Deep learning-based models are utilized to achieve state-of-the-art performance for recommendation systems. A key challenge for these models is to work with millions of categorical classes or tokens. The standard approach is to learn end-to-end, dense latent representations or embeddings for each token. The resulting embeddings require large amounts of memory that blow up with the number of tokens. Training and inference with these models create storage and memory bandwidth bottlenecks leading to significant computing and energy consumption when deployed in practice. To this end, we present the problem of Memory Allocation under budget for embeddings and propose a novel formulation of memory shared embedding, where memory is shared in proportion to the overlap in semantic information. Our formulation admits a practical and efficient randomized solution with Locality sensitive hashing based Memory Allocation (LMA). We demonstrate a significant reduction in the memory footprint while maintaining performance. In particular, our LMA embeddings achieve the same performance compared to standard embeddings with a 16× reduction in memory footprint.

Moreover, LMA achieves an average improvement of over 0.003 AUC across different memory regimes than standard DLRM models on Criteo and Avazu datasets.
as large as tens of millions for each feature, embedding tables can take up over 99.9% of the total memory. Namely, memory footprint can be multiple gigabytes or even terabytes [6, 7, 8]. In practice, deploying these large models often requires the model to be decomposed and distributed across different machines due to memory capacity restrictions [9]. Extensive memory utilization creates memory bandwidth issues due to the highly irregular locality of accesses making training and inference considerably slower [1]. This issue exacerbates further when multiple models need to be co-located on a single machine [9].

**Impact of improving memory usage of Embedding Tables:** Memory consumption of embedding tables is a severe problem in recommendation models. Improving memory utilization can improve recommendation systems on various fronts. 1) It has been observed that larger embedding size in the model leads to better performance [10]. Better memory utilization would imply scope to train and deploy complex models. 2) Lower memory footprint will improve the training and inference speed. With changing consumer interests, recommendation data inherently suffers from concept shift [11], requiring a frequent refresh of models. With faster training, models can be retrained more frequently, improving their average performance. Hence, memory utilization forms a critical problem requiring attention.

Deep learning recommendation model (DLRM) [1] gave rise to an increased interest in constructing more memory-efficient embeddings. Recent SOTA works include compositional embedding using complementary partition [12] and mixed dimension embeddings [7]. A simple memory sharing scheme for weight matrices in deep learning models was proposed by [13]. However, embeddings for tokens are weight matrices that have a structure for which we can reduce the memory burden in an intelligent manner by sharing memory for similar concept tokens. For example, if two tokens represent the concepts “Nike” and “Adidas,” we would expect the pair to share more weights compared to “Nike” and “Jaguar.” In this paper, we approach the problem of learning embedding tables under memory budget by solving a generic problem, which we refer to as **semantically constrained memory allocation (SCMA)** for embeddings. SCMA is a hard combinatorial allocation problem since it involves millions of variables and constraints. Surprisingly, as we show later, that there is a very neat dynamic allocation of memory using locality sensitive hashing which solves SCMA in approximation. The memory allocation can be done in an online consistent manner for each token with negligible overhead.

This paper is organized as follows: we first formally describe the problem and notation in section 2, followed by a recap of hashing schemes in section 3 and a generic solution to the problem in section 4. Section 5 onward we focus on applying the solution to DLRM problem. Section 6 discusses some related work and we present experimental evaluation in section 7.

**2 Semantically Constrained Memory Allocation (SCMA) for Embeddings**

Let $S = \{0, 1, \ldots, |S|−1\}$ denote the set of all values for all categorical features in the dataset. The embedding table $E \in \mathbb{R}^{[S] \times d}$ is a matrix such that each row represents embeddings, each with dimension $d$, for all values in $S$. An embedding, $e_v$, for a particular value, is retrieved by $e_v = E[v, :]$ (see Figure 1). The embedding table $E$ imposes a similarity structure $S$ w.r.t a similarity measure obtained by a kernel function $\phi(\ldots)$. $S \in \mathbb{R}^{[S] \times [S]}$ and each entry of this matrix $S[v, v] = \phi(E[v_1, :], E[v_2, :])$ also denoted as $\phi(v_1, v_2)$.

The problem of end-to-end learning of full embedding table $E$ is well-known [1]. In this paper, we consider the problem of learning $E$ under a memory budget $m$. Let $\mathcal{M}$ be the memory such that $|\mathcal{M}| = m$. If $|S| \times d > m$, then multiple elements of the embedding table have to share memory locations in $\mathcal{M}$. We formally define allocation function $A$ as the mapping from elements of $E$ to the actual memory locations in $\mathcal{M}$.

**Definition 1 (Allocation Function)** Allocation function $A$ for an embedding table $E \in \mathbb{R}^{[S] \times d}$ using the memory $\mathcal{M}$ with budget $|\mathcal{M}| = m$, is defined as the following map.

$$A : \{0, \ldots, |S|−1\} \rightarrow \{0, 1\}^{d \times m} \quad \text{s.t.} \quad \forall v \in S, \forall i \in \{0, \ldots, d−1\}, \sum_j (A(v)[i, j]) = 1.$$  

The $i^\text{th}$ row of the matrix output by $A$, for any value $v$ is a one-hot encoding of the location to which the $i^\text{th}$ element of the embedding vector $v$ maps to in $\mathcal{M}$. Using the allocation function, we can retrieve the embedding by $E[v, i] = \mathcal{M}[A(v)[i, :]]$ where we assume mask-based retrieval on $\mathcal{M}$. We next define the notion of shared memory between two embeddings under a allocation function $A$.

**Definition 2 (Consistent Memory Sharing)** The fraction of Consistently Shared Memory ($f_A$) between two embeddings of size $d$ for values $v_1$ and $v_2$ from $E$ under allocation $A$ is defined as

$$f_A(v_1, v_2) = \frac{1}{d} \langle A(v_1), A(v_2) \rangle_F$$

where $\langle ., . \rangle_F$ is the Frobenius inner product.
We can describe various allocation schemes in terms of the allocation function. For example, \( A_{full} \) describes full embedding which is possible when \( m > |S|d \). \( A_n \) describes the naïve hashing trick (hash function \( h \)) based allocation (see section 6 for details) which works with any memory budget. Specifically, \( \forall v \in S \) and \( i \in \{0, .., d-1\} \), we have the following:

\[
A_{full}(v)[i, j] = 1, \quad \text{iff } j = mv + i.
\]

\[
A_n(v)[i, j] = 1, \quad \text{iff } j = h(v, i).
\]

It is easy to verify that for every pair of values \( v_1 \) and \( v_2 \)
\[
f_{A_{full}}(v_1, v_2) = 0 \quad \text{whereas } \quad f_{A_n}(v_1, v_2) = 0 \quad \text{is random variable with expected value } \frac{1}{m}.
\]

We now define the notation we will use throughout the paper.

**Definition 3 (General Memory Allocation (GMA))**

Let \( E^* \) be the true embedding table which encodes the similarity structure \( S^* \). Let \( E \) be an embedding table recovered from \( M \) with budget \( m \) under allocation \( A \) and \( S \) be the semantic similarity encoded by \( E \). Let both similarities be encoded by the kernel function \( \phi(\cdot) \). We will refer to this as General Memory Allocation (GMA) setup.

**Thought experiment:** We can define the problem of optimal allocation under memory budget when \( m < |S|d \), under GMA setup as:

\[
\arg\min_{A} \min_{M} \sum_{v \in S} \zeta(S^*, S, A)
\]

where \( \zeta \) is a metric on \( \mathbb{R}^{|S| \times |S|} \) (e.g., Euclidean). In order to incorporate the learning of \( M \), we pose the problem as finding \( A \) with best possible associated \( M \) considering that if we choose \( A \) beforehand and then learn \( M \), the learning process will choose the best \( M \). Solving this exact problem appears to be hard. Instead, let us think about an allocation scheme for which we have some evidence of the existence of suitable \( M \). Let us consider \( M \) with each element independently initialized using Bernoulli(0.5, \{−1, +1\}). If we choose an allocation with constraints based on similarity structure \( S^* \) as

\[
f_A(v_1, v_2) = S^*[v_1, v_2] \quad \forall v_1, v_2 \in S
\]

then one can verify (as we will show in Theorem 2 that under this random initialization of memory, pairwise cosine similarity of embeddings of any two values \( v_1 \) and \( v_2 \) retrieved from \( M \) via \( A \), denoted as \( C_A(v_1, v_2) \), is a random variable with expectation \( \mathbb{E}(C_A(v_1, v_2)) = S^*[v_1, v_2] \) and variance \( \text{Var}(C_A(v_1, v_2)) \approx \frac{1}{2} \). Hence, this provides evidence that for a semantic similarity based shared allocation, \( A_{S^*} \), there is an assignment to \( M \), which can produce reasonably small \( \zeta(S^*, S, A) \). We interpret the \( \pm 1 \) assignment to each element of embedding as membership to particular concepts, and the overlap in membership determines the similarity. Following this insight, we formally define the semantically constrained memory problem allocation (SCMA) for embeddings as follows.

**Definition 4 (SCMA Problem)** Under the GMA setup (see Def. 3), Semantically Constrained Memory Allocation (SCMA) is a problem to find allocation \( A \), under the constraints that for every pair \( i, j \in S \), we have \( f_A(i, j) = S^*[i, j] \).
We call parameter \( v \) of collision of values kernel be to this RSCMA. Let the LSH family corresponding to this semantic structure. Consider the standard GMA setup (see Def. 3) with the \( J \) measures the similarity between two given sets \( A, B \). The minwise family is as defined below.

The probability of collision defines a kernel function \( \phi(x, y) \), which is bounded \( 0 \leq \phi(x, y) \leq 1 \), symmetric \( \phi(x, y) = \phi(y, x) \), and reflective \( \phi(x, x) = 1 \). We can create multiple LSH families parameterized by \( k, L_k \), from a given LSH family \( L \) by using \( k \)-independently drawn functions from \( L \). Let \( \{l_i\}^{k}_{i=1} \) be \( k \) independently drawn functions from \( L \). The new LSH function \( \psi \in L_k \) and its kernel function is defined as

\[
\psi(x) = (l_1(x), l_2(x), \ldots, l_k(x)); \quad \phi(x, y)_L = \phi(x, y)_L^k.
\]

We call parameter \( k \), the power of LSH functions. The range of certain LSH functions, particularly functions with large power, can be extremely large and needs rehashing into a range, say \( \{0, \ldots, r-1\} \). This is generally achieved using additional universal hash functions, say \( h \) with range \( r \). Let the rehashed version of the function \( L \) be denoted by \( L_r \). Then, the kernel of this rehashed LSH is

\[
l_r(x) = h(l(x)); \quad \phi(x, y)_L = \phi(x, y)_L + \frac{1 - \phi(x, y)_L}{r}.
\]

### 3.3 Minwise Hashing

The minwise hash function is a LSH function that take sets as inputs. The minwise family is as defined below.

\[
L_{mh} = \{l_r \mid r \rightarrow U, \pi \text{ is a permutation}\},
\]

\[
l_r(A) = \min(\{\pi(x) \mid x \in A\}) \quad \text{ where } A \subseteq U
\]

For a particular function, \( l_r \), the hash value of \( A \) is the minimum of the permuted values of elements in \( A \). As it turns out, the kernel function defined by the collision probability of the minwise hash family is the Jaccard Similarity (J).

\[
J(A, B) = |A \cap B|/|A \cup B|. \text{ It is easy to check that } \phi(A, B)_{L_{mh}} = J(A, B).
\]

### 4 LSH based Memory Allocation (LMA): Solution to RSCMA

Consider the standard GMA setup (see Def. 3) with the semantic structure \( S^* \) defined by a LSH kernel \( \phi(.) \). We provide an LSH based memory allocation (LMA) solution to this RSCMA. Let the LSH family corresponding to this kernel be \( L \). As defined in Section 3.2, the probability of collision of values \( v_1 \) and \( v_2 \) for corresponding LSH function \( l \) and rehashed LSH function \( l_r \), can be written as

\[
\text{Pr}_{l \in L}(l(v_1) = l(v_2)) = \phi(v_1, v_2),
\]

\[
\text{Pr}_{l \in L}(l_r(v_1) = l_r(v_2)) = \phi(v_1, v_2) + \frac{1 - \phi(v_1, v_2)}{m}.
\]

We use \( d \) independently drawn LSH functions \( \{l^{(i)}\}^{d}_{i=1} \).

The LMA solution defines the allocation \( A_L \) as

\[
A_L(v)[i] = \text{one-hot}(l^{(i)}(v)) \quad \forall v \in S,
\]

where one-hot : \( \{0, \ldots, m-1\} \rightarrow \{0, 1\}^m \) such that for any arbitrary \( i \in \{0, \ldots, m-1\}, i \neq j \), one-hot(i)[j] = 0 and one-hot(i)[i] = 1. LMA scheme is illustrated in Figure 2.

In the following theorem, we prove that LMA with the allocation defined by \( A_L \) indeed solves the RSCMA problem. The proof of this theorem is present in the Supplementary.

**Theorem 1 (LMA solves RSCMA) Under the GMA setup (see Def. 3), for any two values \( v_1 \) and \( v_2 \), the fraction of consistently shared memory \( f_{A_L} \) as per allocation \( A_L \) proposed by LMA is a random variable with distribution,

\[
\mathbb{E}(f_{A_L}(v_1, v_2)) = \Gamma = \phi(v_1, v_2) + \frac{1 - \phi(v_1, v_2)}{m},
\]

\[
\forall (f_{A_L}(v_1, v_2)) = \frac{\Gamma (1 - \Gamma)}{d},
\]

\[
\text{Pr}
\left[
\left|f_{A_L}(v_1, v_2) - \phi(v_1, v_2)\right| > \eta \Gamma + \frac{1 - \phi(v_1, v_2)}{m}
\right]
\leq 2 \exp\left\{-\frac{-d \eta^2}{3}\right\},
\]

for all \( \eta > 0 \). Hence, LMA solves RSCMA with \( \epsilon = \eta \Gamma + \frac{1 - \phi(v_1, v_2)}{m} \) and \( \delta = 2 \exp\left\{-\frac{-d \eta^2}{3}\right\} \).

**Proof sketch:** Proof consists of analyzing the random variable for the fraction \( f_{A_L}(v_1, v_2) \) and applying Chernoff concentration inequality to obtain the tail bounds.

**Interpretation:** A reasonable memory \( M \) of 10Mb would have \( |M| > 10^6 \). Hence for all practical purposes, we can ignore the \( 1/m \) terms above. Then, the consistently shared fraction has expected value \( \phi(v_1, v_2) \) and variance that is proportional to \( 1/d \). A good way to visualize the fact that LMA indeed gives a solution to RSCMA problem is to see the 95% confidence interval of the fraction, \( f_{A_L}(v_1, v_2) \),
against the value of $\phi(v_1, v_2)$ as shown in Figure 3. The parameter $\eta$ is a standard parameter that controls the trade-off between error ($\epsilon$) and confidence $1 - \delta$. We can choose $\eta$ very small to reduce error, but then we lose confidence, and instead if we choose $\eta$ large enough to reduce $\delta$, hence increase confidence, then we have more error.

Next, we prove that if we use LMA to solve RSCMA, we indeed provide an allocation, for which there is an assignment of values to $\mathcal{S}$ which can lead to an embedding table $E$, whose associated similarity $S$ as measured by cosine similarity is closely distributed around $S^*$ and hence gives smaller $\zeta(S, S^*)$ (notation as introduced in section 2).

**Theorem 2 (Existence of $\mathcal{M}$ with LMA for $S^*$) Under the GMA setup (see Def. [1]), let us initialize each element of $\mathcal{M}$ independently from a Bernoulli(0.5, {−1, +1}). Then, the embedding table $E$ generated via LMA on this memory, has, for every pair of values $v_1$ and $v_2$, the cosine similarity $C_s(E[v_1,:], E[v_2,:])$, denoted by $C_s(v_1, v_2)$ is distributed as

$$E(C_s(v_1, v_2)) = \Gamma = \phi(v_1, v_2) + \frac{1 - \phi(v_1, v_2)}{m},$$

$$\text{Var}(C_s(v_1, v_2)) = \frac{1 - \Gamma^2}{d} + \frac{2(1 - \Gamma)(d - 1)}{dm^2},$$

$$\Pr \left( |\phi(v_1, v_2) - \phi(v_1, v_2)| \geq \eta \Gamma + \frac{1 - \phi(v_1, v_2)}{m} \right) \leq 1 - \frac{\Gamma^2}{d \Gamma^2 + 2}, \text{ for any } \eta > 0.$$

**Proof sketch:** Proof consists of analyzing the random variable for the cosine similarity $C_s(E[v_1,:], E[v_2,:])$ and applying Chebyshev’s concentration inequality to obtain the tail bounds.

**Interpretation:** We can ignore $1/m$ for any reasonably large memory. Then, the expected value of cosine similarity is exactly $\phi(v_1, v_2)$ and it is closely distributed around it. Chebyshev’s inequality gives a looser bound and that is apparent from the probable region shown in Figure 3.

In this formulation, again $\eta$ is the parameter controlling error and confidence. Theorem 3 shows that if we have such a randomly initialized memory, then LMA will lead to intended similarities in approximation.

### 4.1 LMA: Dynamic Solution to RSCMA

Unlike any static solution (e.g. MIP) to SCMA, LMA solution to RSCMA is highly relevant in a real-world setting. The addition of new features values to datasets is generally frequent, particularly in recommendation data. Any static solution to SCMA will need re-computation every time a value is added. LSH based LMA solution is unaffected by this and can gracefully incorporate new values.

### 5 LMA for Recommendation Systems

Let us now consider the application of LMA for embeddings of categorical values in ML recommendation systems. DLRM can be used as a running example. However, our approach applies to any system that uses embedding tables. When learning from data in practical settings, $S^*$ is often unknown as $E^*$ is not known. However, we can use the Jaccard similarity between pairs of values and use the similarity structure to obtain a proxy for $S^*$. We compute the Jaccard similarity as follows. Let $D_v$ be the set of all sample ids in data in which the value $v$ for some categorical feature appears (e.g., a row in a term-document matrix). Then, we can define the similarity $S^*$ as

$$S^*(v_1, v_2) = |J(D_{v_1}, D_{v_2})|.$$

The resulting kernel is the Jaccard kernel, which is an LSH kernel with the minwise LSH function. To hash a value, say $v$, with the minwise hash function, we will use the corresponding set $D_v$. We can then use the general LMA setup described in Section 2 to adjust our training algorithms.

**Size of data required:** It may appear that storing the data for minhash computations would diminish memory savings. However, recall that we need data only to obtain a good estimate of Jaccard. We find that the actual data required, $D'$, is significantly less than the total data. Namely, $|D'| \ll |S|/d$. Therefore, to formalize, we bound the number of non-zeros required for Jaccard computation.

**Theorem 3 ($D'$ required is small) Assuming a uniform sparsity of each value, the Jaccard Similarity of two values $x$ and $y$, say $J = J(D_x, D_y)$ when estimated from a i.i.d subsample $D' \subseteq D$, $|D'| = n$ is distributed around $J$ as follows.

$$|E(\hat{J}) - J| \leq \epsilon J,$$

$$|\text{Var}(\hat{J}) - A| \leq 2\varepsilon(A + 2J^2),$$

where $A = \frac{1}{n^2} + 2 - 2sJ$ with probability $1 - \delta$ where $\delta = \frac{1 + J - 2sJ}{2n^2}\varepsilon^2$.

**Interpretation:** The bound given by Theorem 3 is loose due to the approximations done to analyze this random variable (see Supplemental). Nonetheless, it can be seen that for a given value of $\delta$, we can control $\epsilon$ and $A$ through $n.s$, which is the number of non-zeros per value. As $n.s$ increases, both the variance and deviation of the expected value from $J$ decrease rapidly. In practice, Section 4 shows we only need around 100K samples for the Criteo dataset out of 4.5M samples. Below we discuss a few considera-
tions in relation to LMA applied to DLRM model.

**Memory Comparison:** The size of the memory used by full embeddings is $O(|S|d)$. The linear dependence on $d$ and the typical very large size of set $S$ makes it difficult to train and deploy large dimensional embedding models. LMA makes it possible to simulate the embeddings of size $O(|S|d)$ using any memory budget $m$. The memory cost with LMA procedure comprises of: 1) Memory budget: $O(m) = |\mathcal{M}|$; 2) Cost of storing LSH function: $O(kd + k')$ required to store $d$ minhash functions with power $k$ and $k'$-universal hash function for rehashing. Generally these values are very small compared to $O(|S|d)$ and can be ignored; and 3) The size of Data $D'$ stored and used for minhash functions. Generally, size of $D'$ is much smaller than $|S|d$. For example in Criteo, the subsample we used had around 3M integers (when using 125K samples) as compared to the range of 50M-540M floating parameters of the models we train. We empirically analyze the effect on various sizes of $D'$ in the experimental section. This requirement of smaller $D'$ is also an effect of the way we handle very sparse features, which is discussed next. To summarize, the memory cost of LMA is $O(m + kd + k' + |D'|) \approx O(m)$.

**Handling very sparse features:** For very sparse features, the embedding quality does not significantly affect the accuracy of the model [6]. Also, it is difficult to get a good estimate of Jaccard similarity for this using a small subsample. Due to these reasons, for very sparse features, we randomly map each element of its embedding into $\mathcal{M}$. Essentially, we revert to $A_k$ (naive hashing trick) for such values.

**Common Memory:** We use a single common memory $\mathcal{M}$ across all embedding tables in DLRM. The idea is to fully utilize all similarities to share maximum memory and hence get the best memory utilization.

**Forward Pass:** The forward pass requires retrieving embeddings from $\mathcal{M}$. Let $V_{batch}$ be the set of all values in a batch. We collect the set $\{D_v \mid v \in V_{batch}\}$, apply GPU friendly implementation of $d$-minhashes to it to obtain a matrix of locations, $I \in \mathbb{R}^{V_{batch} \times d}$. Using this we get $E_{batch} = \mathcal{M}[I]$.

**Backward Pass:** The memory $\mathcal{M}$ contains the parameters which are used to construct embeddings. Hence, in the backward pass, the gradient of parameters in $\mathcal{M}$ is computed and these parameters are updated. The exact functional dependence of the result on a parameter in $\mathcal{M}$ is complex as it is implemented via LSH mappings. Auto gradient computation packages of deep learning libraries (e.g., PyTorch and TensorFlow) are used for gradient computation.

### 6 Related Work

We focus on related works that significantly reduce the size of the embedding matrix for recommendation and click-through prediction systems. Namely, hashing trick [13], compositional embedding using complementary partitions [12], and mixed dimension embedding [6].

**Naive hashing trick:** Given the embedding table $E \in \mathbb{R}^{|S| \times d}$, two basic approaches that leverage hashing trick are presented.

- Vector-wise (or row-wise): let $\hat{E} \in \mathbb{R}^{m \times d}$ such that $m \times d = |\mathcal{M}|$, the memory budget, and $m \ll |S|$ be the reduced size embedding table. We use a hash function $h : \{0, 1, \ldots, |S| - 1\} \rightarrow \{0, 1, \ldots, m - 1\}$ that maps the (row-wise) indices of the embeddings from the full embedding table $E$ to the reduced embedding table $\hat{E}$. The size of the embedding table is reduced from $O(|S|d)$ to $O(md)$.

- Element-wise (or entry-wise): let $\hat{E} \in \mathbb{R}^m$ such that $m = |\mathcal{M}|$. We use a hash function $h : \{(i, j)\}_{i, j = 0}^{d} \rightarrow \{0, 1, \ldots, m - 1\}$ that maps each element $E_{k, j}$ to an element in $\hat{E}$. The size of the embedding table is reduced from $O(|S|d)$ to $O(m)$ [13].

**Compositional embedding using complementary partitions:** In the vector-wise hashing trick, multiple embedding vectors are mapped to the same index, which results in loss of information on the unique categorical values and reduction in expressiveness of the model. To overcome this issue, [12] proposes to construct compositional partitions of $E$, from set theory, and apply compositional operators on embedding vectors from each partition table to generate unique embeddings. One example of a compositional partition is the so-called quotient-remaider (QR) trick. Two embedding tables $E_1 \in \mathbb{R}^{m \times d}$ and $E_2 \in \mathbb{R}^{|S|d \times d}$ are created, and two hash functions $h_1$ and $h_2$ respectively are used for mapping. $h_1$ maps the $i$-th row of $E$ to the $j$-th row of $\hat{E}_1$, using the remainder function: $j = i \mod m$. $h_2$ maps the $i$-th row of $E$ to the $k$-th row of $\hat{E}_2$ using the function $k = i \mod d$, where $\mod$ denotes integer division (quotient). Taking the embeddings $e_j \in \hat{E}_1$ and $e_k \in \hat{E}_2$ and applying element-wise multiplication $e_j \odot e_k$ results in a unique embedding vector. The resulting memory complexity is $O(\frac{|S|d + md}{m})$. In general, the optimal memory complexity is $O(k|S|^{1/k}d)$ with $k$ compositional partitions of sizes $\{m_i \times d\}_{i=1}^{k}$ such that $|S| \leq \Pi_{i=1}^{k} m_i$.

**Mixed dimension (MD) embedding:** Frequency of categorical values are often skewed in real-world applications. Instead of a fixed (uniform) embedding dimension for all categories, [6] proposes that embedding dimensions scale according to popularity of categorical values. Namely, more popular values are set to higher dimension embeddings (i.e., allocate more memory) and vice versa. The idea is to create embedding tables, along with a projection matrix, of the form $\{(\hat{E}_i, \hat{P}_i)\}_{i=1}^{k}$, such that $\hat{E}_i \in \mathbb{R}^{|S| \times d}$ (MD embedding), $\hat{P}_i \in \mathbb{R}^{d_i \times d}$ (projection), $d_i \geq d$, and $|S| = \sum_{i=1}^{k} |S_i|$, $k$ and $d = (d_1, \ldots, d_k)$ are input parameters. One approach proposed to set the parameters is based on a power-law sizing scheme using a meta temperature parameter $\alpha$. Let $p = \{p_1, \ldots, p_k\}$, where $p_i$ is a probability value (e.g., $p_i = 1/|S_i|$). Then, $\lambda = d |p|^{-\alpha}$ is the scaling factor and $d = \lambda p^\alpha$ is the component-wise
Table 1: Description of datasets. cat: categorical, int: integer.

| Dataset | #Samples | #Features (cat+int) | Positive rate | #Values |
|---------|----------|---------------------|---------------|---------|
| Criteo  | 46M      | 26 + 13             | 26%           | 33.76M  |
| Avazu   | 41M      | 21 + 0              | 17%           | 9.45M   |

Comparison with LMA: Both QR Trick and MD Trick change the embedding design. LMA does not affect embedding design directly. Instead, it solves the abstract problem of RSCMA. We can apply LMA in conjunction with any such embedding design tricks to obtain better memory utilization.

7 Experiments

Datasets: To evaluate the performance of LMA on DLRM (LMA-DLRM), we use two public click-through rate (CTR) prediction datasets: Criteo and Avazu. Criteo is used in the DLRM paper [1] and related works focused on memory-efficient embeddings [6, 12]. Avazu is a mobile advertisement dataset. A summary of dataset properties is presented in Table 1. Values represent the number of all categorical values. The number of values per feature varies a lot; for example, some lower values are 10K and they go as high as 10M.

Metrics: We use loss, accuracy, and ROC-AUC as metrics for our experiments. For imbalanced datasets like these, AUC is a better choice of metric than accuracy. We describe these parameters and qualitatively analyze their effect on performance. The results of changing one parameter while keeping the other two fixed are shown in Figure 5. We use 270M budget for varying α and 35M budget for others.

- **Power of LSH** (n_h): The n_h used per LSH mapping (i.e., power in Section 3.2) controls the probability of collision (i.e., J^{n_\alpha}) of corresponding elements of different embeddings as discussed in Section 3.2. Higher power leads to a lower probability of collision, making the hashed LSH function behave more like naïve hashing trick. Very low power will increase memory sharing and might lead to its under-utilization. This phenomenon can be observed in Figure 5a, where n_h=1 gives the worst performance. Increasing n_h improves the performance until n_h = 8. Performance worsens when n_h = 32 and tends towards the hashing trick performance as expected.

- **Expansion rate** (α): LMA-DLRM can simulate embedding tables of any dimension d. We define expansion rate as the ratio of simulated memory to actual memory. Using GMA notation, α = |S|d/|M|. Figure 5b shows that 16 works best for memory budget of ~270M parameters. So increasing α will not improve the performance indefinitely.

- **Samples in D′ (n_s):** Figure 5c shows that the performance boost saturates after the representation size reaches 75K data points, which means that most of the frequently appearing values (v) get decent representations (D_v) from a small number of samples. For very sparse values, we revert to element-wise naïve hashing trick based mapping. We also show the size of the samples in terms of the number of non-zeros integers to be stored. As compared to 540M parameter networks we train a sample of 125K only requires us to store 3.2M integers.

7.2 Main Experiment

We compare LMA-DLRM against full embedding (embedding tables used in DLRM [1]), HashedNet [13] embedding (naïve element-wise hashing trick based), and QR embeddings [12] across different memory budgets. Hyper-parameters n_h=4, α=16, and n_s=125, 000 were used for all LMA-DLRM experiments in this section. For baselines, we use the configurations in their open source code. Training of models was cutoff at 15 epochs. We did not perform
### Results

Figure 6 shows AUC, accuracy, and loss against different memory regimes (number of parameters) on both datasets. Figure 7 shows the evolution of some models for the first 5 epochs.

- **LMA-DLRM outperforms all baselines across different memory regimes, including those reported in [1], achieving average AUC improvement of **0.0032** in Criteo and **0.0034** in Avazu across memory budgets and an average improvement in Accuracy of **0.0017** on Criteo. Recall that an improvement of 0.001 is significant.

- The AUC and accuracy of full embedding models with 540M parameters can be achieved by LMA-DLRM with only 36M parameters (16× reduction) on Criteo. On Avazu, results with 10M parameter LMA model are much better than full embedding model with 150M parameters (15× reduction).

- LMA-DLRM achieves best AUC **0.805** as opposed to the best of full embedding **0.802** on Criteo. On Avazu, we see improvement of **0.0025** on best AUCs as well.

- The typical evolution of AUC metric for Criteo and Avazu on models of different sizes for LMA-DLRM and full embedding models clearly supports the better performance of LMA-DLRM over full embeddings.

### 8 Conclusion

We define two problems namely SCMA (Semantically Constrained Memory Allocation) and Randomized SCMA for efficient utilization of memory in Embedding tables. We propose a neat LSH-based Memory Allocation (LMA) which solves the dynamic version of RSCMA under any memory budget with negligible memory overhead. LMA was applied to an important problem of heavy memory tables in widely used recommendation models and found tremendous success. In this paper, we focus on the memory aspect of LMA. In future work, we would like to benchmark the LMA method for its time efficiency for training and inference for recommendation models.
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A  LMA solves RSCMA

Under the GMA setup (see def [3]), for any two values $v_1$ and $v_2$, the fraction of consistently shared memory $f_{A_L}$ as per allocation $A_L$ proposed by LMA is a random variable with distribution,

$$E(f_{A_L}(v_1, v_2)) = \Gamma = \phi(v_1, v_2) + \frac{1-\phi(v_1, v_2)}{m},$$

$$V(f_{A_L}(v_1, v_2)) = \frac{\Gamma(1-\Gamma)}{d},$$

$$Pr\left( |f_{A_L}(v_1, v_2) - \phi(v_1, v_2)| > \eta \Gamma + \frac{1-\phi(v_1, v_2)}{m} \right) \\ \leq 2 \exp \left\{ \frac{-d\eta^2}{3} \right\},$$

for all $\eta > 0$. Hence, LMA solves RSCMA with $\epsilon = \eta \Gamma + \frac{1-\phi(v_1, v_2)}{m}$ and $\delta = 2 \exp \left\{ \frac{-d\eta^2}{3} \right\}$.

Proof sketch: Proof consists of analyzing the random variable for the fraction $f_{A_L}(v_1, v_2)$ and applying Chernoff concentration inequality to obtain the tail bounds.

Proof

The probability that a particular location is shared is exactly the probability of collision of the rehashed lsh function $l_r$. (using the notation from section [4])

$$Pr(l_r(x) == l_r(y)) = \phi(x, y) + \frac{1-\phi(x, y)}{m} \quad (1)$$

We can write the indicator for fraction of consistently shared memory as,

$$f_{A_L} = \sum_{i=1}^{d} I(l_r^{(i)}(x) == l_r^{(i)}(y)) \quad (2)$$

$f_{A_L}$ is a sum of independent bernoulli variables. It is easy to check that the expected value of consistently shared fraction is,

$$E(f_{A_L}) = \Gamma = \phi(x, y) + \frac{1-\phi(x, y)}{m} \quad (3)$$

$$V(f_{A_L}) = \frac{\Gamma(1-\Gamma)}{d} \quad (4)$$

We can apply the chernoff’s bound to get tail bound. Let $\eta > 0$ be any positive real number

$$Pr \left( |f_{A_L}(v_1, v_2) - \Gamma| > \eta \Gamma \right) \leq 2 \exp \left\{ \frac{-d\eta^2}{3} \right\} \quad (5)$$

$$Pr \left( |f_{A_L}(v_1, v_2) - \phi(v_1, v_2)| > \eta \Gamma + \frac{1-\phi(v_1, v_2)}{m} \right) \leq 2 \exp \left\{ \frac{-d\eta^2}{3} \right\} \quad (6)$$

B  Existence of $M$ with LMA for $S^*$

Under the GMA setup (see Def. [3]), let us initialize each element of $M$ independently from a Bernoulli$(0.5, \{-1, +1\})$. Then, the embedding table $E$ generated via LMA on this memory, has , for every pair of values $v_1$ and $v_2$, the cosine similarity $C_s(E[v_1], E[v_2])$, denoted by $C_s(v_1, v_2)$ is distributed as

$$E(C_s(v_1, v_2)) = \Gamma = \phi(v_1, v_2) + \frac{1-\phi(v_1, v_2)}{m},$$

$$V(C_s(v_1, v_2)) = \frac{1-\Gamma^2}{d} + \frac{2(1-\Gamma)(d-1)}{dm^2},$$

$$Pr \left( |C_s(v_1, v_2) - \phi(v_1, v_2)| \geq \eta \Gamma + \frac{1-\phi(v_1, v_2)}{m} \right) \leq \frac{1-\Gamma^2}{d\eta^2 \Gamma^2} \text{ for any } \eta > 0.$$
Analysis of Variance is a bit involved due to interdependence between each of the terms in the summation above that makes it difficult to analyze independently.

The norm of any embedding of dimension $d$ drawn from this distribution is: \[ \| \mathbf{V} \| = \sqrt{\mathbb{E}[\sum_{i=1}^{d} (I(l_i(x) == l_i(y)), \mathcal{M}[l_i(x)]^2 + I(l_i(x) != l_i(y)), \mathcal{M}[l_i(x)], \mathcal{M}[l_i(y)])]} \]

Let $\Gamma = \phi(x, y) + \frac{1 - \phi(x, y)}{m}$

\[ E(\langle \mathbf{E}_x, \mathbf{E}_y \rangle) = d\Gamma \]

\[ E(Cosine(\mathbf{E}_x, \mathbf{E}_y)) = \Gamma \]

Analysis of Variance is a bit involved due to interdependence between each of the terms in the summation above that makes it difficult to analyze independently.

**Case 1: Assume independence**

\[ \mathbb{V}(\langle \mathbf{E}_x, \mathbf{E}_y \rangle) = d(\mathbb{V}(I(l_i(x) == l_i(y))) + I(l_i(x) != l_i(y)), \mathcal{M}[l_i(x)], \mathcal{M}[l_i(y)])] \]

\[ \mathbb{V}(\langle \mathbf{E}_x, \mathbf{E}_y \rangle) = d(\Gamma(1 - \Gamma) + (1 - \Gamma)) \]

\[ \mathbb{V}(\langle \mathbf{E}_x, \mathbf{E}_y \rangle) = d(1 + \Gamma)(1 - \Gamma) \]

**Case 2: Don’t assume independence.**

\[ Cosine(\mathbf{E}_x, \mathbf{E}_y) = \frac{1}{d} \langle \mathbf{E}_x, \mathbf{E}_y \rangle \]

\[ Cosine(\mathbf{E}_x, \mathbf{E}_y) = \frac{1}{d} \sum_{i=1}^{d} (I(l_i(x) == l_i(y)), \mathcal{M}[l_i(x)]^2 + I(l_i(x) != l_i(y)), \mathcal{M}[l_i(x)], \mathcal{M}[l_i(y)])] \]

\[ E(Cosine(\mathbf{E}_x, \mathbf{E}_y)) = \Gamma \]

\[ E(\langle Cosine(\mathbf{E}_x, \mathbf{E}_y) \rangle^2) = \frac{1}{d^2} E(\langle \mathbf{E}_x, \mathbf{E}_y \rangle^2) \]

\[ E(\langle Cosine(\mathbf{E}_x, \mathbf{E}_y) \rangle^2) = \frac{1}{d^2} E(\sum_{i=1}^{d} (ex_i ey_i)^2 + \sum_i \neq j (ex_i ey_i)(ex_j ey_j)) \]

\[ E(\langle Cosine(\mathbf{E}_x, \mathbf{E}_y) \rangle^2) = \frac{1}{d^2} E(\langle ex_i ey_i \rangle^2) + (d - 1) E(ex_i ey_i(ex_j ey_j)) \]

\[ E(\langle Cosine(\mathbf{E}_x, \mathbf{E}_y) \rangle^2) = \frac{1}{d^2} (1 + (d - 1)E\mathcal{M}[l_i(x)], \mathcal{M}[l_j(x)], \mathcal{M}[l_i(y)], \mathcal{M}[l_j(x)]) \]
Using table\[\text{2}\]

\[E((\text{Cosine}(ex, ey))^2) = \frac{1}{d}(1 + (d - 1)(\Gamma^2 + 2(1 - \Gamma)) \frac{1}{m^2})\]  

(24)

\[\text{Var}((\text{Cosine}(E_x, E_y))) = E((\text{Cosine}(E_x, E_y))^2) - E((\text{Cosine}(E_x, E_y))^2)\]  

(25)

\[\text{Var}((\text{Cosine}(ex, ey))) = \frac{1}{d}(1 + (d - 1)(\Gamma^2 + 2(1 - \Gamma)) \frac{1}{m^2}) - \Gamma^2\]  

(26)

Collecting terms

\[\text{Var}((\text{Cosine}(ex, ey))) = \frac{1}{d}(1 - \frac{2(1 - \Gamma)}{m^2} - \Gamma^2) + 2d(1 - \Gamma)\frac{1}{m^2}\]  

(27)

\[\text{Var}((\text{Cosine}(ex, ey))) = \frac{1 - \Gamma^2}{d} + 2(1 - \Gamma)(d - 1) \frac{1}{dm^2} \approx \frac{1}{d}(1 - \Gamma^2)\]  

(28)

C Procedures requires small data sample

Assume that the real dataset is of size N, the sparsity of each feature value is s, i.e. the probability of a feature appearing in an example is s. For simplicity, let us assume that each feature value has the same sparsity. This may not be generally true. But nonetheless it helps us draw an idea of data sample needed. Let us consider the situation where the two features have jaccard similarity J. The venn diagram below shows the distribution of samples in our case.

Figure 8: Venn diagram for two features f1, f2

Let the events be as follows:

- \(A_i\): \(i^{th}\) sample has feature f1
- \(B_i\): \(i^{th}\) sample has feature f2

The estimated jaccard similarity after drawing n random i.i.d samples would be,

\[\hat{J} = \frac{\Sigma_{i=1}^{n} I(A_i \land B_i)}{\Sigma_{i=1}^{n} I(A_i \lor B_i)} = \frac{X}{Y}\]  

(29)
Let $X$ and $Y$ and $C$ be the following

$$C = \frac{(1 + J)}{2n s}$$  \hspace{1cm} \text{(30)}$$

$$X = C \sum_{i=1}^{n} \mathbb{I}(A_i \cap B_i)$$  \hspace{1cm} \text{(31)}$$

$$Y = C \sum_{i=1}^{n} \mathbb{I}(A_i \cup B_i)$$  \hspace{1cm} \text{(32)}$$

$$E(X) = C n \frac{2sJ}{1 + J} \implies E(X) = \frac{(1 + J)}{2n s} \frac{n}{1 + J} \quad \text{(33)}$$

$$E(Y) = C n \frac{2s}{1 + J} \implies E(Y) = \frac{(1 + J)}{2n s} \frac{2s}{1 + J} = 1 \quad \text{(34)}$$

$$Var(X) = C^2 n \frac{2sJ}{1 + J} \left(1 - \frac{2sJ}{1 + J} \right)$$  \hspace{1cm} \text{(35)}$$

$$Var(Y) = C^2 n \frac{2s}{1 + J} \left(1 - \frac{2s}{1 + J} \right)$$  \hspace{1cm} \text{(36)}$$

Let us look at Variance of $X$

$$Var(X) = \frac{(1 + J)^2}{4n^2 s^2} \frac{2sJ}{1 + J} \left(1 - \frac{2sJ}{1 + J} \right)$$  \hspace{1cm} \text{(37)}$$

$$Var(X) = \frac{J}{2n s} \left(1 + J - 2sJ \right)$$  \hspace{1cm} \text{(38)}$$

Let us look at Variance of $(Y)$

$$Var(Y) = \frac{(1 + J)^2}{4n^2 s^2} \frac{2s}{1 + J} \left(1 - \frac{2s}{1 + J} \right)$$  \hspace{1cm} \text{(39)}$$

$$Var(Y) = \frac{1}{2n s} \left(1 + J - 2s \right)$$  \hspace{1cm} \text{(40)}$$

Let us look at Variance of $(Y)$

$$Var(Y) = \frac{(1 + J)^2}{4n^2 s^2} \frac{2s}{1 + J} \left(1 - \frac{2s}{1 + J} \right)$$  \hspace{1cm} \text{(39)}$$

$$Var(Y) = \frac{1}{2n s} \left(1 + J - 2s \right)$$  \hspace{1cm} \text{(40)}$$

Using Chebysev’s inequality,

$$P(|X - 1| \geq \epsilon) < \frac{Var(Y)}{\epsilon^2}$$  \hspace{1cm} \text{(42)}$$

Hence, $1 - \epsilon \leq Y \leq 1 + \epsilon$ with probability $1 - \delta$ where $\delta = \frac{1 + J - 2s}{2n s \epsilon^2}$

Hence we can write, with probability $1 - \delta$

$$\frac{X}{1 + \epsilon} \leq \hat{J} \leq \frac{X}{1 - \epsilon}$$  \hspace{1cm} \text{(43)}$$

Hence, with probability $1 - \delta$

$$\frac{J}{1 + \epsilon} \leq E(\hat{J}) \leq \frac{J}{1 - \epsilon}$$  \hspace{1cm} \text{(44)}$$

$$\frac{X^2}{(1 + \epsilon)^2} \leq \hat{J}^2 \leq \frac{X^2}{(1 - \epsilon)^2}$$  \hspace{1cm} \text{(45)}$$

$$\frac{E(X^2)}{(1 + \epsilon)^2} \leq E(\hat{J}^2) \leq \frac{E(X^2)}{(1 - \epsilon)^2}$$  \hspace{1cm} \text{(46)}$$

$$\frac{Var(X) + E(X)^2}{(1 + \epsilon)^2} \leq E(\hat{J}^2) \leq \frac{Var(X) + E(X)^2}{(1 - \epsilon)^2}$$  \hspace{1cm} \text{(47)}$$

$$\frac{J^2}{(1 + \epsilon)^2} \leq E(\hat{J}^2) \leq \frac{J^2}{(1 - \epsilon)^2}$$  \hspace{1cm} \text{(48)}$$

$$\frac{Var(X) + E(X)^2}{(1 + \epsilon)^2} - \frac{J^2}{(1 - \epsilon)^2} \leq E(\hat{J}^2) - E(\hat{J})^2 \leq \frac{Var(X) + E(X)^2}{(1 - \epsilon)^2} - \frac{J^2}{(1 + \epsilon)^2}$$  \hspace{1cm} \text{(49)}$$
Semantically Constrained Memory Allocation (SCMA) for Embedding in Efficient Recommendation Systems

\[ \text{Var}(X) + E(X)^2 = \frac{J}{2ns}(1 + J - 2sJ + 2nsJ) \]  

The above is the actual result. However for simplicity we simplify assuming \( \epsilon \) is small.

Let

\[ A = \text{Var}(X) + E(X)^2 - J^2 \]  
\[ B = \text{Var}(X) + E(X)^2 + J^2 \]  

\[ A = \frac{J}{2ns}(1 + J - 2sJ + 2nsJ) - J^2 \]  
\[ B = \frac{J}{2ns}(1 + J - 2sJ + 2nsJ) + J^2 \]  

\[ A = \frac{J}{2ns}(1 + J - 2sJ) \]  
\[ B = A + 2J^2 \]  

\[ A - 2\epsilon(A + 2J^2) \leq \text{Var}(\hat{J}) \leq A + 2\epsilon(A + 2J^2) \]  
\[ (1 - 2\epsilon)A - 4\epsilon J^2 \leq \text{Var}(\hat{J}) \leq (1 + 2\epsilon)A + 4\epsilon J^2 \]  

Also under small \( \epsilon \)

\[ J(1 - \epsilon) \leq E(\hat{J}) \leq J(1 + \epsilon) \]  

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