On an approach for evaluating certain trigonometric character
sums using the discrete time heat kernel

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Abstract

In this article we develop a general method by which one can explicitly evaluate certain
sums of \( n \)-th powers of products of \( d \geq 1 \) elementary trigonometric functions evaluated
at \( \mathbf{m} = (m_1, \ldots, m_d) \)-th roots of unity. Our approach is to first identify the individual
terms in the expression under consideration as eigenvalues of a discrete Laplace operator
associated to a graph whose vertices form a \( d \)-dimensional discrete torus \( G_m \) which
depends on \( \mathbf{m} \). The sums in question are then related to the \( n \)-th step of a Markov chain
on \( G_m \). The Markov chain admits the interpretation as a particular random walk, also
viewed as a discrete time and discrete space heat diffusion, so then the sum in question
is related to special values of the associated heat kernel. Our evaluation follows by deriving
a combinatorial expression for the heat kernel, which is obtained by periodizing the heat
kernel on the infinite lattice \( \mathbb{Z}^d \) which covers \( G_m \).

1 Introduction

1.1 A motivating example

There are some intriguing trigonometric identities which would catch the attention of almost
any mathematician, or mathematics student for that matter. For example, for any positive
integers \( n \) and \( m \), we have that

\[
\sum_{j=0}^{m-1} \cos^n \left( \frac{2\pi j}{m} \right) = 2^{-n} m \sum_{k=-[n/m], \ km+n \ even}^{[n/m]} \frac{n}{(km+n)/2},
\]

where \( [x] \) stands for the largest integer less than or equal to \( x \). Of course, for small \( m \), the
values of \( \cos(2\pi j/m) \) can be explicitly computed. For instance, if \( m = 5 \) then

\[
\{\cos(2\pi j/5)\}_{j=1}^4 = \{(\pm 1 \pm \sqrt{5})/4\}.
\]

With this, the verification of (1) for small \( n \) becomes (or at least should be) an enjoyable
exercise for students in high school mathematics courses. The analogous expression in the
case of \( m = 7 \) is even more pleasant, though challenging.

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As is often true in mathematics, there is much more to (1) than the formula itself. For example, if one multiplies both sides of (1) by $2^n$, the left-hand-side is

$$\sum_{j=0}^{m-1} \left(e^{2\pi ij/m} + e^{-2\pi ij/m}\right)^n.$$  \hspace{1cm} (2)

Each summand in (2) is an algebraic integer in the cyclotomic field $\mathbb{Q}(\zeta_m)$ where $\zeta_m$ is a primitive $m$-th root of unity. It is elementary to prove that the sum (2) is invariant under the action of the Galois group $\text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$; hence, (2) is an integer in $\mathbb{Q}$. Therefore, (1) can be viewed as giving an explicit evaluation of the rational integer (2). Indeed, one can obtain (1) from (2) by applying the binomial theorem.

1.2 The general question

The proof of (1) described above is specific to the series in question and does not admit a complete understanding of the mathematical structure behind similar questions. With this assertion in mind, the purpose of this article (and the follow-up article [CHJSV22a]) is to address the following general problem.

Let $d \geq 1$ be an integer, and let $P(x_1, \cdots, x_d; y_1, \cdots, y_d)$ be any polynomial in $2d$ variables with complex coefficients. Let $\{m_j\}_{j=1}^d$ and $\{a_j\}_{j=1}^d$ be sets of positive integers, and $\{\beta_j\}_{j=1}^d$ a set of real numbers. Can one determine an efficient and effective algorithm which evaluates

$$\sum_{k_1=0}^{m_1-1} \cdots \sum_{k_d=0}^{m_d-1} P\left(\cos\left(\frac{2\pi k_1 a_1}{m_1} + \beta_1\right), \cdots, \cos\left(\frac{2\pi k_d a_d}{m_d} + \beta_d\right)\right);$$

$$\sin\left(\frac{2\pi k_1 a_1}{m_1} + \beta_1\right), \cdots, \sin\left(\frac{2\pi k_d a_d}{m_d} + \beta_d\right)$$

as a finite sum involving the degree and coefficients of $P$, binomial coefficients involving the given data, and exponentials which are linear in $\beta_j$, $j = 1, \ldots, d$?

Specific instances of (3), as well as similar series which include additive or multiplicative characters, or trigonometric realization of important sums, such as Dedekind or Hardy sums have attracted considerable interest, in part because such series appear in different mathematical and physical settings; see, for example, [Ve88], [Do89], [Do92], [CS12] and [Ho18] for sums within physical settings or [MS20], [Ro19], [dFGK18] and [So18] for other mathematical research on related sums. As a result, one can see the potential significance in obtaining closed-form evaluations of such sums. However, as stated in [BY02], many of those sums seem intractable or simply do not have known evaluations, however, it is possible to establish reciprocal relations for such sums; see also [BC13], [Ch18] and [MS20].

Various approaches have been used to evaluate certain finite trigonometric sums. Here is a partial list of articles we found particularly interesting.

In [BY02], the authors used contour integration, and the article includes an interesting discussion of the history of this method. In [CM99], certain series were evaluated using a generating series and a partial fraction decomposition. In [AH18], the authors proved numerous relations by appealing to results in the theory of special functions. In [Ha08], some trigonometric sums were computed by using a discrete form of the sampling theorem associated with certain second-order discrete eigenvalue problems. In [AZ22], the authors show that many trigonometric identities have been “re-discovered” many times in the vast literature on the topic.
and describe an interesting “automated approach” for proving some types of trigonometric identities.

Each of the above mentioned approaches has proved interesting formulas in specific instances or with certain types of series. Still, it would be of mathematical value to develop any other means by which one can compute (3) in closed form.

1.3 Our approach

The aim of this note is to present a unified approach to the numerical evaluation of a wide class of finite trigonometric sums involving multiple sums of products of powers of sine and cosine functions evaluated at rational multiples of \( \pi \), possibly twisted by a character or shifted by an arbitrary additive constant. In effect, we consider the setting of (3) in the case \( P \) is a monomial.

More precisely, we express the discrete time heat kernel on a \( d \)-dimensional discrete torus \( G_d \), twisted by a character of \( \mathbb{Z} \) in two different ways: First, through a combinatorial approach, stemming from the realization of \( G_d \) as a quotient of \( \mathbb{Z}^d \) by a sublattice; and second, through a spectral theoretic approach, which comes from the expansion of the heat kernel in terms of eigenfunctions and eigenvalues of the graph Laplacian. The eigenvalues of the discrete Laplacian are, in effect, the trigonometric terms we seek to evaluate, and the identities follow from the general result that the discrete time heat kernel is unique, hence any two evaluations are equal. It is important to note that we consider the setting where the edges of \( G_d \) are not just “nearest neighbor”, which allows us considerable flexibility in our analysis.

Our approach is similar in spirit to the method undertaken by Ejsmont and Lehner in [EL21] who also obtain an evaluation of a trigonometric sum by employing spectral theory. For our approach is more general since we are looking at the complete spectral expansion on a specially designed weighted Cayley graph. For example, for positive integers \( m_1, m_2, a, b, k \) and real numbers \( \alpha_1, \alpha_2 \) the trigonometric sum

\[
\sum_{j=0}^{2m_1-1} \sum_{\ell=0}^{m_2-1} (-1)^j \cos^{k} \left( \frac{\pi j a}{m_1} + \alpha_1 \right) \sin^{k} \left( \frac{2\pi \ell b}{m_2} + \alpha_2 \right)
\]

equals \( 2m_1m_2 \exp(-\pi i \alpha_1 m_1/a) \) times a specific value of the heat kernel associated to the weighted Cayley graph \( \mathcal{C}(\mathbb{Z}^2/m\mathbb{Z}^2, \pi_S) \) where \( m = (2m_1, m_2), \ S = \{ (\pm a, \pm b) \} \) is a set of four elements and \( \pi_S(s) = 1/4 \) for all \( s \in S \). Furthermore, the discrete Laplacian is twisted by a character \( \chi_{\beta} \) with \( \beta = \left( \frac{\alpha_1 m_1}{\pi a}, \frac{m_2 (\alpha_2 - \pi/2)}{2\pi b} \right) \). With all this, the heat kernel in question is evaluated at the space variables \( x = (m_1, 0), \ y = (0, 0) \) and at discrete time \( k \), from which we can evaluate (4). For additional details, see Section 2.3 which gives a detailed explanation of the notation. The combinatorial evaluation of the discrete time heat kernel, as given in Proposition 2, combined with Proposition 2 yields an explicit and closed evaluation of this sum; namely, it is proved in Example 2 that

\[
(4) = \frac{2m_1m_2}{4^k} e^{i(\alpha_1 + \alpha_2)}, \sum_{d_1,d_2 \in \{0,\ldots,k\}} \left( \binom{k}{d_1} \binom{k}{d_2} \right)^2 e^{2\pi i d_2} e^{-2i(\alpha_1 d_1 + \alpha_2 d_2)} a(2d_1-k)/m_1 - 1/2 \in \mathbb{Z}, \ b(2d_2-k)/m_2 \in \mathbb{Z}.
\]
In other words, by first looking at a series in question, as in (4), one can consider a type of “reverse engineering” to construct the graph from which our approach will evaluate the sum. For example, in Section 5.4 below we describe the setting from which one will obtain an explicit evaluation of the sum

\[
\sum_{k_1=0}^{m_1-1} \cdots \sum_{k_d=0}^{m_d-1} \left( \cos^h \left( \frac{2\pi k_1 a_1}{m_1} + \beta_1 \right) + \cdots + \cos^h \left( \frac{2\pi k_d a_d}{m_d} + \beta_2 \right) \right)^n.
\]

(6)

For the sake of space, we do not write the evaluation; however, we are able to give a reasonably brief yet thorough description as to how one can apply our general result in order to obtain a closed form evaluation of (6) in terms of binomial coefficients and exponentials.

1.4 Outline of the paper and overview of results

This article is organized as follows. In Section 2 we present background material from the literature and establish the notation and language needed throughout the article. The main geometric object of study is a Cayley graph \( X = \mathcal{C}(G, S, \pi_S) \) associated to a finite abelian group \( G \) with a reasonably general set of edges generated by \( S \subseteq G \) with corresponding edge weights \( \pi_S \). The only requirement on \( S \) is that it is symmetric, so then two vertices are connected by an edge if and only if their difference belongs to \( S \). An edge \( s \) connecting \( x, y \in G \) where \( x - y = s \in S \) has associated weight \( \pi_S(s) \). The edge weights \( \pi_S \) are assumed to be positive. For every vertex, the sum of all weights of the joining edges is one. Therefore, the weights can be viewed as defining a probability distribution \( \pi_S \) on \( S \). For this article, we take \( G \) to be a discrete torus, and we write \( G_m = \mathbb{Z}^d / m \mathbb{Z}^d \). Also, we can identify \( G_m \) as the cosets of the action of the group of translations by a vector \( m = (m_1, \ldots, m_d) \in \mathbb{Z}^d \) on the lattice \( \mathbb{Z}^d \). Of course, \( m_j > 0 \) for all \( j \). For a Cayley graph \( X \), one has a naturally defined Laplacian matrix, and the heat kernel is another operator defined from the Laplacian matrix.

In Section 3, we consider fixed but arbitrary \( m, S \) and \( \pi_S \). We then derive one of the two evaluations of the discrete time heat kernel on \( X = \mathcal{C}(G_m, S, \pi_S) \). We begin by evaluating the discrete time heat kernel on the Cayley graph \( Y = \mathcal{C}(\mathbb{Z}^d, S, \pi_S) \) as a single combinatorial coefficient, from which we prove that the discrete time heat kernel on \( X \) can be expressed in terms of a sum of combinatorial coefficients; see Theorem 3. The heat kernel evaluated in Lemma 1 and Theorem 2 may be viewed as a solution to a discrete time dynamic diffusion equation on the \( d \)-dimensional lattice \( \mathbb{Z}^d \) and the discrete torus \( G_m \), respectively, where the coefficients of the equation are given by the probability distribution \( \pi_S \).

The results of Section 3 can also be viewed as a generalization of discrete time results of [SS14] and [SS15] to the case when the space variable belongs to the \( d \)-dimensional lattice or a \( d \)-dimensional discrete torus. Also, the papers [SS14] and [SS15] consider the diffusion process related to the weighted Laplacian on the graph \( Y = \mathcal{C}(\mathbb{Z}^d, S, w_S) \), where \( w_S \) is more general, not necessarily non-negative weight function on the symmetric subset \( S \) of \( \mathbb{Z} \). Though we are dealing with probability distribution only, we assert that our methods can be adopted to treat the general setting which involve non-negative weights.

In Section 4 we prove the main theorem of the article, which is Theorem 7 by obtaining a second expression for the discrete time heat kernel. As stated above, the second formula for the discrete time heat kernel follows form the spectral theorem of the associated Laplacian matrix. In Section 5 we develop several examples of Theorem 7 which include reproving some results in the literature by utilizing special instances of our main theorem. Also, we obtain a new proof of the evaluation of classical Gauss sums.
Of the examples given, we find Corollary 13 particularly intriguing. Our result yields a new combinatorial expression for the trigonometric sum, which is evaluated in the main theorem of [dFK13]. Specifically, the evaluation of $S(100, 13)$ in [dFK13] follows from their algorithm, whereas our method yields a specific formula. The numerical expressions, which of course must coincide, are explicitly given and certainly are not a value which anyone would guess.

Going further, in Section 5, Corollary 14 we derive two explicit evaluations of sums of powers of sines and cosines twisted by a primitive (multiplicative) Dirichlet character, related to results of [BH10] (Theorem 1.2.), [BZ04], [BBCZ05] and [ZW18].

In Section 6 we prove a general formula which amounts to showing that the discrete time heat kernel on the product of Cayley graphs is equal to the corresponding product of discrete time heat kernels. In some sense, one could view the result as “contained in the mathematical folklore”. However, for the sake of completeness in this article as well as for future studies, the mathematical statement is presented and proved. As an application of the general formula in Section 6 we prove (5) and deduce some further special formulas, such as the identity

$$
\sum_{\ell_1=0}^{m_1-1} \cdots \sum_{\ell_d=0}^{m_d-1} \cos \left( \frac{2\pi (\ell_1 + \beta_1)}{m_1} \right) \cdots \cos \left( \frac{2\pi (\ell_d + \beta_d)}{m_d} \right)
$$

where the sum on the right-hand side of the above equation is taken only over $k_j$ for which the sum $n + k_j m_j$, $j = 1, \ldots, d$ is even.

Finally, in Section 7, we offer a concluding remark in which we describe how to use the results given in detail in this article in order to circle back and evaluate (3). As the discussion shows, it is vital that our previous results are proved by allowing for the real character twist.

In summary, we devise a complete and general approach by which one can evaluate sums of the form (3) in terms of the data associated to the polynomial $P$, such as its degree and coefficients, in terms of combinatorial coefficients. Each specific instance of the general result yields an identity, and each identity is just as enjoyable as (1). However, formulas such as (1) are not isolated, but rather part of the general mathematical structure coming from the discrete time heat kernel associated to the Cayley graph of finite abelian groups.

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2 Preliminaries

2.1 Weighted Cayley graphs of abelian groups

Let $G$ be a finite or countably infinite additive abelian group. Let $S \subseteq G$ be a finite symmetric subset of $G$. The symmetry condition means that if $s \in S$ then $-s \in S$. We will not assume that $S$ generates $G$. Let $\alpha: S \to \mathbb{R}_{>0}$ be a function such that $\alpha(s) = \alpha(-s)$. The weighted Cayley graph $X = \mathcal{C}(G, S, \alpha)$ of $G$ with respect to $S$ and $\alpha$ is constructed as follows. The vertices of $X$ are the elements of $G$, and two vertices $x$ and $y$ are connected with an edge if and only if $x - y \in S$. The weight $w(x, y)$ of the edge $(x, y)$ is defined to be $w(x, y) := \alpha(x - y)$. 
One can show that $X$ is a regular graph of degree

$$d = \sum_{s \in S} \alpha(s).$$

If $\alpha$ is a probability distribution on $S$, then the degree of the graph $X$ equals 1. In this case, we will denote $\alpha$ by $\pi_S$.

A function $f : G \to \mathbb{C}$ is an $L^2$-function if $\sum_{x \in G} |f(x)|^2 < \infty$. The set of $L^2$-functions on $G$ is a Hilbert space $L^2(G, \mathbb{C})$ with respect to the classical scalar product of functions

$$\langle f_1, f_2 \rangle = \sum_{x \in G} f_1(x) \overline{f_2(x)}.$$

We will denote by $\delta_x$ the standard delta function given by $\delta_x(x) = 1$ and $\delta_x(y) = 0$ for $x \neq y$.

The adjacency operator $A_X : L^2(G, \mathbb{C}) \to L^2(G, \mathbb{C})$ of the graph $X$ is defined as

$$(A_X f)(x) = \sum_{x-y \in S} \alpha(x-y) f(y).$$

When $X$ is finite, the adjacency operator written in the standard basis is called the adjacency matrix $A_X$ of the graph $X$. The $(x, y)$-entry of the adjacency matrix is $A_X(x, y) = \alpha(x-y)$. Since $\alpha(x-y) = \alpha(y-x)$, the matrix $A_X$ is symmetric. Moreover, when $\alpha = \pi_S$, it has the property that the elements of its columns, and rows since $A_X$ is symmetric, sum up to 1.

Given $x \in G$, let $\chi_x$ denote the character of $G$ corresponding to $x$; see, for example, [CR62] and Section 2.3 below. As proved in Corollary 3.2 of [Ba79], the character $\chi_x$ is an eigenfunction of the adjacency operator $A_X$ of $X$ with corresponding eigenvalue

$$\eta_x = \sum_{s \in S} \alpha(s) \chi_x(s).$$

### 2.2 Discrete time heat kernel on weighted Cayley graphs

Let $X$ denote the weighted Cayley graph $\mathbb{C}(G, S, \pi_S)$. The standard, or random walk, Laplacian $\Delta_X$ is defined to be the operator on $L^2(G, \mathbb{C})$ given by

$$\Delta_X f(x) = f(x) - \sum_{x-y \in S} \pi_S(x-y) f(y).$$

When $G$ is finite, the random walk Laplacian can be represented by the matrix $I - A_X$.

The discrete time heat kernel $K_X : X \times X \times \mathbb{Z}_{\geq 0} \to \mathbb{R}$ on $X$ is defined as the unique solution of the equation

$$(\partial_n + \Delta_X)K_X(x, y; n) = 0, \quad n \geq 0,$$

viewed as a function of $x \in G$ for a fixed $y \in G$, and with initial condition $K_X(x, y; 0) = \delta_x(y)$.

It can be shown that this also holds if we interchange the roles of $x$ and $y$. Here $\partial_n$ denotes the discrete time derivative, which is defined as

$$\partial_n K_X(x, y; n) = K_X(x, y; n+1) - K_X(x, y; n).$$

A straightforward computation shows that (8) is equivalent to

$$K_X(x, y; n+1) = \sum_{z-x \in S} \pi_S(x-z) K_X(z, y; n) = A_X^{n+1} K_X(x, y; 0),$$
In this section we will study (10), which is the discrete time heat kernel on the Cayley graph $X$. More intrinsically, $K_X(x, y; n)$ corresponds to $A_X^t(\delta_y)(x)$. Moreover, one can also show that $K_X(x, y; n) = K_X(x - y, 0; n)$ when $x$ and $y$ are in the same connected component.

The heat kernel $K_X(x, y; n)$ can be interpreted as the probability that the random walk, starting at $y$ arrives at $x$ after $n$ steps. For this random walk we assume that a particle at $z$ moves to an adjacent vertex $w$ with probability $\pi_S(z - w)$.

### 2.3 Twisted discrete time heat kernel on discrete tori

For fixed integers $d \geq 1$, $n > 0$, and a vector $m = (m_1, \ldots, m_d) \in \mathbb{Z}_d$, we denote by $G_m$ the group $\mathbb{Z}_d/m\mathbb{Z}_d = \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_d\mathbb{Z}$, which is a discrete torus, i.e. a product of $d$ discrete circles. Let $S$ be an arbitrary (not necessarily generating) symmetric subset of $G_m$ and $\pi_S$ a probability distribution on $S$. Let $X = \mathcal{C}(G_m, S, \pi_S)$ be the corresponding Cayley graph.

For an arbitrary vector $\beta = (\beta_1, \ldots, \beta_d)$ of real numbers, consider the character $\chi_\beta : \mathbb{Z}^d \rightarrow \mathbb{C}$ of $\mathbb{Z}^d$ defined as $\chi_\beta(x) = \exp(2\pi i \beta \cdot x)$. Under this definition, it suffices to take $\beta \in \mathbb{R}^d/\mathbb{Z}^d$.

We are interested in the space $L^2(\mathbb{Z}^d, \mathbb{C}, \chi_\beta)$ of $L^2$-functions $f : \mathbb{Z}^d \rightarrow \mathbb{C}$ satisfying the transformation property

$$f(x + km) = \exp(2\pi i \beta \cdot k) f(x)$$

under the (additive) action of the group $m_1\mathbb{Z} \times \cdots \times m_d\mathbb{Z} = m\mathbb{Z}^d$. Here we use the notation $km = (k_1m_1, \ldots, k_d m_d)$, for $k = (k_1, \ldots, k_d)$ and $m = (m_1, \ldots, m_d)$, and $\beta \cdot k = \beta_1 k_1 + \cdots + \beta_d k_d$ for the classical dot product.

A function $f$ satisfying (9) will be called a $\chi_\beta$-twisted $m$-periodic function on $\mathbb{Z}^d$. When $\beta = (0, \ldots, 0) \in \mathbb{Z}^d$, the function $f$ is periodic with period $m$. When $\beta = (1/2, \ldots, 1/2)$, functions $f$ satisfying transformation property (9) are sometimes called anti-periodic.

The $\chi_\beta$-twisted discrete time heat kernel on $X$ is defined to be a function

$$K_{X, \beta}(x, y; n) : \mathbb{Z}^d \times \mathbb{Z}^d \times \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R},$$

and it has the following properties. For a fixed $y \in \mathbb{Z}^d$, and viewed as a function of $x \in \mathbb{Z}^d$, (10) satisfies (9), and also as a function of $y \in \mathbb{Z}^d$ for a fixed $x \in \mathbb{Z}^d$. Additionally, when viewed as a function of $n$, (10) satisfies the discrete analogue of the heat, or diffusion, equation

$$(\Delta_X + \partial_n)K_{X, \beta}(x, y; n) = 0,$$

with initial condition $K_{X, \beta}(x, y; 0) = \delta_y(x)$, for $x, y \in \mathbb{Z}^d/m\mathbb{Z}^d$.

### 3 A combinatorial formula for the twisted heat kernel

In this section we will study (10), which is the discrete time heat kernel on the Cayley graph $X = \mathcal{C}(G_m, S, \pi_S)$ twisted by $\chi_\beta$. The heat kernel on $X$ can be computed from the heat kernel $K_Y(x, y; n)$ on the Cayley graph $Y = \mathcal{C}(\mathbb{Z}^d, S, \pi_S)$ by using the methods which are common in the theory of automorphic forms, and is analogous to the construction of the continuous time heat kernel on the discrete torus; see [KN06], [CJK10] and [Do12].

We start by determining $K_Y(x, y; n)$. Since $K_Y(x, y; n) = K_Y(x - y, 0; n)$, it suffices to compute $K_Y(x, 0; n)$. A general formula is presented in the lemma below.

**Lemma 1.** Let $S$ be a symmetric, finite subset of $\mathbb{Z}^d$ and let $\pi_S$ a probability distribution on $S$. For $j \in \{1, \ldots, d\}$ let $S_j := \max\{s_j : s = (s_1, \ldots, s_d) \in S\}$. By $t$ we denote the $d$-tuple $(t_1, \ldots, t_d)$, and for $q = (q_1, \ldots, q_d) \in \mathbb{Z}^d$ we let $t^q$ denote $t_1^{q_1} \cdots t_d^{q_d}$.
a. If \(|x_j| > nS_j\), for some \(j \in \{1, \ldots, d\}\), then

\[ K_Y(x, 0; n) = K_Y((x_1, \ldots, x_d), 0; n) = 0. \]

b. If \(x_j \in \{-nS_j, 0, \ldots, nS_j\}\), for all \(j \in \{1, \ldots, d\}\), then \(K_Y(x, 0; n)\) equals the coefficient multiplying \(t^s\) in the expansion of the polynomial

\[ \left( \sum_{s \in S} \pi_S(s)t^s \right)^n. \] (11)

Proof. First, notice that \(S_j \geq 0\) because \(S\) is assumed to be symmetric; hence, the conditions given on coordinates \(x_j\) of \(x\) are well-defined. The proof of the lemma follows immediately by induction, using the observation that, for a fixed \(x \in \mathbb{Z}^d\), \(K_Y(x, 0; n)\) is the probability that the random walk, starting at \(0\), ends up at \(x\) after \(n\) steps. Therefore, at each step, a particle at a point \(y \in \mathbb{Z}^d\) can move only to points \(y + s\), for \(s \in S\) with probability \(\pi_S(s)\).

In the proposition below, the heat kernel \(K_X(x, y; n)\) twisted by a character \(\chi\) is derived from the heat kernel on \(Y\) using what could be called the method of twisted images; see, for example, [Do12].

**Proposition 2.** With notation as above, the discrete time heat kernel on \(X\) twisted by the character \(\chi\) on \(\mathbb{Z}^d\) is given by

\[ K_{X, \beta}(x, y; n) = \sum_{k \in \mathbb{Z}^d} \exp(-2\pi i k \cdot \beta) K_Y(x - y + km, 0; n), \]

where \(K_Y(x, y; n)\) denotes the discrete time heat kernel on the graph \(Y = \mathcal{C}(\mathbb{Z}^d, S, \pi_S)\).

Proof. Notice that the sum on the right-hand side of (12) is finite, and therefore that sum is convergent and well-defined. Next, let us check the transformation property under the action of \(m\mathbb{Z}^d\). For any \(l \in \mathbb{Z}^d\) we have

\[ K_{X, \beta}(x + lm, y; n) = \sum_{k \in \mathbb{Z}^d} \exp(-2\pi i (k + l) \cdot \beta) \exp(2\pi i l \cdot \beta) K_Y(x - y + (k + l)m, 0; n) \]

\[ = \exp(2\pi i l \cdot \beta) \sum_{k' \in \mathbb{Z}^d} \exp(-2\pi i k' \cdot \beta) K_Y(x - y + k'm, 0; n) = \exp(2\pi i l \cdot \beta) K_{X, \beta}(x, y; n), \]

where \(k' = k + l\). The transformation property in the second variable is proved analogously. It only remains to prove that \(K_{X, \beta}(x, y; n)\) satisfies the diffusion equation for the weighted Cayley graph \(X = \mathcal{C}(G_m, S, \pi_S)\). This follows trivially from the definition of the random walk Laplacian acting on the space of functions initially defined on \(X\) and satisfying the transformation property [3].

Now, we develop the main result of this section, which is a general combinatorial formula to compute the heat kernel \(K_{X, \beta}(x, 0; n)\) of the weighted graph \(X = \mathcal{C}(G_m, \pi_S, S)\). We assume \(S\) is presented as a disjoint union \(S = S_1 \cup (-S_1)\), where \(-S_1\) denotes the set of all the inverses of the elements of \(S_1\). We denote by \(l > 0\) the number of elements of \(S_1\). These elements will be denoted as \(s_1 = (s_{11}, \ldots, s_{dl}), \ldots, s_l = (s_{1l}, \ldots, s_{dl})\).
In view of Proposition 2 and Lemma 1 in order to determine $K_{X, \beta}(x, 0; n)$ it suffices to count the number of monomials in the variables $t_1, \ldots, t_d$ which have the same power mod $m$ in the polynomial
\[
\left( \sum_{s \in S_i} (\pi(s)t^s + \pi(-s)t^{-s}) \right)^n = \sum_{(a_1, \ldots, a_l) \in \mathbb{Z}_{>0}^l \atop a_1 + \cdots + a_l = n} \frac{n!}{a_1! \cdots a_l!} \prod_{j=1}^l (s_j)^{a_j} (t^{s_j} + t^{-s_j})^{a_1} \cdots (t^{s_j} + t^{-s_j})^{a_l}.
\]

Note that we have used the fact that $\pi(s) = \pi(-s)$. This polynomial can be written as
\[
P(t_1, \ldots, t_d) = \sum_{a=(a_1, \ldots, a_l) \in \mathbb{Z}_{>0}^l \atop a_1 + \cdots + a_l = n} \frac{n!}{a_1! \cdots a_l!} \prod_{j=1}^l (s_j)^{a_j} (t^j)^{(2j_1-a_1)s_1} \cdots (t^j)^{(2j_l-a_l)s_l}
\]
where $t_1 = t_1^{s_1} \cdots t_1^{s_d}$, $t_l = t_1^{s_1} \cdots t_1^{s_d}$. Hence, one must determine for any fixed $l$-tuple $a = (a_1, \ldots, a_l) \in \mathbb{Z}_{>0}^l$ with $a_1 + \cdots + a_l = n$, all possible choices for the vector $(j_1, \ldots, j_l)$ such that
\[
\left( \begin{array}{cccc}
  s_{11} & \cdots & s_{1l} \\
  \vdots & & \vdots \\
  s_{d1} & \cdots & s_{dl}
\end{array} \right) \left( \begin{array}{c}
  2j_1 - a_1 \\
  \vdots \\
  2j_l - a_l
\end{array} \right) = \left( \begin{array}{c}
  x_1 + k_1 m_1 \\
  \vdots \\
  x_d + k_d m_d
\end{array} \right),
\]
for some $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$, under the restrictions $0 \leq j_r \leq a_r$.

Let us denote by $\sigma$ the $\mathbb{Z}$-linear homomorphism $\sigma : \mathbb{Z}^l \to \mathbb{Z}^d$ determined by the matrix $S = (s_{ij})_{l \times l}$. First, we determine all possible vectors $z \in \mathbb{Z}^l$ such that $p(\sigma(z)) = 0$, where $p$ denotes the canonical surjection $p : \mathbb{Z}^d \to \mathbb{Z}^d/\mathbb{m}\mathbb{Z}^d = G_m$.

All possible solutions in $G_m$ of this problem can be obtained as follows. By selecting appropriate bases $B_{Z^l} = \{v_i : i = 1, \ldots, l\}$ and $B_{Z^d} = \{w_j : j = 1, \ldots, d\}$ for $\mathbb{Z}^l$ and $\mathbb{Z}^d$, respectively, the matrix $S$ has diagonal form when written in these bases
\[
(\bar{S}) = \left( \begin{array}{cccc}
  q_1 & \cdots & 0 & \cdots \\
  \vdots & \ddots & \vdots & \ddots \\
  0 & \cdots & q_t & \cdots
\end{array} \right)_{t \times l}
\]
Here $t = \max\{d, l\}$ and $\{q_i\}$ denotes the set of invariant factors of $\sigma$. In case $d < l$, we define $q_{d+1} = \ldots = q_t = 0$. Finding these basis amounts to finding invertible matrices $U$ and $V$ such that $USV = (\bar{S})$ is the Smith normal form of $S$; see [Sm1861].

Hence, for $i = 1, \ldots, l$ one has $\sigma(v_i) = q_i w_i$. Using the matrix $U = (u_{ij})_{d \times d}$ we can write $w_j = u_{1j} e_1 + \cdots + u_{dj} e_d$, where $e_1, \ldots, e_d$ denotes the canonical basis for $\mathbb{Z}^d$. Let $(z_1, \ldots, z_l)$ be the coordinates of $z \in \mathbb{Z}^l$ in the basis $\{v_1, \ldots, v_l\}$. Solving for $p(\sigma(z)) = p(\sigma(\sum_{i=1}^l z_i v_i)) = 0$ is equivalent to solving in $G_m$ the system of $d$ equations
\[
\left[ \begin{array}{c}
  z_1 c_1 \equiv 0 \mod m_1 \\
  \vdots \\
  z_d c_d \equiv 0 \mod m_d
\end{array} \right],
\]
where \( c_i = \sum_{j=1}^{d} (a_{ij}q_i) \), \( i = 1, \ldots, d \). Each equation \( c_iz_i \equiv 0 \mod m_i \), \( i = 1, \ldots, d \) in the system \([15]\) can be readily solved in \( \mathbb{Z}/m_i\mathbb{Z} \) by elementary number theory. Thus, we assume we have already computed all possible solutions in \( G_m \) of \([15]\) and they are given as the \( N \) elements of the set

\[
L_{S,m} = \{ z_\mu = (z_{1\mu}, \ldots, z_{d\mu}) : \mu = 1, \ldots, N \}.
\]

Now, in order to solve \([14]\) we can take each solution \((z_{1\mu}, \ldots, z_{d\mu}) \in L_{S,m} \) and find all possible \( k = (k_1, \ldots, k_d) \) such that

\[
2j_1 - a_1 = z_{1\mu} + x_1 + k_1m_1
\]

\[
\vdots
\]

\[
2j_d - a_d = z_{d\mu} + x_d + k_dm_d
\]

where the original restrictions \( 0 \leq j_r \leq a_r \) are equivalent to choosing \( k \) such that

\[
0 \leq a_r + z_{r\mu} + x_r + k_rm_r \leq 2a_r, \quad \text{for } r = 1, \ldots, d,
\]

\[
0 \leq j_r \leq a_r, \quad \text{for } r = d + 1, \ldots, l, \text{ in case } d < l.
\]

Thus, \( k \) must satisfy for each \( \mu = 1, \ldots, N \) that

\[
|z_{r\mu} + x_r + k_rm_r| \leq a_r, \quad \text{for } r = 1, \ldots, d.
\]

With all this discussion, we have proved the following theorem.

**Theorem 3.** With notation as above, let

\[
A_{S,m}(a_1, \ldots, a_l; n; k) = e^{-\pi ik} \frac{n!}{a_1! \cdots a_l!} \prod_{j=1}^{l} \pi(s_j)^{a_j} \prod_{r=1}^{d} \left( \frac{a_r}{(a_r + x_r + z_{r\mu} + k_rm_r)/2} \right).
\]

Then the heat kernel \( K_{G_m,S}(x,0;n) \) is given by the following formulas.

1. If \( d \geq l \):

\[
\sum_{(a_1, \ldots, a_l) \in \mathbb{Z}_{\geq 0}^l} \sum_{a_1+\cdots+a_l=n} \sum_{\mu \in L_{S,m}} \sum_{k \in \mathbb{Z}^d} A_{S,m}(a_1, \ldots, a_l; n; k),
\]

2. If \( d < l \)

\[
\sum_{(a_1, \ldots, a_l) \in \mathbb{Z}_{\geq 0}^l} \sum_{a_1+\cdots+a_l=n} \sum_{\mu \in L_{S,m}} \prod_{r=d+1}^{l} \left( \frac{a_r}{j_r} \right) A_{S,m}(a_1, \ldots, a_l; n; k).
\]

**Proof.** It readily follows from \([12]\), \([15]\) and \([17]\). \( \square \)

**Example 4.** Let \( d = 1, S = \{ \pm 1 \}, \pi_S(s) = 1/|S| = 1/2 \) for \( s \in S \). Then an application of Lemma \([11]\) and the binomial theorem yields that the heat kernel \( K_Y(x,0;n) \) for \( n \geq 0 \) on the Cayley graph \( Y = \mathbb{C}(\mathbb{Z}, S, \pi_S) \) is given by

\[
K_Y(x,0;n) = \begin{cases} 
2^{-n} \left( \binom{n}{(x+n)/2} \right), & \text{if } |x| \leq n \text{ and } x + n \text{ is even}; \\
0, & \text{if } |x| > n \text{ or } x + n \text{ odd}.
\end{cases}
\]

This is a well-known result from probability theory; see \([\text{Wo00}]\) as well as \([\text{SS14}], \text{Example 3.3}\) for a related, more general result.
Example 5. Let $m$ be a positive integer, and let $b \in \{1, \ldots, m-1\}$. Consider the weighted Cayley graph $X = \mathcal{C}(\mathbb{Z}/m\mathbb{Z}, S, \pi_S)$, where $S = \{-b, b\}$ is the symmetric subset of $\mathbb{Z}/m\mathbb{Z}$ and $\pi_S(s) = 1/2$, for $s \in S$. Let $Y$ be the weighted Cayley graph $\mathcal{C}(\mathbb{Z}, S, \pi_S)$. Then, from Proposition 2 with $d = l = 1$ and arbitrary $\beta \in \mathbb{R}$, the discrete time heat kernel on $X$ twisted by the character $\chi_\beta$ on $\mathbb{Z}$ is given by

$$K_{X, \beta}(x, y; n) = \sum_{\ell = -\infty}^\infty e^{-2\pi i \ell \beta} K_Y(x - y + \ell m, 0; n),$$

where

$$K_Y(z + \ell m, 0; n) = \begin{cases} 2^{-n}((z + \ell m + bn)/2b), & \text{if } |z + \ell m| \leq bn \text{ and } 2b \mid (z + \ell m + bn) \\ 0 & \text{otherwise.} \end{cases}$$

Remark 6. An application of Theorem 3 with $d = 1$, $\beta = 0$ and $S = \{\pm 1, \ldots, \pm m\}$ yields the combinatorial expression for the discrete time and discrete space heat kernel $u(x, t)$ for $x \in \mathbb{Z}$ and $t \in \mathbb{Z}_{\geq 0}$, which is a solution to the diffusion equation

$$u(x, t + 1) - u(x, t) = \sum_{i = -m}^m a_i u(x + i, t) \quad \text{with} \quad \sum_{i = -m}^m a_i = 0, \ a_0 \leq 0, \ a_i \geq 0 \ \text{for } i \neq 0$$

as studied in \cite{FSS14}. Moreover, Theorem 3 provides a means by which one can explicitly evaluate solutions to the diffusion equation on $\mathbb{Z}_d$ for any $d > 1$.

4 The main result

In this section we present our main result, which is equation (21). The formula relates a sum of products of binomial coefficients to a sum of $n$-th powers of trigonometric functions, after each side of the equation is twisted by a character. The proof is based on the existence of two different expressions for the discrete time twisted heat kernel: First, from a purely combinatorial expression, given in Proposition 2 and Theorem 3; and second, using spectral theory, stemming from the fact that the eigenvalues of the adjacency matrix are expressed in terms of the characters of the underlying group of the corresponding Cayley graph as proved in Corollary 3.2 of \cite{Ba79}. Because the heat kernel is unique, any two distinct formulas for the heat kernel must be equal.

Theorem 7. Let $d \geq 1$ be an integer, $m = (m_1, \ldots, m_d) \in \mathbb{Z}_{\geq 0}^d$ and $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{R}^d$. Let $G_m$ denote the discrete torus $\mathbb{Z}_m^d$ with $M = \prod_{j=1}^d m_j$ vertices, and let $S$ be a symmetric subset of $G_m$ with an associated positive probability distribution $\pi_S$. Then, for any positive integer $n$ and $x, y \in \mathbb{Z}_d$, we have that

$$\sum_{\mathbf{k} \in \mathbb{Z}_d} \exp(-2\pi i \mathbf{k} \cdot \mathbf{\beta}) K_Y(x - y + \mathbf{k} m, 0; n) = \frac{1}{M} \sum_{\mathbf{n} \in G_m} (\lambda_n^m \beta^n \lambda_n^m) \chi_{\mathbf{n}}^m \beta^\mathbf{n} \lambda_{\mathbf{n}}^m \beta^\mathbf{n}, \quad (21)$$

where $K_Y$ is defined in Lemma \ref{lem:heat_kernel}, $\{\lambda_n^m \beta^n \}_{\mathbf{n} = (n_1, \ldots, n_d) \in G_m}$ is the set of additive characters on $\mathbb{Z}_d$ defined by

$$\chi_{\mathbf{n}}^m \beta^\mathbf{n}(x) = \prod_{j=1}^d \exp \left(2\pi i n_j + \frac{\beta_j}{m_j} x_j\right), \quad \text{for all } x = (x_1, \ldots, x_d) \in \mathbb{Z}_d, \quad (22)$$
and \( \lambda_n^{m, \beta} \), for \( n \in G_m \) is given by
\[
\lambda_n^{m, \beta} = \sum_{s \in S} \pi_S(s) \chi_n^{m, \beta}(s).
\] (23)

**Proof.** The proof follows from the spectral expansion of the discrete time twisted heat kernel. Namely, the eigenfunctions of the adjacency matrix of the Cayley graph \( X = \mathcal{C}(G_m, S, \pi_S) \) correspond to the characters of the abelian group \( G_m \). Therefore, it is straightforward that \( \left\{ \frac{1}{\sqrt{M}} \lambda_n^{m, \beta} \right\}_{n \in G_m} \) is a set of \( L^2 \)-normalized eigenfunctions of the adjacency matrix \( A_X \) which satisfy the transformation property (9). The corresponding eigenvalues are \( \left\{ \lambda_n^{m, \beta} \right\}_{n \in G_m} \); there are \( M \) of them, counted with multiplicities.

The matrix which corresponds to the random walk on the weighted graph \( X \) acting on the set of functions satisfying the transformation property (9) is diagonalized when using the eigenfunctions \( \frac{1}{\sqrt{M}} \lambda_n^{m, \beta} \) as a basis. Hence, we may write the discrete time heat kernel twisted by \( \chi_\beta \) at time \( n \geq 1 \) as
\[
K_{X, \beta}(x, y; n) = \frac{1}{M} \sum_{n \in G_m} \left( \lambda_n^{m, \beta} \right)^n \lambda_n^{m, \beta}(x) \overline{\lambda_n^{m, \beta}(y)}.
\] (24)

Thus, equation (21) follows from the uniqueness of the heat kernel, once we combine equation (24) and Proposition 2.

A particularly simple combinatorial identity can be deduced by taking \( x = y \) in (21). Indeed, each character takes values in the set of complex numbers of absolute value equal to one, so then
\[
\lambda_n^{m, \beta}(x) \overline{\lambda_n^{m, \beta}(x)} = 1,
\]
for all \( x, n, m \) and \( \beta \). Therefore, we get the following corollary.

**Corollary 8.** With notation as above and any positive integer \( n \), we have that
\[
\sum_{n \in G_m} \left( \lambda_n^{m, \beta} \right)^n = M \sum_{k \in \mathbb{Z}^d} \exp(-2\pi i k \cdot \beta) K_Y(k m, 0; n),
\] (25)
where \( K_Y \) is defined in Lemma 1 and \( \lambda_n^{m, \beta} \) is given by (23).

Formula (25) shows how to compute explicitly the trigonometric sum which is the trace of the discrete time heat kernel of the random walk Laplacian, twisted by \( \chi_\beta \), in terms of a twisted sum of binomial coefficients. Further simplifications of formula (25) can be obtained by taking \( \beta \in m \mathbb{Z}^d \) or \( \beta \in \frac{1}{2} m \mathbb{Z}^d \). Namely, when \( \beta \in m \mathbb{Z}^d \), we have \( \exp(-2\pi i k \cdot \beta) = 1 \), while for \( \beta \in \frac{1}{2} m \mathbb{Z}^d \) we have \( \exp(-2\pi i k \cdot \beta) = (-1)^k m \). The expressions for \( \lambda_n^{m, \beta} \) can be significantly simplified in those cases as well.

**Remark 9.** Assume \( \pi_S \) is the uniform distribution and that there is no character twist, meaning \( \beta = 0 \). Then the left hand side of equation (25) must be an integer. To see this, let \( E \) denote the normal closure over \( \mathbb{Q} \) containing all the primitive \( m_i \)-th roots of unity \( \omega_i \) for \( i = 1, \ldots, d \). Any element \( \phi \) of the Galois group \( \text{Gal}(E/\mathbb{Q}) \) maps any \( \omega_i \) into another primitive \( m_i \)-th root of unity, meaning \( \phi(\omega_i) = \omega_i^{r_i} \) where \( 0 < r_i < m_i \) is relatively prime to \( m_i \). Therefore,
\[
\phi\left(|S|\lambda_n^{m, 0}\right) = \sum_{s=(s_1, \ldots, s_d) \in S} \phi\left(\lambda_n^{m, 0}\right) = \sum_{(s_1, \ldots, s_d) \in S} \phi(\omega_1^{n_1 s_1} \cdots \omega_d^{n_d s_d}) = \sum_{(s_1, \ldots, s_d) \in S} \omega_1^{n_1 s_1} \cdots \omega_d^{n_d s_d} = |S| \lambda_n^{m, 0},
\]
where \(|S|\) is the cardinality of \(S\), \(n' = rn\), and \(r = (r_1, \ldots, r_d)\). Since each \(r_i\) is relatively prime to \(m_i\), multiplication by \(r\) permutes the elements of \(\mathbb{Z}_m\). Consequently, \(\phi\) permutes the set of eigenvalues \(\lambda_{m,0}^{n,m}\). Hence, if \(a = |S|^n \sum_{m \in G_m} (\lambda_{m,0}^{n,m})^n\) then

\[
\phi(a) = |S|^n \sum_{m' \in G_m} (\lambda_{m',0}^{n,m})^n = a,
\]

and, therefore, \(a\) must be a rational number. Clearly \(a\) is an algebraic integer, so then \(a\) must be an integer. This shows that one interesting consequence of our main theorem is that it provides an explicit formula to compute the integer \(a\) in terms of binomial coefficients.

Implicit in this argument is the fact that the left-hand side of (25) is a symmetric polynomial in the eigenvalues \(\lambda_1, \ldots, \lambda_M\) defined in (23), where \(M\) denotes the order of \(G_m\). In this case the symmetric polynomial is \(p_m(x_1, \ldots, x_M) = x_1^n + \cdots + x_M^n\). It is a well known fact that any symmetric polynomial in the variables \(x_1, \ldots, x_M\) can be written as a polynomial with rational coefficients in the \(p_k\)'s, where \(p_k(x_1, \ldots, x_M) = x_1^k + \cdots + x_M^k\), with \(k = 1, \ldots, n\) [STF99]. Therefore, for any symmetric polynomial with rational coefficients \(q(x_1, \ldots, x_M)\), the value of \(q(\lambda_1, \ldots, \lambda_M)\) is a rational number.

5 Trigonometric sums stemming from the twisted heat kernel on a discrete torus

In this section, we present corollaries of our main result that can be deduced from (21) by an appropriate choice of \(d, m\) and \(\beta\). We derive many interesting identities, and show that special instances of those identities are main results from e.g. [dFGK17], [dFK13], [Me12] and [BH10].

5.1 Trigonometric sums twisted by an additive character

The first corollary in this section is a consequence of the computation of the trace of the heat kernel, as in Corollary 8.

**Corollary 10.** For arbitrary positive integers \(m, n\) and arbitrary \(\beta \in \mathbb{R}\),

\[
\sum_{j=0}^{m-1} \cos^n \left( \frac{2\pi(j + \beta)}{m} \right) = 2^{-n} \sum_{k=-[n/m], \ km+n \ even}^{[n/m]} e^{-2\pi ik\beta} \left( \binom{n}{km+n}/2 \right). \tag{26}
\]

Here \(\lfloor x \rfloor\) stands for the largest integer less than or equal to \(x\).

**Proof.** We apply Corollary 8 with \(d = 1, m = m = M, \beta = \beta, S = \{-1, 1\} \) and \(\pi_S(s) = 1/2\) for \(s \in S\). Then, for a fixed, arbitrary integer \(n > 0\), an application of formula (18) with \(x = km\) shows that, for any \(\beta \in \mathbb{R}\), the right hand side of (25) equals

\[
2^{-n} \sum_{k=-[n/m], \ km+n \ even}^{[n/m]} e^{-2\pi ik\beta} \left( \binom{n}{km+n}/2 \right).
\]

For \(j \in \{0, \ldots, m\}\) the eigenvalues \(\lambda_{j,\beta}^{m,0}\) of the adjacency matrix are

\[
\lambda_{j,\beta}^{m,0} = \frac{1}{2} \left( e^{2\pi i (j+\beta)/m} + e^{-2\pi i (j+\beta)/m} \right) = \cos(2\pi (j + \beta)/m).
\]

Thus, formula (25) yields identity (26). \(\Box\)
Next, we examine a few special choices of $\beta$, in order to show that (26) generalizes the main results in [dFGK17] and [Me12].

1. For $\beta = 0$, specializing (26) to the case where $m = 2m_1$ and $n = 2n_1$ are even natural numbers we get
\[
\sum_{j=0}^{2m_1-1} \cos^{2n_1} \left( \frac{\pi j}{m_1} \right) = 2^{-2n_1} 2m_1 \sum_{k=-|n/m|}^{n/m} \left( \frac{2n_1}{k m_1 + n_1} \right),
\]
which can be simplified to give equation (11) which is the first main result of [dFGK17].

2. When $\beta = -m/4$, we have $\cos \left( \frac{2\pi (j + \beta)}{m} \right) = \sin(2\pi j/m)$, and (26) becomes
\[
\sum_{j=0}^{m-1} \sin^n \left( \frac{2\pi j}{m} \right) = 2^{-n} m \sum_{k=-|n/m|}^{n/m} i^{km} \left( \frac{n}{(km + n)/2} \right).
\]
By specializing to the case where $m = 2m_1$ and $n = 2n_1$ are even natural numbers, analogously as above we obtain
\[
\sum_{j=0}^{m_1-1} \sin^{2n_1} \left( \frac{\pi j}{m_1} \right) = 2^{-2n_1} m_1 \sum_{k=-|n_1/m_1|}^{n_1/m_1} (-1)^{km} \left( \frac{2n_1}{k m_1 + n_1} \right).
\]
This proves the second main result of [dFGK17].

3. By choosing $\beta = 0$, $m = 2m_1$ and $n = 2p$ even natural numbers, and applying trigonometric identities to reduce the range of summation, one easily deduces Theorem 1 from [Me12]. Theorem 3 from [Me12] follows by the same procedure, after taking $m = 2m_1$ and $n = 2p$ even natural numbers and $\beta = 1/2$ in (26).

In the following corollary we apply the main theorem for $d = 1$ and show how to deduce a combinatorial expression for powers of sines and cosines twisted by an additive character.

**Corollary 11.** For arbitrary positive integers $m, n$, any integer $b \in \{1, \ldots, m - 1\}$, arbitrary $\alpha, \varphi \in \mathbb{R}$ and any character $\chi$ of the additive group $\mathbb{Z}/m\mathbb{Z}$, we have that
\[
\sum_{j=0}^{m-1} \chi(j) \cos^n \left( \frac{2\pi jb}{m} + \alpha \right) = \frac{m}{2^n} \sum_{d \in \{0, \ldots, n\} \mod (2d-n)b-r} \binom{n}{d} \exp \left( i\alpha(n - 2d) \right), \tag{27}
\]
and
\[
\sum_{j=0}^{m-1} \chi(j) \sin^n \left( \frac{2\pi jb}{m} + \varphi \right) = \frac{m}{2^n} \sum_{d \in \{0, \ldots, n\} \mod (2d-n)b-r} \binom{n}{d} \exp \left( i\varphi(n - 2d) \right) \exp \left( -i\frac{\pi}{2}(n - 2d) \right). \tag{28}
\]
Here $r \in \{1, \ldots, m - 1\}$ is an integer such that $\chi(x) = \exp(2\pi irx/m)$ for all $x \in \mathbb{Z}/m\mathbb{Z}$ and the empty sum equals zero.

**Proof.** We use the setting of Example 5 in which the discrete time heat kernel on $X$ twisted by the character $\chi_\beta$ on $Z$ is given for positive integers $n$ and $x, y \in \mathbb{Z}/m\mathbb{Z}$ by (19) and (20). Therefore, the left hand side of formula (21) becomes:
\[
K_{X,\beta}(x,y;n) = \sum_{d \in \{0, \ldots, n\} \mod (2d-n)b-(x-y)} 2^{-n} \binom{n}{d} \exp(2\pi i(x - y + (n - 2d)b)\beta/m). \tag{29}
\]
For $j \in \{0, \ldots, m\}$ the product of twisted characters $\chi_{j}^{m,\beta}$ on the right-hand side of formula (21) is

$$\chi_{j}^{m,\beta}(x)\chi_{j}^{m,\beta}(y) = \exp(2\pi i (j + \beta)(x - y)/m)$$

and the associated eigenvalues are

$$\lambda_{j}^{m,\beta} = \frac{1}{2} \left( \exp\left(\frac{2\pi i (j + \beta)b}{m}\right) + \exp\left(-\frac{2\pi i (j + \beta)b}{m}\right) \right) = \cos\left(\frac{2\pi jb}{m} + \frac{2\pi \beta b}{m}\right).$$

Equation (21) now yields the following identity, valid for any real number $\beta$, positive integers $n$ and $x, y \in \mathbb{Z}/m\mathbb{Z}$:

$$\sum_{j=0}^{m-1} \exp\left(2\pi i \frac{j(x - y)}{m}\right) \cos^n \left(\frac{2\pi jb}{m} + \frac{2\pi j\beta b}{m}\right) = \frac{m}{2^n} \sum_{d \in \{0, \ldots, n\}} \binom{n}{d} \exp\left(\frac{2\pi i (n - 2d)\beta b}{m}\right).$$

We take $x, y$ above so that $r = x - y$, and deduce, by choosing $\beta = \frac{ma}{2\pi}$, the identity (27).

By choosing $\alpha = \varphi - \frac{\pi}{2}$, equation (27) becomes (28). This completes the proof of the corollary.

The following example illustrates the results which are stated in Corollary 11.

**Example 12.** For some positive integer $k$, let $m = 3k$ and $r = k$, and set $\alpha = 0$ in Corollary 11. Then, one gets the following interesting formula for the sum of powers of cosines twisted by the third root of unity $\omega = \exp(2\pi i/3)$:

$$\sum_{j=0}^{3k-1} \omega^j \cos^n \left(\frac{2\pi j}{3k}\right) = \frac{3k}{2^n} \sum_{d \in \{0, \ldots, 2n\}} \binom{n}{d} \cos\left(\frac{2\pi (n - 2d)\beta b}{m}\right).$$

We find it fascinating that the sum on the left hand side is a rational number. For example, when $m = 102$, so $k = 34$, and taking $n = 100$ one has that

$$\sum_{j=0}^{101} \omega^j \cos^{100} \left(\frac{\pi j}{51}\right) = \frac{102}{2^{100}} \left(\binom{100}{16} + \binom{100}{67}\right) = \frac{7514656923394238847040235025}{316912650057057350347175801344}.$$

Formulas (27) and (28) can be combined with simple trigonometric identities needed to reduce the range of summation to yield the new, succinct expression for the evaluation of the alternating sum of powers of cosines with fractional multiples of $\pi/2$ evaluated in [DFK13]. Namely, we have the following corollary.

**Corollary 13.** For two positive integers $m, n$, let

$$S(n, m) := \sum_{k=1}^{m} (-1)^k \cos^{2n} \left(\frac{k\pi}{2m} + \frac{2}{2m + 2}\right).$$

Then

$$S(n, m) = -\frac{1}{2} + \frac{m + 1}{4^n} \sum_{d \in \{0, \ldots, 2n\}} \binom{2n}{d},$$

with the convention that the empty sum equals zero.
Proof. Let $M := m + 1$. A simple change of variables in the summation gives that

\[
\sum_{k=0}^{4M-1} (-1)^k \cos^{2n} \left( \frac{k\pi}{2M} \right) = 2 + 2S(n,m) + 2 \sum_{k=0}^{M-1} (-1)^{k+M} \sin^{2n} \left( \frac{k\pi}{2M} \right) = 2 + 4S(n,m)
\]

where the second equality follows by expressing $S(n,m)$ in terms of sums of sines (as in [dFK13], page 357). On the other hand, equation (27) with $m = 4M$, $b = 1$, $\alpha = 0$, $n = 2n$ and $r = 2M$ yields

\[
\sum_{k=0}^{4M-1} (-1)^k \cos^{2n} \left( \frac{k\pi}{2M} \right) = \frac{4M}{4^n} \sum_{d \in \{0, \ldots, 2n\}} \left( \frac{2n}{d} \right).
\]

Combining the last two displays yields identity (30). \(\square\)

Let us note that the procedure to compute $S(n,m)$ using the identity (30) is straightforward. For example, when $n = 100$ and $m = 13$, it suffices to determine the set of $d \in \{0, \ldots, 200\}$ for which $28 \mid (d - 114)$; this set equals $\{2, 30, 58, 86, 114, 142, 170, 198\}$. Hence,

\[
S(100, 13) = -\frac{1}{2} + \frac{14}{4^{100}} \cdot 2 \left( \binom{200}{2} + \binom{200}{30} + \binom{200}{58} + \binom{200}{86} \right).
\]

Inserting this expression into the Wolfram Alpha online calculator, and after a few seconds, one obtains

\[
S(100, 13) = -\frac{2782014441650476614943952313424972252178190684153272739503}{100433627766186892221372630771322266265763768711142455220636},
\]

which exactly matches the evaluation of the same sum on p. 373 of [dFK13]. As a further example (not computed in [dFGK17]) we get that

\[
S(110, 18) = -\frac{1}{2} + \frac{19}{4^{110}} \cdot 2 \left( \binom{220}{15} + \binom{220}{53} + \binom{220}{91} \right).
\]

The Wolfram Alpha calculator produces the rational number

\[
S(110, 18) = -\frac{89182248821791030451854720643493491799318739888463936472767026811637}{2106245833711437339583605536734086463779019081098222508621955072}.
\]

### 5.2 Trigonometric sums twisted by a multiplicative character

Equation (28) yields the main result of [BH10], their Theorem 1.2, which is also related to [BZ04] and [BBCZ05]. Actually, in this section we prove a more general evaluation of a finite trigonometric sum for any multiplicative primitive Dirichlet character modulo $m$.

**Corollary 14.** Choose a positive integer $m$, and let $\tilde{\chi}$ be a primitive Dirichlet character modulo $m$. Then for any integer $b \in \{1, \ldots, m - 1\}$, arbitrary $\alpha, \varphi \in \mathbb{R}$, we have that

\[
\sum_{j=0}^{m-1} \tilde{\chi}(j) \cos \left( \frac{2\pi jb}{m} + \alpha \right) = \frac{m}{2^n \tau(\tilde{\chi})} \sum_{d \in \{0, \ldots, n\}} \tilde{\chi}(r) \left( \frac{n}{d} \right) \exp \left(i\alpha(n - 2d)\right) \tag{31}
\]
\[
\sum_{j=0}^{m-1} \overline{\chi(j)} \sin^n \left( \frac{2\pi jb}{m} + \varphi \right) = \frac{m}{2^n \tau(\overline{\chi})} \sum_{r=0}^{m-1} \sum_{d \in \{0, \ldots, n\} \atop m|(2d-n)b-r} \overline{\chi(r)} \left( \frac{n}{d} \right) i^{2d-n} \exp \left( i\varphi(n-2d) \right),
\]

where \( \tau(\overline{\chi}) \) denotes the Gauss sum associated to the Dirichlet character \( \overline{\chi} \) modulo \( m \).

**Proof.** Let us prove the first identity. The second one follows by taking \( \alpha = \varphi - \pi/2 \) in (31).

We multiply the identity (27) (where \( \chi(j) = \exp(2\pi ij/r) \)) by \( \overline{\chi}(r) \), and take the sum over all \( r \in \{0, \ldots, m-1\} \) to get

\[
\sum_{j=0}^{m-1} \cos^n \left( \frac{2\pi jb}{m} + \alpha \right) \sum_{r=0}^{m-1} \overline{\chi(r)} e^{2\pi ijr/m} = \frac{m}{2^n} \sum_{r=0}^{m-1} \overline{\chi(r)} \sum_{d \in \{0, \ldots, n\} \atop m|(2d-n)b-r} \left( \frac{n}{d} \right) \exp \left( i\alpha(n-2d) \right).
\]

Then, a direct application of the formula

\[
\sum_{r=0}^{m-1} \overline{\chi(r)} e^{2\pi ijr/m} = \overline{\chi(j)} \tau(\overline{\chi})
\]

completes the proof of the first identity. \( \square \)

When \( \overline{\chi} \) is an odd, real multiplicative character modulo \( m \), then (see Chapter 1 of [BEW98])

\[
\sum_{r=0}^{m-1} \overline{\chi(r)} e^{2\pi ijr/m} = i\sqrt{m} \overline{\chi(j)}.
\]

Hence, for odd \( n \), equation (32) becomes the identity

\[
\sum_{j=0}^{m-1} \overline{\chi(j)} \sin^n \left( \frac{2\pi jb}{m} + \varphi \right) = \frac{\sqrt{m}}{2^n} \sum_{r=0}^{m-1} \overline{\chi(r)} \sum_{d \in \{0, \ldots, n\} \atop m|(2d-n)b-r} \left( \frac{n}{d} \right) \exp \left( i\alpha(n-2d) \right) \left(-1\right)^{d-(n-1)/2}.
\]

Next, we observe that for an odd \( n \) and when \( d \) runs through the integers \( 0, \ldots, n \), the difference \( n - 2d \) runs through the odd integers from \( -n \) to \( n \). Hence, by applying the formula \( \exp(ix) + \exp(-ix) = 2 \cos x \) one easily deduces (under the assumption that \( nb < m \)) that the right-hand side of the above formula may be expressed as

\[
\frac{\sqrt{m}}{2^{n-1}} \sum_{d,r \geq 0 \atop r+2db=nb} (-1)^{d-(n-1)/2} \left( \frac{n}{d} \right) \cos \left( \varphi(n-2d) \right) \overline{\chi(r)},
\]

which proves Theorem 1.2. of [BH10].

**Remark 15.** By taking \( n = 1 \) and \( b = 1 \) and even/odd real character \( \overline{\chi} \) in Corollary 14 one may deduce the Gauss theorems for the evaluation of the Gauss sum \( \tau(\overline{\chi}) \).

Namely, when \( \overline{\chi} \) is an even primitive character modulo \( m \), Equation (31) yields the identity

\[
\sum_{j=0}^{m-1} \overline{\chi(j)} \cos \left( \frac{2\pi j}{m} + \alpha \right) = \frac{m}{\tau(\overline{\chi})} \cos \alpha.
\]
When $\alpha = 0$ the left-hand side of the above equation equals $\tau(\tilde{\chi})$, because we assume $\tilde{\chi}$ is even. Hence, one immediately gets $\tau^2(\tilde{\chi}) = m$. If $\tilde{\chi}$ is an odd, primitive character modulo $m$, Equation (31) becomes

$$
\sum_{j=0}^{m-1} \tilde{\chi}(j) \cos \left( \frac{2\pi j}{m} + \alpha \right) = -\frac{im}{\tau(\tilde{\chi})} \sin \alpha.
$$

When $\alpha = -\pi/2$, using the fact that the character $\tilde{\chi}$ is odd, the left-hand side of the above display reduces to $\frac{1}{i}\tau(\tilde{\chi})$. Therefore, the above identity yields the Gauss formula $\tau^2(\tilde{\chi}) = -m$ for the odd character.

### 5.3 Sums which include powers of linear combinations

Let us now consider an example of a trigonometric identity for a power of a linear combination of two cosines, which stems from an application of Theorem 7 when $d = 2$.

**Example 16.** Let us take $d = 2$, $m = (m_1, m_2)$ for positive integers $m_1, m_2$ and let $S = \{\pm(1, 0), \pm(0, 1)\}$ with $\pi_S(s) = 1/4$, for all four elements $s \in S$. Let $\beta = (\beta_1, \beta_2) \in \mathbb{R}^2$. We apply results of Corollary 8 to deduce a trigonometric identity for the sum of powers of a linear combination of cosine functions with shifted arguments.

The eigenvalues $\lambda_{n}^{m, \beta}$ for any $n = (a, b)$, $a \in \{0, \ldots, m_1 - 1\}$ and $b \in \{0, \ldots, m_2 - 1\}$ are given by:

$$
\lambda_{(a,b)}^{(m_1,m_2),(\beta_1,\beta_2)} = \frac{1}{2} \left( \cos \left( 2\pi \frac{a + \beta_1}{m_1} \right) + \cos \left( 2\pi \frac{b + \beta_2}{m_2} \right) \right).
$$

On the other hand, in view of Proposition 2 the right-hand side of (25), for a fixed positive integer $n$, can be simplified as

$$
4^{-n} m_1 m_2 \sum_{v=0}^{n/2} \sum_{k_1=\left[\frac{2v}{m_1}\right]}^{\left\lfloor \frac{n}{2} - \frac{2v}{m_1}\right\rfloor} \sum_{k_2=\left[\frac{n-2v}{m_2}\right]}^{\left\lfloor \frac{n}{2} - \frac{2v}{m_2}\right\rfloor} e^{2\pi i (\beta_1 k_1 + \beta_2 k_2)} \left( \begin{array}{c} n \\ 2v \end{array} \right) \left( \begin{array}{c} 2v \\ v + k_1 m_1 \end{array} \right) \left( \begin{array}{c} n - 2v \\ (n - 2v + m_2 k_2) \end{array} \right),
$$

which yields the following trigonometric identity:

$$
\sum_{a=0}^{m_1-1} \sum_{b=0}^{m_2-1} \left( \cos \left( 2\pi \frac{a + \beta_1}{m_1} \right) + \cos \left( 2\pi \frac{b + \beta_2}{m_2} \right) \right)^n = \frac{m_1 m_2}{2^n} \sum_{v=0}^{n/2} \sum_{k_1=\left[\frac{2v}{m_1}\right]}^{\left\lfloor \frac{n}{2} - \frac{2v}{m_1}\right\rfloor} \sum_{k_2=\left[\frac{n-2v}{m_2}\right]}^{\left\lfloor \frac{n}{2} - \frac{2v}{m_2}\right\rfloor} e^{2\pi i (\beta_1 k_1 + \beta_2 k_2)} \left( \begin{array}{c} n \\ 2v \end{array} \right) \left( \begin{array}{c} 2v \\ v + k_1 m_1 \end{array} \right) \left( \begin{array}{c} n - 2v \\ (n - 2v + m_2 k_2) \end{array} \right).
$$

### 5.4 Further sums which include powers of linear combinations

Let us take $d = 2$ and $S = \{(\pm 1, 0), (0, 0), (0, \pm 2)\}$ with probabilities 1/4 for the elements $(\pm 1, 0)$ and $(0, 0)$ and probabilities 1/8 for the elements $(0, \pm 2)$. In this instance, Corollary 8 will yield an evaluation of the series

$$
\sum_{k_1=0}^{m_1-1} \sum_{k_2=0}^{m_2-1} \left( \cos \left( \frac{2\pi k_1 a_1}{m_1} + \beta_1 \right) + \cos^2 \left( \frac{2\pi k_2 a_2}{m_2} + \beta_2 \right) \right)^n.
$$
In a similar vein, one can consider general $d$ and devise the set $S$ with corresponding probabilities so that our Corollary yields an evaluation of the series

$$
\sum_{k_1=0}^{m_1-1} \cdots \sum_{k_d=0}^{m_d-1} \left( \cos^{h_1} \left( \frac{2\pi k_1 a_1}{m_1} + \beta_1 \right) + \cdots + \cos^{h_d} \left( \frac{2\pi k_d a_d}{m_d} + \beta_d \right) \right)^n
$$

for any sequence $\{h_j\}$ of positive integers.

By choosing probabilities on $S$ in a suitable way, one can also evaluate weighted multiple trigonometric sums with nonnegative weights. (For an occurrence of such a sum in mathematical modelling, see e.g. [F-DG-D14].) We will leave the complete development of these formulas to the interested reader.

Finally, let us note that some multidimensional trigonometric sums can, of course, be written in terms of products of one-dimensional sums and binomial coefficients. However, our approach provides a physical interpretation of this multidimensional sum as a trace of a certain random walk on a graph after $n$ steps; see [OP20] for a related result.

6 The heat kernel for a product of groups

In this section we want to study the heat kernel of the Cayley graph of a product of finite abelian groups $G_1 \times G_2$ with symmetric subset $S_1 \times S_2$, where $S_i \subset G_i$ are symmetric subsets of $G_i$, and $\pi_i$ are probability distributions in $S_i$, $i = 1, 2$. We notice that in this case $S_1 \times S_2$ must also be symmetric and $\pi_1 \times \pi_2$ is a probability distribution in $S_1 \times S_2$.

6.1 The uniform probability distribution

We first deal with the case in which each $\pi_i$, $i = 1, 2$ is the uniform distribution. We will omit it from the notation. We will denote the Cayley graph $\mathcal{C}(G_1 \times G_2, S_1 \times S_2)$ simply as $X_{G_1 \times G_2}$. Similarly, we will write $X_{G_i}$ for $\mathcal{C}(G_i, S_i)$. We start by relating the Hilbert space of $L^2$ function on $G_1 \times G_2$ with those of $L^2(G_i, \mathbb{C}), i = 1, 2$.

**Proposition 17.** There is a canonical isomorphism of Hilbert spaces between $L^2(G_1 \times G_2, \mathbb{C})$ and $L^2(G_1, \mathbb{C}) \otimes L^2(G_2, \mathbb{C})$ given by

$$
\phi : L^2(G_1, \mathbb{C}) \otimes \mathbb{C} L^2(G_2, \mathbb{C}) \rightarrow L^2(G_1 \times G_2, \mathbb{C})
$$

$$
f \otimes g \mapsto f \times g.
$$

**Proof.** The proof follows by applying standard arguments, so we omit it here. $\Box$

Let $A_{G_1}$ and $A_{G_1 \times G_2}$ be the adjacency operators of $X_{G_1}$ and $X_{G_1 \times G_2}$, respectively. As one naturally would expect one has $A_{G_1} \otimes \mathbb{C} A_{G_2} \simeq A_{G_1 \times G_2}$ canonically. In fact, one has the following general result.

**Proposition 18.** There is a canonical isomorphism between $A_{G_1 \times G_2}$ and $A_{G_1} \otimes A_{G_2}$.

**Proof.** Consider the diagram

$$
\begin{array}{ccc}
L^2(G_1, \mathbb{C}) \otimes \mathbb{C} L^2(G_2, \mathbb{C}) & \xrightarrow{A_{G_1} \otimes A_{G_2}} & L^2(G_1, \mathbb{C}) \otimes \mathbb{C} L^2(G_2, \mathbb{C}) \\
\downarrow \phi & & \downarrow \phi \\
L^2(G_1 \times G_2, \mathbb{C}) & \xrightarrow{A_{G_1} \otimes \mathbb{C} A_{G_2}} & L^2(G_1 \times G_2, \mathbb{C})
\end{array}
$$
where the vertical arrows are the isomorphisms defined in Proposition 17. We claim this diagram is commutative.

First, the composite map \( \phi \circ A_G \otimes A_G \) takes any basis vector \( \delta_{\mathbf{x}_1} \otimes \delta_{\mathbf{x}_2} \) into the function given by

\[
A_G(\delta_{\mathbf{x}_1}) \times A_G(\delta_{\mathbf{x}_2})(a, b) = \sum_{s_1 \in S_1} \delta_{\mathbf{x}_1}(a + s_1) \sum_{s_2 \in S_2} \delta_{\mathbf{x}_2}(b + s_2)
\]

(33)

which is equal to 1, if and only if \( a + s_1 = x_1 \) and \( b + s_2 = x_2 \).

On the other hand, \( A_G \times A_G \circ \phi \) takes the basis vector \( \delta_{\mathbf{x}_1} \otimes \delta_{\mathbf{x}_2} \) into the function given by

\[
A_G(\delta_{\mathbf{x}_1, \mathbf{x}_2})(a, b) = \sum_{(s_1, s_2) \in S_1 \times S_2} \delta_{(x_1, x_2)}(a + s_1, b + s_2).
\]

(34)

From this, it is obvious that (33) and (34) are the same.

This proves the commutativity of the diagram, meaning that

\[
A_G \otimes A_G = \phi^{-1} \circ A_G \times A_G \circ \phi,
\]

(35)

as asserted. \( \square \)

The previous proposition allows us to relate the heat kernel of \( X_{G_i \times G_2} \) with the heat kernels of \( G_i, i = 1, 2 \). These operators correspond to random walks on \( G_1 \times G_2 \) and on each \( G_i, i = 1, 2 \). We know that \( K_{X_{G_i}}(x_i, y_i; n) = A^n_{G_i}(\delta_{x_i})(y_i) \). From (33) we see that

\[
(A_G \otimes A_G)^n = A^n_{G_1} \otimes A^n_{G_2} = \phi^{-1} \circ A^n_{G_1 \times G_2} \circ \phi.
\]

Thus,

\[
A^n_{G_1 \times G_2} = \phi \circ A^n_{G_1} \otimes A^n_{G_2} \circ \phi^{-1}.
\]

(36)

On the other hand,

\[
K_{X_{G_1 \times G_2}}((x_1, x_2), (y_1, y_2); n) = A^n_{G_1 \times G_2}(\delta_{x_1} \times \delta_{x_2})(y_1, y_2).
\]

By (36) one has

\[
(A^n_{G_1 \times G_2}(\delta_{x_1} \times \delta_{x_2}))(y_1, y_2) = A^n_{G_1}(\delta_{x_1})(y_1) \times A^n_{G_2}(\delta_{x_2})(y_2).
\]

Thus,

\[
K_{X_{G_1 \times G_2}}((x_1, x_2), (y_1, y_2); n) = K_{X_{G_1}}(x_1, y_1; n) K_{X_{G_2}}(x_2, y_2; n).
\]

Using a straightforward induction argument, one then obtains the following corollary.

**Corollary 19.** Let \( G_i, i = 1, \ldots, d \), be finite abelian groups, and let \( S_i \subset G_i \) be symmetric subsets. Then the heat kernel of the Cayley graph \( \mathcal{C}(G_1 \times \cdots \times G_d, S_1 \times \cdots \times S_d) \) is the product of the heat kernels of \( \mathcal{C}(G_i, S_i), i = 1, \ldots, d \).

### 6.2 Arbitrary probability distribution

The fact that the heat kernel of a product is the product of the heat kernels follows from the fact that the adjacency operator of the Cayley graph \( X \) is canonically identifiable with the adjacency operators of each \( X_{G_i} \). This identification readily extends to the general case of a product of weighted graphs \( \mathcal{C}(G_i, S_i, \pi_i), i = 1, \ldots, d \), where each heat kernel is taken with a \( \chi_{\beta_i} \)-twist.

For this, we first notice that if \( G = G_1 \times \cdots \times G_d \), \( S = S_1 \times \cdots \times S_d \) and \( \pi = \pi_1 \cdots \pi_d \) then \( X_G = \mathcal{C}(G, S, \pi) \) is a weighted graph since \( \pi = \pi_1 \cdots \pi_d \) is a probability distribution in \( S \).
Proposition 20. Let $X_{G_i} = C(G_i, S_i, \pi_i)$, $i = 1, \ldots, d$ be the Cayley graphs of the abelian groups $G_i$ with symmetric subset $S_i \subset G_i$ and probability distributions $\pi_i$. Let $K_{X_{G_i}}(x_i, y_i; n)$ be the discrete time heat kernel of each $X(G_i, S_i, \pi_i)$, twisted by $\chi_{\beta_i}$, $i = 1, \ldots, d$. Then, the heat kernel on $X_{G_i}$, twisted by $\beta = (\beta_1, \ldots, \beta_d)$ is given for $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ by

$$K_{X_{G_i}}(x, y; n) = \prod_{i=1}^{d} K_{X_{G_i}}(x_i, y_i; n).$$

6.3 Multidimensional examples

We start with the proof of formula (7), which is deduced by the following application of Proposition 20.

Example 21. Let $m$ denote the $d$-tuple $(m_1, \ldots, m_d)$ and let $G_i = \mathbb{Z}_{m_i}$ and $G = G_m$. Let $S_i = \{\pm 1\}$ so that $S = S_1 \times \cdots \times S_d$ is the set

$S = \{(\epsilon_1, \ldots, \epsilon_d) : \epsilon_i = \pm 1, i = 1, \ldots, d\}.$

Let $\pi_S(s) = 2^{-d}$, for all $s \in S$. Then, for any $\beta \in \mathbb{R}^d$, and any $l = (\ell_1, \ldots, \ell_d) \in G_m$ one has

$$\lambda_{l}^{m, \beta} = \prod_{k=1}^{d} \cos \left(2\pi(l_k + \beta_k)/m_k\right).$$

By Proposition 20 and using Example 4 we see that

$$K_{X_{G_i}}(km, 0; n) = 2^{-dn} \left(\frac{n}{(n + k_1m_1)/2}\right) \cdots \left(\frac{n}{(n + k_dm_d)/2}\right),$$

under the assumption that $n + k_jm_j$, $j = 1, \ldots, d$ are even natural numbers and $|k_jm_j| \leq n$, $j = 1, \ldots, d$. In all other cases $K_{X_{G_i}}(km, 0; n) = 0$.

Therefore, application of (25) yields the formula (7).

The following are two explicit examples of computation of triple sums, using Maple.

Example 22. For any $n, m_1, m_2, m_3 \in \mathbb{Z}_{>0}$, let

$$S(n, m_1, m_2, m_3) = \sum_{a_1=0}^{m_1-1} \sum_{a_2=0}^{m_2-1} \sum_{a_3=0}^{m_3-1} \cos^{n} \left(\frac{2\pi a_1}{m_1}\right) \cos^{n} \left(\frac{2\pi a_2}{m_2}\right) \cos^{n} \left(\frac{2\pi a_3}{m_3}\right).$$

A computation with Maple shows that $S(100, 40, 60, 80) = \frac{A}{B}$, where

$$A = 3^6 \times 5^3 \times 11^3 \times 13 \times 19^2 \times 29 \times 31^3 \times 83 \times 89 \times 97^3 \times 173 \times 2699 \times 110714391 \times 13231313 \times 54570781 \times 60580339 \times 2008565421593524568089$$

and $B = 2^{283}$.

For $n = 1000$ and $m_1 = 4, m_2 = 6, m_3 = 8$ one gets $S(1000, 4, 6, 8) = \frac{A}{B}$, where

$$A = 87686552760850968990668969900702144983100397008270720601894950619774096 \times 2640830509142702216462423154132876016988593793498236372356748993854275 \times 9729506532575008420457937016441367000686948672606926982141225340930469 \times 061982186126624381189674140721945180423650252562631800324801976874415916 \times 70197162587469026757597845127615839457156412853584822107912966688994077 \times 40190348617924002782019024043632897153590480491978931583944336917869593 \times 3770866195399966721.$$
and $B = 2^{1495}$. We were unable to compute the prime factorization of $A$.

Let us note that, though it is well known that the sum $S(n,m_1,m_2,m_3)$ is a rational number, numerical computation of the sum becomes quite difficult for large values of $n$ because $n$th powers of cosines are very small numbers so the precision in the numerical computation must be extremely high.

In our last example we show how to prove (5).

**Example 23.** Let $X_G$ denote the weighted Cayley graph $G = \mathbb{C}((\mathbb{Z}/m\mathbb{Z})^2, S, \pi_S)$, where $m = (2m_1,m_2)$, $S = \{\pm a, \pm b\}$ is a set of four elements and $\pi_S(s) = \frac{1}{4}$ for all $s \in S$. Then, the heat kernel on $X_G$ twisted by a character $\chi_\beta$, where $\beta = (\alpha_1 m_1, \frac{m_2(\alpha_2 - \pi/2)}{2\pi})$, equals a product of the heat kernel $K_{X_{G_1}}$ on $G_1 = \mathbb{C}(\mathbb{Z}/2m_1\mathbb{Z}, S_1, \pi_{S_1})$, where $S_1 = \{-a, a\}$, $\pi_{S_1}(s) = \frac{1}{2}$ for all $s \in S_1$ twisted by $\chi_{\alpha_1 m_1}$, and the heat kernel $K_{X_{G_2}}$ on $G_2 = \mathbb{C}(\mathbb{Z}/m_2\mathbb{Z}, S_2, \pi_{S_2})$, where $S_2 = \{-b, b\}$, $\pi_{S_2}(s) = \frac{1}{2}$ for all $s \in S_2$ twisted by $\chi_{\frac{m_2(\alpha_2 - \pi/2)}{2\pi}}$.

Proposition 20 yields that

$$K_G((m_1,0),(0,0);k) = K_{G_1}(m_1,0;k)K_{G_2}(0,0;k).$$

Therefore, when combining formulas (24), (22), and (23) in this setting, we deduce that

$$\sum_{j=0}^{2m_1-1} \sum_{\ell=0}^{m_2-1} (-1)^j \cos \left( \frac{\pi j a}{m_1} + \alpha_1 \right) \sin \left( \frac{2\pi \ell b}{m_2} + \alpha_2 \right) = 2m_1 m_2 \exp(-\pi i \alpha_1/m_1)K_{G_1}(m_1,0;k)K_{G_2}(0,0;k).$$

Next, we apply the results of Example 23 to compute $K_{G_1}(m_1,0;k)$ and $K_{G_2}(0,0;k)$ twisted by $\chi_{\alpha_1 m_1}$ and $\chi_{\frac{m_2(\alpha_2 - \pi/2)}{2\pi}}$, respectively. In doing so, we get that

$$K_{G_1}(m_1,0;k) = 2^{-k} e^{i \alpha_1 m_1/a} \sum_{d_1 \in \{0,\ldots,k\}} \binom{k}{d_1} \exp(i \alpha_1(k - 2d_1));$$

and

$$K_{G_2}(0,0;k) = 2^{-k} \sum_{d_2 \in \{0,\ldots,k\}} \binom{k}{d_2} \exp(i \alpha_2(k - 2d_2)) i^{2d_2 - k}.$$

With all this, we have proved the explicit evaluation (5).

### 7 Concluding remarks

From Proposition 20 and the discussion in Example 21 and Example 23 one sees that our methodology applies to yield an explicit evaluation of the sum of terms of the type

$$\left( \prod_{j=1}^d \cos \left( \frac{2\pi k_j a_j}{m_j} + \beta_j \right) \right)^k.$$  

(37)

It is elementary that

$$\left( \frac{\partial^2}{\partial \beta_j^2} + k^2 \right) \prod_{j=1}^d \cos \left( \frac{2\pi k_j a_j}{m_j} + \beta_j \right) = k(k-1) \cos^{k-2} \left( \frac{2\pi k_1 a_1}{m_1} + \beta_1 \right) \prod_{j=2}^d \cos^{k} \left( \frac{2\pi k_j a_j}{m_j} + \beta_j \right).$$
Therefore, from our evaluation of \((37)\), we can repeatedly apply differential operators of the form \(\partial_{\beta_j}^2 + c\), for appropriate \(c\), and derive an evaluation for sums of terms of the type
\[
\prod_{j=1}^d \cos^{2h_j} \left( \frac{2\pi k_j a_j}{m_j} + \beta_j \right)
\]
for any set of integers \(\{h_j\}\). Furthermore, one can employ the Pythagorean identity in order to obtain sums of terms of the type
\[
\prod_{j=1}^d \cos^{2h_{1,j}} \left( \frac{2\pi k_j a_j}{m_j} + \beta_j \right) \sin^{2h_{2,j}} \left( \frac{2\pi k_j a_j}{m_j} + \beta_j \right)
\]
for any set \(\{h_{1,j}, h_{2,j}\}\) of non-negative integers. It is not clear how tractable such expressions may be; nonetheless, the above discussion does provide a guide to a somewhat straightforward algorithm.

Finally, let us describe a means by which one can extend the above assertion to a general monomial whose exponents are not necessarily even. To do so, one needs to establish formulas for the sums of terms of the form
\[
\cos^{h_1} \left( \frac{2\pi k a}{m} + \beta \right) \sin^{h_2} \left( \frac{2\pi k a}{m} + \beta \right)
\]
for general non-negative integers \(h_1\) and \(h_2\). The cases \(h_1 = 0\) or \(h_2 = 0\) follow from our main results, as illustrated by Corollary \[10\] with \(\beta\) replaced by \(m\beta/2\pi\) in the case \(h_2 = 0\) and \(\beta\) replaced by \(m(\beta - \pi/2)/2\pi\) in the case \(h_1 = 0\). Through repeated use of the Pythagorean identity, it suffices to consider the case \(h_2 = 1\). In that instance, such evaluations follow from the case \(h_2 = 0\) and that
\[
\frac{\partial}{\partial \beta} \cos^h \left( \frac{2\pi k a}{m} + \beta \right) = -h \cos^{h-1} \left( \frac{2\pi k a}{m} + \beta \right) \sin \left( \frac{2\pi k a}{m} + \beta \right).
\]
It remains to be seen if the resulting formulas, though obtainable, are manageable.

In a more direct approach one can take in Proposition \[20\] a set \(S\) equals to a product of \(2d\) sets of the form \(S_j = \{\pm a_j\}\), for \(j = 1, \ldots , d\), and \(S_j = \{\pm b_j\}\), for \(j = d + 1, \ldots , 2d\). In this case one obtains a sum of products of terms of the form
\[
\prod_{j=1}^d \cos \left( \frac{2\pi k_j a_j}{m_j} \right) \prod_{j=d+1}^{2d} \cos \left( \frac{2\pi k_j b_j}{m_j} \right).
\]
Then one can use the Chebyshev’s polynomials of the first type
\[
T_n(x) = \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(-1)^r}{n-r} \binom{n-r}{r} (2x)^{n-2r}
\]
to express factors of the form \(\cos^n(\alpha)\) in terms of \(T_n(\cos(n\alpha))\) and lower powers \(\cos^{n-2r}(\alpha)\). Those factors involving sine functions can be obtained by a character twist with \(\beta_j = 0\), for \(j = 1, \ldots , d\), and equal to \(-m_j/(2\pi)\), for \(j = d + 1, \ldots , 2d\). Then one can proceed in a recursive manner to compute sums of mixed terms as in \(39\). An explicit algorithm to do this will be developed and implemented in the subsequent article \[CHJSV22a\].
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