Bose-Einstein Condensation and Ferromagnetism of Low Density Bose gas of Particles with Arbitrary Spin.

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ABSTRACT

Properties of the ground state and the spectrum of elementary excitations are investigated for the low density ultracold spinor 3D Bose gas of particles with arbitrary nonzero spin. Gross-Pitaevskii equations are derived. Within the framework of the considering interaction Hamiltonian it is shown that the ground state spin structure and spin part of the chemical potential is determined by the renormalized interaction, being defined by the contribution of the virtual large momenta. The ferromagnetic structure of the ground state, and the equation of the phase, density, and spin dynamics are obtained from Gross-Pitaevskii equations.

Spinor Bose gas (SBG) at low temperatures is a quantum system in which in addition to the superfluidity the magnetic order can take place. The study of such systems gives rise to the great interest from both theoretical and experimental points of view \cite{6}, \cite{7}, \cite{8}, \cite{4}, \cite{5}, \cite{9}, \cite{10}, \cite{11}, \cite{12}, \cite{13}. In such quantum system a magnetic order and a superfluidity may be revealed simultaneously. In this paper low density nonideal 3D SBG of particles with arbitrary spin $S \neq 0$ is investigated at zero temperature. The basis of the spin coherent states (SCS) for spin $S$ \cite{1}, \cite{2}, \cite{3} is used to expand the creation and annihilation operators of particles $\hat{\psi}^\dagger, \hat{\psi}$ instead of the states being the eigen states of the spin operator $z$-projection $\hat{s}_z$. Hitherto, SCS were employed, as a rule, for investigation of many-body systems of interacting localized spins \cite{15}, \cite{16}, \cite{14}, \cite{17}, \cite{20}, \cite{19}, \cite{18}, \cite{21}. In present paper SCS are used for the investigation of Bose system consists of moving particles with spin $S$. The bare interaction between particles is supposed to consist of short range and long range parts. The short range bare interaction depending on the difference between coordinates of two interacting particles is supposed to have the delta-function dependence on
the coordinates, i.e., the short range interaction is considered as independent of momentum transfer. The short range interaction is divided into two parts. The first one is independent of spin interaction with the coupling constant $g^{(0)}$. The second one is the spin dependent part with the coupling constant $\gamma_{sh}^{(0)}$. The delta-function dependence on the difference between coordinates of two interacting particles for short range interaction is conditioned by the smallness of the interaction radius $a_0$. The radius $a_0$ is supposed to be the smallest length scale parameter. Properties of low density Bose gas is considered in the present paper so the length $a_0$ obeys the inequality $a_0 << \rho^{-1/3}$, where $\rho$ is the density of the system. The radius $a_0$ is supposed to be of the same order as for the spin independent and the spin dependent interactions. The long range part of the interaction is supposed to be of the dipole-dipole type spin-spin interaction. The dipole-dipole form of the long range interaction is valid for sufficiently large distances $|\vec{R}| > a_s >> a_0$, where $\vec{r}, \vec{r}'$ are coordinates of interacting particles. The characteristic length $a_s$ is supposed to obey the inequality $a_0 << a_s << \rho^{-1/3}$. Gross-Pitaevskii equations (GPE) for low energy degrees of freedom are derived. A spin structure of the Bose-Einstein condensate (BEC) is found. It is shown that for the model of the interaction Hamiltonian under consideration the spin structure of the system has a ferromagnetic order, at that, the renormalization of the bare short range and long range interactions between long wave Bose fields plays the key role for the formation of the ground state spin structure. This renormalization is connected with the contribution of large virtual momenta during the scattering process. The equation describing the spin density dynamics is obtained. It is shown that this equation is analogous to Landau-Lifshits equation for Heisenberg ferromagnet system of the localized spins. Green functions and spectra of the elementary excitations of two types are found. These excitations are the density-phase fluctuations and the spin fluctuations.

The model of the bare Hamiltonian under consideration is

$$\hat{H} = \hat{H}^{(0)} + \hat{H}_{int}^{(0)(sh)} + \hat{H}_{int}^{(0)(sp|sh)} + \hat{H}_{int}^{(0)(l)}$$  \hfill (1)

The term $\hat{H}^{(0)}$ is the Hamiltonian of noninteracting particles

$$\hat{H}^{(0)} = \int d^3r \sum_{m=-S}^{S} \hat{\psi}^+_{m}(t, \vec{r}) \left[ -\frac{1}{2m_0} \vec{\nabla}^2 - \mu \right] \hat{\psi}_m(t, \vec{r}) \quad (2)$$

where $\mu$ is a chemical potential, $m_0$ is a mass of the particle, $\hat{\psi}_m, \hat{\psi}_m^+$ are annihilation and creation operators, where $m$ is the integer eigenvalues of the $z$-projection of the spin operator.
\( \hat{s}^{(z)} \), such that \( -S \leq m \leq S \), and the states \( |m> \) are the eigenstates of this operator. The term \( \hat{H}_{int}^{(0)(sh)} \) is the Hamiltonian of the short range independent of spin interaction. This interaction has the form

\[
\hat{H}_{int}^{(0)(sh)} = g^{(0)} \int d^3r \sum_{m_1, m_2} \hat{\psi}_{m_2}^+(t, \overrightarrow{r}) \hat{\psi}_{m_1}^+ (t, \overrightarrow{r}) \hat{\psi}_{m_1} (t, \overrightarrow{r}) \hat{\psi}_{m_2} (t, \overrightarrow{r})
\]

(3)

The value \( g^{(0)} \) is the bare coupling constant of the independent of spin short range interaction. The radius of the short range interaction is of the order of the length \( a_0 \), such that \( a_0 \ll \rho^{-1/3} \) due to the smallness of the density \( \rho \) of the system being considered. Moreover, the coupling constants \( g^{(0)} \) is supposed to be positive and sufficiently large, so that it obeys the inequality \( g^{(0)} a_0 > 1 \). Note, that the inequality \( g^{(0)} a_0 > 1 \) means that the bare coupling constant \( g^{(0)} \) should be much larger than the renormalized spin independent scattering amplitude being of the order of \( a_0 / m_0 \).

The term \( \hat{H}_{int}^{(0)(spish)} \) is the Hamiltonian of the short range spin dependent interaction. This part of the Hamiltonian is taken in the form

\[
\hat{H}_{int}^{(0)(spish)} = \gamma^{(0)}_{sh} \int d^3r \sum_{m_1, m_2} \sum_{m_1', m_2'} \hat{s}^{(i)}_{m_2, m_2'} \hat{s}^{(i)}_{m_1, m_1'} \hat{\psi}_{m_1'}^+ (t, \overrightarrow{r}) \hat{\psi}_{m_2'}^+ (t, \overrightarrow{r}) \hat{\psi}_{m_2} (t, \overrightarrow{r}) \hat{\psi}_{m_1} (t, \overrightarrow{r})
\]

(4)

The radius of the short range spin dependent interaction is supposed to be of the order of \( a_0 \). The term \( \hat{H}_{int}^{(0)(l)} \) is the Hamiltonian of the long range spin-spin interaction of the dipole-dipole type

\[
\hat{H}_{int}^{(0)(l)} = \int d^3r d^3r' \sum_{m_1, m_2, m_1', m_2'} \hat{s}^{(i)}_{m_1, m_1'} \hat{s}^{(j)}_{m_2, m_2'} V^{(0)(l)}_{ij} (\overrightarrow{R}) \hat{\psi}_{m_1}^+ (t, \overrightarrow{r}) \hat{\psi}_{m_2}^+ (t, \overrightarrow{r}) \hat{\psi}_{m_2'} (t, \overrightarrow{r}') \hat{\psi}_{m_1'} (t, \overrightarrow{r}')
\]

(5)

where \( V^{(0)(l)}_{ij} (\overrightarrow{R}) = \gamma^{(0)}_{ij} |\overrightarrow{R}|^2 \delta_{ij} - 3 R^{(i)} R^{(j)} / |\overrightarrow{R}|^5 \), \( \gamma^{(0)}_{ij} > 0 \), \( \overrightarrow{R} = \overrightarrow{r} - \overrightarrow{r}' \), and \( \overrightarrow{r}, \overrightarrow{r}' \) are the space coordinates of the interacting particles, \( \nabla^{(i)}_{\overrightarrow{r}} \) is the gradient operator over the \( i \) component of the coordinate \( \overrightarrow{r} \). The space dependence of \( V^{(0)(l)}_{ij} (\overrightarrow{R}) \) is valid for \( |\overrightarrow{R}| \gg a_s \). The length \( a_s \) is a characteristic length of the long range spin-spin interaction. The length \( a_s \) is supposed to obey the inequality \( \rho^{-1/3} \gg a_s \gg a_0 \).

In what follows, the Hamiltonian \( \hat{H} \) is written in terms of the system of spin coherent
states (SCS). The basis of the spin coherent states (SCS) $|\vec{n}\rangle$ for the spin of the particle $S$ is used to expand the creation and annihilation operators of particles $\hat{\psi}^+, \hat{\psi}$ instead of the states $|m\rangle$. SCS $|\vec{n}\rangle$ is characterized by a unit vector $\vec{n} = (\sin \theta \cos \varphi; \sin \theta \sin \varphi; \cos \theta)$, where $\theta$ and $\varphi$ are spherical angles \[1, \ 2, \ 3\]. The SCS $|\vec{n}\rangle$ can be obtained by the action of the operator $\hat{D}(\vec{n})$ on the eigenstate $|\vec{S}\rangle$ of the operator $\hat{s}^z$ with the minimal eigenvalue $m = -S$.

$$|\vec{n}\rangle = \hat{D}(\vec{n}) |\vec{S}\rangle,$$
(6)

where

$$\hat{D}(\vec{n}) = \exp \left( i\theta (\vec{n} \cdot \vec{s}) \right).$$
(7)

Here $\vec{n} = (\sin \varphi, -\cos \varphi, 0)$, where $\theta$ and $\varphi$ are spherical angles of the vector $\vec{n}$. Thus, $\vec{\kappa} \perp \vec{n}$ and $\vec{\kappa} \perp \vec{n}_0^{(c)}$, where $\vec{n}_0^{(c)} = (0; 0; 1)$. The spherical angle $\theta = 0$ for the vector $\vec{n}_0^{(c)}$, thus $\hat{D}(\vec{n}_0^{(c)}) = \hat{1}$, where $\hat{1}$ is the unit operator, and, as the result, $|\vec{n}_0^{(c)}\rangle = |\vec{S}\rangle$.

The system of SCS forms the overcomplete system and its completeness relation is

$$\sum_{\vec{n}} |\vec{n}\rangle <\vec{n}| = \hat{1},$$
(8)

where $\sum_{\vec{n}}$ means the integration over the spherical angles $\theta$ and $\varphi$ and $\sum_{\vec{n}} = \frac{(2S+1)}{4\pi} \int \sin \theta d\theta d\varphi$.

The spin operator in the SCS representation can be represented in the form

$$\vec{s} = - (S + 1) \sum_{\vec{n}} |\vec{n}\rangle >\vec{n}| <\vec{n}|.$$  
(9)

The transfer from the anihilation and creation operators $\hat{\psi}_m, \hat{\psi}_m^+$ in $|m\rangle$ state representation to SCS representation $\hat{\psi}_{\vec{n}}^-, \hat{\psi}_{\vec{n}}^+$ is defined by equations

$$\hat{\psi}_{\vec{n}}^- = \sum_{m=-S}^S <\vec{n}| m > \hat{\psi}_m, \ \hat{\psi}_{\vec{n}}^+ = \sum_{m=-S}^S \hat{\psi}_m^+ <m| \vec{n}>,$$
(10)

while the inverse transformation has the form

$$\hat{\psi}_m = \sum_{\vec{n}} <m| \vec{n} > \hat{\psi}_{\vec{n}}^-, \ \hat{\psi}_m^+ = \sum_{\vec{n}} \hat{\psi}_{\vec{n}}^+ <\vec{n}| m>.$$  
(11)

Note, that in the space of the functions being defined by Eq. (10) bra-ket product $<\vec{n}_1|\vec{n}_2>$ is analogous to delta function $\delta (\vec{n}_1 - \vec{n}_2)$.

The transition to SCS representation gives the Hamiltonians $\hat{H}^{(0)}$ and bare interaction Hamiltonians $\hat{H}^{(0)(sh)}_{int}, \hat{H}^{(0)(sp|sh)}_{int}, \hat{H}^{(0)(l)}_{int}$ \[2, \ 3, \ 4, \ 5\] in the form
\[ \hat{H}^{(0)} = \int d^3r \sum_{\vec{n}_1} \hat{\psi}_{\vec{n}_1}^+(t, \vec{r}) \left[ -\frac{1}{2m_0} \nabla^2 - \mu \right] \hat{\psi}_{\vec{n}_1}(t, \vec{r}) \]

\[ \hat{H}_{\text{int}}^{(0)(sh)} = g^{(0)} \int d^3r \sum_{\vec{n}_1, \vec{n}_2} \hat{\psi}_{\vec{n}_1}^+(t, \vec{r}) \hat{\psi}_{\vec{n}_2}(t, \vec{r}) \hat{\psi}_{\vec{n}_2}^+(t, \vec{r}) \hat{\psi}_{\vec{n}_1}(t, \vec{r}) \]

\[ \hat{H}_{\text{int}}^{(0)(sp,sh)} = \gamma_{sh}^{(0)} (S + 1)^2 \sum_{\vec{n}_1, \vec{n}_2} (\vec{n}_1 \cdot \vec{n}_2) \int d^3r \hat{\psi}_{\vec{n}_1}^+(t, \vec{r}) \hat{\psi}_{\vec{n}_2}^+(t, \vec{r}) \hat{\psi}_{\vec{n}_2}(t, \vec{r}) \hat{\psi}_{\vec{n}_1}(t, \vec{r}) \]

\[ \hat{H}_{\text{int}}^{(0)(l)} = (S + 1)^2 \sum_{\vec{n}_1, \vec{n}_2} \int d^3r d^3r' \binom{n_1}{n_2} \binom{j}{i} V_{i,j}^{(0)(l)} (\vec{r}, \vec{r}') \hat{\psi}_{\vec{n}_1}^+(t, \vec{r}) \hat{\psi}_{\vec{n}_2}^+(t, \vec{r}') \hat{\psi}_{\vec{n}_2}(t, \vec{r}') \hat{\psi}_{\vec{n}_1}(t, \vec{r}) \]

Further, a low-energy behavior of the system, determined by the excitations with small momenta \(|\vec{p}| \ll \sqrt{2m_0\mu}\), is considered. The contribution of virtual large momenta \(|\vec{p}| \gg \sqrt{2m_0\mu}\) renormalize essentially the bare interaction between low energy particles. In the case of the low density Bose gas the renormalized interaction is defined by the sum of the ladder diagrams. First, the ladder diagrams with small external momenta involving only the short range interactions \(\hat{H}_{\text{int}}^{(0)(sh)}, \hat{H}_{\text{int}}^{(0)(sp,sh)}\) are summarized. Due to the independence of the momentum transfer of these interactions each link of the ladder diagram has the divergence due to the integration over the internal momentum corresponding to the internal bare Green functions \(G^{(0)}(p)\), where \(G^{(0)}(p) = \left(\omega + \mu - \frac{1}{2m_0} \vec{p}^2 + i\delta\right)^{-1}\). This divergence is cut off by a momentum \(\Lambda_0 = 1/a_0\), while the cutoff momentum \(\Lambda_0\) as well as \(a_0\) are supposed independent of the spins of interacting particles. The summation of the ladder diagrams with the interactions \(\hat{H}_{\text{int}}^{(0)(sh)}\) and \(\hat{H}_{\text{int}}^{(0)(sp,sh)}\) gives the renormalization of the short range interaction \(\hat{H}_{\text{int}}^{(sh)}\) in the form

\[ \hat{H}_{\text{int}}^{(sh)} = \int d^3r \sum_{i=1}^3 \sum_{\vec{n}_1, \vec{n}_2} \Gamma_{\vec{n}_1, \vec{n}_2}^{(sh)} \hat{\psi}_{\vec{n}_1}^+(t, \vec{r}) \hat{\psi}_{\vec{n}_2}^+(t, \vec{r}) \hat{\psi}_{\vec{n}_2}(t, \vec{r}) \hat{\psi}_{\vec{n}_1}(t, \vec{r}) \]

where

\[ \Gamma_{\vec{n}_1, \vec{n}_2}^{(sh)} = \frac{g^{(0)} + \gamma_{sh}^{(0)} (S + 1)^2 (\vec{n}_1 \cdot \vec{n}_2)}{1 - \left[ g^{(0)} + \gamma_{sh}^{(0)} (S + 1)^2 (\vec{n}_1 \cdot \vec{n}_2) \right] T_0} \]
and

\[ T_0 = i \int \frac{d^4p_1}{(2\pi)^4} G^{(0)}(p_1) G^{(0)}(-p_1) = -\frac{m_0}{2\pi^2a_0} \]

In the present paper we assume that \( g^{(0)} + \gamma^{(0)}_{sh} (S + 1)^2 (\vec{n}_1 \cdot \vec{n}_2) \) is positive and sufficiently large, so that \( (g^{(0)} + \gamma^{(0)}_{sh} (S + 1)^2) | T_0 | >> 1 \). In this case the interaction constant \( \Gamma^{(sh)}_{\vec{n}_1, \vec{n}_2} \) can be put independent of \( \vec{n}_1, \vec{n}_2 \) and equal to

\[ \Gamma^{(sh)}_{\vec{n}_1, \vec{n}_2} = g = \frac{1}{| T_0 |} \]

To find the total renormalized interaction the diagrams with the bare long range interaction \( \hat{H}^{(0)(l)}_{int} \), besides the diagram’s block of ladder diagrams with the interaction vertexes \( \hat{H}^{(sh)}_{int} \), should be taken into account for the summation of the total sequence of ladder diagrams. The smallness of the coupling constant \( \gamma^{(0)}_l \) compared with the coupling constant \( g \) is supposed, so that \( g >> \gamma^{(0)}_l S^2 \) or \( \frac{m_0 \gamma^{(0)}_l S^2}{2\pi^2 a_s} << 1 \). In this case taking into account that \( a_0/a_s << 1 \) and \( R >> a_s \) the renormalization of the long range spin-spin interaction can be considered in the first order over \( \hat{H}^{(0)(l)}_{int} \). As a result, the renormalized long range spin-spin interaction gives the attractive short range term and takes the form

\[ \hat{H}^{(l)}_{int} = (S + 1)^2 \sum_{\vec{n}_1, \vec{n}_2} \int d^3r d^3r' \left( n_1^{(i)} n_2^{(j)} \right) V_{i,j}^{(r)(l)}(\vec{R}) \cdot \psi_{\vec{n}_1}^\dagger(t, \vec{r}) \psi_{\vec{n}_2}^\dagger(t, \vec{r'}) \psi_{\vec{n}_1}(t, \vec{r}) \psi_{\vec{n}_2}(t, \vec{r'}) \]

where

\[ V_{ij}^{(r)(l)}(\vec{R}) = -\gamma^{(0)}_l \nabla^{(i)} \nabla^{(j)} \frac{1}{| \vec{R} |} - \frac{8}{3} \frac{a_0}{a_s} \gamma^{(0)}_l \delta(\vec{R}) \delta_{i,j}, \]

Note, that the zero momentum transfer Fourier transformation of the bare interaction \( V_{ij}^{(0)(l)}(\vec{R}) \) vanishes, if one put \( i = j \) and regularizes the interaction \( V_{ij}^{(0)(l)}(\vec{R}) \) by the factor \( \exp(-\varkappa R) \) supposing that \( \varkappa \rightarrow 0 \). For this reason, the second term in Eq. (12) should be kept in spite of the fact that it is small over the parameter \( a_0/a_s \).

Using (8), (9), (10) one can determine the generating functional \( Z[J, \bar{J}] \) in the SCS representation as

\[ Z[J, \bar{J}] = \int D\psi D\bar{\psi} \exp \left( iS_0 + iS^{(c)}_{int} + iS^{(s)}_{int} + iS_J \right), \]
where $S_0$ is a free action, $S^{(c)}_{int}$ and $S^{(s)}_{int}$ are the parts of the action determined by the spin independent and spin dependent renormalized interactions, respectively, the term $S_J$ is the part of the action connected with infinitesimal sources $J(t, \vec{r})$. Here

$$S_0 = \int dt d^3 r \sum_n \left\{ \bar{\psi}(t, \vec{r}) \left( i \partial_t + \mu + \frac{1}{2m} \Delta \right) \psi(t, \vec{r}) \right\}$$  \hspace{1cm} (14)

$$S^{(\rho)}_{int} = -g \int dt d^3 r \rho^2(t, \vec{r})$$  \hspace{1cm} (15)

$$S^{(s)}_{int} = - \int dt d^3 r d^3 r' \left\{ S^{(i)}(t, \vec{r}) V^{(i)(l)}_{ij}(\vec{r} - \vec{r}') S^{(j)}(t, \vec{r}') \right\}$$  \hspace{1cm} (16)

$$S_J = \int dt d^3 r \sum_{\vec{r}} J(\vec{r}) \psi(t, \vec{r}) + J(\vec{r}) \bar{\psi}(t, \vec{r})$$  \hspace{1cm} (17)

The density of particles $\rho(t, \vec{r})$ and the spin density vector $\vec{S}(t, \vec{r})$ in SCS representation have the form

$$\rho(t, \vec{r}) = \sum_n \bar{\psi}(t, \vec{r}) \psi(t, \vec{r}),$$  \hspace{1cm} (18)

$$\vec{S}(t, \vec{r}) = \sum_n \vec{\psi}(t, \vec{r}) \vec{\psi}(t, \vec{r}).$$  \hspace{1cm} (19)

Using the equality

$$\int d^3 r d^3 r' \sum_{i,j=1}^3 S^{(i)}(\vec{r}') \left( \nabla^{(i)} \nabla^{(j)} \frac{1}{| \vec{R} |} \right) S^{(j)}(\vec{r}') =$$

$$= 4\pi \int d^3 r \left\{ \left( \text{curl} \vec{S}(\vec{r}) \cdot \hat{\Delta}^{-1} \text{curl} \vec{S}(\vec{r}) \right) + \left( \vec{S}(\vec{r}) \cdot \vec{S}(\vec{r}) \right) \right\}$$

, where $\hat{\Delta}^{-1}$ is the inverse Laplace operator, we can write the spin dependent part of the action as the sum of two terms. The first one is the long range part and the second one is the short range part.

$$S^{(s)}_{int} = S^{(s1)}_{int} + S^{(s2)}_{int}$$

$$S^{(s1)}_{int} = -4\pi \gamma_{l}^{(0)} g \int d^3 r \left\{ \text{curl} \vec{S}(t, \vec{r}) \hat{\Delta}^{-1} \text{curl} \vec{S}(t, \vec{r}) \right\}$$  \hspace{1cm} (20)

$$S^{(s2)}_{int} = -4\pi \gamma_{l}^{(0)} g \int d^3 r \left\{ \left( 1 - \frac{2}{3} a_0 \right) \left( \vec{S}(t, \vec{r}) \cdot \vec{S}(t, \vec{r}) \right) \right\}$$  \hspace{1cm} (21)

Using the Habbard-Stratanovich transformation, the long range interaction (20) can be represented as the interaction via the exchange by the virtual magnetic field $\vec{H} = \text{curl} \vec{A}$,
where $\vec{A}$ is a vector potential obeying the gauge $\text{div} \, \vec{A} = 0$. Due to this transformation the generation functional $Z[J, \vec{J}]$ takes the form

$$
Z[J, \vec{J}] = \int \prod_n D\psi_n D\bar{\psi}_n D\vec{A} \exp \left\{ i \left[ S_{\text{act}} \left[ \psi_n; \bar{\psi}_n; \vec{A} \right] + S_J \right] \right\}
$$

(22)

where

$$
S_{\text{act}} = \int dt d^3r \left\{ \sum_n \left( \bar{\psi}_n (G^{(0)})^{-1} \psi_n \right) - \frac{1}{2} g \rho^2 - 4\pi \gamma \left( 1 - \frac{2}{3} \frac{m}{a_s} \right) \left( \vec{S} \cdot \vec{S} \right) + \sqrt{2\gamma} \left( \vec{A} \cdot \text{curl} \, \vec{S} \right) + \frac{1}{8\pi} \left( \vec{A} \cdot \Delta \vec{A} \right) \right\}
$$

(23)

and

$$
\text{div} \, \vec{A} = 0; \quad \vec{H} = \text{curl} \, \vec{A}
$$

$$
(G^{(0)})^{-1} = i \partial_t + \mu + \frac{1}{2m_0} \hat{\Delta}
$$

Gross - Pitaevskii (GPE) equations can be obtained from the action [23] by equating to zero the first variation of this action with respect to the fields $\psi_n, \bar{\psi}_n, \vec{A}$. These equations take the form

$$
((G_0)^{-1} - g \rho) \psi_n - (S + 1) \sqrt{2\gamma} \sum \vec{n}_1 | < \vec{n}_1 | \vec{n}_1 > \left( \vec{n}_1 \cdot \vec{H} \right) \psi_{\vec{n}_1} = 0
$$

(24)

$$
\frac{1}{4\pi} \hat{\Delta} \vec{A} + \sqrt{2\gamma} \text{curl} \, \vec{S} = 0,
$$

(25)

where

$$
\vec{H} = \vec{H} - \frac{8\pi (\gamma - \tilde{\gamma})}{\sqrt{2\gamma}} \vec{S}
$$

and $(G_0)^{-1} = i \partial_t + \mu + \frac{1}{2m_0} \hat{\Delta}; \tilde{\gamma} = 2 \frac{m}{3a_s} \gamma$. From Eq. [25] it can be easily obtained that the magnetic field $\vec{H}$ in the momentum representation takes the form

$$
\vec{H}_k = 4\pi \sqrt{2\gamma} \left[ \vec{S}_k - \vec{e}_k \left( \vec{e}_k \cdot \vec{S}_k \right) \right],
$$

(26)
where $\mathbf{c}_k = \mathbf{k} / |\mathbf{k}|$. Using (26), one obtains from (24) GPE in the momentum representation

$$i\partial_t + \mu - \frac{1}{2m} \mathbf{k}^2 - g\rho_{-\mathbf{k}} \right) \psi_{\mathbf{n}; \mathbf{k}}(\mathbf{r}) - 8\pi (S + 1) \sum_{i=1}^{3} \sum_{\mathbf{n}_1} <\mathbf{n}_1 | \mathbf{n}_1 > \mathbf{\Gamma}^{(i)}_{\mathbf{k}} \mathbf{\Sigma}^{(i)}_{-\mathbf{k}} \psi_{\mathbf{n}_1; \mathbf{k}} = 0. \tag{27}$$

where $S^{(i)}, n^{(i)}_1, c^{(i)}_k$ are $x, y, z$ components of the vectors $\mathbf{S}_{-\mathbf{k}}, \mathbf{n}_1, \mathbf{c}_k$, and

$$\mathbf{\Gamma}^{(i)}_{\mathbf{k}} = \gamma n^{(i)}_1 - \gamma (\mathbf{n}_1 \cdot \mathbf{c}_k) \mathbf{c}^{(i)}_k \tag{28}$$

To find the condensate field $\psi_{\mathbf{n}}$ one should transfer to the limit $\mathbf{k} \rightarrow 0$ in Eqs. (27), (26). Yet, the limit of $\mathbf{c}_k$ for $\mathbf{k} \rightarrow 0$ is not defined. Taking into account that $(\mathbf{k} \cdot \mathbf{H}_0) = 0$ for arbitrary $\mathbf{k}$, the limit of $\mathbf{c}_k$ for $\mathbf{k} \rightarrow 0$ can be extended to the unit vector $\mathbf{c}_0$ obeying the condition $(\mathbf{c}_0 \cdot \mathbf{H}_0) = 0$, where $\mathbf{H}_0 = \mathbf{H}_{k=0}$. The condition $(\mathbf{c} \cdot \mathbf{H}_0) = 0$ is valid for $\mathbf{k} \neq 0$. This condition is extended to $\mathbf{k} = 0$ and the values $\mathbf{c}_0$ and $\mathbf{H}_0$ are supposed to obey a similar condition $(\mathbf{c}_0 \cdot \mathbf{H}_0) = 0$. If $\mathbf{H}_0$ is nonzero $\mathbf{H}_0 \neq 0$ the spin density $\mathbf{S}_0 = \mathbf{S}_{k=0}$ is parallel to $\mathbf{H}_0$ and $(\mathbf{c}_0 \cdot \mathbf{S}_0) = 0$. Thus, in the case $\mathbf{k} = 0$ Eq. (26) gives

$$\mathbf{H}_0 = 4\pi \sqrt{2\gamma \mathbf{S}_0} \tag{29}$$

The field of BEC $\psi^{(c)}_{\mathbf{n}}$ is spatially homogeneous and time independent. It obeys Eq. (27) for $\mathbf{k} = 0$, and in SCS representation can be found in the form

$$\psi^{(c)}_{\mathbf{n}} = \sqrt{\rho}^{(c)} <\mathbf{n} | \mathbf{n}^{(c)}_0 > \tag{30}$$

here $\rho^{(c)}$ is the density of BEC. Both the spin density of BEC $\mathbf{S}^{(c)}$ and the effective magnetic field of BEC $\mathbf{H}^{(c)} = 4\pi \sqrt{2\gamma \mathbf{S}^{(c)}}$ are spatially independent. $\mathbf{S}^{(c)}$ is obtained after substitution of Eq. (30) into Eq. (19)

$$\mathbf{S}^{(c)} = -(S + 1) \sum_{\mathbf{n}} \overline{\psi}^{(c)}_{\mathbf{n}} \mathbf{n} \psi^{(c)}_{\mathbf{n}} = -S \rho^{(c)} \mathbf{n}^{(c)}_0 \tag{31}$$

Substituting Eq. (31) into (27) and taking into account that $(\mathbf{c}_0 \cdot \mathbf{H}_0) = (\mathbf{c}_0 \cdot \mathbf{S}_0) = 0$ one obtains

$$[i\partial_t + \mu - (g - 8\pi\gamma S^2) \rho^{(c)}] \psi_{\mathbf{n}} = 0, \tag{32}$$
Requiring time independence of the field of BEC $\psi^{(c)}_{\vec{n}}$, one obtains from Eq. (32) the expression for the chemical potential $\mu$

$$\mu = (g - 8\pi \gamma S^2) \rho^{(c)}$$ \hfill (33)

The substitution of Eq. (31) to Eq. (29) gives

$$\overline{H}^{(c)} = 4\pi \sqrt{2\gamma S^2}$$

The equation of the spin density dynamics can be obtained from GPE after three simple operations. The first one is the multiplication of Eq. (24) by $\overline{n} \psi \psi$ and further summation of this equality over $\overline{n}$. The second one is the multiplication of the equation complex conjugate to Eq. (24) by $\overline{n} \psi \psi$ and further summation over $\overline{n}$. The third one is the subtraction of these two obtained equalities. As the result, the equations describing the dynamics of the spin density is obtained

$$\partial_t S_i + \frac{1}{2m} \sum_{j=1}^{3} \nabla^j \Pi^{ij} + \sqrt{2\gamma} \left[ \overline{H} \times \overline{S} \right]^i = 0,$$ \hfill (34)

here $\Pi^{ij}$ is a spin flux tensor

$$\Pi^{ij} = i (S + 1) \sum_{n} \left[ \overline{\psi} n^i \left( \nabla^j \overline{\psi} \right) - \left( \nabla^j \overline{\psi} \right) n^i \overline{\psi} \right].$$ \hfill (35)

Note that in $|m> -$ representation the tensor $\Pi^{ij}$ takes the form

$$\Pi^{ij} = -i \sum_{m_1;m_2} \left[ \overline{\psi}_{m_1} \overline{\psi}_{m_2} \left( \nabla^j \psi_{m_2} \right) - \left( \nabla^j \psi_{m_1} \right) \psi_{m_2} \right].$$ \hfill (36)

Eqs. (34) describing the ferromagnetic system of moving Bose particles with nonzero spin are analogous to the Landau-Lifshits equations which describe the spin dynamics of localised at lattice sites spins.

To obtain Green functions and spectra of the elementary excitations the second order expansion of the action Eq. (23) over the field fluctuations $\delta \psi \overline{n}, \delta \overline{\psi} \overline{n}, \delta \overline{A}$ is considered. The field fluctuations are $\delta \psi \overline{n} = \psi \overline{n} - \psi^{(c)} \overline{n}, \delta \overline{\psi} \overline{n} = \overline{\psi} \overline{n} - \overline{\psi}^{(c)} \overline{n}, \delta \overline{A} = \overline{A} - \overline{A}^{(c)}$, where $\text{curl} \overline{A}^{(c)} = \overline{H}^{(c)}$, and the second order expansion of the action over them takes the form of the functional $\delta S_{act} \left[ \delta \psi \overline{n}; \delta \overline{\psi} \overline{n}; \delta \overline{A} \right]$, where
\[
\delta S_{\text{act}} = \int dt d^3r \left\{ \begin{array}{l}
\sum \frac{\delta \psi}{n} \left( G^{(0)} \right)^{-1} \delta \psi - g \rho^{(2)} \rho^{(c)} - \frac{1}{3} g \rho^{(1)} \delta \rho^{(1)} - 8 \pi \gamma \left( 1 - \frac{2}{3} m \right) \left( \delta S^{(2)} \cdot \vec{S}^{(c)} \right) - 4 \pi \gamma \left( 1 - \frac{2}{3} m \right) \left( \delta S^{(1)} \cdot \delta S^{(1)} \right) + 8 \pi \gamma \left( \delta S^{(2)} \cdot \vec{S}^{(c)} \right) + \sqrt{2} \gamma \left( \vec{A} \cdot \text{curl} \delta S^{(1)} \right) + \frac{1}{8 \pi} \left( \vec{A} \cdot \vec{A} \cdot \vec{A} \right) \\
\end{array} \right\} 
\] (37)

and \( \delta \rho^{(2)} = \sum_n \delta \psi_n \delta \psi_n; \ \delta \rho^{(1)} = \sum_n \left( \delta \psi_n \varphi_n^{(c)} + \varphi_n^{(c)} \delta \psi_n \right); \ \delta S^{(2)} = - (S + 1) \sum_n \delta \psi_n \delta \psi_n; \ \delta S^{(1)} = - (S + 1) \sum_n \left( \delta \psi_n \varphi_n^{(c)} + \varphi_n^{(c)} \delta \psi_n \right). \)

The generation functional \( Z \) of the fluctuations can be written as

\[
Z \left[ J_n; \vec{J}_n \right] = \int \prod_n D\delta \psi_n D\delta \psi_n D\delta \vec{A} \exp \left\{ i \delta S_{\text{act}} \left[ \delta \psi_n; \delta \psi_n; \delta \vec{A} \right] + \delta S_J \right\} 
\] (38)

where \( \delta S_J = \int dt d^3r \sum_n ( \vec{J}_n \delta \psi_n + J_n \delta \psi_n ). \)

The integration of Eq. (38) over \( \delta \vec{A} \) gives

\[
\delta S_{\text{act}} \left[ \delta \psi; \delta \vec{\psi} \right] = \int dt d^3r \left\{ \begin{array}{l}
\sum \frac{\delta \psi}{n} \left( G^{(0)} \right)^{-1} \delta \psi + 8 \pi \gamma \left( \delta S^{(2)} \cdot \vec{S}^{(c)} \right) - g \delta \rho^{(2)} \rho^{(c)} + 4 \pi \gamma \left( \delta S^{(1)} \cdot \delta S^{(1)} \right) - \frac{1}{2} g \rho^{(1)} \delta \rho^{(1)} - 4 \pi \gamma \left( \delta S^{(1)} \cdot \delta S^{(1)} \right) + 4 \pi \gamma \left( \delta S^{(1)} \cdot \delta S^{(1)} \right) - 4 \pi \gamma \left( \vec{A} \cdot \text{curl} \delta S^{(1)} \right) \cdot \vec{A} \cdot \vec{A} \cdot \vec{A} \\
\end{array} \right\} 
\]

Using the equality
\[
\int d^3r \left\{ \left( \text{curl} \vec{S} \cdot \vec{A} \cdot \text{curl} \vec{S} \right) + \left( \vec{S} \cdot \vec{S} \right) \right\} = \int \frac{d^3k}{(2\pi)^3} \left( \vec{e}_k \cdot \vec{S}_k \right) \left( \vec{e}_k \cdot \vec{S}_k \right) 
\]

we can transform the action of fluctuations \( \delta \psi_n, \delta \psi_n \) to the form \( \delta S_{\text{act}} \left[ \delta \psi_n; \delta \psi_n \right] \)

\[
\delta S_{\text{act}} = \int \frac{dt d^3k}{(2\pi)^3} \left\{ \begin{array}{l}
\sum \frac{\delta \psi}{n, k} \left( i \partial_t + \mu - \frac{1}{2\nu} \vec{k} \cdot \vec{k} - g \rho^{(c)} + 8 \pi \gamma S (S + 1) \rho^{(c)} (\vec{n} \cdot \vec{n}) \right) \delta \psi_{n, k} - \frac{1}{2} g \rho^{(1)} \delta \rho^{(1)} - 4 \pi \gamma \left( \delta S^{(1)}_k \cdot \delta S^{(1)}_k \right) + 4 \pi \gamma \left( \vec{e}_k \cdot \delta S^{(1)}_k \right) \left( \vec{e}_k \cdot \delta S^{(1)}_k \right) \\
\end{array} \right\} 
\] (39)
The fields $\psi_\vec{n} (t, \vec{r})$, $\overline{\psi}_\vec{n} (t, \vec{r})$ taking into account the small fluctuations $\delta \psi_\vec{n} (t, \vec{r})$, $\delta \overline{\psi}_\vec{n} (t, \vec{r})$ near the stationary and homogeneous condensate field $\psi^{(c)}_\vec{n}$ can be represented in the form

$$
\psi_\vec{n} (t, \vec{r}) = \Phi (t, \vec{r}) < \vec{n} | \vec{n}_0 (t, \vec{r}) >
$$

$$
\overline{\psi}_\vec{n} (t, \vec{r}) = \overline{\Phi} (t, \vec{r}) < \vec{n}_0 (t, \vec{r}) | \vec{n} >
$$

The function $\Phi (t, \vec{r})$ determines small density-phase fluctuations of the field. It can be represented as

$$
\Phi (t, \vec{r}) = \sqrt{\rho^{(c)}} + \delta \Phi (t, \vec{r})
$$

(40)

where $\delta \Phi (t, \vec{r})$ is a small fluctuation of $\Phi (t, \vec{r})$. If there is no fluctuations $\delta \Phi (t, \vec{r}) = 0$ the function $\Phi (t, \vec{r})$ does not depend on $t$ and $\vec{r}$ and takes the form $\Phi (t, \vec{r}) = \sqrt{\rho^{(c)}}$. The smallness of the fluctuations $\delta \Phi (t, \vec{r})$ means that $| \delta \Phi (t, \vec{r}) | << \sqrt{\rho^{(c)}}$. The ket-vector $| \vec{n}_0 (t, \vec{r}) >$ is a spin coherent state, and the function $\vec{n}_0 (t, \vec{r})$ is a field of unit vectors. Due to the smallness of the fluctuations the field $\vec{n}_0 (t, \vec{r})$ can be represented as

$$
\vec{n}_0 (t, \vec{r}) = \vec{n}_0^{(c)} + \delta \vec{n}_0 (t, \vec{r})
$$

(41)

The state $| \vec{n}_0 (t, \vec{r}) >$ is the fluctuating one near the stationary and homogenous state of the condensate $| \vec{n}_0^{(c)} > = | -S >$. The vector $\delta \vec{n}_0 (t, \vec{r})$ determines the small fluctuations of the spin direction near the constant unit vector $\vec{n}_0^{(c)}$ which determines the direction of the condensate’s spin density $\vec{S}^{(c)}$. The smallness of the spin fluctuations $\delta \vec{n}_0 (t, \vec{r})$ means that $| \delta \vec{n}_0 (t, \vec{r}) | << 1$, $\delta \vec{n}_0 (t, \vec{r}) \perp \vec{n}_0^{(c)}$. The spin coherent state $| \vec{n}_0 (t, \vec{r}) >$ can be represented in the form $[3], [1], [2]$

$$
| \vec{n}_0 (t, \vec{r}) > = \hat{D} (\vec{n}_0 (t, \vec{r})) | \vec{n}_0^{(c)} >
$$

(42)

The operator $\hat{D} (\vec{n}_0 (t, \vec{r})) = \hat{D} (\vec{n}_0 (t, \vec{r})) = \exp \left( i \theta \frac{\vec{r}}{\vec{n}_0} \right)$, where the vectors $\vec{n}$ and $\vec{n}_0$ can be represented via the spherical angles $\theta$ and $\varphi$ which depend on $t$ and $\vec{r}$, at that, $\vec{n} = (\sin \varphi; -\cos \varphi; 0)$, $\vec{n}_0 = (\sin \theta \cos \varphi; \sin \theta \sin \varphi; \cos \theta)$, $\vec{n}_0^{(c)} = (0; 0; 1)$. From the form of $\vec{n}$ and $\vec{n}_0$ it is clear that $\vec{n} \perp \vec{n}_0$ and $\vec{n} \perp \vec{n}_0^{(c)}$. The smallness of the fluctuations $\delta \vec{n}_0$ means the smallness of $\theta$. As the result, for the small fluctuations the
vector of the spin direction $\vec{n}_0$ takes the form $\vec{\theta} = \begin{pmatrix} \delta n_0^{(x)}; -\delta n_0^{(x)}; 0 \end{pmatrix}$, here the terms proportional to $\theta^2$ are neglected. Note that the fields $\delta n_0^{(+)} = \delta n_0^{(x)} + i\delta n_0^{(y)}$, $\delta n_0^{(-)} = \delta n_0^{(x)} - i\delta n_0^{(y)}$ can be written as $\delta n_0^{(+)} = \theta \cdot e^{i\varphi}$, $\delta n_0^{(-)} = \theta \cdot e^{-i\varphi}$.

Due to the smallness of $\theta$ the operator $\hat{D}(\vec{\theta}_0)$ can be rewritten as

$$\hat{D}(\vec{\theta}_0) = \hat{1} + \frac{1}{2} \left( \delta n_0^{(+) \cdot \vec{s}^{(-)} - \delta n_0^{(-) \cdot \vec{s}^{(+)}} \right)$$

(43)

where $\vec{s}^{(\pm)} = \vec{s}^{(x)} \pm i\vec{s}^{(y)}$. As a result, the state $|\vec{\theta}_0\rangle$ can be written via the spin fluctuations $\delta \vec{n}_0$ as

$$|\vec{\theta}_0\rangle = \hat{D}(\vec{\theta}_0) |\vec{\theta}_0^{(c)}\rangle = | -S > -\sqrt{\frac{S}{2}} \delta n_0^{(-)} | -S + 1 >$$

(44)

Due to Eq.(44) the field fluctuations $\delta \psi_{\vec{\theta}}$ and $\bar{\delta} \psi_{\vec{\theta}}$ are written as

$$\delta \psi_{\vec{\theta}} = \delta \Phi < -\vec{n} | -S > -\sqrt{\frac{\rho^{(c)}S}{2}} \delta n_0^{(-)} < -\vec{n} | -S + 1 >$$

(45)

$$\bar{\delta} \psi_{\vec{\theta}} = \delta \bar{\Phi} < -S | \vec{n} > -\sqrt{\frac{\rho^{(c)}S}{2}} \delta n_0^{(-)} < -S + 1 | \vec{n} >$$

(46)

These equations can be rewritten as

$$\delta \psi_{\vec{\theta}} = \delta \Phi < -\vec{n} | -S > -\eta < -\vec{n} | -S + 1 >$$

(47)

$$\bar{\delta} \psi_{\vec{\theta}} = \delta \bar{\Phi} < -S | \vec{n} > -\eta < -S + 1 | \vec{n} >$$

(48)

where

$$\eta = \sqrt{\frac{S\rho^{(c)}}{2}} \delta n_0^{(-)}; \quad \bar{\eta} = \sqrt{\frac{S\rho^{(c)}}{2}} \delta n_0^{(+)\cdot}$$

(49)

The substitution of Eqs.(47), (48) to Eq.(39) transforms the action of the field fluctuations $\delta S_{act} [\delta \psi_{\vec{\theta}}; \bar{\delta} \psi_{\vec{\theta}}]$ to the form $\delta S_{act} [\delta \Phi; \delta \bar{\Phi}; \eta; \bar{\eta}]$

$$\delta S_{act} = \delta S_{act}^{(\Phi)} + \delta S_{act}^{(\eta)} + \delta S_{act}^{(\Phi, \eta)}$$

(50)

where
\[ \delta S_{\text{act}}^{(\Phi)} = \int \frac{d\omega d^3k}{(2\pi)^4} \left( \frac{\delta \Phi}{\sqrt{2}} \left( \omega - \frac{1}{2m_0} k^2 - \sigma_\rho \right) \delta \Phi - \frac{1}{2} \sigma_\rho \left( \delta \Phi_\perp \delta \Phi - \delta \Phi \delta \Phi_\perp \right) \right) \]  
(51)

\[ \delta S_{\text{act}}^{(\eta)} = \int \frac{d\omega d^3k}{(2\pi)^4} \left( \frac{\pi_\eta}{\sqrt{2m_0}} \left( \omega - \frac{1}{2m_0} k^2 - \sigma_\rho e_k^{(+)} e_k^{(-)} \right) \eta_{\perp} - \frac{1}{2} \sigma_\rho \left( e_k^{(+)} \eta_{\perp} \eta_k + e_k^{(-)} \eta_{k\perp} \eta_k \right) \right) \]  
(52)

\[ \delta S_{\text{act}}^{(\psi, \eta)} = -\sqrt{\frac{S}{2}} \sigma_\rho \int \frac{d\omega d^3k}{(2\pi)^4} e_k^{(c)} \left( e_k^{(-)} \eta_{\perp} \delta \Phi_{\perp -} + e_k^{(+)} \eta_{\perp} \delta \Phi_{\perp +} + e_k^{(+)} \eta_{k\perp} \delta \Phi_{k\perp} + e_k^{(-)} \eta_{k\perp} \delta \Phi_{k\perp} \right) \]  
(53)

\[ \sigma_\rho = \left( g - 8\pi \gamma S^2 + 8\pi \gamma S^2 \left( e_k^{(c)} \right)^2 \right) \rho^{(c)} \]  
(54)

\[ \sigma_s = 4\pi \gamma S \rho^{(c)} \]  
(55)

\[ e_k^{(\pm)} = e_k^{(x)} \pm e_k^{(y)}; \quad \vec{e}_k = \frac{\vec{k}}{|k|} \]  
(56)

The action \( \delta S_{\text{act}} \left[ \delta \Phi; \delta \Phi; \eta; \bar{\eta} \right] \) Eq. (50) can be rewritten in the matrix form

\[ \delta S_{\text{act}} = \int \frac{d\omega d^3k}{(2\pi)^4} \sum_{\alpha, \beta = 1}^4 \chi_{\alpha} (k) \left( \hat{G}^{-1} \right)_{i,j} \chi_{\beta} (k) \]  
(57)

The components of the field \( \chi_{\beta} (k) \) are defined as \( \chi_1 (k) = \delta \Phi_k, \chi_2 (k) = \delta \Phi_{\perp -}, \chi_3 (k) = \eta_k, \chi_4 (k) = \bar{\eta}_{\perp -} \), and \( \chi_1 (k) = \delta \Phi_k, \chi_2 (k) = \delta \Phi_{\perp -}, \chi_3 (k) = \eta_k, \chi_4 (k) = \bar{\eta}_{\perp -} \). The matrix \( \hat{G}^{-1} \) is the inverse Green function of the excitations

\[ \hat{G}^{-1} = \begin{pmatrix} f_{\Phi} & -\sigma_\rho & -\sigma_{\Phi\eta}^{(+)} & -\sigma_{\Phi\eta}^{(-)} \\ -\sigma_\rho & f_{\Phi} & -\sigma_{\Phi\eta}^{(+)} & -\sigma_{\Phi\eta}^{(-)} \\ -\sigma_{\Phi\eta}^{(-)} & -\sigma_{\Phi\eta}^{(-)} & f_{\eta} & -\sigma_{s}^{(-)} \\ -\sigma_{\Phi\eta}^{(+)} & -\sigma_{\Phi\eta}^{(+)} & -\sigma_{s}^{(-)} & f_{\eta} \end{pmatrix} \]  
(58)

where the following denotations have the form \( \sigma_{s}^{(\pm)} = \sigma_s e_k^{(\pm)} e_k^{(\pm)}, \sigma_{s}^{(--) = \sigma_s e_k^{(-)} e_k^{(-)},} \sigma_{s}^{(++) = \sigma_s e_k^{(+) e_k^{(+)}}}, \sigma_{\Phi\eta}^{(\pm)} = \sqrt{\frac{s}{2}} \sigma_s e_k^{(\pm)} e_k^{(\pm)} \) , and
For small momenta $\frac{\vec{k}^2}{2m_0}$ the condensate module-phase and has the form

$$
\hat{d} = \omega - \frac{1}{2m_0} \vec{k}^2 - \sigma_\rho = \hat{f}_\psi(0) - \sigma_\rho
$$

$$
\hat{d} = \omega - \frac{1}{2m_0} \vec{k}^2 - \sigma_s e_k^{(+)} e_k^{(-)} = \hat{f}_\eta(0) - \sigma_s e_k^{(+)} e_k^{(-)}
$$

$$
\vec{f}_\psi(\omega) = f_\psi(-\omega); \quad \vec{f}_\eta(\omega) = f_\eta(-\omega)
$$

The spectrum of the elementary excitations can be found from the equality to zero of the determinant of the matrix $\hat{G}^{-1}$. The calculation of this determinant gives

$$
\det \hat{G}^{-1} = \left\{ \begin{array}{l}
\left[ \hat{f}_\psi(0) - \sigma_\rho \left( \hat{f}_\eta(0) + \hat{\eta}(0) \right) \right] \left[ \hat{f}_\eta(0) - \sigma_s^2 \left( e_k^{(+)} e_k^{(-)} \right)^2 \right] - \\
- \frac{1}{2} \left( \sigma_s e_k^{(z)} \right)^2 S e_k^{(+)} e_k^{(-)} \left( \hat{f}_\psi(0) + \hat{\eta}(0) \right) \left( \hat{f}_\eta(0) + \hat{\eta}(0) \right)
\end{array} \right.
$$

or

$$
\det \hat{G}^{-1} = \left[ \omega^2 - \left( \frac{\vec{k}^2}{2m_0} \right)^2 - 2\sigma_\rho \frac{\vec{k}^2}{2m_0} \right] \left[ \omega^2 - \left( \frac{\vec{k}^2}{2m_0} + \sigma_s \left( e_k^{(+)} e_k^{(-)} \right) \right)^2 + \sigma_s^2 \left( e_k^{(+)} e_k^{(-)} \right)^2 \right] - \\
- 2\sigma_s^2 S \left( \frac{\vec{k}^2}{2m_0} \right)^2 \left( e_k^{(+)} e_k^{(-)} \right)
$$

Thus, the spectrum of the elementary excitations $\varepsilon_k^{(\pm)}$ is defined by the equality

$$
\varepsilon_k^{(\pm)} = \left( \frac{\vec{k}^2}{2m_0} \right) \left\{ \begin{array}{l}
\left( \sigma_\rho + \sigma_s e_k^{(+)} e_k^{(-)} + \frac{\vec{k}^2}{2m_0} \right) \pm \\
\left( \sigma_\rho - \sigma_s e_k^{(+)} e_k^{(-)} + \frac{\vec{k}^2}{2m_0} \right)
\end{array} \right. \\
\left( \sigma_\rho + \sigma_s e_k^{(+)} e_k^{(-)} + \frac{\vec{k}^2}{2m_0} \right)^2 - \\
- \left( 2\sigma_\rho + \frac{\vec{k}^2}{2m_0} \right) \left( \frac{\vec{k}^2}{2m_0} + 2\sigma_s e_k^{(+)} e_k^{(-)} \right) + \\
+ 2 \left( \sigma_s e_k^{(z)} \right)^2 S e_k^{(+)} e_k^{(-)}
\right\}
$$

In the case $e_k^{(+)} = e_k^{(-)} = 0$, $e_k^{(z)} = 1$ the spectrum has two independent branches. For small momenta $\frac{\vec{k}^2}{2m_0} \ll \sigma_\rho$ the first branch is the sound like oscillations of the condensate module-phase and has the form

$$
\varepsilon_k^{(1)} = u_\rho \left| \vec{k}^2 \right|
$$

where $u_\rho = \sqrt{\sigma_\rho/m_0}$. The second branch is the oscillations of the spin direction of the condensate and has the form

$$
\varepsilon_k^{(2)} = \frac{1}{2m_0} \vec{k}^2
$$

In the case of nonzero $e_k^{(+)}$, $e_k^{(-)}$ and small momenta $\frac{\vec{k}^2}{2m_0} \ll \sigma_\rho; \frac{\vec{k}^2}{2m_0} \ll \sigma_s e_k^{(+)} e_k^{(-)}$ the spectrum has two branches of the sound type.
\[ \varepsilon^{(1,2)}_k = | \vec{k} | \sqrt{\frac{(\sigma_\rho + \sigma_s e_k^{(+)} e_k^{(-)}) \pm \sqrt{(\sigma_\rho - \sigma_s e_k^{(+)} e_k^{(-)})^2 + 2(\sigma_s e_k^{(z)} e_k^{(-)})^2 S e_k^{(+)} e_k^{(-)}}}{2m_0}} \]  

(61)

Due to smallness of \( \gamma \), so that \( \gamma S^2 \ll g \), these two branches of the spectrum take the form

\[ \varepsilon^{(1,2)}_k = u^{(1,2)} | \vec{k} | \]  

(62)

where \( u^{(1)} = \sqrt{\frac{\sigma_\rho}{m_0}} \), \( u^{(2)} = \sqrt{\frac{\sigma_s e_k^{(+)} e_k^{(-)}}{m_0}} \).

In the case of nonzero \( e_k^{(+)} \), \( e_k^{(-)} \) and small momenta, such that \( \sigma_s e_k^{(+)} e_k^{(-)} \ll \frac{1}{2m_0} \vec{k}^2 \ll \sigma_\rho \), the spectrum has the same two branches as in the case for \( e_k^{(+)} = e_k^{(-)} = 0, e_k^{(z)} = 1 \), i.e.,

\[ \varepsilon^{(1)}_k = u_\rho | \vec{k} |, \varepsilon^{(2)}_k = \frac{1}{2m_0} \vec{k}^2. \]

To find the Green function of the elementary exitsations \( \hat{G} \), the inveerse to \( \hat{G}^{-1} \) matrix should be found. The elements of this matrix have the form

\[ G_{ij} = (-1)^{i+j} \frac{\det (\hat{A}_{ij})}{\det (\hat{G}^{-1})} \]  

(63)

where \( \hat{A}_{ij} \) is the corresponding minor of the matrix \( \hat{G}^{-1} \), and \( 1 \leq i, j \leq 4 \).

In the case \( e_k^{(+)} = e_k^{(-)} = 0, e_k^{(z)} = 1 \) and small momenta \( | \vec{k} |^2 \ll \mu m_0 \) the nonzero Green functions \( G_{11} = G_{\Phi\Phi}, G_{21} = G_{\Phi\bar{\Phi}}, G_{33} = G_{\eta\eta}, G_{34} = G_{\eta\bar{\eta}} \) take the form

\[ G_{\Phi\Phi} = \frac{\sigma_\rho}{\omega^2 - u_\rho^2 | \vec{k} |^2 + i\delta} \]  

(64)

\[ G_{\Phi\bar{\Phi}} = \frac{-\sigma_\rho}{\omega^2 - u_\rho^2 | \vec{k} |^2 + i\delta} \]

\[ G_{\eta\eta} = \frac{1}{\omega - \frac{1}{2m_0} | \vec{k} |^2 + i\delta} \]  

(65)

\[ G_{\eta\bar{\eta}} = \frac{1}{-\omega - \frac{1}{2m_0} | \vec{k} |^2 + i\delta} \]

In the case \( e_k^{(+)} \neq 0, e_k^{(-)} \neq 0 \) and small momenta \( \frac{1}{2m_0} \vec{k}^2 \ll \sigma_\rho; \frac{1}{2m_0} \vec{k}^2 \ll \sigma_s e_k^{(+)} e_k^{(-)} \) these Green functions take the form
\[ G_{\Phi \Phi} = \frac{\sigma_\rho \left( \omega^2 - (u^{(2)})^2 \right)}{\left( \omega^2 - (u^{(1)})^2 \right)} \left( \omega^2 - (u^{(2)})^2 \right) + S \sigma_s^2 \left( e_k^{(+)} e_k^{(-)} \right) \left( \frac{\kappa^2}{2m_0} \right) \]

\[ G_{\Phi \Phi} = \frac{\sigma_\rho \left( \omega^2 - (u^{(2)})^2 \right)}{\left( \omega^2 - (u^{(1)})^2 \right)} \left( \omega^2 - (u^{(2)})^2 \right) + S \sigma_s^2 \left( e_k^{(+)} e_k^{(-)} \right) \left( \frac{\kappa^2}{2m_0} \right) \]

For \( e_k^{(+)} \neq 0, e_k^{(-)} \neq 0 \) and small momenta, such that \( \sigma_s e_k^{(+)} e_k^{(-)} < \frac{1}{2m_0} \kappa^2 < \sigma_\rho \), the Green functions has the same form as in the case for \( e_k^{(+)} = e_k^{(-)} = 0, e_k^{(z)} = 1 \). Note that if \( e_k^{(+)} \neq 0, e_k^{(-)} \neq 0 \) there are nonzero components of \( \hat{G} \) which define the correlations between module-phase excitations and spin direction excitations. They are \( G_{\Phi \eta} \) or \( G_{\Phi \eta} \), for example.

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