Pairwise mode-locking in dynamically-coupled parametric oscillators

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We demonstrate a novel source of bright, broadband, parametric radiation with unique coherence properties. Despite being broadband, the emission is not pulsed and lacks first-order coherence, yet possesses a very high degree of second-order coherence, somewhat similar to broadband two-mode squeezed vacuum, but with classical power levels. Our configuration is comprised of two coupled parametric oscillators within identical multimode cavities, where the coupling between the oscillators is modulated in time at the repetition rate of the cavity modes, with some analogy to active mode-locking in lasers. We therefore term our configuration “pairwise mode-locking”, which we demonstrate in a radio-frequency (RF) experiment, covering over an octave of bandwidth with approximately 20 resonant mode-locked pairs, filling most of the available bandwidth between DC and the pump frequency. We accompany our experiment with an analytic model that accounts for the properties of the coupled parametric oscillators near threshold.

The parametric oscillator (PO) is a central device in modern quantum optics - a fundamental type of oscillator whose internal parameters are modulated by an external drive, leading to parametric amplification [1–4]. Below the oscillation threshold, parametric oscillators are extensively used as sources of squeezed non-classical light, where the quantum fluctuations of one quadrature of the field are reduced below the vacuum (shot-noise) level, at the expense of increased fluctuations in the orthogonal quadrature [5–8], with applications in metrology [9–12], basic quantum information [13–16] and quantum communication [17, 18]. Recently, parametric oscillators have been proposed as scalable sources for continuous-variable (CV) cluster states [19] - the central resource for CV one-way quantum computation, by mixing and coupling between the modes of the frequency comb of a parametric oscillator [20]. Configurations of coupled parametric oscillators were explored in various contexts of quantum information [21–23], quantum computing [19] and coherent computing, e.g. Coherent Ising Machines (CIMs) which employ a network of coupled single-mode degenerate parametric oscillators in order to simulate a network of coupled Ising spins [24–27].

In the terminology of nonlinear optics, a parametric amplifier converts a pump field at frequency $\omega_p$ into a pair of signal and idler fields ($s$ and $i$) such that $\omega_s + \omega_i = \omega_p$. When the process is seeded by spontaneous emission, each field on its own appears as an incoherent thermal source [28], but the radiation produced is two-photon coherent [29–31], i.e., the spectral phase of each frequency mode is random, but the sum of phases $\phi_s + \phi_i = \phi_p$ of each signal-idler pair is well defined and highly coherent (quantum mechanically, even beyond the shot-noise limit) [32, 33]. Normally, non-linear effects are weak, indicating that intense pump fields are required to obtain appreciable down-conversion powers. As such, parametric oscillators, where the parametric amplifier is incorporated inside a high-finesse cavity [6], are used to critically reduce the required pump power. Above the oscillation threshold mode competition drastically narrows the down-conversion bandwidth, ultimately to a single signal-idler pair [34, 35].

Following our previous work [36, 37], we present here a novel configuration of two multi-mode AC-coupled parametric oscillators that produce a highly multi-mode coherent oscillation above threshold. The key concept in our configuration is the modulation of the coupling between the oscillators in time at a frequency that is an integer multiple of the repetition rate $\omega_{rep}$ of the cavities, as illustrated in Fig. 1. This actively couples frequency modes of one oscillator to neighbouring modes of the other oscillator. This scheme is similar to active mode-locking in lasers, where either the gain or the loss is actively modulated in time at the cavity repetition frequency (or integers multiples of), stimulating a bright, broadband parametric oscillation involving all of the signal and idler mode pairs, in sharp contrast to the narrowband oscillation that is normally produced by a continuously-pumped doubly-resonant parametric oscillator. Although this oscillation is not pulsed and totally incoherent in first-order, it demonstrates a very high degree of second-order coherence, where all corresponding signal-idler pairs are complex conjugates of each other. Therefore, our oscillation shows high coherence as pairs of modes but not as single modes, hence we term our method “pairwise” mode-locking.

We demonstrate our configuration in a radio-frequency (RF) experiment, using off-the-shelf components. The reason for the choice of RF is that it provides an ideal and simple platform to explore coherent phenomena that allows direct observation of the oscillation in both time and frequency. In addition, setups that are difficult to design and implement in optics can sometimes be trivially implemented in RF, with substantially less resources, while capturing most (if not all) of the coherent physics involved. Our experimental setup is illustrated in
Two resonators with a repetition rate $\omega_{\text{rep}}$ are locked together when coupled. We realize the two parametric oscillators (PO1 and PO2) are identical in their components and dependent coupling mechanism. The oscillators, labelled broadband RF cavities, coupled together with a time-dependent coupling modes of the same frequency between the cavities experiment, the coupling additionally has a DC component which we implement using yet another mixer, and inject it into the other oscillator using a power combiner denoted by $\omega_0$ (Mini-Circuits ZAPD-2-252-S+).

Our findings imply that our technique is profoundly different from active mode-locking in lasers. In mode-locked lasers, the dynamics of loss and gain within the cavity act to lock all modes in-phase to produce a short pulse [38, 39]. In a rough sense, it is most efficient for the laser to store energy within the medium during most of the repetition time, and then release it in a short intense pulse when the losses are minimal. This type of carrier frequency $\omega_0 = \omega_p/2$, where $\omega_p$ is the pump frequency, in pairs with conjugate phases that sum up to the pump phase all across the oscillation spectrum, therefore showing high pairwise second-order coherence. In time, this spectral symmetry indicates that the oscillation is on a single quadrature of the electric field, i.e. $E^{(j)}(t) = X^{(j)}(t) \cos(\omega_0 t) + \lambda \sin(\omega_0 t)$ quadrature component. Although the oscillation is very broadband, we measure completely random spectral phases of different signal-idler pairs [Fig. 3(b),(d)], which is a clear indication of the lack of first-order coherence and therefore absence of pulsed oscillations in time. This can be also directly observed in time by monitoring the fields PO1 and PO2 on an oscilloscope (Fig. 4).

FIG. 2. Experimental scheme. Each oscillator is comprised of the following main components (Fig. 2 and Ref. [36]): The nonlinear parametric amplifier (paramp) is realized by a RF frequency mixer (Mini-Circuits ZFBT-42G-FT+), denoted by “$\oplus$”. Since the parametric gain of the mixer was insufficient to cross oscillation threshold, a broadband (linear) low-noise amplifier “G” (Mini-Circuits ZX60-P105LN) was added to each cavity. We emphasize that the linear gain was tuned only to mitigate some of the losses in the cavities, and can not induce oscillations on its own. An output coupler (OC) (Mini-Circuits ZFDC-15-5) allows to couple the oscillation out for observation on an oscilloscope or a spectrum analyzer. The time-dependent coupling mechanism at frequency $\omega_{\text{rep}}$ is represented in the red-dashed box, with $\omega_{\text{rep}}$ indicating the modulation of the coupling at the repetition rate of the cavity. In order to control the amount of power coupled into the other cavity, we first pass the signal through a voltage-controlled variable attenuator which we implement using yet another mixer, and inject it into the other oscillator using a power combiner denoted by “$\ominus$” (Mini-Circuits ZAPD-2-252-S+).
mechanism, however, does not exist in a parametric oscillator, since the nonlinear gain is instantaneous and the gain medium lacks the ability to store energy. Thus, any pump energy that is not immediately converted to signal-idler pairs is lost. As a consequence, it is inefficient for a parametric oscillator (in the absence of additional non-linearities, e.g. Kerr non-linearity) to support a pulsed oscillation when the pump is continuous [40, 41]. The pairwise coherent oscillation, although broadband, is first-order incoherent and appears continuous in time, and thus efficiently utilizes the pump resource.

Despite the conceptual difference, a useful and elegant analogy between pairwise and standard mode-locking does exist. We model our system by two identical cavities, driven by a pump field that is resonant on the $N$-th mode of the oscillators, at frequency $\omega_p = N \omega_{rep}$, where $N$ is a positive integer (Fig. 1). For simplicity, in the following, we set $\omega_{rep} = 1$. By the parametric interaction, the injected pump field can be down-converted into a pair of signal and idler modes, at frequencies $\omega$ and $N - \omega$, respectively. The dynamics of the modes in the two cavities is captured by the complex slow-varying amplitudes $A^{(j)}_N$ and $A^{(j)}_0$, at frequency $\omega$ in cavities PO1 and PO2, respectively. Since the energy of the pump is converted into signal-idler pairs, the signal and idler modes are complex conjugates of each other [42, 43]:

$$A^{(j)}_N = (A^{(j)}_N)^*, \text{ for all } \omega \text{ and } j = 1, 2.$$ These relations reflect the fact that the sum of the phases of each signal and idler pair sum up to the pump phase, taken to be zero. In addition, the lack of first-order coherence for different modes implies that they are uncorrelated $\langle A^{(j)}_N A^{(j)^*}_{N'} \rangle = I_j^{(\omega)} \delta_{j,j'} \delta_{\omega,\omega'}$, where we define the steady-state power spectrum $I^{(j)}_\omega \equiv \langle |A^{(j)}_N|^2 \rangle$ [Fig. 3(b),(c)], where the expectation value is an ensemble average over all possible spectral phases.

The modes in the two cavities are then dynamically coupled by a time-dependent coupling, modulated at $m \omega_{rep}$, with $m$ integer. While in our experiment we are interested only in the simplest case of single frequency modulation at the fundamental rep-rate $m = 1$, which corresponds to nearest-neighbor coupling, we note that in general the coupling could be modulated at any integer multiple of the repetition rate, and not necessarily be monochromatic. General couplings could be devised to create many different non-trivial coupling topologies, other than the simple nearest-neighbor coupling that we discuss here [44, 45].

The dynamics of the complex amplitudes in PO1 and PO2 are described by the following set of $2N$ coupled first-order ordinary differential equations ($j \neq k$):

$$\frac{d}{dt} A^{(j)}_N = \left[ G_\omega - \beta \sum_{\omega'} A^{(j)}_{N-\omega'} A^{(j)}_{\omega'} \right] (A^{(j)}_{N-\omega})^* + \frac{\delta}{2} \left[ A^{(k)}_{\omega+m} + A^{(k)}_{\omega-m} \right], \quad (1)$$

for $j,k = 1, 2$, where $m < \omega \leq N - m$ with $N$ odd, and $G_\omega = h/8 - g_{\omega}/2$ is the net gain per round-trip of the $\omega$-mode, dictated by the parametric gain $h$ and the loss term $g_{\omega}$. $\beta$ stands for the gain saturation due to the pump depletion by the entire set of modes, which is responsible for the mode competition between all mode.
pairs in the uncoupled case. We denote by \( \delta \) the strength of the coupling between mode \( \omega \) in PO1 and modes \( \omega \pm m \) in PO2.

As in our experiment, we set \( m = 1 \). The first terms in the right-hand side of Eq. (1) describe the parametric amplification for each individual signal-idler pair at frequency \( \omega \) and \( N - \omega \). In the uncoupled case (\( \delta = 0 \)), these independent pairs compete for the gain resources and only the one associated with the largest net gain \( G_\omega \) oscillates. However, when \( \delta \neq 0 \), the coupling connects all signal and idler pairs at different frequencies. As a consequence, the most efficient mode is now a combination of all signal-idler pairs, whose spectral distribution approximately mimics the spectral loss function. As such, broadband parametric light is generated. To show this theoretically, we derive from Eq. (1) an equation for the spectrum \( I^{(j)}(\omega) \) of the oscillation in steady-state near threshold, where gain saturation is negligible [46]

\[
\frac{\delta^2 \omega^2}{2} \frac{\partial^2}{\partial \omega^2} I^{(j)}(\omega) + (4G_\omega^2 + 2\delta^2) I^{(j)}(\omega) = 0, \tag{2}
\]

which is in a direct analogy to the spectral amplitude of pulses in active mode-locking of lasers [38]. This further establishes the equivalence between pairwise and standard mode-locking. Assuming a simple spectral dependence of the net gain \( G_\omega^2 \approx G_0^2 - \omega^2/2\sigma^2 \), where \( \sigma \) is the gain bandwidth, Eq. (S11) yields a Gaussian spectrum \( I^{(j)}(\omega) \approx e^{-\omega^2/\Delta^2} \) with spectral width \( \Delta^2 = \delta \omega_{\text{rep}} \sigma \) and steady-state gain of \( G_0^2 = \delta \omega_{\text{rep}} \sigma / 4 \sigma \), again in direct similarity to active mode-locking in lasers.

Last, let us consider the implications of the fact that the oscillation is both broadband and on a single quadrature. Normally, the quadrature content of a signal is evaluated by homodyning against a narrowband local oscillator (LO) at the center carrier frequency \( \omega_0 \), which acts as a quadrature reference, so for example \( X^{(j)}(t) = \langle E^{(j)}(t) \cos(\omega_0 t) \rangle \), where the angle brackets denote averaging over fast-varying terms and the phase of the local oscillator determines which quadrature is measured. Here, since the signals \( E^{(j)}(t) \) are broadband, so is the homodyne result, as shown in Fig. 5(a), which shows the spectrum of the homodyne output for the stretched \( \langle |X_\Omega|^2 \rangle \) and the squeezed \( \langle |Y_\Omega|^2 \rangle \) quadratures, with > 20dB difference between them over a broad spectrum. However, the quadrature reference does not need to be narrowband necessarily [47]. In fact, since both signals of PO1 and PO2 are of a single quadrature, they could serve as local oscillators for one another \( X(t) = \langle E^{(1)}(t)E^{(2)}(t) \rangle \), where the delay between them determines the measured quadrature. Indeed, the mixing of the two oscillator outputs also shows a clear difference of > 15dB between the two quadrature phases [Fig. 5(b)], but its spectral structure is inherently different from the standard homodyne with a single frequency LO. Here, each frequency of the product output is the result of the spectral cross-correlation between the two spectral amplitudes of the two oscillators \( X_\Omega = \sum_\omega A^{(j)}(\omega) (A^{(k)}(\omega+\Omega) + A^{(k)}(\omega-\Omega)) \), where \( X_\Omega \) is the Fourier-transform of the product signal. Thus, every frequency within this cross-homodyne result is a collective result of mixing the entire oscillation frequency combs with the relevant frequency offset.

Our results demonstrate the difference in power between the two quadratures, but does not demonstrate squeezing. While quantum noise properties and squeezing cannot be explored in a room-temperature RF experiment, our experiment can still simulate “semi-classical squeezing”, where the thermal noise plays the part of the quantum vacuum fluctuations. This can capture many important properties of squeezing at high powers, near and above threshold [48], and acts as a powerful semi-classical simulator for quantum optics. However, we stress that while our configuration provides a very convenient platform for demonstrating squeezing of thermal noise, it is not demonstrated in this work.
In summary, we presented a new kind of broadband parametric source, comprising of a pair of parametric oscillators, coupled with a time-modulated coupling. We showed in an RF experiment that the generated oscillation is very broadband, yet lacks first-order coherence. This source of bright, broadband parametric light can be a key enabler for a variety of applications in both classical and quantum information [21, 22], from communication protocols [13, 17, 49, 50] to interferometry, sensing and quantum metrology [9–12, 51, 52]. For example, one obvious application would be to use this source as a bright quantum frequency comb [53–55], or in order to generate CV cluster states [22, 56–58]. Another interesting application would be to use this source as a bright narrow-band parametric source, comprising of a pair of parametric oscillators, coupled with a time-modulated coupling. We actually discovered that this source is sent to identify a remote target while maintaining a noise-radar scheme, where a quasi-continuous noise signal would be to use pairwise mode-locking as a part of a key enabler for a variety of applications in both classical and quantum information. This source of bright, broadband parametric light can be very broadband, yet lacks first-order coherence.

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Supplemental material for “Pairwise mode-locking in dynamically-coupled parametric oscillators”

DERIVATION OF THE PAIR-WISE MODE-LOCKED SPECTRUM

We here show the derivation of the pair-wise mode-locked spectrum [Eq. (2) of the main text], obtained from the equation of motion [Eq. (1) of the main text], which is reported below for completeness:

$$\frac{d}{dt} A^{(j)}_\omega = \left( G_\omega - \beta \sum_{\omega} |A^{(j)}_\omega|^2 \right) A^{(j)}_\omega + \frac{\delta}{2} \left[ A^{(k)}_{\omega+m} + A^{(k)}_{\omega-m} \right]. \quad \text{(S1)}$$

where, \( j, k = 1, 2 \) and \( k \neq j \). The goal is showing that the spectrum of the pair-wise correlation \( A^{(j)}_\omega A^{(j)}_{N-\omega} = |A^{(j)}_\omega|^2 \) in parametric oscillators and the spectral amplitude of the mode-locked pulse in lasers obey the same dynamics, therefore establishing the equivalence between these two mechanisms. To reach this goal, we proceed as follows. We compute the second derivative with respect to time of the pair-wise correlation. Since \( d|A^{(j)}_\omega|^2 / dt = A^{(j)}_\omega (d(A^{(j)}_\omega)^*/dt) + (A^{(j)}_\omega)^* (dA^{(j)}_\omega / dt) \), one has

$$\frac{d^2}{dt^2} |A^{(j)}_\omega|^2 = (A^{(j)}_\omega)^* \frac{d^2 A^{(j)}_\omega}{dt^2} + A^{(j)}_\omega \frac{d^2 (A^{(j)}_\omega)^*}{dt^2} + 2 \frac{dA^{(j)}_\omega}{dt} \frac{d(A^{(j)}_\omega)^*}{dt}. \quad \text{(S2)}$$

We now compute one by one the terms on the right-hand side of Eq. (S2). We first compute the first term. By using Eq. (S1), assuming near threshold operation and neglecting the saturation term, we have

$$\frac{d^2 A^{(j)}_\omega}{dt^2} = G_\omega \frac{dA^{(j)}_\omega}{dt} + \frac{\delta}{2} \left( \frac{dA^{(k)}_{\omega+m}}{dt} + \frac{dA^{(k)}_{\omega-m}}{dt} \right)$$

$$= G_\omega \left[ G_\omega A^{(j)}_\omega + \frac{\delta}{2} \left( A^{(k)}_{\omega+m} + A^{(k)}_{\omega-m} \right) \right] + \frac{\delta}{2} \left[ G_\omega A^{(k)}_{\omega+m} + \frac{\delta}{2} \left( A^{(j)}_{\omega+2m} + A^{(j)}_{\omega} \right) + G_\omega A^{(k)}_{\omega-m} + \frac{\delta}{2} \left( A^{(j)}_{\omega} + A^{(j)}_{\omega-2m} \right) \right]$$

$$= \left( G^2 + \frac{\delta^2}{2} \right) A^{(j)}_\omega + \ldots, \quad \text{(S3)}$$

where the ellipsis (\( \ldots \)) accounts for terms at frequencies different from \( \omega \). Each amplitude \( A^{(j)}_\omega \) has a random phase, due to the intrinsic avoided mode-locked regime of parametric amplification. Therefore, by multiplying both sides of Eq. (S3) by \( (A^{(j)}_\omega)^* \) and by performing an ensemble average over all possible phases, one has

$$\left\langle (A^{(j)}_\omega)^* \frac{d^2 A^{(j)}_\omega}{dt^2} \right\rangle \simeq \left( G^2 + \frac{\delta^2}{2} \right) \left\langle |A^{(j)}_\omega|^2 \right\rangle, \quad \text{(S4)}$$

where the terms in the ellipsis in Eq. (S3) vanish, i.e., \( \langle (A^{(j)}_\omega)^* A^{(k)}_{\omega+m} \rangle = \langle (A^{(j)}_\omega)^* A^{(j)}_{\omega+2m} \rangle = 0 \), because of lack of first-order coherence. For the second term on the right-hand side of Eq. (S2), when performing the ensemble average, we have

$$\left\langle A^{(j)}_\omega \frac{d^2 (A^{(j)}_\omega)^*}{dt^2} \right\rangle = \left\langle (A^{(j)}_\omega)^* \frac{d^2 A^{(j)}_\omega}{dt^2} \right\rangle. \quad \text{(S5)}$$

Lastly, for the third term on the right-hand side of Eq. (S2)

$$\frac{dA^{(j)}_\omega}{dt} \frac{d(A^{(j)}_\omega)^*}{dt} = \left[ G_\omega A^{(j)}_\omega + \frac{\delta}{2} \left( A^{(k)}_{\omega+m} + A^{(k)}_{\omega-m} \right) \right] \left[ G_\omega (A^{(j)}_\omega)^* + \frac{\delta}{2} \left( A^{(k)}_{\omega+m}^* + A^{(k)}_{\omega-m}^* \right) \right]$$

$$= G^2 |A^{(j)}_\omega|^2 + \frac{\delta^2}{4} \left( |A^{(k)}_{\omega+m}|^2 + |A^{(k)}_{\omega-m}|^2 \right) + \ldots, \quad \text{(S6)}$$

where the ellipsis (\( \ldots \)) accounts for terms like \( A^{(j)}_\omega (A^{(k)}_{\omega+m})^* \), \( (A^{(k)}_{\omega+m})^* (A^{(j)}_\omega)^* \), and \( A^{(k)}_{\omega-2m} (A^{(j)}_\omega)^* \). As before, one has

$$\left\langle \frac{dA^{(j)}_\omega}{dt} \frac{d(A^{(j)}_\omega)^*}{dt} \right\rangle \simeq G^2 \left\langle |A^{(j)}_\omega|^2 \right\rangle + \frac{\delta^2}{4} \left( \left\langle |A^{(k)}_{\omega+m}|^2 \right\rangle + \left\langle |A^{(k)}_{\omega-m}|^2 \right\rangle \right). \quad \text{(S7)}$$
Eventually, by taking the ensemble average in Eq. (S2) and by using Eqs. (S4)-(S7), we have (we define $I_{\omega}^{(j)} = \langle |A_{\omega}^{(j)}|^2 \rangle$ from now on to ease the notation)

$$\frac{d^2}{dt^2} \langle |A_{\omega}^{(j)}|^2 \rangle \equiv \frac{d^2 I_{\omega}^{(j)}}{dt^2} = \left\langle A_{\omega}^{(j)} \frac{d^2 (A_{\omega}^{(j)})^*}{dt^2} \right\rangle + \left\langle (A_{\omega}^{(j)})^* \frac{d^2 A_{\omega}^{(j)}}{dt^2} \right\rangle + 2 \left\langle \frac{dA_{\omega}^{(j)}}{dt} \frac{d(A_{\omega}^{(j)})^*}{dt} \right\rangle$$

$$= 2 \left( G_{\omega}^2 + \frac{\delta^2}{2} \right) I_{\omega}^{(j)} + 2 G_{\omega}^2 I_{\omega}^{(j)} + \frac{\delta^2}{2} \left( I_{\omega+m}^{(k)} + I_{\omega-m}^{(k)} \right)$$

$$= (4G_{\omega}^2 + \delta^2) I_{\omega}^{(j)} + \frac{\delta^2}{2} \left( I_{\omega+m}^{(k)} + I_{\omega-m}^{(k)} \right).$$

(S8)

One can now set $m = 1$ (in units of the free-spectral range $\omega_{\text{rep}}$). Assuming the spectrum is smooth within the interval $\omega_{\text{rep}}$ one can approximate,

$$I_{\omega \pm m}^{(k)} \approx I_{\omega}^{(k)} \pm \omega_{\text{rep}} \frac{d}{d\omega} I_{\omega}^{(k)} + \frac{\omega_{\text{rep}}^2}{2} \frac{d^2}{d\omega^2} I_{\omega}^{(k)}.$$  

(S9)

In the steady-state, in Eq. (S8), the time derivative vanishes $dI_{\omega}^{(j)}/dt = 0$, and therefore

$$(4G_{\omega}^2 + \delta^2) I_{\omega}^{(j)} + \delta^2 I_{\omega}^{(k)} + \frac{\delta^2 \omega_{\text{rep}}^2}{2} \frac{d^2}{d\omega^2} I_{\omega}^{(k)} = 0.$$  

(S10)

The fact that the two cavities are identical allows us to write $I_{\omega}^{(j)} = I_{\omega}^{(k)}$. From Eq. (S10), one has

$$\frac{\delta^2 \omega_{\text{rep}}^2}{2} \frac{d^2}{d\omega^2} I_{\omega}^{(j)} + (4G_{\omega}^2 + 2\delta^2) I_{\omega}^{(j)} = 0.$$  

(S11)

and the same for $I_{\omega}^{(k)}$. This proves the claim of the main text.

As a concrete example, one can take the gain-loss term $G_{\omega}$ to be sufficiently peaked around $N\omega_{\text{rep}}/2 = \omega_p/2$ (half the pump frequency), one can expand $G_{\omega} \approx G_0^2 - (\omega - \omega_p/2)^2/4\sigma^2$, where $\sigma$ is the gain bandwidth. Therefore, Eq. (S11) becomes (after setting the carrier frequency $\omega_p/2$ as the zero of the spectrum)

$$\frac{\delta^2 \omega_{\text{rep}}^2}{2} \frac{d^2}{d\omega^2} I_{\omega}^{(j)} - \left( \frac{\omega}{\sigma} \right)^2 I_{\omega}^{(j)} + (4G_0^2 + 2\delta^2) I_{\omega}^{(j)} = 0.$$  

(S12)

These results establish the equivalence between standard mode-locking in lasers, and pair-wise mode locking in parametric oscillators.