A Reallocation Algorithm for Online Split Packing of Circles

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Abstract
The Split Packing algorithm [14, 25, 24] is an offline algorithm that packs a set of circles into triangles and squares up to critical density. In this paper, we develop an online alternative to Split Packing to handle an online sequence of insertions and deletions, where the algorithm is allowed to reallocate circles into new positions at a cost proportional to their areas. The algorithm can be used to pack circles into squares and right angled triangles. If only insertions are considered, our algorithm is also able to pack to critical density, with an amortised reallocation cost of $O(c \log \frac{1}{c})$ for squares, and $O(c(1+s^2) \log(1+s^2) \frac{1}{c})$ for right angled triangles, where $s$ is the ratio of the lengths of the second shortest side to the shortest side of the triangle, when inserting a circle of area $c$. When insertions and deletions are considered, we achieve a packing density of $(1 - \epsilon)$ of the critical density, where $\epsilon > 0$ can be made arbitrarily small, with an amortised reallocation cost of $O(c(1 + s^2) \log(1 + s^2) \frac{1}{c} + c \frac{1}{\epsilon})$.

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1 Introduction
A common class of problems in data structures requires handling a sequence of online requests. In a dynamic resource allocation problem, one has to handle a sequence of allocation and deallocation requests. A good allocation algorithm would run fast, and allocate items in a way that uses as few resources as possible to store the items.

Classical solutions disallow the algorithm from reallocating already-placed items except during deletion. In our problem, the algorithm is allowed to reallocate already-placed items with additional cost. Instead of bounding the running time of the algorithm, we focus on trying to bound the reallocation costs required to handle the sequence of requests, while minimising the amount of space required to handle all the requests.

We look at online circle packing, where we try to dynamically pack a set of circles of unequal areas into a unit square while allowing reallocations. Our work builds on insights from the Split Packing papers by Fekete, Morr, and Scheffer [14, 25, 24]. The Split Packing papers derive the critical density $a$ of squares and obtuse triangles (triangles with an internal angle of at least 90°). The critical of density a region is the largest value $a$ such that any set of circles with total area at most $a$ can be packed into the region. To simplify our discussion, we sometimes refer to the critical density of a region as the “capacity” of the region.

As shown in the paper [25, 24], the critical density for a square is equal to the combined area of the two circles in the configuration in Figure 3c. The critical density for an obtuse triangle is the area of its incircle. The Split Packing Algorithm, presented in the paper, is
an offline algorithm that packs circles into squares and obtuse triangles to critical density. Figure 1 shows example circle packings produced by the Split Packing Algorithm and our Online Split Packing Algorithm.

Figure 1 Sample packings produced by the two algorithms.

1.1 The Problem of Online Circle Packing

In online circle packing, we are given a fixed region, and an online sequence of insertion and deletion requests for circles of varying sizes into the region. In between requests, all circles need to be packed within the bounds of the region such that no two circles overlap. While traditionally circles are not to be moved once placed, in our setting, at any point of time, the algorithm is allowed to reallocate sets of circles, at a cost proportional to the sum of the areas of the circles reallocated (volume cost). We do not require reallocations to be in place, meaning that when a set of circles is to be reallocated to new positions, we can see them as being simultaneously removed from the packing, then placed in their new locations. (This is in contrast to requiring circles to be moved one at a time to their new locations.) We aim to bound the total reallocation cost incurred by the algorithm throughout the packing.

An allocation algorithm is said to have a packing density of $A$ on a region if it can handle any sequence of insertion and deletion requests into the region, as long as the total area of the circles in the region at any point of time is at most $A$. The algorithm packs to critical density on a region if it attains a packing density equal to the region’s capacity.

1.2 Results

Our main result is the Online Split Packing Algorithm, an dynamic circle packing algorithm based on the original Split Packing Algorithm. The Online Split Packing Algorithm can be used to pack circles into squares and right angled triangles.

1. For insertions only into a square, the algorithm packs to critical density, with an amortised reallocation cost of $O(c(\log_2 1/c))$ for inserting a circle of area $c$.

2. For insertions only into a right angled triangle of side lengths $\ell$, $s\ell$ and $\ell\sqrt{1+s^2}$, where $s \geq 1$, the algorithm can achieve critical density, with an amortised reallocation cost of $O(c(1+s^2) \log_{1+s^2} 1/c)$ for inserting a circle of area $c$.

3. For insertions and deletions into a region (a square or a right-angled triangle) of capacity $a$, the algorithm achieves a packing density of $a(1-\epsilon)$, where $\epsilon$ can be defined to be arbitrarily small. Consequently, we add an amortised reallocation cost of $O(c(\frac{1}{2}))$ for inserting a circle of area $c$. In other words, we achieve an amortised reallocation cost of $O(c(\log_2 1/c + \frac{1}{2}))$ for squares, and $O(c((1+s^2) \log_{1+s^2} 1/c + \frac{1}{2}))$ for right angled triangles.
1.3 Challenges of Online Circle Packing

The problem of packing circles into a square is a difficult problem in general. The decision problem of whether a set of circles of possibly unequal areas can be packed into a square has been shown to be NP-hard [9]. Given \( n \) equal circles, the problem of finding the smallest square that can fit these circles only has proven optimal solutions for \( n \leq 35 \) [23]. Packing a square to critical density has only been recently solved with Split Packing in 2017.

The best known algorithm for online packing of squares into squares is given by Brubach [8], with a packing density of \( \frac{2}{5} \). By embedding circles in squares, we obtain an online circle packing algorithm with a packing density of \( \frac{\pi}{4} \times \frac{2}{5} \approx 0.3142 \) [24]. The critical density for offline packing of circles into squares, as given by Split Packing, is \( \frac{\pi}{3 + 2\sqrt{2}} \approx 0.5390 \).

The required arrangement of circles in a tight packing is highly dependent on the distribution of circle sizes. The Split Packing Algorithm [25, 24] packs to critical density by packing circles in a top-down, divide and conquer fashion that starts with sorting and partitioning the circles by size. Small changes to the inputs may thus result in very different packings. An example of this is shown in Figure 2.

![Figure 2 Split Packing Algorithm differences when the size of the circle X is altered slightly.](image)

Our algorithm packs an online sequence of circles tightly by allowing reallocations. When packing with reallocations, we aim to repack only a part of the configuration at a time. We do this by making use of a binary tree structure similar to the original Split Packing Algorithm. By designing a new packing strategy, we maintain a tree where left children are packed tightly, while extra slack is kept in the rightmost node of each level (this refers to nodes on the right spine of the binary tree) to allow for circle movement. New circles are recursively inserted into the rightmost node of each level, where it eventually triggers a repack of an entire subtree at a certain level to allocate the circle.

1.4 Related Work

There are many examples of existing work in online resource allocation which do not allow items to be reallocated once placed. Online square packing into squares has been studied by Han et al. [17] and Fekete and Hoffman [12]. Other examples are squares in unbounded strips [13], rectangle packing [1] and multiple variants of online bin packing [8, 10, 11, 21, 28].

Allowing reallocations can lead to better results than if reallocations were not permitted. Ivković and Lloyd [20] provide an algorithm for bin packing that is \( \frac{4}{3} \)-competitive with the best practical offline algorithms by allowing reallocations. This beats the lower bound of \( \frac{4}{3} \) (proven in the same paper) if reallocations are not considered. Berndt et al. [7] study this problem further and achieves better upper and lower bounds. There is also work that aims to bound reallocation costs rather than running times. Fekete et al. [15] study online square packing with reallocations. Bender et al. study dynamic resource allocation with reallocation costs in problems like scheduling [2], memory allocation [3] and maintaining modules on an FPGA [5]. We also take ideas from packed memory arrays [2, 6].
Most existing work on circle packing focuses on the global, offline optimization problem of packing either equal or unequal circles into various shapes. Many existing results are collected in the packomania website [26]. Many heuristic methods have also been developed for offline circle packing. An example is [16], which explores a mix of different strategies, like nonlinear mixed integer programming and genetic algorithms to fit circles of unequal sizes into a rectangle. Offline packings of equal-size spheres into a cube are looked into in [27]. We refer the reader to [18] for a review of other circle and sphere packing methods.

Recent work on online circle packing focuses on packing an online sequence of circles into the minimum number of square bins. Hokama et al. [19] provide an asymptotic competitive ratio of $2.4394$, and gives a lower bound of $2.2920$ for the problem. The upper bound was improved to $2.3536$ in [22]. We have not found existing work that takes into account the possibility of reallocations for online circle packing.

2 Background - The Split Packing Algorithm

We briefly describe the original Split Packing Algorithm. Given a square or obtuse triangle as the region, and any set of circles with total area at most the capacity of the region, the Split Packing Algorithm can pack the set of circles into the region.

The algorithm works in a top down, divide-and-conquer fashion. In the simple case, if the Split Packing Algorithm is given a single circle to be packed into a region, the circle is simply placed into the center of the region. If given at least two circles to be packed, the algorithm partitions the circles into two sets, splits the region into two smaller subregions, and recursively packs each set of circles into its corresponding subregion. Using a binary tree analogy for this algorithm, these subregions are the two children of the original region.

![Figure 3](image.png) Splitting an obtuse triangle or square into two right angled triangles.

The Split Packing papers [25, 24] describe how a region of capacity $a$ is to be split into two smaller regions of capacities $a_1$ and $a_2$ respectively, where $a_1 + a_2 \leq a$. If the original region is an obtuse triangle, $a$ is its incircle area. As shown in Figure 3b, the two subregions will be right angled triangles, defined by squeezing circles of areas $a_1$ and $a_2$ respectively into the left and right corners of the original triangle, which has been oriented such that the obtuse angle is at the top. If the original region is a square, $a$ is as shown in Figure 3c. As shown in Figure 3d, the two subregions will be isosceles right angled triangles, defined by squeezing circles of areas $a_1$ and $a_2$ respectively into opposite corners of the square.

In both cases, the capacities of the two subregions will then be $a_1$ and $a_2$ respectively. The Split Packing papers prove that as long as $a_1 + a_2 \leq a$, the two subregions will not overlap each other. Note that the two subregions may not be contained within the original region, as shown in Figure 3. To account for this, Split Packing rounds the corners of triangles to form hats, which will remain within the bounds of their parent regions.

To decide on the values of $a_1$ and $a_2$ for the split, the Split Packing Algorithm calls SPLIT$(C, F)$ (Algorithm 1) to partition the set of circles $C$ into two sets, $C_1$ and $C_2$, to be
packed into the left and right children respectively. \( a_1 \) and \( a_2 \) are then determined to be the sums of the areas of the circles in \( C_1 \) and \( C_2 \) respectively.

SPLIT\((C, F)\) has a second parameter, the ideal split key \( F = (f_1, f_2) \), which is a function of the shape we are packing the circles into. Intuitively, the ideal split key is represents the “ideal split” of the shape into two shapes of smaller capacity. The ratio \( f_1 : f_2 \) would be the ratio of the capacities of the left and right children in this ideal split.

**Algorithm 1** SPLIT, a greedy algorithm to partition the set \( C \)

1. **procedure** SPLIT\((C, F = (f_1, f_2))\)
2. \( C_1 \leftarrow \emptyset \)
3. \( C_2 \leftarrow \emptyset \)
4. for each \( c \in C \) in decreasing order of area do
5. \[ \text{if } \frac{\text{sum}(C_1)}{f_1} < \frac{\text{sum}(C_2)}{f_2} \text{ then} \]
6. \( C_1 \leftarrow C_1 \cup \{c\} \)
7. else
8. \( C_2 \leftarrow C_2 \cup \{c\} \)
9. return \((C_1, C_2)\)

As seen in the SPLIT\((C, F)\) algorithm, the largest circle in \( C \) will always be placed into \( C_1 \), the child corresponding to \( f_1 \) in the split key. Thus, by swapping the two children if needed, we have the freedom to choose which of the two children the largest circle in \( C \) will be packed into. We take advantage of this in the proofs of Lemmas 10 and 23 later on.

The paper goes further to describe how the same method can be used to pack objects of similar shapes into the same regions. However, the Split Packing Algorithm described is strictly offline, and a method to do this Split Packing dynamically has not been explored.

### 3 Definitions

Similar to the original Split Packing Algorithm, the Online Split Packing Algorithm has a binary tree structure, where each region is recursively split into two smaller nonoverlapping subregions. The root node is the original region we are packing circles into, which is either a square or a right angled triangle. A square splits into two right angled triangles, and a right angled triangle splits into two smaller right angled triangles.

![Figure 4](image-url) An \( s \)-right angled triangle, and its ideal split in to two smaller \( s \)-right angled triangles.

#### Definition 1 \((s\text{-Right Angled Triangle})\)
For any \( s \geq 1 \), we refer to a right angled triangle with side lengths \( \ell \), \( s\ell \) and \( \ell\sqrt{1 + s^2} \) as an \( s \)-right angled triangle. \( s \) is the ratio of the lengths of the two shortest sides. In Figure 4 a right-angled triangle is oriented such that the hypotenuse is at the bottom. We can split the triangle vertically into two sides. The long side is closer to the side with length \( s\ell \), and the short side is closer to the side with length \( \ell \).
Definition 2 (s-shape). To simplify our discussion of the amortised reallocation cost for both right angled triangles and squares later on, we define an s-shape to refer to:
1. A 1-right angled triangle or a square if \( s = 1 \)
2. An \( s \)-right angled triangle if \( s > 1 \)

Splitting an \( s \)-shape will form two more \( s \)-shapes, for the same value of \( s \). Notably, a square is a 1-shape, and all descendant nodes of a square will be 1-right angled triangles.

Definition 3. We define the following terms:
1. \((b\text{-curve})\): We say the long/short side (Definition 1) of a triangle has a \( b \)-curve if its corresponding corner has been trimmed to an arc of a circle with area \( b \), so that such a circle fits snugly into the corner. Figure 5(a) is a triangle with a \( b \)-curve on the long side.
2. \((b\text{-semihat})\): A triangle with a \( b \)-curve on the long side.
3. \((b\text{-hat})\): A triangle where both sides have a \( b \)-curve.
4. \((b\text{-}b'\text{-semihat})\): The intersection between a \( b' \)-hat and a \( b \)-semihat. It is a triangle where the short side has a \( b' \)-curve, and the long side has a \( \max\{b,b'\} \)-curve. A \( b' \)-hat can also be thought of as a 0-\( b' \)-semihat.
5. \((\text{Full triangle})\): A 0-hat. A full triangle has no rounded corners.
6. \((\text{Capacity})\): The capacity of a square or a right angled triangle is the largest value \( a \) such that any set of circles with total area at most \( a \) can be packed within its bounds. (This is the same as the critical density of its shape defined in Section 1). The capacity of a hat or semihat is the capacity of its underlying triangle.
7. \((\text{Total Size})\): The total size of a shape is the sum of the areas of the circles within the shape’s bounds. We write \( \text{totalSize}(C) \) to represent the total area of a set of circles \( C \).

In this paper, we use the term “triangles” to refer to full triangles, hats and semihats formed from right angled triangles. We also extend our definition of \( s \)-shapes (Definition 2) to also refer to hats and semihats formed from \( s \)-right angled triangles.

We also introduce the concept of the ideal capacities \( a_{tg}^* \) and \( a_{sh}^* \), which represent the “ideal split” of a triangle into two smaller triangles. We make reference to these ideal capacities in many parts of our algorithm. Intuitively, we are concerned with the ideal splits as a triangle split by its ideal capacities (Figure 3) divides perfectly into two smaller triangles, with capacities adding up to the original triangle’s capacity. In non-ideal splits, subregions may require rounding their corners to remain within the bounds of the original triangle, with the degree of rounding depending on how much their capacities deviate from the ideal.

Definition 4 (Ideal Capacities of a Triangle). Consider splitting a triangle by its split ratio as defined in [25, 24]. This refers to drawing a vertical line from the right-angled corner to perpendicularly intersect the hypotenuse as shown in Figure 4. When an \( s \)-right angled triangle is split this way, the subregions on the long and short sides have capacities \( a_{tg}^* \) and \( a_{sh}^* \) respectively. We refer to these as their ideal capacities.
4 The Online Split Packing Algorithm

This section describes the Online Split Packing Algorithm. Our algorithm relies on insights from the Split Packing Algorithm for packing circles into a square. We first discuss a version of the algorithm that only handles insertions. Deletions will be discussed in Section 8.

We make use of a binary tree structure to represent our packing. Each node of the binary tree represents a subregion of the original shape, with the root of the binary tree being the original shape we are packing circles into. Every non-root node in the tree is a subregion of its parent node. Nodes of the tree can contain circles. These contained circles are the circles we intend to pack within the bounds of the node. The circles contained in a node includes all circles contained within its descendants. A node is empty if it contains no circles.

(a) Initial structure containing no circles
(b) After circles have been inserted

Figure 6 Binary tree structure used by the Online Split Packing Algorithm.

The root of the binary tree is the original region we are packing circles into. Each level of the binary tree has a rightmost node, which is split into two nonoverlapping subregions that become the left and right children of the node. The left child is always packed with the original Split Packing Algorithm (described later in Lemma 10), while the right child, as the rightmost node of the next level, is again recursively split into two nonoverlapping subregions. The exact details of this split is explained in Section 5.3. We always maintain this structure between insertions. An example of this can be seen in Figure 6b. Because of this structure, we primarily focus on the rightmost node of each level. (We simply refer to them as “rightmost nodes” in the rest of this paper for brevity)

The initial configuration of the binary tree is the result of running the Repack procedure defined in Section 5 on the original region (the root shape) with no circles. This would build a binary tree which is an infinitely long sequence of “ideal splits” (Figure 6a). Each empty square will be split along its diagonal into two right angled triangles, and each empty right angled triangle will be split by its ideal capacities (Definition 4) into two right angled triangles as seen in Figure 4. As the tree extends indefinitely, in an actual implementation of the algorithm, the data structure for the tree only needs to be built as deep as the deepest inserted circle in the tree.

Definition 5 (Packing Invariant). Let S be a rightmost node of capacity a, and let C be a set of circles. The packing invariant is said to hold on S with circles C if:
1. S is either a square or a b-semihat for some b ≥ 0.
2. The total size of the circles in C is at most a.
3. If S is a b-semihat with b > 0, then there exists a circle in C with size at least b.

The packing invariant is said to hold on the packing if on each level of the binary tree, the packing invariant holds on its rightmost node S with the circles C contained in S.

Definition 6 (Tight and Slack Shapes). A rightmost node S of the binary tree is called a:
1. **Tight Shape** if the total size of the circles contained within the left child of $S$ is equal to the capacity of the left child.
2. **Slack Shape** if $S$ is not a tight shape and the right child of $S$ is a full triangle.

**Definition 7 (Shape Invariant).** The Shape Invariant holds on a rightmost node $S$ if it is either a tight shape or a slack shape. The Shape Invariant holds on a binary tree if it holds on the rightmost node of every level of the tree.

Both invariants (Definition 5, 7) are maintained between all insertion requests. The Packing Invariant is used for the proof of correctness (that the packing is valid), and will be shown to always hold in Section 6.3. The Shape Invariant is used in the proof of the reallocation cost bound and algorithm termination, and will be shown to always hold in Section 7.

### 4.1 Inserting a Circle

**Algorithm 2** Inserting a circle $c$ into a shape $S$

1. procedure $\text{InsertCircle}(c, S)$
2. $R \leftarrow S.\text{rightChild}$
3. if $S$ is a tight shape or $\text{TOTALSize}(R) + c.\text{size} < R.\text{capacity}$ then
4. $\text{InsertCircle}(c, R)$ \textcolor{red}{\triangleright} recurse on right child
5. else
6. $C \leftarrow S.\text{containedCircles} \cup \{c\}$
7. $\text{Repack}(C, S)$ \textcolor{red}{\triangleright} repack current node, end recursion

Algorithm 2 inserts a new circle $c$ into a shape $S$. To insert a new circle $c$, we run Algorithm 2 on the root of the binary tree. Algorithm 2 recursively inserts the circle into the right child of each subsequent node, until a single repack is triggered at some level on its rightmost node $S$. $\text{Repack}(C, S)$ completely rebuilds a subtree rooted at $S$ to include the new circle in the packing, and incurs a reallocation cost proportional to the total size of the circles within the subtree. Details on the $\text{Repack}$ algorithm are given in Section 5.

**Lemma 8 (Insertion Invariant).** If $\text{InsertCircle}(c, S)$ (Algorithm 2) is called when the total size of circle $c$ and the circles already in $S$ is at most the capacity of $S$, then Algorithm 2 will only (recursively) call $\text{InsertCircle}(c, R)$ (on Line 4) when the total size of the circle $c$ and the circles already in $R$ is at most the capacity of $R$.

**Proof.** Let $C_R$ be the set of circles to be packed into the right child. Let $C_L$ be the set of circles packed into the left child. Let $L$ and $R$ represent the left and right children respectively. There are two possible cases where we use the recursive call $\text{InsertCircle}(c, R)$. If $\text{TOTALSize}(R) + c.\text{size} < R.\text{capacity}$, it is clear the invariant is maintained. On the other hand, if $S$ is a tight shape, suppose that $\text{TOTALSize}(C_R) > R.\text{capacity}$. As $S$ is a tight shape, $\text{TOTALSize}(C_L) = L.\text{capacity}$, so $\text{TOTALSize}(C = C_L \cup C_R) > R.\text{capacity} + L.\text{capacity} = S.\text{capacity}$, which contradicts the original assumption. \hfill \blacktriangleright

The only precondition when calling $\text{InsertCircle}(c, S)$ (Algorithm 2) is for the total size of the new circle $c$ and the circles already in $S$ to be at most the capacity of $S$. Lemma 8 states that the precondition will continue to be met when the algorithm recursively calls itself. This is used to show Lemma 9 which states that whenever $\text{Repack}(C, S)$ is called, the Packing Invariant holds on $S$ and its ancestors when the new circle $c$ is included in their
sets of contained circles. This property is used for the algorithm’s proof of correctness in Section 6.3.

Lemma 9. Suppose Algorithm \ref{alg:repack} is called to insert a new circle \( c \) into an existing packing where the Packing Invariant (Definition \ref{def:packing-invariant}) holds. Assume that the total size of \( S \) in node \( S \) is at most the capacity of the root node. Suppose Algorithm \ref{alg:repack} calls \textsc{Repack}(\( C, S \)) to repack some shape \( S \). Just before \textsc{Repack}(\( C, S \)) is called, for any node \( S' \) that will now contain the new circle \( c \) (\( S' \) can be \( S \) or any ancestor of \( S \)), the Packing Invariant will hold on \( S' \) with the circles \( C' \cup \{c\} \), where \( C' \) is current set of circles contained in \( S' \).

Proof. Statements 1 and 3 of the Packing Invariant will continue to hold as they previously held on \( S' \) with its original set of circles \( C' \). By our assumption and applying Lemma \ref{lem:packing-invariant}, inductively through Algorithm \ref{alg:repack}’s recursive calls, the total size of the circles in \( C' \cup \{c\} \) is at most the capacity of \( S' \) when Algorithm \ref{alg:repack} calls \textsc{Repack}(\( C, S \)).

5 The Repack Operation

The \textsc{Repack} operation is a recursive algorithm that packs a set of circles \( C \) into a single shape \( S \), by completely rebuilding the subtree rooted at \( S \). Calling \textsc{Repack}(\( C, S \)) assumes that the Packing Invariant (Definition \ref{def:packing-invariant}) holds on the shape \( S \) with the circles \( C \). To pack these circles \( C \) into \( S \) (\( S \) is a square or some \( b \)-semihat of capacity \( a \)), the algorithm splits the shape \( S \) into the left and right children, two smaller shapes of capacities \( a_L \) and \( a_R \) respectively, where \( a_L + a_R = a \). The set of circles \( C \) is also partitioned into two sets \( C_L \) and \( C_R \), to be packed into the left and right children respectively. Before we can split the shape \( S \), the \textsc{Repack} algorithm, given \( C \) and \( S \), must first decide \( a_L, a_R, C_L \) and \( C_R \).

We describe how \textsc{Repack} works for \( b \)-semihats and squares. A square splits into two 1-right angled triangles, and an \( s \)-right angled triangle splits into two \( s \)-right angled triangles with the same value of \( s \). In Section \ref{sec:shapes} we define the ideal capacities \( a_{ig}^* \) and \( a_{sh}^* \) (Definition \ref{def:ideal-capacities}), where \( a_{ig}^* + a_{sh}^* = a \), which represents the best possible split of an \( s \)-right angled triangle.

5.1 Repacking a Triangle

Let \( S \) be a \( b \)-semihat (for some \( b \geq 0 \)) of capacity \( a \), formed from an \( s \)-right angled triangle. Suppose the ideal capacities of its long and short sides are \( a_{ig}^* \) and \( a_{sh}^* \) respectively. To decide \( a_L, a_R, C_L \) and \( C_R \), there are four cases, with Case 1 having the highest priority and Case 4 having the lowest. For Cases 3 and 4, we make use of \( \delta_{sh} \) defined by the following expression:

\[
\delta_{sh} := \left( 1 - \frac{1}{2\sqrt{1 + s^2}} \right)^2
\]

Intuitively, \( \delta_{sh} \) is the largest value of \( \delta \) where Case 3 can pack correctly and maintain the packing invariant. A proper explanation for \( \delta_{sh} \) is given in Lemma \ref{lem:delta-sh} and the proof of correctness of Case 3 in Lemma \ref{lem:correctness-case-3}.

Case 1: There exists a circle in \( C \) of size > \( a_{ig}^* \)

Place the largest circle in \( C_L \), and the remaining circles in \( C_R \). The left child will be packed tightly (\( a_L = \text{TotalSize}(C_L) \)), while the right child has the remaining space.

Case 2: \( C \) has total size \( \leq a_{ig}^* \)

Place all the circles in \( C_L \), while \( C_R \) remains empty. The capacities \( a_L \) and \( a_R \) of the left and right children will be their respective ideal capacities, \( a_{ig}^* \) and \( a_{sh}^* \).
Case 3: We first iterate through the circles $C$ from the largest to the smallest, and greedily pack the circles into the left side while keeping its total size below $a_l^\delta$, as shown in Algorithm 3. This results in the total size of $C_L$ being of the form $a_l^\delta - \delta a$, for some $\delta > 0$. We keep the result of Algorithm 3 if $\delta < \delta_{sh}$. The left child will be packed tightly ($a_L = \text{TotalSize}(C_L)$), while the right child has the remaining space. Note that this means $a_L = a_l^* - \delta a$ and $a_R = a_{sh}^* + \delta a$. If $\delta \geq \delta_{sh}$, we move on to Case 4 instead.

Algorithm 3 TriangleCase3Packing

1: $C_L \leftarrow \emptyset$
2: $C_R \leftarrow \emptyset$
3: for each circle $c$ in $C$ from largest to smallest do
4: if $\text{TotalSize}(C_L) + c.$size $\leq a_l^\delta$ then
5: add $c$ to $C_L$
6: else
7: add $c$ to $C_R$

Case 4: $\delta \geq \delta_{sh}$

We place the two largest circles in $C_L$, and the remaining circles in $C_R$. The left child will be packed tightly ($a_L = \text{TotalSize}(C_L)$), while the right child has the remaining space. Now that $a_L$ and $a_R$ are determined, the algorithm splits the semihat $S$ into two smaller semihats with capacities $a_L$ and $a_R$. Section 5.3.1 details how the semihat is divided up, to create two children termed the long child and the short child. The long child, of capacity $a_L$, will be the left child, and the short child, of capacity $a_R$, will be the right child. The children are then packed as follows:

1. The circles $C_L$ are packed into the left child using the Split Packing Algorithm for Semihats in Lemma 10.
2. The circles $C_R$ are recursively packed into the right child with the Repack algorithm.

The following lemma is used for the Repack Algorithm on triangles. Lemma 10 describes how the original Split Packing Algorithm can be modified to pack circles into a semihat. (The Split Packing Algorithm works for $b$-hats, but is not defined on semihats)

> **Lemma 10** (Split Packing Algorithm for Semihats). Suppose that the Split Packing Algorithm in [25, 24] is able to pack a set of circles $C$ into a $b'$-hat with capacity $a$. If at least one circle in $C$ of size at least $b$, then the circles $C$ can be packed into a $b$-$b'$-semihat of capacity $a$.

![Figure 7](A b-$b'$-semihat. The dotted lines show the long child from the split at each level (the $b$-curve always ends up in the long child). The short children are not drawn.)

**Proof.** A $b$-$b'$-semihat is a $b'$ hat where the $b'$-curve of the long side has been enlarged to form a $b$-curve. We treat the $b$-$b'$-semihat as a $b'$ hat and use the Split Packing Algorithm to pack circles into it. As described in Section 2, the Split Packing Algorithm recursively splits
the hat into smaller hats, and terminates with one circle in each leaf node of the resulting binary tree. At each level, we find that the long child will always be the hat containing the $b$-curve (Figure 7). Due to how the Split algorithm (Algorithm 1) works, we can force the Split Packing Algorithm to always assign the largest circle $c$ to the long child. This way, we end up with a packing of the original $b'$ hat, where the largest circle has been packed snugly into the $b$-curve of the original $b$-$b'$-semihat. Thus all circles in $C$ will also be within the bounds of the $b$-$b'$-semihat.

\[\text{\Rightarrow}\]

5.2 Repacking a Square

Let $S$ be a square of capacity $a$. To decide $a_L$, $a_R$, $C_L$ and $C_R$, there are two cases.

Case 1: $C$ has total size $> 0.5a$

We start with all the circles in the left side. We iterate through the circles $C$ from the largest to the smallest, and greedily remove circles from the left side while keeping its total size above $0.5a$, as shown in Algorithm 4. This results in the total size of $C_L$ being of the form $a(0.5 + \delta)$, for some $\delta \geq 0$.

\begin{algorithm}
\begin{algorithmic}
\State $C_L \leftarrow C$
\State $C_R \leftarrow \emptyset$
\For {each} circle $c$ in $C$ from largest to smallest
\If {TOTALSize($C_L$) - $c$.size $\geq 0.5a$}
\State remove $c$ from $C_L$
\State add $c$ to $C_R$
\EndIf
\EndFor
\end{algorithmic}
\end{algorithm}

The left child will be packed tightly ($a_L = \text{TOTALSize}(C_L)$), while the right child has the remaining space. Note that this means $a_L = a(0.5 + \delta)$ and $a_R = a(0.5 - \delta)$.

Case 2: $C$ has total size $\leq 0.5a$

Place all the circles in $C_L$, while $C_R$ remains empty. The capacities $a_L$ and $a_R$ of the left and right children will each be $0.5a$.

Section 5.3.2 describes how a square is split into two triangles given $a_L$ and $a_R$. The left and right children have capacities $a_L$ and $a_R$ respectively. They are then packed as follows:

1. The circles $C_L$ are packed into the left child with the Split Packing Algorithm.
2. The circles $C_R$ are recursively packed into the right child with the Repack algorithm.

5.3 Splitting Shapes

Splitting shapes refers to subdividing a shape into two nonoverlapping smaller shapes, given target capacities $a_L$ and $a_R$. These smaller shapes will be of capacities $a_L$ and $a_R$ respectively and stay within the bounds of the original shape. These smaller shapes, as the children of the original shape in the binary tree, can then be recursed into in the Repack algorithm.

5.3.1 Splitting Triangles

The Split Packing papers [25, 24] describe how a triangle (hat) splits into two smaller triangles (hats). Details are given in Section 2. We do the same thing with semihats (Figures 8a, 8b). On an $s$-right angled triangle, we call the two smaller semihats the long and short children, corresponding to whether they came from the long or short side.
Lemma 11. The ideal capacities (Definition 4) $a_{lg}^*$ and $a_{sh}^*$ of an $s$-right angled triangle of capacity $a$ obey the following properties:

1. $a_{lg}^* = a \frac{s^2}{1 + s^2}$
2. $a_{sh}^* = a \frac{1}{1 + s^2}$
3. $a_{lg}^* + a_{sh}^* = a$

Proof. Referring to Figure 4, as the three triangles are similar, the shortest side of the long child will be of length $\frac{s}{\sqrt{1 + s^2}}$, so $a_{lg}^* = \left( \frac{s}{\sqrt{1 + s^2}} \right)^2 = \frac{s^2}{1 + s^2}$. Similarly, the shortest side of the short child will be of length $\frac{\ell}{\sqrt{1 + s^2}}$, so $a_{sh}^* = \left( \frac{1}{\sqrt{1 + s^2}} \right)^2 = \frac{1}{1 + s^2}$.

Finally, $a_{lg}^* + a_{sh}^* = a \frac{s^2}{1 + s^2} + a \frac{1}{1 + s^2} = a$.  

Suppose a semihat with capacity $a$ is split into long and short children with capacities $a_{lg}$ and $a_{sh}$ respectively. The Split Packing papers [25, 24] show that if $a_{lg} + a_{sh} \leq a$, the two children will not overlap. In our algorithm, we always have $a_{lg} + a_{sh} = a$. If $a_{lg} \geq a_{lg}^*$, a $b$-semihat splits into a $b_{lg}$-semihat on the long side (for some $b_{lg} \geq 0$), and a full triangle on the short side (Figure 8a). If $a_{lg} \leq a_{lg}^*$, a $b$-semihat splits into a $b$-semihat on the long side, and a $b_{sh}$-semihat on the short side (for some $b_{sh} \geq 0$, Figure 8b). Structurally, this ensures the first property of the Packing Invariant (Definition 5), as the right child is always a $b$-semihat for some value of $b$ (a full triangle is a $0$-semihat).

5.3.2 Splitting Squares

Our strategy for splitting squares is also similar to Split Packing, which is explained in Section 2. A square splits into two isosceles right angled triangles (1-right angled triangles), as shown in Figure 9.

6 Proof of Correctness

To prove the correctness of our algorithm (for inserts only), assuming that the total size of all inserted circles is at most the capacity of the root node, we need to show that:

1. Every circle will be assigned a position in the packing.
2. All circles are packed within the bounds of the region.
3. No two circles overlap in the packing.

Statement 1 will be proved in Section 6.1. The latter two will be proved in Section 6.3.
Figure 9: Splitting a square into a $b$-hat and a full triangle of capacities $a_L$ and $a_R$ respectively.

6.1 Termination

Algorithm 2 is a recursive algorithm that only terminates upon calling Repack. The Repack is a process which never terminates, instead recursively building an infinite sequence of empty nodes after the last circle has been allocated. To prove that the insertion algorithm will always allocate the new circle (as well as the displaced existing circles) as position in the packing (contained in some node), we show that, whenever we want to insert a new circle,

1. Algorithm 2 will always terminate (by calling Repack) after a finite number of steps.
2. Every circle that is to be repacked when Repack is called will be assigned a position after a finite number of steps.

These two statements will be proven as Corollary 14 and Theorem 15. Before we prove these properties, we first show that the Shape Invariant, along with some other properties, always hold on the binary tree (Lemma 12).

Lemma 12. The Shape Invariant (Definition 7) will hold on the initial configuration of the binary tree, as well as after each insertion. Furthermore, the following properties will also always hold on all rightmost nodes:

1. If a node $S$ is a slack shape, the capacity of the right child is $\frac{1}{1+s^2}$ of the capacity of $S$.
2. If a node $S$ is a tight shape, there will be at least one circle in its left child.

Proof. The Repack algorithm can only produce tight or slack shapes. Only Case 2 of Repack on both triangles and squares can generate slack shapes, as the right child will be a full triangle. All other cases produce tight shapes. Let $a$ be the capacity of the original shape and $a_R$ be the capacity of the right child. For Case 2 on triangles, we have $a_R = a_{sh}^* = a \frac{1}{1+s^2}$ (Lemma 11). For Case 2 on squares, we have $a_R = 0.5a = a \frac{1}{1+s^2}$. Thus for all cases which produce slack shapes, the capacity of the slack shape’s right child is $\frac{1}{1+s^2}$ of the capacity of the original shape. For all cases which produce tight shapes, at least one circle will be placed in the shape’s left child.

As stated in Section 4, the initial configuration of the binary tree is the result of running the Repack algorithm on the root shape with no circles. Thus the Shape Invariant as well as both properties would hold on the initial configuration of the binary tree.

The binary tree can only be modified through calling the Repack algorithm on some shape $S$, which rebuilds the subtree from $S$ and below. Thus the Shape Invariant, as well as both properties, will hold on $S$ and its descendant rightmost nodes. For each ancestor of $S$, the left child will not be affected by a repack of $S$, and the shape of the right child (which might be $S$) will not be changed by a repack of $S$. The Shape Invariant as well as both properties, which held previously, will continue to hold on them.

By Lemma 12, the capacity of the right child of a slack $s$-shape is $\frac{1}{1+s^2}$ of its parent’s capacity. This allows us to prove Lemma 13 and Corollary 14.
Lemma 13. When inserting a circle of size $c$ into an $s$-shape of capacity $a$, Algorithm 2 passes through at most $\left\lfloor \log_{1+s^2} \frac{c}{a} \right\rfloor + 1$ slack shapes before terminating by calling Repack.

Proof. If the current shape is a slack shape, the ratio of the capacity of the right child to its parent’s is $\frac{1}{1+s^2}$. As each slack shape reduces the shape size by a factor of $1 + s^2$, the circle passes through at most $\left\lfloor \log_{1+s^2} \frac{c}{a} \right\rfloor + 1$ slack shapes before the circle itself must be larger than the right child’s capacity.

Corollary 14. Algorithm 2 always terminates after a finite number of steps.

Proof. Algorithm 2 terminates when it calls Repack. By Lemma 12, each tight shape has at least one circle in its left child. Thus, there at most $n$ tight shapes, where $n$ is the amount of circles inserted so far. By Lemma 13, a circle passes through a finite number of slack shapes. As every rightmost node is either tight or slack, the algorithm will terminate and call Repack after a finite number of steps.

Theorem 15. Every circle that is to be repacked when Repack is called will be assigned a position after a finite number of steps.

Proof. The Repack procedure assigns a circle a position when it allocates the circle to the left child instead of the right. This is as the left child is packed with the original Split Packing Algorithm. Every time the Repack procedure creates a tight shape, at least one circle will be assigned to the left child. This limits the number of tight shapes created to the number of circles to be packed. Therefore, Repack will create a slack shape after a finite number of recursions, and in all cases where a slack shape is created, all remaining circles will be packed into the left child. Therefore, all circles will be assigned a position after a finite number of steps.

6.2 Properties of Triangle and Square Splitting

In this section, we show some important relationships relating to the sizes of the $b$-curves generated when our triangles (including semihats) and squares are split in the manner described in Section 5.3. The Lemmas in this section are used to prove the validity of our packing in the following sections.

6.2.1 Triangles

![Figure 10](image-url) Necessary $b$-curves needed for child nodes to fit within their parent shapes.

Lemma 16 (Relationship between capacity of a child and its $b$-curve (triangles)). Consider an $s$-right angled triangle $S$ of capacity $a$ split into long and short children of capacities $a_{lg}$
and $a_{sh}$ respectively. Let $a_{lg}^*$ and $a_{sh}^*$ be the ideal capacities of the triangle (Definition 4). If $a_{lg} \geq a_{lg}^*$, we write $a_{lg} = a_{lg}^* + \delta a$ for some $\delta \geq 0$. Let $b_{tg}^\delta$ be the smallest $b$ such that a long child of capacity $a_{lg}$ with a $b$-curve on the short side can fit within the bounds of the triangle $S$ (Figure 10a). Then $b_{tg}^\delta$ has the following expression in terms of $\delta$:

$$b_{tg}^\delta = a \left( \frac{\sqrt{s^2 + \delta(1 + s^2)} - s}{\sqrt{1 + s^2} - s} \right)^2$$ (2)

Similarly, if we instead have $a_{sh} \geq a_{sh}^*$, we can write $a_{sh} = a_{sh}^* + \delta a$ for some $\delta \geq 0$. Let $b_{sh}^\delta$ be the smallest $b$ such that a short child of capacity $a_{sh}$ with a $b$-curve on the long side can fit within the bounds of the triangle $S$ (Figure 10b). Then $b_{sh}^\delta$ has the following expression in terms of $\delta$:

$$b_{sh}^\delta = a \left( \frac{\sqrt{1 + \delta(1 + s^2)} - 1}{\sqrt{1 + s^2} - 1} \right)^2$$ (3)

Proof. Appendix A.1

Lemma 17 (Triangle Packing Lemma (Short Child)). Let $b_{sh}^\delta$ be the expression defined in terms of $\delta \geq 0$ (short child) in Lemma 16 (where $s \geq 1$, $a > 0$). Define $\delta_{sh}$ as follows:

$$\delta_{sh} := \left( 1 - \frac{1}{2\sqrt{1 + s^2} - 1} \right)^2$$ (4)

Then for all $\delta \in [0, \delta_{sh}]$, we have $\delta \geq b_{sh}^\delta/a$.

Proof. Appendix A.2

6.2.2 Squares

Lemma 18 (Relationship between capacity of a child and its $b$-curve (squares)). Consider a square $S$ of capacity $a$ with a child of capacity $a_L$. If $a_L \geq 0.5a$, we write $a_L = 0.5a + \delta a$ for some $\delta \geq 0$. Let $b_{sq}^\delta$ be the smallest $b$ such that a child of capacity $a_L$ and a $b$-curve on both sides can fit within the bounds of the square $S$ (Figure 10c). Then $b_{sq}^\delta$ has the following expression in terms of $\delta$:

$$b_{sq}^\delta = a \left( \frac{\sqrt{1 + 2\delta} - 1}{\sqrt{2} - 2} \right)^2$$ (5)

Proof. Appendix A.3

Lemma 19 (Square Packing Lemma). Let $b_{sq}^\delta$ be the expression defined in terms of $\delta \geq 0$ in Lemma 18 (where $a > 0$). We then have $\delta \geq b_{sq}^\delta/a$ for $\delta \in [0, 0.5]$.

Proof. Appendix A.4

6.3 Correctness of Repack

We prove Theorem 21 to hold after every step of the packing. Theorem 21 necessarily implies that the root node of the tree is validly packed. As we have already shown that every circle will be allocated to some node, this would conclude our proof of correctness.
Definition 20 (Valid Packing within Node $S$). A node $S$ is said to be validly packed if:
1. All circles contained in $S$ are placed within the bounds of the node.
2. No two circles contained in $S$ overlap.

Theorem 21. Assume that the total size of all inserted circles is within the root node’s capacity. Then on the binary tree’s initial configuration, and after every subsequent insertion,
1. Every node in the tree is validly packed (Definition 20).
2. The Packing Invariant (Definition 5) holds on the packing.

The key idea behind the proof for Theorem 21 is that our binary tree is only modified through calls to Repack($C, S$), and only when the Packing Invariant (Definition 5) holds the shape $S$ to be repacked on the circles $C$ to be packed into it. Before we prove Theorem 21, we first prove some properties of a binary tree that is only modified through Repack calls, in the form of Lemmas 22, 23 and 26. We prove Theorem 21 at the end of this section.

Lemma 22. If Repack($C, S$) is only called when the Packing Invariant holds on $S$ with the circles $C$, then for each node in the binary tree, the children of the node are non-overlapping and within the bounds of the node.

Proof. The binary tree is only modified through calls to Repack. Calling Repack($C, S$) does not change the shape of $S$, and only rebuilds $S$’s descendants. From the description of the triangle/square splitting procedures in Section 5.3, the only requirement for the shapes to be nonoverlapping and within the bounds of the original shape is for $a_L + a_R \leq a$ and for $a_L$ and $a_R$ to both be nonnegative and no larger than $a$. These two requirements can be easily seen to be met in each of the cases given in Sections 5.1 and 5.2 (in fact, in most of the cases, we define $a_R$ to be $a - a_L$.)

Lemma 23 (Properties of Repack on Triangles ($b$-semihats)). Assume that the Packing Invariant (Definition 5) holds on a $b$-semihat $S$ with a set of circles $C$. Suppose that we call Repack($C, S$). Let $C_R \subseteq C$ be the set of circles that Repack($C, S$) will pack into the right child of $S$. Then,
1. All circles in the left child of $S$ will be packed within the bounds of the left child, and no two of these circles will overlap.
2. The packing invariant holds on the right child of $S$ with the circles $C_R$ (i.e. it holds when Repack recurses into the right child)

Proof. The algorithm packs a set of circles $C$ into a $b$-semihat with capacity $a$. The left child will be a $b_L$-semihat, for some $b_L \geq 0$, with capacity $a_L$. The right child will be a $b_R$-semihat, for some $b_R \geq 0$, with capacity $a_R$. We do the proof for each of the four cases of the repacking algorithm.

To prove statement (1), by Lemma 10 we only need to show that:
1. If the $b$-semihat was instead a full triangle, the circles $C_L$ can be packed into the left child with the Split Packing Algorithm.
2. The largest circle in $C_L$ has size at least $b$.
To prove statement (2), we show that $\text{totalSize}(C_R) \leq a_R$ and that there exists a circle in $C_R$ with size at least $b_R$.

Case 1:
Consider an imaginary circle $c'$ of size $a - \text{totalSize}(C)$. Running Repack on $C$ is equivalent to a Split Packing of $C \cup \{c'\}$ into a $b$-hat (see Algorithm 1), where the largest circle is placed into the left child.
1. By the Split Packing analogy, the largest circle alone can be packed into the left child using Split Packing. By the invariant, the largest circle also has size of at least b.

2. It is clear that the circles in $C_R$ have total size at most $a_R$. The right child is a full triangle (as $a_R \leq a_{sh}^*$), so $b_R = 0$.

**Case 2:**

1. If the original $b$-semihat was a full triangle, then the left child is also a full triangle with capacity $a_{tg}^*$. As $\text{totalSize}(C_L) \leq a_{tg}^*$, the Split Packing Algorithm can pack them into the left child. By the invariant, the largest circle, which will be in $C_L$, also has size of at least $b$.

2. There are no circles in $C_R$, and the right child is a full triangle ($b_R = 0$), so the invariant trivially holds.

**Case 3:**

1. If the original $b$-semihat was a full triangle, then the left child is also a full triangle with capacity $\text{totalSize}(C_L)$. Thus the Split Packing Algorithm can pack them into the left child.

   Due to Case 1, the largest circle will have size $\leq a_{tg}^*$, and so will be the first circle to be placed in $C_L$. By the invariant, this circle has size at least $b$.

2. It is clear that $\text{totalSize}(C_R) \leq a_{tg}^* + \delta a$, the capacity of the right child.

   Take any circle $c$ in $C_R$. If $c$.size $\leq \delta a$, then at the point circle $c$ was visited by the greedy algorithm, it would have been added to $C_L$. Thus we must have $c$.size $> \delta a$.

   For $\delta \geq 0$, let $b_{sh}^\delta$ be the expression in terms of $\delta$ defined in Lemma 16. The capacity of the right child is $a_{tg}^* + \delta a$, so $b_R = b_{sh}^\delta$. By Lemma 17, as $\delta \leq \delta_{sh}$, we have $c$.size $> \delta a \geq b_R$, so the invariant holds.

**Case 4: ($\delta_{sh}$ is defined in Lemma 17)**

We first show that the two largest circles have sizes in $[\delta_{sh}a, a_{tg}^* - \delta_{sh}a]$.

Take the largest circle $c \in C$. If $c$.size $\leq a_{tg}^*$ or Case 1 would apply. So $c$.size $\leq a_{tg}^* - \delta_{sh}a$ otherwise it would be greedily packed into $L$, and Case 3 would apply. Thus all circles in $C$ have sizes at most $a_{tg}^* - \delta_{sh}a$.

$C_R$ has at least one circle, or Case 2 would apply. Any circle in $C_R$ must have size at least $\delta a$ or the greedy algorithm would have packed it into $L$. As the largest circle is in $L$ by the greedy algorithm, there are at least two circles with size at least $\delta a \geq \delta_{sh} a$.

Thus the two largest circles must have sizes in $[\delta_{sh}a, a_{tg}^* - \delta_{sh}a]$.

Now let the largest two circles be $c_1, c_2$. We show that $c_1$.size + $c_2$.size $> a_{tg}^*$.

By Lemma 24, $3\delta_{sh} \geq a_{tg}^*/a$ for all $s \geq 1$. Thus $c_1$.size + $c_2$.size $\geq 2\delta_{sh} a \geq a_{tg}^* - \delta_{sh}a$.

If $c_1$.size + $c_2$.size $\leq a_{tg}^*$, then the greedy algorithm would place these two circles into $L$, and $L$ would then have total size at least $a_{tg}^* - \delta_{sh}a$, so Case 3 would apply. Thus we must have $c_1$.size + $c_2$.size $> a_{tg}^*$.

1. Because $a_L = c_1$.size + $c_2$.size $> a_{tg}^*$, the left child will not be a full triangle. We show that these two circles fit into the left child (which is the long child).

As both $c_1$.size, $c_2$.size $\leq a_{tg}^* - \delta_{sh}a$, we have $c_1$.size + $c_2$.size $\leq 2a_{tg}^* - 2\delta_{sh}a = a_{tg}^* + (a_{tg}^*/a - 2\delta_{sh})a$.

Let $b_{tg}^\delta$ be the expression in terms of $\delta \geq 0$ defined in Lemma 16. As $b_{tg}^\delta$ is monotonically increasing with $\delta$, we have $b_L \leq b_{tg}^\delta + a_{tg}^*/a - 2\delta_{sh}$.

As $c_1$.size $\in [\delta_{sh}a, a_{tg}^* - \delta_{sh}a]$, we must have $\delta_{sh}a \leq a_{tg}^* - \delta_{sh}a$. Thus $2\delta_{sh} \leq a_{tg}^*/a$, so by Lemma 25 we have $\delta_{sh} \geq b_{tg}^\delta + a_{tg}^*/a - 2\delta_{sh}$. Thus $c_1$.size, $c_2$.size $\geq \delta_{sh}a \geq b_{tg}^\delta + a_{tg}^*/a - 2\delta_{sh} \geq b_L$, so the Split Packing Algorithm can pack them into the left child.
By the invariant, the largest circle, which will be in $C_L$, has size of at least $b$. 
2. It is clear that the circles in $C_R$ have total size at most $a_R$. As $a_L = c_1.size + c_2.size > a^*_i$, we must have $a_R < a^*_R$, the right child is a full triangle, so $b_R = 0$.

The following lemmas were used in the proof in Case 4. Their proofs are in Appendix A.

\section*{Lemma 24} 3$d_{sh}^i \geq a^*_i/a$ for all $s \geq 1$.

\section*{Proof} Appendix A.5

\section*{Lemma 25} If $2d_{sh} \leq a^*_i/a$, then we have $d_{sh} > b_{sq}^{a^*_i/a-2b_{sh}}/a$, where $b_{sq}^{a^*_i/a-2b_{sh}}$ is defined in Lemma A.10 for the long child (by letting $\delta := a^*_i/a - 2d_{sh}$).

\section*{Proof} Appendix A.5

\section*{Lemma 26} \textbf{(Properties of Repack on Squares).} Assume that the Packing Invariant (Definition 5) holds on a square with a set of circles $C$. Suppose that we call Repack($C, S$). Let $C_R \subseteq C$ be the set of circles that Repack($C, S$) will pack into the right child of $S$. Then,

1. All circles in the left child of $S$ will be packed within the bounds of the left child, and no two of these circles will overlap.
2. The packing invariant holds on the right child of $S$ with the circles $C_R$ (i.e. it holds when Repack recurses in to the right child).

\section*{Proof} The algorithm packs a set of circles $C$ into a square with capacity $a$. The left child will be a $b_L$-hat of capacity $a_L$, for some $b_L \geq 0$, while the right child will be a full triangle of capacity $a_R$. We do the proof for each of the two cases of the packing algorithm.

To prove (1), we note that the Split Packing Algorithm can pack a set of circles $C$ into a $b_L$-hat of capacity $a_L$ if totalSize$(C_L) \leq a_L$ and every circle in $C_L$ has size at least $b_L$. To prove (2), we show that totalSize$(C_R) \leq a_R$ and that there exists a circle in $C_R$ with size at least $b_R$.

\textbf{Case 1:}

1. We have totalSize$(C_L) = a_L$. Take any $c \in C_L$. If $c.size \leq \delta a$, then as totalSize$(C_L) = a(0.5 + \delta)$, at the point $c$ is visited in Algorithm A.4, it would have been moved to $C_R$. Thus we can say that $c.size > \delta a$. Let $b_{sq}^i$ be the expression defined in terms of $\delta \geq 0$ in Lemma A.18. We have $b_L = b_{sq}^i$. totalSize$(C_L)$ cannot exceed $a$, so we always have $\delta \leq 0.5$. Thus by the Square Packing Lemma (Definition 7), we have $\delta a \geq b_L$, so $c.size > b_L$.

2. It is clear that totalSize$(C_R) \leq a_R$. The right child is a full triangle, so $b_R = 0$.

\textbf{Case 2:}

1. It is clear that totalSize$(C_L) \leq a_L$. The left child is a full triangle, so $b_L = 0$.

2. There are no circles in $C_R$, and the right child is a full triangle ($b_R = 0$), so the invariant trivially holds.

\section*{Proof of Theorem 21} In the binary tree’s initial configuration, every node in the tree is validly packed (Definition 20) as there are no circles in any of the nodes. Similarly, the Packing Invariant (Definition 5) holds on the initial empty packing described in Section 4.

The binary tree can only be modified through calls to the Repack Algorithm. Repack($C, S$) can only be called once per insertion, through Algorithm 2. Inductively assuming
that the Packing Invariant held on the packing after the previous insertion, by Lemma 9,
\textsc{Repack}(C,S) will only be called when the Packing Invariant (Definition 5) holds on
the circles \(C\). By then applying Lemmas 23 and 26 inductively, the statements outlined in
these two theorems will hold on all rightmost nodes \(S\) and below.

One consequence of this is that the Packing Invariant will now hold on \(S\) and all its
descendant rightmost nodes. For each ancestor \(S'\) of \(S\), statement 1 of the Packing Invariant
continues to hold as \(S'\) is not changed, statement 2 holds by Lemma 9 and statement 3
continues to hold as no circles are removed from its set of contained circles. Thus, the
Packing Invariant continues to hold throughout the packing.

We can then show inductively that every node is validly packed. By Theorem 15, every
circle is packed at some finite height. Let \(H\) be the largest such height over all circles
currently packed. All nodes at heights greater than \(H\) are validly packed as there are no
circles contained in these nodes. Assume inductively that all nodes at heights greater than \(h\)
are validly packed, and consider any node \(N\) at height \(h\).

If \(N\) is not a rightmost node, then \(N\) will be the left child of some node. \(N\) cannot be
\(S\). If \(N\) is a descendant of \(S\), then \(N\) is validly packed by Lemmas 23 and 26. If \(N\) is not a
descendant of \(S\), then \(N\) will be unchanged by the repack, and thus remains validly packed.

If \(N\) is a rightmost node, By Lemma 22, the left and right children are within the bounds
of \(N\) and their bounds do not overlap. Both children are packed validly by the induction
hypothesis, thus \(N\) is also validly packed.

Inductively, this shows that all nodes \(N\), up to and including the root, are validly packed.

\section{Proof of Cost Bound}

In this section, we show an amortised reallocation cost of \(O(c(1 + s^2) \log_{1 + s^2}(\frac{1}{c}))\) when
inserting a circle of size \(c\) into an \(s\)-shape. If the region is a square, the cost bound becomes
\(O(c \log_2(\frac{1}{c}))\). Note that \(s\) depends only on the shape of the root node, as the children of
\(s\)-shapes are also \(s\)-shapes.

\begin{lemma}
When the \textsc{Repack} algorithm instantiates a new slack shape \(S\), its right
child will be empty (contains no circles within its bounds).
\end{lemma}

\begin{proof}
Slack shapes are only produced by Case 2 for \textsc{Repack} on both triangles and squares.
Both cases produce empty right children.
\end{proof}

Algorithm 2 only repacks slack shapes. Thus, we define a potential function that allocates
potential only to slack shapes. As we use circle areas as our cost metric, the reallocation cost
of repacking a shape \(S\) is at most the capacity of \(S\). When a slack shape is newly instantiated
after a repack, the right child is initialised as empty (Lemma 27). When the shape is to be
repacked (when \textsc{Repack}(\(C,S\)) is called), the right child, including the newly-inserted circle,
would be over capacity. As the capacity of the right child of a slack shape is always exactly
\(\frac{1}{1 + s^2}\) of its parent’s capacity (Lemma 12), we define the potential allocated to a slack shape
as \( (1 + s^2) \times \text{TOTALSIZE(rightChild)} \).

With this amount of potential, the shape only needs to draw potential from itself to
repack itself. Immediately after a repack, the shape, as well as all its descendants, will store
0 potential. This is as all newly instantiated slack shapes have empty right children (Lemma 27).
A newly-inserted circle only contributes to the potential of the slack shapes it passes
through, including the one it eventually repacks. As a newly-inserted circle of size \(c\) only
passes through at most \(\lfloor \log_{1 + s^2}(\frac{2}{c}) \rfloor + 1\) slack shapes before a repack is called (Lemma 13),
the amortised cost of inserting a circle of size $c$ is $c(1 + s^2)(\lfloor \log_{1+s^2} \frac{a}{c} \rfloor + 1)$. Thus we have the following result (Theorem 28):

Theorem 28. In the case of insertions only, using the Online Split Packing Algorithm, we obtain an amortised cost of $O(c(1 + \frac{s^2}{c} \log_{1+s^2} \frac{a}{c} + \frac{1}{c}))$ to insert a circle of size $c$ into an $s$-shape.

8 Online Split Packing for Insertions and Deletions

Given any tight online packing algorithm $A$ for insertions only, there is a simple way to extend it to a packing algorithm where both insertions and arbitrary deletions are allowed, by allowing an arbitrarily small amount of slack space. For any fixed $\epsilon \in (0, 1)$, the algorithm

Algorithm 5 Insertions and Deletions with Slack

The basic idea is to perform deletions lazily, only actually removing deleted items through repacking when we run out of space. For the INSERT operation in Algorithm 5, removing and repacking all the active objects in $S$ has a reallocation cost at most the capacity of $S$. After the repack, the total size of $S$ is at most $(1 - \epsilon) \times S.capacity$, so between repacks, at least $\epsilon \times S.capacity$ of insertions must have been done. Thus we need an additional amortised cost of $c \times \frac{1}{c}$ when inserting an object of size $c$. Suppose that the original insertion algorithm has an amortised reallocation cost of $O(f(c))$ when inserting an object of size $c$. When we allow deletions using Algorithm 5, we then have an amortised cost of $O(f(c) + c \frac{1}{c})$ per insertion. We note that Algorithm 5 does not need to know the value of $\epsilon$ being used.

Applying this method in the context of circle packing with the earlier described Online Split Packing Algorithm for insertions only, we obtain the following result (Theorem 29):

Theorem 29. When allowing both insertions and deletions into an $s$-shape of capacity $a$, for any fixed $\epsilon > 0$, the Online Split Packing Algorithm achieves a packing density of $(1 - \epsilon)a$, with an amortised reallocation cost of $O(c(1 + \frac{s^2}{c} \log_{1+s^2} \frac{a}{c} + \frac{1}{c}))$ for inserting a circle of area $c$. More specifically, for the insertion and deletion of circles into a square, we obtain an amortised reallocation cost of $O(c(\log_2 \frac{1}{c} + \frac{1}{c}))$.

9 Conclusion

We have adapted the Split Packing Algorithm to handle an online sequence of insertions and deletions and pack arbitrarily close to critical density by allowing reallocations. Our cost bound is asymptotically equal across $s$-shapes with differing values of $s$.

The Split Packing algorithm has also been shown to work for packing shapes other than circles, like squares and octagons, into triangles. The paper goes on to define a new type of shape, a “Gem”, which represents the most general type of shape that the Split Packing
Algorithm can handle. Due to the close relationship between the algorithms, it is likely that a similar of generalisation would apply to the Online Split Packing Algorithm.

We note that the deletion procedure defined in Section 8 works for any tight packing algorithm for insertions only into a fixed space. The deletion procedure can also be shown to work not only for tight packing algorithms, but also for packing algorithms that achieve packing densities that are arbitrarily close to the critical density.

The current reallocation cost bounds apply only to volume costs. A possible direction of future work would be to understand how the cost bounds differ for the other cost models like unit cost (constant cost for each circle reallocation).

A Appendix: Proofs of Lemmas

This appendix contains the proofs which have been omitted from the main description of the Online Split Packing Algorithm in the paper.

![Figure 11](image.png)

(a) Long Side  
(b) Short Side

**Figure 11** Measurements for computing a triangle’s $b$-curve.

### A.1 Lemma 16 Triangle $b$-curve size

**Proof.**

Suppose that the bounds causes a $b$-curve to be formed for some $b > 0$.

For the case of the long child, we have three similar triangles, corresponding to the incircle areas of $a_{lg}^*$, $a_{lg}$ and $b$ respectively. In the case of the short child, the incircles are of areas $a_{sh}^*$, $a_{sh}$ and $b$ instead.

Let the length of the shortest sides of these triangles be $\ell^*$, $\ell_A$ and $\ell_b$ respectively.

**Long Child:** (Note $a_{lg} := a_{lg}^* + \delta a$, for some $\delta \geq 0$)

From Figure 11a, by computing the length of the part of the shape extending out of the original triangle, we get:

$$
\ell_A \sqrt{1 + s^2} - \ell^* \sqrt{1 + s^2} = \ell_b \sqrt{1 + s^2} - \ell_b \times s
$$

$$
\left( \frac{\ell_A}{\ell^*} - 1 \right) \left( \frac{\sqrt{1 + s^2}}{\sqrt{1 + s^2} - s} \right) = \ell_b
$$

$$
\left( \sqrt{\frac{a_{lg}}{a_{lg}^*}} - 1 \right) \left( \frac{\sqrt{1 + s^2}}{\sqrt{1 + s^2} - s} \right) = \sqrt{\frac{b}{a_{lg}^*}}
$$
\( b = \left( \frac{\sqrt{1 + s^2}}{1 + s^2 - s} \right)^2 \)

\( b = a \left( \frac{\sqrt{1 + s^2}}{1 + s^2 - s} \right)^2 \)

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\( b = a \left( \frac{\sqrt{1 + s^2}}{1 + s^2 - s} \right)^2 \)

**Short Child:** (Note \( a_{sh} := a_{sh}^* + \delta a \), for some \( \delta \geq 0 \))

From Figure [11b] by computing the length of the part of the shape extending out of the original triangle, we get:

\[
\ell_A \sqrt{1 + s^2} - \ell_A \sqrt{1 + s^2} = \ell_b \sqrt{1 + s^2} - \ell_b
\]

\[
\left( \frac{2}{\ell^*} - 1 \right) \left( \frac{\sqrt{1 + s^2}}{1 + s^2 - 1} \right) = \frac{b}{\ell^*}
\]

\[
\left( \frac{a_{sh}}{\ell_{sh}} - 1 \right) \left( \frac{\sqrt{1 + s^2}}{1 + s^2 - 1} \right) = \frac{b}{\ell_{sh}}
\]

\[
b = \left( \frac{\sqrt{1 + s^2}}{1 + s^2 - 1} \right)^2
\]

\[
b = a \left( \frac{\sqrt{1 + s^2}}{1 + s^2 - 1} \right)^2
\]

\[
b = a \left( \frac{\sqrt{1 + s^2}}{1 + s^2 - 1} \right)^2
\]

\[
b = a \left( \frac{\sqrt{1 + s^2}}{1 + s^2 - 1} \right)^2
\]

\[
b = a \left( \frac{\sqrt{1 + \delta(1 + s^2)}}{1 + s^2 - 1} \right)^2
\]

\[
A.2 \text{ Lemma} [17]: \text{Triangle Packing Lemma}
\]

Proof.

**Short Child:**

\[
b_{sh}^* / a = \left( \frac{\sqrt{1 + \delta(1 + s^2)}}{1 + s^2 - 1} \right)^2
\]

Define \( y := \sqrt{1 + s^2} \), and \( h := \sqrt{1 + \delta(1 + s^2)} \), we have \( \delta = \frac{h^2 - 1}{1 + s^2} \)

For \( \delta > b_{sh}^* / a \), we need \( \delta > \left( \frac{\sqrt{1 + \delta(1 + s^2)}}{1 + s^2 - 1} \right)^2 \)

\[
\Longleftrightarrow \frac{h^2 - 1}{y} \geq \left( \frac{h - 1}{y} \right)^2
\]

\[
\Longleftrightarrow (h^2 - 1)(y - 1)^2 > (h - 1)^2 y^2 \text{ as } y > 1
\]

\[
\Longleftrightarrow (h^2 - 1)(y^2 - 2y + 1) > (h^2 - 2h + 1) y^2 \text{ as } y > 1
\]

\[
\Longleftrightarrow h^2(y^2 - 2y + 1) + (y^2 - 2y + 1) > h^2 y^2 - 2h y^2 + y^2
\]

\[
\Longleftrightarrow h^2(-2y + 1) + h(2y^2) + (2y^2 + 2y - 1) > 0
\]

As \( y > 1 \), we have \(-2y + 1 < 0\), so the quadratic curve is concave downwards.

Using the quadratic formula, we obtain the following expression for the roots:
\[
\begin{align*}
    h &= -2y^2 \pm \sqrt{4y^4 - 4(-2y + 1)(-2y^2 + 2y - 1)} \\
    &= -2y^2 \pm 2\sqrt{y^4 - 4y^3 + 6y^2 - 4y + 1} \\
    &= -y^2 \pm \sqrt{(y-1)^4} \\
    &= -2y + 1 \\
    &= -y^2 \pm (y^2 - 2y + 1) \\
    &= -2y + 1
\end{align*}
\]

As \((y-1)^2 \geq 0\) and \(-2y + 1 < 0\), the expression is positive for
\[
\frac{-2y + 1}{-2y + 1} \leq h \leq \frac{-2y^2 + 2y - 1}{-2y + 1}
\]
which is equivalent to:
\[
1 \leq h \leq \frac{1}{2}(2y - 1 + \frac{1}{2y - 1})
\]
\[
\Leftrightarrow 1 \leq \sqrt{1 + \delta y^2} \leq \frac{1}{2}(2y - 1 + \frac{1}{2y - 1})
\]
\[
\Leftrightarrow 0 \leq \delta \leq \frac{1}{y^2} (\frac{1}{4} ((2y - 1) + \frac{1}{2y - 1})^2 - 1)
\]
\[
\Leftrightarrow 0 \leq \delta \leq \frac{1}{y^2} (\frac{1}{4} ((2y - 1)^2 + 2 + \frac{1}{(2y - 1)^2}) - 1)
\]
\[
\Leftrightarrow 1 \leq \delta \leq \frac{1}{y^2} ((2y - 1) - \frac{1}{2y - 1})^2
\]
\[
\Leftrightarrow 0 \leq \delta \leq \frac{1}{2y} ((2y - 1)^2 - \frac{1}{2y - 1})^2
\]
\[
\Leftrightarrow 0 \leq \delta \leq \frac{1}{2y - 1}^2
\]
\[
\Leftrightarrow 0 \leq \delta \leq (1 - \frac{1}{2y - 1})^2
\]
\[
\Leftrightarrow 0 \leq \delta \leq \frac{1}{2\sqrt{1 + \epsilon^2 - 1}}^2
\]

**A.3 Lemma 18:** Square \(b\)-curve size

Proof.
Suppose that the bounds causes a \(b\)-curve to be formed for some \(b \geq 0\).
We have three similar triangles, corresponding to the incircle areas of \(a^* := a/2\), \(A\) and \(b\) respectively. Let the length of the shortest sides of these triangles be \(\ell^*\), \(\ell_A\) and \(\ell_b\) respectively.

![Diagram](image_url)
Let \( r_b \) be the radius of a circle of area \( b \).
First, we compute \( r_b \) in terms of \( \ell_b \) for a square. From Figure 12a,
\[
\begin{align*}
\sqrt{2} + r_b &= \frac{1}{\sqrt{2}} \ell_b \\
r_b &= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2} + 1} \ell_b \\
&= (1 - \frac{1}{\sqrt{2}}) \ell_b \\
\end{align*}
\]
(Note \( A := a^* + \delta a \), for some \( \delta \geq 0 \))
From Figure 12b, by computing the length of the part of the shape extending out of the
original square, we get:
\[
\begin{align*}
\ell_A - \ell^* &= \ell_b - 2r_b \\
\ell_A - \ell^* &= \ell_b - 2(1 - \frac{1}{\sqrt{2}}) \ell_b \\
\frac{\ell_A}{\ell^*} - 1 &= \ell_b(\sqrt{2} - 1) \\
\sqrt{\frac{A}{a^*}} - 1 &= \sqrt{\frac{b}{a^*}}(\sqrt{2} - 1) \\
\sqrt{\frac{A}{a^*}} - 1 &= \sqrt{\frac{b}{a^*}}(\sqrt{2} - 1) \\
b &= \frac{(\sqrt{A} - a^*)^2}{\sqrt{2} - 1} \\
b &= a(\sqrt{\frac{A}{a^*} - \sqrt{a^*/a}})^2 \\
b &= a\left(\frac{\sqrt{0.5 + \delta} - \sqrt{0.5}}{\sqrt{2} - 1}\right)^2 \\
b &= a\left(\frac{\sqrt{1 + 2\delta} - 1}{2 - \sqrt{2}}\right)^2 \\
\end{align*}
\]
\[\triangleright\]

A.4 Lemma [19] Square Packing Lemma

Proof.
Letting \( h := \sqrt{1 + 2\delta} \), we have \( \delta = \frac{h^2 - 1}{2} \)
\[
\delta \geq b^2_{sq} / a = \left(\frac{\sqrt{1 + 2\delta} - 1}{\sqrt{2} - 2}\right)^2 \\
\iff \frac{h^2 - 1}{2} \geq \left(\frac{h - 1}{\sqrt{2} - 2}\right)^2 \\
\iff (h^2 - 1)(\sqrt{2} - 2)^2 \geq 2(h - 1)^2 \\
\iff (h^2 - 1)(6 - 4\sqrt{2}) \geq 2h^2 - 4h + 2 \\
\iff h^2(4 - 4\sqrt{2}) + 4h + (4\sqrt{2} - 8) \geq 0 \\
\iff h^2(1 - \sqrt{2}) + h + (\sqrt{2} - 2) \geq 0 \\
\]
As \( 4 - 4\sqrt{2} < 0 \), the quadratic curve is concave downwards.
Using the quadratic formula, the roots are:
\[
h = \frac{-1 \pm \sqrt{1 - 4(1 - \sqrt{2})(\sqrt{2} - 2)}}{2(1 - \sqrt{2})} \\
= \frac{-1 \pm \sqrt{17 - 12\sqrt{2}}}{2(1 - \sqrt{2})} = \frac{-1 \pm \sqrt{(3 - 2\sqrt{2})^2}}{2(1 - \sqrt{2})} = \frac{-1 \pm (3 - 2\sqrt{2})}{2 - 2\sqrt{2}} \\
\]
Thus the above inequality holds if and only if
\[
\begin{align*}
2 - 2\sqrt{2} \leq h & \leq \frac{2\sqrt{2} - 4}{2 - 2\sqrt{2}} \\
1 \leq \sqrt{1 + 2\delta} & \leq \sqrt{2} \\
\end{align*}
\]
\[ \iff 0 \leq \delta \leq 0.5 \]

### A.5 Lemma 24

**Proof.**
Let \( y = \sqrt{1 + s^2} \)
Thus we have:
\[
\delta_{sh} = \left( \frac{2y - 2}{2y - 1} \right)^2 = 4 \left( \frac{y - 1}{2y - 1} \right)^2
\]
\[
a^*_t / a = \frac{y^2 - 1}{y^2}
\]
Therefore,
\[
3\delta_{sh} > a^*_t / a \iff 3 \times 4 \left( \frac{y - 1}{2y - 1} \right)^2 > \frac{(y + 1)(y - 1)}{y^2}
\]
\[
\iff 12 \left( \frac{y - 1}{2y - 1} \right)^2 > \frac{y + 1}{y^2}
\]
\[
\iff 12(y^3 - y^2) > (y + 1)(2y - 1)^2
\]
\[
\iff 12(y^3 - y^2) > (y + 1)(4y^2 - 4y + 1)
\]
\[
\iff 12y^3 - 12y^2 > 4y^3 - 3y + 1
\]
\[
\iff 8y^3 - 12y^2 - 3y - 1 > 0
\]
Which is true for all \( y \geq \sqrt{2} \), i.e. for all \( s \geq 1 \).

### A.6 Lemma 25

**Proof.**
Suppose \( 2\delta_{sh} \leq a^*_t / a \)
Then we have \( (a^*_t / a - 2\delta_{sh})(1 + s^2) \geq 0 \), so
\[
\sqrt{s^2 + (a^*_t / a - 2\delta_{sh})(1 + s^2)} - s \geq 0.
\]
We want to show that \( \delta_{sh} > b_{l} a^*_t / a - 2\delta_{sh} / a \) on the long child.
We have:
\[
\delta_{sh} > b_{l} a^*_t / a - 2\delta_{sh} / a
\]
\[
\iff \delta_{sh} > \left( \frac{\sqrt{s^2 + (a^*_t / a - 2\delta_{sh})(1 + s^2)} - s}{\sqrt{1 + s^2} - s} \right)^2
\]
(by Lemma 16, where \( b \) is defined on the long child)
\[
\iff \left( 1 - \frac{1}{2\sqrt{1 + s^2} - 1} \right)^2 > \left( \frac{\sqrt{s^2 + (a^*_t / a - 2\delta_{sh})(1 + s^2)} - s}{\sqrt{1 + s^2} - s} \right)^2
\]
\[
\iff \frac{2\sqrt{1 + s^2} - 2}{2\sqrt{1 + s^2} - 1} > \left( \frac{\sqrt{s^2 + (a^*_t / a - 2\delta_{sh})(1 + s^2)} - s}{\sqrt{1 + s^2} - s} \right)^2
\]
\[
\iff \frac{2\sqrt{1 + s^2} - 2}{2\sqrt{1 + s^2} - 1} > \frac{2\sqrt{1 + s^2} - 2}{2\sqrt{1 + s^2} - 1}
\]
\[
\iff \frac{2\sqrt{1 + s^2} - 2}{2\sqrt{1 + s^2} - 1} > \frac{2\sqrt{1 + s^2} - 2}{2\sqrt{1 + s^2} - 1}
\]
(As \( s \geq 1 \), i.e. the numerators and denominators of both sides are nonnegative)
Letting \( y = \sqrt{1 + s^2} \), we have
\[
\iff \frac{y - 1}{2y - 1} > \left( 2(y^2 - 1) - 8(\frac{y - 1}{2y - 1})^2 y^2 - s \right)\frac{y - s}{y - s}
\]
\[
\iff 2(y - 1)(y - s) > \left( 2(y^2 - 1) - 8(\frac{y - 1}{2y - 1})^2 y^2 - s \right)(2y - 1)
\]
(as the denominators are always positive)
A Reallocation Algorithm for Online Split Packing of Circles

\[ 2(y - 1)(y - s) > \sqrt{2(y - 1)^2(y^2 - 1) - 8(y - 1)^2y^2 - s(2y - 1)} \]
\[ 2(y^2 - y - ys + s) + s(2y - 1) > \sqrt{8y^3 - 14y^2 + 8y - 2} \]
\[ 2y^2 - 2y + s > \sqrt{8y^3 - 14y^2 + 8y - 2} \]

As \( s \geq 1 \), it suffices to show that:
\[ 2y^2 - 2y + 1 > \sqrt{8y^3 - 14y^2 + 8y - 2} \]

As \( 2y^2 - 2y + 1 \) is always nonnegative, we have:
\[ (2y^2 - 2y + 1)^2 > 8y^3 - 14y^2 + 8y - 2 \]
\[ 4y^4 - 16y^3 + 22y^2 - 12y + 3 > 0 \]

Which is always true as this polynomial has no roots.

Therefore \( 2\delta_{sh} \leq a^*_t/a \) implies that \( \delta_{sh} > b^*_t/a - 2\delta_{sh}l/a \) on the long child.

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