THE METRIC PROJECTIONS ONTO CLOSED CONVEX CONES IN
A HILBERT SPACE

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Abstract We study the metric projection onto the closed convex cone in a real Hilbert space \( \mathcal{H} \) generated by a sequence \( \mathcal{V} = \{v_n\}_{n=0}^{\infty} \). The first main result of this article provides a sufficient condition under which the closed convex cone generated by \( \mathcal{V} \) coincides with the following set:

\[
\mathcal{C}[\mathcal{V}] := \left\{ \sum_{n=0}^{\infty} a_n v_n \mid a_n \geq 0, \text{the series } \sum_{n=0}^{\infty} a_n v_n \text{ converges in } \mathcal{H} \right\}.
\]

Then, by adapting classical results on general convex cones, we give a useful description of the metric projection onto \( \mathcal{C}[\mathcal{V}] \). As an application, we obtain the best approximations of many concrete functions in \( L^2([-1, 1]) \) by polynomials with nonnegative coefficients.

Keywords: closed convex cones; metric projections; best approximation; polynomials with nonnegative coefficients

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1. Introduction

1.1. Closed convex cone generated by a sequence

Recall that a nonempty subset of a vector space over the field \( \mathbb{R} \) of real numbers is a convex cone if it is closed under linear combinations with nonnegative coefficients. It is a classical problem to find the best approximation of a given vector by elements in certain closed convex subset; cf. [5, 7, 8, 10, 12, 14, 16]. The best approximation of functions on an interval by polynomials with nonnegative coefficients is particularly interesting; for instance, it plays a crucial role in the spectral analysis of self-adjoint operators on real Hilbert spaces [12, 14].
In this article, we are interested in the convex cone generated by a sequence in a real Hilbert space. More precisely, let $\mathcal{H}$ be a Hilbert space over $\mathbb{R}$ and let $\mathcal{V} = \{v_n\}_{n=0}^{\infty} \subset \mathcal{H}$ be a sequence. The algebraic convex cone $C[\mathcal{V}]$ generated by $\mathcal{V}$ is the set of all nonnegative linear combinations of the vectors in $\mathcal{V}$ (here and after, we denote by $\mathbb{N}$ the set of all nonnegative integers: $\mathbb{N} = \{0, 1, 2, \ldots\}$):

$$C[\mathcal{V}] := \left\{ \sum_{n=0}^{N} a_n v_n \bigg| N \in \mathbb{N}, a_n \geq 0 \right\}.$$ 

Denote by $\overline{C}[\mathcal{V}]$ the norm closure of $C[\mathcal{V}]$ inside $\mathcal{H}$. Then $\overline{C}[\mathcal{V}]$ is a closed convex cone.

The convex cone $C[\mathcal{V}]$ and its closure $\overline{C}[\mathcal{V}]$ are useful objects in functional analysis, mathematical optimisation and many other fields; cf. [1, 2, 8, 13]. One can consult [4, 11] for the basic algebraic theory of $C[\mathcal{V}]$ when $\mathcal{V}$ is a finite sequence. Usually, it is more convenient to work with the closure $\overline{C}[\mathcal{V}]$. In fact, the closedness of a convex subset is important in the best approximation theory [5, Chapter 5].

For any element $w \in \mathcal{H}$, let $d(w, \overline{C}[\mathcal{V}])$ denote the distance of $w$ to $\overline{C}[\mathcal{V}]$:

$$d(w, \overline{C}[\mathcal{V}]) := \inf \left\{ \|w - u\| : u \in \overline{C}[\mathcal{V}] \right\} = \inf \left\{ \left\| w - \sum_{n=0}^{N} a_n v_n \right\| : N \in \mathbb{N}, a_n \geq 0 \right\}.$$ 

For a nonzero vector $w \in \mathcal{H}$, we also introduce the relative distance $\lambda(w, \overline{C}[\mathcal{V}])$:

$$\lambda(w, \overline{C}[\mathcal{V}]) := \frac{d(w, \overline{C}[\mathcal{V}])}{\|w\|} \in [0, 1].$$ 

Denote by $\angle(w, v)$ the angle between two nonzero vectors $w, v$. If $\lambda(w, \overline{C}[\mathcal{V}]) < 1$, then it is easy to see that it satisfies the equality

$$\lambda(w, \overline{C}[\mathcal{V}]) = \sin \left( \inf \left\{ \angle(w, v) : v \in \overline{C}[\mathcal{V}] \setminus \{0\} \right\} \right).$$ 

In other words, the quantity $\lambda(w, \overline{C}[\mathcal{V}])$, when belonging to the open interval $(0, 1)$, measures how far the direction of the vector $w$ is away from the directions of the vectors in $\overline{C}[\mathcal{V}]$.

By a classical result on closed convex subsets of Hilbert spaces (see, e.g., [16, p. 239] and [5, Theorem 3.5]), for any $w \in \mathcal{H}$ there exists a unique vector $w^* \in \overline{C}[\mathcal{V}]$ that is closest to $w$:

$$\|w - w^*\| = d(w, \overline{C}[\mathcal{V}]).$$

This unique element $w^*$ is called the metric projection of $w$ onto $\overline{C}[\mathcal{V}]$ and will be denoted by $P_{\overline{C}[\mathcal{V}]}(w)$. In most situations, the computation of $d(w, \overline{C}[\mathcal{V}])$ or $\lambda(w, \overline{C}[\mathcal{V}])$ is then reduced to the computation of $P_{\overline{C}[\mathcal{V}]}(w)$.

To study the best approximation of a given vector by elements in $\overline{C}[\mathcal{V}]$, one may first try to understand the closed convex cone $\overline{C}[\mathcal{V}]$ better. In general, $\overline{C}[\mathcal{V}]$ can be quite complicated. The first aim of this article is to present an explicit description of $\overline{C}[\mathcal{V}]$ under some conditions on the sequence $\mathcal{V}$.
For the sequence $V$, define a set $C[|V|] \subset \mathcal{H}$ as follows:

$$C[|V|] := \left\{ \sum_{n=0}^{\infty} a_n v_n \bigg| a_n \geq 0 \text{ and the series } \sum_{n=0}^{\infty} a_n v_n \text{ converges in } \mathcal{H} \right\}.$$

Clearly, we have

$$C[|V|] \subset C[[V]] \subset C[V]. \quad (1.1)$$

We shall also denote $C[[V]]$ by $C_{\mathcal{H}}[|V|]$ when it is necessary.

**Remark.** In the definition of $C[[V]]$, that the series $\sum_{n=0}^{\infty} a_n v_n$ converges in $\mathcal{H}$ in general does not imply that it converges unconditionally. Therefore, the definition of $C[[V]]$ may depend on the order of the vectors in the sequence $V$.

Observe that $C[[V]]$ is a convex cone. By (1.1), it is closed if and only if $C[[V]] = \overline{C}[V]$. Therefore, the closedness of $C[[V]]$ implies a relatively simple description of $\overline{C}[V]$. However, in general, $C[[V]]$ is not closed. For instance, consider the two-dimensional Euclidean space $\mathbb{R}^2$ and let $v_n = (1, n) \in \mathbb{R}^2$ for $n \in \mathbb{N}$. Then

$$\overline{C}[V] = \{(x, y) | x, y \geq 0\}.$$

But $C[[V]] \neq \overline{C}[V]$ because $(0, 1) \notin C[[V]]$.

So we are going to investigate the following.

**Problem.** When is $C[[V]]$ closed?

It is easy to see that $C[[V]]$ is closed if the sequence $V$ satisfies the following condition: There exist two constants $c, C > 0$ and a sequence of positive numbers $\lambda_n > 0$ such that the inequalities

$$c \left( \sum_{n=0}^{\infty} \lambda_n |c_n|^2 \right)^{1/2} \leq \left\| \sum_{n=0}^{\infty} c_n v_n \right\| \leq C \left( \sum_{n=0}^{\infty} \lambda_n |c_n|^2 \right)^{1/2}$$

hold for all finitely supported sequences $\{c_n\}_{n=0}^{\infty}$ in $\mathbb{R}$. A less obvious sufficient condition for the closedness of $C[[V]]$ is given in Theorem 1.1.

We say that a positive Radon measure $\mu$ on $\mathbb{R}$ has finite moments of all orders if for all $n \in \mathbb{N}$ we have

$$\int_{\mathbb{R}} |t|^n \, d\mu(t) < \infty.$$

Let us denote

$$\mathbb{R}_+ = [0, \infty) \text{ and } \mathbb{R}_- = \mathbb{R} \setminus \mathbb{R}_+ = (-\infty, 0).$$

Given a positive Radon measure $\mu$ on $\mathbb{R}_+ = [0, \infty)$, we denote by $\text{supp}(\mu)$ the topological support of $\mu$ and by $s_{\mu}$ the supremum of $\text{supp}(\mu)$:

$$s_{\mu} := \sup \{ x | x \in \text{supp}(\mu) \} \in [0, \infty].$$
Theorem 1.1. Let $\mathcal{H}$ be a real Hilbert space and let $\mathcal{V} = \{v_n\}_{n=0}^{\infty} \subset \mathcal{H}$ be a sequence. Assume that there exists a positive Radon measure $\mu$ on $\mathbb{R}_+$, a real Hilbert space $\mathcal{K}$, a sequence $\{w_n\}_{n=0}^{\infty} \subset \mathcal{K}$ and a sequence $\{\lambda_n\}_{n=0}^{\infty}$ of positive numbers such that the following conditions are fulfilled:

- The measure $\mu$ has finite moments of all orders and $\mu(\mathbb{R}_+ \setminus [0,s_\mu)) = 0$.
- There exists a constant $C > 0$ such that the inequalities
  $$\left\| \sum_{n=0}^{\infty} a_n w_n \right\|_{\mathcal{H}} \leq C \left\| \sum_{n=0}^{\infty} a_n t^n \right\|_{L^2(\mu)}$$  
  (1.2)

hold for all finite supported sequences $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}_+$.
- For all $n, m \in \mathbb{N}$, we have
  $$\langle \lambda_n v_n, \lambda_m v_m \rangle_{\mathcal{H}} = \int_{\mathbb{R}_+} t^{m+n} \ d\mu(t) + \langle w_n, w_m \rangle_{\mathcal{H}}.$$  
  (1.3)

Then $C[|\mathcal{V}|]$ is a closed convex cone in $\mathcal{H}$ and thus

$$C[|\mathcal{V}|] = C[|\mathcal{V}|].$$

Remark. The condition $\mu(\mathbb{R}_+ \setminus [0,s_\mu)) = 0$ means that either $s_\mu = \infty$ or $\mu([s_\mu)) = 0$ if $s_\mu < \infty$. This condition in general cannot be removed in Theorem 1.1. For instance, let $\nu$ be the Lebesgue measure on $[0,1]$ and let $\mu = \nu + \delta_1$, where $\delta_1$ is the Dirac mass at the point 1, then the set

$$\left\{ \sum_{n=0}^{\infty} a_n t^n \mid a_n \geq 0, \text{the series} \sum_{n=0}^{\infty} a_n t^n \text{converges in} L^2(\mu) \right\}$$

is not closed. Indeed, the sequence $\{t^n\}_{n=0}^{\infty}$ converges in $L^2(\mu)$ to the Dirac function $\delta_1 \in L^2(\mu)$. But, clearly, this limit function $\delta_1$ is not of the form $\sum_{n=0}^{\infty} a_n t^n$.

Remark. By modifying the proof of Theorem 1.1, we can replace the sequence of functions $\{t^n\}_{n=0}^{\infty}$ by any sequence $\{h_n(t)\}_{n=0}^{\infty}$ of continuous nondecreasing nonnegative functions on $\mathbb{R}_+$ satisfying the property

$$\sum_{n=0}^{\infty} h_n(t) < \infty \quad \text{for any pair} \ (s,t) \ \text{with} \ 0 \leq t < s.$$

Theorem 1.1 has the following useful corollary. Before stating the corollary, let us note that if there exists a constant $C > 0$ such that

$$\langle w_n, w_m \rangle \leq C \int_{\mathbb{R}_+} t^{m+n} \ d\mu(t) \quad \text{for all} \ m, n \in \mathbb{N},$$

then the condition (1.2) is satisfied with the constant $\sqrt{C}$. 
Corollary 1.2. Let \( \nu \) be a positive Radon measure on \( \mathbb{R} \) having finite moments of all orders and let the restriction \( \mu = \nu|_{\mathbb{R}^+} \) satisfy the condition
\[
\mu(\mathbb{R}^+ \setminus [0, s]) = 0.
\]
If there exists a constant \( C > 0 \) such that
\[
\int_{\mathbb{R}^-} t^{2n} dv(t) \leq C \int_{\mathbb{R}^+} t^{2n} dv(t) \quad \text{for all } n \in \mathbb{N},
\]
then
\[
C_{L^2(\nu)}[\{t^n\}_{n=0}^{\infty}] = \left\{ \sum_{n=0}^{\infty} a_n t^n \bigg| a_n \geq 0, \text{ the series } \sum_{n=0}^{\infty} a_n t^n \text{ converges in } L^2(\nu) \right\}
\]
is a closed convex cone in \( L^2(\nu) \).

Remark. In general, the condition (1.4) cannot be removed in Corollary 1.2. For instance, consider the Lebesgue measure on the interval \( [-1, 0] \) and the associated Hilbert space \( L^2([-1, 0]) \). Then the set
\[
C_{L^2([-1, 0])}[\{t^n\}_{n=0}^{\infty}] = \left\{ \sum_{n=0}^{\infty} a_n t^n \bigg| a_n \geq 0, \text{ the series } \sum_{n=0}^{\infty} a_n t^n \text{ converges in } L^2([-1, 0]) \right\}
\]
is not closed in \( L^2([-1, 0]) \). See the Appendix for details.

1.2. Metric projection onto a closed convex cone

Assume that the convex cone \( C[\mathcal{V}] \) is closed; that is, \( C[\mathcal{V}] = \overline{C}[\mathcal{V}] \). Applying the classical results (cf. [16, Lemma 1.1]) on the metric projection onto a closed convex cone, we obtain in Proposition 1.3 a description of the metric projection onto \( C[\mathcal{V}] \). We shall see that Proposition 1.3 can be useful in computing explicitly the metric projections of given vectors.

Definition. We say that a sequence \( \mathcal{V} = \{v_n\}_{n=0}^{\infty} \) in a Hilbert space \( \mathcal{H} \) has no positive relations if the coincidence of two convergent series with nonnegative coefficients
\[
\sum_{n=0}^{\infty} a_n v_n = \sum_{n=0}^{\infty} b_n v_n
\]
implies the equalities \( a_n = b_n \) for all \( n \in \mathbb{N} \).

Remark. Note that if \( \mathcal{V} = \{v_n\}_{n=0}^{\infty} \) has no positive relations, then the vectors \( v_n \)s are linearly independent.

By convention, in what follows, we set
\[
\sum_{n \not\in \emptyset} a_n v_n := 0.
\]
Proposition 1.3. Let $\mathcal{V} = \{v_n\}_{n=0}^{\infty} \subset \mathcal{H}$ be a sequence without positive relations and assume that $\mathcal{C}[|\mathcal{V}|]$ is closed. Then for any $w \in \mathcal{H}$, there exists a unique subset $S \subset \mathbb{N}$ such that

$$P_{\mathcal{C}[|\mathcal{V}|]}(w) = \sum_{n \in S} a_n v_n \quad \text{with } a_n > 0 \text{ for all } n \in S,$$

where $(S, \{a_n\}_{n \in S})$ is uniquely determined by

$$\begin{cases}
\sum_{n \in S} a_n v_n \text{ converges in } \mathcal{H} \text{ and } a_n > 0 \text{ for all } n \in S; \\
\sum_{k \in S} a_k \langle v_k, v_n \rangle \geq \langle w, v_n \rangle \text{ for all } n \in \mathbb{N}; \\
\sum_{k \in S} a_k \langle v_k, v_n \rangle = \langle w, v_n \rangle \text{ for all } n \in S.
\end{cases}$$

Remark. Note that in Proposition 1.3, by saying that $\sum_{n \in S} a_n v_n$ converges in $\mathcal{H}$, we mean that the following limit exists in $\mathcal{H}$:

$$\lim_{N \to \infty} \sum_{n \in S, n \leq N} a_n v_n.$$

For any subset $S \subset \mathbb{N}$, define a subset $\mathcal{H}(\mathcal{V}, S) \subset \mathcal{H}$ by

$$\mathcal{H}(\mathcal{V}, S) := \left\{ w \in \mathcal{H} \mid P_{\mathcal{C}[|\mathcal{V}|]}(w) = \sum_{n \in S} a_n v_n \text{ with } a_n > 0 \text{ for all } n \in S \right\}.$$ 

In particular, we have $\mathcal{H}(\mathcal{V}, \emptyset) = (P_{\mathcal{C}[|\mathcal{V}|]})^{-1}(0)$.

By noting that the conditions in (1.6) are stable under addition and multiplication by a positive constant, we obtain the following corollary of Proposition 1.3.

Corollary 1.4. Let $\mathcal{V} = \{v_n\}_{n=0}^{\infty} \subset \mathcal{H}$ be a sequence without positive relations and assume that $\mathcal{C}[|\mathcal{V}|]$ is closed. Then we have a partition of the whole Hilbert space $\mathcal{H}$:

$$\mathcal{H} = \bigsqcup_{S \subset \mathbb{N}} \mathcal{H}(\mathcal{V}, S).$$

Moreover, for any subset $S \subset \mathbb{N}$, the subset $\{0\} \cup \mathcal{H}(\mathcal{V}, S)$ is a convex cone and the restriction of the metric projection

$$P_{\mathcal{C}[|\mathcal{V}|]}|_{\mathcal{H}(\mathcal{V}, S)} : \mathcal{H}(\mathcal{V}, S) \to \mathcal{C}[|\mathcal{V}|]$$

is affine. That is, for any $\lambda_1, \lambda_2 > 0$ and any $w_1, w_2 \in \mathcal{H}(\mathcal{V}, S)$, we have

$$P_{\mathcal{C}[|\mathcal{V}|]}(\lambda_1 w_1 + \lambda_2 w_2) = \lambda_1 P_{\mathcal{C}[|\mathcal{V}|]}(w_1) + \lambda_2 P_{\mathcal{C}[|\mathcal{V}|]}(w_2).$$

By considering the analogue of Proposition 1.3 for a convex cone generated by a finite sequence, we obtain in Corollary 1.5 a result for positive definite matrices. This result seems to be known in the literature. We include it because we believe that our proof may be of interest.
Corollary 1.5. Assume that $A$ is a nonsingular positive definite real coefficient $n \times n$ matrix. Then for any $(c_1, \ldots, c_n) \in \mathbb{R}^n$, there exists a unique subset $S \subset \{1, 2, \ldots, n\}$ and a unique $x \in \mathbb{R}^S$ such that

\[
\begin{cases}
  x_i > 0, & \text{for all } i \in S; \\
  \sum_{j \in S} a_{ij} x_j = c_i, & \text{for all } i \in S; \\
  \sum_{j \in S} a_{ij} x_j \geq c_i, & \text{for all } i \in \{1, 2, \ldots, n\}.
\end{cases}
\]

Remark. If $c_i \leq 0$ for all $1 \leq i \leq n$, then we take $S = \emptyset$ in Corollary 1.5.

1.3. Computation of the metric projections

We shall see in Subsection 1.4 that Proposition 1.3 may be used to compute explicitly the metric projection of a vector onto the closed convex cone generated by a sequence. Note that our method is different from the one presented in [6].

The general scheme is given as follows (note that although we focus on the case of an infinite sequence $\mathcal{V} = \{v_n\}_{n=0}^{\infty}$, the same scheme is clearly still valid for a finite sequence).

The main assumptions for Proposition 1.3 are the following:

(i) The sequence $\mathcal{V} = \{v_n\}_{n=0}^{\infty}$ has no positive relations.

(ii) The closed convex cone $\overline{C}[\mathcal{V}]$ is given by $\overline{C}[\mathcal{V}] = C[\mathcal{V}]$.

Under the above assumptions, assume that $w \in \mathcal{H}$ is a given vector and we want to compute the metric projection $P_{\overline{C}[\mathcal{V}]}(w)$. By Proposition 1.3, we only need to determine a unique subset $S \subset \mathbb{N}$ (we will denote this subset by $S(w; \mathcal{V}, \mathcal{H})$ when it is necessary) and a unique sequence $(a_n)_{n \in S}$ with $a_n > 0$ for all $n \in S$ such that (1.6) is satisfied. For further reference, let us denote

\[
\Gamma(w; \mathcal{V}, \mathcal{H}) := \left\{ v_n \bigg| n \in S(w; \mathcal{V}, \mathcal{H}) \right\}. \tag{1.8}
\]

The main difficulty is to determine the unique subset $S$ or, equivalently, to determine the set $\Gamma(w; \mathcal{V}, \mathcal{H})$. In general, it is not known to the authors whether there is an efficient way for determining such a subset $S \subset \mathbb{N}$ for an arbitrarily given vector $w \in \mathcal{H}$. However, for a given vector $w \in \mathcal{H}$ and a given subset $S \subset \mathbb{N}$, by Proposition 1.3 it is relatively easier to determine whether the equality

\[
\Gamma(w; \mathcal{V}, \mathcal{H}) = \left\{ v_n \bigg| n \in S \right\}
\]

holds or not (this is equivalent to determining whether $w \in \mathcal{H}(\mathcal{V}, S)$ or not, where $\mathcal{H}(\mathcal{V}, S)$ is defined as in (1.7)). Let us explain how to do so when $S \subset \mathbb{N}$ is a given finite subset. Set

\[
M := \left( (v_m, v_n) \right)_{m,n \in \mathbb{N}}
\]
and let $M_S$ be the submatrix indexed by $S \times S$:

$$M_S := \left( \langle v_m, v_n \rangle \right)_{m, n \in S}.$$ 

Then we need to solve the linear equation

$$M_S x = y,$$  \hspace{1cm} (1.9)

where $x = (x_n)_{n \in S} \in \mathbb{R}^S$ is a column vector to be determined and $y = (\langle w, v_n \rangle)_{n \in S}$ is the column vector defined by $y = (\langle w, v_n \rangle)_{n \in S}$. The assumption that the sequence $V$ has no positive relations implies that the matrix $M_S$ is nonsingular (here we use the assumption that $S$ is finite) and thus the linear equation (1.9) has a unique solution, denoted by $\widehat{x} \in \mathbb{R}^S$.

Now it remains to check whether the following conditions are satisfied:

$$\begin{cases} 
\widehat{x}_k > 0 & \text{for all } k \in S \\
\sum_{k \in S} \widehat{x}_k \langle v_k, v_n \rangle \geq \langle w, v_n \rangle & \text{for all } n \in \mathbb{N} \setminus S. 
\end{cases}$$  \hspace{1cm} (1.10)

If (1.10) is satisfied, then $w \in \mathcal{H}(V, S)$; moreover, we obtain the desired metric projection $P_{C[\|\cdot\|]}(w)$:

$$P_{C[\|\cdot\|]}(w) = \sum_{k \in S} \widehat{x}_k v_k.$$

Otherwise, if (1.10) is not satisfied, then $w \notin \mathcal{H}(V, S)$ and we shall try other subsets $S \subset \mathbb{N}$ for computing $P_{C[\|\cdot\|]}(w)$.

1.4. Applications in function theory

Consider the Lebesgue measure on $[-1,1]$ and the associated Hilbert space $L^2([-1,1])$. For ease of notation, set

$$\mathcal{A}_+ := \left\{ \sum_{n=0}^{\infty} a_n t^n \bigg| a_n \geq 0, \text{ the series } \sum_{n=0}^{\infty} a_n t^n \text{ converges in } L^2([-1,1]) \right\}.$$

By Corollary 1.2, the set $\mathcal{A}_+$ is a closed convex cone in $L^2([-1,1])$. As before, the associated metric projection is denoted by $P_{\mathcal{A}_+} : L^2([-1,1]) \to \mathcal{A}_+$.

1.4.1. Power functions. For any $\beta \in (0, \infty)$, let $h_\beta \in L^2([-1,1])$ be the power function defined by

$$h_\beta(t) = |t|^\beta, \quad t \in [-1,1].$$

It turns out that the best approximation of $h_\beta$ by elements in $\mathcal{A}_+$ is given by a linear combination of two elements $t^{2m}, t^{2m+2}$ with $2m$ and $2m + 2$ the closest two even numbers to $\beta$. For this reason, in what follows, it is convenient for us to write

$$\beta = 2m + \alpha \quad \text{with } m \in \mathbb{N} \text{ and } \alpha \in [0,2).$$
The metric projections onto closed convex cones in a Hilbert space

Figure 1. The best approximation of $|t|^3$ by polynomials with nonnegative coefficients: the graphs in the first quadrant.

**Proposition 1.6.** Let $\alpha \in [0, 2)$ and $m \in \mathbb{N}$. Then

$$P_{A_+}(h_{2m+\alpha}) = a_m t^{2m} + b_m t^{2m+2},$$

where $a_m$ and $b_m$ are given by

$$
\begin{align*}
  a_m &= a_m(\alpha) := \frac{(4m+1)(4m+3)}{(4m+1+\alpha)(4m+3+\alpha)} \frac{2-\alpha}{2}, \\
  b_m &= b_m(\alpha) := \frac{(4m+3)(4m+5)}{(4m+1+\alpha)(4m+3+\alpha)} \frac{\alpha}{2}.
\end{align*}
$$

Moreover, the distance $d(h_{2m+\alpha}, A_+)$ is given by

$$d(h_{2m+\alpha}, A_+) = \frac{\sqrt{2\alpha(2-\alpha)}}{(4m+\alpha+1)(4m+\alpha+3)\sqrt{4m+2\alpha+1}}$$

and the relative distance $\lambda(h_{2m+\alpha}, A_+)$ is given by

$$\lambda(h_{2m+\alpha}, A_+) = \frac{\alpha(2-\alpha)}{(4m+\alpha+1)(4m+\alpha+3)}.$$

**Remark.** Let $\alpha \in (0, 2)$. Then the coefficients $a_m(\alpha)$ and $b_m(\alpha)$ in Proposition 1.3 satisfy

$$a_m(\alpha) + b_m(\alpha) = \frac{(4m+3)(4m+2\alpha+1)}{(4m+1+\alpha)(4m+3+\alpha)} > 1.$$ 

Hence, the best approximation in $A_+$ of the function $h_{2m+\alpha}$ is not a convex combination of $t^{2m}$ and $t^{2m+2}$. In Figure 1 we draw the graphs in the first quadrant of the best approximation in $A_+$ of $h_3$:

$$h_3(t) = |t|^3, \quad P_{A_+}(h_3) = \frac{35}{96} t^2 + \frac{21}{32} t^4.$$
The best approximation in $C(\theta_1, \theta_2, \theta_3)$ of the function $g$ is not given by a positive combination of the two functions $\theta_1, \theta_2$ but is given by a positive combination of all three functions $\theta_1, \theta_2, \theta_3$.

Remark. Let $\alpha \in (0, 2)$. Recall the definition (1.8). Proposition 1.3 implies the equality

$$\Gamma\left(|t|^{2m+\alpha}; \{t^n\}_{n=0}^{\infty}, L^2([-1,1])\right) = \{t^{2m}, t^{4m}\}. \quad (1.14)$$

As we shall see in the proof of Proposition 1.3, it requires substantial efforts to prove the equality (1.14).

To more clearly explain the subtlety of the equality (1.14), let us consider the Hilbert space $L^2([0,1])$. It is easy to see that the equality (1.14) is equivalent to the equality

$$\Gamma\left(t^{2m+\alpha}; \{t^{2n}\}_{n=0}^{\infty}, L^2([0,1])\right) = \{t^{2m}, t^{4m}\}. \quad (1.15)$$

Note that the sequence $\{t^n\}_{n=0}^{\infty}$ now is replaced by the sequence $\{t^{2n}\}_{n=0}^{\infty}$. That is, in the Hilbert space $L^2([0,1])$, the best approximation of the function $t^{2m+\alpha}$ by elements in the closed convex cone generated by $\{t^{2n}\}_{n=0}^{\infty}$ is given by a positive combination of the two functions $t^{2m}$ and $t^{2m+2}$. One may think that the equality (1.15) is a consequence of the following observation: Among the graphs of all functions $t^0, t^2, t^4, \cdots$ on $[0,1]$, the closest ones to that of the function $t^{2m+\alpha}$ are exactly those of $t^{2m}$ and $t^{2m+2}$. However, let us point out that such an observation in general is not sufficient to derive the equality (1.15). For instance, as shown in Figure 2, among the three graphs of $\theta_1, \theta_2, \theta_3$, the graphs of $\theta_1$ and $\theta_2$ are the closest to the graph of $g$. However, because

$$g = \frac{1}{2} \theta_1 + \frac{1}{4} \theta_2 + \frac{1}{4} \theta_3 \neq \lambda_1 \theta_1 + \lambda_2 \theta_2 \quad \text{for any } \lambda_1 \geq 0, \lambda_2 \geq 0,$$

we have

$$\Gamma\left(g; \{\theta_1, \theta_2, \theta_3\}, L^2([0,1])\right) = \{\theta_1, \theta_2, \theta_3\} \neq \{\theta_1, \theta_2\}.$$
Corollary 1.7. For any \( m \in \mathbb{N} \) and any positive Radon measure \( \nu \) on \([0, 2]\), we have
\[
P_{A^+} \left( \int_{[0,2]} h_{2m+\alpha} d\nu(\alpha) \right) = t^{2m} \int_{[0,2]} a_m(\alpha) d\nu(\alpha) + t^{2m+2} \int_{[0,2]} b_m(\alpha) d\nu(\alpha),
\]
where \( a_m(\alpha), b_m(\alpha) \) are defined in (1.12).

Remark. The functions in Corollary 1.7 can be quite complicated: For instance, take \( \nu \) the Lebesgue measure on \([0, \alpha]\) with \( 0 < \alpha \leq 2 \), then we have
\[
\int_{[0,\alpha]} h_{2m+\alpha}(t) d\alpha' = \frac{|t|^{2m+\alpha} - t^{2m}}{\log |t|}.
\]

1.4.2. Signed power functions. For any \( \gamma \in (0, \infty) \), let \( f_\gamma \in L^2([-1, 1]) \) be the signed power function defined by
\[
f_\gamma(t) = \text{sgn}(t)|t|^\gamma, \quad t \in [-1, 1].
\]

Proposition 1.8. Let \( \alpha \in [0, 2) \) and \( m \in \mathbb{N} \). Then
\[
P_{A^+}(f_{2m+1+\alpha}) = c_m t^{2m+1} + d_m t^{2m+3},
\]
where \( c_m \) and \( d_m \) are given by
\[
\begin{align*}
c_m = c_m(\alpha) & := \frac{(4m+3)(4m+5)}{(4m+3+\alpha)(4m+5+\alpha)} \frac{2 - \alpha}{2}, \\
d_m = d_m(\alpha) & := \frac{(4m+5)(4m+7)}{(4m+3+\alpha)(4m+5+\alpha)} \frac{\alpha}{2}.
\end{align*}
\]

Moreover, the distance \( d(f_{2m+1+\alpha}, A^+) \) is given by
\[
d(f_{2m+1+\alpha}, A^+) = \frac{\sqrt{2}\alpha(2 - \alpha)}{(4m+\alpha+3)(4m+\alpha+5)\sqrt{4m+2\alpha+3}}
\]
and the relative distance \( \lambda(f_{2m+1+\alpha}, A^+) \) is given by
\[
\lambda(f_{2m+1+\alpha}, A^+) = \frac{\alpha(2 - \alpha)}{(4m+\alpha+3)(4m+\alpha+5)}.
\]

1.4.3. Indicator functions and nondecreasing functions. For any \( a \in [-1, 1) \), let \( \psi_a \) be the function defined by
\[
\psi_a(t) = \mathbb{1}(t \geq a) = \begin{cases} 0 \quad \text{if } t \in [-1, a) \\ 1 \quad \text{if } t \in [a, 1] \end{cases}.
\]

Recall the definition (1.8).

Proposition 1.9. Assume that \( a \in [-1, 1) \). Then the equality
\[
\Gamma \left( \psi_a; \{t^n\}_{n=0}^\infty, L^2([-1, 1]) \right) = \{t^n | n = 0, 1, 2\}
\]
holds if and only if
\[ 0 < a \leq \frac{1}{\sqrt{5}}. \]

Moreover, for any \( a \in (0, 1/\sqrt{5}] \), the metric projection \( P_{A_+}(\psi_a) \) is given by
\[ P_{A_+}(\psi_a) = \frac{1}{8} (4 - 9a + 5a^3) + \frac{3}{4} (1 - a^2) t + \frac{15}{8} (a - a^3) t^2. \] (1.20)

**Proposition 1.10.** Assume that \( a \in [-1, 1) \). Then the equality
\[ \Gamma \left( \psi_a; \{ t^n \}_{n=0}^\infty, L^2([-1,1]) \right) = \{ t^n | n = 0, 1 \} \] (1.21)
holds if and only if
\[ -\frac{1}{\sqrt{5}} \leq a \leq 0. \]

Moreover, for any \( a \in [-1/\sqrt{5}, 0] \), the metric projection \( P_{A_+}(\psi_a) \) is given by
\[ P_{A_+}(\psi_a) = \frac{1}{2} (1 - a) + \frac{3}{4} (1 - a^2) t. \] (1.22)

**Proposition 1.11.** Assume that \( a \in [-1, 1) \). Then the equality
\[ \Gamma \left( \psi_a; \{ t^n \}_{n=0}^\infty, L^2([-1,1]) \right) = \{ t^n | n = 0, 1, 2, 3 \} \] (1.23)
holds if and only if
\[ \frac{1}{\sqrt{5}} < a < \frac{\sqrt{105} - 5}{10}. \]

Moreover, for any \( a \in (\frac{1}{\sqrt{5}}, \frac{\sqrt{105} - 5}{10}) \), the metric projection \( P_{A_+}(\psi_a) \) is given by
\[ P_{A_+}(\psi_a) = \frac{1}{8} (1 - a)(4 - 5a + 5a^2) + \frac{15}{32} (1 - a^2)(3 - 7a^2) t \]
\[ + \frac{15}{8} a(1 - a^2) t^2 + \frac{35}{32} (1 - a^2)(5a^2 - 1) t^3. \] (1.24)

**Remark.** Propositions 1.9 and 1.11 may lead one to guess that there exists a sequence of critical points \( \{ b_k \}_{k=0}^\infty \) with \( 0 < b_0 < b_1 < \cdots < b_k < \cdots < 1 \) such that
\[ \Gamma \left( \psi_a; \{ t^n \}_{n=0}^\infty, L^2([-1,1]) \right) = \{ t^n | n = 0, 1, 2, \cdots, k + 1, k + 2 \} \] for all \( a \in (b_{k-1}, b_k) \).

This is, however, not clear to the authors at the time of writing. Indeed, the situation becomes more involved when \( a \) is close to 1. On the other hand, for negative \( a \), the situation seems to be different. By [15], because \( A_+ \) is a closed convex cone in a Hilbert space, the metric projection \( P_{A_+} : L^2([-1,1]) \rightarrow A_+ \) is a continuous map. Therefore, by the formula (1.21), there exists \( \epsilon > 0 \) such that
\[ \{0, 1\} \subset \Gamma \left( \psi_a; \{ t^n \}_{n=0}^\infty, L^2([-1,1]) \right) \] for all \( a \in \left( -\frac{1}{\sqrt{5}} - \epsilon, -\frac{1}{\sqrt{5}} \right) \).
But by Propositions 1.9, 1.10 and 1.11, for any $a \in (-\frac{1}{\sqrt{5}} - \epsilon, -\frac{1}{\sqrt{5}})$, we know that the set 
$$\Gamma\left(\psi_a; \{t^n\}_{n=0}^{\infty}, L^2([-1,1])\right)$$
cannot be any one of the three sets $\{0,1\}$, $\{0,1,2\}$ or $\{0,1,2,3\}$.

Propositions 1.9, 1.10 and 1.11 allow us to compute the metric projections onto the closed convex cone $A_+$ for functions in three large classes. Let us state the consequence of Proposition 1.9 in Corollary 1.12; the consequences of Propositions 1.10 and 1.11 are similar and will be omitted.

Let $\mathcal{M}_{[0,1/\sqrt{5}]}$ denote the class of functions on $[-1,1]$ consisting of all nondecreasing right-continuous nonnegative functions $\varphi : [-1,1] \to \mathbb{R}_+$ such that 
$$\varphi|_{[-1,0]} \equiv 0 \quad \text{and} \quad \varphi|_{[1/\sqrt{5},1]} \equiv \text{constant}.$$ 

Note that any $\varphi \in \mathcal{M}_{[0,1/\sqrt{5}]}$ uniquely determines a nonnegative Radon measure, denoted by $d\varphi$, on the interval $[-1,1]$, by the formula 
$$d\varphi([-1,t]) := \varphi(t) \quad \text{for any} \ t \in [-1,1].$$ 

Moreover, the support $\text{supp}(d\varphi)$ of the Radon measure $d\varphi$ satisfies $\text{supp}(d\varphi) \subset [0,1/\sqrt{5}]$ and we have 
$$\varphi(t) = \int_{[0,1/\sqrt{5}]} \psi_a(t) d\varphi(a) = \int_{[0,1/\sqrt{5}]} \mathbb{1}(t \geq a) d\varphi(a), \quad t \in [-1,1].$$ 

On the other hand, for any Radon measure on $[-1,1]$ with support $\text{supp}(\nu) \subset [0,1/\sqrt{5}]$, the function $\varphi_\nu$, defined by the formula 
$$\varphi_\nu(t) := \nu([-1,t]) \quad \text{for all} \ t \in [-1,1],$$ 

belongs to the class $\mathcal{M}_{[0,1/\sqrt{5}]}$.

**Corollary 1.12.** For any function $\varphi \in \mathcal{M}_{[0,1/\sqrt{5}]}$, we have 
$$P_{A_+}(\varphi) = A_\varphi + B_\varphi t + C_\varphi t^2,$$

with 

$$\begin{align*}
A_\varphi &= \int_{[0,1/\sqrt{5}]} \frac{1}{8} (4 - 9a + 5a^3) d\varphi(a) \\
B_\varphi &= \int_{[0,1/\sqrt{5}]} \frac{3}{4} (1 - a^2) d\varphi(a) \\
C_\varphi &= \int_{[0,1/\sqrt{5}]} \frac{15}{8} (a - a^3) d\varphi(a)
\end{align*}.$$ 

2. Closedness of convex cones

In this section, we prove Theorem 1.1 and Corollary 1.2.

**Proof of Theorem 1.1.** Let $\mathcal{H}_\mathcal{V} \subset \mathcal{H}$ denote the closed linear span of the sequence $\mathcal{V}$: 
$$\mathcal{H}_\mathcal{V} := \overline{\text{span}}(\mathcal{V}) \subset \mathcal{H}.$$
Let $\mathscr{H}(\mu, \mathcal{W})$ denote the closed linear span of the sequence $\{(t^n, w_n)\}_{n=0}^{\infty}$ in the real Hilbert space $L^2(\mu) \oplus \mathcal{K}$:

$$\mathscr{H}(\mu, \mathcal{W}) := \operatorname{span}\{(t^n, w_n) \mid n \in \mathbb{N}\} \subset L^2(\mu) \oplus \mathcal{K}.$$ 

The equalities (1.3) imply that the map $(t^n, w_n) \mapsto \lambda_n v_n$ can be extended to a linear isometric isomorphism between $\mathscr{H}(\mu, \mathcal{W})$ and $\mathcal{K}_\nu$. To complete the proof of Theorem 1.1, we shall prove that the following set

$$C(\mu, \mathcal{W}) := \left\{ \sum_{n=0}^{\infty} a_n (t^n, w_n) \bigg| a_n \geq 0, \text{the series} \sum_{n=0}^{\infty} a_n (t^n, w_n) \text{converges in} L^2(\mu) \oplus \mathcal{K} \right\}$$

is closed in $L^2(\mu) \oplus \mathcal{K}$.

Note first that for any element $(f, u) \in C(\mu, \mathcal{W}) \subset L^2(\mu)$ with $f = \sum_{n=0}^{\infty} a_n t^n$, $a_n \geq 0$, the series $\sum_{n=0}^{\infty} a_n (t^n, w_n)$ converges in $L^2(\mu)$ and the equality is understood as elements in $L^2(\mu)$, there exists a subsequence $\{N_k\}_{k=0}^{\infty}$ of positive integers with $0 < N_0 < N_1 < \cdots$ such that

$$\lim_{k \to \infty} \sum_{n=0}^{N_k} a_n t^n = f(t) < \infty \quad \text{for } \mu\text{-almost everywhere (a.e.) } t \in [0, s_\mu).$$

(2.26)

By the assumption $a_n \geq 0$ for all $n \in \mathbb{N}$, for any $t_0 \in [0, s_\mu)$ such that the limit equality (2.26) holds, we have

$$\lim_{k \to \infty} \sum_{n=0}^{N_k} a_n t_0^n = \sum_{n=0}^{\infty} a_n t_0^n := \lim_{N \to \infty} \sum_{n=0}^{N} a_n t_0^n.$$

It follows that we have the following $\mu$-a.e.:

$$f(t) = \sum_{n=0}^{\infty} a_n t^n < \infty \quad \text{for } \mu\text{-a.e. } t \in [0, s_\mu).$$

(2.27)

By the definition of $s_\mu$ and the assumption $a_n \geq 0$ for all $n \in \mathbb{N}$, we even have

$$\sum_{n=0}^{\infty} a_n t^n < \infty \quad \text{for all } t \in [0, s_\mu).$$

Now let $\{(f_k, u_k)\}_{k=0}^{\infty}$ be a sequence in $C(\mu, \mathcal{W})$,

$$(f_k, u_k) = \left( \sum_{n=0}^{\infty} a_n^{(k)} t^n, \sum_{n=0}^{\infty} a_n^{(k)} w_n \right) \in C(\mu, \mathcal{W}),$$

and assume that $(f_\infty, u_\infty) \in L^2(\mu) \oplus \mathcal{K}$ is the limit of the sequence $\{(f_k, u_k)\}_{k=0}^{\infty}$:

$$(f_k, u_k) \xrightarrow{k \to \infty} (f_\infty, u_\infty).$$

(2.28)
We want to show that \((f_\infty, u_\infty) \in C(\mu, W)\). Indeed, (2.28) implies, up to passing to a subsequence if necessary, that

\[
f_k(t) = \sum_{n=0}^{\infty} a_n^{(k)} t^n \rightarrow f_\infty(t) \quad \text{for } \mu\text{-a.e. } t \in \mathbb{R}_+.
\]

(2.29)

Note that the condition \(\mu(\mathbb{R}_+ \setminus [0, s_\mu]) = 0\) implies in particular that the support \(\text{supp} \mu\) is an infinite subset of \(\mathbb{R}_+\). By (2.29), there exists a sequence \(\tau_0 < \tau_1 < \cdots \in [0, s_\mu)\) such that

\[
\lim_{m \to \infty} \tau_m = s_\mu
\]

and

\[
f_k(\tau_m) = \sum_{n=0}^{\infty} a_n^{(k)} \tau_m^n \rightarrow f_\infty(\tau_m) \in [0, \infty) \quad \text{for all } m \in \mathbb{N}.
\]

It follows that

\[
M_m := \sup_k \sum_{n=0}^{\infty} a_n^{(k)} \tau_m^n \in [0, \infty) \quad \text{for all } m \in \mathbb{N}.
\]

Because all coefficients \(a_n^{(k)} \geq 0\), we have

\[
0 \leq a_n^{(k)} \tau_m^n \leq M_m \quad \text{for all } m, k \in \mathbb{N}.
\]

(2.30)

Using the compactness of \([0, M_m]\) and the canonical Cantor’s diagonal method, we may extract a subsequence of positive integers, denoted by \(0 < k_1 < k_2 < k_3 < \cdots\), such that for all \(m, n \in \mathbb{N}\), the following limits exist:

\[
\lim_{i \to \infty} a_n^{(k_i)} \tau_m^n \in [0, M_m].
\]

(2.31)

But this means that the limits \(\lim_{i \to \infty} a_n^{(k_i)}\) exist for all \(n \in \mathbb{N}\) and, moreover,

\[
a_n^{(\infty)} := \lim_{i \to \infty} a_n^{(k_i)} \in [0, \frac{M_m}{\tau_m}] \quad \text{for all } m, n \in \mathbb{N}.
\]

(2.32)

Now for any \(t \in [0, \tau_{m-1}]\), we have

\[
0 \leq \frac{t}{\tau_m} \leq \frac{\tau_{m-1}}{\tau_m} < 1
\]

and hence by (2.30),

\[
\sum_{n=0}^{\infty} \sup_i |a_n^{(k_i)} t^n| \leq \sum_{n=0}^{\infty} \frac{M_m}{\tau_m} t^n \leq \sum_{n=0}^{\infty} M_m \left(\frac{\tau_{m-1}}{\tau_m}\right)^n < \infty \quad \text{for all } t \in [0, \tau_{m-1}].
\]

Therefore, by the dominated convergence theorem,

\[
\lim_{i \to \infty} \sum_{n=0}^{\infty} a_n^{(k_i)} t^n = \sum_{n=0}^{\infty} \lim_{i \to \infty} a_n^{(k_i)} t^n = \sum_{n=0}^{\infty} a_n^{(\infty)} t^n \quad \text{for all } t \in [0, \tau_{m-1}].
\]

(2.33)
Combining (2.27), (2.29) and (2.33), we obtain
\[ f_\infty(t) = \sum_{n=0}^{\infty} a_n^{(\infty)} t^n \quad \text{for } \mu_{m-1}-\text{a.e. } t \in [0, \tau_{m-1}], \]
where \( \mu_{m-1} = \mu|_{[0, \tau_{m-1}]} \) is the restriction of the measure \( \mu \) on \([0, \tau_{m-1}]\). Because \( m \) is arbitrary, we have
\[ f_\infty(t) = \sum_{n=0}^{\infty} a_n^{(\infty)} t^n \quad \text{for } \mu-\text{a.e. } t \in [0, s_\mu). \quad (2.34) \]
Combining (2.34) with the condition \( \mu(\mathbb{R}_+ \setminus [0, s_\mu)) = 0 \), we obtain
\[ f_\infty(t) = \sum_{n=0}^{\infty} a_n^{(\infty)} t^n \quad \text{for } \mu-\text{a.e. } t \in \mathbb{R}_+. \quad (2.35) \]
We then need to show that the \( \mu \)-a.e. equality (2.35) implies the following \( L^2 \)-norm convergence:
\[ \lim_{N \to \infty} \left\| \sum_{n=0}^{N} a_n^{(\infty)} t^n - f_\infty \right\|_{L^2(\mu)} = 0. \quad (2.36) \]
But this again follows from the dominated convergence theorem. Indeed, because \( a_n^{(\infty)} \geq 0 \), we have
\[ \sup_N \left( \sum_{n=N+1}^{\infty} a_n^{(\infty)} t^n \right)^2 \leq \left( \sum_{n=0}^{\infty} a_n^{(\infty)} t^n \right)^2 = f_\infty(t)^2. \]
The above inequality combined with the assumption \( f_\infty \in L^2(\mathbb{R}_+, \mu) \) implies
\[ \lim_{N \to \infty} \left\| \sum_{n=0}^{N} a_n^{(\infty)} t^n - f_\infty \right\|_{L^2(\mu)}^2 = \lim_{N \to \infty} \int_{\mathbb{R}_+} \left( \sum_{n=N+1}^{\infty} a_n^{(\infty)} t^n \right)^2 d\mu(t) = 0. \]
Finally, it remains to show that
\[ \lim_{N \to \infty} \left\| \sum_{n=0}^{N} a_n^{(\infty)} w_n - u_\infty \right\| = 0. \quad (2.37) \]
By the assumption (1.2) and the fact that \( a_n^{(\infty)} \geq 0 \) for all \( n \in \mathbb{N} \), we have
\[ \left\| \sum_{n=N}^{M} a_n^{(\infty)} w_n \right\| \leq C \left\| \sum_{n=N}^{M} a_n^{(\infty)} t^n \right\|_{L^2(\mu)} \quad \text{for all } N, M \in \mathbb{N} \text{ with } N \leq M. \]
Thus, the convergence of the series \( \sum_{n=0}^{\infty} a_n^{(\infty)} t^n \) in the space \( L^2(\mu) \) implies the convergence of the series \( \sum_{n=0}^{\infty} a_n^{(\infty)} w_n \) in \( \mathcal{K} \). By (2.28), we have
\[ u_\infty = \lim_{k \to \infty} \sum_{n=0}^{\infty} a_n^{(k)} w_n. \quad (2.38) \]
Let \( \{k_i\}_i \) be the subsequence of positive integers chosen as above for obtaining (2.31). For any \( N \in \mathbb{N} \) and any \( i \in \mathbb{N} \), by the assumption (1.2) and the fact that \( a_n^{(k_i)} \geq 0 \), we have
\[
\left\| \sum_{n=0}^{N} a_n^{(\infty)} w_n - u_{\infty} \right\| \leq \left\| \sum_{n=0}^{N} a_n^{(\infty)} w_n - \sum_{n=0}^{N} a_n^{(k_i)} w_n \right\| + \left\| \sum_{n=N+1}^{\infty} a_n^{(k_i)} w_n \right\|
\leq \sum_{n=0}^{N} |a_n^{(\infty)} - a_n^{(k_i)}| \cdot \left\| w_n \right\| + C \left\| \sum_{n=N+1}^{\infty} a_n^{(k_i)} t^n \right\|_{L^2(\mu)}
+ \left\| \sum_{n=0}^{\infty} a_n^{(k_i)} w_n - u_{\infty} \right\|.
\]
(2.39)
Combining (2.32), (2.38) and (2.39), for any \( N \in \mathbb{N} \), we have
\[
\left\| \sum_{n=0}^{N} a_n^{(\infty)} w_n - u_{\infty} \right\| \leq C \liminf_{i \to \infty} \left\| \sum_{n=N+1}^{\infty} a_n^{(k_i)} t^n \right\|_{L^2(\mu)}.
\]
Note also that (2.28) and (2.36) together imply
\[
\lim_{i \to \infty} \left\| \sum_{n=0}^{\infty} a_n^{(k_i)} t^n - \sum_{n=0}^{\infty} a_n^{(\infty)} t^n \right\|_{L^2(\mu)} = 0.
\]
Therefore, for any fixed \( N \in \mathbb{N} \), we have
\[
\liminf_{i \to \infty} \left\| \sum_{n=N+1}^{\infty} a_n^{(k_i)} t^n \right\|_{L^2(\mu)} = \left\| \sum_{n=N+1}^{\infty} a_n^{(\infty)} t^n \right\|_{L^2(\mu)}
\]
and hence
\[
\left\| \sum_{n=0}^{N} a_n^{(\infty)} w_n - u_{\infty} \right\| \leq C \left\| \sum_{n=N+1}^{\infty} a_n^{(\infty)} t^n \right\|_{L^2(\mu)}.
\]
Thus, we have
\[
\limsup_{N \to \infty} \left\| \sum_{n=0}^{N} a_n^{(\infty)} w_n - u_{\infty} \right\| \leq C \limsup_{N \to \infty} \left\| \sum_{n=N+1}^{\infty} a_n^{(\infty)} t^n \right\|_{L^2(\mu)} = 0.
\]
This completes the proof of the limit relation (2.37).

**Remark.** The implication (2.25) \( \implies \) (2.27) relies heavily on the nonnegativity of the functions \( a_n t^n \) on \([0, s_\mu)\). In general, the equality \( f = \sum_{n=0}^{\infty} f_n \) in the Hilbert space \( L^2(\mu) \) does not imply the \( \mu \)-a.e. equality \( f(t) \equiv \sum_{n=0}^{\infty} f_n(t) \).

**Proof of Corollary 1.2.** By writing \( \mu = \nu|_{\mathbb{R}^+} \), we have
\[
L^2(\mathbb{R}, \nu) = L^2(\mathbb{R}^+, \mu) \oplus L^2(\mathbb{R}^-, \nu).
\]
The assumption (1.4) implies that for any finitely supported sequence \( \{a_n\}_{n=0}^{\infty} \) of nonnegative numbers, we have
\[
\left\| \sum_{n=0}^{\infty} a_n t^n \right\|_{L^2(\mathbb{R}_-, \nu)}^2 = \sum_{n, m \geq 0} a_n a_m \int_{\mathbb{R}_-} t^{m+n} d\nu(t)
\leq \sum_{m, n \geq 0, m+n \text{ is even}} a_n a_m \int_{\mathbb{R}_+} t^{m+n} d\mu(t)
\leq C \sum_{m, n \geq 0} a_n a_m \int_{\mathbb{R}_+} t^{m+n} d\mu(t) = C \left\| \sum_{n=0}^{\infty} a_n t^n \right\|_{L^2(\mathbb{R}_+, \mu)}^2,
\]
where in the first inequality we have used the inequalities
\[
\int_{\mathbb{R}_-} t^{2k+1} d\nu(t) \leq 0 \quad \text{for all } k \in \mathbb{N}.
\]
Now Corollary 1.2 follows immediately from Theorem 1.1.

3. Characterisation of the metric projection

3.1. Proof of Proposition 1.3

Recall that \( \mathcal{C}[\mathcal{V}] \) is assumed to be closed. For any \( w \in \mathcal{H} \), by a classical result on the metric projections onto a closed convex set (cf. [16, Lemma 1.1]), \( P_{\mathcal{C}[\mathcal{V}]}(w) \) is uniquely determined by
\[
\begin{cases}
P_{\mathcal{C}[\mathcal{V}]}(w) \in \mathcal{C}[\mathcal{V}], \\
\langle w - P_{\mathcal{C}[\mathcal{V}]}(w), u - P_{\mathcal{C}[\mathcal{V}]}(w) \rangle \leq 0, \quad \text{for all } u \in \mathcal{C}[\mathcal{V}].
\end{cases}
\tag{3.40}
\]

By the assumption that \( \mathcal{C}[\mathcal{V}] \) is a closed convex cone, we may use [9, Lemma 3] to obtain
\[
\langle w - P_{\mathcal{C}[\mathcal{V}]}(w), P_{\mathcal{C}[\mathcal{V}]}(w) \rangle = 0.
\]

This combined with (3.40) implies that \( P_{\mathcal{C}[\mathcal{V}]}(w) \) is uniquely determined by
\[
\begin{cases}
P_{\mathcal{C}[\mathcal{V}]}(w) \in \mathcal{C}[\mathcal{V}], \\
\langle w - P_{\mathcal{C}[\mathcal{V}]}(w), P_{\mathcal{C}[\mathcal{V}]}(w) \rangle = 0, \\
\langle w - P_{\mathcal{C}[\mathcal{V}]}(w), u \rangle \leq 0, \quad \text{for all } u \in \mathcal{C}[\mathcal{V}].
\end{cases}
\tag{3.41}
\]

By the definition of \( \mathcal{C}[\mathcal{V}] \), the condition
\[
\langle w - P_{\mathcal{C}[\mathcal{V}]}(w), u \rangle \leq 0 \quad \text{for any } u \in \mathcal{C}[\mathcal{V}]
\]
is satisfied if and only if
\[
\langle w - P_{\mathcal{C}[\mathcal{V}]}(w), v_n \rangle \leq 0 \quad \text{for all } n \in \mathbb{N}.
\tag{3.42}
By writing
\[ P_C(\mathcal{V})(w) = \sum_{n=0}^{\infty} a_n v_n \quad \text{with } a_n \geq 0 \quad \text{for all } n \in \mathbb{N}, \]
we have
\[ 0 = \langle w - P_C(\mathcal{V})(w), P_C(\mathcal{V})(w) \rangle = \sum_{n=0}^{\infty} a_n \langle w - P_C(\mathcal{V})(w), v_n \rangle. \tag{3.43} \]
Combining (3.42) and (3.43), we obtain
\[ \langle w - P_C(\mathcal{V})(w), v_n \rangle = 0 \quad \text{for all those } n \in \mathbb{N} \text{ such that } a_n > 0. \tag{3.44} \]
On the other hand, (3.44) clearly implies the equality (3.43). Therefore, the condition (3.41) is equivalent to
\[
\begin{align*}
P_C(\mathcal{V})(w) &= \sum_{n=0}^{\infty} a_n v_n \in \mathcal{C}[\mathcal{V}], \\
\sum_{k \in \mathbb{N}} a_k \langle v_k, v_n \rangle &= \langle w, v_n \rangle, \quad \text{for all } n \in \mathbb{N} \text{ such that } a_n > 0, \\
\sum_{k \in \mathbb{N}} a_k \langle v_k, v_n \rangle &\geq \langle w, v_n \rangle, \quad \text{for all } n \in \mathbb{N}.
\end{align*}
\]
This completes the proof of Proposition 1.3.

3.2. Proof of Corollary 1.5
If \( \mathcal{V} \subset \mathcal{H} \) is a finite set, then the convex cone \( \mathcal{C}[\mathcal{V}] \) generated by \( \mathcal{V} \) is always closed; cf., for example, [16, p. 236] and [3, p. 25].

**Lemma 3.1.** Let \( X_1, \ldots, X_n \) be linear independent real-valued random variables, all of which are of finite second moment. Then for any \( c = (c_1, \ldots, c_n) \in \mathbb{R}^n \), there exists a real-valued random variable \( Y \) of finite second moment such that
\[ c_i = E[YX_i], \quad 1 \leq i \leq n. \tag{3.45} \]
**Proof.** Set
\[ A = (E(X_iX_j))_{1 \leq i, j \leq n} = E \left[ \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} (X_1 \cdots X_n) \right]. \]
Because \( X_1, \ldots, X_n \) are linear independent, by Schmidt’s orthogonalisation method, there exists a nonsingular matrix \( P \) such that the random variables \( Z_i \)'s defined by
\[ \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} = P \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \]
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are orthogonal and $E(Z^2_i) > 0$ for all $i \in \{1, \cdots, n\}$. Note that the condition (3.45) can be written as

$$
\begin{pmatrix}
c_1 \\
\vdots \\
c_n
\end{pmatrix}
= E\left[
\begin{pmatrix}
X_1 \\
\vdots \\
X_n
\end{pmatrix}
Y\right].
$$

(3.46)

Because $P$ is nonsingular, (3.46) is equivalent to

$$
\begin{pmatrix}
c_1' \\
\vdots \\
c_n'
\end{pmatrix}
= P\begin{pmatrix}
c_1 \\
\vdots \\
c_n
\end{pmatrix}
= PE\left[
\begin{pmatrix}
X_1 \\
\vdots \\
X_n
\end{pmatrix}
Y\right].
$$

(3.47)

If we set

$$
Y = \sum_{i=1}^{n} \frac{c'_i Z_i}{E(Z^2_i)},
$$

then $Y$ satisfies the condition (3.47). This completes the whole proof.

\textbf{Proof of Corollary 1.5.} Because $A$ is a nonsingular positive definite matrix, there exist real-valued square-integrable and linear independent random variables $X_1, \cdots, X_n$ such that

$$
E(X_i) = 0 \; \text{and} \; A = (E(X_i X_j))_{1 \leq i, j \leq n}.
$$

By Lemma 3.1, there exists a real-valued square-integrable random variable $Y$ such that

$$
c_i = E(X_i Y), \; \; 1 \leq i \leq n.
$$

The convex cone in the associated Hilbert space of square-integrable random variables generated by $X_1, \cdots, X_n$ is

$$
C(X_1, \cdots, X_n) = \left\{ \sum_{i=1}^{n} b_i X_i \left| b_i \geq 0 \right. \right\}.
$$

Because $C(X_1, \cdots, X_n)$ is closed (cf. [16, p. 236] and [3, p. 25]), there exists a unique $Z \in C(X_1, \cdots, X_n)$ closest to $Y$. Write

$$
Z = \sum_{i=1}^{n} b_i X_i = \sum_{i \in S} b_i X_i \; \; \text{with} \; b_i > 0 \; \text{for all} \; i \in S.
$$

By Proposition 1.3, $S$ and the coefficients $(b_i)_{i \in S}$ are uniquely determined by

$$
\begin{cases}
    b_i > 0, & \text{for all} \; i \in S, \\
    \left( \sum_{i \in S} b_i X_i, X_j \right) = \left( Y, X_j \right) = c_j, & \text{for all} \; j \in S, \\
    \left( \sum_{i \in S} b_i X_i, X_j \right) \geq \left( Y, X_j \right) = c_j, & \text{for all} \; j \in \{1, 2, \cdots, n\}.
\end{cases}
$$
In other words, \( S \) and \((b_j)_{j \in S}\) are uniquely determined by

\[
\begin{align*}
&b_i > 0, \quad \text{for all } i \in S; \\
&\sum_{i \in S} a_{ij} b_i = c_j, \quad \text{for all } j \in S; \\
&\sum_{i \in S} a_{ij} b_i \geq c_j, \quad \text{for all } j \in \{1, 2, \ldots, n\}.
\end{align*}
\]

By noting \( a_{ij} = a_{ji} \), we complete the whole proof. \( \Box \)

4. Applications in function theory

4.1. Power functions and signed power functions

**Proof of Proposition 1.6.** Let \( \alpha \in [0, 2), m \in \mathbb{N} \) and \((a_m, b_m)\) be defined as (1.12). By Proposition 1.3, to prove the equality (1.11), it suffices to verify

\[
\left\langle a_m t^{2m} + b_m t^{2m+2}, t^j \right\rangle_{L^2([-1,1])} \geq \left\langle |t|^{2m+\alpha}, t^j \right\rangle_{L^2([-1,1])}
\]

for all \( j \in \mathbb{N} \) \quad (4.48)

and

\[
\left\langle a_m t^{2m} + b_m t^{2m+2}, t^j \right\rangle_{L^2([-1,1])} = \left\langle |t|^{2m+\alpha}, t^j \right\rangle_{L^2([-1,1])}
\]

for \( j \in \{2m, 2m+2\} \). \quad (4.49)

If \( j \) is an odd number, then (4.48) holds because both sides of (4.48) vanish; the same is true for (4.49). So we now focus on even numbers \( j = 2k \) with \( k \geq 0 \). Note that \((a_m, b_m)\) defined by (1.12) is in fact the solution of the linear equation

\[
\begin{pmatrix}
\frac{2}{4m+1} & \frac{2}{4m+3} \\
\frac{2}{4m+3} & \frac{2}{4m+5}
\end{pmatrix}
\begin{pmatrix}
a_m \\
b_m
\end{pmatrix} = \begin{pmatrix}
\frac{2}{4m+1+\alpha} \\
\frac{2}{4m+3+\alpha}
\end{pmatrix}.
\]

(4.50)

Because for any even number \( j = 2k \), we have

\[
\left\langle |t|^{\beta}, t^{2k} \right\rangle_{L^2([-1,1])} = \frac{2}{\beta + 2k + 1}
\]

for all \( \beta \geq 0 \)

and

\[
\left\langle t^{2\ell}, t^{2k} \right\rangle_{L^2([-1,1])} = \frac{2}{2\ell + 2k + 1}
\]

for all \( k, \ell \in \mathbb{N} \),

the equality (4.50) is equivalent to the equality (4.49). It remains to show the inequalities (4.48) for all even numbers \( j = 2k \) with \( k \geq 0 \). That is, we need to show

\[
\frac{2a_m}{2m+2k+1} + \frac{2b_m}{2m+2k+3} \geq \frac{2}{2m+2k+\alpha+1}
\]

for all \( k \in \mathbb{N} \).

Set

\[
D_k := (4m+1+\alpha)(4m+3+\alpha)(2m+2k+1)(2m+2k+3)(2m+2k+\alpha+1)
\]

\[
\times \left( \frac{2a_m}{2m+2k+1} + \frac{2b_m}{2m+2k+3} - \frac{2}{2m+2k+\alpha+1} \right).
\]
Then we only need to show that
\[ D_k \geq 0 \quad \text{for all} \quad k \in \mathbb{N}. \quad (4.51) \]

Write
\[ \tau = 4m + 3 \quad \text{and} \quad x = 4k, \quad (4.52) \]
then
\[
D_k = (2 - \alpha)(4m + 1)(4m + 3)(2m + 2k + 3)(2m + 2k + \alpha + 1) \\
+ \alpha(4m + 3)(4m + 5)(2m + 2k + 1)(2m + 2k + \alpha + 1) \\
- 2(4m + 1 + \alpha)(4m + 3 + \alpha)(2m + 2k + 1)(2m + 2k + \alpha + 1) \\
= (2 - \alpha)(\tau - 2)\tau \left( \frac{\tau + 3}{2} + \frac{x}{2} \right) \left( \frac{\tau - 3}{2} + \frac{x}{2} + \alpha + 1 \right) \\
+ \alpha \tau(\tau + 2) \left( \frac{\tau - 3}{2} + \frac{x}{2} + 1 \right) \left( \frac{\tau - 3}{2} + \frac{x}{2} + \alpha + 1 \right) \\
- 2(\tau - 2 + \alpha)(\tau + \alpha) \left( \frac{\tau - 3}{2} + \frac{x}{2} + 1 \right) \left( \frac{\tau - 3}{2} + \frac{x}{2} + 3 \right) \\
= \frac{1}{4}(Ax^2 + Bx + C),
\]
where
\[ A = 2\alpha(2 - \alpha), \quad B = 4\alpha(\alpha - 2)(\tau - 1) \quad \text{and} \quad C = 2\alpha(2 - \alpha)(\tau^2 - 2\tau - 3). \]

Therefore,
\[ D_k = \frac{1}{2}\alpha(2 - \alpha)[(x - (\tau - 1))^2 - 4]. \]

By substituting (4.52) into the above equality, we have
\[ D_k = 2\alpha(2 - \alpha)[(2k - 2m - 1)^2 - 1]. \]

By observing
\[ (2k - 2m - 1)^2 - 1 \geq 0 \quad \text{for any} \quad k, m \in \mathbb{N} \]
and using the assumption \( \alpha \in [0, 2) \), we obtain the desired inequalities (4.51). This completes the proof of the equality (1.11).

Now we proceed to the proof of the equality (1.13). We have
\[
[d(h_{2m+a}, A_+)]^2 = \int_{[-1,1]} (|t|^{2m+a} - a_m t^{2m} - b_m t^{2m+2})^2 dt \\
= 2 \int_{[0,1]} (t^{2m+a} - a_m t^{2m} - b_m t^{2m+2})^2 dt
\]
Therefore, we have

\[ \text{Proof of Corollary 1.7.} \]

Thus, we obtain

\[ \text{where} \]

By substituting \( a_m, b_m \) defined in (1.12) and by writing again \( \tau = 4m + 3 \), we obtain

\[
|d(h_{2m+\alpha}, A_+)|^2 = \frac{2}{\tau + 2\alpha - 2} + \frac{2}{\tau - 2} \left( \frac{(\tau - 2)\tau}{(\tau + \alpha - 2)(\tau + \alpha)} \right)^2
\]

\[
+ \frac{2}{\tau + 2} \left( \frac{(\tau + \alpha - 2)(\tau + \alpha)}{2} \right)^2
\]

\[
+ \frac{4}{\tau} \left( \frac{(\tau - 2)\tau}{(\tau + \alpha - 2)(\tau + \alpha)} \right)^2
\]

\[
- \frac{4}{\tau + \alpha - 2} \left( \frac{(\tau + \alpha - 2)(\tau + \alpha)}{2} \right)^2 - \frac{4}{\tau + \alpha} \left( \frac{(\tau + 2)(\tau + \alpha)}{2} \right)^2
\]

\[
= \frac{2}{\tau + 2\alpha - 2} + \frac{2\tau}{(\tau + \alpha - 2)^2(\tau + \alpha)^2} H(\alpha, \tau),
\]

where

\[
H(\alpha, \tau) = \frac{(2 - \alpha)^2}{4} \tau(\tau - 2) + \frac{\alpha^2}{4} \tau(\tau + 2) + \frac{\alpha(2 - \alpha)}{2} (\tau - 2)(\tau + 2)
\]

\[
- (2 - \alpha)(\tau - 2)(\tau + \alpha) - \alpha(\tau + 2)(\tau + \alpha - 2)
\]

\[
= -\tau^2 + (2 - 2\alpha)\tau + 2\alpha(2 - \alpha) = -\tau(2\alpha - 2) + 2\alpha(2 - \alpha).
\]

Therefore, we have

\[
|d(h_{2m+\alpha}, A_+)|^2 = \frac{2}{(\tau + 2\alpha - 2)(\tau + \alpha - 2)^2(\tau + \alpha)^2} K(\alpha, \tau)
\]

with \( K(\alpha, \tau) \) given by

\[
K(\alpha, \tau) = (\tau + \alpha - 2)^2(\tau + \alpha)^2 + \tau(\tau + 2\alpha - 2) H(\alpha, \tau)
\]

\[
= \left[ \tau(\tau + 2\alpha - 2) - \alpha(2 - \alpha) \right]^2 - \tau^2(\tau + 2\alpha - 2)^2 + 2\alpha(2 - \alpha)\tau(\tau + 2\alpha - 2)
\]

\[
= \alpha^2(2 - \alpha)^2.
\]

Thus, we obtain

\[
|d(h_{2m+\alpha}, A_+)|^2 = \frac{2\alpha^2(2 - \alpha)^2}{(\tau + 2\alpha - 2)(\tau + \alpha - 2)^2(\tau + \alpha)^2} K(\alpha, \tau)
\]

\[
= \frac{2\alpha^2(2 - \alpha)^2}{(4m + 2\alpha + 1)(4m + \alpha + 1)(4m + \alpha + 3)^2}.
\]

This completes the proof of the equality (1.13). \( \square \)

**Proof of Corollary 1.7.** Fix \( m \in \mathbb{N} \) and a positive Radon measure \( \nu \) on \([0, 2]\). Set

\[
g_{m, \nu} := \int_{[0, 2]} h_{2m+\alpha} d\nu(\alpha)
\]
and
\[ A_{m,v} := \int_{[0,2]} a_m(\alpha) \, d\nu(\alpha) \geq 0, \quad B_{m,v} := \int_{[0,2]} b_m(\alpha) \, d\nu(\alpha) \geq 0. \]

By integrating the inequalities (4.48) and the equalities (4.49) against the measure \( \nu \), we obtain
\[ \left( A_{m,v} t^{2m} + B_{m,v} t^{2m+2}, t^j \right)_{L^2([-1,1])} \geq \left( g_{m,v}, t^j \right)_{L^2([-1,1])} \quad \text{for all } j \in \mathbb{N} \]
and
\[ \left( A_{m,v} t^{2m} + B_{m,v} t^{2m+2}, t^j \right)_{L^2([-1,1])} = \left( g_{m,v}, t^j \right)_{L^2([-1,1])} \quad \text{for } j \in \{2m, 2m+2\}. \]

Thus, by Proposition 1.3, we obtain the desired equality
\[ P_{A^+}(g_{m,v}) = A_{m,v} t^{2m} + B_{m,v} t^{2m+2}. \]

**Proof of Proposition 1.8.** Let \( \alpha \in [0,2), \) \( m \in \mathbb{N} \) and \( (c_m, d_m) \) be defined as in (1.17). By Proposition 1.3, to prove the equality (1.16), it suffices to verify
\[ \left( c_m t^{2m+1} + d_m t^{2m+3}, t^j \right)_{L^2([-1,1])} \geq \left( \text{sgn}(t)|t|^{2m+1+\alpha}, t^j \right)_{L^2([-1,1])} \quad \text{for all } j \in \mathbb{N} \quad (4.53) \]
and
\[ \left( c_m t^{2m+1} + d_m t^{2m+3}, t^j \right)_{L^2([-1,1])} = \left( \text{sgn}(t)|t|^{2m+1+\alpha}, t^j \right)_{L^2([-1,1])} \quad \text{for } j \in \{2m+1, 2m+3\}. \quad (4.54) \]

If \( j \) is an even number, then (4.53) holds because both sides of (4.53) vanish; the same is true for (4.54). So we now focus on odd numbers \( j = 2k+1 \) with \( k \geq 0 \). Note that \( (c_m, d_m) \) defined by (1.17) is in fact the solution of the linear equation
\[ \begin{pmatrix} \frac{2}{4m+3} & \frac{2}{4m+5} \\ \frac{2}{4m+5} & \frac{2}{4m+7} \end{pmatrix} \begin{pmatrix} c_m \\ d_m \end{pmatrix} = \begin{pmatrix} \frac{2}{4m+\alpha+3} \\ \frac{2}{4m+\alpha+5} \end{pmatrix} . \quad (4.55) \]

Because for any odd number \( j = 2k+1 \),
\[ \left( \text{sgn}(t)|t|^\gamma, t^{2k+1} \right)_{L^2([-1,1])} = \frac{2}{\gamma+2k+2} \quad \text{for all } \gamma \geq 0 \]
and
\[ \left( t^{2\ell+1}, t^{2k+1} \right)_{L^2([-1,1])} = \frac{2}{2\ell+2k+3} \quad \text{for all } k, \ell \in \mathbb{N}, \]
the equality (4.55) is equivalent to the equality (4.54). It remains to show the inequalities (4.53) for all odd numbers \( j = 2k+1 \) with \( k \geq 0 \). That is, we need to show
\[ \frac{2c_m}{2m+2k+3} + \frac{2d_m}{2m+2k+5} \geq \frac{2}{2m+2k+\alpha+3} \quad \text{for all } k \in \mathbb{N}. \]
The metric projections onto closed convex cones in a Hilbert space

Set

\[ T_k := (2m + 2k + 3)(2m + 2k + 5)(4m + 3 + \alpha)(4m + 5 + \alpha)(2m + 2k + 3 + \alpha) \times \left( \frac{2c_m}{2m + 2k + 3} + \frac{2d_m}{2m + 2k + 5} - \frac{2}{2m + 2k + 3 + \alpha} \right). \]

Then we only need to show

\[ T_k \geq 0 \quad \text{for all } k \in \mathbb{N}. \tag{4.56} \]

By using exactly the same arguments as those in dealing with \( D_k \) in the proof of Proposition 1.6 (a simpler way is to replace everywhere the pair \((m, k)\) in the definition of \( D_k \) by the pair \((m + \frac{1}{2}, k + \frac{1}{2})\) to obtain a reduced form of \( T_k \), we obtain

\[ T_k = 2\alpha(2 - \alpha)[(2k - 2m - 1)^2 - 1] \quad \text{for all } k \in \mathbb{N}. \]

By observing \((2k - 2m - 1)^2 - 1 \geq 0\) for any \( k, m \in \mathbb{N} \) and using the assumption \( \alpha \in [0, 2) \), we obtain the desired inequalities (4.56). This completes the proof of the equality (1.16).

Verification of the equality (1.18) is similar to that of the equality (1.13).

4.2. Indicator functions

For any \( n \in \mathbb{N} \), define an \((n+1) \times (n+1)\)-matrix by

\[ M_n := \left( \langle t^i, t^j \rangle_{L^2([-1,1])} \right)_{0 \leq i,j \leq n}. \]

By the linear independence of the functions \(1, t, t^2, \ldots\) on \([-1,1]\), for any \( n \in \mathbb{N} \), the matrix \( M_n \) is nonsingular.

Note that for any \( n \in \mathbb{N} \) and any \( a \in [-1,1) \), we have

\[ \langle \psi_a, t^n \rangle_{L^2([-1,1])} = \int_{-1}^{1} t^n dt = \frac{1 - a^{n+1}}{n+1}. \]

Let \( v^{(n)}_a \in \mathbb{R}^{n+1} \) be the column vector defined by

\[ v^{(n)}_a = (\langle \psi_a, 1 \rangle_{L^2([-1,1])}, \langle \psi_a, t \rangle_{L^2([-1,1])}, \ldots, \langle \psi_a, t^n \rangle_{L^2([-1,1])})^\top \]

\[ = \left( 1 - a, \frac{1 - a^2}{2}, \ldots, \frac{1 - a^{n+1}}{n+1} \right)^\top. \]

Denote by \( \mathbb{R}_+^* = (0, \infty) \) the set of all positive numbers. Lemmas 4.1 and 4.2 will be used in the proof of Propositions 1.9 and 1.10.

**Lemma 4.1.** Assume that \( a \in [-1,1) \). Then the linear equation

\[ M_2x = v^{(2)}_a \tag{4.57} \]

has a solution in \((\mathbb{R}_+^*)^3\) if and only if

\[ 0 < a < \sqrt{105} - 5 \quad \frac{10}{10}. \]
Proof. The solution \( x = (x_0, x_1, x_2) \) of the linear equation (4.57) is given by
\[
\begin{align*}
x_0 &= \frac{1}{5}(4 - 9a + 5a^3) = \frac{1}{5}(1 - a)(4 - 5a - 5a^2) \\
x_1 &= \frac{2}{5}(1 - a^2) \\
x_2 &= \frac{15}{5} (a - a^3) = \frac{15}{5}a(1 - a^2)
\end{align*}
\] (4.58)

A simple computation shows that, under the assumption \( a \in [-1,1] \), the solution \( x \) belongs to \( (\mathbb{R}_+^*)^3 \) if and only if \( 0 < a < \frac{\sqrt{105} - 5}{10} \). This completes the proof of the lemma.

Lemma 4.2. Assume that \( \rho \in [0,1) \). Then the condition
\[
(1 - 3\rho + 2\rho^{n+1})n + 3\rho^{n+1} \geq 3\rho \quad \text{for all integers } n \geq 1
\] (4.59)
holds if and only if \( \rho \in [0, \frac{1}{5}] \).

Proof. Assume that (4.59) holds. Then by taking \( n = 1 \), we have
\[
1 - 3\rho + 2\rho^2 + 3\rho^2 \geq 3\rho.
\]
This combined with the assumption \( \rho \in [0,1) \) implies that \( \rho \in [0,1/5] \).

Conversely, assume that \( \rho \in [0,1/5] \). Then for \( n = 1 \) we have
\[
1 - 3\rho + 2\rho^2 + 3\rho^2 - 3\rho = (1 - \rho)(1 - 5\rho) \geq 0.
\]
This implies the inequality (4.59) for \( n = 1 \). Now assume that \( n \geq 2 \), and we have
\[
(1 - 3\rho + 2\rho^{n+1})n + 3\rho^{n+1} - 3\rho \geq (1 - \frac{3}{5})n - \frac{3}{5} = \frac{2n - 3}{5} \geq \frac{1}{5} \geq 0.
\]
This implies the inequality (4.59) for all integers \( n \geq 2 \).

Proof of Proposition 1.9. Let \( x_0, x_1, x_2 \) be given as in (4.58) and recall that \( x \) is the solution to the linear equation (4.57). By the discussion in Subsection 1.3 and Proposition 1.3, the equality (1.19) holds if and only if the following conditions are all satisfied:

- \( x \in (\mathbb{R}_+^*)^3 \);
- for all \( j \in \{0,1,2\} \),
\[
\left\langle \sum_{i=0}^{2} x_i t^i, t^j \right\rangle_{L^2([-1,1])} = \langle \psi_a, t^j \rangle_{L^2([-1,1])} = \frac{1 - a^{j+1}}{j+1}; \quad (4.60)
\]
- for all integers \( j \geq 3 \),
\[
\left\langle \sum_{i=0}^{2} x_i t^i, t^j \right\rangle_{L^2([-1,1])} \geq \langle \psi_a, t^j \rangle_{L^2([-1,1])} = \frac{1 - a^{j+1}}{j+1}. \quad (4.61)
\]

Note that the system of linear equations (4.60) for \( j \in \{0,1,2\} \) is equivalent to the linear equation (4.57). Thus, by the definition of \( x \), the equalities (4.60) hold for all \( j \in \{0,1,2\} \). By Lemma 4.1, \( x \in (\mathbb{R}_+^*)^3 \) if and only if \( 0 < a < \frac{\sqrt{105} - 5}{10} \). Now let us analyse (4.61).
If \( j = 2n + 1 \) with \( n \geq 1 \), then
\[
\sum_{i=0}^{2} x_i t^i, t^{2n+1} \right|_{L^2([-1,1])} - \langle \psi_a, t^{2n+1} \rangle_{L^2([-1,1])} = \frac{2}{2n+3} \cdot \frac{3}{4} (1 - a^2) - \frac{1 - a^{2n+2}}{2n+2} \\
= \frac{(1 - 3a^2 + 2a^{2n+2}) n + 3a^{2n+2} - 3a^2}{2(2n + 3)(n + 1)},
\]
and if \( j = 2n + 2 \) with \( n \geq 1 \), then
\[
\sum_{i=0}^{2} x_i t^i, t^{2n+2} \right|_{L^2([-1,1])} - \langle \psi_a, t^{2n+2} \rangle_{L^2([-1,1])} = \frac{2}{2n+3} \cdot \frac{1}{8} (4 - 9a + 5a^3) + \frac{2}{2n+5} \cdot \frac{15}{8} (a - a^3) - \frac{1 - a^{2n+3}}{2n+3} \\
= \frac{a \left[ (3 - 5a^2 + 2a^{2n+2}) n - 5a^2 + 5a^{2n+2} \right]}{(2n + 3)(2n + 5)}.
\]
Therefore, the equality (1.19) holds if and only if the following conditions are all satisfied:
- \( 0 < a < \frac{\sqrt{105} - 5}{10} \);
- for all integers \( n \geq 1 \),
  \[
  (1 - 3a^2 + 2a^{2n+2}) n + 3a^{2n+2} - 3a^2 \geq 0;
  \]
- for all integers \( n \geq 1 \),
  \[
  a \left[ (3 - 5a^2 + 2a^{2n+2}) n - 5a^2 + 5a^{2n+2} \right] \geq 0.
  \]

Observe that
\[
(3 - 5a^2 + 2a^{2n+2}) n + 5a^{2n+2} - 5a^2 = 3 \left[ (1 - 3a^2 + 2a^{2n+2}) n + 3a^{2n+2} - 3a^2 \right] \\
+ 4a^2 (n + 1) (1 - a^{2n}) \\
\geq 3 \left[ (1 - 3a^2 + 2a^{2n+2}) n + 3a^{2n+2} - 3a^2 \right] .
\]

Thus, by Lemma 4.2 and the inequality \( \frac{1}{\sqrt{5}} < \frac{\sqrt{105} - 5}{10} \), the equality (1.19) holds if and only if \( 0 < a \leq \frac{1}{\sqrt{5}} \).

Finally, by Proposition 1.3, the above arguments imply that for \( 0 < a \leq \frac{1}{\sqrt{5}} \), we have
\[
P_{A_a}(\psi_a) = \sum_{i=0}^{2} x_i t^i.
\]
This is the desired equality (1.20).

**Proof of Corollary 1.12.** The proof of Corollary 1.12 is similar to that of Corollary 1.7. \( \square \)
Proof of Proposition 1.10. Set
\[ y_0 := \frac{1 - a}{2}, \quad y_1 := \frac{3}{4}(1 - a^2). \]
Then the assumption \( a \in [-1, 1) \) implies that \( y = (y_0, y_1) \in (\mathbb{R}_+^*)^2 \). Therefore, by Proposition 1.3, the equality (1.22) holds if and only if
\[
\left( \sum_{i=0}^{1} y_i t^i, t^j \right)_{L^2([-1, 1])} = \left( \psi_a, t^j \right)_{L^2([-1, 1])} = \frac{1 - a^{j+1}}{j+1} \quad \text{for } j \in \{0, 1\} \quad (4.62)
\]
and
\[
\left( \sum_{i=0}^{1} y_i t^i, t^j \right)_{L^2([-1, 1])} \geq \left( \psi_a, t^j \right)_{L^2([-1, 1])} = \frac{1 - a^{j+1}}{j+1} \quad \text{for } j \geq 2. \quad (4.63)
\]
The equalities (4.62) can be checked directly by using the definitions of \( y_0 \) and \( y_1 \). Now let us analyse the inequalities (4.63). Note that if \( j = 2n \) for \( n \geq 1 \), then
\[
\left( \sum_{i=0}^{1} y_i t^i, t^{2n} \right)_{L^2([-1, 1])} - \left( \psi_a, t^{2n} \right)_{L^2([-1, 1])} = \frac{2}{2n+1} \left( 1 - \frac{1 - a^{2n+1}}{2n+1} \right) = \frac{a^{2n+1} - a}{2n+1},
\]
and if \( j = 2n + 1 \) with \( n \geq 1 \), then
\[
\left( \sum_{i=0}^{1} y_i t^i, t^{2n+1} \right)_{L^2([-1, 1])} - \left( \psi_a, t^{2n+1} \right)_{L^2([-1, 1])} = \frac{2}{2n+3} \cdot \frac{3}{4} (1 - a^2) - \frac{1 - a^{2n+2}}{2n+2}
\]
\[
= \frac{(1 - 3a^2 + 2a^{2n+2})n + 3a^{2n+2} - 3a^2}{2(2n+3)(n+1)}.
\]
Therefore, (4.63) holds if and only if
\[
\begin{cases}
  a^{2n+1} - a \geq 0 \\
  (1 - 3a^2 + 2a^{2n+2})n + 3a^{2n+2} \geq 3a^2
\end{cases}
\quad \text{for all integers } n \geq 1. \quad (4.64)
\]
By Lemma 4.2, under the assumption \( a \in [-1, 1) \), the condition (4.64) holds if and only if \( -\frac{1}{\sqrt{5}} \leq a \leq 0 \). This completes the proof of the proposition.

Lemmas 4.3 and 4.4 will be used in the proof of Proposition 1.11.

Lemma 4.3. Assume that \( a \in [-1, 1) \). Then the solution to the linear equation
\[
M_3 z = v^{(3)}_a \quad (4.65)
\]
belongs to \((\mathbb{R}_+^*)^4\) if and only if
\[
\frac{1}{\sqrt{5}} < a < \frac{\sqrt{105} - 5}{10}.
\]
Proof. The solution \( z = (z_0, z_1, z_2, z_3) \) of the linear equation (4.65) is given by
\[
\begin{aligned}
z_0 &= \frac{1}{9}(1 - a)(4 - 5a - 5a^2) \\
z_1 &= \frac{15}{32}(1 - a^2)(3 - 7a^2) \\
z_2 &= \frac{15}{8}a(1 - a^2) \\
z_3 &= \frac{35}{32}(1 - a^2)(5a^2 - 1)
\end{aligned}
\tag{4.66}
\]
Therefore, under the assumption \( a \in [-1, 1) \), the solution \( z \) belongs to \( \mathbb{R}_+^4 \) if and only if \( \frac{1}{\sqrt{3}} < a < \frac{\sqrt{105} - 5}{10} \). This completes the proof of the lemma. \( \square \)

Lemma 4.4. Suppose that \( \frac{1}{\sqrt{3}} < a < \frac{\sqrt{105} - 5}{10} \). Then for all integers \( n \geq 2 \), we have
\[
(3a - 5a^3 + 2a^{2n+1})n + 3a^{2n+1} - 3a \geq 0
\]
and
\[
(-3 + 30a^2 - 35a^4 + 8a^{2n+2})n^2 + (3 - 35a^4 + 32a^{2n+2})n + 30a^{2n+2} - 30a^2 \geq 0.
\]
Proof. Assume that \( \frac{1}{\sqrt{3}} < a < \frac{\sqrt{105} - 5}{10} \). For the first inequality, note that, combined with the elementary inequality
\[
\left( \frac{\sqrt{105} - 5}{10} \right)^2 < \frac{3}{10},
\]
the assumption \( \frac{1}{\sqrt{3}} < a < \frac{\sqrt{105} - 5}{10} \) implies
\[
3a - 5a^3 = a(3 - 5a^2) > 0 \text{ and } 3a - 10a^3 = a(3 - 10a^2) > 0.
\]
Thus, for any \( n \geq 2 \), we have
\[
(3a - 5a^3 + 2a^{2n+1})n + 3a^{2n+1} - 3a \geq (3a - 5a^3)n - 3a \\
\geq 2(3a - 5a^3) - 3a = 3a - 10a^3 > 0.
\]
For the second inequality, note that, combined with the elementary inequality
\[
\left( \frac{\sqrt{105} - 5}{10} \right)^4 < \frac{3}{35},
\]
the assumption \( \frac{1}{\sqrt{3}} < a < \frac{\sqrt{105} - 5}{10} \) implies
\[
3 - 35a^4 > 0
\]
and
\[
-3 + 30a^2 - 35a^4 > -3 + 30\left( \frac{1}{\sqrt{5}} \right)^2 - 35a^4 = 3 - 35a^4 > 0.
\]
Note also that
\[
-1 + 15x^2 - 35x^4 \geq 0 \quad \text{provided that } \frac{15 - \sqrt{85}}{70} \leq x^2 \leq \frac{15 + \sqrt{85}}{70}.
\]
Therefore, by noting
\[
\frac{15 - \sqrt{85}}{70} \leq \frac{1}{5} \leq a^2 \leq \left( \frac{\sqrt{105} - 5}{10} \right)^2 \leq \frac{15 + \sqrt{85}}{70},
\]
we have
\[-1 + 15 a^2 - 35 a^4 \geq 0.\]

It follows that for any integer \( n \geq 2 \), we have
\[
\begin{align*}
&(-3 + 30 a^2 - 35 a^4 + 8a^{2n+2})n^2 + (3 - 35 a^4 + 32a^{2n+2})n + 30a^{2n+2} - 30a^2 \\
\geq& (-3 + 30 a^2 - 35 a^4)n^2 + (3 - 35 a^4)n - 30a^2 \\
\geq& (-3 + 30 a^2 - 35 a^4) \times 4 + (3 - 35 a^4) \times 2 - 30a^2 \\
= &6(-1 + 15 a^2 - 35 a^4) \geq 0.
\end{align*}
\]

The lemma is proved completely. \( \square \)

**Proof of Proposition 1.11.** Let \( z = (z_0, z_1, z_2, z_3) \) be given as in (4.66) and recall that \( z \) is the solution to the linear equation (4.65). By the discussion in Subsection 1.3 and Proposition 1.3, the equality (1.23) holds if and only if the following conditions are all satisfied:

- \( z \in (\mathbb{R}^*_+)^4; \)
- for \( j \in \{0, 1, 2, 3\} \), we have
  \[
  \left\langle \sum_{i=0}^{3} z_i t^i, t^j \right\rangle_{L^2([-1, 1])} = \left\langle \psi_a, t^j \right\rangle_{L^2([-1, 1])} = \frac{1 - a^{j+1}}{j+1}; \tag{4.67}
  \]
- for all integers \( j \geq 4 \), we have
  \[
  \left\langle \sum_{i=0}^{3} z_i t^i, t^j \right\rangle_{L^2([-1, 1])} \geq \left\langle \psi_a, t^j \right\rangle_{L^2([-1, 1])} = \frac{1 - a^{j+1}}{j+1}. \tag{4.68}
  \]

Note that the system of linear equations (4.67) for \( j \in \{0, 1, 2, 3\} \) is equivalent to the linear equation (4.65). Thus, by the definition of \( z \), the equalities (4.67) hold for all \( j \in \{0, 1, 2, 3\} \). By Lemma 4.3, \( z \in (\mathbb{R}^*_+)^4 \) if and only if \( \frac{1}{\sqrt{5}} < a < \frac{\sqrt{105} - 5}{10} \). Now let us analyse the inequalities (4.68) for \( j \geq 4 \). For even numbers \( j = 2n \) with \( n \geq 2 \), we have
\[
\begin{align*}
&\left\langle \sum_{i=0}^{3} z_i t^i, t^{2n} \right\rangle_{L^2([-1, 1])} - \left\langle \psi_a, t^{2n} \right\rangle_{L^2([-1, 1])} \\
= &\frac{2}{2n+1} \cdot \frac{1}{8} (1-a)(4-5a-5a^2) + \frac{2}{2n+3} \cdot \frac{15}{8} a(1-a^2) - \frac{1-a^{2n+1}}{2n+1} \\
= &\frac{(3a-5a^3+2a^{2n+1})n+3a^{2n+1}-3a}{(2n+1)(2n+3)}
\end{align*}
\]
and for odd numbers \( j = 2n + 1 \) with \( n \geq 2 \), we have

\[
\left\langle \sum_{i=0}^{3} z_i t^i, t^{2n+1} \right\rangle_{L^2([-1,1])} - \langle \psi_a, t^{2n+1} \rangle_{L^2([-1,1])}
= \frac{2}{2n+3} \cdot \frac{15}{32} (1-a^2)(3-7a^2) + \frac{2}{2n+5} \cdot \frac{35}{32} (1-a^2)(5a^2-1) - \frac{1-a^{2n+2}}{2n+2} 
= \frac{(-3 + 30a^2 - 35a^4 + 8a^{2n+2})n^2 + (3 - 35a^4 + 32a^{2n+2})n + 30a^{2n+2} - 30a^2}{2(2n+3)(2n+5)(2n+2)}.
\]

Therefore, the equality (1.23) holds if and only if the following conditions are all satisfied:

- \( \frac{1}{\sqrt{5}} < a < \frac{\sqrt{105}-5}{10} \);
- for all integers \( n \geq 2 \),
  \[ (3a - 5a^3 + 2a^{2n+1})n + 3a^{2n+1} - 3a \geq 0; \]
- for all integers \( n \geq 2 \),
  \[ (-3 + 30a^2 - 35a^4 + 8a^{2n+2})n^2 + (3 - 35a^4 + 32a^{2n+2})n + 30a^{2n+2} - 30a^2 \geq 0. \]

Thus, by Lemma 4.4, the equality (1.23) holds if and only if \( \frac{1}{\sqrt{5}} < a < \frac{\sqrt{105}-5}{10} \).

Finally, by Proposition 1.3, the above arguments imply that if \( \frac{1}{\sqrt{5}} < a < \frac{\sqrt{105}-5}{10} \), then

\[ P_{A_+}(\psi_a) = \sum_{i=0}^{3} z_i t^i. \]

This is the desired equality (1.24).

\[ \square \]

5. Appendix

In this appendix, we show that the set

\[ C_{L^2([-1,0])} \left[ \{t^n\}_{n=0}^\infty \right] = \left\{ \sum_{n=0}^\infty a_n t^n \middle| a_n \geq 0, \text{ the series } \sum_{n=0}^\infty a_n t^n \text{ converges in } L^2([-1,0]) \right\} \]

is not closed. Or, equivalently, we show that the set

\[ C_{L^2([0,1])} \left[ \{(-t)^n\}_{n=0}^\infty \right] = \left\{ \sum_{n=0}^\infty a_n (-t)^n \middle| a_n \geq 0, \text{ the series } \sum_{n=0}^\infty a_n (-t)^n \text{ converges in } L^2([0,1]) \right\} \]

is not closed. Indeed, set \( \rho_k = 1 - \frac{1}{k+1} \) for any \( k \in \mathbb{N} \). Let

\[ g_k(t) := \frac{1}{(1 + \rho_k t)^2} = \sum_{n=0}^\infty (n + 1) \rho_k^n (-t)^n, \quad t \in [0,1). \]
Then, clearly, we have \( g_k \in C_{L^2([0,1])}[\{(−t)^{n}\}_{n=0}^{\infty}] \) and

\[
g_k(t) \xrightarrow{k \to \infty} g_\infty(t) = \frac{1}{(1+t)^2}.
\]

Now let us show that \( g_\infty \notin C_{L^2([0,1])}[\{(−t)^{n}\}_{n=0}^{\infty}] \). Otherwise, \( g_\infty \in C_{L^2([0,1])}[\{(−t)^{n}\}_{n=0}^{\infty}] \), then there exists a sequence \( (a_n)_{n=0}^{\infty} \) of nonnegative numbers such that

\[
g_\infty = \sum_{n=0}^{\infty} a_n(-t)^n,
\]

where the equality is understood as

\[
\lim_{N \to \infty} \left\| \sum_{n=0}^{N} a_n(-t)^n - \frac{1}{(1+t)^2} \right\|_{L^2([0,1])} = 0. \tag{5.69}
\]

The above convergence implies

\[
\frac{a_n^2}{2n+1} = \| a_n(-t)^n \|^2_{L^2([0,1])} \leq \sup_n \| a_n(-t)^n \|^2_{L^2([0,1])} = M < \infty. \tag{5.70}
\]

Because the \( L^2 \)-norm convergence implies the a.e. convergence along a subsequence, (5.69) implies that, along a subsequence \( (N_k)_{k=0}^{\infty} \) of positive numbers, we have

\[
\lim_{k \to \infty} \sum_{n=0}^{N_k} a_n(-t)^n = \frac{1}{(1+t)^2} \quad \text{for Lebesgue a.e. } t \in [0,1].
\]

Note that (5.70) implies that the series \( \sum_n a_n z^n \) has a radius of convergence not smaller than 1; hence, we have

\[
\lim_{k \to \infty} \sum_{n=0}^{N_k} a_n(-t)^n = \sum_{n=0}^{\infty} a_n(-t)^n \quad \text{for all } t \in [0,1).
\]

Therefore, we obtain

\[
\sum_{n=0}^{\infty} a_n(-t)^n = \frac{1}{(1+t)^2} \quad \text{for Lebesgue a.e. } t \in [0,1].
\]

By elementary results on analytic functions, the above equality implies that \( a_n = n+1 \) for all \( n \in \mathbb{N} \). Thus, the limit relation (5.69) now reads

\[
\lim_{N \to \infty} \left\| \sum_{n=0}^{N} (n+1)(-t)^n - \frac{1}{(1+t)^2} \right\|_{L^2([0,1])} = 0.
\]

However, for any large integer \( N \) and \( t \in (0,1) \), we have

\[
\sum_{n=0}^{N} (n+1)(-t)^n - \frac{1}{(1+t)^2} = \sum_{n=N+1}^{\infty} (n+1)(-t)^n = \frac{(N+2)(-t)^{N+1}}{1+t} + \frac{(-t)^{N+2}}{(1+t)^2}.
\]
Because
\[ \left\| \frac{(N + 2)(-t)^{N+1}}{1 + t} \right\|_{L^2([0,1])} \leq \frac{N + 2}{2} \left\| (-t)^{N+1} \right\|_{L^2([0,1])} = \frac{N + 2}{2\sqrt{2N + 3}} \]
and
\[ \left\| \frac{(-t)^{N+2}}{(1 + t)^2} \right\|_{L^2([0,1])} \leq 1, \]
we have
\[ \lim_{N \to \infty} \left( \sum_{n=0}^{N} (n + 1)(-t)^n \right) \leq \frac{1}{(1 + t)^2} \left\| L^2([0,1]) \right\| \geq \lim_{N \to \infty} \left( \frac{N + 2}{2\sqrt{2N + 3}} - 1 \right) = \infty. \]
Thus, we obtain a contradiction. Hence, \( C_{L^2([0,1])} \langle ((-t)^n)_{n=0}^\infty \rangle \) is not closed in \( L^2([0,1]) \).

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