The Schrödinger operator in Newtonian space-time

Katarzyna Grabowska¹, Janusz Grabowski², Paweł Urbański¹

¹ Physics Department
University of Warsaw
² Institute of Mathematics
Polish Academy of Sciences

Abstract

The Schrödinger operator on the Newtonian space-time is defined in a way which is independent on the class of inertial observers. In this picture the Schrödinger operator acts not on functions on the space-time but on sections of certain one-dimensional complex vector bundle over space-time. This bundle, constructed from the data provided by all possible inertial observers, has no canonical trivialization, so these sections cannot be viewed as functions on the space-time. The presented framework is conceptually four-dimensional and does not involve any ad hoc or axiomatically introduced geometrical structures. It is based only on the traditional understanding of the Schrödinger operator in a given reference frame and it turns out to be strictly related to the frame-independent formulation of analytical Newtonian mechanics that makes a bridge between the classical and quantum theory.

MSC 2000: 35J10, 70G45.

Key words: Schrödinger operator, space-time, complex vector bundle.

1 Introduction

In the papers [4, 5, 6, 17] we have presented an approach to differential geometry in which sections of a one-dimensional affine bundle over a manifold have been used instead of functions on the manifold. This approach, initiated by W. M. Tulczyjew in [15, 16], has been successfully applied to frame-independent description of different systems, in particular to a frame-independent formulation of Newtonian mechanics [7].

The latter problem is closely related to the problem of frame-independent formulation of wave mechanics in the Newtonian space-time. It is known that a solution of the Schrödinger equation in one inertial frame will not, in general, satisfy the Schrödinger equation in a different frame. The same quantum state of a particle must be represented by a different wave function in reference to a different inertial frame. The corresponding gauge transformation of solutions of the Schrödinger equation was known already to W. Pauli [10]. Many ways of solving this problem have been proposed in the literature. For instance, a general axiomatic theory of quantum bundles, quantum metrics, quantum connections etc. has been developed in [9] to deal with a covariant description of Schrödinger operators in curved space-times. Another general fibre bundle formulation of nonrelativistic quantum mechanics has been proposed in a series of papers [5].

An approach which is the closest to what we propose in this paper is a frame-independent formulation of wave mechanics by extending Newtonian space-time to five-dimensional Galilei space [3, 13, 14]. The corresponding geometry is associated with the Bargmann group - nontrivially extended Galilei group [1].

*Research supported by the Polish Ministry of Scientific Research and Information Technology under the grant No. 2 P03A 036 25.
In the present paper we change this viewpoint a little bit, making the whole theory four-dimensional again. For simplicity, we deal with the flat Newtonian space-time and the very standard Schrödinger operator to show that a frame-independent formulation of wave mechanics is possible in terms of a canonically constructed line bundle. In this picture the Schrödinger operator acts not on functions on the space-time but on sections of certain one-dimensional complex vector bundle over space-time. This bundle, constructed from the data provided by all possible inertial observers, has no canonical trivialization, so these sections cannot be viewed as functions on the space-time.

We want to stress three facts. First, we do not look just for transformations rules for solutions of the Schrödinger equation in different reference frames, but we built the bundle whose sections represent the arguments of the Schrödinger operator and we give to the operator itself a covariant geometric meaning.

Second, we show that the transformation rules, so the bundle, are unique up to unavoidable change by constant factors - the integration is always slightly non-unique.

And last but not least, we prove that the proposed formulation is strictly related to the frame-independent formulation of analytical Newtonian mechanics [7]. This makes a bridge between the classical and quantum theory which, in our opinion, is not understood completely yet and usually not present in the literature.

2 Newtonian space-time

The Newtonian space-time (some authors prefer to call it Galilean space-time, but we follow the terminology of Benenti [2] and Tulczyjew [13]) is a system $(N,\tau,g)$, where $N$ is a four-dimensional affine space for which, say $V$, is the model vector space, where $\tau$ is a non-zero element of $V^*$, and where $g: E_0 \rightarrow E_0^*$ represents an Euclidean metric on $E_0 = \ker \tau$. The corresponding scalar product reads $\langle v, v' \rangle = (g(v))(v')$ and the corresponding norm $\|v\| = \sqrt{\langle v, v \rangle}$. The elements of the space $N$ represent events. The time elapsed between two events is measured by $\tau$: $\Delta t(x, x') = \tau(x - x')$

and the distance between two simultaneous events is measured by $g$: $d(x, x') = \|x - x'\|.$

The space-time $N$ is fibred over the time $T = N/E_0$ which is a one-dimensional affine space modelled on $\mathbb{R}$.

Let $E_1$ be an affine subspace of $V$ defined by the equation $\tau(v) = 1$. The model vector space for this subspace is $E_0$. An element of $E_1$ represents velocity of a particle. The affine structure of $N$ allows us to associate to an element $u$ of $E_1$ the family of inertial observers that move in the space-time with the constant velocity $u$. In this way we can interpret an element of $E_1$ also as a class of inertial reference frames. For a fixed inertial frame $u \in E_1$ and a point $x_0 \in N$, we can identify $N$ with $E_0 \times \mathbb{R}$ by

$$\Phi_{(x_0,u)} : N \rightarrow E_0 \times \mathbb{R}, \quad x \mapsto ((x - x_0) - \tau(x - x_0)u, \tau(x - x_0)).$$

A change of the inertial reference frame results in the change of this identification and it is represented by

$$\Theta_{(x_0,u)}^{(x_0',u')} : E_0 \times \mathbb{R} \rightarrow E_0 \times \mathbb{R},$$

$$\Theta_{(x_0,u)}^{(x_0,u')}(v,t) \mapsto (v - ((x_0' - x_0) - \tau(x_0' - x_0)u') - (u' - u)t, t - \tau(x_0' - x_0)).$$

We can fix linear coordinates $y = (y_i) : E_0 \rightarrow \mathbb{R}^3$ in $E_0$ so that $\|v\|^2 = \sum_i g_i^2(v)$. Then, with every inertial frame $(x_0, u)$, we can associate coordinates $(y, t)$ in $N$, thus $V$, with $(y, t)(x) = \varphi_{(x_0,u)}(x) = (y(x - x_0 - \tau(x - x_0)u), \tau(x - x_0))$, and the change of coordinates is

$$\Theta_{(x_0,u)}^{(x_0,u')}(y, t) = (y - \Delta y - y(t)t, t - \Delta t),$$

where $(\Delta y, \Delta t) \in \mathbb{R}^3 \times \mathbb{R}$ are coordinates of $x_0' - x_0 \in V$ for the observer $(x_0, u)$ and $y(v) \in \mathbb{R}^3$ are coordinates of $v = u' - u \in E_0$. 

K. Grabowska, J. Grabowski, P. Urbański
3 The Schrödinger operator

The classical Schrödinger operator for a particle of mass \( m \) and a potential \( \tilde{U} \in C^\infty(\mathbb{R}^3 \times \mathbb{R}) \) is a second order complex differential operator which, in coordinates described above, reads

\[
S^m_\psi = i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \sum_k \frac{\partial^2 \psi}{\partial y_k^2} - \tilde{U}(y) \psi.
\]

(3.1)

Here, \( \sum_k \frac{\partial^2}{\partial y_k^2} \) is clearly the spatial Laplace-Beltrami operator associated with the metric \( g \). The problem is that, if assumed as acting on functions, the Schrödinger operator (3.1) is not invariant with respect to the change of coordinates (2.4) associated with the choice of another inertial frame. On the other hand, by arguments coming from physics, the form of the Schrödinger operator should be independent on the choice of inertial observer.

The solution we propose is that the Schrödinger operator acts in fact on sections of certain 1-dimensional complex vector bundle \( S_m \) over \( N \) (we will call it Schrödinger bundle) which is trivializable with no canonical trivialization. Thus the situation is completely parallel to the one we encounter for frame-independent description of the standard lagrangian in Newtonian mechanics [7].

Since the bundle \( S_m \) has no canonical trivialization, we have to combine every change of coordinates (2.4) in \( N \) with a linear change in values of wave functions. However, we can simplify this problem a little bit. Since, as can be easily seen, the part corresponding to the potential \( \tilde{U} \) associated with a function \( U \) on \( N \) behaves properly and the Schrödinger operator is invariant with respect to the change of coordinates associated with observers moving with the same velocity, \( u = u' \), we can assume that \( \tilde{U} = 0 \) and \( x'_0 = x_0 \), so we reduce transformations to vertical (spatial) ones. Thus we shall look for an action of the commutative group \( E_0 \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{C} \) of the form

\[
\Psi_y(y_k, t, z) = \left( y_k - y_k(v) t, t, e^{F_v(y, t)} z \right),
\]

(3.2)

corresponding to the representation of \( E_0 \) in the algebra \( C^\infty_c(\mathbb{R}^3 \times \mathbb{R}) \) of complex-valued functions on \( \mathbb{R}^3 \times \mathbb{R} \),

\[
T_v(\psi)(y, t) = e^{F_v(y, t)} \psi(y - y(v)t, t),
\]

(3.3)

such that the "free" Schrödinger operator

\[
S^m_0 \psi = i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \sum_k \frac{\partial^2 \psi}{\partial y_k^2}.
\]

remains unchanged:

\[
S^m_0 \left( e^{F_v(y, t)} \psi(y - y(v)t, t) \right) = e^{F_v(y, t)} S^m_0(\psi)(y - y(v)t, t).
\]

(3.5)

**Remark.** That our spatial part is 3-dimensional is motivated by physics. However, from the mathematical point of view, there is no difference if we use other dimensions. All considerations and proofs remain unchanged if we use \( \mathbb{R}^n \times \mathbb{R} \times \mathbb{C} \) instead of \( \mathbb{R}^3 \times \mathbb{R} \times \mathbb{C} \).

4 The explicit transformations

Let us look what the function \( F_v \) could be in order that (3.5) is satisfied. Straightforward calculations show that \( S^m_0 \) is equivalent to

\[
\psi(y - y(v)t, t) \left( i(\partial_t F)(y, t) + \frac{\hbar}{2m} \left( \sum_k (\partial_{y_k} F)^2(y, t) + \sum_k (\partial_{y_k}^2 F)(y, t) \right) \right) + \sum_k (\partial_{y_k} \psi)(y - y(v)t, t) \left( \frac{\hbar}{2m} (\partial_{y_k} F)(y, t) - i y_k(v) \right) = 0
\]

(4.1)
for all complex functions $\psi$ on $\mathbb{R}^3 \times \mathbb{R}$. Since $\psi$ is arbitrary, this, in turn, is equivalent to the system of equations

\begin{align}
(4.2) \quad & i(\partial_t F)(y, t) + \frac{\hbar}{2m} \left( \sum_k (\partial_{y_k} F)(y, t) + \sum_k (\partial^2_{y_k} F)(y, t) \right) = 0, \\
(4.3) \quad & \frac{\hbar}{2m} (\partial_{y_k} F)(y, t) - i y_k(v) = 0, \ k = 1, 2, 3.
\end{align}

From (4.3) it follows that $\partial^2_{y_k} F = 0$, $k = 1, 2, 3$, so that (4.2) reduces to

\begin{equation}
(4.4) \quad i(\partial_t F)(y, t) - \frac{m}{2\hbar} \sum_k y_k^2(v) = 0.
\end{equation}

The equations (4.3) and (4.4) for partial derivatives determine $F$ up to a constant, so, as can be easily seen,

\begin{equation}
(4.5) \quad F_v(y, t) = \frac{im}{\hbar} \left( \sum_k y_k(v) y_k - \frac{t}{2} \sum_k y_k^2(v) \right) + c.
\end{equation}

But $T_v(\psi)(y, t) = e^{F_v(y, t)} \psi(y - y(t)v, t)$ must be a representation, i.e.

\begin{equation}
(4.6) \quad T_v \circ T_{v'} = T_{v+v'}.
\end{equation}

Direct calculations show that (4.6) is satisfied with $F_v$ as in (4.5) if and only if $c = 0$. In this way we get the following.

**Theorem 4.1.** The map $\mathbb{R}^3 \ni v \mapsto T_v$, where $T_v$ is a linear operator in $C^\infty_c(\mathbb{R}^3 \times \mathbb{R})$ defined by

\begin{equation}
(4.7) \quad T_v(\psi)(y, t) = \exp \left( \frac{im}{\hbar} \left( \sum_k y_k(v) y_k - \frac{t}{2} \sum_k y_k^2(v) \right) \right) \psi(y - y(t)v, t),
\end{equation}

is a representation of $\mathbb{R}^3$ in $C^\infty_c(\mathbb{R}^3 \times \mathbb{R})$ leaving invariant the Schrödinger operator (4.4).

The fact that the above transformations act on solutions of the Schrödinger equation in different reference frames is known (see e.g. [10, p. 100]). Here, independently, we have found these transformations in order to recognize properly the arguments of the Schrödinger operator and the operator itself, and we have proved the uniqueness.

5 The Schrödinger bundle

Let us now fix $x_0 \in N$. In the trivial 1-dimensional complex bundle $E_1 \times N \times \mathbb{C}$ over $E_1 \times N$ we consider the following linear action of the additive group $E_0$:

\begin{equation}
(5.1) \quad R_v(u, x, z) = \left( u + v, x, z \cdot \exp \left[ \frac{m}{i\hbar} \left( v \mid x - x_0 - \tau(x - x_0)u \rangle \right. \right. \right. \left. \left. \left. \left. \left. - \frac{\tau(x - x_0)}{2}\|v\|^2 \right) \right] \right).
\end{equation}

This is indeed an action, since

\begin{equation*}
\langle v + v' \mid x - x_0 - \tau(x - x_0) (u + v) \rangle - \frac{\tau(x - x_0)}{2}\|v + v'\|^2 = \\
\langle v \mid x - x_0 - \tau(x - x_0)u \rangle - \frac{\tau(x - x_0)}{2}\|v\|^2 + \langle v' \mid x - x_0 - \tau(x - x_0)(u + v) \rangle - \frac{\tau(x - x_0)}{2}\|v'\|^2.
\end{equation*}

Since the action is free and linear and since the orbits project onto $E_1 \times \{x\}$, the quotient space, i.e. the family of orbits, forms a new vector bundle $S_m$ over $N$. If $[u, x, z]$ is the class (orbit) through $(u, x, z)$, the addition in the bundle is represented by

$$[u, x, z] + [u + v, x, z'] = [u, x, z + z' \cdot \exp \left[ \frac{im}{\hbar} \left( v \mid x - x_0 - \tau(x - x_0)u \rangle \right. \right. \right. \left. \left. \left. \left. \left. \left. \left. - \frac{\tau(x - x_0)}{2}\|v\|^2 \right) \right] \right].$$
and the multiplication by complex numbers $\alpha$ is given by

$$\alpha[u,x,z] = [u,x,\alpha \cdot z].$$

This bundle we shall call the *Schrödinger bundle* associated with the mass $m$. It is clear that the bundle is trivializable, since, for fixed $u$, the global section $\sigma_u(x) = [u,x,1]$ is nowhere-vanishing. On the other hand, there is no canonical trivialization, since the choice of $u$ is arbitrary.

We have a system of global trivializations $\Psi_u : S_m \to \mathbb{R}^3 \times \mathbb{R} \times \mathbb{C}$, $u \in E_1$,

$$\Psi_u([u+v,x,z]) = \left( \varphi_u(x), z \cdot \exp \left[ \frac{\hbar}{m} \left( \langle v \mid x - x_0 - \tau(x - x_0)u \rangle - \frac{r(x-x_0)}{2} \|v\|^2 \right) \right] \right) =$$

$$\left(y(x - x_0 - \tau(x - x_0)u), \tau(x - x_0), z \cdot \exp \left[ \frac{\hbar}{m} \langle v \mid x - x_0 - \tau(x - x_0)u \rangle - \frac{r(x-x_0)}{2} \|v\|^2 \right] \right).$$

Note first that $\Psi_u$ is well defined, i.e. does not depend on the representant $(u+v,x,z)$ in the class. Moreover, the change of trivializations in coordinates reads

$$(\Psi_{u+v} \circ \Psi_u^{-1})(y,t,z) = \left(y - g(v)t, t, e^{F_v(y,t)}z\right)$$

with

$$F_v(y,t) = \frac{im}{\hbar} \left( \sum_k y_k(v)y_k - \frac{i}{2} \sum_k g_k(v) \right).$$

According to Theorem [43] the differential operator $S^m_0$ on $S_m$ that corresponds to $S^m_0$ on the trivial 1-dimensional vector bundle $\mathbb{R}^3 \times \mathbb{R} \times \mathbb{C}$, does not depend on the trivialization $\Psi_u$, so it gives rise to a well-defined differential operator $S^m_0$ on $S_m$. Choosing a potential $U \in C^\infty_c(N)$ we can write the full Schrödinger operator as $S^m_U\psi = S^m_0\psi + U\psi$ acting on sections of $S_m$. It corresponds, via $\Psi_u$, to $S^m_U = S^m_0 + U$ with $U = U \circ \varphi_{(x_0,u)}^{-1}$. Moreover, the bundle isomorphisms $\Psi_{u+v} \circ \Psi_u^{-1}$ of $S^m_0 + U \circ \varphi_{(x_0,u)}^{-1}$ with $S^m_0 + U \circ \varphi_{(x_0,u+v)}^{-1}$. We can summarize these observations as follows.

**Theorem 5.1.** For any function $U$ on the newtonian space-time $N$ (potential) and every $m \in \mathbb{R}$ (mass) there is a well-defined differential operator $S^m_U$ (the Schrödinger operator), acting on sections of the Schrödinger bundle $S_m$. This operator corresponds, via the trivialization $\Psi_u$, to the differential operator

$$S^m_U\psi = \frac{i\hbar}{m} \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \sum_k \frac{\partial^2 \psi}{\partial y_k^2} - (U \circ \varphi_u^{-1})\psi$$

acting on complex functions $\psi(y,t)$ on $\mathbb{R}^3 \times \mathbb{R}$.

6 Relation to Newtonian mechanics

In addition to the vector bundle operation in the Schrödinger bundle $S_m$ we can define a norm in its fibers by

$$||(u,x,z)|| = |z|.$$ 

Consequently, we can consider the bundle $S_m$ as associated with a principal $U(1)$-bundle $B$. It is a subbundle of normalized vectors in $S_m$. By means of a group homomorphism

$$\mathbb{R} \rightarrow U(1) : r \mapsto \exp \left( \frac{ir}{\hbar} \right),$$

the bundle $B$ can be considered as the reduced $(\mathbb{R},+)$-bundle $Z_m$. It is an AV-bundle in terminology of [5]. An element of $Z_m$ is an equivalence class of triples $(u,x,r) \in E_1 \times N \times \mathbb{R}$, where two triples $(u,x,r)$ and $(u',x',r')$ are equivalent if $x = x'$ and

$$r' = r + m \left( \frac{r(x-x_0)}{2} ||u' - u||^2 - (x-x_0 - r(x-x_0)u, u' - u) \right).$$
The corresponding affine covector (see [5]) is an equivalence class of triples \((u, x, p) \in E_1 \times N \times V^*\). Two such triples \((u, x, p)\) and \((u', x', p')\) are equivalent if \(x = x'\) and

\[
(6.1) \quad p' = p + m \cdot d \left( \frac{\tau(x - x_0)}{2} \|u' - u\|^2 - \langle x - x_0 - \tau(x - x_0)u \mid u' - u \rangle \right).
\]

Since

\[
d \left( \frac{\tau(x - x_0)}{2} \|u' - u\|^2 - \langle x - x_0 - \tau(x - x_0)u \mid u' - u \rangle \right)(v) = \\
\frac{1}{2} \|u' - u\|^2 \tau(v) - \langle v - \tau(v)u \mid u' - u \rangle,
\]

we get

\[
p' = p + m \sigma(u', u),
\]

where

\[
\sigma(u', u)(v) = \langle u' - u \mid v - \tau(v)u' + u \rangle,
\]

which is precisely the relation used in [7] to define the affine phase space for a Newtonian particle of mass \(m\).

7 Concluding remarks

We have proposed an understanding of the Schrödinger operator on the Newtonian space-time in a way which makes it independent on the class of inertial observers. In this picture the Schrödinger operator acts not on functions on the space-time but on sections of certain one-dimensional complex vector bundle over space-time. This bundle, constructed from the data provided by all possible inertial observers, has no canonical trivialization, so these sections cannot be viewed as functions on the space-time. The presented framework is conceptually four-dimensional (the base is the traditional Newtonian space-time but the values of wave functions are not true numbers) and strictly related to the frame-independent formulation of analytical Newtonian mechanics that makes a bridge between the classical and quantum theory.

What we propose can look for the first sight like a formal modification of the five-dimensional approach, but it is in fact a real conceptual change. We insist on using the picture in which wave functions live on the standard space-time, since, in our opinion, finding appropriate geometric structures for physics is not just a pure mathematical game of formal manipulations. It might result in correcting our understanding of the physical content of the notions used in the theory. It is sufficient to mention the impact on physics of the Einstein’s proper geometric formulation of what is space, time, and matter. Moreover, our approach does not involve any \textit{ad hoc} or axiomatically introduced geometrical structures and is based only on the traditional understanding of the Schrödinger operator in a given reference frame. This makes it mathematically simple, demonstrative, and respecting the postulate of Occam’s Razor.

To finish, let us mention that there is a way to understand the Schrödinger operator as a true total Laplace-Beltrami operator – but defined for geometric structures going beyond the standard understanding of metric, de Rham derivative etc. The approach is again a bit similar to the idea of using a Lorenzian metric on the extended space-time as present in [3]. However, this task requires much more advanced geometrical techniques and we postpone it to a separate paper as well as a generalization to curved space-times.

References

[1] V. Bargmann: On unitary ray representations of continuous groups, \textit{Ann. Math.} \textbf{59} (1954), 1–46.

[2] S. Benenti: Fibrés affines canoniques et mécanique newtonienne \textit{Séminaire Sud-Rhodanien de Géometrie, Journées S.M.F, Lyon, 26-30 mai, 1986}. 
The Schrödinger operator in Newtonian space-time

[3] C. Duval, G. Burdet, H. P. Künzle, M. Perrin: Bargmann structures and Newton-Cartan theory, *Phys. Rev. D* **31**, 1841–1853.

[4] K. Grabowska, J. Grabowski and P. Urbański: Lie brackets on affine bundles, *Ann. Global Anal. Geom.* **24** (2003), 101–130.

[5] K. Grabowska, J. Grabowski and P. Urbański: AV-differential geometry: Poisson and Jacobi structures, *J. Geom. Phys.* **52** (2004) no. 4, 398–446.

[6] K. Grabowska, J. Grabowski and P. Urbański: AV-differential geometry: Euler-Lagrange equations, arXiv:math.DG/0604130.

[7] K. Grabowska and P. Urbański: AV-differential geometry and Newtonian mechanics, *Rep. Math. Phys.* **58** (2006), 21–40.

[8] B. Z. Iliev: Fibre bundle formulation of nonrelativistic quantum mechanics. I, II, III, *J. Phys. A* **34** (2001), no. 23, 4887–4918, 4919–4934, 4935–4950.

[9] J. Janyška and M. Modugno: Covariant Schrödinger operator, *J. Phys. A* **35** (2002), no. 40, 8407–8434.

[10] W. Pauli: *Handbuch der Physik* XXIV/1, pp. 83-272, Berlin 1933.

[11] G. Pidello: Una formulazione intrinseca della meccanica newtoniana, *Tesi di dottorato di Ricerca in Matematica*, Consorzio Interuniversitario Nord - Ovest, 1987/1988.

[12] W. M. Tulczyjew: Frame independence of analytical mechanics, *Att. Accad. Sci. Torino* **119** (1985), 273–279.

[13] W. M. Tulczyjew: Mécanique ondulatoire dans l’espace-temps newtonien, *C. R. Acad. Sc. Paris* **301** (1985), 419–421.

[14] W. M. Tulczyjew: An intrinsic formulation of nonrelativistic analytical mechanics and wave mechanics, *J. Geom. Phys.* **2** (1985), 93–105.

[15] W. M. Tulczyjew and P. Urbański: An affine framework for the dynamics of charged particles, *Att. Accad. Sci. Torino Suppl.* n. 2, **126** (1992), 257–265.

[16] W. M. Tulczyjew, P. Urbański, S. Zakrzewski: A pseudocategory of principal bundles, *Att. Accad. Sci. Torino* **122** (1988), 66–72.

[17] P. Urbański: Affine framework for analytical mechanics, in *Classical and Quantum Integrability*, Grabowski, J., Marmo, G., Urbański, P. (eds.), Banach Center Publications, vol. **59** (2003), 257–279.