 34

2X2 Loop for all $N$

- $N=3$
- $N=4$
- $N=6$
- $N=8$
- $N=16$

$W(2,2)$

3X3 Loop for all $N$

- $N=3$
- $N=4$
- $N=6$
- $N=8$
- $N=16$

$W(3,3)$

4X4 Loop for all $N$

- $N=3$
- $N=4$
- $N=6$
- $N=8$
- $N=16$

$W(4,4)$
Volume independence holds up to a very high level of precision if the appropriate boundary conditions are used. The data is consistent with VI$^2$ being valid at all values of $b$. Finite $N$ corrections are small and allow the reduced version to be a practical tool to access the large $N$ world.

- Complete the study in perturbation theory (with M. Garcia Perez)
- Extend the study to other observables and asymmetric boxes (finite temperature?)

**Large $N$ WORKSHOP at IFT (Madrid)**
18 May- June 5 2015

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$^2$VI stands for Volume Independence not the middle name of Felipe VI