Star Products with Separation of Variables
Admitting a Smooth Extension

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Abstract. Given a complex manifold $M$ with an open dense subset $\Omega$ endowed with a pseudo-Kähler form $\omega$ which cannot be smoothly extended to a larger open subset, we consider various examples where the corresponding Kähler–Poisson structure and a star product with separation of variables on $(\Omega, \omega)$ admit smooth extensions to $M$. We give a simple criterion of the existence of a smooth extension of a star product and apply it to these examples.

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1. Introduction

A formal differential star product on a Poisson manifold $(M, \{\cdot, \cdot\})$ is an associative product on the space $C^\infty(M)[[\nu]]$ of smooth complex-valued formal functions on $M$ given by the formula

$$f \ast g = \sum_{r \geq 0} \nu^r C_r(f, g),$$

where $C_r$ are bidifferential operators on $M$, $C_0(f, g) = fg$ and $C_1(f, g) - C_1(g, f) = i \{f, g\}$ (see [1]). It was proved by Kontsevich [8] that deformation quantizations exist on any Poisson manifold.

We will assume that the unit constant function $1$ is the unity with respect to the star product: $f \ast 1 = 1 \ast f = f$ for all $f \in C^\infty(M)[[\nu]]$. Given functions $f, g \in C^\infty(M)[[\nu]]$, we will denote by $L_f$ and $R_g$ the left star multiplication operator by $f$ and the right star multiplication operator by $g$, respectively, so that $f \ast g = L_f g = R_g f$. The associativity of the star product $\ast$ is equivalent to the statement that $[L_f, R_g] = 0$ for all $f, g \in C^\infty(M)[[\nu]]$. A star-product on a Poisson manifold $M$ can be restricted to any open subset of $M$. 
We call a Poisson tensor on a complex manifold $M$ a Kähler–Poisson tensor if it is of type $(1,1)$ with respect to the complex structure. In local coordinates $\{z^k, \bar{z}^l\}$ it is written as $g^{lk}$. The tensor $g^{lk}$ defines a global Poisson bivector field $ig_{kl}\frac{\partial}{\partial z^k} \wedge \frac{\partial}{\partial \bar{z}^l}$ and the corresponding Poisson bracket $\{\cdot, \cdot\}$ on $M$. If a Kähler–Poisson tensor $g^{lk}$ is nondegenerate, the inverse matrix $g_{kl}$ is a pseudo-Kähler metric tensor which defines a pseudo-Kähler form $\omega = ig_{kl}dz^k \wedge d\bar{z}^l$. We call a complex manifold $M$ endowed with a Kähler–Poisson tensor a Kähler–Poisson manifold. Any pseudo-Kähler manifold is a Kähler–Poisson manifold. In this paper, we give several examples of Kähler–Poisson manifolds with the Kähler–Poisson tensor degenerate on the complement of an open dense subset.

A star product $(\star)$ on a Kähler–Poisson manifold defines a deformation quantization with separation of variables if the operators $C_r$ differentiate their first argument in antiholomorphic directions and the second argument in holomorphic ones. Under the assumption that the unit constant 1 is the unity with respect to the star product, the condition of separation of variables can be equivalently stated as follows: for any local holomorphic function $a$ and a local antiholomorphic function $b$ the identities $a \star f = af$ and $f \star b = bf$ hold. Otherwise speaking, $L_a = a$ and $R_b = b$ are pointwise multiplication operators. In this case,

$$C_1(u, v) = g^{lk} \frac{\partial u}{\partial \bar{z}^l} \frac{\partial v}{\partial z^k}.$$

It is not known whether there exists a star product with separation of variables on an arbitrary Kähler–Poisson manifold. However, star products with separation of variables exist on any pseudo-Kähler manifold $M$ (see [2,5]).

Given a star product with separation of variables $\star$ on a Kähler–Poisson manifold $M$, the formal Berezin transform of the star product $\star$ is a formal differential operator $B = 1 + \nu B_1 + \nu^2 B_2 + \cdots$ globally defined on $M$ by the condition that

$$B(ab) = b \star a$$

for any local holomorphic function $a$ and a local antiholomorphic function $b$. In particular,

$$B_1 = g^{lk} \frac{\partial^2}{\partial z^k \partial \bar{z}^l}$$

is a Laplace–Beltrami operator. A star product with separation of variables can be recovered from its Berezin transform.

A deformation quantization with separation of variables on a pseudo-Kähler manifold $M$ equipped with a pseudo-Kähler form $\omega$ is called standard if its restriction to any contractible coordinate chart $(U, \{z^k\})$ has the property that

$$L_{\frac{\partial}{\partial z^k}} = \frac{\partial \Phi}{\partial z^k} + \nu \frac{\partial \Phi}{\partial z^k} \quad \text{and} \quad R_{\frac{\partial}{\partial \bar{z}^l}} = \frac{\partial \Phi}{\partial \bar{z}^l} + \nu \frac{\partial \Phi}{\partial \bar{z}^l},$$