Statistical Mechanics of MAP Estimation: General Replica Ansatz

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Abstract—The large-system performance of maximum-a-posterior estimation is studied considering a general distortion function when the observation vector is received through a linear system with additive white Gaussian noise. The analysis considers the system matrix to be chosen from the large class of rotationally invariant random matrices. We take a statistical mechanical approach by introducing a spin glass corresponding to the estimator, and employing the replica method for the large-system analysis. In contrast to earlier replica based studies, our analysis evaluates the general replica ansatz of the corresponding spin glass and determines the asymptotic distortion of the estimator for any structure of the replica correlation matrix. Consequently, the replica symmetric as well as the replica symmetry breaking ansatz with steps of breaking is deduced from the given general replica ansatz. The generality of our distortion function lets us derive a more general form of the maximum-a-posterior decoupling principle. Based on the general replica ansatz, we show that for any structure of the replica correlation matrix, the vector-valued system decouples into a bank of equivalent decoupled scalar systems followed by maximum-a-posterior estimators. The structure of the decoupled system is further studied under both the replica symmetry and the replica symmetry breaking assumptions. For steps of symmetry breaking, the decoupled system is found to be an additive system with a non-Gaussian noise term given as the sum of an independent Gaussian random variable with non-Gaussian impairment terms which depend on the input symbol. The general decoupling property of the maximum-a-posterior estimator leads to the idea of a replica simulator which represents the replica ansatz through the state evolution of a transition system described by its corresponding decoupled system. As an application of our study, we investigate large compressive sensing systems by considering the \ell_p norm minimization recovery schemes. Our numerical investigations show that the replica symmetric ansatz for \ell_p norm recovery fails to give an accurate approximation of the mean square error as the compression rate grows, and therefore, the replica symmetry breaking ansätze are needed in order to assess the performance precisely.

Index Terms—Maximum-a-posterior estimation, linear vector channel, decoupling principle, equivalent single-user system, compressive sensing, zero norm, replica method, statistical physics, replica symmetry breaking, replica simulator.

I. INTRODUCTION

Consider a vector-valued additive white Gaussian noise (AWGN) channel specified by

\[ y = Ax + z \]

(1)

where the independent and identically distributed (i.i.d.) source vector \( x_{n \times 1} \), with components in a support set \( X \subseteq \mathbb{R} \), is measured by the random system matrix \( A_{k \times n} \in \mathbb{A}^{k \times n} \), with \( A \subseteq \mathbb{R} \), and corrupted by the i.i.d. zero-mean Gaussian noise vector \( z_{1 \times 1} \), with variance \( \lambda_0 \), i.e., \( z \sim \mathcal{N}(0, \lambda_0 I) \). The source vector can be estimated from the observation vector \( y_{1 \times 1} \) using a maximum-a-posterior (MAP) estimator. For a given system matrix \( A \), the estimator maps the observation vector to the estimated vector \( \hat{x}_{n \times 1} \in \mathbb{X}^n \) via the estimation function \( g(\cdot | A) \) defined as

\[ g(y|A) = \underset{v \in \mathbb{X}^n}{\arg \min} \left[ \frac{1}{2\lambda} \|y - Av\|^2 + u(v) \right] \]

(2)

for some utility function \( u(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^+ \) and estimation parameter \( \lambda \in \mathbb{R}^+ \). In (2), \( \| \cdot \| \) denotes the Euclidean norm, and it is assumed that the minimum is not degenerate so that \( g(\cdot | A) \) is well-defined, at least for almost all \( y \) and \( A \).

In order to analyze the performance of this system in the large-system limit, i.e., \( k, n \uparrow \infty \), one considers a general distortion function \( d(\cdot, \cdot) : \mathbb{X} \times \mathbb{X} \mapsto \mathbb{R} \). For some choices of \( d(\cdot, \cdot) \), the distortion function determines the distance between the source and estimated vector, e.g. \( d(\hat{x}; x) = |\hat{x} - x|^2 \). The asymptotic distortion

\[ D = \lim_{n \uparrow \infty} \frac{1}{n} \sum_{i=1}^{n} d(\hat{x}_i; x_i), \]

(3)

then, expresses the large-system performance with respect to (w.r.t.) the distortion function \( d(\cdot, \cdot) \). The performance analysis of the estimator requires (2) to be explicitly computed, and then \( \hat{x} = g(y|A) \) substituted in the distortion function. This task is however not trivial for many choices of the utility function and source support \( \mathbb{X} \), and becomes unfeasible as \( n \) grows large. As basic analytic tools fail, we take a statistical mechanical approach and investigate the large-system performance by studying the macroscopic parameters of the corresponding spin glass. This approach enables us to use the
replica method which has been developed in the context of statistical mechanics.

A. Corresponding Spin Glass

Consider a thermodynamic system which consists of \( n \) particles with each having a microscopic parameter \( v_i \in \mathbb{V} \). The vector \( v_{n \times 1} = [v_1, \ldots, v_n]^T \), collecting the microscopic parameters, represents the microscopic state of the system and is called the microstate. The main goal of statistical mechanics is to excavate the macroscopic parameters of the system, such as energy and entropy through the analysis of the microstate in the thermodynamic limit, i.e., \( n \uparrow \infty \). Due to the large dimension of the system, statistical mechanics proposes a stochastic approach in which the microstate is supposed to be randomly distributed over the support \( \mathbb{V}^n \) due to some distribution \( p_v \). For this system, the Hamiltonian \( \mathcal{E}(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^+ \) assigns to each realization of the microstate a non-negative energy level, and

\[
\mathcal{H} := -\frac{1}{n} \mathbb{E}_{p_v} \log p_v
\]

denotes the normalized entropy\(^1\) of the system. The normalized free energy of the thermodynamic system at the inverse temperature \( \beta \) is then defined as

\[
\mathcal{F}(\beta) := -\frac{1}{n} \mathbb{E}_{p_v} \mathcal{E}(v) - \frac{1}{\beta} \mathcal{H}.
\]

The second law of thermodynamics states that the microstate at thermal equilibrium takes its distribution such that the free energy meets its minimum. Thus, the microstate’s distribution at thermal equilibrium reads

\[
p_v^\beta(v) = \left[ Z(\beta) \right]^{-1} e^{-\beta \mathcal{E}(v)}
\]

where \( Z(\beta) \) is a normalization factor referred to as the partition function, and the superscript \( \beta \) indicates the distribution’s dependence on the inverse temperature. The distribution in (6) is known as the Boltzmann-Gibbs distribution and covers many distributions on \( \mathbb{V}^n \) by specifying \( \mathcal{E}(\cdot) \) and \( \beta \) correspondingly. Substituting the Boltzmann-Gibbs distribution in (5), the normalized free energy at thermal equilibrium and inverse temperature \( \beta \) reads

\[
\mathcal{F}(\beta) = -\frac{1}{\beta n} \log Z(\beta).
\]

The average normalized energy and the normalized entropy of the system at thermal equilibrium are further determined by

\[
\bar{\mathcal{E}}(\beta) := \frac{1}{n} \mathbb{E}_{p_v^\beta} \mathcal{E}(v),
\]

\[
\bar{\mathcal{H}}(\beta) := -\frac{1}{n} \mathbb{E}_{p_v^\beta} \log p_v^\beta(v),
\]

which can be calculated in terms of the free energy via

\[
\bar{\mathcal{E}}(\beta) = \frac{d}{d\beta} [\beta \mathcal{F}(\beta)],
\]

\[
\bar{\mathcal{H}}(\beta) = \beta^2 \frac{d}{d\beta} [\mathcal{F}(\beta)].
\]

In spin glasses [3], the Hamiltonian assigns the energy levels randomly using some randomizer \( Q \) resulting from random interaction coefficients. In fact, each realization of \( Q \) specifies a thermodynamic system represented by the deterministic Hamiltonian \( \mathcal{E}(\cdot|Q) \). In statistical mechanics, \( Q \) is known to have quenched randomness. This means that \( Q \) is generated randomly once and remains unchanged afterwards.\(^2\) The analysis of spin glasses takes similar steps as those taken for analysis of deterministic thermodynamic systems considering the given realization of the quenched randomizer. Hence, as the system converges to its thermal equilibrium at the inverse temperature \( \beta \), the microstate’s conditional distribution given \( Q \), i.e., \( p_v^\beta|Q \), is a Boltzmann-Gibbs distribution specified by \( \mathcal{E}(\cdot|Q) \). Consequently, the normalized free energy reads

\[
\mathcal{F}(\beta|Q) = -\frac{1}{\beta n} \log Z(\beta|Q).
\]

where \( Z(\beta|Q) \) is the partition function w.r.t. the Hamiltonian \( \mathcal{E}(\cdot|Q) \).

Here, the normalized free energy, as well as other macroscopic parameters of the system, is random due to the quenched randomizer. However, the physical intuition behind the analyses suggests that these random macroscopic parameters converge to deterministic values in the thermodynamic limit. This property is known as the self averaging property and has been rigorously justified for some particular Hamiltonians, e.g., [4]–[7]. Nevertheless, in cases where a mathematical proof is still lacking, the property is supposed to hold during the analysis. According to the self averaging property, the normalized free energy of a spin glass for almost all realizations of the quenched randomizer \( Q \) converges to its expected value in the thermodynamic limit.

As mentioned earlier, the MAP estimator in (2) can be investigated using a corresponding spin glass. To see that, consider a spin glass whose microstate is taken from \( \mathbb{X}^n \), and whose Hamiltonian is defined as

\[
\mathcal{E}(v|y, A) = \frac{1}{2\lambda} \| y - Av \|^2 + u(v).
\]

Here, the system matrix \( A \) and the observation vector \( y \) are the quenched randomizers of the spin glass. In this case, for given realizations of \( A \) and \( y \), the conditional distribution of the microstate is given by

\[
p_v^\beta_{y|A}(v|y, A) = \left[ Z(\beta|y, A) \right]^{-1} e^{-\beta \mathcal{E}(v|y, A)}.
\]

Taking the limit when \( \beta \uparrow \infty \) and using the Laplace method of integration [8], the zero temperature distribution, under the assumption that the minimizer is unique, reduces to

\[
p_v^\infty_{y|A}(v|y, A) = \left[ \mathcal{E}(\cdot|y, A) \right] \left[ g(y|A) \right],
\]

where \( 1\{\cdot\} \) denotes the indicator function, and \( g(\cdot|A) \) is defined in (2). From (13b), it is observed that the microstate of this spin glass converges to the estimated vector of the MAP

\(^2\)Note that the order of randomness in a quenched randomizer is different from randomness in the microstate. In statistical mechanics, the microstate is often said to be an annealed random variable.
estimator, i.e., \( \hat{x} = g(y|A) \), in the zero temperature limit. Invoking this connection, we study the corresponding spin glass instead of the MAP estimator. We represent the input-output distortion of the system w.r.t. a general distortion function as a macroscopic parameter of the spin glass. Consequently, the replica method developed in statistical mechanics is employed to determine the defined macroscopic parameter of the corresponding spin glass. The replica method is a nonrigorous but effective method developed in the physics literature to study spin glasses. Although the method lacks rigorous mathematical proof in some particular parts, it has been widely accepted as an analysis tool and utilized to investigate a variety of problems in applied mathematics, information processing, and coding [9]–[12].

The use of the replica method for studying multiuser estimators goes back to [13] where Tanaka determined the asymptotic bit error rate of marginal-posterior-mode (MPM) estimators by employing the replica method. The study demonstrated interesting large-system properties of multiuser estimators. As a result, the statistical mechanical approach received more attention in the context of multiuser systems. This approach was employed in the literature to study multiple estimation problems in large vector-valued linear systems, e.g., [14]–[16]. The method was also utilized to analyze the asymptotic properties of multiple-input multiple-output (MIMO) systems in [17] considering an approach similar to [13]. Regarding multiuser estimators, the earlier studies mainly considered the cases in which the entries of the source vector are binary or Gaussian random variables. The results were later extended to a general source distribution in [16].

The statistical mechanical approach was further employed to address mathematically similar problems in vector precoding, compressive sensing, and analysis of superposition codes [18]–[20], to name just a few. Despite the fact that the replica method lacks mathematical rigor, a body of work, such as [7], [21]–[27], has shown the validity of several replica-based results in the literature, e.g., Tanaka’s formula in [13], using some alternative rigorous approaches. We later discuss these rigorous results with more details by invoking the literature of compressive sensing.

### B. Decoupling Principle

Considering the MAP estimator defined in (2), the entries of the estimated vector \( \hat{x} \) are correlated in general, since the system matrix couples the entries of \( x \) linearly, and \( g(x|A) \) performs several nonlinear operations on \( y \). In large-system performance analysis, the marginal joint distribution of two corresponding input-output entries \( x_j \) and \( \hat{x}_j \), \( 1 \leq j \leq n \), is of interest. To clarify our point, consider the case in which a linear estimator is employed instead of (2), i.e., \( \hat{x} = G^T y \). Denote the matrices \( A \) and \( G \) as \( A = [a_1, \ldots, a_n] \) and \( G = [g_1, \ldots, g_n] \), respectively with \( a_i \) and \( g_j \) being \( k \times 1 \) vectors for \( 1 \leq i \leq n \). Therefore, \( \hat{x}_j \) is written as

\[
\hat{x}_j = g_j^T y = g_j^T \left[ \sum_{i=1}^{n} x_i a_i + z \right] = (g_j^T a_j) x_j + \sum_{i=1, i \neq j}^{n} (g_j^T a_i) x_i + g_j^T z. \tag{14b}
\]

Here, the right hand side (r.h.s.) of (14b) can be interpreted as the linear estimation of a single-user system indexed by \( j \) in which the symbol \( x_j \) is corrupted by an additive impairment given by the last two summands in the r.h.s. of (14b). The impairment term is not necessarily independent and Gaussian. For some classes of matrix ensembles, and under a set of assumptions, it is shown that the dependency of the performance of the derived single-user systems on index \( j \) vanishes, and the distribution of the impairment terms converges to a Gaussian distribution in the large-system limit [28]. As a result, one can assume the vector-valued system described by (1) followed by the linear estimator \( G \) to be a set on \( n \) additive scalar systems with Gaussian noise which have been employed in parallel. In other words, the vector system can be considered to decouple into a set of similar scalar systems. Each of them relates an input entry \( x_j \) to its corresponding estimated one \( \hat{x}_j \). This asymptotic property of the estimator is referred to as the decoupling property and can be investigated through the large-system performance analysis.

The decoupling property was first studied for linear estimators. Tse and Hanly noticed this property while they were determining the multiuser efficiency of several linear multiuser estimators in the large-system limit [29]. They showed that for an i.i.d. system matrix, the effect of impairment is similar to the effect of some modified Gaussian noise when the dimension tends to infinity. This asymptotic property was then investigated further by studying the asymptotics of different linear receivers and their large-system distributions [30], [31]. In an independent work, Verdú and Shamai also studied the minimum mean square error (MMSE) estimator and showed that the conditional output distribution is asymptotically Gaussian [32]. In [33], the authors studied the asymptotics of the impairment term when a family of linear estimators is employed and proved that it converges in distribution to a Gaussian random variable. The latter result was further extended to a larger class of linear estimators in [28].

Regarding linear estimators, the main analytical tool is random matrix theory [34], [35]. In fact, invoking properties of large random matrices and the central limit theorem, the decoupling property is rigorously proved, e.g., [36], [37]. These tools however fail for nonlinear estimators as the source symbol and impairment term do not decouple linearly due to nonlinear operations at the estimators. In [38], Müller and Gerstacker employed the replica method and studied the capacity loss due to the separation of detection and decoding. The authors showed that the additive decoupling of the spectral efficiency, reported in [37] for Gaussian inputs, also holds for binary inputs. As a result, it was conjectured that regardless of input distribution and linearity, the spectral efficiency always decouples in an additive form [39]. In [16], Guo and Verdú justified this conjecture for a family of nonlinear MMSE estimators, and showed that for an i.i.d. system matrix, the estimator decouples into a bank of single-user MMSE estimators under the replica symmetry (RS) assumption. In [40], Rangan et al. studied the asymptotic performance of a class of MAP estimators. Using standard large deviation arguments, the authors represented the MAP estimator as the limit of an indexed MMSE estimators’ sequence.
Consequently, they determined the estimator’s asymptotics employing the results from [16] and justified the decoupling property of MAP estimators under the RS ansatz for an i.i.d. $A$.

Regarding the decoupling property of MAP estimators, there are still two main issues which need further investigations: 1) cases in which the system matrix is not i.i.d., and 2) analysis of the estimator under replica symmetry breaking (RSB). The first issue was partially addressed in [41] where, under the RS assumption, the authors studied the asymptotics of a MAP estimator employed to recover the support of a source vector from observations received through noisy sparse random measurements. They considered a model in which a sparse Gaussian source vector is first randomly measured by a square matrix $V$, and then, the measurements are sparsely sampled by a diagonal matrix $B$ whose non-zero entries are i.i.d. Bernoulli random variables. For this setup, the input-output information rate and support recovery error rate were investigated assuming measuring matrix $V$ to belong to a larger set of matrix ensembles. These results, moreover, could address the decoupling property of the considered setting. Although the ensemble of system matrices is broadened in [41], it cannot be considered as a complete generalization of the property presented in [16] and [40], since it is restricted to cases with a sparse Gaussian source and loading factors less than one, i.e., $k/n < 1$.

Vehkaperä et al. also tried to investigate the first issue for a similar formulation in compressive sensing [42]. In fact, the authors considered a linear sensing model as in (1) for rotationally invariant random matrices and under RS determined the asymptotic mean squared error (MSE) for several regularized least-squares recovery schemes which are equivalently represented by the formulation in (2). The large-system results in [42], however, did not address the asymptotic marginal joint input-output distribution, and the emphasis was on the MSE.

Regarding the second issue, the MAP estimator has not yet been investigated under RSB ansätze in the literature. Nevertheless, the necessity of such investigations was mentioned for various similar settings in the literature; see for example [43]–[45]. In [43], the performance of code division multiple access (CDMA) detectors was investigated by studying both the RS and one-step RSB ansätze and the impact of symmetry breaking onto the results for low noise scenarios were discussed. The authors in [45] further studied the performance of vector precoding under both RS and RSB and showed that the analysis under RS yields a significantly loose bound on the true performance. However, the replica ansatz with one-step of RSB was shown to lead to a tighter bound which is consistent with rigorous performance bounds available in the literature. A similar observation was recently reported for the problem of least-square-error precoding in [46], [47]. The replica analyses of compressive sensing in [44], [48], moreover, discussed the necessity of investigating the performance of $\ell_p$ minimization recovery schemes under RSB ansätze for some choices of $p$.

C. Compressive Sensing

The MAP estimation of a source vector from a set of noisy linear observations arises in several applications, such as MIMO and sampling systems. To address one, we consider large compressive sensing [49]–[51] systems and employ our asymptotic results to analyze the large-system performance. In context of compressive sensing, (1) represents a noisy sampling system in which the source vector $x$ is being sampled linearly via $A$ and corrupted by additive noise $z$. In the noise-free case, i.e., $\lambda_0 = 0$, the source vector $x$ is exactly recovered from the observation vector $y$, if the number of observations $k$ is as large as the source length $n$ and the sampling matrix $A$ is full rank. As the number of observations reduces, possible answers to the exact reconstruction problem may increase depending on the source support $X$. Therefore, the recovered source vector from the observation vector is not necessarily unique. In this case, one needs to enforce some extra constraints on the properties of the source vector in order to recover it uniquely among all possible solutions. In compressive sensing, the source vector is supposed to be sparse, i.e., a certain fraction of entries are zero. This property of the source imposes an extra constraint on the solution which allows for exact recovery in cases with $k < n$.

In fact, in this case, one should find a solution to $y = Av$ over

$$S = \{ v_{n \times 1} \in X^n : \| v \|_0 < an \}$$

where $\| \cdot \|_0$ denotes the $\ell_0$ norm defined as

$$\| v \|_0 := \sum_{i=1}^{n} I(v_i \neq 0),$$

and $a \leq 1$ is the source’s sparsity factor defined as the fraction of non-zero entries. Depending on $A$ and $X$, the latter problem can have a unique solution even for $k < n$ [52]–[54]. Searching for this solution optimally over $S$ is however an NP-hard problem and therefore intractable. The main goal in noise-free compressive sensing is to study feasible reconstruction schemes and derive tight bounds on the sufficient compression rate, i.e., $k/n$, for exact source recovery via these schemes.

In noisy sampling systems, exact recovery is only possible for some particular choices of $X$. Nevertheless, considering either cases in which exact recovery is not possible or choices of $X$ for which the source vector can be exactly recovered from noisy observations, the recovery approaches in these sensing systems need to take the impact of noise into account. The classical strategy in this case is to find a vector in $S$ such that the recovery distortion is small. Consequently, a recovery scheme for noisy sensing based on the $\ell_0$ norm is given by

$$\hat{x} = \arg\min_{v \in \mathbb{X}^n} \left[ \frac{1}{2\lambda} \| y - Av \|^2 + \| v \|_0 \right]$$

which is the MAP estimator defined in (2) with $u(v) = \| v \|_0$. It is straightforward to show that for $\lambda_0 = 0$, i.e., zero noise variance, (16) reduces to optimal noise-free recovery as $\lambda \downarrow 0$. Similar to the noise-free case, (16) results in a NP-hard

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3It means that the entries of the source vector are of the form $c_i b_i$ where $x_i$ and $b_i$ are Gaussian and Bernoulli random variables, respectively.

4The class of rotationally invariant random matrices is precisely defined later throughout the problem formulation.
The performance is then calculated by taking the expectation over the output of a reconstruction scheme is analyzed by determining the probability of exact recovery with high probability [60]. In this case, the performance of a reconstruction scheme is analyzed by determining the considered performance metric, e.g., MSE and probability of exact recovery in the noisy and noise-free case, respectively, for a given realization of the sensing matrix. The average performance is then calculated by taking the expectation over the matrix distribution. Comparing (16) with (2), one can utilize the MAP formulation, illustrated at the beginning of the section, to study the large-system performance of several reconstruction schemes. This similarity was considered in a series of papers, e.g., [19], [40], and therefore, earlier replica results were employed to study compressive sensing systems.

The extension of analyses from the context of multiuser estimation has the disadvantage that the assumed sampling settings were limited to those setups which are consistent with the estimation problems in the literature. However, compressive sensing systems might require a wider set of assumptions, and thus, a large class of settings could not be addressed by earlier investigations. As a result, a body of work deviated from this approach and applied the replica method directly to the compressive sensing problem; see for example [41], [42], [44], [61]–[63].

Although in general the replica method is considered to be mathematically non-rigorous, several recent studies have justified the validity of the results derived via the replica method in the context of compressive sensing by using some alternative tools for analysis. A widely investigated approach is based on the asymptotic analysis of approximate message passing (AMP) algorithms. In compressive sensing, AMP algorithms were initially introduced to address iteratively the convex reconstruction schemes based on $\ell_1$ norm minimization, such as LASSO and basis pursuit, with low computational complexity [64], [65]. This algorithmic approach was later extended to a large variety of estimation problems including MAP and MMSE estimation; see for example [66], [67]. The primary numerical investigations of AMP demonstrated that for large sensing systems the sparsity-compression rate tradeoff of these iterative algorithms, as well as the compression rate-distortion tradeoff in noisy cases, is derived by the fixed-points of state evolution and recovers the asymptotics of convex reconstruction schemes [65]. This observation was then rigorously justified for i.i.d. sub-Gaussian sensing matrices in [68] by using the conditioning technique developed in [69]. The study was recently extended to cases with rotationally invariant system matrices in [70], [71]. The investigations in [72], [73] moreover showed that using AMP algorithms for spatially coupled measurements, the fundamental limits on the required compression rate [74], [75] can be achieved in the asymptotic regime. The methodology proposed by AMP algorithms and their state evolution also provided a justification for validity of several earlier studies based on the replica method. In fact, the results given by the replica method were recovered through state evolution of the corresponding AMP algorithms. Invoking this approach along with other analytical tools, the recent study in [25] further approved the validity of the replica prediction for the asymptotic MMSE and mutual information of the linear system in (1) with i.i.d. Gaussian measurements. Similar results were demonstrated in [23] using a different approach.

D. Contribution and Outline

In this paper, we determine the asymptotic distortion for a general distortion function for cases where the MAP estimator is employed to estimate the source vector from the observation given in (1). We represent the asymptotic distortion in (3) as a macroscopic parameter of the corresponding spin glass and study this spin glass via the replica method. The general replica ansatz is then given for an arbitrary replica correlation matrix, and its special cases are studied considering the RS and RSB assumptions. The asymptotic distortion is determined for rotationally invariant random system matrices invoking results for asymptotics of spherical integrals [76], [77].

Using our asymptotic results, we derive a more general form of the decoupling principle by restricting the distortion function to be of a special form and employing the moment method [78], [79]. We show that the vector-valued system in (1) estimated by (2) decouples into a bank of similar noisy single-user systems followed by single-user MAP estimators. This result holds for any replica correlation matrix; however, the structure of the decoupled single-user system depends on the supposed structure of the correlation matrix. Under the RSB assumption with $b$ steps of breaking ($b$RSB), the noisy single-user system is given in the form of an input term added by an impairment term. The impairment term, moreover, is expressed as the sum of an independent Gaussian random variable and $b$ correlated non-Gaussian interference terms. By reducing the assumption to RS, the result reduces to the formerly studied RS decoupling principle of the MAP estimators [40] for rotationally invariant system matrix ensembles. In fact, our investigations collect the previous results regarding the decoupling principle in addition to a new set of setups under a single umbrella given by a more general form of the decoupling principle. More precisely, we extend the scope of the decoupling principle to systems whose measuring matrix belongs to the class of rotationally invariant random matrices, and the replica ansatz with general replica correlations which includes the RS and RSB ansätze.

To address a particular application, we study large-system performance of compressive sensing systems with the $\ell_p$ norm minimization recovery schemes. We address the linear reconstruction, as well as the LASSO and $\ell_0$ norm minimization
considering both the sparse Gaussian and discrete alphabet sources. Our general setting allows to investigate the asymptotic performance w.r.t. different metrics and for multiple sensing matrices such as the random i.i.d. and projector matrix. The numerical investigations show that the RS ansatz becomes unstable for some regimes of system parameters and predicts the performance of $\ell_0$ norm minimization loosely within a large range of compression rates. This observation agrees with the earlier discussions on the necessity of RSB investigations reported in [44], [48]. We therefore study the performance under RSB and discuss the impact of the symmetry breaking. Throughout the numerical investigations, it is demonstrated that the performance enhancement obtained via random orthogonal measurements, reported in [42], also holds for sparse discrete alphabet sources in which sensing via random projector matrices results in phase transitions at higher rates.

The rest of the manuscript is organized as follows. In Section II, the problem is formulated. We illustrate our statistical mechanical approach in Section III and explain briefly the replica method. The general replica ansatz, as well as the general decoupling principle is given in Section IV. The ansatz under the RS and RSB assumptions are expressed in Sections V and VI, respectively. Invoking bRSB decoupling, we propose the idea of a replica simulator in Section VII and describe the derivation of the given replica ansätze in terms of their corresponding decoupled systems. To address an application of our study, we consider large compressive sensing systems in Section VIII and discuss several examples. The numerical investigations of the examples are then given in Section IX. Finally, we conclude the manuscript in Section X.

E. Notations

Throughout the manuscript, we represent scalars, vectors and matrices with lower, bold lower, and bold upper case letters, respectively. $\mathbb{Z}$ and $\mathbb{R}$ represent the set of integer and real numbers, respectively, and the superscript $+$, e.g. $\mathbb{R}^+$, indicates the corresponding subset of all non-negative numbers. $\mathbf{A}^T$ and $\mathbf{A}^H$ indicate the transposed and Hermitian of $\mathbf{A}$. $\mathbf{I}_m$ is the $m \times m$ identity matrix and $\mathbf{I}_m$ is an $m \times m$ matrix with all entries equal to 1. For a random variable $x$, $p_x$ represents either the probability mass function (PMF) or probability density function (PDF), and $F_x$ represents the cumulative distribution function (CDF). We denote the expectation over $x$ by $E_x$, and an expectation over all random variables involved in a given expression by $E$. For sake of compactness, the set of integers $\{1, \ldots, n\}$ is abbreviated as $[1:n]$, the zero-mean and unit-variance Gaussian PDF is denoted by $\phi(\cdot)$, and Gaussian averaging is expressed as

$$\int \phi(\cdot) \, d\mathbf{r} := \int_{-\infty}^{+\infty} \phi(r) \, dr.$$  \hfill (17)

In many cases, we drop the set on which a sum, minimization or an integral is taken. Whenever needed, we consider the entries of $x$ to be discrete random variables, namely the support $X$ to be discrete. The results of this paper are however in full generality and hold also for continuous distributions.

F. Announcements

Although the results have a mathematical flavor, the stress is not on investigating the rigor of available tools such as the replica method, but rather to employ them for deriving formulas which can be used in different problems. We hence represent the main results of this paper under the title of Result, in order to indicate the lack of mathematical rigor in some steps of derivations.

II. Problem Formulation

Consider the vector-valued linear system described by (1). Let the system satisfy the following properties:

(a) $\mathbf{x}_{n \times 1}$ is an i.i.d. random vector with each entry being distributed with $p_x$ over $\mathbb{R} \subseteq \mathbb{R}$.

(b) $\mathbf{A}_{k \times n}$ is randomly generated over $\mathbb{A}^{k \times n}$ with $\mathbb{A} \subseteq \mathbb{R}$ from rotationally invariant random ensembles. The random matrix $\mathbf{A}$ is said to be rotationally invariant when its Gramian, i.e., $\mathbf{J} = \mathbf{A}^T \mathbf{A}$, has the eigendecomposition

$$\mathbf{J} = \mathbf{U} \mathbf{D} \mathbf{U}^T$$  \hfill (18)

with $\mathbf{U}_{n \times n}$ being an orthogonal Haar distributed matrix and $\mathbf{D}_{n \times n}$ being a diagonal matrix. For a given $n$, we denote the empirical CDF of $\mathbf{J}$’s eigenvalues (cumulative density of states) with $F^{\mathbf{J}}$ and define it as

$$F^{\mathbf{J}}(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(\lambda_i^{\mathbf{J}} < x).$$  \hfill (19)

where $\lambda_i^{\mathbf{J}}$ for $i \in [1:n]$ denotes the $i$th diagonal entry of $\mathbf{D}$. We assume that $F^{\mathbf{J}}$ converges, as $n \uparrow \infty$, to some deterministic CDF $F^{\mathbf{J}}$.

(c) $\mathbf{z}_{k \times 1}$ is a real i.i.d. zero-mean Gaussian random vector in which the variance of each entry is $\mathbf{z}_0$.

(d) The number of observations $k$ is a deterministic function of the transmission length $n$, such that

$$\lim_{n \uparrow \infty} \frac{k(n)}{n} = \frac{1}{r} < \infty.$$  \hfill (20)

For sake of compactness, we drop the explicit dependence of $k$ on $n$.

(e) $x, A$ and $z$ are independent.

The source vector $x$ is reconstructed from the observation vector $y$ with the help of the system matrix $A$ that is known at the estimator. Thus, for a given $A$, the source vector is recovered by $\hat{x} = \mathbf{g}(y|A)$ where $\mathbf{g}(\cdot|A)$ is given in (2). Here, the non-negative scalar $\hat{\lambda}$ is the estimation parameter. The utility function $u(\cdot)$ is supposed to decouple which means that it takes arguments with different lengths, i.e., $u(\cdot) : \mathbb{R}^{\ell} \mapsto \mathbb{R}^+$ for any positive integer $\ell$, and

$$u(x) = \sum_{i=1}^{n} u(x_i).$$  \hfill (21)

In order to use the estimator in (2), one needs to guarantee the uniqueness of the estimation output. Therefore, we impose the following constraint on our problem:

(f) For almost all observation vectors $y$, there exists a unique essential minimizer to the optimization problem in (2).
A. MAP Estimator

The MAP estimator in (2) can be considered as the optimal estimator in the sense that it minimizes the reconstruction’s error probability postulating a source prior distribution proportional to $e^{-u(x)}$ and noise variance $\lambda$. To clarify the argument, assume $X$ is a discrete set. In this case, we can define the re-construction error probability as

$$P_e = Pr[x \neq \hat{g}(y|A)]$$

(22)

for some estimator $\hat{g}(|A)$. In order to minimize $P_e$, $\hat{g}(|A)$ is chosen such that the posterior distribution over the input support $X^n$ is maximized, i.e.,

$$\hat{g}(y|A) = \arg\max_{v} p_{y|x,A}(v|y,A)$$

(23a)

$$= \arg\max_{v} p_{y|x,A}(y|v,A)p_{x|A}(v|A)$$

(23b)

$$\triangleq \arg\min_{v} [- \log p_{y|x,A}(y|v,A) - \log p_{x}(v)]$$

(23c)

where $\ast$ comes from the independency of $x$ and $A$. Here, $p_{y|x,A}(y|v,A) = p_{x}(y - Ax)$, and $p_{x}$ is the prior distribution of the source vector. Now, let the estimator postulate the noise variance to be $\lambda$ and the prior to be

$$p_{x}(v) = \frac{e^{-u(v)}}{\sum_{v'} e^{-u(v')}}$$

(24)

for some non-negative function $u(\cdot)$. Substituting into (23c), the estimator $\hat{g}(A)$ reduces to $\hat{g}(A)$ defined in (2). The estimator in (2) models several particular reconstruction schemes in compressive sensing. We address some of these schemes later on in Section VIII.

Remark II.1. In the case that entries of $x$ are continuous random variables, the above argument needs some modifications. In fact, in this case the error probability as defined in (22) is always one, and therefore, it cannot be taken as the measure of error. The MAP estimator in such scenarios is considered to maximize the posterior PDF postulating $p_{x}(v) \propto e^{-u(v)}$ and noise variance $\lambda$.

B. Asymptotic Distortion and Conditional Distribution

In many applications, the distortion is given in terms of the average MSE, while in some others the average symbol error rate is considered. In fact, the former takes the $\ell_2$ norm as the distortion function, and the latter considers the $\ell_0$ norm. In general, the distortion function can be of a generic form. Here, we study the asymptotic performance by considering a general distortion function which determines the imperfection level of the estimation. Thus, we consider a distortion function $d(\cdot; \cdot)$ which

$$d(\cdot; \cdot) : X \times X \to \mathbb{R}$$

(25)

The term average distortion usually refers to the case when the averaging weights are uniform meaning that the distortion of each tuple $(x_i, \hat{x}_i)$ is weighted equally when the distortion is averaged over all tuples. It is however possible to average the distortion by a non-uniform set of weights. In the following,

5i.e., the entries of $x$ are discrete random variables.

we define the average distortion for a class of binary weights which includes the case of uniform averaging, as well.

Definition II.1 (Asymptotic distortion) Consider the vectors $x_{n \times 1}$ and $\hat{x}_{n \times 1}$ defined over $X^n$, and let $d(\cdot; \cdot)$ be a distortion function defined in (25). Define the index set $\mathbb{W}(n)$ as a subset of $[1 : n]$, and let $|\mathbb{W}(n)|$ grow with $n$ such that

$$\lim_{n \to \infty} \frac{|\mathbb{W}(n)|}{n} = \eta$$

(26)

for some $\eta \in [0, 1]$. Then, the average distortion of $\hat{x}$ and $x$ over index set $\mathbb{W}(n)$ w.r.t. distortion function $d(\cdot; \cdot)$ is

$$D^{\mathbb{W}(n)}(\hat{x}; x) := \frac{1}{|\mathbb{W}(n)|} \sum_{w \in \mathbb{W}(n)} d(\hat{x}_w; x_w).$$

(27)

Assuming the limit of (27) when $n \to \infty$ exists, we denote

$$D^\mathbb{W}(\hat{x}; x) := \lim_{n \to \infty} D^{\mathbb{W}(n)}(\hat{x}; x)$$

(28)

and refer to it as the asymptotic distortion over the limit of the index set $\mathbb{W}(n)$.

Definition II.1 is moreover utilized to investigate the asymptotic conditional distribution of the estimator which plays a key role in the derivation of the decoupling principle. For further convenience, we define the asymptotic conditional distribution of the MAP estimator as follows.

Definition II.2 (Asymptotic conditional distribution) Consider the source vector $x_{n \times 1}$ passed through the linear system in (1), and let $\hat{x}_{n \times 1}$ be its MAP estimation as given in (2). For a given $n$, we take an index $j \in [1 : n]$ and denote the conditional distribution of $\hat{x}_j$ given $x_j$ by $p_{\hat{x}|x}(\hat{x}_j|x)$ which at the mass point $(\hat{v},v) \in X \times X$ reads

$$p_{\hat{x}|x}(\hat{v}|v) := \frac{p_{\hat{x},x}(\hat{v},v)}{p_{x}(v)}$$

(29)

with $p_{\hat{x},x}(\hat{v},v)$ being the marginal joint distribution of $x_j$ and $\hat{x}_j$ at the mass point $(\hat{v},v)$. Then, we define the asymptotic conditional distribution of $\hat{x}_j$ given $x_j$ at $(\hat{v},v)$ as

$$p_{\hat{x}|x}(\hat{v}|v) := \lim_{n \to \infty} p_{\hat{x}|x}(\hat{v}|v).$$

(30)

We study the asymptotic distortion over the limit of a desired index set $\mathbb{W}(n)$ and distortion function $d(\cdot; \cdot)$ by defining it as a macroscopic parameter of the corresponding spin glass and employing the replica method to evaluate it. Using the result for the asymptotic distortion, we then determine the asymptotic conditional distribution and investigate the decoupling property of the estimator.

III. Statistical Mechanical Approach

The Hamiltonian in (11) introduces a spin glass which corresponds to the MAP estimator. The spin glass at zero temperature describes the asymptotics of the MAP estimator. For further convenience, we formally define the corresponding spin glass as follows:

Definition III.1 (Corresponding spin glass) Consider an integer $n \in \mathbb{Z}^+$. The corresponding spin glass with the microstate
\( \mathbf{v}_{n \times 1} \in \mathbb{X}^n \) and the quenched randomizers \( \mathbf{y}_{n \times 1} \) and \( \mathbf{A}_{k \times n} \) is specified by the Hamiltonian

\[
\mathcal{E}(\mathbf{v}; \mathbf{y}, \mathbf{A}) = \frac{1}{2\beta} ||\mathbf{y} - \mathbf{A}\mathbf{v}||^2 + u(\mathbf{v}).
\]  

(31)

For the corresponding spin glass, at the inverse temperature \( \beta \), the following properties are directly concluded:

- The conditional distribution of the microstate reads
  \[
p^{\beta}_{\mathbf{v}|\mathbf{y}, \mathbf{A}}(\mathbf{v}|\mathbf{y}, \mathbf{A}) = \frac{e^{-\beta \mathcal{E}(\mathbf{v}; \mathbf{y}, \mathbf{A})}}{\mathcal{Z}(\beta|\mathbf{y}, \mathbf{A})}
\]
  with \( \mathcal{Z}(\beta|\mathbf{y}, \mathbf{A}) \) being the partition function

\[
\mathcal{Z}(\beta|\mathbf{y}, \mathbf{A}) = \sum_{\mathbf{v}} e^{-\beta \mathcal{E}(\mathbf{v}; \mathbf{y}, \mathbf{A})}.
\]  

(32)

- The normalized free energy is given by

\[
\mathcal{F}(\beta) = -\frac{1}{\beta} \mathbb{E} \log \mathcal{Z}(\beta|\mathbf{y}, \mathbf{A}),
\]

(33)

where expectation is taken over the quenched randomizers.

- The normalized free entropy of the spin glass is determined as

\[
\mathcal{H}(\beta) = \beta^2 \frac{d}{d\beta} [\mathcal{F}(\beta)] .
\]

(35)

Regarding the MAP estimator, one can represent the asymptotic distortion as a macroscopic parameter of the corresponding spin glass. More precisely, using Definition II.2, the asymptotic distortion reads

\[
\mathcal{D}^{\mathcal{W}}(\hat{\mathbf{x}}; \mathbf{x}) = \lim_{\beta \uparrow \infty} \lim_{n \uparrow \infty} \mathbb{E}_{\mathbf{v}}^{\beta} \mathcal{D}^{\mathcal{W}(n)}(\mathbf{v}; \mathbf{x})
\]

(36)

where \( \mathbb{E}_{\mathbf{v}}^{\beta} \) indicates expectation over \( \mathbf{v} \in \mathbb{X}^n \) w.r.t. the conditional Boltzmann-Gibbs distribution \( p_{\mathbf{v}|\mathbf{y}, \mathbf{A}}^{\beta} \) defined in (32). In fact, by introducing the macroscopic parameter \( \mathcal{D}^{\mathcal{W}}(\beta) \) at the temperature \( \beta \) as

\[
\mathcal{D}^{\mathcal{W}}(\beta) := \lim_{n \uparrow \infty} \mathbb{E}_{\mathbf{v}}^{\beta} \mathcal{D}^{\mathcal{W}(n)}(\mathbf{v}; \mathbf{x}),
\]

(37)

the asymptotic distortion can be interpreted as the macroscopic parameter at zero temperature. Here, we take a well-known strategy in statistical mechanics which modifies the partition function to

\[
\mathcal{Z}(\beta, h) = \sum_{\mathbf{v}} e^{-\beta \mathcal{E}(\mathbf{v}; \mathbf{y}, \mathbf{A}) + h \mathbb{E} D^{\mathcal{W}(n)}(\mathbf{v}; \mathbf{x})}.
\]

(38)

In this case, the expectation in (37) is taken as

\[
\mathcal{D}^{\mathcal{W}}(\beta) = \lim_{n \uparrow \infty} \lim_{h \downarrow 0} \frac{1}{n} \frac{1}{\partial h} \log \mathcal{Z}(\beta, h).
\]

(39)

The macroscopic parameter defined in (37) is random, namely depending on the quenched randomizer \( \mathbf{y}, \mathbf{A} \). As discussed in Section I-A, under the self averaging property, the macroscopic parameter is supposed to converge in the large-system limit to its expected value over the quenched random variables. For the corresponding spin glass defined in Definition III.1, the self averaging property has not been rigorously justified, and the proof requires further mathematical investigations as in [6]. However, as it is widely accepted in the literature, we assume that the property holds at least for the setting specified here. Therefore, we state the following assumption.

**Assumption 1 (Self Averaging Property)** Consider the corresponding spin glass defined in Definition III.1. For almost all realizations of the quenched randomizers \( \mathbf{A} \) and \( \mathbf{y} \),

\[
\mathcal{D}^{\mathcal{W}}(\beta) = \mathbb{E}_{\mathbf{y}, \mathbf{A}} D^{\mathcal{W}}(\beta).
\]

(40)

Using the self averaging property of the system, the asymptotic distortion is written as

\[
\mathcal{D}^{\mathcal{W}}(\hat{\mathbf{x}}; \mathbf{x}) = \lim_{\beta \uparrow \infty} \lim_{n \uparrow \infty} \frac{1}{\partial h} \mathbb{E} \log \mathcal{Z}(\beta, h).
\]

(41)

Evaluation of (41), as well as the normalized free energy defined in (34), confronts the nontrivial problem of determining a logarithmic expectation. This task can be bypassed by using the Riesz equality [80] which for a given random variable \( t \) states that

\[
\mathbb{E} \log t = \lim_{m \uparrow 0} \frac{1}{m} \log \mathbb{E} t^m.
\]

(42)

Using the Riesz equality, the asymptotic distortion is finally written as

\[
\mathcal{D}^{\mathcal{W}}(\hat{\mathbf{x}}; \mathbf{x}) = \lim_{\beta \uparrow \infty} \lim_{h \downarrow 0} \frac{1}{\partial h} \log \mathbb{E} [\mathcal{Z}(\beta, h)]^m .
\]

(43)

Equation (43) expresses the asymptotic distortion in terms of the moments of the modified partition function; however, it does not yet simplify the problem. In fact, one faces two main difficulties when calculating the r.h.s. of (43): 1) the moment needs to be evaluated for real values of \( m \) (at least within a right neighborhood of 0), and 2) the limits need to be taken in the order stated. Here is where the replica method plays its role. The replica method suggests to determine the moment for an arbitrary non-negative integer \( m \) as an analytic function in \( m \) and then assume the two following statements:

1) The moment function analytically continues from the set of integer numbers onto the real axis (at least for some \( m \) at a right neighborhood of 0) which means that an analytic expression found for integer moments directly extends to all (or some) real moments. Under this assumption, the expression determined for integer moments is replaced in (43), and the limit w.r.t. \( m \) taken when \( m \downarrow 0 \). This assumption is the main part where the replica method lacks rigor. It is known as the replica continuity.

2) In (43), the limits w.r.t. \( m \) and \( n \) exchange. We refer to this assumption as the limit exchange.

In order to employ the replica method, we assume the validity of the above two statements; therefore, we consider the following assumption:

**Assumption 2 (Replica Continuity and Limit Exchange)** For the spin glass defined in Definition III.1, the replica continuity and limit exchange assumptions hold.

By means of Assumption 2, the calculation of asymptotic distortion reduces to the evaluation of integer moments of the
modified partition function which is written as
\[
\tilde{Z}(m) := E[\mathcal{Z}(\beta, h)]^m
\]
\[
= E \prod_{a=1}^{m} \sum_{v_a} e^{-\beta \mathcal{E}(v_a|y, A)+hnD^{(0)}(v_a; x)}
\]
\[
= E_x E_A E_{\{v_a\}} \sum_{v_a} e^{-\beta \mathcal{E}(v_a|y, A)+hn \sum_{a=1}^{m} D^{(0)}(v_a; x)}.
\]

Here, we refer to \(v_a \in \mathbb{X}^n\) for \(a \in [1 : m]\) as the replicas, and define \(\{v_a\} := \{v_1, \ldots, v_m\} \in \mathbb{X}^n \times \cdots \times \mathbb{X}^n\) as the set of replicas. After taking the expectation w.r.t. \(z\) and \(A\), it is further observed that, in the large-system limit, the expectation w.r.t. \(x\) can be dropped due to the law of large numbers. By inserting the final expression for \(\tilde{Z}(m)\) in (43) and taking the limits, the asymptotic distortion is determined as in Result IV.1.

**Remark III.1.** Although Assumptions 1 and 2 are the main assumptions of the replica method, they are not the only points in which the method lacks mathematical rigor. In fact, the derivations comprise of several analytical tricks whose validity is either intractable or complicated to check. We hence refer to predictions given via this method as Results.

**IV. MAIN RESULTS**

Result IV.1 states the general replica ansatz. The term general is emphasized here, in order to indicate that no further assumption needs to be considered for derivation. Using Result IV.1 along with results in the classical moment problem \([79]\), a general form of the decoupling principle is justified for the MAP estimator.

Before stating the general replica ansatz, let us define the R-transform of a probability distribution. Considering a random variable \(r\), the corresponding Stieltjes transform over the upper complex half plane is defined as
\[
G_r(s) = E (t - s)^{-1}.
\]
Denoting the inverse w.r.t. composition by \(G_r^{-1}(\cdot)\), the R-transform is given by
\[
R_r(\omega) = G_r^{-1}(-\omega) - \omega^{-1}
\]
such that \(\lim_{\omega \downarrow 0} R_r(\omega) = E r\).

The definition is also extended to matrix arguments. Assume matrix \(M_{n \times n}\) has the decomposition \(M = U \Lambda U^{-1}\) where \(\Lambda_{n \times n}\) is a diagonal matrix whose nonzero entries represent the eigenvalues of \(M\), i.e., \(\Lambda = \text{diag}[\lambda_1, \ldots, \lambda_n]\), and \(U_{n \times n}\) is the matrix of eigenvectors. \(R_r(M)\) is then an \(n \times n\) matrix defined as
\[
R_r(M) = U \text{diag}[R_r(\lambda_1), \ldots, R_r(\lambda_n)] U^{-1}.
\]

**A. General Replica Ansatz**

Result IV.1 expresses the macroscopic parameters of the corresponding spin glass, including the asymptotic distortion, in terms of the parameters of a new spin glass of finite dimension. It is important to note that the new spin glass, referred to as spin glass of replicas, is different from the corresponding spin glass defined in Definition III.1. In fact, the spin glass of replicas is the projection of the corresponding spin glass on the reduced support \(\mathbb{X}^m\) with \(m\) indicating the number of replicas. The macroscopic parameters of the spin glass of replicas can therefore readily be determined.

**Definition IV.1 (Spin glass of replicas)** For the finite integer \(m\), the spin glass of replicas with the microstate \(v_{m \times 1} \in \mathbb{X}^m\) and quenched randomizer \(x_{m \times 1}\) is defined as follows:

- All the entries of \(x\) equal to \(x\) where \(x\) is a random variable distributed with the source distribution \(p_x\).
- For a given realization of \(x\), the Hamiltonian reads
\[
\mathcal{E}^R(\mathbf{v}|\mathbf{x}) = (\mathbf{x} - \mathbf{v})^T \mathbf{R}_J(\mathbf{x} - \mathbf{v}) + \mu(\mathbf{v})
\]
where \(\mathbf{R}_J(\cdot)\) is the R-transform corresponding to \(\mathbf{f}_J\), \(\mathbf{T}\) is an \(m \times m\) matrix defined as
\[
\mathbf{T} := \frac{1}{2\beta} \left[ \mathbf{I}_m - \frac{\beta \lambda_0}{\lambda + m \beta \lambda_0} \mathbf{I}_m \right],
\]
and \(Q_{m \times m}\) is the so-called replica correlation matrix defined as
\[
Q = \mathbf{E}(\mathbf{x} - \mathbf{v})(\mathbf{x} - \mathbf{v})^T.
\]
with expectation taken over \(\mathbf{x}\) and \(\mathbf{v}\) at thermal equilibrium.
- At thermal equilibrium, the microstate is distributed according to the Boltzmann-Gibbs distribution \(p^\beta_{\mathbf{v}|\mathbf{x}}\)
\[
p^\beta_{\mathbf{v}|\mathbf{x}}(\mathbf{v}|\mathbf{x}) = \frac{e^{-\beta \mathcal{E}^R(\mathbf{v}|\mathbf{x})}}{Z^R(\beta|\mathbf{x})}
\]
where \(Z^R(\beta|\mathbf{x})\) denotes the partition function of the system defined as
\[
Z^R(\beta|\mathbf{x}) := \sum_{\mathbf{v}} e^{-\beta \mathcal{E}^R(\mathbf{v}|\mathbf{x})}.
\]
- The normalized free energy of the system at the inverse temperature \(\beta\) is given by
\[
\mathcal{F}^R(\beta, m) = -\frac{1}{\beta m} \mathbf{E} \log Z^R(\beta|\mathbf{x}),
\]
where expectation is taken over \(\mathbf{x}\) w.r.t. \(p_x\). The average energy and entropy at the inverse temperature \(\beta\) are further found using (9a) and (9b).
- For the system at thermal equilibrium, the replicas’ average distortion w.r.t. the distortion function \(D(\cdot, \cdot)\) at the inverse temperature \(\beta\) is
\[
D^R(\beta, m) = \frac{1}{m} \mathbf{E} \sum_{a=1}^{m} d(v_a, x),
\]
with expectation taken over \(\mathbf{x}\) and \(\mathbf{v}\) w.r.t. \(p_x p^\beta_{\mathbf{v}|\mathbf{x}}\).

Considering Definition IV.1, the evaluation of the system parameters such as the replicas’ average distortion \(D^R(\beta, m)\) or the normalized free energy \(\mathcal{F}^R(\beta, m)\) needs the replica correlation matrix \(Q\) to be explicitly calculated first. In fact, (50) describes a fixed point equation in terms of \(Q\) when one writes out the expectation using the conditional distribution in (51). The solution can then be substituted in the distribution and the parameters of the system can be calculated.
via (52)-(54). The fixed point equation, however, may have several solutions and thus can result in multiple outputs for the system. Nevertheless, we express the asymptotic distortion of the MAP estimator in terms of a single output of the spin glass of replicas for which the limits exist and the free energy is minimized.

**Result IV.1** Let linear system (1) fulfill the constraints of Section II. Suppose Assumptions 1 and 2 hold, and consider the spin glass of replicas as defined in Definition IV.1 for a finite integer m. Then, the free energy of the corresponding spin glass of replicas as defined in Definition III.1 is determined via the replica method as

\[
\mathcal{F}(\beta) = \lim_{m \to 0} \frac{1}{m} \int_{0}^{1} \text{tr}(TQ_{RJ}(-2\beta TQ))d\omega - \text{tr}(Q^{T}TR_{J}(-2\beta TQ)) + \mathcal{F}^{R}(\beta, m) \tag{55}
\]

where \(Q\) is the replica correlation matrix satisfying (50) and \(\text{tr}()\) denotes the trace operator.

The asymptotic distortion of the MAP estimator w.r.t. distortion function \(d(\cdot); \cdot\) is then determined as

\[
D^{W}(x; x) = \lim_{\beta \uparrow \infty} D^{R}(\beta, m). \tag{56}
\]

In case that multiple solutions are available for the replica correlation matrix, the replica’s average distortion in (56) is evaluated via that correlation matrix which minimizes the free energy at zero temperature, i.e., \(\beta \uparrow \infty\).

**Proof.** The proof is given in Appendix A. However, we explain briefly the strategy in the following:

Starting from (44c), the expectation w.r.t. the noise term is straightforwardly taken. Using the results from [77], the expectation w.r.t. the system matrix is further taken as discussed in Appendix D. Then, by considering the following variable exchange,

\[
|Q|_{ab} = \frac{1}{n}(x - v_{a})^{T}(x - v_{b}). \tag{57}
\]

\(\tilde{Z}(m)\) is determined in terms of the replica correlation matrix \(Q\). Finally, by employing the law of large numbers, the \(m\)th moment of the partition function is given as

\[
\tilde{Z}(m) = E_{x} \int e^{-n[G(TQ^{T}R_{J}(-2\beta TQ))] + \epsilon_{a} Q} dQ \tag{58}
\]

where \(dQ := \prod_{a,b} d|Q|_{ab}\) with integral taken over \(\mathbb{R}^{m \times n}\), \(\epsilon_{a}\) tends to zero as \(m \uparrow \infty\) and \(T\) is given by (49). Moreover, \(e^{\epsilon_{a} Q}\) denotes the non-normalized probability weight of the vectors of replicas with a same correlation matrix and is explicitly determined in Appendix A, and \(G(\cdot)\) reads

\[
G(M) = \int_{0}^{\beta} \text{Tr}[M R_{J}(-2\beta M)]d\omega \tag{59}
\]

for some diagonal matrix \(M\). In (58), \(e^{\epsilon_{a} Q} dQ\) is a probability measure which satisfies the large deviations property. Using results from large deviations [81], the integral in (58) for large values of \(n\) is written as the integrand at the saddle point \(Q\) multiplied by some bounded coefficient \(K_{n}\). This results in

\[
\tilde{Z}(m) = K_{n} e^{-n[Q^{T}R_{J}(-2\beta TQ)]} \tag{60}
\]

with \(\cong\) denoting the asymptotic equivalence in exponential scale.\(^7\) Consequently, by substituting \(\tilde{Z}(m)\) in (53), and exchanging the limits w.r.t. \(n\) and \(m\), as suggested in Assumption 2, the asymptotic distortion is found as in Result IV.1 where (50) determines the saddle point of the integrand function in (58). The free energy is further determined as in (55) by substituting (60) in (34). Finally, by noting that the free energy is minimized at the equilibrium, the proof is concluded.

Using Result IV.1, the asymptotics of the MAP estimator are determined tractably. To pursue the analysis, one needs to solve the fixed point equation (50) for the replica correlation matrix \(Q\) and calculate the parameters of the spin glass of replicas explicitly. The direct approach to find \(Q\), however, raises both complexity and analyticity issues. In fact, finding the saddle point by searching over the set of all possible choices of \(Q\) is a hard task to do; moreover, several solutions may not be of use since they do not lead to analytic \(Z_{R}^{R}(\beta, m)\) and \(D^{R}(\beta, m)\) in \(m\), and hence they cannot be continued analytically to the real axis via Assumption 2.

To overcome these two issues, the conventional approach in the replica method is to restrict our search into a parameterized set of replica correlation matrices and find the solution within this set. The choices for the parameterized set are roughly divided into RS and RSB schemes. The former considers the \(m\) replicas to interact symmetrically while the latter recursively breaks this symmetry in a systematic manner. The RSB scheme was first introduced due to some symmetric properties observed in the analysis of spin glasses [82]. However, these properties do not force the correlation matrix to have a symmetric structure. Several examples were found later on in the literature showing that RS may lead to wrong conclusions. For these examples, the RSB scheme was devised as an extension of the symmetric structure of the correlation matrix to a larger set. We consider both the RS and RSB schemes in this manuscript; however, before pursuing our study, let us first investigate the general decoupling property of the estimator which is concluded from Result IV.1.

**B. General Decoupling Property of MAP Estimator**

Regardless of any restriction on \(Q\), the general ansatz leads to the decoupling property of the MAP estimator. Using Result IV.1, it can be shown that for almost any tuple of input-output entries, the marginal joint distribution converges to a deterministic joint distribution which does not depend on the entries’ indices. The explicit term for the joint distribution, however, depends on the assumptions imposed on the correlation matrix.

\(^7\)\(\alpha()\) and \(b()\) are asymptotically equivalent in exponential scale over a non-bounded set \(X\), if for \(x \in X\) we have \(\lim_{x \uparrow \infty} \frac{\alpha(x)}{b(x)} = 0\).
Result IV.2 (General Decoupling Principle) Let linear system (1) fulfill the constraints of Section II. Suppose Assumptions 1 and 2 hold, and consider the spin glass of replicas as defined in Definition IV.1 with the replica correlation matrix \(Q\). Then, for \(j \in [1 : n]\), the asymptotic conditional distribution of the MAP estimator \(\hat{x}^{j}_{\text{MAP}}\) derived via the replica method is independent of \(j\) almost sure in \(A\), namely
\[
p^{j}_{\text{MAP}}(\hat{v}|v) = p^{j}_{\text{MAP}}(v|v) \tag{61}
\]
for some conditional distribution \(p^{j}_{\text{MAP}}\) at the mass point \((\hat{v}, v) \in \mathbb{X} \times \mathbb{X}\). Consequently, the marginal joint distribution of the entries \(x_j\) and \(\hat{x}_j\) is identical to the input-output joint distribution of the single-user system described by the conditional distribution \(p^{j}_{\text{MAP}}\) and the input \(x \sim p_x\). The explicit form of \(p^{j}_{\text{MAP}}\) depends on \(Q\).

**Proof.** The proof follows Result IV.1 and the moment problem. 

Proof.

To determine the joint moments of input and output entries, consider the distortion function
\[
d(\hat{x}, x) = x^k \hat{x}^\ell \tag{63}
\]
in Result IV.1 and evaluate the asymptotic distortion over the limit of \(\mathcal{W}(n) = \{j : j + \eta n\}\) for some \(\eta\) in a right neighborhood of zero. The \((k, \ell)\)-th joint moment of \((\hat{x}_j, x_j)\) is then determined by taking the limit \(\eta \downarrow 0\).

Substituting the distortion function and the index set in Result IV.1, it is clear that the asymptotic distortion is independent of \(\eta\) and \(j\), and therefore, the limit w.r.t. \(\eta\) exists and is independent of \(j\) as well. Noting that the evaluated moments are uniformly bounded, it is inferred that the asymptotic joint distribution of \((\hat{x}_j, x_j)\) is uniquely specified and does not depend on the index \(j\). Finally, by using the fact that the source vector is i.i.d. and the distribution of the entry \(j\) is independent of the index, we conclude that the asymptotic conditional distribution \(p^{j}_{\text{MAP}}\) is independent of \(j\). The exact expression for \(p^{j}_{\text{MAP}}\) is then found by determining the solution \(Q\) to the fixed point equation and determining the joint moments.

Result IV.2 is a generalized form of the RS decoupling principle of the MAP estimator studied in [40]. In fact, Result IV.2 indicates that the asymptotic characterization of a vector system followed by a MAP estimator via the replica method always decouples into a bank of identical single-user systems regardless of any restriction on the replica correlation matrix.

**C. Consistency Test**

Several methods were introduced in the literature to check the consistency of a solution derived via the replica method. A classic method is to calculate the entropy of the corresponding spin glass at zero temperature: As the temperature goes to zero, the distribution of the microstate tends to an indicator function at the point of the estimated vector, and consequently, the entropy of the corresponding spin glass converges to zero.\(^8\) One consistency check is therefore the zero temperature entropy of a given solution.

Several works invoked this consistency test and showed that for those settings in which the RS ansatz fails to give a tight bound on the exact solution, the zero temperature entropy determined from the RS ansatz does not converge to zero; see for example [45]. This observation illustrates the invalidity of the RS assumption and hints at RSB ansätze giving better bounds on the true solution. Inspired by the aforementioned results, we evaluate the zero temperature entropy of the corresponding spin glass as a measure of consistency.

To determine the zero temperature entropy, we use (35) which determines the entropy in terms of the free energy at inverse temperature \(\beta\). Considering the free energy of the corresponding spin glass as given in Result IV.1, the normalized entropy \(H(\beta)\) reads
\[
H(\beta) = \lim_{\beta \to 0} \frac{\beta^2}{\mathcal{W}(n)} \frac{\partial}{\partial \beta} \left[\int_0^1 \text{tr}(TQ) \left(\phi \omega TQ\right) d\omega \right. \\
- \left. \text{tr}(Q^T TQ) \right] + H^R(\beta, m) \tag{64}
\]
where \(H^R(\beta, m)\) denotes the normalized entropy of the spin glass of replicas. As \(H^R(\beta, m)\) determines the entropy of a thermodynamic system, for any \(m \in \mathbb{R}^+\) we have
\[
\lim_{\beta \to 0} H^R(\beta, m) = 0 \tag{65}
\]
and therefore, the zero temperature entropy is given by
\[
H^0 = \lim_{\beta \to 0} \frac{\beta^2}{\mathcal{W}(n)} \frac{\partial}{\partial \beta} \left[\int_0^1 \text{tr}(TQ) \left(\phi \omega TQ\right) d\omega \right. \\
- \left. \text{tr}(Q^T TQ) \right] \tag{66}
\]
which depends on the structure of the replica correlation matrix. In [45], the authors determined the zero temperature entropy for the spin glass which corresponds to the vector precoding problem considering the RS and 1RSB assumptions, and observed that it takes the same form under both assumptions. They then conjectured that the zero temperature entropy is of a similar form for the general RSB structure regardless of the number of breaking steps.\(^9\) Using (66), we later show that the conjecture in [45] is true.

\(^8\)Remember that the entropy of the spin glass at temperature \(\beta^{-1}\) denotes either the conditional entropy (for discrete supports) or the conditional differential entropy (for continuous supports) of a random vector \(x\) distributed conditioned to \(y\) and \(A\) with Boltzmann-Gibbs distribution \(p^y_{\beta}(\cdot | y, A)\).

\(^9\)In fact in this case, the dependence of the zero temperature entropy on the correlation matrix is completely described via the scalar \(\gamma\) which corresponds to the diagonal entries of the correlation matrix. See Assumption 3-5 for more illustrations.
V. RS ANSATZ AND RS DECOUPLING PRINCIPLE

The most elementary structure imposed on the replica correlation matrix is RS. Here, one assumes the correlation matrix to be of a symmetric form which means that the replicas of the spin glass defined in Definition IV.1 are invariant under any permutation of indices. Using the definition of the replica correlation matrix as given in (50), it consequently reads that

\[(x - v_a)(x - v_b) = \begin{cases} q_0 & \text{if } a \neq b, \\ q_1 & \text{if } a = b. \end{cases} \quad (67)\]

Assumption 3 (RS Structure) Considering the spin glass defined as in Definition IV.1, the RS replica correlation matrix is of the form

\[Q = \frac{\chi}{\beta} I_m + q I_m \quad (68)\]

where \(\chi\) and \(q\) are some non-negative real scalars.

Assumption 3 considers \(q_0 = q\) and \(q_1 = \chi \beta^{-1} + q\) in (67). Substituting (68) in Definition IV.1, the spin glass of replicas is specified by the scalars \(\chi\) and \(q\). These scalars are related via a set of saddle point equations obtained from (50). Finally, using Result IV.1, the asymptotics of the system are found.

Result V.1 (RS Ansatz) Let linear system (1) fulfill the constraints of Section II. Suppose Assumptions 1 and 2, as well as Assumption 3 hold. Let \(x \sim p_x\), and

\[g = \arg \min_v \left\{ \frac{1}{2\lambda^2} (x + \sqrt{\lambda^2 z - v})^2 + u(v) \right\} \quad (69)\]

with \(v \in \mathbb{R}\), and \(\lambda^2\) being defined as

\[\lambda^2_0 := \left[ R_{J} \left( \frac{-\chi}{\lambda} \right) \right]^2 \frac{\partial}{\partial \chi} \left\{ (\lambda_0 \chi - \lambda q) R_{J} \left( \frac{-\chi}{\lambda} \right) \right\} \quad (70a)\]

\[\lambda^2 := \left[ R_{J} \left( \frac{-\chi}{\lambda} \right) \right]^{-1} \lambda \quad (70b)\]

for some non-negative real \(\chi\) and \(q\) and \(z \in \mathbb{R}\). Then, the asymptotic distortion determined via the replica method reads

\[D^W = E \int d(g, x) Dz, \quad (71)\]

for \(\chi\) and \(q\) which satisfy

\[\sqrt{\lambda^2_0} \chi = \lambda^2 \quad \text{and} \quad \lambda^2 = \left( \int (g - x)^2 Dz \right)^{-1} \quad (72a)\]

\[q = E \int (g - x)^2 Dz \quad (72b)\]

and minimize the zero temperature free energy \(F^0_{rs}\) which is given by

\[F^0_{rs} = \frac{1}{2\lambda^2} \left[ \int_0^1 F(\omega) d\omega - F(1) \right] + E \int \frac{1}{2\lambda^2} \left[ (x + \sqrt{\lambda^2_0 z - g})^2 - \lambda^2 z^2 \right] + u(g) Dz \quad (73)\]

with \(F(\cdot)\) being defined as

\[F(\omega) = \left[ q \frac{\lambda_0}{\lambda} \frac{d}{d\omega} \omega R_J \left( \frac{-\chi}{\lambda} \omega \right) \right]. \quad (74)\]

Proof. See Appendix B. \(\square\)

The asymptotic distortion under the RS ansatz is equivalent to the average distortion of a scalar additive white Gaussian noise (AWGN) channel followed by a single-user estimator as shown in Fig. 1. In this block diagram, the single-user estimator \(\hat{g}_{map}(\cdot; \lambda^8, u)\) maximizes the posterior probability considering a postulated scalar AWGN channel. We refer to this estimator as the decoupled MAP estimator and define it as follows:

Definition V.1 (Decoupled MAP estimator) The decoupled MAP estimator \(\hat{g}_{map}(\cdot; \lambda^8, u)\) with the estimation parameter \(\lambda^8\) and the utility function \(u(\cdot)\) is defined as

\[\hat{g}_{map}(y; \lambda^8, u) := \arg \max_v q_{ls}(v|y) \quad (75a)\]

\[= \arg \min_v \left[ \frac{1}{2\lambda^8} (y - v)^2 + u(v) \right], \quad (75b)\]

where \(q_{ls}(v|y)\) denotes the decoupled posterior distribution postulated by the estimator which reads

\[q_{ls}(v|y) = K e^{-\frac{1}{2\lambda^8} (y - v)^2 + u(v) / \lambda^8} \quad (76)\]

for some real constant \(K\).

Using the definition of the decoupled MAP estimator, the RS decoupled system is defined next.

Definition V.2 (RS decoupled system) The RS decoupled system is consistent with the block diagram in Fig. 1 in which

- source symbol \(x\) is distributed with \(p_x\) over support \(\mathbb{X}\),
- \(z\) is a zero-mean and unit-variance Gaussian random variable,
- \(x\) and \(z\) are independent,
- \(\hat{x}\) is estimated from the observation \(y := x + \sqrt{\lambda^2_0} z\) as
  \[\hat{x} = \hat{g}_{map}(y; \lambda^8, u). \quad (77)\]

\(\hat{g}_{map}(\cdot; \lambda^8, u)\) is the decoupled MAP estimator with the estimation parameter \(\lambda^8\) and the utility function \(u(\cdot)\) as defined in Definition V.1.

\(\lambda^8\) and \(\lambda^8\) are defined as in Result V.1.

Using Result IV.2, the equivalency in the asymptotic distortion can be extended to the asymptotic conditional distribution as well.

Result V.2 (RS Decoupling Principle) Let the linear system (1) fulfill the constraints of Section II and be estimated via the MAP estimator in (2). Consider the RS decoupled system as defined in Definition V.2, and suppose Assumptions 1, 2 and 3 hold. Then, for any \(j \in [1 : n]\), the tuple \((\hat{x}_j, x_j)\) converges in distribution to \((\hat{x}, x)\) if \(p_x = p_x\), almost surely in \(A\).
Proof. Using Result IV.2, for any two different indices \( j, q \in [1 : n] \) we have
\[
p_{i|x}^j(\hat{g}|\hat{v}) = p_{i|x}^q(\hat{g}|\hat{v})
\] (78)
at the mass point \((\hat{g}, \hat{v})\). Therefore for any index \( j \), we have
\[
E \chi_j k x_{j}^\ell = \lim_{n \to \infty} \frac{1}{E} \sum_{i=1}^{n} \chi_i k x_{j}^\ell.
\] (79)
Consequently, the asymptotic \((k, \ell)\)-th joint moment of tuple \((\hat{x}_j, x_j)\) under the RS assumption is determined by letting \(\forall (n) = [1 : n]\) and the distortion function in the RS ansatz
\[
d(\hat{x}; x) = \chi k x_{\ell}.
\] (80)
and determining the asymptotic distortion. Substituting in Result V.1, the asymptotic joint moment reads
\[
M_{k,\ell}^j = E \int g^k x^\ell D\xi
\] (81)
where \(g\) is defined in (69). Considering Definition V.2 and assuming \(p_x = p_x^*, (81)\) describes the \((k, \ell)\)-th joint moment of \((\hat{x}, x)\) as well. Noting that \(M_{k,\ell}^j\) is uniformly bounded for any pair of indices \(k\) and \(\ell\), it is concluded that the asymptotic joint distribution of \((\hat{x}_j, x_j)\) and the joint distribution of \((\hat{x}, x)\) are equivalent. \(\square\)

Result V.2 gives a more general form of the RS decoupling principles investigated in [40] and [41]. In fact, by restricting the system matrix and source distribution as in [40] and [41], one can recover the formerly studied RS decoupling principles.

RS Zero Temperature
To have a basic measure of the RS ansatz consistency, we evaluate the zero temperature entropy under the RS assumption following the discussion in Section IV-C. Substituting (68) in (66) and taking the same steps as in Appendix B, the RS zero temperature entropy is determined as
\[
\mathcal{H}_{\text{rs}}^0 = \lim_{\beta \to \infty} \frac{\beta^2}{2} \frac{\partial}{\partial \beta} \left[ \int_0^1 F^\beta(\omega) d\omega - F^\beta(1) \right]
\] (82)
where the function \(F^\beta(\cdot)\) is defined as
\[
F^\beta(\omega) = \frac{\chi}{\beta} R_{\beta}(-\frac{\chi}{\beta} \omega) + \left[ q - \frac{\lambda_0}{\chi} \right] \frac{d}{d\omega} R_{\beta}(-\frac{\chi}{\beta} \omega).
\] (83)
Taking the derivative first and then the limit, it finally reads
\[
\mathcal{H}_{\text{rs}}^0 = \frac{\chi}{2\lambda} \left[ R_{\beta}(-\frac{\chi}{\beta} \omega) - \int_0^1 R_{\beta}(-\frac{\chi}{\beta} \omega) d\omega \right].
\] (84)
We later see that the zero temperature entropy takes the same form under the RSB assumptions.

VI. RSB ANSÄTZE AND RSB DECOUPLING PRINCIPLE
In [83], Parisi introduced a breaking scheme to broaden the restricted set of correlation matrices. The scheme recursively extends the set of matrices. This breaking scheme was employed to broaden the RS structure, and therefore the obtained structure was identified as the broken RS or, RSB structure. The key feature of Parisi’s breaking scheme is that, by starting from the RS structure, the new structure after breaking can be reduced to the structure before breaking. Thus, the set of fixed point solutions found by assuming the broken structure includes the solutions of the previous structure as well.

Definition VI.1 (Parisi’s breaking scheme) Let \(m\) be a multiple of the integer \(\xi\), and \(Q^{(\ell)}\) represent an \(n \times n\) correlation matrix. Parisi’s breaking scheme then constructs a new \(m \times m\) correlation matrix \(Q^{(\ell+1)}\) as
\[
Q^{(\ell+1)} = I_{\xi} \otimes Q^{(\ell)} + \kappa I_m
\] (85)
for some real scalar \(\kappa\).

By choosing \(Q^{(0)}\) to be an RS correlation matrix in Definition VI.1, the matrix \(Q^{(1)}\) finds the RSB structure with one step of breaking (1RSB). The steps of breaking can be further increased recursively by inserting \(Q^{(1)}\) in the breaking scheme, determining the new correlation matrix \(Q^{(2)}\), and repeating the procedure. We start with the 1RSB correlation matrix, and then, extend the result to higher RSB ansätze with more steps of breaking.

Assumption 4 (1RSB Structure) Considering the spin glass of replicas as defined in Definition IV.1, the 1RSB replica correlation matrix is of the form
\[
Q = \frac{\chi}{\beta} I_m + p \frac{1}{\beta} \otimes 1_m + q 1_m
\] (86)
where \(\chi, p, q\) and \(\mu\) are some non-negative real scalars.

Assumption 4 considers \(Q = Q^{(1)}\) by letting \(Q^{(0)}\) have the RS structure with parameters \(\chi \beta^{-1}\) and \(p, \xi = \mu \beta^{-1}\) and \(\kappa = q\) in Parisi’s breaking scheme. Here, the 1RSB structure reduces to RS by setting \(p = 0\). Therefore, the set of 1RSB correlation matrices contains those given by Assumption 3.

Result VI.1 (1RSB Ansatz) Let linear system (1) fulfill the constraints of Section II. Suppose Assumptions I and 2, as well as Assumption 4 hold. Let \(x \sim p_x\) and
\[
g = \arg\min_{v} \left[ \frac{1}{2\lambda} (x + \sqrt{\lambda \delta_0} z_0 + \sqrt{\lambda \delta_1} - v)^2 + u(v) \right]
\] (87)
with \(v \in \mathbb{R}\) and \(\delta_0^\lambda, \delta_1^\lambda\) and \(\delta\) being defined by
\[
\delta_0^\lambda = \frac{\partial}{\partial \lambda} \left[ (\lambda \delta_0 + \lambda \delta_1) R_{\lambda}(-\frac{\lambda \delta_1}{\lambda}) \right],
\] (88a)
\[
\delta_1^\lambda = \frac{\lambda^2}{\mu R_{\lambda}(-\frac{\lambda \delta_1}{\lambda})},
\] (88b)
\[
\delta^\lambda = \frac{\lambda}{\mu R_{\lambda}(-\frac{\lambda \delta_1}{\lambda})}
\] (88c)
for some non-negative real \(\lambda\), \(p\), \(q\) and \(\mu\) and real variables \(z_0\) and \(z_1\). Then, the asymptotic distortion determined via the replica method reads
\[
D^{W} = E \int d(g, x) \, \hat{\lambda}(z_1|z_0, x) Dz_1Dz_2,
\] (89)
with $\tilde{\Lambda}(z_1|z_0, x) := \int \Lambda(z_1, z_0, x)\,Dz_1^{-1} \Lambda(z_1, z_0, x)$ and $\Lambda(z_1, z_0, x)$ being defined by

$$\Lambda(z_1, z_0, x) = \frac{e^{-\mu\left[\mathcal{F}(\chi + \nu) + \mathcal{F}(\chi - \nu)\right]}}{e^{-\mu\left[\mathcal{F}(\chi + \nu) + \mathcal{F}(\chi - \nu)\right]}}.$$  \hfill (90)

The scalars $\chi$, $p$, and $q$ satisfy

$$\sqrt{\lambda_r^2}(\chi + \mu p) = \mathcal{F}(\chi + \nu)\int (g - x)z_0 \tilde{\Lambda}(z_1|z_0, x)\,Dz_1\,Dz_0 \tag{91a}$$

$$\sqrt{\lambda_r^2}(\chi + \mu q + \mu p) = \mathcal{F}(\chi + \nu)\int (g - x)z_1 \tilde{\Lambda}(z_1|z_0, x)\,Dz_1\,Dz_0 \tag{91b}$$

$$q + p = \mathcal{E}\int (g - x)^2 \tilde{\Lambda}(z_1|z_0, x)\,Dz_1\,Dz_0 \tag{91c}$$

and $\mu$ is a solution of the fixed point equation

$$\frac{\mu}{2\lambda^2} \left[ \frac{\lambda_r^2}{\lambda^2} q + p \right] - \frac{1}{2\lambda} \int \mathcal{F}(\chi + \nu)\,d\nu = \mathcal{E}\int \log \tilde{\Lambda}(z_1|z_0, x) \tilde{\Lambda}(z_1|z_0, x)\,Dz_1\,Dz_0. \tag{92}$$

$\chi$, $p$, $q$, and $\mu$ minimize the zero temperature free energy $\mathcal{F}^{[0]}$ which reads

$$\mathcal{F}^{[0]} = \frac{1}{2\lambda} \left[ \int \mathcal{F}(\chi + \nu)\,d\nu - \mathcal{F}(1) \right]$$

$$- \frac{1}{\mu} \mathcal{E}\int \log \left[ \int \Lambda(z_1, z_0, x)\,Dz_1 \right] \,Dz_0 \tag{93}$$

with $\mathcal{F}(\cdot)$ being defined as

$$\mathcal{F}(\chi) = \frac{1}{\mu} \left[ \int \mathcal{F}(\chi + \nu)\,d\nu \right]$$

$$+ \left[ q - \lambda_0 \frac{\chi + \mu p}{\lambda} \right] \frac{d}{d\nu} \left[ \mathcal{F}(\chi + \nu)\,d\nu \right]. \tag{94}$$

Proof. See Appendix C. \hfill \Box

Similar to our approach under the RS ansatz, we employ Result VI.1 to introduce the decoupled 1RSB single-user system which describes the statistical behavior of the MAP estimator’s input-output entries under the 1RSB assumption. The decoupled system under 1RSB differs from RS by a new impairment term which is correlated with the source and noise symbols through a joint distribution. The impairment term intuitively plays the role of a correction factor which compensates the possible inaccuracy of the RS ansatz. The decoupled MAP estimator, however, follows the same structure as for RS.

**Definition VI.2 (1RSB decoupled system)** Fig. 2 defines the 1RSB decoupled system in which

- source symbol $x$ is distributed with $p_x$ over support $\mathcal{X}$,
- $z_0$ is a zero-mean and unit-variance Gaussian random variable,
- $z_1$ is a random variable dependent on $x$ and $z_0$.

$$P_{z_1|x, z_0} = \tilde{\Lambda}(z_1|z_0, x)\psi(z_1) \tag{95}$$

with $\tilde{\Lambda}$ defined in Result VI.1.

- $\hat{x}$ is estimated from the observation $y := x + \sqrt{\lambda_r^2}z_0 + \sqrt{\lambda_r^2}z_1$ as
  $$\hat{x} = \mathcal{G}(\cdot; \lambda^R, u). \tag{96}$$

- $\mathcal{G}(\cdot; \lambda^R, u)$ is the decoupled MAP estimator with estimation parameter $\lambda^R$ and utility function $u(\cdot)$ as defined in Definition VI.1.

- $\lambda^R$, $\lambda_1^R$, and $\lambda^R$ are defined as in Result VI.1.

**Result VI.2 (1RSB Decoupling Principle)** Let linear system (1) fulfill the constraints of Section II and be estimated via the MAP estimator in (2). Consider the 1RSB decoupled system as defined in Definition VI.2, and suppose Assumptions 1, 2 and 4 hold. Then, for all $j \in \{1 : n\}$, the tuple $(\hat{x}_j, x_j)$ converges in distribution to $(\tilde{x}, x)$ if $p_x = p_X$.

Proof. The proof takes exactly same steps as for the RS decoupling principle using Result VI.1. \hfill \Box

The 1RSB decoupled system, in general, provides a more accurate approximation of the estimator’s asymptotics by searching over a larger set of solutions which include the RS ansatz. To investigate the latter statement, consider the case of $p = 0$. In this case, the 1RSB structure reduces to RS. Setting $p = 0$ in Result VI.1, $\lambda_1$ becomes zero, and consequently $\tilde{\Lambda}(z_1|z_0, x) = 1$. Moreover, the fixed point equations in (92) hold for any choice of $\mu$, and the scalars $\chi$ and $q$ couple through the same set of equations as in the RS ansatz. The zero temperature free energy of the system, furthermore, reduces to its RS form under the assumption of $p = 0$. Denoting the parameters of the RS ansatz by $[\chi_{rs}, q_{rs}]$, it is then concluded that $[\chi, p, q, \mu] = [\chi_{rs}, 0, q_{rs}, 0]$ is a solution to the 1RSB fixed point equations, when an stable solution to the RS fixed point exists. The solution, however, does not give necessarily the 1RSB ansatz, the stable solution to the 1RSB fixed point equations with minimum free energy may occur at some other point. We investigate the impact of replica breaking for some examples later through numerical results.

Parisi's breaking scheme can be employed to extend the structure of the correlation matrix to the RSB structure with more steps of breaking by recursively repeating the scheme. In fact, one can start from an RS structured $Q^{[0]}$ and employ the breaking scheme for $b$ steps to determine the correlation matrix $Q^{[b]}$. In this case, the replica correlation matrix is referred to as the $b$RSB correlation matrix.
Assumption 5 (bRSB Structure) Considering the spin glass of replicas as defined in Definition IV.1, the bRSB replica correlation matrix is of the form

$$Q = \frac{\chi}{\beta} I_m + \sum_{\kappa=1}^{b} p_\kappa I_{m,\kappa} \boxtimes I_{m,\kappa} + q I_m$$

(97)

where scalars $\chi$ and $q$, and sequences $\{p_\kappa\}$ and $\{\mu_\kappa\}$ with $\kappa \in [1 : b]$ are non-negative reals, and $\{\mu_\kappa\}$ satisfies the following constraint

$$\frac{\mu_{\kappa+1}}{\mu_\kappa} \in \mathbb{Z}^+$$

(98)

for $\kappa \in [1 : b - 1]$.

Considering the correlation matrix in Result IV.1 to be of the form indicated in Assumption 5 the previous ansätze are extended to a more general ansatz which can reduce to the 1RSB as well as RS ansatz. Result VI.3 expresses the replica ansatz under the bRSB assumption.

Result VI.3 (bRSB Ansatz) Let linear system (1) fulfill the constraints of Section II. Suppose Assumptions 1 and 2, as well as Assumption 5 hold. For $\kappa \in [0 : b]$, define the sequence $\{\tilde{\kappa}_\kappa\}$, such that $\tilde{\kappa}_0 = \chi$ and

$$\tilde{\kappa}_\kappa := \chi + \sum_{\varsigma=1}^\kappa \mu_\varsigma p_\varsigma$$

(99)

for $\kappa \in [1 : b]$. Let $x \sim p_x$, and

$$g = \arg \min_v \left[ \frac{1}{2\lambda_0^0} (\chi + \sum_{\kappa=0}^b \sqrt{\lambda_0^0 z_{\kappa}^0} - v)^2 + u(v) \right]$$

(100)

with $v \in \chi$ and $\lambda_0^0$, $\lambda_0^b$ and $\lambda_0^b$ being defined as

$$\lambda_0^0 = \left[ \frac{\lambda_0 \tilde{b}_0}{\lambda_0} \right] \frac{\chi - \frac{1}{2} \lambda_0 \tilde{b}_0}{\lambda_0^0}$$

and

$$\lambda_0^b = \left[ \frac{\lambda_0 \tilde{b}_0}{\lambda_0} \right] \frac{\chi - \frac{1}{2} \lambda_0 \tilde{b}_0}{\lambda_0^0}$$

(101b)

(101a)

$$\lambda_0^b = \left[ \frac{\lambda_0 \tilde{b}_0}{\lambda_0} \right] \frac{\chi - \frac{1}{2} \lambda_0 \tilde{b}_0}{\lambda_0^0}$$

(101c)

for some non-negative real scalar $q$, sequences $\{\tilde{\kappa}_\kappa\}$ and $\{\mu_\kappa\}$ and the sequence of real variables $\{z_\kappa\}$. Then, the asymptotic distortion determined via the replica method reads

$$D^{\text{F}} = E \left[ \int d(g ; x) \sum_{\kappa=1}^b \hat{A}_\kappa(z_\kappa) \left[ \frac{1}{2} \lambda_0 \tilde{b}_0 + \frac{1}{2} \lambda_0 \tilde{b}_0 \right] \right] D_{z_\kappa} D_{z_\kappa}$$

(102)

where $\{z_\kappa\} = \{z_\kappa, \ldots, z_b\}$ and $\hat{A}_\kappa(z_\kappa) = \frac{1}{2} \int A_\kappa((z_\kappa)_{\kappa+1}^b, z_\kappa, 0) D_{z_\kappa} D_{z_\kappa}$

$$A_\kappa((z_\kappa)_{\kappa+1}^b, z_\kappa, 0) = \left[ \int A_\kappa((z_\kappa)_{\kappa+1}^b, z_\kappa, 0) D_{z_\kappa} D_{z_\kappa} \right]^{-1} D_{z_\kappa} D_{z_\kappa}$$

(103)

and $\{A_\kappa((z_\kappa)_{\kappa+1}^b, z_\kappa, 0)\}$ for $\kappa \in [2 : b]$ being recursively determined by

$$A_\kappa((z_\kappa)_{\kappa+1}^b, z_\kappa, 0) = \left[ \int A_{\kappa-1}((z_\kappa)_{\kappa+1}^b, z_\kappa, 0) D_{z_\kappa} D_{z_\kappa} \right]^{\mu_{\kappa-1}}.$$

(104)

The scalar $q$ and sequences $\{\tilde{\kappa}_\kappa\}$ and $\{p_\kappa\}$ satisfy the fixed-point equations in (105a)-(105c), as shown at the bottom of this page, and sequence $\{\mu_\kappa\}$ is given by (106), as shown at the bottom of this page. In (106), the function $F(\omega; \{\mu_\kappa\})$ is defined as

$$F(\omega; \{\mu_\kappa\}) := \frac{b}{\mu_\kappa} \int_0^1 \frac{d\omega}{\mu_\kappa} \int \tilde{\kappa}_\kappa \omega^t R_\kappa(-\tilde{z}_\kappa) d\omega$$

(107)

$$[q - \frac{z_0}{\lambda_0} \tilde{b}_0 \tilde{b}_0] \frac{d}{d\omega} \left[ \omega R_\kappa(-\tilde{z}_\kappa) \right]$$

(108)

$$\Lambda(\cdot)$$ is defined as

$$\Lambda(\{\mu_\kappa\}) = \frac{b}{\mu_\kappa} \left[ \zeta_0 \tilde{b}_0 - \zeta_{\kappa-1} \tilde{b}_0 \right] + \zeta_b q - \frac{\lambda_0}{\lambda_0^b} \tilde{b}_0$$

(109)

with $\zeta_0 = 1$, and $\zeta_\kappa$ for $\kappa \in [1 : b]$ denoting

$$\zeta_\kappa := 1 - \sum_{\varsigma=1}^\kappa \mu_\varsigma \frac{\lambda_0^\varsigma}{\lambda_0^\varsigma}$$

and $\{\tilde{\mu}_\kappa\} \in S_{\mu}$ in which

$$S_{\mu} := \left\{ \{\mu_1, \ldots, \mu_b\} \ni \mu_\kappa \in \mathbb{R}^+ \land \frac{\mu_{\kappa+1}}{\mu_\kappa} \in \mathbb{Z}^+ \forall \kappa \in [1 : b - 1] \right\}.$$

(110)
In the case of multiple solutions for $\chi$, $q$, $\{p_\kappa\}$ and $\{\mu_\kappa\}$, the ansatz is chosen such that the free energy at zero temperature

$$F^{(\beta)}_{rsb} = \frac{1}{2\lambda} \left[ \int_0^1 F(\omega; \{\mu_\kappa\})d\omega - F(1; \{\mu_\kappa\}) \right]$$

$$- \frac{1}{\mu_b} E \log \left[ \int \Lambda_b(z_b, z_0, x) Dz_b \right] Dz_0$$

(111)

is minimized.

**Proof.** See Appendix D.

One can simply observe that Result VI.3 reduces to Results VI.1 and VI.1 by letting $b = 1$ and $p_\kappa = 0$ for $\kappa \in [1 : b]$, respectively. The ansatz, moreover, extends the corresponding decoupled single-user system of the estimator considering the general decoupling principle investigated in Result IV.2. By taking the same steps as in Result IV.2, the decoupled bRSB single-user system is found which represents the extended version of the 1RSB system with $b$ additive impairment taps. In fact, considering the impairment terms to intuitively play the role of correction factors, the bRSB ansatz takes more steps of correction into account. The decoupled MAP estimator, moreover, remains unchanged.

**Definition VI.3 (bRSB decoupled system)** The bRSB decoupled system is consistent with the diagram in Fig. 3 in which

- source symbol $x$ is distributed with $p_x$ over support $X$,
- $z_0$ is a zero-mean and unit-variance Gaussian random variable,
- $z_\kappa$ is a random variable dependent on $x$, $z_0$ and $\{z_{\kappa+1}, \ldots, z_b\}$
- $x$ and $z_0$ are independent, and
- $p_{z_{\kappa+1}, \ldots, z_b, z_0, x} = \tilde{\Lambda}_\kappa(z_\kappa | [z_{\kappa+1}, z_0, x]) \phi(z_\kappa)$

(112)

with $\tilde{\Lambda}$ defined in Result VI.3.

- $\hat{x}$ is estimated from observation $y := x + \sum_{\kappa=0}^b \sqrt{\lambda_\kappa} z_\kappa$ as

$$\hat{x} = \mathcal{g}_{\text{map}}[y; \lambda^b, u]$$

(113)

- $\mathcal{g}_{\text{map}}[\cdot; \lambda^b, u]$ is the decoupled MAP estimator with estimation parameter $\lambda^b$ and utility function $u(\cdot)$ as defined in Definition VI.1, and

- $\lambda^0_\kappa$ and $\lambda_\kappa^b$ for $\kappa \in [1 : b]$ are as in Result VI.3.

**Result VI.4 (bRSB Decoupling Principle)** Let linear system (1) fulfill the constraints of Section II and be estimated via the MAP estimator in (2). Consider the bRSB decoupled system as defined in Definition VI.3, and suppose Assumptions 1, 2 and 5 hold. Then, for all $j \in [1 : n]$, the tuple $(\hat{x}_j, x_j)$ converges in distribution to $(\hat{x}, x)$ if $p_\kappa = p_x$.

**Proof.** Using Result VI.1, it takes the same steps as for Result V.2.

**RSB Zero Temperature**

In Appendix D, it is shown that under the bRSB assumption on the replica correlation matrix the free energy of the corresponding spin glass at the inverse temperature $\beta$ reads

$$F(\beta) = \frac{1}{2\lambda} \left[ \int_0^1 F^\beta(\omega)d\omega - F^\beta(1) \right] + F^R(\beta).$$

(114)

Here, $F^R(\beta)$ denotes the normalized free energy of the spin glass of replicas defined in (53) in the limit $m \downarrow 0$, and the function $F^\beta(\cdot)$ is defined as

$$F^\beta(\omega) = \sum_{\kappa=1}^b \frac{1}{\mu_\kappa} \frac{d}{d\omega} J_{\omega}^{\kappa \omega} R_{J\omega}(-\frac{\omega}{\lambda}) + \frac{X}{\beta} R_{J\omega}(-\frac{X}{\lambda})$$

$$+ \left[ q - \frac{\lambda_0}{\lambda} \frac{d}{d\omega} \right] \frac{d}{d\omega} \left[ R_{J\omega}(-\frac{X}{\lambda}) \right].$$

(115)

Following the discussion in Section IV-C, the entropy at the zero temperature reads

$$H^{(b)}_{rsb} = \lim_{\beta \uparrow \infty} \frac{\beta^2}{2\lambda} \left[ \int_0^1 F^\beta(\omega)d\omega - F^\beta(1) \right]$$

(116)

which reduces to

$$H^{(b)}_{rsb} = \frac{1}{2\lambda} \left[ R_{J\omega}(\frac{X}{\lambda}) - \int_0^1 R_{J\omega}(\frac{X}{\lambda})d\omega \right].$$

(117)

This equation justifies the conjecture in [45] and states that the zero temperature entropy under any number of breaking steps, including the RS case, is of the similar form and only depends on scalar $\chi$. In fact, the Hamiltonian in (11) reduces to the one considered in vector precoding by considering $x$ to be the deterministic vector of zeros, $\lambda_0 = 0$, $\lambda = 1$ and $u(\nu) = 0$. Substituting in (117), the zero temperature entropy reduces to the one determined in [45] within a factor of 2. The factor comes from the difference in the prior assumption on the support of microstate.\(^{10}\)

**Remark VI.1.** From (117), it is observed that by setting $\chi = 0$, the zero temperature entropy reads $H^{(b)}_{rsb} = 0$. This means that $\chi = 0$ is a sufficient condition for having a vanishing zero temperature entropy. In [45, Appendix E], it is shown that for any probability distribution different from a single mass point, $x$.

\(^{10}\)In [45], the authors considered $\nu$ to be a complex vector.
the R-transform, wherever its derivative w.r.t. a real argument exists, is a strictly increasing function. As a result, for any \( A \neq \theta I \), we have

\[
R_f(-\frac{\chi}{\lambda}) \leq \int_0^1 R_f(-\frac{\chi}{\lambda}) d\omega
\]

where the equality occurs if and only if \( \chi = 0 \). This means that \( \chi = 0 \) is also a necessary condition for \( \mathcal{H}_b^{(\text{RSB})} = 0 \). Hence, one concludes that \( \chi \) determines the consistency of a given ansatz. In other words, the ansatz whose fixed point equations result in smaller value of \( \chi \) is more reliable.

VII. REPLICA SIMULATOR: CHARACTERIZATION VIA THE SINGLE-USER REPRESENTATION

The \( b \text{RSB} \) decoupling principle determines an equivalent single-user system which describes the input-output statistics of the MAP estimator under the \( b \text{RSB} \) ansatz. In order to specify the exact parameters of the decoupled single-user system, we need to solve the fixed point equations given in (106). The proposed approach enables us to investigate the properties of the RS and RSB ansätze by studying the replica simulator. In order to clarify the idea of the replica simulator, let us define a set of input-output statistics for the \( b \text{RSB} \) decoupled system.

Definition VII.1 Consider the single-user system consistent with the block diagram in Fig. 3. Denote the joint distribution of the source and impairment terms with \( p_{x,z} \). For this system,

- the \( \kappa \)-th noise-error correlation is defined as
  \[
  C_{\kappa} = \frac{1}{\sqrt{\lambda_\kappa}} E(\hat{x} - x)z_\kappa
  \]
  \[\text{(118)}\]
  for \( \kappa \in [0:b] \), and
- the MSE is denoted by
  \[
  \text{MSE} = E(\hat{x} - x)^2
  \]
  \[\text{(119)}\]

Invoking Definition VII.1, the \( b \text{RSB} \) ansatz is completely represented in terms of the input-output statistics of the decoupled system. In fact, by means of Definition VII.1, the fixed point equations in (105a)-(105c) can be expressed as

\[
\sum_{\kappa=1}^{b} \frac{\theta_{b}}{\lambda_\kappa} + \mu_{\kappa} \left( \sum_{\kappa=0}^{b} p_{\kappa} + q \right) = \lambda^{b} C_{\kappa},
\]

\[\text{(120b)}\]

\[
\hat{\lambda}_{b-1} + \mu_{\kappa} \left( \sum_{\kappa=0}^{b} p_{\kappa} + q \right) = \lambda^{b} C_{\kappa},
\]

\[\text{(120c)}\]

for \( \kappa \in [1:b] \); moreover, the factor \( \Lambda_1 \) is given as

\[
\Lambda_1((z_{\kappa}^b)_{\kappa=1}^b, z_0, x) = \frac{e^{-\mu_{1}^b \frac{1}{2\lambda_1} (y-\hat{y})^2 + u(\hat{y})}}{e^{-\mu_{1}^b \frac{1}{2\lambda_1} (y-x)^2}},
\]

\[\text{(121)}\]

which reduces to

\[
\Lambda_1((z_{\kappa}^b)_{\kappa=1}^b, z_0, x) = p_{x}(x)^{\mu_{1}^b} \left[ \frac{q_{RSB}(\hat{y}|y)}{q_{RSB}(x|y)} \right]^{\mu_{1}^b},
\]

\[\text{(122)}\]

with \( q_{RSB}(\cdot|y) \) indicating the decoupled posterior distribution defined in Definition V.1. The second term on the r.h.s. of (122) is an extended form of the likelihood ratio. By defining

\[
\Gamma_1((z_{\kappa}^b)_{\kappa=1}^b, z_0, x) = \left[ \frac{q_{RSB}(\hat{y}|y)}{q_{RSB}(x|y)} \right]^{\mu_{1}^b},
\]

\[\text{(123)}\]

(122) reads

\[
\Lambda_1((z_{\kappa}^b)_{\kappa=1}^b, z_0, x) = p_{x}(x)^{\mu_{1}^b} \Gamma_1((z_{\kappa}^b)_{\kappa=1}^b, z_0, x),
\]

\[\text{(124)}\]

and \( \Lambda_b((z_{\kappa}^b)_{\kappa=0}^b, z_0, x) \) for \( \kappa \in [2:b] \) are determined by

\[
\Lambda_{\kappa}((z_{\kappa}^b)_{\kappa=0}^\kappa, z_0, x) = p_{x}(x)^{\mu_{\kappa}} \Gamma_{\kappa}((z_{\kappa}^b)_{\kappa=0}^\kappa, z_0, x),
\]

\[\text{(125)}\]

where \( \Gamma_{\kappa}((z_{\kappa}^b)_{\kappa=0}^\kappa, z_0, x) \) are recursively defined as

\[
\Gamma_{\kappa}((z_{\kappa}^b)_{\kappa=0}^\kappa, z_0, x) = \left( \int \Gamma_{\kappa-1}((z_{\kappa}^b)_{\kappa-1}^{\kappa-1}, z_0, x) Dz_{\kappa-1} \right)^{\mu_{\kappa}^{\kappa-1}}.
\]

\[\text{(126)}\]

The fixed point in (106) is further rewritten accordingly.

The above alternative representation of the \( b \text{RSB} \) ansatz leads us to a new interpretation. In fact, one can define a transition system in which the vector of replica parameters denotes the state, and the decoupled single-user system defines the transition rule [84], [85]. We refer to this transition system as the \textit{replica simulator}, and define it formally as follows:

Definition VII.2 (Replica simulator) Let \( b \) be a non-negative integer. Define the vector \( s \) as

\[
s := [\chi, \mu_1, \ldots, \mu_b, p_1, \ldots, p_b, q],
\]

\[\text{(127)}\]

with entries satisfying the corresponding constraints in Result VI.3, and denote its support by \( \mathbb{S}_b \). The transition rule \( \mathbb{T}_b : \mathbb{S}_b \mapsto \mathbb{S}_b \) maps the prior state \( s_i \in \mathbb{S}_b \) to the posterior state \( s_f \in \mathbb{S}_b \) in the following way: \( \mathbb{T}_b \) realizes the \( b \text{RSB} \) decoupled system considering the entries of \( s_i \) as the replica parameters. It then constructs the entries of \( s_f \) by determining a new set of replica parameters from the statistics of the decoupled system using the equivalent representation of the fixed point equations given in (120a)-(126).

The replica simulator of breaking order \( b \) is then defined as the transition system \( \text{Sim}^{\text{RSB}}[b] := (\mathbb{S}_b, \mathbb{T}_b) \).

The structure of the replica simulator is illustrated in Fig. 4. For the replica simulator of breaking order \( b \), a sequence of states \( \{s_t\} \) is considered to be a \textit{process}, if for \( t \in [1: \infty] \)

\[
s_t \xrightarrow{\mathbb{T}_b} s_{t+1}.
\]

\[\text{(128)}\]
The state \( s^* \) is then called the steady state, if setting \( s_t = s^* \) results in \( s_{t+1} = s^* \). Considering Result VI.3, the bRSB ansatz is in fact the steady state of the replica simulator which minimizes the free energy function. Our conclusion also extends to the RS case, if we set \( b = 0 \).

Considering Definition VII.2, as well as the above discussions, the bRSB ansatz can be numerically investigated using the methods developed in the literature of transition systems. This approach may reduce the complexity of numerical analysis; however, it does not impact the computational complexity. In fact, assuming that one realizes the bRSB decoupled system for any desired state vector denoted in (127) via some methods of realization, e.g., Monte Carlo simulation, the bRSB ansatz can be found by means of an iterative algorithm which has been designed to find the steady state of a transition system.

The latter statement can be clarified as in Scheme 1.

In Scheme 1, \( T^b \) in step A can be realized via various methods. One may determine the input-output distribution of the single-user system analytically or simulate the system by generating impairment and source samples numerically via Monte Carlo techniques. Another degree of freedom is in step B where different mapping rules with different convergence speeds can be employed. For algorithms designed based on Scheme 1, the computational complexity depends on the realization method while the convergence speed is mainly restricted with some given mapping rule \( IM(\cdot, \cdot) \).

The replica simulator introduces a systematic approach for investigating the replica anàtse based on the decoupling principle. Moreover, it gives an intuition about the impact of symmetry breaking. To clarify the latter statement, let us consider an example.

Example. (RS vs. 1RSB ansatz) Let \( b = 0 \); thus, the RS fixed point equations read

\[
q = \text{MSE}, \\
\chi = \lambda^b C_0.
\]

The equations under the 1RSB assumption are further given by

\[
q + p = \text{MSE}, \\
\chi + \mu p = \lambda^b C_0, \\
\chi + \mu p + \mu q = \lambda^b C_1,
\]

and \( \mu \) is determined through the fixed point equation

\[
\frac{\mu}{2} x^b \left[ \frac{\lambda^b}{\lambda} q + p \right] - \frac{1}{2\lambda} \int_0^{\mu p} R_1(-\chi + \omega) d\omega = I(z_1; z_0, x) + D_{\text{KL}}(p_{z_1} \parallel \phi)
\]

where \( I(\cdot; \cdot) \) and \( D_{\text{KL}}(\cdot \parallel \cdot) \) denote the mutual information and Kullback-Leibler divergence, respectively. Assuming the system matrix to be i.i.d. and setting \( z_1 \) to be independent of \( z_0 \) and \( x \), the r.h.s. of (131) tends to zero, and therefore, the solutions \( \mu = 0 \) and \( p = 0 \) are concluded. Consequently, (130c) becomes ineffective, and the fixed point equations reduce to RS. The latter observation can be interpreted in terms of the state evolution of the replica simulator. More precisely, assume that the initial state of the replica simulator with breaking order one is chosen such that in the corresponding decoupled system, \( z_1 \) is sufficiently correlated with the source and noise symbols. In this case, by assuming the mapping rule \( IM(\cdot, \cdot) \) to be converging, the correlation in each iteration of Scheme 1 reduces, and thus, at the steady state, \( z_1 \) becomes independent of \( z_0 \) and \( x \).

The above discussion can be extended to replica simulators with larger breaking orders. Moreover, further properties of the RSB ansätze could be studied using methods developed in the literature of transition systems. We leave the further investigations as a possible future work.

VIII. LARGE COMPRESSIVE SENSING SYSTEMS

Considering the setting represented in Section II, a large compressive sensing system can be studied through our results by restricting the source’s CDF to be of the form

\[
F_x(x) = (1 - \alpha) \mathbf{1} \{x \geq 0\} + \alpha \tilde{F}_x(x).
\]

In the large-system limit, the source vector distributed as (132) has \((1 - \alpha)n\) entries equal to zero while the remaining \( an \) entries are distributed with \( \tilde{F}_x \). In this case, \( x \) is an \( an \)-sparse vector, and thus, (1) is considered to represent a large compressive sensing system with the sensing matrix \( A \).

Considering the prior as in (132), different recovery schemes are then investigated by restricting the prior setup of the system, correspondingly. In this section, we study the asymptotics of several recovery schemes using our bRSB decoupling principle for both the cases of continuous and discrete alphabet sources.
A. Continuous Sources

For \( X = \mathbb{R} \), (132) describes a continuous random variable multiplied by an \( \alpha \)-Bernoulli random variable. In this case, by varying utility function \( u(\cdot) \), different reconstruction schemes are considered. Here, we address the linear, LASSO and \( \ell_0 \) norm recovery schemes. However, the results can be employed to investigate a general \( \ell_p \) norm recovery scheme [63].

Example 1. (linear recovery scheme) The MAP estimation is reduced to the linear recovery scheme when the utility function is set to be

\[
u(u) = \frac{u^2}{2},
\]

(133)

In fact, in this case, the MAP estimator postulates the prior distribution to be a zero-mean and unit-variance Gaussian and performs considerably inefficient when the source is sparse.

Using the bRSB decoupling principle, we conclude that in the large-system limit source entry \( x_j \), for any \( j \in [1 : n] \), converge in probability to a sparse random variable \( x \) distributed as in (132) and the estimated symbol \( \hat{x} = \hat{g}_{\text{map}}(y; \lambda^s, u) \), respectively, where the decoupled system reduces to

\[
\hat{g}_{\text{map}}(y; \lambda^s, u) = \frac{y}{1 + \lambda^s},
\]

(134)

with \( y \) being given by

\[
y = x + \sum_{k=0}^{b} \sqrt{\lambda^s_k} \epsilon_k,
\]

(135)

and scalars \( \lambda^s \) and \( \{\lambda^s_k\} \) for \( k \in [0 : b] \) are defined as in Result VI.3. By letting \( b = 0 \), the result reduces to the RS decoupling principle reported in the literature, see [40], [41]; however, the result here holds for a wider set of sensing matrices and source distributions.

Example 2. (LASSO recovery scheme) To study the LASSO recovery scheme, we set

\[
u(u) = |u|.
\]

(136)

From the bRSB decoupling principle, the prior distribution of the decoupled input \( x \) is postulated to be Laplacian\(^{13} \) with unit variance. This postulation results in a better performance of the recovery scheme in many cases, since the Laplacian PDF could be a more realistic approximation of the sparse source distribution. Consequently, the decoupled system’s output is found as \( \hat{x} = \hat{g}_{\text{map}}(y; \lambda^s, u) \) with

\[
\hat{g}_{\text{map}}(y; \lambda^s, u) = |y - \lambda^s \text{sgn}(y)| w_{\lambda^s}(y)
\]

(137)

where \( y \) is denoted as in (135), \( w_{\lambda^s}(\cdot) \) is the null window function with window width \( \lambda^s \) defined as

\[
w_{\lambda^s}(y) = \begin{cases} 1 & |y| > \lambda^s \\ 0 & |y| \leq \lambda^s \end{cases},
\]

(138)

and \( \text{sgn}(y) \) is the sign indicator. The decoupled single-user estimator in (138) which is often referred to as the soft-thresholding operator recovers the earlier RS results by setting \( b = 0 \) and the sensing matrix to be i.i.d. [40].

Example 3. (\( \ell_0 \) norm recovery scheme) The \( \ell_0 \) norm recovery scheme which considers

\[
u(u) = 1_{\{u \neq 0\}}
\]

(139)

can perform significantly better than the latter schemes when the sparsity increases. In this case, the prior distribution of the bRSB decoupled system’s input \( x \) is found as the limit of

\[
P_{x}^{0,\delta}(x) = \frac{1}{2\theta + 2\delta(e - 1)} \begin{cases} e & |x| \leq \delta \\ 1 & \delta < |x| \leq \theta \end{cases},
\]

(140)

when \( \theta \uparrow \infty \) and \( \delta \downarrow 0 \). For finite values of \( \theta \) and \( \delta \), \( P_{x}^{0,\delta} \) can be considered as a sparse distribution in which non-zero symbols occur uniformly. This prior explains the better performance of the \( \ell_0 \) norm recovery scheme compared to the linear and LASSO schemes. For this case, the output of the decoupled single-user system reads \( \hat{x} = \hat{g}_{\text{map}}(y; \lambda^s, u) \), such that

\[
\hat{g}_{\text{map}}(y; \lambda^s, u) = y w_{\vartheta}(y)
\]

(141)

where \( y \) is denoted as in (135), and \( w_{\vartheta}(\cdot) \) is the null window function with \( \vartheta = \sqrt{2\lambda^s} \). Here, \( \hat{g}_{\text{map}}(y; \lambda^s, u) \) is the hard-thresholding operator and recovers the analysis in [40] for a wider class of settings.

The above examples have been also studied in earlier replica based studies, e.g., [40], [42]. The given results are analytically derived from the above expressions by considering the RS ansatz and properly substituting the corresponding R-transforms. We address the results of two important cases reported in the literature in following.

Special Case 1: In [40], the authors addressed the case of an i.i.d. sparse source which is sampled by an i.i.d. sensing matrix where the matrix entries are zero-mean random variables with the variance vanishing proportional to \( k^{-1} \). The asymptotic performance of the estimator was then addressed when the linear, LASSO, and \( \ell_0 \) norm recovery schemes are employed using the RS MAP decoupling principle. The results reported in [40] are derived directly by setting the R-transform in Result V.2 to be

\[
R_f(\omega) = \frac{1}{1 - \tau \omega}.
\]

(142)

Special Case 2: The results of [40] extended in [42] to a larger set of sensing matrices, and the RS prediction of the asymptotic MSE was determined for sparse Gaussian sources. The given results can be recovered by considering the distortion function

\[
\text{d}(\hat{x}; x) = ||\hat{x} - x||^2,
\]

(143)

and the source distribution to be (132) with \( \hat{F}_x \) representing the zero-mean and unit-variance Gaussian CDF.

B. Discrete Alphabet Sources

Our result can be further employed to study the sampling problem of discrete alphabet sources. Considering

\[
\mathbb{X} = \{0, t_1, \ldots, t_{\ell-1}\},
\]

(144)
in which the symbol 0 occurs with probability $1 - \alpha$, and other $\ell - 1$ outcomes are distributed due to $p_i$. Consequently, the source distribution reads

$$p_\ell(x) = (1 - \alpha) 1{x = 0} + \alpha 1{x \neq 0}p_i(x) \quad (145)$$

which can be interpreted as the multiplication of the non-zero discrete random variable $t$ distributed with $p_i$ and an $\alpha$-Bernoulli random variable. For sake of brevity, we denote the sorted version of the symbols in the support $X$ by $c_1, \ldots, c_\ell$ in which

$$-\infty < c_1 < c_2 < \ldots < c_\ell < +\infty. \quad (146)$$

For notational compactness, we further define $c_0$ and $c_{\ell + 1}$ to be $-\infty$ and $+\infty$, respectively. Similar to the continuous case, different choices of the utility function address different types of reconstruction schemes which we investigate in the sequel.

Example 4. (linear recovery scheme) We consider the case in which the discrete alphabet source is reconstructed via the linear recovery scheme as introduced in Example 1. Using the bRSB decoupling principle, the source and estimated symbols $(x_j, \hat{x}_j)$ converge to the random variables $x$ and $\tilde{x}$ for all $j \in [1 : n]$ where $x$ is distributed with $p_x$ defined in (145), and $\hat{x}$ is found as $\hat{x} = g_{\text{map}}(y); \lambda^g, u]$ with

$$g_{\text{map}}(y); \lambda^g, u] = c_k \quad \text{if} \quad y \in \left( v_k^{f_2}, v_{k+1}^{f_2} \right) \quad (147)$$

for $k \in [1 : \ell]$. The scalar $y$ indicates the observation symbol in the equivalent decoupled system, and the boundary point $v_k^{f_2}$ is defined as

$$v_k^{f_2} := \frac{1 + \lambda^g}{2}(c_k + 1 - c_k) \quad (148)$$

Example 5. (LASSO recovery scheme) Replacing the reconstruction scheme in Example 4 with LASSO, the single-user estimator of the bRSB decoupled system is of the form

$$g_{\text{map}}(y); \lambda^g, u] = c_k \quad \text{if} \quad y \in \left( v_k^{f_1}, v_{k+1}^{f_1} \right) \quad (149)$$

for $k \in [1 : \ell]$ where the boundary point $v_k^{f_1}$ reads

$$v_k^{f_1} := \frac{1}{2} (c_k - 1 + c_k) + \lambda^g \frac{|c_k| - |c_k - 1|}{c_k - c_k - 1}. \quad (150)$$

Example 6. ($\ell_0$ norm recovery scheme) For discrete alphabet sources, the $\ell_0$ norm recovery scheme is optimal in terms of symbol error rate, since it realizes the sparse uniform distribution. In fact, for the case in which the non-zero symbols of the source are uniformly distributed, the $\ell_0$ norm utility function exactly models the source’s true prior, and therefore, can be considered as the optimal scheme. Under the $\ell_0$ norm recovery scheme, the bRSB decoupled system reduces to

$$g_{\text{map}}(y); \lambda^g, u] = c_k \quad \text{if} \quad y \in \left( v_k^{f_0}, v_{k+1}^{f_0} \right) \quad (151)$$

with the boundary point $v_k^{f_0}$ being defined as

$$v_k^{f_0} := \frac{1}{2} (c_k + 1 + c_k) + \lambda^g \frac{1{|c_k| \neq 0} - 1{|c_k - 1| \neq 0}}{c_k - c_k - 1}. \quad (152)$$

for $k \in [1 : \ell]$.

IX. NUMERICAL RESULTS FOR LARGE COMPRESSIVE SENSING SYSTEMS

In this section, we numerically investigate examples of large compressive sensing systems for some known setups. For this purpose, we simulate the decoupled systems by setting the source distribution and sensing matrix to a specific form and determine the expected distortion of the equivalent scalar system. We then discuss the validity of the RS and RSB assumptions for these examples.

A. Simulation Setups

The settings and distortion functions being considered in the numerical investigations are as follows:

1) Sensing Matrices: Throughout the numerical investigations, we consider the two important cases of random i.i.d. and projector matrices.

- I.I.D. Random Matrix: In this case, the entries of $A_{k \times n}$ are supposed to be generated i.i.d. from an arbitrary distribution $p_a$. Without loss of generality, we assume that the entries are zero-mean random variables with variance $k^{-1}$. This structure is the most primary and also the most frequently discussed case in random matrix theory. For this matrix, regardless of $p_a$, it is well known that the asymptotic empirical eigenvalue CDF of the Gramian $J$ follows the Marchenko-Pastur law which states

$$F_J(\lambda) = \int_0^{|\lambda|} \frac{x \sqrt{\nu^2 + \nu^2 - 1}}{2\nu^2} \, d\nu \quad (153)$$

where $|x|$ returns $x$ when $x$ is non-negative and is zero otherwise [34], [35], [86]. Using the definition of the R-transform, it is straightforward to show that $R_J(\cdot)$ reads

$$R_J(\omega) = \frac{1}{1 - r\omega}. \quad (154)$$

- Projector Matrix: Here, the only constraint on the sensing matrix is that the row vectors are orthogonal. The matrices are also referred to as the row orthogonal matrices. For the sensing matrix $A_{k \times n}$, we assume the case that the row vectors are normalized by the number of rows and $k \leq n$; thus, the outer product $AA^T$ reads

$$AA^T = \frac{n}{k}I_k. \quad (155)$$

Consequently, Gram matrix $J$ takes two different eigenvalues: $\lambda = 0$ with multiplicity $n - k$, and $\lambda = nk^{-1}$ with multiplicity $k$. Considering the definition of factor $r$ in (20), the asymptotic empirical CDF of the eigenvalues reads

$$F_J(\lambda) = \left[ 1 - \frac{1}{r} \right] 1{|\lambda| > 0} + \frac{1}{r} I{|\lambda| > r} \quad (156)$$

which results in the R-transform of the form

$$R_J(\omega) = \frac{r\omega^2 + \sqrt{(r\omega^2 - 1)^2 + 4\omega^2}}{2\omega}. \quad (157)$$
2) Source Model: We consider continuous and discrete alphabet sources with sparse Gaussian and sparse uniform distributions, respectively. More precisely, we assume that the entries of the continuous and discrete alphabet sources are generated from a Gaussian and uniform distribution, respectively, and multiplied by a Bernoulli random variable with probability \( \alpha \) to take 1 and \( 1 - \alpha \) to take 0. We further assume the nonzero outcomes of the discrete alphabet source to be of the symmetric form
\[
\{ \pm a, \ldots, \pm \kappa a \}
\]
for some positive real \( a \) and integer \( \kappa \).

3) Distortion Function: For the continuous sources, we determine the performance of different estimators by considering the MSE as the distortion function, i.e.,
\[
d(\hat{x}; x) = \| \hat{x} - x \|^2.
\]
For the discrete alphabet sources, moreover, we determine the probability of the error as measure for which
\[
d(\hat{x}; x) = \sum_{i=1}^{n} 1 \{ \hat{x}_i \neq x_i \}
\]
as well as the MSE.

Considering either case, it is clear that the MSE obtained by any \( \ell_0 \) norm recovery scheme is bounded from below by the MMSE bound reported in the literature [23], [25].

B. Numerical Results for Continuous Sources

Considering Examples 1-3, we consider the case in which a sparse Gaussian source with sparsity factor \( \alpha = 0.1 \) is sampled via a random sensing matrix. Fig. 5 shows the RS prediction of normalized MSE, defined as
\[
\text{MSE}^0 = \frac{\text{MSE}}{E|y|^2} = a^{-1} \text{MSE},
\]
as a function of estimation parameter \( \lambda \). The compression rate is set to \( r = 2 \), and both i.i.d. random and projector sensing matrices are considered. The curves match the results reported in [42] and [40]. As it is seen, the \( \ell_0 \)-norm recovery scheme with the optimal choice of estimation parameter outperforms the LASSO scheme; however, the non-optimal choice of the estimation parameter can make the \( \ell_0 \) norm’s performance even worse than the LASSO. In contrast to the noiseless case, the projector matrix is always outperforming the i.i.d. matrix in the noisy case; this fact has been also reported in [42].

In [44], the authors showed that in the noiseless sampling case with an i.i.d. Gaussian matrix, the RS ansatz for linear and LASSO recovery schemes is locally stable against perturbations that break the symmetry of the replica correlation matrix. This result in fact agrees with the general belief that convex optimization problems do not exhibit RSB [87]. The result in [44], however, indicated that for the \( \ell_0 \) norm reconstruction, the RS ansatz becomes unstable, and therefore RSB ansätze are needed for accurately assessing the performance.

In order to investigate the observation of [44], we have plotted the normalized MSE of LASSO recovery predicted by the RS ansatz in terms of estimation parameter \( \lambda \) in Fig. 6 considering various compression rates. It is observed that for a given estimation parameter \( \lambda \), the normalized MSE increases as the compression rate grows. For large compression rates, the normalized MSE converges to 0 dB which agrees with the fact that for asymptotically large compression rates, the source and observation vectors are independent, and thus, the MSE converges to the source power.

For \( \ell_0 \) norm recovery, Fig. 7 shows the normalized MSE predicted by the RS ansatz as a function of the estimation parameter for two different compression rates. The system setup has been set to be similar to the one considered in Fig. 6, and the curves have been plotted for both the i.i.d. and projector measurements. In contrast to the LASSO recovery scheme, the RS ansatz starts to give invalid predictions for the \( \ell_0 \) norm scheme as the compression rate increases. As it is observed, the normalized MSE drops unexpectedly down for an interval of the estimation parameters when the compression rate grows.
In fact, in this interval, the RS fixed point equations have either an unstable solution or no solution. To illustrate this result further, let us consider Examples 2 and 3 under the RS ansatz when an i.i.d. sensing matrix is employed. In this case, the equivalent noise power and estimation parameter $\lambda^*$ and $\lambda_0^*$ read

\begin{align}
\lambda^* &= \lambda + r\chi \\
\lambda_0^* &= \lambda_0 + rq.
\end{align}

(162a)  (162b)

By increasing the compression rate, the interference increases, and thus, the MSE takes larger values. Therefore, for small $\lambda$ and $\lambda_0$, one can consider $r\chi \gg \lambda$ and $rq \gg \lambda_0$ as $r$ takes large values and write

\begin{align}
\lambda^* &\approx r\chi \\
\lambda_0^* &\approx rq.
\end{align}

(163a)  (163b)

Considering Example 2, by substituting (163a) and (163b) in the RS ansatz, the fixed point equations, as $r$ grows large, are written in the following form

\begin{align}
ru^2 \phi(u\sqrt{r}) &\approx \int_u^\infty \varepsilon^2 D\varepsilon + \epsilon_r \\
q &\approx 2\alpha \int_0^r \varepsilon D\varepsilon + \epsilon_r.
\end{align}

(164a)  (164b)

for some $\epsilon_r$ tending to zero as $r \uparrow \infty$, and the bounded real scalar $u$ defined as

\begin{align}
u &:= \frac{\chi}{\sqrt{q}}.
\end{align}

(165)

Taking the limit $r \uparrow \infty$, (164a) is valid for any bounded real value of $u$ and (164b) reduces to $q \approx \alpha$ for large compression rates. Noting that for this setup $q = \text{MSE}$, one concludes that $\text{MSE}^0 \approx 1$ which agrees with the results given in Fig. 6. A similar approach for the $\ell_0$ norm recovery scheme in Example 3, however, results in the following contradicting equations

\begin{align}
\int_u^\infty \varepsilon^2 D\varepsilon &\approx \epsilon_r \\
\int_0^r \varepsilon D\varepsilon &\approx \epsilon_r.
\end{align}

(166a)  (166b)

Clearly, the set of equations in (166a) and (166b) have no solution as $r \uparrow \infty$. The approximated fixed point equations explain the invalidity of the RS predicted performance of the $\ell_0$ norm recovery scheme for large compression rates.

In order to further investigate the RS ansatz, we plot the optimal normalized MSE as a function of the compression rate in Fig. 8. Here, we consider the case with i.i.d. sensing matrix when the sparsity factor is set to be $\alpha = 0.1$ and source to noise power ratio to be 10 dB. The normalized MSE is minimized numerically over estimation parameter $\lambda$. As the figure illustrates, the MSE of the RS ansatz starts to drop in moderate compression rates. The observation confirms the discussion on the stability of the RS saddle point given in [43], [44]. In fact, the unexpected drop of the RS ansatz is caused by the limited stability region of the RS fixed point solutions. More precisely, for a given source to noise power ratio and estimation parameter, the RS fixed point equations have stable solutions within a certain interval of compression rates. The interval widens as $\lambda$ grows.

Fig. 9 compares the RS and 1RSB ansätze for $\ell_0$ norm recovery. In this figure, the optimal normalized MSE is plotted for an i.i.d. sensing matrix. The sparsity factor is $\alpha = 0.1$ and the noise variance is set to $\lambda_0 = 0.01$. For the RS ansatz, we have considered two cases, namely when $\text{MSE}^0$ is minimized over 1) all possible estimation parameters and
In fact, we consider \( \lambda > \lambda_d \), where \( \lambda_d \) is the point in Fig. 7 in which the RS ansatz deviates from the prediction given by the 1RSB ansatz.

In this case, the RS \( \epsilon \)-validity region \( \mathcal{V}_{\text{br}}(\text{snr}, \lambda; \epsilon) \) is the area enclosed by both the \( \lambda \) and \( \text{snr} \) axes as well as the \( r_{\text{br}}(\text{snr}, \lambda; \epsilon) \) surface.

Fig. 11 illustrates the intersection of the RS \( \epsilon \)-validity region and the planes \( \text{snr} = 9 \) and \( \text{snr} = 12 \) for \( \epsilon = 10^{-3} \). The break compression rate increases w.r.t. both \( \lambda \) and \( \text{snr} \) which agrees with the intuition. Moreover, \( r_{\text{br}}(\text{snr}, \lambda; \epsilon) \) starts to saturate as \( \lambda \) grows. Another extreme case is when the estimation parameter tends to zero in which the MAP estimator reduces to

\[
g(y) = \arg \min_{\|y - \lambda v\| \leq \epsilon} \|v\|_0.
\]

with \( \epsilon_0 = \mathcal{O}(\frac{1}{\sqrt{\text{snr}}} \). For this case, the break compression rate converges to the minimum compression rate \( r = 1 \).

2) the interval of \( \lambda \) in which the RS ansatz is valid within.\(^{14}\)

The latter case is referred to as the RS restricted curve. As the figure depicts, the difference between the RS and the 1RSB ansatz is quite small for low compression rates. The 1RSB prediction however deviates from RS at larger compression rates. This observation indicates that the performance analysis of the \( \ell_0 \) norm recovery exhibits RSB. For sake of comparison, we also plot the curve for the LASSO recovery scheme as well as the MMSE bound as shown in Fig. 9.

In this case, the RS \( \epsilon \)-validity region \( \mathcal{V}_{\text{br}}(\text{snr}, \lambda; \epsilon) \) as functions of the compression rate considering the spin glass which corresponds to the \( \ell_0 \) norm recovery scheme. The setting is considered to be the same as in the setup investigated in Fig. 8 and Fig. 9.
Moreover, occurs in a larger interval of SNR. The deviation from the single-user bound in the i.i.d. case, meets the single-user bound in a relatively large SNR figure illustrates, for either sensing matrix, the error probability of the RS ansatz reported in [44].

This observation in the large SNR limit agrees with the instability of the RS ansatz reported in [44].

\[ v_{k}^{\ell_{2}} = \pm \left( \frac{2k - 1}{2} a(1 + \lambda^{2}) \right) \]  
for \( k \in [1 : \kappa] \). In the case of LASSO recovery, the boundary points are given by

\[ v_{k}^{\ell_{1}} = \pm \left( \frac{2k - 1}{2} a + \lambda^{2} \right) \]  
for \( k \in [1 : \kappa] \). Finally, by employing \( \ell_{0} \) norm recovery, we have \( v_{k}^{\ell_{0}} = \pm \left( \frac{1}{2} a + \frac{\lambda^{2}}{\kappa} \right) \) and

\[ v_{k}^{\ell_{0}} = \pm \frac{2k - 1}{2} a \]  
for \( k \in [2 : \kappa] \). The source to noise power ratio SNR is further defined as

\[ \text{SNR} := \frac{E_{X}^{2}}{\lambda_{0}} = \frac{a a^{2}(\kappa + 1)(2\kappa + 1)}{6\lambda_{0}}. \]  

Fig. 12 shows the RS predicted error probability of the discrete alphabet system as a function of SNR for the unit compression rate when the \( \ell_{0} \) norm recovery scheme is employed. The cases with \( \kappa = 2 \) and \( \kappa = 4 \) are considered for both random i.i.d. and orthogonal measurements. Here, the sparsity factor is set to be \( a = 0.1 \) and \( \lambda = 0.1 \). As the figure illustrates, for either sensing matrix, the error probability meets the single-user bound in a relatively large SNR regime. The deviation from the single-user bound in the i.i.d. case, moreover, occurs in a larger interval of SNR as \( \kappa \) grows. In contrast to i.i.d. measurements, the coincidence occurs for almost every SNR and \( \kappa \) when a projector sensing matrix is employed. This observation is intuitively justified due to the orthogonality of the rows in the latter case.

For the MSE, similar behavior as in Fig. 12 is observed in Fig. 13. As the compression rate increases, considering either the MSE or error probability, a deviation from the single-user bound is observed for both random i.i.d. and orthogonal measurements. Considering a fixed SNR, it is observed that the MSE-compression rate has a discontinuity point in which the MSE suddenly jumps from a lower value to an upper value at a certain compression rate. This phenomenon is known as phase transition in which the macroscopic parameters suddenly change the phase within a minor variation of the setting. The compression rate in which the phase transition occurs is referred to as transition rate which increases as SNR grows. Phase transitions were reported in the literature of communications and information theory for several problems such as turbo coding and CDMA systems [13], [88].

Fig. 13 illustrates the phase transition under the RS assumption for a sparse binary source, i.e., \( \kappa = 1 \), when the source...
of the system. We deviate from the earlier approaches, e.g., [40]–[42], by evaluating the general replica ansatz of the corresponding spin glass. The general ansatz lets us derive the RS, as well as the bRSB ansatz considering the class of rotationally invariant random matrices. The results recover the previous studies [40]–[42] in special cases and justifies the uniqueness of the zero temperature entropy’s expression under the bRSB assumption conjectured in [45].

The replica ansatz evaluated here leads us to this finding that in the large-system limit, the marginal joint distribution of the entries determined by any replica ansatz decouples into a set of identical joint distributions. The form of this asymptotic decoupled distribution depends on the structure imposed on the ansatz. For the bRSB ansatz, the vector-valued AWGN channel estimated by a MAP estimator decouples into a bank of single-user noisy channels with effective additive noise followed by scalar MAP estimators. The noise term is in general non-Gaussian and depends on the decoupled input sample. The derived form of the bRSB decoupled system contains correlated impairment terms. These terms intuitively model the interference of the system. The general decoupling principle justified here confirms the conjecture that decoupling is a generic property of MAP estimators, since its validity relies only on the replica continuity assumption. Recent results in statistical mechanics have shown that failures in finding the exact solution via the replica method are mainly caused by the structure imposed on the ansatz, and not replica continuity [5], [89]–[91]. The decoupling property further enables us to represent the alternative replica simulator interpretation of the replica ansatz in which the bRSB fixed point equations are completely described by the statistics of the corresponding decoupled system.

The investigation of large compressive sensing systems via the RS and RSB ansätze demonstrates that the linear and LASSO recovery schemes are accurately characterized via the RS ansatz within a large interval of compression rates. Nevertheless, the RS ansatz fails to predict performance of the $\ell_0$ norm scheme for moderate to high compression rates. Numerical investigations under 1RSB ansatz for this scheme show more reliable prediction for high compression rates. Such an observation agrees with the earlier findings in the literature, e.g., [44], [87]. For sparse discrete alphabet sources, the RS prediction of the error probability and MSE exhibits phase transition w.r.t. the compression rate. Similar to the case with continuous source symbols, the random orthogonal measurements perform better than random i.i.d. sensing.

The current work can be pursued in several directions. As an example, the replica simulator introduced in Section VII can be studied further by methods developed in the context of transition systems. The analysis may result in proposing a new framework which simplifies the evaluation of fixed point equations. Another direction is the analysis of the conditional distribution of the correlated impairment terms in the bRSB decoupled system. The study can lead to understanding the necessary or sufficient conditions under which the RS ansatz gives an accurate approximation of performance. Inspired by the Sherrington-Kirkpatrick model of spin glasses, for which
the full RSB ansatz, i.e., $b \uparrow \infty$, has been proven to give a stable solution for all system parameters [82], our conjecture is that the exact solution at all large compression rates, for the cases exhibiting RSB, is given by a large number of breaking steps. Nevertheless, as our numerical investigations depict, even in those cases, the RS ansatz, or the bRSB ansatz with small $b$, can give good approximations of the solution up to some moderate compression rates. The latter study further illustrates the accuracy of the approximation given by a finite number of RSB steps. Investigating the connection between the replica simulator and message passing based algorithms is another interesting topic for future work.

**APPENDIX A**

**PROOF OF RESULT IV.1**

Starting from (43), we define the moment function $\tilde{Z}(m)$ as in (44a). Therefore,

$$D^W(\hat{x}; x) = \lim_{\beta \uparrow \infty} \lim_{n \uparrow \infty} \lim_{\bar{h} \downarrow 0} \lim_{m \uparrow \infty} \frac{1}{nh} \frac{\partial}{\partial h} \log \tilde{Z}(m).$$

(176)

Taking the expectation w.r.t. the noise term first, the moment function reduces to (177c), as shown at the bottom of this page, where $\tilde{v}_a = x - v_a$ for $a \in [1 : m]$ is the unbiased15 replica vector, $J$ is the Gramian of the system matrix defined as $J := A^T A$ and satisfies the properties stated in Section II. $*$ comes from taking expectation over $z$, and the factor $\zeta_{ab}$ is defined as

$$\zeta_{ab} := \frac{1}{2\lambda} \left[ \mathbf{1}(a = b) - \frac{\lambda_0}{\lambda + m\beta\lambda_0} \right].$$

(178)

with $\lambda_0$ being the true noise variance specified in Section II.

**Remark A.1.** In (177b), as shown at the bottom of this page, one could drop the term $\frac{1}{2\lambda} \|z\|^2$, since it plays no role in the optimization problem (2). More precisely, one could redefine the Hamiltonian in (11) to be

$$\mathcal{E}_{\text{new}}(\mathbf{v}|y, A) = \mathcal{E}(\mathbf{v}|y, A) - \frac{\lambda_0}{2\lambda} \|z\|^2$$

(179)

without loss of generality. In this case, the coefficient at the r.h.s. of (177c) reduces to 1, and $\zeta_{ab}$ reads

$$\zeta_{ab} := \frac{1}{2\lambda} \left[ \mathbf{1}(a = b) - \frac{\lambda_0}{\lambda} \right].$$

(180)

It is, however, clear from (177c) that as $m$ tends to zero, both the approaches result in a same result.

15We call it unbiased, since it expresses the deviation of the replicas from the source vector.

Considering the expression in (177c), we define the random variable $\tilde{Z}(m; x)$ as

$$\tilde{Z}(m; x) := E_J \sum_{\{v_a\}} \left[ e^{-\beta \left( \sum_{n=1}^m \tilde{v}_n^T J \tilde{v}_n + \sum_{n=1}^m u(v_a) \right) + \bar{h} n} \sum_{n=1}^m D^{W(n)}(v_a, x) \right].$$

(181)

Consequently, the moment function is given by taking the expectation of $\tilde{Z}(m; x)$ w.r.t. $x$ and multiplying with the scalar $(x + m\beta\lambda_0)^\frac{1}{2}$. We later show that for almost all realizations of the source vector, $\tilde{Z}(m; x)$ converges to a deterministic value, and therefore, the expectation w.r.t. $x$ can be dropped. For $\tilde{Z}(m; x)$, we have

$$\tilde{Z}(m; x) = E_J \sum_{\{v_a\}} \left[ e^{-\beta \sum_{n=1}^m \tilde{v}_n^T J \tilde{v}_n} + \bar{h} n \sum_{n=1}^m D^{W(n)}(v_a, x) \right]$$

(182a)

$$= \sum_{\{v_a\}} \left[ \left( E_J e^{-n\beta|\mathbf{G}|} \right) \sum_{n=1}^m \tilde{v}_n \mathbf{G} \tilde{v}_n + \bar{h} n \sum_{n=1}^m D^{W(n)}(v_a, x) \right]$$

(182b)

where $G_{n \times n}$ is defined as

$$G := \frac{1}{n} \sum_{a,b=1}^m \tilde{v}_b \tilde{v}_a^T \zeta_{a,b},$$

(183)

Considering the eigendecomposition of the Gramian $J = U^T D U$, the expectation in (182b) can be expressed in terms of a spherical integral where the integral measure is the probability measure of $U$. Regarding the system setup specified in Section II, the matrix $U$ is distributed over the orthogonal group $O_n$ with Haar probability distribution. Therefore, the corresponding spherical integral is the so-called Harish-Chandra or Itzykson & Zuber integral. This integral has been extensively studied in the physics and mathematics literature, see for example [92], [93] and [76]. A brief discussion on the spherical integral and its closed form solution has been given in Appendix F. Invoking Theorem 1.2 and 1.7 in [77], as long as rank($G$) = $O(\sqrt{n})$, the expectation in (182b) can be written as

$$E_J e^{-n\beta|\mathbf{G}|} = e^{-n \sum_{a=1}^m \frac{\beta}{2} \lambda_0 R_{(2 \pi \lambda_0) a_a} + \epsilon_n}$$

(184)

16$O(^\cdot)$ denotes growth order w.r.t. composition, i.e., $\lim_{n \uparrow \infty} O(f(n))/f(n) = K < \infty$. 

$$\tilde{Z}(m) = E_x E_A E_b \sum_{\{v_a\}} e^{-\beta \sum_{n=1}^m E(v_a)|y, A)} + \bar{h} n \sum_{n=1}^m D^{W(n)}(v_a; x)$$

(177a)

$$= E_x E_A E_b \sum_{\{v_a\}} e^{-\beta \sum_{n=1}^m \left( \frac{\lambda}{2\lambda} (x - v_a)^T J (x - v_a) + \frac{\lambda}{2\lambda} \|z\|^2 + u(v_a) \right) + \bar{h} n \sum_{n=1}^m D^{W(n)}(v_a; x)}$$

(177b)

$$\simeq \left( \frac{\lambda}{\lambda + m\beta\lambda_0} \right)^\frac{1}{2} E_x E_A \sum_{\{v_a\}} e^{-\beta \sum_{n=1}^m \tilde{v}_n^T J \tilde{v}_n + \sum_{n=1}^m u(v_a)} + \bar{h} n \sum_{n=1}^m D^{W(n)}(v_a; x)$$

(177c)
with $G$ being defined in (183), and $\epsilon_n \downarrow 0$ as $n \uparrow \infty$. In order to employ the above result and substitute it in (182b), we need to check the rank condition.

**Lemma A.1** Considering $G$ to be defined as in (183), the following argument holds.

$$\text{rank}(G) = O(\sqrt{n}).$$

(185)

**Proof.** First, we rewrite $G$ as

$$G = \frac{1}{2\pi n} \left[ \sum_{a=1}^{m} \tilde{v}_a \tilde{v}_a^T - \frac{\lambda_0}{\lambda + \beta \lambda_0} \left( \sum_{a=1}^{m} \tilde{v}_a \right) \left( \sum_{b=1}^{m} \tilde{v}_b \right) \right]$$

$$= \frac{1}{2\pi n} \tilde{V} \left( I_m - \frac{\lambda_0}{\lambda + \beta \lambda_0} 1_m \right) \tilde{V}^T$$

(186)

where $\tilde{V} = [\tilde{v}_1, \ldots, \tilde{v}_m]$ is an $n \times m$ matrix with the columns being the unbiased replicas. Then, by considering (186), it is obvious that $G$ could be, at most, of rank $m$. As Assumption 2 indicates, $\tilde{Z}(m)$ analytically continues to the real axis, and the limit w.r.t. $m$ is taken in a right neighborhood of 0. Therefore, for all values of $n$ there exists a constant $K \in \mathbb{R}^+$, such that $m \leq K$. Consequently, one can write

$$\lim_{n \uparrow \infty} \frac{\text{rank}(G)}{\sqrt{n}} \leq \lim_{n \uparrow \infty} \frac{m}{\sqrt{n}} \leq \lim_{n \uparrow \infty} \frac{K}{\sqrt{n}} = 0$$

(187)

which concludes that $\text{rank}(G) = O(\sqrt{n})$. \hfill \Box

Lemma A.1 ensures that (184) always holds; therefore, noting the fact that $G$ has only $m$ non-zero eigenvalues, the expectation in the r.h.s. of (182b) reduces to

$$\mathbb{E}_I e^{-n\beta \text{tr}(|G|)} = e^{-n\tilde{\gamma}(TQ)} + \epsilon_n$$

(188)

where the function $\tilde{\gamma}(\cdot)$ is defined as

$$\tilde{\gamma}(M) := \int_0^\beta \text{Tr}[\text{MR}_J(-2\omega M)]d\omega$$

(189)

for some square matrix $M$, $T$ is an $m \times m$ deterministic matrix given by

$$T := \frac{1}{2\lambda} \left[ I_m - \frac{\lambda_0}{\lambda + \beta \lambda_0} \beta 1_m \right],$$

(190)

and $Q^v$ is the $m \times m$ correlation matrix defined as

$$Q^v = \frac{1}{n} \tilde{V}^T \tilde{V}.$$  

(191)

**Remark A.2.** Note that although $Q^v$ is symmetric, $TQ^v$ is not a symmetric matrix, in general; however, due to the symmetry of $G$, the eigenvalues of $TQ^v$ are real; therefore, the sequence of integrals over the real axis in (184) exists for all indices. By substituting (188) in (182b), $\tilde{Z}(m; x)$ is given as

$$\tilde{Z}(m; x) = \sum_{\{v_a\}} \left[ e^{-n\tilde{\gamma}(TQ^v)} - \beta \sum_{a=1}^{m} u(v_a) \right]$$

$$\times e^{hn \sum_{a=1}^{m} D^{(\cdot)}(v_a; x) + \epsilon_n}.$$ 

(192)

In order to determine the sum in (192), we follow the technique which has been employed in [18] and [45]. We split the space of all replicas into subshells defined by the correlation matrices in which all the vectors of replicas in each subshell have a same correlation matrix. More precisely, for a given source vector $x$, the subshell of the matrix $Q_{m \times m}$ is defined as

$$\mathcal{S}(Q) = \{ v_1, \ldots, v_m | (x - v_a)^T(x - v_b) = nq_{ab} \}$$

(193)

with $q_{ab} = [Q_{ab}]$ denoting the entry $(a, b)$ of $Q$. The sum in (192) is determined first over each subshell, and then, over all the subshells as the following.

$$\tilde{Z}(m; x) = \sum_{\{v_a\}} \left[ \int e^{-n\tilde{\gamma}(TQ)} \delta(Q^v - Q) dQ \right]$$

$$\times \left[ e^{-\beta \sum_{a=1}^{m} u(v_a) + hn \sum_{a=1}^{m} D^{(\cdot)}(v_a; x) + \epsilon_n} \right]$$

$$= \int e^{-n\tilde{\gamma}(TQ)} + \epsilon_n \left[ \sum_{\{v_a\}} \delta(Q^v - Q) \right.$$ 

$$\times \left[ e^{-\beta \sum_{a=1}^{m} u(v_a) + hn \sum_{a=1}^{m} D^{(\cdot)}(v_a; x)} \right] dQ$$

$$= \int e^{-n\tilde{\gamma}(TQ)} + \epsilon_n \left[ \prod_{a,b=1}^{m} \delta(\tilde{u}_a^T \tilde{v}_b - nq_{ab}) \right].$$

(194a)

(194b)

where $dQ := \prod_{a,b=1}^{m} dq_{ab}$, the integral is taken over $\mathbb{R}^{m \times m}$, and the term $e^{n\tilde{\gamma}(\cdot)}$ which determines the probability weight of the subshell $\mathcal{S}(Q)$ is defined as

$$e^{n\tilde{\gamma}(Q)} := \sum_{\{v_a\}} \left[ \delta(Q^v - Q) \right.$$ 

$$\times e^{-\beta \sum_{a=1}^{m} u(v_a) + hn \sum_{a=1}^{m} D^{(\cdot)}(v_a; x)} \left. \right].$$

(195)

**Remark A.3.** One may define the subshells over the transferred correlation matrix $TQ$ instead of correlation matrix $Q$. In this case the subshells over $Q$ defined in (193) only rotate in the $m$-dimensional space w.r.t. $T$. The rotation, however, does not have any impact on the analysis.

The last step is to determine $e^{n\tilde{\gamma}(\cdot)}$. To do so, we represent the Dirac impulse function using its inverse Laplace transform. By defining $s_{ab}$ as the complex frequency corresponding to $\delta(\tilde{u}_a^T \tilde{v}_b - nq_{ab})$,

$$\delta(\tilde{u}_a^T \tilde{v}_b - nq_{ab}) = \int e^{s_{ab} \tilde{u}_a^T \tilde{v}_b - nq_{ab}} \frac{ds_{ab}}{2\pi j}$$

(197)

where the integral is taken over the imaginary axis $\mathbb{I} = (t - j\infty, t + j\infty)$, for some $t \in \mathbb{R}$. Consequently, by defining the frequency domain correlation matrix $\mathcal{S}$ to be an $m \times m$ matrix with $[\mathcal{S}]_{ab} = s_{ab}$, (195) reads

$$\delta(Q^v - Q) = \int e^{-n\tilde{\gamma}(\mathcal{S}^v)} d\mathcal{S}$$

(198a)

$$= \left[ e^{-n\tilde{\gamma}(\mathcal{S}^v)} \right] e \prod_{a,b=1}^{m} s_{ab} \tilde{u}_a^T \tilde{v}_b d\mathcal{S}$$

(198b)
with $dS$ being defined as $dS := \prod_{a,b=1}^{m} \frac{d\nu_{ab}}{\nu_{ab}}$, and the integral being taken over $[m]$. Substituting in (196), $e^{n\mathcal{Z}(Q)}$ reduces to
\[
e^{n\mathcal{Z}(Q)} = \int e^{-n\text{tr}(s^TQ)} \prod_{\{u_a\}} \left[ e^{\sum_{a=1}^{m} \sum_{x,y} x_{ab} - \beta \sum_{x} u(x)} \right] dS = \int e^{-n\text{tr}(s^TQ)} e^{n\mathcal{M}(S)} dS
\]  
with $\mathcal{M}(S)$ being defined as
\[
\mathcal{M}(S) = \frac{1}{n} \log \prod_{\{u_a\}} \left[ e^{\sum_{a=1}^{m} \sum_{x,y} x_{ab} - \beta \sum_{x} u(x)} \right] 
\]  
It follows that $\mathcal{M}(S)$ is a deterministic asymptotic as $S \to \infty$, and thus, the expectation is taken over $\{v_a\}$. Hence, one needs to evaluate the expectation of $\mathcal{M}(S)$ using the law of large numbers, and the decoupling property of the functions $u(\cdot)$ and $d(\cdot; \cdot)$.

**Lemma A.2** Consider the system specified in Section II, and let Assumption 2 hold. Then, as $n \uparrow \infty$, $\mathcal{M}(S)$ defined in (200) is given by
\[
\mathcal{M}(S) = \mathbb{E} \left[ \frac{1}{n} \log \sum_{v} e^{(x-v)^T S (x-v) - \beta u(v)} \right] + \frac{\eta}{n} \sum_{v} e^{(x-v)^T S (x-v) - \beta u(v) + h \eta^{-1} d(v; x)}
\]  
where $v_{m \times 1} \in \mathbb{R}^m$, $x_{m \times 1}$ is a vector with all the elements being the random variable $x$ which is distributed with the source distribution $p_x$, and $d(\cdot; \cdot)$ is defined as $d(v; x) := \sum_{a=1}^{m} d(v_a; x_a)$.

**Proof.** Consider the decoupling property of the functions $u(\cdot)$ and $d(\cdot; \cdot)$. Define the vector $v_{m \times 1}$ over the support $\mathbb{R}^m$, and the coefficients $\{w_i\}$ for $i \in [1 : n]$ as
\[
\mathcal{W}_i = \begin{cases} 0 & \text{if } i \notin \mathcal{W}(n) \\ \frac{1}{n} & \text{if } i \in \mathcal{W}(n) \end{cases}
\]  
Then, $\mathcal{M}(S)$ is expanded as in (204a)-(204c), as shown at the bottom of this page, where the functions $\mathcal{M}_0(\cdot; \cdot)$ and $\mathcal{M}_1(\cdot; \cdot)$ are defined as
\[
\mathcal{M}_0(S; x_i) = \log \sum_{v} e^{(x-v)^T S (x-v) - \beta u(v)}
\]  
\[
\mathcal{M}_1(S; x_i) = \log \sum_{v} e^{(x-v)^T S (x-v) - \beta u(v) + h \mathcal{W}(v; x)/(\mathcal{W}(n))}
\]  
where $x_{m \times 1}$ is a vector with all the elements being $x_i$, and we define $d(v; x) := \sum_{a=1}^{m} d(v_a; x_a)$ for compactness. Assume Assumption 2 suggests the limits w.r.t. $n$ and $m$ can be exchanged in (176); thus, one can consider the asymptotics of $\mathcal{M}(S)$ for a given $m$ when $n$ tends to its large limit. Regarding the fact that $x$ is collected from an i.i.d. distribution, the term in the r.h.s. of (204c) converges to the expectation over the distribution $p_x$ due to the law of large numbers; more precisely, as $n \uparrow \infty$
\[
\frac{1}{n} \sum_{i \notin \mathcal{W}} \mathcal{M}_0(S; x_i) \to (1 - \eta) \mathbb{E}_x \mathcal{M}_0(S; x) 
\]  
\[
\frac{1}{n} \sum_{i \notin \mathcal{W}} \mathcal{M}_1(S; x_i) \to \eta \mathbb{E}_x \mathcal{M}_1(S; x)
\]  
with $\eta$ being defined as in (26). Substituting (206a) and (206b) in (204c), Lemma A.2 is concluded.

**Remark A.4.** Considering Lemma A.2, it eventually says that the probability weight $e^{n\mathcal{Z}(Q)}$ for a given correlation matrix $Q$ converges to a deterministic weight as $n$ tends to infinity. This statement equivalently states that for almost any given realization of the source vector, the correlation matrix converges to its expectation. In fact, considering the correlation matrix $Q^v$, as defined in (191), the entries are functions of $x$, and therefore, variate randomly due to the source distribution. As an alternative approach, one could study the convergence property of the correlation matrix $Q^v$ by means of the law of large numbers first, and then,
conclude Lemma A.2 by rewriting $\mathcal{M}(\mathbf{S})$ in proper way, and replacing it with the expectation using the fact that the probability weight $e^{nI(\mathbf{Q})}$ needs to converge deterministically as $n \uparrow \infty$. Nevertheless, the approach taken here seems to be more straightforward.

Using Lemma A.2, we drop the expectation w.r.t. $\mathbf{x}$ in (177c). Replacing in (176), the asymptotic distortion is found by taking the limits. As Assumption 2 suggests, we exchange the order of the limits and take the limit w.r.t. $n$ at first. Denoting that the probability measure defined with $e^{nI(\mathbf{Q})}d\mathbf{Q}$ satisfy the large deviation properties [81], we can use the saddle point approximation to evaluate the integral in (201) which says that as $n \uparrow \infty$

$$
\tilde{Z}(m) = \frac{1}{\beta + m} \int e^{-n[G(\mathbf{TQ}) + \text{tr}(\mathbf{S}^T \mathbf{Q}) - \mathcal{M}(\mathbf{S})]} d\mathbf{S} d\mathbf{Q} \\
\approx K_n e^{-n[G(\mathbf{TQ}) + \text{tr}(\mathbf{S}^T \mathbf{Q}) - \mathcal{M}(\mathbf{S})]},
$$

where we drop $\epsilon_n$ given in (201) regarding the fact that it vanishes in the large limit. Here, $(\mathbf{Q}, \mathbf{S})$ is the saddle point of the integrand function’s exponent, $K_n$ is a bounded coefficient, and $\bar{=} \cdot$ indicates the asymptotic equivalency in exponential scale defined as the following.

**Definition A.1** The functions $a(\cdot)$ and $b(\cdot)$ defined over the non-bounded set $\mathcal{X}$ are said to be asymptotically equivalent in exponential scale, if

$$
\lim_{n \to \infty} \frac{a(x_n)}{b(x_n)} = 0.
$$

for an unbounded sequence $(x_n \in \mathcal{X})$.

As $n \uparrow \infty$, the $m$th moment can be replaced with its asymptotic equivalent in (176). Consequently, by substituting the equivalent term and exchanging the limits’ order, we have

$$
\begin{align*}
\text{D}^W(\bar{x}; x) &= \lim_{\beta \uparrow \infty} \lim_{m \downarrow 0} \lim_{n \to \infty} \frac{1}{m} \frac{\partial}{\partial h} \left[ -G(\mathbf{TQ}) - \text{tr}(\mathbf{S}^T \mathbf{Q}) \right] \\
&\quad + \mathcal{M}(\mathbf{S}) + \frac{\log K_n}{n} \tag{209a}
\end{align*}
$$

and (201b) results in

$$
\tilde{Q} = \mathbb{E} \sum_v (\mathbf{x} - v)^T e^{(\mathbf{x} - v)^T S(\mathbf{x} - v) - \beta u(v)} \\
\sum_v e^{(\mathbf{x} - v)^T S(\mathbf{x} - v) - \beta u(v)}.
$$

By replacing (211) in (209c) and (212), the expression for the asymptotic distortion and the saddle point correlation matrix can be considered as expectations over conditional Boltzmann-Gibbs distribution $p_{\mathbf{v}x}$ defined as

$$
P_{\mathbf{vX}}(\mathbf{v}|\mathbf{x}) := \frac{e^{\beta (\mathbf{x} - v)^T T(\mathbf{J})(\mathbf{-2\beta TQ})(\mathbf{x} - v) + u(v)}}{\sum_v e^{-\beta (\mathbf{x} - v)^T T(\mathbf{J})(\mathbf{-2\beta TQ})(\mathbf{x} - v) + u(v)}}
$$

which simplifies the expressions in (209c) and (212) to those given in Result IV.1.

In general the fixed point equation (212) can be satisfied with several saddle points, and therefore, multiple asymptotic distortions might be found. In this case, one should note that the valid solution is the one which minimizes the free energy of the spin glasses at the zero temperature, i.e., $\beta \uparrow \infty$. Using the $m$th moment, the free energy of the system reads

$$
\mathcal{F}(\beta) = -\lim_{n \to \infty} \lim_{m \downarrow 0} \frac{1}{\beta m} \log \tilde{Z}(m)
$$

and

$$
\beta \lim_{m \downarrow 0} \frac{1}{\beta m} \left[ G(\mathbf{TQ}) + \text{tr}((\mathbf{S}^T \mathbf{Q}) - \mathcal{M}(\mathbf{S})) \right]
$$

(214b)

where $\therefore$ comes from the facts that $K_n$ is bounded and the limits w.r.t. $m$ and $n$ are supposed to exchange, and $\therefore$ is deduced from (211) and Lemma A.2. Finally by considering the definition of $G(\cdot)$, Result IV.1 is concluded.

**APPENDIX B**

**PROOF OF RESULT V.1**

Starting from Assumption 3, the replica correlation matrix is

$$
\mathbf{Q} = \frac{\chi}{\beta} \mathbf{I}_m + \mathbf{q} \mathbf{I}_m
$$

for some non-negative real $\chi$ and $q$. Considering Definition IV.1, the Hamiltonian of the spin glass of replicas is given by

$$
\mathcal{H}(\mathbf{v}|\mathbf{x}) = (\mathbf{x} - v)^T T(\mathbf{J})(\mathbf{-2\beta TQ})(\mathbf{x} - v) + u(v)
$$

with $\mathbf{T}$ being defined in (49). Denoting $\mathbf{R} := T(\mathbf{J})(\mathbf{-2\beta TQ})$, it is shown in Appendix E that $\mathbf{R}$ has the same structure as the correlation matrix; thus, one can write

$$
\mathbf{R} = \mathbf{eI}_m - \beta f^2 \frac{1}{2} \mathbf{1}_m,
$$

for some real $f$ and $e$ which are functions of $\chi$ and $q$. Denoting the eigendecomposition of $\mathbf{Q}$ as $\mathbf{Q} = \mathbf{VD}^0\mathbf{V}^\top$, we have

$$
\mathbf{T} = \mathbf{V} \mathbf{D}^T
$$

(218a)

$$
\mathbf{R} = \mathbf{D}^2 \mathbf{V}^\top
$$

(218b)

Note that $\mathbf{Q}$ is full-rank and symmetric.
where $D^Q$, $D^T$ and $D^R$ are the diagonal matrices of eigenvalues. Therefore, we have

$$D^R = D^T R_f(-2\beta D^T D^Q)$$

(219)

which equivalently states that for $a \in [1 : m]$

$$\lambda_a^R = \lambda_a^T R_f(-2\beta \lambda_a^T Q_a^Q)$$

(220)

with $\lambda_a^R$, $\lambda_a^Q$ and $\lambda_a^T$ being the eigenvalue of $R$, $Q$ and $T$ corresponding to the $a$th column of $V$. The matrices $R$, $Q$ and $T$ have two different corresponding eigenvalues, namely

$$\lambda_1^R = \lambda_1^Q = e, \quad \lambda_1^T = \frac{1}{2\lambda},$$

(221a, 221b, 221c)

which occur with multiplicity $m-1$, and

$$\lambda_2^R = e - \beta m f^2, \quad \lambda_2^Q = \frac{\lambda}{\beta} + mq, \quad \lambda_2^T = \frac{1}{2\lambda} \left[ 1 - \frac{m\lambda_0}{\lambda + m\lambda_0} \right],$$

(222a, 222b, 222c)

which occur with multiplicity 1. Substituting in (220) and taking the limit when $m \downarrow 0$, $e$ and $f$ are found as

$$e = \frac{1}{2\lambda} R_f(-\frac{\lambda}{\lambda}),$$

(223a)

$$f^2 = \frac{1}{\lambda^2} \frac{\partial}{\partial \lambda} \left\{ [\lambda \lambda_0 - \lambda q] R_f(-\frac{\lambda}{\lambda}) \right\}. \quad \text{(223b)}$$

To pursue the analysis, we rewrite the Hamiltonian using (217)$

$$\mathcal{E}^R(v|x) = e\|x - v\|^2 - \beta \frac{f}{2} \text{tr}(x - v)(x - v)^T I_m + u(v),$$

and therefore, the partition function $Z^R(\beta|x)$ is given by

$$Z^R(\beta|x) = \sum_{\{v_n\}} \left[ e^{-\beta e\|x-v\|^2} \times e^{\beta \frac{f}{2} \text{tr}(x-v)(x-v)^T I_m} - \beta u(v) \right].$$

(225)

Using the Gaussian integral, we have

$$e^{\beta \frac{f^2}{2} \text{tr}(x-v)(x-v)^T I_m} = \int e^{-\beta f \sum_{a=1}^m (x-v_a)^2} \text{D} z, \quad \text{(226)}$$

and thus, the partition function reduces to

$$Z^R(\beta|x) = \int \left[ \sum_{v} e^{-\beta e(x-v)^2 + f(x-v)z + u(0)} \right]^m \text{D} z \quad \text{(227)}$$

with $v \in X$. The parameters of the spin glass of replicas are then determined using the partition function. Starting with the normalized free energy, it reads

$$\mathcal{F}^R(\beta, m) = -\frac{1}{\beta m} E_x \left\{ \log \int \left[ \sum_{v} e^{-\beta e(x-v)^2 + f(x-v)z + u(0)} \right]^m \text{D} z \right\}.$$ 

(228)

Noting that $\int \text{D} z$ takes expectation over the Gaussian distribution, one can use the Riesz equality in (42) to show that when $m$ varies in a vicinity of 0

$$\mathcal{F}^R(\beta, m) = -\frac{1}{\beta} E \left\{ \int \log \sum_{v} e^{-\beta e(x-v)^2 + f(x-v)z + u(0)} \text{D} z + \epsilon_m \right\}$$

(229)

where $\epsilon_m \rightarrow 0$ as $m \downarrow 0$ and the expectation is taken over $x \sim p_X$. Consequently, as $m \downarrow 0$ the normalized free energy reads

$$\mathcal{F}^R(\beta) = \lim_{m \downarrow 0} \mathcal{F}^R(\beta, m) = -\frac{1}{\beta} E \left\{ \int \log \sum_{v} e^{-\beta e(x-v)^2 + f(x-v)z + u(0)} \text{D} z \right\},$$

(230)

The next parameters to be specified are $\chi$ and $q$. By determining the conditional distribution $p^R_{\beta|x}$ and substituting in (50), the following fixed point equations are deduced

$$\left[ \frac{\chi}{\beta} + q \right] = E_x \sum_v \|x - v\|^2 p^R_{\beta|x}(v|x), \quad \text{(231a)}$$

$$\left[ \frac{\chi}{\beta} + mq \right] = E_x \sum_v \left[ \text{tr}((x - v)(x - v)^T I_m) \right] p^R_{\beta|x}(v|x).$$

(231b)

where (231a) and (231b) are found by taking the trace and sum over all the entries of the both sides of (50), respectively. One can directly evaluate the r.h.s.s of (231a) and (231b); however, considering (225), it is straightforward to show that

$$E_x \sum_v \|x - v\|^2 p^R_{\beta|x}(v|x) = \frac{m \partial}{\partial \epsilon} \mathcal{F}^R(\beta, m), \quad \text{(232a)}$$

$$E_x \sum_v \left[ \text{tr}((x - v)(x - v)^T I_m) \right] p^R_{\beta|x}(v|x) = \frac{m \partial}{\partial f} \mathcal{F}^R(\beta, m).$$

(232b)

After substituting and taking the limit $m \downarrow 0$, the fixed point equations finally read

$$\frac{\chi}{\beta} + q = E \left\{ \sum_{v} (v - x)^2 e^{-\beta e(x-v)^2 + f(x-v)z + u(0)} \right\} \text{D} z \quad \text{(233a)}$$

$$\chi = \frac{1}{f} E \left\{ \sum_{v} (v - x)z e^{-\beta e(x-v)^2 + f(x-v)z + u(0)} \right\} \text{D} z \quad \text{(233b)}$$

with $f$ and $e$ defined in (223a) and (223b).

In order to determine the replicas’ average distortion defined in (54) regarding the distortion function $d(\cdot, \cdot)$, we replace the Hamiltonian by

$$\mathcal{E}^R_{\beta|x} = \mathcal{E}^R(v|x) + h \sum_{a=1}^m d(v_a; x) \quad \text{(234)}$$
with $\mathcal{E}^R(v|x)$ given in (224), and take the steps as in (225)-(228) to find the modified form of the normalized free energy, i.e. $\mathcal{F}^R(\beta, h, m)$. The replicas’ average distortion is then evaluated as

$$D^R(\beta, m) = \frac{\partial}{\partial h} \mathcal{F}^R(\beta, h, m)_{h = 0}$$

which does not depend on $m$, and thus, taking the limit $m \downarrow 0$ is not needed.

The last step is to take the zero temperature limit. Using the Laplace method of summation, as $\beta \uparrow \infty$ the fixed point equations reduce to

$$q = \mathbb{E} \int (g - x)^2 \, Dz,$$

$$\chi = \frac{1}{f} \mathbb{E} \int (g - x)z \, Dz,$$

with $g$ being defined as

$$g := \arg\min_v \left[ (x - v)^2 + f(x - v)z + u(v) \right].$$

Taking the same approach, the replicas’ average distortion at zero temperature reads

$$D^W = \mathbb{E} \int d(g; x) \, Dz.$$ (238)

In order to avoid multiple solutions, we need to find the normalized free energy of the corresponding spin glass as given in Result IV.1. In fact, the fixed point equations in (236a) and (236b) may have different solutions, and therefore, the several asymptotics for the distortion can be obtained. In this case, the fixed point solution which minimizes the zero temperature free energy of the system and its corresponding asymptotic distortion are taken. Substituting in Result IV.1, the free energy of the corresponding spin glass at the inverse temperature $\beta$ is found as

$$\mathcal{F}(\beta) = \frac{1}{2\lambda} \left[ \int_0^1 F^\beta(\omega) d\omega - F^\beta(1) \right] + \mathcal{F}^R(\beta)$$

where the function $F^\beta(\cdot)$ is defined as

$$F^\beta(\omega) = \frac{\chi}{\beta} R J(-\frac{\chi}{\lambda} \omega) + \left[ q - \frac{\lambda_0}{\lambda} \right] \frac{d}{d\omega} \left[ \omega R J(-\frac{\chi}{\lambda} \omega) \right].$$

By taking the limit as $\beta \uparrow \infty$, the zero temperature free energy reads

$$\mathcal{F}^0 = \frac{1}{2\lambda} \left[ \int_0^1 F^\infty(\omega) d\omega - F^\infty(1) \right]$$

$$+ \mathbb{E} \int e(x - g)^2 + f(x - g)z + u(g) \, Dz$$

with $g$ being defined in (237) and $F^\infty(\omega) := \lim_{\beta \uparrow \infty} F^\beta(\omega)$. By defining $\lambda^\beta := [2\epsilon]^{-1}$ and $\lambda^\infty_0 := [4\epsilon^2]^{-1} f^2$, Result V.1 is concluded.

### APPENDIX C

**PROOF OF RESULT VI.1**

We take the same approach as in Appendix B. Considering the replica correlation matrix to be of the form

$$Q = \frac{z}{\beta} I_m + \rho I_{\beta, \mu} \otimes I_{s, T} + q I_m,$$ (242)

for some non-negative real $\chi$, $p$, $q$, and $\mu$, we need to evaluate the parameters of the spin glass of replicas defined in Definition IV.1. Starting with the Hamiltonian,

$$\mathcal{E}^R(v|x) = (x - v)^T R J(-2\beta TQ)(x - v) + u(v)$$

where $T$ is given in (49). As discussed in Appendix E, for a given $\mu$ the matrix $R := TR J(-2\beta TQ)$ is of the following form

$$R = eI_m - \beta g^2 I_{\beta, \mu} \otimes I_{s, T} - \beta z^2 I_m$$

where $e$, $g$ and $f$ can be found in terms of $\chi$, $p$ and $q$. Using the eigendecomposition of $Q$, $R$ and $T$, it is then straightforward to show that for $a \in [1 : m]$

$$\lambda^R_a = \lambda^Q_a \chi^T_a$$

where $\lambda^R_a$, $\lambda^Q_a$ and $\lambda^T_a$ denote the $a$th eigenvalues of $Q$, $R$ and $T$, respectively. Regarding the structure of $Q$ and $R$, there are three different sets of corresponding eigenvalues for $Q$, $R$ and $T$, namely

$$\lambda^R_1 = e,$$ (246a)
$$\lambda^Q_1 = \frac{\chi}{\beta},$$ (246b)
$$\lambda^T_1 = \frac{1}{2\lambda},$$ (246c)

which occur with multiplicity $m - m \beta \mu^{-1}$

$$\lambda^R_2 = e - \mu g^2,$$ (247a)
$$\lambda^Q_2 = \frac{\chi + \mu p}{\beta},$$ (247b)
$$\lambda^T_2 = \frac{1}{2\lambda},$$ (247c)

occurring with multiplicity $m \beta \mu^{-1} - 1$, and

$$\lambda^R_3 = e - \mu g^2 - m \beta z^2,$$ (248a)
$$\lambda^Q_3 = \frac{\chi + \mu p}{\beta} + mq,$$ (248b)
$$\lambda^T_3 = \frac{1}{2\lambda} \left[ 1 - \frac{m \beta \lambda_0}{\lambda + m \beta \lambda_0} \right],$$ (248c)

which occur with multiplicity 1. Thus, by substituting in (245) and taking the limit when $m \downarrow 0$, $e$, $g$ and $f$ are given as

$$e = \frac{1}{2\lambda} R J(-\frac{\chi}{\lambda}),$$ (249a)
$$g^2 = \frac{1}{\lambda \mu} \left[ R J(-\frac{\chi}{\lambda}) - R J(-\frac{\chi + \mu p}{\lambda}) \right],$$ (249b)
The next step is to evaluate the partition function. Substituting (244) in (243), the Hamiltonian reads
\[
\mathcal{E}^R(v|x) = e\|x - v\|^2 - \beta \frac{g^2}{2} \operatorname{tr}(\mathbf{(x - v)(x - v)^T I_{m}\beta \otimes I_1}) - \beta \frac{r^2}{2} \operatorname{tr}(\mathbf{(x - v)(x - v)^T I_m}) + u(v).
\] (250)

The partition function is then determined as in (52). Substituting in (52) and using the identities
\[
e^{-\beta \frac{g^2}{2} \operatorname{tr}(\mathbf{(x - v)(x - v)^T I_{m}\beta \otimes I_1})} = \prod_{k=0}^{\infty} \int e^{-\beta g \sum_{x \neq \psi_k} (x - \psi_k)} \mathcal{D}z_1, \tag{251b}
\]

where \( \psi_k = k \mu \beta^{-1} + 1, \) \( \psi_k = (k + 1) \mu \beta^{-1}, \) and \( \Xi = m \mu \beta^{-1} - 1, \) the partition function is found as
\[
\mathcal{Z}^R(\beta; \mu|x) = \int \left[ \int \left[ \sum_v e^{-\beta [e(x-v)^2 + f(z_0 + g z_1) (x-v) + \mu v]} \right]^\frac{1}{p} \mathcal{D}z_1 \right] \frac{\mu h}{p} \mathcal{D}z_0.
\] (252)

with \( v \in \mathbb{X} \) where we denoted \( \mu \) in the argument of the partition function to indicate that the expression is determined for a given \( \mu. \) The normalized free energy of the spin glass of replicas then reads
\[
\mathcal{F}^R(\beta; m; \mu) = -\frac{1}{\beta m} E \log \int \left[ \int \left[ \sum_v e^{-\beta [e(x-v)^2 + f(z_0 + g z_1) (x-v) + \mu v]} \right]^\frac{1}{p} \mathcal{D}z_1 \right] \frac{\mu h}{p} \mathcal{D}z_0. \tag{253}
\]

Using the Riesz equality and taking the limit \( m \downarrow 0, \) the normalized free energy is given as in (254b), as shown at the bottom of this page.

To find the fixed point equations, we use (50) and write
\[
\left[ \mathcal{Z}^R(\beta; m; \mu) \right]^{-1} = \frac{1}{E} \int \log \left[ \int \left[ \sum_v e^{-\beta [e(x-v)^2 + f(z_0 + g z_1) (x-v) + \mu v]} \right]^\frac{1}{p} \mathcal{D}z_1 \right] \frac{\mu h}{p} \mathcal{D}z_0, \tag{254b}
\]

Taking derivative and determining the limit \( m \downarrow 0, \) the fixed point equations finally reduce to what given in (257a)-(257c), as shown at the bottom of this page.

Here, \( \Lambda^\beta := \left[ \int \Lambda^\beta \mathcal{D}z_1 \right]^{-1} \Lambda^\beta \) and \( \Lambda^\beta \) being defined as
\[
\Lambda^\beta = \left[ \sum_v e^{-\beta [e(x-v)^2 + f(z_0 + g z_1) (x-v) + \mu v]} \right]^\frac{1}{p} \mathcal{D}z_1 \mathcal{D}z_0. \tag{258}
\]
Remark C.1. In general, $\Lambda^\beta$ and $\tilde{\Lambda}^\beta$ are functions of $z_1, z_0$ and $x$. This point is explicitly indicated in the presentation of Result VI.1 in Section VI. Nevertheless, we drop the arguments here for sake of brevity.

The replicas’ average distortion w.r.t. distortion function $d(\cdot; \cdot)$ is further determined by modifying the Hamiltonian as

$$\mathcal{E}_h^R(v|x) = \mathcal{E}_R^R(v|x) + h \sum_{a=1}^m d(v_a; x)$$

with $\mathcal{E}_R^R(v|x)$ given in (250), and taking the steps as in (250)-(253) to find the modified form of the normalized free energy, i.e. $\mathcal{F}^R(\beta, h, m; \mu)$. The replicas’ average distortion then reads

$$\mathcal{D}_R^R(\beta; \mu) = \lim_{m \to 0} \frac{\partial}{\partial h} \mathcal{F}_R(\beta, h, m; \mu)|_{h=0}$$

$$= E \sum_{a} d(v; x) e^{-\beta [\epsilon(x-v)^2 + (fz_0 + gz_1)(x-v) + u(v)]} \times \tilde{\Lambda}^\beta \Delta z_1 \Delta z_0.$$  

(260b)

The analysis is concluded by taking the zero temperature limit. As $\beta \uparrow \infty$, (257a)-(257c) read

$$q + p = E \int (g - x)^2 \tilde{\Lambda} \Delta z_1 \Delta z_0,$$

$$\chi + \mu q + \mu p = \frac{1}{g} E \int (g - x) \tilde{\Lambda} \Delta z_1 \Delta z_0,$$

$$\chi + \mu p = \frac{1}{f} E \int (g - x) \tilde{\Lambda} \Delta z_1 \Delta z_0,$$

(261a)

(261b)

(261c)

where $g$ is defined as

$$g \equiv \arg \min_v \epsilon(x-v)^2 + (fz_0 + gz_1)(x-v) + u(v)$$

and $\tilde{\Lambda} \equiv \int \Lambda \Delta z_1^{-1} \Lambda$ with $\Lambda$ denoting

$$\Lambda \equiv \lim_{\beta \uparrow \infty} \Lambda^\beta$$

$$= e^{-\mu [\epsilon(x-g)^2 + (fz_0 + gz_1)(x-g) + u(g)]}$$

(263a)

(263b)

Moreover, the asymptotic distortion for a given $\mu$ reads

$$D^W = E \int d(g; x) \tilde{\Lambda} \Delta z_1 \Delta z_0.$$  

(264)

The equation in (254b), as well as those given in (261a)-(264), are determined in terms of $\mu$. Moreover, for a given $\mu$, multiple solution to the fixed point equations can be found. Result IV.1 suggests us to choose the solution which minimizes the free energy. Therefore, one needs to find the optimal $\mu$, and its corresponding $\chi$, $p$ and $q$, such that the free energy meets its minimum value. As the second law of thermodynamics is satisfied at any inverse temperature, we should initially search for the optimal $\mu$ considering a given $\beta$. We, then, find the corresponding $\chi$, $p$, and $q$ which minimize the zero temperature free energy. Using Result IV.1, the free energy at the inverse temperature $\beta$ for a given $\mu$ is written as

$$\mathcal{F}(\beta; \mu) = \frac{1}{2\beta} \int_0^1 \mathcal{F}^\beta(\omega; \mu) d\omega - \mathcal{F}^\beta(1; \mu)$$

$$+ \mathcal{F}_R^R(\beta; \mu)$$

(265)

where the function $\mathcal{F}_R^\beta(\cdot; \mu)$ is defined as

$$\mathcal{F}_R^\beta(\omega; \mu) = \frac{1}{2\beta} \int_0^1 \mathcal{F}^\beta(\omega; \mu) d\omega - \mathcal{F}^\beta(1; \mu)$$

$$+ \mathcal{F}_R^R(\beta; \mu)$$

(266)

To find $\mu$ at the thermal equilibrium, we let

$$\frac{\partial}{\partial \mu} \mathcal{F}(\beta; \mu) = 0$$

(267)

Using (249a)-(249c), (267) concludes that $\mu$ satisfies

$$\frac{1}{2\beta} \left[ p \mathcal{R}_J(\beta, \mu) - q \mathcal{R}_J(\beta, \mu) - q \mathcal{R}_J(-\beta, \mu) \right]$$

$$= \mathcal{F}_R^R(\beta; \mu) + \frac{1}{2\beta} \int_{\beta}^{\beta+\mu} \mathcal{R}_J(-\beta, \mu) d\beta + E \frac{1}{\beta} \int \log \left[ \sum \mathcal{F}_R^R(\beta; \mu) \right]$$

(268)

which as $\beta \uparrow \infty$ reduces to

$$\frac{1}{2\beta} \left[ p \mathcal{R}_J(\beta, \mu) - q \mathcal{R}_J(\beta, \mu) - q \mathcal{R}_J(-\beta, \mu) \right]$$

$$= \frac{1}{2\beta} \mathcal{F}_R^R(\beta; \mu) + \frac{1}{\mu} \int \log \left[ \sum \mathcal{F}_R^R(\beta; \mu) \right]$$

(269)

Denoting the solution to (269) by $\mu^*$, the free energy of the corresponding spin glass is then given as $\mathcal{F}(\beta) = \mathcal{F}(\beta; \mu^*)$ which at the zero temperature reads

$$\mathcal{F}^0 = \frac{1}{2\beta} E \int_0^1 \mathcal{F}^\beta(\omega; \mu^*) - \mathcal{F}^\beta(1; \mu^*)$$

$$- \frac{1}{\beta} E \int \log \left[ \sum \mathcal{F}_R^R(\beta; \mu^*) \right]$$

(270)

with

$$\mathcal{F}^\beta(\omega; \mu^*) = \lim_{\beta \uparrow \infty} \mathcal{F}^\beta(\omega; \mu^*).$$

(271)

Finally, by defining $\lambda^\beta := [2e]^{-1}$, $\lambda_0^\beta := [4e^2]^{-1} f^2$ and $\lambda_0^\beta := [4e^2]^{-1} g_1^2$, Result VI.1 is concluded.

APPENDIX D

PROOF OF RESULT VI.3

The strategy here is to extend the approach in Appendix C to a general number of breaking steps. Following Appendix E and considering $Q$ as

$$Q = \frac{\chi}{\beta} I_m + \sum_{k=1}^b \left( \frac{p_k I_{2\mu} \otimes I_{2\mu} + q I_m} \right)$$

(272)

the frequency domain correlation matrix $R := T \mathcal{R}_J(-2\beta T) Q$ is written as

$$R = eI_m - \beta \sum_{k=1}^b \frac{g^2_k}{2} I_{2\mu} \otimes I_{2\mu} - \beta \frac{f^2}{2} I_m$$

(273)

considering a given vector $\mu = [\mu_1, \ldots, \mu_b]^T$, such that

$$\mu_{k+1} = \partial_{k+1} \mu_k,$$
with \( \{ \vartheta_k \} \) being non-negative integers, \( e, f \) and \( \{ g_k \} \) are then found in terms of \( \varphi, q \) and \( \{ p_k \} \) by letting

\[
\lambda_a^R = \frac{\zeta}{\lambda} R_{\frac{\lambda}{\lambda}} \left(-2\beta \frac{\lambda}{\lambda} Q_0\right)
\]

for \( a \in \{ 1 : m \} \) where \( \lambda_a^R, \lambda_a^Q \) and \( \lambda_a^T \) denote the \( a \)th corresponding eigenvalues of \( R, Q \) and \( T \). As long as the constraint in (274) holds, \( Q, T \) and \( R \) have \( b + 2 \) different sets of corresponding eigenvalues:

\[
\begin{align*}
\lambda_a^R &= e \\
\lambda_1^Q &= \frac{\varphi}{\lambda} \\
\lambda_1^T &= \frac{1}{2\lambda}
\end{align*}
\]

with multiplicity \( \Theta_1(m) = m - m\beta\mu_1^{-1} \),

\[
\begin{align*}
\lambda_k^R &= e - \sum_{\varphi = 1}^{k-1} p_{\varphi} \frac{\mu_{\varphi}}{\beta} \\
\lambda_k^Q &= \frac{\varphi}{\lambda} + \sum_{\varphi = 1}^{k-1} p_{\varphi} \frac{\mu_{\varphi}}{\beta} \\
\lambda_k^T &= \frac{1}{2\lambda},
\end{align*}
\]

for \( k \in \{ 2 : b \} \) with multiplicity \( \Theta_k(m) = m\beta \left( \mu_{k-1}^{-1} - \mu_k^{-1} \right) \),

\[
\begin{align*}
\lambda_{b+1}^R &= e - \sum_{\varphi = 1}^{b} p_{\varphi} \frac{\mu_{\varphi}}{\beta} \\
\lambda_{b+1}^Q &= \frac{\varphi}{\lambda} + \sum_{\varphi = 1}^{b} p_{\varphi} \frac{\mu_{\varphi}}{\beta} \\
\lambda_{b+1}^T &= \frac{1}{2\lambda}
\end{align*}
\]

with multiplicity \( \Theta_{b+1}(m) = m\beta\mu_b^{-1} - 1 \), and

\[
\begin{align*}
\lambda_{b+2}^R &= e - \sum_{\varphi = 1}^{b} p_{\varphi} \frac{\mu_{\varphi}}{\beta} \\
\lambda_{b+2}^Q &= \frac{\varphi}{\lambda} + \sum_{\varphi = 1}^{b} p_{\varphi} \frac{\mu_{\varphi}}{\beta} + mq \\
\lambda_{b+2}^T &= \frac{1}{2\lambda} \left[ 1 - \frac{m\beta\lambda_0}{\lambda + m\beta\lambda_0} \right].
\end{align*}
\]

with multiplicity \( \Theta_{b+1}(m) = 1 \).

Substituting in (275) \( e, f \) and \( \{ g_k \} \) are determined in terms of \( \varphi, q \) and \( \{ p_k \} \) as

\[
\begin{align*}
e &= \frac{1}{2\beta} R_{\frac{\lambda}{\lambda}} \left(-\frac{\varphi}{\lambda} \right), \\
g_k^2 &= \frac{1}{\lambda\mu_k} \left[ R_{\frac{\lambda}{\lambda}} \left(-\frac{\varphi}{\lambda} \right) - R_{\frac{\lambda}{\lambda}} \left(-\frac{\varphi}{\lambda} \right) \right].
\end{align*}
\]

where we define \( \tilde{\varphi}_0 := \varphi \) and

\[
\tilde{\varphi}_k := \varphi + \sum_{\varphi = 1}^{k} \mu_\varphi p_\varphi
\]

for \( k \in \{ 1 : b \} \). The Hamiltonian of the spin glass of replicas is then determined as in (48).

\[
\begin{align*}
e^{-\frac{1}{2} \varphi u(x,v) \cdot (-\varphi u(x,v))} T_u &= \int \exp \left[ -\beta f \left( \sum_{m=1}^{\varphi} (x_{m,v}) \right) \right] D\varphi, \\
e^{-\frac{1}{2} \varphi u(x,v) \cdot (-\varphi u(x,v))} T_u &= \int \exp \left[ -\beta f \left( \sum_{m=1}^{\varphi} (x_{m,v}) \right) \right] D\varphi.
\end{align*}
\]

Consequently, one evaluates the free energy as in (53) which by using the Riesz equality when \( m \downarrow 0 \) reduces to (285), as shown at the bottom of this page, where we have defined \( \mu_0 = \beta \) for sake of compactness. The fixed point equations are, moreover, found via (50) where we have

\[
\begin{align*}
e^{-\frac{1}{2} \varphi u(x,v) \cdot (-\varphi u(x,v))} m &= \text{E}_{x} \sum_{v} \| x - v \|^{2} p_{v|x}(v|x), \\
&= \int \left[ \sum_{m=1}^{\varphi} (x_{m,v}) \right] D\varphi.
\end{align*}
\]
Thus, the fixed point equations are finally concluded as in (286):

\[
\begin{aligned}
    \tilde{\xi}_{k-1} + \frac{\mu_k}{\beta} \left( \sum_{\zeta=\kappa}^b p_\zeta + q \right) m &= E_x \left\{ \sum_v \text{tr}((x-v)(x-v)^\top T_{\frac{m}{\mu_k}} \otimes 1_{\frac{m}{\mu_k}}) p_{v|x}(v|x) \right\}, \\
    \tilde{\eta}_b + m \xi &= E_x \left\{ \sum_v \text{tr}((x-v)(x-v)^\top T_m) p_{v|x}(v|x) \right\},
\end{aligned}
\]  

(286b)

and \([\Lambda_k^a] \) for \( \kappa \in [2:b] \) being recursively defined as

\[\Lambda_k^a := \int \Lambda_{k-1}^a Dz_{k-1} \frac{\mu_k}{\mu_{k-1}} \]  

(290)

The replicas’ average distortion w.r.t. \( d(\cdot; \cdot) \) is further determined using the Hamiltonian modification technique employed in Appendix B and C. After modifying the Hamiltonian and taking the derivatives, the average distortion at the inverse temperature \( \beta \) is given by (291), as shown at the bottom of this page.

Finally, by taking the limit \( \beta \uparrow \infty \), we find

\[D^W = E \int d(g; x) \prod_{k=1}^b \tilde{\Lambda}_k Dz_k Dz_0 \]  

(292)

where \( g \) is defined as

\[g := \arg \min_{\nu} \left[ e(x-v)^2 + (f \sum_{k=1}^b g_k z_k)(x-v) + u(\nu) \right], \]

(293)

and \( \tilde{\Lambda}_k \) denotes the limit \( \tilde{\Lambda}_k^\infty \), i.e., when \( \beta \uparrow \infty \). Considering the definition of \( \tilde{\Lambda}_k^a \), \( \tilde{\Lambda}_k \) reads

\[\tilde{\Lambda}_1 := e^{-\mu_1 \left[ e(x-g)^2 + (f \sum_{k=1}^b g_k z_k)(x-g) + u(g) \right] \]  

(294)

and \([\Lambda_k] \) for \( \kappa \in [2:b] \)

\[\Lambda_k := \int \Lambda_{k-1} Dz_{k-1} \frac{\mu_k}{\mu_{k-1}} \]  

(295)

Moreover, the fixed point equations reduce to

\[\sum_{k=1}^b p_k + q = E \int (g-x)^2 \prod_{k=1}^b \tilde{\Lambda}_k Dz_k Dz_0 \]  

(289a)

and

\[\tilde{\xi}_{k-1} + \mu_k \left( \sum_{\zeta=\kappa}^b p_\zeta + q \right) = \frac{1}{\xi_k} E \int \sum_{v} (v-x)^2 \left[ -\beta \left( e(x-v)^2 + (f \sum_{k=1}^b g_k z_k)(x-v) + u(\nu) \right) \right] Dz_k Dz_0 \]  

(288b)

\[\tilde{\eta}_b = \frac{1}{\eta_b} E \int \sum_{v} (v-x) z_0 \left[ -\beta \left( e(x-v)^2 + (f \sum_{k=1}^b g_k z_k)(x-v) + u(\nu) \right) \right] Dz_k Dz_0 \]  

(288c)

\[D^R(\beta; \mu) = E \int \sum_{v} d(v; x) e^{-\beta \left( e(x-v)^2 + (f \sum_{k=1}^b g_k z_k)(x-v) + u(\nu) \right) \prod_{k=1}^b \tilde{\Lambda}_k Dz_k Dz_0} \]  

(291)
\[ \tilde{\chi}_{\kappa-1} + \mu \kappa \left( \sum_{\zeta=0}^{b} p_{\zeta} + q \right) \]
\[ = \frac{1}{g_{\kappa}} \mathbf{E} \int (g - x) z_{\kappa} \prod_{\kappa=1}^{b} \tilde{\Lambda}_{\kappa} \, Dz_{\kappa} Dz_{0}, \quad (296b) \]
\[ \tilde{\chi}_{b} = \frac{1}{f} \int (g - x) z_{0} \prod_{\kappa=1}^{b} \tilde{\Lambda}_{\kappa} \, Dz_{\kappa} Dz_{0}, \quad (296c) \]
for \( \kappa \in [1 : b] \).

As in the 1RSB ansatz, we set \( \mu \) to be the extreme point of the free energy at a given inverse temperature \( \beta \), in order to satisfy the second law of thermodynamics. The solution needs to be found over the set of non-negative real vectors which satisfy the constraint in (274). The parameters of the ansatz, however, are finally taken such that the zero temperature free energy is minimized.

Using Result IV.1, the free energy of the corresponding spin glass for a given vector \( \mu \) is written as
\[ \mathcal{F}(\beta; \mu) = \frac{1}{2\lambda} \left[ \int_{0}^{1} F^{\beta}(\omega; \mu) d\omega - F^{\beta}(1; \mu) \right] + \mathcal{F}^{R}(\beta; \mu) \]
where the function \( F^{\beta}(\cdot; \mu) \) is defined as
\[ F^{\beta}(\omega; \mu) = \sum_{\kappa=1}^{b} \frac{1}{b_{\kappa}} \frac{d}{d\omega} \int_{\tilde{\chi}_{\kappa}-\omega}^{\tilde{\chi}_{\kappa}} R_{\mathbf{J}}(-t_{\lambda}) dt + \frac{\chi_{\kappa}}{\lambda} R_{\mathbf{J}}(-\frac{\lambda}{\lambda} \omega) \]
\[ + \left[ q - \frac{\lambda_{0}}{\lambda} \tilde{\chi}_{b} \right] \frac{d}{d\omega} \omega R_{\mathbf{J}}(-\frac{\lambda}{\lambda} \omega). \]
Therefore, the vector \( \mu^{*} \), for a given \( \beta \), is set as
\[ \mu^{*} = \arg \min_{\mu} \mathcal{F}(\beta; \mu) \]
with \( \mu \in \mathbb{S}_{\mu} \) where \( \mathbb{S}_{\mu} \) is the set of non-negative real vectors satisfying the constraint in (274). By substituting (280a)-(280c) in (298), \( \mu^{*} \) reduces to
\[ \mu^{*} = \arg \min_{\mu} \left\{ \frac{1}{2\lambda} \left[ \int_{0}^{1} F^{\beta}(\omega; \mu) d\omega \right] + \mathcal{F}^{R}(\beta; \mu) - e \Delta(\mu) \right\} \]
with \( \mu \in \mathbb{S}_{\mu} \) where \( \Delta(\cdot) \) reads
\[ \Delta(\mu) := \frac{1}{e} \left( \sum_{\kappa=1}^{b} \frac{1}{b_{\kappa}} \left[ \tilde{e}_{\kappa} \tilde{\chi}_{\kappa} - \tilde{e}_{\kappa-1} \tilde{\chi}_{\kappa-1} \right] + \left[ \tilde{e}_{0} \tilde{\chi}_{0} + \tilde{e}_{b} q - \frac{f^{2}}{2} \tilde{\chi}_{b} \right] \right) \]
for \( \kappa \in [1 : b] \). The vector \( \mu^{*} \) is then determined such that it minimizes the free energy. Finally by taking the limit \( \beta \uparrow \infty \), the zero temperature free energy is evaluated as
\[ \mathcal{F}^{0} = \frac{1}{2\lambda} \left[ \int_{0}^{1} F^{\infty}(\omega)d\omega - F^{\infty}(1) \right] - \frac{1}{\mu_{b}} \mathbf{E} \log \left[ \int \Lambda_{b} Dz_{b} \right] Dz_{0} \]
where we define
\[ F^{\infty}(\omega) := \lim_{\beta \uparrow \infty} F^{\beta}(\omega; \mu^{*}). \]

By defining scalars \( \lambda^{\infty} := [2e]^{1} \), \( \lambda_{0}^{b} := [4e^{2}]^{-1} f^{2} \) and \( \lambda^{b}_{\kappa} := [4e^{2}]^{-1} g^{2} \) for \( \kappa \in [1 : b] \), and defining sequence \( \{\zeta_{\kappa}\} \) such that \( \zeta_{0} = 1 \) and
\[ \zeta_{\kappa} := 1 - \sum_{i=1}^{\kappa} \mu_{i} \frac{\lambda^{b}_{\kappa}}{\lambda^{b}_{\infty}} \]
for \( \kappa \in [1 : b] \), Result VI.3 is concluded.

### APPENDIX E

**General RSB Frequency Domain Correlation Matrix**

Consider the spin glass of replicas defined in Definition IV.1, the Hamiltonian reads
\[ \mathcal{E}^{R}(v|x) = (x - v)^{T} \mathbf{R}(x - v) + u(v). \]
where \( \mathbf{R} := \text{TR}_{T}(-2\beta \mathbf{T}) \) is referred to as the frequency domain correlation matrix. In this appendix, we show that under the general RSB assumption on \( \mathbf{Q} \), including the RS case, the frequency domain correlation matrix has the same structure with different scalar coefficients. To show that, let the correlation matrix be of the form
\[ \mathbf{Q} = q_{0} \mathbf{1}_{m} + \sum_{i=1}^{b} q_{i} \mathbf{1}_{\kappa_{i}} \otimes \mathbf{1}_{\kappa_{i}} + q_{b+1} \mathbf{1}_{m} \]
for some integer \( b \) where \( q_{0}, q_{b+1} \neq 0 \). (307) represents the bRSB as well as RS structures by setting the coefficients correspondingly. Considering \( \mathbf{T} \) as defined in (49), \( \mathbf{TQ} \) is then written as
\[ \mathbf{TQ} = \frac{1}{2\lambda} \left[ \mathbf{Q} - \frac{\beta \lambda_{0}}{\lambda + m \beta \lambda_{0}} \mathbf{1}_{m} \mathbf{Q} \right]. \]

Defining the vector \( u_{m \times 1} \) as a vector with all entries equal to 1, it is clear that \( u \) is an eigenvector of \( \mathbf{Q} \), and therefore, by denoting the eigendecomposition of \( \mathbf{Q} \) as \( \mathbf{VD}^{\frac{1}{2}} \mathbf{V}^{T} \), \( \mathbf{1}_{m} \) reads
\[ \mathbf{1}_{m} = u \mathbf{1}_{m} \]
which states that \( \mathbf{TQ} \) and \( \mathbf{Q} \) span the same eigenspace. The eigenvalues of \( \mathbf{TQ} \) and \( \mathbf{Q} \) are also distributed with the same frequencies. In fact, as the eigenvalue corresponding to \( u \) occurs with multiplicity 1, the second term on the r.h.s. of
(308) does not change the distribution of eigenvalues and only modifies the eigenvalue corresponding to $u$. Therefore, $TQ$ can be also represented as in (307) with different scalar coefficient.

To extend the scope of the analysis to $R$, we note that the function $R_T(\cdot)$ is strictly increasing for any $F_j$ different from the single mass point CDF. Therefore, $TQ$ becomes a constant function which results in $R_T(-2\beta TQ) = K I_m$ for some constant $K$. Therefore, $R = K T$ which is again represented as in (311) by setting $r_i = 0$ for $i \in [1 : b]$. This concludes that $R$ has the same structure as $Q$ for any $F_j$.

\section*{APPENDIX F

\section*{ASYMPTOTICS OF SPHERICAL INTEGRAL}

Consider $\mu_n$ to be the Haar measure on the orthogonal group $O_n$ for $\zeta = 1$, and on the unitary group $U_n$ for $\zeta = 2$. Let $G_n$ and $D_n$ be $n \times n$ matrices; then, the integral of the form

$$I_n^\zeta(G_n, D_n) := \int e^{n Tr(U G_n U^* D_n)} d\mu_n(U),$$

is known as the spherical integral. This integral has been extensively studied in the mathematics literature, as well as physics where it is often called \textit{Harish-Chandra} or \textit{Itzykson & Zuber} integral. In a variety of problems, such as ours, the evaluation of spherical integrals in asymptotic regime is interesting, and therefore, several investigations have been done on this asymptotics. In [76], the asymptotics of the integral has been investigated when the matrices $G_n$ and $D_n$ have $n$ distinct eigenvalues with converging spectrums, and under some assumptions, a closed form formula has been given; however, the final formula in [76] is too complicated and hard to employ. In [77], the authors showed that, for a low-rank $G_n$, the asymptotics of the integral can be written directly in terms of the $R$-transform corresponding to the asymptotic eigenvalue distribution of $D_n$. As long as the replica analysis is being considered, we can utilize the result from [77], since the number of replicas can be considered to be small enough.

In [77], Theorem 1.2, it is shown that when $G_n$ is a rank-one matrix, under the assumption that the spectrum of $D_n$ asymptotically converges to a deterministic CDF $F_D$ with compact and finite length support, the asymptotics of the integral can be written in terms of the $R$-transform $R_D(\cdot)$ as

$$\lim_{n \to \infty} \frac{1}{n} \log I_n^\zeta(G_n, D_n) = \int_0^\infty R_D(\frac{2\omega}{\epsilon}) d\omega,$$

in which $\theta$ denotes the single nonzero eigenvalue of $G_n$. The authors further showed in Theorem 1.7 that in the case of rank($G_n$) = $O(\sqrt{n})$, under the same assumption as in Theorem 1.2, the spherical integral asymptotically factorizes

$$\lim_{n \to \infty} \frac{1}{n} \log I_n^\zeta(G_n, D_n) = \sum_{i=1}^{m} \int_0^\infty R_D(\frac{2\omega}{\epsilon}) d\omega,$$

with $\{\theta_i\}$ denoting the nonzero eigenvalues of $G_n$ for $i \in [1 : m]$, and $m = \text{rank}(G_n)$.

In Appendix A, one can employ (314) in order to evaluate the asymptotics over the system matrix consistent to the system setup illustrated in Section II. Moreover, by using the above discussion, the investigations in Appendix A can be extended to the case of complex variables. More about the spherical integral and its asymptotics can be found in [77], and the references therein.

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