$p$-Adic interpolation of square roots of central $L$-values of modular forms

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Abstract We construct a meromorphic function on part of the eigencurve that interpolates a square root of a ratio of quadratic twists of the central modular $L$-value.

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1 Introduction

The theme of $p$-adic interpolation of special values of $L$-functions has become a standard one in algebraic number theory. Serre [23] gave a construction of the Kubota–Leopoldt $p$-adic $L$-function by introducing $p$-adic modular forms and studying their coefficients. Since the work of Serre, this relationship between $p$-adic variation of $L$-values and $p$-adic variation of automorphic forms has become increasingly clear. In the case of modular forms for $GL_2$, Coleman and Mazur [5] (and later in greater generality Buzzard [4]and, independently, Chenevier [6]) have constructed a rigid-analytic eigencurve that is the universal $p$-adic family of (overconvergent, finite-slope)
modular eigenforms. This is the natural domain for the study of questions related to the $p$-adic variation of these modular forms.

Let $p$ be an odd prime and let $N$ be a positive integer that is not divisible by $p$. We will denote by $D(N)$ the tame level $N$ cuspidal eigencurve. The rigid-analytic space $D(N)$ is constructed (as outlined in Sect. 2) so that its points parameterize finite-slope systems of eigenvalues of the Hecke operators $T_\ell$ for $\ell \nmid np$ and $U_p$, as well as the diamond operators $\langle d \rangle_N$ for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, acting on cuspidal modular forms of tame level $N$. In particular, any classical modular form $f$ of level $Np^m$ that is an eigenform for the Hecke operators away from the level, is an eigenform for $U_p$ with non-zero eigenvalue, and has a nebentypus character, furnishes a point on $D(N)$. Moreover, the collection of such classical points is Zariski-dense. By multiplicity one, we may attach to each classical point $x \in D(N)$ a unique normalized newform $f_0 = f_0(x)$ having the system of Hecke eigenvalues prescribed by $x$. The aim of this paper is to interpolate a square-root of a certain ratio of twists of the central $L$-value of $f_0$ across the “even” part of $D(N)$.

In the spirit of simultaneous $p$-adic variation of $L$-values and automorphic forms, a number of authors have constructed “several variable” $p$-adic $L$-functions that interpolate classical critical values of an $L$-function associated to one or more modular forms varying in a $p$-adic family. In particular, in the ordinary case, a two-variable $p$-adic $L$-function interpolating the classical critical $L$-values in a Hida family of ordinary forms has been constructed in various places (see for example [10,12,14]). A similar result was proven for Coleman families of arbitrary (fixed) finite slope by Panchishkin [17]. Additionally, two-variable $p$-adic $L$-functions of at least two sorts have been constructed on the eigencurve in [1] and [9].

Of particular interest are the central values of the $L$-functions of modular forms, as these are the values connected to the Birch and Swinnerton-Dyer conjecture and its analogs in higher weight. In many instances, these central values (appropriately normalized) are squares in a certain number field. In such cases, it is natural to ask whether a square-root can be interpolated in a $p$-adic family of forms. In particular, one may ask when the “diagonal” $L$-function obtained by specializing a two-variable $p$-adic $L$-function to the central point is a square.

Koblitz [15] studies the problem of descending congruences between modular forms of integral weight to half-integral weight through the Shimura lifting (a problem that he attributes to a question of Hida—see also the work of Maeda [16]). Waldspurger [26] has famously established a connection between squares of the Fourier coefficients of half-integral weight modular forms and the central $L$-values of their Shimura liftings. Accordingly, Koblitz also considers, largely conjecturally, consequences of these descended congruences for congruences between square roots of the central $L$-values of modular forms. Inspired by the conjectures of Koblitz, Sofer [24] proves that one can interpolate a square root of the central $L$-value in a family of Hecke $L$-functions of imaginary quadratic fields. Her arguments utilize $p$-adic properties of modular forms in an essential way. Interpolation of square roots has also been studied by Harris and Tilouine [11] where these authors interpolate a central square root of a triple product $L$-function in which one of the three modular forms varies in a Hida family of ordinary forms. Hida [13] apparently returns to his own question and proves a $\Lambda$-adic version of Waldspurger’s result by interpolating a square-root of the ratio of the central values.
of two quadratic twists of a form lying in a Hida family. The results of the present paper are essentially a generalization of this work of Hida to higher-slope, placed in the context of the eigencurve.

Our interpolation takes place across the “even” part of the eigencurve, that is, the union of connected components on which the $p$-adic weight character and tame nebentypus characters are squares. An even classical weight character is equal to $\kappa^2$, where

$$\kappa : \mathbb{Z}_p^\times \rightarrow \mathbb{C}_p^\times$$

is of the form

$$\kappa(t) = t^{(k-1)/2}\kappa'(t)$$

for an odd positive integer $k$ and finite-order character $\kappa'$. The tame nebentypus of such a point is of the form $\chi^2$ for some character $\chi$. In this context, $\chi_0$ will denote the character $\chi_0(n) = \chi(n) \cdot (-1/n)^{(k-1)/2}$.

For a Dirichlet character $\psi$ modulo $N$, let $D(N)_\psi$ denote the union of connected components of the underlying reduced space $D(N)_{\text{red}}$ parameterizing forms of tame nebentypus $\psi$. For a square-free positive integer $n$, $\chi_n$ will denote the quadratic character associated to the field $\mathbb{Q}(\sqrt{n})$.

As is typical, we must make a choice to relate the complex-analytic world of holomorphic modular forms and their $L$-values and the $p$-adic world of rigid-analytic modular forms and the interpolating functions that we construct. Thus, we fix once and for all an isomorphism $\mathbb{C} \cong \mathbb{C}_p$. However, in the interest of notational brevity, this identification is kept implicit in the arguments and results that follow. The reader familiar with $p$-adic interpolation of this flavor should have no difficulty explicating the identification should the need arise.

The following theorem is the main result of the paper.

**Theorem 1** Let $p$ be an odd prime and let $N$ be a square-free positive integer that is relatively prime to $2p$. Suppose that the character $\psi$ modulo $N$ is a square and let $n$ and $m$ be square-free positive integers with $m/n \in (\mathbb{Q}_\ell^\times)^2$ for all $\ell \mid 2Np$. There exists a character $\chi$ modulo $8N$ with $\chi^2 = \psi$ such that on the union of irreducible components of $D(N)_\psi$ with even weight on which

$$L\left(f_0 \otimes \chi_0^{-1}\kappa'^{-1}\chi_n, (k - 1)/2\right)$$

does not vanish generically, there exists a meromorphic function $\Phi_{m,n}$ such that

$$\Phi_{m,n}(x)^2 = \frac{L(f_0 \otimes \chi_0^{-1}\kappa'^{-1}\chi_m, (k - 1)/2)}{L(f_0 \otimes \chi_0^{-1}\kappa'^{-1}\chi_n, (k - 1)/2)} \chi(m/n)\kappa(m/n)(m/n)^{-1/2}$$

holds at non-critical slope classical points $x$ outside of a discrete set.
Remark 1  1. We have suppressed the dependence of $f_0$, $\kappa$, and $k$ on the point $x$ on the right side of the equation in this statement for brevity.

2. The value $L(f_0 \otimes \chi_0^{-1}k^{-1}\chi_n,(k - 1)/2)$ is only defined at classical points. To say that it “vanishes generically” may be taken to mean that, among such points, it is supported on set of classical points that meets each irreducible component in a proper analytic set (in the sense of, for example, [2]). Of course, if $L(f_0 \otimes \chi_0^{-1}k^{-1}\chi_n,(k - 1)/2)$ (appropriately normalized so that $p$-adic interpolation is sensible) is the value of a globally-defined analytic function, then this condition simply amounts by the density of classical points to saying that this function vanishes identically.

3. In general, there is a sort of interplay between the choice of “nebentypus square-root” $\chi$, the square-free integer $n$ of the denominator, and the irreducible component on which we seek to interpolate. Theorem 1 is set up to interpolate with a fixed choice of $n$ with just one $\chi$ on as many components of $D(N)\psi$ as possible. In fact, the analysis that leads to the theorem allows for somewhat greater flexibility. For example, if one was instead bound to a particular $\chi$ and working on a particular union of irreducible components of $D(N)\chi^2$, this analysis shows that there infinitely many pairs $(m,n)$ for which the function $\Phi_{m,n}$ as above can be constructed when the denominator does not vanish generically. On the other hand, if one fixes $m$ as well, then there exist infinitely many such $n$ if and only if there exists a single one.

The proof of Theorem 1 relies on the aforementioned work of Waldspurger that relates the central special values of modular $L$-functions to Fourier coefficients of half-integral weight modular forms. The other major input is the work of the author [20–22] on $p$-adic families of such forms. We begin by giving a detailed construction in Sect. 3 of a coherent sheaf on an eigenvariety constructed using Buzzard’s [4] eigenvariety machine that interpolates the duals of the eigenspaces associated to the systems of eigenvalues parameterized by that eigenvariety. On the half-integral weight eigencurve constructed in [21], Fourier coefficients furnish sections of this sheaf. In Sect. 5, we establish a general criterion for linear dependence of sections of a coherent sheaf on a rigid space. When applied to a pair of Fourier coefficient sections, this criterion implies under suitable hypotheses that these sections are related by a meromorphic function. Using the interpolation of the Shimura lifting constructed in [22], this function can be moved to the integral weight eigencurve where it has interpolation property stated in Theorem 1.

Generic vanishing of the $L$-value in the denominator in Theorem 1 is an obvious impediment to interpolation across an irreducible component. As we will see, there are other impediments that preclude interpolation on additional components on the half-integral weight side. Fortunately, these will be ameliorated in passage to integral-weight by choosing the half-integral weight component mapping to the desired integral weight component judiciously. Doing so requires a detailed analysis of Waldspurger’s description of the preimage of the Shimura lifting that we carry out in the special case of square-free tame level.
2 The eigenvariety machine

Buzzard [4] introduces a systematic way to construct an eigenvariety given a certain system of Banach modules and commuting endomorphisms. The details of this construction will be used in what follows, so we briefly recall them here. For a more detailed account with proofs, see [4].

Let \( W \) be a reduced rigid space over a complete and discretely-valued extension \( K \) of \( \mathbb{Q}_p \). Fix a set \( T \) with a distinguished element \( \phi \). Suppose that, for each admissible open affinoid \( X \subseteq W \), we are given a Banach module \( M_X \) over \( \mathcal{O}(X) \) satisfying a certain technical hypothesis (called \( (Pr) \) in [4]) and a map

\[
T \rightarrow \text{End}_{\mathcal{O}(X)}(M_X)
\]

\[
t \mapsto t_X
\]

whose image consists of commuting endomorphisms such that \( \phi_X \) is compact for all \( X \). Suppose also that for each pair \( X_1 \subseteq X_2 \subseteq W \) we are given a continuous \( \mathcal{O}(X_1) \)-linear injection

\[
\alpha_{12} : M_{X_1} \rightarrow M_{X_2} \hat\otimes_{\mathcal{O}(X_2)} \mathcal{O}(X_1)
\]

that is a “link” in the sense of [4]. Finally suppose that the links are equivariant for the endomorphisms associated to elements of \( T \) in the obvious sense and satisfy the cocycle condition \( \alpha_{13} = \alpha_{23} \circ \alpha_{12} \) for any triple \( X_1 \subseteq X_2 \subseteq X_3 \subseteq W \).

Out of this data, Buzzard constructs rigid spaces \( D \) and \( Z \) called the eigenvariety and spectral variety, respectively, together with canonical maps

\[
D \rightarrow Z \rightarrow W.
\]

The construction of \( Z \) is straightforward. For each admissible affinoid open \( X \subseteq W \) we let \( Z_X \) be the zero locus of the Fredholm determinant

\[
P_X(T) = \det(1 - \phi_X T \mid M_X)
\]

inside \( X \times \mathbb{A}^1 \). The links ensure that this determinant is independent of \( X \) in the sense that if \( X_1 \subseteq X_2 \) then \( P_{X_1}(T) \) is the image of \( P_{X_2} \) in \( \mathcal{O}(X_1)[[T]] \). It follows that the \( Z_X \) glue to a space \( Z \) equipped with a canonical projection map \( Z \rightarrow W \).

The construction of \( D \) is more difficult. The starting point is the following theorem proven in [4].

**Theorem 2** Let \( R \) be a reduced affinoid algebra over \( K \), let \( P(T) \) be a Fredholm series over \( R \), and let \( Z \subseteq \text{Sp}(R) \times \mathbb{A}^1 \) denote the hypersurface cut out by \( P(T) \) equipped with the projection \( \pi : Z \rightarrow \text{Sp}(R) \). Define \( \mathcal{C}(Z) \) to be the collection of admissible affinoid opens \( Y \) in \( Z \) such that

- \( Y' = \pi(Y) \) is an admissible affinoid open in \( \text{Sp}(R) \),
- \( \pi : Y \rightarrow Y' \) is finite, and
- there exists \( e \in \mathcal{O}(\pi^{-1}(Y')) \) such that \( e^2 = e \) and \( Y \) is the zero locus of \( e \).

Then \( \mathcal{C}(Z) \) is an admissible cover of \( Z \).
We will often take \( Y' \) to be connected in what follows. This is not a serious restriction, since \( Y \) is the disjoint union of the parts lying over the various connected components of \( Y' \).

Let \( X \subseteq \mathcal{W} \) and let \( Y \in \mathcal{C}(Z_X) \) with connected image \( Y' \). To the choice of \( Y \) we can associate a factorization

\[
\det(1 - (\phi_X \otimes 1) T | M_X \otimes_{\mathcal{O}(X)} \mathcal{O}(Y')) = Q(T)Q'(T)
\]

into relatively prime factors with constant term 1. Here, \( Y \) is the zero locus of \( Q \) while \( Q' \) cuts out the complement of \( Y \) in \( \pi^{-1}(Y') \). The factor \( Q \) is actually a polynomial of degree equal to the degree of the finite map \( Y \to Y' \) and has a unit for leading coefficient. Associated to this factorization there is a unique decomposition

\[
M_X \otimes_{\mathcal{O}(X)} \mathcal{O}(Y') \cong N \oplus F
\]

into closed \( \mathcal{O}(Y') \)-submodules \( N \) and \( F \) with the property that \( Q^*(\phi) \) vanishes on \( N \) and is invertible on \( F \). Moreover, \( N \) is projective of rank equal to the degree of \( Q \) and the characteristic power series of \( \phi \) on \( N \) is \( Q(T) \).

The projectors onto the submodules \( N \) and \( F \) lie in the closure of \( \mathcal{O}(Y')[\phi] \), so it follows from the commutativity assumption that \( N \) and \( F \) are preserved by the endomorphisms \( t_X \otimes 1 \). Let \( T(Y) \) denote the \( \mathcal{O}(Y') \)-subalgebra of \( \text{End}_{\mathcal{O}(Y')}(N) \) generated by these endomorphisms. This algebra is finite over \( \mathcal{O}(Y') \) and hence affinoid, so we may define \( D_Y = \text{Sp}(T(Y)) \). Since the polynomial \( Q \) is the characteristic power series of \( \phi \) acting on \( N \) and has a unit for leading coefficient, \( \phi \) is invertible on \( N \) and \( Q(\phi^{-1}) = 0 \) on \( N \). Thus there is a well-defined map \( D_Y \to Y \) given by

\[
\mathcal{O}(Y')[T]/(Q(T)) \to T(Y)
\]

\[
T \mapsto \phi^{-1}
\]

on the underlying affinoid algebras. Now one simply glues over varying \( Y \) to obtain a space \( D \) equipped with a map \( D \to Z \).

**Definition 1** Let \( L/K \) be a complete and discretely-valued extension of \( K \). A pair \((\kappa, \gamma)\) consisting of a point \( \kappa \in \mathcal{W}(L) \) and a map \( T \to L \) is called an \( L \)-valued system of eigenvalues of \( T \) acting on the \( \{ M_X \} \) if there exists an admissible affinoid open \( X \subseteq \mathcal{W} \) containing \( \kappa \) and a nonzero form \( f \in M_X \otimes_{\mathcal{O}(X)} L \) such that \((t_X \otimes 1)f = \gamma(t)f\) for all \( t \in T \). The system of eigenvalues \( (\kappa, \gamma) \) is called \( \phi \)-finite if in addition \( \gamma(\phi) \neq 0 \).

Let \( x \) be an \( L \)-valued point of \( D \). Let \( \kappa_x \) denote the image of \( x \) in \( \mathcal{W} \) and pick an admissible open \( X \subseteq \mathcal{W} \) containing \( \kappa \). Then \( x \in D_Y \) for some \( Y \in \mathcal{C}(Z_X) \) so we can associate to \( x \) a map \( \gamma_x : T \to L \). The following is Lemma 5.9 of [4].

**Lemma 1** The association \( x \to (\kappa_x, \gamma_x) \) is a well-defined bijection between \( D(L) \) and the set of \( \phi \)-finite \( L \)-valued systems of eigenvalues of \( T \) acting on the \( \{ M_X \} \). The image of the system of eigenvalues \( (\kappa, \gamma) \) in \( Z \) is \( (\kappa, \gamma(\phi)^{-1}) \).
3 The coherent sheaf $N^*$

Let $\{M_X\}$ be a system of Banach modules in the sense of [4] as above. We wish to construct a coherent sheaf on $D$ whose fiber at a point is the linear dual of the eigenspace for the system of eigenvalues corresponding to this point by Lemma 1.

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Fix $X \subseteq \mathcal{W}$ and $Y = \text{Sp}(A) \in \mathcal{C}(Z_X)$ with connected image $Y' = \text{Sp}(A') \subseteq X$. Let $T(Y)$ and $N(Y)$ be as in the previous section. We will build the sheaf $N^*$ by gluing the finite $T(Y)$-modules $\text{Hom}_{A'}(N(Y), A')$. For a general $Y \in \mathcal{C}(Z_X)$, we will denote by $T(Y)$ the product of the algebras obtained from the various connected components of the image $Y'$ of $Y$ in $X$. For such $Y$ we will denote by $N(Y)$ the projective $\mathcal{O}(Y')$-module given by the product of the modules obtained in the previous section from the various connected components of $Y'$. Finally, for such $Y$, $N(Y)^*$ will denote the $\mathcal{O}(Y')$-linear dual of $N(Y)$, or equivalently the product of the duals of the projective modules corresponding to the various connected components of $Y'$.

**Lemma 2.** Let $Y_1, Y_2 \in \mathcal{C}(Z_X)$. Then $Y = Y_1 \cap Y_2 \in \mathcal{C}(Z_X)$ and there is a canonical identification

$$N(Y)^* \cong N(Y_i)^* \otimes_{T(Y_i)} T(Y).$$

**Proof.** Let $A_i$ denote the affinoid algebra of $Y_i$ or $i = 1, 2$ and $A$ denote that of $Y_1 \cap Y_2$. Lemma 5.2 of [4] gives the first assertion as well as the fact that restriction induces an isomorphism

$$T(Y_i) \otimes_{A_i} A \xrightarrow{\sim} T(Y).$$

Thus, it suffices to construct an identification

$$N(Y)^* \cong N(Y_i)^* \otimes_{A_i} A. \quad (1)$$

Each object in this purported isomorphism of $A$-algebras breaks up into factors associated to the various connected components of the images of $Y_i$ and $Y$ in $X$. Let $Y'$ be a connected component of the image of $Y = Y_1 \cap Y_2$, and let $Y_i'$ denote the unique connected component of the image of $Y_i$ containing $Y'$. Now re-define $Y_i$ to be its part lying over $Y_i'$ for $i = 1, 2$. Each $Y_i'$ lies in $\mathcal{C}(Z_X)$ with connected image $Y_i'$ so the preimages $Y_i \cap \pi^{-1}(Y') = Y_i \times_{Y_i'} Y'$ lie in $\mathcal{C}(Z_X)$ with image $Y'$. Now re-define $Y$ to be the intersection of these preimages, so that $Y$ is also in $\mathcal{C}(Z_X)$ with image $Y'$, as it is the intersection of two unions of connected components of $\pi^{-1}(Y')$.

Thus it suffices to prove (1) with these new $Y_i$ and $Y$ where now each image $Y_i'$ and $Y'$ is connected. In the interest of notational brevity we adopt the convention that objects associated with $Y_i$ carry a subscript $i$ and objects associated with $Y$ carry no subscript at all. Also, we will let $A_i$ denote the affinoid algebra of $Y_i$, let $A$ denote that of $Y$, let $A_i'$ denote that of $Y_i'$, and let $A'$ denote that of $Y'$.
Note that
\[ N_i^* \otimes_{A_i} A \cong (N_i^* \otimes_{A_i} (A_i \otimes_{A'_i} A')) \otimes_{A_i \otimes_{A'_i} A'} A \]
(2)
and since \( N_i \) is projective over \( A'_i \) we have
\[ N_i^* \otimes_{A'_i} A' = \text{Hom}_{A'_i}(N_i, A') \otimes_{A'_i} A' \cong \text{Hom}_{A'_i}(N_i \otimes_{A'_i} A', A'). \]
(3)

By definition, we have isomorphisms
\[ M \cong M_i \hat{\otimes}_{A'_i} A' \]
(4)
that are equivariant with respect to the Hecke actions on both sides. Tensoring the decomposition \( M_i \cong N_i \oplus F_i \) with \( A' \) gives a decomposition
\[ M_i \hat{\otimes}_{A'_i} A' \cong (N_i \otimes_{A'_i} A') \oplus (F_i \hat{\otimes}_{A'_i} A') \]
(5)
in which the first summand is the kernel of \( Q_i^*(\phi) \). Note that we are abusing notation slightly here by identifying \( Q_i \) with its image in \( A'[T] \). This sort of abuse will persist throughout the argument. The equivariance of (4) identifies \( N \) with the submodule of (5) on which \( Q_i^*(\phi) = 0 \). Since \( Y \) is a union of connected components of \( Y_i \times_{Y'_i} Y' \), there is a factorization \( Q_i = Q \tilde{Q}_i \) into relatively prime factors in \( A'[T] \) with constant term 1. In particular, (4) identifies \( N \) with the submodule of \( N_i \otimes_{A'_i} A' \) on which \( Q^*(\phi) = 0 \) since \( Q^*|_{Q_i^*} \).

This factorization of \( Q_i \) induces a canonical decomposition
\[ A_i \otimes_{A'_i} A' \cong A'[T]/(Q_i(T)) \cong A'[T]/(Q(T)) \times A'[T]/(\tilde{Q}_i(T)) \cong A \times \tilde{A}_i \]
where \( \tilde{A}_i \cong A'[T]/(\tilde{Q}_i(T)) \) is the affinoid algebra of \( Y_i \times_{Y'_i} Y' \setminus Y \). This in turn decomposes the \( A_i \otimes_{A'_i} A' \)-module \( N_i \otimes_{A'_i} A' \) into the direct sum of the kernels of \( Q(\phi^{-1}) \) and \( \tilde{Q}_i(\phi^{-1}) \) [equivalently, the kernels of \( Q^*(\phi) \) and \( \tilde{Q}^*(\phi) \)]. As the former is naturally identified with \( N \) via the isomorphism (4), we have a decomposition
\[ N_i \otimes_{A'_i} A' \cong N \oplus \tilde{N}_i \]
which, with (3), gives
\[ N_i^* \otimes_{A'_i} A' \cong N^* \oplus \tilde{N}_i^*. \]
Combining this with (2) yields an isomorphism
\[ N_i^* \otimes_{A_i} A \cong (N^* \otimes_{A_i \otimes_{A'_i} A'} A) \oplus (\tilde{N}_i^* \otimes_{A_i \otimes_{A'_i} A'} A). \]
As $A$ acts as zero on $\overline{N}_i$, the second summand vanishes. By contrast, the action of $A_i \otimes A_i' A'$ on $N$ factors through $A$, so the first factor is simply identified with $N^*$, and we have exhibited a canonical isomorphism $N_i^* \otimes_{A_i} A \cong N^*$.

The isomorphisms exhibited in this proof satisfy the cocycle condition because those of Lemma 5.2 of [4] do (as observed in the comments following that lemma), as do the isomorphisms (4) for trivial reasons. Thus we may glue the $N(Y)^*$ for varying $Y \in \mathcal{C}(Z_X)$ to obtain a coherent sheaf on $D(X)$. A similar argument to that in Lemma 2 using the links in place of the trivial isomorphisms (4) and Lemma 5.6 of [4] in place of Lemma 5.2 of [4] allows us to glue these sheaves for varying $X \subseteq \mathcal{W}$ to obtain a coherent sheaf $N^*$ on all of $D$.

Let us now compute the completed stalks and fibers of $N^*$. Fix an admissible open $X \subseteq \mathcal{W}$ and $Y = \text{Sp}(A) \in \mathcal{C}(Z_X)$ with connected image $Y' = \text{Sp}(A') \subseteq X$. In the interest of notational brevity we will drop the $Y$ from the notation of the module $N$ and the algebra $T(Y)$ and refer to them simply as $N$ and $T$. The conflict in notation with the set $T$ should cause no confusion. For each $y \in Y'$ we have

$$T \otimes_{A'} \widehat{A}_y' \cong \prod_x \widehat{T}_x$$

where the product is taken over the fiber of the finite map $D_Y \longrightarrow Y'$ over $y$. The module $N \otimes_{A'} \widehat{A}_y'$ is finite and projective over $\widehat{A}_y'$, and therefore free. This module moreover carries a faithful action of $T \otimes_{A'} \widehat{A}_y'$ since $A' \longrightarrow \widehat{A}_y'$ is flat, and therefore breaks up as

$$N \otimes_{A'} \widehat{A}_y' \cong \prod_x N \otimes_T \widehat{T}_x$$

and each factor of (6) acts faithfully on the corresponding factor of (7). It follows that

$$\text{Hom}_{A'}(N, A') \otimes_{A'} \widehat{A}_y' \cong \prod_x \text{Hom}_{\widehat{A}_y'} \left( N \otimes_T \widehat{T}_x, \widehat{A}_y' \right).$$

Extending scalars of either of these modules from $T \otimes_{A'} \widehat{A}_y'$ to $\widehat{T}_x$ for a particular $x$ simply picks out the factor corresponding to $x$, so we have canonical identifications

$$\text{Hom}_{A'}(N, A') \otimes_T \widehat{T}_x \cong \left( \text{Hom}_{A'}(N, A') \otimes_T (T \otimes_{A'} \widehat{A}_y') \right) \otimes_{T \otimes_{A'} \widehat{A}_y'} \widehat{T}_x$$

$$\cong \left( \text{Hom}_{A'}(N, A') \otimes_{A'} \widehat{A}_y' \right) \otimes_{T \otimes_{A'} \widehat{A}_y'} \widehat{T}_x$$

$$\cong \text{Hom}_{\widehat{A}_y'} \left( N \otimes_T \widehat{T}_x, \widehat{A}_y' \right),$$

which completes the description of the completed stalk of $N^*$ at $x$.

We now turn to the fiber. Again let $y \in Y'$ and let $L/K$ be a finite extension containing the residue field of $y$. Further extending scalars to $L$ in (6) and (7) we arrive at
\[ T \otimes_{A'} L \cong \prod_x \hat{T}_x \otimes_{\hat{A}_y} L \]  \hspace{1cm} (8)

and

\[ N \otimes_{A'} L \cong \prod_x (N \otimes_T \hat{T}_x) \otimes_{\hat{A}_y} L \]  \hspace{1cm} (9)

If we fix an \( L \)-valued point \( \lambda : T \rightarrow L \) of \( D_Y \) corresponding to a point \( x \) in the fiber over \( y \) (perhaps after enlarging \( L \)), then the image of this point in \( Y' \) gives a map \( \hat{A}'_y \rightarrow L \) and we may decompose as above. The following lemma characterizes the factor corresponding to \( x \) in terms of \( \lambda \).

**Lemma 3** There exists a positive integer \( e \) such that \((t - \lambda(t))^e = 0\) on the factor corresponding to \( x \) in (8). Moreover, for \( x' \neq x \) in the fiber over \( y \), there exists \( t \in T \) such that \( t - \lambda(t) \) is invertible on the factor corresponding to \( x' \).

**Proof** Denote the maximal ideals of \( \hat{A}_y' \) and \( \hat{T}_x \) by \( m_y \) and \( m_x \), respectively. Since \( \hat{T}_x/m_y \hat{T}_x \) is finite-dimensional Noetherian local algebra over the field \( \hat{A}_y'/m_y \), the Krull intersection theorem implies that there exists an integer \( e \) such that \( m_x^e \subseteq m_y \hat{T}_x \). Since \( t - \lambda(t) \in m_x \) for all \( t \), the first claim follows immediately.

Now pick \( x' \neq x \) in the fiber over \( y \). Pick a finite extension \( L'/L \) containing the residue field of the local algebra \( \hat{T}_{x'} \otimes_{\hat{A}_y} L \) and let

\[ \mu : \hat{T}_{x'} \otimes_{\hat{A}_y} L \rightarrow L' \]

realize this inclusion. Suppose that, for all \( t \in T \), we have \( \mu(t - \lambda(t)) = 0 \). Then \( \mu(t) = \lambda(t) \) for all \( t \) and it follows from Lemma 1 that \( x' = x \). Thus there exists \( t \) such that \( \mu(t - \lambda(t)) \neq 0 \), which implies that \( t - \lambda(t) \) is a invertible element of the local algebra \( \hat{T}_x \otimes_{\hat{A}_y} L \). \( \square \)

Let \( x \in D \) and let \( \lambda \) be the corresponding system of eigenvalues. Pick \( X \subseteq W \) containing the image of \( x \) and let

\[ M_\lambda = \{ f \in M \otimes_{\hat{O}(X)} L \mid (t \otimes 1)f = \lambda(t)f \text{ for all } t \in T \} \]

be the corresponding eigenspace. Note that this space is independent of the choice of \( X \) in the sense that the links that are part of the data of the system \( \{ M_X \} \) identify the eigenspace obtained from any two choices of \( X \) containing the image of \( x \). Note also that \( x \in D_Y \) implies that \( M_\lambda \) in fact lies in the summand \( N \otimes_{A'} L \) of \( M \otimes_{\hat{O}(X)} L \). The following is an immediate consequence of Lemma 3.

**Corollary 1** In the decomposition (9), the eigenspace \( M_\lambda \) is contained in the summand \((N \otimes_T \hat{T}_x) \otimes_{\hat{A}_y} L \) of \( N \otimes_{A'} L \)
Using the above description of the completed stalk of $\mathcal{N}^*$ at $x$, we can compute the fiber by extending scalars via $\widehat{T}_x \longrightarrow L$ to get

$$
\text{Hom}_{\widehat{A}_y}(N \otimes_{\widehat{T}_x} \lambda_{\widehat{A}_y}) \otimes_{\widehat{T}_x} L
\cong (\text{Hom}_{\widehat{A}_y}(N \otimes_{\widehat{T}_x} \lambda_{\widehat{A}_y}) \otimes_{\widehat{T}_x} (\widehat{T}_x \otimes \lambda_{\widehat{A}_y} L)) \otimes_{\widehat{T}_x \otimes \lambda_{\widehat{A}_y} L} L
\cong (\text{Hom}_{\widehat{A}_y}(N \otimes_{\widehat{T}_x} \lambda_{\widehat{A}_y} L, L) \otimes_{\widehat{T}_x \otimes \lambda_{\widehat{A}_y} L} L
\cong \text{Hom}_L((N \otimes_{\widehat{T}_x} \lambda_{\widehat{A}_y} L, L) \otimes_{\widehat{T}_x \otimes \lambda_{\widehat{A}_y} L} L
$$

where the last isomorphism follows because $N \otimes_{\widehat{T}_x} \lambda_{\widehat{A}_y}$ is free of finite rank over $\lambda_{\widehat{A}_y}$.

Now by Corollary 1, $M_\lambda$ is contained in $(N \otimes_{\widehat{T}_x} \lambda_{\widehat{A}_y} L$, and the following lemma shows that the restriction map induces an isomorphism

$$
\text{Hom}_L((N \otimes_{\widehat{T}_x} \lambda_{\widehat{A}_y} L, L) \otimes_{\widehat{T}_x \otimes \lambda_{\widehat{A}_y} L} L \longrightarrow \text{Hom}_L(M_\lambda, L).
$$

**Lemma 4** Let $V$ be a finite-dimensional vector space over a field $L$ equipped with the action of a commutative $L$-algebra $A$ and let $\lambda : A \longrightarrow L$ be an $L$-algebra homomorphism. Then the restriction map induces an isomorphism

$$
V^* \otimes_A L \longrightarrow (V_\lambda)^*
$$

where $V_\lambda$ is the $\lambda$-eigenspace of $V$ for the action of $A$.

**Proof** First observe that the indicated map is well-defined since by definition $A$ acts on $V_\lambda$ via $\lambda$. It suffices to check that the dual map

$$
V_\lambda \longrightarrow (V^* \otimes_A L)^*
$$

is an isomorphism. The surjection

$$
A \longrightarrow L
$$

induces a surjection

$$
V^* \longrightarrow V^* \otimes_A L.
$$

Dualizing we arrive at an injection

$$
(V^* \otimes_A L)^* \longrightarrow V.
$$

It is clear that this injection has image in $V_\lambda$ and it is easy to check that the map so obtained is the inverse of the dual of the map in the statement. □
The upshot is that we have constructed an identification

\[ N^*(x) := N^* \otimes O_D L \cong \text{Hom}_L(M_{\lambda}, L) \]  

(10)

for any \( L \)-valued point \( x \) of \( D \).

4 Comments on the Shimura lifting and irreducible components

In this section, we recall the main constructions and results of the author’s papers [21] and [22]. Fix an odd prime \( p \) and a positive integer \( N \) that is relatively prime to \( p \). We begin with \( p \)-adic weight space \( \mathcal{W} \) as usual, whose points correspond continuous characters of \( \mathbb{Z}^\times_p \). This space is the admissible disjoint union of \( p-1 \) spaces \( \mathcal{W}^i \) for \( 0 \leq i \leq p-1 \) corresponding the canonical decomposition

\[ \mathbb{Z}^\times_p \cong (\mathbb{Z}/p\mathbb{Z})^\times \times (1+p\mathbb{Z}_p) \]

given by the Teichmuller character \( \tau \). In particular, \( \mathcal{W}^i \) is the connected component of \( \mathcal{W} \) containing \( \tau^i \).

The Banach modules used in the construction of the half-integral weight eigen-curve are defined similarly to those used in integral weight, where to define families of forms over an admissible open, one essentially uses families of analytic functions on that open, the idea being that one has divided the “actual” families of forms by the Eisenstein family to obtain families of functions. Then one fixes up the Hecke action using the Eisenstein family so that it agrees with the usual one on \( q \)-expansions on classical forms. Of course, one must take care to avoid Eisenstein zeros, which serves to limit the domain on the modular curve over which one may define forms (depending on how big a family one wants over weight space). The latter issue is the source of the sequence \( \{r_n\} \) in [4] and [21]; there is a natural increasing exhaustion \( \{\mathcal{W}^i_n\}_n \) of \( \mathcal{W}^i \) by admissible affinoids such that one can define families of forms over an admissible open \( X \subseteq \mathcal{W}^i_n \) only on the affinoid subdomain \( X_1(Np)^{an}_{\geq p-r} \) of limited overconvergence in the modular curve.

In the half-integral weight case, the same strategy is employed, but the Eisenstein family has been multiplied by the usual Jacobi theta function

\[ \theta(q) = \sum_{n \in \mathbb{Z}} q^{n^2}. \]

As this function has zeros at some cusps on \( X_1(4Np) \), one must replace families of analytic function by families of sections of \( \mathcal{O}(\Sigma_{4Np}) \) for an appropriate divisor \( \Sigma_{4Np} \).

Fix a connected admissible open \( X \subseteq \mathcal{W}^i \) and overconvergence \( 0 \leq r \leq r_n \). In [21], we define the module of families of overconvergent half-integral weight cuspidal modular forms of tame level \( 4N \) over \( X \) with growth condition \( p^{-r} \) to be

\[ \tilde{M}_X = H^0(X_1(4Np)^{an}_{p-r}, \mathcal{O}(\Sigma_{4Np} - C_{4Np})) \otimes O(X), \]
Interpolation of square roots of central $L$-values

where the superscript denotes the $\tau^i$ isotypic component for the action of the diamond operators at $p$, and the divisor $C_{4Np}$ is there to ensure cuspidality (but is not generally simply the divisor of cusps—see [21]). To define the Hecke action one proceeds as in integral-weight by twisting the usual “pull-back/push-forward” construction by multiplying by an appropriate function on the source modular curve of the Hecke correspondence.

The space $\tilde{M}_X$ is a Banach module over $\mathcal{O}(X)$, and the specialization of this module to a point $\kappa \in X$ is (essentially by definition—see [21]) the space of overconvergent half-integral weight cuspidal modular forms of the above tame level and overconvergence and weight $\kappa$. Recall from [21] that the weight character book-keeping in half-integral weight is such that classical weight $k/2$ corresponds to the weight character $\kappa(t) = t^{(k-1)/2}$, and that the $p$-part of the nebentypus character of a classical form is packaged as part of the weight character. By a classical weight in $\mathcal{W}$ we shall mean one of the form $\kappa(t) = t^{(k-1)/2}\kappa'(t)$ where $k$ is an odd positive integer and $\kappa'$ is of finite order. In particular, at classical weights, the specialization above contains the entire space of classical forms of this weight and tame level (or, strictly speaking, these forms divided by $\theta$ times an Eisenstein series). In [21], we also prove a Coleman-style “low slope implies classical” Lemma in the half-integral weight setting.

Using these modules and the machinery of Sect. 2, we construct in [21] a rigid space $\tilde{D}(4N)$ (the “half-integral weight cuspidal eigencurve”) parameterizing finite-slope systems of eigenvalues of the Hecke operators $T_{\ell^2}$ for $\ell \nmid 2Np$, $U_{p^2}$, and the diamond operators $\langle d \rangle_{4N}$ for $d \in (\mathbb{Z}/4N\mathbb{Z})^\times$, acting on cuspidal modular forms of half-integral weight and tame level $4N$.

Remark 2 Strictly speaking, the constructions of [21] and [22] outlined above and below were carried out in the case where Hecke eigenconditions are imposed at all primes. However, the constructions contained in these papers adapt very easily to the present situation where such conditions are imposed only for $\ell \nmid 2N$.

Let $D(N)$ denote the tame level $N$ eigencurve parameterizing finite-slope systems of eigenvalues of the Hecke operators $\tilde{T}_\ell$ for $\ell \nmid Np$, $U_p$, and the diamond operators $\langle d \rangle_{N}$ for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, acting on cuspidal modular forms of integral weight and tame level $N$. In [22], the author constructs a map $\text{Sh}$ on underlying reduced rigid spaces fitting into a diagram

\[
\begin{array}{ccc}
\tilde{D}(4N)_{\text{red}} & \xrightarrow{\text{Sh}} & D(2N)_{\text{red}} \\
\downarrow & & \downarrow \\
\mathcal{W} & \xrightarrow{\kappa \mapsto \kappa^2} & \mathcal{W}
\end{array}
\]

that interpolates the classical Shimura lifting at classical points in the sense that it preserves the system of eigenvalues of Hecke and diamond operators in the evident sense. This map takes each irreducible component of $\tilde{D}(4N)_{\text{red}}$ isomorphically onto an irreducible component of $D(2N)_{\text{red}}$.

Recall that a classical weight shall mean one of the form $\kappa(t) = t^{(k-1)/2}\kappa'(t)$ where $k$ is an odd positive integer and $\kappa'$ is of finite order. A point $x \in \tilde{D}(4N)$
with corresponding system of eigenvalues $\lambda_x$ will be called \textit{classical} if there exists a classical form of the associated weight having this system of eigenvalues. The point $x$ will be called \textit{strictly classical} if there exists a classical form of the associated weight having this system of eigenvalues. The point $x$ will be called \textit{strictly classical} if the entire $\lambda_x$-eigenspace at this weight consists of classical forms. Finally, a point $x$ lying over a classical weight $\kappa(t) = t^{(k-1)/2} \kappa'(t)$ of finite order is called \textit{low-slope} if $v(\lambda_x(U_p^2)) < k-2$. In [21] it is proven that the low-slope points are strictly classical, and in [22] it is shown that they comprise a Zariski-dense set in $\tilde{D}(4N)$. Note that the notion of low-slope is compatible with that used in the integral weight setting in the sense that the Shimura lifting takes weight $k/2$ to weight $k - 1$.

The system of eigenvalues given by a classical point in $D(N)$ corresponds to a unique normalized newform $f_0$ that generates an automorphic representation $\pi = \bigotimes_v \pi_v$ of $GL_2(\mathbb{A}_{\mathbb{Q}})$. Given a property of this representation (e.g. \textquotedblright $\pi_\ell$ is Steinberg\textquotedblright) we will say that a point on $D(N)$ has this property if the corresponding representation does. Similarly, given a point of $\tilde{D}(4N)$, we will say that it has such a property if its image under the map $Sh$ does.

We now make a few observations about how some of these properties sit on the eigencurve $D(N)$ [and hence on $\tilde{D}(4N)$]. A more comprehensive study of this topic can be found in the papers of Paulin [18, 19]. We content ourselves here with some self-contained observations at square-free tame level that will suffice for our purposes.

First, we look at $\ell = p$. By the finite-slope condition, a classical point in $D(N)$ is either irreducible principal series or (an unramified twist of) Steinberg at $p$. The Steinberg points with weight $k - 1$ satisfy

$$\lambda_x(U_p)^2 = \chi(p)^2 p^{k-3},$$

where $\chi$ is the tame nebentypus of the point. In particular, these points lie in the preimage of the discrete analytic set $\mu_N(\mathbb{C}_p) \cdot p^\mathbb{Z}$ under the analytic function

$$D(N) \rightarrow \mathbb{G}_m, \quad x \mapsto \lambda_x(U_p)^2.$$

The Steinberg at $p$ locus is accordingly analytic in $D(N)$ and cannot contain an irreducible component, and thus meets each irreducible component in a proper analytic set.

Now suppose that $N$ is square-free and $\ell \mid N$. The values of $\lambda_x(\langle d \rangle_N)$ are constant on connected components of $D(N)$, so the tame nebentypus character is as well. Let $x \in D(N)$ be a point with weight $\kappa(t) = t^{k-1} \kappa'(t)$ (with $\kappa'$ of finite order) and tame nebentypus character $\chi$. Since $N$ is square-free, a classical point $x \in D(N)$ is either irreducible principal series or an unramified twist of Steinberg at $\ell$. The ramified irreducible principal series among these points are precisely those lying on connected components with non-trivial nebentypus at $\ell$.

Let us restrict now to the connected components with trivial nebentypus at $\ell$. We wish to determine the nature of the locus in $D(N)$ that is Steinberg at $\ell$. To do this we first consider the auxiliary eigencurve $D(N)^\ell$ obtained by further imposing an eigenvalue for $U_{\ell}$ (that is, by adding $U_{\ell}$ to the set $T$ used in the construction of $D(N)$). Let $x \in D(N)^\ell$ be a classical point of weight $\kappa(t) = t^{k-1} \kappa'(t)$ and tame nebentypus $\chi$. 

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Then $x$ is in the Steinberg locus if and only if
\[
\lambda_x(U_\ell)^2 = \kappa'(\ell)\chi(\ell)\ell^{k-3} = \chi(\ell)\ell^{-2}\kappa(\ell),
\]
and this equation defines an analytic set in $D(N)^\ell$. We conclude that an irreducible component of $D(N)^\ell$ on which $\chi$ is trivial at $\ell$ is either generically Steinberg at $\ell$ or generically unramified principal series at $\ell$. Moreover, a component that is generically Steinberg is in fact everywhere Steinberg, while a generically unramified principal series component may have a proper analytic subset of Steinberg points. Now, exploiting the evident finite map $D(N)^\ell \rightarrow D(N)$, we may draw these very same conclusions for $D(N)$.

Assume now that $N$ is square-free and odd. Using the Shimura lifting, we get a similar classification of the components of $\tilde{D}(4N)$. Here, however, we can further divide the Steinberg components into two types. Let us again adjoin $\ell$ to the list of eigenconditions (via the operator $U_{\ell^2}$) to obtain the curve $\tilde{D}(4N)^\ell$. Let $x$ denote a classical point of $\tilde{D}(4N)^\ell$ with weight $\kappa(t) = t^{(k-1)/2}\kappa'(t)$ and tame nebentypus $\chi$. Factor $\chi = \chi^{(\ell)}\chi^{(\sim)}$ into a character modulo a power of $\ell$ and a character of conductor prime to $\ell$. In particular, if $x$ lies in a Steinberg component, then $\chi^{(\ell)}$ is quadratic, and the Steinberg condition can be written
\[
\lambda_x(U_{\ell^2})^2 = (\chi^{(\sim)}(\ell))^2\ell^{-2}\kappa(\ell)^2
\]
so that
\[
\lambda_x(U_{\ell^2}) = \pm \chi^{(\sim)}(\ell)\ell^{-1}\kappa(\ell).
\]
Each sign defines an analytic set in $\tilde{D}(4N)^\ell$, so a Steinberg component here can be labeled as either a \textquotedblleft $+$\textquotedblright component or a \textquotedblleft $-$\textquotedblright component. Once again, we can exploit the finite map $\tilde{D}(4N)^\ell \rightarrow \tilde{D}(4N)$ to get such a labeling of the components of $\tilde{D}(4N)$.

Owing to the need to have a component on the half-integral weight side with tame nebentypus of conductor divisible by 8, we will later work on the eigencurve $\tilde{D}(8N)$ with $N$ odd and square-free. Here, the Shimura lifting maps to $D(4N)$, which contains $D(2N)$ as a union of irreducible components along with some other (e.g. supercuspidal at 2) components. However, will only be interested in those components that lie in $D(2N)$, and we label an irreducible component of $\tilde{D}(8N)$ that maps under the Shimura lifting to an irreducible component of $D(2N)$ in the manner described above.

5 A criterion for linear dependence of sections of a coherent module

We denote the sheaf of meromorphic functions on a rigid space $X$ by $\mathcal{M}_X$. This is the localization of $\mathcal{O}_X$ at the subsheaf of regular (non-zerodivisor) sections. For a coherent module $\mathcal{F}$ on $X$, we denote its $\mathcal{O}_X$-dual by
\[
\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X).
\]
This module is also coherent and there is a canonical map $\mathcal{F} \mapsto \mathcal{F}^{\vee\vee}$ as usual.

The proofs of the following two basic lemmas concerning meromorphic functions can be found in the discussion in Section 2 of [7].

**Lemma 5** Suppose that $\text{Sp}(A) \subseteq X$ is an admissible affinoid open. The choice of a finite $A$-module $M$ and an identification $\mathcal{F}|_{\text{Sp}(A)} = M$ induce an identification

$$\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}_X) \cong M \otimes_A \text{Frac}(A).$$

**Lemma 6** Let $X$ be a reduced rigid space and let $\pi : \tilde{X} \rightarrow X$ be the normalization of $X$. For coherent sheaves $\mathcal{F}$ on $X$, there is a natural identification

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}_X \xrightarrow{\sim} \pi_* (\pi^* \mathcal{F} \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{M}_{\tilde{X}})$$

that is functorial in $\mathcal{F}$.

The next Lemma gives several equivalent characterizations of “generic vanishing” of a section of a coherent sheaf on a reduced rigid space.

**Lemma 7** Let $X$ be a reduced rigid space and let $\mathcal{F}$ be a coherent sheaf on $X$. For a section $f \in \Gamma(X, \mathcal{F})$, the following are equivalent:

(a) $f$ lies in the kernel of the natural map

$$\mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}_X;$$

(b) $f$ lies in the kernel of the natural map

$$\mathcal{F} \rightarrow \mathcal{F}^{\vee\vee};$$

(c) $f$ vanishes on a nonempty admissible open in $X$ that meets each irreducible component of $X$;

(d) $f$ is supported on a nowhere-dense analytic set in $X$.

**Proof** The fact that (a) and (b) are equivalent follows from the facts that the right and bottom arrows in the natural diagram

$$\mathcal{F} \xrightarrow{\sim} \mathcal{F}^{\vee\vee}$$

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}_X \xrightarrow{\sim} \mathcal{F}^{\vee\vee} \otimes_{\mathcal{O}_X} \mathcal{M}_X$$

are injections. Suppose that a section $f$ satisfies condition (a). Then there exists a non-zerodivisor $a$ on $X$ with $af = 0$. It follows that the support of $f$ lies in the zero locus of $a$, which is a nowhere-dense analytic set in $X$ since $a$ is not a zero-divisor, so (a) implies (d). That (d) implies (c) follows because the complement of an analytic set is an admissible open.
We claim that (c) implies (a), which will complete the proof. By Lemma 6, it suffices to prove this after pulling back to the normalization. Looking at connected components, we see that it suffices to prove the claim when \( X \) is moreover normal and connected. But then any two points in \( X \) can be connected by a finite chain of nonempty admissible affinoid opens, so it suffices to check that for an inclusion \( \text{Sp}(B) \subseteq \text{Sp}(A) \) of such opens, the restriction map

\[
\Gamma(\text{Sp}(A), \mathcal{F} \otimes_{\mathcal{O}_{X}} \mathcal{M}_{X}) \longrightarrow \Gamma(\text{Sp}(B), \mathcal{F} \otimes_{\mathcal{O}_{X}} \mathcal{M}_{X})
\]

is injective. By normality, \( A \) and \( B \) are domains, and this injectivity follows readily from the description of sections in Lemma 5.

Recall that a subset \( S \) of a rigid space \( X \) is Zariski-dense if the only analytic set containing \( S \) is all of \( X \).

**Lemma 8** Let \( X \) be a normal and connected rigid-analytic curve, let \( \mathcal{F} \) be a coherent sheaf on \( X \), and let \( s \) be a global section of \( \mathcal{F} \). If \( s \) vanishes at a Zariski-dense set in \( X \), then the image of \( s \) in \( \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X}} \mathcal{M}_{X} \) is zero.

**Proof** With the hypotheses on \( X \), the sheaf \( \mathcal{F}^{\vee\vee} \) is locally-free since it is torsion-free, so the zero-locus of a section is an analytic set. As the image of \( s \) in this sheaf vanishes on a Zariski-dense set, this images vanishes and the result follows from Lemma 7.

**Lemma 9** Let \( X \) be a reduced rigid-analytic space of pure dimension 1, let \( \mathcal{F} \) be a coherent sheaf on \( X \), and let \( s_{1} \) and \( s_{2} \) be global sections of \( \mathcal{F} \) such that

1. the section \( s_{2} \) does not satisfy the equivalent conditions of Lemma 7 on any irreducible component of \( X \), and
2. there exists a Zariski-dense subset \( S \subseteq X \) such that \( s_{1}(x) \) and \( s_{2}(x) \) are linearly dependent in the fiber \( \mathcal{F}(x) \) for all \( x \in S \).

Then there exists a global meromorphic function \( \Phi \) on \( X \) such that \( s_{1} \otimes 1 = s_{2} \otimes \Phi \) in \( \mathcal{F} \otimes_{\mathcal{O}_{X}} \mathcal{M}_{X} \).

**Proof** Let \( \pi : \widetilde{X} \longrightarrow X \) denote the normalization of \( X \). By Lemma 6, it suffices to show that there exists a section \( \Psi \in \Gamma(\widetilde{X}, \mathcal{M}_{\widetilde{X}}) \) such that

\[
\pi^{*}(s_{1}) \otimes 1 = \pi^{*}(s_{2}) \otimes \Psi \quad \text{in} \quad \pi^{*}\mathcal{F} \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{M}_{\widetilde{X}},
\]

which we may check on each connected component of \( \widetilde{X} \) separately. Note that the hypotheses on \( s_{1} \) and \( s_{2} \) in the statement are satisfied on each such component by the pull-backs \( \pi^{*}(s_{1}) \) and \( \pi^{*}(s_{2}) \) since \( \pi \) is finite.

Thus we are reduced to proving the lemma in the case where the curve \( X \) is moreover normal and connected. In this case, the torsion-free coherent sheaf \( \mathcal{F}^{\vee\vee} \) is locally-free. By Lemma 8 applied to \( \bigwedge_{\mathcal{O}_{X}}^{2} \mathcal{F} \), the section \( s_{1} \wedge s_{2} \) maps to zero in

\[
\left( \bigwedge_{\mathcal{O}_{X}}^{2} \mathcal{F} \right) \otimes_{\mathcal{O}_{X}} \mathcal{M}_{X} \cong \bigwedge_{\mathcal{M}_{X}}^{2} (\mathcal{F} \otimes_{\mathcal{O}_{X}} \mathcal{M}_{X}).
\]
We will construct the meromorphic function $\Phi$ locally and glue. Let $\text{Sp}(A) \subseteq X$ be an admissible affinoid open. Then $A$ is a domain, say with field of fractions $K$. Choose a finite $A$-module $M$ with $\mathcal{F} \cong \tilde{M}$ on $\text{Sp}(A)$, so that

$$\Gamma(\text{Sp}(A), \mathcal{F} \otimes_{\mathcal{O}_X} M_X) \cong M \otimes_A K$$

by Lemma 5. The section $s_1 \otimes 1$ restricts to a nonzero element of this finite-dimensional $K$-vector space by Lemma 7. Since $(s_1 \otimes 1) \land (s_2 \otimes 1) = 0$ on $\text{Sp}(A)$, there exists $\psi \in K$ such that $s_1 \otimes 1 = s_2 \otimes \psi$. That these $\psi$ glue to a global section of $M_X$ follows from the non-vanishing of $s_2 \otimes 1$ on a nonempty admissible open intersection of two admissible opens. □

### 6 Some results of Waldspurger

Waldspurger [25, 26] recasts the Shimura lifting in terms of automorphic representations and uses this perspective to prove some powerful theorems relating the coefficients of half-integral weight modular forms to special values of the $L$-functions of their integral weight Shimura lifts. In this section, we recall a key construction and two of the main results of [26]. We will use the notation of [26] freely here, and we will take $N$ to be any positive integer for the remainder of this section.

Let $\chi$ be a Dirichlet character modulo $4N$ and let $f_0$ be a newform of weight $k - 1$ and nebentypus $\chi^2$. Following Waldspurger, we let $\lambda_\ell$ denote the eigenvalue of $T_\ell$ or $U_\ell$ on $f_0$, and we set

$$S_{k/2}(4N, \chi, f_0) = \{ F \in S_{k/2}(4N, \chi) \mid T_\ell^2 F = \lambda_\ell F \text{ for almost all } \ell \nmid 2N \}.$$ 

Here, $S_{k/2}(4N, \chi)$ is the space of cusp forms of level $4N$ and weight $k/2$ with nebentypus character $\chi$ modulo $4N$. Waldspurger [26] provides a recipe for building the space $S_{k/2}(4N, \chi, f_0)$ from local data associated to $f_0$ and $\chi$. The details of this recipe are far too long to reproduce here. Nonetheless, we provide an outline that will allow the motivated reader to compare the details of the arguments below to the construction in [26].

For each prime number $\ell$, Waldspurger defines a certain non-negative integer $\tilde{n}_\ell \leq \text{ord}_\ell(4N)$. Then, for each integer $e$ with $\tilde{n}_\ell \leq e \leq \text{ord}_\ell(4N)$, he defines a certain set of functions $U_\ell(e)$ that we may take to be defined on the set of positive integers. The nature of $\tilde{n}_\ell$ and the functions in $U_\ell(e)$ is dictated by local (at $\ell$) data associated to the integral weight newform $f_0$ and the half-integral weight nebentypus $\chi$.

For a positive integer $n$, let $n_{sf}$ denote the square-free part of $n$, and let $A$ be a function defined on the set of square-free positive integers. Now, for each prime $\ell$, choose an integer $e$ as above and an element $c_\ell \in U_\ell(e)$, and from these data we form the $q$-expansion

$$f(A, \{c_\ell\}) = \sum_{n=1}^{\infty} A(n_{sf}) \prod_v c_\ell(n) q^n,$$

where we take $c_\infty(n) = n^{(k-2)/4}$. The following is Théorème 1 of [26].

\[ Springer \]
Theorem 3 (Waldspurger) Let $k \geq 5$ be an odd positive integer and suppose that $f_0$ satisfies hypotheses (H1) and (H2). There exists a function $A$ defined on the set of square-free positive integers satisfying the following two conditions:

(a) $A(n)^2 = L(f_0 \otimes \chi_0^{-1} \chi_n, (k - 1)/2) \varepsilon(\chi_0^{-1} \chi_n, 1/2)$, and

(b) the collection of $f(A, \{c_v\})$ defined above spans the space $S_{k/2}(4N, \chi, f_0)$.

Remark 3 1. Let $\pi = \otimes \pi_v$ be the automorphic representation attached to $f_0$. Hypothesis (H1) is the assertion that, for all $\ell$ with $\pi_\ell$ irreducible principal series associated to the characters $\mu_1, \mu_2$ of $\mathbb{Q} \times \mathbb{F}_\ell$, we have $\mu_1(-1) = \mu_2(-1) = 1$. By a theorem of Flicker, this is equivalent to $f_0$ being in the image of the Shimura lifting at some level. We observe for later use that if the local central character $\mu_1 \mu_2$ is even and the conductor of $\pi_\ell$ divides $\ell$, then hypothesis (H1) is automatically satisfied since at most one of $\mu_1$ and $\mu_2$ can be ramified.

2. Hypothesis (H2) pertains to $\pi_2$. For our purposes, it suffices to note that (H2) is satisfied if $\pi_2$ is not supercuspidal.

3. While we have chosen to use the more classical normalization of the argument of the $L$-function, we have preserved Waldspurger’s normalization for the $\varepsilon$ factor above. This choice is irrelevant for our purposes, as all we use about this value is that it is non-vanishing.

Corollary 2 With notation as in Theorem 3, if $n$ is a square-free positive integer such that

$$L(f_0 \otimes \chi_0^{-1} \chi_n, (k - 1)/2) = 0$$

then $a_n \equiv 0$ on $S_{k/2}(4N, \chi, f_0)$.

The converse of Corollary 2 is false. However, Theorem 3 can be used to analyze the vanishing of $a_n$ in general. This analysis is carried out in the case $N$ is odd and square-free in Proposition 2 in order to characterize in Corollary 4 the degenerate irreducible components (see Definition 2) for a given coefficient section $a_n$.

Theorem 4 (Waldspurger) Let $n$ and $m$ be square-free positive integers such that $n/m \in (\mathbb{Q} \times \mathbb{F}_\ell)^2$ for all $\ell | 2N$. If $F \in S_{k/2}(4N, \chi, f_0)$ then

$$a_n(F)^2 L(f_0 \otimes \chi_0^{-1} \chi_m, (k - 1)/2) \chi(m/n) m^{k/2 - 1}$$

$$= a_m(F)^2 L(f_0 \otimes \chi_0^{-1} \chi_n, (k - 1)/2) n^{k/2 - 1}$$

Proof If $F = 0$ then the statement is trivial. Otherwise hypothesis (H1) is satisfied by Proposition 2 of [26], and this result is Corollaire 2 of [26] applied to $F$. □

7 Interpolation of square roots

In this section, we establish the basic interpolation result on the half-integral weight side. Here, we make no assumption on the tame level. The trade-off is that we will have little control over which irreducible components we may interpolate. In the next
section, we impose the square-free condition and determine these components, as well as move the interpolation to the integral weight eigencurve.

Fix a a primitive \( n \)th root of unity \( \zeta_{4Np} \in \mathbb{C}_p \) thought of as a point on the Tate curve \( \text{Tate}(q) \). As defined in Section 5 of [21], half-integral weight modular forms and families thereof have \( q \)-expansions associated to this point on \( \text{Tate}(q) \). For a fixed positive integer \( n \), the \( n \)th coefficient in this expansion gives an element of the dual of the Banach space of families of forms. It is a simple matter to verify that these elements glue to a section

\[ a_n \in \Gamma(\tilde{D}(4N), N^*) , \]

and that the image of this element in the fiber \( N^*(x) \) is the \( n \)th coefficient map on the eigenspace associated to \( x \) under the identification (10).

**Definition 2** We will say that an \( a_n \) is degenerate on an irreducible component \( C \) of \( \tilde{D}(4N)_{\text{red}} \) if \( a_n \) satisfies the equivalent conditions of Lemma 7 on \( C \).

**Proposition 1** Let \( N \) be a positive integer that is relatively prime to \( p \) and let \( n \) and \( m \) be square-free positive integers such that \( n/m \in (\mathbb{Q}_p^*)^2 \) for all \( \ell | 2Np \). Let \( C \) be an irreducible component of \( \tilde{D}(8N)_{\text{red}} \), and assume that \( C \) is generically principal series or Steinberg at the prime 2. Then, there exists a Zariski-dense subset \( S \) of \( C \) such that \( a_n(x) \) and \( a_m(x) \) are linearly dependent in the fiber \( N^*(x) \) for all \( x \in S \).

**Proof** The set of low-slope (and hence strictly classical) points is Zariski-dense in \( C \). By hypothesis, the same is then true of the set of low-slope points that are not supercuspidal at the prime 2. Let \( x \) be such a point of weight \( \kappa(t) = t^{(k-1)/2}\chi_{\kappa'}(t) \) with \( k \) odd and at least 5 and \( \chi' \) of conductor \( p^r \). Let \( \lambda_x \) be the corresponding system of eigenvalues and let \( S_x \) denote the eigenspace of forms for \( \lambda_x \). In particular \( S_x \subseteq S_{k/2}(8Np^r, \chi\kappa', f_0) \) where \( f_0 \) denotes the integral weight newform associated to \( x \).

If

\[ L \left( f_0 \otimes \chi_0^{-1}\chi_n^{-1}, (k - 1)/2 \right) = 0 \]

then \( a_n \equiv 0 \) on \( S_x \) by Corollary 2. Thus \( a_n(x) = 0 \), so \( a_n(x) \) and \( a_m(x) \) are trivially linearly dependent. On the other hand, suppose that

\[ L \left( f_0 \otimes \chi_0^{-1}\chi_n^{-1}, (k - 1)/2 \right) \neq 0 . \]

By Lemma 4, if \( F \in S_x \) and \( a_n(F) = 0 \), then \( a_m(F) = 0 \) as well. Thus the kernel of the linear functional \( a_n(x) \) is contained in that of \( a_m(x) \), and again we see that these two are linearly dependent.

**Corollary 3** With notation as in Proposition 1, suppose further that \( a_n \) is not degenerate on \( C \). Then there exists a meromorphic function \( \Phi_{m,n} \) on \( C \) with the property that \( a_m \otimes 1 = a_n \otimes \Phi_{m,n} \) holds in \( N^*_{\text{red}} \otimes \mathcal{M}_C \).

**Proof** This follows from the Proposition 1 and Lemma 9. \( \square \)
By §1.2 of [8], the singular (equivalently, non-normal) locus in the reduced component $C$ is a proper analytic set.

**Theorem 5** With notation as in Proposition 1, suppose that $x \in C$ is a point of weight $\kappa(t) = t^{(k-1)/2\kappa'}(t)$, with $k \geq 5$ and $\kappa'$ torsion, satisfying

- $x$ is strictly classical,
- $x$ is not supercuspidal at 2,
- $C$ is smooth at $x$,
- $N^*_{\text{red}}$ is torsion-free on $C$ at $x$, and
- $a_n(x) \neq 0$.

Then $\Phi_{m,n}$ is regular at $x$ and we have

$$\Phi_{m,n}(x)^2 = \frac{L(f_0 \otimes \chi_0^{-1} \kappa^{-1} \chi_m, (k-1)/2 \chi_{m}, (m/n)(m/n)(m/n)^{-1/2}}{L(f_0 \otimes \chi_0^{-1} \kappa^{-1} \chi_n, (k-1)/2 \chi_{n}, (m/n)(m/n)(m/n)^{-1/2}}.$$ 

**Remark 4** The smoothness assumption is likely known to be superfluous at classical points, but the author knows of no reference in which this is completely proven for the eigencurves used here.

**Proof** Let $x$ be as in the statement and choose a normal admissible open affinoid $\text{Sp}(A) \subseteq C$ containing $x$. Choose a finite $A$-module $M$ with $N^*_{\text{red}} \cong \tilde{M}$ on $\text{Sp}(A)$ and let $K$ denote the fraction field of $A$. Restricting to $\text{Sp}(A)$, we may regard $a_n$ and $a_m$ as elements of $M$, and $\Phi_{m,n}$ as an element of $K$. In particular, we have

$$a_m \otimes 1 = a_n \otimes \Phi_{m,n}$$

in $M \otimes_A K$.

Write $\Phi_{m,n} = f/g$ for $f, g \in A$ and clear denominators to get

$$(ga_m - f a_n) \otimes 1 = 0.$$ 

Let $m$ denote the maximal ideal of $A$ corresponding to the point $x$. By faithful flatness of completion, $M$ is torsion-free at $x$, so the equality $ga_m = f a_n$ actually holds in the localization $M_m$. By normality, $A_m$ is a DVR, so we may write $f = \pi^\alpha u$ and $g = \pi^\beta v$ where $\pi$ is a uniformizer for $A_m$ and $u$ and $v$ are units in this local ring. If $\beta > \alpha$, then we have

$$a_n = vu^{-1} \pi^{\beta-\alpha} a_m \in \pi M_m,$$

contrary to the hypothesis that $a_n(x) \neq 0$. Thus $\beta \leq \alpha$, and $\Phi_{m,n}$ is regular at $x$.

Since $a_n(x) \neq 0$, there exists a nonzero classical form $F$ in the eigenspace corresponding to $x$ with the property that $a_n(F) \neq 0$. By the previous paragraph, we may write

$$a_m(F) = a_n(F) \Phi_{m,n}(x).$$
By Corollary 2 we have \( L(f_0 \otimes \chi_0^{-1} \chi_n, (k - 1)/2) \neq 0 \), and Theorem 4 gives

\[
\Phi_{m,n}(x)^2 = \frac{a_m(F)^2}{a_n(F)^2} = \frac{L(f_0 \otimes \chi_0^{-1} \chi_m, (k - 1)/2)}{L(f_0 \otimes \chi_0^{-1} \chi_n, (k - 1)/2)} \chi(m/n) \kappa'(m/n)(m/n)^{(k-2)/2} \\
= \frac{L(f_0 \otimes \chi_0^{-1} \chi_m, (k - 1)/2)}{L(f_0 \otimes \chi_0^{-1} \chi_n, (k - 1)/2)} \chi(m/n) \kappa(m/n)(m/n)^{-1/2}.
\]

\( \square \)

8 Characterization of the degenerate components for square-free conductor

Let \( N \) be a square-free positive integer that is relatively prime to \( 2p \). In this section we determine necessary and sufficient conditions under which \( a_n(x) = 0 \) for strictly classical points \( x \in \widetilde{D}(8N) \) whose Shimura lifting is not supercuspidal at the prime 2. We then use these conditions to characterize the components on which \( a_n \) is degenerate. Suppose that \( x \) is a strictly classical point of weight \( \kappa(t) = t(k-1)/2 \kappa'(t) \) with \( k \geq 5 \) and \( \kappa' \) of conductor \( p^\ell \) and tame nebentypus \( \chi \) modulo \( 8N \). Let \( f_0 \) denote the newform associated to this point and observe that the eigenspace \( S_x \) corresponding to \( x \) satisfies

\[
S_x \subseteq S_{k/2}(8Np^\ell, \chi \kappa', f_0)
\]

since \( x \) is strictly classical.

**Proposition 2** With notation as above, let \( \pi = \bigotimes_v \pi_v \) denote the automorphic representation generated by the integral weight newform \( f_0 \) associated to \( x \), and suppose that \( \pi_2 \) has conductor at most 2. The fiber \( N^*(x) \) has dimension

\[
\dim N^*(x) = \prod_{\ell \mid 2N} d_\ell
\]

where

\[
d_\ell = \begin{cases} 
1 + v_2(\ell) & \text{\( \pi_\ell \) ramified} \\
2 + v_2(\ell) & \text{\( \pi_\ell \) unramified}
\end{cases}
\]

Furthermore, \( a_n(x) = 0 \) if and only if one of the following holds at \( x \):

(i) \( L(f_0 \otimes \chi_0^{-1} \chi_n, (k - 1)/2) = 0 \)

(ii) for some \( \ell \mid 2N \), \( \pi_\ell \) is Steinberg with

\[
\lambda_\ell = (\ell, n) \ell \chi_0^{(\sim, \ell)}(\ell) \kappa(\ell)
\]
and

- if \( \ell = 2 \), then \((n, -1)_2 = \chi_0^{(2)}(-1) \) and either \( 2 \nmid n \) and \( \chi_0 \equiv 1 \) on \( 1 + 4\mathbb{Z}_2 \) or \( 2 \nmid n \) and \( \chi_0 \equiv 1 \) on \( 1 + 4\mathbb{Z}_2 \).

- if \( \ell \nmid N \), then either \( \ell \nmid n \) and \( \chi \) is unramified at \( \ell \) or \( \ell \mid n \) and \( \chi \) is ramified at \( \ell \).

(iii) \( \pi_p \) is Steinberg with

\[
\frac{\lambda_p}{\pi_p} = \lambda_x(U_{p^2}) = (p, n)_p \chi_0(p)p^{(k-3)/2}
\]

and either \( p \mid n \) and \( \kappa \) is unramified at \( p \) (which is to say \( \kappa = 1 \)) or \( p \mid n \) and \( \kappa' \) is ramified at \( p \).

**Proof** Let \( \kappa(t) = t^{(k-1)/2}\kappa'(t) \) denote the weight of \( x \). In addition to using the notation in the preceding paragraphs, we borrow the following notation from [26] for the duration of the proof: for a prime number \( \ell \), \( \lambda_\ell \) will continue to denote the Hecke eigenvalue of \( f_0 \) for the appropriate operator (either \( T_\ell \) or \( U_\ell \)) and we let \( \lambda_\ell = \ell^{1-k/2}\lambda_\ell \).

If \( \ell \mid 2Np \), we denote by \( \alpha_\ell \) and \( \alpha'_\ell \) the roots of the polynomial

\[
X^2 - \lambda_\ell X + (\chi(\ell)\kappa'(\ell))^2.
\]

For convenience, we also introduce the notation \( \alpha_\ell = \ell^{k/2-1}\alpha_\ell \).

The following table, compiled from Section VIII of [26], lists the elements that arise in the sets \( U_\ell(e) \) from the construction outlined in Sect. 6 for the local conditions that can arise with our restrictions on the tame level. The notation is that of Waldspurger, who gives explicit formulas for all of the functions below. We have not listed the value of \( \bar{n}_\ell \) and the particular integers \( e \) that give rise to these elements, as these will be of no use to us here.

| \( \ell \) | Local condition | \( c_\ell \in U_\ell(e) \) |
|-----|-----------------|-----------------------------|
| \( \ell \mid 2Np \) | None | \( c_\ell^0[\lambda] \) |
| \( \ell = 2 \) | \( \pi_2 \) Unramified, \( \chi_0 \equiv 1 \) on \( 1 + 4\mathbb{Z}_2 \), \( \alpha_2 \neq \alpha'_2 \) | \( c_\ell^0[\lambda_2] \) |
| \( \pi_2 \) Unramified, \( \chi_0 \equiv 1 \) on \( 1 + 4\mathbb{Z}_2 \), \( \alpha_2 = \alpha'_2 \) | \( c_\ell^0[\lambda_2] \) |
| \( \pi_2 \) Unramified, \( \chi_0 \equiv 1 \) on \( 1 + 4\mathbb{Z}_2 \), \( \alpha_2 \neq \alpha'_2 \) | \( c_\ell^0[\lambda_2] \) |
| \( \pi_2 \) Unramified, \( \chi_0 \equiv 1 \) on \( 1 + 4\mathbb{Z}_2 \), \( \alpha_2 = \alpha'_2 \) | \( c_\ell^0[\lambda_2] \) |
| \( \pi_2 \) Ramified principal series | \( c_\ell^0[\lambda_2] \) |
| \( \pi_2 \) Steinberg, \( \chi_0 \equiv 1 \) on \( 1 + 4\mathbb{Z}_2 \) | \( c_\ell^0[\lambda_2] \) |
| \( \pi_2 \) Steinberg, \( \chi_0 \equiv 1 \) on \( 1 + 4\mathbb{Z}_2 \) | \( c_\ell^0[\lambda_2] \) |
| \( \ell \mid Np \) | \( \pi_\ell \) Unramified, \( \chi\kappa' \) Unramified | \( c_\ell^0[\lambda_\ell] \) |
| \( \pi_\ell \) Unramified, \( \chi\kappa' \) Ramified, \( \alpha_\ell \neq \alpha'_\ell \) | \( c_\ell^0[\lambda_\ell] \) |
| \( \pi_\ell \) Unramified, \( \chi\kappa' \) Ramified, \( \alpha_\ell = \alpha'_\ell \) | \( c_\ell^0[\lambda_\ell] \) |
| \( \pi_\ell \) Ramified principal series | \( c_\ell^0[\lambda_\ell] \) |
| \( \pi_\ell \) Steinberg, \( \chi\kappa' \) Unramified | \( c_\ell^0[\lambda_\ell] \) |
| \( \pi_\ell \) Steinberg, \( \chi\kappa' \) Ramified | \( c_\ell^0[\lambda_\ell] \) |
As outlined in Sect. 6, Waldspurger uses these local functions to construct the space \( S_{k/2}(8Np^r, \chi \kappa', f_0) \). From the explicit formulas for the \( c_\ell \) in [26], the effect of the appropriate Hecke operator at \( \ell \) on the form \( f(A, \{c_v\}) \) can be read off from the element \( c_\ell \). In our setting, this operator is \( T_{\ell} \) only in the case \( c_\ell = c_\ell^0[\alpha_\ell] \), and a form constructed using this element is a \( T_{\ell} \)-eigenform with eigenvalue \( \lambda_\ell \). In all other cases, the relevant Hecke operator is \( U_{\ell} \). With the exception of the functions \( c_\ell^0[\sqrt{\lambda_\ell}], \) \( c_\ell'[\alpha_\ell] \), and \( c_\ell[\alpha_\ell] \), a function constructed from an element in the above table is a \( U_{\ell} \)-eigenform whose eigenvalue is the argument of \( c_\ell \) in brackets times \( \ell^{k/2-1} \) (i.e. the “underlined version” of this argument). In the case \( c_\ell^0[\sqrt{\lambda_\ell}] \), the same is true except the eigenvalue is \( \lambda_\ell \). If \( c_\ell = c_\ell''[\alpha_\ell] \) then we have

\[
U_{\ell} f(A, \{c_v\}) = \alpha_\ell f(A, \{c_v\}) + \alpha_\ell f(A, \{b_v\})
\]

where

\[
b_v = \begin{cases} 
c_v & v \neq \ell 
c_\ell'[\alpha_\ell] & v = \ell 
\end{cases}
\]

The behavior in the \( c_\ell'' \) case is the same as this case with all the primes switched to the left side.

Since \( c_\ell = c_\ell^0[\lambda_\ell] \) for \( \ell \notdivides 4Np \), forms in \( S_{k/2}(8Np^r, \chi \kappa', f_0) \) satisfy \( T_{\ell} F = \lambda_\ell F \) for all \( \ell \notdivides 4Np \). Thus, the subspace \( S_\chi \) is obtained by further requiring only that the \( F \in S_{k/2}(8Np^r, \chi \kappa', f_0) \) also satisfy \( U_{\ell} F = \lambda_\ell(U_{\ell}^2) F \). In the ramified cases where there is just one \( c_p \) in the above table, we have \( \lambda_\chi(U_{\ell}^2) = \lambda_p \), and this condition is automatic. In the unramified cases where there are two functions, this condition picks out exactly one of the functions, namely, \( c_p'[\alpha_p] \) or \( c_p[\alpha_p] \) depending on whether \( \kappa' \) is unramified or ramified, respectively. Here, in case \( \alpha_p \neq \alpha_p' \), the choice of \( \alpha_p \) is dictated by the value of \( \lambda_\chi(U_{\ell}^2) \).

Exploiting these properties of the Hecke operators acting on the forms \( f(A, \{c_v\}) \), it is a simple exercise to show that, for a fixed \( f_0 \) in our setting, the collection of forms \( f(A, \{c_v\}) \) constructed from the elements in the above table are linearly dependent if an only if one of them vanishes identically. That the latter does not occur follows from well-known results about non-vanishing of quadratic twists of central modular \( L \)-values (see, for example, [3]). This linear independence establishes the dimension claim of the proposition.

By Theorem 3, for a square-free positive integer \( n \), we have \( a_n(x) = 0 \) if and only if the \( L \)-value in Theorem 3 vanishes or, for some \( \ell \), we have \( c_\ell(n) = 0 \) for all local functions appearing in the above table at \( \ell | 2Np \) that occur on the subspace \( S_\chi \). Again examining the explicit formulas in [26] for the \( c_\ell \) listed in this table (and using the fact that \( \lambda_\ell \neq 0 \) for \( \ell \) dividing the square-free conductor of \( f_0 \) one concludes that, for \( \ell | 2Np \), all \( c_\ell(n) \) occurring in \( S_\chi \) vanish if and only if \( \pi_\ell \) is Steinberg with

\[
\lambda_\ell = (\ell, n)_\ell(\chi_0^{\kappa'})^\sim(\ell)\ell^{(\ell-3)/2} = \begin{cases} 
(\ell, n)_\ell\chi_0^{(\ell)}(\ell)\ell^{-1}\kappa(\ell) & \ell \neq p 
(p, n)_p\chi_0(p)p^{(k-3)/2} & \ell = p 
\end{cases}
\]
and
- if \( \ell = 2 \), then we have \((n, -1) \equiv 2 = \chi_0^{(2)}(-1) \) and either \( 2 \nmid n \) and \( \chi_0 \equiv 1 \) on \( 1 + 4\mathbb{Z}_2 \)
or \( 2 \mid n \) and \( \chi_0 \not\equiv 1 \) on \( 1 + 4\mathbb{Z}_2 \)
- if \( \ell \mid Np \) then either \( \ell \nmid n \) and \( \chi \kappa' \) is unramified at \( \ell \) or \( \ell \mid n \) and \( \chi \kappa' \) is ramified at \( \ell \).

**Corollary 4** Let \( C \) be an irreducible component of \( \tilde{D}(8N)_{\text{red}} \) and suppose that \( C \) maps to a component of \( D(2N)_{\text{red}} \) under the Shimura lifting. Then the section \( a_n \) is degenerate on \( C \) if and only if one of the following holds:
(a) \( C \) contains a Zariski-dense set of classical points \( x \) at which
\[
L(f_0 \otimes \chi_0^{-1} \kappa'-1 \chi_n, (k - 1)/2) = 0
\]
(b) for some \( \ell \mid 2N \), \( C \) is Steinberg at \( \ell \) of sign \((\ell, n)_{\ell}\) and one of the following holds
- \( \ell = 2 \), \( (n, -1) \equiv 2 = \chi_0^{(2)}(-1), 2 \nmid n, \) and \( \chi_0 \equiv 1 \) on \( 1 + 4\mathbb{Z}_2 \)
- \( \ell = 2 \), \( (n, -1) \equiv 2 = \chi_0^{(2)}(-1), 2 \mid n, \) and \( \chi_0 \not\equiv 1 \) on \( 1 + 4\mathbb{Z}_2 \)
- \( \ell \mid N, \ell \nmid n, \) and \( \chi \) is unramified at \( \ell \)
- \( \ell \mid N, \ell \mid n, \) and \( \chi \) is ramified at \( \ell \).

**Proof** The “if” portion is clear from Proposition 2. Suppose that \( a_n \) is degenerate on \( C \) and condition (b) in the statement does not hold. The collection of points that either satisfy condition \((ii)\) of Proposition 2 or at which \( \pi_\rho \) is Steinberg lies in a proper analytic set in \( C \). By the results of Section 4 of [22], the collection of low-slope (hence strictly classical) points is Zariski-dense in the complement of this proper analytic set in \( C \). By Proposition 2, we must have \( L(f_0 \otimes \chi_0^{-1} \kappa'-1 \chi_n, (k - 1)/2) = 0 \) at these points.

**Corollary 5** Suppose that \( x \in \tilde{D}(8N)_{\text{red}} \) is a strictly classical point lying on an irreducible component \( C \) that maps to a component of \( D(2N)_{\text{red}} \) under the Shimura lifting. The sheaf \( N^*_{\text{red}} \) is torsion-free on \( C \) at \( x \).

**Proof** Let \( x \) be as in the statement. The local ring of \( C \) at \( x \) is a domain, so it suffices to show that the fiber rank and generic rank of \( N^* \) coincide at \( x \). By hypothesis, \( C \) is either generically irreducible principal series or generically Steinberg at each \( \ell \mid 2N \). As we have observed, if \( C \) is generically ramified principal series or Steinberg then it is entirely so, while if it is generically unramified principal series, then \( C \) may contain a proper analytic set of Steinberg points.

By Proposition 2, the generic rank of \( N^* \) on \( C \) is \( 2^r \cdot s \) where \( r \) is the number of \( \ell \mid N \) at which \( C \) is generically unramified principal series and \( s = 2 \) or 3 according to whether \( C \) is generically ramified (i.e. ramified principal series or Steinberg) or generically unramified principal series at the prime 2, respectively. But by Proposition 2 and the previous paragraph (which in effect says that “ramified does not degenerate to unramified”), the dimension of the fiber does not increase at a strictly classical point. By upper-semicontinuity of fiber dimension, this dimension cannot decrease from the generic rank either. Thus the generic rank and fiber rank coincide at smooth strictly classical points of \( C \).
In the following Theorem, we move the interpolation to the integral weight eigencurve via the Shimura lifting. In so doing, we may effectively eliminate components that are degenerate for reasons other than the vanishing of $L$-values by choosing a component on the half-integral weight side with some care.

**Theorem 6** Let $N$ be a square-free positive integer that is relatively prime to $2p$ and let $n$ and $m$ be square-free positive integers with $m/n \in (\mathbb{Q}_\ell^\times)^2$ for all $\ell\mid 2Np$. Fix a square Dirichlet character $\psi$ modulo $2N$. There exists a character $\chi$ modulo $8N$ such that $\chi^2 = \psi$ with the following property: for any irreducible component $C$ of $D(2N)_\text{red}$ of even weight and tame nebentypus $\psi$ on which $L(f_0 \otimes \chi^{-1}\kappa^{-1} \chi_n, (k - 1)/2)$ does not vanish generically, there exists a meromorphic function $\Phi_{m,n}$ on $C$ such that

$$\Phi_{m,n}(x)^2 = \frac{L(f_0 \otimes \chi^{-1}\kappa^{-1} \chi_m, (k - 1)/2)}{L(f_0 \otimes \chi^{-1}\kappa^{-1} \chi_n, (k - 1)/2)} \chi(m/n)^{k(m/n)(m/n)-1/2}$$

holds for strictly classical points $x \in C$ outside of the discrete set at which $C$ is singular, condition (iii) of Proposition 2 is satisfied, or the denominator vanishes.

**Proof** Let $\ell\mid 2N$ be a prime at which $\psi$ is unramified. Utilizing quadratic characters of prime conductor, we may choose $\chi$ to satisfy the following conditions at each such $\ell$:

- for $\ell\neq 2$, $\chi$ is unramified at $\ell$ if and only if $\ell\mid n$
- at $\ell = 2$, $\chi_0^{(2)} \equiv 1$ on $1 + 4\mathbb{Z}_2$ if an only if $2\mid n$

We remark that it is in satisfying the second of these conditions for odd $n$ that necessitates working at level $8N$ instead of $4N$.

Let $C$ be as in the statement and let $x_0 \in C$ be a classical point of weight character $t^{k-1}\kappa'(t)^2$ where $\kappa'$ is a character modulo $p^r$. The even hypothesis combined with finite-slope and conductor considerations ensure that hypothesis (H1) is satisfied [see Remark 3(1)]. Suppose that $x_0$ lies on at most one component of the image of $\tilde{D}(8N)$ in $D(4N)$ under the Shimura lifting. Since $\pi_2$ is not supercuspidal at $x_0$, hypothesis (H2) is satisfied, and the construction of [26] outlined in Sect. 6 implies that $S_{k/2}(8Np^r, \chi\kappa', f_0(x_0)) \neq 0$ and that this space moreover contains a nonzero eigenform for $U_{p^2}$ with eigenvalue $\lambda_{x_0}(U_p)$. This form gives rise to a point on $\tilde{x}_0 \in \tilde{D}(8N)$ that maps to $x_0$ under the Shimura lifting. Let $\widetilde{C}$ denote the unique irreducible component of $\tilde{D}(8N)_{\text{red}}$ containing this point (so that, in particular, $\widetilde{C}$ maps isomorphically to $C$ under the Shimura lifting). Since the tame nebentypus is constant on $\widetilde{C}$, condition (b) of Corollary 4 does not hold on $\widetilde{C}$ by our choice of $\chi$. By hypothesis, condition (a) does not hold either, and we conclude that $a_n$ is non-degenerate on $\widetilde{C}$.

Let $\Phi_{m,n}$ be the meromorphic function on $\widetilde{C}$ given by Corollary 3. Let $x \in C$ be a smooth point of low slope. Then, $x$ is strictly classical regarded as a point on $\widetilde{C}$ under the Shimura lifting. By Corollary 5, $N^*_{\text{red}}(x)$ is torsion-free at $x$. If neither condition (i) nor condition (iii) of Proposition 2 hold at $x$, then $a_n(x) \neq 0$ and Theorem 5 implies that $\Phi_{m,n}$ is regular at $x$ and its square has the claimed value. \hfill \Box
This finishes the proof of Theorem 1 in the case of even tame level. If \( N \) is odd, the result follows simply by pulling back through the natural map \( D(N) \rightarrow D(2N) \).

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**References**

1. Bellaïche, J.: Critical \( p \)-adic \( L \)-functions. Invent. Math. **189**(1), 1–60 (2012)
2. Bosch, S., Güntzer, U., Remmert, R.: Non-Archimedean analysis, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. In: A systematic approach to rigid analytic geometry, vol. 261, Springer, Berlin (1984)
3. Buzzard, K.: Eigenvarieties. \( L \)-functions and Galois representations. In: London Math. Soc. Lecture Note Series, vol. 320, pp. 59–120. Cambridge University Press, Cambridge (2007)
4. Coleman, R., Mazur, B.: The eigencurve. In: Galois representations in arithmetic algebraic geometry (Durham 1996), London Math. Soc. Lecture Note Ser., vol. 254, pp. 1–113. Cambridge University Press, Cambridge (1998)
5. Conrad, B.: Moishezon spaced in rigid geometry (2013)
6. Harris, M., Tilouine, J.: The half-integral weight eigencurve. In: Galois representations in arithmetic algebraic geometry (Durham 1996), London Math. Soc. Lecture Note Ser., vol. 254, pp. 1–113. Cambridge University Press, Cambridge (1998)
7. Hida, H.: A congruence between modular forms of half-integral weight. Math. Ann. **274**(2), 621–639 (1986)
8. Hida, H.: \( p \)-adic families of Eisenstein series and congruences of cusp forms. In: Proc. Internat. Congress Math. (Vol. 2, Vancouver, B.C., 1986), pp. 555–561. AMS, Providence (1987)
9. Hida, H.: \( p \)-adic measures and square roots of special values of triple product \( L \)-functions. Invent. Math. **114**(2), 407–447 (1993)
10. Hida, H.: On standard \( p \)-adic \( L \)-functions of families of elliptic cusp forms. In: \( p \)-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991). Contemp. Math., vol. 165, pp. 81–110. Amer. Math. Soc., Providence (1994)
11. Koblitz, N.: \( p \)-Adic congruences and modular forms of half integral weight. Math. Ann. **274**(2), 199–220 (1986)
12. Maeda, Y.: A congruence between modular forms of half-integral weight. Hokkaido Math. J. **12**(1), 64–73 (1983)
13. Panchishkin, A.A.: Two variable \( p \)-adic \( L \) functions attached to eigenfamilies of positive slope. Invent. Math. **154**(3), 551–615 (2003)
14. Paulin, A.G.M.: Local to global compatibility on the eigencurve. Proc. Lond. Math. Soc. (3) **103**(3), 405–440 (2011)
15. Ramsey, N.: The overconvergent Shimura lifting. Int. Math. Res. Not. IMRN (2), 193–220 (2009)
16. Serre, J.-P.: Formes modulaires et fonctions zêta \( p \)-adiques. In: Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972). Lecture Notes in Math, vol. 350, pp. 191–268. Springer, Berlin (1973)
24. Sofer, A.: $p$-adic interpolation of square roots of central values of Hecke $L$-functions. Duke Math. J. 83(1), 51–78 (1996)
25. Waldspurger, J.-L.: Correspondance de Shimura. J. Math. Pures Appl. (9) 59(1), 1–132 (1980)
26. Waldspurger, J.-L.: Sur les coefficients de Fourier des formes modulaires de poids demi-entier. J. Math. Pures Appl. (9) 60(4), 375–484 (1981)