Quantum inequalities for massless spin-3/2 field in Minkowski spacetime

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Quantum inequalities have been established for various quantum fields in both flat and curved spacetimes. In particular, for spin-3/2 fields, Yu and Wu have explicitly derived quantum inequalities for massive case. Employing the similar method developed by Fewster and colleagues, this paper provides an explicit formula of quantum inequalities for massless spin-3/2 field in four-dimensional Minkowski spacetime.

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I. INTRODUCTION

As is well known, seemingly reasonable energy conditions such as weak energy condition play a special role in classical general relativity. It has proved profitable to only require one or more energy conditions satisfied by the energy momentum tensor rather than to know the specific expression of the energy momentum tensor for matters. For example, the singularity theorem and the positive mass conjecture are proved under such assumptions.

However, all the pointwise energy conditions are violated in the framework of quantum field theory. Even there exist such series of quantum states in which the energy density at a given point may approach arbitrary negative values. If the magnitude and duration of such negative energy densities were unconstrained, various exotic phenomena might occur. These result in serious ramifications such as the violation of the second law of thermodynamics, and the existence of traversable wormholes, and even time machines.

Interestingly, there exist some mechanisms in quantum field theory to restrict the extent of negative energy densities: the weighted average of energy densities by non-negative sampling functions satisfies the quantum weak energy inequalities, simply called quantum inequalities. Since the pioneering work by Ford, who obtained a quantum inequality for massless minimally-coupled scalar field in Minkowski spacetime with a Lorentzian sampling function\(^1\), progress has been made toward generalizing to various quantum fields in both flat and curved spacetimes\(^2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\). Especially, for spin-3/2 field, using the method developed by Fewster and colleagues\(^6, 7, 10, 13\), Yu and Wu have given an explicit derivation of quantum inequalities for massive case in four-dimensional Minkowski spacetime\(^15\). As a further step along this line, this paper will provide quantum inequalities for massless spin-3/2 field in four-dimensional Minkowski spacetime for arbitrary non-negative sampling functions by the same approach.

The result obtained here is

\[
\int_{-\infty}^{\infty} dx^0 \langle \hat{\rho}(x^0, x) \rangle g^2(x^0) \geq -\frac{1}{24\pi^3} \int_{0}^{\infty} du |\tilde{g}(u)|^2 u^4. 
\]  

(1)

Here notations and conventions follow those in\(^19\). Especially, the metric signature takes \((+,-,-,-)\), and \(\{\sigma^{\mu\Sigma\Sigma'} = \frac{1}{\sqrt{2}}(I, \sigma) |\mu = 0, 1, 2, 3; \Sigma(\Sigma') = 1, 2\} \) with \(\sigma\) Pauli matrices. In addition, the Fourier transformer of a function \(g\) is
defined by
\[ \tilde{g}(\omega) = \int_{-\infty}^{\infty} dx^0 g(x^0)e^{-i\omega x^0}. \] (2)

II. EQUATION OF MOTION AND ENERGY MOMENTUM TENSOR FOR MASSLESS SPIN-3/2 FIELD FROM RARITA-SCHWINGER LAGRANGIAN

This section will present a brief review of the theory of massless spin-3/2 field in four-dimensional Minkowski spacetime, which provides a concise foundation for later work. For more details, please refer to [19].

Start with Rarita-Schwinger Lagrangian [19, 20]
\[ \mathcal{L} = -i\sqrt{2}[\bar{\psi}^a B' \sigma^b B' B \nabla_b \psi^a B - \frac{1}{3}(\bar{\psi}^a B' \sigma_{aB'} B \nabla_b \psi^B B + \bar{\psi}^a B' \sigma_{B'B} B \nabla_a \psi^B B) + \frac{2}{3} \bar{\psi}^a B' \sigma_{aB'} B B'C \sigma_{C'C} \nabla_b \psi^C C], \] (3)
where the bar denotes the Hermitian conjugation. From here, Euler-Lagrange equation leads to
\[ \sigma^b B' B \nabla_b \psi^a B - \frac{1}{3}(\sigma_{aB'} B \nabla_b \psi^B B + \sigma_{B'B} B \nabla_a \psi^B B) + \frac{2}{3} \sigma_{aB'} B B'C \sigma_{C'C} \nabla_b \psi^C C = 0. \] (4)

With the covariant derivative and the soldering form action on the equation of motion, respectively, we have
\[ \sigma^b B' B \nabla_b \nabla^a \psi^a B = 0, \]
\[ \nabla^a \psi^a B = 0, \] (5)
where the identity \( \sigma_{aC} B' \sigma_{bB'} B' + \sigma_{bC} B' \sigma_{aD} B' = \delta_{ab} \epsilon_{CD} \) has been employed [19]. Taking into account Rarita-Schwinger constraint condition, i.e.,
\[ \sigma^a B' B \psi^a B = 0, \] (6)
the equation of motion is simplified as
\[ \sigma^b B' B \nabla_b \psi^a B = 0. \] (7)

Eqn. (6) and Eqn. (7) are just our familiar Rarita-Schwinger equations for massless spin-3/2 field [19, 20]. Furthermore, by Belinfante’s construction and after a straightforward calculation, the energy momentum tensor for massless spin-3/2 field reads [19]
\[ T^{ab}_B = -i\sqrt{2} [\bar{\psi}^a D' E \sigma^b D' E \nabla^a \psi^D E - \nabla^a \bar{\psi}^b D' E \sigma^a D' E \psi^D E + (\nabla^a \bar{\psi}^b D' E \sigma^a D' E \nabla^a \psi^b D E - \bar{\psi}^C D' \sigma^a D' E \nabla^a \psi^b D E)], \] (8)
which is equivalent with that obtained by the variational principle [21], thus acts as the source of Einstein’s gravitational field equation.

It is worth noting that Rarita-Schwinger field equations are invariant under the following gauge transformation [19, 20]
\[ \psi^a B \rightarrow \psi^a B + \nabla^a \varphi^B \] (9)
with
\[ \sigma^b B' B \nabla_b \varphi^B = 0. \] (10)

However, the energy momentum tensor (8) is not gauge invariant. Thus in the following discussions we will restrict ourselves to Coulomb gauge, i.e.,
\[ \psi^0 B = 0. \] (11)

Obviously, the energy density in Coulomb gauge is given by
\[ \rho = T^{00} = -i\sqrt{2} [\bar{\psi}^a D' E \sigma^0 D' E \nabla^0 \psi^D E - \nabla^0 \bar{\psi}^a D' E \sigma^0 D' E \psi^D E]. \] (12)
III. CANONICAL QUANTIZATION AND QUANTUM INEQUALITY FOR MASSLESS SPIN-3/2 FIELD IN COULOMB GAUGE

To obtain quantum inequalities for massless spin-3/2 field in Coulomb gauge, we need first quantize massless spin-3/2 field. A consistent massless spin-3/2 quantum field can be constructed by the plane wave basis in Coulomb gauge as

\[\hat{\psi}_a^B(x) = \int d^4p [a(p)\psi_{pa}^B(x) + a^+(p)\psi_{-pa}^B(x)], p_0 > 0.\]  

(13)

Here the annihilation and creation operators satisfy the anti-commutation relations as follows

\[\{a(p), a(p')\} = 0,\]
\[\{a(p), a^+(p')\} = \delta^3(p - p'),\]
\[\{a^+(p), a^+(p')\} = 0,\]
\[\{c(p), c(p')\} = 0,\]
\[\{c(p), c^+(p')\} = \delta^3(p - p'),\]
\[\{c^+(p), c^+(p')\} = 0.\]  

(14)

The plane wave solutions to Rarita-Schwinger equations in Coulomb gauge read

\[\psi_{pa}^B(x) = \frac{1}{\sqrt{(2\pi)^3}} \frac{1}{\sqrt{2|p_0|}} \tilde{\psi}_\mu^\Sigma(p)(dx^\nu)\sigma(a\varepsilon\Sigma)^B e^{-ip_0 x^b}.\]  

(15)

where

\[\tilde{\psi}(0, 0, 0, 1) = (0, 1, i, 0) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix},\]  

(16)

and

\[\tilde{\psi}_\mu^\Sigma(p = e^{-\lambda}, e^{-\lambda}\sin \theta \cos \phi, e^{-\lambda}\sin \theta \sin \phi, e^{-\lambda}\cos \theta) = \tilde{\psi}_\mu^\Sigma(-p) = (\Lambda^{-1})^\nu_\mu L_{\Gamma} \tilde{\psi}_\nu^\Gamma (1, 0, 0, 1)\]  

(17)

with

\[\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \phi & 0 & 0 \\ 0 & \sin \phi & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\]

\[L = \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 & 0 & e^{i\frac{\phi}{2}} \\ 0 & \cos \frac{\phi}{2} & -\sin \frac{\phi}{2} & 0 \\ 0 & \sin \frac{\phi}{2} & \cos \frac{\phi}{2} & 0 \end{pmatrix}.\]  

(18)

Next substituting the quantum massless spin-3/2 field \(\psi\) into Eqn. 12 and taking the normal order, the expectation value of the quantum energy density operator in an arbitrary quantum state can be written as

\[\langle \hat{\rho} \rangle = \langle : \hat{T}^{00} : \rangle = \frac{1}{2} \frac{1}{(2\pi)^3} \int \frac{d^3p}{\sqrt{2p_0}} \frac{d^3p'}{\sqrt{2p'_0}} (-\sqrt{2}) \]

\[\langle (p_0 + p'_0) [(a^+(p')a(p))\tilde{\psi}_\mu^\Sigma(p)\sigma^0_\Sigma\tilde{\psi}_\mu^\Sigma(p')e^{i(p_0 - p'_0)x^b} + (c^+(p')c(p))\tilde{\psi}_\mu^\Sigma(-p)\sigma^0_\Sigma\tilde{\psi}_\mu^\Sigma(-p')e^{-i(p_0 - p'_0)x^b} \]

\[+ (p_0 - p'_0) [(a^+(p')c(p))\tilde{\psi}_\mu^\Sigma(p)\sigma^0_\Sigma\tilde{\psi}_\mu^\Sigma(p')e^{i(p_0 + p'_0)x^b} - (c^+(p)c(p))\tilde{\psi}_\mu^\Sigma(-p)\sigma^0_\Sigma\tilde{\psi}_\mu^\Sigma(-p')e^{-i(p_0 + p'_0)x^b}].\]  

(19)
Now consider the sampled energy density measured by an inertial observer at the spatial position \( \mathbf{x} \), i.e.,

\[
\langle \hat{\rho} \rangle_f = \int_{-\infty}^{\infty} dx^0 \langle \hat{\rho}(x^0, \mathbf{x}) \rangle f(x^0)
\]

where \( f \) is a non-negative sampling function. Let \( f = g^2 \) and introduce a family of operators

\[
\hat{O}_\mu^\Sigma(\omega) = \frac{1}{\sqrt{(2\pi)^3}} \int \frac{d^3p}{\sqrt{2p_0}} \hat{g}(-p_0 + \omega) a(\mathbf{p}) \tilde{\psi}_\mu^\Sigma(p) e^{-i\mathbf{p} \cdot \mathbf{x}} + \hat{g}(p_0 + \omega) c(\mathbf{p}) \tilde{\psi}_\mu^\Sigma(-p) e^{i\mathbf{p} \cdot \mathbf{x}}.
\]

Using

\[
\sqrt{2} \tilde{\psi}_\mu^\Sigma(-p) \sigma^0 \Sigma \tilde{\psi}_\mu^\Sigma(\mathbf{p}) = \sqrt{2} \Lambda^0_{\nu} \tilde{\psi}_\mu^\Sigma(1, 0, 0, 1) \sigma^\nu \Sigma \tilde{\psi}_\mu^\Sigma(1, 0, 0, 1) = 2p_0,
\]

and \( \hat{g}(-\omega) = \bar{\hat{g}}(\omega) \), it can be shown that

\[
-\sqrt{2} \tilde{\psi}_\mu^\Sigma(\mathbf{p}) \sigma^0 \Sigma \tilde{\psi}_\mu^\Sigma(\omega) = Z(\omega) + \frac{1}{(2\pi)^3} \int d^3p \bar{\hat{g}}(p_0 + \omega) |\hat{g}(p_0 + \omega)|^2.
\]

Employing the identity\( \ref{eq:identity1}, \ref{eq:identity2} \),

\[
\langle \hat{\rho} \rangle_f = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega [\langle -\sqrt{2} \tilde{\psi}_\mu^\Sigma(\omega) \sigma^0 \Sigma \tilde{\psi}_\mu^\Sigma(\omega) \rangle - Z(\omega)] \omega
\]

we obtain

\[
\langle \hat{\rho} \rangle_f = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega [Z(-\omega) + Z(\omega) + (\sqrt{2} \tilde{\psi}_\mu^\Sigma(\omega) \sigma^0 \Sigma \tilde{\psi}_\mu^\Sigma(\omega)) - Z(\omega) \omega]
\]

\[
= \frac{1}{2\pi} \int_0^{\infty} d\omega [Z(-\omega) + (\sqrt{2} \tilde{\psi}_\mu^\Sigma(\omega) \sigma^0 \Sigma \tilde{\psi}_\mu^\Sigma(\omega)) |\omega]
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega [Z(-\omega) + (\sqrt{2} \tilde{\psi}_\mu^\Sigma(\omega) \sigma^0 \Sigma \tilde{\psi}_\mu^\Sigma(\omega)) |\omega]
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega [Z(-\omega) + (\sqrt{2} \tilde{\psi}_\mu^\Sigma(\omega) \sigma^0 \Sigma \tilde{\psi}_\mu^\Sigma(\omega)) |\omega]
\]
By \{\hat{O}_n^{\Sigma}(\omega) = 0| \Sigma = 1, 2\} in Coulomb gauge, we have
\[
\langle \sqrt{2}\hat{O}_{\mu}^{\Sigma}(\omega)\sigma^{0\Sigma} \Sigma \hat{O}_{\mu}^{\Sigma}(\omega) \rangle = -[\langle \hat{O}_{1}^{1}(\omega)\bar{\hat{O}}_{1}^{1}(\omega) \rangle + \langle \hat{O}_{2}^{1}(\omega)\bar{\hat{O}}_{2}^{1}(\omega) \rangle + \langle \hat{O}_{2}^{2}(\omega)\bar{\hat{O}}_{2}^{2}(\omega) \rangle + \langle \hat{O}_{3}^{1}(\omega)\bar{\hat{O}}_{3}^{1}(\omega) \rangle + \langle \hat{O}_{3}^{2}(\omega)\bar{\hat{O}}_{3}^{2}(\omega) \rangle] \leq 0, \tag{28}
\]
and similarly
\[
\langle -\sqrt{2}\hat{O}_{\mu}^{\Sigma}(\omega)\sigma^{0\Sigma} \Sigma \hat{O}_{\mu}^{\Sigma}(\omega) \rangle \geq 0 \tag{29}
\]
hold for arbitrary quantum states. Therefore
\[
\langle \hat{\rho} \rangle_f \geq \frac{1}{2\pi} \int_{-\infty}^{0} d\omega Z(-\omega)\omega - \int_{0}^{\infty} d\omega Z(\omega)\omega \\
= -\frac{1}{\pi} \int_{0}^{\infty} d\omega Z(\omega)\omega \\
= -\frac{1}{2\pi^3} \int_{0}^{\infty} d\omega \int_{0}^{\infty} dp_0 \int_{0}^{\infty} dp \int_{0}^{\infty} du |\hat{g}(p_0 + \omega)|^2 \\
= -\frac{1}{2\pi^3} \int_{0}^{\infty} du \int_{0}^{\infty} dp_0 \int_{0}^{\infty} dp \int_{0}^{\infty} du |\hat{g}(u)|^2 u^4. \tag{30}
\]

IV. DISCUSSIONS

In summary, based on the technique developed by Fewster and colleagues, we have obtained an explicit formula of quantum inequalities for massless spin-3/2 field by arbitrary non-negative sampling function in Coulomb gauge. It is worth noting that the bound here is weaker, by a factor of 4, than that obtained by taking massless limit of quantum inequalities for massive spin-3/2 field in four-dimensional Minkowski spacetime. This seems to originate from the fact that massive spin-3/2 field has the degrees of freedom with four times as many as massless one. Thus like bosonic fields, the quantum inequalities derived so far for fermionic fields in four-dimensional Minkowski spacetime can also be written in terms of a unified form
\[
\int_{-\infty}^{\infty} dx^0 \langle \hat{\rho}(x^0, x) \rangle g^2(x^0) \geq -\frac{s}{48\pi^3} \int_{m}^{\infty} du |\hat{g}(u)|^2 u^4 Q_3^F \left( \frac{u}{m} \right), \tag{31}
\]
where \(s\) denotes the degrees of freedom for fields, and
\[
Q_3^F(z) = 4(1 - \frac{1}{2z^2})^3 - 3[(1 - \frac{1}{2z^2})^4 \left( 1 - \frac{1}{2z^2} \right) - \frac{1}{2\pi^4} \ln(z + \sqrt{z^2 - 1})], \tag{32}
\]
which is replaced by \(\lim_{z \to \infty} Q_3^F(z) = 1\) in the massless case.

We conclude with an important question. Different from electromagnetic field, there is no gauge invariant energy momentum tensor for massless spin-3/2 field, which is also shared by linear gravitational field indeed. Since we here choose a particular gauge, it is a natural question whether the quantum inequalities obtained here for massless spin-3/2 field is gauge independent. However, the answer is not obvious, thus worthy of further investigation, which is expected to be reported elsewhere.

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[1] L. H. Ford, Phys. Rev. D 43: 3972(1991).
[2] L. H. Ford and T. A. Roman, Phys. Rev. D 55: 2082(1997).
[3] M. J. Pfenning and L. H. Ford, Phys. Rev. D 55: 4813(1997).
[4] E. E. Flanagan, Phys. Rev. D 56: 4922(1997).
[5] M. J. Pfenning and L. H. Ford, Phys. Rev. D 57: 3489(1998).
[6] C. J. Fewster and S. P. Eveson, Phys. Rev. D 58: 084010(1998).
[7] C. J. Fewster and E. Teo, Phys. Rev. D 59: 104016(1999).
[8] C. J. Fewster, Class. Quant. Grav. 17: 1897(2000).
[9] D. N. Vollick, Phys. Rev. D 61: 084022(2000).
[10] C. J. Fewster and R. Verch, Commun. Math. Phys. 225: 331(2002).
[11] M. J. Pfenning, Phys. Rev. D 65: 024009(2002).
[12] E. E. Flanagan, Phys. Rev. D 66: 104007(2002).
[13] C. J. Fewster and B. Mistry, Phys. Rev. D 68: 105010(2003).
[14] C. J. Fewster and M. J. Pfenning, J. Math. Phys. 44: 4480(2003).
[15] H. Yu and P. Wu, Phys. Rev. D 69: 064008(2004).
[16] C. J. Fewster, Phys. Rev. D 70: 127501(2004).
[17] C. J. Fewster and S. Hollands, Rev. Math. Phys. 17: 577(2005).
[18] S. P. Dawson, Class. Quantum Grav. 23: 287(2006).
[19] F. Sun and H. Zhang, [hep-th/0601011]
[20] W. Rarita and J. Schwinger, Phys. Rev. 60: 61(1941).
[21] H. Zhang, Commun. Theor. Phys. 44: 1007(2005).