Two paradoxical results in linear models: the variance inflation factor and the analysis of covariance

Peng Ding*

Abstract

A result from a standard linear model course is that the variance of the ordinary least squares (OLS) coefficient of a variable will never decrease if we add additional covariates. The variance inflation factor (VIF) measures the increase of the variance. Another result from a standard linear model or experimental design course is that including additional covariates in a linear model of the outcome on the treatment indicator will never increase the variance of the OLS coefficient of the treatment at least asymptotically. This technique is called the analysis of covariance (ANCOVA), which is often used to improve the efficiency of treatment effect estimation. So we have two paradoxical results: adding covariates never decreases the variance in the first result but never increases the variance in the second result. In fact, these two results are derived under different assumptions. More precisely, the VIF result conditions on the treatment indicators but the ANCOVA result requires random treatment indicators. In a completely randomized experiment, the estimator without adjusting for additional covariates has smaller conditional variance at the cost of a larger conditional bias, compared to the estimator adjusting for additional covariates. Thus, there is no real paradox.

Keywords: Conditioning; Design-based inference; Gauss–Markov model; Potential outcomes; Randomization; Unit-treatment additivity

*Peng Ding (Email: pengdingpku@berkeley.edu) is Assistant Professor in the Department of Statistics, University of California, Berkeley, CA 94720, USA. The author thanks Ugur Yildirim for raising this question in his class of “Linear Models” at UC Berkeley, and thanks Xinran Li and Liyun Chen for helpful discussions.
1. Variance inflation factor

Consider the following linear regression:

\[ y_i = \tau z_i + \beta' x_i + \varepsilon_i, \quad (i = 1, \ldots, n) \] (1)

where the regressors \( z_i \) is a scalar and \( x_i \) is a vector containing 1. Using the Frisch–Waugh Theorem, we can write the OLS estimator for \( \tau \) as

\[ \hat{\tau}_a = \frac{\sum_{i=1}^{n} \tilde{z}_i y_i}{\sum_{i=1}^{n} \tilde{z}_i^2}, \]

where \( \tilde{z}_i \) is the residual from the OLS fit of \( z_i \) on \( x_i \). If the regressors \((z_i, x_i)\)'s are all fixed and the \( \varepsilon_i \)'s are IID with mean 0 and variance \( \sigma^2 \), then we can express the variance of \( \hat{\tau}_a \) as

\[ \text{var}(\hat{\tau}_a) = \frac{\sum_{i=1}^{n} \tilde{z}_i^2 \text{var}(y_i)}{(\sum_{i=1}^{n} \tilde{z}_i^2)^2} = \frac{\sigma^2}{(\sum_{i=1}^{n} \tilde{z}_i^2)} = \frac{\sum_{i=1}^{n} (z_i - \bar{z})^2}{\sum_{i=1}^{n} \tilde{z}_i^2} \times \frac{\sum_{i=1}^{n} (z_i - \bar{z})^2}{\sum_{i=1}^{n} \tilde{z}_i^2}. \] (2)

The first term of (2) is the variance of

\[ \hat{\tau} = \frac{\sum_{i=1}^{n} (z_i - \bar{z}) y_i}{\sum_{i=1}^{n} (z_i - \bar{z})^2}, \]

i.e., the coefficient of \( z_i \) in the OLS fit of \( y_i \) on \((z_i, 1)\). The second term of (2) is the VIF, no smaller than 1, because it is the total sum of squares divided by the residual sum of squares in the OLS fit of \( z_i \) on \( x_i \). See Faraway (2016), Fox (2015) and Agresti (2015) for textbook discussions.

Thus, from (2), the variance of \( \text{var}(\hat{\tau}_a) \) will never decrease with more covariates in (1), because the residual sum of squares \( \sum_{i=1}^{n} \tilde{z}_i^2 \) will decrease while the total sum of squares \( \sum_{i=1}^{n} (z_i - \bar{z})^2 \) is fixed. An immediate result is that \( \text{var}(\hat{\tau}_a) \geq \text{var}(\hat{\tau}) \).

2. Analysis of covariance

Now we view (1) in a slightly different way: the \( x_i \)'s are pretreatment covariates, the \( z_i \)'s are the binary treatment indicators, and the \( y_i \)'s are the outcomes of interest. Then (1) is the standard ANCOVA model, and the parameter \( \tau \) is the treatment effect of interest. Let \( n_1 = \sum_{i=1}^{n} z_i \) and
\[ n_0 = \sum_{i=1}^{n} (1 - z_i). \] Because \( z_i \) is binary, we can simplify the expressions of \( \hat{\tau} \) to

\[
\hat{\tau} = n_1^{-1} \sum_{i=1}^{n} z_i y_i - n_0^{-1} \sum_{i=1}^{n} (1 - z_i) y_i
\]

\( \equiv \tau + \beta' \hat{\delta}_x + \hat{\delta}_\varepsilon, \quad (3) \)

and the expression of \( \hat{\tau}_a \) to

\[
\hat{\tau}_a = n_1^{-1} \sum_{i=1}^{n} z_i (y_i - \hat{\beta}' x_i) - n_0^{-1} \sum_{i=1}^{n} (1 - z_i) (y_i - \hat{\beta}' x_i)
\]

\[ = \tau + (\beta - \hat{\beta}) \hat{\delta}_x + \hat{\delta}_\varepsilon \approx \tau + \hat{\delta}_\varepsilon, \quad (4) \]

where \( \hat{\delta}_x = n_1^{-1} \sum_{i=1}^{n} z_i x_i - n_0^{-1} \sum_{i=1}^{n} (1 - z_i) x_i \) and \( \hat{\delta}_\varepsilon = n_1^{-1} \sum_{i=1}^{n} z_i \varepsilon_i - n_0^{-1} \sum_{i=1}^{n} (1 - z_i) \varepsilon_i \) are the differences-in-means of \( x \) and \( \varepsilon \), and \( \hat{\beta} \) is the OLS estimator for \( \beta \) in (1). We ignore the term \( (\beta - \hat{\beta})' \hat{\delta}_x \) in (4) because with large samples, \( \hat{\beta} \to \beta \) in probability.

As in Section (1), we assume that the \( \varepsilon_i \)'s are IID with mean 0 and variance \( \sigma^2 \). We further assume that the \( z_i \)'s are IID Bernoulli(\( \pi \)), and if we condition on \((n_1, n_0)\), then \((z_1, \ldots, z_n)\) is a permutation of \( n_1 \) 1's and \( n_0 \) 0's. We can show that \( E(\hat{\delta}_\varepsilon) = 0, E(\hat{\delta}_x) = 0, \) and

\[
\text{var}(\hat{\delta}_\varepsilon) = \frac{n}{n_1 n_0} \sigma^2, \quad \text{var}(\hat{\delta}_x) = \frac{n}{n_1 n_0} S_x^2, \quad \text{cov}(\hat{\delta}_\varepsilon, \hat{\delta}_x) = 0, \quad (5)
\]

where \( S_x^2 = (n - 1)^{-1} \sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x})' \) is the finite population covariance of the \( x_i \)'s. The first variance and the third covariance in (5) follow from standard variance and covariance calculations by first conditioning on all \( z_i \)'s, and the second variance in (5) follows from Neyman’s (1923) result on the difference-in-means from a completely randomized experiment (c.f. Imbens and Rubin 2015; Li and Ding 2017). Then \( E(\hat{\tau}) = \tau \) and \( E(\hat{\tau}_a) \approx \tau \), i.e., \( \hat{\tau} \) is unbiased and \( \hat{\tau}_a \) is consistent for \( \tau \).

Their variances satisfies

\[
\text{var}(\hat{\tau}) - \text{var}(\hat{\tau}_a) \approx \text{var}(\beta' \hat{\delta}_x) = \frac{n}{n_1 n_0} \beta' S_x^2 \beta \geq 0.
\]

Thus, if \( \beta \neq 0 \) then ANCOVA improves estimation efficiency, at least asymptotically. See Kempthorne (1952), Hinkelmann and Kempthorne (2007) and Cox and Reid (2000) for textbook discussions.
3. From conflict to unification

From the VIF result, we see that adding more covariates will never decrease the variance of an OLS coefficient. In contrast, from the ANCOVA result, we see that adding more covariates will never increase the variance of an OLS coefficient at least asymptotically. These two results are both standard in textbooks of linear models or experimental designs. However, they seem to give opposite conclusions. Both results are derived under the linear model (1), and therefore, these two conflicting results seems paradoxical.

If we go back to the derivations above carefully, we will find that Section 1 assumes that the \( z_i \)'s and \( x_i \)'s are both fixed, but Section 2 assumes that the \( z_i \)'s are random and the \( x_i \)'s are fixed. Therefore, the VIF and the ANCOVA results hold under different model assumptions. This vaguely explains the paradox. Below, we give a more unified discussion.

Consider the following data generating process: for \( i = 1, \ldots, n \),

(a) the \( x_i \)'s are fixed constants with the first component being 1;

(b) generate the potential outcomes under control as \( y_i(0) = \beta'x_i + \varepsilon_i \), where \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \) are IID with mean 0 and variance \( \sigma^2 \);

(c) generate the potential outcomes under treatment as \( y_i(1) = y_i(0) + \tau \), i.e., the individual treatment effect \( y_i(1) - y_i(0) \) is constant \( \tau \);

(d) generate \( Z = (z_1, \ldots, z_n) \) IID from Bernoulli(\( \pi \));

(e) the observed outcome is

\[
y_i = z_i y_i(1) + (1 - z_i) y_i(0) = \tau z_i + y_i(0) = \tau z_i + \beta'x_i + \varepsilon_i.
\]  

(6)

In (b) and (c), I use the potential outcomes notation (Neyman 1923). Readers who are uncomfortable with \( y_i(1) \) and \( y_i(0) \) can ignore (b) and (c) and view (6) as the data generating process with random \( \varepsilon_i \)'s and \( z_i \)'s. Then \( \tau \) is the average treatment effect parameter of interest.
Conditional on \(Z\), (6) is a linear model with fixed \((z_i, x_i)\)’s and homoskedastic errors \(\varepsilon_i\)’s. The discussion in Section 1 applies in this case. Then from the VIF result, we know that \(\text{var}(\hat{\tau}_a \mid Z) \geq \text{var}(\hat{\tau} \mid Z)\), i.e., the estimator adjusting for covariates \(x_i\)’s has larger variance. However, \(\hat{\tau}_a\) is an unbiased estimator, but \(\hat{\tau}\) is not an unbiased estimator. From the classic OLS theory, \(E(\hat{\tau}_a \mid Z) = \tau\), and from (3), the bias of \(\hat{\tau}\) is \(E(\hat{\tau} \mid Z) - \tau = \delta_x\). Therefore, the smaller conditional variance of \(\hat{\tau}\) comes at the cost of having a larger conditional bias.

Conditional on \(E\) and \((n_1, n_0)\), we have fixed potential outcomes and completely randomized \(Z\). The classic results from randomization inference applies in this case. Neyman (1923) shows that \(E(\hat{\tau} \mid E, n_1, n_0) = \tau\), and Freedman (2008) and Lin (2013) show that \(E(\hat{\tau}_a \mid E, n_1, n_0) \approx \tau\). Freedman (2008) further shows that asymptotically \(\text{var}(\hat{\tau} \mid E, n_1, n_0)\) is at least as large as \(\text{var}(\hat{\tau}_a \mid E, n_1, n_0)\). Thus, under a constant treatment effect model, ANCOVA improves efficiency asymptotically.

Conditional only on \((n_1, n_0)\), we have random potential outcomes and random treatment indicators. The discussion in Section 2 applies in this case. We have shown that \(E(\hat{\tau} \mid n_1, n_0) = \tau\) and \(E(\hat{\tau}_a \mid n_1, n_0) \approx \tau\), and moreover, asymptotically, \(\text{var}(\hat{\tau} \mid n_1, n_0)\) is at least as large as \(\text{var}(\hat{\tau}_a \mid n_1, n_0)\).

4. Some final remarks

I have shown that the seemingly paradoxical results of VIF and ANCOVA are due to different statistical assumptions. The key difference is whether the treatment indicators \(Z\) are random or not. In a model with fixed \(Z\), the unadjusted estimator has smaller variance but has larger bias. In a model with random \(Z\), both unadjusted and adjusted estimators are consistent for \(\tau\) but the variance of the adjusted estimator is no larger than the variance of the unadjusted estimator. In a randomized experiments, we still prefer using ANCOVA.

In (a), I fix the \(x_i\)’s. With random covariates, we can condition on them and obtain the same results. Again, the key is whether \(Z\) is random or not. I do not focus on the conditions for asymptotic analyses. See Freedman (2008), Lin (2013) and Li and Ding (2017) for more details.

In this short note, I do not focus on more general potential outcomes models. The data generating process in (a)–(c) assumes constant treatment effect. It yields the standard ANCOVA model (1) or (6). The literature of randomization-based causal inference often does not assume constant
treatment effect (Neyman 1923; Freedman 2008; Lin 2013; Li and Ding 2017). In those general cases, ANCOVA may increase or decrease the efficiency (Freedman 2008), but simply adding the interaction term $z_i \times x_i$ with centered $x_i$’s gives an estimator that is asymptotically as efficient as the unadjusted estimator (Lin 2013).

References

Agresti, A. (2015). *Foundations of Linear and Generalized Linear Models*. New York: John Wiley & Sons.

Cox, D. R. and Reid, N. (2000). *The Theory of the Design of Experiments*. Chapman and Hall/CRC.

Faraway, J. J. (2016). *Linear Models with R*. Boca Raton: Chapman and Hall/CRC.

Fox, J. (2015). *Applied Regression Analysis and Generalized Linear Models*. Sage Publications.

Freedman, D. A. (2008). On regression adjustments to experimental data. *Advances in Applied Mathematics*, 40:180–193.

Hinkelmann, K. and Kempthorne, O. (2007). *Design and Analysis of Experiments, Volume 1, Introduction to Experimental Design, 2nd Edition*. New York: John Wiley & Sons.

Imbens, G. W. and Rubin, D. B. (2015). *Causal Inference for Statistics, Social, and Biomedical Sciences: An Introduction*. New York: Cambridge University Press.

Kempthorne, O. (1952). *The Design and Analysis of Experiments*. Wiley.

Li, X. and Ding, P. (2017). General forms of finite population central limit theorems with applications to causal inference. *Journal of the American Statistical Association*, 112:1759–1769.

Lin, W. (2013). Agnostic notes on regression adjustments to experimental data: Reexamining Freedman’s critique. *Annals of Applied Statistics*, 7:295–318.

Neyman, J. (1923). On the application of probability theory to agricultural experiments: Essay on principles, Section 9. Masters Thesis. Portions translated into english by D. Dabrowska and T. Speed (1990). *Statistical Science*, 5:465–472.