Rotating quantum states

Victor E. Ambruș and Elizabeth Winstanley
Consortium for Fundamental Physics, School of Mathematics and Statistics,
University of Sheffield, Hicks Building, Hounsfield Road, Sheffield, S3 7RH, United Kingdom
(Dated: January 27, 2014)

We revisit the definition of rotating thermal states for scalar and fermion fields in unbounded Minkowski space-time. For scalar fields such states are ill-defined everywhere, but for fermion fields an appropriate definition of the vacuum gives thermal states regular inside the speed-of-light surface. For a massless fermion field, we derive analytic expressions for the thermal expectation values of the fermion current and stress-energy tensor. These expressions may provide qualitative insights into the behaviour of thermal rotating states on more complex space-time geometries.

PACS numbers: 03.70.+k,04.62.+v

INTRODUCTION

In the canonical quantization of a free field, an object of fundamental importance is the vacuum state, from which states containing particles are constructed. For fields of all spins, the process starts by expanding the classical field in terms of an orthonormal basis of field modes, which are split into positive and negative frequency modes. The expansion coefficients are promoted to operators, the expansion coefficients of the positive frequency modes being particle annihilation operators [1]. The vacuum state is defined as that state annihilated by all the particle annihilation operators. The definition of a vacuum state is therefore dependent on how the field modes are split into positive and negative frequency modes. This split is restricted for a quantum scalar field by the fact that positive frequency modes must have positive Klein-Gordon norm. For a quantum fermion field, both positive and negative frequency fermion modes have positive Dirac norm, so the split of the field modes into positive and negative frequency is less constrained compared with the scalar field case. There is therefore more freedom in how the vacuum state is defined for a fermion field, leading to more freedom in how states containing particles are defined.

In this letter we explore this difference between scalar and fermion quantum fields by considering the definition of rotating vacuum and thermal states in Minkowski space. This toy model reveals that there are quantum states which can be defined for a fermion field but which have no analogue for scalar fields.

ROTATING SCALARS

We consider Minkowski space in cylindrical coordinates \((t_{\text{Mink}}, \rho, \varphi_{\text{Mink}}, z)\) [2]. We wish to define quantum states which are rigidly rotating with angular velocity \(\Omega\). Choosing the \(z\) axis of the coordinate system along the angular velocity vector \(\Omega\), the line element of the rotating space-time can be found by making the transformation \(\varphi = \varphi_{\text{Mink}} - \Omega t_{\text{Mink}}, t = t_{\text{Mink}}\) in the usual Minkowski line element, giving:

\[
ds^2 = -(1 - \rho^2 \Omega^2)dt^2 + 2\rho^2 \Omega dt d\varphi + \rho^2 d\varphi^2 + dz^2.
\]

The Killing vector \(\partial_t\), which defines the co-rotating Hamiltonian \(H = i\partial_t\), becomes null on the speed-of-light surface (SOL), defined as the surface where \(\rho = \Omega^{-1}\). The Klein-Gordon equation for a massless scalar field on the space-time (1) is:

\[
-\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial \varphi^2} - \frac{\partial^2}{\partial z^2}\right) \Phi(x) = 0,
\]

where \(P_\rho = -i\partial_\rho\) and \(L_z = -i\partial_z\) are the \(z\) components of the momentum and angular momentum operators, respectively. The mode solutions of (2) are:

\[
f_{\omega km}(x) = \frac{1}{\sqrt{8\pi \omega |\omega|}} e^{-\tilde{\omega}t + im\varphi + ikz} J_m(q\rho),
\]

where \(J_m(q\rho)\) is the Bessel function of the first kind of order \(m\), \(m\) is the eigenvalue of \(L_z\), \(k\) is the eigenvalue of \(P_\rho\), and \(q\) is the longitudinal component of the momentum and \(\omega = \pm \sqrt{k^2 + q^2}\) gives the Minkowski energy of the mode. The eigenvalue of the Hamiltonian, \(\tilde{\omega} = \omega - \Omega m\), represents the energy of the mode as seen by a co-rotating observer. It is convenient to introduce the shorthand \(j = (\omega_j, k_j, m_j)\) and

\[
\delta(j, j') = \delta_{m, m'} \delta(k_j - k_{j'}) \frac{\delta(\omega_j - \omega_{j'})}{|\omega_j|}.
\]

Using the Klein-Gordon inner product:

\[
\langle f, g \rangle = -i \int d^3x \sqrt{-g} (f^* \partial^t g - g \partial^t f^*),
\]

the norm of the modes (3) can be calculated:

\[
\langle f_j, f_{j'} \rangle = \frac{\omega}{|\omega|} \delta(j, j').
\]

As discussed by Letaw and Pfautsch [3], particles must be described by modes with positive norm (\(\omega > 0\), implying...
the following expansion for the scalar field operator:

\[ \Phi(x) = \sum_{m,j=-\infty}^{\infty} \int_0^\infty \omega_j d\omega_j \int_{-\omega_j}^{\omega_j} dk_j \left[ f_j(x)a_j + f_j^*(x)a_j^\dagger \right], \]

(7)

where \( a_j \) and \( a_j^\dagger \) are one-particle annihilation and creation operators, respectively, satisfying the canonical commutation relations \( [a_j, a_j^\dagger] = \delta(j, j') \). The induced vacuum state \( \langle 0 \rangle \), satisfying \( a_j |0\rangle = 0 \), coincides with the Minkowski vacuum [3].

At finite inverse temperature \( \beta = T^{-1} \), Vilenkin [4] gives the following thermal expectation value (t.e.v.):

\[ \langle a_j^\dagger a_j \rangle_\beta = \frac{\delta(j, j')}{e^{\beta \omega_j} - 1}. \]

(8)

The above expression cannot hold when \( \omega < 0 \), since it would imply that the vacuum expectation value of \( a_j^\dagger a_j \), obtained by taking the limit \( \beta \to \infty \), is non-zero, contradicting the definition of the vacuum. Furthermore, the divergent behaviour of the thermal weight factor of modes with \( \omega \) close to zero renders t.e.v.s infinite, causing rotating thermal states for scalar fields to be ill-defined everywhere in the space-time [4, 5]. As discussed by Duffy and Ottewill [5], a resolution to these problems is to enclose the system inside a boundary located inside or on the SOL, restricting wavelengths such that \( \omega \) stays positive for all values of \( m \).

**ROTATING FERMIONS**

In the Cartesian gauge [6], a natural frame for the metric [1] can be chosen to be:

\[ e_i = \partial_i - \Omega \partial_x, \quad e_i = \partial_i. \]

(9)

In the following, hats shall be used to indicate tensor components with respect to the tetrad, i.e. \( A^\mu = A^\mu e^\mu_i \). The massless Dirac equation for fermions takes the form:

\[ \left[ \gamma^i (H + \Omega M_z) - \gamma \cdot P \right] \psi(x) = 0, \]

(10)

where the gamma matrices are in the Dirac representation [7] and the covariant derivatives are given by:

\[ iD_i = H + \Omega M_z, \quad -iD_j = P_j. \]

(11)

The momentum operators \( P_j \) and angular momentum operator \( M_z \) are:

\[ P_j = -i\partial_j, \quad M_z = -i\partial_x + \frac{1}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}. \]

(12)

The Dirac equation (10) admits the following solutions:

\[ U_{Ekm}^\lambda(x) = \frac{1}{\sqrt{8\pi^2}} e^{-iE \hat{t} + ikz} \left( \begin{array}{c} \phi_{Ekm}^\lambda \\ 2\Omega E m \phi_{Ekm}^\lambda \end{array} \right), \]

(13)

where the two-spinor \( \phi_{Ekm}^\lambda \) is defined as:

\[ \phi_{Ekm}^\lambda(\rho, \varphi) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} e^{i\varphi m \varphi} J_m(\rho) \\ 2i\lambda \sqrt{1 - 2\Omega E} (m+1) \varphi J_{m+1}(\rho) \end{array} \right), \]

(14)

where \( \lambda \) is the helicity [4, 7] and \( E = \sqrt{k^2 + q^2} \) controls the sign of the Minkowski energy of the mode. The eigenvalues of the Hamiltonian are \( E = -\Omega(m + \frac{1}{2}) \), representing, as in the scalar case, the energy seen by a co-rotating observer. The notation \( j = (E_j, k_j, m_j, \lambda_j) \) and

\[ \delta(j, j') = \delta_{\lambda_j \lambda_j'} \delta_{m_j, m_j'} \delta(k_j - k_j') \frac{\delta(E_j - E_j')}{|E_j|} \]

(15)

is useful to refer to modes and their norms. The latter can be computed using the Dirac inner product:

\[ \langle \psi, \chi \rangle = \int d^3x \sqrt{-g} \psi^\dagger(x) \chi(x). \]

(16)

It can be shown that \( \langle U_j, U_{j'} \rangle = \delta(j, j') \) for all possible labels \( j, j' \). After choosing a suitable definition for particle modes (i.e. a range for the labels in \( j \)), the anti-particle modes can be constructed using charge conjugation [6, 7]:

\[ V_j = i\gamma^j U_j^*. \]

Hence, \( V_j \) automatically inherits the same normalisation as \( U_j \): \( \langle V_j, V_{j'} \rangle = \delta(j, j') \). Therefore there is no restriction on how we perform the split into particle and anti-particle modes, as long as the charge conjugation symmetry is preserved.

According to Vilenkin [4], the definition of particles for co-rotating observers should be the same as for inertial Minkowski observers, with the field operator written as:

\[ \psi_V(x) = \sum_{\lambda_j = \pm \frac{1}{2}} \sum_{m_j = -\infty}^{\infty} \int_0^\infty E_j dE_j \int_{-E_j}^{E_j} dk_j \]

\[ \times \left[ U_j(x) b_{j;V} + V_j(x) d_{j;V}^\dagger \right]. \]

(17)

Vilenkin’s quantisation is equivalent to the one suggested by Letaw and Pfautsch [3] for the scalar field, yielding a vacuum state equivalent to the Minkowski vacuum. In contrast, Iyer [8] argues that the modes which represent particles for a co-rotating observer have positive frequency with respect to the co-rotating Hamiltonian, implying the following expression for the field operator:

\[ \psi_I(x) = \sum_{\lambda_j = \pm \frac{1}{2}} \sum_{m_j = -\infty}^{\infty} \int_0^\infty E_j dE_j \int_{-E_j}^{E_j} dk_j \]

\[ \times \left[ U_j b_{j;I} + V_j(x) d_{j;I}^\dagger \right], \]

(18)

with the integral with respect to \( E_j \) running over both positive and negative values of \( E_j \), as long as \( E_j \geq 0 \). Both quantisation methods lead to the canonical anti-commutation relations \( \{b_j, b_{j'}^\dagger\} = \{d_j, d_{j'}^\dagger\} = \delta(j, j') \).
The ensuing quantum field theory differs in the two pictures, as the vacuum state corresponding to Iyer’s quantization differs from the Minkowski vacuum. This can be seen by looking at the connection between the Iyer and Vilenkin one-particle operators:

\[
b_{j,t} = \begin{cases} b_{j,V} & E_j > 0, \\ (2m+1)^2 b_{j,V} & E_j < 0, \end{cases}
\]  

(19)

and similarly for \(d_{j,t}\), where \(\mathcal{J} = (-E_j, -k_j, -m_j - 1, \lambda_j)\). Thus, the Vilenkin vacuum state (i.e. the non-rotating Minkowski vacuum) contains particles as defined according to Iyer’s quantization.

Vilenkin [4] also considered rotating thermal states for fermions. In analogy with [5], he gives the following t.e.v.s relative to the Minkowski vacuum [4]:

\[
\langle b_j^\dagger b_{j'} \rangle_\beta = \langle d_{j}^\dagger d_{j'} \rangle_\beta = \frac{\delta(j,j')}{e^{\beta E_j} + 1}.
\]  

(20)

As in the scalar case, (20) is not valid when \(E_j < 0\). However, in contrast to the scalar case, the modes with negative \(E_j\) can be eliminated from the set of particle modes, by using Iyer’s quantisation. Furthermore, unlike the thermal factor for scalars [5], the Fermi-Dirac density of states factor (20) is regular for all \(E_j\).

Eq. (20) can be used to construct the t.e.v. of the neutrino charge current operator

\[
J^\hat{\alpha}(x) = \frac{1}{2} \left[ \psi, \gamma^\hat{\alpha} \frac{1 + \gamma^5}{2} \psi \right].
\]  

(21)

Using the Vilenkin quantization, we find the following t.e.v. relative to the Minkowski vacuum:

\[
\langle J^\hat{\alpha} \rangle_\beta = -\frac{1}{2\pi^2} \sum_{m=-\infty}^{\infty} \int_0^\infty \frac{E^2 dE}{e^{\beta E} + 1} \times \int_0^E dk \left[ J^\hat{\alpha}_m(qp) - J^\hat{\alpha}_{m+1}(qp) \right].
\]  

(22)

To evaluate (22), we expand the Fermi-Dirac density of states factor in powers of \(\Omega\), after which the sum over \(m\) and the integral with respect to \(k\) are performed, yielding:

\[
\langle J^\hat{\alpha} \rangle_\beta = -\frac{\Omega}{2\pi^2} \sum_{j=0}^{\infty} (\rho \Omega)^2 j \sum_{n=0}^{\infty} \frac{(2j + 2)!}{(2n + 2j + 1)!} c_{n,j} \times \int_0^\infty dE E \Omega^{2n} \frac{d^{2n}}{dE^{2n}} \left( \frac{1}{e^{\beta E} + 1} \right),
\]  

(23)

where the coefficients \(c_{n,j}\) are defined to be [9]:

\[
c_{n,j} = \frac{1}{(2j + 1)!} \left[ \frac{d^{2n+2j+1}}{d\alpha^{2n+2j+1}} \left( \frac{2\sinh \frac{\alpha}{2} \overline{2j+1} \right) \right]_{\alpha=0}.
\]  

(24)

The integral with respect to \(E\) in (23) vanishes for all \(n > 1\). Hence, we arrive at the following result:

\[
\langle J^\hat{\alpha} \rangle_\beta = -\frac{\Omega}{12\beta^2 \varepsilon^2} - \frac{\Omega^3}{48\pi^2 \varepsilon^3} (4 - 3\varepsilon),
\]  

(25)

where \(\varepsilon = 1 - \rho^2\Omega^2\). Vilenkin [4] calculates \(\langle J^\hat{\alpha} \rangle_\beta\) on the axis of rotation, where only the \(m = -1\) and \(m = 0\) terms in (22) contribute. Our result (25) agrees with [4] on the axis of rotation when \(\rho = 0\).

We can apply the same method for the construction of the t.e.v. of the stress-energy tensor (SET) operator, defined in terms of tetrad components as:

\[
T_{\alpha\beta}(x) = -\frac{i}{4} \left\{ \left[ \bar{\psi}, \gamma_{(\alpha} D_{\beta)} \psi \right] - \left[ D_{(\alpha} \bar{\psi} \gamma_{\beta)} \right] \right\}.
\]  

(26)

We obtain the following results for the co-ordinate basis components of the SET:

\[
\langle T_{\mathrm{V:tt}} \rangle_\beta = \frac{7\pi^2}{60\beta^4 \varepsilon^2} + \frac{\Omega^2}{8\beta^2 \varepsilon^2} \left( \frac{4}{3} - \frac{1}{3} \varepsilon \right) + \frac{\Omega^4}{64\pi^2 \varepsilon^3} \left( \frac{8}{9} + \frac{56}{45} \varepsilon - \frac{17}{15} \varepsilon^2 \right),
\]  

(27a)

\[
\langle T_{\mathrm{V:tt}} \rangle_\rho = -\rho \Omega \left\{ \frac{7\pi^2}{60\beta^4 \varepsilon^2} + \frac{13\Omega^2}{72\beta^2 \varepsilon^3} \left( \frac{16}{13} - \frac{3}{13} \varepsilon \right) + \frac{119\Omega^4}{960\pi^2 \varepsilon^4} \left( \frac{200}{119} - \frac{64}{119} \varepsilon - \frac{1}{7} \varepsilon^2 \right) \right\},
\]  

(27b)

\[
\langle T_{\mathrm{V:tp}} \rangle_\beta = \frac{7\pi^2}{180\beta^4 \varepsilon^2} + \frac{\Omega^2}{24\beta^2 \varepsilon^3} \left( \frac{9}{3} - \frac{1}{3} \varepsilon \right) + \frac{\Omega^4}{192\pi^2 \varepsilon^4} \left( \frac{8}{15} - \frac{88}{15} \varepsilon - \frac{17}{15} \varepsilon^2 \right),
\]  

(27c)

\[
\langle T_{\mathrm{V:tp}} \rangle_\rho = \frac{7\pi^2}{180\beta^4 \varepsilon^2} + \frac{\Omega^2}{24\beta^2 \varepsilon^3} \left( \frac{9}{3} - \frac{1}{3} \varepsilon \right) + \frac{\Omega^4}{192\pi^2 \varepsilon^4} \left( \frac{8}{15} - \frac{88}{15} \varepsilon - \frac{17}{15} \varepsilon^2 \right),
\]  

(27d)

\[
\langle T_{\mathrm{V:zz}} \rangle_\beta = \frac{7\pi^2}{180\beta^4 \varepsilon^2} + \frac{\Omega^2}{24\beta^2 \varepsilon^3} \left( \frac{9}{3} - \frac{1}{3} \varepsilon \right) + \frac{\Omega^4}{192\pi^2 \varepsilon^4} \left( \frac{8}{15} - \frac{88}{15} \varepsilon - \frac{17}{15} \varepsilon^2 \right).
\]  

(27e)
The analytic results for $\langle \hat{J}^\perp \rangle_\beta$ [25] and $\langle \hat{T}_{V,\sigma} \rangle_\beta$ [27] reveal a number of physical features. Firstly, they all contain contributions which are independent of the temperature (equivalently, independent of $\beta$). These terms are unphysical as the t.e.v.s should vanish when the temperature is set to zero ($\beta \to \infty$). These temperature-independent terms are generated by modes with $E < 0$ and arise because the Vilenkin quantization [4] has been used. If the Iyer quantization [8] is employed, the corresponding t.e.v.s $\langle \hat{J}^\perp \rangle_\beta$ and $\langle \hat{T}_{V,\sigma} \rangle_\beta$ are equal to the expressions in [25] and [27] respectively without the temperature-independent parts, for example $\langle \hat{J}^\perp \rangle_\beta = -\frac{\Omega}{(12\beta^2\varepsilon^2)}$. Thus we conclude that the Iyer vacuum is the appropriate choice when considering thermal rotating states. The temperature-independent contributions to [25] and [27] are the expectation values of the Iyer vacuum relative to the Vilenkin vacuum. Since these are non-zero and depend on $\rho$, we see that the Iyer vacuum is not equivalent to, and has fewer symmetries than, the maximally symmetric Vilenkin vacuum.

If $\Omega = 0$, there is no rotation and the t.e.v.s [25] and [27] reduce to the usual Minkowski t.e.v.s. On the axis of rotation, we have $\varepsilon = 1$ and the expressions in round brackets in [27] evaluate to unity. In this case the t.e.v.s take the form of the Minkowski values plus an $\Omega$-dependent correction. A similar effect is found for rotating thermal states for a scalar field inside a reflecting cylinder [9].

The t.e.v.s [25] and [27] are finite as long as $\varepsilon > 0$ but diverge as $\varepsilon \to 0$ and the SOL is approached. Using the Iyer quantization (when the $\beta$-independent terms are absent), the current $\langle \hat{J}^\perp \rangle_\beta$ diverges as $\varepsilon^{-2}$ and the SET components $\langle \hat{T}_{I,\sigma} \rangle_\beta$ diverge as $\varepsilon^{-4}$.

The discussion so far has been limited to massless fermions. Although analytic calculations in the massive case are less tractable, numerical methods can be employed. In Fig. 1 we show the t.e.v.s $\langle \hat{T}_{I,I} \rangle_\beta$ using the Iyer quantization for massive fermions. As expected, the energy t.e.v. is smaller for massive fermions compared with the massless case and decreases as the temperature decreases. It can be seen from plot (b) that as the SOL is approached, the t.e.v.s in the massive case diverge at exactly the same rate as for massless fermions.

CONCLUSIONS

We have studied the construction of rotating states for scalar and fermion fields in four-dimensional Minkowski space. Our analysis has demonstrated that the definition of fermion quantum states is less constrained than for scalar fields. This is due to more freedom in how the split into particle and anti-particle modes is performed for fermion fields, since all fermion modes have positive norm (the Dirac norm is positive definite but the Klein-Gordon norm is not). For fermion fields we have shown that the Iyer quantization [8] gives a natural definition of a rotating vacuum and is suitable for the construction of rotating thermal states. This rotating vacuum state is inequivalent to the Minkowski vacuum, but cannot be defined for a quantum scalar field (which is restricted to the Minkowski vacuum). While rotating thermal states for scalar fields are ill-defined everywhere on the unbounded space-time, we have constructed fermion rotating thermal states which are regular inside the speed-of-light surface.

In this paper we have considered the toy model of rotating states in Minkowski space-time. However, the main physical features extend to curved space-times. For example, recent work on the construction of quantum states for fermion fields on a rotating Kerr black hole [10] has also demonstrated the existence of fermion states which have no analogue for scalar fields. Furthermore, the simplicity of our toy model has enabled us to derive analytic expressions for the thermal expectation values which could provide qualitative insights into the behaviour of thermal rotating states on more complex space-time geometries, for example, the nature of the divergence as the speed-of-light surface is approached.
ACKNOWLEDGEMENTS

This work is supported by the Lancaster-Manchester-Sheffield Consortium for Fundamental Physics under STFC grant ST/J000418/1, the School of Mathematics and Statistics at the University of Sheffield and the European Cooperation in Science and Technology (COST) action MP0905 “Black Holes in a Violent Universe”.

∗app10vea@sheffield.ac.uk
†e.winstanley@sheffield.ac.uk
[1] The adjoints of the expansion coefficients of the negative frequency modes are also particle annihilation operators. For a scalar field, these annihilation operators are the same as the expansion coefficients of the positive frequency modes; for a fermion field they are different.

[2] Throughout this paper we use units in which $c = \hbar = k_B = 1$.

[3] J. Letaw and J. Pfautsch, Phys. Rev. D 22, 1345 (1980).
[4] A. Vilenkin, Phys. Rev. D 21, 2260 (1980).
[5] G. Duffy and A. Ottewill, Phys. Rev. D 67, 044002 (2003).
[6] I. I. Cotăescu, J. Phys. A: Math. Gen. 33, 9177 (2000).
[7] C. Itzykson and J.-B. Zuber, Quantum field theory (Dover Publications, inc., 1980).
[8] B. Iyer, Phys. Rev. D 26, 1900 (1982).
[9] V. Ambruș and E. Winstanley, paper in preparation.
[10] M. Casals, S. R. Dolan, B. C. Nolan, A. C. Ottewill, and E. Winstanley, Phys. Rev. D 87, 064027 (2013).