Maximizing spectral radius and number of spanning trees in bipartite graphs

Ravindra B. Bapat
Indian Statistical Institute
New Delhi 110 016, India.
email: rbb@isid.ac.in

Abstract

The problems of maximizing the spectral radius and the number of spanning trees in a class of bipartite graphs with certain degree constraints are considered. In both the problems, the optimal graph is conjectured to be a Ferrers graph. Known results towards the resolution of the conjectures are described. We give yet another proof of a formula due to Ehrenborg and van Willigenburg for the number of spanning trees in a Ferrers graph. The main tool is a result which gives several necessary and sufficient conditions under which the removal of an edge in a graph does not affect the resistance distance between the end-vertices of another edge.

Key words. spectral radius, Ferrers graph, spanning trees, bipartite graph, resistance distance, Laplacian

AMS Subject Classifications. 05C50
1 Introduction

We consider simple graphs which have no loops or parallel edges. Thus a graph $G = (V, E)$ consists of a finite set of vertices, $V(G)$, and a set of edges, $E(G)$, each of whose elements is a pair of distinct vertices. We will assume familiarity with basic graph-theoretic notions, see, for example, Bondy and Murty [5].

There are several matrices that one normally associates with a graph. We introduce some such matrices which are important. Let $G$ be a graph with $V(G) = \{1, \ldots, n\}$. The adjacency matrix $A$ of $G$ is an $n \times n$ matrix with its rows and columns indexed by $V(G)$ and with the $(i, j)$-entry equal to 1 if vertices $i, j$ are adjacent and 0 otherwise. Thus $A$ is a symmetric matrix with its $i$-th row (or column) sum equal to $d(i)$, which by definition is the degree of the vertex $i, i = 1, 2, \ldots, n$. Let $D$ denote the $n \times n$ diagonal matrix, whose $i$-th diagonal entry is $d(i), i = 1, 2, \ldots, n$. The Laplacian matrix of $G$, denoted by $L$, is the matrix $L = D - A$.

By the eigenvalues of a graph we mean the eigenvalues of its adjacency matrix. Spectral graph theory is the study of the relationship between the eigenvalues of a graph and its structural properties. The spectral radius of a graph is the largest eigenvalue, in modulus, of the graph. It is a topic of much investigation. It evolved during the study of molecular graphs by chemists. We refer to [12] for the subject of spectral graph theory.

A connected graph without a cycle is called a tree. Trees constitute an important subclass of graphs both from theoretical and practical considerations. A spanning tree in a graph is a spanning subgraph which is a tree. Spanning trees arise in several applications. If we are interested in establishing a network of locations with minimal links, then it corresponds to a spanning tree. We may also be interested in the spanning tree with the least weight, where each edge in the graph is associated a weight and the weight of a spanning tree is the sum of the weights of its edges.

If $G$ is connected, then $L$ is singular with rank $n - 1$. Furthermore, the well-known Matrix-Tree Theorem asserts that any cofactor of $L$ equals the number of spanning trees $\tau(G)$ in $G$. For basic results concerning matrices associated with a graph we refer to [2].

A graph $G$ is bipartite if its vertex set can be partitioned as $V(G) = X \cup Y$ such that no two vertices in $X$, or in $Y$, are adjacent. We often denote the bipartition as $(X, Y)$. A graph is bipartite if and only if it has no cycle of odd length.

The adjacency matrix of a bipartite graph $G$ has a particularly simple form viewed as a partitioned matrix

$$A(G) = \begin{bmatrix} 0 & B \\ B' & 0 \end{bmatrix}.$$
This form is especially useful in dealing with matrices associated with a bipartite graph.

In this paper we consider two optimization problems over bipartite graphs under certain constraints. One of the problems is to maximize the spectral radius, while the other is to maximize the number of spanning trees.

We now describe the contents of this paper. In Section 2 we introduce the class of Ferrers graphs which are bipartite graphs such that the edges of the graph are in direct correspondence with the boxes in a Ferrers diagram. This class is of interest in both the maximization problems that we consider.

The problem of maximizing the spectral radius of a bipartite graph is considered in Section 3. We give a brief survey of the problem and provide references to the literature containing results and open problems.

In Section 4 we state an elegant formula for the number of spanning trees in a Ferrers graph due to Ehrenborg and van Willigenburg [13]. We give references to the proofs of the formula available in the literature. The formula leads to a conjectured upper bound for the number of spanning trees in a bipartite graph and is considered in Section 5. A reformulation of the conjecture in terms of majorization due to Slone is described in Section 6.

Sections 7 and 8 contain new results. The concept of resistance distance [17] between two vertices in a graph captures the notion of the degree of communication in a better way than the classical distance. The resistance distance can be defined in several equivalent ways, see, for example [3]. It is known, and intuitively obvious, that the resistance distance between any two vertices does not decrease when an edge, which is not a cut-edge, is deleted from the graph. In Section 7 we first give an introduction to resistance distance. We then examine the situation when the removal of an edge in a graph does not affect the resistance distance between the end-vertices of another edge. Several equivalent conditions are given for this to hold. This result, which appears to be of interest by itself, is then used in Section 8 to give another proof of the formula for the number of spanning trees in a Ferrers graph. Ehrenborg and van Willigenburg [13] also use electrical networks and resistances in their proof of the formula but our approach is different.

## 2 Ferrers graphs

A Ferrers graph is defined as a bipartite graph on the bipartition \((U, V)\), where \(U = \{u_1, \ldots, u_m\}, V = \{v_1, \ldots, v_n\}\) such that

- if \((u_i, v_j)\) is an edge, then so is \((u_p, v_q)\), where \(1 \leq p \leq i\) and \(1 \leq q \leq j\).
• \((u_1, v_n)\) and \((u_m, v_1)\) are edges.

For a Ferrers graph \(G\) we have the associated partition \(\lambda = (\lambda_1, \ldots, \lambda_m)\), where \(\lambda_i\) is the degree of vertex \(u_i, i = 1, \ldots, m\). Similarly we have the dual partition \(\lambda' = (\lambda'_1, \ldots, \lambda'_n)\) where \(\lambda'_j\) is the degree of vertex \(v_j, j = 1, \ldots, n\). Note that \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m\) and \(\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n\). The associated Ferrers diagram is the diagram of boxes where we have a box in position \((i, j)\) if and only if \((u_i, v_j)\) is an edge in the Ferrers graph.

**Example 2.1** The Ferrers graph with the degree sequences \((3, 3, 2, 1)\) and \((4, 3, 2)\) is shown below.

The associated Ferrers diagram is

The definition of Ferrers graph is due to Ehrenborg and van Willigenburg [13]. Chestnut and Fishkind [10] defined the class of bipartite graphs called *difference graphs*. A bipartite graph with parts \(X\) and \(Y\) is a difference graph if there exists a function \(\phi : X \cup Y \to \mathbb{R}\) and a threshold \(\alpha \in \mathbb{R}\) such that for all \(x \in X\) and \(y \in Y\), \(x\) is adjacent to \(y\) if and only if \(\phi(x) + \phi(y) \geq \alpha\). It turns out that the class of Ferrers graphs coincides with the class of difference graphs, as shown by Hammer et al. [16]. A more direct proof of this equivalence is given by Cheng Wai Koo [18]. The same class is termed *chain graphs* in [4].
3 Maximizing the spectral radius of a bipartite graph

We introduce some notation. Let $G = (V \cup W, E)$ be a bipartite graph, where $V = \{v_1, \ldots, v_m\}, W = \{w_1, \ldots, w_n\}$ are the two partite sets. We view the undirected edges $E$ of $G$ as a subset of $V \times W$. Let

$$D(G) = d_1(G) \geq d_2(G) \geq \cdots \geq d_m(G)$$

be the rearranged set of the degrees of $v_1, \ldots, v_m$. Note that $e(G) = \sum_{i=1}^{m} d_i(G)$ is the number of edges in $G$. Recall that the eigenvalues of $G$ are simply the eigenvalues of the adjacency matrix of $G$. Since the adjacency matrix is entrywise nonnegative, it follows from the Perron-Frobenius Theorem that the spectral radius of the adjacency matrix is an eigenvalue of the matrix. Denote by $\lambda_{\text{max}}(G)$ the maximum eigenvalue of $G$. It is known [4] that

$$\lambda_{\text{max}}(G) \leq \sqrt{e(G)}$$

and equality occurs if and only if $G$ is a complete bipartite graph, with possibly some isolated vertices.

We now consider refinements of (1) for noncomplete bipartite graphs. For positive integers $p, q$, let $K_{p,q}$ be the complete bipartite graph $G = (V \cup W, E)$ where $|V| = p, |W| = q$. Let $K(p, q, e)$ be the family of subgraphs of $K_{p,q}$ with $e$ edges, with no isolated vertices, and which are not complete bipartite graphs. The following problem was considered in [4]:

**Problem 3.1** Let $2 \leq p \leq q, 1 < e < pq$ be integers. Characterize the graphs which solve the maximization problem

$$\max_{G \in K(p, q, e)} \lambda_{\text{max}}(G).$$

Motivated by a conjecture of Brualdi and Hoffman [7] for nonbipartite graphs, which was proved by Rowlinson [20], the following conjecture was proposed in [4]:

**Conjecture 3.2** Under the assumptions of Problem [3.1], an extremal graph that solves the maximal problem is obtained from a complete bipartite graph by adding one vertex and a corresponding number of edges.

As an example, consider the class $K(3, 4, 10)$. There are two graphs in this class which satisfy the description in Conjecture [3.2]. The graph $G_1$ obtained from the complete bipartite graph $K_{2,4}$ by adding an extra vertex

...
of degree 2, and the graph $G_2$, obtained from $K_{3,3}$ by adding an extra vertex of degree 1. The graph $G_1$ is associated with the Ferrers diagram

```
  O  O  O  O  O
  |  |  |  |  |
  O  O  O  O  O
  |  |  |  |  |
  O  O  O  O  O
  |  |  |  |  |
  O  O  O  O  O
```

while $G_2$ is associated with the Ferrers diagram

```
  O  O  O  O  O
  |  |  |  |  |
  O  O  O  O  O
  |  |  |  |  |
  O  O  O  O  O
  |  |  |  |  |
  O  O  O  O  O
  |  |  |  |  |
  O  O  O  O  O
  |  |  |  |  |
  O  O  O  O  O
```

It can be checked that $\lambda_{\text{max}}(G_2) = 3.0592 > \lambda_{\text{max}}(G_1) = 3.0204$. Thus according to Conjecture 3.2, $G_2$ maximizes $\lambda_{\text{max}}(G)$ over $G \in \mathcal{K}(3, 4, 10)$.

Conjecture 3.2 is still open, although some special cases have been settled, see [4, 14, 21, 23]. We now mention a result from [4] toward the solution of Problem 3.1 which is of interest by itself, and is related to Ferrers graphs.

Let $D = \{d_1, d_2, \ldots, d_m\}$ be a set of positive integers where $d_1 \geq d_2 \geq \cdots \geq d_m$ and let $\mathcal{B}_D$ be the class of bipartite graphs $G = (X \cup Y, E)$ with no isolated vertices, with $|X| = m$, and with degrees of vertices in $X$ being $d_1, \ldots, d_m$. Then it is shown in [4] that $\max_{G \in \mathcal{B}_D} \lambda_{\text{max}}(G)$ is achieved, up to isomorphism, by the Ferrers graph, with the Ferrers diagram having $d_1, d_2, \ldots, d_m$ boxes in rows $1, 2, \ldots, m$, respectively.

It follows that an extremal graph solving Problem 3.1 is a Ferrers graph.
4 The number of spanning trees in a Ferrers graph

Definition 4.1 Let $G = (V, E)$ be a bipartite graph with bipartition $V = X \cup Y$. The Ferrers invariant of $G$ is the quantity

$$F(G) = \frac{1}{|X||Y|} \prod_{v \in V} \deg(v).$$

Recall that we denote the number of spanning trees in a graph $G$ as $\tau(G)$. Ehrenborg and van Willigenburg [13] proved the following interesting formula.

Theorem 4.2 If $G$ is a Ferrers graph, then $\tau(G) = F(G)$.

Let $G$ be the Ferrers graph with bipartition $(U, V)$, where $|U| = m$, $|V| = n$. We assume $U = \{u_1, \ldots, u_m\}$, $V = \{v_1, \ldots, v_n\}$. Let $d_1 \geq \cdots \geq d_m$ and $d'_1 \geq \cdots \geq d'_n$ be the degrees of $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ respectively. We may assume $G$ to be connected, since otherwise, $\tau(G) = 0$. If $G$ is connected, then $d_1 = |V|$ and $d'_1 = |U|$. Thus according to Theorem 4.2

$$\tau(G) = d_2 \cdots d_m d'_2 \cdots d'_n.$$

As an example, the Ferrers graph in Example 2.1 has degree sequences $(3, 3, 2, 1)$ and $(4, 3, 2)$. Thus, according to Theorem 4.2, it has $3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 = 36$ spanning trees.

The complete graph $K_{m,n}$ has $m^{n-1}n^{m-1}$ spanning trees, and this can also be seen as a consequence of Theorem 4.2.

Theorem 4.2 can be proved in many ways. The proof given by Ehrenborg and van Willigenburg [13] is based on electrical networks. A purely bijective proof is given by Burns [8]. We give yet another proof based on resistance distance, which is different than the one in [13], see Section 8.

It is tempting to attempt a proof of Theorem 4.2 using the Matrix-Tree Theorem. As an example, the Laplacian matrix of the Ferrers graph in Example 2.1 is given by

$$L = \begin{bmatrix}
3 & 0 & 0 & 0 & -1 & -1 & -1 \\
0 & 3 & 0 & 0 & -1 & -1 & -1 \\
0 & 0 & 2 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 \\
-1 & -1 & -1 & -1 & 4 & 0 & 0 \\
-1 & -1 & -1 & 0 & 0 & 3 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 2
\end{bmatrix}.$$ 

Let $L(1|1)$ be the submatrix obtained from $L$ be deleting the first row and column. According to the Matrix-Tree Theorem, the number of spanning
trees in the graph is equal to the determinant of $L(1|1)$. Thus Theorem 4.2 will be proved if we can evaluate the determinant of $L(1|1)$. But this does not seem easy in general.

A weighted analogue of Theorem 4.2 has also been given in [13] which we describe now. Consider the Ferrers graph $G$ on the vertex partition $U = \{u_0, \ldots, u_n\}$ and $V = \{v_0, \ldots, v_m\}$. For a spanning tree $T$ of $G$, define the weight $\sigma(T)$ to be
\[
\sigma(T) = \prod_{p=0}^{n} x^{\deg_T(u_p)} \prod_{q=0}^{m} y^{\deg_T(v_q)},
\]
where $x_0, \ldots, x_n, y_0, \ldots, y_m$ are indeterminates.

For a Ferrers graph $G$ define $\Sigma(G)$ to be the sum $\Sigma(G) = \sum_T \sigma(T)$, where $T$ ranges over all spanning trees $T$ of $G$.

**Theorem 4.3** [13] Let $G$ be the Ferrers graph corresponding to the partition $\lambda$ and the dual partition $\lambda'$. Then
\[
\Sigma(G) = x_0 \cdots x_n \cdot y_0 \cdots y_m \prod_{p=1}^{n} (y_0 + \cdots + y_{\lambda_p-1}) \prod_{q=1}^{m} (x_0 + \cdots + x_{\lambda'_q-1}).
\]

Theorem 4.2 follows from Theorem 4.3 by setting $x_0 = \cdots = x_n = y_0 = \cdots = y_m = 1$.

5 Maximizing the number of spanning trees in a bipartite graph

For general bipartite graphs the following conjecture was proposed by Ehrenborg [18, 22].

**Conjecture 5.1** (Ferrers bound conjecture). Let $G = (V, E)$ be a bipartite graph with bipartition $V = X \cup Y$. Then
\[
\tau(G) \leq \frac{1}{|X||Y|} \prod_{v \in V} \deg(v),
\]
that is, $\tau(G) \leq F(G)$.

Conjecture 5.1 is open in general. In this section we describe some partial results towards its solution, mainly from [15] and [18]. The following result has been proved in [15].
Theorem 5.2  Let $G$ be a connected bipartite graph for which Conjecture 5.1 holds. Let $u$ be a new vertex not in $V(G)$, and let $v$ be a vertex in $V(G)$. Let $G'$ be the graph obtained by adding the edge $\{u, v\}$ to $G$. Then Conjecture 5.1 holds for $G'$ as well.

Note that Conjecture 5.1 clearly holds for the graph consisting of a single edge. Any tree can be constructed from such a graph by repeatedly adding a pendant vertex. Thus as an immediate consequence of Theorem 5.2 we get the following.

Corollary 5.3  Conjecture 5.1 holds when the graph is a tree.

Using explicit calculations with homogeneous polynomials, the following result is also established in [15].

Theorem 5.4  Let $G$ be a bipartite graph with bipartition $X \cup Y$. Then Conjecture 5.1 holds when $|X| \leq 5$.

The following result is established in [18].

Proposition 5.5  Let $G$ and $G'$ be bipartite graphs for which Conjecture 5.1 holds. Let $X$ and $Y$ be the parts of $G$, and let $X'$ and $Y'$ be the parts of $G'$. Choose vertices $x \in X$ and $x' \in X'$. Define the graph $H$ with $V(H) = V(G) \cup V(G')$ and $E(H) = E(G) \cup E(G') \cup \{xx'\}$. Then the conjecture holds for $H$ also.

It may be remarked that Corollary 5.3 can be proved using Proposition 5.5 and induction as well. The following bound has been obtained in [6].

Theorem 5.6  Let $G$ be a bipartite graph on $n \geq 2$ vertices. Then

$$
\tau(G) \leq \frac{\prod_u d_u}{|E(G)|},
$$

with equality if and only if $G$ is complete bipartite.

Since there can be at most $|X||Y|$ edges in a bipartite graph with parts $X$ and $Y$, if Conjecture 5.1 were true, then Theorem 5.6 would follow. Thus the assertion of Conjecture 5.1 improves upon Theorem 5.6 by a factor of $|E(G)|/(|X||Y|)$. This motivates the following definition introduced in [18].

Definition 5.7  Let $G$ be a bipartite graph with parts $X$ and $Y$. The bipartite density of $G$, denoted $\rho(G)$, is the ratio $E(G)/(|X||Y|)$. Equivalently, $G$ contains $\rho(G)$ times as many edges as the complete bipartite graph $K_{|X|,|Y|}$. 

9
Let $G$ be a graph with $n$ vertices. Let $A$ be the adjacency matrix of $G$ and let $D$ be the diagonal matrix of vertex degrees of $G$. Note that $L = D - A$ is the Laplacian of $G$. The matrix $K = D^{-\frac{1}{2}}LD^{-\frac{1}{2}}$ is termed as the normalized Laplacian of $G$. If $G$ is connected, then $K$ is positive semidefinite with rank $n - 1$. Let $\mu_1 \geq \mu_2 \cdots \geq \mu_{n-1} > \mu_n = 0$ denote the eigenvalues of $K$. It is known, see [11], that $\mu_{n-1} \leq 2$, with equality if and only if $G$ is bipartite. Conjecture 5.1 can be shown to be equivalent to the following, see [18].

**Conjecture 5.1** Let $G$ be a bipartite graph on $n \geq 3$ vertices with parts $X$ and $Y$. Then
\[ \prod_{i=1}^{n-2} \mu_i \leq \rho(G). \]

Yet another result from [18] is the following.

**Lemma 5.9** Let $G$ be a bipartite graph on $n \geq 3$ vertices with parts $X$ and $Y$. Suppose, for some $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$ we have
\[ \prod_{i=1}^{k} \mu_i (2 - \mu_i) \leq \rho(G). \]

Then Conjecture 5.1 holds for $G$.

We conclude this section by stating the following result [18]. It asserts that Conjecture 5.1 holds for a sufficiently edge-dense graph with a cut-vertex of degree 2.

**Theorem 5.10** Let $G$ be a bipartite graph. Suppose that $\rho(G) \geq 0.544$ and that $G$ contains a cut vertex $x$ of degree 2. Then Conjecture 5.1 holds for $G$.

### 6 A reformulation in terms of majorization

This section is based on [22]. Call a bipartite graph $G$ Ferrers-good if $\tau(G) \leq F(G)$. Thus Conjecture 5.1 may be expressed more briefly as the claim that all bipartite graphs are Ferrers-good.

In 2009, Jack Schmidt (as reported in [22]) computationally verified by an exhaustive search that all bipartite graphs on at most 13 vertices are Ferrers-good. For a bipartite graph, we refer to the vertices in the two parts as red vertices and blue vertices. In 2013, Praveen Venkataramana proved an inequality weaker than Conjecture 5.1 valid for all bipartite graphs:
Proposition 6.1 (Venkataramana). Let $G$ be a bipartite graph with red vertices having degrees $d_1, \ldots, d_p$ and blue vertices having degrees $e_1, \ldots, e_q$. Then

$$\tau(G) \leq \prod_{i=1}^{p} (d_i + \frac{1}{2}) \prod_{j=1}^{q} (e_j + \frac{1}{2}) \sqrt{e_1}.$$ 

Conjecture 5.1 can be expressed in terms of majorization, for which the standard reference is [19]. For a vector $a = (a_1, \ldots, a_n)$ the vector $(a\{1\}, \ldots, a\{n\})$ denotes the rearrangement of the entries of $a$ in nonincreasing order. Recall that a vector $a = (a_1, \ldots, a_n)$ is majorized by another vector $b = (b_1, \ldots, b_n)$, written $a \prec b$, provided that the inequality

$$\sum_{i=1}^{k} a[i] \leq \sum_{i=1}^{k} b[i]$$

holds for $1 \leq k \leq n$ and holds with equality for $k = n$.

Given a finite sequence $a$, let $\ell(a)$ denote its number of parts and $|a|$ denote its sum. For example, if $a = (4, 3, 1)$, then $\ell(a) = 3$ and $|a| = 8$.

Definition 6.2 (Conjugate sequence). Let $a$ be a partition of an integer. The conjugate partition of $a$ is the partition $a^*$

$$a^*_i = \#\{j : 1 \leq j \leq \ell(a) \text{ and } a_j \geq i\}.$$ 

For example, $(5, 5, 4, 2, 2, 1)^* = (6, 5, 3, 3, 2)$.

Definition 6.3 (Concatenation of sequences). Let $a = (a_1, \ldots, a_p)$ and $b = (b_1, \ldots, b_q)$ be sequences. Then their concatenation is the sequence

$$a \oplus b = (a_1, \ldots, a_p, b_1, \ldots, b_q).$$

With this notation, we can now state the following conjecture.

Conjecture 6.4 Let $d$ be a partition with $\ell(d) = n$, and let $\lambda$ be a non-increasing sequence of positive real numbers with $\ell(\lambda) = n - 1$. Suppose $d = a \oplus b$ for some $a, b$ with $\ell(a) = p$ and $\ell(b) = q$. If $a \prec b^*$ and $d \prec \lambda \prec d^*$, then

$$\frac{1}{n} \prod_{i=1}^{n-1} \lambda_i \leq \frac{1}{pq} \prod_{i=1}^{n} d_i.$$ 

Conjecture 6.4 implies Conjecture 5.1 in view of the following two theorems.

Theorem 6.5 (Gale-Ryser). Let $a$ and $b$ be partitions of an integer. There is a bipartite graph whose blue degree sequence is $a$ and whose red degree sequence is $b$ if and only if $a \prec b^*$. 

11
Theorem 6.6 (Grone-Merris conjecture, proved in [1]) The Laplacian spectrum of a graph is majorized by the conjugate of its degree sequence.

Now let us show that Conjecture 6.4 implies Conjecture 5.1. Assume Conjecture 6.4 is true. Let $G$ be a bipartite graph on $n$ vertices, with $p$ blue vertices and $q$ red vertices. Let $d$ be its degree sequence, with blue degree sequence $a$ and red degree sequence $b$, and let $\lambda$ be its Laplacian spectrum. By Theorem 6.5, $a \prec b^*$. Since the Laplacian is a Hermitian matrix, $d \prec \lambda$, and by Theorem 6.6, $\lambda \prec d^*$. Hence the assumptions of Conjecture 2 apply. We conclude that

$$\frac{1}{n} \prod_{i=1}^{n-1} \lambda_i \leq \frac{1}{pq} \prod_{i=1}^{n} d_i.$$  

(4)

By the Matrix-Tree Theorem, the left-hand side of (4) is $\tau(G)$. Hence Conjecture 5.1 holds as well.

7 Resistance distance in $G$ and $G \setminus \{f\}$

We recall some definitions that will be useful. Given a matrix $A$ of order $m \times n$, a matrix $G$ of order $n \times m$ is called a generalized inverse (or a g-inverse) of $A$ if it satisfies $AGA = A$. Furthermore $G$ is called Moore-Penrose inverse of $A$ if it satisfies $AGA = A, GAG = G, (AG)' = AG$ and $(GA)' = GA$. It is well-known that the Moore-Penrose inverse exists and is unique. We denote the Moore-Penrose inverse of $A$ by $A^+$. We refer to [9] for background material on generalized inverses.

Let $G$ be a connected graph with vertex set $V = \{1, \ldots, n\}$ and let $i, j \in V$. Let $H$ be a g-inverse of the Laplacian matrix $L$ of $G$. The resistance distance $r(i,j)$ between $i$ and $j$ is defined as

$$r_G(i,j) = h_{ii} + h_{jj} - h_{ij} - h_{ji}.$$  

(5)

It can be shown that the resistance distance does not depend on the choice of the g-inverse. In particular, choosing the Moore-Penrose inverse, we see that

$$r_G(i,j) = \ell^+_{ii} + \ell^+_{jj} - 2\ell^+_{ij}.$$  

Let $G$ be a connected graph with $V(G) = \{1, \ldots, n\}$. We assume that each edge of $G$ is given an orientation. If $e = \{i,j\}$ is an edge of $G$ oriented from $i$ to $j$, then the incidence vector $x_e$ of $e$ is and $n \times 1$ vector with 1(−1) at $i$-th ($j$-th) place and zeros elsewhere. The Laplacian $L$ of $G$ has rank $n - 1$ and any vector orthogonal to 1 is in the column space of $L$. In particular, $x_e$ is in the column space of $L$.

For a matrix $A$, we denote by $A(i|j)$ the matrix obtained by deleting row $i$ and column $j$ from $A$. We denote $A(i|i)$ simply as $A(i)$. Similar
notation applies to vectors. Thus for a vector $x$, we denote by $x(i)$ the vector obtained by deleting the $i$-th coordinate of $x$. Let $L$ be the Laplacian matrix of a connected graph $G$ with vertex set $\{1, \ldots, n\}$. Fix $i, j \in \{1, \ldots, n\}, i \neq j$, and let $H$ be the matrix constructed as follows. Set $H(i) = L(i)^{-1}$ and let the $i$-th row and column of $H$ be zero. Then $H$ is a g-inverse of $L$ ([2], p.133). It follows from (5) that $r(i, j) = h_{jj}$. For basic properties of resistance distance we refer to [2, 3].

In the next result we give several equivalent conditions under which deletion of an edge does not affect the resistance distance between the end-vertices of another edge. This result, which appears to be of interest by itself, will be used in Section 8 to give another proof of Theorem 4.2. We denote an arbitrary g-inverse of the matrix $L$ by $L^{-}$.

**Theorem 7.1** Let $G$ be a graph with $V(G) = \{1, \ldots, n\}, n \geq 4$. Let $e = \{i, j\}, f = \{k, \ell\}$ be edges of $G$ with no common vertex such that $G \setminus \{e\}$ and $G \setminus \{f\}$ are connected subgraphs. Let $L, L_e$ and $L_f$ be the Laplacians of $G, G \setminus \{e\}$ and $G \setminus \{f\}$, respectively. Let $x_e, x_f$ be the incidence vector of $e, f$ respectively. Then the following statements are equivalent:

(i) $r_G(i, j) = r_{G \setminus \{f\}}(i, j)$
(ii) $r_G(k, \ell) = r_{G \setminus \{e\}}(k, \ell)$
(iii) $\tau(G \setminus \{e\})\tau(G \setminus \{f\}) = \tau(G \setminus \{e, f\})$
(iv) The $i$-th and the $j$-th coordinates of $L^+ x_f$ are equal
(v) The $i$-th and the $j$-th coordinates of $L^- x_f$ are equal for any $L^-$
(vi) The $i$-th and the $j$-th coordinates of $L^+_e x_f$ are equal
(vii) The $i$-th and the $j$-th coordinates of $L^-_e x_f$ are equal for any $L^-_e$
(viii) The $k$-th and the $\ell$-th coordinates of $L^+ x_e$ are equal
(ix) The $k$-th and the $\ell$-th coordinates of $L^- x_e$ are equal for any $L^-$
(x) The $k$-th and the $\ell$-th coordinates of $L^+_e x_e$ are equal
(xi) The $k$-th and the $\ell$-th coordinates of $L^-_e x_e$ are equal for any $L^-_e$.

**Proof** Let $u = L^+_f x_f, w = L^+ x_f$. Since $x_f$ is in the column space of $L_f$, we have $x_f = L_f z$ for some $z$. It follows that $L_f u = L_f L^+_f x_f = L_f L^+_f L_f z = L_f z = x_f$. Similarly $L w = x_f$. Since $L = L_f + x_f x'_f$, then $L w = L_f w + x_f x'_f w$ and hence

$$L_f(u - w) = x_f x'_f w. \quad (6)$$
Also, 
\[(x'f)wL_fu = x_f(x'f)w. \tag{7}\]

Subtracting \((7)\) from \((8)\) gives 
\[L_f(u - w - (x'f)w)u = 0,\]
which implies 
\[u - w - (x'f)w = \alpha 1\]
for some \(\alpha\). It follows that 
\[(1 - x'f)w = w + \alpha 1.\]
If 
\[1 - x'f w = 0,\]
then all coordinates of \(w\) are equal, which would imply 
\[Lw = 0,\]
contradicting 
\[x_f = Lw.\]
Thus any two coordinates of \(u\) are equal if and only if the corresponding coordinates of \(w\) are equal. This implies the equivalence of \((iv)\) and \((vi)\). A similar argument shows that \((iv) - (vii)\) are equivalent and that \((viii) - (xii)\) are equivalent.

Note that 
\[r_G(i, j) = \frac{\det L_i(x_j)}{\det L_i(x)} = \frac{\tau(G \setminus \{e\})}{\tau(G \setminus \{f\})} r_G(f \setminus \{i,j\}, i, j) = \frac{\det L_i(x_j)}{\det L_i(x)} = \frac{\tau(G \setminus \{e,f\})}{\tau(G \setminus \{e\})}.\]
Thus \((i), (ii)\) and \((iii)\) are equivalent.

We turn to the proof of \((iv) \Rightarrow (i)\). Let 
\[w = L^+ x_f,\]
and suppose 
\[w_i = w_j.\]
Since the vector \(1\) is in the null space of \(L^+\), we may assume, without loss of generality, that 
\[w_i = w_j = 0.\]
As seen before, 
\[Lw = x_f.\]
Since 
\[L(i) = L_f(i) + x_f(i)x_f(i')'\],
by the Sherman-Morrison formula,
\[L(i)^{-1} = (L_f(i) + x_f(i)x_f(i')')^{-1} = L_f(i)^{-1} - \frac{L_f(i)^{-1}x_f(i)x_f(i)'L_f(i)^{-1}}{1 - x_f(i)'L_f(i)^{-1}x_f(i)}. \tag{8}\]

Since 
\[x_f = Lw, w_i = 0 \text{ and } (x_f(i))_j = 0,\]
we have
\[\begin{align*}
(x_f(i))_j &= (L(i)w(i))_j \\
&= ((L_f(i) + x_f(i)x_f(i')')w(i)_j \\
&= (L_f(i)w(i))_j + x_f(i)'w(i)(L_f(i)x_f(i))_j.
\end{align*}\]

Hence 
\[(L_f(i)^{-1}x_f(i))_j = 0.\]
It follows from \((8)\) that the \((j, j)\)-th element of \(L(i)^{-1}\) and \(L_f(i)^{-1}\) are identical. In view of the observation preceding the Theorem, the \((j, j)\)-element of \(L(i)^{-1}\) (respectively, \(L_f(i)^{-1}\)) is the resistance distance between \(i\) and \(j\) in \(G\) (respectively, \(G \setminus \{f\}\)). Therefore the resistance distance between \(i\) and \(j\) is the same in \(G\) and \(G \setminus \{f\}\) if the \(i\)-th and the \(j\)-th coordinates of \(L^+ x\) are equal.

Before proceeding we remark that if \((v)\) holds for a particular \(g\)-inverse, then it can be shown that it holds for any \(g\)-inverse. Similar remark applies to \((vi)\), \((ix)\) and \((x)\).

Now suppose \((i)\) holds. Then 
\[\begin{align*}
(L(i))_{j,j}^{-1} &= (L_f(i))_{j,j}^{-1},
\end{align*}\]
and using \((8)\) we conclude that 
\[\begin{align*}
(L_f(i)^{-1}x_f(i)x_f(i')L_f(i))_{jj} &= 0,
\end{align*}\]which implies
\[\begin{align*}
(L_f(i)^{-1}x_f(i))_{j,j} &= 0. \tag{9}\]
If we augment $L_f(i)^{-1}$ by introducing the $i$-th row and $i$-th column, both equal to zero vectors, then we obtain a g-inverse $L_f^-$ of $L_f$. Since the $i$-th coordinate of $x_f$ is zero, we conclude from (9) that $(L_f^- x_f)_j = 0$. Since the $i$-th row of $L_f^-$ is zero, $(L_f^- x_f)_i = 0$. It follows that the $i$-th and the $j$-th coordinates of $L_f^- x_f = 0$ and thus (vii) holds (for a particular g-inverse and hence for any g-inverse). Similarly it can be shown that (ii) ⇒ (xi). This completes the proof.

8 The number of spanning trees in Ferrers graphs

We now prove a preliminary result.

Lemma 8.1 Consider the Ferrers graph $G$ with bipartition $(U, V)$, where $U = \{u_1, \ldots, u_m\}, V = \{v_1, \ldots, v_n\}$. Let $\lambda_i$ be the degree of $u_i$, $i = 1, \ldots, m$ and let $\lambda'_j$ be the degree of $v_j$, $j = 1, \ldots, n$. Let $p \in \{1, \ldots, m-1\}$ be such that $\lambda_i = n, i = 1, \ldots, p$ and $\lambda_{p+1} = n < n$. Let $f$ be the edge $\{u_p, v_n\}$. Then

$$r_G(u_{p+1}, v_k) = r_G(\{f\})(u_{p+1}, v_k).$$

(10)

Proof The bipartite adjacency matrix of $G$ is given by

$$M = \begin{pmatrix} 1 & 2 & \cdots & \cdots & n \\ 1 & 1 & \cdots & \cdots & 1 \\ 2 & 1 & \cdots & \cdots & 1 \\ \vdots & 1 & \cdots & \cdots & 1 \\ p & 1 & \cdots & 0 & 0 \\ p+1 & 1 & \cdots & 0 & 0 \\ \vdots & 1 & \cdots & 0 & 0 \\ m & 1 & \cdots & \cdots & 0 \end{pmatrix},$$

and the Laplacian matrix $L$ of $G$ is given by

$$L = \text{diag}(\lambda_1, \ldots, \lambda_m, \lambda'_1, \ldots, \lambda'_n) - \begin{bmatrix} 0 & M \\ M' & 0 \end{bmatrix}.$$

Let

$$w = \frac{1}{p} \left( -\frac{1}{n}, \ldots, -\frac{1}{n}, \frac{p-1}{n}, 0, \ldots, 0, -1 \right).$$

It can be verified that $Lw$ is the $(m+n) \times 1$ vector with 1 at position $p$, $-1$ at position $m+n$ and zeros elsewhere. Thus $Lw = x_f$, the incidence vector of the edge $f = \{u_p, v_n\}$. 

15
It follows from basic properties of the Moore-Penrose inverse \[9\] that

\[ L^+ L = \left( I - \frac{1}{m+n} \textbf{1}\textbf{1}' \right) . \]

Hence

\[ L^+ x_f = L^+ Lw = \left( I - \frac{1}{m+n} \textbf{1}\textbf{1}' \right) w = w - \alpha \textbf{1}\textbf{1}', \quad (11) \]

where \( \alpha = \textbf{1}'w/(m+n) \). Let \( e \) be the edge \( \{u_{p+1}, v_k\} \). Since the coordinates \( p+1 \) and \( m+k \) of \( w \) are zero, it follows from (11) and the implication \((iv) \Rightarrow (i)\) of Theorem 7.1 that (10) holds. This completes the proof.

Let \( G \) be a connected graph with \( V(G) = \{1, \ldots, n\} \), and let \( i, j \in V(G) \). Let \( L \) be the Laplacian of \( G \). We denote by \( L(i,j) \) the submatrix of \( L \) obtained by deleting rows \( i,j \) and columns \( i,j \). Recall that \( \tau(G) \) denotes the number of spanning trees of \( G \). It is well-known that

\[ r_G(i,j) = \frac{\det L(i|j)}{\tau(G)}. \quad (12) \]

Furthermore, \( \det L(i,j) \) is the number of spanning forests of \( G \) with two components, one containing \( i \) and the other containing \( j \). Now suppose that \( i \) and \( j \) are adjacent and let \( f = \{i,j\} \) be the corresponding edge. Let \( \tau'(G) \) and \( \tau''(G) \) denote the number of spanning trees of \( G \), containing \( f \), and not containing \( f \), respectively. Then in view of the preceding remarks, \( \tau'(G) = \det L_1(i,j) \), where \( L_1 \) is the Laplacian of \( G \setminus \{e\} \).

\textbf{Theorem 8.2} \[13\] Let \( G \) be the Ferrers graph with the bipartition \((U,V)\), where \( U = \{u_1, \ldots, u_m\}, V = \{v_1, \ldots, v_n\} \) and let \( \lambda = (\lambda_1, \ldots, \lambda_m), \lambda' = (\lambda'_1, \ldots, \lambda'_n) \) be the associated partitions. Then the number of spanning trees in \( G \) is

\[ \frac{1}{mn} \prod_{i=1}^{m} \lambda_i \prod_{i=1}^{n} \lambda'_i. \]

\textbf{Proof} We assume \( \lambda_m, \lambda'_n \) to be positive, for otherwise, the graph is disconnected and the result is trivial. We prove the result by induction on the number of edges. Let \( e = \{p+1, m+k\}, f = \{p, m+n\} \) be edges of \( G \).

By the induction assumption we have

\[ \tau(G \setminus \{e\}) = \frac{1}{mn} \prod_{i=1}^{m} \lambda_i \prod_{i=1}^{n} \lambda'_i (\lambda'_{p+1} - 1)(\lambda_k - 1) \lambda_{p+1} \lambda'_k, \quad (13) \]

\[ \tau(G \setminus \{f\}) = \frac{1}{mn} \prod_{i=1}^{m} \lambda_i \prod_{i=1}^{n} \lambda'_i (\lambda_p - 1)(\lambda'_n - 1) \lambda_p \lambda'_n, \quad (14) \]

16
\[ \tau(G \setminus \{e, f\}) = \frac{1}{mn} \prod_{i=1}^{m} \lambda_i \prod_{i=1}^{n} \lambda'_i \frac{(\lambda_{p+1} - 1)(\lambda_p - 1)(\lambda'_n - 1)}{\lambda_{p+1} \lambda'_p \lambda'_n} \] (15)

It follows from (13), (14), (15) and Theorem 7.1 that

\[ \tau(G) = \frac{\tau(G \setminus \{e\}) \tau(G \setminus \{f\})}{\tau(G \setminus \{e, f\})} = \frac{1}{mn} \prod_{i=1}^{m} \lambda_i \prod_{i=1}^{n} \lambda'_i, \]

and the proof is complete.

Acknowledgment I sincerely thank Ranveer Singh for a careful reading of the manuscript. Support from the JC Bose Fellowship, Department of Science and Technology, Government of India, is gratefully acknowledged.

References

1. Hua Bai, The Grone-Merris conjecture, Transactions of the American Mathematical Society, 363(8) (2011), 4463-4474.
2. R.B. Bapat, Graphs and matrices, Second edition, Hindustan Book Agency, New Delhi and Springer, 2014.
3. R.B. Bapat, Resistance distance in graphs. Math. Student 68 (1999), no. 1-4, 87-98.
4. Amitava Bhattacharya, Shmuel Friedland and Uri N. Peled, On the first eigenvalue of bipartite graphs, The Electronic Journal of Combinatorics, 15 (2008) # R144.
5. J.A. Bondy and U.S.R. Murty, U. S. R., Graph theory, Graduate Texts in Mathematics, 244, Springer, New York, 2008.
6. S. Bozkurt and Burcu, Upper bounds for the number of spanning trees of graphs, J. Inequal. Appl. 2012:269 (2012).
7. R.A. Brualdi and A.J. Hoffman, On the spectral radius of (0, 1)-matrices, Linear Algebra Appl., 65 (1985), 133-146.
8. Jason Burns, Bijective proofs for "Enumerative Properties of Ferrers Graphs", arXiv: math/0312282v1 [math CO] 15 Dec 2003.
9. S.L. Campbell and C.D. Meyer, Jr., Generalized inverses of linear transformations, Pitman, London, 1979.
10. Stephen R. Chestnut and Donniell E. Fishkind, Counting spanning trees in threshold graphs, arXiv:1208.4125v2, (2013).

11. R.K. Fan Chung, Spectral Graph Theory, CBMS Regional Conference Series in Mathematics, American Mathematical Society, 1997.

12. D.M. Cvetković, Michael Doob and Horst Sachs, Spectra of graphs. Theory and applications, Third edition, Johann Ambrosius Barth, Heidelberg, 1995.

13. Richard Ehrenborg and Stephanie van Willigenburg, Enumerative properties of Ferrers graphs, Discrete Comput. Geom. 32 (2004), no. 4, 481–492.

14. S. Friedland, Bounds on the spectral radius of graphs with $e$ edges, Linear Algebra Appl. 101 (1988), 8186.

15. Fintan Garrett and Steven Klee, Upper bounds for the number of spanning trees in a bipartite graph. Preprint, http://fac-staff.seattleu.edu/klees/web/bipartite.pdf, 2014.

16. Peter L. Hammer, Uri N. Peled, and Xiaorong Sun, Difference graphs, Discrete Applied Mathematics, 28(1) (1990) 35-44.

17. D.J. Klein and M. Randić, Resistance distance, J. Math. Chem., 12 (1993), no. 1-4, 81–95.

18. Cheng Wai Koo, A bound on the number of spanning trees in bipartite graphs, 2016. Senior thesis, https://www.math.hmc.edu/ckoo/thesis/.

19. Albert W. Marshall, Ingram Olkin, and Barry C. Arnold. Inequalities: Theory of Majorization and Its Applications. Springer, New York, 2011.

20. P. Rowlinson, On the maximal index of graphs with a prescribed number of edges, Linear Algebra Appl., 110 (1988), 4353.

21. Miroslav Petrović and Slobodan K. Simić, A note on connected bipartite graphs of fixed order and size with maximal index, Linear Algebra Appl., 483 (2015), 21-29.

22. Michael Slone, A conjectured bound on the spanning tree number of bipartite graphs, arXiv:1608.01929v2 [math.CO] 10 Aug 2016.

23. R.P. Stanley, A bound on the spectral radius of graphs with $e$ edges, Linear Algebra Appl. 87 (1987), 267269.