Lorentz and Galilei Invariance on Lattices

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We show that the algebraic aspects of Lie symmetries and generalized symmetries in nonrelativistic and relativistic quantum mechanics can be preserved in linear lattice theories. The mathematical tool for symmetry preserving discretizations on regular lattices is the umbral calculus.

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I. INTRODUCTION.

One of the recognized difficulties in the study of quantum systems on space–time lattices is the description of fundamental space–time symmetries, such as Lorentz invariance, Galilei invariance, conformal invariance, etc. The usual statement is that “a lattice formulation severely mutilates Lorentz invariance at the outset” [1] and that continuous symmetries are only recovered in the continuous limit of a lattice theory.

There are many reasons to consider quantum physics on a lattice. In addition to providing a convenient tool for obtaining quark confinement and regularizing divergences in quantum theories [2, 3, 4, 5], an elementary approach to quantum gravity [6, 7, 8, 9]. Extensions of the standard model violating Lorentz invariance have been suggested [10]. Sensitive tests of Lorentz invariance violations and related CPT violations have been proposed [11, 12].

The aim of this article is somewhat complementary to the above studies. Namely, we show that it is possible to formulate physical theories on lattices in a manner that preserves very many of the symmetry properties of continuous theories. In particular, we demonstrate that the existence of Lorentz and Galilei symmetry algebras for classical linear field theories is perfectly compatible with an intrinsically discrete space–time geometry.

The mathematical approach that we wish to apply is the so-called ”umbral calculus”. Its modern form is due to G. C. Rota and collaborators (see [27, 28] and also [29] for an up–to–date review on the recent developments of umbral calculus). The main applications of umbral calculus have been in combinatorics. In this article we focus on an application to difference equations and to an ”umbral correspondence”, relating algebraic properties of difference equations (on regular equally spaced lattices) to those of differential equations. For applications in physics and more details on the umbral calculus, see e.g. [30, 31]. For a direct difference operator approach to symmetries of linear difference equations, see also [32, 33].

In Section 2 we give a brief review of some relevant aspects of umbral calculus. These then are applied in Section 3 to study symmetries of the Schrödinger and Klein–Gordon equations in discrete space–time. Section 4 is devoted to integrability, superintegrability and generalized symmetries on a lattice. Some conclusions and open problems are presented in the final Section 5.

II. THEORY OF FINITE DIFFERENCE OPERATORS

Symmetries in quantum mechanics are expressed in terms of linear self–adjoint operators $X_j$ commuting with a Hamiltonian $H$. The operators $X_j$, as well as the Hamiltonian itself, can be viewed as elements of the enveloping algebra of the Heisenberg algebra $\{p_j, x_j, h\}$,
with the commutation relations
\[ [p_j, x_k] = -i\hbar \delta_{jk}, \quad (1) \]
and all other commutators vanishing.

We shall use the following aspects of the umbral calculus. For each coordinate \( x_j \) (including time \( t = x_0 \)) we introduce a shift operator \( T_j \), satisfying \( T_j [f(x_j)] = f(x_j + \sigma_j) \) where \( \sigma_j \) is the lattice spacing in the direction \( j \). A delta operator \( Q_j \) is a linear operator characterized by two properties, namely
\[ [Q_j, T_k] = 0, \quad Q_j x_k = \delta_{jk}. \quad (2) \]

We shall be interested in two types of delta operators. One is the usual (continuous) derivative \( \partial_{x_j} \), the other are difference operators
\[ \Delta_j = \frac{1}{\sigma} \sum_{k=l}^{m} a_k^l (T_j)^k, \quad \sum_{k=l}^{m} a_k^l = 0, \quad \sum_{k=l}^{m} k a_k^l = 1. \quad (3) \]

We can impose additional conditions on the coefficients \( a_k^l \) if we have \( (m - l) \geq 2 \). We impose
\[ \sum_{k=l}^{m} a_k^l k^q = 0, \quad q = 2, 3, \ldots, m - l \quad (4) \]
and then we shall say that the order of the operator \( \Delta_j \) is \( m - l \) since it provides an approximation of order \( \sigma^{m-l} \) of the continuous derivative \( \partial_{x_j} \). For each operator \( \Delta_j \) of the type \( \delta \) there exists a unique conjugate operator \( \beta_j \) such that the equation
\[ [\Delta_j, x_k \beta_k] = \delta_{jk} \quad (5) \]
is satisfied, namely
\[ \beta_j^{-1} = \sum_{k=l}^{m} a_k^l (T_j)^k. \quad (6) \]

All summations are given explicitly, e.g. there is no summation over repeated indices in eq. \( \delta \) and in the rest of this letter.

For each delta operator \( Q \) there exists a unique sequence of basic polynomials \( P_n (x) \) satisfying
\[ Q P_n (x) = n P_{n-1} (x), \quad P_0 (x) = 1, \]
\[ P_n (0) = 0, \quad n \geq 1. \quad (7) \]

In general, we have \( P_n (x) = (x \beta)^n \cdot 1 \). For example,
\[ Q = \partial_{x_j}, \quad \beta = 1, \quad P_n = x^n, \]
\[ Q = \Delta^+ = \frac{T - 1}{\sigma}, \quad \beta = T^{-1}, \]
\[ P_n (x) = [x]^n = x (x + \sigma) (x + 2 \sigma) \ldots (x + (n - 1) \sigma), \]
\[ Q = \Delta^- = \frac{1 - T^{-1}}{\sigma}, \quad \beta = T, \quad (8) \]

Our basic tool for constructing relativistic and nonrelativistic linear quantum theories in discrete space–time is the "umbral correspondence". In our specific case, this is a mapping that takes
\[ \partial_x \rightarrow \Delta_x, \quad x \rightarrow x \beta. \quad (9) \]
Consequently the sequence of basic polynomials \( P_n (x) = x^n \) for \( \partial_x \) goes into the basic sequence for \( \Delta_x \). Since this mapping preserves the Heisenberg relations \( \delta \), it will preserve the commutation relations between formal power series in \( x \) and \( \partial_x \).

III. LIE SYMMETRIES IN DISCRETE QUANTUM MECHANICS

Let us consider a linear partial differential equation of order \( N \) on \( \mathbb{R}^n \),
\[ Lu = 0, \quad L = \sum_{j=1}^{n} \sum_{k=1}^{N} a_j^k (x) \partial_{x_j}^k. \quad (10) \]
Its Lie point symmetries can be expressed in terms of first order linear differential operators
\[ X_a = \sum_{j=1}^{n} \xi_j^a (x) \partial_{x_j} + \phi^a (x) \quad (11) \]
satisfying
\[ [L, X_a] = \lambda_a (x) L, \quad (12) \]
where \( \lambda_a (x) \) is an arbitrary function. The operators \( X_a \) form a Lie algebra \( \mathfrak{l} \) : \( [X_a, X_b] = f_{ab}^c X_c \).

The umbral correspondence provides us with the following prescription: replace all derivatives \( \partial_{x_j} \) and independent variables \( x_k \) in \( L \) and \( X_a \) by the difference operators \( \Delta_{x_k} \) and expressions \( x_k \beta_k \), respectively. From eq. \( \delta \), we obtain a difference equation \( L^D u(x \beta) = 0 \) and from eq. \( \delta \) a set of difference operators
\[ X_a^D = \sum_{k=1}^{n} \xi_k^a (x \beta) \Delta_{x_k} + \phi^a (x \beta) \quad (13) \]
that commute with \( L^D \) on its solution set and that realize a Lie algebra isomorphic to \( \mathfrak{l} \).

What is given up in this approach are the global aspects of point symmetries. Vector fields corresponding
to differential operators of the form \( (\mathbf{l}) \) can be integrated to provide global, or at least local (finite) group transformations. This is no longer true for the difference operators \( (\mathbf{D}) \). For this reason we are, in the present article, always considering Lie algebras and their representations, rather than Lie groups. In this sense, the "discrete" point symmetries \( (\mathbf{D}) \) are similar to generalized symmetries \( (\mathbf{D}) \) given by higher order operators. These do not provide (local) group transformations even in the continuous limit. However we can still use the symmetries \( (\mathbf{D}) \) to perform symmetry reduction and possibly implement separation of variables.

Let us now apply this result to discrete quantum mechanics. As a first example, consider a free nonrelativistic particle. The discrete time dependent Schrödinger equation is

\[
L_0^D \psi = 0, \quad L_0^D = i \Delta_t - \frac{1}{2} \sum_{k=1}^{n} (\Delta x_k)^2.
\]

Let us impose the commutation relations \( (\mathbf{D}) \) between \( L_0^D \) and \( X_0^D \) of eq. \( (\mathbf{D}) \). We obtain a set of \( d = (n^2 + 3n + 10)/2 \) linear difference operators, satisfying the commutation relations of the Schrödinger algebra \( \text{sch} (n) \). A basis for this Lie algebra is given by the following difference operators (we drop the superscript \( D \))

\[
P_0 = \Delta_t, P_k = \Delta x_k, L_{jk} = (x_j \partial_x) \Delta x_k - (x_k \partial_x) \Delta x_j,
\]

\[
B_k = (t \partial_x) \Delta x_k - \frac{i}{2} (x_k \partial_x),
\]

\[
D = 2 (t \partial_x) \Delta_t + \sum_{k=1}^{n} (x_k \partial_x) \Delta x_k + \frac{1}{2},
\]

\[
C = (t \partial_x)^2 \Delta_t + \sum_{k=1}^{n} (t \partial_x) (x_k \partial_x) \Delta x_k + \frac{1}{2} (t \partial_x - \frac{i}{4} \sum_{k=1}^{n} (x_k \partial_x)^2),
\]

\[
W = i, \quad M = 1.
\]

In the continuous case, these operators correspond to translations, rotations, Galilei transformations, dilations \( (D) \), ”expansions” \( (C) \), the possibility of changing the phase of the wave function \( (W) \) and the norm of the wave function \( (M) \). We mention that the transformations corresponding to \( M \) and \( C \) change the norm of the wave function and should hence be excluded for physical reasons. However, the operator \( C \) itself can be useful. It can be diagonalized together with the Hamiltonian. This will provide quantum numbers and facilitate the calculations of wave functions. The point is that the Lie algebra can be extremely useful, even if the corresponding group transformations may be disallowed for other reasons, or, e.g. in the discrete case, may not exist at all. In the discrete case, the difference operators in \( (\mathbf{D}) \) do not generate point transformations. They all act at least at two points, more points if they involve the operators \( \beta \). We shall call them "generalized point symmetries". The algebra \( (\mathbf{D}) \) could have been obtained directly from the standard realization \( (\mathbf{D}) \), via the umbral correspondence.

By way of an example, let us write the explicit expression of the operator of angular momentum for some choices of the operators \( \Delta \) and \( \beta \). If \( \Delta = \Delta^+ \) then \( \beta = T^{-1} \), and correspondingly

\[
L_{jk} = x_j \Delta^+ x_k - x_k \Delta^+ x_j,
\]

since \( T^{-1} \Delta^+ = \Delta^- \). If \( \Delta = \Delta^- \), \( \beta = T \) and we obtain

\[
L_{jk} = x_j \Delta^- x_k - x_k \Delta^- x_j.
\]

The presence of a potential in the discrete Schrödinger equation \( (\mathbf{D}) \) will break the symmetry, just as in the continuous case. However, if the potential is obtained in a manner consistent with umbral calculus, there will exist a subalgebra of the symmetry algebra of the discrete equation which is isomorphic to that of the continuous one. For instance, consider the Schrödinger equation \( (\mathbf{D}) \) with the potential \( V = V (\hat{\rho}) \), \( \hat{\rho} = [(x_1 \partial_x)^2 + \ldots + (x_n \partial_x)^2]^{1/2} \). The operator \( L^D = L_0^D + V (\hat{\rho}) \) will commute with a subalgebra of \( \text{sch} (n) \), namely \( \{ P_0, L_{jk}, W \} \), just as in the continuous case. We mention that if \( V (\hat{\rho}) \) is not a polynomial in \( \hat{\rho} \), then it should be interpreted as a formal power series in \( x_1 \beta_1, \ldots, x_n \beta_n \)

\[
V = \sum_{j_1, \ldots, j_n=0}^{\infty} \alpha_{j_1, \ldots, j_n} (x_1 \beta_1)^{j_1} \ldots (x_n \beta_n)^{j_n}.
\]

As a further example, let us consider a free relativistic particle with spin \( s = 0 \), and mass \( m > 0 \). The discrete Klein–Gordon equation is

\[
(\Box_D - m) \varphi = 0, \quad \Box_D = (\Delta x_0)^2 - \sum_{k=1}^{n} (\Delta x_k)^2.
\]

On the solution set of eq. \( (\mathbf{D}) \) the operator \( (\Box_D - m) \) commutes with the difference operators

\[
P_0 = \Delta x_0, P_j = \Delta x_j, L_{jk} = (x_j \partial_x) \Delta x_k - (x_k \partial_x) \Delta x_j,
\]

\[
L_{0k} = (x_0 \partial_x) \Delta x_k + (x_k \partial_x) \Delta x_0.
\]

These operators provide a realization of the Lie algebra of the Poincaré group. For massless particles \( (m = 0) \) we would obtain a “discrete” realization of the conformal Lie algebra \( o (n + 1, 2) \)
IV. INTEGRABILITY AND SUPERINTEGRABILITY ON A LATTICE

In (continuous) nonrelativistic quantum mechanics on \( \mathbb{R}^n \) a system is completely integrable if there exists a set of \( n \) self-adjoint differential operators \( \{ X_1, \ldots, X_n \} \), including the Hamiltonian \( H \), that are algebraically independent and commute amongst each other. The system is superintegrable if some further independent operators \( Y_1 \) exist that commute with the Hamiltonian (but not with all operators \( X_j \)). If a system is integrable, its wave functions can be described by a complete set of \( n \) self–adjoint differential operators (like those of the Coulomb atom, or the harmonic oscillator). If the operators \( X_j \) are polynomials in the momenta and coordinates, or at least formal power series, then the umbral correspondence takes an integrable or superintegrable system from continuous into discrete space.

As an example, let us consider the generalized isotropic harmonic oscillator (a harmonic oscillator plus inverse squared terms) \([32]\). Its discrete version is given by the Hamiltonian

\[
H^D = -\frac{1}{2} \sum_{k=1}^{3} (\Delta_{x_k})^2 + \frac{\omega^2}{2} \sum_{k=1}^{3} (x_k \beta_k)^2 + \frac{1}{2} \sum_{k=1}^{3} A_k (x_k \beta_k)^{-2}.
\]

Inverse monomials of the type \((x\beta)^{-n}\) are well defined finite objects. For instance, \((x\beta)^{-2} = (x\beta)^{-1} (x\beta)^{-1}\) and \((x\beta)^{-1} = \beta^{-1} x^{-1}\). A similar discretization was proposed in Ref. \([32]\) using the Fock space formalism. The corresponding \( 5 \) independent operators commuting with \( H^D \) can be chosen to be

\[
X_a = (\Delta_{x_a})^2 - \omega^2 (x_a \beta_a)^2 - A_a^2 (x_a \beta_a)^{-2}, a = 1, \ldots, 3,
\]

\[
Y_1 = L_{12}^2 + L_{23}^2 + L_{31}^2 - \sum_{k=1}^{3} (x_k \beta_k)^2 \sum_{k=1}^{3} A_k (x_k \beta_k)^{-2},
\]

\[
Y_2 = L_{12}^2 - \sum_{k=1}^{2} (x_k \beta_k)^2 \sum_{k=1}^{2} A_k (x_k \beta_k)^{-2}.
\]

This system, and its \( n \)-dimensional generalization is maximally superintegrable. It allows \( 2n - 1 = 5 \) independent integrals of motion, all of them second order in the momenta. This holds both in the continuous, and in the discrete case.

Second and higher order operators commuting with the Hamiltonian, like those in eq. \([24]\), are examples of generalized Lie symmetries in both discrete and continuous quantum mechanics \([31]\).

V. CONCLUSIONS

We have shown that umbral calculus makes it possible to transfer all algebraic features of symmetry theory from continuous space–time to a discrete one. Since the symmetry algebras of continuous and discrete linear systems are isomorphic, this allows us to apply the standard tools of (abstract) Lie algebra representation theory to discrete quantum mechanics. For instance, a free relativistic particle in discrete space–time can be described by solutions of relativistic difference equations, or alternatively, by irreducible representations of the Poincaré Lie algebra. In particular, such basic characteristics as the particle mass and spin still have the standard interpretation as eigenvalues of the Casimir operators of the Poincaré algebra in a given irreducible representation.

An important question is that of the spectra of difference operators in discrete quantum mechanics. For simplicity, consider the one–dimensional discrete Schrödinger equation

\[
\left[ -\frac{1}{2} (\Delta_x)^2 + V(x,\beta) \right] \psi^D(x) = E \psi^D(x) .
\]

Here \( \psi^D \) is assumed to belong to a Hilbert space \( l_2 \) of square summable functions on a lattice. Let us assume that in the continuous limit the Schrödinger equation has an eigenfunction (for the given value of \( E \)), that can be expanded into a formal power series \( \psi(x) = \sum_{n=0}^{\infty} A_n x^n \). Then the difference equation \([24]\) will, for the same energy \( E \), have the formal solution

\[
\psi^D = \sum_{n=0}^{\infty} A_n P_n(x) ,
\]

where \( P_n(x) \) are the basic polynomials corresponding to \( \Delta_x \). We call a solution of the type \([24]\) an ”umbral solution”. In general, the umbral solutions admitted by a linear difference equation are obtained from the solutions of its continuous limit (when it exists) via the umbral correspondence.

Physics in a discrete world can be richer than in a continuous one. Thus, in addition to umbral solutions, difference equations can have additional solutions that cannot be expanded into a formal power series and do not have a continuous limit. They appear when the order of the discrete derivatives involved in a difference equation is greater than one. For example in the case when the delta operator \( Q \) is represented by a symmetric discrete derivative

\[
Q = \Delta^* = \frac{T - T^{-1}}{2\sigma}, \quad \beta^* = \left( \frac{T + T^{-1}}{2} \right)^{-1}
\]

we have extra solutions \([30]\). An analogous phenomenon has been observed in discrete models of general relativity \([13]\). By the same token, the determining equations for symmetries in the discrete case can have ”non–umbral”
solutions and hence additional "purely discrete" Lie symmetries may exist. Furthermore, the representation theory of Lie algebras is richer than that of Lie groups, since representations exist that cannot be integrated to representations of groups.

A discussion of the functional analysis aspects of umbral calculus in quantum theory is beyond the scope of the present article. This includes questions like the convergence of the formal power series used, the relevant measures for imposing square-integrability and the unitarity of representations of the corresponding Lie algebras. In particular, an interesting question is how the spectra of quantum systems are influenced by the proposed discretization procedure. It is clear that, if $A$ is a self-adjoint (bounded or unbounded) operator on the Hilbert space $l_2$, then the operator $U(t) = e^{iAt}$ is a unitary operator satisfying $U(t+s) = U(t)U(s)$ and generating a strongly continuous one parameter unitary group. Therefore, the quantum-mechanical evolution of a state $\psi$ is given by $\psi(t) = U(t)\psi(t_0)$. Nevertheless, as pointed out in [31], the self-adjointness properties of quantum Hamiltonians are not automatically preserved on a lattice. Rather, the discrete analog of a self-adjoint Hamiltonian may have many different self-adjoint extensions on the lattice, each of them describing a different physics. In such situations, an open problem deserving further investigation is how to determine unambiguously the spectral properties of discrete Hamiltonians by a choice of suitable physical constraints on the lattice.

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