Pitfalls of the theory of Bose-Einstein correlations in multiple particle production processes.

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Abstract

Some basic difficulties and ambiguities on the way from the measured momentum distributions to the inferred properties of the interaction regions are reviewed and discussed.

1 Introduction

The Bose-Einstein correlations in multiple particle production processes are studied mainly in order to get information about the interaction regions, i.e. about the regions where the final state hadrons are created, and about the hadron’s last scattering surfaces which by definition are the boundaries in space-time of the interaction regions. There is a number of things one would like to know about an interaction region: its size and shape, its orientation with respect to the event axis and in the case of the heavy ion collisions also with respect to the impact parameter vector; the flows of matter, if any, within the region; the nature of the matter filling the interaction region, its equation of state and its phase transitions, if any. This list could be made longer. One could try to obtain this information either from first principles, or using phenomenological models, or in some model independent way from the data. The approach from first principles is for the moment too difficult. In the present paper we show some of the difficulties in deriving information about the interaction region directly from the data without
model assumptions. The conclusion is that little reliable information can be
derived without a trustworthy model. The data in practice means momentum
distributions: single particle, two-particle, three-particle etc. There is
some additional information, e.g. every experiment gives some information
about the positions of the interaction vertices, but when we are interested
in distances of the order of tens of fermis or less this additional information
does not contribute significantly. The \( k \)-particle momentum distribution is
given by the diagonal elements of the \( k \)-particle density matrix in the mo-
momentum representation. Most models assume that this can be expressed as
a symmetrized product of single particle density matrices \( [1] \):

\[
\rho(p_1, \ldots, p_k; p_1, \ldots, p_k) = \sum_P \prod_{j=1}^k \rho_1(p_j; p_{P_j}),
\]

where the summation is over all the permutations of the momenta \( p_1, \ldots, p_k \).
Thus, e.g. the single particle momentum distribution is \( \rho_1(p; p) \) and for the
two-particle distribution we get the well-known formula

\[
\rho(p_1, p_2; p_1, p_2) = \rho_1(p_1; p_1)\rho_1(p_2; p_2) + |\rho_1(p_1; p_2)|^2.
\]

2 Emission function

A popular strategy when studying the Bose-Einstein correlations is to evalu-
ate the emission function \( S(K, X) \) (cf. \([2]\) and references given there) related
to the single particle density matrix by the formula

\[
\rho_1(p_1; p_2) = \int dX S(K, X)e^{iqX}.
\]

Here \( K, q \) and \( X \) are four-vectors defined by

\[
K = \frac{1}{2}(p_1 + p_2); \quad q = p_1 - p_2; \quad X = \frac{1}{2}(x_1 + x_2).
\]

Note that \( Kq = 0 \). It is usual to interpret \( S(X, K) \), for given \( K_0 \) and \( x_0 = t \),
in the spirit of the interpretation of the Wigner function, as the phase space
distribution of particles. Then \( S(K, X) \) gives directly all the information
about the geometry and the evolution in time of the interaction region.
Moreover, it gives the momentum distribution of the produced particles and
information about the momentum-position correlations. The emission func-
tion can be calculated when the production amplitudes are known \([3], [4]\).
Let us consider now the prospects for calculating the emission function from
the data using formula \([3]\).
On both sides of equation (3) the four-vector $K$ has the same, fixed value. The right-hand-side is a Fourier transform of $S$ changing the variables $X^\mu$ into the variables $q^\mu$. This Fourier transformation, however, cannot be inverted, because the left hand side is known only for the four-vectors $q$ satisfying the condition $Kq = 0$, i.e. for only one value of $q_0$ at each $q$. This implies that for any given density matrix $\rho_1$ there is an infinity of different emission functions satisfying relation (3). They correspond to the infinite variety of possible continuations of function $\rho(p_1; p_2)$ to unphysical values of $q_0$. The data without additional information from theory, or from a model, are unable to tell us which of these emission functions is the good one. In order to see how this affects the conclusions about the interaction region let us consider the following example.

We replace equation (3) by the equation

$$\rho_1(p_1; p_2) = \int dX' S(K, X') e^{iq(X' - cK)},$$

where $c$ is an arbitrary real constant. This is, of course, legitimate, because besides renaming the integration variable $X'$ we have just added to the exponent $icKq = 0$. Changing the integration variable to $X = X' - cK$ we get

$$\rho_1(p_1; p_2) = \int dX S(K, X + cK) e^{iqX}.$$  

Thus $S(K, X + cK)$ is another solution of equation (3) for the same density matrix $\rho_1$. The space-time distribution of the sources (more precisely of the particles produced by the sources) is obtained by integrating the emission function over $K$. Suppose now that for $c = 0$ there are no $X-K$ correlations, i.e. for every $K$ the space time region occupied by the interaction region is the same. Then the $K$ integration does not affect the space time distribution — one can say that the interaction regions for all the $K$s are piled on top of each other. When $c$ becomes different from zero the interaction regions corresponding to the different $K$ values get different shifts. As a result the size of the overall space time region occupied by the interaction region increases. With sufficiently large $c$ it can be made as large as one wishes. For instance, one could make it the size of a football, or living for an hour. Such extreme choices of $c$ can be eliminated using what we know about the vertex positions in space-time, but using momentum distributions only, they are just as good as $c = 0!$

The obvious question is: what information besides the momentum measurements is necessary to obtain reasonably accurate descriptions of the interaction regions? One could try to evade the problem by saying: just forget
about the emission functions and your problem will disappear. We will show in the next section that this is not a satisfactory answer.

3 Density matrix and Wigner function

Let us assume that all the hadrons have been produced in their final states (scattered for the last time) instantaneously and simultaneously at some time which we may chose as $t = 0$. Then, for any $t > 0$, all these hadrons can be described by standard, time independent (in the interaction representation) density matrices. According to assumption (1) all these density matrices can be expressed in terms of the single particle density matrix $\rho_1$. Our key observation [5] is that replacing $\rho_1$ by

$$
\rho_{1\alpha}(p_1; p_2) = e^{i\alpha(p_1)}\rho_1(p_1; p_2)e^{-i\alpha(p_2)},
$$

(7)

where $\alpha(p)$ is any real-valued function of momentum, does not change the momentum distributions. This is obvious for $k = 1$ and $k = 2$. In general, however, on the right-hand side of equation (1), in every monomial every momentum $p_i$ occurs exactly once as the first argument of $\rho_1$ and exactly once as the second argument of $\rho_1$. Thus, when $\rho_1$ gets replaced by $\rho_{1\alpha}$, it brings two factors $e^{i\alpha(p_1)}$ and $e^{-i\alpha(p_2)}$ which cancel.

As a corollary let us note that this makes the definition of the emission function, if obtained from the data using equation (3), even more ambiguous. Not only for given $\rho_1$ there is an infinity of different solutions for $S$, but moreover $\rho_1$ can be replaced by any of the infinity of functions $\rho_{1\alpha}$ without affecting the fits to the experimental data. Using the density matrix formalism instead of the emission functions, we have the second ambiguity, but not the first. Let us see how it affects the inferences concerning the interaction regions.

Let us make the simplifying assumption that

$$
\rho_1(p_1; p_2) = \frac{1}{(\sqrt{2\pi}\Delta^2)^3} \exp\left[ -\frac{K^2}{2\Delta^2} - \frac{1}{2}R^2q^2 \right].
$$

(8)

Using the general formulae for the diagonal elements of the density matrix in the coordinate representation and for the Wigner function in terms of the elements of the density matrix in the momentum representation:

$$
\tilde{\rho}_1(x; x) = \int dK dq \ e^{i\mathbf{q} \cdot \mathbf{X}} \rho_1(p_1; p_2),
$$

(9)

$$
W(K, X) = \int \frac{dq}{(2\pi)^3} e^{i\mathbf{q} \cdot \mathbf{X}} \rho_1(p_1; p_2)
$$

(10)
one gets

\[ \tilde{\rho}_1(x; x) = \frac{1}{\sqrt{2\pi R^2}} \exp \left[ -\frac{x^2}{2R^2} \right], \]  

(11)

\[ W(K, X) = \frac{1}{(2\pi R\Delta)^3} \exp \left[ -\frac{\Delta^2}{2\Delta^2} - \frac{X^2}{2R^2} \right], \]  

(12)

The diagonal elements of the density matrix in the coordinate representation yield the distribution of particles in space at \( t = 0 \) (keep in mind that we are using the interaction representation). This is a spherically symmetric Gaussian with the root mean square width in every coordinate equal \( R \). The Wigner function is usually interpreted as an approximation to the phase space distribution. In our case it confirms the result for the space distribution and, moreover, gives the information that the momentum distribution is uncorrelated to the space distribution and is also Gaussian with the root mean square width \( \Delta \).

Let us now replace the Gaussian \( \rho_1 \) by the corresponding \( \rho_{1\alpha} \) with

\[ \alpha(p) = \frac{1}{2}cp^2, \]  

(13)

where \( c \) is an arbitrary real number. A simple calculation gives

\[ \tilde{\rho}_{1\alpha}(x; x) = \frac{1}{\sqrt{2\pi(R^2 + c^2\Delta^2)}} \exp \left[ -\frac{x^2}{2(R^2 + c^2\Delta^2)} \right], \]  

(14)

\[ W_{\alpha}(K, X) = \frac{1}{(2\pi R\Delta)^3} \exp \left[ -\frac{K^2}{2\Delta^2} - \frac{(X + cK)^2}{2R^2} \right]. \]  

(15)

The first formula shows that the distribution in space remains spherically symmetric and Gaussian, but the mean square radius can be any number not less than \( R \). The second equation shows that momentum-position correlations have been introduced and that the smearing of the interaction region is due to a mechanism similar to that described in the preceding section: the interaction regions for particles with different values of the momentum \( K \) shift with respect to each other.

Note that, if we have a model which predicts unambiguously the density matrix \( \rho_1 \), there is no problem. The problem arises if one insists that the conclusions about the interaction region should be deduced from the
data without additional assumptions. An interesting question, however, is: suppose that we have two models based on completely different physical assumptions, one giving a matrix $\rho_1$ which agrees with experiment, and the other a corresponding matrix $\rho_{1\alpha}$ which, as follows from the preceding discussion, agrees with experiment just as well; how can we tell which of the two models is the good one? Experimental data on momentum distributions cannot help however precise they are. The standard advice to look at more particle distributions is also useless.

4 Distribution of distance between points where identical particles are produced

Let us consider a pair of identical particles with total momentum $2K$ and denote the distance between their production points by $x$. An important strategy is to study the distribution of the distances ($x$), i.e. $S_K(x)$ \[4], \[2], \[6]. The relation of the function $S_K$ to the data is

$$\frac{dN/dp_1 dp_2}{dN/dp_1 dN/dp_2} = \int d^3 x S_K(x') \left[ |\phi_q^*(x')|^2 - 1 \right],$$

(16)

where the primed quantities are defined in the center of mass frame of the pair and $\phi_q^*(x')$ is the two-particle wave function of the pair in its center of mass frame. When two particles are far from each other their relative momentum is well defined and equals $q'$. In particular for free particles the content of the square bracket equals $\cos(q' \cdot x')$, but one can include final state interactions by using more complicated wave functions.

The solution of equation (16) for the free particle case can be easily done by inverting the Fourier transformation. For more complicated cases one has to use more advanced methods known as imaging \[7], \[8], \[9], but there is no difficulty of principle. In this approach function $S_K$ is unambiguously given by the data. This is a nice change after the two methods described in the preceding two sections and, therefore, the method is rapidly gaining popularity.

Let us note, however, that the left hand side of equality (16), in approximation (11) is

$$\frac{dN/dp_1 dp_2}{dN/dp_1 dN/dp_2} = \frac{|\rho_1(p_1; p_2)|^2}{\rho_1(p_1; p_1) \rho_1(p_2; p_2)} + 1.$$ 

(17)

The right hand side is clearly invariant with respect to the transformation discussed in Section 3. Therefore, there is an ambiguity concerning the con-
clusions about the interaction region as discussed there. We infer that function $S_K$ can be evaluated unambiguously from the data, but its implications for the interaction region are highly ambiguous. Thus, again there is no simple way from the data to the information about the interaction region. It is plausible that each of the three approaches discussed here, and certainly many others, share essentially the same difficulty, which just changes its form when the formalism is being changed.

5 Conclusions

The analysis presented in the previous three sections strongly suggests that little information about the interaction region can be obtained from the data on the momentum distributions without making additional assumptions. Usually this problem is solved by using a model. Every model contains assumptions impossible to prove by just using the data on the momentum distributions. Within the model these assumptions could be justified by arguments of beauty, simplicity, or by invoking some general principles. This rises the following important problem, however.

Suppose that two models differ only in these additional assumption and thus they give exactly the same predictions for all the momentum distributions. Suppose further that they correspond to widely different and conflicting pictures of the interaction region and of what happens there. How should one decide which model is the realistic one? This is not an academic problem. It has been already noticed that widely different model can fit the same data, cf. e.g. [10], [11].

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