Soliton Solutions and Nontrivial Scattering in an Integrable Chiral Model in (2+1) Dimensions.

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Abstract. The behaviour of solitons in integrable theories is strongly constrained by the integrability of the theory; i.e. by the existence of an infinite number of conserved quantities which these theories are known to possess. One usually expects the scattering of solitons in such theories to be rather simple, i.e. trivial. By contrast, in this paper we generate new soliton solutions for the planar integrable chiral model whose scattering properties are highly nontrivial; more precisely, in head-on collisions of $N$ indistinguishable solitons the scattering angle (of the emerging structures relative to the incoming ones) is $\pi/N$.

We also generate soliton-antisoliton solutions with elastic scattering; in particular, a head-on collision of a soliton and an antisoliton resulting in $90^0$ scattering.
I. Introduction.

This paper studies certain exact soliton solutions of an integrable system. Before any detailed discussion, and to avoid confusion later on, it is worthwhile clearing up a small point of terminology: the word *solitons* was introduced by mathematicians to describe lumps of energy which were stable to perturbations and did not change either velocity or shape when colliding with each other. However, in recent literature all sorts of localized energy configurations have been called solitons. We shall go along this looser definition. By a soliton we shall mean a lump of energy that moves but we shall not imply stability of the shape or the velocity or a simple behaviour in collision.

An interesting problem is to look at the scattering properties of two or more solitons colliding. In some known systems with nontrivial topology, the collision of two solitons is inelastic (some radiation is emitted) and nontrivial (a head-on collision results in $90^\circ$ scattering); all this has been observed analytically [1, 2] and numerically [3]-[6]. One can construct explicit time-dependent solutions only in very special, so-called integrable models. Usually in these models extended objects interact trivially, in the sense that they pass through each other with no lasting change in velocity or shape (i.e., they behave as genuine solitons). Some examples in (2+1) dimensions are the Kadomtsev-Petviashvili equation [7] and the integrable chiral model [8]. The last system is the subject of this paper and will be described below. Until now, nontrivial scattering of solitons occurs mostly in nonintegrable systems which is far from simple. The question that arises is whether this type of scattering can occur in integrable models too. There are some limited examples of integrable systems where soliton dynamics can be nontrivial. In (1+1) dimensions there many models which possess nontrivial soliton-like solutions (cf. [9]); like the boomeron solutions [10], which are solitons with time dependent velocities. In (2+1) dimensions there are the dromion solutions [11] of the Davey-Stewartson equation, which decay exponentially in both spatial coordinates and interact in a nontrivial manner [12]; and the soliton solutions [13] of the Kadomtsev-Petviashvili equations, whose scattering properties are highly nontrivial.

In the present work we are going to construct families of soliton solutions for the integrable (2+1)-dimensional chiral model and observe the occurrence of different types of behaviour. This happens since the solitons in this system have internal degrees of freedom which
determine their orientation in space; do not affect the initial energy density; and are important in understanding the evolution as a whole. Therefore, they can interact either trivially or nontrivially, depending on the orientation of these internal parameters and on the values of the impact parameter defined as the distance of closest of approach between their centres in the absence of interaction. Namely, if two initial soliton-like structures are sent towards each other at zero impact parameter, then, as most numerical simulations have shown, the outgoing structures emerge at $90^\circ$.

To proceed further let us specify the system. The modified SU(2) chiral model studied by Ward [8] is given by the field equation

$$\partial^\mu (J^{-1} J_\mu) - \frac{1}{2} V_\alpha \varepsilon^{\alpha\mu\nu} [J^{-1} J_\mu, J^{-1} J_\nu] = 0. \quad (1)$$

Here $J$ takes values in SU(2) group and is thought of as a $2 \times 2$ unitary matrix of functions of the spacetime coordinates on $\mathbb{R}^{2+1}$: $x^\mu = (x^0, x^1, x^2) = (t, x, y)$ with $\det J = 1$. Greek letters are spacetime indices, taking values 0, 1, 2, $\partial_\mu$ denotes partial differentiation with respect to $x^\mu$, while $J_\mu \equiv \partial_\mu J$. The quantity $\varepsilon^{\alpha\mu\nu}$ is the alternating tensor of three indices with $\varepsilon^{012} = 1$. Finally, $V_\alpha$ is a unit vector in spacetime. The conformal properties of $V_\alpha$ determine whether the symmetry group is SO(2) or SO(1,1) (depending on whether $V_\alpha$ is timelike or spacelike).

Ward [8] chooses $V_\alpha$ to have the components $V_\alpha = (0, 1, 0)$, the spacelike case, so that (1) is a chiral equation with torsion term and has the same conserved energy-momentum vector as the chiral field equation. In fact, the corresponding energy density is

$$\mathcal{E} = -\frac{1}{2} \text{tr} \left[ (J^{-1} J_t)^2 + (J^{-1} J_x)^2 + (J^{-1} J_y)^2 \right]. \quad (2)$$

Here tr denotes the matrix trace. It should be emphasized that $\mathcal{E}$ is a positive-defined functional of $J$, and hence a conserved energy exists which is the integral of the energy density over the spacelike plane $x^0 = \text{const}$. The boundary conditions are chosen so that the field configuration has finite energy. Consequently, we require that $J$ be everywhere smooth and that

$$J = J_0 + J_1(\theta) r^{-1} + O(r^{-2}), \quad (3)$$

at spatial infinity, with $x + iy = r \, e^{i\theta}$. Here $J_0$ is a constant matrix, and $J_1$ depends only on $\theta$ (no time dependence).
The ensuing system when $V_\alpha$ is $i$ times a timelike vector instead of spacelike has been studied in [14]. Equation (1) admits solitons, localized in two dimensions, with trivial scattering, i.e. each soliton suffers no change in velocity and no phase shift upon scattering [8, 14]. It is the purpose of this paper to construct new soliton solutions for (1), and investigate their scattering behaviour. Such solutions are localized along the direction of motion; they are not however, of constant size: their height, which corresponds to the maximum of the energy density $E$, is time dependent.

The rest of the paper is arranged as follows. In the next section we shall briefly discuss the integrability properties of (1), and write down a family of multisoliton solutions as configurations that are the limiting cases of the ones already obtained using the standard method of Riemann problem with zeros [8]. In section 3 we construct two families of multisoliton solutions with nontrivial scattering; in particular, for the first one we prove that in all head-on collisions the $N$ moving structures undergo $\pi/N$ scattering. In section 4 we construct a mixture of soliton-antisoliton solutions, and in section 5 we discuss their dynamics and scattering properties. We finish the paper with a short section containing our conclusions.

II. Construction of Soliton Solutions.

The integrable nature of equation (1) means that there is a variety of methods for constructing exact solutions. Together with Riemann problem with zeros [8], both twistor techniques [13] and a full inverse scattering formalism [16] have been applied to the model. This section indicates a general method for constructing soliton solutions of the integrable chiral model (1). The technique is a variation of that in [8, 17], following a pioneering idea of Zakharov and his collaborators [18].

The nonlinear equation (1) is integrable in a sense that it may be written as the compatibility condition for the following linear system

\[ L\psi \equiv (\lambda\partial_x - \partial_u)\psi = A\psi, \]
\[ M\psi \equiv (\lambda\partial_v - \partial_x)\psi = B\psi. \] (4)

Here $\lambda \in \mathbb{C}$, $(u, v, x)$ are coordinates on $\mathbb{R}^{2+1}$ with $u = (t + y)/2$, $v = (t - y)/2$, $A$ and $B$ are $2 \times 2$ anti-hermitean trace-free matrices depending only on $(u, v, x)$, and $\psi(\lambda, u, v, x)$ is an unimodular $2 \times 2$ matrix function satisfying the reality condition

\[ \psi(\lambda, u, v, x) \psi(\bar{\lambda}, u, v, x)^\dagger = I, \] (5)
where bar denotes complex conjugate, † denotes the complex conjugate transpose matrix and \( I \) is the \( 2 \times 2 \) identity matrix. The system (3) is overdetermined, and in order for a solution \( \psi \) to exist, \( A \) and \( B \) have to satisfy the integrability conditions, which are

\[
B_x = A_v, \quad A_x - B_u - [A, B] = 0. \tag{6}
\]

If we put \( J(u, v, x) = \psi(\lambda = 0, u, v, x)^{-1} \) where \( \psi \) is a solution of the system (3), we get by comparing (3) and (6) that

\[
A = J^{-1} J_v, \quad B = J^{-1} J_x. \tag{7}
\]

Therefore, the integrability condition for (3) implies that there exists a field \( J \) which satisfies the equation of motion (1); and moreover, the reality condition on \( \psi \) ensures that \( J \) is unitary.

Using the standard method of Riemann problem with zeros, in order to construct multi-soliton solution one may assume that the function \( \psi \) has simple poles in \( \lambda \), or in other words must possess the form

\[
\psi(\lambda) = I + \sum_{k=1}^{n} \frac{M_k}{\lambda - \mu_k}, \tag{8}
\]

where \( M_k \) are \( 2 \times 2 \) matrices independent of the complex parameter \( \lambda \), \( n \) is the number of solitons, and the complex parameter \( \mu_k \) determines the velocity of the \( k \)-th soliton. The components of the matrix \( M_k \) are given in terms of a rational function \( f_k \) of the complex variable \( \omega_k = x + \mu_k u + \mu_k^{-1} v \). (Roughly speaking, \( f_k(\omega_k) \) describes the shape of the \( k \)-th soliton). In fact, the matrix \( M_k \) (cf. [8]) has the form

\[
M_k = -\sum_{l=1}^{n} (\Gamma^{-1})^{kl} \bar{m}_a^l m_b^k, \tag{9}
\]

with \( \Gamma^{-1} \) the inverse of

\[
\Gamma^{kl} = \sum_{a=1}^{2} (\bar{\mu}_k - \mu_l)^{-1} \bar{m}_a^k m_a^l. \tag{10}
\]

Here \( m_a^k \) are holomorphic functions of \( \omega_k \), given by \( m_a^k = (m_1^k, m_2^k) = (1, f_k) \). These solitons pass each other without any change of direction or phase shift. Infinite energy extended wave solutions [19] may be constructed by taking \( f_k \) to be an exponential function of \( \omega_k \). Such extended wave solutions suffer a phase shift upon scattering, although again there is no change in velocity.
All this assumes that the parameters $\mu_k$ are distinct, and also $\bar{\mu}_k \neq \mu_l$ for all $k, l$. In this paper examples are given of two generalizations of these constructions: one involving higher-order poles in $\mu_k$, and the other where $\bar{\mu}_k \neq \mu_l$.

Let us look at an example in which the function $\psi$ has a double pole in $\lambda$, and no other poles. So we take $\psi$ to have the form

$$\psi = I + \sum_{k=1}^{2} \frac{R_k}{(\lambda - \mu)^k},$$

where $R_k$ are $2 \times 2$ matrices independent of $\lambda$. [This hypothesis can be generalized by taking the function $\psi$ to have a pole of order $n$ in $\lambda$.]

It has been proved [17] that $\psi$ given by (11) satisfies the reality condition (5) if and only if it factorizes as

$$\psi(\lambda) = \left( I - \frac{(\bar{\mu} - \mu)}{(\lambda - \mu)} q_1^\dagger \otimes q_1 \right) \left( I - \frac{(\bar{\mu} - \mu)}{(\lambda - \mu)} q_2^\dagger \otimes q_2 \right),$$

where $q_k$ are two-dimensional row vectors and $\|q_k\|^2 = q_k \cdot q_k^\dagger$.

The $q_k$ have to satisfy a condition, which amounts to saying the matrices $A = (L\psi)\psi^{-1}$ and $B = (M\psi)\psi^{-1}$ are independent of $\lambda$. One way of obtaining $q_k$ with this property is as a limit of the simple-pole case (8) with $n = 2$. The idea is to take a limit $\mu_k \to \mu$. In order to end up with a smooth solution $\psi$ for all $(u, v, x)$, it is necessary that $f_2(\omega) - f_1(\omega) \to 0$ in this limit.

In our case, with $n = 2$, we put $\mu_1 = \mu + \varepsilon$, $\mu_2 = \mu - \varepsilon$ and write $f_1(\omega_1) = f(\omega_1)$, $f_2(\omega_2) = f(\omega_2)$, with $f$ being a rational function of one variable. In the limit $\varepsilon \to 0$, $\psi$ has the form (12), with

$$q_1 = (1 + |f|^2)(1, f) + \varphi (\bar{\mu} - \mu)(\bar{f}, -1),$$

$$q_2 = (1, f).$$

Here $f$ is a rational function of $\omega = x + \mu u + \mu^{-1} v$, $\varphi = (u - \mu^{-2} v) f'(\omega)$, while $f'(\omega)$ denotes the derivative of $f(\omega)$ with respect to its argument. As a result, we have a solution $J = \psi(\lambda = 0)^{-1}$ depending on the complex parameter $\mu$ and on the arbitrary function $f$.

In fact, it has the form of the following product

$$J = \left( I + \frac{(\bar{\mu} - \mu)}{\mu} q_2^\dagger \otimes q_2 \right) \left( I + \frac{(\bar{\mu} - \mu)}{\mu} q_1^\dagger \otimes q_1 \right),$$

where $R_k$ are $2 \times 2$ matrices independent of $\lambda$. [This hypothesis can be generalized by taking the function $\psi$ to have a pole of order $n$ in $\lambda$.]
with $q_k$ given by (13). Notice that $J$ takes values in $\text{SU}(2)$; is smooth everywhere on $\mathbb{R}^{2+1}$ (mainly because, the two vectors $q_1$ and $q_2$ are nowhere zero); it satisfies the boundary condition (3); and the equation of motion (1).

To start with, and in order to illustrate the above family of soliton solutions, let us examine two simple cases, by giving specific values to the parameters $\mu$ and $f(\omega)$. [The complex parameter $\mu$ determines the velocity of the “centre-of-mass” of the system.]

- Let us take $\mu = i$ (which corresponds to the “centre-of-mass” of the system being stationary) and $f(\omega) = \omega$, thus $\omega = z$ and $\varphi = t$, where $z = x + iy$; $r^2 = z\bar{z}$. Therefore the row vectors (13), become

$$
q_1 = (1 + r^2)(1, z) - 2i t(z, -1), \\
q_2 = (1, z).
$$

In this time-dependent solution, for $t$ negative, a ring structure with reducing radius is obtained, which deforms to a single peak at $t = 0$ and thereafter expands again to a ring. Figure 1 presents few pictures of the corresponding energy density at some representative values of time. Ring structures occur in the soliton scattering of many nonintegrable planar systems [3, 5] and are an approximation of two solitons.

This picture can be confirmed by looking at the energy density of the solution, which is

$$
E = 16 \frac{r^4 + 2r^2 + 4t^2(2r^2 + 1) + 1}{[r^4 + 2r^2 + 4t^2 + 1]^2}.
$$

Notice that the energy density is time-reversible and rotationally symmetric (see below). For large (positive) $t$, the height of the ring (maximum of $E$) is proportional to $1/t$, while its radius is proportional to $\sqrt{t}$.

- Accordingly, let us take $\mu = i$ and $f(\omega) = \omega^2$. Thus, the row vectors (13) are

$$
q_1 = (1 + r^2)(1, z^2) - 4it(z^2, -1), \\
q_2 = (1, z^2).
$$

Here, for negative $t$, a single peak occurs with an additional ring, which changes to a ring structure at $t = 0$ and reverts back to the original form, for positive $t$ (see Figure 2). However, these rings are not radiation since they travel with speed less than that of light. In fact, for large (positive) $t$, their velocity is approximately proportional to $t^{-2/3}$. [Note that we have set the velocity of the light, $c$, equal to the unity, so that in all our calculations we can use dimensionless quantities.]
This leads to an energy density, which is

\[ \mathcal{E} = 64 \frac{r^{10} + 18t^2r^8 + 2r^6 + 4t^2r^4 + r^2 + 2t^2}{[r^8 + 2r^4 + 16t^2r^2 + 1]^2}. \] (18)

Again, \( \mathcal{E} \) has the same symmetries as in (16). For large (positive) \( t \), the height of the soliton peak is proportional to \( t^2 \) and its radius is proportional to \( 1/t \); while the soliton ring spread out, becoming broader and broader, with height proportional to \( t^{-2/3} \) and radius proportional to \( t^{1/3} \).

Finally, a general concluding remark should be made. Although (11) is not rotationally symmetric in the \( xy \)-plane; when \( f(z) = z^p \) the field \( J \) (13,14) is invariant under the transformation \( z \to e^{i\phi}z \), since

\[ J \to J' = \begin{pmatrix} e^{i\phi p/2} & 0 \\ 0 & e^{-i\phi p/2} \end{pmatrix} J \begin{pmatrix} e^{-i\phi p/2} & 0 \\ 0 & e^{i\phi p/2} \end{pmatrix}. \] (19)

This transformation does not affect the equation of motion (11) due to the chiral symmetry \( J \to \kappa J \tau \) where \( \kappa \) and \( \tau \) are constant SU(2) matrices. The main features of this time-dependent solution may be inferred as follow. If \( r \) is large, the field \( J \) is close to its asymptotic value \( J_0 \), as long as \( 2t f'/|f|^2 \to 0 \). But as \( 2t |f'|/|f|^2 \approx 1 \), \( J \) departs from its asymptotic value \( J_0 \) and a ring structure emerge with radius proportional to \( (2tp)^{1/(p+1)} \).

III. Soliton-Soliton Scattering.

We now move on to the more interesting question of scattering processes. In fact, we will use the method of section 2 to construct solutions of (11) representing scattering solitons. We will see that, in all head-on collisions of \( N \) moving solitons the scattering angle is \( \pi/N \). Moreover, when the \( N \) solitons are very close together, and in particular, when they are on top of each other the \( N \) lumps which represent them merge together to form a ring-like structure. Then, instead of moving towards the centre, they emerge from the ring in a direction that bisects the angle formed by the incoming ones. As we have already mentioned this nontrivial scattering is not usual in an integrable theory, but is exceptional.

The scattering solutions arise if we take a solution of the simple-pole case (3) with \( n = 2 \), put \( \mu_1 = \mu + \varepsilon, \mu_2 = \mu - \varepsilon \) and take the limit \( \varepsilon \to 0 \). The constraint \( f_2(\omega_2) - f_1(\omega_1) \to 0 \) as \( \varepsilon \to 0 \) has to be imposed, in order for the resulting solution \( \psi \) to be smooth for all
So let us write \( f_1(\omega_1) = f(\omega_1) + \varepsilon h(\omega_1) \), \( f_2(\omega_2) = f(\omega_2) - \varepsilon h(\omega_2) \), where \( f \) and \( h \) are both rational functions of one variable (the examples of the previous section had \( h = 0 \)). Once again \( J \) is given by (14), with the two-vectors \( q_k \) given by

\[
q_1 = (1 + |f|^2)(1, f) + \vartheta (\bar{\mu} - \mu)(\bar{f}, -1), \\
q_2 = (1, f),
\]

where \( \vartheta = \varphi + h(\omega) \). So this solution belongs to a large family, since one may take \( f \) and \( h \) to be any rational meromorphic functions of \( \omega \). Note that \( J \) is smooth on \( \mathbb{R}^{2+1} \) and satisfies its boundary condition, irrespective of the choice of \( f \) and \( h \).

It may seem strange that one can take the limit of a family of soliton solutions with trivial scattering, and obtain a new one with nontrivial scattering. Thus, it is interesting to study how the solitons are affected by varying \( \varepsilon \). To do so, let us take a solution of the simple-pole case (8) with \( n = 2 \), put \( \mu_1 = i + \varepsilon, \mu_2 = i - \varepsilon \), while taking \( f_k = \omega_k \); and study how the configuration of the two initial well separated solitons changes as \( \varepsilon \to 0 \) at a fixed time \( t = -15 \). Figure 3 shows that as \( \varepsilon \to 0 \) the solitons disperse, shift and interact with each other. In other words, their internal degrees of freedom as well as the impact parameter change in this limit, making the process highly nontrivial.

As an example, let us present two typical cases.

- Let us take \( \mu = i, f(\omega) = \omega \) and \( h(\omega) = \omega^3 \); thus \( \vartheta = t + z^3 \). For \( r \) large, \( J \) is equal to its asymptotic value \( J_0 \), as long as \( \vartheta/z^3 = 1 + t/z^3 \approx 1 \), but as \( z \) approaches any of the three cube roots of \(-t\), then \( \vartheta \to 0 \), while \( J \) departs from \( J_0 \), and three localized solitons emerge. For \( t \) negative, the three solitons are approximately at the points: \((-t)^{1/3}, 0\), \((-(-t))^{1/3}, \pm \sqrt{3}(-t)^{1/3}\); while for \( t \) positive, the solitons are at \((-t)^{1/3}, 0\), \((t^{1/3}, \pm \sqrt{3} t^{1/3})\).

More information can be deduced from the energy density, which is

\[
E = 16 \left[ 2r^8 + 16r^6 + 19r^4 + 2r^2(1 + 8xy^2t) + 4t^2(1 + 2r^2) + 1 + 8xy^4t - 8x^5t - 16tx(x^2 - y^2))]/[4r^6 + r^4 + 2r^2 + 4t^2 + 1 + 8tx(x^2 - 3y^2)]^2.
\]

The density \( E \) is symmetric under the interchange \( t \mapsto -t, x \mapsto -x \) and \( y \mapsto -y \). For small (negative) \( t \), the solitons form an intermediate state having the shape of a ring with three maxima on the direction of the incoming solitons which deforms to a circularly-symmetric ring at \( t = 0 \) and then energy seems to flow around, until three other maxima are formed in the transverse direction, for small (positive) \( t \).
Figure 4 shows clearly the intermediate states with three maxima. The three new maxima then give rise to three new solitons emerging at 60° to the original direction of motion. During the intermediate phase solitons lose their identity.

Finally something has to be said about their size. For large (positive) \(t\), their height is proportional to \(t^{-4/3}\), their radius is proportional to \(t^{1/3}\), while their speed is proportional to \(t^{-2/3}\); therefore, they spread out and slow down.

Accordingly, let us take \(\mu = i\) while choose \(f(\omega) = \omega^2\) and \(h(\omega) = \omega^3\). Here \(J\) departs from its asymptotic value \(J_0\), when \(z\) approaches the values \(\pm \sqrt{-2t}\) or zero (since \(\vartheta = z(2t + z^2) \to 0\); and (again) three localized solitons emerge. In this case though, if \(t\) is negative, all three of them are on the \(x\)-axis at \(x \approx \pm \sqrt{-2t}\) and at the origin; while if \(t\) is positive, they are on the \(y\)-axis at \(y \approx \pm \sqrt{2t}\) and at the origin. So the picture consists of three solitons: a static one at the origin, with the other two accelerating towards the origin, scattering at right angles and then decelerating as they separate.

This can be observed from the energy density, which is

\[
E = 32r^{12} + 2r^2(r^8 + r^6 + 1) + 36t^2r^8 + 4r^6 + 9r^4 + 8t^2r^4 + 4t^2 + 12t(x^{10} - y^{10}) + 4t(x^2 - y^2)(3 + 2xy^2 + 6x^2y^4) + 4t(x^6 - y^6)(9x^2y^2 - 2) - y^{10})/[r^8 + 4r^6 + 2r^4 + 16tr^2(t + x^2 - y^2) + 1]^2.
\]

Here \(E\) is symmetric under the interchange \(t \leftrightarrow -t, x \leftrightarrow y\); therefore the collision is time symmetric, with the only effect the 90° scattering (no phase shift; no radiation). For large (positive) \(t\), the height of the static soliton is proportional to \(t^2\) and its radius is proportional to \(1/t\); while the moving solitons expand with height proportional to \(t^{-2/3}\) and radius proportional to \(t^{1/3}\).

In Figure 5 we present some pictures of the total energy densities of three solitons during a typical nontrivial evolution.

In principle one should be able to visualize the emerging soliton structures when \(f(\omega) = \omega^p\) and \(h(\omega) = \omega^q\), i.e. are rational of degree \(p, q \in \mathbb{N}\), respectively. In fact, for \(q > p\) the configuration consists of \((p - 1)\) static solitons at the “centre-of-mass” of the system (if more than one, a ring structure is formed) accompanied by \(N = q - p + 1\) solitons accelerating towards the ones in the middle, scattering at an angle of \(\pi/N\), and then decelerating as they separate. This follows from the fact that the field \(J\) departs from its
asymptotic value \( J_0 \) when \( \vartheta = \omega^{(p-1)}(p(u - \mu^{-2}\nu) + \omega^N) \to 0 \), which is true when either \( \omega^{(p-1)} = 0 \) or \( \omega^N + p(u - \mu^{-2}\nu) = 0 \); and this is approximately where the solitons are located.

We conclude this section by investigating the corresponding case where \( \psi(\lambda) \) has a triple pole (and no others). Therefore, it is taken to have the form

\[
\psi(\lambda) = I + \sum_{k=1}^{3} \frac{R_2}{(\lambda - \mu)^k}.
\]  

(23)

As we have already mentioned, the reality condition (3) is satisfied if and only if \( \psi \) factorizes into three simple factors of the following type

\[
\psi(\lambda) = i \left( I - \frac{(\bar{\mu} - \mu)}{(\lambda - \mu) \|q_1\|^2} q_1 \otimes q_3 \right) \left( I - \frac{(\bar{\mu} - \mu)}{(\lambda - \mu) \|q_2\|^2} q_2 \otimes q_3 \right) \left( I - \frac{(\bar{\mu} - \mu)}{(\lambda - \mu) \|q_3\|^2} q_3 \otimes q_1 \right),
\]  

(24)

for some two-vectors \( q_k \). The requirement that the matrices \( A = (L\psi)\psi^{-1} \) and \( B = (M\psi)\psi^{-1} \) should be independent of \( \lambda \) imposes differential equations on \( q_k \); which are three nonlinear equations, and it seems difficult to find their general solution.

One way of proceeding is to take a solution for the simple-pole case (8) with \( n = 3 \), put \( \mu_1 = i + \varepsilon, \mu_2 = i, \mu_3 = i - \varepsilon \) and take the limit \( \varepsilon \to 0 \). In order to obtain a smooth solution \( \psi \) for all \( (u, v, x) \), it is necessary that \( f_1(\omega_1) - f_2(\omega_2) \to 0, f_1(\omega_1) - f_3(\omega_3) \to 0, f_2(\omega_2) - f_3(\omega_3) \to 0 \) as \( \varepsilon \to 0 \). So let us write \( f_1(\omega_1) = f(\omega_1) + \varepsilon h(\omega_1) + \varepsilon^2 g(\omega_1), f_2(\omega_2) = f(\omega_2), f_3(\omega_3) = f(\omega_3) - \varepsilon h(\omega_3) + \varepsilon^2 g(\omega_3) \), where \( f, h \) and \( g \) are rational functions of one variable. On taking the limit, we obtain a \( \psi \) of the form (24), smooth on \( \mathbb{R}^{2+1} \) and such that the matrices \( A \) and \( B \) be independent of \( \lambda \).

Consequently, \( J = \psi(0)^{-1} \) is a smooth solution of (11) of the form

\[
J = i \left( I - \frac{2q_3^\dagger \otimes q_3}{\|q_3\|^2} \right) \left( I - \frac{2q_2^\dagger \otimes q_2}{\|q_2\|^2} \right) \left( I - \frac{2q_1^\dagger \otimes q_1}{\|q_1\|^2} \right),
\]  

(25)

with \( q_k \) being in terms of \( f(z), h(z) \) and \( g(z) \) by

\[
q_1 = (1 + |f|^2)^2(1,f) - 4i(b + id)(1 + |f|^2)(\bar{f}, -1) - 4b^2(\bar{f}^2, -\bar{f} - 2ib) - 8id\bar{b}(1, f),
\]

\[
q_2 = (1 + |f|^2)(1, f) - 2ib(\bar{f}, -1),
\]

\[
q_3 = (1, f),
\]  

(26)

where \( b = tf'(z) + h(z) \) and \( d = t^2f''(z)/2 + i(t - y)f'(z)/2 + th'(z) + g(z) \). Note that the two-vectors \( q_2, q_3 \) here correspond to the ones given by (20) for \( \mu = i \), respectively.

Let us examine a sample example of this solution, since we may take \( f, h \) and \( g \) to be any rational meromorphic function of \( z \).
Let us take \( f(z) = 0, h(z) = z \) and \( g(z) = z^2 \); thus \( b = z \) and \( d = t + z^2 \). This solution consists of two solitons coming in along the \( y \)-axis merging to form a peak at the origin and then two new solitons emerging along the \( x \)-axis. Figure 6 illustrates what happens near \( t = 0 \).

The energy density of the system is,

\[
E = 32 \frac{80r^4 + 32(t^2 + r^2) + 256t^2r^2 - 64t(x^2 - y^2) + 128tyr^2 - 8y + 3}{32r^4 + 12r^2 - 16yr^2 + 16t^2 + 16ty + 32t(x^2 - y^2) + 1},
\]

which has a reflection symmetry around the \( x \)-axis. For large (positive) \( t \), \( E \) is peaked at two points on the \( y \)-axis, namely \( y \approx \pm \sqrt{t} \). Moreover, the height of the corresponding solitons is proportional to \( 1/t \), and their radius is proportional to \( \sqrt{t} \); which means that the \( y \)-axis asymmetry vanishes at \( t \to \infty \).

IV. Construction of Soliton-Antisoliton Solutions.

In this section we construct a large family of solutions which as we will argue later, can be thought of as representing soliton-antisoliton field configurations. Roughly speaking, solitons correspond to \( f \) being a function of the variable \( z \), and antisolitons correspond to a function of \( \bar{z} \).

One way to generate a soliton-antisoliton solution of (1), is to assume that \( \psi(\lambda) \) has the form

\[
\psi(\lambda) = I + \frac{n^1 \otimes m^1}{(\lambda - i)} + \frac{n^2 \otimes m^2}{(\lambda + i)}.
\]

Here \( n^k, m^k \) for \( k = 1, 2 \) are complex-valued two-vector functions of \((t, z, \bar{z})\) (not depending on \( \lambda \)).

The idea is to find the \( n^1_1, ..., n^1_l, ... \) such that the reality condition (3) holds, and such that the matrices \( A = (L\psi)\psi^{-1} \) and \( B = (M\psi)\psi^{-1} \) are independent of \( \lambda \). One way of proceeding is to take the solution (8) with \( n = 2 \), put \( \mu_1 = i + \varepsilon, \mu_2 = -i - \varepsilon \) and take the limit \( \varepsilon \to 0 \). In order for the resulting \( \psi \) to be smooth on \( \mathbb{R}^{2+1} \) it is necessary to take \( f_1 = f(\omega_1), f_2 = -1/\bar{f}(\omega_2) - \varepsilon h(\omega_2) \), where \( f \) and \( h \) are rational functions of one variable. On taking the limit \( \varepsilon \to 0 \), we then obtain a \( \psi \) as in (28) with \( m^k = (m^k_1, m^k_2) \) being holomorphic functions of \( z \) (or \( \bar{z} \)), through the relations \( m^1 = (1, f), m^2 = (-\bar{f}, 1) \), while

\[
n^1 = \frac{2i(1 + |f|^2)}{(1 + |f|^2)^2 + |w|^2} m^1 + \frac{2\bar{w}}{(1 + |f|^2)^2 + |w|^2} m^2,
\]
\[ n^2 = -\frac{2w}{(1 + |f|^2)^2 + |w|^2} \bar{m}^1 - \frac{2i(1 + |f|^2)}{(1 + |f|^2)^2 + |w|^2} \bar{m}^2, \tag{29} \]

with
\[ w \equiv \bar{h}f^2 + 2t f'. \tag{30} \]

So we generate a solution \( J = \psi(\lambda = 0)^{-1} \), which depends on the two arbitrary rational functions \( f = f(z) \) and \( h = h(\bar{z}) \). This solution has the form
\[ J = \frac{1}{(1 + |f|^2)^2 + |w|^2} \begin{pmatrix} |w|^2 + 2i(f \bar{w} + \bar{f} w) - (1 + |f|^2)^2 & 2i(w - f^2 \bar{w}) \\ 2i(-\bar{w} - \bar{f}^2 w) & |w|^2 - 2i(f \bar{w} + \bar{f} w) - (1 + |f|^2)^2 \end{pmatrix}, \tag{31} \]

with \( w \) given by (30). In general, by taking \( f(z) = z^p \) and \( h(\bar{z}) = \bar{z}^q \) where \( p \) is a positive integer and \( q \) is a non-negative integer; the energy, obtained by integrating (2), is \( E = (2p + q)8\pi \). Roughly speaking, the solution looks like \((2p + q)\) lumps at arbitrary positions in the \( xy \)-plane; which as we are going to see are a combination of solitons and antisolitons.

A topological charge may be defined for the field \( J \) (31) by exploiting the connection of it with the \( O(3) \) \( \sigma \)-model. The unmodified chiral model [i.e., (1) with \( V^\alpha = (0, 0, 0) \)] is equivalent to the \( O(4) \) \( \sigma \)-model through the relation
\[ J = I \phi_0 + i\sigma \cdot \phi, \tag{32} \]

where \( \sigma \) are the usual Pauli matrices, and \( (\phi_0, \phi) = (\phi_0, \phi_1, \phi_2, \phi_3) \) is a four vector of real fields that are constrained to lie on \( S^3 \), i.e. \( \phi_0^2 + \phi \cdot \phi = 1 \). The only static finite energy solutions of the \( O(4) \) \( \sigma \)-model correspond to the embedding of the \( O(3) \) \( \sigma \)-model [21]. Therefore the only static solutions of (1) are the \( O(3) \) embeddings that we shall describe. This is because for the one-soliton solution (static or Lorentz boosted in the \( y \)-axis) the term in (11) proportional to \( V^\alpha \) is zero, so the system behaves like the \( O(4) \) model, for which the \( O(3) \) embedding is totally geodesic. [However, for time-dependent configurations, the term proportional to \( V^\alpha \) is non-zero and will affect the evolution of the field, which will in general not lie in an \( O(3) \) subspace of \( O(4) \).]

To proceed further, let us mention the topological aspects of the \( O(3) \) and \( O(4) \) \( \sigma \)-models. In studying soliton-like solutions, we require that the field configuration has finite energy. This implies that the field must take the same value at all points of spatial infinity, so that space is compactified from \( \mathbb{R}^2 \) to \( S^2 \). At fixed time, the field is a map from \( S^2 \) into
the target space. Now for the O(3) model, the field is a map \( \phi : S^2 \rightarrow S^2 \), and due to the homotopy relation

\[
\pi_2(S^2) = \mathbb{Z},
\]

such maps are classified by an integer winding number \( N \) which is a conserved topological charge. An expression for this charge is given by

\[
N = (8\pi)^{-1} \int \epsilon_{ij} \phi \cdot (\partial_i \phi \wedge \partial_j \phi) \, d^2x,
\]

where \( i = 1, 2 \) with \( x^i = (x, y) \).

Although, for the O(4) model [the same argument is valid for (\ref{eq:0}) due to the topological aspects of the theory] the field at fixed time is a map \( (\phi_0, \phi) : S^2 \rightarrow S^3 \) and the corresponding homotopy relation is

\[
\pi_2(S^3) = 0,
\]

so there is no winding number. However, for soliton solutions that correspond to some initial embedding of O(3) space into O(4), there is a useful topological quantity, as we are going to see.

Consider the O(4) configuration which at some time corresponds to an O(3) embedding, which we choose to be \( \phi_0 = 0 \) for definiteness. At this time the field is restricted to an \( S^2 \) equator of the possible \( S^3 \) target space. Suppose that the field never maps to the anti-podal points \( \{A_1, A_2\} = \{\phi_0 = 1, \phi_0 = -1\} \) at any time, so the target space is \( S^3_0 = S^3 - \{A_1, A_2\} \). Now \( S^3_0 \approx S^2 \times \mathbb{R} \), and thus we have the homotopy relation

\[
\pi_2(S^3_0) = \pi_2(S^2 \times \mathbb{R}) = \pi_2(S^2) \oplus \pi_2(\mathbb{R}) = \mathbb{Z},
\]

and therefore a topological winding number exists. An expression for this winding number is easy to give, since it is the winding number of the map after projection onto the chosen \( S^2 \) equator, i.e.

\[
N' = (8\pi)^{-1} \int \epsilon_{ij} \phi' \cdot (\partial_i \phi' \wedge \partial_j \phi') \, d^2x,
\]

where \( \phi' = \phi/|\phi| \). If the field does map to the anti-podal points \( \{A_1, A_2\} \) at some time the winding number is ill defined at this time and if considered as a function of time \( N' \) will be integer valued but may suffer discontinuous jumps as the field moves through the anti-podal points. In the following examples, before comparing the solution \( J \) given by (\ref{eq:1}) with the O(3) embedding it is convenient to perform the transformation \( J \rightarrow MJ \).
with $M = (\sqrt{2})^{-1} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ so that the evolution of the field remains close to the O(3) embedding.

**V. Soliton-Antisoliton Scattering.**

Usually in the nonintegrable models, there is an attractive force between solitons of opposite topological charge. In fact, if the solitons and antisolitons are well separated, then they attract each other and eventually annihilate into a wave of pure radiation which spreads with the velocity of light $[3, 4]$. However, the interaction forces between solitons and antisolitons do depend on their configuration; in particular, they depend on the relative orientation between them in the internal space. Therefore, the cross section for the soliton-antisoliton elastic scattering is non-zero. [In the real world, the proton-antiproton elastic scattering is seen in a reasonable fraction of the cases.] This is the first example for which there has been constructed an explicit (since the system is integrable) solution of elastic soliton-antisoliton scattering in either integrable or nonintegrable model. As a result, it provides a major link between soliton dynamics in integrable and nonintegrable systems.

The evolution is initially similar to the numerical results obtained through the connection of the integrable chiral model (1) with the O(3) $\sigma$-model $[20]$. In particular, a soliton and an antisoliton are moving along the $x$-axis towards each other at an accelerating rate until they merge at the origin and form a peak. Note that a peak is formed rather than a ring since the energy is mainly kinetic when a soliton and an antisoliton merge. However, rather than the peak dissipating in a wave of radiation it now reforms into two new structures which undergo 90$^\circ$ scattering. In general, in all head-on collisions of $N$ moving soliton and antisoliton objects, the scattering angle is $\pi/N$ degrees relative to the initial direction of motion.

Next we looked at two cases corresponding to the mixtures of solitons and antisolitons. [The configurations given by (31) when $h(\bar{z}) = 0$ are equivalent to the ones obtained from (33,4) when $f(z) = z^p$.]

- First, let us take $f(z) = z$ and $h(\bar{z}) = 1$. Roughly speaking, if $r$ is large, $J$ is close to its asymptotic value $J_0$, as long as $w/z^2 = 1 + 2t/z^2 \approx 1$; but as $z$ approaches
±√−2t then \( w \to 0 \), and \( J \) departs from its asymptotic value: this is where the two structures are located. More precisely, for negative \( t \), the two objects are on the \( x \)-axis, approximately at \( x \approx ±\sqrt{−2t} \); while for positive \( t \), they are on the \( y \)-axis, approximately at \( y \approx ±\sqrt{2t} \). Figure 7 illustrates what happens near \( t = 0 \).

The picture is consistent with the properties of the energy density of the solution, which is

\[
E = 16 \frac{2r^4 + 4r^2 + 4t^2(1 + 2r^2) - 4t(x^2 - y^2) + 1}{[2r^4 + 2r^2 + 4t(x^2 - y^2) + 4t^2 + 1]^2}. \tag{38}
\]

Note the symmetry of \( E \) under the interchange \( t \mapsto -t, x \leftrightarrow y \); the time symmetry of the density confirms the lack of radiation. The corresponding localized structures are not however of constant size: for large (positive) \( t \), their height is proportional to \( 1/t \), while their radius is proportional to \( \sqrt{t} \).

The projected topological charge \( \mathcal{N}' \) is zero throughout the scattering process; while the projected topological density \( q' \), i.e.

\[
\mathcal{N}' = \int q' \, dx \, dy, \tag{39}
\]

has an almost identical distribution (up to a scale) to that of the energy density (see Figure 8(a)). Therefore, the configuration represents a soliton and an antisoliton which are clearly visible as distinct structures having respectively +1 and -1 units of topological charge concentrated in a single lump.

Equation (I) is not Lorentz invariant and indeed is not even radially symmetric due to the presence of the vector \( V_a \) which picks out a particular direction in space, and therefore one may expect to find different scattering behaviour for more general solutions; e.g., when the soliton and the antisoliton are moving along the \( x \)-axis rather than the \( y \)-axis. However, this is not true since (I) is a reduction of the self-dual Yang-Mills equation in \( \mathbb{R}^{2+2} \) which does have an \( \text{SO}(1,2) \) symmetry. Therefore, the \( \text{SO}(2) \) symmetry of the Yang-Mills system means that any given solution \( J \), can in principle be converted to gauge fields by performing a coordinate rotation (together with a gauge transformation) and then recover the corresponding \( J' \) which will describe the same solution as \( J \) but with a rotated coordinate system. Indeed, this is what happens by taking

\[
f(z) = e^{(2i\phi)}z, \quad h(\bar{z}) = 1, \tag{40}
\]

where \( \phi \) is an angle in the \( xy \)-plane. This picture presents a rotated version through any angle \( \phi \) in the \( xy \)-plane of the original one (i.e., Figure 7).
Finally, let us take $f(z) = z$ and $h(\bar{z}) = \bar{z}$. The corresponding configuration consists of one soliton and two antisolitons (see Figure 8(b)).

It is interesting to look at the time dependence of various energies in each process. The total energy, of course, is constant and it is the spatial integral of the following energy density

$$
E = 8[r^8 + 8r^6 + 11r^4 + 4r^2 - 8x^5t + 16ty^2(x^3 + t) + 8t^2 + 48xy^2t + 2 -16x^2t(x - t) + 24xy^4]/[r^6 + r^4 + 2r^2 + 4t^2 + 4tx^3 - 12xy^2t + 1]^2.
$$

Obviously, the energy density $E$ is symmetric under the interchange $t \mapsto -t$, $x \mapsto -x$ and $y \mapsto -y$, only. Again all three structures come together forming a bell-like structure and then emerge at an angle of $60^\circ$ with respect to the original direction. However, by looking at the maximum of $E$ we observe that, for large (positive) $t$, the height of the localized structures is proportional to $t^{-4/3}$, while their radius is proportional to $t^{1/3}$; thus they spread out as they move apart.

Figure 9 shows the results of a head-on collision of the one-soliton two-antisoliton system.

Let us conclude with the observation that, by taking $f(z) = z^p$ and $h(\bar{z}) = \bar{z}^q$, $J$ departs from its asymptotic value $J_0$ when $w = z^{p-1}(2tp + z^N) \to 0$ with $N = p + q + 1$, which is true when either $z^{p-1} = 0$ or $2tp + z^N = 0$: this is approximately where the lumps are located. Therefore, $J$ represents a family of soliton-antisoliton solution which consists of $(p - 1)$ static soliton-like objects at the origin, with $N$ others accelerating towards them, scattering at an angle of $\pi/N$, and then decelerating as they separate.

**VI. Conclusion.**

The infinite number of conservation laws associated with a given integrable system place severe constraints upon possible soliton dynamics. The construction of exact analytic multisoliton solutions with trivial scattering properties is a result of such integrability properties. In this paper new soliton and soliton-antisoliton solutions have been obtained for the planar integrable chiral model (I). These structures travel with non-constant velocity; their size is non-constant; and they interact non-trivially. Such results might be useful for connecting integrable and nonintegrable systems which possess soliton solutions. In addition, they indicate the likely occurrence of new phenomena in higher dimensional soliton theory that are not present in (1+1) dimensions.
It seems likely that there are many more interesting solutions still to be found; an open question being what is the general form of the function $\psi$ when it has a higher-order pole in $\lambda$. One could, for example, investigate the case $n = 3$ for $\psi(\lambda)$ with a single and a double pole; and determine the scattering properties of the emerging structures, in terms of their initial velocity and of the values of the impact parameter. Finally, it would be of great interest to deduce the general form of the function $\psi(\lambda)$ for the soliton-antisoliton case (28) with the only constraint to satisfy the reality condition (3) and the requirement that the matrices $A = (L\psi)\psi^{-1}$ and $B = (M\psi)\psi^{-1}$ be independent of $\lambda$.

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Figure Captions.

**Figure 1:** The energy density $\mathcal{E}$ (16) at increasing times.

**Figure 2:** The energy density $\mathcal{E}$ (18) at various times.

**Figure 3:** Energy density at various values of $\varepsilon$ for a system of two solitons ($t = -15$).

**Figure 4:** Energy density at increasing times for a system of three solitons with $60^\circ$ angle scattering.

**Figure 5:** Energy density at various times for the scattering of three solitons, with one being static at the origin.

**Figure 6:** Energy density at increasing times when $\psi(\lambda)$ has a triple pole (and no others).

**Figure 7:** Energy density at increasing times showing a $90^\circ$ scattering between a soliton and an antisoliton.

**Figure 8:** Topological charge density at increasing times for (a) soliton-antisoliton scattering, (b) one-soliton two-antisoliton scattering.

**Figure 9:** Energy density of a system consisting of a soliton and two antisolitons at various times.
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