On a structure of the one-loop divergences in 4D harmonic superspace sigma-model

I. L. Buchbinder\textsuperscript{1,2,3,a}, A. S. Budekhina\textsuperscript{1,2,b}, B. S. Merzlikin\textsuperscript{3,1,c}

\textsuperscript{1} Center for Theoretical Physics, Tomsk State Pedagogical University, 634061 Tomsk, Russia
\textsuperscript{2} National Research Tomsk State University, 634050 Tomsk, Russia
\textsuperscript{3} Tomsk State University of Control Systems and Radioelectronics, 634050 Tomsk, Russia

Received: 3 November 2021 / Accepted: 3 January 2022 / Published online: 10 January 2022
© The Author(s) 2022

Abstract We study the quantum structure of four-dimensional $\mathcal{N} = 2$ superfield sigma-model formulated in harmonic superspace in terms of the omega-hypermultiplet superfield $\omega$. The model is described by harmonic superfield sigma-model metric $g_{ab}(\omega)$ and two potential-like superfields $L_{++}^a(\omega)$ and $L^{(+4)}(\omega)$. In the bosonic component sector this model describes some hyper-Kähler manifold. The manifestly $\mathcal{N} = 2$ supersymmetric covariant background-quantum splitting is constructed and the superfield proper-time technique is developed to calculate the one-loop effective action. The one-loop divergences of the superfield effective action are found for arbitrary $g_{ab}(\omega)$, $L_{++}^a(\omega)$, $L^{(+4)}(\omega)$, where some specific analogy between the algebra of covariant derivatives in the sigma-model and the corresponding algebra in the $\mathcal{N} = 2$ SYM theory is used. The component structure of divergences in the bosonic sector is discussed.

1 Introduction

Nonlinear sigma-models have been studied extensively over the years and take a special place in many areas of the field theory. One attractive class of such models is given by the two-dimensional conformal field theories, which are well known to be exactly solvable in the sense that S-matrix, the correlation functions and the anomalous dimensions are completely determined by the conformal invariance [1]. Such models have non-trivial UV behaviour being finite in the case of Ricci-flat target spaces [2,3]. Besides, in the worldsheet field theory, the correlations between conformal invariance, (extended) supersymmetry and geometry of the complex manifolds in the full quantum theory lead to restrictions on the background fields geometry in every order of the perturbation theory [4–7].

Supersymmetric two-dimensional nonlinear sigma-models caused the great interest due to their remarkable properties and inspired the construction of the geometrical non-polynomial theories of the supersymmetric matter in four-dimensional space-time (see, e.g. [8–11]). The formulation of supersymmetric nonlinear sigma-models has been worked out both for the simple and extended supersymmetries (see, e.g., [12–14] for review). The examination of these theories led to the discovery of fascinating relations between extended supersymmetry and complex manifolds. It was shown that the supersymmetric sigma-models in four dimensions possess the Kähler manifolds as a target [15–18] in case of $\mathcal{N} = 1$ supersymmetry and the hyper-Kähler manifolds for the $\mathcal{N} = 2$ case [4,17,19,20] (see also [21] for a review). A number of supersymmetric sigma-models and their chiral truncations, constructed within ten-dimensional superstring theory [22], where the extra dimensions are wrapped up into a coset space, has been of a certain interest. Also, note the construction of the one-dimensional $\mathcal{N} = 4$ sigma-model in harmonic superspace [23].

The present paper studies the quantum aspects of the four-dimensional $\mathcal{N} = 2$ supersymmetric sigma-model in harmonic superspace [14]. We introduce the harmonic superfield sigma-model given in terms of analytic superfield $\omega$-hypermultiplet. As well known, there are two types of hypermultiplets in harmonic superspace, the $q$-hypermultiplet and $\omega$-hypermultiplet (see, e.g. [14]). Formulated in terms of unconstrained $\mathcal{N} = 2$ superfields, such a model, as well as the general $q$-hypermultiplet theory (see, e.g. [14]), has two manifest symmetries: reparametrization invariance in the harmonic superspace and $\mathcal{N} = 2$ supersymmetry. However, use of $\omega$-hypermultiplet leads to certain simplifications in constructing the quantum effective action, since $\omega$-hypermultiplet is an uncharged superfield. Here we would...
like to make a comment on terminology. According to [19] (see also [14]) the 4D hyper-Kähler sigma-model is associated with general $q$-hypermultiplet theory in harmonic superspace. Such a general theory is formulated in terms of functions $L^+_{±}(q^+, u)$ and $L^+_{¬}(q^+, u)$ of hypermultiplet analytic superfields $q^+\alpha$ and harmonics $u$. The theory is manifestly invariant under the $\mathcal{N} = 2$ supersymmetry and arbitrary reparameterizations of the hypermultiplet superfield. After going to components and eliminating the auxiliary fields, we get in bosonic sector an action of the 4D hyper-Kähler sigma-model. In our paper we introduce a model which is the superfield sigma-model with harmonic superfield target space and harmonic superspace metric. Such a model is of course related of general $q$-hypermultiplet theory. To avoid ambiguities, we call the considered theory the harmonic superfield sigma-model. This theory is interesting by itself, since it is internally consistent and similar in many aspects of the conventional sigma model due to the presence superfield metrics in harmonic superspace. As well as the general $q$-hypermultiplet theory, the harmonic superspace sigma-model is manifestly invariant under the $\mathcal{N} = 2$ supersymmetry and arbitrary reparameterizations of the hypermultiplet superfield. The classical aspects of $\mathcal{N} = 2$ hypermultiplet models in various dimensions are widely studied (see, e.g., [14,24,25] for a review). The quantum structure of harmonic superspace sigma-models has never been studied in detail. The aim of this paper is to fill this gap.

The main object of our work is the quantum effective action of the harmonic superfield sigma-model. Since the classical model under consideration possesses two manifest symmetries, it is natural to develop such a scheme for constructing the effective action that preserves the same manifest symmetries. As a result, we are faced with the problem of a manifest covariant formulation of the effective action and the problem of its manifest covariant computation. The solution to the first problem is realized within the harmonic superspace background field method based on superfield background-quantum splitting that generalizes the known background-quantum splitting procedure in the conventional sigma-models (see, e.g., [4,6,7,26–29]). The second problem is solved with the help of superfield proper-time technique, which is a powerful tool for manifest covariant analysis of the effective actions in the supersymmetric field theories (see the applications of this technique in the various superfield models e.g., in [13,30–38]). As we will see, just use of the $ω$-hypermultiplet allows to use simply enough the superfield proper-time technique. Note also that the harmonic superspace approach is the only manifestly $\mathcal{N} = 2$ supersymmetric formalism that may preserve the explicit off-shell supersymmetry on all steps of quantum computations (see, e.g., [12,39]).

The divergences of the effective action in four-dimensional $\mathcal{N} = 1$ supersymmetric sigma-models are studied in [29] in the case of vanishing (anti-)chiral potentials and in [39] in the general case. Some assumptions about a structure of the possible one-loop divergences in $\mathcal{N} = 2$ sigma-models on the base of $\mathcal{N} = 1$ divergences were considered in [29]. In the present paper, we calculate the one-loop divergent contributions to the effective action of $\mathcal{N} = 2$ supersymmetric sigma-model in manifestly covariant and $\mathcal{N} = 2$ supersymmetric manner, which as far as we know, was not held directly in terms of $\mathcal{N} = 2$ superfields. It is worth pointing out that the calculation of divergences in $\mathcal{N} = 2$ supersymmetric sigma-models has a common difficulty: the absence of analytic normal coordinates on the generic Kähler manifolds does not allow to develop a higher-loop expansion preserving all symmetries of the theory. This fact has already been mentioned in the pioneering papers [2,3]. Some papers were directly addressed to the treatment of the above difficulty [40–42]. However, this problem is unessential for one-loop calculations because the above difficulty arises only in the higher loops.

The paper is organized as follows. In Sect. 2, we discuss the basic properties of $\mathcal{N} = 2$ harmonic superspace and formulate the harmonic superfield sigma-model. In Sect. 3, we develop the covariant background field method to study the one-loop effective action of the model. For this aims, we consider the covariant harmonic and spinor derivatives and study their algebra. Section 4 is devoted to the calculation of the one-loop divergent contributions of the effective action. Section 5 includes the discussion of the main obtained results and the directions of further studies. In the Appendix we discuss how the harmonic superfield sigma-model is connected to general $q$-hypermultiplet theory.

2 Harmonic superfield sigma-model

Throughout the paper, we use the notations and conventions from [14]. Namely, we denote by

$$\begin{align*}
(z, u) = (x^M, \theta^\alpha_i, \bar{\theta}_i^\beta, u^±i), & \quad M = 0, \ldots, 3, \quad \alpha = 1, 2 \quad i = 1, 2, \quad (1)
\end{align*}$$

the central basis coordinates of the $\mathcal{N} = 2$ harmonic superspace. The additional harmonic variables $u^±i$ correspond to the coset of the R-symmetry group $SU(2)/U(1)$ of the $\mathcal{N} = 2$ Poincare superalgebra in four dimensions. The analytic harmonic superspace involves the coordinates
\[(\zeta, u) = (x^{M}_{A}, \theta^{+}, \bar{\theta}^{+}, u^{\pm i}), \]
\[x^{M}_{A} = x^{M} - 2i\theta^{a}(\sigma^{M})_{a}^{} \bar{\theta}^{i} u^{i}, \quad \theta^{+\sigma} = u^{i} \theta^{a i}. \quad (2)\]

The analytic harmonic superspace involves only half of the original Grassmann coordinates and is closed under the \(N = 2\) supersymmetry superspace [14].

Let us introduce the spinor and harmonic derivatives in the analytic basis
\[D^{+}_{\theta^{+}} = \frac{\partial}{\partial \theta^{+}}, \quad D_{\theta^{+}} = \frac{\partial}{\partial \theta^{-}}, \quad (3)\]
\[D_{\theta^{-}} = -\frac{\partial}{\partial \theta^{+}} + 2i\theta^{a} \partial_{a}, \quad (4)\]

where \(\partial_{a} = (\sigma^{M})_{a}^{} \partial_{M}\). The spinor derivatives (3) and (4) together with harmonic derivatives
\[D^{\pm} = u^{i} \frac{\partial}{\partial u^{i}} - 2i\theta^{a} \bar{\theta}^{+} \partial_{a}, \quad (5)\]
\[D^{0} = u^{i} \frac{\partial}{\partial u^{i}} - u^{-i} \frac{\partial}{\partial u^{-i}} + \theta^{+a} \frac{\partial}{\partial \theta^{a}}, \quad (6)\]

satisfy the algebra
\[\{D^{++}, D^{-}\} = D^{0}, \quad \{D^{\pm}, D_{a}^{\pm}\} = 0, \quad \{D^{\pm}, D^{\pm}_{a}\} = 0, \quad \{D^{\pm}_{a}, D^{\pm}_{a}\} = D^{0}_{a}, \quad \{D^{+}_{a}, D^{-}_{a}\} = -\{D^{+}_{a}, D^{-}_{a}\} = 2i\theta_{a}. \quad (7)\]

The following conventions for the full and the analytic superspace integration measures are used
\[d^{12}z = d^{4}x_{A}(D^{+})^{4}(D^{-})^{4}, \quad d^{8}z = d^{4}x_{A}(D^{+})^{2}(D^{-})^{2}, \quad d^{4}x_{A}(D^{-})^{4}du, \quad (8)\]

where we have assumed the notation
\[(D^{+})^{4} = (D^{+})^{2}(D^{+})^{2}, \quad (D^{-})^{4} = (D^{-})^{2}(D^{-})^{2}, \quad (9)\]
\[(D^{\pm})^{2} = \frac{1}{4} D^{\pm a} D_{a}^{\pm}, \quad (D^{\pm}_{a})^{2} = \frac{1}{4} D_{a}^{\pm} D^{\pm}_{a}. \quad (10)\]

We consider a four-dimensional \(N = 2\) supersymmetric sigma-model in the analytic harmonic superspace. The model is described in terms of analytic harmonic superfields \(\phi^{a}(\zeta, u)\), which parameterize the \(n\)-dimensional target space, \(a = 1, \ldots, n\). The classical action for the model is written in the form
\[\begin{align*}
S[\phi] &= \int d^{4}x_{A}(D^{+})^{4}(D^{-})^{4} \left\{-\frac{1}{2} g_{ab}(\omega) D^{++a} D^{++b} + L^{++}(\omega) D^{++a} + L^{(4)}(\omega)\right\}, \\
&= \int d^{4}x_{A}(D^{-})^{2}(D^{+})^{2} \left\{-\frac{1}{2} g_{ab}(\omega) D^{++a} D^{++b} + L^{++}(\omega) D^{++a} + L^{(4)}(\omega)\right\}, \quad (11)\]
\end{align*}\]

where the target space metric \(g_{ab}\), and \(L^{++}\) and \(L^{(4)}\) are the arbitrary analytic functions of the \(\omega^{a}\)-superfields. The action (11) is invariant under reparameterizations transformations
\[\omega^{a} \to \omega^{a} + \lambda^{a}(\omega, u), \quad (12)\]

in the assumption that superfields \(g_{ab}\), \(L^{++}\) and \(L^{(4)}\) transform under (12) as a tensor of the corresponding rank. The equations of the motion corresponding to action (11) looks like
\[\begin{align*}
(D^{++})^{2} \omega^{a} + \Gamma^{a}_{bc}(\omega) D^{++b} \omega^{c} + g^{ab} L^{++}(\omega) D^{++c} + g^{ab} \partial_{b} L^{(4)}(\omega) &= 0, \\
&= \int d^{4}x_{A}(D^{+})^{4}(D^{-})^{4} \left\{-\frac{1}{2} g_{ab}(\omega) D^{++a} D^{++b} + L^{++}(\omega) D^{++a} + L^{(4)}(\omega)\right\}, \quad (13)\]
\end{align*}\]

where we have introduced the harmonic superspace Christoffel symbols \(\Gamma^{a}_{bc}(\omega)\) and the inverse metric \(g^{ab}\). Also we have denoted \(L^{++}_{ab} = \partial_{a} L^{++}_{b} - \partial_{b} L^{++}_{a}\). The superfield model with action (11) as well as conventional sigma-model contains the sigma-model-type metric and two derivatives. Therefore, the study of quantum structure of this model can be based on a generalization of the methods developed for conventional sigma models.

3 Background-quantum splitting

In this section, we develop the covariant background field method in \(N = 2\) harmonic superspace to study the effective action for the model (11). It is known that the linear background-quantum splitting to construct the loop expansion of the quantum effective action for the nonlinear sigma-models leads to breaking the reparameterization invariance. To preserve the above invariance in quantum theory for non-linear sigma-models, the manifestly covariant background field formalism was developed (see, e.g., [26,27]). Here we adopt such a formalism for the \(\omega\)-hypermultiplet in harmonic superspace.

Let us introduce the analytic superfield \(\rho^{a}(s)\) that satisfies the harmonic superspace geodesic equation
\[\begin{align*}
\frac{d^{2}\rho^{a}(s)}{ds^{2}} + \Gamma^{a}_{bc}(\rho) \frac{d\rho^{b}(s)}{ds} \frac{d\rho^{c}(s)}{ds} &= 0, \\
&= \int d^{4}x_{A}(D^{+})^{4}(D^{-})^{4} \left\{-\frac{1}{2} g_{ab}(\omega) D^{++a} D^{++b} + L^{++}(\omega) D^{++a} + L^{(4)}(\omega)\right\}, \quad (14)\]
\end{align*}\]

with the conditions
\[\rho^{a}(0) = \Omega^{a}, \quad \rho^{a}(1) = \Omega^{a} + \pi^{a}, \quad \frac{d\rho^{a}(s)}{ds}igg|_{s=0} = \xi^{a}. \quad (15)\]

Since the omega-hypermultiplet is uncharged, the functions \(L^{++}\) and \(L^{(4)}\) must obligatory include the explicit dependence on harmonics.
The analytic superfield $\xi^a = \xi^a(\zeta, u)$ denotes the tangent vector to the geodesic at $z = 0$ and plays the role of quantum field in the background-quantum splitting.

The solution to the Eq. (14) with the initial conditions (15) can be written in the form of the series

$$\rho^a(s) = \Omega^a + \sum_{n=1}^{\infty} \frac{s^n}{n!} \rho^{a(n)}(s),$$

where the functions $\rho^{a(n)}$ depend on the background fields $\Omega^a$ and quantum $\xi^a$ ones and are obtained directly from the equation (14)

$$\rho^{a(1)} = \xi^a, \quad \rho^{a(2)} = -\Gamma^a_{bc} \xi^b \xi^c, \quad \rho^{a(3)} = -(\partial_d \Gamma^a_{bc} - 2\Gamma^a_{bk} \Gamma^k_{cd}) \xi^b \xi^c \xi^d, \quad \text{etc.} \quad (17)$$

The decomposition of the classical action (11) under (60)

$$S[\rho] = S[\Omega] + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^n S[\rho]}{d s^n} \bigg|_{s=0} = S[\Omega] + S_1 + S_2 + \cdots.$$  

will be manifestly covariant. Our aim is to study the one-loop quantum correction to the classical action (11). For this aim, only the quadratic in quantum fields $\xi^a$ in action $S_2$ in (18) should be taken into account. The explicit expression for $S_2$ is written as follows

$$S_2 = \frac{1}{2} \int d\xi^4 \xi^a \left( g_{cd}(\nabla^+ \nabla^+) \right)_c^d (\nabla^+ \nabla^+)^b \left( \nabla^+ \nabla^+ \right)^b \xi^a \left( \nabla^+ \right)^b + \nabla_a \nabla_b \left( L_c^+ \nabla^+ \nabla^+ \right)^b + \nabla_a \nabla_b L_c^+ \right) \xi^b,$$  

where we have introduced the harmonic covariant derivative

$$(\nabla^+ \nabla^+) \xi = (\nabla^+ \nabla^+) \xi^b \xi^b,$$

and $\nabla_a$ is a covariant derivative along the curve $\rho^a(s)$ in the target space. We also assume the symmetrization without one half in the expression (19), Riemann tensor $R^d_{abc}$ and the metric tensor $g_{ab}$, which appeared above in (19), depend on the background superfields $\Omega^a$. The following useful property

$$(\nabla^+ g)_{ab} = 0,$$  

can be derived from the vanishing of the covariant derivative of the metric tensor.

### 4 One-loop divergences

The quadratic over quantum fields $\xi^a$ action $S_2$, explicitly derived in (19), determines the one-loop quantum correction $\Gamma^{(1)}$ to the classical action (11). After integrating over quantum fields $\xi^a$, one obtains

$$\Gamma^{(1)}[\Omega] = \frac{i}{2} \text{Tr}_{(4,0)} \ln S_2^2[\Omega],$$  

where the functional trace includes the matrix trace and integration over harmonic superspace

$$\text{Tr}_{(q,A-q)} \mathcal{O} = \int d\xi(4-q) d\xi(4-q) \delta^{(4-q)}(2|1) \mathcal{O}(q,A-q)(1|2).$$

The delta-function $\delta^{(4-q)}(2|1)$ introduced above is an analytic function on both arguments [14]. Also we have denoted by $\mathcal{O}(q,A-q)$ the kernel of the operator $\mathcal{O}$ acting in the space of analytic superfields with the harmonic U(1) charge $q$.

The expression (22) is expressed through the second variation derivative of the action $S_2$ (19) with respect to the quantum superfields and can be represented in the schematic form

$$\Gamma^{(1)}[\Omega] = \frac{i}{2} \text{Tr}_{(4,0)} \ln \left( (\nabla^+)^2 + (\nabla L^+) \nabla^+ + X^{(+4)} \right),$$  

where the operator $(\nabla L^+) \nabla^+$ means $\nabla_a L_c^{++} (\nabla^+)^b_c$. The expression $X^{(+4)}$ in (23) is written as follows

$$X^{(+4)} = -R^d_{abc} D^{++} \Omega^d D^{++} \Omega_d + \nabla_a \nabla_b L_c^{++} D^{++} \Omega^c + L_c^{++} R^d_{abc} D^{++} \Omega^d + \nabla_a \nabla_b L^{(+4)}$$  

and is symmetrized over the target space indices without one-half.

For further analysis, it is convenient to rewrite the operator $(\nabla^+)^2 + (\nabla L^+) \nabla^+ + X^{(+4)}$ in expression (23) in another form, where the term is linear in the operator $\nabla^+$ is eliminated with help of redefinition of covariant derivative. It can be done if to introduce the new covariant derivative $D^{++} = D^{++} + \nabla^+ = \nabla^+ + \bar{\Gamma}^{++}$ in terms of new analytic connection $\bar{\Gamma}^{++} = \Gamma^{++} + \bar{\Gamma}^{++}$. Here $\bar{\Gamma}^{++} a_b = \Gamma^{a}_{bc}(\Omega) D^{++} \Omega^c$ and $\bar{\Gamma}^{++} a_b = g^{ac} \nabla_c L_b^{++}$. As a result we obtain

$$\Gamma^{(1)} = \frac{i}{2} \text{Tr}_{(4,0)} \ln \left( (D^{++})^2 + \bar{X}^{(+4)} \right).$$  

The superfield $\bar{X}^{(+4)}$ in (25) reads

$$\bar{X}^{(+4)} = -R^d_{abc} D^{++} \Omega^d D^{++} \Omega_d + L_c^{++} R^d_{abc} D^{++} \Omega^d + \nabla^a \nabla^b L_c^{++} (\nabla^+) a_b + \nabla_a \nabla_b L^{(+4)}$$  

The term linear in covariant derivative $D^{++}$ is absent in expression (25).

The covariant derivative $D^{++} = D^{++} + \nabla^+$ is similar to the covariant derivative for the hypermultiplet coupled to gauge superfield, where the connection $\nabla^+$ is analogous to gauge superconnection. The only difference is that in our case, the connection $\nabla^+$ is constructed from the background.
hypermultiplet while the gauge superconnection is independent analytic superfield. Note that such kind of connection was observed and discussed for the first time in the paper [17]. Taking into account such an analogy, the calculations of the effective action (25) can be carried out using the methods developed earlier for the study of the effective action in \( N = 2 \) supersymmetric quantum gauge theory (see, e.g., [30–36]).

Evaluation of the effective action in \( N = 2 \) supersymmetric quantum gauge theory is based on the algebra of the covariant derivative. Therefore we begin the calculation of effective action with a discussion how the initial algebra of the harmonic derivative \( D^{++}\). First of all the covariant derivative \( D^{++}\) preserves the analyticity and hence the commutators \( [D^+_a, D^{++}] = [\bar{D}^+_a, D^{++}] = 0 \) should be held. Then we define the nonanalytic covariant harmonic derivative \( D^{+-}\). Similar to the \( N = 2 \) SYM theory in harmonic superspace [14], we use the zero curvature condition

\[
[(D^{++})^i_a, (D^{-})^j_b] = \delta^i_b D^j_0.
\]

Assuming \( D^{-} = D^{--} + \bar{V}^{--} \) we obtain like in \( N = 2 \) SYM theory [14]

\[
\bar{V}^{--} = \sum_{n=1}^{\infty} (-1)^n \int du_1 \cdots du_n \frac{\mathcal{V}^{++1} \mathcal{V}^{++2} \cdots \mathcal{V}^{++n}}{(u_1^+ u_1^-) \cdots (u_n^+ u_n^-)}.
\]

In the rest, the algebra of covariant derivatives looks similar to the algebra of \( N = 2 \) supersymmetric gauge theory in four dimensions [14]. Omitting the target space indices, one gets

\[
\begin{align*}
[D^+_a, D^+_b] & = 2\epsilon_{ab}\bar{W}, \quad \{D^+_a, \bar{D}^+_b\} = 2\epsilon_{ab}\bar{W}, \\
\{D^+_a, D^-_b\} & = -\{D^+_a, \bar{D}^-_b\} = 2iD_{a\bar{a}}, \\
[D^+_a, D^-_{\bar{b}}] & = \bar{D}^+_a \epsilon_{a\bar{b}} W, \\
[D^+_a, \bar{D}^-_{\bar{b}}] & = \bar{D}^+_a \epsilon_{a\bar{b}} W, \\
[D^{++}, D^-] & = \bar{D}^{++}, \quad [D^{--}, D^+] = D^-.
\end{align*}
\]

Here we have denoted

\[
\begin{align*}
\mathcal{D}^+_a & = D^+_a - D^{++}_a \bar{V}^{--}, \quad \mathcal{D}^+_{\bar{a}} = \delta_{\bar{a}a} - \frac{1}{2} D^{++}_a D^{++}_a \bar{V}^{--}, \\
\mathcal{W} & = (D^{++})^2 \bar{V}^{--}, \quad W = (D^+)^2 \bar{V}^{--}.
\end{align*}
\]

Using the above algebra, we introduce the analytic covariant d’Alembertian

\[
\begin{align*}
\hat{\square} & = \frac{1}{2}(D^{++})^4(D^{--})^2 \\
& = \mathcal{D} M \mathcal{D}^M - \frac{1}{4}(D^{++})^2 \mathcal{W} D^{--} - \frac{1}{2} D^{++}_a \mathcal{W} D^{--}_a.
\end{align*}
\]

The covariant d’Alembertian \( \hat{\square} \) depends on the background \( \Omega \) superfield through the analytic connection \( \mathcal{V}^{++}\). Acting in the space of analytic superfields, this operator preserves analyticity, \( [D^+_a, \hat{\square}] = 0 \).

To calculate the one-loop divergent contributions to the effective action (25), we represent it as a sum of two terms

\[
\Gamma^{(1)}[\Omega] = i \operatorname{Tr}(2,2) \ln D^{++} + \frac{1}{2} \operatorname{Tr}(4,0) \ln \left( \delta^{(4,4)} + G^{(0,0)} \hat{X}^{++4} \right),
\]

where the Green function \( G^{(0,0)} \) satisfies the equation

\[
(D^{++})^2 G^{(0,0)}(1,2) = \delta^{(4,4)}(1,2).
\]

Explicit solution this equation has the form [14]

\[
G^{(0,0)}(1,2) = -\frac{1}{\square_1} (D^+_1)^4(D^+_2)^4 \delta^{12}(z_1 - z_2) \frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3},
\]

where \( \delta^{12}(z_1 - z_2) \) is a full \( N = 2 \) superspace delta-function. It should be noted that the operator \( (D^{++})^2 \) commute with the covariant d’Alembertian \( \hat{\square} \) acting on the analytic superfields with the zero harmonic \( U(1) \) charge. Dependence on background hypermultiplet superfields \( \Omega \) is contained in the operator \( \hat{\square} \).

4.1 Calculating the divergences

The first term in (32) is a trace of first order operator acting in the space of analytic superfields with the harmonic \( U(1) \) charge \( +2 \). To extract the divergent contribution from this term, we vary \( \delta \Gamma_{\mathcal{W}}^{(1)}[\Omega] \) with respect to the superfield \( \mathcal{V}^{++}\)

\[
\begin{align*}
\delta \Gamma_{\mathcal{W}}^{(1)}[\Omega] & = i \delta \operatorname{Tr}(2,2) \ln D^{++} + i \operatorname{Tr} \mathcal{V}^{++} G^{(1,1)} \\
& = i \operatorname{Tr} \int d^4z_1 \delta \mathcal{V}^{++} G^{(1,1)}(1|2) \bigg|_{z_2 = 1},
\end{align*}
\]

where the trace has been taken over target space indices and \( G^{(1,1)} \) is the Green function of the operator \( D^{++}\)

\[
\begin{align*}
(D^{++})^2 G^{(1,1)}(1|2) & = \delta^{(4,4)}(1|2), \\
G^{(1,1)}(1,2) & = -\frac{1}{\square_1} (D^+_1)^4(D^+_2)^4 \delta^{12}(z_1 - z_2) \frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3},
\end{align*}
\]

For further analysis, we use in (35) the proper-time representation for the operator \( \hat{\square}^{-1} \) in the Green function (36). It leads to

3 Here and after, we follow a similar consideration in case of the four-dimensional \( N = 2 \) supersymmetric gauge theory, which was carried out thoroughly using the harmonic superspace formulation in the works [30,36].
\[ \delta \Gamma^{(1)}_{\mathcal{W}} = -i\hbar \int d\zeta^{-4} \delta \mathcal{W}^{++} \]
\[ \times \int_0^\infty d(is)(s \mu^2)^{\frac{1}{2}} e^{-is\square_1} (D_1^+)^4 (D_2^+)^4 \frac{\delta^{12}(z_1 - z_2)}{(u_1^+ u_2^+)^3} \big|_{2=1}. \]  
\hspace{1cm} (37)

Here we have introduced the integration over proper-time \( s \) and \( \mu \) and \( \varepsilon \) are the parameters related to dimensional regularization. The divergences of effective action appear in (37) as a pole \( \frac{1}{\varepsilon} \).

To extract the divergent contributions from the expression (37), we first of all, count a coincident points limit, \( \theta_2 \rightarrow \theta_1 \). The integrand in the expression (37) contains a 4-D-factors acting on the Grassmann delta-function \( \delta^8(\theta_1 - \theta_2) \). Using the identity
\[ (D_1^+)^4 (D_2^+)^4 \delta^8(\theta_1 - \theta_2) \big|_{2=1} = (u_1^+ u_2^+)^4 \]  
one rewrites the expression (37) in the form
\[ \delta \Gamma^{(1)}_{\mathcal{W}, \text{div}} = -i\hbar \int d\zeta^{-4} \delta \mathcal{W}^{++} \]
\[ \times \int_0^\infty d(is)(s \mu^2)^{\frac{1}{2}} e^{-is\square_1} (u_1^+ u_2^+)^4 \delta^4(\chi_1 - \chi_2) \big|_{2=1, \text{div}}. \]  
\hspace{1cm} (39)

Then we commute the operator \( e^{-is\square} \) with the factor \( (u_1^+ u_2^+) \), taking into account that the operator \( \square_1 \) contains the harmonic derivative \( \nabla^{-} \). In the coincident harmonic limit, \( u_1^+ \rightarrow u_1^+ \), the only non-zero contribution appears due to the identity \( (D_1^-)^4 |_{2=1} = (u_1^+ u_2^+) |_{2=1} = -1 \).

After that one obtains
\[ e^{-is\square}(u_1^+ u_2^+) e^{is\square} \big|_{2=1} = \frac{i}{4}(D^+)^2 \mathcal{W} + \cdots \]  
\hspace{1cm} (40)

where we omit all terms with high power of the proper-time \( s \). Such contributions correspond to the finite corrections to the effective action. Next, we pass to the momentum representation for the space-time delta-function and calculate the integral over proper-time. It leads to
\[ \delta \Gamma^{(1)}_{\mathcal{W}, \text{div}} = \frac{1}{4(4\pi)^2} \mathrm{tr} \int d\xi^{-4} \delta \mathcal{W}^{++} (D^+)^2 \mathcal{W}. \]  
\hspace{1cm} (41)

Hence, from Eq. (41) one can read off the divergent part of \( \Gamma^{(1)}_{\mathcal{W}} \)
\[ \Gamma^{(1)}_{\mathcal{W}, \text{div}} = \frac{1}{4(4\pi)^2} \mathrm{tr} \int d\zeta \mathcal{W}^2, \]  
\hspace{1cm} (42)

where the superfield \( \mathcal{W} \) was defined in (30).

Now let us consider the second term in the effective action (32). This part of effective action is defined as a series
\[ \Gamma^{(1)}_X = \frac{i}{2} \mathrm{Tr}_{(4,0)} \ln \left( 1 + (G^{(0,0)})^{ac} \tilde{X}^{(++)}_{eb} \right) \]
\[ = \frac{i}{2} \mathrm{Tr} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left( (G^{(0,0)})^{ac} \tilde{X}^{(++)}_{eb} \right)^n, \]  
\hspace{1cm} (43)

where the matrix power in the last expression includes the integration over analytic subspace as well. The Green function \( G^{(0,0)} \) contains the inverse power of the d’Alembertian \( \square \). By power counting the divergent contributions correspond to the term with \( n = 2 \) in the decomposition (43)

\[ \Gamma^{(1)}_{X, \text{div}} = \frac{i}{4} \int d\zeta^{-4} d\zeta^{-4} (G^{(0,0)})^{ab} (12) \]
\[ \times \tilde{X}^{(++)}_{ba} (12) \tilde{X}^{(++)}_{ab} (12). \]  
\hspace{1cm} (44)

The further strategy is as follows. The Green function \( G^{(0,0)} \) (23) contains the \( D \)-factors, which can be used to restore the full superspace measure. Then, we integrate over Grassmann variable \( \theta_2 \), taking into account the delta-function \( \delta^8(\theta_2 - \theta_1) \) from the Green function \( G^{(0,0)} \), and use the identity (38). After that, one gets

\[ \Gamma^{(1)}_{X, \text{div}} = \frac{1}{4(4\pi)^2} \mathrm{tr} \int d\zeta \mathcal{W}^2 \]
\[ \times \int d\zeta d\zeta d\zeta d\zeta (u_1^+ u_2^+) (u_1^+ u_2^+) \tilde{X}^{(++)}_{ba} (12) \tilde{X}^{(++)}_{ab} (12), \]  
\hspace{1cm} (45)

where we have used the divergent part of the momentum integral

\[ \int \frac{d^4q}{q^2(\xi - p)^2} = \frac{i\pi^2}{\varepsilon}, \varepsilon \rightarrow 0. \]  
\hspace{1cm} (46)

Finally, we combine both contributions (41) and (45), and obtain the divergent contribution of the effective action (32) in the form

\[ \Gamma^{(1)}_{\text{div}} = \frac{1}{2(4\pi)^2} \mathrm{tr} \int d\zeta \mathcal{W}^2 \]
\[ - \frac{1}{4(4\pi)^2} \mathrm{tr} \int d\zeta d\zeta d\zeta d\zeta (u_1^+ u_2^+) (u_1^+ u_2^+) \tilde{X}^{(++)}_{ba} (12) \tilde{X}^{(++)}_{ab} (12) \]  
\hspace{1cm} (47)

The relation (47) is the general result for the divergences of the one-loop effective action (23). The function \( \tilde{X}^{(++)}_{ba} \) is given by (26). It is interesting to note two points. First, the expression (47) is invariant under the reparameterizations transformations (12) by construction. Second, the divergent part of effective action is non-local in harmonics. This should not come as much of a surprise. For example, the classical action of \( \mathcal{N} = 2 \) SYM theory is non-local in harmonics as well [14]. However, the corresponding component action is completely local. Later we will show that the expressions of the form (47) are also local in the components.

### 4.2 Special cases

Let us consider the expression (47) in more detail. The superfield \( \tilde{X}^{(++)} \) was introduced in (26) and included two types of
contributions. The first one depends on the Riemann tensor, constructed in the background metric tensor $g_{ab}(\Omega)$. Such terms vanish in the case of the constant metric, $g_{ab}(\Omega) = h_{ab}$.

The second type of contributions contains both the background metric and the potential functions $L^{++}$ and $L^{(+4)}$. Such type of contributions vanish if we assume $L^{++} = 0$ and $L^{(+4)} = 0$. Using the expression (47), one can extract them independently.

First, let us assume $L^{++} = 0$ and $L^{(+4)} = 0$ and mark the corresponding effective action by symbol $R$. In this case the one-loop contribution to effective action (32) is reduced to

$$
\Gamma^{(1)}_R[\Omega] = i \text{Tr}_{(2,2)} \ln \nabla^{++} + \frac{i}{2} \text{Tr}_{(4,0)} \ln \left(1 - (G^{(0,0)})^{\alpha\beta} R^{\alpha\beta} D^{++} \bar{\Omega}^{+} D^{++} \Omega_{\alpha \beta}\right),
$$

(48)

where the covariant derivative $\nabla^{++}$ was introduced in (20).

The divergent contribution, in this case, can be found from the general relation (47) and has the form

$$
\Gamma^{(1)}_{R, \text{div}}[\Omega] = \frac{1}{2(4\pi)^2} \text{tr} \int d^8 z \ W^2 - \frac{1}{2(4\pi)^2} \int d^1 z u_1 u_2 \left(\frac{u_1^+ u_2^-}{(u_1^+ u_2^-)^2}\right) R^{ab} a(1) R^{e b a}(2) \times D^{++} \bar{\Omega}^{+}(1) D^{++} \bar{\Omega}^{+}(2) D^{++} \Omega_{e}(2).
$$

(49)

The superfield $W$ in the expression (49) is obtained from $W$ (30) in case of vanishing potential functions $L^{++}$ and $L^{(+4)}$. The superfield connection $\nabla^{++}$ in such a case coincides with the Levi–Civita analytic connection $(\Gamma^{++})_b^a = \Gamma^{ab}(\Omega) D^{++} \Omega^e$.

Now we return to the effective action (32) and consider the case, then the background metric, $g_{ab}(\Omega) = h_{ab}$, does not depend on the superfield $\Omega$ and superspace point $z$. In this case, all terms containing Christoffel symbols and Riemann curvature tensor obviously disappear, and we have only parts depending on potentials $L_a^{++}$, $L_a^{(+4)}$. We mark the corresponding effective action by symbol $L$. In this case, the one-loop effective action (32) is reduced to

$$
\Gamma^{(1)}_L[\Omega] = i \text{Tr}_{(2,2)} \ln \tilde{\nabla}^{++} + \frac{i}{2} \text{Tr}_{(4,0)} \ln \left(1 + (G^{(0,0)})^{L^{(+4)}}\right),
$$

(50)

where we have introduced the notations

$$
\nabla^{++}_{ab} = h_{ab} D^{++} + \tilde{\Gamma}^{++} \nabla^{++}_{ab},
$$

(51)

$$
L^{(+4)}_{ab} = \partial_a \partial_b L^{(+4)} + \partial_a \partial_b L^{++} D^{++} \bar{\Omega}^e - D^{++} \tilde{\Gamma}^{++} + (\tilde{\Gamma}^{++})^2_{ab}.
$$

(52)

The analytic superfield $\tilde{\Gamma}^{++}_{ab} = \partial_a L^{++}_{b} + \partial_a L^{++}_{b}$ can be considered as an analytic connection $\nabla^{++}$ corresponding to the case under consideration. The superfield $L^{++}_{ab}$ was introduced earlier in (13). The divergent part of the effective action (50) is obtained from general relation (47) and has the form

$$
\Gamma^{(1)}_{L, \text{div}} = \frac{1}{2(4\pi)^2} \text{tr} \int d^8 z \ W^2 - \frac{1}{2(4\pi)^2} \int d^1 z u_1 u_2 \left(\frac{u_1^+ u_2^-}{(u_1^+ u_2^-)^2}\right) L^{(+4)}_{ab}(1) L^{(+4)ba}(2).
$$

(53)

The superfield $W$ is defined by the superfield $W$ (30) for the case of the constant metric $g_{ab}$, when superfield connection $\nabla^{++}$ reducing to $\tilde{\Gamma}^{++}$. The structure of the contribution above is similar to (47) and (49).

4.3 The component structure of divergences

Let us briefly discuss the component structure of (49) and (53). We are not going to consider the whole component form of the one-loop divergences and will only demonstrate that the non-local in harmonics expressions of the form (47) are local in components. To see that it is sufficient to derive the component form of the expression (48) in bosonic sector since it completely reflects the general expression (47).

The component expansion of the background field $\Omega^{e}(\zeta, u)$, contains both the physical fields and the set of auxiliary fields. Using the classical equations of motion (13), one can exclude auxiliary unphysical components and formulate the model in terms of the physical fields (see the [14] for details). The divergent contributions (49) and (53) to the one-loop effective action were calculated without any restriction on the background superfields $\Omega^{e}(\zeta, u)$. Hence, we can use the general expression for the decomposition of the analytic superfield $\Omega$ in series over Grassmann variables $\theta^+$ and $\bar{\theta}^+$

$$
\Omega(\zeta, u) = \frac{1}{\sqrt{2}} \omega(\zeta, u) + \omega^{ij}(\zeta, u) u^i_j + \theta^{+\alpha} \psi^{i\alpha}(\zeta, u) u^i_j + \bar{\theta}^{+\alpha} \bar{\psi}^{i\alpha}(\zeta, u) u^i_j + \ldots,
$$

(54)

where ellipses stand for the unessential (for us) terms. The fields $\omega$ and $\omega^{ij} = \omega^{ji}$ are the physical bosonic scalar fields, and $\psi^{i\alpha}(\zeta, u)$ is the spinor one. In the expression (54), we are assuming that all fields have the target space indices.

The general scheme of passing to component action in the divergent contribution (49) is as follows. First, we substitute the component expansion 54 for the superfield $\Omega$ and collect the terms with the fourth power of Grassmann variables $\theta$ and $\bar{\theta}$. Then we use the relations $u^i_j u^*_{-j} = \delta_{ij}$ for the explicit dependence of the integrand in (49) on the harmonic variable $u_2$ and integrate over $u_2$ using the general rules of integration over harmonics [14]. After that the integrals over anticommuting variables $\theta$ and $\bar{\theta}$ are evaluated and for the result in the bosonic sector of the expression (49) we have
\( \Gamma^{(1)}_{\text{div}}[\omega] = - \frac{1}{128 \pi^2} \times \int d^4 x \, R_{\alpha \beta \gamma \delta}^e R_{\delta \beta \alpha}^e \partial \alpha \partial \beta \partial \gamma \partial \delta \omega + \cdots , \)

(55)

where we omit all terms with permutations on spinor indices and terms with the triplet of scalars of bosonic component \( \omega_{ij} \) of the superfield \( \Omega \). The expression (55) is evidently local.

The component expression of the divergent contribution 53 can be obtained in the same manner. In this case, the explicit form of the component action depends on the potential-like functions \( L^\alpha_{\alpha} \) and \( L^{(4)} \).

5 Summary

In the present paper we have developed the manifestly covariant approach for studying the quantum structure of the harmonic superspace sigma-model in four dimensions. Specific feature of this approach is the exploration of the analogies between one-loop effective action for such a theory and one-loop effective action for hypermultiplet in external non-Abelian \( \mathcal{N} = 2 \) supersymmetric gauge superfield.

The harmonic superspace sigma-model (11), is formulated in \( \mathcal{N} = 2 \) harmonic superspace in terms of analytic omega-hypermultiplet superfields. Such a formulation provides both manifest reparameterization invariance in harmonic superspace and manifest \( \mathcal{N} = 2 \) supersymmetry. The one-loop effective action for such a model is constructed in the framework of the manifestly covariant and manifestly \( \mathcal{N} = 2 \) supersymmetric background-quantum splitting in \( \mathcal{N} = 2 \) harmonic superspace. Such an one-loop effective action was presented as the functional determinant (22) of the special differential operator acting in the analytic subspace of harmonic superspace. We developed the harmonic superspace proper-time technique that allows us to calculate such functional determinants and found the divergent part of the one-loop effective action in a general form. The result is given by the expression (47) for arbitrary background hypermultiplet \( \Omega \). Taking into account this general result, we calculated the one-loop divergences in two special cases. First, when the potential-like functions \( L^\alpha_{\alpha} \) and \( L^{(4)} \) are absent in action (11) and the sigma-model metric \( g_{ab}(\Omega) \) is arbitrary. Second, when the sigma-model metric \( g_{ab} \) in (11) is flat, the functions \( L^\alpha_{\alpha} \) and \( L^{(4)} \) are arbitrary. The background hypermultiplet is still arbitrary in both cases. It is interesting to note that the one-loop divergences given by the expression (47) are non-local in harmonics but are space-time local. We emphasize that the developed technique of finding the effective action is manifestly covariant and preserves the manifest \( \mathcal{N} = 2 \) supersymmetry at all steps calculations.

As far as we know, there was only one attempt in literature to compute the one-loop divergences in some \( \mathcal{N} = 2 \) supersymmetric sigma-model [29], which is a special partial case of the model (11) considered here. The work [29] approach was based on the construction of divergences of the gauged \( \mathcal{N} = 2 \) supersymmetric sigma-model. The result of such an indirect approach is not manifestly \( \mathcal{N} = 2 \) supersymmetric and we believe it requires justification.

As we noted, the calculation of one-loop divergences in this paper was based on analogies between one-loop effective action for the general \( \mathcal{N} = 2 \) model and one-loop effective action for the hypermultiplet in \( \mathcal{N} = 2 \) gauge superfield. However, these analogies are indirect. For example, the structure of the one-loop effective action (23) differs from the corresponding effective actions in \( \mathcal{N} = 2 \) supersymmetric gauge theories (see, e.g., [30,31,35,36]) due to the presence of superfield \( X^{(+4)} \). Such a superfield leads to the additional divergent term in the effective action in comparison with one in SYM theories [43–45].

The approach to calculating the effective action for \( \mathcal{N} = 2 \) supersymmetric sigma-models developed in the paper is general enough and can be applied to study the various quantum aspects of the hypermultiplet theories. In particular, it would be interesting to apply this general method to calculation of the finite contributions to the effective action of the model and to study the corresponding deformation of the initial geometry by quantum corrections. We believe that the finite contributions can be evaluated by the same harmonic superfield proper-time method as the divergent ones. We plan to consider these issues in the forthcoming works.

Acknowledgements The authors are grateful to E.A. Ivanov and S.M. Kuzenko for valuable comments and discussions and to S.A. Fedoronk for drawing our attention to the Ref. [23]. The work is partially supported by the Ministry of Education of the Russian Federation, project No. FFWF-2020-003. I.L.B is thankful to the Laboratory of Theoretical Physics, JINR where the work has been finalized for hospitality.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: We do not use any experimental data sets.]

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

Funded by SCOAP³.
Appendix: Derivation of the harmonic superspace sigma-model (11) from the general $q^+$–hypermultiplet theory

Let us consider the most general $\mathcal{N}=2$ sigma-model with the action

$$S[q^+] = \int d^4 \xi \left( \mathcal{L}^{+A} D^{+a} q^{+a}_A + \mathcal{L}^{(+4)} \right),$$  

where $A=1,2$ and $\alpha = 1, \ldots , n$. The superfield function $\mathcal{L}^{+A}(q^{+}, u)$ and $\mathcal{L}^{(+4)}(q^{+}, u)$ are arbitrary function of the superfield $q^{+a}_A$ and harmonics $u$. To derive the action for the $\omega$-hypermultiplet (11), we make the change of variables

$$q^{+a}_A = u^{+A}_A \omega^a + u^{+}_A q^{+a},$$  

substitute (57) into (56) and eliminate the function $f^{+a}$ using the equation of motion. The decomposition of the functions $\mathcal{L}^{+A}(q^{+}, u)$ and $\mathcal{L}^{(+4)}(q^{+}, u)$ over $f^{+a}$ reads

$$\mathcal{L}^{+A}(u^{+A}_A \omega^a + u^{+}_A f^{+a})$$

$$= \mathcal{L}^{+A}(u^{+A}_A \omega^a) + \frac{\partial \mathcal{L}^{+A}(u^{+A}_A \omega^a)}{\partial (u^{+A}_B \omega^b)} u^{+B}_B f^{+b}$$

$$+ \frac{\partial^2 \mathcal{L}^{+A}(u^{+A}_A \omega^a)}{\partial (u^{+A}_B \omega^b) \partial (u^{+A}_C \omega^c)} u^{+B}_B f^{+b} u^{+C}_C f^{+c} + \cdots ,$$

$$\mathcal{L}^{(+4)}(u^{+A}_A \omega^a + u^{+}_A f^{+a})$$

$$= \mathcal{L}^{(+4)}(u^{+A}_A \omega^a) + \frac{\partial \mathcal{L}^{(+4)}(u^{+A}_A \omega^a)}{\partial (u^{+A}_B \omega^b)} u^{+B}_B f^{+b}$$

$$+ \frac{\partial^2 \mathcal{L}^{(+4)}(u^{+A}_A \omega^a)}{\partial (u^{+A}_B \omega^b) \partial (u^{+A}_C \omega^c)} u^{+B}_B f^{+b} u^{+C}_C f^{+c} + \cdots .$$

Substituting the last expressions into (56) one obtains

$$S[\omega, f^{+a}] = \int d^4 \xi \left( \mathcal{L}^{+A} u^{+A}_A \omega^a + \mathcal{L}^{+A} D^{+a} u^{+}_A f^{+a} \right)$$

$$+ h^{AB}_{ab} u^{+A}_A u^{+B}_B \omega^a f^{+b} + h^{+A}_{ab} u^{+A}_A u^{+}_B f^{+b}$$

$$+ h^{+A}_{ab} u^{+A}_B u^{+B}_a f^{+b} + \mathcal{L}^{(+4)}$$

$$+ h^{(3)+A}_{a} u^{+A}_a f^{+a} + \cdots ,$$

where we have denoted

$$h^{AB}_{ab}(\omega) = \frac{\partial \mathcal{L}^{+A}(u^{+A}_A \omega^a)}{\partial (u^{+A}_B \omega^b)},$$

$$h^{(3)+A}_{a}(\omega) = \frac{\partial \mathcal{L}^{(+4)}(u^{+A}_A \omega^a)}{\partial (u^{+A}_B \omega^b)} .$$

In the expression (60) the ellipsis stands for the higher derivatives terms of the functions $\mathcal{L}^{+A}$ and $\mathcal{L}^{(+4)}$. To obtain the action under consideration (11) we exclude the superfield $f^{+a}$ from (60) using the corresponding equation of motion. Note that the action (60) contains all powers of the auxiliary superfield $f^{+a}$. The equation of motion for the superfield $f^{+a}$ is the non-linear algebraic one, however its can be in general analyzed. After exclusion of the superfield $f^{+a}$ from the equation of motion we immediately obtain the the action of the $\omega$-hypermultiplet model that contains nothing more then the terms with two $D^{+}$ derivatives of $\omega$, terms with one such a derivative and terms without derivatives. The coefficients at the terms with derivative are the functions of $\omega$, the terms without derivatives also are functions of $\omega$. All the above functions are expressed through the $\mathcal{L}^{+A}$ and $\mathcal{L}^{(+4)}$ and their derivatives. As a result, we arrive at exactly the considered harmonic superfield sigma model (11).

To conclude, in the Appendix we have demonstrated the relation between the general $q^+$-hypermultiplet theory and harmonic superfield sigma-model introduced in (11).

Emphasize that the superfield model (11) is internally consistent, possesses reparameterization invariance and manifest $\mathcal{N}=2$ supersymmetry and has a structure analogous to conventional sigma-model. It is completely non-contradictory in itself and can be used for studying the quantum exactive action.

References

1. A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory. Nucl. Phys. B 241, 333 (1984)
2. L. Alvarez-Gaume, D.Z. Freedman, Ricci flat Kahler manifolds and supersymmetry. Phys. Lett. 94B, 171 (1980)
3. L. Alvarez-Gaume, D.Z. Freedman, S. Mukhi, The background field method and the ultraviolet structure of the supersymmetric nonlinear sigma model. Ann. Phys. 134, 85 (1981)
4. L. Alvarez-Gaumé, D.Z. Freedman, Geometrical structure and ultraviolet finiteness in the supersymmetric sigma model. Commun. Math. Phys. 80, 443 (1981)
5. J. Bagger, E. Witten, The gauge invariant supersymmetric nonlinear sigma model. Phys. Lett. B 118, 193p (1982)
6. L. Alvarez-Gaume, S.R. Coleman, P.H. Ginsparg, Finiteness of Ricci flat supersymmetric nonlinear sigma models. Commun. Math. Phys. 102, 311p (1985)
7. L. Alvarez-Gaume, S.R. Coleman, P.H. Ginsparg, Finiteness of Ricci flat $N=2$ supersymmetric $\sigma$ models. Commun. Math. Phys. 103, 423p (1986)
8. I.L. Buchbinder, S.V. Ketov, The fourth order nonlinear sigma models and asymptotic freedom in four-dimensions. Theor. Math. Phys. 77, 1032 (1988)
9. I.L. Buchbinder, S.V. Ketov, Fortsch. Phys. 39, 1 (1991)
10. A.A. Deriglazov, S.V. Ketov, Renormalization of the $N=1$ supersymmetric four-dimensional nonlinear sigma model with higher derivatives. Theor. Math. Phys. 77, 1160 (1988)
11. S.V. Ketov, Quantum nonlinear sigma models: from quantum field theory to supersymmetry, conformal field theory, black holes and strings (Springer, Berlin, 2000)
12. S.J. Gates Jr., M.T. Grisaru, M. Rocek, W. Siegel, Superspace, or a walk through superspace (IOP Publ., Bristol, 1998), p. 643
13. I.L. Buchbinder, S.M. Kuzenko, Ideas and methods of supersymmetry and supergravity or a walk through superspace (IOP Publ., Cambridge University Press, Cambridge, 2001), p. 306
15. B. Zumino, Supersymmetry and Kähler manifolds. Phys. Lett. B 
787, 203 (1979)
16. N.J. Hitchin, A. Karlhede, U. Lindström, M. Roček, Hyperkähler
metrics and supersymmetry. Commun. Math. Phys. 108, 535
(1987)
17. A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, Hyperkahler
metrics and harmonic superspace. Commun. Math. Phys. 103, 515
(1986)
18. J. Bagger, E. Witten, Matter couplings in N = 2 supergravity. Nucl.
Phys. B 222, 1 (1983)
19. A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, Gauge field
geometry from complex and harmonic analyticities, hyperkahler
case. Ann. Phys. 185, 22 (1988)
20. G. Sierra, P.K. Townsend, The hyperkähler supersymmetric σ-
model in six-dimensions. Phys. Lett. B 124, 497 (1983)
21. S.M. Kuzenko, Lectures on nonlinear sigma-models in projective
superspace. J. Phys. A 43, 443001 (2010). arXiv:1004.0880 [hep-
th]
22. M.B. Green, J.H. Schwarz, E. Witten, Superstring theory (Univer-
sity Press, Cambridge, 1987)
23. S. Fedoruk, E. Ivanov, A. Smilga, Generic HKT geometries in the
harmonic superspace approach. J. Math. Phys. 59, 083501 (2018).
arXiv:1802.09675 [hep-th]
24. S.M. Kuzenko, I.N. McArthur, Hypermultiplet effective action:
N = 2 superspace approach. Phys. Lett. B 513, 213 (2001).
arXiv:hep-th/0105121
25. I.L. Buchbinder, N.G. Pletnev, Induced effective action in 6D
hypermultiplet theory on a vector/tensor background. Phys. Part.
Nucl. Lett. 14(2), 266 (2017)
26. S. Mukhi, The geometric background field method, renormaliza-
tion and the Wess–Zumino term in nonlinear sigma models. Nucl.
Phys. B 264, 640 (1986)
27. P.S. Howe, G. Papadopoulos, K.S. Stelle, The background field
method and the nonlinear σ model. Nucl. Phys. B 296, 26 (1988)
28. A. Blasi, F. Delduc, S.P. Sorella, The background-quantum split
symmetry in two-dimensional σ-models: a regularization-independent
proof of its renormalizability. Nucl. Phys. B 314, 409 (1989)
29. B. Spence, Non-renormalizability of supersymmetric non-linear
sigma models in four dimensions. Nucl. Phys. B 260, 531–544
(1985)
30. I.L. Buchbinder, E.I. Buchbinder, E.A. Ivanov, S.V. Kuzenko, B.A.
Ovrut, Effective action of the N = 2 Maxwell multiplet in har-
monic superspace. Phys. Lett. B (1997). arXiv:hep-th/9703147
31. I.L. Buchbinder, E.I. Buchbinder, S.M. Kuzenko, B.A. Ovrut, The
background field method for N = 2 super Yang-Mills theories in
harmonic superspace. Phys. Lett. B 417, 61 (1998). arXiv:hep-th/9704214
32. I.L. Buchbinder, S.M. Kuzenko, A.A. Tseytlin, On low-energy
effective actions in N = 2, 4 superconformal theories in four dimen-
sions. Phys. Rev. D 62, 045001 (2000). arXiv:hep-th/9911221
33. E.I. Buchbinder, B.A. Ovrut, I.L. Buchbinder, E.A. Ivanov, S.M.
Kuzenko, Low-energy effective action in N = 2 supersymmetric
field theories. Phys. Part. Nucl. 32, 641 (2001)
34. E.I. Buchbinder, B.A. Ovrut, I.L. Buchbinder, E.A. Ivanov, S.M.
Kuzenko, Fiz. Elem. Chast. Atom. Yadra 32, 1222 (2001)
35. S.M. Kuzenko, I.N. McArthur, Effective action of N = 4 super
Yang-Mills: N = 2 superspace approach. Phys. Lett. B 506, 140
(2001). arXiv:hep-th/0101127
36. S.M. Kuzenko, I.N. McArthur, Hypermultiplet effective action:
N=2 superspace approach. Phys. Lett. B 513, 213 (2001).
arXiv:hep-th/0105121
37. S.M. Kuzenko, Exact propagators in harmonic superspace. Phys.
Lett. B 600, 163 (2004). arXiv:hep-th/0407242
38. S.M. Kuzenko, Five-dimensional supersymmetric Chern–Simons
action as a hypermultiplet quantum correction. Phys. Lett. B 644,
88 (2007). arXiv:hep-th/0609078
39. A.T. Banin, I.L. Buchbinder, N.G. Pletnev, On quantum properties
of the four-dimensional generic chiral superfield model. Phys. Rev.
D 74, 045010 (2006). arXiv:hep-th/0606242
40. T.E. Clark, S.T. Love, Supersymmetric effective actions for anom-
alous internal chiral symmetries. Phys. Lett. B 138, 289 (1984)
41. K. Higashijima, M. Nitta, Kahler normal coordinate expansion
in supersymmetric theories. Prog. Theor. Phys. 105, 243 (2001).
arXiv:hep-th/0006027
42. K. Higashijima, M. Nitta, Normal coordinates in Kahler mani-
folds and the background field method. Prog. Theor. Phys. 108,
185 (2002). arXiv:hep-th/0203081
43. I.N. McArthur, T.D. Gargett, A Gaussian approach to computing
supersymmetric effective actions. Nucl. Phys. B 497, 525 (1997).
arXiv:hep-th/9705200
44. N.G. Pletnev, A.T. Banin, Covariant technique of derivative expan-
sion of one-loop effective action. Phys. Rev. D 60, 105017 (1999).
arXiv:hep-th/9811031
45. I.L. Buchbinder, E.A. Ivanov, B.S. Merzlikin, K.V. Stepanyantz,
One-loop divergences in the 6D, N = (1, 0) abelian gauge theory.
Phys. Lett. B 763, 375 (2016). arXiv:1609.00975 [hep-th]