On the Classification of Real Forms of Non-Abelian Toda Theories and $\mathcal{W}$-algebras

J.M. Evans, DAMTP, University of Cambridge
and
J.O. Madsen, Universidade de Santiago de Compostela

Abstract
We consider conformal non-Abelian Toda theories obtained by hamiltonian reduction from Wess-Zumino-Witten models based on general real Lie groups. We study in detail the possible choices of reality conditions which can be imposed on the WZW or Toda fields and prove correspondences with $sl(2, \mathbb{R})$ embeddings into real Lie algebras and with the possible real forms of the associated $\mathcal{W}$-algebras. We devise a method for finding all real embeddings which can be obtained from a given embedding of $sl(2, \mathbb{C})$ into a complex Lie algebra. We then apply this to give a complete classification of real embeddings which are principal in some simple regular subalgebra of a classical Lie algebra.
1 Introduction

Toda field theories provide a rich set of examples of integrable models in two dimensions which can be of either conformal or massive type. In each case integrability rests on some underlying Lie algebra (or superalgebra) which renders the theory tractable despite highly non-trivial self-interactions. In this paper we shall consider conformally-invariant Toda theories [1] which can be obtained by hamiltonian reduction from a Wess-Zumino-Witten (WZW) model [2],[3],[4]. More precisely, one can construct a Toda theory by taking the lagrangian for a WZW model based on some real, non-compact Lie group $G$ and gauging a suitable nilpotent subalgebra of its Kac-Moody symmetry. This produces a new field theory in which a WZW lagrangian for some subgroup $G_0 \subset G$ is obtained, though modified by the addition of a potential term of a special kind. The resulting Toda theory is usually referred to as abelian or non-abelian depending on the nature of $G_0$ (the original group $G$ is always non-abelian).

Perhaps the most important aspect of conformally-invariant Toda models is that they provide natural realizations of a certain class of infinite-dimensional chiral symmetry algebras which contain a Virasoro subalgebra. These are referred to generically as extended conformal algebras, or $W$-algebras. The simplest examples are the superconformal algebras and the Kac-Moody/Virasoro semi-direct products, which have generators with spins at most two that close on themselves in the conventional sense of Lie algebras. In general, however, $W$-algebras may have generators of spin higher than two and they are usually non-linear in the sense that classical Poisson brackets or quantum commutators close only onto expressions that are polynomial in the generators. The great interest in $W$-algebras stems from the fact that they greatly extend the domain of rational models, in which the Hilbert space of the theory splits into finitely many irreducible representations of the chiral symmetry algebra. For a comprehensive review see [5].

In the hamiltonian reduction approach mentioned above, the gauging of a nilpotent subgroup of the original WZW model corresponds to imposing a certain set of first-class constraints on the Kac-Moody currents. It is precisely the gauge-invariant polynomials in these currents which become generators of the $W$-symmetry in the Toda theory. This allows one to understand rather easily the spin content of the $W$-algebra in terms of some simple group theory. Moreover, the Toda point-of-view has proved crucial to the study of $W$-algebras, not just because it has provided Lagrangian realizations of known $W$-algebras, but also because it has lead to the construction of many new examples which are more difficult to obtain by other means. It is a plausible conjecture that hamiltonian reduction can provide the basis for a complete classification of $W$-algebras.

Our aim in this paper is to carry out a detailed study of hamiltonian reduction and the Toda theories and $W$-algebras which emerge when $G$ is a general real Lie group. Previously in the literature, attention has been confined largely to examples in which $G$ is maximally non-compact (i.e. the split real form) and, although the necessary general apparatus has been in existence for some time, no systematic analysis of other possibilities seems to have been undertaken. As indicated in our earlier preliminary investigation [6], the consideration of Toda models based on general real forms turns out to be interesting for a number of inter-related reasons.

- Different choices of real form for $G$ yield different real forms for the $W$-algebra which appears in the resulting Toda theory. This suggests a systematic way of generating new real forms of $W$-algebras and of classifying them.
- The existence of inequivalent real forms is clearly crucial as regards the representation theory of $\mathcal{W}$-algebras. We can compare with the case of ordinary Lie algebras, where the representation theory of compact and non-compact real forms is very different. Recently, this question has been raised for the simpler cases of finite $\mathcal{W}$-algebras \cite{7} (finite $\mathcal{W}$-algebras can be constructed from finite dimensional Lie algebras in the same way the “affine” $\mathcal{W}$-algebras are constructed from affine Lie algebras).

- The preceding remarks take on special significance when one bears in mind that the reduced theory often has a residual Kac-Moody symmetry corresponding to a subgroup $K \subset G$. Different real forms for $G$ will lead to inequivalent real forms of $K$. For many applications of Kac-Moody algebras we are most interested in the compact real form; we may therefore expect that when we consider $\mathcal{W}$-algebras it will be of great interest to determine whether we are able to choose a real form of $G$ so as to make $K$ compact.

The conformal reduction of a WZW model based on a group $G$ can be phrased in terms of choosing an $sl(2,\mathbb{R})$ subalgebra of the Lie algebra $G$. Our approach can therefore be interpreted as a systematic search for $sl(2,\mathbb{R})$ embeddings into real Lie algebras. We shall present a method for determining all possible real forms of a given Lie algebra compatible with the existence of a given $sl(2,\mathbb{R})$ subalgebra $S \subset G$, and we shall use this method to classify a natural subset of the embeddings allowed when $G$ is classical. Note that the compact real form of $G$ is never compatible with hamiltonian reduction in this sense, since there are no embeddings of $sl(2,\mathbb{R})$ into a compact real Lie algebra.

It is also important to draw a clear distinction between what we are trying to achieve here and the theory and classification of embeddings of $sl(2,\mathbb{C})$ into complex Lie algebras, as studied by Dynkin about 40 years ago \cite{8}. His classic results can be transferred immediately to deal with embeddings of $sl(2,\mathbb{R})$ in maximally non-compact or split real Lie algebras but, as far as we know, very little has been done regarding embeddings of $sl(2,\mathbb{R})$ into general real Lie algebras, and no form of classification seems to have been carried out until now. This actually goes some way towards explaining why so little has been done in the area of hamiltonian reduction based on general real Lie groups, which we hope to rectify here.

In a recent paper \cite{9}, Bina and Günaydin have carried out the classification of real forms of simple non-linear superconformal algebras, as well as others known as ‘quasi-superconformal’ and ‘super-quasi-superconformal’ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ superconformal algebras \cite{10}, using an approach based on quaternionic and superquaternionic symmetric spaces of simple Lie groups and supergroups. Although we consider exclusively bosonic algebras in this paper, we have already indicated in \cite{6} how our methods can be extended to the hamiltonian reduction of superalgebras, and it would be interesting to apply them to the special cases of superconformal and $\mathbb{Z}_2 \times \mathbb{Z}_2$ algebras discussed in \cite{4}. The quasi-superconformal algebras (such as the Bershadsky algebra $\mathcal{W}_{3 \frac{1}{2}}^2$) involve hamiltonian reductions with half-integer gradings, which again falls outside our present aims. Nevertheless our methods are in principle applicable to these cases too.

The paper is organized in the following way: In section 2 we start by giving a brief account of hamiltonian reduction as applied to a WZW model based on a general real Lie group. In section 3 we recall some facts about the real forms of a complex Lie algebra before stating and proving some general results. These establish precise correspondences
between real forms of a Lie algebra consistent with an $sl(2, \mathbb{R})$ embedding, real forms of non-abelian Toda theories, and real forms of extended conformal algebras. In section 4 we illustrate our general results by discussing in some detail the reductions and real forms for integral embeddings in algebras of rank two. In the following section 5 we further develop our these techniques to cope with algebras of arbitrary rank. We apply them to classify all integral embeddings which are principal in some regular subalgebra of one of the classical algebras of type $A_n$, $B_n$, $C_n$, or $D_n$. In section 5.3 we give a summary of the classification, followed by a discussion of our results and some suggestions for future work. Two appendices contains some selected calculations.

2 Non-Abelian Toda Theories and Conformal Hamiltonian Reduction.

In this section we give a brief review of the conformal hamiltonian reduction of a WZW model based on a general real Lie group $G$ and how this gives rise to a Toda theory (non-abelian in general) realizing a certain $\mathcal{W}$-algebra. This is in principle a special case of the general reduction schemes described in sections 2 and 3 of [4]. However, this very fact will allow us to make some drastic simplifications in the presentation and to concentrate on the points that are most essential in order to understand the work that is to follow.

2.1 WZW models and Toda theories

Our starting point is a WZW model based on a real Lie group $G$. We have the standard action

$$S_{\text{WZW}}(g) = \frac{1}{2} \int d^2x \, \text{tr}(\partial_+ g \partial_- g^{-1}) + \frac{1}{3} \int_D \text{tr}(g^{-1}dg)^3 \quad (2.1)$$

where the field $g(x^+, x^-) \in G$ is a function of light-cone coordinates $x^\pm$ on two-dimensional spacetime; $D$ is some three-dimensional disc whose boundary is spacetime; and ‘tr’ denotes the usual invariant inner-product on the Lie algebra $\mathcal{G}$. The equations of motion are equivalent to the conservation conditions

$$\partial_- J_+ = \partial_+ J_- = 0$$

for the Kac-Moody (KM) currents defined by

$$J_+ = k(\partial_+ g)g^{-1}, \quad J_- = -kg^{-1}\partial_- g,$$

which take values in $\mathcal{G}$. The constant $k$ is the level. If we introduce a basis $\{t_a\}$ for $\mathcal{G}$ and expand the current $J_\pm$ on this basis:

$$J_\pm(x^\pm) = \sum_a J^a_\pm(x^\pm)t_a$$

then the Poisson-brackets of the functions $J^a_\pm(x^\pm)$ can be given in the form (suppressing $\pm$ indices for convenience)

$$\{J^a(x), J^b(y)\}_{PB} = k\eta^{ab}\delta'(x - y) + f^{abc}J^c(x)\delta(x - y)$$
where the Killing form $\eta^{ab}$ has been used to raise and lower indices on the structure constants $f_{abc}$ (our conventions for Lie algebras will be specified in more detail in the next section). This is the standard form for the Kac-Moody algebra $\mathcal{G}$. The Kac-Moody symmetry generated by these currents is

$$g(x^+, x^-) \to a(x^+)g(x^+, x^-)b(x^-)^{-1} \quad \text{where} \quad a(x^+), b(x^-) \in G.$$ (2.2)

The model is also conformally invariant, with the fields $g$ behaving as scalars. The traceless energy-momentum tensor takes the well-known Sugawara form

$$T_{\pm\pm} = \frac{1}{2k} \text{tr}(J_\pm J_\pm).$$

The first step in the hamiltonian reduction of the WZW theory is to choose an integral grading of the Lie algebra $\mathcal{G}$ by specifying a generator $M \in \mathcal{G}$ for which $\text{ad}_M$ has integer eigenvalues. This defines a decomposition

$$\mathcal{G} = \sum_n \mathcal{G}_n \quad \text{where} \quad X \in \mathcal{G}_n \Leftrightarrow \text{ad}_M(X) = [M, X] = nX,$$

so that the integers $n$ label the eigenspaces of $\text{ad}_M$. It follows that

$$X \in \mathcal{G}_m, Y \in \mathcal{G}_n \implies [X, Y] \in \mathcal{G}_{m+n} \quad \text{and} \quad \text{tr}(XY) = 0 \text{ if } m + n \neq 0.$$

This allows $\mathcal{G}$ to be written as a direct sum of subalgebras of zero, strictly positive, or strictly negative grade:

$$\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_0 \oplus \mathcal{G}_-, \quad \mathcal{G}_+ = \sum_{n>0} \mathcal{G}_n, \quad \mathcal{G}_- = \sum_{n<0} \mathcal{G}_n.$$

We emphasize that $\mathcal{G}_0$ and $\mathcal{G}_\pm$ are closed under the Lie bracket—in fact the latter two are nilpotent subalgebras—and we can therefore associate with them subgroups $\mathcal{G}_0$ and $\mathcal{G}_\pm$ respectively. The restriction of the inner-product is non-degenerate on $\mathcal{G}_0$; it is completely degenerate on both $\mathcal{G}_+$ and $\mathcal{G}_-$ separately but defines a non-singular pairing between these subalgebras.

The second step is to impose constraints on the Kac-Moody currents of the form

$$\text{tr} \{ X_\mp (J_\pm - kM_\pm) \} = 0, \quad X_\pm \in \mathcal{G}_\pm,$$ (2.3)

where $M_\pm$ are specific elements chosen in $\mathcal{G}_\pm$, but $X_\pm$ are arbitrary elements in $\mathcal{G}_\pm$. These constraints fix exactly those parts of the currents $J_\pm$ living in the subalgebras $\mathcal{G}_\pm$ to be the constant elements $M_\pm$. The new theory defined in this way is invariant under a modification of the original conformal symmetry which is constructed so as to commute with the constraints. This is achieved by shifting the energy-momentum tensor by a term proportional to the grading operator, so that it becomes

$$T_{\pm\pm} = \frac{1}{2k} \text{tr}(J_\pm J_\pm) - \text{tr}(M_\pm \partial J_\pm).$$

The constraints above are first-class, corresponding to a gauged KM symmetry for the nilpotent subgroups $\mathcal{G}_\pm$. In what follows we shall always assume in addition the non-degeneracy condition

$$\mathcal{G}_\pm \cap \ker(\text{ad}_{M_\pm}) = \{0\}.$$ (2.4)
The importance of this can be explained in a number of different ways, but in particular it guarantees that the residual KM symmetry in the presence of the constraints becomes a $W$-algebra with a basis of primary fields. At a technical level, this is revealed through the fact that the non-degeneracy condition allows us to impose highest weight or Drinfeld-Sokolov (DS) gauges, which reveal a direct connection between the KM and $W$-algebra generators, as we shall indicate below.

Let us now recall how to make manifest the gauge symmetry corresponding to the constraints (2.3). We introduce gauge fields $A_\pm$ taking values in $G_\pm$ respectively and then consider the action

$$S(g, A_+, A_-) = S_{\text{WZW}}(g) + \int d^2x \, \text{tr} \left\{ A_- (M_+ - (\partial_+ g) g^{-1}) + A_+ (M_- + g^{-1} \partial_- g) + A_- g A_+ g^{-1} \right\}$$

which we claim is equivalent to the constrained WZW model. The gauged action above is invariant under the local transformations

$$g \to u g v^{-1}, \quad A_+ \to v A_+ v^{-1} - (\partial_+ v) v^{-1}, \quad A_- \to u A_- u^{-1} - (\partial_- u) u^{-1}, \quad \text{where} \quad u(x^+, x^-) \in G_-, \quad v(x^+, x^-) \in G_+.$$ 

Although this is easy to check, it relies on a number of unusual features: the light-cone components $A_\pm$ are defined in different subalgebras of the full Lie algebra $G$; the terms involving $M_\pm$ are put in by hand, being gauge-invariant by themselves; and invariance of these and other terms in the lagrangian depends crucially on properties of the grading. Now any single component of a gauge field can always be set to zero (locally) by a gauge transformation and since $A_\pm$ live, by definition, in disjoint subalgebras, we are always free to make the gauge choice $A_\pm = 0$. On doing so, we recover the previous action $S_{\text{WZW}}$, but in addition we must impose the equations of motion for $A_\pm$, which in this gauge give exactly the desired constraints (2.3).

We have thus established that the constrained WZW model is equivalent to the gauged lagrangian (2.5) on making a specific gauge choice. But there is another natural gauge choice which leads directly to the description in terms of a non-abelian Toda theory. Given the grading of $G$ introduced earlier, we can define (locally) a Gauss decomposition of an arbitrary group element:

$$g = a g_0 b, \quad \text{where} \quad a \in G_-, \quad g_0 \in G_0, \quad b \in G_+.$$ 

Now we can clearly use the gauge freedom (2.4) to set $a = b = 1$. Unlike the previous gauge choice, the fields $A_\pm$ no longer vanish, but they can be eliminated algebraically from the lagrangian. The result is a (non-abelian) Toda theory for the field $g_0 \in G_0$ governed by the action

$$S(g_0) = S_{\text{WZW}}(g_0) - \int d^2x \, \text{tr} \left\{ M_+ g_0 M_- g_0^{-1} \right\}.$$ 

The integrability of these models was first discussed in [1] from a rather different point of view.

We emphasize that the first term in the Toda action is again of WZW type, though now for the zero-grade subgroup $G_0$. This part of the action has a KM symmetry $\tilde{G}_0$, but the potential term involving $M_\pm$ breaks this down to a smaller symmetry. If $G_0$ is non-Abelian, the theory as a whole will in general have a residual Kac-Moody invariance $\tilde{K}$ contained within its chiral symmetry algebra, where the subalgebra $K \subset G_0$ is the centralizer of $M_\pm$. 

5
2.2 \( sl(2) \) embeddings

The data specifying the grading and constraints on the original WZW model can be expressed in terms of an embedding of \( sl(2, \mathbb{R}) \) into \( \mathcal{G} \). We consider a set of generators \( \{M_0, M_+, M_-\} \) which obey

\[
[M_0, M_\pm] = \pm M_\pm , \\
[M_+, M_-] = 2M_0 ,
\]

and which are therefore a basis for an \( sl(2, \mathbb{R}) \) subalgebra \( S \subset \mathcal{G} \). As the notation suggests, \( M_\pm \) should be identified with the specific Lie algebra elements used earlier in our definition of the constraints (2.3). We can decompose \( \mathcal{G} \) into irreducible representations of \( S \), with \( \text{Ad}_{M_0} \) having integer or half-integer eigenvalues, and we then say that \( S \subset \mathcal{G} \) is an integral or half-integral embedding respectively. Given an integral embedding, we can immediately define an integral grading \( M = M_0 \), which always satisfies the non-degeneracy condition (2.4). This is the class of reduced WZW models we shall focus on in this paper. We should emphasize that we will always understand the embedded subalgebra \( S \) to have a preferred set of generators as above, so that we are really specifying not just the subalgebra but also the particular element \( M_0 \) which defines the grading.

There is actually a more general possibility for the choice of grading whenever there exists a generator \( Y \in \mathcal{G} \) that commutes with the entire subalgebra \( S \). Then we can also consider an integral grading operator of the form \( M = M_0 + Y \), provided (2.4) still holds. One can show [1] that taking a non-zero \( Y \) does not change the \( W \)-algebra in the reduced theory, so from this point of view one gains nothing new. In certain cases, however, the addition of \( Y \) can be used to define an integral grading from a half-integral embedding, giving a more general class of lagrangian realizations. It can also be shown that to any choice of integer grading \( M \) and constraints \( M_\pm \) which together satisfy (2.4), one can always associate an embedded subalgebra \( S \) in the manner described above (see e.g. sections 3.3, 3.4 and appendix B of [1] for more details). Notwithstanding these comments, it is natural to restrict attention in the first instance to the case \( Y = 0 \). In fact it is also true that \( Y \neq 0 \) leads to no new possibilities as far as the classification given in section 5 is concerned.

From what we have said so far, we are confronted with the task of understanding the possible \( sl(2, \mathbb{R}) \) subalgebras \( S \) in \( \mathcal{G} \). Embeddings of \( sl(2, \mathbb{C}) \) in complex Lie algebras have been classified by Dynkin [8], and one may extend his results immediately to the case where \( \mathcal{G} \) is real but maximally non-compact. The classification states that any embedding of \( sl(2) \) into \( \mathcal{G} \) can be realized as a principal embedding in a regular subalgebra \( \mathcal{H} \subset \mathcal{G} \), up to a few exceptions which can be enumerated. We recall that a regular subalgebra is one for which \( \Delta(\mathcal{H}) \subset \Delta(\mathcal{G}) \), where \( \Delta \) denotes the set of all roots, while a principal \( sl(2) \) embedding is one in which \( M_\pm \) are taken to be linear combinations of step operators for all the positive/negative simple roots in some chosen basis. We shall denote an embedding of this important kind simply by the pair \( (\mathcal{G}, \mathcal{H}) \). Abelian Toda models correspond to reductions of the form \( (\mathcal{G}, \mathcal{G}) \), and are so called because the group \( G_0 \) in this case just a maximal torus in \( G \). In section 5 we shall classify the integral embeddings \( (\mathcal{G}, \mathcal{H}) \) with \( \mathcal{G} \) classical and \( \mathcal{H} \) simple. The general results given in this section, however, will be independent of any such more detailed information about an embedding \( S \subset \mathcal{G} \).

The exceptions are \( \frac{n-2}{2} \) embeddings in \( D_n \), one embedding in \( E_6 \), and two embeddings in each of \( E_7 \) and \( E_8 \), which are not principal in any regular subalgebra [8].

\[1\]
2.3 $\mathcal{W}$-algebras and conformal field theory

We have seen above that to every integral embedding $\mathcal{S} \subset \mathcal{G}$ we can associate a non-abelian Toda model. We now recall how this leads to a realization of an extended conformal algebra which we denote by $\mathcal{W}_G^S$. We need to clarify how the generators of the $\mathcal{W}$-algebra appear, and how this relates to the usual approach in CFT.

Up until now we have been working in Minkowski spacetime, with Kac-Moody currents $J_\pm(x^\pm)$ taking values in the real Lie algebra $\mathcal{G}$. If we introduce a basis of generators $t_a$ for $\mathcal{G}$ then we can write $J_\pm = \sum_a J^{a}_\pm t_a$ and the components $J^{a}_\pm$ are real classical fields, or hermitian operators upon quantization. To make contact with the usual language of conformal field theory, we take our model to be defined on a circle in the spatial direction, with all fields being periodic. After complexifying the coordinates $(x^\pm) \rightarrow (w, \bar{w})$ we use a conformal transformation to map from the cylinder to the plane $(w, \bar{w}) \rightarrow (z, \bar{z}) = (e^{iw}, e^{-i\bar{w}})$. Now our KM currents are the familiar holomorphic and anti-holomorphic quantities $J(z) = J^a(z) t_a, \quad \bar{J}(\bar{z}) = \bar{J}^a(\bar{z}) t_a$

and for our purposes we may set the level $k = 1$ without loss of generality. The energy-momentum tensor similarly has components $T(z)$ and $\bar{T}(\bar{z})$.

An important point which deserves emphasis is that all fields regarded as functions on the complexified plane $(z, \bar{z})$ have a natural notion of hermitian conjugation which is inherited from their original definition as fields on Minkowski spacetime. For a holomorphic field $\Phi$ which is quasi-primary with conformal weight $h$, the hermitian conjugate $\Phi^*$ is defined by

$$\Phi^*(z) = (i/z)^{2h} \Phi(1/z). \quad (2.8)$$

We stress that $z$ and $\bar{z}$ are independent complex variables and that the conjugation operation on fields is defined entirely within each holomorphic or anti-holomorphic sector. Real fields in Minkowski space become self-conjugate, satisfying $\Phi = \Phi^*$. In particular this applies to the Kac-Moody currents $J^a(z)$ with $h = 1$. It also applies to the energy-momentum tensor $T(z)$ with $h = 2$, giving the standard hermiticity properties for the Virasoro generators.

One very powerful aspect of the conformal reduction of WZW models is that much of the structure of $\mathcal{W}$-algebras which emerge can be understood simply by analyzing how the adjoint representation of $\mathcal{G}$ decomposes into irreducible representations of the embedded subalgebra $\mathcal{S}$. We shall therefore choose a basis of generators $\{t_a\}$ to be states of definite total spin $j$ and $M_0$ eigenvalue $m$ with respect to $\mathcal{S}$. In addition, we need a label $\alpha$ to distinguish representations of the same spin. Our generators will now be written $\{t^\alpha_{(j,m)}\}$. The subalgebra $\mathcal{S}$ itself is of course a representation of spin $j = 1$ which we label by $\alpha = 0$, so that $t^0_{(1,0)} = M_0$ and $t^0_{(1,\pm1)} = M_{\pm}$. Concentrating on the holomorphic sector, the Kac-Moody current can be expanded $J(z) = J^a_{(j,m)}(z) t^\alpha_{(j,m)}$ with respect to this basis, and the constraints (2.3) then take the form:

$$J^a_{(j,m)}(z) = \begin{cases} 1 & \text{if } \alpha = 0 \quad (j, m) = (1, -1) \\ 0 & \text{else} \end{cases} \quad \text{for } m < 0. \quad (2.9)$$

As mentioned before, these constraints are first-class, corresponding to a gauge symmetry of the constrained action. The KM currents themselves transform under this gauge symmetry, reflecting its non-abelian nature. The generators of the symmetry algebra $\mathcal{W}_G^S$
appearing in the constrained model can be defined as the set of gauge-invariant differential polynomials in the KM currents. It is more enlightening to make this concrete, however, by introducing some specific way of fixing the gauge symmetry. Any particular gauge choice allows the gauge-invariant polynomials to be identified with certain components of the KM currents, and the classical $W$-algebra is then given by the Dirac brackets of these quantities, defined using the full set of constraints.

A very convenient gauge choice in which the generators have a particularly simple structure is the highest-weight Drinfeld-Sokolov gauge, where the only non-zero current components correspond to generators of highest weight with respect to $S$, i.e.

$$J_{hw}(z) = M_{-} + \sum J_{(j,j)}^{\alpha}(z) t_{(j,j)}^{\alpha}. \quad (2.10)$$

One can show that there is exactly one independent gauge-invariant polynomial for each of the highest weight components, so that the independent generators of $W_{G}^{\alpha}$ are simply $W_{j+1}^{\alpha}(z) = J_{(j,j)}^{\alpha}(z)$. Moreover, it is easy to show that the fields $W_{h}^{\alpha}$ have (holomorphic) conformal weight $h = j + 1$, so that the spin-content of the $W$-algebra can be simply read off from the representation theory associated with the embedding $S \subset G$, see [10].

Two examples are worth mentioning. First, the energy-momentum tensor $T$ of the extended conformal algebra appears as the gauge-invariant polynomial corresponding to the highest weight generator $M_{+}$—i.e. to the representation carried by $S$ itself, which we have labelled $\alpha = 0$. In this case $j = 1$ (the adjoint representation) and of course $T$ has $h = 2$ (although it is only quasi-primary). Second, consider the centralizer $K \subset G_{0}$ of $S$ in $G$. Since it commutes with $S$, its generators must appear as singlets, with $j = 0$, in the highest weight decomposition. Consequently, these generators have conformal weight $h = 1$ and we expect them to be KM currents. This corresponds exactly to the residual $\hat{K}$ Kac-Moody symmetry of the non-abelian Toda action to which we drew attention earlier.

It is important to stress that the simple relation between highest-weight generators and gauge-invariant polynomials holds only in the highest weight gauge. In a general DS gauge, the gauge-invariant polynomials take the form:

$$W_{j+1}^{\alpha}(z) = J_{(j,j)}^{\alpha}(z) + \ldots, \quad (2.11)$$

where the dots denote terms which are non-zero in general, but which are higher order in currents and/or derivatives. It has been shown [12] that this form of the generators holds also in the quantized algebra. This gives the generators of the $W$-algebra as differential polynomials in all the currents in $\hat{G}$.

Apart from the DS class of gauges, another important choice is the diagonal gauge, where only currents corresponding to generators with zero grade are non-vanishing. In this case we can write the current as:

$$J_{\text{diag}}(z) = M_{-} + \sum J_{(j,0)}^{\alpha}(z) t_{(j,0)}^{\alpha}. \quad (2.14)$$

Using this gauge it is possible to see that if we restrict the general differential polynomial expressions (2.11) involving the $\hat{G}$ currents to just those with zero grade, i.e. we keep only the currents from $\hat{G}_{0}$, then this gives an equivalent realization of $W$. This restricted realization is clearly the more natural point-of-view when considering the non-abelian Toda action, with fields in $G_{0}$, rather than the original gauged WZW theory based on $G$. The equivalence between these two different sets of variables is essentially the so-called generalized Miura transformation.
Finally, we recall that the structure of the original Lie algebra \( \mathcal{G} \) is always present in algebra \( \mathcal{W}^G_\sigma \) in the following way. Given an extended conformal algebra, we can define the (linearized) vacuum preserving algebra (VPA) by expanding each generator \( W^\alpha(z) \) of conformal weight \( h \) in modes: 
\[
W^\alpha(z) = \sum_n W_n^\alpha z^{-n-h}
\]
and considering the finite-dimensional subalgebra which is generated by \( W_n^\alpha \) with \( |n| \leq (h-1) \) on dropping all quadratic and higher-order terms. This vacuum-preserving algebra is isomorphic to the original Lie algebra \( \mathcal{G} \), as explained in [13]. Furthermore, the VPA always contains the subalgebra \( \mathcal{S} \), whose generators \( \{M_0, M_\pm\} \) correspond precisely to the usual \( sl(2) \) subalgebra of the Virasoro algebra with generators \( \{L_0, L_\pm\} \). These remarks will prove important later.

3 Real Forms

3.1 Real Forms of Lie algebras and \( sl(2) \) embeddings

We first recall some basic facts concerning the relationship between a real Lie algebra \( \mathcal{G} \) and its complexification, the complex Lie algebra \( \mathcal{G}^c = \mathcal{G} \otimes \mathbb{C} \) (for more details see e.g. [14, 13]). We then give methods for constructing real forms which we shall later use extensively.

For any real Lie algebra \( \mathcal{G} \) or complex Lie algebra \( \mathcal{G}^c \), the Killing form will be denoted by \( \eta(X,Y) = \lambda \text{tr}(XY) \) with \( \lambda \) a suitable constant. As a basis for either of these algebras we introduce an orthonormal set of generators \( \{t_a\} \) obeying
\[
[t_a, t_b] = f_{ab}^c t_c \quad \text{where} \quad f_{ab}^c \in \mathbb{R} \quad (3.12)
\]
\[
\eta(t_a, t_b) = \eta_{ab} = \sigma_a \delta_{ab} \quad \text{where} \quad \sigma_a = \pm 1 \quad (3.13)
\]
If \( d_\pm \) denotes the number of basis elements for which \( \sigma_a = \pm 1 \) then the real dimension of the real Lie algebra \( \mathcal{G} \) is \( \dim(\mathcal{G}) = d_+ + d_- \). The character of \( \mathcal{G} \) is defined to be the signature of the Killing form, \( \sigma(\mathcal{G}) = \sum_a \sigma_a = d_+ - d_- \). The character provides a way of distinguishing between different real forms and hence of classifying them.

We shall reserve the symbols \( A_n, B_n, C_n, D_n \) for complex Lie algebras, while their real forms will be denoted by the symbols given below together with their characters.

| \( \mathcal{G}^c \) | \( \mathcal{G} \) | \( \sigma \) |
|---|---|---|
| \( A_n \) | \( sl(n+1) \equiv sl(n+1, \mathbb{R}) \) | \( n \) |
| \( \text{su}(p,q) \) | \( p+q = n+1 \) | \( 1 - (p-q)^2 \) |
| \( \text{su}^*(n+1) \) | \( n \) odd | \( -n - 2 \) |
| \( B_n \) | \( so(p,q) \) | \( p+q = 2n+1 \) | \( n + \frac{1}{2} - \frac{1}{2}(p-q)^2 \) |
| \( C_n \) | \( sr(n) \equiv sp(2n, \mathbb{R}) \) | \( n \) |
| \( sp(p,q) \) | \( p+q = n \) | \( -2[n + (p-q)^2] \) |
| \( D_n \) | \( so(p,q) \) | \( p+q = 2n \) | \( n - \frac{1}{2}(p-q)^2 \) |

It is clear from the table that the character \( \sigma \) usually specifies the real form \( \mathcal{G} \) uniquely given the complex type \( \mathcal{G}^c \). The only exceptions occur in the \( A_n \) and \( D_n \) series. For \( n = 4k^2 - 3 \), the algebras \( \text{su}^*(n+1) \) and \( \text{su}(k^2+k-1,k^2-k-1) \) have the same character but are not isomorphic for \( k > 1 \). For \( n = k^2 \) the algebras \( \text{so}^*(2n) \) and \( \text{so}(n+k,n-k) \) also have the same character without being isomorphic. In these cases the algebras can be distinguished by their maximal compact subalgebras.
We shall use the symbol $U_1$ to mean a complex Lie algebra with a single generator. This has only one distinct real form in any intrinsic sense, of course, but it is nevertheless useful to distinguish a ‘compact’ real form $u(1)$ and a ‘non-compact’ form $gl(1)$ which arise through an embedding into a non-abelian algebra whose Killing form is negative-definite or positive-definite respectively when restricted to these abelian subalgebras.

A convenient way to define a real form of a given complex Lie algebra $G^c$ is via a conjugation map, by which we mean an anti-linear automorphism $\tau$ of order two on $G^c$:

$$\tau(X), \tau(Y) = \tau([X,Y]), \quad X, Y \in G^c.$$

$$\tau(aX + bY) = a^* \tau(X) + b^* \tau(Y), \quad a, b \in \mathbb{C}.$$  

$$\tau^2 = 1$$

An automorphism of this type essentially corresponds to a notion of complex conjugation on $G^c$, which explains its name (we follow [15] in this). Given a conjugation map, we can define a corresponding real Lie algebra $G^\tau$ consisting of the elements it leaves invariant:

$$G^\tau = \left\{ X \in G^c \mid \tau(X) = X \right\}.$$  

(3.15)

The statements that $G^\tau$ is a real subalgebra of $G^c$ and that $\tau$ is an anti-linear automorphism of $G^c$ are equivalent, since $x, y \in G$ means that $[x, y] = [\tau(x), \tau(y)] = \tau([x, y]) \in G$. It is also easy to see that any real form of $G^c$ arises in this way [14, 15]. (We shall often omit the label $\tau$ and write the resulting real Lie algebra simply as $G$ when no possible confusion can arise.)

A conjugation map $\tau$ can be defined by its action on some basis $\{t_a\}$ for $G^c$ as introduced in (3.12) and (3.13) above, the extension to the whole algebra being fixed by the property of anti-linearity. We write

$$\tau(t_a) = \sum_b t_b \tau_{ba}.$$  

(3.16)

where $\tau_{ab}$ is an $(n \times n)$ matrix which must satisfy $\tau_{ab}^* \tau_{bc} = \delta_{ac}$. A general element of the corresponding real Lie algebra $G^\tau$ is then

$$\Phi = \sum_a \phi_a t_a, \quad \text{where} \quad (\phi_a)^* = \sum_b \tau_{ab} \phi_b.$$  

(3.17)

It is sufficient for our purposes to consider a basis $\{t_a\}$ for $G^c$ as introduced in (3.12) and (3.13) above, the extension to the whole algebra being fixed by the property of anti-linearity. We write

$$\tau(t_a) = \sum_b t_b \tau_{ba}.$$  

(3.16)

where $\tau_{ab}$ is an $(n \times n)$ matrix which must satisfy $\tau_{ab}^* \tau_{bc} = \delta_{ac}$. A general element of the corresponding real Lie algebra $G^\tau$ is then

$$\Phi = \sum_a \phi_a t_a, \quad \text{where} \quad (\phi_a)^* = \sum_b \tau_{ab} \phi_b.$$  

(3.17)

It is sufficient for our purposes to consider a basis $\{t_a\}$ for which the Killing form and the conjugation map are simultaneously diagonalized, so that in addition to (3.13) we have:

$$\tau_{ab} = \epsilon_a \delta_{ab}, \quad \text{where} \quad \epsilon_a = \pm 1.$$  

(3.18)

Notice that $\tau(t_a) = t_a$ automatically extends to an anti-linear automorphism because the structure constants $f_{abc}$ are real. The resulting real form $G$ is simply the real span of the basis $\{t_a\}$ with character $\sigma(G) = \sum_a \sigma_a$ as discussed in the first paragraph. More generally, however, we see from (3.18) that the real form $G^\tau$ has character

$$\sigma(G^\tau) = \sum_a \sigma_a \epsilon_a.$$  

(3.19)

Of particular importance for any semi-simple complex Lie algebra $G^c$ is the Cartan-Weyl basis. Let $H$ be a choice of Cartan subalgebra and $\Delta$ the corresponding set of
non-vanishing roots. For each \( \alpha \in \Delta \) we have a Cartan generator \( h_\alpha \in H \) and step operators \( e_{\pm \alpha} \) obeying

\[
[h_\alpha, e_\beta] = (\alpha \cdot \beta)e_\beta, \quad [e_\alpha, e_{-\alpha}] = h_\alpha, \quad [e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha + \beta} \quad (\alpha + \beta \neq 0) \tag{3.20}
\]

where \( N_{\alpha\beta} \) is non-vanishing if \( \alpha + \beta \in \Delta \). Let \( \{\alpha_j\} \) be a set of simple roots with \( j = 1, \ldots, \text{rank}(\mathcal{G}^c) \); it is convenient to set \( h_j = h_{\alpha_j} \). For each root we have

\[
\alpha = \sum_j \ell_j(\alpha)\alpha_j, \quad h_\alpha = \sum_j \ell_j(\alpha)h_j, \quad \ell(\alpha) = \sum_j \ell_j(\alpha) \tag{3.21}
\]

where \( \ell_j(\alpha) \) are integers and \( \ell(\alpha) \) is the height of the root. The subset of positive roots \( (\ell(\alpha) \geq 0) \) will be denoted \( \Delta_+ \). With this normalization for the generators the non-trivial entries of the Killing form are \( \eta(h_\alpha, h_\beta) = (\alpha_\cdot \alpha_j), \) a positive-definite matrix, and \( \eta(e_\alpha, e_{-\alpha}) = 1 \) for each root \( \alpha \). It follows that we can write an orthogonal decomposition of the space of generators \( \{h_j\} \oplus \{e_\alpha + e_{-\alpha}\} \oplus \{e_\alpha - e_{-\alpha}\} \) and by taking appropriate real combinations within each of these subspaces the Killing form can be diagonalized with eigenvalues +1, +1, −1 respectively.

There are two real forms which arise in a uniform way for any complex Lie algebra \( \mathcal{G}^c \) by using the Cartan-Weyl basis. Consider first a conjugation map which fixes each of the Cartan-Weyl generators:

\[
\tau(h_j) = h_j, \quad \tau(e_{\pm \alpha}) = e_{\pm \alpha} \tag{3.22}
\]

This defines a real Lie algebra consisting of the real span of the Cartan-Weyl generators. The character is \( \sigma = \text{rank}(\mathcal{G}^c) \): this is the maximally non-compact or split real form. Alternatively, if we consider the conjugation map

\[
\tau(h_j) = -h_j, \quad \tau(e_{\pm \alpha}) = -e_{\mp \alpha} \iff \tau(e_\alpha \pm e_{-\alpha}) = \mp(e_\alpha \pm e_{-\alpha}) \tag{3.23}
\]

then the resulting real Lie algebra is the real span of \( \{ih_j, i(e_\alpha + e_{-\alpha}), (e_\alpha - e_{-\alpha})\} \). From the remarks above it is clear that in this case the Killing form is negative-definite and the character is \( \sigma = -\text{dim}(\mathcal{G}^c) \); this is the compact real form.

In each of these two cases the conjugation map was chosen to fix the Cartan subalgebra \( H \). In fact there is no loss of generality in assuming this, since any automorphism is conjugate to one with this property. Furthermore, the fact that \( \tau(H) = H \) means that the algebra automorphism specifies a root automorphism, by which we mean an element of \( \text{Aut}(\Delta) \), the group of linear transformations which fix the set of roots \( \Delta \). This root automorphism will, with a slight abuse of notation, also be denoted by \( \tau \). The root automorphism does not quite specify the algebra automorphism uniquely, however; the most general possibility is

\[
\tau(h_{\alpha_j}) = h_{\tau(\alpha_j)}, \quad \tau(e_\alpha) = \chi_\alpha e_{\tau(\alpha)}, \quad \tau(e_{-\alpha}) = \chi_\alpha e_{\tau(-\alpha)} \quad \text{where} \quad \chi_\alpha = \pm 1. \tag{3.24}
\]

The constants \( \chi_\alpha \) must satisfy the consistency condition

\[
\chi_{\alpha + \beta} = \frac{N_{\tau(\alpha)\tau(\beta)}N_{\alpha\beta}}{N_{\alpha\beta}} \chi_\alpha \chi_\beta \tag{3.25}
\]

but we may choose \( \chi_\alpha \) independently for each simple root, and then the remaining values are determined. Finally, the algebra automorphism \( \tau \) defined in this way need not
necessarily be of order two, even if the root automorphism in Aut(Δ) is. To ensure this one must have in addition

\[ \chi_\alpha = \chi_{\tau(\alpha)} \quad (3.26) \]

for all \( \alpha \). It is not difficult to see that it is sufficient to impose this condition for each of the simple roots.

We should now draw attention to the relationship between Aut(Δ), defined above, and the Weyl group \( W(\Delta) \), which is generated by reflections in the (hyper)planes perpendicular to each root. The Weyl group \( W(\Delta) \) is a normal subgroup of Aut(Δ) and the quotient Aut(Δ)\!/W(Δ) can be identified with outer-automorphisms of the algebra, or more pictorially with the symmetry group of its Dynkin diagram. It is therefore non-trivial only for the (simple) algebras \( A_n, D_n \) and \( E_6 \). We shall refer to any representative of this quotient group as a \textit{diagram automorphism}, for obvious reasons.

We can now set the split and compact real forms in a more general context. The split real form corresponds to the root automorphism \( \tau(\alpha) = \alpha \), which is simply the identity in the Weyl group. It is clearly consistent to choose \( \chi_\alpha = 1 \) as in (3.22). The compact real form corresponds to the root automorphism \( \tau(\alpha) = -\alpha \). For this element of Aut(Δ), the general solution of the consistency condition (3.25) is

\[ \chi_\alpha = (-1)^{\ell(\alpha)+1} \prod_j (\chi_{\alpha_j})^{\ell_j(\alpha)} \quad (3.27) \]

The compact form corresponds to the particular choice \( \chi_\alpha = -1 \) for all roots, but different choices of the \( \chi_\alpha \) could lead to other conjugation maps and real forms.

We are now in a position to calculate the character of a real form defined by any conjugation map of the general kind (3.24).

**Theorem 1** Let \( \tau \) be a conjugation map on \( G^c \) which fixes a Cartan subalgebra \( H \) and which is defined from a root automorphism by the quantities \( \chi_\alpha \) as in (3.24). The character of the corresponding real Lie algebra \( G \) is

\[ \sigma = \text{tr}(\tau_H) + 2 \sum_{\alpha \in \Gamma_-} \chi_\alpha \quad (3.28) \]

where \( \tau_H \) is the restriction of \( \tau \) to \( H \) and \( \Gamma_- = \{ \alpha \in \Delta_+ | \tau(\alpha) = -\alpha \} \).

Proof: The key idea is to calculate the contributions to the character from combinations of root generators which are diagonal with respect to both \( \tau \) and \( \eta \). If \( \tau(\alpha) = \alpha \) and \( \tau(e_\alpha) = \chi_\alpha e_\alpha \) then the combinations \( (e_\alpha \pm e_{-\alpha}) \) have equal eigenvalues \( \chi_\alpha \) under \( \tau \) but opposite eigenvalues \( \pm 1 \) for the Killing form. As a result we see from (3.19) that their contributions to the character always cancel. If \( \tau(\alpha) = -\alpha \), on the other hand, then \( \tau(e_{\pm \alpha}) = -\chi_\alpha e_{\mp \alpha} \) and now the combinations \( (e_\alpha \pm e_{-\alpha}) \) have opposite eigenvalues \( \pm \chi_\alpha \) under \( \tau \). The total contribution to the character from these generators is therefore \( 2\chi_\alpha \) for each \( \alpha \in \Delta_+ \). Finally we consider those roots for which \( \tau(\alpha) \neq \pm \alpha \). In this case there are four linearly-independent combinations \( e_\alpha + v_1 e_{\tau(\alpha)} + v_2 e_{-\alpha} + v_1 v_2 e_{-\tau(\alpha)} \) with \( v_i = \pm 1 \). The eigenvalue of \( \tau \) is \( v_1 \chi_\alpha \) while the eigenvalue of \( \eta \) is \( v_2 \). The contributions to the character from all four combinations therefore cancel. The result now follows. \( \square \)

Note that the formula in Theorem 1 successfully reproduces the two cases we encountered earlier. If the root automorphism \( \tau \) is the identity in the Weyl group then \( \Gamma_- \) is
empty and $\sigma = \text{rank}(G)$ as required for the split form. If we take instead the root automorphism $\tau(\alpha) = -\alpha$ then for the general case given by (3.27) we find that eq. (3.28) becomes

$$\sigma = -\text{rank}(G) + 2 \sum_{\alpha \in \Delta^+} (-1)t(\alpha) + 1 \prod_j (\chi_{\alpha_j})^{\ell_j(\alpha)}$$

(3.29)

For the choice $\chi_\alpha = -1$ we find $\sigma = -\dim(G)$, as required for the compact real form.

Having developed some technology to discuss the real forms of Lie algebras, we recall that our chief concern in this paper is with the integrable models and $W$-algebras which are associated to embeddings of $\text{sl}(2, \mathbb{R})$ into real Lie algebras, with a preferred basis $\{M_0, M_+, M_-\}$ which selects the grading operator $M_0$. There is a very simple idea which we shall use later to obtain from the well-known classification of complex embeddings a classification of (a large subset of) real embeddings. An embedding $S \subset G$ extends automatically to an embedding of complex algebras $S^c \subset G^c$, and the preferred basis $\{M_0, M_+, M_-\}$ for the real algebra $S \cong \text{sl}(2, \mathbb{R})$ becomes a preferred basis for the complex algebra $S^c \cong \text{sl}(2, \mathbb{C})$. Since much more is known about complex embeddings, we would like to reverse this procedure, i.e., to take an embedding $S^c \subset G^c$ with a preferred basis and obtain from it an embedding of real forms $S \subset G$. It is clear that what we require is the following.

**Definition 1** An embedding $S^c \subset G^c$ is compatible with a real form $G$ or the conjugation map $\tau$ which defines it iff $\tau$ fixes the preferred $\text{sl}(2)$ generators:

$$\tau(M_0) = M_0, \quad \tau(M_{\pm}) = M_{\pm}$$

(3.30)

When this condition is met we clearly have an embedding $S \subset G$ of the type we are seeking.

We remark that the idea of a specific grading, as defined by $M_0$ above, arises naturally in the work of Dynkin on embeddings $S^c \subset G^c$. We shall make use later of a particular result which underlies Dynkin’s classification, which states that for any integral complex embedding it is possible to choose a Cartan subalgebra of grade zero elements, and a basis of step operators for simple roots which have gradings 0 or $\pm \frac{1}{2}$.

In the special case of a principal embedding $S^c \subset G^c$, all the raising and lowering operators corresponding to the simple roots have grades +1 and -1 respectively. This means that $M_{\pm}$ are linear combinations of these step operators with non-zero coefficients. It is easy to see that for such an embedding the only compatible conjugation maps correspond to diagram automorphisms. We shall refer to the resulting real embedding $S \subset G$ as being principal, irrespective of whether the diagram automorphism involved is trivial or not. Nearly all complex embeddings are actually of the type $(G^c, H^c)$, meaning that $S^c \subset H^c$ is a principal embedding in the regular subalgebra $H^c \subset G^c$. By extension of what we have already said, any compatible real form $(G, H)$ corresponds to a conjugation map which must act as a diagram automorphism on $H^c$. These observations will prove important in section 5.

Before moving on to Toda theories proper, we also note the following Corollary of Theorem 1.

---

2We repeat that we deal only with integral embeddings in this paper; half-integral embeddings would allow half-integral gradings of some step operators. Note also that in Dynkin’s original work the definition of the grading is double that used here, so what we call integral gradings would correspond to even integral gradings in Dynkin’s terminology.
Corollary 1 Consider a real embedding $S \subset G$ defined by a compatible conjugation map from a complex embedding $S^c \subset G^c$. Let $G_0$ be the real form of the zero-grade subalgebra $G_0^c$. The characters are related by

$$\sigma(G) = \sigma(G_0)$$

Proof: The automorphism $\tau$ respects the gradation, i.e. the grade of $\tau(e_\alpha)$ is equal to the grade of $e_\alpha$. Furthermore, the grade of $e_{-\alpha}$ is minus the grade of $e_\alpha$. This implies that if $\tau(e_\alpha) = \pm e_{-\alpha}$, then $e_\alpha \in G_0$. Since furthermore the Cartan subalgebra of $G$ is contained in $G_0$, the result follows immediately from eq. (3.28).

3.2 Real Forms and Toda Theories

The WZW models which are most obviously relevant for physical applications are those defined on a real compact group manifold, since these will have positive-definite actions and the corresponding KM algebras will have highest-weight, unitary representations. Nevertheless, WZW models for non-compact groups have attracted some attention in the past, for example as models of string-theory in curved space-time, see e.g. [16], or as a starting point for the construction of solvable black hole models, see e.g. [17].

Consider a WZW model based on a group $G$ whose real Lie algebra $G$ can be defined from the complex Lie algebra $G^c$ by a conjugation map $\tau$. The group elements appearing in the action can be parametrized locally (in a neighbourhood of the identity) by $g = \exp(\sum a \phi_a(x) t_a)$, where the fields $\phi_a(x)$ satisfy the reality-conditions (3.17). The currents of this theory are also elements of the Lie algebra, and so they satisfy analogous reality conditions:

$$J(z) = J^a(z) t_a, \quad (J^a)^* = \sum_b \tau_{ab} J^b$$

(3.31)

We want to investigate how these possible reality conditions for the fields and the currents generalize to the case of a non-Abelian Toda model with $W^G_S$ symmetry.

The action (2.7) for the non-Abelian Toda model is defined for an embedding $S \subset G$ which specifies the real Lie algebra $G_0$. Now the the first term in the action $S_{WZW}(g_0)$ makes sense whatever real form of $G_0^c$ we choose, but it is natural to ask which of these real forms is consistent with reality of the potential,

$$V = \text{tr} \{ M_+ g_0 M_- g_0^{-1} \}$$

(3.32)

and which of them arise from a choice of real form for $G^c$. The answer is provided by the following:

Theorem 2 Consider a complex embedding $S^c \subset G^c$ with zero-grade subalgebra $G_0^c$. A real form $G_0$ for this subalgebra can be extended uniquely to a real form $G$ for the whole algebra which is compatible with the real embedding $S \subset G$ if and only if the Toda potential $\text{tr} \{ M_+ g_0 M_- g_0^{-1} \}$ is real for any $g_0 \in G_0$.

Proof: The implication is immediate in one direction—since for a real form $G$ with subalgebras $S$ and $G_0$ the quantity (3.32) is manifestly real.

To show the converse is more involved. It is useful to begin by thinking of $G_0$ as defined by a conjugation map $\tau_0$ on $G_0^c$. We want to show that we can extend this uniquely to a
conjugation map $\tau$ on $G^c$ which satisfies (3.30). We note first that if such a map exists it is certainly unique, because any $x \in G^c$ can be written in the form

$$x = (\text{ad}_{M_{\pm}})^m x_0, \quad x_0 \in G^c_0 \quad \Rightarrow \quad \tau(x) = (\text{ad}_{M_{\pm}})^m \tau_0(x_0)$$

in order to satisfy (3.30). We need to show that this actually defines a conjugation map on all of $G^c$.

Rather than doing this directly, however, the argument can be simplified by focusing on elements at grades 0 and $\pm 1$. We can make use of a result of Dynkin mentioned earlier, which states that for any integral complex embedding $S^c \subset G^c$ we can choose a Cartan subalgebra of grade zero elements and a basis of simple root generators with grades 0 or $\pm 1$. Since any element of the algebra can be expressed in terms of these in a canonical way, it is enough to check that we have an automorphism acting on elements with these specific grades. Thus it suffices to show that:

$$[\tau(x), \tau(y)] = \tau([x, y]) \quad \text{for} \quad \left\{ \begin{array}{l}
(i) \quad x, y \in G^c_0 \\
(ii) \quad x \in G^c_0, y \in G^c_{\pm 1} \\
(iii) \quad x \in G^c_{\pm 1}, y \in G^c_{\mp 1}
\end{array} \right.$$  

In fact, we shall recast this in yet another equivalent form. Recall that the invariant subset of a conjugation map is a real Lie algebra precisely because the conjugation map is an automorphism, as discussed in the last section. If we consider the sets $G_n$ consisting of the elements fixed by $\tau$ at each specific grade, then it is clear that the conditions above are equivalent to showing that:

$$[x, y] \in G_n \quad \text{for} \quad \left\{ \begin{array}{l}
(i) \quad x, y \in G_0 \\
(ii) \quad x \in G_0, y \in G_{\pm 1} \\
(iii) \quad x \in G_1, y \in G_{-1}
\end{array} \right.$$  

(i) True by assumption.

(ii) Assume that (ii) is not true, i.e. that we can find $t_1 \in G_1$ and $t_0 \in G_0$ such that $[t_1, t_0] \notin G_1$. This implies that $[t_1, t_0]$, and for generic real $x$ also $e^{xt_0}t_1e^{-xt_0}$, is a complex linear combination of elements of $G$.

It is clear that any element in $G_1$ can be written in the form $\text{ad}(M_+)(t_0)$, $t_0 \in G_0$. This implies that $\text{ad}(G_0)(M_+)$ is dense in $G_1$ and we can therefore assume that we can find $g_0 \in G_0$ such that $t_1 = g_0 M_+(g_0)^{-1}$. Collecting these statements, we find that $e^{xt_0}g_0 M_+(g_0)^{-1}e^{-xt_0}$ is a complex linear combination of elements of $G$; it is now easy to find an element $g \in G_0$ such that $V = \text{tr} \{ g M_+ g^{-1} M_- \} \notin \mathbb{R}$, which is inconsistent with the assumptions.

(iii) The proof of (iii) is very similar to the proof of (ii) above. Assume (iii) does not hold, i.e. that we have $t_1 \in G_1$ and $t_{-1} \in G_{-1}$, such that $[t_1, t_{-1}] \notin G$. This implies that $[t_1, t_{-1}]$ is a complex linear combination of elements in $G_0$.

We can then find an element $t_0 \in G_0$ such that

$$\text{tr} \{ [t_1, t_{-1}]t_0 \} = \text{tr} \{ [t_0, t_1]t_{-1} \} \notin \mathbb{R}$$

and therefore, for generic real $x$, also

$$\text{tr} \{ e^{xt_0}t_1e^{-xt_0}t_{-1} \} \notin \mathbb{R}$$
Using the same arguments as in (ii), we can assume that we can find \( g_0 \) and \( h_0 \) in \( G_0 \) such that
\[
t_1 = g_0 M_+(g_0)^{-1} \quad t_{-1} = h_0 M_-(h_0)^{-1},
\]
and it is now easy to see that we can find a group element \( g \in G_0 \) such that
\[
V = \text{tr} \left\{ g M_+ g^{-1} M_- \right\} \not\in \mathbb{R}
\]
which contradicts the assumptions.

We have shown that (i), (ii), and (iii) are true, and we conclude that \( \tau \) is an automorphism; we have therefore proven theorem 2.

The result we have just proved generalizes an observation first made some time ago [18] regarding the possible reality conditions for abelian Toda theories.

**Corollary 2** Consider a principal complex embedding \( S^c \subset G^c \) so that \( G^c_0 \) is a Cartan subalgebra. A conjugation map defining a compatible real form for \( G^c \) is given by a diagram automorphism, and the reality conditions for the Toda fields which ensure a real lagrangian are therefore in one-to-one correspondence with symmetries of the Dynkin diagram.

### 3.3 Real Forms and W-algebras

The WZW model based on a real form \( G \) of a complex Lie algebra \( G^c \) has a Kac-Moody symmetry \( \hat{G} \), which is in turn a real form of the complex Kac-Moody algebra \( \hat{G} \otimes \mathbb{C} \). In a similar way, an embedding \( S \subset G \) gives rise to a real algebra \( W^G_S \) which is a real form of the complex extended conformal algebra \( W^G_S \otimes \mathbb{C} \), which can be thought of as associated with the complex embedding \( S^c \subset G^c \). Although it is often convenient algebraically to work with this complexified algebra, one should not lose sight of the fact that it is some specific real form which is truly the symmetry of the reduced WZW or non-abelian Toda theory.

For the Virasoro subalgebra of a \( W \)-algebra there is a standard set of hermiticity conditions which correspond to the fact that the energy-momentum tensor of the theory is real in the sense of (2.8). In principle one can imagine the possibility of rather complicated real forms of a given complex \( W \)-algebra which might change these hermiticity properties. In this paper, however, we shall always understand a \( W \)-algebra to come with a preferred Virasoro subalgebra, and we shall assume as part of its definition that a real form of a complex \( W \)-algebra always restricts to the standard real form for this preferred Virasoro subalgebra.

Since the generators of the Virasoro algebra \( \{ L_{-1}, L_0, L_1 \} \) correspond to the preferred generators \( \{ M_-, M_0, M_+ \} \) discussed earlier, any such real form corresponds to a compatible conjugation map. This really amounts to saying that for the complex algebras \( W^G_S \otimes \mathbb{C} \), the only real forms we allow are of type \( W^G_S \).

Before we can state the main result of this section, we must set up some notation. In section 2.3 we described how the generators of \( W^G_S \) can be associated with the irreducible representations occurring in the decomposition of the adjoint for the embedding \( S \subset G \). If we introduce the same basis of generators \( \{ t^\alpha_{(j,m)} \} \) for \( G^c \), where \( \alpha \) labels irreducible representations of \( S^c \) as before, then the compatible conjugation map \( \tau \) which defines \( G \)
will in general act non-trivially. However, by virtue of (3.30), the action must be of the form
\[ \tau(t_{(j,m)}) = \sum_{\beta} t^\beta_{(j,m)} \tau^\beta_{\alpha} \] (3.33)
where the \((j, m)\) labels are unaffected and in particular \(\tau_{\alpha\beta}\) can have non-zero entries only for representations \(\alpha\) and \(\beta\) of the same total spin.

**Theorem 3** Consider a real embedding \(S \subset \mathcal{G}\) defined from \(S^c \subset \mathcal{G}^c\) by a compatible conjugation map \(\tau\). The generators of \(\mathcal{W}_S^G\), labelled by \(\alpha\), obey
\[ (W^\alpha)^* = \sum_{\beta} \tau_{\alpha\beta} W^\beta \] (3.34)
with \(\tau_{\alpha\beta}\) defined above in (3.33); and in particular the energy-momentum tensor is always real. Furthermore, the conformal weights of the generators of \(\mathcal{W}_S^G\) are independent of the choice of real form \(\mathcal{G}\).

Proof: Given the basis of Lie algebra generators in (3.33), the \(\mathcal{W}\)-algebra generators \(W^\alpha\) can be identified with the Kac-Moody currents \(J^\alpha_{(j,j)}\) in the highest-weight DS gauge. The required condition (3.34) follows from (3.31) given (3.33). Since the embedding is compatible with the real form, and the energy-momentum tensor is associated with the generator \(M_+\), it follows that it is always real. Finally, the matrix for the conjugation map in (3.33) only mixes \(sl(2)\) representations of the same spin. As explained in section 2, the spin determines the conformal weight of the corresponding \(\mathcal{W}\)-generator uniquely. \(\square\)

We have mentioned that the algebra \(\mathcal{G}\) can always be recovered from \(\mathcal{W}_S^G\) as its VPA. But we also know that there can be inequivalent embeddings in a given \(\mathcal{G}\) which produce distinct \(\mathcal{W}\)-algebras. A manifestation of this is that when we consider a complex embedding \(S^c \subset \mathcal{G}^c\) and then take real forms by using compatible conjugation maps \(\tau\) and \(\tau'\), we may well find real embeddings \(S \subset \mathcal{G}\) and \(S \subset \mathcal{G}'\) which are inequivalent, even when \(\mathcal{G} \cong \mathcal{G}'\). The reason is that this isomorphism of real algebras need not fix the embedded subalgebra \(S\). If the two real forms \(\mathcal{G}\) and \(\mathcal{G}'\) have zero-grade subalgebras \(\mathcal{G}_0\) and \(\mathcal{G}'_0\) and centralizers \(\mathcal{K}\) and \(\mathcal{K}'\), then a necessary condition for the isomorphism \(\mathcal{G} \to \mathcal{G}'\) to fix \(S\) is that it should restrict to isomorphisms \(\mathcal{G}_0 \to \mathcal{G}'_0\) and \(\mathcal{K} \to \mathcal{K}'\). In fact it is quite easy to find examples where \(\mathcal{G} \cong \mathcal{G}'\) and \(\mathcal{G}_0 \cong \mathcal{G}'_0\) but \(\mathcal{K} \not\cong \mathcal{K}'\) (see section 5). In this case the algebras \(\mathcal{W}_S^G\) and \(\mathcal{W}_S^{G'}\) will certainly not be isomorphic. Moreover we see that there are inequivalent non-abelian Toda models based on the same \(\mathcal{G}_0\), but with Kac-Moody symmetries corresponding to different real forms of the same complex algebra \(\mathcal{K}^c\).

One may wonder whether the real forms \(\mathcal{G}_0\) and \(\mathcal{K}\) are together enough to determine the non-abelian Toda model and its \(\mathcal{W}\)-algebra completely. If \(\mathcal{G}_0\) and \(\mathcal{G}'_0\) are isomorphic they must have the same character, and we know from Corollary 1 (page 14), that the characters of \(\mathcal{G}\) and \(\mathcal{G}'\) are then also equal. This means in most cases that \(\mathcal{G}\) and \(\mathcal{G}'\) are actually isomorphic (with the possible exceptions of the few real forms mentioned in section 3.1 with degenerate values of the character ). But we would like to know, in addition, whether this isomorphism can always be chosen to fix \(S\), in which case it would certainly be necessary to have \(\mathcal{K}\) and \(\mathcal{K}'\) isomorphic too. We shall not attempt to answer this question in all generality, but we mention that for the wide variety of cases to be
considered in the classification of section 5, the real forms $G_0$ and $K$ actually do specify the embedding $S \subset G$ and hence $W^S$ uniquely.

A related point arises when we recall that most embeddings $S^c \subset G^c$ are (with a numerable set of exceptions) of the type $(G^c, H^c)$. An important fact is that such an $sl(2, \mathbb{C})$ embedding is unique up to conjugation once the regular subalgebra $H^c$ is given up to isomorphism. The analogous result does not hold for real embeddings, however. If we consider real embeddings $(G, H)$ and $(G', H')$ defined by compatible conjugation maps $\tau$ and $\tau'$ as before, then our previous remarks imply that we may have $G \cong G'$ and $H \cong H'$, but $K \not\cong K'$. Hence, specifying $H$ up to isomorphism does not determine (up to conjugation) a generalized principal $sl(2, \mathbb{R})$ embedding.

Finally, it is also interesting to note that while a real form of an affine, or infinite-dimensional, $W$-algebra always corresponds to a real form of the corresponding finite $W$-algebra, the opposite statement is false. In fact, let $T$ be the energy-momentum tensor in a given affine algebra $W$ and denote by $\rho$ the map from $W$ to the corresponding finite algebra $W_{\text{fin}}$; then $\rho(T)$ is in the center of $W_{\text{fin}}$. This implies that we can have an automorphism on a finite $W$-algebra which does not leave $\rho(T)$ invariant, and therefore we may have a conjugation map of $W_{\text{fin}}$ which does not descend from a conjugation map of $W$.

4 Real forms and non-abelian Toda theories for algebras of rank 2

We now illustrate some of our general results and show how they can be used to find all integral embeddings into real Lie algebras of rank 2. These embeddings are each of the type $(G, H)$ which we shall obtain by conjugation maps compatible with the complex embeddings $(G^c, H^c)$. The regular subalgebras of any $G^c$ are easily found by first drawing its extended Dynkin diagram and then removing nodes to obtain all the possible Dynkin diagrams for $H^c$. For the algebras $A_2$, $B_2 \cong C_2$ or $G_2$, the results obtained by this procedure are shown in the table below. Each black node denotes a short root, and in the last column we have marked with a * those embeddings $(G^c, H^c)$ which are integral.

We shall examine each of the integral embeddings in detail below. Although we could discuss the half-integral embeddings in a similar way, these are of less direct interest from the point of view of integrable Toda theories (at least of the simplest kind). The second and third embeddings for both $B_2$ and $G_2$ are not really distinct. It is a well-known feature of Dynkin’s method that it can give rise to embeddings which are conjugate to one another, even when the regular subalgebras concerned are clearly not conjugate, and this is what happens in these two examples. Representations of Lie algebras will be denoted, as usual, by their dimension, except that for $A_1$ we shall also write $(j)$ to mean the irreducible representation of spin $j$. 
4.1 Embeddings in $A_2$

The only integral embedding in $A_2$ is the principal one

$$(A_2, A_2) : \quad M_0 = h_1 + h_2, \quad M_\pm = \sqrt{2}(e_{\pm\alpha_1} + e_{\pm\alpha_2})$$

(4.35)

The decomposition of the adjoint representation is $8 \rightarrow (2) \oplus (1)$ and the zero-grade subalgebra is simply the Cartan subalgebra $U_1 \oplus U_1$. This reduction corresponds to the Abelian Toda theory of type $A_2$. Introducing fields

$$\Phi = \phi_1 h_1 + \phi_2 h_2$$

to parametrize the zero-grade subalgebra, we see that the WZW part of the action is essentially trivial, while the potential term reads

$$V = e^{-2\phi_1 + \phi_2} + e^{-2\phi_2 + \phi_1}$$

The $W$-algebra of the theory is generated by the energy-momentum theory $T$ of spin 2 and one additional primary $W$ of spin 3, these being even and odd respectively under the interchange $\phi_1 \leftrightarrow \phi_2$. This is the famous $W_3$ algebra of Zamalodchikov.

The split real form of $A_2$ is $sl(3, \mathbb{R})$ corresponding to the identity in the Weyl group and a conjugation map which fixes all the Cartan-Weyl generators. The fields $\phi_j$ and the generators $T$ and $W$ are then all real. There is only one other conjugation map (up to isomorphism) which is compatible with the embedding (4.35). It is given by the root automorphism $\tau : \alpha_1 \leftrightarrow \alpha_2$ which acts as a reflection symmetry of the Dynkin diagram. This defines the real form $su(2,1)$.
When we impose the conditions (3.17) corresponding to the new real form we find

\[(\phi_1)^* = \phi_2 \iff \phi_1 = u + iv, \quad \phi_2 = u - iv\] (4.36)

with \(u\) and \(v\) real. The Toda potential is

\[V = 2e^{-u}\cos(3v)\]

when written in terms of the new fields, which is manifestly real, in keeping with Theorem 2. By virtue of their behaviour under \(\phi_1 \leftrightarrow \phi_2\), the generators of the \(W\)-algebra now obey

\[T^* = T \quad W^* = -W\]

in accordance with Theorem 3. This constitutes a new real form of the algebra \(W_3\) which has in fact been considered previously in [19].

4.2 Embeddings in \(B_2\)

The positive roots of \(B_2\) are \(\alpha_1\) (long), \(\alpha_2\) (short), \(\alpha_1 + \alpha_2\) and \(\alpha_1 + 2\alpha_2 = \psi\), the highest root.

The principal embedding is integral, as always, and results in an abelian Toda model. The adjoint representation decomposes as \(10 \rightarrow (3) + (1)\) and the corresponding \(W\)-algebra therefore has a single generator of spin 4 in addition to the energy-momentum tensor. The standard reality conditions for this Toda theory correspond to the maximally non-compact form \(so(3,2)\). There is no other real form of \(B_2\) compatible with the principal embedding (by corollary 2, since there is no symmetry of the Dynkin diagram) and we omit further discussion of this case.

Now we turn to the more interesting case of non-principal embeddings. Consider first the embedding defined by the short root

\[(B_2, A_1) : \quad M_0 = h_2, \quad M_\pm = \sqrt{2}e^{\pm\alpha_2}\]

The decomposition of the adjoint is

\[10 \rightarrow (1) \oplus (1) \oplus (1) \oplus (0)\]

and the zero-grade subalgebra is

\[A_1 \oplus U_1 = \{h_1 + h_2, \sqrt{2}e^{(\alpha_1 + \alpha_2)}, \sqrt{2}e^{-(\alpha_1 + \alpha_2)}\} \oplus \{h_2\}\] (4.37)

The other integral embedding in the table is defined by a pair of orthogonal long roots

\[(B_2, A_1 \oplus A_1) : \quad M_0 = h_2, \quad M_\pm = e^{\mp\alpha_1} + e^{\pm\psi}\]

In this case the adjoint representation decomposes via representations of \(A_1 \oplus A_1\) in the pattern

\[10 \rightarrow (1,0) \oplus (0,1) \oplus (1/2,1/2) \rightarrow (1) \oplus (1) \oplus (1) \oplus (0)\]

The zero-grade subalgebra is obviously again given by (4.37). The coincidence of the final pattern of representations and the zero-grade subalgebras confirm our assertion that these embeddings are equivalent; in fact it can be shown that they are conjugate (by
exponentials involving the generators $e_{\pm(a_1+a_2)}$ but we shall omit the details. The pattern of representations reveals that we have a $\mathcal{W}$-algebra consisting of an abelian Kac-Moody generator $U$ of spin 1 and two primary fields $W^\pm$ of spin 2 in addition to the energy-momentum tensor $T$.

An efficient and quite general way to calculate the Toda potential is to first identify the representations of the zero-grade subalgebra to which the generators $M_{\pm}$ belong, since the potential term involves precisely the adjoint action of this subalgebra on these generators. In the present case it is clear that the sets $\{e_{\pm a_1}, e_{\pm a_2}, e_{\pm(a_1+2a_2)}\}$ carry spin-1 representations of the $A_1$ factor in (4.37) and that they have opposite weight under the $U_1$ generator. The potential term can now be readily calculated for either of the descriptions $(B_2, A_1)$ or $(B_2, A_1 \oplus A_1)$ given above (some details are given in appendix B).

Introducing a parametrization

$$\Phi = uM_0 + 2\phi_0(h_1+h_2) + \sqrt{2}\phi_+ e_{a_1+a_2} + \sqrt{2}\phi_- e_{-(a_1+a_2)}$$

(4.38)

for $G_0$ and taking $g = \exp \Phi$ we find (up to an overall constant)

$$(B_2, A_1) : \quad V = \frac{e^{-u}}{\rho^2}(\phi_0^2 + \phi_+\phi_- \cosh 2 \rho)$$

$$(B_2, A_1 \oplus A_1) : \quad V = \frac{e^{-u}}{4\rho^2}\{((\phi_+ - \phi_-)^2 - 4\phi_0^2) \cosh 2 \rho - (\phi_+ + \phi_-)^2\}$$

where $\rho^2 = \phi_0^2 + \phi_+\phi_-$. The apparent difference in these potentials is really just an artifact of the way we are parametrizing $G_0$; they are related by exchanging $(\phi_+ + \phi_-) \leftrightarrow 2\phi_0$. This corresponds precisely to the action on $G_0$ of the isomorphism on $G$ which relates the embeddings.

Let us now consider the possible real forms consistent with these embeddings. For the split real form $G = so(3, 2)$ the Toda fields and the $\mathcal{W}$-algebra generators discussed above are all real and the zero-grade subalgebra is $G_0 = su(1, 1) \oplus gl(1)$. There is only one other real form which is compatible. This can be defined by the conjugation map

$$\tau(h_1) = -(h_1 + 2h_2), \quad \tau(h_2) = h_2, \quad \tau(e_{\pm a_1}) = e_{\mp(a_1 + 2a_2)}, \quad \tau(e_{\pm a_2}) = e_{\pm a_2}$$

which in the $(B_2, A_1 \oplus A_1)$ description simply amounts to exchanging the two $A_1$’s. The resulting real forms are $G = so(4, 1)$ and $G_0 = su(2) \oplus gl(1)$, rendering the non-abelian factor compact in the latter case.

In terms of the parametrization (4.38) the effect, according to eq. (3.14), is to take

$$(\phi_0)^* = -\phi_0 \quad (\phi_\pm)^* = -\phi_\mp \quad \iff \quad \phi_0 = i\phi_3, \quad \phi_\pm = \pm\phi_1 + i\phi_2$$

with $\phi_1$, $\phi_2$ and $\phi_3$ real. The Toda potentials can now of course be written in terms of these new real fields. For the $(B_2, A_1)$ case we find

$$V = \frac{e^{-u}}{\rho^2}((\phi_1^2 + \phi_2^2) \cos 2\phi + \phi_3^2)$$

with $\phi^2 = \phi_1^2 + \phi_2^2 + \phi_3^2$, while for the $(B_2, A_1 \oplus A_1)$ embedding we have $\phi_2 \leftrightarrow \phi_3$ in the above, in keeping with the relationship between the coordinate systems mentioned earlier.
The reality properties of the \( \mathcal{W} \)-algebra now change to

\[
T^* = T, \quad U^* = -U, \quad (W^\pm)^* = \pm W^\pm \iff U = i\tilde{U}, \quad W^\pm = \pm W^1 + iW^2
\]

where \( \tilde{U} \) and \( W^1, W^2 \) are real. The detailed form of this classical \( \mathcal{W} \)-algebra can be found in our previous paper [6]. The representation theory of the algebra has also been analyzed in [20]. It is interesting that it is the second (non-split) real form of the group which results in a compact (though abelian) Kac-Moody symmetry, and which is therefore the real form of the \( \mathcal{W} \)-algebra possessing highest-weight representations of the standard kind.

### 4.3 Embeddings in \( G_2 \)

The positive roots of \( G_2 \) are \( \alpha_1 \) (long), \( \alpha_2 \) (short), \( \alpha_1 + \alpha_2 \), \( \alpha_1 + 2\alpha_2 \), \( \alpha_1 + 3\alpha_2 \) and \( 2\alpha_1 + 3\alpha_2 = \psi \), the highest root.

For the principal embedding the adjoint decomposes as \( 14 \rightarrow (5) \oplus (1) \) which gives rise to an abelian Toda model whose \( \mathcal{W} \)-algebra contains a primary field of spin 6 in addition to the energy-momentum tensor. Once again there are no non-trivial choices for a conjugation map in this case, since there is no symmetry of the \( G_2 \) Dynkin diagram.

There are two superficially different non-principal integral embeddings listed in the table. First we have

\[
(G_2, A_2): \quad M_0 = -(h_1 + 3h_2), \quad M^\pm = \sqrt{2}(e\pm\alpha_1 + e\mp\psi)
\]

for which the adjoint of \( G_2 \) decomposes via representations of \( A_2 \) as:

\[
14 \rightarrow 8 \oplus 3 \oplus \bar{3} \rightarrow (2) \oplus (1) \oplus (1) \oplus (1)
\]

Alternatively we have

\[
(G_2, A_1 \oplus A_1): \quad M_0 = -(h_1 + 3h_2), \quad M^\pm = e\pm\alpha_1 + \sqrt{3}e\mp(\alpha_1 + 2\alpha_2)
\]

for which the adjoint decomposes via representations of \( A_1 \oplus A_1 \) as

\[
14 \rightarrow (3/2, 1/2) \oplus (1, 0) \oplus (0, 1) \rightarrow (2) \oplus (1) \oplus (1) \oplus (1)
\]

In both cases the zero-grade subalgebra is

\[
A_1 \oplus U_1 = \left\{ \frac{3}{2}(h_1 + h_2), \sqrt{3}e_{\alpha_1 + \alpha_2}, \sqrt{3}e_{-(\alpha_1 + \alpha_2)} \right\} \oplus \{ h_1 + 3h_2 \} \quad (4.39)
\]

As in the example of \( B_2 \), these embeddings can be shown to be conjugate in \( G_2 \) but we shall not discuss the details. For definiteness, we deal from now on with the \( (G_2, A_2) \) description.

From the decomposition of the adjoint representation we see that the resulting \( \mathcal{W} \)-algebra has primary fields of spins 3, 2, 2 in addition to the energy-momentum tensor. As in the \( B_2 \) example, we can calculate the Toda potential by identifying the representations to which \( M^\pm \) belong. It is easy to see that \( \{ e\pm\alpha_1, e\pm\alpha_2, e\pm(\alpha_1 + 2\alpha_2), e\pm(2\alpha_1 + 3\alpha_2) \} \) form spin-3/2 representations. Adopting a similar explicit parametrization for the zero-grade subalgebra as in the \( B_2 \) case (4.38) we find the Toda potential (up to a numerical constant)

\[
V = \frac{e^{-u}}{\rho^3} \left\{ 2\rho^3 \cosh^3 \rho + 6\phi_0^2 \rho \cosh \rho \sinh^2 \rho + (\phi_+^3 + \phi_-^3) \sinh^3 \rho \right\} \quad (4.40)
\]
with $\rho^2 = \phi_0^2 + \phi_+\phi_-$ (see appendix B for further details).

Turning our attention now to the possible real forms, we have as usual the maximally non-compact form for $G_2$, in which case the Toda fields and the $\mathcal{W}$-algebra generators referred to above are all real. To look for other possibilities compatible with the non-principal embedding, we seek a root automorphism $\tau \in \text{Aut}(\Delta)$ which could be used to construct a conjugation map $\tau(e_\alpha) = \chi_\alpha e_{\tau(\alpha)}$ and $\tau(h_\alpha) = h_{\tau(\alpha)}$, with $\chi_\alpha = \pm 1$, as explained in the previous section. To leave $M_+$ invariant we have just two possibilities. Either $\tau$ fixes both $\alpha_1$ and $\psi$, in which case we quickly reduce to the trivial case, or else it exchanges them: $\tau(\alpha_1) = -\psi$ and $\tau(\psi) = -\alpha_1$ (this is reflection in the root $\alpha_1 + \alpha_2$). The constants $\chi_\alpha$ must satisfy the consistency conditions (3.25), which leads to $\chi_\psi = -\chi_{\alpha_2}$ and $\chi_{\alpha_1 + \alpha_2} = -\chi_{\alpha_1}\chi_{\alpha_2}$. We must ensure $\tau(M_+) = M_+$ which implies that $\chi_{\alpha_1} = \chi_\psi = 1$ and therefore $\chi_{\alpha_1 + \alpha_2} = 1$. But it then turns out (for example by computing characters) that this conjugation map again defines the split real form of $G_2$.

Despite appearances, therefore, there is actually no other real form compatible with the non-principal embedding, and hence no alternative real form of the $\mathcal{W}$-algebra.

It is instructive to reconcile these observations with our formula for the potential. The reality conditions corresponding to the automorphism $\tau$ which exchanges $\alpha_1$ and $\psi$ would be $(\phi_0)^* = -\phi_0$ and $(\phi_1)^* = \sigma\phi_2$, where an easy calculation shows that in fact $\sigma = \chi_{\alpha_1 + \alpha_2}$. Taking $\sigma = +1$ leads once again to the split real form. If we try to take $\sigma = -1$, however, we see that this choice leads to a potential with a non-zero imaginary part. The fact that this real form is not compatible with the embedding is therefore entirely consistent with our earlier Theorem 2.

5 Embeddings $(\mathcal{G}, \mathcal{H})$ with $\mathcal{G}$ classical and $\mathcal{H}$ simple

In this section we describe the classification of real forms compatible with integral $sl(2, \mathbb{C})$ embeddings of the type $(\mathcal{G}^c, \mathcal{H}^c)$ where $\mathcal{G}^c$ is one of the classical algebras and $\mathcal{H}^c$ is a simple, regular subalgebra. The general strategy which we outline below could in principle be applied to any integral embedding in a regular subalgebra. Embeddings in the exceptional algebras could certainly be dealt with in a similar fashion and there is also nothing technically which constrains us to consider only simple subalgebras, although this is clearly the most natural case to investigate first and there would be a vast proliferation of additional examples for non-simple subalgebras.

The first step is to find concrete expressions for root automorphisms and conjugation maps which we can use to define all the real forms of the classical algebras. Once we have done this, the next step will be to examine each of the simple regular subalgebras which correspond to integral embeddings and then determine the subset of conjugation maps which leave invariant the relevant principal embedding. This second step is greatly simplified by the fact that the choice of subalgebra $\mathcal{H}^c \subset \mathcal{G}^c$ fixes the complex embedding uniquely up to conjugation. This means that we can make convenient assumptions about which simple roots of $\mathcal{G}^c$ are also simple roots of $\mathcal{H}^c$ with no loss of generality. We shall also find it useful to describe each embedding by giving its equivalent characteristic diagram, for which one assigns the grade 0 or $\pm 1$ to each simple root of $\mathcal{G}^c$. The third and last step is to calculate the resulting real forms $\mathcal{G}_0$ and $\mathcal{K}$ which are the aspects of the embedding most directly associated with the properties of the non-abelian Toda theory.

In the following subsections we shall discuss in turn embeddings in $A_n$, $D_n$, and then
the related cases $B_n$ and $C_n$. We shall certainly not include every detail of the arguments, since this would make our account prohibitively lengthy. We shall also take the liberty of omitting more and more routine technicalities as we work our way through the list of possibilities. The final results of the classification are collected together as Theorem 5 and summarized in the table on page 34.

5.1 Conjugation Maps and Root Spaces

Recall that an automorphism of $\mathcal{H}^c$ which fixes a principal $sl(2, \mathbb{C})$ subalgebra must fix the Cartan subalgebra $H_\mathcal{H}$ and must furthermore be a diagram automorphism of $\mathcal{H}^c$. We therefore seek all automorphisms of $\mathcal{G}^c$ which act as diagram automorphisms on some regular subalgebra $\mathcal{H}^c$. We claim that it is sufficient to consider automorphisms that fix a Cartan subalgebra $H_{\mathcal{G}}$ of the larger algebra $\mathcal{G}^c$. This can be regarded as a slight generalization of the well-known result that any automorphism of a complex Lie algebra is conjugate by an inner automorphism to one which fixes a given Cartan subalgebra.

**Theorem 4** Consider a complex simple Lie algebra $\mathcal{G}^c$ and a regular subalgebra $\mathcal{H}^c$. Choose Cartan subalgebras $H_{\mathcal{G}}$ and $H_\mathcal{H}$ respectively with $H_\mathcal{H} \subset H_{\mathcal{G}}$. Any automorphism of $\mathcal{G}^c$ which restricts to an automorphism of $\mathcal{H}^c$ and which fixes $H_\mathcal{H}$ is conjugate to an automorphism with all these properties but which in addition fixes $H_{\mathcal{G}}$.

Proof: Define $Z^c$ to be the centralizer of $\mathcal{H}^c$ in $\mathcal{G}^c$. We may choose a Cartan subalgebra $H_Z$ for this centralizer with the property that $H_{\mathcal{G}} \subset H_\mathcal{H} + H_Z$ (in fact the spaces $H_\mathcal{H}$ and $H_Z$ must be orthogonal). Let $\tau$ be the automorphism fixing $H_\mathcal{H}$ and $\mathcal{H}^c$. We will show that $\tau$ is conjugate to an automorphism $\tilde{\tau}$ which fixes not only $H_\mathcal{H}$ but also $H_Z$, which is clearly enough to show that it fixes $H_{\mathcal{G}}$. First note that $\tau$ fixes $Z^c$, simply because it also fixes $\mathcal{H}^c$. But this means that $\tau$ restricts to an automorphism on $Z^c$ and hence that there is an inner automorphism $\sigma$ on $Z^c$ such that $\sigma^{-1}\tau\sigma$ fixes $H_Z$. Since inner-automorphisms are nothing but conjugation by some group element, $\sigma$ is actually automatically defined as an inner-automorphism on all of $\mathcal{G}^c$. Furthermore, $\sigma$ restricts to the identity on $\mathcal{H}^c$ precisely because $Z^c$ is its centralizer. The desired automorphism of $\mathcal{G}^c$ is therefore $\tilde{\tau} = \sigma^{-1}\tau\sigma$. \hfill \Box

To write down conjugation maps we follow the procedure explained in section 3 for constructing algebra automorphisms from the root automorphisms $\text{Aut}(\Delta)$. The root automorphisms are in turn conveniently described by regarding the roots as living in $\mathbb{R}^m$ with its standard basis vectors $\{e_i\}$ obeying $e_i \cdot e_j = \delta_{ij}$ where $i, j = 1, \ldots, m$. For the algebras $B_n$, $C_n$ and $D_n$ we take $m = n$ so that this is exactly the span of the set of roots $\Delta$. For $A_n$, however, we take $m = n + 1$, and the span of $\Delta$ is the subspace of co-dimension one orthogonal to $\sum_{i=1}^{n+1} e_i$. The precise definitions of the roots are given below.

The important point for our purposes is that the group $\text{Aut}(\Delta)$ always acts as a subgroup of the *signed permutations* of the basis vectors $\{e_i\}$; by which we mean permutations accompanied by a change of sign of any number of the basis vectors. We can of course decompose any permutation into disjoint cycles, and we shall use the phrase $m$-cycle to mean one of length or order $m$. We shall also refer to an *inverted* $m$-cycle, meaning an $m$-cycle accompanied by a change of sign of all the basis vectors it permutes. Thus the map $\tau(e_i) = e_{i+1}$ for $i = 1, 2, \ldots, m-1$ and $\tau(e_m) = e_1$ is a typical $m$-cycle, while
5.2 Embeddings in $A_n$

The root space of $A_n$ is the subspace of $\mathbb{R}^{n+1}$ orthogonal to $\sum_{i=1}^{n+1} e_i$. The set of roots is $\Delta = \{e_k - e_l : k \neq l\}$ and the simple roots are $\alpha_k = e_k - e_{k+1}$ with $k = 1, \ldots, n$. The Weyl group $W(\Delta)$ is the group of permutations of the vectors $e_i$; $\text{Aut}(\Delta)$ is the group of permutations with a possible sign-change of all basis vectors. Note that the quotient is then indeed $\mathbb{Z}_2$, the symmetry of the Dynkin diagram.

5.2.1 Real forms of $A_n$

We will show how to construct all possible real forms of $A_n$ by defining conjugation maps from the elements of $\text{Aut}(\Delta)$ as in eq. (3.24).

Consider a conjugation map constructed from an element $\tau$ of $W(\Delta)$. Suppose that $\tau$ can be decomposed into $p$ 2-cycles and $n+1-2p$ invariant basis vectors. One such map is:

\[
\begin{align*}
\tau(\alpha_{2i-1}) &= -\alpha_{2i-1} & 1 \leq i \leq p \\
\tau(\alpha_{2i}) &= \alpha_{i-1} + \alpha_i + \alpha_{i+1} & 1 < i < p \\
\tau(\alpha_{2p}) &= \alpha_{2p-1} + \alpha_{2p} \\
\tau(\alpha_i) &= \alpha_i & i > 2p
\end{align*}
\]

We want to calculate the characters of the possible real forms which result from this by using equation (3.28) of Theorem 1. The trace of $\tau$ restricted to the Cartan subalgebra is $\text{tr}(\tau_H) = n-2p$. Next we consider those roots which satisfy $\tau(\alpha) = -\alpha$, namely $\alpha_1, \alpha_3, \ldots, \alpha_{2p-1}$. The parameters $\chi_{\alpha}$ which enter in equation (3.28) are strongly restricted in this case by the consistency condition $\chi_{\alpha} = \chi_{\tau(\alpha)}$, which implies $\chi_{\alpha_1} = \chi_{\alpha_3} = \cdots = \chi_{\alpha_{2p-1}} \equiv \chi$. If $2p = n + 1$ then we may choose $\chi = \pm 1$, but if $2p \neq n + 1$ then we have the unique possibility $\chi = 1$. The result of the calculation for the character is therefore

\[
\sigma = n - 2p + 2\chi p = \begin{cases} 
-n - 2 & \text{if } 2p = n + 1 \text{ and } \chi = -1 \\
2p & \text{all other cases}
\end{cases}
\]

From the table of real forms and characters (3.14) we see that any element of the Weyl group which leaves a basis vector invariant can give rise only to conjugation maps defining the split real form $sl(n+1)$ of $A_n$. The only elements of the Weyl group capable of producing a different real form are those consisting entirely of 2-cycles. This is only possible when $n$ is odd, and the result is the real form $su^*(n+1)$. To obtain the remaining real forms of $A_n$ we must use root automorphisms which are not elements of the Weyl group.

Consider an element $\tau \in \text{Aut}(\Delta) \setminus W(\Delta)$ which consists of $p$ inverted 2-cycles and $n+1-2p$ inverted basis vectors. One such $\tau$ has an action on the simple roots just as in (5.41) except for a change of sign on the right hand side of every equation. Now we have $\text{tr}(\tau_H) = -(n-2p)$. The roots that satisfy $\tau(\alpha) = -\alpha$ are now the roots in the $A_{n-2p}$
subalgebra generated by the \( n + 1 - 2p \) inverted basis vectors. For each positive root of this subalgebra \( \alpha \in \Delta_+(A_{n-2p}) \) we can consistently choose \( \chi_\alpha = -1 \) and then we find the character

\[ \sigma = 2p - n - 2|\Delta_+(A_{n-2p})| = -\dim(A_{n-2p}) = 1 - (n + 1 - 2p)^2 \]

Comparing this result with table 3.14 we find that these are exactly the characters of \( su(n+1-p,p) \) for \( p = 0, \ldots, \left\lceil \frac{n+1}{2} \right\rceil \).

### 5.2.2 Invariant regular subalgebras

The simple, regular subalgebras of \( A_n \) are \( A_m \) with \( m \leq n \). The embedding is integral when \( m \) is even or when \( m = n \) is odd, the latter giving the principal embedding. The embedding is half-integral—and hence irrelevant for our purposes—for odd \( m < n \).

We may describe a grading by assigning a number \( j_k \) to each basis vector \( e_k \), so that the grade of any root \( e_k - e_l \) is \( j_k - j_l \). It is then simple to identify the zero-grade subalgebra as the set of all Cartan generators together with those step-operators for roots \( e_k - e_l \) with \( j_k = j_l \). We can alternatively define a grading by specifying an integer \( p_i \) for each simple root \( \alpha_i \) and we denote this on the Dynkin diagram by writing \( p_i \) above the appropriate node. For example, we can take an embedded \( A_m \) subalgebra generated by the first \( m \) simple roots of \( A_n \), and then the grading corresponds to the diagram:

![Diagram](image)

It is always possible to make a Weyl transformation to a basis where the gradings of the simple roots are 0 or ±1; the resulting characteristic Dynkin diagram is:

![Diagram](image)

Note that in the case of the principal embedding of \( sl(2, \mathbb{C}) \) into an algebra \( \mathcal{G} \), \( (\mathcal{G}, \mathcal{G}) \) in our notation, the grades of all simple positive roots are 1 by definition.

Now the zero-grade subalgebra is generated by all Cartan generators and by the step operators corresponding to simple roots that have grade zero. In this case we see that \( \mathcal{G}_0^c = A_{n-m} \oplus mU_1 \). It will be important that exactly one of the basis elements \( e_i \) is common to the root-spaces of both \( A_m \) and \( \mathcal{G}_0^c \); this is \( e_{\frac{m}{2}+1} \) in the basis where \( A_m \) is generated by the first \( m \) simple roots of \( A_n \).

We wish to consider automorphisms of \( A_n \) which leave \( A_m \) invariant up to a diagram automorphism. In the previous subsection we found that elements of \( W(\Delta) \) lead only to the real forms \( sl(n+1) \) or \( su^*(n+1) \). This first real form can of course be realized by the identity in the Weyl group, which certainly leaves the \( A_m \) subalgebra invariant. For the second real form, however, we saw above that the corresponding elements of the Weyl group must consist entirely of 2-cycles, and this implies that the only regular subalgebra it can leave invariant is one of type \( A_1 \). In fact if we now consider more general root automorphisms in \( \text{Aut}(\Delta) \setminus W(\Delta) \), then we find likewise that the only simple, regular subalgebras that can be invariant are again those of type \( A_1 \). We conclude that only the
split real form of $A_n$ corresponds to a conjugation map which restricts to the identity on $A_n$. This clearly results in the split real form of the zero-grade subalgebra.

The remaining possibility which we must investigate is that the conjugation map on $A_n$ acts as a non-trivial diagram automorphism on $A_m$. Any such non-trivial diagram automorphism $\tau$ acts on $A_m$ by interchanging $e_i$ and $-e_{m+2-i}$, i.e. it consists of $\frac{m}{2}$ inverted 2-cycles and one inverted basis vector $e_{m+2}$.

Let us now turn to the action of such an automorphism on $G^c_0 = A_{n-m} \oplus mU_1$. The abelian part $mU_1$ is composed of the Cartan generators of $A_m$ and the action of $\tau$ on it is therefore completely determined; the resulting real form is $\frac{m}{2}gl(1) \oplus \frac{m}{2}u(1)$. When we come to the non-abelian part, we find that we can choose freely the action of $\tau$ on the basis vectors of the root-space of $A_{n-m}$, except for the fact that it must invert the basis vector $e_{m+1}$ which, as we recall, lies in the root-spaces of both $G^c_0$ and $A_m$. We conclude that the only restriction on the real form of the simple part of $G^c_0$ is that it be constructed using an automorphism which inverts at least one basis vector. In other words, it cannot consist entirely of (inverted) 2-cycles. From the results of the previous subsection, we find that by using such elements of $\text{Aut}(\Delta)$ we can define conjugation maps giving each of the real forms $su((n-m)+1-p,p)$ with $p = 0, \ldots, \left[\frac{n-m+1}{2}\right]$ of $A_{n-m}$. It is not possible, however, to get the real form $su^*(n-m+1)$.

Finally we come to the centralizer which is of complex type $K^c = A_{n-m-1} \oplus U_1$. The split real form of $A_n$, corresponding to the identity map on $A_m$, obviously results in the real form $K = sl(n-m) \oplus gl(1)$ for the centralizer. Aside from this, however, we must consider the broader set of possibilities arising from non-trivial diagram automorphisms of $A_m$.

We first note that the abelian factor $U_1$ in $K^c$ is generated by the projection of the vector $\sum_{i=1}^m e_i$. It follows that any non-trivial diagram automorphism on $A_m$ produces the compact form $u(1)$ of this abelian factor. Now the relevant real forms of the simple, non-abelian factor $A_{n-m-1}$ are clearly $su((n-m)-r,r)$ with $r = 0, \ldots, \left[\frac{n-m}{2}\right]$. It is quite possible for a given real form $G_0$ to give rise to a number of different real forms $K$. In other words, there may be more than one value of $r$ in this list which is allowed for a given value of $p$ in our earlier list of real forms $G_0$. Nevertheless, the values of $r$ are severely restricted simply by the fact that $K \subset G_0$. In the present case it is clear that this is satisfied only if $r = p$ or $r = p-1$ (assuming ranges of values where these equations make sense).

To show that these possibilities can indeed occur, we can exhibit the relevant conjugation maps. Set $k = n-m$ for convenience and take the simple subalgebras $A_{k-1} \subset K^c$ and $A_k \subset G^c_0$ to be generated by the first $k$ and first $k+1$ basis vectors respectively. As earlier, we consider an automorphism $\tau$ on $G^c_0$ which inverts the first $k-2p$ basis vectors, and consists of inverted 2-cycles for the rest. We choose $\chi_{a_2} = \chi_{a_3} = \ldots = \chi_{a_{k-2p-1}} = -1$. It is then easy to show that the non-abelian factor of $G_0$ is $su(k+1-p,p)$, while the non-abelian factor of $K$ is $su(k-p,p)$ if $\chi_{a_1} = -1$ and $su(k+1-p,p-1)$ if $\chi_{a_1} = 1$.

The statements made in the last few paragraphs should strictly be modified very slightly for the special case $p = 0$, since then we obviously cannot take $r = p - 1$, and there is a unique possibility for the centralizer $K = su(k) \oplus u(1)$. The fact that the result is a compact group makes this one of the more interesting possibilities.

We now summarize what we have found for even $m \leq n$. Only the split real form $G = sl(n+1)$ corresponds to the identity map on the regular subalgebra, with $G_0 = sl(n-m+1) \oplus m gl(1)$ and $K = sl(n-m) \oplus gl(1)$. Other allowed real forms $G = su(n+1-$
\( \frac{m}{2} - p, \frac{m}{2} + p \) correspond to a non-trivial diagram automorphism of the regular subalgebra and result in \( \mathcal{G}_0 = su(n-m+1-p, p) \oplus \frac{m}{2} gl(1) \oplus u(1) \) and \( \mathcal{K} = su(n-m-r, r) \oplus u(1) \) where \( r = p-1 \) or \( r = p \) (when these values make sense).

The remaining isolated case \( m = n \) admits just two possibilities corresponding to the split form or a non-trivial diagram automorphism.

### 5.2.3 An example: \((A_4, A_2)\)

The general discussion we have just given will probably seem more intelligible if we supplement it with a simple example. Consider then the regular \( A_2 \) subalgebra generated by the first two simple roots \( \alpha_1 \) and \( \alpha_2 \) of \( A_4 \) or, equivalently, by the basis vectors \( e_1, e_2, \) and \( e_3 \). The grading is given by the diagram

\[
\begin{array}{cccc}
1 & 1 & -1 & 0 \\
\end{array}
\]

and the characteristic diagram for the embedding is

\[
\begin{array}{cccc}
\end{array}
\]

which can be found by applying a Weyl transformation to the original basis (exchange of \( e_3 \) with \( e_5 \)). From this we identify the complex zero-grade algebra \( \mathcal{G}_0^c = A_2 \oplus 2U_1 \), where basis vectors of the root space of \( A_2 \) are \( e_2, e_4 \) and \( e_5 \). The complex centralizer is \( \mathcal{K}^c = A_1 \oplus U_1 \), where \( A_1 \) has root space spanned by \( e_4 \) and \( e_5 \).

The possible real forms of \( A_2 \) are \( sl(3) \), \( su(2,1) \) and \( su(3) \). As always, the first can be obtained by using the trivial element of the Weyl group. An element of \( \text{Aut}(\Delta_{A_4}) \) which does not lie in the Weyl group must either exchange \( e_4 \) with \( -e_5 \), or else invert \( e_4 \) and \( e_5 \) individually. As shown in subsection 5.2.1, the first possibility yields \( su(2,1) \), while the second gives rise to the compact real form \( su(3) \) if we choose \( \chi_{e_2-e_4} = \chi_{e_4-e_5} = -1 \).

If we want to find which possible real forms \( \mathcal{K} \) are allowed for each real form of \( \mathcal{G}_0 \), we need to investigate the automorphisms a little more closely. In this example we consider all possible choices of \( \chi_1 = \chi_{e_2-e_4} \) and \( \chi_2 = \chi_{e_4-e_5} \) for the automorphism which inverts \( e_4 \) and \( e_5 \). This results in the real forms:

\[
\begin{array}{cccccc}
\chi_1 & \chi_2 & A_2^c \subset \mathcal{G}_0 & A_1^c \subset \mathcal{K} \\
-1 & -1 & su(3) & su(2) \\
1 & -1 & su(2,1) & su(2) \\
-1 & 1 & su(2,1) & sl(2) \\
1 & 1 & su(2,1) & sl(2) \\
\end{array}
\]

The total set of possibilities can be summarized:

| \( \mathcal{G} \) | \( \mathcal{G}_0 \) | \( \mathcal{K} \) |
|------------------|------------------|------------------|
| \( sl(5) \) | \( sl(3) \oplus gl(1) \oplus gl(1) \) | \( sl(2) \oplus u(1) \) |
| \( su(3, 2) \) | \( su(2,1) \oplus gl(1) \oplus u(1) \) | \( su(2) \oplus u(1) \) |
| \( su(3, 2) \) | \( su(2,1) \oplus gl(1) \oplus u(1) \) | \( su(2) \oplus u(1) \) |
| \( su(4, 1) \) | \( su(3) \oplus gl(1) \oplus u(1) \) | \( su(2) \oplus u(1) \) |
5.3 Embeddings in \( D_n \)

The roots of \( D_n \) are \( \Delta = \{ \pm(e_k \pm e_l) : k \neq l \} \); the simple roots are \( \alpha_i = e_i - e_{i+1} \) for \( i = 1, \ldots, n-1 \) and \( \alpha_n = e_{n-1} + e_n \). For \( n > 4 \) the Weyl group \( W(\Delta) \) acts as permutations on the basis vectors \( e_i \) together with a sign change (inversion) of any even number of them. The group \( \text{Aut}(\Delta) \) acts as permutations together with a sign change (inversion) of an arbitrary number of them. This is clearly consistent with \( \text{Aut}(\Delta)/W(\Delta) \) acting as a \( \mathbb{Z}_2 \) symmetry of the Dynkin diagram. The case \( n = 4 \) has well-known exceptional symmetry properties (e.g. \( \text{Aut}(\Delta)/W(\Delta) \) is the group of permutations on three objects) but in fact these are not relevant for our purposes. We shall therefore present arguments below for the general case \( n > 4 \); the special case \( n = 4 \) can be confirmed to fit into the pattern despite its peculiarities.

5.3.1 Real forms of \( D_n \)

Consider an element \( \tau \in \text{Aut}(\Delta) \) for which the first \( n-p \) basis vectors are invariant and the last \( p \) are inverted. This gives an action on the simple roots:

\[
\begin{align*}
\tau(\alpha_i) &= \alpha_i & i < n-p \\
\tau(\alpha_{n-p}) &= \alpha_{n-p} + 2\alpha_{n-p+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \\
\tau(\alpha_i) &= -\alpha_i & n-p < i \leq n
\end{align*}
\]

where \( \alpha_{n-p} + 2\alpha_{n-p+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n = \alpha_{n-p} + \alpha_{n-p+1} + \psi \), and \( \psi \) is the highest root of the \( D_p \) subalgebra generated by the last \( p \) basis vectors. The roots of this subalgebra are exactly those roots which satisfy \( \tau(\alpha) = -\alpha \). For this subalgebra we can consistently choose \( \chi_\alpha = -1 \), and using eq. (3.28) we find the character:

\[
\sigma = (n - 2p) - 2|\Delta_+(D_p)| = n - p - \dim(D_p) = n - 2p^2, \quad p = 0, \ldots, n \quad (5.42)
\]

where \( \text{tr}(\tau_H) = n - 2p \). This is the character of the real form \( so(n+p, n-p) \).

In order to construct the real form \( so^\ast(2n) \) we could consider a more complicated element of \( \text{Aut}(\Delta) \) with \( \frac{n}{2} \) 2-cycles if \( n \) is even, and \( \frac{n-1}{2} \) 2-cycles and one inverted basis vector if \( n \) is odd. It is easy to check that this gives the correct real form with character \(-n\). An alternative method which will be more useful in what follows is to take the same root automorphism written in (5.42) above with \( p = n \), so that all the basis vectors are inverted and \( \tau(\alpha) = -\alpha \). From this we can define a different conjugation map by taking \( \chi_{\alpha_{n-1}}\chi_\alpha = -1 \). Using the formula \( \chi_\alpha = (-1)^{\ell+1} \prod \chi_{\ell_i} \), where \( \alpha = \sum \ell_i \alpha_i \) has height \( \ell = \sum \ell_i \), we find that the contributions to the character from the roots \( e_k - e_l \) and \( e_k + e_l \) always cancel for \( k < l \). The character is therefore determined solely by the action of the conjugation map on the Cartan subalgebra, giving the desired result \( \sigma = -n \).

We have already seen that we can obtain the split real form \( so(n, n) \) by taking \( p = n \) in the construction (5.42). This is simply the usual statement that the split real form arises from the identity in the Weyl group. We can also obtain the split real form from a conjugation map which inverts some basis vectors, however, which will be significant shortly. To see how to do this we take the root automorphism (5.42) with \( p = n-2 \), so that \( e_{n-1} \rightarrow -e_{n-1} \) and \( e_n \rightarrow -e_n \) while all other basis vectors are invariant. In contrast to what we did earlier, however, we define a different conjugation map by choosing \( \chi_{\alpha_{n-1}} = \chi_\alpha = 1 \). An easy calculation then shows that the resulting character in \( \sigma = n \), as required.
5.3.2 Invariant Regular Subalgebras

The simple regular subalgebras of $D_n$ are $A_m$ with $m < n$ and $D_m$ with $m \leq n$. Integral embeddings arise for $A_m$ with $m$ even or $m = n - 1$, and for all $D_m$.

- $(D_n, A_m)$. Consider first even $m < n - 1$. We can assume without loss of generality that $A_m$ is generated by the first $m$ simple roots of $D_n$ and the grading is specified by

$$1 \quad 1 \quad -\frac{m}{2} \quad 0 \quad 0 \quad 0 \quad 0$$

or in terms of the characteristic Dynkin diagram:

$$0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0$$

(5.43)

From this we see that the complex zero-grade subalgebra is $G_0^c = D_{n-m} \oplus \frac{m}{2} A_1 \oplus \frac{m}{2} U_1$ and the complex centralizer is $K^c = D_{n-m-1} \oplus U_1$. There is exactly one basis vector which belongs to the root-spaces of both $A_m$ and of $G_0^c$, namely $e_{\frac{m}{2} + 1}$.

Any automorphism $\tau$ which restricts to a diagram automorphism on $A_m$ is completely determined on $\frac{m}{2} A_1 \oplus \frac{m}{2} U_1 \subset G_0^c$ and gives the real form $\frac{m}{2} sl(2) \oplus \frac{m}{2} gl(1)$. This is similar to results which hold for the embeddings $(B_n, A_m)$ and $(C_n, A_m)$, the first of which is discussed in appendix A; we therefore omit further details.

Turning to $D_{n-m} \subset G_0^c$, we can freely choose the action of the automorphism $\tau$ on the basis vectors which generate this subalgebra, except for the action on the basis vector it has in common with $A_m$. This basis vector must be invariant if $\tau$ leaves $A_m$ invariant, but it must be inverted if $\tau$ acts as a non-trivial diagram automorphism on $A_m$. From the results above, we find that from any automorphism which acts as the identity on $A_m$ we can construct conjugation maps leading to the real forms $so(n-m+p, n-m-p)$ with $p = 0, \ldots, n-m-1$. From an automorphism which restricts to a non-trivial diagram automorphism on $A_m$ we can in fact get all these and also the additional remaining real form $so^*(2n-2m)$. In order to show this we need to use the conjugation maps given earlier which involve at least one inverted basis vector.

The possible real forms $K$ are $so(2n-2m-2) \oplus gl(1)$ and $so(n-m-1+p, n-m-1-p) \oplus gl(1)$ with $p = 0, \ldots, n-m-2$ if $\tau$ leaves $A_m$ invariant, and $so^*(2n-2m-2) \oplus u(1)$ and $so(n-m-1+p, n-m-1-p) \oplus u(1)$, $p = 0, \ldots, n-m-1$ if $\tau$ acts non-trivially on $A_m$.

Finally, if $m = n - 1$ then the grading is integral whether $n$ is odd or even. The outstanding case $n$ even has $G_0^c = \frac{n}{2} A_1 \oplus \frac{n}{2} U_1$ and $K^c = U_1$. We simply quote the final results for the real forms in the table on 34.
• \((D_n, D_m)\). We can assume that \(D_m\) is generated by the last \(m\) simple roots of \(D_n\). The grading is:

\[
\begin{array}{cccccc}
0 & 0 & 1-m & 1 & \cdots & 1 \\
& & & m & \end{array}
\]

and the corresponding characteristic Dynkin diagram is:

\[
\begin{array}{ccccccc}
1 & & 0 & & & & 0 \\
& & & m & \end{array}
\]

From this we find \(G_0 = D_{n+1-m} \oplus (m-1)U_1\), while the complex centralizer is \(K^c = D_{n-m}\).

As in previous cases, it is important that \(e_n\) is a basis vector of the root spaces of both \(D_m\) and \(G_0\).

An automorphism \(\tau\) on \(D_n\) which restricts to a diagram automorphism on \(D_m\) will have the first \((m-1)\) basis vectors of \(D_m\) as fixed points, while \(e_n\) is a fixed point if \(\tau\) restricts to the identity on \(D_m\), but is inverted if \(\tau\) is a non-trivial diagram automorphism on \(D_m\).

We can freely choose the action of the automorphism on the simple part \(D_{n+1-m} \subset G_0\), provided that it fulfills this criterion.

Referring back to eq. (5.42) we see that with at least one invariant basis vector in \(D_{n+1-m}\) we can obtain the real forms \(so((n-m+1) + p, (n-m+1) - p)\) with \(p = 0, \ldots, n-m\). Alternatively, with one inverted basis vector we can get the compact real form \(so((n-m+1)+p, (n-m+1)-p)\) with \(p = 1, \ldots, n-m\). For the centralizer \(K\) we find that the possible real forms are respectively \(so(2n-2m)\) and \(so((n-m)+r, (n-m)-r)\) with \(r = p-1, p, p+1\).

### 5.4 Embeddings in \(B_n\) and \(C_n\)

The cases of \(B_n\) and \(C_n\) are very similar to each other; in fact the groups \(W(\Delta) = \text{Aut}(\Delta)\) are isomorphic. The roots of \(B_n\) are \(\Delta = \{\pm e_j; \pm (e_k \pm e_l) : k \neq l\}\) and the simple roots are \(\alpha_i = e_i - e_{i+1}\) for \(i = 1, \ldots, n-1\) and \(\alpha_n = e_n\). The roots of \(C_n\) are \(\Delta = \{\pm 2e_j; \pm (e_k \pm e_l) : k \neq l\}\) and the simple roots are \(\alpha_i = e_i - e_{i+1}\) for \(i = 1, \ldots, n-1\) and \(\alpha_n = 2e_n\). In either case the Weyl group \(W(\Delta)\) is the group of permutations of the basis vectors \(e_i\) with a sign change of an arbitrary number of them.

#### 5.4.1 Real Forms of \(B_n\) and \(C_n\)

In the case of \(B_n\) we consider again a conjugation map with the first \((n-p)\) basis vectors invariant and the last \(p\) inverted. The result is

\[
\begin{align*}
\tau(\alpha_i) &= -\alpha_i & i < n-p \\
\tau(\alpha_{n-p}) &= \alpha_{n-p} + 2\alpha_{n-p+1} + \cdots + 2\alpha_n \\
\tau(\alpha_i) &= -\alpha_i & n-p < i \leq n
\end{align*}
\]

The set of characters that we obtain via this construction is:

\[
\sigma = n-p - \dim(B_p) = n-2p(p+1),
\]
corresponding to the complete set of real forms of $B_n$.

As with the case of $D_n$, it will also be important that we can obtain the split real form from a conjugation map which inverts one basis vector. The roots of $B_n$ are:

$$
e_k - e_l = \alpha_k + \cdots + \alpha_{l-1}
$$
$$
e_k + e_l = \alpha_k + \cdots + \alpha_{l-1} + 2\alpha_l + \cdots + 2\alpha_n
$$
$$
e_k = \alpha_k + \cdots + \alpha_n
$$

We consider the conjugation map defined by $\alpha \to -\alpha$, and we choose $\chi_\alpha = 1$ for all $\alpha$. Using equations (3.27) and (3.29), a brief calculation shows that the resulting character is $\sigma = n$, i.e the resulting real Lie algebra is indeed the split real Lie algebra.

In the case of $C_n$ we find that any automorphism which leaves at least one basis vector invariant leads to the split real form with character $n$ (details are given in appendix A). We therefore consider an automorphism consisting of $p$ 2-cycles acting on the first $2p$ basis vectors and the rest of the basis vectors inverted. This automorphism acts on the simple roots as:

$$
\tau(\alpha_{2i-1}) = -\alpha_{2i-1} \quad i \leq p
$$
$$
\tau(\alpha_{2i}) = \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} \quad i < p
$$
$$
\tau(\alpha_{2p}) = \alpha_{2p-1} + \alpha_{2p} + 2\alpha_{n-p+1} - \cdots - 2\alpha_{n-1} + \alpha_n
$$
$$
\tau(\alpha_i) = -\alpha_i \quad n - p < i \leq n
$$

Choosing $\chi_\alpha = -1$ for the last $n - 2p$ simple roots generating a $C_{n-2p}$ subalgebra, we get the characters:

$$
\sigma = -(n - 2p) - 2p - 2|\Delta_+(C_{n-2p})| = -2p - \dim(C_{n-2p}) = -n - 2(n - 2p)^2
$$

with $p = 0, \ldots, [\frac{n}{2}]$. These are the characters of the remaining real forms $sp(p, n-p)$ of $C_n$.

### 5.4.2 Invariant Regular Subalgebras

The simple regular subalgebras of $B_n$ are: $A_m$, $m < n$; $B_m$, $m \leq n$; and $D_m$, $m \leq n$. For an integral embedding we require $A_m$ with $m$ even; but no additional restrictions on $B_m$ or $D_m$.

The simple regular subalgebras of $C_n$ are: $A_m$, $m < n$; and $C_m$, $m \leq n$. For an integral embedding we require $A_m$ with $m$ even or $m = n - 1$; and $C_m$ with $m = n$, giving the principal embedding.

- $(B_n, A_m)$. We consider the regular $A_m$ subalgebra generated by the first $m$ (even) simple roots of $B_n$, with grading:

```
1 1 -m/2 0 0 0
```

Corresponding to the characteristic Dynkin diagram:

```
0 1 0 1 0 0 0
```

32
The complex zero-grade subalgebra is $G_0^c = B_{n-m} \oplus \frac{m}{2}A_1 \oplus \frac{m}{2}U_1$, while the complex centralizer is $K^c = B_{n-m-1} \oplus U_1$.

Any automorphism which acts as a diagram automorphism on $A_m$ will have a restriction to $\frac{m}{2}A_1 \oplus \frac{m}{2}U_1 \subset G_0^c$ which is completely fixed; the corresponding conjugation map defines the real form $\frac{m}{2}sl(2) \oplus \frac{m}{2}gl(1)$. Details of this are given in appendix A.

The basis vector $e_{m+2}^m$ is common to the root-space of both $A_m$ and of $B_{n-m} \subset G_0^c$. An automorphism which restricts to the identity on $A_m$ must leave this basis vector invariant, while an automorphism which restricts to a non-trivial diagram automorphism on $A_m$ will invert $e_{m+2}^m$ so we are free to choose the action of an automorphism on $B_{n-m} \subset G_0^c$ subject only to this constraint. Using one or the other of these possibilities we can define conjugation maps leading to any real form of $B_{n-m} \subset G_0^c$. Similarly we can obtain all consistent real forms of the simple part of the centralizer $B_{n-m-1} \subset K^c$. The abelian factor $U_1 \subset K^c$ has real forms $gl(1)$ or $u(1)$, depending on whether the automorphism on $A_m$ is trivial or not.

• $(B_n, B_m)$. We consider the regular $B_m$ subalgebra generated by the last $m$ simple roots of $B_n$, with grading:

\[
\begin{array}{cccccc}
0 & 0 & -m & 1 & 1 & 1 \\
\end{array}
\]

which is already a characteristic Dynkin diagram. We find $G_0^c = B_{n-m} \oplus mU_1$ and $K^c = D_{n-m}$.

An automorphism which is trivial when restricted to $B_m$ will also be trivial on $mU_1 \subset G_0^c$. We can, however, choose the action of the automorphism on $B_{n-m} \subset G_0^c$ and on $K^c$ freely, since in this case the root spaces of $B_m$ and $G_0^c$ do not intersect. We can therefore obtain any real form of these subalgebras.

• $(B_n, D_m)$. We consider the regular $D_m$ subalgebra generated by the first $(m-1)$ simple roots of $B_n$ and by the root $e_{m-1} + e_m$. This gives the grading:

\[
\begin{array}{cccccc}
1 & 1 & 0 & 0 & 0 \\
\end{array}
\]

which is already a characteristic Dynkin diagram. We find $G_0^c = B_{n+1-m} \oplus (m-1)U_1$ and $K^c = B_{n-m}$.

The basis vector $e_m$ is common to the root-spaces of both $D_m$ and $G_0^c$. An automorphism $\tau$ which restricts to a diagram automorphism on $D_m$ will leave the first $m-1$ basis vectors of $D_m$ invariant, while fixing $e_m$ if $\tau$ is the identity on $D_m$, and inverting it if it is a non-trivial diagram automorphism. We can therefore choose the action of the automorphism on $B_{n+1-m} \subset G_0^c$ subject only to this one constraint. From these two classes of automorphisms we can define conjugation maps which give any desired real form of $B_{n+1-m} \subset G_0^c$ and any consistent real form of $K^c$. 

33
We consider first even $m < n - 1$. The Dynkin diagram is identical to that for the reduction $(B_n, A_m)$ with white and black nodes exchanged. The zero-grade subalgebra is $G_0^c = C_{n-m} \oplus \frac{m}{2} A_1 \oplus \frac{m}{2} U_1$ and the centralizer is $K^c = C_{n-m-1} \oplus U_1$.

The restriction of an automorphism to $\frac{m}{2} A_1 \oplus \frac{m}{2} U_1 \subset G_0^c$ is completely determined when we impose that the restriction to $A_m$ must be a diagram automorphism, and the real form will be $\frac{m}{2} su(2) \oplus \frac{m}{2} gl(1)$ (the proof is similar to one given in appendix A for embeddings in $B_n$). The basis vector $e_{m+2}$ is common to the root spaces of both $A_m$ and $G_0^c$. The action of an automorphism on $C_{n-m} \subset G_0^c$ is constrained only by the fact that it must leave $e_{m+2}$ invariant if it acts trivially on $A_m$, while it must invert $e_{m+2}$ if it acts non-trivially on $A_m$.

Any automorphism on $C_{n-m}$ which leaves at least one basis vector invariant defines a conjugation map which results in the split real form of $C_{n-m}$. This requires some justification and we again consign the details to the appendix so as not to interrupt our summary. It follows that an automorphism which restricts to the identity on $A_m$ can result only in the split real form of $C_{n-m} \subset G_0^c$. An automorphism which acts non-trivially on $A_m$ can result in any real form of $C_{n-m} \subset G_0^c$. We can furthermore find conjugation maps which give any consistent real form of $C_{n-m-1} \subset K^c$. The abelian factor $U_1 \subset K^c$ always has real form $u(1)$ for an automorphism which acts as a non-trivial diagram automorphism on $A_m$.

Finally we mention the special case $m = n - 1$, which also gives an integer grading when $m$ is odd. $G_0^c = \frac{n}{2} A_1 \oplus \frac{n}{2} U_1$ and the possible real forms are $\frac{n}{2} sl(2) \oplus \frac{n}{2} gl(1)$ and $\frac{n}{2} su(2) \oplus \frac{n}{2} gl(1)$. The centralizer is $K^c = U_1$ with real forms $gl(1)$ or $u(1)$.

### 5.5 Summary of classification

**Theorem 5** The real forms $(G, H)$ compatible with an integral embedding $(G^c, H^c)$ where $G^c$ is classical of rank $n$ and $H^c$ is simple of rank $m$ are as shown in the Table. For each such embedding we give the complex types $G_0^c$ and $K^c$ of the zero-grade subalgebra and the centralizer, followed by their corresponding real forms $G_0$ and $K$. We set $k = n - m$ throughout to simplify notation.

Proof: Sketched in the foregoing sections 5.1 to 5.4 with some additional technical details in appendix A.

Note that in order to write the results in the most compact way we have made free use of the standard identifications $U_1 \cong D_1$; $A_1 \cong B_1 \cong C_1$; $B_2 \cong C_2$; $D_2 \cong A_1 \oplus A_1$; $A_3 \cong D_3$ and besides the equivalence of the real forms $sl(2) \cong su(1,1)$ and $so^*(8) \cong so(6,2)$. 

34
| $G$ compatible with $(A_n, A_m)$ $m$ even | $G_0$ real form of $A_k \oplus mU_1$ | $K$ real form of $A_{k-1} \oplus U_1$ |
|-------------------------------------------|---------------------------------|---------------------------------|
| $sl(n+1)$                                 | $sl(k+1) \oplus mgl(1)$         | $sl(k) \oplus gl(1)$           |
| $su\left(n+1-\frac{m}{2}, \frac{m}{2}\right)$ | $su(k+1) \oplus \frac{m}{2}gl(1) \oplus \frac{m}{2}u(1)$ | $su(k) \oplus u(1)$           |
| $su\left(n+1-\frac{m}{2}-p, \frac{m}{2}+p\right)$ | $su(k+1-p, p) \oplus \frac{m}{2}gl(1) \oplus \frac{m}{2}u(1)$ | $su(k+1-p, p-1) \oplus u(1)$ |
| $G$ compatible with $(A_n, A_n)$ $n$ odd | $G_0$ real form of $nU_1$       | $K$ trivial                     |
| $sl(n+1)$                                 | $ngl(1)$                        |                                |
| $su\left(\frac{n+1}{2}, \frac{n+1}{2}\right)$ | $\frac{n+1}{2}gl(1) \oplus \frac{n-1}{2}u(1)$ |                                |
| $G$ compatible with $(B_n, A_m)$ $m$ even | $G_0$ real form of $B_k \oplus \frac{m}{2}A_1 \oplus \frac{m}{2}U_1$ | $K$ real form of $B_{k-1} \oplus U_1$ |
| $so(n+1, n)$                              | $so(k+1, k) \oplus \frac{m}{2}sl(2) \oplus \frac{m}{2}gl(1)$ | $so(k, k-1) \oplus gl(1)$     |
| $so(2n+1-m, m)$                           | $so(2k+1) \oplus \frac{m}{2}sl(2) \oplus \frac{m}{2}gl(1)$ | $so(k, k-1) \oplus u(1)$      |
| $so(n+1+p, n-p)$                          | $so(k+1+p, k-p) \oplus \frac{m}{2}sl(2) \oplus \frac{m}{2}gl(1)$ | $so(k+1+p, k-p) \oplus u(1)$ |
| $p = 1, \ldots, k-2$                     |                                | $so(k+1+p, k-1-p) \oplus u(1)$ |
| $G$ compatible with $(B_n, B_m)$          | $G_0$ real form of $B_k \oplus mU_1$ | $K$ real form of $D_k$         |
| $so(n+1, n)$                              | $so(k+1, k) \oplus mgl(1)$      | $so(k, k)$                     |
| $so(2n+1-m, m)$                           | $so(2k+1) \oplus mgl(1)$        | $so(2k)$                      |
| $so(n+1+p, n-p)$                          | $so(k+1+p, k-p) \oplus mgl(1)$  | $so(k+p, k-p)$                 |
| $p = 1, \ldots, k-1$                     |                                | $so(k+1+p, k-1-p)$             |
| $G$ compatible with $(B_n, D_m)$          | $G_0$ real form of $B_{k+1} \oplus (m-1)U_1$ | $K$ real form of $B_k$         |
| $so(n+1, n)$                              | $so(k+2, k+1) \oplus (m-1)gl(1)$ | $so(k+1, k)$                  |
| $so(2n+2-m, m-1)$                         | $so(2k+3) \oplus (m-1)gl(1)$    | $so(2k+1)$                    |
| $so(2n+1-m, m)$                           | $so(2k+2, 1) \oplus (m-1)gl(1)$ | $so(2k, 1)$                   |
| $so(2n+1-p, n-p)$                         | $so(2k+2, p, k+1-p) \oplus (m-1)gl(1)$ | $so(2k+1)$                    |
| \(G\) compatible with \((C_n, A_m)\) \(m\) even | \(G_0\) real form of \(C_k \oplus \frac{m}{2} A_1 \oplus \frac{m}{2} U_1\) | \(K\) real form of \(C_{k-1} \oplus U_1\) |
|-----------------|------------------|------------------|
| \(sr(n)\) | \(sr(k) \oplus \frac{m}{2} sl(2) \oplus \frac{m}{2} gl(1)\) | \(sr(k-1) \oplus gl(1)\) |
| \(sp\left(\frac{m}{2}, n - \frac{m}{2}\right)\) | \(sp(k) \oplus \frac{m}{2} su(2) \oplus \frac{m}{2} gl(1)\) | \(sp(k-1) \oplus u(1)\) |
| \(sp\left(\left[\frac{n}{2}\right] - p, \left[\frac{n+1}{2}\right] + p\right)\) \(p = 1, \ldots, \left[\frac{k}{2}\right] - 1\) | \(sp\left(\left[\frac{k}{2}\right] - p, \left[\frac{k+1}{2}\right] + p\right) \oplus \frac{m}{2} su(2) \oplus \frac{m}{2} gl(1)\) | \(sp\left(\left[\frac{k}{2}\right] - p-1, \left[\frac{k+1}{2}\right] + p\right) \oplus u(1)\) |

| \(G\) compatible with \((C_n, A_{n-1})\) \(n\) even | \(G_0\) real form of \(n U_1\) | \(K\) real form of \(U_1\) |
|-----------------|------------------|------------------|
| \(sr(n)\) | \(\frac{n}{2} sl(2) \oplus \frac{n}{2} gl(1)\) | \(gl(1)\) |
| \(sp\left(\frac{n}{2}, \frac{n}{2}\right)\) | \(\frac{n}{2} su(2) \oplus \frac{n}{2} gl(1)\) | \(u(1)\) |

| \(G\) compatible with \((C_n, C_n)\) | \(G_0\) real form of \(n U_1\) | \(K\) trivial |
|-----------------|------------------|------------------|
| \(sr(n)\) | \(n gl(1)\) | — |

| \(G\) compatible with \((D_n, A_m)\) \(m\) even | \(G_0\) real form of \(D_k \oplus \frac{m}{2} A_1 \oplus \frac{m}{2} U_1\) | \(K\) real form of \(D_{k-1} \oplus U_1\) |
|-----------------|------------------|------------------|
| \(so(n, n)\) | \(so(k, k) \oplus \frac{m}{2} sl(2) \oplus \frac{m}{2} gl(1)\) | \(so(k-1, k-1) \oplus gl(1)\) |
| \(so^*(2n)\) | \(so^*(2k) \oplus \frac{m}{2} su(2) \oplus \frac{m}{2} gl(1)\) | \(so^*(2k-2) \oplus u(1)\) |
| \(so(2n-m, m)\) | \(so(2k) \oplus \frac{m}{2} sl(2) \oplus \frac{m}{2} gl(1)\) | \(so(2k-2) \oplus u(1)\) |
| \(so(2n-m-1, m+1)\) | \(so(2k-1, 1) \oplus \frac{m}{2} sl(2) \oplus \frac{m}{2} gl(1)\) | \(so(2k-3, 1) \oplus u(1)\) |
| \(so(n+p, n-p)\) \(p = 1, \ldots, k-2\) | \(so(k+p, k-p) \oplus \frac{m}{2} sl(2) \oplus \frac{m}{2} gl(1)\) | \(so(k-2+p, k-p) \oplus u(1)\) |

| \(G\) compatible with \((D_n, A_{n-1})\) \(n\) even | \(G_0\) real form of \(\frac{n}{2} A_1 \oplus \frac{n}{2} U_1\) | \(K\) real form of \(U_1\) |
|-----------------|------------------|------------------|
| \(so(n, n)\) | \(\frac{n}{2} sl(2) \oplus \frac{n}{2} gl(1)\) | \(gl(1)\) |
| \(so^*(2n)\) | \(\frac{n}{2} su(2) \oplus \frac{n}{2} gl(1)\) | \(u(1)\) |

| \(G\) compatible with \((D_n, D_m)\) | \(G_0\) real form of \(D_{k+1} \oplus (m-1)U_1\) | \(K\) real form of \(D_k\) |
|-----------------|------------------|------------------|
| \(so(n, n)\) | \(so(k+1, k+1) \oplus (m-1) gl(1)\) | \(so(k, k)\) |
| \(so(2n-m+1, m-1)\) | \(so(2k+2) \oplus (m-1) gl(1)\) | \(so(2k)\) |
| \(so(2n-m, m)\) | \(so(2k+1, 1) \oplus (m-1) gl(1)\) | \(so(2k-1, 1)\) |
| \(so(n+p, n-p)\) \(p = 1, \ldots, k-1\) | \(so(k+1+p, k+1-p) \oplus (m-1) gl(1)\) | \(so(k-1+p, k+1-p)\) |

| \(G\) compatible with \((D_n, D_m)\) | \(G_0\) real form of \(D_{k+1} \oplus (m-1)U_1\) | \(K\) real form of \(D_k\) |
|-----------------|------------------|------------------|
| \(so(n, n)\) | \(so(k+1, k+1) \oplus (m-1) gl(1)\) | \(so(k, k)\) |
| \(so(2n-m+1, m-1)\) | \(so(2k+2) \oplus (m-1) gl(1)\) | \(so(2k)\) |
| \(so(2n-m, m)\) | \(so(2k+1, 1) \oplus (m-1) gl(1)\) | \(so(2k-1, 1)\) |
| \(so(n+p, n-p)\) \(p = 1, \ldots, k-1\) | \(so(k+1+p, k+1-p) \oplus (m-1) gl(1)\) | \(so(k-1+p, k+1-p)\) |

| \(G\) compatible with \((D_n, D_m)\) | \(G_0\) real form of \(D_{k+1} \oplus (m-1)U_1\) | \(K\) real form of \(D_k\) |
|-----------------|------------------|------------------|
| \(so(n, n)\) | \(so(k+1, k+1) \oplus (m-1) gl(1)\) | \(so(k, k)\) |
| \(so(2n-m+1, m-1)\) | \(so(2k+2) \oplus (m-1) gl(1)\) | \(so(2k)\) |
| \(so(2n-m, m)\) | \(so(2k+1, 1) \oplus (m-1) gl(1)\) | \(so(2k-1, 1)\) |
| \(so(n+p, n-p)\) \(p = 1, \ldots, k-1\) | \(so(k+1+p, k+1-p) \oplus (m-1) gl(1)\) | \(so(k-1+p, k+1-p)\) |
6 Concluding Remarks

We hope the results we have described in this paper will be of interest and of use from three related but distinct points of view. First, for future work on integrable lagrangian field theories. Second, for the continued investigation of \( \mathcal{W} \)-algebras, their representation theory, and their eventual classification. Third, for the purely mathematical problem of classifying all embeddings of \( sl(2, \mathbb{R}) \) into real Lie algebras. One may anticipate, given the wide-ranging importance of Lie algebras in many branches of mathematics and physics, that such a classification may well have unexpected applications in quite new areas.

There are also two more specific topics which immediately suggest themselves. One very interesting question is the corresponding classification in the case of Lie superalgebras. The significance of these ideas for supersymmetric models was originally pointed out in our earlier paper [6], where the choice of the appropriate real form was shown to be essential for the correct realization of \( N = 4 \) superconformal symmetry in a particular model. It remains to set this isolated example in a more general context, which should also include the superconformal algebras discussed in [9] as special cases. Finally, it would also be very interesting to extend the study carried out in this paper to the case of affine Toda theories.

Acknowledgements

The research of JME is supported by a PPARC Advanced Fellowship, while JOM is supported by the TMR European network contract number FMRX-CT96-0012, and is grateful to the Danish Research Council for additional support. One of the authors (JOM) would like to thank L. Fehér, E. Ragoucy, P. Sorba, and I. Tsutsui for useful discussions, and Laboratoire de Physique Théorique ENSLAPP, where much of this work was done, for financial support and kind hospitality.

A Technical Results for the Classification

In this appendix we give some technical details of results which were quoted in our classification of real embeddings in Theorem 5.

A.1 \( D_n \)

We will show that an automorphism on \( D_n \) for \( n > 4 \) which leaves at least one basis vector invariant cannot give the real form \( so^*(2n) \) (note that \( so^*(8) \) is isomorphic to \( so(6, 2) \)). In our proof of this we will use the character to distinguish real forms. This is not sufficient by itself for the particular case when \( n = k^2 \), since the real forms \( so^*(2k^2) \) and \( so(n+k, n-k) \) happen to have the same character. To complete the proof we should therefore also consider the maximal compact subalgebra as a means of distinguishing these real forms. We shall omit these details however.

Consider an element \( \tau \in \text{Aut}(\Delta) \) of order two, consisting of \( p \) 2-cycles, \( q \) invariant basis vectors, and \( r \) inverted basis vectors, so \( n = 2p + q + r \). Taking into account the consistency equation \( \chi_\alpha = \chi_{\tau(\alpha)} \) given in (3.26), we find the character

\[
\sigma = (q - r) + 2(p + \sum_{\alpha \in \Delta_+(D_r)} \chi_\alpha) = (n - r) + (-r + 2 \sum_{\alpha \in \Delta_+(D_r)} \chi_\alpha),
\]
where $D_r$ is the subalgebra corresponding to the $r$ inverted basis vectors. The term $q - r$ is the trace of $\tau_H$ and $2p$ is the contribution from the $p$ positive roots that are inverted by the $p$ two-cycles. Note that, remarkably, the character depends only on $r$. The term $-r + 2\sum_{\alpha \in \Delta^+_t(D_r)} \chi_\alpha$ is the character $\sigma_r$ of the real form of the $D_r$ subalgebra, which means that:

$$-r + 2\sum_{\alpha \in \Delta^+_t} \chi_\alpha = \sigma_r = \begin{cases} (r - 2j^2), & j = 1, \ldots, \left[\frac{r}{2}\right] \\ -r & \end{cases}$$

To obtain the real form $so^*(2n)$ the character should be

$$\sigma = (n-r) + \sigma_r = -n$$

This has solutions if $n = k^2$ or if $r = n$. In the latter case, the automorphism leaves no basis vector invariant. In the former case we cannot distinguish, using the character alone, between $su^*(n)$ and $so(n+k, n-k)$. But consideration of the maximal compact subalgebra shows that it is the latter real form which is defined by this conjugation map.

### A.2 $B_n$

For the reduction $(B_n, A_m)$ discussed in section 5.4 we found the zero-grade subalgebra to be of complex type $G_0^c = \frac{m}{2}U_1 \oplus \frac{m}{2}A_1 \oplus B_{n-m}$. We claimed that any automorphism that leaves $A_m$ invariant up to a diagram automorphism will result in the real form $\frac{m}{2}sl(2) \subset G_0$ of $\frac{m}{2}A_1 \subset G_0^c$. We now justify this statement.

The roots of $\frac{m}{2}A_1$ are $e_1 + e_{m+1}$, $e_2 + e_m$, $\ldots$, $e_{m+1} + e_m$. Consider e.g. $e_1 + e_{m+1}$. If $\tau$ acts as a non-trivial diagram automorphism on $A_m$ then $\tau(e_1 + e_{m+1}) = -(e_1 + e_{m+1})$ and $\chi_{\alpha_1} = \chi_{\alpha_2} = \ldots = \chi_{\alpha_m}$. In order to find the character of the real form of the corresponding $A_1$ we must find $\chi_{e_1+e_{m+1}}$. Considering the roots $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_m = e_1 - e_{m+1}$ and $\beta = e_{m+1}$ we see that $e_1 + e_{m+1} = \alpha + 2\beta$. Now $\tau(\alpha) = \alpha$ and $\tau(\beta) = -(\alpha + \beta)$, so

$$\chi_{\alpha+2\beta} = \frac{N_{\alpha+2\beta}}{N_{\alpha-(\alpha+\beta)}N_{\beta-(\alpha+\beta)}} \chi_\alpha(\chi_\beta)^2$$

We see immediately that $N_{\alpha+2\beta} = N_{\beta-(\alpha+\beta)}$. Also $\chi_\alpha = -1$ because

$$\alpha = (\alpha_1 + \ldots + \alpha_{m+1}) + (\alpha_{m+1} + \ldots + \alpha_m) \equiv \beta_1 + \beta_2$$

with $\tau(\beta_1) = \beta_2$. By symmetry we have $\chi_{\beta_1} = \chi_{\beta_2}$, so $\chi_\alpha = \frac{N_{\beta_1}\beta_2}{N_{\beta_2}\beta_1} \chi_1 \chi_2 = -1$. Finally, $N_{\alpha+\beta} = -N_{\alpha-(\alpha+\beta)}$ because

$$N_{\alpha+\beta}N_{\alpha-(\alpha+\beta)} = N_{\alpha+\beta}N_{\alpha-\alpha} = \alpha \cdot \beta = -1$$

where we have used the Jacobi identity. We conclude that $\chi_{e_1+e_{m+1}} = 1$ and we can now calculate the character of the corresponding $A_1$ to be 1, i.e. the character of $sl(2)$.

Similar (but slightly more complicated) calculations can be done in the cases of $(C_n, A_m)$ and $(D_n, A_m)$ referred to in the text.
A.3  $C_n$

We show that any conjugation map on $C_n$ with at least one invariant basis vector leads to a real form with character $\sigma = n$, i.e. to the split real Lie algebra. We will limit ourselves to show this in the case where the automorphism has only invariant and inverted basis vectors. The calculation when the automorphism also contains 2-cycles is not significantly different.

Consider a conjugation map on $C_n$ which leaves the first $p$ basis vectors invariant and which inverts the remaining $n-p$ basis vectors. The action of $\tau$ on the simple roots is:

\[
\begin{align*}
\tau(\alpha_i) &= \alpha_i & i < p \\
\tau(\alpha_j) &= -\alpha_j & j > p \\
\tau(\alpha_p) &= \tau(e_p - e_{p+1}) &= e_{p+1}
\end{align*}
\]

Setting $\gamma = e_{p+1} - e_n$ and writing

\[
e_p + e_{p+1} = (e_p - e_{p+1}) + (e_{p+1} - e_n) + (e_{p+1} - e_n) + 2e_n = \alpha_p + \gamma + \gamma + \alpha_n
\]

we get the consistency condition

\[
\chi_{\alpha_p} = \chi_{e_{p+1}} = \frac{N_{\alpha_p\gamma}}{N_{\alpha_p+2\gamma+\alpha_n-\gamma}} \frac{N_{\gamma\alpha_n}}{N_{-\gamma-\alpha_n}} \chi_{\alpha_p} \chi_{\alpha_n} = \chi_{\alpha_p} \chi_{\alpha_n}
\]

since $N_{\alpha_p\gamma}/N_{\alpha_p+2\gamma+\alpha_n-\gamma} = N_{\gamma\alpha_n}/N_{-\gamma-\alpha_n} = -1$. This implies $\chi_{\alpha_n} = 1$.

Corresponding to the inverted basis vectors, there is a subalgebra of type $C_{n-p}$ with roots satisfying $\tau(\alpha) = -\alpha$, and these can be written in the form:

\[
\begin{align*}
e_k - e_l &= \alpha_k + \cdots + \alpha_{l-1} \\
e_k + e_l &= \alpha_k + \cdots + \alpha_{l-1} + 2\alpha_l + \cdots + 2\alpha_{n-1} + \alpha_n \\
2e_k &= 2\alpha_k + \cdots + 2\alpha_{n-1} + \alpha_n
\end{align*}
\]

This, together with the equation (3.27), shows that when $\chi_{\alpha_n} = 1$ the contribution to the character from the roots $e_k - e_l$ cancels the contribution from the roots $e_k + e_l$, and that $\chi_{2e_k} = \chi_{\alpha_n}$. The total character is then:

\[
\sigma = (p - (n - p)) + 2(n - p) = n
\]

where the first term arises from $\text{tr}(\tau_H)$, while the second term is the contribution from the roots $2e_{p+1}, 2e_{p+2}, \ldots, 2e_n$. 

39
B Calculating Toda Potentials

In this appendix we indicate how to calculate the non-abelian Toda potentials

\[ V = \text{tr}(M_+ g M_- g^{-1}) \]

for the cases of non-principal embeddings in \( B_2 \) and \( G_2 \) discussed in section 4. The general idea is simply to identify the representations of \( G_0 \) in which \( M_\pm \) transform under conjugation in \( G \), since this is what occurs in \( V \) above. The actual calculations are particularly easy to carry out when the only non-abelian factors in \( G_0 \) are of the simple type \( A_1 \).

Consider the spin-1/2 representation of \( sl(2) \) acting on a pair of states labelled \( \pm \) to indicate their eigenvalues \( \pm 1/2 \) under the diagonal generator. With a standard choice of coordinates \( \phi_0 \) and \( \phi_\pm \) we have

\[
g \equiv \begin{pmatrix} U_{++} & U_{+-} \\ U_{-+} & U_{--} \end{pmatrix} = \exp \left( \begin{pmatrix} \phi_0 & \phi_+ \\ \phi_- & -\phi_0 \end{pmatrix} \right) = \frac{1}{\rho} \begin{pmatrix} \rho \cosh \rho + \phi_0 \sinh \rho & \phi_+ \sinh \rho \\ \phi_- \sinh \rho & \rho \cosh \rho - \phi_0 \sinh \rho \end{pmatrix}
\]

where \( \rho = \phi_0 + \phi_+ \phi_- \).

In the case of \( B_2 \) the generators \( M_\pm \) transform in the spin-1 representation, which we may regard as a symmetrized product of two spin-1/2 representations. The relevant matrix acting on states labelled by \( +1, 0, -1 \) is then

\[
\begin{pmatrix}
U_{++}^2 & \frac{1}{2}(U_{++}U_{--} + U_{+-}U_{-+}) & U_{+-}^2 \\
U_{++}U_{--} & \frac{1}{2}(U_{++}U_{--} + U_{+-}U_{-+}) & U_{+-}U_{-+} \\
U_{++}^2 & U_{+-}^2 & \frac{1}{2}(U_{++}U_{--} + U_{+-}U_{-+})
\end{pmatrix}
\]

Taking the \( (B_2, A_1) \) description of the embedding, we have \( M_\pm \) appearing as states of spin 0. The contribution to the potential is therefore given by the ‘centre’ element of this matrix:

\[
(U_{++}U_{--} + U_{+-}U_{-+}) = \frac{1}{\rho^2}(\rho \cosh \rho - \phi_0^2 \sinh^2 \rho + \phi_+ \phi_- \sinh^2 \rho) = \frac{1}{\rho^2}(\phi_0^2 + \phi_+ \phi_- \cosh 2\rho)
\]

as given in the text. The potential for the \( (B_2, A_1 \oplus A_1) \) embedding can be calculated similarly, but involves the components with spins \( \pm 1 \) instead.

In the case of \( G_2 \) the generators \( M_\pm \) transform in the spin-3/2 representation, with states labelled by \( +3/2, +1/2, -1/2, -3/2 \). Once again we can construct this as the totally symmetrized product of three spin-1/2 representations. The matrix in question is

\[
\begin{pmatrix}
U_{++}^3 & U_{++}^2 & U_{++} \\
U_{++}^2 & \frac{1}{3}(U_{++}U_{--} + 2U_{+-}U_{-+}U_{++}) & \frac{1}{3}(U_{++}U_{--} + 2U_{+-}U_{-+}U_{++}) \\
U_{++} & \frac{1}{3}(U_{++}U_{--} + 2U_{+-}U_{-+}U_{++}) & \frac{1}{3}(U_{++}U_{--} + 2U_{+-}U_{-+}U_{++})
\end{pmatrix}
\]

For the \( (G_2, A_2) \) embedding the elements \( M_\pm \) involve the states of extreme spin \( \pm 3/2 \). It is easy to see that the contribution to the potential is therefore given by the sum of the ‘corner’ entries in the matrix above:

\[
U_{++}^3 + U_{+-}^3 + U_{-+}^3 + U_{++} = \frac{1}{\rho^3}(2\rho^3 \cosh^3 \rho + 6\rho^2 \phi_0^2 \cosh \rho \sinh^2 \rho + \phi_+^3 \sinh^3 \rho + \phi_-^3 \sinh^3 \rho)
\]

where \( \rho = \phi_0^2 + \phi_+ \phi_- \), which is again the result given in the text.
References

[1] A.N. Leznov and M.V. Saveliev, *Comm. Math. Phys.* **89** (1983) 59; *Comm. Math. Phys.* **74** (1980) 11.

[2] L. Fehér, L. O’Raifeartaigh, P. Ruelle, I. Tsutsui, *Ann. Phys.* **213** (1992) 1; J. Balog, L. Fehér, L. O’Raifeartaigh, P. Forgács and A. Wipf, *Ann. Phys.* **203** (1990) 76.

[3] F.A. Bais, T. Tjin and P. van Driel, *Nucl. Phys.* **B357** (1991) 632.

[4] L. Fehér, L. O’Raifeartaigh, P. Ruelle, I. Tsutsui, A. Wipf, *Phys. Rep.* **222** (1992) 1.

[5] P. Bouwknegt and K. Schoutens, *Phys. Rep.* **223** (1993) 183.

[6] J.M. Evans and J.O. Madsen, *Phys. Lett.* **B384** (1996) 131, hep-th/9605126.

[7] J. de Boer, F. Harmsze, T. Tjin, *Phys. Rep.* **272** (1996) 139.

[8] E. Dynkin, *Amer. Math. Soc. Transl. Ser.* **2**, 6 (1957) 111.

[9] B. Bina and M. Günaydin, *Nucl. Phys.* **B502** (1997) 713; hep-th/9703188.

[10] L. Frappat, E. Ragoucy and P. Sorba, *Comm. Math. Phys.* **157** (1993) 499.

[11] J.O. Madsen and E. Ragoucy, *Comm. Math. Phys.* **185** (1997) 509, hep-th/9503042.

[12] J.O. Madsen and E. Ragoucy, *Phys. Lett.* **B279** (1992) 319.

[13] P. Bowcock and G.M.T. Watts, *Nucl. Phys.* **B379** (1992) 63; L. Fehér, L. O’Raifeartaigh, and I. Tsutsui, *Phys. Lett.* **B316** (1993) 275.

[14] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces* (Academic Press, New York, 1978).

[15] A.A. Sagle and R.E. Walde, *Introduction to Lie Groups and Lie Algebras* (Academic Press, New York, 1973).

[16] J. Balog, L. O’Raifeartaigh, P. Forgács and A. Wipf, *Nucl. Phys.* **B325** (1989) 225; S. Hwang, *Nucl. Phys.* **B354** (1991) 100, *Phys. Lett.* **B276** (1992) 451; I. Bars, *Phys. Rev.* **D53** (1996) 3308; J.M. Evans, M.R. Gaberdiel and M.J. Perry, *The No-ghost Theorem for AdS$_3$ and the Stringy Exclusion Principle*, hep-th/9806024, to appear in *Nuclear Physics B*.

[17] E. Witten, *Phys. Rev.* **D44** (1991) 314; J-L. Gervais and M.V. Saveliev, *Phys. Lett.* **B286** (1992) 271; S. Chaudhuri, J.D. Lykken, *Nucl. Phys.* **B396** (1993) 270; A. Bilal, *Nucl. Phys.* **B422** (1994) 258.

[18] J.M. Evans, *Nucl. Phys.* **B390** (1993) 225.

[19] A. Honecker, *Nucl. Phys.* **B400** (1993) 574.

[20] P. Bowcock, *Representation Theory of a W-algebra from Generalized DS Reduction*, preprint DTP 94-5, hep-th/9403154.