HYERS-ULAM-RASSIAS STABILITY OF HIGH-DIMENSIONAL QUATERNION IMPULSIVE FUZZY DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this paper, the Hyers-Ulam-Rassias stability of high-dimensional quaternion fuzzy dynamic equations with impulses is first considered on time scales. Some fundamental calculus results of the high-dimensional fuzzy quaternion functions in fuzzy quaternion space are established. Based on it, some sufficient conditions are obtained to guarantee the Hyers-Ulam-Rassias stability of the quaternion impulsive fuzzy dynamic equations in high-dimensional case. Moreover, several examples are provided to show the feasibility of our main results on various types of time scales.

1. Introduction. Many phenomena in the real world does exist inherent inaccuracy and can only be precisely described by fuzzy dynamic equations. As a focus area of research, fuzzy dynamic equations are an important branch of the current information science research and have wide applications (see [9,19]). The notion of strongly generalized differentiability which was introduced in the literature [2] has a broad application of studying fuzzy differential equations (see [3]). Along with the deepening of the development of the applications in fuzzy dynamic equations, many literatures focused on this hot topic and many innovative results have been published covering fuzzy differential equations with fractional derivatives (see [1,24]), initial and boundary value problems (see [29,33]), periodic problems (see [17]), and almost periodic problems (see [39]), etc. Furthermore, some optimized fuzzy arithmetics were introduced such as the generalized Hukuhara difference and division for interval and the unidimensional and multidimensional “boxes” were put forward to solve the fuzzy problems in high-dimensional cases (see [35]). Meanwhile, the discrete time in the fuzzy equations leads to the other important form called fuzzy difference equations, some nice results were established and used to the qualitative analysis of the dynamical behavior of fuzzy iteration solutions, for instance, the stability and existence analysis, and they turn out to be very effective in the fuzzy
control process (see [18]). To unify the various time-hybrid analysis, in 1988, the
time scale theory was proposed by Hilger and it can be applied to unify the study
of various types of dynamic equations including quantum difference equations (or
$q$-difference equations, and the fuzzy $q$-difference equations was discussed in [39]) if
the time scale is equal to $\mathbb{Q} := \{ q^n : q > 1, n \in \mathbb{Z} \} \cup \{0\}$. As is well known that
a time scale is an arbitrary nonempty subset of the real line, for more theory and
applications in this field, we refer the readers to the literatures [7, 12, 15, 40–43, 46]
and the books [4, 44] for more details.

On the other hand, the dynamical behavior of solutions to dynamic equations
does not always stay in a single pattern and may be perturbed by a sudden change.
This rapid impact will alter the previous dynamical path and such a sudden change
is portrayed by the “impulsive term” of the dynamic equation of state (see [25, 45]).
The introduction of the impulsive term in the fuzzy dynamic equations is the key
point in this paper. In the field of applications, the impulsive effect will bring a more
obvious advantages and remove some drawbacks which the non-impulsive dynamic
equations will lead to. Through this optimizing method, many real phenomena with
sudden change evolution process can be accurately depicted by the instantaneous
external terms and their reflected dynamical behavior are perfectly appropriate for
the control processes in many application areas (see [34]).

Since Buckley first introduced the notion of fuzzy complex number (see [5]), it
was applied to investigate the complex fuzzy sets and complex membership grade
(see [30, 37]). As proved in these literatures, the advantages of this expansive def-
dition of complex fuzzy sets is its capacity to accommodate fuzzy cycles. To ex-
tend fuzzy complex number, Moura et al. (see [26]) introduced the notion of the
fuzzy quaternion number and studied some main concepts such as infimum, supre-
mum, distance and arithmetic properties, etc. Beyond that, it is powerful to use
quaternions to model rotation and orientation in engineering, physics and molecular
biology, etc. (see [8, 10, 11, 20–22, 48, 49] and the book [6] of Topics in Clifford
Analysis), many superiorities over real-valued vectors in the applications of these
related fields were demonstrated adequately in the fuzzy environment (see [50, 51]).

As early as in 1940, the famous stability theory of the linear functional equation
was introduced by Hyers (see [13]). Since then a series of mathematical questions
related to this stability theory was collected in the book and studied by Ulam
(see [38]) and improved by Rassias (see [31]). From then on, many new results were
reported on this topic and a new comprehensive stability theory of various types of
differential equations was developed under the framework of Hyers-Ulam-Rassias’s
stable structure including ordinary and impulsive ordinary differential equations (see
[32, 47]), functional equations (see [14, 16]), fuzzy differential equations (see [23, 36]),
differential operators (see [27]) and quaternion differential equations (see [50]).

However, there is no work on the Hyers-Ulam-Rassias stability for quaternion
high-dimensional impulsive fuzzy dynamic equations on time scales. Motivated by
the above, under the fuzzy quaternion space, the Hyers-Ulam-Rassias stability of
three types of impulsive fuzzy dynamic equations on time scales is first considered
in Section 3 and some sufficient conditions to guarantee the Hyers-Ulam-Rassias
stability of their fuzzy solutions are derived. The paper is organized as four sections.
In Section 2, some fundamental results of calculus of fuzzy quaternion functions
are established in the high-dimensional fuzzy quaternion space. In Section 3, we
obtain the sufficient conditions of the Hyers-Ulam-Rassias stability for the high-
dimensional quaternion impulsive fuzzy dynamic equations and several examples
are provided to illustrate the effectiveness of the obtained results. In Section 5, we present a conclusion to end this paper.

2. High-dimensional fuzzy quaternion space and calculus of fuzzy quaternion functions on time scales. For convenience, we denote the set of quaternion by \( \mathbb{Q} \) and introduce some notations, \( \mathbb{Q}^n \) the \( n \)-dimension quaternion vectors, \( \mathbb{Q}_{n \times n} \) the \( n \times n \)-quaternion-matrices set, \( \mathbb{R} \) the real line, \( \mathbb{R}^n \) the \( n \)-dimension real vectors space, \( \mathbb{N} \) the natural numbers set, \( \mathbb{Z} \) the integers set for some \( n \in \mathbb{N} \). Moreover, \( i, j, k \) are the quaternion unites and satisfy the following multiplication

\[
i^2 = j^2 = k^2 = -1, \quad jk = -kj = i, \quad ki = -ik = j, \quad ij = -ji = k.
\]

Let \( x_n \in \mathbb{Q}^n, x_n = (x_1, x_2, \ldots, x_n)^T = (x_{10} + x_{11}i + x_{12}j + x_{13}k, x_{20} + x_{21}i + x_{22}j + x_{23}k, \ldots, x_{n0} + x_{n1}i + x_{n2}j + x_{n3}k)^T, x_{hm} \in \mathbb{R} \) for \( 0 \leq m \leq 3 \) and \( 1 \leq h \leq n \), we define \( \|x_n\| = \sum_{m=1}^{n} \sum_{h=0}^{3} |x_{hm}|. \) Similarly, for \( A \in \mathbb{Q}_{n \times n}, A = [a_{h,m}]_{n \times n} = [a_{h,m,0} + a_{h,m,1}i + a_{h,m,2}j + a_{h,m,3}k]_{n \times n}, \) \( \|A\| \) is defined by \( \|A\| = \sum_{m=1}^{n} \sum_{h=1}^{3} \sum_{v=0}^{3} |a_{h,m,v}|. \)

**Definition 2.1** (\cite{9}). Let \( T = [a, b] \subset \mathbb{R} \) be a compact interval and denote \( E^n = \{u : \mathbb{R}^n \rightarrow [0, 1] \} \) \( u \) satisfies \((i) - (iv))\, \text{where} \,(i) \, u \text{ is a normal, i.e. there exists} \, x_0 \in \mathbb{R}^n \text{such that} \, u(x_0) = 1; \,(ii) \, u \text{ is fuzzy convex}; \,(iii) \, u \text{ is upper semi-continuous}; \,(iv) \, \|u\| = \{x \in \mathbb{R}^n : u(x) > 0\} \text{ is compact. Moreover,} \, |u|^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\} \text{ for} \, \alpha \in (0, 1], \, |u|^\alpha \text{ is a nonempty compact convex subset of} \, \mathbb{R}^n, \, |u+v|^\alpha = |u|^\alpha + |v|^\alpha \text{ and} \, |cu|^\alpha = c^\alpha |u|^\alpha \text{ for any} \, u, v \in E^n \text{ and} \, c > 0. \)

**Lemma 2.1** (\cite{9}). Let \( u, v, w \in E^n \) and \( \lambda \in \mathbb{R} \). Then \( D_{R^n}(u + v, w) = D_{R^n}(u, v) \) and \( D_{R^n}(\lambda u, \lambda v) = |\lambda| D_{R^n}(u, v) \), recall that the Hausdorff metric is defined as \( d(A, B) = \inf \{\varepsilon : A \subset N(B, \varepsilon), B \subset N(A, \varepsilon)\} \), where \( A, B \in \mathcal{P}_k(\mathbb{R}^n) \) and \( N(A, \varepsilon) = \{x \in \mathbb{R}^n : \|x - y\| < \varepsilon \text{ for some} \, y \in A\} \), the symbol \( \mathcal{P}_k(\mathbb{R}^n) \) denotes the family of all nonempty compact convex subsets of \( \mathbb{R}^n \). Define the addition and scalar multiplication in \( \mathcal{P}_k(\mathbb{R}^n) \) as usual. Then \( \mathcal{P}_k(\mathbb{R}^n) \) is a commutative semigroup under addition, i.e., if \( \alpha, \beta \in \mathbb{R} \), \( A, B \in \mathcal{P}_k(\mathbb{R}^n) \), then \( \alpha(A + B) = \alpha A + \beta B, \, \alpha(\beta A) = \alpha \beta A, \, IA = A \text{ and if} \, \alpha, \beta \geq 0 \text{ then} \, (\alpha + \beta) A = \alpha A + \beta A. \)

Next, we will introduce the addition and scalar multiplication of the \( n \)-dimensional quaternion vectors.

**Definition 2.2**. Let \( A \in \mathbb{Q}_{n \times n}, w \in \mathbb{Q}, Q_1^n, Q_2^n \subset \mathbb{Q}^n \). Define the addition and scalar multiplication by \( Q_1^n + Q_2^n = \{q_1 + q_2 : q_1 \in Q_1^n, q_2 \in Q_2^n\}, Q_1^n A = \{q_1 : q_1 \in Q_1^n\} \) and \( wQ_1^n = \{wq_1 : q_1 \in Q_1^n\} \). The symbol \( \mathcal{P}_k(\mathbb{Q}^n) \) denotes the family of all nonempty convex subsets of \( \mathbb{Q}^n \).

**Theorem 2.1**. Let \( A_1, A_2 \in \mathbb{Q}_{n \times n}, w \in \mathbb{Q}, Q_1^n, Q_2^n \subset \mathbb{Q}^n \). Then \((i) \, A_1(Q_1^n + Q_2^n) = A_1Q_1^n + A_1Q_2^n \) and \( w(Q_1^n + Q_2^n) = wQ_1^n + wQ_2^n; \,(ii) \, (\alpha + \beta) Q_1^n = \alpha Q_1^n + \beta Q_1^n \) for \( \alpha, \beta > 0 \) and \( Q_1^n \) is a convex subset of \( Q_2^n \); \((iii) \, (A_1 + A_2)Q_1^n \subset A_1Q_1^n + A_2Q_1^n. \) Moreover, \( (A_1 + A_2)Q_1^n = A_1Q_1^n + A_2Q_1^n \) if for any \( q_1, q_2 \in Q_1^n, \) there exists \( q_3 \in Q_1^n \) such that \( A_1q_1 + A_2q_2 = (A_1 + A_2)q_3. \)

**Proof. Case (i)**. By Definition 2.2, we have \( A_1(Q_1^n + Q_2^n) = A_1\{q_1 + q_2 : q_1 \in Q_1^n, q_2 \in Q_2^n\} \} = \{A_1q_1 + A_1q_2 : q_1 \in Q_1^n, q_2 \in Q_2^n\} = \{A_1q_1 : q_1 \in Q_1^n\} + \{A_1q_2 : q_2 \in Q_2^n\} = A_1Q_1^n + A_1Q_2^n. \) Similarly, we can obtain \( w(Q_1^n + Q_2^n) = wQ_1^n + wQ_2^n. \)

**Case (ii)**. Since \( Q_1^n \) is a convex set, then for any \( q_1, q_2 \in Q_1^n \) and \( \lambda \in [0, 1], \) there exists \( q_3 \) such that \( q_3 = \lambda q_1 + (1 - \lambda)q_2 \in Q_1^n \) and for any \( q_4 \in \alpha Q_1^n + \beta Q_1^n, \) there
exists \( q_1, q_2 \in Q^n \) such that \( q_4 = \alpha q_1 + \beta q_2 \). Thus \( q_4 = (\alpha + \beta)(\lambda q_1 + (1 - \lambda)q_2) \) for \( \lambda = \frac{\alpha}{\alpha + \beta} \) and \( \lambda q_1 + (1 - \lambda)q_2 \in Q^n_1 \), which implies \( q_4 \in (\alpha + \beta)Q^n_1 \), i.e., \( (\alpha + \beta)Q^n_1 \supset \alpha Q^n_1 + \beta Q^n_1 \). Next, we will prove \( (\alpha + \beta)Q^n_1 \subset \alpha Q^n_1 + \beta Q^n_1 \). For any \( q_0 \in (\alpha + \beta)Q^n_1 \), there exists \( q_1 \in Q^n_1 \) such that \( q_0 = (\alpha + \beta)q_1 = \alpha q_1 + \beta q_1 \in \alpha Q^n_1 + \beta Q^n_1 \), i.e., \( (\alpha + \beta)Q^n_1 \subset \alpha Q^n_1 + \beta Q^n_1 \). Therefore, \( (\alpha + \beta)Q^n_1 = \alpha Q^n_1 + \beta Q^n_1 \).

**Case(iii).** For any \( q_0 \in (A_1 + A_2)Q^n_1 \), there exists \( q_1 \in Q^n_1 \) such that \( q_0 = (A_1 + A_2)q_1 = A_1q_1 + A_2q_1 \in A_1Q^n_1 + A_2Q^n_1 \). Hence \( (A_1 + A_2)Q^n_1 \subset A_1Q^n_1 + A_2Q^n_1 \). Moreover, if for any \( q_1, q_2 \in Q^n_1 \), there exists \( q_3 \in Q^n_1 \) such that \( A_1q_1 + A_2q_2 = (A_1 + A_2)q_3 \), then for any \( q_4 \in A_1Q^n_1 + A_2Q^n_1 \), there exists \( q_5, q_6 \in Q^n_1 \) such that \( q_4 = A_1q_5 + A_2q_6 \). Therefore, \( (A_1 + A_2)Q^n_1 = A_1Q^n_1 + A_2Q^n_1 \). The proof is completed.

2.1. The quaternion interval sets. In this subsection, we will provide the equivalence between the \( n \)-dimensional quaternion vector space \( Q^n \) with \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \) and introduce a notion of the quaternion interval sets.

Now, the definition of the convex in \( Q^n \) is defined as follows.

**Definition 2.3.** Let \( Q_1 \subset Q^n \), we call \( Q_1 \) a convex set in \( Q^n \) if

\[
\lambda q_1 + (1 - \lambda)q_2 \in Q_1
\]

for any \( q_1, q_2 \in Q_1 \) and \( \lambda \in [0, 1] \).

The relationship between the convex in \( Q^n \) and the convex in \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \) will be built as follows.

**Theorem 2.2.** Let \( Q_1 \subset Q^n \), \( Q_1 = \{q_{10} + q_{11}i + q_{12}j + q_{13}k| q_{10} \in Q_{10}, q_{11} \in Q_{11}, q_{12} \in Q_{12}, q_{13} \in Q_{13}\} \) for \( Q_{10}, Q_{11}, Q_{12}, Q_{13} \subset \mathbb{R}^n \). Then \( Q_1 \) is convex in \( Q^n \) if and only if \( Q_{10}, Q_{11}, Q_{12}, Q_{13} \) are convex in \( \mathbb{R}^n \).

**Proof.** Step (I). (Necessity) Since \( Q_{10}, Q_{11}, Q_{12}, Q_{13} \) are convex in \( \mathbb{R}^n \), then we have

\[
\lambda q_{1l} + (1 - \lambda)q_{2l} \in Q_{1l}
\]

for any \( q_{1l}, q_{2l} \in Q_{1l}, \ l = 0, 1, 2, 3 \) and \( \lambda \in [0, 1] \). Hence for any \( q_1, q_2 \in Q_1 \), we have

\[
q_l = q_{l0} + q_{l1}i + q_{l2}j + q_{l3}k \quad \text{for} \ l = 1, 2, \text{and} \ q_{1l}, q_{2l} \in Q_{1l}, \text{and}
\]

\[
\lambda q_{1l} + (1 - \lambda)q_{2l} \in Q_{1l}.
\]

Thus \( \lambda q_1 + (1 - \lambda)q_2 \in Q_1 \), i.e., \( Q_1 \) is convex in \( Q^n \).

Step (II). (Sufficiency) Since \( Q_1 \) is convex in \( Q^n \), then \( \lambda q_1 + (1 - \lambda)q_2 \in Q_1 \) for any \( q_1, q_2 \in Q_1 \), where \( \lambda \in [0, 1] \). On the other hand, \( Q_1 = \{q_{10} + q_{11}i + q_{12}j + q_{13}k| q_{10} \in Q_{10}, q_{11} \in Q_{11}, q_{12} \in Q_{12}, q_{13} \in Q_{13}\} \) for \( Q_{10}, Q_{11}, Q_{12}, Q_{13} \subset \mathbb{R}^n \), i.e., \( q_{10} + q_{11}i + q_{12}j + q_{13}k, q_{20} + q_{21}i + q_{22}j + q_{23}k \in Q_1 \) for any \( q_{1l}, q_{2l} \in Q_{1l} \) and \( l = 0, 1, 2, 3 \). Hence \( \lambda q_{1l} + (1 - \lambda)q_{2l} \in Q_{1l} \), i.e., \( Q_{10}, Q_{11}, Q_{12}, Q_{13} \) are convex in \( \mathbb{R}^n \). The proof is completed.

The equivalence of the compactness of the sets in \( Q^n \) and \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \) will be obtained as follows.

**Theorem 2.3.** Let \( Q_1 \subset Q^n \), \( Q_1 = \{q_{10} + q_{11}i + q_{12}j + q_{13}k| q_{10} \in Q_{10}, q_{11} \in Q_{11}, q_{12} \in Q_{12}, q_{13} \in Q_{13}\} \) for \( Q_{10}, Q_{11}, Q_{12}, Q_{13} \subset \mathbb{R}^n \). Then \( Q_1 \) is compact if and only if \( Q_{10}, Q_{11}, Q_{12}, Q_{13} \) are compact.

**Proof.** \( Q_{10}, Q_{11}, Q_{12}, Q_{13} \) are compact if and only if there exists some \( M > 0 \) such that

\[
\sqrt{\|q_{10}\|^2 + \|q_{11}\|^2 + \|q_{12}\|^2 + \|q_{13}\|^2} < M
\]
if and only if \(q_{10}, q_{11}, q_{12}, q_{13}\) are bounded and if only if \(Q_1\) is compact. The proof is completed.

**Definition 2.4** ([28]). Let \(I_R = \{[a, b] | a, b \in \mathbb{R}, a < b\}\) with the following operation:

\[
(a, b) + [c, d] = [a + c, b + d]; \quad [a, b] - [c, d] = [a - d, b - c], \text{ in which } -[c, d] = [-d, -c];
\]

\[
[a, b] \cdot [c, d] = \min\{ac, ad, bc, bd\}, \text{ and } \max\{ac, ad, bc, bd\}; \quad [c, d]^{-1} = \frac{1}{[c, d]} = \left[\frac{1}{c}, \frac{1}{d}\right] \text{ if } 0 \notin [c, d]; \quad [a, d] = [a, b][c, d]^{-1} \text{ if } 0 \notin [c, d].
\]

The quaternion interval set \(I_Q\) and the quaternion vector interval set \(I_Q^b\) will be introduced as follows.

**Definition 2.5.** (I). We define the quaternion interval set by \(I_Q = \{[p, q] | q = q_0 + q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k}, p = p_0 + p_1 \mathbf{i} + p_2 \mathbf{j} + p_3 \mathbf{k} \in Q, p_l < q, l = 0, 1, 2, 3\}\) with the operation as follows: \([p, q] + [h, w] = [p + h, q + w]; -[p, q] = [-q, -p]; [p, q]. [h, w] = \min\{ph, pw, qh, qw\}, [p, q]^{-1} = [q^{-1}, p^{-1}]\) for \(0 \notin [p, q]\), where \(\min\{p, q\} = \min\{p_0, q_0\} + \min\{p_1, q_1\} + \min\{p_2, q_2\} + \min\{p_3, q_3\}\) and \(\max\{p, q\} = \max\{p_0, q_0\} + \max\{p_1, q_1\} + \max\{p_2, q_2\} + \max\{p_3, q_3\}\). For convenience, we denote \([P, Q]^n \in I_Q^n\) with the property that \([p_m, q_m] \in I_Q\) for \(P = (p_1, p_2, \ldots, p_n)^T\) and \(Q = (q_1, q_2, \ldots, q_n)^T\). The quaternion vector interval set \(I_Q^b\) with the operation as follows: \([P, Q]^n \in [P, Q]^n \in I_Q^n \times I_Q^n = \prod_{m=1}^n [q_m, h_m] = \prod_{m=1}^n [p_m + h_m, q_m + w_m] - [P, Q]^n = \prod_{m=1}^n [-q_m, -p_m]; [P, Q]^n \cdot [H, W]^n = \prod_{m=1}^n [q_m \cdot h_m] = \prod_{m=1}^n [p_m, q_m] \cdot [h_m, w_m], ([P, Q]^n)^{-1} = \prod_{m=1}^n q_m^{-1} = \prod_{m=1}^n [p_m, q_m]^{-1}\). The equivalence of \(I_Q\) with \(I_R \times I_R \times I_R \times I_R\) will be given as follows.

**Theorem 2.4.** Let \((z_1, z_2, z_3, z_4)^T, (v_1, v_2, v_3, v_4)^T \in \prod_{m=1}^4 I_R, \lambda \in \mathbb{R}, \) with the following 4-dimensional vector operations:

\[
(z_1, z_2, z_3, z_4)^T + (v_1, v_2, v_3, v_4)^T = (z_1 + v_1, z_2 + v_2, z_3 + v_3, z_4 + v_4)^T,
\]

\[
\lambda(z_1, z_2, z_3, z_4)^T = (\lambda z_1, \lambda z_2, \lambda z_3, \lambda z_4)^T,
\]

\[
(z_1, z_2, z_3, z_4)^T \cdot (v_1, v_2, v_3, v_4)^T = (z_1 v_1 - z_2 v_2 - z_3 v_3 - z_4 v_4, z_1 v_2 + z_2 v_1 + z_3 v_4 - z_4 v_3, \allowbreak z_1 v_3 - z_2 v_4 + z_3 v_1 + z_4 v_2, z_1 v_4 + z_2 v_3 - z_3 v_2 + z_4 v_1)^T,
\]

then \(I_Q\) and \(\prod_{m=1}^4 I_R\) are equivalent.

**Proof.** By Definitions 2.4 and 2.5, the desired result follows immediately. \(\square\)
2.2. The fuzzy quaternion vector in high-dimensional space. Now, we will introduce a notion of fuzzy quaternion vector in high-dimensional case.

**Definition 2.6.** A function $h': Q^n \to [0, 1]$ is said to be a fuzzy quaternion vector if $h'(a_n + b_n i + c_n j + d_n k) = \min\{A(a_n), B(b_n), C(c_n), D(d_n)\}$, where $a_n, b_n, c_n, d_n \in \mathbb{R}^n, A, B, C, D \in E^n$. The set of all fuzzy quaternion vector is denoted by $Q^n_F$.

**Theorem 2.5.** Let $s', h' \in Q^n_F, A, B, C, D \in E^n$, $\alpha \in (0, 1]$, then (i) $|h'|^\alpha = [A]_\alpha \times [B]_\alpha \times [C]_\alpha \times [D]_\alpha$; (ii) $h'$ is a fuzzy convex for any $h' \in Q^n_F$; (iii) $|h'|^\alpha \geq |h'|^{\alpha_2}$ if $0 < \alpha_1 \leq \alpha_2 \leq 1$; (iv) $|h'|^0 = [A]^0 \times [B]^0 \times [C]^0 \times [D]^0$ is compact, where the $\alpha$-levels $[A]^{\alpha} = \{x_n \in \mathbb{R}^n : A(x_n) \in [\alpha, 1]\}$, $[A]^0 = \{x_n \in \mathbb{R}^n : A(x_n) \in (0, 1]\}$. Similarly, we can define the level sets $[B]^{\alpha}, [C]^{\alpha}, [B]^0, [C]^0, [D]^0$.

**Proof.** **Case (i).** Let $z = a_n + b_n i + c_n j + d_n k \in [h']^{\alpha}$, i.e.,

$$h'(z) = \min\{A(a_n), B(b_n), C(c_n), D(d_n)\} \geq \alpha,$$

then $A(a_n) \geq \alpha, B(b_n) \geq \alpha, C(c_n) \geq \alpha, D(d_n) \geq \alpha$, i.e., $a_n \in [A]^{\alpha}, b_n \in [B]^{\alpha}, c_n \in [C]^{\alpha}, d_n \in [D]^{\alpha}$. Hence $z \in [A]^{\alpha} \times [B]^{\alpha} \times [C]^{\alpha} \times [D]^{\alpha}$. On the other hand, if $z = a_n + b_n i + c_n j + d_n k \in [A]^{\alpha} \times [B]^{\alpha} \times [C]^{\alpha} \times [D]^{\alpha}$, then $h'(z) = \min\{A(a_n), B(b_n), C(c_n), D(d_n)\} \geq \alpha$, which implies $z \in [h']^{\alpha}$. Thus $|h'|^{\alpha} = [A]^{\alpha} \times [B]^{\alpha} \times [C]^{\alpha} \times [D]^{\alpha}$.

**Case (ii).** Let $x_1 = x_{10} + x_{11} i + x_{12} j + x_{13} k$ and $x_2 = x_{20} + x_{21} i + x_{22} j + x_{23} k$, $x_{hm} \in \mathbb{R}^n$. By Definition 2.1 (ii), we have

$$A(\lambda x_{10} + (1 - \lambda)x_{20}) \geq \min\{A(x_{10}), A(x_{20})\},$$

$$B(\lambda x_{11} + (1 - \lambda)x_{21}) \geq \min\{B(x_{11}), B(x_{21})\},$$

$$C(\lambda x_{12} + (1 - \lambda)x_{22}) \geq \min\{C(x_{12}), C(x_{22})\},$$

$$G(\lambda x_{13} + (1 - \lambda)x_{23}) \geq \min\{G(x_{13}), G(x_{23})\}.$$

Hence

$$h'(x_1) = \min\{A(a_1), B(b_1), C(c_1), D(d_1)\}.$$

**Case (iii).** For $0 < \alpha_1 \leq \alpha_2 \leq 1$, if $x \in [h']^{\alpha_2}$, then $h'(x) \geq \alpha_2 \geq \alpha_1$, i.e., $x \in [h']^{\alpha_1}$. Hence $|h'|^{\alpha_1} \geq |h'|^{\alpha_2}$.

**Case (iv).** Let $h' = A_m \times B_m \times C_m \times G_m$ be a sequence on $Q^n_F$. Since $[u]^0$ is compact on $\mathbb{R}^n$ for any $u \in E^n$ and $A_m \in E^n$, then there exists a subsequence $\{A_{m_1}\}$ of the sequence $\{A_m\}$ such that $A_{m_1}$ is convergent on $[A]^0$ and it follows that the sequence $\{A_{m_1} \times B_{m_1} \times C_{m_1} \times G_{m_1}\}$ is a subsequence of $\{A_m \times B_m \times C_m \times G_m\}$. Similarly, there exists a subsequence $\{B_{m_2}\}$ of the sequence $\{B_{m_1}\}$ such that $B_{m_2}$ is convergent on $[B]^0$, i.e., $A_{m_2}, B_{m_2}$ is convergent on $[A]^0 \times [B]^0$. Similarly, we can obtain a subsequence $\{A_{m_4} \times B_{m_4} \times C_{m_4} \times G_{m_4}\}$ of $\{h'_m\}$ such that $A_{m_4} \times B_{m_4} \times C_{m_4} \times G_{m_4}$ is convergent, i.e., $\{h'_m\}$ is convergent, which implies that for any sequence $\{h'_m\}$ there exists a subsequence $\{h'_{m_4}\}$ such that $\{h'_{m_4}\}$ is convergent on $[h']^{\alpha}$. Hence, $[h']^0 = [A]^0 \times [B]^0 \times [C]^0 \times [D]^0$ is compact. The proof is completed. \qed
Example 2.1. Let
\[
A(x) = \begin{cases} x, & x \in [0,1], \\ 0, & \text{otherwise}, \end{cases} \quad B(x) = \begin{cases} \frac{\sin \frac{x}{n}}{n}, & x \in [0,1], \\ 0, & \text{otherwise}, \end{cases} \\
C(x) = \begin{cases} \sin \frac{\pi x}{2}, & x \in [0,1], \\ 0, & \text{otherwise}, \end{cases} \quad G(x) = \begin{cases} x^2, & x \in [0,1], \\ 0, & \text{otherwise}. \end{cases}
\]
Then
\[
h'(a + bi + cj + dk) = \begin{cases} \max\{a, \frac{e^i}{n}, \sin \frac{\pi}{2}, b^2\}, & a, b, c, d \in [0,1], \\ 0, & \text{otherwise}, \end{cases}
\]
h'[1 + i + j + k] = 1, [h]0 = [0,1] × [0,1] × [0,1] × [0,1], [h]' = \frac{[1]}{[1]} × [1 - \ln 2,1] × \frac{[1]}{[1]} × [2,1]. In fact, [h]'^\alpha = [A]^\alpha × [B]^\alpha × [C]^\alpha × [G]^\alpha, where [A]^\alpha = [\alpha,1], [B]^\alpha = \{x \in [0,1] : 1 \geq \frac{\sin \frac{\pi}{2}}{n} \geq \alpha\}, [C]^\alpha = \{x \in [0,1] : 1 \geq x^2 \geq \alpha\}.

Now we will introduce a notion of $H$-difference in the fuzzy quaternion high-dimensional space.

Definition 2.7. Let $s', h' \in \mathbb{Q}_F^n$. If there exists $q' \in \mathbb{Q}_F^n$ such that $s' = h' \oplus q'$, then $q'$ is called the $H$-difference of $s', h'$ and it can be expressed by $s' \oplus h'$. Moreover, we define $0'$ by
\[
0'(x_n) = \begin{cases} 1, & \text{for } x_n = (x_{n1}, x_{n2}, \ldots, x_{nn})^T, \quad x_{n1} = x_{n2} = \ldots = x_{nn} = 0, \\ 0, & \text{otherwise}. \end{cases}
\]
For $h' \in \mathbb{Q}_F^n$, if there exists $h'' \in \mathbb{Q}_F^n$ such that $h' \oplus h'' = 0'$, then we denote such $h''$ by $\ominus h'$.

Example 2.2. Let $n = 2, \alpha \in [0,1], h'(t_n) = \begin{cases} 1, & t_n = (1 + 2i, 3i + k)^T, \\ 0, & \text{otherwise}, \end{cases}$
\[
\ominus h'(t_n) = \begin{cases} 1, & t_n = (-1 - 2j, -3i - k)^T, \\ 0, & \text{otherwise}. \end{cases}
\]
In fact, $[h']^\alpha = \{(1 + 2i, 3i + k)^T\}$ and $[\ominus h']^\alpha = \{(-1 - 2j, -3i - k)^T\}$, i.e., $[h']^\alpha + [\ominus h']^\alpha = \{(0,0)^T\}$.

Now, based on the properties of fuzzy quaternion vectors, we introduce some fuzzy arithmetics in the high-dimensional fuzzy quaternion space which will be used later.

Definition 2.8. Let $s', h' \in \mathbb{Q}_F^n$, $s' = (X,Y,Z,W)$ and $h' = (A,B,C,G)$, $X, Y, Z, W, A, B, C, G \in \mathbb{E}^\alpha$. Then
(i): the addition is defined by $s' \oplus h' = (X \oplus A, Y \oplus B, Z \oplus C, W \oplus G);$ (ii): the scalar multiplication is defined by $\omega h' = (xA \oplus yB \ominus zC \ominus wG, xB \ominus yA \ominus zG \ominus wC, xC \ominus yG \ominus zA \ominus wB, xG \ominus yC \ominus zB \ominus wA)$ for $\omega = x + yi + zj + wk$ and $x, y, z, w \in \mathbb{R}$; (iii): $[Qh']^\alpha$ is defined by $[Qh']^\alpha = Q[h']^\alpha$ for any $\alpha \in [0,1]$ and $Q \in \mathbb{Q}_{n \times n}$; (iv): the metric between $s'$ and $h'$ is defined by $D(s', h') = \sup_{\alpha \in [0,1]} \left\{d([X]^\alpha, [A]^\alpha) + d([Y]^\alpha, [B]^\alpha) + d([Z]^\alpha, [C]^\alpha) + d([W]^\alpha, [G]^\alpha)\right\}$, where $d([X]^\alpha, [A]^\alpha) = \inf\{r : [X]^\alpha \subset N([A]^\alpha, r), [A]^\alpha \subset N([X]^\alpha, r)\}$, $N([A]^\alpha, r) = \{x_n \in \mathbb{E}^\alpha : ||x_n - y_n|| < r\}$.
\( r \) for some \( y_n \in [A]^{\alpha} \). We can also define \( d([Y]^{\alpha}, [B]^{\alpha}) \), \( d([Z]^{\alpha}, [C]^{\alpha}) \) and \( d([W]^{\alpha}, [G]^{\alpha}) \) similarly.

**Theorem 2.6.** Let \( x, y, z \in Q_{x,n}^d, Q \in Q_{n,n}^d \). Then (i) \( D(x, y) \leq D(x, z) + D(z, y) \); (ii) \( D(x \oplus z, y \oplus z) = D(x, y) \); (iii) \( D(Qx, Qy) \leq \|Q\|D(x, y) \); (iv) if \( x \oplus (\ominus y) = x \ominus y \) and \( x \oplus (\ominus y) = x \oplus y \), then \( y = z \ominus Qx \) if \(-Q\)x \( y = z\).

**Proof.** Let \( x = x_0 + x_1 + x_2 + x_3 \), \( y = y_0 + y_1 + y_2 + y_3 \), \( z = z_0 + z_1 + z_2 + z_3 \), \( \alpha \in [0, 1] \), where \( x_m, y_m, z_m \in E^\alpha, m = 0, 1, 2, 3 \).

(i). Since \( \|x_n - y_n\| \leq \|x_n - z_n\| + \|z_n - y_n\| \) for \( x_n, y_n, z_n \in R^n \), then \( d([x_0]^{\alpha}, [y_0]^{\alpha}) \leq d([x_0]^{\alpha}, [z_0]^{\alpha}) + d([z_0]^{\alpha}, [y_0]^{\alpha}) \) and by \( \sup_{\alpha \in [0, 1]} \{f(\alpha) + g(\alpha)\} \leq \sup_{\alpha \in [0, 1]} f(\alpha) + \sup_{\alpha \in [0, 1]} g(\alpha) \), we have

\[
D(x, y) = \sup_{\alpha \in [0, 1]} \{d([x_0]^{\alpha}, [y_0]^{\alpha}) + d([x_1]^{\alpha}, [y_1]^{\alpha}) + d([x_2]^{\alpha}, [y_2]^{\alpha}) + d([x_3]^{\alpha}, [y_3]^{\alpha}) \}
\]

(ii) By Lemma 2.1 and Definition 2.8, we have

\[
D(x \oplus z, y \oplus z) = D_{R^\alpha}(x_0 \oplus z_0, y_0 \oplus z_0) + D_{R^\alpha}(x_1 \oplus z_1, y_1 \oplus z_1) + D_{R^\alpha}(x_2 \oplus z_2, y_2 \oplus z_2) + D_{R^\alpha}(x_3 \oplus z_3, y_3 \oplus z_3)
\]

(iii) By Lemma 2.1 and Definition 2.8, we have

\[
D(Qx, Qy) = \sup_{\alpha \in [0, 1]} \{d([Qx_0]^{\alpha}, [Qy_0]^{\alpha}) + d([Qx_1]^{\alpha}, [Qy_1]^{\alpha}) + d([Qx_2]^{\alpha}, [Qy_2]^{\alpha}) + d([Qx_3]^{\alpha}, [Qy_3]^{\alpha}) \}
\]

From Definition 2.8, we have

\[
d([Qx_0]^{\alpha}, [Qy_0]^{\alpha}) = \inf \{r : Q[x_0]^{\alpha} \subset N(Q[y_0]^{\alpha}, r),
\]

\[
Q[y_0]^{\alpha} \subset N(Q[x_0]^{\alpha}, r)\}
\]

\( N(Q[x_0]^{\alpha}, r) = \{Qx_n \in R^n : \|Qx_n - Qy_n\| < r \) for some \( y_n \in [x_0]^{\alpha} \), it follows that \( \|Qx_n - Qy_n\| \leq \|Q\||x_n - y_n|| \), i.e.,

\[
d([Qx_0]^{\alpha}, [Qy_0]^{\alpha}) \leq \|Q\|d([x_0]^{\alpha}, [y_0]^{\alpha})
\]

which implies \( D(Qx, Qy) \leq \|Q\|D(x, y) \).

(iv) Let \( w = x \ominus y \), by Definition 2.1 and Theorem 2.5 (i), we have \( x = w \oplus y \), i.e. \( [x]^{\alpha} = [w]^{\alpha} + [y]^{\alpha} \) and \( [w]^{\alpha} = [x]^{\alpha} - [y]^{\alpha} \). On the other hand, \( [x \oplus (\ominus y)]^{\alpha} = [x]^{\alpha} + [\ominus y]^{\alpha} \) and \( [y]^{\alpha} + [\ominus y]^{\alpha} = [0]^{\alpha} \). Hence, \( [x \oplus (\ominus y)]^{\alpha} = [x]^{\alpha} - [y]^{\alpha} \), i.e.,
Let \([x \oplus (\ominus y)]^\alpha = [x \ominus y]^\alpha\). Hence \(x \oplus (\ominus y) = x \ominus y\). Next, we will show \(x \oplus (\ominus y) = x \ominus y\). Let \(z = x \ominus (\ominus y)\), then \(x = z \ominus (\ominus y) = z \ominus y\), i.e., \(z = x \ominus y\). Thus \(x \ominus (\ominus y) = x \ominus y\).

(v) Since \((-Q)x \ominus y = z\), then \([(-Q)x \ominus y]^\alpha = [z]^\alpha = [(-Q)x]^\alpha + [y]^\alpha = -Q[x]^\alpha + [y]^\alpha\). Hence, \([y]^\alpha = [z]^\alpha + Q[x]^\alpha = [z \oplus Qx]^\alpha\). Thus \(y = z \oplus Qx\). The proof is completed.

In what follows, we will establish some necessary basic results of the calculus of \(\Delta\)-differentiable and \(\Delta\)-differentiable quaternion functions in high-dimensional fuzzy quaternion space.

**Definition 2.9.** Let \(H : \mathbb{T} \to \mathbb{Q}_F^\alpha\), and \(t_0 \in \mathbb{T}^\alpha\), we say that \(H\) is \(\Delta\)-differentiable at \(t_0\), if there exists \(H^\Delta(t_0) \in \mathbb{Q}_F^\alpha\) such that either

\[(E_1)\] \(H(\sigma(t_0)) \ominus H(t_0) \in \mathbb{Q}_F^\alpha\) and \(H(t_0) \ominus H(\rho(t_0)) \in \mathbb{Q}_F^\alpha\) exist at \(t_0 \in \mathbb{T}^\alpha\) and for any \(\epsilon > 0\), there is a neighborhood \(U\) of \(t_0\) (i.e. \(U = (t_0 - \delta, t_0 + \delta)\) for some \(\delta > 0\)) such that \(D(H(\sigma(t_0)) \ominus H(s), H^\Delta(t_0)) \ominus H(\sigma(t_0) - s) < \epsilon|\sigma(t_0) - s|\) for all \(s \in U\), i.e., the limit \(H^\Delta(t_0) = \lim_{s \to t_0} \frac{H(\sigma(t_0)) \ominus H(s)}{\sigma(t_0) - s}\) exists; or

\[(E_2)\] \(H(t_0) \ominus H(\sigma(t_0)) \in \mathbb{Q}_F^\alpha\) and \(H(\rho(t_0)) \ominus H(t_0) \in \mathbb{Q}_F^\alpha\) exist at \(t_0 \in \mathbb{T}^\alpha\) and for any \(\epsilon > 0\), there is a neighborhood \(U\) of \(t_0\) (i.e. \(U = (t_0 - \delta, t_0 + \delta)\) for some \(\delta > 0\)) such that \(D(H(t) \ominus H(t), H^\Delta(t)) \ominus H(t) - \sigma(t)\) \(H^\Delta(t_0)) < \epsilon|t - t_0|\) for all \(t \in U\), i.e., the limit \(H^\Delta(t_0) = \lim_{s \to t_0} H(t) \ominus H(\sigma(t_0)) \ominus H(t) - \sigma(t)\) exists.

Then \(H\) is said to be \((E_1)\)-differentiable at \(t_0\) if \((E_1)\) is satisfied. Similarly, if \((E_2)\) is satisfied, then \(H\) is said to be \((E_2)\)-differentiable at \(t_0\).

**Remark 2.1.** Note that for \(H : \mathbb{T} \to \mathbb{Q}_F^\alpha\), \(H(\sigma(t)) \ominus H(t) = \mu(t)H^\Delta(t)\) if \(H\) is \((E_1)\)-differentiable and \(H(t) \ominus H(\sigma(t)) = (-\mu(t))H^\Delta(t)\) if \(H\) is \((E_2)\)-differentiable.

**Example 2.3.** Let \(n = 2, \alpha \in [0, 1]\).

**Case (i).** \(t \in \mathbb{T} = \bar{h}_{\mathbb{Z}}, \bar{h} > 0\), \(H(t) = h'(t_n)(t)\),

\[
h'(t_n)(t) = \begin{cases} 
1, & t_n = (x, x)^T, \ x \in [t_0, t] \text{ for } t < 0 \text{ and } x \in [0, t] \text{ for } t > 0, \\
0, & \text{otherwise}.
\end{cases}
\]

Then \(H(t)\) is \((E_1)\)-differentiable for \(t > 0\) and \(H(t)\) is \((E_2)\)-differentiable for \(t < 0\).

In fact, \([H(t)]^\alpha = \{(x, x)^T : x \in [t_0, t]\}\) for \(t < 0\) and \([H(t)]^\alpha = \{(x, x)^T : x \in [0, t]\}\) for \(t > 0\), \([H(\sigma(t)) \ominus H(t)]^\alpha = \{(x, x)^T : x \in [\bar{h}, 0]\}\) for \(t < 0\) and \([H(\sigma(t)) \ominus H(t)]^\alpha = \{(x, x)^T : x \in [0, \bar{h}]\}\) for \(t > 0\), \([H(t) \ominus H(\sigma(t))]^\alpha = \{(x, x)^T : x \in [0, \bar{h} - \bar{h}]\}\) for \(t < 0\) and \([H(t) \ominus H(\sigma(t))]^\alpha = \{(x, x)^T : x \in [\bar{h}, 0]\}\) for \(t > 0\).

**Case (ii).** \(t \in \mathbb{T} = \bar{t}_{\mathbb{Z}}, H(t) = h'(t_n)(t)\),

\[
h'(t_n)(t) = \begin{cases} 
1, & t_n = (x, x)^T, \ x \in [t_1, 1] \text{ for } t < \frac{1}{2} \text{ and } x \in [\frac{1}{2}, t] \text{ for } t > \frac{1}{2}, \\
0, & \text{otherwise}.
\end{cases}
\]

Then \(H(t)\) is \((E_1)\)-differentiable. In fact, \([H(t)]^\alpha = \{(x, x)^T : x \in [t_1, 1]\}\) for \(t < 1\) and \([H(t)]^\alpha = \{(x, x)^T : x \in [\frac{1}{2}, t]\}\) for \(t > 1\), \([H(\sigma(t)) \ominus H(t)]^\alpha = \{(x, x)^T : x \in [t, 1]\}\) for \(t < 1\) and \([H(\sigma(t)) \ominus H(t)]^\alpha = \{(x, x)^T : x \in [\frac{1}{2}, t]\}\) for \(t > 1\), \([H(t) \ominus H(\sigma(t))]^\alpha = \{(x, x)^T : x \in [-t, -\frac{1}{2}] = \emptyset\}\) for \(t < 1\) and \([H(t) \ominus H(\sigma(t))]^\alpha = \{(x, x)^T : x \in [-\frac{1}{2}, -t] = \emptyset\}\) for \(t > 1\).

Now we introduce the fuzzy Aumann \(\Delta\)-integral in high-dimensional fuzzy quaternion space as follows.
Definition 2.10. Let \( a, b \in T \), \( Q_1 : T \to Q^n_F \), then the \( \alpha \)-level of the Aumann \( \Delta \)-integral (or \( \Delta \)-integral for short) of \( Q_1 \) is defined by
\[
\int_a^b [Q_1(\tau)]^\alpha \Delta \tau = \int_a^b \{ q_1(\cdot) : q_1(\cdot) \in [Q_1(\tau)]^\alpha \} \Delta \tau
\]
\[
= \left\{ \int_a^b q_1(\tau) \Delta \tau : q_1(\cdot) \in [Q_1(\tau)]^\alpha \right\},
\]
where \([Q_1(\tau)]^\alpha \in \mathcal{P}_\alpha(Q^n_F)\) for any \( \tau \in [a, b] \) and \( \alpha \in [0, 1] \). The symbol \( \mathcal{P}_\alpha(Q^n_F) \) denotes the family of all nonempty convex subsets of \( Q^n_F \).

Theorem 2.7. Let \( H : T \to Q^n_F \), \( b, t \in T^c \), the \( \Delta \)-derivative \( H^\Delta \) is integrable over \( T \). We have (i) if \( H \) is \((E_1)\)-differentiable, then \( H(t) = H(b) \oplus \int_b^t H^\Delta(\tau) \Delta \tau \); (ii) if \( H \) is \((E_2)\)-differentiable, then \( H(t) = H(b) \oplus \int_b^t H^\Delta(\tau) \Delta \tau \); (iii) \( \int_a^b H(\tau) \Delta \tau = \int_a^c H(\tau) \Delta \tau + \int_c^b H(\tau) \Delta \tau \) for \( a, b, c \in T, a < c < b \).

Proof. Let \( H^\Delta(t) = h(t) \), then \( \int_a^b h(\tau) \Delta \tau = \int_b^b h(\tau) \Delta \tau = \left\{ \int_b^b h_1(\tau) \Delta \tau : h_1(\cdot) \in [h(\tau)]^\alpha \right\} \).

Case (i). Since \( H \) is \((E_1)\)-differentiable, \( [\mu(t)h(t)]^\alpha = \mu(t)[h(t)]^\alpha = [H(\sigma(t))]^\alpha + [H(t)]^\alpha = [H_1(\sigma(t)) \cdot H_2(t)]^\alpha = [H_1(\sigma(t)) \cdot H_2(t) : H_1(\cdot), H_2(\cdot) \in [H(t)]^\alpha] \). Hence, case (1), if there exists a right dense point \( \tau \) on \([b, t] \cap T\), then for \( \mu(\tau)[h(\tau)]^\alpha = [H_1(\sigma(\tau)) \cdot H_2(t)]^\alpha = [H_1(\sigma(\tau)) \cdot H_2(t) : H_1(\cdot), H_2(\cdot) \in [H(\tau)]^\alpha] \), we have \( H_1(\tau) = H_2(t) \), i.e., \( [\mu(\tau)]^\alpha = [H_1(\sigma(\tau)) \cdot H_2(t)]^\alpha \), which implies \( H(t = H(b) \oplus \int_b^t H^\Delta(\tau) \Delta \tau \); case (2), if \( \tau \) is right-scattered point for any \( \tau \in [b, t] \cap T \), then for any \( H^\Delta(t) = \sum_{i=1}^N h_0(\tau_i) \mu(\tau_i) = \sum_{i=1}^N (H_1(\sigma(\tau_i)) - H_2(t)) = H_1(t) - H_2(t) \) for some finite natural number \( N \), \( \tau_1 = b \) and \( \sigma(N) = t \). Hence \( H(t) = H(b) \oplus \int_b^t H^\Delta(\tau) \Delta \tau \).

Case (ii). Similar to the proof process of Case (i), we can obtain (ii) immediately, we omit it here.

Case (iii). By Definition 2.10, we have \( [\int_a^c H(\tau) \Delta \tau]^\alpha = \left\{ \int_a^c H(\tau) \Delta \tau : H_1(\cdot) \in [H(\tau)]^\alpha, \tau \in [a, b] \right\} \) and
\[
[\int_a^c H(\tau) \Delta \tau]^\alpha + [\int_c^b H(\tau) \Delta \tau]^\alpha
= \left\{ \int_a^c H_1(\tau_1) \Delta \tau_1 : H_1(\cdot) \in [H(\tau)]^\alpha, \tau_1 \in [a, c] \right\}
+ \left\{ \int_c^b H_2(\tau_2) \Delta \tau_2 : H_2(\cdot) \in [H(\tau)]^\alpha, \tau_2 \in [c, b] \right\}
= \left\{ \int_a^c H_1(\tau_1) \Delta \tau_1 + \int_c^b H_2(\tau_2) \Delta \tau_2 : H_1(\cdot) \in [H(\tau)]^\alpha,
H_2(\cdot) \in [H(\tau)]^\alpha, \tau_1 \in [a, c], \tau_2 \in [c, b] \right\}
= \left\{ \int_a^b H_1(\tau) \Delta \tau : H_1(\cdot) \in [H(\tau)]^\alpha, \tau \in [a, b] \right\}.
\]
Hence \( \int_a^b H(\tau) \Delta \tau = \int_a^c H(\tau) \Delta \tau \oplus \int_c^b H(\tau) \Delta \tau \). The proof is completed. \( \square \)

Let \( H : T \to Q^n_F \), \( A : T \to Q_n \times n \) and \( A(t) \) is \( \Delta \)-differentiable, \( \alpha \in [0, 1] \). We introduce the following conditions:
\((C_1)\): for given \(t \in \mathbb{T}\), \(H(\sigma(t)) \sqcup H(t)\) and \(H(t) \sqcup H(\rho(t))\) exist;
\((C_2)\): for given \(t \in \mathbb{T}\), \(H(t) \sqcup H(\sigma(t))\) and \(H(\rho(t)) \sqcup H(t)\) exist;
\((C_3)\): for any \(h_1(\cdot), h_2(\cdot) \in [H(t)]^\alpha\), there exists \(h_3(\cdot) \in [H(t)]^\alpha\) such that \(A(h_1(t) + \mu(t)A^\Delta(t)h_2(t)) = (A(t) + \mu(t)A^\Delta(t))h_3(t)\), for convenience, we denote such a relationship between \(A(t)\) and \(H(t)\) by \((A(\cdot), H(\cdot)) \in (\mathfrak{F}, \mathfrak{G})\);
\((C_4)\): for any \(h_1(\cdot), h_2(\cdot) \in [H(t)]^\alpha\), there exists \(h_3(\cdot) \in [H(t)]^\alpha\) such that \(A(\sigma(t))h_1(t) - \mu(t)A^\Delta(t)h_2(t) = (A(\sigma(t)) - \mu(t)A^\Delta(t))h_3(t)\), for convenience, we denote such a relationship between \(A(t)\) and \(H(t)\) by \((A(\cdot), H(\cdot)) \in (\mathfrak{F}, \mathfrak{G})\).

\textbf{Lemma 2.2.} Let \(Q_1, Q_2 : \mathbb{T} \to \mathbb{Q}_{n \times n}, H_1, H_2 : \mathbb{T} \to \mathbb{Q}_{n}^\alpha\), \(\alpha \in [0, 1], t \in \mathbb{T}\), if for any \(q_1(\cdot), q_2(\cdot) \in [H_1(t)]^\alpha\), there exists \(q_3(\cdot) \in [H_1(t)]^\alpha\) such that \(Q_1(t)q_1(t) + Q_2(t)q_2(t) = (Q_1(t) + Q_2(t))q_3(t)\). Then \((Q_1(t) + Q_2(t))H_1(t) = Q_1(t)H_1(t) \oplus Q_2(t)H_1(t)\) and \(Q_1(t)(H_1(t) \oplus H_2(t)) = Q_1(t)H_1(t) + Q_1(t)H_2(t)\).

\textbf{Proof.} By Definition 2.2 and Theorem 2.1, we have \([(Q_1(t) + Q_2(t))H_1(t)]^\alpha = (Q_1(t) + Q_2(t))[H_1(t)]^\alpha \subset Q_1(t)[H_1(t)]^\alpha + Q_2(t)[H_1(t)]^\alpha\). Next we will prove \((Q_1(t) + Q_2(t))[H_1(t)]^\alpha \supset Q_1(t)[H_1(t)]^\alpha + Q_2(t)[H_1(t)]^\alpha\).

For any \(q(\cdot) \in Q_1(t)[H_1(t)]^\alpha + Q_2(t)[H_1(t)]^\alpha\), there exist \(q_1(\cdot), q_2(\cdot), q_3(\cdot) \in [H_1(t)]^\alpha\) such that \(Q(t) = Q_1(t)q_1(t) + Q_2(t)q_2(t) = (Q_1(t) + Q_2(t))q_3(t)\), which implies \(q(\cdot) \in (Q_1(t) + Q_2(t))[H_1(t)]^\alpha\), i.e., \((Q_1(t) + Q_2(t))[H_1(t)]^\alpha \supset Q_1(t)[H_1(t)]^\alpha + Q_2(t)[H_1(t)]^\alpha\). Hence \((Q_1(t) + Q_2(t))H_1(t) = Q_1(t)H_1(t) \oplus Q_2(t)H_1(t)\). Similar to the proof process of Theorem 2.1, we have \(Q_1(t)(H_1(t) \oplus H_2(t)) = Q_1(t)H_1(t) \oplus Q_1(t)H_2(t)\). The proof is completed. \(\square\)

\textbf{Example 2.4.} Let \(n = 2, \alpha \in [0, 1]\).

\textbf{Case (i).} \(t \in \mathbb{T} = h\mathbb{Z}, h > 0, \)
\(Q_1(t) = \begin{bmatrix} t & 2ti \\ 4tj & 8tk \end{bmatrix}, \quad Q_2(t) = \begin{bmatrix} \frac{1}{2}t & ti \\ 2tj & 4tk \end{bmatrix}, \quad H(t) = h'(t_n)(t), \quad h'(t_n)(t) = \begin{cases} 1, & t_n = (2x, x)^T, \quad x \in [-|t|, |t|], \\ 0, & \text{otherwise}. \end{cases}\)

Then \((Q_1(t) + Q_2(t))H(t) = Q_1(t)H(t) \oplus Q_2(t)H(t)\). In fact, \([H(t)]^\alpha = [h'(t_n)(t)]^\alpha = \{(2x, x)^T : x \in [-|t|, |t|]\}, \quad [(Q_1(t) + Q_2(t))H(t)]^\alpha = \{(3tx + 3txi, 12tx + 8txk)^T : x \in [-|t|, |t|]\}, \quad [Q_1(t)H(t)]^\alpha = \{(2tx + 2txi, 8txj + 8txk)^T : x \in [-|t|, |t|]\}, \quad [Q_2(t)H(t)]^\alpha = \{(tx + txi, 4txj + 4txk)^T : x \in [-|t|, |t|]\}\).

\textbf{Case (ii).} \(t \in \mathbb{T} = \mathbb{R}, \)
\(Q_1(t) = \begin{bmatrix} 2t & t+j \\ tk & tj \end{bmatrix}, \quad Q_2(t) = \begin{bmatrix} t & 2tj \\ 3tk & 2tj \end{bmatrix}, \quad H(t) = t^2h'(t_n)(t), \quad h'(t_n)(t) = \begin{cases} 1, & t_n = (x, x)^T, \quad x \in S, \\ 0, & \text{otherwise}, \end{cases}\)

where \(S = [t, \frac{1}{2}]\) if \(t < 1\) and \(S = [\frac{1}{2}, t]\) if \(t > 1\). Then \((Q_1(t) + Q_2(t))H(t) = Q_1(t)H(t) \oplus Q_2(t)H(t)\). In fact, \([H(t)]^\alpha = [t^2h'(t_n)(t)]^\alpha = \{(xt^2, xt^2)^T : x \in S\}, \quad [(Q_1(t) + Q_2(t))H(t)]^\alpha = \{(3tx^3 + 3tx^3i + 3tx^3j + 3tx^3k + 3tx^3j)^T : x \in S\}, \quad [Q_1(t)H(t)]^\alpha = \{(2tx^3 + 2tx^3i + 2tx^3j)^T : x \in S\}, \quad [Q_2(t)H(t)]^\alpha = \{(xt^3 + 2tx^3j, 3tx^3k + 2tx^3j)^T : x \in S\}.\)
Theorem 2.8. Let \( A : \mathbb{T} \to \mathbb{Q}_{n \times n}^{\alpha} \) and \( H : \mathbb{T} \to \mathbb{Q}_{F}^{n} \) be two \( \Delta \)-differentiable functions, \( t \in \mathbb{T} \), \( \alpha \in [0, 1] \). Then

(i): if \( H \) is \((E_1)\)-differentiable, \( A(t)H(t) \) satisfies \((C_1)\) and \((A(\cdot), H(\cdot)) \in (\mathfrak{g}, \Omega)\), then \( A(t)H(t) \) is \((E_1)\)-differentiable and

\[
(A(t)H(t))^\Delta = A^\Delta(t)H(t) \oplus A(\sigma(t))H^\Delta(t);
\]

(ii): if \( H \) is \((E_1)\)-differentiable, \( A(t)H(t) \) satisfies \((C_2)\) and \((A(\cdot), H^\sigma(\cdot)) \in (\mathfrak{B}, \Omega)\), then \( A(t)H(t) \) is \((E_2)\)-differentiable and

\[
(A(t)H(t))^\Delta = A^\Delta(t)H(\sigma(t)) \oplus (-A(t))H^\Delta(t);
\]

(iii): if \( H \) is \((E_2)\)-differentiable, \( A(t)H(t) \) satisfies \((C_1)\) and \((A(\cdot), H(\cdot)) \in (\mathfrak{g}, \Omega)\), then \( A(t)H(t) \) is \((E_1)\)-differentiable and

\[
(A(t)H(t))^\Delta = A^\Delta(t)H(t) \oplus (-A(\sigma(t)))H^\Delta(t);
\]

(iv): if \( H \) is \((E_2)\)-differentiable, \( A(t)H(t) \) satisfies \((C_2)\) and \((A(\cdot), H^\sigma(\cdot)) \in (\mathfrak{B}, \Omega)\), then \( A(t)H(t) \) is \((E_2)\)-differentiable and

\[
(A(t)H(t))^\Delta = A^\Delta(t)H(\sigma(t)) \oplus A(t)H^\Delta(t),
\]

where \( H^\sigma(t) = H(\sigma(t)) \).

Proof. Case (i). Since \( H \) is \((E_1)\)-differentiable, then there exists \( \mu(\cdot)H^\Delta(\cdot) \in \mathbb{Q}_{F}^{n} \) such that \( H(\sigma(t)) \oplus H(t) = \mu(t)H^\Delta(t) \), so \( H(\sigma(t)) = H(t) \oplus \mu(t)H^\Delta(t) \). By Lemma 2.2 and \((A(\cdot), H(\cdot)) \in (\mathfrak{g}, \Omega)\), we can obtain

\[
A(\sigma(t))H(\sigma(t)) = (A(t) + \mu(t)A^\Delta(t))(H(t) \oplus \mu(t)H^\Delta(t))
\]

\[
= (A(t) + \mu(t)A^\Delta(t))H(t)
\]

\[
\oplus (A(t) + \mu(t)A^\Delta(t))\mu(t)H^\Delta(t)
\]

\[
= A(t)H(t) \oplus \mu(t)A^\Delta(t)H(t) \oplus A(\sigma(t))\mu(t)H^\Delta(t).
\]

Notice Remark 2.2 and \( A(t)H(t) \) satisfies \((C_1)\), we have \( A(\sigma(t))H(\sigma(t)) \oplus A(t)H(t) = \mu(t)[A^\Delta(t)H(t) \oplus A(\sigma(t))H^\Delta(t)] \).

Case (ii). Since \( H \) is \((E_1)\)-differentiable, then there exists \( \mu(\cdot)H^\Delta(\cdot) \in \mathbb{Q}_{F}^{n} \) such that \( H(\sigma(t)) \oplus H(t) = \mu(t)H^\Delta(t) \), i.e., \( H(t) = H(\sigma(t)) \oplus \mu(t)H^\Delta(t) \). By Lemma 2.2 and \((A(\cdot), H^\sigma(\cdot)) \in (\mathfrak{B}, \Omega)\), we have

\[
A(t)H(t) = (A(\sigma(t)) - \mu(t)A^\Delta(t))(H(\sigma(t)) \oplus \mu(t)H^\Delta(t))
\]

\[
= (A(\sigma(t)) - \mu(t)A^\Delta(t))H(\sigma(t))
\]

\[
\oplus (A(\sigma(t)) - \mu(t)A^\Delta(t))\mu(t)H^\Delta(t) = A(\sigma(t))H(\sigma(t))
\]

\[
\oplus (-\mu(t)A^\Delta(t))H(\sigma(t)) \oplus A(t)\mu(t)H^\Delta(t).
\]

Note Remark 2.2 and \( A(t)H(t) \) satisfies \((C_2)\), we have \( A(t)H(t) \oplus A(\sigma(t))H(\sigma(t)) = (-\mu(t))[A^\Delta(t)H(\sigma(t)) \oplus (-A(t))H^\Delta(t)] \).

Case (iii). Since \( H \) is \((E_2)\)-differentiable, then there exists \((-\mu(\cdot))H^\Delta(\cdot) \in \mathbb{Q}_{F}^{n} \) such that \( H(t) \oplus H(\sigma(t)) = (-\mu(t))H^\Delta(t) \), i.e., \( H(\sigma(t)) = H(t) \oplus (-\mu(t))H^\Delta(t) \).
By Lemma 2.2 and \((A(\cdot), H(\cdot)) \in (\mathfrak{G}, \Omega)\), we have
\[
A(\sigma(t))H(\sigma(t)) = (A(t) + \mu(t)A^\Delta(t))(H(t) \oplus (-\mu(t))H^\Delta(t))
\]
\[
= (A(t) + \mu(t)A^\Delta(t))H(t)
\]
\[
\oplus (A(t) + \mu(t)A^\Delta(t))(-\mu(t))H^\Delta(t)
\]
\[
= A(t)H(t) \oplus \mu(t)A^\Delta(t)H(t) \oplus A(\sigma(t))(-\mu(t))H^\Delta(t).
\]
By Remark 2.2 and \(A(t)H(t)\) satisfies \((C_1)\), we can obtain
\[
A(\sigma(t))H(\sigma(t)) \oplus A(t)H(t) = \mu(t)[A^\Delta(t)H(t) \oplus (-A(\sigma(t)))H^\Delta(t)].
\]

Case (iv). Since \(H\) is \((E_2)\)-differentiable, then there exists \((-\mu(\cdot))H^\Delta(\cdot) \in \mathbb{Q}_T^\mu\) such that \(H(t) \oplus H(\sigma(t)) = (-\mu(t))H^\Delta(t)\), i.e., \(H(t) = H(\sigma(t)) \oplus (-\mu(t))H^\Delta(t)\). Through Lemma 2.2 and \((A(\cdot), H^\sigma(\cdot)) \in (\mathfrak{G}, \Omega)\), we have
\[
A(t)H(t) = (A(\sigma(t)) - \mu(t)A^\Delta(t))(H(\sigma(t)) \oplus (-\mu(t))H^\Delta(t))
\]
\[
= (A(\sigma(t)) - \mu(t)A^\Delta(t))H(\sigma(t))
\]
\[
\oplus (A(\sigma(t)) - \mu(t)A^\Delta(t))(-\mu(t))H^\Delta(t)
\]
\[
= A(\sigma(t))H(\sigma(t)) \oplus ((-\mu(t))A^\Delta(t))H(\sigma(t)) \oplus A(t)(-\mu(t))H^\Delta(t).
\]
Notice Remark 2.2 and \(A(t)H(t)\) satisfies \((C_2)\), we can obtain
\[
A(t)H(t) \oplus A(\sigma(t))H(\sigma(t)) = (-\mu(t))[A^\Delta(t)H(t) \oplus A(t)H^\Delta(t)].
\]
The proof is completed.

Let \(I\) be an identity matrix. We will introduce the following quaternion matrix exponential function which satisfies the dynamic equations \(e_A^\Delta(t, t_0) = A(t)e_A(t, t_0)\).

**Definition 2.11.** Let \(A : T \rightarrow \mathbb{Q}_{n \times n}, t_0, t \in T\), then define the exponential function \(e_A(t, t_0)\) by
\[
e_A(t, t_0) = I + \sum_{n=1}^{\infty} \int_{t_0}^{t} A(\tau_1) \int_{t_0}^{\tau_1} A(\tau_2) \ldots \int_{t_0}^{\tau_{n-1}} A(\tau_n) \Delta \tau_1 \ldots \Delta \tau_{n-1} \Delta \tau_n.
\]

**Remark 2.2.** Note that the exponential function \(e_A^\Delta(t, t_0) = A(t)e_A(t, t_0)\) is the solution of the dynamic equation \(X^\Delta(t) = A(t)X(t)\) with the initial value \(X(t_0) = I\), where \(X, A : T \rightarrow \mathbb{Q}_{n \times n}\). Hence, \(e_A(t, s)e_A(s, t_0)\) is the solution of the dynamic equation \(X^\Delta(t) = A(t)X(t)\) with the initial value \(X(s) = e_A(s, t_0)\), which implies \(e_A(t, t_0) = e_A(t, s)e_A(s, t_0)\), where \(t_0 < s < t\). Moreover \(e_A^{-1}(t, t_0) = (e_A(t, s)e_A(s, t_0))^{-1} = e_A^{-1}(s, t_0)e_A^{-1}(t, s)\), where \(e_A^{-1}(t, t_0)\) is the inverse of \(e_A(t, t_0)\).

3. **Hyers-Ulam-Rassias stability analysis.** In this section, we shall consider the Hyers-Ulam-Rassias stability of the following three types of the quaternion impulsive fuzzy dynamic equations on time scales:

\[
\begin{align*}
\frac{y(\Delta(t)) = A(t)y(t) \oplus f(t, y(t)), \quad t \in \bar{T} \setminus \{t_m\},}
\frac{\Delta y(t_m) = I_m(y(t_m)), \quad m \in M;}
\end{align*}
\]

\[
\begin{align*}
\frac{y(\Delta(t)) \oplus A(t)y(t) = f(t, y(t)), \quad t \in \bar{T} \setminus \{t_m\},}
\frac{\Delta y(t_m) = I_m(y(t_m)), \quad m \in M;}
\end{align*}
\]
\begin{equation}
\left\{ \begin{array}{l}
y^\Delta(t) + f(t, y(t)) = A(t)y(t), \quad t \in \hat{T}\backslash \{t_m\}, \\
\Delta y(t_m) = I_m(y(t_m)), \quad m \in \mathbb{M},
\end{array} \right. 
\tag{3}
\end{equation}

where $A : \mathbb{T} \to \mathbb{Q}_{n \times n}, f : \mathbb{T} \times \mathbb{Q}_p^n \to \mathbb{Q}_p^n, I_m : \mathbb{Q}_p^n \to \mathbb{Q}_p^n, \hat{T} = \{t \in \mathbb{T} : t_0 \leq t < T\}$ for some finite numbers $t_0, T \in \mathbb{T}, t_m \in \hat{T}, t_0 < t_m < t_{m+1} < T, m \in \mathbb{M} = \{n \in \mathbb{N} : 1 \leq n \leq k\}, k$ is a finite natural number, $\Delta y(t_m) = y(\sigma(t_m^+)) \oplus y(t_m)$. Note that $y(\sigma(t_m^+)) = y(t_m^+)$ if $t_m$ is a right dense point on $\mathbb{T}$ and $y(\sigma(t_m^+)) = y(\sigma(t_m))$ if $t_m$ is a right scattered point on $\mathbb{T}$. Denote $\overline{T} = \{t \in \mathbb{T} : t_0 \leq t \leq T\}$.

For $\forall \epsilon > 0$, $\phi : \mathbb{T} \to [0, +\infty], \varphi : \mathbb{M} \to [0, +\infty)$, consider the mapping $y : \overline{T} \to \mathbb{Q}_p^n$ with the condition $(C_1)$ (or $(C_2)$) and corresponds to (1), (2) and (3), respectively, satisfying the following conditions:

\begin{equation}
\begin{aligned}
&D(y^\Delta(t), A(t)y(t) \oplus f(t, y(t))) \leq \phi(t)\epsilon, \quad t \in \hat{T}\backslash \{t_m\}, \\
&D(\Delta y(t_m), I_m(y(t_m))) \leq \varphi(m)\epsilon, \quad m \in \mathbb{M}, \\
&D(I_m(y), I_m(z)) \leq LD(y, z), \quad \text{for } y, z \in \mathbb{Q}_p^n \text{ and } L > 0;
\end{aligned}
\tag{4}
\end{equation}

\begin{equation}
\begin{aligned}
&D(y^\Delta(t) \oplus A(t)y(t), f(t, y(t))) \leq \phi(t)\epsilon, \quad t \in \hat{T}\backslash \{t_m\}, \\
&D(\Delta y(t_m), I_m(y(t_m))) \leq \varphi(m)\epsilon, \quad m \in \mathbb{M}, \\
&D(I_m(y), I_m(z)) \leq LD(y, z), \quad \text{for } y, z \in \mathbb{Q}_p^n \text{ and } L > 0;
\end{aligned}
\tag{5}
\end{equation}

\begin{equation}
\begin{aligned}
&D(y^\Delta(t), f(t, y(t)), A(t)y(t)) \leq \phi(t)\epsilon, \quad t \in \hat{T}\backslash \{t_m\}, \\
&D(\Delta y(t_m), I_m(y(t_m))) \leq \varphi(m)\epsilon, \quad m \in \mathbb{M}, \\
&D(I_m(y), I_m(z)) \leq LD(y, z), \quad \text{for } y, z \in \mathbb{Q}_p^n \text{ and } L > 0.
\end{aligned}
\tag{6}
\end{equation}

**Definition 3.1.** (1) (or (2); (3), respectively) is said to be Hyers-Ulam-Rassias stable under $(C_1)$ (or $(C_2)$) if for each $\epsilon > 0$ and each solution of (4) (or (5); (6), respectively), there exists a $\tilde{y} \in \mathbb{Q}_p^n$ satisfying $(C_1)$ (or $(C_2)$) and $\tilde{y}$ is a solution of (1) (or (2); (3), respectively) with $D(y(t), \tilde{y}(t)) \leq \tilde{Q}(\varphi(t) + \phi(t))\epsilon$ for $\tilde{Q} > 0, t \in \mathbb{T}$.

Now, we will consider the stability of (1) under the condition $(C_1)$. Through the iterative method of solution for the impulsive dynamic equations, the solution of (1) under the condition $(C_1)$ can be given by:

\[ y_1(t, t_0, y_1(t_0)) = \]
\[ \left\{ \begin{array}{l}
\prod_{n=m-1}^{1} e_A(t_{n+1}, \sigma(t_n^+)) \left[ e_A(t_{n+1}, t_0) y_1(t_0) \\
\sum_{l=2}^{m-1} \prod_{v=m-1}^{1} e_A(t_{v+1}, \sigma(t_v^+)) \left[ \int_{\sigma(t_v^+)}^{t_l} e_A(t, \sigma(t)) f(\tau, y_1(\tau)) d\tau \oplus I_l(y_1(t_l)) \right] \\
\sum_{l=2}^{m-1} \sum_{v=m-1}^{1} e_A(t_{v+1}, \sigma(t_v^+)) \left[ \int_{\sigma(t_v^+)}^{t_l} e_A(t, \sigma(t)) f(\tau, y_1(\tau)) d\tau \oplus I_l(y_1(t_l)) \right] \\
\prod_{n=m-1}^{1} e_A(t_{n+1}, \sigma(t_n^+)) \left[ e_A(t_{n+1}, t_0) y_1(t_0) \\
\sum_{l=2}^{m-1} \prod_{v=m-1}^{1} e_A(t_{v+1}, \sigma(t_v^+)) \left[ \int_{\sigma(t_v^+)}^{t_l} e_A(t, \sigma(t)) f(\tau, y_1(\tau)) d\tau \oplus I_l(y_1(t_l)) \right] \\
\prod_{n=m-1}^{1} e_A(t_{n+1}, \sigma(t_n^+)) \left[ e_A(t_{n+1}, t_0) y_1(t_0) \\
\sum_{l=2}^{m-1} \prod_{v=m-1}^{1} e_A(t_{v+1}, \sigma(t_v^+)) \left[ \int_{\sigma(t_v^+)}^{t_l} e_A(t, \sigma(t)) f(\tau, y_1(\tau)) d\tau \oplus I_l(y_1(t_l)) \right] \\
\end{array} \right. \]
Theorem 3.1. Let \( y_1 \) be a solution of (4) and \((E_1)\)-differentiable, satisfying
\[
e_A(t, \sigma(\tau)) f(\tau, y_1(\tau))
\]
integrable on \([\sigma(t^+_m), t]\) for \( t \in (\sigma(t^+_m), t_{m+1}] \). Moreover, \( e_A^-1(t, \sigma(t^+_m)) y_1(t) \) satisfies (C1) and \((e_A^-1(t, \sigma(t^+_m)), y_1(\cdot)) \in (\mathcal{G}, \Omega), m \in \mathbb{M}\), \( y_1 \) is a solution of (1). If
\[
e_A^{-1}(t, \sigma(t^+_m))[y_1(t) \oplus \int_{\sigma(t^+_m)}^t e_A(t, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau]
\]
exists for all \( s, t \). For \( t \in (\sigma(t^+_m), t_{m+1}] \), \( e_A(t, \sigma(t^+_m)) y_1(t) \) satisfies (C1) and \((e_A^{-1}(t, \sigma(t^+_m)), y_1(\cdot)) \in (\mathcal{G}, \Omega), m \in \mathbb{M}\), \( y_1 \) is a solution of (1). If
\[
e_A^{-1}(t, \sigma(t^+_m))[y_1(t) \oplus \int_{\sigma(t^+_m)}^t e_A(t, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau]
\]
exists for all \( t \in n\backslash\{t_m\} \). Then there exists an unique \( y_0 \in Q^n_F \) with
\[
D(y_1(t), \tilde{y}_1(t)) \leq \varepsilon \|e_A(t, \sigma(t^+_m))\| \int_{s}^{t} \|e_A^{-1}(\sigma(\tau), \sigma(t^+_m))\| \phi(\tau) \Delta \tau
\]
for \( s, t \in (\sigma(t^+_m), t_{m+1}], s < t, m \in \mathbb{M} \) and
\[
D(y_1(\sigma(t^+_m+1)), \tilde{y}_1(\sigma(t^+_m+1)))
\]
\[
\leq \varepsilon (1 + L) \|e_A(t_{m+1}, \sigma(t^+_m))\| \int_{s}^{t} \|e_A^{-1}(\sigma(\tau), \sigma(t^+_m))\| \phi(\tau) \Delta \tau + \varepsilon \varphi(m + 1).
\]
Proof. For \( t \in (\sigma(t^+_m), t_{m+1}] \), we have
\[
\tilde{y}_1(t) = e_A(t, \sigma(t^+_m)) \left\{ \prod_{n=m-1}^{1} e_A(t_{n+1}, \sigma(t^+_n))[e_A(t_1, t_0)y_1(t_0)]
\right\}
\]
\[
\oplus \int_{t_0}^{t_1} e_A(t_1, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau
\]
\[
\oplus \left[ I_1(y_1(t_1)) \right] \oplus \sum_{l=2}^{m-1} \prod_{n=m-1}^{1} e_A(t_{n+1}, \sigma(t^+_n)) \left[ \int_{\sigma(t^+_n)}^{t} e_A(t, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau \right]
\]
\[
\oplus \left[ I_m(y_1(t_m)) \right] \oplus \int_{\sigma(t^+_m)}^{t} e_A(t, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau
\]
\[
\oplus \int_{\sigma(t^+_m)}^{t} e_A(t, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau
\]
\[
\oplus \int_{\sigma(t^+_m)}^{t} e_A(t, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau.
\]
\[
\text{Let } q(t) = e_A^{-1}(t, \sigma(t^+_m))[y_1(t) \oplus \int_{\sigma(t^+_m)}^{t} e_A(t, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau]
\]
\[
\oplus \sum_{l=2}^{m-1} \prod_{n=m-1}^{1} e_A(t_{n+1}, \sigma(t^+_n))
\]
For any $t, s \in (\sigma(t_m^+), t_{m+1}]$ and $t > s$, we have

$$D(q(t), q(s)) = D\left(\epsilon_A^{-1}(t, \sigma(t_m^+)) [y_1(t) \oplus \int_{\sigma(t_m^+)}^t \epsilon_A(t, \sigma(\tau)) f(\tau, y_1(\tau)) d\tau \oplus I_1(y_1(t_1))]\right)$$

$$\oplus \sum_{l=2}^{m-1} \prod_{v=m-1}^l \epsilon_A(t_v+1, \sigma(t_v^+)) \times \left[ \int_{\sigma(t_{v-1}^+)}^{t_{v+1}} \epsilon_A(t_l, \sigma(\tau)) f(\tau, y_1(\tau)) d\tau \oplus I_1(y_1(t_1)) \right],$$

$$\epsilon_A^{-1}(s, \sigma(t_m^+)) [y_1(s) \oplus \int_{\sigma(t_m^+)}^s \epsilon_A(s, \sigma(\tau)) f(\tau, y_1(\tau)) d\tau]$$

$$\oplus \sum_{l=2}^{m-1} \prod_{v=m-1}^l \epsilon_A(t_v+1, \sigma(t_v^+)) \times \left[ \int_{\sigma(t_{v-1}^+)}^{t_{v+1}} \epsilon_A(t_l, \sigma(\tau)) f(\tau, y_1(\tau)) d\tau \oplus I_1(y_1(t_1)) \right],$$

$$\epsilon_A^{-1}(s, \sigma(t_m^+)) [y_1(s) \oplus \int_{\sigma(t_m^+)}^s \epsilon_A(s, \sigma(\tau)) f(\tau, y_1(\tau)) d\tau]$$

$$\oplus \sum_{l=2}^{m-1} \prod_{v=m-1}^l \epsilon_A(t_v+1, \sigma(t_v^+)) \times \left[ \int_{\sigma(t_{v-1}^+)}^{t_{v+1}} \epsilon_A(t_l, \sigma(\tau)) f(\tau, y_1(\tau)) d\tau \oplus I_1(y_1(t_1)) \right]$$

$$= D\left(\epsilon_A^{-1}(t, \sigma(t_m^+)) [y_1(t) \oplus \int_{\sigma(t_m^+)}^t \epsilon_A(t, \sigma(\tau)) f(\tau, y_1(\tau)) d\tau], \epsilon_A^{-1}(s, \sigma(t_m^+)) [y_1(s)] \oplus \int_{\sigma(t_m^+)}^s \epsilon_A(s, \sigma(\tau)) f(\tau, y_1(\tau)) d\tau\right)$$

$$\oplus \int_{\sigma(t_m^+)}^s \epsilon_A(s, \sigma(\tau)) f(\tau, y_1(\tau)) d\tau$$

$$= D(\epsilon_A^{-1}(t, \sigma(t_m^+)) y_1(t)$$
\[ \int_{\sigma(t_m^+)}^t e_A^{-1}(\sigma(\tau), \sigma(t_m^+)) f(\tau, y_1(\tau)) \Delta \tau, e_A^{-1}(s, \sigma(t_m^+)) y_1(s) \]
\[ \int_{\sigma(t_m^+)}^t e_A^{-1}(\sigma(\tau), \sigma(t_m^+)) f(\tau, y_1(\tau)) \Delta \tau \]
\[ = D(e_A^{-1}(t, \sigma(t_m^+)) y_1(t), e_A^{-1}(s, \sigma(t_m^+)) y_1(s) + \int_s^t e_A^{-1}(\sigma(\tau), \sigma(t_m^+)) f(\tau, y_1(\tau)) \Delta \tau). \]

Since \( e_A(t, t_0)e_A^{-1}(t, t_0) = I \), then
\[ (e_A(t, t_0)e_A^{-1}(t, t_0))^\Delta = e_A(t, t_0)e_A^{-1}(t, t_0) + e_A(t_0, t_0)(e_A^{-1}(t, t_0))^\Delta = 0, \]
i.e.,
\[ (e_A^{-1}(t, t_0))^\Delta = -e_A^{-1}(\sigma(t), t_0) A(t). \]

Since \( y_1(t) \) is \((E_1)\)-differentiable and \( e_A^{-1}(t, \sigma(t_m^+)) y_1(t) \) satisfies \((C_1)\) and
\[ (e_A^{-1}(\cdot, \sigma(t_m^+)), y_1(\cdot)) \in (\delta, \Omega), \]
by Theorem 2.8, we have
\[ (e_A^{-1}(t, \sigma(t_m^+)) y_1(t))^\Delta = -e_A^{-1}(\sigma(t), \sigma(t_m^+)) A(t) y_1(t) + e_A^{-1}(\sigma(t), \sigma(t_m^+)) y_1^\Delta(t). \]

Hence
\[ e_A^{-1}(t, \sigma(t_m^+)) y_1(t) = e_A^{-1}(s, \sigma(t_m^+)) y_1(s) \]
\[ + \int_s^t -e_A^{-1}(\sigma(\tau), \sigma(t_m^+)) A(\tau) y_1(\tau) + e_A^{-1}(\sigma(\tau), \sigma(t_m^+)) y_1^\Delta(\tau) \Delta \tau. \]

Thus we have
\[ D(y(t), q(s)) \]
\[ = D(e_A^{-1}(s, \sigma(t_m^+)) y_1(s)) + \int_s^t -e_A^{-1}(\sigma(\tau), \sigma(t_m^+)) A(\tau) y_1(\tau) \]
\[ + e_A^{-1}(\sigma(\tau), \sigma(t_m^+)) y_1^\Delta(\tau) \Delta \tau, \]
\[ e_A^{-1}(s, \sigma(t_m^+)) y_1(s) + \int_s^t e_A^{-1}(\sigma(\tau), \sigma(t_m^+)) f(\tau, y_1(\tau)) \Delta \tau \]
\[ = D(\int_s^t -e_A^{-1}(\sigma(\tau), \sigma(t_m^+)) A(\tau) y_1(\tau) + e_A^{-1}(\sigma(\tau), \sigma(t_m^+)) y_1^\Delta(\tau) \Delta \tau, \]
\[ \int_s^t e_A^{-1}(\sigma(\tau), \sigma(t_m^+)) f(\tau, y_1(\tau)) \Delta \tau) \]
\[ \leq \int_s^t D(-e_A^{-1}(\sigma(\tau), \sigma(t_m^+)) A(\tau) y_1(\tau) + e_A^{-1}(\sigma(\tau), \sigma(t_m^+)) y_1^\Delta(\tau), e_A^{-1}(\sigma(\tau), \sigma(t_m^+)) f(\tau, y_1(\tau)) \Delta \tau \]
\[ \leq \int_s^t ||e_A^{-1}(\sigma(\tau), \sigma(t_m^+))|| D(-A(\tau) y_1(\tau) + y_1^\Delta(\tau), f(\tau, y_1(\tau))) \Delta \tau \]
\[ \leq \int_s^t ||e_A^{-1}(\sigma(\tau), \sigma(t_m^+))|| D(y_1^\Delta(\tau), A(\tau) y_1(\tau) + f(\tau, y_1(\tau))) \Delta \tau \]
\[ \leq \epsilon \int_s^t ||e_A^{-1}(\sigma(\tau), \sigma((t_m^+))|| \phi(\tau) \Delta \tau. \]
Hence

\[
\begin{align*}
D(y_1(t), \tilde{y}_1(t)) &= D\left(y_1(t), e_A(t, \sigma(t_m^+))\right) \left\{ \prod_{n=m-1}^{1} e_A\left(t_{n+1}, \sigma(t_n^+)\right) \left[e_A(t_1, t_0) y_1(t_0) \right] \right\} + \int_{t_0}^{t_1} e_A(t_1, \sigma(t_m^+)) f(\tau, y_1(\tau)) \Delta \tau + I_1(y_1(t_1)) + \sum_{l=2}^{m-1} \prod_{n=m-1}^{l} e_A\left(t_{n+1}, \sigma(t_n^+)\right) \\
&\times \left[ \int_{\sigma(t_{n-1}^-)}^{t_1} e_A(t_1, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau \right] \\
&\oplus \int_{\sigma(t_{n-1}^-)}^{t_1} e_A(t_1, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau \\
&\oplus I_1(y_1(t_1)) + \sum_{l=2}^{m-1} \prod_{n=m-1}^{l} e_A\left(t_{n+1}, \sigma(t_n^+)\right) \\
&\times \left[ \int_{\sigma(t_{n-1}^-)}^{t_1} e_A(t_1, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau + I_1(y_1(t_1)) \right] \\
&\oplus I_1(y_1(t_1)) + \sum_{l=2}^{m-1} \prod_{n=m-1}^{l} e_A\left(t_{n+1}, \sigma(t_n^+)\right) \\
&\times \left[ \int_{\sigma(t_{n-1}^-)}^{t_1} e_A(t_1, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau \oplus I_1(y_1(t_1)) \right] \\
&\oplus \sum_{l=2}^{m-1} \prod_{n=m-1}^{l} e_A\left(t_{n+1}, \sigma(t_n^+)\right) \left[ \int_{\sigma(t_{n-1}^-)}^{t_1} e_A(t_1, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau \oplus I_1(y_1(t_1)) \right] \\
&\oplus \int_{\sigma(t_{n-1}^-)}^{t_1} e_A(t_1, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau \oplus I_1(y_1(t_1)) + \prod_{n=m-1}^{1} e_A\left(t_{n+1}, \sigma(t_n^+)\right) \\
&\times \left[ \int_{t_0}^{t_1} e_A(t_1, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau \oplus I_1(y_1(t_1)) \right] \right\}.
\end{align*}
\]

Since the space $Q^p_\mathbb{F}$ is complete and the same type inequality holds for any $t \in (\sigma(t_{m})$, $t_{m+1}$), which implies $[q(s)]$ is a Cauchy sequence in $Q^p_\mathbb{F}$. Hence there exists a $y_0 = y_1(t_0) \in Q^p_\mathbb{F}$ such that $q(s)$ converges to $\prod_{n=m-1}^{1} e_A\left(t_{n+1}, \sigma(t_n^+)\right) e_A(t_1, t_0) y_1(t_0)$. Hence
\[ e_A(t, \sigma(t_{+}^n)) \prod_{\tilde{n}=m-1}^{1} e_A(t_{\tilde{n}+1}, \sigma(t_{+}^\tilde{n})) e_A(t_1, t_0) y_1(t_0) \]

\[ = D(e_A(t, \sigma(t_{+}^m)) q(t), e_A(t, \sigma(t_{+}^m)) \prod_{\tilde{n}=m-1}^{1} e_A(t_{\tilde{n}+1}, \sigma(t_{+}^\tilde{n})) e_A(t_1, t_0) y_1(t_0)) \]

\[ \leq D(e_A(t, \sigma(t_{+}^m)) q(s), e_A(t, \sigma(t_{+}^m)) \prod_{\tilde{n}=m-1}^{1} e_A(t_{\tilde{n}+1}, \sigma(t_{+}^\tilde{n})) e_A(t_1, t_0) y_1(t_0)) \]

\[ + D(e_A(t, \sigma(t_{+}^m)) q(t), e_A(t, \sigma(t_{+}^m)) q(s)) \]

\[ \leq \epsilon \| e_A(t, \sigma(t_{+}^m)) \| \int_s^t \| e_A^{-1}(\sigma(\tau), \sigma(t_{+}^m)) \| \phi(\tau) \Delta \tau. \]

For \( t = \sigma(t_{+}^m) \), we have

\[ D(y_1(\sigma(t_{+}^m)), \tilde{y}_1(\sigma(t_{+}^m))) \]

\[ = D(y_1(\sigma(t_{+}^m)) \oplus y_1(t_{m+1}) \oplus \tilde{y}_1(t_{m+1}) \oplus I_{m+1}(\tilde{y}_1(t_{m+1}))) \]

\[ = D(\tilde{\Delta} y_1(t_{m+1}) \oplus y_1(t_{m+1}) \oplus \tilde{y}_1(t_{m+1}) \oplus I_{m+1}(\tilde{y}_1(t_{m+1}))) \]

\[ \leq D(\tilde{\Delta} y_1(t_{m+1}) \oplus y_1(t_{m+1}) \oplus \tilde{y}_1(t_{m+1})) \]

\[ + D(\tilde{\Delta} y_1(t_{m+1}) \oplus \tilde{y}_1(t_{m+1}) \oplus I_{m+1}(\tilde{y}_1(t_{m+1}))) \]

\[ = D(y_1(t_{m+1}), \tilde{y}_1(t_{m+1})) + D(\tilde{\Delta} y_1(t_{m+1}), I_{m+1}(\tilde{y}_1(t_{m+1}))) \]

\[ + D(I_{m+1}(y_1(t_{m+1})), I_{m+1}(\tilde{y}_1(t_{m+1}))) \]

\[ = (1 + L) D(y_1(t_{m+1}), \tilde{y}_1(t_{m+1})) + D(\tilde{\Delta} y_1(t_{m+1}), I_{m+1}(\tilde{y}_1(t_{m+1}))) \]

\[ \leq \epsilon (1 + L) \| e_A(t_{m+1}, \sigma(t_{+}^m)) \| \int_s^t \| e_A^{-1}(\sigma(\tau), \sigma(t_{+}^m)) \| \phi(\tau) \Delta \tau + \epsilon \varphi(m + 1). \]

Now we prove the uniqueness of \( y_0 = y_1(t_0) \). Assume that there exists a \( y_{10} \) such that

\[ D(y_1(t), \tilde{y}_1(t)) \leq \epsilon \| e_A(t, \sigma(t_{+}^m)) \| \int_s^t \| e_A^{-1}(\sigma(\tau), \sigma(t_{+}^m)) \| \phi(\tau) \Delta \tau, \]

then

\[ D(y_0, y_{10}) \]

\[ = D(e_A^{-1}(t_1, t_0) \prod_{n=1}^{m-1} e_A^{-1}(t_{n+1}, \sigma(t_{+}^n)) e_A^{-1}(t, \sigma(t_{+}^n)) e_A(t, \sigma(t_{+}^m))) \]

\[ \times e_A(t_{n+1}, \sigma(t_{+}^n)) \]

\[ \times e_A(t_1, t_0) y_0 e_A^{-1}(t_1, t_0) \prod_{n=1}^{m-1} e_A^{-1}(t_{n+1}, \sigma(t_{+}^n)) e_A(t, \sigma(t_{+}^n)) e_A(t, \sigma(t_{+}^m)) \]

\[ \times e_A(t_{n+1}, \sigma(t_{+}^n)) e_A(t_1, t_0) y_{10} \]
\[ D\left(e_A^{-1}(t_1, t_0) \prod_{n=1}^{m-1} e_A^{-1}(t_{n+1}, \sigma(t_n^+)) e_A^{-1}(t, \sigma(t_m^+)) \right) \]
\[
\left\{ e_A(t, \sigma(t_m^+)) \left[ \prod_{n=m-1}^{1} e_A(t_{n+1}, \sigma(t_n^+)) \right] \right\} \times [e_A(t_1, t_0) y_1(t_0) \oplus \int_{t_0}^{t_1} e_A(t_1, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau \oplus I_1(y_1(t_1))]
\]
\[
\oplus \sum_{l=2}^{m-1} \prod_{v=m-l}^{l} e_A(t_{v+1}, \sigma(t_v^+)) \left( \int_{\sigma(t_{v-1}^+)}^{t_l} e_A(t_l, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau \oplus I_1(y_1(t_l)) \right)
\]
\[
\oplus \int_{\sigma(t_{m-1}^+)}^{t_m} e_A(t_m, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau
\]
\[
\oplus I_m(y_1(t_m)) \oplus \int_{\sigma(t_{m-1}^+)}^{t} e_A(t, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau
\]
\[
\leq D\left(e_A^{-1}(t_1, t_0) \prod_{n=1}^{m-1} e_A^{-1}(t_{n+1}, \sigma(t_n^+)) e_A^{-1}(t, \sigma(t_m^+)) \right) \]
\[
\times [e_A(t_1, t_0) y_1(t_0) \oplus \int_{t_0}^{t_1} e_A(t_1, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau \oplus I_1(y_1(t_1))]
\]
\[
\oplus \sum_{l=2}^{m-1} \prod_{v=m-l}^{l} e_A(t_{v+1}, \sigma(t_v^+)) \left( \int_{\sigma(t_{v-1}^+)}^{t_l} e_A(t_l, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau \oplus I_1(y_1(t_l)) \right)
\]
\[
\oplus \int_{\sigma(t_{m-1}^+)}^{t_m} e_A(t_m, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau \oplus I_m(y_1(t_m)) \]
\[
\oplus \int_{\sigma(t_{m-1}^+)}^{t} e_A(t, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau
\]},
Theorem 3.2. Let $y_1$ be a solution of (4) satisfying $(C_2)$ and $e_A(t, \sigma(\tau))f(\tau, y_1(\tau))$ is integrable on $[\sigma(t_m^+), t]$, for $t \in [\sigma(t_m^+), t_{m+1}]$, $m \in \mathbb{M}$. Moreover, $e_A^{-1}(t, \sigma(t_m^+))y_1(t)$ satisfies $(C_2)$ and $(e_A^{-1}(\cdot, \sigma(t_m^+)), y_1^*(\cdot)) \in (\mathfrak{B}, \Omega)$, $\tilde{y}_1$ is a solution of (1). If

$$e_A^{-1}(t_1, t_0) \prod_{n=1}^{m-1} e_A^{-1}(t_{\tilde{n}+1}, \sigma(t_{\tilde{n}}^+))e_A^{-1}(t, \sigma(t_m^+))y_1(t)$$

$$+ \int_{t_0}^{t_1} e_A(t, \sigma(t_m^+))y_1(t), e_A^{-1}(t_1, t_0) \prod_{n=1}^{m-1} e_A^{-1}(t_{\tilde{n}+1}, \sigma(t_{\tilde{n}}^+))e_A^{-1}(t, \sigma(t_m^+))\{e_A(t, \sigma(t_m^+))$$

$$\times \left[ \prod_{n=m-1}^{1} e_A(t_{\tilde{n}+1}, \sigma(t_{\tilde{n}}^+)) \right] e_A(t_1, t_0) y_{10} + \int_{t_0}^{t_1} e_A(t, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau$$

$$+ I_1(y_1(t_1))$$

$$+ \sum_{l=2}^{t} \sum_{v=m-1}^{1} e_A(t_{v+1}, \sigma(t_v^+)) \left( \int_{\sigma(t_{v-1}^+)}^{t_l} e_A(t_1, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau \oplus I_l(y_1(t_l)) \right)$$

$$\oplus \int_{\sigma(t_{m-1}^+)}^{t} e_A(t, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau$$

$$\oplus I_m(y_1(t_m)) \oplus \prod_{n=m-1}^{1} e_A(t_{\tilde{n}+1}, \sigma(t_{\tilde{n}}^+)) \left[ \int_{\sigma(t_{\tilde{n}-1}^+)}^{t} e_A(t_1, \sigma(\tau)) f(\tau, y_1(\tau)) \Delta \tau \oplus I_{\tilde{n}}(y_1(t_1)) \right]$$

$$\leq 2\varepsilon \|e_A^{-1}(t_1, t_0)\| \prod_{n=m-1}^{1} \|e_A^{-1}(t_{\tilde{n}+1}, \sigma(t_{\tilde{n}}^+))\| \|e_A^{-1}(t, \sigma(t_m^+))\| \|e_A(t, \sigma(t_m^+))\|$$

$$\times \int_s^t \|e_A^{-1}(\sigma(\tau), \sigma(t_m^+))\| \phi(\tau) \Delta \tau.$$
for \( s, t \in (\sigma(t_m^+), t_{m+1}) \), \( s < t \), \( m \in \mathbb{M} \) and
\[
D(y_1(\sigma(t_{m+1}^+)), \tilde{y}_1(\sigma(t_{m+1}^+))) \leq \epsilon(1 + L)\|e_A(t_{m+1}, \sigma(t_m^+))\| \int_s^{t_{m+1}} \|e_A^{-1}(\sigma(\tau), \sigma(t_m^+))\| \phi(\tau) \Delta \tau + \epsilon \varphi(m + 1).
\]

Next we will consider the stability of (2) under the condition \((C_1)\). According to the iterative method of solution for the impulsive dynamic equations, the solution of (2) is as follows:

\[
y_2(t, t_0, y(t_0)) = \left\{ \begin{array}{ll}
\prod_{\tilde{n} = m-1}^{n} e_{\mathcal{A}}(t, t_0(\tilde{n})) + \int_{t_0}^{t} e_{\mathcal{A}}(t, \sigma(\tau)) f(\tau, y_2(\tau)) \Delta \tau, & t \in [t_0, t_1], \\
\prod_{l = m-1}^{m} e_{\mathcal{A}}(t_{l+1}, \sigma(t_l^+)) \prod_{l = m-1}^{m} e_{\mathcal{A}}(t, \sigma(\tau)) f(\tau, y_2(\tau)) \Delta \tau + \int_{t_0}^{t} e_{\mathcal{A}}(t, \sigma(\tau)) f(\tau, y_2(\tau)) \Delta \tau \end{array} \right.
\]

[Equation (7)]

where \( e_{\mathcal{A}}(t, \sigma(\tau)) y = y \oplus \sum_{n=1}^{m} \int_{e_{\mathcal{A}}(t, t_0(\tau)) \cap A(\tau_n) \cap A(\tau_{n-1}) \ldots \cap A(\tau_1) \cap A(\tau_0) \ldots \int_{e_{\mathcal{A}}(t, \sigma(\tau)) f(\tau, y(t_2(\tau)) \Delta \tau \oplus I_m(y_2(t_m))) (t, t_0, y(t_0)) = \right. \]

By using (7), the proofs of Theorems 3.3-3.4 are very similar to Theorem 3.1, we will state them as follows and omit their proofs.

**Theorem 3.3.** Let \( y_2 \) be a solution of (5) and \((E_1)\)-differentiable, satisfying
\[
e_{\mathcal{A}}(t, \sigma(\tau)) f(\tau, y_2(\tau))
\]

integrable on \( [\sigma(t_m^+), t] \) for \( t \in (\sigma(t_m^+), t_{m+1}) \), \( m \in \mathbb{M} \). Moreover, \( e_{\mathcal{A}}^{-1}(t, \sigma(t_m^+)) y_2(t) \) satisfies \((C_1)\) and \((e_{\mathcal{A}}^{-1}(t, \sigma(t_m^+)) y_2(t)) \), \( y_2(\cdot) \in (\mathcal{H}, \Omega) \), \( \tilde{y}_2 \) is a solution of (2). If
\[
e_{\mathcal{A}}^{-1}(t, \sigma(t_m^+)) \left[ y_2(t) \oplus \int_{t_0}^{t} e_{\mathcal{A}}(t, \sigma(\tau)) f(\tau, y_2(\tau)) \Delta \tau \right]
\]

\[
\oplus \sum_{l = m-1}^{t} \prod_{l = m-1}^{t} e_{\mathcal{A}}(t, \sigma(\tau)) f(\tau, y_2(\tau)) \Delta \tau \oplus I_i(y_2(t_i)) \oplus \int_{t_0}^{t} e_{\mathcal{A}}(t, \sigma(\tau)) f(\tau, y_2(\tau)) \Delta \tau \oplus I_1(y_2(t_1))
\]

\[
\times \left[ \int_{t_0}^{t} e_{\mathcal{A}}(t, \sigma(\tau)) f(\tau, y_2(\tau)) \Delta \tau \oplus I_m(y_2(t_m)) \right]
\]
exists for all \( t \in n \setminus \{ t_m \} \). Then there exists an unique \( y_0 \in \mathbb{Q}^n_F \) with

\[
D(y_2(t), \bar{y}_2(t)) \leq \epsilon \| e_{-A}(t, \sigma(t_m^+)) \| \int_0^t \| e_{-A}(\sigma(\tau), \sigma(t_m^+)) \| \phi(\tau) \, \Delta \tau
\]

for \( s, t \in (\sigma(t_m^+), t_{m+1}], s < t, m \in \mathbb{M} \) and

\[
D(y_2(\sigma(t_{m+1}^+)), \bar{y}_2(\sigma(t_{m+1}^+))) \leq \epsilon(1 + L) \| e_{-A}(t_{m+1}, \sigma(t_m^+)) \| \int_0^{t_{m+1}} \| e_{-A}(\sigma(\tau), \sigma(t_m^+)) \| \phi(\tau) \, \Delta \tau + \epsilon \varphi(m + 1).
\]

**Theorem 3.4.** Let \( y_2 \) be a solution of (5) and satisfy (C2), \( e_{\sigma_A}(t, \sigma(\tau)) f(\tau, y_2(\tau)) \) integrable on \( [0, t_{m+1}] \) for \( t \in (\sigma(t_m^+), t_{m+1}], m \in \mathbb{M} \). Furthermore, \( e_{-A}(t, \sigma(t_m^+)) y_2(t) \) satisfies (C2) and \( (e_{-A}(t, \sigma(t_m^+)), y_2(t)) \in (\mathcal{B}, \mathcal{O}) \). \( \bar{y}_2 \) is a solution of (2). If

\[
e_{-A}(t, \sigma(t_m^+))[y_2(t) \oplus \int_0^t -e_{\sigma_A}(t, \sigma(\tau)) f(\tau, y_2(\tau)) \, \Delta \tau] \\
\oplus \sum_{l=2}^{m-1} \prod_{e=m-1}^{l} e_{\sigma_A}(t_{e+1}, \sigma(t_{e}^+)) \times \left[ \oplus \int_0^{t_{e+1}^+} -e_{\sigma_A}(t_{e+1}, \sigma(\tau)) f(\tau, y_2(\tau)) \, \Delta \tau \\
\oplus I_1(y_2(t_{e+1})) \right] \oplus \int_0^{t_{m}^+} -e_{\sigma_A}(t_m, \sigma(\tau)) f(\tau, y_2(\tau)) \, \Delta \tau \\
\oplus I_m(y_2(t_m)) \oplus \prod_{\bar{n}=m-1}^{1} e_{\sigma_A}(t_{\bar{n}+1}, \sigma(t_{\bar{n}}^+)) \\
\times \left[ \oplus \int_0^{t_{1}^+} -e_{\sigma_A}(t_{1}, \sigma(\tau)) f(\tau, y_2(\tau)) \, \Delta \tau + I_1(y_2(t_{1})) \right]
\]

exists for all \( t \in n \setminus \{ t_m \} \). Then there exists an unique \( y_0 \in \mathbb{Q}^n_F \) with

\[
D(y_2(t), \bar{y}_2(t)) \leq \epsilon \| e_{-A}(t, \sigma(t_m^+)) \| \int_0^t \| e_{-A}(\sigma(\tau), \sigma(t_m^+)) \| \phi(\tau) \, \Delta \tau
\]

for \( s, t \in (\sigma(t_m^+), t_{m+1}], s < t, m \in \mathbb{M} \) and

\[
D(y_2(\sigma(t_{m+1}^+)), \bar{y}_2(\sigma(t_{m+1}^+))) \leq \epsilon(1 + L) \| e_{-A}(t_{m+1}, \sigma(t_m^+)) \| \int_0^{t_{m+1}} \| e_{-A}(\sigma(\tau), \sigma(t_m^+)) \| \phi(\tau) \, \Delta \tau + \epsilon \varphi(m + 1).
\]

In the following, we will consider the stability of (3) under the condition (C1). Through the iterative method of solution for the impulsive dynamic equations, the solution of (3) is as follows:

\[
y_2(t, t_0, y_0(t_0)) =
\]
Theorem 3.1. Let $y_3$ be a solution of (6) and (E1)-differentiable, $e_A(t, \sigma(\tau)) f(\tau, y_3(\tau))$ integrable on $[\sigma(t^+_n), t_m]$, for $t \in [\sigma(t^+_n), t_m]$, $m \in \mathbb{N}$. Furthermore, $e_A(t, \sigma(t^+_n)) y_3(t)$ satisfies (C1) and $(e_A^{-1}(\cdot, \sigma(t^+_n)), y_3(\cdot)) \in (\mathfrak{F}, \mathfrak{G})$, $\tilde{y}_3$ is a solution of (3). If

$$e_A(t, t_0) y_3(t_0) + \int_{t_0}^{t} e_A(t_0, \sigma(\tau)) f(\tau, y_3(\tau)) \Delta \tau, \quad t \in [t_0, t_1],$$

$$\prod_{n=m-1}^{l} e_A(t_{n+1}, \sigma(t^+_n)) e_A(t_0, t_0) y_3(t_0)$$

$$= \int_{t_0}^{t} e_A(t_0, \sigma(\tau)) f(\tau, y_3(\tau)) \Delta \tau + I_1(y_3(t_1))$$

exists for all $t \in [t_0, t_1]$. Then there exists an unique $y_0 \in Q^m_F$ with

$$D(y_3(t), \tilde{y}_3(t)) \leq \epsilon \|e_A(t, \sigma(t^+_n))\| \int_{s}^{t} \|e_A^{-1}(\sigma(\tau), \sigma(t^+_n))\| \phi(\tau) \Delta \tau$$

for $s, t \in (\sigma(t^+_n), t_m)$, $s < t$, $m \in \mathbb{N}$ and

$$D(y_3(\sigma(t^+_m)), \tilde{y}_3(\sigma(t^+_m)))$$

$$\leq \epsilon(1 + L) \|e_A(t_{m+1}, \sigma(t^+_m), \sigma(t^+_m))\| \int_{s}^{t_m} \|e_A^{-1}(\sigma(\tau), \sigma(t^+_n))\| \phi(\tau) \Delta \tau + \epsilon \varphi(m + 1).$$

Theorem 3.6. Let $y_3$ be a solution of (6) and satisfy (C2), $e_A(t, \sigma(\tau)) f(\tau, y_3(\tau))$ integrable on $[\sigma(t^+_n), t]$, for $t \in (\sigma(t^+_n), t_m]$, $m \in \mathbb{N}$. Moreover, $e_A^{-1}(t, \sigma(t^+_n)) y_3(t)$
satisfies (C2) and \((e^{-1}_A(\cdot, \sigma(t^+_m)), y^\sigma_3(\cdot)) \in (\mathcal{B}, \Omega)\), \(\tilde{y}_3\) is a solution of (3). If
\[
e^{-1}_A(t, \sigma(t^+_m))[y_3(t) \ominus \int_{t}^{t^+_m} e_A(t, \sigma(\tau)) f(\tau, y_3(\tau)) d\tau]
\] and the solution of (4) can be given as follows:
\[
\begin{align*}
&\ominus \sum_{l=2}^{m-1} \prod_{v=m-1}^{l} e_A(t_{v+1}, \sigma(t^+_m)) \left[ \ominus \int_{\sigma(t^-_{l-1})}^{t_{l+1}} e_A(t, \sigma(\tau)) f(\tau, y_3(\tau)) d\tau \right] \\
&\ominus I_l(y_3(t_l)) \ominus \int_{\sigma(t^-_{m+1})}^{t_m} e_A(t, \sigma(\tau)) f(\tau, y_3(\tau)) d\tau \\
&\ominus I_m(y_3(t_m)) \ominus 1 \prod_{\tilde{n}=m-1}^{1} e_A(t_{\tilde{n}+1}, \sigma(t^+_m)) \\
&\times \left[ \ominus \int_{t_0}^{t_1} e_A(t, \sigma(\tau)) f(\tau, y_3(\tau)) d\tau \ominus I_1(y_3(t_1)) \right]
\end{align*}
\] exists for all \(t \in n \setminus \{t_m\}\). Then there exists an unique \(y_0 \in \mathbb{Q}_F^n\) with
\[
D(y_3(t), \tilde{y}_3(t)) \leq \epsilon\|e_A(t, \sigma(t^+_m))\| \int_s^t \|e^{-1}_A(\sigma(\tau), \sigma(t^+_m))\| d\tau
\] for \(s, t \in (\sigma(t^+_m), t_{m+1}), s < t, m \in \mathbb{M}\) and
\[
\begin{align*}
D(y_3(\sigma(t^+_m)), \tilde{y}_3(\sigma(t^+_m))) \\
&\leq \epsilon(1 + L)\|e_A(t_{m+1}, \sigma(t^+_m))\| \int_s^{t_{m+1}} \|e^{-1}_A(\sigma(\tau), \sigma(t^+_m))\| d\tau + \epsilon\varphi(m + 1).
\end{align*}
\]

4. **Example.** Let \(n = 2, T = \mathbb{Z}, \tilde{T} = \{t \in T : 0 \leq t < 8\}, t_0 = 0, t_1 = 1, t_2 = 4, \mathbb{M} = \{1, 2\}, f(t, y(t)) = t^2 y(t), \phi(t) = t + \frac{1}{2}, \varphi(m) = m, I_m(y(t_m)) = my(t_m), A(t) = \left[\frac{t}{2}, \frac{t^3}{2}, \frac{t^4}{2}, \frac{t^5}{2}\right]\) and satisfies the inequalities (4). Case (i). Assume that all the conditions of Theorem 3.1 are satisfied. Then the solution of (1) can be given by:
\[
\tilde{y}(t, 0, \tilde{y}(0)) =
\begin{cases}
e_A(t, 0)\tilde{y}(0) + \int_0^t e_A(t, \sigma(\tau)) f(\tau, \tilde{y}(\tau)) d\tau, & t \in [0, 1], \\
2e_A(1, 0)\tilde{y}(0) + \int_0^1 e_A(1, \sigma(\tau)) f(\tau, \tilde{y}(\tau)) d\tau, & t = 2, \\
2A(2)\left(2e_A(1, 0)\tilde{y}(0) + \int_0^1 e_A(1, \sigma(\tau)) f(\tau, \tilde{y}(\tau)) d\tau\right) \\
\ominus \left(9e_A(1, 0)\tilde{y}(0) + \int_0^1 e_A(1, \sigma(\tau)) f(\tau, \tilde{y}(\tau)) d\tau, & t = 4, \\
6A(2)\left(2e_A(1, 0)\tilde{y}(0) + \int_0^1 e_A(1, \sigma(\tau)) f(\tau, \tilde{y}(\tau)) d\tau\right) \\
\ominus \left(27e_A(1, 0)\tilde{y}(0) + \int_0^1 e_A(1, \sigma(\tau)) f(\tau, \tilde{y}(\tau)) d\tau, & t = 8
\end{cases}
\]
and the solution of (4) can be given as follows:
\[
y(t, 0, \tilde{y}(0)) =
\begin{cases}
e_A(t, 0)\tilde{y}(0) + \int_0^t e_A(t, \sigma(\tau)) f(\tau, \tilde{y}(\tau)) d\tau \ominus g_0(t), & t \in [0, 1], \\
2e_A(1, 0)\tilde{y}(0) + \int_0^1 e_A(1, \sigma(\tau)) f(\tau, \tilde{y}(\tau)) d\tau \ominus g_1(t), & t = 2, \\
2A(2)\left(2e_A(1, 0)\tilde{y}(0) + \int_0^1 e_A(1, \sigma(\tau)) f(\tau, \tilde{y}(\tau)) d\tau\right) \\
\ominus \left(9e_A(1, 0)\tilde{y}(0) + \int_0^1 e_A(1, \sigma(\tau)) f(\tau, \tilde{y}(\tau)) d\tau, & t = 4, \\
6A(2)\left(2e_A(1, 0)\tilde{y}(0) + \int_0^1 e_A(1, \sigma(\tau)) f(\tau, \tilde{y}(\tau)) d\tau\right) \\
\ominus \left(27e_A(1, 0)\tilde{y}(0) + \int_0^1 e_A(1, \sigma(\tau)) f(\tau, \tilde{y}(\tau)) d\tau, & t = 8
\end{cases}
\]
where
\[
D(0', g'_0(t)) \leq c\|e_A(t, 0)\| \int_0^t \|e^{-1}_A(\sigma(\tau), 0)\|\phi(\tau) \Delta \tau,
\]
\[
D(0', g'_2(t)) \leq 5c\|e_A(4, 2)\||e^{-1}_A(4, 2)||,
\]
\[
D(0', g'_1(t)) \leq (1 + L)c\|e_A(1, 0)\| \int_0^t \|e^{-1}_A(\sigma(\tau), 0)\|\phi(\tau) \Delta \tau + \epsilon,
\]
\[
D(0', g'_3(t)) \leq 5(1 + L)c\|e_A(4, 2)\||e^{-1}_A(4, 2)|| + 2c.
\]

Case (ii). Assume that all the conditions of Theorem 3.2 are fulfilled. Then the solution of (1) can be given by:
\[
\hat{y}(t, 0, \hat{y}(0)) = \begin{cases}
    e_A(t, 0)\hat{y}(0) \oplus \int_0^t -e_A(t, \sigma(\tau)) f(\tau, \hat{y}(\tau)) \Delta \tau, & t \in [0, 1], \\
    2e_A(1, 0)\hat{y}(0) \oplus \int_0^1 -e_A(1, \sigma(\tau)) f(\tau, \hat{y}(\tau)) \Delta \tau, & t = 2, \\
    4A(2)(2e_A(1, 0)\hat{y}(0) \oplus \int_0^1 -e_A(1, \sigma(\tau)) f(\tau, \hat{y}(\tau)) \Delta \tau) \\
    \oplus \int_0^1 (2e_A(1, 0)\hat{y}(0) \oplus 2\int_0^1 -e_A(1, \sigma(\tau)) f(\tau, \hat{y}(\tau)) \Delta \tau) \\
    \oplus \int_0^2 6A(2)(2e_A(1, 0)\hat{y}(0) \oplus \int_0^1 -e_A(1, \sigma(\tau)) f(\tau, \hat{y}(\tau)) \Delta \tau) \\
    \oplus \int_0^4 27(2e_A(1, 0)\hat{y}(0) \oplus \int_0^1 -e_A(1, \sigma(\tau)) f(\tau, \hat{y}(\tau)) \Delta \tau) & t = 8,
\end{cases}
\]

and the solution of (4) can be given as:
\[
y(t, 0, \hat{y}(0)) = \begin{cases}
    e_A(t, 0)\hat{y}(0) \oplus \int_0^t -e_A(t, \sigma(\tau)) f(\tau, \hat{y}(\tau)) \Delta \tau \oplus g'_0(t), & t \in [0, 1], \\
    2e_A(1, 0)\hat{y}(0) \oplus \int_0^1 -e_A(1, \sigma(\tau)) f(\tau, \hat{y}(\tau)) \Delta \tau \oplus g'_1, & t = 2, \\
    4A(2)(2e_A(1, 0)\hat{y}(0) \oplus \int_0^1 -e_A(1, \sigma(\tau)) f(\tau, \hat{y}(\tau)) \Delta \tau) \\
    \oplus \int_0^1 (2e_A(1, 0)\hat{y}(0) \oplus \int_0^1 -e_A(1, \sigma(\tau)) f(\tau, \hat{y}(\tau)) \Delta \tau) \\
    \oplus \int_0^3 6A(2)(2e_A(1, 0)\hat{y}(0) \oplus \int_0^1 -e_A(1, \sigma(\tau)) f(\tau, \hat{y}(\tau)) \Delta \tau) \\
    \oplus \int_0^6 27(2e_A(1, 0)\hat{y}(0) \oplus \int_0^1 -e_A(1, \sigma(\tau)) f(\tau, \hat{y}(\tau)) \Delta \tau) \oplus g'_3, & t = 8,
\end{cases}
\]

where
\[
D(0', g'_0(t)) \leq c\|e_A(t, 0)\| \int_0^t \|e^{-1}_A(\sigma(\tau), 0)\|\phi(\tau) \Delta \tau,
\]
\[
D(0', g'_2(t)) \leq 5c\|e_A(4, 2)\||e^{-1}_A(4, 2)||,
\]
\[
D(0', g'_1(t)) \leq (1 + L)c\|e_A(1, 0)\| \int_0^t \|e^{-1}_A(\sigma(\tau), 0)\|\phi(\tau) \Delta \tau + \epsilon,
\]
\[
D(0', g'_3(t)) \leq 5(1 + L)c\|e_A(4, 2)\||e^{-1}_A(4, 2)|| + 2c.
\]

5. Conclusion. In this paper, we conduct the first discussion of the Hyers-Ulam-Rassias stability for the high-dimensional quaternion fuzzy dynamic equations with impulses on time scales. Through establishing some basic results of the calculus of fuzzy quaternion functions in the high-dimensional fuzzy quaternion space, some sufficient conditions to guarantee the Hyers-Ulam-Rassias stability of the fuzzy iteration solutions for these impulsive fuzzy dynamic equations are obtained on time scales. The results established in this paper fill the gap of Hyers-Ulam-Rassias stability theory of the high-dimensional quaternion impulsive fuzzy dynamic equations on time scales.
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