Scale invariant elliptic operators with singular coefficients

G. Metafune ∗ N. Okazawa † M. Sobajima ‡ C. Spina §

Abstract

We show that a realization of the operator

$$L = |x|^\alpha \Delta + c|x|^\alpha - 1 \cdot \nabla - b|x|^\alpha - 2$$

in $L^p(\mathbb{R}^N)$ if and only if $D_c = b + (N - 2 + c)^2/4 > 0$ and $s_1 + \min\{0, 2 - \alpha\} < N/p < s_2 + \max\{0, 2 - \alpha\}$, where $s_i$ are the roots of the equation $b + s(N - 2 + c - s) = 0$, or $D_c = 0$ and $s_0 + \min\{0, 2 - \alpha\} \leq N/p \leq s_0 + \max\{0, 2 - \alpha\}$, where $s_0$ is the unique root of the above equation. The domain of the generator is also characterized.

Mathematics subject classification (2010): 47D07, 35B50, 35J25, 35J70.

Keywords: elliptic operators, unbounded coefficients, generation results, analytic semigroups.

1 Introduction

In this paper we make a systematic investigation of the operator

$$L = |x|^\alpha \Delta + c|x|^\alpha - 1 \cdot \nabla - b|x|^\alpha - 2$$

in $L^p(\mathbb{R}^N)$, $N \geq 1, 1 < p < \infty$. Here $\alpha, b, c$ are unrestricted real numbers. Operators of the form $L(s) = (s + |x|^\alpha)\Delta + c|x|^\alpha - 1 \cdot \nabla, s = 0, 1$, or operators containing a more general diffusion matrix in the second order part have been already studied in literature. See for example [7], [16], [15], [18], [19], [25], where generation results, domain characterization and spectral properties have been proved and [17], [27], where kernel estimates have been deduced via weighted Nash inequalities.

Operators of the form (1) with $\alpha = 0$ have been studied in $L^p$-spaces with weight $|x|^{-\beta}$ for real $\beta$ (see [1], [22]).

In order to treat the singularity at zero we introduce $\Omega = \mathbb{R}^N \setminus \{0\}$ and define $C_0^\infty(\Omega)$ as the space of infinitely continuously differentiable functions with compact support in $\Omega$. We define $L_{\min}$ as the closure in $L^p(\mathbb{R}^N)$ of $(L, C_0^\infty(\Omega))$ and $L_{\max} = (L, D_{\max}(L))$ where

$$D_{\max}(L) = \{ u \in W^{2,p}_{\text{loc}}(\Omega) \cap L^p(\mathbb{R}^N) : Lu \in L^p(\mathbb{R}^N) \}.$$  

The domain of $L_{\min}$ will be denoted by $D_{\min}(L)$. Note that if $u \in D_{\max}(L)$ and $f = Lu$ the equation $Lu = f$ is satisfied in the sense of distributions in $\Omega$ rather than in $\mathbb{R}^N$. We study when

---

∗Dipartimento di Matematica “Ennio De Giorgi”, Università del Salento, C.P.193, 73100, Lecce, Italy. e-mail: giorgio.metafune@unisalento.it
†Department of Mathematics, Tokyo University of Science, Japan. email: okazawa@ma.kagu.tus.ac.jp
‡Department of Mathematics, Tokyo University of Science, Japan and Dipartimento di Matematica “Ennio De Giorgi”, Università del Salento, C.P.193, 73100, Lecce, Italy. email: msobajima1984@gmail.com
§Dipartimento di Matematica “Ennio De Giorgi”, Università del Salento, C.P.193, 73100, Lecce, Italy. e-mail: chiara.spina@unisalento.it
suitable realizations of $L$ between $L_{\min}$ and $L_{\max}$ generate a semigroup in $L^p(\mathbb{R}^N)$. The introduction of $C^\infty_c(\Omega)$ instead of $C^\infty_c(\mathbb{R}^N)$ is unavoidable to treat the singularity at 0 but sometimes leads to unnatural difficulties. For example, if $\alpha = b = c = 0$ and $N \geq 3$, then the Laplacian with domain $W^{2,p}(\mathbb{R}^N)$ coincides with $\Delta_{\min}$ if and only if $p \leq N/2$ and with $\Delta_{\max}$ if and only if $p \geq N/(N-2)$. Similar problems happen when $C^\infty_c(\mathbb{R}^N) \subset D_{\max}(L)$ (depending on $\alpha, b, c, p$) and this explains why we need also intermediate operators between $L_{\min}$ and $L_{\max}$. When $\alpha = c = 0$, $L$ becomes the Schrödinger operator with inverse square potential which is widely studied in the literature. A famous result in [2] shows that the parabolic equation $u_t = Lu$ presents instantaneous blow-up for positive solutions when $D_0 := b + (N-2)^2/4 < 0$, where $4D_0$ is the discriminant of the quadratic equation

$$f_0(s) := -s^2 + (N-2)s + b = 0.$$  

In the general case we show that the elliptic equation $\lambda u - Lu = f$, with $\lambda, f \geq 0$, has no positive solution if $\alpha \neq 2$ and $D_c := b + (N-2+c)^2/4 < 0$. The case $\alpha = 2$ is special in the whole paper and the above restriction is not necessary. We obtain positive results under the assumption $D_c \geq 0$.

In order to formulate our main results we introduce the quadratic function

$$f(s) = b + s(N-2+c-s) = -s^2 + (N-2+c)s + b$$

whose discriminant is $4D_c$. Its roots are $s_1, s_2$ ($s_1 < s_2$) given by

$$s_1 = \frac{N-2+c}{2} - \sqrt{b + \left(\frac{N-2+c}{2}\right)^2}, \quad s_2 = \frac{N-2+c}{2} + \sqrt{b + \left(\frac{N-2+c}{2}\right)^2}. \quad (4)$$

Note that $f$ has the maximum at $s_0 = (N-2+c)/2$ with $f(s_0) = D_c$.

Our main result in the case $D_c > 0$ is the following which summarizes Theorems 4 and 5.

**Theorem 1.1** Let $1 < p < \infty$, $\alpha \neq 2$, $D_c = b + (N-2+c)^2/4 > 0$. Then a suitable realization of $L_{\min} \subset L_{\int} \subset L_{\max}$ generates a semigroup in $L^p(\mathbb{R}^N)$ if and only if

$$s_1 + \min\{0, 2-\alpha\} < N/p < s_2 + \max\{0, 2-\alpha\}.$$  

In this case the generated semigroup is bounded analytic and positive. The domain of $L_{\int}$ is given by equation (42).

In general the semigroup is not contractive. The case $\alpha = 2$ is special and much simpler: no restriction on $N/p$ is needed, see Proposition 6.3.

We observe that $L$ generates a semigroup in some $L^p(\mathbb{R}^N)$ if and only if the open intervals $(s_1 + \min\{0, 2-\alpha\}, s_2 + \max\{0, 2-\alpha\})$ and $(0, N)$ intersect. This is always the case when $b > 0$ since $s_1$ and $s_2$ have opposite signs but easy examples show that the contrary can happen if $b \leq 0$, see the last section of this paper. In such cases no realization of $L$ between $L_{\min}$ and $L_{\max}$ is a generator but it can happen that $L$ endowed with a suitable domain is a generator. We refer the reader to [20] where it is shown that for every $b \in \mathbb{R}$ a suitable realization of $\Delta - b|x|^{-2}$ is self-adjoint and non-positive in $L^2(\mathbb{R}^N)$.

In the critical case $D_c = 0$ we prove the following result in Section 6.

**Theorem 1.2** Let $1 < p < \infty$, $\alpha \neq 2$, $D_c = b + (N-2+c)^2/4 = 0$ and $s_0 = \frac{N-2+c}{2}$. Then a suitable realization of $L_{\min} \subset L_{\int} \subset L_{\max}$ generates a semigroup in $L^p(\mathbb{R}^N)$ if and only if

$$s_0 + \min\{0, 2-\alpha\} \leq N/p \leq s_0 + \max\{0, 2-\alpha\}.$$

2
In this case the generated semigroup is bounded analytic and positive. The domain of \( L_{\text{int}} \) is given by equations \((48)\), \((50)\).

Note that the endpoints are included in Theorem \([12]\) but excluded in Theorem \([11]\). We also point out that the validity of the equalities \( L_{\text{int}} = L_{\text{min}} \) and \( L_{\text{int}} = L_{\text{max}} \) is also characterized through the paper.

The paper is organized as follows. In Section 2 we prove and recall some preliminary results. In Section 3 we partially generalize the results in \([2]\) by showing that if \( b + (N - 2 + c)^2/4 < 0 \) the equation \( u - Lu = f \) has no positive distributional solutions for certain positive \( f \) with compact support. In Section 4 we show that \( L_{\text{min}} \) generates an analytic semigroup when \( s_1 + 2 - \alpha < N/p < s_2 + 2 - \alpha \) and characterize its domain, using Rellich inequalities from \([12]\). The proof is done first for very large \( b > 0 \) showing sectoriality and then extended to the precise range above using a perturbation argument in \([24]\), as stated in the Appendix. Generation results for \( L_{\text{max}} \) are deduced by duality. The sharpness of the above intervals is then shown using the asymptotics of special radial solutions: in particular the "only if" part of Theorem 1.1 is proved in Theorem 4.1.

The operator \( L_{\text{int}} \) is introduced in Section 5. Using the results of Section 4 for \( L_{\text{min}} \) we give a proof of the "if" part of Theorem 1.1, see Theorem 5.4 for a more precise formulation. The critical case \( D_c = 0 \) is studied in Section 6, using the methods of Section 5 but adding a logarithmic term in the weighted estimates. In contrast with Section 5, we do not prove directly the resolvent estimates in \( \mathbb{R}^N \) but first show a weaker form in the unit ball and then improve them in the whole space by scaling. In Section 7 we present some examples. It is worth mentioning that our main results, specialized to the case of Schrödinger operators with inverse square potentials, yields more precise results than those already known. In particular we show that the semigroup exists in the same range of \( p \) as in \([11]\) when \( D_c > 0 \) but we are able to characterize the domain of the generator in addition to the domain of the form.

The precise range of existence of the semigroup is also given in the critical case and seems to be new.

Our result are valid when \( N = 1 \) with \([0, \infty]\) instead of \( \mathbb{R} \). In the statements, however, we keep the notation \( \mathbb{R}^N \) even when \( N = 1 \). Accordingly \( \Omega = [0, \infty] \) and all balls \( B_r \) should be replaced by the intervals \([0, r]\). With these (formal) changes all proofs hold in the one-dimensional case with, at most, some simplifications.

**Notation.** We use \( \Omega \) for \( \mathbb{R}^N \setminus \{0\} \) and for \([0, \infty]\) when \( N = 1 \). \( C_c^\infty(V) \) denotes the space of infinitely continuously differentiable functions with compact support in \( V \). We adopt standard notation for \( L^p \) and Sobolev spaces. The unit sphere in \( \mathbb{R}^N \) is denoted by \( S^{N-1} \) and \( B_r \) stands for the ball with center at 0 and radius \( r \).

## 2 Preliminary results

Here we collect some known or simple fact necessary to our analysis. Observe that if \( I_\lambda u(x) = u(\lambda x) \) for \( \lambda > 0 \), then \( (I_\lambda)^{-1} LI_\lambda = \lambda^{2-\alpha} L \). Note that \( L \) is scale invariant when \( \alpha = 2 \). Other symmetry properties follow from the use of the Kelvin transform. Let \( Tu(x) = |x|^{2-N} u(|x|^{-2}) \).

A straightforward but tedious computation shows that

\[
T^{-1}LT = |x|^{4-\alpha} \Delta - c|x|^{3-\alpha} \frac{x}{|x|} \cdot \nabla + (c(2-N) - b) |x|^{2-\alpha}.
\]

In particular the power \( \alpha \) is changed into \( 4-\alpha \). Many proofs will be subdivided according to \( \alpha < 2 \) and \( \alpha > 2 \). If \( \alpha < 2 \) the degeneracy at infinity is easy to treat but that at the origin is the real
source of the difficulties. Conversely when $\alpha > 2$, using the Kelvin transformation and noticing that it maps the unit ball into its complement, one can study only case, e.g., $\alpha < 2$ and reduce the other to it. Observe however that the Kelvin transform is an isomorphism in $L^p$ if and only if $p = 2N/(N - 2)$. Let us show the closedness of $L_{\min}$ and $L_{\max}$.

**Proposition 2.1** The operator $L_{\max}$ is closed and $(L, C^\infty_c(\Omega))$ is closable.

**Proof.** The closedness of $L_{\max}$ is an immediate consequence of local elliptic regularity, since $L$ has regular coefficients outside the origin. Since $C^\infty_c(\Omega) \subset D_{\max}(L)$, the closability of $(L, C^\infty_c(\Omega))$ follows from the closedness of $L_{\max}$. □

Next we introduce the formal adjoint

$$\tilde{L} = |x|^\alpha \Delta + \tilde{c}|x|^{\alpha-1} \frac{x}{|x|} \cdot \nabla - \tilde{b}|x|^{\alpha-2}$$

with

$$\begin{cases} \tilde{c} = 2\alpha - c \\ \tilde{b} = b + (c - \alpha)(\alpha - 2 + N) \end{cases}$$

acting on $L^p(\mathbb{R}^N)$. Observe that the function

$$\tilde{f}(s) = \tilde{b} + s(N - 2 + \tilde{c} - s) = f(N + \alpha - 2 - x)$$

defined as in (3) and relative to $\tilde{L}$, has roots $\tilde{s}_i = s_i + \alpha - c$, $i = 1, 2$, where $s_1, s_2$ are defined in (3) and that its discriminant $\tilde{b} + (N - 2 + \tilde{c})^2/4$ coincides with that of $f$ (that is with $b + (N - 2 + c)^2/4$). Then we have

**Proposition 2.2**

$$(\tilde{L}_{\max}) = (L_{\min})^* \text{ and } (\tilde{L}_{\min}) = (L_{\max})^*$$

**Proof.** The first identity is immediate consequence of the definitions and of interior elliptic regularity. Taking the adjoint in the equality $(\tilde{L}_{\max}) = (L_{\min})^*$ one obtains $(\tilde{L}_{\max})^* = (L_{\max})^*$, by the closedness of $L_{\min}$, which is the second one (with the roles of $L$ and $\tilde{L}$ interchanged). □

As pointed out in the Introduction, the case $\alpha = 2$ is quite special. Let us state the result in the next proposition (see [12, Section 6]) for the proof.

**Proposition 2.3** Consider the operator $L$ defined in (7) with $\alpha = 2$ and let $1 < p < \infty$. Then $L_{\max} = L_{\min}$ generates an analytic semigroup of positive operators $(T(t))_{t \geq 0}$ in $L^p$ satisfying

$$\|T(t)\|_p \leq e^{(b - \omega_p)t}, \quad \omega_p = f(N/p) - b = \frac{N}{p} \left( \frac{N}{p} - 2 + c \right).$$

Finally

$$D_{\max}(L) = \{ u \in L^p(\mathbb{R}^N) \cap W^{2,p}(\Omega), \ |x|^{\alpha}D^2u, |x|^{\alpha-1}\nabla u, |x|^2D^2u \in L^p(\mathbb{R}^N) \}.$$ 

When $\alpha \in \mathbb{R}$ we introduce the domain

$$D_{p,\alpha} = \{ u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{\text{loc}}(\Omega), \ |x|^\alpha D^2u, |x|^\alpha-1\nabla u, |x|^2D^2u \in L^p(\mathbb{R}^N) \}$$

endowed with its canonical norm and note that it to that in the above proposition when $\alpha = 2$. Note that extra integrability condition for $u$ is relevant near 0 when $\alpha < 2$ and near infinity when $\alpha > 2$. 

4
Lemma 2.4 The space $C^\infty_c(\Omega)$ is dense in $D_{p,\alpha}$. The following interpolation property holds in $D_{p,\alpha}$: there exist $C, \varepsilon_0$ depending on $N, p, \alpha$ such that for every $u \in D_{p,\alpha}$ and $\varepsilon \leq \varepsilon_0$

$$\|\|x|^{\alpha-1}\nabla u\|_p \leq \varepsilon \|Lu\|_p + C_{\varepsilon} \|x|^{\alpha-2}u\|_p. \tag{10}$$

Proof. Let us first observe that a function $u \in W^{2,p}(\Omega)$ with compact support in $\Omega$ can be approximated by a sequence of $C^\infty$ functions with compact support in $\Omega$ in the $D_{p,\alpha}$ norm, by using standard mollifiers. Let $u \in D_{p,\alpha}$ and $\eta_n$ be smooth functions such that $\eta_n = 1$ in $B_n \setminus B_{1/n}$, $\eta_n = 0$ in $\mathbb{R}^N \setminus (B_{2n} \cup B_{1/2n})$, $0 \leq \eta_n \leq 1$ and $|\nabla \eta_n(x)| \leq C|x|^{-1}$, $|D^2 \eta_n(x)| \leq C|x|^{-2}$. If $u \in D_{p,\alpha}$, then $u_n = \eta_n u$ are compactly supported functions in $W^{2,p}(\Omega)$, $u_n \to u$ in $L^p(\mathbb{R}^N)$.

As before $|x|^{\alpha-1}\eta_n(x)\nabla u \to |x|^{\alpha-1}\nabla u$ in $L^p(\mathbb{R}^N)$. For the left term, since $\nabla \eta_n(x)$ can be different from zero only for $1/2n \leq |x| \leq 1/n$ or $n \leq |x| \leq 2n$ we have

$$|x|^{\alpha-1}|\nabla \eta_n(x)||u| \leq C|x|^{\alpha-2}|u|(\chi_{(2n)^{-1} \leq |x| \leq n^{-1}} + \chi_{n \leq |x| \leq 2n}),$$

and the right hand side tends to 0 as $n \to \infty$. A similar argument shows the convergence of the second order derivatives in the weighted $L^p$ norm and the proof of the density is complete. Concerning (10) we observe that the weaker inequality

$$\|\|x|^{\alpha-1}\nabla u\|_p \leq \varepsilon \|x|^{\alpha}D^2 u\|_p + C_{\varepsilon} \|x|^{\alpha-2}u\|_p \tag{11}$$

holds in $C^\infty_c(\Omega)$ by [12] Lemma 4.4. By applying the classical Calderón-Zygmund $\|D^2 v\|_p \leq C\|\Delta v\|_p$ to $v = |x|^{\alpha}u$ and using (11) to interpolate the gradient terms we get

$$\|\|x|^{\alpha}D^2 u\|_p \leq C(\|D^2(|x|^{\alpha}u)\|_p + \|x|^{\alpha-1}\nabla u\|_p + \|x|^{\alpha-2}u\|_p),$$

$$\leq C(\|\Delta(|x|^{\alpha}u)\|_p + \|x|^{\alpha-1}\nabla u\|_p + \|x|^{\alpha-2}u\|_p) \leq C(\|x|^{\alpha}Lu\|_p + \|x|^{\alpha-1}\nabla u\|_p + \|x|^{\alpha-2}u\|_p) \leq C(\|x|^{\alpha}Lu\|_p + \varepsilon \|x|^{\alpha}D^2 u\|_p + C_{\varepsilon} \|x|^{\alpha-2}u\|_p).$$

Taking $\varepsilon$ small, (10) follows. By the density of $C^\infty_c(\Omega)$ in $D_{p,\alpha}$ the proof is complete.

The following lemma is useful to study the equality $L_{\min} = L_{\max}$.

Lemma 2.5 For every $\alpha \neq 2$, $D_{max}(L) \cap D(|x|^{\alpha-2}) = D_{p,\alpha} \subset D_{min}(L)$.

Proof. The inclusion $D_{p,\alpha} \subset D_{max}(L) \cap D(|x|^{\alpha-2})$ is evident and $D_{p,\alpha} \subset D_{min}(L)$ follows from the density of $C^\infty_c(\Omega)$ in $D_{p,\alpha}$. Let $u \in D_{max}(L) \cap D(|x|^{\alpha-2})$, we define $v = |x|^{\alpha-2}u \in L^p(\mathbb{R}^N)$ and note that $Lu = \tilde{L}u = \tilde{L}u - bv$ where

$$\tilde{L} = |x|^2\Delta + (4 - 2\alpha + c)x \cdot \nabla + (2 - \alpha)(N - \alpha + c).$$

Then $v \in D_{max}(\tilde{L})$ and therefore, by Proposition 2.3 $|x|\nabla v, |x|^2D^2v \in L^p(\mathbb{R}^N)$. This yields $u \in D_{p,\alpha}$ and concludes the proof.
We need also the asymptotic behavior of the solutions of certain singular ordinary differential equations related to Bessel equations. We recall that the numbers $s_1$, $s_2$ are defined in [4].

**Lemma 2.6** Let $\alpha \neq 2$, $b, c \in \mathbb{R}$ and $\lambda > 0$ and assume that $k := b + (\frac{N-2+c}{2})^2 \geq 0$. The differential equation

$$
\lambda u - r^\alpha \left( u'' + \frac{N-1+c}{r} u' - \frac{b}{r^2} u \right) = 0
$$

(12)

has two positive solutions $u_1$ and $u_2$ with the following behavior: if $\alpha < 2$ and $k > 0$, then

$$
u_1(r) \approx r^{-s_1} \quad \text{near } 0, \quad u_1(r) \approx r^{-\frac{N-2+c}{2} + \frac{\alpha}{2} \frac{2}{\lambda + \frac{2}{r} \frac{\alpha}{2}}} e^\frac{r}{r^\alpha} \quad \text{near } \infty, \quad (13)
$$

$$
u_2(r) \approx r^{-s_2} \quad \text{near } 0, \quad u_2(r) \approx r^{-\frac{N-2+c}{2} + \frac{\alpha}{2} \frac{2}{\lambda + \frac{2}{r} \frac{\alpha}{2}}} e^\frac{r}{r^\alpha} \quad \text{near } \infty; \quad (14)
$$

if $\alpha < 2$ and $k = 0$, then

$$
u_1(r) \approx r^{-\frac{N-2+c}{2}} \quad \text{near } 0, \quad u_1(r) \approx -r^{-\frac{N-2+c}{2}} \log r \quad \text{near } 0, \quad (15)
$$

and the behavior at $\infty$ is as above. When $\alpha > 2$, [13], [14] and [15] hold with $0$ and $\infty$ interchanged in each of them.

**Proof.** Defining $\tilde{u}(r) = r^{-\frac{N-2+c}{2}} u(r)$ we obtain

$$
\nu_2 \tilde{u}''(r) + r \tilde{u}'(r) = r^{\frac{N-2+c}{2}} \left( r^2 u''(r) + r(N-1+c)u'(r) + \left( \frac{N-2+c}{2} \right)^2 u(r) \right)
$$

$$
= r^{\frac{N-2+c}{2}} \left( \lambda r^{2-\alpha} + b + \left( \frac{N-2+c}{2} \right)^2 \right) u(r) = (\lambda r^{2-\alpha} + k) \tilde{u}(r).
$$

Setting $v(r) = \tilde{u}(cr^\gamma)$, we have

$$
\nu_2 v''(r) + r v'(r) = \gamma^2 \left[ 2 \nu_2 \tilde{u}''(cr^\gamma) + cr^\gamma \frac{2}{\nu} \tilde{u}'(cr^\gamma) \right] = \gamma^2 \left( \lambda c^{2-\alpha} r^{(2-\alpha)} + k \right) v(r).
$$

Choosing

$$
\gamma = \frac{2}{2 - \alpha} \quad c = \left( \frac{2-\alpha}{4\lambda} \right)^{\frac{1}{2-\alpha}} \quad (16)
$$

it follows that $v$ satisfies the Bessel equation

$$
\nu_2 v''(r) + r v'(r) = (\nu^2 + r^2) v(r) \quad (17)
$$

with $\nu^2 = \frac{(2-\alpha)^2}{4\lambda}$, for which the modified Bessel functions $I_\nu$ and $K_\nu$ constitute a basis. We note that both $I_\nu$ and $K_\nu$ are positive, $I_\nu$ is monotone increasing and $K_\nu$ is monotone decreasing. Moreover, by [3] Section 7.5,

$$
I_\nu(r) \approx r^\nu \quad \text{near } 0, \quad I_\nu(r) \approx \frac{e^r}{\sqrt{r}} \quad \text{near } \infty
$$

$$
K_\nu(r) \approx r^{-\nu} \quad (\nu > 0), \quad K_\nu \approx -\log r \quad (\nu = 0) \quad \text{near } 0, \quad K_\nu(r) \approx \frac{e^{-r}}{\sqrt{r}} \quad \text{near } \infty.
$$

Since

$$
u_1(r) := r^{-\frac{N-2+c}{2}} I_\nu \left( 1 + \frac{\alpha}{2} \frac{2}{\lambda + \frac{2}{r} \frac{\alpha}{2}} \right), \quad \nu_2(r) := r^{-\frac{N-2+c}{2}} K_\nu \left( 1 + \frac{\alpha}{2} \frac{2}{\lambda + \frac{2}{r} \frac{\alpha}{2}} \right).
$$

all the assertions readily follow. □
The following elementary consequence of Hölder inequality will be used several times; we state it here to fix the parameters.

**Lemma 2.7** Assume that $\mu$ is a measure and that all powers are integrable with respect to $\nu$. If $\gamma_1 \leq \gamma_2 \leq \gamma_3$, then $$\| |x|^{\gamma_2} \|_p \leq \| |x|^{\gamma_1} \|_p \| |x|^{\gamma_3} \|_p^{1-\tau}$$ with $\tau = \frac{\gamma_2 - \gamma_3}{\gamma_3 - \gamma_1}$ and the norms are taken in $L^p$ with respect to $\mu$.

### 3 Non existence of positive solutions for $b + \left( \frac{N-2+c}{2} \right)^2 < 0$

A famous result in [2], see also [5], [4] for different proofs, states that the equation $u_t = \Delta u - b|x|^{-2}u$ does not admit positive solution if $b + (N-2)^2/4 < 0$. Note that $b_0 := (N-2)^2/4$ is the best constant in Hardy inequality in $L^2(\mathbb{R}^N)$. A detailed analysis of the solution for $b \geq b_0$ is done in [28], including an investigation of oscillating solutions for $b < b_0$. In fact in [26] it is proved that for every $b \in \mathbb{R}$ the operator above, endowed with a suitable domain, generates a self-adjoint semigroup of positivity preserving operators. However the semigroup solution so produced, satisfies the parabolic equation in a weaker sense than in [2], namely it is a distributional solution in a set $\mathbb{R}^N \setminus F$ where $F$ is a closed set of measure zero. In this section we show that a phenomenon similar to that of [2] occurs, independently of $\alpha$. We prove it for the elliptic problem rather than for the parabolic one.

**Theorem 3.1** Let $\alpha \neq 2, b + \left( \frac{N-2+c}{2} \right)^2 < 0$. Then, for every $\lambda > 0$, there exists a radial function $0 \leq \phi \in C_0^\infty(\Omega), \phi \neq 0$, such that the problem

$$\lambda u - Lu = \phi$$

(18)

does not admit any positive distributional solution in $\Omega$.

**Proof.** Assume that $\alpha < 2$ and that there exists $u \geq 0$ satisfying (18) as a distribution in $\Omega$. By local elliptic regularity, $u \in C^\infty(\Omega)$. Set

$$v(r) = \int_{S^{N-1}} u(r \omega) d\omega.$$ 

Since $u \geq 0$, then $v \geq 0$ and, by the divergence theorem, we have for $r > \delta > 0$

$$v'(r) = \int_{S^{N-1}} \nabla u(r \omega) \cdot \omega \ d\omega = r^{1-N} \int_{|\eta|=r} \nabla u(\eta) \cdot \frac{\eta}{r} \ d\eta = r^{1-N} \int_{B_r \setminus B_\delta} \Delta u(x) \ dx$$

$$+ \ r^{1-N} \int_{|\eta|=\delta} \nabla u(\eta) \cdot \frac{\eta}{\delta} \ d\eta$$

hence

$$\frac{d}{dr} \left( r^{N-1} v'(r) \right) = \int_{|\eta|=r} \Delta u(\eta) \ d\eta = r^{N-1} \int_{S^{N-1}} \Delta u(r \omega) \ d\omega$$

and therefore

$$v'' + (N-1+c) \frac{v'}{r} - \frac{b}{r^2} v = \int_{S^{N-1}} \left( \Delta u(r \omega) + \frac{c}{r} \nabla u(r \omega) \cdot r \omega - \frac{b}{r^2} u(r \omega) \right) \ d\omega = \int_{S^{N-1}} r^{-\alpha} Lu(r \omega) \ d\omega$$

7
Then it follows from (18) that \( v \) satisfies

\[ \lambda v - r^\alpha \left[ v'' + (N - 1 + c) \frac{v'}{r} - \frac{b}{r^2} v \right] = \phi(r). \]

Setting \( w(s) = e^{(\frac{N-2+c}{2})s} v(e^s) \) we get

\[ w''(s) = (k + \lambda e^{(2-\alpha)s} w(s) - e^{(2-\alpha)s} \phi(e^s), \ s \in \mathbb{R} \]

where

\[ k = b + \frac{(N - 2 + c)^2}{4} < 0. \]

We choose \( m \in \mathbb{R} \) such that \((k + \lambda e^{(2-\alpha)s}) \leq \frac{k}{2} < 0 \) for \( s \leq m \). By the Sturm Comparison Theorem all non-zero solutions of the homogeneous equation

\[ \zeta''(s) = (k + \lambda e^{(2-\alpha)s}) \zeta(s) \]  

are oscillating for \( s \leq m \). By variation of parameters we write

\[ w(s) = u_2(s) \int_s^\infty u_1(t) g(t) dt + u_1(s) \int_s^\infty u_2(t) g(t) dt + c_1 u_1(s) + c_2 u_2(s), \]

where \( c_1, c_2 \in \mathbb{C}, \ g(s) = e^{(2-\alpha)s} \phi(e^s) \) and \( u_i, i = 1, 2 \) are linearly independent solutions of (20) with Wronskian equal to 1. Since \( g \) is compactly supported we have for \( s \) near \(-\infty\)

\[ w(s) = u_1(s) \int_{supp \, g} u_2(t) g(t) dt + c_1 u_1(s) + c_2 u_2(s). \]

However \( w \) is non-negative, because \( v \geq 0 \), and also oscillating near \(-\infty\) since solves (20). Hence \( w = 0 \) near \(-\infty\) and therefore

\[ c_1 = -\int_{supp \, g} u_2(t) g(t) dt, \ c_2 = 0. \]

This gives

\[ w(s) = u_2(s) \int_{-\infty}^s u_1(t) g(t) dt + u_1(s) \int_s^\infty u_2(t) g(t) dt - u_1(s) \int_{supp \, g} u_2(t) g(t) dt \]

\[ = u_2(s) \int_{-\infty}^s u_1(t) g(t) dt - u_1(s) \int_{-\infty}^s u_2(t) g(t) dt = \int_{-\infty}^s (u_1(t) u_2(s) - u_1(s) u_2(t)) g(t) dt. \]

For fixed \( s \) the function \( t \mapsto G(s,t) = u_1(t) u_2(s) - u_1(s) u_2(t) \) is also oscillating near \( t = -\infty \). Therefore, if we choose \( g \neq 0 \) such that \( G(s,t) < 0 \) on \( supp \, g \), we get \( w(s) < 0 \) and this contradicts \( v \geq 0 \). The case \( \alpha > 2 \) is similar arguing near \(+\infty\) instead of 0.

4 The case \( b + \left( \frac{N-2+c}{2} \right)^2 > 0 \): \( L_{min} \) and \( L_{max} \)

We always assume that \( \alpha \neq 2 \). We recall the function \( f \) defined in Introduction

\[ f(s) = b + s (N - 2 + c - s) \]
and note that
\[
\max_{s \in \mathbb{R}} f(s) = b + \left( \frac{N - 2 + c}{2} \right)^2 > 0
\]
Its roots are \( s_1, s_2 \) defined in \([3]\) and \( f(s) > 0 \) if and only if \( s_1 < s < s_2 \). Observe that the equation \( Lu = 0 \) has the two radial solutions \( |x|^{-s_1}, |x|^{-s_2} \).

**Definition 4.1** In order to approximate \( L \) with uniformly elliptic operators we set for \( \varepsilon (0 < \varepsilon < 1 \leq \varepsilon^{-1}) \), \( \Omega_\varepsilon = B_{\varepsilon^{-1}} \setminus B_\varepsilon \) and \( L_\varepsilon = L \) in \( \Omega_\varepsilon \) with Dirichlet boundary conditions. Since \( L_\varepsilon \) is uniformly elliptic it follows that \( D(L_\varepsilon) = W^{2,p}(\Omega_\varepsilon) \cap W_0^{1,p}(\Omega_\varepsilon) \). To shorten the notation we also write \( L_0 \) for \( L_{\min} \).

### 4.1 Positive results for \( L_{\min} \)

We first prove necessary and sufficient conditions under which \( L_\varepsilon \) and \( L_{\min} \) are sectorial in the sense of \([9, Definitions 1.5.8]\); note that the sectoriality (or more precisely, sectorial-valuedness) in a Hilbert space was originally introduced in \([10, Section V.3.10]\).

**Proposition 4.2** Let \( 1 < p < \infty \). If
\[
f \left( \frac{N - 2 + \alpha}{p} \right) = b + \frac{(N + \alpha - 2)^2}{p'} + \frac{(N + \alpha - 2)(c - \alpha)}{p} > 0,
\]
or equivalently,
\[
s_1 + \frac{2 - \alpha}{p} < N \frac{p}{p'} < s_2 + \frac{2 - \alpha}{p},
\]
then the operators \( L_\varepsilon, L_{\min} \) are sectorial in \( L^p(\Omega_\varepsilon), L^p(\mathbb{R}^N) \), respectively, with sectoriality constants independent of \( \varepsilon \). Moreover \( L_{\min} \) is dissipative in \( L^p(\mathbb{R}^N) \) if and only if
\[
b + \frac{(N + \alpha - 2)^2}{p'} + \frac{(N + \alpha - 2)(c - \alpha)}{p} \geq 0. \tag{21}
\]
Furthermore, if \( f \left( \frac{N - 2 + \alpha}{p} \right) = 0 \) and \( \frac{N - 2 + \alpha}{p} = \frac{N - 2 + c}{2} \), then \( L_\varepsilon \) and \( L_{\min} \) are sectorial.

The dissipativity of \( L_{\min} \) with \( \alpha = 0 \) is independently proved in \([21]\) with constant \( f((N - 2)/p) \).

**Proof.** Let \( u \in D(L_\varepsilon) \) for \( \varepsilon > 0 \) or \( u \in C_c^\infty(\Omega) \) when \( \varepsilon = 0 \). Multiply \( Lu \) by \( \pi |u|^{p - 2} \) and integrate it over \( \mathbb{R}^N \). The integration by parts is straightforward when \( p \geq 2 \). For \( 1 < p < 2 \), \( |u|^{p - 2} \) becomes singular near the zeros of \( u \). It is possible to prove that the integration by parts is allowed also in this case (see \([14]\)). Put \( v = |x|^{-\frac{N - 2 + \alpha}{p}} u \). Then
\[
Lu = |x|^{\alpha} \text{div} \left( |x|^{-\frac{N + \alpha - 2}{p}} \nabla v - \frac{N + \alpha - 2}{p} |x|^{-\frac{N + \alpha - 2}{p}} x v \right) \tag{22}
+ c |x|^{\alpha} \frac{N + \alpha - 2}{p} x \cdot \nabla v - \left( b + \frac{c(N + \alpha - 2)}{p} \right) |x|^{\alpha - \frac{N + \alpha - 2}{p}} v. \tag{23}
\]
Setting $v^* = \overline{v}|v|^{p-2}$, by integration by parts we have

$$\int_{\Omega_*} (-Lu)|u|^{p-2} \, dx$$

$$= \int_{\Omega_*} |x|^{2-N} \left( \nabla v - \frac{N + \alpha - 2}{p} \frac{x v}{|x|^2} \right) \cdot \left( \nabla v^* - \left( N - 2 - \frac{N + \alpha - 2}{p} \right) \frac{x v^*}{|x|^2} \right) \, dx$$

$$- c \int_{\Omega_*} |x|^{-N} (x \cdot \nabla v^*) \, dx + \left( b + c \frac{(N + \alpha - 2)}{p} \right) \int_{\Omega_*} |x|^{-N} u v^* \, dx$$

$$= \int_{\Omega_*} |x|^{2-N} \left( \nabla v \cdot \nabla v^* \right) \, dx - 2 \left( N - 2 + c - \frac{N + \alpha - 2}{p} \right) \int_{\Omega_*} |x|^{-N} (x \cdot \nabla v) |v|^{p-2} \, dx$$

$$+ \left[ b + \frac{N + \alpha - 2}{p} \left( N - 2 + c - \frac{N + \alpha - 2}{p} \right) \right] \int_{\Omega_*} |x|^{-N} v^* \, dx.$$ 

By taking real and imaginary parts of both sides of the equality, and since $\text{div}(x|x|^{-N}) = 0$ we have

$$\text{Re} \left( \int_{\Omega_*} (-Lu)|u|^{p-2} \, dx \right)$$

$$= (p-1) \int_{\Omega_*} |x|^{2-N} |u|^{p-4} \text{Re}(\overline{v} \nabla v)|^2 \, dx + \int_{\Omega_*} |x|^{2-N} |v|^{p-4} |\text{Im}(\overline{v} \nabla v)|^2 \, dx$$

$$- 2 \left( N - 2 + c - \frac{N + \alpha - 2}{p} \right) \int_{\Omega_*} |x|^{-N} (x \cdot \text{Re}(\overline{v} \nabla v)) |v|^{p-2} \, dx$$

$$+ f \left( \frac{N + \alpha - 2}{p} \right) \int_{\Omega_*} |x|^{-N} |v|^p \, dx$$

$$= (p-1) \int_{\Omega_*} |x|^{2-N} |u|^{p-4} \text{Re}(\overline{v} \nabla v)|^2 \, dx + \int_{\Omega_*} |x|^{2-N} |v|^{p-4} |\text{Im}(\overline{v} \nabla v)|^2 \, dx$$

$$+ f \left( \frac{N + \alpha - 2}{p} \right) \int_{\Omega_*} |x|^{-N} |v|^p \, dx$$

$$= \int_{\Omega_*} |x|^{2-N} (\nabla v \cdot \nabla v^*) \, dx + f \left( \frac{N + \alpha - 2}{p} \right) \int_{\Omega_*} |x|^{-N} |v|^p \, dx,$$

$$\text{Im} \left( \int_{\Omega_*} (-Lu)|u|^{p-2} \, dx \right) = (p-2) \int_{\Omega_*} |x|^{2-N} |u|^{p-4} \text{Re}(\overline{v} \nabla u) \cdot \text{Im}(\overline{v} \nabla u) \, dx$$

$$- 2 \left( N - 2 + c - \frac{N + \alpha - 2}{p} \right) \int_{\Omega_*} |x|^{-N} (x \cdot \text{Im}(\overline{v} \nabla v)) |v|^{p-2} \, dx.$$ 

Therefore setting

$$B^2 = \int_{\Omega_*} |v|^{p-4} |x|^{2-N} |\text{Re}(\overline{v} \nabla v)|^2 \, dx,$$

$$C^2 = \int_{\Omega_*} |v|^{p-4} |x|^{2-N} |\text{Im}(\overline{v} \nabla v)|^2 \, dx,$$

$$D^2 = \int_{\Omega_*} |x|^{\alpha-2} |u|^p \, dx = \int_{\Omega_*} |x|^{-N} |v|^p \, dx,$$
we see that
\[ Re \left( \int_{\Omega} (-Lu)\overline{u}|u|^{p-2} \, dx \right) = (p-1)B^2 + C^2 + f \left( \frac{N+\alpha-2}{p} \right) D^2 \]  
(25)
and
\[ Im \left( \int_{\Omega} (-Lu)\overline{u}|u|^{p-2} \, dx \right) \]
\[ \leq |p-2| \left( \int_{\Omega} |v|^{p-4} |x|^{2-N} |Re(\nabla v)|^2 \, dx \right) \frac{1}{2} \left( \frac{1}{|p-2|} \int_{\Omega} |v|^{p-4} |x|^{2-N} |Im(\nabla v)|^2 \, dx \right) \frac{1}{2} \]
\[ + 2 \left| \frac{N-2+c}{p} - \frac{N+\alpha-2}{p} \right| \int_{\Omega} |v|^{p-4} |x|^{1-N} |Im(\nabla v)| \, dx \]
\[ \leq |p-2|BC + 2 \left| \frac{N-2+c}{p} - \frac{N+\alpha-2}{p} \right| CD. \]
By condition (21) or condition \( \frac{N-2+\alpha}{p} = \frac{N-2+c}{p} \), we see that
\[ Im \left( \int_{\Omega} (-Lu)\overline{u}|u|^{p-2} \, dx \right) \leq l_{\alpha} \left\{ Re \left( \int_{\Omega} (-Lu)\overline{u}|u|^{p-2} \, dx \right) \right\}, \]  
(26)
where
\[ l_{\alpha} = \sqrt{\frac{(p-2)^2}{4(p-1)} + \left| \frac{N-2+c}{p} - \frac{N+\alpha-2}{p} \right|^2} f \left( \frac{N+\alpha-2}{p} \right)^{-1} \]

(0/0 = 0). This shows the sectoriality of \( L_\varepsilon \) and \( L_{min} \), with sectoriality constants independent of \( \varepsilon \). Assume now that \( L_{min} \) is dissipative. Then, by (25) for real-valued functions, the inequality
\[ (p-1) \int_{\mathbb{R}^N} |x|^{2-N} |\nabla v|^2 |v|^{p-2} \, dx + f \left( \frac{N+\alpha-2}{p} \right) \int_{\mathbb{R}^N} |x|^{-N} |v|^p \, dx \geq 0 \]
holds. By (6) Corollary 2.3 (ii) (with \( b = (N-2)/2 \) and \( u = |v|^{p/2} \)) we obtain \( f \left( \frac{N+\alpha-2}{p} \right) \geq 0. \]

**Remark 4.3** We remark that the above proposition holds also when \( b + (N-2+c)^2/4 = 0 \) and \( (N-2+c)/p = (N-2+c)/2 \). In this case \( s_1 = s_2 = (N-2+c)/2 \), hence the condition \( f \left( \frac{N-2+\alpha}{p} \right) = 0 \) is satisfied. We also remark that the choice of the power in the substitution \( v = |x|^{-2+c} |u| \) is the only one which leads to the term \( x|x|^{-N} \) in (24) which has zero divergence.

We can state the main result of this subsection.

**Theorem 4.4** Let \( p, \alpha, c, b \) satisfy
\[ f \left( \frac{N}{p} - 2 + \alpha \right) = \left( \frac{N}{p} - 2 + \alpha \right) \left( \frac{N}{p'} - \alpha + c \right) + b > 0, \]
or equivalently,
\[ s_1 + 2 - \alpha < \frac{N}{p} < s_2 + 2 - \alpha. \]

Then the operator \( L_{min} \) generates a bounded positive analytic semigroup in \( L^p(\mathbb{R}^N) \), coherent with respect to all \( p \) satisfying the above inequalities. Moreover, \( D_{min}(L) \) coincides with
\[ D_{p,\alpha} = \{ u \in W^{2,p}_{loc}(\Omega), |x|^\alpha D^2 u, |x|^{\alpha-1} \nabla u, |x|^{\alpha-2} u \in L^p(\mathbb{R}^N) \}. \]
Proof. Put\( u \) and note that the generation interval \( s_1 \) differs from the contractivity interval \( s_1 + (2 - \alpha)/p, s_2 + (2 - \alpha)/p \). In particular, for certain values of \( p, L_{min} \) generates a non-contractive semigroup. The proof of the theorem above is based upon the perturbation results stated in Theorem A.1. The next lemma provides the validity of its assumptions for the operators \( L_\varepsilon \) introduced in Definition 4.4 with constants independent of \( \varepsilon \).

**Lemma 4.5** Let \( p, \alpha, c, b \) as in Theorem 4.4. Put

\[
M := f \left( \frac{N}{p} - 2 + \alpha \right) = b + \left( \frac{N}{p} - 2 + \alpha \right) \left( \frac{N}{p} - \alpha + c \right) > 0
\]

as in Theorem 4.3. Then for \( V(x) = |x|^\alpha - 2 \),

\[
-\text{Re} \int_{\Omega_\varepsilon} (Lu)V^{p-1}|u|^{p-2} \, dx \geq M\|Vu\|_p^p
\]

for every \( u \in C_c^\infty(\Omega) \).

**Proof.** Put \( \beta := \alpha + (p - 1)(\alpha - 2) \). Then since \( \frac{N-2+\beta}{p} = \frac{N}{p} + \alpha - 2 \), we have

\[
f \left( \frac{N-2+\beta}{p} \right) = f \left( \frac{N}{p} + \alpha - 2 \right) > 0.
\]

Now let \( u \in D(L_\varepsilon) \) for \( \varepsilon > 0 \) or \( u \in C_c^\infty(\Omega) \) when \( \varepsilon = 0 \). Then

\[
-\text{Re} \int_{\Omega_\varepsilon} (Lu)V^{p-1}|u|^{p-2} \, dx
\]

\[
= -\text{Re} \int_{\Omega_\varepsilon} \left( |x|^\alpha \Delta u + c|x|^{\alpha-1} \frac{x}{|x|} \nabla u - b|x|^{\alpha-2} u \right) |x|^{\alpha-2(p-1)}|u|^{p-2} \, dx
\]

\[
= -\text{Re} \int_{\Omega_\varepsilon} \left( |x|^{\alpha} \Delta u + c|x|^{\alpha-1} \frac{x}{|x|} \nabla u - b|x|^{\alpha-2} u \right) |u|^{p-2} \, dx,
\]

where \( \beta = \alpha + (p - 1)(\alpha - 2) \) is as defined above. By applying (28) with \( \beta = \alpha \), we get

\[
-\text{Re} \int_{\Omega_\varepsilon} (Lu)V^{p-1}|u|^{p-2} \, dx \geq f \left( \frac{N - 2 + \beta}{p} \right) \int_{\Omega_\varepsilon} |x|^{\beta-2}|u|^p \, dx.
\]

Since \( \beta - 2 = p(\alpha - 2) \), this is nothing but the desired inequality with \( M = f \left( \frac{N}{p} + \alpha - 2 \right) \).  

**Proof of Theorem 4.4** Step 1. First assume that \( b \) is sufficiently large so that the conditions of Proposition 4.2 and Lemma 4.5 are satisfied. Then, by (20), there exists \( 0 < \theta < \pi/2 \) such that \( \lambda - L_\varepsilon \) is injective for \( \lambda \in \Sigma_{\pi/2+\theta} \) for every \( \varepsilon \geq 0 \) (take such a \( \theta \) with \( \tan \theta < l_0 \)). For \( \varepsilon > 0 \), \( L_\varepsilon \) is uniformly elliptic, hence generates an analytic semigroup. By (20) again, \( \lambda - L_\varepsilon \) is invertible and satisfies \( \|\lambda - L_\varepsilon\|^{-1} \leq C|\lambda|^{-1} \) for \( \lambda \in \Sigma_{\pi/2+\theta} \) with \( C \) independent of \( \varepsilon \) (actually, \( C = 1 \) in \( \Sigma_\theta \)). Let \( f \in L^p(\mathbb{R}^N) \) and \( u_\varepsilon = (\lambda - L)^{-1}f \) for \( \lambda \in \Sigma_{\pi/2+\theta} \). Since \( \|u_\varepsilon\| \leq C|\lambda|^{-1} \) and \( \lambda u_\varepsilon - Lu_\varepsilon = f \), by local elliptic regularity we can find a sequence \( \varepsilon_n \) such that \( u_{\varepsilon_n} \to u \) weakly in \( W_{loc}^{2,p}(\Omega) \), strongly in \( L^p_{loc}(\Omega) \) and pointwise. By Lemma 4.4, \( M\|Vu_\varepsilon\| \leq \|Lu_\varepsilon\| \leq (C + 1)\|f\|_p V(x) = |x|^{-2} \) and therefore \( \|\lambda\|_p \leq C\|f\|_p, \|Vu_\varepsilon\| \leq (C + 1)\|f\|_p \) and \( \lambda u_\varepsilon - Lu_\varepsilon = f \). Since \( u \in D_{max}(L) \cap D(V) \), by Lemma 2.5, \( u \in D_{min}(L) \) and this shows that \( \lambda - L_{min} \) is invertible for \( \lambda \in \Sigma_{\pi/2+\theta} \) and that \( \|\lambda - L_{min}\|^{-1} \leq C|\lambda|^{-1} \). If \( \lambda > 0 \) and \( f \geq 0 \), then \( u_\varepsilon \geq 0 \) by the classical maximum principle,
hence \( u \geq 0 \). Finally, if \( f \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) \), then the solutions \( u_\varepsilon \) do not depend on \( p,q \) and we can select the same sequence \((u_\varepsilon)\) convergent both in \( L^p \) and in \( L^q \). Therefore \( u \) is the same in \( L^p, L^q \) and this shows the coherence of the resolvents.

**Step 2.** Assume now that \( b \) satisfies only the condition in the statement and let \( A = -L + kV \) with \( k \) large enough to satisfy also the conditions of Step 1. Then \((-A)_{\min} \) generates an analytic semigroup of positive contractions in \( L^p(\mathbb{R}^N) \), coherent with respect to all \( p \). It is sufficient to write the conditions of Theorem 4.4 for the operator \( -L + kV \).

**Proof.** For \( L(\mathbb{R}^N) \), where \( L = (-A)_{\min} \) generates a bounded positive analytic semigroup in \( L^p(\mathbb{R}^N) \), see Lemma 2.3. Conversely, let \( u \in D_{\max}(L) \cap D(|x|^{\alpha-2}) \). By Lemma 2.3, \( u \in D_{p,\alpha} \).

### 4.2 Positive results for \( L_{\max} \)

We consider the adjoint operator

\[
\tilde{L} = |x|^\alpha \Delta + \tilde{c}|x|^{\alpha-1} \frac{x}{|x|} \cdot \nabla - \tilde{b}|x|^{\alpha-2}
\]

where \( \tilde{c} = 2\alpha - c \), \( \tilde{b} = b + (c-\alpha)(\alpha - 2 + N) \) on \( L^p(\mathbb{R}^N) \), see (6), (7). Since, by Proposition 2.2, \( L_{\max} = (\tilde{L}_{\min})^* \), we deduce generation results for \( L_{\max} \) by duality.

**Theorem 4.6** Let \( p, \alpha, c, b \) such that

\[
f \left( \frac{N}{p} \right) = b + \omega_p = b + \frac{N}{p} \left( \frac{N}{p'} - 2 + c \right) > 0,
\]

or equivalently,

\[
s_1 < \frac{N}{p} < s_2.
\]

Then the operator \( L_{\max} \) generates a bounded positive analytic semigroup in \( L^p(\mathbb{R}^N) \), coherent with respect to all \( p \) satisfying the above inequalities.

**Proof.** It is sufficient to write the conditions of Theorem 4.4 for the operator \( \tilde{L} \) in \( L^p(\mathbb{R}^N) \), to recall that \( \hat{s}_i = s_i + \alpha - c, i = 1, 2 \), \( s_1 + s_2 = N - 2 + c \), and then to argue by duality.
Observe that the condition in the above theorem is independent of $\alpha$. Observe also that if $p$ satisfies both the conditions of Theorems 4.4 and 4.6, then $L_{\min} = L_{\max}$.

Remark 4.7 In the next sections we shall see what happens in Theorems 4.4, 4.6 when $N/p$ coincides with one of the endpoints. For example, if $\alpha < 2$, then $L_{\min}$ generates if and only if $s_1 + 2 - \alpha \leq N/p < s_2 + 2 - \alpha$, see Propositions 4.8, 4.9, 5.5, but for the equality $D(L_{\min}) = D_p$ one needs $s_1 + 2 - \alpha < N/p < s_2 + 2 - \alpha$, see Proposition 5.6. By duality $L_{\max}$ is a generator if and only if $s_1 < N/p \leq s_2$. The case $\alpha > 2$ is similar with the roles of $s_1, s_2$ interchanged.

### 4.3 Negative results for $L_{\min}$

We prove that the generation conditions for $L_{\min}$ given in Theorem 4.4 are sharp.

**Proposition 4.8** If $\alpha < 2$ and $N/p < s_1 + 2 - \alpha$ or $\alpha > 2$ and $N/p > s_2 + 2 - \alpha$ then, for every $\lambda > 0$, $R_q(\lambda - L_{\min}) \not\in L^p(\mathbb{R}^N)$. Therefore $L_{\min}$ does not generate a semigroup in $L^p(\mathbb{R}^N)$.

**Proof.** We focus on the case $\alpha < 2$, the other being similar. We consider the adjoint operator $\tilde{L}$ defined in (10), see Proposition 2.2, and we prove that $N(\lambda - \tilde{L}_{\max}) \not\in \{0\}$ in $L^p(\mathbb{R}^N)$ by exhibiting a radial function $u \in D_{\max}(\tilde{L})$ in $L^p(\mathbb{R}^N)$ satisfying

$$\lambda u - \tilde{L}u = 0.$$  

(27)

By Lemma 2.1 with $b$ and $c$ respectively replaced with $\tilde{b}$ and $\tilde{c}$ defined in (7), $u$ can be written by $u = c_1 u_1 + c_2 u_2$, where $u_j$ is defined in Lemma 2.1. In order to have integrability of $u(\rho)$ in $L^p(\mathbb{R}^N)$ for large $\rho$, we consider the solution $u_2$ ($c_1 = 0$ and $c_2 = 1$). This choice will lead to an additional assumption to insure also the integrability near the origin: the solution $u_2$ is in $L^p(B_1)$ if and only if

$$\left(\frac{N - 2 + 2\alpha - c}{2} + \sqrt{k}\right)^p < \frac{N}{p},$$

or equivalently,

$$\frac{N}{p} < s_1 + 2 - \alpha.$$  

(28)

**Proposition 4.9** If $\alpha < 2$ and $N/p \geq s_2 + 2 - \alpha$ or $\alpha > 2$ and $N/p \leq s_1 + 2 - \alpha$ then for every $\lambda > 0$, $N(\lambda - L_{\min}) \not\in \{0\}$. Therefore no extension of $L_{\min}$ generates a semigroup in $L^p(\mathbb{R}^N)$.

**Proof.** As in the proof of Proposition 4.8, we focus on the case $\alpha < 2$ and we prove the existence of a radial function $u \in D_{\min}(L) \setminus \{0\}$ satisfying

$$u - Lu = 0.$$  

We write $u = c_1 u_1 + c_2 u_2$, where $u_j$ is defined in Lemma 2.1 for $j = 1, 2$. The integrability of $u$ near $\infty$ implies that $u = c_2 u_2$ with $c_2 \neq 0$.

We prove that $u_2 \in D_{\min}(L)$. We first assume that $s_2 < N/p + \alpha - 2$. In this case from (14) we have $u_1, \{|x|^{\alpha - 2} u_1\} \in L^p(\mathbb{R}^N)$. By Lemma 2.6 we obtain $u_2 \in D_{\min}(L)$.

Next, we assume that $s_2 = N/p + \alpha - 2$. Let $\varepsilon > 0$ with $\alpha + \varepsilon < 2$. Then using (14), we have

$$\|x|^{\alpha - 2 + \varepsilon} u_2\| = \int_{B_1} |x|^{(\alpha - 2)\varepsilon} |x|^{(\alpha - 2 + \varepsilon)\varepsilon} u_2| dx + \int_{R^N \setminus B_1} |x|^{(\alpha - 2 + \varepsilon)\varepsilon} u_2| dx$$

$$\leq C_1 \int_{B_1} |x|^{(\alpha - 2 + \varepsilon)\varepsilon} u_2| dx + C_2 \int_{R^N \setminus B_1} \exp\left\{-C_3 |x|^{2-\alpha} \right\} dx$$

$$\leq C_1 \omega_N \int_0^1 r^{p-1} dr + C_2 \leq \frac{C_1 \omega_N}{\varepsilon p} + C_2.$$  

14
We apply Lemma 2.3 to \( |x|^\alpha L \) (with \( \alpha + \varepsilon \) instead of \( \alpha \)) to deduce that \( u_2 \in D_{p, \alpha + \varepsilon} \). Moreover the interpolation inequality (10) yields
\[
\| |x|^{\alpha - 1 + \varepsilon} \nabla u_2 \| \leq C' \left( \||x|^\alpha L u_2 \| + \||x|^{\alpha - 2 + \varepsilon} u_2 \| \right) \leq C'' \left( \||x|^\alpha u_2 \| + \||x|^{\alpha - 2 + \varepsilon} u_2 \| \right) \leq C'' \left( 1 + \varepsilon^{-\frac{1}{p}} \right).
\]
Hence we have
\[
L (|x|^\alpha u_2) = |x|^\alpha u_2 + 2 \varepsilon |x|^{\alpha - 2 + \varepsilon} x \cdot \nabla u_2 + \varepsilon (N - 2 + c + \varepsilon) |x|^{\alpha - 2} u_2 \in L^p (\mathbb{R}^N).
\]
This implies that \( |x|^\alpha u_2 \in D_{\max}(L) \cap D(|x|^{\alpha - 2}) \subset |x|^\alpha u_2 \in D_{\min}(L) \), by Lemma 2.3. Moreover, by the above estimates
\[
\| L_{\min} (|x|^\alpha u_2) - u_2 \| \leq C'' \left( \varepsilon + \varepsilon^{-\frac{1}{p}} \right).
\]
The closedness of \( L_{\min} \) yields \( u_2 \in D_{\min}(L) \) and \( L_{\min} u_2 = u_2 \).

4.4 Negative results for \( L_{\min} \subset L \subset L_{\max} \)

Since \( L_{\max} = (L_{\min})^* \), from Propositions 1.8, 1.9 we obtain the following result.

**Proposition 4.10** (i) If \( \alpha < 2 \) and \( N/p \leq s_1 \) or \( \alpha > 2 \) and \( N/p \geq s_2 \), then for every \( \lambda > 0 \), \( R(\lambda - L) \neq L^p (\mathbb{R}^N) \). Therefore no restriction of \( L_{\max} \) generates a semigroup in \( L^p (\mathbb{R}^N) \).

(ii) If \( \alpha < 2 \) and \( N/p > s_2 \) or \( \alpha > 2 \) and \( N/p < s_1 \), then for every \( \lambda > 0 \), \( R(\lambda - L_{\max}) \neq L^p (\mathbb{R}^N) \). Therefore \( L_{\max} \) does not generate a semigroup in \( L^p (\mathbb{R}^N) \).

**Proof.** We prove (i) and assume \( \alpha < 2 \). By Proposition 1.9, \( N(\lambda - L_{\min}) \neq 0 \) in \( L^p (\mathbb{R}^N) \), that is \( R(\lambda - L_{\max}) \neq L^p (\mathbb{R}^N) \), if \( N/p' \geq s_2 + 2 - \alpha \). Here \( s_i \) are the roots of the function \( f \) defined in 5) and relative to \( L \). Since \( s_i = s_i + \alpha - c \) the above condition reads \( N/p \leq N - c - 2 - s_2 = s_1 \). The case \( \alpha > 2 \) is similar. The proof of (ii) follows similarly from Proposition 1.8.

Finally we state the following negative result for any realization \( (L, D) \) such that \( D_{\min}(L) \subset D_{\max}(L) \), \( L_{\min} \subset L \subset L_{\max} \), in short. This proves the "only if" part in Theorem 1.1.

**Theorem 4.11** Let \( \alpha \neq 2 \) and assume that \( N/p \leq s_1 + \min \{ 0, 2 - \alpha \} \) or \( N/p \geq s_2 + \max \{ 0, 2 - \alpha \} \). Then no realization of the operator \( L \) between \( L_{\min} \) and \( L_{\max} \) generates a semigroup in \( L^p (\mathbb{R}^N) \).

**Proof.** This follows immediately from Propositions 1.8, 1.9 (i). In fact, if \( \alpha < 2 \) and \( N/p \leq s_1 \) no restriction of \( \lambda - L_{\max} \) can be surjective whereas if \( N/p \geq s_2 + 2 - \alpha \) no extension of \( \lambda - L_{\min} \) can be injective.

5 The case \( b + (\frac{N-2+\varepsilon}{2})^2 > 0 \): \( L_{\min} \subset L \subset L_{\max} \)

We always assume \( \alpha \neq 2 \) and show that a suitable realization of \( L_{\min} \subset L \subset L_{\max} \) generates a semigroup in \( L^p (\mathbb{R}^N) \) if and only if
\[
s_1 + \min \{ 0, 2 - \alpha \} < \frac{N}{p} < s_2 + \max \{ 0, 2 - \alpha \} \tag{29}
\]
To explain the meaning of the above condition let us fix \( \alpha < 2 \). By the results of the previous section \( L_{\max} \) generates if \( s_1 < N/p < s_2 \) and \( L_{\min} \) when \( s_1 + 2 - \alpha < N/p < s_2 + 2 - \alpha \).

15
and \( L_{\min} = L_{\max} \) if both conditions are satisfied. Therefore we have generation under (29) if \( s_2 < s_1 + 2 - \alpha \). However this last condition is not always verified: this is the case when when \( \alpha \) is very negative but also for \( N = 3, 4 \) and \( \alpha = b = c = 0 \): as already pointed out in the Introduction \( \Delta_{\min} \) generates for \( p \leq N/2 \) and \( \Delta_{\max} \) for \( p \geq N/(N - 2) \). We also remark that under the condition \( s_1 + (2 - \alpha)/p \leq N/p \leq s_2 + (2 - \alpha)/p \), see Proposition 4.2 \( L \) is dissipative in the annulus \( B_{\varepsilon - 1} \setminus B_{\varepsilon} \) when endowed with Dirichlet boundary conditions. A semigroup can therefore be constructed via approximation as in Step 1 of the proof of Theorem 4.4. We do not follow this approach since it does not cover all cases considered in (29).

5.1 The operator \( L_{\text{int}} \)

We define an intermediate operator \( L_{\text{int}} \) between \( L_{\min} \) and \( L_{\max} \) as

\[
D_{\text{int}}(L) = \{ u \in L_{\max}(L) \subset L^p(\mathbb{R}^N) : |x|^\theta(\alpha - 2) u \in L^p(\mathbb{R}^N) \text{ for every } \theta \in I \},
\]

where \( I \) is the interval of all \( \theta \in [0, 1] \) such that

\[
f (\frac{N}{p} + \theta(\alpha - 2)) > 0.
\]

Note that (29) is equivalent to the existence of some \( \theta \in [0, 1] \) satisfying (31). First we show the injectivity of \( \lambda - L_{\text{int}} \) for \( \text{Re} \lambda > 0 \).

**Lemma 5.1** For every \( \lambda \) such that \( \text{Re} \lambda > 0 \), \( \lambda - L_{\text{int}} \) is injective.

**Proof.** We fix \( \theta \in I \), \( \text{Re} \lambda > 0 \) and suppose that \( u \in D_{\text{int}}(L) \) satisfies \((\lambda u - L_{\text{int}})u = 0 \). Set

\[
A = |x|^{\theta(\alpha - 2)} L |x|^{-\theta(\alpha - 2)} = |x|^\alpha \Delta + c_A |x|^\alpha - 2 \cdot \nabla - b_A |x|^\alpha - 2,
\]

where \( c_A = c - 2\theta(\alpha - 2) \) and \( b_A := b + \theta(\alpha - 2)(N - 2 + c - \theta(\alpha - 2)) \). Then

\[
(\lambda - A)|x|^{\theta(\alpha - 2)} u = |x|^{\theta(\alpha - 2)}(\lambda - L)u = 0.
\]

Setting \( v := |x|^{\theta(\alpha - 2)} u \in L^p(\mathbb{R}^N) \), we have \( v \in D(A_{\max}) \) and \((\lambda - A_{\max})v = 0 \). On the other hand, \( A \) satisfies the hypothesis of Theorem 4.4.

\[
\begin{align*}
\frac{N}{p} + \theta(\alpha - 2) &= b + \theta(\alpha - 2)(N - 2 + c - \theta(\alpha - 2)) + \frac{N}{p} \left( \frac{N}{p'} - 2 + c - 2\theta(\alpha - 2) \right) \\
&= b + \left( \frac{N}{p} + \theta(\alpha - 2) \right) \left( \frac{N}{p'} - 2 + c - \theta(\alpha - 2) \right) > f \left( \frac{N}{p} + \theta(\alpha - 2) \right) > 0.
\end{align*}
\]

Then \( A_{\max} \) generates a bounded analytic semigroup on \( L^p(\mathbb{R}^N) \) and, in particular, \( \lambda \in \rho(A_{\max}) \). Hence \( v = (\lambda - A_{\max})^{-1}(\lambda - A_{\max})v = 0 \) and, by the definition of \( v \), \( u = 0 \), too.

We approximate \( L_{\text{int}} \) through the operators

\[
\begin{align*}
L_t &:= |x|^{\alpha \Delta} + c|x|^{\alpha - 2} \cdot \nabla - (b + k)|x|^{\alpha - 2} + k \min\{t, |x|^{\alpha - 2}\}, \\
D(L_t) &:= D_{p,\alpha}
\end{align*}
\]

where \( t > 0 \), \( D_{p,\alpha} \) is defined in (11) and \( k \) is a large fixed nonnegative constant for which the conditions of Proposition 4.2 and Theorems 4.4, 4.5 are satisfied for every \( p > 1 \). Observe that, in particular \( D_{p,\alpha} = D_{\min}(L) = D_{\max}(L) \). Then we have
Lemma 5.2 For every $1 < p < \infty$, $L_t$ generates an analytic semigroup of positive operators in $L^p(\mathbb{R}^N)$, coherent with respect to $p$. Moreover $(kt, \infty) \subset \rho(L_t)$ and $C_c^\infty(\Omega)$ is a core for $L_t$.

Proof. Because of the assumption on $k$ we see from Theorem [1.4] that $L - k|x|^{\alpha - 2}$ with domain $D_p(L)$ generates an analytic semigroup of positive operators in $L^p(\mathbb{R}^N)$ for every $1 < p < \infty$, coherent with respect to $p$. Since $k \min\{t, |x|^{\alpha - 2}\}$ is bounded the same is true for $L_t$ and moreover $(kt, \infty) \subset \rho(L_t)$.

We show weighted and unweighted resolvent estimates for $L_t$ with constants independent of $t$.

Lemma 5.3 Let $\theta$ satisfy (31). Then there exist constants $C, C' > 0$ such that for every $\lambda \in \mathbb{C}_+$, $t > 0$ and $u \in C_c^\infty(\Omega)$,

$$
\|u\|_p \leq \frac{C}{|\lambda|} \|\lambda u - L_t u\|_p. 
$$

(32)

$$
\|u\|_p \leq \frac{C'}{|\lambda| - \sigma} \|\lambda u - L_t u\|_p. 
$$

(33)

Therefore $\mathbb{C}_+ \subset \rho(L_t)$ and $\|\lambda - L_t\|^{-1} \leq C|\lambda|^{-1}$.

Proof. First we prove (33) when $\theta \in [\frac{1}{p}, 1]$. We observe that the assumptions of Proposition [4.2] are satisfied if we replace $\alpha$ with $\beta = \alpha + (p\theta - 1)(\alpha - 2)$, hence $\beta - 2 = p\theta(\alpha - 2)$. Therefore we consider the operator $|x|^{(p\theta - 1)(\alpha - 2)}L_t$. For $u \in C_c^\infty(\Omega)$ we have

$$
\text{Re} \left[ e^{i\omega} \int_{\mathbb{R}^N} (-Lu) |x|^{(p\theta - 1)(\alpha - 2)} |u|^{p - 2} dx \right] \geq 0, \quad \omega \in \left[-\frac{\pi}{2} + \omega_1, \frac{\pi}{2} - \omega_1\right],
$$

where $\pi/2 - \omega_1 > 0$ is the angle of sectoriality of $|x|^{(p\theta - 1)(\alpha - 2)}L_t$. Since $L_t = L_t - V_t$ with $V_t \geq 0$, the same inequality holds for $L_t$, thus for $\text{Re} \lambda > 0$

$$
\text{Re}(\lambda e^{i\omega}) \int_{\mathbb{R}^N} |x|^{(p\theta - 1)(\alpha - 2)} |u|^{p} dx \leq \text{Re} \left[ e^{i\omega} \int_{\mathbb{R}^N} (\lambda - L_t u) |x|^{(p\theta - 1)(\alpha - 2)} |u|^{p - 2} dx \right] 
$$

(34)

and, by choosing $\omega \in [-\frac{\pi}{2} + \omega_1, \frac{\pi}{2} - \omega_1]$ such that $\text{Re}(\lambda e^{i\omega}) = |\lambda| \cos \omega_1$, H"{o}lder inequality yields

$$
|\lambda| \cos \omega_1 \left\| |x|^{(\frac{\theta - 1}{p})(\alpha - 2)} u \right\|_p^p \leq \left\| |x|^{\frac{p\theta - 1}{p - 1} \cdot (\alpha - 2)} u \right\|_p^{p - 1} \left\| \lambda u - L_t u \right\|_p. 
$$

(35)

Noting that

$$
0 \leq \theta - \frac{1}{p} < \frac{p\theta - 1}{p - 1} \leq \theta,
$$

we apply Lemma [2.7] with respect to the measure $|u|^p dx$ to get

$$
\left\| |x|^{\frac{p\theta - 1}{p - 1} \cdot (\alpha - 2)} u \right\|_p \leq \left\| |x|^{(\frac{\theta - 1}{p})(\alpha - 2)} u \right\|_p^\theta \left\| \left| |x|^{(\theta - \frac{1}{p})(\alpha - 2)} u \right|_p \right\|_p^{1 - \frac{p\theta - 1}{p - 1}} 
$$

(36)

and

$$
|\lambda| \cos \omega_1 \left\| |x|^{(\frac{\theta - 1}{p})(\alpha - 2)} u \right\|_p^p \leq \left\| |x|^{(\theta - \frac{1}{p})(\alpha - 2)} u \right\|_p^\theta \left\| \lambda u - L_t u \right\|_p. 
$$

(37)
On the other hand, (39) applied again to \(|x|^{(pθ-1)(α-2)} L\) implies that

\[
M \left\| x^{(θ(α-2))} u \right\|^p_p \leq \text{Re} \int_{\mathbb{R}^N} (λu - Lu)|x|^{(pθ-1)(α-2)}|u|^{p-2} dx
\]

(38)

\[
\leq \text{Re} \int_{\mathbb{R}^N} (λu - L_t u)|x|^{(pθ-1)(α-2)}|u|^{p-2} dx
\]

\[
\leq \left\| \left[ x^{(θ(α-2))} u \right] \left( p^{-1}_p \right) \right\| \| λu - L_t u \|_p.
\]

where \(M = f(\frac{N}{p} + θ(α - 2)) > 0\). Combining (36) and (37) with the above estimate, we have

\[
\left\| x^{(θ(α-2))} u \right\|^p_p \leq \frac{1}{M} \left\| x^{(θ(α-2))} u \right\|^{pθ-1}_p \left\| x^{(θ-\frac{1}{p})(α-2)} u \right\|^{p-pθ}_p \| λu - L_t u \|_p
\]

\[
\leq \frac{1}{M} \left( \frac{1}{|λ| \cos ω_1} \right)^{\frac{1-θ}{p}} \left\| x^{(θ(α-2))} u \right\|^{pθ-1}_p \| λu - L_t u \|_p.
\]

Therefore we obtain

\[
\left\| x^{(θ(α-2))} u \right\|^p_p \leq \frac{1}{Mθ} \left( \frac{1}{|λ| \cos ω_1} \right)^{1-θ} \| λu - L_t u \|_p.
\]

(39)

Next we prove (32). From Proposition 4.2, we have

\[
\text{Re} \left[ e^{iω} \int_{\mathbb{R}^N} (-Lu + k|x|^{α-2} u)|u|^{p-2} dx \right] \geq 0, \quad ω \in [-\frac{π}{2} + ω_2, \frac{π}{2} - ω_2]
\]

where \(π/2 - ω_2\) is the angle of sectoriality of \(L - k|x|^{α-2}\). Since \(L_t = L - V_t\) with \(V_t \geq 0\), the same inequality holds for \(L_t\), thus for \(Re λ > 0\) arguing as for (34)

\[
|λ| \cos ω_2 ||u||^p_p \leq k \left\| x^{(θ(α-2))} u \right\|^p_p + ||u||^{p-1}_p \| λu - L_t u \|_p.
\]

Since \(pθ \geq 1\) we may apply Hölder inequality to obtain the estimate

\[
\left\| x^{(θ(α-2))} u \right\|^p_p \leq \left\| u \right\|^{1-\frac{θ}{p}}_p \left\| x^{(θ(α-2))} u \right\|^{\frac{θ}{p}}_p.
\]

Then we have

\[
\left\| u \right\|^{\frac{1}{p}}_p \leq \frac{k}{|λ| \cos ω_2} \left\| x^{(θ(α-2))} u \right\|^{\frac{1}{p}}_p + \frac{1}{|λ| \cos ω_2} \left\| u \right\|^{\frac{1-θ}{p}}_p \| λu - L_t u \|_p
\]

\[
\leq \frac{k}{|λ| \cos ω_2} \left\| x^{(θ(α-2))} u \right\|^{\frac{1}{p}}_p + (1 - θ) \left\| u \right\|^{\frac{1}{p}}_p + θ \left( \frac{1}{|λ| \cos ω_2} \| λu - L_t u \|_p \right)^{\frac{1}{θ}}.
\]

and hence using (39), we have

\[
\left\| u \right\|_p \leq \left[ \frac{k}{θ \cos ω_2} \frac{1}{M} \left( \frac{1}{\cos ω_1} \right)^{\frac{1-θ}{p}} + \left( \frac{1}{\cos ω_2} \right)^\frac{1}{θ} \frac{1}{|λ|} \right] \| λu - Lu \|_p.
\]

This a-priori estimate implies that \(C_+ \subset ρ(L_t)\) and that (32), (33) hold for every \(λ \in C_+\).
To deal with the case $\theta \in (0, \frac{1}{p})$ we consider the adjoint operator in $L^p(\mathbb{R}^N)$

$$(L_t)^* v = |x|^\alpha \Delta + \tilde{c}|x|^{\alpha-2}x \cdot \nabla - (\tilde{b} + k)|x|^{\alpha-2} + k \min\{t, |x|^{\alpha-2}\},$$

where $\tilde{c} = 2\alpha - c$ and $\tilde{b} = b + (c - \alpha)(N + \alpha - 2)$, see (33) and (36). Then, taking $\tilde{\theta} := 1 - \theta \in (\frac{1}{p}, 1)$, we see from (33) that

$$\tilde{f} \left( \frac{N}{p} + \tilde{\theta}(\alpha - 2) \right) = f \left( \frac{N}{p} + \theta(\alpha - 2) \right) > 0.$$ 

Thus applying (32) to $L_t^*$, we get

$$\left\| \langle \lambda - L_t^* \rangle^{-1} \right\| \leq \frac{C}{|\lambda|^\kappa}, \quad \lambda \in \mathbb{C}_+.$$ 

By duality we have (32) for $L_t$. Finally, let $\chi \in C^\infty(\mathbb{R}^N)$ satisfy

$$\begin{cases} 
\chi \equiv 1 \text{ in } B_1 & \text{ and } \chi \equiv 0 \text{ and } B_2^c \quad \text{ if } \alpha < 2, \\
\chi \equiv 0 \text{ in } B_{1/2} \text{ and } \chi \equiv 1 \text{ and } B_1^c \quad \text{ if } \alpha > 2.
\end{cases}$$ 

Then noting that $p\theta < 1$, we obtain from (38)

$$M \int_{\mathbb{R}^N} |x|^{p\theta(\alpha-2)}|\chi u|^p \, dx \leq \text{Re} \int_{\mathbb{R}^N} (\lambda \chi u - L_t(\chi u))|x|^{p\theta-1}(\alpha-2)\chi|u|^{p-2} \, dx$$

$$\leq \left\| |x|^{p\theta-1}(\alpha-2)\chi u \right\|^{p-1} \|\lambda \chi u - L_t(\chi u)\|$$

$$\leq 2^{\frac{1}{1-p\theta}(\alpha-2)} \|u\|^{p-1} \|\lambda u - L_t u\| + C_1 \|u\|_p + C_2 \|\nabla u\|_{L^p(\text{supp } \chi)}.$$ 

We note that $\text{supp } \nabla \chi \subset B_2 \setminus B_{1/2}$, that first and second order coefficients of $L_t$ are independent of $t$, and that the zero-order coefficients of $L_t$ are uniformly bounded with respect to $t$ in the annulus $D_4 = B_4 \setminus B_1$.

Therefore the interior gradient estimates

$$\|\nabla u\|_{L^p(B_2 \setminus B_{1/2})} \leq C_5(\|L_t u\|_{L^p(D_4)} + \|u\|_{L^p(D_4)}) \leq C_5(\|\lambda u - L_t u\|_p + (1 + |\lambda|)\|u\|_p)$$

hold with $C$ independent of $t > 0$. Using these estimates, (38) and (32) we obtain for $\lambda \in \mathbb{C}_+$, $|\lambda| \geq 1$

$$\left\| |x|^{\theta(\alpha-2)}u \right\|_{L^p(B_1)} \leq C_4 \|\lambda u - L_t u\|_p.$$ 

if $\alpha < 2$,

$$\left\| |x|^{\theta(\alpha-2)}u \right\|_{L^p(B_1^c)} \leq C_4 \|\lambda u - L_t u\|_p.$$ 

if $\alpha > 2$

with $C_4$ independent of $\lambda, t$. Combining the above estimate with (32) we obtain

$$\left\| |x|^{\theta(\alpha-2)} u \right\|_p \leq C_5 \|\lambda u - L_t u\|_p$$

(41)

for $\lambda \in \mathbb{C}_+$, $|\lambda| \geq 1$ and $C_5$ independent of $\lambda, t$. Finally, applying (11) with $\lambda = e^{i\omega}$ to $u(x) = v(sx)$ we get

$$\left\| |x|^{\theta(\alpha-2)} u \right\|_p \leq C_5 s^{(1-\theta)(2-\alpha)} \|s^{\alpha-2}e^{i\omega} u - L_{ts^{\alpha-2}} u \|_p$$

or, with $\eta = s^{\alpha-2}e^{i\omega}$ and $\tau = ts^{\alpha-2}$,

$$\left\| |x|^{\theta(\alpha-2)} (\eta - L_\tau)^{-1} \right\| \leq \frac{C_5}{|\eta|^{1-\theta}}.$$ 

□

19
We are now in a position to state and prove the main result of this section, that is the "if" part of Theorem 1.1. We recall that \( I \) is the interval of all \( \theta \in [0,1] \) such that (31) is satisfied. For every \( \theta \in I \) we set \( \alpha' = \alpha'(\theta) = \theta(\alpha - 2) + 2 \) and define

\[
D_{\text{reg}}(L) = \begin{cases} 
\{ u \in D_{\max}(L) : |x|^\alpha D^2 u, |x|^{\alpha'-1} \nabla u, |x|^{\alpha'-2} u \in L^p(B) \} & \text{for every } \theta \in I \\
|x|^\alpha D^2 u, |x|^{\alpha-1} \nabla u \in L^p(B^c) & \text{if } \alpha < 2; \\
\{ u \in D_{\max}(L) : |x|^\alpha D^2 u, |x|^{\alpha-1} \nabla u, |x|^{\alpha'-2} u \in L^p(B^c) \} & \text{for every } \theta \in I \\
|x|^\alpha D^2 u, |x|^{\alpha-1} \nabla u \in L^p(B) & \text{if } \alpha > 2.
\end{cases}
\]

where \( B = B_1 \). Note that the maximum of regularity is achieved when 1 \( \in I \), that is when Theorem 1.4 applies.

**Theorem 5.4** If (29) is satisfied, then \( L_{\text{int}} \) generates a positive analytic semigroup in \( L^p(\mathbb{R}^N) \) which is coherent with respect to all \( p \) satisfying (29). Moreover, \( D_{\text{int}}(L) \) defined in (30) coincides with \( D_{\text{reg}}(L) \) defined above.

**Proof.** Fix \( \lambda \) with \( \Re \lambda > 0 \) and recall that, by Lemma 5.2 \( \lambda - L_{\text{int}} \) is injective. To show the surjectivity we fix \( f \in L^p(\mathbb{R}^N) \) and define \( u_n = (\lambda - L_n)^{-1} f \in D_{\text{reg}} \). By Lemma 5.3 \( |\lambda||u_n||_p \leq C||f||_p \) and \( |\lambda|^{\theta-1}||x|^{\theta(\alpha - 2)}u_n||_p \leq C||f||_p \) with \( C \) independent of \( \lambda, n \). Note that the operators \( L_n \) differ only for the zero-order coefficients which are uniformly bounded on every compact subset of \( \Omega \). By local elliptic regularity, the sequence \( (u_n) \) is therefore bounded in \( W^{2,p}_{\text{loc}}(\Omega) \) and, passing a subsequence, we may assume that \( (u_n) \to u \) weakly in \( W^{2,p}_{\text{loc}}(\Omega) \), strongly in \( L^p_{\text{loc}}(\Omega) \) and pointwise.

Then \( \lambda u - Lu = f \) and \( |\lambda||u||_p \leq C||f||_p \). Moreover \( |\lambda|^{\theta-1}||x|^\theta(\alpha-2)u||_p \leq C||f||_p \), hence \( u \in D_{\text{int}}(L) \). Note that the injectivity of \( \lambda - L_{\text{int}} \) actually implies that the whole sequence \( (u_n) \) converges to \( u \), that is \( (\lambda - L_n)^{-1} f \to (\lambda - L_{\text{int}})^{-1} f \). If \( \lambda > 0, f \geq 0 \), then \( u_n \geq 0 \) by Lemma 6.2 hence \( u \geq 0 \). Moreover, if \( f \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) \) the solution \( u \) is independent of \( p, q \). Since \( \alpha < 2 \), so are the \( u_n \), by Lemma 6.2 again.

Finally we prove the equality \( D_{\text{int}}(L) = D_{\text{reg}}(L) \) and focus, as usual, on the case \( \alpha < 2 \), the other being similar. The inclusion \( D_{\text{reg}}(L) \subset D_{\text{int}}(L) \) is obvious. Let now \( u \in D_{\text{int}}(L) \) and write \( u = u_1 + u_2 \) where \( u_1 = u \phi \), \( u_2 = u(1 - \phi) \) and \( \phi \in C_0^\infty(\mathbb{R}^N) \) with support in \( B_2 \) and equal to 1 in \( B_1 \). We introduce the operator \( L_2 \) on \( \mathbb{R}^N \) in this way: the coefficients of \( L_2 \) coincide with those of \( L \) in \( B_1 \) whereas in \( B_1 \) they take the (constant) value that they have on \( \partial B_1 \). \( L_2 \) is therefore uniformly elliptic with Lipschitz coefficients in \( B_1 \) and satisfies Hypothesis 2.1 of [27]. By construction the function \( u_2 \) belongs to the maximal domain of \( L_2 \) and, by Proposition 2.9), \( |x|^\alpha D^2 u_2, |x|^{\alpha'-1} \nabla u_2 \in L^p(B^c) \), that is \( |x|^\alpha D^2 u, |x|^{\alpha'-1} \nabla u \in L^p(B^c) \). To treat \( u_1 \) we consider the operator \( L_1 = |x|^\alpha - \alpha L \). Since \( \alpha < 2 \) then \( \alpha' \geq \alpha \) and then \( u_1 \in D_{\text{max}}(L_1) \) and, by the definition of \( L_{\text{int}} \), \( |x|^{\alpha'-2} u_1 \in L^p(\mathbb{R}^N) \). By Lemma 2.3 \( u_1 \in D_{p,\alpha'} \). It follows that \( |x|^\alpha D^2 u_1, |x|^{\alpha'-1} \nabla u_1, |x|^{\alpha'-2} u_1 \in L^p(B) \), hence the same holds for \( u \).

We observe that \( L_{\text{int}} = L_{\text{min}} \) if the conditions of Theorem 1.3 are satisfied and \( L_{\text{int}} = L_{\text{max}} \) if the conditions of Theorem 4.6 hold. In both cases the equality \( D_{\text{int}}(L) = D_{\text{reg}}(L) \) yields a better description of \( D_{\text{min}}(L) \) and \( D_{\text{max}}(L) \), respectively.

**5.2 Some consequences**

In the next proposition we show that \( L_{\text{min}} \) is a generator when \( N/p \) coincides with one of the endpoints of the interval \( (s_1 + 2 - \alpha, s_2 + 2 - \alpha) \) of Theorem 1.4.



20
Proposition 5.5 Let $\alpha < 2$ and $N/p = s_1 + 2 - \alpha$ or $\alpha > 2$ and $N/p = s_2 + 2 - \alpha$. Then $L_{\text{int}}$ coincides with $L_{\min}$.

Proof. We only treat the case $\alpha < 2$ and we first show that $Rg(I - L_{\min}) = L^p(\mathbb{R}^N)$. Let $v \in L^p(\mathbb{R}^N)$ and suppose that for every $\varphi \in C_c^\infty(\Omega)$,

$$\int_{\mathbb{R}^N} \nabla(\varphi - L\varphi) \, dx = 0. \quad (43)$$

Fix $\varepsilon > 0$. Since $b + \varepsilon + (N/p + \alpha - 2)(N/p' - \alpha + c) = \varepsilon$ it follows from Theorem 4.2 that the minimal realization of $L - \varepsilon|x|^{\alpha - 2}$ generates analytic semigroup in $L^p(\mathbb{R}^N)$ and its domain is $D_{p,\alpha}$. Hence (43) holds for in $D_{p,\alpha}$ and we deduce from the invertibility of $I - L + \varepsilon|x|^{\alpha - 2}$ that for every $f \in L^p(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \nabla f \, dx = \int_{\mathbb{R}^N} \nabla (\varepsilon|x|^{\alpha - 2}(I - L + \varepsilon|x|^{\alpha - 2})^{-1} f) \, dx. \quad (44)$$

Choosing $f = |v|^{p'}^2 v$ and setting $w_\varepsilon = (I - L + \varepsilon|x|^{\alpha - 2})^{-1} f$, we have

$$\int_{\mathbb{R}^N} |v|^{p'} \, dx = \int_{\mathbb{R}^N} \nabla (\varepsilon|x|^{\alpha - 2}w_\varepsilon) \, dx. \quad (44)$$

Observe that Lemma 4.5 implies that

$$\left\| |x|^{\alpha - 2}w_\varepsilon \right\|_p^p + \varepsilon \left\| |x|^{\alpha - 2}w_\varepsilon \right\|_p^p \leq \int_{\mathbb{R}^N} |x|^{(p-1)(\alpha - 2)}w_\varepsilon |x|^{\alpha - 2}w_\varepsilon \, dx \leq \|f\|_p \left\| |x|^{\alpha - 2}w_\varepsilon \right\|_p^{p-1}. \quad (44)$$

This yields

$$\left\| \varepsilon|x|^{\alpha - 2}w_\varepsilon \right\|_p \leq \|f\|_p \quad \text{and} \quad \left\| \varepsilon |x|^{\alpha - 2}w_\varepsilon \right\|_p^p \leq \|f\|_p^p. \quad (45)$$

Using the above estimates, we obtain

$$\varepsilon|x|^{\alpha - 2}w_\varepsilon \rightarrow 0 \quad \text{weakly in } L^p(\mathbb{R}^N). \quad (45)$$

In fact, for every $\phi \in C_c^\infty(\Omega)$, we have

$$\left\| \int_{\mathbb{R}^N} (\varepsilon|x|^{\alpha - 2}w_\varepsilon) \phi \, dx \right\|_p \leq \|f\|_p \left\| |x|^{\alpha - 2} \phi \right\|_p \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (45)$$

as $\{\varepsilon|x|^{\alpha - 2}w_\varepsilon\}$ is bounded in $L^p(\mathbb{R}^N)$, a density argument implies (45). Consequently, combining (44), with (45), we obtain $v = 0$. This means that $Rg(I - L_{\min}) = L^p(\mathbb{R}^N)$.

Finally, we prove $L_{\text{int}} = L_{\min}$. The inclusion $L_{\text{int}} \supset L_{\min}$ is obvious. Conversely, let $u \in D_{\text{int}}(L)$. Since $\overline{Rg(I - L_{\min})} = L^p(\mathbb{R}^N)$, we can choose $u_n \in D_{\text{min}}(L) \subset D_{\text{int}}(L)$ such that $(I - L_{\min})u_n \rightarrow (I - L_{\text{int}})u$. Since $I - L_{\text{int}}$ is invertible we have

$$\|u_n - u\|_p \leq C\|(I - L_{\text{int}})(u_n - u)\|_p \rightarrow 0$$

and the closedness of $L_{\min}$ implies $u \in D(L_{\min})$. \qed
Remark 5.6 The equality $\text{Rg}(I - L_{\min}) = L^p(\mathbb{R}^N)$ is true even when $\alpha < 2$ and $N/p = s_2 + 2 - \alpha$ or $\alpha > 2$ and $N/p = s_1 + 2 - \alpha$, by the same proof as above. However, in these cases, the injectivity of $I - L_{\min}$ breaks down, see Proposition 4.5.

By duality one obtains a similar result for $L_{\max}$, see Remark 4.7.

Proposition 5.7 Let $\alpha < 2$ and $N/p = s_2$ or $\alpha > 2$ and $N/p = s_1 + 2 - \alpha$. Then $L_{\text{int}}$ coincides with $L_{\max}$.

We end this section with some remarks on $L_{\text{int}}$. We fix $\theta \in I$, that is satisfying (31), and define $L_{\theta}$ through the domain

$$D_{\theta}(L) = \{u \in D_{\max}(L) \subset L^p(\mathbb{R}^N) ; |x|^{\theta(\alpha - 2)}u \in L^p(\mathbb{R}^N)\}.$$ 

Clearly $L_{\text{int}} \subset L_{\theta}$. However, since $I - L_{\text{int}}$ is invertible and $I - L_{\theta}$ is injective, by Lemma 5.1 (whose proof works for any fixed $\theta$), then both operators coincide and $L_{\text{int}} = L_{\theta}$. This means that the extra integrability condition $|x|^{\theta(\alpha - 2)}u \in L^p(\mathbb{R}^N)$, $u \in D_{\max}(L)$, for a fixed $\theta \in I$ extends automatically to every $\theta \in I$.

In the next proposition we show that, unless $1 \in I$, this integrability condition does not hold for $\theta_0 = \sup I$. Note that $1 \in I$ is equivalent to say that Theorem 4.3 applies and is more restrictive than requiring that $L_{\min}$ generates. Note also that $\theta_0$ can be equal to 1 even though $1 \not\in I$.

Proposition 5.8 Assume that (29) holds and that $1 \not\in I$. Set $\theta_0 = \sup I$ and $\alpha_0 = \theta_0(\alpha - 2) + 2$. Then there exists $u \in D_{\text{int}}(L)$ such that $|x|^\alpha u / \notin L^p(\mathbb{R}^N)$.

Proof. We give a proof only $\alpha < 2$. In this case (29) reads $s_1 < N/p < s_2 + 2 - \alpha$. Since $1 \not\in I$, then $f(N/p + \theta_0(\alpha - 2)) = 0$ and then $s_1 = N/p + \theta_0(\alpha - 2)$. We set $u(x) = |x|^{-s_1} \zeta(x)$, where $\zeta \in C^\infty_c(\mathbb{R}^N)$ is one in the unit ball $B_1$ and zero outside the ball $B_2$. Then for every $\theta \in (0, \theta_0)$, we have

$$|x|^{\theta(\alpha - 2)}u = |x|^{\theta(\alpha - 2) - s_1} \zeta(x) = |x|^{-\frac{\alpha}{2} + (\theta_0 - \theta)(2 - \alpha)} \zeta(x) \in L^p(\mathbb{R}^N).$$

Since $L|x|^{-s_1} = 0$ then $Lu \in C^\infty_c(\Omega)$ and therefore $u \in D_{\text{int}}(L)$. However $|x|^{\alpha_0 - 2}u \notin L^p(\mathbb{R}^N)$.

Finally let us show that for $\lambda > 0$, $f \geq 0$, $(\lambda - L_{\text{int}})^{-1}f$ is the minimal among the positive solutions $u \in D_{\max}(L)$ of the equation $Lu - Lu = f$. This characterizes the generated semigroup as the minimal one and is important when $L_{\text{int}}$ differs both from $L_{\min}$ and $L_{\max}$. First prove a maximum principle for the operator $L$ restricted to the annulus $\Omega_\varepsilon$. Note that the classical maximum principle does not hold when $\theta_0 < 0$.

Lemma 5.9 Let $\lambda > 0$, $g \leq 0$ and let $u \in W^{2,p}(\Omega_\varepsilon)$ solve $\lambda u - Lu = g$ in $\Omega_\varepsilon$ with $u \leq 0$ at the boundary. Then $u \leq 0$ in $\Omega_\varepsilon$.

Proof. Let $\theta$ be such that $f \left(\frac{N}{p} + \theta(\alpha - 2)\right) > 0$. We multiply the equation $\lambda u - Lu = g$ by $|x|^{(p\theta - 1)(\alpha - 2)}(u^+)^{p-1}g$ and integrate over $\Omega_\varepsilon$. We proceed as in Proposition 4.2 with $\alpha$ replaced by $\beta = \alpha + (p\theta - 1)(\alpha - 2)$ and observe that, since $u \leq 0$ on the boundary, no boundary terms appear after integration by parts. Setting $v = |x|^\frac{N+2-\beta}{p-\beta} u$ we obtain the analogous of (24)

$$\lambda \int_{\Omega_\varepsilon} |x|^{(p\theta - 1)(\alpha - 2)}(u^+)p + (p - 1) \int_{\Omega_\varepsilon} |x|^{-N} |v u^+|^2(u^+)^{p-2} + f \left(\frac{N}{p} + \theta(\alpha - 2)\right) \int_{\Omega_\varepsilon} |x|^{-N}(u^+)^p$$

$$= \int_{\Omega_\varepsilon} |x|^{(p\theta - 1)(\alpha - 2)} g(u^+)^{p-1} \leq 0.$$ 

It follows that $\int_{\Omega_\varepsilon} |x|^{(p\theta - 1)(\alpha - 2)}(u^+)^p \leq 0$ and therefore $u^+ = 0$ in $\Omega_\varepsilon$. 

\[22\]
Proposition 5.10 Let \( \lambda > 0, f \geq 0 \) and let \( 0 \leq u \in D_{\text{max}}(L) \) satisfy \( \lambda u - Lu = f \). Then \( (\lambda - L_{\text{int}})^{-1} f \leq u \).

**Proof** Let \( u_\varepsilon \in W^{2,p}(\Omega_\varepsilon) \cap W^{1,p}_0(\Omega_\varepsilon) \) be such that \( \lambda u_\varepsilon - Lu_\varepsilon = f \) in \( \Omega_\varepsilon \). Then \( u_\varepsilon \geq 0, \lambda(u_\varepsilon - u) - L(u_\varepsilon - u) = 0 \) and \( u_\varepsilon - u \leq 0 \) at the boundary. By Lemma 5.10, \( u_\varepsilon \leq u \) in \( \Omega_\varepsilon \). If \( v = (\lambda - L_{\text{int}})^{-1} f \) the same argument shows that \( \varepsilon_1 \leq \varepsilon_2 \), then \( u_{\varepsilon_2} \leq u_{\varepsilon_1} \) in \( \Omega_{\varepsilon_2} \), by the above lemma again. Then \( (u_\varepsilon) \) converges pointwise as \( \varepsilon \to 0 \) to some \( u_0 \leq v \). By dominated convergence \( u_\varepsilon \to u_0 \) in \( L^p(\mathbb{R}^N) \) and, by elliptic interior estimates, also in \( W^{2,p}(\mathbb{R}^N) \). Then \( \lambda u_0 - Lu_0 = f \) and \( u_0 \in D_{\text{max}}(L) \). Since \( 0 \leq u_0 \leq v \), then \( u_0 \in D_{\text{int}}(L) \), hence \( u_0 = v \), by uniqueness. Since \( u_\varepsilon \leq u \), letting \( \varepsilon \to 0 \), it follows that \( v \leq u \).

\[ \square \]

6 The critical case: \( b + \left(\frac{N-2+c}{2}\right)^2 = 0 \)

We always assume \( \alpha \neq 2 \) and show that a suitable realization of \( L_{\text{min}} \subset L \subset L_{\text{max}} \) generates a semigroup in \( L^p(\mathbb{R}^N) \) if and only if

\[ s_1 + \min\{0, 2 - \alpha\} \leq \frac{N}{p} \leq s_2 + \max\{0, 2 - \alpha\}. \tag{46} \]

Note that the endpoints above are included, whereas they are excluded in \([29]\). In this case the function \( f \) defined in \([3]\) is negative except for \( s = s_0 = (N - 2 + c)/2 \) where it vanishes and both \( s_1 \) and \( s_2 \) coincide with \( s_0 \). We first consider the case \( \alpha < 2 \) and we give full proofs following the method of Section 5, but adding logarithmic weights in the resolvent estimates. Moreover, we consider the operator first in the unit ball \( B_1 \) and then we use a gluing procedure to treat the case of the whole space. The case \( \alpha > 2 \) will be shortly considered in Subsection 6.2.

6.1 Positive results for \( \alpha < 2 \)

We always assume \( \alpha < 2 \) in this subsection and fix \( \theta_0 \in [0, 1] \) such that \( \frac{N}{p} = s_0 + \theta_0(2 - \alpha) \), \( s_0 = \frac{N - 2 + c}{2} \).

**Theorem 6.1** Assume that

\[ s_0 \leq \frac{N}{p} \leq s_0 + 2 - \alpha \tag{47} \]

and define \( L_{\text{int}} \) through the domain

\[ D_{\text{int}}(L) = \{ u \in D_{\text{max}}(L) : |x|^{\theta_0(\alpha - 2)}|\log |x||^{-\frac{2}{\alpha}}u \in L^p(B_1/2)\}. \tag{48} \]

Then \( L_{\text{int}} \) generates a positive analytic semigroup in \( L^p(\mathbb{R}^N) \) which is coherent with respect to all \( p \) satisfying \([17]\).

For technical reasons we need also the the operator \( L \) in the unit ball \( B_1 \) (with Dirichlet boundary conditions) defined on the domain

\[ D_{\text{int}}^1(L) = \{ u \in L^p(B_1) \cap W^{2,p}(B_1 \setminus B_2) \; \forall \; 0 < \varepsilon < 1, Lu \in L^p(B_1) ; \; |x|^{\theta_0(\alpha - 2)}|\log |x||^{-\frac{2}{\alpha}}u \in L^p(B_1/2) \}, \tag{49} \]

where \( \theta \) is as before. We denote this operator by \( L^1_{\text{int}} \). Most computations will be performed on the set

\[ D_1 := \{ u \in C^\infty_c(\overline{B_1} \setminus \{0\}) : u = 0 \; \text{on} \; \partial B_1 \} \]

In the next proposition we prove the injectivity of \( \lambda - L_{\text{int}} \) and \( \lambda - L^1_{\text{int}} \) for positive \( \lambda \).
Proposition 6.2 The operators $\lambda - L_{\text{int}}$ and $\lambda - L^1_{\text{int}}$ are injective for $\lambda > 0$.

Proof. We start with $L_{\text{int}}$. We denote by $\Delta_{S^{n-1}}$ the Laplace Beltrami on the unit sphere $S^{n-1}$. If $Q$ is a spherical harmonic of order $n \geq 0$, then $-\Delta_{S^{n-1}}Q = \lambda_n Q$ with $\lambda_n = n(n + N - 2)$. If $v \in N(\lambda - L_{\text{int}})$ we set

$$ v_Q(r) = \int_{S^{n-1}} v(r, \omega)Q(\omega) \, d\omega, $$

where $Q$ is a spherical harmonic of order $n$. Then

$$ r^{p\delta_0 - 1} |\log |x||^{-2} |v_Q(r)|^p \in L^1(0, 1/2). \quad (50) $$

Observe that

$$ v''_Q + \frac{N - 1}{r} v'_Q = \int_{S^{n-1}} \left( v_{r r} + \frac{N - 1}{r} v_r \right) Q(\omega) \, d\omega = \int_{S^{n-1}} \left( \Delta v - \frac{\Delta_{S^{n-1}} v}{r^2} \right) Q \, d\omega $$

$$ \int_{S^{n-1}} \left( Q \Delta v - \frac{\Delta_{S^{n-1}} Q}{r^2} \right) \, d\omega = \int_{S^{n-1}} \left( \Delta v + \frac{\lambda_n v}{r^2} \right) Q \, d\omega. $$

This implies that $v_Q$ satisfies

$$ \lambda v_Q - r^\alpha \left( v''_Q + (N - 1 + c) \frac{v'_Q}{r} - (b + \lambda_n) \frac{v_Q}{r^2} \right) = 0. $$

We use Lemma 2.6 to show that $v_Q = 0$. The integrability of $v_Q$ at $r = \infty$ and (43) imply that $v_Q = c u_2$. If $n > 0$, by (14) and (50) we see that $c = 0$, that is $v_Q = 0$. If $n = 0$, $u_2$ behaves like $r^{-\delta_0} \log r$ near 0, see (15), and hence (50) is not satisfied unless $v_Q = 0$. The density of spherical harmonics in $L^p(S^{n-1})$ yields $v(r, \cdot) = 0$ for every $r$, hence $v = 0$ and this concludes the proof for $L_{\text{int}}$. In the case of $L^1_{\text{int}}$ the proof is similar: if $v_Q \neq 0$ then $v_Q = c_1 u_1 + c_2 u_2$ with $c_1 \neq 0$, $c_2 \neq 0$ since $v_Q(1) = 0$ and $u_1, u_2$ are positive. Hence $v_Q$ behaves like $u_2$ near 0 and (50) is not satisfied.

The following weighted estimates will be crucial in what follows.

Proposition 6.3 For every $v \in D_1$

$$ \Re \int_{B_1} |x|^{2-N} \nabla v \cdot \nabla (\overline{v} |v|^{p-2}) \, dx \geq \frac{p-1}{p^2} \int_{B_1} |x|^{-N} |\log |x||^{-2} |v|^p \, dx. \quad (51) $$

In particular, if $u \in D_1$, and $v = |x|^{\frac{N-2+\alpha}{2}} u$, then

$$ \Re \int_{B_1} (-Lu)|x|^{(p\theta_0-1)(\alpha-2)} |u|^{p-2} \, dx \geq \frac{p-1}{p^2} \int_{B_1} |x|^{p\delta_0(\alpha-2)} |\log |x||^{-2} |u|^p \, dx. \quad (52) $$

Noting that $r^\alpha |\log r| \leq (\varepsilon) r^{-1}$ if $r \in (0, 1)$ and $\varepsilon > 0$, we obtain from (52)

Lemma 6.4 For every $\delta > 0$, $\theta_0 > 0$ and $u \in D_1$,

$$ \Re \int_{B_1} (-Lu)|x|^{(p\theta_0-1)(\alpha-2)} |u|^{p-2} \, dx \geq \frac{p-1}{4} \frac{(2-\alpha)^2 \delta^2}{4} \int_{B_1} |x|^{p(\theta_0-\delta)(\alpha-2)} |u|^p \, dx. \quad (53) $$
Proof. If \( v \in D_1 \) integrating by parts we obtain

\[
\int_{B_1} |x|^{2-N} \nabla v \cdot \nabla (|v|^{p-2}) \, dx = - \int_{B_1} |x|^{-N} (|x|^2 \Delta v + (2 - N)x \cdot \nabla v) |v|^{p-2} \, dx.
\]

Observe that in spherical coordinates

\[
|x|^2 \Delta v + (2 - N)x \cdot \nabla v = r^2 \frac{\partial^2 v}{\partial r^2} + r \frac{\partial v}{\partial r} + \Delta_{S^{N-1}} v,
\]

and

\[
\int_{B_1} |x|^{2-N} \nabla v \cdot \nabla (|v|^{p-2}) \, dx = - \int_{B_1} r^{-N} \left( r^2 \frac{\partial^2 v}{\partial r^2} + r \frac{\partial v}{\partial r} + \Delta_{S^{N-1}} v \right) |v|^{p-2} \, dx
\]

\[
= - \int_{S^{N-1}} \int_0^1 \left( r^2 \frac{\partial^2 v}{\partial r^2} + r \frac{\partial v}{\partial r} \right) |v|^{p-2} \, dr + \int_0^1 \left( \int_{S^{N-1}} (-\Delta_{S^{N-1}} v) |v|^{p-2} \, d\omega \right) \frac{dr}{r}.
\]  

(54)

Since \( \Delta_{S^{N-1}} \) is dissipative in \( L^p(S^{N-1}) \) we see that for every \( r > 0 \)

\[
Re \int_{S^{N-1}} (-\Delta_{S^{N-1}} v) |v|^{p-2} \, d\omega \geq 0.
\]  

(55)

On the other hand, fix \( \omega \in S^{N-1} \) and set \( w(s) = v(e^s, \omega) \). Then

\[
\int_0^1 \left( r^2 \frac{\partial^2 v}{\partial r^2} \right) (r, \omega) + r \frac{\partial v}{\partial r} (r, \omega) \, v \, \overline{|v|^{p-2}} \, r \, d\omega = \int_{-\infty}^0 (-w'') \, |v|^{p-2} \, ds.
\]

Using Hardy’s inequality we have

\[
Re \int_{-\infty}^0 (-w'') \, |v|^{p-2} \, ds \geq \frac{p-1}{p^2} \int_{-\infty}^0 s^{-2} |w|^p \, ds = \frac{p-1}{p^2} \int_0^1 |\log r|^{-2} |v(\gamma, \omega)|^p \, dr.
\]

Therefore we deduce that

\[
- Re \int_{S^{N-1}} \int_0^1 \left( r^2 \frac{\partial^2 v}{\partial r^2} + r \frac{\partial v}{\partial r} \right) |v|^{p-2} \, dr \, d\omega \geq \frac{p-1}{p^2} \int_{S^{N-1}} \int_0^1 |\log r|^{-2} |v(r, \omega)|^p \, dr \, d\omega
\]

\[
= \frac{p-1}{p^2} \int_{B_1} |x|^{-N} |\log |x||^{-2} |v|^p \, dx.
\]  

(56)

Combining (55) and (56) with (54), we obtain (51). To prove (52), we consider the operator \( x^\gamma L \) with \( \gamma = (p\theta_0 - 1)(\alpha - 2) \). Then \( (N - 2 + \alpha + \gamma) = (N - 2 + \alpha) / 2 = s_0 \) and Proposition 4.2 applies. In particular, since \( f(s_0) = 0 \), (24) with \( v = |x|^{s_0} u \) yields

\[
Re \int_{B_1} (-Lu)|x|^{(p\theta_0 - 1)(\alpha - 2)} |u|^{p-2} = Re \int_{B_1} |x|^{2-N} \nabla v \cdot \nabla (|v|^{p-2}) \, dx
\]

and (52) follows from (51).

To prove Theorem 6.1, as in Section 5, we introduce the operator

\[
L_t u := Lu - k|x|^\alpha - k \min \{ |x|^\alpha, t \},
\]  

25
defined in $D_{p,\alpha}$, see (9), and, in order to consider separately the singularities at infinity and near the origin, we introduce also the operators $L^1_p$ and $L^2_p$ in $B_1$ and $B^1_{1/2}$ defined as $L_t$ on the domains

$$D^1_{p,\alpha} = \{ u \in L^p(B_1) \cap W^{2,p}(B_1 \setminus B_2) \forall 0 < \varepsilon < 1, |x|^\alpha D^2 u, |x|^{\alpha-1} \nabla u, |x|^{\alpha-2} u \in L^p(B_1) \}$$

and

$$D^2_{p,\alpha} = \{ u \in W^{2,p}(B^1_{1/2}) \cap |x|^\alpha D^2 u, |x|^{\alpha-1} \nabla u, \in L^p(B^1_{1/2}) \},$$

respectively. Here $k$ is a fixed constant large enough, $t > 0$ is a parameter and both the operators are endowed with Dirichlet boundary conditions. As in Theorem 4.4 or Proposition 4.2 we have,

**Lemma 6.5** For every $t > 0$, $L^1_t$ generates an analytic semigroup on $L^p(\mathbb{R}^N)$. Moreover, $D_1$ is a core for $L^1_t$.

Now we state a-priori estimates for $L^1_p$. Due to the presence of the logarithmic term, we cannot directly prove that (59) holds with $\gamma = 1$, as in Lemma 5.3.

**Proposition 6.6** For every $\gamma \in (0,1)$ there are constants $C_\gamma, C' > 0$ such that for every $t > 0$, $\lambda \in \mathbb{C}^+$ and $u \in D^1_{p,\alpha}$

$$\|u\|_{L^p(B_1)} \leq \frac{C_\gamma}{|\lambda|^\gamma} \|\lambda u - L^1_t u\|_{L^p(B_1)}$$

and

$$\left\| |x|^{\theta_0(\alpha-2)}|\log |x||^{-\frac{2}{p}} u \right\|_{L^p(B_1)} \leq C\|\lambda u - L^1_t u\|_{L^p(B_1)}.$$

If $\theta_0 = \frac{1}{p}$, then $\gamma = 1$ is allowed.

**Proof.** We proceed as in Lemma 5.3 in $B_1$ rather than $\mathbb{R}^N$, and consider first the case when $\theta_0 \geq \frac{1}{p}$. By density we may assume that $u \in D_1$. The estimate

$$|\lambda| \left\| |x|^{\theta_0(\alpha-2)} u \right\|_{L^p(B_1)}^p \leq C\|\lambda u - L^1_t u\|_{L^p(B_1)} \left\| |x|^{\theta_0(\alpha-2)} u \right\|_{L^p(B_1)}^{p-1}.$$

is identical to (55) and obtained as in Lemma 5.3 recalling that Proposition 4.2 holds in the critical case. Observe that if $\theta_0 = \frac{1}{p}$ (or $\frac{N-2+\alpha}{2} = \frac{N-2+\alpha}{2}$) the above inequality gives (59) with $\gamma = 1$. If $\theta_0 > \frac{1}{p}$ we note that

$$0 < \theta_0 - \frac{1}{p} < \frac{p\theta_0 - 1}{p-1} = \theta_0 - \frac{1 - \theta_0}{p-1}.$$

We choose $\delta \in [0, \frac{1 - \theta_0}{p-1}]$ and we see from Lemma 2.7

$$\left\| |x|^{\theta_0(\alpha-2)} u \right\|_{L^p(B_1)} \leq \left\| |x|^{(\theta_0 - \delta)(\alpha-2)} u \right\|_{L^p(B_1)}^{\tau} \left\| |x|^{(\theta_0 - \frac{1}{p})(\alpha-2)} u \right\|_{L^p(B_1)}^{1-\tau},$$

where

$$\tau = \frac{p\theta_0 - 1}{(p-1)(1 - p\delta)}.$$

Using the above inequality in (61) we have

$$|\lambda| \left\| |x|^{(\theta_0 - \frac{1}{p})(\alpha-2)} u \right\|_{L^p(B_1)}^{1+(p-1)\tau} \leq C\|\lambda u - L^1_t u\|_{L^p(B_1)} \left\| |x|^{(\theta_0 - \delta)(\alpha-2)} u \right\|_{L^p(B_1)}^{(p-1)\tau}.\ (62)$$
Combining the previous estimate with (62), we have

\[
\frac{(p - 1)(\alpha - 2)^2e^{2\delta^2}}{4} \left\| \frac{\partial}{\partial t} \left( \theta_0 - \delta \right) u \right\|_{L^p(B_1)}^p \leq \left\| \lambda u - L_1^u \right\|_{L^p(B_1)} \left\| \frac{\partial}{\partial t} \left( \theta_0 - \delta \right) u \right\|_{L^p(B_1)}^{(p - 1)(1 - \tau)}.
\]

On the other hand, by Lemma 6.3, we see that

\[
\left\| \lambda u - L_1^u \right\|_{L^p(B_1)} \leq \left\| \lambda u - L_1^u \right\|_{L^p(B_1)} \left\| \frac{\partial}{\partial t} \left( \theta_0 - \delta \right) u \right\|_{L^p(B_1)}^{(p - 1)(1 - \tau)}.
\]

This yields

\[
\left\| \frac{\partial}{\partial t} \left( \theta_0 - \delta \right) u \right\|_{L^p(B_1)} \leq \left( \frac{4}{(p - 1)(\alpha - 2)^2e^{2\delta^2}} \right)^\frac{1}{p} \left( \frac{C}{\lambda} \right)^\frac{1}{p} \left\| \lambda u - L_1^u \right\|_{L^p(B_1)}.
\]

Next, we prove (59). By Proposition 4.2, we have

\[
\text{Re} \left[ e^{i\omega} \int_{B_1} \left( -L_1^u u + k|x|^{(\alpha - 2)u} \right) \sin \frac{t}{p} dx \right] \geq 0, \quad \omega \in \left[ -\frac{\pi}{2} + \omega_0, \frac{\pi}{2} - \omega_0 \right],
\]

for some \( \omega_0 \in (0, \frac{\pi}{2}) \). Thus we see that

\[
\left\| u \right\|_{L^p(B_1)}^p \leq \frac{1}{|\lambda| \cos \omega_0} \left\| \lambda u - L_1^u \right\|_{L^p(B_1)} \left\| u \right\|_{L^p(B_1)}^{p - 1} + \frac{k}{|\lambda| \cos \omega_0} \left\| \frac{\partial}{\partial t} \left( \theta_0 - \delta \right) u \right\|_{L^p(B_1)}^p.
\]

Fix \( \delta < \theta_0 - \frac{1}{p} \). Then \( p(\theta_0 - \delta) > 1 \) and Hölder’s inequality yields

\[
\left\| \frac{\partial}{\partial t} \left( \theta_0 - \delta \right) u \right\|_{L^p(B_1)} \leq \left\| u \right\|_{L^p(B_1)}^{1 - \frac{1}{p(\theta_0 - \delta)}} \left\| \frac{\partial}{\partial t} \left( \theta_0 - \delta \right) u \right\|_{L^p(B_1)}^{\frac{1}{p(\theta_0 - \delta)}}
\]

Hence using Young’s inequality and (63) we obtain

\[
\left\| u \right\|_{L^p(B_1)}^{\frac{1}{p(\theta_0 - \delta)}} \leq \frac{1}{|\lambda| \cos \omega_0} \left\| \lambda u - L_1^u \right\|_{L^p(B_1)} \left\| u \right\|_{L^p(B_1)}^{\frac{1}{p(\theta_0 - \delta)}} + \frac{k}{|\lambda| \cos \omega_0} \left\| \frac{\partial}{\partial t} \left( \theta_0 - \delta \right) u \right\|_{L^p(B_1)}^p.
\]
Consequently, setting $\gamma(\delta) := \theta_0 - \delta + \frac{(p-1)(1-\gamma)}{p}$, we have
\[
\|u\|_{L^p(B_t)} \leq \left( \frac{C_0}{|\lambda|} + \frac{C_\rho}{|\lambda|^{\gamma(\delta)}} \right) \|\lambda u - L_t^1 u\|_{L^p(B_t)}
\]
and we note that $\lim_{\delta \to 0} \gamma(\delta) = 1$. Next, we prove (60). By Lemma 6.3 since $L_t^1 = L - V_t$ with $V_t \geq 0$, we have
\[
\frac{p-1}{p^2} \left\| |x|^{\theta_0(\alpha-2)}|\log |x||^{-\frac{2}{p}} u \right\|_{L^p(B_t)}^p \leq \left( \int_{B_t} (\lambda u - L_t^1 u)|x|^{\left(\frac{p\theta_0-1}{\alpha-2}\right)} dx \right)^{\frac{p-1}{p}} \leq \|\lambda u - L_t^1 u\|_{L^p(B_t)} \left\| |x|^{\theta_0(\alpha-2)}|\log |x||^{-\frac{2}{p}} u \right\|_{L^p(B_t)}^p.
\]
From estimate (63) with $\delta = \frac{1-\theta_0}{p-1}$, hence $\tau = 1$, we obtain
\[
\left\| |x|^{\theta_0(\alpha-2)}|\log |x||^{-\frac{2}{p}} u \right\|_{L^p(B_t)}^p \leq C\|\lambda u - L_t^1 u\|_{L^p(B_t)}^p.
\]
Finally, we consider the case $0 \leq \theta_0 < \frac{1}{p}$ by using the dual operator $(L_t^1)^*$ in $L^{p'}(B_t)$ and proceeding as in Lemma 5.3. One verifies that $(L_t^1)^*$ satisfies $\frac{N-\frac{2}{p}+\theta}{p} = \frac{N}{p} + \theta(\alpha-2)$ with $\theta 1-\theta > \frac{1}{p}$. Therefore applying (59) to $(L_t^1)^*$ in $L^{p'}(B_t)$, we have $\|\lambda - (L_t^1)^*\|^{-1} \leq C(1 + |\lambda|^{-1})$ and by duality we obtain (59) for $L_t^1$ in $L^{p'}(B_t)$. Moreover, since $(p\theta_0-1)(\alpha-2) \geq 0$, Lemma 6.3 implies that
\[
\frac{p-1}{p^2} \left\| |x|^{\theta_0(\alpha-2)}|\log |x||^{-\frac{2}{p}} u \right\|_{L^p(B_t)}^p \leq \|\lambda u - L_t^1 u\|_{L^p(B_t)} \left\| |x|^{\theta_0(\alpha-2)}|\log |x||^{-\frac{2}{p}} u \right\|_{L^p(B_t)}^p \leq \|\lambda u - L_t^1 u\|_{L^p(B_t)} \left\| |x|^{\theta_0(\alpha-2)}|\log |x||^{-\frac{2}{p}} u \right\|_{L^p(B_t)}^p \leq C\|\lambda u - L_t^1 u\|_{L^p(B_t)}^p.
\]
This completes the proof.

Next we prove a-priori estimates for $L_t$, for large $t$, by gluing the resolvents of $L_t^1$ and $L_t^2$.

**Proposition 6.7** For every $\gamma \in (0,1)$, there are constants $\tau, \rho, C_\gamma, C' > 0$ such that if $t \geq \tau$, $\lambda \in \mathbb{C}_+, |\lambda| \geq \rho$, and $u \in D_{p,\alpha}$
\[
\|u\|_{L^p} \leq \frac{C_\gamma}{|\lambda|^{\gamma}} \|\lambda u - L_t u\|_{L^p}.
\]
and
\[
\left\| |x|^{\theta_0(\alpha-2)}|\log |x||^{-\frac{2}{p}} u \right\|_{L^{p,(1/2)}(B_t)} \leq C'\|\lambda u - L_t u\|_{L^p}.
\]

**Proof.** Observe that if $t \geq 2^{\gamma-\alpha}$ then $L_t^2$ coincides with $L$ in $B_{1/2}^\alpha$. Since $\alpha < 2$, the function $a(x) = |x|^\alpha$ satisfies the inequality $|\nabla a|^{1/2} \leq C$ in $B_{1/2}^\alpha$, therefore, by [7], the operator $L_t^2$ generates an analytic semigroup and the resolvent estimate
\[
\|(\lambda - L_t^2)^{-1}\|_p \leq \frac{C}{|\lambda|}
\]
holds for $\lambda \in \mathbb{C}_+, |\lambda| \geq \rho$. By virtue of [17, Section 5] we can represent the resolvent of $L_t$ by gluing together the resolvents of $L_t^1$ and $L_t^2$. In order to do this we need gradient estimates for $L_t^1$
and \( L^2_t \) in an annulus \( \Sigma_{r_1, r_2} = B_{r_2} \setminus B_{r_1} \subset \mathbb{R}^2 \setminus B_{1/2} \). We fix \( \frac{1}{2} < s_1 < r_1 < r_2 < s_2 < 1 \) and we use the classical interior estimates for uniformly elliptic operators. Since the coefficients of \( L^2_t \) are uniformly bounded with respect to \( t \) in the annulus \( \Sigma_{s_1, s_2} \), there exists \( C > 0 \), independent of \( t > 0 \) such that for every \( u \in D^1_{p, \alpha} \)

\[
\|\nabla u\|_{p, \Sigma_{s_1, s_2}} \leq C \left( \varepsilon \|L^1_t u\|_{p, \Sigma_{s_1, s_2}} + \frac{1}{\varepsilon} \|u\|_{p, \Sigma_{s_1, s_2}} \right).
\]

Using (59) it follows that, for every \( s \) with \( \|f\|_p \leq C \|\lambda u - L^1_t u\|_p + \varepsilon|\lambda|^{1-\gamma} \|\lambda u - L^1_t u\|_p + \frac{1}{\varepsilon} |\lambda|^{-\gamma} \|\lambda u - L^1_t u\|_p \).

By choosing \( \varepsilon = |\lambda|^{-\delta} \) with \( 1 - \gamma < \delta < \gamma \), we get for \( r = \min\{-\delta, \delta - 1, \delta - \gamma\} < 0 \)

\[
\|\nabla u\|_{p, \Sigma_{s_1, s_2}} \leq C|\lambda|^r \|\lambda u - L^1_t u\|_p.
\]

In a similar way one proves gradient estimates for \( L^2_t \). Following the method of [17] Section 5 one constructs an approximate resolvent \( R(\lambda) \) for \( L_t \),

\[
R(\lambda) = \eta_1 (\lambda - L^1_t)^{-1} \eta_1 + \eta_2 (\lambda - L^2_t)^{-1} \eta_2
\]

where \( \eta_1, \eta_2 \) are smooth functions supported in \( B_{r_2}, B_{r_1} \) respectively and such that \( \eta_1^2 + \eta_2^2 = 1 \). The operator \( R(\lambda) \) satisfies \( (\lambda - L_t) R(\lambda) = I + S(\lambda) \) and \( \|S(\lambda)\| \leq 1/2 \) for \( |\lambda| \) large, because of the gradient estimates. Then, for \( \lambda \in \mathbb{C}_+, |\lambda| \) large, we have

\[
(\lambda - L_t)^{-1} = R(\lambda)(I + S(\lambda))^{-1}
\]

and hence (59) follows from (55) and (59). Estimate (60) follows similarly from (60), since \( \eta_2 \) vanishes near 0.

**Proof (Theorem 6.1).** We consider the operator \( L_{\text{int}} \) defined in (59). For \( \lambda > 0 \), \( \lambda - L_{\text{int}} \) is injective, by Proposition 6.2. As in the proof of Theorem 5.3 one sees that for \( \lambda \geq \rho > 0 \), \( f \in L^p(\mathbb{R}^N) \), \( (\lambda - L_n)^{-1} f \to (\lambda - L_{\text{int}})^{-1} f \). By (61) it follows that if \( \lambda \in \mathbb{C}_+, |\lambda| \geq \rho \), \( \lambda - L_{\text{int}} \) is invertible and, for every \( \gamma < 1 \)

\[
\| (\lambda - L_{\text{int}})^{-1} \| \leq \frac{C}{|\lambda|^\gamma}.
\]

For \( s > 0 \) let \( I_s : L^p \to L^p \) defined by \( I_s u(x) = u(sx) \). Clearly \( I_s \) is invertible with inverse \( I_{s^{-1}} \) and \( \|I_s u\|_p = s^{-N/p} \|u\|_p \). Since \( L = s^{2-\alpha} I_s L I_s^{-1} \), if \( \lambda \in \mathbb{C}_+, \lambda = r \omega \) with \( |\omega| = \rho \) (hence \( \omega \) belongs to the resolvent set) then the equality

\[
\lambda - L_{\text{int}} = I_s r \left( \omega - \frac{s^{2-\alpha} L_{\text{int}}}{r} \right) I_s^{-1}
\]

with \( s = r^{\frac{1}{2-N}} \) shows that \( \mathbb{C}_+ \) is in the resolvent set and yields the decay

\[
\| (\lambda - L_{\text{int}})^{-1} \|_p \leq \frac{C}{|\lambda|} \max\{ \| (\omega - L_{\text{int}})^{-1} \|_p : |\omega| = \rho, \omega \in \mathbb{C}_+ \}.
\]

For \( \lambda > 0 \), positivity and coherence with respect to \( p \) of \( (\lambda - L_{\text{int}})^{-1} \) follow since \( (\lambda - L_{\text{int}})^{-1} f = \lim_{n \to \infty} (\lambda - L_n)^{-1} f \).
6.2 Positive results for $\alpha > 2$

The generation result proved in the critical case for $\alpha < 2$ can be extended by using similar arguments to the case $\alpha > 2$. Recall that $s_0 = \frac{N-2+\alpha}{2}$.

**Theorem 6.8** Assume that

$$s_0 + 2 - \alpha \leq \frac{N}{p} \leq s_0$$

(67)

and define $L_{\text{int}}$ through the domain

$$D_{\text{int}}(L) = \{ u \in D_{\text{max}}(L) ; |x|^{\theta_0(\alpha-2)}|\log |x||^{-\frac{2}{p}} u \in L^p(B_1^c) \},$$

(68)

where $\theta_0 \in [0, 1]$ satisfies $\frac{N}{p} = s_0 + \theta_0(2 - \alpha)$. Then $L_{\text{int}}$ generates a positive analytic semigroup in $L^p(\mathbb{R}^N)$ which is coherent with respect to all $p$ satisfying (67).

We only state the main steps.

**Proposition 6.9** For $\lambda > 0$ the operator $\lambda - L_{\text{int}}$ is injective.

**Proposition 6.10** Set

$$\tilde{D}_1 := \{ u \in C_c(\mathbb{R}^N \setminus B_1) ; u = 0 \text{ on } \partial B_1 \}.$$

For every $v \in \tilde{D}_1$

$$\text{Re} \int_{B_1^c} |x|^{2-N}\langle \nabla v, \nabla (|v|^{p-2}) \rangle \, dx \geq \frac{p-1}{p^2} \int_{B_1^c} |x|^{-N}|\log |x||^{-2}|v|^p \, dx.$$  \hspace{1cm} (69)

In particular, if $u \in \tilde{D}_1$, and $v = |x|^{\frac{N-2+\alpha}{2}}u$, then

$$\text{Re} \int_{B_1^c} (-Lu)|x|^{(p\theta_0-1)(\alpha-2)}|\nabla |x|^{p-2}| u|^p \, dx \geq \frac{p-1}{p^2} \int_{B_1^c} |x|^{p\theta_0(\alpha-2)}|\log |x||^{-2}|u|^p \, dx.$$  \hspace{1cm} (70)

**Proposition 6.11** For every $\gamma \in (0, 1)$ there are constants $C_\gamma, C' > 0$ such that for every $t > 0$, $\lambda \in \mathbb{C}_+$ and $u \in \tilde{D}_1$,

$$\|u\|_{L^p(B_1^c)} \leq \frac{C_\gamma}{|\lambda|^\gamma} \|\lambda u - L_tu\|_{L^p(B_1^c)}$$  \hspace{1cm} (71)

and

$$\left\| |x|^{\theta_0(\alpha-2)}|\log |x||^{-\frac{2}{p}} u \right\|_{L^p(B_1^c)} \leq C'\|\lambda u - L_tu\|_{L^p(B_1^c)}.$$  \hspace{1cm} (72)

where $L_t = L - k|x|^{\alpha-2} + k \min\{t, |x|^{\alpha-2}\}$.

By using the propositions stated above, we deduce Theorem 6.8 arguing as for Theorem 6.1.
6.3 The equalities $L_{\text{int}} = L_{\text{min}}$ and $L_{\text{int}} = L_{\text{max}}$

Here we investigate when $L_{\text{int}}$ coincides with $L_{\text{min}}$ or $L_{\text{max}}$.

**Proposition 6.12** Assume $\alpha < 2$ and $s_0 \leq \frac{N}{p} \leq s_0 + 2 - \alpha$. Then

(i) If $\frac{N}{p} = s_0$, then $L_{\text{int}} = L_{\text{max}}$;

(ii) If $\frac{N}{p} \neq s_0$, then $N(\lambda - L_{\text{max}}) \neq \{0\}$, hence $L_{\text{int}} \neq L_{\text{max}}$.

**Proof.** By the definition of $L_{\text{int}}$, see (14), (i) is obvious since $\theta_0 = 0$. We show (ii), that is, $N(\lambda - L_{\text{max}}) \neq \{0\}$. We use Lemma 2.6 with $k = \hat{k} = 0$ and take $v(x) = u_2(|x|)$. Since $\frac{N}{p} > s_0$, we see from (I4) that $v \in L^p(\mathbb{R}^N)$. This means that $N(\lambda - L_{\text{max}}) \neq \{0\}$. By duality, the following proposition directly follows from Proposition 6.13.

**Proposition 6.13** Assume $\alpha < 2$ and $s_0 \leq \frac{N}{p} \leq s_0 + 2 - \alpha$. Then

(i) If $\frac{N}{p} = s_0 + 2 - \alpha$, then $L_{\text{int}} = L_{\text{min}}$, that is, $C^\infty_c(\Omega)$ is a core for $L_{\text{int}}$;

(ii) If $\frac{N}{p} \neq s_0 + 2 - \alpha$, then $R(\lambda - L_{\text{min}}) \neq L^p(\mathbb{R}^N)$, hence $L_{\text{int}} \neq L_{\text{min}}$.

The case $\alpha > 2$ is similar.

**Proposition 6.14** Assume $\alpha > 2$ and $s_0 + 2 - \alpha \leq \frac{N}{p} \leq s_0$. Then

(i) If $\frac{N}{p} = s_0$, then $L_{\text{int}} = L_{\text{max}}$;

(ii) If $\frac{N}{p} \neq s_0$, then $N(\lambda - L_{\text{max}}) \neq \{0\}$, hence $L_{\text{int}} \neq L_{\text{max}}$.

**Proposition 6.15** Assume $\alpha > 2$ and $s_0 + 2 - \alpha \leq \frac{N}{p} \leq s_0$. Then

(i) If $\frac{N}{p} = s_0 + 2 - \alpha$, then $L_{\text{int}} = L_{\text{min}}$, that is, $C^\infty_c(\Omega)$ is a core for $L_{\text{int}}$;

(ii) If $\frac{N}{p} \neq s_0 + 2 - \alpha$, then $R(\lambda - L_{\text{min}}) \neq L^p(\mathbb{R}^N)$, hence $L_{\text{int}} \neq L_{\text{min}}$.

Integrability of first and second derivatives for $u \in D_{\text{int}}(L)$ can be established as in Theorem 5.4. For every $\theta < \theta_0$ we set $\alpha' = \alpha'(\theta) = \theta(\alpha - 2) + 2$ and define for $\theta_0 > 0$,

$$D_{\text{reg}}(L) = \left\{ \begin{array}{ll}
\{ u \in D_{\text{max}}(L) : |x|^\theta \log |x| |\nabla^\alpha u| \in L^p(B_{1/2}), & \text{if } \alpha < 2; \\
|\nabla |\nabla^\alpha D^2 u| \in L^p(B) & \forall \theta \in (0, \theta_0), \\
|\nabla^\alpha D^2 u| \in L^p(B_c) & \forall \theta \in (0, \theta_0), \\
\{ u \in D_{\text{max}}(L) : |x|^\theta \log |x| |\nabla^\alpha u| \in L^p(B_{1/2}), & \text{if } \alpha > 2; \\
|\nabla |\nabla^\alpha D^2 u| \in L^p(B_{1/2}) & \forall \theta \in (0, \theta_0), \\
|\nabla^\alpha D^2 u| \in L^p(B_{1/2}) & \forall \theta \in (0, \theta_0), \\
|\nabla^\alpha D^2 u| \in L^p(B_c) & \forall \theta \in (0, \theta_0), \\
\{ u \in D_{\text{max}}(L) : |x|^\theta \log |x| |\nabla^\alpha u| \in L^p(B_{1/2}), & \text{if } \alpha > 2; \\
|\nabla |\nabla^\alpha D^2 u| \in L^p(B_{1/2}) & \forall \theta \in (0, \theta_0), \\
|\nabla^\alpha D^2 u| \in L^p(B_{1/2}) & \forall \theta \in (0, \theta_0), \\
|\nabla^\alpha D^2 u| \in L^p(B_{1/2}) & \forall \theta \in (0, \theta_0), \\
\} \right. \right\}$$

where $B = B_1$.  

31
**Proposition 6.16** If \( \theta_0 > 0 \), that is \( N/p \neq s_0 \), then the domains \( D_{\text{int}}(L) \) and \( D_{\text{reg}}(L) \) coincide.

**Proof.** Assume \( \alpha < 2 \) and let \( u \in D_{\text{int}}(L) \). We write \( u = u_1 + u_2 \) where \( u_1 = u\phi, u_2 = u(1 - \phi) \) and \( \phi \in C_0^\infty(\mathbb{R}^N) \) with support in \( B_2 \) and equal to 1 in \( B_1 \). We introduce the operator \( L_2 \) on \( \mathbb{R}^N \) in this way: the coefficients of \( L_2 \) coincide with those of \( L \) in \( B_1^c \) whereas in \( B_1 \) they take the (constant) value that they have on \( \partial B_1 \). \( L_2 \) is therefore uniformly elliptic with Lipschitz coefficients in \( B_1 \) and satisfies Hypothesis 2.1 of [1].

By construction the function \( u_2 \) belongs to the maximal domain of \( L_2 \) and, by [7, Proposition 2.9], \( |x|^\alpha D^2 u_2, |x|^\alpha - 1 \nabla u_2 \in L^p(B' \setminus \{0\}) \), that is \( |x|^\alpha D^2 u, |x|^\alpha - 1 \nabla u \in L^p(B^c) \). To treat \( u_1 \) we consider the operator \( L_1 = |x|^\alpha - \alpha L \). Since \( \alpha < 2 \), \( \alpha' \geq \alpha \) and then \( u_1 \in D_{\text{max}}(L_1) \). Since \( \alpha' - 2 = \theta_0(\alpha - 2) > \theta_0(\alpha - 2) \), by the definition of \( L_{\text{int}} \), \( |x|^\alpha - 2 u_1 \in L^p(\mathbb{R}^N) \).

By Lemma 2.3, \( u_1 \in D_{p,\alpha'} \). It follows that \( |x|^\alpha D^2 u_1, |x|^\alpha - 1 \nabla u_1, |x|^\alpha - 2 u_1 \in L^p(B) \), hence the same holds for \( u \).

**Remark 6.17** The case \( \theta_0 = 0 \) or \( N/p = s_0 \) is quite special and we recall that \( L_{\text{int}} = L_{\text{max}} \). Integrability of first and second derivatives can be obtained directly using Proposition 2.3. If \( \alpha < 2 \) and \( u \in D_{\text{int}}(L) \), then \( |x|^2 D^2 u, |x| \nabla u \in L^p(B) \) and \( |x|^\alpha D^2 u, |x|^\alpha - 1 \nabla u \in L^p(B^c) \) and conversely if \( \alpha > 2 \). To see this we proceed as in the proposition above splitting \( u = u_1 + u_2 \) and treating \( u_2 \) in the same way. Finally we note that \( u_1 \in D_{\text{max}}(|x|^2 - \alpha L) \) and then apply Proposition 2.3.

As in Section 5 one shows the minimality of \( (\lambda - L_{\text{int}})^{-1} \), noting that the proof of Lemma 5.9 extends to the critical case, choosing \( \theta_0 \) such that \( N/p + \theta_0(\alpha - 2) = s_0 \).

**Proposition 6.18** Let \( \lambda > 0 \), \( f \geq 0 \) and let \( 0 \leq u \in D_{\text{max}}(L) \) satisfy \( \lambda u - Lu = f \). Then \( (\lambda - L_{\text{int}})^{-1} f \leq u \).

### 6.4 Negative results

We show that if \( \frac{N}{p} \) falls outside the closed interval \([10]\), then no realization \( L_{\text{min}} \subset L \subset L_{\text{max}} \) generates a semigroup in \( L^p(\mathbb{R}^N) \).

**Theorem 6.19**

(i) If \( \alpha < 2 \) and \( \frac{N}{p} > s_0 + 2 - \alpha \), or \( \alpha > 2 \) and \( \frac{N}{p} < s_0 + 2 - \alpha \). Then \( N(\lambda - L_{\text{min}}) \neq \{0\} \).

(ii) If \( \alpha < 2 \) and \( \frac{N}{p} < s_0 \), or \( \alpha > 2 \) and \( \frac{N}{p} > s_0 \). Then \( R(\lambda - L_{\text{max}}) \neq L^p(\mathbb{R}^N) \).

**Proof.** (i) We give a proof only for \( \alpha < 2 \). As in Proposition 1.3 we consider radial solutions of the equation

\[ \lambda v - Lv = 0. \]

We use Lemma 2.6 with \( k = 0 \) and choose \( v = u_2 \) so that \( v \) satisfies (15). Since \( s_0 + 2 - \alpha \) \( \leq \frac{N}{p} \) this implies that \( v, |x|^\alpha - 2 v \in L^p(\mathbb{R}^N) \) and hence, by Lemma 2.3, we have \( v \in D_{\text{min}}(L) \) and \( \lambda v - Lv = 0 \). The proof of (ii) follows from (i), by duality.

### 7 Examples

In this section we specialize our results to particular operators.
Example 7.1 We consider Schrödinger operators with inverse square potential \( L = \Delta - \frac{b}{|x|^2} \) (that is \( \alpha = c = 0 \)) assuming \( b + \left( \frac{N-2}{2} \right)^2 > 0 \). In this case

\[
s_1 = \frac{N - 2}{2} - \sqrt{b + \left( \frac{N - 2}{2} \right)^2}, \quad s_2 = \frac{N - 2}{2} + \sqrt{b + \left( \frac{N - 2}{2} \right)^2}.
\]

Theorem 4.4 shows that \( L_{int} \) endowed with the domain \( D \) generates a positive analytic semigroup in \( L^p(\mathbb{R}^N) \) if and only if

\[
s_1 < \frac{N}{p} < s_2 \quad \text{or} \quad \frac{N}{p} - \frac{N}{2} < 1 + \sqrt{b + \left( \frac{N - 2}{2} \right)^2}.
\]

Observe that this improves the results in [2] and [4]. We point out that although generation results of analytic semigroup for \( p \) in the sharp range above have already been proved in [11, Section 4], the description of domain of the generator \( D_{int}(L) \) seems to be new. Let us analyze it in more detail. By Theorem 4.4 and Proposition 5.5 if \( s_1 + 2 \leq \frac{N}{p} \leq s_2 + 2 \) then \( L_{int} \) coincides with \( L_{min} \). In particular, if

\[
s_1 + 2 < \frac{N}{p} < s_2 + 2 \quad \text{or} \quad \frac{N}{p} - \frac{N}{2} - 1 < \sqrt{b + \left( \frac{N - 2}{2} \right)^2},
\]

then the domain is given by

\[
D_{p,\alpha} = \{ u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{\text{loc}}(\Omega), \: D^2u, \: |x|^{-1}\nabla u, \: |x|^{-2}u \in L^p(\mathbb{R}^N) \}.
\]

We remark that in this case the generation result for \( L_{min} \) is also stated in [31, Section 3]. If \( s_1 < \frac{N}{p} \leq s_1 + 2 \), then setting \( \alpha'_1 = s_1 - \frac{N}{p} + 2 \in [0, 2) \) we have from Theorem 5.4

\[
D_{int}(L) = \left\{ u \in D_{max}(L) \cap W^{2,p}(B^c) : |x|^{\alpha}D^2u, |x|^{\alpha - 1}\nabla u, |x|^{\alpha - 2}u \in L^p(B) \quad \forall \alpha > \alpha'_1 \right\}.
\]

In Example 7.3 we show that, when \( p = 2 \), then \( L_{int} \) coincides with the Friedrich’s extension of \( L_{min} \).

When \( N = 1, L \) is the so called Calogero Hamiltonian and the above results are of \( L^p \) generalizations of the well-known properties of the Calogero operator in \( L^2 \). In fact, specializing to the case \( N = 1 \) (where we recall that \( \mathbb{R} \) should be substituted by \([0, \infty[) \) and \( p = 2 \), we obtain: if \( b \geq \frac{3}{4} \), then \( L_{min} \) is nonnegative and selfadjoint and coincides with \( L_{max} \); if \( -\frac{1}{4} \leq b < \frac{3}{4} \), then \( L_{min} \) and \( L_{max} \) are not selfadjoint but there exists a selfadjoint extension of \( L_{min} \). The critical case \( b = -\frac{1}{4} \) is explained in next example.

Example 7.2 Consider now the Schrödinger operators with inverse square potential \( L = \Delta - \frac{b}{|x|^2} \) in the critical case \( b + \left( \frac{N-2}{2} \right)^2 = 0 \), where \( s_0 = \frac{N-2}{2} \). If \( s_0 \leq \frac{N}{p} \leq s_0 + 2 \) or (for \( N \geq 3 \))

\[
\frac{2N}{N + 2} \leq p \leq \frac{2N}{N - 2}
\]

then, by Theorem 6.1 there exists an operator \( L_{int} \) such that \( L_{min} \subseteq L_{int} \subseteq L_{max} \) and \( L_{int} \) generates a positive analytic semigroup in \( L^p(\mathbb{R}^N) \) which is coherent with respect to all \( p \). It is worth noticing that the interval of generation is closed, in contrast with the case \( b + \left( \frac{N-2}{2} \right)^2 > 0 \). Observe
that this improves the results in [2, 3] and [11 Section 4] where, although generation results of analytic semigroup for \( p \) in the sharp range above have already been proved, the description of domain of the generator \( D_{int}(L) \) seems to be new. More precisely, by Proposition 6.13 if \( \frac{N}{p} = s_0 + 2 \) then \( L_{int} \) coincides with \( L_{\min} \) and, by Proposition 6.12 if \( \frac{N}{p} = s_0 \) then \( L_{int} \) coincides with \( L_{\max} \). If
\[
s_0 \leq \frac{N}{p} \leq s_0 + 2,
\]
then, by Proposition 6.2 the domain is given by
\[
D_{int}(L) = \{ u \in D_{\max}(L) : |x|^{\theta_0(\alpha - 2)}|\log |x||^{-\frac{2}{p}} u \in L^p(B_{1/2}) \},
\]
where \( \theta_0 \in [0, 1] \) satisfies \( \frac{N}{p} = s_0 + 2\theta_0 \). Integrability of first and second derivatives for \( u \in L_{int} \) is given in Proposition 6.16.

If \( N = 1 \), then \( b = \frac{1}{2} \) and \( L \) is the Calogero operator in \([0, \infty[\). Then \( L_{\min} \) and \( L_{\max} \) are not selfadjoint but there exists a selfadjoint extension of \( L_{\min} \). In this case \( L_{int} \) coincides with the Friedrich’s extension of \( L_{\min} \) (this fact in a more general context is explained in Example 7.3).

Example 7.3 More generally, we can consider the formally selfadjoint operators
\[
L = \text{div}(|x|^\alpha \nabla) - b|x|^{\alpha - 2},
\]
which corresponds to \( c = \alpha \) in [1] and we focus our attention to \( p = 2 \). If \( \alpha = 2 \), \( L_{\min} = L_{\max} \) are selfadjoint and their domain is already given in [12], see also Proposition 7.3.

We consider the case \( \alpha \neq 2 \) and \( D_{\alpha} = b + (\frac{N - 2 + \alpha}{2})^2 \geq 0 \). Since \( c = \alpha \) the function \( f \) defined in [3] satisfies
\[
f(s) = b + s(N - 2 + \alpha - s), \quad f \left( \frac{N - 2 + \alpha}{2} \right) = D_{\alpha}.
\]

If \( D_{\alpha} > 0 \), then condition (73), which is equivalent to (29), is satisfied with \( \theta = \frac{1}{\alpha} \). If \( D_{\alpha} = 0 \), then
the assumption in Theorem 6.1 is satisfied with \( \theta_0 = \frac{1}{2} \). Therefore, \( L_{int} \) defined by (33) if \( D_{\alpha} > 0 \) and (48) if \( D_{\alpha} = 0 \) generates an analytic semigroup in \( L^2(\mathbb{R}^N) \) and its domain is characterized by (12) if \( D_{\alpha} > 0 \), which gives a precise regularity. Moreover, \( L_{int} \) is the limit of \( L_t \) in the resolvent sense (see Subsection 5.1 and Section 6) and each \( L_t \) is nonnegative and selfadjoint for every \( t > 0 \), since it coincides with \( (L_{int})_{min} \). This yields that \( L_{int} \) is also selfadjoint.

It is worth noticing that, since \( L_{\min} \) is symmetric, \( L_{\min} \) is selfadjoint if and only if \( L_{\max} \) is selfadjoint. This means that the conditions on generation by \( L_{\min} \) and \( L_{\max} \) given in Theorems 4.1 and 4.6 coincide. This fact can be easily found via the identity
\[
f \left( \frac{N}{2} \right) = f \left( \frac{N}{2} + \alpha - 2 \right) = D_{\alpha} - \left( \frac{\alpha - 2}{2} \right)^2
\]
(74)
Moreover, (14) implies that if \( D_{\alpha} \geq (\frac{\alpha - 2}{2})^2 \), that is,
\[
b \geq - \left( \frac{N - 2 + \alpha}{2} \right)^2 + \left( \frac{\alpha - 2}{2} \right)^2,
\]
then \( L_{\min} \) is nonnegative and selfadjoint and coincides with \( L_{\max} \) (see also Proposition 5.5), hence with \( L_{int} \). On the contrary, if \( 0 \leq D_{\alpha} < (\frac{\alpha - 2}{2})^2 \), that is,
\[
- \left( \frac{N - 2 + \alpha}{2} \right)^2 \leq b < - \left( \frac{N - 2 + \alpha}{2} \right)^2 + \left( \frac{\alpha - 2}{2} \right)^2,
\]
then \( L_{\min} \) (and \( L_{\max} \)) does not generate a semigroup in \( L^2(\mathbb{R}^N) \) but its self-adjoint extension \( L_{int} \) does it. \( L_{int} \) is the unique among the infinitely many self-adjoint extensions of \( L_{\min} \) which has the minimality property with respect to positive solutions, as explained in Propositions 5.10, 6.18.

Note that the constant \((\frac{4}{N-2})^2\) with \( \alpha = 0 \) coincides with the difference of the optimal constants in the usual Hardy and Rellich inequalities \( \frac{N(N-4)}{4} - (\frac{N-2}{2})^2 = 1. \)

When \( 0 \leq D_\alpha \) a self-adjoint extension of \( L_{\min} \) can be constructed by closing the nonnegative form

\[
a(u, v) = \int_{\mathbb{R}^N} |x|^{\alpha} \nabla u \cdot \nabla v dx + b \int_{\mathbb{R}^N} |x|^{\alpha-2} u v dx, \quad D(a) = C_c^\infty(\Omega).
\]

This extension is called the Friedrich's extension \( L_F \) of \( L_{\min} \) and is one of nonnegative selfadjoint extensions of \( L_{\min} \) (not unique, unless \( L_{\min} \) itself is self-adjoint). In general, \( D(L_F) \), the domain of \( L_F \), is given only formally. However, we can characterize \( D(L_F) \) by showing that \( L_F \) and \( L_{int} \) coincide. Since both operators generate semigroups, it suffices to observe that \( D(L_F) \subset D_{int}(L) \) and we divide the proof accordingly to \( D_\alpha > 0 \) and \( D_\alpha = 0 \).

(The case \( D_\alpha > 0 \)). We see from (26) with \( p = 2 \) and \( c = \alpha \) that for every \( u \in C_c^\infty(\Omega) \),

\[
a(u, u) \geq D_\alpha \int_{\mathbb{R}^N} |x|^{\alpha-2} |u|^2 dx = D_\alpha \| |x|^{\alpha-2} u \|_2^2.
\]

This implies that \( D(\overline{a}) \subset D(|x|^{\frac{\alpha-2}{2}}) \), where \( \overline{a} \) is the closure of the form \( a \). Hence, since \( \theta = 1/2 \), by the definition of \( L_{int} \), see [13], we have

\[
D(L_F) \subset D_{\max}(L) \cap D(\overline{a}) \subset D_{int}(L).
\]

(The case \( D_\alpha = 0 \)). Set

\[
U_0 := \begin{cases} B_{1/2} & \text{if } \alpha < 2, \\ B_2^c & \text{if } \alpha > 2, \end{cases} \quad U_1 := \begin{cases} B_1 & \text{if } \alpha < 2, \\ B_1^c & \text{if } \alpha > 2 \end{cases}
\]

and let \( \eta \in C_c^\infty(\mathbb{R}^N) \) satisfy \( 0 \leq \eta \leq 1, \eta \equiv 1 \) in \( U_0 \) and \( \eta \equiv 0 \) in \( U_1^c \). Then using [52] if \( \alpha < 2 \) and [63] if \( \alpha > 2 \), with \( p = 2 \) and \( \theta_0 = 1/2 \), we see that

\[
\| \chi_{U_0} |x|^{\frac{\alpha-2}{2}} \log |x|^{-1/2} u \|_2^2 \leq \int_{U_1} |x|^{\alpha-2} |\log |x||^{-2} |\eta u|^2 dx \leq 4a(\eta u, \eta u).
\]

Thus, we have

\[
a(\eta u, \eta u) = \int_{\mathbb{R}^N} \eta^2 (|x|^{\alpha} |\nabla u|^2 + b|x|^{\alpha-2} |u|^2) dx
\]

\[
+ 2\Re \int_{U_0^c} |x|^{\alpha} (\eta \nabla \eta) \cdot (\pi \nabla u) dx + \int_{U_0^c} |x|^{\alpha} |\nabla \eta|^2 |u|^2 dx
\]

\[
\leq a(u, u) + 2 \| |x|^{\frac{\alpha}{2}} \nabla \eta \|_{\infty} \| u \|_2 \| |x|^{\frac{\alpha}{2}} \nabla u \|_{L^2(U_1 \setminus U_0)} + \| |x|^{\frac{\alpha}{2}} \nabla \eta \|_{L^2(U_1 \setminus U_0)} \| u \|_2^2.
\]

Since

\[
\| |x|^{\frac{\alpha}{2}} \nabla u \|_{L^2(U_1 \setminus U_0)} \leq \int_{U_1 \setminus U_0} (|x|^{\alpha} |\nabla u|^2 + b|x|^{\alpha-2} |u|^2) dx + |b| \int_{U_1 \setminus U_0} |x|^{\alpha-2} |u|^2 dx
\]

\[
\leq a(u, u) + 2^{\alpha-2} |b| \| u \|_2^2,
\]

35
where $C$ is a constant independent of $u$. This implies that
\[ D(\mathfrak{F}) \subset D \left( \chi u_0 |x|^{\frac{\alpha}{2}} \log |x|^{-1} u \right). \]

Therefore, from (48) we have
\[ D(L_F) \subset D_{\text{max}}(L) \cap D(\mathfrak{F}) \subset D_{\text{int}}(L). \]

**Example 7.4** Let $b = c = 0$, that is $L = |x|^{\alpha} \Delta$ and assume first that $N \neq 2$ so that $b + (\frac{N-2+c}{2})^2 = (\frac{N-2}{2})^2 > 0$. Since $s_1 = 0$, $s_2 = N - 2$, $L_{\text{int}}$ endowed with the domain \((142)\) generates a positive analytic semigroup in $L^p(\mathbb{R}^N)$ if and only if
\[ \min\{0, 2 - \alpha\} < \frac{N}{p} < N - 2 + \max\{0, 2 - \alpha\}. \]

If $\alpha < 2$, the condition reads $\frac{N}{p} < N - \alpha$ and, if $\alpha > 2$, $\frac{N}{p} < N - 2$.

By Theorem $4.4$ $L_{\text{int}} = L_{\text{min}}$ if $2 - \alpha < N/p < N - \alpha$ and the domain is given by
\[ D_{p,\alpha} = \{ u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{\text{loc}}(\Omega), |x|^\alpha D^2 u, |x|^{\alpha-1} \nabla u, |x|^\alpha u \in L^p(\mathbb{R}^N) \}. \]

By Theorem $4.4$ $L_{\text{int}} = L_{\text{max}}$ if $N/p < N - 2$. We observe also that, when $N = 1$ and $\alpha \geq 2$, the interval of admissible $p$ is contained in the negative axis and the operator is not a generator, as proved in [15] for the operator $(1 + |x|^\alpha) \Delta$.

Observe that this improves the results of [16] Section 8]. Indeed here we get a more precise description of the domain of $L$. Moreover we establish here non existence results for semigroups outside the above interval whereas in [16] only the non existence of a positive semigroups is proved.

Let us consider the critical case $N = 2$ where we have $s_0 = 0$. If $\alpha > 2$, the interval $[s_0 + \min\{0, 2 - \alpha\}, s_2 + \max\{0, 2 - \alpha\}] = [2 - \alpha, 0]$ is contained in the negative real axis and therefore the operator $L$ is not a generator in $L^p(\mathbb{R}^2)$. We point out that the same result has been obtained in [15] for the operator $(1 + |x|^\alpha) \Delta$.

When $\alpha < 2$, the operator $L_{\text{int}}$, endowed with the domain
\[ D_{\text{int}}(L) = \{ u \in D_{\text{max}}(L) : |x|^\theta_0 \alpha^{-2} |x|^{-\frac{\alpha}{p}} u \in L^p(B_{1/2}) \}, \]

where $\theta_0 \in (0, 1]$ satisfies $\frac{2}{p} = \theta_0 (2 - \alpha)$, generates an analytic semigroup for $p > \frac{2}{2 - \alpha}$. In particular, if $\alpha < 0$, the operator $L_{\text{int}}$ generates an analytic semigroup for every $1 < p < \infty$.

**Example 7.5** Let $b = 0$, that is $L = |x|^{\alpha} \Delta + c|x|^{\alpha-1} \frac{\nabla}{|x|} \nabla$. If $N - 2 + c \neq 0$ then $(\frac{N-2+c}{2})^2 > 0$ and $s_1 = 0$, $s_2 = N - 2 + c$. By Theorem $4.4$ if
\[ \min\{0, 2 - \alpha\} < \frac{N}{p} < N - 2 + c + \max\{0, 2 - \alpha\}, \]

$L_{\text{int}}$ endowed with the domain \((12)\) generates a positive analytic semigroup in $L^p(\mathbb{R}^N)$. In particular, if $\alpha < 2$, the condition reads $\frac{N}{p} < N - \alpha + c$ and, if $\alpha > 2$, $\frac{N}{p} < N - 2 + c$. By Theorem $4.4$ $L_{\text{int}} = L_{\text{min}}$ when $2 - \alpha < N/p < N - \alpha + c$ and the domain is given by
\[ D_{p,\alpha} = \{ u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{\text{loc}}(\Omega), |x|^\alpha D^2 u, |x|^{\alpha-1} \nabla u, |x|^\alpha u \in L^p(\mathbb{R}^N) \}. \]
By Theorem 4.3, \( L_{\text{int}} = L_{\text{max}} \) if \( N/p < N - 2 + c \).

Observe that this improves the results in [15]. Indeed here the degeneracy near the origin is also allowed and the domain description is more precise.

The critical case \( b + (N - 2 + c) = 0 \) occurs for \( c = 2 - N \) and we have \( s_0 = 0 \). As in the previous example, if \( \alpha > 2 \), the interval \([s_0 + \min\{0, 2 - \alpha\}, s_0 + \max\{0, 2 - \alpha\}] = [2 - \alpha, 0] \) is contained in the negative real axis, therefore the operator \( L \) is not a generator in any \( L^p(\mathbb{R}^N) \). The same phenomenon has been already proved in [16] for the operator \( (1 + |x|^\alpha)\Delta + c|x|^{\alpha - 1}|x|\nabla \). When \( \alpha < 2 \), the operator \( L_{\text{int}} \), endowed with the domain

\[
D_{\text{int}}(L) = \{ u \in D_{\text{max}}(L) ; \ |x|^\theta_0(\alpha - 2) \log |x|^{-\frac{2}{\alpha}} u \in L^p(B_{1/2}) \},
\]

where \( \theta_0 \in (0, 1] \) satisfies \( \frac{2}{\alpha} = \theta_0(2 - \alpha) \), generates an analytic semigroup for \( p > \frac{2}{2 - \alpha} \). In particular, if \( \alpha < 0 \), the operator \( L_{\text{int}} \) generates an analytic semigroup for every \( 1 < p < \infty \).

**Example 7.6** For certain choices of the parameters \( \alpha \), \( b \) and \( c \), \( L_{\text{int}} \) can generate an analytic semigroup in \( L^p(\mathbb{R}^N) \) even though \( L \) is not dissipative for any \( 1 < q < \infty \) and \( L_{\text{max}} \) does not generate for any \( 1 < q < \infty \). Similarly, \( L_{\text{int}} \) can generate an analytic semigroup in \( L^p(\mathbb{R}^N) \) even though \( L_{\text{min}} \) and \( L_{\text{max}} \) do not generate for any \( 1 < q < \infty \).

(a) Assume that \( b = 0 \) and \( N - 2 + c < 0 \). It follows that \( s_1 = N - 2 + c < s_2 = 0 \). Therefore the operator \( L_{\text{max}} \) does not generate an analytic semigroup for any \( 1 < q < \infty \). If, in addition, we assume that \( N \geq 2 \) and \( 0 \leq \alpha < 2 \), the dissipativity condition

\[
s_1 \leq \frac{N + \alpha - 2}{p} \leq s_2
\]

is never satisfied but the generation condition for \( L_{\text{int}} \) is valid for some \( p \) since \( s_2 + 2 - \alpha > 0 \).

(b) We keep the conditions \( b = 0 \) and \( N - 2 + c < 0 \) so that \( s_1 < s_2 = 0 \) and the operator \( L_{\text{max}} \) does not generate for any \( 1 < q < \infty \) but we assume \( \alpha < 2 \) and \( N \leq s_1 + 2 - \alpha \), that is \( \alpha \leq c \). It follows that \( L_{\text{min}} \) never generates an analytic semigroup. Finally observe that, since \( s_1 \leq 0 \) and \( s_2 + 2 - \alpha > N \), the operator \( L_{\text{int}} \) generates an analytic semigroup for every \( 1 < p < \infty \).

(c) If \( L_{\text{int}} \) generates for some \( p \) one can always find a \( 1 < q < \infty \) such that \( L \) is dissipative in \( L^q \) or \( L_{\text{min}} \) generates or \( L_{\text{max}} \) generates in \( L^q \).

In fact, assume that \( \alpha < 2 \) and that the generation condition for \( L_{\text{int}} \) is true for some \( 1 < p < \infty \) that is

\[
s_1 < \frac{N}{p} < s_2 + 2 - \alpha
\]

In order to violate the generation condition for \( L_{\text{max}} \) for every \( 1 < q < \infty \) we should have \( s_1 < s_2 \leq 0 \). Indeed, if \( s_1 < 0 < s_2 \), then we can find some \( q \) such that \( s_1 < \frac{N}{q} < s_2 \). If \( s_1 \) and \( s_2 \) are positive, the generation condition for \( L_{\text{max}} \) is violated only if \( s_1 \geq \frac{N}{q} \) but this is not possible since \( s_1 < \frac{N}{\alpha} \).

Therefore we have: \( s_1 < s_2 \leq 0 \) and \( s_2 + 2 - \alpha > 0 \). If \( L_{\text{min}} \) does not generate in any \( L^q \), then \( s_1 + 2 - \alpha \geq N \). If we choose \( q \) such that

\[
\frac{N + \alpha - 2}{s_1} < q < \frac{N + \alpha - 2}{s_2}
\]

then \( 1 < q < \infty \), \( s_1 < (N + \alpha - 2)/q < s_2 \) and \( L \) is dissipative in \( L^q \).

37
Example 7.7 Let \( L_1 = |x|^\alpha \Delta + c|x|^{\alpha-1} - \nabla \cdot b|x|^{-2} \) in the ball \( B_R \), with Dirichlet boundary conditions. We define the domain of \( L_1 \) and deduce generation results for this operator in the ball by those in the whole space.

Definition 7.8
\[
D_{\max}(L_1) = \{ u \in L^p(B_R) \cap W^{2,p}(B_R \setminus B_\varepsilon) \mid \forall \varepsilon > 0 : u(x) = 0 \text{ if } |x| = R, \quad L_1 u \in L^p(B_R) \}.
\]
Observe that the Dirichlet boundary condition \( u(x) = 0 \) for \( |x| = R \) makes sense, since \( u \) has second derivatives in \( L^p \) in a neighborhood of the boundary of \( B_R \). By elliptic regularity \( L_1 \) is closed on its maximal domain. If \( \alpha \geq 2 \) the function \( a(x) = |x|^\alpha \) satisfies the inequality \( |\nabla a^{1/2}| \leq C \) in the ball \( B_R \) even though not globally in \( \mathbb{R}^N \) when \( \alpha > 2 \). In analogy with [7] we define the domain of \( L_1 \) as follows.

Definition 7.9 If \( \alpha \geq 2 \) we set
\[
D_p(L_1) = \{ u \in L^p(B_R) \cap W^{2,p}(B_R \setminus B_\varepsilon) \mid \forall \varepsilon > 0 : u(x) = 0 \text{ if } |x| = R, \quad |x|^{\alpha/2} \nabla u, |x|^{\alpha} D^2 u \in L^p(B_R) \}.
\]
By the results in [7] and in [10], we immediately get generation for every \( 1 < p < \infty \) when \( \alpha \geq 2 \).

Proposition 7.10 If \( \alpha \geq 2 \), then \( D_{\max}(L_1) = D_p(L_1) \), the operator \( L_1 \) is closed on its domain and generates an analytic semigroup in \( L^p(B_R) \) for every \( 1 < p < \infty \).

Concerning the case \( \alpha < 2 \), the result proved in \( \mathbb{R}^N \) is still true. It can be deduced by Theorem 5.4 by using the methods of [10] Proposition 5.7.

Proposition 7.11 Let \( 1 < p < \infty, \alpha < 2 \). If \( b + (N - 2 + c)^2/4 > 0 \), then a suitable realization of \( L_{1,\min} \subset L_{1,\int} \subset L_{1,\max} \) generates a semigroup in \( L^p(B_R) \) if and only if \( s_1 < N/p < s_2 + 2 - \alpha \). The semigroup is analytic and positive. Moreover, setting \( \alpha' = \theta(\alpha - 2) + 2, D_{\int}(L_1) \) is given by
\[
\left\{ u \in D_{\max}(L_1) : |x|^{\alpha'} D^2 u, |x|^{\alpha'-1} \nabla u, |x|^{\alpha'-2} u \in L^p(B_R) \text{ for every } \theta \in I \right\}
\]
where \( I \) is the interval of all \( \theta \in [0, 1] \) such that \( f \left( \frac{N}{p} + \theta(\alpha - 2) \right) > 0 \). If \( b + (N - 2 + c)^2/4 = 0 \), set \( s_0 = \frac{N - 2 + c}{2} \), a suitable realization of \( L_{1,\min} \subset L_{1,\int} \subset L_{1,\max} \) generates a semigroup in \( L^p(B_R) \) if and only if \( s_0 \leq N/p \leq s_0 + 2 - \alpha \). The semigroup is analytic and positive. Moreover,
\[
D_{\int}(L_1) = \{ u \in D_{\max}(L_1) : |x|^{\theta_0(\alpha - 2)} \log |x| - \frac{2}{p} u \in L^p(B_{1/2}) \},
\]
where \( \theta_0 \in [0, 1] \) satisfies \( \frac{N}{p} = s_0 + \theta_0(2 - \alpha) \).

Let us show the compactness of the resolvent for \( \alpha < 2 \).

Proposition 7.12 Let \( \alpha < 2 \) and the assumptions of Proposition 7.11 be satisfied. Then the resolvent of \( L_{1,\int} \) is compact.

Let us prove that \( D_{\int}(L_1) \) is compactly embedded into \( L^p(B_R) \). Consider the case \( b + (N - 2 + c)^2/4 > 0 \). Let \( \mathcal{U} \) be a bounded subset of \( D(L_{1,\int}) \). By the domain characterization, we obtain \( \int_{B_R} |x|^{\alpha'-2} u^p \leq M \) for some positive \( M \) and for every \( u \in \mathcal{U} \). Since \( \alpha' < 2 \), given \( \varepsilon > 0 \), there exists \( 0 < r < R \) such that \( \int_{|x|<r} |u|^p < \varepsilon^p \) for every \( u \in \mathcal{U} \). Let \( \mathcal{U}' \) be the set of the restrictions of
the functions in $U$ to $B_R \setminus B_r$. Since the embedding of $W^{2,p}(B_R \setminus B_r)$ into $L^p(B_R \setminus B_r)$ is compact, the set $U'$ which is bounded in $W^{2,p}(B_R \setminus B_r)$ is totally bounded in $L^p(B_R \setminus B_r)$. Therefore there exist $n \in \mathbb{N}$, $f_1, \ldots, f_n \in L^p(B_R \setminus B_r)$ such that

$$U' \subseteq \bigcup_{i=1}^{n} \{ f \in L^p(B_R \setminus B_r) : \|f - f_i\|_{L^p(B_R \setminus B_r)} < \varepsilon \}.$$  

Set $f_i = f_i$ in $B_R \setminus B_r$ and $f_i = 0$ in $B_r$. Then $f_i \in L^p(B_R)$ and

$$U \subseteq \bigcup_{i=1}^{n} \{ f \in L^p(B_R) : \|f - f_i\|_{L^p(B_R)} < 2\varepsilon \}.$$ 

It follows that $U$ is relatively compact in $L^p(B_R)$.

If $b + (N - 2 + c)^2/4 = 0$ and $\frac{N}{p} \neq s_0$ (that is $\theta_0 \neq 0$), the proof follows in similar way since the weight $|x|^\theta \log |x|$ tends to $0$ as $x \to 0$. If $\frac{N}{p} = s_0$, we consider the adjoint operator $L_{1, int}$ in $L^p(B_R)$ whose domain is given as above with $\theta_0 = 1 - \theta = 0$, see the proof of Lemma 5.3.

Then the resolvent of $L_{1, int}$ is compact in $L^p(B_R)$, hence the semigroup, since it is analytic. By duality the semigroup generated by $L_{1, int}$ is compact in $L^p(B_R)$, hence the resolvent. \hfill \Box

**Example 7.13** Let $L_2 = |x|^\alpha \Delta + c|x|^{\alpha-1} \nabla - b |x|^{\alpha-2}$ in the exterior domain $B_R^c$, with Dirichlet boundary conditions. We proceed as in the previous Example.

**Definition 7.14**

$$D_{max}(L_2) = \{ u \in L^p(B_R^c) \cap W^{2,p}(B_R^c \setminus B_r) \forall r > 0 : u(x) = 0 \text{ if } |x| = R, \ L_2 u \in L^p(B_R^c) \}.$$ 

As before, the Dirichlet boundary condition $u(x) = 0$ for $|x| = R$ makes sense, since $u$ has second derivatives in $L^p$ in a neighborhood of the boundary of $B_R^c$. By local elliptic regularity, $L_2$ is closed on its maximal domain. Observe that, when $\alpha \leq 2$, the function $a(x) = |x|^\alpha$ satisfies the inequality $|\nabla a^{1/2}| \leq C$ in the exterior domain $B_R^c$. By following \[7\], we can also define the domain $D_p(L_2)$ as follows.

**Definition 7.15** If $\alpha \leq 2$ we set

$$D_p(L_2) = \{ u \in L^p(B_R^c) \cap W^{2,p}(B_R^c \setminus B_r) \forall r > 0 : u(x) = 0 \text{ if } |x| = R, \ |x|^{\theta/2} \nabla u, |x|^\alpha D^2 u \in L^p(B_R^c) \}.$$ 

As before the first generation results immediately follows from \[7\] and the results in \[16\].

**Proposition 7.16** If $\alpha \leq 2$, then $D_{max}(L_2) = D_p(L_2)$, the operator $L_2$ is closed on its domain and, for every $1 < p < \infty$ generates an analytic semigroup in $L^p(B_R^c)$.

In the case $\alpha > 2$, the following result can be deduced from Theorem \[6\] through \[16\] Proposition 5.6.

**Proposition 7.17** Let $1 < p < \infty$, $\alpha > 2$. If $b + (N - 2 + c)^2/4 > 0$, a suitable realization of $L_{2, min} \subset L_{2, int} \subset L_{2, max}$ generates a semigroup in $L^p(B_R^c)$ if and only if $s_1 + 2 - \alpha < N/p < s_2$. The semigroup is analytic and positive. Moreover, setting $\alpha' = \alpha - 2$, $D_{int}(L_2)$ is given by

$$\left\{ u \in D_{max} : |x|^\alpha D^2 u, |x|^\alpha \nabla u, |x|^\alpha \Delta u \in L^p(B_R^c) \text{ for every } \theta \in I \right\}$$

39
where $I$ is the interval of all $\theta \in [0, 1]$ such that $f \left( \frac{N}{p} + \theta (\alpha - 2) \right) > 0$. If $b + (N - 2 + c)^2/4 = 0$, set $s_0 = \frac{N - 2 + c}{2}$, a suitable realization of $L_{2, \min} \subset L_{2, \text{int}} \subset L_{2, \max}$ generates a semigroup in $L^p(B_R)$ if and only if $s_0 + 2 - \alpha \leq N/p \leq s_0$. The semigroup is analytic and positive. Moreover,\[ D_{\text{int}}(L_2) = \{ u \in D_{\max}(L_2) ; |x|^{\theta_0 (\alpha - 2)} |\log |x||^{-\frac{\theta}{p}} u \in L^p(B^c_R) \}, \]
where $\theta_0 \in [0, 1]$ satisfies $\frac{N}{p} = s_0 + \theta_0 (2 - \alpha)$.

**Proposition 7.18** Let $\alpha > 2$ and the conditions of Theorem 7.17 be satisfied. Then the resolvent of $L_2$ is compact in $L^p(B^c_R)$.

**Appendix**

**Theorem A.1** ([24, Theorem 1.1]) Let $A$ and $B$ be densely defined operators in Banach space $X$. Assume that

i) $-A$ generates a bounded analytic semigroup on $X$;

ii) $D(A) \subset D(B)$ and there exists $\beta_0 > 0$ such that for every $u \in D(A)$,

\[ \text{Re} \left( Au, F(Bu) \right)_{X,X'} \geq \beta_0 \| Bu \|^2, \tag{75} \]

\[ (u, F(Bu))_{X,X'} \geq 0, \tag{76} \]

where $F$ is single-valued duality map from $X$ to the dual space $X'$. Then for every $k \in \mathbb{C}$ satisfying $\text{Re} k > -\beta_0$, $-(A + kB)$ with domain $D(A)$ also generates a bounded analytic semigroup on $X$. Moreover, if $A$ has a compact resolvent, then $A + kB$ also has a compact resolvent. Assume further that $X$ is a Banach lattice, $A$ has positive resolvent and $B$ is also positive. Then $A + kB$ is also positive resolvent for every $k \in (-\beta_0, 0)$.

**References**

[1] W. ARENDT, J.A. GOLDSTEIN, G.R. GOLDSTEIN: Outgrowths of Hardy’s inequality, Recent advances in differential equations and mathematical physics, 51–68, Contemp. Math. 412, Amer. Math. Soc. Providence, RI, 2006.

[2] P. BARAS, J.A. GOLDSTEIN: The heat equation with a singular potential, Trans. Amer. Math. Soc. 284 (1984), 121–139.

[3] R. BEALS, R. WONG: Special Functions, Cambridge Studies in Advances Mathematics, 126 (2010).

[4] H. BREZIS, J.L. VAZQUEZ: Blow-up solutions of some nonlinear elliptic problems, Rev. Mat. Univ. Compl. Madrid 10 (1997), 443–469.

[5] X. CABRE, Y. MARTEL: Existence versus explosion instantanée pour des équationes de la chaleur lineaires avec potentiel singularies C.R. Acad. Sci. Paris 329 (1999), 973–978.

[6] D. G. COSTA: Some new and short proofs for a class of Caffarelli-Kohn-Nirenberg type inequalities J. Math. Anal. Appl. 337 (2008) 311–317.
[7] S. Fornaro, L. Lorenzi: Generation results for elliptic operators with unbounded diffusion coefficients in $L^p$ and $C_0$-spaces, Discrete and Continuous Dynamical Systems A18 (2007), 747–772.

[8] D. M. Gitman, I. V. Tyutin and B. L. Voronov: Self-adjoint extensions and spectral analysis in the Calogero problem, J. Phys. A: Math. Theor. 43 (2010), 145–205.

[9] J.A. Goldstein: Semigroups of Linear Operators and Applications , Oxford Mathematical Monographs, Oxford University Press, New York (1985).

[10] T. Kato: Perturbation Theory for Linear Operators, Grundlehren der math. Wissenschaten 132, Springer-Verlag, Berlin and New York (1976).

[11] V. Liskevich, Z. Sobol, H. Vogt: On the $L^p$-theory for $C_0$-semigroups associated with second-order elliptic operators II, Journal of Functional Analysis 193 (2002), 55–76.

[12] G. Metafune, M. Sobajima, C. Spina: Weighted Calderón-Zygmund and Rellich inequalities in $L^p$, preprint (2013).

[13] G. Metafune, M. Sobajima, C. Spina: Spectral properties of operators obtained by localization methods, preprint (2014).

[14] G. Metafune, C. Spina: An integration by parts formula in Sobolev spaces, Mediterranean Journal of Mathematics 5 (2008), 359–371.

[15] G. Metafune, C. Spina: Elliptic operators with unbounded diffusion coefficients in $L^p$ spaces, Annali Scuola Normale Superiore di Pisa Cl. Sc. (5), 11 (2012), 303–340 .

[16] G. Metafune, C. Spina: A degenerate elliptic operators with unbounded coefficients, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei, to appear.

[17] G. Metafune, C. Spina: Kernel estimates for some elliptic operators with unbounded coefficients, Discrete and Continuous Dynamical Systems A32 (6) (2012), 2285–2299.

[18] G. Metafune, C. Spina, C. Tacelli: Elliptic operators with unbounded diffusion and drift coefficients in $L^p$ spaces, Advances Diff. Equat., (to appear).

[19] G. Metafune, C. Spina, C. Tacelli: On a class of elliptic operators with unbounded diffusion coefficients, preprint (2014).

[20] N. Okazawa: $L^p$-theory of Schrödinger operators with strongly singular potentials, Japan. J. Math. 22 (1996), 199–239.

[21] N. Okazawa, M. Sobajima, $L^p$-theory for Schrödinger operators perturbed by singular drift terms, Proceedings of the conference “Differential Equations, Inverse Problems and Control Theory (2013)”, (to appear).

[22] N. Okazawa, M. Sobajima, T. Yokota: Existence of solutions to heat equations with singular lower order terms, J. Differential Equations, 256 (2014), 3568-3593.

[23] M. Sobajima: $L^p$-theory for second-order elliptic operators with unbounded coefficients, J. Evol. Equ. 12 (2012), 957-971.
[24] M. Sobajima: A class of relatively bounded perturbations for generators of bounded analytic semigroups in Banach spaces, *J. Math. Anal. Appl.* (416) (2014), 855-861.

[25] M. Sobajima, C. Spina: Generation results for some elliptic second order operators with unbounded diffusion coefficients, preprint (2014).

[26] M. Sobajima, S. Watanabe: Landau-Lifschitz conjecture about the motion of a quantum mechanical particle under the inverse square potential, preprint (2013).

[27] C. Spina: Kernel estimates for some elliptic elliptic operators with unbounded diffusion coefficients in the one-dimensional and bi-dimensional cases, *Semigroup Forum* 86 (1) (2013), 67–82.

[28] J.L. Vazquez, E. Zuazua: The Hardy Inequality and the Asymptotic Behaviour of the Heat Equation with an Inverse Square Potential, *Journal of Functional Analysis* 173 (2000), 103–153.