Unidimensional time domain quantum optics

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Choosing the right first quantization basis in quantum optics is critical for the interpretation of experimental results. The usual frequency basis is, for instance, inappropriate for short, subcycle waveforms. We derive first quantization in time domain, and apply the results to ultrashort pulses propagating along unidimensional waveguides. We show how to compute the statistics of the photon counts, or that of their times of arrival. We also extend the concept of quadratures to the time domain, making use of the Hilbert transform.

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![Figure 1](image.png)

**FIG. 1.** Voltage vs. time [arb. units] for quasi monochromatic (a) and quasi time-localized (b) signals. First quantization in frequency is well-suited for the signal on the left. It is not the case for the ultrabroadband signal on the left, for which first quantization in time is more natural.

Introduction. Quantization of the free electromagnetic (EM) field is carried out in two steps: i) first quantization [1] sorts various parts of the field into modes and associates a complex amplitude to each of them; ii) second quantization [2] upgrades the status of the amplitude and its complex conjugate to that of a couple of hermitian conjugate operators obeying bosonic commutation relations.

Textbook treatments of first quantization enclose the free electromagnetic (EM) field in a fictitious cavity and expands it in terms of resonant frequencies [3–5]. This is first quantization in the frequency domain, and it is perfectly appropriate for quasi-monochromatic modes of light, such as the modes of a laser. With shorter pulses and larger spectra, the same type of first quantization can be used, as long as there exists a well defined central carrier frequency (see Fig. 1, a). However, one might wonder whether first quantization in the frequency domain is the right basis for pulse widths of the order of one optical cycle (as in Fig. 1, b).

Progress in ultrafast optics [6] has seen generation of femtosecond pulses of subcycle durations [7, 8], thus motivating different approaches in the treatment of the quantum properties of such radiation. Recent measurements of mid-infrared fields have demonstrated experimental capability to subcycle sample quantum fields localized in space and time [9–11]. Subcycle signals can also readily be generated, propagated and measured in the microwave part of the electromagnetic spectrum. The only restriction with the exploration of such signals in the microwave quantum regime is that the temperature of the conductors must be lowered to a few tens of mK, so that blackbody radiation, i.e. thermal noise, does not overwhelm the signal (a temperature $T=1\text{K}$ corresponds to a frequency $k_B T / h \approx 21 \text{GHz}$). This regime can be easily achieved in dilution refrigerators, and quantum optical properties of microwave signals are the object of intense scrutiny [12–27], in particular within the framework of circuit quantum electrodynamics [28].

Some observable quantities, like the amount of energy carried by a pulse, are independent of the first quantization basis, in the sense that they have as much validity in the many-cycle and subcycle regimes. Conversely, it is not immediately clear what the number of photons in a subcycle pulse is. There is, in particular, no direct link between energy and photon number $n$, contrary to the narrow bandwidth regime, where the mean energy is almost $\langle n \rangle h \nu$, with $\nu$ the center frequency and $\langle n \rangle$ the average number of photons. As a result, the photon is sometimes described as the quantum of energy of the EM field. This contrasts with the viewpoint expressed in earlier seminal papers [29, 30]. We take the view that a good operational definition of the photon is simply that of a “click” on a cascaded photodetector. This definition has the advantage of preserving well-known photocount distributions, such as that of the thermal state, which verifies, for instance, the relation $\langle \Delta n^2 \rangle = \langle n \rangle (\langle n \rangle + 1)$ between the variance and the average of the photocount number. This relation is verified for any cascaded photodetector, whatever its bandwidth [31]. The photon is then a fully time-resolved event with a completely uncertain energy, and first quantization in time domain is most natural. The object of this Letter is to answer questions like: how many photons are expected in the pulse shown in right panel of Fig. 1, and what are their expected energies and times of arrival?

This Letter is organized as follows. We first describe first quantization for unidimensional waveguides in the time domain limit. Here we introduce a new observable, the weighted time of arrival, which is a time domain...
dual of the Hamiltonian. In a second part, we show how to compute the statistics of all important observables of microwave signals with simple transforms of continuous voltages recorded by ultrahigh-bandwidth oscilloscopes. Among those observables are quadratures, for which we extend the definition to the time domain. We finally show that the choice of first quantization basis must be tailored to the experimental setup to adequately explain results.

First quantization in time domain. We consider a unidimensional waveguide parameterized by a position \( z \). Classically, a signal is a real function of position and time, \( s(z,t) \). Supposing that the propagation velocity is \( v \) for all such signals, we can change the notation and define a time-like variable \( \tau = z/v \) instead of the original position parameter. We thus use \( s(\tau,t) \), with \( \tau \) a time-like label for position and \( t \) the real time.

Considering the signal in a quantum setting, we start with first quantization in \( \tau \)-time domain, and for the moment set \( t = 0 \). Although photons have been described as non localizable \cite{32}, several arguments have been made that photonic modes, more broadly considered, can be non localizable \cite{36,37}, several arguments have been made.

Directional modes cannot be fully localized, just because of their directionality. Thus, the \( a(\tau) \) operators do not obey bosonic commutation relations of the form

\[
[a(\tau), a(\tau')^\dagger] = \delta(\tau - \tau'),
\]

where \( \delta \) is the delta function. In the remainder of the text, it can also stand for \( \pm 1 \) when required.

In \( \tau \)-time domain, we define

\[
a_{\nu,\sigma} = \int_{-\infty}^{\infty} d\tau \ a(\tau) e^{i\nu \tau},
\]

where \( \nu \) is a spatial frequency, the equivalent of a wavevector. Positive (negative) spatial frequencies represent signals propagating in the +z (-z) direction. Note that \( a_\nu \) is not the Fourier transform of \( a^\dagger \).

We impose the usual bosonic commutation relation

\[
[a_{\nu}, a_{\nu'}^\dagger] = \delta(\nu - \nu'),
\]

which translates to

\[
[a_{\tau}, a_{\tau'}^\dagger] = \delta(\tau - \tau'),
\]

The \( a_\nu^{(t)} \) modes are localized and thus are not the usual (propagating) photonic modes of quantum optics. They are in fact a superposition of \( \pm \tau \)-propagating directional modes \( a_{\tau,\pm}^{(t)} \), which are the usual modes \cite{36,37} and correspond to the analytic parts \( a_\pm \) of the localized modes \( a_\nu \), i.e.

\[
a_{\nu,+} = H(\nu) \ a_{\nu,+}; \ a_{\nu,+}^\dagger = H(\nu) \ a_{\nu,+}^\dagger; \ a_{\nu,-} = H(-\nu) \ a_{\nu,-}; \ a_{\nu,-}^\dagger = H(-\nu) \ a_{\nu,-}^\dagger,
\]

where \( H(\nu) \) is the Heaviside function taking the value one for positive argument, and zero for negative argument. Thus, we have

\[
a_{\nu}^{(t)} = a_{\nu,+}^{(t)} + a_{\nu,-}^{(t)} = \sum_{\sigma} a_{\nu,\sigma}^{(t)},
\]

where \( \sigma \) stands for “+” or “−”. Among those observables is quadratures, for which we can show that the choice of first quantization basis must be tailored to the experimental setup to adequately explain results.
form (3). However, we have the physically meaningful relations [31]

$$\begin{align*}
[a_{\tau,\sigma}(t), a^\dagger_{\tau',\sigma'}(t')] = \delta_{\sigma,\sigma'} \delta[(t - t') - \sigma(\tau - \tau')].
\end{align*}$$

(11)

For instance, a directional photon created at \((\tau', t')\) and propagating in the \(+z\) direction can only be annihilated at \((\tau, t)\) if \(t - t' = \tau - \tau'\). The clear physical meaning of these commutation relations derives from the separation of the position and time variables. It simply reflects causality.

The Fourier transform of \(a_{\tau,\sigma}(t)\), with respect to real time, is \(a_{\tau,\sigma}(f)\). Its hermitian conjugate is \(a^\dagger_{\tau,\sigma}(f)\).

Using the relations (4), we find that \(a^{(1)}_{\tau,\sigma}(f) = 0\) when \(f < 0\). With this formalism, the real frequency is naturally always positive. The commutation relation for the Fourier transformed operators is

$$\begin{align*}
[a_{\tau,\sigma}(f), a^\dagger_{\tau',\sigma'}(f')] = \delta_{\sigma,\sigma'} \delta(f - f') e^{i\sigma 2\pi f(\tau - \tau')}.
\end{align*}$$

(12)

The operators \(a^{(1)}_{\tau,\sigma}(t)\) and \(a^{(1)}_{\tau,\sigma}(f)\) are the most relevant for the description of experiments, as they relate to detection at a fixed location, in real time and frequency domains. We focus the remainder of the text on these operators and we define the directional photon number and directional Hamiltonian operators at location \(\tau\) as

$$\begin{align*}
N_{\tau,\sigma} &= \int_{-\infty}^{+\infty} df \, a^\dagger_{\tau,\sigma}(f) a_{\tau,\sigma}(f) \\
&= \int_{-\infty}^{+\infty} dt \, a^\dagger_{\tau,\sigma}(t) a_{\tau,\sigma}(t);
\end{align*}$$

(13)

$$\begin{align*}
H_{\tau,\sigma} &= \int_{-\infty}^{+\infty} df \, h f a^\dagger_{\tau,\sigma}(f) a_{\tau,\sigma}(f).
\end{align*}$$

(14)

The Hamiltonian has a simple form in frequency domain (it is a weighted average of energies), but requires derivatives of the ladder operators in time domain [31]. In a dual way, we define the “weighted average time” operator

$$\theta_{\tau,\sigma} = \int_{-\infty}^{+\infty} dt \, t \, a^\dagger_{\tau,\sigma}(t) a_{\tau,\sigma}(t)$$

(15)

which is well defined in terms of ladder operators in time domain but requires their derivatives in frequency domain. This observable is associated with the time of arrival of photons that would be detected by an infinitely fast detector. If this quantity is observed, little information can be gained on the energy. In contrast, when measuring the energy (e.g., with a bolometer), the time of photon arrival is ill-defined. Indeed, for fixed \(\tau\), the commutator [31]

$$\begin{align*}
[H_{\tau,\sigma}, \theta_{\tau,\sigma}] = i\hbar \delta_{\sigma,\sigma'} N_{\tau,\sigma},
\end{align*}$$

(16)

leads to the uncertainty relation

$$\begin{align*}
\sqrt{\langle \Delta H^2_{\tau,\sigma} \rangle} \langle \Delta \theta^2_{\tau,\sigma} \rangle \geq \frac{\hbar}{2} \langle N_{\tau,\sigma} \rangle,
\end{align*}$$

(17)

where \(\langle \Delta \Omega^2 \rangle\) is the variance and \(\langle \Omega \rangle\) the expected value of any observable \(\Omega\). Hence, the number of photons possesses a physical meaning as a number of action quanta, and not as a number of energy quanta. Here, we show that time and energy are on the same footing. It is also important to note that the commutator is not \([H, \theta] = i\hbar\). A classic argument provided by Pauli shows that if such a relation existed, the energy spectrum could not be bounded [38, p. 63]. But Pauli’s argument does not apply here.

**Application to microwaves.** In this section, we consider measurements at a fixed location \(\tau\) of a signal propagating in a given direction. For clarity, we drop the \(\tau\) and \(\sigma\) subscripts, although we do consider a directional photonic mode. In the microwave domain, the main observable is the voltage. One way of measuring voltage is to record a trace on an oscilloscope. The measured value \(v(t)\) corresponds to a measurement of the observable [36]

$$\begin{align*}
v(t) = -i \sqrt{\frac{Z\hbar}{2}} \int_{-\infty}^{+\infty} df \, \sqrt{f} \, [a(f) e^{-i2\pi ft} - \text{h.c.}],
\end{align*}$$

(18)

with \(Z\) the characteristic impedance of the transmission line, considered independant of the frequency. We can take the integral from \(-\infty\) to \(+\infty\), remembering that \(a^{(1)}(f) = 0 \forall f < 0\).

Second quantized EM fields are usually described in terms of discrete (e.g. photon number) or continuous observables. The main continuous observables in the frequency domain are the quadratures [3-5].

We now use the results of the previous section and extend the quadratures in time domain, as

$$\begin{align*}
x_{\alpha}(t) &= \frac{a(t) e^{-i\alpha} + a^\dagger(t) e^{i\alpha}}{\sqrt{2}},
\end{align*}$$

(19)

where \(\alpha\) is an angle between 0 and \(2\pi\).

In order to reconstruct two orthogonal quadratures from the voltage alone, we use the transforms

$$\begin{align*}
\sqrt{\frac{2}{Z\hbar}} \int_{-\infty}^{+\infty} dt' \, \frac{v(t - t')}{\sqrt{|t'|}} = x_0(t) = q(t),
\end{align*}$$

(20)

and

$$\begin{align*}
\sqrt{\frac{2}{Z\hbar}} \int_{-\infty}^{+\infty} dt' \, \frac{v(t - t') \text{sgn}(t')}{\sqrt{|t'|}} = x_\pm(t) = p(t),
\end{align*}$$

(21)

where \(\text{sgn}\) is the sign function. Quadratures of the signals of Fig. 1 are shown in the second row of Fig. 2.

All sets of orthogonal quadratures are Hilbert transforms of one another [31]. Since the Hilbert transform
exchanges sine and cosine in the frequency domain, this is a very natural extension of quadratures. From this (or any) set of orthogonal quadratures, we can construct the “instantaneous photon flux” operator \( n(t) \) as

\[
\frac{1}{2} [q^2(t) + p^2(t)] = \frac{1}{2} [a^\dagger(t)a(t) + a(t)a^\dagger(t)] = n(t) + \text{v.c.,}
\]

(22)

with v.c. vacuum contributions that should not generate photocounts. All the cumulants from these vacuum contributions can always be subtracted in actual measurements.

Transforms (20) and (21) are non-local in time and necessitate both past and future amplitudes. This is due to the non-local nature of the directional modes (even in time [39]) and does not imply that relativistic causality is violated [40, 41]. However, it means that the instantaneous photon flux cannot be obtained on the fly and must be recovered using past and future input. In the laboratory, recording traces on an oscilloscope enables the computation of all transforms, and the recovery a posteriori of the instantaneous power and photon flux.

Using the definition (22), we find that the instantaneous photon flux associated with an infinite sinusoidal voltage is constant at all times, as expected. It is reasonable to expect, as shown in Fig. 2 (a),(b), that the instantaneous photon flux resembles the envelope of the pulse. Of note is the fact that it is possible for the photon flux to be at a maximum when the voltage is zero, as shown in Fig. 2 (b).

Eq. (22) allows to compute the full counting statistics of photons at instant \( t \). As an example, for a thermal state, we find \( \langle \Delta n^2 \rangle = \langle n \rangle (\langle n \rangle + 1) \). This remarkable result, that the statistics of ultrabroadband thermal light is the same as that obtained in the monochromatic limit, stems from the fact that we chose the first quantization basis adapted to our detection method. Indeed, we considered a detector sensitive to the instantaneous value of the EM field and the first quantization limits are useful to understand the electromagnetic modes of a unidimensional waveguide. Noticeable operators in time domain are the “weighted time of arrival” (equivalent of the Hamiltonian in the frequency domain) and the extended quadrature observables that constitute Hilbert transform pairs.

The probability to observe a “click” on a detector depends on non-local features of the voltage and can be maximal when the instantaneous voltage vanishes.

From Eq. (25), we can compute the statistics of time of arrival. To better understand the physical meaning of \( \theta \), we consider a single pulse for which we can measure both \( \theta \) and \( N \), as they commute. Repeating the experiment, the quantity \( \langle \theta \rangle / \langle N \rangle \) is the mean time of arrival of pulses, while \( \sqrt{\langle \Delta \theta^2 \rangle / \langle N \rangle} \) (which involves the two-time correlator \( \langle n(t_1)n(t_2) \rangle \) is the uncertainty in the time of arrival, or jitter. In contrast, the pulse width (or intra-pulse time uncertainty) involves the quantity \( \langle f dt \, t^2 \, n(t) \rangle \).

**Conclusion.** Both frequency domain and time domain first quantization limits are useful to understand the electromagnetic modes of a unidimensional waveguide. Noticeable operators in time domain are the “weighted time of arrival” (equivalent of the Hamiltonian in the frequency domain) and the extended quadrature observables that constitute Hilbert transform pairs.

The probability to observe a “click” on a detector depends on non-local features of the voltage and can be maximal when the instantaneous voltage vanishes.

The full picture of EM field measurements is dependent on the choice of first quantization basis. It is thus crucial to choose the right basis for a particular measurement. In particular, ultrabroadband quantum experiments with mesoscopic devices such as tunnel and Josephson junctions [20, 24, 25, 42] are likely to be best modeled in time domain.

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LADDER OPERATORS

Throughout, we use $\int \equiv \int_{-\infty}^{+\infty}$, $\int_+ = \int_0^{+\infty}$, $\int_- = \int_{-\infty}^0$.

Relationships between ladder operators

The “standing wave” ladder operators for a unidimensional waveguide are

$$a^{(1)}_\nu,$$  \hspace{1cm} (1)

which are a sum of the “propagating wave” ladder operators

$$a^{(1)}_{\nu,\sigma} = \mathcal{H}(\sigma \nu) a^{(1)}_\nu \quad \sigma \in \{+, -\}. \hspace{1cm} (2)$$

The counterparts of the $a^{(1)}_{\nu, \sigma}$ in real space are

$$a_{\tau, \sigma} = \int d\nu \ a^{(1)}_{\nu, \sigma} e^{i 2\pi \nu \tau};$$
$$a^\dagger_{\tau, \sigma} = \int d\nu \ a^{\dagger(1)}_{\nu, \sigma} e^{-i 2\pi \nu \tau}, \hspace{1cm} (3)$$

and the inverse relationships are

$$a^{(1)}_{\nu, \sigma} = \int d\tau \ a_{\tau, \sigma} e^{-i 2\pi \nu \tau};$$
$$a^{\dagger(1)}_{\nu, \sigma} = \int d\tau \ a^{\dagger(1)}_{\tau, \sigma} e^{i 2\pi \nu \tau}. \hspace{1cm} (4)$$

Evolution of these operators in time is

$$a^{(1)}_{\nu, \sigma} \rightarrow a^{(1)}_{\nu, \sigma}(t) = a^{(1)}_{\nu, \sigma} e^{-i \sigma 2\pi \nu t};$$
$$a^{\dagger(1)}_{\nu, \sigma} \rightarrow a^{\dagger(1)}_{\nu, \sigma}(t) = a^{\dagger(1)}_{\nu, \sigma} e^{i \sigma 2\pi \nu t}, \hspace{1cm} (5)$$

from which we also get $a^{(1)}_{\tau, \sigma}(t)$ by applying the transforms (3).

The usual operators of quantum optics in frequency domain are the $a^{(1)}_{\rho, \sigma}(f)$, obtained through

$$a_{\rho, \sigma}(f) = \int dt \ a^{(1)}_{\rho, \sigma}(t) e^{i 2\pi f t};$$
$$a^\dagger_{\rho, \sigma}(f) = \int dt \ a^{\dagger(1)}_{\rho, \sigma}(t) e^{-i 2\pi f t}. \hspace{1cm} (6)$$

and the inverse relationships

$$a_{\rho, \sigma}(t) = \int_{+} df \ a_{\rho, \sigma}(f) e^{-i 2\pi f t};$$
$$a^\dagger_{\rho, \sigma}(t) = \int_{+} df \ a^{\dagger(1)}_{\rho, \sigma}(f) e^{i 2\pi f t}. \hspace{1cm} (7)$$

for $\rho \in \{\nu, \tau\}$.

The relations (2) imply that $a_{\rho, \sigma}(f) = 0 \ \forall f < 0.$
Commutators

The main commutator is

\[
\left[ a_\nu, a^\dagger_{\nu'} \right] = \delta(\nu - \nu').
\] (8)

It yields immediately

\[
\left[ a_\tau, a^\dagger_{\tau'} \right] = \int\int_{-\infty}^{+\infty} d\nu d\nu' \left[ a_\nu, a^\dagger_{\nu'} \right] e^{i2\pi \nu \tau} e^{-i2\pi \nu' \tau'}
= \int_{-\infty}^{+\infty} d\nu e^{i2\pi \nu (\tau - \tau')} = \delta(\tau - \tau').
\] (9)

We have

- \( a_{\nu,+} = a_\nu \) if \( \nu > 0 \) and \( a_{\nu,+} = 0 \) if \( \nu < 0 \);
- \( a_{\nu,-} = a_\nu \) if \( \nu < 0 \) and \( a_{\nu,-} = 0 \) if \( \nu > 0 \).

In conjunction with the main commutator (8), this yields

\[
\left[ a_\nu,\sigma, a^\dagger_{\nu',\sigma'} \right] = \delta_{\sigma,\sigma'} \delta(\nu - \nu').
\] (10)

Since \( a_{\nu,\sigma}(t) = a_{\nu,\sigma} e^{-i2\pi \sigma \nu t} \) and \( a^\dagger_{\nu,\sigma}(t) = a^\dagger_{\nu,\sigma} e^{i2\pi \sigma \nu t} \), we get

\[
\left[ a_{\nu,\sigma}(t), a^\dagger_{\nu',\sigma'}(t') \right] = \delta_{\sigma,\sigma'} \delta(\nu - \nu') e^{-i2\pi \sigma \nu (t - t')}.
\] (11)

Now, we have

\[
\left[ a_{\tau,\sigma}(t), a^\dagger_{\tau',\sigma'}(t') \right] = \int\int_{-\infty}^{+\infty} d\nu d\nu' \left[ a_{\nu,\sigma}(t), a^\dagger_{\nu',\sigma'}(t') \right] e^{i2\pi \nu \tau} e^{-i2\pi \nu' \tau'}
= \delta_{\sigma,\sigma'} \int_{-\infty}^{+\infty} d\nu e^{i2\pi \nu (\tau - \tau' - \sigma t + \sigma t')}
= \delta_{\sigma,\sigma'} \delta[(t - t') - \sigma(\tau - \tau')].
\] (12)

Finally, we have

\[
\left[ a_{\tau,\sigma}(f), a^\dagger_{\tau',\sigma'}(f') \right] = \int\int_{-\infty}^{+\infty} df df' \left[ a_{\tau,\sigma}(f), a^\dagger_{\tau',\sigma'}(f') \right] e^{i2\pi f \tau} e^{-i2\pi f' \tau'}
= \delta_{\sigma,\sigma'} \int_{-\infty}^{+\infty} df e^{i2\pi (f - f') t} e^{i\sigma 2\pi f' (\tau - \tau')}
= \delta_{\sigma,\sigma'} \delta(f - f') e^{i\sigma 2\pi f (\tau - \tau')}.
\] (13)

Weighted average time operator

We define the weighted average time operator as

\[
\theta_{\tau,\sigma} = \int dt \, t \, a^\dagger_{\tau,\sigma}(t) a_{\tau,\sigma}(t).
\] (14)

The Hamiltonian is defined as

\[
H_{\tau,\sigma} = \int df \, h(f) \, a^\dagger_{\tau,\sigma}(f) a_{\tau,\sigma}(f).
\] (15)
Making use of the commutator (13), we have
\[
[H_{\tau,\sigma}, \theta_{\tau',\sigma'}]
\]
\[
= \oint df \, df_1 \, dt \, e^{i2\pi(f_1-f_2)t} \left\{ a_{\tau,\sigma}(f) a_{\tau',\sigma'}(f_1) \right\} \left( a_{\tau,\sigma}(f), a_{\tau',\sigma'}(f_1) \right) a_{\tau,\sigma}(f)
\]
\[
\equiv \sqrt{\hbar} \left( \int df \, e^{i\pi/2 - i\sigma f} + \int df \, e^{i\pi/2 + i\sigma f} \right)
\]
\[
\equiv \sqrt{\hbar} \frac{\pi}{2} x_{\pi/2 - i\sigma}.
\]  

For \( \tau = \tau' \), this leads to
\[
[H_{\tau,\sigma}, \theta_{\tau,\sigma'}] = -i \hbar \delta_{\sigma,\sigma'} \int df \, \partial_f [a_{\tau,\sigma}(f)] a_{\tau,\sigma}(f) \]  
\[
= i \hbar \delta_{\sigma,\sigma'} \int df \, a_{\tau,\sigma}(f) a_{\tau,\sigma}(f) \]  
\[
= i \hbar \delta_{\sigma,\sigma'} N_{\tau,\sigma}.
\]

after integration by parts.

The uncertainty relation for noncommuting variables is
\[
\sqrt{\langle \Delta A^2 \rangle \langle \Delta B^2 \rangle} = \Delta A \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle|.
\]  

Applied to Eq. (17), this yields
\[
\sqrt{\langle \Delta H^2_{\tau,\sigma} \rangle \langle \Delta \theta^2_{\tau,\sigma} \rangle} \geq \frac{\hbar}{2} \langle N_{\tau,\sigma} \rangle.
\]  

**Quadratures**

Quadratures can be computed from the transforms
\[
\int_{\sigma} \frac{dt'}{\sqrt{|t'|}} v(t-t')
\]
\[
= \int dt' \frac{\mathcal{H}(\sigma t')}{\sqrt{|t'|}} v(t-t')
\]
\[
= -i \sqrt{\frac{Z\hbar}{2}} \int dt' \int df \, \int df_1 \frac{[1 + i \sigma \text{sgn}(f') \sqrt{|f'|}]}{2 \sqrt{|f'|}} a(f) e^{i2\pi f(t-t')} e^{i2\pi f(t-t')} \]  
\[
= i \sqrt{\frac{Z\hbar}{2}} \int df' \int df \, \sqrt{|f'|} e^{i\sigma \text{sgn}(f')} \frac{1}{2} a(f) e^{i2\pi f t} \int dt' e^{i2\pi f f'} e^{i2\pi f f'} \]  
\[
= -i \sqrt{\frac{Z\hbar}{2}} \int df \left[ a(f) e^{-i2\pi ft} e^{i\frac{\pi}{2} - i\sigma f} - a^+(f) e^{i2\pi ft} e^{i\frac{\pi}{2} + i\sigma f} \right]
\]
\[
= \sqrt{\frac{Z\hbar}{2}} \int df \left[ a(f) e^{-i2\pi ft} e^{-i\left(\frac{\pi}{2} - i\sigma f\right)} + a^+(f) e^{i2\pi ft} e^{i\left(\frac{\pi}{2} + i\sigma f\right)} \right]
\]
\[
= \sqrt{\frac{Z\hbar}{2}} \left[ a_+ e^{-i\left(\frac{\pi}{2} - i\sigma f\right)} + a^+ e^{i\left(\frac{\pi}{2} + i\sigma f\right)} \right]
\]
\[
= \sqrt{\frac{Z\hbar}{2}} \left[ a_+ e^{-i\left(\frac{\pi}{2} - i\sigma f\right)} + a^+ e^{i\left(\frac{\pi}{2} + i\sigma f\right)} \right]
\]  
\[
\equiv \sqrt{\frac{Z\hbar}{2}} x_{\frac{\pi}{2} - i\sigma}.
\]
We note that
\[
\int_+ dt' \frac{v(t-t')}{|t'|} = \int d\nu \frac{1 + i \text{sgn}(\nu)}{|\nu|} e^{-i2\pi \nu t},
\]
\[
\int_- dt' \frac{v(t-t')}{|t'|} = \int d\nu \frac{1 - i \text{sgn}(\nu)}{|\nu|} e^{-i2\pi \nu t},
\]
so that the Fourier transforms of orthogonal quadratures only differ by a factor of \(\pm i \text{sgn}(\nu)\). As the Fourier transform of \(i \text{sgn}(\nu)\) is \(1/\pi t\), this means that they are Hilbert transforms of one another.

In order to compute quadratures associated with the modes \(b_0^\dagger(\tau)\) defined in Eq. (67), we use the transforms
\[
x_{0,b}(t) = \int dt_1 dt_2 \frac{\text{sgn}(t_2)\beta(t_1-t_2)}{\sqrt{|t_2|}} v(t-t_1),
\]
and
\[
x_{\pi/2,b}(t) = \int dt_1 dt_2 \frac{\beta(t_1-t_2)}{\sqrt{|t_2|}} v(t-t_1),
\]
which are also Hilbert transform of one another as \(\frac{1}{\sqrt{|t|}}\) and \(\frac{\text{sgn}(t)}{\sqrt{|t|}}\) are a Hilbert pair.

**WAVELETS AND SECOND QUANTIZATION STATES**

In this section we consider a real signal that can be represented by a wavelet that is part of a basis as defined in Ref. [1]. We only consider one wavelet at a time, and not the whole basis. We thus define a complex function
\[
\chi_{b}(\lambda, \lambda^*, \rho) = \text{Tr} \left[ \rho e^{A^\dagger(\tau')} e^{-A(\tau')} \right],
\]
where \(k\) is Boltzmann’s constant and Tr is the trace operator. To better illustrate the absence of any phase relation between first quantization components, the density matrix can be rewritten in terms of the tensor product integral
\[
e^{-\hbar/kT} = \bigotimes_{-\infty}^{+\infty} [\rho_{th}(f, T)]^{df},
\]
with \(\rho_{th}(f, T) = \exp[-\hbar |f| A^\dagger(f) A(f)/kT] \).

A useful tool for characterizing possible measurements on a state is the quantum characteristic function [3]. It is intimately linked to the Q, P and Wigner functions [4]. Its determination is sufficient to predict full statistics for a number of observables and is particularly adapted to the statistics associated with photon counting. It is defined as
\[
\chi_{b}(\lambda, \lambda^*, \rho) = \text{Tr} \left[ \rho e^{A^\dagger(\tau')} e^{-A(\tau')} \right],
\]
with $\rho_b$ the density matrix of the input state.

For the thermal state, we have (see below)

$$\chi_b[\lambda, \lambda^*, \rho_{\text{th}}(T)] = \exp \left[ \frac{-|\lambda|^2}{\bar{f}_b/kT - 1} \right],$$

(28)

where $\bar{f}_b = \int df f |\beta_\tau(f)|^2$. Thus, the characteristic function of the thermal state will depend both on the temperature and on the measurement basis. The fact that the characteristic function depends on a simple weighted average of the frequencies in the measurement basis is not obvious a priori.

We now turn to signals that are second quantized states of first quantized wavelets. Hence, they are generated by applying a function of the creation operator $b_{\tau,\sigma}^\dagger$ (defined by Eqs. (24)) on the vacuum. The most common states of this type are respectively the coherent, Fock, and squeezed vacuum states,

$$|\alpha\rangle_b = \exp \left[ \alpha b_{\tau,\sigma}^\dagger - \alpha^* b_{\tau,\sigma} \right] |\text{vac}\rangle;$$

(29)

$$|n\rangle_b = \frac{1}{\sqrt{n!}} \left[ b_{\tau,\sigma}^\dagger \right]^n |\text{vac}\rangle;$$

(30)

$$|\xi\rangle_b = \exp \left[ -\frac{\xi}{2} b_{\tau,\sigma}^\dagger b_{\tau,\sigma} + \frac{\xi^*}{2} b_{\tau,\sigma}^\dagger b_{\tau,\sigma} \right] |\text{vac}\rangle,$$

(31)

where $|\text{vac}\rangle$ is the vacuum state and $\alpha$ and $\xi$ are complex numbers.

Characterization of state measurements is usually done using the same first quantization basis for the generation and the measurement of the signal. Here, we assume more generally that the modes are measured in the $c_{\tau',\sigma'}$ basis, where

$$c_{\tau',\sigma'} = \int df' \gamma_{\tau',\sigma'}^{\ast}(f') a_{\tau',\sigma'}(f').$$

(32)

The general commutation relation

$$[c_{\tau',\sigma'}, b_{\tau,\sigma}^\dagger] = \delta_{\tau',\sigma'} \int df \gamma_{\tau'}^{\ast}(f) \beta_\sigma(f) e^{i\sigma 2\pi f(\tau' - \tau)}$$

$$= \delta_{\tau',\sigma'} \int dt \gamma_{\tau'}^{\ast}(t) \beta_\sigma[t - \sigma(\tau' - \tau)]$$

(33)

then plays a central role in the determination of possible measurement outcomes. Indeed, the characteristic function in the $c$ basis verifies (see below)

$$\chi_c(\lambda, \lambda^*, \rho_b) = \chi_b(\lambda \eta_{bc} \lambda^* \eta_{bc}, \rho_b),$$

(34)

with $\eta_{bc} = [c, b^\dagger] = [b, c^\dagger]^\ast$ and $|\eta_{bc}| \leq 1$ because of the normalization of the $b$ and $c$ modes.

Thanks to this relation, we can use well-known results for measurements in the initial basis [5]. For instance,

$$\chi_c(\lambda, \lambda^*, |\alpha\rangle_b) = e^{\lambda \eta_{bc} \alpha^* - \lambda^* \eta_{bc} \alpha},$$

(35)

which is the regular characteristic function for the coherent $|\eta_{bc} \alpha \rangle_c$ state. This shows that a coherent state will remain coherent, whatever the measurement basis. The normally ordered full photon counting statistics is

$$\langle \alpha | (c^\dagger)^m c^m | \alpha \rangle_b = |\eta_{bc} \alpha|^{2m} \quad \forall m.$$

(36)

For Fock states, we have

$$\chi_c(\lambda, \lambda^*, |n\rangle_b) = \mathcal{L}_n \left( |\lambda \eta_{bc}|^2 \right),$$

(37)

with $\mathcal{L}_n$ the Laguerre polynomial of degree $n$. This leads to the normally ordered full photon counting statistics

$$\langle n | (c^\dagger)^m c^m | n \rangle_b = \frac{n!}{(n - m)!} |\eta_{bc}|^{2m} \quad \forall m \leq n.$$
In particular, the variance \( \langle \Delta n^2 \rangle = n |\eta_{bc}|^2 (1 - |\eta_{bc}|^2) \) is not zero, contrary to that of Fock states measured in the same basis as that of their creation. But the degree of second order coherence \[ g^{(2)}(0) = 1 - \frac{1}{n} \] (39)
is the same as that of the Fock state \(|n\rangle_c\). In particular, the degree of second order coherence vanishes for a single photon, whatever the bases for creation and measurement. Experimentally, a single photon created in any mode and sent onto a Hanbury Brown and Twiss setup consisting of a beam splitter and two photodetectors \[7\] cannot yield coincidences on the detectors, whatever their measurement basis. This means that a single photonic excitation of any given mode cannot project onto several excitations in other modes.

For squeezed vacuum states, we have

\[
\chi_e(\lambda, \lambda^*, |\xi\rangle_b) = \exp \left[ -S^2_{\xi}\lambda \eta_{bc}^2 - \frac{1}{2}C_{\xi}S_{\xi}(\lambda \eta_{bc})^2 e^{-i\phi_{\xi}} - \frac{1}{2}C_{\xi}S_{\xi}(\lambda^* \eta_{bc})^2 e^{i\phi_{\xi}} \right],
\]
(40)

with \( C_{\xi} = \cosh(|\xi|) \), \( S_{\xi} = \sinh(|\xi|) \), and \( \phi_{\xi} = \arg(\xi) \). The most interesting observables for this state are the quadratures

\[
x_{e,\theta} = \frac{c e^{-i\theta} + c^\dagger e^{i\theta}}{\sqrt{2}}
\]
(41)

We compute their variance from the characteristic function (see below) as

\[
\langle \xi | \Delta x_{e,\theta}^2 | \xi \rangle_b = \frac{1}{2} + |\eta_{bc}|^2 \left[ S^2_{\xi} - C_{\xi}S_{\xi} \cos(2\theta - \phi_{\xi} - 2\phi_{\eta}) \right],
\]
(42)

where we have defined \( \phi_{\eta} = \arg(\eta_{bc}) \). The second part of the left-hand side sum reflects the squeezed characteristic of the measurement. Not surprisingly, measured squeezing is thus a function of \( |\eta_{bc}|^2 \). Hence, squeezing is maximal when the creation and measurements are the same.

**PROPERTIES OF THE QUANTUM CHARACTERISTIC FUNCTION**

**Definition and useful relations**

The quantum characteristic function \[3\] is

\[
\chi_e(\lambda, \lambda^*, \rho_b) = \text{Tr} \left\{ \rho_b e^{\lambda c^\dagger} e^{-\lambda^* c} \right\} = \sum_{mn} \rho_{mn} \langle n | e^{\lambda c^\dagger} e^{-\lambda^* c} | m \rangle_b,
\]
(43)

where \( \rho_b = \sum_{mn} \rho_{mn} |n\rangle_b \langle n|_b \). We also define the c-number commutator \( \eta \equiv [c, b^\dagger] = [b, c^\dagger]^* \).

From the characteristic function, we get all the normally ordered moments of the ladder operators

\[
\text{Tr} \left[ \rho_b (c^\dagger)^k c^\ell \right] = (-1)^\ell \frac{\partial^{k+\ell}}{\partial \lambda^k \partial \lambda^\ell} \chi_e(\lambda, \lambda^*, \rho_b) \bigg|_{\lambda=0}
\]
(44)

We now proceed to show that

\[
b \langle n | e^{\lambda c^\dagger} e^{-\lambda^* c} | m \rangle_b = \sum_{k=0}^{m} \sum_{\ell=0}^{n} \frac{(-\eta \lambda^*)^k (\eta^* \lambda)^\ell}{k! \ell! (m-k)! (n-\ell)!} \delta_{(n-m)-(\ell-k)} \sqrt{m! n!}.
\]

(45)

We start with

\[
e^{-\lambda^* c} |m\rangle_b = \sum \frac{(-\lambda^*)^k}{k! \sqrt{m!}} c^k (b^\dagger)^m |\text{vac}\rangle.
\]

(46)
Each term of the form $c^k (b^\dagger)^m \ket{\text{vac}}$ yields
\[
c^k (b^\dagger)^m \ket{\text{vac}} = c^{k-1} c (b^\dagger)^{m-1} \ket{\text{vac}} + \eta c^{k-1} (b^\dagger)^{m-1} \ket{\text{vac}}
\]
\[= \ldots\]
\[= m\eta c^{k-1} (b^\dagger)^{m-1} \ket{\text{vac}}
\]
\[= m(m-1)\eta^2 c^{k-2} (b^\dagger)^{m-2} \ket{\text{vac}}
\]
\[= \ldots\]
\[= \frac{m!}{(m-k)!} \eta^k (b^\dagger)^{m-k} \ket{\text{vac}} \quad m \geq k
\]
\[= m! \eta^m c^{k-m} \ket{\text{vac}} = 0 \quad m < k,
\]
so that in the end
\[
e^{-\lambda^* c} \ket{m}_b = \sum_{k=0}^m \frac{(-\eta \lambda^*)^k \sqrt{m!}}{k! \sqrt{(m-k)!}} \ket{m-k}_b.
\]
Eq. (45) is recovered from Eq. (48) and its hermitian conjugate. If we take $c = b$, we also have
\[
b \langle n | e^{\lambda b^\dagger} e^{-\lambda^* b} | m \rangle_b = \sum_{k=0}^m \sum_{\ell=0}^n \frac{(-\lambda^*)^k \lambda^\ell \sqrt{m! n!}}{k! \ell! \sqrt{(m-k)! (n-\ell)!}} \delta_{(n-m)-(\ell-k)},
\]
so that
\[
b \langle n | e^{\lambda c^\dagger} e^{-\lambda^* c} | m \rangle_b = b \langle n | e^{\lambda^* b^\dagger} e^{-\lambda^* b} | m \rangle_b,
\]
or
\[
\chi_c(\lambda, \lambda^*, \rho_b) = \chi_b(\lambda^* \rho_c, \lambda^* \rho_c, \rho_b).
\]

**Thermal state**

The thermal state is defined as
\[
\rho_{\text{th}} = \frac{e^{-h/kT}}{\text{Tr}[e^{-h/kT}]}.
\]
Its decomposition in the Fock state basis associated with the mode $b$ is
\[
\rho_{\text{th}} = \sum_{mn} \frac{\rho_{mn}}{\sum_n \rho_{nn}} \frac{b^\dagger_n b_n \langle n | \langle m \rangle_b}{\rho_{nn}},
\]
with $\rho_{mn} = b \langle m | e^{-h/kT} | n \rangle_b$.

We have
\[
\rho_{mn} = \frac{1}{\sqrt{m! n!}} \sum_{\ell=0}^{+\infty} \left( \frac{-1}{kT} \right)^\ell \frac{\langle \text{vac} | b^\dagger_m b^{\ell \dagger n} | \text{vac} \rangle}{\ell!}
\]
Now, $h^\ell$ can be written in antinormally ordered form, with terms that always contains the same number of $a$ and $a^\dagger$ operators. Hence, the expected value is zero if $m \neq n$, and we have (see Eq. (60))
\[
\rho_{mn} = \delta_{m,n} \sum_{\ell=0}^{+\infty} \left( \frac{-1}{kT} \right)^\ell \frac{\langle \text{vac} | b^\dagger_m h^{\ell \dagger n} b_n | \text{vac} \rangle}{\ell!}
\]
\[= \delta_{m,n} e^{-nhf_{\text{th}}/kT},
\]
with
\[ \bar{f}_b = \int df \, \beta_\sigma(f)^2. \]  
(56)

Finally, we have
\[ \rho^\text{th} = \left(1 - e^{-\hbar \bar{f}_b / kT} \right) \sum_n e^{-n \hbar \bar{f}_b / kT} b|n\rangle \langle n|_b = \sum_n \rho_{nn} b|n\rangle \langle n|_b. \]  
(57)

For a diagonal state, the characteristic function is [5]
\[ \chi(\lambda, \lambda^*, \rho) = \sum_n \rho_{nn} L_n(|\lambda|). \]  
(58)

Using the properties of Laguerre polynomials, we get
\[ \chi_b(\lambda, \lambda^*, \rho^\text{th}) = \exp \left[ -\frac{|\lambda|^2}{e^{\hbar \bar{f}_b / kT} - 1} \right]. \]  
(59)

We now proceed to prove Eq. (55). We have
\[
\frac{1}{n!} \langle \text{vac} | b^n \hbar f b^n | \text{vac} \rangle = \frac{1}{n!} \int dv_i' \cdots dv_n' dv_1 \cdots dv_n df_1 \cdots df_\ell \beta_\sigma^*(\nu_i') \cdots \beta_\sigma^*(\nu_n') \beta_\sigma^*(\nu_1) \cdots \beta_\sigma^*(\nu_n) \hbar f_1 \cdots \hbar f_\ell
\]
\[
\langle \text{vac} | a_{\nu_i'} a_{\nu_i'}^\dagger a_{f_i'} a_{f_i'}^\dagger \cdots a_{f_{\ell-1}} a_{f_{\ell-1}}^\dagger a_{\nu_n'}^\dagger a_{\nu_n'} | \text{vac} \rangle = \frac{1}{n!} \int dv_i' \cdots dv_n' dv_i \cdots dv_n \beta_\sigma^*(\nu_i') \cdots \beta_\sigma^*(\nu_n') \beta_\sigma^*(\nu_1) \cdots \beta_\sigma^*(\nu_n)
\]
\[
\langle \text{vac} | a_{\nu_i'} a_{\nu_i'}^\dagger a_{f_i'} a_{f_i'}^\dagger \cdots a_{f_{\ell-1}} a_{f_{\ell-1}}^\dagger a_{\nu_n'}^\dagger a_{\nu_n'} | \text{vac} \rangle \sum_{i=1}^n \hbar \nu_i
\]
\[
= \cdots
\]
\[
= \frac{1}{(n-1)!} \int dv_i' \cdots dv_{n-1}' dv_1 \cdots dv_{n-1} \beta_\sigma^*(\nu_i') \cdots \beta_\sigma^*(\nu_{n-1}') \beta_\sigma^*(\nu_1) \cdots \beta_\sigma^*(\nu_{n-1}) | \beta_\sigma(\nu_n)|^2
\]
\[
\langle \text{vac} | a_{\nu_i'} a_{\nu_i'}^\dagger a_{f_i'} a_{f_i'}^\dagger \cdots a_{f_{n-1}}^\dagger a_{\nu_{n-1}}^\dagger | \text{vac} \rangle \sum_{i=1}^n \hbar \nu_i \cdots \hbar \nu_{n-1}
\]
\[
= \cdots
\]
\[
= \int dv_1 \cdots dv_n | \beta_\sigma(\nu_1)|^2 \cdots |\beta_\sigma(\nu_n)|^2 \sum_{i=1}^n \hbar \nu_i \cdots \hbar \nu_{n}
\]
\[
= \left\{ \int dv_1 \cdots dv_n | \beta_\sigma(\nu_1)|^2 \cdots |\beta_\sigma(\nu_n)|^2 \sum_{i=1}^n \hbar \nu_i \right\}^\ell
\]
\[
= \left\{ n \int dv \beta_\sigma(\nu)|^2 \hbar \nu \left[ \int dv' \beta_\sigma(\nu')|^2 \right]^{n-1} \right\}^\ell
\]
\[
= \left\{ n \int df \beta_\sigma(f)|^2 \hbar f \right\}^\ell
\]
\[
= [n \hbar \bar{f}_b]^\ell.
\]
Squeezed vacuum state

We define the quadrature operator

\[
x_{\chi, \theta} = \frac{c e^{-i\theta} + c^\dagger e^{i\theta}}{\sqrt{2}}.
\]

We thus have

\[
\langle x_{\chi, \theta} \rangle = \frac{1}{\sqrt{2}} [\langle c \rangle e^{-i\theta} + \langle c^\dagger \rangle e^{i\theta}];
\]

\[
\langle x_{\chi, \theta}^2 \rangle = \frac{1}{2} [\langle c^2 \rangle e^{-i2\theta} + \langle c^\dagger 2 \rangle e^{i2\theta}] + \langle c^\dagger c \rangle + \frac{1}{2}
\]

From [5, Eqs. (4.4.1) and (4.4.42)] and Eq. (51), we have

\[
\chi_c(\lambda, \lambda^*, |\xi| b) = \exp \left[ -S_\xi^2|\lambda \eta_{be}|^2 - \frac{1}{2} C_\xi S_\xi (\lambda \eta_{be})^2 e^{-i\phi_\xi} - \frac{1}{2} C_\xi S_\xi (\lambda^* \eta_{be})^2 e^{i\phi_\xi} \right],
\]

from which we compute

\[
\langle \Delta x_{\chi, \theta}^2 \rangle = \frac{1}{2} + |\eta_{be}|^2 S_\xi^2 - C_\xi S_\xi \eta_{be}^2 e^{i(2\theta-\phi_\xi)} + \eta_{be}^2 e^{-i(2\theta-\phi_\xi)}
\]

\[
= \frac{1}{2} + |\eta_{be}|^2 \left[ S_\xi^2 - C_\xi S_\xi \cos(2\theta - \phi_\xi - 2\phi_\eta) \right],
\]

where we have defined \( \phi_\eta = \text{arg}(\eta_{be}) \).

Coherent state

The voltage operator is

\[
v_{\sigma}(\tau, t) = i \sqrt{\frac{Z}{2}} \int df \sqrt{h |f|} \left[ a_{\chi, \sigma}^\dagger (f) e^{i2\pi f t} - a_{\chi, \sigma} (f) e^{-i2\pi f t} \right].
\]

In this section, we use the wavelet mode

\[
b_{\chi, \sigma} = \int df \beta_{\sigma}^* (f) a_{\chi, \sigma} (f); \\
b_{\chi, \sigma}^\dagger = \int df \beta_{\sigma} (f) a_{\chi, \sigma}^\dagger (f),
\]

and we define an associated coherent state by applying the associated displacement operator on the vacuum, i.e.

\[
|\alpha\rangle_b = e^{\alpha b_{\chi, \sigma}^\dagger - \alpha^* b_{\chi, \sigma}} |\text{vac}\rangle.
\]

We first show that \(|\alpha\rangle_b\) is an eigenvector of all destruction operators \(a_{\chi, \sigma}(f')\). We start with the fact that the displacement operator can also be written as

\[
e^{\alpha b_{\chi, \sigma}^\dagger - \alpha^* b_{\chi, \sigma}} = e^{-\frac{|\alpha|^2}{2}} e^{\alpha b_{\chi, \sigma}^\dagger} e^{-\alpha^* b_{\chi, \sigma}},
\]

so that

\[
|\alpha\rangle_b = e^{-\frac{|\alpha|^2}{2}} \left\{ I + \sum_{n=1}^{+\infty} \frac{\alpha^n}{n!} \left[ \int df \beta_{\sigma}(f) a_{\chi, \sigma}(f) \right]^n \right\} \left\{ I + \sum_{n=1}^{+\infty} \frac{(-\alpha^*)^n}{n!} \left[ \int df \beta_{\sigma}^*(f) a_{\chi, \sigma}^*(f) \right]^n \right\} |\text{vac}\rangle
\]

\[
= e^{-\frac{|\alpha|^2}{2}} \left\{ I + \sum_{n=1}^{+\infty} \frac{\alpha^n}{n!} \left[ \int df \beta_{\sigma}(f) a_{\chi, \sigma}(f) \right]^n \right\} |\text{vac}\rangle.
\]
and in the end,

\[
aτ,σ′(f′)|α⟩b = e^{-|α|^2} aτ,σ′(f′) \left\{ \mathbf{I} + \sum_{n=1}^{+∞} \frac{α^n}{n!} \left[ \int df \beta_σ(f) a_τ,σ(f) \right]^n \right\} |\text{vac}\rangle. \tag{71}
\]

In this expression, we sum terms of the form \(aτ,σ′(f′) a_τ,σ(f_1) \cdots a_τ,σ(f_n)|\text{vac}\rangle\). We can calculate those terms as

\[
aτ,σ′(f′) a_τ,σ(f_1) \cdots a_τ,σ(f_n)|\text{vac}\rangle = \left\{ \delta_σ,σ′ δ(f′ - f_1) e^{iσ2πf(τ′-τ)} + aτ,σ′(f_1)a_τ,σ′(f′) \right\} a_τ,σ(f_2) \cdots a_τ,σ(f_n)|\text{vac}\rangle = \cdots
\tag{72}
\]

so that

\[
aτ,σ′(f′)|α⟩b = e^{-|α|^2} \sum_{n=1}^{+∞} \frac{α^n}{n!} \left\{ n \delta_σ,σ′ e^{iσ2πf(τ′-τ)} β_σ(f′) \left[ \int df \beta_σ(f) a_τ,σ(f) \right]^{n-1} \right\} |\text{vac}\rangle
\]

\[
= δ_σ,σ′ e^{-|α|^2} β_σ(f′) e^{iσ2πf(τ′-τ)} \left\{ \sum_{n=1}^{+∞} \frac{α^n}{n!} \left[ \int df \beta_σ(f) a_τ,σ(f) \right]^{n-1} \right\} |\text{vac}\rangle
\]

\[
= δ_σ,σ′ α β_σ(f′) e^{iσ2πf(τ′-τ)} e^{-|α|^2} \left\{ \mathbf{I} + \sum_{n=1}^{+∞} \frac{α^n}{n!} \left[ \int df \beta_σ(f) a_τ,σ(f) \right]^n \right\} |\text{vac}\rangle
\]

\[
= δ_σ,σ′ α β_σ(f′) e^{iσ2πf(τ′-τ)} |α⟩b,
\]

which is what we wanted.

For the expectation of the voltage observable, this yields

\[
⟨α| vτ,σ′(t) |α⟩b = δ_σ,σ′ i \sqrt{Z} \int df \sqrt{hf} \left[ α^∗ β_σ^∗(f) e^{i2πf[t+σ(τ-τ′)]} - α β_σ(f) e^{-i2πf[t+σ(τ-τ′)]} \right]. \tag{74}
\]

We want this expectation to equal the expected classical voltage, or

\[
⟨α| vτ,σ′(t) |α⟩b = vτ,σ′(t) = \int df \left[ vτ,σ′(f) e^{i2πf t} + vτ,σ′(f) e^{-i2πf t} \right]. \tag{75}
\]

This yields the equality

\[
vτ,σ′(f) = -δ_σ,σ′ i \sqrt{Z \hbar |f|} α β_σ(f) e^{iσ2πf(τ′-τ)}, \tag{76}
\]

or

\[
α β_σ(f) = i δ_σ,σ′ \sqrt{\frac{2}{Z \hbar |f|}} e^{iσ2πf(τ-τ′)} vτ,σ′(f). \tag{77}
\]

We now simplify and consider the case where \(σ = σ′\) and \(τ = τ′\). Eq. (77) then becomes

\[
α β_σ(f) = i \sqrt{\frac{2}{Z \hbar |f|}} v(f). \tag{78}
\]

Moreover, the normalization condition

\[
\int df |β(f)|^2 = 1 \tag{79}
\]
leads to

$$|\alpha|^2 = \frac{2}{2\hbar} \int df \frac{|v(f)|^2}{|f|}. \quad (80)$$

As $\alpha$ can be defined up to a phase anyway, we have

$$\beta_\sigma(f) = B_\sigma v(f) \frac{\mathcal{H}(f)}{\sqrt{|f|}}. \quad (81)$$

with

$$B_\sigma = \frac{1}{\sqrt{\int_\sigma df |v(f)|^2}}. \quad (82)$$

In time domain, this means that $\beta_\sigma(t)$ is the convolution

$$\beta_\sigma(t) = \int df \frac{\mathcal{H}(f)}{\sqrt{|f|}} e^{-i2\pi ft}$$

$$= \int df \int dt' \int dt'' \frac{1 - i \text{sgn}(t')}{2\sqrt{|t'|}} v(t'') e^{i2\pi f(t'+t''-t)}$$

$$= \int dt'' v(t-t') \frac{e^{-i \text{sgn}(t') \pi/4}}{\sqrt{2}} \frac{1}{\sqrt{|t'|}}. \quad (83)$$

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