DIAMETER ESTIMATE FOR CLOSED MANIFOLDS WITH POSITIVE SCALAR CURVATURE

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Abstract. For a simply connected closed Riemannian manifold with positive scalar curvature, we prove an upper diameter bound in terms of its scalar curvature integral, the Yamabe constant and the dimension of the manifold. When a manifold has a conformal immersion into a sphere, the dependency on the Yamabe constant is not necessary. The power of scalar curvature integral in these diameter estimates is sharp and it occurs at round spheres with canonical metric.

1. Introduction

One of central problems in Riemannian geometry is to investigate the relationship between curvature and topology of Riemannian manifolds. The classical Myers’ theorem [11] states that if the Ricci curvature of an n-dimensional complete connected Riemannian manifold $(M, g)$ satisfies $\text{Ric} \geq (n - 1)K$ for some constant $K > 0$, then $(M, g)$ is compact and its diameter is at most $\pi/\sqrt{K}$. Here the diameter of $(M, g)$ is defined by

$$\text{diam}(M) := \max \{\text{dist}(x, y) \mid \forall x, y \in M\},$$

where $\text{dist}(x, y)$ denotes the geodesic distance from point $x$ to point $y$. A natural question about the Myers’ result is that if one can get an upper diameter estimate under some scalar curvature assumption instead of the Ricci curvature condition. In this paper, we will study this question and obtain an upper diameter estimate for any closed (i.e. compact without boundary) manifold with positive scalar curvature. Our diameter upper estimate depends on the $L^{\frac{n}{n-1}}$-norm of scalar curvature, the Yamabe constant and the dimension of the manifold. In particular, if a closed manifold has a conformal immersion into $n$-sphere, the dependency on the Yamabe constant is not necessary.

To state the main result, we start to recall some basic facts about the Yamabe constant. For a closed differential manifold $M$ of dimension $n \geq 3$, the normalized Einstein-Hilbert functional $\mathcal{E}$ assigning to each Riemannian metric $g$ is defined by

$$\mathcal{E}(g) = \frac{\int_M R_g dv_g}{(\int_M dv_g)^{\frac{n-2}{n}}}$$

where $R_g$ is the scalar curvature of $(M, g)$ and $dv_g$ is the volume element associated to the metric $g$. This functional is scale-invariant and can be regarded as measuring the average scalar curvature of metric $g$ over $M$. It was conjectured by Yamabe that every conformal
class on any smooth closed manifold contains a metric of constant scalar curvature (the so-called Yamabe problem). This problem was proved by Yamabe [21], Trudinger [18], Aubin [4] and Schoen [15] that a minimum value of $E$ is attained in each conformal class of metrics, and this minimum is achieved by a metric of constant scalar curvature. In particular, each conformal class $[g]$ of $g$ has an associated Yamabe constant $Y(M, [g])$, given by

$$
Y(M, [g]) := \inf_{0 < \varphi \in C^\infty(M)} \left\{ E(\tilde{g}) \bigg| \tilde{g} = \varphi^{4 \over n-2} g \in [g] \right\}
$$

(1.1)

$$
= \inf_{0 < \varphi \in C^\infty(M)} \frac{4(n-1)}{n-2} \int_M |\nabla \varphi|^2 dv_g + \int_M R \varphi^2 dv_g.
$$

The above infimum is finite and not to be negative infinite. Moreover, $Y(M, [g]) > 0$ on a closed Riemannian manifold $(M, g)$ if and only if the conformal class $[g]$ contains a conformal metric with positive scalar curvature everywhere. For more results about the Yamabe problem, the interested reader are referred to surveys [10, 1].

The main result of this paper is that

**Theorem 1.1.** Let $(M, g)$ be an $n$-dimensional ($n \geq 3$) simply connected closed manifold with the scalar curvature $R > 0$. There exists a constant $C(n, Y)$ depending on $n$ and the Yamabe constant $Y := Y(M, [g])$ such that

$$\text{diam}(M) \leq C(n, Y) \int_M R^{n-1 \over 2} dv.$$

In particular, we can take

$$C(n, Y) = 4 \max \left\{ w_n^{-1}, (2cn^{-1}Y)^{-n \over 2} e^{17n^{-1}18 \cdot 2^n} \right\},$$

where $w_n$ is the volume of the unit $n$-sphere $S^n$.

**Remark 1.2.** A different expression (but the same in spirit) of the diameter upper estimate was proved by K. Akutagawa [2] by using a different approach. Our estimate form may be likely to reflect the dependence on the energy of scalar curvature.

The exponent $n-1 \over 2$ of scalar curvature integral in Theorem 1.1 is sharp, which can be achieved by the $n$-sphere $S^n(r)$ of large radius $r$ with the canonical metrics $g_0$. Indeed, in this special case, its diameter is equivalent to $r$, i.e., $\text{diam}_{g_0}(M) \approx r$, while the right hand side of our estimate essentially equals to $c(n)r$ because $R(g_0) \approx r^{-2}$, $\text{Vol}(S^n(r)) \approx r^n$ and $Y(M, [g]) \approx c(n)$. The coefficient $C(n, Y)$ is of course not optimal, which might be sharpened by choosing a delicate cut-off function in Theorem 2.2 below.

For a 4-dimensional closed simply connected Riemannian manifold $(M, g)$, the Yamabe constant $Y(M, [g])$ could be replaced by some other curvature integrals via the following Gursky’s formula (see [7])

$$\int_M R^2 dv - 12 \int_M |\overset{\circ}{\text{Ric}}|^2 dv \leq Y^2(M, [g]),$$

where $\overset{\circ}{\text{Ric}}$ denotes the traceless part of the Ricci tensor.

Zhang [22] proved a diameter estimate depending on the $L^{n-1 \over 2}$-norm of the scalar curvature, the volume of manifold and the positive Yamabe constant. Deng [6] also proved a
similar diameter bound depending on the $L^{n-1}$-norm of the scalar curvature, the volume of manifold and the positive Yamabe constant. But our result indicates that the dependency on the volume of manifold is not necessary.

Recall that if $(M, g)$ and $(\tilde{M}, \tilde{g})$ are Riemannian manifolds, an immersion $\Psi : (M, g) \to (\tilde{M}, \tilde{g})$ is said to be conformal if there exists $f \in C^\infty(M)$ such that

$$\Psi^*\tilde{g} = e^fg.$$

Below we will see that if a closed manifold has a conformal immersion into $n$-sphere, then the diameter estimate does not depend on the Yamabe constant.

**Theorem 1.3.** Let $(M, g)$ be an $n$-dimensional ($n \geq 3$) simply connected closed manifold. Assume that there exists a conformal immersion

$$\Psi : (M, g) \to (S^n, g_0),$$

where $(S^n, g_0)$ denotes the standard unit sphere in $\mathbb{R}^{n+1}$. There exists a constant $C(n)$ depending only on $n$ such that

$$\text{diam}(M) \leq C(n) \int_M R_+^{\frac{n-1}{2}} \, dv,$$

where $R_+$ denotes the positive part of the scalar curvature $R$.

**Remark 1.4.** On any simply connected conformally flat Riemannian manifold, there exists a conformally immersion from $(M, g)$ to $(S^n, g_0)$; see for instance the explanation in [9]. Hence the assumption of Theorem 1.3 includes the conformally flat manifold as a special case.

Our proof is motivated by the argument of Topping’s papers [16, 17]. Recall that Topping [16] used the Perelman’s $W$-functional to prove an upper diameter bound for a closed manifold in terms of scalar curvature integral under the Ricci flow. In [17], Topping applied the Michael-Simon Sobolev inequality to get an upper diameter estimate for a closed connected manifold immersed in the Euclidean space in terms of its mean curvature integral. In our setting, we first apply the Yamabe constant to get the Yamabe-Sobolev inequality and furthermore get a logarithmic Sobolev inequality on closed manifolds with positive scalar curvature. Then we use the logarithmic Sobolev inequality and cut-off functions to prove a new functional inequality. This functional inequality relates a maximal function of scalar curvature and the volume ratio (see Theorem 2.2). Later, we apply the functional inequality to prove an alternative theorem, which states that the maximal function and the volume ratio cannot be simultaneously smaller than a fixed constant on a geodesic ball (see Theorem 3.1). Finally, we use the alternative theorem and a Vitali-type covering lemma to prove the diameter estimate. When a closed manifold has a conformal immersion into a sphere, we can utilize another Sobolev inequality (see Proposition 5.1) and follow the above procedure to prove Theorem 1.3.

In the past few years, Topping’s results have been generalized by Zheng and the second author [20], Zhang [22], Deng [6] and the second author [19], etc.. In fact, in [20], Zheng and the second author proved an upper diameter bound for a closed manifold immersed in the ambient manifold in terms of its mean curvature integral. In [22], Zhang applied the uniform Sobolev inequality along the Ricci flow to obtain an upper diameter bound depending only on the $L^{(n-1)/2}$ bound of the scalar curvature, volume and the Sobolev
constant (or the Yamabe constant) under the Ricci flow. In [6], Deng applied the Yamabe-
Sobolev inequality to detect the compactness of a class of complete manifolds and also
proved an upper diameter estimate for such manifolds. Recently, the second author [19]
applied the Perelman’s entropy functional to prove a sharp upper diameter bound for a
compact shrinking Ricci soliton. In this direction, further development can be referred to
[3, 12, 13, 14] and references therein. Besides, we would like to mention that Bakry and
Ledoux [5] applied a sharp Sobolev inequality to give an alternative proof of the Myers’
diameter estimate.

The structure of this paper is as follows. In Section 2 we give a (logarithmic) Yamabe-
Sobolev inequality on closed manifolds with positive scalar curvature. By a suitable cut-off
function, we reduce the Sobolev inequality into a new functional inequality. In Section 3,
we apply the functional inequality to give an alternative lower bound between the maximal
function of scalar curvature and volume ratio. In Section 4 we apply the alternative theorem
to prove Theorem 1.1. In Section 5 we adopt the same argument of Theorem 1.1 to prove
Theorem 1.3.

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2. Functional inequalities

In this section, we will discuss some functional inequalities on closed manifolds with
positive scalar curvature. We first apply the Yamabe constant to get a logarithmic Yamabe-
Sobolev inequality on closed manifolds with positive scalar curvature. Then we apply the
Sobolev inequality to prove a new functional inequality, which will be used in the proof of
Theorem 1.1.

On a closed manifold \((M, g)\), we have a fact that \([g]\) contains a conformal metric with
positive scalar curvature everywhere if and only if the Yamabe constant is positive. So if
scalar curvature \(R > 0\) on \((M, g)\), then

\[ Y(M, [g]) > 0. \]

Therefore the definition (1.1) yields the Yamabe-Sobolev inequality

\[
\left( \int_M \varphi^{2n/(n-2)} dv \right)^{n-2} \leq Y^{-1}(M, [g]) \left( \int_M \frac{4(n-1)}{n-2} |\nabla \varphi|^2 dv + \int_M R \varphi^2 dv \right)
\]

for any positive \(\varphi \in C^\infty(M)\). The Yamabe-Sobolev type inequality implies much geometric
information, such as first eigenvalue estimates, gap theorems, etc. We refer the interested
reader to [8] and references therein. Here we will see that (2.1) immediately implies a
logarithmic Yamabe-Sobolev inequality.

Lemma 2.1. Let \((M, g)\) be an \(n\)-dimensional \((n \geq 3)\) closed manifold with positive scalar
curvature. For any positive \(\varphi \in C^\infty(M)\) with

\[ \int_M \varphi^2 dv = 1 \]

and any real number \(\tau > 0\),

\[
\frac{n}{2} \ln \frac{2eY(M, [g])}{n} \leq \tau \int_M \left( \frac{4(n-1)}{n-2} |\nabla \varphi|^2 + R \varphi^2 \right) dv - \int_M \varphi^2 \ln \varphi^2 dv - \frac{n}{2} \ln \tau,
\]
where $R$ is the scalar curvature of $(M, g)$ and $Y(M, [g])$ is the Yamabe constant.

Proof of Lemma 2.1. Assume that the Yamabe-Sobolev inequality (2.1) holds on a closed manifold $(M, g)$. For any smooth function $\varphi$ with $\|\varphi\|_2 = 1$, we consider the weighted measure $d\mu = \varphi^2 dv$ on $(M, g)$, and then

$$\int_{M} d\mu = 1.$$ 

Note that smooth function $\ln \Phi$ is concave with respect to positive parameter $\Phi$. Applying the standard Jensen inequality

$$\int_{M} \ln \Phi d\mu \leq \ln \left( \int_{M} \Phi d\mu \right)$$

to positive function $\Phi = \varphi^{q-2}$, where $q = \frac{2n}{n-2}$, we get that

$$\int_{M} (\ln \varphi^{q-2}) \varphi^2 dv \leq \ln \left( \int_{M} \varphi^{q-2} \varphi^2 dv \right) = \ln \| \varphi \|_q^q.$$

In other words,

$$\int_{M} \varphi^2 \ln \varphi dv \leq \frac{q}{q-2} \ln \| \varphi \|_q = \frac{n}{2} \ln \| \varphi \|_q.$$

Combining this with (2.1), we have the following estimate:

$$\int_{M} \varphi^2 \ln \varphi^2 dv \leq \frac{n}{2} \ln \| \varphi \|_q^2$$

$$\leq \frac{n}{2} \ln \left[ Y^{-1}(M, [g]) \left( \int_{M} \frac{4(n-1)}{n-2} |
\nabla \varphi|^2 dv + \int_{M} R \varphi^2 dv \right) \right]$$

$$= -\frac{n}{2} \ln Y(M, [g]) + \frac{n}{2} \ln \left[ \int_{M} \left( \frac{4(n-1)}{n-2} |
\nabla \varphi|^2 + R \varphi^2 \right) dv \right].$$

Using the elementary inequality $\ln x \leq \sigma x - (1 + \ln \sigma)$ for any real number $\sigma > 0$, the above estimate can be simplified as

$$\int_{M} \varphi^2 \ln \varphi^2 dv \leq -\frac{n}{2} \ln Y(M, [g]) + \frac{n\sigma}{2} \int_{M} \left( \frac{4(n-1)}{n-2} |
\nabla \varphi|^2 + R \varphi^2 \right) dv - \frac{n}{2} (1 + \ln \sigma).$$

Setting $\tau = \frac{n\sigma}{2}$, then

$$\int_{M} \varphi^2 \ln \varphi^2 dv \leq \tau \int_{M} \left( \frac{4(n-1)}{n-2} |
\nabla \varphi|^2 + R \varphi^2 \right) dv - \frac{n}{2} \ln Y(M, [g]) - \frac{n}{2} \ln (2\tau) + \frac{n}{2} \ln \frac{n}{e}$$

and hence the result follows by arranging some terms. \qed

Now we will show that Lemma 2.1 indeed implies an important functional inequality by adapting the arguments of [16, 19]. This inequality is linked with some maximal function of scalar curvature and the volume ratio.
Theorem 2.2. Let \((M, g)\) be an \(n\)-dimensional \((n \geq 3)\) closed manifold with positive scalar curvature. For any point \(p \in M\) and for any \(r > 0\),

\[
\frac{n}{2} \ln \frac{2eY(M, [g])}{n} \leq \frac{16(n - 1)}{n - 2} \cdot \frac{V(p, r)}{V(p, \frac{r}{2})} + \frac{r^2}{V(p, \frac{r}{2})} \int_{B(p, r)} Rdv + \ln \frac{V(p, r)}{r^n},
\]

where \(R\) is the scalar curvature of \((M, g)\), \(Y(M, [g])\) is the Yamabe constant and \(V(p, r)\) is the volume of geodesic ball \(B(p, r)\) with radius \(r\) and center \(p\).

Proof of Theorem 2.2. We choose a smooth cut-off function \(\psi : [0, \infty) \to [0, 1]\) supported in \([0, 1]\) such that \(\psi(t) = 1\) on \([0, 1/2]\) and \(|\psi'| \leq 2\) on \([0, \infty)\). For any point \(p \in M\), let

\[
\varphi(x) := e^{-\frac{\lambda}{2}} \psi\left(\frac{d(p, x)}{r}\right),
\]

where \(\lambda\) is some constant determined by the constraint condition \(\int_M \varphi^2 dv = 1\). Obviously, \(\lambda\) satisfies

\[
V\left(p, \frac{r}{2}\right) \leq e^\lambda \int_M \varphi^2 dv = e^\lambda
\]

and

\[
e^\lambda = e^\lambda \int_M \varphi^2 dv = \int_M \psi^2(d(p, x)/r)dv \leq V(p, r).
\]

That is, the constant \(\lambda\) has upper and lower bounds as follows

\[
V\left(p, \frac{r}{2}\right) \leq e^\lambda \leq V(p, r).
\]

We now apply the above cut-off function \(\varphi\) to simplify the logarithmic Yamabe-Sobolev inequality in Lemma 2.1. Notice that \(\varphi\) satisfies

\[
|\nabla \varphi| \leq \frac{2}{r} \cdot e^{-\frac{\lambda}{2}},
\]

which is supported in \(B(p, r)\). Let us estimate each term of the right hand side of (2.2).

For the first term of the right hand side of (2.2), we have that

\[
\frac{4(n - 1)}{n - 2} r \int_M |\nabla \varphi|^2 dv = \frac{4(n - 1)}{n - 2} r \int_{B(p, r)} |\nabla \varphi|^2 dv
\]

\[
\leq \frac{4(n - 1)}{n - 2} r V(p, r) \frac{4}{r^2} e^{-\lambda}
\]

\[
\leq \frac{16(n - 1) \tau}{(n - 2) r^2} \frac{V(p, r)}{V(p, \frac{r}{2})}.
\]

For the second term of the right hand side of (2.2), we estimate

\[
\tau \int_M R\varphi^2 dv \leq \tau e^{-\lambda} \int_{B(p, r)} Rdv
\]

\[
\leq \frac{\tau}{V(p, \frac{r}{2})} \int_{B(p, r)} Rdv.
\]
Next we will estimate the third term of the right hand side of (2.2). We see that continuous function \( \Psi(t) := -t \ln t \) is concave with respect to \( t > 0 \) and the Riemannian measure \( dv \) is supported in \( B(p, r) \). Using the Jensen’s inequality
\[
\int_{B(p,r)} \Psi(\phi^2) dv \leq \Psi\left( \frac{\int_{B(p,r)} \phi^2 dv}{\int_{B(p,r)} dv} \right)
\]
and the definition of \( \Psi \), we get that
\[
-\int_{B(p,r)} \phi^2 \ln \phi^2 dv \leq \ln \left( \frac{\int_{B(p,r)} \phi^2 dv}{\int_{B(p,r)} dv} \right).
\]
Since \( \int_{B(p,r)} \phi^2 dv = 1 \), the above inequality can be simplified as
\[
-\int_{B(p,r)} \phi^2 \ln \phi^2 dv \leq \ln V(p,r).
\]
By the definition of \( \varphi(x) \), therefore
\[
\int_M \varphi^2 \ln \phi^2 dv = -\int_{B(p,r)} \varphi^2 \ln \phi^2 dv \leq \ln V(p,r).
\]
Putting (2.3), (2.4) and (2.5) into (2.2), we arrive at
\[
\frac{n}{2} \ln \frac{2eY(M,[g])}{n} \leq \frac{16(n-1)\tau}{(n-2)r^2} \frac{V(p,r)}{V(p,\frac{r}{2})} + \frac{\tau}{V(p,\frac{r}{2})} \int_{B(p,r)} R dv + \ln \frac{V(p,r)}{\tau^2}
\]
for any \( \tau > 0 \). The conclusion then follows by letting \( \tau = r^2 \). \(\square\)

3. Maximal function and volume ratio

In this section, we will give an alternative property of uniformly lower bounds between the maximal function of scalar curvature and the volume ratio in a geodesic ball.

Inspired by Topping’s arguments in [16, 17], on an \( n \)-dimensional \( (n \geq 3) \) Riemannian manifold \((M,g)\), for any point \( p \in M \) and \( r > 0 \), we consider the maximal function
\[
Mf(p,r) := \sup_{s \in (0,r]} s^{-\frac{n-3}{2}} \left[ V(p,s) \right]^{-\frac{n-3}{2}} \left( \int_{B(p,s)} |f| dv \right)^{\frac{n-1}{2}}
\]
for \( f \in C^\infty(M) \), and the volume ratio
\[
\kappa(p,r) := \frac{V(p,r)}{r^n}.
\]

With the help of Theorem 2.2, we will show that the maximal function of scalar curvature and the volume ratio in closed manifolds with positive scalar curvature cannot be simultaneously smaller than a fixed constant.

**Theorem 3.1.** Let \((M,g)\) be an \( n \)-dimensional \( (n \geq 3) \) closed manifold with positive scalar curvature. Then there exits a constant \( \delta > 0 \) depending only on \( n \) and \( Y := Y(M,[g]) \) such that for any point \( p \in M \) and for any \( r > 0 \), at least one of the following is true:

1. \( MR(p,r) > \delta \);
2. \( \kappa(p,r) > \delta \).
Here $R(p, r)$ denotes the scalar curvature in the geodesic ball $B(p, r)$. In particular, we can take
\[
\delta = \min \left\{ w_n, \left(2en^{-1}Y\right)^\frac{n}{n-1} e^{-\frac{17n-18}{n-2} \frac{2}{\sqrt{n}}} \right\},
\]
where $Y := Y(M, [g])$ is the Yamabe constant and $w_n$ is the volume of the unit $n$-sphere $S^n$.

**Proof of Theorem 3.1.** The proof is similar to the proof of Theorem 3.1 in [19]. We give its detailed proof here for the sake of completeness. Suppose that there exist a point $p \in (M, g)$ and $r > 0$ such that $M \supseteq B(p, r) \subseteq \delta$ for some constant $\delta > 0$. For any $0 < \epsilon < 1$, we define constant $\delta$ as follows:
\[
\delta := \min \left\{ \left(1 - \epsilon\right) w_n, \left(2en^{-1}Y\right)^\frac{n}{n-1} e^{-\frac{17n-18}{n-2} \frac{2}{\sqrt{n}}} \right\}.
\]

In the following our aim is to prove $\kappa(p, r) > \delta$. If it is not true, we make the following

**Claim.** Suppose there exist a point $p \in M$ and $r > 0$ such that $M \supseteq B(p, r) \subseteq \delta$ for some constant $\delta > 0$. If $\kappa(p, s) \leq \delta$, then $\kappa(p, s/2) \leq \delta$ for any $s \in (0, r]$.

This claim will be proved later. We now continue to prove Theorem 3.1. We repeatedly use the claim and finally have that
\[
(3.1) \quad \kappa \left( p, \frac{r}{2m} \right) \leq \delta \leq (1 - \epsilon) w_n
\]
for any $m \in \mathbb{N}$, where $\epsilon$ is the sufficiently small positive constant. But if we let $m \rightarrow \infty$, then
\[
\kappa \left( p, \frac{r}{2m} \right) \rightarrow w_n,
\]
which contradicts (3.1). Therefore $\kappa(p, r) > \delta$ and the theorem follows. The desired constant $\delta$ is obtained by letting $\epsilon \rightarrow 0^+$.

In the rest, we only need to check the above claim. We adopt the argument from [19].

**Proof of Claim.** According to the relative sizes of $V(p, s/2)$ and $V(p, s)$, we may prove the claim by two cases.

**Case one.** Suppose that
\[
V \left( p, \frac{s}{2} \right) \leq \delta^2 2^{-\alpha_0} s^{\frac{2n}{n-1}} [V(p, s)]^{\frac{n-3}{n-1}}.
\]
Then,
\[
\kappa \left( p, \frac{s}{2} \right) := \frac{2^n}{s^n} V \left( p, \frac{s}{2} \right)
\leq \delta^2 2^{-\alpha_0} s^{\frac{2n}{n-1}} [V(p, s)]^{\frac{n-3}{n-1}}
= \delta^2 (\kappa(p, s))^{\frac{n-3}{n-1}}
\leq \delta^2 \delta^{\frac{n-3}{n-1}}
= \delta,
\]
which proves the claim.

**Case Two.** Suppose that
\[
V \left( p, \frac{s}{2} \right) > \delta^2 2^{-\alpha_0} s^{\frac{2n}{n-1}} [V(p, s)]^{\frac{n-3}{n-1}}.
\]
Since \( MR(p, r) \leq \delta \) and \( R > 0 \), according to the definition of \( MR(p, r) \), we indeed have
\[
\int_{B(p, s)} R \, dv \leq \delta^{\frac{n^2-4}{n^2-r}} \, [V(p, s)]^{\frac{n-4}{n-1}}
\]
for all \( s \in (0, r] \). Using the assumption of Case Two, the above estimate can be reduced to
\[
\int_{B(p, s)} R \, dv \leq 2^n s^{-2} \, V\left(p, \frac{s}{2}\right)
\]
for all \( s \in (0, r] \). Substituting this into Theorem 2.2
\[
\frac{n}{2} \ln \frac{2eY(M, [g])}{n} \leq \frac{16(n-1)}{n-2} \cdot \frac{V(p, s)}{V(p, \frac{s}{2})} + \frac{s^2}{V(p, \frac{s}{2})} \int_{B(p, s)} R \, dv + \ln \kappa(p, s)
\]
\[
\leq \frac{16(n-1)}{n-2} \cdot \frac{V(p, s)}{V(p, \frac{s}{2})} + 2^n + \ln \delta
\]
for all \( s \in (0, r] \), where in the above second line we used \( \kappa(p, s) \leq \delta \).

On the other hand, the definition of \( \delta \) implies that
\[
\ln \delta \leq \frac{n}{2} \ln \frac{2eY(M, [g])}{n} - \frac{17n-18}{n-2} \cdot 2^n.
\]
Substituting this into (3.2),
\[
\frac{V(p, s)}{V(p, \frac{s}{2})} \geq 2^n
\]
for all \( s \in (0, r] \). Therefore, for any \( s \in (0, r] \), we have
\[
\kappa\left(p, \frac{s}{2}\right) := \frac{2^n \cdot V\left(p, \frac{s}{2}\right)}{s^n}
\]
\[
\leq \frac{V(p, s)}{s^n}
\]
\[
= \kappa(p, s)
\]
\[
\leq \delta
\]
and the claim follows. \( \square \)

4. Diameter estimate

In this section we will apply Theorem 3.1 to prove Theorem 1.1 by adapting the arguments of [16, 19]. We need to carefully examine the explicit coefficients of the diameter estimate in terms of the positive Yamabe constant in (2.1).

To prove Theorem 1.1, we will need a Vitali-type covering lemma (see [16], or [20]), which is the key step to prove our theorem.

**Lemma 4.1.** Let \( \gamma \) be a shortest geodesic connecting any two points \( x \) and \( y \) in \( (M, g) \), and \( s \) be a nonnegative bounded function defined on \( \gamma \). If \( \gamma \subset \{ B(p, s(p)) \mid p \in \gamma \} \), then for any \( \rho \in (0, \frac{1}{2}) \), there exists a countable (possibly finite) set of points \( \{ p_i \in \gamma \} \) such that

1. \( B(p_i, s(p_i)) \) are disjoint;
2. \( \gamma \subset \bigcup_i B(p_i, s(p_i)) \);
(3) \( \rho \text{dist}(x, y) \leq \sum_i 2s(p_i) \), where \( \text{dist}(p_1, p_2) \) denotes the distance between \( x \) and \( y \) in \((M, g)\).

Now we can finish the proof of Theorem 1.1.

Proof of Theorem 1.1. For the fixed constant \( \delta \) defined in Theorem 3.1 and the closed manifold \( M \), we could choose \( r_0 > 0 \) sufficiently large so that the total volume of \( M \) is less than \( \delta r_0^n \) because the total volume of \( M \) is finite. For any point \( p \in M \), we conclude that

\[
\kappa(p, r_0) = \frac{V(p, r_0)}{r_0^n} \leq \frac{V(M)}{r_0^n} \leq \delta,
\]

where \( V(M) \) denotes the volume of \( M \). By Theorem 3.1, it indicates that \( MR(p, r_0) > \delta \). That is, there exists \( s = s(p) > 0 \) such that

\[
(4.1) \quad \delta < s^{-1}[V(p, s)]^{-\frac{n-1}{2}} \left( \int_{B(p, s)} R dv \right)^{\frac{n-1}{2}}.
\]

Notice that the following Hölder inequality holds

\[
\int_{B(p, s)} R dv \leq \left( \int_{B(p, s)} R^{\frac{n-1}{2}} dv \right)^{\frac{n}{n-1}} \left( \int_{B(p, s)} dv \right)^{\frac{n-3}{n-1}}.
\]

Applying this, we can estimate (4.1) by

\[
\delta < s^{-1} \int_{B(p, s)} R^{\frac{n-1}{2}} dv.
\]

In the other words,

\[
(4.2) \quad s(p) < \delta^{-1} \int_{B(p, s(p))} R^{\frac{n-1}{2}} dv.
\]

In the next step, we shall pick appropriate points \( p \) such that (4.2) will be used in these points. Assume that \( p_1 \) and \( p_2 \) are two extremal points in the closed manifold \((M, g)\) such that \( \text{diam}(M) = \text{dist}(p_1, p_2) \). Let \( \gamma \) be a shortest geodesic connecting \( p_1 \) and \( p_1 \). Then we obviously have \( \gamma \subset \{B(p, s(p)) \mid p \in \gamma \} \). By Lemma 4.1, there exists a countable (possibly finite) set of points \( \{p_i \in \gamma \} \) such that geodesic balls \( \{B(p_i, s(p_i))\} \) are disjoint and

\[
\rho \text{diam}(M) = \rho \text{dist}(p_1, p_2) \leq \sum_i 2s(p_i).
\]

Substituting (4.2) into the above inequality,

\[
\text{diam}(M) \leq \frac{2}{\rho} \sum_i s(p_i)
\]

\[
(4.3) \quad < \frac{2}{\rho} \delta^{-1} \sum_i \int_{B(p_i, s(p_i))} R^{\frac{n-1}{2}} dv \leq \frac{2}{\rho} \delta^{-1} \int_M R^{\frac{n-1}{2}} dv,
\]
where $\delta > 0$ is a constant depending only on $n$ and $Y(M, [g])$. Letting $\rho \to \frac{1}{2}$, we have
\[
\text{diam}(M) \leq 4\delta^{-1} \int_M R^{\frac{n-1}{2}} dv,
\]
where $\delta$ is chosen as in Theorem 3.1. The desired estimate follows. □

In the course of proving Theorem 1.1, we indeed prove a quantitative estimate for the diameter of some collapsed geodesic ball. It states that if there exist a point $p \in (M, g)$ and a real number $r_0 > 0$ such that
\[
Y = Y(M, [g]) > 0
\]
in $B(p, 2r_0)$ and
\[
\frac{V(p, r_0)}{r_0^n} < \delta;
\]
where $\delta := \min\{w_n, (2en^{-1}Y)^\frac{n}{2} e^{\frac{48-17n}{n-2}2^n}\}$, then
\[
\text{diam}(B(p, r_0)) \leq 4\delta^{-1} \int_{B(p, 2r_0)} R^{\frac{n-1}{2}}.
\]

5. PROOF OF THEOREM 1.3

When a closed manifold has a conformal immersion into a sphere, we can prove another upper diameter bound (i.e. Theorem 1.3) which depends only on the positive part of scalar curvature integral.

The proof of Theorem 1.3 is almost the same as the argument of Theorem 1.1. Indeed, under the assumption of Theorem 1.3 we have the following Sobolev inequality which does not depend on the Yamabe constant; see Proposition 3.25 in [8].

**Proposition 5.1.** Let $(M, g)$ be an $n$-dimensional ($n \geq 3$) simply connected closed manifold. Assume that there exists a conformal immersion
\[
\Psi : (M, g) \to (S^n, g_0),
\]
where $(S^n, g_0)$ denotes the standard unit sphere in $\mathbb{R}^{n+1}$. For any positive $\varphi \in C^\infty(M)$,
\[
\left( \int_M \varphi^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}} \leq \left[ n(n-1)w_n^2 \right]^{-1} \left( \int_M \frac{4(n-1)}{n-2} |\nabla \varphi|^2 dv + \int_M R^+ \varphi^2 dv \right),
\]
where $w_n$ is the volume of the unit $n$-sphere $(S^n, g_0)$ and $R^+$ is the positive part of the scalar curvature $R$.

With the help of Proposition 5.1, Theorem 1.3 easily follows by adapting the lines of proving Theorem 1.1. We omit the repeated discussion.

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