Polarisation of SKT Calabi-Yau $\partial\bar{\partial}$-manifolds by Aeppli classes

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Abstract. Given a $\partial\bar{\partial}$-manifold $X$ with trivial canonical bundle and carrying a metric $\omega$ such that $\partial\bar{\partial}\omega = 0$, we introduce the concept of small deformations of $X$ polarised by the Aeppli cohomology class $[\omega]_A$ of an SKT metric $\omega$. There is a correspondence between the manifolds polarised by $[\omega]_A$ in the Kuranishi family of $X$ and the Bott-Chern classes that are primitive in a sense that we define. We also investigate the existence of a primitive element in an arbitrary Bott-Chern primitive class and compare the metrics on the base space of the subfamily of manifolds polarised by $[\omega]_A$ within the Kuranishi family.

1. Introduction

Let $X$ be a compact complex manifold of dimension $n$. Recall that a manifold is said to be a $\partial\bar{\partial}$-manifold if for any pure-type $d$-closed form $u$, we have the following equivalences:

\[ u \text{ is } d\text{-exact} \iff u \text{ is } \partial\text{-exact} \iff u \text{ is } \bar{\partial}\text{-exact} \iff u \text{ is } \partial\bar{\partial}\text{-exact}. \]

A complex manifold is called Calabi-Yau if its canonical bundle $K_X$ is trivial.

A Hermitian metric $\omega$, seen as a positive definite $C^\infty (1,1)$-form, on a complex manifold $X$ is called strong Kähler with torsion (SKT for short) if $\partial\bar{\partial}\omega = 0$ and it is called Hermitian-symplectic [ST10] if $\omega$ is the component of bidegree $(1,1)$ of a real smooth $d$-closed 2-form on $X$. Obviously, on a $\partial\bar{\partial}$-manifold, a metric is SKT if and only if it is Hermitian-symplectic. The study of SKT metrics (also called pluriclosed, see [ST10]) has received a lot of attention over recent years. A necessary condition for the existence of a smooth family of SKT metrics on a differentiable family of complex manifolds is given in [PS21]. The existence of a left-invariant SKT structure on any even-dimensional compact Lie group $G$ is obtained in [MS11].

The $\partial\bar{\partial}$-property is open under holomorphic deformations of the complex structure by [Wu06]. Namely, if $\pi : \mathcal{X} \rightarrow B$ is a proper holomorphic submersion between complex manifolds, and $X_0 := \pi^{-1}(0)$ is a $\partial\bar{\partial}$-manifold, then $X_t := \pi^{-1}(t)$ is a $\partial\bar{\partial}$-manifold for all $t$ in a small neighbourhood of 0. Moreover, the Hodge numbers are independent of $t$ in this case. If $X_0$ is a Calabi-Yau manifold and $h^{n,0}$ does not jump, which means that $h^{n,0}(t) = h^{n,0}(0)$ for $t$ in a small neighbourhood of 0, then $X_t$ is again a Calabi-Yau manifold for $t$ close to 0. Though the SKT condition is not deformation open, if $X_0$ is an SKT $\partial\partial$-manifold, then $X_t$ is again an SKT $\partial\bar{\partial}$-manifold for $t$ in a small neighbourhood of 0. Indeed, the Hermitian-symplectic condition is deformation open by [Yan15] (see also [Bel20]). Putting these things together, we get that if $X_0$ is
an SKT Calabi-Yau $\partial\bar{\partial}$-manifold, then for all $t$ close to 0, $X_t$ is again an SKT Calabi-Yau $\partial\bar{\partial}$-manifold.

Now, fix an SKT metric $\omega$ on a Calabi-Yau SKT $\partial\bar{\partial}$-manifold $X$. Let $(X_t)_{t \in B}$ be the Kuranishi family of $X$. By [Bog78], [Tia87], [Tod89] (see also [Pop19]), the Kuranishi family is unobstructed. In particular, $B$ can be seen as a ball about 0 in $H^{0,1}(X_0, T^{1,0}X_0)$, where $T^{1,0}X_0$ is the holomorphic tangent bundle of $X_0$. We can define the notion of $X_t$ being polarised by the SKT Aeppli class $[\omega]_A$ by requiring the canonical image $\{\omega\}_{DR}$ of $[\omega]_A$ in $H^2_{DR}(X, \mathbb{C})$, where $X$ is the $C^\infty$ manifold underlying the fibres $X_t$, to be of type $(1,1)$ for the complex structure of $X_t$ (see Definition 1). Somehow we can view the set $B_{[\omega]}$ of all the fibres $X_t$ polarised by $[\omega]_A$ as the intersection of $B$ and $H^{0,1}(X, T^{1,0}X)_{[\omega]}$, the latter being defined in Lemma 3 as a vector subspace of $H^{0,1}(X_0, T^{1,0}X_0)$.

In section 4, we define the primitivity of certain Bott-Chern classes w.r.t. $[\omega]_A$. Then we can identify the space of $H^{0,1}(X_0, T^{1,0}X_0)_{[\omega]}$ with the space of Bott-Chern primitive classes.

In section 6, we compare the Weil-Petersson metric and the metric induced by the period map on the base space $B_{[\omega]}$ of the family of $[\omega]$-polarised small deformations of $X$. In the case of Kähler polarised deformations, these two metrics coincide with each other by [Tia87, Theorem 2].

This work was inspired by [Pop19] where the notion of small deformations co-polarised by a balanced class was introduced and studied. Besides, deformations co-polarised by a Gauduchon class was studied in [Bel21].

Acknowledgements. The author is supported by China Scholarship Council. He is grateful to his Ph.D. supervisor Dan Popovici for suggesting the problem and for his constant guidance and support.

2. Preliminaries

Recall that a compact complex manifold $X$ is a $\partial\bar{\partial}$-manifold if and only if there are canonical isomorphisms between the Bott-Chern, Aeppli and Dolbeaut cohomology groups, i.e. the canonical maps

$$H^{p,q}_{BC}(X, \mathbb{C}) = \frac{\ker \partial \cap \ker \bar{\partial}}{\text{Im} \partial \bar{\partial}} \implies H^{p,q}_{\partial\bar{\partial}}(X, \mathbb{C}) = \frac{\ker \bar{\partial}}{\text{Im} \partial \bar{\partial}} \implies H^{p,q}_{A}(X, \mathbb{C}) = \frac{\ker \partial \bar{\partial}}{\text{Im} \partial \bar{\partial} + \text{Im} \partial \bar{\partial}}$$

are isomorphisms for all $0 \leq p, q \leq n$. For $\partial\bar{\partial}$-manifolds, we have a Hodge decomposition in the sense that the canonical map with $d\alpha^{p,q} = 0, \forall p, q = k - p$

$$\bigoplus_{p+q=k} H^{p,q}_{A}(X, \mathbb{C}) \cong H^k_{DR}(X, \mathbb{C})$$

$([\alpha^{p,q}]_A)_{p+q=k} \mapsto \{ \sum_{p+q=k} \alpha^{p,q} \}$

is an isomorphism for all $0 \leq k \leq 2n$, where all the forms $\alpha^{p,q}$ are $d$-closed. The $\partial\bar{\partial}$-assumption on $X$ guarantees that every Aeppli cohomology class can be represented by a $d$-closed form. We could use any of the Bott-Chern, Aeppli or Dolbeaut cohomologies here. As in [KS60] and [Sch07], Bott-Chern and
Aeppli Laplacians $\Delta_{BC}, \Delta_A : C^\infty_{p,q}(X, \mathbb{C}) \to C^\infty_{p,q}(X, \mathbb{C})$ are defined as:

$$\Delta_{BC} = \partial^* \partial + \bar{\partial}^* \bar{\partial} + (\partial \bar{\partial})(\partial \bar{\partial})^* + (\partial \bar{\partial})(\partial \bar{\partial})^* + (\partial^* \bar{\partial})(\partial^* \bar{\partial}),$$

$$\Delta_A = \partial \partial^* + \bar{\partial} \bar{\partial}^* + (\partial \bar{\partial})(\partial \bar{\partial})^* + (\partial \bar{\partial})(\partial \bar{\partial})^* + (\partial^* \bar{\partial})(\partial^* \bar{\partial})^*,$$

and then we have

(1) \( \ker \Delta_{BC} = \ker \partial \cap \ker \bar{\partial} \cap \ker (\partial \bar{\partial})^* \),

(2) \( \ker \Delta_A = \ker (\partial \bar{\partial}) \cap \ker \partial^* \cap \ker \bar{\partial}^* \).

Let \( \omega \) be an SKT metric on a \( \partial \bar{\partial} \)-manifold \( X \). By the \( \partial \bar{\partial} \)-property, there exists a form \( \alpha \in C^\infty_{0,1}(X, \mathbb{C}) \), such that

(3) \( \bar{\partial} \omega = \partial \alpha \).

Note that \( d(\omega + \partial \alpha + \bar{\partial} \alpha) = 0 \), so \( \omega + \partial \alpha + \bar{\partial} \alpha \) is a \( d \)-closed representative of \( [\omega]_A \). We define \( \{ \omega \}_b \) (resp. \( [\omega]_\bar{\partial} \)) to be the image of \( [\omega]_A \) under the canonical injection \( H^1_A(X, \mathbb{C}) \hookrightarrow H^2_{DR}(X, \mathbb{C}) \) (resp. the isomorphism \( H^1_A(X, \mathbb{C}) \cong H^1_{\bar{\partial}}(X, \mathbb{C}) \)), which means that \( \{ \omega \}_b = \{ \omega + \partial \alpha + \bar{\partial} \alpha \}_b \) (resp. \( [\omega]_{\bar{\partial}} = [\omega + \partial \alpha]_{\bar{\partial}} \)).

Given a Calabi-Yau manifold \( X \), we fix a non-vanishing holomorphic \( n \)-form \( u \) on \( X \). Note that \( u \) exists and is unique up to a multiplicative constant since \( K_X \) is trivial. It defines the Calabi-Yau isomorphism:

$$T[u] : H^{0,1}(X, T^{1,0}X) \to H^{n-1,1}_{\bar{\partial}}(X, \mathbb{C})$$

$$[\theta] \mapsto [\theta \cdot u],$$

where the operator \( \cdot \cdot \cdot \cdot \) combines the contraction of \( u \) by the vector field component of \( \theta \) with the multiplication by the \( (0,1) \)-form component.

For a primitive form \( v \) of bidegree \( (p,q) \), we will often use the following formula (see [Voi02, Proposition 6.29])

(4) \( \star v = (-1)^{\frac{p+q(p+q+1)}{2}} p^{1-q} \omega^{n-p-q} \wedge v \)

for the Hodge star operator \( \star \).

3. Polarisation by SKT classes

Let \( X \) be a compact SKT Calabi-Yau \( \partial \bar{\partial} \)-manifold and let \( \omega \) be an SKT metric on it. Let \( \pi : X \to B \) be the Kuranishi family of \( X \). In a small neighbourhood of 0, \( X_t := \pi^{-1}(t) \) is again a Calabi-Yau SKT \( \partial \bar{\partial} \)-manifold, and we have the following Hodge decomposition by [Sch07]

(5) \( H^2_{DR}(X, \mathbb{C}) \cong H^2_A(X_t, \mathbb{C}) \oplus H^1_A(X_t, \mathbb{C}) \oplus H^0_A(X_t, \mathbb{C}), \quad t \sim 0, \)

where “\( \cong \)” stands for the canonical isomorphism whose inverse is defined by \( ([\alpha^2,0], [\alpha^1,1], [\alpha^0,2]) \mapsto \{ \alpha^2,0 + \alpha^1,1 + \alpha^0,2 \}_b \), where \( d\alpha^{p,2-p} = 0 \) for \( p = 0, 1, 2 \).

Definition 1. Fix the Aeppli class \( [\omega]_A \in H^1_A(X, \mathbb{C}) \) of an SKT metric \( \omega \) on \( X_0 = X \). For \( t \in B \), we say that \( X_t \) is polarised by \( [\omega]_A \) if the projection \( [\omega]^{0,2}_{A,t} \) of \( \{ \omega \}_b \) onto \( H^0_A(X_t, \mathbb{C}) \) w.r.t \( (5) \) is 0.

Denote by \( B(\omega) \) the set of \( t \in B \) such that \( X_t \) is polarised by \( [\omega]_A \), namely

\( B(\omega) = \{ t \in B | [\omega]^{0,2}_{A,t} = 0 \in H^0_A(X_t, \mathbb{C}) \} \).
This means that \( \{\omega\}_{DR} \) is of \( J_t \)-pure-type \((1,1)\) for \( t \in B_{[\omega]} \) since \( [\omega]^{2,0}_{A,t} = 0 \) if and only if \( [\omega]_{A,t}^{0,2} = 0 \). Indeed, \( \{\omega\}_{DR} \) being real, \( [\omega]^{2,0}_{A,t} \) is the conjugate to \([\omega]^{0,2}_{A,t}\).

**Theorem 2.** Let \( \omega \) be an SKT metric on a compact \( \partial \overline{\partial} \)-manifold \( X \) and let \( \pi : X \to B \) be its Kuranishi family. Consider \( \gamma_t^{1,1} \in H^{1,1}_A(X_t, C) \) the Aeppli component of \( J_t \)-type \((1,1)\) of \( \{\omega\}_{DR} \) w.r.t. \((5)\). Then there exists an SKT metric \( \omega_t \in \gamma_t^{1,1} \) for all \( t \) in a small neighbourhood of 0.

**Proof.** By the definition \((3)\) of \( \alpha \), we construct two \( d \)-closed forms:

\[
\tilde{\omega} = \omega + d\alpha + \overline{d\alpha}, \\
\tilde{\omega} = -\partial\overline{\alpha} + \omega - \partial\alpha.
\]

We know that \( \tilde{\omega} \) is of \( J_0 \)-bidegree \((1,1)\) and \( \omega \) is the \( J_0 \)-(1,1)-component of the 2-form \( \tilde{\omega} \). Since \( \tilde{\omega} = \omega + d(\alpha + \overline{\alpha}) \), we have that \( \{\tilde{\omega}\}_{DR} = \{\omega\}_{DR} \).

Decompose \( \tilde{\omega} \) into components of pure \( J_t \)-type:

\[
\tilde{\omega} = \Omega_t^{2,0} + \omega_t^{1,1} + \Omega_t^{0,2}.
\]

We know that \( \omega_t^{0,1} = \omega \). Moreover, \( \omega_t^{1,1} \) is real and \( \partial_t \overline{\partial} \)-closed because \( \tilde{\omega} \) is real and \( \partial_t \)-closed. By the continuity of \( (\omega_t^{1,1})_t \), with respect to \( t \) and \( \omega_t^{0,1} = \omega > 0 \), we also have \( \omega_t^{1,1} > 0 \) for \( t \) near 0. This ensures that \( \omega_t^{1,1} \) is an SKT metric on \( X_t \). Moreover, we have \( \omega_t^{1,1} \in \gamma_t^{1,1} \), \( \forall t \sim 0 \). We put \( \omega_t := \omega_t^{1,1} \), \( \forall t \sim 0 \) and we are done.

Recall that \( H^{0,1}(X_0, T^{1,0}X_0) \) admits a decomposition:

\[
\ker \overline{\partial} : C^\infty_{0,1}(X_0, T^{1,0}X_0) \to C^\infty_{0,2}(X_0, T^{1,0}X_0) \\
\text{Im} \overline{\partial} : C^\infty_{0,0}(X_0, T^{1,0}X_0) \to C^\infty_{0,1}(X_0, T^{1,0}X_0)
\]

where \( \overline{\partial} \) is the holomorphic structure of \( T^{1,0}X_0 \). Because the Kodaira-Spencer map gives an isomorphism between \( T_0B \) and \( H^{0,1}(X_0, T^{1,0}X_0) \), after possibly shrinking \( B \), we can view the base space \( B \) as an open subset of \( H^{0,1}(X_0, T^{1,0}X_0) \).

Then we have the following lemma:

**Lemma 3.** Consider the following subspace of \( H^{0,1}(X, T^{1,0}X) \):

\[
H^{0,1}(X, T^{1,0}X)_{[\omega]} := \{[\theta] \in H^{0,1}(X, T^{1,0}X) \mid \theta \cdot \zeta \}_{A} = 0 \in H^{0,2}_{A}(X, C)\}
\]

where \( \zeta \) is an arbitrary representative in \([\omega]_{\overline{\partial}}\). It is well defined and \( T_0^{1,0}B_{[\omega]} = H^{0,1}(X, T^{1,0}X)_{[\omega]} \).

Or locally we can view this as \( B_{[\omega]} = B \cap H^{0,1}(X, T^{1,0}X)_{[\omega]} \).

**Proof.** By

\[
\overline{\partial}(\theta \cdot \beta) = \overline{\partial} \theta \cdot \beta + (-1)^q \theta \cdot \overline{\partial} \beta, \quad \forall \theta \in C^\infty_{0,q}(X, T^{1,0}X), \beta \in C^\infty_{1,q'}(X, C),
\]

we see that for \( \theta \) a representative in \([\theta]\), \( \overline{\partial}(\theta \cdot \zeta) = 0 \) and

\[
\theta \zeta + \overline{\partial} \zeta = \theta \zeta + \overline{\partial}(\theta \zeta')
\]

for \( \zeta' \in C^\infty(X, T^{1,0}X), \zeta' \in C^\infty_{1,0}(X, C) \). Hence the classes \([\theta \cdot \zeta]_{\overline{\partial}}\) and \([\theta \cdot \zeta]_{A}\) are independent of the choices of representatives of \([\theta]\) and \([\zeta]_{\overline{\partial}}\). Therefore, the
space \( H^{0,1}(X,T^{1,0}X)[\omega] \) is well defined. By the \( \partial\bar{\partial} \)-property, the classes \([\theta,\zeta]_A\) and \([\theta,\zeta]_B\) correspond to each other under the isomorphism \( H^{0,2}_A(X,\mathbb{C}) \overset{\cong}{\to} H^{0,2}_B(X,\mathbb{C}) \), so we also have

\[
\{[\theta] \in H^{0,1}(X,T^{1,0}X)| [\theta,\zeta]_A = 0 \in H^{0,2}_A(X,\mathbb{C})\} = \{[\theta] \in H^{0,1}(X,T^{1,0}X)| [\theta,\zeta]_B = 0 \in H^{0,2}_B(X,\mathbb{C})\}
\]

For \( t \) near 0, \( X_t \) is a \( \partial\bar{\partial} \)-manifold, so we have the Hodge decompositions:

\[
H^2_{DR}(X,\mathbb{C}) = H^{0,0}_A(X_t,\mathbb{C}) \oplus H^{1,1}_A(X_t,\mathbb{C}) \oplus H^{0,2}_A(X_t,\mathbb{C}) \\
\cong H^2_{\bar{\partial}}(X_t,\mathbb{C}) \oplus H^{1,1}(X_t,\mathbb{C}) \oplus H^{0,2}(X_t,\mathbb{C}).
\]

Take a vector in \( T_{0,0}^1 B[\omega] \), say \( \frac{\partial}{\partial t}|_{t=0} \). Denote by \([\theta]\) the image of it under the Kodaira-Spencer map \( \rho : T_{0,1}^0 B \to H^{0,1}(X_0,T^{1,0}X_0) \). Then we have \( \nabla_{\frac{\partial}{\partial t}}|_{t=0} [\omega]_A^{0,2} = [\theta \cdot \zeta]_A \), where \( \nabla \) is the Gauss-Manin connection. Then by the definition of \( B[\omega] \), we have

\[
T_{0,1}^0 B[\omega] = \{[\theta] \in H^{0,1}(X,T^{1,0}X)| [\theta,\zeta]_A = 0 \in H^{0,2}_A(X,\mathbb{C})\} = H^{0,1}(X,T^{1,0}X)[\omega].
\]

\[\square\]

**Remark 4.** If \( \omega \) is moreover \( \bar{\text{K"ahler}} \), \( \alpha \) can be taken as 0. Therefore everything here coincides with the case of \( \bar{\text{K"ahler}} \) polarised deformation.

4. **Primitive classes**

**Lemma and Definition 5.** Let \( X \) be a compact complex manifold of dimension \( n \), and \( \omega \) be an SKT metric on \( X \). Then the map

\[
L[\omega] : H^{p,q}_{BC}(X,\mathbb{C}) \to H^{p+1,q+1}_A(X,\mathbb{C}) \\
[\gamma]_{BC} \mapsto [\omega \wedge \gamma]_A
\]

is well-defined and only depends on the Aeppli class of \( \omega \).

We say that a Bott-Chern class \([\gamma]_{BC}\) of bidegree \((p,n-p)\) is primitive (or \([\omega]_{A\text{-primitive}}\)) if \( L[\omega](\gamma)_{BC} = 0 \). We denote the space of primitive Bott-Chern classes of bidegree \((n-1,1)\) by \( H^{n-1,1}_{BC,\text{prim}}(X,\mathbb{C}) \).

**Proof.** Since \( \partial\bar{\partial} = \bar{\partial}\partial = 0 \), we get

\[
\partial\bar{\partial}(\omega \wedge \gamma) = \partial\bar{\partial}\omega \wedge \gamma = 0,
\]

so \( \omega \wedge \gamma \) represents an Aeppli class. By

\[
\omega \wedge (\gamma + \partial\bar{\partial}\beta) = \omega \wedge \gamma + \partial(\omega \wedge \bar{\partial}\beta) + \bar{\partial}(\omega \wedge \beta) + \partial\bar{\partial}(\omega \wedge \beta) = \omega \wedge \gamma + \partial(\omega \wedge \bar{\partial}\beta) + \bar{\partial}(\omega \wedge \beta),
\]

we have \([\omega \wedge (\gamma + \partial\bar{\partial}\beta)]_A = [\omega \wedge \gamma]_A \). Hence the map \( L[\omega] \) is well-defined. From

\[
(\omega + \partial\beta_1 + \bar{\partial}\beta_2) \wedge \gamma = \omega \wedge \gamma + \partial(\beta_1 \wedge \gamma) + \bar{\partial}(\beta_2 \wedge \gamma),
\]

we see that the map \( L[\omega] \) only depends on the Aeppli class of \( \omega \). \( \square \)
Let $X$ be a $\partial\bar{\partial}$-manifold. We denote by
$$j : H^{p,q}_\partial(X, \mathbb{C}) \longrightarrow H^n_\partial(X, \mathbb{C})$$
the canonical isomorphism and by
$$\tilde{T}_{[u]} : H^{0,1}(X, T^{1,0}X) \longrightarrow H^{n-1,1}_\partial(X, \mathbb{C})$$
the composition of canonical isomorphism $i$ and Calabi-Yau isomorphism.

More precisely, for $[\theta] \in H^{0,1}(X, T^{1,0}X)$, we have $T_{[u]}([\theta]) = [\theta, u] \in H^{n-1,1}_\partial(X, \mathbb{C})$.

Then the isomorphism $i$ maps $[\theta, u]$ to $[\theta, u + \bar{\partial}\eta]_{BC}$, where $\eta$ is an $(n-1, 0)$-form such that $\partial(\theta, u + \bar{\partial}\eta) = 0$. Such a form $\eta$ exists because of the $d$-closedness, $\partial$-exactness of $\partial(\theta, u)$ and the $\partial\bar{\partial}$-property of $X$. The class $[\theta, u + \bar{\partial}\eta]_{BC}$ is independent of the choice of $\eta$ such that $\partial(\theta, u + \bar{\partial}\eta) = 0$, again by the $\partial\bar{\partial}$-property of $X$.

**Lemma 6.** The following map
$$f_{[u]} : H^{0,q}_\partial(X, \mathbb{C}) \longrightarrow H^{n,q}_\partial(X, \mathbb{C})$$
$$[\xi] \longmapsto [u \wedge \xi]$$
is well-defined and an isomorphism for all $q = 1, \cdots, n$. As a consequence, we have the equality between the Hodge numbers $h^{0,q} = h^{n,q}$.

**Proof.** To check that this is an isomorphism, we first check that
$$f_{u} : C^\infty_{0,q}(X, \mathbb{C}) \longrightarrow C^\infty_{n,q}(X, \mathbb{C})$$
$$\xi \longmapsto u \wedge \xi$$
is an isomorphism. In local coordinates, let
$$\xi = \sum_{|J|=q} \xi_J dz_J$$
and $u = gdz_1 \wedge \cdots \wedge dz_n$,
where $g$ does not vanish.

$$u \wedge \xi = \sum_{|J|=q} g\xi_J dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_J.$$ 

Hence
$$C^\infty_{0,q}(X, \mathbb{C}) \longrightarrow C^\infty_{n,q}(X, \mathbb{C})$$
$$\xi \longmapsto u \wedge \xi$$
is an isomorphism because $g$ does not vanish.

It is easy to check $f_{u}(\ker \partial) = \ker \partial$ and $f_{u}(\Im \partial) \subset \Im \bar{\partial}$ since $\partial u = 0$, which means that $f_{[u]}$ is well-defined and injective. Now we already have $h^{n,q} \geq h^{0,q}$ for all $q = 1, \cdots, n$ by injectivity. Take any $\partial$-exact form $\partial\eta \in C^\infty_{n,q}(X, \mathbb{C})$. In local coordinates, write
$$\eta = \sum_{|I|=n-1} \eta_I dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_I.$$ 

Let $\xi = \sum_{|I|=n-1} \eta_I d\bar{z}_I$, it is easy to check that $u \wedge \partial\bar{\partial}\xi = \partial\bar{\partial}\eta$ and this implies the surjectivity of $f_{[u]}$. 

\[\square\]
Theorem 7. If $\omega$ is an SKT metric on a Calabi-Yau $\partial \bar{\partial}$-manifold $X$, then
\[
\widetilde{T}_{[\omega]} : H^{0,1}(X, T^{1,0}X)_{[\omega]} \longrightarrow H^{n-1,1}_{BC,\text{prim}}(X, \mathbb{C})
\]
is an isomorphism.

Proof. For $[\theta] \in H^{0,1}(X, T^{1,0}X)$, we have $\widetilde{T}_{[\omega]}([\theta]) = [\theta \cdot u + \bar{\partial}\eta]_{BC}$, where $\eta$ is some $(n-1,0)$-form such that $\partial(\theta \cdot u + \bar{\partial}\eta) = 0$.

Let $\zeta$ be an arbitrary representative in $[\omega]_\beta$. Then there exist $\beta_1 \in C^\infty_{0,1}(X, \mathbb{C})$ and $\beta_2 \in C^\infty_{1,0}(X, \mathbb{C})$, such that $\zeta - \omega = \partial\beta_1 + \bar{\partial}\beta_2$. By
\[
0 = \theta \cdot (\omega \wedge u) = (\theta \cdot \omega) \wedge u + \omega \wedge (\theta \cdot u),
\]
we get
\[
L_{[\omega]}(\widetilde{T}_{[\omega]}([\theta])) = \omega \wedge (\theta \cdot u + \bar{\partial}\eta) \big|_A
= -(\theta \cdot \omega) \wedge u + \omega \wedge \bar{\partial}\eta \big|_A
= [(\theta \cdot (\zeta - \omega)) \wedge u + \omega \wedge \bar{\partial}\eta]_A - [(\theta \cdot \zeta) \wedge u]_A.
\]
Moreover,
\[
[(\theta \cdot (\zeta - \omega)) \wedge u + \omega \wedge \bar{\partial}\eta]_A
= -(\zeta - \omega) \wedge (\theta \cdot u) + \omega \wedge \bar{\partial}\eta]_A
= -(\partial\beta_1 + \bar{\partial}\beta_2) \wedge (\theta \cdot u) + \omega \wedge \bar{\partial}\eta]_A
= [\beta_1 \wedge (\theta \cdot u) + \omega \wedge \bar{\partial}\eta]_A
= [\beta_1 \wedge \partial(\theta \cdot u) + \omega \wedge \bar{\partial}\eta]_A
= [\omega + \partial\beta_1 + \bar{\partial}\beta_2 + \bar{\partial}\beta_1 \wedge \partial\eta]_A
= [\zeta \wedge \bar{\partial}\eta]_A
= 0.
\]
Hence we have
\[
(j \circ L_{[\omega]})(\widetilde{T}_{[\omega]}([\theta])) = j(\omega \wedge (\theta \cdot u + \bar{\partial}\eta)) = -(\theta \cdot \zeta) \wedge u]_\beta.
\]
By Lemma 6, we have that $(j \circ L_{[\omega]})(\widetilde{T}_{[\omega]}([\theta])) = 0$ if and only if $[\theta] \in H^{0,1}(X, T^{1,0}X)_{[\omega]}$, i.e. $\widetilde{T}_{[\omega]}$ is an isomorphism.

Corollary 8. The map
\[
L_{[\omega]} : H^{n-1,1}_{BC}(X, \mathbb{C}) \longrightarrow H^{n,2}_A(X, \mathbb{C})
\]
is surjective.

Proof. For $\partial \bar{\partial}$-manifolds, we know the equality between the dimensions of corresponding Dolbeaut, Bott-Chern, Aeppli cohomology classes $h^{p,q} := h^{p,q}_\bar{\partial} = h^{p,q}_{BC} = h^{p,q}_A$. Therefore we have
\[
\dim_{\mathbb{C}} H^{0,1}(X, T^{1,0}X)_{[\omega]} = \dim_{\mathbb{C}} H^{0,1}(X, T^{1,0}X) - \dim_{\mathbb{C}} H^{0,2}(X, \mathbb{C})
= h^{n-1,1} - h^{0,2}
\]
by the surjectivity of

\[ H^{0,1}(X, T^{1,0}X) \to H^{0,2}_\partial (X, \mathbb{C}) \]

\[ [\theta] \mapsto [\theta, \omega]. \]

To prove this surjectivity, we write the map in local coordinates \((z_1, \cdots, z_n)\). The metric can be written as \(\omega = \sum_{j,k=1}^{n} i \omega_{jk} dz_j \wedge d\bar{z}_k\). Then the map is

\[ [\theta = \sum_{i,j=1}^{n} \theta^i_j d\bar{z}_i \otimes \frac{\partial}{\partial z_j}] \mapsto [\theta, \omega = \sum_{i,j,k=1}^{n} i \theta^i_j \omega_{jk} dz_i \wedge d\bar{z}_k]. \]

The surjectivity is derived from the invertibility of \((\omega_{jk})_{1 \leq j, k \leq n}\).

By Theorem 7, we get

\[ \dim_{\mathbb{C}} \ker L_\omega = \dim_{\mathbb{C}} H^{n-1,1}_{BC,prim}(X, \mathbb{C}) = \dim_{\mathbb{C}} H^{0,1}(X, T^{1,0}X)|_\omega = h^{n-1,1} - h^{0,2}. \]

Hence we have

\[ \dim_{\mathbb{C}} \text{Ker} L_\omega = \dim_{\mathbb{C}} H^{n-1,1}_{BC}(X, \mathbb{C}) - \dim_{\mathbb{C}} \ker L_\omega = h^{0,2} = \dim_{\mathbb{C}} H^{n-2}(X, \mathbb{C}), \]

which means that \(L_\omega|_{H^{n-1,1}_{BC}(X, \mathbb{C})}\) is surjective.

In the Kähler case, every primitive Dolbeault class has one and only one \(d\)-closed primitive representative, which is the \(\Delta''\)-harmonic element. In general, we have the following:

**Lemma 9.** For a primitive form \(v\) of degree \(n\), the following are equivalent:

(a) \(d\)-closed,
(b) \(d^*\)-closed,
(c) \(\Delta v = 0\),
(d) \(\Delta_A v = 0\),
(e) \(\Delta_{BC} v = 0\).

**Proof.** Recall that

\[ \partial^* = -\ast \bar{\partial} \ast, \bar{\partial}^* = -\ast \partial \ast, \]

\[ d^* = -\ast d \ast. \]

Therefore by formula (4), \(v\) and \(\ast v\) are proportional. Hence, (a) and (b) are equivalent; \(\partial v = 0\) if and only if \(\bar{\partial}^* v = 0\); \(\partial^* v = 0\) if and only if \(\bar{\partial} v = 0\). Thus by equations (1) and (2), we have that (c), (d) and (e) are equivalent. The equivalence between (a) and (c) is trivial now.

In the SKT case, there is at most one \(d\)-closed primitive form in a cohomology class of degree \(n\). If we look for a \(d\)-closed primitive form in a primitive class on an SKT \(\partial\bar{\partial}\)-manifold, it does not matter whether we search in a Dolbeaut or Bott-Chern class.

We analyse an example of SKT manifold given in [TT17] from one point of view. Let \(X = S^3 \times S^3\). Recall that the 3-sphere is diffeomorphic to the special unitary group \(SU(2)\). We know that \(\mathfrak{su}(2)\), the Lie algebra of \(SU(2)\), has a basis \(\{e_1, e_2, e_3\}\) with the following relations:

\[ [e_1, e_2] = 2e_3, \quad [e_1, e_3] = -2e_2, \quad [e_2, e_3] = 2e_1. \]

Then by the Cartan formula, we have the following for the dual co-frame \(\{e^1, e^2, e^3\}\):

\[ de^1 = -2e^2 \wedge e^3, \quad de^2 = 2e^1 \wedge e^3 \quad de^3 = -2e^1 \wedge e^2. \]
On $X$, we take $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ to be two copies of this basis of $\mathfrak{su}(2)$, and the corresponding co-frames $\{e^1, e^2, e^3\}$ and $\{f^1, f^2, f^3\}$. Then we define a complex structure on $X$, namely the Calabi-Eckmann complex structure by:

$$Je_1 = e_2, \quad Jf_1 = f_2, \quad Je_3 = f_3.$$ 

We have

$$Je^1 = -e^2, \quad Jf^1 = -f^2, \quad Je^3 = -f^3.$$ 

For the complex co-frame of $(1, 0)$-forms, we set

$$\varphi^1 = e^1 + ie^2, \quad \varphi^2 = f^1 + if^2, \quad \varphi^3 = e^3 + if^3.$$ 

Thus, we have

$$d\varphi^1 = i\varphi^1 \wedge \varphi^3 + i\varphi^1 \wedge \bar{\varphi}^3,$$

$$d\varphi^2 = \varphi^2 \wedge \varphi^3 - \varphi^2 \wedge \bar{\varphi}^3,$$

$$d\varphi^3 = -i\varphi^1 \wedge \bar{\varphi}^1 + \varphi^2 \wedge \bar{\varphi}^2.$$ 

Equivalently,

$$\partial \varphi^1 = i\varphi^1 \wedge \varphi^3,$$

$$\partial \varphi^2 = \varphi^2 \wedge \varphi^3,$$

$$\partial \varphi^3 = 0,$$

$$\bar{\partial} \varphi^1 = i\varphi^1 \wedge \varphi^3,$$

$$\bar{\partial} \varphi^2 = -\varphi^2 \wedge \varphi^3,$$

$$\bar{\partial} \varphi^3 = -i\varphi^1 \wedge \bar{\varphi}^1 + \varphi^2 \wedge \bar{\varphi}^2.$$ 

We define a Hermitian metric

$$\omega := \frac{i}{2} \sum_{j=1}^{3} \varphi^j \wedge \bar{\varphi}^j.$$ 

By direct calculation we know that $\bar{\partial} \omega = 0$, which means that $\omega$ is an SKT metric on $X$. We calculate the Bott-Chern cohomology groups (see [TT17]):

$$H_{BC}^{0,0}(X, \mathbb{C}) = \langle [1] \rangle,$$

$$H_{BC}^{1,1}(X, \mathbb{C}) = \langle [\varphi^{11}], [\varphi^{22}] \rangle,$$

$$H_{BC}^{2,1}(X, \mathbb{C}) = \langle [\varphi^{22} + i\varphi^{13}] \rangle,$$

$$H_{BC}^{1,2}(X, \mathbb{C}) = \langle [\varphi^{22} - i\varphi^{13}] \rangle,$$

$$H_{BC}^{2,2}(X, \mathbb{C}) = \langle [\varphi^{12}] \rangle,$$

$$H_{BC}^{3,2}(X, \mathbb{C}) = \langle [\varphi^{12} + i\varphi^{13}] \rangle,$$

$$H_{BC}^{2,3}(X, \mathbb{C}) = \langle [\varphi^{12}] \rangle,$$

$$H_{BC}^{3,3}(X, \mathbb{C}) = \langle [\varphi^{12}] \rangle,$$

where all the representatives above are the $\Delta_{BC}$-harmonic ones. The other Bott-Chern cohomology groups are trivial.

Note that

$$[\omega \wedge (\varphi^{22} + i\varphi^{13})]_A = \frac{-1 + i}{2} [\varphi^{12}]_A = \frac{-1 + i}{2} [\partial \varphi^{13}]_A = 0.$$
Hence $[\varphi^{2}[\pi] + i\varphi^{1}[\pi]]_{BC} \in H^{2,1}_{BC}(X, \mathbb{C})$ is a primitive class but has no primitive representative. By conjugation, $[\varphi^{2}[\pi] - i\varphi^{1}[\pi]]_{BC} \in H^{1,2}_{BC}(X, \mathbb{C})$ is again a primitive class but has no primitive representative.

5. Period map

In this section, we recall the definition of the period map and the local Torelli theorem. This section closely follows [Pop19].

Fix a Hermitian metric $\omega$ on $X$. We have the Hodge star operator

$$\star : C^{\infty}_{n}(X, \mathbb{C}) \rightarrow C^{\infty}_{n}(X, \mathbb{C})$$

and $\star^{2} = (-1)^{n} id$, where $n = \dim_{\mathbb{C}} X$. When $n$ is even, the eigenvalues of $\star$ is 1 and $-1$. When $n$ is odd, the eigenvalues of $\star$ is $i$ and $-i$. This induces a decomposition

$$C^{\infty}_{n}(X, \mathbb{C}) = \Lambda^{n}_{+} \oplus \Lambda^{n}_{-},$$

where $\Lambda^{n}_{+}$ (resp. $\Lambda^{n}_{-}$) is the eigenspace corresponding to 1 or $i$ (resp. $-1$ or $-i$).

Since $\Delta = dd^{*} + d^{*}d$ commutes with $\star$, we have $\star(\mathcal{H}^{n}_{\Delta}(X, \mathbb{C})) = \mathcal{H}^{n}_{\Delta}(X, \mathbb{C})$. By $\mathcal{H}^{n}_{\Delta}(X, \mathbb{C}) = H^{n}_{DR}(X, \mathbb{C})$, we get a decomposition

$$H^{n}_{DR}(X, \mathbb{C}) = H^{n}_{+}(X, \mathbb{C}) \oplus H^{n}_{-}(X, \mathbb{C}).$$

The Hodge-Riemann bilinear form can be defined on $H^{n}_{DR}(X, \mathbb{C})$ without any assumption on $\omega$:

$$Q : H^{n}_{DR}(X, \mathbb{C}) \times H^{n}_{DR}(X, \mathbb{C}) \rightarrow \mathbb{C}$$

$$(\{\alpha\}, \{\beta\}) \mapsto (-1)^{\frac{n(n+1)}{2}} \int_{X} \alpha \wedge \beta.$$

It is non-degenerate. Indeed for every class $\{\alpha\}$, if $\alpha$ is the $\Delta$-harmonic representative, then $\star \alpha$ is also $\Delta$-harmonic. We have

$$(-1)^{\frac{n(n-1)}{2}} Q(\{\alpha\}, \{\star \alpha\}) = \int_{X} \alpha \wedge \star \alpha = \|\alpha\|^{2},$$

which is not 0 as long as $\alpha$ is not 0. Therefore we define a non-degenerate sesquilinear form

$$H : H^{n}_{DR}(X, \mathbb{C}) \times H^{n}_{DR}(X, \mathbb{C}) \rightarrow \mathbb{C}$$

$$(\{\alpha\}, \{\beta\}) \mapsto (-1)^{\frac{n(n+1)}{2}} l^{n} \int_{X} \alpha \wedge \bar{\beta}.$$

Then one can define the period domain as follows.

**Definition 10.** The period domain is defined as a subset of $\mathbb{P} H^{n}(X, \mathbb{C})$:

- If $n$ is even,

  $$D = \{ \text{complex line } l \in \mathbb{P} H^{n}(X, \mathbb{C}) \mid \forall \varphi \in l \setminus \{0\}, Q(\varphi, \varphi) = 0 \text{ and } H(\varphi, \varphi) > 0 \};$$

- If $n$ is odd,

  $$D = \{ \text{complex line } l \in \mathbb{P} H^{n}(X, \mathbb{C}) \mid \forall \varphi \in l \setminus \{0\}, Q(\varphi, \varphi) = 0 \text{ and } H(\varphi, \varphi) < 0 \}.$$

We prove that $H^{n,0}(X, \mathbb{C})$ is a subset of the period domain by the following two lemmas:
Lemma 11. Let $X$ be a compact complex $\partial\bar{\partial}$-manifold, then

- if $n$ is even, $H^{n,0}(X, \mathbb{C}) \subset H^n_{\bar{\partial}}(X, \mathbb{C})$;
- if $n$ is odd, $H^{n,0}(X, \mathbb{C}) \subset H^n_{\partial}(X, \mathbb{C})$.

Proof. Because every element $\alpha \in C^n_{0,0}(X, \mathbb{C})$ is primitive for bidegree reasons, we have $\ast \alpha = i^{n(n+2)} \alpha$ by (4). Hence $\alpha$ is $\bar{\partial}$-closed if and only if $\ast \alpha$ is $\bar{\partial}$-closed. We have $i^{n(n+2)} = 1$ if $n$ is even, and $i^{n(n+2)} = -i$ if $n$ is odd. This proves the lemma. \hfill \Box

Lemma 12. We have the following properties:

$H(\{\alpha\}, \{\alpha\}) > 0$ for every class $\{\alpha\} \in H^n_{\bar{\partial}}(X, \mathbb{C}) \setminus \{0\}$,

$H(\{\alpha\}, \{\alpha\}) < 0$ for every class $\{\alpha\} \in H^n_{\partial}(X, \mathbb{C}) \setminus \{0\}$.

Proof. If $n$ is even, for every class $\{\alpha\} \in H^n_{\bar{\partial}}(X, \mathbb{C}) \setminus \{0\}$, we have $\ast \alpha = \alpha$. Then

$$H(\{\alpha\}, \{\alpha\}) = i^{n(n+2)} \int_X \alpha \wedge \bar{\alpha} = \int_X \alpha \wedge \ast \alpha = \|\alpha\|^2 > 0.$$  

If $n$ is odd, for every class $\{\alpha\} \in H^n_{\partial}(X, \mathbb{C}) \setminus \{0\}$, we have $\ast \alpha = i\alpha$. We still have

$$H(\{\alpha\}, \{\alpha\}) = i^{n(n+2)} \int_X \alpha \wedge \bar{\alpha} = \int_X \alpha \wedge \ast \alpha = \|\alpha\|^2 > 0.$$  

One can prove the second statement similarly. \hfill \Box

Theorem 13 (Local Torelli Theorem [Pop19, Thm 5.4]). Let $X$ be a compact Calabi-Yau $\partial\bar{\partial}$-manifold of dimension $n$, and $\pi : X \rightarrow B$ be its Kuranishi family. Then the associated period map

$$\mathcal{P} : B \rightarrow D \subset \mathbb{P} H^n(X, \mathbb{C})$$  

$$t \mapsto H^{n,0}(X_t, \mathbb{C})$$  

is a local holomorphic immersion.

Proof. Because $X = X_0$ is Calabi-Yau $\partial\bar{\partial}$-manifold, we know that $X_t$ is a Calabi-Yau $\partial\bar{\partial}$-manifold for all $t$ in a neighbourhood of 0. Then $H^{n,0}(X_t, \mathbb{C})$ is a point in $H^{n,0}(X_t, \mathbb{C})$.

Besides, there is a holomorphic family of nowhere vanishing $n$-forms $(u_t)_t \in B$ on $X$, such that $u_t$ is a $J_t$-holomorphic $(n, 0)$-form. Then we know $H^{n,0}(X_t, \mathbb{C}) = \mathbb{C} u_t$. Hence the period map $\mathcal{P}$ is holomorphic.

If $\mathcal{P}$ is not a local immersion, then we can choose a point in $B$, say 0, and a tangent vector $\partial/\partial t \in T_0 B$, such that $(d\mathcal{P})_0(\partial/\partial t) = 0$. Because $X$ is $\partial\bar{\partial}$-manifold, we can choose a representative $\theta$ in $H^{0,1}(X, T^{1,0}X)$ such that $d(\theta, u_0) = 0$, where $\rho$ is the Kodaira-Spencer map.

By Ehresmann’s lemma, we have a smooth family of diffeomorphisms $\Phi_t^{-1} : X_0 \rightarrow X_t$. Taking a set of $J_t$-holomorphic coordinates $z_1(t), \ldots, z_n(t)$, we write $u_t = f_t(z_1(t)) \wedge \cdots \wedge (t).$ Therefore, we have

$$(6) \quad \frac{\partial(\Phi_t^{-1})^* u_t}{\partial t}_{t=0} = \theta^{\delta} u_0 + \frac{\partial f_t}{\partial t}_{t=0} dz_1(t) \wedge \cdots \wedge dz_n(t),$$

where $v := \frac{\partial f_t}{\partial t}_{t=0} dz_1(t) \wedge \cdots \wedge dz_n(t)$ is a $(n, 0)$-form and $\theta u_0$ is a $(n-1, 1)$-form. Then we know that $\theta = 0$. Hence the period map $\mathcal{P}$ is a local immersion. \hfill \Box
6. Metrics on $B$

In this section we compare two versions of Weil-Petersson metric and the metric induced by the period map given in the previous section.

We use the definition of $\omega$-minimal $d$-closed representative given in [Pop19]. The $\omega$-minimal $d$-closed representative of a Dolbeaut cohomology class $[\beta]$ is $\beta_{\min} = \beta + \partial \bar{\partial} v_{\min}$, where $\beta$ is the $\Delta''$-harmonic representative in $[\beta]$ and $v_{\min}$ is the solution of minimal $L^2$-norm of $\partial \beta = -\partial \bar{\partial} v$.

**Definition 14.** Let $(u_t)_{t \in B}$ be a fixed holomorphic family of non-vanishing holomorphic $n$-forms on the fibres $(X_t)_{t \in B}$ and let $(\omega_t)_{t \in B_{[\omega]}}$ be a smooth family of SKT metrics on the fibres $(X_t)_{t \in B_{[\omega]}}$ such that $\omega_t \in \{\omega\}$ for any $t$ and $\omega_0 = \omega$. The Weil-Petersson metrics $G^{(1)}_{WP}$ and $G^{(2)}_{WP}$ are defined on $B_{[\omega]}$ by

$$G^{(1)}_{WP}([\theta_t], [\eta_t]) := \frac{\langle\langle \theta_t, \eta_t \rangle\rangle_{\omega_t}}{\int_{X_t} dV_{\omega_t}}$$

$$G^{(2)}_{WP}([\theta_t], [\eta_t]) := \frac{\langle\langle \theta_t, \eta_t \rangle\rangle_{\omega_t}}{\int_{X_t} u_t \land \bar{u}_t}$$

for any $t \in B_{[\omega]}$, $[\theta_t], [\eta_t] \in H^{0,1}(X_t, T^{1,0} X_t)_{[\omega]}$. Here $\theta_t$ (resp. $\eta_t$) is chosen such that $\theta_{t, u_t}$ (resp. $\eta_{t, u_t}$) is the $\omega_t$-minimal $d$-closed representative of the class $[\theta_{t, u_t}] \in H^{n-1,1}(X_t, \mathbb{C})$ (resp. $[\eta_{t, u_t}] \in H^{n-1,1}(X_t, \mathbb{C})$).

**Remark 15.** Denote the $(1, 1)$-forms associated with $G^{(1)}_{WP}, G^{(2)}_{WP}$ by $\omega^{(1)}_{WP}, \omega^{(2)}_{WP}$. If $\text{Ric}(\omega_t) = 0$ for all $t \in B_{[\omega]}$, we have $\omega^{(1)}_{WP} = \omega^{(2)}_{WP}$.

Let $L = O_{\mathbb{P}H^n(X, \mathbb{C})}(-1)$ be the tautological line bundle on $\mathbb{P}H^n(X, \mathbb{C})$. We set:

$$C_+ := \{\{\alpha\} \in H^n(X, \mathbb{C})| H(\{\alpha\}, \{\alpha\}) > 0\};$$

$$C_- := \{\{\alpha\} \in H^n(X, \mathbb{C})| H(\{\alpha\}, \{\alpha\}) < 0\};$$

$$U^n_+ := \{[l] \in \mathbb{P}H^n(X, \mathbb{C})| l \text{ is a complex line such that } l \subset C_+\};$$

$$U^n_- := \{[l] \in \mathbb{P}H^n(X, \mathbb{C})| l \text{ is a complex line such that } l \subset C_-\}.$$

Then $\text{Im} \mathcal{P}$ is a subset of $U^n_+$ when $n$ is even, $\text{Im} \mathcal{P}$ is a subset of $U^n_-$ when $n$ is odd. We can get a Hermitian fibre metric $h^+_L$ on $L|_{U^n_+}$ from $H$. Then the associated Fubini-Study metric on $U^n_+$ is

$$\omega^+_{FS} = -i \Theta_{h^+_L}(L|_{U^n_+}).$$

Similarly, we get the associated Fubini-Study metric on $U^n_-$:

$$\omega^-_{FS} = -i \Theta_{h^-_L}(L|_{U^n_-}).$$

Then we have a Hermitian metric $\gamma$ on $B$:

$$\gamma := \begin{cases} \mathcal{P}^* \omega^+_{FS} & \text{if } n \text{ is even,} \\ \mathcal{P}^* \omega^-_{FS} & \text{if } n \text{ is odd.} \end{cases}$$
Lemma 16. The Kähler metric $\gamma$ defined on $B$ is independent of the choice of metrics on $(X_t)_{t \in B}$ and is explicitly given by the formula:

$$\gamma_t([\theta_t], [\theta_t]) = \frac{-\int_X (\theta_t, \omega_t) \wedge (\overline{\theta_t}, \omega_t)}{i^n \int_X \omega_t \wedge \overline{\omega_t}} = \frac{-H(\theta_t, \omega_t, \overline{\theta_t}, \omega_t)}{i^n \int_X \omega_t \wedge \overline{\omega_t}}, \text{ if } n \text{ is even},$$

$$\gamma_t([\theta_t], [\theta_t]) = \frac{-\int_X (\theta_t, \omega_t) \wedge (\overline{\theta_t}, \omega_t)}{i^n \int_X \omega_t \wedge \overline{\omega_t}} = \frac{H(\theta_t, \omega_t, \overline{\theta_t}, \omega_t)}{i^n \int_X \omega_t \wedge \overline{\omega_t}}, \text{ if } n \text{ is odd},$$

for every $t \in B$ and every $[\theta_t] \in H^{0,1}(X_t, T^{1,0}X_t)$.

Proof. Because $X$ is a Calabi-Yau manifold, there is a family of nowhere vanishing $J_t(n, 0)$-forms $(u_t)_{t \in B}$, such that $u_t$ is $J_t$-holomorphic.

When $n$ is even, we have $|u_t|^2_{\omega_{FS}} = H(u_t, u_t)$. Now we know that

$$\omega_{FS}^* = -\overline{\partial_t \partial_t} \log(H(u_t, u_t)).$$

Taking a class $[\theta]$ in $H^{0,1}(X, T^{1,0}X)$, assume $[\theta]$ is the image of $\frac{\partial}{\partial t}|_{t=0}$ under the Kodaira-Spencer map. Then

$$\gamma_0([\theta], [\theta]) = -\frac{\partial^2 \log(H(u_t, u_t))}{\partial t \partial t}|_{t=0} = -\frac{\partial}{\partial t} \left( \frac{H(u_t, \overline{u_t})}{H(u_t, u_t)} \right)|_{t=0}.$$

Because the left hand side of (6) is $d$-closed, $v$ is a $J_0$-holomorphic $(n, 0)$-form. Thus, $v$ is $C^0 u_0$ for some constant $C$. So we have

$$-\frac{\partial}{\partial t} \left( \frac{H(u_t, \overline{u_t})}{H(u_t, u_t)} \right)|_{t=0}$$

$$= -\frac{H(\overline{u_t}|_{t=0}, \overline{u_t}|_{t=0})H(u_0, u_0) - H(\overline{u_t}|_{t=0}, u_0)H(u_0, \overline{u_t}|_{t=0})}{H^2(u_0, u_0)}$$

$$= -\frac{H(\theta \wedge u_0, \theta \wedge u_0)H(u_0, u_0) + CCH^2(u_0, u_0) - C\overline{C}H^2(u_0, u_0)}{H^2(u_0, u_0)}$$

$$= -\frac{H(\theta \wedge u_0, \theta \wedge u_0)}{H(u_0, u_0)}.$$

The calculation of the case that $n$ is odd differs with only a $(-1)$ factor. □

Denote the space of global smooth forms of bidegree $(n - 1, 1)$ by $\Lambda^{n-1,1}$. Then we have two decompositions. The first one is Lefschetz decompositon:

$$\Lambda^{n-1,1} = \Lambda_{\text{prim}}^{n-1,1} \oplus (\omega \wedge \Lambda^{n-2,0}).$$

The second one is the decomposition into the eigenspaces of Hodge star oper-ator:

$$\Lambda^{n-1,1} = \Lambda^{n-1,1}_+ \oplus \Lambda^{n-1,1}_-.$$

Lemma 17. These two decompositions coincide up to order. Specifically, we have

- $\Lambda^{n-1,1}_{\text{prim}} = \Lambda^{n-1,1}_- \wedge \Lambda^{n-2,0} = \Lambda^{n-1,1}_+$ if $n$ is even,
- $\Lambda^{n-1,1}_{\text{prim}} = \Lambda^{n-1,1}_+ \wedge \Lambda^{n-2,0} = \Lambda^{n-1,1}_-$ if $n$ is odd.
Proof. For a primitive form $u$ of bidegree $(n-1, 1)$, we have $\ast u = (-1)^{n(n+1)/2} i^{n-2} u$. Therefore, we get $\Lambda_{prim}^{n-1,1} \subset \Lambda_{prim}^{n-1,1}$ if $n$ is even, and $\Lambda_{prim}^{n-1,1} \subset \Lambda_{prim}^{n-1,1}$ if $n$ is odd.

Now, it suffices to prove that for a form $v$ of bidegree $(n-2, 0)$, $\ast (\omega \wedge v) = \omega \wedge v$ when $n$ is even, and $\ast (\omega \wedge v) = -i \omega \wedge v$ when $n$ is odd. Firstly, for every form $u$ of bidegree $(n-1, 1)$ we have a decomposition $u = u_{prim} + \omega \wedge u_1$. Besides, $v$ is primitive because it is of bidegree $(n-2, 0)$. Hence, we get $\ast v = \frac{i^{n(n-2)}}{n(n-2)} v \wedge \omega^2$. Then

$$\int_X u \wedge \ast (\omega \wedge v) = \langle \langle u, \omega \wedge \bar{v} \rangle \rangle$$

$$= \langle \langle \omega \wedge u_1, \omega \wedge \bar{v} \rangle \rangle$$

$$= 2 \langle \langle u_1, \bar{v} \rangle \rangle$$

$$= 2 \int_X u_1 \wedge \ast v$$

$$= i^{n(n-2)} \int_X u_1 \wedge \omega^2 \wedge v$$

$$= i^{n(n-2)} \int_X u \wedge \omega \wedge v.$$ 

Therefore we have $\ast (\omega \wedge v) = i^{n(n-2)} \omega \wedge v$. □

By Lemma 17, for any $\theta \in C_{0,1}^{\infty}(X, T^{1,0}X)$, we have the decomposition:

$$\theta \wedge u = \theta' \wedge u + \omega \wedge \zeta.$$ 

By orthogonality, we have

$$G^{(2)}_{WP}([\theta_t], [\bar{\theta}_t]) = \frac{\langle \langle \theta_t \wedge u_t, \theta_t \wedge u_t \rangle \rangle}{i^{n^2} \int_X u_t \wedge \bar{u}_t} = \frac{\|\theta_t' \wedge u_t\|^2 + 2\|\zeta_t\|^2}{i^{n^2} \int_X u_t \wedge \bar{u}_t}.$$ 

If $n$ is even, by Lemma 17, we have $\ast (\theta' \wedge u) = -\theta' \wedge u$ and $\ast (\omega \wedge \zeta) = \omega \wedge \zeta$. As a consequence, we have

$$\int_X (\theta_t \wedge u_t) \wedge (\bar{\theta}_t \wedge \bar{u}_t) = \int_X (\theta_t' \wedge u_t + \omega_t \wedge \zeta_t) \wedge \ast (-\theta_t' \wedge u_t + \omega_t \wedge \zeta_t)$$

$$= -\|\theta_t' \wedge u_t\|^2 + 2\|\zeta_t\|^2.$$ 

Then we have

$$\gamma_t([\theta_t], [\bar{\theta}_t]) = \frac{\|\theta_t' \wedge u_t\|^2 - 2\|\zeta_t\|^2}{i^{n^2} \int_X u_t \wedge \bar{u}_t}.$$ 

Similarly, we get the same expression for $n$ odd.

Remark 18. For all $[\theta_t] \in H^{0,1}(X_t, T^{1,0}X_t)[\omega] \setminus \{0\}$, we have

$$(G^{(2)}_{WP} - \gamma)_t([\theta_t], [\bar{\theta}_t]) = \frac{4\|\zeta_t\|^2}{i^{n^2} \int_X u_t \wedge \bar{u}_t} \geq 0.$$ 

Hence if every class in $H_{prim}^{n-1,1}(X_t, \mathbb{C})$ has a $d$-closed and primitive representative, or equivalently, $H_{BC,prim}^{n-1,1}(X_t, \mathbb{C})$ has a primitive representative, we would get $G^{(2)}_{WP} = \gamma$. The polarised deformation of a Kähler manifold satisfies this condition in [Tia87].
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