Light Propagation in Nonlinear Waveguide 
and Classical Two-Dimensional Oscillator 

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Abstract 

The quantum optical problem of the propagation of electromagnetic waves in a 
nonlinear waveguide is related to the solutions of the classical nonstationary harmonic oscillator using the method of linear integrals of motion [Malkin et.al., Phys Rev.2D (1970) 1371]. An explicit solution of the classical oscillator with a varying frequency, corresponding to the light propagation in an anisotropic waveguide is obtained using the expressions for the quantum field fluctuations. 

Substitutions have been found which allow to establish connections of the linear and quadratic invariants of Malkin et.al. to several types of invariants of quadratic systems, considered in later papers. These substitutions give the opportunity to relate the corresponding quantum problem to that of the classical two-dimensional nonstationary oscillator, which is physically more informative. 

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1 Introduction 

The generation of subfluctuant (squeezed) and cofluctuant (covariance) states of various systems has stimulated a great deal of interest due to the important prospective applications of these states in optical communications and super sensitive detection systems. Many schemes of processes which generate the nonclassical states have been suggested. A considerable attention has been paid to the investigations of the nonclassical states of an oscillator with time-dependent frequency or mass. Recently, an oscillator with a linear sweep of restoring force [10], and an oscillator which describes the quantum motion in the Paul trap [11,12] have been investigated. F.L. Li and coauthors have also studied the squeezing property of an oscillator with a time-dependent frequency [13]-[15]. There are some other recent articles, considering explicit solutions of quantum systems with quadratic Hamiltonians [16]-[20]. We would like to note that two overcomplete families of solutions for general quadratic Hamiltonians are built in [16,17] by the method of linear invariants.
One of the aims of this article is to establish a relationship between the results in later works with those in [2]-[5].

Solving the Quantum – Optical problem [22] for an anisotropic waveguide, we have found a new solution of the equation for the classical nonstationary harmonic oscillator with a time-dependent frequency:

\[ \ddot{\epsilon} + \Omega^2(t)\epsilon = 0. \]  

(1)

The second aim of this paper is to determine the classical nonstationary harmonic oscillator, which corresponds to the quantum problem of the light propagation in a second order nonlinear waveguide at the degenerated parametric down conversion \(\chi^{[2]}(2\omega = \omega + \omega)\), i.e. to find the time-dependent frequency \(\Omega(t)\) and the solution \(\epsilon(t)\) of equation (1) corresponding to this case.

In the next section we show that according to the principle of correspondence between classical mechanics and quantum mechanics there exist exactly 2\(s\) quantum linearly independent invariants [4]. For quadratic Hamiltonians these 2\(s\) integrals of motion are linear in terms of \(\hat{q}\) and \(\hat{p}\), therefore any problem with an one-dimensional quadratic Hamiltonian (\(\hat{H}_{quad}\)) is related to the two-dimensional classical harmonic oscillator. The found substitutions allow to transform some classical equations considered in later papers, to the equation of the two-dimensional nonstationary harmonic oscillator. In the third section we apply the method of linear invariants to anisotropic waveguides. In the fourth section we consider the explicit solution for the classical nonstationary two-dimensional harmonic oscillator for the anisotropic waveguide.

### 2 Method of Linear Integrals of Motion

Let us consider a classical system with \(s\) degree of freedom and let \(u = u(q_1, ..., q_s, p_1, ..., p_s, t)\) be one dynamic variable of this system. Expressed in terms of Poisson brackets \(\{,\}\), the full derivative of \(u\) with respect to \(t\) is:

\[ \frac{du}{dt} = \frac{\partial u}{\partial t} + \{u, H_{class}\} = 0. \]  

(2)

By definition \(u^{inv}\) will be an integral of motion (invariant) iff \(du^{inv}/dt = 0\), i.e.

\[ \frac{\partial u^{inv}}{\partial t} + \{u^{inv}, H_{class}\} = 0. \]  

(3)

It is well known [28] that there are 2\(s\) independent integrals of motion, which are linear with respect to \(q_1, ..., q_s, p_1, ..., p_s\). For example, one can choose the coordinates of the initial point \((q_1(0), ..., q_s(0), p_1(0), ..., p_s(0))\) of the trajectory in the phase space:

\begin{align*}
Q_k^{inv} &= q_k(0) \\
P_k^{inv} &= p_k(0) \quad (k = 1, ..., s).
\end{align*}

(4)

As far as there is a principle of correspondence between classical and quantum mechanics, the analogy requires the existence of 2\(s\) Hermitian operators - integrals of motion for any quantum system, and the relevant equations of the quantum invariants are [4]:

\[ \frac{\partial \hat{I}_\nu^{inv}}{\partial t} + \frac{1}{i\hbar}[\hat{I}_\nu^{inv}, \hat{H}] = 0, \quad \nu = 1, ..., 2s. \]  

(5)
The equations for the invariants (5) are different from the Heisenberg equations: $\frac{d\hat{A}}{dt} - \frac{i}{\hbar} [\hat{A}, \hat{H}] = 0$. The same difference exists in classical mechanics between (5) and the Hamilton equations written in terms of Poisson brackets: $\frac{du}{dt} - \{u, H_{\text{class}}\} = 0, u = q_k, p_k$ [25]. The independent solutions of (5) for any quantum system are also $2s$: $\hat{I}_{\nu}^{\text{inv}}(t) = U(t)\hat{I}_{\nu}^{\text{inv}}(0)U^+(t)$. In particular, the invariants could be operators of the coordinates and the moments at $t = 0$,

\[
\hat{Q}_k^{\text{inv}}(t) = \hat{U}(t)\hat{q}_k(0)\hat{U}^+(t) = \sqrt{\frac{\hbar}{2m\omega}}(\hat{A}_k^1 + \hat{A}_k)
\]

\[
\hat{P}_k^{\text{inv}}(t) = \hat{U}(t)\hat{p}_k(0)\hat{U}^+(t) = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{A}_k^1 - \hat{A}_k),
\]

where for the further convenience they are presented as real and imaginary parts of non-hermitian operators $\hat{A}_k$ ( $m$ and $\omega$ are constants with dimension of mass and frequency respectively).

If $\hat{A}$ is an invariant, it will be easy to verify that any function presented as power series $F(\hat{A})$ is an invariant too. The invariants for the general quadratic quantum system with $\hat{H}_{\text{quad}}$ are published in [4, 5], and it is worth noting that they reproduce any quadratic or higher degree invariants. Generally, using the formula

\[
I_g(t) = \alpha_1\hat{A}_1^2 + \alpha_2\hat{A}_1^3\hat{A} + \alpha_3\hat{A}_2^2 + \alpha_4\hat{A}_3^3 + \alpha_5\hat{A} + \alpha_6,
\]

any quadratic integrals of motion for $\hat{H}_{\text{quad}}$ could be built up with the help of $\hat{A}(t)$. Here, $\alpha_i, i = 1,..6$ are time-independent coefficients. See [4, 13, 17, 20, 21] where the linear invariants from [3, 4] are not mentioned at all.

Following [2, 3], we will shortly recall the method of linear invariants and will apply it in the next sections in the case of an one-mode waveguides to find the explicit solution of the two-dimensional harmonic oscillator. Without losing generality, let us consider the one-dimensional case (i.e. $s = 1$) of a nonstationary quantum system described by a quadratic Hamiltonian (the general case is considered in the first paper [4], eq.(1))

\[
\hat{H} = a(t)p^2 + b(t)\{p,q\}_+ + c(t)\hat{q}^2 + d(t)p + e(t)q + f(t),
\]

where $a(t), b(t), c(t), d(t), e(t), f(t)$ are arbitrary real functions of $t$, and $[\ , \ , \ ]_+$ denotes an anti-commutator. For this Hamiltonian (see [4], eq.(3)) we can construct $s = 1$ non-Hermitian linear integral (2) of motion (and its Hermitian conjugate, respectively)

\[
\hat{A} = \frac{i}{\sqrt{ha(t)}} \left\{ a(t)\epsilon(t)\hat{p} + \left[ b(t)\epsilon(t) - \frac{\epsilon(t)}{2} - \frac{1}{4a(t)}\epsilon(t) \right] \hat{q} \right\} + \delta(t),
\]

\[
\hat{A}^\dagger = -\frac{i}{\sqrt{ha(t)}} \left\{ a(t)\epsilon^*(t)\hat{p} + \left[ b(t)\epsilon^*(t) - \frac{\epsilon^*(t)}{2} - \frac{1}{4a(t)}\epsilon^*(t) \right] \hat{q} \right\} + \delta(t).
\]

Here $\epsilon(t)$ is a complex function satisfying (4) and $\delta(t)$ is expressed in terms of $\epsilon$ and $\dot{\epsilon}$ similar to [4]. Since we want to deal with hermitian Hamiltonians (i.e. real $a, b, c, d, e, f$) let us introduce real functions $\epsilon_1(t), \epsilon_2(t)$ and $\delta_1(t), \delta_2(t)$ so that

\[
\epsilon(t) = \epsilon_1(t) + i\epsilon_2(t), \quad \delta(t) = \delta_1(t) + i\delta_2(t).
\]
We shall show that the classical nonstationary two-dimensional harmonic oscillator which corresponds to this general quadratic system with one degree of freedom and quadratic Hamiltonian (9) has a Lagrangian

\[
L = \frac{m}{2}(\xi_1^2 + \xi_2^2) - \frac{m}{2}\Omega^2(t)(\xi_1^2 + \xi_2^2).
\]

(12)

The time-dependent frequency \(\Omega = \Omega(t)\) of this classical harmonic oscillator achieves different forms for the different quantum systems. We shall recall the general form [2] of this frequency and we will apply it to find the explicit expression for the special case of the quantum propagation of light in a nonlinear waveguide.

Let us construct the second type Lagrange equations [28] for this two-dimensional nonstationary harmonic oscillator \((m = 1;\) the case with a time-dependent mass \(M(t)\) could be reduced to this one by a transformation of time, see for example eq.(122) of the second paper in [4]):

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\xi}_\nu} - \frac{\partial L}{\partial \xi_\nu} = 0, \quad k = 1, 2.
\]

(13)

These two real equations can be written as

\[
\dot{\xi}_\nu + \Omega^2(t)\xi_\nu = 0, \quad \nu = 1, 2
\]

(14)

which differ from the equation for the simple harmonic oscillator by the fact that the frequency is time-dependent \(\Omega = \Omega(t)\).

Without losing generality in deriving the expression for the frequency \(\Omega(t)\), we assume that \(d(t) = e(t) = f(t) = 0 = \delta_1(t) = \delta_2(t)\). Since the operators [10] are integrals of motion, they obey the (necessary and sufficient) conditions [5]. Hence

\[
\frac{\partial}{\partial \tau} \left( \frac{i}{\sqrt{\lambda(t)}} \left\{ a(t)\epsilon(t)\dot{p} + \left[ b(t)\epsilon(t) - \frac{i\dot{\epsilon}(t)}{2} - \frac{i\dot{\lambda}(t)}{4a(t)}\epsilon(t) \right] \dot{q} \right\} \right) - \frac{i}{\hbar} \left[ \frac{\partial}{\partial \tau} \left( \frac{i}{\sqrt{\lambda(t)}} \left\{ a(t)\epsilon(t)\dot{p} + \left[ b(t)\epsilon(t) - \frac{i\dot{\epsilon}(t)}{2} - \frac{i\dot{\lambda}(t)}{4a(t)}\epsilon(t) \right] \dot{q} \right\} \right) \right] = 0.
\]

(15)

It reduces to

\[
\left\{ 0 \cdot \dot{p} + \left[ -\ddot{\epsilon} + 0 \cdot \dot{\epsilon} + \left( -4a(t)c(t) - 2\frac{\dot{\lambda}(t)}{a(t)}b(t) - \frac{\ddot{\lambda}(t)}{2a(t)} + \frac{3\dot{\lambda}(t)^2}{4a(t)^2} + 4b(t)^2 + 2\ddot{b}(t) \right) \epsilon \cdot \dot{q} \right] \right\} = 0.
\]

(16)

From this equation we see, that \(\hat{A}\) and \(\hat{A}^\dagger\) will be the quantum invariants for the system with the general nonstationary Hamiltonian (8), if the frequency \(\Omega(t)\) is connected with time-dependent coefficients of the Hamiltonian in the following way [2]:

\[
\Omega^2(t) = 4a(t)c(t) + 2\frac{\dot{\lambda}(t)}{a(t)}b(t) + \frac{\dot{\lambda}(t)}{2a(t)} - \frac{3\dot{\lambda}(t)^2}{4a(t)^2} - 4b(t)^2 - 2\ddot{b}(t),
\]

(17)

and \(\epsilon_1\) and \(\epsilon_2\) are solutions of classical Lagrange equations [13] for the two-dimensional nonstationary oscillator, or respectively, the solution \(\epsilon = \epsilon_1 + i\epsilon_2\) of the classical complex nonstationary harmonic oscillator. This solution can be represented in the form

\[
\epsilon(t) = \rho(t) e^{i \int_0^t \frac{d\tau}{\rho^2(\tau)}},
\]

(18)
where \( \rho(t) \equiv |\epsilon(t)| \). 

The efforts of many investigations have been directed to the quadratic quantum system whose evolutions are described by a classical differential equation like the equation (14). In the literature several equations [1, 6, 9, 17, 19] connected with the harmonic oscillator or with its generalizations are published. In Table 1, we give the substitutions which transform these equations to the classical nonstationary complex oscillator (or the same – to the two-dimensional oscillator). The first column presents the references, where the equations are taken from. More details on these equations and the relevant substitutions are presented in Appendix 1.

We would like to note another independent approach to this method presented by Toledo de Piza [19], where the evolution of the quantum system in pure and mixed states is described by classical Hamilton equations. In the case of pure states, such two systems of Hamilton equations for two pairs of real physical parameters \( \{q, p\} \) and \( \{\sigma, \Pi\} \) have been derived earlier in [23] to describe completely the class of the Schrödinger minimum uncertainty states. The fourth substitution in Table 1 shows the explicit connection between the classical Hamilton equation and the two-dimensional harmonic oscillator, see Appendix 1.

The method of linear integrals of motion is very powerful and can be used in different directions: defining the evolution of each quantum state, transition probabilities, Berry phase etc. [8, 1, 11, 18, 23].

With these comments on the general quadratic Hamiltonian and the two-dimensional harmonic oscillator we have shortly made a resume of the method of linear invariants, developed in series of papers [2, 3, 4, 5]; for any quantum system, described by Hamiltonian in form (9), there exists a classical two-dimensional isotropic nonstationary harmonic oscillator (or the same – complex nonstationary harmonic oscillator (11)) with a Lagrangian (12) and equations of motion (13) with nonstationary frequency \( \Omega(t) \) (17).

### 3 Light propagation in anisotropic waveguide and quadratic Hamiltonians

The problem of a propagation of light from the quantum-mechanical point of view has been investigated in series of papers [23, 26, 22].

In [22] it has been shown that the Hamiltonian,

\[
\hat{H} = \hbar \omega (\hat{a}^{\dagger} \hat{a} + \frac{1}{2}) + \hbar s \left[ e^{i\omega t} \hat{a} \hat{a} + e^{-i\omega t} \hat{a}^{\dagger} \hat{a}^{\dagger} \right]
\]

describes the light propagation in the nonlinear Ti : LiNbO\(_3\) waveguide in the case of a degenerated parametric down conversion. In the above mentioned paper it has been found an explicit form of Heisenberg equations, and are also calculated quantum fluctuations of the electromagnetic field \( \sigma_q(t), \sigma_p(t) \) and their cofluctuation \( c_{qp}(t) \) (the three independent second moments in probability theory [23]):

\[
\sigma_q^2(t) = \frac{\hbar}{2\omega} \left( 1 + 2 \sinh^2(st) + \sinh(2st) \sin(2\omega t) \right),
\]

\[
\sigma_p^2(t) = \frac{\hbar \omega}{2} \left( 1 + 2 \sinh^2(st) - \sinh(2st) \sin(2\omega t) \right),
\]

(20)
\[ e_{qp}^2(t) = \frac{\hbar^2}{4} \sinh^2(2st) \cos^2(2\omega t). \]

The Hamiltonian (19) has the same form in volume nonlinear materials [27] though with different coefficients since in our case the squeezed parameter \( s \) depends on the waveguide parameters and phase-matching conditions for this experimental geometry [22].

Now we are going to apply this method in the opposite direction; since we have solved the quantum problem for the nonlinear anisotropic waveguide with particular Hamiltonian of type (9) we will now find the explicit form of the frequency \( \Omega(t) \) and solutions \( \epsilon_1(t) \) and \( \epsilon_2(t) \) for this harmonic oscillator.

In order to define the time-dependent coefficients \( a(t) \), \( b(t) \), and \( c(t) \) we express the Hamiltonian (19) in terms of canonical operators \( \hat{q} \) and \( \hat{p} \) in the Schrödinger picture,

\[
\hat{H} = (1 - \frac{2s}{\omega} \cos(2\omega t))\hat{p}^2 + (-2s\omega \sin(2\omega t))\frac{[\hat{p}, \hat{q}]}{\omega} + (\omega^2 - \frac{2s}{\omega} \cos(2\omega t))\hat{q}^2.
\]

Hence, for the time-dependent coefficients of (9) we obtain,

\[ a(t) = \frac{1}{2} - \frac{s}{\omega} \cos(2\omega t), \]
\[ b(t) = -s \sin(2\omega t), \]
\[ c(t) = \frac{\omega^2}{2} + s\omega \cos(2\omega t), \]

Using these results for the quantum system describing the light propagation in an anisotropic waveguide, we are able to express in explicit form the solution of equation (1) and the frequency \( \Omega(t) \).

Many authors in their papers, concerning nonstationary harmonic oscillator for different cases of time-dependent frequencies [6]-[17], have described frequencies which are particular cases of general formulae received in [3, 4]. Besides, up to now no explicit solution of equation (1) has been found for the case of parametric down-conversion in a nonlinear waveguide, which is described by the Hamiltonian (19), (21). The subject of the following section is to consider that frequency for the classical equation of the harmonic oscillator, whose solutions determine the quantum dynamics of the electromagnetic field for such waveguides.

4 Explicit solutions

In this section we shall derive the formula for the nonstationary frequency, with the help of which we shall give the explicit solution of the equation (1) (respectively (14)) in the case of an anisotropic waveguide. Using equations (22) and substituting in the equation (17) we obtain the frequency \( \Omega(t) \) in the following explicit form:

\[
\Omega^2(t) = \omega^2 - 4s^2 + 4\omega \cos(2\omega t) + \frac{4s\omega^2 \cos(2\omega t) - 8s^2 \omega \sin^2(2\omega t)}{\omega - 2\omega \cos(2\omega t)} - \frac{12s^2 \omega^2 \sin^2(2\omega t)}{(\omega - 2s \omega \cos(2\omega t))^2}.
\]
It is very important in our consideration that we have solved the quantum problem in an independent way, i.e. we have determined the evolution of the operators \( \hat{p} \) and \( \hat{q} \) and their second moments (20) by a direct calculation [22]. On the other hand, a connection exists between the modulus \( \rho(t) = |\epsilon(t)| \) and the quantum mechanical fluctuations of \( \hat{q} \) and \( \hat{p} \), as shown in [3]. In order to calculate the fluctuations we express \( \hat{q} \) and \( \hat{p} \) in terms of linear invariants \( \hat{A}^\dagger(t) \) and \( \hat{A}(t) \) (10) and take their mean value with respect to the eigenstates of the linear invariants (\( \hat{A}(t)|z; \alpha >= z|z; \alpha \)). The result is:

\[
\sigma^2_q(t) = \hbar \, a(t) \, \rho^2(t),
\]

\[
\sigma^2_p(t) = \frac{\hbar}{a(t)} \left[ \frac{1}{4\rho^2(t)} + \left( b(t)\rho(t) - \frac{\dot{\rho}(t)}{2} - \frac{\dot{a}(t)}{4a(t)}\rho(t) \right)^2 \right],
\]

\[
c_{qp}^2(t) = \hbar^2 \rho^2(t) \left( b(t)\rho(t) - \frac{\dot{\rho}(t)}{2} - \frac{\dot{a}(t)}{4a(t)}\rho(t) \right)^2.
\]

The last formula in (24) presents the third independent second moment (cofluctuation) \( c_{qp}(t) \) expressed in terms of the modulus \( \rho(t) \equiv |\epsilon(t)| \). This formula for \( c_{qp}(t) \) has been derived on the base of the Schrödinger uncertainty relation [24] (see Appendix 2).

Now we are able to define \( \rho(t) \) as a function of the time and parameters \( \omega \) and \( s \), using the expression for \( \sigma_q(t) \) from (20) and (24) to eliminate it. Once we have an expression for \( \rho(t) \) we can get the final solution \( \epsilon(t) \) by the general formula (18). Thus, we have obtained the explicit solution of the classical equation for the nonstationary harmonic oscillator (1) with frequency (23):

\[
\epsilon(t) = \sqrt{\frac{1 + 2 \sinh^2(st) - \sinh(2st) \sin(2\omega t)}{\omega - 2s \cos(2\omega t)}} \times
\]

\[
x \exp \left[ i \int_0^t \frac{\omega - 2s \cos(2\omega \tau)}{1 + 2 \sinh^2(s\tau) - \sinh(2s\tau) \sin(2\omega \tau)} d\tau \right].
\]

The eigenstates of the linear invariants (10) satisfy \( \hat{A}|z; \alpha >= z|z; \alpha \rangle \), at the same time they are Schrödinger Minimum Uncertainty States (i.e. \(|\text{SMUS} >= |z; \alpha \rangle \)) as shown in [23]. Because \(|\text{SMUS} \rangle \) form an over-complete system of eigenfunctions

\[
\frac{1}{\pi} \int |z; \alpha >\langle \alpha; z|d^2z = 1, \quad d^2z \equiv d(Rez)d(Imz),
\]

and remain stable in the time evolution, governed by Hamiltonians (9) [23], the class of two-dimensional isotropic oscillators (i.e. \( \epsilon_{1,2}(t) \)) determines the dynamics of any quantum state \( \psi(t) \); in particular it determines the propagation of light in nonlinear waveguides with second order nonlinearity \( \chi^{(2)}(\omega_1 = \omega_2 + \omega_3) \) (see [22]).

5 Conclusion

It is well known that solutions for the equation of time-dependent harmonic oscillator are found only in few particular cases [29]. The present work gives one more contribution to the class of solutions of the particular oscillator’s equation.
We have found a solution of the equation for the classical nonstationary harmonic oscillator with a time-dependent frequency (23) in integral form (25). We have shown that the quantum-optical problem for the propagation of electromagnetic waves in the nonlinear waveguide [22] is related to the solutions of the classical nonstationary harmonic oscillator. This method could be used for obtaining other solutions of the classical harmonic oscillator with different time-dependent frequency, if we have already determined the fluctuations $\sigma_q$ and $\sigma_p$ by some independent methods.

We have also found the relations between the general method of linear invariants and the equations given in the second column in Table 1. This method gives us the substitutions which transform the classical equations to the physically more informative equation of the two-dimensional nonstationary harmonic oscillator. We should also note that for any quantum system described by a nonstationary quadratic Hamiltonian, there is not only a correspondence to a complex oscillator, but also a correspondence to a real two-dimensional isotropic oscillator with nonstationary frequency.

Using the Schrödinger uncertainty relation (discovered in 1930 year) we have obtained the evolution of the cofluctuation $c_{qp}$, expressed by the solutions of the equation for the classical nonstationary harmonic oscillator. The cofluctuation plays significant role in the so called Cofluctuant States, which differ from widely studied Coherent and Squeezed States.

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Appendix 1

Here we shall prove the substitution from Table 1, which gives us the connection between the equation of the two-dimensional nonstationary harmonic oscillator and other classical equations, describing the quantum evolution of a system with quadratic Hamiltonians.

First, we shall show that the Ricatti equation for $c_1(t)$ from [4] can be transformed to equation (13), and vice versa, with the help of the second substitution from Table 1. Here we shall use the same functions of time $c_2(t)$, $c_3(t)$, $a_1'(t)$, $a_2'(t)$ and $a_3'(t)$ as in [4]:

$$c_2(t) = \int_0^t [a_2'(@) + a_3'(@)c_1(@)]d@ \quad c_3(t) = \int_0^t a_3'(@)e^{c_2(@)}d@. \quad (27)$$

The time-dependent coefficients in the Hamiltonian from [4] $a_1'(t)$, $a_2'(t)$ and $a_3'(t)$ are real, and are connected with our coefficients as follows:

$$a_1'(t) = -2c(t)$$
$$a_2'(t) = -4b(t)$$
$$a_3'(t) = -2a(t). \quad (28)$$
With these notations we can express the common solution of the complex equation for a nonstationary harmonic oscillator in the following way

\[ \epsilon(t) = -\frac{e^{-\frac{c_2(t)}{2}}}{\sqrt{m_0\omega_0a_3(t)}} (m_0\omega_0c_3(t) + i), \]

(29)

where to simplify expressions we have used \( c_2(t) \) and \( c_3(t) \) from [9]. After some calculations the second derivative of (29) becomes:

\[ \ddot{\epsilon} = e^{-\frac{c_2(t)}{2}} \sqrt{m_0\omega_0a_3(t)} \times \left[ a_3'(-a_2'c_1 - a_3'c_1^2 + \dot{c}_1) - a_3'a_1' + a_3'a_1' - \frac{1}{2} \frac{a_3'}{a_3}a_2' - \frac{3}{4} \frac{a_3'}{a_3}a_3' - \frac{1}{2} \frac{a_3'}{a_3}a_3' - \frac{a_3'^2}{2} + \frac{a_3'^2}{2} \right]. \]

We have obtained exactly the frequency \( \Omega^2(t) \) (17) expressed through \( a_1'(t), a_2'(t) \) and \( a_3'(t) \) from [9]. Taking into account the relations (29) and (28) we receive:

\[ \ddot{\epsilon} = -\epsilon \left[ a_3'(-a_1' - a_2'c_1 - a_3'c_1^2 + \dot{c}_1) + \Omega^2(t) \right]. \]

(31)

As far as the Ricatti equation (eq.(9) from [9]) should be fulfilled

\[ \dot{c}_1 = a_1' + a_2'c_1 + a_3'c_1^2 \]

(32)

the expression in parentheses is zero. As a result, we obtain the equation (11). Presenting \( \epsilon(t) \) as a real and an imaginary part, we obtain the classical system of equations (13) of a two-dimensional nonstationary harmonic oscillator. These two equations could also be derived directly, working separately with the real and imaginary parts of the second substitution from Table 1. The opposite is also true; if we want eq. (1 ) to be fulfilled then using (31) we shall receive the Ricatti equation for \( c_1(t) \).

Second, we shall show that the equation (8) for \( f(t) \) from [17]

\[ \ddot{f} + \frac{m(t)}{m(t)} \dot{f} + \omega^2(t)f = 0 \]

(33)

can be also transformed to equation (11) or (13). Using the well known substitution [29]

\[ f(t) = \frac{\epsilon(t)}{\sqrt{m(t)}}, \]

(34)

we again obtain the equation

\[ \ddot{\epsilon} + \left[ \omega^2(t) + \frac{1}{4} \frac{m(t)^2}{m(t)^2} - \frac{1}{2} \frac{m(t)}{m(t)} \right] \epsilon = 0 \]

(35)

with a frequency \( \Omega(t) \), which is a particular case of (17); the coefficient in front of the anti-commutator \([\hat{p}, \hat{q}]_+\) in first article in [17] is zero, which simplifies the expression for the frequency. Note that \( \epsilon(t) \) has no imaginary part, i.e. equations (13) are identical and the oscillator becomes one-dimensional.
Third, we shall show that Hamilton equations (32) from [23] for the pair of canonical variables \( \sigma(t) \equiv \sigma_q \) and \( \Pi \equiv c/\sigma_p \), where \( c \equiv \sigma_{qp} \) is the third independent second momentum – the so called cofluctuation. Namely,
\[
\dot{\sigma} = \frac{\partial <H_1>}{\partial \sigma} = 2b\sigma + 2a\Pi \\
\dot{\Pi} = -\frac{\partial <H_1>}{\partial \Pi} = -2c\sigma - 2b\Pi + \frac{\hbar^2 a}{2} \frac{1}{\sigma^3}
\]
can also be transformed to the equation (1) with the help of the following classical Hamiltonian:
\[
H_1(\sigma(t), \Pi(t)) = a(t) \left( \frac{1}{4\sigma^2} + \Pi^2 \right) + 2b(t)\sigma\Pi + c(t)\sigma^2.
\](37)
This classical Hamilton function is the second part of the quantum mean value of the Hamiltonian (9) taken in Schrödinger minimum uncertainty states \(|z; \alpha>\):
\[
<\hat{H}(\hat{p}, \hat{q})> = \tilde{H}_0(p, q) + \tilde{H}_1(\sigma, \Pi),
\]
where \( p = <\hat{p}> \), \( q = <\hat{q}> \). If we differentiate the substitution for \( \sigma \) from Table 1, and replace it in the first Hamilton equation (36), (here \( \Pi = \Pi(\epsilon, \dot{\epsilon}) \) is implicit function of \( t \)) we receive a first order differential equation for \( \epsilon(t) \). Differentiating it again and replacing \( \dot{\Pi} \) from the second Hamilton equation (36), we obtain
\[
\ddot{\epsilon} + (4a(t)c(t) + 2\frac{\dot{a}(t)}{a(t)}b(t) + \frac{\dot{a}(t)}{2a(t)} - \frac{3\dot{a}(t)^2}{4a(t)^2} - 4b(t)^2 - 2b(t))\epsilon +
\]
\[
\frac{i}{\epsilon^2\epsilon} - \frac{1}{\epsilon^2\epsilon} \frac{1}{\epsilon^3e^{-4i\int_0^t \frac{d\tau}{\epsilon}}} = 0,
\]
where the substitution for \( \Pi \) from Table 1 has been also used. Taking into account that the Wronskian for eq. (11) is \( W(\epsilon, \dot{\epsilon}) = \dot{\epsilon}\epsilon^* - \epsilon\dot{\epsilon}^* = 2i \), and relation \( \epsilon^2 = \epsilon^2e^{-4i\int_0^t \frac{d\tau}{\epsilon}} \), we obtain (11). Therefore, we have received the same frequency \( \Omega(t) \) as in the equation (17).

It is easily to show that classical equations \( \dot{q} = \frac{\partial \tilde{H}_0(p, q)}{\partial p} \) and \( \dot{p} = -\frac{\partial \tilde{H}_0(p, q)}{\partial q} \) could also be transformed to (11) or to Lagrange equations (13) with the following substitutions
\[
q = \sqrt{\hbar a} \ 2Re(ze^*)
\]
\[
p = -\sqrt{\frac{\hbar}{a}} \ 2Re \left[ z(b\epsilon - \frac{\dot{\epsilon}}{2} - \frac{1}{4a}\epsilon^*) \right],
\]
where the complex \( z \) are the eigenvalues of Schrödinger minimum uncertainty states (see [20]).

**Appendix 2**
Here we calculate the cofluctuation (third eq. in (24)). By definition, the third independent second moment \( c_{qp}(t) \) in the probability theory is

\[
c_{qp}(t) = \frac{1}{2} < \hat{q}\hat{p} + \hat{p}\hat{q} > - < \hat{q} > < \hat{p} > .
\]  

(42)

We can directly calculate it as we have done for \( \sigma_q \) and \( \sigma_p \) in (24). This derivation involves again the use of the Wronskian \( W(\epsilon, \dot{\epsilon}) \) and formula \( \epsilon(t)\dot{\epsilon}^*(t) = \rho(t)\dot{\rho}(t) - i \), following from (18). But there is an easy way (using a physical argument - uncertainty principle) to derive the cofluctuation, since we have already expressed the quantum fluctuations by \( \rho(t) \). As we have mentioned in (26), the eigenstates of the linear invariants \( |z; \alpha> \) are Schrödinger minimum uncertainty states. The Schrödinger uncertainty relation \( [24] \)

\[
\sigma_q^2 \sigma_p^2 \geq \frac{1}{4} |[\hat{q}, \hat{p}]|^2 + c_{qp}^2, \quad [\hat{q}, \hat{p}] = i\hbar
\]  

(43)

becomes equality for these states \( |z; \alpha> \). Expressing \( c_{qp}^2 \) from this relation, we obtain

\[
c_{qp}^2 = \sigma_q^2 \sigma_p^2 - \frac{\hbar^2}{4}.
\]  

(44)

Taking the fluctuations \( \sigma_q \) and \( \sigma_p \) from (24), the final result for the cofluctuation becomes:

\[
c_{qp}^2 = \hbar^2 \rho^2(t) \left( b(t)\rho(t) - \frac{\dot{\rho}(t)}{2} - \frac{\dot{a}(t)}{4a(t)} \rho(t) \right)^2 .
\]  

(45)

References

[1] H.R. Lewis, W.B. Risenfeld, J.Math. Phys., v.10, p.1458 (1969);

[2] D.A.Trifonov, Coherent States and Uncertainty Relation, Phys. Lett. 48A, no.3, pp.165-66 (1974);

[3] D.A.Trifonov, Coherent States of Quantum Systems, Bulgarian J. Phys, v.2, no.4, pp.303-311 (1975),
D.A.Trifonov, Coherent States and Evolution of Uncertainty Products, Preprint ICTP IC/75/2 (1975).

The quantum fluctuations \( \sigma_q^2 \) and \( \sigma_p^2 \) should be replaced in their expressions in formulas (18) in the first reference (respectively in (15) in the second reference). Note that the difference between these formulas and the first two expressions in (24) in our article is due to the different definitions of the Hamiltonian’s coefficients;

[4] I.A.Malkin, V.I.Manko, D.A.Trifonov, J.Math. Phys., v.14, no.5, p.576 (1973);
I.A.Malkin, V.I.Manko, D.A.Trifonov, Phys.Rev. D, v.2, no.8, pp.1371-85 (1970);
I.A.Malkin, V.I.Manko, D.A.Trifonov, Nuovo Cimento A, v.4, p.773 (1971);

[5] A. Holz, Nuovo Cimento Lett. A, v.4, p.1319 (1970)

[6] C.F.Lo, Phys.Rev. A, v.43, no.1, pp.404-409 (1991);

[7] H.W.Lee, Phys.Lett. A, v.153, no.4, pp.219-223 (1991);
[8] C.M.Cheng, P.C.Fung, J. Phys. A, v.21, no.22, p.4115-4131 (1988);

[9] C.F.Lo, Nuovo Cimento B, v.105, no.5, pp.497-506 (1990);
    C.F.Lo, J. Phys. A, v.23, p.1155 (1990);
    C.F.Lo, Nuovo Cimento D, v.13, p.1279 (1991);
    C.F.Lo, Europhys. Lett., v.24, p.319 (1993);
    C.F.Lo, Nuovo Cimento, ser2, no.9, p.1015 (1995);

[10] G.S.Agarwal, S.A.Kumar, Phys.Rev. Lett., v.67, no.26, pp.3665-68 (1991);

[11] L.S.Brown, Phys.Rev. Lett., v.66, no.5, pp.527-529 (1991);

[12] W.Paul, Rev. Mod. Phys., v.62, no.3, pp.531-540 (1990);

[13] F.L. Li, Phys.Lett. A, v.168, p.400 (1992);

[14] S.J.Wang, F.L.Li, A.Weiguny, Phys.Lett. A, v.180, pp.189-96 (1993);

[15] F.L. Li, S.J.Wang, A.Weiguny, D.L.Lin, J. Phys. A, v.27, pp.985-92 (1994);

[16] A.L. de Brito, A.N.Chaba, B.Baseia, Phys.Rev., A, v.52, no.2, pp.1518-24 (1995);

[17] J.-Y. Ji, J.K.Kim, S.P.Kim, Phys.Rev., A, v.51, no.5, pp.4268-71 (1995);
    H.-C. Kim, M.-H.Lee, J.-Y. Ji, J.K.Kim, Phys. Rev. A, v.53, no.6, pp.3767-72 (1996);

[18] A.N. Seleznyova, Phys. Rev. A, v.51, no.2, pp.950-959 (1995);

[19] A.F.R. de Toledo Piza, Phys. Rev. A, v.51, no.2, pp.1612-1616 (1995); The second
    pair variables in this work relevant to our, are: \{Q = \sigma, P = \Pi \}.

[20] Y.-Z.Lai, J.-Q.Liang, H.J.W. Müller-Kirtsten, J.-G.Zhou, Phys. Rev. A, v.53, no.5,
    pp.3691-93 (1996);

[21] N.J.Gunther, P.G.Leach, J. Math. Phys., v.18, no.4, p.572 (1977);
    A.K.Rajagopal, J.Marshal, Phys. Rev.A, v.26, p.2977 (1982);
    J.Hartley, H.Ray, Phys. Rev.D, v.25, p.382 (1982);
    X.Jing-Bo, Q.T.Zheng, G.X.Chun, Europhys. Lett., v.15, no.2, p.119 (1991);
    J.B.Xu, T.Z.Quang, X.C.Gao, Phys. Lett.A, v.159, p.113 (1991);

[22] A.Angelow, D.A.Trifonov, Schrödinger Covariance States in Anisotropic Waveguide,
    Preprint ICTP, IC/95/44 Trieste, Italy, (1995);

[23] Cofluctuant ( or Covariance ) states with a non-zero cofluctuation are a sub-class
    of Schrödinger minimum uncertainty states, generalized squeezed states, two-photon
    coherent states e.c. Equivalence between them has been proved for first time in:
    D.A.Trifonov, On the stable evolution of squeezed and correlated states, J. Sov. Laser
    Research, v.12, no.5, pp.414-420 (1991)

[24] E. Schrödinger, Um Heisenbergschen unscharfeprinzip, Sonderausgabe aus den sit-
    zungsberichten der preussischen akademie der wissenschaften, Phys.-Math. Klasse,
    pp.348-356 (1930), v.XIX ;
[25] M.D. Levenson, R.M. Shelby, A. Aspect, M. Reid, D.F. Walls, Phys. Rev. A, v.32, no.3, pp.1550-62 (1985);
[26] P.D. Drummond, S.J. Carter, J.O.S.A., vol.4, no.10, pp.1565-73 (1987);
      P.D. Drummond, Phys. Rev., A, v.42, no.11, pp.6845-57 (1990);
[27] A. Yariv, Quantum Electronics, John Wiley and Sons, Inc., 3rd Ed., New York (1988);
      J. Perina, Quantum statistic of linear and nonlinear optical phenomena, D. Reidel Publishing Company, Boston (1984);
[28] I. Zlatev, A. Nikolov, Theoretical Mechanics, v.1, N.I., Sofia (1981);
[29] E. Kamke, Differentialgleichungen, Lösungsmethoden und Lösungen, Gewöhnliche differentialgleichungen, vol.I., Leipzig, Akademische Verlagsgesellschaft (1959)
Table 1.

|   | Refer. | Equations | Substitution to get eq. $\dot{\epsilon} + \Omega^2 \epsilon = 0$; $\epsilon = \epsilon_1 + i \epsilon_2$ |
|---|--------|-----------|---------------------------------------------------------------------------------------------------|
| 1 | [1, 4] | $\dot{\rho} - \frac{1}{\rho^2} + \Omega^2(t) \rho = 0$ | $\rho(t) = \epsilon(t) e^{i \int_0^t \frac{d\tau}{\epsilon^*(\tau) \epsilon(\tau)}}$ follows from (18) |
| 2 | [8] | $\dot{c}_1 = a_1' + a_2' c_1 + a_3' c_1^2$ | $\epsilon(t) = -\frac{1}{2} \int_0^t [a_2'(\tau) + a_3'(\tau) c_1(\tau)] d\tau$ $\times$ $(m_0 \omega_0 \int_0^t a_3'(\tau) e^{i \int_0^\tau (a_2'(\tilde{\tau}) + a_3'(\tilde{\tau}) c_1(\tilde{\tau})) d\tilde{\tau}} d\tau + i)$ |
| 3 | [17, 29] | $\dot{f} + \frac{m(t)}{m(t)} \dot{f} + \omega^2(t) f = 0$ | $f(t) = \epsilon(t) / \sqrt{m(t)}$ |
| 4 | [23, 19] | $\dot{\sigma} = \frac{\partial <H_1>}{\partial \sigma} = 2b \sigma + 2a \Pi$, $\dot{\Pi} = -\frac{\partial <H_1>}{\partial \Pi} = -2c \sigma - 2b \Pi + \frac{h^2}{2} \frac{1}{\sigma^2}$ | $\sigma = \sqrt{h a(t)} \epsilon(t) e^{-i \int_0^t \frac{d\tau}{\epsilon^*(\tau) \epsilon(\tau)}}$, $\Pi = \sqrt{h a(t)} \left[ \frac{1}{2} \epsilon(t) - \frac{i}{\epsilon^*(t)} \right] - (\frac{1}{2} \frac{a(t)}{a(t)} - b(t)) \times e^{-i \int_0^t \frac{d\tau}{\epsilon^*(\tau) \epsilon(\tau)}}$ |