THERMODYNAMIC FORMALISM FOR TRANSIENT DYNAMICS ON THE REAL LINE

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ABSTRACT. We develop a new thermodynamic formalism to investigate the transient behaviour of maps on the real line which are skew-periodic \( \mathbb{Z} \)-extensions of expanding interval maps. Our main focus lies in the dimensional analysis of the recurrent and transient sets as well as in determining the whole dimension spectrum with respect to \( \alpha \)-escaping sets. Our results provide a one-dimensional model for the phenomenon of a dimension gap occurring for limit sets of Kleinian groups. In particular, we show that a dimension gap occurs if and only if we have non-zero drift and we are able to precisely quantify its width as an application of our new formalism.

1. INTRODUCTION

The main motivation of this article is the connection between transient phenomena of dynamical systems and its manifestation in dimensional quantities. Since transience can impose major obstructions to an ergodic-theoretic description of (fractal-)geometric features its further understanding is vital and has attracted a lot of attention. For instance, it has a strong tradition in complex dynamics with landmark results like the ones obtained for complex quadratic polynomials in [37] or [2]. A closely related and parallelising line of research established corresponding results for families of transcendental functions, see for example [42, 30, 21, 26]. In both cases, a particular striking effect revealing the preponderance of transience is the occurrence of a so-called dimension gap. In fact, the origin of this phenomenon goes back to the rich field of geometric group theory which we explain in more detail further below.

In the framework of thermodynamic formalism, transient effects in topological Markov chains have been seminally studied by Sarig [34, 35]. Directly related to this are fractal-geometric applications of thermodynamic formalism for infinite conformal graph directed Markov systems which have been systematically worked out by Mauldin and Urbanski in [25]. In there, strong mixing conditions were introduced to guarantee that the recurrent behaviour governs the system. One main goal of this paper is to set up a new thermodynamic formalism in the absence of such strong mixing conditions to provide a systematic approach to the geometric phenomenon of a dimension gap. More precisely, we introduce the concept of fibre-induced pressure which allows us to express the occurrence and the width of a dimension gap for skew-periodic \( \mathbb{Z} \)-extensions of expanding interval maps exclusively in terms of this newly developed pressure. Furthermore, we obtain effective analytic relations between the fibre-induced pressure and the classical pressure of the base transformation allowing us to determine the crucial dimensional quantities in a number of examples explicitly.

Key words and phrases. Transient dynamics, skew products, thermodynamic formalism, random walks and multifractals.

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Let us now illustrate the phenomenon of a dimension gap in the setting of actions of non-elementary Kleinian groups $G$ on the hyperbolic space $\mathbb{H}^n$. By a general result of Bishop and Jones \cite{4} we know that the Hausdorff dimension of both the radial limit set $\Lambda_r(G)$ and the uniformly radial limit set $\Lambda_{ur}(G)$ of $G$ are equal to the Poincaré exponent of $G$ given by

$$\delta_G := \inf \left\{ s \geq 0 : \sum_{g \in G} e^{-s d_H(0,g0)} < \infty \right\},$$

where $d_H$ denotes the hyperbolic distance on $\mathbb{H}^n$. Recall that $\Lambda_r(G)$ and $\Lambda_{ur}(G)$ represent recurrent dynamics of the geodesic flow on $\mathbb{H}^n/G$. Clearly, for a normal subgroup $N < G$ we have that $\delta_G \geq \delta_N$ and moreover, $\dim_H(\Lambda_r(G)) = \delta_G > \delta_N = \dim_H(\Lambda_r(N)) \iff G/N$ is non-amenable, with $\dim_H(\cdot)$ the Hausdorff dimension of the corresponding set. This was first proved by Brooks for certain Kleinian groups fulfilling $\delta_G > (n-1)/2$ in \cite{6} and later generalised to a wider class of groups without this restriction by Stadlbauer \cite{39}. Note that by a result of Falk and Stratmann $\delta_N \geq \delta_G/2$, see \cite{10}. If $G$ is additionally geometrically finite, then the strict inequality $\delta_N > \delta_G/2$ holds by a result of Roblin \cite{31} (see also \cite{15}). Furthermore, if $\Lambda(G)$ denotes the limit set of the Kleinian group $G$, then $\delta_G = \dim_H(\Lambda(G))$ and, since $\Lambda(N) = \Lambda(G)$, this implies the following criterion for the occurrence of a dimension gap:

$$\dim_H(\Lambda_r(N)) = \dim_H(\Lambda_{ur}(N)) < \dim_H(\Lambda(N)) \iff G/N \text{ is non-amenable.}$$

In other words, a certain amount of transient behaviour causes a dimension gap from the dimension of the full limit set compared to the restriction of the limit set to certain recurrent parts. It is remarkable that for Kleinian groups the presence of a dimension gap depends only on the group-theoretic property of amenability. Accordingly, a natural example for the occurrence of a dimension gap is given by a Schottky group $G = N \rtimes \mathbb{F}_2$ where $\mathbb{F}_2$ denotes the free group with two generators. Nevertheless, only little is known in the literature concerning the concrete size of this dimension gap.

The occurrence of a dimension gap is closely related to the decay of certain return probabilities. In fact, Kesten \cite{23,24} has shown for symmetric random walks on countable groups that exponential decay of return probabilities is equivalent to non-amenability. However, for amenable groups exponential decay can also be caused by non-symmetric random walks. To be more precise, for groups admitting a recurrent random walk (e.g. $\mathbb{Z}$) it is shown in \cite{16} that exponential decay of return probabilities is equivalent to a lack of certain symmetry condition on the thermodynamic potential related to the random walk (see also Remark \cite{39} for further details).

We are aiming at investigating these closely linked phenomena for a class of maps on $\mathbb{R}$ which can be considered as models of $\mathbb{Z}$-extensions of Kleinian groups. In fact, if the Kleinian group $G = N \rtimes \mathbb{Z}$ is a Schottky group, then the elements in $\Lambda_r(N)$ can be characterised as the limits of $G$-orbits for which the $\mathbb{Z}$-coordinate returns infinitely often to some point in $\mathbb{Z}$ (compare this with our definition of a recurrent set, see Section \cite{1.1.1}). Since $\mathbb{Z}$ is amenable, these limit points have full Hausdorff dimension by Brooks’ amenability criterion. We will see later (end of Section \cite{5}) that this property also follows from the fact that the $\mathbb{Z}$-coordinate has zero drift with respect to a canonical invariant measure obtained from the Patterson-Sullivan construction.
Our models witness drift behaviour and we show that indeed non-zero drift is equivalent to the occurrence of a dimension gap, see Theorem 1.4. It is therefore also very natural to consider subsets of the transient dynamics with fixed drift in more detail. This motivates the definitions of various escaping sets in our one-dimensional models, see Section 1.1.2. The related dimension spectra will allow us to determine the size of the dimension gap explicitly. Similar results will be shown in the forthcoming paper [13] on \( \mathbb{Z} \)-extensions with reflective boundaries allowing us to illuminate earlier results in [40, 7, 14] which studied a family suggested by van Strien modelling induced maps of Fibonacci unimodal maps. Let us point out that drift arguments where also prominent in the proofs of [7].

Our leading motivating example for which we obtain dimensional results on the transient behaviour stems from the family of (a-)symmetric random walks, see Example 1.1. More precisely, let \( F \) be an expanding interval map with finitely many \( C^{1+\varepsilon} \) full branches, say \( F|_{l_i} : l_i \rightarrow [0,1] \) with disjoint intervals \( l_i \subset [0,1] \) with non-empty interior, \( i \in I := \{1, \ldots, m\} \), \( m \geq 2 \). Set \( h_i := H_i^{-1} : [0,1] \rightarrow I \) for the continuous continuation \( H_i \) of \( F|_{l_i} \) to \( I \) and define the corresponding coding map \( \pi : \Sigma := \hat{I}^\mathbb{N} \rightarrow \{0,1\} \) by \( \pi((\omega_1, \omega_2, \ldots)) := x \) for \( \{h(n) \circ \cdots \circ h_0(0,1)\} = \{x\} \). The repeller of \( F \) is then \( \pi(\Sigma) \subset [0,1] \) and we set \( \mathcal{D} := \pi(\Sigma) \setminus \mathcal{D} + \mathbb{Z} \), where we subtract the countable set \( \mathcal{D} \) to avoid technical problems stemming from possible discontinuities at the boundaries of the \( I_i \)'s, see [2, 1] for the precise definition of \( \mathcal{D} \). We assume that the \textit{step length function} \( \Psi : [0,1] \rightarrow \mathbb{Z} \) is constant on each of the intervals \( I_i \) and consider the \( \Psi \)-lift of \( F \) given by

\[
F_\Psi : \mathcal{D} \rightarrow \mathcal{D} : \quad x \mapsto \sum_{k \in \mathbb{Z}} (k + F(x) - k + \Psi(x)) \mathbbm{1}_{[0,1]}(x). 
\]

Since this function is periodic up to a certain skewness induced by \( \Psi \), we refer to this map as a \textit{skew-periodic interval map}. In order to study the various escaping sets of \( F_\Psi \) later it will be necessary to consider \( \mathbb{R} \)-extensions rather than \( \mathbb{Z} \)-extensions. This is the reason why in our abstract set-up we will consider \textit{skew product dynamical systems} defined via

\[
\sigma \times f : \Sigma \times \mathbb{R} \rightarrow \Sigma \times \mathbb{R}, \quad (\sigma \times f)(\sigma, x) := (\sigma(x), x + f(\sigma(x))),
\]

for some function \( f : \Sigma \rightarrow \mathbb{R} \). We want to stress that after neglecting a countable set \( F_\Psi \) is topological conjugate to the \( \mathbb{Z} \)-extension \( \sigma \times (\Psi \circ \pi) : \Sigma \times \mathbb{Z} \rightarrow \Sigma \times \mathbb{Z} \), see Lemma 2.1.

\textbf{Example 1.1 (One-dimensional random walk).} We model a classical one-step random walk via a skew-periodic interval map \( F_\Psi \). For this fix \( c_1, c_2 \in (0,1) \) with \( c_1 + c_2 \leq 1 \) and consider the map

\[
F : x \mapsto \begin{cases} 
 c_1^{-1} x & \text{for } x \in [0, c_1] \\
 c_2^{-1} x & \text{for } x \in (1 - c_2, 1]
\end{cases},
\]

with code space \( \Sigma := \{1,2\}^\mathbb{N} \) and set \( \Psi := -\mathbbm{1}_{(0,c_1]} + \mathbbm{1}_{(1-c_2,1]} \) (see Figure 1.1). In this setting we will also refer to \( F_\Psi : \mathcal{D} \rightarrow \mathcal{D} \) as the \textit{random walk model}. For any \( (p_1, p_2) \)-Bernoulli measure \( \mu \) on \( \Sigma \), we can model the classical (a-)symmetric random walk on \( \mathbb{Z} \) with transition probability \( p_1 \) to go one step left and probability \( p_2 \) one step right. The random walk starting in \( 0 \in \mathbb{Z} \) would then be given by the stochastic process \( \{F_\Psi^n \}_{n \in \mathbb{N}} \) with respect to the probability measure \( \mu \circ \pi^{-1} \) on the repeller \( \pi(\Sigma) \). Note that \( \pi(\Sigma) \) is a Cantor set with Hausdorff dimension \( \delta < 1 \) if and only if \( c_1 + c_2 < 1 \) where \( \delta \) is the unique number \( s \) with \( c_1^s + c_2^s = 1 \). Otherwise, \( \pi(\Sigma) \) is the unit interval and hence, \( \mathcal{D} = \mathbb{R} \setminus (\mathcal{D} + \mathbb{Z}) \).
Crucial for our analysis will be the fibre-induced pressure $P$ and its close connection to the so-called $\alpha$-Poincaré exponent $\delta_\alpha$ defined in the next section. Indeed, the new pressure $P$ is a natural generalisation of the notion of Gurevich pressure (cf. [34] and Remark 3.7) and is necessary to perform our analysis for general escaping rates, see Section 1.1.2. In particular, Gurevich pressure is defined for topological Markov chains whereas our new notion is defined for more general $\mathbb{R}$-extensions. Further, we will see that our new quantity can also be deduced from the classical pressure $\mathcal{P}$. In fact, we will show in Theorem 3.1 below that

$$P(f, \psi) = \inf_{s \in \mathbb{R}} \mathcal{P}(s\psi + f),$$

for $f, \psi : \Sigma \rightarrow \mathbb{R}$ Hölder continuous and $\psi$ satisfying natural conditions fulfilled in our setting. Generalising the concept of the classical Poincaré exponent, the $\alpha$-Poincaré exponent will reveal a natural connection to the analysis of limit sets of Kleinian groups as well as the dimension theory of Birkhoff averages, see the remark after Theorem 1.5 and Remark 4.2.

1.1. Main results. We define the geometric potential $\varphi : \Sigma \rightarrow (-\infty, 0)$ in a Hölder continuous way such that $\varphi(\omega) := -\log |F'(\pi(\omega))|$ except possibly on a finite set. Denote by $\delta > 0$ the unique $s$ such that $\mathcal{P}(s\varphi) = 0$. Here, $\mathcal{P}(f)$ refers to the classical topological pressure defined for any continuous function $f : \Sigma \rightarrow \mathbb{R}$, see end of Section 2 Then by Bowen’s formula we have for the repeller $\pi(\Sigma)$ of $F$ that

$$\delta = \dim_H(\pi(\Sigma)).$$

Further, we can associate to $\Psi$ a symbolic step length function $\psi : \Sigma \rightarrow \mathbb{Z}$ which is constant on one-cylinder sets such that $\psi = \Psi \circ \pi$ except possibly on a finite set. Let us introduce the $\alpha$-Poincaré exponent

$$\delta_\alpha := \inf \left\{ s \geq 0 \mid \sum_{\omega \in F^s} e^{s\psi} < \infty \right\},$$

Figure 1.1. The graph of $F_\psi$ modelling the (a-)symmetric random walk with parameters $c_1 = 0.4$ and $c_2 = 0.6.$
where \( I^* \) denotes the set of all finite words over the alphabet \( I \) and \( K > 0 \) is sufficiently large (the term \( S_m f, \omega \in I^* \) is defined in (2.2) for any \( f : \Sigma \to \mathbb{R} \)). For this exponent we observe
\[
\alpha \in \mathbb{R} \setminus \left[ \underline{\psi}, \overline{\psi} \right] \implies \delta_\alpha = 0
\]
(1.5)
and for \( \alpha \in \left[ \underline{\psi}, \overline{\psi} \right] \) the critical exponent \( \delta_\alpha \) can be expressed as the unique zero of the fibre-induced pressure for suitable potentials and is positive on \( (\underline{\psi}, \overline{\psi}) \) (see Theorem 4.1 as well as (3.1) for the definition of \( \underline{\psi} \) and \( \overline{\psi} \)). We will also see that the map \( \alpha \mapsto \delta_\alpha \) is real-analytic on \( (\underline{\psi}, \overline{\psi}) \) and unimodal but not necessarily concave (cf. Section 4 and Section 1.1.3 for examples).

1.1.1. Recurrent and transient sets and dimension gap. For \( F_\Psi \) we define the recurrent set
\[
\mathcal{R} := \{ x \in \mathcal{S} \mid \exists K \in \mathbb{R} \text{ such that } |F_\Psi^n(x)| \leq K \text{ for infinitely many } n \in \mathbb{N} \}
\]
and the uniform recurrent set
\[
\mathcal{R}_u := \{ x \in \mathcal{S} \mid \exists K \in \mathbb{R} \forall n \in \mathbb{N} \text{ } |F_\Psi^n(x)| \leq K \}. 
\]
The positive (resp. negative) transient set is given by
\[
\mathcal{T}_1^+ := \left\{ x \in \mathcal{S} \mid \lim_{n \to \infty} F_\Psi^n(x) = \pm \infty \right\}.
\]
We also consider the supersets
\[
\mathcal{T}_2^+: \equiv \left\{ x \in \mathcal{S} \mid \limsup_{n \to \infty} +F_\Psi^n(x) < \infty \right\}, \quad \mathcal{T}_3^+: (r) := \{ x \in \mathcal{S} \mid \forall n \geq 0 \pm F_\Psi^n(x) > r \},
\]
for some \( r \in \mathbb{R} \). Observe that
\[
\mathcal{T}_1^+ \subseteq \mathcal{T}_2^+ = \bigcup_{k \in \mathbb{Z}} \mathcal{T}_3^+(k).
\]
(1.6)

The following theorems will be a consequence of our general multifractal decomposition for escaping sets presented in the next subsection. The corresponding proofs can be found in Section 4. The first theorem is the analogue of the result of Bishop and Jones in our setting. Last, recall that for the Hölder continuous function \( \delta \phi : \Sigma \to \mathbb{R} \) there exists a unique Gibbs measure \( \mu_{\delta \phi} \), see end of Section 2.

**Theorem 1.2.** Let \( F_\Psi : \mathcal{S} \to \mathcal{S} \) be the \( \Psi \)-lift of an expanding interval map \( F \). Then we have for the recurrent and uniformly recurrent sets
\[
\dim_H(\mathcal{R}) = \dim_H(\mathcal{R}_u) = \delta_0.
\]

We say the system \( F_\Psi \) has a dimension gap if the Hausdorff dimension \( \delta_0 \) of the (uniformly) recurrent set is strictly less than the Hausdorff dimension \( \delta \) of \( \mathcal{S} \). In fact, this is the case if and only if the system has a drift \( \mu_{\delta \phi}(\psi) \neq 0 \), see Theorem 1.4 below. Furthermore, we are able to provide direct methods to determine \( \delta_0 \), see Theorem 4.1. This allows us to easily calculate \( \delta_0 \) for our examples and to precisely quantify the dimension gap, see for instance Figure 1.3.

**Theorem 1.3.** For the transient sets the following implications hold:
- \( \mu_{\delta \phi}(\psi) \geq 0 \) implies \( \dim_H(\mathcal{T}^-) = \delta_0 \) and \( \dim_H(\mathcal{T}^+) = \delta \),
- \( \mu_{\delta \phi}(\psi) \leq 0 \) implies \( \dim_H(\mathcal{T}^+) = \delta_0 \) and \( \dim_H(\mathcal{T}^-) = \delta \).

where \( \mathcal{T}^\pm \) may be chosen to be one of the sets \( \mathcal{T}_1^+, \mathcal{T}_2^+ \) or \( \mathcal{T}_3^+(r) \) for any \( r \in \mathbb{R} \).
1.1.2. Escaping sets. For $\alpha \in \mathbb{R}$ let us now define the $\alpha$-escaping set for $F_\Psi$ by

$$E(\alpha) := \{ x \in \mathbb{R} \mid \exists K > 0 \mid |F^n_\Psi(x) - n\alpha| \leq K \text{ for infinitely many } n \in \mathbb{N} \},$$

and the uniformly $\alpha$-escaping set for $F_\Psi$ by

$$E_u(\alpha) := \{ x \in \mathbb{R} \mid \exists K > 0 \forall n \in \mathbb{N} \mid |F^n_\Psi(x) - n\alpha| \leq K \}.$$

As stated above, the following result proves our statement on the occurrence of a dimension gap.

**Theorem 1.4.** We have $\delta_\alpha = \delta$ if and only if $\mu_{\delta \phi}(\psi) = \alpha$. In particular, a dimension gap occurs if and only if $\mu_{\delta \phi}(\psi) \neq 0$.

**Theorem 1.5** (Multifractal decomposition with respect to $\alpha$-escaping). For $\alpha \in \mathbb{R}$ we have

$$\dim_H(E(\alpha)) = \dim_H(E_u(\alpha)) = \delta_\alpha.$$

We note that $\delta_\alpha$ also allows for a multifractal spectral interpretation of certain Birkhoff averages (cf. Remark 4.2). However, the results of Theorem 1.5 need more subtle ideas adopted from the analysis of Kleinian groups and we believe that this connection is also of independent interest.

1.1.3. First examples and further consequences. The previous theorems applied to Example 1.1 with fixed $c_1, c_2 \in (0, 1)$ lead to the following observations. We determine the graph of $\alpha \mapsto \dim_H(E(\alpha)) = \dim_H(E_u(\alpha)) = \delta_\alpha(c_1, c_2)$ for the random walk model, see Figure 1.2.

![Figure 1.2](image-url)

**Figure 1.2.** The escape rate spectrum $\alpha \mapsto \delta_\alpha(c_1, c_2)$ for the random walk model for different values of $c_1 = 0.5$ (solid line, symmetric case), $c_1 = 0.3$ (dashed line), and $c_1 = 0.9$ (dotted line) and $c_2 = 1 - c_1$.

Moreover, the Hausdorff dimension of the (uniformly) recurrent set for the random walk model given by Theorem 1.2 is

$$\delta_0(c_1, c_2) = \frac{\log 4}{\log(1/c_1) + \log(1/c_2)},$$

see Figure 1.3 for a one-parameter plot of $c \mapsto \delta_0(c, 1 - c)$.

As an extension of Example 1.1 we will consider asymmetric step widths and interval maps with more than two branches. The corresponding calculations and precise formulas are postponed to Section 5.
We end the introduction by providing an application of our results to the above mentioned $\mathbb{Z}$-extensions of Kleinian groups. More precisely, we partially recover a result of [29] and provide a direct alternative proof, see end of Section 5. Recall that a group $G$ is of divergence type if the series in (1.1) is infinite for the critical exponent $\delta_G$. 

Corollary 1.6. If $G = N \rtimes \mathbb{Z}$ is a Schottky group, then $\delta_N = \delta_G$ and $N$ is of divergence type.

2. Preliminaries and basic definitions

Let $I$ be a finite set, $I^* := \bigcup_{k=1}^{\infty} I^k \cup \{\emptyset\}$ the set of all finite words over $I$ containing the empty word $\emptyset$ and set $\Sigma := I^\mathbb{N}$. For $\omega \in I^*$ we denote by $|\omega|$ the unique $k \in \mathbb{N}$ such that $\omega \in I_k$, and we refer to $|\omega|$ as the length of $\omega$. Note that $\emptyset$ is the unique word of length zero. For $\omega = (\omega_1, \ldots, \omega_n) \in I^n$ and $1 \leq k \leq n$, or $\omega \in \Sigma$ and $k \in \mathbb{N}$ resp., we set $\omega|_k := (\omega_1, \ldots, \omega_k)$. For $\omega = (\omega_1, \ldots, \omega_n) \in I^n$ and $\nu = (\nu_1, \ldots, \nu_k) \in I^k$, or $\nu \in \Sigma$ resp., we define the concatenation $\omega \nu := (\omega_1, \ldots, \omega_n, \nu_1, \ldots, \nu_k)$, or $\omega \nu := (\omega_1, \ldots, \omega_n, \nu_1, \nu_2, \ldots)$.

We denote by $\sigma : \Sigma \rightarrow \Sigma$ the (left) shift map given by $\sigma(\omega)_i := \omega_{i+1}$ for every $i \in \mathbb{N}$. We endow $\Sigma$ with the metric $d(\omega, \tau) := \exp(-\max \{k \geq 0 \mid \omega_k = \tau_k\})$. In this way we obtain $(\Sigma, \sigma)$ a continuous dynamical system over the compact metric space $(\Sigma, d)$.

Recall the definition of a skew-periodic interval map $F_{\Psi}$ with $\Psi : [0, 1] \rightarrow \mathbb{Z}$ and of a skew product dynamical system $(\Sigma \times \mathbb{R}, \sigma \times f)$ with $f : \Sigma \rightarrow \mathbb{R}$, see (1.2). Let us now give the precise definition of the auxiliary countable set, where we set $h_{\emptyset} := id_{[0,1]}$.

\[
\mathcal{D} := \bigcup_{(\omega_1, \ldots, \omega_n) \in I^*} h_{\emptyset} \circ \cdots \circ h_{\emptyset} (\{0,1\}).
\]

(2.1)

Note that $\pi^{-1}(\mathcal{D})$ is the set of all sequences in $\Sigma$ which are eventually constant.

Lemma 2.1. $F_{\Psi}$ and $\sigma \times (\Psi \circ \pi)|_{\Sigma \times \pi^{-1}(\mathcal{D}) \times \mathbb{Z}}$ are topological conjugate.
Proof. Let us define the map \( \tilde{\pi} : \Sigma \times \mathbb{Z} \to \mathbb{R} \) by \( \tilde{\pi}(\omega, \ell) := \pi(\omega) + \ell \) and restrict its domain to \( \tilde{\pi}^{-1}(\mathcal{D}) \) (which equals \( \Sigma \setminus \pi^{-1}(\mathcal{D}) \times \mathbb{Z} \)). Then for \((\omega, \ell) \in \tilde{\pi}^{-1}(\mathcal{D})\) we have

\[
F_{\Psi}(\tilde{\pi}(\omega, \ell)) = \sum_{k \in \mathbb{Z}} (k + F(\pi(\omega) + \ell - k) + \Psi(\pi(\omega) + \ell - k)) \mathbb{I}_{[0,1]}(\pi(\omega) + \ell - k).
\]

Since \( F \circ \pi = \pi \circ \sigma \) on \( \Sigma \setminus \pi^{-1}(\mathcal{D}) \), we have

\[
F_{\Psi}(\tilde{\pi}(\omega, \ell)) = F(\pi(\omega)) + \ell + \Psi(\pi(\omega)) = \tilde{\pi}(\sigma \times (\Psi \circ \pi)(\omega, \ell)).
\]

\[\square\]

For \( n \in \mathbb{N} \) and \( \omega \in I^\ast \), we set

\[
S_n f := \sum_{k=0}^{n-1} f \circ \sigma^k, \quad S_0 f := 0 \quad \text{as well as} \quad S_{\omega,f} := \sup_{x \in [\omega]} S_{[\omega],f}(x),
\]

where \([\omega] := \{ x \in \Sigma : x|_{[\omega]} = \omega \}\) denotes the cylinder set in \( \Sigma \) over \( \omega \). We say \( f : \Sigma \to \mathbb{R} \) is H"older continuous if there exists \( \alpha > 0 \) such that

\[
\sup_{\omega, \tau \in I^\ast} |f(\omega) - f(\tau)| / d(\omega, \tau)^\alpha < \infty.
\]

Note that by the H"older continuity there exists a constant \( D_f \) such that for all \( n \in \mathbb{N} \) and \( \omega \in I^n \) we have

\[
\sup_{x,y \in [\omega]} |S_n f(x) - S_n f(y)| \leq D_f.
\]

We recall an important result from \([1]\) adopted to our situation.

**Lemma 2.2.** Suppose \( \mu \) is a \( \sigma \)-invariant ergodic Borel probability measure and \( f : \Sigma \to \mathbb{R} \) is \( \mu \)-integrable. We have

\[
\mu(f) = 0 \quad \iff \quad \liminf_{\ell \to \infty} |S_{\ell,f}| = 0 \quad \mu\text{-a.e.}
\]

**Proof.** If \( \mu \) is non-atomic, we apply the result from \([1]\). For this we consider the natural extension of \((\Sigma, \sigma)\) which is given by the two-sided shift \( I^Z \) making \( \sigma : I^Z \to I^Z \) a bi-measurable map with respect to the unique \( \sigma \)-invariant measure \( \tilde{\mu} \) on \( I^Z \) such that \( \mu = \tilde{\mu} \circ h^{-1} \). Here, \( h \) denotes the canonical projection from \( I^Z \) to \( \Sigma \). Then we can transfer the corresponding result from \([1]\) for \( \tilde{f} := f \circ h \) back to our situation.

If \( \mu \) has atoms, then by \( \sigma \)-invariance and ergodicity we in fact have \( \mu = n^{-1} \sum_{k=0}^{n-1} \delta_{\sigma^k x} \), where \( x \in \Sigma \) is some periodic point with period \( n \). With this at hand the equivalence is obvious. \(\square\)

The following fact puts the previous lemma into the context of infinite ergodic theory.

**Fact ([56] Theorem 5.5).** Let \( \mu \) be a \( \sigma \)-invariant Borel probability measure and assume that \( f : \Sigma \to \mathbb{R} \) is \( \mu \)-integrable. Then the \((\sigma \times f)\)-invariant measure \( \mu \times \lambda \) (with \( \lambda \) the Lebesgue measure on \( \mathbb{R} \)) is conservative if and only if \( \mu \)-a.e. \( \liminf_{\ell \to \infty} |S_{\ell,f}| = 0 \).

Next, let us recall the classical thermodynamical formalism for full shifts. On the compact subspace \( J^\mathbb{N} \subset \Sigma \) the classical topological pressure \( \mathcal{P}(f,J) \) of the continuous potential \( f : \Sigma \to \mathbb{R} \) is given by

\[
\mathcal{P}(f,J) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in J^n} \exp(S_{\omega,f}) \quad \text{and} \quad \mathcal{P}(f) := \mathcal{P}(f,I),
\]

in here the limit always exists (see for example \([43]\)). If \( f \) is Hölder continuous and if \( J \subset I \) is not a singleton, then there exists a unique \( \sigma \)-invariant Borel probability measure \( \mu_{f,J} \) on \( J^\mathbb{N} \) called the Gibbs
measure (for \( f \) restricted to \( J^n \)) – fulfilling the Gibbs property, that is, there exists \( c \geq 1 \) such that for all \( n \in \mathbb{N} \), \( \omega \in J^n \) and \( x \in [\omega] \cap J^n \) we have

\[
    c^{-1} \leq \frac{\mu_{f,J}(\{\omega\})}{\exp(S_n f(x) - n \mathbb{P}(f,J))} \leq c.
\]  

(2.4)

If \( J = \{i\} \), then \( \mu_{f,J} \) denotes the Dirac measure \( \delta_i \) on the constant sequence \( x = (i, i, \ldots) \). Moreover, for \( J = I \) we set \( \mu_f := \mu_{f,I} \).

3. Fibre-induced Pressure and recurrence properties

In this section we will always assume that \( \psi, f : \Sigma \rightarrow \mathbb{R} \) are Hölder continuous functions and let

\[
    \psi := \inf_{\omega \in \Sigma} \liminf_{n \rightarrow \infty} S_n \psi(\omega) / n \quad \text{and} \quad \bar{\psi} := \sup_{\omega \in \Sigma} \limsup_{n \rightarrow \infty} S_n \psi(\omega) / n. \tag{3.1}
\]

This gives rise to a skew product dynamical system \( \sigma \times \psi : \Sigma \times \mathbb{R} \rightarrow \Sigma \times \mathbb{R} \) and we aim at defining the new notion of fibre-induced pressure for functions \( f \circ \pi_1 : \Sigma \times \mathbb{R} \rightarrow \mathbb{R} \) depending only on the first coordinate where \( \pi_1 : \Sigma \times \mathbb{R} \rightarrow \Sigma \). Recall that \( D_\psi \) denotes the energy of \( \psi \) defined in 2.3.

For \( K > 0 \) and \( n \in \mathbb{N} \) we define

\[
    \mathcal{C}_n(K) := \mathcal{C}_n(\psi, K) := \{ \omega \in I^n \mid |S_n \psi| \leq K \}, \quad \mathcal{C}(K) := \bigcup_{n \in \mathbb{N}} \mathcal{C}_n(K),
\]

\[
    \zeta_n(\psi, K) := \sum_{\omega \in \mathcal{C}_n(K)} e^{S_n f}
\]

as well as

\[
    \mathcal{P}(f, \psi, K) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log \zeta_n(f, \psi, K) \quad \text{and} \quad \mathcal{P}(f, \psi) := \lim_{K \rightarrow \infty} \mathcal{P}(f, \psi, K).
\]

We will call \( \mathcal{P}(f, \psi) \) the fibre-induced pressure of \( f \) with respect to \( \psi \).

**Theorem 3.1** (Fibre-induced vs. classical pressure). Let \( \psi, f : \Sigma \rightarrow \mathbb{R} \) be Hölder continuous functions and fix \( K > D_\psi \). If \( 0 \in (\underline{\psi}, \overline{\psi}) \), then there exists a unique number \( t(f) \in \mathbb{R} \) with \( f \psi d\mu_t(f)\psi + f = 0 \). For this number we have

\[
    \mathcal{P}(f, \psi, K) = \mathcal{P}(t(f) \psi + f) = \min_{s \in \mathbb{R}} \mathcal{P}(s \psi + f). \tag{3.2}
\]

Further, if we assume that \( \psi \) is constant on one-cylinder sets, allowing also for \( 0 \notin (\underline{\psi}, \overline{\psi}) \), we have

\[
    \mathcal{P}(f, \psi, K) = \inf_{s \in \mathbb{R}} \mathcal{P}(s \psi + f)
\]

and the value of the fibre-induced pressure will be finite if and only if \( 0 \in [\underline{\psi}, \overline{\psi}] \). For \( 0 \in (\underline{\psi}, \overline{\psi}) \) and setting \( I_0 := \{ i \in I : \psi(i, \ldots) = 0 \} \) we have in this situation

\[
    \mathcal{P}(f, \psi, K) = \mathcal{P}(f, I_0).
\]

In any case if \( 0 \in (\underline{\psi}, \overline{\psi}) \) or \( \psi \) is constant on one-cylinder sets and \( 0 \in [\underline{\psi}, \overline{\psi}] \), then we have

\[
    \sum_{\omega \in \mathcal{C}(K)} e^{S_n f - |\omega| \mathcal{P}(f, \psi, K)} = \infty.
\]

**Proof.** We first consider the case \( 0 \in (\underline{\psi}, \overline{\psi}) \). It is well known that the function \( H : t \mapsto \mathcal{P}(t \psi + f) \) is real-analytic and convex (133). By our assumption on \( \psi \), we have that \( \lim_{t \rightarrow \pm \infty} H(t) = \infty \). It follows
that $H$ is not affine and hence, $H$ is strictly convex. We conclude that there is a unique $r(f) \in \mathbb{R}$ such that $H'(t(f)) = \int \psi d\mu_{r(\psi)} = 0$ and $H(t(f))$ must be the unique minimum of $H$. We have $\mu_{r(\psi)} + f$-a.e. that $\liminf_{t \to \infty} |S_t \psi| = 0$, by Lemma 2.2, and as a consequence of the Borel-Cantelli Lemma we have $\sum_{t \geq 1} \mu_{r(\psi) + f} \{ |S_t \psi| \leq \epsilon \} = \infty$ for every $\epsilon > 0$. This fact combined with the Gibbs property (2.4) implies for $\epsilon := K - D_{\psi} > 0$ and $c \geq 1$ from (2.4),

$$\infty = \sum_{t \geq 1} \mu_{r(\psi) + f} \{ |S_t \psi| \leq \epsilon \} \leq c \sum_{\omega \in \mathcal{F}(K)} e^{S_\omega (\psi + f)} - |\omega| \frac{P\psi}{P\psi + f}$$

$$\leq c e^{H(f)R} \sum_{\omega \in \mathcal{F}(K)} e^{S_\omega f - |\omega| P\psi}. $$

From this it is easy to see that $\mathcal{P}(f, \psi, K) = \mathcal{P}(\psi + f)$, hence, for the boundary cases $0 \in \{ \psi, \psi \}$, we are left to verify that $\mathcal{P}(f, \psi, K) \geq \inf_{s \in \mathbb{R}} \mathcal{P}(s\psi + f)$. Let us only consider the case $\psi = 0$ and $\psi$ is constant on one-cylinders. Then $\inf_{s \in \mathbb{R}} \mathcal{P}(s\psi + f) = \lim_{s \to +\infty} \mathcal{P}(s\psi + f)$. Let us now consider the ergodic invariant Gibbs measure, respectively atomic measure, $\mu_{f, I_0}$. Since $\mu_{f, I_0}(\psi) = 0$ and by applying Lemma 2.2, we obtain for $c \geq 1$ from (2.4),

$$\infty = \sum_{t \geq 1} \mu_{f, I_0} \{ |S_t \psi| \leq K \} = \sum_{t \geq 1} \sum_{\omega \in \mathcal{F}(K) \setminus I_0} \mu_{f, I_0} \{ \omega \}$$

$$\leq c \sum_{t \geq 1} \sum_{\omega \in \mathcal{F}(K) \setminus I_0} e^{S_\omega f - |\omega| \mathcal{P}(f, I_0)}$$

$$\leq c \sum_{t \geq 1} \sum_{\omega \in \mathcal{F}(K)} e^{S_\omega f - |\omega| \mathcal{P}(\psi, I_0)}.$$  

From this it is easy to see that $\mathcal{P}(f, \psi, K) - \mathcal{P}(f, I_0) \geq 0$. To see that $\inf_{s \in \mathbb{R}} \mathcal{P}(s\psi + f) = \mathcal{P}(f, I_0)$ we first note that obviously $\mathcal{P}(s\psi + f) \geq \mathcal{P}(f, I_0)$ for any $s \in \mathbb{R}$. By the sub-additivity of the classical pressure, we have for any $n \in \mathbb{N}$ and $s > 0$ that

$$\mathcal{P}(s\psi + f) \leq \frac{1}{n} \log \sum_{\omega \in \mathcal{F}} e^{S_\omega (s\psi + f)} \to \frac{1}{n} \log \sum_{\omega \in I_0} e^{S_\omega f} \text{ for } s \to \infty.$$  

Hence,

$$\inf_{s \in \mathbb{R}} \mathcal{P}(s\psi + f) \leq \frac{1}{n} \log \sum_{\omega \in I_0} e^{S_\omega f} \to \mathcal{P}(f, I_0) \text{ for } n \to \infty.$$  

In particular, we have $\mathcal{P}(f, \psi, K) > -\infty$.

Finally, if $0 \notin [\psi, \psi]$, then $\inf_{s \in \mathbb{R}} \mathcal{P}(s\psi + f) = -\infty$ and by the obvious estimate $\mathcal{P}(f, \psi, K) = -\infty = \inf_{s \in \mathbb{R}} \mathcal{P}(s\psi + f)$. \qed

**Corollary 3.2.** If $0 \in (\psi, \psi)$ or if $\psi$ is constant on one-cylinder sets, then the fibre-induced pressure is given by

$$\mathcal{P}(f, \psi) = \mathcal{P}(f, \psi, K),$$

for $K > D_{\psi}$.

We can now characterise when the fibre-induced pressure and classical pressure coincide.
Corollary 3.3. Let \( f : \Sigma \to \mathbb{R} \) and \( \psi : \Sigma \to \mathbb{R} \) be Hölder continuous functions. Then we have
\[
\mathcal{P}(f, \psi) = \mathcal{P}(f) \quad \text{if and only if} \quad \mu_f(\psi) = 0.
\]

Proof. First, assume \( \mu_f(\psi) = 0 \). With \( H \) defined as in the proof of Theorem 3.1, we have \( H(0) = 0 \).
We distinguish two cases. If \( H \) is strictly convex, then \( 0 \in (\psi, \bar{\psi}) \) and we have \( t(f) = 0 \). Thus, by
Theorem 3.1, \( \mathcal{P}(f, \psi) = \mathcal{P}(f) \). If \( H \) is affine, then \( \psi \) is cohomologous to zero. Hence, also in this case,
we have \( \mathcal{P}(f, \psi) = \mathcal{P}(f) \).
Now, suppose \( \mu_f(\psi) \neq 0 \). Again by the proof of Theorem 3.1, \( \mathcal{P}(f, \psi) \leq \inf_{s \in \mathbb{R}} \mathcal{P}(s\psi + f) \).
Since \( \frac{\partial}{\partial t} \mathcal{P}(t\psi + f)|_{t=0} = \mu_f(\psi) \neq 0 \), we conclude \( \mathcal{P}(f, \psi) < \mathcal{P}(f) \). \( \square \)

For later use we need the following auxiliary statement.

Lemma 3.4. If \( f < 0 \) and \( 0 \in [\psi, \bar{\psi}] \), then \( s \mapsto \mathcal{P}(sf, \psi) \) is finite, continuous, strictly decreasing and we have
\[
\lim_{s \to \pm\infty} \mathcal{P}(sf, \psi) = \mp\infty.
\]

Proof. By definition, we have for \( K > D_\psi \) and \( s < t \),
\[
\mathcal{P}(tf, \psi, K) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{a \in \mathcal{E}_n(K)} e^{(t-s)+s} \mathcal{P}(sf, \psi, K).
\]
The claim follows because \( \mathcal{P}(sf, \psi, K) \in \mathbb{R}, s \in \mathbb{R} \), which follows from the arguments given in the proof
of Theorem 3.1. \( \square \)

Remark 3.5. For \( f, \psi \) Hölder continuous functions, due to the correspondence between the fibre-induced and the classical pressure as stated in Theorem 3.1, we obtain that \( (s,a) \mapsto \mathcal{P}(sf, \psi - a) \) is real-analytic with respect to \( s \in \mathbb{R} \) and \( a \in (\psi, \bar{\psi}) \).

We end this section introducing the new notion of \( \psi \)-recurrent potentials. As already mentioned in the introduction this clarifies the connection of our setting to the ideas developed in [34] and [16].

Let \( f, \psi : \Sigma \to \mathbb{R} \) be Hölder continuous. We say that \( f \) is a \( \psi \)-recurrent potential if for all \( K > D_\psi \)
\[
\sum_{a \in \mathcal{E}(K)} e^{S_\psi f - |a| \mathcal{P}(f, \psi)} = \infty.
\]
The following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.6. Let \( \psi : \Sigma \to \mathbb{R} \) be a Hölder continuous function with \( \underline{\psi} < 0 < \bar{\psi} \) or \( \psi \) is constant on
one-cylinders. Then any Hölder continuous \( f : \Sigma \to \mathbb{R} \) is a \( \psi \)-recurrent potential.

Remark 3.7. We would like to point out that our definition of recurrent potentials is analogous to the corresponding definition for topological Markov chains ([34]). For this suppose that \( \psi : \Sigma \to \mathbb{R} \) is constant on 1-cylinders such that \( \underline{\psi} = \min \psi < 0 < \max \psi = \bar{\psi} \). Then we have \( D \psi = 0 \) and \( \mathcal{P}(f, \psi) = \mathcal{P}(f, \psi, K) \) for every \( K > 0 \). Moreover, \( G := \{ S_\psi \omega | \omega \in \Sigma \} \) is a countable semi-group and if \( G \) is a discrete subset of \( \mathbb{R} \), by Lemma 3.8 below, \( G \) is a countable group. Further, \( \sigma \times \psi : \Sigma \times G \to \Sigma \times G \) is a transitive topological Markov chain with alphabet \( I \times G \). Now, Sarig’s definition in [34] of the Gurevich pressure of \( f \circ \pi_1 \) with respect to this Markov chain coincides with our fibre-induced pressure and Sarig’s definition of a recurrent potential also coincides with ours (see also [9, 8, 38]). For the closely related concept of induced pressure for Markov shifts we refer to [19].
Lemma 3.8. Assume that $\psi : \Sigma \to \mathbb{R}$ is constant on one-cylinder sets and $0 \in (\psi, \psi)$. Then for any $N > 0$ and for every $\varepsilon > 0$ there is a finite set $\Lambda \subset I^*$ such that for any $\omega \in \mathcal{C}(N)$ there exists $\nu \in \Lambda$ such that $\omega \nu \in \mathcal{C}(\varepsilon)$.

Proof. If $0 \in (\psi, \psi)$, the map $t \mapsto \mathcal{P}(t \psi)$ is real-analytic, strictly convex and we have $\lim_{t \to \pm \infty} \mathcal{P}(t \psi) = \infty$. Hence, it has a unique minimum in $t_0 \in \mathbb{R}$. Then for the unique Gibbs measure $\mu_{t_0, \psi}$ with respect to the potential $t_0 \psi$ we get

$$\mu_{t_0, \psi}(\psi) = \frac{d}{dt} \mathcal{P}(t \psi) \mid_{t = t_0} = 0$$

and this Gibbs measure is also non-atomic and ergodic with respect to $\sigma$. By Lemma 2.2 we obtain

$$\lim_{t \to \infty} |S_t \psi| = 0 \quad \mu_{t_0, \psi} \text{-a.e.} \quad (3.3)$$

For $k = 1, \ldots, m := \lceil 2N/\varepsilon \rceil$ we set

$$I_k := [(k-1)\varepsilon/2, k\varepsilon/2) \text{ and } I_{-k} := [-k\varepsilon/2, (-k+1)\varepsilon/2).$$

We can ignore all $k \in \{-1, \ldots, m\}$ such that $S_\omega \psi \notin I_k$ for all $\omega \in I^*$. For the remaining $k$ we find $\omega_k \in I^*$ such that $S_\omega \psi \in I_k$. Since every non-empty cylinder $[\omega]$ with $\omega \in I^*$ carries positive $\mu_{t_0, \psi}$-measure, using (3.3) we find $x \in [\omega_k]$ and $n \geq |\omega_k|$ such that $|S_n \psi(x)| < \varepsilon/2$. With $v_k := (x_{\omega_k+1}, \ldots, x_n)$ we have $|S_{\omega_k v_k} \psi| < \varepsilon/2$ applying (2.3). Hence using (2.3) once more, for an arbitrary $\omega \in I^*$ with $S_{\omega} \psi \in I_k$, we have for all $x \in [\omega v_k]$

$$|S_{[\omega v_k]} \psi(x)| = |S_{[\omega v_k]} \psi(x) - S_{[\omega v_k]} \psi(\omega_k \sigma^{[\omega]}(x))| + |S_{[\omega v_k]} \psi(\omega_k \sigma^{[\omega]}(x))| + |S_{[\omega v_k]} \psi(\omega_k \sigma^{[\omega]}(x))|$$

$$\leq |S_{[\omega v_k]} \psi(x) - S_{[\omega v_k]} \psi(\omega_k \sigma^{[\omega]}(x))| + |S_{[\omega v_k]} \psi(\omega_k \sigma^{[\omega]}(x))| + |S_{[\omega v_k]} \psi(\omega_k \sigma^{[\omega]}(x))| + \varepsilon/2$$

$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

proving the desired statement. \qed

Remark 3.9. For a subgroup $G \subset \mathbb{Z}$ and $\psi : \Sigma \to G$ it is shown in [16] that $f \circ \pi_1$ is recurrent (according to Sarig, see [34]) if and only if there is a character $c : G \to (0, \infty)$ such that $\mu_{f - \log c \circ \psi} \times \lambda_G$ is the conservative equilibrium measure of $f \circ \pi_1$, where $\lambda_G$ denotes the counting measure on $G$ (for the definition of an equilibrium measure in this setting, see [35]). If $\psi$ is constant on one-cylinders and $0 \in (\psi, \psi)$, then $f - \log c \circ \psi$ is of the form $f + \ell(f)\psi$ as stated in Theorem 3.1. Further, we can derive (3.2) from [16, Theorem 1(1)] where we note again that the fibre-induced and Gurevich pressure coincide in this setting, see the previous remark. Moreover, by [16, Proposition 1.4] we have that $\mathcal{P}(f, \psi) = \mathcal{P}(f)$ if and only if $f$ is symmetric on average, i.e.

$$\sup_{m \in \mathbb{Z}} \limsup_{n \to \infty} \frac{\sum_{|\omega| \leq n, S_\omega \psi = m} e^{S_\omega f - |\omega| \mathcal{P}(f, \psi)}}{\sum_{|\omega| \leq n, S_\omega \psi = -m} e^{S_\omega f - |\omega| \mathcal{P}(f, \psi)}} < \infty.$$
4. Multifractal decomposition with respect to $\alpha$-escaping sets

In this section $\psi: \Sigma \to \mathbb{Z}$ denotes the symbolic step length function which is constant on one-cylinder sets and such that $\psi = \Psi \circ \pi$ except possibly on a finite set. Note that under these assumption on $\psi$ we have $D_\psi = 0$. For a given drift parameter $\alpha \in [\psi, \overline{\psi}]$ let us set

$$\psi_\alpha := \psi - \alpha.$$

Note that we have

$$E(\alpha) \cap [0, 1] = \pi \{ \omega \in \Sigma \mid \exists K > 0 \ |S_n \psi_\alpha(\omega)| \leq K \text{ for infinitely many } n \in \mathbb{N} \} \setminus \mathcal{D}$$

and

$$E_a(\alpha) \cap [0, 1] = \pi \{ \omega \in \Sigma \mid \exists K > 0 \forall n \in \mathbb{N} \ |S_n \psi_\alpha(\omega)| \leq K \} \cup \mathcal{D},$$

for the countable set $\mathcal{D}$, see (2.1). Recall the definition of the $\alpha$-Poincaré exponent $\delta_\alpha$ with $K > D_\psi$, see (1.5). Next we give the proof of the implication (1.5) stated in the introduction.

**Proof of implication (1.5).** For $\alpha \in \mathbb{R} \setminus [\psi, \overline{\psi}]$ we have that $S_n \psi_\alpha$ diverges uniformly to either $+\infty$ or $-\infty$ and hence, being a finite sum, $\sum_{\omega \in \mathcal{T}} |S_n(\psi - \alpha)| \leq K e^{\psi_\alpha}$ is finite for all $s \geq 0$. \hfill $\square$

The following assertion will be crucial for determining the multifractal decomposition with respect to our escaping sets, and to derive explicit formulas for dimension gaps.

**Theorem 4.1.** We have the following three different characterisations of the $\alpha$-Poincaré exponent $\delta_\alpha$:

(i) For $\alpha \in \psi, \overline{\psi}$ we have that $\delta_\alpha$ is uniquely determined by

$$\mathcal{P}(\delta_\alpha \varphi, \psi_\alpha) = 0.$$

(ii) For $\alpha \in (\psi, \overline{\psi})$ we have that $\delta_\alpha$ is determined by the unique solution $(\delta_\alpha, q_\alpha) \in \mathbb{R}^2$ of the equations

$$\Psi(\delta_\alpha \varphi + q_\alpha \psi_\alpha) = 0 \quad \text{and} \quad \frac{\partial}{\partial q} \Psi(\delta_\alpha \varphi + q \psi_\alpha)|_{q = q_\alpha} = 0.$$

In particular, we have

$$\int \psi \, d\mu_{\delta_\alpha \varphi + q_\alpha \psi_\alpha} = \alpha.$$

(iii) Let $s : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the implicitly defined function given by $\Psi(s(q, a) \varphi + q \psi + a1) = 0$. Then $s$ is convex and for the Legendre transform of $s$ defined by

$$\hat{s}(\alpha_1, \alpha_2) := \sup_{(q, a) \in \mathbb{R}^2} q \alpha_1 + a \alpha_2 - s(q, a)$$

we have for $\alpha \in (\psi, \overline{\psi})$

$$\delta_\alpha = - \inf_{r \in \mathbb{R}} \hat{s}(\alpha r, r) = \inf_{r \in \mathbb{R}} s(r, -\alpha r).$$

In particular, we have that on the line $g$ through the origin with slope $-\alpha$ in the $(q, a)$-plane there exists exactly one point $(r, s(\alpha r))$ such that the gradient of $s$ in this point is perpendicular to $g$. The height of $s$ in $(r, -\alpha r)$ is $\delta_\alpha$ and the plane which is tangential to the graph of $s$ in $(r, -\alpha r, s(r, -\alpha r))$ intersects the $s$-axis in $-\delta_\alpha$ (cf. Figure 4.1).
Figure 4.1. The contour plot for the random walk model with $c_1 = 0.1$, $c_2 = 0.5$ of $s(q,a)$ which is implicitly defined by $\mathcal{P}(s(q,a)\varphi + q\psi + a1) = 0$. The fourth contour line is determined by $s(q,a(q)) = 1$, i.e. corresponds to height $\delta$ (defined in (1.3)). The line $g$ goes through the origin with slope $-\alpha$.

Proof. (i) To prove that the $\alpha$-Poincaré exponent coincides with the zero of the fibre-induced pressure first note that for all $\alpha \in [\varphi,\overline{\varphi}]$ we have that there is a unique zero $\tilde{\delta}_\alpha \in \mathbb{R}$ of the function $s \mapsto \mathcal{P}(s\varphi,\psi_\alpha)$, by Lemma 3.4. By definition, we have $
abla \varphi (q) \geq \sum_{\omega \in \mathcal{E}_n(\psi_\alpha,K)} e^{S_\omega \varphi} = \sum_{n \in \mathbb{N}} \exp \left( \frac{1}{n} \log \sum_{\omega \in \mathcal{E}_n(\psi_\alpha,K)} e^{S_\omega \varphi} \right) < \infty$

implying $s \geq \tilde{\delta}_\alpha$, and by the arbitrariness of $s > \tilde{\delta}_\alpha$ we have $\tilde{\delta}_\alpha \geq \delta_\alpha$. For the reverse inequality assume $s < \delta_\alpha$. Then $\mathcal{P}(s\varphi,\psi_\alpha) > 0$. Again by definition of the pressure, there exists a sequence $n_j \to \infty$ such that $\frac{1}{n_j} \log \sum_{\omega \in \mathcal{E}_n(\psi_\alpha,K)} e^{S_\omega \varphi} > 0$ for all $j \in \mathbb{N}$. Hence,

this gives $s \leq \delta_\alpha$. Since this holds for every $s < \delta_\alpha$, we conclude $\delta_\alpha \leq \delta_\alpha$.

(ii) By the Hölder continuity of $\varphi$ and $\psi$ we have that the function $p : (s,t) \mapsto \mathcal{P}(s\varphi + t\psi_\alpha)$ is convex and real-analytic with respect to both coordinates. For fixed $q \in \mathbb{R}$ the function $s \mapsto p(s,q)$ is strictly monotonically decreasing from $+\infty$ to $-\infty$, hence there is a unique number $s(q)$ such that $p(s(q),q) = 0$. Also the function $s : q \mapsto s(q)$ is real-analytic and convex by the Implicit Function Theorem and is tending to infinity for $q \to \pm \infty$. Hence, there is a unique number $q_\alpha \in \mathbb{R}$ minimising $s(q)$, or equivalently, such that $s'(q_\alpha) = 0 = -\frac{\partial}{\partial q} p(s(q_\alpha),q_\alpha) = \int \psi_\alpha d\mu_{s(q_\alpha)\varphi + q_\alpha\psi_\alpha}$. To see that $\delta_\alpha$ in fact coincides with $s(q_\alpha)$ we combine this observation, our first alternative characterisation of $\delta_\alpha$ and Theorem 3.1 (with $f = s(q_\alpha)\varphi$)}
Before we proceed with the proof of Theorem 1.5, let us fix some further properties of increasing on Proposition 4.3.

If following proposition.

First, for $\alpha \in \mathbb{R}$, we have that the mapping $T$ with equality if $\alpha = 0$.

By definition of $\alpha$, we also find $\int \psi d\mu(\alpha) = \alpha$.

As a consequence of Remark 3.5, the fact that $\delta$.

Now, the gradient of $s$ in $(q_\alpha, -\alpha q_\alpha)$ is $(\alpha t, q_\alpha t)$ with $t_\alpha := 1 / \int \phi d\mu_{\alpha \phi + q_\alpha \psi}$. Hence, for all $t \in \mathbb{R}$, we find that for $s(\alpha t, q_\alpha t)$, we have by (ii) that the line through the origin with slope $-\alpha$ is tangential to the contour line $a_{\delta_\alpha}$ of height $\delta_\alpha$ in the point $(q_\alpha, -\alpha q_\alpha)$ (see Figure 4.1). This shows that

$$\delta_\alpha = \inf_{r \in \mathbb{R}} s(r, -\alpha r).$$

Now, the gradient of $s$ in $(q_\alpha, -\alpha q_\alpha)$ is $(\alpha t, q_\alpha t)$ with $t_\alpha := 1 / \int \phi d\mu_{\alpha \phi + q_\alpha \psi}$ and hence, for all $t \in \mathbb{R}$

$$s(q_\alpha, -\alpha q_\alpha) \geq -\hat{s}(\alpha, t),$$

with equality if $t = t_\alpha$ (see e.g. [32, Theorem 23.5]). This proves $\delta_\alpha = \sup_{t \in \mathbb{R}} -\hat{s}(\alpha, t)$.}

Remark 4.2. As a consequence of Remark 3.5, the fact that $\phi < 0$, and the Implicit Function Theorem, we have that the mapping $\alpha \mapsto \delta_\alpha$ is real-analytic on $[\psi, \Psi]$. Also note that the characterisations in Theorem 4.1 in particular show that we have the following multifractal spectral identity

$$\delta_\alpha = \dim_H \left\{ \pi(\omega) : \omega \in \Sigma, \lim_{n \to \infty} \frac{S_n(\psi \circ \pi)(\omega)}{n} = \alpha \right\}.$$

See also [28, 3, 12, 11, 20] and Remark 4.4.

We are now in the position to prove Theorems 1.4 and 1.5 stated in the introduction.

Proof of Theorem 1.4. By definition of $\delta$ and Theorem 4.1, we have $\delta_\alpha = \delta > 0$ if and only if $\Psi(\delta \phi) = 0 = \mathcal{P}(\delta \phi, \psi_\alpha)$. Hence by Corollary 3.3, this holds if and only if $\mu_{\delta \phi}(\psi_\alpha) = 0$ which is equivalent to $\mu_{\delta \phi}(\psi) = \alpha$.

Before we proceed with the proof of Theorem 1.5, let us fix some further properties of $\alpha \mapsto \delta_\alpha$ in the following proposition.

Proposition 4.3. If $\psi < \Psi$, then the function $\alpha \mapsto \delta_\alpha$ given by the $\alpha$-Poincaré exponent is strictly increasing on $(\psi, \alpha_{\max})$ and strictly decreasing on $(\alpha_{\max}, \Psi)$ and hence has a unique maximum in $\alpha_{\max} := \mu_{\delta \phi}(\psi)$ with value $\delta$. Further, we have

$$\frac{d}{d\alpha} \delta_\alpha = \frac{q_\alpha}{\mu_{\delta \phi + q_\alpha \psi}}(\phi)$$

and in particular,

$$\lim_{\alpha \to -\infty} \frac{d}{d\alpha} \delta_\alpha = -\infty \quad \text{and} \quad \lim_{\alpha \to \psi} \frac{d}{d\alpha} \delta_\alpha = +\infty.$$

Proof. First, for $\alpha \in \mathbb{R}$, we check that with $\mu = \mu_{\delta \phi + q_\alpha \psi}$,

$$\frac{\partial}{\partial \alpha} (\mathcal{P}(\delta_\alpha \phi + q_\alpha \psi)) = \frac{d}{d\alpha} \delta_\alpha \int \phi d\mu + \frac{d}{d\alpha} q_\alpha \int \psi d\mu - \frac{d}{d\alpha} q_\alpha \phi q_\alpha - q_\alpha.$$
By Theorem 4.1, we conclude that \( \frac{\partial}{\partial t} \left( \Psi(\delta_\alpha \varphi + q_\alpha \psi_\alpha) \right) = 0 \). Hence, \( q_\alpha = \frac{d}{d\alpha} \delta_\alpha \int \varphi d\mu \). We are left to prove the monotonicity. We find that the contour lines of \( s \) given by the function \( q \mapsto a_c(q) \) implicitly via \( s(q, a_c(q)) = c \) intersect the \( a \)-axis in \( -\Psi(c \varphi) \) for all \( c \in \Omega \). Since \( a_\delta(0) = 0, a'_\delta(0) = -\alpha_{\max} \) and \( -\Psi(c \varphi) < 0 \) for \( c < \delta \), we find – using the characterisation of \( \delta_\alpha \) from Theorem 4.1 (iii) – that \( q_\alpha < 0 \) for \( \alpha \in \left( \psi, \alpha_{\max} \right) \) and \( q_\alpha > 0 \) for \( \alpha \in \left( \alpha_{\max}, \Psi \right) \).

Proof of Theorem 4.5 For the upper bound we use a straightforward covering argument and the definition of the \( \alpha \)-Poincaré exponent. The lower bound uses the fibre-induced pressure and a concrete construction of a Frostman measure inspired by [4]. Let \( \alpha \) for any \( \Gamma \) and \( \varepsilon \) such that \( \sum_{\omega \in \mathcal{E}_\varepsilon(\psi_\alpha, k)} |\pi(\omega)|^s \leq \sum_{n \geq N} \sum_{\omega \in \mathcal{E}_\varepsilon(\psi_\alpha, k)} e^{c_{\omega_\alpha} \varphi} \leq \sum_{\omega \in \mathcal{E}_\varepsilon(\psi_\alpha, k)} e^{c_{\omega_\alpha} \varphi} < \infty \). This shows that the \( s \)-dimensional Hausdorff measure of \( L(\alpha, k) \) is finite and hence \( \dim_H(L(\alpha, k)) \leq s \) for every \( s > \delta_\alpha \) giving the upper bound.

For the lower bound we start with the case \( \alpha \in \left( \psi, \Psi \right) \). For \( I_0 := \{ i \in I : \psi_\alpha(i, \ldots) = 0 \} \) we clearly have \( \pi(I_0^s) \subset \mathcal{E}_\varepsilon(\alpha) \). By Theorem 3.1, we have \( 0 = \mathscr{P}(\delta_\alpha \varphi + \psi_\alpha) = \Psi(\delta_\alpha \varphi, I_0) \) and hence by Bowen’s formula \( \dim_H \left( \pi(I_0^s) \right) = \delta_\alpha \) providing the lower bound for this case. For \( \alpha \in \left( \psi, \Psi \right) \) we consider only the case \( \delta_\alpha > 0 \). It then suffices to construct for every positive \( s < \delta_\alpha \) a subset \( C \subset \mathcal{E}_\varepsilon(\alpha) \) such that \( \dim_H C > s \). Fix such positive \( s < \delta_\alpha \) and let \( \varepsilon > 0 \). Then \( \mathcal{P}(s \varphi, \psi_\alpha, k) > 0 \) by Theorem 4.1 (i). Hence, for \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} \sup_{\omega \in \mathcal{E}_\varepsilon(\psi_\alpha, k)} e^{c_{\omega_\alpha} \varphi} = \infty.
\]

Hence, for arbitrarily large \( M > 0 \) we find \( \ell \in \mathbb{N} \) with \( \sum_{\omega \in \mathcal{E}_\varepsilon(\psi_\alpha, k)} e^{c_{\omega_\alpha} \varphi} > M \). Set \( \Gamma := \mathcal{E}_\varepsilon(\psi_\alpha, k) \). Further, let \( \Lambda \subset \mathcal{E}_\varepsilon(\psi_\alpha, k) \) be the finite set of words of length less than \( r \in \mathbb{N} \). We find \( \Gamma = \mathcal{E}_\varepsilon(\psi_\alpha, k) \). Then for every \( \alpha \in \Gamma \) we have \( |S_\omega \psi_\alpha| \leq \varepsilon \). Suppose we have defined \( \Gamma_n \) such that \( |S_\omega \psi_\alpha| \leq \varepsilon \) for all \( \omega \in \Gamma_n \). Then we define inductively \( \Gamma_{n+1} \) as follows: Since for \( \omega \in \Gamma_n \) we have \( |S_\omega \psi_\alpha| \leq \varepsilon \) it follows that \( |S_{\omega \nu} \psi_\alpha| \leq 2\varepsilon \) for every \( \nu \in \Gamma \). Then we find an element \( \tau_{\omega \nu} \in \Lambda \) such that \( |S_{\omega \nu} \psi_\alpha| \leq \varepsilon \) and we set

\[
\Gamma_{n+1} := \{ \omega \tau_{\omega \nu} \omega : \omega \in \Gamma_n, \nu \in \Gamma \}.
\]
Next, using Hölder continuity of \( \varphi \) and bounded distortion, we find suitable constants \( c_i, i = 1, \ldots, 5 \), such that for all \( \omega \in \Gamma_n, n \in \mathbb{N} \),

\[
\sum_{\nu \in \tau_{\omega} \cap \tau_{\nu} \in \Gamma_{n+1}} \left| \pi[\omega \nu \tau_{\nu}] \right|^s \geq c_1 \sum_{\nu \in \tau_{\omega} \cap \tau_{\nu} \in \Gamma_{n+1}} e^{s\nu \tau_{\nu}} \varphi \geq c_2 e^{s\nu \tau_{\nu}} \sum_{\nu \in \tau_{\omega} \cap \tau_{\nu} \in \Gamma_{n+1}} e^{s\nu \tau_{\nu}} \varphi \geq c_3 e^{s\nu \tau_{\nu}} \sum_{\gamma \in \Gamma} e^{s\nu \tau_{\nu}} \varphi \geq c_4 M e^{s\nu \tau_{\nu}} \geq c_5 M |\pi[\omega]|^s \geq |\pi[\omega]|^s,
\]

where the last inequality holds for \( \ell \), and hence for \( M \), chosen sufficiently large.

In the next step we define the Cantor set

\[
C := \limsup_{n \to \infty} \bigcup_{\omega \in \Gamma_n} \pi[\omega]
\]

and a probability measure \( \mu \) supported on \( C \) defined by its marginals as follows: \( \mu ([0,1]) = 1 \) and for \( \omega \in \Gamma_n \) and \( \omega \nu \tau_{\nu} \in \Gamma_{n+1} \)

\[
\mu(\pi[\omega \nu \tau_{\nu}]) := \frac{\left| \pi[\omega \nu \tau_{\nu}] \right|^s}{\sum_{\omega \in \Gamma_{n+1} \cap \omega' \subseteq [\omega]} |\pi[\omega']|^s} \mu(\pi[\omega]).
\]

Note that the existence of \( \mu \) is guaranteed by Kolmogorov’s Consistency Theorem. Inductively, using the definition of \( \mu \) in tandem with inequality (4.1) we verify that for all \( n \in \mathbb{N} \) and \( \omega \in \Gamma_n \) we have

\[
\mu(\pi[\omega]) \leq |\pi[\omega]|^s.
\]

For each interval \( L \subset [0,1] \) we let

\[
\Gamma(L) := \left\{ \omega \in \Gamma_n : n \in \mathbb{N}, \pi[\omega] \cap L \neq \emptyset, |\pi[\omega]| \leq |L| \right\}.
\]

Since

\[
\mu(L) \leq \sum_{\omega \in \Gamma(L)} \mu(\pi[\omega]) \leq \sum_{\omega \in \Gamma(L)} |\pi[\omega]|^s \leq \text{card}(\Gamma(L)) |L|^s
\]

and \( \text{card}(\Gamma(L)) \leq 2 \cdot (I)^{1+r} \), the Mass Distribution Principle gives

\[
\dim_H(C) \geq s.
\]

Since \( s < \delta_\alpha \) was arbitrary, and \( \mu(C) = \mu(C \setminus \emptyset) \) the desired lower bound is established. \( \square \)

**Remark 4.4.** Let \( \mu_\alpha := \mu_{\delta_\alpha \varphi + \psi_\alpha} \circ \pi^{-1} \). Since \( \mu_{\delta_\alpha \varphi + \psi_\alpha}(\psi_\alpha) = 0 \), Lemma 2.2 implies that \( \mu_\alpha(\mathbf{E}(\alpha)) = 1 \). By Young’s formula we have \( \dim_H(\mu_\alpha) = \delta_\alpha \), which then gives an alternative proof for the lower bound of the Hausdorff dimension of \( \mathbf{E}(\alpha) \).

**Proof of Theorem 1.2.** First suppose that \( 0 \in [\psi, \bar{\psi}] \). Then this corollary is a special case of Theorem 1.5 with \( \alpha = 0 \) by noting that \( R = \mathbf{E}(0) \) and \( R_\alpha = \mathbf{E}_\alpha(0) \). If \( 0 \notin [\psi, \bar{\psi}] \), then \( R = \mathbf{E}(0) = R_\alpha = \mathbf{E}_\alpha(0) = \emptyset \) and \( \delta_0 = 0 \). \( \square \)
Theorem 4.1. By the general multifractal formalism, for the set $T$, let us consider the implicitly defined function $\alpha$. For each $\alpha > 0$, we obtain
$$\delta = \dim_H (E_\alpha (\alpha)) = \sup_{\alpha' > \alpha} \dim_H (E_{\alpha'} (\alpha')) \leq \dim_H (T_1^+) \leq \dim_H (T_2^+) \leq \delta.$$

For the set $T_3^+(r), r \in \mathbb{R}$ note that
$$\bigcup_{\ell \geq 0} F_{\ell}^{-\ell} (T_3^+(r)) \supseteq T_1^+.$$

Since for each $\ell \in \mathbb{N}$ the set $F_{\ell}^{-\ell} (T_3^+(r))$ is a countable union of bi-Lipschitz images of $T_3^+(r)$,
$$\delta = \dim_H (T_1^+) \leq \dim_H \left( \bigcup_{\ell \geq 0} F_{\ell}^{-\ell} (T_3^+(r)) \right) = \sup_{\ell \geq 0} \dim_H \left( F_{\ell}^{-\ell} (T_3^+(r)) \right) \leq \dim_H (T_3^+(r)) \leq \delta.$$

Similarly, we have $E(\alpha') \subset T_1^- \subset T_2^-$ for all $\alpha' > 0$ and hence, by Theorem 4.5, giving the lower bound
$$\dim_H (E(0)) = \delta_0 = \sup_{\alpha' < 0} \delta_{\alpha'} = \sup_{\alpha' < 0} \dim_H (E(\alpha')) \leq \dim_H (T_1^-) \leq \dim_H (T_2^-).$$

And for $T_3^-(r), r \in \mathbb{R}$ similarly as above
$$\dim_H (E(0)) \leq \dim_H (T_1^-) \leq \sup_{\ell \geq 0} \dim_H \left( F_{\ell}^{-\ell} (T_3^- (r)) \right) = \dim_H (T_3^- (r)).$$

For the upper bound we assume without loss of generality that $\delta_0 < \delta$ (otherwise nothing is to be shown) and fix $0 < \varepsilon < \delta - \delta_0$. By Proposition 4.3 we have $q_0 \leq 0$. Hence, we obtain the following bound for $n \in \mathbb{Z}$,
$$\sum_{|\alpha| > N \atop S_\alpha \psi < -n} |\pi [\omega]|^{\delta_0 + \varepsilon} \leq \sum_{|\alpha| > N \atop S_\alpha \psi < -n} e^{(\delta_0 + \varepsilon) S_\alpha \psi + q_0 S_\alpha \psi} e^{-q_0 S_\alpha \psi} \leq e^{-q_0 n} \sum_{|\alpha| > N \atop S_\alpha \psi < -n} e^{(\delta_0 + \varepsilon) S_\alpha \psi + q_0 S_\alpha \psi} \leq e^{-q_0 n} \sum_{|\alpha| > N} e^{(\delta_0 + \varepsilon) S_\alpha \psi + q_0 S_\alpha \psi} < \infty,$$

which shows $\dim_H (T_3^- (r) \cap [n-r-1, n-r]) \leq \delta_0$, for all $n \in \mathbb{Z}$. As a consequence of the countable stability of the Hausdorff dimension we obtain $\dim_H (T_3^- (r)) \leq \delta_0$. In particular, for $r \in \mathbb{Z}$ we use (1.6) and the countable stability of the Hausdorff dimension once more to finish the proof.

Remark 4.5. The last upper bound in the above proof could also been seen via the general multifractal formalism for limiting behaviour of $(S_n \psi / S_n \phi)$ provided e.g. in [18,17] by observing the inclusion, for $r \in \mathbb{R}$ and $n \in \mathbb{Z}$,
$$T_3^- (r) \cap [n,n+1] \subset \pi \left\{ \omega \in \Sigma: \liminf_{\ell \to \infty} \frac{S_\omega \psi}{S_\omega \phi} \leq 0 \right\} + n.$$

Let us consider the implicitly defined function $s : \mathbb{R} \to \mathbb{R}$ by $\mathcal{B} (s(q) \phi + q \psi) = 0$ as in the proof of Theorem 4.1. By the general multifractal formalism, for $\alpha = \mu_{S_\phi} (\psi) > 0$, we have
$$\dim_H \left( \pi \left\{ \omega \in \Sigma: \liminf_{\ell \to \infty} \frac{S_\omega \psi}{S_\omega \phi} \leq 0 \right\} \right) = \inf_{q \in \mathbb{R}} s(q).$$
This is to say that we have to find \( s_0 \) and \( q_0 \) with \( \Psi(s_0 \psi + q_0 \psi) = 0 \) and \( s'(q_0) = \frac{\partial}{\partial q} \Psi(s_0 \psi + q \psi) \big|_{q=q_0} = 0 \). By Theorem 4.1 this shows \( \inf_{q \in \mathbb{R}} s(q) = \delta_0 \). The upper bound then follows from the countable stability of the Hausdorff dimension.

5. Examples and an application to Kleinian groups

Let us consider an interval map \( F \) with two expansive branches with slopes \( 1/c_1 \) and \( 1/c_2 \), respectively, where \( c_1, c_2 \in (0,1) \) with \( c_1 + c_2 \leq 1 \) (see for instance Example 1.1 in the introduction). Then the corresponding geometric potential is given by \( \phi(\omega) := \log(c_{a_1}) \) for \( \omega = (a_1, a_2, \ldots) \). Moreover, note that \( \delta \) solves the Moran-Hutchinson formula \( c_1^{\delta} + c_2^{\delta} = 1 \). We determine the dimension spectrum \( \alpha \mapsto \delta_\alpha \) of the escaping sets for \( F_\psi \) for different parameters \( c_1, c_2 \) and step length functions \( \Psi \). In the following we also make the convention that \( 0 \cdot \log 0 = 0 \).

Example 5.1. First, we consider arbitrary \( c_1, c_2 \in (0,1) \) and pick a symmetric step length function \( \Psi \) such that \( \psi(\omega) = (-1)^{a_0} \). Then we have to solve the two equations for \( \alpha \in (-1,1) \):

\[
1 = \exp(\Psi(s\psi + q\psi)) = e^{-q\alpha}(c_1^1e^{-q} + c_2^1e^{q}) =: z_\alpha(s,q)
\]

and

\[
0 = \frac{\partial z_\alpha}{\partial q}(s,q) = -\alpha + e^{-q\alpha}(-c_1^1e^{-q} + c_2^1e^{q}).
\]

Using Theorem 4.1(i)&(ii), this gives for \( \alpha \in [-1,1] \)

\[
\delta_\alpha = \delta_\alpha(c_1, c_2) = \frac{(1+\alpha) \log(1+\alpha) + (1-\alpha) \log(1-\alpha)}{(1+\alpha) \log(1/c_1) + (1-\alpha) \log(1/c_2)}.
\]

Note that this expression is in fact the quotient of the measure-theoretic entropy over the Lyapunov exponent of \( F \) with respect to the \((\frac{1+\alpha}{2},\frac{1-\alpha}{2})\)-Bernoulli measure. Moreover,

\[
\lim_{\alpha \to -1} \frac{d}{d\alpha} \delta_\alpha = \infty \quad \text{and} \quad \lim_{\alpha \to 1} \frac{d}{d\alpha} \delta_\alpha = -\infty,
\]

see Figure 1.2 for the graph of \( \alpha \mapsto \delta_\alpha \). In order to determine the Hausdorff dimension of the (uniformly) recurrent set we need to determine \( \delta_0 \) depending on the parameters \( (c_1, c_2) \). We obtain

\[
\delta_0(c_1, c_2) = \frac{\log 4}{\log(1/c_1) + \log(1/c_2)},
\]

see Figure 1.3 for a one-parameter plot of \( c \mapsto \delta_0(c, 1-c) \).

Finally, for \( \alpha = 0 \), using Theorem 3.1 we obtain for the fibre-induced pressure

\[
\mathcal{P}(t\phi, \psi) = \log 2 + t \cdot (\log c_1 + \log c_2)/2.
\]

Example 5.2. Here, we set \( c_1 = c_2 = c \) and consider \( \Psi \) such that \( \psi(\omega) = m_1(2 - \omega_1) + m_2(\omega_1 - 1) \) with \( m_1 < m_2 \) (\( \alpha \))-symmetric. A similar calculation as in the first example leads us to the dimension spectrum

\[
\delta_\alpha = \delta_\alpha(c, m_1, m_2) = \frac{(\frac{\alpha - m_1}{m_2 - m_1}) \log(\frac{\alpha - m_1}{m_2 - m_1}) + (\frac{m_2 - \alpha}{m_2 - m_1}) \log(\frac{m_2 - \alpha}{m_2 - m_1})}{\log(1/c)},
\]

for \( \alpha \in [m_1, m_2] \). Again, note that this expression is the quotient of the measure-theoretic entropy over the Lyapunov exponent of \( F \) with respect to the \((\frac{\alpha - m_1}{m_2 - m_1}, \frac{m_2 - \alpha}{m_2 - m_1})\)-Bernoulli measure.
Example 5.3. For the last example we change the map $F$. Assume that $F$ has $g_1 + g_2$ intervals where $g_1, g_2 \geq 1$ such that on each interval the slope $1/c \geq g_1 + g_2$. Further, we choose $\Psi$ in such a way that on $g_1$ intervals it is $-1$ and on the others it is $+1$. An analogous calculation as in the previous examples gives for $\alpha \in [-1, 1]$ the following dimension spectrum

$$\delta \alpha = \delta \alpha (c, g_1, g_2) = - \frac{g_1 \cdot \left( \frac{1-\alpha}{2g_1} \right) \log \left( \frac{1-\alpha}{2g_1} \right) + g_2 \cdot \left( \frac{1+\alpha}{2g_2} \right) \log \left( \frac{1+\alpha}{2g_2} \right)}{\log(1/c)}.$$ 

Once more, observe that this expression is a quotient of the measure-theoretic entropy over the Lyapunov exponent of $F$ for a corresponding Bernoulli measure. For a particular choice consider $g_1 = 1, g_2 = 2$ and $c = 1/3$, see also Figure 5.1. We have $\dim_H (E(1)) = \log 2/\log 3$ and this means that the Hausdorff dimension of $E(1)$ coincides with the dimension of the $1/3$-Cantor set.

Finally, we provide an application of our results partially recovering a result of [29] about extensions of Kleinian groups. Recall that if $G = \langle g_1, \ldots, g_k \rangle$ is a Schottky group, the limit set of $G$ can be represented by the subshift of finite type $\Sigma_G := \{ \omega = (\omega_1, \omega_2, \ldots) \in I^\mathbb{N} \mid \omega_i \neq -\omega_{i+1}, \ i \in \mathbb{N} \}$ with alphabet $I = \{ \pm 1, \cdots \pm k \}$. Then $\pi$ denotes the natural coding map of the limit set with respect to $\Sigma_G$. Further, the $\sigma$-invariant $\delta_G$-dimensional Patterson-Sullivan measure of $G$ is given by the $\sigma$-invariant Gibbs measure on $\Sigma_G$ with respect to the Hölder continuous geometric potential $\delta_G \cdot \varphi : \Sigma_G \rightarrow \mathbb{R}$ associated with $G$ (5). Concerning the general theory of the Patterson-Sullivan measure and its $\sigma$-invariant version, see [27] and [41], respectively.

**Proposition 5.4.** Let $G = \langle g_1, \ldots, g_k \rangle$ be a Schottky group. Let $\mu$ be the $\sigma$-invariant version of the $\delta_G$-dimensional Patterson-Sullivan measure on the subshift of finite type $\Sigma_G$ with alphabet $I = \{ \pm 1, \cdots \pm k \}$. Then we have $\mu ([i]) = \mu ([{-i}])$ for all $i \in I$. 

![Figure 5.1](image_url)
Proof. Denote by $m$ the $\delta_G$-dimensional Patterson-Sullivan measure. Recall that $|\xi - \eta|^{-2\delta_G} dm(\xi) \times dm(\eta)$ defines a $G$-invariant measure on the geodesics on $\mathbb{H}^n \setminus G$ represented by $\Lambda(G) \times \Lambda(G)$. By disintegration of this measure, we obtain the $\sigma$-invariant version $\mu$ on $\Sigma_G$ which satisfies for all $i \in I$,

$$\mu([i]) = \int_{\pi([i])} \sum_{j \neq i} \int_{\pi([j])} |\xi - \eta|^{-2\delta_G} dm(\eta) dm(\xi).$$

Since

$$g_i \left( \sum_{j \neq i} \pi[i] \times \bigcup_{j \neq i} \pi[j] \right) = \left( \bigcup_{j \neq -i} \pi[j] \right) \times \pi[-i],$$

the $G$-invariance of $|\xi - \eta|^{-2\delta_G} dm(\xi) \times dm(\eta)$ implies

$$\mu([i]) = \int_{\pi([i])} \sum_{j \neq i} \int_{\pi([j])} |\xi - \eta|^{-2\delta_G} dm(\eta) dm(\xi)$$

$$= \int_{\pi([-i])} \sum_{j \neq -i} \int_{\pi([j])} |\xi - \eta|^{-2\delta_G} dm(\eta) dm(\xi) = \mu([-i]).$$

This shows the assertion. \hfill $\square$

Now, let $G$ be a Schottky group and $N < G$ a normal subgroup such that $G/N = \langle g \rangle \cong \mathbb{Z}$ and without loss of generality $g \in \{g_1, \ldots, g_k\}$. Then, using Kronecker delta notation, we set $\psi : \Sigma_G \to \mathbb{Z}$, $\psi(\omega) := \delta_{g_{n_1}, g} - \delta_{g_{n_2}, g}^{-1}$. Relating the hyperbolic distance to $\varphi$ as in [22] as well as replacing the fullshift $\Sigma_G$ with $\Sigma_N$ in the definition of Poincaré exponents, we deduce that $\delta = \delta_G$ and $\delta_0 = \delta_N$. Further, note that $0 \in \left( \frac{\psi, \psi}{\psi, \psi} \right) = (-1, 1)$ and that the first and the last assertion of Theorem 3.1 and Corollary 3.3 remain valid also for mixing subshifts of finite type. Hence, combining the previous proposition with Corollary 3.3 and Theorem 3.1 applied to $\delta_G \varphi$, gives Corollary 1.6.

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