The stability of standard homogeneous Einstein manifolds

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Abstract
Back in 1985, Wang and Ziller obtained a complete classification of all homogeneous spaces of compact simple Lie groups on which the standard or Killing metric is Einstein. The list consists, beyond isotropy irreducible spaces, of 12 infinite families (two of them are actually conceptual constructions) and 22 isolated examples. We study in this paper the nature of each of these Einstein metrics as a critical point of the scalar curvature functional.

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1 Introduction

There is a canonical Riemannian metric on any homogeneous space \( M^d = G / K \) of a compact semisimple Lie group \( G \) naturally provided by the Killing form \( B_\mathfrak{g} \) of the Lie algebra \( \mathfrak{g} \) of \( G \). It is called the standard or Killing metric and will be denoted by \( g_B \) in this paper. Any simply connected compact homogeneous manifold \( M \) therefore admits one standard metric for each of its presentations \( M = G / K \) as a homogeneous space with \( G \) compact semisimple.

At this point, one might wonder how special is the standard metric from a geometric point of view. When \( M = G / K \) is an irreducible symmetric space, or more in general, an isotropy irreducible homogeneous space (see [31] or [2, 7.47]), \( g_B \) is the unique \( G \)-invariant metric on \( M \) up to scaling and it is automatically Einstein. Standard metrics have nonnegative sectional curvature, as actually any normal metric (i.e., defined by any bi-invariant inner product on \( \mathfrak{g} \)) does. All these metrics belong to the much wider class of naturally reductive metrics (with respect to \( G \)), that is when there is an \( \text{Ad}(K) \)-invariant decomposition \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) such that the one-parameter subgroups \( \exp t X \cdot \mathfrak{p}, X \in \mathfrak{p} \) are all the geodesics through any point \( p \in M \).

For \( G \) simple, the three notions are known to coincide.

The Einstein condition for \( g_B \) turns out to be quite strong: the Casimir operator of the isotropy \( K \)-representation must act with the same multiple on each of the irreducible summands. Indeed, the complete list obtained by Wang and Ziller [29] in the case when \( G \) is simple, a real tour de force in representation theory, looks short considering the jungle of all possibilities. The list consists, beyond isotropy irreducible spaces, of:

- \( G \) classical (see Table 1):
  - 10 infinite families parametrized by the natural numbers,
  - 2 constructions parametrized by \( l \)-tuples \( (l \geq 2) \) of certain irreducible symmetric spaces listed in Table 7,
  - and 2 isolated examples;

- \( G \) exceptional (see Tables 2, 3, 4): 20 isolated examples.

The classification of standard Einstein metrics is still open for non-simple groups (see [21, Section 4.14]).

It is also natural to ask how special is a standard Einstein metric among the space \( \mathcal{M}_G \) of all \( G \)-invariant metrics on \( M \). Since \( G \)-invariant Einstein metrics are precisely the critical points of the scalar curvature functional

\[
\text{Sc} : \mathcal{M}_G^1 \rightarrow \mathbb{R},
\]

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where \( \mathcal{M}_1^G \subset \mathcal{M}^G \) is the codimension one submanifold of all metrics of some fixed volume, the nature of \( g_B \) as a critical point may give an insight.

More in general, Einstein metrics on a compact differentiable manifold \( M \) are the critical points of the total scalar curvature functional

\[
\tilde{\text{Sc}}(g) := \int_M \text{Sc}(g) \, d\text{vol}_g,
\]

on the space \( \mathcal{M}_1 \) of all Riemannian metrics on \( M \) of some fixed volume (see [2, 4.21]). Moreover, if \( \tilde{\text{Sc}} \) is further restricted to the submanifold

\[
C_1 := \{ g \in \mathcal{M}_1 : \text{Sc}(g) \text{ is a constant function on } M \},
\]

then the nullity and coindex of critical points are both finite (see [2, 4.60]) and so the possibility of having a local maxima for \( \tilde{\text{Sc}}|_{C_1} \) comes into play. Remarkably, certain compact irreducible symmetric spaces are the only known local maxima of \( \tilde{\text{Sc}}|_{C_1} \) with \( \text{Sc} > 0 \).

**Remark 1.1** After the first version of the present paper was uploaded to arXiv, the \( G \)-stable Einstein metric on \( E_7/\text{SO}(8) \) found in [17] was proved to be stable in [25], giving the first example of a non-symmetric local maxima of \( \tilde{\text{Sc}}|_{C_1} \) with \( \text{Sc} > 0 \) (see [18] for further information).

The tangent space \( T_g C_1 \) coincides, modulo trivial variations, with the space \( T T_g \) of all divergence-free (or transversal) and traceless symmetric 2-tensors, so-called TT-tensors.

Moreover, if \( \text{Rc}(g) = \rho g \), then the Hessian of \( \tilde{\text{Sc}} \) is given by

\[
\tilde{\text{Sc}}''(T,T) = \frac{1}{4}((2\rho \text{id} - \Delta)T, T)_{g}, \quad \forall \, T \in T T_g,
\]

where \( \Delta \) is the Lichnerowicz Laplacian of \( g \) (see [2, 4.64]). Analogously, in the \( G \)-invariant context, the tangent space \( T_g \mathcal{M}_1^G \) at a metric \( g \in \mathcal{M}_1^G \) is precisely the finite dimensional vector subspace \( T T_g^G \) of \( G \)-invariant TT-tensors, modulo the trivial variations defined by \( N_G(K) \subset \text{Diff}(M) \), the normalizer of \( K \) in \( G \) (see [15, Section 3.4]). All this naturally motivates the definition of the following concepts. Let \( \lambda_L \) (resp. \( \lambda_L^G \)) denote the smallest eigenvalue of \( \Delta_L|_{T T_g} \) (resp. of \( L_p := \Delta_L|_{T T_g^G} \)).

**Definition 1.2** (see Fig. 1). An Einstein metric \( g \in \mathcal{M}_1 \) (resp. \( g \in \mathcal{M}_1^G \)) with \( \text{Rc}(g) = \rho g \) is said to be,

- **stable** (resp. G-stable): \( \tilde{\text{Sc}}''|_{T T_g} < 0 \), i.e., \( 2\rho < \lambda_L \) (resp. \( 2\rho < \lambda_L^G \)). In particular, \( g \) is a local maximum of \( \tilde{\text{Sc}}|_{C_1} \) (resp. of \( \text{Sc}|_{\mathcal{M}_1^G} \)).
- **semistable** (resp. G-semistable): \( \tilde{\text{Sc}}''|_{T T_g} \leq 0 \), i.e., \( 2\rho \leq \lambda_L \) (resp. \( 2\rho \leq \lambda_L^G \)); and otherwise unstable (resp. G-unstable).
- **neutrally stable** (resp. G-neutrally stable): \( \tilde{\text{Sc}}''|_{T T_g} \leq 0 \) and has nonzero kernel, i.e., \( 2\rho = \lambda_L \) (resp. \( 2\rho = \lambda_L^G \)).
- **non-degenerate** (resp. G-non-degenerate): \( \tilde{\text{Sc}}''|_{T T_g} \) (resp. \( \text{Sc}'|_{T T_g^G} \)) is non-degenerate, i.e., \( 2\rho \notin \text{Spec}(\Delta_L|_{T T_g}) \) (resp. \( 2\rho \notin \text{Spec}(\Delta_L|_{T T_g^G}) \)); otherwise, degenerate (resp. G-degenerate). In particular, if \( G \) is non-degenerate (resp. G-non-degenerate) then \( g \) is rigid (resp. G-rigid), in the sense that it is an isolated critical point up to the action of \( \text{Diff}(M) \) (resp. of \( N_G(K) \)).
- **dynamically stable**: for any metric \( g_0 \) near \( g \), the normalized Ricci flow starting at \( g_0 \) exists for all \( t \geq 0 \) and converges modulo diffeomorphisms, as \( t \to \infty \), to an Einstein metric near \( g \) (see [13, 14]).
• \( \nu\)-stable: \( \nu''_g|_{C^\infty(M)} < 0 \) and \( \nu''_g|_{TT_g} < 0 \), where \( \nu \) is the \( \nu \)-entropy functional introduced by Perelman, which is strictly increasing along Ricci flow solutions unless the solution consists of a shrinking gradient Ricci soliton like an Einstein metric with \( \rho > 0 \) (see [5, 22, 28]).

The study of the stability types of irreducible compact symmetric spaces was initiated by Koiso in [12] and recently concluded in [26, 27].

In the \( G \)-invariant setting, the graph theorem [4, Theorem 3.3] and its generalization, the simplicial complex theorem [3, Theorem 1.5], suggest that \( G \)-instability is an expected behavior. Indeed, all the Einstein metrics provided by these powerful existence results have positive augmented coindex (i.e., coindex plus nullity of \( Sc''_g|_{TT_g} \)). It is worth pointing out that, according to [4, Theorem 5.1], a \( G \)-unstable Einstein metric does not realize the Yamabe invariant of \( M \) (i.e., \( Sc(g) \) is not the supreme among all Yamabe metrics of \( M \); recall that a metric is called Yamabe when it has the smallest scalar curvature in its unit volume conformal class). There are several examples of unstable Einstein manifolds in the literature (see [17] and references therein).

The main result of this paper is the following.

**Theorem 1.3** Let \( M = G/K \) be a homogeneous space which is not isotropy irreducible and assume that \( G \) is simple and \( g_g \) is Einstein. Then the \( G \)-stability and critical point types of \( g_g \) are given as in Tables 1, 2, 3 and 4, with the only exception of the space \( \text{SO}(4n^2)/\text{Sp}(n) \times \text{Sp}(n) \) in Table 1, 3b.

Some cases were previously solved in [15, 17–19]. The last column on the right of the tables provides a reference to the place in the paper (or to another paper) where the case was proved. The smallest eigenvalue \( \lambda^G_L \) of \( L_p = \Delta_L|_{TT} \) is denoted by \( \lambda_p \) and the greatest one by \( \lambda^\text{max}_p \) in the tables (see Sect. 3.3 for further information on the notation used in the tables).

A main ingredient in the proof of Theorem 1.3 is the following stability criterion for \( g_g \) in terms of only the extremal Casimir eigenvalues \( \lambda_\tau \) and \( \lambda^\text{max}_\tau \) of the representation \( \text{sym}(\mathfrak{g}) \) of \( \mathfrak{g} \) (see Table 5), without even involving the Einstein constant \( \rho \).

**Theorem 1.4** (See Theorem 4.3, (i)). The standard metric \( g_B \) on \( G/K \) is \( G \)-stable if

\[
\frac{\dim K}{\dim G} < \frac{\lambda_\tau - 1}{2\lambda^\text{max}_\tau}.
\]

The proof of this purely Lie theoretical criterion is based on the formula for \( L_p \) given in [15, Section 5] and most cases with \( \mathfrak{g} \) exceptional follow from it. We also strongly use throughout the paper the formula for the matrix of \( L_p \) in terms of structural constants given in [15, 17].

We now discuss some geometric consequences of Theorem 1.3. When \( \dim \mathcal{M}^G_1 > 1 \), local maxima of \( Sc|_{\mathcal{M}^G_1} \) are not common. The local maxima of \( Sc|_{\mathcal{M}^G_1} \) (not necessarily standard) which had been known before all have either that \( K \) is a maximal subgroup of \( G \) or \( \dim \mathcal{M}^G_1 \leq 5 \) (see [17]). As an application of Theorem 1.4, we have found many unexpected new examples of standard local maxima, including the exceptional full flag manifolds of \( E_6 \), \( E_7 \), and \( E_8 \), which respectively have \( \dim \mathcal{M}^G_1 = 35, 63, 119 \) (see Tables 3 and 4).

**Corollary 1.5** Among the 20 spaces with exceptional \( G \) in Theorem rm1.3, \( g_B \) is a local maximum on 14 of them and a local minimum on 3 of them.
In contrast, the standard metric is $G$-unstable on most of the spaces with $G$ classical. Actually, $g_B$ is even a local minimum on many of them (see Table 1).

It is still an open problem whether there are only finitely many $G$-invariant Einstein metrics (up to homothety) on a given compact homogeneous space $M = G/K$. This has been conjectured to hold in the multiplicity-free isotropy representation case in [4]. It follows from the compactness theorem [4, Theorem 1.6] that the conjecture is equivalent to the $G$-rigidity (i.e., it is an isolated point in the moduli space of all $G$-invariant unit volume Einstein metrics on $M$ = $G/K$ -invariant Einstein metric (see [15, Section 3.1] for a more detailed treatment).

Except for a few spaces, $g_B$ is $G$-non-degenerate in all the cases where Spec$(L_p)$ was computed (i.e., in 23 cases of a total of 32, see also Table 9), which implies the following on $G$-rigidity.

**Corollary 1.6**  The standard metric is an isolated point in the space of all $G$-invariant unit volume Einstein metrics on $M = G/K$ for all the spaces in Theorem 1.3 where Spec$(L_p)$ is known, except possibly for the spaces $SO(2n)/T^n$, $n \geq 4$, $E_6/SO(3)^3$ and $E_8/SO(9)$.

We also obtained that the standard metric is $G$-degenerate on $SU(4)/T^3$ (see Table 1, 1a.2) and $E_6/SU(2) \times SO(6)$ (see Table 2, 4), but it is known to be $G$-rigid on these spaces (see [24, pp.78] and [7, IV.16], respectively). We do not know whether the standard metric is indeed $G$-degenerate or not on the spaces $E_6/SO(3)^3$ (see Table 2, 2) and $E_8/SO(9)$ (see Table 3, 9). Concerning $SO(2n)/T^n$, $n \geq 4$ (see Table 1, 1b.2), on which the standard metric has nullity $n - 1$, it was proved in [18] that one of such null directions gives rise to an inflection point of $S_c$, showing that it is not a local maximum.

We next highlight some other general consequences of the theorem.

- The Lichnerowicz Laplacian restricted to $TT_g^G$, has either only one or two eigenvalues in all the cases where Spec$(L_p)$ was computed, with the only exception of the full flag manifolds $SO(2n)/T^n$, $n \geq 4$, where $L_p$ has three different eigenvalues.
- Among the homogeneous spaces considered in Theorem 1.3, there are exactly four cases where $K$ is a maximal subgroup of $G$:

  $$SO(n^2)/SO(n) \times SO(n), \quad SO(4n^2)/Sp(n) \times Sp(n), \quad E_8/SO(5), \quad E_8/SU(5) \times SU(5),$$

  they all have two isotropy summands (see Table 1, items 3a and 3b and Table 3, items 8 and 11, respectively). For these spaces, it is well known that there exists a global maximum (see [30]) and at most two other $G$-invariant Einstein metrics (see [7]). We obtained that $g_B$ is indeed the global maximum and it is the only $G$-invariant Einstein metric for the first and fourth spaces and that $g_B$ is a local maximum on $E_8/SO(5)$. However, we were not able to figure out whether $g_B$ is a local maximum or not on $SO(4n^2)/Sp(n) \times Sp(n)$ due to the huge length of computations needed. What we know in this case is that either $g_B$ is a global maximum and the unique $G$-invariant Einstein metric or $g_B$ is a local minimum and there exist other two $G$-invariant Einstein metrics.

Concerning the rest of the proof of Theorem 1.3, the nine spaces with two isotropy summands which are not covered by any criterion are case by case worked out in Sect. 6, by combining the three formulas for $\rho$ in terms of the structural constants, the Killing form constants and the Casimir constant, respectively.

The most difficult case is by far the conceptual construction $SO(m)/K$ listed in Table 1, items 4 and 5, where the inclusion of $K$ is defined by the isotropy representation of a certain symmetric space (see Sect. 7). We compute the structural constants and detect eigenvectors.
of $L_p$ with eigenvalue $\frac{m}{2(m-\frac{1}{2})}$, which is less than $2\rho$ and so $g_B$ is always $G$-unstable (see Theorem 7.8 for more results on this class of spaces, including the computation of Spec($L_p$) in several cases).

The proof of Theorem 1.3 concludes in Sect. 8 with the space $E_7/SU(2)^7$, for which the uniform behavior of their structural constants allows to obtain the matrix of $L_p$ and consequently $\lambda_G^{L_p}$ explicitly.

Finally, for completeness, we compute in Sect. 9 the Einstein constants and the spectrum of $L_p$ for 10 spaces which were solved without needing these data by the criteria (except Table 2, 4). This information is provided in Table 9 for the spaces in Table 1 and in the same tables for the exceptional cases.

## 2 Preliminaries

We consider a compact connected differentiable manifold $M$ of dimension $d$ and assume that $M$ is homogeneous. We also fix an almost-effective transitive action of a compact Lie group $G$ on $M$. The $G$-action determines a presentation $M = G/K$ of $M$ as a homogeneous space, where $K \subset G$ is the isotropy subgroup at some point $o \in M$. Let $\mathcal{M}_G^G$ denote the space of all $G$-invariant Riemannian metrics on $M$, it is an open cone in $S^2(M)^G$, the finite-dimensional vector space of all $G$-invariant symmetric 2-tensors. Thus $\mathcal{M}_G^G$ is a differentiable manifold of dimension between 1 and $\frac{d(d+1)}{2}$.

As well known, $g \in \mathcal{M}_G^G$ is Einstein if and only if $g$ is a critical point of the scalar curvature functional $Sc : \mathcal{M}_G^G \rightarrow \mathbb{R}$, where $\mathcal{M}_1^G \subset \mathcal{M}_G^G$ is the codimension one submanifold of all unit volume metrics. The stability type of a $G$-invariant Einstein metric on $M$ as a critical point of $Sc |_{\mathcal{M}_1^G}$ is therefore encoded in the signature of the second derivative or Hessian $Sc''$. We refer to [15, Section 3] and [17, Section 2] for more detailed treatments.

According to [15, Section 3.4], the tangent space decomposes as

$$T_g \mathcal{M}_1^G = T_g N_G(K)^* g \oplus T T_g^G,$$
where $N_G(K) \subset \text{Diff}(M)$ is the normalizer of $K$ and $TT^G_g$ is the space of $G$-invariant TT-tensors. Note that $T_g N_G(K) \cdot g$ is the space of trivial variations, that is, the tangent space of the $G$-equivariant isometry class of $g$. This gives rise to the different notions of $G$-stability given in Definition 1.2. We note that $G$-semistability must hold for any local maximum of $\text{Sc}|_{\mathcal{M}^G_1}$ and that a $G$-neutrally stable Einstein metric may or may not be a local maximum. On the other hand, any $G$-unstable metric is a saddle point unless $\text{Sc}''_g | TT^G_g > 0$, in which case $g$ is a local minimum of $\text{Sc}|_{\mathcal{M}^G_1}$. The coindex is the dimension of the maximal subspace of $TT^G_g$ on which $\text{Sc}''_g$ is positive definite and the nullity is $\dim \ker \text{Sc}''_g | TT^G_g$.

Let $g = \mathfrak{t} \oplus \mathfrak{p}$ be any reductive decomposition for $M = G/K$, where $\mathfrak{g}$ and $\mathfrak{t}$ are the Lie algebras of $G$ and $K$, respectively. This provides the usual identifications $T_0 M \equiv \mathfrak{p}$ and

$$\text{sym}(\mathfrak{p})^K := \{ A \in \text{sym}(\mathfrak{p}) : [\text{Ad}(K), A] = 0 \} \equiv S^2(M)^G,$$

where $\text{sym}(\mathfrak{p}) := \{ A \in gl(\mathfrak{p}) : A^t = A \}$. Furthermore, any $g \in \mathcal{M}^G$ is determined by the $\text{Ad}(K)$-invariant inner product $\langle \cdot, \cdot \rangle := g_o$ on $\mathfrak{p}$.

The second variation of the total scalar curvature at any Einstein metric $g$ on $M$ (not necessarily homogeneous), say with $\text{Rc}(g) = \rho g$, is given on $TT^G_g$ by

$$\text{Sc}''_g = \frac{1}{2} (2 \rho \text{id} - \Delta_L),$$

(2)

where $\Delta_L$ is the Lichnerowicz Laplacian of $g$ (see [2, 4.64]). For each $g \in \mathcal{M}^G$, we consider the self-adjoint operator

$$L_p = L_p(g) : \text{sym}(\mathfrak{p})^K \rightarrow \text{sym}(\mathfrak{p})^K,$$

such that

$$\Delta_L T = \langle L_p A, \cdot \rangle, \quad \forall T \in TT^G_g, \quad T = \langle A, \cdot \rangle \in S^2(M)^G, \quad A \in \text{sym}(\mathfrak{p})^K,$$

where $\langle A, B \rangle := \text{tr} AB$.

According to (2), the $G$-stability type of $g$ is therefore determined by how is the constant $2\rho$ suited relative to the spectrum of $L_p$. Let $\lambda_p = \lambda_p(g)$ and $\lambda^\max_p = \lambda^\max_p(g)$ denote, respectively, the minimum and maximum eigenvalues of $L_p$ restricted to the subspace $TT^G_g$. Recall that $\lambda_p$ was denoted by $\lambda^G_p$ in the introduction. The characterizations of the $G$-stability types in terms of $\lambda_p$ and the Einstein constant $\rho$ were given in Definition 1.2. We note that an Einstein metric $g \in \mathcal{M}^G_1$ is a local minimum of $\text{Sc}|_{\mathcal{M}^G_1}$ if $\lambda^\max_p < 2\rho$ and that $L_p | T_g N_G(K)^* g = 2 \rho \text{id}$, since $\text{Sc}''_g$ vanishes on trivial variations (cf. [15, (30)]).

Assume from now on that the metric $g \in \mathcal{M}^G$ is naturally reductive with respect to $G$ and $\mathfrak{p}$, i.e., the map $\text{ad}_\mathfrak{p} X : \mathfrak{p} \rightarrow \mathfrak{p}, \ Y \mapsto [X, Y]_\mathfrak{p}$ is skew-symmetric for any $X \in \mathfrak{p}$, where $[X, Y]_\mathfrak{p}$ denotes the projection onto $\mathfrak{p}$ of the Lie bracket $[X, Y]$ relative to $g = \mathfrak{t} \oplus \mathfrak{p}$. In that case,

$$T_g \mathcal{M}^G_1 = TT^G_g = \text{sym}_\mathfrak{p}(\mathfrak{p})^K := \{ A \in \text{sym}(\mathfrak{p})^K : \text{tr} A = 0 \},$$

and the Lichnerowicz Laplacian $L_p$ is given by

$$L_p A := -\frac{1}{2} \sum [\text{ad}_\mathfrak{p} X_i, [\text{ad}_\mathfrak{p} X_i, A]], \quad \forall A \in \text{sym}(\mathfrak{p})^K,$$

(3)

where $\{ X_i \}$ is any $g$-orthonormal basis of $\mathfrak{p}$ (see [15, Section 5]).

Example 2.1 If $g_\mathfrak{B}$ is the Killing left-invariant metric on any compact simple Lie group $G$, which always satisfies $\text{Rc}(g_\mathfrak{B}) = \frac{1}{4} g_\mathfrak{B}$, then $L_p(g_\mathfrak{B}) = \frac{1}{2} C_\tau$, where $C_\tau$ is the Casimir acting
on the representation \( \text{sym}(\mathfrak{g}) \) of \( \mathfrak{g} \) relative to the Killing form. Thus the \( G \)-stability type of \( g_B \) follows from the spectrum of \( C_\tau \) (see [15, Table 1]), which is well known and can be computed using representation theory (see Table 5). Thus \( g_B \) is \( G \)-stable, except for \( SU(n), \) \( n \geq 3 \) and \( \text{Sp}(n), n \geq 2, \) where it is \( G \)-neutrally stable and \( G \)-unstable, respectively. It is proved in [18] that on \( SU(n), n \geq 3 \) the metric \( g_B \) is not a local maximum of \( \text{Sc} |_{\mathcal{M}_1^G} \) (see also [19, Theorem 3, 4]).

Any orthogonal decomposition \( p = p_1 \oplus \cdots \oplus p_r \) in \( \text{Ad}(K) \)-invariant subspaces \( p_1, \ldots, p_r \) \((d_i := \dim p_i)\) determines structural constants given by,

\[
[ijk] := \sum_{\alpha, \beta, \gamma} \langle [X^i_\alpha, X^j_\beta], X^k_\gamma \rangle^2,
\]

where \( \{X^i_\alpha\} \) is an orthonormal basis of \( p_i \). Note that the number \( [ijk] \) is invariant under any permutation of \( ijk \) by natural reductivity. Consider the orthonormal subset \( \{I_1, \ldots, I_r\} \) of \( \text{sym}(p)^K \) given by \( I_k|_{p_i} := \delta_{ki} \frac{1}{\sqrt{d_k}} I_k \), where \( I_k \) will always denote the identity map on the vector space \( v \).

According to [15, Theorem 5.3] (see also [17, Theorem 3.1]),

\[
\langle L_p I_k, I_k \rangle = \frac{1}{d_k} \sum_{j \neq k; i} [ijk], \quad \forall k,
\]

\[
\langle L_p I_k, I_m \rangle = -\frac{1}{\sqrt{d_kd_m}} \sum_i [ikm], \quad \forall k \neq m.
\]

In the case when \( G/K \) is multiplicity-free, i.e., the subspaces \( p_k \)'s are \( \text{Ad}(K) \)-irreducible and pairwise inequivalent, \( \{I_1, \ldots, I_r\} \) is an orthonormal basis of \( \text{sym}(p)^K \) and so the above numbers are precisely the entries of the matrix of \( L_p \). This provides a useful tool to compute the whole spectrum of \( L_p \), in order to establish the \( G \)-stability type of the Einstein metric \( g \). Note that the vector \( (\sqrt{d_1}, \ldots, \sqrt{d_r}) \) is always in the kernel of \( L_p \) since these are precisely the coordinates of the identity map \( I_p \in \text{sym}(p)^K \) (see (3)).

In the general case, i.e., the subspaces \( p_k \)'s are only assumed to be \( \text{Ad}(K) \)-invariant, the spectrum of the symmetric \( r \times r \) matrix defined as in (5) (restricted to the hyperplane \( \sum d_i a_i = 0 \) and intersected with \( \text{TT}^G_g \) if necessary) is still contained in \( [\lambda_p, \lambda_p^{\max}] \). In particular, the \( G \)-instability of \( g \) (say with \( \text{Rc}(g) = \rho g \)) follows as soon as some eigenvalue of such matrix is less than \( 2\rho \) (see [17, Remark 3.3]).

### 3 Standard Einstein metrics

Any homogeneous space \( M^d = G/K \) with \( G \) compact semisimple admits a canonical \( G \)-invariant metric \( g_B \), called the standard metric and given by \( \langle \cdot, \cdot \rangle = -B_g|_{\mathfrak{k} \times \mathfrak{p}} \), where \( B_g \) is the Killing form of \( \mathfrak{g} \) and \( g = \mathfrak{k} \oplus \mathfrak{p} \) is the \( B_g \)-orthogonal decomposition. Note that \( g_B \) is clearly naturally reductive with respect to \( G \) and \( \mathfrak{p} \). We consider any orthogonal decomposition \( p = p_1 \oplus \cdots \oplus p_r \) in \( \text{Ad}(K) \)-irreducible subspaces and denote by \( \chi \) the isotropy representation of \( K \) and \( \mathfrak{t} \) on \( \mathfrak{p} \). We refer to [29, Chapter 1] and [2, Chapter 7, Section G] for more detailed treatments on standard metrics.
### 3.1 Einstein equation

The Ricci operator of the standard metric is given by

\[ \text{Ric}(g_B) = M_{\mathfrak{g}} + \frac{1}{2} I_p = \frac{1}{2} I_p + \frac{1}{2} C_\chi, \]  

(6)

where \( M_{\mathfrak{g}} = \frac{1}{2} \sum (\text{ad}_p X_i)^2 \) and \( C_\chi = -\sum (\text{ad} Z_j)^2 \) is the Casimir acting on the isotropy representation \( \chi \) with respect to \( -B_\mathfrak{g} |_\mathfrak{k} \) (see [15, (16), (31)] and [29, Corollary 1.7], respectively). Here \( \{Z_j\} \) and \( \{X_i\} \) are respectively \( -B_\mathfrak{g} \)-orthonormal basis of \( \mathfrak{k} \) and \( \mathfrak{p} \). Note that \( M_{\mathfrak{g}} \leq 0 \) and \( C_\chi \geq 0 \). The following conditions are therefore equivalent:

- \( g_B \) is Einstein, say \( \text{Ric}(g_B) = \rho g_B \) (i.e., \( \text{Ric}(g_B) = \rho I_p \)).
- \( M_{\mathfrak{g}} = (\rho - \frac{1}{2}) I_p \) (since \( \text{tr} M_{\mathfrak{g}} = -\frac{1}{2} |\mathfrak{k}|^2 \), it follows that \( \rho \leq \frac{1}{2} \), where equality holds if and only if \( \{p, p\} \subset \mathfrak{k} \), i.e., \( (G/K, g_B) \) is a locally symmetric space).
- \( C_\chi = (2\rho - \frac{1}{2}) I_p \) (in particular, \( \frac{1}{2} \leq \rho \), where equality holds if and only if \( \mathfrak{k} = 0 \)).
- \( C_{\chi_k} = (2\rho - \frac{1}{2}) I_p \) for every \( k = 1, \ldots, r \), where \( C_{\chi_k} \) is the Casimir operator with respect to \( -B_\mathfrak{g} |_\mathfrak{k} \) acting on the irreducible representation \( \mathfrak{p}_k \) of \( \mathfrak{k} \) (note that \( C_{\chi_k} \) is always a multiple of the identity).

We write the adjoint map by

\[ \text{ad} X = \begin{bmatrix} 0 & a(X) \\ -a(X)^t & \text{ad}_p \end{bmatrix}, \quad \forall X \in \mathfrak{p}, \]  

(7)

where \( a(X)^t : \mathfrak{k} \to \mathfrak{p} \) is the adjoint operator of \( a(X) : \mathfrak{p} \to \mathfrak{k} \) with respect to \( -B_\mathfrak{g} \), i.e., \( \langle a(X)^t Z, Y \rangle = -B_\mathfrak{g}(Z, [X, Y]) \) for all \( Z \in \mathfrak{k}, X, Y \in \mathfrak{p} \).

Most of the following lemma is contained in [29] (see also [2, Chapter 7, Section G]); we include a proof for completeness.

**Lemma 3.1** The following formulas hold:

(i) \( B_\chi = \sum a(X_i) a(X_i)^t \), where \( B_\chi : \mathfrak{k} \to \mathfrak{k} \) is defined by \( -B_\mathfrak{g}(B_\chi Z, Z) = \text{tr} \chi(Z)^2 \) for all \( Z \in \mathfrak{k} \).

(ii) \( C_\chi = \sum a(X_i)^t a(X_i) \). In particular, \( \text{tr} C_\chi = -\text{tr} B_\chi \).

(iii) \( \text{C}_\text{ad} = \mathfrak{k}_\mathfrak{t} + B_\chi \), where \( \text{C}_\text{ad} \) is the Casimir acting on the adjoint representation of \( \mathfrak{k} \) with respect to \( -B_\mathfrak{g} |_\mathfrak{t} \).

(iv) \( \sum a(X_i) \text{ad}_p X_i = 0 \).

(v) \( 2C_\chi - 4 M_{\mathfrak{g}} = I_p \).

**Remark 3.2** Using part (ii), each of the formulas for the Ricci operator given in (6) follows from the other one. It can be easily shown that \( A = \langle C_\chi, \cdot, \cdot \rangle \) is precisely the symmetric bilinear form defined in [29, (1.1)].

**Proof** Part (i) follows from

\[ \text{tr} \chi(Z)^2 = \sum \langle [Z, [Z, X_i]], X_i \rangle = -\sum \langle [X_i, Z], [X_i, Z] \rangle = \sum -B_\mathfrak{g}((\text{ad} X_i)^2 Z, Z) = \sum -B_\mathfrak{g}(-a(X_i) a(X_i)^t Z, Z), \]

and part (ii) from

\[ \langle C_\chi X, X \rangle = -\sum \langle (\text{ad} Z_j)^2 X, X \rangle = \sum \langle [Z_j, X], [X, X] \rangle^2 = \sum -B_\mathfrak{g}([X_i, X] \mathfrak{k}, [X_i, X] \mathfrak{k}) = \sum \langle a(X_i)^t a(X_i) X, X \rangle. \]
On the other hand, it is easy to see that the Casimir acting on the adjoint representation of \( g \) with respect to \( -B_g \), \( C_{\text{ad}} = -\sum (\text{ad} \ Z_j)^2 - \sum (\text{ad} \ X_i)^2 \), is given by

\[
C_{\text{ad}} = \left[ \sum \frac{C_{\text{ad}}}{2 \chi} - \sum a(X_i) \text{ad}_p X_i \right].
\]

The remaining parts therefore follow from the fact that \( C_{\text{ad}} = I_g \).

### 3.2 Einstein constant

We take a decomposition \( k = k_1 \oplus \cdots \oplus k_s \) in ideals of \( k \) and assume that \( B_{k_i} = c_i B_g \ | k_i \) for some \( c_i \in \mathbb{R} \), for each \( i = 1, \ldots, s \). Note that \( 0 \leq c_i \leq 1 \), \( c_i = 0 \) if and only if \( k_i \) is abelian and \( c_i = 1 \) if and only if \( k_i \) is an ideal of \( g \) (see [6, Theorem 11, pp.35] for more information on these constants \( c_i \)'s). This assumption in particular holds if each \( k_i \) is either simple or abelian. It is easy to see that \( C_{\text{ad}} k_i \) is the block map \([c_1 I_{k_1}, \ldots, c_s I_{k_s}]\) and so by Lemma 3.1, (iii),

\[
B_{\chi} = [(c_1 - 1) I_{k_1}, \ldots, (c_s - 1) I_{k_s}]. \tag{8}
\]

If \( g_B \) is Einstein, then

\[
\sum_{i=1}^{s} (1 - c_i) \dim k_i = - \text{tr} B_{\chi} = \text{tr} C_{\chi} = (2 \rho - \frac{1}{2})d, \tag{9}
\]

from which the following useful formula for the Einstein constant follows (cf. [6, pp.34], [29, pp.568] and [2, (7.94)], where the following typo appears: the sum should be from \( i = 0 \) to \( r \)):

\[
\rho = \frac{1}{4} + \frac{1}{2d} \sum_{i=1}^{s} (1 - c_i) \dim k_i. \tag{10}
\]

In particular, if \( \mathfrak{k} \) is abelian then \( \rho = \frac{1}{4} + \frac{1}{2d} \dim \mathfrak{k} \) and, if \( [p, p] \subset \mathfrak{k} \) (i.e. \( M = G/K \) is a symmetric space) then \( c_i = \frac{2 \dim \mathfrak{k} - d}{2 \dim \mathfrak{k}} \) by [6, Theorem 11, (i)] and so \( \rho = \frac{1}{2} \).

A well-known formula for the Ricci eigenvalues of \( g_B \) in terms of the structural constants defined in (4) (see e.g. [15, (34)] or [17, (18)]) is given by

\[
\rho_k = \frac{1}{2} - \frac{1}{4d_k} \sum_{i,j} [ijk], \quad \forall k = 1, \ldots, r, \tag{11}
\]

and it follows from [30, Lemma (1.5)] that

\[
\sum_{i,j} [ijk] = d_k (1 - 2a_k), \quad \forall k = 1, \ldots, r, \tag{12}
\]

where \( C_{\chi k} = a_k I_{p_k} \). Recall that \( g_B \) is Einstein if and only if \( a_1 = \cdots = a_r \) (in that case, \( \rho = \frac{1}{4} + \frac{1}{2}a_1 \)), and that \( d = d_1 + \cdots + d_r = \dim p = \dim g - \dim \mathfrak{k} \).

### 3.3 \( G \)-stability

A complete classification of standard metrics which are Einstein was obtained by Wang and Ziller in [29] in the case when \( G \) is simple. Besides isotropy irreducible spaces and assuming that \( G \) and \( K \) are both connected, the list consists of 12 infinite families (two of them are
The following is the main result of this paper, providing a complete picture of the $G$-stability of these standard Einstein metrics.

**Theorem 3.3** The $G$-stability types of all standard Einstein metrics on non-isotropy irreducible homogeneous spaces $M = G/K$ with $G$ simple are given as in Tables 1, 2, 3 and 4, except for the space $\text{SO}(4n^2)/\text{Sp}(n) \times \text{Sp}(n)$.

The proof of the theorem is worked out in Sects. 4 through 8.

**Remark 3.4** The following remarks are in order:

1. The only cases for which the $G$-stability type had been known before are the flag manifolds listed in Table 1, items 1a.1, 1a.2, 1a.3, 2a, 2b, 2c (see [15, Table 2]) and the generalized Wallach spaces given in Table 2, items 1, 3, 5, 6 and Table 3, item 14 (see [17, Table 1]).

2. The number $r$ of irreducible isotropy summands is provided in the tables. Recall that $\dim \mathcal{M}^G_1 \geq r - 1$, where equality holds if and only if the space is multiplicity-free.
Table 2 [29, Table IB, pp.578]. Excepting the item 2, in all cases $\lambda_p$ is the only nonzero eigenvalue of $L_p$ and has multiplicity $r - 1$.

| No. | $g/k$ | $r$ | $\rho$ | $\lambda_p$ | $G$-stab. type | C1 | C2 | References |
|-----|-------|-----|--------|-------------|----------------|----|----|------------|
| 1   | $f_4$ | 3   | $\frac{4}{7}$ | $\frac{1}{2}$ | G-unst., loc.min. | ✓ | ✓ | [17], Cor.4.5, [19] |
| 2   | $so(3) \oplus so(3) \oplus so(3)$ | 5 | $\frac{5}{16}$ | G-semistab. | ✓ | ✓ | Cor. 4.5 |
| 3   | $spin(8) \oplus \mathbb{R}^2$ | 3 | $\frac{5}{17}$ | $\frac{1}{2}$ | G-unst., loc.min. | ✓ | No | [17], Cor. 4.6, [19] |
| 4   | $su(3) \oplus so(6)$ | 2 | $\frac{3}{5}$ | $\frac{1}{4}$ | G-neut.stab., saddle | No | No | Section 6.6 |
| 5   | $so(8)$ | 3 | $\frac{13}{30}$ | $\frac{5}{6}$ | G-stab., loc.max. | No | No | [17] |
| 6   | $spin(8) \oplus 5 \cdot su(2)$ | 3 | $\frac{7}{18}$ | $\frac{2}{3}$ | G-unst., loc.min. | No | No | [17] |

Table 3 [29, Table IB, pp.579]. In all the cases where it is provided, $\lambda_p$ is the only nonzero eigenvalue of $L_p$ and has multiplicity $r - 1$. ✓* means that the criteria only imply that $2\rho \leq \lambda_p$.

| No. | $g/k$ | $r$ | $\rho$ | $\lambda_p$ | $\lambda_{\text{max}}$ | $G$-stab. type | C1 | C2 | References |
|-----|-------|-----|--------|-------------|----------------|----------------|----|----|------------|
| 7   | $\eta_7$ | 7 | $\frac{1}{3}$ | $\frac{7}{9}$ | mult=7 | G-stab., loc.max. | No | No | Section 8 |
| 8   | $\epsilon_8$ | 2 | | | | G-stab., loc.max. | ✓ | ✓ | Cor.4.5 |
| 9   | $\epsilon_8$ | 3 (nmf) | $\frac{13}{36}$ | $\frac{5}{6}$ | mult=6 | G-semistab. | ✓ | ✓ | Cor.4.5 |
| 10  | $\epsilon_8$ | 2 | $\frac{7}{20}$ | $\frac{4}{5}$ | G-stab., glob.max. | No | No | Section 6.8 |
| 11  | $\epsilon_8$ | 4 | $\frac{19}{60}$ | $\frac{4}{5}$ | G-stab., loc.max. | ✓ | ✓ | Cor. 4.5 |
| 12  | $\epsilon_8$ | 9 (nmf) | $\frac{11}{40}$ | mult=7 | G-stab., loc.max. | ✓ | ✓ | Cor. 4.5 |
| 13  | $\epsilon_8$ | 3 | $\frac{11}{30}$ | $\frac{4}{5}$ | G-stab., loc.max. | No | No | [17] |

Table 4 [29, Table IB, pp.580].

| No. | $g/k$ | $r$ | $\rho$ | $\lambda_p$ | $\lambda_{\text{max}}$ | $G$-stab. type | C1 | C2 | References |
|-----|-------|-----|--------|-------------|----------------|----------------|----|----|------------|
| 15  | $\epsilon_8$ | 14 | $\frac{3}{10}$ | $\frac{4}{5}$ | mult=7 | G-stab., loc.max. | ✓ | ✓ | Cor.4.5 |
| 16  | $\epsilon_8$ | 6 (nmf) | $\frac{7}{24}$ | mult=20 | G-stab., loc.max. | ✓ | ✓ | Cor.4.5 |
| 17  | $\epsilon_8$ | 5 (nmf) | $\frac{17}{60}$ | mult=27 | G-stab., loc.max. | ✓ | ✓ | Cor.4.5 |
| 18a | $\epsilon_6$ | 36 | $\frac{7}{24}$ | $\frac{3}{1}$ | mult=15 | G-stab., loc.max. | ✓ | ✓ | Cor.4.5 |
| 18b | $\epsilon_7$ | 63 | $\frac{5}{18}$ | $\frac{7}{9}$ | mult=35 | G-stab., loc.max. | ✓ | ✓ | Cor.4.5 |
| 18c | $\epsilon_8$ | 120 | $\frac{4}{15}$ | $\frac{4}{3}$ | mult=35 | G-stab., loc.max. | ✓ | ✓ | Cor.4.5 |
There are only six cases which are not multiplicity-free; they have been marked with ‘(nmf)’ on the right of the number $r$.

(3) Additional information on what kind of critical point of $\mathcal{S}_c |_{\mathcal{M}_i^G}$ is the standard metric $g_B$ is also given, including the coindex (if positive and $< r - 1$) and the nullity (if positive).

(4) The last column on the right of the tables provides a reference to the place in the paper (or to another paper) where the $G$-stability type was proved.

(5) Einstein constants (see Table 9 for the spaces in Table 1) appearing with an asterisk were computed in this paper; the authors were not able to find any of them in the literature.

(6) The eigenvalues $\lambda_p$ and $\lambda_p^{\max}$ of the Lichnerowicz Laplacian computed here were also marked with an asterisk and the corresponding multiplicity was added below the value in the cases when it is different from $r - 1$ (see Table 9 for the spaces in Table 1). Recall that $TT_g = \text{sym}_0(p)^K$ for any standard metric $g$.

(7) All the eigenvalues of $L_p |_{\text{sym}_0(p)^K}$ we found are nonzero, which implies that the standard metric is Ricci locally invertible (see [16]) in all the cases where the whole spectrum of $L_p$ was computed.

(8) The meaning of the columns C1 and C2 is explained at the end of Sect. 4.

(9) We do not know whether each of the two $G$-semistable cases $E_6/\text{SO}(3)^3$ (see Table 2, 2) and $E_8/\text{SO}(9)$ (see Table 3, 9) is either $G$-stable or $G$-neutrally stable.

(10) In the case $E_8/\text{SO}(5)$ (see Table 3, 8), $g_B$ is a local maximum as it is $G$-stable. On the other hand, since $K$ is a maximal subgroup of $G$, we know that there exists a global maximum (see [30, Theorem (2.2)]), but we were not able to figure out whether $g_B$ is the global maximum or not without more information on the structural constants.

### 4 Stability criteria for standard Einstein metrics

In this section, we give sufficient conditions for the $G$-stability types of $g_B$ on $G/K$ to hold involving only $\dim G$, $\dim K$ and the Casimir eigenvalues $\lambda_\tau$ and $\lambda_\tau^{\max}$ attached to $\mathfrak{g}$ (see Table 5). Several cases, specially when $\mathfrak{g}$ is exceptional, follow from these purely Lie theoretical criteria.

Let $C_\tau$ denote the Casimir operator acting on the representation $\text{sym}(\mathfrak{g})$ of $\mathfrak{g}$ given by $\tau(X)A := [\text{ad} X, A]$ with respect to $- B_\mathfrak{g}$, i.e.,

$$C_\tau = - \sum \tau(Z_j)^2 - \sum \tau(X_i)^2.$$

The eigenvalues $\lambda_\tau < \lambda_\tau^{\mid \text{mid}} < \lambda_\tau^{\max}$ of $C_\tau$ restricted to $\text{sym}_0(\mathfrak{g})$ are given in Table 5 for each compact simple Lie algebra $\mathfrak{g}$ (it is only one eigenvalue for $\mathfrak{su}(2)$ and only two, say $\lambda_\tau < \lambda_\tau^{\max}$, for any exceptional $\mathfrak{g}$). The following result paves the way to estimate the eigenvalues $\lambda_p$ and $\lambda_p^{\max}$ of the Lichnerowicz Laplacian $L_p = L_p(g_B)$ in terms of $\lambda_\tau$ and $\lambda_\tau^{\max}$.

**Lemma 4.1** $C_\tau$ leaves the subspace $\text{sym}(\mathfrak{g})^K$ invariant and

\[
C_\tau \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 B X & B \\ 0 & -2 \sum a(X_i)^t B a(X_i) \end{bmatrix}, \quad \forall B \in \text{sym}(\mathfrak{t})^K,
\]

\[
C_\tau \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} = \begin{bmatrix} -2 \sum a(X_i) A a(X_i)^t & 2 \sum a(X_i) A \text{ad}_p(X_i) \\ -2 \sum \text{ad}_p(X_i) A a(X_i)^t & C_\chi A + A C_\chi + 2 L_p A \end{bmatrix}, \quad \forall A \in \text{sym}(\mathfrak{p})^K,
\]

where $L_p$ is the Lichnerowicz Laplacian of $g_B$. 

[ Springer ]
Finally, using Lemma 3.1, (i), (ii) and (iv), the formulas stated in the lemma follow. \(\square\)

The following are stability criteria in which the Einstein constant is not involved.

### Remark 4.2

Most of the times, the \(K\)-irreducible subspaces of \(\mathfrak{t}\) are pairwise non-equivalent as \(K\)-representations to those of \(\mathfrak{p}\), which implies that

\[
\text{sym}(\mathfrak{g})^K = \left\{ \begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix} : B \in \text{sym}(\mathfrak{k})^K, A \in \text{sym}(\mathfrak{p})^K \right\}.
\]

A simple counterexample of a standard Einstein \(G/K\) with \(G\) simple to the above property is \(\text{SO}(2kl)/K^l\), where \(K\) is any simple Lie group and the inclusion is defined as \(l\) copies of \(\Delta(K) \subset K \times K\) (see Table 1, 5).

### Proof

If \(\tau\) also denotes the corresponding \(G\)-representation, i.e., \(\tau(x)A = \text{Ad}(x)A \text{Ad}(x)^{-1}\) for \(x \in G\), then every \(\tau(x)\) commutes with \(C_{\mathfrak{z}}\). Indeed, \(C_{\mathfrak{z}}\) is induced by the Casimir element, which belongs to the center of the enveloping algebra. Since \(\text{sym}(\mathfrak{g})^K\) is precisely the intersection of all the kernels of \(\tau(z), z \in K\), one obtains that \(C_{\mathfrak{z}} \text{sym}(\mathfrak{g})^K \subset \text{sym}(\mathfrak{g})^K\).

Concerning the formulas, we observe that \(\tau(Z_j) \begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix} = 0\) for all \(j\) by the \(K\)-invariance of \(B\) and \(A\), and from (7) one easily obtains that

\[
\tau(X_i) \begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix} = \begin{bmatrix} B(a(X_i) + a(X_i)A) \\ -Ba(X_i) + a(X_i)A \end{bmatrix}.
\]

This implies that

\[
\tau(X_i)^2 \begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix} = \begin{bmatrix} -a(X_i)a(X_i)^tB + Ba(X_i)a(X_i)^t & Ba(X_i)\text{ ad}_p X_i \\ B(a(X_i) + a(X_i)A) & -2a(X_i)^tBa(X_i) \end{bmatrix}.
\]

and \(\tau(X_i)^2\) equals

\[
\begin{bmatrix} 2a(X_i)a(X_i)^t & a(X_i)[\text{ ad}_p (X_i), A] - a(X_i)A \text{ ad}_p (X_i) \\ -a(X_i)^t[a(X_i), A] - Aa(X_i)^t[a(X_i) + [\text{ ad}_p (X_i), [\text{ ad}_p (X_i), A]] \end{bmatrix}.
\]

Finally, using Lemma 3.1, (i), (ii) and (iv), the formulas stated in the lemma follow. \(\square\)

### Table 5

| \(\mathfrak{g}\) | \(\dim \mathfrak{g}\) | \(\lambda_{\tau}\) | \(\lambda_{\text{mid}}^{\mathfrak{g}}\) | \(\lambda_{\text{max}}^{\mathfrak{g}}\) |
|-----------------|----------------|----------------|----------------|----------------|
| \(\mathfrak{su}(2)\) | 3 | 3 | – | – |
| \(\mathfrak{su}(n), n \geq 3\) | \(n^2 - 1\) | 1 | \(\frac{2(n-1)}{n}\) | \(\frac{2(n+1)}{n}\) |
| \(\mathfrak{so}(7)\) | 21 | \(\frac{6}{5}\) | \(\frac{7}{5}\) | \(\frac{13}{5}\) |
| \(\mathfrak{so}(n), n \geq 8\) | \(\frac{n(n-1)}{2}\) | \(\frac{n}{n-2}\) | \(\frac{2(n-4)}{n-2}\) | \(\frac{2(n-1)}{n-2}\) |
| \(\mathfrak{sp}(n), n \geq 2\) | \(n(2n+1)\) | \(\frac{n}{n+1}\) | \(\frac{2n+1}{n+1}\) | \(\frac{2n+4}{n+1}\) |
| \(\mathfrak{e}_6\) | 78 | \(\frac{3}{2}\) | – | \(\frac{13}{6}\) |
| \(\mathfrak{e}_7\) | 133 | \(\frac{14}{9}\) | – | \(\frac{19}{9}\) |
| \(\mathfrak{e}_8\) | 248 | \(\frac{8}{3}\) | – | \(\frac{31}{12}\) |
| \(\mathfrak{f}_4\) | 52 | \(\frac{13}{9}\) | – | \(\frac{20}{9}\) |
| \(\mathfrak{g}_2\) | 14 | \(\frac{7}{6}\) | – | \(\frac{5}{2}\) |
Theorem 4.3  Suppose that \( g_B \) is Einstein.

(i) If \( \dim \mathfrak{t} < \frac{\dim g(\lambda_{\max} - 1)}{2\lambda_{\max}} \), then \( 2\rho < \lambda_p \) and \( g_B \) is \( G \)-stable.

(ii) If \( \dim \mathfrak{t} = \frac{\dim g(\lambda_{\max} - 1)}{2\lambda_{\max}} \), then \( 2\rho \leq \lambda_p \) and \( g_B \) is \( G \)-semistable.

(iii) If \( \dim \mathfrak{t} > \frac{\dim g(\lambda_{\max} - 1)}{2\lambda_{\max}} \), then \( \lambda_p^{\max} < 2\rho \) and \( g_B \) is \( G \)-unstable and a local minimum of \( \text{Sc} \mid_{\mathcal{M}} \).

(iv) If \( \dim \mathfrak{t} = \frac{\dim g(\lambda_{\max} - 1)}{2\lambda_{\max}} \), then \( \lambda_p^{\max} \leq 2\rho \). In particular, \( g_B \) is \( G \)-unstable if \( \lambda_p < \lambda_p^{\max} \), and it is either \( G \)-unstable and a local minimum or \( G \)- neutrally stable if \( \lambda_p = \lambda_p^{\max} \).

Proof  Recall that \( \langle A, B \rangle = \text{tr} AB \) for all \( A, B \in \text{sym}(\mathfrak{p})^{K} \). If \( \text{Rc}(g_B) = \rho g_B \), or equivalently, \( C_{\chi} = (2\rho - \frac{1}{2}) I_p \), then it follows from Lemma 4.1 that

\[
\langle I_p A, A \rangle = \frac{1}{2} (C_{\chi} A, A) - (2\rho - \frac{1}{2}) |A|^2, \quad \forall A \in \text{sym}(\mathfrak{p})^{K}, \quad \overline{A} := \left[ \begin{array}{cc} 0 & 0 \\ 0 & A \end{array} \right].
\]

Since \( \lambda_\tau \leq \langle C_{\chi} A, A \rangle \leq \lambda_\tau^{\max} \), \( \forall A \in \text{sym}_0(\mathfrak{p})^{K}, \quad |A| = 1 \), we obtain the following estimates:

\[
\frac{1}{2}(\lambda_\tau - (8\rho - 1)) + 2\rho \leq \lambda_p \leq \lambda_p^{\max} \leq \frac{1}{2}(\lambda_\tau^{\max} - (8\rho - 1)) + 2\rho.
\]  

(13)

This already proves the criteria given in Corollary 4.6 below. To conclude the proof, we need to estimate \( \lambda_p \) and \( \lambda_p^{\max} \) in terms of \( \rho \).

We consider the orthogonal decomposition

\[
\mathbb{R} I_0 \oplus \text{sym}_0(\mathfrak{t})^{K} \oplus \text{sym}_0(\mathfrak{p})^{K} \subset \text{sym}_0(\mathfrak{g})^{K},
\]

(14)

where

\[
I_0 := \left[ \begin{array}{cc} I_p & 0 \\ 0 & -\frac{\dim \mathfrak{t}}{d} I_p \end{array} \right], \quad \text{sym}_0(\mathfrak{t})^{K} := \left[ \begin{array}{ccc} \text{sym}_0(\mathfrak{t})^{K} & 0 \\ 0 & 0 \end{array} \right], \quad \text{sym}_0(\mathfrak{p})^{K} := \left[ \begin{array}{cc} 0 & 0 \\ 0 & \text{sym}_0(\mathfrak{p})^{K} \end{array} \right].
\]

Again by Lemma 4.1,

\[
C_{\chi} I_0 = \left[ \begin{array}{cc} -\frac{\dim g}{d} B_{\chi} & 0 \\ 0 & -\frac{\dim g}{d} (2\rho - \frac{1}{2}) I_p \end{array} \right],
\]

(15)

and so by (9),

\[
\langle C_{\chi} I_0, I_0 \rangle = -\frac{\dim g}{d} \text{tr} B_{\chi} + \frac{\dim g}{d} \frac{\dim \mathfrak{t}}{d} (2\rho - \frac{1}{2}) = 2 \dim g (2\rho - \frac{1}{2}) + \frac{\dim g}{d} \frac{\dim \mathfrak{t}}{d} (2\rho - \frac{1}{2})
\]

\[
= 2 \dim g (1 + \frac{\dim \mathfrak{t}}{d}) (2\rho - \frac{1}{2}) = \frac{2(\dim g)^2}{d} (2\rho - \frac{1}{2}).
\]

Since

\[
\langle I_0 \rangle^2 = \dim \mathfrak{t} + \left(\frac{\dim \mathfrak{t}}{d}\right)^2 = \frac{\dim \mathfrak{t} \dim g}{d},
\]

we obtain that

\[
\frac{\langle C_{\chi} I_0, I_0 \rangle}{\langle I_0 \rangle^2} = \frac{\dim g}{\dim \mathfrak{t}} (4\rho - 1).
\]

(16)

Thus

\[
\lambda_\tau \leq \frac{\dim g}{\dim \mathfrak{t}} (4\rho - 1) \leq \lambda_\tau^{\max},
\]
Table 6 Relevant quantities to apply Theorem 4.3

| $\dim g(\lambda_\tau - 1)$ | $s(8)$ | $f_4$ | $e_6$ | $e_7$ | $e_8$ |
|-----------------------------|--------|-------|-------|-------|-------|
| $\frac{2\dim g(\lambda_\tau - 1)}{2\lambda_\tau^\text{max}}$ | 26     | 5     | 9     | 35    | 2     | 36    |
| $\frac{\dim g(\lambda_\tau^\text{max} - 1)}{2\lambda_\tau}$ | 22     | 14    | 91    | 95    | 248   |

or equivalently,

\[ \frac{2 \dim \mathfrak{k}}{\dim \mathfrak{g}} \lambda_\tau + 1 \leq 8 \rho - 1 \leq \frac{2 \dim \mathfrak{k}}{\dim \mathfrak{g}} \lambda_\tau^\text{max} + 1. \]

The condition in part (i) therefore implies that $8 \rho - 1 < \lambda_\tau$, and so by (13) we conclude that $2 \rho < \lambda_\tau$, i.e., $g_B$ is $G$-stable. The other parts follow in much the same way.

Example 4.4 As a first example of the usefulness of the above criteria we consider the full flag manifold $E_8/T^8$. It follows from Table 5 that $\dim \mathfrak{k} = 8 < 36 = \frac{\dim g(\lambda_\tau - 1)}{2\lambda_\tau^\text{max}}$, and so according to Theorem 4.3, (i), the standard metric $g_B$ on $E_8/T^8$ is $G$-stable and in particular a local maximum of $\mathfrak{sc}|_{\mathcal{M}_1^G}$. This is remarkable, considering that $\dim \mathcal{M}_1^G = 119$.

On the other hand, consider the space $E_8/SU(2)^8$. The Einstein constant $\rho$ is not available in the literature and its computation is far from straightforward (see Sect. 9.7), but since $\dim \mathfrak{k} = 24 < 36$, we also obtain from Theorem 4.3, (i) that $g_B$ is $G$-stable and a local maximum. Note that $\dim \mathcal{M}_1^G = 13$ in this case.

The numbers on the right in each of the conditions in Theorem 4.3 only depend on $\mathfrak{g}$ and are given in Table 6 for some Lie algebras (see Table 5 and Figure 2). A simple comparison between these numbers and $\dim \mathfrak{k}$ gives the following corollary of Theorem 4.3, which contains the totality of the cases where the criteria can be successfully applied.

Corollary 4.5 The standard metric is

- $G$-stable for the spaces numbered as 8, 12, 13, 15, 16, 17, 18a, 18b and 18c in Tables 3 and 4,
- $G$-unstable and a local minimum for case 1 in Table 2,
- $G$-semistable for case 2 in Table 2 and case 10 in Table 3.

The following stability criteria involving the Einstein constant follows from (13), recall that $1 \leq 8 \rho - 1 \leq 3$ (see Fig. 2).

Corollary 4.6 Assume that $\text{Rc}(g_B) = \rho g_B$.

(i) If $8 \rho - 1 < \lambda_\tau$, then $2 \rho < \lambda_\tau$ and $g_B$ is $G$-stable.
(ii) If $8 \rho - 1 = \lambda_\tau$, then $2 \rho \leq \lambda_\tau$ and $g_B$ is $G$-semistable.
(iii) If $\lambda_\tau^\text{max} < 8 \rho - 1$, then $\lambda_\tau^\text{max} < 2 \rho$ and $g_B$ is $G$-unstable and a local minimum.
(iv) If $\lambda_\tau^\text{max} = 8 \rho - 1$, then $\lambda_\tau^\text{max} \leq 2 \rho$ (cf. Theorem 4.3, (iv)).

We note that each of the parts of the above corollary is a necessary condition for the corresponding part in Theorem 4.3 to hold. The following example shows that the estimates (13) are not in general sharp. This warns us that all the above criteria may fail to find the $G$-stability type in many cases.
The authors recently became aware of the following stability criterion given in Remark 4.8 which works if $C_2$ does it.

**Example 4.7** For the full flag manifold $M = SU(n)/T^{n-1}$, $n \geq 3$, one has that $\rho = \frac{n+2}{4n}$, $\lambda_p = \frac{1}{2}$ and $\lambda_p^{max} = \frac{n-1}{n}$ (see [15, Section 6]). From Table 5 we know that $\lambda_\tau = 1$ and $\lambda_\tau^{max} = \frac{2(n+1)}{n}$, which implies that

$$\frac{1}{2} \lambda_\tau - 2 \rho + \frac{1}{2} = \frac{n-2}{2n} < \frac{1}{2} = \lambda_p,$$

$$\lambda_p^{max} = \frac{n-1}{n} < 1 = \frac{1}{2} \lambda_\tau^{max} - 2 \rho + \frac{1}{2}.$$ 

It is also easy to see that $\lambda_p < \lambda_\tau^{mid}$ if and only if $n \geq 9$ and that $\lambda_p = \lambda_\tau^{mid}$ if and only if $n = 8$. Furthermore, since $\lambda_\tau^{max} > \frac{n+4}{n} = 8 \rho - 1$, the $G$-instability of $g_B$ (i.e., $\lambda_p < 2\rho$) does not follow from Corollary 4.6, (iii) or (iv).

The following are the only cases, beyond those covered by Corollary 4.5, where we can apply Corollary 4.6 after computing the Einstein constant $\rho$:

(i) $sp(3n-1)/sp(n) \oplus u(2n-1)$ (see Table 1, 7a): we have that $d_1 = 2n(2n-1)$, $d_2 = 4n(2n-1)$ (see [7, III.6]), and it follows from [6, Table 1, pp.37] that $B_{sp(n)} = \frac{n+1}{n} B_{sp(3n+2)} |_{sp(n)}$. On the other hand, since $B_{su(n)} = \frac{4n}{8(n+1)} B_{sp(n)}$, we obtain that

$$B_{su(2n-1)} = \frac{4(2n-1)}{16n} B_{sp(2n-1)} = \frac{2n-1}{4n} \frac{2}{3} B_{sp(3n-1)} = \frac{2n-1}{6n} B_{sp(3n-1)}.$$ 

Formula (10) therefore gives that $\rho = \frac{5}{12}$. This implies that $8 \rho - 1 = \frac{7}{3} \geq \frac{2(3n+1)}{3n} = \lambda_\tau^{max}$ if and only if $n \geq 2$, where equality holds if and only if $n = 2$. Thus $g_B$ is $G$-unstable and a local minimum for any $n \geq 3$ (see Sect. 6.4 for $n = 1, 2$).

(ii) $so(3n+2)/so(n) \oplus u(n+1)$ (see Table 1, 7b): since $d_1 = n(n+1)$ and $d_2 = 2n(n+1)$ (see [7, I.18]), one obtains from [6, Table 1, pp.37] that

$$B_{so(n)} = \frac{n-2}{3n} B_{so(3n+2)} |_{so(n)} , \quad B_{su(n+1)} = \frac{n+1}{3n} B_{so(3n+2)} |_{su(n+1)}.$$ 

Thus $\rho = \frac{5}{12}$ by formula (10) and so $8 \rho - 1 = \frac{7}{3} \geq \frac{2(3n+1)}{3n} = \lambda_\tau^{max}$ for any $n \geq 3$, which implies that $g_B$ is $G$-unstable and a local minimum.

(iii) $e_6/spin(8) \oplus \mathbb{R}^2$ (see Table 2, 3): it is known that $\rho = \frac{4}{17}$, so $8 \rho - 1 = \frac{7}{3} > \frac{13}{6} = \lambda_\tau^{max}$ and hence $g_B$ is $G$-unstable and a local minimum (cf. [17, Table 2, W7]).

The information on the applicability of the criteria given in Corollaries 4.6 and 4.5 was respectively added on the columns C1 and C2 in Tables 2-4 and 9. Note that C1 necessarily works if C2 does it.

**Remark 4.8** The authors recently became aware of the following stability criterion given in [19, Theorem 2]: if $\rho > \frac{5}{12}$ then $g_B$ is a local minimum of $\mathcal{M}_{4}^{1}$. It is easy to check that this works for 6 of the 9 local minima exhibited in Tables 1-4, the three exceptions are the cases Table 1, 6,8 and Table 2, 6. We note that $\rho > \frac{5}{12}$ implies that $\frac{11}{5} < 8 \rho - 1$, which in turn gives that $\lambda_\tau^{max} < 8 \rho - 1$ for $e_6$, $e_7$, $e_8$ and some low dimensional classical simple Lie algebras, establishing the local minimality of the standard metric by Corollary 4.6, (iii) in the case when $\mathfrak{g}$ is one of these Lie algebras.
5 Full flag manifolds $SO(2n)/T^n$

We study in this section the homogeneous spaces $SO(2n)/T^n$, $n \geq 3$ (see Table 1, 1b.1 and 1b.2), where $T^n$ is the usual maximal torus of $SO(2n)$, by following the lines of [15, Section 6]. The standard block matrix reductive decomposition is given by

$$g = \mathfrak{t} \oplus p, \quad p = p_{12} \oplus p_{13} \oplus \cdots \oplus p_{(n-1)n},$$

where every block $p_{ij} = p_{ji}$ (note that always $i \neq j$) has dimension 4 and is $\text{Ad}(T^n)$-invariant. It is easy to see that each of these subspaces in addition decomposes in the sum of two $\text{Ad}(T^n)$-irreducible and pairwise inequivalent 2-dimensional subspaces. Thus $SO(2n)/T^n$ is multiplicity-free and $\dim \mathcal{M}^G = n(n-1)$.

It is easy to check that $[p_{ij}, p_{kl}]_p = 0$ if $\{i, j\}$ and $\{k, l\}$ are either equal or disjoint, and that there exist decompositions $p_{ij} = p_{ij}^1 \oplus p_{ij}^2$ such that

$$[p_{ij}^1, p_{ik}^1]_p \subset p_{jk}^1, \quad [p_{ij}^2, p_{ik}^2]_p \subset p_{jk}^2, \quad [p_{ij}^1, p_{ik}^2]_p \subset p_{jk}^1, \quad [p_{ij}^2, p_{ik}^1]_p \subset p_{jk}^2,$$

for all $j \neq k$. Moreover, it is also easy to prove that all the above triples produces the same nonzero structural constant $\frac{1}{2(n-1)}$. It follows from (5) that the Lichnerowicz Laplacian restricted to $S^2(M)^G$ of the standard metric $g_B$, which is Einstein with $\rho = \frac{n}{4(n-1)}$, is given by

$$[L_p]_{(ij)^2(ij)^2} = \frac{n-2}{n-1}, \quad [L_p]_{(ij)^4} = [L_p]_{(ij)^2(ik)^2} = -\frac{1}{4(n-1)},$$

for all $\#\{i, j, k\} = 3$, and zero otherwise. This implies that

$$[L_p] = \frac{1}{4(n-1)} \left( 4(n - 2)I - \begin{bmatrix} \text{Adj}(X) & \text{Adj}(X) \\ \text{Adj}(X) & \text{Adj}(X) \end{bmatrix} \right), \quad (17)$$

where $X = J(n, 2, 1)$ is the Johnson graph with parameters $(n, 2, 1)$ (see [8, Section 1.6]) and $\text{Adj}(X)$ denotes its adjacency matrix.

Since this graph is strongly regular with parameters $\left(\frac{n(n-1)}{2}, 2(n-2), n-2, 4\right)$ for any $n \geq 4$ (see [8, Section 10.1]), it follows from [8, Section 10.2] that the spectrum of $\text{Adj}(X)$ is given by

$$2(n - 2), \quad n - 4, \quad -2, \quad \text{with multiplicities } 1, \quad n - 1, \quad \frac{n(n-3)}{2},$$

respectively, and thus the matrix $\begin{bmatrix} \text{Adj}(X) & \text{Adj}(X) \\ \text{Adj}(X) & \text{Adj}(X) \end{bmatrix}$ has eigenvalues

$$4(n - 2), \quad 2(n - 4), \quad -4, \quad 0, \quad \text{with multiplicities } 1, \quad n - 1, \quad \frac{n(n-3)}{2}, \quad \frac{n(n-1)}{2},$$

respectively. It follows from (17) that

$$\text{Spec}(L_p) = \left\{ 0, \frac{n}{2(n-1)}, 1, \frac{n-2}{n-1}, \frac{n-3}{2} \right\},$$

with multiplicities $1, n - 1, \frac{n(n-3)}{2}$ and $\frac{n(n-1)}{2}$, respectively. Thus $\lambda_p = \frac{n}{2(n-1)}$, $\lambda_p^{\text{mid}} = \frac{n-2}{n-1}$ and $\lambda_p^{\text{max}} = 1$ for any $n \geq 4$.

For $n = 3$, $X$ is the complete graph on 3 vertices and so the spectrum of $\text{Adj}(X)$ equals $\{2, -1\}$, with multiplicities 1 and 2, respectively. Thus $\lambda_p = \frac{1}{2}$ and $\lambda_p^{\text{max}} = \frac{3}{4}$ with multiplicities 3 and 2, respectively.

Since $2\rho = \frac{n}{2(n-1)}$, we conclude that $g_B$ is $G$-neutrally stable of nullity $n - 1$ for any $n \geq 4$ and it is $G$-unstable of coindex 3 and $G$-degenerate of nullity 2 for $n = 3$. 
6 Two isotropy summands

We study in this section standard Einstein metrics on homogeneous spaces with only two irreducible isotropy summands, i.e., $p = p_1 \oplus p_2$. These spaces were classified in [7], and some corrections were added in [10, Appendix A].

Remark 6.1 The authors take the opportunity to note that the space $E_8/\text{Spin}(9)$ (see Table 3, 10) was missed in the above two papers.

We assume that the summands are inequivalent since, according to [7], the only case where this is not the case is $\mathfrak{so}(8)/\mathfrak{g}_2$ (see Table 1, 9), which will be treated at the end of the section. Since $\dim \mathcal{M}_1^G = 1$, we have that $\text{Sc}|_{\mathcal{M}_1^G}$ is a one-variable function and so $\text{Spec}(L_p) = \{0, \lambda_p\}$. It follows from [17, Section 3.1] that for the standard metric $g_B$ one has

$$2\rho = 1 - \frac{[111]+[122]+2[112]}{2d_1} = 1 - \frac{[222]+[112]+2[122]}{2d_2},$$

and

$$\lambda_p = \frac{d_1+d_2}{d_1d_2} ([112] + [122]).$$

If $K$ is not a maximal subgroup of $G$, then it can be assumed that $[112] = 0$ (i.e., $\mathfrak{k} \oplus p_1$ is a subalgebra) and it is known that there exist at most two $G$-invariant Einstein metrics on $G/K$ (see [30, Theorem (3.1)] or [7, Theorem 3.1]).

On the other hand, if $K$ is a maximal subgroup of $G$, then $[112], [122] > 0$, there exist at least one (a global maximum, see [30, Theorem (2.2)]) and at most three $G$-invariant Einstein metrics on $G/K$ (see [7, Section 3.2]) and $g_B$ is Einstein in exactly the following four cases (see [7, Section 6]): Table 1, items 3a and 3b and Table 3, items 8 and 11. However, we do not know a priori whether $g_B$ is a global maximum or not.

Lemma 6.2 Assume that $g_B$ is Einstein, $d_1 = d_2$, $[111] = [222]$ and $[112] = [122] > 0$. Then, the following conditions are equivalent:

(i) $g_B$ is a global maximum.

(ii) $g_B$ is the unique $G$-invariant Einstein metric.

Otherwise, $g_B$ is a local minimum and there exist other two Einstein metrics (both necessarily local maxima and at least one of them global maximum).

Proof If $a := \frac{d_1}{2} - \frac{1}{4}[111] - \frac{1}{2}[112]$ and $b := \frac{1}{4}[112]$ (note that $a > b > 0$), then the scalar curvature $\text{Sc}(x)$ of the metric $(x, \frac{1}{x})$ satisfies (see e.g. [17, (19)])

$$\text{Sc}(x) = a \left( x + \frac{1}{x} \right) - b \left( x^3 + \frac{1}{x^3} \right),$$

$$\text{Sc}'(x) = a \left( 1 - \frac{1}{x^2} \right) - 3b \left( x^2 - \frac{1}{x^2} \right),$$

$$\text{Sc}''(x) = 2a \frac{1}{x^2} - 6bx - 12b \frac{1}{x^3}.$$ 

Thus $\text{Sc}(x) \to -\infty$ as $x \to 0$ or $x \to \infty$. Since $\text{Sc}''(1) = 2a - 18b$, $g_B$ is $G$-stable if and only if $a < 9b$ (note that this is equivalent to $2\rho < \lambda_p$ and to $d_1\rho < 2[112]$).

It also follows that the squares of the other potential critical points of $\text{Sc}$ are zeroes of $\varepsilon^2 + \frac{3b-a}{3b}z + 1$, so either $a \leq 9b$ and $g_B$ is the unique critical point (necessarily a global maximum, and $G$-degenerate if and only if $a = 9b$) or $a > 9b$ and there exist other two Einstein metrics $x_1 < 1 < x_2$ (both necessarily local maxima and at least one of them global maximum) and $g_B$ is a local minimum, concluding the proof. \qed
In what follows, we find the $G$-stability type of the standard metric in many cases.

### 6.1 $\mathfrak{so}(n^2)/\mathfrak{so}(n) \oplus \mathfrak{so}(n)$

(See Table 1, 3a). The inclusion $K \subset G$ is defined via the representation $\mathbb{R}^n \otimes \mathbb{R}^n$ of $\text{SO}(n) \times \text{SO}(n)$, so $d_1 = d_2 = \frac{n(n-1)^2(n+2)}{4}$. According to [29, Section 2], the Casimir constant is given by $E(\chi)_{ij}/\alpha_\epsilon$ using the notation in that paper, so in this case, it follows from [29, Tables in pp.602 and pp.583] that it is $\frac{2n-1}{n(n+2)}$. The Einstein constant is therefore given by $\rho = \frac{1}{4} + \frac{n-1}{n(n+2)}$.

The automorphism $(X, Y) \mapsto (Y, X)$ of $\mathfrak{so}(n) \oplus \mathfrak{so}(n)$ forces $[111] = [222]$ and $[112] = [122]$. Furthermore, (18) gives an expression for $[111] + 3[112]$, thus it is sufficient (and necessary) to determine any of these two nonzero structural constants to know the whole picture.

**Lemma 6.3** We have that $[112] = \frac{n(n-1)^2(n-2)(n+2)^2}{16(n^2-2)}$.

**Proof** The proof involves tedious but straightforward calculations. We next explain the strategy leaving the details to the reader.

We use the identification $\mathbb{R}^{n^2} = \mathbb{R}^n \otimes \mathbb{R}^n$ with basis $e_i \otimes e_j$ for $1 \leq i, j \leq n$. Since $B_{\mathfrak{so}(n^2)}(X, Y) = (n^2 - 2) \text{tr}(XY)$ for all $X, Y \in \mathfrak{so}(n^2)$, $X_{ij}, (kl) := \frac{1}{\sqrt{2(n^2-2)}}(E_{ij}, (kl) - E_{kl}, (ij))$ for $i < k$ and $j < l$ form a basis of $\mathfrak{so}(n^2)$. Here, $E_{(ij), (kl)}$ is the $n^2 \times n^2$ matrix with a 1 in the entry $((ij), (kl))$ and zero otherwise. Let $h$ be in $[1, 2]$. It is not difficult to see that an orthonormal basis for $\mathfrak{p}_h$ is given by $B_{(h)}^1 := B_{11}^1 \cup B_{21}^1$, where $B_{11}^1 = \{\frac{1}{\sqrt{2}}(X_{(ij), (kl)} - (-1)^h X_{(kl), (ij)}) : i < k, j < l\}$, $B_{21}^1 = \{(\sum_{t=1}^{n-1}(\delta_{t,j} - \frac{1}{n+\sqrt{n}})X_{(it), (kt)} - \frac{1}{\sqrt{n}}X_{(it), (tk)} - \frac{1}{\sqrt{n}}X_{(itn), (tk)} : i < k, j < n\}$, and $B_{22}^1 = \{(\sum_{t=1}^{n-1}(\delta_{t,j} - \frac{1}{n+\sqrt{n}})X_{(it), (tk)} - \frac{1}{\sqrt{n}}X_{(itn), (tk)} : i < k, j < n\}$. It follows that

$$[12h] = \sum_{1 \leq p, q \leq 2} \sum_{X_1 \in B_{p1}^1} \sum_{X_2 \in B_{q1}^1} \sum_{X_3 \in B_{q2}^1} B([X_1, X_2], X_3)^2 : \sum_{1 \leq p, q \leq 2} B(p, q).$$

Tiresome computations give $B(1, 1) = \frac{n^3(n-1)^2(n-2)}{16(n^2-2)}$, $B(1, 2) = B(2, 1) = \frac{n^2(n-1)^2(n-2)}{8(n^2-2)}$, and $B(2, 2) = \frac{n(n-1)^2(n-2)}{4(n^2-2)}$, and the assertion follows.

Since $\lambda_p = \frac{4}{4\psi} [112] = \frac{n^2-4}{n^2-2}$ by (19), it is easy to see that $\lambda_p > 2\rho$ if and only if $n^3 - 10n + 4 > 0$, which is true for every $n \geq 3$. We conclude that $g_B$ is $G$-stable.

**Remark 6.4** To the best of authors’ knowledge, the classification of $G$-invariant Einstein metrics on $G/K$ was unknown. Lemma 6.2 implies that $g_B$ is the unique $G$-invariant Einstein metric and a global maximum.

### 6.2 $\mathfrak{so}(4n^2)/\mathfrak{sp}(n) \oplus \mathfrak{sp}(n)$

(See Table 1, 3b). The inclusion $K \subset G$ is defined via the real representation of $\text{Sp}(n) \times \text{Sp}(n)$ whose complexification is $\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$, so $d_1 = d_2 = n(n-1)(2n+1)^2$. In much the same way as the above case, one obtains from [29, Tables in pp.602 and pp.583] that the Casimir constant...
is \( \frac{2n+1}{2n(2n^2-1)} \), so the Einstein constant is given by \( \rho = \frac{1}{4} + \frac{2n+1}{4n(2n^2-1)} \). Again, \([111] = [222], [112] = [222], 2\rho = 1 - \frac{1}{2d_1}([111] + 3[112])\) and \( \lambda_p = \frac{d_1}{2d_1} [112] \).

Although it is sufficient to compute \([111]\) or \([112]\) to determine \( \lambda_p \) and consequently the \( G \)-stability type of \((G/K, g_B)\), the authors decided to leave this case open due to the great difficulty they encountered in the computations.

### 6.3 \( \text{su}(pq+l)/\text{su}(p) \oplus \text{su}(q) \oplus \text{u}(l) \)

(See Table 1, 6). The inclusion \( K \subset G \) is defined via the representation \((\mathbb{C}^p \otimes \mathbb{C}^q) \oplus \mathbb{C}^l \) of \((U(p) \times U(q)) \times U(l)\) and so \( \mathfrak{k} \) is isomorphic to \( \mathfrak{s}(u(p) \oplus u(q) \oplus u(l))/\mathbb{R}Z \) for a certain \( Z \in \mathfrak{z} \), the center of \( \mathfrak{s}(u(p) \oplus u(q) \oplus u(l)) \). It is proved in [29, Example 8, pp. 576] that the standard metric is Einstein if and only if \( l = \frac{p^2 + q^2 + 1}{pq} \), for which there are infinitely many integer solutions. Note that \( \dim \mathfrak{k} = p^2 + q^2 + l^2 - 2 \) and \( d = p^2q^2 + p^2 + q^2 + 3 \). We have that \( p = p_1 \oplus p_2 \), where \( p_2 \) is the off diagonal block of \( \text{su}(pq+l) \) and \( p_1 \) is given by \( p_1 = \tilde{p}_1 \oplus \mathbb{R}Z \), where \( \text{su}(pq) = \mathfrak{s}(u(p) \oplus u(q)) \oplus \tilde{p}_1 \) and \( \{Z, Z^\perp\} \) is an orthogonal basis of the center of \( \mathfrak{z} \), that is,

\[
\text{su}(pq+l)/\mathfrak{z} = \left[ \begin{array}{ccc} \mathfrak{s}(u(p) \oplus u(q)) & \tilde{p}_1 & p_2 \\ p_2 & \text{su}(l) \end{array} \right].
\]

Thus \( d_1 = p^2q^2 - p^2 - q^2 + 1, d_2 = 2pq = 2p^2 + 2q^2 + 2 \) and the only nonzero structural constants are \([111]\) and \([122]\). It follows from (18) and (19) that

\[
2\rho = 1 - \frac{1}{d_2}[122], \quad \lambda_p = \frac{d}{d_1d_2}[122].
\]

On the other hand, since \( B_{\text{su}(p)} = \frac{1}{q^2} B_{\text{su}(pq)} \mid \text{su}(p) \) and \( B_{\text{su}(q)} = \frac{1}{p^2} B_{\text{su}(pq)} \mid \text{su}(q) \), we have that

\[
B_{\text{su}(p)} = \frac{p}{q(pq+l)} B_{\text{su}(pq+l)} \mid \text{su}(p), \quad B_{\text{su}(q)} = \frac{q}{p(pq+l)} B_{\text{su}(pq+l)} \mid \text{su}(q),
\]

\[
B_{\text{su}(l)} = \frac{1}{pq+l} B_{\text{su}(pq+l)} \mid \text{su}(l),
\]

and hence according to (10), \( \rho = \frac{1}{4} + \frac{1}{2d_1} \alpha \), where

\[
\alpha = \left( 1 - \frac{p}{qq(pq+l)} \right) (p^2 - 1) + \left( 1 - \frac{q}{pp(pq+l)} \right) (q^2 - 1) + \left( 1 - \frac{l}{pq+l} \right) (l^2 - 1) + 1
\]

\[
= \frac{p^2q^2 + 2pq^2 + q^2 + 2 + p^2q^2 + q^2 + 1}{pq(pq+l)} = \frac{(p^2q^2 + 2pq^2 + q^2 + 3)(p^2 + q^2)}{pqq^2 + p^2q^2 + q^2 + 1},
\]

which implies that

\[
\rho = \frac{p^2q^2 + 2p^2 + 3q^2 + 1}{4(p^2q^2 + p^2 + q^2 + 1)}, \quad \lambda_p = \frac{p^2q^2 + p^2 + q^2 + 3}{2(p^2q^2 + p^2 + q^2 + 1)},
\]

since \([122] = \frac{d(p-2a)}{2d_1} \) and so \( \lambda_p = \frac{d-2a}{2d_1} \). Now a straightforward calculation gives that \( \lambda_p < 2\rho \) if and only if \( 1 < p^2 + q^2 \), and hence \( g_B \) is always \( G \)-unstable as \( p, q \geq 2 \).

### 6.4 \( \text{sp}(3n - 1)/\text{sp}(n) \oplus \text{u}(2n - 1) \)

(See Table 1, 7a). This case was already solved for \( n \geq 3 \) in part (i) at the end of Sect. 4. The intermediate subalgebra \( \mathfrak{k} \subset \mathfrak{h} := \text{sp}(n) \oplus \text{sp}(2n - 1) \subset \mathfrak{g} \) gives the reductive decomposition \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \), where \( \mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_1 \). Since \( H/K \) and \( G/H \) are symmetric spaces, we obtain
that the only nonzero structural constant is $[122]$. It follows from (18) that $d_2 = 2d_1$ and $[122] = (1 - 2\rho)2d_1$, and from (19) that

$$\lambda_p = \frac{3d_1}{2d_1^2} (1 - 2\rho)2d_1 = 3(1 - 2\rho) < 2\rho,$$

since $2\rho = \frac{5}{6}$. Hence $g_B$ is $G$-unstable and a local minimum for any $n \geq 1$.

6.5 $\text{so}(26)/\text{sp}(1) \oplus \text{sp}(5) \oplus \text{so}(6)$

(See Table 1, 8). For the intermediate subalgebra $\mathfrak{t} \subset \mathfrak{h} := \text{so}(20) \oplus \text{so}(6) \subset \mathfrak{g}$, we obtain the reductive decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$, where $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{p}_1$, $H/K$ is isotropy irreducible and $G/H$ is symmetric. Note that $d_1 = 132$ and $d_2 = 120$. The only nonzero structural constants are therefore $[111]$ and $[122]$.

In order to apply formula (10) to compute $\rho$, we set $\mathfrak{t}_1 := \text{sp}(1)$, $\mathfrak{t}_2 := \text{sp}(5)$ and $\mathfrak{t}_3 := \text{so}(6)$. We have that $c_3 = \frac{1}{60}$ by [6, pp.37] and by following the method given in [6, pp.38–40] one obtains that $c_1 = \frac{1}{60}$ and $c_2 = \frac{1}{4}$. Thus $\rho = \frac{29}{60}$ and it therefore follows from (18) that $[122] = 33$. Now (19) gives that

$$\lambda_p = \frac{21}{40} < \frac{29}{60} = 2\rho,$$

and hence $g_B$ is $G$-unstable and a local minimum.

6.6 $\text{e}_6/\text{su}(2) \oplus \text{so}(6)$

(See Table 2, 4). According to [7, IV.16], $g_B$ is the only $G$-invariant Einstein metric, hence it is necessarily an inflection point of $\text{Sc}|_{\mathcal{M}_1^G}$. In particular, it is $G$-neutrally stable.

6.7 $\text{e}_8/\text{spin}(9)$

(See Table 3, 10). The embedding of $\mathfrak{t}$ into $\mathfrak{e}_8$ is via the spin representation $\mathfrak{t} \rightarrow \text{so}(16) \subset \mathfrak{e}_8$, which implies that the only nonzero structural constants are $[111]$ and $[122]$. According to (10), $\rho = \frac{1}{4} + \frac{36(1-c)}{424}$, where $B_{\text{spin}(9)} = c B_{\mathfrak{e}_8}|_{\text{spin}(9)}$. Furthermore, one can see that $c = \frac{7}{60}$ by using [6, pp.38–40], thus $\rho = \frac{13}{60}$. We have that $d_1 = 84$ and $d_2 = 128$, so (18) gives that $[122] = \frac{224}{5}$. By (19) we conclude that

$$\lambda_p = \frac{53}{60} > \frac{13}{20} = 2\rho,$$

and thus $g_B$ is $G$-stable.

6.8 $\text{e}_8/\text{su}(5) \oplus \text{su}(5)$

(See Table 3, 11). It is straightforward to prove that $B_{\text{su}(5)} = \frac{1}{6} B_{\mathfrak{e}_8}|_{\text{su}(5)}$ for both copies, which implies that $\rho = \frac{7}{20}$ by using formula (10). It can be shown that the only nonzero structural constants are $[112]$ and $[122]$, and since $d_1 = d_2 = 100$ and $\rho = \frac{7}{20}$, one obtains that $[112] = [122] = 20$ by (18). This implies that $\lambda_p = \frac{4}{5}$. Since $2\rho < \lambda_p$ we obtain that $g_B$ is $G$-stable; moreover, it is the only Einstein metric on $G/K$ and a global maximum by Lemma 6.2.
6.9 $\mathfrak{so}(8)/\mathfrak{g}_2$

(See Table 1, 9). Note that \(\text{SO}(8)/\mathfrak{g}_2 = S^7 \times S^7\) as a manifold. It follows from [11, Section 5] that \(d_1 = d_2 = 7, [112] = [222] = 0\) and \([111] = [122] = \frac{7}{6}\), thus \(\rho = \frac{5}{12}\) by (18). We now use (19) and [17, Remark 3.3] to conclude that

\[
\lambda_p \leq \frac{1}{3} = \frac{2}{7} < \frac{5}{6} = 2\rho,
\]

and hence \(g_B\) is \(G\)-unstable. It can be proved that \(g_B\) is actually a local minimum (see [9]).

7 Case \(\text{SO}(m)/K_1 \times \cdots \times K_l\)

In this section, we show that all the standard Einstein metrics constructed in [29, Example 3], which correspond to cases 4 and 5 in Table 1, are \(G\)-unstable and compute the spectrum of \(L_p\) together with the multiplicities.

Let \(l_1 \leq l_2 \leq l\) be non-negative integers and assume \(2 \leq l\). For each \(1 \leq i \leq l\), we choose an irreducible symmetric space \(G_i/K_i\) according to the following constraints:

- \(\text{SO}(2n_i)/\text{SO}(n_i) \times \text{SO}(n_i)\) or \(\text{Sp}(2n_i)/\text{Sp}(n_i) \times \text{Sp}(n_i)\) for any \(1 \leq i \leq l_1\) (Grassmannian spaces),
- \(\text{SO}(n_i + 1)/\text{SO}(n_i) = S^{n_i}\) for any \(l_1 + 1 \leq i \leq l_2\) (spheres),
- \(K_i\) is simple and \(G_i/K_i\) is not a sphere (as above) for any \(l_2 + 1 \leq i \leq l\).

We will see below that the construction only depends on the spaces \(G_i/K_i\) up to covering. In Table 7, all possible \(G_i/K_i\) are listed without repetition.

Remark 7.1 The spaces \(\text{SO}(4)/\text{SO}(2) \times \text{SO}(2), \text{Sp}(2)/\text{Sp}(1) \times \text{Sp}(1), \text{SU}(4)/\text{Sp}(2)\) and \((H \times H)/\Delta(H)\) with \(H = \text{SU}(2)\) or \(\text{SO}(3)\) were all omitted in Table 7 for being locally isometric to \(\text{SO}(3)/\text{SO}(2) \times \text{SO}(3)/\text{SO}(2), \text{SO}(5)/\text{SO}(4), \text{SO}(6)/\text{SO}(5)\) and \(\text{SO}(4)/\text{SO}(3)\), respectively.

For each \(G_i/K_i\), we consider the reductive decomposition \(\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{m}_i\), set \(m_i = \dim \mathfrak{m}_i = \dim G_i/K_i\), and let \(\pi_i : K_i \to \mathfrak{so}(m_i)\) denote its isotropy representation. Consider the homogeneous space

\[
\text{SO}(m_i)/K_i := \text{SO}(m_i)/\pi_i(K_i).
\]

If \(\mathfrak{so}(m_i) = \mathfrak{k}_i \oplus \mathfrak{v}_i\) is its reductive decomposition, then \(\lambda^2 \mathfrak{m}_i \simeq \text{ad}_{\mathfrak{k}_i} \mathfrak{v}_i\) as \(K_i\)-modules.

It turns out that \(\mathfrak{v}_i = 0\) if and only if \(l_1 + 1 \leq i \leq l_2\) (i.e. \(G_i/K_i\) is a sphere). Furthermore, as a representation of \(K_i\), \(\mathfrak{v}_i\) is irreducible for any \(i > l_2\) and \(\mathfrak{v}_i \simeq V_i^1 \oplus V_i^2\) with \(V_i^1, V_i^2\) inequivalent and irreducible submodules of the same dimension for any \(i \leq l_1\) (see Sects. 6.1 and 6.2). Note that the each space \(\text{SO}(m_i)/K_i\) with \(i \leq l_1\) corresponds to case 3 in Table 1.

We set \(G = G_1 \times \cdots \times G_i\) and \(K = K_1 \times \cdots \times K_i\) and now consider the isotropy representation \(\pi\) of \(G/K\) given by

\[
\pi := \pi_1 + \cdots + \pi_i, \quad m := \dim m = m_1 + \cdots + m_i.
\]

We study in this section the homogeneous space

\[
M = \text{SO}(m)/K = \text{SO}(m)/\pi(K),
\]

i.e., the case Table 1, 5.
Table 7 Irreducible symmetric spaces admitted as $G_i/K_i$ in Sect. 7

| $G_i/K_i$                  | Cond. | $\dim \xi_i$ | $m_i$ | $\dim \xi_i/m_i$ |
|---------------------------|-------|--------------|-------|------------------|
| $SO(2n)/SO(n) \times SO(n)$ | $n \geq 3$ | $n(n-1)$     | $n^2$ | $\frac{n-1}{n}$  |
| $Sp(2n)/Sp(n) \times Sp(n)$ | $n \geq 2$ | $2n(2n+1)$   | $4n^2$ | $\frac{2n+1}{2n}$ |
| $SO(n+1)/SO(n)$           | $n \geq 2$ | $\frac{n(n-1)}{2}$ | $n$  | $\frac{n-1}{n}$  |
| $SU(n)/SO(n)$             | $n \geq 3$ | $\frac{n(n-1)}{2}$ | $(n-1)(n+2)$ | $\frac{n}{n+2}$ |
| $SU(2n)/Sp(n)$            | $n \geq 3$ | $n(2n+1)$    | $(n-1)(2n+1)$ | $\frac{n}{n-1}$ |
| $E_6/Sp(4)$               |       | 36           | 42    | 9                |
| $E_6/F_4$                 |       | 52           | 26    | 2                |
| $E_7/SU(8)$               |       | 63           | 70    | 9                |
| $E_8/Spin(16)$            |       | 120          | 128   | 15               |
| $F_4/Spin(9)$             |       | 36           | 16    | 9                |
| $(H \times H)/\Delta H$   |       | dim $H > 3$  | dim $H$ | dim $H$ |

$H$ is any compact simple Lie group in the last row

Remark 7.2 This space belongs to the case Table 1, 4 if and only if $G_i/K_i = H_i \times H_i/\Delta H_i$ for any $i = 1, \ldots, l$, where $H_1, \ldots, H_l$ are compact simple Lie groups (in particular, $l_1 = l_2 = 0$) and so $K \simeq H_1 \times \cdots \times H_l, m = \dim \xi$ and $(m, \pi)$ is equivalent to the adjoint representation of $K$. On the other hand, setting $G_i/K_i = SO(n+1)/SO(n)$ for all $i = 1, \ldots, l$ gives $M = SO(ln)/SO(n)^l$, which are (for $l \geq 3$) the cases Table 1, 1b and 2c for $n = 2$ and $n \geq 3$, respectively.

Consider the reductive decomposition $\mathfrak{so}(m) = \mathfrak{\tau} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is the $K$-representation such that $\wedge^2 \mathfrak{m} = \text{ad}_{\mathfrak{\tau}} \oplus \mathfrak{p}$. Thus $\mathfrak{p}$ decomposes in pairwise inequivalent $K$-irreducible subspaces as

$$\mathfrak{p} = \bigoplus_{i \leq l_1} (\mathfrak{p}_{(ii)})^l \oplus \bigoplus_{i > l_2} (\mathfrak{p}_{(ii)})^l \oplus \bigoplus_{i < j} (\mathfrak{p}_{(ij)}),$$

(20)

where $\mathfrak{p}_{(ii)}^l := V_i^l \subset \mathfrak{so}(m_i)$ for $\epsilon = 1, 2$ and $i \leq l_1$, $\mathfrak{p}_{(ii)} := V_i \subset \mathfrak{so}(m_i)$ for $i > l_2$, and $\mathfrak{p}_{(ij)} := m_i \otimes m_j$ for all $i < j$ (we set $\mathfrak{p}_{(jj)} := \mathfrak{p}_{(ii)}$). In particular, $\mathfrak{so}(m)/K$ is multiplicity-free. We have that $d_{(ii)}^\epsilon = \dim V_i^\epsilon$ for $i \leq l_1$ (recall that $d_{(ii)}^1 = d_{(ii)}^2$), $d_{(ii)} = \dim V_i$ for $i > l_2$, $d_{(ij)} = m_i m_j$ for $i < j$, and the number of irreducible terms in the isotropy representation is given by

$$r = 2l_1 + l - l_2 + \frac{l(l-1)}{2}.$$  

(21)

The situation for $l_1 = 1, l_2 = 2, l = 3$ is described as follows:

$$\mathfrak{so}(m) = \mathfrak{so}(m_1 + m_2 + m_3) = \begin{bmatrix} \xi_1 \oplus \mathfrak{p}_{(11)}^1 & \mathfrak{p}_{(12)}^2 & \mathfrak{p}_{(13)}^2 & \mathfrak{p}_{(23)}^2 & \mathfrak{p}_{(33)}^2 \\ \mathfrak{p}_{(12)} & \xi_2 & \mathfrak{p}_{(13)} & \mathfrak{p}_{(23)} & \mathfrak{p}_{(33)} \\ \mathfrak{p}_{(13)} & \mathfrak{p}_{(23)} & \xi_3 & \mathfrak{p}_{(33)} \end{bmatrix}.$$
One can check that $[p_{i(i)}, p_{jj}]_p \subset \delta_{i,j} p_{i(i)}$, $[p_{i(i)}, p_{i(i)}]_p \subset p_{ij}$ if $i \neq j$, $[p_{ij}, p_{ij}]_p = 0$ if $k \neq l$ and $\{i, j\}$ and $\{k, l\}$ are either equal or disjoint, and $[p_{ij}, p_{i(k)]}_p \subset p_{ik}$ if $\#\{i, j, k\} = 3$. The only nonzero structural constants are therefore given by

\[
[[i(j)(k)](ik)], \quad \forall \#\{i, j, k\} = 3, \quad \[(ii)(i)(ij)], \quad \forall i \leq l_1, i \neq j, \quad \[(ii)(ij)(ij)], \quad \forall i > l_2, i \neq j, \quad \[(ii)(iii)(i)], \quad \forall i \leq l_1, \quad (22) \\
[(ii)(ii)(ii)], \quad \forall i > l_2, \quad \[(ii)(ii)(ii)], \quad \forall i \leq l_1.
\]

It is proved in [29, Example 3] that the Killing metric $g_B$ on $M = SO(m)/K$ is Einstein if and only if

\[
\frac{\dim \mathfrak{t}_1}{m_1} = \ldots = \frac{\dim \mathfrak{t}_f}{m_f},
\]

which will be assumed from now on. According to Table 7, the numbers $\kappa$ with $\kappa = \frac{\dim \mathfrak{t}_i}{m_i}$ having more than one solution are listed in Table 8 (see also the table in [29, pp.575]). Note that $SO(4)/SO(3)$ is present in the row $\frac{\dim \mathfrak{t}_1}{m_1} = 1$ as $(SU(2) \times SU(2))/\Delta SU(2)$.

**Remark 7.3** It follows immediately from Table 8 that $l_1(l_2 - l_1) = 0$, that is, Grassmannian spaces and spheres cannot take part simultaneously. Furthermore, $l_1 > 0$ (resp. $l_1 < l_2$) forces $G_i/K_1 = G_1/K_1$ for all $i \leq l_1$ (resp. $G_i/K_i = G_{l_2}/K_{l_2}$ for all $l_1 < i \leq l_2$). If $l_1 > 0$, we denote by [111], [112], [122], [222] the structural constants of $SO(m_1)/K_1$, which were considered in Sects. 6.1 and 6.2. Although we do not know their values in the case Sect. 6.2, we know that $[111] = [222], [112] = [122]$, and we have an expression for $[111] + 3[112]$.

**Remark 7.4** Table 8 also implies that if $G_i/K_i = SO(n + 1)/SO(n)$ with $n \neq 4, 5$ for some $i$, then $G_i/K_i$ is independent of $i$ giving $M = SO(In)/SO(n)^l$ already considered.

It is also shown in [29, pp.574] that the Casimir constant on any $p_{ij}$ relative to $-B_{SO(m)}$ equals $a := \frac{2\dim \mathfrak{t}_i}{m_i(m-2)}$ and, since the Einstein constant for $g_B$ is $\rho = \frac{1}{4} + \frac{1}{2}a$ (see (6)), one obtains that

\[
\rho = \frac{1}{4} + \frac{\dim \mathfrak{t}_i}{m_i(m-2)}, \quad \forall i.
\]
Lemma 7.5 If \( g_B \) is Einstein, then the nonzero structural constants of the homogeneous space \( \text{SO}(m)/K \) (see (22)) are given as follows:

\[
(ii)(ii)(ii) = \frac{d_{ii}}{m-2} \left( m_i - 2 - \frac{4 \dim \xi_i}{m_i} \right), \quad i > l_2,
\]

\[
(ii)^1(ii)(i) = \frac{m_i - 2}{m - 2} \left[ 111 \right], \quad i \leq l_1,
\]

\[
(ii)(i)^1(ii) = \frac{m_i - 2}{m - 2} \left[ 112 \right], \quad i \leq l_1,
\]

\[
(ij)(ij)(ii) = \frac{m_i d_{ii}}{m - 2}, \quad i \neq j, \quad i > l_2,
\]

\[
(ij)(ij)(i) = \frac{m_i d_{ii}}{2(m-2)}, \quad i \neq j, \quad i \leq l_1,
\]

\[
(ij)(ik)(jk) = \frac{m_i m_j m_k}{2(m-2)}, \quad \#\{i, j, k\} = 3.
\]

Proof Recall that the constants \([111] \) and \([112] \) were introduced in Remark 7.3. We consider the usual \(-B_{\text{so}(m)}\)-orthonormal basis of \( \text{so}(m) \) (recall that \( B_{\text{so}(m)}(X, Y) = (m-2) \text{tr}(XY) \) given by \( X_{\alpha, \beta} := \frac{1}{\sqrt{2(m-2)}} \left( E_{\alpha, \beta} - E_{\beta, \alpha} \right) : 1 \leq \alpha < \beta \leq m \), which provides, for each \( p_{(ii)} \) with \( i \neq j \), the \(-B_{\text{so}(m)}\)-orthonormal basis \( \left\{ X_{\alpha, \beta} : 1 \leq \alpha - \tilde{m}_i \leq m_i, \ 1 \leq \beta - \tilde{m}_j \leq m_j \right\} \), where \( \tilde{m}_k := m_1 + \cdots + m_k - 1 \). Using that \( X_{\alpha, \beta}, X_{\beta, \gamma} = \frac{1}{\sqrt{2(m-2)}} X_{\alpha, \gamma} \) for all \( \#\{\alpha, \beta, \gamma\} = 3 \), a straightforward computation gives the formula for \( (ij)(ik)(jk) \).

Suppose that \( i \leq l_1 \). Let \( \left\{ Y_{h}^{(ii)} \right\} \) be a \(-B_{\text{so}(m)}\)-orthonormal basis of \( \text{so}(m_i) \subset \text{so}(m) \). Since \( B_{\text{so}(m_i)} = \frac{m_i - 2}{m_i} B_{\text{so}(m) | \text{so}(m_i)} \), we have that \( \left\{ \sqrt{\frac{m_i - 2}{m_i}} Y_{h}^{(ii)} \right\} \) is \(-B_{\text{so}(m_i)}\)-orthonormal. This implies that

\[
(ii)^1(ii)(i) = \frac{m_i - 2}{m - 2} \left[ 111 \right], \quad \quad (ii)(i)^1(ii) = \frac{m_i - 2}{m - 2} \left[ 112 \right],
\]

\[
(ii)^2(ii)(i) = \frac{m_i - 2}{m - 2} \left[ 222 \right], \quad \quad (ii)(i)^2(ii) = \frac{m_i - 2}{m - 2} \left[ 122 \right].
\]

The corresponding assertions for these terms follow since \([111] = [222] \) and \([112] = [122] \) (see Remark 7.3). Similarly, for \( i > l_2 \), one has that

\[
(ii)(ii)(ii) = \frac{m_i - 2}{m - 2} \left[ 111 \right],
\]

where \([111]_i \) is the structural constant of the isotropically irreducible space \( \text{SO}(m_i)/K_i \), which can be computed using formulas (10) and (11) for its Einstein constant as follows:

\[
\frac{1}{4} + \frac{\dim \xi_i}{m_i(m_i - 2)} = \frac{1}{2} - \frac{1}{4d_{ii}} \left[ 111 \right].
\]

This proves the formula for \( (ii)(ii)(ii) \).

Let \( p = (ii) \) for some \( i \leq l_1 \) or \( p = (i) \) for some \( i > l_2 \). We have that

\[
(ii)(ij)p = \sum_{1 \leq \alpha - \tilde{m}_i \leq m_i} \sum_{1 \leq \gamma - \tilde{m}_i \leq m_i} \sum_{h=1} d_p \left( -B_{\text{so}(m)} \right) \left( [X_{\alpha, \beta}, X_{\gamma, \delta}]_h, Y_{h}^{p} \right)^2
\]

\[
= \frac{1}{2(m-2)} \sum_{1 \leq \alpha - \tilde{m}_i \leq m_i} \sum_{1 \leq \beta - \tilde{m}_i \leq m_i} \sum_{h=1} d_p \left( -B_{\text{so}(m)} \right) \left( [X_{\alpha, \gamma}, Y_{h}^{p}] \right)^2
\]

\[
= \frac{m_j}{2(m-2)} \sum_{1 \leq \alpha - \tilde{m}_i \leq m_i} \sum_{h=1} d_p \left( -B_{\text{so}(m)} \right) \left( [X_{\alpha, \gamma}, Y_{h}^{p}] \right)^2 = \frac{m_j d_p}{m - 2},
\]

concluding the proof. \( \square \)
We are now ready to compute the \( r \times r \) matrix \( [L_p] \) of the Lichnerowicz Laplacian restricted to \( S^2(M)^{SO(m)} \) (see (5)). Recall that the set of \( r \) indexes, which will be denoted by \( \mathbb{I} \), where \( r = 2l_1 + l - l_2 + \frac{(l-1)}{2} \), consists of \((ii)^1, (ii)^2\) for \( 1 \leq i \leq l_1 \), \((ii)^3\) for \( l_2 + 1 \leq i \leq l \) and \((ij)\) for each pair \( 1 \leq i < j \leq l \).

**Proposition 7.6** The nonzero coefficients of the matrix \( [L_p] \) are given by

\[
[L_p]_{(ij),(ij)} = \frac{1}{m_i m_j} \left( m_i - m_j + \frac{d_{ii}}{m_i} + \frac{d_{jj}}{m_j} \right), \quad i \neq j,
\]

\[
[L_p]_{(ii),(ii)} = \frac{m_i - m_j}{m_i - m_j}, \quad i > l_2,
\]

\[
[L_p]_{(ii)^2,(ii)^2} = [L_p]_{(ii),(ii)^2} = \frac{m_i - m_j}{m_i - m_j} + \frac{4[112]}{d_{ii}} \frac{m_i - m_j}{m_i - m_j}, \quad i \leq l_1,
\]

\[
[L_p]_{(ii)^1,(ii)^1} = -\frac{4[112]}{d_{ii}} \frac{m_i - m_j}{m_i - m_j}, \quad i \leq l_1,
\]

\[
[L_p]_{(ij),(ii)} = -\frac{1}{m_i} \sqrt{\frac{d_{ii}}{m_i}}, \quad i \neq j, \quad i > l_2,
\]

\[
[L_p]_{(ij),(ii)^3} = [L_p]_{(ij),(ii)^3} = -\frac{1}{m_i} \sqrt{\frac{d_{ii}}{m_i}}, \quad i \neq j, \quad i \leq l_1,
\]

\[
[L_p]_{(ij),(ik)} = -\frac{1}{m_i m_j m_k} \sum_{k \neq i, j} [(ii)(ij)(jk)], \quad \#\{i, j, k\} = 3.
\]

**Proof** It follows from (5) and (22) that the only nonzero coefficients are

\[
[L_p]_{(ij),(ij)} = \frac{1}{m_i m_j} \left( [(ii)(ij)(ii)] + [(ij)(ij)(jj)] \right) + \frac{2}{m_i m_j} \sum_{k \neq i, j} [(ii)(ik)(jk)], \quad \forall i \neq j,
\]

\[
[L_p]_{(ii),(ii)} = \frac{1}{d_{ii}} \sum_{j \neq i} [(ii)(ij)(ij)], \quad \forall i > l_2,
\]

\[
[L_p]_{(ii)^2,(ii)^2} = \frac{2}{d_{ii}} \left( [(ii)^2(ii)^2] + [(ii)^2(ii)^2] + \sum_{j \neq i} [(ii)(ij)(ii)] \right), \quad \forall i \leq l_1,
\]

\[
[L_p]_{(ii)^1,(ii)^1} = -\frac{2}{d_{ii}} \left( [(ii)^2(ii)^1] \right) + [(ii)^1(ii)^2(ii)^2], \quad \forall i \leq l_1,
\]

\[
[L_p]_{(ij),(ii)} = -\frac{(i)(ij)(ii))}{d_{ii} \sqrt{m_i m_j}}, \quad \forall i \neq j, \quad i > l_2,
\]

\[
[L_p]_{(ij),(ii)^3} = -\frac{\sqrt{2}(ii)(ij)(ii)^3}{d_{ii} \sqrt{m_i m_j}}, \quad \forall i \neq j, \quad i \leq l_1,
\]

\[
[L_p]_{(ij),(ik)} = -\frac{1}{m_i \sqrt{m_j m_k}} [(ii)(ij)(jk)], \quad \forall \#\{i, j, k\} = 3.
\]

Thus each of the formulas in the proposition can be proved by a straightforward computation using the expressions for the structural constants given in Lemma 7.5. \(\square\)

The following proposition provides a large amount of eigenvectors of \( L_p \). We recall from Sect. 2 that \( [L_p] \) stands for the matrix of \( L_p : \text{sym}(p) \rightarrow \text{sym}(p) \) with respect to the orthonormal basis \( \{I_r\}_{r \in \mathbb{I}} \) of \( \text{sym}(p)^K \) given by \( I_r \big|_{p_q} = \delta_{r,q} \frac{1}{\sqrt{d_e}} I_{p_r} \), where \( I_{p_q} \) is the identity map on \( p_q \).
Proposition 7.7. Given indexes 1 ≤ i, j ≤ l with i ≠ j, let \( A^{ij} = \sum_{r \in \mathbb{I}_r} a_r^{ij} I_r \) and \( B^{ij} = \sum_{r \in \mathbb{I}_r} b_r^{ij} I_r \) be elements in \( \text{sym}(p)^K \) given by

\[
a_r^{ij} = \begin{cases} 
  m_j \sqrt{d_{(ii)}} & r = (ii), i > l_2, \\
  m_j \sqrt{d_{(ii)}}/2 & r = (ii)^e, i \leq l_1, \\
  -m_i \sqrt{d_{(jj)}} & r = (jj), j > l_2, \\
  -m_i \sqrt{d_{(jj)}}/2 & r = (jj)^e, j \leq l_1, \\
  \frac{m_i - m_j}{2} \sqrt{m_i m_j} & r = (ij), \\
  \frac{m_i^2}{2} \sqrt{m_i m_h} & r = (ih), h \neq i, j, \\
  \frac{m_i}{2} \sqrt{m_j m_h} & r = (jh), h \neq i, j, \\
  0 & \text{otherwise}, 
\end{cases}
\]

\[
b_r^{ij} = \begin{cases} 
  \sqrt{d_{(jj)}} & r = (ii), i > l_2, \\
  \sqrt{d_{(jj)}}/2 & r = (ii)^e, i \leq l_1, \\
  \sqrt{d_{(jj)}}/2 & r = (jj), j > l_2, \\
  \sqrt{d_{(jj)}}/2 & r = (jj)^e, j \leq l_1, \\
  -2 \sqrt{d_{(ii)} d_{(jj)}} & r = (ij), \\
  0 & \text{otherwise}.
\end{cases}
\]

Then, \( L_p(A^{ij}) = \frac{m}{2(m-2)} A^{ij} \) and \( L_p(B^{ij}) = \lambda B^{ij} \), where

\[
\lambda = \frac{m-1-\frac{2 \dim t_i}{m_i}}{m-2} = \frac{m-\frac{2 d_{(ii)}}{m_i}}{m-2}.
\]

**Proof.** By applying Proposition 7.6, lengthy but straightforward computations give the assertions. We next give some details in a simple case.

Suppose that \( i, j > l_2 \), so \( B^{ij} = \sqrt{d_{(jj)}} I_{(ii)} + \sqrt{d_{(ii)}} I_{(jj)} - 2 \sqrt{d_{(ii)} d_{(jj)}} / \sqrt{m_i m_j} I_{(jj)} \).

Proposition 7.6 and simple manipulations give

\[
L_p(B^{ij}) = \sum_{r \in \mathbb{I}_r} \left( \sqrt{d_{(jj)}} [L_p] r, (ii) I_r + \sqrt{d_{(ii)}} [L_p] r, (jj) I_r - \frac{2 \sqrt{d_{(ii)} d_{(jj)}}}{\sqrt{m_i m_j}} [L_p] r, (ij) I_r \right)
\]

\[
= I_{(ii)} \frac{\sqrt{d_{(jj)}}}{m-2} \left( m - m_i + \frac{d_{(ii)}}{m_i} \right) + I_{(jj)} \frac{\sqrt{d_{(ii)}}}{m-2} \left( m - m_j + \frac{d_{(jj)}}{m_j} \right)
\]

\[
- I_{(ij)} \frac{2 \sqrt{d_{(ii)} d_{(jj)}}}{\sqrt{m_i m_j} (m-2)} \left( m - \frac{m_j}{2} - \frac{m_i}{2} + \frac{d_{(ii)}}{m_i} + \frac{d_{(jj)}}{m_j} \right) = \lambda B^{ij}.
\]

It is a simple matter to extend the above proof when \( i \leq l_1 \) or \( j \leq l_1 \). Furthermore, \( B^{ij} = 0 \) if \( l_1 < i \leq l_2 \) or \( l_1 < j \leq l_2 \), so the assertion holds trivially. \( \square \)

We are finally in a position to prove the main result of this section.

**Theorem 7.8.** Assume that the standard metric \( g_B \) on the homogeneous space \( SO(m)/K \) constructed above is Einstein.

(a) \( g_B \) is always \( G \)-unstable and has coindex \( \geq l_1 + (l-l_2) - 1 \).

(b) If \( l_1 + (l-l_2) \geq 2 \) (i.e., at least two symmetric spaces different from a sphere are involved), then \( g_B \) is a saddle point of \( \text{Sc} \mid M_i \).

(c) If \( l_1 = l_2 = 0 \), then

\[
\text{Spec}(L_p) = \{ 0, \lambda_p, \ldots, \lambda_p, \lambda_p^{\max}, \ldots, \lambda_p^{\max} \}_{(l-1) \text{-times}} \}
\]

where \( \lambda_p = \frac{m}{2(m-2)}, \lambda_p^{\max} = \frac{1}{m-1} \left( m - 1 - \frac{2 \dim t_i}{m_i} \right) \). Moreover, \( \lambda_p < 2 \rho < \lambda_p^{\max} \).

**Proof.** It follows from (24) that \( \frac{m}{2(m-2)} < 2 \rho \) if and only if \( \frac{1}{2} < \frac{\dim t_i}{m_i} \). A simple inspection to Table 7 yields that this condition always holds with the only exception of \( G_i/K_i = SO(3)/SO(2) \) for all \( i \), but this case can be omitted according to Remark 7.4. Proposition 7.7
now implies that $g_B$ is $G$-unstable. Moreover, the coindex is at least $l_1 + (l - l_2) - 1$ because $\{A^{ij} : j \leq l_1 \text{ or } j > l_2, j \neq i\}$ is clearly linearly independent for any $i$.

**Remark 7.9** Actually, the coindex is at least $l_1 + (l - l_2)$ when $l_1 < l_2$ by choosing $i$ such that $l_1 < i \leq l_2$.

Under the assumption of part (b), Proposition 7.7 ensures the existence of a second nonzero eigenvalue $\lambda$, which satisfies that $\lambda > 2\rho$ if and only if $\frac{\dim \mathfrak{t}_i}{m_i} < \frac{m}{8}$, which always holds (see Table 7) proving part (b).

Concerning part (c), we first observe that the set $\{B^{ij} : i < j\}$ is clearly linearly independent, thus the corresponding eigenvalue $\lambda$ has multiplicity at least $\binom{l}{2}$ by Proposition 7.7. Similarly, $\frac{m}{2(m-2)}$ has multiplicity at least $l-1$ because $\{A^{ij} : i > 2\}$ is linearly independent. Since the kernel is at least 1-dimensional and the size of $L_p$ is $\binom{l+1}{2}$, the spectrum has to have the form as stated. □

### 8 Case $E_7/\text{SU}(2)$

This space is listed in Table 3, 7. We consider the root space decomposition

$$\mathfrak{g}_C = \mathfrak{t}_C \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g}_C, \mathfrak{t}_C)} \mathfrak{g}_\alpha,$$

where

$$\Delta(\mathfrak{g}_C, \mathfrak{t}_C) = \{\pm \varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq 6\} \cup \{\pm (\varepsilon_7 - \varepsilon_8)\} \cup \left\{\frac{1}{2} \left(\sum_{i=1}^{6} (-1)^{n(i)} \varepsilon_i \pm (\varepsilon_7 - \varepsilon_8)\right) : \sum_i n(i) \text{ is odd}\right\}.$$

Inside the maximal subalgebra $\mathfrak{h} := \mathfrak{so}(12) \oplus \mathfrak{su}(2)$ of maximal rank of $\varepsilon_7$ associated to $\Delta(\mathfrak{h}_C, \mathfrak{t}_C) = \{\pm \varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq 6\} \cup \{\pm (\varepsilon_7 - \varepsilon_8)\}$, we describe (the complexification of) the subalgebra $\mathfrak{k} = 7 \cdot \mathfrak{su}(2)$ as follows:

$$\mathfrak{k}_C = C H_{e_1-e_2} \oplus C g_{-e_1-e_2} \oplus C g_{-(e_1-e_2)} \oplus C H_{e_1+e_2} \oplus C g_{e_1+e_2} \oplus g_{-(e_1+e_2)} \oplus C H_{e_3-e_4} \oplus C g_{e_3-e_4} \oplus C g_{-(e_3-e_4)} \oplus C H_{e_3+e_4} \oplus C g_{e_3+e_4} \oplus g_{-(e_3+e_4)} \oplus C H_{e_5-e_6} \oplus C g_{e_5-e_6} \oplus C g_{-(e_5-e_6)} \oplus C H_{e_5+e_6} \oplus C g_{e_5+e_6} \oplus g_{-(e_5+e_6)} \oplus C H_{e_7-e_8} \oplus C g_{e_7-e_8} \oplus g_{-(e_7-e_8)},$$

where $H_0$ is any non-trivial element in $[g_\alpha, g_{-\alpha}] \subset \mathfrak{t}_C$. Note that each triple $C H_\alpha \oplus g_\alpha \oplus g_{-\alpha}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Thus the $B_{e_7}$-orthogonal complement $\mathfrak{p}$ of $\mathfrak{k}$ in $\mathfrak{g}$ is given by $\mathfrak{p} = p_1 \oplus \cdots \oplus p_7$, where $(p_k)_C = \bigoplus_{\alpha \in \Delta_k} g_\alpha$ and

- $\Delta_1 = \{\pm \varepsilon_i \pm \varepsilon_j : i = 1, 2; j = 3, 4\}$, $\Delta_4 = \left\{\frac{1}{2} (\pm (\varepsilon_1 - \varepsilon_2) \pm (\varepsilon_3 - \varepsilon_4) \pm (\varepsilon_5 - \varepsilon_6) \pm (\varepsilon_7 - \varepsilon_8))\right\}$,
- $\Delta_2 = \{\pm \varepsilon_i \pm \varepsilon_j : i = 1, 2; j = 5, 6\}$, $\Delta_5 = \left\{\frac{1}{2} (\pm (\varepsilon_1 + \varepsilon_2) \pm (\varepsilon_3 + \varepsilon_4) \pm (\varepsilon_5 - \varepsilon_6) \pm (\varepsilon_7 - \varepsilon_8))\right\}$,
- $\Delta_3 = \{\pm \varepsilon_i \pm \varepsilon_j : i = 3, 4; j = 5, 6\}$, $\Delta_6 = \left\{\frac{1}{2} (\pm (\varepsilon_1 - \varepsilon_2) \pm (\varepsilon_3 + \varepsilon_4) \pm (\varepsilon_5 + \varepsilon_6) \pm (\varepsilon_7 - \varepsilon_8))\right\}$,
- $\Delta_7 = \left\{\frac{1}{2} (\pm (\varepsilon_1 + \varepsilon_2) \pm (\varepsilon_3 - \varepsilon_4) \pm (\varepsilon_5 + \varepsilon_6) \pm (\varepsilon_7 - \varepsilon_8))\right\}$. 

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It is easy to check that each \( p_i \) is \( \text{Ad}(K) \)-irreducible and equivalent to the sum of four copies of standard representations of four of the seven \( su(2) \)'s, which naturally produces the notation

\[
p = p_{(1234)} \oplus p_{(1357)} \oplus p_{(1256)} \oplus p_{(2457)} \oplus p_{(3456)} \oplus p_{(1467)} \oplus p_{(2367)},
\]

where every summand has dimension 16 (see [29, pp. 579]). Note that each pair of these seven 4-tuples of indexes intersects in exactly two numbers. It turns out that the Lie bracket between any two summands only projects on the summand obtained after deleting their intersection (e.g., \([p_{(1234)}, p_{(1357)}] \subset p_{(2457)}\)) and the structural constants are all equal, say to a number \( c \).

It follows from (5) that \([L_p]_{pp} = \frac{3c}{8}\) and \([L_p]_{pq} = -\frac{c}{16}\) for all \( p \neq q \), so

\[
L_p = \frac{7c}{16} I - \frac{c}{16} \begin{bmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \end{bmatrix},
\]

and thus \( L_p \) has a unique nonzero eigenvalue \( \lambda_p = \frac{7c}{16} \) with multiplicity 6.

On the other hand, formula (11) implies that \( 2\rho = 1 - \frac{3c}{16} \). It is not hard to show that \( B_{su(2)} = \frac{1}{9} B_{e7 | su(2)} \), so \( \rho = \frac{1}{9} \) by (10), which gives \( c = \frac{16}{9} \). We conclude that

\[
\lambda_p = \frac{7}{9} > \frac{2}{3} = 2\rho,
\]

that is, \( g_B \) is \( G \)-stable.

**Remark 8.1** The Casimir operator acts on any irreducible component \( \chi_i = 4 \cdot \pi_1 + 3 \cdot \pi_0 \) of \( \chi \) as \( C_{\chi_i} = \frac{4}{9} C_{\pi_1} - B_{su(2)} = \frac{4}{9} I_{\pi_1} = \frac{1}{9} I_{\pi_1} \) (here \( \pi_1 \) and \( \pi_0 \) respectively denote the standard and trivial representations of each of the seven copies of \( su(2) \)), which confirms that \( \rho = \frac{1}{4} + \frac{1}{12} = \frac{1}{3} \) (see (6)).

### 9 Computing Einstein constants and the spectrum of \( L_p \)

There are ten spaces for which it was not necessary to know \( \rho \) nor \( \lambda_p \) to obtain their \( G \)-stability types. Except for the case Table 2, 4, they were all solved using the stability criteria given in Sect. 4. In this section, for the standard metrics on these spaces, we compute the Einstein constants and in most cases also the full spectrum of \( L_p \), as this may be useful in studying other problems.

#### 9.1 \( so(3n + 2)/so(n) \oplus u(n + 1) \)

(See Table 9, 7b, and case (ii) at the end of Sect. 4). The intermediate subalgebra \( \xi \subset h := so(n) \oplus so(2n + 2) \subset g \) gives that the only nonzero structural constant is \([122] = n(n + 1)/3 \) and so \( \lambda_p = \frac{1}{2} \) by (19).

#### 9.2 \( e_6/so(3) \oplus so(3) \oplus so(3) \)

(See Table 2, 2). Let \( h_C \) be the maximal subalgebra of \( (e_6)_C \) constructed after deleting the only node with three edges in the extended Dynkin diagram. It turns out that \( \xi \subset h \simeq \mathfrak{h} \)

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su(3) ⊕ su(3) ⊕ su(3), where \( \mathfrak{s}\mathfrak{o}(3) \subset \mathfrak{su}(3) \) is the standard embedding for each term, thus \( B_{\mathfrak{s}\mathfrak{o}(3)} = \frac{1}{6} B_{\mathfrak{su}(3)} |_{\mathfrak{s}\mathfrak{o}(3)} = \frac{1}{24} B_{\mathfrak{e}3} |_{\mathfrak{s}\mathfrak{o}(3)} \). Now formula (10) implies that \( \rho = \frac{5}{16} \).

9.3 \( \mathfrak{e}_6/\mathfrak{su}(2) \oplus \mathfrak{s}\mathfrak{o}(6) \)

(See Table 2, 4). The intermediate subalgebra \( \mathfrak{k} \subset \mathfrak{h} := \mathfrak{su}(6) \oplus \mathfrak{su}(2) \subset \mathfrak{g} \) gives that the only nonzero structural constant is \( [122] \). Since \( B_{\mathfrak{su}(6)} = \frac{1}{2} B_{\mathfrak{e}6} |_{\mathfrak{su}(6)} \) (see [29, pp.38]) and \( B_{\mathfrak{s}\mathfrak{o}(2n)} = \frac{n-1}{24} B_{\mathfrak{su}(2n)} |_{\mathfrak{s}\mathfrak{o}(2n)} \), we obtain that \( B_{\mathfrak{s}\mathfrak{o}(6)} = \frac{1}{6} B_{\mathfrak{su}(6)} |_{\mathfrak{s}\mathfrak{o}(6)} = \frac{1}{72} B_{\mathfrak{e}8} |_{\mathfrak{s}\mathfrak{o}(6)} = \frac{1}{6} B_{\mathfrak{e}6} |_{\mathfrak{s}\mathfrak{o}(6)} \), and it is not hard to see that \( B_{\mathfrak{su}(2)} = \frac{1}{6} B_{\mathfrak{e}6} |_{\mathfrak{su}(2)} \). Thus formula (10) gives that \( \rho = \frac{3}{5} \).

On the other hand, we have that \( d_1 = 20, d_2 = 40 \), thus \( [122] = 10 \) by (18). Now (19) gives that \( \lambda_p = \frac{3}{4} \).

9.4 \( \mathfrak{e}_8/\mathfrak{s}\mathfrak{o}(9) \)

(See Table 3, 9). Formula (10) gives that \( \rho = \frac{13}{40} \) by using that \( B_{\mathfrak{s}\mathfrak{o}(9)} = \frac{7}{180} B_{\mathfrak{su}(9)} |_{\mathfrak{s}\mathfrak{o}(9)} = \frac{7}{180} B_{\mathfrak{e}8} |_{\mathfrak{s}\mathfrak{o}(9)} \).

9.5 \( \mathfrak{e}_8/4 \cdot \mathfrak{su}(3) \)

(See Table 3, 12). Let \( \mathfrak{h} = \mathfrak{e}_6 \oplus \mathfrak{su}(3) \) be the maximal subalgebra of \( \mathfrak{e}_8 \). One has that \( \mathfrak{k} \subset \mathfrak{h} \) by embedding \( 3 \cdot \mathfrak{su}(3) \) into \( \mathfrak{e}_6 \) as explained in Sect. 9.2. We have that \( r = 4 \) and \( d_i = 54 \) for all \( i = 1, \ldots, 4 \). One can check that \( B_{\mathfrak{su}(3)} = \frac{1}{10} B_{\mathfrak{e}8} |_{\mathfrak{su}(3)} \) for each of the four copies and so \( \rho = \frac{19}{60} \) by (10). Via a detailed construction of the \( p_i \)'s using the root system of \( \mathfrak{e}_8 \), it can be shown that \( [iii] = \frac{36}{5} \) for all \( i \), \( [ijk] = \frac{27}{5} \) if \(#\{i, j, k\} = 3 \), and \( [ijk] = 0 \) otherwise. This in particular confirms the value \( \rho = \frac{19}{60} \) using formula (11).

It follows from (5) that \( [L_p]_{k,k} = \frac{3}{5} \) and \( [L_p]_{j,k} = -\frac{1}{5} \) for all \( j \neq k \), which gives

\[
[L_p] = \frac{4}{5} \text{Id} - \frac{1}{5} \begin{pmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1
\end{pmatrix}.
\]

Thus \( \lambda_p = \frac{4}{5} \) and has multiplicity 3.

9.6 \( \mathfrak{e}_8/4 \cdot \mathfrak{s}\mathfrak{o}(3) \)

(See Table 3, 13). Let \( \mathfrak{h} \) be the intermediate subalgebra \( 4 \cdot \mathfrak{su}(3) \) considered in Sect. 9.5. We have that \( B_{\mathfrak{s}\mathfrak{o}(3)} = \frac{1}{6} B_{\mathfrak{su}(3)} |_{\mathfrak{s}\mathfrak{o}(3)} = \frac{1}{10} B_{\mathfrak{e}8} |_{\mathfrak{s}\mathfrak{o}(3)} = \frac{1}{60} B_{\mathfrak{e}8} |_{\mathfrak{s}\mathfrak{o}(3)} \), which implies that \( \rho = \frac{11}{40} \) by (10).

9.7 \( \mathfrak{e}_8/8 \cdot \mathfrak{su}(2) \)

(See Table 4, 15). This case is very similar to the space treated in Sect. 8. If we view \( \mathfrak{k} \) as \( 4 \cdot \mathfrak{s}\mathfrak{o}(4) \), then \( c_i = \frac{1}{15} \) for all \( i = 1, \ldots, 4 \) since \( B_{\mathfrak{s}\mathfrak{o}(4)} = \frac{1}{5} B_{\mathfrak{su}(12)} |_{\mathfrak{s}\mathfrak{o}(4)} = \frac{1}{15} B_{\mathfrak{e}8} |_{\mathfrak{s}\mathfrak{o}(4)} \). According to (10), we obtain that \( \rho = \frac{3}{10} \).
The subalgebra $\mathfrak{k}$ can be described in much the same way as in Sect. 8 as a subalgebra of the maximal subalgebra $\mathfrak{so}(16)$ of $\mathfrak{e}_8$ by just adding the copy $\mathbb{C}H_{e_7+e_8} \oplus \mathfrak{g}_{e_7+e_8} \oplus \mathfrak{g}_{e_7-e_8}$. There are 14 Ad$(K)$-irreducible subspaces of the same dimension 16,

$$
\mathfrak{p}(1234), \quad \mathfrak{p}(5678), \quad \mathfrak{p}(1256), \quad \mathfrak{p}(3478), \quad \mathfrak{p}(1278), \quad \mathfrak{p}(3456), \quad \mathfrak{p}(1458), \\
\mathfrak{p}(2367), \quad \mathfrak{p}(1467), \quad \mathfrak{p}(2358), \quad \mathfrak{p}(1357), \quad \mathfrak{p}(2468), \quad \mathfrak{p}(1368), \quad \mathfrak{p}(2457),
$$
satisfying that

$$
\mathfrak{p}(a_1a_2a_3a_4), \mathfrak{p}(b_1b_2b_3b_4) \subset \begin{cases} 0 & \text{if } a_i \neq b_j \quad \forall i, j, \\
\mathfrak{p}(a_ia_jb_kb_l) & \text{if } \{i, j\} = \{h : a_k \neq b_m \quad \forall m\}, \\
\{k, l\} = \{h : b_h \neq a_m \quad \forall m\}. 
\end{cases}
$$

Moreover, all nonzero structural constant are the same, say $b$. It follows easily from (5) that $[L_p]_{j,k} = \frac{3b}{4}$ for all $k$ and, for $j \neq k$, $[L_p]_{j,k} = 0$ if $[p_j, p_k] = 0$ and $[L_p]_{j,k} = \frac{b}{16}$ otherwise. Hence

$$
L_p = \frac{3b}{4} \text{Id} - \frac{b}{16}.
$$

It is a simple matter to conclude that

$$
\text{Spec}(L_p) = \left\{ 0, \underbrace{\lambda_p, \ldots, \lambda_p}_{7\text{-times}}, \underbrace{\lambda_p^{\max}, \ldots, \lambda_p^{\max}}_{6\text{-times}} \right\} \quad \text{where} \quad \lambda_p = \frac{3b}{4}, \quad \lambda_p^{\max} = \frac{7b}{8}.
$$

On the other hand, since $\rho = \frac{3}{16}$, it follows from (11) that $b = \frac{16}{15}$ and so $\lambda_p = \frac{4}{5}$ and $\lambda_p^{\max} = \frac{14}{5}$.

### 9.8 $\mathfrak{e}_8/\mathfrak{so}(5) \oplus \mathfrak{so}(5)$

(See Table 4, 16). The intermediate subalgebra $\mathfrak{so}(5) \oplus \mathfrak{so}(5) \subset \mathfrak{so}(16) \subset \mathfrak{e}_8$ satisfies that $B_{\mathfrak{so}(16)} = \frac{7}{15} B_{\mathfrak{e}_8} |_{\mathfrak{so}(16)}$, and following the lines of [6, pp.38–40], one can prove that $B_{\mathfrak{so}(5)} = \frac{3}{24} B_{\mathfrak{so}(16)} |_{\mathfrak{so}(5)}$, concluding that $B_{\mathfrak{so}(5)} = \frac{1}{70} B_{\mathfrak{e}_8} |_{\mathfrak{so}(5)}$. Now formula (10) gives that $\rho = \frac{3}{24}$.

### 9.9 $\mathfrak{e}_8/\mathfrak{su}(3) \oplus \mathfrak{su}(3)$

(See Table 4, 17). We argue in much the same way as in the previous case with the intermediate subalgebra $\mathfrak{su}(3) \oplus \mathfrak{su}(3) \subset \mathfrak{su}(9) \subset \mathfrak{e}_8$. One has that $B_{\mathfrak{su}(9)} = \frac{3}{10} B_{\mathfrak{e}_8} |_{\mathfrak{su}(9)}$ and it can be shown that $B_{\mathfrak{su}(3)} = \frac{1}{9} B_{\mathfrak{su}(9)} |_{\mathfrak{su}(3)}$ by using [6, pp.38–40]. Thus $B_{\mathfrak{su}(3)} = \frac{1}{30} B_{\mathfrak{e}_8} |_{\mathfrak{su}(3)}$ and so $\rho = \frac{17}{60}$ by (10).
Table 9 [29, Table I A, pp. 577]. Einstein constants and spectra of $L_{p}$

| No. | $g/\mathfrak{t}$ | $\rho$ | $\lambda_p$ | $\lambda_{p}^{\text{mid}}$ | $\lambda_{p}^{\text{max}}$ | $C_1$ | $C_2$ |
|------|-----------------|--------|-------------|-------------------------|--------------------------|-------|-------|
| 1a.1 | $\mathfrak{su}(3)$ | 5 | 1 | 2 | – | – | No | No |
| 1a.2 | $\mathfrak{su}(4)$ | 3 | 1 | 2 | – | – | No | No |
| 1a.3 | $\mathfrak{su}(n)$ | $n+2$ | 1 | – | $\frac{n-1}{n}$ | No | No |
| 1b.1 | $\mathfrak{so}(6)$ | 5 | 1 | – | 2 | No | No |
| 1b.2 | $\mathfrak{so}(2n)$ | $\frac{n}{4(n-1)}$ | $\frac{n}{2(n-1)}$ | $\frac{n-2}{n-1}$ | 1 | No | No |
| 2a | $\mathfrak{su}(nk)$ | $\frac{n}{4n}$ | 1 | – | $\frac{n-1}{n}$ | No | No |
| 2b | $\mathfrak{sp}(nk)$ | $(n+2)k+2$ | $(nk+1)$ | – | $\frac{(n-1)k}{nk+1}$ | No | No |
| 2c | $\mathfrak{so}(nk)$ | $(n-2)k-4$ | $(nk-2)$ | – | $\frac{(n-1)k}{nk-2}$ | No | No |
| 3a | $\mathfrak{so}(n^2)$ | $\frac{n^2-2}{n^2-4}$ | $\frac{n^2-4}{n^2-2}$ | – | – | No | No |
| 3b | $\mathfrak{sp}(n)$ | $\frac{2n^2+2}{4n(n^2-1)}$ | – | – | No | No |
| 4 | $\mathfrak{so}(m)$ | $\frac{n}{4(m-2)}$ | $\frac{n}{2(m-2)}$ | – | 1 | No | No |
| 5 | $\mathfrak{su}(p+q)$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{m-1}$ | $\frac{m}{m+1}$ | – | No |
| 6 | $\mathfrak{sp}(sp(2n-1))$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{m+1}$ | $\frac{m}{m+1}$ | – | No |
| 7a | $\mathfrak{so}(3n+2)$ | $\frac{5}{8}$ | $\frac{5}{8}$ | $\frac{1}{2}$ | 2 | – | No |
| 7b | $\mathfrak{so}(3n+2)$ | $\frac{5}{8}$ | $\frac{5}{8}$ | $\frac{1}{2}$ | 2 | – | No |
| 8 | $\mathfrak{sp}(26)$ | $\frac{29}{40}$ | $\frac{21}{40}$ | – | – | No | No |
| 9 | $\mathfrak{su}(5)$ | $\frac{5}{12}$ | $\frac{5}{12}$ | – | – | No | No |

In case 7a, $\checkmark$ means that the criterion $C_1$ works for any $n \geq 3$, it gives $2\rho \leq \lambda_p$ for $n = 2$ and it does not work if $n = 1$. $\checkmark^*$ means that the criteria only imply that $\lambda_{p}^{\text{max}} \leq 2\rho$. In case 5, we know that the information on the spectra is valid assuming that $l_1 = l_2 = 0$ (see Theorem 7.8).

### 9.10 Full flag manifolds

(See Table 9, 1a.3, 1b.1, 1b.2 and Table 4, 18a, 18b, 18c). Let $G$ be a compact simple Lie group and consider a maximal torus $T \subset G$ with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. We study in this subsection the full flag manifold $M = G/T$. As usual, we have the root space decomposition

$$\mathfrak{g}_C = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha},$$

and for each $\alpha \in \Delta$, we take $E_{\alpha} \in \mathfrak{g}_\alpha$ such that $B(E_{\alpha}, E_{-\alpha}) = 1$ and $[E_{\alpha}, E_{-\alpha}] = H_{\alpha}$, where $H_{\alpha} \in \mathfrak{t}_C$ is defined by $B(H, H_{\alpha}) = \alpha(H)$ for all $H \in \mathfrak{t}_C$. The $-\mathfrak{g}_\theta$-orthogonal reductive decomposition for $G/T$ is given by

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{p}_\alpha, \quad (25)$$
Using that \( \rho \) is Ad(T)-invariant and irreducible and it is generated by the \(-\mathbf{B}_G\)-orthonormal basis

\[
X_1^\alpha := \frac{1}{\sqrt{2}} (E_\alpha - E_{-\alpha}), \quad X_2^\alpha := \frac{1}{\sqrt{2}} (E_\alpha + E_{-\alpha}).
\]

Note that \( G/T \) is therefore multiplicity-free, \( r = \#\Delta^+ = \frac{\dim g - \dim t}{2} \) and \( d_1 = \cdots = d_r = 2 \). Using that \( [E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} \) and that these numbers satisfy that \( N_{\alpha,\beta} = -N_{-\alpha,-\beta} \) and \( N_{\alpha,\beta} = -N_{\beta,\alpha} \), it is straightforward to show (see [3, p. 89], [24, Section 3] or [1, Proposition 2.3]) that for all \( \alpha, \beta \in \Delta^+ \)

\[
[p_\alpha, p_\beta] = p_{\alpha+\beta} \oplus p_{\alpha-\beta}, \quad (26)
\]

where \( p_\gamma = 0 \) if \( \gamma \notin \Delta \) and \( p_{-\gamma} = p_\gamma \) if \( \gamma \in \Delta^+ \). Moreover, for all \( \alpha, \beta, \gamma \in \Delta^+ \), the structural constant is given by

\[
[\alpha\beta\gamma] = \begin{cases} 2(N_{\alpha,\beta})^2 & \text{if } \gamma = \alpha + \beta, \\ 2(N_{\alpha,-\beta})^2 & \text{if either } \gamma = \alpha - \beta \text{ or } \gamma = -\alpha + \beta, \\ 0 & \text{otherwise}. \end{cases} \quad (27)
\]

It follows from (6) that \( g_B \) is Einstein on \( G/T \) if and only if all the roots have the same length, which holds precisely in the following cases:

\[
\text{SU}(n)/T^{n-1}, \quad \text{SO}(2n)/T^n, \quad E_6/T^6, \quad E_7/T^7, \quad E_8/T^8.
\]

We assume from now on that \( G/T \) is one of these spaces. Thus all the nonzero structural constants \( [\alpha\beta(\alpha \pm \beta)] \) are equal to a number \( b_\beta \) (see [24, (9)] or [1, Remark 2.4]). On the other hand, given \( \alpha \in \Delta^+ \), it is easy to check that the number

\[
\kappa_\beta := \# \{ \beta \in \Delta^+ : \alpha + \beta \in \Delta \text{ or } \alpha - \beta \in \Delta \},
\]

does not depend on \( \alpha \) and its value is as in Table 10. It follows from (11) that \( \rho = \frac{1}{2} - \frac{\kappa_\beta b_\beta}{8} \), and since alternatively, \( \rho = \frac{1}{4} + \frac{\dim t}{2d} \) by (10), we obtain the value of \( b_\beta \) as given in Table 10.

In particular, the Einstein constant \( \rho \) of \( g_B \) is respectively given by \( \frac{n+2}{2n} \cdot \frac{7}{12} \cdot \frac{15}{8} \cdot \frac{15}{8} \). We now consider the \( r \times r \) matrix \( A_g := [a_{\alpha,\beta}]_{\alpha,\beta \in \Delta^+} \), where \( a_{\alpha,\beta} = 1 \) if \( \alpha + \beta \in \Delta \) or \( \alpha - \beta \in \Delta \) and \( a_{\alpha,\beta} = 0 \) otherwise.

**Lemma 9.1** *The Lichnerowicz Laplacian of \((G/T, g_B)\) restricted to \(TT^G_{g_B}\) is given by*

\[
[L_p] = \frac{b_\beta}{2} \left( \kappa_\beta I - A_g \right).
\]

**Proof** It follows from (5) that \( [L_p]_{\alpha,\alpha} = \frac{\kappa_\beta b_\beta}{2} \) for all \( \alpha \in \Delta^+ \) and, for \( \alpha \neq \beta \) in \( \Delta^+ \),

\[
[L_p]_{\alpha,\beta} = \begin{cases} -\frac{b_\beta}{2} & \text{if } \alpha + \beta \in \Delta \text{ or } \alpha - \beta \in \Delta, \\ 0 & \text{otherwise}, \end{cases}
\]

concluding the proof. \( \square \)
In order to compute \( \text{Spec}(L_p) \), it only remains to know \( \text{Spec}(A_g) \), which has already been computed for \( \text{su}(n) \) and \( \text{so}(2n) \) in [15] and Sect. 5, respectively. The exceptional cases were worked out with the software Sage [23].

**Lemma 9.2** The spectrum \( \text{Spec}(A_g) \) of \( A_g \) is given as follows: \( 2, -1, -1 \) if \( g = \text{su}(3) \), \( \{ 4, -2, -2, 0, 0, 0 \} \) if \( g = \text{so}(6) \),

\[
\begin{align*}
&\{ 2(n - 2), n - 4, \ldots, n - 4, -2, \ldots, -2 \} \quad \text{if } g = \text{su}(n), n \geq 4, \\
&\{ 4(n - 2), 2(n - 4), \ldots, 2(n - 4), -4, \ldots, -4 \} \quad \text{if } g = \text{so}(2n), n \geq 4,
\end{align*}
\]

\( (n-1)\)-times \( n(n-3) \)-times \( n(n-3) \)-times

\[
\begin{align*}
&\{ 20, 2, \ldots, 2, -4, \ldots, -4 \} \quad \text{if } g = \epsilon_6, \\
&\{ 32, 4, \ldots, 4, -4, \ldots, -4 \} \quad \text{if } g = \epsilon_7, \\
&\{ 56, 8, \ldots, 8, -4, \ldots, -4 \} \quad \text{if } g = \epsilon_8.
\end{align*}
\]

\( 20\)-times \( 15\)-times \( 27\)-times \( 35\)-times \( 35\)-times \( 84\)-times

The following corollary follows from the above two lemmas.

**Corollary 9.3** The spectrum of the Lichnerowicz Laplacian of \( (G/T, g_B) \) restricted to \( \mathcal{T}T^G \)

\( \mathcal{G}_B \)

is given as in Table 9, 1 and Table 4, 18.

**Remark 9.4** Alternatively, the number \( b_g \) (see Table 10), which has been computed in [24] in all the classical cases, can be obtained as follows for \( \epsilon_6, \epsilon_7 \) and \( \epsilon_8 \). For \( g = \epsilon_8 \), we have

\[
\Delta(g_C, t_C) = \{ \pm \epsilon_i \pm \epsilon_j : 1 \leq i < j \leq 8 \} \cup \left\{ \frac{1}{2} \sum_{i=1}^{8} (-1)^{n(i)} \epsilon_i : \sum_{i=1}^{8} n(i) \text{ is even} \right\},
\]

and let \( \text{so}(12) \) denote the subalgebra of \( g \) attached to the subset \( \{ \pm \epsilon_i \pm \epsilon_j : 1 \leq i < j \leq 6 \} \).

It follows from [6, pp.37] that \( B_{\epsilon_7} = \frac{3}{5} B_{\epsilon_8} |_{\epsilon_7} \) and \( B_{\text{so}(12)} = \frac{5}{8} B_{\epsilon_7} |_{\text{so}(12)} \), so \( B_{\text{so}(12)} = \frac{5}{3} B_{\epsilon_8} |_{\text{so}(12)} = \frac{1}{3} B_{\epsilon_8} |_{\text{so}(12)} \).

This implies that

\[
b_g = \frac{1}{27} \sum_{i,j,k} (-3 B_{\text{so}(12)})([\sqrt{3} X_i^{\epsilon_1-\epsilon_2}, \sqrt{3} X_j^{\epsilon_2-\epsilon_3}, \sqrt{3} X_k^{\epsilon_3-\epsilon_1}])^2 = \frac{1}{3} B_{\text{so}(12)} = \frac{1}{30}.
\]

In much the same way, for \( g = \epsilon_7 \) one considers the same subalgebra \( \text{so}(12) \) and use that \( B_{\text{so}(12)} = \frac{5}{8} B_{\epsilon_7} |_{\text{so}(12)} \) and for \( g = \epsilon_6 \), the subalgebra \( \text{so}(10) \) attached to \( \{ \pm \epsilon_i \pm \epsilon_j : 1 \leq i < j \leq 5 \} \), for which one has that \( B_{\text{so}(10)} = \frac{2}{3} B_{\epsilon_6} |_{\text{so}(10)} \).

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**References**

1. Arvanitoyeorgos, A., Chrysikos, I., Sakane, Yu.: Homogeneous Einstein metrics on \( G_2/T \). Proc. Am. Math. Soc. 141, 2485–2499 (2013)
2. Besse, A.: Einstein Manifolds, Ergeb Math, vol. 10. Springer, Berlin (1987)
3. Böhm, C.: Homogeneous Einstein metrics and simplicial complexes. J. Diff. Geom. 67, 79–165 (2004)
4. Böhm, C., Wang, M.Y., Ziller, W.: A variational approach for compact homogeneous Einstein manifolds. Geom. Funct. Anal. 14, 681–733 (2004)
5. Cao, Huai-Dong., He, Chenxu: Linear stability of Perelman’s $\nu$-entropy on symmetric spaces of compact type. J. Reine Angew. Math. 709, 229–246 (2015)
6. J. D’Atri, W. Ziller, Naturally reductive metrics and Einstein metrics on compact lie groups. Mem. Am. Math. Soc. 215 (1979)
7. Dickinson, W., Kerr, M.: The geometry of compact homogeneous spaces with two isotropy summands. Ann. Global Anal. Geom. 34, 329–350 (2008)
8. Godsil, C., Royle, G.: Algebraic Graph Theory, GTM, vol. 207. Springer, New York (2001)
9. M.V. Gutiérrez, personal communication
10. He, Chenxu: Cohomogeneity one manifolds with a small family of invariant metrics. Geom. Dedicata 157, 41–90 (2012)
11. Kerr, M.: New Examples of Homogeneous Einstein Metrics. Michigan Math. J. 45, 115–134 (1998)
12. Koiso, N.: Rigidity and stability of Einstein metrics: the case of compact symmetric spaces. Osaka J. Math. 17, 51–73 (1980)
13. K. Kröncke, Stability of Einstein Manifolds, Ph.D. thesis, Universität Potsdam (2013)
14. Kröncke, K.: Stability and instability of Ricci solitons. Calc. Var. PDE. 53, 265–287 (2015)
15. J. Lauret, On the stability of homogeneous Einstein manifolds. Asian J. Math. (2022) (in press) (arXiv:2105.06336)
16. Lauret, J., Will, C.E.: Prescribing Ricci curvature on homogeneous manifolds. J. Reine Angew. Math. 783, 95–133 (2022)
17. J. Lauret, C.E. Will, On the stability of homogeneous Einstein manifolds II. J. Lond. Math. Soc. (2022) (in press) (arXiv:2107.00354)
18. Lauret, J., Will, C.E.: Homogeneous Einstein metrics and local maxima of the Hilbert action. J. Geom. Phys. 178, 104544 (2022)
19. Yu.G. Nikonorov, On a characterization of critical points of the scalar curvature functional (Russian), Tr. Rubtsovsk. Ind. Inst. 7, 211–217 (2000). Translation: arXiv:2112.00993. This article consists of [N2, Section 1.2, pp. 25–34
20. Yu.G. Nikonorov, Analytical methods in the theory of homogeneous Einstein manifolds (Russian), Dissertation (thesis) for the degree of Doctor of Physical and Mathematical Sciences, Rubtsovsk, (2002)
21. Nikonorov, Yu.G., Rodionov, E.D., Slavskii, V.V.: Geometry of homogeneous Riemannian manifolds. J. Math. Sci. (N.Y.) 146, 6313–6390 (2007)
22. G. Perelman, The entropy formula for the Ricci flow and its geometric applications. preprint 2002 (arXiv:math/0211159)
23. SageMath, the Sage Mathematics Software System (Version 9.4), The Sage Developers, 2021, https://www.sagemath.org
24. Yu. Sakane, Homogeneous Einstein metrics on flag manifolds. Lobachevskii J. Math. (1999), 71–87
25. Schwahn, P., Semmelmann, U., Weingart, G.: Stability of the Non-Symmetric Space $E_7/PSO(8)$. (2022). arXiv:2203.10138v2
26. Schwahn, P.: Stability of Einstein metrics on symmetric spaces of compact type. Ann. Global Anal. Geom. 61, 333–357 (2022)
27. Semmelmann, U., Weingart, G.: Stability of Compact Symmetric Spaces. J. Geom. Anal. 32, 137 (2022)
28. Changliang Wang, M.Y. Wang, Instability of some Riemannian manifolds with real Killing spinors, preprint 2018 (arXiv:1810.04526)
29. Wang, M.Y., Ziller, W.: On normal homogeneous Einstein manifolds. Ann. Sci. École Norm. Sup. 18, 563–633 (1985)
30. Wang, M.Y., Ziller, W.: Existence and nonexistence of homogeneous Einstein metrics. Invent. Math. 84, 177–194 (1986)
31. Wolf, J.A.: The geometry and structure of isotropy irreducible homogeneous spaces. Acta Math. 120, 59–148 (1968). (Correction: Acta Math. 152 (1984), 141–142)

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