Warm inflation with a generalized Langevin equation scenario

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In this paper, we discuss the warm inflation model with the Langevin equation and a generalized Langevin equation scenario. As a brief picture to illustrate the basic properties of stochastic differential equation in warm inflation, we start from the simple condition with constant dissipative coefficient. In this model, we prove the perturbed inflaton field exhibits a stationary process on large scale, so the perturbed field has a scale-invariant power spectrum. Then we study the warm inflation with a generalized Langevin equation scenario. The perturbed field in such model also shows a stationary process and the power spectrum is quite similar to the one in cold inflation. If choosing an appropriate fluctuation-dissipation relation, we can get a spectrum same as the cold inflation. In a word, we attempt to show the rationality of warm inflationary scenario via statistical physics method.

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I. INTRODUCTION

Warm inflation model was established as a candidate scenario to overcome some defects in cold inflation [1, 2]. However, it was realized a few years after its original proposal that the idea of warm inflation was not easy to be realized in concrete models and even is simply not possible in relevant works [3, 4]. Some problems mentioned were suspected in such scenario. Shortly afterwards many successful models of the warm inflation have been established, in which the inflaton indirectly interacts with the light degrees of freedom though a heavy mediator fields instead of being coupled with a light field directly [1, 5–7]. The evolution of the inflaton field can be properly determined in the context of the in-in, or the Schwinger closed-time path functional formalism [8]. This equation not only displays both dissipation and non-Markovian stochastic noise terms, but also can be regarded as a generalized Langevin-like equation of motion [9, 10].

Compared with the predictions of the cold inflation that primordial density fluctuations mostly from quantum fluctuation and thermal bath are only generated at the end of inflation [11], the warm inflation model suggests that our universe is hot during the whole inflation when inflaton fields couple with the thermal bath and the primary source of density fluctuations comes from thermal fluctuations [12, 13]. The equation of motion for warm inflation can be written as a stochastic Langevin equation, in which there is a dissipation term to describe the inflaton fields coupling with the thermal bath and there is also a fluctuation term described by a stochastic noise term [9, 15]. The fundamental principles of the warm inflation have been described and reviewed in [16].

The Langevin equation is widely used in the dynamics of a Brownian particle in phase space which is described by the Markovian set of the differential equations [17–19]. If we consider the dissipative coefficient as a constant, this stochastic differential equation can be casted as the Langevin equation which represents a Markov process. Otherwise if the dissipative coefficient is a function (called integral kernel, damping kernel or memory kernel), the stochastic equation, as a differential integral equation, can be called a generalized Langevin equation corresponding to a non-Markov process [20, 21]. Both the dissipative coefficient and integral kernel, obviously, yield the fluctuation-dissipation theory [22].

In this paper, we attempt to illustrate the rationality of explanation with the warm inflationary scenario via statistical physics method. To achieve that goal, we need to prove the process described by the (generalized) Langevin equations are stationary process on large scale. Thus we can examine the scale-invariant power spectrum for the reason that a stationary process means invariant expectation variance [23]. In this way, we can also get the power spectrum at the horizon-crossing scale. The spectrum from the Langevin scenario is the same as the one via the Green’s function [12], while the spectrum from the generalized Langevin scenario is similar to the cold inflation. The isotropy and homogeneity of cosmic microwave background shows a near thermal equilibrium state must be hold for our early universe [24, 25]. Based on this observational fact, we get the freeze-out wave number and the approximate conditions satisfied in warm inflationary model.

This paper is organized as follows: In Sec. II, we give a brief introduction to Langevin equations and warm inflation together with their properties. In Sec. III we study the warm inflation with a Langevin equation scenario and get some suggestive results to prepare further discussions in next section. In Sec. IV we study the warm inflation with generalized Langevin equation scenario and get some important conclusions. Finally, in
II. LANGEVIN EQUATION AND WARM INFLATION

Before starting the discussion to the thermodynamic properties of the warm inflation, it’s necessary to have a brief introduction for several thermodynamics foundations. In physics, the Langevin equation is a stochastic differential equation describing the statistical properties of particles with irregular motion. The Langevin dynamics method has the form [19]

\[ m\ddot{x} + \beta \dot{x} + U'(x) = \xi(t), \quad (1) \]

where \( \beta \) is a constant (called dissipative coefficient) which describes the damped effect of a particle coupling with other particles and \( \xi(t) \) is a stochastic force which denotes the fluctuate effect of a particle driven by other particles nearby. The dissipative constant and fluctuate force follow the fluctuation-dissipation theorem

\[ \langle \xi(t)\xi(t') \rangle = m \kappa B T \beta \delta(t-t'). \quad (2) \]

The stochastic differential equation of Eq. (1) describes a Markov progress, which means the stochastic properties of a thermodynamics system at time \( t \) are independent on its previous time \( t' < t \). If a stochastic variable depends on its previous state, this variable is called a non-Markov progress which has the form [22]:

\[ m\ddot{x}(t) + m \int_0^t \gamma(t-t')v(t')dt' + U'(x) = \xi(t), \quad (3) \]

where \( \gamma(t-t') \) is called damped kernel and \( \xi(t) \) is also named stochastic force. The damped kernel and stochastic force, obviously, follow the fluctuation-dissipation theorem.

\[ \langle \xi(t)\xi(t') \rangle = m k_B T \gamma(|t-t'|). \quad (4) \]

The stochastic differential integrate equation of Eq. (3) is a non-Markov progress.

In warm inflation model, the equation of motion of background field is often written as the Langevin equation

\[ \left[ \frac{\partial^2}{\partial t^2} + (3H + \Gamma) \frac{\partial}{\partial t} - \frac{1}{a^2} V' \right] \Phi + \frac{\partial V(\Phi)}{\partial \Phi} = \xi(x,t), \quad (5) \]

where \( \Gamma \) is the dissipation coefficient and \( \xi \) is the thermal noise fluctuation. In this paper, we consider only in the case of de Sitter space-time, where \( a(t) = \exp(HT) \) and a constant \( H \). According to the fluctuation-dissipation theorem, dissipation coefficient \( \Gamma \) and fluctuation noise \( \xi \) have the relation

\[ \langle \xi(x,t)\xi^*(x',t') \rangle = 2 \Gamma T a^{-3} \delta^3(x-x')\delta(t-t'). \quad (6) \]

The Fourier transformation of Eq. (6) is

\[ \langle \xi(k,t)\xi^*(k',t') \rangle = 2(2\pi^3)\Gamma Ta^{-3} \delta^3(k - k')\delta(t-t'). \quad (7) \]

Usually \( \Gamma \) is a function of both background homogeneous inflaton field \( \Phi \) and temperature \( T \).

The inflaton field operator \( \Phi(x, t) \) is often separated into the parts as follow

\[ \Phi(x, t) = \phi(t) + \delta \phi(x, t), \quad (8) \]

where \( \delta \phi(x, t) \) is the perturbed part of inflaton, and \( \phi(t) \) is the background homogeneous inflaton field defended as

\[ \phi(t) = \frac{1}{\Omega} \int_\Omega d^3 x \Phi(x, t). \quad (9) \]

Here, \( \Omega \) is particle horizon size \( \Omega = 1/H \). With this relation, Eq. (5) reads

\[ \frac{\partial^2 \phi}{\partial t^2} + (3H + \Gamma) \frac{\partial \phi}{\partial t} + V, \phi'(\phi) = 0, \quad (10) \]

\[ \left\{ \frac{\partial^2}{\partial t^2} + (3H + \Gamma(\phi)) \frac{\partial}{\partial t} + \frac{k^2}{a^2} + \Gamma_\phi(\phi) \frac{\partial \phi}{\partial t} + V_{,\phi}(\phi) \right\} \delta \phi = \xi(k, t). \quad (11) \]

It is also necessary to define some slow-roll parameters for warm inflation,

\[ \varepsilon = \frac{1}{16\pi G} \left( \frac{V_{,\phi}}{V} \right)^2 \ll 1 + r, \quad (12) \]

\[ \eta = \frac{1}{8\pi G} \frac{V_{,\phi \phi}}{V} \ll 1 + r, \quad (13) \]

and

\[ \beta = \frac{1}{8\pi G} \frac{\Gamma_{,\phi} V_{,\phi}}{TV} \ll 1 + r, \quad (14) \]

where \( r \) is the ratio of dissipation coefficient \( \Gamma \) and Hubble parameter \( H \), i.e., \( r \equiv \Gamma/3H \).

III. WARM INFLATION WITH CONSTANT DISSIPATIVE COEFFICIENT

We first consider the condition of dissipative coefficient with a constant. Although the dissipative term may be very complex as a function of inflationary fields or cosmic time \( t \), such a simple model would help us to get some preliminary conclusions and illustrate some useful properties of warm inflationary scenario.

With the slow-roll approximation, we treat the dissipative coefficient as a constant so that the Langevin equation of Eq. (11) approximately reads [20]

\[ (3H + \Gamma) \frac{d \delta \phi(k, t)}{dt} + [k^2_p + V''(\phi)] \delta \phi(k, t) \approx \xi(k, t), \quad (15) \]
where \( V''(\phi) = d^2V(\phi)/d\phi^2 \) and \( \phi \) is defined in Eq. (9), \( k_p = k/a \) is the physical wave number and \( k \) is the conformal wave number. The process in the equation above is a stationary, Markov, and Gaussian process. The solution of Eq. (15) is

\[
\delta \phi(k, t) \approx \frac{1}{3H + \Gamma} e^{-(t-t_0)/\tau(\phi)} \delta \phi(k, t_0) + \frac{1}{3H + \Gamma} \int_{t_0}^{t} e^{-(t-t_0)/\tau(\phi)} \xi(k, t') dt'.
\] (16)

where \( t_0 \) is any coordinate time during inflation and

\[
\tau(\phi) = \frac{3H + \Gamma}{k_p^2 + V''(\phi)}. \] (17)

In statistical mechanics, \( \tau(\phi) \) is called relaxation time which means a time during which a thermodynamic system returns its perturbed states into equilibrium. The observation of isotropy and homogeneous cosmic micro background, of course, requires that the relaxation time must be much smaller than the cosmic time, i.e., \( \tau(\phi) \ll 1/3H \), which yields

\[
k_p \gg 3H(1 + r)^{1/2}, \] (18)

where we have used the relation of slow-roll parameter \( V''/H^2 = 3\eta \ll 1 + r \). Define the freeze-out number

\[
k_F \equiv H(1 + r)^{1/2}. \] (19)

From Eq. (19), it’s easy to find the freeze-out number in warm inflation degenerates to that in cold inflation with weak dissipative condition while it approximately equals to \( (H\Gamma)^{1/2} \) with strong dissipative condition \( \Gamma \gg 3H \). In warm inflation, the freeze-out wave number is always larger than the Hubble crossing wave number \( k = aH \), which means the freeze-out time will always precede the Hubble crossing time. The evolution of the inflaton is mainly deterministic during time \( t > t_F \).

The autocorrelation function of the perturbed inflation field is

\[
\{\delta \phi(k, t_1)\delta \phi^*(k', t_2)\} = \delta \phi(k, t_0)\delta \phi(k', t_0) e^{-(t_1+t_2)/\tau(\phi)} + \frac{2(2\pi)^3 k_B T T \delta^3(k - k')}{(3H + \Gamma)^2} e^{-(t_1+t_2)/\tau(\phi)}
\]
\[
\times \int_{t_0}^{t_1} \int_{t_0}^{t_2} e^{(s_1+s_2)/\tau(\phi)} a^{-3}(s_1)\delta(s_1 - s_2) ds_1 ds_2
\]
\[
\] (20)

where we have used Eq. (17), \( \langle \cdots \rangle \) denotes stochastic averaging and \( \{ \cdots \} \) denotes the stochastic averaging on initial state \( \delta \phi(k, t_0) \). The double integral in Eq. (20) contains a \( \delta \) function, so we need to integrate first to the lager one in \( t_1 \) and \( t_2 \). Thus,

\[
\langle \delta \phi(k, t_1)\delta \phi^*(k', t_2)\rangle
\]
\[
= \delta \phi(k, t_0)\delta \phi(k', t_0) e^{-(t_1+t_2)/\tau(\phi)} + \frac{2(2\pi)^3 k_B T T \delta^3(k - k')}{(3H + \Gamma)^2} e^{-(t_1+t_2)/\tau(\phi)}
\]
\[
\times e^{-(t_1+t_2)/\tau(\phi)} \int_{t_0}^{\min(t_1,t_2)} \int_{t_0}^{\max(t_1,t_2)} e^{(s_1+s_2)/\tau(\phi)} a^{-3}(s_1)\delta(s_1 - s_2) ds_1 ds_2
\]
\[
= \delta \phi(k, t_0)\delta \phi^*(k', t_0) e^{-(t_1+t_2)/\tau(\phi)} + \frac{2(2\pi)^3 k_B T T \delta^3(k - k')}{(3H + \Gamma)^2} e^{-(t_1+t_2)/\tau(\phi)}
\]
\[
\times \left[ e^{-(t_1+t_2)/\tau(\phi)} a^{-3}(t_m) - e^{-(t_1+t_2)/\tau(\phi)} a^{-3}(t_0) \right]
\]
\[
= \delta \phi(k, t_0)\delta \phi^*(k', t_0) e^{-(t_1+t_2)/\tau(\phi)} + \frac{2(2\pi)^3 k_B T T \delta^3(k - k')}{(3H + \Gamma)^2} e^{-(t_1+t_2)/\tau(\phi)}
\]
\[
\times \left[ e^{-(t_1+t_2)/\tau(\phi)} a^{-3}(t_m) - e^{-(t_1+t_2)/\tau(\phi)} a^{-3}(t_0) \right].
\] (21)

By using slow-roll approximation Eq. (13) and semi thermal equilibrium approximation of Eq. (18), we can do more detailed calculation of the second term in final equality:

\[
\Gamma \tau(\phi) / (3H + \Gamma)^2 (2 - 3H \tau(\phi)) a^3 = \frac{\Gamma}{a^3(3H + \Gamma)^2} \frac{(3H + \Gamma) / (k_p^2 + V'')}{2 - 3H (3H + \Gamma) / (k_p^2 + V'')}
\]
\[
= \frac{1}{(1 + r)} \frac{1}{a^3(3H)^2} \frac{1}{2k_p^2/9H^2 - 3\eta - 1 - r} \approx \frac{r}{(1 + r)} \frac{k_F}{2^3 a^2 k_p^2} = \frac{r H \bar{z}}{2(1 + r)^{1/2} k^3},
\] (22)
where we have defined a new parameter $\tilde{z} = k_p/k_F = k/aH(1+r)^2$. Finally, the autocorrelation function reads

$$\{\delta\varphi(k, t_{1})\delta\varphi^*(k', t_{2})\} = \{\delta\varphi(k, t_{0})\delta\varphi^*(k', t_{0})\} - \frac{(2\pi)^{3}THr\tilde{z}}{(1+r)^{3/2}k^{3}}\delta^{3}(k - k')e^{-(t_{1} + t_{2})/\tau(\phi)} + \frac{(2\pi)^{3}THr\tilde{z}}{(1+r)^{3/2}k^{3}}\delta^{3}(k - k')e^{-|t_{1} - t_{2}|/\tau(\phi)}. \tag{23}$$

The correlation function is defined as

$$P_{\delta\varphi}(x - y, t_{1}, t_{2}) = \langle \delta\varphi(x, t_{1})\delta\varphi(y, t_{2})\rangle, \tag{24}$$

whose Fourier transformation is

$$P_{\delta\varphi}(k, t_{1}, t_{2}) = \int \frac{d^{3}k'}{(2\pi)^{3}}\langle \delta\varphi(k, t_{1})\delta\varphi^*(k', t_{2})\rangle. \tag{25}$$

Using the definition of Eq.\,(25), we can simplify the autocorrelation function of Eq.\,(23) as

$$P_{\delta\varphi}(k, t_{1}, t_{2}) \equiv \left[P_{\delta\varphi}(k, t_{0}, t_{0}) - k_{B}THr\tilde{z}_{s}\right]e^{-(t_{1} + t_{2})/\tau(\phi)} + k_{B}THr\tilde{z}_{s}e^{-|t_{1} - t_{2}|/\tau(\phi)}, \tag{26}$$

where we have set $t_{1} < t_{2}$ and $\tilde{z}_{s}$ represents the freeze-out that occurs at $t_{1}$ when $k_{B}(t_{1}) = k_F$. The autocorrelation function $P(k, t_{1}, t_{2})$ is obviously dependent on the initial state $\delta\varphi(k, t_{0})$. If $t_{1} + t_{2} \gg \tau(\phi)$, the memorability on initial state is no longer important. Thus the autocorrelation $P_{\delta\varphi}(k, t_{1}, t_{2})$ is a function as $|t_{1} - t_{2}|$, i.e., $P_{\delta\varphi}(k, t_{1}, t_{2}) = \chi(|t_{1} - t_{2}|)$, which means the thermal system evolves toward a stationary process. In statistical physics, a stationary process represents that the variance and expectation of a system does not change when shifted in time. If we choose an appropriate time as the initial time $t_{0}$ when $P_{\delta\varphi}(k, t_{0}, t_{0}) = k_{B}THr/(1+r)^{3/2}k^{3}$, the system of Eq.\,(13) is totally a stationary process, in which way, we can get the power spectrum of warm inflationary scenario. With the definition of power spectrum

$$P_{\delta\varphi}(k, t) = \frac{k^{3}}{2\pi^{2}}\int \frac{d^{3}k'}{(2\pi)^{3}}\langle \delta\varphi(k, t_{0})\delta\varphi^*(k', t_{0})\rangle, \tag{27}$$

it is now possible to use the definition of Eq.\,(27) to obtain the power spectrum. Now make average on initial state and set $t_{1} = t_{2} = t$, so correlation function can be written as

$$P_{\delta\varphi}(k, t) = \left(P_{\delta\varphi}(k, t_{0}) - \frac{k_{B}THr}{2\pi^{2}(1+r)^{3/2}}\right)e^{-2t/\tau(\phi)} + \frac{k_{B}THr}{2\pi^{2}(1+r)^{3/2}}. \tag{28}$$

Let’s consider the strong dissipative condition $r \gg 1$ which yields a scale-invariant power spectrum \[14\]

$$P_{\delta\varphi}(k, t) = (\Gamma H/3)^{1/2}k_{B}T/2\pi^{2}. \tag{29}$$

If the power spectrum of the initial state is also scale-invariant same as in Eq.\,(29), the spectrum $P_{\delta\varphi}(k, t)$ is also scale-invariant and totally the same with $P_{\delta\varphi}(k, t_{0})$. In other words, $P_{\delta\varphi}$ is stable in this condition which means system is on thermal equilibrium during time interval $t > t_{0}$. This is quite similar with the condition of correlation function for particles with Brown motion \[27\]. Now, consider that $P_{\delta\varphi}(k, t_{0})$ is not a scale-invariant spectrum and assume $P_{\delta\varphi}(k, t_{0}) = A(k/k_{0})^{n' - 1}$ where $n'$ is an arbitrary number. Then $P_{\delta\varphi}(k, t)$ becomes also dependent on $k$, i.e., $P_{\delta\varphi}(k, t) = A(k/k_{0})^{n - 1}$. However, this term damps with the increase of time, which means $n$ tends to unit as the time evolution. This is an effect dominated by non-equilibrium dynamics. In this way, we can say that scalar index $n_{s}$ and slow-roll parameter $\beta$ are parameters that illustrate the deviation from equilibrium state. Probe on cosmic microwave background shows that our universe is almost in thermal equilibrium \[29\] if considering our universe in early epoch as a model in thermal bath. The relation of Eq.\,(28) indicates that $P_{\delta\varphi}(k, t_{0})$ (the initial condition of universe) becomes not that important even though we cannot give an accurate description till now.

Finally, let’s have a brief conclusion on this section. From Eq.\,(21), we know that the autocorrelation function of inflaton is stationary on super-horizon scale, which means the variance or power spectrum tends to a constant during inflation. This result is the same as cold inflation. In this way, we can get the power spectrum of warm inflationary scenario which is also the same as the one via Green’s function method \[14\] \[28\]. Similar results have been accomplished in relevant previous references \[29\] \[30\].

IV. WARM INFLATION WITH NON-MARKOV DISSIPATIVE COEFFICIENT

In quantum field theory, evolution equation of field is a differential integrate equation \[31\]. In finite temperature condition, the evolution equation of field becomes a stochastic equation of motion \[32\]:

$$[\partial^{2} + \omega^{2}(x)]\Phi(x) + \int_{0}^{t}dt'\Sigma(x - x')\Phi(x') = \xi(x). \tag{30}$$

From Eq.\,(11), $\xi(x)$ can be interpreted as a Gaussian stochastic noise two-point statistical correlation function

$$\langle \xi(x)\xi(x')\rangle = \frac{3}{2}\Sigma(|x - x'|), \tag{31}$$

in which the coefficient $3/2$ comes from the three dimension of space similarly with the result in molecule statistical dynamics. In de Sitter space-time, we led to the following equation for the perturbed inflaton field $\delta\varphi(x)$
in Eq. [3] in momentum space:

\[
\left[ \frac{d^2}{dt^2} + 3H \frac{d}{dt} + \frac{k^2}{a^2(t)} + V''(\phi) \right] \delta\varphi(k, t) + \int_0^t dt' a^3(t') \Sigma(k, t', t') \delta\varphi(k, t') = \xi(k, t).
\]

In Eq. [32], the integral kernel (self-energy) \(\Sigma(k; t, t')\) is a function of \(t\) and \(t'\) instead of \(t - t'\). However, the new term with conformal transformation

\[
\bar{\Sigma}(k; t - t') = a^{3/2}(t)a^{3/2}(t')\Sigma(k; t, t')
\]

(33)
is an integrate kernel as a function of \(t - t'\). According to the fluctuation-dissipation theory, together with the principle of general relativity, \(\xi(k, t)\) and \(\bar{\Sigma}(k; t - t')\) follow the relation

\[
\langle \xi(k, t) \xi^*(k', t') \rangle = 2(2\pi)^3 k_B T \delta^3(k - k') \frac{\bar{\Sigma}(k, |t - t'|)}{a^{3/2}(t)a^{3/2}(t')}.
\]

(34)

Now, we need to do some further treatment on Eq. [32]. Define \(\delta\varphi(k, t) = a^{3/2}\bar{\delta}\varphi(k, t)\) and set \(t \to t/H\). With the slow-roll approximation, then Eq. [32] becomes

\[
\frac{d}{dt} \bar{\delta}\varphi(k, t) + \int_{t_0}^t dt' \gamma(t - t') \bar{\delta}\varphi(k, t') + \left[ \bar{z}^2 + 3\eta - \frac{3}{2} a^{-1}(t) \right] \bar{\delta}\varphi(k, t) = \bar{\xi}(k, t),
\]

(35)

where \(t\) is a dimensionless variable, \(\gamma(t - t') \equiv \bar{\Sigma}(k, t - t')/3H^2\) and we have neglected the label of momentum \(k\), \(\bar{z} \equiv k/aH = k_B\), and \(\xi \equiv a^{3/2}\xi/3H^2\) with

\[
\langle \bar{\xi}(k, t) \bar{\xi}^*(k', t') \rangle = \frac{2(2\pi)^3 k_B T}{3H^2} \delta^3(k - k') \gamma(|t - t'|).
\]

(36)

Although Eq. [36] contains a parameter \(a^{-1}\), we do not care about it too much for the reasons as follow:

- The arbitrariness of initial time;
- When \(t \gg 1/H\), this term can be neglected;
- In this paper, we only consider the state of system when tending to thermal equilibrium and studying the initial state will benefit us nothing.

Define \(\omega^2 \equiv \bar{z}^2 + 3\eta\) and we choose \(\bar{z}\) as the value at the horizon crossing. Thus

\[
\bar{\delta}\varphi(k, t) + \int_{t_0}^t dt' \gamma(t - t') \delta\varphi(k, t') + \omega^2 \bar{\delta}\varphi(k, t) = \bar{\xi}(k, t),
\]

(37)
The process in the equation above is a stationary, non-Markov and Gaussian process. Following the standard method used in stochastic physics [22], we apply the Laplace transformation on the both sides of Eq. [37], and get the solution

\[
\delta\varphi(k, z) = \hat{\chi}(z) \left( \delta\varphi(k, t_0) + \hat{\xi}(k, z) \right),
\]

(38)

where \(\delta\varphi(k, t_0)\) is the initial value of perturbed inflaton field. The function \(\hat{\chi}(z)\) is

\[
\hat{\chi}(z) = \frac{1}{z + \gamma(z) + \omega^2}.
\]

(39)

with \(\gamma(z)\) being the Laplace transformation of integrate kernel

\[
\hat{\chi}(z) = \mathcal{L}\left[ \gamma(t) \right] \equiv \int_0^\infty dt \gamma(t)e^{-zt}.
\]

(40)

Then applying the inverse Laplace transformation on both sides of Eq. [38], we get the solution of Eq. [37] as function of time \(t\):

\[
\delta\varphi(k, t) = \chi(t) \delta\varphi(k, t_0) + \int_0^t \chi(t - s) \hat{\xi}(s) ds,
\]

(41)

where \(\chi(t)\) is the inverse Laplace transformation of \(\hat{\chi}(z)\):

\[
\chi(t) = \mathcal{L}^{-1}[\hat{\chi}(z)] \equiv \int_{c-i\infty}^{c+i\infty} dz \ e^{zt} \hat{\chi}(z)
\]

(42)

with \(c > \max\{\text{Re Res}[\hat{\chi}(z)]\}\).

A. Proof of stationary process

As discussed in Sec. [11], if we want to get the power spectrum of the warm inflaton, we need to prove the system of Eq. [37] is a stationary process. Now we can do the proof. The autocorrelation function of the warm inflaton is
The last double integration in Eq. (44) also contains a double Laplace transform, the double integration on the right-hand side in Eq. (43) could be calculated by applying the double Laplace transformation on region \[ 0 \leq \tau_1, \tau_2 \leq \infty \]

\[
\left\{ \left\{ \delta \varphi (k, t_1) \delta \varphi^* (k', t_2) \right\} \right\} \chi^*(t_2) + \frac{2(2\pi)^3 k_B T \delta^3(k - k')}{3H^2} \int_0^{t_1} \int_0^{t_2} \chi(t_1 - s_1) \chi^*(t_2 - s_2) \gamma(\tau_1 - \tau_2) ds_1 ds_2.
\]

(43)

The double integration on the right-hand side in Eq. (43) could be calculated by applying the double Laplace transform

\[
\int_0^\infty \int_0^\infty dt_1 \int_0^\infty dt_2 e^{-z_1 t_1} e^{-z_2 t_2} \int_0^{t_1} ds_1 \int_0^{t_2} ds_2 \chi(t_1 - s_1) \chi^*(t_2 - s_2) \gamma(|s_1 - s_2|)
\]

\[
= \int_0^\infty ds_1 \int_0^\infty ds_2 \int_0^\infty dt_1 \int_0^\infty dt_2 e^{-z_1 (t_1 - s_1)} e^{-z_2 (t_2 - s_2)} \chi(t_1 - s_1) \chi^*(t_2 - s_2) e^{-z_1 s_1} e^{-z_2 s_2} \gamma(|s_1 - s_2|)
\]

\[
= \hat{\chi}(z_1) \hat{\chi}^*(z_2) \int_0^\infty ds_2 e^{-z_1 s_1} e^{-z_2 s_2} \gamma(|s_2 - s_1|)
\]

(44)

The last double integration in Eq. (44) also contains a double Laplace transform,

\[
\int_0^\infty ds_2 \int_0^\infty ds_1 e^{-z_1 s_1} e^{-z_2 s_2} \gamma(|s_2 - s_1|)
\]

\[
= \left( \int_0^\infty ds_2 \int_0^\infty ds_1 + \int_0^\infty ds_1 \int_0^\infty ds_2 \right) e^{-z_1 s_1} e^{-z_2 s_2} \gamma(|s_2 - s_1|)
\]

\[
= \int_0^\infty ds_2 \int_0^\infty d\tau e^{-(z_1 + z_2) s_1} e^{-z_2 \tau} \gamma(\tau) + \int_0^\infty d\tau e^{-(z_1 + z_2) s_1} e^{-z_2 \tau} \gamma(\tau')
\]

\[
= \frac{\hat{\gamma}(z_1) + \hat{\gamma}(z_2)}{z_1 + z_2}.
\]

(45)

In second equality, we separate the integration into two parts: the integration on region \( s_1 > s_2 \) and the integration on region \( s_2 > s_1 \). Inserting Eq. (45) into Eq. (44), we have

\[
\int_0^\infty dt_1 \int_0^\infty dt_2 e^{-z_1 t_1} e^{-z_2 t_2} \int_0^{t_1} ds_1 \int_0^{t_2} ds_2 \chi(t_1 - s_1) \chi^*(t_2 - s_2) \gamma(|s_1 - s_2|) = \hat{\chi}(z_1) \hat{\chi}^*(z_2) \frac{\hat{\gamma}(z_1) + \hat{\gamma}(z_2)}{z_1 + z_2}
\]

(46)

According to Eq. (39), it follows the relation

\[
\hat{\chi}(z_1) \hat{\gamma}(z_1) = \frac{\hat{\gamma}(z_1)}{z_1 + \hat{\gamma}(z_1) + \omega^2} = 1 - \hat{\chi}(z_1)(z_1 + \omega^2).
\]

(47)

We should notice that \( \gamma(t - s) \) is a symmetry function (matrix), while \( \omega^2 = -i\Omega \) is an asymmetry parameter (matrix)

(34), which leads to

\[
\hat{\chi}^*(z_2) \hat{\gamma}(z_2) = 1 - \hat{\gamma}^*(z_2)(z_2 - \omega^2).
\]

(48)

Then we have

\[
\hat{\chi}(z_1) \hat{\chi}^*(z_2) \frac{\hat{\gamma}(z_1) + \hat{\gamma}(z_2)}{z_1 + z_2} = \frac{\hat{\chi}(z_1) + \hat{\chi}^*(z_2)}{z_1 + z_2} - \hat{\chi}(z_1) \hat{\chi}^*(z_2).
\]

(49)
Based on the converse calculation of Eq. (45), the first term in Eq. (49) follows the inversion Laplace transform relation as
\[
\mathcal{L}^{-1}\left\{\frac{\hat{\chi}(z_1) + \hat{\chi}^*(z_2)}{z_1 + z_2}\right\} = \tilde{\chi}(t_1 - t_2)\theta(t_1 - t_2)\chi(t_1 - t_2) + \theta(t_2 - t_1)\chi^*(t_2 - t_1).
\] (50)

Using Eqs. (43), (46) and (50), the autocorrelation function of \(\tilde{\phi}(k, t)\) is given by
\[
\{\langle \delta \varphi(k, t_1)\delta \varphi^*(k', t_2) \rangle\}
= \left[\{\delta \varphi(k, t_0)\delta \varphi^*(k', t_0)\} - \frac{2(2\pi)^3 k_B T \delta^3(k - k')}{3\mathcal{H}^2}\right]\chi(t_1)\chi^*(t_2)
+ \frac{2(2\pi)^3 k_B T \delta^3(k - k')}{3\mathcal{H}^2}\left[\theta(t_1 - t_2)\chi(t_1 - t_2) + \theta(t_2 - t_1)\chi^*(t_2 - t_1)\right].
\] (51)

If we choose an appropriate initial time and the variance on \(\delta \varphi(k, t_0)\) satisfies the relation
\[
\langle \delta \varphi(k, t_0)\delta \varphi^*(k', t_0) \rangle = \frac{2(2\pi)^3 k_B T \delta^3(k - k')}{3\mathcal{H}^2},
\] (52)
the autocorrelation function is just a function as variable \(|t_1 - t_2|\), which exhibits the stationarity of the process. It’s worth noting that the first term on the right-hand in Eq. (51) contains two functions dependent on \(t_1\) and \(t_2\) instead of \(|t_1 - t_2|\). If this term does not vanish, \(\chi(t_1)\chi^*(t_2)\) will contribute a dramatically damping trend to autocorrelation function. As a brief illustration, in first panel on Fig. 4 we plot the portraits of \(|\chi(t)|\) with different values of \(\Gamma\) (set \(\mathcal{H} = 1\)) in a simple memory kernel
\[
\gamma(t - t') = \Gamma e^{-\Gamma|t-t'|},
\] (53)
Together, the second panel includes three lines with different values of \(\omega^2\). In third panel, the damping kernel function is
\[
\gamma(t - t') = D[\Gamma_1 e^{-\Gamma_1|t-t'|} - \Gamma_2 e^{-\Gamma_2|t-t'|}],
\] (54)
where \(D\) is called Markov friction strength. The damping kernel function above describes a noise whose spectrum density function vanishes at both low and high frequency. Another damping kernel is an oscillating damping mode
\[
\gamma(t - t') = \frac{\Omega \cos(2\Omega|t-t'|) + \Gamma \sin(2\Omega|t-t'|)}{2\Omega^2(\Gamma^2 + \Omega^2)} e^{-2\Omega|t-t'|}.
\] (55)

B. Power spectrum

With the proof on stationary process, now we compute the power spectrum of Eq. (37). Take the derivative of Eq. (52) with respect to \(t\), there has
\[
\dot{\chi}(t) = \int_{c-i\infty}^{c+i\infty} \frac{dz}{z + \Gamma(z) + \omega^2} e^{zt}
= -\omega^2\chi(t) - \int_0^t \chi(t-s)\gamma(s)ds.
\] (56)
Define a stochastic perturbation variable
\[
\dot{y}(t) = \delta \phi(k, t) - \chi(t)\delta \phi(k, t_0)
= \int_0^t \chi(t-s)\xi(s)ds.
\] (57)
The variance of \(A\) is
\[
A(t) \equiv \langle y(t)\bar{y}(t) \rangle = \int_0^t \int_0^t \chi(t-s_1)\chi^*(t-s_2)\gamma(|s_1 - s_2|)ds_1ds_2.
\] (58)
Using Eq. (36), we have
\[
A(t) = \frac{2(2\pi)^3 k_B T \delta^3(k - k')}{3\mathcal{H}^2}
\times \int_0^t \int_0^t \chi(t-s_1)\chi^*(t-s_2)\gamma(|s_1 - s_2|)ds_1ds_2
= \frac{2(2\pi)^3 k_B T \delta^3(k - k')}{3\mathcal{H}^2}
\times \int_0^t \int_0^t \chi(\tau)\gamma(|\tau - \tau'|)\chi^*(\tau')d\tau d\tau'.
\] (59)
Take the derivative with respect to \(t\) of \(A(t)\), we have
\[
\dot{A}(t) = \frac{4(2\pi)^3 k_B T \delta^3(k - k')}{3\mathcal{H}^2}\chi^*(t)\int_0^t \gamma(t - \tau')\chi(\tau)d\tau
= \frac{4(2\pi)^3 k_B T \delta^3(k - k')}{3\mathcal{H}^2}\chi^*(t)\mathcal{L}^{-1}[\gamma(z)\chi(z)]
= \frac{4(2\pi)^3 k_B T \delta^3(k - k')}{3\mathcal{H}^2}\left[\chi^*(t)\dot{\chi}(t) + \omega^2|\chi(t)|^2\right]
= -\frac{2(2\pi)^3 k_B T \delta^3(k - k')}{3\mathcal{H}^2}\left[\frac{d}{dt}|\chi(t)|^2 + 2\omega^2|\chi(t)|^2\right].
\] (60)
FIG. 1: Different types of propagating functions $|\chi(t)|$. (a) Propagating functions with red noise friction of Eq. (53). The different modes of $|\chi(t)|$ with $\omega^2 = 2$ and different values of $\Gamma$ (we have set $H = 1$). (b) Different modes of $|\chi(t)|$ with $\Gamma = 5$ and different values of $\omega^2$. (c) Different modes of $|\chi(t)|$ with coloured noise of Eq. (54) together with the Markov friction strength $D = 5$ and different combinations of $\Gamma_1$ and $\Gamma_2$. (d) Different modes of $|\chi(t)|$ with oscillating damping kernel of Eq. (55) and different combinations of $\Omega$, $\Gamma$.

Thus,

$$A(t) = \frac{2(2\pi)^3 k_B T \delta^3}{qH^4} \left[ 1 - |\chi(t)|^2 - 2\omega^2 B(t) \right], \quad (61)$$

since $A(0) = 0$, $\chi(0) = 1$. The expression of $B(t)$ is

$$B(t) = \int_0^t |\chi(t')|^2 dt' \quad (62)$$

with $B(0) = 0$.

$$\{ \langle \delta \varphi(k, t) \delta \varphi^*(k', t) \rangle \}$$

$$= \left\{ \delta \varphi(k, t_0) \delta \varphi^*(k', t_0) - \frac{2(2\pi)^3 k_B T \delta^3(k - k')}{3H^2} \right\} |\chi(t)|^2$$

$$+ \frac{2(2\pi)^3 k_B T \delta^3(k - k')}{3H^2} \left[ 1 - 2\omega^2 B(t) \right]. \quad (63)$$

From above, we can see that the first term on the right-hand in Eq. (63) contain two functions dependent on $t_1$ and $t_2$ instead of $|t_1 - t_2|$. If this term does not vanish, although there exist a time dependent function in second term on the right-hand, this term could almost be ignored under larger scale limit $\bar{z} = k/aH \ll 1$ and slow-roll condition $\eta \ll 1 + r$, so $\omega^2 \ll 1$. Besides, the first panel and the second panel in Fig. 1 show that the portrait of $|\chi(t)|^2$ has a sharp distribution near $t = 0$ with $\Gamma/H \gg 1$. The integral on $|\chi(t)|^2$ will not generate a large value and is even much smaller than unit, i.e., $B(\infty) \ll 1$. Thus $1 - 2\omega^2 B(t)$ can be regarded as unit since both $\omega^2$ and $B(t)$ are much smaller than the unit. Based on the discussion in this section, the memory kernel $\gamma(t)$ drives the system evolving to an equilibrium state, while $\omega^2$ compels the system deviating from the equilibrium state which means the slow-roll parameter $\eta$ must be much smaller than the unit. According to the definition of Eq. (27), we finally get the power spectrum
of $\delta \varphi(k, t)$ at horizon crossing:

$$P_{\delta \varphi}(k, t) \simeq \frac{k_B T H}{3\pi^2 H^3 a^3 k^{-3}} = \frac{k_B T H}{3\pi^2},$$

(64)

which is quite analogous to the one in cold inflation. This result is based on the fluctuation-dissipation relation of Eqs. (11) and (36) instead of Eq. (31) (one should notice the difference between Eqs. (35) and (36), which leads to the difference between fluctuation-dissipation relation of Eqs. (35) and (31)). If we follow the relation of Eq. (31) and repeat the computation in this section, we obtain the power spectrum of warm inflaton

$$P_{\delta \varphi}(k, t) \simeq \frac{H^2}{4\pi^2},$$

(65)

which is exactly the same as the one in cold inflation [37]!

### C. Approximate condition

As usual, cold inflationary model needs two approximate parameters $\varepsilon$ and $\eta$ as seen in Eqs. (12) and (13). The warm inflationary model with Langevin scenario also need another approximate parameter $\beta$ as discussed in Sec. [11] where the thermal equilibrium approximation requires that the relaxation time $\tau(\psi)$ is much smaller then the inverse of expansion rate $3H$. This parameter describes the departure of the thermodynamic system from its equilibrium state. The generalized Langevin equation scenario as a stochastic differential equation of Eq. (37) is a slow variation function namely $\dot{\gamma}/\gamma \ll 1$. Thus, the propagating function reads

$$\chi(t) \approx \mathcal{L}^{-1} \{ \left[ \Gamma(z) \right]^{-1} \} \approx \tilde{\chi}(t) \Gamma^2 e^{-\Gamma t},$$

(69)

which leads to the Langevin equation of Eq. (11). Using Eq. (69) and $\tau_{\text{ex}} \ll 1/3H$, we have

$$\frac{\dot{\gamma}}{3H\gamma} \ll 1 + r.$$  

(71)

Specially, if we set $\tilde{\gamma}(t)$ is the average background inflaton field $\phi(t)$, i.e., $\tilde{\gamma}(t) = \langle \phi(t) \rangle$ and using the slow-roll condition, we also have

$$\beta = \frac{1}{8\pi G} \frac{\Upsilon_{\phi} V_{\phi}}{V} \ll 1 + r,$$  

(72)

which is just the slow-roll approximation of Eq. (14).

### V. CONCLUSION AND DISCUSSION

In this paper, we discuss the Markovian and non-Markovian statistical dynamical problem of the warm inflationary scenario via a Langevin language. In Sec. [11] we study a simple condition with constant dissipative coefficient. In this model, if we reckon the initial state which have been already on a thermal equilibrium state, the perturbed inflaton field exhibits a stationary process on superhorizon scale. In other words, the variance of the perturbed field does not change with the time on large scale, which is similar to the cold inflationary scenario. If the initial state is not on equilibrium, the variance exponentially tends to a constant in an exponentially damping manner, and this constant is the spectrum of perturbed warm inflaton field. The non-equilibrium initial state leads to a time dependent spectrum which corresponds to the spectrum index. Using the semi-thermal equilibrium approximation, we also derive a freeze-out scale in warm inflation which is always smaller than that in cold inflation.

In Sec. [11] we study the warm inflation with a generalized Langevin equation scenario as a stochastic differential integral equation in which the dissipative effect
is described by a time dependent integral kernel as long as the initial state is on equilibrium. We also prove that the stochastic process is also a stationary process. With the general fluctuation-dissipation theory, we derive the power spectrum of the perturbed field as well, but it is a time dependent one. If we consider the large scale limit and the slow-roll approximation, we can reckon that this power spectrum is also time independent. So, in this method, we get a scale-invariant power spectrum, which is quite analogous to the one in cold inflation. If we choose the fluctuation-dissipation relation of Eq. [31], we obtain a scale-invariant power spectrum which is the same as that in cold inflation. These results show us that the warm inflation model is a extremely possible scenario to substitute the cold inflation model. As the discussion on warm inflation with Langevin equation, we also treat the early universe satisfies the semi-thermal equilibrium condition. As a result, this condition leads to a approximation on damping kernel analogous to Eq. [14].

With the discussion above, we strongly believe that the warm inflationary model is a alternative scenario to cold inflation. In this paper, we only show a brief picture to illustrate the rationality of explanation of the warm inflation via statistical physics method. There are still many questions which deserve further discussion, such as the initial condition problem, the spectrums from different choice on potential function \( V(\phi) \) and so on.

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