THE TWISTED SYMMETRIC SQUARE \( L \)-FUNCTION OF \( \text{GL}(r) \)

SHUICHIRO TAKEDA

Abstract. In this paper, we consider the (partial) symmetric square \( L \)-function \( L^S(s, \pi, \text{Sym}^2 \otimes \chi) \) of an irreducible cuspidal automorphic representation \( \pi \) of \( \text{GL}_r(\mathbb{A}) \) twisted by a Hecke character \( \chi \). In particular, we will show that the \( L \)-function \( L^S(s, \pi, \text{Sym}^2 \otimes \chi) \) is holomorphic for the region \( \Re(s) > 1 - \frac{1}{r} \) with the exception that, if \( \chi \omega^2 = 1 \), a pole might occur at \( s = 1 \), where \( \omega \) is the central character of \( \pi \). Our method of proof is essentially a (nontrivial) modification of the one by Bump and Ginzburg in which they considered the case \( \chi = 1 \).

Introduction

Let \( \pi \cong \otimes_v \pi_v \) be an irreducible cuspidal automorphic representation of \( \text{GL}_r(\mathbb{A}) \) and \( \chi \) a unitary Hecke character on \( \mathbb{A}^\times \), where \( \mathbb{A} \) is the ring of adeles over a number field \( F \). By the local Langlands correspondence by Harris-Taylor [HT] and Henniart [He], each \( \pi_v \) corresponds to an \( r \)-dimensional representation \( \text{rec}(\pi_v) \) of the Weil-Deligne group \( WD_{F_v} \) of \( F_v \). We can also consider the twist of \( \text{rec}(\pi_v) \) by \( \chi_v \), namely,

\[
\text{rec}(\pi_v) \otimes \chi_v : WD_{F_v} \to \text{GL}_r(\mathbb{C}),
\]

where \( \chi_v \) is viewed as a character of \( WD_{F_v} \) via local class field theory. Now for each homomorphism \( \rho : \text{GL}_r(\mathbb{C}) \to \text{GL}_N(\mathbb{C}) \), one can associate the local \( L \)-factor \( L_v(s, \pi_v, \rho \circ \text{rec}(\pi_v) \otimes \chi_v) \) of Artin type. Then one can define the automorphic \( L \)-function by

\[
L(s, \pi, \rho \otimes \chi) := \prod_v L_v(s, \pi_v, \rho \circ \text{rec}(\pi_v) \otimes \chi_v).
\]

In particular in this paper, we consider the case where \( \rho \) is the symmetric square map

\[
\text{Sym}^2 : \text{GL}_r(\mathbb{C}) \to \text{GL}_{r(r+1)}(\mathbb{C}),
\]

namely we consider the twisted symmetric square \( L \)-function \( L(s, \pi, \text{Sym}^2 \otimes \chi) \). By the Langlands-Shahidi method, it can be shown that the \( L \)-function \( L(s, \pi, \text{Sym}^2 \otimes \chi) \) admits meromorphic continuation and a functional equation. (See [Sh1, Theorem 7.7].)

The Langlands-Shahidi method, however, is unable to determine the locations of the possible poles of \( L(s, \pi, \text{Sym}^2 \otimes \chi) \). The main theme of this paper is to determine them to some extent, though we consider only the partial \( L \)-function \( L^S(s, \pi, \text{Sym}^2 \otimes \chi) \). To be more specific, let \( S \) be the finite set of places that contains all the archimedean places and non-archimedean places where \( \pi \) or \( \chi \) ramifies. For \( v \notin S \), each \( \pi_v \) is parameterized by a set of \( r \) complex numbers \( \{\alpha_{v,1}, \ldots, \alpha_{v,r}\} \) known as the Satake parameters. Then we have

\[
L_v(s, \pi_v, \text{Sym}^2 \otimes \chi_v) = \prod_{i \leq j} \frac{1}{(1 - \chi_v(\varpi_v)\alpha_{v,i} \alpha_{v,j} q_v^{-s})},
\]

where \( \varpi_v \) is the uniformizer of \( F_v \) and \( q_v \) is the order of the residue field. And we set

\[
L^S(s, \pi, \text{Sym}^2 \otimes \chi) = \prod_{v \notin S} L_v(s, \pi_v, \text{Sym}^2 \otimes \chi_v).
As our main theorem (Theorem \[5.1\]) we will prove

**Theorem \[5.1\].** Let \( \pi \) be a cuspidal automorphic representation of \( \text{GL}_r(\mathbb{A}) \) with unitary central character \( \omega \) and \( \chi \) a unitary Hecke character. Then for each archimedean \( v \), there exists an integer \( N_v \geq 0 \) such that the product

\[
L^S(s, \pi, \text{Sym}^2 \otimes \chi) \prod_{v \mid \infty} L_v(rs - r + 1, \chi^r_\omega^2)^{-N_v}
\]

is holomorphic everywhere except at \( s = 0 \) and \( s = 1 \). Moreover there is no pole if \( \chi^r \omega^2 \neq 1 \).

Here the factor \( L_v(rs - r + 1, \chi^r_\omega^2)^{-N_v} \) at each archimedean place is a kind of compensation factor, which stems from a very subtle issue in the theory of asymptotic expansions of matrix coefficients of real Lie groups, which will be explained in detail in the proof of Proposition \[5.3\].

Notice that by this theorem the possible poles of \( L^S(s, \pi, \text{Sym}^2 \otimes \chi) \) other than at \( s = 0 \) and \( s = 1 \) come from the poles of the archimedean \( L \)-factors \( L_v(rs - r + 1, \chi^r_\omega^2)^{N_v} \), which are gamma functions. Hence in particular, we have

**Corollary \[5.8\].** The (incomplete) twisted symmetric square \( L \)-function \( L^S(s, \pi, \text{Sym}^2 \otimes \chi) \) is holomorphic everywhere in the region \( \text{Re}(s) > 1 - \frac{1}{2r} \) except at \( s = 1 \). Moreover there is no pole at \( s = 1 \) if \( \chi^r \omega^2 \neq 1 \).

The reason we can show the holomorphy only for the region \( \text{Re}(s) > 1 - \frac{1}{2r} \) is the issue at the archimedean places pointed out above. However we believe that this can be removed and that we can prove the following stronger version

**Conjecture \[5.9\].** The (incomplete) twisted symmetric square \( L \)-function \( L^S(s, \pi, \text{Sym}^2 \otimes \chi) \) is holomorphic everywhere except at \( s = 0 \) and \( s = 1 \). Moreover there is no pole if \( \chi^r \omega^2 \neq 1 \).

We will take up this issue in our later work (\[12\]).

Let us also note that the above corollary does not tell us that the \( L \)-function \( L^S(s, \pi, \text{Sym}^2 \otimes \chi) \) does have a pole at \( s = 1 \) if \( \chi^r \omega^2 = 1 \). However, based on an observation made by Shahidi, one can show that if \( r \) is odd, then the \( L \)-function \( L^S(s, \pi, \text{Sym}^2 \otimes \chi) \) has a pole at \( s = 1 \) if and only if \( \tilde{\pi} = \pi \otimes \chi \), where \( \tilde{\pi} \) is the contragredient of \( \pi \). (See Corollary \[5.11\]).

Our method of proof is by Rankin-Selberg convolution with what we call the exceptional representation of the metaplectic double cover \( \tilde{\text{GL}}_r(\mathbb{A}) \) of \( \text{GL}_r(\mathbb{A}) \), which is viewed as a natural generalization of theta series for \( r = 2 \). Indeed for \( r = 2 \), the same result has been obtained by Gelbart and Jacquet (\[GJ\]) already in the late 70’s, whose method in turn has its origin in the work by Shimura (\[Shi\]), where he considered the analogous problem in the classical context of elliptic modular forms. Later Patterson and Piatetski-Shapiro (\[PP\]) generalized the method to \( r = 3 \) though this time they considered only the non-twisted case, \textit{i.e.} \( \chi = 1 \). Afterwards, Bump and Ginzburg (\[BG\]) generalized the method to arbitrary \( r \) but again only for \( \chi = 1 \). For the twisted case, Banks worked out the case \( r = 3 \) in (\[B2\]).

Bump and Ginzburg in (\[BG\]) used the exceptional representation constructed by Kazhdan and Patterson (\[KP\]), which is a representation of the metaplectic cover \( \tilde{\text{GL}}_r \) of \( \text{GL}_r \) both locally and globally. In order to incorporate character twist into the work of Bump and Ginzburg, one needs to obtain the twisted version of the exceptional representation of Kazhdan and Patterson, which we call the twisted exceptional representation. It turns out that one needs the twisted exceptional
representation of $\tilde{\text{GL}}_{2q}$, where $2q$ is such that $r = 2q$ or $r = 2q + 1$. If $q = 1$, the (twisted) exceptional representation is simply the (twisted) Weil representation of $\text{GL}_2$, which is precisely what is used by Gelbart and Jacquet ([GJ]) for $r = 2$ and by Banks ([B2]) for $r = 3$. For higher ranks, one needs to construct the twisted exceptional representation. Locally this is the Langlands quotient of an induced representation whose inducing representation is essentially $q$ copies of the (twisted) Weil representation of $\text{GL}_2$ for the local case, and globally the residues of the Eisenstein series constructed from the corresponding global induced representation. This construction for the non-archimedean local field of odd residual characteristic is carried out as a main part of the Ph.D thesis by Banks ([B1]) supervised by Bump. And part of the reason that Bump and Ginzburg only considered the non-twisted case is that the twisted exceptional representation is essentially a (constituent) of restriction of the exceptional representation of $\tilde{\text{GL}}_{2q}$.

Notations

Throughout the paper, $F$ will be either a local or global field of characteristic 0. If $F$ is global, we denote the ring of adeles by $\mathbb{A}$. If $F$ is a non-archimedean local field $F$, we denote the ring of integers by $\mathcal{O}_F$, and the uniformizer by $\varpi_F$ or simply by $\varpi$ when the field is clear from the context.

We fix the non-trivial additive character $\psi$ on $F\setminus \mathbb{A}$ if $F$ is a number field or on $F$ if $F$ is a local field. Though we often use the same symbol $\psi$ both for the local and global cases, this will cause no confusion. Whether $F$ is local or global, for each $a \in F^\times$ we denote by $\psi_a$ the additive character defined by $\psi_a(x) = \psi(ax)$. If $F$ is local and $\chi$ is a character on $F^\times$, by $L(\chi)$ we mean the local Tate factor for $\chi$. In particular for non-archimedean $F$, $L(\chi) = (1 - \chi(\varpi_F))^{-1}$ (resp. $L(\chi) = 1$) if $\chi$ is unramified (resp. ramified). If $F$ is global and $\chi = \otimes_v \chi_v$, we let $L(\chi) = \prod_v L(\chi_v)$.

For the group $\text{GL}_r$, we often consider the two cases: $r$ is even and $r$ is odd. For the former we let $r = 2q$ and for the latter $r = 2q + 1$. If $P$ is a parabolic subgroup of $\text{GL}_r$, we denote the Levi part by $M_P$ and the unipotent radical by $N_P$. We always assume that the Levi part $M_P = \text{GL}_{r_1} \times \cdots \times \text{GL}_{r_k}$ sits in $\text{GL}_r$ diagonally. We often denote each element $\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{pmatrix} \in M_P$ by $(g_1, \ldots, g_k)$ or $\text{diag}(g_i)$ for $g_i \in \text{GL}_{r_i}$ whenever it is convenient. Also we denote the maximal torus of $M_P$ by $T_P$. We denote the Borel subgroup by $B$ and we denote $T_Q$ simply by $T$. Also we let $\delta_P$ be the modular character of $P$. We let $W$ be the Weyl group of $\text{GL}_r$ and we choose each element $w \in W$ in such a way that each entry in $w$ is either 0 or 1. We denote the $r \times r$ identity matrix by $I_r$. 

Finally, let us mention that the result of this paper will be used in a work by Asgari and Shahidi ([AS2]) for determination of the image of the Langlands transfers from the general spin groups to $\text{GL}_r$, which they obtained in their earlier paper ([ASI]).
For an algebraic group $G$ over $F$, we sometimes write simply $G$ for the $F$-rational points, when there is no danger of confusion. Also for a global $F$ we sometimes denote each element in $G(F)$ by $\prod_v g_v$ where $g_v \in G(F_v)$. If $A$ is a locally compact abelian group, we denote its Pontryagin dual by $\hat{A}$.

Let $G$ be any group and $H \subseteq G$ a subgroup. For each $g \in G$ and $h \in H$ we let $g h = ghg^{-1}$ and $g H = \{ g h : h \in H \}$. If $\pi$ is a representation of $H$, we define the twist $g \pi$ of $\pi$ with $g$ to be the representation of $g H$ given by $g \pi(h) = \pi(g^{-1}hg)$ for $h \in H$. We use the symbol $\text{Ind}$ for normalized induction and ind for unnormalized one.

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1. The metaplectic double cover $\widetilde{GL}_r$ of $GL_r$

In this section, we review the theory of the metaplectic double cover $\widetilde{GL}_r$ of $GL_r$ for both local and global cases, which was originally constructed by Kazhdan and Patterson in [KP].

1.1. The local metaplectic double cover $\widetilde{GL}_r$. Let $F$ be a (not necessarily non-archimedean) local field of characteristic 0. In this paper, by the metaplectic double cover $\widetilde{GL}_r(F)$ of $GL_r(F)$, we mean the central extension of $GL_r(F)$ by $\{ \pm 1 \}$ as constructed in [KP] by Kazhdan and Patterson. (Kazhdan and Patterson considered more general cover $GL_r^{(c)}(F)$ with a twist by $c \in \{ 0, 1 \}$. But we only consider the non-twisted case, i.e. $c = 0$.) Later, Banks, Levy, and Sepanski ([BLS]) gave an explicit description of a 2-cocycle 

$$\sigma_r : GL_r(F) \times GL_r(F) \to \{ \pm 1 \}$$

which defines $\widetilde{GL}_r(F)$ and shows that their 2-cocycle is “block-compatible”, by which we mean the following property of $\sigma_r$: For the standard $(r_1, \ldots, r_k)$-parabolic $P$ of $GL_r$, so that its Levi $M_P$ is of the form $GL_{r_1} \times \cdots \times GL_{r_k}$ which is embedded diagonally into $GL_r$, we have

$$\sigma_r \left( \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix}, \begin{pmatrix} g'_1 \\ \vdots \\ g'_k \end{pmatrix} \right) = \prod_{i=1}^k \sigma_{r_i}(g_i, g'_i) \prod_{1 \leq i < j \leq k} (\det(g_i), \det(g'_j))_F,$$

for all $g_i, g'_i \in GL_{r_i}(F)$ ([BLS Theorem 11, §3]), where $(-, -)_F$ is the Hilbert symbol for $F$. The 2-cocycle of [BLS] generalizes the well-known cocycle given by Kubota ([Ku]) for the case $r = 2$. Note that $\widetilde{GL}_r(F)$ is not the $F$-rational points of an algebraic group, but this notation seems to be standard.
We need to recall how this cocycle is constructed. Let $G_r = SL_{r+1}$. Matsumoto in [Mat] constructed the metaplectic double cover $\tilde{G}_r$ of $G_r$. A cocycle $\sigma_\tau$, defining the cover $\tilde{G}_r$ is explicitly computed in [BLS], and satisfies the block-compatibility in a much stronger sense ([BLS, Theorem 7, §2]). Consider the embedding

$$l : GL_r(F) \to G_r(F), \quad g \mapsto \left( g, \det(g)^{-1} \right).$$

Then the cocycle $\sigma_r$ is defined by

$$\sigma_r(g, g') = \sigma_{G_r}(l(g), l(g'))(\det(g), \det(g'))_F.$$

(See [BLS p.146].) We define $\sigma GL_r(F)$ to be the group whose underlying set is

$$\sigma GL_r(F) = GL_r(F) \times \{ \pm 1 \} = \{ (g, \xi) : g \in GL_r(F), \xi \in \{ \pm 1 \} \},$$

and the group law is defined by

$$(g_1, \xi_1) \cdot (g_2, \xi_2) = (g_1g_2, \sigma_r(g_1, g_2)\xi_1\xi_2).$$

Since we would like to emphasize the cocycle being used, we write $\sigma GL_r(F)$ instead of $\tilde{G}_r(F)$.

To use the block-compatible 2-cocycle of [BLS] has obvious advantages. In particular, it can be explicitly computed and, of course, it is block-compatible. However it does not allow us to construct the embedding $\sigma$. Then the cocycle $\sigma_\tau$ (See [BLS, p.146].) to use the block-compatible 2-cocycle of [BLS] has obvious advantages. In particular, it can be explicitly computed and, of course, it is block-compatible. However it does not allow us to construct the embedding $\sigma$.

For this reason, we will use a different 2-cocycle $\tau_r$, which works nicely with the global metaplectic cover $GL_r(A)$. To construct such $\tau_r$, first assume $F$ is non-archimedean. It is known that an open compact subgroup $K$ splits in $GL_r(F)$, and moreover if the residue characteristic of $F$ is odd, $K = GL_r(O_F)$. (See [KF] Proposition 0.1.2.) Also for $k_1, k_2 \in K$, we have $(\det(k_1), \det(k_2))_F = 1$. Hence one has a continuous map $s_r : GL_r(F) \to \{ \pm 1 \}$ such that $\sigma_r(g_1, g_2) s_r(g_1) s_r(g_2) = s_r(g_1g_2)$ for all $g_1, g_2 \in K$. Then define our 2-cocycle $\tau_r$ by

$$\tau_r(g_1, g_2) := \sigma_r(g_1, g_2) s_r(g_1) s_r(g_2) / s_r(g_1g_2)$$

for $g_1, g_2 \in GL_r(F)$. If $F$ is archimedean, we set $\tau_r = \sigma_r$.

The choice of $s_r$ and hence $\tau_r$ is not unique. However when the residue characteristic of $F$ is odd, there is a canonical choice with respect to the splitting of $K$ in the following sense. Since the cocycle $\sigma_r$ is the restriction of $\sigma_{G_r}$, to the image of the embedding $l$, and it is known that the compact group $G_r(O_F)$ also splits in $\tilde{G}_r(F)$, there is a map $s_r : G_r(F) \to \{ \pm 1 \}$ such that the section $G_r(F) \to \tilde{G}_r(F)$ given by $(g, s_r(g))$ is a homomorphism on $G_r(O_F)$. (Here we assume $\tilde{G}_r(F)$ is realized as $G_r(F) \times \{ \pm 1 \}$ as a set and the group structure is defined by the cocycle $\sigma_{G_r}$.) Moreover $s_r|_{G_r(O_F)}$ is determined up to twists by the elements in $H^1(G_r(O_F), \{ \pm 1 \}) = Hom(G_r(O_F), \{ \pm 1 \})$. But $Hom(G_r(O_F), \{ \pm 1 \}) = 1$ since $G_r(O_F)$ is perfect, and hence $s_r|_{G_r(O_F)}$ is unique. (See [KP] p. 43 for this matter.) We choose $s_r$, so that

$$s_r = s_r|_{l(GL_r(O_F))}.$$ 

With this choice, we have the commutative diagram

$$\begin{array}{ccc}
\sigma \tilde{G}_r(O_F) & \longrightarrow & \tilde{G}_r(O_F) \\
k \mapsto (k, s_r(k)) & & k \mapsto (k, s_r(k)) \\
K & \longrightarrow & G_r(O_F)
\end{array}$$
where the top arrow is \((g, \xi) \mapsto (l(g), \xi)\) and all the arrows can be seen to be homomorphisms. This choice of \(s_r\) is crucial for constructing the metaplectic tensor product of automorphic representations in Appendix A. Also note that the left vertical arrow in the above diagram is what is called the canonical lift in [KP] and denoted by \(\kappa^*\) there.

Also when \(r = 2\), we assume that \(\tau_2\) is chosen to be the cocycle \(\beta\) used in [E, p.125], which can be shown to be block-compatible, and equal to the choice we made above when the residue characteristic of \(F\) is odd.

Using \(\tau_r\), we realize \(\tilde{\text{GL}}_r(F)\) to be

\[
\tilde{\text{GL}}_r(F) = \text{GL}_r(F) \times \{\pm 1\},
\]

as a set and the group law is given by

\[
(g_1, \xi_1) \cdot (g_2, \xi_2) = (g_1g_2, \tau_r(g_1, g_2)\xi_1\xi_2).
\]

Note that we have the exact sequence

\[
0 \longrightarrow \{\pm 1\} \longrightarrow \tilde{\text{GL}}_r(F) \xrightarrow{p_r} \text{GL}_r(F) \longrightarrow 0
\]
given by the obvious maps, where we call \(p_r\) the canonical projection.

We define a set theoretic section

\[
\kappa : \text{GL}_r(F) \to \tilde{\text{GL}}_r(F), \quad g \mapsto (g, 1).
\]

Note that \(\kappa\) is not a homomorphism. But by our construction of the cocycle \(\tau_r\), \(\kappa|_K\) is a homomorphism if \(F\) is non-archimedean and \(K\) is a sufficiently small open compact subgroup, and moreover if the residue characteristic is odd, one has \(K = \text{GL}_r(\mathcal{O}_F)\).

Also we define another set theoretic section

\[
s : \text{GL}_r(F) \to \tilde{\text{GL}}_r(F), \quad g \mapsto (g, s_r(g)^{-1})
\]

where \(s_r(g)\) is as above. We sometimes write \(s\) for \(s_r\) when we would like to emphasize the rank of the group. We have the isomorphism

\[
\tilde{\text{GL}}_r(F) \overset{\sigma}{\longrightarrow} \sigma\tilde{\text{GL}}_r(F), \quad (g, \xi) \mapsto (g, s_r(g)\xi),
\]

which gives rise to the commutative diagram

\[
\begin{array}{ccc}
\tilde{\text{GL}}_r(F) & \overset{\sigma}{\longrightarrow} & \sigma\tilde{\text{GL}}_r(F) \\
\downarrow{s} & & \downarrow{g \mapsto (g, 1)} \\
\text{GL}_r(F) & & \text{GL}_r(F)
\end{array}
\]

of set theoretic maps, \(i.e.,\) maps which are not necessarily homomorphisms. Also note that the elements in the image \(s(\text{GL}_r(F))\) "multiply via \(\sigma_r\)" in the sense that for \(g_1, g_2 \in \text{GL}_r(F)\), we have

\[
(g_1, s_r(g_1)^{-1})(g_2, s_r(g_2)^{-1}) = (g_1g_2, \sigma_r(g_1, g_2)s_r(g_1, g_2)^{-1}).
\]

For a subgroup \(H \subseteq \text{GL}_r(F)\), whenever the cocycle \(\sigma_r\) is trivial on \(H \times H\), the section \(s\) splits \(H\) by \([13]\). We often denote the image \(s(H)\) by \(H^*\) or sometimes simply by \(H\) when it is clear from the context. Particularly important is that by [BLS] Theorem 7 (f), \(s\) splits \(N_B\), the unipotent radical of the Borel subgroup \(B\) of \(\text{GL}_r(F)\), and accordingly we denote \(s(N_B)\) by \(N_B^*\). (Note that in [BG] and [KP], the notation \(H^*\) seems to be used whenever \(H\) splits in \(\text{GL}_r(F)\) via any section. But we avoid this abuse of notation. For example, if \(F\) is non-archimedean of odd residual characteristic, \(\text{GL}_r(\mathcal{O}_F)\) splits via \(\kappa\) but not via \(s\), and hence the notation \(\text{GL}_r(\mathcal{O}_F)^*\) does not make sense in this paper.)
Assume $F$ is non-archimedean of odd residue characteristic. By [KP] Proposition 0.1.3 we have
\begin{equation}
\kappa|_{T \cap K} = s|_{T \cap K}, \quad \kappa|_{W} = s|_{W}, \quad \kappa|_{N_{B} \cap K} = s|_{N_{B} \cap K},
\end{equation}
where $W$ is the Weyl group and $K = \text{GL}_{r}(O_{F})$. In particular, this implies $s_{r}|_{T \cap K} = s_{r}|_{W} = s_{r}|_{N_{B} \cap K} = 1$. Also note that $s_{r}(1) = 1$. In particular the section $s$ splits the Weyl group $W$. If the residue characteristic of $F$ is not odd, however, $s$ does not split $W$. Indeed, $s$ splits $W$ if and only if $(-1, -1)_{F} = 1$. (See [BLS] §5.) Yet in either case, for each element $w \in W$, we denote $s(w)$ simply by $w$, when it is clear from the context.

Note that $\text{GL}_{1} = \text{GL}_{1}(F) \times \{ \pm 1 \}$, where the product is the direct product, i.e. $\sigma_{1}$ is trivial. (See [BLS] Corollary 8, §3.) Also we define $\widehat{F}^\times$ to be $\widehat{F}^\times = \widehat{F}^\times \times \{ \pm 1 \}$ as a set but the product is given by $(a_{1}, \xi_{1}) \cdot (a_{2}, \xi_{2}) = (a_{1}a_{2}, (a_{1}, a_{2})_{F}\xi_{1}\xi_{2})$. (It is known that $\widehat{F}^\times$ is isomorphic to $\widetilde{\text{GL}}_{1}$ if and only if $(-1, -1)_{F} = 1$. It is our understanding that this is due to J. Klose ([KP, p.42]), though we do not know where his proof is written. See [Ad] for a proof for a more general statement.)

For each subgroup $H(F) \subseteq \text{GL}_{r}(F)$, we denote the preimage $p_{r}^{-1}(H(F))$ of $H(F)$ via the canonical projection $p_{r}$ by $\widetilde{H}(F)$ or sometimes simply by $\widetilde{H}$ when the base field is clear from the context. We call it the “metaplectic preimage” of $H(F)$.

If $P$ is a parabolic subgroup of $\text{GL}_{r}$ whose Levi is $M_{P} = \text{GL}_{r_{1}} \times \cdots \times \text{GL}_{r_{k}}$, we often write
$$\widetilde{M}_{P} = \text{GL}_{r_{1}} \times \cdots \times \text{GL}_{r_{k}}$$
for the metaplectic preimage of $M_{P}$. One can check
$$\widetilde{P} = \widetilde{M}_{P} N_{P}^{\ast}$$
because each element in $\widetilde{N}_{P}$ is written in the form $(1, \xi)n^{\ast}$ for $n^{\ast} \in N_{P}^{\ast}$ and $\xi \in \{ \pm 1 \}$, and $(1, \xi) \in \widetilde{M}_{P}$. Moreover, one can check by using [BLS] Theorem 7 (f), §3 that $N_{P}^{\ast}$ is normalized by $\widetilde{M}_{P}$. Also we have $\widetilde{M}_{P} \cap N_{P}^{\ast} = \{(1, 1)\}$. Hence if $\pi$ is a representation of $\widetilde{M}_{P}$, one can consider the induced representation $\text{Ind}_{\widetilde{M}_{P} N_{P}^{\ast}}^{\text{GL}_{r}}\pi$ as usual by letting $N_{P}^{\ast}$ act trivially. This is the reason we prefer to write $\widetilde{P} = \widetilde{M}_{P} N_{P}^{\ast}$ rather than $\widetilde{P} = \widetilde{M}_{P} \widetilde{N}_{P}$.

Next let
$$\text{GL}_{r}^{(2)} = \{ g \in \text{GL}_{r} : \det g \in (\widehat{F}^\times)^{2} \},$$
and $\widetilde{\text{GL}}_{r}^{(2)}$ its metaplectic preimage. Also we define
$$M_{P}^{(2)} = \{ (g_{1}, \ldots, g_{k}) \in M_{P} : \det g_{i} \in (\widehat{F}^\times)^{2} \}$$
and often denote its preimage by $\widetilde{M}_{P}^{(2)} = \widetilde{\text{GL}}_{r_{1}}^{(2)} \times \cdots \times \widetilde{\text{GL}}_{r_{k}}^{(2)}$. We write $P^{(2)} = M_{P}^{(2)} N_{P}$ and denote its preimage by $\widetilde{P}^{(2)}$. Then we have
$$\widetilde{P}^{(2)} = \widetilde{M}_{P}^{(2)} N_{P}^{\ast}.$$
is the center of $\widetilde{\text{GL}}_r$, where $I_r$ is the identity matrix. From (1.1), one can compute

$$\sigma_r(a_1I_r,a_2I_r) = \prod_{1 \leq i < j \leq r} (a_1, a_2)_F = (a_1, a_2)_{F}^{\frac{1}{r}(r-1)}.$$ 

Hence for either $r = 2q$ or $r = 2q + 1$, $\bar{Z}$ is isomorphic to $F^\times$ if $q$ is odd, and isomorphic to $\widetilde{\text{GL}}_1$ if $q$ is even. Also note that for $r = 2q$ we have $\bar{Z} \subset \text{GL}_r^{(2)}$ and it is the center of $\text{GL}_r^{(2)}$.

Let $\pi$ be an admissible representation of a subgroup $\bar{H} \subseteq \widetilde{\text{GL}}_r$. We say $\pi$ is “genuine” if each element $(1, \xi) \in \bar{H}$ acts as multiplication by $\xi$, so if $\pi$ is genuine, it does not descend to a representation of $H$ via the canonical projection $\bar{H} \to H$. On the other hand, if $\pi$ is a representation of $H$, one can always view it as a (non-genuine) representation of $\bar{H}$ by pulling back $\pi$ via the canonical projection $\bar{H} \to H$, which we denote by the same symbol $\pi$. In particular, for a parabolic subgroup $P$, we view the modular character $\delta_P$ as a character on $\bar{F}$ in this way.

1.2. The global metaplectic double cover $\widetilde{\text{GL}}_r$. In this subsection we consider the global metaplectic group. So we let $F$ be a number field and $\mathbb{A}$ the ring of adeles. We shall define the 2-fold metaplectic cover $\widetilde{\text{GL}}_r(\mathbb{A})$ of $\text{GL}_r(\mathbb{A})$. (Just like the local case, we write $\widetilde{\text{GL}}_r(\mathbb{A})$ even though it is not the adelic points of an algebraic group.) The construction of $\widetilde{\text{GL}}_r(\mathbb{A})$ has been done in various places such as [KP, FK].

First define the adelic 2-cocycle $\tau_r$ by

$$\tau_r(g_1, g_2) := \prod_v \tau_{r,v}(g_{1,v}, g_{2,v}),$$

for $g_1, g_2 \in \text{GL}_r(\mathbb{A})$, where $\tau_{r,v}$ is the local cocycle defined in the previous subsection and $g_{i,v}$ is the $v$-component of $g_i$ as usual. By definition of $\tau_{r,v}$, we have $\tau_{r,v}(g_{1,v}, g_{2,v}) = 1$ for almost all $v$, and hence the product is well-defined.

We define $\widetilde{\text{GL}}_r(\mathbb{A})$ to be the group whose underlying set is $\text{GL}_r(\mathbb{A}) \times \{\pm 1\}$ and the group structure is defined as in the local case, i.e.

$$(g_1, \xi_1) \cdot (g_2, \xi_2) = (g_1 g_2, \tau_r(g_1, g_2) \xi_1 \xi_2),$$

for $g_i \in \text{GL}_r(\mathbb{A})$, and $\xi_i \in \{\pm 1\}$. Just as the local case, we have

$$\begin{array}{c}
0 \to \{\pm 1\} \xrightarrow{\psi} \widetilde{\text{GL}}_r(\mathbb{A}) \xrightarrow{p_r} \text{GL}_r(\mathbb{A}) \to \{\pm 1\},
\end{array}$$

where we call $p_r$ the canonical projection. Define a set theoretic section $\kappa : \text{GL}_r(\mathbb{A}) \to \widetilde{\text{GL}}_r(\mathbb{A})$ by $g \mapsto (g, 1)$.

It is well-known that $\text{GL}_r(F)$ splits in $\widetilde{\text{GL}}_r(\mathbb{A})$. However the splitting is not via $\kappa$. In what follows, we will write the splitting $\text{GL}_r(F) \to \widetilde{\text{GL}}_r(\mathbb{A})$ explicitly.

Let us start with

**Proposition 1.7.** For $g_1, g_2 \in \text{GL}_r(F)$, we have $\sigma_{r,v}(g_1, g_2) = 1$ for almost all $v$, and further

$$\prod_v \sigma_{r,v}(g_1, g_2) = 1.$$ 

**Proof.** From the explicit description of the cocycle $\sigma_{r,v}(g_1, g_2)$ given at the end of §4 of [BLS], one can see that $\sigma_{r,v}(g_1, g_2)$ is written as a product of Hilbert symbols of the form $(t_1, t_2)_{F_v}$ for $t_i \in F_v^\times$. This proves the first part of the proposition. The second part follows from the product formula for the global Hilbert symbol. \qed

This “product formula” of the block-compatible 2-cocycle implies
Proposition 1.8. If \( g \in \GL_r(F) \), then for almost all \( v \), we have \( s_{r,v}(g) = 1 \), where \( s_{r,v} \) is the map \( s_{r,v} : \GL(F_v) \to \{ \pm 1 \} \) defining the section \( s : \GL(F_v) \to \GL_r(F_v) \).

Proof. By the Bruhat decomposition we have \( g = bwb' \) for some \( b, b' \in B(F) \) and \( w \in W \). Then for each place \( v \)

\[
s_{r,v}(g) = s_{r,v}(bw) s_{r,v}(w) s_{r,v}(b) / \tau_{r,v}(w, b')
\]

for all \( v \). By [BLS, Theorem 7(f)], and (1.2) we know that \( s_{r,v}(b) = s_{r,v}(b') = 1 \) for almost all \( v \). Finally by definition of \( \tau_{r,v} \), \( \tau_{r,v}(w, b') = \tau_{r,v}(w, b') = 1 \) for almost all \( v \).

This proposition implies that the expression

\[
s_r(g) := \prod_v s_{r,v}(g)
\]

makes sense for all \( g \in \GL_r(F) \), and one can define the map

\[
s : \GL_r(F) \to \GL_r(A), \quad g \mapsto (g, s_r(g)^{-1})\]

Moreover, unfortunately, however, the expression \( \prod_v s_{r,v}(g_v) \) does not make sense for every \( g = \prod_v g_v \in \GL_r(A) \) because one does not know whether \( s_{r,v}(g_v) = 1 \) for almost all \( v \). But whenever the product \( \prod_v s_{r,v}(g_v) \) makes sense we denote the element \( (g, \prod_v s_{r,v}(g_v)^{-1}) \) by \( s(g) \). This defines a partial global section \( s : \GL_r(A) \to \GL_r(A) \). For example, if \( g \in B(A) \), \( s(g) \) is defined thanks to (1.4). (See the last paragraph of [BG, p.150] as well.) Also we know that \( s \) splits \( N_B(A) \) thanks to [BLS, Theorem 7(f), §3].

Analogously to the local case, if the partial global section \( s \) is defined on a subgroup \( H \subseteq \GL_r(A) \) and \( s|_H \) is a homomorphism, we denote the image \( s(H) \) by \( H^* \) or simply by \( H \) when there is no danger of confusion. This applies to, for example, \( H = \GL_r(F) \) or \( N_B(A) \). But let us emphasize that we reserve this notation only for the subgroup split by \( s \).

Moreover we have

Lemma 1.9. For \( g \in \GL_r(F) \) and \( n \in N_B(A) \), both \( s(gn) \) and \( s(n) \) are defined and moreover \( s(gn) = s(g)s(n) \) and \( s(n) = s(n)s(g) \).

Proof. To show \( s(gn) \) is defined, it suffices to show \( s_r(gn) \) is defined. We know both \( s_r(g) \) and \( s_r(n) \) are defined. Moreover for all places \( v \), we have \( s_{r,v}(g_v, n_v) = 1 \) by [BLS, Theorem 7(f), §3]. Hence for all \( v \), \( s_{r,v}(g n_v) = s_{r,v}(g) s_{r,v}(n_v) / \tau_{r,v}(g, n_v) \). For almost all \( v \), the right hand side is 1. Hence the global \( s_r(gn) \) is defined. Also this equality shows that \( s(gn) = s(g)s(n) \). The same argument works for \( ng \).
via the projection $\tilde{H}(\mathbb{A}) \to H(\mathbb{A})$. On the other hand, any representation of $H(\mathbb{A})$ is viewed as a representation of $\tilde{H}(\mathbb{A})$ by pulling it back by $\rho_r$, which we also denote by $\pi$. In particular, this applies to the modular character $\delta_F$ for each parabolic $P(\mathbb{A})$.

We can also describe $\tilde{\mathrm{GL}}_r(\mathbb{A})$ as a quotient of a restricted direct product of the groups $\tilde{\mathrm{GL}}_r(F_v)$ as follows. Consider the restricted direct product $\prod_v \tilde{\mathrm{GL}}_r(F_v)$ with respect to the groups $\kappa(K_v) = \kappa(\tilde{\mathrm{GL}}_r(O_{F_v}))$ for all $v$ with $v \nmid 2$ and $v \nmid \infty$. If we denote each element in this restricted direct product by $\Pi_v(g_v, \xi_v)$ so that $g_v \in K_v$ and $\xi_v = 1$ for almost all $v$, we have the surjection

\[ \rho : \prod_v \tilde{\mathrm{GL}}_r(F_v) \to \tilde{\mathrm{GL}}_r(\mathbb{A}), \quad \Pi_v(g_v, \xi_v) \mapsto (\Pi_v g_v, \Pi_v \xi_v). \]

This is a group homomorphism by our definition of $\tilde{\mathrm{GL}}_r(F_v)$ and $\tilde{\mathrm{GL}}_r(\mathbb{A})$. Of course

\[ \prod_v \tilde{\mathrm{GL}}_r(F_v)/\ker \rho \cong \tilde{\mathrm{GL}}_r(\mathbb{A}), \]

where $\ker \rho$ consists of the elements of the form $\Pi_v(1, \xi_v)$ with $\xi_v = -1$ at an even number of $v$.

Suppose we are given a collection of irreducible admissible representations $\pi_v$ of $\tilde{\mathrm{GL}}_r(F_v)$ such that $\pi_v$ is $\kappa(K_v)$-spherical for almost all $v$. Then we can form an irreducible admissible representation of $\prod_v \tilde{\mathrm{GL}}_r(F_v)$ by taking a restricted tensor product $\otimes'_v \pi_v$ as usual. Suppose further that $\ker \rho$ acts trivially on $\otimes'_v \pi_v$, which is always the case if each $\pi_v$ is genuine. Then it descends to an irreducible admissible representation of $\tilde{\mathrm{GL}}_r(\mathbb{A})$, which we denote by $\cong \otimes'_v \pi_v$, and call it the “metaplectic restricted tensor product”. Let us emphasize that the space for $\cong \otimes'_v \pi_v$ is the same as that for $\otimes'_v \pi_v$. Conversely, if $\pi$ is an irreducible admissible representation of $\tilde{\mathrm{GL}}_r(\mathbb{A})$, it is written as $\cong \otimes'_v \pi_v$ where $\pi_v$ is an irreducible admissible representation of $\tilde{\mathrm{GL}}_r(F_v)$, and for almost all $v$, $\pi_v$ is $\kappa(K_v)$-spherical. (To see it, view $\pi$ as a representation of the restricted product $\prod_v \tilde{\mathrm{GL}}_r(F_v)$ by pulling it back by $\rho$ and apply the usual tensor product theorem for the restricted product, which gives $\otimes'_v \pi_v$, and it descends to $\cong \otimes'_v \pi_v$.) Note that though the restricted tensor product (metaplectic or not) is far from canonical, each local component $\pi_v$ is uniquely determined up to equivalence.

2. The exceptional representations of $\tilde{\mathrm{GL}}_r$

In this section, we first review the theory of the (non-twisted) exceptional representation of $\tilde{\mathrm{GL}}_r$ of Kazhdan-Patterson ([KP]), and after that we construct the twisted version of it. Throughout the section we write

\[ r = \begin{cases} 2q \\ 2q + 1 \end{cases} \]

depending on the parity of $r$.

2.1. The non-twisted exceptional representation of $\tilde{\mathrm{GL}}_r$. Let us consider the non-twisted exceptional representation of $\tilde{\mathrm{GL}}_r$ developed by Kazhdan and Patterson in [KP]. We treat both the $r = 2q$ and $2q + 1$ cases at the same time. Also in this subsection, all the groups are over the local field $F$ (non-archimedean or archimedean) or the adeles $\mathbb{A}$, and most of the time we consider the local and global case at the same time.

Roughly speaking, this exceptional representation is the Langlands quotient of a certain induced representation of $\tilde{\mathrm{GL}}_r$ induced from the metaplectic preimage $\tilde{B}$ of the Borel subgroup $B$, which we will define now.
First for the maximal torus $T \subseteq B$, we let
\[ T^e = \left\{ \left( \begin{array}{cc}
t_1 & \\
& \ddots \\
& & t_r \end{array} \right) \in T : t_1t_2^{-1}, t_3t_4^{-1}, \ldots, t_{2q-1}t_{2q}^{-1} \text{ are squares} \right\}. \]

The metaplectic preimage $\widetilde{T}^e$ of $T^e$ is a maximal abelian subgroup of $\widetilde{T}$. Also we denote $T^eN_B$ by $B^e$.

To define the exceptional representation, we need to recall the notion of the Weil index attached to each (local or global) additive character $\psi$, which was first defined by Weil in his important paper ([We]). A good reference (for the local case) is [R, Appendix]. First consider the local case. For the additive character $\psi$ on $F$, the map $F \to \mathbb{C}^\times$ defined by $x \mapsto \psi(x^2)$ is what Weil called a character of second degree. Weil attached to any character of second degree $f$ an eight root of unity $\gamma(f)$, which is called the Weil index of $f$. In particular, we denote by $\gamma(\psi)$ the Weil index of $x \mapsto \psi(x^2)$, which we call the Weil index of $\psi$. Of course, we can also define $\gamma(\psi_a)$ for each $a \in F$ analogously. We let
\[ \mu_\psi(a) := \frac{\gamma(\psi_a)}{\gamma(\psi)}. \]

Various properties of $\mu_\psi$ as well as those of $\gamma(\psi)$ are reviewed in [R, Appendix]. In particular, one has
\[ \mu_\psi(ab) = \mu_\psi(a)\mu_\psi(b)(a, b)_F. \]

This property implies that the map $\widetilde{F}^\times \to \mathbb{C}^\times$ defined by $(a, \xi) \mapsto \xi\mu_\psi(a)$ is a homomorphism. Let us also mention that
\[ \mu_\psi(a) = \mu_\psi_0 \text{ if and only if } a \equiv b \mod (F^\times)^2. \]

Next assume $F$ is global and $\psi$ is an additive character on $\mathbb{A}$. We define $\mu_\psi := \prod_v \mu_\psi_v$. By [R, Proposition A.11], $\mu_\psi_v = 1$ on $\mathcal{O}_F^\times$ for almost all $v$, and hence the product is well-defined. As in the local case $\mu_\psi$ defines a character on $\mathbb{A}^\times$.

The non-twisted exceptional representation of $\tilde{GL}_r$ is the unique irreducible quotient of the induced representation $\text{Ind}_{\widetilde{T},N_B}^{\tilde{GL}_r} \omega_\chi \otimes \delta_B^{1/4}$, where $\omega_\chi$ is the character on $\tilde{T}$ defined as follows: Let $\chi$ be a unitary character of $F^\times$ if $F$ is local, and a unitary Hecke character of $\mathbb{A}^\times$ if $F$ is global. Define a character $\omega_\chi$ on $\tilde{T}$ by
\[ \omega_\chi((1, \xi)s(t)) = \xi\chi(\det t)\mu_\psi(t_1)\mu_\psi(t_3)\mu_\psi(t_5) \cdots \mu_\psi(t_{2q-1}). \]

Here even when $F$ is global, the section $s$ is defined on $T(\mathbb{A})$ and the expression $s(t)$ makes sense. Note that if $t = \text{diag}(t_1, t') = \text{diag}(t''_1) \in T^e$, then one can see from (1.1) together with basic properties of the Hilbert symbol that
\[ \sigma_r(t, t') = (t_1t'_1)(t_3t'_3)(t_5t'_5) \cdots (t_{2q-1}t'_{2q-1}). \]

Then (2.1) implies that $\omega_\chi$ is indeed a character on $\tilde{T}^e$.

It is shown in [KP] that

**Proposition 2.4.** The induced representation $\text{Ind}_{\widetilde{T},N_B}^{\tilde{GL}_r} \omega_\chi \otimes \delta_B^{1/4}$ has a unique irreducible quotient, which we denote by $\theta_\chi^e$. For the local case, it is the image of the intertwining integral
\[ \text{Ind}_{\tilde{T},N_B}^{\tilde{GL}_r} \omega_\chi \otimes \delta_B^{1/4} \to \text{Ind}_{(\omega_0 T),N_B}^{\tilde{GL}_r} \omega_0(\omega_\psi \otimes \delta_B^{1/4}), \]

where $w_0$ is the longest Weyl group element. (See the notation section for the notations for the superscript $w_0$. Also note that $w_0$ is actually $s(w_0)$ when viewed as an element in $\tilde{GL}_r$. ) For the
global case, it is generated by the residues of the Eisenstein series for this induced space, and \( \theta_\chi^\psi \) is a square integrable automorphic representation of \( \tilde{\text{GL}}_r(A) \). Moreover for the global \( \theta_\chi^\psi \), one has the decomposition
\[
\theta_\chi^\psi = \tilde{\otimes}_v \theta_\chi^\psi_v.
\]

We call the representation \( \theta_\chi^\psi \) the non-twisted exceptional representation of \( \tilde{\text{GL}}_r \) with the determinantal character \( \chi \).

Remark 2.5. Assume \( F \) is local. Define
\[
\Omega^\psi_\chi := \text{Ind}_{T^F \rightarrow \tilde{T}}^\tilde{\text{GL}}_r \omega^\psi_\chi.
\]
This is irreducible (\cite{KP}, p.55). Also if \( r \) is even, this is independent of \( \psi \). This is because each element in \( T^F \setminus \tilde{T} \) is represented by \( s(a_1, \ldots, a_{2q}) \) with \( a_i \in (F^\times)^{2q} \setminus F^\times \) and by direct computation one can check that the twists of \( \omega^\psi_\chi \) by \( s(a_1, \ldots, a_{2q}) \) are all distinct by using (2.2).

By inducing in stages, one can see that
\[
\text{Ind}_{F^N_B}^{\tilde{\text{GL}}_r} \omega^\psi_\chi \otimes \delta^{1/4}_B = \text{Ind}_{B}^{\tilde{\text{GL}}_r} \Omega^\psi_\chi \otimes \delta^{1/4}_B,
\]
which implies \( \theta^\psi_\chi \) is independent of \( \psi \) if \( r \) is even.

One of the important properties of the exceptional representation is that the constant term is again an exceptional representation, which can be called the “periodicity” of Jacquet module for the non-archimedean case and the periodicity of constant terms for the global case. Namely, locally we have

**Proposition 2.6 (Local Periodicity).** Assume \( F \) is non-archimedean. Let \( (\theta^\psi_\chi)_N_B \) be he Jacquet module of \( \theta^\psi_\chi \) along the parabolic \( \tilde{B} \). Then
\[
(\theta^\psi_\chi)_N_B = w_0(\Omega^\psi_\chi) \otimes \delta^{1/4}_B = \Omega^\psi_\chi \otimes \delta^{1/4}_B,
\]
where \( w_0 \) is the longest element in the Weyl group.

**Proof.** The first equality is \cite{KP} Theorem I.2.9(e) with the notations adjusted to ours. The second equality follows because the metaplectic tensor products behaved in the expected way under conjugation by a Weyl group element as proven in \cite{L1}. \( \square \)

Globally, we have

**Proposition 2.7 (Global Periodicity).** Assume \( F \) is a number field. Let \( (\theta^\psi_\chi)_N_B \) be the space generated by the constant terms of the automorphic forms in \( \theta^\psi_\chi \) along the Borel \( \tilde{B}(A) \). Then as a representation of \( \tilde{T}(A) \), we have
\[
(\theta^\psi_\chi)_N_B = w_0(\Omega^\psi_\chi) \otimes \delta^{1/4}_B = \Omega^\psi_\chi \otimes \delta^{1/4}_B,
\]
where \( w_0 \) is the longest element in the Weyl group.

**Proof.** This is not proven in \cite{KP}. But it can be proven by using the theory of Eisenstein series developed in \cite{MW}. We will give the detailed argument later for the twisted case, and one may simply mimic the argument there. \( \square \)

Finally let us mention that (locally or globally) if \( r = 2q + 1 \), under \( \theta^\psi_\chi \) the center \( \tilde{Z} \) acts by the character
\[
(1, \xi) \mapsto \xi_\chi(a)^{2q+1} \mu_\psi(a)^q, \quad z = \begin{pmatrix} a & \cdots & \cdots & a \\ & & & \\ & & & \end{pmatrix} \in \text{GL}_{2q+1}.
\]
As we see in Section 12, if \( q \) is odd, \( \tilde{Z} \cong \tilde{F}^\times \) or \( \tilde{\mathbb{A}}^\times \), and hence certainly \( z \mapsto \mu_\psi(a)^q \) is a character on \( \tilde{Z} \). If \( q \) is even, \( \tilde{Z} \cong \tilde{\text{GL}}_1 \) (trivial extension) but by (2.1) one can see that the map \( z \mapsto \mu_\psi(a)^q \) is also a character.

2.2. The Weil representation of \( \tilde{\text{GL}}_2 \). To construct the twisted exceptional representation of \( \tilde{\text{GL}}_r \), one needs the Weil representation of \( \tilde{\text{GL}}_2 \) both for the local and global cases. In this subsection, we review the basics of the theory of the Weil representation of \( \tilde{\text{GL}}_2 \). The definitive references for this are [G] and [GPS].

Local case:

Let us consider the local case, and hence \( F \) will be a (not necessarily non-archimedean) local field of characteristic 0. Everything stated below without any specific reference is found in [GPS] §2 for the non-archimedean case and in [G] §4 for the archimedean case. Let \( S(F) \) be the space of Schwartz-Bruhat functions on \( F \), i.e. smooth functions with compact support if \( F \) is non-archimedean, and functions with all the derivatives rapidly decreasing if \( F \) is archimedean. Let \( \rho^\psi \) denote the representation of \( \tilde{\text{SL}}_2(F) \) on \( S(F) \) such that

\[
\rho^\psi(s \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) f(x) = \gamma(\psi) \hat{f}(x),
\]

\[
\rho^\psi(s \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}) f(x) = \psi(bx^2) f(x), \quad b \in F
\]

\[
\rho^\psi(s \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}) f(x) = |a|^{1/2} \mu_\psi(a) f(ax), \quad a \in F^\times
\]

where \( \hat{f}(x) = \int f(y) \psi(2xy) dy \) with the Haar measure \( dy \) normalized in such a way that \( \hat{f}(x) = f(-x) \). Also \( \gamma(\psi) \) is the Weil index of \( \psi \), and \( \mu_\psi(a) = \gamma(\psi_a)/\gamma(\psi) \). It is well-known that \( \rho^\psi \) is reducible and written as \( \rho^\psi = \rho^\psi_+ \oplus \rho^\psi_- \), where \( \rho^\psi_+ \) (resp. \( \rho^\psi_- \)) is an irreducible representation realized in the subspace of even functions (resp. odd functions) in \( S(F) \).

Let \( a \in F^\times \). For each \( g \in \tilde{\text{SL}}_2(F) \) let us write

\[
g^a = s \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}^{-1} g s \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}.
\]

**Lemma 2.9.** Let \( \epsilon \in \{\pm\} \) be fixed. For each \( a \in F^\times \), let \( \sigma^\psi \) be the representation of \( \tilde{\text{SL}}_2(F) \) defined by \( \sigma^\psi_g = \rho^\psi_+(g^n) \) for all \( g \in \tilde{\text{SL}}_2(F) \). Then

\[
\sigma^\psi = \rho_\chi^\psi.
\]

**Proof.** This is [G] Proposition 2.27].

Let \( \chi \) be a unitary character on \( F^\times \). If \( \chi(-1) = 1 \) (resp. \( \chi(-1) = -1 \)), one can extend \( \rho^\psi_+ \) (resp. \( \rho^\psi_- \)) to a representation \( \rho^\psi_\chi \) of \( \tilde{\text{GL}}_2^{(2)}(F) \) by letting

\[
\rho^\psi_\chi(s \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix}) f(x) = \chi(a)|a|^{-1/2} f(a^{-1}x).
\]

This is indeed a well-defined irreducible representation of \( \tilde{\text{GL}}_2^{(2)}(F) \) and we call it the Weil representation of \( \tilde{\text{GL}}_2^{(2)}(F) \) associated with \( \chi \). We denote by \( S_\chi(F) \) the subspace of \( S(F) \) in which \( \rho^\psi_\chi \) is realized.
which is the space of even functions if \( \chi(-1) = 1 \) and odd functions if \( \chi(-1) = -1 \). Note that

\[
(2.10) \quad r^\psi_\chi(s \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix})f(x) = \chi(a)\mu_\psi(a)f(x).
\]

Lemma 2.9 implies

**Lemma 2.11.** For \( a \in F^\times \), let \( a^r r^\psi_\chi \) be the representation of \( \widetilde{GL}_2^{(2)}(F) \) obtained by conjugating \( r^\psi_\chi \) by \( s \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \). Then

\[
a^r r^\psi_\chi = r^\psi_{a^r \chi}.
\]

Also note

**Lemma 2.12.** \( r^\psi_{a^r \chi} \) and \( r^{\psi b}_\chi \) are equivalent if and only if \( a \equiv b \mod (F^\times)^2 \).

**Proof.** See [GPS (1.3)]. \( \square \)

The Weil representation \( r_\chi \) of \( \widetilde{GL}_2(F) \) is defined by

\[
r_\chi = \text{Ind}_{\tilde{GL}_2(F)}^{\tilde{GL}_2(2)(F)} r^\psi_\chi.
\]

Then \( r_\chi \) is irreducible and independent of the choice of \( \psi \), and hence our notation. By Mackey theory together with \( a^r r^\psi_\chi = r^\psi_{a^r \chi} \), we have

\[
(2.13) \quad r_\chi|_{\tilde{GL}_2(2)(F)} = \bigoplus_{\alpha \in \Sigma} r^\psi_\chi,
\]

where \( \Sigma \) is a set of representatives of \( (F^\times)^2 \backslash F^\times \), because \( \tilde{GL}_2(2)(F) \backslash \tilde{GL}_2(F) = \Sigma \).

If \( \chi(-1) = 1 \), one can check that \( r_\chi \) is the exceptional representation of Kazhdan-Patterson for \( r = 2 \) with the determinantal character \( \chi^{1/2} \). (See [GPS Proposition 2.3.3] for the non-archimedean case, and [GPS §6] for the archimedean case.) Namely, we have the embedding

\[
(2.14) \quad r_\chi \hookrightarrow \text{Ind}_{B}^{\tilde{GL}_2} s(\Omega_{\chi^{1/2}} \otimes \delta_B^{1/4}) = \text{Ind}_{(\tilde{T}_e)N_F}^{\tilde{GL}_2} s(\omega_{\chi^{1/2}} \otimes \delta_B^{1/4}),
\]

where \( s \) is the Weyl group element \( s = (1 \ 1) \). Similarly we have the embedding

\[
(2.15) \quad r^\psi_\chi \hookrightarrow \text{Ind}_{(\tilde{T}_e)N_F}^{\tilde{GL}_2} s(\omega_{\chi^{1/2}} \otimes \delta_B^{1/4}).
\]

Let us mention that one can choose any \( \chi^{1/2} \) because in general for any quadratic character \( \epsilon \) and character \( \eta \), we have \( \omega_{\eta}^{\psi} = \omega_{\eta}^\psi \) for a character of \( \tilde{T}_r \subset \tilde{GL}_r \) as long as \( r \) is even.

If \( \chi(-1) = -1 \), then \( r_\chi \) is supercuspidal for the non-archimedean case ([GPS Proposition 3.3.3]), is a discrete series representation of lowest weight \( 3/2 \) for the real case ([GPS §6]) and is identified with a certain induced representation for the complex case ([GPS §6]).

**Proposition 2.16.** The Weil representation \( r^\psi_\chi \) of \( \tilde{GL}_2^{(2)}(F) \) is \( \psi_{\mu} \)-generic if and only if \( a = b^2 \). Also in this case, the \( \psi_{\mu_2} \)-Whittaker functional on \( S_\chi(F) \) is (a scalar multiple of) the functional given by \( f \mapsto f(b) \).

**Proof.** This seems to be folkloric, though the author does not know any reference for it. So we will give a brief proof here. First of all, since \( r^\psi_\chi \) is extended from the representation \( r^\psi_\chi \) of \( SL_2(F) \), it suffices to show the corresponding statement for \( r^\psi_\chi \). From the explicit description of the action of \( SL_2(F) \), it is immediate that the functional given by \( f \mapsto f(b) \) is a \( \psi_{\mu_2} \)-Whittaker functional. This shows one direction.
The non-obvious part is the converse. One way to prove this is to invoke the theory of Waldspurger developed in [W1, W2], according to which an irreducible admissible representation \( \pi \) of \( \text{SL}_2(F) \) has a non-zero theta lift with respect to \( \psi_a \) to \( \text{PGL}_2(F) \) if and only if \( \pi \) has a \( \psi_a \)-Whittaker functional. But from the explicit theta correspondences obtained in [W2, Theorem 1], one can see that this is possible only when \( r^\psi_a \) is isomorphic to \( r^\psi_2 \), which implies \( a \in (F^\times)^2 \). (Apparently to use the theory of Waldspurger is overkill and too indirect. One can directly prove it by using a theory of distributions. But in the interest of space, we only give this indirect proof here.)

This proposition together with [2,13] implies that the Weil representation \( r_\chi \) of \( \widetilde{\text{GL}}_2(F) \) is \( \psi_a \)-generic for any \( a \).

Global case:

Next we consider the global Weil representation. So we let \( F \) be a number field, \( \mathbb{A} \) the ring of adeles and \( \chi \) a unitary Hecke character on \( \mathbb{A}^\times \). We define the global Weil representation \( r_\chi \) of \( \text{GL}_2(\mathbb{A}) \) as the restricted tensor product of the local Weil representations, i.e.

\[
r_\chi = \mathcal{G}(\mathbb{A}) .
\]

It is shown in [GPS, §8] that \( r_\chi \) is a square integrable automorphic representation of \( \widetilde{\text{GL}}_2(\mathbb{A}) \), and moreover it is cuspidal if and only if \( \chi^{1/2} \) does not exist. Also one can see that if \( \chi^{1/2} \) exists, then just like the local case, \( r_\chi \) is the exceptional representation of Kazhdan-Patterson for \( r = 2 \), namely \( r_\chi = \theta_{\chi^{1/2}} \). (Again as in the local case, it is independent of the choice of \( \chi^{1/2} \).)

We also define the global Weil representation \( r^\psi_\chi \) of \( \widetilde{\text{GL}}_2(\mathbb{A}) \) by

\[
r^\psi_\chi = \mathcal{G}(\mathbb{A}) .
\]

Then \( r^\psi_\chi \) can be realized in the subspace \( S_\chi(\mathbb{A}) = \otimes' S_\chi(F_v) \) of the space \( S(\mathbb{A}) \) of Schwartz-Bruhat functions on \( \mathbb{A} \) with the action given by the same formulas as the local case.

The two representations \( r_\chi \) and \( r^\psi_\chi \) are related by

**Proposition 2.17.** Let \( r^{(2)}_\chi \) be the representation of \( \widetilde{\text{GL}}_2(\mathbb{A}) \) whose space is \( \{ f |_{\widetilde{\text{GL}}_2(\mathbb{A})} : f \in r_\chi \} \), namely the space of restrictions to \( \widetilde{\text{GL}}_2(\mathbb{A}) \) of automorphic forms in \( r_\chi \). Then as a representation of \( \widetilde{\text{GL}}_2(\mathbb{A}) \), we have

\[
r^{(2)}_\chi = \bigoplus_{a \in (F^\times)^2 \setminus F^\times} r^\psi_{\chi a}.
\]

**Proof.** As in the proof of [GPS, Proposition 8.1.1], a typical element in the space of \( r_\chi \) is written as \( \Phi = (\Phi_a)_{a \in \Sigma} \), where the indexing set is \( \Sigma = (\mathbb{A}^\times)^2 \setminus \mathbb{A}^\times \), and each \( \Phi_a \) is in \( S(\mathbb{A}) \), on which \( \widetilde{\text{GL}}_2(\mathbb{A}) \) acts as \( \otimes' \mathcal{G}(\mathbb{A}) \). Then the function \( \varphi_\Phi \) on \( \widetilde{\text{GL}}_2(\mathbb{A}) \) defined by

\[
\varphi_\Phi(g) = \sum_{a \in (F^\times)^2 \setminus F^\times} \sum_{\xi \in \mathbb{F}} (r_\chi(g)\Phi)_a(\xi)
\]

gives an automorphic realization of \( r_\chi \). Here note that the natural map \( (F^\times)^2 \setminus F^\times \to (\mathbb{A}^\times)^2 \setminus \mathbb{A}^\times \) is an injection by the Hasse-Minkowski theorem.

Notice that the representation \( \otimes' \mathcal{G}(\mathbb{A}) \) is an automorphic representation of \( \widetilde{\text{GL}}_2(\mathbb{A}) \) if and only if \( a \in (F^\times)^2 \setminus F^\times \). Then one can see that for \( a \in (F^\times)^2 \setminus F^\times \), the function on \( \widetilde{\text{GL}}_2(\mathbb{A}) \) given by

\[
g \mapsto \sum_{\xi \in \mathbb{F}} (r_\chi(g)\Phi)_a(\xi)
\]

is an automorphic form on \( \widetilde{\text{GL}}_2(\mathbb{A}) \), which is in the space of \( r^\psi_{\chi a} \). Hence \( r^\psi_{\chi a} \)
is a constituent of $r_\chi^{(2)}$. Since we know $r_\chi$ is square integrable and so $r_\chi^{(2)}$ is in the space of square integrable automorphic forms on $\tilde{\text{GL}}_2^{(2)}(\mathbb{A})$, it is completely reducible. The proposition follows. □

The following is the global analogue of Proposition 2.16

**Proposition 2.18.** The Weil representation $r_\psi\chi$ of $\tilde{\text{GL}}_2^{(2)}(\mathbb{A})$ is $\psi_a$-generic if and only if $a = b^2$ for $b \in F^\times$.

**Proof.** This is implied by the local case, or one may directly compute the $\psi_a$-Whittaker coefficient for the automorphic realization $\sum_{\xi \in F}(r_\chi(g)\Phi)(\xi)$ of $r_\psi\chi$ as in the proof of the above proposition. □

2.3. The Weil representation of $\tilde{M}_P$. In this subsection, we assume $r = 2q$ and $P$ is the $(2, \ldots, 2)$-parabolic, so that $M_P = \text{GL}_2 \times \cdots \times \text{GL}_2$. Recall from Section 1.2 that we write $\tilde{\text{M}}_P = \tilde{\text{GL}}_2 \times \cdots \times \tilde{\text{GL}}_2$ and $\tilde{M}_P^{(2)} = \tilde{\text{GL}}_2^{(2)} \times \cdots \times \tilde{\text{GL}}_2^{(2)}$. Let $R = F$ if $F$ is local and $R = \mathbb{A}$ if $F$ is global. Then we let $$(M_P)^{(2)} := M_P \cap \text{GL}^{(2)}_{2q} = \{(g_1, \ldots, g_q) \in M_P : \prod \det(g_i) \in (R^\times)^2\}.$$ We let $(\tilde{M}_P)^{(2)}$ be the metaplectic preimage of $(M_P)^{(2)}$. Let us note the inclusions $$\tilde{M}_P^{(2)} \subseteq (\tilde{M}_P)^{(2)} \subseteq \tilde{M}_P.$$ Also note that $\tilde{M}_P^{(2)} \subseteq \tilde{M}_P$. Then we have $$\tilde{M}_P^{(2)}(R) \setminus \tilde{M}_P(R) = (R^\times)^2 \setminus R^\times \times \cdots \times (R^\times)^2 \setminus R^\times$$ \[q \text{ times}\] and $$\tilde{M}_P^{(2)}(R) \setminus \tilde{M}_P(R) = (R^\times)^2 \setminus R^\times.$$ In this subsection, we extend the theory of the Weil representation as discussed in the previous subsection to the groups $\tilde{M}_P^{(2)}$, $(\tilde{M}_P)^{(2)}$, and $\tilde{M}_P$. Naively, the Weil representations of those groups are simply the tensor products of $q$ copies of the Weil representation for $\text{GL}_2$ or $\text{GL}_2^{(2)}$.

To construct a representation of $\tilde{M}$ or $\tilde{M}_P^{(2)}$ out of representations of $\text{GL}_2$, it is convenient to consider the groups $\tilde{M}_P$ and $\tilde{M}_P^{(2)}$ constructed by the block-compatible cocycle $\tau_P$ in Appendix [A]. Since in this subsection we often use the results and notations from Appendix [A] the reader is advised to read Appendix [A] before moving on.

**Local case:**

Let us consider the local case, so $F$ is a local field and $\chi$ is a unitary character on $F^\times$. We would like to work with $q$ different additive characters. For this purpose, we let $$\bar{a} = (a_1, \ldots, a_q) \in F^\times \times \cdots \times F^\times$$ be a $q$-tuple of elements of $F^\times$. 
For each $i \in \{1, \ldots, q\}$, let $r^{\psi_{a_i}}_\chi$ be the Weil representation of $\widetilde{GL}_2^{(2)}$. We define the Weil representation $\pi^{\psi_a}_\chi$ of $\widetilde{cMp}^{(2)}$ with respect to $\chi, \psi$ and $\bar{a}$ by the metaplectic tensor product

$$\pi^{\psi_a}_\chi := r^{\psi_{a_1}}_\chi \otimes \cdots \otimes r^{\psi_{a_q}}_\chi.$$  

In particular, the space of $\pi^{\psi_a}_\chi$ is the usual tensor product of the spaces of $r^{\psi_{a_i}}_\chi$, i.e. $S_\chi(F) \otimes \cdots \otimes S_\chi(F)$ (q-times). If $\bar{a} = (1, \ldots, 1)$, we simply write $\pi^\psi_\chi$ for $\pi^{\psi_a}_\chi$.

**Lemma 2.19.** Let $\bar{a} = (a_1, \ldots, a_q), \bar{b} = (b_1, \ldots, b_q) \in F^\times \times \cdots \times F^\times$. Then $\pi^{\psi_a}_\chi \cong \pi^{\psi_b}_\chi$ if and only if $a_i \equiv b_i \mod (F^\times)^2$ for each $i$.

**Proof.** The if-part follows from Lemma 2.12. For the converse, recall from Appendix A that the metaplectic tensor product is defined in terms of the tensor product representation $r^{\psi_{a_1}}_\chi \otimes \cdots \otimes r^{\psi_{a_q}}_\chi$ of the group $GL_2^{(2)} \times \cdots \times GL_2^{(2)}$ (direct product). But if $r^{\psi_{a_1}}_\chi \otimes \cdots \otimes r^{\psi_{a_q}}_\chi \cong r^{\psi_{a_1}}_\chi \otimes \cdots \otimes r^{\psi_{a_q}}_\chi$, then $r^{\psi_{a_i}}_\chi \cong r^{\psi_{b_i}}_\chi$ for each $i$, which implies $a_i \equiv b_i \mod (F^\times)^2$ by Lemma 2.12.

Let $\pi$ be a representation of $\widetilde{cMp}^{(2)}$. For each $m \in \widetilde{cMp}$, recall from the notation section that $m_\pi$ is the representation of $\widetilde{cMp}^{(2)}$ twisted by $m$. The set of the elements of the form

$$m = ((\begin{pmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_q \end{pmatrix}), \ldots, \begin{pmatrix} 1 \\ a_q \end{pmatrix} \bar{1}) \in \widetilde{cMp},$$

where each $a_i$ is chosen modulo $(F^\times)^2$, is a complete set of the representatives of $\widetilde{cMp}^{(2)} \setminus \widetilde{cMp}$. For each such $m$, we have

$$m r^{\psi}_\chi = r^{\psi_{a_1}}_\chi \otimes \cdots \otimes r^{\psi_{a_q}}_\chi$$

because for each $i$ we have $a_i r^{\psi}_\chi = r^{\psi_{a_i}}_\chi$ by Lemma 2.11. By Lemma 2.19, $m r^{\psi}_\chi \cong r^{\psi}_\chi$ if and only if $m \in \widetilde{cMp}^{(2)}$. Thus Mackey’s irreducibility criterion is satisfied and hence the induced representation

$$\Pi_\chi := \text{Ind}_{\widetilde{cMp}^{(2)}}^{\widetilde{cMp}} \pi^{\psi_a}_\chi$$

is irreducible. This is independent of the choice of $\bar{a}$ and $\psi$. Indeed, this is the metaplectic tensor product of $q$ copies of $r^{\psi}_\chi$ in the sense of [Mc].

For our purposes, we would like to consider the representation

$$\omega^{\psi_a}_\chi := \text{Ind}_{\widetilde{cMp}^{(2)}}^{\widetilde{cMp}^{(2)}} \pi^{\psi_a}_\chi,$$

where $\widetilde{(cMp)}^{(2)}$ is the subgroup of $\widetilde{cMp}$ whose underlying set is $(cMp)^{(2)} \times \{\pm 1\}$ and the group law is defined via the block-compatible cocycle $\tau_p$ as defined in Appendix A. This induced representation is irreducible because $\Pi_\chi$ is, but is dependent on $\psi$ and $\bar{a}$. Also note that by inducing in states, we have

$$\Pi_\chi = \text{Ind}_{\widetilde{cMp}^{(2)}}^{\widetilde{cMp}} \pi^{\psi_a}_\chi = \text{Ind}_{\widetilde{cMp}^{(2)}}^{\widetilde{cMp}^{(2)}} \omega^{\psi_a}_\chi.$$

Now the set

$$\{(((\begin{pmatrix} 1 \\ a \end{pmatrix}), \begin{pmatrix} 1 \\ a \end{pmatrix}), \ldots, \begin{pmatrix} 1 \\ 1 \end{pmatrix})\} \in \widetilde{cMp} : a \in (F^\times)^2 \setminus F^\times \}$$

is a complete set of the representatives of $\widetilde{(cMp)}^{(2)} \setminus \widetilde{cMp}$. For an element of the form $\bar{a} = (a, 1, \ldots, 1)$, we denote $\omega^{\psi_a}_\chi$ simply by $\omega^{\psi}_\chi$. By Mackey theory

$$\Pi_\chi|_{\widetilde{(cMp)}^{(2)}} = \bigoplus_{a \in (F^\times)^2 \setminus F^\times} \omega^{\psi_a}_\chi.$$
Also the set
\[
\{((1\ a_1), (1\ a_2), \ldots, (1\ a_q)), 1) \in (\widetilde{cM}_P^{(2)}) : a_i \in (F^\times)^2 \setminus F^\times, a_1 \cdots a_q \in (F^\times)^2\}
\]
is a complete set of the representatives of \(\widetilde{cM}_P^{(2)} \setminus (\widetilde{cM}_P^{(2)})\), and again by Mackey theory, one sees that
\[
\varpi^a \big|_{\widetilde{cM}_P^{(2)}} = \bigoplus_{(a_1, \ldots, a_q)} r_{\chi}^{a_1} \otimes r_{\chi}^{a_2} \otimes \cdots \otimes r_{\chi}^{a_q},
\]
where the sum is over the elements of the form \((a_1, \ldots, a_q) \in (F^\times)^2 \setminus F^\times \times \cdots \times (F^\times)^2 \setminus F^\times\) and \(a_1 \cdots a_q \in (F^\times)^2\).

Also from the above decomposition of \(\varpi^a \big|_{\widetilde{cM}_P^{(2)}}\), we have

**Lemma 2.20.** The induced representation \(\varpi^a\) is realized in the space \(\bigoplus_{a} S_\chi(F^q)\), where \(a = (a_1, \ldots, a_q)\) runs through the elements in \(((F^\times)^2 \setminus F^\times)^q\) with \(a_1 \cdots a_q \in (F^\times)^2\) and each summand \(S_\chi(F^q)\) realizes the representation \(r_{\chi}^{a_1} \otimes \cdots \otimes r_{\chi}^{a_q}\).

Further by the above decomposition of \(\Pi_{\chi} \big|_{\widetilde{cM}_P^{(2)}}\), we have

**Lemma 2.21.** The representation \(\Pi_{\chi}\) is realized in the space \(\bigoplus_{a \in (F^\times)^2 \setminus F^\times} \bigoplus_{a} S_\chi(F^q)\), where \(a = (a_1, \ldots, a_q)\) runs through the elements in \(((F^\times)^2 \setminus F^\times)^q\) with \(a_1 \cdots a_q \in (F^\times)^2\) and each summand \(S_\chi(F^q)\) realizes the representation \(r_{\chi}^{a_1} \otimes \cdots \otimes r_{\chi}^{a_q}\).

Finally for the local case, let us mention the genericity of \(\varpi^a\). Recall from Proposition 2.10 that the Weil representation \(r_{\chi}^1\) is \(\psi_b\)-generic if and only if \(b \equiv a\) mod \((F^\times)^2\). Hence if we define the additive character \(\psi_{(a_1, \ldots, a_q)}\) on the unipotent part \(N_B \cap M_P\) of \(M_P\) by
\[
\psi_{(a_1, \ldots, a_q)}(n) = \psi(a_1 x_1 + \cdots + a_q x_q),
\]
where
\[
n = \begin{pmatrix}
1 & x_1 \\
& \ddots \\
& & 1 \\
1 & x_q
\end{pmatrix},
\]
we have

**Lemma 2.23.** The Weil representation \(r_{\chi}^{a_1} \otimes \cdots \otimes r_{\chi}^{a_q}\) is \(\psi_{(b_1, \ldots, b_q)}\)-generic if and only if \(b_i \equiv a_i\) mod \((F^\times)^2\) for each \(i\).

Then we have

**Proposition 2.24.** The representation \(\varpi^a\) is \(\psi_{(b_1, \ldots, b_q)}\)-generic if and only if \(b_1 \cdots b_q \in (F^\times)^2\).

**Proof.** Assume \(\varpi^a\) is \(\psi_{(b_1, \ldots, b_q)}\)-generic. Then some \(r_{\chi}^{a_1} \otimes \cdots \otimes r_{\chi}^{a_q}\) in the decomposition of \(\varpi^a \big|_{\widetilde{cM}_P^{(2)}}\) is \(\psi_{(b_1, \ldots, b_q)}\)-generic. Hence by the above lemma, we have \(b_i \equiv a_i\) mod \((F^\times)^2\). But since \(a_1 \cdots a_q \in (F^\times)^2\), we also have \(b_1 \cdots b_q \in (F^\times)^2\).

Conversely assume \(b_i \cdots b_q \in (F^\times)^2\). Then in the decomposition of \(\varpi^a \big|_{\widetilde{cM}_P^{(2)}}\), there is a constituent \(r_{\chi}^{a_1} \otimes \cdots \otimes r_{\chi}^{a_q}\) which is \(\psi_{(b_1, \ldots, b_q)}\)-generic. Hence \(\varpi^a\) is \(\psi_{(b_1, \ldots, b_q)}\)-generic.
Let us mention how the Weil representation $\Pi_\chi$ is related to the non-twisted exceptional representation of Kazhdan-Patterson when $\chi^{1/2}$ exists, which we fix once and for all. Recall from (2.15) that the Weil representation $r_\chi^\psi$ embeds into the induced representation $\text{Ind}_{B_2}^{GL_2} s_\chi^\psi \otimes \delta_{B_2}^{-1/4}$, where by $B_2$ we mean the Borel subgroup of $GL_2$ with the maximal torus $T_2$ and $\omega_\chi^\psi$ is the character on the group $\widetilde{T}_2^e$ as defined in (2.3). Hence we have the embedding
\[
\pi_\chi^\psi \mapsto (\text{Ind}_{B_2}^{GL_2} s_\chi^\psi \otimes \delta_{B_2}^{-1/4}) \otimes \cdots \otimes (\text{Ind}_{B_2}^{GL_2} s_\chi^\psi \otimes \delta_{B_2}^{-1/4}).
\]

Now let $B_{2,...,2} = B_2 \times \cdots \times B_2$ (resp. $B_{2,...,2}^e = B_2^e \times \cdots \times B_2^e$) be the product of $q$ copies of $B_2$ (resp. $B_2^e$), and view them as subgroups of $M_P$. Also let $\widetilde{B}_{2,...,2}$ (resp. $\widetilde{B}_{2,...,2}^e$) be the metaplectic preimage of $B_{2,...,2}$ (resp. $B_{2,...,2}^e$) in $\widetilde{M}_P$, so in particular we assume that their group structures are given by the cocycle $\tau_P$. Also for the maximal torus $T_2 \subseteq B_2$, we have $T_2^e = T_2^{(2)}$, and for the maximal torus $T$ of $B_{2,...,2}$, which is the same as the maximal torus of $GL_{2q}$, we have $T^e = T_2^{(2)} \times \cdots \times T_2^{(2)}$ ($q$ times). We view both $\widetilde{T}^e$ and $\widetilde{T}$ as subgroups of $\widetilde{B}_{2,...,2}$, so in particular the group structures are given by the cocycle $\tau_P$. (Actually one can check that the restriction of the cocycle $\tau_P$ to $T$ is the same as $\sigma_e$.)

With those notations, one can check
\[
(\text{Ind}_{B_2}^{GL_2} s_\chi^\psi \otimes \delta_{B_2}^{-1/4}) \otimes \cdots \otimes (\text{Ind}_{B_2}^{GL_2} s_\chi^\psi \otimes \delta_{B_2}^{-1/4}) = \text{Ind}_{B_{2,...,2}}^{\widetilde{M}_P^{(2)}} w_1 \omega_\chi^\psi \otimes \delta_{B_2 \times \cdots \times B_2},
\]
where $\omega_\chi^\psi$ is the character on $\widetilde{T}_2^e$ associated with $\chi^{1/2}$ as defined by (2.3) and $w_1$ is the Weyl group element of the form
\[
w_1 = \begin{pmatrix} s \\ \vdots \\ s \end{pmatrix}
\]
where $s = (1, 1)$.

To sum up, we have the embedding
\[
(2.25) \quad \pi_\chi^\psi \mapsto \text{Ind}_{B_{2,...,2}}^{\widetilde{M}_P^{(2)}} w_1 \omega_\chi^\psi \otimes \delta_{B_2 \times \cdots \times B_2},
\]
and, by inducing both sides to $\widetilde{M}_P$, we have the embedding
\[
(2.26) \quad \Pi_\chi \mapsto \text{Ind}_{B_{2,...,2}}^{\widetilde{M}_P} w_1 \omega_\chi^\psi \otimes \delta_{B_2 \times \cdots \times B_2}.
\]

**Global case:**

Next we consider the global case, so $F$ is global, $\chi$ is a unitary Hecke character and $\psi$ is our fixed additive character on $F \setminus \mathbb{A}$. As we did in the local case, it is convenient to consider the groups $\widetilde{M}_P(\mathbb{A}), \widetilde{M}_P^{(2)}(\mathbb{A})$ and $\epsilon(\widetilde{M}_P^{(2)}(\mathbb{A})$ instead of $\widetilde{M}_P(\mathbb{A}), \widetilde{M}_P^{(2)}(\mathbb{A})$ and $(\widetilde{M}_P^{(2)}(\mathbb{A})$ for the sake of constructing metaplectic tensor products. (See Appendix A)

As we did in the local case we would like to consider the tensor product of the Weil representations with respect to possibly different additive characters. Namely, we let
\[
\tilde{a} = (a_1, \ldots, a_q) \in F^\times \times \cdots \times F^\times,
\]
and we define
\[
\pi_{\chi, \psi^{\tilde{a}}} = r_{\chi}^{\psi_{a_1}} \otimes \cdots \otimes r_{\chi}^{\psi_{a_q}}.
\]
to be the metaplectic tensor product representation of $\tilde{M}_{r}^{(2)}(A)$, where $r_{\chi}^{\psi}$ is the global Weil representation of $\tilde{GL}^{(2)}(A)$ with respect to the additive character $\psi_{a_{i}}$. As in Appendix A the space of this metaplectic tensor product is the same as that of the usual tensor product $r_{\chi}^{\psi_{a_{1}}} \otimes \cdots \otimes r_{\chi}^{\psi_{a_{q}}}$, and moreover since each $r_{\chi}^{\psi_{a_{i}}}$ is automorphic, so is the metaplectic tensor product by Proposition A.8.

Note that

$$\pi_{\chi}^{a_{i}} = \otimes_{\nu} \pi_{\chi_{\nu}}^{a_{i}},$$

where at each $\nu$ we view $a = (a_{1}, \ldots, a_{q})$ naturally as in $((F_{\nu}^{\times})^{2} \backslash F_{\nu}^{\times})^{q}$.

Next we let

$$v_{\chi}^{a_{i}} = \otimes_{\nu} v_{\chi_{\nu}}^{a_{i}}.$$  

To see its automorphy, recall from Section 2.2 that each $r_{\chi}^{\psi_{v}}$ is realized in the subspace $S_{\chi_{v}}(F_{v})$ and accordingly $r_{\chi}^{\psi}$ is realized in a subspace $S_{\chi}(A)$ of the space of Schwartz functions on $A$. Hence the representation $\pi_{\chi}^{v_{a}}$ is realized in a subspace $S_{\chi}(A^{g}) = S_{\chi}(A) \otimes \cdots \otimes S_{\chi}(A)$ of the space of Schwartz functions on $A^{g}$. (Once again, the space of the metaplectic tensor product is the same as that of the usual tensor product.) Now let

$$\Sigma_{v} = \{ \bar{a} = (a_{1}, \ldots, a_{q}) \in ((F_{v}^{\times})^{2} \backslash F_{v}^{\times})^{q} : a_{1} \cdots a_{q} \in (F_{v}^{\times})^{2} \}.$$  

From Lemma 2.20 the representation $v_{\chi}^{v_{a}}$ is realized in the space $\bigoplus_{\bar{a} \in \Sigma_{v}} S_{\chi_{v}}(F_{v}^{q})$, where each $S_{\chi_{v}}(F_{v}^{q})$ realizes the representation $r_{\chi_{v}}^{\psi_{a_{1}}} \otimes \cdots \otimes r_{\chi_{v}}^{\psi_{a_{q}}}$ for each $(a_{1}, \ldots, a_{q}) \in \Sigma_{v}$. Then as we have seen for the Weil representation of $\tilde{GL}_{2}$ in Section 2.2 the global representation $v_{\chi}^{v_{a}}$ is realized in the space of elements of the form $\Phi = (\Phi_{a})_{\bar{a} \in \Sigma_{a}}$, where the indexing set $\Sigma_{a}$ is given by

$$\Sigma_{a} = \{ \bar{a} = (a_{1}, \ldots, a_{q}) \in ((A^{\times})^{2} \backslash A^{\times})^{q} : a_{1} \cdots a_{q} \in (A^{\times})^{2} \}.$$  

Now the representation $v_{\chi}^{v_{a}}$ has an automorphic realization similarly to the Weil representation of $\tilde{GL}_{2}(A)$. Namely for each element $\Phi = (\Phi_{a})$, we put

$$\varphi_{\Phi}(g) = \sum_{\bar{a} \in \Sigma_{\Phi}} \sum_{\xi \in F} (v_{\chi}^{v_{a}}(g) \Phi_{a})(\xi),$$

where $g \in (\tilde{M}_{r}(A))^{(2)}$ and

$$\Sigma_{\Phi} = \{ \bar{a} = (a_{1}, \ldots, a_{q}) \in ((F^{\times})^{2} \backslash F^{\times})^{q} : a_{1} \cdots a_{q} \in (F^{\times})^{2} \}.$$  

Then as in [GPS] Proposition 8.1.1, one sees that the map $\Phi \mapsto \varphi_{\Phi}$ defines an embedding of $v_{\chi}^{v_{a}}$ into the space of automorphic forms on $(\tilde{M}_{r}(A))^{(2)}$.

Once we obtain this automorphic realization of $v_{\chi}^{v_{a}}$, the following global analogue of Proposition 2.24 follows just as Proposition 2.18.

**Proposition 2.27.** Let $(b_{1}, \ldots, b_{q}) \in ((F^{\times})^{2} \backslash F^{\times})^{q}$. Then $v_{\chi}^{v_{a}}$ is $\psi(b_{1}, \ldots, b_{q})$-generic if and only if $b_{1} \cdots b_{q} \in (F^{\times})^{2}$, where the additive character $\psi(b_{1}, \ldots, b_{q})$ is defined analogously to the local case.

**Proof.** One can prove it in the same way as Proposition 2.18. □

Essentially this says that many of the Whittaker-Fourier coefficients for the forms in $v_{\chi}^{v_{a}}$ vanish. This proposition will play a crucial role in our computation for unfolding of our Rankin-Selberg integral for the case $r = 2q$.

Finally, we define the global Weil representation $\Pi_{\chi}$ of $\tilde{M}_{r}(A)$ by

$$\Pi_{\chi} := \otimes_{\nu} \Pi_{\chi_{\nu}},$$
where each $\Pi_{\chi}$ is the local Weil representation of $^c\tilde{M}_P(F_v)$ as defined previously. One can prove the automorphy of $\Pi_{\chi}$ in the same way as the automorphy of $\varpi_{\chi}^v$. (Let us mention that this is precisely the metaplectic tensor product of $q$ copies of the Weil representation in the sense defined in [TI].)

Analogously to Proposition 2.17, we have

**Proposition 2.28.** Let $\Pi^{(2)}_{\chi}$ be the representation of $c(\tilde{M}_P)^{(2)}(\mathbb{A})$ whose space is $\{ f|_{c(\tilde{M}_P)^{(2)}(\mathbb{A})} : f \in \Pi_{\chi} \}$, namely the space of restrictions to $c(\tilde{M}_P)^{(2)}(\mathbb{A})$ of automorphic forms in $\Pi_{\chi}$. Then as a representation of $c(\tilde{M}_P)^{(2)}(\mathbb{A})$, we have

$$\Pi^{(2)}_{\chi} = \bigoplus_{a \in (F^\times)^2 \setminus F^\times} \varpi_{\chi}^a.$$ 

**Proof.** The proof is essentially identical to Proposition 2.17. \qed

### 2.4. The twisted exceptional representation of $\tilde{GL}_{2q}$

We construct the twisted exceptional representation of $\tilde{GL}_r$ when $r = 2q$ for both the local and global cases. But for a non-archimedean local field of odd residual characteristic, this is one of the main achievements of the Ph.D thesis by Banks ([B1]). The basic idea for the local case is that just like the non-twisted exceptional representation of Kazhdan-Patterson the twisted one is constructed as a quotient of the induced representation $\text{Ind}(\tilde{GL}_r, \Pi_{\chi} \otimes \delta_P^{-1})$, where $\Pi_{\chi}$ is the Weil representation of $\tilde{M}_P$ constructed in the previous subsection and extended trivially on $N_P$. For this purpose, Banks explicitly computed the local coefficients for intertwining operators on this induced representation and showed that it has a unique irreducible quotient, which is the image of an intertwining operator. This quotient is precisely the twisted exceptional representation. But for technical reasons, Banks treated only the case of odd residual characteristic.

However, thanks to the recent work by Ban and Jantzen ([BJ]) that proves the Langlands quotient theorem for metaplectic covers over the $p$-adic field, the construction of the twisted exceptional representation for the non-archimedean case is very simple. (But let us mention that the approach taken by [BJ] gives more information about the induced representation such as the point of reducibility.) Also let us note that over the archimedean field the Langlands quotient theorem has been already available for groups like $\tilde{GL}_{2q}$ ([BW Chapter IV]). Note that the groups $\tilde{GL}_{2q}(\mathbb{R})$ and $\tilde{GL}_{2q}(\mathbb{C})$ are real reductive groups in the sense of [BW 0.3.1], which are also called real reductive groups in the Harish-Chandra class in [Wa] p.289] to which the general theory of [BW applies.) Hence the construction of the twisted exceptional representation for the archimedean case is very simple as well. Indeed, Kazhdan-Patterson constructed the non-twisted exceptional representation over the archimedean field by the Langlands quotient theorem as well. The twisted case can be treated in the same way. The global case is a standard argument in the Langlands theory of Eisenstein series, which is also the method employed by Kazhdan-Patterson for the non-twisted case.

Throughout this subsection, $r = 2q$ and $P$ is the $(2, \ldots, 2)$-parabolic whose Levi $M_P$ is $GL_2 \times \cdots \times GL_2$ ($q$-times). Also we need to view the group $^c\tilde{M}_P$ as the subgroup $\tilde{M}_P$ of $\tilde{GL}_r$ via the embedding $\tilde{\varphi}_P : ^c\tilde{M}_P \to \tilde{GL}_r$ (See Appendix A) In other words when we treat the group $\tilde{M}_P$ by itself, we always mean $^c\tilde{M}_P$ and when we would like to view it as a subgroup of $\tilde{GL}_r$ we consider it as the image of the embedding $\tilde{\varphi}_P$. Accordingly, we view the Weil representation $\Pi_{\chi}$ constructed in the previous subsection as a representation of $\tilde{M}_P$ via $\tilde{\varphi}_P$, namely $\Pi_{\chi} \circ \tilde{\varphi}_P^{-1}$. But we simply write $\Pi_{\chi}$ for $\Pi_{\chi} \circ \tilde{\varphi}_P^{-1}$ since this does not produce any confusion. The same applies to the representations $\varpi_{\chi}$ and $\pi_{\chi}$.

Let us set up some general notations. For each standard parabolic subgroup $Q$ of $\tilde{GL}_{2q}$, we let $T_Q$ be the maximal torus in $Q$, and $\Phi_Q$ the set of roots of $GL_r$ relative to $T_Q$. The choice of $Q$ determines the positive roots in $\Phi_Q$. We have the natural inclusion $\Phi_Q(\mathbb{C}) \subset \Phi_B(\mathbb{C})$ via the inclusion $M_B \hookrightarrow M_Q$. 


We write $\rho_Q$ for half the sum of positive roots in $\Phi_Q$. Assume the Levi part $M_Q$ of $Q$ is of the form $\text{GL}_{r_1} \times \cdots \times \text{GL}_{r_k}$. Each root $\beta \in \Phi_Q$ is identified with a pair of integers $\beta = (i, j)$ for $1 \leq i, j \leq k$ with $i \neq j$, and $\beta = (i, j)$ is positive if $i < j$. To be precise, let us denote each element in $M_Q$ by $\text{diag}(g_l)$ where each $g_l \in \text{GL}_{r_l}$ for $l = 1, \ldots, k$. Then for $\beta = (i, j)$, we have $\beta(\text{diag}(g_l)) = \det(g_l) \det(g_j)^{-1}$.

Now assume $Q = P$, i.e. the $(2, 2, 2)$-parabolic subgroup of $\text{GL}_2q$. Define $W_P$ to be the set of block matrices

$$\Pi = \{(\delta_{w(i,j)}I_2) : w \in S_q\},$$

where $S_q$ is the symmetric group on $q$ letters and $\delta_{i,j}$ is the Kronecker delta function. Then $W_P$, which is isomorphic to $S_q$, is a subgroup of the Weyl group $W_B$ of $\text{GL}_2q$. For each elements $\text{diag}(h_k) \in M_P$ and $w \in W_P$, we have $w \text{diag}(h_k)w^{-1} = \text{diag}(h_{wk})$. We often view each element $w \in W_P$ as the element $s(w)$ in $\tilde{\text{GL}}_2q$. For each root $\beta = (i, j) \in P$, we let $\beta^\vee$ be the corresponding coroot, so that for each $t \in F^\times$ we have $\beta^\vee(t) = \text{diag}(g_l)$ where $g_l = I, tI_2$ for $l \neq i, j$, $g_i = tI_2$ and $g_j = t^{-1}I_2$, i.e.

$$\beta^\vee(t) = \begin{pmatrix} \cdots & tI_2 & \cdots \\ \cdots & \cdots & \cdots \\ t^{-1}I_2 & \cdots & \cdots \end{pmatrix},$$

where $tI_2$ and $t^{-1}I_2$ are in the $i$th and $j$th entries respectively and all the other diagonal entries are $I_2$.

The space $\Phi_P(\mathbb{C}) := \Phi_P \otimes \mathbb{C}$ is identified with $\mathbb{C}^{\nu-1}$ by choosing a basis to be the set of the simple roots i.e. the roots of the form $(i, i+1)$. For each $\nu \in \Phi_P(\mathbb{C})$ and a representation $\Pi$ of $\tilde{M}_P$ (locally or globally), we define the representation

$$\Pi' := \Pi \otimes \exp(\nu, H_P(\mathbb{C}))$$

of $\tilde{P}$ where $\Pi$ is extended trivially to the unipotent part, and $H_P$ is the Harish-Chandra map as usual (or strictly speaking the Harish-Chandra map composed with the canonical projection $\text{GL}_2q \to \text{GL}_2q$). If $\nu = \rho_P/2 \in \Phi_P(\mathbb{C})$, then $\Pi' = \Pi \otimes \delta_P^{1/4}$, where $\delta_P$ is the modular character of $P$. We often write $\delta_P := \exp(\nu, H_P(\mathbb{C}))$.

Note that $\delta_P$ can be computed as

$$\delta_P(g_1, \ldots, g_q) = |\det(g_1)|^{2(q-1)} |\det(g_2)|^{2(q-3)} \cdots |\det(g_q)|^{-2(q-1)}$$

for the element $(g_1, \ldots, g_q) \in M_P$.

By following the notation of [KP, p.62], for each irreducible representation $\Pi$ of $\tilde{M}_P(\mathbb{A})$ (resp. $\tilde{M}_P(F)$) if $F$ is global (resp. local) and for each root $\beta \in \Phi(\mathbb{C})$, we define the character on $\mathbb{A}^\times$ (resp. $F^\times$) by

$$\Pi|_\beta(t) = \Pi((\beta^\vee(t^2)), 1)$$

for $t \in \mathbb{A}^\times$ (resp. $F^\times$). Note that the map $t \mapsto (\beta^\vee(t^2), 1)$ is indeed a homomorphism from $\mathbb{A}^\times$ to $\tilde{M}_P(\mathbb{A})$ (resp. $F^\times$ to $\tilde{M}_P(F)$). Namely this is just the central character of $\Pi$ evaluated at $(\beta^\vee(t^2), 1)$. In particular, by considering $\pi^{\psi}_\chi = r^{\psi}_\chi \otimes \cdots \otimes r^{\psi}_\chi$ and the central character of $r^{\psi}_\chi$ is given by (2.10), one can see that

$$\Pi|_\beta(t) = \delta_P^{1/4}(\beta^\vee(t^2)),$$

and if $\nu = \rho_P/2$, so $\delta_P = \delta_P^{1/4}$, and $\beta = (i, j)$, then by using (2.29)

$$\delta_P^{1/4}(\beta^\vee(t^2)) = |t|^{4(j-i)}.$$
Let us recall the notion of intertwining integrals. First assume $F$ is local and $\Pi$ is an irreducible admissible representation of $\widetilde{M}_P$. For $w_1, w_2 \in W_P$, we define the intertwining integral

\begin{equation}
A(w_1\nu, w_1\Pi, w_2) : \text{Ind}_{\widetilde{P}}^{\text{GL}_2q} w_1(\Pi') \to \text{Ind}_{\widetilde{P}}^{\text{GL}_2q} w_2 w_1(\Pi')
\end{equation}

by

\[ A(w_1\nu, w_1\Pi, w_2)f(g) = \int_{\widetilde{N}_P} f(w_2^{-1}ng) \, dn \]

for $f \in \text{Ind}_{\widetilde{P}}^{\text{GL}_2q} w_1(\Pi')$.

Next assume $F$ is global and $\Pi$ an irreducible automorphic representation of $\widetilde{M}_P(A)$. By following [MW] we view the induced representation $\text{Ind}_{\widetilde{P}(A)}^{\text{GL}_2q} \Pi$ as a space of automorphic forms on $N_{P}(A)^* M_{P}(F)^* \backslash \text{GL}_2q(A)$. Then the global intertwining integral $M(w_1\nu, w_1\Pi, w_2)$ is defined in the completely analogous way as the local case.

**Local case:**

Let us consider the local case. But as we mentioned at the beginning of this subsection, the construction of the twisted exceptional representation is quite simple thanks to the Langlands quotient theorem. But first we should mention

**Lemma 2.34.** Let $\pi$ be an irreducible admissible representation of $\widetilde{M}_P^{(2)}$ such that $\Pi := \text{Ind}_{\widetilde{M}_P^{(2)}}^{\widetilde{M}_P^{(2)}} \pi$ is irreducible, so $\varpi := \text{Ind}_{\widetilde{M}_P^{(2)}}^{\widetilde{M}_P^{(2)}} \pi$ is irreducible as well. Then

\[ \text{Ind}_{\widetilde{M}_P N_P}^{\text{GL}_r} \Pi = \text{Ind}_{\widetilde{M}_P^{(2)} N_P}^{\text{GL}_r} \varpi = \text{Ind}_{\widetilde{M}_P^{(2)} N_P}^{\text{GL}_r} \pi. \]

**Proof.** The proof is straightforward. See [B1] Proposition 4.1 as well.

By this lemma, together with the fact that $\widetilde{M}_P^{(2)}$ is better behaved in the sense that each $\text{GL}_2^{(2)}$-factor in the Levi $\widetilde{M}_P^{(2)}$ commutes with each other, it is easier to work with $\text{Ind}_{\widetilde{M}_P^{(2)} N_P}^{\text{GL}_r} \pi$ than $\text{Ind}_{\widetilde{M}_P N_P}^{\text{GL}_r} \Pi$.

With this said, the local twisted exceptional representation is constructed as follows:

**Proposition 2.35.** The induced representation $\text{Ind}_{\widetilde{P}(2)}^{\text{GL}_r} \varpi \chi \otimes \delta_P^{1/4}$ has a unique irreducible quotient, which we denote by $\partial_\chi$. It is the image of the intertwining integral

\[ \text{Ind}_{\widetilde{P}(2)}^{\text{GL}_r} \varpi \chi \otimes \delta_P^{1/4} \to \text{Ind}_{\widetilde{P}(2)}^{\text{GL}_r} w_0(\varpi \chi \otimes \delta_P^{1/4}), \]

where $w_0$ is the longest Weyl group element relative to $P$. (Recall from Section 1.2 that $\widetilde{P}(2) = \widetilde{M}_P^{(2)} N_P^{*}$.)

**Proof.** Let us first note that if $\chi^{1/2}$ exists, from the embedding [2.25], one can see that the situation boils down to the non-twisted case of Kazhdan-Patterson. Hence we assume that $\chi^{1/2}$ does not exist.

Let us consider the non-archimedean case. As we noted in Section 2.2 the Weil representation $\varpi_\chi^{(2)}$ is supercuspidal, and hence in particular tempered. Thus $\varpi_\chi^{(2)}$ and so $\Pi_\chi$ are tempered. Then the Langlands quotient theorem for metaplectic covers ([B1] Theorem 4.1) applies to this situation, and implies that the induced representation $\text{Ind}_{\widetilde{P}}^{\text{GL}_r} \Pi_\chi \otimes \delta_P^{1/4} = \text{Ind}_{\widetilde{P}(2)}^{\text{GL}_r} \varpi_\chi \otimes \delta_P^{1/4}$ has a unique irreducible quotient, which is also obtained as a unique irreducible subrepresentation of $\text{Ind}_{\widetilde{P}(2)}^{\text{GL}_r} w_0(\varpi_\chi \otimes \delta_P^{1/4})$. 
One needs to show that this irreducible quotient is indeed obtained as the image of the intertwining integral. (Unlike the usual Langlands quotient theorem, it is not shown in [BJ] that the Langlands quotient is indeed obtained as the image of the intertwining integral.) But this can be easily proven for the case at hand because in exactly in the same way as the proof of [KP] Proposition I.2.2 one can show that
$$\dim \text{Hom}_{\widetilde{\GL}_2(\mathbb{A})} (\text{Ind}_{\widetilde{P}(2)}^{\widetilde{\GL}_2(\mathbb{A})} \pi^\psi \otimes \delta_P^{1/4}, \text{Ind}_{\widetilde{P}(2)}^{\widetilde{\GL}_2(\mathbb{A})} \nu \otimes (\pi^\psi \otimes \delta_P^{1/4})) \leq 1$$
by standard computations of the Jacquet modules of the induced representations. (Also see [BJ] Corollary 6.7.)

The archimedean case follows from the Langlands quotient theorem, which is available for the groups like $\GL_n$ as we mentioned at the beginning of this subsection. \hfill $\square$

We call the representation $\vartheta_\chi$ the “twisted exceptional representation” of $\widetilde{\GL}_2$. By Lemma 2.34, $\vartheta_\chi$ is also the quotient of $\text{Ind}_{\widetilde{P}}^{\widetilde{\GL}_2} \Pi_\chi \otimes \delta_P^{1/4}$. Since $\Pi_\chi$ is independent of the choice of $\psi$, so is $\vartheta_\chi$, and hence our notation. If $\chi^{1/2}$ exists, then the twisted and non-twisted ones are related as $\vartheta_\chi = \theta_\chi^{1/2}$. (As we mentioned in Remark 2.4 if $r$ is even, $\vartheta_\chi^\psi$ is independent of the choice of $\psi$.)

Finally, we have the analogue of Proposition 2.6.

**Proposition 2.36 (Local Periodicity).** Assume $F$ is non-archimedean. Let $(\vartheta_\chi)_{N_P}$ be the Jacquet module of $\vartheta_\chi^\psi$ along the parabolic $\widetilde{P}$. Then
$$(\vartheta_\chi)_{N_P} = \nu \Pi_\chi \otimes \delta_P^{1/4} = \Pi_\chi \otimes \delta_P^{1/4}.$$

**Proof.** This proof is completely analogous to the proof the non-twisted case ([KP] Theorem I.2.9(e)) and left to the reader. \hfill $\square$

**Global case:**

We construct the global twisted exceptional representation of $\widetilde{\GL}_v(\mathbb{A})$, so $F$ is a number field, $\chi$ is a unitary Hecke character and $\psi$ is our fixed additive character on $F \backslash \mathbb{A}$. The construction is analogous to the local case in that the exceptional representation is obtained as a unique irreducible quotient of the global induced space $\text{Ind}_{\widetilde{P}(\mathbb{A})}^{\widetilde{\GL}_v(\mathbb{A})} \Pi_\chi \otimes \delta_P^{1/4}$, where $\Pi_\chi$ is the global Weil representation of $\widetilde{M}_P(\mathbb{A})$. (Strictly speaking $\Pi_\chi$ is the pullback of the Weil representation of $\widetilde{cM}_P(\mathbb{A})$ via the map $\overline{\varphi_P^{-1}} : \widetilde{M}_P(\mathbb{A}) \rightarrow \widetilde{cM}_P(\mathbb{A})$, which is also automorphic. See Corollary A.7.) Moreover the exceptional representation is generated by the residues of certain Eisenstein series to be defined below.

Let us start with the definition of the Eisenstein series. Although this might be already quite familiar to experts, let us repeat some of the essential points of the theory of Eisenstein series. The best reference (probably the only one for metaplectic groups) for the theory of Eisenstein series is MW. For a (cuspidal) automorphic representation $\Pi$ of $\widetilde{M}_P(\mathbb{A})$, the induced representation $\text{Ind}_{\widetilde{P}(\mathbb{A})}^{\widetilde{\GL}_2(\mathbb{A})} \Pi$ is realized in a space of automorphic forms on $N_P(\mathbb{A})M_P(F) \backslash \widetilde{\GL}_2(\mathbb{A})$. (Here of course we are viewing $N_P(\mathbb{A})$ and $M_P(F)$ as subgroups of $\widetilde{\GL}_2(\mathbb{A})$ via the splitting $s$ and writing simply $N_P(\mathbb{A})$ and $M_P(F)$ for $N_P(\mathbb{A})^*$ and $M_P(F)^*$, respectively.) To be precise, we have the Iwasawa decomposition
$$\widetilde{\GL}_2(\mathbb{A}) = N(\mathbb{A})M_P(\mathbb{A})K,$$
where $K \subseteq \GL_2(\mathbb{A})$ is the usual maximal compact subgroup of $\GL_2(\mathbb{A})$, namely $K = \prod_v K_v$, where $K_v$ is $\GL_2(O_{F,v})$ at non-archimedean $v$, $O(2q)$ for real $v$ and $U(2q)$ for complex $v$. Then
$$N_P(\mathbb{A})M_P(F) \backslash \widetilde{\GL}_2(\mathbb{A}) = (M_P(F) \backslash \widetilde{M}_P(\mathbb{A})) \cdot K.$$
Hence for each automorphic form \( \phi \) on \( N_P(\A)M_P(F) \backslash \G\!L_{2q}(\A) \) and each \( k \in \tilde{K} \), the function \( \phi_k \) on \( M_P(F) \backslash \tilde{M}_P(\A) \) defined by \( \phi_k(m) = \phi(mk) \) is an automorphic form on \( \tilde{M}_P(\A) \). Each \( f^\nu \in \text{Ind}_{\tilde{P}(\A)}^{\tilde{G}\!L_{2q}(\A)} \Pi^\nu \) is of the form

\[
(2.37) \quad f^\nu = \phi \otimes \exp(\nu + \rho_P, H_P(\)),
\]

where \( \phi : N_P(\A)M_P(F) \backslash \G\!L_{2q}(\A) \to \C \) is such that for each \( k \in \tilde{K} \), the function \( \phi_k \) is in the space of \( \Pi \). Also note that our induction is normalized so that we have the shift by \( \rho_P \). Also note

\[
(2.38) \quad f^\nu|_{\tilde{M}_P(\A)} \in \Pi^{\nu + \rho_P},
\]

i.e. the restriction of \( f^\nu \) to \( \tilde{M}_P(\A) \) is in the space of \( \Pi^{\nu + \rho_P} \). For each \( f^\nu \), we define the Eisenstein series by

\[
E(g, \Pi, f^\nu) = \sum_{\gamma \in P(F) \backslash \G\!L_{2q}(F)} f^\nu(\gamma g),
\]

where \( g \in \G\!L_{2q}(\A) \). It converges absolutely when \( \nu \) is in a sufficiently positive part of the Weyl chamber, and admits meromorphic continuation. This is an automorphic form on \( \G\!L_{2q}(\A) \) whenever it is holomorphic. If the inducing representation \( \Pi \) is cuspidal and \( \nu \) is in the positive chamber, the poles of \( E(g, \Pi, f^\nu) \) are at most simple, and when it has a (simple) pole, the residue is an automorphic form on \( \G\!L_{2q}(\A) \) and the space generated by the residues is a space of a square integrable automorphic representation of \( \G\!L_{2q} \).

The twisted exceptional representation to be constructed is generated by the residues of the Eisenstein series \( E(g, \Pi, f^\nu) \) associated with the induced representation \( \text{Ind}_{\tilde{P}(\A)}^{\tilde{G}\!L_{2q}(\A)} \Pi^\nu \) at \( \nu = \rho_P/2 \). To see it, one needs to know this Eisenstein series indeed has a simple pole at \( \nu = \rho_P/2 \). But to study poles of the Eisenstein series, one should look at the global intertwining operator

\[
M(\nu, \Pi, w) : \text{Ind}_{\tilde{P}(\A)}^{\tilde{G}\!L_{2q}(\A)} \Pi^\nu \otimes \delta^\nu_P \to \text{Ind}_{\tilde{P}(\A)}^{\tilde{G}\!L_{2q}(\A)} w(\Pi^\nu \otimes \delta^\nu_P),
\]

where \( w \in W_P \). (See [MW, Proposition IV.1.11].) We will show that the global intertwining operator \( M(\nu, \Pi, w) \) (and hence the Eisenstein series \( E(g, \Pi, f^\nu) \)) has a pole at \( \nu = \rho_P/2 \) if and only if \( w = w_0 \).

The computation of poles of the global intertwining operator essentially boils down the computation of the “normalizing factor” for the corresponding local intertwining operator

\[
A(\nu, \Pi, w) : \text{Ind}_{\tilde{P}(F)}^{\tilde{G}\!L_{2q}(F)} \Pi^\nu \otimes \delta^\nu_P \to \text{Ind}_{\tilde{P}(F)}^{\tilde{G}\!L_{2q}(F)} w(\Pi^\nu \otimes \delta^\nu_P)
\]

at the unramified place \( v \). Namely

**Lemma 2.39.** Assume \( v \) is a place where all the data defining \( \text{Ind}_{\tilde{P}(F)}^{\tilde{G}\!L_{2q}(F)} \Pi^\nu \otimes \delta^\nu_P \) are unramified. Let \( f^\nu \in \text{Ind}_{\tilde{P}(F)}^{\tilde{G}\!L_{2q}(F)} \Pi^\nu \otimes \delta^\nu_P \) be such that \( f^\nu(1) = 1 \). Then

\[
A(\nu, \Pi, w_0) f^\nu(1) = \prod_{\beta > 0} \frac{L(|\nu^\nu(\Pi^\nu_{\chi_v})\beta|^{1/2}) \cdot L(|\nu^\nu(\Pi^\nu_{\chi_v})\beta|^{1/2})}{L(|\nu^\nu(\Pi^\nu_{\chi_v})\beta|^{1/2}) \cdot L(|\nu^\nu(\Pi^\nu_{\chi_v})\beta|^{1/2})},
\]

where recall the character \( (\Pi^\nu_{\chi_v})\beta \) is defined in (2.30), and \( L \) is the local Tate factor as defined in the notation section.
Proof. Since all the data are unramified, we have $\chi_v(-1) = 1$. Then the embedding (2.26) gives us the commutative diagram

\[
\begin{array}{ccc}
\text{Ind}^\mathbb{GL}_r(F_v)_{\mathcal{B}_2^{\nu},(F_v)} & w_1 \omega_\chi^{1/2} & \text{Ind}^\mathbb{GL}_r(F_v)_{\mathcal{B}_2^{\nu},(F_v)} \\
\uparrow & & \uparrow \\
\text{Ind}^\mathbb{GL}_r(F_v)_{\mathcal{P}(F_v)} & (\nu, \Pi_\chi^{\nu}) & \text{Ind}^\mathbb{GL}_r(F_v)_{\mathcal{P}(F_v)} \\
\end{array}
\]

where $w_1$ is the Weyl group element in $W_P$ of the form $((1, 1), \ldots, (1, 1))$ and the top arrow is the intertwining operator for the corresponding induced representations, which is studied by Kazhdan and Patterson in [KP]. Hence

\[
A(\nu, \Pi_\chi^{\nu}, w_1) f_\chi(1) = A(\nu, \Pi_\chi^{\nu}, \omega_\chi^{1/2}, w_1) f_\chi(1).
\]

But the right hand side is computed by Kazhdan and Patterson in [KP] Proposition I.2.4. Then one can see that this formula by Kazhdan-Patterson is rewritten as in the lemma.

\[\square\]

Remark 2.40. The inverse of the product of the Tate factors appearing in the above lemma can be used as a normalizing factor of the corresponding intertwining operator. A similar expression can be obtained for all places $v$, which give more refined results on the twisted exceptional representation. The author has carried out this computation, which will appear elsewhere.

Next we need

Lemma 2.41. Just for the sake of this lemma, let us assume $F$ is local. Then the local intertwining operator

\[
A(\nu, \Pi_\chi^{\nu}, w) : \text{Ind}^\mathbb{GL}_r(F_v)_{\mathcal{P}(F_v)} \Pi_\chi \otimes \delta_P \rightarrow \text{Ind}^\mathbb{GL}_r(F_v)_{\mathcal{P}(F_v)} w(\Pi_\chi \otimes \delta_P)
\]

is holomorphic for $\nu$ in the positive chamber, in particular at $\nu = \rho_P/2$.

Proof. This is a general fact for the region of the convergence for intertwining integrals when the inducing representation is tempered, at least for non-metaplectic groups. For archimedean $F$, it is indeed known ([BW] Lemma 4.2, p.84) even for the metaplectic case. (A proof for the $p$-adic non-metaplectic case is also available in [BW] Proposition 2.6, p.217.) But at this moment, it is not known for the non-archimedean metaplectic case, or at least to the best of our knowledge, a proof is not written anywhere, though the author has been informed by D. Ban that this might be included in [BM] for a future revision. But here, to be absolutely rigorous, we will give an alternate indirect proof, which works at least for the case of our interest.

The idea is to use a global argument. Namely one can always globalize the character $\chi$ to a Hecke character $\tilde{\chi}$ in such a way that at two places $v_1 \neq v_2$, $\tilde{\chi}_{v_1} = \tilde{\chi}_{v_2} = \chi$ and there is at least one place $v$ at which $\tilde{\chi}_v(-1) = -1$, so that $\Pi_{\tilde{\chi}}$ is cuspidal. Now if the local intertwining operator $A(\nu, \Pi_\chi^{\nu}, w)$ has a pole at positive $\nu$, the global operator $M(\nu, \Pi_{\tilde{\chi}}^{\nu}, w) = \otimes_v A(\nu, \Pi_{\tilde{\chi}}^{\nu}, w)$ must have at least a double pole. But in the positive chamber the global intertwining operator has at most a simple pole by [MW] Proposition IV.1.11]. Hence $A(\nu, \Pi_\chi^{\nu}, w)$ cannot have a pole.

\[\square\]

Proposition 2.42. The global intertwining operator $M(\nu, \Pi_\chi^{\nu}, w)$ has a pole at $\nu = \rho_P/2$ if and only if $w = w_0$. 

Proof. This is a general fact for the region of the convergence for intertwining integrals when the inducing representation is tempered, at least for non-metaplectic groups. For archimedean $F$, it is indeed known ([BW] Lemma 4.2, p.84) even for the metaplectic case. (A proof for the $p$-adic non-metaplectic case is also available in [BW] Proposition 2.6, p.217.) But at this moment, it is not known for the non-archimedean metaplectic case, or at least to the best of our knowledge, a proof is not written anywhere, though the author has been informed by D. Ban that this might be included in [BM] for a future revision. But here, to be absolutely rigorous, we will give an alternate indirect proof, which works at least for the case of our interest.

The idea is to use a global argument. Namely one can always globalize the character $\chi$ to a Hecke character $\tilde{\chi}$ in such a way that at two places $v_1 \neq v_2$, $\tilde{\chi}_{v_1} = \tilde{\chi}_{v_2} = \chi$ and there is at least one place $v$ at which $\tilde{\chi}_v(-1) = -1$, so that $\Pi_{\tilde{\chi}}$ is cuspidal. Now if the local intertwining operator $A(\nu, \Pi_\chi^{\nu}, w)$ has a pole at positive $\nu$, the global operator $M(\nu, \Pi_{\tilde{\chi}}^{\nu}, w) = \otimes_v A(\nu, \Pi_{\tilde{\chi}}^{\nu}, w)$ must have at least a double pole. But in the positive chamber the global intertwining operator has at most a simple pole by [MW] Proposition IV.1.11]. Hence $A(\nu, \Pi_\chi^{\nu}, w)$ cannot have a pole. 

\[\square\]
Proof. Assume \(M(\nu, \Pi_X, w)\) has a pole at \(\nu = \rho_P/2\). Let \(f^\nu = \otimes' f^\nu_v \in \text{Ind}_{P(\mathbb{A})}^{\text{GL}_{2a}(\mathbb{A})} \Pi_X^\nu\), where for almost all \(v\), \(f^\nu_v(1) = 1\). By Lemma 2.39, we can write

\[
M(\nu, \Pi_X, w)f^\nu = \prod_{\beta > 0 \atop w\beta < 0} \left( \frac{L(\| |1| \beta^{-1}((\Pi_X^\nu)_{\beta})^{1/2}) \cdot L((\Pi_X^\nu)_{\beta})^{1/2}}{L(\| |1| \beta((\Pi_X^\nu)_{\beta})^{1/2}) \cdot L(\| \beta((\Pi_X^\nu)_{\beta})^{1/2})} \right) \otimes' \prod_{\nu, w_\beta < 0} \frac{L(\| |1| \beta^{-1}((\Pi_X^\nu)_{\beta})^{1/2}) \cdot L((\Pi_X^\nu)_{\beta})^{1/2}}{L(\| |1| \beta((\Pi_X^\nu)_{\beta})^{1/2}) \cdot L((\Pi_X^\nu)_{\beta})^{1/2})} A(\nu, \Pi_X, w)f^\nu_v.
\]

Note that Lemma 2.39 guarantees that the restricted tensor product in the right hand side makes sense.

At \(\nu = \rho_P/2\), by (2.32) the local part in the above decomposition is written as

\[
\frac{L(\| |1| \beta^{-1}((\Pi_X^\nu)_{\beta})^{1/2}) \cdot L((\Pi_X^\nu)_{\beta})^{1/2}}{\prod_{\nu, w_\beta < 0} \frac{L(\| |1| \beta^{-1}((\Pi_X^\nu)_{\beta})^{1/2}) \cdot L((\Pi_X^\nu)_{\beta})^{1/2}}{L(\| |1| \beta((\Pi_X^\nu)_{\beta})^{1/2}) \cdot L((\Pi_X^\nu)_{\beta})^{1/2})} A(\nu, \Pi_X, w_\beta)f^\nu_v
\]

for each \(\beta = (i, j)\). By Lemma 2.41 together with the fact that all the local Tate factors appearing here have no pole, we conclude that, if \(M(\nu, \Pi_X, w)f^\nu\) has a pole at \(\nu = \rho_P/2\), it comes from the global factor. But for each \(\beta = (i, j)\), (2.32) gives

\[
(2.43) \quad \frac{L(\| |1| \beta^{-1}((\Pi_X^\nu)_{\beta})^{1/2}) \cdot L((\Pi_X^\nu)_{\beta})^{1/2}}{L(\| \beta((\Pi_X^\nu)_{\beta})^{1/2})} = \frac{L(\| |1| \beta^{-1}((\Pi_X^\nu)_{\beta})^{1/2}) \cdot L((\Pi_X^\nu)_{\beta})^{1/2})}{L(\| \beta((\Pi_X^\nu)_{\beta})^{1/2})},
\]

at \(\nu = \rho_P/2\). This has a pole if \(j = i + 1\), i.e. \(\beta\) is simple. Now in order for the product over all \(\beta > 0\) with \(w_\beta < 0\) to have a pole at \(\nu = \rho_P/2\), it must be the case that (2.43) has a pole for all simple \(\beta > 0\). This is because by a pole, we mean a pole of a meromorphic function for \((q - 1)\) variables on \(\Phi(S) = \mathbb{C}^{q-1}\), which are indexed by the simple roots, and hence to have a pole at \(\nu = \rho_P/2\), it must have a pole at all simple \(\beta > 0\). But if \(w \neq w_0\), there is a simple \(\beta > 0\) such that \(w_\beta > 0\). Hence we must have \(w = w_0\).

Conversely assume \(w = w_0\). By reversing the reasoning, one can see that \(M(\nu, \Pi_X, w_\beta)\) has a pole at \(\nu = \rho_P/2\). (Let us note that one can always choose the local \(f^\nu_v\) so that \(A(\nu, \Pi_X, w_\beta)f^\nu_v \neq 0\), and hence the pole of (2.43) is not cancelled by the local factors.) \(\square\)

Now we are ready to construct the twisted exceptional representation as follows:

**Theorem 2.44.** At \(\nu = \rho_P/2 \in \Phi_P(S)\), the global induced space \(\text{Ind}_{P(\mathbb{A})}^{\text{GL}_{2a}(\mathbb{A})} \Pi_X^\nu\) has a unique irreducible quotient. Moreover, this irreducible quotient is (equivalent to) a square integrable automorphic representation realized in the space of automorphic forms on \(\text{GL}_{2a}(\mathbb{A})\), which are generated by the residues of the Eisenstein for series \(E(-, \Pi_X, f^\nu)\) at \(\nu = \rho_P/2\). Let us denote this irreducible quotient by \(\vartheta_X\). Then

\[
\vartheta_X = \otimes' \vartheta_{X_v},
\]

where \(\vartheta_{X_v}\) is the local twisted exceptional representation. We call \(\vartheta_X\) the global twisted exceptional representation.

**Proof.** First of all, let us note that if \(\chi^{1/2}\) exists, then just as in the local case this theorem is subsumed under the Kazhdan-Patterson construction of the global exceptional representation which is discussed in Part II of [KP]. Also in this case, one can see that our \(\vartheta_X\) is the exceptional representation of Kazhdan-Patterson with the determinantal character \(\chi^{1/2}\), i.e. \(\vartheta_X = \vartheta_{\chi^{1/2}}\). The way to reduce this case to [KP] is completely analogous to the local case, and the detail is left to the reader.
Hence we consider the case where $\chi^{1/2}$ does not exist, and so by [GPS] Proposition 8.1.1 we know that $\Pi_\chi$ is cuspidal. However, even for this case the argument is essentially the same as [KP], which is a reworking of the Langlands theory of Eisenstein series for metaplectic groups. Of course, thanks to [MW], this theory has been completely worked out.

By ([MW] Proposition IV.1.11), our Eisenstein series $E(g, \pi_\psi^\nu, f^\nu)$ has meromorphic continuation and has a pole when the global intertwining operator $M(\nu, \Pi_\chi, w_0)$ has a pole, which is simple. By the above proposition, this happens at $\nu = \rho_p/2$. Then the residues make up the residual spectrum. Note that our inducing representation is cuspidal. Thus the residues are square integrable automorphic forms. (See [MW] Theorem (iii), V.3.13)

Let us write

$$E_{-1}(-, \Pi_\chi, f) = \text{Res}_{\nu=\rho_p/2} E(-, \Pi_\chi, f^\nu).$$

The map $f^\nu \mapsto E_{-1}(-, \pi_\psi^\nu, f)$ defines a $\widetilde{GL}_{2q}(\mathbb{A})$ intertwining operator

$$E_{-1} : \text{Ind}_{\widetilde{P}(\mathbb{A})}^{\widetilde{GL}_{2q}(\mathbb{A})}(\Pi^\rho_p/2_\chi) \rightarrow \mathcal{A}_2(\widetilde{GL}_{2q}(\mathbb{A})),$$

where $\mathcal{A}_2(\widetilde{GL}_{2q}(\mathbb{A}))$ is the space of square integrable automorphic forms on $\widetilde{GL}_{2q}(\mathbb{A})$. Also let

$$M_{-1}(\Pi_\chi, w_0, f) = \text{Res}_{\nu=\rho_p/2} M(\nu, \Pi_\chi, w_0) f^\nu$$

be the residue of the intertwining operator. The map $f^\nu \mapsto M_{-1}(\Pi_\chi, w_0, f)$ defines a $\widetilde{GL}_{2q}(\mathbb{A})$ intertwining operator

$$M_{-1} : \text{Ind}_{\widetilde{P}(\mathbb{A})}^{\widetilde{GL}_{2q}(\mathbb{A})}(\Pi^\rho_p/2_\chi) \rightarrow \text{Ind}_{\widetilde{P}(\mathbb{A})}^{\widetilde{GL}_{2q}(\mathbb{A})}(w_0(\Pi^\rho_p/2_\chi)).$$

That the global induced space $\text{Ind}_{\widetilde{P}(\mathbb{A})}^{\widetilde{GL}_{2q}(\mathbb{A})}(\Pi^\rho_p/2_\chi)$ has a unique irreducible quotient follows from the corresponding statement for the local induced representations. The image of $M_{-1}$ is the unique irreducible quotient, which we denote by $\vartheta_\chi$. By decomposing the intertwining operator into local constituents, we see $\vartheta_\chi \cong \otimes \vartheta_{\chi_i}$.

It remains to show that $\vartheta_\chi$ is generated by the residues of the Eisenstein series. This follows from the inner product formula of pseudo-Eisenstein series ([MW] Theorem II.2.1), which implies (up to a suitable normalization of inner products) that

$$\langle E_{-1}(-, \Pi_\chi, f_1), E_{-1}(-, \Pi_\chi, f_2) \rangle = \langle f_1, M_{-1}(\Pi_\chi, w_0, f_2) \rangle,$$

where the inner product on the left hand side is the usual inner product on $\mathcal{A}_2(\widetilde{GL}_{2q}(\mathbb{A}))$ and the one on the right hand side is the usual pairing on $\text{Ind}_{\widetilde{P}(\mathbb{A})}^{\widetilde{GL}_{2q}(\mathbb{A})}(\Pi^\rho_p/2_\chi) \times \text{Ind}_{\widetilde{P}(\mathbb{A})}^{\widetilde{GL}_{2q}(\mathbb{A})}(w_0(\Pi^\rho_p/2_\chi))$. (For the derivation of this inner product, see the proof of [KP] Theorem II.1.4, which is based on the argument by Langlands in [L].) This inner product formula implies

$$\ker M_{-1} \subseteq \ker E_{-1},$$

and so the image of $E_{-1}$ is equivalent to a quotient of the image of $M_{-1}$ i.e. $\vartheta_\chi$. But since $\vartheta_\chi$ is the unique irreducible quotient, the image of $E_{-1}$ is indeed isomorphic to $\vartheta_\chi$. This completes the proof of the theorem. 

Finally in this section, let us give a proof of the global periodicity of $\vartheta_\chi$, which is the twisted analogue of Proposition 2.7.
Proposition 2.45 (Global Periodicity). Let $(\vartheta_\chi)_{N_p}$ be the space generated by the constant terms of the automorphic forms in $\vartheta_\chi$ along the parabolic $\tilde{P}$. Then
\[(\vartheta_\chi)_{N_p} = w_0(\Pi_\chi) \otimes \delta_p^{1/4} = \Pi_\chi \otimes \delta_p^{1/4}.\]

Proof. This follows from Proposition 2.42 and a well-known computation of constant terms of Eisenstein series. Namely by [MW] Proposition II.1.7], the constant term $E_P(-, \Pi_\chi, f^\nu)$ can be computed as
\[E_P(-, \Pi_\chi, f^\nu) = \sum_{w \in W_p} M(\nu, \Pi_\chi, w)f^\nu(-),\]
where both sides are viewed as automorphic forms on $\tilde{M}_p(A)$. Proposition 2.42 implies
\[\text{Res}_{\nu=\rho_p/2} E_P(-, \Pi_\chi, f^\nu) = \text{Res}_{\nu=\rho_p/2} M(\nu, \Pi_\chi, w_0)f^\nu(-).\]

But the space generated by \(\left(\text{Res}_{\nu=\rho_p/2} M(\nu, \Pi_\chi, w_0)f^\nu\right)\big|_{\tilde{M}_p(A)}\) is equal to $w_0(\Pi_\chi^{\rho_p/2})\rho_p$ because the residue $\text{Res}_{\nu=\rho_p/2} M(\nu, \Pi_\chi, w_0)f^\nu$ is in the space of $\text{Ind}_{\tilde{M}_p(A)}^{\tilde{M}_p} w_0(\Pi_\chi^{\rho_p/2})$. (Recall how $f^\nu$ is defined in (2.37) as well as (2.38)). Finally the residue of the constant term of the Eisenstein series is the same as the constant term of the residue of the Eisenstein series. (To see this, note that the constant term is obtained by an integral over the compact set $N_p/F\setminus N_p(A)$, and the residue is obtained by an integral over a closed path around the singularity, which is again an integration over a compact set. Two integrations over compact sets can be interchanged.) This completes the proof.

\[\square\]

2.5. The exceptional representation of $\tilde{\text{GL}}_{2q}^{(2)}$. For our purposes, especially for taking care of the issue raised by Kable in his thesis ([K1]) for the case $r = 2q$, we need to construct the exceptional representation of $\tilde{\text{GL}}_{2q}^{(2)}$ both for the local and global cases. This exceptional representation is simply a constituent of the restriction of the twisted exceptional representation $\vartheta_\chi$ of $\tilde{\text{GL}}_{2q}$ constructed in the previous subsection. The important property of those representations (especially the global one) is the vanishing of many of the Whittaker-Fourier coefficients, which is essentially a generalization of the analogous property of the Weil representations of $\tilde{\text{GL}}_{2}$ as stated in Proposition 2.16 and 2.18. The use of this property of the exceptional representations is one of the key points for our unfolding argument for the Rankin-Selberg integral to be considered in the next section.

Local case:

For the local case, let us first note the following. We have the intertwining operator
\[A(\rho_p/2, \Pi_\chi, w_0) : \text{Ind}_{M_p^{(2)}N_p^{(2)}}^{\tilde{M}_p N_p^{(2)}} \Pi_\chi \otimes \delta_p^{1/4} \to \text{Ind}_{M_p^{(2)}N_p^{(2)}}^{\tilde{M}_p N_p^{(2)}} w_0(\Pi_\chi \otimes \delta_p^{1/4})\]
given by the integral as in (2.33). Also we have the identity
\[(\text{Ind}_{M_p^{(2)}N_p^{(2)}}^{\tilde{M}_p N_p^{(2)}} \Pi_\chi \otimes \delta_p^{1/4})|_{\text{Ind}_{\tilde{\text{GL}}_{2q}^{(2)}}^{\tilde{\text{GL}}_{2q}^{(2)}} \Pi_\chi \otimes \delta_p^{1/4}} = \bigoplus_{a \in \Sigma} \text{Ind}_{(M_p^{(2)}N_p^{(2)})^{(2)}}^{(M_p^{(2)}N_p^{(2)})^{(2)}} \varphi_a^{w_0} \otimes \delta_p^{1/4},\]
where for the first space the restriction of representation actually coincides with the restriction of functions in the induced space, and $\Sigma = (F^\times)^2 \setminus F^\times$. Hence by composing $A(\rho_p/2, \Pi_\chi, w_0)$ with restriction to $\tilde{\text{GL}}_{2q}^{(2)}$, we obtain a $\tilde{\text{GL}}_{2q}^{(2)}$ intertwining map
\[\bigoplus_{a \in \Sigma} \text{Ind}_{(M_p^{(2)}N_p^{(2)})^{(2)}}^{(M_p^{(2)}N_p^{(2)})^{(2)}} \varphi_a^{w_0} \otimes \delta_p^{1/4} \to \bigoplus_{a \in \Sigma} \text{Ind}_{(M_p^{(2)}N_p^{(2)})^{(2)}}^{(M_p^{(2)}N_p^{(2)})^{(2)}} w_0(\varphi_a^{w_0} \otimes \delta_p^{1/4}).\]
Moreover one can check by direct computation that the “a component” for each $a \in \Sigma$ on the left hand side maps into the component for the same $a$. Namely we have

$$A(\rho_p/2, \Pi_\chi, w_0) = \bigoplus_{a \in \Sigma} A(\rho_p/2, \varphi^a_\chi, w_0).$$

With this said, one can prove

**Proposition 2.46.** For the local twisted exceptional representation $\vartheta_\chi$ of $\widetilde{\text{GL}}_{2q}$, we have

$$\vartheta_\chi|_{\text{GL}_{2q}^{(2)}} = \bigoplus_{a \in \Sigma} \varphi^a_\chi,$$

where $\Sigma = (F^\times)^2 \setminus F^\times$ and $\varphi^a_\chi$ is a unique irreducible quotient of $\text{Ind}_{(M_P)^{(2)} N_P}^{(2)} \varphi^a_\chi \otimes \delta_{P}^{1/4}$. Moreover $\vartheta^a_\chi$ is the image of the intertwining integral $A(\rho_p/2, \varphi^a_\chi, w_0)$.

**Proof.** We have the commutative diagram

$$
\begin{array}{ccc}
(\text{Ind}_{(M_P)^{(2)} N_P}^{(2)} \Pi_\chi \otimes \delta_{P}^{1/4})|_{\text{GL}_{2q}^{(2)}} & = & \bigoplus_{a \in \Sigma} \text{Ind}_{(M_P)^{(2)} N_P}^{(2)} \varphi^a_\chi \otimes \delta_{P}^{1/4} \\
A(\rho_p/2, \Pi_\chi, w_0) & \downarrow & \bigoplus_{a \in \Sigma} \text{Ind}_{(M_P)^{(2)} N_P}^{(2)} \varphi^a_\chi \otimes \delta_{P}^{1/4} \\
\vartheta_\chi|_{\text{GL}_{2q}^{(2)}} & = & \bigoplus_{a \in \Sigma} \text{Ind}_{(M_P)^{(2)} N_P}^{(2)} \varphi^a_\chi \otimes \delta_{P}^{1/4}.
\end{array}
$$

Hence each irreducible constituent of $\vartheta_\chi|_{\text{GL}_{2q}^{(2)}}$ is an irreducible quotient of $\text{Ind}_{(M_P)^{(2)} N_P}^{(2)} \varphi^a_\chi \otimes \delta_{P}^{1/4}$ for some $a \in \Sigma$, which is the image of the intertwining integral $A(\rho_p/2, \varphi^a_\chi, w_0)$. Moreover, one and only one of the constituents of $\vartheta_\chi|_{\text{GL}_{2q}^{(2)}}$ appears as a quotient of each $\text{Ind}_{(M_P)^{(2)} N_P}^{(2)} \varphi^a_\chi \otimes \delta_{P}^{1/4}$ because the number of irreducible constituents of $\vartheta_\chi|_{\text{GL}_{2q}^{(2)}}$ is at most the size of $\Sigma$. This shows that for each $a \in \Sigma$, there is a unique subrepresentation $\varphi^a$ of $\text{Ind}_{(M_P)^{(2)} N_P}^{(2)} \varphi^a_\chi \otimes \delta_{P}^{1/4}$ that is obtained as the image of the intertwining integral $A(\rho_p/2, \varphi^a_\chi, w_0)$ such that $\vartheta_\chi|_{\text{GL}_{2q}^{(2)}} = \bigoplus_{a \in \Sigma} \varphi^a$.

To show the uniqueness, assume there exists $a_0 \in \Sigma$ such that $\text{Ind}_{(M_P)^{(2)} N_P}^{(2)} \varphi^{a_0} \otimes \delta_{P}^{1/4}$ has more than two quotients, say $\sigma_1$ and $\sigma_2$. For each $i = 1, 2$ and $a \in \Sigma$, let $s(i,a)$ be the representation of $\widetilde{\text{GL}}_{2q}$ obtained by twisting $\sigma_i$ by $s((1 \ 1), I_2, \cdots, I_2)$. Then the representation $\bigoplus_{a \in \Sigma} s(i,a)$ extends to a representation of $\widetilde{\text{GL}}_{2q}$ which can be seen as a quotient of $\text{Ind}_{(M_P)^{(2)} N_P}^{(2)} \varphi^{a_0} \otimes \delta_{P}^{1/4}$. But this induced representation has a unique irreducible quotient, namely $\vartheta_\chi$. Hence $\sigma_1 = \sigma_2$. \(\square\)

We call the representation $\vartheta^a_\chi$ constructed above “the exceptional representation” of $\widetilde{\text{GL}}_{2q}^{(2)}$.

This exceptional representation also has the periodicity property.

**Proposition 2.47** (Local Periodicity). Assume $F$ is non-archimedean. Let $(\vartheta_\chi)_N$ be he Jacquet module of $\vartheta_\chi$ along the parabolic $P^{(2)}$. Then

$$(\vartheta^a_\chi)_N = w^a(\varphi^a_\chi) \otimes \delta_{P}^{1/4} = \varphi^a_\chi \otimes \delta_{P}^{1/4}.$$
Global case:

Let us consider the global case, and construct the (twisted) exceptional representation \( \vartheta^\psi_\chi \) of \( \widetilde{\text{GL}}_{2q}(\mathbb{A}) \). In a way, the construction should be completely analogous to the \( \widetilde{\text{GL}}_2(\mathbb{A}) \) case and one would like to obtain \( \vartheta^\psi_\chi \) as the representation generated by the residues of the Eisenstein series. But a key ingredient missing for this construction is the Langlands theory of Eisenstein series for the group of the form \( \widetilde{\text{GL}}_{2q}(\mathbb{A}) \). Probably, there is no danger to assume that the theory of Eisenstein series for the metaplectic group as developed in [MW] can be carried over to \( \widetilde{\text{GL}}_{2q}(\mathbb{A}) \) at least to the extent necessary for the construction of the exceptional representation. If one takes this for granted, the exceptional representation \( \vartheta^\psi_\chi \) of \( \widetilde{\text{GL}}_{2q}(\mathbb{A}) \) can be constructed in the same way as the exceptional representation \( \vartheta_\chi \) of \( \text{GL}_{2q}(\mathbb{A}) \). However here we give an alternate approach, in which we will show that the exceptional representation \( \vartheta^\psi_\chi \) of \( \widetilde{\text{GL}}_{2q}(\mathbb{A}) \) is simply a constituent of the restriction to \( \widetilde{\text{GL}}_{2q}(\mathbb{A}) \) of the exceptional representation \( \vartheta_\chi \) of \( \text{GL}_{2q}(\mathbb{A}) \). Here by restriction we mean the restriction of automorphic forms as functions, not restriction of abstract representation. Namely we have

**Proposition 2.48.** For the global exceptional representation \( \vartheta_\chi \) of \( \widetilde{\text{GL}}_{2q}(\mathbb{A}) \), let \( \vartheta^{(2)}_\chi \) be the representation of \( \widetilde{\text{GL}}_{2q}(\mathbb{A}) \) whose space is \( \{ f|_{\widetilde{\text{GL}}_{2q}(\mathbb{A})} : f \in \vartheta_\chi \} \), namely the space of restrictions to \( \widetilde{\text{GL}}_{2q}(\mathbb{A}) \) of automorphic forms in \( \vartheta_\chi \). Then we have

\[
\vartheta^{(2)}_\chi = \bigoplus_{a \in \Sigma} \vartheta^{\psi_a}_\chi,
\]

where \( \vartheta^{\psi_a}_\chi \) is an irreducible quotient of the global induced representation

\[
\text{Ind}_{\text{GL}_{2q}(\mathbb{A})}^{\widetilde{\text{GL}}_{2q}(\mathbb{A})} \vartheta^{(2)}_\chi \otimes \delta^{3/4}_P,
\]

and \( \Sigma = (F^\times)^2 \setminus F^\times \).

**Proof.** Recall from our construction of \( \vartheta_\chi \) in the previous subsection that \( \vartheta_\chi \) is constructed as the residual representation of the Eisenstein series on \( \text{Ind}_{\text{GL}_{2q}(\mathbb{A})}^{\text{GL}_{2q}(\mathbb{A})} \Pi^{\nu}_\chi \) at \( \nu = \rho_P/2 \). For each \( f^{\nu} \) in this space, we have defined the Eisenstein series by

\[
E(g, \Pi^{\nu}_\chi, f^{\nu}) = \sum_{\gamma \in P(F) \setminus \text{GL}_{2q}(F)} f^{\nu}(\gamma g).
\]

But note that

\[
P(F) \setminus \text{GL}_{2q}(F) = (M_P)^{(2)}(F) N_P(F) \setminus \text{GL}_{2q}^{(2)}(F),
\]

and so one can write

\[
E(g, \Pi^{\nu}_\chi, f^{\nu}) = \sum_{\gamma \in (M_P)^{(2)}(F) N_P(F) \setminus \text{GL}_{2q}^{(2)}(F)} f^{\nu}(\gamma g).
\]

Hence we have

\[
E(-, \Pi^{\nu}_\chi, f^{\nu})|_{\widetilde{\text{GL}}_{2q}(\mathbb{A})} = E(-, \Pi^{\nu}_\chi, f^{\nu})|_{\text{GL}_{2q}(\mathbb{A})},
\]

where the latter may be called the “Eisenstein series” on \( \widetilde{\text{GL}}_{2q}(\mathbb{A}) \).
To consider \( f' \mid \widetilde{\Gamma}_{2q}(\mathbb{A}) \), note that the induced representation \( \text{Ind}_{\dot{\GL}_{2q}^x(\mathbb{A})}^{\text{GL}_{2q}^x(\mathbb{A})} \varpi^{\psi, \nu} \) is also realized in a space of automorphic forms on \( N_P(\mathbb{A})M_P(\mathbb{A})^{(2)}(F) \backslash \widetilde{\text{GL}}_{2q}^x(\mathbb{A}) \). The map \( f'' \mapsto f'' \mid \widetilde{\Gamma}_{2q}(\mathbb{A}) \) gives a \( \text{GL}_{2q}^x(\mathbb{A}) \) surjection to see this, recall from (2.37) and (2.38) that \( f'' = \phi \otimes \exp(\nu + \rho_P, H_P(\ )\ ) \) where \( \phi \) is such that the function \( \phi_k \) is in \( \Pi_\chi \). Then \( f'' \mid \Gamma_{2q}(\mathbb{A}) = \phi \mid \Gamma_{2q}(\mathbb{A}) \otimes \exp(\nu + \rho_P, H_P(\ )) \), where the Harish-Chandra map is also restricted to \( \widetilde{\text{GL}}_{2q}^x \). Hence the map \( m \mapsto \phi \mid \text{Ind}_{\Gamma_{2q}(\mathbb{A})}^{\text{GL}_{2q}^x(\mathbb{A})} (mk) \) where \( m \in (\widetilde{M}_P(\mathbb{A})) \) and \( k \in \widetilde{K} \cap \widetilde{\text{GL}}_{2q}^x(\mathbb{A}), \) is in \( \Pi_\chi^{(2)} \) in the notation of Proposition 2.28. Then use Proposition 2.28.

Now if one chooses \( f'' \in \text{Ind}_{\dot{\Gamma}_{2q}^x(\mathbb{A})}^{\text{GL}_{2q}^x(\mathbb{A})} \Pi_\chi \) so that its image under the above restriction map is in \( \text{Ind}_{\dot{\Gamma}_{2q}^x(\mathbb{A})}^{\text{GL}_{2q}^x(\mathbb{A})} \varpi^{\psi, \nu} \) for a fixed \( a \in \Sigma \), then the restriction of the Eisenstein series \( E(g, \Pi_\chi, f'') \) is the Eisenstein series associated to \( \text{Ind}_{\dot{\Gamma}_{2q}^x(\mathbb{A})}^{\text{GL}_{2q}^x(\mathbb{A})} \varpi^{\psi, \nu} \).

Therefore at \( \nu = \rho_P/2 \), by taking the residues, we have the commutative diagram of \( \text{GL}_{2q}^x(\mathbb{A}) \)-intertwining maps

\[
\begin{array}{c}
\text{Ind}_{\dot{\Gamma}_{2q}^x(\mathbb{A})}^{\text{GL}_{2q}^x(\mathbb{A})} \Pi_\chi \\
\downarrow E_{-1} \\
\vartheta_\chi \\
\downarrow \oplus E_{-1} \\
\bigoplus_{a \in \Sigma} V^a,
\end{array}
\]

where the vertical arrows are given by residue of Eisenstein series, the horizontal arrows given by restriction of functions, and each \( V^a \) is the image of each \( \text{Ind}_{\dot{\Gamma}_{2q}^x(\mathbb{A})}^{\text{GL}_{2q}^x(\mathbb{A})} \varpi^{\psi, \nu} \).

We need to show that each \( V^a \) is \( \vartheta_\chi^{\psi^a} \) as claimed in the statement of the proposition, namely we need to show it is irreducible. But the square integrability of \( \vartheta_\chi \) implies that of \( V^a \), which implies complete reducibility. Hence if \( V^a \) is not irreducible, there is a place \( v \) at which the \( v \)-component \( V^a_v \) is a direct sum of more than two irreducible representations of \( \text{GL}_{2q}(F_v) \). But since \( V^a_v \) is a quotient of the corresponding local induced representation, this contradicts the uniqueness part of Proposition 2.46. Hence \( V^a \) is irreducible.

Note that the representation \( \vartheta_\chi \) is dependent on \( \psi \), and hence the notation.

Finally we need to prove the global periodicity property of \( \vartheta_\chi \).

**Proposition 2.49 (Global Periodicity).** Let \( \vartheta_\chi \mid N_P \) be the space generated by the constant terms of automorphic forms in \( \vartheta_\chi \) along the parabolic \( \dot{P}(\mathbb{A}) \cap \text{GL}_{2q}^x(\mathbb{A}) = (\dot{M}_P(\mathbb{A})) N_P(\mathbb{A}) \). Then we have

\[
(\vartheta_\chi \mid N_P) = u^a(\varpi^{\psi}_\chi \otimes \delta^1_P \otimes \delta^1_P) = \varpi^{\psi}_\chi \otimes \delta^1_P.
\]
Proof. This follows from the above proposition and Proposition 2.43 or strictly speaking from their proofs. Namely from the proof of the above proposition, one knows that each element in $\vartheta_\chi^N$ is written as

$$ \operatorname{Res}_{\nu=\rho P/2} E(-, \Pi_\chi, f^\nu|_{\widetilde{GL}_{2q}(\mathbb{A})}) $$

for some $f^\nu$ so that $f^\nu|_{\widetilde{GL}_{2q}(\mathbb{A})} \in \text{Ind}_{\widetilde{GL}_r(\mathbb{A})}^{\widetilde{GL}_{2q}(\mathbb{A})} \varpi_\chi^a \otimes \vartheta^\nu$, and since the constant term is computed by integrating along $N_P(F)\backslash N_P(\mathbb{A})$, each element in $(\vartheta_\chi^N)^N_P$ is generated by the elements of the form

$$ \operatorname{Res}_{\nu=\rho P/2} E_P(-, \Pi_\chi, f^\nu|_{\widetilde{GL}_{2q}(\mathbb{A})}), $$

where the notation $E_P$ is as in the proof of Proposition 2.43. But from the proof of Proposition 2.43 we have

$$ \operatorname{Res}_{\nu=\rho P/2} E_P(-, \Pi_\chi, f^\nu|_{\widetilde{GL}_{2q}(\mathbb{A})}) = \operatorname{Res}_{\nu=\rho P/2} M(\nu, \Pi_\chi, w_0)f^\nu|_{\widetilde{GL}_{2q}(\mathbb{A})}, $$

where both sides are viewed as forms on $(\widetilde{M}_P)^{(2)}(\mathbb{A})$. Here note that $M(\nu, \Pi_\chi, w_0)(f^\nu|_{\widetilde{GL}_{2q}(\mathbb{A})}) = (M(\nu, \Pi_\chi, w_0)f^\nu)|_{\widetilde{GL}_{2q}(\mathbb{A})}$ and that is why we can simply write $M(\nu, \Pi_\chi, w_0)f^\nu|_{\widetilde{GL}_{2q}(\mathbb{A})}$. Viewed as forms on $(\widetilde{M}_P)^{(2)}(\mathbb{A})$, we have

$$ \operatorname{Res}_{\nu=\rho P/2} M(\nu, \Pi_\chi, w_0)f^\nu|_{\widetilde{GL}_{2q}(\mathbb{A})} \in w_0(\varpi_\chi^a \otimes \delta_p^{1/4}) \otimes \delta_p^{1/4}, $$

and $w_0(\varpi_\chi^a \otimes \delta_p^{1/4}) \otimes \delta_p^{1/4} = w_0(\varpi_\chi^a) \otimes \delta_p^{1/4} = \varpi_\chi^a \otimes \delta_p^{1/4}$. 

Finally, in this subsection let us mention that under $\vartheta_\chi^N$, the center $\mathbb{Z}$ of $\widetilde{GL}_{2q}^{(2)}$ acts by the character

$$ (2.50) \quad (1, \xi)s(z) \mapsto \xi\chi(a)^q \mu_\psi(a)^q, \quad z = \begin{pmatrix} a \\ \vdots \\ a \end{pmatrix}. $$

This follows from (2.10).

2.6. **The semi-Whittaker functional on the exceptional representation.** One of the key properties that we need for the exceptional representations $\theta_\chi, \vartheta_\chi$ and $\vartheta_\psi^N$ is that they do not possess Whittaker functionals (unless $r = 2$), but instead they possess what Bump and Ginzburg call the semi-Whittaker functionals. This fact essentially follows from the periodicity property for those exceptional representations. To recall this notion, let us define the character $\psi_\chi^N$ on $N$ by

$$ \psi_\chi^N \left( \begin{array}{cccc} 1 & x_{12} & \cdots & x_{1r} \\ 1 & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 1 & & & 1 \end{array} \right) = \psi(x_{r-1,r} + x_{r-3,r-2} + x_{r-5,r-4} + \cdots). $$

Then

**Proposition 2.51.** Assume $F$ is non-archimedean, and $\theta$ is any of the exceptional representations of $\widetilde{GL}_r(F)$ (or $\widetilde{GL}_r^{(2)}(F)$). Then there is a unique (up to scalar multiple) semi-Whittaker functional $L$ on $\theta$, i.e. a linear functional $L$ on $\theta$ such that

$$ L(\theta(s(n))v) = \psi_\chi^N(n)L(v) $$

for $v \in V$ and $n \in N(F)$. 
Proof. For the non-twisted case $\theta = \theta_\chi$, this is [BG] Theorem 1.4. For the twisted case $\theta = \vartheta_\chi$ or $\vartheta_\psi$, this can be shown in the same way as [BG] by the periodicity property of $\vartheta_\chi$ and $\vartheta_\psi$ (Proposition 2.36 and 2.47). For example, assume $\theta = \vartheta_\psi$. Then the semi-Whittaker functional is simply the composite of the surjection $\vartheta_\psi \rightarrow \varpi_{\psi} \otimes \delta_{p}^{1/2}$ with the Whittaker functional on the Weil representation $\varpi_{\psi}$. The uniqueness follows from the uniqueness of the Whittaker functional of the Weil representation. □

Remark 2.52. The important remark we have to make here is that the uniqueness of the semi-Whittaker functional can be shown only for non-archimedean $F$, because the proof requires the periodicity of the Jacquet module of $\theta$, which is available only for the non-archimedean $F$. Though this might hold for the archimedean case as well, at this moment the author does not know if the same technique can be applied to the archimedean case. (In [BG] it is simply stated without any proof or reference that the periodicity holds for the archimedean case as well.) Because of this lack of the uniqueness of the semi-Whittaker functional for the archimedean places, it does not seem to be possible to prove the Euler product of the Rankin-Selberg integral. But to get around this, we obtain the “almost Euler product”, which is enough for our proof of the main theorem of this paper.

2.7. Exceptional representations of $\widetilde{\text{GL}}_{r-1} \times \text{GL}_1$. Lastly in this section, we consider the exceptional representation of $\widetilde{\text{GL}}_{r-1} \times \text{GL}_1$. Indeed the notion of the exceptional representation can be generalized to the group $\widetilde{\text{GL}}_{r} \times \cdots \times \text{GL}_k$ both for twisted and non-twisted cases following the method discussed in [BG] p.142-143]. But here we specialize only to the case $\widetilde{\text{GL}}_{r-1} \times \text{GL}_1$.

Let $Q$ be the $(r-1,1)$-parabolic subgroup of $\text{GL}_r$ whose Levi part is $M_Q = \text{GL}_{r-1} \times \text{GL}_1$. Naively speaking, the exceptional representation of $\widetilde{M}_Q$ is the tensor product of the exceptional representation of $\widetilde{\text{GL}}_{r-1}$ and a character on $\text{GL}_1$. Things will slightly differ, depending on the parity of $r$.

Even case ($r = 2q$)

Assume $r$ is even, so $r = 2q$. Fix $a \in F^\times$, where $F$ is either local or global. Define a character $\psi_a$ on $T^\circ$ by

$$\psi_a((1, \xi)s(t)) = \xi(\det t)\mu_{\psi_a}(t_1)^{-1}\mu_{\psi}(t_3)^{-1}\mu_{\psi}(t_5)^{-1}\cdots\mu_{\psi}(t_{2q-1})^{-1}.$$ (2.53)

Here unlike (2.3), we use $\mu_{\psi}^{-1}$. Also for the first factor we use the character $\psi_a$. This modification is needed for later purposes.

Let $B'$ be the Borel subgroup of $M_Q$, namely $M_Q \cap B$. For each $\nu \in \Phi_{B'}$, we define $\omega_{\chi}^{\psi_a, \nu} := \omega_{\chi}^{\psi_a} \otimes \exp(\nu, H_{B'}(\big)).$ Then we have

Proposition 2.54. The induced representation $\text{Ind}_{\widetilde{\text{F}}^{\times}N^{\times}}^{\widetilde{\text{M}}_{Q}} \omega_{\chi}^{\psi_a, \nu}$ has a unique irreducible quotient at $\nu = \rho_{B'}/2$, which we denote by $\widetilde{\theta}_{\chi}$. This is independent of $\psi$ and $a$. If $F$ is global, it is a square integrable automorphic representation in the residual spectrum of $\widetilde{M}_Q(k)$.

Proof. This is nothing but what Bump and Ginzburg call the exceptional representation of $\widetilde{\text{GL}}_{r-1} \times \text{GL}_1$ in [BG] p.142-143]. To show that it is independent of $\psi$ and $a$, one can argue as we did for $\theta_\psi$ for the even case. □

For a (local or global) character $\eta$, define the character on $\widetilde{M}_Q$ by

$$((g,a),\xi) \mapsto \eta(a)$$
for \((g,a) \in M_Q\), namely the composition of \(\eta\) with the projections \(\overline{M}_Q^{(2)} \rightarrow M_Q \rightarrow \text{GL}_1\). We denote this character again by \(\eta\). We let
\[
\theta_{\chi,\eta} := \tilde{\theta}_{\chi} \otimes \eta,
\]
and call it the exceptional representation of \(\overline{M}_Q\) associated with the characters \(\chi\) and \(\eta\).

Let \((\overline{M}_Q)^{(2)}\) be the metaplectic preimage of
\[
(M_Q)^{(2)} := \{(g, a) \in \text{GL}_{r-1} \times \text{GL}_1 : (\det g)a \text{ is a square}\}.
\]

As we did before for the exceptional representations of \(\text{GL}_r\), the restriction of \(\theta_{\chi,\eta}\) to \((\overline{M}_Q)^{(2)}\) is described as follows.

**Proposition 2.55.** Assume \(F\) is local. Then we have the decomposition
\[
\theta_{\chi,\eta}|_{(\overline{M}_Q)^{(2)}(F)} = \bigoplus_{a \in \Sigma} \theta_{\chi,\eta}^{\psi_a},
\]
where \(\theta_{\chi,\eta}^{\psi_a}\) is a unique irreducible quotient of the induced representation \(\text{Ind}_{T^{(2)}(F),N(F)}^{(\overline{M}_Q)^{(2)}(F)} \psi_a \otimes \delta_B^{-1/4}\), and \(\Sigma = (F^\times)^2 \setminus F^\times\).

Assume \(F\) is global and let \((\theta_{\chi,\eta})^{(2)}\) be the space of the restrictions \((\overline{M}_Q)^{(2)}(\mathfrak{A})\) of the automorphic forms in \(\theta_{\chi,\eta}\). As representations of \((\overline{M}_Q)^{(2)}(\mathfrak{A})\) we have the decomposition
\[
(\theta_{\chi,\eta})^{(2)} = \bigoplus_{a \in \Sigma} \theta_{\chi,\eta}^{\psi_a},
\]
where \(\theta_{\chi,\eta}^{\psi_a}\) is an irreducible quotient of the global induced representation \(\text{Ind}_{T^{(2)}(\mathfrak{A}),N^{\mathfrak{A}}(\mathfrak{A})}^{(\overline{M}_Q)^{(2)}(\mathfrak{A})} \omega_B^{\psi_a} \otimes \delta_B^{-1/4}\), and \(\Sigma = (F^\times)^2 \setminus F^\times\).

**Proof.** This can be proven in the same way as Proposition 2.46 and 2.48. \(\square\)

We call \(\theta_{\chi,\eta}^{\psi_a}\) the exceptional representation of \((\overline{M}_Q)^{(2)}\). We are mainly interested in \(\theta_{\chi,\eta}^{\psi_a}\), i.e. \(a = 1\). Also let us note that \(\tilde{Z}\) is in the center of \((\overline{M}_Q)^{(2)}\), and each element \((1, \xi)s(z)\) acts as
\[
\theta_{\chi,\eta}^{\psi_a}((1, \xi)s(z)) = \xi \chi^q(a) \eta(a) \mu_{\psi}(a)^{-q}, \quad z = \begin{pmatrix} a \\ \vdots \\ a \end{pmatrix} \in \text{GL}_{2q}.
\]

As we mentioned in Note 2.8, whether \(q\) is even or odd, the map \(z \mapsto \mu_{\psi}(a)^{-q}\) is indeed a character.

**Odd case** \((r = 2q + 1)\)

Next we will consider the odd case. But this case is much simpler because \(\tilde{Z}\) is the center of \(\text{GL}_{2q+1}\). First consider \(\overline{M}_Q\) as in Appendix A. Note that \(\tilde{Z}\) is in the center of \(\overline{M}_Q\), and \(\text{GL}_{2q}\) naturally embeds into \(\overline{M}_Q\) by \((g, \xi) \mapsto ((g, 1), \xi)\). So we have \(\tilde{Z}\text{GL}_{2q} = \overline{M}_Q\). Moreover inside \(\overline{M}_Q\), we have \(\tilde{Z} \cap \text{GL}_{2q} = \{(1, \xi)\}\).

Let \(\vartheta_{\chi}\) be the (local or global) exceptional representation of \(\text{GL}_{2q}\), where we include the case \(\chi^{1/2}\) exists, and \(\eta\) a (local or global) character. We define the representation
\[
\vartheta_{\chi,\eta}^{\psi_a}
\]
of $\hat{\mathcal{M}}_Q$ by extending $\vartheta_\chi$ to a representation of $\hat{\mathbb{G}}L_{2q} = \hat{\mathcal{M}}_Q$ by letting the center $\hat{\mathbb{Z}} \subseteq \hat{\mathcal{M}}_Q$ act as

\begin{equation}
(1, \xi)s_Q(z) \mapsto \xi(\chi(a))\eta(a)\mu_\psi(a)^{-q}, \quad z = \begin{pmatrix} a & \cdots \\ \cdots & a \end{pmatrix},
\end{equation}

where $s_Q : \mathcal{M}_Q \to \hat{\mathcal{M}}_Q$ is the map defined by $(g, a) \mapsto ((g, a), s_2q(g)^{-1}s_1(a)^{-1})$ for $g \in \mathbb{G}L_{2q}$ and $a \in \mathbb{G}L_1$. (Strictly speaking it is a partial map if $F$ is global, whose domain includes $B(\mathbb{A})$ and hence $Z(\mathbb{A})$.) As in the even case, the map $z \mapsto \mu_\psi(a)^{-q}$ is a indeed a character for any $q$.

Now we identify $\hat{\mathcal{M}}_Q$ with the subgroup $\hat{\mathcal{M}}_Q$ of $\mathbb{G}L_{2q+1}$ via $\varphi_Q : \hat{\mathcal{M}}_Q \to \hat{\mathcal{M}}_Q$. (See Appendix A.) Then we call the representation $\vartheta_\chi,\eta \circ \varphi_Q^{-1}$ the exceptional representation of $\hat{\mathcal{M}}_Q$, which we simply denote by $\vartheta_\chi,\eta$ by abuse of notation. The central character acts in the same way as in (2.57) with $s_Q$ replaced by $s$.

Finally for this subsection let us mention

**Lemma 2.58.** Let $r = 2q$ or $2q + 1$. Also assume $F$ is a non-archimedean local field of odd residue characteristic. Further assume that all of $\chi, \eta$ and $\psi$ are unramified. Consider the intertwining operators

\[ A(s, \theta^\psi_{\chi,\eta}, w_0) : \text{Ind}_{\hat{\mathcal{M}}_Q}^{\mathbb{G}L_{2q}^{(2)}(\mathbb{A})} \theta^\psi_{\chi,\eta} \otimes \delta_Q \to \text{Ind}_{\hat{\mathcal{M}}_Q}^{\mathbb{G}L_{2q}^{(2)}(\mathbb{A})} w_0(\theta^\psi_{\chi,\eta} \otimes \delta_Q^{-s}), \quad (r = 2q) \]

\[ A(s, \vartheta_{\chi,\eta}, w_0) : \text{Ind}_{\hat{\mathcal{M}}_Q}^{\mathbb{G}L_{2q+1}^{(2)}(\mathbb{A})} \vartheta_{\chi,\eta} \otimes \delta_Q \to \text{Ind}_{\hat{\mathcal{M}}_Q}^{\mathbb{G}L_{2q+1}^{(2)}(\mathbb{A})} w_0(\vartheta_{\chi,\eta} \otimes \delta_Q^{-s}), \quad (r = 2q + 1), \]

where $w_0 = \left( \begin{array}{c} I_{r-1} \\ 1 \end{array} \right)$ and $N_{(1, r-1)}$ is the unipotent radical of the standard $(1, r-1)$-parabolic.

If $f_0^s \in \text{Ind}_{\hat{\mathcal{M}}_Q}^{\mathbb{G}L_{2q}^{(2)}(\mathbb{A})} \theta^\psi_{\chi,\eta} \otimes \delta_Q$ (or $\text{Ind}_{\hat{\mathcal{M}}_Q}^{\mathbb{G}L_{2q+1}^{(2)}(\mathbb{A})} \vartheta_{\chi,\eta} \otimes \delta_Q$) is the spherical section such that $f_0^s(1) = 1$, then

\[ A(s, \theta^\psi_{\chi,\eta}, w_0)f_0^s(1) = \frac{L(r(2s + 1) - r + 1, \eta^{-2})}{L(r(2s + q + 1, \eta^{-2})}, \quad (r = 2q) \]

\[ A(s, \vartheta_{\chi,\eta}, w_0)f_0^s(1) = \frac{L(r(2s + 1) - r + 1, \chi \eta^{-2})}{L(r(2s + q + 1, \chi \eta^{-2})}, \quad (r = 2q + 1). \]

**Proof.** This is derived from the unramified computation by Kazhdan and Patterson just as we did for Lemma 2.39. Since the computation is straightforward, though quite tedious, we will omit the details. Also this generalizes [BG, Proposition 5.6]. See the proof there as well. \[ \square \]

### 3. The Rankin-Selberg integrals for the case $r = 2q$

In this section, we consider the Rankin-Selberg integral for the cuspidal representation $\pi$ of $\mathbb{G}L_{r}(\mathbb{A})$ when $r$ is even. So throughout this section, we assume that

\[ r = 2q = \text{even}, \]

and $F$ is a number field, unless stated otherwise. We let $\chi$ be a unitary Hecke character on $\mathbb{A}^\times$ and $\omega$ the central character of $\pi$.

We let

\[ \theta := \psi^\psi_{\chi \omega^{-2}} \]
be the global twisted exceptional representation of $GL_r(\hat{A})$ associated with the character $\chi \omega^{-2}$. Also we let
\[ \theta' := \theta_{\omega, \omega^{-1} \chi^{-q}} \]
be the global exceptional representation of $\tilde{M}_Q(\hat{A}) = GL_{r-1}(\hat{A}) \times GL_1(\hat{A})$ associated with the characters $\omega$ and $\omega^{-1} \chi^{-q}$.

The global decomposition in Proposition [2.54] implies that we have $\tilde{\omega} = GL_2(\hat{A})$ intertwining operator
\[ \text{ind}_{\tilde{Q}(\hat{A})}^{GL_r(\hat{A})} \theta' \otimes \delta_Q \longrightarrow \bigoplus_{a \in \Sigma} \text{ind}_{\tilde{Q}(\hat{A})}^{GL_r(\hat{A})} (\tilde{M}_Q(\hat{A})) \cdot \theta_{\omega, \omega^{-1} \chi^{-q}}^a \otimes \delta^r_Q \]
by restriction of functions. Here we assume the induction is NOT normalized to be consistent with the convention in [BG].

In this section, unlike what we did in the previous section, we view each section $f^s$ as a map
\[ f^s : GL_r(\hat{A}) \longrightarrow \text{space of } \theta' \otimes \delta_Q^s, \]
and hence for each $\tilde{g} \in \tilde{GL}_r(\hat{A})$, $f^s(\tilde{g})$ is a function on $\tilde{M}_Q(\hat{A})$. We sometimes need to evaluate $f^s(\tilde{g})$ for each $\tilde{m} \in \tilde{M}_Q(\hat{A})$. But we avoid the notation $f^s(\tilde{g})(\tilde{m})$, but rather use the notation $f^s(\tilde{g}; \tilde{m})$. The advantage of this notation is that if we have another $\tilde{m}_1 \in \tilde{M}_Q(\hat{A})$, then the translation of $\tilde{m}_1$ from the first variable to the second is more naturally written like $f^s(\tilde{m}_1; \tilde{g}; \tilde{m}_1)$.

Choose a section $f^s$ so that its image under the above surjection is in $\text{ind}_{\tilde{Q}(\hat{A})}^{GL_r(\hat{A})} \theta_{\omega, \omega^{-1} \chi^{-q}}^1 \otimes \delta_Q^1$, i.e. the component for $a = 1$. Let $E(-, s, f^s)$ be the Eisenstein series on $\tilde{GL}_r(\hat{A})$ associated with $f^s$. To be precise,
\[ E(\tilde{g}, s, f^s) = \sum_{\gamma \in Q(F) \cap GL_r(F)} f^s(s(\gamma) \tilde{g}; e), \]
where $\tilde{g} \in \tilde{GL}_r(\hat{A})$ and $e$ is the identity element in $GL_r(\hat{A})$. Note that the group $GL_r(F)$ is viewed as a subgroup of $GL_r(\hat{A})$ via the splitting $s$ and we simply write $GL_r(F)$ for $GL_r(\hat{A})$. By an easy calculation, one sees that
\[ Q(F) \cap GL_r(F) = Q_{r-1}(F) \cap GL_r^{(2)}(F), \]
where
\[ Q_{r-1}(F) := Q(F) \cap GL_r^{(2)}(F) = \{ g \in Q : \det g \in (F^x)^2 \} = \{ \begin{pmatrix} h & * \\ 0 & a \end{pmatrix} : (\det h)a \in (F^x)^2 \}, \]
and hence
\[ E(\tilde{g}, s, f^s) = \sum_{\gamma \in Q_{r-1}(F) \cap GL_r^{(2)}(F)} f^s(s(\gamma) \tilde{g}; e). \]
(The reason for the notation $Q_{r-1}$ will be clear in due course.) Hence the restriction of the Eisenstein series $E(-, s, f^s)$ to $GL_r(\hat{A})$ is the “Eisenstein series” on $\tilde{GL}_r(\hat{A})$ associated with the induced representation $\text{ind}_{\tilde{Q}(\hat{A})}^{GL_r^{(2)}(\hat{A})} \theta_{\omega, \omega^{-1} \chi^{-q}} \otimes \delta_Q^a$.

Let $\Theta$ be an automorphic form in the space of $\theta$. Since $\Theta(\tilde{g})$ and $E(\tilde{g}, s, f^s)$ are genuine automorphic forms on $\tilde{GL}_r(\hat{A})$, their product is a function on $GL_r(\hat{A})$ in the sense that if $\tilde{g} \in \tilde{GL}_r(\hat{A})$ is any of the preimages of $g \in GL_r(A)$, then the function $g \mapsto \Theta(\tilde{g})E(\tilde{g}, s, f^s)$ is independent of the choice of $\tilde{g}$.

Next let us consider how the center $\tilde{Z}(\hat{A})$ acts. Let $(1, \xi)s(z) \in \tilde{Z}(\hat{A})$ with $z = aI_r$ and $a \in \hat{A}^\times$. By (2.50),
\[ \Theta((1, \xi)s(z)) = \xi \chi(a)^q \omega(a)^{-2q} \mu_\psi(a)^a \Theta(e), \]
and by \eqref{eq:2.56},
\begin{equation}
E((1, \xi)s(z), s, f^*) = \xi \omega(a)^{2q-1} \chi^{-q}(a) \mu_\psi(a)^{-q} E(e, s, f^*),
\end{equation}
where \( e \) is the identity element in \( \GL_r(\mathbb{A}) \). Hence on the product \( \Theta(-)E(-, s, f^*) \), the center acts as \( \omega^{-1} \).

Now for a cusp form \( \phi \in \pi \), the Rankin-Selberg integral we consider is
\begin{equation}
Z(\phi, \Theta, f^*) = \int_{Z(\mathbb{A}) \GL^{(2)}(F) \setminus \GL^{(2)}(\mathbb{A})} \phi(g)\Theta(\kappa(g))E(\kappa(g), s, f^*)dg.
\end{equation}
Note that since \( \omega \) is the central character of \( \pi \) and as we have seen, on the product \( \Theta(-)E(-, s, f^*) \) the center acts by \( \omega^{-1} \), this integral is well-defined in the sense that the center \( Z(\mathbb{A}) \) acts trivially for the integrand. (Strictly speaking, one needs to use the fact that the product \( \Theta(\tilde{g})E(\tilde{g}, s, f^*) \) is independent of the choice of \( \tilde{g} \) to check that the center acts by \( \omega^{-1} \).) The reader should notice that our integral differs from the one in \[BG\] (3.4)]. (As is pointed out by Kable \[K1, Appendix\], the integral in \[BG\] for the case \( r = 2q \) cannot be well-defined.)

However, if we define \( Z(\phi, \Theta, f^*) \) in this way, we cannot obtain the desired Euler product simply by following the computation of \[BG\]. Instead, we have to take an alternate approach. But first note that
\begin{equation}
Z(\phi, \Theta, f^*) = \int_{Z(\mathbb{A}) \GL^{(2)}(F) \setminus \GL^{(2)}(\mathbb{A})} \phi(g)\Theta(\kappa(g)) \sum_{\gamma \in Q_{r-1}(F) \setminus \GL^{(2)}(F)} f^*(s(\gamma)\kappa(g); e)dg
= \int_{Z(\mathbb{A}) \GL^{(2)}(F) \setminus \GL^{(2)}(\mathbb{A})} \sum_{\gamma \in Q_{r-1}(F) \setminus \GL^{(2)}(F)} \phi(\gamma g)\Theta(s(\gamma)\kappa(g))f^*(s(\gamma)\kappa(g); e)dg.
\end{equation}

Now we would like to collapse the sum as usual using \( \gamma \). To do it, we would like to write
\begin{equation}
s(\gamma)\kappa(g) = \kappa(\gamma g).
\end{equation}
But this equality does not hold in general. Yet, the fact that both \( \Theta \) and \( f^* \) are genuine allows one to do this manipulation. Let us explain this more in detail. For each \( \gamma \) and \( g \), there is \( \xi = \xi(\gamma, g) \in \{ \pm 1 \} \), depending on \( \gamma \) and \( g \), such that
\begin{equation}
s(\gamma)\kappa(g) = (1, \xi)\kappa(\gamma g).
\end{equation}
Since the induced representation is genuine, we have
\begin{equation}
f^*(s(\gamma)\kappa(g); e) = f^*((1, \xi)\kappa(\gamma g); e) = \xi f^*(\kappa(\gamma g); e).
\end{equation}
The same consideration regarding \( s(\gamma)\kappa(g) \) and \( \kappa(\gamma g) \) applies to \( \Theta \). Then we have two \( \xi \), one from \( f^* \) and the other from \( \Theta \), and they get cancelled out when \( f^* \) and \( \Theta \) are multiplied. Hence
\begin{equation}
\Theta(s(\gamma)\kappa(g))f^*(s(\gamma)\kappa(g); e) = \Theta(\kappa(\gamma g))f^*(\kappa(\gamma g); e).
\end{equation}
Namely the “genuineness” of \( \Theta \) and \( f^* \) takes care of the discrepancy between \( s(\gamma)\kappa(g) \) and \( \kappa(\gamma g) \). This trick allows one to exchange \( s \) and \( \kappa \) freely, as long as one does the same to both \( \Theta \) and \( f^* \) at the same time. Since we need this trick regularly, we call it the “\( s - \kappa \) trick”.

Now we are allowed to collapse the sum and obtain
\begin{equation}
Z(\phi, \Theta, f^*) = \int_{Z(\mathbb{A})Q_{r-1}(F) \setminus \GL^{(2)}(\mathbb{A})} \phi(g)\Theta(\kappa(g))f^*(\kappa(g); e)dg.
\end{equation}
To get the desired Euler product, [BG] used the well-know Fourier expansion of the cusp form $\phi$ and collapsed the sum in the Fourier expansion. But in our case, their method does not work because the integral is over $GL_r^{(2)}$ instead of $GL_r$, or roughly put, the group $Q_{r-1}(F)$ is not large enough to collapse the sum in the Fourier expansion of $\phi$. To get around this, we carry out “column-by-column” computations of the Fourier coefficients of $\Theta$ and $f^s$ together with some of the properties of the exceptional representations we developed in the previous section.

But before going into the computation, let us fix notations. Let $N$ be the unipotent radical of the Borel subgroup $B$ of $GL_r$. For an integer $1 \leq m \leq r-1$, we define $N_m$ to be the subgroup of $N$ consisting of the elements whose only non-zero entries off the diagonal are in the $m+1$st column, i.e.

$$N_m = \{ \begin{pmatrix} I_m & * \\ 1 & 0 \\ I_{r-m-1} \end{pmatrix} \}.$$ 

Note that

$$N = N_{r-1}N_{r-2}\cdots N_1.$$ 

Also note that $N_m(F) \setminus N_m(\mathbb{A})$ is a compact abelian group, which is isomorphic to $(F\setminus \mathbb{A})^m$. Since we use this group so frequently, let us put

$$|N_m| = N_m(F) \setminus N_m(\mathbb{A}).$$

Also each element in $|N_m|$ is often denoted by the symbol $n_m$. Now for our fixed additive character $\psi$ on $F\setminus \mathbb{A}$ and $a \in F^\times$, we define the character $\psi^a_N$ on $N(\mathbb{A})$ by

$$\psi^a_N \left( \begin{array}{cccc} 1 & x_{12} & \cdots & x_{1r} \\ & 1 & \vdots & \\ & \ddots & \vdots & \\ & & x_{r-1} & 1 \end{array} \right) = \psi(ax_{12} + \sum_{i=2}^{r-1} x_{i,i+1}).$$

We write $\psi^1_N = \psi_N$, which is the one we usually use. We often consider $\psi_N$ restricted to $N_m(\mathbb{A})$, which we also denote by $\psi_N$.

We let

$$H_m = \{ \begin{pmatrix} g_m \\ aI_{r-m} \end{pmatrix} \in GL_r^{(2)} : g_m \in GL_m, a \in GL_1 \},$$

so the product $(\det g_m)a^{r-m}$ is a square. Note $H_{m-1} \subseteq H_m$. We let

$$Q_{m-1} = H_{m-1}N_{m-1} = \{ \begin{pmatrix} g_{m-1} & n \\ a & aI_{r-m} \end{pmatrix} \} \subseteq H_m,$$

where we assume $Q_0 = \{ aI_r : a \in GL_1 \}$. Also notice that $Q_{r-1} = Q \cap GL_r^{(2)}$, and hence our previous notation for this group.

For our cusp form $\phi$ we write the “partial Whittaker coefficient” by

$$W_m(g) := \int_{[N_m][N_{m+1}]} \cdots \int_{[N_{r-1}]} \phi(n_{r-1}n_{r-2}\cdots n_m g) \psi_N(n_{r-1}n_{r-2}\cdots n_m) dn_{r-1}dn_{r-2}\cdots dn_m.$$  

Strictly speaking $W_m$ depends on the choice of $\psi$ and $\phi$ but we use this notation since it will not produce any confusion. The following property of this partial Whittaker coefficient will be necessary for our computation.
Lemma 3.4. For any $h_{m-1} \in H_{m-1}(F)$, one has

$$W_m(h_{m-1}g) = W_m(g).$$

Proof. This follows from the automorphy of $\phi$ and the fact that $h_{m-1}$ fixes $\psi_N$ in the sense that $\psi_N(h_{m-1}n_r-1n_r-2\cdots n_mh_{m-1}^{-1}) = \psi_N(n_{r-1}n_{r-2}\cdots n_m).$  

For our unfolding argument we compute the Fourier expansions of $\Theta$ and $f^s$ along $N_m$ and $N_{m-1}$ “alternatingly”. Namely first we consider the Fourier expansion of $\Theta$ along $N_{r-1}$, and then that of $f^s$ along $N_{r-2}$, and then $\Theta$ along $N_{r-3}$ and then $f^s$ along $N_{r-4}$, etc. For this computation, the following lemma plays a pivotal role.

Lemma 3.5. Assume $m \geq 2$. The group $H_m(F)$ acts on the dual space $[N_m] \cong F^m$ by conjugation as

$$(h_m \cdot \psi)(n_m) = \psi(h_m^{-1}n_mh_m), \quad h_m \in H_m(F), n_m \in N_m(\AA).$$

with two orbits: the zero orbit and the orbit of $\psi_N$, where $\psi_N$ is actually the restriction of $\psi_N$ to $N_m(\AA)$. Moreover the stabilizer of $\psi_N$ is $Q_{m-1}(F)$, and hence the orbit of $\psi_N$ is indexed by $Q_{m-1}(F) \setminus H_m(F)$.

Proof. Straightforward computation.  

Now we are ready to work out our Rankin-Selberg integral to obtain the desired Euler product. Recall we have obtained

$$Z(\phi, \Theta, f^s) = \int_{Z(\AA)Q_{r-1}(F) \setminus GL_r^{(2)}(\AA)} \phi(g)\Theta(\kappa(g))f^s(\kappa(g); e)dg. \quad (3.6)$$

In what follows, one should keep in mind Lemma 3.4 along with the fact that the partial section $s : GL_r(F) \to GL_r(\AA)$ is not only defined but also is a homomorphism on both of the groups $GL_r(F)$ and $N_B(\AA)$.

**Unfolding Step 1**

The first step starts with computing the Fourier expansion of $\Theta$ along the “last column” $N_{r-1}$. Consider $n_{r-1} \mapsto \Theta(s(n_{r-1})\kappa(g))$ as a function on $[N_{r-1}]$, and expand it. Then one has

$$\Theta(\kappa(g)) = \sum_{\psi \in [N_{r-1}] [N_{r-1}]} \int_{[N_{r-1}]} \Theta(s(n_{r-1})\kappa(g))\psi(n_{r-1}^{-1}dn_{r-1}.$$  

By Lemma 3.6 we obtain

$$\Theta(\kappa(g)) = \int_{[N_{r-1}]} \Theta(s(n_{r-1})\kappa(g))dn_{r-1} + \sum_{h_{r-1} \in Q_{r-2}(F) \setminus H_{r-1}(F)} \int_{[N_{r-1}]} \Theta(s(n_{r-1}h_{r-1})\kappa(g))\psi_N(n_{r-1}^{-1}dn_{r-1}.$$
By substituting this expression of $\Theta(\kappa(g))$ in (3.6), one obtains

$$Z(\phi, \Theta, f^*) = \int_{Z(\mathbb{A})Q_{r-1}(F) \backslash GL(2)(\mathbb{A})} \phi(g)\left(\int_{[N_{r-1}]} \Theta(s(n_{r-1})\kappa(g)) dn_{r-1} + \sum_{h_{r-1} \in Q_{r-2}(F) \backslash H_{r-1}(F)} \int_{[N_{r-1}]} \Theta(s(n_{r-1}h_{r-1})\kappa(g)) \psi_N(n_{r-1}^{-1} dn_{r-1}) f^*(\kappa(g); e) dg \right).$$

One of the key points in our computation is that the term coming from the zero orbit (“zero orbit term”) vanishes because of the cuspidality of $\phi$. To see it, we would like to multiply out the large parentheses and write out the zero orbit term separately as

$$\int_{Z(\mathbb{A})Q_{r-1}(F) \backslash GL(2)(\mathbb{A})} \phi(g) \int_{[N_{r-1}]} \Theta(s(n_{r-1})\kappa(g)) dn_{r-1} f^*(\kappa(g); e) dg.$$

But we need justification for this process because we need to know that the product

$$\phi(g) \int_{[N_{r-1}]} \Theta(s(n_{r-1})\kappa(g)) dn_{r-1} f^*(\kappa(g); e),$$

viewed as a function on $g$, is indeed invariant on $Z(\mathbb{A})Q_{r-1}(F)$ so that we can carry out the integration for $dg$. This is not immediately clear. To see it, let $h \in Q_{r-1}(F)$. First of all, by the $s - \kappa$ trick we introduced before, we have

$$\int_{[N_{r-1}]} \Theta(s(n_{r-1})\kappa(hg)) dn_{r-1} f^*(\kappa(hg); e) = \int_{[N_{r-1}]} \Theta(s(n_{r-1})s(h)\kappa(g)) dn_{r-1} f^*(s(h)\kappa(g); e).$$

Now $f^*(s(h)\kappa(g); e) = f^*(\kappa(g); s(h)) = f^*(\kappa(g); e)$ by the automorphy of $f^*$. Also

$$\int_{[N_{r-1}]} \Theta(s(n_{r-1})s(h)\kappa(g)) dn_{r-1} = \int_{[N_{r-1}]} \Theta(s(n_{r-1}h)\kappa(g)) dn_{r-1} \quad \text{by Lemma 1.9}$$

$$= \int_{[N_{r-1}]} \Theta(s(hn_{r-1})\kappa(g)) dn_{r-1} \quad \text{by change of variable for } n_{r-1}$$

$$= \int_{[N_{r-1}]} \Theta(s(h)s(n_{r-1})\kappa(g)) dn_{r-1} \quad \text{by Lemma 1.9}$$
The cuspidality of 

By the invariance of the measure $Z$ and zero. For this, note that we can write

where we also used the fact that $s$ spectively, so all the occurrences of $g$

Therefore indeed (3.7) viewed as a function of $g$ is left invariant on $Q_{r-1}(F)$. Similarly one can see that it is left invariant on $Z(A)$ by using the $s - \kappa$ trick and the actions of the center on $\Theta$ and $f^*$.

Thus the expression (3.7) makes sense and we can work on this integral. Indeed, we will show it is zero. For this, note that we can write

Therefore we obtain

Thus the expression (3.7) is written as

Then we can write the outer integral of (3.7) as an integral over those two sets $N_{r-1}(F)\backslash N_{r-1}(A)$ and $Z(A)H_{r-1}(F)N_{r-1}(A)\backslash GL_1^{(2)}(A)$, whose corresponding variables we denote by $n'_{r-1}$ and $g$ respectively, so all the occurrences of $g$ in the integrant are replaced by $n'_{r-1}g$, and $dg$ is replaced by $dn'_{r-1}dg$. Then the zero orbit term (3.7) is written as

By using the $s - \kappa$ trick, this is written as

where we also used the fact that $s(N_B(A))$ acts trivially on $f^*$, and $s$ is a homomorphism on $N_B(A)$. By the invariance of the measure $dn_{r-1}$ for the integral for $\Theta$, the two inner integrals are written as

The cuspidality of $\phi$ makes this term vanish.

Therefore we obtain

$$Z(\phi, \Theta, f^*) = \int_{Z(A)Q_{r-1}(F)\backslash GL_1^{(2)}(A)} \phi(g)$$
\[
\left( \sum_{h_{r-1} \in Q_{r-2}(F) \setminus H_{r-1}(F)} \int_{[N_{r-1}]} \Theta(s(n_{r-1} h_{r-1}) \kappa(g)) \psi_N(n_{r-1})^{-1} dn_{r-1} \right) f^*(\kappa(g); e) dg.
\]

For each \( h_{r-1} \in H_{r-1}(F) \) one sees that \( f^*(\kappa(g); e) = f^*(\kappa(g); s(h_{r-1})) = f^*(s(h_{r-1}) \kappa(g); e) \) by the automorphy. We move around \( s \) and \( \kappa \) by the \( s - \kappa \) trick, and one can see that the above integral is written as

\[
\int_{Z(\mathbb{A}) Q_{r-1}(F) \setminus \text{GL}^{(2)}(\mathbb{A})} \phi(h_{r-1} g) \left( \sum_{h_{r-1} \in Q_{r-2}(F) \setminus H_{r-1}(F)} \int_{[N_{r-1}]} \Theta(s(n_{r-1}) \kappa(h_{r-1} g)) \psi_N(n_{r-1})^{-1} dn_{r-1} \right) f^*(\kappa(h_{r-1} g); e) dg,
\]

where we used the automorphy of \( \phi \) as well as \( s(n_{r-1} h_{r-1}) = s(n_{r-1}) s(h_{r-1}) \) by Lemma 1.9. Then we are allowed to collapse the sum by using \( Q_{r-1}(F) \), and obtain

\[
Z(\phi, \Theta, f^*) = \int_{Z(\mathbb{A}) N_{r-1}(F) Q_{r-2}(F) \setminus \text{GL}^{(2)}(\mathbb{A})} \phi(g) \left( \int_{[N_{r-1}]} \Theta(s(n_{r-1}) \kappa(g)) \psi_N(n_{r-1})^{-1} dn_{r-1} \right) f^*(\kappa(g); e) dg.
\]

Note that

\[
\int_{Z(\mathbb{A}) N_{r-1}(F) Q_{r-2}(F) \setminus \text{GL}^{(2)}(\mathbb{A})} = \int_{Z(\mathbb{A}) Q_{r-2}(F) N_{r-1}(\mathbb{A}) \setminus \text{GL}^{(2)}(\mathbb{A})} \int_{N_{r-1}(F) \setminus N_{r-1}(\mathbb{A})}.
\]

So we have

\[
Z(\phi, \Theta, f^*) = \int_{Z(\mathbb{A}) Q_{r-2}(F) N_{r-1}(\mathbb{A}) \setminus \text{GL}^{(2)}(\mathbb{A})} \int_{[N_{r-1}]} \phi(n'_{r-1} g) \left( \int_{[N_{r-1}]} \Theta(s(n_{r-1} \kappa(n'_{r-1} g)) \psi_N(n_{r-1})^{-1} dn_{r-1} f^*(\kappa(n'_{r-1} g); e) dn'_{r-1} dg.\right)
\]

By using the \( s - \kappa \) trick, this is written as

\[
Z(\phi, \Theta, f^*) = \int_{Z(\mathbb{A}) Q_{r-2}(F) N_{r-1}(\mathbb{A}) \setminus \text{GL}^{(2)}(\mathbb{A})} \int_{[N_{r-1}]} \phi(n'_{r-1} g) \left( \int_{[N_{r-1}]} \Theta(s(n_{r-1} n'_{r-1} \kappa(g)) \psi_N(n_{r-1})^{-1} dn_{r-1} f^*(\kappa(g); e) dn'_{r-1} dg,\right)
\]

where we also used \( f^*(s(n'_{r-1}) \kappa(g); e) = f^*(\kappa(g); e) \). The change of variable \( n_{r-1} n'_{r-1} \mapsto n_{r-1} \) in the inner most integral gives

\[
Z(\phi, \Theta, f^*) = \int_{Z(\mathbb{A}) Q_{r-2}(F) N_{r-1}(\mathbb{A}) \setminus \text{GL}^{(2)}(\mathbb{A})} \int_{[N_{r-1}]} \phi(n'_{r-1} g) \psi(n'_{r-1}) dn'_{r-1} \left( \int_{[N_{r-1}]} \Theta(s(n_{r-1} \kappa(g)) \psi_N(n_{r-1})^{-1} dn_{r-1} f^*(\kappa(g); e) dg.\right)
\]
So we have

\[
Z(\phi, \Theta, f^s) = \int_{Z(\mathbb{A})Q_2(F)\mathbb{A}^1} W_{r-1}(g) \int_{[N_{r-1}]} \Theta(s(n_{r-1})\kappa(g))\psi_N(n_{r-1})^{-1} \ dn_{r-1} f^s(\kappa(g); e) \ dg,
\]

recalling the notation for \( W_{r-1}(g) \) from \([3.5]\).

**Unfolding Step 2**

The second step starts with computing the Fourier expansion of \( f^s(\kappa(g); -) \) along the “\( r \)-1st column” \( N_{r-2} \). Recall that \( f^s(\kappa(g); -) \) is an automorphic form on \( \widetilde{GL}_{r-1} \times \widetilde{GL}_1 \). By viewing the function \( n_{r-2} \mapsto f^s(\kappa(g); s(n_{r-2})) \) as a function on \([N_{r-2}]\), and expanding it by using Lemma \([3.5]\) we obtain

\[
f^s(\kappa(g); e) = \int_{[N_{r-2}]} f^s(s(n_{r-2})\kappa(g); e) \ dn_{r-2} + \sum_{h_{r-2} \in Q_{r-3}(F) \backslash H_{r-2}(F)} \int_{[N_{r-2}]} f^s(\kappa(g); s(n_{r-2}h_{r-2})) \psi_N(n_{r-2})^{-1} \ dn_{r-2}.
\]

Now the first term (the zero orbit), when integrated with the cusp form \( \phi \), vanishes by the cuspidality of \( \phi \) as we have seen in Step 1. The idea is essentially the same but the computation is not completely identical. Hence we will give the detailed computation here.

First if the above Fourier expansion of \( f^s \) is plugged in to the formula for the \( Z(\phi, \Theta, f^s) \) we obtained at the end of Step 1, the product of \( \int_{[N_{r-1}]} \Theta(s(n_{r-1})\kappa(g))\psi_N(n_{r-1})^{-1} \ dn_{r-1} \) with each term in the Fourier expansion of \( f^s \) viewed as a function on \( g \) is invariant on \( Z(\mathbb{A})Q_2(F)\mathbb{A}^1 \). (To see this, once again we need the \( s - \kappa \) trick.) Hence the expression for \( Z(\phi, \Theta, f^s) \) after the Fourier expansion of \( f^s \) is plugged in can be expanded and we can take out the zero orbit term separately as

\[
\int_{Z(\mathbb{A})Q_2(F)\mathbb{A}^1} W_{r-1}(g) \int_{[N_{r-1}]} \Theta(s(n_{r-1})\kappa(g))\psi_N(n_{r-1})^{-1} \ dn_{r-1} \int_{[N_{r-2}]} f^s(s(n_{r-2})\kappa(g); e) \ dn_{r-2} \ dg.
\]

Since

\[
\int_{Z(\mathbb{A})Q_2(F)\mathbb{A}^1} W_{r-1}(g) \int_{[N_{r-1}]} \Theta(s(n_{r-1})\kappa(g))\psi_N(n_{r-1})^{-1} \ dn_{r-1} \int_{[N_{r-2}]} f^s(s(n_{r-2})\kappa(g); e) \ dn_{r-2} \ dg = \int_{Z(\mathbb{A})N_{r-2}(F)\mathbb{A}^1} f^s(s(n_{r-2})\kappa(g); e) \ dn_{r-2} \ dg.
\]

the outer integral can be written as integrals over those two sets

\[
Z(\mathbb{A})N_{r-2}(F)\mathbb{A}^1 \text{ and } N_{r-2}(F)\mathbb{A}^1.
\]
whose corresponding variables we denote by $g$ and $n_{r-2}'$ respectively. Then all the occurrences of $g$ are replaced by $n_{r-2}'g$. Namely the zero orbit term is written as

$$\int \int W_{r-1}(n_{r-2}'g)$$

$$\int \Theta(s(n_{r-1})\kappa(n_{r-2}'g))\psi_N(n_{r-1})^{-1} dn_{r-1} \int f^*(s(n_{r-2})\kappa(n_{r-2}'g); e) dn_{r-2} dn_{r-2}' dg,$n_{r-1}$$

where the outer integral is over the set $Z(\mathbb{A}) H_{r-2}(F) N_{r-2}(\mathbb{A}) N_{r-1}(\mathbb{A}) \setminus GL_r^{(2)}(\mathbb{A})$. By using the $s - \kappa$ trick, we can write $s(n_{r-1})\kappa(n_{r-2}'g) = s(n_{r-1}n_{r-2}')\kappa(g)$ inside $\Theta$ and $f^*$. So the integral becomes

$$\int \int W_{r-1}(n_{r-2}'g)$$

$$\int \Theta(s(n_{r-1}n_{r-2}')\kappa(g))\psi_N(n_{r-1})^{-1} dn_{r-1} \int f^*(s(n_{r-2}n_{r-2}')\kappa(g); e) dn_{r-2} dn_{r-2}' dg.$n_{r-1}$$

By the invariance of the measure $dn_{r-2}$, this is written as

$$\int \int W_{r-1}(n_{r-2}'g)$$

$$\int \Theta(s(n_{r-1}n_{r-2}')\kappa(g))\psi_N(n_{r-1})^{-1} dn_{r-1} dn_{r-2}' \int f^*(s(n_{r-2})\kappa(g); e) dn_{r-2} dg.$n_{r-1}$$

Hence to show the vanishing of the zero orbit term, it suffices to show

$$(3.10) \int_{[N_{r-2}]} W_{r-1}(n_{r-2}'g) \int_{[N_{r-1}]} \Theta(s(n_{r-1}n_{r-2}')\kappa(g))\psi_N(n_{r-1})^{-1} dn_{r-1} dn_{r-2}' = 0.$n_{r-1}$$

To proceed, we need following crucial property of the exceptional representations.

**Lemma 3.11.** Let $\Theta$ be an automorphic form in the space of the exceptional representation $\vartheta_\chi^\psi$ or $\theta_\chi$ of $GL_r(\mathbb{A})$, $GL_r^{(2)}(\mathbb{A})$ or $GL_r(\mathbb{A})$, respectively, where $r$ can be either $2q$ or $2q + 1$. For an integer $1 \leq m \leq q$, the integral

$$\int_{[N_{r-2m+1}]} \cdots \int_{[N_{r-2}]} \int_{[N_{r-1}]} \Theta(s(n_{r-1}n_{r-2}n_{r-3} \cdots n_{r-2m})\kappa(g))\psi_N(n_{r-1}n_{r-2}n_{r-3} \cdots n_{r-2m+1})$$

$$dn_{r-1}dn_{r-2} \cdots dn_{r-2m+1}$$

is independent of $n_{r-2m} \in N_{r-2m}(\mathbb{A})$.

Consequently, by integrating over $[N_{r-2m}]$, this integral is equal to

$$\int_{[N_{r-2m}]} \cdots \int_{[N_{r-2}]} \int_{[N_{r-1}]} \Theta(s(n_{r-1}n_{r-2}n_{r-3} \cdots n_{r-2m})\kappa(g))\psi_N(n_{r-1}n_{r-2}n_{r-3} \cdots n_{r-2m+1})$$

$$dn_{r-1}dn_{r-2} \cdots dn_{r-2m},$$

provided the measure is so chosen that the volume of $[N_{r-2m}]$ is 1.
Proof. The case for $\theta_\chi$ is Proposition 2.4 and 2.5 of [BG]. The case for $\theta_\chi$ can be proven identically. The key ingredient for the proof is the non-existence of the Whittaker functional for the exceptional representation, which implies Proposition 2.1 of [BG]. Once the case for $\theta_\chi$ is taken care of, the case for $\theta_\chi^*$ trivially follows because any automorphic form in the space of $\theta_\chi^*$ is simply the restriction of an automorphic form in the space of $\theta_\chi$. 

By applying the first part of the lemma with $m = 1$, the left hand side of (3.10) is written as

$$
\int_{[N_{r-1}]} W_{r-1}(n'_{r-2}g) \, dn'_{r-2} \int_{[N_{r-1}]} \Theta(s(n_{r-1})\kappa(g))\psi_N(n_{r-1})^{-1} \, dn_{r-1}.
$$

By definition of $W_{r-1}$, which is given in (3.9), we have

$$
\int_{[N_{r-2}]} W_{r-1}(n'_{r-2}g) \, dn'_{r-2} = \int_{[N_{r-2}]} \int_{[N_{r-1}]} \phi(n_{r-1}n'_{r-2}g)\psi_N(n_{r-1}) \, dn_{r-1} \, dn'_{r-2}
$$

$$
= \int_{[A]} \left( \int_{[N_{(r-2,2)}]} \phi(n_{(r-2,2)}ag) \, dn_{(r-2,2)} \right) \psi_N(a) \, da,
$$

where $N_{(r-2,2)}$ is the unipotent radical of the $(r-2,2)$-parabolic, and $A$ is the set of the matrices of the form

$$
a = \begin{pmatrix} I_{r-2} & 1 & \ast \\ 1 & \ast & \end{pmatrix}.
$$

By the cuspidality of $\phi$, the inner integral is zero. This shows that the zero orbit term vanishes.

Hence we obtain

$$
Z(\phi, \Theta, f^*)
= \int_{Z(\mathbb{A})Q_{r-2}(F)\, N_{r-1}(\mathbb{A})\backslash GL_r^{(2)}(\mathbb{A})} W_{r-1}(g) \int_{[N_{r-1}]} \Theta(s(n_{r-1})\kappa(g))\psi_N(n_{r-1})^{-1} \, dn_{r-1}
$$

$$
\sum_{h_{r-2} \in Q_{r-3}(F)\backslash H_{r-2}(F)} f^*(s(n_{r-2}h_{r-2})\kappa(g); e)\psi_N(n_{r-2})^{-1} \, dn_{r-2} \, dg
$$

$$
= \int_{Z(\mathbb{A})Q_{r-2}(F)\, N_{r-1}(\mathbb{A})\backslash GL_r^{(2)}(\mathbb{A})} \sum_{h_{r-2} \in Q_{r-3}(F)\backslash H_{r-2}(F)} W_{r-1}(h_{r-2}g)
$$

$$
\int_{[N_{r-1}]} \Theta(s(h_{r-2}n_{r-1}h_{r-2}^{-1}h_{r-2})\kappa(g))\psi_N(n_{r-1})^{-1} \, dn_{r-1}
$$

$$
\int_{[N_{r-2}]} f^*(s(n_{r-2}h_{r-2})\kappa(g); e)\psi_N(n_{r-2})^{-1} \, dn_{r-2} \, dg,
$$

where for the second equality we used $W_{r-1}(h_{r-2}g) = W_{r-1}(g)$ by Lemma 3.14 and the automorphy of $\Theta$. 
Now by the change of variable $h_{r-2}n_{r-1}^{-1}h_{r-2}^{-1} \mapsto n_{r-1}$ for the integral for $\Theta$, the zeta integral becomes

$$
\int_{Z(\mathbb{A})Q_{r-2}(F)N_{r-1}(\mathbb{A}) \backslash GL_{r-2}^{(2)}(\mathbb{A})} \sum_{h_{r-2} \in Q_{r-2}(F) \backslash H_{r-2}(F)} W_{r-2}(h_{r-2}g)
$$

$$
\int_{[N_{r-1}]} \Theta(s(n_{r-1}h_{r-2})\kappa(g))\psi_N(n_{r-1})^{-1} \, dn_{r-1} \int_{[N_{r-2}]} f^*(s(n_{r-2}h_{r-2})\kappa(g); e)\psi_N(n_{r-2})^{-1} \, dn_{r-2} \, dg
$$

by using $\psi_N(h_{r-2}^{-1}n_{r-1}h_{r-2}) = \psi_N(n_{r-1})$.

Then one can collapse the sum with the outer integral and obtain

$$Z(\phi, \Theta, f^*) = \int W_{r-1}(g) \int_{[N_{r-1}]} \Theta(s(n_{r-1})\kappa(g))\psi_N(n_{r-1})^{-1} \, dn_{r-1} \int_{[N_{r-2}]} f^*(s(n_{r-2})\kappa(g); e)\psi_N(n_{r-2})^{-1} \, dn_{r-2} \, dg,$$

where the outermost integral is over

$$Z(\mathbb{A})N_{r-2}(F)Q_{r-3}(F)N_{r-1}(\mathbb{A}) \backslash GL_{r}^{(2)}(\mathbb{A}).$$

By applying the second part of Lemma 3.11 with $m = 1$, one obtains

$$Z(\phi, \Theta, f^*) = \int W_{r-1}(g) \int_{[N_{r-2}][N_{r-1}]} \Theta(s(n_{r-1}n_{r-2})\kappa(g))\psi_N(n_{r-1})^{-1} \, dn_{r-1}dn_{r-2} \int_{[N_{r-2}]} f^*(s(n_{r-2})\kappa(g); e)\psi_N(n_{r-2})^{-1} \, dn_{r-2} \, dg.$$

By

$$\int Z(\mathbb{A})N_{r-2}(F)Q_{r-3}(F)N_{r-1}(\mathbb{A}) \backslash GL_{r}^{(2)}(\mathbb{A}) = \int Z(\mathbb{A})Q_{r-3}(F)N_{r-1}(\mathbb{A})N_{r-2}(\mathbb{A}) \backslash GL_{r}^{(2)}(\mathbb{A}) \int Z(\mathbb{A})N_{r-2}(F) \backslash N_{r-2}(\mathbb{A})$$

together with the $s - \kappa$ trick, one obtains

$$Z(\phi, \Theta, f^*) = \int \int W_{r-1}(n_{r-2}g) \int_{[N_{r-2}]} \Theta(s(n_{r-1}n_{r-2}n'_{r-2})\kappa(g))\psi_N(n_{r-1})^{-1} \, dn_{r-1}dn_{r-2} \int_{[N_{r-2}]} f^*(s(n_{r-2}n'_{r-2})\kappa(g); e)\psi_N(n_{r-2})^{-1} \, dn_{r-2}dn'_{r-2} \, dg,$$

where the outermost integral is over

$$Z(\mathbb{A})Q_{r-3}(F)N_{r-1}(\mathbb{A})N_{r-2}(\mathbb{A}) \backslash GL_{r}^{(2)}(\mathbb{A}).$$

The variable $n'_{r-2}$ inside $\Theta$ goes away by the invariance of the measure $dn_{r-2}$. By the change of variable $n_{r-2}n'_{r-2} \mapsto n_{r-2}$ inside $f^*$, the character $\psi_N(n'_{r-2})$ comes out, and one obtains

$$Z(\phi, \Theta, f^*) = \int \int W_{r-1}(n_{r-2}g)\psi(n'_{r-2}) \, dn'_{r-2} \int_{[N_{r-2}][N_{r-1}]} \Theta(s(n_{r-1}n_{r-2})\kappa(g))\psi_N(n_{r-1})^{-1} \, dn_{r-1}dn_{r-2}$$
\[
\int_{[N_{r-2}]} f^* (n_{r-2} g) (e) \psi_N (n_{r-2})^{-1} \, dn_{r-2} dg \\
= \int_{[N_{r-2}][N_{r-1}]} W_{r-2} (g) \int \Theta (s (n_{r-1} n_{r-2}) \kappa (g)) \psi_N (n_{r-1})^{-1} \, dn_{r-1} dn_{r-2} \\
\int_{[N_{r-2}]} f^* (s (n_{r-2}) \kappa (g); e) \psi_N (n_{r-2})^{-1} \, dn_{r-2} dg,
\]

where the outermost integrals are over

\[Z (\mathbb{A}) Q_{r-3} (F) N_{r-1} (\mathbb{A}) N_{r-2} (\mathbb{A}) \setminus \text{GL}_r^{(2)} (\mathbb{A}).\]

**Unfolding Step 3**

The third step starts with computing the Fourier expansion of \(\Theta\) along the "\(r - 2^{nd}\) column" \(N_{r-3}\), i.e. consider the Fourier expansion of the function

\[n_{r-3} \mapsto \int_{[N_{r-2}][N_{r-1}]} \Theta (s (n_{r-1} n_{r-2} n_{r-3}) \kappa (g)) \psi_N (n_{r-1})^{-1} \, dn_{r-1} dn_{r-2} \]
on \([N_{r-3}]\). Again by Lemma 3.11 we have

\[\int_{[N_{r-2}][N_{r-1}]} \Theta (s (n_{r-1} n_{r-2} \kappa (g)) \psi_N (n_{r-1})^{-1} \, dn_{r-1} dn_{r-2} = (\text{zero orbit})\]

By substituting this in (3.12), one can show that, first of all, the zero orbit vanishes thanks to the cuspidality of \(\phi\), and second of all, the sum can be collapsed with the outermost integral by using the \(s - \kappa\) tick and the change of variable \(h_{r-3} n_{r-2} h_{r-3}^{-1} \mapsto n_{r-2}\) for the integral of \(f^*\) along with \(\psi_N (h_{r-3} n_{r-2} h_{r-3}^{-1}) = \psi_N (n_{r-2})\) and Lemma 3.3. The computations are essentially the same as the previous steps, and left to the reader.

By applying Lemma 3.11 to \(f^*\), one obtains

\[Z (\phi, \Theta, f^*) = \int_{[N_{r-3}]} W_{r-3} (g) \]

\[\int_{[N_{r-3}][N_{r-2}][N_{r-1}]} \Theta (s (n_{r-1} n_{r-2} n_{r-3}) \kappa (g)) \psi_N (n_{r-1} n_{r-3})^{-1} \, dn_{r-1} dn_{r-2} dn_{r-3} \]

\[\int_{[N_{r-3}][N_{r-2}]} f^* (s (n_{r-2} n_{r-3}) \kappa (g); e) \psi_N (n_{r-2})^{-1} \, dn_{r-2} dn_{r-3} dg,\]

where the outermost integral is over

\[Z (\mathbb{A}) Q_{r-4} (F) N_{r-1} (\mathbb{A}) N_{r-2} (\mathbb{A}) N_{r-3} (\mathbb{A}) \setminus \text{GL}_r^{(2)} (\mathbb{A}).\]
Unfolding Step 4 and further

We repeat this process. Namely the next step (Step 4) is to apply the Fourier expansion formula (Lemma 3.5) to the function

\[ n_{r-4} \mapsto \int \int f^*(s(n_{r-2}n_{r-3}n_{r-4})\kappa(g); e)\psi_N(n_{r-2})^{-1} \ dn_{r-2} \ dn_{r-3} \]

on \([N_{r-4}]\) and one sees that the zero orbit vanishes by the cuspidality of \(\phi\), and collapse the sum by using Lemma 3.4. Then apply Lemma 3.11 to \(\Theta\), which gives

\[ Z(\phi, \Theta, f^*) \]

\[ = \int W_{r-4}(g) \]

\[ \int \int \cdots \int \Theta(s(n_{r-1}n_{r-2}n_{r-3}n_{r-4})\kappa(g))\psi_N(n_{r-1}n_{r-3})^{-1} \ dn_{r-1} \ dn_{r-2} \ dn_{r-3} \ dn_{r-4} \]

\[ \int \int \cdots \int f^*(s(n_{r-2}n_{r-3}n_{r-4})\kappa(g); e)\psi_N(n_{r-2})^{-1} \ dn_{r-2} \ dn_{r-3} \ dn_{r-4} \ dg, \]

where the outermost integral is over

\[ Z(\mathbb{A})Q_{r-5}(F)N_{r-1}(\mathbb{A})N_{r-2}(\mathbb{A})N_{r-3}(\mathbb{A})N_{r-4}(\mathbb{A}) \ \text{GL}_2^{(2)}(\mathbb{A}). \]

For the next step (Step 5) one needs to compute the Fourier expansion for \(\Theta\) along \(N_{r-5}\) using Lemma 3.5 (the zero orbit goes away by the cuspidality of \(\phi\), then collapse the sum using Lemma 3.4 and then apply Lemma 3.11 to \(f^*\). Then the next step (Step 6) is to switch the roles of \(\Theta\) and \(f^*\) and use those three lemmas, Lemma 3.5, 3.4 and 3.11 in this order, and then proceed to the next step, and so on.

Unfolding Final Step

After finishing step \(r-2\), which is done by computing the Fourier expansion of \(f^*\), one obtains

\[ Z(\phi, \Theta, f^*) \]

\[ = \int W_2(g) \]

\[ \int \int \cdots \int \Theta(s(n_{r-1}n_{r-2} \cdots n_2)\kappa(g))\psi_N(n_{r-1}n_{r-3} \cdots n_3)^{-1} \ dn_{r-1} \ dn_{r-2} \cdots \ dn_2 \]

\[ \int \int \cdots \int f^*(s(n_{r-2}n_{r-3} \cdots n_2)\kappa(g); e)\psi_N(n_{r-2}n_{r-4} \cdots n_2)^{-1} \ dn_{r-2} \ dn_{r-3} \cdots \ dn_2 \ dg, \]

where the outermost integral is over

\[ Z(\mathbb{A})Q_1(F)N_{r-1}(\mathbb{A})N_{r-2}(\mathbb{A}) \cdots N_3(\mathbb{A})N_2(\mathbb{A}) \ \text{GL}_r^{(2)}(\mathbb{A}). \]

The final step (Step \(r-1\)) does not work out as before because the key Lemma 3.5 does not hold for \(m = 1\). Namely, for \(m = 1\), though \(H_1\) acts on \([N_1]\) as before, the number of orbits is not 2 but
rather the nonzero orbits are indexed by $(F^\times)^2 \backslash F^\times$, and indeed
\[ [N_1] = \text{(zero orbit)} + \sum_{a \in (F^\times)^2 \backslash F^\times} H_1 \psi_N^a, \]
where the stabilizer of each $\psi_N^a$ in $H_1$ is $Q_0$. But everything else is the same as the previous steps, and we obtain
\[ Z(\phi, \Theta, f^s) = \int W_1(g) \]
\[ \sum_{a \in (F^\times)^2 \backslash F^\times} \int_{[N_1][N_2]} \cdots \int_{[N_{r-1}]} \Theta(s(n_{r-1}n_{r-2} \cdots n_1)\kappa(g)) \psi_N^a(n_{r-1}n_{r-3} \cdots n_2)^{-1} dn_{r-1}dn_{r-2} \cdots dn_1 \]
\[ \int_{[N_1][N_2]} \cdots \int_{[N_{r-2}]} f^s(s(n_{r-2}n_{r-3} \cdots n_1)\kappa(g); e) \psi_N(n_{r-2}n_{r-4} \cdots n_2)^{-1} dn_{r-2}dn_{r-3} \cdots dn_1 \ dg, \]
where the outermost integral is over
\[ Z(\mathbb{A})N_{r-1}(\mathbb{A})N_{r-2}(\mathbb{A}) \cdots N_1(\mathbb{A}) \backslash GL_r^{(2)}(\mathbb{A}), \]
which is the same as
\[ Z(\mathbb{A})N(\mathbb{A}) \backslash GL_r^{(2)}(\mathbb{A}) \]
because $N_{r-1}(\mathbb{A})N_{r-2}(\mathbb{A}) \cdots N_1(\mathbb{A}) = N(\mathbb{A})$.

**Almost Euler product**

Now we are ready to obtain the (almost) Euler product from this last expression. But as we have noted before, we are not able to obtain the full Euler product. This is due to the lack of the uniqueness result for the semi-Whittaker functional at the archimedean places. Although such uniqueness result might hold at the archimedean places, at this moment the author does not know how to prove it. Hence the best we can do is to obtain the “almost Euler product”, or the Euler product at the finite places.

First notice that $W_1(g)$ is the usual Whittaker coefficient with respect to $\psi^{-1}$, so let us write
\[ W_1(g) = W^\psi_{\phi^{-1}}(g) = W(g), \]
where again we ignore the dependence of $W(g)$ on $\phi$ and $\psi$. Also by following [BG], we define
\[ Q^\phi(\kappa(g)) = \int_{[N_1]} \cdots \int_{[N_{r-1}]} \Theta(s(n_{r-1}n_{r-2}n_{r-3} \cdots n_1)\kappa(g)) \]
\[ \psi_N^a(n_{r-1}n_{r-3}n_{r-5} \cdots)dn_{r-1} \cdots dn_1 \]
and
\[ R^s(\kappa(g)) = \int_{[N_1]} \cdots \int_{[N_{r-2}]} f^s(s(n_{r-2}n_{r-3} \cdots n_1)\kappa(g); e) \]
\[ \psi_N(n_{r-2}n_{r-4} \cdots)dn_{r-2} \cdots dn_1. \]
With this notation, the last formula we obtained for $Z(\phi, \Theta, f^*)$ is written as

$$Z(\phi, \Theta, f^*) = \int_{Z(\mathbb{A})N(\mathbb{A}) \setminus GL(2)(\mathbb{A})} W(g) \left( \sum_{\alpha \in (F^x)^2 \setminus F^x} Q^a(\kappa(g)) \right) R^s(\kappa(g)) dg,$$

We need to take care of the sum $\sum_{\alpha \in (F^x)^2 \setminus F^x}$. First for each fixed $g \in GL_r(\mathbb{A})$ consider the map

$$\psi^0_{\chi^0} \rightarrow \mathbb{C}$$

defined by

$$\Theta \mapsto Q^a(\kappa(g)).$$

This map is non-zero, because it is (a scalar multiple of) the composite of the constant term map $\psi^0_{\chi^0} \rightarrow \omega_{\chi}^0 \otimes \delta_p^{1/4}$ along the $(2, \ldots, 2)$-parabolic $P$ with the $\psi(a, 1, \ldots, 1)$-Whittaker functional of $\omega_{\chi}^0 \otimes \delta_p^{1/4}$. (See Proposition 2.49 for the constant term and (2.22) for the notation $\psi(a, 1, \ldots, 1)$. With this said, one can see that Proposition 2.27 implies

**Lemma 3.13.** The map $\Theta \mapsto Q^a(\kappa(g))$ is not identically zero if and only if $a \equiv 1 \mod (F^x)^2$.

This gives

$$Z(\phi, \Theta, f^*) = \int_{Z(\mathbb{A})N(\mathbb{A}) \setminus GL(2)(\mathbb{A})} W(g) Q(\kappa(g)) R^s(\kappa(g)) dg,$$

where we wrote $Q^1(\kappa(g)) = Q(\kappa(g))$. This is precisely the analogue of [BG] (3.5).

Now it would be ideal if we could show both $Q$ and $R_s$ decompose into products of local components like $Q(\kappa_v(g)) = \prod_v Q_v(\kappa_v(g_v))$ and $R^s = \prod_v R_v^s(\kappa_v(g_v))$ and hence by choosing $\phi$ so that the Whittaker function $W$ decomposes into a product $\prod_v W_v$, we could obtain the Euler product. However, to achieve this, one needs the uniqueness of the semi-Whittaker functional on the local exceptional representation for both archimedean and non-archimedean cases. For the non-archimedean case, the uniqueness of the semi-Whittaker functional follows from the periodicity of the Jacquet module of the exceptional representation (Proposition 2.51), which seems to be available only for the non-archimedean case, and the author does not know if such uniqueness is available for the archimedean case. (See Remark 2.52). Because of this issue, we need to compromise with the almost Euler product, which is, nonetheless, enough for proving our main theorem.

First let

$$L : \theta^0_{\chi} \rightarrow \mathbb{C}, \quad \Theta \mapsto Q(\kappa(\epsilon)),$$

where $\epsilon \in GL_r(\mathbb{A})$ is the identity element. This is a global semi-Whittaker functional. Note that $L$ is not identically zero by Lemma 3.13.

Next let us define $\widetilde{GL}_r(F_{\infty})$ to be the image of the map

$$\prod_{v \mid \infty} \widetilde{GL}(F_v) \rightarrow \widetilde{GL}_r(\mathbb{A}), \quad (g_v, \xi_v) \mapsto (\prod_{v \mid \infty} g_v, \prod_{v \mid \infty} \xi_v),$$

which may be called “the archimedean component” of $\widetilde{GL}_r(\mathbb{A})$. Then we can write $\theta = \theta_{\infty} \otimes' \prod_{v \mid \infty} \theta_v$, where

$$\theta_{\infty} = \otimes_{v \mid \infty} \theta_v$$

is the metaplectic tensor product of $\theta_v$ for all archimedean $v$, which is a representation of $\widetilde{GL}_r(F_{\infty})$. And we write each simple tensor in $\theta_{\infty} \otimes' \prod_{v \mid \infty} (\theta_v \otimes \psi_{\epsilon})$ as $x_{\infty} \otimes \otimes' x_v$, where $x_{\infty} \in \theta_{\infty}$ and $\otimes' x_v \in \widetilde{\otimes}_v \theta_v$. (Since the space of restricted metaplectic tensor product is the same as the usual restricted tensor product, we use the notation $\otimes$ rather than $\widetilde{\otimes}$ to denote each vector.) Let us fix the vector
Proposition 2.51, this is equal $cL$.

This is a semi-Whittaker functional for $L$ for any

Then $S$ by induction on the cardinality of $\mathcal{V}$.

Assume the statement holds of all vectors $x_v = x_v^0$ such that $x_v \neq x_v^0$, so $L_v(x_v) = Q_v(\kappa_v(e)) = 1$ if $v \notin S$. The proof is by induction on the cardinality of $S$. Namely assume $S$ is empty. Then $x_v = x_v^0$ for all finite $v$. Then $L(x_v) = L_v(x_v) = Q_v(\kappa_v(e)) = Q_v(e)$. This is the base step of induction.

Assume the statement holds of all vectors $x_v \otimes (\otimes_v x_v)$ whose $S$ has cardinality for some $n$. Now assume $y_v \otimes (\otimes_v y_v)$ is such that the corresponding $S$ has cardinality $n + 1$. Let $w$ be a place where $y_w \neq y_w^0$. Consider the map

$$\theta_w : C, \quad y_w' = L(y_w \otimes y_w^0 \otimes \bigotimes_{v \neq w} y_v).$$

This is a semi-Whittaker functional for $\theta_w$. By the uniqueness of the local semi-Whittaker functional (Proposition 2.51), this is equal $cL_w(y_w^0)$ for some scalar. Let $y_w^0 = x_w^0$, so that by the induction hypothesis,

$$cL_w(x_w^0) = L_w(y_w) \prod_{v \neq w} L_v(y_v).$$

But $L_w(x_w^0) = 1$, which gives $c = L_w(y_w^0) \prod_{v \neq w} L_v(y_v)$. Thus we have

$$L(y_w \otimes y_w^0 \otimes \bigotimes_{v \neq w} y_v) = L_w(y_w) \prod_{v \neq w} L_v(y_v)$$

for any $y_w'$. By letting $y_w' = y_w$, the induction is complete.

Similarly we can obtain the decomposition

$$R^s(\kappa(g)) = R^s(\kappa_v(g_v)) \prod_{v \neq w} R^s(\kappa_v(g_v))$$

for a decomposable $f^s = f^s_\infty \otimes (\otimes_v f^s_v)$ by defining $L_\mathcal{V}' : \theta'_\mathcal{V} \to C$ as before and setting

$$R^s(\kappa_v(g_v)) = L'_v(\theta'_v(\kappa_v(g_v))f^s_v)$$
and for \( v < \infty \)
\[
R_v^s(\kappa(g_v)) = L_v'(\theta'_v(\kappa(g_v))f_v^s)
\]
where \( L_v' : \theta'_v \to \mathbb{C} \) is (a scalar multiple of) the semi-Whittaker functional on \( \theta'_v \). Hence we obtain the almost Euler product of the zeta integral
\[
Z(W, Q, R^s) = Z_\infty(W_\infty, Q_\infty, R^s_\infty) \prod_{v < \infty} Z_v(W_v, Q_v, R_v^s),
\]
where
\[
Z_\infty(W_\infty, Q_\infty, R^s_\infty) = \int_{Z(F_\infty)N(F_\infty) \backslash GL_2^r(F_\infty)} W_\infty(g_\infty)Q_\infty(\kappa_\infty(g_\infty))R^s_\infty(\kappa_\infty(g_\infty)) \, dg_\infty
\]
and for \( v < \infty \)
\[
Z_v(W_v, Q_v, R_v^s) = \int_{Z(F_v)N(F_v) \backslash GL_2^r(F_v)} W_v(g_v)Q_v(\kappa_v(g_v))R_v^s(\kappa_v(g_v)) \, dg_v.
\]
Here note that \( F_\infty = \prod_{v | \infty} F_v \), which is a product of copies of \( \mathbb{R} \) and/or \( \mathbb{C} \).

**Unramified factor**

We will compute the unramified factor here. For this we need the following “Iwasawa decomposition” of \( GL_2^r \).

**Lemma 3.15.** Assume \( F \) is a non-archimedean local field and \( P \) is any parabolic subgroup of \( GL_r(F) \). We have the decomposition
\[
GL_2^r(F) = P(F)^\# K^\#,
\]
where \( P^\#(F) = P(F) \cap GL_2^r(F) \) and \( K^\# = GL_r(O_F) \cap GL_2^r(F) \).

For a measurable function \( f \) on \( G = GL_2^r(F) \), we have
\[
\int_G f(g) \, dg = \int_{P^\#} \int_{K^\#} f(pk) \, dp \, dk
\]
where \( dp \) is the left Haar measure on \( P^\# \) and \( dk \) is the Haar measure on \( K^\# \).

**Proof.** By the usual Iwasawa decomposition of \( GL_r \), each element \( g \in GL_2^r(F) \) is written as \( g = pk \) for \( p \in P(F) \) and \( k \in K \) such that \( \det(p) \det(k) \in (F^\times)^2 \). We may assume \( \det(p) = \varpi^n \) where \( \varpi \) is a uniformizer of \( F \) and \( n \in \mathbb{Z} \). For if \( \det(p) = \varpi^n u \) for some \( u \in O_F^\times \), let \( k_1 \) be an element in \( K \cap P \) with \( \det(k_1) = u^{-1} \), for example \( k_1 = \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} \). Then since \( pk = (pk_1)(k_1^{-1}k) \) and \( pk_1 \in P \), we may simply replace \( p \) by \( pk_1 \).

Thus \( \det(pk) = \varpi^n \det(k) \in (F^\times)^2 \). But \( \det(k) \in O_F^\times \). Hence we must have \( \det(p) \in (F^\times)^2 \). (Indeed this implies that \( n \) is even.) Then \( \det(k) \in (F^\times)^2 \) as well.

The decomposition of the measure is [Bu Proposition 2.1.5 (ii)]. (The assumptions of [Bu Proposition 2.1.5 (ii)] are satisfied by \( P^\# \) and \( K^\# \).)

**Proposition 3.16.** At each unramified place \( v \),
\[
Z_v(W_v, Q_v, R_v^s) = L(2s - \frac{1}{2}, \pi_v, Sym^2 \otimes \chi_v)L(r(2s - \frac{1}{2}), \chi_v^r \omega_v^{-2})^{-1}.
\]
Proof. The computation is almost identical to [BG, Theorem 4.1], and hence we only give the key points. Also we omit the subscript $v$ in our notation and simply write $GL_r^{(2)} = GL_r^{(2)}(F), N = N_B(F_v), Z = Z(F_v), T = T(F_v), B = B(F_v)$ and $K = GL_r(O_{F_v})$.

We will work on the integral
$$\int_{ZN \setminus GL_r^{(2)}} W(g)Q(\kappa(g))R^s(\kappa(g))dg,$$
where all the data are unramified. By the above lemma, this is written as
$$\int_{ZN \setminus B^\#} \int_{K^\#} W(bk)Q(\kappa(bk))R^s(\kappa(bk))dkdb,$$
where $B^\#$ is as in the above lemma with $P = B$. By the $s - \kappa$ trick (or strictly speaking it should be called “$\kappa - \kappa$” trick in this case), this is written as
$$\int_{ZN \setminus B^\#} \int_{K^\#} W(bk)Q(\kappa(b)\kappa(k))R^s(\kappa(b)\kappa(k))dkdb.$$

By the $K$ invariance of the integrand, we have
$$\int_{ZN \setminus B^\#} W(b)Q(\kappa(b))R^s(\kappa(b))db.$$

Since the integrand is left $N$ invariant, this is written as
$$\int_{Z \setminus T^\#} W(t)Q(\kappa(t))R^s(\kappa(t))\delta_B(t)^{-1}dt,$$
where $T^\# = T \cap B^\#$ and $\delta_B$ is the modular character of the Borel subgroup $B$. (Once again, one also need the $s - \kappa$ trick for this formulation.) For each
$$\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r,$$
we write
$$t_\lambda = \left( \begin{smallmatrix} \overline{\omega}^{\lambda_1} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \overline{\omega}^{\lambda_r} \end{smallmatrix} \right).$$

Then the integral is equal to
$$\sum_{\lambda \in \mathbb{Z}^r, t_\lambda \in T^\#} W(t_\lambda)Q(\kappa(t_\lambda))R^s(\kappa(t_\lambda))\delta_B(t_\lambda)^{-1},$$
where $\lambda$ runs through the elements of the form $(\lambda_1, \ldots, \lambda_{r-1}, 0)$ with $\sum_{i=1}^{r-1} \lambda_i = \text{even}$. (Since we mod out by $Z$, we always have $\lambda_r = 0$.)

We have
$$R^s(\kappa(t_\lambda)) = Q'(\kappa(t_\lambda))\delta_Q(t_\lambda),$$
where $Q'$ is the semi-Whittaker functional for the inducing representation $\vartheta^{\overline{\omega}^{r-1}\kappa}$, and $\delta_Q$ is the modular character for the parabolic $Q$. (Unfortunately we have two different $Q$ here, but we assume it should not create any confusion.)
Let $\alpha = \{\alpha_1, \ldots, \alpha_r\}$ be the Satake parameter of the $v$ component $\pi_v$ of our cuspidal representation $\pi$. By Shintani’s formula (Shim),

$$W(t_\lambda) = \begin{cases} 
\delta_B(t_\lambda)^{1/2}s_\lambda(\alpha), & \text{if } \lambda_1 \geq \cdots \geq \lambda_{r-1} \geq 0; \\
0, & \text{otherwise}, 
\end{cases}$$

where $s_\lambda$ is the symmetric function of $r$ variables as defined in [Mac Section I.3], and $s_\lambda(\alpha)$ is the value of the function evaluated at the Satake parameter.

Following [BG], we call $\lambda$ even if all the components $\lambda_i$ are even. Since $\chi^{1/2}$ certainly exists in the unramified situation, which we fix, one can see that

$$Q(\kappa(t_\lambda)) = \begin{cases} 
\delta_B^{1/2}(t_\lambda)\chi^{1/2}\omega^{-1}(\det(t_\lambda)), & \lambda \text{ is even}; \\
0, & \text{otherwise}.
\end{cases}$$

One can also see

$$Q'(\kappa(t_\lambda)) = \begin{cases} 
\delta_B^{1/2}(t_\lambda)\omega(\det(t_\lambda)), & \lambda \text{ is even}; \\
0, & \text{otherwise},
\end{cases}$$

where $\delta_B'$ is the modulus character of the Borel subgroup $B'$ of $\GL_{r-1}$ viewed as a subgroup of $\GL_r$ with the embedding $h \mapsto (h_1)$. By multiplying all those, one can see that the local zeta integral is equal to

$$\sum_{\text{even } \lambda \in \mathbb{Z}^r \atop \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r-1} \geq 0, \lambda_r = 0} s_\lambda(\alpha)\delta_B(t_\lambda)^{s-\frac{1}{2}}\chi^{1/2}(\det(t_\lambda)), $$

which is precisely the twisted analogue of [BG] (4.7)]. Hence the computation in the proof of [BG] Theorem 4.1] can be directly applied to our integral, which yields the proposition. Namely, as in p.171 of [BG], we have

$$\prod_{1 \leq i \leq j \leq r} (1 - \alpha_i\alpha_jX)^{-1} = \sum_{\text{even } \lambda \in \mathbb{Z}^{r-1} \atop \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r-1} \geq 0} s_\lambda(\alpha)X^\left(\sum_{i=1}^{r-1} \lambda_i\right)/2 \times (1 - \omega(\varpi)^2X^r)^{-1}. $$

(Here notice that in the exponent of $X$ in the first factor of the corresponding formula in [BG], there is a typo.) By taking $X = \chi(\varpi)q^{-2s+1/2}$, we obtain our proposition. \qed

4. The Rankin-Selberg integrals for the case $r = 2q + 1$

We consider

$$r = 2q + 1 = \text{odd}. $$

Note that for the $r = 2q + 1$ case there is no issue raised by Kable (K1) for the Rankin-Selberg integral of Bump and Ginzburg. But in order to incorporate the character twist into the Bump-Ginzburg integral, we need to choose

$$\theta = \theta_{\omega^{-1}},$$

where $\theta_{\omega^{-1}}$ is the global non-twisted exceptional representation of $\widetilde{\GL}_r(\mathbb{A})$ with determinantal character $\omega^{-1}$, and

$$\theta' = \theta'_{\chi^{\omega^2}\chi^{-q}}$$

for the exceptional representation of $\widetilde{\GL}_{r-1}(\mathbb{A}) \times \GL_1(\mathbb{A}) \subseteq \widetilde{\GL}_r(\mathbb{A})$ associated with $\chi^{\omega^2}$ and $\chi^{-q}$. Notice that the central character of $\theta$ is

$$(1, \xi)\varphi(z) \mapsto \xi \omega^{-2q-1}(a)\mu_\varphi(a)^{q}$$
where $W,Q_r$ like the case the almost Euler product the integral is well-defined. By following the computation of $[BG]$, the global integral decomposes into

$$Z(\phi, \Theta, f^*) = \int_{Z(\mathbb{A}) \backslash GL_v(F) \backslash GL_v(A)} \phi(g) \Theta(\kappa(g)) E(\kappa(g), s, f^*) \ dg,$$

where $\phi \in \pi$ is a cusp form and $E(-, s, f^*)$ is the Eisenstein series as before. Note that the product $\Theta(\kappa(g)) E(\kappa(g), s, f^*)$ is not genuine, on which the center $z \in Z(\mathbb{A})$ acts as the character $\omega^{-1}$, and hence the integral is well-defined. By following the computation of $[BG]$, the global integral decomposes into the almost Euler product

$$Z(\phi, \Theta, f^*) = Z(W, Q, R^s) = Z_\infty(W_\infty, Q_\infty, R^s_\infty) \prod_{v < \infty} Z_v(W_v, Q_v, R^s_v),$$

where $W,Q$ and $R^s$ and their local components are just as the $r = 2q$ case. Note again that just like the case $r = 2q$ because of the issue on the uniqueness of the semi-Whittaker functional at the archimedean places, we need to compromise with this almost Euler product in place of the full Euler product.

We can compute the unramified factor as follows.

**Proposition 4.1.** At each unramified place $v$,

$$Z_v(W_v, Q_v, R^s_v) = L(2s - \frac{1}{2}, \pi_v, \text{Sym}^2 \otimes \chi_v)L(r(2s - \frac{1}{2}), \chi_v' \omega^2_v)^{-1}.$$

**Proof.** This is even more straightforward modification of $[BG]$ than the $r = 2q$ case. Also see $[B2]$ Theorem 7] for the case $r = 3$. \qed

5. The poles of $L^S(s, \pi, \text{Sym}^2 \otimes \chi)$

Now we are ready to prove the following main theorem of this paper.

**Theorem 5.1.** Let $\pi$ be a cuspidal automorphic representation of $GL_r(A)$ with central character $\omega$ and $\chi$ a unitary Hecke character. Then for each archimedean $v$, there exists an integer $N_v \geq 0$ such that the product

$$L^S(s, \pi, \text{Sym}^2 \otimes \chi) \prod_{v \mid \infty} L_v(r s - r + 1, \chi_v' \omega_v^2)^{-N_v}$$

is holomorphic everywhere except at $s = 0$ and $s = 1$. Moreover there is no pole if $\chi' \omega^2 \neq 1$.

**Proof.** The proof is a modification of the one given by Bump and Ginzburg $[BG]$ Theorem 7.5]. Since the essential points are already in $[BG]$, we only give a sketch of the proof for most of the time. Our Rankin-Selberg integral gives

$$L(r(2s - \frac{1}{2}), \chi' \omega^2) Z(\phi, \Theta, f^*)$$

$$= L^S(2s - \frac{1}{2}, \pi, \text{Sym}^2 \otimes \chi)L_\infty(r(2s - \frac{1}{2}), \chi_\infty' \omega_\infty^2)Z_\infty(W_\infty, Q_\infty, R^s_\infty)$$

$$\prod_{v \in S, v < \infty} L_v(r(2s - \frac{1}{2}), \chi_v' \omega_v^2)Z_v(W_v, Q_v, R^s_v)$$

(5.2)
for the factorizable $f^s = f^s_\infty \otimes (\otimes' f^s_\check{s})$. Recall that
\[ Z(\phi, \Theta, f^s) = \int \phi(g) \Theta(\kappa(g)) E(\kappa(g), s, f^s) dg, \]
where the integral is over $Z(\mathbb{A}) \text{GL}_r^{(2)}(F) \setminus \text{GL}_r^{(2)}(\mathbb{A})$ if $r$ is even and $Z(\mathbb{A}) \text{GL}_r(F) \setminus \text{GL}_r(\mathbb{A})$ if $r$ is odd.

Let us define the normalized Eisenstein series by
\[ E^*(g, s, f^s) := L^S(r(2s - \frac{1}{2}), \chi^r \omega^2) E(g, s, f^s). \]

Let us note the following proposition, whose proof will be given after the proof of this main theorem.

**Proposition 5.3.** Let $f^s$ be a flat section. Then for each archimedean $v$, there exists an integer $N_v \geq 0$ such that the product
\[ E^*(g, s, f^s) \prod_{v \mid \infty} L_v(r(2s - \frac{1}{2}) - r + 1, \chi^r \omega^2)^{-N_v} \]
is entire except that, if $\chi^r \omega^2 = 1$, it has simple poles at $s = 1/4$ and $s = 3/4$.

**Remark 5.4.** Let us note that we are not able to show that the normalized Eisenstein series $E^*(g, s, f^s)$ has the desired analytic properties, but we need to multiply a kind of compensation factor $L_v(r(2s - \frac{1}{2}) - r + 1, \chi^r \omega^2)^{-N_v}$ at each archimedean place. This is because of a subtle issue about asymptotic expansions of matrix coefficients to be explained later.

We also have

**Proposition 5.5.** The local zeta integral $Z_v(W_v, Q_v, R^*_v)$ (resp. the archimedean $Z_\infty(W_\infty, Q_\infty, R^*_\infty)$) has meromorphic continuation as a function in $s \in \mathbb{C}$. Moreover, for each fixed $s = s_0$, one may choose the local data so that $Z_v(W_v, Q_v, R^*_v)$ (resp. $Z_\infty(W_\infty, Q_\infty, R^*_\infty)$) does not have a zero at $s = s_0$.

**Proof.** For the non-archimedean zeta integral, the first part is proven in the same way as [BG Proposition 5.2] and the second part is as [BG Theorem 7.2]. For the archimedean zeta integral, we can apply their arguments to the product of copies of $\text{GL}_r(\mathbb{R})$ and/or $\text{GL}_r(\mathbb{C})$ instead of just one copy of each.

But since our zeta integrals are not identical to those treated in [BG], we repeat the essential points by making clear how the proofs have to be modified. First for the meromorphic continuation ([BG Proposition 5.2]), there are two key ingredients. One is the asymptotic expansion of the Whittaker functions ([JS §4]) for $W_v$ and the product $Q_v R^*_v$. (Since $Q_v$ and $R^*_v$ are semi-Whittaker functionals, in which the alternating entries one above the diagonal come out via the additive character, the product $Q_v R^*_v$ is a Whittaker functional.) Note that for $r = 2q$, all the data $W_v, Q_v$ and $R^*_v$ are restrictions to $\text{GL}_r^{(2)}(F_v)$ of those defined over $\text{GL}_r(F_v)$, and hence there is no issue for applying this theory. The second ingredient is the Iwasawa decomposition. For the non-archimedean case, this is Lemma [3.15]. For the archimedean case, we also have the analogous decomposition. Namely if $F_v = \mathbb{C}$, then $\text{GL}_r^{(2)}(\mathbb{C}) = \text{GL}_r(\mathbb{C})$, so there is no issue here. If $F_v = \mathbb{R}$, then $\text{GL}_r^{(2)}(\mathbb{R}) = \text{GL}_r(\mathbb{R})^+ = \{ g \in \text{GL}_r(\mathbb{R}) : \det(g) > 0 \}$, and we have the Iwasawa decomposition with $K^# = \text{SO}(n)$. Using those two ingredients, one can reduce the problem to meromorphic continuation of a torus integral of a finite sum of a product of a Schwartz function and a finite function (see [BG p.178]), where by torus we mean $T \cap B^#$ when $r = 2q$. The rest of the computation is identical.

For the second part of the proposition, which corresponds to [BG Theorem 7.2], again the key ingredient is the Iwasawa decomposition. With it, one can reduce the problem to a problem on a integral over $\text{GL}_{r-1}(F_v)$, where $\text{GL}_{r-1}(F_v)$ sits in the Levi part of the $(r-1,1)$-parabolic of $\text{GL}_r(F_v)$, and show the non-vanishing of the integral by induction. For the case $r = 2q$, one can argue in the same way using $\text{GL}_r^{(2)}(F_v)$.

\[ \square \]
Hence by taking all those into account, we know that the poles of \( L^S(2s - \frac{1}{2}, \pi, Sym^2 \otimes \chi) \) are among the poles of the normalized Eisenstein series \( E^*(g, s, f^*) \) because by canceling the local factors \( L_v(r(2s - \frac{1}{2}), \chi_\nu^0 \omega_\nu^2) \) for all \( v \in S \) in (5.2), we have
\[
Z^*(\phi, \Theta, f^*) = L^S(2s - \frac{1}{2}, \pi, Sym^2 \otimes \chi)Z_\infty(W_\infty, Q_\infty, R_{\infty}^s) \prod_{v \in S, v < \infty} Z_v(W_v, Q_v, R_v^s)
\]
where
\[
Z^*(\phi, \Theta, f^*) := \int \phi(g)\Theta(\kappa(g))E^*(\kappa(g), s, f^*)dg.
\]
Thus the theorem follows. \( \Box \)

We give a proof of Proposition 5.3.

**Proof of Proposition 5.3** The proof is almost identical to the one given by [BG, Theorem 7.4] except a subtle issue about asymptotic expansions of matrix coefficients at the archimedean places. Since the proof is essentially the same as in [BG] for most of the part, we will reproduce only the main points. Moreover, since the case of our main interest is the case for \( r = 2q \), the case \( r = 2q + 1 \) being more similar to [BG], we only consider \( r = 2q \). (Also the twisted case for \( r = 3 \) is treated by [B2].)

First let us note that as we explained at the beginning of the previous section, the Eisenstein series \( E(-, s, f^*) \) on \( \widetilde{GL}_{2q}(\mathbb{A}) \) is simply the restriction of the Eisenstein series on \( \widetilde{GL}_{2q}(\mathbb{A}) \). Hence one can apply the theory of Eisenstein series \( (MW) \) to this case.

As the proof in [BG] p.195-196, the computation of the poles boils down to determining the poles of the intertwining operator
\[
M(s) : \text{ind}_{\widetilde{GL}_{2q}(\mathbb{A})}^{\text{GL}_{2q}(\mathbb{A})} \theta_{\omega, \omega^{-1}} \chi^{-q} \otimes \delta_{Q} \rightarrow \text{ind}_{\widetilde{GL}_{2q}(\mathbb{A})} Q(\mathbb{A})_{N(1, r-1)(\mathbb{A})} w_0 (\theta_{\omega, \omega^{-1}} \chi^{-q} \otimes \delta_{Q}'),
\]
where the induction is NOT normalized and \( w_0 = (1_{2q-1}, \frac{1}{2}) \).

For each factorizable section \( f^* = \otimes' f_v^* \), we know from Lemma 2.68 that
\[
M(s)f^* = \frac{L(r(2s - \frac{1}{2}) - r + 1, \chi_\nu^r \omega_\nu^2)}{L(r(2s - \frac{1}{2}), \chi_\nu^r \omega_\nu^2)} (\otimes'_{v \notin S} \frac{L_v(r(2s - \frac{1}{2}), \chi_\nu^r \omega_\nu^2)}{L_v(r(2s - \frac{1}{2}) - r + 1, \chi_\nu^r \omega_\nu^2)} M_v(s)f_v^*),
\]
where \( M_v(s) \) is the corresponding local intertwining operator. (Note that in Lemma 2.68 the induction is normalized, and hence we need to shift \( s \) by 1/2.)

Hence the poles of the normalized Eisenstein series \( E^*(g, s, f^*) \) are the poles of
\[
L^S(r(2s - \frac{1}{2}), \chi_\nu^r \omega_\nu^2) M(s)f^*
\]
\[
= L(r(2s - \frac{1}{2}) - r + 1, \chi_\nu^r \omega_\nu^2) \otimes'_{v \notin S} \frac{L_v(r(2s - \frac{1}{2}), \chi_\nu^r \omega_\nu^2)}{L_v(r(2s - \frac{1}{2}) - r + 1, \chi_\nu^r \omega_\nu^2)} M_v(s)f_v^*.
\]

The Hecke \( L \)-function \( L(r(2s - \frac{1}{2}) - r + 1, \chi_\nu^r \omega_\nu^2) \) has no pole unless \( \chi_\nu^r \omega_\nu^2 = 1 \). (Note that if \( \chi_\nu^r \omega_\nu^2 = 1 \), then this \( L \)-function has poles at \( r(2s - \frac{1}{2}) - r + 1 = 1 \) i.e. \( s = 3/4 \) and \( r(2s - \frac{1}{2}) - r + 1 = 0 \) i.e. \( s = 1/4 \). This is why the normalized Eisenstein series could have a pole at \( s = 1/4 \) and \( s = 3/4 \) for this case.) Also for almost all \( v \), we know from Lemma 2.68 that
\[
\frac{L_v(r(2s - \frac{1}{2}), \chi_\nu^r \omega_\nu^2)}{L_v(r(2s - \frac{1}{2}) - r + 1, \chi_\nu^r \omega_\nu^2)} M_v(s)f_v^*.
is the spherical section in \( \text{ind}_{\widetilde{GL}_2(F_v)}^{GL_2(F_v)} \theta_{\omega^{-1} \chi^{-1} v} \otimes \delta_Q \), which means that this has no pole.

By taking all those into account, what we have to prove is that for non-archimedean \( v \) the local intertwining operator

\[
\frac{1}{L_v(r(2s - \frac{1}{2}) - r + 1, \chi_v^2)} M_v(s) : \text{ind}_{\widetilde{GL}_2(F_v)}^{GL_2(F_v)} \theta_{\omega^{-1} \chi^{-1} v} \otimes \delta_Q \rightarrow \text{ind}_{\widetilde{GL}_2(F_v)}^{GL_2(F_v)} \theta_{\omega^{-1} \chi^{-1} v} \otimes \delta_Q
\]

has no pole, and for archimedean \( v \) it has no pole except those which are canceled by the compensation factor \( L_v(r(2s - \frac{1}{2}) - r + 1, \chi_v^2) \).

First assume \( v \) is non-archimedean. But our situation is identical to [BG], because our \( \theta_{\omega^{-1} \chi^{-1} v} \) is the same as theirs, except that ours has the twist \( \omega^{-1} \chi^{-1} v \) by the \( GL_1 \) factor of the parabolic \( \tilde{Q} \), which does no harm when one applies the method of [BG]. To show the above intertwining operator \((5.6)\) has no pole, Bump and Ginzburg considered the inner product of the induced representations \( \text{ind}_{\tilde{Q}(F_v)}^{GL_2(F_v)} \theta_{\omega^{-1} \chi^{-1} v} \otimes \delta_Q \) and \( \text{ind}_{\tilde{Q}(F_v)}^{GL_2(F_v)} \theta_{\omega^{-1} \chi^{-1} v} \delta_Q^{-s} \), and reduced the problem to a computation of the asymptotic behavior of the matrix coefficients given by

\[
\int_{F_v^{-1}} \int_{F_v} \theta_v \left( s \begin{pmatrix} y & I_{r-2} \ y^{-1} \end{pmatrix} \right) [u_1], \theta_v \left( s \begin{pmatrix} Z & \ -1 \ 
 & 1 \end{pmatrix} \right) [u_2] \phi(y, Z) |y|^{r(s-1)} dy dZ,
\]

where \( \theta_v = \theta_{\omega^{-1} \chi^{-1} v}, u_i \) is a vector in the space of \( \theta_v \), and \( \phi(y, Z) \) is a Schwartz function on \( F_v^{-1} \).

This is equation (7.14) of [BG], and so the details can be found there.) So it suffices to show that this integral has no pole. It is shown by [B] (Casselman’s theorem applied to the metaplectic group) that the asymptotic of the matrix coefficients as \( |y| \rightarrow 0 \) is determined by the Jacquet module of \( \theta_v \) along the Borel subgroup. But since the representation \( \theta_v \) is the exceptional representation for which we know the exact expression for the Jacquet module by Proposition 2.6, one can explicitly compute the asymptotic expansion, which is carried out in [BG, p.200].

Assume \( v \) is real. (Let us mention that what follows is explained to the author by N. Wallach, and the author would like to thank him for it.) Unlike the non-archimedean case, we do not have such description of the Jacquet module. But instead, we (and Bump-Ginzburg) use the theory of Harish-Chandra ([BG, p.200-201]). The basic idea is essentially analogous to the non-archimedean case in that one needs to consider the analogous integral of the matrix coefficient, and instead of Casselman’s theorem, one needs to use the asymptotic expansion of the matrix coefficient due to Harish-Chandra. Then one obtains

\[
\left( \theta_v \left( s \begin{pmatrix} y & I_{r-2} \ y^{-1} \end{pmatrix} \right) [u_1], \theta_v \left( s \begin{pmatrix} Z & \ -1 \ 
 & 1 \end{pmatrix} \right) [u_2] \right) \sim \sum_{n=0}^{\infty} a_n(Z) |y|^{n+(r-2)/4} \omega_v \chi_v^q(y) P(\log |y|) \quad \text{as } |y| \rightarrow 0,
\]

where \( P(\log |y|) \) is some polynomial in \( \log |y| \). (The reader is advised to compare it with the formula in [BG, p.201]. In [BG], the factor \( P(\log |y|) \) is missing.) Then as in [BG, p.201] if one carries out the integration, one obtains the Mellin transform of a function in \( y \), which vanishes for \( |y| \) large, and the possible poles are determined by the asymptotic as \( |y| \rightarrow 0 \). Indeed, for example if \( \chi_v^{1/2} \) exists, the
possible poles are at
\[ s = \frac{3}{4} - \frac{1}{2r} - \frac{\rho}{r} - \frac{n}{r} \quad \text{for } n \geq 0 \]
where \( \rho \) is the purely imaginary number so that \( \chi^{1/2} \omega_v(y) = (y/|y|)^\epsilon |y|^\rho \) where \( \epsilon = 0 \) or \( 1 \) as in [BG]. (This computation is done by integration by parts.) And those are precisely where the local archimedean factor \( L_v(r(2s - \frac{1}{2}) - r + 1, \chi^r \omega_v^2) \) has poles. However, this theory only tells us the locations of the possible poles, but the orders of the possible poles cannot be shown to be simple. Indeed, this theory only tells that the order of each possible pole is at most
\[(\text{the degree of the polynomial } P) + 1.\]

All those issues are explained quite in detail in [Wa, p.361-362]. Hence unless one can show that the degree of \( P \) is 0, one can not conclude that the possible poles are canceled with the poles of \( L_v(r(2s - \frac{1}{2}) - r + 1, \chi^r \omega_v^2) \). Although it might be still possible that the polynomial \( P \) indeed has degree 0, at least the author does not know how to show it. Hence it should be considered that even after the factor \( L_v(r(2s - \frac{1}{2}) - r + 1, \chi^r \omega_v^2)^{-1} \) is multiplied to the intertwining operator \( M_v(s) \), we still have the possible poles at the above locations. Hence the best we have is the product
\[ L_v(r(2s - \frac{1}{2}) - r + 1, \chi^r \omega_v^2)^{-N_v} M_v(s) \]

is holomorphic, where \( N_v \) is the degree of the polynomial \( P \).

Hence by taking all those into account, we can show the holomorphy of the product
\[ E^*(g, s, f^*) \prod_{v \mid \infty} L_v(r(2s - \frac{1}{2}) - r + 1, \chi^r \omega_v^2)^{-N_v} \]
as in the proposition. \( \square \)

From the main theorem (Theorem 5.1), it is immediate that the possible poles other than at \( s = 0 \) and \( s = 1 \) come from the poles of the local archimedean factors \( L_v(r(2s - \frac{1}{2}) - r + 1, \chi^r \omega_v^2) \), which are just gamma functions. Hence we have

**Corollary 5.8.** The (incomplete) twisted symmetric square \( L \)-function \( L^S(s, \pi, \text{Sym}^2 \otimes \chi) \) is holomorphic everywhere in the region \( \text{Re}(s) > 1 - \frac{1}{2r} \) except at \( s = 1 \). Moreover there is no pole at \( s = 1 \) if \( \chi^r \omega_v^2 \neq 1 \).

The reason we can have our result only for \( \text{Re}(s) > 1 - \frac{1}{2r} \) is the archimedean issue pointed out above. But we believe this issue can be resolved and hope to prove

**Conjecture 5.9.** The (incomplete) twisted symmetric square \( L \)-function \( L^S(s, \pi, \text{Sym}^2 \otimes \chi) \) is holomorphic everywhere except at \( s = 0 \) and \( s = 1 \). Moreover there is no pole if \( \chi^r \omega_v^2 \neq 1 \).

We hope this can be done in our forthcoming paper [T2].

Finally let us note that this corollary does NOT tell us that the \( L \)-function \( L^S(s, \pi, \text{Sym}^2 \otimes \chi) \) has a pole at \( s = 1 \) if \( \chi^r \omega_v^2 = 1 \). We only know it might have a pole at \( s = 1 \), but it might not. We are not able to determine this. However if \( r \) is odd, the following theorem due to Jacquet-Shalika and Shahidi allows one to tell exactly when \( L^S(s, \pi, \text{Sym}^2 \otimes \chi) \) has a pole at \( s = 1 \).

**Theorem 5.10.** Assume \( r \) is odd. Then the (complete) twisted exterior square \( L \)-function \( L(s, \pi, \wedge^2 \otimes \chi) \) is non-zero holomorphic at \( s = 1 \).

**Proof.** The non-vanishing part is the main theorem of [Sh2], and the holomorphy is [JS] Theorem 9.6.2. \( \square \)
This theorem implies

**Corollary 5.11.** Assume \( r \) is odd. Then the \( L \)-function \( L^S(s, \pi, \text{Sym}^2 \otimes \chi) \) has a pole at \( s = 1 \) if and only if \( \tilde{\pi} = \pi \otimes \chi \), where \( \tilde{\pi} \) is the contragredient of \( \pi \).

**Proof.** Recall

\[
L^S(s, \pi \times \pi \otimes \chi) = L^S(s, \pi, \Lambda^2 \otimes \chi)L^S(s, \text{Sym}^2 \otimes \chi),
\]

and the Rankin-Selberg \( L \)-function \( L^S(s, \pi \times \pi \otimes \chi) \) has a pole at \( s = 1 \) if and only if \( \tilde{\pi} = \pi \otimes \chi \).

Hence the corollary follows from the above theorem. \( \square \)

**Appendix A. Metaplectic tensor product**

In this appendix, we will recall the notion of metaplectic tensor product for \( \tilde{\text{GL}}_r^{(2)} \) both locally and globally. For the local case, if one uses the block-compatible cocycle \( \sigma_r \), the formulation of metaplectic tensor product is done in several places. (See \[B1, K2, M\].) But since we use our \( \tau_r \), which works both for the local and global cases, we need another formulation. Let us mention that this appendix is a portion of \[T1\] in which we developed the theory of metaplectic tensor products for automorphic representations of the \( n \)-fold cover of \( \text{GL}_r(\mathbb{A}) \), and in the interest of space, we only recall the basic facts necessary for our purposes and we will occasionally omit the details of the proofs, all of which are available in \[T1\].

Let \( P \) be a parabolic subgroup of \( \text{GL}_r \) whose Levi is

\[
M_P = \text{GL}_{r_1} \times \cdots \times \text{GL}_{r_k}.
\]

Of course we assume \( M_P \) sits in \( \text{GL}_r \) diagonally. Let us denote by \( \tilde{M}_P \) the metaplectic preimage of \( M_P \), and write

\[
\tilde{M}_P = \widetilde{\text{GL}}_{r_1} \times \cdots \times \text{GL}_{r_k},
\]

where the group structure of \( \tilde{M}_P \) is defined via the restriction of the cocycle \( \tau_r \).

**A.1. The group \( \tilde{\epsilon}M_P \).** One difficulty to work with \( \tau_r \) is that it is not known that it is block-compatible unless \( r = 2 \). To get around it, let us define a cocycle

\[
\tau_P : M_P \times M_P \to \{ \pm 1 \},
\]

both locally and globally, by

\[
\tau_P\left( \begin{pmatrix} g_1 & \cdots & g_k \end{pmatrix}, \begin{pmatrix} g'_1 & \cdots & g'_k \end{pmatrix} \right) = \prod_{i=1}^k \tau_{r_i}(g_i, g'_i) \prod_{1 \leq i < j \leq k} (\det(g_i), \det(g'_j)),
\]

where \((-,-)\) is the local or global Hilbert symbol. Note that the definition makes sense both locally and globally. Moreover the global \( \tau_P \) is the product of the local ones.

We define the group \( \tilde{\epsilon}M_P \) to be

\[
\tilde{\epsilon}M_P = M_P \times \{ \pm 1 \}
\]

as a set and the group structure is given by \( \tau_P \). The superscript \( \epsilon \) is for “compatible”. One advantage to work with \( \tilde{\epsilon}M_P \) is that each \( \text{GL}_{r_i} \) embeds into \( \tilde{\epsilon}M_P \) via the natural map

\[
(g_i, \xi) \mapsto \left( \begin{pmatrix} I_{r_1+\cdots+r_i-1} & & \cr & g_i & \\
 & & I_{r_{i+1}+\cdots+r_k} \end{pmatrix}, \xi \right).
\]

Or rather, the cocycle \( \tau_P \) is so chosen that we have this embedding.
Also recall our notation
\[ M_P^{(2)} = GL^{(2)}_{r_1} \times \cdots \times GL^{(2)}_{r_k}, \]
and
\[ \tilde{M}_P^{(2)} = \tilde{GL}^{(2)}_{r_1} \times \cdots \times \tilde{GL}^{(2)}_{r_k}. \]
We define \( \tilde{c}M_P^{(2)} \) analogous to \( \tilde{c}M_P \), namely the group structure of \( \tilde{c}M_P^{(2)} \) is defined via the cocycle \( \tau_P \). Of course, \( \tilde{c}M_P^{(2)} \) is a subgroup of \( \tilde{c}M_P \). Note that each \( \tilde{GL}^{(2)}_{r_i} \) naturally embeds into \( \tilde{c}M_P^{(2)} \) as above.

**Lemma A.1.** The subgroups \( \tilde{GL}^{(2)}_{r_i} \) and \( \tilde{GL}^{(2)}_{r_j} \) in \( \tilde{c}M_P^{(2)} \) commute pointwise for \( i \neq j \).

**Proof.** Locally or globally, it suffices to show \( \tau_P(\tilde{g}_i, \tilde{g}_j) = \tau_P(\tilde{g}_j, \tilde{g}_i) \). But since the global \( \tau_r \) is the product of local ones, it suffices to show the local case. So assume our groups are over a local field. By the relation between \( \tau_P \) and \( \sigma_r \), it suffices to show \( \sigma_r(\tilde{g}_i, \tilde{g}_j) = \sigma_r(\tilde{g}_j, \tilde{g}_i) \). But this follows from the block-compatibility of the 2-cocycle \( \sigma_r \) as in (1.1). (See also [BG, p.141].) \( \square \)

**Lemma A.2.** There is a surjection
\[ \tilde{GL}^{(2)}_{r_1} \times \cdots \times \tilde{GL}^{(2)}_{r_k} \rightarrow \tilde{c}M_P^{(2)} \]
given by the map
\[ ((g_1, \xi_1), \ldots, (g_k, \xi_k)) \mapsto \left( \begin{array}{c} g_1 \\ \cdot \cdot \cdot \\ g_k \end{array} \right), \]
whose kernel is
\[ K_P := \{ ((1, \xi_1), \ldots, (1, \xi_k)) : \xi_1 \cdots \xi_k = 1 \}, \]
so that \( \tilde{c}M_P^{(2)} \cong \tilde{GL}^{(2)}_{r_1} \times \cdots \times \tilde{GL}^{(2)}_{r_k} / K_P \).

**Proof.** The above lemma together with the block-compatibility of \( \tau_P \) guarantees that the map is indeed a group homomorphism. The description of the kernel is immediate. \( \square \)

Note that for the group \( \tilde{M}_P \), the group structure is defined by the restriction of \( \tau_r \) to \( M_P \times M_P \), and hence each \( \tilde{GL}_{r_i} \) might not embed into \( GL_{r_i} \) in the natural way because of the possible failure of the block-compatibility of \( \tau_r \) unless \( r = 2 \). To make explicit the relation between \( \tilde{c}M_P \) and \( M_P \), the discrepancy between \( \tau_r |_{M_P \times M_P} \) (which we denote simply by \( \tau_r \)) and \( \tau_P \) has to be clarified.

**Local case:**
Assume \( F \) is local. Then we have
\[
\tau_P \left( \begin{array}{c} g_1 \\ \cdot \cdot \cdot \\ g_k \end{array} \right), \left( \begin{array}{c} g'_1 \\ \cdot \cdot \cdot \\ g'_k \end{array} \right) = \sigma_r \left( \begin{array}{c} g_1 \\ \cdot \cdot \cdot \\ g_k \end{array} \right), \left( \begin{array}{c} g'_1 \\ \cdot \cdot \cdot \\ g'_k \end{array} \right) \prod_{i=1}^k s_{r_i}(g_i)s_{r_i}(g'_i)/s_{r_i}(g_i) s_{r_i}(g'_i),
\]
so \( \tau_P \) and \( \sigma_r |_{M_P \times M_P} \) are cohomologous via the function \( \prod_{i=1}^k s_{r_i} \). Here recall from Section 1.2 that the map \( s_{r_i} : GL_{r_i} \rightarrow \{ \pm 1 \} \) relates \( \tau_{r_i} \) with \( \sigma_{r_i} \), by
\[
\sigma_{r_i}(g_i, g'_i) = \tau_{r_i}(g_i, g'_i) s_{r_i}(g_i) s_{r_i}(g'_i),
\]
for \( g_i, g_i' \in \text{GL}_r \). Moreover if the residue characteristic is odd, \( s_r \) is chosen to be “canonical” in the sense that (1.3) is satisfied.

The block-compatibility of \( \sigma_r \) implies

\[
\tau_r(m, m') \cdot \frac{s_r(mm')}{s_r(m)s_r(m')} = \tau_P(m, m') \prod_{i=1}^{k} \frac{s_r(g_1 g_2)}{s_r(g_1) s_r(g_2)},
\]

for \( m = \begin{pmatrix} g_1 & \cdots & g_k \\ \vdots & \ddots & \vdots \\ g_k \end{pmatrix} \) and \( m' = \begin{pmatrix} g_1' & \cdots & g_k' \\ \vdots & \ddots & \vdots \\ g_k' \end{pmatrix} \). Hence if we define \( \hat{s}_P : M_P \to \{ \pm 1 \} \) by

\[
\hat{s}_P(m) = \prod_{i=1}^{k} s_r(g_i),
\]

we have

\[\tau_P(m, m') = \tau_r(m, m') \cdot \frac{\hat{s}_P(m) \hat{s}_P(m')}{\hat{s}_P(mm')}\]

(A.3)

Therefore we have the isomorphism

\[\tilde{\varphi}_P : \tilde{c}_P \to \tilde{M}_P, \quad (m, \xi) \mapsto (m, \hat{s}_P(m) \xi).\]

An important fact about the map \( \hat{s}_P \) is

**Lemma A.4.** Assume \( F \) is non-archimedean of odd residual characteristic. Then for all \( k \in M_P(\mathcal{O}_F) \), we have \( \hat{s}_P(k) = 1 \).

**Proof.** This is \[T1\] Lemma 3.5 and essentially follows from the “canonicality” of \( s_r \) and \( s_r \), so that \( s_r \) has been chosen to satisfy \( s_r = s_r|_{\text{GL}_r(\mathcal{O}_F)} \), where \( s_r \) is the map on \( \mathcal{G}_r(F) \) that makes the diagram (1.4) commute, and from the fact that the cocycle for \( \mathcal{G}_r \) is block-compatible for a very strong sense as in \[BLS\] Lemma 5, Theorem 7 §2. See \[T1\] for the detail. \( \square \)

**Global case:**

Assume \( F \) is global. Define \( \hat{s}_P : M_P(\mathbb{A}) \to \{ \pm 1 \} \) by

\[\hat{s}_P(\prod_v m_v) := \prod_v \hat{s}_P(m_v)\]

for \( \prod_v m_v \in M_P(\mathbb{A}) \). The product is finite thanks to Lemma A.4 Since both of the cocycles \( \tau_r \) and \( \tau_P \) are the products of the corresponding local ones, one can see that the relation (A.3) holds globally as well.

Thus analogously to the local case, we have the isomorphism

\[\tilde{\varphi}_P : \tilde{c}_P \to \tilde{M}_P, \quad (m, \xi) \mapsto (m, \hat{s}_P(m) \xi).\]

**Lemma A.5.** The splitting of \( M_P(F) \) into \( \tilde{c}_P(\mathbb{A}) \) is given by

\[s_P : M_P(F) \to \tilde{c}_P(\mathbb{A}), \quad \left( \begin{array}{ccc} g_1 & \cdots & g_k \\ \vdots & \ddots & \vdots \\ g_k & \end{array} \right) \mapsto \left( \begin{array}{ccc} g_1 & \cdots & g_k \\ \vdots & \ddots & \vdots \\ g_k & \end{array} \right) \prod_{i=1}^{k} s_i(g_i)^{-1}.\]
Proof. For each $i$ the splitting $s_{r_i} : \text{GL}_{r_i}(F) \to \tilde{\text{GL}}_{r_i}(\mathbb{A})$ is given by $g_i \mapsto (g_i, s_{r_i}(g_i)^{-1})$, where \( \text{GL}_{r_i}(\mathbb{A}) \) is defined via the cocycle $\tau_{r_i}$. Then the lemma follows by the block-compatibility of $\tau_P$ and the product formula for the Hilbert symbol. 

This splitting is related to the splitting $s : \text{GL}_r(F) \to \text{GL}_r(\mathbb{A})$ by

**Proposition A.6.** We have the following commutative diagram:

$$
\begin{array}{ccc}
\tilde{c}M_P(\mathbb{A}) & \overset{\tilde{\varphi}_P}{\longrightarrow} & \tilde{\text{GL}}_r(\mathbb{A}) \\
\uparrow s_P & & \uparrow s_r \\
M_P(F) & \overset{\pi}{\longrightarrow} & \text{GL}_r(F).
\end{array}
$$

**Proof.** Note that for the elements in $\text{GL}_r(F)$, all of $s_{r_i}$ and $s_r$ are defined globally, and then the proposition follows from the definition of $s_P$ and $s_r$. 

This proposition implies

**Corollary A.7.** Assume $\pi$ is an automorphic representation of $\tilde{c}M_P(\mathbb{A})$. The representation of $\tilde{M}_P(\mathbb{A})$ defined by $\pi \circ \tilde{\varphi}_P^{-1}$ is also automorphic.

**Proof.** If $\pi$ is realized in a space $V$ of automorphic forms on $\tilde{c}M_P(\mathbb{A})$, then $\pi \circ \tilde{\varphi}_P^{-1}$ is realized in the space of functions of the form $f \circ \tilde{\varphi}_P^{-1}$ for $f \in V$. Then the automorphy follows from the commutativity of the diagram in the above lemma. 

A.2. Metaplectic tensor product. We are ready to define the notion of metaplectic tensor product. We treat both local and global cases at the same time. Let $\pi_1, \ldots, \pi_k$ be irreducible admissible representations of $\text{GL}_{r_1}^{(2)}, \ldots, \text{GL}_{r_k}^{(2)}$, respectively, where each $\pi_i$ is realized in the space $V_i$. Further assume each $\pi_i$ is genuine. Consider the usual tensor product representation $\pi_1 \otimes \cdots \otimes \pi_k$ of the direct product $\tilde{\text{GL}}_{r_1}^{(2)} \times \cdots \times \tilde{\text{GL}}_{r_k}^{(2)}$ realized in the space $V_1 \otimes \cdots \otimes V_k$. Since each $\pi_i$ is genuine, the kernel $K_P$ of the above lemma acts trivially on $\pi_1 \otimes \cdots \otimes \pi_k$. Thus this tensor product representation descends to a representation of $\tilde{c}M_P^{(2)}$, which we denote by

$$\pi_1 \tilde{\otimes} \cdots \tilde{\otimes} \pi_k,$$

and we call it the metaplectic tensor product representation of $\tilde{c}M_P^{(2)}$. Let us emphasize that the space of the metaplectic tensor product representation is the same as that of the tensor product.

Of course one can pullback the metaplectic tensor product $\pi_1 \tilde{\otimes} \cdots \tilde{\otimes} \pi_k$ of $\tilde{c}M_P^{(2)}$ to a representation of $\tilde{M}_P^{(2)}$ via the map $\tilde{\varphi}_P^{-1}$, which we often denote by the same symbol $\pi_1 \tilde{\otimes} \cdots \tilde{\otimes} \pi_k$, when there is no danger of confusion, and we call it the metaplectic tensor product representation of $\tilde{M}_P^{(2)}$.

**Proposition A.8.** Assume $F$ is global, and $\pi_1, \ldots, \pi_k$ are genuine irreducible automorphic representations of $\tilde{\text{GL}}_{r_1}^{(2)}(\mathbb{A}), \ldots, \tilde{\text{GL}}_{r_k}^{(2)}(\mathbb{A})$, respectively. Then the metaplectic tensor product representation $\pi_1 \tilde{\otimes} \cdots \tilde{\otimes} \pi_k$ of $\tilde{c}M_P^{(2)}(\mathbb{A})$ is also automorphic.

**Proof.** This is [T1] Proposition 5.2]. The proof is quite straightforward by viewing each function $f_1 \otimes \cdots \otimes f_k \in \pi_1 \otimes \cdots \otimes \pi_k$ naturally as a function on $\tilde{c}M_P^{(2)}(\mathbb{A})$ by

$$
(f_1 \otimes \cdots \otimes f_k)(\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{pmatrix}, \xi) = \xi f_1(g_1,1) \cdots f_k(g_k,1).
$$
The automorphy follows from the definition of $s_p$ and $s_r$, along with the block compatibility of $\tau_P$. □

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211