Quantum fluctuations and Gross-Pitaevskii theory

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Using the linearized version of the time dependent Gross-Pitaevskii equation we calculate the dynamic response of a Bose-Einstein condensate to periodic density and particle perturbations. The zero temperature limit of the fluctuation-dissipation theorem is used to evaluate the corresponding quantum fluctuations induced by the elementary excitations in the ground state. In uniform conditions the predictions of Bogoliubov theory, including the infrared divergency of the particle distribution function and the quantum depletion of the condensate, are exactly reproduced by Gross-Pitaevskii theory. Results are also given for the crossed particle-density response function.

The extension of the formalism to non uniform systems is finally discussed.

Introduction.

Bogoliubov \cite{1} and Gross-Pitaevskii \cite{2,3} theories represent basic approaches to the physics of a weakly interacting Bose gas. While Bogoliubov theory is based on a quantum description where the particle operators are transformed into quasi-particle operators, allowing for an explicit diagonalization of the quantum Hamiltonian, Gross-Pitaevskii theory consists of an equation for the order parameter, a classical field associated with the spontaneous breaking of gauge symmetry.

The main purpose of the present paper is to show that the quantum fluctuations exhibited by an interacting Bose-Einstein condensate, can be properly calculated using the linearized version of the time dependent Gross-Pitaevskii equation, recovering the results of Bogoliubov theory and allowing for applications to non uniform configurations. In addition to the density fluctuations an important case considered in this work concerns the particle fluctuations whose knowledge give access to the momentum distribution and to the quantum depletion of the condensate.

We will make explicit use of the fluctuation dissipation theorem \cite{4}, which relates the fluctuations associated with a given physical operator \( \hat{F} \) to the imaginary part of the corresponding dynamic polarizability. At zero temperature the theorem takes the form (see, for example, \cite{5})

\[
\langle \{ (F^\dagger - \langle F^\dagger \rangle), (F - \langle F \rangle) \} \rangle = \frac{\hbar}{\pi} \int_{-\infty}^{+\infty} d\omega \chi''_F(\omega) \text{sign}(\omega)
\]

where \( \{ \hat{A}, \hat{B} \} \equiv \hat{A}\hat{B} + \hat{B}\hat{A} \) is the anticommutator between the two operators. Identity (1) emphasizes the quantum nature of the fluctuations \cite{6}. Equivalently, one can also write (again at zero temperature)

\[
\langle (F^\dagger - \langle F^\dagger \rangle)(F - \langle F \rangle) \rangle = \frac{\hbar}{\pi} \int_{0}^{+\infty} d\omega \chi''_F(\omega)
\]

The crucial ingredient entering Eqs. (1,2) is the dynamic polarizability, defined by the fluctuations

\[
\delta(\hat{F}^\dagger) = \lambda e^{\eta t} [e^{-i\omega t} \chi_F(\omega) + e^{i\omega t} \chi_F^*(-\omega)]
\]

induced by an external time dependent perturbation of the form

\[
H_{pert} = -\lambda e^{\eta t} [\hat{F} e^{-i\omega t} + \hat{F}^\dagger e^{i\omega t}]
\]

with \( \eta \) positive and small, ensuring that at \( t = -\infty \) the system is governed by the unperturbed Hamiltonian. Perturbation theory yields the following result for the dynamic polarizability \cite{4}:

\[
\chi_F(\omega) \equiv \chi_{F,\hat{F}} = -\frac{1}{\hbar Q^{-1}} \sum_{m,n} e^{-\beta E_m} \left\{ \frac{\langle m|\hat{F}^\dagger|n\rangle\langle n|\hat{F}|m\rangle}{\omega - \omega_{nm} + i\eta} - \frac{\langle m|\hat{F}^\dagger|n\rangle\langle n|\hat{F}^\dagger|m\rangle}{\omega + \omega_{nm} + i\eta} \right\}
\]

with \( Q = \sum_m e^{\beta E_m} \) the partition function. If the excitation operator \( \hat{F} \) does not conserve the total number of particles it is convenient to use the grand canonical formalism, adding the term \( -\mu \hat{N} \) to the unperturbed Hamiltonian.

The time dependent Gross-Pitaevskii theory is well suited to calculate the response function \( \chi(\omega) \) and consequently provides direct access to the quantum fluctuations of the operator \( \hat{F} \), through the use of Eqs. (1,2). An important example are the density fluctuations associated with the \( q \)-component \( \hat{\rho}_q = \sum_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}-\mathbf{q}} \hat{a}_{\mathbf{p}} \) of the density operator, where \( \hat{a}^\dagger \) and \( \hat{a} \) the usual creation and annihilation particle operators. In this case Eq. (1) gives access to the density fluctuations and in particular to the inelastic static structure factor

\[
S(q) = \frac{1}{N} \langle \hat{\rho}^\dagger_q \hat{\rho}_q \rangle - \frac{1}{N} \langle |\hat{\rho}^\dagger_q|^2 \rangle.
\]

Another important case that will be discussed in the paper concerns the fluctuations of the particle operator \( \hat{a}_p \), where \( \mathbf{p} \) is the momentum of the particle. In this case the left hand side of Eq.(2) allows for the calculation of the particle distribution

\[
n_p = \langle \hat{a}^\dagger_p \hat{a}_p \rangle
\]

which, in the presence of Bose-Einstein condensation, is known to exhibit an infrared divergent behavior at
small momenta [8] and whose integral provides the quantum depletion of the condensate. At first sight it may look surprising that an apparently classical approach, like Gross-Pitaevskii theory, accounts for these crucial quantum fluctuations. Actually the quantum nature of TDGP is the most famous Bogoliubov spectrum of the elementary excitations fixed by the interaction coupling constant $g$, with $n$ the density of the system, while $E_0$ is the grand canonical ground state energy, whose evaluation requires a proper renormalization of the coupling constant in order to avoid the occurrence of ultraviolet divergencies [7]. The excitation spectrum $\epsilon(p)$ exhibits the typical phononic dispersion $\epsilon(p) = cp$ at small momenta with the sound velocity given by $c = \sqrt{gn/m}$ and the single particle dispersion $p^2/2m$ at high momenta. The values of the Bogoliubov amplitudes which diagonalize the Hamiltonian, are given by

$$u_p, v_p = \pm \sqrt{\frac{p^2/2m + gn}{2\epsilon(p)}} \pm \frac{1}{2}$$

and satisfy the normalization condition $|u_p|^2 - |v_p|^2 = 1$. In the Bogoliubov approach the elementary excitation carrying momentum $p$ is created by the operator $b_p^\dagger$ applied to the ground state, which is defined as the vacuum of quasi-particles:

$$\hat{b}_p(0)_{Bog} = 0$$

for any $p \neq 0$. As a consequence, the density and particle fluctuations in the ground state are straightforwardly calculated by using the Bogoliubov transformations (11) and the commutation rule $[\hat{b}_p^\dagger, \hat{b}_q^\dagger] = 1$. For example, using the Bogoliubov prescription and approximating $N_0$ with $N$ we can write the density operator in the form $\hat{F} = \rho_{\text{q}} = \sqrt{N} (\hat{a}_p + \hat{a}_p^\dagger)$ with $p = k\mathbf{q}$, yielding the result

$$\langle \hat{a}_p^\dagger \hat{a}_p \rangle = N \frac{\hbar^2 g^2/(2m)}{\epsilon(\mu)}$$

for the density fluctuations in uniform conditions. Choosing $\hat{F} = \hat{a}_p$, with $p \neq 0$, one instead finds the result

$$n_p = \langle \hat{a}_p^\dagger \hat{a}_p \rangle = \frac{p^2/2m + gn}{2\epsilon(p)} - \frac{1}{2}$$

for the particle distribution function. Notice that $n_p$ identically vanishes in the absence of interactions ($g = 0$). It gives rise to the infrared divergent behavior [8, 9] $n_p \to mc/2p$ as $p \to 0$ and yields the result $\delta N_0/N = (\delta/3\sqrt{\pi})(na^0)^{1/2}$ for the quantum depletion of the condensate. The quantum depletion has been recently measured in a uniform 3D Bose Einstein condensed gas [10], confirming the prediction of Bogoliubov theory.

**Equation for the field operator.**

As already mentioned in the introduction, time dependent Gross-Pitaevskii theory is well suited to study the dynamic response of the system to space and time dependent external fields. In order to formulate the problem in the general context it is useful to derive the Gross-Pitaevskii theory starting from the Heisenberg equation

$$i\hbar \frac{\partial}{\partial t} \hat{\Psi}(\mathbf{r}, t) = [\hat{\Psi}(\mathbf{r}, t), \hat{H} + \hat{H}_{\text{pert}}]$$

for the time evolution of the field operator, where $\hat{H}_{\text{pert}}$ is the perturbative term (4).

The commutator involving the unperturbed Hamiltonian (9) gives the result

$$[\hat{\Psi}(\mathbf{r}, t), \hat{H}] = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}} + g\hat{\Psi}^\dagger(\mathbf{r}, t)\hat{\Psi}(\mathbf{r}, t) - \mu \right] \hat{\Psi}(\mathbf{r}, t)$$
where, for sake of generality, we have included an external trapping potential.

In order to include the effect of the perturbation it is convenient to write \( \hat{H}_{\text{pert}} \) in terms of the field operator. In the case of the coupling with the \( q \)-component of the density operator one writes \( \tilde{F} = \rho_\alpha = \int d\mathbf{r} \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}) e^{i \mathbf{q} \cdot \mathbf{r}} \) and the relevant commutator takes the form

\[
[\hat{\Psi}(\mathbf{r}, t), \hat{H}_{\text{pert}}] = -\lambda \hbar e^{it} \left( e^{-i\mathbf{q} \cdot \mathbf{r} - \omega t} + e^{i\mathbf{q} \cdot \mathbf{r} - \omega t} \right) \hat{\Psi}(\mathbf{r}, t).
\]

For the density response in Gross-Pitaevskii theory

\[
\langle \hat{H}(t) \rangle = \lambda \left\langle \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \right\rangle e^{i \mathbf{q} \cdot \mathbf{r} - \omega t}.
\]

We are now ready to study the response function in the framework of Gross-Pitaevskii theory, where the field operator is replaced by a classical field, following the Bogoliubov prescription.

**Density response in Gross-Pitaevskii theory**

By replacing the field operator \( \hat{\Psi}(\mathbf{r}) \) with the classical field \( \Psi(\mathbf{r}) \) in Eqs. (18,19,20) one obtains the time dependent Gross-Pitaevskii equation

\[
i \hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}} + \mu \right) \Psi(\mathbf{r}, t) - \lambda \sqrt{n(\mathbf{r})} e^{it} \left( e^{i\mathbf{q} \cdot \mathbf{r} - \omega t} + e^{-i\mathbf{q} \cdot \mathbf{r} - \omega t} \right)
\]

in the presence of the density perturbation, where in the last term of the equation we have taken the unperturbed value \( \Psi(\mathbf{r}, t) = \sqrt{n(\mathbf{r})} \), consistent with the rules of perturbation theory.

In uniform conditions the ansatz

\[
\Psi(\mathbf{r}, t) = \Psi_0 + e^{it} \left( u e^{i\mathbf{q} \cdot \mathbf{r} - \omega t} + v^* e^{-i\mathbf{q} \cdot \mathbf{r} - \omega t} \right)
\]

solves the time dependent Gross-Pitaevskii equation both in the absence and in the presence of the external density perturbation. In the above equation \( \Psi_0 \) is the order parameter calculated at equilibrium. In the absence of the external perturbation one finds the well known oscillating solutions with frequency

\[
\omega(q) = \sqrt{\frac{gnq^2 + \hbar^2 (q^2/2m)^2}{m}}.
\]

This result is fully consistent with the dispersion relation (13) predicted by Bogoliubov theory, after adopting the de Broglie quantization rules \( \epsilon(p) = \hbar \omega(q) \) and \( p = \hbar q \). In the presence of the periodic density perturbation, Eq. (22) can be also solved analytically, yielding the following result for the amplitudes \( u \) and \( v \):

\[
u = -\lambda \sqrt{n} \frac{\hbar \omega - p^2/2m}{(\hbar \omega + i\eta)^2 - \epsilon^2(p)}
\]

where \( \epsilon(p) \) is the Bogoliubov dispersion dispersion law (13) and we have used \( \mu = gn \), with \( n = N/V \) the density of the system. Evaluating the variation \( \delta n \) = \( \sqrt{\pi} \int d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r} - \omega t} \delta \hat{\rho}(\mathbf{r}, t) \) induced by the perturbation and using definition (3), we finally obtain the result

\[
\chi_{\text{density}}(\mathbf{q}, \omega) = -N \frac{p^2/m}{(\hbar \omega + i\eta)^2 - \epsilon^2(p)}
\]

for the density-density response function of the uniform gas (\( p = \hbar q \)). By taking the imaginary part of the response function and using the fluctuation-dissipation theorem (1) one immediately recovers the Bogoliubov result (16) for the density fluctuations. Result (26) keeps the same form in the canonical and in the grand canonical formalism since the excitation operator \( \hat{\rho}_\alpha \) commutes with \( \hat{N} \). In the canonical case the ansatz for the order parameter satisfying the time-dependent Gross-Pitaevskii equation is simply obtained by multiplying Eq.(23) by the factor \( \exp(-i\mu t) \).

**Particle response in Gross-Pitaevskii theory**

By replacing the field operator \( \hat{\Psi}(\mathbf{r}) \) with the classical field \( \Psi(\mathbf{r}, t) \) in Eqs. (18,19,21) one instead obtains the time dependent Gross-Pitaevskii equation

\[
i \hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}} + g |\Psi(\mathbf{r}, t)|^2 - \mu \right) \Psi(\mathbf{r}, t) - \lambda \frac{1}{(2\pi\hbar)^{3/2}} e^{-i\mathbf{p} \cdot \mathbf{r} - \omega t} e^{i\mu t}.
\]

accounting for the coupling with the \( p \)-component \( \hat{F} = \hat{\psi}(\mathbf{p}) \) of field operator. We can still use the ansatz (23) to solve the GP equation and in this case we obtain the following result for the amplitudes \( u \) and \( v \):

\[
u = \frac{\lambda \hbar \omega - p^2/2m - gn}{(2\pi\hbar)^{3/2} (\hbar \omega + i\eta)^2 - \epsilon^2(p)}.
\]

The response function is then determined by evaluating the fluctuations induced in the \( p \)-component of the classical field \( \delta \Psi^\dagger(\mathbf{p}, t) = (2\pi\hbar)^{-3/2} \int d\mathbf{r} e^{-i\mathbf{p} \cdot \mathbf{r} - \omega t} \hat{\rho}_\alpha(\mathbf{r}, t) \). In uniform conditions it is convenient to write the \( p \)-component \( \hat{\psi}(\mathbf{p}) \) of the field operator in terms of the particle annihilation operator as

\[
\hat{\Psi}(\mathbf{p}) = \frac{\sqrt{V}}{(2\pi\hbar)^{3/2}} \hat{\rho}_\alpha \tag{29}
\]

so that the response function \( \chi_{\text{field}}(\mathbf{p}, \omega) \) relative to field operator \( \hat{F} = \hat{\psi}(\mathbf{p}) \) in momentum space can be expressed
in terms of the response function $\chi_{\text{particle}}(p, \omega)$ relative to the particle operator $\hat{F} = \hat{a}_p$ as

$$\chi_{\text{field}}(p, \omega) = \frac{V}{(2\pi\hbar)^3} \chi_{\text{particle}}(p, \omega).$$  \hspace{1cm} (30)

Using results (28) for $u$ and $v$ one finally finds the following result for the particle response function:

$$\chi_{\text{particle}}(p, \omega) = \frac{\hbar \omega - p^2/2m - gn}{(\hbar + i\eta)^2 - \epsilon^2(p)} = -\frac{1}{2\epsilon(p)} \left( \frac{p^2/2m + gn - \epsilon(p)}{\hbar + i\eta - \epsilon(p)} - \frac{p^2/2m + gn + \epsilon(p)}{\hbar + i\eta + \epsilon(p)} \right),$$  \hspace{1cm} (31)

yielding the expression

$$A(p, \omega) = \frac{1}{2\epsilon(p)} \left[ (p^2/2m + gn - \epsilon(p)) \delta(\hbar \omega - \epsilon(p)) - (p^2/2m + gn + \epsilon(p)) \delta(\hbar \omega + \epsilon(p)) \right]$$  \hspace{1cm} (32)

for the spectral function, corresponding to the imaginary part of $\chi$. Result (31) shows that in the grand canonical formalism, the particle response function shares the same poles of the density response function (26). Equations (31,32) can be easily recast in the canonical form, by simply replacing the frequency $\omega$ with $\omega + \mu$. This reflects the fact that the operator $\hat{a}_p^\dagger$ ($\hat{a}_p$) add (remove) a particle, in addition to exciting an elementary mode in the system. In the canonical formalism the solution for the order parameter would actually take the form

$$\Psi(r, t) = e^{-i\mu t} \Psi_0 + e^{i\eta} \left( u e^{-2i\mu t} e^{i(q \cdot r - \omega t)} + v^* e^{-i(q \cdot r - \omega t)} \right).$$  \hspace{1cm} (33)

For large values of $\omega$ the response function approaches the value $1/(\hbar \omega)$ in agreement with the general result

$$\chi_F(\omega \rightarrow \infty) = \frac{1}{\hbar \omega} \langle [\hat{F}, \hat{F}^\dagger] \rangle$$  \hspace{1cm} (34)

holding for the dynamic polarizability in the large $\omega$ limit [5], involving the commutator between $\hat{F}$ and $\hat{F}^\dagger$ (see Eq.(6).

Using the fluctuation dissipation theorem (2) one exactly recovers the result (17) predicted by Bogoliubov theory for the particle distribution function, characterized by the infrared divergence $n_p \rightarrow mc/p$ at small $p$ and accounting for the quantum depletion of the condensate.

Analogously, one can also derive the expression for the mixed particle-density response function, providing the fluctuations induced in the average of the particle operator $\hat{a}_p^\dagger$ by the presence of an external perturbation coupled to the density operator $\hat{\rho}_q$ with $q = p/\hbar$. Such a perturbation modifies the wave function of the condensate according to Eqs. (23,25). One finally finds

$$\chi_{\text{particle-density}}(p, \omega) = -\frac{1}{\hbar} \sum_n \left[ \frac{(0|\hat{a}_p^\dagger n|n)\hat{\rho}_q 0}{\omega - \omega + \eta} - \frac{(0|\hat{\rho}_q n|n\hat{a}_p^\dagger 0) 0}{\omega + \omega + \eta} \right] = \sqrt{N} \frac{\hbar \omega - p^2/2m}{(\hbar \omega + \eta)^2 - \epsilon^2(p)}. \hspace{1cm} (35)$$

Result (35) is consistent with the large $\omega$ result $\chi_{\text{particle-density}} \rightarrow -\langle [\hat{a}_p^\dagger, \hat{\rho}_q] \rangle/(\hbar \omega)$, derivable from sum rule arguments [11]. In the canonical formalism the physical meaning of Eq. (35) would correspond to replacing the operator $\hat{a}_p^\dagger$ with the number conserving operator $\hat{a}_p^\dagger \hat{a}_0/\sqrt{N_0}$.

In the static limit the result $\chi_{\text{particle-density}}(p, \omega = 0) = \sqrt{N}(p^2/2m)/\epsilon^2(p)$ can be used to investigate the effect of a static periodic perturbation of the form $H_{\text{pert}} = -\lambda(\hat{\rho}_q + \hat{\rho}_{-q}) = -2\lambda \sum_j \cos(qz_j)$ on the momentum distribution of the system, which turns out to be characterized by the occurrence of the macroscopic occupation $N_p = N(\lambda(p^2/2m)/\epsilon^2(p))^2$ of the single particle state with momentum $p$ (and analogously for $-p$). The effect should be easily observable experimentally also for relatively small values of the coupling $\lambda$ in systems exhibiting a pronounced roton minimum as happens, under proper conditions, in the case of long-range dipolar interactions [12–14]. The coupling between density and particle excitations accounted for by Eq. (35) reflects a peculiar property of a Bose-Einstein condensate and disappears in the absence of coherence, as proven experimentally for large intensities of the external density coupling when the system enters the insulator phase [15].

**Response function in non uniform systems**

The above results can be straightforwardly generalized to the case of a non uniform trapped Bose-Einstein condensed gas, where the Hamiltonian contains an external static potential $V_{\text{ext}}$. In this case the density response function takes the form:

$$\chi_{\text{density}}(q, \omega) = -\sum_n \int dr (u_n(r) + v_n(r)) \Psi_0(r) e^{-iqr} \frac{| e^{-iqr} |^2}{(\hbar \omega + i\eta)^2 - \epsilon_n^2}, \hspace{1cm} (36)$$

while the response to the field operator $F = \hat{\Psi}(p)$ in momentum space reads

$$\chi_{\text{field}}(p) = -\frac{1}{(2\pi\hbar)^3} \sum_n \int dr u_n(r) e^{-i pr/\hbar} \frac{| e^{-iqr} |^2}{(\hbar \omega + i\eta) - \epsilon_n} - \int dr v_n(r) e^{-i pr/\hbar} \frac{| e^{-iqr} |^2}{(\hbar \omega + i\eta) - \epsilon_n}. \hspace{1cm} (37)$$

In both Eqs. (36) and (37) $u_n$, $v_n$ and $\epsilon_n$ are provided by the solutions of the coupled Gross-Pitaevskii equations

$$\epsilon_n u_n = (\hbar 0 - \mu + 2gn(r)) u_n(r) + gn(r) v_n(r),$$  \hspace{1cm} (38)

$$-\epsilon_n v_n = (\hbar 0 - \mu + 2gn(r)) v_n(r) + gn(r) u_n(r)$$.  \hspace{1cm} (39)
which are the analogs of the Bogoliubov equations of uniform matter and the sum over \( n \) includes all the excitations of the system. Here \( H_0 = -(\hbar^2/2m)\nabla^2 + V_{ext} \) is the single-particle Hamiltonian. The amplitudes \( u_n \) and \( v_n \) satisfy the ortho-normalization condition \( \int d\mathbf{r}(u_n^* u_m - v_n^* v_m) = \delta_{nm} \) and in a uniform gas take the form \( u_n = u_k e^{i k \mathbf{r}}, \) and analogously for \( v_n \).

Equation (37), together with result (2), allows for the calculation of the momentum distribution

\[
n(p) = \langle \hat{\Psi}^\dagger(p) \hat{\Psi}(p) \rangle = \frac{1}{(2\pi\hbar)^3} \sum_n \int d\mathbf{r} v_n(\mathbf{r}) e^{-i p \cdot \mathbf{r}/\hbar} \langle \hat{\Psi}(\mathbf{r}) \rangle^2.
\]

(39)

In addition to the mean field contribution \( |\Psi_0(p)|^2 \), fixed by the Fourier transform \( \Psi_0(p) = (2\pi\hbar)^{-3/2} \int d\mathbf{r} e^{i p \cdot \mathbf{r}/\hbar} \Psi_0(\mathbf{r}) \) of the order parameter at equilibrium and providing the leading contribution for \( p < \hbar/R \) with \( R \) the typical size of the condensate, Eq. (39) accounts for the quantum fluctuations caused by the elementary excitations of the system and provides the leading contribution at larger values of \( p \). The experimental determination of \( n(p) \) for large values of \( p \) has been the object of a recent time-of-flight experimental investigation [16]. The presence of interactions during the expansion does not however allow, in this experiment, for a safe identification of the in-situ momentum distribution [17].

Another instructive example concerns the calculation of the fluctuations of the field operator \( \hat{F} = \hat{\Psi}(\mathbf{r}) \) in coordinate space. In this case one finds the result

\[
n(r) = \langle \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \rangle = |\Psi_0(\mathbf{r})|^2 + \sum_n |v_n(\mathbf{r})|^2,
\]

(40)

which provides a natural decomposition of the density into the Gross-Pitaevskii value \( |\Psi_0(\mathbf{r})|^2 \) and the contribution arising from the fluctuations of the condensate. In uniform configurations the values of \( v_{\mathbf{p}} \) are fixed by Eq.(25) and the decomposition corresponds to writing \( N = N_0 + \delta N_0 \) with \( \delta N_0 = \sum_{\mathbf{p}} |v_{\mathbf{p}}|^2 = N(8/3\sqrt{\pi})(n_0^3)^{1/2} \). In non uniform configurations the use of Eq.(40) requires more careful considerations. In fact while the fluctuations of the field operator are proportional to the perturbation parameter that scales as \( a^{3/2} \), the order parameter \( \Psi_0 \), calculated in GP theory, ignores corrections of the same order arising from the renormalization of the coupling constant, as predicted by the theory of Lee-Huang-Yang [7]. By evaluating the order parameter \( \Psi_0 \) using the Gross-Pitaevskii theory in the Thomas-Fermi (LDA) approximation, one can in fact easily show that the prediction of (40) differs from the total density derivable by including the LHY correction in the equation of state [18] (see also [5], Sect. 11.5). It is worth noticing that both the LHY and the fluctuation correction affect the density profile in the same physical region where \( r < R_{TF} \) and the density significantly differs from zero. This differs from the case of the momentum distribution where, as already pointed out, the fluctuations of the condensate modify the momentum distribution in the region \( p > \hbar/R_{TF} \) where the value of \( \Psi_0(\mathbf{p}) \) is negligible.

**Conclusions.** In conclusion we have shown that the use of the \( T = 0 \) limit of the fluctuation dissipation theorem allows for the calculation of the quantum fluctuations of both the density and particle operators of a Bose-Einstein condensed gas, employing the time dependent Gross-Pitaevskii equation for the wave function of the condensate, a classical field describing the order parameter of the system. This approach enlightens the deep equivalence between the Bogoliubov and Gross-Pitaevskii approaches, despite their different theoretical formulation. The suitability of the GP approach to describe non uniform configurations might offer novel possibilities for investigating the nature of the fluctuations in the presence of quantum defects, like solitons and quantized vortices.

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