Abstract—In some countries and regions, water distribution and treatment consume a considerable amount of electric energy. This paper investigates the water network’s potential ability to provide demand response services to the power grid for the management of renewable resources under the framework of a distribution-level water–energy nexus (micro WEN). In particular, the hidden controllable water loads, such as irrigation systems, were closely studied as virtual energy storage to improve the flexibility of electrical grids. An optimization model is developed for the demand-side management (DSM) of micro WEN, and the simulation results assert that grid flexibility indeed benefits from taking controllable water loads into account. Although the proposed optimal DSM model is a computationally intractable mixed-integer nonlinear programming (MINLP) problem, quasi-convex hull techniques were developed to relax the MINLP into a mixed-integer convex programming (MICP) problem. The numerical study shows that the quasi-convex hull relaxation is tight and that the resulting MICP problem is computationally efficient.

Index Terms—Electric power networks, hybrid systems, optimization, water–energy nexus (WEN).

NOMENCLATURE

Parameters

\( A \)
Incidence matrix of water network.

\( a_{i,k} \)
Coefficients of pump characteristics in pipe \( k \).

\( C \)
Cost function of the whole system.

\( c_t \)
Locational marginal price at PCC.

\( E_{i,0}^{ES} \)
Initial state of charging of the battery energy storage system (BESS) at bus \( i \).

\( h_i \)
Elevation at node \( i \) of the water network.

\( P_{i,t}^{L} \)
Active electric load at bus \( i \).

\( P_{i,t}^{RE} \)
Active power output of renewable at bus \( i \).

\( Q_{i,t}^{L} \)
Reactive electric load at bus \( i \).

\( Q_{i,t}^{RE} \)
Reactive power output of renewable at bus \( i \).

\( r_{ij} \)
Loss coefficient related to the converter of the BESS.

\( R_k \)
Resistance of line \( kj \).

\( S_{i,t} \)
Binary variable denoting the on/off state of the pump.

\( \alpha_{i,t} \)
Loss coefficient related to the battery of the BESS.

\( \beta_{i,t} \)
Loss coefficient related to the battery of the BESS.

\( \eta_{i,t} \)
Head loss coefficient of pipe \( k \).

\( \eta_{i} \)
Reactance of line \( ij \).

Sets

\( E_i \)
Edge set of the water network.

\( N_W \)
Node set of the water network.

\( N_W^S \)
Set of nodes connected to a tank.

\( E_E \)
Edge set of the electricity network.

\( E_P \)
Set of pipes with a pump installed.

\( N_E \)
Bus set of the electricity network.

\( N_E^G \)
Set of bus with controllable generations.

\( N_E^P \)
Set of bus with a pump connected.

\( N_E^ES \)
Set of bus with a BESS connected.

Variables

\( \alpha_{i,t} \)
Binary variable denoting the on/off state of the pump at bus \( i \).

\( Q_{i,t}^{ES} \)
Reactive power of the BESS at bus \( i \).

\( Q_{i,t}^{pump} \)
Reactive consumptions of the pump at bus \( i \).

\( P_{i,t}^{ES} \)
Active power of the BESS at bus \( i \).

\( P_{i,t}^{pump} \)
Active consumptions of the pump at bus \( i \).

\( L_{ij,t}^{E} \)
Active power loss of the BESS at bus \( i \).

\( f_i \)
Vector of water flow in pipes.

\( f_{CT}^{ij} \)
Water flow to the customer-owned water tank at node \( i \).

\( f_{UT}^{ij} \)
Water flow to the utility-owned water tank at node \( i \).

\( f_{ij,t}^{G} \)
Water flow injected from the water source in pipe \( i \).

\( f_{ij,t}^{S} \)
Water flow to the utility-owned water tank at node \( i \).

\( T_{ij,t} \)
Square of current magnitude in line \( ij \).

\( V_{i,t} \)
Square of voltage magnitude at bus \( i \).

\( x_{ij} \)
Node set of the electricity network.
This paper introduces a distribution-level water–energy network (WEN). The network modeling and optimization of such a water–energy system is also investigated. In this section, the background and motivation behind this paper are presented in Section I-A, related work is introduced in Section I-B, and the contribution and organization of this paper are presented in Section I-C.

A. Background and Motivation

Modern-day water and power systems are closely intertwined as a coupled system, which is commonly referred to as the water–energy nexus (WEN) [1]–[5]. On one hand, most of the water facilities consume electrical energy. For instance, ground water pumping and seawater desalination account for roughly 12% of the total electric energy consumption in the Arabian Gulf regions [2]. On the other hand, water usage is inevitable in refining fuels and generating electric power. Despite the two networks’ inevitable interdependence, water and energy networks have traditionally been operated independently from one another, and the idea of cooperating the two in parallel has long been glossed over. Therefore, this paper aims at investigating an optimal strategy for the cooperation of the water and energy networks.

In recent years, researchers have started to direct their attention to water system’s potential ability to provide demand response (DR) services [6]. It is believed that a higher cost efficiency can be achieved by cooperating the water and power systems. In 2016, PG & E built up an efficiency partnership with the water utility in the city of San Luis Obispo, which is the first-of-its-kind.

The power grid suffers from an unprecedented amount of network uncertainty with the ever-increasing penetration of intermittent renewable energy and electromobility [7], and the idea of exploiting the flexibility of the water network has come under the spotlight as a possible solution. The cooperation of two systems allows the water system to fast and accurately respond to the energy imbalance caused by the uncertain renewable energy generation or even contingencies on the electricity side. With this solution, the overall security and reliability of water and power systems will benefit from the nexus operation mode.

Approximately 1.8 billion people worldwide will be living in areas with the issue of the water shortage by 2025 [8]. The urgent need for improving the efficiency of the water utilization requires that the function of a future water distribution system (WDS) should include the water treatment, water recycling, water purification, cooling, metering/monitoring, funds/revenue generated, etc. [9]. As a result, the WDS will be more electrified and have higher capability for DR.

B. Related Work

This paper introduces a micro WEN, wherein both the water and energy networks are at distribution levels, to the countries and regions with heavy electricity-driven water loads. Much of the micro WEN’s flexibility will rise from the water system, as electricity-driven water services—pumping, cooling, desalination, and water treatment—are time flexible. Further flexibility can be introduced by including the controllable water loads such as irrigation services, which this paper uses to demonstrate hidden DR capabilities of the water network. The flexibility of pumps is constrained by water tank capacities that are physically limited by nature. However, incorporating controllable discharging scheme for water tanks should, to some extent, resolve the issue. Moreover, pumps, tanks, and irrigation systems are to be jointly used to create virtual energy storages to power grids to alleviate the stress on the micro WEN in case of energy imbalance.

We aim at developing a mathematical tool to assess the flexibility and responsiveness of a given micro WEN to the energy imbalance. For this purpose, a mathematical model for the introduced micro WEN is built in this paper. Compared with the models developed for the transmission-level WENs in the literature [3]–[5], the power network in the micro WEN is captured by an ac power flow model integrated with battery energy storage systems (BESSs), water pumps, and high penetration of renewable energy sources [10]. The water pipe network model is characterized by the directed Darcy–Weishchak equation [11] and allow for water flow directions to reverse, and the ON/OFF status of pumps are represented by integer variables.

The built mathematical model of the micro WEN is mixed-integer nonlinear, which will introduce nonconvexity and NP-hardness for solving the optimal demand-side management problem on micro WENs. To mitigate its intractability, a quasi-convex hull relaxation [10], [12] is developed for the micro WEN model. An optimal water-power flow (OWPF) problem was solved in [13] where a feasible point pursuit-successive convex approximation algorithm developed in [14] was adopted to obtain a near-optimal solution to the OWPF problem at the price of running time. In contrast, the quasi-convex hull relaxation proposed in this paper is more computationally effective and can guarantee an optimal solution with high tightness.

C. Contribution and Paper Organization

An important contribution of this paper lies in considering the water system as a resource to manage renewable generation. This paper introduces a novel engineering problem, i.e., the network optimization of micro WEN, which is a mix-integer nonlinear program (MINLP). Such a problem is hard to solve due to the nonconvexity of a large number of system components, including electrical feeders, water pipes, pumps, and BESSs, and the discreteness related to OFF/ON state of pumps.
Another main contribution of this paper is to solve the network optimization of the micro WEN with high efficiency. The numerical experiment shows that the computational time is dramatically reduced with the implementation of the proposed approach. The basic idea of the proposed solution method is to relax the original MINLP problem into a mixed-integer convex program (MICP) by carefully developing a quasi-convex hull (QCH) relaxation for each nonconvex system component. The contribution on the aspect of convexification is that we customize the QCH relaxation for various type of nonconvex constraints and the resulting QCH relaxations are tight.

The rest of this paper is organized as follows: The physical and mathematical models of the micro WEN systems are introduced in Section II. A cooptimization framework of the micro WEN is developed in the same section. In Section III, we propose the quasi-convex-hull relaxation of the cooptimization problem. Section IV presents a flexible irrigation scheme. A case study is provided in Section V, while conclusion is drawn in Section VI.

### II. PROBLEM FORMULATIONS

#### A. Physical Model of Micro WEN

The schematic of the micro WEN is given in Fig. 1. The electricity side is a distribution network, or a microgrid, integrated with renewable energy and BESSs. The water side consists of a pipe network, pumps, utility- and customer-owned tanks, and irrigation systems. Electric vehicles (EV’s) and water treatment facilities—including desalination, water and waste water treatment, and recycling—are not considered in this paper for the sake of simplicity, but those two elements can be easily incorporated and will be considered in future research.

It has been discussed in [1] that such a micro WEN is a fundamental infrastructure of a smart community, such as smart buildings [15]/cities [16]/villages [17]. Under the environment of a smart community, all infrastructures will be connected through the emerging Internet of Things techniques. In order to operate and control such a connected physical system, it is essential to develop the mathematical model and design the optimization algorithms for optimal resource allocation.

#### B. Mathematical Model of the Micro WEN

This section introduces a multiperiod mathematical model of the micro WEN. Unless otherwise stated, the subscript \( t \) denotes the discrete time period. The structure of an ac microgrid or an electric distribution system is usually radial. Consequently, we use a DistFlow model [18], [19] integrated with renewable generation, pumps, and batteries to describe the electricity network. The detailed model is given as

\[
\begin{align*}
\mathcal{I} & = \mathcal{I}^{\text{G}} + \mathcal{I}^{\text{RE}} - \mathcal{I}^{\text{Pump}} - \mathcal{I}^{\text{L}} + \mathcal{I}^{\text{ES}} \\
\mathcal{Q} & = \mathcal{Q}^{\text{G}} + \mathcal{Q}^{\text{RE}} - \mathcal{Q}^{\text{Pump}} - \mathcal{Q}^{\text{L}} + \mathcal{Q}^{\text{ES}} \\
\mathcal{V} & = \mathcal{V}^{\text{G}} + \mathcal{V}^{\text{RE}} - \mathcal{V}^{\text{Pump}} - \mathcal{V}^{\text{L}} + \mathcal{V}^{\text{ES}} \\
\end{align*}
\]

where \( i \in \mathcal{N}_E \) and \( ik \in \mathcal{E}_E \). Constraint (1g) is valid for all buses except the reference bus, which can be the point of common coupling (PCC) or any diesel generator bus. The renewable generators, such as photovoltaics (PVs), are considered available for reactive power support. For the sake of simplicity, this paper only consider the three-phase balanced cases. It has been proved in the literature that the convex relaxations of the DistFlow for balanced networks can be easily extended to the unbalanced cases under some mild approximations. Therefore, the proposed
approach in this paper can be easily leveraged to the cases of three-phase unbalanced distribution networks or microgrids.

The following nonlinear model of a battery energy storage unit is incorporated into the overall mathematical model of the micro WEN. Please refer to [10] for more details about this BESS model. For all \( \forall i \in \mathcal{N}_E \), we have

\[
\begin{align*}
(i^2 + i^2 C_v E) + (Q^2 E) = L^E \forall i, t. \\
(P^E) + (Q^E)^2 \leq (S^E)^2 \\
E^{E} E_{l+t} - i \in 0 \sum_i (P^E + L^E) \leq E^{E}.
\end{align*}
\]

We make the following assumptions for the water distribution networks.

1) The pipe network is a directed graph \( G = (\mathcal{N}_W, E_W) \) with incidence matrix \( A \) such that \( A_{ik} \in \{-1, 0, 1\} \) for all \( i, k \).
2) A pump is considered as a type of pipe that imposes a head gain when the pump is ON and closed otherwise.
3) The pump converts the electric power into a mechanical power at a constant efficiency of \( \eta \).
4) The power factors of pumps are fixed, namely \( P_{\text{pump}}^2/Q_{\text{pump}}^2 \) is constant.

The resulting model can be expressed as

\[
\begin{align*}
\sum_i A_{ik} f_{i, t} = f_{i, t}^G - f_{i, t}^U - f_{i, t}^C, (i \in \mathcal{N}_W) \quad \text{(a)}
\end{align*}
\]

\[
\begin{align*}
y_i, t - y_j, t + h_i - h_j = R^p f_{i, t}^C f_{i, t}^2, (k \in E_W \setminus E_{W}) \quad \text{(b)}
\end{align*}
\]

\[
\begin{align*}
y_{k, t} - y_{k, t} + h_i - h_j = R^p \text{sgn}(f_{k, t}) f_{k, t}^2, (k \in E_W \setminus E_{W}) \quad \text{(c)}
\end{align*}
\]

\[
\begin{align*}
S_W \leq S_{W,i} + \sum_{i=0}^{t} f_{i, t}^U/C_T \leq S_{W}, (i \in \mathcal{N}_W^S) \quad \text{(d)}
\end{align*}
\]

where \( A \) is a \( |\mathcal{N}_W| \times |\mathcal{E}_W| \) incidence matrix and pipe \( k \) connects nodes \( i \) and \( j \). Equation (a) represents the mass balance of the water network; constraints (b) and (c) formulate the hydraulic characteristics of a normal pipe and the pipe with a pump installed, respectively; constraint (d) states the nature of charging/discharging of the water tanks; (3e)–(3i) are system constraints, where, for instance, \( y_i \) denotes the minimum allowable pressure head at node \( i \); \( \text{sgn}(f) = -1 \) if \( f \leq 0 \) or, otherwise \( \text{sgn}(f) = 1 \). When \( \alpha_{k, t} = 1 \), the quantity \( f_{k, t} \) in constraint (3c) is nonnegative. In model (3), the quantity \( f_{i, t}^C \) represents the uncontrolled water load. Similar to the uncontrolled electric load, it is a given value at each period.

The pumps are considered as constant-speed motors in this paper. The hydraulic characteristics of a constant-speed pump is generally approximated by a quadratic function of the water flow across the pump, i.e., \( y^P = a_2 f^2 + a_1 f + a_0 \) [22], [23]. Contribution of the nonlinear \( a_2 f^2 \) is usually very small compared to the linear ones \( a_1 f + a_0 \). As a result, in constraint (3c), we have the head gain

\[
y_{k, t} = a_1 k f_{k, t} + a_0 k.
\]

The following constraints act as the mathematical link between the distribution network (1)–(2) and the WDS (3):

\[
\eta P_{\text{pump}}^2 = f_{k, t} y_{k, t} = a_1 k f_{k, t}^2 + a_0 k f_{k, t}, \quad \text{(4)}
\]

where \( i \in \mathcal{N}_E \) and \( k \in E_W^i \).

### C. Cooptimization Framework of Water and Electricity

Based on the mathematical model of the micro WEN introduced previously, this subsection introduces a cooptimization framework for water and electric energy networks. The objective of this cooptimization problem is to minimize the total energy cost for meeting the demands of both electricity and water. We formulate the energy cost as

\[
C(P_{i, t}^{G}) = \sum_i \left( c_i P_{i, t}^{G} + \sum_{i=0}^{t} \left( c_{i, j} P_{i, j}^{G} + c_{2, i} \left( P_{i, t}^{G} \right)^2 \right) \right) \quad \text{(5)}
\]

where \( P_{1, t}^{G} \) denotes the power from the grid via PCC (i.e., the serial number of PCC is 1); \( c_i \) can be considered as the nodal prices at PCC, which are obtained by solving the security constrained economic dispatch by ISOs/RTOs. As a result, the cooptimization model is

\[
\min_{P_{i, t}^{G}} \quad \text{CO-OPT} \quad \text{(5)}
\]

s.t. \( (1)-(4) \)

where is an MINLP. In the hierarchical control framework of droop-controlled microgrids [24], [25], the solutions of (CO-OPT) are adopted as the set points for the tertiary control. Water networks and power networks are generally owned by different utilities. A potential mechanism for implementing the (CO-OPT) framework is that the water utilities could receive a subsidy from electric utilities if they agree to transfer the operation right of pumps to the electric utilities.

### III. QUASI-CONVEX HULL RELAXATIONS

The MINLP problem is computationally intractable, especially for large-scale systems. To reduce the computational burden, this section relaxes the MINLP into a mixed-integer convex programming problem [26] with high tightness.

#### A. Convex Hull Relaxations of Constraints (1d) and (2a)

Within the circular bounds (1e) and (2b), the feasible sets of (1d) and (2a) can be captured by the following general
formulation:

$$\Omega_0 = \left\{ \begin{array}{lcl} ax_1^2 + bx_2^2 & = & x_3 x_4 \\ x_1^2 + x_2^2 & \leq & c \\ x_3, x_4 & \leq & \bar{x}_3, \bar{x}_4 \end{array} \right\}$$

where $x = [x_1, x_2, x_3, x_4]^T$, $a \geq b$, and $x_3, x_4 \leq c \leq \bar{x}_3, \bar{x}_4$. By generalizing the theorem presented in [10], we have the following Lemma.

**Lemma:** The convex hull of nonconvex set $\Omega_0$ is

$$\Omega_1 = CH(\Omega_0) = \left\{ x \; | \; \begin{array}{l} ax_1^2 + bx_2^2 \leq x_3 x_4 \\ (a-b)x_2^2 + x_3 x_4 \leq ac \\ (a-b)x_2^2 + x_3 x_4 \leq ac \\ D^T x - d \leq 0 \\ x_1^2 + x_2^2 \leq c \\ x_3, x_4 \leq \bar{x}_3, \bar{x}_4 \end{array} \right\}$$

where $D = [0 \; 0 \; k_1 \; k_2]^T$ is a coefficient vector, $d$ is a scalar, and their values are given by

$$\begin{cases} k_1 = ac, k_2 = x_3 x_4, d = ac(x_3 + \bar{x}_3), & \text{if } x_3, x_4 \leq ac \leq x_3, \bar{x}_4 \\ k_1 = \bar{x}_3 x_4, k_2 = ac, d = ac(\bar{x}_3 + \bar{x}_4), & \text{if } x_3, x_4 \leq \bar{x}_3, \bar{x}_4 \\ k_1 = \bar{x}_3, k_2 = x_3, d = ac + x_3 \bar{x}_4, & \text{if } x_3, x_4 \leq x_3, \bar{x}_4 \\ k_1 = x_3, k_2 = \bar{x}_3, d = ac + x_3 \bar{x}_4, & \text{if } ac \leq x_3, \bar{x}_4, \bar{x}_3 \bar{x}_4 \end{cases}$$

**Proof:** See appendix.

For the case of (1d), $a = b = 1$, $c = \bar{x}_3$, and $(\bar{x}_3, \bar{x}_4, \bar{x}_3, \bar{x}_4) = (Y, 0, \bar{Y}, 1, \bar{Y})_i$, assume that $\bar{Y}^2 \leq Y$. Within the system bounds (1e)–(1g), the convex hull of (1d) is given as

$$\begin{cases} P_{i,k,j}^2 + Q_{i,k,j}^2 \leq V_{i,j} \bar{L}_{i,k,j} \leq \bar{Y}_i (\bar{V}_j + \bar{V}_i) \end{cases}$$

(6)

For the case of (2a), $a = r_i^{\text{Batt}} + r_i^{\text{Cvt}}$, $b = r_i^{\text{Cvt}}$, $c = (\bar{S}_i^{\text{ES}})^2$, and $(\bar{x}_3, \bar{x}_4, \bar{x}_3, \bar{x}_4) = (Y, 0, \bar{V}, 1, \bar{V})_i$. It is obvious that $V_i \bar{L}_{i} \leq (\bar{S}_i^{\text{ES}})^2 (r_i^{\text{Batt}} + r_i^{\text{Cvt}}) \leq Y_i L_{i}^{\text{ES}}$. Hence, within the system bounds (1g) and (2b), the convex hull of (2a) is

$$\begin{cases} (r_i^{\text{Batt}} + r_i^{\text{Cvt}}) \left(\frac{P_{i,k,j}^{\text{ES}}}{V_{i,j}}\right)^2 + r_i^{\text{Cvt}} (Q_{i,k,j}^{\text{ES}})^2 \leq L_{i,k,j}^{\text{ES}} \bar{V_i} \\ r_i^{\text{Batt}} (Q_{i,k,j}^{\text{ES}})^2 + V_i \bar{L}_{i,k,j} \leq (\bar{S}_i^{\text{ES}})^2 (r_i^{\text{Batt}} + r_i^{\text{Cvt}}) \\ (\bar{S}_i^{\text{ES}})^2 \bar{V_i} + V_i \bar{L}_{i,k,j} \leq (\bar{S}_i^{\text{ES}})^2 (\bar{V}_j + \bar{V}_i) \end{cases}$$

(7)

**B. Quasi-Convex Hull Relaxation of (3b)**

With the left-hand side replaced by an auxiliary variable $y$ and the right-hand side replaced by a general term $R_k^p \text{sign}(x) x_i^2$, function (3b) yields the blue curve, as shown in Fig. 2, in the $(x, y)$-plane. It is relaxed into the red polygon as shown in the figure. The red polygon is not exactly the convex hull of (3b). However, it is very close to the convex hull of constraint (3b), and therefore, called a quasi-convex hull relaxation. Its mathematical formulation in the $(f_i, y_i)$-domain is given as

$$y_i = y_j + h_i - h_j + y_j^{\text{Cvt}} + M \cdot (1 - \alpha)$$

$$\begin{cases} R_k^p f_{k,i}^2 - y_i \leq 0 \\ Y_j - R_k^p f_{k,j}^2 \leq 0 \\ 0 \leq f_{k,i} \leq M \cdot \alpha \end{cases}$$

(9)

(9b)

(9c)

where $Y_i = y_i - y_j + h_i - h_j + y_j^{\text{Cvt}} + M \cdot (1 - \alpha)$ and $Y_j = y_i - y_j + h_i - h_j + y_j^{\text{Cvt}} + M \cdot (\alpha - 1)$. Note that, ignoring the binary variable $\alpha$, the expression (9a) is convex, while (9b) is a concave constraint. The convex hull of (9b) can be obtained through a geometric approach as shown in Fig. 3. Its mathematical expression is given as

$$Y_j - R_k^p f_{k,j}^2 \leq 0.$$

(10)
It can be observed, by comparing Figs. 2 and 3, that one can construct a tighter relaxation for the hydraulic characteristic of a pipe if the direction of the water flow is given. A planning problem of gas networks is discussed in [27] where the authors introduced additional binary variables and bilinear equations to relax the Weymouth equation. The Weymouth equation is similar to constraint (3b). However, the convex relaxation of the Weymouth equation developed in [27] is not necessarily tighter than the proposed relaxation (8) due to the introduced bilinear equations. Moreover, the auxiliary binary variables and constraints in [27] are not desirable for an operation problem that is sensitive to computational time.

D. Convex Hull Relaxation of Constraint (4)

Constraint (4) is a quadratic equation that can be considered as the intersection of a convex inequality and a concave inequality. The geometric approach introduced in Fig. 3 can be used to construct the convex hull of the concave inequality. As a result, the convex hull constraint (4) is given by

\[ \eta P^{\text{Pump}}_{k,t} \geq a_k f_{k,t}^2 + b_k f_{k,t} \] (11a)
\[ \eta P^{\text{Pump}}_{k,t} \leq (a_k f_{k,t} + b_k) f_{k,t} \] (11b)

where \( f_{k,t} \) is nonnegative since the direction of pump flows is determined.

E. Quasi-Convex Hull Relaxation of (CO-OPT)

To sum up, the quasi-convex hull relaxation of the overall cooptimization problem (CO-OPT) is

\[
\begin{align*}
\min_{\alpha_{i,t}} & \quad S_i^w \\
\text{s.t.} & \quad (1\text{a,c, e-i}), (2\text{b-c}), (3\text{a, d-i}), (6) \quad \text{(C-CO-OPT)} \\
& \quad (7), (8), (9\text{a, c}), (10) \text{ and } (11),
\end{align*}
\]

which is a mixed-integer convex quadratically constrained quadratic programming (MICQCQP) problem.

The basic idea of the quasi-convex hull relaxation of an optimization problem is replacing the nonconvex constraints with their convex hulls or quasi-convex hulls. As discussed in [10] and [12], the concept of the convex hull is attractive since the extreme points of a convex hull generally belong to its original nonconvex set. If the objective function is a convex function and monotonic over the convex hull, the optimal solution is usually located at one of the extreme points, implying that the optimal solution obtained by solving the convex relaxation is most likely the exact globally optimal solution of the original problem. Unfortunately, for many of the nonlinear nonconvex sets, it is extremely hard to formulate their convex hulls. For such cases, an interesting alternative is to construct a convex inner approximation of a nonconvex set [28]. Compared with the convex relaxation, the foremost advantage of the convex inner approximation is guaranteeing the feasibility of the obtained solutions to the original nonconvex set.

A characteristic of the MINLP problem (CO-OPT) is that the integer variables only exist in linear terms of constraints. For the purpose of improving the computational efficiency, we suggest to relax such a mixed-integer problem into a mixed-integer convex problem. The convex relaxations that are tight for the continuous cases are equivalently tight for the discrete cases since the nonconvex terms that need to be relaxed do not contain integer variables.

IV. FLEXIBLE IRRIGATION SCHEME

It is straightforward to improve the grid flexibility by allowing for controllability of electric loads. From a different angle, this subsection explores opportunities for improving the DR capacity of water systems by investigating the flexibility of customer-owned water tanks and irrigation systems that are water loads rather than electric loads. An intuitive interpretation is that customers can use the superfluous energy to pump and store water. However, there are some current volume limits on tanks. Thus, it is necessary to develop a coordination strategy for charging and discharging tanks based on the multiperiod energy imbalance and the flexible water consumption.

Assuming that the crops growth is not sensitive to the watering time, the irrigation process is considered flexible. To develop a mathematical model of such a flexible irrigation system, we have the following assumptions: 1) the irrigation flow is fixed and the irrigation volume is a function of the watering time; and 2) for a given season, the daily amount of water is fixed, which can be represented as turning ON the irrigation system for totally \( N \) hours per day. Consequently, the mathematical model is given by

\[
\begin{align*}
S_i^w & \leq S_{i,0}^w + \sum_{t=0}^{N} (f_{i,t}^\text{CT} - f_{i,t}^\text{D} - k_{i,t}^2) \\ & \leq S_{i,t}^w \\
\sum_{t} \alpha_{i,t} & = N
\end{align*}
\]

which is mixed-integer linear.

By incorporating the model (12) of flexible irrigation systems into (C-CO-OPT), we have the following DSM scheme

\[
\begin{align*}
\min_{\alpha_{i,t}} & \quad S_i^w \\
\text{s.t.} & \quad (1\text{a-c, e-i}), (2\text{b-c}), (3\text{a, d-i}), (6) \quad \text{(C-DSM)} \\
& \quad (7), (8), (9\text{a, c}), (10) \text{ and } (11),
\end{align*}
\]

which is also an MICQCQP problem.

V. CASE STUDY

A. Introduction to the Test System

The micro WEN for the case study is composed of the IEEE 13-bus system and an eight-node WDS from the EPANET manual [20]. The topology of the test micro WEN system is given in Fig. 4. We assume that the 13-bus microgrid is integrated with high penetration of PV resources. Fig. 5 shows the shape of a typical average load in summer [21]. The 24-h load profile of the 13-bus system is generated by applying this load shape with the load provided by IEEE as the load at 9 AM. The total electrical load at 9 AM is 3.26 MW. The 24-hour nodal prices at PCC is shown in Fig. 6. Further detailed information about the PV systems, BESSs, and pumps are given in Table I.
where $f$ is the coefficient of surface resistance, $D$ and $L$ are the diameter and length of pipe, respectively, and $g$ is the gravitational acceleration.

### B. Tightness of the QCH Relaxation

The tightness of the proposed convex relaxation is first evaluated by comparing solutions obtained by solving (CO-OPT) and (C-CO-OPT), respectively. Using the JuMP package of Julia [29], the optimization problems were solved in a MAC computer with a 64-bit Intel i7 dual core CPU at 2.40 GHz and 8 GB of RAM. The MINLP problem (CO-OPT) and its quasi-convex hull relaxation (C-CO-OPT) are solved by calling BONMIN [30] and GUROBI [31] solvers, respectively.

The simulation results are tabulated in Table II. The first and foremost improvement brought by the convexification lies in the computational efficiency. The required CPU time has been significantly reduced by solving the quasi-convex hull relaxation. Note that BONMIN is an open source solver with a limited computational capacity. However, MINLP problems are NP-hard to solve. Even using some well-designed commercial solvers, like KNITRO [32], the CPU time of solving an MINLP problem is still not comparable to that of solving an MICQCQP problem of a similar size.

The proposed quasi-convex hull relaxation is exact for the test case in this paper. It means that the optimal solution obtained by solving (C-CO-OPT) is the exact global optimal solution of its original nonconvex problem (CO-OPT) with zero optimality gap. The numerical results in this subsection demonstrate the potential of the proposed quasi-convex hull relaxation for convexifying similar MINLP problems with high accuracy.

### C. Improvement in System Security

In current practice, the electrical and water systems are operated separately by the electrical and water utilities, respectively. First, the water utility tries to minimize the energy consumption by doing a day-ahead optimal pump scheduling based on the day-ahead water demand forecast (see [1, formulation (OPS)]). Then, with the schedule of energy consumption reported by the water utility, the power operator solves a multiperiod optimization problem (see [1, formulation (UC)]) to minimize the total energy cost for meeting all electricity demands based on the day-ahead forecast of electricity demands, where the diesel generators and BESS units are control devices. It can be observed from Table III that the cooptimization produces a lower cost solution under the same penetration level.

In this section, we also evaluate the improvement in system security introduced by cooptimizing the electrical and water networks. There is a certain limit on the penetration of PV that the distribution system can accommodate. A PV generation that exceed this limit will cause security problem to the system. The simulation results tabulated in Table III demonstrate that, by considering the pumps as controllable loads, the PV penetration that can be accommodated by the power distribution system is increased by 33%. In reality, the proposed cooperation scheme of the micro WEN allows the electricity-driven water facilities
response to energy uncertainties in the power grid, and consequently, improve the system’s security.

D. Efficiency of the DSM Scheme of Tanks

This subsection evaluates the efficiency of the DSM of the flexible irrigation system (12) by comparing the results of (C-CO-OPT) and (C-DSM). Both problems are MICQCQP and solved by GUROBI on the computer mentioned in the previous subsection. The results are tabulated in Table IV. Problem (C-DSM) has more integer decision variables than (C-CO-OPT). However, the CPU time for solving (C-DSM) is not significantly larger than that for solving (C-CO-OPT) due to the property that the nonlinear terms are convex.

In the studied case, 30% of the total water load is for irrigation, and namely, flexible. It can be observed from Table IV that the total operational costs of the micro WEN can be reduced by considering the flexibility of the irrigation systems. A considerable cost saving can be expected if the proposed approach is applied to a larger system and 2) the flexibility of other water facilities, such as water/waste water treatment, desalination, recycling, and cooling, is also included.

VI. CONCLUSION AND FUTURE WORK

This paper introduces a mixed-integer nonlinear mathematical model for the distribution-level WEN, or the micro WEN. A cooptimization framework for water and energy distribution networks is built upon this mathematical model. Based on the convex hulls or quasi-convex hulls of the system components, a tight mixed-integer convex relaxation is developed to improve the computational efficiency of solving the cooptimization framework. When the proposed approach is applied to solve the cooptimization problem of a micro WEN that consists of a 13-bus distribution system and an eight-node water distribution network, the CPU time reduces from nearly 2 h to less than 1 s.

To further explore the capacity of WDSs for providing DR service to the power grids, this paper developed an optimal DR framework considering a flexible irrigation system. Simulation results on the test micro WEN demonstrated the DR potential of water systems. In the future work, we will consider the flexibility of other water facilities, such as water/waste water treatment, desalination, recycling, and cooling, in the DSM model.

The WEN is a fundamental infrastructure in the building/city/village as both water and electricity are lifelines of humans. The findings of this research should be a good fit to the research framework of the smart building/city/village.

APPENDIX

The proof for only the case, where \( k_1 = ac, k_2 = x_3 + x_4, \) and \( d = ac(x_3 + x_4) \), is provided due to the page limit. The set \( \Omega_1 \) represents a nonlinear convex solid in the \( x \)-space that consists of five (linear) facets and four (nonlinear) surfaces. The relation \( \Omega_1 = CH(\Omega_0) \) means \( CH(\Omega_0) \subseteq \Omega_1 \) and \( \Omega_1 \subseteq CH(\Omega_0) \).

1) \( CH(\Omega_0) \subseteq \Omega_1 \)

Let

\[
\Omega_2 = \{ x \mid ax_1^2 + bx_2^2 \leq x_3 x_4 \}
\]

\[
\Omega_3 = \{ x \mid (a-b)x_2^2 + x_3 x_4 \leq ac \}
\]

\[
\Omega_4 = \{ x \mid x_2^2 + x_3^2 - x_3 x_4 \leq 0 \}
\]

\[
\Omega_5 = \{ x \mid x_2^2 + x_3^2 \leq c \}
\]

which are convex sets. It is straightforward to know that \( \Omega_0 \subseteq \Omega_2 \). Under the condition \( x_1^2 + x_2^2 \leq c \), equation \( ax_1^2 + bx_2^2 = x_3 x_4 \) implies \( (a-b)x_2^2 + x_3 x_4 = (a^2 + b^2) \leq ac \). Any point \((x_2, x_3)\) that satisfies \( (a-b)x_2^2 + x_3 x_4 \leq ac \) will also satisfy \( (a-b)x_3^2 + x_2 x_4 \leq ac \). Therefore, \( \Omega_0 \subseteq \Omega_3 \). Similarly, we can prove that \( \Omega_0 \subseteq \Omega_4 \). Equation \( ax_1^2 + bx_2^2 = x_3 x_4 \) also implies \( x_3 x_4 \leq ac \) since \( ax_1^2 + bx_2^2 \leq ac \). From Fig. 7, it suffices to show that \( \Omega_0 \subseteq \Omega_5 \). The convex hull of \( \Omega_0 \) is defined as...
the intersection of all convex relaxations of $\Omega_0$ [33]. Hence, $C H (\Omega_0) \subseteq (\Omega_2 \cap \Omega_3 \cap \Omega_4 \cap \Omega_5) = \Omega_1$. 

2) $\Omega_1 \subseteq C H (\Omega_0)$

If a linear cut is valid for the convex set $C H (\Omega_0)$, it will also be valid for any subset of $C H (\Omega_0)$. Note that “a linear inequality is valid for a set” means the inequality is satisfied by all its feasible solutions [34]. On the other hand, $\Omega_1$ is a subset of $C H (\Omega_0)$ if any valid cut of $C H (\Omega_0)$ is also valid for $\Omega_1$ according to the properties of supporting hyperplanes [33]. Let $a^T x \leq \beta$ denote any given valid linear cut for $C H (\Omega_1)$, it should also be valid for $\Omega_0$. It is straightforward to know that the cut $a^T x \leq \beta$ is valid for $\Omega_1$ if it is valid for all the surfaces of $\Omega_1$.

The mathematical formulation of a surface can be obtained by changing one inequality constraint in $\Omega_1$ into an equality. For instance, Surf$_1$ and Surf$_2$ are two surfaces of solid $\Omega_1$. It suffices to show that $a^T x \leq \beta$ is valid for Surf$_1$ since, in reality, Surf$_1$ = $\Omega_0$. In this appendix, we prove that $a^T x \leq \beta$ is valid for Surf$_2$ as an example.

Let $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4)$ denote any given point on Surf$_2$, where $(a-b)\hat{x}_2 + \hat{x}_4 = ac$. Let us consider the two chosen points $\hat{x}_1 = (\hat{x}_{1,1}, \hat{x}_2, \hat{x}_3, \hat{x}_{1,4})$ and $\hat{x}_2 = (\hat{x}_{2,1}, \hat{x}_2, \hat{x}_3, \hat{x}_{2,4})$ that are located in the original feasible set $\Omega_0$. That means $a\hat{x}_{i,1}^2 + b\hat{x}_{i,2}^2 = \hat{x}_3\hat{x}_{i,4}$ for $i = 1, 2$. By carefully choosing the values of $\hat{x}_{1,1}$ and $\hat{x}_{2,1}$ inside the required bounds, we can make conditions $\hat{x}_{1,1} \leq \hat{x}_1 \leq \hat{x}_{2,1}$ and $\hat{x}_{1,4} \leq \hat{x}_4 \leq \hat{x}_{2,4}$ hold. Thus, it suffices to show that the following condition holds:

$$\hat{x} = \gamma_1 \hat{x}_1 + \gamma_2 \hat{x}_2$$

where $\gamma_1$ and $\gamma_2$ are nonnegative, and $\gamma_1 + \gamma_2 = 1$. There always exist two points $\hat{x}_1$ and $\hat{x}_2$ that satisfy the aforementioned conditions for any given point $\hat{x}$ on Surf$_2$ as long as the intersection of $\Omega_0$ and Surf$_2$ is nonempty. In reality, Surf$_2$ is redundant if its intersection with $\Omega_0$ is an empty set. We do not need to consider the case that Surf$_2$ is redundant.

Given that $\hat{x}_1$ and $\hat{x}_2$ belong to the original set $\Omega_0$, the linear cut $a^T x \leq \beta$ is valid for both of them. Therefore, we have

$$a^T \hat{x} = \gamma_1 a^T \hat{x}_1 + \gamma_2 a^T \hat{x}_2 \leq \gamma_1 \beta + \gamma_2 \beta = \beta$$

which means $a^T x \leq \beta$ is also valid for $\hat{x}$, and consequently, Surf$_2$ since $\hat{x}$ represents any point on Surf$_2$. We do not provide the proof showing that the linear cut is also valid for the rest of surfaces due to the page limit. The readers could complete the proofs themselves by following the aforementioned method for the rest of surfaces of $\Omega_1$ and the rest of the cases in the Lemma.

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Qifeng Li (S’14–M’16) received the Ph.D. degree in electrical engineering from Arizona State University, Tempe, AZ, USA, in 2016. He is currently an Assistant Professor with the Department of Electrical and Computer Engineering, University of Central Florida (UCF), Orlando FL, USA, and a Visiting Assistant Professor with the Department of Mechanical Engineering, Massachusetts Institute of Technology (MIT), Cambridge, MA, USA. Before joining the UCF, he was a Postdoctoral Associate with the Massachusetts Institute of Technology (MIT), Cambridge, MA, USA. His main research interests include modeling, control, and optimization of microgrids and water–energy nexus, integration of renewable generation, and battery energy storage.

Suhyoun Yu received the B.A. degree in mathematics form Wellesley College, Wellesley, MA, USA, in 2014, and the M.Sc. degree in mechanical engineering from the Massachusetts Institute of Technology, Cambridge, MA, USA, in 2017, where she is currently working toward the Ph.D. degree.

Her research interests include Q-learning and general machine learning in the context of water and energy micro nexus.

Ameena S. Al-Sumaiti received the B.Sc. degree in electrical engineering from United Arab Emirates University, Al Ain, United Arab Emirates (UAE), in 2008, and the M.A.Sc. and Ph.D. degrees in electrical and computer engineering (power and energy systems) from the University of Waterloo, Waterloo, ON, Canada, in 2010 and 2015, respectively.

She was a Visiting Assistant Professor with the Massachusetts Institute of Technology, Cambridge, MA, USA, in 2017. She is currently an Assistant Professor with the Department of Electrical and Computer Engineering, Khalifa University, Abu Dhabi, UAE. Her research interests include energy-management systems and power system optimization for smart grids and microgrids.

Konstantin Turitsyn (M’09) received the M.Sc. degree in physics from the Moscow Institute of Physics and Technology, Dolgoprudy, Russia, and the Ph.D. degree in physics from the Landau Institute for Theoretical Physics, Moscow, Russia, in 2007.

He is currently an Associate Professor with the Mechanical Engineering Department, Massachusetts Institute of Technology (MIT), Cambridge, MA, USA. Before joining the MIT, he held the position of Oppenheimer Fellow with Los Alamos National Laboratory, and Kadanoff–Rice Postdoctoral Scholar with the University of Chicago. His research interests include a broad range of problems involving nonlinear and stochastic dynamics of complex systems. His main research interests include energy-related fields including stability and security assessment, and integration of distributed and renewable generation.