LIOUVILLE PROPERTY AND EXISTENCE OF ENTIRE SOLUTIONS OF HESSIAN EQUATIONS

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Abstract. In this paper, we establish the existence and uniqueness theorem for entire solutions of Hessian equations with prescribed asymptotic behavior at infinity. This extends the previous results on Monge-Ampère equations. Our approach also makes the prescribed asymptotic order optimal within the range preset in exterior Dirichlet problems. In addition, we show a Liouville type result for $k$-convex solutions. This partly removes the $(k+1)$- or $n$-convexity restriction imposed in existing work.

1. Introduction

In this paper, we study the Liouville property of Hessian equations

$$\sigma_k(\lambda(D^2u)) = 1$$

and the existence of solutions of

$$\sigma_k(\lambda(D^2u)) = f$$

in $\mathbb{R}^n$ with $f$ a perturbation of 1 near infinity. Here $\lambda(D^2u)$ denotes the eigenvalue vector $\lambda := (\lambda_1, \cdots, \lambda_n)$ of the Hessian matrix $D^2u$,

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}$$

is the $k$-th elementary symmetric function, $k = 1, \cdots, n$. Note that for $k = 1$, (1.2) corresponds to Possion’s equation $\Delta u = f$, which is a linear elliptic equation. For $2 \leq k \leq n$, (1.2) is an important class of fully nonlinear elliptic equations. Especially, for $k = n$, it corresponds to the famous Monge-Ampère equation $\det(D^2u) = f$.

For the case that the right hand side $f \equiv 1$, continuing attention is paid to Liouville properties of (1.1). For $k = 1$, the classical Liouville theorem for harmonic functions shows that any convex entire classical solution of (1.1) must be a quadratic polynomial. For $k = n$, the celebrated theorem of Jörgens [21], Calabi [10] and Pogorelov [24] for the Monge-Ampère equation states that any convex entire classical solution of (1.1) must be a quadratic polynomial. Different proofs of Jörgens-Calabi-Pogorelov theorem were given by Cheng-Yau [13], Caffarelli [6] and Jost-Xin [22]. For $k = 2$, Chang-Yuan [12] proved that if $u$ is an entire solution of

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(1.1) with
\[ D^2 u \geq \left( \delta - \sqrt{\frac{2}{n(n-1)}} \right) I \]
for some \( \delta > 0 \), then \( u \) is a quadratic polynomial. Recently, Shankar-Yuan [25] improved this result to general semiconvex solutions, i.e.
\[ D^2 u \geq -KI \]
for a large \( K > 0 \). For general \( k \), assuming a lower quadratic growth condition, Bao-Chen-Guan-Ji [2] demonstrated that any strictly convex solution of (1.1) is a quadratic polynomial. Afterwards, Li-Ren-Wang [23] relaxed the strict convexity restriction in [2] to \((k+1)\)-convexity. However, to the best of our knowledge, there is little known about the Liouville type result for \( k \)-convex solutions of (1.1). We refer to [1], where Bao established such a result by introducing a integral growth condition for \( D^2 u \).

For the general right hand side, Caffarelli-Li [7] generalized the Jörgens-Calabi-Pogorelov theorem. They considered
\[ \det(D^2 u) = f \quad \text{in} \quad \mathbb{R}^n, \]  
where \( f \in C^0(\mathbb{R}^n) \) satisfies
\[ \inf_{\mathbb{R}^n} f > 0 \quad \text{and} \quad \text{supp}(f - 1) \text{ is bounded}. \]

For \( n \geq 3 \), they proved that any convex viscosity solution of (1.3) approaches a quadratic polynomial at infinity with
\[ \lim_{|x| \to \infty} |x|^{n-2} \left| u(x) - \left( \frac{1}{2} x^T Ax + b \cdot x + c \right) \right| < \infty, \]  
where \( A \) is a symmetric positive definite matrix with \( \det(A) = 1 \), \( b \in \mathbb{R}^n \) and \( c \in \mathbb{R} \). With such prescribed asymptotic behavior near infinity, they also established an existence and uniqueness theorem for solutions of (1.3). Extending [7], Bao-Li-Zhang [4] considered
\[ \inf_{\mathbb{R}^n} f > 0 \quad \text{and} \quad f \in C^3(\mathbb{R}^n \setminus D), \]  
\[ \exists \beta > 2 \text{ such that } \lim_{|x| \to \infty} |x|^\beta |D^m(f(x) - 1)| < \infty, \quad m = 0, 1, 2, 3, \]
where \( D \subset \mathbb{R}^n \) is a bounded open set in (1.5a). In the same spirit as [7], they derived asymptotic behavior of solutions near infinity,
\[ \begin{cases} 
\lim_{|x| \to \infty} |x|^\min(\beta,n)-2 \left| u(x) - \left( \frac{1}{2} x^T Ax + b \cdot x + c \right) \right| < \infty, & \text{if } \beta \neq n, \\
\lim_{|x| \to \infty} |x|^{n-2} \ln|x|^{-1} \left| u(x) - \left( \frac{1}{2} x^T Ax + b \cdot x + c \right) \right| < \infty, & \text{if } \beta = n.
\end{cases} \]

Also, with this prescribed asymptotic behavior, they proved the existence and uniqueness theorem for solutions of (1.3) (the case \( \beta = n \) was missed in the original paper [4]). Note that the above asymptotic results are extensions for Jörgens-Calabi-Pogorelov theorem. In dimension two, the asymptotic results were studied by Ferrer–Martínez–Milán [16] and Bao-Li-Zhang [4]. See also Bao-Xiong-Zhou [5] for the existence of solutions of (1.3) with prescribed asymptotic behavior.

We would like to mention a further extension of Jörgens-Calabi-Pogorelov theorem. In another paper [8], Caffarelli-Li proved that any classical convex solution
of \((1.3)\) with a periodic positive \(f\) must be sum of a quadratic polynomial and a periodic function. See also the work of Teixeira-Zhang [26] for a perturbation of periodic positive functions \(f\).

In this paper, we focus our attention on Hessian equations. We obtain a Liouville property for \(k\)-convex entire solutions of \((1.1)\). Under an additional upper quadratic growth condition, this removes the assumption of \((k+1)\)- or \(n\)-convexity in [23] and [2], respectively. Furthermore, we prove the existence and uniqueness for entire solutions of \((1.2)\) with prescribed asymptotic behavior \((1.6)\) at infinity. This generalizes the previous corresponding results (see [7] and [4]) on Monge-Ampère equations to Hessian equations.

Recall that (see [9]) for an open set \(\Omega \subset \mathbb{R}^n\), we say that a function \(u \in C^2(\Omega)\) is \(k\)-convex, if \(\lambda(D^2u(x)) \in \Gamma_k\) for every \(x \in \Omega\), where \(\Gamma_k\) is an open convex symmetric cone in \(\mathbb{R}^n\) with its vertex at the origin, given by

\[
\Gamma_k = \{ (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n : \sigma_j(\lambda) > 0, \forall j = 1, \ldots, k \}.
\]

Our first main result is

**Theorem 1.1.** Let \(u \in C^{3,1}(\mathbb{R}^n)\) be a \(k\)-convex solution of \((1.1)\) in \(\mathbb{R}^n\). If there exist positive constants \(A_1, A_2, B\) and \(R_0\) such that

\[
A_1|x|^2 \leq u(x) \leq A_2|x|^2 + B \quad \text{for } |x| \geq R_0.
\]

Then \(u\) is a quadratic polynomial.

We remark that it is of interest to investigate the Liouville property for \(k\)-convex solutions due to the natural ellipticity class for Hessian equations. Benefiting from the fact that level sets of convex functions are between two balls after some affine transformation (see [15]), level set method is an effective way to study Liouville properties of Monge-Ampère equations (see e.g., [4], [7] and [27]). However, the level sets of \(k\)-convex functions could even be unbounded. It is at this point where main changes have to be made to control the level sets of \(k\)-convex solutions. For this purpose, based on the lower quadratic growth condition in [2] and [23], we further introduce the upper quadratic growth condition in Theorem 1.1.

Denote

\[
\mathcal{A}_k = \{ A : A \text{ is real } n \times n \text{ symmetric positive definite matrix and } \sigma_k(\lambda(A)) = 1 \}.
\]

The second main result of the paper is

**Theorem 1.2.** Let \(n \geq 3\) and \(f \in C^{2,\gamma}_{\text{loc}}(\mathbb{R}^n)\) satisfy \((1.5a)-(1.5b)\) for some \(0 < \gamma < 1\). Then for any \(A \in \mathcal{A}_k\), \(b \in \mathbb{R}^n\) and \(c \in \mathbb{R}\), there exists a unique \(k\)-convex solution \(u \in C^{2,\gamma}_{\text{loc}}(\mathbb{R}^n)\) of \((1.2)\) in \(\mathbb{R}^n\) satisfying \((1.6)\).  

**Remark 1.3.** For the special cases that \(k = n = 2\) and that \(k = n \geq 3\), the corresponding results have been proved in [5] and [4], respectively.

Now let us comment the order in \((1.6)\) that the solution of \((1.2)\) approaches the quadratic polynomial near infinity. For the exterior Dirichlet problems for \((1.2)\), the prescribed order is approximate to that in \((1.6)\) with \(n\) replaced by \(\theta n\), where \(\theta \in \left[\frac{2}{5}, 1\right]\) (see, e.g., [3] and [11]). Therefore, we give a finer order that may be attained for solutions of \((1.2)\) defined in exterior domains. A key step in proving Theorem 1.2 is to analyze the asymptotic behavior of solutions near infinity. We deal with it in Proposition 3.1.
The paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we investigate the asymptotic behavior of solutions of (1.2) near infinity, which plays a crucial role in the proof of Theorem 1.2. Section 4 is devoted to proving Theorem 1.2.

In the rest of the paper, we denote $B_r$ as the ball in $\mathbb{R}^n$ centered at 0 of radius $r$, and $C(m_1, \cdots, m_j)$ as some positive constant depending only on $m_1, \cdots, m_j$ that may vary from line to line. For a real $n \times n$ symmetric matrix $\xi = (\xi_{ij})$, we denote by $\lambda(\xi)$ the eigenvalue vector of $\xi$. When $\lambda(\xi) \in \Gamma_k$, we denote $F(\xi) = [\sigma_k(\lambda(\xi))]^{1/k}$ and $F_{ij}(\xi) = D_{\xi} F_{ij}(\xi)$.

2. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 by making use of some known interior estimates for solutions of Hessian equations.

Proof of Theorem 1.1. Without loss of generality, we may assume $R_0^2 \geq 2B$. For $R > R_0$, let

$$\Omega_R = \{ x \in \mathbb{R}^n : u(Rx) < R^2 \},$$

and

$$v(x) = \frac{u(Rx) - R^2}{R^2}.$$ 

Clearly, $v$ satisfies

$$\begin{cases}
\sigma_k(\lambda(D^2v)) = 1 & \text{in } \Omega_R, \\
v = 0 & \text{on } \partial \Omega_R.
\end{cases}$$

By (1.7), we have

$$B_{\sqrt{\frac{x_2}{x_2+B}}} \subseteq \Omega_R \subseteq B_{\sqrt{\frac{x_2}{x_1}}} \subseteq B_{1+\sqrt{\frac{x_1}{x_2}}}$$

(2.1)

and

$$A_1 |x|^2 - 1 \leq v(x) \leq A_2 |x|^2 - 1 + \frac{B}{R^2} \leq A_2 |x|^2 - \frac{1}{2}.$$ 

It follows that

$$\|v\|_{L^\infty(B_{1+\sqrt{\frac{x_1}{x_2}}})} \leq C(A_1, A_2).$$

In view of (2.1), we apply the interior gradient estimate in [14, Theorem 3.2] to $v$ in $B_{1+\sqrt{\frac{x_1}{x_2}}}$ and obtain

$$\|Dv\|_{L^\infty(\Omega_R)} \leq C.$$ 

Here and in the following, $C \geq 1$ denotes some constant depending only on $n, k, A_1, A_2, B$ and $R_0$ unless otherwise stated.

For $M < 0$, let

$$O_M = \{ x \in \Omega_R : v(x) < M \}.$$ 

Taking $M = -\frac{1}{4}$, we apply the second derivative estimate in [14, Theorem 1.5] to $v$ in $O_M$ and obtain

$$\left(v + \frac{1}{4}\right)^4 |D^2v| \leq C \quad \text{in } O_{-\frac{1}{4}}.$$ 

This yields that

$$|D^2v| \leq C \quad \text{in } O_{-\frac{1}{4}}.$$
Note that } v(x) \leq A^2 |x|^2 - \frac{1}{2}. \text{ We thus get } B \sqrt{\frac{1}{8} A^2} \subset O_{-\frac{1}{4}}.

It follows from the Evans-Krylov theorem that 

\| D^2 v \|_{C^\alpha(\sqrt{\frac{1}{8} A^2})} \leq C, \quad \forall \ 0 < \alpha < 1.

where \( C \) depends in addition on \( \alpha \). Hence 

\| D^2 u \|_{C^\alpha(\sqrt{\frac{1}{8} A^2})} = R^{-\alpha} \| D^2 v \|_{C^\alpha(\sqrt{\frac{1}{8} A^2})} \leq CR^{-\alpha}.

Since \( R > R_0 \) is arbitrary, by letting \( R \to \infty \), we obtain 

\| D^2 u \|_{C^\alpha(B^n)} = 0.

Therefore, \( u \) is a quadratic polynomial. This finishes the proof of Theorem 1.1. \( \square \)

3. Asymptotic behavior of solutions near infinity

In this section we analyze the behavior of solutions near infinity. We will show that any solution that satisfies a certain growth condition approaches a quadratic polynomial at a certain speed near infinity. This key result will be used to construct entire solutions in Section 4. Our proof employs iterative arguments.

Throughout this section, we always denote \( A = \text{diag}(a_1, a_2, \ldots, a_n) \) and \( a = (a_1, a_2, \ldots, a_n) \).

**Proposition 3.1.** Let \( n \geq 3 \) and \( r_0 \) be a positive number. Suppose that \( f \in C^3(\mathbb{R}^n \setminus B_{r_0}) \) satisfies \( \inf_{\mathbb{R}^n \setminus B_{r_0}} f \geq \frac{1}{c_0} \) and

\[ |D^m(f(x) - 1)| \leq c_0 |x|^{-\beta - m}, \quad \forall \ |x| > r_0, \ m = 0, 1, 2, 3 \tag{3.1} \]

for some \( \beta > 2 \) and \( c_0 > 0 \). Let \( u \in C^{k, \gamma}_{\text{loc}}(\mathbb{R}^n \setminus B_{r_0}) \) be a \( k \)-convex solution of 

\[ \sigma_k(\lambda(D^2 u)) = f \quad \text{in} \ \mathbb{R}^n \setminus B_{r_0}, \]

satisfying 

\[ \left| u(x) - \frac{1}{2} x^T A x \right| \leq c_1 |x|^{2-\varepsilon}, \quad \forall \ |x| > r_0 \tag{3.2} \]

for some \( 0 < \gamma < 1 \), \( A \in \mathcal{A}_k \) and \( \varepsilon, c_1 > 0 \). Then there exist \( b \in \mathbb{R}^n \) and \( c \in \mathbb{R} \) such that (1.6) holds.

Before the proof we first derive the power decay of derivatives.

**Lemma 3.2.** Under the assumptions of Proposition 3.1, let

\[ w(x) = u(x) - \frac{1}{2} x^T A x \quad \text{for} \ |x| > r_0. \]

Then there exist constants \( C = C(n, k, A, r_0, \varepsilon, \beta, m, c_0, c_1) > 0 \) and \( r_1 = r_1(A, \varepsilon, c_1) > r_0 \) such that for \( m = 0, 1, 2, 3, 4, \)

\[ |D^m w(x)| \leq C |x|^{2-\min(\varepsilon, \beta)-m} \quad \text{for} \ |x| > r_1. \tag{3.3} \]
Proof. For $s > 0$, let
\[ D_s = \left\{ x \in \mathbb{R}^n : \frac{1}{2}x^TAx < s \right\}. \]
For $x$ with $|x| > 2r_0$ and $R = (\min_{1 \leq i \leq n} a_i)^{\frac{1}{2}}|x|$, let
\[ u_R(y) = \left( \frac{4}{R} \right)^2 u \left( x + \frac{R}{4} y \right) \quad \text{for} \quad y \in D_2, \]
and
\[ w_R(y) = \left( \frac{4}{R} \right)^2 w \left( x + \frac{R}{4} y \right) \quad \text{for} \quad y \in D_2. \]
Then
\[ w_R(y) = u_R(y) - \frac{8}{R^2} (x + \frac{R}{4} y)^T A \left( x + \frac{R}{4} y \right) \]
\[ = u_R(y) - \left( \frac{1}{2} y^T Ay + \frac{4}{R} x^T Ay + \frac{8}{R^2} x^T Ax \right) \quad \text{for} \quad y \in D_2. \]
Let
\[ \bar{u}_R(y) = u_R(y) - \frac{4}{R} x^T Ay \left( y^T Ay \right) \quad \text{for} \quad y \in D_2. \]
It is clear that
\[ \sigma_k(\lambda(D^2\bar{u}_R)) = f_R(y) := f \left( x + \frac{R}{4} y \right) \quad \text{in} \quad D_2. \]
By (3.1), we have for $m = 0, 1, 2, 3$,
\[ \|f_R - 1\|_{C^m(D_{2})} \leq C(A, \beta, m, c_0) R^{-\beta}. \]
By (3.2), we have for $y \in D_2$,
\[ \left| \bar{u}_R(y) - \frac{1}{2} y^T Ay \right| = |w_R(y)| \leq \frac{16c_1}{R^2} \left| x + \frac{R}{4} y \right|^{2-\epsilon} \leq C(A, \epsilon, c_1) R^{-\epsilon}, \tag{3.4} \]
which particularly yields
\[ \|\bar{u}_R\|_{L^\infty(D_2)} \leq C(A, r_0, \epsilon, c_1). \]
For $M \leq 2$, let
\[ \Omega_{M,R} = \{ y \in D_2 : \bar{u}_R(y) < M \}. \]
In view of (3.4), we have
\[ \Omega_{1.5,R} \subset D_{1.6} \]
for $R > r_1$ with $r_1 = r_1(A, \epsilon, c_1) > r_0$ sufficiently large. Applying the interior gradient estimate in [14, Theorem 3.2] to $\bar{u}_R$ in $D_2$, we obtain
\[ \|\nabla \bar{u}_R\|_{L^\infty(D_{1.6})} \leq C. \]
Here and in the following, $C \geq 1$ denotes some constant depending only on $n, k, A, r_0, \epsilon, \beta, m, c_0$ and $c_1$ unless otherwise stated. Applying the interior second derivatives estimate in [14, Theorem 1.5] to $\bar{u}_R$ in $\Omega_{1.5,R}$, we further obtain
\[ (\bar{u}_R(y) - 1.5)^4 |D^2\bar{u}_R(y)| \leq C \quad \text{for} \quad y \in \Omega_{1.5,R}, \]
and so
\[ \|D^2\bar{u}_R\|_{L^\infty(\Omega_{1.2,R})} \leq C. \]
It follows that
\[ D^2u_R = D^2\bar{u}_R \leq CI \quad \text{in} \quad D_{1.1}. \]
This implies that the operator $F$ is uniformly elliptic in $D_{1.1}$. Combining its concavity, the interior derivative estimate (see, e.g., [18, chapter 17.4]) yields that
\[ \|u_R\|_{C^{4,\alpha}(\overline{D}_{1})} \leq C, \quad \forall \ 0 < \alpha < 1, \]
where $C$ depends in addition on $\alpha$.

It is easy to see
\[ \|w_R\|_{C^{4,\alpha}(\overline{D}_{1})} \leq C \quad \text{and} \quad A + D^2 w_R \leq CI \quad \text{in} \ D_{1}. \quad (3.5) \]
Clearly, $w_R$ satisfies
\[ a_{ij}^R(y)D_{ij}w_R(y) = F(A + D^2 w_R(y)) - F(A) = \bar{f}_R(y) - 1 \quad \text{for} \ y \in D_{1}, \]
where
\[ a_{ij}^R(y) = \int_0^1 F_{ij}(A + sD^2 w_R(y)) \, ds. \]

By (3.5), $(a_{ij}^R)$ is uniformly elliptic in $D_{1}$ and
\[ \|a_{ij}^R\|_{C^{2,\alpha}(\overline{D}_{1})} \leq C. \]

By the Schauder estimates, we have for $m = 0, 1, 2, 3, 4$,
\[ |D^m w_R(0)| \leq C(\|w_R\|_{L^\infty(D_{1.1})} + \|\bar{f}_R - 1\|_{C^{2,\alpha}(\overline{D}_{1})}) \leq CR^{-\min\{\varepsilon, \beta\}}. \]

It follows that
\[ |D^m w(x)| = \left( \frac{R}{4} \right)^{2-m} |D^m w_R(0)| \leq C|x|^{2-\min\{\varepsilon, \beta\} - m} \quad \text{for} \ |x| > r_1. \]
This finishes the proof. \hfill \Box

Next we give the power decay of solutions of linear elliptic equations on exterior domains. The following lemma will be used in the proof of this section several times.

**Lemma 3.3.** Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain containing the origin. Let $h \in C^2(\mathbb{R}^n \setminus \Omega)$ be a solution of
\[ a^{ij}D_{ij}h(x) = g(x) \quad \text{for} \ x \in \mathbb{R}^n \setminus \overline{\Omega}, \]
where $(a^{ij})$ is a real $n \times n$ symmetric positive definite constant matrix, and $g \in C^{0,\gamma}(\mathbb{R}^n \setminus \Omega)$ satisfies
\[ |g(x)| \leq c_2|x|^{-\delta} \quad \text{for} \ x \in \mathbb{R}^n \setminus \Omega, \quad (3.6) \]
for some $0 < \gamma < 1$, $c_2 > 0$ and $\delta > 2$. Suppose there exists a constant $h_\infty$ such that
\[ h(x) \to h_\infty \quad \text{as} \ |x| \to \infty. \]

Then
\[ h(x) - h_\infty = \begin{cases} O(|x|^{2-\min\{\delta, n\}}), & \text{if} \ \delta \neq n, \\ O(|x|^{2-n \ln |x|}), & \text{if} \ \delta = n, \end{cases} \]
as $|x| \to \infty$. 


Then \((\omega, \nabla)\) follows from the elementary estimates.

Indeed, for each \(y \in \mathbb{R}^n \setminus E\), let

\[
E_1 = \left\{ z \in \mathbb{R}^n \setminus E : |z| \leq \frac{|y|}{2} \right\}, \\
E_2 = \left\{ z \in \mathbb{R}^n \setminus E : |z - y| \leq \frac{|y|}{2} \right\}, \\
E_3 = \left\{ z \in \mathbb{R}^n \setminus E : |z - y| \geq |z| \right\}, \\
E_4 = (\mathbb{R}^n \setminus E) \setminus (E_1 \cup E_2 \cup E_3).
\]

Then (3.7) follows from the elementary estimates.

By the above, we conclude that

\[
\begin{cases}
\Delta (\tilde{h} - \hat{h}) = 0 & \text{in } \mathbb{R}^n \setminus \overline{E}, \\
|h(y) - h_{\infty} - \hat{h}(y)| \to 0 & \text{as } |y| \to \infty.
\end{cases}
\]

Then the maximum principle implies

\[
|h(y) - h_{\infty} - \hat{h}(y)| = O(|y|^{2-n}) \quad \text{as } |x| \to \infty.
\]

The conclusion follows from (3.7). \(\square\)

To continue we prove a lemma that improves the estimates in Lemma 3.2. It is a key step towards Proposition 3.1. In the rest of this section, we write \(g = f^+\).

**Lemma 3.4.** Under the assumptions of Lemma 3.2, if \(2 \varepsilon < 1\), then there exist constants \(C = C(n,k,A,r_0,\varepsilon,\beta,m,c_0,c_1) > 0\) and \(r_2 = r_2(n,k,A,r_0,\varepsilon,\beta,m,c_0,c_1) > r_1\) such that for \(m = 0,1,2,3,4\),

\[
|D^m w(x)| \leq C|x|^{2-2\varepsilon-m} \quad \text{for } |x| > r_2,
\]

where \(r_1\) is as in Lemma 3.2.
Proof. Differentiating the equation
\[ F(D^2u) = g \quad \text{in } \mathbb{R}^n \setminus B_{r_0} \]
with respect to \( x_r \) gives
\[ a_{ij}(x)D_{ij}(D_r u)(x) = D_r g(x) \quad \text{for } |x| > r_0, \quad \] (3.8)
where \( a_{ij}(x) = F_{ij}(D^2 u(x)) \). Lemma 3.2 implies for \( |x| > r_1 \)
\[ |a_{ij}(x) - \bar{a}_i \delta_{ij}| \leq C|x|^{-\varepsilon} \quad \text{and} \quad |D a_{ij}(x)| \leq C|x|^{-1-\varepsilon}, \] (3.9)
where
\[ \bar{a}_i = F_{ii}(A) = \frac{1}{k} \sigma_{k-1;i}(a) \quad \text{and} \quad \sigma_{k-1;i}(a) = \sigma_{k-1}(a)_{a_i=0} > 0. \]
Moreover, differentiating (3.8) with respect to \( x_r \) and \( x_s \) gives
\[ a_{ij} D_{ij}(D_r u) = D_{rs} g - D_s (a_{ij}) D_{ij}(D_r u) \quad \text{in } \mathbb{R}^n \setminus B_{r_1}. \]
Letting \( h_1 = D_{rs} u \), we rewrite the above equation as
\[ \bar{a}_i D_{ij} h_1 = g_1 := D_{rs} g - D_s (a_{ij}) D_{ij}(D_r u) - (a_{ij} - \bar{a}_i \delta_{ij}) D_{ij} h_1. \]
It follows from (1.5b), (3.10) and Lemma 3.2 that
\[ |g_1(x)| \leq C|x|^{-2-2\varepsilon} \quad \text{for } |x| > r_1. \] (3.11)
By Lemma 3.2, we have \(|D^2 w(x)| \to 0\) as \( |x| \to \infty \), thus \( h_1(x) \to a_r \delta_{rs} \) as \( |x| \to \infty \).
Since \( 2\varepsilon < 1 \), Lemma 3.3 yields that
\[ |h_1(x) - a_r \delta_{rs}| \leq C|x|^{-2\varepsilon} \quad \text{for } |x| > r_2, \]
and so \(|D^2 w(x)| \leq C|x|^{-2\varepsilon}\). We thus get
\[ |w| \leq C|x|^{2-2\varepsilon} \quad \text{for } |x| > r_2. \]
Applying Lemma 3.2 to \( w \), we have the result. \( \square \)

Now we proceed with the proof of Proposition 3.1.

Proof of Proposition 3.1. It suffices to prove for \( \varepsilon > 0 \) small. Let \( k_0 \) be a positive integer such that \( 2^{k_0+1} < 1 \) and \( 2^{k_0+1} > 1 \). Let \( \varepsilon_1 = 2^{k_0} \varepsilon \), then \( 1 < 2\varepsilon_1 < 2 \).
Applying Lemma 3.4 \( k_0 \) times, we obtain for \( m = 0, 1, 2, 3, 4, \)
\[ |D^m w(x)| \leq C|x|^{2-\varepsilon_1-m} \quad \text{for } |x| > r_3. \] (3.12)
Here and in the following, \( r_3 \) denotes some sufficiently large constant depending only on \( n, k, A, r_0, \varepsilon, \beta, m, c_0 \) and \( c_1 \).
Let \( h_1 \) and \( g_1 \) be as in Lemma 3.4, i.e. \( h_1 = D_{rs} u \) and
\[ g_1 = D_{rs} g - D_s (a_{ij}) D_{ij}(D_r u) - (a_{ij} - \bar{a}_i \delta_{ij}) D_{ij} h_1. \]
In view of (3.12), corresponding to the proof in Lemma 3.4, we have
\[ |a_{ij}(x) - \bar{a}_i \delta_{ij}| \leq C|x|^{-\varepsilon_1} \quad \text{and} \quad |D a_{ij}(x)| \leq C|x|^{-1-\varepsilon_1}. \] (3.13)
It follows from (1.5b), (3.12) and (3.13) that
\[ |g_1(x)| \leq C(|x|^{-2-\beta} + |x|^{-2-2\varepsilon_1}) \leq C|x|^{-2-2\varepsilon_1} \quad \text{for } |x| > r_3. \]
Applying Lemma 3.3 to \( h_1 \) and \( g_1 \), we get
\[ |h_1(x) - a_r \delta_{rs}| = O([|x|^{-2\varepsilon_1} + |x|^{2-n}]) \quad \text{as } |x| \to \infty. \]
and so
\[ |D^2 w(x)| = O(|x|^{-1}) \quad \text{as } |x| \to \infty. \]
Recalling that $D_r u$ satisfies the equation (3.9), [17, Theorem 4] yields that

$$D_r w(x) \to b_r \quad \text{as } |x| \to \infty$$

for some constant $b_r$. Denote $b = \lim_{|x| \to \infty} \nabla w(x)$.

Let

$$\bar{w}(x) = w(x) - b \cdot x \quad \text{for } |x| > r_3,$$

Then

$$\bar{a}_{ij} D_{ij} \bar{w} = F(A + D^2 \bar{w}) - F(A) = g(x) - 1 \quad \text{for } |x| > r_3,$$

where

$$\bar{a}_{ij}(x) = \int_0^1 F_{ij}(A + sD^2 \bar{w}(x)) ds.$$

For $e \in S^{n-1}$, applying $D_e$ to $F(A + D^2 \bar{w}(x)) = g(x)$, we have

$$a_{ij} D_{ij}(D_e \bar{w}) = D_e g \quad \text{in } \mathbb{R}^n \setminus \overline{B}_{r_3}.$$

Let $h_2 = D_e \bar{w}$, we rewrite the above equation as

$$\bar{a}_i D_i h_2 = g_2 := D_e g - (a_{ij} - \bar{a}_i \delta_{ij}) D_{ij}(D_e \bar{w}) \quad \text{for } |x| > r_3.$$

Here $a_{ij}$ and $\bar{a}_i$ are as in Lemma 3.4. It follows from (1.5b), (3.12) and (3.13) that

$$|g_2(x)| \leq C(|x|^{-\beta - 1} + |x|^{-1 - 2\varepsilon_1}) \leq C|x|^{-1 - 2\varepsilon_1} \quad \text{for } |x| > r_3.$$

Applying Lemma 3.3 to $h_2$ and $g_2$, we get

$$|h_2(x)| \leq C(|x|^{1 - 2\varepsilon_1} + |x|^{2 - n}) \leq C|x|^{1 - 2\varepsilon_1} \quad \text{for } |x| > r_3.$$

That is

$$|\nabla \bar{w}(x)| \leq C|x|^{1 - 2\varepsilon_1} \quad \text{for } |x| > r_3.$$

Hence

$$|\bar{w}(x)| \leq C|x|^{2 - 2\varepsilon_1} \quad \text{for } |x| > r_3.$$

Applying Lemma 3.2 to $\bar{w}$, we obtain for $m = 0, 1, 2, 3, 4,$

$$|D^m \bar{w}(x)| \leq C|x|^{2 - 2\varepsilon_1 - m} \quad \text{for } |x| > r_3.$$

This implies

$$|a_{ij}(x) - \bar{a}_i \delta_{ij}| \leq C|x|^{-2\varepsilon_1} \quad \text{and} \quad |Da_{ij}(x)| \leq C|x|^{-1 - 2\varepsilon_1}.$$

From (1.5b), (3.15) and (3.16), we have the new estimate of $g_2$

$$|g_2(x)| \leq C(|x|^{-1 - \beta} + |x|^{-1 - 4\varepsilon_1}).$$

Since $h_2(x) \to 0$ as $|x| \to \infty$, we apply Lemma 3.3 to $h_2$ and $g_2$ and obtain the new estimate of $h_2$

$$|h_2(x)| \leq \begin{cases} C(|x|^{1 - \beta} + |x|^{1 - 4\varepsilon_1} + |x|^{2 - n}), & \text{if } \min\{\beta, 4\varepsilon_1\} \neq n - 1, \\ C|x|^{2 - n} \ln |x|, & \text{if } \min\{\beta, 4\varepsilon_1\} = n - 1. \end{cases}$$

We thus have

$$|\nabla \bar{w}(x)| = O(|x|^{-1}) \quad \text{as } |x| \to \infty.$$

In view of (3.14), we apply [17, Theorem 4] to $\bar{w}$ and obtain

$$\lim_{|x| \to \infty} \bar{w}(x) = c$$

for some constant $c$. We thus conclude that

$$\lim_{|x| \to \infty} \left| w(x) - \left( \frac{1}{2} x^T A x + b \cdot x + c \right) \right| = 0.$$
Let $$\overline{w}(x) = \tilde{w}(x) - c$$ for $$|x| > r_3$$.

Then $$|\overline{w}(x)| \leq C$$ for $$|x| > r_3$$. Applying Lemma 3.2 to $$\overline{w}$$, we obtain for $$m = 0, 1, 2, 3, 4,$$

$$|D^m \overline{w}(x)| \leq C|x|^{-m} \quad \text{for } |x| > r_3.$$  \hfill (3.17)

This implies

$$|a_{ij}(x) - \bar{a}_i \delta_{ij}| \leq C|x|^{-2}.$$  \hfill (3.18)

Clearly,

$$a_{ij} D_{ij} \overline{w} = F(A + D^2 \overline{w}) - F(A) = g - 1.$$  

We rewrite the above equation as

$$\bar{a}_i D_{ii} \overline{w} = g_3 := g - 1 - (a_{ij} - \bar{a}_i \delta_{ij}) D_{ij} \overline{w}.$$  

It follows from (1.5b), (3.17) and (3.18) that

$$|g_3(x)| \leq C(|x|^{-\beta} + |x|^{-4}) \quad \text{for } |x| > r_3.$$  

Applying Lemma 3.3 to $$\overline{w}$$ and $$g_3$$, we obtain

$$|\overline{w}(x)| = \begin{cases} O(|x|^{2-\beta} + |x|^{-2} + |x|^{2-n}), & \text{if } \min\{\beta, 4\} \neq n, \\ O(|x|^{2-n} \ln |x|), & \text{if } \min\{\beta, 4\} = n, \end{cases}$$  \hfill (3.19)

as $$|x| \to \infty$$. It is easy to see that the proof is finished for $$\min\{\beta, n\} \leq 4$$. If $$\min\{\beta, n\} > 4$$, then

$$|\overline{w}(x)| = O(|x|^{-2}) \quad \text{as } |x| \to \infty.$$  

Applying Lemma 3.2 to $$\overline{w}$$, we obtain for $$m = 0, 1, 2, 3, 4,$$

$$|D^m \overline{w}(x)| \leq C|x|^{-2-m} \quad \text{for } |x| > r_3.$$  \hfill (3.20)

This implies

$$|a_{ij}(x) - \bar{a}_i \delta_{ij}| \leq C|x|^{-4}.$$  \hfill (3.21)

From (1.5b), (3.20) and (3.21), we have the new estimate of $$g_3$$

$$|g_3(x)| \leq C(|x|^{-\beta} + |x|^{-8}) \quad \text{for } |x| > r_3.$$  

Applying Lemma 3.3 to $$\overline{w}$$ and $$g_3$$, we obtain

$$|\overline{w}(x)| = \begin{cases} O(|x|^{2-\beta} + |x|^{-6} + |x|^{2-n}), & \text{if } \min\{\beta, 8\} \neq n, \\ O(|x|^{2-n} \ln |x|), & \text{if } \min\{\beta, 8\} = n, \end{cases}$$  

as $$|x| \to \infty$$. Apply the same argument as above finite times, we can remove the term $$|x|^{-2}$$ from (3.19). Eventually, we get

$$|\overline{w}(x)| = \begin{cases} O(|x|^{2-\beta} + |x|^{2-n}), & \text{if } \beta \neq n, \\ O(|x|^{2-n} \ln |x|), & \text{if } \beta = n, \end{cases}$$  

as $$|x| \to \infty$$. This completes the proof of Proposition 3.1.  \hfill □
4. Proof of Theorem 1.2

In this section, we shall prove Theorem 1.2. We mainly follow the idea of [7] and [20], and suitably modify for our situation.

Let $A = \text{diag}(a_1, \cdots, a_n) \in \mathcal{A}_k$. Following the notations used in [3], we call $u$ a generalized symmetric function with respect to $A$ if it is a function of $s = \frac{1}{2} \sum_{i=1}^{n} a_ix_i^2$, that is

$$u(x) = u(s) := u\left(\frac{1}{2} \sum_{i=1}^{n} a_ix_i^2\right).$$

We denote $u'(s) = \frac{du}{ds}$ and $u''(s) = \frac{d^2u}{ds^2}$.

Since $f$ satisfies (1.5a)-(1.5b), there exist $C_0, s_0 > 1$ such that for $s \geq s_0$,

$$f(x) \leq \overline{f}(x) = \overline{f}(s) := 1 + C_0s^{-\frac{\beta}{k}},$$

and

$$f(x) \geq \underline{f}(x) = \underline{f}(s) := 1 - C_0s^{-\frac{\beta}{k}} > 0.$$

For $s > 0$, let

$$D_s = \left\{ x \in \mathbb{R}^n : \frac{1}{2}x^TAx < s \right\}.$$

For the case that $f \in C^\infty(\mathbb{R}^n)$, let $u_s \in C^\infty(D_s)$ be the unique $k$-convex solution of

$$\begin{cases}
\sigma_k(\lambda(D^2u_s)) = f & \text{in } D_s, \\
u_s = s & \text{on } \partial D_s.
\end{cases}$$

Here the existence of $u_s$ is guaranteed by [9, Theorem 3].

**Lemma 4.1.** Let $n \geq 3$ and $f \in C^\infty(\mathbb{R}^n)$ satisfy (1.5a)-(1.5b). Then for $s \geq s_0$, there exists a positive constant $C_1$ such that

$$\sup_{D_s} \left| u_s(x) - \frac{1}{2} \sum_{i=1}^{n} a_ix_i^2 \right| < C_1,$$

(4.1)

where $C_1$ depends only on $n, k, A, C_0, s_0, \beta$ and $\|f\|_{L^\frac{n}{k}(D_{s_0})}$.

*Proof.* We begin by constructing subsolutions and supersolutions. Let $\eta$ be a nonnegative smooth function supported in $D_{s_0}$ satisfying $\|\eta\|_{L^\frac{n}{k}(D_{s_0})} = 1$ and $v_1 \in C^\infty(\overline{D}_{s_0})$ be the $k$-convex solution of

$$\begin{cases}
\sigma_k(\lambda(D^2v_1)) = f + c_0\eta & \text{in } D_{s_0}, \\
v_1 = 0 & \text{on } \partial D_{s_0},
\end{cases}$$

where $c_0 > 0$ will be chosen later. Let $v_2 \in C^\infty(\overline{D}_{s_0})$ be the convex solution of

$$\begin{cases}
\det(D^2v_2) = (f + c_0\eta)^\frac{n}{k} & \text{in } D_{s_0}, \\
v_2 = 0 & \text{on } \partial D_{s_0}.
\end{cases}$$

By the Maclaurin inequality, we have

$$\left(\det(D^2v_2)\right)^\frac{1}{k} \leq \left(\frac{1}{C_n} \sigma_k(\lambda(D^2v_2))\right)^\frac{1}{k},$$
where $C_n^k = \frac{n!}{k!(n-k)!}$. Thus

$$\sigma_k(\lambda(D^2 v_2)) \geq f + c_0\eta \quad \text{in } D_{s_0}. $$

By the comparison principle, we have $v_1 \geq v_2$ in $D_{s_0}$. It follows from the Alexandrov’s maximum principle (see, e.g., [19, Theorem 1.4.2]) that

$$v_2 \geq -C(n, A, s_0) \left( \int_{D_{s_0}} (f(x) + c_0\eta(x)) \frac{1}{r} dr \right)^\frac{1}{n-1}$$

$$\geq -C(n, A, s_0) \left( \|f\|_{L^\infty(D_{s_0})} + c_0 \right)^\frac{1}{n-1} =: -c_1 \quad \text{in } D_{s_0}.$$

Let

$$\tau = \frac{1}{2} \sum_{i=1}^{n} a_i x_i^2, \quad c_2 = \int_{s_0}^t \int_{t}^r n r^{n-1} \frac{1}{r} (r) dr dt, \quad H_1 = \frac{c_1}{c_2}$$

and

$$v_3(x) = \begin{cases} 
-c_1, & \text{for } 0 \leq \tau < \frac{s_0}{2}, \\
H_1 \int_{s_0}^\tau \int_{t}^r n r^{n-1} \frac{1}{r} (r) dr dt, & \text{for } \frac{s_0}{2} \leq \tau \leq s_0.
\end{cases}$$

Firstly, $v_2 \geq v_3$ in $\overline{D_{s_0}}$. Secondly, using the explicit formula [3, Lemma 1.3] that Hessian operators act on generalized symmetric functions, direct computation shows

$$\det(D^2 v_3) = \det(A)(v_{3}')(n) + v_{3}''(v_{3}')(n-1) \sum_{i=1}^{n} \sigma_{n-1;i}(a)(a_i) x_i^2$$

$$\geq 2\det(A)v_{3}''(v_{3}')(n-1)$$

$$\geq \det(A) H_1^\tau \frac{1}{r} \text{ in } D_{s_0} \setminus \overline{D_{s_0}},$$

where $a = (a_1, \cdots, a_n)$ and $\sigma_{n-1;i}(a) = \sigma_{n-1}(a)|_{a_i=0}$. By taking $c_0$ sufficiently large such that $c_1 \geq c_2(\det(A))^{-\frac{1}{n-1}}$, we have

$$\det(D^2 v_3) \geq \frac{1}{r} \geq f \frac{1}{r} = \det(D^2 v_2) \quad \text{in } D_{s_0} \setminus \overline{D_{s_0}},$$

and $v_2 = v_3 = 0$ on $\partial D_{s_0}$. By the comparison principle, we have $v_2 \geq v_3$ in $D_{s_0} \setminus \overline{D_{s_0}}$, and so $v_2 \geq v_3$ in $D_{s_0}$.

Let

$$u(x) = \begin{cases} 
v_1(x), & \text{for } 0 \leq \tau < s_0, \\
\int_{s_0}^\tau \left( t^{-\kappa} \left( \int_{s_0}^\tau \kappa r^{\kappa-1} \frac{1}{r} (r) dr + H_2 \right) \right) dt, & \text{for } \tau \geq s_0.
\end{cases}$$

where $\kappa = \frac{k}{2h_k(a)}$, $h_k(a) = \max_{1 \leq i \leq n} a_i \sigma_{k-1;i}(a)$, and $H_2 > 0$ will be chosen later. Then $u \in C^0(\mathbb{R}^n) \cap C^\infty(\overline{D_{s_0}}) \cap C^\infty(\mathbb{R}^n \setminus D_{s_0})$,  

$$\sigma_k(\lambda(D^2 u)) \geq f \quad \text{in } D_{s_0}. \quad (4.2)$$

For $\tau > s_0$, it follows that

$$u'(r) = \left( \frac{\tau^{-\kappa}}{\left( \int_{s_0}^\tau \kappa r^{\kappa-1} \frac{1}{r} (r) dr + H_2 \right) \frac{1}{r}} \right)^{\frac{1}{n-1}} > 0.$$
and
\[ u''(\tau) = -\frac{\tau^{-\frac{1}{2}}}{2h_k(\tau)}(u')^{1-k}\left(\int_{s_0}^{\tau} \kappa r^{\kappa-1} f(r) dr + H_2 - \tau^{\kappa} f(\tau)\right) \]
\[ =: -\frac{\tau^{-\frac{1}{2}}}{2h_k(\tau)}(u')^{1-k} \mathcal{H}(\tau), \]
where
\[ \mathcal{H}(\tau) = \begin{cases} H_2 + \frac{C_0 \beta}{2\kappa - \beta} r^{\frac{\kappa}{2}} - \frac{2C_0 \kappa}{2\kappa - \beta}s_0^{\frac{\kappa}{2}} - s_0, & \text{if } \kappa \neq \frac{\beta}{2}, \\ H_2 + C_0 \kappa \ln \tau - C_0 \kappa \ln s_0 - s_0^\beta - C_0, & \text{if } \kappa = \frac{\beta}{2}. \end{cases} \]

Therefore, there exists \( \tilde{H} > 0 \), depending only on \( C_0, s_0, \kappa \) and \( \beta \), such that \( u'' < 0 \) when \( H_2 \geq \tilde{H} \). Using [3, Lemma 1.3] again, we have
\[ \sigma_k(\lambda(D^2 u)) = (u')^k + u''(u')^{k-1} \sum_{i=1}^{\kappa} (\kappa-1;i) (a_i x_i^2) \]
\[ \geq (u')^k + 2h_k(a) u''(u')^{k-1} \tau \]
\[ = \tilde{f} \text{ in } \mathbb{R}^n \setminus D_{s_0}. \]

Moreover, we have
\[ u \geq v_3 \text{ in } D_{s_0} \text{ and } u = v_3 \text{ on } \partial D_{s_0}. \]

Then
\[ \lim_{t \to 0^+} \frac{u(x) - u(x - t\nu)}{t} \leq \lim_{t \to 0^+} \frac{v_3(x) - v_3(x - t\nu)}{t} \text{ for } x \in \partial D_{s_0}, \]
where \( \nu \) is the unit outer normal vectors of \( \partial D_{s_0} \). By taking \( H_2 \geq \tilde{H} \) sufficiently large, we have
\[ \lim_{\tau \to s_0^+} v'_3(\tau) = H_1 \left( \int_{s_0^\gamma}^{s_0} n r^{n-1} \tilde{f}(r) dr \right)^{\frac{1}{n}} \leq (H_2 s_0^\gamma)^{\frac{1}{n}} \leq \lim_{\tau \to s_0^+} u'(\tau). \]
Hence
\[ \lim_{t \to 0^+} \frac{u(x) - u(x - t\nu)}{t} \leq \lim_{t \to 0^+} \frac{u(x + t\nu) - u(x)}{t} \text{ for } x \in \partial D_{s_0}. \]

Also, by a simple computation, we have
\[ \sup_{\mathbb{R}^n} \left| u(x) - \frac{1}{2} \sum_{i=1}^{n} a_i x_i^2 \right| \leq C, \]
for some \( C \geq 1 \) depending only on \( n, k, A, C_0, s_0, \beta \) and \( \|f\|_{L^p(D_{s_0})} \).

We proceed to define
\[ \overline{u}(x) = \begin{cases} 0, & \text{for } 0 \leq \tau < s_0, \\ \int_{s_0}^{\tau} \left( t^{-\kappa} \int_{s_0}^{t} \kappa r^{\kappa-1} f(r) dr \right)^{\frac{1}{\kappa}} dt, & \text{for } \tau \geq s_0. \end{cases} \]

Then \( \overline{u} \in C^0(\mathbb{R}^n) \cap C^\infty(\overline{D_{s_0}}) \cap C^\infty(\mathbb{R}^n \setminus D_{s_0}) \) and
\[ \lim_{\tau \to s_0^+} \overline{u}(\tau) = \lim_{\tau \to s_0^+} \underline{u}(\tau) = 0. \]
Arguing as above, we infer that
\[ \sigma_k(D^2\pi) \leq f \text{ in } D_{s_0}, \]
\[ \sigma_k(D^2\pi) \leq f \text{ in } \mathbb{R}^n \setminus D_{s_0}, \]
and
\[ \sup_{\mathbb{R}^n} \left| u(x) - \frac{1}{2} \sum_{i=1}^{n} a_i x_i^2 \right| \leq C. \quad (4.7) \]
We conclude from (4.5) and (4.7) that
\[ \beta_- := \inf_{\mathbb{R}^n} \left( \frac{1}{2} \sum_{i=1}^{n} a_i x_i^2 - u(x) \right) > -\infty, \]
and
\[ \beta_+ := \sup_{\mathbb{R}^n} \left( \frac{1}{2} \sum_{i=1}^{n} a_i x_i^2 - \overline{u}(x) \right) < \infty. \]
Next, we will show that for \( s > s_0 \),
\[ u(x) + \beta_- \leq u_s(x) \leq \overline{u}(x) + \beta_+ \quad \text{for } x \in D_s. \quad (4.8) \]
To establish the first inequality, let \( \overline{x} \) be a maximum point of the function
\[ h(x) := u(x) + \beta_- - u_s(x) \]
in \( \overline{D}_s \). It follows that (4.2) and (4.3) that
\[ \sigma_k(\lambda(D^2u)) \geq \sigma_k(\lambda(D^2u_s)) \text{ in } D_{s_0}, \]
and
\[ \sigma_k(\lambda(D^2u)) \geq \sigma_k(\lambda(D^2u_s)) \text{ in } \mathbb{R}^n \setminus \overline{D}_{s_0}. \]
Then we have, by the strong maximum principle, \( \overline{x} \in \partial D_{s_0} \) or \( \overline{x} \in \partial D_s \). If \( \overline{x} \in \partial D_s \), then in view of the boundary data of \( u_s \) and the definition of \( \beta_- \),
\[ h(x) \leq h(x) = u(x) + \beta_- - u_s(x) \leq \frac{1}{2} \sum_{i=1}^{n} a_i x_i^2 - s = 0 \quad \text{for } x \in \overline{D}_s, \]
and so the inequality holds. If \( x \in \partial D_{s_0} \), then
\[ \lim_{t \to 0^+} \frac{h(x) - h(x - t \nu)}{t} \geq 0 \geq \lim_{t \to 0^+} \frac{h(x + t \nu) - h(x)}{t}, \]
which contradicts to (4.4) by recalling the smoothness of \( u_x \). Hence, the first inequality in (4.8) holds. For the second inequality, let \( \overline{\pi} \) be a minimum point of the function
\[ \overline{h}(x) := \overline{u}(x) + \beta_+ - u_s(x) \]
in \( \overline{D}_s \). Again the strong maximum principle yields that \( \overline{\pi} \in \partial D_{s_0} \) or \( \overline{\pi} \in \partial D_s \). If \( \overline{\pi} \in \partial D_s \), then by the definition of \( \beta_+ \), we have
\[ \overline{h}(x) \geq \overline{h}(\overline{\pi}) = \overline{u}(\overline{\pi}) + \beta_+ - u_s(\overline{\pi}) \geq \frac{1}{2} \sum_{i=1}^{n} a_i \overline{\pi}_i^2 - s = 0 \quad \text{in } \overline{D}_s, \]
and so the second inequality holds. If \( \overline{x} \in \partial D_{s_0} \), in view of (4.6) and the smoothness of \( u_s \), we have
\[ \nabla u_s(\overline{x}) = 0, \]
which is impossible due to Hopf’s Lemma. Hence, the second inequality in of (4.8) holds. This completes the proof. \( \square \)
Note that Lemma 4.1 gives an estimate that depends only on \( L^p \) norm of \( f \) instead of some stronger norm of \( f \).

**Proof of Theorem 1.2.** We only show the existence part as the uniqueness part follows immediately from the comparison principle. For the existence part, by an orthogonal transformation and subtracting a linear function, we only need to prove for the case that \( A = \text{diag}(a_1, \cdots, a_n), b = 0 \) and \( c = 0 \). Without loss of generality, we may assume \( D \subset D_{20} \).

First, we explore the proof under the hypothesis that \( f \in C^\infty(\mathbb{R}^n) \). We will show that along a sequence \( s \to \infty \), \( u_s \) converges to a solution \( u \) of (1.2) satisfying

\[
\sup_{\mathbb{R}^n} \left| u(x) - \frac{1}{2} \sum_{i=1}^n a_i x_i^2 \right| \leq C_1,
\]

where \( C_1 \) is as in Lemma 4.1. For this purpose, we are going to derive the locally uniform estimates of \( u_s \).

For any fixed compact subset \( K \) of \( \mathbb{R}^n \), let

\[
K_1 = \{ x \in \mathbb{R}^n : \text{dist}(x, K) \leq 1 \},
\]

\[
M = \sup_{K_1} \left( \frac{1}{2} \sum_{i=1}^n a_i x_i^2 \right) + C_1 + 1,
\]

and

\[
\Omega_{s,M} = \{ x \in D_s : u_s(x) < M \}.
\]

It is easy to check that

\[
K_1 \subset \Omega_{s,M-1} \subset \Omega_{s,M} \subset D_{s_1} \quad \text{for } s > s_1
\]

with \( s_1 > s_0 \) sufficiently large. Indeed, taking \( s_1 \geq M + C_1 \), it follows from Lemma 4.1 that

\[
u_s(x) < \frac{1}{2} \sum_{i=1}^n a_i x_i^2 + C_1 \leq M - 1 \quad \text{in } K_1,
\]

and

\[
u_s(x) > \frac{1}{2} \sum_{i=1}^n a_i x_i^2 - C_1 \geq s_1 - C_1 \geq M \quad \text{in } D_s \setminus D_{s_1}.
\]

Moreover, Lemma 4.1 yields that

\[
|\|u_s\|_{L^\infty(D_{s_1+1})}| \leq C_1 + s_1 + 1 \quad \text{for } s \geq s_1 + 1.
\]

By (4.10), we apply the interior gradient estimate in [14, Theorem 3.2] to \( u_s \) in \( D_{s_1+1} \) and obtain

\[
|\|\nabla u_s\|_{L^\infty(\Omega_{s,M})} \leq C(n, k, A, C_1, K, \|f\|_{C^{0,1}(D_{s_1+1})}) \quad \text{for } s \geq s_1 + 1.
\]

The second derivative estimate in [14, Theorem 1.5] further yields that

\[
|\|D^2 u_s\| \leq C(n, k, A, C_1, K, \|f\|_{C^{1,1}(D_{s_1+1})}) \quad \text{in } \Omega_{s,M}.
\]

It follows that

\[
|\|D^2 u_s\| \leq C \quad \text{in } \Omega_{s,M-1}.
\]

Hence, the operator \( F \) is uniformly elliptic with respect to \( u_s \) in \( \Omega_{s,M-1} \). Combining its concavity, the Evans-Krylov estimates and Schauder theory imply that for \( m \geq 4 \) and \( 0 < \alpha < 1 \),

\[
|\|u_s\|_{C^{m,\alpha}(K)} \leq C(n, k, A, m, \alpha, C_1, K, \|f\|_{C^{m-2,\alpha}(D_{s_1+1})}).
\]
Up to a subsequence $s_i \to \infty$, we get
\[ u_{s_i} \to u_\infty \text{ in } C^m_{\text{loc}}(\mathbb{R}^n), \quad \forall \ m \geq 4. \]
This particularly implies that $u_\infty \in C^\infty(\mathbb{R}^n)$ is a $k$-convex solution of
\[ \sigma_k(\lambda(D^2u_\infty)) = f \text{ in } \mathbb{R}^n, \]
satisfying
\[ \sup_{\mathbb{R}^n} \left| u_\infty(x) - \frac{1}{2} \sum_{i=1}^n a_i x_i^2 \right| \leq C_1 \]
and for $m \geq 4$ and $0 < \alpha < 1$,
\[ \|u_\infty\|_{C^{m,\alpha}(K)} \leq C(n, k, A, m, \alpha, C_1, K, \|f\|_{C^{m-2,\alpha}(D_{\epsilon_{i+1}})}). \]

For general $f \in C^2_{\text{loc}}(\mathbb{R}^n)$, let $f^\varepsilon = \rho_\varepsilon * f$, where $\rho_\varepsilon$ is the standard mollifier. Let $u^\varepsilon_\infty$ be the solution found above for $f^\varepsilon$. From the proof, we see that
\[ \sup_{\mathbb{R}^n} \left| u^\varepsilon_\infty(x) - \frac{1}{2} \sum_{i=1}^n a_i x_i^2 \right| \leq C_1 \]
and
\[ \|u^\varepsilon_\infty\|_{C^{2,\gamma}(K)} \leq C(n, k, A, \gamma, C_1, K, \|f^\varepsilon\|_{C^{2,\gamma}(D_{\epsilon_{i+1}})}), \quad \forall \ K \Subset \mathbb{R}^n. \]

Since $f \in C^2_{\text{loc}}(\mathbb{R}^n)$, we have, up to a subsequence $\varepsilon_i \to 0$,
\[ u^\varepsilon_i \to u_\infty \text{ in } C^4_{\text{loc}}(\mathbb{R}^n) \]
for some $k$-convex function $u_0^\infty$. Hence $u_0^\infty \in C^4_{\text{loc}}(\mathbb{R}^n)$ is a solution of (1.2) and satisfies
\[ \sup_{\mathbb{R}^n} \left| u_0^\infty(x) - \frac{1}{2} \sum_{i=1}^n a_i x_i^2 \right| \leq C_1. \quad (4.11) \]

It follows from Proposition 3.1 and (4.11) that there exists $\tilde{c} \in \mathbb{R}$ such that
\[ \begin{cases} 
\limsup_{|x| \to \infty} \left| x^{\min\{\beta, n\} - 2} \left( u^\infty_0(x) - \left( \frac{1}{2} \sum_{i=1}^n a_i x_i^2 + \tilde{c} \right) \right) \right| < \infty, & \text{if } \beta \neq n, \\
\limsup_{|x| \to \infty} |x|^{-2} \min\{\beta, n\} \left( u^\infty_0(x) - \left( \frac{1}{2} \sum_{i=1}^n a_i x_i^2 + \tilde{c} \right) \right) < \infty, & \text{if } \beta = n.
\end{cases} \]

Then
\[ u := u_0^\infty - \tilde{c} \text{ in } \mathbb{R}^n \]
is the desired solution. This completes the proof of Theorem 1.2. \qed

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