ON EQUICONTINUITY AND TIGHTNESS OF BI-CONTINUOUS SEMIGROUPS

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Abstract. In contrast to classical strongly continuous semigroups, the study of bi-continuous semigroups comes with some freedom in the properties of the associated locally convex topology. This paper aims to give minimal assumptions in order to recover typical features like tightness and equicontinuity with respect to the mixed topology as well as to carefully clarify on mutual relations between previously studied variants of these notions. The abstract results—exploiting techniques from topological vector spaces—are thoroughly discussed by means of several example classes, such as semigroups on spaces of bounded continuous functions.

1. Introduction

Strongly continuous semigroups of operators are a well-established framework in the study of evolution equations. In many applications, however, the semigroups are not strongly continuous ($C_0$) with respect to the norm $\|\cdot\|$ of the underlying Banach space but strongly continuous with respect to a weaker Hausdorff locally convex topology $\tau$. Typical examples emerge from the fact that norm-strong continuity is in general not preserved when dual semigroups are considered [68]. Further examples are implemented semigroups [3, Sect. 3.2], transition semigroups on the space $C_0(\Omega)$ of bounded continuous functions on e.g. a Polish space $\Omega$ like the Ornstein–Uhlenbeck semigroup [19, 30, 51] and even on the space of bounded Hölder continuous functions on a separable Hilbert space [26]. In addition, also certain Koopman semigroups on $C_0(\Omega)$, i.e. semigroups induced by a semiflow on a completely regular Hausdorff space $\Omega$ [22, 33], require such a relaxed notion.

All these examples belong to the general framework of $\tau$-bi-continuous semigroups, which are $\tau$-strongly continuous and locally bi-equicontinuous, and were first studied by Kühnemund in [48, 49]. On the other hand, the theory of $C_0$-semigroups was extended beyond Banach spaces to equicontinuous $C_0$-semigroups on Hausdorff locally convex spaces [1, 42, Chap. IX], quasi-eqicontinuous $C_0$-semigroups [1, 42, 72], locally equicontinuous $C_0$-semigroups [20, 42, 60] and sequentially (locally) equicontinuous semigroups [45]. The corresponding notion of local bi-equicontinuity of a bi-continuous semigroup is weaker than local $\tau$-equicontinuity due to [49, Examples 6 (a), p. 209-210]. Nevertheless, it was already observed in [45, Theorem 7.4, p. 180] that $\tau$-bi-continuous semigroups are locally $\gamma$-equicontinuous under some mild assumptions on the mixed topology $\gamma := \gamma(\|\cdot\|, \tau)$.

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The mixed topology $\gamma$ was introduced in [72] and is the finest Hausdorff locally convex topology — even finest linear topology — that coincides with $\tau$ on $|\cdot|$-bounded sets.

In the context of $\tau$-bi-continuous semigroups another notion related to local $\gamma$-equicontinuity emerged, namely, the notion of locally equitight $\tau$-bi-continuous semigroups (or more generally of families of operators), which are sometimes simply referred to as “tight” or “local” in [26, 29]. Equicontinuity and local equicontinuity are crucial ingredients for perturbation results for $C_0$-semigroups on Hausdorff locally convex spaces like dissipative perturbations or Desch–Schappacher perturbations [11, 34]. Local equitightness plays a similar role in perturbation theory for bi-continuous semigroups, see [26, Theorem 1.2, p. 669] (cf. [29, Theorem 3.2.3, p. 47]), [30, Theorems 2.4, 3.2, p. 92, 94-95], [11, Remark 4.1, p. 101], [11, Theorem 5, p. 8] (cf. [10, Theorem 5.19, p. 81]) and [12, Theorem 3.3, p. 582].

In this paper we study $\gamma$-equicontinuity and equitightness for $\tau$-bi-continuous semigroups and their relation. The space $(X, \gamma)$ is usually neither barrelled nor bornological, since this would imply that $\gamma$ coincides with the $|\cdot|$-topology by [13, I.1.15 Proposition, p. 12]. Thus automatic local equicontinuity results for strongly continuous semigroups like [43, Proposition 1.1, p. 259] are not applicable. In [51, Corollary 3.10, p. 553], [51, Theorem 4.10, p. 555-556] and [45, Proposition 3.4, p. 161] automatic local equicontinuity results for strongly continuous semigroups were extended to strong Mackey spaces. In particular, barrelled or sequentially complete bornological Hausdorff locally convex spaces belong to the class of strong Mackey spaces by [45, Proposition 3.3 (a), (b), p. 160-161], which is not helpful for $(X, \gamma)$. We take a different route as $(X, \gamma)$ being a strong Mackey space is quite a strong assumption. This route is more in the spirit of [45, Proposition 3.3 (c), p. 160-161] and [45, Theorem 7.4, p. 180], exploiting the assumption that $(X, \gamma)$ is a $C$-sequential space, which means that every convex sequentially open subset of $(X, \gamma)$ is already open.

After fixing some notions and preliminaries on $\tau$-bi-continuous semigroups and the mixed topology $\gamma$ in Section 2 we derive sufficient conditions on the interplay of $|\cdot|$, $\tau$ and $\gamma$ that imply the quasi-$\gamma$-equicontinuity of a $\tau$-bi-continuous semigroup in Section 3. Moreover, we deduce sufficient conditions that guarantee the equivalence between $(\text{local, quasi-})\gamma$-equicontinuity and $(\text{local, quasi-})$-equitightness, Theorem 3.17 in combination with Proposition 3.16. It turns out that these conditions are satisfied for most of the classical examples of $\tau$-bi-continuous semigroups we mentioned before. Indeed, these applications are covered by the results in Section 4.

2. Notions and preliminaries

For a continuous map $f:(X_1, \tau_1) \to (X_2, \tau_2)$ from a topological space $(X_1, \tau_1)$ to a topological space $(X_2, \tau_2)$ we typically write that it is $\tau_1$-$\tau_2$-continuous and, if $(X_2, \tau_2) = (X_1, \tau_1)$, we just write that it is $\tau_1$-continuous. For two topologies $\tau_1$ and $\tau_2$ on a space $X$, we write $\tau_1 \leq \tau_2$ if the topology $\tau_1$ is coarser than $\tau_2$. For a vector space $X$ over the field $\mathbb{R}$ or $\mathbb{C}$ with a Hausdorff locally convex topology $\tau$ we denote by $(X, \tau')$ the topological linear dual space and just write $X' := (X, \tau')$ if $(X, \tau)$ is a Banach space. We use the symbol $L(X; Y) := L((X, \|\cdot\|_X); (Y, \|\cdot\|_Y))$ for the space of continuous linear operators from a Banach space $(X, \|\cdot\|_X)$ to a Banach space $(Y, \|\cdot\|_Y)$ and denote by $\|\cdot\|_{L(X; Y)}$ the operator norm on $L(X; Y)$. If $X = Y$, we set $L(X) := L(X; X)$. 

[72]
2.1. Assumption ([10, Assumptions 1, p. 206]). Let \((X, \| \cdot \|, \tau)\) be a triple where \((X, \| \cdot \|)\) is a Banach space, and

(i) \(\tau\) is a coarser Hausdorff locally convex topology than the \(\| \cdot \|\)-topology,
(ii) \(\tau\) is sequentially complete on \(\| \cdot \|\)-bounded sets, i.e. every \(\| \cdot \|\)-bounded \(\tau\)-Cauchy sequence is \(\tau\)-convergent,
(iii) the dual space \((X, \tau)'\) is norming, i.e.
\[
\|x\| = \sup_{y \in \tau'} |y(x)|, \quad x \in X.
\]

For what follows, the mixed topology, [72, Section 2.1], and the notion of a Saks space [13, 1.3.2 Definition, p. 27-28] will be crucial.

2.2. Definition. Let \((X, \| \cdot \|)\) be a Banach space and \(\tau\) a Hausdorff locally convex topology on \(X\) that is coarser than the \(\| \cdot \|\)-topology \(\tau_{\|\|}\). Then

(a) the mixed topology \(\gamma := \gamma(\| \cdot \|, \tau)\) is the finest linear topology on \(X\) that coincides with \(\tau\) on \(\| \cdot \|\)-bounded sets and such that \(\tau \leq \gamma \leq \tau_{\|\|}\);
(b) the triple \((X, \| \cdot \|, \tau)\) is called a Saks space if there exists a directed system of seminorms \(P_{\tau}\) that generates the topology \(\tau\) such that
\[
\|x\| = \sup_{p \in P_{\tau}} p(x), \quad x \in X.
\]

In the next remark we collect some observations from [47, Remark 5.3, p. 2680], [47, Lemma 5.5 (b), p. 2681] and [13, Remark 5.2, p. 338] (see [45, Lemma 4.4, p. 163] as well) concerning the previous assumptions.

2.3. Remark. (a) The mixed topology is Hausdorff locally convex and our definition is equivalent to the one from the literature [72, Section 2.1] due to [72, Lemmas 2.2.1, 2.2.2, p. 51].
(b) Let \(\tau\) be a Hausdorff locally convex topology on \(X\) that is coarser than the \(\| \cdot \|\)-topology and \((X, \tau)'\) norming. Then the sequential completeness of \(\tau\) on \(\| \cdot \|\)-bounded sets is equivalent to the sequential completeness of \((X, \gamma)\).
(c) The existence of a system of seminorms \(P_{\tau}\) generating the topology \(\tau\) and satisfying \([\Omega]\) is equivalent to the property that \((X, \tau)'\) is norming. We note however that not every system of seminorms that induces the topology \(\tau\) need to fulfil \([\Omega]\) even if \((X, \tau)'\) is norming (see [72, Example B, p. 65]).
(d) Assumption 2.1 (iii) may be weakened to
\[
\|x\| = \sup_{y \in \Phi(\tau)} |y(x)|, \quad x \in X,
\]
where \(\Phi(\tau)\) is the set all linear functionals \(y \in X'\) with \(|y|_{X'} \leq 1\) whose restriction to the unit ball \(B_{[\|\|]} := \{ x \in X \mid \|x\| \leq 1\}\) is \(\tau\)-sequentially continuous by [51, Remarks 2.2, 2.4, p. 4]. However, no concrete example is known where this is strictly weaker than Assumption 2.1 (iii).
(e) With the previously stated facts in this remark, we can rephrase Assumption 2.1 by saying that \((X, \| \cdot \|, \tau)\) is a Saks space such that \((X, \gamma)\) is sequentially complete.

2.4. Example. (a) Let \(\Omega\) be a completely regular Hausdorff space. We recall that a topological space \(\Omega\) is called completely regular if for any non-empty closed subset \(A \subset \Omega\) and \(x \in \Omega \setminus A\) there is a continuous function \(f: \Omega \to [0,1]\) such that \(f(x) = 0\) and \(f(z) = 1\) for all \(z \in A\) (see [38, Definition 11.1, p. 180]). Let \(C_{h}(\Omega)\) be the space of bounded real- or complex-valued continuous functions on \(\Omega\) and
\[
\|f\|_{\infty} := \sup_{x \in \Omega} |f(x)|, \quad f \in C_{h}(\Omega).
\]
The compact-open topology, i.e. the topology $\tau_{co}$ of uniform convergence on compact subsets of $\Omega$, is induced by the directed system of seminorms $\mathcal{P}_{\tau_{co}}$ given by

$$p_K(f) := \sup_{x \in K} |f(x)|, \quad f \in C_b(\Omega),$$

for compact $K \subset \Omega$. Then $(C_b(\Omega), \| \cdot \|_{\infty}, \tau_{co})$ is a Saks space by [72, Example D], p. 65-66.

Let $V$ denote the set of all non-negative bounded functions $\nu$ on $\Omega$ that vanish at infinity, i.e. for every $\varepsilon > 0$ the set $\{ x \in \Omega \mid |\nu(x)| \geq \varepsilon \}$ is compact. Let $\beta_0$ be the Hausdorff locally convex topology on $C_b(\Omega)$ that is induced by the seminorms

$$|f|_\nu := \sup_{x \in \Omega} |f(x)|\nu(x), \quad f \in C_b(\Omega),$$

for $\nu \in V$. Due to [63, Theorem 2.4, p. 316] we have $\gamma(\| \cdot \|_{\infty}, \tau_{co}) = \beta_0$. If $\Omega$ is locally compact, then $V$ may be replaced by the functions in $C_0(\Omega)$ that are non-negative by [63, Theorem 2.3 (b), p. 316] where $C_0(\Omega)$ is the space of real-valued continuous functions on $\Omega$ that vanish at infinity.

(b) Let $(X, \| \cdot \|)$ be a Banach space and recall

$$\|y\|_{X'} = \sup_{x \in B_{X'}} |y(x)|, \quad y \in X'.$$

The weak$^*$-topology $\sigma^* := \sigma(X', X)$ is induced by the directed system of seminorms given by

$$p_K(y) := \sup_{x \in K} |y(x)|, \quad y \in X',$$

for finite $K \subset X$.

Then $(X', \| \cdot \|_{X'}, \sigma^*)$ is a Saks space and $\gamma(\| \cdot \|_{X'}, \sigma^*) = \tau_c(X', X)$ by [72, Example E], p. 66 where $\tau_c(X', X)$ is the topology of uniform convergence on compact subsets of $X$.

(c) Let $(X, \| \cdot \|)$ be a Banach space. The dual Mackey-topology $\mu^* := \mu(X', X)$ is induced by the directed system of seminorms given by

$$p_K(y) := \sup_{x \in K} |y(x)|, \quad y \in X',$$

for $\sigma(X, X')$-compact absolutely convex $K \subset X$. Since $(X', \mu^*)' = X$ is norming, $(X', \| \cdot \|_{X'}, \mu^*)$ is a Saks space by Remark 2.3 (c). Furthermore, we have $\gamma(\| \cdot \|_{X'}, \mu^*) = \mu^*$ by the second example in [17, p. 593] as $\gamma(\| \cdot \|_{X'}, \mu^*)$ is, in particular, the finest Hausdorff locally convex topology that coincides with $\mu^*$ on $\| \cdot \|_{X'}$-bounded sets by Remark 2.3 (a).

(d) Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be Banach spaces and recall

$$\|R\|_{\mathcal{L}(X; Y)} = \sup_{x \in B_{X}} \|Rx\|_Y, \quad R \in \mathcal{L}(X; Y).$$

The weak operator topology $\tau_{wot}$ on $\mathcal{L}(X; Y)$ is induced by the directed system of seminorms given by

$$p_{N, M}(R) := \sup_{x \in N, y \in M} |y(Rx)|, \quad R \in \mathcal{L}(X; Y),$$

for finite $N \subset X$ and finite $M \subset Y'$. The strong operator topology $\tau_{sot}$ on $\mathcal{L}(X; Y)$ is induced by the directed system of seminorms given by

$$p_N(R) := \sup_{x \in N} \|Rx\|_Y, \quad R \in \mathcal{L}(X; Y),$$

for finite $N \subset X$. Due to [48, p. 75] and [23, VI.1.4 Theorem, p. 477] $(\mathcal{L}(X; Y), \tau_{wot})' = (\mathcal{L}(X; Y), \tau_{sot})'$ is norming for $(\mathcal{L}(X; Y), \| \cdot \|_{\mathcal{L}(X; Y)})$ and

$\mathcal{P}_{\tau_{co}}$.
thus \((L(X;Y),\|\cdot\|_{L(X;Y)},\tau_{wot})\) and \((L(X;Y),\|\cdot\|_{L(X;Y)},\tau_{wot})\) are Saks spaces by Remark 2.3 (c).

Concerning example (a), we note that examples of completely regular Hausdorff spaces are locally compact Hausdorff spaces by [27, 3.3.1 Theorem, p. 148], uniformisable, particularly metrisable, spaces by [38, Proposition 11.5, p. 181] and Hausdorff locally convex spaces by [27, Proposition 3.27, p. 95].

2.5. Definition ([49, Definition 3, p. 207]). Let \((X,\|\cdot\|,\tau)\) be a triple satisfying Assumption 2.4. A family \((T(t))_{t\geq 0}\) in \(L(X)\) is called a \(\tau\)-bi-continuous semigroup if

(i) \((T(t))_{t\geq 0}\) is a semigroup, i.e. \(T(t+s) = T(t)T(s)\) and \(T(0) = id\) for all \(t,s \geq 0\),

(ii) \((T(t))_{t\geq 0}\) is \(\tau\)-strongly continuous, i.e. the map \(T_x: [0,\infty) \to (X,\tau), T_x(t) := T(t)x\), is continuous for all \(x \in X\),

(iii) \((T(t))_{t\geq 0}\) is exponentially bounded (of type \(\omega\)), i.e. there exist \(M \geq 1\) and \(\omega \in \mathbb{R}\) such that \(\|T(t)\|_{L(X)} \leq M e^{\omega t}\) for all \(t \geq 0\),

(iv) \((T(t))_{t\geq 0}\) is locally bi-equicontinuous, i.e. for every sequence \((x_n)_{n\in\mathbb{N}}\) in \(X\), \(x \in X\) with \(\sup_{n\in\mathbb{N}}\|x_n\| < \infty\) and \(\tau\)-lim \(x_n = x\) it holds that

\[\tau\text{-lim}_{n\to\infty} (T(t)(x_n - x)) = 0\]

locally uniformly for all \(t \in [0,\infty)\).

For a \(\tau\)-bi-continuous semigroup \((T(t))_{t\geq 0}\) we call

\[\omega_0 := \omega_0(T) := \inf \{\omega \in \mathbb{R} \mid \exists M \geq 1 \forall t \geq 0 : \|T(t)\|_{L(X)} \leq M e^{\omega t}\} \]

divid its growth bound (see [48, p. 7]).

2.6. Remark. Let \((X,\|\cdot\|)\) be a Banach space, \(\tau\) a Hausdorff locally convex topology on \(X\) that is coarser than the \(\|\cdot\|\)-topology, and \(\gamma := \gamma(\|\cdot\|,\tau)\) the mixed topology.

(a) Due to [18, 1.1.10 Proposition, p. 9] a sequence \((x_n)_{n\in\mathbb{N}}\) in \(X\) is \(\gamma\)-convergent if and only if it is \(\tau\)-convergent and \(\|\cdot\|\)-bounded. This implies that \(\gamma\)-bi-continuous semigroups are \(\gamma\)-strongly continuous and locally sequentially \(\gamma\)-equicontinuous.

(b) The condition of exponential boundedness in Definition 2.5 is superfluous. Indeed, if \((T(t))_{t\geq 0}\) is a semigroup of linear operators on \(X\) that is \(\gamma\)-strongly continuous and locally sequentially \(\gamma\)-equicontinuous, then \((T(t))_{t\geq 0}\) is exponentially bounded by [52, Proposition 3.6 (ii), p. 1137] because a set in \(X\) is \(\gamma\)-bounded if and only if it is \(\|\cdot\|\)-bounded by 2.4.1 Corollary, p. 56].

(c) Let \((X,\|\cdot\|,\tau)\) satisfy Assumption 2.4. It follows from (a) and (b) that a family of operators \((T(t))_{t\geq 0}\) in \(L(X)\) is a \(\tau\)-bi-continuous semigroup if and only if it is a \(\gamma\)-strongly continuous and locally sequentially \(\gamma\)-equicontinuous semigroup. This remains valid if \(\gamma\) is replaced by any other Hausdorff locally convex topology on \(X\) that has the same convergent sequences as \(\gamma\) (cf. [29, Lemma A.1.2, p. 72] and [32, Proposition 1.6, p. 313]).

3. Continuity, equicontinuity and tightness

We briefly recall the different types of equicontinuity that emerged in the context of semigroups.

3.1. Definition. Let \((X_1,\tau_1)\) and \((X_2,\tau_2)\) be Hausdorff locally convex spaces. A family \((T(t))_{t\in\mathbb{R}}\) of maps from a set \(M_1 \subset X_1\) to \(X_2\) is called \(\tau_1,M_1,\tau_2\)-equicontinuous at \(x \in M_1\) if for every \(\tau_2\)-neighbourhood \(U_2\) of zero in \(X_2\) there is a \(\tau_1\)-neighbourhood
3.5. If \( x \in X_1 \) such that \( T(t)(U_1 \cap M_1) \subset T(t)(x) + U_2 \) for all \( t \in I \). The family \( (T(t))_{t \in I} \) is called \( \tau_{1|M_1, \tau_2}\text{-equicontinuous on } M_1 \) if it is \( \tau_{1|M_1, \tau_2}\text{-equicontinuous at every } x \in M_1 \). If \((X_2, \tau_2) = (X_1, \tau_1)\) and \( M_1 = X_1 \), then we just write \( \tau_{1}\text{-equicontinuous instead of } \tau_{1|M_1, \tau_2}\text{-equicontinuous on } M_1 \).

### 3.2. Remark.

Let \( \mathcal{P}_{\tau_1} \) and \( \mathcal{P}_{\tau_2} \) be directed systems of seminorms inducing the topologies \( \tau_1 \) and \( \tau_2 \), respectively. If \( (T(t))_{t \in I} \) is a family of linear maps \( T(t) : X_1 \to X_2 \), then it is \( \tau_1\text{-}\tau_2\text{-equicontinuous if and only if}

\[
\forall \, p \in \mathcal{P}_{\tau_2} \exists \, \tilde{p} \in \mathcal{P}_{\tau_1} \quad C \geq 0 \forall \, t \in I, \, x \in X_1 : p(T(t)x) \leq C \tilde{p}(x).
\]

### 3.3. Definition.

Let \((X, \tau)\) be a Hausdorff locally convex space and \((T(t))_{t \geq 0}\) a family of linear maps \( X \to X \).

1. \((T(t))_{t \geq 0}\) is called locally \( \tau\text{-equicontinuous if } (T(t))_{t \in [0, t_0]} \) is \( \tau\text{-equicontinuous for all } t_0 \geq 0 \).
2. \((T(t))_{t \geq 0}\) is called quasi-\( \tau\text{-equicontinuous if there exists } \alpha \in \mathbb{R} \text{ such that } (e^{-\alpha t} T(t))_{t \geq 0} \text{ is } \tau\text{-equicontinuous.}

\( \tau\text{-equicontinuity of } (T(t))_{t \geq 0} \) implies quasi-\( \tau\text{-equicontinuity, which implies local } \tau\text{-equicontinuity, which again implies the } \tau\text{-continuity of all } T(t), \, t \geq 0 \). Due to \cite{49} Example 3.2, p. 549 the left translation semigroup on \( C_0(\mathbb{R}) \) is locally \( \tau_{\mathbb{R}}\text{-equicontinuous but not quasi-} \tau_{\mathbb{R}}\text{-equicontinuous. The Gauß–Weierstraß semigroup on } C_0(\mathbb{R}^d) \text{ is } \tau_{\mathbb{R}}\text{-bi-continuous but not locally } \tau_{\mathbb{R}}\text{-equicontinuous by } \cite{49} \text{ Examples } 6 \text{ (a), p. 209-210}. \) However, we will see in Example 4.2 that both semigroups are locally \( \beta_0\text{-equicontinuous, even quasi-} \beta_0\text{-equicontinuous, where we recall that } \beta_0 = \gamma(\| \cdot \|, \tau_{\mathbb{R}}) \).

### 3.4. Definition.

Let \((X, \| \cdot \|, \tau)\) be a Saks space and \( \mathcal{P}_\tau \) a directed system of seminorms generating the topology \( \tau \). A family of linear maps \((T(t))_{t \in I}\) from \( X \) to \( X \) is called \((\| \cdot \|, \tau)\text{-equitight if}

\[
\forall \, \varepsilon > 0, \, p \in \mathcal{P}_\tau \exists \, \tilde{p} \in \mathcal{P}_\tau, \, C \geq 0 \forall \, t \in I, \, x \in X : p(T(t)x) \leq C \tilde{p}(x) + \varepsilon \|x\|.
\]

If \( I \) is a singleton, i.e. \( I = \{t\} \) for some \( t \), we just call \( T(t) \) \((\| \cdot \|, \tau)\text{-tight. If no confusion seems to be likely, we just write equitight and tight instead of } (\| \cdot \|, \tau)\text{-equitight and } (\| \cdot \|, \tau)\text{-tight, respectively.}

We note that the definition of (equi-)tightness does not depend on the choice of the directed system of seminorms \( \mathcal{P}_\tau \) that generates the topology \( \tau \).

Tight operators \( T \in \mathcal{L}(X) \) as well as families of equitight operators \((T(t))_{t \in [0, t_0]}\) in \( \mathcal{L}(X) \) were introduced in \cite{19} Definitions 1.2.20, 1.2.21, p. 12 under the name \textit{local} (see \cite{14} \textbf{Definition} 5.13, p. 79] and \cite{11} \textbf{Definition} 5, p. 5] as well). In the context of \( \tau\text{-}\text{bi-continuous semigroups } (T(t))_{t \geq 0} \) the notion of tightness is used in \cite{10} \textbf{Definition} 1.1, p. 668], meaning that \((T(t))_{t \in [0, t_0]}\) is equitight (or local) for all \( t_0 \geq 0 \). The name tightness stems from the relation to tight measures in the context of \( \tau_{\mathbb{R}}\text{-bi-continuous semigroups on } C_0(\Omega), \text{ } \Omega \text{ Polish (i.e. completely metrisable and separable), see } \cite{33} \text{ Lemma 2.3, Theorem 2.4, p. 92}, \cite{32} \text{ Proposition 3.3, p. 317} \text{ and Remark } 3.20 \text{ (a). We prefer the notion of tightness to localness in varying degrees in correspondence to (equi-)continuity and thus introduce the following notions of local and quasi-equitightness.

### 3.5. Definition.

Let \((X, \| \cdot \|, \tau)\) be a Saks space and \((T(t))_{t \geq 0}\) a family of linear maps \( X \to X \).

1. \((T(t))_{t \geq 0}\) is called \textit{locally equitight} if \((T(t))_{t \in [0, t_0]}\) is equitight for all \( t_0 \geq 0 \).
2. \((T(t))_{t \geq 0}\) is called \textit{quasi-equitight} if there is \( \alpha \in \mathbb{R} \) such that \((e^{-\alpha t} T(t))_{t \geq 0}\) is equitight.
Equitightness of \((T(t))_{t \geq 0}\) implies quasi-equitightness, which implies local equitightness, which again implies the tightness of all \(T(t), t \geq 0\). We remark that the notion of tightness is not only relevant for bi-continuous semigroups but for other operators as well. In [23, Lemma 1.2.23, p. 12] it is shown that the resolvent family \(\{R(\lambda, A) \mid \lambda \in [\alpha, \beta]\}\) for \(\beta \geq \alpha > \omega_0\) of a locally equitight \(\tau\)-bi-continuous semigroup \((T(t))_{t \geq 0}\) of type \(\omega\) is equitight as well.\footnote{The assumption that \((T(t))_{t \geq 0}\) should be locally equitight (i.e. local in terms of \([23]\)) is missing in [23, Lemma 1.2.23, p. 12] but is used in its proof.} We note the following slight improvement (and correction) of this statement.

3.6. Proposition. Let \((X, |\cdot|, \tau)\) be a triple satisfying Assumption \([\text{24}]\) and \((T(t))_{t \geq 0}\) a \(\tau\)-bi-continuous semigroup. If \((T(t))_{t \geq 0}\) is locally equitight, then \(\{R(\lambda, A) \mid \Re \lambda \in [\alpha, \beta]\}\) is equitight for all \(\beta \geq \alpha > \omega_0(T)\).

Proof. First, we note that [29, Lemma 1.2.23, p. 12], which is based on representing the resolvent as a Laplace transform \([49, \text{Definition 9, p. 213}]\), still holds if the type \(\omega\) is replaced by the growth bound \(\omega_0(T)\) and the condition \(\lambda > \omega\) by \(\Re \lambda > \omega_0(T)\) since the estimates in the proof on [29, p. 13] still hold if \(\lambda\) is replaced by \(\Re \lambda\) after the first “\(<\)” sign. Then it follows from this updated version of [23, Lemma 1.2.23, p. 12] and [29, Definition 1.2.21, p. 12] that \(\{R(\lambda, A) \mid \Re \lambda \in [\alpha, \beta]\}\) is equitight for all \(\beta \geq \alpha > \omega_0(T)\).

The following example shows that one cannot drop the condition of local equitightness of \((T(t))_{t \geq 0}\) in Proposition 3.6.

3.7. Example. We use the example of a \(\tau_{\text{co}}\)-bi-continuous semigroup \((T(t))_{t \geq 0}\) on the space \(C_b(\Omega)\) of bounded \(\mathbb{R}\)-valued continuous functions on \(\Omega\) from [32, Example 4.1, p. 320] where \(\Omega := [0, \omega_1]\) is equipped with the order topology and \(\omega_1\) is the first uncountable ordinal. Let \(\beta \Omega\) be the Stone–Čech compactification of \(\Omega\), i.e. \(\beta \Omega = [0, \omega_1]\), and choose \(x \in \beta \Omega\) as constructed in [32, Example 4.1, p. 320], i.e. \(x = \omega_1\). Then the operator \(A := 1 \otimes \delta_x\), i.e. \(Af(z) = f(x), z \in \Omega\), for \(f \in C_b(\Omega)\) with \(f\) continuously extended to \(\beta \Omega\), belongs to \(\mathcal{L}(C_b(\Omega))\) and generates the \(\tau_{\text{co}}\)-bi-continuous semigroup on \(C_b(\Omega)\) given by

\[
T(t) := \text{id} - A + \varepsilon f(A), \quad t \geq 0,
\]

by [32, Example 4.1, p. 320]. Further, the operators \(T(t)\) for \(t > 0\) are not \(\tau_{\text{co}}\)-continuous on \(\|\cdot\|_\infty\)-bounded sets by [32, Example 4.1, p. 320] and thus the semigroup \((T(t))_{t \geq 0}\) is not locally \((\|\cdot\|_\infty, \tau_{\text{co}})\)-equitight by Proposition 3.6 below. We have

\[
\|T(t)\|_{\mathcal{L}(C_b(\Omega))} = \sup_{f \in C_b(\Omega)} \|T(t)f\|_\infty = \sup_{f \in C_b(\Omega)} \sup_{z \in \Omega} |f(z) - f(x) + \varepsilon f(x)| \leq 3e^t, \quad t \geq 0,
\]

thus \(\omega_0(T) \leq 1\), and

\[
R(\lambda, A)f = \int_0^\infty e^{-\lambda s}T(s)f(t)ds = \int_0^\infty e^{-\lambda s}(f - f(x) + \varepsilon f(x))ds
\]

\[
= (f - f(x)) \int_0^\infty e^{-\lambda s}ds + f(x) \int_0^\infty e^{(1-\lambda)s}ds = \frac{f - f(x)}{\lambda} + \frac{f(x)}{\lambda - 1}
\]

for \(\Re \lambda > 1\) and \(f \in C_b(\Omega)\) by [49, Definition 9, p. 213] where the first two integrals are to be understood as improper \(\tau_{\text{co}}\)-Riemann integrals.

Now, suppose that \(\{R(\lambda, A) \mid \lambda \in [2, 3]\}\) is \((\|\cdot\|_\infty, \tau_{\text{co}})\)-equitight. Let \(K \subset \Omega\) be compact and choose \(\varepsilon := \frac{1}{n}\) for \(n \in \mathbb{N}\). Then there are a compact set \(\bar{K}_{K,n} \subset \Omega\) and
$C_{K,n} \geq 0$ such that for all $\lambda \in [2,3]$ and $f \in C_b(\Omega)$ it holds that

$$\sup_{z \in K} \left| \frac{1}{\lambda} f(z) + \frac{1}{\lambda(\lambda - 1)} f(x) \right| = \sup_{z \in K} |R(\lambda, A)f(z)| \leq C_{K,n} \sup_{z \in K} |f(z)| + \frac{1}{n} |f|_\infty. \quad (2)$$

The set $A_{K,n} := (K \cup \overline{K}_{K,n}) \subset \Omega = [0,\omega_1)$ is a compact subset of $\beta\Omega = [0,\omega_1]$ for every $n \in \mathbb{N}$ and $x = \omega_1 \notin [0,\omega_1)$. Since $[0,\omega_1)$ is a completely regular Hausdorff space by Part II, 43. Closed ordinal space $[0,\omega]; 4, p. 69$, every $n \in \mathbb{N}$ there exists a continuous function $f_n: [0,\omega_1) \rightarrow [0,1]$ such that $f_n = 0$ on $A_{K,n}$ and $f_n(x) = 1$ by [8, (2.1.5) Proposition, p. 17]. This implies that

$$\frac{1}{6} \leq \frac{1}{\lambda(\lambda - 1)} = \sup_{z \in K} \left| \frac{1}{\lambda} f_n(z) + \frac{1}{\lambda(\lambda - 1)} f_n(x) \right| \leq C_{K,n} \sup_{z \in K} |f_n(z)| + \frac{1}{n} |f_n|_\infty = \frac{1}{n}$$

for all $n \in \mathbb{N}$ and $\lambda \in [2,3]$, which is a contradiction.

Local equitightness plays a crucial role in perturbation results for bi-continuous semigroups, see [26, Theorem 1.2, p. 669] (cf. [29, Theorem 3.2.3, p. 47], [31, Theorems 2.4, 3.2, p. 92, 94-95], [32, Remark 4.1, p. 101], [11, Theorem 5, p. 8] (cf. [10, Theorem 5.19, p. 81]) and [12, Theorem 3.3, p. 582]. Unfortunately, the missing assumption of local equitightness in [29, Lemma 1.2.3, p. 12] has some consequences for one of the cited references above.

### 3.8. Remark

We refer to [11] for the relevant notions. In [11] Theorem 5, p. 8] (cf. [10, Theorem 5.19, p. 81]) we have the following assumptions. Let $(X, |\cdot|, \tau)$ be a triple satisfying Assumption 2.1 (A, D(A)) the generator of a positive, locally equitight (local in terms of [11]) $\tau$-bi-continuous semigroup $(T(t))_{t \geq 0}$ on a bi-AL space $X$ with $\eta$-bi-dense domain $D(A)$ for some $1 < \eta < 2$. Let $B: D(A) \rightarrow X$ be a positive operator, i.e. $Bx \geq 0$ for each $x \in D(A) \cap X_+$, and assume that $BR(\lambda, A)$ is tight and $(A + B, D(A))$ is resolvent positive.

In the proof of [11] Theorem 5, p. 8] it is used in [11, p. 14] that for $s \in [0,1]$ the tight operator $sBR(\lambda, A) \in L(X)$ generates a $\tau$-bi-continuous and $|\cdot|$-strongly continuous semigroup $(E(t))_{t \geq 0}$ on $X$ given by

$$E(t) = e^{tsBR(\lambda, A)} = \sum_{n=0}^{\infty} \frac{t^n(sBR(\lambda, A))^n}{n!}, \quad t \geq 0.$$ 

Then [29, Lemma 1.2.23, p. 12] is applied to the semigroup $(E(t))_{t \geq 0}$ to conclude that $R(1, sBR(\lambda, A))$ is tight. But it seems not to be guaranteed that this conclusion is true due to the missing assumption in [29, Lemma 1.2.23, p. 12]. However, the conclusion is true if $(E(t))_{t \geq 0}$ is supposed to be locally equitight by Proposition 3.6. For instance, this additional assumption can be ensured by Theorem 3.17 (b) below which implies that every $\tau$-bi-continuous semigroup on $X$ is locally equitight if the space $(X, \gamma)$ with mixed topology $\gamma := \gamma(\|\cdot\|, \tau)$ is $C$-sequential, meaning that every convex sequentially open subset of $(X, \gamma)$ is already open, and if $\gamma$ coincides with the submixed topology $\gamma_s$ (see Definition 3.9 below).

Concerning the application of [11], Theorem 5, p. 8], the [11, Example 3.1, p. 14-15] of rank-one perturbations can be repaired, at least, under the additional assumption that $(X, \gamma)$ is $C$-sequential and $\gamma = \gamma_s$. The [11, Example 3.2, p. 15-21] of the Gauß–Weierstraß semigroup on the space $X = M(\mathbb{R})$ of bounded Borel measures on $\mathbb{R}$ can be saved as well since $M(\mathbb{R})$ coincides with the space $M_c(\mathbb{R})$ of bounded Radon measures on the Polish space $\mathbb{R}$ and $\gamma := \gamma(\|\cdot\|_{M_c(\mathbb{R})}, \sigma(M_c(\mathbb{R}), C_b(\mathbb{R})))$ fulfills that $(X, \gamma)$ is $C$-sequential and $\gamma = \gamma_s$ by Corollary 3.23. Here $\|\cdot\|_{M_c(\mathbb{R})}$ denotes the total variation norm on $M_c(\mathbb{R})$ (see e.g. [51, p. 543]).

We will study the relation between $\gamma$-(equi-)continuity and (equi-)tightness in the forthcoming. For that purpose we introduce another kind of mixed topology (see [18, p. 41]).
3.9. **Definition.** Let \((X, \| \cdot \|, \tau)\) be a Saks space and \(\mathcal{P}_\tau\) a directed system of seminorms that generates the topology \(\tau\) and fulfills (11). For a sequence \((p_n)_{n \in \mathbb{N}}\) in \(\mathcal{P}_\tau\) and a sequence \((a_n)_{n \in \mathbb{N}}\) in \((0, \infty)\) with \(\lim_{n \to \infty} a_n = \infty\) we define the seminorm
\[
\|x\|_{(p_n), (a_n)} := \sup_{n \in \mathbb{N}} p_n(x)a_n^{-1}, \quad x \in X.
\]
We denote by \(\gamma_s := \gamma_s(\| \cdot \|, \tau)\) the locally convex Hausdorff topology that is generated by the system of seminorms \(\{\|x\|_{(p_n), (a_n)}\}_{n \in \mathbb{N}}\) and call it the **submixed topology**.

3.10. **Remark.** Let \((X, \| \cdot \|, \tau)\) be a Saks space, \(\mathcal{P}_\tau\) a directed system of seminorms that generates the topology \(\tau\) and fulfills (11), \(\gamma := \gamma(\| \cdot \|, \tau)\) the mixed and \(\gamma_s := \gamma_s(\| \cdot \|, \tau)\) the submixed topology.

(a) We have \(\tau \leq \gamma_s \leq \gamma\) since \(\gamma_s\) is stronger than \(\tau\) by definition and coarser than \(\gamma\) on \(X\) by the first part of the proof of [18, I.4.5 Proposition, p. 41-42] where \(\gamma_s\) is called \(\widehat{\gamma}\). Moreover, \(\gamma_s\) has the same convergent sequences as \(\gamma\) by [13, I.1.10 Proposition, p. 9] and [29, Lemma A.1.2, p. 72] where \(\gamma_s\) is called \(\tau_m\).

(b) If
(i) for every \(x \in X\), \(\varepsilon > 0\) and \(p \in \mathcal{P}_\tau\) there are \(y, z \in X\) such that \(x = y + z\), \(p(z) = 0\) and \(\|y\| \leq p(x) + \varepsilon\), or
(ii) the \(\| \cdot \|\)-unit ball \(B_1^\tau = \{x \in X \mid \|x\| \leq 1\}\) is \(\tau\)-compact, then \(\gamma = \gamma_s\) due to [13, I.4.5 Proposition, p. 41-42].

The submixed topology \(\gamma_s\) was introduced in [72, Theorem 3.1.1, p. 62]. It also appears under the name mixed topology with symbol \(\tau_m\) or \(\tau_0^M\) in [29, Definition A.1.1, p. 72], [12, Definition 2.4, p. 579] and [36, Definition 2.1, p. 21] but usually in a context where \(\gamma = \gamma_s\) holds.

3.11. **Example.** (a) Let \(\Omega\) be a completely regular Hausdorff space. The Saks space \((C_b(\Omega), \| \cdot \|_{\infty}, \tau_{co})\) fulfills condition (i) of Remark 3.10 (b) and \(\gamma(\| \cdot \|_{\infty}, \tau_{co}) = \gamma_s(\| \cdot \|_{\infty}, \tau_{co})\) by [72, Example D], p. 65-66).

(b) Let \((X, \| \cdot \|)\) be a Banach space. The Saks space \((X', \| \cdot \|_{X', \sigma'})\) fulfills condition (ii) of Remark 3.10 (b) with \(B_{\| \cdot \|_{X', \sigma'}}\) and \(\gamma(\| \cdot \|_{X', \sigma'}) = \gamma_s(\| \cdot \|_{X', \sigma'})\) by [22, Example E], p. 66).

(c) Let \((X, \| \cdot \|)\) be a Banach space. Due to [62, Proposition 3.1, p. 275] \(B_{\| \cdot \|_{X'}}\) is \(\mu^\sigma\)-compact if and only if \(X\) is a **Schur space**, i.e. every \(\sigma(X, X')\)-convergent sequence is \(\| \cdot \|\)-convergent [23, p. 253]. Thus the Saks space \((X', \| \cdot \|_{X', \mu^\sigma})\) satisfies condition (ii) of Remark 3.10 (b) and \(\gamma(\| \cdot \|_{X', \mu^\sigma}) = \gamma_s(\| \cdot \|_{X', \mu^\sigma})\) if \(X\) is a Schur space.

(d) Let \((X, \| \cdot \|)\) and \((Y, \| \cdot \|)\) be Banach spaces. \(B_{\| \cdot \|_{L(X,Y)}}\) is \(\tau_{\text{wot}}\)-compact if and only if \(X\) is reflexive by [16, Theorem 2.19, p. 1689]. \(B_{\| \cdot \|_{L(X,Y)}}\) is \(\tau_{\text{wot}}\)-compact if and only if \(Y\) is finite-dimensional by [16, Theorem 3.15, p. 1699]. Thus for the Saks spaces \((L(X,Y), \| \cdot \|_{L(X,Y), \tau_{\text{wot}}})\) and \((L(X,Y), \| \cdot \|_{L(X,Y), \tau_{\text{wot}}})\) condition (ii) of Remark 3.10 (b) holds and \(\gamma(\| \cdot \|_{L(X,Y), \tau_{\text{wot}}}) = \gamma_s(\| \cdot \|_{L(X,Y), \tau_{\text{wot}}})\) if \(Y\) is reflexive, and \(\gamma(\| \cdot \|_{L(X,Y), \tau_{\text{wot}}}) = \gamma_s(\| \cdot \|_{L(X,Y), \tau_{\text{wot}}})\) if \(Y\) is finite-dimensional, respectively. If \(Y\) is the scalar field of \(X\), then the last case is already covered by example (b) above.

Concerning example (c), we note that the space of absolutely summable sequences \(\ell^1 := \ell^1(\mathbb{N})\) is a Schur space by [27, Theorem 5.36, p. 252]. Another example is the **Lipschitz-free or Arens-Eells space** \(F(M, \omega \circ d)\) by [41, Theorem 4.6, p. 186] where \((M, d)\) is a metric space and \(\omega : [0, \infty) \to [0, \infty)\) a continuous increasing subadditive function with \(\omega(0) = 0\), \(\omega(t) \geq t\) for \(0 \leq t \leq 1\) and \(\lim_{t \to \infty} \omega(t)/t = \infty\), e.g. \(\omega(t) := t^\alpha\) when \(0 < \alpha < 1\). If \(M\) contains a distinguished point (the origin), denoted by 0, then
\( \mathcal{F}(M, \omega \circ d) \) is the canonical predual of the space \( \text{Lip}(M) \) of Lipschitz continuous functions \( f: M \to \mathbb{R} \) with \( f(0) = 0 \) equipped with the norm

\[ \|f\|_{\text{Lip}} := \sup_{x, y \in M, x \neq y} \frac{|f(x) - f(y)|}{\omega(x, y)} \]

(see [11, p. 179]). If \( \omega(t) := t^\alpha \) for some \( 0 < \alpha < 1 \), then \( \text{Lip}(M) \) is the space of \( \alpha \)-Hölder continuous functions on \( (M, d) \) that vanish at \( 0 \).

For the next proposition we recall our notation that \( \mathcal{L}(X) = \mathcal{L}(X, \| \cdot \|; (X, \| \cdot \|)) \).

3.12. Proposition. Let \( (X, \| \cdot \|, \tau) \) be a Saks space, \( \gamma := \gamma(\| \cdot \|, \tau) \) the mixed and \( \gamma_s := \gamma_s(\| \cdot \|, \tau) \) the submixed topology. Consider the following assertions for a linear map \( S: X \to X \):

- (a) \( S \) is \( \gamma_s \)-continuous.
- (b) \( S \) is \( \gamma_s \)-\( \tau \)-continuous.
- (c) \( S \) is \( (\| \cdot \|, \tau) \)-tight.
- (d) \( S \) is \( \tau \)-continuous on \( \| \cdot \| \)-bounded sets, i.e. the restricted map \( S|_{B} : (B, \tau_B) \to (X, \tau) \) is continuous on \( B \) for every \( \| \cdot \| \)-bounded set \( B \subset X \).
- (e) \( S \) is \( \tau \)-continuous at zero on the \( \| \cdot \| \)-unit ball \( B_{\| \cdot \|} \).
- (f) \( S \) is \( \gamma \)-\( \tau \)-continuous.
- (g) \( S \) is \( \gamma \)-continuous.

Then \( (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Leftrightarrow (e) \Leftrightarrow (f) \Leftrightarrow (g) \). Moreover, if \( S \in \mathcal{L}(X) \), then \( (f) \Rightarrow (g) \). If \( S \in \mathcal{L}(X) \) and \( \gamma = \gamma_s \), then all seven assertions are equivalent.

Proof. In the following proof \( \mathcal{P}_\tau \) denotes the directed system of seminorms that generates the \( \tau \)-topology and fulfills (1), i.e. \( \|x\| = \sup_{p \in \mathcal{P}_\tau} px \) for all \( x \in X \), from the definition of the submixed topology \( \gamma_s \) (see the note below Definition 3.4).

- (a) \( \Rightarrow (b) \): Obvious as \( \tau \) is coarser than \( \gamma_s \) and \( \gamma \).

- (b) \( \Rightarrow (c) \): Let \( S: (X, \gamma_s) \to (X, \tau) \) be continuous and \( p \in \mathcal{P}_\tau \). Then there are \( (p_n)_{n \in \mathbb{N}} \in \mathcal{P}_\tau \), a sequence \( (a_n)_{n \in \mathbb{N}} \) in \( (0, \infty) \) with \( \lim_{n \to \infty} a_n = \infty \) and \( C > 0 \) such that for any \( x \in X \)

\[ p(Sx) \leq C \|x\|_{(p_n),(a_n)} = C \sup_{n \in \mathbb{N}} p_n(x) a_n^{-1}. \]

Let \( \varepsilon > 0 \). Then there is \( N \in \mathbb{N} \) such that \( a_n^{-1} \leq \varepsilon/C \) for all \( n \geq N \). Since \( \mathcal{P}_\tau \) is directed, there are \( \overline{p} \in \mathcal{P}_\tau \) and \( \overline{C} \geq 0 \) such that

\[ p(Sx) \leq C \sup_{1 \leq n \leq N} p_n(x) a_n^{-1} + C \sup_{n \geq N} p_n(x) a_n^{-1} \leq C \overline{C} \sup_{1 \leq n \leq N} \overline{p}(x) + \varepsilon \|x\| \]

for all \( x \in X \) and thus (c) holds.

- (c) \( \Rightarrow (e) \): Let (c) hold and \( \overline{\varepsilon} > 0 \). For \( p \in \mathcal{P}_\tau \) and \( \varepsilon = \overline{\varepsilon}/2 \) there are \( \overline{p} \in \mathcal{P}_\tau \) and \( C \geq 0 \) by (c) such that

\[ p(Sx) \leq C \overline{p}(x) + \varepsilon \|x\| \leq C \overline{p}(x) + \frac{\overline{\varepsilon}}{2} \]

for all \( x \in B_{\| \cdot \|} \). We set \( U_{\overline{p}}(0; \frac{\overline{\varepsilon}}{2}) := \{ x \in X \mid \overline{p}(x) \leq \frac{\overline{\varepsilon}}{2} \} \) and note that

\[ p(Sx) \leq C \overline{p}(x) + \frac{\overline{\varepsilon}}{2} \leq \overline{\varepsilon} \]

for all \( x \in U_{\overline{p}}(0; \frac{\overline{\varepsilon}}{2}) \cap B_{\| \cdot \|} \), implying the \( \tau \)-continuity of \( S \) at zero on \( B_{\| \cdot \|} \).

- (d) \( \Rightarrow (e) \) \( \Leftrightarrow (f) \): The first equivalence is [18, I.1.8 Lemma p. 8] combined with the fact that a subset \( B \subset X \) is \( \| \cdot \| \)-bounded if and only if there is \( n \in \mathbb{N} \) such that \( B \subset nB_{\| \cdot \|} \). Due to [18, I.1.7 Corollary p. 8] the equivalence (d) \( \Leftrightarrow (f) \) holds.

- (f) \( \Rightarrow (g) \) if \( S \in \mathcal{L}(X) \): Let (f) hold and \( S \in \mathcal{L}(X) \). We know that (f) and (e) are equivalent. Let \( \varepsilon > 0 \) and \( p \in \mathcal{P}_\tau \). Due to (e) there are \( \overline{p} \in \mathcal{P}_\tau \) and \( \overline{\delta} > 0 \) such that \( p(Sx) \leq \varepsilon \) for all \( x \in U_{\overline{p}}(0; \overline{\delta}) \cap B_{\| \cdot \|} \). Since \( S \in \mathcal{L}(X) \), we have that \( \|Sx\| \leq \|S\|_{\mathcal{L}(X)} \|x\| \) for all \( x \in B_{\| \cdot \|} \). It follows that \( \|Sx\| \leq \|S\|_{\mathcal{L}(X)}B_{\| \cdot \|} \) for all
$x \in U_p(0;\delta) \cap B_{\parallel}$. Due to \cite[1.1.7 Corollary p. 8]{15} and \cite[1.1.8 Lemma p. 8]{15} this means that $S:(X,\gamma) \to (X,\gamma)$ is continuous. 

Let us recall that a topological vector space $(X,\tau)$ is called convex-sequential or C-sequential (see \cite[p. 273]{16}) if every convex sequentially open subset of $(X,\tau)$ is already open. For our next proofs we need a classification of C-sequential Hausdorff locally convex spaces. Let $(X,\tau)$ be a Hausdorff locally convex space and $U^+$ be the collection of all absolutely convex subsets $U \subset X$ which satisfy the condition that every sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ converging to 0 is eventually in $U$. Then $U^+$ is a zero neighbourhood basis for a Hausdorff locally convex topology $\tau^+$ on $X$, which is the finest Hausdorff locally convex topology on $X$ with the same convergent sequences as $\tau$ by \cite[Proposition 1.1, p. 342]{17}. In the view of Remark \ref{remark:3.10}(a) we have $\gamma^+ = \gamma^+$. 

3.13. \textbf{Proposition} \cite[Theorem 7.4, p. 52]{18}. Let $(X,\tau)$ be a Hausdorff locally convex space. Then the following assertions are equivalent:

(a) $X$ is C-sequential.

(b) $\tau^+ = \tau$.

(c) For any Hausdorff locally convex space $Y$ a linear map $f:X \to Y$ is continuous if and only if it is sequentially continuous.

Every bornological topological vector space is C-sequential by \cite[Theorem 8, p. 280]{16}. However, if $(X,\gamma)$ is bornological (or barrelled) for the mixed topology $\gamma := \gamma(\cdot,\cdot,\tau)$, then $\gamma$ coincides with the $\|\cdot\|$-topology by \cite[1.1.15 Proposition, p. 12]{18}. Hence in the interesting cases (for us) the space $(X,\gamma)$ is neither bornological nor barrelled. We recall the following sufficient conditions for being C-sequential given in \cite[Proposition 5.7, p. 2681-2682]{19} and \cite[Corollary 7.6, p. 52]{18}.

3.14. \textbf{Proposition}. Let $(X,\|\cdot\|)$ be a Banach space and $\tau$ a Hausdorff locally convex topology on $X$ that is coarser than the $\|\cdot\|$-topology and $\gamma := \gamma(\cdot,\cdot,\tau)$ the mixed topology. If

(i) $\tau$ is metrisable on the $\|\cdot\|$-unit ball $B_{\|\|} := \{x \in X \mid \|x\| \leq 1\}$, or

(ii) $(X,\gamma)$ is a Mackey–Mazur space, then $(X,\gamma)$ is a C-sequential space.

Recall that a Hausdorff locally convex space $(X,\tau)$ with scalar field $K = \mathbb{R}$ or $\mathbb{C}$ is called Mazur space (see \cite[40]{20} or weakly semi-bornological (see \cite[p. 337]{6}) if

$$(X,\tau') = \{y : X \to K \mid y \text{ is } \tau\text{-sequentially continuous}\}.$$ 

Due to Proposition \ref{proposition:3.13} every C-sequential space is a Mazur space. Mackey–Mazur spaces are also called semi-bornological spaces (see \cite[p. 337]{6}).

3.15. \textbf{Theorem}. Let $(X,\|\cdot\|)$ be a triple satisfying Assumption \cite[2.7]{21} $\gamma := \gamma(\cdot,\cdot,\tau)$ the mixed and $\gamma_s := \gamma_s(\cdot,\cdot,\tau)$ the submixed topology and $(T(t))_{t \geq 0}$ a $\tau$-bi-continuous semigroup.

(a) If $(X,\gamma)$ is C-sequential, then $T(t)$ is $\gamma$-continuous for every $t \geq 0$.

(b) If $(X,\gamma)$ is C-sequential and $\gamma = \gamma_s$, then $T(t)$ is $(\|\cdot\|,\tau)$-tight for every $t \geq 0$.

\textbf{Proof}. Due to Proposition \ref{proposition:3.12} and since $T(t) \in \mathcal{L}(X)$ for every $t \geq 0$ we only need to prove that $T(t)$ is $\gamma$-$\tau$-continuous for every $t \geq 0$. Due to Proposition \ref{proposition:3.13} and $(X,\gamma)$ being C-sequential the continuity of $T(t):(X,\gamma) \to (X,\tau)$ for $t \geq 0$ is equivalent to $\gamma$-$\tau$-sequential continuity. By Remark \ref{remark:2.6}(a) a sequence is $\gamma$-convergent if and only if it is $\tau$-convergent and $\|\cdot\|$-bounded. Thus the bi-equicontinuity of $(T(t))_{t \geq 0}$ implies that $T(t)$ is $\gamma$-$\tau$-sequentially continuous for every $t \geq 0$, which proves our claim. \hfill $\square$
Note that Theorem 3.15 asserts a “pointwise” statement for every $t \geq 0$ whose proof does neither require the semigroup property nor the $\tau$-strong-continuity of the trajectories, Definition 2.7 (ii). We proceed with an analogue of Proposition 3.12 for families of linear maps.

3.16. Proposition. Let $(X, \| \cdot \|, \tau)$ be a Saks space, $\gamma := \gamma(\| \cdot \|, \tau)$ the mixed and $\gamma_s := \gamma_s(\| \cdot \|, \tau)$ the submixed topology. Consider the following assertions for a family $S := (S(t))_{t \in I}$ of linear maps $X \to X$:

(a) $S$ is $\gamma_s$-equicontinuous.
(b) $S$ is $\gamma_s$-$\tau$-equicontinuous.
(c) $S$ is $(\| \cdot \|, \tau)$-equitight.
(d) $S$ is $\tau$-equicontinuous on $\| \cdot \|$-bounded sets, i.e. the restricted family $S_{|B} := (S(t)|_{B})_{t \in I}$ is $\tau_{|B}$-$\tau$-equicontinuous on $B$ for every $\| \cdot \|$-bounded set $B \subset X$.
(e) $S$ is $\tau$-equicontinuous at zero on the $\| \cdot \|$-unit ball $B_{\| \cdot \|}$.
(f) $S$ is $\gamma$-$\tau$-equicontinuous.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\iff$ (e) $\iff$ (f). Moreover, if $S \in \mathcal{L}(X)$ and $\sup_{t \in I} \| S(t) \|_{\mathcal{L}(X)} < \infty$, then (f) $\Rightarrow$ (g). If $S \in \mathcal{L}(X)$, $\sup_{t \in I} \| S(t) \|_{\mathcal{L}(X)} < \infty$ and $\gamma = \gamma_s$, then all seven assertions are equivalent.

Proof. This follows analogously to Proposition 3.12 since [18, I.1.7 Corollary p. 8] and [18, I.1.8 Lemma p. 8] cover the equicontinuous case as well. □

As an application of the preceding proposition we get our main theorem.

3.17. Theorem. Let $(X, \| \cdot \|, \tau)$ be a triple satisfying Assumption 2.7, $\gamma := \gamma(\| \cdot \|, \tau)$ the mixed and $\gamma_s := \gamma_s(\| \cdot \|, \tau)$ the submixed topology and $(T(t))_{t \geq 0}$ a $\tau$-bi-continuous semigroup.

(a) If $(X, \gamma)$ is C-sequential, then $(T(t))_{t \geq 0}$ is quasi-$\gamma$-equicontinuous.
(b) If $(X, \gamma)$ is C-sequential and $\gamma = \gamma_s$, then $(T(t))_{t \geq 0}$ is quasi-$\| \cdot \|, \tau$-equitight.

Proof. Part (a) is just a combination of results that are already known. If $(X, \gamma)$ is C-sequential, then $(T(t))_{t \geq 0}$ is an SCLE semigroup w.r.t. $\gamma$ by [45, Theorem 7.4, p. 180] combined with Proposition 3.13 which means that the semigroup $(T(t))_{t \geq 0}$ is $\gamma$-strongly continuous and locally $\gamma$-equicontinuous (see [45, p. 160]). Due to [45, Corollary 4.7, p. 165], [45, Definition 4.8, p. 165], [45, Proposition 7.3, p. 179] in combination with $(X, \gamma)$ being C-sequential and [45, Corollary 6.5, p. 176] the semigroup $(T(t))_{t \geq 0}$ is quasi-$\gamma$-equicontinuous.

Let us turn to part (b). By part (a) there is $\alpha_1 \in \mathbb{R}$ such that the rescaled semigroup $(e^{-\alpha_1 t}T(t))_{t \geq 0}$ is $\gamma$-equicontinuous. Choosing $\alpha_2 > \omega_0(T)$, we note that $(e^{-\alpha_2 t}T(t))_{t \geq 0}$ is bounded w.r.t. $\| \cdot \|_{\mathcal{L}(X)}$ by the exponential boundedness of $(T(t))_{t \geq 0}$. Hence we get with $\alpha := \max(\alpha_1, \alpha_2)$ that $(S(t))_{t \geq 0} := (e^{-\alpha t}T(t))_{t \geq 0}$ is $\gamma$-equicontinuous and $\sup_{t \geq 0} \| S(t) \|_{\mathcal{L}(X)} < \infty$. It follows from Proposition 3.17 that $(S(t))_{t \geq 0}$ is $(\| \cdot \|, \tau)$-equitight and thus $(T(t))_{t \geq 0}$ is quasi-$\| \cdot \|, \tau$-equitight. □

Part (b) gives a partial answer to the question raised in [31, Remark 2.5, p. 92-93], namely, which other $\tau$-bi-continuous semigroups apart from the $\tau_{\text{c}}$-$\text{bi-continuous}$ semigroups on $c_0(\Omega)$, with $\Omega$ Polish, are locally $(\| \cdot \|, \tau)$-equitight. We remark that the latter semigroups are actually quasi-$(\| \cdot \|, \tau_{\text{c}})$-equitight as well (see Theorem 4.3 below).

The proof of Theorem 3.17 heavily relies on the quite involved theory of [45]. Next, we present a simpler and independent proof. On the one hand this proof only works under the stronger assumption that $\tau$ is metrisable on the $\| \cdot \|_{\infty, \tau_{\text{c}}}$-unit ball $B_{\| \cdot \|_{\infty, \tau_{\text{c}}}}$, on the other hand this stronger condition is actually the one that we check.
in almost every example to get that \((X, \gamma)\) is C-sequential (see Remark 3.19). We state this special case of Theorem 3.17 in the following proposition.

3.18. **Proposition.** Let \((X, \| \cdot \|, \tau)\) be a triple satisfying Assumption 2.2, \(\gamma := \gamma(\| \cdot \|, \tau)\) the mixed and \(\gamma_s := \gamma_s(\| \cdot \|, \tau)\) the submixed topology and \((T(t))_{t \geq 0}\) a \(\tau\)-bi-continuous semigroup.

(a) If \(\tau\) is metrisable on the \(\| \cdot \|\)-unit ball \(B_{\| \cdot \|}\), then \((T(t))_{t \geq 0}\) is quasi-(\(\| \cdot \|, \tau\))-equicontinuous.

(b) If \(\tau\) is metrisable on the \(\| \cdot \|\)-unit ball \(B_{\| \cdot \|}\) and \(\gamma = \gamma_s\), then \((T(t))_{t \geq 0}\) is quasi-(\(\| \cdot \|, \tau\))-equitight.

**Proof.** Since \(\tau\) is metrisable on \(B_{\| \cdot \|}\), there is a metric \(d : B_{\| \cdot \|} \times B_{\| \cdot \|} \to [0, \infty)\) which induces the \(\tau\)-topology on \(B_{\| \cdot \|}\). We choose \(\alpha > \omega_0(T)\) and note that the rescaled semigroup \((S(t))_{t \geq 0} := (e^{-\alpha t}(T(t)))_{t \geq 0}\) is globally \(\tau\)-bi-equicontinuous on \([0, \infty)\) and \(\sup_{t \geq 0} \|S(t)\|_{L_2(X)} < \infty\) by Proposition 1.4 (b), p. 88 and the exponential boundedness of \((T(t))_{t \geq 0}\). Due to Proposition 3.16 we only need to show that \((S(t))_{t \geq 0}\) is \(\gamma\)-equicontinuous at zero on \(B_{\| \cdot \|}\). We prove this by contradiction. Suppose that \((S(t))_{t \geq 0}\) is not \(\gamma\)-equicontinuous at zero on \(B_{\| \cdot \|}\). This implies that there is \(\varepsilon > 0\), a sequence \((x_n)_{n \in \mathbb{N}}\) in \(B_{\| \cdot \|}\) and a sequence \((t_n)_{n \in \mathbb{N}}\) in \([0, \infty)\) such that \(d(x_n, 0) < \frac{\gamma}{\varepsilon}\) and \(d(S(t_n)x_n, 0) > \varepsilon\) for all \(n \in \mathbb{N}\). We deduce that \((x_n)_{n \in \mathbb{N}}\) is \(\| \cdot \|\)-bounded and \(\tau\)-convergent to zero and that \(\tau\)-lim\(_{n \to \infty}\) \(S(t_n)x_n = 0\). Thus \((S(t))_{t \geq 0}\) is not globally \(\tau\)-bi-equicontinuous, which is a contradiction.

Let us turn to the sufficient conditions from Proposition 3.14 ensuring that \((X, \gamma)\) is C-sequential.

3.19. **Remark.** (a) Let \(\Omega\) be a completely regular Hausdorff space and \(X := C_0(\Omega)\). Then the compact-open topology \(\tau_{co}\) is metrisable on \(B_{\| \cdot \|}\) if and only if \(\Omega\) is hemicompact by Remark II.1.24 1), p. 88, and Definition 2.22 (a). We recall that a Hausdorff space \(\Omega\) is called hemicompact if there is a sequence \((K_n)_{n \in \mathbb{N}}\) of compact sets in \(\Omega\) such that for every compact set \(K \subset \Omega\) there is \(N \in \mathbb{N}\) such that \(K \subset K_N\) [23, Exercises 3.4.E, p. 165]. In particular, \((C_0(\Omega), \gamma, \gamma_s)\) is C-sequential if \(\Omega\) is hemicompact by Proposition 3.14 (i) where \(\gamma := \gamma(\| \cdot \|, \| \cdot \|_{co}, \tau_{co}) = \beta_0\).

Furthermore, \((C_0(\Omega), \gamma, \gamma_s)\) is also a C-sequential space if \(\Omega\) is a Polish space by [63, Theorem 2.4 (a), p. 316], [63, Theorem 9.1 (a), p. 332], [63, Theorem 8.1 (a), p. 330] and Proposition 3.13.

(b) Let \(X\) be a Banach space. Then the weak*-topology \(\sigma(X', X)\) is metrisable on \(B_{\| \cdot \|_{X'}}\) if and only if \(X\) is separable by [23, V.5.2 Theorem, p. 426]. In particular, \((X', \gamma)\) is C-sequential if \(X\) is separable by Proposition 3.14 (i) where \(\gamma := \gamma(\| \cdot \|_{X'}, \sigma(X', X)) = \tau_{se}(X', X)\).

(c) Let \((X, \| \cdot \|)\) be a Banach space. Then the dual Mackey topology \(\mu(X', X)\) is metrisable on \(B_{\| \cdot \|_{X'}}\) if and only if \(X\) is an SWCG space by [61, 2.1 Theorem, p. 388-389]. We recall that a Banach space \((X, \| \cdot \|)\) is called strongly weakly compactly generated (SWCG) if there exists a \(\sigma(X, X')\)-compact set \(K \subset X\) such that for every \(\sigma(X, X')\)-compact set \(L \subset X\) and \(\varepsilon > 0\) there is \(n \in \mathbb{N}\) with \(L \subset (nK + \varepsilon B_{1,1})\) by [61, p. 387]. In particular, \((X', \gamma)\) is C-sequential if \(X\) is an SWCG space by Proposition 3.14 (i) where \(\gamma := \gamma(\| \cdot \|_{X'}, \mu(X', X)) = \mu(X', X)\).

(d) Let \(X, Y\) be Banach spaces. Then the weak operator topology \(\tau_{wot}\) is metrisable on \(B_{\| \cdot \|_{L(X,Y)}}\) if \(X\) and \(Y\) are separable (\(\tau_{wot}\) is metrisable on \(B_{\| \cdot \|_{L(X,Y)}}\) by \(d(S, T) := \sum_{m,n \in \mathbb{N}} 2^{-m+n}\|y_m^* T x_n - S x_n\|\) for \(S, T \in B_{\| \cdot \|_{L(X,Y)}}\) where...
(x_n)_{n \in \mathbb{N}} \text{ is a } \| \cdot \|_X \text{-dense sequence in } B_1 \|_X \text{ and } (y_n')_{m \in \mathbb{N}} \text{ a } \| \cdot \|_{Y'} \text{-dense sequence in } B_1 \|_{Y'}). \text{ In particular, } (L(X; Y), \gamma) \text{ is C-sequential if } X \text{ and } Y' \text{ are separable by Proposition 3.14 (i) where } \gamma := \gamma(\| \cdot \|_{L(X; Y)}, \tau_{\text{swot}}).

The strong operator topology \( \tau_{\text{so}} \) is metrisable on \( B_1 \|_{L(X; Y)} \) if \( X \) is separable (\( \tau_{\text{swot}} \) is metrisable on \( B_1 \|_{L(X; Y)} \) by \( d(S, T) := \sum_{n=1}^{\infty} 2^{-n} |T x_n - S x_n|_Y \) for \( S, T \in B_1 \|_{L(X; Y)} \) where \( (x_n)_{n \in \mathbb{N}} \text{ is a } \| \cdot \|_X \text{-dense sequence in } B_1 \|_X \). \text{ In particular, } (L(X; Y), \gamma) \text{ is C-sequential if } X \text{ is separable by Proposition 3.14 (i) where } \gamma := \gamma(\| \cdot \|_{L(X; Y)}, \tau_{\text{swot}}).

Concerning example (a), every hemicompact space is \( \sigma \)-compact by 25. Exercises 3.8.C (a), p. 194] and this implication is strict, every first-countable hemicompact space is locally compact by 25. Exercises 3.4.E (a), p. 165], and a locally compact Hausdorff space is hemicompact if and only if it is \( \sigma \)-compact by 25. Exercises 3.8.C (b), p. 195.

Concerning example (c), examples of SWCG spaces are reflexive Banach spaces, separable Schur spaces, the space \( \mathcal{N}(H) \) of operators of trace class on a separable Hilbert space \( H \) and the space \( L^1(\Omega, \nu) \) of (equivalence classes of) absolutely integrable functions on \( \Omega \) w.r.t. a \( \sigma \)-finite measure \( \nu \) by [61 2.3 Examples, p. 389-390]. Let us look at the second sufficient condition from Proposition 3.14 that also guarantees that \( (X, \gamma) \) is C-sequential, namely that \( (X, \gamma) \) is a Mackey–Mazur space. Some of the spaces we already considered fulfil this one as well.

3.20. Remark. (a) Condition (ii) of Proposition 3.14 is fulfilled in Remark 3.19 (a) if \( \Omega \) is a hemicompact Hausdorff \( k_\sigma \)-space or a Polish space since \( (C_b(\Omega), \beta_0) \) is C-sequential and

\[
\gamma(\| \cdot \|_\infty, \tau_{\text{co}}) = \beta_0 = \mu(C_b(\Omega), M_0(\Omega))
\]

by 58. Theorem 5.2, p. 884 and [63, Theorem 9.1 (a), (d), p. 332] where \( M_t(\Omega) = (C_b(\Omega), \beta_0)' \) is the space of bounded Radon measures on \( \Omega \) by [63, Theorem 4.4, p. 320]. Here we use the definition of a Radon measure given in [51, p. 543] where the space \( M_0(\Omega) \) is called \( M_0(\Omega) \), in [63, p. 312] this space is just called \( M_t \) and the index \( t \) stands for \( t \)-tight. We recall that a completely regular space \( \Omega \) is called \( k_\sigma \)-space if any map \( f: \Omega \to \mathbb{R} \) (or equivalently, for any completely regular space \( Y \) and any map \( f: \Omega \to Y \)) whose restriction to each compact \( K \subset \Omega \) is continuous, the map is already continuous on \( \Omega \) (see [54, p. 487] and [8, (2.3.7) Proposition, p. 22]).

With regard to strong Mackey spaces we mentioned in the introduction, we note that \( (C_b(\Omega), \beta_0) \) is a strong Mackey space, i.e. \( \sigma(M_t(\Omega), C_b(\Omega)) \)-compact subsets of \( M_t(\Omega) \) are \( \beta_0 \)-equicontinuous (see [63, p. 317]), and that \( \Omega \) is \( \beta \)-simple, i.e. the strict topologies \( \beta_0, \beta \) and \( \beta_1 \) of Sentilles [63 p. 314-315] coincide on \( C_b(\Omega) \) (see [58, Definition 2.12, p. 877]), if \( \Omega \) is a hemicompact, Hausdorff \( k_\sigma \)-space or a Polish space by [58, Theorem 5.2, p. 884] and [63 Theorems 5.7, 5.8 (b), 9.1 (a), p. 325, 332].

(b) Clearly, condition (ii) of Proposition 3.14 is also fulfilled in Remark 3.19.

(c) If \( X \) is an SWCG space.

(e) Let \((X, \| \cdot \|)\) be a Banach space. If \( X \) is weakly sequentially complete, i.e. \((X, \sigma(X, X'))\) is sequentially complete (see [4, Definition 2.3.10, p. 38]), and has an almost shrinking basis, which means that \( X \) has a Schauder basis such that its associated sequence of coefficient functionals forms a Schauder basis of \((X', \mu(X', X))\) (see [40, p. 75]), then \((X', \mu(X', X)) = (X', \gamma)\) with \( \gamma := \gamma(\| \cdot \|_{X'}, \mu(X', X)) = \mu(X', X) \) is a Mackey–Mazur space by 71. Theorem 7.1, p. 51] and thus condition (ii) of Proposition 3.14 is fulfilled.
Concerning example (a) above, we will take a closer look at $k_{\mathbb{R}}$-spaces in the next section. Concerning example (c), SWCG spaces and the space $L^1(\Omega, \nu)$ for an arbitrary measure $\nu$ are weakly sequentially complete by \cite[2.5 Theorem, p. 390]{1} and \cite[Exercise 5.85, p. 286]{2}. Moreover, if a Banach space has an unconditional basis (see \cite[Definition 3.3.1, p. 51]{3}), then this basis is almost shrinking by \cite[Proposition 4, p. 78]{4}. The space $L^1([0,1], \lambda)$, $\lambda$ the Lebesgue measure, has no unconditional basis by \cite[Theorem 6.5.3, p. 144]{5}, but the separable Schur, thus SWCG, space $\ell^1$ has an unconditional basis, namely, the canonical unit sequences. Hence $(\ell^\infty, \mu(\ell^\infty, \ell^1))$ is a Mackey–Mazur space (cf. \cite[Example 7.3, p. 51]{6} and \cite[E.1.1 Proposition, p. 338]{7}). An example of a weakly sequentially complete Banach space with an unconditional basis that is not an SWCG space is given in \cite[2.6 Example, p. 391]{8}.

3.21. Remark. Let $(X, \| \cdot \|, \tau)$ be a triple satisfying Assumption \cite[2.4]{7}$\gamma := \gamma(\| \cdot \|, \|, \tau)$ the mixed and $\gamma_s := \gamma_s(\| \cdot \|, \tau)$ the submixed topology and $(T(t))_{t \geq 0}$ a $\tau$-bi-continuous semigroup. In \cite[Assumption 2.6, p. 579]{9} the assumption is made that

(i) $(X, \gamma_s)$ is complete, and

(ii) $(T(t))_{t \geq 0}$ locally $\gamma_s$-equicontinuous $(\gamma_s$ is called $\gamma$ in \cite{10}).

With regard to condition (i) we note that the space $(X, \gamma)$ is complete if and only if $B_1$ is $\tau$-complete by \cite[I.1.14 Proposition, p. 11]{11}. In particular, $(X, \gamma)$ is complete if $B_1$ is $\tau$-compact, which is a sufficient condition for $\gamma = \gamma_s$ by Remark \cite[3.10]{12}, and fulfilled in Example \cite[5.11]{13} (b)-(d).

In Example \cite[5.11]{13} (a) we have $\beta_0 = \gamma = \gamma_s$ as well and the space $(X, \gamma_s) = (C_0(\Omega), \beta_0)$ for a completely regular Hausdorff space $\Omega$ is complete if and only if $\Omega$ is a $k_{\mathbb{R}}$-space by \cite[I.1.9 Corollary, p. 81]{14} (cf. \cite[3.6.9 Theorem, p. 72]{15}).

In Remark \cite[3.20]{7} (c) the space $(X, \gamma) := (X'_0, \gamma) = (X'_0, \mu(X'_0, X_0))$ for a Banach space $(X_0, \| \cdot \|_0)$ is also complete by \cite[p. 74]{16}. If, in addition, $\gamma_s = \mu(X'_0, X_0)$ holds, then $(X, \gamma_s) = (X'_0, \gamma_s)$ is complete, too.

Condition (ii) is fulfilled, $(T(t))_{t \geq 0}$ is even quasi-$(\| \cdot \|, \tau)$-equitight, by Theorem \cite[3.17]{13} (b) whenever $(X, \gamma)$ is C-sequential and $\gamma = \gamma_s$ (thus for the examples in the intersection of Example \cite[5.11]{13} and Remark \cite[3.19]{13} resp. Remark \cite[3.20]{16}).

In the last part of this section we tackle the problem from Remark \cite[4.3]{17} to show that the space $M_\mu(\mathbb{R})$ equipped with the mixed topology is C-sequential and that the mixed topology and submixed topology coincide on $M_\mu(\mathbb{R})$.

3.22. Proposition. Let $(X, \| \cdot \|, \tau)$ be a Saks space, $\gamma := \gamma(\| \cdot \|, \|, \tau)$ the mixed topology, $X_\gamma := (X, \gamma)$ and $| \cdot |_{X_\gamma}$ the restriction of $\| \cdot \|_{X_\gamma}$ to $X_\gamma'$. Then $(X_\gamma', \| \cdot \|_{X_\gamma'}, \sigma(X_\gamma', X))$ is a Saks space, $B_{1\| X_\gamma'}$ is $\sigma(X_\gamma', X)$-compact and

\[ \gamma' := \gamma(\| \cdot \|_{X_\gamma'}, \sigma(X_\gamma', X)) = \gamma_s(\| \cdot \|_{X_\gamma'}, \sigma(X_\gamma', X)) = \tau_c(X_\gamma', (X, \| \cdot \|)) \]

where $\tau_c(X_\gamma', (X, \| \cdot \|))$ is the topology of uniform convergence on compact subsets of $(X, \| \cdot \|)$.

(b) If $(X, \gamma)$ is complete and

(i) $(X, \gamma)$ is a strong Mackey space, or

(ii) $(X_\gamma', \gamma')$ is a C-sequential space and every $\gamma'$-null sequence in $X_\gamma'$ is $\gamma'$-equicontinuous, then

\[ \gamma' = \tau_c(X_\gamma', X_\gamma). \]

Proof. (a) The space $X_\gamma'$ is a closed subspace of $(X', \| \cdot \|_{X'})$ by \cite[I.1.18 Proposition, p. 15]{14} and thus $(X_\gamma', \| \cdot \|_{X_\gamma'})$ is a Banach space. Since $(X_\gamma', \sigma(X_\gamma', X))'$ is $X$ is
norming for \((X'_\gamma, \| \cdot \|_{X'_\gamma})\), we deduce from Remark 2.8(c) (c) that \((X'_\gamma, \| \cdot \|_{X'_\gamma}, \sigma(X'_\gamma, X))\) is a Saks space. Hence the unit-ball \(B_{\| \cdot \|_{X'_\gamma}} = \{ y \in X'_\gamma \mid \|y\|_{X'_\gamma} \leq 1 \}\) is \(\sigma(X'_\gamma, X)\)-closed by 18 I.3.1 Lemma, p. 27. Furthermore, the unit-ball \(B_{\| \cdot \|_{X'_\gamma}}\) is \(\sigma(X', X)\)-compact by the Banach–Alaoglu theorem, which implies that \(B_{\| \cdot \|_{X'_\gamma}}\) is \(\sigma(X'_\gamma, X)\)-compact because

\[
B_{\| \cdot \|_{X'_\gamma}} = B_{\| \cdot \|_{X'_\gamma}} \cap X'_\gamma \quad \text{and} \quad \sigma(X'_\gamma, X) = \sigma(X', X)|_{X'_\gamma}.
\]

Therefore condition (ii) of Remark 3.10 (b) is fulfilled for \(\gamma'\) and we obtain

\[
\gamma' = \gamma(\| \cdot \|_{X'_\gamma}, \sigma(X'_\gamma, X)) = \tau_c(X'_\gamma, (X, \| \cdot \|))
\]

as well as

\[
\gamma' = \gamma(\| \cdot \|_{X'_\gamma}, \sigma(X'_\gamma, X)) \in (a) resp. (b).
\]

First, we prove \(\gamma' \leq \tau_c(X'_\gamma, X)\). A compact set in \((X, \| \cdot \|)\) is also compact in \(X'\) because \(\gamma\) is coarser than the \(\| \cdot \|\)-topology. This yields that the identity map \(id:(X'_\gamma, \tau_c(X'_\gamma, X)) \to (X', \tau_c(X, X))\) is continuous and therefore we have

\[
\gamma' = \tau_c(X'_\gamma, (X, \| \cdot \|)) \leq \tau_c(X'_\gamma, X)) \quad \text{by part (a)}.
\]

Next, we show \(\gamma' \geq \tau_c(X'_\gamma, X)\), i.e. that id: \((X'_\gamma, \gamma') \to (X'_\gamma, \tau_c(X'_\gamma, X))\) is continuous.

(i) Suppose that \((X, \gamma)\) is a strong Mackey space. Due to 18, 1.1.8 Lemma p. 8 and 18, 1.1.7 Corollary p. 8 we only need to show that the restriction of id to \(B_{\| \cdot \|_{X'_\gamma}}\) is \(\sigma(X'_\gamma, X)-\tau_c(X'_\gamma, X)-\gamma\)-equicontinuous because \((X, \gamma)\) is a strong Mackey space. Since \(X'_\gamma = (X, \gamma)\) is complete, the topology \(\tau_c(X'_\gamma, X)\) coincides with the topology of uniform convergence on precompact subsets of \(X\) by 33 Theorem 3.5.1, p. 64. Hence \(\sigma(X'_\gamma, X)\) and \(\tau_c(X'_\gamma, X)\) coincide on the \(\gamma\)-equicontinuous set \(B_{\| \cdot \|_{X'_\gamma}}\) by 33 Theorem 8.5.1 (b), p. 156), implying that the restriction of id to \(B_{\| \cdot \|_{X'_\gamma}}\) is \(\sigma(X'_\gamma, X)-\tau_c(X'_\gamma, X)-\gamma\)-continuous.

(ii) Suppose that \((X'_\gamma, \gamma')\) is \(C\)-sequential and every \(\gamma'\)-null sequence in \(X'_\gamma\) is \(\gamma\)-equicontinuous. We only need to show that id: \((X'_\gamma, \gamma') \to (X'_\gamma, \tau_c(X'_\gamma, X))\) is sequentially continuous at zero. Then linearity yields that id is sequentially continuous, which implies the continuity of id by virtue of \((X'_\gamma, \gamma')\) being \(C\)-sequential.

Let \((y_n)_{n \in \mathbb{N}}\) be a \(\gamma'\)-null sequence in \(X'_\gamma\). It follows that \((y_n)_{n \in \mathbb{N}}\) is a \(\sigma(X', X)\)-null sequence and thus a \(\tau_c(X'_\gamma, X)\)-null sequence as well because \(\sigma(X', X)\) and \(\tau_c(X'_\gamma, X)\) coincide on the \(\gamma\)-equicontinuous set \((y_n)_{n \in \mathbb{N}}\). Hence id is sequentially \(\gamma'\)-continuous at zero.

We note that one cannot use 72 Example E, p. 66) (or 18 I.2 Examples A, p. 20-21) directly to show that \(\gamma' = \tau_c(X'_\gamma, (X, \| \cdot \|))\) in (a) resp. \(\gamma' = \tau_c(X'_\gamma, X)\) in (b) since the space \((X, \gamma)\) need not be a Banach (or Fréchet) space. For the next corollary we recall that \(\| \cdot \|_{\mathcal{M}_b(\Omega)}\) denotes the total variation norm on the space \(\mathcal{M}_b(\Omega)\) of bounded Radon measures on a completely regular Hausdorff space \(\Omega\).

3.23. Corollary. Let \(\Omega\) be a completely regular Hausdorff space.

(a) Then \(\mathcal{M}_b(\Omega), \sigma(\mathcal{M}_b(\Omega), C_0(\Omega)))\) is a Saks space, \(B_{\| \cdot \|_{\mathcal{M}_b(\Omega)}}\) is \(
\sigma(\mathcal{M}_b(\Omega), C_0(\Omega)))\)-compact and

\[
\gamma' = \gamma(\| \cdot \|_{\mathcal{M}_b(\Omega)}, \sigma(\mathcal{M}_b(\Omega), C_0(\Omega))) = \tau_c(\mathcal{M}_b(\Omega), (C_0(\Omega), \| \cdot \|_{\infty})).
\]
(b) If \( \Omega \) is a hemicompact \( k_2 \)-space or Polish, then \( (M_b(\Omega), \beta'_0) \) is sequentially complete and

\[
\beta'_0 = \tau_{C}(M_b(\Omega), (C_b(\Omega), \beta_0)).
\]

(c) If \( \Omega \) is Polish, then \( \sigma(M_b(\Omega), C_b(\Omega)) \) is metrisable on \( B_{\|M_b(\Omega)\|} \) and the space \( (M_b(\Omega), \beta'_0) \) is \( C \)-sequential.

Proof. (a) The triple \((C_b(\Omega), \|\cdot\|_{\infty}, \tau_{co})\) is a Saks space by Example 3.24 (a) and it holds \( M_b(\Omega) = (C_b(\Omega), \beta_0)' \) by [62, Theorem 4.4, p. 320] for completely regular Hausdorff spaces \( \Omega \) with \( \beta_0 = \gamma(\|\cdot\|_{\infty}, \tau_{co}) \). Further, we have

\[
\|p\|_{M_b(\Omega)} = \|p\|_{(C_b(\Omega), \|\cdot\|_{\infty})'}, \quad \nu \in M_b(\Omega),
\]

by [52, Example 2.4, p. 441]. Hence Proposition 3.22 (a) implies our statement.

(b) The space \((C_b(\Omega), \beta_0)\) is complete if \( \Omega \) is Hausdorff \( k_2 \)-space (see Remark 3.21). In particular, Polish spaces are Hausdorff \( k_2 \)-spaces by the comments above Theorem 1.1. In Remark 3.20 (a) we observed that \((C_b(\Omega), \beta_0)\) is a strong Mackey space if \( \Omega \) is a hemicompact Hausdorff \( k_2 \)-space or Polish. Thus we get

\[
\beta'_0 = \tau_{C}(M_b(\Omega), (C_b(\Omega), \beta_0))
\]

by condition (ii) of Proposition 3.22 (b). The sequential completeness of \((M_b(\Omega), \beta'_0)\) follows from the sequential completeness of \((M_b(\Omega), \sigma(M_b(\Omega), C_b(\Omega)))\) by [62, Theorems 4.4, 8.7 (a), p. 320, 331] and the fact that a hemicompact Hausdorff \( k_2 \)-space or Polish space \( \Omega \) is \( \beta \)-simple (see Remark 3.21 (a)).

(c) By part (a) we know that the unit-ball \( B_{\|M_b(\Omega)\|} \) is \( \sigma(M_b(\Omega), C_b(\Omega)) \)-compact. Hence we obtain that \( \sigma(M_b(\Omega), C_b(\Omega)) \) is metrisable on \( B_{\|M_b(\Omega)\|} \) by [46, Lemma 2.5, p. 185] because the Polish space \( \Omega \) is \( \beta \)-simple. Thus condition (i) of Proposition 3.21 is satisfied and we deduce that \((M_b(\Omega), \beta'_0)\) is \( C \)-sequential.

If \( \Omega \) is Polish, then \( \Omega \) is \( \beta \)-simple and it follows from part (b) of Corollary 3.23 and [46, Theorem 1.7, p. 183] that \( \beta'_0 \) is the finest topology on \( M_b(\Omega) \) that coincides with \( \sigma(M_b(\Omega), C_b(\Omega)) \) on all \( \beta_0 \)-equicontinuous subsets of \( M_b(\Omega) \). It is an open question whether this is still true if \( \Omega \) is a hemicompact Hausdorff \( k_2 \)-space. Another open question is whether Corollary 3.23 (c) holds for hemicompact Hausdorff \( k_2 \)-spaces instead of Polish spaces.

3.24. Remark. Let \( \Omega \) be a hemicompact Hausdorff \( k_2 \)-space or Polish, and set

\[
C_b(\Omega)^\circ := \{ y \in C_b(\Omega)' \mid \text{\( y \) \( \tau_{co} \)-sequentially continuous on \( \|\cdot\|_{-\text{bounded}} \)} \}.
\]

Then \((C_b(\Omega), \beta_0)\) is \( C \)-sequential, in particular a Mazur space, by Remark 3.19 (a) and thus

\[
C_b(\Omega)^\circ = (C_b(\Omega), \beta_0)' = M_b(\Omega)
\]

by [13 I.1.18 Proposition, p. 15]. A sequence is a \( \beta'_0 \)-null sequence in \( M_b(\Omega) \) if and only if it is a \( \sigma(M_b(\Omega), C_b(\Omega)) \)-null sequence and \( \|\cdot\|_{M_b(\Omega)} \)-bounded by [18 I.1.10 Proposition, p. 9]. It follows that a \( \beta'_0 \)-null sequence in \( M_b(\Omega) \) is \( \beta_0 \)-equicontinuous since \((C_b(\Omega), \beta_0)\) is a strong Mackey space and \( B_{\|M_b(\Omega)\|} \) is \( \sigma(M_b(\Omega), C_b(\Omega)) \)-compact (see the proof of Corollary 3.23). Therefore every \( \beta'_0 \)-null sequence in \( M_b(\Omega) \) is \( \beta_0 \)-equicontinuous. Hence we conclude that [52, Hypothesis C, p. 315], namely, that every \( \|\cdot\|_{M_b(\Omega)} \)-bounded \( \sigma(M_b(\Omega), C_b(\Omega)) \)-null sequence in \( C_b(\Omega)^\circ \) is \( \tau_{co} \)-equicontinuous on \( \|\cdot\|_{-\text{bounded}} \) sets, is satisfied by [3] and [18 I.1.7 Corollary, p. 8].
4. Applications

For our first application of the results of Section 3 we consider $\tau_{co}$-bi-continuous semigroups on $C_b(\Omega)$ for Hausdorff $k_\beta$-spaces $\Omega$. Let us recall some facts on $k_\beta$-spaces from general topology. Examples of $k_\beta$-spaces are completely regular $k$-spaces by [25, 3.3.21 Theorem, p. 152]. A topological space $\Omega$ is called $k$-space (compactly generated space) if it satisfies the following condition: $A \subset \Omega$ is closed if and only if $A \cap K$ is closed in $K$ for every compact $K \subset \Omega$. Every locally compact Hausdorff space is a completely regular $k$-space by [25, p. 152]. Further, every first-sequence Hausdorff space is a $k$-space by [25, 3.3.20 Theorem, p. 152], in particular, every first-countable Hausdorff space. Thus metrisable spaces are completely regular $k$-spaces. However, there are Hausdorff $k_\beta$-spaces that are not $k$-spaces by [32, 2.3 Construction, p. 392-393]. From our observations here and after Remark 3.19, it follows that metrisable spaces are $k_\beta$-spaces and that $\sigma$-compact locally compact Hausdorff spaces are hemicompact $k_\beta$-spaces. Moreover, hemicompact $k_\beta$-spaces are $k$-spaces by [32, Lemme 1 (2.3), p. 55] (cf. [58, Lemma 5.1, p. 884]). Further, there are hemicompact Hausdorff $k_\beta$-spaces that are neither locally compact nor metrisable by [64, p. 267], namely the completely regular Hausdorff $k$-space $\Omega := (E', \tau(E', E))$ for any infinite-dimensional Fréchet space $E$.

Let $\Omega$ be a completely regular Hausdorff space. The triple $(C_b(\Omega), \| \cdot \|_{\infty}, \tau_{co})$ is a Saks space by Example 2.4 (a) and thus Assumption 2.1 (i) and (iii) are fulfilled. Further, the space $(C_b(\Omega), \beta_0)$ is complete if and only if $\Omega$ is a $k_\beta$-space (see Remark 3.19 (b)). Therefore it follows from Remark 2.3 (b) that Assumption 2.1 (ii) is also fulfilled if $\Omega$ is a $k_\beta$-space.

4.1. Theorem. Let $\Omega$ be a Hausdorff $k_\beta$-space and $(T(t))_{t \geq 0}$ a $\tau_{co}$-bi-continuous semigroup on $C_b(\Omega)$. If $\Omega$ is hemicompact or Polish, then $(T(t))_{t \geq 0}$ is quasi-$\beta_0$-equicontinuous and quasi-$\langle \| \cdot \|_{\infty}, \tau_{co} \rangle$-equitight.

Proof. By virtue of Example 2.4 (a) we have $\gamma(\| \cdot \|_{\infty}, \tau_{co}) = \beta_0$. If $\Omega$ is a hemicompact Hausdorff $k_\beta$-space, our statement follows from Proposition 3.18 combined with Remark 3.19 (a) and Example 3.11 (a). If $\Omega$ is a Polish space, our statement follows from Theorem 3.17 combined with Remark 3.19 (a) and Example 3.11 (a).

The preceding theorem improves [30, Theorem 2.4, Remark 2.5, p. 92-93] where it is observed that every $\tau_{co}$-bi-continuous semigroup on $C_b(\Omega)$, $\Omega$ Polish, is locally $\beta_0$-equicontinuous and locally $\langle \| \cdot \|_{\infty}, \tau_{co} \rangle$-equitight. In [32, Example 4.1, p. 320] (see Example 5.7 an example of a $\tau_{co}$-bi-continuous semigroup $(T(t))_{t \geq 0}$ on $C_b([0, \omega_1])$, where $\omega_1$ is the first uncountable ordinal and $[0, \omega_1)$ is equipped with the order topology, is constructed such that no $T(t), t > 0$, is $\beta_0$-continuous. This is no contradiction to Theorem 4.1 because $[0, \omega_1)$ is locally compact by [25, 3.3.14 Examples, p. 151], but neither $\sigma$-compact nor separable by [65, Part II, 42. Open ordinal space $[0, \Omega]$; 3, 10, p. 69-70].

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2For instance, the space $\Omega := Y$ in [32, 2.3 Construction, p. 392-393] constructed from a completely regular Hausdorff non-$k$-space $X$ due to the comments above this construction. For $X$ take e.g. the Fortissimo space which is completely normal by [65, Part II, 25. Fortissimo space; 1, p. 53-54] and easily seen to be Hausdorff, thus completely regular. Further, finite sets are the only compact sets in the Fortissimo space $X$ by [65, Part II, 25. Fortissimo space; 2, p. 54] and hence $X$ is not a $k$-space, which follows from the choice of the non-closed set $A := X \setminus \{p\}$ in the definition of a $k$-space with $p$ from the definition of the Fortissimo space.

3This means that [33, Proposition 2.7 (iv), p. 4] is not correct since there are Hausdorff $k_\beta$-spaces which are not $k$-spaces. However, this does have no impact on the validity of the other results in [33].
ON EQUICONTINUITY AND TIGHTNESS OF BI-CONTINUOUS SEMIGROUPS

4.2. Example. (a) The left translation semigroup \( (T(t))_{t \geq 0} \) on \( C_b(\mathbb{R}) \) given by

\[
T(t)f(x) := f(x+t), \quad x \in \mathbb{R}, \quad f \in C_b(\mathbb{R}), \quad t \geq 0,
\]
is a \( \tau_{co} \)-bi-continuous semigroup by \cite[Examples 6 (b), p. 6-7]{44}. Due to \cite[Example 3.2, p. 549]{44} it is \( \tau_{co} \)-locally equicontinuous but not quasi-\( \tau_{co} \)-equicontinuous. However, by Theorem 4.1 \( T(t) \) is quasi-\( \beta_0 \)-equicontinuous and quasi-(\( \| \cdot \|_{\infty}, \tau_{co} \))-equitight.

(b) The Gauß–Weierstraß semigroup \( (T(t))_{t \geq 0} \) on \( C_b(\mathbb{R}^d) \) given by \( T(0)f := f \) and

\[
T(t)f(x) := \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} f(y)e^{-\|y-x\|^2/4t} \, dy, \quad x \in \mathbb{R}^d, \quad f \in C_b(\mathbb{R}^d), \quad t > 0,
\]
is \( \tau_{co} \)-continuous by \cite[Examples 6 (a), p. 5-6]{44}. It is also quasi-\( \beta_0 \)-equicontinuous and quasi-(\( \| \cdot \|_{\infty}, \tau_{co} \))-equitight by Theorem 4.1.

(c) Next, we consider the Ornstein–Uhlenbeck semigroup on \( C_b(H) \) \( H \) a separable Hilbert space. We refer the reader to \cite[Section 3.3]{45} and \cite[Section 4]{44} for details. The Ornstein–Uhlenbeck semigroup is \( \tau_{co} \)-bi-continuous by \cite[Proposition 3.10, p. 63]{44}. Due to Theorem 4.1 this semigroup is quasi-\( \beta_0 \)-equicontinuous and quasi-(\( \| \cdot \|_{\infty}, \tau_{co} \))-equitight as well.

(d) The transition semigroups \( (T(t))_{t \geq 0} \) on \( C_b(\Omega) \) given by

\[
T(t)f := \int_{\Omega} f(\xi)\mu_t(d\xi), \quad f \in C_b(\Omega), \quad t \geq 0,
\]

for a separable Banach space \( \Omega \) in \cite[Proposition 6.2, p. 1162]{44} resp. by

\[
T(t)f(\xi) = E[f(X(t, \xi))], \quad \xi \in \Omega, \quad f \in C_b(\Omega), \quad t \geq 0,
\]

for a separable Hilbert space \( \Omega \) in \cite[Proposition 6.3, p. 1164]{44} are \( \tau_{co} \)-bi-continuous by \cite[Eq. (6.3), p. 1162]{44}, \cite[Proposition 5.14, p. 1161]{44} and Remark 2.6 (c) \( (\beta_0 = \gamma_4(\| \cdot \|_{\infty}, \tau_{co}) \) is called \( \tau_M \) in \cite{45}). It follows from Theorem 4.1 that \( (T(t))_{t \geq 0} \) is quasi-\( \beta_0 \)-equicontinuous and quasi-(\( \| \cdot \|_{\infty}, \tau_{co} \))-equitight in both cases. Interestingly, this is in general not true for the Hausdorff locally convex topology \( \tau_C \) on \( C_b(\Omega) \) defined in \cite[pp. 1154]{45}. The topology \( \tau_C \) has the same convergent sequences as the (sub)mixed topology \( \beta_0 \) by \cite[Proposition 5.4 (i), p. 1154]{44} and Remark 2.6 (a), but \( \tau_C \) is strictly coarser than \( \beta_0 \) by \cite[Proposition 4.23, p. 38]{44} if \( \dim(\Omega) \geq 1 \). In \cite[Example 6.4, p. 1165]{44} a special case of the semigroup \( \tau_C \) is constructed which is \( \chi_C \)-strongly continuous and locally sequentially \( \tau_C \)-equicontinuous but not locally \( \tau_C \)-equicontinuous in contrast to its behaviour w.r.t. \( \beta_0 \).

Keeping the remark before Theorem 4.1 in mind that there are Hausdorff \( kرز \)-spaces which are not \( k \)-spaces, we generalise and improve \cite[Proposition 2.10, p. 5]{44} on the \textit{Koopman semigroup} by weakening the assumptions from \( k \)-spaces to \( kرز \)-spaces as well as strengthening its implications from local \( \beta_0 \)-equicontinuity to local (\( \| \cdot \|_{\infty}, \tau_{co} \))-equitightness resp. even quasi-(\( \| \cdot \|_{\infty}, \tau_{co} \))-equitightness.

4.3. Corollary. Let \( \Omega \) be a completely regular Hausdorff space and \( \phi: [0, \infty) \times \Omega \rightarrow \Omega \) a semiflow, i.e. with \( \phi_t(x) := \phi(t, x) \) it holds \( \phi_0 = \text{id} \) and \( \phi_{t+s} = \phi_t \phi_s \) for \( t, s \geq 0 \). Consider the linear Koopman semigroup \( (T_{\phi}(t))_{t \geq 0} \) on \( C_b(\Omega) \) given by

\[
T_{\phi}(t)f(x) := f(\phi_t(x)), \quad x \in \Omega, \quad f \in C_b(\Omega), \quad t \geq 0.
\]

(a) If \( \phi \) is (jointly) continuous, then \( (T_{\phi}(t))_{t \geq 0} \) is \( \beta_0 \)-strongly continuous, locally \( \beta_0 \)-equicontinuous and locally (\( \| \cdot \|_{\infty}, \tau_{co} \))-equitight.

(b) If \( \Omega \) is a \( kرز \)-space and \( (T_{\phi}(t))_{t \geq 0} \) is \( \beta_0 \)-strongly continuous, then \( \phi \) is (jointly) continuous.
(c) If $\Omega$ is a $k_{\mathbb{R}}$-space and $\phi$ (jointly) continuous, then $(T_\phi(t))_{t \geq 0}$ is a $\tau_{co},$-bi-
continuous semigroup.

(d) If $\Omega$ is a hemicompact $k_{\mathbb{R}}$-space or Polish space, and $\phi$ (jointly) continuous,
then $(T_\phi(t))_{t \geq 0}$ is quasi-$\beta_0$, equicontinuous and quasi-$(\cdot, \| \cdot \|_{\infty}, \tau_{co})$-equitight.

Proof. The proof of part (a) and (b), except for the local $(\cdot, \| \cdot \|_{\infty}, \tau_{co})$-equitightness,
is the same as in Proposition 2.10, p. 5. We only have to replace the fact that $[0, \infty) \times \Omega$ is a completely regular $k$-space for completely regular $k$-spaces $\Omega$ by the fact that $[0, \infty) \times \Omega$ is a $k_{\mathbb{R}}$-space for $k_{\mathbb{R}}$-spaces $\Omega$, which follows from a comment after the proof of Theorem (2.1), p. 54-55 since $[0, \infty)$ is a locally compact Hausdorff space.

Let us turn to the local $(\cdot, \| \cdot \|_{\infty}, \tau_{co})$-equitightness in part (a). We note that
$$|T_\phi(t)|_{\infty} = \sup_{x \in \Omega} |f(\phi_t(x))| \leq \|f\|_{\infty}, \quad f \in C_0(\Omega),$$
for all $t \geq 0$. Hence we get that $(S(t))_{t \in [0, t_0]} := (T_\phi(t))_{t \in [0, t_0]}$ is $(\cdot, \| \cdot \|_{\infty}, \tau_{co})$-equitight for any $t_0 \geq 0$ by Proposition 2.10 Example 3.11 (a) and the local $\beta_0$-equicontinuity of $(T_\phi(t))_{t \geq 0}$. Thus $(T_\phi(t))_{t \geq 0}$ is locally $(\cdot, \| \cdot \|_{\infty}, \tau_{co})$-equitight.

We turn to part (c). The triple $(C_0(\Omega), \| \cdot \|_{\infty}, \tau_{co})$ fulfills Assumption 2.1 if $\Omega$ is a
$k_{\mathbb{R}}$-space (see the remarks above Theorem 4.1). Now, part (a) and (b) and Remark 2.6 (c) yield that $(T_\phi(t))_{t \geq 0}$ is $\tau_{co}$-continuous if $\Omega$ is a $k_{\mathbb{R}}$-space and $\phi$ (jointly) continuous.

Finally, part (d) follows from part (c) and Theorem 4.1.

Part (d) improves 2.3 Theorem, p. 5 where it was already observed that the Koopman semigroup is quasi-$\beta_0$-equicontinuous if $\Omega$ is Polish. Furthermore, the left translation semigroup in Example 4.2 (a) is a special case of part (d) with
$\phi(t, x) := t + x$ and $\Omega := \mathbb{R}$.

Now, we focus on bi-continuous semigroups on the space $M_1(\Omega)$ of bounded Radon measures on a completely regular Hausdorff space $\Omega$ and generalise Theorems 3.5, 3.6, p. 318. We recall that for a semigroup $(T(t))_{t \geq 0}$ in $\mathcal{L}(X)$,
X Banach, the dual semigroup $(T'(t))_{t \geq 0}$ in $\mathcal{L}(X')$ is defined by $(T'(t)y, x) := y(T(t)x)$ for $y \in X'$ and $x \in X$. Due to Corollary 3.29 (a) we get that the triple
$(M_1(\Omega), \| \cdot \|_{M_1(\Omega)}, \sigma(M_1(\Omega), C_0(\Omega)))$ is a Saks space, which yields that Assumption 2.3 (i) and (iii) are fulfilled. If $\Omega$ is a hemicompact Hausdorff $k_{\mathbb{R}}$-space or Polish, then Assumption 2.3 (ii) is also fulfilled by Remark 2.3 (b) because $(M_1(\Omega), \beta_0)'$ is sequentially complete by Corollary 3.29 (b).

4.4. Theorem. Let $\Omega$ be a hemicompact Hausdorff $k_{\mathbb{R}}$-space or a Polish space, and $(T(t))_{t \geq 0}$ a $\tau_{co}$-bi-continuous semigroup on $C_0(\Omega)$. Then the semigroup $(T'(t))_{t \geq 0}$ defined by $T'(t) := T'(t)_{M_1(\Omega)}$ for $t \geq 0$ is a $\sigma(M_1(\Omega), C_0(\Omega))$-bi-continuous semigroup on $M_1(\Omega)$.

Proof. Due to the sequential completeness of $(M_1(\Omega), \beta_0)'$ and 3 we have that
3. Hypothesis B, p. 314] is satisfied. By Remark 3.24 we already know that 3. Hypothesis C, p. 315] is fulfilled. Hence we may apply 3. Proposition 2.4, p. 315], yielding our statement.

So we get examples of such semigroups $(T'(t))_{t \geq 0}$ by using the semigroups $(T(t))_{t \geq 0}$ from Example 4.2 and Corollary 3.3 (d). Actually, all $\sigma(M_1(\Omega), C_0(\Omega))$-bi-continuous semigroups on $M_1(\Omega)$ are of this form if $\Omega$ is a hemicompact Hausdorff $k_{\mathbb{R}}$-space or Polish since we have the following converse of Theorem 4.4 which is already known in the case that $\Omega$ is Polish (see 3. Theorem 3.6, p. 318).

4.5. Theorem. Let $\Omega$ be a hemicompact Hausdorff $k_{\mathbb{R}}$-space or a Polish space, and $(S(t))_{t \geq 0}$ a $\sigma(M_1(\Omega), C_0(\Omega))$-bi-continuous semigroup on $M_1(\Omega)$. Then there is
a $\tau_{\alpha_0}$-bi-continuous semigroup $(T(t))_{t \geq 0}$ on $C_0(\Omega)$ such that $T^\circ(t) = S(t)$ for all $t \geq 0$.

**Proof.** The proof is almost the same as for [32, Theorem 3.6, p. 318]. We only need to replace the applications of [32, Theorem 3.5, p. 318] and [32, Theorem 3.1 c), p. 316-317] by Theorem [4.3] and Remark [3.24] respectively, if $\Omega$ is a hemicompact Hausdorff $k_2$-space. □

4.6. **Theorem.** Let $\Omega$ be a Polish space and $(T(t))_{t \geq 0}$ a $\sigma(M(\Omega),C_b(\Omega))$-bi-continuous semigroup on $M_0(\Omega)$. Then $(T(t))_{t \geq 0}$ is quasi-$\beta_0$-equicontinuous and quasi-$\{\cdot|_{M(\Omega)},\sigma(M(\Omega),C_b(\Omega))\}$-equitight.

**Proof.** Our statement is a consequence of Proposition [3.18] and Corollary [3.23] □

Let us turn to dual semigroups of norm-strongly continuous semigroups. If $(X,\|\cdot\|)$ is a Banach space, then $(X',\|\cdot\|_{X'},\sigma(X',X))$ is a Saks space by Example [61] (b). Thus Assumption [2.1] (i) and (iii) are fulfilled. Assumption [2.1] (ii) is also fulfilled by the Banach–Steinhaus theorem [see e.g. [67, Corollary, p. 348]]

4.7. **Theorem.** Let $(X,\|\cdot\|)$ be a Banach space and $(T(t))_{t \geq 0}$ be a $\|\cdot\|_{X'}$-strongly continuous semigroup on $X$. If $X$ is separable, then the dual semigroup $(T^\ast(t))_{t \geq 0}$ on $X'$ is quasi-$\tau_c(X',X)$-equicontinuous and quasi-$\{\cdot|_{X'},\sigma(X',X)\}$-equitight.

**Proof.** By [48, Proposition 3.18, p. 78] $(T^\ast(t))_{t \geq 0}$ is $\sigma(X',X)$-bi-continuous on $X'$. Due to Example [2.4] (b) we have $\tau_c(\cdot|_{X'},\sigma(X',X)) = \tau_c(X',X)$. For separable $X$ our statement follows from Proposition [3.18] combined with Remark [3.19] (b) and Example [3.11] (b).

The theorem above actually covers all $\sigma(X',X)$-bi-continuous semigroups on $X'$ for separable $X$ due to [61, Theorem 4.2, p. 158], which says that we only need $\sigma(X',X)$-continuity on $\|\cdot\|_{X'}$-bounded sets of all members of the $\sigma(X',X)$-bi-continuous semigroup for the conclusion that a $\sigma(X',X)$-bi-continuous semigroup is the dual semigroup of a $\|\cdot\|_{X'}$-strongly continuous semigroup on $X$. Due to Theorem [3.15] (a) we have a nice condition on $(X',\tau_c(X',X))$ that guarantees this kind of continuity. In particular this condition is satisfied for separable $X$ by Remark [3.19] (b).

4.8. **Proposition.** Let $(X,\|\cdot\|)$ be a Banach space and $(T(t))_{t \geq 0}$ a $\sigma(X',X)$-bi-continuous semigroup. If $(X',\tau_c(X',X))$ is C-sequential, then there is a $\|\cdot\|_{X'}$-strongly continuous semigroup $(S(t))_{t \geq 0}$ on $X$ such that $S'(t) = T(t)$ for all $t \geq 0$.

**Proof.** If $(X',\tau_c(X',X))$ is C-sequential, then by Theorem [3.15] (a) and the equivalence (d)$\iff$(g) of Proposition [3.12] every $T(t)$ is $\sigma(X',X)$-continuous on $\|\cdot\|_{X'}$-bounded sets. Hence the existence of $(S(t))_{t \geq 0}$ follows from [61, Theorem 4.2, p. 158]. □

Let us turn to $\mu(X',X)$-bi-continuous semigroups. If $(X,\|\cdot\|)$ is a Banach space, then $(X',\|\cdot\|_{X'},\mu(X',X))$ is a Saks space by Example [5.11] (c). Thus Assumption [2.4] (i) and (iii) are fulfilled. Again, Assumption [2.4] (ii) is fulfilled by a consequence of the Banach–Steinhaus theorem [see e.g. [67, Corollary 2, p. 356]].

4.9. **Theorem.** Let $(X,\|\cdot\|)$ be a Banach space and $(T(t))_{t \geq 0}$ a $\mu(X',X)$-bi-continuous semigroup on $X'$.

(a) If $X$ is an SWCG space, or a weakly sequentially complete space with an almost shrinking basis, then $(T(t))_{t \geq 0}$ is quasi-$\mu(X',X)$-equicontinuous.

(b) If $X$ is an SWCG Schur space, e.g. a separable Schur space, then $(T(t))_{t \geq 0}$ is quasi-$\mu(X',X)$-equicontinuous and quasi-$\{\cdot|_{X'},\mu(X',X)\}$-equitight.
Proof. Due to Example 4.10 (c) we have \( \gamma(\| \cdot \|_X, \mu(X', X)) = \mu(X', X) \). Part (a) follows from Proposition 3.18 (a) combined with Remark 3.19 (c) if \( X \) is an SWCG space. If \( X \) is a weakly sequentially complete space with an almost shrinking basis, then Theorem 3.17 (a) in combination with Remark 3.20 (c) yields part (a).

Part (b) is a consequence of Proposition 3.18 (b) combined with Remark 3.19 (c) and Example 3.11 (c) if \( X \) is an SWCG Schur space, and the fact that a separable Schur space is an SWCG space by [61, 2.3 Examples (b), p. 389]. □

Clearly, a Schur space with an almost shrinking basis is already separable. Necessary and sufficient conditions such that the dual semigroup of a \( \| \cdot \| \)-strongly continuous semigroup is \( \mu(X', X) \)-bi-continuous are given in [48, Propositions 3.19, 3.20, 3.24, p. 78-83]. Due to [51, Corollary 3.10 (1), p. 553] such a \( \mu(X', X) \)-bi-continuous dual semigroup is always quasi-\( \mu(X'^*, X') \)-equicontinuous since it is \( \mu(X', X) \)-strongly continuous and

\[
\mathcal{L}(X) = \mathcal{L}(X, \mu(X, X')) = \mathcal{L}(X, \sigma(X, X'))
\]

where the first equality follows from \( (X, \| \cdot \|) \) being a Mackey space, the second from [44, §21.4 (6), p. 262] and \( \mathcal{L}(X, \tau) \) denotes the space of \( \tau \)-continuous linear operators from \( X \) to \( X \). However, in return for more restrictions on \( X \) Theorem 4.9 gives a stronger implication in part (b) and holds for all \( \mu(X', X) \)-bi-continuous semigroups, not only for the ones that are dual semigroups of \( \| \cdot \| \)-strongly continuous semigroups (even though that are the examples we consider).

4.10. Example.

(a) Let \((S(t))_{t \geq 0}\) be the left translation semigroup on \( L^1(\mathbb{R}) := L^1(\mathbb{R}, \lambda) \)

\[
S(t) f(x) := f(x + t), \quad x \in \mathbb{R}, \; f \in L^1(\mathbb{R}), \; t \geq 0,
\]

which is \( \| \cdot \|_{L^1} \)-strongly continuous by [24, Chap. I, 5.4 Example, p. 39]. Its dual semigroup \((T(t))_{t \geq 0} := (S'(t))_{t \geq 0}\) is the right translation semigroup on \( L^\infty(\mathbb{R}) := L^1(\mathbb{R})' \), which fulfills

\[
T(t) f(x) = f(x - t), \quad x \in \mathbb{R}, \; f \in L^\infty(\mathbb{R}), \; t \geq 0,
\]

and is a \( \mu(\mathbb{R}^\infty, L^1) \)-bi-continuous semigroup by [49, Examples 3.22, p. 82]. \( L^1(\mathbb{R}) \) is an SWCG space by [61, 2.3 Examples (d), p. 390] as the Lebesgue measure \( \lambda \) is \( \sigma \)-finite. Hence \((T(t))_{t \geq 0}\) is quasi-\( \mu(\mathbb{R}^\infty, L^1) \)-equicontinuous by Theorem 4.9 (a). Moreover, \((T(t))_{t \geq 0}\) is quasi-\( \{ \| \cdot \|_{L^\infty}, \sigma(\mathbb{R}^\infty, L^1) \} \)-equitight and quasi-\( \tau(\mathbb{R}^\infty, L^1) \)-equicontinuous by Theorem 4.7.

(b) Let \((S(t))_{t \geq 0}\) be the multiplication semigroup on \( L^1(\mathbb{R}) \)

\[
S(t) f(x) := e^{\nu(x)t} f(x), \quad x \in \mathbb{R}, \; f \in L^1(\mathbb{R}), \; t \geq 0,
\]

for some measurable and locally integrable function \( \nu : \mathbb{R} \to \mathbb{R} \), which is \( \| \cdot \|_{L^1} \)-strongly continuous by [24, Chap. I, 4.11 Proposition, p. 32]. Then the dual semigroup \((T(t))_{t \geq 0} := (S'(t))_{t \geq 0}\) is a \( \mu(\mathbb{R}^\infty, L^1) \)-bi-continuous semigroup on \( L^\infty(\mathbb{R}) \) by [49, Remark 3.2, p. 83] and quasi-\( \mu(\mathbb{R}^\infty, L^1) \)-equicontinuous by Theorem 4.9 (a). Moreover, \((T(t))_{t \geq 0}\) is quasi-\( \{ \| \cdot \|_{L^\infty}, \sigma(\mathbb{R}^\infty, L^1) \} \)-equitight and quasi-\( \tau(\mathbb{R}^\infty, L^1) \)-equicontinuous by Theorem 4.7.

(c) Let \((S(t))_{t \geq 0}\) be the multiplication semigroup on \( \ell^1 = \ell^1(\mathbb{N}) \)

\[
S(t) f_n := e^{\nu(n)t} f_n, \quad n \in \mathbb{N}, \; f \in \ell^1, \; t \geq 0,
\]

for some function \( \nu : \mathbb{N} \to \mathbb{C} \) with \( \nu(n) < \infty \) for some \( \| \cdot \|_{\ell^1} \)-strongly continuous by [24, Chap. I, 4.11 Proposition, p. 32]. Next, we verify that the dual semigroup \((T(t))_{t \geq 0} := (S'(t))_{t \geq 0}\) is a \( \mu(\mathbb{R}^\infty, \ell^1) \)-bi-continuous semigroup on \( \ell^\infty = \ell^\infty(\mathbb{N}) \), using [48, Propositions 3.19, 3.20, p. 78-81]. We proceed like in [48, Example 3.22, p. 82]. By [21, Theorem III.2.15, p. 76] we have
a set \( M \subset \ell^1 \) is \( \sigma(\ell^1, \ell^\infty) \)-compact if and only if \( M \) is \( \| \cdot \|_{\ell^1} \)-bounded and uniformly absolutely summable, i.e.

\[
\forall \varepsilon > 0 \exists \delta > 0 \forall \Omega \subset \mathbb{N}, \| \Omega \| < \delta, f \in M : \sum_{n \in \Omega} |f_n| < \varepsilon,
\]

where \( \| \Omega \| \) denotes the cardinality of \( \Omega \). Let \( M \subset \ell^1 \) be \( \sigma(\ell^1, \ell^\infty) \)-compact. Then we have with \( C := \sup_{n \in \mathbb{N}} \| \text{Re} q(n) \| \) that

\[
\| S(t)f \|_{\ell^1} = \sum_{n=1}^{\infty} e^{t\text{Re} q(n)}|f_n| \leq e^C \| f \|_{\ell^1} \leq e^C \sup_{g \in M} \| g \|_{\ell^1} < \infty
\]

for all \( f \in M \) and \( 0 \leq t \leq 1 \), which implies that

\[
M_1 := \bigcup_{0 \leq t \leq 1} S(t)M
\]

is \( \| \cdot \|_{\ell^1} \)-bounded. Next, we need to show that \( M_1 \) is uniformly absolutely summable. Let \( \varepsilon > 0 \). Since \( M \) is uniformly absolutely summable, there is \( \delta > 0 \) such that for all \( \Omega \subset \mathbb{N} \) with \( \| \Omega \| < \delta, f \in M \) and all \( 0 \leq t \leq 1 \) it holds

\[
\sum_{n \in \Omega} |S(t)f_n| = \sum_{n \in \Omega} e^{t\text{Re} q(n)}|f_n| \leq e^C \sum_{n \in \Omega} |f_n| < e^C \varepsilon e^{tC} \leq \varepsilon,
\]

meaning that \( M_1 \) is uniformly absolutely summable. Hence \( M_1 \) is \( \sigma(\ell^1, \ell^\infty) \)-compact and it follows from [48, Propositions 3.19, 3.20, p. 78-81] that (\( T(t) \))\(_{t \geq 0} \) is a \( \mu(\ell^\infty, \ell^1) \)-bi-continuous semigroup on \( \ell^\infty \). The space \( \ell^1 \) is a separable Schur space. Thus (\( T(t) \))\(_{t \geq 0} \) is quasi-\( \mu(\ell^\infty, \ell^1) \)-equicontinuous and quasi-(\( \| \cdot \|_{\ell^\infty}, \| \cdot \|_{\ell^1} \))-equitight due to Theorem 4.15(b). Furthermore, (\( T(t) \))\(_{t \geq 0} \) is quasi-(\( \| \cdot \|_{\ell^\infty}, \sigma(\ell^\infty, \ell^1) \))-equitight and quasi-\( \tau_c(\ell^\infty, \ell^1) \)-equicontinuous by Theorem 4.17.

Next, we consider implemented semigroups [27, Sect. 3.2]. If \( X \) and \( Y \) are Banach spaces, then \((\mathcal{L}(X;Y), \| \cdot \|_{\mathcal{L}(X;Y)}, \tau_{\text{wot}})\) and \((\mathcal{L}(X;Y), \| \cdot \|_{\mathcal{L}(X;Y)}, \tau_{\text{sot}})\) are Saks spaces by Example 3.11(d). Thus Assumption 2.1(i) and (iii) are fulfilled in both cases. First, by [48, p. 75] \( \tau_{\text{sot}} \) is sequentially complete on \( \| \cdot \|_{\mathcal{L}(X;Y)} \)-bounded sets. Hence Assumption 2.1(ii) is fulfilled for \( \tau_{\text{sot}} \). Second, \((\mathcal{L}(X;Y), \gamma(\| \cdot \|_{\mathcal{L}(X;Y)}), \tau_{\text{wot}})\) is complete by [18, I.1.14 Proposition, p. 11] if \( X \) is reflexive as the unit ball \( B_{\| \cdot \|_{\mathcal{L}(X;Y)}} \) is \( \tau_{\text{wot}} \)-compact by Example 3.11(d) under this condition. Since \((\mathcal{L}(X;Y), \| \cdot \|_{\mathcal{L}(X;Y)}, \tau_{\text{wot}})\) is a Saks space, the completeness of \((\mathcal{L}(X;Y), \gamma(\| \cdot \|_{\mathcal{L}(X;Y)}), \tau_{\text{wot}})\) yields that Assumption 2.1(ii) is fulfilled for \( \tau_{\text{wot}} \) by Remark 2.3(b) if \( Y \) is reflexive.

4.11. **Theorem.** Let \((X, \| \cdot \|_X), (Y, \| \cdot \|_Y)\) be Banach spaces, \((T(t))_{t \geq 0}\) a \( \| \cdot \|_Y \)-strongly continuous semigroup on \( Y \) and \((S(t))_{t \geq 0}\) a \( \| \cdot \|_X \)-strongly continuous semigroup on \( X \). We consider the implemented semigroup \((\mathcal{U}(t))_{t \geq 0}\) on \( \mathcal{L}(X;Y) \) given by

\[
\mathcal{U}(t)R := T(t)R S(t), \quad t \geq 0, R \in \mathcal{L}(X;Y).
\]

(a) If \( X \) is separable, then \((\mathcal{U}(t))_{t \geq 0}\) is quasi-\( \gamma(\| \cdot \|_{\mathcal{L}(X;Y)}, \tau_{\text{sot}}) \)-equicontinuous.

(b) If \( X \) is separable and \( Y \) finite-dimensional, then \((\mathcal{U}(t))_{t \geq 0}\) is quasi-\( \gamma(\| \cdot \|_{\mathcal{L}(X;Y)}, \tau_{\text{sot}}) \)-equicontinuous.

(c) If \( X \) and \( Y \) are separable, \( Y \) is reflexive and \((\mathcal{U}(t))_{t \geq 0}\) is locally \( \tau_{\text{wot}} \)-bi-equicontinuous, then \((\mathcal{U}(t))_{t \geq 0}\) is quasi-\( \gamma(\| \cdot \|_{\mathcal{L}(X;Y)}, \tau_{\text{wot}}) \)-equicontinuous and quasi-\( \tau_c(\| \cdot \|_{\mathcal{L}(X;Y)}, \tau_{\text{wot}}) \)-equitight.

**Proof.** By [48, Proposition 3.16, p. 75] \((\mathcal{U}(t))_{t \geq 0}\) is a \( \tau_{\text{wot}} \)-bi-continuous semigroup on \( \mathcal{L}(X;Y) \). Since \( \tau_{\text{wot}} \leq \tau_{\text{sot}} \), it is also \( \tau_{\text{wot}} \)-strongly continuous and under the assumption of local \( \tau_{\text{wot}} \)-bi-equicontinuity in (b) \( \tau_{\text{wot}} \)-bi-continuous as well.

Parts (a) and (b) follow from Proposition 3.18 combined with Remark 3.19(d) and and Example 3.11(d). Let us turn to part (c). If \( Y \) is reflexive, then the
separability of $Y$ implies the separability of $Y'$ by Propositions 6.13, 7.3, p. 49, 53. Hence we deduce part (c) from Proposition 6.18 (b) combined with Remark 6.19 (d) and Example 6.11 (d).

4.12 Example. Let $H$ be a separable Hilbert space and $\mathcal{K}(H)$ the space of compact operators in $\mathcal{L}(H)$. We denote by $\beta$ the Hausdorff locally convex topology on $\mathcal{L}(H)$ induced by the directed system of seminorms

$$p_M(R) := \max_{Q \in M} \sup_{\|RQ\|_{\mathcal{L}(H)}, \|QR\|_{\mathcal{L}(H)}} R \in \mathcal{L}(H),$$

for finite $M \subset \mathcal{K}(H)$. Due to Theorem 3.9, p. 84 and Corollary 2.8, p. 638 we have $\beta = \mu(\mathcal{L}(H), \mathcal{N}(H))$ where $\mathcal{N}(H)$ is the space of trace class operators in $\mathcal{L}(H)$. We deduce part (c) from Proposition 3.18 (b) combined with Remark 3.11 (d).

Further, we denote by $\tau_{\text{sot}}$, the symmetric strong operator topology, i.e. the Hausdorff locally convex topology on $\mathcal{L}(H)$ induced by the directed system of seminorms

$$p_N(R) := \max_{x \in N} \sup_{\|Rx\|_H, \|Rx\|_H} R \in \mathcal{L}(H),$$

for finite $N \subset H$, and by $\beta_{\text{sot}}$, the mixed topology $\gamma(\|\cdot\|_{\mathcal{L}(H)}, \tau_{\text{sot}})$ (see p. 204 where $\beta_{\text{sot}}$ is called $\beta_\gamma$). The triple $(\mathcal{L}(H), \|\cdot\|_{\mathcal{L}(H)}, \tau_{\text{sot}})$ is a Saks space and $(\mathcal{L}(H), \beta_{\text{sot}})$ is complete by Example 2.3 by Remark 2.3. Due to, Proposition 3.11 (p. 211) we have $\beta_{\text{sot}} = \mu(\mathcal{L}(H), \mathcal{N}(H))$ as well and hence

$$\beta_{\text{sot}} = \beta(\mathcal{L}(H), \mathcal{N}(H)).$$

(5)

Since $\mathcal{N}(H)$ is an SWCG space by Examples (c), p. 389-390, we get that $(\mathcal{L}(H), \beta_{\text{sot}}) = (\mathcal{L}(H), \beta)$ is a C-sequential space by Remark 5.10 (c), which gives a different proof of the $\beta_\gamma = \beta$ statement in Proposition 8.5, p. 182.

Let $(T(t))_{t \geq 0}$ be a $\beta$-strongly continuous semigroup on $H$. One checks as in the case of $\tau_{\text{sot}}$ (see the proof of Proposition 3.16, p. 75-76) that the implemented semigroup $(U(t))_{t \geq 0}$ on $\mathcal{L}(H)$ given by

$$U(t)R := T^*(t)RT(t), \quad t \geq 0, R \in \mathcal{L}(H),$$

(6)
is $\tau_{\text{sot}}$-bi-continuous, where $T^*(t)$ is the adjoint of $T(t)$. Since $\beta_{\text{sot}}$ and $\tau_{\text{sot}}$ coincide on $\|\cdot\|_{\mathcal{L}(H)}$-bounded sets and $(\mathcal{L}(H), \beta_{\text{sot}})$ fulfills Assumption 2.2 by Definition 2.2 (a), Remark 2.2 (b) and Lemma 5.5 (a), p. 2680, the semigroup $(U(t))_{t \geq 0}$ is $\beta_{\text{sot}}$-bi-continuous as well. Therefore $(U(t))_{t \geq 0}$ is quasi-$\beta_{\text{sot}}$-equicontinuous by Theorem 4.10 (a) because $\beta_{\text{sot}} = \mu(\mathcal{L}(H), \mathcal{N}(H))$ and $\mathcal{N}(H)$ is an SWCG space. In addition, we have that $(U(t))_{t \geq 0}$ is quasi-$\gamma(\|\cdot\|_{\mathcal{L}(H)}, \tau_{\text{sot}})$-equicontinuous by Theorem 4.11 (a) as $H$ is separable.

The preceding example gives an independent proof of Proposition 8.6, p. 182, where it is shown that $(U(t))_{t \geq 0}$ is an SCLE semigroup w.r.t. $\beta$, due to [4]. The implemented semigroup [5] is a special case of a so-called quantum dynamical group by Example 3.1, p. 30, which are defined to be strongly continuous w.r.t. the ultraweak topology (also called $\sigma$-weak topology).

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