Graded Lie algebras, Representation theory, 
Integrable mappings and Systems

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Abstract

A new class of integrable mappings and chains is introduced. Corresponding $(1+2)$ integrable systems invariant, with respect to such discrete transformations, are presented in an explicit form. Their soliton-type solutions are constructed in terms of matrix elements of fundamental representations of semisimple $A_n$ algebras for a given group element. The possibility of generalizing this construction to multi-dimensional case is discussed.
1 Introduction

In an old paper \cite{1} of the author the effective method (based on two-dimensional zero-curvature condition) was proposed for constructing of exactly integrable systems in two dimensions together with their general solutions. During the last fifteen years situation became more clear and understandable.

It turned out that by the method of \cite{1} it is possible to construct the integrable mappings \cite{2,3} responsible for the existence of the hierarchies of integrable systems. Each equation of a given hierarchy is invariant with respect to the transformation of the corresponding mapping (or substitution).

Moreover, it has became clear that the formalism of $L-A$ pair is not the principal point of the whole construction. There exists a more direct way for obtaining the integrable mappings and technique of the $L-A$ pair is not more than one of its consequences. The situation with respect to the explicit solution of the quantum two-dimensional Toda lattice \cite{4} for Heisenberg operators is the most important argument for the necessity of the development of a new way disconnected with the representation of the zero curvature. In letter case the $L-A$ representation is absent at all and nevertheless the method of the present paper works as well. In this connection it is necessary remind also about the general solution of periodical Toda lattice, which can be obtained in the form of absolutely convergent infinite series without any using of the $L-A$ pair formalism, but with the help of below construction \cite{5}.

The scheme proposed in the present paper is the following one.

At the first step we introduce two integrable in quadratures equations of $S$–matrix type for the two group-valued functions, depending on two different arguments. The coefficient functions of these equations are determined by the structure of the corresponding Lie algebra and the choice of definite grading in it. We make no difference between algebra and super-algebra cases, recalling only that even (odd) elements of super-algebras are always multiplied by even (odd) elements of the Grassman space.

By help of these two group elements we construct a new composite one. Arising relations of equivalence between its matrixes elements lead to integrable substitution.

At the second step we assume additional dependence on arbitrary functions, determining general solution of the integrable substitution (with the fixed ends), on some ”time–like” parameters. This is achieved by two additional equations for the above mentioned group elements $M^\pm$ in such a way that the condition of their selfconsistency leads to finding the explicit dependence of arbitrary functions on both the space and time like parameters. With the help of these time parameters we present the hierarchy of completely integrable systems each one of which is invariant with respect to the transformation of the integrable mapping (constructed at the first step).

At last, (this is the third step which is, in fact, not in the close connection with the previous ones) we observe that in the framework of the method \cite{1} there exists some hidden previously omitted non–trivial possibility for the generalization of the whole construction on the multi-dimensional case \cite{1}.

What is the most remarkable in this approach is the fact that dimension of the possible ”multi-generalization” is uniquely determined by the properties of the algebra and the choice of the grading in it. Sometimes ”multi-generalization” is equivalent to the trivial change of variables, sometimes it leads to nontrivial new possibilities. Nevertheless, in all of the cases

\footnote{About the other possibility to enlarge the class of integrable systems which was not discussed in \cite{1}, see \cite{6}.}
it is possible to obtain only particular (but not the general) solutions of arising in this way systems and equations.

The present paper is organized in the following way. In section 2 we briefly repeat the content of first section of the paper \[1\] but in “opposite direction” compared to the original. In section 3 for convenience of the reader, we present the most important for further consideration results from the theory of representation of (super) semisimple algebras and groups. In section 4 we show how to avoid the relatively cumbersome procedure of resolution of the Gauss decomposition (in the cases when it is equivalent to solution of the whole problem); we present the integrable systems together with their general solutions in terms of the matrix elements of the various fundamental representations of semisimple algebras. In this way we construct \(UToda(m_1, m_2)\) integrable mappings or substitutions. In section 5 we demonstrate the way of introducing of the evolution parameters in order to obtain the hierarchies of integrable systems invariant with respect to transformation of the constructed integrable mappings. In section 6 we discuss the possibility containing in this construction, for its generalization to the multi-dimensional case (with not arbitrary dimensions!). The concluding remarks are concentrated in section 7.

2 Moving in the opposite direction

Let us have some arbitrary finite dimensional graded algebra \(\mathcal{G}\). This means that \(\mathcal{G}\) may be represented as a direct sum of subspaces with the different grading indexes

\[
\mathcal{G} = \left( \bigoplus_{k=1}^{N_{-}} \mathcal{G}_{-k} \right) \mathcal{G}_0 \left( \bigoplus_{k=1}^{N_{+}} \mathcal{G}_{+k} \right).
\]  

(1)

The generators with the integer graded indexes are called bosonic, while with half-integer indexes – the fermionic ones. Positive (negative) grading corresponds to upper (lower) triangular matrices.

Let \(M_+(y), M_-(x)\) be the elements of the group (when it exists) corresponding to algebra \(\mathcal{G}\) and only some solutions of the equations of \(S\)-matrix type in some given finite dimensional representation of initial algebra \(\mathcal{G}\):

\[
\frac{\partial M_-}{\partial x} = L_{m_1}^-(x)M_-, \quad \frac{\partial M_+}{\partial y} = L_{m_2}^+(y)M_+ = \sum_{s=0}^{m_2} B^{+s}(y)M_+\]

(2)

where \(A^{-s}, B^{+s}\) are arbitrary functions of their arguments taking values in corresponding graded subspaces; \(s\) is an integer or half-integer number.

Let us introduce the group element \(K\)

\[
K = M_+ M_-^{-1}.
\]  

(3)

By the logic of Ref. \[1\] it is necessary to represent \(K\) in the form of the Gauss decomposition

\[
K = M_+ M_-^{-1} = N_-^{-1} g_0 N_+
\]

(\(N_+\) are elements of positive (negative) nilpotent subgroups, \(g_0\) – of the group with the algebra of the zero subspace) and to consider the group element \(G\)

\[
G = N_- M_+ = g_0 N_+ M_-
\]  

(4)
As a direct consequence of above definitions and equations for $M^\pm$ elements (2) we obtain the following relations:

$$G_xG^{-1} = (N_-)_xN_-^{-1} = \sum_s R^{(-s)}(x, y)$$

$$s = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$$

(5)

$$G_yG^{-1} = (g_0)_yg_0^{-1} + g_0(N_+)_yN_+^{-1}(g_0)^{-1} = (g_0)_yg_0^{-1} + \sum_s R^{(+s)}(x, y)$$

The Maurer–Cartan identity applied to (5) leads to the equations of exactly integrable system with the general solution determined by (4) and the above formulae.

All that has been said so far was the almost literal repetition of [1], but in the opposite direction to the original one. The main technique difficulty under such a approach consists in explicit resolving of the Gauss decomposition – finding from (4) group elements $N^\pm, g_0$ when $M^\pm$ are known. This is sufficiently cumbersome problem under direct attempts of its solution.

At this place we would like to emphasize, that connection between the equations (2) and the further chain of equalities connected with the element $K$ leading to zero-curvature representation encoded in (5) is true only in the case when the algebra $G$ can be integrated up to the corresponding group. In the case of Lie algebras as it well-known it is always possible [7]. But there are exist many other interesting problems ( and among them exactly the cases of the quantum Toda chain and general solution of the periodically one mentioned in introduction [4], [5]), when solution of equations (2) may be presented in quadrature but to pass to representation of zero-curvature is impossible due to the absent of the corresponding group element.

Fortunately, there exists a more direct way allowing us, using the definition of the element $K$ (not always having the group ones sense), to reconstruct the form of equations of the integrable system as well as its general solution. We will demonstrate this way applying it to the case of semisimple algebras when it is possible to present in explicit form as integrable mappings and as their general solutions ( in the interrupted version) in terms of matrix elements of the various fundamental representations.

Exactly the same technique is applicable to quantum version of two-dimensional Toda lattice and general solution of the periodical one, when corresponding group element is absent ( simultaneously with the rigorous representation of zero-curvature).

3 Some facts from the representation theory of the semisimple algebras

In this section we restrict ourselves to the case of semisimple algebras. In the general case the grading operator $H$ may be presented as linear combination of elements of commutative Cartan subalgebra (taking the unit or zero values on the generators of the simple roots in the form):

$$H = \sum_{i=1}^r (K^{-1}c)_ih_i$$

(6)

Here, $(K^{-1})_{j,i}$ is the inverse Cartan matrix, $K^{-1}K = KK^{-1} = I$ and $c$ the column consisting of zeros and unities in arbitrary order.
As usually the generators of the simple roots $X^\pm_i$ (raising, lowering operators) and Cartan elements $h_i$ satisfy the system of commutation relations:

$$[h_i, h_j] = 0, \quad [h_i, X^\pm_j] = \pm K_{ij} X^\pm_j, \quad [X^+_i, X^-_j] = \delta_{i,j} h_j, \quad (1 \leq i, j \leq r),$$

(7)

where $K_{ij}$ is the Cartan matrix, and the brackets $[,]$ denote the graded commutator, $r$ is the rank of the algebra.

The highest vector $|j\rangle$ ($\langle j | \equiv | j\rangle^\dagger$) of the $j$-th fundamental representation possesses the following properties:

$$X^+_i |j\rangle = 0, \quad h_i |j\rangle = \delta_{i,j} |j\rangle, \quad \langle j | |j\rangle = 1.$$  

(8)

The representation is exhibited by repeated applications of the lowering operators $X^-_i$ to the $|j\rangle$ and extracting all linear-independent vectors with non-zero norm. Its first few basis vectors are

$$|j\rangle, \quad X^-_i |j\rangle, \quad X^-_i X^-_j |j\rangle, \quad K_{i,j} \neq 0, \quad i \neq j$$

(9)

In the fundamental representations, matrix elements of the arbitrary group element $G$ satisfy the following important identity \[10\]

$$s\text{det} \left( \frac{\langle j | X^+_i G X^-_j | j \rangle}{\langle j | G X^-_j | j \rangle}, \frac{\langle j | X^+_j G | j \rangle}{\langle j | G | j \rangle} \right) = \prod_{i=1, i \neq j}^r \langle i | G | i \rangle^{-K_{ji}},$$

(10)

where $K_{ji}$ are the elements of the Cartan matrix. The identity \[10\] represents the generalization of the famous Jacobi identity connecting determinants of $(n-1)$, $n$ and $(n+1)$ orders of some special matrixes to the case of arbitrary semisimple Lee super-group. As we will see in the next section, this identity is so important at the deriving of the integrable mappings, that one can even say that it is responsible for their existence. We conserve for \[10\] the name of the first Jacobi identity. Besides \[10\], there exists no more less important independent identity \[3\] :

$$K_{i,j} (-1)^P \frac{\langle j | X^+_i X^+_j G | j \rangle}{\langle j | G | j \rangle} + K_{j,i} \frac{\langle i | X^+_j X^+_i G | i \rangle}{\langle i | G | i \rangle} + K_{i,j} K_{j,i} (-1)^P \frac{\langle j | X^+_j G | j \rangle}{\langle j | G | j \rangle} \frac{\langle i | X^+_i G | i \rangle}{\langle i | G | i \rangle} = 0, \quad K_{i,j} \neq 0$$

(11)

which will be called as a second Jacobi identity. This identity is responsible (in the above sense) for the fact of existence of hierarchy of integrable systems each one of which is invariant with respect to transformations of constructed integrable mapping.

As from \[10\] either from \[11\] it is possible to construct many useful recurrent relations which will be used under further consideration.

### 4 Integrable mappings and chains

In this section using the apparat of the previous one we will show that matrix elements of group element $K$ satisfy the closed system of equations of equivalence which can be interpreted as

\[2\] Let us remind the definition of the superdeterminant, $s\text{det} \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \equiv \text{det}(A - BD^{-1} C)(\text{det}D)^{-1}$. 

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5
the integrable mapping or the chain like system with the known general solution. We restrict ourselves by the case $A_n$ semisimple algebra with the main embedding in it. All other cases try for their consideration a little more cumbersome (only from the technical point of view) calculations.

Under the main embedding the grading operator $\tilde{L}^m$ takes unity values on all generators of the simple roots. Generators of $\pm m$-graded subspaces have the form

$$Y_i^{\pm m} = [X_i^{\pm m_1} \ldots [X_i^{\pm 1}, X_i^\pm] \ldots]$$

The action of right ”Lagrangian“ $\tilde{L}_m^-$ (sign $-$ means that terms with the graded index zero are interrupted from $L^m$) on the state vector $| i \rangle$ may be presented in form

$$\tilde{L}_m^- | i \rangle = \sum_{n=1}^{m_1} \sum_{s=0}^{n-1} (-1)^s \phi^m_{i-s} T_s^- (X_{i-1}^+) T_{n-s-1}^+ (X_{i+1}^-) X_i^- | i \rangle$$

where

$$T_m^+(X_i^+) = T_{m-1}^+(X_{i+1}^+) X_i^- \quad T_0^+(X_i^-) = 1, \quad T_m^-(X_i^-) = T_{m-1}^-(X_{i-1}^+) X_i^- \quad T_0^-(X_i^-) = 1$$

To understand (12) is not so difficult. In fact, in the $m$-graded subspace there are exactly $m - 1$ generators containing generator of the given simple root $i$. In connection with (6) the different from zero contribution may be arised only in the case if this generator occurs on the last place. Exactly about this tell us formula (12).

By the same reasons the action of ”Lagrangian“ $\tilde{L}_m^+$ on the left state vector $\langle i |$ takes the form

$$\langle i | \tilde{L}_m^+ = \langle i | X_i^+ \sum_{n=1}^{m_2} \sum_{s=0}^{n-1} (-1)^s \phi^m_{s-i} R_s^- (X_i^- + i - 1) R_{n-s-1}^+ (X_{i+1}^-)$$

where

$$R_m^+(X_i^+) = X_i^+ R_{m-1}^+(X_{i+1}^+) \quad R_0^+(X_i^+) = 1, \quad R_m^-(X_i^+) = X_i^+ R_{m-1}^-(X_{i+1}^-) \quad R_0^-(X_i^+) = 1$$

The following definitions will be used:

$$\alpha_i^{\pm m} = \frac{\langle i | R_m^\pm (X_i^+) K | i \rangle}{\langle i | K | i \rangle}, \quad \alpha_i^{\pm m} = \frac{\langle i | K T_m^\pm (X_i^-) | i \rangle}{\langle i | K | i \rangle}$$

$$< i > \equiv \langle i | K | i \rangle, \quad \theta_i \equiv \frac{< i - 1 > - < i + 1 >}{< i >^2}$$

In what follows for concrete calculations we will use many times the same trick. The matrix element $\langle a | X_i^+ G X_k^- | b \rangle$, were $\langle a |, | b \rangle$ are arbitrary state vectors of some representation of the group may be rewritten as:

$$\langle a | X_i^+ G X_k^- | b \rangle \equiv \langle \hat{X}_i^+ |_{left} \hat{X}_k^- |_{right} \langle a | G | b \rangle$$

where now $(\hat{X}_i^+ |_{left})$, $(\hat{X}_k^- |_{right})$ are the corresponding generators of the left (right) regular representation acting on group element $G$. This approach allows to lead the most part of the calculations below only to repeatedly using of the first and second Jacobi identities (10), (11).

The simplest example of such calculations for the case of arbitrary semisimple algebra reader can find in the Appendix of this paper.
As a first example let us consider the proof of the following recurrent relations for function $Q_{a;b;\pm 1} = (R_{a}^{\pm}(X_{i;\pm 1})t(i \pm 1 | K | i \pm 1)(T_{b}^{\pm}(X_{i;\pm 1}))_{r}$:

$$Q_{a;b;\pm 1} = (R_{a-1}^{\pm}(X_{i;\pm 2})t(i \pm 1 | X_{i;\pm 1}KX_{i;\pm 1} | i \pm 1)(T_{b-1}^{\pm}(X_{i;\pm 2}))_{r} =$$

$$= \frac{1}{\langle i \pm 1 | K | i \pm 1 \rangle}Q_{a-1,\pm 1;i;\pm 2} + \langle i \pm 1 | K | i \pm 1 \rangle \bar{\alpha}_{i;\pm 1}^{\pm a} \bar{\alpha}_{i;\pm 1}^{\pm b} \tag{16}$$

In the process of performing the last transformations we have expressed the matrix element $\langle i \pm 1 | X_{i;\pm 1}KX_{i;\pm 1} | i \pm 1 \rangle$ with the help of the first Jacobi identity (10) and used a definition for the $\alpha$, $\bar{\alpha}$ functions (14).

Now we are ready to pass to calculation of the equivalence relations among introduced above values.

We present here the main typical steps of calculation the derivative of the $\bar{\alpha}$ (14) functions with respect to $x$ coordinate. We have consequently:

$$(\bar{\alpha}_{j}^{+m})_{x} = (\langle j | K | j \rangle)^{-2}det \left( \begin{array}{cc} \langle j | R_{m}^{+}(X_{j}^{+})K \tilde{L}_{m}^{+} | j \rangle, & \langle j | R_{m}^{+}(X_{j}^{+})K | j \rangle \\ \langle j | K \tilde{L}_{m}^{+} | j \rangle, & \langle j | K | j \rangle \end{array} \right) =$$

$$(\langle j | K | j \rangle)^{-2}(\tilde{T}_{m-1}^{+}(X_{j+1}^{+})t(i \pm 1 | K | j - 1)(j + 1 | K | j + 1)$$

Up to now in the process of the last evaluation we have used many times the mentioned above trick (13) and first Jacobi identity (10). Further evaluation is connected with the repeatedly application of the recurrent relation (16) to the expression

$$(\tilde{T}_{m-1}^{+}(X_{j+1}^{+})t(i \pm 1 | K | j - 1)(j + 1 | K | j + 1))$$

and following from definition (14) relation:

$$(\tilde{T}_{s}^{-}(X_{i-1}^{+}))_{r}(\langle j - 1 | K | j - 1 \rangle) = (\langle j - 1 | K | j - 1 \rangle)\alpha_{j-1}^{-s}$$

Performing all calculations having pure algebraic character we come to final expression for interesting for us derivative:

$$(\bar{\alpha}_{j}^{+m})_{x} = \sum_{q=0}^{m-1} \Theta_{i}^{q+1} p_{i}^{(q+1)\bar{\alpha}_{i+q+1}^{m-1-q}} \tag{17}$$

where

$$p_{i}^{r} = \sum_{n=1}^{m_{1}} \sum_{s=0}^{n-1} (-1)^{s} \phi_{i-s}^{n} \alpha_{i-1}^{-s} \alpha_{i+1}^{n-s-r} \quad \Theta^{+p} = \prod_{r=0}^{p} \theta_{i;\pm r}, \quad p_{i}^{(m_{1})} \equiv 1, \quad p_{i}^{(r)} = 0, \quad m_{1} + 1 \leq r$$

By the same technique we obtain also

$$(\bar{\alpha}_{i}^{-m})_{x} = \sum_{q=0}^{m-1} (-1)^{q} \Theta_{i}^{-q} p_{i}^{(q+1)\bar{\alpha}_{i-q-1}^{-(m-1-q)}}$$
and corresponding expressions for derivatives of \(\alpha\)–functions with respect to variable \(y\) :

\[
(\alpha_i^\pm m)_y = \sum_{q=0}^{m-1} \Theta_{i+q}^\pm p_i^{q+1} \alpha_{i+q+1}^{m-1-q} \quad (\alpha_i^- m)_y = \sum_{q=0}^{m-1} (-1)^q \Theta_{i-q}^- p_i^{q+1} \alpha_{i-q-1}^{m-1-q}
\]  

(18)

where

\[
p_i^{(r)} = \sum_{n=1}^{m_2} \sum_{t=0}^{n-1} (-1)^t \phi_n^{\bar{t}} \alpha_{i-s}^{n-s-t}, \quad \bar{p}_i^{(m_2)} \equiv 1, \quad \bar{p}_i^{(r)} = 0, \quad m_2 + 1 \leq r
\]

Now we describe the main steps of calculations of the mixed derivative \(\frac{\partial^2 \ln(i | K | i)}{\partial x \partial y}\). Using the main equations (14), it is possible (similar to calculations above) to present this derivative in the form of determinant of the second order and apply to it the first Jacobi identity (10). In such way we obtain:

\[
\frac{\partial^2 \ln(i | K | i)}{\partial x \partial y} = \langle i > - 2 \sum_{n=1}^{m_2} \sum_{s=0}^{n-1} \sum_{m=1}^{m_2} \sum_{t=0}^{m-1} (-1)^{s+t} \phi_{i-s}^{n} \phi_{i-t}^{\bar{m}}
\]

\[
(\hat{R}_m^{+} (X_{i+1}^{+}))(\hat{T}_n^{+} (X_{i-1}^{+})), (j + 1 | K | j + 1)
\]

\[
(\hat{R}_m^{-} (X_{i+1}^{-}))(\hat{T}_n^{+} (X_{i-1}^{-})), (j - 1 | K | j - 1)
\]

Application to both factors of the last sum recurrent relations (16) allows to transform it to the form:

\[
\sum_{n=1}^{m_1} \sum_{s=0}^{n-1} \sum_{m=1}^{m_2} \sum_{t=0}^{m-1} (-1)^{s+t} \phi_{i-s}^{n} \phi_{i-t}^{\bar{m}} (\theta_i)^{-1} \Theta_i^{-p} \phi_{i-s}^{(t-p)} \phi_{i-t}^{\bar{m}} \sum_{q=0}^{p} \Theta_i^{\bar{t}+q} \alpha_{i-t}^{(m-t-q-1)} \alpha^{(n-s-q-1)}
\]

and at last changing in the last sum the order of summation we come to the final expression:

\[
(\theta_i)^{-1} \sum_{p=0}^{p+q \leq M \in \{m_1-1,m_2-1\}} \Theta_i^{-p} \Theta_i^{(p+q+1)} p_i^{(p+q+1)} p_i^{(p+q+1)}
\]

where all functions involved are defined above.

The knowledge of the explicit expressions for derivatives of \(\alpha\) and \(\bar{\alpha}\) functions with respect correspondingly to \(y, x\) coordinates allow to calculate the derivatives of \(\bar{p}_i^{(r)}\) and \(\bar{p}_i^{(r)}\) functions and obtain the closed system of equations ( or identities) for unknown functions \(p_i^{(r)}, \bar{p}_i^{(r)}\) and \(< i >\) (with the corresponding boundary conditions):

\[
\frac{\partial \bar{p}_i^{(r)}}{\partial x} = \sum_{q=1}^{m_2-r} (\Theta_{i+q}^{+} p_i^{q+r} - \Theta_{i-q}^{-} p_i^{r-q})
\]

\[
\frac{\partial^2 \ln(i | K | i)}{\partial x \partial y} = \theta_i^{-1} \sum_{p=0}^{p+q \leq M \in \{m_1-1,m_2-1\}} \Theta_i^{-p} \Theta_i^{(p+q+1)} p_i^{(p+q+1)} p_i^{(p+q+1)}
\]

(19)

\[
\frac{\partial \bar{p}_i^{(r)}}{\partial y} = \sum_{q=1}^{m_1-r} (\Theta_{i+q}^{+} p_i^{q+r} - \Theta_{i-q}^{-} p_i^{r-q})
\]

In paper [3] the system (13) was called as UToda\((m_1, m_2)\) system and we preserve for it this name, keeping in mind necessary modifications which was done in [3].
We can forget about the boundary conditions and consider the lattice system (19) as infinite one, where index \( i \) takes all natural values (positive and negative ones).

In this case (19) can be considered as some mapping – the law with the help of which some number of initial functions are connected with the same number of the finally ones. We would like to clarify situation on the concrete examples of UToda(\( m_1, m_2 \)) lattices with the lowest numbers of \( m_1, m_2 \).

4.1 UToda(1, 1)

In this case (19) is equivalent to the chain of equations for the single unknown function \(< i >\) or \( \theta_i \) (there are many other equivalent forms of the usual Toda lattice):

\[
\frac{\partial^2 \ln < i >}{\partial x \partial y} = \frac{(i - 1) < i + 1 >}{< i >^2}, \quad \frac{\partial^2 \ln \theta_i}{\partial x \partial y} = \theta_{i+1} - 2\theta_i + \theta_{i-1}
\]

The initial functions in this case are the two functions \( \phi_1 = \theta_i, \phi_2 = \theta_{i-1} \). The final ones are \( \tilde{\phi}_1 = \theta_{i+1}, \tilde{\phi}_2 = \theta_i \) and the corresponding mapping takes the form:

\[
\tilde{\phi}_1 = \frac{\partial^2 \ln \phi_1}{\partial x \partial y} + 2\phi_1 - \phi_2, \quad \tilde{\phi}_2 = \phi_1
\]

4.2 UToda(1, 2)

In this case (19) takes the form of two chain equation for two unknown functions \( \theta_i, p_i^{(1)} \) in each point of the lattice:

\[
\frac{\partial^2 \ln \theta_i}{\partial x \partial y} = \theta_{i+1} p_{i+1}^{(1)} - 2\theta_i p_i^{(1)} + \theta_{i-1} p_{i-1}^{(1)}, \quad \frac{\partial p_i^{(1)}}{\partial y} = \theta_{i+1} - \theta_{i-1}
\]

The last system may be rewritten in the mapping form. Four initial functions \( \phi_1 = \theta_i, \phi_2 = \theta_{i-1}, \phi_3 = p_i^{(1)}, \phi_4 = p_{i-1}^{(1)} \) are connected with the four final ones \( \tilde{\phi}_1 = \theta_{i+1}, \tilde{\phi}_2 = \theta_i, \tilde{\phi}_3 = p_{i+1}^{(1)}, \tilde{\phi}_4 = p_i^{(1)} \) with the help of the rule:

\[
\tilde{\phi}_4 = \phi_3, \quad \tilde{\phi}_3 \tilde{\phi}_1 = \frac{\partial^2 \ln \phi_1}{\partial x \partial y} - \phi_2 \phi_4 + 2\phi_1 \phi_3, \quad \tilde{\phi}_2 = \phi_1, \quad \tilde{\phi}_1 = \phi_2 + \frac{\partial \phi_3}{\partial y}
\]

In the general case UToda(\( m_1, m_2 \)) substitution connect \( 2^{m_1+m_2-1} \) initial functions with the same number of the final ones.

The remarkable property of UToda(\( m_1, m_2 \)) mappings consist in their integrability. This means that corresponding to them symmetry equation possesses the infinite number of non-trivial solutions. Each solution of the symmetry equation initiate the completely integrable system invariant with the respect of transformation of UToda(\( m_1, m_2 \)) substitution [4]. By this property all such systems are united into corresponding integrable hierarchy.

5 Evolution parameters and the Integrable Hierarchies

In this section we introduce the parameters of evolution and explain the way of constructing the evolution type systems of equations together with their explicit soliton-like solutions.
such constructed systems are invariant with the respect to UToda($m_1, m_2$) substitutions and belong to the integrable hierarchy with the same title.

We will assume that arbitrary functions, which enter in equations for $M^\pm$ elements (2) in their turn depend on some additional $t_{n_2}, \bar{t}_{n_1}$ (left-right time) parameters in such a way that $M^\pm$ elements satisfy additional equations selfconsistent with (3):

$$\frac{\partial M_+}{\partial t_{n_1}} = \sum_{s=0}^{n_1} P^+_s M_+, \quad \frac{\partial M_-}{\partial t_{n_2}} = \sum_{s=0}^{n_2} R^{-s} M_-$$

(20)

and where now $P^+_s(y, \bar{t}_{n_1}), R^{-s}(x, t_{n_2})$ are the functions of their arguments taking values in subspaces with $\pm s$ graded indexes. Of course, now in (2) dependence on involved in it values is the following $A^+_s(y, \bar{t}_{n_1}), B^{-s}(x, t_{n_2})$.

Further content of the present section we divide on four parts. In two first ones we demonstrate in what connection are equations (2) and (20) with the systems of integrable hierarchy. In the third one we present the explicit solution of the problem of self-consistency of (2) and (20). In fourth part we briefly describe the construction of multi-soliton type solutions of the systems of Darboux–Toda (D–T) hierarchy.

To make material more comprehensive and understandable we begin from the two-dimensional Darboux–Toda hierarchy (by the name of corresponding integrable mapping) all systems of which are invariant with respect to transformation of usual Toda chain (UToda(1,1)) for which solution of the problem was obtained before [8] with the help of the direct solution of the corresponding symmetry equation.

5.1 The integrable systems of D–T hierarchy

Comparing (3) and (20) we conclude, that with respect to the pair of the space coordinates $(x, y)$ the corresponding matrix elements of group element $K$ satisfy equations of the usual Toda lattice:

$$\frac{\partial^2 \ln < i >}{\partial x \partial y} = \theta_i$$

(21)

With respect to argument pair $(\bar{t}_k, x)$ (time and space, respectively) we have UToda $(k,1)$ chain describing by the system:

$$\frac{\partial p^{(s)}_i}{\partial x} = \theta_{i+s} p^{(s+1)}_i - \theta_{i-1} p^{(s+1)}_{i-1}, \quad \frac{\partial^2 \ln < i >}{\partial x \partial \bar{t}_k} = \theta_i p^{(1)}_i, \quad 1 \leq s \leq (k-1), \quad p^{(k)}_i = 1$$

(22)

And finally with respect to argument pair $(y, t_k)$ we have UToda(1, $k$) chain with the corresponding equations:

$$\frac{\partial p^{(s)}_i}{\partial x} = \theta_{i+s} p^{(s+1)}_i - \theta_{i-1} p^{(s+1)}_{i-1}, \quad \frac{\partial^2 \ln < i >}{\partial y \partial t_k} = \theta_i p^{(1)}_i, \quad 1 \leq s \leq (k-1), \quad p^{(k)}_i = 1$$

Resolving the equations of zero curvature with respect to remaining pairs $(x, t_{n_2})$ and $(y, \bar{t}_{n_1})$ will be done in the third subsection.

Here we would like to emphasize that in all three examples above the explicit dependence of the functions $p^{(s)}_i, \bar{p}^{(s)}_i$ on matrix elements of the single group element $K$ is not the same and determines by the corresponding formulae (19) of the previous section.

Instead of the general consideration we consider several simplest examples from which the situation in the general case become absolutely clear (we also restrict ourselves by the choice of the left time parameter).
5.1.1  \( k=2 \)

Integrating over the argument \( x \) the first and second equations (22) we obtain consequently:

\[
\frac{\partial \ln \langle i \rangle}{\partial \bar{t}_k} = \int^{x'} dx' \theta_i \bar{p}_i^{(1)}(\theta_{i+1} - \theta_{i-1})
\]

Introducing the functions \( v_i = \langle i+1 \rangle - \langle i \rangle, u_i = \langle i-1 \rangle - \langle i \rangle \), we obtain the following system of equalities for them \((\theta_i = u_i v_i)\):

\[
\frac{\partial u_i}{\partial \bar{t}_2} = \int^{x'} dx' \theta_i \int^{x'} dx''(\theta_{i+1} - \theta_{i-1}) - \int^{x'} dx' \theta_i \int^{x'} dx''(\theta_i - \theta_{i-2})
\]

\[
\frac{\partial v_i}{\partial \bar{t}_k} = \int^{x'} dx' \theta_{i+1} \int^{x'} dx''(\theta_{i+2} - \theta_{i}) - \int^{x'} dx' \theta_i \int^{x'} dx''(\theta_{i+1} - \theta_{i-1})
\]

(23)

The last system is exactly Davey-Stewartson one [9] rewritten in terms of discrete transformations shifts.

Indeed, the UToda(1,1) integrable mapping (21), rewritten in the terms of \((u, v)\) functions, takes the form:

\[
\tilde{u} = v^{-1} \quad \tilde{v} = v(\ln v)_{xy}
\]

(24)

and was called before as Darboux–Toda integrable mapping [8].

Performing all necessary changes of variables in (23) with the help of (24), we come finally to \((u_i \rightarrow u, v_i \rightarrow v)\):

\[
-\dot{u} + u_{yy} + 2u \int dx(uv)_y = 0 \quad \dot{v} + v_{yy} + 2v \int dx(uv)_y = 0
\]

(25)

This is exactly Davey-Stewartson system in its original form [3]. In one-dimensional limit - usual nonlinear Schrodinger equation.

In [8] it was obtained the sequence of the solutions of symmetry equation corresponding to Darboux-Toda integrable substitution and (23) is one them.

5.1.2  \( k=3 \)

Literally repeating the calculations of the the last subsubsection, we have consequently:

\[
\frac{\partial \ln \langle i \rangle}{\partial \bar{t}_3} = \int^{x'} dx' \theta_i \bar{p}_i^{(1)} = \int^{x'} dx' \theta_i \int^{x'} dx''(\theta_{i+2} \bar{p}_i^{(2)} - \theta_{i-1} \bar{p}_i^{(2)})
\]

Substituting into the last expression \( \bar{p}_i^{(2)} \) in terms of \( \bar{p}_i^{(3)} \) and keeping in mind that it is necessary to put \( \bar{p}_i^{(3)} = 1 \) in this case, we finally obtain for derivative \( \frac{\partial \ln \langle i \rangle}{\partial \bar{t}_3} \) expression consisting of four terms. Equations (equalities) for \( u_i, v_i \) in its turn contain eight terms with three repeated integrals and exactly coincide with those from the paper [8].

Now the strategy of calculations in the case of arbitrary \( k \) is absolutely clear and final result is the same as in the cited paper [3].

We bring to the attention of the reader the fact that constructed systems are satisfied under arbitrary choice of the index \( i \) in them. This means that constructed evolitional-like systems all are invariant with respect to transformation of Darboux-Toda substitution (24).
5.2 The integrable systems of UToda($m_2, m_1$) hierarchy

In the general case as a consequence of the condition of self–consistency of the equations (2) and (20) (for definiteness we keep in mind the case of “left” time) with the respect the pair of the “space” coordinates ($y, x$) usual UToda($m_2, m_1$) substitution (19) arises and with the respect of the space–time pair ($t_k, x$) the UToda($k, m_1$) ones (it is necessary in (19) only change $m_2 \rightarrow k$).

Explicit solution of both these systems (with fixed ends) are constructed from the matrix elements of the single group element $K$ by the rules of the previous section. From the corresponding formulae reader can see that the functions $< i >, p_i^{(s)}$ dependence on matrix elements of $K$ respect of the space–time pair ($\bar{t}_k, x$) the UToda($k, m_1$) ones (it is necessary in (19) only change $m_2 \rightarrow k$).

We rewrite UToda($k, m_1$) system in useful for us notations (integrated by space coordinate $x$ first two its equations):

$$\bar{p}_i^{(r)} = \int^x dx' \sum_{q=1}^{m_2-r} (\Theta_{i+r}^{(q-1)} p_i^{(q)} - \Theta_{i}^{-1} p_i^{(q-r)})$$

$$\frac{\partial \ln(< i | K | i>)}{\partial t_k} = \int^x dx' \Theta_i^{p+q} \sum_{p=0, q=0}^{p+q \leq Min(k-1, m_1-1)} \Theta_i^{-p} \Theta_i^{q} p_i^{(p+q+1)} p_i^{(p+q+1)}$$

(26)

Keeping in mind the condition $\bar{p}_i^{(k)} = 1$ and "nilpotent" character of the first system of equations we can resolve the last one and obtain the explicit expressions for all $\bar{p}_i^{(s)}$ functions in form of repeated integrals on space coordinate $x$ with integrand functions always be some functionals of the functions $< i >, p_i^{(s)}$. Substituting obtained in such way expressions for $\bar{p}_i^{(s)}$ functions into two last systems of (26), we find the explicit form of derivatives of $< i >, p_i^{(s)}$ functions with respect to the time argument.

In the same way it is possible to resolve first system of equations of UToda($m_2, m_1$) substitution with respect to $\bar{p}_i^{(s)}$ functions as functionals of the same type, as has made above on $< i >, p_i^{(s)}$ functions. Now in the last iteration procedure $\bar{p}_i^{(m_2)} = 1(!)$. Knowledge of the time derivatives of $< i >, p_i^{(s)}$ functions allows to reconstruct the time derivatives of all $\bar{p}_i^{(s)}$ functions.

So we have the time derivatives of all functions involved into UToda($m_2, m_1$) substitution in terms of functionals of themselves and their discrete shifts of the correspondingly (limited) order. But with the help of equations of UToda($m_2, m_1$) mapping (19) it is always possible to present these shifts in term of exactly of $2^{m_1+m_2-1}$ initial functions and its derivatives up to the definite order.

So we have obtained the system of equalities between the time derivatives of $2^{m_1+m_2-1}$ functions expressed in sufficiently cumbersome functional form (nonlinear and nonlocal simultaneously) on their space derivatives.

Reminding about the way of obtaining we can consider the last system as completely integrable one with known sequence of its soliton–like solutions.
5.3 Solution of the Nilpotent chain system

Up to now we have not resolved only two pairs of nilpotent systems on the point of their self-consistency. For definitly let us consider the time-space pair \((y, \bar{t}_k)\) and restrict ourselves by the case of \(UToda(1,k)\) substitution. We rewrite the corresponding systems (as the combination of the components of the equations (2) and (20)):

\[
(m_+)_i = (\bar{g}_0)^{-1}(\bar{g}_0)_{ik} + \sum_{s=1}^{k} p^{(s)} m_+ \quad (m_+)_y = ((\bar{g}_0)^{-1}(\bar{g}_0)_y + I^{(+1)}) m_+ \quad (27)
\]

where \(\bar{g}_0 \equiv \exp \sum_{k=1}^{r}(h_k \tau_k)\). Maurer-Cartan identity applied to this pair of equations (27), is equivalent to chain like system for unknown functions \(\bar{g}_0\), \(\pi^{(s)} = \bar{g}_0 p^{(s)} \bar{g}_0^{-1}\) (differentiation with respect to argument \(\bar{t}_k\) we denote by ', with respect to \(y\) by ' and for a time put \(\bar{g}_0 \to g\)):

\[
\pi^{(1)}' = g I^{(+1)} g^{-1}, \quad \pi^{(s)}' = [\pi^{(s-1)} g, g I^{(+1)} g^{-1}] \quad (28)
\]

where \(\pi_i^{(+k)} = 1\) and \(I^{(+s)}\) means that in \(p^{(s)}\) all \(p_i^{(+s)} = 1\).

Reminding that \(p^{(s)}\) (and correspondingly \(\pi^{(s)}\)) may be presented as a direct sum of components, we rewrite (28) in component form:

\[
(\pi_i^{(1)})' = (g_i g_i^{-1}) (g_i g_i^{-1})' = \pi_i^{(k-1)} (g_i g_i^{-1}) \pi_i^{(k-1)}
\]

\[
(\pi_i^{(s)})' = \pi_i^{(s-1)} (g_i g_i^{-1}) \pi_i^{(s-1)} \quad 2 \leq s \leq (k-1)
\]

And at the last after identification \(G_i \equiv g_i g_i^{-1} = \exp(\tau_{i+1} - 2\tau_i + \tau_{i-1})\), we come to the final system of equations for determining of the unknown functions \(G_i, \pi_i^{(+s)}\):

\[
(\pi_i^{(1)})' = \dot{G}_i \quad (G_i G_{i+1} \ldots G_{i+k-1})' = \pi_i^{(k-1)} G_i G_{i+k-1} - G_i \pi_i^{(k-1)}
\]

\[
(\pi_i^{(s)})' = \pi_i^{(s-1)} G_i G_{i+s-1} - G_i \pi_i^{(s-1)} \quad 2 \leq s \leq (k-1)
\]

In spite of very complicate on the first look structure of the last chain-like system the solution of it is possible to find in explicit form.

For this purpose let us consider the following linear equation for unknown function \(X\):

\[
\dot{X} = X^{(k)} + A^{(2)} X^{(k-2)} + \ldots + A^{(k)} X \quad (30)
\]

where \(A^{(s)}\) are arbitrary functions of two arguments \(y, \bar{t}_k\).

The following assertion takes place:

Let

\[
X_1 = \phi_1, \quad X_2 = \phi_1 \int^y dy' \phi_2(y'), \quad X_3 = \phi_1 \int^y dy' \phi_2(y') \int^y dy'' \phi_3(y''), \ldots
\]

different solutions of linear equation (30) presented in Frobenious-like form. Then the solution of chain-like system (29) may be written in terms of these solutions as follows:

\[
G_i = \phi_i \equiv \frac{Det_{i+1}(X)}{Det_i(X)}, \quad X_{p,q} = \frac{\partial^{p-1} X_q}{\partial y^{p-1}} \quad (31)
\]

where matrix \(X\) by the form coincides with the matrix of Vronsky determinant. The functions \(\pi_i^{(+s)}\) in its turn may be expressed in terms of repeated integrals with the known integrands \(G_i, \dot{G}_i\) after consequent integration of corresponding "nilpotent" system (29) for them.

Moreover for only ones necessary for further consideration functions \(g_i\) we obtain:

\[
g_i \equiv (\bar{g}_0)_i = Det_i^{-1}(X) \quad (32)
\]

To prove this assertion in the general form we will substitute by the detailed consideration of two examples from which the general case become absolutely clear.
\[ k=2 \]

The system \((29)\) takes the form:

\[
(\pi_i^{(i+1)})' = \dot{G}_i \quad (G_i G_{i+1})' = \pi_i^{(i+1)} G_{i+1} - G_i \pi_{i+1}^{(i+1)}
\]  \( (33) \)

or the system of chain-like equations for only one unknown function \(G_i\):

\[
(G_i G_{i+1})' = \int dy(\dot{G}_i) G_{i+1} - G_i \int dy(\dot{G}_{i+1})
\]  \( (34) \)

Now let us consider the equation \((30)\) for \(k=2\). For \(\phi_{1,2}\) we obtain consequently:

\[
\dot{\phi}_1 = \phi''_1 + A^2 \phi_1, \quad \dot{\phi}_2 = (2\phi^{-1}_1 \phi_2 + \phi_2)'
\]

The fact that \(X_3 = \phi_1 \int dy' \phi_2(y') x_2 \int dy'' \phi_3(y'')\) is equivalent to the substitution \(\phi_2 \rightarrow \phi_2 \int dy \phi_3\). After trivial algebraical manipulations we come to equality:

\[
\phi_2 \int dy \dot{\phi}_3 - \int dy \dot{\phi}_2 \phi_3 = (\phi_2 \phi_3)'
\]

Further substitution \(\phi_3 \rightarrow \phi_3 \int dy \phi_4\) (\(X_4\) is also the solution of the equation \((30)\)) leads to the result:

\[
\phi_3 \int dy \dot{\phi}_4 - \int dy \dot{\phi}_3 \phi_4 = (\phi_3 \phi_4)'
\]

By induction it is not difficult to show that each two consequent functions \(\phi\) are connected by relation:

\[
\phi_i \int dy \dot{\phi}_{i+1} - \int dy \dot{\phi}_i \phi_{i+1} = (\phi_i \phi_{i+1})'
\]

Comparison of the last relation with \((34)\) leads to conclusion, that in the case under consideration:

\[
G_i = \phi_i, \quad \pi_i^{(i+1)} = \int dy (\dot{G}_i)
\]

and the assertion \((31)\) is proved.

\[ k=3 \]

In this case system \((29)\) takes the form:

\[
(\pi_i^{(i+2)})' = \dot{G}_i \quad (G_i G_{i+1} G_{i+2})' = \pi_i^{(i+2)} G_{i+2} - G_i \pi_{i+1}^{(i+1)}
\]  \( (35) \)

or after excluding all functions \(\phi_i^{(1,2)}\) we come to a chain (nonlinear and nonlocal simultaneously) system of equations for functions \(G_i\):

\[
G_{s+2}(y) [ \int^{y'} dy' G_{s+1}(y') \int^{y''} dy'' \dot{G}_s(y'') - \int^{y'} dy' \dot{G}_s(y') \int^{y''} dy'' \dot{G}_{s+1}(y'') ] -
\]

\[
G_s(y) [ \int^{y'} dy' G_{s+2}(y') \int^{y''} dy'' \dot{G}_s(y'') - \int^{y'} dy' G_{s+1}(y') \int^{y''} dy'' \dot{G}_{s+2}(y'') ] = (G_s G_{s+1} G_{s+2})'
\]  \( (36) \)
Now let us consider equation (30) in the case \( k = 3 \):

\[
\dot{X} = X''' + A^2 X' + A^3 X
\]  

(37)

Substituting in it the form of the solution \( X_1, X_2 \) proposed by the assertion (31) we obtain:

\[
\dot{\phi}_1 = \phi_1''' + A^2 \phi_1' + A^3 \phi_1, \quad \dot{\phi}_2 = (3\phi_1^{-1} \phi_1' \phi_2 + 3\phi_1^{-1} \phi_1' \phi_2' + \phi_2'' + A^2 \phi_2)'
\]

The fact that \( X_3 \) is also solution of the same equation equivalent to the change \( \phi_2 \rightarrow \phi_2 \int \phi_3 \) and leads to the following equality:

\[
\phi_2 \int \dot{\phi}_3 - \phi_3 \int \dot{\phi}_2 = (3\phi_1^{-1} \phi_1' \phi_2 \phi_3 + 2\phi_2' \phi_3 + \phi_2'')'
\]

And at last after substitution \( \phi_3 \rightarrow \phi_3 \int \phi_4 \), which is equivalent to proposition that \( X_4 \) is also solution of the same linear equation, we come to equality of our interest:

\[
\phi_2(y)[\int^y dy' \phi_3(y')] \int^y dy' \dot{\phi}_4(y'') - \int^y dy' \dot{\phi}_4(y') \int^y dy' \phi_3(y'')]
\]

\[
[\int^y dy' \phi_2(y') \int^y dy' \phi_3(y'') - \int^y dy' \phi_3(y') \int^y dy' \phi_2(y'')] \phi_4(y) = (\phi_2(y) \phi_3(y) \phi_4(y))'
\]

By the induction this equality can be continued on all three arbitrary consequent functions \( \phi_s(y), \phi_{s+1}(y), \phi_{s+2}(y) \), solving in explicit form the system (36) and proving the proposed above assertion (31).

### 5.4 Multi-soliton like solutions of D–T hierarchy

From the results of the last section it follows the deep connection between the general solution of two-dimensional Toda lattice with fixed ends (what is equivalent to exploiting of finite-dimensional \( A_n \) algebra) and the particular soliton-like solutions of the systems of D–T hierarchy.

To obtain such kind of solutions of the systems of D–T hierarchy, which may be enumerated by index \( k \) in (30), it is necessary only substitute into the general solution of Toda chain encoded in functions \( \langle i \rangle \) instead of arbitrary functions \( \bar{g}_0(y), (g_0(x)) \) the functions \( \bar{g}_0(y, \tilde{t}_k) \) from (32) and corresponding expression for \( g_0(x, t_k) \).

As a corollary each pair of functions

\[
\begin{align*}
\text{\textbf{u}}^k & \equiv u^k_i = \langle i - \frac{1}{k} \rangle_i, \\
\text{\textbf{v}}^k & \equiv v^k_i = \langle i + \frac{1}{k} \rangle_i
\end{align*}
\]

satisfy the \( k \)-th system of the integrable D–T hierarchy simultaneously with respect to left and right time parameters. The first nontrivial example of this hierarchy \( (k = 2) \) is the D–S system (23) in the form of the discrete shifts or (25) in the form of the usual derivatives.

Some additional consideration it is necessary to extract the solutions invariant with respect to some inner authomorphism of the problem or more precisely of D–T substitution (in this connection see [12], [13], [14]).
6 Possible generalization on the multi-dimensional case

Let for some gradings (this can be satisfied far not always) it is possible represent “lagrangians” $L^{(+m_1)}(y), L^{(-m_2)}(x)$ from (2) in the block form $L^{(+m_1)} = \sum_s L_s^{(+m_1)}$, where elements of different blocks are mutually commutative $[L_s^{(+m_1)}(y), L_s^{(+m_1)}(y')] = 0$. And the same may be true in some decomposition is true with respect to the second lagrangian.

Then the solution of (2) may be presented in the form of the product of $p_1$ ($p_1$ is the number of blocks of the above type) mutually commutative factors:

$$M^+(y) = \prod_s M_s^+(y)$$

Let us instead of $M^+(y)$ consider the new element $M^+(y_1,...y_{p_1})$ depending on $p_1$ arguments $y_s$, determined by relation:

$$M^+(y_1,...y_{p_1}) = \prod_s M_s^+(y_s)$$

The same procedure it is possible to realize with $M^-$ and change it on the group-valued function on $p_2$ independent arguments:

$$M^-(x_1,...x_{p_2}) = \prod_s M_s^-(x_s)$$

Now we construct as in the second section the group-valued element

$$G = N_-M_+ = g_0N_+M_-$$

and with respect to $p_1$ and $p_2$ algebra-valued functions

$$G_{x_i}G^{-1} = (N_-)_{x_i}N^{-1} = \sum_s R_{x_i}^{-s}(x,y), \quad 1 \leq s \leq m_2$$

$$G_{y_n}G^{-1} = (g_0)_{y_n}g_0^{-1} + g_0(N_+)_{y_n}N_+^{-1}(g_0)^{-1} = (g_0)_{y_n}g_0^{-1} + \sum_s R_{y_n}^{+s}(x,y), \quad 1 \leq s \leq m_1$$

come to the same conclusions as it was done before with respect to the case of the single $x$ and $y$ arguments.

Since to go further it is necessary to understand from pure algebraical point of view the possible block structure of described above type as a direct corollary of the properties of the algebra and chosen grading in it. Now we are not ready to solve this problem in the whole measure.

7 Outlook and further perspectives

The main result of the present paper consists in proposition that the theory of integrable systems in $(1+2)$ dimensions (and of course in $(1+1)$ case as a direct reduction of the previous one) is nothing more than the equations of equivalence from the representation theory of semisimple
algebras and groups encoded in some nontrivial way. It is true at least in the framework of integrable substitutions considered here.

We have the chain of the following consequent steps: Graded Lee algebras – Representation theory – Integrable mappings (substitutions) and, at last, evolution–type Hierarchies of Integrable systems together with their (soliton-like) solutions. This chain after its consequent realization is nothing more than the theory of integrable systems belonging to the same hierarchy in \((2 + 2)\) dimensions (keeping in mind the existence of the "left" and "right" time parameters).

On the other side, the theory of integrable systems, in particular, in \((1 + 1)\) and \((1 + 2)\) dimensions, was up to the latest time, the independent branch of mathematical physics with its own technique and methods of investigations [15]. This means only that there are many independent methods for investigation of the representation theory of the semisimple algebras and groups. The methods applied in the theory of integrable systems were ones among many other possible ones.

The researches worked in this area have rediscovered for many times and by absolutely independent methods the different forms of the two Jacobi identities (10) and (11) and numerous corollaries from them. Of course, in the case of \(A_n\) algebra and the principal grading in it it is possible to perform all of this job in the language of Jacobi identities in the determinant form, which on the first look have no connection to representation theory and may be performed without any mention about its existence.

In context of the material of the present paper it arised many other problems some ones of which we want to emphasize.

First of all (and this is sufficiently obvious) to try to generalize the presented construction to the case of arbitrary semisimple algebra together with the principle (embedding) grading in it. This problem may be identified as the case of Abelian Integrable Mapping in the framework of semisimple algebras. For solution of this problem it is necessary the more detailed information about the structure of the algebra in the case of the existence of the repeated roots as for its fundamental representations in the case of arbitrary semisimple algebra. In this connection see Appendix.

Secondly the problem of generalization to the case of arbitrary grading of semisimple algebras arises. In general case the subspace with zero graded index become noncommutative algebra by itself and it leads to additional technical difficulties. The typical example is the simplest case of so called matrix Toda lattice considered from the different points of view in the papers of the different groups of authors [11], [13], [14].

And the last problem in framework of our comments, how this construction works in the case of Lie algebras of the general position and to what kind of integrable substitutions it leads? This is a subject of additional and non trivial (as it possible to assume) investigation.

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Appendix: GToda(2, 2; s, ¯¯) lattice

The title of this section is decoded as follows: we consider arbitrary semisimple algebra together with the principle grading in it and in the main equations (2) restrict ourselves by the choice \( m_1 = m_2 = 2 \). The sense of additional parameters become clear from what follows (see [6]).

In this case “Lagrangians” take the form

\[
L^2_{\pm} = (h\tau_{\pm} + \sum X_{\alpha}^\pm \varphi_{\alpha}^{\pm 1} + \frac{1}{2} \sum [X_{\alpha}^\pm, X_{\beta}^\pm] \varphi_{\alpha,\beta}^{\pm 2}) \varphi_{\alpha,\beta}^{\pm 2} = -\varphi_{\beta,\alpha}^{\pm 2}
\]

The action of \( L^2_\pm \) on the state vector \(|i\rangle\) is as follows:

\[
L^2_\pm |i\rangle = (\varphi_{i}^{-1} + \sum_{k_{a},i \neq 0} \varphi_{a,i}^{-2} X_{a}^{-}) X_{i}^{-} |i\rangle
\]

Calculation of derivatives of \( \alpha^1, \bar{\alpha}^1 \) (14) by the same technique as in the main text (section 4) leads to the result:

\[
\frac{\partial \bar{\alpha}^1_i}{\partial x} = \alpha_i^{-1} \frac{\partial \alpha^1_i}{\partial x} \quad \frac{\partial \bar{\alpha}^1_i}{\partial y} = \alpha_i^{-1} \frac{\partial \alpha^1_i}{\partial y}
\]

we obtain the following closed system of equalities for these functions together with \( <i> \):

\[
\frac{\partial p^1_i}{\partial y} = \sum_{j} K_{j,i} \varphi_{j,i}^{\pm 2} \theta_j p^1_j \quad \frac{\partial p^1_i}{\partial x} = \sum_{j} K_{j,i} \varphi_{j,i}^{\pm 2} \theta_j p^1_j
\]

\[
A.1
\]

The second mixed derivatives of \( \ln <i> \) is calculated without any difficulties by the same way as first equations of (A.1).

Comparing (19) (in the case \( m_1 = m_2 = 2 \)) with (A.1) after substitution in the last system the Cartan matrix of \( A_n \) algebra, shows the whole identity of this systems under additional choice of arbitrary functions \( \varphi_{j,i}^{\pm 2} \).

For simplicity in the main text of the paper we put \( \bar{p}^1_i = p^1_i = 1 \). In fact this is some additional assumption and we want now to get rid of it.

Indeed the main equations (2) are obviously invariant with respect to gauge transformation with the group element \( \exp(h\tau_{\pm}) \) \( (\tau_+ \equiv \tau_+(y) \quad \tau_- \equiv \tau_-(x)) \). With the help of such transformations all \( \varphi_{a,\beta}^{\pm 2} \) may be evaluated to a constant values. Let us work in such gauge, where they take zero and unity values in arbitrary order and denote such sequences of these parameters by the symbol \( \bar{s}, s \). The last finally explain the notation GToda(2, 2; s, ¯¯) in the title of this Appendix. The arising systems are essentially different as by the form of the equations by itself also as by the form of their general solutions (see in this connection [3]).
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