Destabilization of the U(1) Dirac spin liquid phase on the triangular lattice by quenched disorder

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It has been recently demonstrated [Song et al., Nature Comm. 10, 4254 (2019), Iqbal et al., Phys. Rev. B 93, 444411 (2016)] that the staggered π-flux Dirac spin liquid phase on the non-bipartite triangular lattice may be stable in the clean limit. However, quenched disorder plays a crucial role in determining whether such a phase can be experimentally viable. The effective low-energy description of Dirac spin liquids in (2 + 1) dimensions is given by the compact quantum electrodynamics (cQED_{2+1}) which admits monopoles. It is already known that generic quenched random perturbations to the non-compact version of QED_{2+1} (where monopoles are absent) lead to strong-coupling instabilities. In this work we study cQED_{2+1} in the presence of a class of time-reversal invariant quenched disorder perturbations. We show that in this model, random non-abelian vector potentials make the symmetry-allowed monopole operators more relevant. The disorder induced under-screening of monopoles thus generically makes the gapless spin liquid phase fragile.

I. INTRODUCTION

Quantum spin-liquids with their topologically ordered ground state, fractionalized excitations and long-range entanglement offer a fascinating insight into many-body quantum correlations\textsuperscript{1-3}. Experimentally, the observation of a spin-liquid phase has been fraught with the complications arising from spatial inhomogeneities in real materials, which often leads to symmetry breaking towards a spin glass ground state\textsuperscript{4-6}. While the role of quenched disorder in a frustrated spin system may vary considerably\textsuperscript{7-10}, in a number of examples\textsuperscript{11-13} it has been shown that the topological properties of frustrated systems are considerably affected by quenched disorder. In this paper we consider the gapless Dirac spin liquid (DSL) state with 2N flavors of matter fermions and compact U(1) gauge symmetry and investigate its stability in the presence of random gauge fluctuations.

As a prototypical spin liquid state with linearly dispersing, gapless, fractionalized spinons and minimally coupled compact U(1) gauge fields, the DSL state has been discussed in the context of high-T\textsubscript{c} cuprates\textsuperscript{14}, as a parent state for different competing orders\textsuperscript{15}, deconfined quantum critical points between topological phases\textsuperscript{16} and as the prospective ground state of the kagome lattice Heisenberg model\textsuperscript{17,18} and the triangular lattice Heisenberg model with next-nearest neighbor exchange interaction\textsuperscript{19}. The variational DSL state can be derived from the mean-field decomposition of an SU(2) Heisenberg Hamiltonian, \[ H = J \sum_{\langle ij \rangle} S_i^z \cdot S_j^z \] in terms of fermionic spinons. In this picture, a spin-1/2 operator at site \( i \) is rewritten as \( \bar{S}_i = (1/2) f_{i,\alpha}^\dagger \sigma_{\alpha\beta} f_{i,\beta} \) modulo the physical constraint \( \sum_\alpha f_{i,\alpha}^\dagger f_{i,\alpha} = 1 \). Here \( f_{i,\alpha} \) are fractionalized fermionic spinons with \( \alpha = \uparrow, \downarrow \) being the spin indices. The mean-field decomposition with bond variables \( t_{ij} = -(1/2) (f_{i,\alpha}^\dagger f_{j,\alpha}) \) reduces the Hamiltonian to \( H_{\text{MF}} = J \sum_{\langle ij \rangle} t_{ij} f_{i,\alpha}^\dagger f_{j,\alpha} + h.c. \), with the mean-field ansatz of bond variables \( t_{ij} \) chosen suitably to minimize the variational energy. In the spinon decomposition, the compact U(1) gauge symmetry is manifest with the transformation \( f_{i,\alpha} \rightarrow e^{iA_i} f_{i,\alpha} \), which ultimately leads to the emergence of dynamical U(1) gauge fields. In the triangular lattice the mean-field ansatz \( t_{ij} = \pm 1 \) with no fluxes through the lower triangular plaquettes, \( \prod_{\langle ij \rangle \in \mathcal{V}} t_{ij} = 1 \) and \( \pi \) fluxes through the upper triangular plaquettes \( \prod_{\langle ij \rangle \in \mathcal{A}} t_{ij} = -1 \) yield a DSL state with four gapless Dirac cones, i.e two Dirac nodes (valleys) for each spin flavor. In the long-wavelength and low-energy limit the DSL state and its gauge fluctuations are described by the action of the 2 + 1 dimensional compact quantum electrodynamics\textsuperscript{15,20-23}.

\[ S_{\text{cQED}} = \int d\tau d^2r \left[ \bar{\psi}_i \gamma^\mu (\partial_\mu + iA_\mu) \psi_i + \frac{1}{4e^2} F_{\mu\nu}^2 \right], \] where \( \psi \) are \( 2N \) copies of two-component fermionic fields which are descendents of the fermionic spinors \( f_{i,\alpha} \), with \( i \in 1, \ldots, 2N \) and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the usual field strength tensor for a Maxwell gauge theory. The number \( N \) is determined by the number of Dirac nodes of the microscopic dispersion per spin, i.e. \( N = 2 \) for triangular lattice DSL. In the following, we shall suppress all the fermionic flavor indices. Here the three Dirac gamma matrices \( \gamma^\mu \) are taken to be two-component\textsuperscript{24,25} and they obey the usual Clifford algebra, \( \{ \gamma^\mu, \gamma^\nu \} = 2g^\mu_\nu I_3 \). The gauge charge has scaling dimension \( [e^2] = +1 \) and it flows to infinity in the deep infrared. Consequently, the infrared fixed point of the action as written has conformal symmetry in the large-\( N \) limit\textsuperscript{23}. The action also has an emergent SU(2N) symmetry under which the fermions \( \psi \) transform as vectors.

From the lattice regularization, the gauge fields \( A_\mu \) are periodic in the clean limit and therefore it can be shown that the cQED\textsubscript{2+1} action must admit monopole operators of charge \( q \) which insert \( 4\pi q \) units of magnetic flux locally\textsuperscript{26-28}. It was shown originally by Ref. 24 that the proliferation of these monopoles strongly confines the electric charges of the pure compact Maxwell gauge theory. As the spinons carry U(1) gauge charge, the fate of the compact U(1) gauge theories minimally coupled to gapless fermion spinons with Dirac dispersion has been controversial\textsuperscript{27,28}, but it has been demonstrated that there is indeed a stable deconfined phase for a large number for fermionic flavors\textsuperscript{23,25}. In particular, within a large 2N approximation the scaling dimension of the
monopole operators are found to be of the form
\[ \Delta_{M(q)} = (2N)\lambda_0^{(q)} + \lambda_1^{(q)} + O(1/2N). \]

For the lowest charge \( q = 1/2 \) monopole operators, computation using state-operator correspondence\cite{26,30} have yielded, \( \lambda_0^{(1/2)} = 0.265 \) and \( \lambda_1^{(1/2)} = -0.0383 \). Therefore, if the monopole operators of the lowest charge are allowed, the minimum number of fermionic flavors needed to avoid confinement is \( 2Nc \geq 12 \). This number is more than the number of fermionic flavors obtained in the known mean-field DSL states (\( 2N = 4 \) for the kagome\cite{18} and triangular lattice DSL\cite{19}; and \( 2N = 4 \) and \( 2N = 8 \) respectively for the staggered and \( \pi \) flux DSL states in square-lattices\cite{15,31}). However, for the Dirac spin liquid states in non-bipartite triangular and Kagome lattice geometries, it has been recently shown that the monopole operators of the lowest charges are prohibited by lattice symmetries\cite{12,32}. Ref. 22 demonstrated that for the triangular lattice Dirac spin-liquid only monopole operators with charges \( q \geq 3/2 \) are allowed by microscopic symmetries and these higher charge monopole operators are all irrelevant if the large \( 2N \) approximated monopole scaling dimension \( \Delta_{M(q)}\) is extrapolated to \( 2N = 4 \). This indicates the possibility of a stable deconfined DSL phase in the triangular lattice. The same analysis found that for the kagome lattice the smallest allowed monopole operators are very close to being marginal but relevant within the large \( 2N \) approximation. Indeed, on the triangular lattice next-nearest neighbor \( J_1 - J_2 \) Heisenberg model a DSL phase is observed in variational Monte Carlo simulations\cite{19,33} and density matrix renormalization (DMRG) calculations\cite{34}. A spin liquid phase was found to be stable for \( 0.07 < J_2/J_1 < 0.15 \) by a separate DMRG study\cite{35}. In other reports\cite{36,37}, a chiral spin liquid is found in the same parameter range in the presence of a time-reversal symmetry breaking perturbation, which is consistent with a viable DSL phase in the time-reversal symmetric limit.

In our treatment, we examine the fate of the cQED\(_{2+1}\) action [Eq. (1)] as an effective theory of the DSL in the presence of time-reversal symmetric microscopic perturbations. Microscopically, triangular lattice DSLs shall be our focus as a promising candidate in the clean limit. Theoretical efforts\cite{38,39,40} to study QED\(_{2+1}\) in the presence of various quenched random perturbations has so far been focused on the non-compact limit which neglects the monopole operators. It has been established that random perturbations which break the time-reversal symmetry and/or break completely the emergent SU(2) symmetry of the cQED\(_{2+1}\) action drive a renormalization group (RG) flow to a strong disorder coupling fixed point, which in the microscopic sense indicates the destruction of the spin liquid phase\cite{18}. In this paper, we therefore focus on the compact nature of the effective theory and perturbatively calculate the disorder induced modification to the scaling dimension of the monopole operator to further clarify the fate of the algebraic DSL phase.

In Sec. II we introduce the RG marginal random couplings that we consider as a perturbation to the \( e^2 \to \infty, N \to \infty \) conformal fixed point of the theory and discuss their microscopic origin. Adapting the state-operator correspondence method described in Sec. III, in Sec. IV we calculate the scaling dimension of the monopole operators in the dirtied cQED\(_{2+1}\) within a controlled expansion in large-\( N \) and perturbative disorder strength. We find that disorder significantly reduces the scaling dimension of the monopole operators and enhances the possibility of confinement of the spinons which carry electric gauge charges. In Sec. V we consider the combined flow of the monopole fugacity and the perturbative disorder couplings and show that even when disorder in itself remains marginal, the monopole fugacity may flow to strong coupling and confine the theory. In the concluding Sec. VI, we comment on the instabilities introduced by disorder driven spinon confinement within the context of the DSL phase and argue why among other possibilities a glassy random-singlet like ground state is a likely outcome for even small to moderate disorder in this scenario.

II. QUENCHED DISORDER IN cQED\(_{2+1}\)

The QED\(_{2+1}\) action is an effective low-energy description and the spatial inhomogeneities in the lattice translate to random coupling perturbations to the theory. Ref. 38 showed that there are no relevant random perturbations to QED\(_{2+1}\) in the large \( 2N \) limit and the only marginal random couplings are the various conserved currents and mass operators associated with the SU(2N) symmetry of the fermions\cite{38}. In our discussion, we choose \( \sigma^\alpha \) and \( \tau^b \) as the \( 2N^2 - 1 \) generators of SU(2\( N \)) where \( \sigma^\alpha \) are \( 2 \times 2 \) Pauli matrices with \( \alpha = x, y, z \) and \( \tau^b \) are \( N^2 - 1 \) traceless \( N \times N \) Hermitian matrices with the normalization \( \text{tr} [\sigma^\alpha \sigma^{\alpha'}] = \delta^{\alpha \alpha'}/2 \). The generators satisfy the usual commutation relations \( [\sigma^\alpha, \sigma^\beta] = i\epsilon_{\alpha\beta\gamma} \gamma \) and \( [\tau^b, \tau^c] = i\epsilon_{abc} \tau^e \), where \( \epsilon_{abc} \) are structure constants of the corresponding SU(2N) Lie-algebra. In this notation \( \sigma^\alpha \) operate on the spin-space and \( \tau^b \) operate on the fermion-doubled valley-space originating from the Dirac node structure of the parent mean-field state. Associated with these symmetry generators are SU(2N) current,

\[
J^{\alpha\mu} = i\bar{\psi}\gamma^{\mu}\sigma^\alpha\gamma_\mu\psi, \quad J^{\alpha0} = i\bar{\psi}\sigma^\alpha\gamma_\mu\bar{\psi}, \quad J^{0\mu} = i\bar{\psi}\tau^b\gamma_\mu\psi
\]

and mass terms,

\[
M^{\alpha0} = \bar{\psi}\sigma^\alpha\gamma^\nu, \quad M^{\alpha0} = \bar{\psi}\sigma^\alpha\psi, \quad M^{0b} = \bar{\psi}\tau^b\psi.
\]

It is to be noted that terms not containing \( \sigma^\alpha \) are related to the spin-singlet local (bilinear) operators of the microscopic model whereas the rest maps to the spin-triplet operators. In the clean limit the conserved SU(2N) currents, e.g. \( i\bar{\psi}\sigma^\alpha\gamma^\mu\psi \) have the scaling dimension \( \Delta = 2 \) to all order in \( 1/(2N) \) but the SU(2N) mass terms e.g. \( i\bar{\psi}\sigma^\alpha\psi \) acquire anomalous scaling dimensions \( \propto 1/(2N) \). Let us consider quenched random coupling to one such operator \( O(\bar{\tau}^\alpha \tau) \) such that the
perturbation action is \( S_{\text{dis}} = \int d\tau d^2r \ h(\vec{r}) \mathcal{O}(\vec{r}, \tau) \) with uncorrelated random conjugate fields \( h(\vec{r}) h(\vec{r'}) = \rho_0 \delta^{(2)}(\vec{r} - \vec{r'}) \). Following standard replica technique, \( T = \log Z = \lim_{n \to 0} (Z^n - 1)/n \sim \lim_{n \to 1} \prod_{r=1}^n Z_r \), a replicated partition sum emerges,

\[
Z_{\text{replica}} = \int \mathcal{D}[\psi_r, A_r] \exp \left( -\sum_r \int d\tau d^2r \ \psi_r [\partial^2 + i A_r] \psi_r, \rho_0 \sum_r \int d\tau d^2r' \ \mathcal{O}_r(\vec{r}, \tau) \mathcal{O}_s(\vec{r'}, \tau) \right).
\]

(5)

From power-counting it clearly follows that \( \Delta_{\rho_0} = 2 + 2z - 2\Delta_\mathcal{O} \) where \( z = -|\tau| \) is the dynamical critical exponent. Therefore in the large-\( N \) limit when \( z = 1 \), random couplings to the various SU(2\(N \)) current and mass terms are marginal at the tree level. Similarly, the random couplings to simple mass terms \( \sim \bar{\psi} \psi \) are also RG marginal but such mass terms break time reversal symmetry in (2 + 1) dimension. Quenched disorder breaks Lorentz invariance and consequently the scaling dimension of both the random SU(2\(N \)) mass and current disorder couplings are modified beyond the tree level. In the absence of monopoles, previous works\(^{38,40,42} \) have established that if such random couplings break the fermionic SU(2\(N \)) symmetry or the time-reversal symmetry the combined RG flow generically moves to a strong coupling fixed point. However, Ref.\(^{38} \) has shown that for time reversal symmetric random perturbations, if the symmetry is only partially broken to U(1) \( \times \) SU(N), a finite disorder conformal fixed line is obtained, parametrized by the corresponding coupling strengths. Technically this fixed line is demarcated by the breakdown of the microscopic SU(2) symmetry down to U(1). The scenario we discuss presently may diverge from this narrative due to the inclusion of monopoles.

In keeping with the goal of calculating the scaling dimension of the monopole operators by invoking the state-operator correspondence of radial quantization\(^{43} \), we shall presently only consider the SU(N) symmetric, random vector potential (RVP) perturbations,

\[
S_{\text{dis}} = \int d\tau d^2r \ V_{\alpha j}(\vec{r}) \ i\bar{\psi} \sigma^\alpha \gamma_j \psi(\vec{r}, \tau),
\]

\( P[V] = e^{-\frac{1}{\rho_0} \int d^2r \ V_0^2(\vec{r})}, \)

(6)

where a Gaussian distribution for the conjugate random field \( V_{\alpha j} \) has been considered for convenience with \( \rho_0 \) being the corresponding disorder strength. Also, the index \( j \) of Dirac matrices here runs strictly over the spatial components. The cases of scalar potentials with \( \gamma^a \) and random SU(2\(N \)) mass terms have been left out of our current study as these terms introduce non-trivial infrared divergences in the disorder averaged radially quantized version of the theory.

Microscopically, the time-reversal invariant local random perturbations are usually either random bond type, \( \mathcal{P}_j = \vec{S}_i \cdot \vec{S}_j \) or vector chirality type, \( \mathcal{C}_{ij} = \vec{S}_i \times \vec{S}_j \). The former behaves as scalars in spin-space and therefore is associated with the spin-singlet mass and current terms, \( J_\mu^{(0)}, M^{(0)} \) and random U(1) vector potentials. While random abelian vector potential is known to be\(^{38,40} \) an irrelevant perturbation for non-compact QED\(_{2+1} \), Ref.\(^{38} \) showed that the random spin-singlet SU(2\(N \)) current and mass terms are however relevant perturbation, and therefore it can be surmised that random bond like perturbations are destructive to the DSL phase. On the other hand, the same treatment revealed the presence of a fixed line for U(1) \( \times \) SU(N) symmetric random couplings to the terms \( J_\mu^{(0)} \) and \( M^{(0)} \), the former of which we presently consider. In (2 + 1) dimension random the random current terms, \( J_\mu^{(0)} = i\bar{\psi} \gamma^\mu \gamma_5 \psi \) preserve the time reversal symmetry\(^{38} \). Additionally, these current terms (and also the mass terms \( M^{(0)} \)) are spin-triplet operators and therefore their microscopic origin lies in random vector chirality like perturbations.

III. MONOPOLE SCALING DIMENSION OF CLEAN cQED\(_{2+1} \)

In the absence of monopole operators, the cQED\(_{2+1} \) action [Eq. (1)] has an additional topological symmetry U(1)\(_{\text{topo}} \) attributed to the conserved current \( J_\mu^a = (1/2\pi) e^{\alpha/\lambda_\chi} \partial_\mu A_\lambda \). However, there exists static, stable and singular gauge field configurations which carry \( q \) units of the U(1)\(_{\text{topo}} \) charge. These are the monopole operators that spontaneously breaks the topological symmetry to create \( 4\pi q \) magnetic flux locally while satisfying the Dirac quantization constraint, \( 2q \in \mathbb{Z}^2 \). Although these are local operators, they can not be constructed as polynomials of the fundamental fields of the theory, which makes it difficult to calculate their scaling dimension using direct methods of feynman diagrams. However, the \( c^2 \to \infty, N \to \infty \) fixed point of cQED\(_{2} \) is conformal and for conformal field theories (CFT) the scaling dimension of local operators can be determined using state-operator correspondence of the radial quantization picture. In this picture, a local operator \( \mathcal{O} \) of a CFT inserted at the origin of flat \( \mathbb{R}^3 \) space-time has a one-to-one correspondence to normalizable states of the CFT on \( S^2 \times \mathbb{R} \). Further, the scaling dimension \( \Delta_\mathcal{O} \) of the operator on \( \mathbb{R}^3 \) is equal to the energy of the corresponding state on \( S^2 \times \mathbb{R}^3 \). In this scenario the energy eigenvalue of the state corresponding to the monopole operator \( \mathcal{M}(q) \) of charge \( q \) at the origin amounts diagonalizing the cQED\(_{2+1} \) action on \( S^2 \times \mathbb{R} \) in the presence of \( 4\pi q \) unit of magnetic flux\(^{26} \). An alternate strategy is to compute the free energy \( F(q) \) of flux inserted action such that the scaling dimension is obtained as\(^{30} \),

\[
\Delta_{\mathcal{M}(q)} = F(q) - F(0), \quad \text{with} \quad F(q) = -\log Z_{S^2 \times \mathbb{R}}^{(q)} = -\lim_{\beta \to \infty} \frac{1}{\beta} \log Z_{S^2 \times S^1}^{(q)}.
\]

(7)

The above expression for the scaling dimension subtracts a potentially divergent background free energy in the absence of any monopoles which does not affect physical quantities.

We have to consider the cQED\(_{2+1} \) action in the curved \( S^2 \times \mathbb{R} \) space-time. From the euclidean signature the vierbein \( e_{\mu}^a \) can be introduced to get a curved space metric \( g_{\mu\nu} = e_{\mu}^a e_{\nu}^a \). Eliminating any spin-connection by performing appropriate unitary rotation, the cQED\(_{2+1} \) action in the
curved space-time can be written as

$$S_{\text{cQED}} = \int d^3r \sqrt{g} \psi^\dagger \gamma^a \left[ \partial_\mu + i A_\mu \right] \psi,$$

where $\gamma^a$ are the three spinor matrices defined on the flat space time. Insertion of a monopole of charge $q$ amounts to embedding $q$ unit of magnetic flux at the origin by introducing a singular gauge field configuration. The static gauge field contribution due to the monopole at the center is $A_\mu = q \frac{1-cos^2 \theta^2}{2sin \theta} \phi$. Mapping to the cylindrical space-time $S^2 \times \mathbb{R}$ with the metric $ds^2 = dr^2 + (d\theta^2 + \sin \theta^2 d\phi^2)$ from the usual spherical co-ordinates in $\mathbb{R}^3$ with the metric $ds^2 = dr^2 + (d\theta^2 + \sin \theta^2 d\phi^2)$ obtained by putting $r = e^r$ and performing a Weyl rescaling.

$$g_{\mu\nu} \rightarrow e^{-2r}g_{\mu\nu}, \quad \psi, \tilde{\psi} \rightarrow e^{-\tau} \psi, e^{-\tau} \tilde{\psi}, \quad e^{\mu}_a \rightarrow e^{-\tau} e^{\mu}_a, \quad A_\mu \rightarrow A_\mu.$$

The transformed Dirac operator in the presence of magnetic monopole $q$ in the cylindrical space-time is given by

$$D = \gamma_r \left[ \frac{\partial}{\partial r} - \left( j^2 - L^2 + \frac{1}{4} \right) + q \gamma_r \right],$$

where $\gamma_r = \hat{r} \cdot \hat{\gamma}$. Here, $J$ and $L$ are respectively the generalised total and orbital angular momentum in the presence of the monopole magnetic flux. At this level dynamical contribution towards the gauge fields are ignored. This is strictly valid in the large-$N$ limit and their sub-leading effect can be incorporated back within a controlled $1/(2N)$ expansion.

Following earlier work by Ref. 44, it was shown by Ref. 26 that the Dirac operator in the presence of a monopole generating background gauge field can be diagonalized by a special monopole harmonics basis. In the presence of a monopole of charge $q$, the monopole harmonics are defined as $L^2 Y_{l,m} = (l+1) Y_{l+1,m} = m Y_{l,m}$, with $l = |q|, |q| + 1, |q| + 2, \ldots$ and $m = -l, -l+1, \ldots l$. The Dirac equation is not diagonal in the monopole harmonics basis. Instead a basis involving two separate modes of the total angular momentum $j = l \pm \frac{\delta}{2}$ needs to be considered,

$$T_{q,lm}(\theta, \varphi) = \begin{cases} \sqrt{\frac{1+ l + m + 1/2}{2l + 1}} Y_{q,l+1,m}(\theta, \varphi) & j = l + \frac{1}{2}, \\ \sqrt{\frac{1- l + m + 1/2}{2l + 1}} Y_{q,l,m+1/2}(\theta, \varphi) & j = l - \frac{1}{2}. \end{cases}$$

$$S_{q,lm}(\theta, \varphi) = \begin{cases} \sqrt{\frac{1+ l - m + 1/2}{2l + 1}} Y_{q,l,m}(\theta, \varphi) & j = l + \frac{1}{2}, \\ \sqrt{\frac{1- l - m + 1/2}{2l + 1}} Y_{q,l,m-1/2}(\theta, \varphi) & j = l - \frac{1}{2}. \end{cases}$$

which brings the monopole Dirac equation to an almost diagonal form. Following the notation of Ref. 30 we can write down the $2 \times 2$ eigenvalue equation of the Dirac operator in the basis $(T_{q,lm}, S_{q,lm})^T$,

$$S_{q,lm} e^{-i \omega \tau} = d_{q,l}(\omega) \left( T_{q,lm} e^{-i \omega \tau} \right),$$

where $d_{q,l}(\omega) = A_{q,l}(-i \omega + B_{q,l})$ is the eigenvalue matrix given by

$$A_{q,l} = \begin{pmatrix} -\frac{q}{\tau} & -\sqrt{1 - \frac{q^2}{\tau^2}} \\ -\sqrt{1 - \frac{q^2}{\tau^2}} & \frac{q}{\tau} \end{pmatrix},$$

$$B_{q,l} = \begin{pmatrix} l (1 - \frac{q^2}{\tau^2}) & -q \sqrt{1 - \frac{q^2}{\tau^2}} \\ -q \sqrt{1 - \frac{q^2}{\tau^2}} & l (1 - \frac{q^2}{\tau^2}) \end{pmatrix}.$$
restores the homogeneity of space-time. Quenched randomness is static in time and if we seek to make a connection with the radial quantization picture, such random couplings must be parametrized by the coordinates on the two-sphere.

In the standard treatments, quenched random couplings are parametrized on the planar, spatial \( \mathbb{R}^2 \) submanifold of the \((2 + 1)\) dimensional space-time manifold. To obtain the correspondence between the disorder strengths of random couplings defined on a space-like plane and a space-like sphere we consider a one-point compactification of the two-dimensional plane

\[
(x, y) = (\tan \frac{\theta}{2} \cos \phi, \tan \frac{\theta}{2} \sin \phi),
\]

which transforms the planar spatial metric \( ds^2 = dx^2 + dy^2 \) into \( ds^2 = \frac{1}{2} \sec \frac{\theta}{2} d\theta^2 + \tan \frac{\theta}{2} d\phi^2 \). Naturally, the induced metric on the sphere \( ds^2 \) differs from the usual spherical metric \( ds^2 = d\theta^2 + \sin^2 \theta d\phi^2 \). In the compactified space the Gaussian weight for the random couplings in (6) therefore becomes

\[
P[V] = e^{-\frac{1}{2\pi} \int d^2 r \sqrt{g} V^2 j_{\alpha j}},
\]

In the \( S^2 \times \mathbb{R} \) space, the disordered perturbation to the clean cQED\(_{2+1}\) action is given by

\[
S_{\text{dis}} = \int d^3 r \sqrt{g} \left( iV_{\alpha j} \bar{\sigma}^a e^a_{\gamma} \gamma^a \right) \psi.
\]

The Weyl rescaling leaves the RVP fields \( V_{\alpha j} \) unchanged. This distinguishes them from random mass perturbations (which do not stay invariant under the Weyl rescaling) and allows us to move forward with the radial quantization technique. With the modified distribution, the on-site correlation between the random couplings is given by,

\[
V_{\alpha j}(\theta, \phi) V_{\beta k}(0,0) = \rho_\alpha \delta_{\alpha \beta} \delta_{jk} \delta(\theta) \delta(\phi) / \sqrt{g}. \]

In the integrated form it yields

\[
\int d^2 r \sqrt{g} V_{\alpha j}(\theta, \phi) V_{\beta k}(0,0) = 4 \rho_\alpha \delta_{\alpha \beta} \delta_{jk}.
\]

This establishes a correspondence between the strengths of the random couplings on the two-sphere and the plane. In the following we shall use the usual metric on the two-sphere and carefully account for this additional factor of 4 to obtain the disorder averaged monopole free energy. Later we shall use the known expression\(^{18,40}\) for the RG flow of the random couplings in non-compact disordered QED\(_{2+1}\) defined on flat space to characterize the disorder driven instabilities through monopole proliferation. Following disorder averaging the scaling dimension of the monopole operator can be simply extracted from the difference of the free energies, \( \Delta M(\theta) = F(\theta)(V_{\alpha j}) - F(0)(V_{\alpha j}) \).

As the RVP couple quadratically to fermionic fields, we can formally integrate out the fermions from the generic disordered cQED\(_{2+1}\) action and similar to what is done with the dynamical gauge fields, perturbatively expand the resulting expression in random coupling strength and perform direct disorder averaging. It is convenient to exploit the homogeneity of the \( S^2 \times \mathbb{R} \) space-time post disorder averaging and compute the functional trace in the space-time basis as, \( \text{Tr} \mathcal{A} = \mathcal{V}(S^2) / (2\beta) \text{Tr}(A(r_0)) \), where \( \mathcal{V} \) denotes the volume of the space, \( r_0 \) is any given point in the space, and \( \text{Tr} \) is a trace within the Dirac spinor and SU(2N) flavor space. In the following we choose the north pole co-ordinates \( r_0 = (\tau = 0, \theta = 0, \phi = 0) \) and obtain,

\[
F(\theta)(V) = -\text{Tr} \log \left[ \mathcal{D}^2 + iA + iV_{\alpha j} \sigma^a e^a_{\gamma} \gamma^a \right] + O(1/(2N))
\]

\[
= (2N) \lambda_0(q) + \lambda_1(q) + \lim_{\beta \to \infty} \frac{\mathcal{V}(S^2) \mathcal{V}(\mathbb{R})}{2\beta} \int d^3 r \sqrt{g} \text{Tr} \left[ G(\theta)(r_0, r) V_{\alpha j}(\theta, \phi) \sigma^a e^a_{\gamma} \gamma^a G(\theta)(r, r_0) V_{\beta k}(0,0) \sigma^b e^b_{\gamma} \gamma^b \right] + O(1/(2N), V^4)
\]

\[
= (2N) \lambda_0(q) + \lambda_1(q) + 2\pi(4\rho_\alpha) \int d\tau \text{Tr} \left[ G(\tau)(\tau) i\sigma^a e^a_{\gamma} \gamma^a G(\tau)(\tau) i\sigma^b e^b_{\gamma} \gamma^b \right] + O(1/(2N), \rho_\alpha^2),
\]

where \( G(\tau) \) is the monopole Greens function between coincident angles, \( G(\tau) = \langle r_0 | D^{-1} | r \rangle \) with \( r = (\tau, 0, 0) \). For a controlled perturbative double expansion we must have, \( \rho_\alpha \sim 1/(2N) \). In that way, following the trace over the vertex matrices, the first order disorder contribution is an \( O(1) \) perturbation to the zeroth order free energy. The spectral decomposition of the Greens function matrix is expressed in the \( 2 \times 2 \) monopole spherical harmonic basis as

\[
G(\tau) = \int \frac{d\omega}{2\pi} e^{-i\omega \tau} \sum_{l=0}^{\infty} \sum_{m=-l}^{l-1} (T_{q,l-1,m} S_{q,l,m}) \frac{d q}{d \omega} (\omega)^{-1} \left[ T_{q,l-1,m} S_{q,l,m}^\dagger \right]_r.
\]

(21)
The full expression of the Greens function is given in Ref. 30 and involves complicated special functions. The present scenario however, is simpler and at the north pole using the property $Y_{q,l,m} = \delta_{q,-m} \sqrt{(2l+1)/(4\pi)^2}$, we have a much simpler expression for the co-incident angle Greens function,

$$G(\tau) = -\frac{\text{sgn}(\tau)}{2} \left[ \frac{q}{4\pi} \left( I + \gamma^0 \right) + \sum_{l=q+1}^{\infty} \frac{l}{2\pi} e^{-\sqrt{t^2-q^2} |\tau|} \gamma^0 \right].$$

(22)

where $I$ is a $2 \times 2$ identity matrix. In the contribution to the free energy, the Greens function convolution includes a sum over the Dirac matrices. Evaluated at north pole the summand remains

For the first sum, $I(\tau)$ then the summand remains

For the present case we have an expression for the free energy [Eq. (24)] with formally divergent summations over the angular momentum indices. However, the finite part of these summations can be obtained using analytic continuation of Hurwitz Zeta function

$$\sum_{l=0}^{\infty} (l+z)^{s} = \zeta(-s, z) = \frac{B_{s+1}(z)}{s+1} \quad \forall \ s \neq 1.$$

(25)

For the first sum, $I_1(q) = q \sum_{l=q+1}^{\infty} \sqrt{l^2 - q^2}$ we notice that the summand remains $\propto 1$ for large $l$. The regularized

$$I_1(q)/q = \lim_{x \to 1} \left[ \sum_{l=q+1}^{\infty} \left( \frac{l}{(l^2 - q^2)^{s/2} - l^{1-s}} + \sum_{l=q+1}^{\infty} \frac{l^{1-s}}{l^{1-s}} \right) \right] = R_1(q) + \zeta(0, q+1),$$

(26)

where in the second step we can take the limit $s = 1$ by using the Hurwitz Zeta function identity for the second formally divergent term. Here $R_1(q)$ is a perfectly convergent summation which can be evaluated upto arbitrary numerical accuracy. The same technique can be extended to obtain the finite contribution from the second sum, $I_2(q) =$
the lowest allowed charge \( q = 3/2 \). The proliferation of the monopoles leads to the confinement of fractionalized spinons of the destruction of the spin liquid phase. For generic couplings to random SU(2N) currents, the disorder strengths \( \rho_\alpha \) also renormalize. In that case the fate of the conformal fixed point is governed by the combined renormalization of all the couplings of the problem.

\[ \sum_{\alpha=1}^{\infty} \rho_\alpha \frac{\mu'}{\sqrt{\rho_\alpha^2 + \mu'^2}}, \]

where, owing to the presence of a double summation, the resulting expression is cumbersome. The complete expression for the double summation contribution with an unimportant \( q \) independent piece subtracted, \( I_2(q) = I_2(q) - I_2(0) \), has bee provided in Appendix A.

From the disorder averaged free energy the scaling dimension of the monopole operator of charge \( q \) is therefore given by

\[
\Delta_{M(q)} = (2N)\lambda_0^{(q)} + \lambda_1^{(q)} - \frac{2\rho(2N)}{\pi} \left[ I_1(q) + I_2(q) \right] + O(1/(2N), \rho_\alpha^2),
\]

where the disorder contribution, \( \rho = \sum_\alpha \rho_\alpha \), is to be summed over the SU(2) indices \( \alpha \), depending on the residual symmetries of the RVP disorder. The unitarity bound dictates that the scaling dimension of a conformal scalar operator has to be \( \geq 0.5 \) in \( D = 2 + 1 \). It clearly follows that at the critical disorder strength \( \rho^* = \pi((2N)\lambda_0^{(q)} + 0.5)/(2(2N)(I_1(q) + I_2(q))) \), the bound is saturated for the monopole operator of charge \( q \). This signals the breakdown of the conformal symmetry of the infrared fixed point.

However, more importantly, it is the RG relevance of the monopole fugacity operator, which has the scaling dimension \( 3 - \Delta_{M(q)} \) in three-space-time dimension that dictates the suppression or proliferation of the monopoles. From Fig. 1, where we have e.g. considered a particular RVP perturbation, \( i\bar{\psi}\gamma^a\sigma^a\gamma^b\psi \) in the context of the triangular lattice staggered \( \pi \)-flux DSL phase with allowed monopole charges \( q \geq 3/2 \), it emerges that with increasing disorder strength, higher charged monopole fugacities become relevant but the instability is still instigated by the proliferation of the monopoles of the lowest allowed charge \( q = 3/2 \). The proliferation of the monopoles leads to the confinement of fractionalized spinons of the destruction of the spin liquid phase. For generic couplings to random SU(2N) currents, the disorder strengths \( \rho_\alpha \) also renormalize. In that case the fate of the conformal fixed point is governed by the combined renormalization of all the couplings of the problem.

V. DISORDERED RG FLOW WITH MONOPOLES

To study the fate of the deconfined fixed point of the cQED\(_{2+1} \) action, the renormalization of all the associated couplings and the related instabilities need to be considered together. In our case, this boils down to the renormalization group flow of the monopole fugacity, \( y^{(q)} \) (which has tree-level scaling dimension \( 3 - \Delta_{M(q)} \)) and the random disorder couplings. The infrared flow of the gauge electric charge \( e^2 \rightarrow \infty \) is to the leading order unaffected by the disorder couplings and can be ignored for the present discussion.

The RG flow equation of RVP couplings \( \rho_\alpha \) were obtained in Ref. 38 and in the absence any other form of disorder we can adapt their expression to write, \( \frac{d\rho_\alpha}{dt} = 2|\epsilon_{\alpha\beta\gamma}|\rho_\beta \rho_\gamma \), where \( \epsilon_{\alpha\beta\gamma} \) is the usual Levi-Civita tensor. This equation describes a flow of the genetic disorder strength to its strong coupling fixed point and therefore such random couplings introduce instability to the conformal fixed point of cQED\(_{2+1} \). Including monopoles in the picture we have a combined RG flow equa-
SU(3) The combined RG flow of $SU(N)$ symmetric random vector potential coupling $\rho_x = \rho_y = \rho_z = \rho$ and monopole fugacity $y^{(q)}$ of the $q = 3/2$ monopole operator which is allowed on the triangular lattice.

\[ \frac{dy^{(q)}}{dl} = \left( 3 - \sum_\alpha \Delta_{M^{(q)}}(\rho_\alpha) \right) y^{(q)}, \]
\[ \frac{d\rho_\alpha}{dl} = 2|\epsilon_{\alpha\beta\gamma}|\rho_\beta\rho_\gamma, \tag{28} \]

where the contribution to the scaling dimension of the monopole operator from all the disorder couplings have been added together. Presence of all three $\rho_\alpha$ couplings would indicate that the emergent $SU(2N)$ flavor symmetry has been broken down to a reduced $SU(N)$. From the coupled flow equation it is clear that the $SU(N)$ symmetric disorder moves to a strong coupling fixed point. Consequently, due to its linear regressive dependence on the disorder strength, the monopole fugacity also flows to strong coupling (see Fig. 3). This observation leads to the clear indication that the DSL phase is destroyed by $SU(N)$ symmetric RVP disorder as magnetic monopoles proliferate and confinement ensues. From Fig. 1 we understand that the fugacity of the monopoles with the lowest microscopically allowed charge turns relevant first as the disorder strength flows to its strong coupling limit.

Ref. 38 further showed that if the random perturbations to the action obey more symmetries, the effect of disorder may be less drastic. Particularly, for a random coupling $\rho_z$ to only one of the three components of $SU(2)$ sub-group of $SU(2N)$ vector currents ($\rho_x = \rho_y = 0$), the RVP is $U(1) \times SU(N)$ symmetric and in this case, following the same RG equation from above, it turns out that the disorder coupling $\rho_z$ is marginal under RG. However, as it is demonstrated in Fig. 2, for the case of the triangular lattice where monopole operators of charge $q < 3/2$ are prohibited, a confinement transition ensues at a finite disorder strength and the spin liquid phase is destabilized at a finite critical value $\rho_z^c \sim 1.75$. As per our definition of the RVP perturbation [Eq. (6)], this critical disorder strength is a dimensionless phenomenological number and the precise form of its magnitude as a function of the inhomogeneities present in the lattice depends on the microscopic details.

VI. CONCLUSION AND OUTLOOK

From the available studies\textsuperscript{38–40} it is known that the most generic local random perturbations to the $(2 + 1)$ dimensional Dirac spin liquid phase most likely lead to strong disorder instabilities and consequently the demise of the spin liquid phase. This eventuality can however be affected by two key aspects, the symmetric nature of the random perturbations\textsuperscript{38} and the effects of the microscopic monopole excitations of the phase. Our work extends the findings of the earlier works on this problem to consider the interaction between quenched disorder and the monopole operators. By computing the normalized scaling dimension of the monopole operators of the effective $c$QED\textsubscript{2+1} description we establish that the spin liquid phase is further destabilized due to the disorder induced under-screening of the monopole operators. In the absence of monopoles, Ref. 38 found that the random vector chirality like perturbations which break the microscopic $SU(2)$ spin symmetry down to an $U(1)$ drives the effective theory of the phase to a finite disorder fixed point where the spin liquid phase may yet survive despite some quantitative modifications\textsuperscript{31}. We find that this finite disorder fixed point is in fact also fragile once the monopole operators are considered. On the other hand, more generic forms of random perturbations are seen to drive the effective description towards a strong coupling fixed point where both the monopole fugacity and disorder strength turns relevant.

Observations of spin liquid like signatures in certain triangular lattice organic salts such as $\kappa = (Et)_2Cu_2(CN)_3$\textsuperscript{50,51} and $EtMe_3Sb[Pd(dmit)$_2$]_2$\textsuperscript{52,53} have triggered discussions surrounding the viability of a stable $U(1)$ Dirac spin liquid phase on the triangular lattice. However, many of the prospective triangular lattice spin liquid compounds are also noted to include significant quenched randomness effect\textsuperscript{54} which may mimic spin liquid behaviours. Our study indicates that time-reversal invariant random vector chirality perturbations generically turn the symmetry allowed monopole operators relevant on the triangular lattice. Proliferation of monopole operators on the clean triangular lattice has been associated with spiralling Néel magnetic order and valence bond solid (VBS) order\textsuperscript{22}. However, in the current scenario the lattice lacks translational symmetry. Separate works applicable to the frustrated triangular lattice have highlighted that both $120^\circ$ spiral Néel and VBS orders are unstable against small random exchange perturbations and ultimately give rise to glassy phases\textsuperscript{5,10,13}. Following the work in Ref. 55, the same inference can be extended to random vector chirality like perturbations effects at least for the case of the spiral Néel order. Altogether it is therefore likely that in the presence of generic random perturbations the triangular lattice DSL state is destroyed in favor of a random-signlet like glass phase with monopole operators confining while disorder flows to strong coupling. This observation is consistent with the quantum critical behavior seen in the compound $\kappa = (ET)_2Cu_2(CN)_3$ where it is antic-
ipated that a gapless spin liquid state enters a glassy phase in the presence of random Dzyaloshinskii-Morya and multi-spin chiral interaction at low temperatures\textsuperscript{56}.

On the technical side, we have used radial quantization techniques to calculate the scaling dimension of a (2 + 1) dimensional CFT with quenched random couplings. The scheme can be adapted to a number of (2 + 1) dimensional U(1) conformal gauge theories\textsuperscript{47} perturbed by quenched disorder. Among them, the $\mathbb{CP}^{N_b-1}$ theory of unit-norm $N_b$-component complex bosonic spinons constitute a viable example. The monopole operators of this model are interpreted as the order parameter of VBS order in quantum antiferromagnets\textsuperscript{45}. Therefore, it will be worthwhile to apply the methodology of our paper within the context of the Néel-VBS transition in the presence of quenched disorder.

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**Appendix A: Regularisation of the double summation in the disordered QED$_{2+1}$ free energy**

The second contribution appearing in the expression for the free energy Eq. (20) involves a formally divergent double summation over angular momentum indices. In this Appendix, the divergent summation is regularised using the Zeta function method. It is convenient to split the double summation in two parts,

\[
I_2(q) = \sum_{l, l' = q+1}^{\infty} \frac{ll'}{\sqrt{l^2 - q^2} + \sqrt{l'^2 - q^2}}.
\]

The first single sum grows as $\propto l$ asymptotically. Following the same regularisation technique as in the main text a perfectly converged sum may instead be considered, \(1/2\sum_{l = q+1}^{\infty} l^2/(l^2 - q^2)^{1/2 + s}\). It is possible to subtract and then add back from this expression its asymptotic dependence and then analytically continue the result to $s \to 0$ using the identities of Hurwitz Zeta function.

However, in certain cases where the resulting expression contains essential singularities in the limit $s \to 0$, a modification to this regularization scheme is better suited\textsuperscript{57}. Let us consider $A(s)$ as a quantity which we want to analytically continue to $\lim_{s \to 0} A(s) = a_0$ but $A(s) = a_{-m} s^{-m} + a_{-(m-1)} s^{-(m-1)} + \ldots + a_0 + a_1 s + \ldots$ is singular. We can instead take the operator

\[
D \left[ \frac{d^n}{ds^n} (s^n A(s)) \right] = a_0,
\]

and consider any $n > m$ such that the regularized finite part $a_0 = \lim_{s \to 0} A(s)$ is obtained without encountering any singularities. The original scheme corresponds to $n = 0$.

Now following this strategy the manipulated summand yields the regularized finite contribution

\[
\begin{align*}
&= \frac{1}{2} \sum_{l = q+1}^{\infty} \left[ \frac{l^2}{(l^2 - q^2)^{1/2 + s}} - \left[ l^{1-2s} + \left( s + \frac{1}{2} \right) \frac{q^2}{2} l^{1-2s} \right] \right]_{s \to 0} \\
&= \frac{1}{2} \sum_{l = q+1}^{\infty} \left[ l^{1-2s} + \left( s + \frac{1}{2} \right) \frac{q^2}{2} l^{1-2s} \right]_{s \to 0} \\
&= R_2(q) + \frac{1}{2} \left[ \zeta(-1, q+1) - \frac{q^2}{2} \psi(q+1) \right].
\end{align*}
\]

Here $R_2(q) = \frac{1}{2} \sum_{l = q+1}^{\infty} \left( \frac{q^2}{l^2 - q^2} - \left[ l + \frac{q^2}{l^2 - q^2} \right] \right)$ is now a convergent summation even after taking the limit $s \to 1$.

The remaining double sum offers more difficulty. To make progress the sum may be cast in a different form,

\[
\sum_{l \neq l'} \frac{ll'}{(l^2 - l'^2)} \left( \frac{l^2}{l'^2} - \frac{l'^2}{l^2} \right) = \sum_{l \neq l'} \frac{2ll'}{(l^2 - l'^2)^{3/2}} - \frac{2ll'}{(l^2 - l'^2)^{1/2}}
\]

which is helpful to obtain the sum over one of the indices in a purely analytical form. Thus, the sum over $l'$ of the quantity $\frac{2ll'}{(l^2 - l'^2)^{3/2}}$ is first considered. The summand grows as $\propto -\frac{2}{l}$ and a finite value to it can be assigned by considering the modified regularization $D[\ldots]$. For $l \geq q + 2$ it follows that,

\[
\begin{align*}
&= \sum_{l' = q+1}^{\infty} \frac{2l'}{l^2 - l'^2} = \sum_{l' = q+1}^{\infty} \left( \frac{1}{l' - l} + \frac{1}{l' + l} \right) \\
&= \frac{1}{2l} + \psi(l - q) + \psi(l + q + 1) \quad \forall \: l \geq q + 2,
\end{align*}
\]

which then reduces the double summation to a single summa-
tion over $l$,
\[
\sum_{l,l'=q+1 \atop l \neq l'} \frac{l l'}{\sqrt{l^2 - q^2} + \sqrt{l'^2 - q^2}}
\]
\[
= \sum_{l=q+2}^{\infty} \frac{2(l+1) \sqrt{2q+1}}{(q+1)^2 - l^2} \left( \frac{2l l'}{l^2 - l'^2} \sqrt{l^2 - q^2} \right)
\]
\[
+ \sum_{l=q+2}^{\infty} l \sqrt{l^2 - q^2} \left( \frac{1}{2l} + \psi(l - q) + \psi(l + q + 1) \right).
\]
\tag{A6}

The first term in the above expression can be computed similarly as a principle value to yield the finite contribution,
\[
- (q + 1) \sqrt{2q + 1} \sum_{l=q+2}^{\infty} \left( \frac{1}{l - (q + 1)} + \frac{1}{l + (q + 1)} \right)
\]
\[
= (q + 1) \sqrt{2q + 1} [\gamma + \psi(2q + 3)]
\]
\tag{A7}

where $\gamma = 0.577216$ is the Euler-Mascheroni constant.

The second term is also formally divergent due to its asymptotic growth, $\propto 2l^2 \log l + \frac{1}{2} - q^2 \log l - \frac{1}{2} (1/6 + q + q^2) - \frac{q^2}{4l}$. The logarithmically growing portion can be regularised by using the identity, $\log l = -\frac{d}{ds} l^{-s}|_{s=0}$, such that one has,
\[
\sum_{l=q+2}^{\infty} \log l = - \frac{d}{ds} \zeta(s, q + 2)|_{s=0} = -\zeta'(0, q + 2).
\]
\tag{A8}

In a similar manner the other logarithmically growing term gets the finite expression, $\sum_{l=q+2}^{\infty} l^2 \log l = -\zeta'(-2, q + 2)$.

The terms which grows as $l$ and $1/l$ can be regularized using the various identities already invoked above. Subtracting the diverging part from the summand and adding its regularized value back to the summation as above, the regularized double summation is thus obtained to be
\[
\sum_{l,l'=q+1 \atop l \neq l'} \frac{l l'}{\sqrt{l^2 - q^2} + \sqrt{l'^2 - q^2}}
\]
\[
=(q + 1) \sqrt{2q + 1} [\gamma + \psi(2q + 3)] + R_3(q)
\]
\[
+ \left[ -2 \zeta'(-2, q + 2) + \zeta(-1, q + 2) + q^2 \zeta'(0, q + 2) \right]
\]
\[
- \left( \frac{1}{6} + q + q^2 \right) \zeta(0, q + 2) + \frac{q^2}{4} \psi(q + 2) \right],
\]
\tag{A9}

where,
\[
R_3(q) = \sum_{l=q+2}^{\infty} \left( l \sqrt{l^2 - q^2} \left[ \frac{1}{2l} + \psi(l - q) + \psi(l + q + 1) \right] - \left[ 2l^2 \log l + \frac{1}{2} - q^2 \log l - \frac{1}{2} (1/6 + q + q^2) - \frac{q^2}{4l} \right] \right)
\]
\tag{A10}

is once again a convergent sum. Putting together all of these pieces the complete and finite regularized expression for the second contribution to the disorder averaged scaling dimension [Eq. (27)] is found to be
\[
I_2(q) - I_2(0) = R_2(q) + [R_3(q) - R_3(0)] + [f(q) - f(0)],
\]
\tag{A11}

where $f(q)$ combines the contributions with an analytical expression,
\[
f(q) = (q + 1) \sqrt{2q + 1} H_{2q+2}
\]
\[
+ \frac{q^2}{2} \log \frac{\Gamma(2 + q)^2}{2\pi}
\]
\[
- 4 - q (4 + 35q + 12 (3 + q) q^2) \frac{12(1 + q)}{12(1 + q)}
\]
\[
- 2 \zeta'(-2, q + 2).
\]
\tag{A12}

Here $H_z$ is the Harmonic number and $\Gamma(z)$ is the Gamma function and both of these special functions can be evaluated up to arbitrary precision.

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