THE QUANTUM CAUCHY FUNCTIONAL AND SPACE-TIME APPROACH TO RELATIVISTIC QUANTUM MECHANICS

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ABSTRACT. We show that the quantum complex Cauchy functional on bump functions (see [16]), yields a generalized complex Cauchy process, which is a generalized functional on the bump functionals on Borel cylindrical sets of a real Hilbert space, whose support is locally compact for the uniform convergence topology with derivatives in $L^2(0,t)$.

The retarded Green’s functions of the Dirac electron and Einstein photon viewed as complex matrix-valued functionals on bump functionals, also yield generalized complex matrix-valued processes of Cauchy-Dirac and Cauchy-Maxwell, which are generalized functional on the bump functionals on Borel cylindrical sets of a real Hilbert space, but their supports are compact for the uniform convergence topology with derivatives in $L^2(0,t)$.

We study the way the classical relativistic mechanics of particle comes from the quantum mechanics of the free Dirac particle.

INTRODUCTION

The present work is based on results of author’s work [16], and we assume that the reader is acquainted with it.

We study one-and three-dimensional quantum Cauchy functionals, as defined in [16], that are complex-valued functionals on bump functions, and are non-Gaussian. The next important property of these functionals, which are fundamental solutions for integral evolution equations, as well as measurable functions, plays important role: They generate compatible complex measures of cylindrical subsets with Borel bases in $R^{(n)}$ that are Lebesgue’s integrals over the bases.

The complex pre-measures corresponding to the one-and three-dimensional quantum Cauchy functionals, viewed as functionals on bump functions, are such that the measure of the complement to a ball of large enough radius $R$ in a Euclidean space of any dimension is purely imaginary of fixed sign and tends to zero when $R \to \infty$.

Therefore these complex pre-measures in $R^{(n)}$ can be extended to cylindrical Borel sets of a real Hilbert space, whose support is locally compact for the uniform convergence topology with derivatives in $L^2(0,t)$.

The 3-dimensional problems for quantum Cauchy functionals are not generated by 1-dimensional ones. In the exposition we use the Radon-Gelfand transform describing in details the solution of the problem in 1-dimensional case and explaining simultaneously the modifications needed to treat the 3-dimensional case.

Key words and phrases. the momentum and position presentations of functionals, semicontinuous functions, the extended Lebesgue’s integral, a continuation of functionals and measures, the support of a functional, Zitterbewegung, the Foldy-Wouthuysen transform, Parseval’s identity.

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At the end of the article we show that, using an isomorphism between the spaces of solutions of the Dirac equation for free electron and the Maxwell photon equation and the respective presentations of Foldy-Wouthuysen (which are diagonalized solutions that can be reduced to quantum Cauchy functionals, see [16]) all the results render to these physical processes of relativistic quantum mechanics.

For a short exposition of the results see [20].

1. The quantum Cauchy functional; the constructions of pre-measures

In [16] we constructed the next position coordinate presentation of the one-dimensional quantum Cauchy functional

\[ C_{it}(x) = \frac{1}{2} (\delta(x - t) + \delta(x + t)) + \frac{i}{\pi} \cdot \frac{t}{t^2 - x^2}. \]

on the space of bump test functions \( \varphi(x) \in K^{(1)} \) (see [2]) which is the fundamental solution of the equation

\[ \frac{\partial}{\partial t} C_{it}(x) = -\frac{i}{\pi x^2} * C_{it}(x) \]

(here \( * \) is the convolution of generalized functions, see [2]) with the momentum coordinate presentation \( \tilde{C}_{it}(p) = \exp(it|p|) \) viewed as a functional on analytic functions \( \psi(p) \in Z^{(1)} \).

Article [16] also provides the position coordinate presentation of the space quantum Cauchy functional (which is a spherically symmetric functional on bump functions \( \varphi \in K^{(3)} \))

\[ C_{it}(x) = \frac{\delta_{St}}{4\pi t^2} + i \cdot \frac{t}{\pi^2 (t^2 - r^2)^2}; \]

here \( \delta_{St} \) is the delta-function of the sphere of radius \( t \), and \( r = \sqrt{x_1^2 + x_2^2 + x_3^2} \). This functional is a fundamental solution of the equation

\[ \frac{\partial}{\partial t} C_{it}(x) = \frac{i}{\pi^2 r^4} * C_{it}(x), \]

and its momentum coordinate presentation \( \tilde{C}_{it}(p) = \exp(it\rho) \), where \( \rho = \sqrt{p_1^2 + p_2^2 + p_3^2} \), is a functional on \( \psi(p) \in Z^{(3)} \).

Our aim is first to construct and study complex-valued measures of cylindrical sets of trajectories that correspond to \( C_{it}(x) \), with \( n \)-dimensional Borel bases whose points \( a_k, k = 1, \ldots, n \), are defined by conditions

\[ a_k = \sum_{1 \leq j \leq n} \alpha_{jk} \Delta x_j; \quad k \leq l \leq n, \]

where \( \alpha_{jk} \in \mathbb{R}^{n'} \) are real numbers (or \( a_k = \int_0^t \alpha_k(\tau)dx_\tau \), where \( \alpha_k(\tau) \) are piecewise linear functions). We assume that \( \det \alpha_{jk} \neq 0 \).
The next aim is to find a topological vector space $X$ (with $\alpha_k(\tau) \in X'$) on whose cylindrical subsets one can extend the constructed measure of cylindrical subsets of $R^{(n)}$ (see [3], [14]).

We first construct retarded Green's functions $\mathcal{C}_t(a_1, \ldots, a_n)$ that correspond to $C_t(x)$ (see (1)) as functionals on $\varphi(a) \in K^{(n)}$ on $n$-dimensional base of a cylindrical set of trajectories (5). We will see that these Green's functions define measures in $C_R$ of cylindrical subsets one can extend the constructed measure of cylindrical subsets where $0 = t_0 < t_1 < \ldots < t_n = t$, $\Delta t_j = t_j - t_{j-1}$, $j = 1, \ldots, n$. For a proof of the existence of these convolutions, see below. Equality (6) can be also understood as a method of construction of a "new" generalized function $\mathcal{C}_t(x)$ of the simplest linear functional of $\Delta x_1, \ldots, \Delta x_n$ from $C_{t_1}(\Delta x_1), \ldots, C_{t_n}(\Delta x_n)$.

Indeed, consider the l.h.s. of equality (6) and write the bump function $\varphi$ via the Fourier integral

$$\varphi(\Delta x_1 + \ldots + \Delta x_n) = \frac{1}{2\pi} \int \exp(-i(\Delta x_1 + \ldots + \Delta x_n)p)\psi(p)dp,$$

$\psi(p) \in Z^{(1)}$ (see [2]); then the l.h.s. becomes

$$\frac{1}{2\pi} \int (\Pi_{j=1}^n C_{t_j}(\Delta x_j)) \exp(-i\Delta x_jp)d\Delta x_j\psi(p)dp.$$

But we know already each of the inner integrals: it equals $\exp(-i\Delta t_j|p|)$.

Thus (6) determines the functional $C_t(\Sigma_{j=1}^n \alpha_j \Delta x_j)$ on $\varphi(x) \in K^{(1)}$. At that the shape of $\mathcal{C}_t(p)$ implies the existence of the convolutions in the Chapman-Kolmogorov equation (6).

Therefore one can say that the retarded Green's function $C_t(x)$ is defined on the base of the simplest cylindrical set of functions $\Sigma_{j=1}^n \alpha_j \Delta x_j = x$ with fixed value of the position coordinate at time $t$.

To construct retarded Green's function $\mathcal{C}_t(a)$ on the line in general case that corresponds to case $\Sigma_{j=1}^n \alpha_j \Delta x_j = a$ (see (5), (6)), one considers

$$\int (\Pi_{j=1}^n C_{t_j}(\Delta x_j))\varphi(\Sigma_{j=1}^n \alpha_j \Delta x_j)d\Delta x_1 \ldots d\Delta x_n.$$

Repeating literally the case we have considered, we get

$$\frac{1}{2\pi} \int \exp(i|p|\Sigma_{j=1}^n |\alpha_j| \Delta t_j)\psi(p)dp = \int \mathcal{C}_t(a)\varphi(a)da,$$

where

$$\mathcal{C}_t(a) = C_{iT}(a), \quad T = \Sigma_{j=1}^n |\alpha_j| \Delta t_j.$$
Remark. Retarded Green’s function $\mathcal{C}_{it}(a)$ is a functional on $\varphi(a) \in K^{(1)}$, and it is a measurable function: this is evident for the real part, and the imaginary part (see (1), (8))

$$\frac{T}{T^2 - a^2} = \frac{1}{2} \left( \frac{1}{T - a} + \frac{1}{T + a} \right),$$

that has non-summable singularities at points $T$ and $-T$ (where $T \neq 0$), is the sum of semi-continuous, hence measurable, functions (see [12], [17]).

Therefore $C_{it}(x)$ yields complex-valued measure $\int_{\mathbb{R}} C_{it}(a) da$ of cylindrical sets $R \in \mathfrak{M}^{(1)} (= \text{Borel sets on the line})$ and generating line that corresponds to piecewise constant function $a_k(\tau)$, see (5), as Lebesgue’s integral (see [17]).

Thus $C_{it}(a)$, being a functional on $K^{(1)}$, may be not determined at some point $a \in \mathbb{R}$ as a function, but the integral $\int_a \mathcal{C}_{it}(x) dx$ over any $\varepsilon$-neighborhood of $a$ is well defined. We call $\mathcal{C}_{it}(a)$ the Lebesgue density.

To construct the retarded Green’s function $\mathcal{C}_{it}(a_1, \ldots, a_n)$ on $n$-dimensional base of cylindrical set (5) $\sum_{1 \leq j \leq n} \alpha_j \Delta x_j = a_k$ ($k = 1, \ldots, n$) as a functional of $\varphi(a) \in K^{(n)}$, consider the integral

$$\int (\Pi_{j=1}^n C_{i\Delta t_j}(\Delta x_j)) \varphi(\Sigma_{j=1}^n \alpha_j \Delta x_j, \ldots, \Sigma_{j=1}^n \alpha_j \Delta x_j) \, d\Delta x_1 \cdots d\Delta x_n,$$

where $\varphi(a_1, \ldots, a_n) \in K^{(n)}$.

Repeating literally the construction of Green’s function $\mathcal{C}_{it}(a)$, see (5), (6), (7), we get momentum coordinates presentation of the promised Green’s function

$$\hat{\mathcal{C}}_{it}(p_1, \ldots, p_n) = \exp(i \Sigma_{j=1}^n \alpha_j p_j) \det(\Pi_{j=1}^n \Delta t_j) = \exp(i \Sigma_{j=1}^n (\alpha_j, p) \Delta t_j)$$

as a functional on $\psi(p_1, \ldots, p_n) \in Z^{(n)}$.

To construct $\mathcal{C}_{it}(a_1, \ldots, a_n)$ explicitly consider Parseval’s identity

$$\int \mathcal{C}_{it}(a) \varphi(a) da = \frac{1}{(2\pi)^n} \int \left( \int \exp(i \Sigma_{j=1}^n (\alpha_j, p) \Delta t_j + i(p, a)) dp \right) \varphi(a) da$$

(here $a$, $p$, and $\alpha_j$ are vectors in the $n$-dimensional Euclidean space). Changing the variables $(\alpha_j, p) = q_j$ or $\hat{A}p = q$, where $A = \alpha_j$ (see (5)), $\hat{A}$ is the transposed matrix, we get

$$\int \mathcal{C}_{it}(a) \varphi(a) da = \frac{1}{(2\pi)^n} \int \left( \int \exp(i \Sigma_{j=1}^n q_j \Delta t_j + i(q, \hat{A}^{-1} q, a)) dq \right) \varphi(a) \det^{-1} Ada =$$

$$= \frac{1}{(2\pi)^n} \int \left( \int \exp(i \Sigma_{j=1}^n q_j \Delta t_j + i(q, A^{-1} a)) dq \right) \varphi(a) \det^{-1} Ada.$$

Therefore (see (1), (8)) one has

$$\int \mathcal{C}_{it}(a) \varphi(a) da = \int (\Pi_{j=1}^n C_{i\Delta t_j}(A^{-1} a_j)) \varphi(a) \det^{-1} Ada =$$

$$= \int (\Pi_{j=1}^n C_{i\Delta t_j}(\Delta x_j)) \varphi(\Delta x) d\Delta x = \int (\Pi_{j=1}^n C_{i\Delta t_j}(\Delta x_j)) \varphi(\Delta x_1, \ldots, \Delta x_n) \Pi_{j=1}^n dx_j,$$
where \((A^{-1}a)_j\) is \(j\)th component of the vector \(A^{-1}a\), or

\[
\mathcal{C}_t(a) = \prod_{j=1}^n C_{itj} \left((A^{-1}a)_j\right) \det^{-1} A.
\]

Here the r.h.s. is the direct product of functionals representing Green’s functions on one-dimensional bases of the cylindrical sets. Therefore \(C_{it}(x)\) defines uniquely the measures of cylindrical sets with \(n\)-dimensional bases as well.

The complex-valued measure of some cylindrical sets may have infinite imaginary part (of different signs), but the measure of all cylindrical sets for every finite dimension of the base is (see (9)) \(\int \mathcal{C}_{it}(a_1, \ldots, a_n) da_1 \ldots da_n = 1\) and also \(\int \mathcal{C}_{it}(a_1, \ldots, a_n) da_k = \mathcal{C}_{it}(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n)\), i.e., our complex-valued functional is normalized and the corresponding pre-measures are compatible.

We will see later (see theorem (II) in §3) that it is enough to consider only those matrices \(\alpha_{jk}\) that imply finiteness of \(\sum_{j=1}^n \sqrt{\alpha_{j1}^2 + \ldots + \alpha_{jn}^2} \Delta t_j\) for all \(n\), which implies (see (19) and §§3,4 below) that \(\alpha_k(\tau) \in L_2(0, t)\). Therefore one has

\[
\int f(a_k) \mathcal{C}_{it}(a_k) da_k |_{k \to \infty} = \int f(a) \mathcal{C}_{it}(a) da
\]

for any continuous bounded function \(f(a)\) of \(n\) variables. Thus the measures of cylindrical sets that correspond to \(\mathcal{C}_{it}(a)\), satisfy the (weak) continuity condition, see [3]. A similar condition holds in the space problem case.

Therefore we have deduced

**Theorem (I):** The quantum Cauchy functionals \(\mathcal{C}_{it}(a_1, \ldots, a_n)\) yield complex-valued normalized continuous measures of cylindrical sets of trajectories with Borel bases (pre-measures), and the measures of different cylindrical sets are mutually compatible.

Let us construct now the smallest space of functionals \(X'\) (where \(\alpha_1(\tau), \ldots, \alpha_n(\tau)\) belong, see (5)) on a topological vector space \(X\).

At that we show that there is a set of infinite matrices \(\alpha_{jk}\) such that the measure, corresponding to \(\mathcal{C}_{it}(a_1, \ldots, a_n)\), of the complement to the ball \(\sum_{j=1}^n \alpha_j^2 = R^2\) tends to zero for \(R \to \infty\) for every \(n\) (in our situation the measure is complex-valued, and it is important that the measure of every subset of the above domain is negative purely imaginary, see below).

In the next section we show that such \(X\) is a real Hilbert space.

This approach to the problem of extension of measures is due to V. D. Erokhin (see [18], [14]) who formulated a criterion for continuation of a probabilistic pre-measure on a countably-normed space \(X\) to all its Borel subsets.

To estimate the measure of the ball complements in the next section, we use an analytic construction due to R. A. Minlos (see [14]).

**2. Estimating the quantum Cauchy functional measures of the complements of balls and their subsets**

We find the measure, corresponding to \(\mathcal{C}_{it}(a)\), of the complement to a ball of radius \(R\) in \(n\)-dimensional space using the Radon-Gelfand transform \(\mathcal{C}_{it}(\xi; r)\) of the generalized function \(\mathcal{C}_{it}(a)\) defined on \(K^{(n)}\), see [19]. At that \(\mathcal{C}_{it}(\xi; r)\) (where
(ξ, a) = r, ξ is a unit vector in \( R^{(n)} \) appears to be determined on \( K^{(1)} \). This method is especially convenient since it uses only the structure of one-dimensional problem to solve both one-dimensional and space problem.

Thus the problem is reduced to construction of a generalized function of single argument \( (ξ, a) \) that corresponds to \( \mathcal{G}_u(a) \). To that end let us consider \( \int \mathcal{G}_u(a) \varphi((ξ, a))da \), where \( \varphi(r) \in K^{(1)} \). Recall that a similar problem was already solved when we constructed the generalized function of argument \( \sum_{j=1}^{n} \alpha_j \Delta x_j = a \) that corresponds to \( \mathcal{G}_u(x) \). Repeating the computation literally, we get

\[
\mathcal{G}_u \sim (ξ; r) = C_i Q(r), \quad Q = \sum_{j=1}^{n} |(ξ, \alpha_j)| \Delta t_j, \quad (ξ, \alpha_j) = \sum_{k=1}^{n} \xi_k \alpha_{jk}.
\]

Therefore

\[
\mathcal{G}_u \sim (ξ; r) = \frac{1}{2} (\delta(r + Q) + \delta(r - Q)) + \frac{i}{\pi} \frac{Q}{Q^2 - r^2}
\]

as a functional on \( K^{(1)} \).

Since \( \mathcal{G}_u \sim (ξ; r) \) is Lebesgue’s density of measure of a cylindrical set with one-dimensiona base, the measure of the half-space \( (a, ξ) \geq R > 0 \) of \( n \)-dimensional space, corresponding to \( \mathcal{G}_u(a) \), equals \( \int_R \mathcal{G}_u(ξ; r)dr \).

But the measure of that half-space equals \( \int \chi_\xi^R(a) \mathcal{G}_u(a)da \), where \( \chi_\xi^R(a) \) is the characteristic function of the half-space. So

\[
\int_R \mathcal{G}_u \sim (ξ; r) \chi_\xi^R \int_0^1 R/|a| (1 - y^2)^{(n-1)/2} dy
\]

where the overline means averaging over the surface of the unit sphere.

Notice that in \( \chi_\xi^R(a)\chi_\xi^S \) vector \( a \) is fixed, and we integrate over vectors \( ξ \) with \( (a, ξ) \geq R \). These vectors form a cap on the sphere cut by the plane that has distance \( R/|a| \) from 0, so (cf. [3], [14]), using the Radon transform on the sphere (see [19]), we see that

\[
\chi_\xi^R \chi_\xi^S = N_n \int_0^1 (1 - y^2)^{(n-1)/2}dy
\]

if \( |a| > R \) and 0 otherwise; here constant \( N_n \) depends on dimension \( n \). Since \( \chi_\xi^S = 1/2 \), one has

\[
N_n = \frac{1}{2} (\int_0^1 (1 - y^2)^{-1} dy)^{-1}.
\]

Using spherical coordinates in (16), we get

\[
\int_R \mathcal{G}_u \sim (ξ; r) \chi_\xi^R \int_0^1 \int_{R/\rho} \int_0^1 (1 - y^2)^{(n-1)/2} dyd\rho \int_{|a|<\rho} \mathcal{G}_u(a)da.
\]

This equality relates mean values of measures of half-spaces and of balls corresponding to \( \mathcal{G}_u(a) \) (compare with similar result for positive measures in [3], [14]).
Consider \( \int_R^{\infty} \mathcal{C}^{-1}_{it}(\xi; r) S_1 dr \). For \( R > Q \) (see (13)) the mean value theorem implies

\[
\int_R^{\infty} \mathcal{C}^{-1}_{it}(\xi; r) S_1 dr = \frac{-i}{\pi} \ln \left| \frac{P + 1}{P - 1} \right|,
\]

where \( P = \frac{1}{R} \sum_{j=1}^{n} |(\hat{\xi}, a)_{\Delta t_j}| > 0 \) and \( \hat{\xi} \) is a fixed point on the unit sphere.

Therefore, using (17) and (18), we get

\[
\int_R^{\infty} \mathcal{C}^{-1}_{it}(\xi; r) S_1 dr = \frac{-i}{2\pi} \ln \left| \frac{P + 1}{P - 1} \right| = \mathcal{N} \int_{R/\rho}^{1} (1 - y^2)^{(n-1)/2} dy d\rho \int_{|a|<\rho} i \mathcal{C}_{it}(a) da.
\]

So the measure of the complement to a ball of large enough radius, that corresponds to \( \mathcal{C}_{it}(a) \), is negative purely imaginary for every \( n \).

Notice that

\[
\mathcal{N} \int_{R/\rho}^{1} (1 - y^2)^{(n-1)/2} dy = \frac{1}{2} \int_{0}^{\sqrt{n}} (1 - x^2)^{(n-1)/2} dx \int_{R/\sqrt{n}\rho}^{\sqrt{n}} (1 - x^2)^{(n-1)/2} dx.
\]

So for \( n \) large enough one has

\[
\mathcal{N} \int_{R/\rho}^{1} (1 - y^2)^{(n-1)/2} dy |_{n \to \infty} \approx \frac{1}{2} \left( \int_{0}^{\infty} \exp(-x^2/2) dx \right)^{-1} \left( \int_{R/\sqrt{n}\rho}^{\infty} \exp(-x^2/2) dx \right)
\]

\[
= \sqrt{1/2\pi} \int_{R/\sqrt{n}\rho}^{\infty} \exp(-x^2/2) dx,
\]

cf. [3], [14].

Returning to (19), one has

\[
\frac{1}{2\pi} \ln \left| \frac{P + 1}{P - 1} \right| |_{n \to \infty} \approx \sqrt{1/2\pi} \int_{R/\rho}^{\infty} \exp(-x^2/2) d\rho \int_{|a|<\rho} i \mathcal{C}_{it}(a) da,
\]

and, since the inner integral in r.h.s. decreases when \( \rho \) decreases (and we deal with equality of positive numbers), we get

\[
\frac{1}{2\pi} \ln \left| \frac{P + 1}{P - 1} \right| |_{n \to \infty} > \sqrt{1/2\pi} \int_{R/\rho}^{\infty} \exp(-x^2/2) dx \int_{|a|\geq R} i \mathcal{C}_{it}(a) da.
\]

This implies (see (12), (13)) that

\[
\lim_{R \to \infty} \int_{|a|\geq R} i \mathcal{C}_{it}(a) da = 0
\]

if \( \alpha_{jk} \) are such that

\[
\sum_{j=1}^{n} |(\hat{\xi}, a)|_{\Delta t_j} \leq \sum_{j=1}^{n} \sqrt{\alpha_{j1}^2 + \ldots + \alpha_{jn}^2} \Delta t_j < N.
\]

Using the above method, we study the local structure of the measure of the complement to that ball that corresponds to \( \mathcal{C}_{it}(a) \), i.e., the measure of a neighborhood of any point \( A \) that lies outside the ball of radius \( R \).
Let us find the measure of the ball of radius $\sigma$ with center at $A$ corresponding to $\mathcal{C}_it(a)$. To that end it is enough to know the Radon-Gelfand transform $\mathcal{C}_it(\xi; r + (a, A))$ of the functional $\mathcal{C}_it(a + A)$ (see [19]). Consider

$$\mathcal{C}_it(\xi; r + (\xi, A)) = \frac{1}{2} (\delta(Q - r - (\xi, A)) + \delta(Q + r + (\xi, A))) + \frac{i}{\pi} \frac{Q}{Q^2 - (r + (\xi, A))^2},$$

where $Q = \sum_{\tau=1}^n |(\xi, \alpha_j)|\Delta t_j$, see (11).

Repeating literally the previous arguments, we get

$$\int_0^\sigma \mathcal{C}_it(\xi; r + (\xi, A)) S_1 dr = N_n \int_0^\sigma \int_0^1 (1 + y^2)^{(n-1)/2} dy d\rho \int_{|a| < \rho} \mathcal{C}_it(a + A) da.$$  

By the mean value theorem, this implies

$$\int_0^\sigma \mathcal{C}_it(\xi; r + (\xi, A)) S_1 dr = \frac{i}{\pi} \int_0^\sigma \frac{\hat{Q}}{\hat{Q}^2 - (r + (\xi, A))^2} dr,$$

where $\hat{Q} = \sum_{\tau=1}^n |(\xi, \alpha_j)|\Delta t_j$, and $\hat{\xi}$ is a fixed point on the unit sphere with center at $A$. So for large enough translations $A$ (precisely, if $|r + (\hat{\xi}, A)| > \hat{Q}$), and for an arbitrary small $\sigma > 0$, the measure $\int_{|a| < \rho < \sigma} \mathcal{C}_it(a + A) da$ of the ball we consider is purely imaginary.

Therefore for $\sum_{\tau=1}^n |(\xi, \alpha_j)|\Delta t_j$ bounded one has

$$\lim_{R \to \infty} \int_\zeta \mathcal{C}_it(a) da = 0,$$

where $\zeta$ is any domain that does not intersect the ball of radius $R$ in $n$-dimensional space.

By a similar argument shows the above condition also implies that

$$\lim_{R \to \infty} \int_\zeta \overline{\mathcal{C}_it(a)} da = 0,$$

where $\int_\zeta \overline{\mathcal{C}_it(a)} da$ is positive purely imaginary.

3. The generalized quantum Cauchy measure of cylindrical subsets of a Hilbert space with Borel bases

Notice that (21) implies that for every $n$ the series $\sum_{\tau=1}^n \alpha_{j\tau}^2$ converges for each $j$ (i.e., $\alpha_{j\tau} \in l_2$ for each $j$). Thus $\sum_{\tau=1}^n \alpha_{j\tau}^2 = \int_0^t A_j^2(\sigma) d\sigma$ where $A_j(\sigma) \in L_2(0, t)$.

Therefore condition (21) holds if the integral $\int_0^t \sqrt{\int_0^t A_j^2(\sigma) d\sigma} d\tau$ is defined, and since

$$\int_0^t \sqrt{\int_0^t A_j^2(\sigma) d\sigma} d\tau \leq \sqrt{t} \int_0^t \int_0^t A_j^2(\sigma) d\sigma d\tau = \sqrt{t} \int_0^t \int_0^t A_j^2(\sigma, \tau) d\sigma d\tau,$$

this follows if $A(\sigma, \tau)$ is any element of the Hilbert space on the square.
Therefore one has

\[
\dot{z}(\sigma) = \int_0^t A(\sigma, \tau) d\tau, \quad \dot{z}(\sigma) \in L_2(0, t).
\]

Notice that for every \( \beta_j(\sigma) \in L_2(0, t) \) one has equalities that are functional analogs of (5)

\[
a_j = \int_0^t \alpha_j(\tau) \dot{\alpha}(\tau) d\tau,
\]

where \( \alpha_j = \int_0^t \beta_j(\tau) \dot{\alpha}(\tau) d\tau \) are numbers, and \( \int_0^t \beta_j(\sigma) A(\sigma, \tau) d\sigma = \alpha_j(\tau) \) are arbitrary elements of \( L_2(0, t) \).

Thus \( \dot{z}(\tau) \in L_2(0, t) \) (by the Riesz theorem about linear functionals on \( L_2(0, t) \), see [12], [12]), and so our Hilbert space is the largest space to whose Borel cylindrical sets with generatrices \( \alpha_j(\tau) \in L_2(0, t) \) the measure \( \mathcal{C}_it(a) \) can be extended, see (10).

At that the Lebesgue measure density of the one-dimensional base of a cylindrical subset of \( L_2(0, t) \) with generatrix \( \alpha(\tau) \in L_2(0, t) \) equals \( C_{it} \delta_{[\alpha(\tau)]d\tau(a)} \), see (8).

**Remark.** If \( \mathcal{C}_it(a) \) were a probability pre-measure then the above result would imply that the pre-measure can be extended to all Borel subsets of \( L_2(0, t) \) and our measure is \( \sigma \)-additive.

Let us discuss now the support \([x_\tau]\) of that complex generalized measure of cylindrical subsets of \( L_2(0, t) \).

It is clear that every trajectory \( x(\tau) = \int_0^\tau \dot{x}(s) ds \) from the support \([x_\tau]\) is continuous.

Notice that for elements of that support one has (here we set \( \dot{x}(s) = 0 \) for \( s \neq (0, t) \) and use (21))

\[
|x(\tau + T) - x(\tau)| = \left| \int_0^{\tau + T} \dot{x}(s) ds - \int_\tau^{\tau + T} \dot{x}(s) ds \right| =
\]

\[
= \left| \int_\tau^{\tau + T} \dot{x}(s) ds \right| \leq \sqrt{T} \sqrt{\int_\tau^{\tau + T} \dot{x}^2(s) ds} \leq \sqrt{T} N, \quad N < \infty,
\]

since \( \dot{x}(\tau) \) are uniformly bounded in \( L_2(0, t) \).

Therefore \( \sup_{0 \leq \tau \leq t} \left| x(\tau + T) - x(\tau) \right| \rightarrow 0 \) simultaneously for all \( x(\tau) \in C(0, t) \) in the support. Thus the support \([x_\tau]\) of measures of cylindrical subsets of \( L_2(0, t) \) that correspond to \( C_{it}(x) \), is locally compact in the uniform convergence topology by Arzela’s theorem (see [17]).

**Theorem (II):** The topological space \( X \), on whose dual the generalized pre-measure on \( R^{(n)} \) that corresponds to quantum Cauchy functional \( C_{it}(x) \) is extendable to a measure (pre-measure) of its Borel cylindrical sets, is the real Hilbert space of functions \( \dot{x}(t) \in L_2(0, t) \). The support \([x_\tau]\) of this generalized complex measure is locally compact for the metric of the space of continuous functions \( C(0, t) \).

4. The quantum generalized Cauchy functional and the quantum generalized Cauchy process on the Borel cylindrical subsets of the Hilbert space

Notice that (10) can be rewritten as

\[
\int \mathcal{C}_it(a_1, \ldots, a_n) \varphi(a_1, \ldots, a_n) da = \int (\Pi_{j=1}^n C_{i\Delta t_j}(\Delta x_j)) \varphi(A\Delta x) d\Delta x =
\]
= \int (\Pi^n_1 C_i \Delta x_i (\Delta x_i)) \phi(\Delta x_1, \ldots, \Delta x_n) \Pi^n_1 d\Delta x_i,

where \( \phi(\Delta x) = \varphi(A \Delta x) \), so \( \phi(\Delta x) \in K^{(n)} \) is any bump function.

Passing in (25) to the limit for \( n \to \infty \) (\( \max \Delta t_j \to 0 \)), we get

\[ \int_{[x_\tau]} (\Pi^\infty_\tau C_{idr}(dx_\tau)) \varphi(\ldots, dx_\tau, \ldots) \Pi^\infty_\tau dx_\tau = \int_{[x_\tau]} C_{it}[dx_\tau] \varphi[dx_\tau] \Pi^\infty_\tau dx_\tau, \]

or, in short, \( \Pi^\infty_\tau C_{idr}(dx_\tau) = C_{it}[dx_\tau] \).

Here \( C_{it}[dx_\tau] \) is the Lebesque density of the measure of a concrete element \( x_\tau \) of the support \( [x_\tau] \) (i.e., of the process \( x_\tau \)), and the support \( [x_\tau] \) has full measure \( \int_{[x_\tau]} C_{it}[dx_\tau] \Pi^\infty_\tau dx_\tau = 1 \).

At that \( C_{it}(x_1, \ldots, x_n) \) is the same generalized functional on the set of those elements of the support that take value \( x_1, \ldots, x_n \) at fixed moments of time \( t_1, \ldots, t_n \), and it is the Lebesque density of the measure of that set.

Consider (27); using (6) we get

\[ \int_{[x_\tau]} (\Pi^\infty_\tau C_{idr}(dx_\tau)) \varphi(\ldots, dx_\tau, \ldots) \Pi^\infty_\tau dx_\tau = \int_{[x_\tau]} C_{it}(x) \varphi(x) dx, \]

or, in short, \( C_{it} = \Pi^\infty_\tau C_{idr} \).

This equality describes the fundamental solution of equation (2) (or (4) in the space case) as a generalized function via the functional integral ("path integral") of the generalized Cauchy functional with the Lebesque density \( C_{it}[dx_\tau] \).

Its generalization to the space case is evident.

**Theorem (III):** The retarded Green's function \( C_{it}(x) \), viewed as a generalized function on \( K^{(3)} \), yields the measure of the generalized Cauchy process with Lebesque density \( \Pi^\infty_\tau C_{idr}(dx_\tau) = C_{it}[dx_\tau] \) on the Borel bases of cylindrical subsets of the real Hilbert space, its support \( [x_\tau] \) is locally compact in \( C(0,t) \) with derivatives \( \dot{x}(\tau) \in L_2(0,t) \).

The result remains true in the space situation in the evident manner.

5. **The Modified Quantum Cauchy Functional and the Modified Generalized Quantum Cauchy Measure**

In [16] we found the next position coordinate presentation of modified quantum Cauchy functional \( C_{it}^m(x) \): this is a spherical symmetric functional on \( K^{(3)} \)

\[ C_{it}^m(x) = C_{it}(x) B_t(x), \]
where $C_{it}(x)$ is as in (3) and

$$B_t(x) = iml^4 \frac{\partial}{\partial(l^2)} \frac{K_1(iml)}{l}$$

where $l = \sqrt{t^2 - r^2}$ and $K_1(z)$ is the Macdonald function (see [10], 3.7, formula (6)).

We show in [16] that $B_t(x)$ is an infinitely differentiable nowhere vanishing function such that $B_t(x)|_{r=1} = 1$.

Therefore

$$\int C_{it}^m(x)\varphi(x)dx = \int C_{it}(x)\Phi_t(x)dx,$$

where $\Phi_t(x) = B_t(x)\varphi(x) \in K^{(3)}$ is an arbitrary bump function and $t$ is a parameter (see [2]). Thus the generalized function $C_{it}^m(x)$ has same properties as $C_{it}(x)$, so we have

**Theorem (IV):** The retarded Green’s function $C_{it}^m(x)$ viewed as a generalized function on $K^{(3)}$ yields a generalized measure of modified Cauchy process with Lebesque’s density $\mathcal{C}_{it}^m[dx] = \Pi_{t=0}^\infty \mathcal{C}_{it}(dx)$. The supports of this measure and the measure corresponding to the generalized quantum Cauchy functional coincide, hence they are mutually absolutely continuous.

It is clear that Green’s functions $C_{it}^m(x)$ and $\overline{C}_{it}^m(x)$ can be represented by the corresponding functional integrals (“path integrals”, see (28)).

6. The fundamental solutions of Dirac equations for the free electron and of Maxwell equations for the Einstein photon.

In [16] we found a relation between the fundamental solutions of the Dirac electron equation in position coordinates $D_{it}^m(x)$ and its Foldy-Wouthuysen presentation $D_{it}^{mF}(x)$ (see [6])

$$D_{it}^m(x) = T^m(x) * D_{it}^{mF}(x) * T^m(x).$$

Here $D_{it}^{mF}(x)$ yields a matrix-valued generalized quantum process having the same properties as $C_{it}^m(x)$, see [16]. Therefore the support of the (diagonal)matrix-valued measure that corresponds to the Foldy-Wouthuysen presentation $D_{it}^{mF}(x)$ of the fundamental solution of the Dirac equation is equal to the support of the scalar measure that corresponds to $C_{it}^m(x)$, viewed as functionals on the Borel cylindrical sets of the Hilbert space.

Recall now that the retarded Green’s function $D_{it}^m(x)$ viewed as a functional on $K^{(3)}$ can be obtained also from the retarded Green’s function of the Klein-Gordon equation $G_t(x)$ (Pauli-Jordan formulas, see [8]). Namely

$$D_{it}^m(x) = \gamma^0(\gamma^0 \frac{\partial}{\partial t} D_{it}^m(x) + (\gamma, \nabla) + im)G_t(x),$$

$$G_t(x) = \frac{1}{2\pi} \delta(t^2 - r^2) - \frac{m}{4\pi} \theta(t - r) \frac{J_1(m\sqrt{t^2 - r^2})}{\sqrt{t^2 - r^2}};$$
here $\theta$ is the Heaviside step function and $J_1$ is the Bessel function. These formulas amount to formula (30) that we use; this can be easily seen by passing to the Fourier transforms. We also see that $D^m_{\alpha}(x)$ is a finite functional, as opposed to $D^m_{\alpha F}(x)$. At that $D^m_{\alpha}(x)$ and $D^m_{\alpha F}(x)$ are mutually isomorphic.

Therefore formula (30) should be interpreted as a result of passage from the Foldy-Wouthuysen variables (where Green’s function $D^m_{\alpha F}(x)$ can be written as a mathematically well-defined functional integral with support $[x_\tau]$ locally compact in $L_2(0,t)$) to the Dirac equation Green’s function $D^m_{\alpha}(x)$ that vanishes away from the 3-sphere $r = t$ (the light cone) hence representable by a functional integral with support $\{x_\tau\}$ compact in $C[0,t]$. And every element of the supports $[x_\tau]$ and $\{x_\tau\}$ satisfies condition $\dot{x}_\tau \in L_2(0,t)$. Therefore we have deduced

**Theorem (V):** The retarded Green’s function $D^m_{\alpha}(x)$ of the Dirac equation, viewed as a complex matrix-valued generalized function on $K^{(3)}$, yields the measure of the generalized quantum process Dirac-Cauchy process with Lebesque’s density

$$ D^m_{\alpha}(dx_\tau) = \Pi_{\tau=0}^t T^m * D^m_{\alpha F}(dx_\tau) * T^m $$

on the Borel cylindrical subsets of the Hilbert space with the support $\{x_\tau\}$ compact in $C[0,t]$ and the derivatives $\dot{x}(\tau) \in L_2(0,t)$.

It is known that the system of Maxwell’s equations for the photon field (see [1], [16])

$$ \frac{\partial}{\partial t} E = \text{rot} H, \quad \text{div} E = 0, \quad \frac{\partial}{\partial t} H = -\text{rot} E, \quad \text{div} H = 0 $$

splits, using Majorana’s coordinates, into two independent subsystems that are complex conjugate. Each of them can be made diagonal $M^F_{\alpha}(x)$ and $\tilde{M}^F_{\alpha}(x)$ using a unitary operator $Q(x)$ (the Foldy-Wouthuysen type presentation of the fundamental solution of the Maxwell equation, see [16]).

Thus, recovering the fundamental solution of the Maxwell equation in Majorana coordinates, we get

$$ M_\tau(x) = Q^+(x) * M^F_{\alpha}(x) * Q(x) \times Q^+(x) * \tilde{M}^F_{\alpha}(x) * Q(x), \tag{31} $$

where $\times$ is the direct product of $3 \times 3$-matrices. Here $M^F_{\alpha}(x) \times \tilde{M}^F_{\alpha}(x)$ yields also a (diagonal)matrix-valued generalized quantum process with the corresponding Lebesque’s density on the bases of cylindrical subsets in $L_2(0,t)$ and support $\{x_\tau\}$. Hence, due to the isomorphism property of the Foldy-Wouthuysen transform of solutions of the Maxwell equation, the generalized matrix-valued measure of cylindrical subsets of the Hilbert space that corresponds to $M_\tau(x)$, due to the argument similar to the one above in case of the Dirac equation, has support $\{x_\tau\}$ compact in the metric of $C(0,t)$ since the speed of light $c$ is constant. We get

**Theorem (VI):** The retarded Green’s function $M_\tau(x)$ of the Maxwell equation, viewed as a matrix-valued complex generalized function on $K^{(3)}$, yields the measure of the generalized quantum Cauchy-Maxwell process with Lebesque’s density

$$ M_\tau(dx_\tau) = \Pi_{\tau=0}^t Q^+ * M^F_{\alpha}(dx_\tau) * Q \times Q^+ * \tilde{M}^F_{\alpha}(dx_\tau) * Q $$

on the Borel cylindrical sets of the real Hilbert space with support $\{x_\tau\}$ compact in $C(0,t)$ and derivatives $\dot{x}_\tau \in L_2(0,t)$.

We will not write down our matrix-valued complex Green’s functions $D^m_{\alpha}(x)$ and $M_\tau(x)$ as functional integrals (“path integrals”); they have similar structure and properties to the presentation of Green’s function $C_\alpha(x)$ given in (28).
It is important to notice that, due to nondegeneracy of the unitary operators $Q(x), T(x), T^m(x)$, the retarded Green's functions of Dirac's particles and photons can be identified by nondegenerate transformations regardless of the fact that these are fermions and bosons.

Following Schrödinger, we point out a peculiarity of the Dirac electron movement (see [5], [15]). Namely, one can measure exactly just one Cartesian component of the velocity, so one may doubt the existence of electron's space trajectories that we have discussed above.

In fact, in the support of the generalized measure that corresponds to $D^m_t(x)$ one has $\frac{dx_\tau}{d\tau} \in L_2(0,t)$, so the velocity of the electron on the support of the measure is not continuous, hence has no exact values at any moment of time (for the 1-dimensional case as well). Thus the peculiarity of Dirac's electron movement observed by Schrödinger does not contradict the existence of the measure on the space of trajectories for the trajectories in the support do not have exact values of velocity.

Therefore the spontaneous micro-movement of Dirac's electron with indeterminately high speed, called Zitterbewegung (see [5], [15]) can be seen as a corollary of the movement along the trajectories in such a support.

**Digression.** Zitterbewegung (literally “twitching movement”), an irregular movement of Dirac’s electron with speed faster than light, was first pointed out by Schrödinger [5].

We point out the general nature of that movement: it occurs for all relativistic quantum particles, both photons (bosons) and Dirac's particles (fermions), and is related to unified quantum micro-structure of space-time characterized by single support $\{x_\tau\}$ and governs their evolution regardless of mass or spin.

Notice that the evolution of state of relativistic quantum particles that occurs due to that micro-structure of space-time, is purely kinematic which amounts to the Dirac or Maxwell equation evolution.

The found alternative description of evolution of relativistic quantum particles by accounting their trajectories (with complex weight) together with solving the corresponding wave equations supports in a deeper way the wave-particle duality of de Broglie in quantum mechanics.

Consider the correspondence principle for the Dirac electron using the “path integral” description of the fundamental solution of the Dirac equation (cf. (28)):

\[
\int D^m_t(x)\varphi(x)dx = \int_{\{x_\tau\}} (\Pi^{m}_{\tau=0} D^m_{d\tau}(dx_\tau))\varphi(\int_0^t dx_\tau)\Pi^{m}_{\tau=0}dx_\tau,
\]

or simply $D^m_t = \Pi^{m}_{\tau=0} * D^m_{d\tau}$.

It is easy to see (see [16]) that (32) means that

\[
\int D^m_t(x)\varphi(x)dx = \int T^m(\alpha)D^{mF}_t(x)T^m(\beta)\varphi(\alpha + x + \beta)d\alpha dx d\beta,
\]

or $D^m_t = T^m * D^{mF} * T^m$, where

\[
\int D^{mF}_t(x)\varphi(x)dx = \int_{\{x_\tau\}} (\Pi^{mF}_{\tau=0} D^{mF}_{d\tau}(dx_\tau))\varphi(\int_0^t dx_\tau)\Pi^{mF}_{\tau=0}dx_\tau,
\]
or $D_{mF}^t = \Pi_{t=0}^t \AST D_{mF}^\tau$.

Since $m = m_0 h^{-1}$ (here $m_0$ is the invariant mass of the electron, see [15]; we still assume that $c = 1$), one has $\lim_{h \to 0} T^m(x) = \delta(x)$, see [16]. Therefore $D_{mF}^m|_{h \to 0} \simeq D_{mF}^t$, so the quasi-classical approximation of a solution of the Dirac equation equals to its quasi-classical approximation in the Foldy-Wouthuysen coordinates.

Since, by Theorem 4 in [16], one has $C_{\Sigma t}^m(x) = \frac{\delta S_t}{4\pi l^2} + \frac{i}{\pi^2} \frac{\partial}{\partial(l_t^2)} \frac{-tmK_1(iml_t)}{l_t}$, where $l_t = \sqrt{t^2 - r^2}$ is the interval, we see, using the asymptotic of the Macdonald function $K_1(z) \simeq \sqrt{\frac{\pi}{z}} \exp(-z)$ (see [10], 7.23, formula (1)) and (33), that every element of the diagonal matrix in (30) can be written in quasi-classical approximation (up to a regular in $h$ factor under the functional integral sign) as

$$\int_{\{x, t\}} \exp(\pm \int_0^t m_0 dl_{\tau}) \varphi(\int_0^t dx_{\tau}) \Pi_{\tau=0}^t dx_{\tau},$$

where $\int_0^t m_0 dl_{\tau}$ is the action of free relativistic particle on any trajectory $\{x, t\}$ inside the light cone. So for $h \to 0$ the principle of stationary action distinguishes, as in Feynman [9], the classical trajectory among all other trajectories in the support. Accordingly, the support of Green’s function of the electron degenerates into a single trajectory of the free relativistic particle. And Zitterbewegung disappears.

Notice also that in quantum relativistic case, as follows from (32), the dependence of the averaged functional from the free particle action is exponential only in quasi-classical approximation, as opposed to non-relativistic situation (see [9]). This is a discriminating trait of the quantization of relativistic particles.

The evolution of state of quantum relativistic particles can be presented as evolution of their scalar description (i.e., the Foldy-Wouthuysen presentation) $D_{mF}^t(x)$ in case of electron and $M_{F}^t(x)$ in case of photon; then, due to interaction of operators $T^m(x)$ and $G(x)$ in the first and the last moments of time respectively, the full Green’s functions of electron $D_{mF}^t(x)$ and of photon $M_{F}^t(x)$ appear, see (30), (31).

**Conclusion**

The present work shows special role played by the Cauchy processes in imaginary time (quantum Cauchy processes) in the analysis of solutions of the quantum mechanics equations, as well as, possibly, in formulation and solving of new problems there.

Using these processes one finds a fundamental connection between solutions of the Maxwell and Dirac equations, where there appear unitary transformations that interchange bosons and fermions which may be related to a possibility of construction of the F. A. Berezin superinteraction theory (see [4]).

We also find that the time derivative of the support of the quantum Cauchy processes (i.e., the velocity of all relativistic quantum particles) belongs to a Hilbert space. This observation might serve as a base for explanation of the known paradox of Einstein-Podolsky-Rosen, see [13], and leads to an adequate mathematical description of such facts as collapse of the wave function at the moment of measurement, entangled states, etc.
We explain a special role of these processes for the analysis of the passage from quantum relativistic problems to their classical relativistic and non-relativistic versions.

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