The Geometry of 6D, $\mathcal{N} = (1, 0)$ Superspace and its Matter Couplings

Thesis presented in candidacy for the degree of Master of Science

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This thesis is dedicated to the study of the geometry of six-dimensional superspace, endowed with the minimal amount of supersymmetry. In the first part of it, we unfold the main geometrical features of such superspace by solving completely the Bianchi identities for the constrained superspace torsion, which allow us to determine the full six-dimensional derivate superalgebra. Next, the conformal structure of the supergeometry is considered. Specifically, it is shown that the conventional torsion constraints remain invariant under super-Weyl transformations generated by a real scalar superfield parameter.

In the second part of this work, the field content and superconformal matter couplings of the supergeometry are explored. The component field content of the Weyl multiplet is presented and the question of how this multiplet emerges in superspace is addressed. Finally, the constraints that conformal invariance imposes on some matter representations are analyzed.
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Chapter 1

Introduction

Since the birth of modern science, the concept of symmetry has been extremely fruitful in every aspect of physics. It is not an accident that, each time we uncover the underlying symmetries that characterize a certain physical system, we can further understand, in a much deeper way, such a system.

Two of the most beautiful realizations of the notion of symmetry, are the concepts of \textit{gauge symmetry} and \textit{supersymmetry}. Gauge symmetry is a remarkable symmetry simply because we can explain almost everything around us, at the fundamental level, in terms of such concept. Three of the four fundamental interactions in nature - the strong, weak, and electromagnetic interactions - can be understood, in a unified way, in terms of a gauge theory: the standard model of particle physics. As an outcome of this model, we know that gauge fields (bosons) mediate forces between particles described by matter fields (fermions).

Supersymmetry \cite{1,2,3}, on the other hand, is a bizarre symmetry linking completely different type of particles. It relates bosons (force carriers) and fermions (matter building blocks), in such a way that every bosonic degree of freedom possesses a fermionic \textit{superpartner}, and vice versa. Although it has not been tested experimentally\footnote{Supersymmetry is not an exact symmetry. The fact that we have not yet found any superpartner particle implies that supersymmetry must be broken at a energy scale above what we have been} supra-
persymmetry represents, without doubt, one of the cornerstones of modern theoretical physics. Its applications run from condensed matter and cosmology to particle phenomenology, superstring theory and mathematical physics, turning it into a central tool in the quest for our understanding of fundamental phenomena.

There are several reasons to pursue the study of supersymmetric theories. First of all, the supersymmetry algebra is the unique nontrivial spacetime extension of the Poincaré algebra consistent with four-dimensional quantum field theory, being the largest possible symmetry of the S-matrix [4]. Within the context of the minimal supersymmetric standard model, it provides a resolution of the hierarchy problem and the gauge coupling unification. In Cosmology, it also provides natural candidates for the particle spectrum of (cold) dark matter. Finally, its local version, supergravity [5, 6], has become an entire field of research mainly because it emerges as the low energy limit of superstring theory, playing a central role in the realization of the AdS/CFT correspondence [7].

There exist two approaches when dealing with supersymmetric theories. The most used one is the standard approach of component-fields, also known as “tensor calculus”. In this case, supersymmetry is not manifest. The second, less used route, is the superfield [8, 9] or superspace formulation, in which supersymmetry is manifest. Superspace emerges as a geometrical realization of supersymmetry where supersymmetry transformations are simply translations in this space which contains, in addition to the familiar bosonic coordinates, fermionic directions. It turns out that all important concepts of differential geometry can be extended to superspace, although the description of these spaces can be quite complicated (see for instance, the standard references in the subject [10, 11, 12]). Nevertheless, this allows for the definition and study of curved supermanifolds.

able to measure. Nevertheless, the current operation of the Large Hadron Collider (LHC), the most extraordinary particle collider ever made, holds the possibility of detecting evidence in favor of it. As of this writing, this has not happened.
In the present work, the geometry of six-dimensional, $\mathcal{N} = (1,0)$ superspace is considered. Recently, superconformal models in six dimensions have captured some interest. There are at least three good reasons to focus on (1,0) superconformal models. Firstly, these models are the maximal off-shell subgroup of $\mathcal{N} = (1,1)$ and (2,0) supersymmetric formulations. This fact allows, for instance, the enhancement of (1,0) supersymmetry to (2,0), through the addition of a collection of (1,0) superfields (hypermultiplets) \cite{13}. These (2,0) theories describe the low energy limit of multiple five-branes, for which no Lagrangian description is known\footnote{Recall that, while perturbative arguments appear to rule out local, unitary QFTs in six dimensions, string theory nevertheless predicts the existence of a fully interacting such theory related to the low energy dynamics of multiple coincident five-branes \cite{14}.}. Also within the context of string theory, the six-dimensional (1,0) theory appears as the target space for the covariant superstring on a K3 surface \cite{15}, as well as playing a central role in the study of the AdS$_7$/CFT$_6$ correspondence.

This thesis is an attempt to collect and further develop the most important results regarding the geometry of six-dimensional (1,0) superspace presented in \cite{16}, and it is organized as follows: In section 2 we will solve the supergravity Bianchi identities subject to a set of conventional torsion constraints. We will elucidate, by consistency of these identities, that the full superalgebra of covariant derivatives can be written in terms of two dimension-1 superfields. Consequently, all torsions and curvatures will be expressed in terms of such fields. In section 3 we will impose the invariance of the conventional constraints under super-Weyl transformations. In particular we will deduce the set of transformation rules that superfields and covariant derivatives must satisfy in order to realize the aforementioned conformal invariance. Section 4 is devoted to the study of the field content of the superspace theory studied in the previous sections. The Weyl (conformal) multiplet \cite{19} is reviewed and the question of how this multiplet emerges in superspace is considered. Finally, in section 5 we investigate the constraints that super-Weyl transformations impose on matter fields. The cases of the
abelian vector and tensor multiplets are studied with some detail. We conclude this work with some final comments in section 6. Notation and conventions are defined in appendix A and a supergeometry summary is presented in appendix B.
Chapter 2

Supergeometry

This chapter is dedicated to the study of the general structure of $\mathcal{N} = (1, 0)$, six-dimensional superspace\textsuperscript{1} suitable for a description of superfield supergravity. A superspace formulation of minimal supergravity corresponds to selecting out a specific subspace from the space of all possible supergeometries, by imposing torsion constraints. Such constraints allow us to solve the supergravity Bianchi identities that covariant derivatives must satisfy. Perhaps, the most important outcome arising from these Bianchi identities is the fact that supercurvature is, in the end, a redundant object. More precisely, after solving the Bianchi identities one is able to express the supercurvature entirely in terms of the supertosion.\textsuperscript{2} Following this reasoning, we derive in detail the new six-dimensional curved superspace geometry presented in \cite{16}, suitable for a superspace description of simple supergravity in six dimensions. In particular, we calculate the full six-dimensional curved superspace derivative algebra, through solving completely the Bianchi identities for the constrained superspace torsion.

\textsuperscript{1}Minimal supersymmetry in six dimensions (8 real supercharges) has the two different formulations, depending on the chirality of the chosen supergenerators. These are denoted by $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (0, 1)$. Both superalgebras are isomorphic.

\textsuperscript{2}In superspace literature, this fact is known as Dragon theorem. For a more detailed discussion see \cite{12}. 

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2.1 The setup

Let us consider a curved six-dimensional superspace $\mathcal{M}^6|8$, parametrized through the supercoordinates

$$z^M = (x^m, \theta^\mu_i), \ m = 0, \cdots, 3; 5, 6, \ \mu = 1, 2, 3, 4, \ i = \frac{1}{2},$$

with $m$ labeling bosonic coordinates $(x^m)$, and $\mu$ labeling fermionic ones $(\theta^\mu_i)$. The index $i$ is related to the R-symmetry of the theory, as indicated below. Further details of conventions and notation are given in appendix A.

Choosing the structure group to be $G = SO(5,1) \times SU(2)$, we expand the covariant derivative $D_A = (D_a, D_{\alpha i})$ as

$$D_A = E_A + \Omega_A + \Phi_A,$$

with $E_A$, $\Omega_A$ and $\Phi_A$ denoting the coframe, and the Lorentz and SU(2) connections, respectively. Each piece can be written in terms of the generators of the superalgebra

$$E_A = E_A^M \partial_M, \ \Omega_A = \frac{1}{2} \Omega_A^{bc} M_{bc}, \ \Phi_A = \Phi_A^{ij} J_{ij},$$

where $\partial_M = \partial/ \partial z^M$, $M_{bc} = -M_{cb}$ is the Lorentz generator and $J^{ij} = J^{ji}$ is the SU(2) R-symmetry generator. These are defined through their action on spinor derivatives as

$$[M_{ab}, D_{\gamma k}] = -\frac{1}{2} (\gamma_{ab})_{\gamma} \delta D_{\delta k}, \ [J^{ij}, D_{\gamma}^k] = \varepsilon^{k(i} D_{\gamma}^j).$$

From the spinor representation of the Lorentz generator, it also follow that

$$[M_{ab}, D_{c}] = 2 \eta_{c[a} D_{b]}.$$
The (anti-)commutation relations of covariant derivatives defines torsion $T_{AB}^C$, Lorentz curvature $R_{AB}^{cd}$, and SU(2) field strength $F_{AB}^{ij}$

$$\{\mathcal{D}_A, \mathcal{D}_B\} = T_{AB}^C \mathcal{D}_C + \frac{1}{2} R_{AB}^{cd} M_{cd} + F_{AB}^{ij} J_{ij}, \quad (2.6)$$

where we use $\{\mathcal{D}_A, \mathcal{D}_B\}$ to denote a graded commutator (anti-commutator if both A and B are fermionic indices, commutator otherwise). Relations (2.6) obey Bianchi identities

$$\{\mathcal{D}_A, \mathcal{D}_{[B}, \mathcal{D}_{C]}\} + (-1)^{\varepsilon_A (\varepsilon_B + \varepsilon_C)} \{\mathcal{D}_{B}, \mathcal{D}_{[C}, \mathcal{D}_{A]}\}$$

$$+ (-1)^{\varepsilon_C (\varepsilon_A + \varepsilon_B)} \{\mathcal{D}_{C}, \mathcal{D}_{[A}, \mathcal{D}_{B]}\} = 0, \quad (2.7)$$

where $\varepsilon_M$ stands for the Grassmann parity function: $\varepsilon_M = 0$ if $M = m$ (bosons) and $\varepsilon_M = 1$ if $M = \mu$ (fermions).

In order to solve the previous identities, we need to impose conventional constraints on the torsion. These fix completely the geometry in the sense that they isolate a specific subspace in the the space of all possible supergeometries. Such constraints are taken to be\footnote{These constraints are formally identical to those of five-dimensional conformal superspace supergravity of [17].}

$$T_{\alpha i\beta j}^c = 2\varepsilon_{ij} (\gamma^c)_{\alpha\beta} \quad \text{(dimension 0)}, \quad (2.8)$$

$$T_{\alpha i\beta j}^\gamma k = 0, \ T_{\alpha i\beta}^c = 0 \quad \text{(dimension} \ \frac{1}{2}), \quad (2.9)$$

$$T_{ab}^c = 0, \ T_{a\beta} (j^\beta k) = 0 \quad \text{(dimension 1)}. \quad (2.10)$$

Once the constraints (2.8)-(2.10) are introduced, Bianchi identities (2.7) can be solved. For this purpose, it is convenient to organize the study of the identities according to the increasing mass-dimension of them. This dimensionality depends on the index combination $(A, B, C)$ that we take in (2.7). The number of possibilities for such
combinations is four, and they give rise to the following set of identities: 2

\[ 0 = 2 [D_{\alpha}, \{D_{\beta}, D_{\gamma}\}] + [D_{\gamma}, \{D_{\alpha}, D_{\beta}\}] \quad ; \quad (A = \alpha, B = \beta, C = \gamma), \quad (2.11) \]

\[ 0 = 2 [D_{\alpha}, [D_{\beta}, D_{c}]] + [D_{c}, \{D_{\alpha}, D_{\beta}\}] \quad ; \quad (A = \alpha, B = \beta, C = c), \quad (2.12) \]

\[ 0 = [D_{\alpha}, [D_{b}, D_{c}]] + 2 [D_{[b}, [D_{c}, D_{a}]} \}} \quad ; \quad (A = \alpha, B = b, C = c), \quad (2.13) \]

\[ 0 = 2 [D_{[a}, [D_{b}, D_{c}]], [D_{c}, [D_{a}, D_{b}]]] \quad ; \quad (A = a, B = b, C = c). \quad (2.14) \]

Furthermore, within each of these four equation, there are four independent pieces: two parts proportional to the covariant derivatives (fermionics and bosonic, \( D_{\alpha i} \) and \( D_{a} \)), as well a two parts proportional to the Lorentz and SU(2) generators, \( M_{ab} \) and \( J_{ij} \), respectively. Table (2.1) below summarizes the splitting just described, together with the mass-dimension of each independent piece within the Bianchi identities.

| \( D_{\alpha i} \) | \( D_{a} \) | \( M_{ab} \) | \( J_{ij} \) |
|---------------|-------|-------|-------|
| [sss]         | 1     | 1/2   | 3/2   | 3/2   |
| [ssv]         | 3/2   | 1     | 2     | 2     |
| [svv]         | 2     | 3/2   | 5/2   | 5/2   |
| [vvv]         | 5/2   | 2     | 3     | 3     |

Table 2.1: Summary of Bianchi identities we study in this section. Here, “s” stands for a spin index, and “v” for a vector one. In this way, for instance, the first row give us the dimensionality of each part within the Bianchi identity (2.11), the second row indicates the dimension of each piece in (2.12), and so forth.

In the next sections, we proceed to solve in detail the Bianchi identities up to dimension-2. The outcome of this procedure will be the full algebra of covariant derivatives which characterizes the curved supergeometry. We will express curvatures and field-strengths completely in terms of the torsion, and we will find the constraints that the supergravity fields entering in the algebra must satisfy.

\(^5\)Here and through this work, we adopt the usual notation for composite indices \( \alpha := \alpha i \).


2.2 Dimension-1 Bianchi identities

Dimension-1 identities arise by taking the part proportional to the spinorial derivative inside the $[sss]$-identity \( (2.11) \), and the piece proportional to the vector derivative within the $[ssv]$-identity \( (2.12) \), as indicated in table \( (2.1) \). As a first step, let us focus on the latter. This is given by

\[
0 = 2i (\gamma^b)_{\alpha \gamma} T_{\beta j a}^\gamma i + 2i (\gamma^b)_{\beta \gamma} T_{\alpha i a}^\gamma j - R_{\alpha i \beta j a}^b .
\] (2.15)

From here, it is clear that we can solve for the dimension-1 curvature in terms of the dimension-1 torsion

\[
R_{\alpha i \beta j a}^b = 2i (\gamma^b)_{\alpha \gamma} T_{\beta j a}^\gamma i + 2i (\gamma^b)_{\beta \gamma} T_{\alpha i a}^\gamma j .
\] (2.16)

Moreover, demanding the antisymmetry of the curvature on its Lorentz indices, that is imposing $R_{\alpha i \beta j}^{(ab)} = 0$, we get

\[
T^{\alpha \beta j \gamma k} (\gamma^b)_{\gamma \delta} + T^{\alpha \delta j \gamma k} (\gamma^b)_{\gamma \beta} = 0 .
\] (2.17)

The above constraint on the dimension-1 torsion is particularly strong, since it implies the general form that such torsion must have. Expanding out the torsion into irreducible pieces\(^6\)

\[
T^{j \gamma k} a \gamma = A_a \varepsilon^{j k} \delta \gamma + B_b \varepsilon^{j k} (\gamma_{ab} \beta) \gamma + C_{abc} \varepsilon^{j k} (\gamma^b \gamma)_{\beta} \gamma + M_{abc} \varepsilon^{j k} (\gamma^c \gamma)_{\beta} \gamma
\] (2.18)

and plugging this general expression back in \( (2.17) \), one finds that, necessarily, the superfields $A_a$ and $B_b$ must vanish, as well as the tensor superfield $M_{abc}$. This means that the torsion and curvature tensors defined in \( (2.6) \) can be expressed entirely in terms of the dimension-1 superfields $N_{abc}$ and $C_{aij}$, and their covariant derivatives. It also follows that these superfields must have the symmetries

\[
N_{abc} = N_{[abc]} ; \quad C_{aij} = C_{a(ij)} .
\] (2.19)

\(^6\)Note that, given the torsion expansion declared here, this theory will not contain Lorentz zero forms.
Therefore, we find that the dimension-1 torsion is defined by

$$T_{\gamma k a}^\delta l \mathbf{D}_{\delta l} := [\mathbf{D}_{\gamma k}, \mathbf{D}_a] = -C_{k l}^{c b} (\gamma_{a b})_\gamma^\delta \mathbf{D}_d^l + N_{a b c} (\gamma^{b c})_\gamma^\delta \mathbf{D}_{d k} \ ,$$

(2.20)

and because of the (spin) traceless of the gamma 2-forms in the above commutator, we indeed have a stronger dimension-1 conventional constraint

$$T_{a \beta j}^\delta \beta k = 0 \ .$$

(2.21)

Following our analysis, the dimension-1/2 covariant derivatives obey an anti-commutation relation which can be expanded over the superfields $C_{aij}$ and $N_{abc}$. The most general form consistent with the dimension-0 and 1/2 torsions is

$$\{ \mathbf{D}_{ai}, \mathbf{D}_{bj} \} = 2i \varepsilon_{ij} (\gamma^a)_{\alpha \beta} \mathbf{D}_a + i a (\gamma^{ab})_{\alpha \beta} C_{aij} M_{bc} + i b \varepsilon_{ij} (\gamma_a)_{\alpha \beta} N_{abc} M_{bc} + i c \varepsilon_{ij} (\gamma_a)_{\alpha \beta} \tilde{N}_{abc} M_{bc} + i d \varepsilon_{ij} (\gamma_a)_{\alpha \beta} C_{a k l} J_{kl} + i e (\gamma^{ab})_{\alpha \beta} N_{abc} J_{ij} \ ,$$

(2.22)

with $a, b, c, d$ and $e$ some coefficient that must be fixed by the consistency of the dimension-1 Bianchi identities. None of these coefficient can be absorbed in the normalization of the fields since this would change the coefficient in the dimension-1 torsion.

Using the expansion (2.22) in the $[ssv]$-identity (2.12) and taking the dimension-1 piece (the part proportional to the vector derivative) gives

$$0 = \left[ -2i a (\gamma^c_{c a b})_{\alpha \beta} C_{aij} + 4i (\gamma_c^{ab})_{\alpha \beta} C_{aij} \right] \mathbf{D}_b - \left[ 2i (b N_c^{a b} + c \tilde{N}_c^{ab}) \varepsilon_{ij} (\gamma_a)_{\alpha \beta} - 8i \varepsilon_{ij} N_c^{a b} (\gamma_a)_{\alpha \beta} \right] \mathbf{D}_b \ ,$$

(2.23)

where the two lines must vanish separately. On the one hand, from the terms involving the $C$ field, it follows that $a = 2$. On the other hand, splitting $N$ into self-dual and anti-self dual parts, the second line in (2.23) implies two equations: $b + c - 4 = 0$ and $b - c - 4 = 0$\footnote{Note that, in principle, it is possible that $N$ have a definite duality property which would eliminate one of these equations. Nevertheless, we consider here the most general case in which $N$ does not obey any duality constraint.}, which determine the values $b = 4$ and $c = 0$. The coefficients $d$ and $e$ are fixed by the normalizations of the fields. Hence, the most general form consistent with the dimension-0 and 1/2 torsions is

$$\{ \mathbf{D}_{ai}, \mathbf{D}_{bj} \} = 2i \varepsilon_{ij} (\gamma^a)_{\alpha \beta} \mathbf{D}_a + 4i (\gamma^{ab})_{\alpha \beta} C_{aij} M_{bc} + 8i \varepsilon_{ij} (\gamma_a)_{\alpha \beta} N_{abc} M_{bc} + i e (\gamma^{ab})_{\alpha \beta} N_{abc} J_{ij} \ .$$

(2.24)
follow from the dimension-1 piece inside the \( [sss] \)-identity. Plugging the expansion (2.22) into (2.11), and taking the part proportional to the spinorial derivative, we get terms of the type \( CD \) and \( ND \), as in (2.23). For simplicity, we analyze each of these terms separately. Beginning with \( CD \), we find

\[
0 = 2i \varepsilon_{ij} [D_{\alpha \beta}, D_{\gamma k}] - \frac{i a}{2} C_{aij} (\gamma^{abc})_{\alpha \beta} (\gamma_{bc})_{\gamma} \delta D_{\delta k} + i d \varepsilon_{ij} C_{akl} (\gamma^a)_{\alpha \beta} D_{\gamma l} + \text{c.p.} ,
\]

where "c.p." stands for "cyclic permutation" of indices. Here, the first term can be re-written using the dimension-1 torsion (2.20) and the identity (A.33) as

\[
2i \varepsilon_{ij} [D_{\alpha \beta}, D_{\gamma k}] = 4i \varepsilon_{ij} C_{akl} \varepsilon_{\alpha \beta \gamma \delta} (\tilde{\gamma}^{a})_{\alpha \beta} \delta D_{\delta l} + 2i \varepsilon_{ij} C_{akl} (\gamma^a)_{\alpha \beta} D_{\gamma l} .
\]

In this last expression, the first term vanishes under cyclic permutation since

\[
\varepsilon_{ij} \varepsilon_{\alpha \beta \gamma \delta} \psi_k + \varepsilon_{jk} \varepsilon_{\beta \gamma \alpha \delta} \psi_i + \varepsilon_{ki} \varepsilon_{\gamma \alpha \beta \delta} \psi_j = \varepsilon_{\alpha \beta \gamma \delta} \varepsilon_{[ij] k} \equiv 0 ,
\]

for any \( \psi \). Now, the second term in (2.24) can be simplified by using (A.35) and cyclic reordering to

\[
- \frac{i a}{2} C_{aij} (\gamma^{abc})_{\alpha \beta} (\gamma_{bc})_{\gamma} \delta D_{\delta k} + \text{c.p.} = -4i a (\gamma^a)_{\alpha \beta} C_{akl} [D_{\gamma l}] + \text{c.p.}.
\]

From this is clear that \( d = -6 \) (recall that \( a = 2 \)). Next, we consider the terms of the type \( ND \) inside the dimension-1 part of the \( [sss] \)-identity. This gives

\[
0 = 2i \varepsilon_{ij} [D_{\alpha \beta}, D_{\gamma k}] - \frac{i b}{2} \varepsilon_{ij} N_{abc} (\gamma^a)_{\alpha \beta} (\gamma^{bc})_{\gamma} \delta D_{\delta k} - i e N_{abc} (\gamma^{abc})_{\alpha \beta} \varepsilon_{k(i} D_{\gamma j]} + \text{c.p.}
\]

\[
= -i (2 + \frac{b}{2}) \varepsilon_{ij} N_{abc} (\gamma^a)_{\alpha \beta} (\gamma^{bc})_{\gamma} \delta D_{\delta k} - i e N_{abc} (\gamma^{abc})_{\alpha \beta} \varepsilon_{k(i} D_{\gamma j]} + \text{c.p.}
\]
This time, the second term rearranges under cyclic permutation as
\[-ieN_{abc}(\gamma^{abc})_{\alpha\beta}\varepsilon_k(D_{\gamma}) + \text{c.p.} = ie\varepsilon_{ij}N_{abc}(\gamma^{abc})_{\gamma[a}D_{\beta]k} + \text{c.p.} \quad (2.30)\]

Plugging the identity (A.47) and using the relation (2.26) we obtain
\[2 + \frac{b}{2} + \frac{3e}{2} = 0 . \quad (2.31)\]

Therefore, substituting our previous result \(b = 4\), we get the value \(e = -\frac{8}{3}\), which fixes all the coefficients in the dimension-1 anti-commutator (2.22). We conclude that
\[\{D_{\alpha i}, D_{\beta j}\} = 2i\varepsilon_{ij}(\gamma^a)_{\alpha\beta}D_a + 2i(\gamma^{abc})_{\alpha\beta}C_{aij}M_{bc} + 4i\varepsilon_{ij}(\gamma_a)_{\alpha\beta}N^{abc}M_{bc} - 6i\varepsilon_{ij}(\gamma_a)_{\alpha\beta}C_{a}^{\,kl}J_{kl} - \frac{8i}{3}(\gamma^{abc})_{\alpha\beta}N_{abc}J_{ij} . \quad (2.32)\]

This calculation completes the analysis of the dimension-1 identities.

### 2.3 Dimension-\(\frac{3}{2}\) Bianchi identities

There are four pieces of the Bianchi identities with dimension-\(\frac{3}{2}\), as we can read off from the table (2.1). None of these is trivially fulfilled. In this section, we will analyze these four parts separately. From this analysis, we will be able to express the dimension-\(\frac{3}{2}\) curvature, torsion and isospin field strength in terms of irreducible pieces. We will also show that Bianchi identities impose constraints on the supergravity fields \(C\) and \(N\), and we will find such constraints.

**Dimension-\(\frac{3}{2}\) curvature**  At dimension-\(\frac{3}{2}\) level, we can write the Lorentz curvature in terms of the torsion. In order to do this, we take the part proportional to the vector derivative \(D_a\) of the \([svv]\)-identity (2.13). This gives
\[R_{\gamma k[c\alpha]b} = iT_{ca}^{\delta} k(\gamma_b)_{\delta\gamma} . \quad (2.33)\]

Adding to this the signed permutation \((cab + bca - abc)\) and using the antisymmetry of \(R\) on its Lorentz indices, we derive that
\[R_{\gamma kca}^{\delta} = -iT_{ab}^{\delta} k(\gamma_c)_{\delta\gamma} + 2iT_{ca}^{\delta} k(\gamma_b)_{\delta\gamma} , \quad (2.34)\]
and thus we have an equation for the curvature in terms of the torsion.

**Dimension-$\frac{3}{2}$ isospin field strength** From the part proportional to the spinorial derivative of the $[ssv]$-identity (2.12), we can get a general expression for the isospin field strength. Although this expression will depend explicitly on the torsion and curvature, it will be enough to write the field strength in terms of irreducibles. The $D_{\gamma k}$-part of (2.12) is given by

\[ 0 = -2i \varepsilon_{ij}(\gamma^d)_{\alpha\beta} T_{dc} \gamma^k_k + (D_{\alpha i} T_{\beta j c} \gamma^k_k + D_{\beta j} T_{\alpha i c} \gamma^k_k) + (\varepsilon_{ki} R_{\beta j c a} \gamma + \varepsilon_{kj} R_{\alpha i c} \gamma) \]

\[ - (\delta_{\alpha}^\gamma F_{\beta j c i k} + \delta_{\beta}^\gamma F_{\alpha i c j k}) \]  \hspace{1cm} (2.35)

Here, we use bold font to indicate that the tensor in question is known in terms of the fields $C$ and $N$. In this case

\[ T_{\alpha i c} \gamma^k_k := C_{ik}^d (\gamma_{cd})_\alpha \gamma - \varepsilon_{ik} N_{cab} (\gamma^{ab})_\alpha \gamma \] \hspace{1cm} (2.36)

Now, taking the trace $\alpha = \gamma$ over (2.35) and noting that $T_{\gamma i c} \gamma^k = 0$, gives

\[ F_{\alpha j c i k} + 4 F_{\alpha i c j k} = 2i \varepsilon_{ij}(\gamma^d)_{\alpha\beta} T_{cd} \gamma^\beta_i + D_{\beta j} T_{\alpha i c} \gamma^\beta_k + \varepsilon_{ki} R_{\beta j c a} \gamma \] \hspace{1cm} (2.37)

We can solve for $F$ by adding to the previous equation the same expression with a factor of $-\frac{1}{4}$, getting

\[ F_{\alpha i c j k} = \frac{2i}{3} \varepsilon_{ij}(\gamma^d)_{\alpha\beta} T_{cd} \gamma^\beta_i + \frac{4}{15} (D_{\beta j} T_{\alpha i c} \gamma^\beta_k - \frac{1}{2} D_{\beta i} T_{\alpha j c} \gamma^\beta_k) \]

\[ + \frac{4}{15} (\varepsilon_{ki} R_{\beta j c a} \gamma - \frac{1}{4} \varepsilon_{kj} R_{\beta i c a} \gamma) \] \hspace{1cm} (2.38)

The resulting expression for the field strength must be symmetric in its isospin indices $(jk)$. Imposing such symmetry we obtain

\[ 0 = 2 (\gamma^d)_{\alpha\beta} T_{cd} \gamma^\beta_i - \frac{1}{4} (\gamma^{ab})_{\alpha\beta} T_{\alpha i c} \gamma^\beta_i - T_{\alpha i c} \gamma \] \hspace{1cm} (2.39)

where we have defined

\[ T_{\alpha i c} := -2i D_{\beta k} T_{\alpha i c} \gamma^k_k + \frac{1}{2} D_{\beta i} T_{\alpha i k} \gamma^k_k = 2i (\gamma_{cd})_{\alpha\beta} D_{\beta j} C_{ij}^d - i (\gamma^{ab})_{\alpha\beta} D_{\beta i} N_{abc} \] \hspace{1cm} (2.40)
Contracting (2.39) with $(\tilde{\gamma}^c)^\gamma_\alpha$, we can isolate the term

$$(\gamma^{ab})_{\beta}^\gamma T_{ab}^{\beta} i = \frac{1}{3}(\tilde{\gamma}^c)^\gamma_\beta T_{\alpha i c} = -\frac{10i}{3}(\tilde{\gamma}_c)^\gamma_\beta D_{\beta j}^\gamma C_{c i j} - \frac{i}{3}(\tilde{\gamma}^{abc})^\gamma_\beta D_{\beta i} N_{abc}.$$ (2.41)

Plugging this back in (2.39), we find

$$(\gamma^d)_{\alpha\beta} T_{cd}^{\beta} i = -\frac{2}{3} \left[ \delta^d_\alpha \delta_c^\beta + \frac{1}{12}(\tilde{\gamma}_c)^\gamma_\delta T_{\gamma i d} \right].$$ (2.42)

With this we can simplify the trace of the Lorentz curvature in Eq. (2.34) to

$$R_{\beta i c}^{\gamma \alpha} = \frac{7i}{3} T_{\alpha i c} + \frac{5i}{18} (\tilde{\gamma}_c)^\gamma_\delta T_{\delta i d}.$$ (2.43)

Therefore, plugging the previous trace into (2.38), we solve for the dimension-$\frac{3}{2}$ isospin field strength

$$F_{\alpha i c j k} = \frac{2i}{3} \varepsilon_{i(j} (\gamma^d)_{\alpha\beta} T_{cd}^{\beta} k) - \frac{4i}{15} \varepsilon_{i(j} R_{\beta k)c^\alpha}^{\delta} \left[ D_{\beta j}^\delta T_{\alpha i c} - \frac{1}{4} D_{\beta i} T_{\alpha (j c) k} \right].$$ (2.44)

**Dimension-$\frac{3}{2}$ torsion**  Next, we focus on the torsion. For this, we go back to the identity (2.35). Performing the contraction with $(\gamma_{ab})^\gamma_\gamma$ isolates the term $(\gamma_{ab})^\gamma_\alpha F_{\beta j c i k}$ which must be symmetric in $(i k)$. Enforcing this condition gives

$$0 = (\gamma_{ab})^\gamma_\beta (D_{\alpha i} T_{\beta j c}^{\gamma i} + D_{\beta j} T_{\alpha i c}^{\gamma i}) - i (\gamma_{ab}^d)_{\alpha\gamma} T_{dc}^{\gamma j} + \frac{i}{2} \varepsilon_{a b}^{d e f} (\gamma_{c f g})_{\alpha\delta} T_{d e}^{\gamma j} + \frac{i}{2} \varepsilon_{a b}^{d e f} (\gamma_{f a}^j)_{\alpha\gamma} T_{b d}^{\gamma j} - 2i (\gamma_{c a}^d)_{\alpha\delta} T_{b j d}^{\gamma j} + 2i (\gamma_{c a}^d)_{\alpha\delta} T_{a b}^{\gamma j}. \quad (2.45)$$

Contracting again with $(\tilde{\gamma}^c)^\delta_\alpha$, we obtain

$$0 = -20i T_{ab}^{\delta} j - 8i (\gamma_{a}^c)^\delta_\gamma T_{b c}^{\gamma j} + (\tilde{\gamma}^c)^\delta_\alpha (\gamma_{ab})^\gamma_\beta (D_{\alpha i} T_{\beta j c}^{\gamma i} + D_{\beta j} T_{\alpha i c}^{\gamma i}) . \quad (2.46)$$

On the other hand, contracting (2.42) with $(\tilde{\gamma}_b)^\gamma_\alpha$ and anti-symmetrizing the resulting expression gives

$$(\gamma_{[b}^d)_{\alpha}^{\gamma_\beta} T_{c d]^{\beta} j} = T_{bc}^{\gamma} j + \frac{5}{9} (\tilde{\gamma}_{[b}^i)^\gamma_\alpha T_{a o j c] + \frac{1}{15} (\tilde{\gamma}_{bc}^d)^\gamma_\delta T_{d j}.$$ (2.47)
which can be plugged back in (2.46) to obtain
\[
0 = -28i T_{a j}^\delta - \frac{4i}{9} (\tilde{\gamma}_{[a})^\delta \beta T_{j b]} - \frac{4i}{9} (\tilde{\gamma}_{ab})^\delta \beta T_{j c}
\]
\[+ (\tilde{\gamma}^c)^\delta \alpha (\gamma_{ab})^\gamma (D_{\alpha i} T_{\beta j c} \gamma_i D_{\gamma j} T_{\alpha i c} \gamma_i), \quad (2.48)
\]

Here, let us compute each of the four last terms independently. The second term is proportional to
\[
(\tilde{\gamma}_{[a})^\delta \beta T_{j b]} = 2i (\tilde{\gamma}_{abc})^\delta \alpha D_{\alpha i} C_{ij} - 4i (\tilde{\gamma}_{[a})^\delta \alpha D_{\alpha i} C_{bij}
\]
\[+ i (\tilde{\gamma}_{[a})^\delta \alpha D_{\alpha j} N_{b|cd} - 4i (\tilde{\gamma}_{[a})^\delta \alpha D_{\alpha j} N_{abc}, \quad (2.49)
\]
while the third is given by
\[
(\tilde{\gamma}_{ab})^\delta \beta C_{|ij} = -6i (\tilde{\gamma}_{abc})^\delta \alpha C_{ij} + 16i (\tilde{\gamma}_{[a})^\delta \alpha D_{\alpha j} N_{b|cd}
\]
\[+ 4i (\tilde{\gamma}_{[a})^\delta \alpha D_{\alpha j} N_{b|cd} + 2i (\tilde{\gamma}_{[a})^\delta \alpha D_{\alpha j} N_{abc} + 6i (\tilde{\gamma}_{[a})^\delta \alpha D_{\alpha j} N_{abc} \cdot (2.50)
\]
The last two terms in (2.48) expand out to give
\[
(\tilde{\gamma}^c)^\delta \alpha (\gamma_{ab})^\gamma (D_{\alpha i} T_{\beta j c} \gamma_i D_{\gamma j} T_{\alpha i c} \gamma_i) = 8 (\tilde{\gamma}_{[a})^\delta \alpha D_{\alpha i} C_{bij} - 8 (\tilde{\gamma}^c)^\delta \alpha D_{\alpha j} N_{abc}
\]
\[+ 2 (\tilde{\gamma}^c)^\delta \alpha D_{\alpha j} N_{(cde}
\]
\[= 8 (\tilde{\gamma}_{[a})^\delta \alpha D_{\alpha i} C_{bij} - 8 (\tilde{\gamma}^c)^\delta \alpha D_{\alpha j} N_{abc} - 12 (\tilde{\gamma}^c)^\delta \alpha D_{\alpha j} (N_{abc} + N_{abc})
\]
\[+ 6 (\tilde{\gamma}_{[a})^\delta \alpha D_{\alpha j} (N_{b|cd} + N_{b|cd}). \quad (2.51)
\]
Putting all these results together, that is, replacing (2.49), (2.50) and (2.51) in (2.48), we can finally solve for the dimension-$\frac{3}{2}$ isospin field strength:
\[
T_{ab} \gamma_k = -\frac{2i}{9} (\tilde{\gamma}_{[a})^\gamma \beta D_{[b}^\gamma \beta C_{ijkl} - \frac{2i}{9} (\tilde{\gamma}_{abc})^\gamma \beta D_{[b}^\gamma \beta C_{kl} + i (\tilde{\gamma}^c)^\gamma \beta D_{[b}^\gamma \beta N_{abc}
\]
\[+ \frac{i}{3} (\tilde{\gamma}^c)^\gamma \beta D_{[b}^\gamma \beta \bar{N}_{abc} - \frac{5i}{42} (\tilde{\gamma}_{[a})^\gamma \beta D_{[b}^\gamma \beta N_{b|cd} - \frac{3i}{14} (\tilde{\gamma}_{[a})^\gamma \beta D_{[b}^\gamma \beta \bar{N}_{b|cd}). \quad (2.52)
\]
At this point, we have studied two of the four dimension-$\frac{3}{2}$ identities; the $[svv]$ and $[ssv]$ $v$ pieces. The remaining two parts will give rise to the constraints on the supergravity fields $C$ and $N$. Recall that these superfields define the dimension-1 torsion according to (2.20).
Dimension-1 torsion constraints  As we mentioned at the beginning of this section, the dimension-$\frac{3}{2}$ identities impose constraints on the supergravity fields $C$ and $N$. In what follows, we will show that these constraints are given by

\begin{align}
\mathcal{D}_\gamma(kC_{\alpha i \beta j}) &= -\frac{1}{3} \varepsilon_{\alpha \beta \gamma \delta} \mathcal{D}_{\epsilon(k)} C^{\delta \epsilon}_{i j}, \\
\mathcal{D}_{(\alpha i} N_{\beta)\gamma} &= -\frac{1}{2} \mathcal{D}_{\gamma i} N_{\alpha \beta},
\end{align}

and

\begin{align}
\mathcal{D}_{(\alpha i} N_{\beta)\gamma} &= -\frac{1}{4} \mathcal{D}_{(\alpha} C_{\beta)\gamma i}.
\end{align}

The last equation implies

\begin{align}
\frac{2}{3} (\gamma^{a b c} \tilde{\gamma}_{d})_{\alpha}^{\beta} \mathcal{D}_{\beta i} N_{a b c} = \tau_{d a}^{\gamma} (5, 1) \mathcal{D}_{\gamma i} C_{c i j} &= -4 (\tilde{\gamma}_{d})_{\beta}^{\gamma} \mathcal{D}_{\beta j} N_{\gamma \alpha}.
\end{align}

with the tensor $\tau$ defined in Eq. (2.60) below.

In order to derive the constraint (2.53), we use the part proportional to the Lorentz generator $M$ within the $[s s s]$-identity (2.11). This has the form

\begin{align}
0 &= 2 i \varepsilon_{ij} (\gamma^{c})_{\alpha \beta} \left[ \frac{1}{2} R_{\gamma k e}^{a b} + 2 \mathcal{D}_{\gamma k} N_{e a b} \right] M_{a b} - 2 i (\mathcal{D}_{\gamma k} C_{c i j}) (\gamma^{a b c})_{\alpha \beta} M_{a b} + c.p.
\end{align}

Completely symmetrizing all three isospin indices implies

\begin{align}
0 &= \mathcal{D}_\gamma(kC_{c i j})(\gamma^{a b c})_{\alpha \beta} + 2 \mathcal{D}_{(\alpha}(kC_{c i j})(\gamma^{a b c})_{\beta)\gamma}.
\end{align}

Contracting this last equation with $(\tilde{\gamma}_{d})_{\beta}^{\gamma}$ we get

\begin{align}
\left[ 5 \delta_{\alpha}^{c} \delta_{\alpha}^{\beta} + (\gamma_{d}^{e})_{\alpha \beta} \right] \mathcal{D}_{\beta(k} C_{c i j)} = 0.
\end{align}

In this last expression, the tensor structure

\begin{align}
\tau_{a a a}^{b \beta}(5, 1) := 5 \delta_{a}^{b} \delta_{\alpha}^{\beta} + (\gamma_{a}^{b})_{\alpha \beta}
\end{align}
is not invertible\footnote{In general, the multiplication of these tensors is given by}

This implies that the totally symmetric term in \( (2.59) \) is proportional to a gamma matrix

\[
\mathcal{D}_\gamma(kC_{\alpha\beta ij}) = (\gamma_a)\gamma^dC^\delta_{ij} \quad \text{with} \quad C^\delta_{ij} := -\frac{1}{6}(\gamma^b)^\delta\beta\mathcal{D}_\beta(kC_{bij}) .
\] (2.62)

This last expression is equivalent to \( (2.53) \).

In order to derive the second constraint \( (5.11) \), it is enough to study the part proportional to the SU(2) generator inside the \( \{sss\} \)-identity \( (2.11) \). We get the following expression

\[
0 = -2i\varepsilon_{ij}(\gamma^d)_{\alpha\beta}(F_{\gamma k}^d_{mn} - 3\mathcal{D}_\gamma kC_{d mn}) + 16i\mathcal{D}_\gamma kN_{\alpha\beta}\delta^{(m \alpha)}_i\delta^{(m \beta)}_j + \text{c.p.} \quad (2.63)
\]

Contracting with \( \delta^i_m\delta^j_n \) gives

\[
0 = 48i\mathcal{D}_\gamma kN_{\alpha\beta} + 48i\mathcal{D}_{(ak)N_{\beta}\gamma} - 4i(\gamma^d)_{\gamma(\alpha}[F_{\beta)j}^d_{k} + 3\mathcal{D}_\gamma jC_{djk}] \quad (2.64)
\]

Using the identity \( (A.44) \), it follows that the first and second terms in this equation are related through

\[
(\gamma^d)_{\gamma(\alpha}(\gamma^{bc})_{\beta)}^\delta\mathcal{D}_{\delta k}N_{dbc} = 2\mathcal{D}_\gamma kN_{\alpha\beta} - 2\mathcal{D}_{(ak)N_{\beta}\gamma} .
\] (2.65)

The third term in \( (2.64) \) contains the (isospin) trace of the field strength. Such a term can be written in terms of derivatives of the superfields \( C \) and \( N \) by taking the trace of Eq. \( (2.38) \) and using the trace of the Lorentz curvature \( (2.43) \). This gives

\[
-4i(\gamma^d)_{\gamma(\alpha}F_{\beta)j}d^j_k = 12i\mathcal{D}_{(\alpha}^jC_{\beta)\gamma j}k + 8i(\gamma^d)_{\gamma(\alpha}(\gamma^{bc})_{\beta)}^\delta\mathcal{D}_{\delta k}N_{dbc} .
\] (2.66)

Replacing \( (2.65) \) and \( (2.66) \) in \( (2.64) \) we get

\[
0 = -3\mathcal{D}_{(\alpha}^jC_{\beta)\gamma ij} + 4\mathcal{D}_{(ai}N_{\beta)\gamma} + 8\mathcal{D}_\gamma kN_{\alpha\beta} .
\] (2.67)
We can now simplify this result by symmetrizing on \((\beta\gamma)\), obtaining

\[
0 = -3 \mathcal{D}_{(\beta^j C_\gamma)^{\alpha}}_{\alpha}\ j + 4 \mathcal{D}_{\alpha i N_{\beta\gamma}} + 20 \mathcal{D}_{(\beta i N_\gamma)^{\alpha}}.
\]  

(2.68)

Finally, manipulating indices and subtracting from (2.67) we get

\[
\mathcal{D}_{(ai N_\beta)^{\gamma}} = -\frac{1}{2} \mathcal{D}_{\gamma i N_{\alpha\beta}},
\]

(2.69)

which is the constraint (5.11). Plugging this expression back into (2.67) we find the constraint (2.55).

**Irreducible decomposition** Once the dimension-1 torsion constraints have been obtained, we may expand the derivative of the fields in their Lorentz- and isospin-irreducible components, in the following way

\[
\mathcal{D}_{\gamma k} C_{\alpha i j} =: C_{\alpha \gamma k i j} + (\gamma_a)_{\gamma \delta} C_{\delta i j} + \varepsilon_{k(i} (\gamma_a)_{\gamma \delta} C_{\delta j)}; \quad (2.70)
\]

\[
\mathcal{D}_{\gamma k} N_{\alpha\beta} =: N_{\gamma k \alpha\beta} + \tilde{N}_{\gamma k \alpha\beta}; \quad (2.71)
\]

\[
\mathcal{D}_{\gamma k} N^{\alpha\beta} =: N_{\gamma k}^{\alpha\beta} + \delta^{(\alpha}_{\gamma} N^{\beta)_{k}}. \quad (2.72)
\]

Under this decomposition, the content of the constraints is given by

\[
C_{\alpha \gamma k i j} = 0, \quad (2.73)
\]

\[
C_{\delta i j}^{\gamma} = -\frac{1}{6} (\tilde{\gamma}^b)_{\delta} \mathcal{D}_{(k C_{b i j})}; \quad (2.74)
\]

\[
C_{\alpha \beta j} = \frac{1}{9} \tau_{\alpha \beta}^{(5, 1)} \mathcal{D}_{\gamma} C_{\alpha i j}; \quad (2.75)
\]

\[
C_{\gamma k} = -\frac{1}{9} \mathcal{D}_{\delta l} C^{\beta}_{\delta l k}. \quad (2.76)
\]

and

\[
N_{\gamma k \alpha\beta} = 0, \quad (2.77)
\]

\[
\tilde{N}_{\gamma k \alpha\beta} = \frac{1}{2} \mathcal{D}_{(\alpha}^{j} C_{\beta)^{\gamma}} i j = -\frac{3}{4} (\gamma_a)_{\gamma (a} C_{\alpha k)}; \quad (2.78)
\]

\[
N_{\gamma k}^{\alpha\beta} = \mathcal{D}_{\gamma k} N^{\alpha\beta} - \frac{2}{3} \delta^{(\alpha}_{\gamma} \mathcal{D}_{\delta k N^{\beta)\delta}}, \quad (2.79)
\]

\[
N_{\alpha i} = \frac{2}{3} \mathcal{D}_{\beta} i N^{\beta\alpha}. \quad (2.80)
\]
Let us focus now on the irreducible decomposition of the dimension-\(3/2\) curvature, torsion and field strength. From (2.34), we note that the curvature is most conveniently expressed in terms of the torsion, so that we do not consider its decomposition. For the torsion (2.52), we expand into its irreducible pieces:

\[
T_{ab}^{\gamma k} = \mathfrak{T}_{ab}^{\gamma k} + (\tilde{\gamma}[a) \gamma^\delta \mathfrak{T}_{b]^{\delta k} + (\gamma_{ab})_{\delta}^{\gamma} \mathfrak{T}_{\delta k} ,
\]

under which we find

\[
\mathfrak{T}_{ab}^{\gamma k} = i 21 (\tilde{\gamma}[a)^{cd} \gamma^\delta D_\delta^{k} N^{(-)}_{b|cd} - \frac{2i}{3} (\tilde{\gamma}[a)^{cd} \gamma^\delta D_\delta^{k} N^{(+)}_{b|cd} + \frac{6i}{5} (\tilde{\gamma})^\delta \gamma D_\delta^{k} N^{(+)}_{abc} ,
\]

\[
\mathfrak{T}_{a\beta}^{\gamma} = - \frac{i}{9} \tau^e_{a \beta} (5, 1) D_x^e \cdot C_{cij} - \frac{3i}{5} (\tilde{\gamma}_a) \gamma^\delta D_\delta^{jk} N^{\gamma}\beta = - i C_{a \beta} + \frac{i}{3} (\tilde{\gamma}_a) \gamma^\delta N_{\gamma \delta \beta} ,
\]

\[
\mathfrak{T}_{\delta k} = i 9 D_{\gamma} C^\gamma^{\delta k} + \frac{i}{15} D_\gamma^{k} N^{\gamma} = - i C^{\delta k} + \frac{i}{6} N^{\delta k} .
\]

Here, the first term in \(\mathfrak{T}_{ab}^{\gamma k}\) vanishes by (A.39). Furthermore, using (A.36) and (A.37) this torsion simplifies to

\[
\mathfrak{T}_{ab}^{\gamma k} = - \frac{2i}{5} (\gamma_{ab})_{\beta} (\gamma^\delta D_\delta^{k} N^{\gamma})^\beta + \frac{3i}{5} (\gamma_{ab})_{\beta}^{[\gamma} D_\delta^{k} N^{\delta]}^\beta = - \frac{i}{2} (\gamma_{ab})_{\beta}^{\delta} N_{\delta \beta}^\gamma .
\]

It is easily verified that this combination is \(\gamma\)-traceless due to the tracelessness of \(N\). Additionally, using the constraint relations (5.11) and (2.55), we obtain

\[
\mathfrak{T}_{a \beta}^{\gamma} = - \frac{7i}{4} C_{a \beta} .
\]

In order to finish the analysis of the dimension-\(3/2\) Bianchi identities, it remains to decompose the field strength (2.44). Expanding

\[
F_{a \gamma}^{ij} = \mathfrak{F}_{a \gamma}^{ij} + (\gamma_{a})_{\gamma}^{\delta} \mathfrak{F}^{\delta}_{ij} + \delta_{k}^{(i} \mathfrak{F}_{a \gamma}^{j)} + \delta_{k}^{(i} (\gamma_{a})_{\gamma}^{\delta} \mathfrak{F}^{\delta_{i}j} )
\]

we may resolve the field strength into its irreducible components, by projections of the equation\(\textsuperscript{10}\)

\[
F_{a(kcij} = F_{a(kcij} - \frac{2}{3} \varepsilon_{k(i}F_{a|c)j}^{t} .
\]

\(\textsuperscript{10}\)This expression follows simply from symmetries arguments.
This gives
\[ F_{\alpha i} = \frac{5}{9} D_{\beta j} C_{\beta \alpha i j} + \frac{2}{3} D_{\beta i} N_{\beta \alpha} = -5 C_{\alpha i} + \frac{5}{3} N_{\alpha i} . \] (2.92)

At this point the only irreducible tensors which have not been simplified are $\mathcal{F}_{\alpha i}$ and $\mathcal{F}_{\alpha i}$. These combinations involve constraints on the self-dual part of the superfield $N$ which, as we will see in chapter 3, is covariant under conformal transformation (and therefore, the superspace version of the Weyl tensor is constructed from it).

### 2.4 Dimension-2 Bianchi identities

In this section we study the dimension-2 Bianchi identities. As indicated in table (2.1), there are four pieces with this dimension: The parts proportional to the Lorentz and SU(2) generators within the $[ssv]$-indentity, the part proportional to the spinorial covariant derivative inside the $[svv]$-identity, and the piece proportional to the vector derivative appearing in the $[vvv]$-identity. The latter is identically fulfilled, giving rise to what is known as “Second Bianchi Identity” for the Riemann tensor
\[ R_{a[bcd]} = 0 . \] (2.94)

Let us proceed with the study of the first three aforementioned identities. The part proportional to the Lorentz generator $M_{ab}$ within the $[ssv]$-indentity (2.12) is given by
\[ 0 = i \varepsilon_{ij} (\gamma^a)_{\alpha \beta} \left[ R_{ca}^{\;\;bd} + 4 D_{c} N_{a}^{\;\;bd} \right] + 2 i (\gamma^{abd})_{\alpha \beta} D_{c} C_{ai j} + D_{(a} R_{b)_{c}^{\;\;bd}}^{\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;} \\
+ 4 i T_{(a}^{\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;} \left[ (\gamma^{abd})_{\beta \gamma} C_{aj}^{\;\;k} + 2 \varepsilon_{j k} (\gamma_{a})_{\beta \gamma} N^{abdf} \right] . \] (2.95)
The above expression is symmetric in composite indices \((\alpha\beta)\). Such a symmetry can be implemented through simultaneous symmetry or antisymmetry of both, spin and isospin indices. Let us first analyze the double antisymmetric case. We can isolate the Riemann tensor multiplying \((2.95)\) by \(\frac{\iota}{8}\epsilon^{ij}(\tilde{\gamma}_e)_{\alpha\beta}\). This gives
\[
R_{ce}^{\quad bd} = -4 \mathcal{D}_c N_e^{\quad bd} - \frac{1}{8}(\tilde{\gamma}_e)_{\alpha\beta} \mathcal{D}_{\alpha i} R_{\beta c}^{\quad i bd} \nonumber \\
- \frac{1}{2}(\tilde{\gamma}_e)_{\alpha\beta} T_{j c}^{\quad \gamma k} [(\gamma^{abcd})_{\beta\gamma} C_{ajk} + 2\epsilon_{jk}(\gamma^a)_{\beta\gamma} N_{a bd}] ,
\]
and plugging the curvature and torsion back in \((2.96)\), we obtain a first expression for the Riemann tensor
\[
R_{ce}^{\quad bd} = -4 \mathcal{D}_c N_e^{\quad bd} + \frac{1}{8}(\tilde{\gamma}_e)_{\delta\gamma} \mathcal{D}_{\alpha i} T^{bd\delta i} - \frac{1}{8}\eta_{ce} \mathcal{D}_{\alpha i} T^{bd\alpha i} - \frac{1}{4}\mathcal{D}_{\alpha i} (\gamma^{[d}_{\quad e} \delta^{b]}_{\quad c})_{\alpha i} + \frac{1}{4}\mathcal{D}_{\alpha i} \delta^{[d}_{\quad e} T^{b]}_{\quad c} \alpha i ,
\]
Symmetries of the curvature tensor \((2.97)\) should be fulfilled. On the one hand, clearly \(R_{ce}^{(bd)} = 0\) identically. On the other hand, demanding \(R_{(ce)}^{\quad bd} = 0\) we get
\[
\mathcal{D}_{(c} N_{e)}^{\quad bd} = -\frac{1}{32}\eta_{ce} \mathcal{D}_{\alpha i} T^{bd\alpha i} - \frac{1}{16}\mathcal{D}_{\alpha i} (\gamma^{[d}_{\quad e} \delta^{b]}_{\quad c})_{\alpha i} + \frac{1}{16}\mathcal{D}_{\alpha i} \delta^{[d}_{\quad e} T^{b]}_{\quad c} \alpha i ,
\]
equation which can be contracted with the metric tensor \(\eta^{ce}\) to obtain the divergence of the superfield \(N\)
\[
\mathcal{D}_c N^{\quad cde} = \frac{1}{16}\mathcal{D}_{\alpha i} (\gamma^{[d}_{\quad e} \delta^{b]}_{\quad c})_{\alpha i} - \frac{1}{4}\mathcal{D}_{\alpha i} T^{bd\alpha i} ,
\]
From the Riemann tensor \((2.97)\), we can obtain the Ricci tensor
\[
R_{cb} = \frac{3}{4}\mathcal{D}_{\alpha i} T^{\quad c b \alpha i} - \frac{1}{4}\mathcal{D}_{\alpha i} (\gamma^{[d}_{\quad c} \delta^{b]}_{\quad d})^{\alpha i} + 8 C^{(j k}_{\quad (c) j k} - 8 \eta_{cb} C^{ajk} C_{ajk} + 16 N_{ad(b} N_{c) ad} .
\]
Here, we note that requiring the symmetry of the Ricci tensor \(R_{[ab]} = 0\) is equivalent to \(\mathcal{D}_{\alpha i} T_{ab \alpha i} = 0\). This reduce the Riemann and the Ricci tensors to
\[
R_{ce}^{\quad bd} = -4 \mathcal{D}_{(c} N_{e)}^{\quad bd} + \frac{1}{8}(\tilde{\gamma}_e)_{\delta\gamma} \mathcal{D}_{\alpha i} T^{bd\delta i} - \frac{1}{4}\mathcal{D}_{\alpha i} (\gamma^{[b}_{\quad e} \delta^{d]}_{\quad c})_{\alpha i} + \frac{1}{4}\mathcal{D}_{\alpha i} \delta^{[b}_{\quad e} T^{d]}_{\quad c} \alpha i \\
- \frac{1}{2}\mathrm{tr}(\gamma^{abcd}(\tilde{\gamma}_e)_{\alpha\beta}) C_{ajk} C^{fjk} + 16 N_{ace} N^{\quad abd} ,
\]
24
\[ R_{cb} = -\frac{1}{4} D_{\alpha i} (\gamma_{d(c)}^i \delta^a T_b) d \delta^i + 8 C_{(b}^j C_{c)j}^k - 8 \eta_{cb} C^{ajk} C_{ajk} + 16 N_{ad(b} N_{c)ad} . \] (2.102)

The Ricci scalar arises directly from (2.100)

\[ R = \frac{1}{4} (\gamma_{ab})^i D_{\alpha i} T^{abdi} - 40 C_{aij} C^{aij} + 16 N_{abc} N^{abc} . \] (2.103)

The above curvature quantities depend on a combination which involves the (spin) derivative of the dimension-\( \frac{3}{2} \) torsion, \( \Delta_{ab}^{cd} := (\gamma_{ab})_i^\alpha D_{\alpha i} C^{abdi} \), and further symmetries and contractions of it. It is, in general, not direct to express such combination in terms of irreducible pieces. For this reason, as we will see, it will be simpler to compute the Riemann tensor from the \( [svv] \)-identity. Nevertheless, it is possible at this moment to write down \( \Delta_{a(bc)}^d \) and \( \Delta_{ab}^{ab} \) in terms of irreducible parts, and therefore the Ricci tensor (2.102), together with the curvature scalar (2.103) are given by

\[ R_{ab} = \frac{i}{8} \eta_{ab} \left[ 10 D_{\alpha i} C^{ai} - \frac{5}{3} D_{\alpha i} N^{ai} + 64 i C_{dij} C_{dij} \right] + 8 C_{aij} C_{aij} + 16 N_{acd} N_{bcd} , \quad (2.104) \]

\[ R = \frac{15 i}{2} D_{\alpha i} C^{ai} - 40 C_{aij} C^{aij} - \frac{5 i}{4} D_{\alpha i} N^{ai} + 16 N_{abc} N^{abc} . \] (2.105)

This completes the analysis of the double antisymmetric part of (2.95). From the double symmetric side, we can isolate the \( DC \) term by contracting (2.95) with the 3-form \( (\tilde{\gamma}^c_{bd})^{\alpha \beta} \). This gives

\[ 0 = 2 i \text{tr}(\tilde{\gamma}^c_{bd}) \gamma^{abk} D_{c} C_{aij} + (\tilde{\gamma}^c_{bd})^{\alpha \beta} D_{a(i} R_{b)j} C^{cd} \]

\[ + 4 i (\tilde{\gamma}^c_{bd})^{\alpha \beta} T_{a(i} \gamma^{k} \left[ (\gamma_{abdi})_{\beta \gamma} C_{aj}^k + 2 \varepsilon_{ijk} (\gamma_{ij}^a)_{\beta \gamma} N_{abc} \right] , \]

which gives the divergence of the \( C \)-field

\[ D_a C^{aij} = \frac{i}{12} (\gamma_a)^{\alpha \beta} D_{a(i} C^{ajk)} = \frac{3}{4} D_a (i C^{aj}) . \] (2.106)

For the sake of completeness, we can also compute the divergence of the \( N \)-field. From the symmetries of the Ricci tensor, we argued that \( D_{\alpha i} T_{ab}^{\alpha i} = 0 \). Combining this constraint with Eq. (2.99) one obtain

\[ D^c N_{abc} = \frac{i}{8} (\gamma_{ab})^\beta \left[ D_{ai} C^{\beta i} + \frac{5}{12} D_{ai} N^{\beta i} \right] . \] (2.107)
This completes the analysis of the piece proportional to the Lorentz generator $M$ within the $[ssv]$-identity.

Next, we focus on the part proportional to the SU(2) generator, $J_{ij}$, in (2.12). This is

$$
0 = 2i \varepsilon_{ij}(\gamma^a)_{\alpha\beta} \left[ F^{im}_{ca} - 3 \mathcal{D}_c C^{lm}_{a} \right] - 16i \delta_i^l \delta_j^m \mathcal{D}_c N_{\alpha\beta} + 2 \mathcal{D}(\alpha(iF_{\beta}j))c^{lm} + 4i \mathcal{T}_c(\alpha(i\gamma^k)(\gamma^a)_{\alpha\beta} C^{alm} + 8 \delta_j^l \delta^m_k N_{\beta\gamma}) .
$$

(2.109)

Naturally, in the same way that the part proportional to the Lorentz generation, equation (2.109) exhibits the symmetry $\alpha\beta$, which may be realized through a double symmetry or antisymmetry of spin and isospin indices. In order to obtain the SU(2)-field strength, we proceed to focus on the double antisymmetry of (2.109). Contracting with $\frac{i}{16} \varepsilon_{ij}(\tilde{\gamma})^{\alpha\beta}$ the second term vanish due the $(ij)$-symmetry and we can isolate $F_{ab}^{ij}$. The resulting expression is not easily expressed explicitly in terms of fundamental superfields. Therefore, as well as for the Riemann tensor, we will see that it will be more manageable to compute the SU(2) field strength from the $[svv]$-identity. Nevertheless, at this point, the antisymmetry $F_{(ab)}^{ij} = 0$ is required, obtaining the divergence of the $C$-field

$$
\mathcal{D}_a C^{alm} = \frac{i}{4} (\tilde{\gamma}_a)^{\alpha\beta} \mathcal{D}^{(lD_{\beta}jC^{alm}))j} = \frac{qi}{4} \mathcal{D}^{(iC^{\alpha j})} .
$$

(2.110)

Then, comparing (2.107) and (2.110) we see that the $C$ superfield is divergence-free, that is $\mathcal{D}_a C^{aij} = 0 = \mathcal{D}^{(iC^{\alpha j})}$.

Finally, the double symmetric combination of spin and isospin indices in (2.109) can be considered by multiplying by the three form $(\tilde{\gamma}^{cde})^{\alpha\beta}$ and contracting isospin indices. This gives

$$
0 = \mathcal{D}^{c} N^{(-)}_{abc} + 8 N^{(+cd} |a N^{(-)}_{b]cd} + \frac{11i}{32} (\gamma_{ab})_{\beta}^{\alpha} \mathcal{D}_{\alpha i} C^{\beta i} - \frac{5i}{96} (\gamma_{ab})_{\beta}^{\alpha} \mathcal{D}_{\alpha i} N^{\beta i} .
$$

(2.111)
This equation can be combined with (2.108) in order to obtain some expressions for the divergence of the superfield $N$.

\[
(\gamma_{ab})^\beta_\alpha D_{\alpha i} C^\beta_i = -\frac{16i}{3} D^c \tilde{N}_{abc},
\]

(2.112)

\[
(\gamma_{ab})^\beta_\alpha D_{\alpha i} \beta_N^{\beta i} = -\frac{96i}{5} \left[ D^c N_{abc} - \frac{2}{3} D^c \tilde{N}_{abc} \right],
\]

(2.113)

\[
16 N^{(+)}_{cd}[a N^{(-)}_{b]cd} = -3 D^c N^{(+)}_{abc} + 5 D^c N^{(-)}_{abc}.
\]

(2.114)

where, $\tilde{N}$ denotes the 3-form dual to $N$. This concludes the study of the dimension-2 part of the Bianchi identity (2.12).

Finally, the last part to be considered in the dimension-2 analysis is the part proportional to the spinorial derivative arising from the $[svv]$-identity (2.13). This is given by

\[
0 = D_{\alpha i} T_{ab}^{\beta j} + \frac{1}{4} \delta^j_i (\gamma_{cd})^\beta_\alpha R_{ab}^{cd} + \delta^\beta_\alpha F_{ab}^{ij} + 2 D_{[a} T_{b]\alpha i}^{\beta j} - 2 (\gamma_{[a}^\gamma T_{b]\alpha i} \gamma^j C_{\alpha i}] + 2 (\gamma_{[a}^\gamma T_{b]\alpha i \gamma^j N_{b]cd}. (2.115)
\]

From here, the Riemann tensor and the SU(2) field strength will be computed.

**The Riemann tensor** is contained in the second term of (2.115). This may be isolated by multiplying the whole expression by $(\gamma^{cf})_\beta^\alpha$ and taking the trace $i = j$.

This yields

\[
R_{ab}^{cd} = -\frac{i}{8} (\gamma_{cd})^\beta_\alpha (\gamma_{ab})^\alpha \beta D_{\alpha i} D_i^{\delta} N^{\beta \gamma} - \varepsilon_{ab}^{cdmn} D^p \left[ N_{mp} - \frac{5}{2} \tilde{N}_{mp} \right] + 8 D_p \delta^{[c}_{[a} N^{d]}_{b]} + 6 \delta^{[c}_{[a} \tilde{N}^{d]}_{b]} + \frac{1}{2} \delta^{[c}_{[a} \delta^{d]}_{b]} D_{\alpha i} C_{\alpha i}] + 8 D_{[a} N^{cd}_{b]}
\]

\[
- 32 N^{[c}_{[a} N^{d]}_{b]} + 8 \delta^{[c}_{[a} C_{bij}^{d]} + 4 \delta^{[c}_{[a} \delta^{d]}_{b]} C_{aij} C^{aij},
\]

(2.116)

\[^{11}\text{Note that it is not possible having a term like } \varepsilon^{cdmnq} N_{amn} N_{bpq} \text{ within the Riemann tensor, because such a term is symmetric in (ab). This argument also makes clear why in 6D, necessarily } \tilde{N}_{abc} N^{abc} = 0.\]
so that the only reducible term is the first one. This can be computed by demanding the exchange symmetry \( R_{ab,cd} = R_{cd,ab} \). From such symmetry, it follows that

\[
(\gamma^{cd})_{\beta}^{\alpha}(\gamma_{ab})_{\gamma} \{ \mathcal{D}_{\alpha i}, \mathcal{D}_{\delta}^{i} \} N^{\beta\gamma} = -128i \mathcal{D}_{\rho} \delta_{[a}^{[c} N_{b]}^{d]} d^{dp} + 96i \mathcal{D}_{\rho} \delta_{[a}^{[c} \tilde{N}_{b]}^{d]} d^{dp} - 64i \mathcal{D}_{[a} N_{b]}^{cd} \\
+ 64i \mathcal{D}^{[c} N_{ab]}^{d]}
\]

(2.117)

With (2.117) in hand, it is simple to compute the reducible term in the Riemann tensor (2.116)

\[
(\gamma^{cd})_{\beta}^{\alpha}(\gamma_{ab})_{\gamma} \mathcal{D}_{\alpha i} \mathcal{D}_{\delta}^{i} N^{\beta\gamma} = - (\gamma^{cd})_{\beta}^{\alpha}(\gamma_{ab})_{\gamma} N_{\alpha\delta}^{\beta\gamma} - 64i \mathcal{D}_{\rho} \delta_{[a}^{[c} N_{b]}^{d]} d^{dp} + 48i \mathcal{D}_{\rho} \delta_{[a}^{[c} \tilde{N}_{b]}^{d]} d^{dp} \\
- 32i \mathcal{D}_{[a} N_{b]}^{cd} + 32i \mathcal{D}^{[c} N_{ab]}^{d]} + 8i \varepsilon_{ab}^{cdmn} d^{dp} \left[ N_{mnp} - \frac{2}{3} \tilde{N}_{mnp} \right] \\
+ \frac{8i}{3} \varepsilon_{ab}^{cdmn} \left[ d^{p} N_{mnp}^{(+)} - 8 N_{npq}^{(-)} N_{mmp}^{(+)} \right] - \delta_{[a}^{[c} d^{pj} \mathcal{D}_{\alpha i} N_{\alpha i}^{\alpha} \right),
\]

(2.118)

where \( N_{\alpha\beta}^{\gamma\delta} \) stands for the Weyl tensor, defined as

\[
N_{\alpha\beta}^{\gamma\delta} = \mathcal{D}_{(\alpha}^{i} N_{\beta)}^{j} \delta_{i}^{\gamma} \delta_{j}^{\delta} - \frac{1}{3} \delta_{(\alpha}^{i} \mathcal{D}_{(\sigma}^{j} N_{\beta)}^{i} \delta_{\sigma)}^{\delta}.
\]

(2.119)

Therefore, replacing (2.118) in (2.116), the Riemann tensor for the supergeometry is obtained

\[
R_{ab,cd} = \frac{i}{8} (\gamma^{cd})_{\beta}^{\alpha}(\gamma_{ab})_{\gamma} N_{\alpha\delta}^{\beta\gamma} + 2 \varepsilon_{ab}^{cdmn} d^{dp} \left[ N_{mnp}^{(+)} - \frac{4}{3} N_{mmp}^{(-)} \right] \\
+ 4 \mathcal{D}_{[a} N_{b]}^{cd} + 4 \mathcal{D}^{[c} N_{ab]}^{d]} - 32 N_{e[a}^{[c} N_{d]}^{b]} e + 8 \delta_{[a}^{[c} C_{bij}^{d]} d^{ij} \\
+ \frac{i}{2} \delta_{[a}^{[c} d^{bf} \mathcal{D}_{\alpha i} C_{\alpha i}^{bf} + 8i C_{nij} C_{\alpha ij}^{n} - \frac{1}{6} \mathcal{D}_{\alpha i} N_{\alpha i}^{\alpha} \right),
\]

(2.120)

where we have used (2.114) in order to write \( N^{(-)} N^{(+)} \) in terms of derivatives of the selfdual and antiselfdual part of \( N \). As a consistency check, straightforward calculation shows that further contraction of the Riemann tensor (2.120) give rise to the Ricci tensor (2.104) and Ricci scalar (2.105).
The SU(2) field strength can be extracted from (2.115) by tracing \(\alpha = \beta\) and rearranging isospin indices

\[
F_{ab}^{ij} = -\frac{1}{4} D_\alpha T_{ab}^{\alpha j} + \frac{1}{2} (\gamma_{\alpha[a})^\beta T_{b]j} i^\beta k \mathcal{C}^{cjk} + \frac{1}{2} (\gamma^{cd})_{\beta}^\alpha T_{a[c} i^\beta j^k N_{b]cd} \, .
\]  

(2.121)

Demanding \(F_{ab}^{[ij]} = 0\), we get \(D_\alpha T_{ab}^{\alpha i} = 0\), in agreement with previous analysis. It also follows, from the symmetric piece in isospin indices, that

\[
F_{ab}^{ij} = -\frac{1}{4} D_\alpha T_{ab}^{\alpha j} - 2 C_{[a} k^{(i} C_{b]}^{j)} k + 8 N_{abc} C^{cij} \, .
\]  

(2.122)

where the first term is not irreducible. Taking the derivative of the dimension \(\frac{3}{2}\)-torsion, this term can expressed as follows

\[
D_\alpha (T_{ab}^{\alpha j}) = -\frac{5i}{3} N_{ab}^{ij} + \frac{11i}{72} C_{ab}^{ij} - \frac{10i}{9} \tilde{C}_{ab}^{ij} - \frac{4i}{9} D_{[a} C_{b]j}^{ij} - \frac{416}{9} C_{[a}^{k(i} C_{b]}^{j)k} + \frac{272}{9} N_{abd}^{(+)} C^{dij} - \frac{512}{9} N_{abd}^{(-)} C^{dij} \, ,
\]  

(2.123)

where we have defined irreducible superfields

\[
N_{ab}^{ij} := D^{(i} \varphi_{ab} N^{j)} \, ,
\]  

(2.124)

\[
C_{ab}^{ij} := D^k \varphi_{ab} D_k C^{cij} \, ,
\]  

(2.125)

\[
\tilde{C}_{ab}^{ij} := D_{[a}^{k(i} C_{b]}^{j)k} = \frac{1}{4} [D_\alpha^{k}, D_\beta^{[i}(\varphi_{ab})^{C^{j)k]} \, .
\]  

(2.126)

Replacing (2.123) into (2.122), we conclude that the SU(2) field strength of the supergeometry will be given by

\[
F_{ab}^{ij} = \frac{5i}{12} N_{ab}^{ij} - \frac{11i}{288} C_{ab}^{ij} + \frac{5i}{18} \tilde{C}_{ab}^{ij} + \frac{10i}{9} D_{[a} C_{b]j}^{ij} + \frac{86i}{9} C_{[a}^{k(i} C_{b]}^{j)k} + \frac{4i}{9} N_{abd}^{(+)} C^{dij} + \frac{200}{9} N_{abd}^{(-)} C^{dij} \, .
\]  

(2.127)

This result concludes that analysis of the Bianchi identities. Summarizing, we have computed completely the geometrical information necessary for the description of simple six-dimensional superspace supergravity. Specifically, we have fixed the dimension-1 and \(-\frac{3}{2}\) (anti)commutators defining the derivative superalgebra, we have expressed the dimension-\(\frac{3}{2}\) curvature, field strength and torsion in terms of irreducible parts,
and we have computed all the relevant curvature quantities which characterize the supergeometry. A summary containing the most relevant results of this chapter can be found in appendix (B). It is important to point out that we have studied the superspace from an off-shell point of view, in the sense that we have isolated its geometry from the dynamics of the supergravity fields entering in the superalgebra.
Chapter 3

Conformal structure

The description of matter-coupled supergravity theories turns out to be rather complicated. In this respect, superconformal methods represent a simpler approach to the study of such matter-coupled systems. These methods exploit the fact that, among the spacetime symmetries, conformal symmetry is the maximal symmetry of a non-trivial field theory [20]. The underlying idea is to formulate a gauge theory of the superconformal algebra (the supersymmetric extension of the conformal algebra). Such theory contain extra fields which are then eliminated by imposing curvature constraints or by gauge fixing the extra symmetries. The result is a gauge theory of the Poincaré supersymmetry algebra where the initial extra symmetries are not visible.

In this chapter, we study the conformal structure of the superspace geometry described in chapter (2). We do so, by following a different route. Instead of considering the superconformal group as the structure group of the theory, we impose the conformal invariance of the conventional constraints (2.8)-(2.10). In particular, we will fix the super-Weyl transformation rules that superfields ($C_{aij}$ and $N_{abc}$) and covariant derivatives ($D_\alpha$ and $D_a$) must obey in order to preserve such set of constraints.

1Superconformal methods to study conformal 4$D$, $\mathcal{N} = 1, 2$ superspace were used in [18]. There, the construction relies on considering the full superconformal group as the structure group of theory. Along this line, one might attempt to construct our 6$D$ superspace by de-gauging a conformal supergeometry.
3.1 Super-Weyl transformations

The super-Weyl (sW) transformations act on the spinorial covariant derivative as

$$\delta D_{\alpha i} = \sigma D_{\alpha i} + a (D_{\beta j} \sigma) M_{\alpha \beta} + b (D_{\alpha } \sigma) J_{ij},$$  \hspace{1cm} (3.1)

where $\sigma = \sigma(z)$ is, \textit{a priori}, an arbitrary scalar superfield and $a, b$ some coefficients that can be determined by requiring the preservation of the the conventional constraints under the above transformation \hbox{(3.1)}. Let us consider the transformation of the dimension-1 commutator

$$\delta \{D_{\alpha i}, D_{\beta j}\} = 2 \{\delta D_{\alpha}, D_{\beta}\}$$  \hspace{1cm} (3.2)

Preservation of the algebra means that the above expression must be equal to

$$\delta \{D_{\alpha i}, D_{\beta j}\} = 2 \{\delta D_{\alpha}, D_{\beta}\} = 2 \delta R_{\alpha \beta}^{\gamma \delta} M_{\gamma \delta} + \delta F_{\alpha \beta}^{ijkl} J_{ijkl}. $$  \hspace{1cm} (3.3)

Therefore, independent pieces in \hbox{(3.2)} and \hbox{(3.3)} should cancel each other. In particular, matching the terms proportional to the spinorial covariant derivative we get

$$0 = -2i \epsilon_{ij} (\gamma^c)_{\alpha \beta} \delta D_{c} + b \epsilon_{ij} (D_{[\alpha} M_{\beta]} + \left( \frac{a}{2} + 2 \right) (D_{\alpha} \sigma) D_{\beta j})$$  \hspace{1cm} (3.4)

The previous equation has the symmetry $(\alpha \beta)$, that can be implemented through simultaneous symmetry or antisymmetry of spin and isospin indices. Taking the symmetric part $(ij)$, we obtain the following condition on the coefficient that parametrize the sW transformation

$$2 - \frac{3a}{2} - b = 0. $$  \hspace{1cm} (3.5)

Taking now the antisymmetric combination, multiplying \hbox{(3.4)} by $\epsilon^{ij}$ we get

$$0 = 4i (\gamma^c)_{\alpha \beta} \delta D_{c} + \left[ \frac{5a}{2} + 3b + 2 \right] (D_{[\alpha} \sigma) D_{\beta]i}. $$  \hspace{1cm} (3.6)
But the transformation of the vector derivative may have a part, besides the homogeneous term, proportional to spinorial covariant derivative, $\delta D_c \propto D_{\gamma k}$. Such a term should have the structure

$$\delta D_c = 2\sigma D_c + i\alpha(\tilde{\gamma}_c)^{\beta\gamma}(D_{\beta k}\sigma)D_{\gamma k} + \cdots ,$$

(3.7)

with $\alpha$ some factor to be determined. Then, plugging (3.7) into (3.6)

$$2 + \frac{5\alpha}{2} + 3b = 16\alpha .$$

(3.8)

Additionally, in order to elucidate the values of the parameters $a$ and $b$, we can compute the preservation of the dimension 1/2 conventional constraint, $T_{\alpha i b}^c = 0$. This is equivalent to setting to zero the part proportional to the vector covariant derivative within the dimension 3/2 commutator transformation, which is given by

$$\delta[D_{\alpha i}, D_b]|_{D_c} = a (D_{\beta i})[M_{\alpha}^{\beta}, D_b] + 2(D_{\alpha i}\sigma)D_b + i\alpha(\tilde{\gamma}_b)^{\beta\gamma}[D_{\alpha i}, (D_{\beta k}\sigma)D_{\gamma k}]|_{D_c}
= -\frac{a}{2}(\gamma_{bc})^{\alpha} (D_{\beta i}\sigma)D_c + 2(D_{\alpha i}\sigma)D_b + 2\alpha(\gamma_{ec}\tilde{\gamma}_b)^{\alpha} (D_{\beta i}\sigma)D_c .$$

(3.9)

Here, the $\gamma\tilde{\gamma}$ product of the last term decompose as the metric tensor (arising from the symmetric part that satisfy the Clifford algebra) and a 2-form (antisymmetric part), and thus the last term in (3.9) combines to the first two. Then

$$\delta[D_{\alpha i}, D_b]|_{D_c} = - \left(\frac{a}{2} + 2\alpha\right)(\gamma_{bc})^{\alpha} (D_{\beta i}\sigma)D_c + 2(1 - \alpha)(D_{\alpha i}\sigma)D_b .$$

(3.10)

Demanding that the above expression vanish, we get

$$\alpha = 1 \text{ and } a = -4 .$$

(3.11)

Therefore, the set of equations (3.5), (3.8) and (3.11) is consistent for

$$b = 8 .$$

(3.12)

At this point, we have got the sW transformation of the spinorial covariant derivative

$$\delta D_{\alpha i} = \sigma D_{\alpha i} - 4(D_{\beta j}\sigma)M_{\alpha}^{\beta} + 8(D_{\alpha i}\sigma)J_{ij} .$$

(3.13)
Let us now focus on the transformations rules for the superfields $C_{aij}$ and $N_{abc}$. For this, we notice that for the values of $a$ and $b$ we just got, the third an fourth term in (3.2) vanish. Thus, requiring preservation of the algebra of (spinorial) covariant derivatives, that is, equating (3.2) and (3.3), yields

\[ 0 = 2i \varepsilon_{ij} (\gamma_c)^{\alpha\beta} \delta \mathcal{D}_c + \frac{1}{2} \delta R_{ai\beta}^{\,cd} M_{cd} + \delta F_{ai\beta j}^{\,kl} J_{kl} - 2\sigma \{ \mathcal{D}_{ai}, \mathcal{D}_{\beta j} \} \]

\[- b \varepsilon_{ij} (\mathcal{D}_{\alpha k} \sigma) \mathcal{D}_{\beta j} - \left[ a (\mathcal{D}_{\beta j} \mathcal{D}_{\gamma i} \sigma) M_{\alpha}^{\gamma} + b (\mathcal{D}_{\beta j} \mathcal{D}_{\alpha}^{\,k} \sigma) J_{ik} + (\alpha \leftrightarrow \beta) \right]. \quad (3.14)\]

Again, the symmetry of the latter equation can be realized in two different ways. Taking the terms symmetric in both, spin and isospin indices gives linearly independent terms proportional to the Lorentz generator $M$ and the SU(2) generator $J$. The part proportional to $M$ gives rise to

\[ 0 = 2i (\gamma_{abc})^{\alpha\beta} (\delta C_{aij} - 2\sigma C_{aij}) M_{bc} - \left[ a (\gamma_{bc})^{\alpha\beta} (\mathcal{D}_{\beta j} \mathcal{D}_{\gamma i} \sigma) M_{\alpha}^{\gamma} + b (\mathcal{D}_{\beta j} \mathcal{D}_{\alpha}^{\,k} \sigma) J_{ik} + (\alpha \leftrightarrow \beta) \right] M_{bc} \quad (3.15)\]

Within the square bracket the $\mathcal{D}\mathcal{D}$ term splits into a commutator and an anticommutator. Since $\sigma$ does not carry any spin or SU(2) charge, it follows that $\{ \mathcal{D}_{\alpha(i}, \mathcal{D}_{\beta j)} \} \sigma = 0$, so that only the commutator part remains. Then we have

\[ 0 = 2i (\gamma_{abc})^{\alpha\beta} (\delta C_{aij} - 2\sigma C_{aij}) M_{bc} + a (\gamma_{bc})^{\alpha\beta} (\mathcal{D}_{\beta j} \mathcal{D}_{\gamma i} \sigma) M_{bc} \quad (3.16)\]

The commutator must be antisymmetric in its spin indices. This allows us to write this term as

\[ (\gamma_{bc})^{\alpha \gamma} (\mathcal{D}_{\beta j} \mathcal{D}_{\gamma i}) \sigma = (\gamma_{bc})^{\alpha \gamma} (\delta_{\beta}^{\mu} \delta_{\gamma}^{\nu}) (\mathcal{D}_{\mu(i}, \mathcal{D}_{\nu j)}) \sigma \]

\[ = -\frac{1}{4} (\gamma_{bc}^{\alpha \gamma} (\mathcal{D}_{\mu(i}, \mathcal{D}_{\nu j)}) \sigma \]

\[ = -\frac{1}{2} (\gamma_{abc})^{\alpha\beta} (\mathcal{D}_{i} \mathcal{D}_{j}) \sigma \quad (3.17)\]

Therefore, plugging (3.17) into (3.16) we obtain

\[ 0 = 2i (\gamma_{abc})^{\alpha\beta} \left[ \delta C_{aij} - 2\sigma C_{aij} + \frac{ia}{16} (\mathcal{D}_{i} \mathcal{D}_{j}) \sigma \right] M_{bc} \quad (3.18)\]

Finally, since this piece must vanish independently of the others, we must set to zero the coefficient of $M$ in the above equation, obtaining the transformation rule for the $C$
superfield
\[
\delta C_{aij} = 2\sigma C_{aij} - \frac{ia}{16} D_{(i} \tilde{\gamma}_{a} D_{j)} \sigma .
\]  
(3.19)

Having obtained the transformation of \( C \), we now focus on the transformation for the superfield \( N \). This rule also arises from (3.14), but this time taking the symmetric part \( (a;\beta) \) and \( (ij) \) proportional to the SU(2) generator, \( J \). This piece gives
\[
0 = -\frac{8i}{3} (\gamma^{abc})_{\alpha\beta} (\delta N_{abc} - 2\sigma N_{abc}) J_{ij} - \left[ b D_{(\beta(j)} D_{\alpha)k} \sigma J_{ij} + (\alpha \leftrightarrow \beta) \right] .
\]  
(3.20)

In the last term, only the commutator part contributes. Due to symmetries, we have that \( [D_{(\alpha}, D_{\beta)}] = -\frac{1}{2} \delta^{k}_{i} [D_{(\alpha}^{l}, D_{\beta)} k] \). Furthermore, using the Fierz identity (A.27) for the 3-form, we can rewrite
\[
[D_{(\alpha}^{l}, D_{\beta)}] = \delta^{\mu}_{(\alpha} \delta^{\nu}_{\beta)} [D_{\mu}^{l}, D_{\nu} k] = \frac{1}{48} (\gamma^{abc})_{\alpha\beta} (\tilde{\gamma}_{abc})^{\mu\nu} [D_{\mu}^{l}, D_{\nu} k]
= \frac{1}{24} (\gamma^{abc})_{\alpha\beta} D^{k} \tilde{\gamma}_{abc} D_{k} .
\]  
(3.21)

With this we obtain
\[
0 = -\frac{8i}{3} (\gamma^{abc})_{\alpha\beta} \left[ \delta N_{abc} - 2\sigma N_{abc} + \frac{ib}{128} D^{k} \tilde{\gamma}_{abc} D_{k} \sigma \right] J_{ij} .
\]  
(3.22)

As argued previously, this term must vanish. Cancelation of the factor of \( J \) yields to the sW-transformation of the superfield \( N \)
\[
\delta N_{abc} = 2\sigma N_{abc} - \frac{ib}{128} D^{k} \tilde{\gamma}_{abc} D_{k} \sigma .
\]  
(3.23)

This completes the analysis of the doubly-symmetric part of the equation (3.14). Next, we proceed to study the doubly-antisymmetric part of it. Tracing with \( \epsilon^{ij} \) gives
\[
0 = -4i (\gamma^{c})_{\alpha\beta} (\delta D_{c} - 2\sigma D_{c}) + 2b D_{[\alpha}^{k} \sigma D_{\beta]} k
- 8i (\gamma_{a})_{\alpha\beta} (\delta N^{abc} - 2\sigma N^{abc}) M_{bc} + \frac{a}{2} (\gamma^{bc})_{\alpha\beta} D_{k} \sigma M_{bc}
- 12i (\gamma^{a})_{\alpha\beta} (\delta C_{a}^{ij} - 2\sigma C_{a}^{ij}) J_{ij} + 2b D_{[\alpha}^{i} D_{\beta]}^{j} \sigma J_{ij} .
\]  
(3.24)
From here, we can isolate the term $\delta \mathcal{D}_c$ contracting spin indices. Multiplying this last equation by the 2-form $(\tilde{\gamma}^d)^{a\beta}$ we get

$$
\delta \mathcal{D}_a = 2\sigma \mathcal{D}_a - \frac{i b}{8} (D^k \sigma) \tilde{\gamma}_a D_k + 3 \left[ \delta C_{aij} - 2\sigma C_{aij} - \frac{i a}{24} (D_i \tilde{\gamma}_a D_j \sigma) \right] J^{ij} 
$$

$$
- 2 \left[ \delta N_{abc} - 2\sigma N_{abc} - \frac{i a}{64} (D^k \tilde{\gamma}_{abc} D_k \sigma) + \frac{i a}{32} \eta_{ab} (D^k \tilde{\gamma}_c D_k \sigma) \right] M^{bc} .
$$

(3.25)

Using our results about transformation laws of $C$ and $N$ this simplifies further to

$$
\delta \mathcal{D}_a = 2\sigma \mathcal{D}_a - \frac{i b}{8} (D^k \sigma) \tilde{\gamma}_a D_k + \frac{i}{64} \left[ (2a + b) (D^k \tilde{\gamma}_{abc} D_k \sigma) - 32i a \eta_{ab} (D_c \sigma) \right] M^{bc}
$$

$$
- \frac{i}{8} \left( \frac{3a}{2} + b \right) (D_i \tilde{\gamma}_a D_j \sigma) J_{ij} ,
$$

(3.26)

where we have also used $(D^k \tilde{\gamma}_a D_k) \sigma = 8i \mathcal{D}_a \sigma$. The values of the parameters above imply, on the one hand, that $2a + b = 0$ and therefore the factor of the 3-form in the first line of (3.26) vanish. This means that there is no $(D^k \tilde{\gamma}_{abc} D_k \sigma) M^{bc}$-term within the sW-transformation rule of the bosonic covariant derivative. On the other hand, $(3a + 2b)/2 = 2$ so that the vector covariant derivative will transform as

$$
\delta \mathcal{D}_a = 2\sigma \mathcal{D}_a - i (D^k \sigma) \tilde{\gamma}_a D_k - 2 (D^b \sigma) M_{ab} - \frac{i}{4} (D^i \tilde{\gamma}_a D_j \sigma) J_{ij} .
$$

(3.27)

This conclude the analysis of the conformal transformations. As a final comment, note that the Weyl transformation rules of the fields (3.19) and (3.23) contain inhomegeneous terms. Such terms can be used to gauge away some of the components of these superfields.
Chapter 4

Field content

In this chapter we focus on the study of the field content of the six-dimensional con-
formal supergeometry presented in the previous chapters. We first review briefly the
construction of the Weyl multiplet of Bergshoeff et alia [19], which emerges as a real-
ization of the conformal supersymmetry algebra. We then explore how this multiplet
appears in superspace.

4.1 The Weyl multiplet

The Weyl multiplet refers to the set of fields on which the six-dimensional superconfor-
mal algebra Osp(6, 2|1) is realized. The generators of this algebra are the usual Poincaré
plus SU(2) generators, as specified at the beginning of chapter [2]

\[ M_{ab} , P_a , J_{ij} \]  \hspace{1cm} (4.1)

together with the supersymmetry generators plus the dilatation, special conformal, and
special supersymmetry generators

\[ Q_{ai} , D , K_a , S_{ai} . \]  \hspace{1cm} (4.2)
As pointed out in [19], the superconformal algebra generated by (4.1) and (4.2) can be realized on the following set of fields

\[
\begin{array}{cccccc}
\varepsilon^a_m & \psi^{ai}_m & \Phi^{ij}_m & B_m & N^{(-)}_{abc} & \chi^{ai} & F \\
14 & -32 & 15 & 0 & 10 & -8 & 1 \\
\end{array}
\]

(4.3)

Here, the first four fields are the gauge fields corresponding to the generators \( P_a, Q_{ai}, J_{ij} \) and \( D \), respectively; \( \varepsilon^a_m \) is the (inverse of the) frame field, \( \psi^{ai}_m \) is the gravitino, \( \Phi^{ij}_m \) the SU(2) connection and \( B_m \) is the dilatation gauge field. The anti-self-dual tensor \( N^{(-)}_{abc} \), the spinor \( \chi^{ai} \) and the scalar \( F \) are matter fields. The number of the off shell degrees of freedom carried by the field is indicated explicitly. For the gauge fields, the counting of these degrees of freedom can be worked out by counting the number of components of each field and then subtracting the gauge transformations:

\[
\begin{align*}
\delta \varepsilon^a_m &= \partial_m \xi^a + \Lambda^a_b \varepsilon^b_m + \sigma \varepsilon^a_m, \\
\delta \psi^{ai}_m &= \partial_m \lambda^{ai} + e^a_m (\tilde{\gamma}_a)^{\alpha \beta} \eta^{i}_\beta, \\
\delta \Phi^{ij}_m &= \partial_m \alpha^{ij}, \\
\delta B_m &= e^a_m b_a.
\end{align*}
\]

(4.4)

(4.5)

(4.6)

(4.7)

In this way, to the 36 components of \( \varepsilon^a_m \), we need to subtract the 6 + 15 + 1 components of the gauge parameters \( \xi^a, \lambda^a_b \) and \( \sigma \), respectively, resulting in \( 36 - 22 = 14 \) off shell degrees of freedom. In the case of the gravitino \( \psi^{ai}_m \), to its \(-48\) components (the minus sign denote fermionic components) we need to subtract the \(-8 - 8\) components of the gauge parameters \( \lambda^{ai} \) and \( \eta^i_\beta \), for a total of \(-48 + 16 = -32\) off shell degrees of freedom. Next, the counting for the SU(2) gauge field \( \Phi^{ij}_m \) is 18 components minus the 3 components of the parameter \( \alpha^{ij} \), giving 15 off shell degrees of freedom. Finally, the dilaton gauge field \( B_m \) is pure gauge, since the gauge parameter \( b_a \) has the same number of components of it (that is, 6).

\footnote{More precisely, the number of the independent degrees of freedom in each gauge parameter entering in the gauge transformations.}
The matter fields, on the other hand, carry just their component degrees of freedom with the exception of the anti-self-dual tensor $N_{abc}^{(-)}$, which carry only a half of the possible 20 carried by a totally antisymmetric tensor $N_{abc}$ (the remaining 10 components are carried by its self-dual counterpart $N_{abc}^{(+)}$).

4.2 The Weyl multiplet in superspace

We would now like to understand how the Weyl multiplet just described appears in superspace. Firstly, the component gauge fields plus the gravitino are related to the $\theta = 0$ components of the superframe field and superconnections, while the matter fields are given by

$$N_{abc}^{(-)} = N_{abc}|_{\theta=0} , \quad \chi^{\alpha i} = N^{\alpha i}|_{\theta=0} , \quad F = D_\alpha N^{\alpha i}|_{\theta=0}$$

(4.8)

Secondly, the definition of $C^{\alpha i}$ and $N^{\alpha i}$ imply that their derivatives decompose as

$$D_\alpha C^{\alpha j} = \frac{8i}{3} D_a C^{\alpha ij} - \frac{1}{2} \epsilon_{ij} D_{ak} C^{\gamma k} ,$$

$$D_\alpha N^{\alpha j} = \frac{1}{2} \delta^j_i D_{\gamma k} N^{\gamma k} .$$

(4.9) (4.10)

It also follows that the supergravity fields obey the relations

$$(\gamma_{ab})^{\alpha \beta} D_{\beta i} C^{\alpha i} = -\frac{32i}{9} \left[ D^c N_{abc}^{(-)} - 8 N^{(+)}_{cd} |a N^{(-)}_{b|cd}} \right] ,$$

$$(\gamma_{ab})^{\alpha \beta} D_{\beta i} N^{\alpha i} = -\frac{32i}{5} \left[ D^c N_{abc}^{(+)} + 8 N^{(+)}_{cd} |a N^{(-)}_{b|cd}} \right] .$$

(4.11) (4.12)

The importance of these expressions is that (4.9) and (4.10) imply that there are no auxiliary iso-triplets $D^{ij}$ in the supergravity multiplet, while from (4.11) and (4.12) follow that there is no new singlet 2-form field strength. However, we do have the 2-forms iso-triplets

$$C_{abij} := D^k \tilde{\gamma}_{abc} D_k C^{ci} ,$$

$$N_{abij} := D_{(i \tilde{\gamma}_{ab} N^{(j)} ,}$$

(4.13) (4.14)
and the isospin components \( C^{ijkl} := D^{(i} \epsilon^{jkl)} \). Thus, at this stage we are left with the following set of components fields:

\[
\begin{array}{cccccccc}
  C^\alpha_{ij} & C^\alpha_{ij} & C_\alpha_{ij} & C_{\alpha ij} & C_{\alpha ij} & C_{ijkl} \\
  N^{\alpha \beta} & N^{\gamma_k \alpha \beta} & N^{\alpha i} & N & N_{abij} & N_{\gamma \delta \alpha \beta} \\
  N_{\alpha \beta} \\
\end{array}
\]  

where we have renamed \( C := D_{\alpha i} C^{\alpha i} \) and \( N := D_{\alpha i} N^{\alpha i} \), and the superfield \( N_{\gamma \delta \alpha \beta} := D_{(\alpha i} N_{\beta) i} \gamma \delta \)-traces, denotes the Weyl tensor. Of these fields, one can use the various components in \( \sigma \) to gauge away \( C^\alpha_{ij}, C^\alpha_{ij}, C_\alpha_{ij}, C_{ijkl} \) and \( N_{\alpha \beta} \). This leaves

\[
\begin{array}{cccccccc}
  C^\alpha & C & C_{\alpha ij} \\
  N^{\alpha \beta} & N_{\gamma_k \alpha \beta} & N^{\alpha i} & N & N_{abij} & N_{\gamma \delta \alpha \beta} \\
\end{array}
\]  

(4.16)

The bottom row contains the correct component content to describe an anti-self-dual tensor, the curl of the gravitino (both the \( \gamma \)-traceless and \( \gamma \)-trace parts), the SU(2) field strength, the curvature scalar, and the Weyl tensor.
Chapter 5

Matter couplings

This chapter is devoted to the study of the possible matter field configurations compatible with the conformal superspace structure developed in the previous chapters. In the final part, it is also shown that the constraints defining the scalar (hyper) and tensor multiplets imply a Weyl-type and scalar equation of motion for the superfield defining each multiplet.

5.1 Abelian vector multiplet

Let $W^{\beta(s)j(f)} := W^{\beta_1 \cdots \beta_s j_1 \cdots j_f}$ be an arbitrary superfield of Weyl-weight $w$, symmetric in $s$ spin and $f$ isospin indices with $s \geq 1$ and $f \geq 1$. The Weyl transformation of the spinorial derivative of the field is given by

$$
\delta \left( D_{\alpha i} W^{\beta_1 \cdots \beta_s j_1 \cdots j_f} \right) = \sigma (1 + 2w) D_{\alpha i} W^{\beta_1 \cdots \beta_s j_1 \cdots j_f} + (2w + s) (D_{\alpha i} \sigma) W^{\beta_1 \cdots \beta_s j_1 \cdots j_f}
- 4 (D_{\gamma i} \sigma) \sum_{q=1}^{s} \delta_{\alpha q}^{\gamma} W^{\beta_1 \cdots \beta_{q-1} \gamma \beta_{q+1} \cdots \beta_s j_1 \cdots j_f}
- 8 \sum_{q=1}^{f} \delta_{(i}^{j_q} W^{\beta_1 \cdots \beta_s j_{q-1} j_{q+1} \cdots j_f} ,
$$

(5.1)

$^1$We assume total symmetry in both kind of indices, spin and isospin.
where we have used the following commutators

\[
[M_\alpha^\gamma, W^{\beta_1\cdots\beta_s}] = -\frac{s}{4} \delta_\alpha^\gamma W^{\beta_1\cdots\beta_s} + \sum_q \delta_\alpha^\beta_q W^{\beta_1\cdots\beta_q-1\gamma\beta_{q+1}\cdots\beta_s}, \tag{5.2}
\]

\[
[J_{ij}, W^{k_1\cdots k_f}] = -\sum_q \delta_{(i}^{k_q} W^{k_1\cdots k_{q-1} j_{k+1}\cdots k_f}. \tag{5.3}
\]

Contracting \( \alpha = \beta_1 \) and \( i = j_1 \), transformation (5.1) becomes (we only need to be careful with the first term of each sum, and split the last isospin sum into its symmetric parts)

\[
\delta \left( D_{\alpha i} W^{\alpha\cdots\beta_s} i_{j_f} \right) = \sigma(1 + 2w)D_{\alpha i} W^{\alpha\cdots\beta_s} i_{j_f} + (2w + s)(D_{\alpha i} \sigma) W^{\alpha\cdots\beta_s} i_{j_f}
- 4(3 + s)(D_{\gamma i} \sigma) W^{\gamma\cdots\beta_s} i_{j_f} + 4(1 + f)(D_{\alpha i} \sigma) W^{\alpha\cdots\beta_s} i_{j_f}
+ 4(D_{\alpha i} \sigma) W^{\alpha\cdots\beta_s} i_{j_f}. \tag{5.4}
\]

Therefore, if we require that the inhomogeneous parts of the expression above cancel, necessarily the Weyl weight should be fixed in terms of the number of spin and isospin indices as

\[
w = \frac{3}{2}s + 2(1 - f). \tag{5.5}
\]

If so, the following constraint

\[
D_{\alpha i} W^{(\alpha\cdots\beta_s)} (i_{j_f}) = 0 \tag{5.6}
\]

transforms homogeneously under Weyl transformations. In particular, for the spinor superfield \( W^{ak} \) (\( s = 1 = f \)) we can consider

\[
D_k W^k = 0 ; \quad w = \frac{3}{2}. \tag{5.7}
\]
Let us now consider the combination \((\tilde{\gamma}_{ab})^\alpha_{\beta_1} D_{\alpha(i} W_{\beta_1 \cdots \beta_s k})^{j_2 \cdots j_f}\). From (5.1) we get (not yet symmetrizing \((ik)\))

\[
\varepsilon_{k j_1} (\tilde{\gamma}_{ab})^\alpha_{\beta_1} \delta_D (D_{\alpha i} W_{\beta_1 \cdots \beta_s j_1 \cdots j_f}) = \sigma (1 + 2w)(\tilde{\gamma}_{ab})^\alpha_{\beta_1} D_{\alpha i} W_{\beta_1 \cdots \beta_s k}^{j_2 \cdots j_f} + (2w + s)(\tilde{\gamma}_{ab})^\alpha_{\beta_1} (D_{\alpha i} \sigma) W_{\beta_1 \cdots \beta_s k}^{j_2 \cdots j_f} - 4(D_{\gamma i} \sigma) \sum_{q=1}^{s} (\tilde{\gamma}_{ab})^\beta_q_{\beta_1} W_{\beta_1 \cdots \beta_q - 1 \beta_q+1 \cdots \beta_s k}^{j_1 \cdots j_f} - 8(\tilde{\gamma}_{ab})^\alpha_{\beta_1} (D_{\alpha i} \sigma) \sum_{q=1}^{f} \varepsilon_{k j_1} \delta_D (\tilde{\gamma}_{ab})^\beta_q_{\beta_1} W_{\beta_1 \cdots \beta_q j_1 \cdots j_f - 1 j_q+1 \cdots j_f} .
\]

(5.8)

We note that, in the first sum, all the terms have the spin index of the covariant derivative and the superfield \(W\) (the \(\gamma\) index) contracted, except the term for \(q = 1\), which is identically zero. This restricts us, for the transformation (5.8) to be homogeneous, to the case \(s = 1\). The last sum (over the isospin indices), can be written as

\[
\sum_{q=1}^{f} \varepsilon_{k j_1} \delta_D (\tilde{\gamma}_{ab})^\beta_q_{\beta_1} W_{\beta_1 \cdots \beta_q j_1 \cdots j_f - 1 j_q+1 \cdots j_f} = \varepsilon_{k(i} W_{\beta_1 \cdots \beta_s l})^{j_2 \cdots j_f} + \sum_{q=2}^{f} \varepsilon_{k j_1} \delta_D (\tilde{\gamma}_{ab})^\beta_q_{\beta_1} W_{\beta_1 \cdots \beta_q j_1 \cdots j_f - 1 j_q+1 \cdots j_f} ,
\]

(5.9)

and again, for the transformation to be homogeneous, we need \(f = 1\) (otherwise there will be terms of the type \(W_{\cdots i \cdots k\cdots}\)). Then, taking the symmetric part in \((ij)\) and combining (5.8) and (5.9), we conclude that the only possibility for this kind of combination is necessarily the case \(s = 1 = f\), that is

\[
\delta_D (D_{\alpha(i} (\tilde{\gamma}_{ab})^\alpha_{\beta} W_{\beta j})) = \sigma (1 + 2w)D_{\alpha(i} (\tilde{\gamma}_{ab})^\alpha_{\beta} W_{\beta j}) + (2w + 1)(D_{\alpha(i} \sigma) (\tilde{\gamma}_{ab})^\alpha_{\beta} W_{\beta j}) - 4D_{\alpha(j} (\tilde{\gamma}_{ab})^\alpha_{\beta} W_{\beta i}) .
\]

(5.10)

Therefore, the only possible homogeneous constraint is

\[
D_{(i} \tilde{\gamma}_{ab} W_{j)} = 0 ,
\]

(5.11)
with Weyl weight $w = \frac{3}{2}$. The abelian vector multiplet was precisely described by this Weyl weight-$\frac{3}{2}$, spinor superfield $W^{\alpha i}$ subject to the constraints (5.7) and (5.11). At the level of components, the first says that the vector multiplet auxiliary fields consist of an iso-triplet of scalars, while the second says that there is only one 2-form field strength (of the four possibles). Together, these constraints can be written as

$$D_{\alpha i} W_{\beta j} = \frac{1}{4} \delta_\alpha^\beta D_{\gamma(i} W_{\gamma j)} - \frac{1}{16} \varepsilon_{ij} (\gamma^{ab})_\alpha^\beta D_k \tilde{\gamma}_{ab} W^k .$$  

(5.12)

5.2 Tensor multiplet

Let $\tilde{\Phi}$ denote a six-dimensional real scalar superfield of Weyl-weight $w$, that is $\delta \tilde{\Phi} = 2w\sigma \Phi$. Then, given such a transformation, its double spinorial derivative transforms into

$$\delta(D_{\alpha i} D_{\beta j} \tilde{\Phi}) = 2(1 + w)\sigma D_{\alpha i} D_{\beta j} \tilde{\Phi} + 2w(D_{\alpha i} D_{\beta j}\sigma)\tilde{\Phi}$$

$$+ 2w(D_{\alpha i} D_{\beta j}\sigma)D_{\alpha j} \tilde{\Phi} + 2w D_{\alpha i} \tilde{\Phi} - 4 \varepsilon_{ij}(D_{\alpha k}\sigma)D_{\beta k} \tilde{\Phi} .$$  

(5.14)

Taking the symmetric part in isospin indices $(ij)$ gives

$$\delta(D_{\alpha(i} D_{\beta j)} \tilde{\Phi}) = 2(1 + w)\sigma D_{\alpha(i} D_{\beta j)} \tilde{\Phi} + 2w(D_{\alpha(i} D_{\beta j)}\sigma)\tilde{\Phi}$$

$$+ 4(w - 2)(D_{\alpha(i}\sigma)D_{\beta j)}\tilde{\Phi} .$$  

(5.16)

In this last equation, the symmetric part in spin indices $(\alpha\beta)$ is trivial. The appearance of the last term means that the anti-symmetric part can not be corrected to transform homogeneously unless $w = 2$. In this case, the second term can be cancelled by adding a connection term (3.19). We define $\tilde{\Phi} = \Phi$ to be a real, weight-2, scalar superfield satisfying the invariant condition

$$D_{(i} \tilde{\gamma}^{a} D_{j)} \Phi + 16i C_{ij} \Phi = 0 .$$  

(5.17)

The same argument that lead to establishing the constraint in Eq. (5.7) implies that if a superfield potential $V^{\alpha i}$ has weight $w = \frac{3}{2}$, the combination $\Phi := D_{\alpha i} V^{\alpha i}$ will be
covariant under Weyl transformations and will have weight $w = 2$. Then, it can be shown that, if $V^{\alpha i}$ satisfies only the constraint

$$\mathcal{D}_{(i}\bar{\gamma}_{ab}V_{j)} = 0 ,$$

then the associated scalar $\Phi$ satisfies the condition \(5.17\). The scalar field $\Phi$ contains the anti-self-dual field strength of a 2-form potential $H_{abc}^{(-)} \sim \mathcal{D}^k \bar{\gamma}_{abc} \mathcal{D}_k \Phi \mid$. In terms of the potential superfield $V^{\alpha i}$, the potential 2-form is $B_{ab} \sim \mathcal{D}_k \bar{\gamma}_{ab} V^k$. Thus, we find that the field $\Phi$ describes a real scalar, an anti-self-dual 3-form field strength, and their superpartners while the field $V^{\alpha i}$ describes the same multiplet in terms of a gauge 2-form potential.

### 5.3 Other multiplets

Let $\Phi_{\alpha(s)i(f)} := \Phi_{\alpha_1...\alpha_i...i_f}$ denote a superfield symmetric in $s$ spin and $f$ isospin indices. Let $w$ denote the Weyl-weight $\delta \Phi = 2w \sigma \Phi$. Then the combination $\mathcal{D}_{(\alpha(i} \Phi_{\beta(s))i(f))}^\gamma$, completely symmetrized on all indices, transforms homogeneously under Weyl transformations if and only if \(^{2}\)

$$w = 2f - \frac{3}{2}s .$$

When this relation between Weyl-weight and spin and SU(2) indices is satisfied, the constraint

$$\mathcal{D}_{(\alpha(i} \Phi_{\beta(s))i(f))}^\gamma = 0 ,$$

can be imposed on the (matter) field $\Phi$. One can further confirm that this constraint is integrable in the sense that the anti-commutator $\{\mathcal{D}_{\alpha i}, \mathcal{D}_{\beta j}\} \Phi_{\gamma(s)k(f)}^\gamma$ vanish identically when symmetrized on all spin and isospin indices \(^{3}\) Examples include the hypermultiplet

\(^{2}\)We reach this result through a similar argument used to derive \((5.5)\).

\(^{3}\)Straightforward calculation shows that the isospin part will always be proportional to the SU(2) anti-symmetric tensor $\varepsilon_{ij}$, while the spin part will always reduce to having one $\gamma$-matrix. Therefore, after symmetrization, the anti-commutator in question vanish identically.
The scalar multiplet can be described by the iso-doublet, Weyl weight-2, scalar superfield $q^i$ subject to the constraint (5.21)

\[ \mathcal{D}_{\alpha(i} q_{j)} = 0 ; \quad w = 2 , \]  

and the superfield $A_{\alpha i}$ subject to the constraint

\[ \mathcal{D}_{(\alpha(i} A_{\beta j))} = 0 ; \quad w = \frac{1}{2} . \]  

Note that since $A_{\alpha i}$ has the same dimension as the covariant derivative $\mathcal{D}_{\alpha i}$, the replacement $\mathcal{D}_{\alpha i} \to \nabla_{\alpha i} = \mathcal{D}_{\alpha i} + i A_{\alpha i}$ corresponds to minimal coupling to a super-1-form. It also follows that since $\{\mathcal{D}_{\alpha(i}, \mathcal{D}_{\beta j)}\} \Omega \equiv 0$ on a scalar superfield $\Omega$, the constraint (5.24) is invariant under the abelian gauge transformation $A_{\alpha i} \to A_{\alpha i} + \mathcal{D}_{\alpha i} \Omega$.

5.4 Tensor calculus

As anticipated at the beginning of this chapter, the constraints defining the scalar and tensor multiplets imply the on-shellness of these multiplets. In this section, we will compute for each of these matter representations the equations of motion arising from such constraints.

The scalar multiplet can be described by the iso-doublet, Weyl weight-2, scalar superfield $q^i$ subject to the constraint (5.21)

\[ \mathcal{D}_{\delta(i} q_{j)} = 0 . \]  

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A Weyl-type equation for $q^i$ arises by contracting the previous constraint with the operator $\varepsilon^{\alpha\beta\gamma\delta} D_{\gamma i} D_{\delta j}$. Then, straightforward calculation leads to

$$0 = D^\alpha D^i q^i + 2 N^\alpha D^i q^i + 48 C^{\alpha i} q^i - 10 N^{\alpha i} q^i . \quad (5.26)$$

Contracting again this equation with the spinorial covariant derivative, we obtain the scalar equation of motion

$$0 = D^a D_a q_i - \frac{17}{3} D_a C^{a ij} q^j + 2 C^{a ij} D_a q^j + \frac{15}{2} C_{a jk} C^{a jk} q_i - 4 N^{\alpha \beta} N_{\alpha \beta} q_i$$
$$- \frac{3i}{2} D_{a j} C^{a j} q_i - \frac{3i}{2} C^a i D_{a j} q^j + \frac{5i}{16} D_{a j} N^{a j} q_i , \quad (5.27)$$

where we have used the decompositions (4.9) and (4.10). These two equations of motion make manifest that the scalar multiplet is on-shell.

**The tensor multiplet** can be described by a real, Weyl weight-2, scalar superfield $\Phi$ subject to the constraint (5.17)

$$D_{(i} \tilde{\gamma}_a D_{j)} \Phi = -16i C_{a ij} \Phi . \quad (5.28)$$

Acting on this expression with the differential operator $(D^j \tilde{\gamma}^a)^{\alpha}$ gives the spinor equation of motion

$$0 = D^\alpha D^i \Phi - C^{\alpha \beta} D^i \Phi - 2 N^{\alpha \beta} D^i \Phi - 12 C^{\alpha i} \Phi . \quad (5.29)$$

Contracting the equation above with a spinor covariant derivative results in a Klein-Gordon-type equation for the scalar $\Phi$

$$0 = D^a D_a \Phi + 8 C_{a ij} C^{a ij} \Phi + \frac{i}{6} N^{abc} D^k \tilde{\gamma}_{abc} D_k \Phi - \frac{3i}{2} D_{a i} C_{a i} - 3i C^{a i} D_{a i} \Phi + \frac{5i}{2} N^{a i} D_{a i} \Phi . \quad (5.30)$$

Again, we conclude that equations (5.29) and (5.30) put the tensor multiplet automatically on-shell.

\footnote{Alternatively, it can be shown that the same equation results from directly contracting the constraint (5.28) with the operator $D^{(i} \tilde{\gamma}^a D^{j)}$.}
Chapter 6

Concluding remarks

In this thesis, we have studied the geometry of six-dimensional, $\mathcal{N} = (1, 0)$ superspace and its matter couplings. In the first part of this work, we fixed the basic ground-work of our formulation by firstly setting the superspace structure group to $G = SO(5, 1) \times SU(2)$. Then, after imposing the set of conventional torsion constraints (2.8), (2.9) and (2.9), we systematically solved the supergravity Bianchi identities up to and including (mass) dimension-2. In doing so, we found that the full derivative algebra can be expressed entirely in terms of a vector iso-triplet $C_{aij}$, and a 3-form superfield $N_{abc}$. These superfields define the dimension-1 torsion, according to (2.20). We further elucidated that consistency of the identities implies the constraints (2.53)-(2.55) on these supergravity fields. At dimension-$\frac{3}{2}$ we worked out the irreducible decomposition of torsion and isospin field strength. At the dimension-2 level, we computed the Riemann curvature tensor (2.120) and the field strength for R-symmetry group (2.127).

Once we had in hand the complete supergeometry, we explored the invariance of the conventional torsion constraints under conformal transformations. In particular, we fixed the set of transformation rules that the superfields and covariant derivatives must satisfy in order to implement the conformal invariance. These transformations are given by (3.13), (3.19), (3.23) and (3.27). One of the important features of the transformation rules we found is that there are inhomogeneous pieces in the Weyl transformation of
the superfields $C$ and $N$, which can be used to gauge away some of their components.

The second part of this thesis was dedicated to the study of the field content of the superspace supergravity presented in the first part, and its superconformal matter couplings. The Weyl multiplet was presented. As we have seen, its gauge fields structure includes the frame field $e^a_m$, the gravitino $\psi^\alpha_i$ and the (pure gauge) dilatation gauge field $B_m$. This set of gauge fields is encoded within the $\theta = 0$ components of the superframe-field and superconnections. The matter field structure, on the other hand, is characterized by the set of fields $\{N^{(a)}_{abc}, \chi^\alpha_i, F\}$, that is, an anti-self-dual tensor field, an auxiliary spinor and a real auxiliary scalar, respectively. These fields arise from the $\theta = 0$ components of the three-form superfield $N_{abc}$ and its spinorial covariant derivative(s), as indicated in (4.8).

Next, we investigated the possible matter fields allowed by conformal invariance. We started by addressing the question of what are the most general conformally invariant constraints on a certain matter superfields. We then used those constraints to further study the (abelian) vector and tensor multiplets. The former turns out to be described by a Weyl-weight-$\frac{3}{2}$ spinor superfield $W^{\alpha i}$ subject to the constraints (5.7) and (5.11), while the latter is characterized by a real, weight-2 scalar superfield $\Phi$ satisfying the condition (5.17). This scalar field admits a weight-$\frac{3}{2}$ potential $V$, defined through $\Phi = D^\alpha_i V^{\alpha i}$, which allows an alternative description of the same tensor multiplet.

We concluded this thesis with the study of the component field equations of motion for the scalar and tensor multiplet. It was shown that, starting with the constraints defining a matter representation, one may further derive Weyl-type and Klein-Gordon-type equations of motion for the component fields defining each multiplet. These equations are given by (5.26), (5.27), (5.29) and (5.30), and imply that both multiplets are realized on-shell.

Summarizing, this thesis may be considered as a companion to reference [16], developing the very first steps and basic results in order to carry out further and deeper explorations of simple six-dimensional superspaces and their applications. There re-
mains, therefore, much work to be done. A natural direction for future research is
the dimensional reduction to five dimensions, with the hope of recovering the five-
dimensional superfield supergravity presented in [21]. It would also be desirable, since
simple six-dimensional supergravity enjoys the same fermionic structure that of four-
dimensional, $\mathcal{N} = 2$ supergravity, to address the issue of how the latter is embedded in
six-dimensional superspace.

The study of supersymmetric backgrounds in superspace is also a open problem.
Along this line, one might attempt to extend early classifications of the geometries
admissible for a six-dimensional supergravity description [22] to superspace. More am-
bitiously, the extension of lower-dimensional rigid supersymmetric backgrounds [23, 24, 25]
to six-dimensional curved superspace may be investigated.
Appendix A

6D notation and conventions

We adopt the 6D superspace conventions established in [16]. The procedure is to first define \( \gamma_m := -\Gamma_mC^{-1} \) and \( \tilde{\gamma}_m = -C\Gamma_m \) for \( m = 0, \ldots, 3; 5 \). Then we take \( \gamma_6 = C^{-1} \) and \( \tilde{\gamma}_6 = -C \). The relative sign has been chosen so that the six \( 8 \times 8 \) Dirac matrices satisfy the Clifford algebra

\[
\{ \Gamma_m, \Gamma_n \} = -2\eta_{mn}1 ,
\]

with \( m, n = 0, \ldots, 5 \) and

\[
\eta_{mn} = \text{diag}(-1, 1, 1, 1, 1, 1) .
\]

The overall sign is chosen so that, in terms of explicit indices, the formulæ are

\[
(\gamma^a)_{\alpha\beta} = (\Gamma^a)_{\alpha\beta} , \quad (\tilde{\gamma}^a)_{\alpha\beta} = -(\Gamma^a)_{\alpha\beta} \quad \text{for } a = 0, 1, 2, 3; 5
\]

\[
(\gamma_6)_{\alpha\beta} = \varepsilon_{\alpha\beta} , \quad (\tilde{\gamma}_6)_{\alpha\beta} = -\varepsilon_{\alpha\beta} .
\]

In terms of Pauli-type matrices, Dirac matrices take the form

\[
\Gamma_m = \begin{pmatrix}
0 & (\gamma_m)_{\alpha\beta} \\
(\tilde{\gamma}_m)^{\beta\alpha} & 0
\end{pmatrix} ,
\]

with \( \alpha = 1, \ldots, 4 \).
It is possible to represent these 6D Pauli-type matrices $\gamma_m$ and $\tilde{\gamma}_m$, in terms of the 4D Pauli matrices, $\sigma_m$ and $\tilde{\sigma}_m$. Denoting the 4D, SL(2, $\mathbb{C}$) spinor indices by $\alpha = 1, 2$ and $\dot{\alpha} = 1, 2$, such representation is given by

$$
\gamma_m = \begin{pmatrix}
0 & - (\sigma_m)_{\alpha \dot{\beta}} \\
(\tilde{\sigma}_m)_{\dot{\alpha} \beta} & 0
\end{pmatrix}
$$

(A.5)

for $m = 0, \ldots, 3$ and

$$
(\gamma_5)_{\alpha \beta} = \begin{pmatrix}
i \varepsilon_{\alpha \beta} & 0 \\
0 & i \varepsilon_{\dot{\alpha} \dot{\beta}}
\end{pmatrix}, \quad (\gamma_6)_{\alpha \beta} = \begin{pmatrix}
\varepsilon_{\alpha \beta} & 0 \\
0 & - \varepsilon_{\dot{\alpha} \dot{\beta}}
\end{pmatrix}.
$$

(A.6)

Defining now

$$
(\tilde{\gamma}_5)_{\alpha \beta} = \begin{pmatrix}
i \varepsilon_{\alpha \beta} & 0 \\
0 & i \varepsilon_{\dot{\alpha} \dot{\beta}}
\end{pmatrix}, \quad (\tilde{\gamma}_6)_{\alpha \beta} = \begin{pmatrix}
- \varepsilon_{\alpha \beta} & 0 \\
0 & \varepsilon_{\dot{\alpha} \dot{\beta}}
\end{pmatrix},
$$

(A.7)

six-dimensional Pauli-type matrices obey the algebra

$$
(\gamma^m)_{\alpha \beta} (\tilde{\gamma}^n)_{\beta \gamma} + (\gamma^n)_{\alpha \beta} (\tilde{\gamma}^m)_{\beta \gamma} = - 2 \eta^{mn} \delta^{\gamma}_{\alpha},
$$

$$
(\tilde{\gamma}^m)_{\alpha \beta} (\gamma^n)_{\beta \gamma} + (\tilde{\gamma}^n)_{\alpha \beta} (\gamma^m)_{\beta \gamma} = - 2 \eta^{mn} \delta^{\gamma}_{\dot{\alpha}}.
$$

(A.8)

Note that the 6-dimensional Pauli-type matrices are antisymmetric

$$
(\gamma_m)_{\alpha \beta} = - (\gamma_m)_{\beta \alpha},
$$

(A.9)

implying an isomorphism between the space of 6-dimensional vectors and antisymmetric $4 \times 4$ matrices

$$
V_{\alpha \beta} := (\gamma^m)_{\alpha \beta} V_m = - V_{\beta \alpha} \iff V_m = \frac{1}{4} (\tilde{\gamma}_m)_{\alpha \beta} V_{\alpha \beta}.
$$

(A.10)

The second relation is a consequence of the analysis below and equation (A.24) in particular. Similarly, six-dimensional 2-forms are in one-to-one correspondence with traceless $4 \times 4$ matrices and (anti-)self-dual 3-forms are in correspondence with symmetric rank-2 spin matrices with their indices (up) down as we now work out in detail.
To begin, it is useful to define the normalized anti-symmetrized products of Pauli-type matrices
\[ \gamma_{m_1 \ldots m_p} := \gamma_{[m_1} \tilde{\gamma}_{m_2} \cdots \tilde{\gamma}_{m_p]} = \frac{1}{p!} \gamma_{m_1} \tilde{\gamma}_{m_2} \cdots \tilde{\gamma}_{m_p} + \text{perm}. \]
(A.11)

With these normalizations, products reduce without factors. For example
\[ \gamma^{ab} \gamma^c = \gamma^{abc} + 2\eta^{[a} \gamma^{b]} \]
\[ \tilde{\gamma}^{ab} \gamma^c = \tilde{\gamma}^{abc} - 2\eta^{[a} \gamma^{b]} \]
(A.12)

Other useful identities are
\[ \gamma^{abc} \tilde{\gamma}^d = -\frac{1}{2} \epsilon^{abdef} \gamma^{ef} - 3\eta^{[a} \gamma^{b]} \]
(A.13)

\[ \tilde{\gamma}^{abc} \gamma^d = \frac{1}{2} \epsilon^{abdef} \gamma^{ef} - 3\eta^{[a} \gamma^{b]} \]
(A.14)

\[ \tilde{\gamma}^c \gamma^{gh} \gamma^f = \frac{1}{2} \epsilon^{cghrs} \tilde{\gamma}^{rs} - 3\delta^{c} \gamma^{gh} - 2\eta^{[g} \gamma^{h]} c + 2\delta^{c} [a \gamma^{b]} \]
(A.15)

\[ \tilde{\gamma}^d \gamma^{abc} \tilde{\gamma}^d = 0 \]
(A.16)

\[ \gamma^{ab} \gamma^{cd} = -\frac{1}{2} \epsilon^{abdef} \gamma^{ef} + 4\delta^{[a} \gamma^{b]} d - 2\delta^{[a} \delta^{b]} \]
(A.17)

\[ \tilde{\gamma}^{abc} \gamma^{de} = \frac{1}{2} \epsilon^{abdef} \gamma^{ef} + \epsilon^{abdef} \gamma^{ef} + \eta^{de} \gamma^{abc} - 3\eta^{[d} \gamma^{b][e]c} + 6\eta^{[d} \gamma^{b][e]} \]
(A.18)

A more commonly used convention regarding the 2-form matrix is as the spinor representation \(\Sigma^{ab}\) of the Lorentz generator \(M_{ab}\) which is related by
\[ (\Sigma^{ab})_{\alpha}^{\beta} = -\frac{1}{2} (\gamma^{ab})_{\alpha}^{\beta}. \]
(A.20)

In terms of these matrices, we define
\[ F_\alpha^\beta := \frac{1}{2} (\Sigma^{mn})_{\alpha}^{\beta} F_{mn} \Rightarrow F_{mn} = - (\Sigma_{mn})_{\beta}^{\alpha} F_\alpha^\beta. \]
(A.21)

The second relation is a consequence of the analysis below and equation (A.26) in particular. Both equations again agree with the five-dimensional conventions. Using the second type of matrix, we can construct \(\tilde{F}^\alpha_{\beta} := \frac{1}{4} (\tilde{\gamma}^{mn})^\alpha_{\beta} F_{mn}\), however
\[ (\tilde{\gamma}^{mn})^\alpha_{\beta} = -(\gamma^{mn})_{\beta}^\alpha, \]
(A.22)
so that this second matrix is not essentially new. Finally, the third-rank antisymmetric
tensors can be separated into (anti-)self-dual parts which are then in one-to-one corre-
spondence with symmetric $4 \times 4$ matrices. To see how this works in detail, we must
first establish some Fierz identities. There is a completeness relation

$$\frac{1}{2}(\gamma^m)_{\alpha\beta}(\gamma_m)_{\gamma\delta} = \epsilon_{\alpha\beta\gamma\delta} . \quad (A.23)$$

Contraction with $\epsilon^{\gamma\delta}\gamma^\gamma$ implies the completeness relation

$$\frac{1}{2}(\gamma^m)_{\alpha\beta}(\tilde{\gamma}_m)_{\gamma\delta} = \delta^\gamma_{\alpha}\delta^\delta_{\beta} - \delta^\gamma_{\beta}\delta^\delta_{\alpha} , \quad (A.24)$$

and

$$\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}(\gamma_m)_{\gamma\delta} = (\tilde{\gamma}_m)_{\alpha\beta} \Rightarrow (\gamma_m)_{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}(\tilde{\gamma}_m)_{\gamma\delta} . \quad (A.25)$$

Contraction of (A.24) with itself gives

$$\frac{1}{4}(\tilde{\gamma}^{mn})_{\alpha\beta}(\gamma_{mn})_{\gamma\delta} = -\frac{1}{2}\delta^\gamma_{\alpha}\delta^\delta_{\beta} + 2\delta^\delta_{\alpha}\delta^\gamma_{\beta} . \quad (A.26)$$

Another contraction with (A.24) gives

$$(\tilde{\gamma}^{abc})_{\alpha\beta}(\gamma_{abc})_{\gamma\delta} = 24(\delta^\alpha_{\gamma}\delta^\beta_{\delta} + \delta^\alpha_{\delta}\delta^\beta_{\gamma}) , \quad (A.27)$$

while contraction with (A.23) shows that

$$(\gamma^{abc})_{\alpha\beta}(\gamma_{abc})_{\gamma\delta} = 0 \quad \text{and} \quad (\tilde{\gamma}^{abc})_{\alpha\beta}(\tilde{\gamma}_{abc})_{\gamma\delta} = 0 . \quad (A.28)$$

Thus we see that $\tilde{\gamma}^{mnp}$ and $\gamma^{mnp}$ correspond to (anti-)self-dual 3-forms. To show that

$$(\tilde{\gamma}^{mnp}) \gamma^{mnp} \text{ is (A)SD},$$

one uses the identities

$$\gamma_0\gamma_1\gamma_2\gamma_3\gamma_5\gamma_6 = +1 \quad \text{and} \quad \tilde{\gamma}_0\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3\tilde{\gamma}_5\tilde{\gamma}_6 = -1 . \quad (A.29)$$

\footnote{This relation follows, up to normalization, from the equal dimensions of the spaces of 6-vectors and antisymmetric $4 \times 4$ matrices.}
to conclude that, for example, \( \gamma_{012} = \epsilon_{012345} \gamma_{345} \) whereas \( \tilde{\gamma}_{012} = -\epsilon_{012345} \tilde{\gamma}_{345} \). The relation (A.19) immediately implies the trace relation on the 3-forms

\[
\text{tr}(\tilde{\gamma}_{abc}\gamma^{def}) = 4! \left( \delta_{[a}^{d} \delta_{b}^{e} \delta_{c]}^{f} - \frac{1}{3!} \epsilon_{abc}^{def} \right),
\]

(A.30)

from which it follows that the (anti-)self-dual parts of a 3-form \( N \) satisfy

\[
N^{(+)}_{\alpha\beta} := \frac{1}{3!} N_{abc} (\gamma_{abc})_{\alpha\beta} \Rightarrow N^{(+)}_{abc} = \frac{1}{8} \text{tr}(N^{(+)}\gamma_{abc}),
\]

\[
N^{(-)}_{\alpha\beta} := \frac{1}{3!} N_{abc} (\gamma_{abc})_{\alpha\beta} \Rightarrow N^{(-)}_{abc} = \frac{1}{8} \text{tr}(N^{(-)}\tilde{\gamma}_{abc}).
\]

(A.31)

Recall that (six-dimensional) Hodge duality on 3-forms is an involution of order 2:

\[
\frac{1}{3!} \epsilon_{abcrst} \epsilon_{defrst} = -3! \delta_{[a}^{d} \delta_{b}^{e} \delta_{c]}^{f}.
\]

(A.32)

Other useful relations resulting from (A.23) are

\[
(\gamma_{a})_{\alpha\beta} (\gamma_{b})_{\delta} = 2 \varepsilon_{\alpha\beta\gamma} (\gamma_{b})_{\delta} + (\gamma_{b})_{\alpha\beta} \delta_{\gamma}^{\delta},
\]

(A.33)

\[
(\gamma_{a})_{\alpha\beta} (\gamma_{bc})_{\gamma} = -2 \varepsilon_{\alpha\beta\delta} (\gamma_{bc})_{\delta} + 2 (\gamma_{b})_{\alpha\beta} (\gamma_{c})_{\gamma},
\]

(A.34)

\[
(\gamma_{abc})_{\alpha\beta} (\gamma_{c})_{\gamma} = -8 (\gamma_{a})_{\gamma} (\gamma_{b})_{\delta}.
\]

(A.35)

Further contractions of these equations give a long list of useful identities, namely

\[
(\gamma_{abc})_{\alpha\beta} (\tilde{\gamma}_{c})_{\gamma} = -4 \delta_{\alpha}^{[\gamma} (\gamma_{ab})_{\beta] \delta},
\]

(A.36)

\[
(\gamma_{acd})_{\alpha\beta} (\tilde{\gamma}_{bd})_{\gamma} = 8 \delta_{(a}^{\gamma} \delta_{(a}^{\delta})_{(\gamma)} - 8 \delta_{(a}^{\gamma} (\gamma_{ab})_{\beta] \delta},
\]

(A.37)

\[
(\gamma_{abc})_{\alpha\beta} (\gamma_{c})_{\gamma} = -4 (\gamma_{a})_{\gamma} (\gamma_{b})_{\beta] \delta},
\]

(A.38)

\[
(\gamma_{acd})_{\alpha\beta} (\gamma_{bd})_{\gamma} = -8 (\gamma_{a})_{\gamma} (\gamma_{b})_{\beta] \delta},
\]

(A.39)

\[
(\tilde{\gamma}_{abc})_{\alpha\beta} (\gamma_{c})_{\gamma} = 4 \delta_{\gamma}^{(\alpha} (\gamma_{ab})_{\beta] \delta},
\]

(A.40)

\[
(\tilde{\gamma}_{abc})_{\alpha\beta} (\gamma_{bc})_{\gamma} = 8 (\gamma_{a})_{\gamma} (\delta^{(\alpha} \delta^{\beta)_{\gamma}}).
\]

(A.41)

Let us conclude deriving some other useful relations. Starting from the second relation above and contracting with \( \varepsilon^{\mu\gamma\delta} \) gives

\[
0 = \delta_{(\alpha}^{\mu} (\gamma_{bc})_{\beta] \nu} - \delta_{(\alpha}^{\nu} (\gamma_{bc})_{\beta] \mu} + (\gamma_{b})_{\alpha\beta} (\tilde{\gamma}_{c})_{\mu}^{\nu},
\]

(A.42)
Next, contraction with \((\gamma^a)_{\gamma\mu}\) gives

\[
(\gamma^{bc}\gamma^a)_{[\alpha|\gamma]\delta_{\beta]} = (\gamma^{[b})_{\alpha\beta}(\gamma^{a]}_{|\gamma})_{\gamma\nu} + (\gamma^a)_{\gamma[\alpha}(\gamma^{bc})_{\beta]\delta^\nu}.
\]

Taking the completely antisymmetric part \([abc]\) gives the identity

\[
(\gamma^{abc})_{\gamma[\alpha\delta}\delta_{\beta]} = - (\gamma^{[a})_{\alpha\beta}(\gamma^{bc])_{\gamma}\delta + (\gamma^a)_{\gamma[\alpha}(\gamma^{bc})_{\beta]\delta^\nu}.
\]

Finally, we can use the fact that

\[
\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\mu\nu\lambda\delta} = 3! \delta_{[\alpha}^{\mu} \delta_{\beta]}^{\nu} \delta_{\gamma}^{\lambda},
\]

to show that

\[
(\gamma^a)_{\gamma[\alpha\psi}\beta] = \frac{1}{4} \varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\mu\nu\lambda\delta}(\gamma^a)_{\mu\nu}\psi_{\lambda} - \frac{1}{2}(\gamma^a)_{\alpha\beta}\psi_{\gamma},
\]

and therefore

\[
(\gamma^{abc})_{\gamma[\alpha\delta\beta]} = - \frac{3}{2} (\gamma^{[a})_{\alpha\beta}(\gamma^{bc]}_{\gamma})_{\gamma}\delta + \frac{1}{4} \varepsilon_{\alpha\beta\gamma\rho}\varepsilon^{\mu\nu\lambda\rho}(\gamma^{[a})_{\mu\nu}(\gamma^{bc})_{\lambda}\delta.
\]
Appendix B

Supergeometry summary

Superalgebra  The dimension-1 and $\frac{3}{2}$ commutators are given by

$$\{D_{\alpha i}, D_{\beta j}\} = 2i \varepsilon_{ij}(\gamma^a)_{\alpha\beta} D_a + 2i (\gamma^{abc})_{\alpha\beta} C_{aij} M_{bc} + 4i \varepsilon_{ij}(\gamma_a)_{\alpha\beta} N^{abc} M_{bc}$$

$$- 6i \varepsilon_{ij}(\gamma^a)_{\alpha\beta} C_a^{kl} J_{kl} - \frac{8i}{3} (\gamma^{abc})_{\alpha\beta} N_{abc} J_{ij}$$  \hspace{1cm} (B.1)

$$[D_{\gamma k}, D_a] = -C^b_{\ k l}(\gamma_{ab})_\gamma \delta_{d}^l + N_{abc}(\gamma_{bc})_\gamma \delta_{d}^k + i \left[ \frac{1}{2} (\gamma_{a})_{\gamma\delta} T_{bc} \delta_{k} - (\gamma_{[b})_{\gamma} T_{c]} \gamma_{\delta} \right] M_{bc}$$

$$- \left[ (\gamma_{a})_{\gamma} \delta_{i} \delta_{j}^k - 6 \delta_{k}^{(i} (\gamma_{a})_{\gamma} \delta_{j)} (C_{ij} - \frac{1}{3} N_{ij}) \right] J_{ij}$$  \hspace{1cm} (B.2)

Irreducibles  Spinorial derivatives of the supergravity fields decompose as

$$D_{\gamma k} C_{a ij} = C_a \gamma_{k ij} + (\gamma_{a})_{\gamma} \delta_{ij}^k + \varepsilon_{k(i} C_{a \gamma j)} + \varepsilon_{k(i} (\gamma_{a})_{\gamma} \delta_{j)}$$  \hspace{1cm} (B.3)

$$D_{\gamma k} N_{\alpha\beta} = N_{\gamma k \alpha\beta} + \bar{N}_{\gamma k \alpha\beta}$$  \hspace{1cm} (B.4)

$$D_{\gamma k} N^{\alpha\beta} = N_{\gamma k} \alpha\beta + \delta^{(a} N_{\gamma}^{(\beta)} k$$  \hspace{1cm} (B.5)

Under this decomposition, dimension-1 torsion constraints are equivalent to

$$C_{a \gamma k ij} = 0 \hspace{2cm} N_{\gamma k \alpha\beta} = 0$$

$$C^b_{ijk} = -\frac{1}{6} (\gamma^b)_{\delta\beta} D_{\beta(k} C_{bij)} \hspace{2cm} \bar{N}_{\gamma k \alpha\beta} = -\frac{3}{4} (\gamma^{(a})_{\gamma} C_a \beta)_{k}$$

$$C_{a \beta j} = \frac{1}{5} \tau_{a \beta}(5,1) D_{\gamma i} C_{c i j} \hspace{2cm} N_{\gamma k \alpha\beta} = D_{\gamma k} N^{\alpha\beta} - \frac{2}{5} \delta((a} D_{\delta k} N^{\beta)\delta}$$

$$C^\gamma k = -\frac{1}{5} D_{\delta l} C^\delta_{\gamma l k} \hspace{2cm} N^{\gamma i} = \frac{2}{5} D_{\beta i} N^{\beta}$$  \hspace{1cm} (B.6)
The irreducible parts of the dimension-$\frac{3}{2}$ torsion and isospin field strength are

\[
T_{\alpha}^{\gamma \mu \kappa} = \mathcal{T}_{\alpha}^{\gamma \mu \kappa} + (\mathcal{T}_{\alpha}^{\gamma \mu \kappa})_{\delta}^\gamma \mathcal{T}_{\delta}^{\gamma \mu \kappa} + (\mathcal{T}_{\alpha}^{\gamma \mu \kappa})_{\delta}^\gamma \mathcal{T}_{\delta}^{\gamma \mu \kappa} \quad (B.7)
\]

\[
F_{\alpha}^{\gamma \kappa} = \mathcal{F}_{\alpha}^{\gamma \kappa} + (\mathcal{F}_{\alpha}^{\gamma \kappa})_{\delta}^\gamma \mathcal{F}_{\delta}^{\gamma \kappa} + (\mathcal{F}_{\alpha}^{\gamma \kappa})_{\delta}^\gamma \mathcal{F}_{\delta}^{\gamma \kappa} \quad (B.8)
\]

where

\[
\mathcal{T}_{\alpha}^{\gamma \mu \kappa} = -\frac{i}{2} (\gamma_{\alpha})_{\beta}^\gamma N_{\delta}^{\mu \beta \kappa} \\
\mathcal{T}_{\alpha}^{\gamma \mu \kappa} = -\frac{\eta_{\alpha}}{4} C_{\alpha} \delta_j^\gamma \\
\mathcal{T}_{\alpha}^{\gamma \mu \kappa} = -i C_{\delta \kappa} + \frac{i}{6} N_{\delta \kappa} \\
\mathcal{F}_{\alpha}^{\gamma \kappa} = 6 C_{\alpha} \delta_j^\gamma \\
\mathcal{F}_{\alpha}^{\gamma \kappa} = -5 C_{\alpha} + \frac{5}{3} N_{\alpha} \quad (B.9)
\]

Riemann and Ricci tensors, Curvature Scalar, SU(2) Field Strength At dimension-2 level, Bianchi identities encode the Riemann tensor

\[
R_{\alpha}^{cd} = \frac{i}{8} (\gamma_{\alpha})_{\beta}^\gamma N_{\alpha}^{\beta \gamma} + 2 \varepsilon_{\alpha}^{cdmn} D^p \left[ N_{mn}^{(+)} - 4 \frac{3}{3} N_{mn}^{(-)} \right] + 4 D_{[a} N_{b]}^{cd} + 4 D_{[c} N_{d]}^{ab} - 32 N_{[a [c} N_{d]}^{d]} N_{b]} + 8 \delta_{[a}^{[c} C_{b]}^{d]} C_{d]}^{ij} + 8 i C_{[a i} C^{a j]} - \frac{1}{6} D_{a i} N_{a i} \quad (B.10)
\]

It also follows that

\[
R_{\alpha} = \frac{i}{8} \eta_{\alpha} \left[ 10 D_{a i} C^{a i} - \frac{5}{3} D_{a i} N^{a i} + 64 i C^{a i j} C_{a i j} \right] + 8 C_{a i} C_{b i j} + 16 N_{a c} N_{b c} \quad (B.11)
\]

\[
R = \frac{16 i}{2} D_{a i} C^{a i} - 40 C_{a i j} C_{a i j} - \frac{5}{4} D_{a i} N^{a i} + 16 N_{a b c} N^{a b c} \quad (B.12)
\]

Finally, the dimension-2 SU(2) field strength is given by

\[
F_{\alpha}^{a b} = \frac{5 i}{12} N_{a b}^{ij} - \frac{111}{288} C_{a b}^{ij} + \frac{5 i}{18} C_{a b}^{ij} + \frac{10}{9} D_{[a} C_{b]}^{ij} + \frac{86}{9} C_{a}^{k(i} C_{b]}^{j)} + \frac{4}{9} N_{a b c} N_{a b c} \quad (B.13)
\]

Super-Weyl transformations Covariant derivatives and superfields transform as

\[
\delta D_{\alpha} = \sigma D_{\alpha} - 4 (D_{[b j \sigma]) M_{a}^{\beta} + 8 (D_{a}^{b}) \sigma) J_{ij} \quad (B.14)
\]

\[
\delta D_{\alpha} = -2 \sigma D_{\alpha} - i (D_{b}^{b} \sigma) \gamma_{a} D_{k} - 2 (D_{b}^{b} \sigma) M_{a b} - \frac{i}{4} (D_{b}^{b} \gamma_{a} D_{j} \sigma) J_{ij} \quad (B.15)
\]

\[
\delta C_{a i j} = 2 \sigma C_{a i j} + \frac{i}{4} D_{(i} \gamma_{a} D_{j)} \sigma \quad (B.16)
\]

\[
\delta N_{a b c} = 2 \sigma N_{a b c} - \frac{i}{16} D_{k} \gamma_{a b c} D_{k} \sigma \quad (B.17)
\]
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