THE MODULI SPACE OF 8 POINTS ON $\mathbb{P}^1$ AND AUTOMORPHIC FORMS

SHIGEYUKI KONDÔ

ABSTRACT. First we give a complex ball uniformization of the moduli space of 8 ordered points on the projective line by using the theory of periods of $K3$ surfaces. Next we give a projective model of this moduli space by using automorphic forms on a bounded symmetric domain of type $IV$ which coincides with the one given by cross ratios of 8 ordered points of the projective line.

Dedicated to Igor Dolgachev on his 60th birthday

1. INTRODUCTION

The main purpose of this paper is to give an application of the theory of automorphic forms on a bounded symmetric domain of type $IV$ due to Gritsenko and Borcherds [Bor] for studying the moduli spaces. We consider the moduli space $P^8_1$ of semi-stable 8 ordered points on the projective line. It is known that $P^8_1$ is isomorphic to the Satake-Baily-Borel compactification of an arithmetic quotient of 5-dimensional complex ball by using the theory of periods of a family of curves which are the 4-fold cyclic covers of the projective line branched at eight points ([DM]). Recently Matsumoto and Terasoma [MT] gave an embedding of $P^8_1$ into $\mathbb{P}^{104}$ by using the theta constants related to the above curves. Their map coincides the one defined by the cross ratios of 8 points on the $\mathbb{P}^1$. Here they used the fact that the complex ball is canonically embedded in a Siegel upper half plane.

In this paper, instead of the periods of curves, we use the periods of $K3$ surfaces. In our case, the complex ball is embedded in a bounded symmetric domain of type $IV$. In fact, to each stable point from $P^8_1$, we associate a $K3$ surface with a non-symplectic automorphism of order 4 ($\S2$). The period domain of these $K3$ surfaces is a 5-dimensional complex ball $B$ ($\S3$). This was essentially given in the paper [Kon1]. By using this, we shall see that $P^8_1$ is isomorphic to the Satake-Baily-Borel compactification $B/\Gamma(1-i)$ of $B/\Gamma(1-i)$ where $\Gamma(1-i)$ is an arithmetic subgroup of a unitary group of a hermitian form of signature $(1,5)$ defined over the Gaussian integers (Theorems 3.7, 4.11). The symmetry group $S_8$ of degree 8 naturally acts on $P^8_1$. On the other hand, there exists an arithmetic subgroup $\Gamma$ acting on $B$ with $\Gamma/\Gamma(1-i) \cong S_8$. The above isomorphism $P^8_1 \cong B/\Gamma(1-i)$ is $S_8$-equivariant.

Next we apply the theory of automorphic forms [Bor] to this situation. The main idea comes from the paper of Allcock and Freitag [AF] in which they studied the same problem in the case of cubic surfaces. We shall show that there exists a 14-dimensional space of automorphic forms on $B$ which gives an $S_8$-equivariant map from the arithmetic quotient $B/\Gamma(1-i)$ into $\mathbb{P}^{13}$ (Theorem 7.6). Under the identification $P^8_1 \cong B/\Gamma(1-i)$ we show that this map coincides with the one defined by the cross ratios of 8 points on the projective line (Theorem 7.5). Thus our map coincides the one given by Matsumoto and Terasoma [MT].

Research of the author is partially supported by Grant-in-Aid for Scientific Research A-14204001, Japan.
Let $P$ be the projective line. Let $\{\lambda_i : 1\}$ be a set of distinct 8 points on the projective line. Let $(x_0 : x_1, y_0 : y_1)$ be the bi-homogenous coordinates on $P^1 \times P^1$. Consider a smooth divisor $C$ in $P^1 \times P^1$ of bidegree $(4, 2)$ given by

\[(2.1) \quad y_0^2 \prod_{i=1}^{4}(x_0 - \lambda_i x_1) + y_1^2 \prod_{i=5}^{8}(x_0 - \lambda_i x_1) = 0.\]

Let $L_0$ (resp. $L_1$) be the divisor defined by $y_0 = 0$ (resp. $y_1 = 0$). Let $\iota$ be an involution of $P^1 \times P^1$ given by

\[(2.2) \quad (x_0 : x_1, y_0 : y_1) \mapsto (x_0 : x_1, y_0 : -y_1)\]

which preserves $C$ and $L_0, L_1$. Note that the double cover of $P^1 \times P^1$ branched along $C + L_0 + L_1$ has 8 rational double points of type $A_1$ and its minimal resolution $X$ is a $K3$ surface. This $K3$ surface $X$ is obtained as follows: First blow up the 8 points which are the intersection of $C$ and $L_0 + L_1$. Then $X$ is the double cover branched along the proper transforms of $C$, $L_0$ and $L_1$. We remark that the isomorphism class of $X$ depends only on the 8 points in $P^1$ (i.e. independent on the order of 8 points) because elementary transformations change the order of 8 points.

The involution $\iota$ lifts to an automorphism $\sigma$ of order 4. We can easily see that $\sigma^* \omega_X = \pm \sqrt{-1} \omega_X$ where $\omega_X$ is a nowhere vanishing holomorphic 2-form on $X$. We denote by $S_0, S_1$ the inverse image of $L_0, L_1$ respectively. The projection

\[(x_0 : x_1, y_0 : y_1) \mapsto (x_0 : x_1)\]

from $P^1 \times P^1$ to $P^1$ induces an elliptic fibration

$\pi : X \rightarrow P^1$

which has 8 singular fibers of type $III$ in the sense of Kodaira [Kod] and two sections $S_0, S_1$. Let $E_i + F_i$ ($1 \leq i \leq 8$) be the 8 singular fibers of $\pi$. Then we may assume that

$E_i \cdot S_0 = F_i \cdot S_1 = 1$.

Put

\[(2.3) \quad H^2(X, \mathbb{Z})^+ = \{ x \in H^2(X, \mathbb{Z}) \mid (\sigma^2)^*(x) = x\};\]

\[(2.4) \quad H^2(X, \mathbb{Z})^- = \{ x \in H^2(X, \mathbb{Z}) \mid (\sigma^2)^*(x) = -x\}.\]

We also denote by $S_X, T_X$ the Picard lattice, the transcendental lattice of $X$ respectively.
2.2. Lemma. (i) $H^2(X, \mathbb{Z})^+ \simeq U(2) \oplus D_4 \oplus D_4$, $H^2(X, \mathbb{Z})^- \simeq U \oplus U(2) \oplus D_4 \oplus D_4$.
(ii) $\sigma^*$ acts on $H^2(X, \mathbb{Z})^+$ trivially.
(iii) The following elements generate $(H^2(X, \mathbb{Z})^+)^*/H^2(X, \mathbb{Z})^+ \simeq (\mathbb{Z}/2\mathbb{Z})^6$:

\[
(F_1 + F_2)/2, (F_1 + F_3)/2, (F_1 + F_4)/2, (F_1 + F_5)/2, (F_1 + F_6)/2, (F_1 + F_7)/2.
\]

(iv) Let $U \oplus A_1^{\oplus 8}$ be the sublattice of $H^2(X, \mathbb{Z})^+$ generated by the classes of a fiber, $S_0$ and $F_i$ ($1 \leq i \leq 8$). Then $H^2(X, \mathbb{Z})^+$ is obtained from $U \oplus A_1^{\oplus 8}$ by adding the vector $(F_1 + \cdots + F_8)/2$.

Proof. For the proof of the assertions (i)–(iii), see [Kon1], Lemma 5.2. The sublattice $U \oplus A_1^{\oplus 8}$ has index 2 in $H^2(X, \mathbb{Z})^+$. The later one is obtained from the former by adding the class of $S_1$. Hence the last assertion follows from the fact that $S_1 = 2F + S_0 + (F_1 + \cdots + F_8)/2$ where $F$ is a fiber. □

It follows that $H^2(X, \mathbb{Z})^+ \subset S_X$ and $T_X \subset H^2(X, \mathbb{Z})^-$.

2.3. A quadratic form. First of all, we define:

\[
L = U^3 \oplus E_8^3, \quad M = U(2) \oplus D_4 \oplus D_4, \quad N = U \oplus U(2) \oplus D_4 \oplus D_4.
\]

Recall that $H^2(X, \mathbb{Z}) \cong L$. We consider $M$ as a sublattice of $L$ and $N$ is the orthogonal complement of $M$ in $L$. It follows from Theorem 1.14.4 in Nikulin [N3] that the embedding of $M$ into $L$ is unique. Let $A_N = N^*/N$ which is isomorphic to a vector space $\mathbb{F}_2^6$ of dimension 6 over $\mathbb{F}_2$ (Lemma 2.2). The discriminant quadratic form

\[
q_N : A_N \longrightarrow \mathbb{Q}/2\mathbb{Z}
\]

is defined by $q_N(x) = \langle x, x \rangle \mod 2\mathbb{Z}$. In our situation, the image of $q_N$ is contained in $\mathbb{Z}/2\mathbb{Z}$ and hence $q_N$ is a quadratic form on $A_N$ defined over $\mathbb{F}_2$, whose associated bilinear form is given by

\[
b_N(x, y) = 2\langle x, y \rangle \mod 2\mathbb{Z}.
\]

Let $u$ be the hyperbolic plane defined over $\mathbb{F}_2$, that is, the quadratic form of dimension 2 defined over $\mathbb{F}_2$ corresponding to the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The quadratic form $q_N$ is isomorphic to the direct sum of 3 copies of $u$: $q_N \cong u^{\oplus 3}$. It is known that $\langle A_M = M^*/M, q_M \rangle \cong (A_N, -q_N)$ ([N3], Corollary 1.6.2).

2.4. Lemma. (i) $O(q_M) \cong O(q_N) \cong O(u^{\oplus 3}) \cong S_8$ where $S_8$ is the symmetric group of degree 8.
(ii) The group $O(q_M)$ naturally isomorphic to the subgroup of $O(M)$ generated by the permutations of the 8 components $A_1$ in $U \oplus A_1^{\oplus 8}$.

Proof. The assertion (i) is well known (e.g. [Atlas], page 22). For (ii), note that the permutations of 8 components $A_1$ can be extended to isometries of $M$ because they preserve $(F_1 + \cdots + F_8)/2$ (see Lemma 2.2 (iv)). Now the assertion is obvious. □

2.5. A fundamental domain. Let $M = U(2) \oplus D_4 \oplus D_4$. Let $P(M)$ be a connected component of \{ $x \in M \otimes \mathbb{R} : \langle x, x \rangle > 0$ \}. Let $W(M)$ be the reflection group generated by $(-2)$-reflections

\[
s_r : x \mapsto x + \langle x, r \rangle r
\]

for any $r \in M$ with $r^2 = -2$. The group $W(M)$ acts on $P(M)$ discretely. Let $C(M)$ be the finite polyhedral cone defined by the 18 $(-2)$-vectors which are corresponding to the 18 smooth rational curves $S_0, S_1, E_i, F_i$ ($1 \leq i \leq 8$) on $X$ under an isomorphism $M \cong H^2(X, \mathbb{Z})^+$. 

THE MODULI SPACE OF 8 POINTS ON $\mathbb{P}^1$
2.6. **Proposition.** (i) The group $W(M)$ is of finite index in the orthogonal group $O(M)$. Moreover the closure $\overline{C(M)}$ of $C(M)$ is a fundamental domain of $W(M)$. The symmetry group of $C(M)$ is isomorphic to $S_8 \times \mathbb{Z}/2\mathbb{Z}$.

(ii) If $S_X \cong M$, then $X$ contains exactly 18 smooth rational curves $S_0, S_1, E_i, F_i$ ($1 \leq i \leq 8$).

**Proof.** (i) The first assertion follows from Nikulin’s classification of such hyperbolic 2-elementary lattices ([N]), Theorem 4.4.1. Moreover $C(M)$ satisfies the condition in Vinberg’s theorem [V]. Theorem 2.6, that is, any maximal extended Dynkin diagram in these $18$ $(-2)$-vectors is either $A_1^{\oplus 8}$ or $\tilde{D}_4^{\oplus 2}$ both of which have the maximal rank 8. Hence $C(M)$ is of finite volume. Let $C(M)'$ be a fundamental domain of $W(M)$ with $C(M)' \subset C(M)$. Then [V], Lemma 2.4 implies that $C(M) = C(M)'$. The last assertion is obvious.

(ii) It follows from a remark in Vinberg [V], p. 335 that $C(M)$ has finite volume iff the polyhedral cone $\overline{C(M)}$ is contained in the closure $\overline{P(M)}$ of $P(M)$. If there exists a smooth rational curve $E$ different from the above 18 curves. Then the intersection number of $E$ and any one of these 18 curves is non negative, that is, the class of $E$ is contained in $\overline{C(M)} \subset \overline{P(M)}$. This implies that $E^2 \geq 0$, which is a contradiction.

2.7. Let $X$ be a $K3$ surface as above. Let $P(X)$ be the component of $\{ x \in S_X \otimes \mathbb{R} : \langle x, x \rangle > 0 \}$ which contains an ample class. Let $\Delta(X)$ be the set of all effective divisors $r$ with $r^2 = -2$. Let $C(X)$ be the polyhedral cone defined by:

$$C(X) = \{ x \in S_X \otimes \mathbb{R} : \langle x, r \rangle > 0, r \in \Delta(X) \}.$$

Note that the integral points in $C(X)$ are nothing but the ample classes. If $S_X \cong M$, then $C(X) = C(M)$.

2.8. **Lemma.** The orthogonal complement of $H^2(X, \mathbb{Z})^+$ in $S_X$ contains no $(-2)$-vectors.

**Proof.** If $r \in (H^2(X, \mathbb{Z})^+) \cap S_X$ with $r^2 = -2$, then $(\sigma^*)^2(r) = -r$. On the other hand, Riemann-Roch theorem implies that $r$ is effective. This is a contradiction.

2.9. **Proposition.** Assume that $S_X = H^2(X, \mathbb{Z})^+$. Then the automorphism group of $X$ is finite. Moreover $X$ has exactly 18 smooth rational curves which are components of singular fibers of $\pi$ and two sections.

**Proof.** Recall that $\text{Aut}(X)$ is isomorphic to $O(S_X)/W(S_X)$ up to finite groups ([PS]). Here $W(S_X)$ is the subgroup generated by $(-2)$-reflections. Hence the assertion follows from Proposition 2.6.

2.10. **The automorphism of order 4.** We shall study the action of $\sigma$ on $H^2(X, \mathbb{Z})^-$. Recall that $D_4 \cong \{ (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid x_1 + x_2 + x_3 + x_4 \equiv 0 \pmod{2} \}$.

Here we consider the standard inner product on $\mathbb{Z}^4$ with the negative sign. Let $\rho_0$ be the isometry of $D_4$ given by

$$\rho_0(x_1, x_2, x_3, x_4) = (x_2, -x_1, x_4, -x_3).$$

Obviously $\rho_0$ is of order 4 and fixes no non-zero vectors in $D_4$. Also an easy calculation shows that $\rho_0$ acts trivially on $D_4^*/D_4$. Next let $e, f$ (resp. $e', f'$) be a basis of $U$ (resp. $\tilde{U}(2)$). Define the isometry $\rho_1$ of $U \oplus \tilde{U}(2)$ by

$$\rho_1(e) = -e - e', \quad \rho_1(f) = f - f', \quad \rho_1(e') = e' + 2e, \quad \rho_1(f') = 2f - f'.$$
Obviously \( \rho_1 \) is of order 4, fixes no non-zero vectors in \( U \oplus U(2) \) and acts trivially on the discriminant group of \( U \oplus U(2) \). Thus we have an isometry \( \rho = \rho_1 \oplus \rho_0 \oplus \rho_0 \) of \( N = U \oplus U(2) \oplus D_4 \oplus D_4 \) which fixes no non-zero vectors in \( N \) and acts trivially on \( N^*/N \). Then \( \rho \) can be extended to an isometry of the \( K3 \) lattice \( L \) acting trivially on \( M \) ([N3], Proposition 1.6.1).

2.11. **Lemma.** The isometry \( \rho \) is conjugate to \( \sigma^* \) under an isomorphism \( H^2(X, \mathbb{Z}) \cong L \).

**Proof.** Let \( \omega \) be an eigenvector of \( \rho \) which is sufficiently general, that is, satisfying the condition \( \omega \perp \cap L = M \). By the surjectivity of the period map for \( K3 \) surfaces, there exists a \( K3 \) surface \( Y \) and an isometry

\[ \alpha_Y : H^2(Y, \mathbb{Z}) \rightarrow L \]

with \( \alpha_Y(\omega_Y) = \omega \) where \( \omega_Y \) is a nowhere vanishing holomorphic 2-form on \( Y \). By the condition \( \omega \perp \cap L = M \), we have \( S_Y \cong M \). Consider the isometry \( \phi = \alpha_Y^{-1} \circ \rho \circ \alpha_Y \) of \( H^2(Y, \mathbb{Z}) \). Since \( \rho \) acts trivially on \( M, \phi \) preserves ample classes. Then it follows from the Torelli theorem [PS] that \( \phi \) is induced from an automorphism \( g \) of \( Y \) of order 4. On the other hand, Proposition 2.9 implies that \( Y \) contains exactly 18 smooth rational curves whose dual graph is the same as that of the smooth rational curves on \( X \). In particular, \( Y \) has an elliptic fibration with two sections and 8 singular fibers each of which is type \( III \) or \( I_2 \). Since \( g \) acts trivially on the Picard lattice \( M, g \) preserves the elliptic fibration and the class of each component of singular fibers of type \( III \) or \( I_2 \). Since the elliptic fibration has 8 singular fibers, \( g \) acts trivially on the base of the elliptic fibration, and hence induces an automorphism of each fiber. Hence all singular fibers are of type \( III \). It follows from Nikulin [NT], Theorem 4.2.2 that the set of fixed points of the involution \( g^2 \) is the disjoint union of two smooth rational curves \( R_0, R_1 \) and a smooth curve \( C \) of genus 3. Since \( g \) acts trivially on the base, \( R_0, R_1 \) are sections of the elliptic fibration. We can easily see that \( C \) passes through singular points of singular fibers of type \( III \). Thus we have the same configuration of smooth rational curves on \( Y \) as that of \( X \). By taking the quotient of \( Y \) by \( g^2 \), we can see that \( Y \) is a deformation of \( X \). Hence we have the assertion.

2.12. **Markings.** Recall that \( H^2(X, \mathbb{Z})^+ \cong M = U(2) \oplus D_4 \oplus D_4 \) (Lemma 2.2). We fix a fundamental domain \( C(M) \) (Proposition 2.6). It follows from Lemma 2.11 that there exists an isometry

\[ \alpha_X : H^2(X, \mathbb{Z}) \rightarrow L \]

satisfying \( \rho = \alpha_X \circ \sigma^* \circ \alpha_X^{-1} \). We call \( \alpha_X \) a marking and the pair \( (X, \alpha_X) \) a marked \( K3 \) surface. Then

2.13. **Proposition.** There exists a marking \( \alpha_X \) such that \( \alpha_X(C(X)) \cap M \otimes \mathbb{R} \subset C(M) \).

**Proof.** It follows from Lemma 2.8 that \( \alpha_X(C(X)) \cap M \otimes \mathbb{R} \) is an open polyhedral cone in \( M \otimes \mathbb{R} \). Hence Proposition 2.6 implies the assertion.

3. A COMPLEX BALL UNIFORMIZATION

In this section we construct an \( S_8 \)-equivariant isomorphism between the moduli space of the projective equivalence classes of the set of distinct 8 ordered points in \( \mathbb{P}^1 \) and an open set of the arithmetic quotient of 5-dimensional complex ball.
3.1. **The period domain.** Let \((X, \alpha_X)\) be a marked K3 surface and let \(\omega_X\) be a nowhere vanishing holomorphic 2-form on \(X\). Then \(\alpha_X(\omega_X)\) is contained in the following domain:

\[
D = \{\omega \in P(N \otimes \mathbb{C}) : \langle \omega, \omega \rangle = 0, \langle \omega, \bar{\omega} \rangle > 0\}.
\]

Note that \(D\) is a disjoint union of two copies of a bounded symmetric domain of type IV and of dimension 10. To get the period domain, we first define:

\[
V_{\pm} = \{z \in N \otimes \mathbb{C} \mid \rho(z) = \pm \sqrt{-1}z\}.
\]

It follows from Nikulin [N2], Theorem 3.1 that \(N \otimes \mathbb{C} = V_+ \oplus V_-\). Now we may assume \(\sigma^*(\omega_X) = \sqrt{-1} \cdot \omega_X\). Then \(\alpha_X(\omega_X)\) is, in fact, contained in \(B\) defined by

\[
B = \{z \in P(V_+) \mid \langle z, \bar{z} \rangle > 0\}.
\]

If \(z \in B\), then

\[
\langle z, z \rangle = \langle \rho(z), \rho(z) \rangle = \langle \sqrt{-1}z, \sqrt{-1}z \rangle = -\langle z, z \rangle,
\]

and hence \(\langle z, z \rangle = 0\). Thus we have

\[
D \cap P(V_+) = B.
\]

We remark that \(B\) is a 5-dimensional complex ball. We call \(\alpha_X(\omega_X)\) the period of \((X, \alpha_X)\). We also define two arithmetic subgroups:

\[
\Gamma = \{\gamma \in O(N) \mid \gamma \circ \rho = \rho \circ \gamma\};
\]

\[
\Gamma(1 - i) = \text{Ker}(\Gamma \to O(q_N)).
\]

We shall see that the quotient \(B/\Gamma\) (resp. \(B/\Gamma(1 - i)\)) is the coarse moduli space of distinct 8 unordered points on \(\mathbb{P}^1\) (resp. distinct 8 ordered points on \(\mathbb{P}^1\)) (see Theorem 3.7).

3.2. **Hermitian form.** We consider \(N\) as a free \(\mathbb{Z}[\sqrt{-1}]\)-module \(\Lambda\) by

\[
(a + b\sqrt{-1})x = ax + bp(x).
\]

Let

\[
h(x, y) = \sqrt{-1}\langle x, \rho(y) \rangle + \langle x, y \rangle.
\]

Then \(h(x, y)\) is a hermitian form on \(\mathbb{Z}[\sqrt{-1}]\)-module \(\Lambda\). With respect to a \(\mathbb{Z}[\sqrt{-1}]\)-basis \((1, -1, 0, 0), (0, 1, -1, 0)\) of \(D_4\), the hermitian matrix of \(h \mid D_4\) is given by

\[
\begin{pmatrix}
-2 & 1 - \sqrt{-1} \\
1 + \sqrt{-1} & -2
\end{pmatrix}.
\]

And with respect to a \(\mathbb{Z}[\sqrt{-1}]\)-basis \(e, e'\) of \(U \oplus U(2)\), the hermitian matrix of \(h \mid U \oplus U(2)\) is given by

\[
\begin{pmatrix}
0 & 1 + \sqrt{-1} \\
1 - \sqrt{-1} & 0
\end{pmatrix}.
\]
Let \( \varphi : \Lambda \to N^* \) be a linear map defined by \( \varphi(x) = (x + \rho(x))/2 \). Note that \( \varphi((1 - \sqrt{-1})x) = \varphi(x - \rho(x)) = x \in N \). Hence \( \varphi \) induces an isomorphism

\[
\Lambda/(1 - \sqrt{-1})\Lambda \cong N^*/N.
\]

3.3. Remark. The hermitian form \( h \) coincides with the one of Matsumoto and Yoshida in [MY], §6. This implies that our groups \( \Gamma, \Gamma(1 - i) \) coincide with the ones of Matsumoto and Yoshida in [MY].

3.4. Reflections. For \( r \in N \) with \( \langle r, r \rangle = -2 \), we define a reflection

\[
s_r(x) = x + \langle r, x \rangle r
\]

which is contained in \( \tilde{O}(N) = \text{Ker}(N \to O(q_N)) \), but not in \( \Gamma \). On the other hand, by considering \( r \) as in \( \Lambda \), we define a reflection

\[
R_{r, \epsilon}(x) = x - (1 - \epsilon)\frac{h(r, x)}{h(r, r)} r
\]

where \( \epsilon \neq 1 \) is a 4-th root of unity. We can easily see that \( R_{r, -1} \) corresponds to the isometry in \( \Gamma \)

\[
x \to x + \langle r, x \rangle r + \langle \rho(r), x \rangle \rho(r)
\]

which coincides with \( s_r \circ s_{\rho(r)} \). Also \( R_{r, \sqrt{-1}} \) corresponds to the isometry in \( \Gamma \)

\[
x \to x + \langle r, x \rangle (r - \rho(r))/2 + \langle \rho(r), x \rangle (r + \rho(r))/2
\]

which induces a transvection of \( A_N \) defined by

\[
t_{\alpha}(x) = x + b_N(x, \alpha) \alpha
\]

where \( \alpha \in A_N \) is a non-isotropic vector \( (r + \rho(r))/2 \mod N \).

3.5. Discriminant. Let \( r \in N \) with \( r^2 = -2 \). We denote by \( H_r \) the hyperplane of \( B \) defined by

\[
H_r = \{ z \in B : \langle z, r \rangle = 0 \}.
\]

Let \( \mathcal{H} \) be the union of all hyperplanes \( H_r \) where \( r \) moves on the set of all \((-2)\)-vectors in \( N \). We call \( \mathcal{H} \) the discriminant locus. By Lemma 2.8, the periods of marked \( K3 \) surfaces as above are contained in \( B \setminus \mathcal{H} \).

Conversely let \( \omega \in B \setminus \mathcal{H} \). Then by the surjectivity of the period map, there exists a marked \( K3 \) surface \((X, \alpha_X)\) with \( \alpha_X(\omega_X) = \omega \). The condition \( \omega \notin \mathcal{H} \) implies that Proposition 2.13 holds for this \( K3 \) surface. Hence, if necessary by replacing \( \alpha_X \), we may assume that the isometry \( \alpha_X^{-1} \circ \rho \circ \alpha_X \) preserves the ample cone of \( X \). It now follows from the Torelli type theorem ([PS]) that there exists an automorphism \( \sigma \) of order 4 satisfying \( \alpha_X^{-1} \circ \rho \circ \alpha_X = \sigma^* \). Moreover the marking defines an elliptic fibration

\[
\pi : X \to \mathbb{P}^1
\]

with a section \( s \).
3.6. Lemma. (i) \( \pi \) has 8 singular fibers of type III and two sections;
(ii) The set of fixed points of \( \sigma^2 \) is the disjoint union of two sections and a smooth curve of genus 3 which passes through 8 singular points of 8 singular fibers.

Proof. It is known that the set of fixed points of the involution \( \sigma^2 \) is the disjoint union of two smooth rational curves \( R_0, R_1 \) and a smooth curve \( C \) of genus 3 (Nikulin [N1], Theorem 4.2.2). Obviously the set \( X^\sigma \) of fixed points of \( \sigma \) is contained in \( R_0 + R_1 + C \). Since \( \sigma^* = \alpha_X^{-1} \circ \rho \circ \alpha_X \), \( X^\sigma \) has the Euler number 12. Since \( \sigma \) acts on \( M \) trivially, it preserves the section \( s \) and the class of a fiber of \( \pi \). We show that \( \sigma \) acts trivially on the base of \( \pi \). Assume otherwise, then \( X^\sigma \) is contained in two invariant fibers, \( F_1, F_2 \). Let \( l \) be the number of irreducible one-dimensional components of \( X^\sigma \) and \( k \) the number of isolated fixed points of \( \sigma \). Then \( 2l + k = 12 \). If we denote by \( U \) the sublattice generated by the classes of a fiber and the section \( s \), then \( U^\perp \cap H^2(X, \mathbb{Z})^{\sigma^*} = A^8_{\mathbb{Z}} \). Hence the divisor \( F_1 + F_2 \) contains at least 10 components. Assume \( F_1 \) contains at least 5 components. Note that \( \sigma \) preserves the component of \( F_1 \) which meets with \( s \). Obviously there are no singular fibers with non-trivial symmetry of order 4. Hence the involution \( \sigma^2 \) preserves each component of \( F_1 \). Then the sublattice generated by components of \( F_1 \) not meeting \( s \) has at least rank 4 and is isomorphic to an indecomposable root lattice \( \hat{R} \). Since \( \sigma^2 \) acts trivially on \( R, \hat{R} \) is contained in \( A^8_{\mathbb{Z}} \) which is impossible.

Thus \( \sigma \) acts trivially on the base. This implies that each fiber has an automorphism of order 4. In particular, singular fibers of \( \pi \) are either of type III, III* or \( I^*_8 \). Since \( U^\perp \cap H^2(X, \mathbb{Z})^{\sigma^*} = A^8_{\mathbb{Z}} \) and the singular fibers of type III* and \( I^*_8 \) have no non-trivial symmetry of order 4, every singular fiber is of type III. Since \( \sigma \) fixes two points on each component of a singular fiber one of which is the singular point, \( R_0, R_1 \) or \( C \) passes through these points. Now we can easily see the assertion (ii). \( \square \)

3.7. Theorem. The period map induces an \( S_8 \)-equivariant isomorphism \( \phi \) between the moduli space \((\mathcal{P}^8)^0\) of distinct ordered 8 points on the projective line and the quotient space \((\mathcal{B} \setminus \mathcal{H})/\Gamma(1 - i)\).

Proof. As in [3.5] for each \( \omega \in \mathcal{B} \setminus \mathcal{H} \), we have a marked K3 surface \((X, \alpha_X)\) with \( \alpha_X(\omega_X) = \omega \). Moreover \( X \) has an automorphism \( \sigma \) of order 4. By Lemma 3.6, \( X \) has an elliptic fibration with two section and 8 singular fibers of type III. By taking the quotient of \( X \) by \( \sigma^2 \) and contracting \((-1)\)-curves, we have the embedding of \( C \) as in [2.1]. This correspondence is the inverse of the period map. \( \square \)

4. Discriminant Locus

In this section we shall determine the discriminant locus \( \mathcal{H} \) of \( \mathcal{B} \).

4.1. Let \( r \) be a \((-2)\)-vector in \( N \). Let \( \omega \in \mathcal{B} \). Then \( \langle r, \omega \rangle = \sqrt{-1} \langle \rho(r), \omega \rangle \). Hence \( r \) and \( \rho(r) \) define the same hyperplane \( H_r \) in \( \mathcal{B} \). Hence \( H_r \) corresponds to an embedding of the lattice \( R_r \cong A_1 \oplus A_1 \) generated by \( r \) and \( \rho(r) \) into \( N \). Obviously \( R_r \cong A_1 \oplus A_1 \). Also every embedding of \( R_r \) into \( N \) is primitive, that is, \( N/R_r \) is torsion free.

4.2. Lemma. Let \( R_r^\perp \) be the orthogonal complement of \( R_r \) in \( N \). Then \( R_r^\perp \cong U \oplus U(2) \oplus D_4 \oplus A^2_1 \). In particular, \( (r + \rho(r))/2 \in N^* \).

Proof. The proof for the first assertion is similar to those of [Kon1], Lemmas 3.2, 3.3. Then \( R_r^\perp \oplus R_r \) is a sublattice of \( N \) of index 2 and \( N \) is obtained from \( R_r^\perp \oplus R_r \) by adding \( (r + \rho(r))/2 + \theta \) where \( \theta \in (R_r^\perp)^* \). We can see that \( \langle (r + \rho(r))/2, x \rangle \in \mathbb{Z} \) for \( x \in R_r \oplus R_r^\perp \) and \( x = (r + \rho(r))/2 + \theta \). Hence the second assertion holds. \( \square \)
4.3. Note that $A_N$ consists of the following 64 vectors:

Type $(00) : \alpha = 0, \#\alpha = 1$ (zero);
Type $(0) : \alpha \neq 0, q(\alpha) = 0, \#\alpha = 35$ (non-zero isotropic vector);
Type $(1) : q(\alpha) = 1, \#\alpha = 28$ (non-isotropic vector).

4.4. By Lemma 4.2, the vector $(r + \rho(r))/2$ is contained in $N^*$. In particular it defines a non-isotropic vector $(r + \rho(r))/2 \mod N$ in $A_N$. Conversely let $\delta$ be a $(-4)$-vector in $N$ with $\delta/2 \in N^*$. Since $\rho$ acts trivially on $A_N = N^*/N$, $\delta - \rho(\delta) \in 2N$. Put $r = (\delta - \rho(\delta))/2 \in N$. Since $\langle \delta, \rho(\delta) \rangle = 0$, $r^2 = -2$. Obviously $\delta = r + \rho(r)$. Thus we have

4.5. Lemma. \{\delta \in N : \delta^2 = -4, \delta/2 \in N^*\} = \{r + \rho(r) : r \in N, r^2 = -2\}.

4.6. Lemma. Any non-isotropic vector in $A_N$ is represented by $(r + \rho(r))/2$ for a suitable $(-2)$-vector $r$ in $N$.

**Proof.** It follows from [N3], Theorem 1.14.2 that the natural map from $O(N)$ to $O(A_N)$ is surjective. The group $O(A_N) \cong S_8$ acts transitively on the set of non-isotropic vectors in $A_N$. Combining these with Lemma 4.5 we have the assertion. □

4.7. Proposition. (i) $\Gamma/\Gamma(1 - i) \cong S_8$.

(ii) $\Gamma$ acts transitively on the set of cusps of $B$ and on the set of $\rho$-invariant $R = A_1 \oplus A_1$.

(iii) $\Gamma(1 - i)$-orbits of $\rho$-invariant $R = A_1 \oplus A_1$ bijectively correspond to non-isotropic vectors in $A_N$. Also $\Gamma(1 - i)$-orbits of cusps of $B$ bijectively correspond to non-zero isotropic vectors in $A_N$.

**Proof.** Recall that the pair $(N, \rho)$ naturally corresponds to the hermitian form $h$ (see Remark 3.3). Hence the assertions follow from [MY]. □

4.8. Stable points. Next we shall construct a $K3$ surface associated to each stable point from $P^8_1$. Recall that 8 points is stable (resp. semi-stable) iff no four points (resp. five points) coincide (e.g. [DO], Chap. I, §4, Example 2 (page 31)). We denote by the symbol $(11111111)$ for distinct 8 points in $P^1$. If two points (resp. three points) coincide, then we denote it by $(21111111)$ (resp. $(3111111)$).

4.9. Example: $(21111111)$. We use the same notation as in Section 2. We assume that $(\lambda_1 : 1)$ has multiplicity 2. Then the curve $C$ in (2.1) degenerates to one of the following two types:

\begin{align*}
C_1 : & \ (x_0 - \lambda_1 x_1)(y_0^2 \prod_{i=2}^{4}(x_0 - \lambda_i x_1)) + y_1^2 \prod_{i=5}^{7}(x_0 - \lambda_i x_1)) = 0; \\
C_2 : & \ y_0^2(x_0 - \lambda_1 x_1)^2 \prod_{i=2}^{3}(x_0 - \lambda_i x_1) + y_1^2 \prod_{i=4}^{7}(x_0 - \lambda_i x_1) = 0.
\end{align*}

The minimal resolution $Y_i$ of the double covering of $P^1 \times P^1$ branched along $C_1 + L_0 + L_1$ is a $K3$ surface and has an elliptic fibration $\pi$ which has 6 singular fibers of type $III$, one singular fiber of type $I_0^*$ and two sections $S_0, S_1$ ($i = 1, 2$). We remark that $Y_1$ and $Y_2$ are isomorphic because $C_1$ and $C_2$ are mutually transformed under elementary transformations. Thus we denote by $Y$ instead of $Y_1, Y_2$. Denote by $2R_0 + R_1 + R_2 + R_3 + R_4$ the singular fiber of type $I_0^*$. Assume that $S_0$ meets $R_1$ and $S_1$ meets $R_2$. Then the normalization $\tilde{C}$ of $C_1$ meets $R_3, R_4$. The involution $\iota$ given in (2.2) induces an automorphism $\sigma'$ of $Y$ of order 4. Note that the restriction $\sigma'$ on $\tilde{C}$ is the hyperelliptic involution of the smooth curve of genus 2. This implies that $\sigma'$ switches $R_3$ and $R_4$. Let $U$ be the
isometry from the sublattice generated by the classes of a fiber and $S_0$. Then 6 components in the fibers of type $III$ not meeting to $S_0$ and $R_2$, $2R_0 + R_2 + R_3 + R_4$ generate the sublattice isomorphic to $A_1^\delta$. This gives an isometry from $M$ into $S_X$ such that $M^{\perp} \cap S_X$ contains $(-2)$ vectors $R_3, R_4$. Thus the period of $Y$ is contained in $\mathcal{H}$. 

For other stable points, the process is similar. We have several types of the branch curve $C$ depending on the order of 8 points, however, they are transformed each other under elementary transformations. Hence the corresponding $K3$ surface is determined by the isomorphism class of 8 points (independent of the order of 8 points).

If three points coincide, then the corresponding elliptic fibration has a singular fiber of type $III^*$. All cases except $(2222)$, the elliptic fibration has two sections. In case of $(2222)$, it has four sections.

The next Table 1 lists the type of 8 stable points on the projective line, type of singular fibers of the elliptic fibration, the Picard lattice and the transcendental lattice of a generic member.

| 8 points | Singular fibers | Picard lattice | Transcendental lattice |
|----------|----------------|----------------|------------------------|
| 1) (11111111) | $8III$ | $U(2) \oplus D_4 \oplus D_4$ | $U \oplus U(2) \oplus D_4 \oplus D_4$ |
| 2) (2111111) | $I_0^*, 6III$ | $U \oplus D_4 \oplus D_4 \oplus A_1^{\delta 2}$ | $U \oplus U(2) \oplus D_4 \oplus A_1^{\delta 2}$ |
| 3) (221111) | $2I_0^*, 4III$ | $U \oplus D_6 \oplus D_4 \oplus A_1^{\delta 2}$ | $U \oplus U(2) \oplus A_1^{\delta 4}$ |
| 4) (22211) | $3I_0^*, 2III$ | $U \oplus D_6 \oplus D_4 \oplus A_1^{\delta 2}$ | $A_1(-1)^{\delta 2} \oplus A_1^{\delta 4}$ |
| 5) (2222) | $4I_0^*$ | $U \oplus D_8 \oplus D_8$ | $(U(2) \oplus U(2))$ |
| 6) (311111) | $III^*, 5III$ | $U \oplus D_8 \oplus D_4$ | $U \oplus U(2) \oplus D_4$ |
| 7) (32111) | $III^*, I_0^*, III$ | $U \oplus E_8 \oplus D_4 \oplus A_1^{\delta 2}$ | $U \oplus U(2) \oplus A_1^{\delta 2}$ |
| 8) (3221) | $III^*, 2I_0^*, III$ | $U \oplus E_8 \oplus D_6 \oplus A_1^{\delta 2}$ | $A_1(-1)^{\delta 2} \oplus A_1^{\delta 2}$ |
| 9) (3311) | $2III^*, 2III$ | $U \oplus E_8 \oplus D_8$ | $U \oplus U(2)$ |
| 10) (332) | $2III^*, I_0^*$ | $U \oplus E_8 \oplus D_{10}$ | $A_1(-1)^{\delta 2}$ |

**Table 1.**

4.10. **Strictly semi-stable points:** $(44)$. In this case, we have the following 3 cases of curves in the quadric corresponding to the strictly semi-stable points with unique minimal closed orbit:

\[
C_3 : (x_0 - \lambda_1 x_1)^2(x_0 - \lambda_2 x_1)^2(y_0^2 + y_1^2) = 0;
\]

\[
C_4 : (x_0 - \lambda_1 x_1)(x_0 - \lambda_2 x_1)(y_0^2(x_0 - \lambda_1 x_1)^2 + y_1^2(x_0 - \lambda_2 x_1)^2) = 0;
\]

\[
C_5 : y_0^2(x_0 - \lambda_1 x_1)^4 + y_1^2(x_0 - \lambda_2 x_1)^4 = 0.
\]

These curves appear in the list of Shah’s classification of semistable $K3$ surfaces of degree 4. See Shah [S], Theorem 4.8, B, Type II, (i)–(iii).

We denote by $\mathcal{B}/\Gamma(1 - i)$ the Satake-Baily-Borel compactification of $\mathcal{B}/\Gamma(1 - i)$ whose boundary consists of 35 cusps. Then we conclude:

4.11. **Theorem.** The $S_8$-equivariant isomorphism $\phi$ in Theorem S7 can be extended to an $S_8$-equivariant isomorphism $\tilde{\phi}$ between $P_8^\delta$ and $\mathcal{B}/\Gamma(1 - i)$. Moreover $\tilde{\phi}$ sends strictly semistable points to cusps and stable but not distinct 8 points into $\mathcal{H}/\Gamma(1 - i)$. 
Proof. We can apply the argument of Horikawa’s proof of the main theorem in [H]. Let \( \mathcal{M} \) be the space of all 8 semi-stable points on \( \mathbb{P}^1 \) and \( \mathcal{M}_0 \) the space of all distinct 8 points on \( \mathbb{P}^1 \). We can easily see that \( \mathcal{M} \setminus \mathcal{M}_0 \) is locally contained in a divisor with normal crossing. By construction, \( \phi \) is locally liftable to \( \mathcal{B} \). It now follows from a theorem of Borel [Bo] that \( \phi \) can be extended to a holomorphic map from \( \mathcal{M} \) to \( \bar{\mathcal{B}}/\Gamma(1-i) \) which induces a holomorphic map
\[
\tilde{\phi} : \mathbb{P}^8_1 \to \bar{\mathcal{B}}/\Gamma(1-i).
\]
By using the same argument as in the proof of [H], Theorem 2.2, we can see that \( \tilde{\phi} \) sends stable, but non-distinct 8 points to \( \mathcal{H} \). More precisely we can choose a marking for \( K_3 \) surfaces corresponding to stable, but non-distinct 8 points, and define the period for them. For each stratification as in Table 1, we can prove the similar statement as in Lemma 3.6. Then, as in the generic case (Theorem 3.7), by case by case argument according to strata, we can see that the map \( \tilde{\phi} \) is injective over \( \mathcal{H} \). Moreover Shah’s classification [S] implies the image of strictly semi-stable points go to the boundaries. Hence the Zariski Main theorem implies that \( \tilde{\phi} \) is an isomorphism. The \( S_8 \)-equivariantness is obvious. \( \square \)

5. The Weil representation

In this section we shall study the quadratic form \((A_N, q_N)\) over \( \mathbb{F}_2 \) given in (2.6) and the Weil representation of \( SL(2, \mathbb{Z}) \) on the group ring \( \mathbb{C}[A_N] \).

5.1. Let

\[
T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

We denote by \( \{ e_\alpha \}_{\alpha \in A_N} \) the standard basis of \( \mathbb{C}[A_N] \). Let \( \rho \) be the Weil representation of \( SL(2, \mathbb{Z}) \) on \( \mathbb{C}[A_N] \) which factors through \( SL(2, \mathbb{Z}/2\mathbb{Z}) \) ([Bo]):

\[
\rho(T)(e_\alpha) = (-1)^{q_N(\alpha)}e_\alpha; \quad \rho(S)(e_\alpha) = \frac{1}{8} \sum_\beta (-1)^{b_N(\beta, \alpha)}e_\beta.
\]

Representatives of the conjugacy classes of \( SL(2, \mathbb{Z}/2\mathbb{Z}) \cong S_3 \) consist of \( E, T, ST \). A direct calculation shows that the traces of the action of \( E, T, ST \) on \( \mathbb{C}[A_N] \) are

\[
tr(E) = 2^6, \quad tr(T) = 8, \quad tr(ST) = 1.
\]

Let \( \chi_i \) (\( 1 \leq i \leq 3 \)) be the characters of irreducible representations of \( SL(2, \mathbb{Z}/2\mathbb{Z}) \): \( \chi_1, \chi_2 \) or \( \chi_3 \) is the trivial, alternating character or the character of 2-dimensional irreducible representation respectively. Let \( \chi \) be the character of the Weil representation of \( SL(2, \mathbb{Z}/2\mathbb{Z}) \) on \( \mathbb{C}[A_N] \) and let \( \chi = \sum m_i \chi_i \) be its decomposition into irreducible characters. Then an elementary calculation shows that

\[
\chi = 15\chi_1 + 7\chi_2 + 21\chi_3.
\]

We call a subspace \( I \) of \( A_N \) totally isotropic if \( q_N \) vanishes on \( I \), and \( I \) maximal if it has dimension 3.
5.2. **Lemma.** For each maximal totally isotropic subspace \( I \) of \( A_N \),
\[
\sum_{\alpha \in I} e_\alpha \in \mathbb{C}[A_N]^{SL(2,\mathbb{Z})}.
\]

*Proof.* The proof is the same as that of [Kon2], Lemma 3.2. \( \square \)

Let \( \alpha \in A_N \) with \( q_N(\alpha) = 1 \). Then
\[
t_\alpha : x \mapsto x + b_N(x, \alpha)\alpha
\]
is called a *transvection* and contained in \( O(q_N) \). Note that \( t_\alpha \) is induced from a reflection \( s_r \) associated with a \((-4)\)-vector \( r \) in \( N \) with \( r/2 \mod N = \alpha \) and these \( t_\alpha \) (\( \alpha \in A_N \) with \( q_N(\alpha) = 1 \)) generate \( O(q_N) \). These 28 transvections in \( O(A_N) \) correspond to the 28 transpositions of \( S_8 \).

5.3. **Definition.** Let \( q_s \) be a quadratic form on \( \mathbb{F}_2^3 \) given by
\[
q_s(x) = \sum_{i=1}^{3} x_i, \quad x = (x_1, x_2, x_3) \in \mathbb{F}_2^3.
\]
Note that the associated bilinear form of \( q_s \) is identically zero. Let \( V \) be a 3-dimensional subspace of \( A_N \). We call \( V \) *maximal totally singular* if \( (V, q_N \mid V) \) is isomorphic to \((\mathbb{F}_2^3, q_s)\). Obviously \( V \) has a basis consisting of 3 mutually orthogonal non-isotropic vectors \( \{\alpha_1, \alpha_2, \alpha_3\} \). We remark that \( V \) consists of 4 non-isotropic vectors \( \alpha_i \) (\( 1 \leq i \leq 3 \)), \( \alpha_1 + \alpha_2 + \alpha_3 \) and 4 isotropic vectors \( 0, \alpha_i + \alpha_j \) (\( 1 \leq i < j \leq 3 \)). For each maximal totally singular subspace \( V \), we define a vector \( f_V \in \mathbb{C}[A_N]^{SL(2,\mathbb{Z})} \) on which the transvection \( t_\alpha \) (\( \alpha \in V \)) acts as \(-1\). Let \( I \) be the kernel of \( q_N \mid V \). Then \( I \) is a totally isotropic subspace of dimension 2 in \( A_N \) and there exist exactly two maximal totally isotropic subspaces \( I^+, I^- \) in \( A_N \) which contain \( I \). We define
\[
f_V = \sum_{\alpha \in I^+} e_\alpha - \sum_{\alpha \in I^-} e_\alpha \in \mathbb{C}[A_N].
\]

5.4. **Theorem.** Let \( V \) be a maximal totally singular subspace. Then \( f_V \) is contained in \( \mathbb{C}[A_N]^{SL(2,\mathbb{Z})} \) satisfying the following condition: \( f_V \) is the unique vector (up to constant) in \( \mathbb{C}[A_N] \) on which transvections \( t_\alpha(\alpha \in V, q_N(\alpha) = 1) \) act as \(-1\).

*Proof.* The proof is the same as that of [Kon2], Theorem 3.4. \( \square \)

5.5. **Remark-Definition.** The group \( O(q_N)(\simeq S_8) \) naturally acts on \( \mathbb{C}[A_N]^{SL(2,\mathbb{Z})} \) with character \( \chi_1 + \chi_{14} \), where \( \chi_1 \) is the trivial character and \( \chi_{14} \) is the character of an irreducible representation of \( S_8 \) of degree 14. This follows from Lemma 5.2 and [Atlas], page 22. Moreover it follows from Theorem 5.4 that the multiplicity of the irreducible representation of degree 14 on \( \mathbb{C}[A_N] \) is one. We denote by \( W \) the subspace of dimension 14 in \( \mathbb{C}[A_N]^{SL(2,\mathbb{Z})} \) with character \( \chi_{14} \).

5.6. **Lemma.** The number of maximal totally singular subspaces of \( A_N \) is equal to 105.

*Proof.* We can easily count the number of mutually orthogonal three non-isotropic vectors in \( A_N \) which is \( 2^3 \cdot 3^2 \cdot 5 \cdot 7 \). On the other hand, the automorphism group of a maximal totally singular subspace has order \( 2^3 \cdot 3 \). Hence the assertion follows. \( \square \)

In the Lemma 7.3 we shall give a geometric interpretation of maximal totally singular subspaces.
5.7. **Heegner divisors.** Let \( \delta \in \mathbb{N} \) be a \((-4)\)-vector with \( \delta/2 \in \mathbb{N}^* \). Let \( D_\delta \) be the hyperplane of \( D \) defined by

\[
D_\delta = \delta^\perp \cap D.
\]

It follows from Lemma 4.5 that

\[
H_r = D_\delta \cap B \quad \text{where} \quad r = (\delta - \rho(\delta))/2 \text{ is a } (-2)\text{-vector in } \mathbb{N} \quad \text{and} \quad H_r \text{ is as in } \text{(5.5)}.
\]

For \( \alpha \in A_N \) with \( q_N(\alpha) = 1 \), we define **Heegner divisors** \( D_\alpha \) and \( H_\alpha \) by

\[
D_\alpha = \sum_\delta D_\delta, \quad H_\alpha = \sum_\delta H_\delta
\]

where \( \delta \) varies over the set of \((-4)\)-vectors in \( \mathbb{N} \) with \( \delta/2 \mod \mathbb{N} = \alpha \). Since \( H_r = D_\delta \cap B = D_{\rho(\delta)} \cap B \), we have

\[
(5.5) \quad 2H_\alpha = D_\alpha \mid B.
\]

6. **Automorphic forms**

6.1. Let \( \{f_\alpha\}_{\alpha \in A_N} \) be a vector valued elliptic modular form of weight \(-4\) and of type \( \rho \), i.e., \( f_\alpha \) is a holomorphic function on the upper half plane satisfying

\[
f_\alpha(\tau + 1) = e^{2\pi i q(\alpha)}f_\alpha(\tau), \quad f_\alpha(-1/\tau) = \frac{\tau^{-4}}{8} \sum_{\beta \in A_N} e^{-2\pi i (\alpha, \beta)}f_\beta.
\]

Recall that there are three types of vectors in \( A_N \) denoted by type 00, 0 or 1 according to zero, non-zero isotropic or non-isotropic respectively (see 4.3). For each \( \alpha \in A_N \), we denote by \( m_0 \) or \( m_1 \) the number of vectors \( \beta \in A_N \) with \( b(\alpha, \beta) = 0 \) or \( 1 \) respectively which is as in the following Table 2:

\[
\begin{array}{cccccccccccc}
\alpha & 00 & 00 & 00 & 0 & 0 & 0 & 1 & 1 & 1 \\
\beta & 00 & 0 & 1 & 00 & 0 & 1 & 00 & 0 & 1 \\
m_0 & 1 & 35 & 28 & 1 & 19 & 12 & 1 & 15 & 16 \\
m_1 & 0 & 0 & 0 & 0 & 16 & 16 & 0 & 20 & 12
\end{array}
\]

**Table 2.**

We shall find a modular form \( h \) such that the components \( h_\alpha \) are given by functions \( h_{00}, h_0, h_1 \) depending only on the type of \( \alpha \). Then it follows from (6.1) and Table 2 that \( h = \{h_\alpha\} \) satisfies:

\[
h_{00}(\tau + 1) = h_{00}(\tau), \quad h_{00}(-1/\tau) = \frac{\tau^{-4}}{8}(h_{00}(\tau) + 35h_0(\tau) + 28h_1(\tau)),
\]

\[
h_0(\tau + 1) = h_0(\tau), \quad h_0(-1/\tau) = \frac{\tau^{-4}}{8}(h_{00}(\tau) + 3h_0(\tau) - 4h_1(\tau)),
\]

\[
h_1(\tau + 1) = -h_1(\tau), \quad h_1(-1/\tau) = \frac{\tau^{-4}}{8}(h_{00}(\tau) - 5h_0(\tau) + 4h_1(\tau)).
\]
6.2. **Lemma.** One solution of these equations is given as follows:

\[
\begin{align*}
  h_0(\tau) &= 56\eta(2\tau)^8/\eta(\tau)^{16} = 56 + 896q + 8064q^2 + \cdots, \\
  h_0(\tau) &= -8\eta(2\tau)^8/\eta(\tau)^{16} = -8 - 128q - 1152q^2 - \cdots, \\
  h_1(\tau) &= 8\eta(2\tau)^8/\eta(\tau)^{16} + \eta(\tau/2)^8/\eta(\tau)^{16} = q^{-1/2} + 36q^{1/2} + 402q^{3/2} + \cdots
\end{align*}
\]

where \(\eta(\tau)\) is the Dedekind eta function and \(q = e^{2\pi\sqrt{-1}\tau}\).

**Proof.** The proof is the same as that of [Kon2], Lemma 4.3. \(\square\)

6.3. By applying Borcherds [Bor], Theorem 13.3, for the vector valued modular form \(h\) given in Lemma 6.2, we have

6.4. **Theorem.** There exists an automorphic form of weight \(28(= 56/2)\) on \(D\) which vanishes exactly on Heegner divisors corresponding to 28 non-isotropic vectors in \(A_N\).

6.5. On the other hand, by Borcherds [Bor], Theorem 14.3, we have an \(S_8\)-equivariant map

\[\phi : W \to A_4(\tO(N))\]

where \(W\) is the 14-dimensional subspace of \(C[A_N]^{SL(2,\mathbb{Z})}\) given in Remark-Definition 5.5. \(A_4(\tO(N))\) is the space of automorphic forms on \(D\) of weight 4 with respect to \(\tO(N)\) and

\(\tO(N) = \text{Ker}(O(N) \to O(q_N))\).

It follows from [N3], Theorem 1.14.2 that the map from \(O(N)\) to \(O(q_N)\) is surjective and hence \(S_8\) (\(\cong O(q_N) \cong O(N)/\tO(N)\)) naturally acts on \(A_4(\tO(N))\). On the other hand, \(S_8\) acts on \(W\). With respect to these actions, \(\phi\) is \(S_8\)-equivariant.

6.6. **Lemma.** The map \(\phi\) is injective.

**Proof.** The proof is the same as that of [Kon2], Lemma 4.1. \(\square\)

6.7. **Theorem.** Let \(V\) be a maximal totally singular subspace of \(A_N\). Let \(F_V\) be an automorphic form of weight 4 on \(D\) associated with \(f_V \in C[A_N]^{SL(2,\mathbb{Z})}\): \(F_V = \phi(f_V)\). Then

\[(F_V) = \sum_{\alpha \in V, \ q(\alpha) = 1} D_\alpha\]

where \(D_\alpha\) is the Heegner divisor associated with \(\alpha\).

**Proof.** Let \(\Phi\) be the product of all \(F_V\) where \(V\) varies over all maximal totally singular subspaces. Then \(\Phi\) is an automorphic form of weight \(105 \times 4\) (Lemma 5.6). By Theorem 5.4 and the \(S_8\)-equivariantness of \(\phi\), we have

\[\sum_{\alpha \in V, \ q(\alpha) = 1} D_\alpha \subset (F_V)\]

Hence \(\Phi\) vanishes along \(D_\alpha\) with vanishing order \(\geq 4 \times 105/28 = 15\). On the other hand, the 15-th power of the automorphic form given in Theorem 6.4 has the same weight and vanishes on \(D_\alpha\) with multiplicity 15. The assertion now follows from the Koecher principle. \(\square\)
7. Cross ratios

7.1. Let

\[
\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\
\tau_{21} & \tau_{22} \\
\tau_{31} & \tau_{32} \\
\tau_{41} & \tau_{42} \end{pmatrix}, \quad \tau_{ij} \in \{1, 2, \ldots, 8\}.
\]

We call \(\tau\) a tableau. A tableau \(\tau\) is called standard if

\[
\tau_{ij} < \tau_{ij+1}, \quad \tau_{ij} \leq \tau_{i+1j}
\]

for any \(i, j\). The number of tableaus is 105 and the number of standard tableaus is 14 (e.g. \[DO\], Chap. I). For each \(\tau\) we define

\[
\mu_\tau = \prod_{1 \leq i \leq 4} \det(v^{\tau_{i1}} v^{\tau_{i2}})
\]

where \(v^i \in \mathbb{C}^2\) is a column vector. If

\[
v^i = \begin{pmatrix} 1 \\ x^i \end{pmatrix},
\]

then,

\[
\mu_\tau = \prod_{1 \leq i \leq 4} (x^{\tau_{i2}} - x^{\tau_{i1}}).
\]

These \(\mu_\tau\) define an \(S_8\)-equivariant map \(\Theta\) from \(P^8_1\) to \(P^{13}\) (e.g. \[DO\]). We identify \(P^8_1\) with \(\bar{B}/\Gamma(1-i)\) under the isomorphism given in Theorem 4.11. In the following, we shall discuss a relation between \(\Theta\) and the map defined by 14-dimensional space \(W\) of automorphic forms given in §6.

7.2. We give a relation between the set of tableaus and the set of totally singular subspaces. Let \(K = U \oplus A^{\oplus 8}_1\) be the sublattice given in Lemma 2.2. Then \(A_K = K^*/K \cong (F_2)^8\) is generated by \(F_i/2\) (1 \(\leq i \leq 8\)) corresponding to 8 points on the projective line. The discriminant quadratic form \(q_K\) of \(K\) is a map

\[
q_K : (F_2)^8 \to \mathbb{Q}/2\mathbb{Z}
\]

defined by \(q_K(x) = \langle x, x \rangle \mod 2\mathbb{Z}\). Let \(\theta = (F_1 + \cdots + F_8)/2\) which is perpendicular to all vectors in \(A_K\). Then it is known (Nikulin \[N3\], Proposition 1.4.1) that the discriminant quadratic form of \(M\) is obtained by

\[
q_M = q_K | \theta^\perp/\theta.
\]

Finally \(q_N\) is canonically isomorphic to \(-q_M\) (\[N3\], Corollary 1.6.2). Thus non-isotropic vectors with respect to \(q_N\) bijectively corresponds to the vectors \((F_i + F_j)/2\) (\(i \neq j\)) in \(A_K\). Each column \((\tau_{i1}, \tau_{i2})\) of \(\tau\) in (7.1) defines a vector in \((F_2)^8\) whose nonzero entries are indexed by \(\tau_{i1}, \tau_{i2}\). Thus four columns of \(\tau\) corresponds to mutually orthogonal 4 non-isotropic vectors which generate a maximal totally singular subspace in \(A_N\). This implies the following:

7.3. Lemma. The set of 105 tableaus \(\tau\) bijectively corresponds to the set of maximal totally singular subspaces of \(A_N\). Under this correspondence, the zero of \(\mu_\tau\) coincides with the zero of \(F_V\) where \(V\) is a maximal totally singular subspace corresponding to \(\tau\).
7.4. Consider the linear system of automorphic forms $F_V$ of dimension 14 defined by $W$ (see 6.5). Note that the divisor $(F_V | B)$ is given by

$$2 \sum_{\alpha \in V, \, q(\alpha) = 1} H_\alpha$$

(see 5.3, Theorem 6.7). Since $B$ is simply connected, we can take a square root of $F_V$. Thus we have an automorphic form $G_V$ of weight 2 on $B$ with

$$(G_V) = \sum_{\alpha \in V, \, q(\alpha) = 1} H_\alpha.$$ 

Then $\{G_V\}_V$ defines a map $\Psi$ from $\overline{B}/\Gamma(1 - i)$ to $\mathbb{P}^{13}$.

7.5. **Theorem.** The map $\Theta$ coincides with $\Psi$.

**Proof.** We shall show that $\mu_{\tau_1}/\mu_{\tau_2}$ coincides with $G_{V_1}/G_{V_2}$ for suitable tableaux $\tau_1, \tau_2$ and the corresponding maximal totally singular subspaces $V_1, V_2$. We consider the following tableaux:

$$\tau_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}, \tau_2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 7 \\ 6 & 8 \end{pmatrix}, \tau_3 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 8 \\ 6 & 7 \end{pmatrix}.$$

We take a decomposition of $A_N = u_1 \oplus u_2 \oplus u_3$ into three hyperbolic planes $u_1, u_2, u_3$ defined over $F_2$. Let $\langle e_i, f_i \rangle$ be a basis of $u_i$ with $\langle e_i, e_i \rangle = 0$, $\langle f_i, f_i \rangle = 0$, $\langle e_i, f_i \rangle = 1$. Let $\alpha_i = e_i + f_i$ be the non-isotropic vector in $u_i$. We may assume that

$$V_1 = \langle \alpha_1, \alpha_2, \alpha_3 \rangle, \quad V_2 = \langle \alpha_1, \alpha_2, \alpha_1 + e_3 \rangle, \quad V_3 = \langle \alpha_1, \alpha_2, \alpha_1 + f_3 \rangle$$

correspond to $\tau_1, \tau_2, \tau_3$ respectively. Then by (7.3), we can see that $(G_{V_1}/G_{V_2}) = (\mu_{\tau_1}/\mu_{\tau_2})$ as divisors. Note that the function $\mu_{\tau_1}/\mu_{\tau_2}$ takes the value 1 on the divisors defined by $x_5 - x_8$ and $x_6 - x_7$. On the other hand, easy calculation shows that $f_{V_1} - f_{V_2} = f_{V_3}$ and hence $F_{V_1} - F_{V_2}$ vanishes on the divisor of $F_{V_3}$. This implies that $G_{V_1}/G_{V_2} = 1$ on the divisors $H_{a_1 + f_3}$ and $H_{a_2 + f_3}$. Hence $G_{V_1}/G_{V_2} = \mu_{\tau_1}/\mu_{\tau_2}$. For any pair of $\tau_1', \tau_2'$, the ratio $\mu_{\tau_1'}/\mu_{\tau_2'}$ can be written as the product of some $\mu_{\tau_1}/\mu_{\tau_2}$ as above type. Hence the assertion follows.

7.6. **Theorem.** $\Psi$ is an embedding from $\overline{B}/\Gamma(1 - i)$ into $\mathbb{P}^{13}$. The image satisfies $2^2 \cdot 3 \cdot 5 \cdot 7$ quartic relations.

**Proof.** It is known that $\Theta$ is embedding (Koike [Koi]). The proof of the second assertion is the same as that of [Kon2], Theorem 7.2, that is, for each non-isotropic vector in $A_N$ we have 15 quartic relations. Since the number of non-isotropic vectors is 28, the second assertion follows.

7.7. **Remark.** In the paper [Koi], by using a computer, he showed that the image is the intersection of 14 quadrics.

7.8. **Remark.** In the paper [MT], Matsumoto and Terasoma constructed an $S_8$-equivariant map from the moduli space of ordered 8 points on $\mathbb{P}^1$ to $\mathbb{P}^{104}$ by using the theta constants related to the curve which is the 4-fold covering of $\mathbb{P}^1$ branched at 8 points. They showed that this map coincides with the map defined by the above 105 $\mu_{\tau}$. 

7.9. **Remark.** Since 8 points on $\mathbb{P}^1$ naturally correspond to hyperelliptic curves of genus three, we can consider that our case is a degenerate one of smooth curves of genus three. The moduli space of non-hyperelliptic curves of genus three can be also described as an arithmetic quotient of a complex ball (\([\text{Kon1}]\)). On the other hand, Coble constructed a map from the moduli space of curves of genus 3 with level 2-structure to $\mathbb{P}^{14}$ by using Göpel functions (Coble \([\text{C}]\), Dolgachev, Ortland \([\text{DO}]\), Chap. IX). It would be interesting to extend the result in this paper to the case of curves of genus three.

**REFERENCES**

[AF] D. Allcock, E. Freitag, *Cubic surfaces and Borcherds products*, Comm. Math. Helv., 77 (2002), 270–296.

[Atlas] J. H. Conway et al., *Atlas of finite groups*, Oxford 1985.

[Bor] R. Borcherds, *Automorphic forms with singularities on Grassmannians*, Invent. Math. 132 (1998), 491–562.

[Bo] A. Borel, *Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem*, J. Diff. Geometry 6 (1972), 543–560.

[C] A. Coble, *Algebraic geometry and theta functions*, Amer. Math. Soc. Coll. Publ. 10 Providence, R.I., 1929 (3rd ed., 1969).

[DM] P. Deligne, G. W. Mostow, *Monodromy of hypergeometric functions and non-lattice integral monodromy*, Publ. Math. IHES, 63 (1986), 5–89

[DO] I. Dolgachev, D. Ortland, *Point sets in projective spaces and theta functions*, Astérisque 165 (1988).

[H] E. Horikawa, *On the periods of Enriques surfaces. II*, Math. Ann. 235 (1978), 217–246.

[Kod] K. Kodaira, *On compact complex analytic surfaces II*, Ann. Math., 77(1963), 563–626. III, Ann. Math., 78(1963), 1–40.

[Koi] K. Koike, *The projective embedding of the configuration space $X(2,8)$*, preprint.

[Kon1] S. Kondō, *A complex hyperbolic structure for the moduli space of curves of genus three*, J. reine angew. Math., 525(2000), 219–232.

[Kon2] S. Kondō, *The moduli space of Enriques surfaces and Borcherds products*, J. Algebraic Geometry, 11 (2002), 601–627.

[MT] K. Matsumoto, T. Terasoma, *Theta constants associated to coverings of $\mathbb{P}^1$ branching at 8 points*, Compositio Math., 140 (2004), 1277–1301.

[MY] K. Matsumoto, M. Yoshida, *Configuration space of 8 points on the projective line and a 5-dimensional Picard modular group*, Compositio Math., 86 (1993), 265–280.

[N1] V. V. Nikulin, *Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections*, J. Soviet Math., 22 (1983), 1401–1475.

[N2] V. V. Nikulin, *Finite automorphism groups of Kähler $K^3$ surfaces*, Trans. Moscow Math. Soc., 38 (1980), 71–135.

[N3] V. V. Nikulin, *Integral symmetric bilinear forms and its applications*, Math. USSR Izv., 14 (1980), 103–167.

[PS] I. Piatetski-Shapiro, I. R. Shafarevich, *A Torelli theorem for algebraic surfaces of type $K^3$*, Math. USSR Izv., 5 (1971), 547–587.

[S] J. Shah, *Degenerations of K3 surfaces of degree 4*, Trans. A. M. S., 263 (1981), 271–308.

[V] E. B. Vinberg, *Some arithmetic discrete groups in Lobachevskii spaces*, in "Discrete subgroups of Lie groups and applications to moduli", Tata-Oxford (1975), 323–348.

**GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, NAGOYA, 464-8602, JAPAN**

**E-mail address:** kondo@math.nagoya-u.ac.jp