SU(2|1) mechanics and harmonic superspace

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Received 28 August 2015, revised 28 October 2015
Accepted for publication 18 November 2015
Published 5 February 2016

Abstract
We define the worldline harmonic SU(2|1) superspace and its analytic sub-space as a deformation of the flat \( \mathcal{N} = 4, d = 1 \) harmonic superspace. The harmonic superfield description of the two mutually mirror off-shell \((4, 4, 0)SU(2|1)\) supermultiplets is developed and the corresponding invariant actions are presented, as well as the relevant classical and quantum supercharges. Whereas the \( \sigma \)-model actions exist for both types of the \((4, 4, 0)\) multiplet, the invariant Wess–Zumino term can be defined only for one of them, thus demonstrating non-equivalence of these multiplets in the \(SU(2|1)\) case as opposed to the flat \( \mathcal{N} = 4, d = 1 \) supersymmetry. A superconformal subclass of general \(SU(2|1)\) actions invariant under the trigonometric-type realizations of the supergroup \(D(2, 1; \alpha)\) is singled out. The superconformal Wess–Zumino actions are shown to possess an infinite-dimensional super-symmetry forming the centerless \( \mathcal{N} = 4 \) super Virasoro algebra. We solve a few simple instructive examples of the \(SU(2|1)\) supersymmetric quantum mechanics of the \((4, 4, 0)\) multiplets and reveal the \(SU(2|1)\) representation contents of the corresponding sets of the quantum states.

Keywords: supersymmetry, superfields, deformation, superconformal mechanics, harmonic superspace

1. Introduction

Motivated by a recent interest in theories with a curved rigid supersymmetry (see e.g. [1–3]), in [4–6] we defined and studied a new type of supersymmetric quantum mechanics (SQM) based on the worldline realizations of the supergroup \(SU(2|1)\) and its central extension. It can be viewed as a deformation of the standard flat \( \mathcal{N} = 4 \) supersymmetric mechanics by a mass
parameter \( m \). We defined the worldline \( SU(2|1) \) superfields, which are carriers of the off-shell \( SU(2|1) \) multiplets with the \( d = 1 \) field contents \((1, 4, 3)\) and \((2, 4, 2)\), constructed the invariant superfield and component actions for them and explored the classical and quantum properties of the corresponding \( SU(2|1) \) invariant \( d = 1 \) systems. A few interesting features of these deformed SQM models were revealed. In particular, the SQM models associated with the multiplet \((1, 4, 3)\) reproduce the ‘weak supersymmetry’ systems studied in [7]. The invariant \( \sigma \)-model Lagrangians of the chiral multiplet \((2, 4, 2)\) contain, along with the standard kinetic, is achieved in the framework of the harmonic multiplets. It is manifested in the property that an \( \sigma \)-model Lagrangians, as well as that of another important multiplet \( \sigma \)-model in the \( (2, 4, 2) \) models. Recently, we studied superconformal properties of the \( \sigma \)-model Lagrangians of the chiral multiplet \((2, 4, 2)\), as compared to the \( \sigma \)-model Lagrangians of the chiral multiplet \((1, 4, 3)\) [6] and found that the formulation in the \( SU(2|1) \) superspace automatically yields the trigonometric realizations of the most general \( \mathcal{N} = 4, d = 1 \) superconformal group \( D(2, 1; \alpha) \) in the conformal subclass of such models. We established a simple criterion under which one or another \( SU(2|1) \) invariant action possesses superconformal \( D(2, 1; \alpha) \) symmetry.

As was argued in [8] in the component approach and in [9] in the superfield language, the basic multiplet of \( \mathcal{N} = 4, d = 1 \) supersymmetry is the so-called ‘root’ multiplet \((4, 4, 0)\). All other multiplets and the SQM models associated with them can be obtained from the root multiplet and the associate SQM models by some well-defined procedures: either by a sort of Hamiltonian reduction with respect to some isometries of the on-shell \((4, 4, 0)\) Lagrangians, or by gauging these isometries in the off-shell superfield approach \([9–11]\). The natural superfield description of the multiplet \((4, 4, 0)\), as well as that of another important multiplet \((3, 4, 1)\), is achieved in the framework of the harmonic \( \mathcal{N} = 4, d = 1 \) superspace \([12]\). In particular, the \( d = 1 \) harmonic superspace approach allows one to establish relations between various \( \mathcal{N} = 4, d = 1 \) multiplets in the manifestly off-shell supersymmetric way and to understand the general target geometry of the \((4, 4, 0)\) superfield models \([13]\).

In view of the crucial role of the harmonic superspace approach \([14, 15]\) in the flat \( \mathcal{N} = 4 \) SQM models it is natural to expect that it could be of equal importance for \( SU(2|1) \) SQM models as deformations of the \( \mathcal{N} = 4 \) ones. The existence of the worldline harmonic coset in the supergroup \( SU(2|1) \) was observed in \([4]\) \(^1\). The main purpose of the present paper is to work out, in full generality, the \( d = 1 \) harmonic superspace formalism for \( SU(2|1) \), to describe the deformed analogs of the flat \((4, 4, 0)\) multiplet within this framework and to construct the relevant SQM models, in both the superfield and the component forms. As in the case of the previously considered \( SU(2|1) \) multiplets, the basic new features of these models as compared to the flat \( \mathcal{N} = 4 \) SQM of the multiplet \((4, 4, 0)\) are the inevitable presence of the oscillator type potential terms in the \( \sigma \)-model actions and the appearance of the induced WZ terms in the case of non-zero intrinsic \( U(1) \) charge. An interesting phenomenon having no analog in the flat case is the failure of the equivalence between the SQM models associated with the mutually mirror \((4, 4, 0)\) multiplets. It is manifested in the property that an \( SU(2|1) \) invariant superfield WZ term can be defined only for one sort of such multiplets.

\(^1\) Some other kind of harmonic extensions of \( SU(2|1) \) (in application to the field theory on the sphere \( S^3 \)) was considered in \([16]\).
The paper is organized as follows. In section 2, we define the harmonic $SU(2|1)$ superspace $\zeta_0$ and its analytic subspace $\zeta_{(A)}$ as the proper supercosets of the harmonic extension of $SU(2|1)$, following the general construction of [15]. We give the explicit expressions for the relevant covariant derivatives, both Grassmann and harmonic, in the basis $\zeta_0$ and define the $SU(2|1)$ analyticity conditions. In sections 3 and 4, we consider the $SU(2|1)$ harmonic superfields $q^{\alpha a}$ and $Y^A$, $\bar{Y}^A$ describing two types of the $(4, 4, 0)$ supermultiplets which can be defined in the $SU(2|1)$ case. The general $\sigma$-model-type terms for both multiplets are constructed and it is shown that an external $SU(2|1)$ invariant WZ term can be defined only for the mirror multiplet $(4, 4, 0)$, in a crucial distinction from the flat $\mathcal{N} = 4, d = 1$ case. We present general expressions for the relevant supercharges, in the classical and quantum cases, and explicitly find the spectrum of the corresponding Hamiltonian for a few simple models. In section 5, we single out the superconformal subclass of the $SU(2|1)$ SQM models for both types of the $(4, 4, 0)$ multiplet and show that they exhibit the trigonometric realization of the superconformal group $D(2, 1; \alpha)$, like the SQM models considered in [6, 17]. The superconformal WZ terms reveal an infinite-dimensional superconformal symmetry corresponding to some centerless $\mathcal{N} = 4$ super Virasoro algebra. In section 6, we summarize the results and outline some further directions of study. Some technical details are collected in appendices A, B and C.

2. $SU(2|1)$ harmonic superspace

2.1. Superalgebra

We start from the standard form of the superalgebra $su(2|1)$:

\[
\{Q^i, \bar{Q}_j\} = 2 m l^i + 2 \delta^i_j \bar{H}, \quad \begin{bmatrix} I^i, J^j \end{bmatrix} = \delta^j_i I^i - \delta^i_j I^j, \\
\begin{bmatrix} I^i, \bar{Q}_j \end{bmatrix} = \frac{1}{2} \delta^i_j \bar{Q}_i - \delta^i_j \bar{Q}_j, \quad \begin{bmatrix} I^i, J^k \end{bmatrix} = \delta^k_i J^k - \frac{1}{2} \delta^k_i Q^k, \\
\begin{bmatrix} \bar{H}, \bar{Q}_i \end{bmatrix} = \frac{m}{2} \bar{Q}_i, \quad \begin{bmatrix} \bar{H}, J^k \end{bmatrix} = -\frac{m}{2} Q^k.
\]

(2.1)

All other (anti)commutators are vanishing. It is a deformation of the standard $\mathcal{N} = 4, d = 1$ ‘Poincaré’ superalgebra. The bosonic subalgebra consists of the $SU(2)$ symmetry generators $I^i$ and the $U(1)$ symmetry generator $\bar{H}$. In the limit $m = 0$, the generators $I^i$ become those of the $SU(2)$ automorphism group of the standard $\mathcal{N} = 4, d = 1$ superalgebra, while $\bar{H}$ turns into the $d = 1$ time-translation generator.

Actually, the flat $\mathcal{N} = 4, d = 1$ superalgebra has an additional automorphism group $SU'(2)$ which rotates the $Q$ and $\bar{Q}$ generators among each other\(^2\). Upon deformation to the superalgebra (2.1), one of these $SU'(2)$ generators, which we denote $F$, still survives as an external $U(1)$ automorphism symmetry ($R$-symmetry) of the deformed superalgebra. So one can extend (2.1) by the generator $F$ possessing non-zero commutation relations with the supercharges only [1]:

\[
\begin{bmatrix} F, \bar{Q}_i \end{bmatrix} = -\frac{1}{2} \bar{Q}_i, \quad \begin{bmatrix} F, Q^k \end{bmatrix} = \frac{1}{2} Q^k.
\]

(2.2)

\(^2\) In what follows, the primed indices $i', j'$ will be associated with the $SU'(2)$ doublets.
After introducing the new basis in this extension of (2.1),
\[ \hat{H} = H - mF, \]
(2.3)
one can pass to the centrally extended superalgebra \( \hat{su}(2|1) \) defined by the following non-
vanishing (anti)commutators:
\[
\{Q^i, \tilde{Q}_j\} = 2mI^j_i + 2\delta^i_j(H - mF),
\quad \left[I^i_j, I^k_l\right] = \delta^i_l I^j_k - \delta^i_k I^j_l, \\
\left[I^i_j, \tilde{Q}_k\right] = \frac{1}{2} \delta^i_j \tilde{Q}_k - \delta^i_k \tilde{Q}_j,
\quad \left[I^i_j, Q^k\right] = \delta^i_j Q^k - \frac{1}{2} \delta^i_k Q^j, \\
\left[F, \tilde{Q}_i\right] = -\frac{1}{2} \tilde{Q}_i,
\quad \left[F, Q^i\right] = \frac{1}{2} Q^i.
\]
(2.4)
Its bosonic sector contains the central charge generator \( H \) (commuting with all other
generators) and the \( U(2)_{\text{int}} \) generators \( I^i_j \) and \( F \). Just this superalgebra was our point of
departure in [4].

2.2. Harmonic \( SU(2|1) \) superspace as a coset superspace

Harmonic \( d = 1 \) superspace for the supergroup \( \tilde{SU}(2|1) \) as a coset manifold of the latter can be
defined using the same tricks as in the construction of the harmonic superspace for the
\( N = 2, d = 4 \) superconformal group in [15]. The basic steps are passing to the new basis in (2.4),
in which all generators are labeled by their \( U(1) \) charges, and introducing an extra automorphism
group \( SU(2)_{\text{ext}} \) which uniformly rotates all the doublet indices of the generators in (2.4).

Using the notations
\[
Q^i \equiv Q^+, \quad Q^2 \equiv Q^-,
\quad \tilde{Q}_1 \equiv \tilde{Q}^-, \quad \tilde{Q}_2 \equiv -\tilde{Q}^+,
\quad I^{++} \equiv I^1_1, \quad I^{--} \equiv I^2_2, \quad I^{00} \equiv I^1_0 - I^2_0 = 2I^0_0,
\]
(2.5)
we can rewrite the relations (2.4) as
\[
\{Q^-, \tilde{Q}^+\} = mI^0 - 2H + 2mF,
\quad \{Q^+, \tilde{Q}^-\} = mI^0 + 2H - 2mF, \\
\{Q^\pm, \tilde{Q}^\pm\} = \mp 2mI^{\pm\pm}, \quad [I^0_i, I^{\pm\pm}] = \pm 2I^{\pm\pm}, \quad [I^{++}, I^{--}] = I^0, \\
[I^0, Q^\pm] = \pm \tilde{Q}^\pm, \quad [I^{++}, Q^+] = \tilde{Q}^+, \quad [I^{--}, Q^-] = \tilde{Q}^-, \\
[I^0, Q^\pm] = \pm Q^\pm, \quad [I^{++}, Q^+] = Q^+, \quad [I^{--}, Q^-] = Q^-,
\quad [F, \tilde{Q}^\pm] = -\frac{1}{2} \tilde{Q}^\pm,
\quad [F, Q^\pm] = \frac{1}{2} Q^\pm.
\]
(2.6)
We can also add, to the \( su(2|1) \) superalgebra (2.6), the automorphism group \( SU(2)_{\text{ext}} \) with the generators \( \{T^0_i, T^{++}, T^{--}\} \) which rotate the supercharges in precisely the same way
as the internal \( SU(2)_{\text{int}} \) generators \( \{I^0_i, I^{++}, I^{--}\} \). For consistency, the \( SU(2)_{\text{ext}} \) generators
should rotate, in the same way, the indices of the \( SU(2)_{\text{int}} \) generators \( I^i_j \), so these two \( SU(2) \) groups form a semi-direct product
\[
[T, I] \propto I.
\]
(2.7)

Then we introduce the following harmonic coset of the extended supergroup:
\[
\tilde{SU}(2|1) \times SU(2)_{\text{ext}} / U(1)_{\text{int}} \times U(1)_{\text{ext}} = \left\{ \left. \begin{array}{c} H, Q^i, \tilde{Q}_k, F, I^i_j, T^{\pm\pm}, T^0 \\ F, T^0 \end{array} \right| \sim \left( t, \theta_i, \vartheta^k, \nu^i_j, u^j_i \right) \right\}.
\]
(2.8)
where the harmonic variables $v^j$ parametrize the group $SU(2)_{\text{ext}}$, while $u^\pm, u^\mp u^- = 1$, are the standard harmonics on the coset $SU(2)_{\text{ext}}/U(1)_{\text{ext}} \sim S^2$ [14]. As the next step, one passes to the ‘minimal’ complex harmonic coset

$$ \{ H, Q^\pm, \bar{Q}^\pm, F, I^{\pm \pm}, I^0, T^{\pm \pm}, T^0 \} \sim \left( \{ t_{(A)}, \theta^\pm, \bar{\theta}^\pm, w^\pm \} \right) = \zeta_H. \quad (2.9) $$

It is a deformation of the standard ‘flat’ $\mathcal{N} = 4, d = 1$ harmonic superspace [12]. The new harmonics $w^\pm$ still satisfy the standard condition $w^\pm w^- = 1$ and are properly constructed from the bi-harmonic set $(v^j, u^\pm)$ [15].

We skip most of the details of the whole construction which basically coincide with those given in section 9.1 of the book [15]. The set of coordinates defined in (2.9) will be referred to as the analytic basis of the $SU(2|1)$ harmonic superspace. The odd $SU(2|1)$ transformations in this basis obtained as left shifts of the relevant supercoset are written as

$$ \begin{align*}
\delta \theta^+ &= e^+ + m \bar{\theta}^+ \theta^-, \quad \delta \bar{\theta}^- = \bar{e}^+ - m \bar{\theta}^- \theta^-, \\
\delta \theta^- &= e^+ - 2m \bar{\theta}^+ \theta^-, \quad \delta \bar{\theta}^- = \bar{e}^+ - 2m \bar{\theta}^- \theta^-, \\
\delta w^+_i &= -m \left( \bar{e}^+ + \bar{\theta}^+ \theta^- \right) w^-_i, \quad \delta w^-_i = 0, \quad \delta t_{(A)} = 2i \left( e^- \bar{\theta}^+ + \bar{e}^- \theta^- \right),
\end{align*} \quad (2.10) $$

where

$$ \begin{align*}
e^\pm &:= e^j w^+_j, \quad \bar{e}^\pm := \bar{e}^k w^-_k, \quad w^+_j w^-_k = w^-_k w^+_j = \delta_{ik}.
\end{align*} \quad (2.11) $$

Notice the asymmetry in the transformations of the harmonic variables $w^+_j$ and $w^-_i$. This is a general feature of the harmonic extensions of curved superspaces [15, 18], and it was already encountered in the $d = 1$ harmonic superspace formalism, when considering realization of the $\mathcal{N} = 4, d = 1$ superconformal group $D(2, 1; \alpha)$ on the harmonic coordinates [12].

It follows from the transformations (2.10) that the $SU(2|1)$ harmonic superspace contains the analytic harmonic subspace parametrized by the reduced coordinate set

$$ \zeta_{(A)} = \left( t_{(A)}, \theta^+, \bar{\theta}^+, w^+ \right), \quad (2.12) $$

which is closed under the action of $SU(2|1)$. It can be identified with the supercoset

$$ \left\{ H, Q^\pm, \bar{Q}^\pm, F, I^{\pm \pm}, I^0, T^{\pm \pm}, T^0 \right\} \sim \zeta_{(A)}'. \quad (2.13) $$

One can define the analytic subspace integration measure

$$ d\zeta^-_{(A)} := dw_{(A)} d\bar{\theta}^+ d\theta^+, \quad (2.14) $$

3 By passing to the new basis in the semi-direct product of two $SU(2)$ groups involved in (2.9), (2.13), $(I, T) \rightarrow (I, J = I - T)$, where the generators $J^i$ commute with all other ones including $I^j$, the supercoset (2.13) is split into the product

$$ \left\{ H, Q^\pm, \bar{Q}^\pm, F, I^{\pm \pm}, I^0, J^{\pm \pm}, J^0 \right\} = \left\{ H, Q^\pm, \bar{Q}^\pm, F, I^{\pm \pm}, I^0 \right\} \otimes \left\{ J^{\pm \pm}, J^0 \right\}. $$

The first factor stands for the pure $SU(2|1)$ harmonic supercoset [4] in which the internal part is the complex coset of $SU(2)_{\text{int}}$ over its parabolic subgroup and which can be parametrized by $(t_{(A)}, \theta^+, \bar{\theta}^+, \chi^{\pm \pm})$ where $\chi^{\pm \pm}$ is a complex ($CP^2$) coordinate of the internal space. The second factor fully decouples since the generators involved in its definition commute with the $SU(2|1)$ ones. While such a ‘minimal’ definition of the harmonic analytic $SU(2|1)$ superspace is in the spirit of the approach of [19], here we prefer to represent the internal sector by the harmonic variables $w^i$, like in [15].
which is invariant under the supersymmetry transformations (2.10). The corresponding full integration measure $d\zeta_H$ in the analytic basis can be written as

$$d\zeta_H := dw\, dt(A)\, d\bar{\theta}^- \, d\theta^- \, d\bar{\theta}^+ \, d\theta^+ \left( 1 + m \, \bar{\theta}^+ \theta^- - m \, \bar{\theta}^- \theta^+ \right),$$

(2.15)

and it transforms as

$$\delta(d\zeta_H) = d\zeta_H \left[ -m \left( \bar{\theta}^- \epsilon^+ + \theta^- \epsilon^+ \right) \left( 1 - m \, \bar{\theta}^+ \theta^- + m \, \bar{\theta}^- \theta^+ \right) \right].$$

(2.16)

One can check that there is no way to achieve the $SU(2|1)$ invariance of this measure: no scalar factor can be picked up to compensate the non-zero variation (2.16). The relation of the analytic basis (2.9) to the central basis containing the standard $SU(2|1)$ superspace [4] is described in the appendix B. In particular, it is shown that the measure (2.15) in the central basis is reduced to the product of the standard invariant integration measure of the $SU(2|1)$ superspace and the non-invariant harmonic measure.

### 2.3. Covariant derivatives

We use the standard notation for the partial harmonic derivatives

$$\partial^{\pm \pm} := w_i^+ \frac{\partial}{\partial w_i^\pm}, \quad \partial^0 := w_i^+ \frac{\partial}{\partial w_i^+} - w_i^- \frac{\partial}{\partial w_i^-},$$

(2.17)

$$\left[ \partial^{++}, \partial^{--} \right] = \partial^0, \quad \left[ \partial^0, \partial^{\pm \pm} \right] = \pm 2\partial^{\pm \pm}.$$

(2.18)

The deformed covariant spinor and harmonic derivatives can be derived by the routine coset techniques applied to the supercoset (2.9). Once again, we skip the details and present the answer

$$\mathcal{D}^- = -\frac{\partial}{\partial \theta^-} - 2i \bar{\theta}^- \partial_{(A)} + 2m \, \bar{\theta}^- \bar{F} - m \bar{\theta}^+ \left( \theta^+ \frac{\partial}{\partial \theta^+} + \bar{\theta}^0 \frac{\partial}{\partial \theta^0} \right) - m \bar{\theta}^- \partial^0 + m \bar{\theta}^+ \bar{\theta}^-,$$

$$\bar{\mathcal{D}}^- = \frac{\partial}{\partial \theta^-} - 2i \theta^- \partial_{(A)} + 2m \theta^- \bar{F} + m \theta^+ \left( \theta^+ \frac{\partial}{\partial \theta^+} + \bar{\theta}^0 \frac{\partial}{\partial \theta^0} \right) + m \theta^- \partial^0 - m \theta^+ \partial^-,$$

$$\mathcal{D}^+ = \frac{\partial}{\partial \theta^+} + m \bar{\theta}^+ \left( 1 + m \theta^+ \bar{\theta}^+ \right) \partial^{++} + 2im \bar{\theta}^- \theta^+ \bar{\theta}^+ \partial_{(A)} - 2m^2 \bar{\theta}^- \theta^+ \bar{\theta}^+ \bar{F} + m \bar{\theta}^+ \partial^+ \partial^- + m \bar{\theta}^- \partial^+ \frac{\partial}{\partial \theta^0},$$

$$\bar{\mathcal{D}}^+ = -\frac{\partial}{\partial \theta^+} - m \theta^- \left( 1 - m \theta^+ \bar{\theta}^+ \right) \partial^{++} - 2im \theta^+ \theta^+ \bar{\theta}^+ \partial_{(A)} + 2m^2 \theta^- \theta^+ \bar{\theta}^+ \bar{F} - m \theta^- \theta^+ \frac{\partial}{\partial \theta^-} - m \theta^+ \theta^+ \frac{\partial}{\partial \theta^0},$$

$$\mathcal{D}^{++} = (1 + m \bar{\theta}^+ \theta^- - m \bar{\theta}^- \theta^+)^{-1} \partial^{++} + 2i \theta^+ \bar{\theta}^+ \partial_{(A)} - 2m \theta^+ \bar{\theta}^+ \bar{F} + \theta^+ \frac{\partial}{\partial \theta^-} + \bar{\theta}^+ \frac{\partial}{\partial \theta^0}. $$

(2.19)
\[ \mathcal{D}^{--} = \left( 1 + m \bar{\theta}^+ \theta^- - m \bar{\theta}^+ \theta^+ \right) \partial^{--} + 2i \bar{\theta}^+ \bar{\theta}^+ \partial_{(A)} \]
\[ - 2m \bar{\theta}^+ \bar{\theta}^+ \bar{F} + \theta^+ \frac{\partial}{\partial \theta^+} + \bar{\theta}^+ \frac{\partial}{\partial \theta^+}, \]
\[ \mathcal{D}^0 = \partial^0 + \left( \theta^+ \frac{\partial}{\partial \theta^+} + \bar{\theta}^+ \frac{\partial}{\partial \theta^+} \right) - \left( \theta^- \frac{\partial}{\partial \theta^-} + \bar{\theta}^- \frac{\partial}{\partial \theta^-} \right), \]
\[ \mathcal{D}_{(A)} = \partial_{(A)}, \quad \partial_{(A)} = \frac{\partial}{\partial \theta_{(A)}}. \quad (2.20) \]

Here, \( \bar{F} \) is a matrix part of the \( U(1)_{\text{int}} \) generator \( F \). One can check that these derivatives are indeed covariant under the transformations (2.10). The (anti)commutation relations among them mimic those of the superalgebra (2.6):

\[
\{ \hat{\mathcal{D}}^+, \mathcal{D}^- \} = m \mathcal{D}^0 - 2m \hat{F} + 2i \mathcal{D}_{(A)}, \quad \{ \hat{\mathcal{D}}^-, \mathcal{D}^+ \} = m \mathcal{D}^0 + 2m \hat{F} - 2i \mathcal{D}_{(A)},
\]
\[
\mathcal{D}^z, \hat{\mathcal{D}}^z = \pm 2m \mathcal{D}^{\pm z}, \quad [\mathcal{D}^{++}, \mathcal{D}^{--}] = \mathcal{D}^0, \quad [\mathcal{D}^0, \mathcal{D}^{\pm z}] = \pm 2 \mathcal{D}^{\pm z},
\]
\[
[\mathcal{D}^{++}, \mathcal{D}^-] = \mathcal{D}^+, \quad [\mathcal{D}^{--}, \mathcal{D}^+] = \mathcal{D}^-, \quad [\mathcal{D}^0, \mathcal{D}^+] = \pm \mathcal{D}^z,
\]
\[
[\mathcal{D}^{++}, \hat{\mathcal{D}}^-] = \hat{\mathcal{D}}^+, \quad [\mathcal{D}^{--}, \hat{\mathcal{D}}^+] = \hat{\mathcal{D}}^-, \quad [\mathcal{D}^0, \hat{\mathcal{D}}^+] = \pm \hat{\mathcal{D}}^z,
\]
\[
(2.21) \]

It will be convenient to represent the derivatives \( \mathcal{D}^+, \hat{\mathcal{D}}^+ \) as

\[ \mathcal{D}^+ = \frac{\partial}{\partial \theta^+} + m \bar{\theta}^- \mathcal{D}^{++}, \quad \hat{\mathcal{D}}^+ = - \frac{\partial}{\partial \theta^-} - m \theta^- \mathcal{D}^{++}. \quad (2.23) \]

These spinor derivatives, together with \( \mathcal{D}^{++} \) and \( \mathcal{D}^0 \), form the so-called CR (‘Cauchy–Riemann’) structure [15]

\[
\{ \mathcal{D}^+, \hat{\mathcal{D}}^+ \} = -2m \mathcal{D}^{++}, \quad \{ \mathcal{D}^+, \mathcal{D}^+ \} = \{ \hat{\mathcal{D}}^+, \hat{\mathcal{D}}^+ \} = 0,
\]
\[
[\mathcal{D}^{++}, \mathcal{D}^+] = [\mathcal{D}^{++}, \hat{\mathcal{D}}^+] = 0,
\]
\[
[\mathcal{D}^0, \mathcal{D}^+] = \mathcal{D}^+, \quad [\mathcal{D}^0, \hat{\mathcal{D}}^+] = \hat{\mathcal{D}}^+, \quad [\mathcal{D}^0, \mathcal{D}^{++}] = 2 \mathcal{D}^{++}. \quad (2.24) \]

All other (anti)commutators can be derived from these ones by applying the harmonic derivative \( \Delta^{--} \) which, together with \( \mathcal{D}^{++} \) and \( \mathcal{D}^0 \), form an \( SU(2) \) algebra. The non-standard feature of the considered case is that the analyticity-preserving covariant harmonic derivative \( \Delta^{--} \) in the CR structure (2.24) does not decouple from the spinorial derivatives \( \mathcal{D}^+, \hat{\mathcal{D}}^+ \). This will entail the essential differences of the \( SU(2|1) \) harmonic formalism and the relevant \( SU(2|1) \) mechanics models from their flat \( N = 4, d = 1 \) harmonic superspace prototypes. Another peculiarity of the \( SU(2|1) \) harmonic formalism is the presence of the additional matrix \( U(1) \) charge \( \hat{F} \) with the non-trivial action (2.22) on the spinor covariant derivatives. One should be careful about taking it correctly into account, while checking various (anti)commutators, in particular those in (2.24).

\[ ^4 \text{The correct result is still obtained, if the lhs in (2.24) is formally replaced by the commutators. However, one should remember that the definite \( \hat{F} \) charge is ascribed to a covariant spinor derivative as a whole, not to its separate constituents.} \]
2.4. Harmonic SU(2|1) superfields

The passive odd transformation of the harmonic superfields in the analytic basis $\Phi(\zeta_H)$ can be written as

$$\delta \Phi = -m \left[ 2 \left( \bar{\theta}^+ e^- - \theta^+ \bar{e}^- \right) \bar{F} + \left( \bar{\theta}^+ e^+ + \theta^+ \bar{e}^+ \right) D^0 + \left( \bar{\theta}^- e^- + \theta^- \bar{e}^- \right) D^{++} \right] \Phi. \quad (2.25)$$

The superfields $\Phi$ are assumed to have definite $U(1)$ charges, $F_{\Phi} \equiv q \Phi$, $D^0 \Phi = q \Phi$. The presence of the derivative $D^{++}$ in (2.25) is necessary for the correct $SU(2|1)$ closure of these variations and for ensuring that various covariant derivatives of $\Phi$, e.g. $D^{++} \Phi$, $D^+ \Phi$ and $D^- \Phi$, transform according to the same generic rule (2.25) as $\Phi$ itself (see appendix A).

Given some set of such superfields $\left\{ \Phi_1, \Phi_2, \ldots \Phi_N \right\}$, we can write the general $\sigma$-model-type action as

$$S = \int dt \ L = \int d\zeta_H K \left( \Phi_1, \Phi_2, \ldots \Phi_N, w_i^\pm \right), \quad D^0 K \left( \Phi_1, \Phi_2, \ldots \Phi_N, w_i^\pm \right) = 0. \quad (2.26)$$

Here $K$ is a real function of superfields and the harmonic coordinates $w_i^\pm$, arbitrary for the moment. Varying (2.26) with respect to the supersymmetry transformations (2.10), (2.25), we obtain

$$\delta S = \int d\zeta_H D^{++} \left\{ -m \left( \bar{\theta}^+ e^- - \theta^+ \bar{e}^- \right) K \left( \Phi_1, \ldots \Phi_N, w_i^\pm \right) \right\} + \int d\zeta_H \left\{ m \left( \bar{\theta}^+ e^+ + \theta^+ \bar{e}^+ \right) \left( 1 + m \bar{\theta}^+ \theta^- - m \bar{\theta}^- \theta^+ \right)^{-1} \delta^{++} \right. \left. - 2 m \left( \bar{\theta}^+ e^- - \theta^+ \bar{e}^- \right) \bar{F} - m \left( \bar{\theta}^+ e^+ + \theta^+ \bar{e}^+ \right) \delta^{--} \right\} K \left( \Phi_1, \ldots \Phi_N, w_i^\pm \right), \quad (2.27)$$

where $\bar{\theta}^\pm$, $\delta^0$ act only on the explicit harmonics in the function $K$. Requiring the variation (2.27) to vanish up to a total derivative under the integral yields the following restrictions on $K$:

$$\delta^{++} K \left( \Phi_1, \Phi_2, \ldots \Phi_N, w_i^\pm \right) = 0 \quad \Rightarrow \quad K = K \left( \Phi_1, \Phi_2, \ldots \Phi_N \right), \quad \bar{F} K \left( \Phi_1, \Phi_2, \ldots \Phi_N \right) = D^0 K \left( \Phi_1, \Phi_2, \ldots \Phi_N \right) = 0. \quad (2.28)$$

Thus the general action is given by

$$S = \int dt \ L = \int d\zeta_H K \left( \Phi_1, \Phi_2, \ldots \Phi_N \right) \quad (2.29)$$

and its variation $\delta S$ is

$$\delta S = \int d\zeta_H D^{++} \left\{ -m \left( \bar{\theta}^+ e^- - \theta^+ \bar{e}^- \right) K \left( \Phi_1, \ldots \Phi_N \right) \right\} = 0. \quad (2.30)$$

In what follows we will be interested in the analytic superfields, i.e. those subjected to the Grassmann Cauchy–Riemann constraints

$$D^+ \phi = D^+ \phi = 0. \quad (2.31)$$

In virtue of the CR structure relations (2.24), these analyticity constraints imply, as their integrability condition, the harmonic Cauchy–Riemann condition

$$D^{++} \phi = 0. \quad (2.32)$$

This is a crucial difference from the flat $N = 4$, $d = 1$ harmonic analytic superfields [12] for which (2.31) do not necessarily imply the appropriate version of (2.32). With taking into
account (2.32), the derivatives \( \mathcal{D}^+ \) and \( \tilde{\mathcal{D}}^+ \) become ‘short’,
\[
\left( \mathcal{D}^+, \tilde{\mathcal{D}}^+ \right) \Rightarrow \left( \frac{\partial}{\partial \theta^+}, - \frac{\partial}{\partial \tilde{\theta}^+} \right).
\] (2.33)
and so (2.31) give
\[
\phi = \phi \left( \zeta_{(A)} \right).
\] (2.34)
The necessity of the harmonic constraint (2.32) for the preservation of Grassmann \( SU(2|1) \) analyticity also directly follows from the general superfield transformation law (2.25). Only under this constraint the variation of the analytic superfield does not contain the coordinates \( \theta^- \), \( \tilde{\theta}^- \).

Finally, we note that the analytic superspace (2.12) can be extended to the two mutually conjugated three-theta analytic superspaces
\[
\zeta^{(3)}_{(A)} := \left( \theta^-, \zeta_{(A)} \right) = \left( t_{(A)}, \theta^-, \tilde{\theta}^+, \theta^+, \omega_i^\pm \right), \quad \tilde{\zeta}^{(3)}_{(A)} = \left( \tilde{\theta}^-, \zeta_{(A)} \right),
\] (2.35)
which are also closed under the coordinate transformations (2.10). The relevant superfields are singled out by the covariant chirality-like conditions\(^5\)
\[
\tilde{\mathcal{D}}^+ \tilde{\phi}_{(1)} = 0 \quad \text{or} \quad \mathcal{D}^+ \tilde{\phi}_{(2)} = 0,
\] (2.36)
which do not require the harmonic constraints (2.32). The existence of analogous extended analytic superspaces in the flat \( N = 4, d = 1 \) case was noticed in [20]. Possible implications of these additional Grassmann analyticities in the \( SU(2|1) \) SQM models will be addressed elsewhere.

3. The multiplet \((4, 4, 0)\)

In the flat \( N = 4, d = 1 \) supersymmetry the multiplet with the field contents \((4, 4, 0)\) is described by an analytic harmonic superfield and it is the basic \( N = 4, d = 1 \) multiplet: all other irreducible multiplets and the related SQM models can be obtained from this multiplet and the SQM models associated with it through different versions of the Hamiltonian reduction [8] or, equivalently, by gauging some isometries of the \((4, 4, 0)\) Lagrangians [9–11].

The \( SU(2|1) \) version of the multiplet \((4, 4, 0)\) is described by the superfield \( q^{+a} \) satisfying the constraints
\[
\tilde{\mathcal{D}}^+ q^{+a} = \mathcal{D}^+ q^{+a} = \mathcal{D}^{++} q^{+a} = 0, \quad \tilde{F} q^{+a} = 0.
\] (3.1)
These constraints look just like those in the flat \( N = 4, d = 1 \) superspace (except for the last one). Their solution reads
\[
q^{+a} \left( \zeta_{(A)} \right) = x^{iu} w^+_i + \theta^+ \psi^a + \tilde{\theta}^+ \tilde{\psi}^a - 2i \theta^+ \tilde{\theta}^+ x^{iu} w^-_i,
\] (3.2)
where
\[
\left( x^{iu} \right)^\dagger = \epsilon_{ab} \epsilon_{ik} \chi^{kb}, \quad \left( \psi^a \right)^\dagger = \tilde{\psi}^a_{\alpha}.
\] (3.3)

\(^5\) These conditions are solved in terms of unconstrained superfields living on \( \psi_{(A)}^{(3)} \) or \( \tilde{\zeta}_{(A)}^{(3)} \) by passing to the new frames where \( \tilde{\mathcal{D}}^+ \) or \( \mathcal{D}^+ \) become short. This passing is accomplished by means of the appropriate invertible intertwining operators, e.g. \( \mathcal{D}^+(1 - m \theta^+ \tilde{\theta}^- \mathcal{D}^{++}) = (1 - m \theta^+ \tilde{\theta}^- \mathcal{D}^{++}) \frac{\partial}{\partial \theta^-} \).
The index \( a = 1, 2 \) is the doublet index of the ‘Pauli–Gürsey’ group \( SU(2)_{PG} \) which commutes with \( SU(2)[1] \). The fermionic fields \( \psi^a, \bar{\psi}^a \) can be combined into a doublet of the external group \( SU(2) \) as \((\psi^a, \bar{\psi}^a) = \psi^a\).

The analytic superfield \( q^{+a} \) has no dependence on \( m \) in its \( \theta \)-expansion, however the non-analytic counterpart of \( q^{+a} \), i.e. \( q^{-a} := \mathcal{D}^{-} q^{+a} \), displays such a dependence:

\[
q^{-a} = \left[ 1 + m \bar{\theta}^{+} \theta^{-} - m \bar{\theta}^{-} \theta^{+} \right] \bar{x}^{ia} w_i^{-} + \bar{\theta}^{-} \bar{\psi}^{a} + \bar{\theta}^{+} \psi^{a} + 2i \left( \bar{\theta}^{+} \theta^{-} + \bar{\theta}^{-} \theta^{+} \right) \bar{x}^{ia} w_i^{-} \\
+ 2i \bar{\theta}^{+} \bar{\theta}^{-} \left[ \bar{x}^{ia} w_i^{+} + \bar{\theta}^{+} \bar{\psi}^{a} + \bar{\theta}^{-} \psi^{a} - 2i \bar{\theta}^{+} \bar{\psi}^{-} \right].
\]

(3.4)

The odd \( SU(2)[1] \) transformation of \( q^{+a} \) is a particular case of the general transformation law (2.25),

\[
\delta q^{+a} = -m \left( \bar{\theta}^{+} \epsilon^{-} + \theta^{+} \epsilon^{-} \right) q^{+a}.
\]

(3.5)

It implies the following transformations for the component fields

\[
\delta x^{ia} = -\epsilon^{i} \psi^{a} - \bar{\epsilon}^{i} \bar{\psi}^{a}, \quad \delta \bar{\psi}^{a} = 2i \epsilon_{k} x^{a}_{k} - m \epsilon_{k} x^{k}_{a}, \quad \delta \psi^{a} = 2i \bar{\epsilon}^{k} x^{a}_{k} + m \bar{\epsilon}^{k} x^{k}_{a}.
\]

(3.6)

Note that the matrix \( U(1) \) generator \( \tilde{F} \) can be ‘activated’ on \( q^{+a} \) by identifying it with some \( U(1) \subset SU(2)_{PG} \), e.g. as

\[
\tilde{F} q^{+a} = \kappa (\tau_{3})_{ib}^{a} q^{+b},
\]

(3.7)

where \( \kappa \) is a new charge. The formulas given above are generalized to the \( \kappa = 0 \) case as

\[
q^{+a} = x^{ia} w_i^{+} + \theta^{+} \psi^{a} + \bar{\theta}^{+} \bar{\psi}^{a} - 2i \theta^{+} \bar{\psi}^{-} \nabla_{(i} x^{ia} w_i^{-},
\]

(3.8)

where

\[
\nabla_{(i} x^{ia} := x^{ia} + i \kappa m (\tau_{3})_{ib}^{a} x^{ib}.
\]

(3.9)

According to the transformation law (2.25), \( q^{+a} \) transforms as

\[
\delta q^{+a} = -m \left( \bar{\theta}^{+} \epsilon^{-} + \theta^{+} \epsilon^{-} \right) q^{+a} - 2i m \left( \bar{\theta}^{+} \epsilon^{-} - \theta^{+} \epsilon^{-} \right) (\tau_{3})_{ib}^{a} q^{+b}.
\]

(3.10)

The component field transformations (3.6) are modified as

\[
\delta x^{ia} = - \epsilon^{i} \psi^{a} - \bar{\epsilon}^{i} \bar{\psi}^{a}, \quad \delta \bar{\psi}^{a} = 2i \bar{\epsilon}^{k} \nabla_{(i} x^{a}_{k} + m \bar{\epsilon}^{k} x^{k}_{a}, \\
\delta \psi^{a} = 2i \epsilon_{k} \nabla_{(i} x^{a}_{k} - m \epsilon_{k} x^{k}_{a}.
\]

(3.11)

3.1. The general \( \sigma \)-model action

For simplicity, we will basically deal with the \( \kappa = 0 \) case, leaving a comment on the \( \kappa \neq 0 \) case for the end of this subsection.

In accordance with the general structure of the harmonic superfield actions (2.29), the \( \sigma \)-model-type action for the multiplet \((4, 4, 0)\) can be written as

\[
S(q^{\pm a}) = \int d\zeta_{H} K(q^{+}, q^{-}),
\]

(3.12)

where \( K \) satisfies the conditions

\[
\bar{F} K(q^{\pm a}) = \mathcal{D}^{0} K(q^{\pm a}) = 0.
\]

(3.13)
Note that the essential difference of \((3.12)\) from its \(\mathcal{N} = 4\) counterpart \([12]\) is that the Lagrangian function \(K\) cannot involve explicit harmonics (see \((2.28)\)). Since the function \(K\) is neutral, it can be written as a function of two neutral superfield arguments, \(K = K(q^2, X^{(ab)})\), with

\[
q^2 = 2 q^a q^b, \quad X^{(ab)} = q^a q^{-b} = \frac{1}{2}((q^a q^{-b} + q^b q^{-a})).
\]

\[
D^{++} q^2 = (D^{++})^2 X^{(ab)} = 0, \quad D^b q^2 = D^b X^{(ab)} = 0.
\]

The function \(K(q^2, X^{(ab)})\) can be represented as a power series in \(X^{(ab)}\):

\[
K(q^2, X^{(ab)}) = K_0(q^2) + \sum_{n=1}^{\infty} C_{a_1 a_2 \ldots a_n b_1 b_2 \ldots b_n} (q^2) X^{(a_1 b_1)} X^{(a_2 b_2)} \ldots X^{(a_n b_n)}.
\]

It can be shown that every term of this expansion, except for the zeroth order one \(K_0(q^2)\), is a total \(D^{++}\) derivative plus a function of \(q^2\) which can be absorbed into \(K_0(q^2)\). We explicitly show this for \(n = 1, 2:\)

\[
X^{(a b)} = \frac{1}{2} D^{++} [q^{-a} q^{-b}],
\]

\[
X^{(a b)} X^{(a b)} = \frac{1}{3} D^{++} [q^{-a} q^{-b} X^{(a b)}] - \frac{1}{96} \epsilon^{a_2 a_3 b_2 b_3} (q^2)^2.
\]

So we have \(K(q^2, X^{(ab)}) = -L(q^2) + D^{++} L^-(q^2, X^{(ab)})\) and the general action \((3.12)\) can be rewritten as

\[
S(q^2) = \int dt \, \mathcal{L} = - \int d\zeta_H \, L(q^2), \quad \mathcal{F}L(q^2) = D^0 L(q^2) = 0,
\]

where the sign minus was chosen for further convenience. The corresponding component Lagrangian reads\(^6\)

\[
\mathcal{L} = G \left[ x^{ab} \partial_{x^{ab}} + \frac{i}{2} \left( \psi^a \psi^a - \psi^a \psi^a \right) + \frac{m}{2} \psi^a \bar{\psi}^a \right] - \frac{i}{2} \partial_{x^{ab}} G \left( \psi^a \bar{\psi}^a + \psi^a \bar{\psi}^a \right)
\]

\[
- \frac{\Delta_4 G(x^2)}{16} \left( \psi^a \bar{\psi}^a \right)^2 + \frac{m}{4} \partial_x G \psi^a \bar{\psi}^a - \frac{m^2}{4} x^2 G,
\]

where

\[
\partial_{x^{ab}} = \partial/\partial x^{ab}, \quad \Delta_4 = \epsilon^{i k} \epsilon^{a b} \partial_{x^{ik}} \partial_{x^{ab}}, \quad x^2 = x^{ab} x_{ab},
\]

\[
G(x^2) = \Delta_4 L(x^2) = 4 x^2 L''(x^2) + 8 L'(x^2), \quad \partial_{x^{ab}} G = 8 x_{ab} \left( 3 L'' + x^2 L'' \right).
\]

We observe that \(SU(2|1)\) supersymmetry imposes rather severe restrictions on the structure of the \((4, 4, 0)\) Lagrangian as compared with the standard \(\mathcal{N} = 4, d = 1\) supersymmetry. Though the bosonic metric is conformally flat in both cases, in the \(SU(2|1)\) case it turns out to be \(SU(2)_{\text{int}} \times SU(2)_{\text{PG}} \sim SO(4)\) symmetric, with the conformal factor being a function of \(x^2\). The extra fermionic \(SU'(2)\) symmetry is broken by the terms \(\sim (\psi^a \bar{\psi}^a)^2\). Note that the whole action respects \(SU(2)_{\text{PG}}\) symmetry.

\(^6\) We use the following convention for Grassmann variables: \((\chi)^2 = \chi_i \chi^i, (\bar{\chi})^2 = \bar{\chi}^i \bar{\chi}_i\).
The simplest case corresponds to the free system with $G = 1$:

$$S_{\text{free}}(q^{1a}) = -\frac{1}{4} \int d\xi_{\mu} \, q^{1a} \partial_{\mu} q^{1a}, \quad (3.21)$$

$$\mathcal{L}_{\text{free}} = \dot{x}^{ia} \dot{x}_{ia} + \frac{i}{2} \left( \bar{\psi}_{\bar{i}} \gamma^{a} \psi_{i} - \bar{\psi}_{i} \gamma^{a} \psi_{\bar{i}} \right) + \frac{m}{2} \bar{\psi}_{i} \gamma^{a} \psi_{i} - \frac{m^{2}}{4} \chi^{ia} \chi_{ia}. \quad (3.22)$$

Let us make a brief comment concerning the case with non-zero external $F_{\mu\nu}$ charge $\kappa \neq 0$ on the example of the free action. The relevant modified Lagrangian can be obtained just via changing $\partial_{\mu} \rightarrow \nabla_{\mu} = \partial_{\mu} + i \kappa m \gamma_{5}$ in (3.22), which yields

$$\mathcal{L}_{\text{free}}^{(\kappa)} = \dot{x}^{ia} \dot{x}_{ia} + \frac{i}{2} \left( \bar{\psi}_{\bar{i}} \gamma^{a} \psi_{i} - \bar{\psi}_{i} \gamma^{a} \psi_{\bar{i}} \right) - 2i \kappa m \left( \gamma_{5} \bar{\psi}_{\bar{i}} \gamma^{a} \psi_{i} - m \left( \delta_{a}^{b} + 2 \kappa (\gamma_{5})^{a}_{b} \right) \bar{\psi}_{i} \gamma^{b} \psi_{\bar{i}} \right) + \left( \kappa^{2} - \frac{1}{4} \right) m^{2} \chi^{ia} \chi_{ia}. \quad (3.23)$$

Note the appearance of the induced WZ term $\sim \kappa$ and the additional contributions to the mass terms in this Lagrangian. The $SU(2)_{PG}$ symmetry gets broken at $\kappa \neq 0$.

3.2. The absence of WZ type actions

The most general WZ (or CS) action [12] is given by the integral over the analytic subspace

$$S_{\text{WZ}}(q^{+a}) = -\frac{i}{2} \int d\zeta_{\bar{\alpha}} L^{++} \left( q^{+a}, \psi_{\bar{i}} \right). \quad (3.24)$$

Since the analytic superfield (3.2) is not deformed by $m$, this action coincides with the non-deformed WZ action for the multiplet $(4, 4, 0)$ given in [12]. The component Lagrangian reads

$$\mathcal{L}_{\text{WZ}} = A_{ia} \dot{x}^{ia} - \frac{i}{2} B_{(ab)} \psi_{i} \bar{\psi}_{\bar{b}}, \quad (3.25)$$

where

$$A_{ia}(x^{ia}) = \int dw_{i} \frac{\partial L^{++}}{\partial x^{+a}}, \quad B_{(ab)}(x^{ia}) = \int dw_{i} \frac{\partial^{2} L^{++}}{\partial x^{+a} \partial x^{+b}}. \quad (3.26)$$

By construction, the external gauge field $A_{ia}$ satisfies the self-duality condition

$$F_{iab} = \partial_{i} A_{ab} - \partial_{a} A_{ib} = \epsilon_{ab} \int du \, \frac{\partial^{2} L^{++}}{\partial x^{+a} \partial x^{+b}} = \epsilon_{ab} B_{(ab)}, \quad (3.27)$$

and the transversal gauge condition

$$\partial_{i} A^{ia} = 0. \quad (3.28)$$

The Lagrangian (3.25) transforms under the $SU(2)_{1}$ transformations (3.6) as

$$\delta \mathcal{L}_{\text{WZ}} = -\partial_{i} \left[ A_{ia} \left( \epsilon^{i} \psi_{a} + \bar{\epsilon}^{i} \bar{\psi}_{a} \right) \right] + \frac{1}{2} m B_{(ab)} \left( \epsilon^{k} \psi_{a} - \bar{\epsilon}^{k} \bar{\psi}_{a} \right) \chi^{b}_{k}, \quad (3.29)$$

and this variation is not reduced to a total derivative because of the term $\sim m$. Thus, the Lagrangian is not $SU(2)_{1}$ invariant for any choice of $L^{++}$, which just means the absence of the WZ action for the multiplet $(4, 4, 0)$ in the case of $SU(2)_{1}$ supersymmetry. The same conclusion is valid for the $\kappa \neq 0$ case as well.
Some further issues regarding the superconformal properties of WZ term will be discussed in section 5.

3.3. Hamiltonian formalism

The classical Hamiltonian obtained as the Legendre transform of the Lagrangian (3.19) reads

\[
H = \frac{1}{4G} \left[ p^a + \frac{i}{2} \left( \bar{\psi}_a \psi^a + \bar{\psi}^b \psi_b \right) \partial^a G \right] \left[ p_a - \frac{i}{2} \left( \bar{\psi}_a \psi^b + \bar{\psi}^b \psi_a \right) \partial_b G \right] + \frac{\Delta G}{16} (\bar{\psi})^2 (\psi)^2 - \frac{m}{4} \left( 2G + x^\nu \partial_\nu G \right) \psi^a \bar{\psi}_a + \frac{m^2}{4} x^{ia} x_{ia} G.
\]

(3.30)

It is also straightforward to find the supercharges $Q^i = \bar{Q}_i$, as well as the remaining bosonic generators,

\[
Q_i = \psi^a \left( p_a + im x_{ia} G + \frac{1}{4} \psi^b \bar{\psi}_b \partial_b G \right), \quad \bar{Q}_i = \psi^a \left( p_a - im x_{ia} G - \frac{1}{4} \psi^b \bar{\psi}_b \partial_b G \right),
\]

\[
F = \frac{1}{2} G \psi^a \bar{\psi}_a, \quad I_{kh} = i x^a (p_{ka})^a.
\]

(3.31)

The Poisson brackets for the bosonic variables and the Dirac brackets for fermions are defined as

\[
\{ p_a, x^{ib} \} = - \delta^k_i \delta^b_a, \quad [ \psi^a, \bar{\psi}_b ] = -i G^{-1} \delta^a_b, \quad [ p_a, \psi^b ] = \frac{1}{2} \psi^b G^{-1} \partial_a G, \quad [ p_a, \bar{\psi}_b ] = \frac{1}{2} \bar{\psi}_b G^{-1} \partial_a G.
\]

(3.32)

To simplify the brackets, it is useful to make the substitutions

\[
\psi^a = G^{-\frac{1}{2}} \xi^a, \quad \bar{\psi}_b = G^{-\frac{1}{2}} \bar{\xi}_b,
\]

whence

\[
\left\{ p_a, x^{ib} \right\} = - \delta^k_i \delta^b_a, \quad \{ \xi^a, \bar{\xi}_b \} = -i \delta^a_b, \quad \{ p_a, \xi^b \} = \{ p_a, \bar{\xi}_b \} = 0.
\]

(3.34)

These brackets can be quantized in the standard way as

\[
p_a = -i \partial_a, \quad \bar{\xi}_a = \partial / \partial \xi^a, \quad \left[ p_a, x^{ib} \right] = -i \delta^k_i \delta^b_a, \quad \{ \xi^a, \bar{\xi}_b \} = \delta^a_b.
\]

(3.35)

The quantum supercharges can be constructed from the classical ones (3.31) according to the prescriptions of [21] which were also applied by the authors in [4]. Their basic steps are Weyl-ordering and the subsequent similarity transformation defined in terms of the target bosonic metric. In the present case this general procedure yields

\[
Q^{(\text{cov})} = -i G^{-\frac{1}{2}} \xi^a \left[ \left( \partial_a - mx_{ia} \right) G - \frac{1}{4} \left( \xi_a \bar{\xi}_b - 2\delta^b_a \right) G^{-1} \partial_b G \right],
\]

\[
\bar{Q}^{(\text{cov})} = -i G^{-\frac{1}{2}} \bar{\xi}_a \left[ \left( \partial_a + mx_{ia} \right) G + \frac{1}{4} \left( \bar{\xi}_a \xi^b + 2\delta^b_a \right) G^{-1} \partial_b G \right].
\]

(3.36)

The quantum Hamiltonian is defined by the anticommutator of the quantum supercharges, and it reads
The rest of the superalgebra generators is

\[ \tilde{F}_i = -\frac{1}{2} \tilde{\xi}_a \zeta^a, \quad I_k = x^a_i \partial_{\bar{k}a}. \]  

(3.38)

Taken together, these generators form the $\tilde{\mathfrak{su}}(2|1)$ superalgebra (2.4).

Note that the quantum covariantized supercharges can be conveniently represented as

\[ Q_{(cov)i} = e^{-m W} Q_{(i=0)} e^{m W}, \quad \tilde{Q}_{(cov)i} = e^{m W} \tilde{Q}_{(i=0)} e^{-m W}, \]  

(3.39)

where the function $W$ is given by the expression

\[ W(x^2) = 2x^2 L'(x^2) + 2L(x^2), \quad \partial_{ia} W(x^2) = -x_{ia} G(x^2), \quad 2W'(x^2) = -G(x^2). \]  

(3.40)

Thus the quantum $SU(2|1)$ supercharges for the multiplet $(4, 4, 0)$ can be obtained from their flat $N = 4$, $d = 1$ counterparts (with the special $SO(4)$ invariant target bosonic metric) through a similarity transformation. This is a particular case of the general phenomenon summarized in [22]. To avoid possible confusion, we note that the transformation (3.39) is not unitary, and for this reason the Hamiltonian (3.37) is by no means equivalent to its $m = 0$ counterpart.

### 3.4. The free model: spectrum and the SU(2|1) Casimirs

Let us consider the simplest case $G = 1$ corresponding to the free Lagrangian (3.22). The relevant quantum Hamiltonian reads

\[ \hat{H} = -\frac{1}{4} \left( \partial^a - mx^a \right) \left( \partial_{ia} + mx_{ia} \right) + \frac{m}{2} \tilde{\xi}_a \xi^a. \]  

(3.41)

The remaining $SU(2|1)$ generators are

\[ Q_i = -i \tilde{\xi}_a \left( \partial_{ia} - mx_{ia} \right), \quad \tilde{Q}_i = -i \tilde{\xi}_a \left( \partial_{ia} + mx_{ia} \right), \]

\[ F = -\frac{1}{2} \tilde{\xi}_a \xi^a, \quad I_k = x^a_i \partial_{\bar{k}a}. \]  

(3.42)

One can also define the generators of the algebra $su(2|G)$ for the considered case

\[ E_{ai} = x^a_i \partial_{ib} - \tilde{\eta}_{ia} \zeta_{bj}, \quad \left[ E_{ai}, E_{bd} \right] = \varepsilon_{cbd} E_{ad} - \varepsilon_{dad} E_{cd}. \]  

(3.43)

They can be checked to commute with all $SU(2|1)$ generators.

Since the spectrum of the Hamiltonian must be bounded from below, we define the ground state $|0\rangle$ by imposing the physical conditions

\[ \xi^a |0\rangle = 0, \quad \left( \partial_{ia} + mx_{ia} \right) |0\rangle = 0. \]  

(3.44)

Solving them, we obtain the ground state wave function annihilated by supercharges (3.42):

\[ |0\rangle = e^{-\frac{1}{2} x^2}, \quad Q^i |0\rangle = \tilde{Q}_i |0\rangle = 0. \]  

(3.45)
All bosonic quantum states can be constructed by action of the creation operators \( \nabla^{ia} := \partial^{ia} - m x^{ia} \) on \( |0\rangle \). The bosonic state \(|\ell; s\rangle\) is defined as

\[
|\ell; s\rangle = A_{(i_1, \ldots, i_{2s})} a_{i_1} a_{i_2} \cdots a_{i_{2s}} \nabla^{i_1 a_1} \nabla^{i_2 a_2} \cdots \nabla^{i_{2s} a_{2s}} \left( \nabla^{ia} \right)_{0s} |0\rangle,
\]

where \( A \) stand for numerical coefficients symmetric in both \( SU(2)_{\text{int}} \) and \( SU(2)_{\text{PG}} \) indices. Here, the quantum number \( s/2 \) is identified with the highest weight (‘isospin’) of the irreducible representation of the group \( SU(2)_{\text{PG}} \). Clearly, this state has the same isospin \( s/2 \) with respect to \( SU(2)_{\text{int}} \).

Acting by the supercharges \( \bar{Q}_i \) on the bosonic parent function \(|\ell; s\rangle\), we obtain the set of its fermionic descendants

\[
\bar{Q}_i |\ell; s\rangle = 2\im \tilde{\xi}^{ia} \left[ 2\ell \nabla_{ia} |\ell - 1; s\rangle + sb_{ia} |\ell; s - 1\rangle \right],
\]

\[
\bar{Q}_i \bar{Q}_i |\ell; s\rangle = -8\ell(2\ell + s)m^2(\tilde{\xi})^2 |\ell - 1; s\rangle,
\]

where \( b_{ia} \) is some coefficient. Thus, \(|\ell; s\rangle\) extends to a super wave function \( \Omega^{(\ell,s)} \) with \( \Omega^{(0,0)} = |0\rangle \). The supercharges \( \bar{Q}_i \) annihilate \(|\ell; s\rangle\), i.e. \( \bar{Q}_i |\ell; s\rangle = 0 \). The spectrum of the Hamiltonian (3.41) is thus given by

\[
H \Omega^{(\ell,s)} = \frac{m}{2}(2\ell + s)\Omega^{(\ell,s)}, \quad m > 0.
\]

Let us analyze the degeneracies of the superwave functions \( \Omega^{(\ell,s)} \), labeling the representations of the group \( SU(2)_{\text{PG}} \times SU(2)_{\text{int}} \) by the pair of indices \((k,n)\). The non-trivial wave function \( \Omega^{(0,0)} \) is a superposition of \((s + 1)^2\) bosonic and \(s(s + 1)\) fermionic states,

\[
|0; s\rangle, \quad |\tilde{\xi}; 0; s - 1\rangle, \quad s > 0,
\]

so revealing the degeneracy \((2s + 1)(s + 1)\). The bosonic states correspond to the representations \((s/2, s/2)\) and the fermionic states to the representations \((s/2, (s - 1)/2)\). On the other hand, the wave functions \( \Omega^{(\ell,s)} \) with \( \ell > 0 \) have \(4(s + 1)^2\)-fold degeneracy, being a superposition of the following states:

\[
|\ell; s\rangle, \quad |\tilde{\xi}; \nabla^{ia} |\ell - 1; s\rangle, \quad |\tilde{\xi}; a |\ell; s - 1\rangle, \quad (\tilde{\xi})^2 |\ell - 1; s\rangle, \quad \ell > 0.
\]

Here, the bosonic states (1st and 4th) span the representation \((s/2, s/2) \oplus (s/2, s/2)\), while the fermionic states span the representation \((s/2, (s + 1)/2) \oplus (s/2, (s - 1)/2)\).

It is interesting to find out the \( SU(2|1) \) representation contents of these wave functions. For the realization (3.42) and (3.43), the \( SU(2|1) \) Casimir operators defined in (C.1), (C.2) acquire the following concise form

\[
m^2 C_2 = H(H + m) - \frac{m^2}{2} E^a_E^b, \quad m^2 C_3 = \left( H + \frac{m}{2} \right) C_2.
\]

The last term in \( C_2 \) is just the Casimir of \( SU(2)_{\text{PG}} \) and it acts on \( \Omega^{(\ell,s)} \) as

\[
\frac{1}{2} E^a_E^b \Omega^{(\ell,s)} = \frac{s}{2} \left( \frac{s}{2} + 1 \right) \Omega^{(\ell,s)}.
\]

Now, based upon (3.51), (3.52) and (3.48), it is easy to find the eigenvalues of \( C_2 \) and to cast them into the general form given in appendix B (equation (C.3)) with

\[
\beta = \frac{1}{2}(2\ell + s + 1), \quad \lambda = \frac{1}{2}(s + 1), \quad \ell > 0.
\]

The relevant quantum states form the so-called typical \( SU(2|1) \) representations. The atypical \( SU(2|1) \) representations correspond to the zero eigenvalues of Casimirs, and so \( \Omega^{(0,0)} \) belong
to this subclass. For them
\[ \ell = 0, \quad \beta = \lambda = \frac{s}{2}. \] (3.54)

The degeneracy of \( \Omega^{(\ell,0)} \) can be computed as the product of the relevant dimensions of \( SU(2)_{\text{PG}} \) and \( SU(2)[1] \) representations (see appendix C). The result coincides with the direct counting given above. In the typical cases, with \( \ell > 0 \), the wave function \( \Omega^{(\ell,0)} \) has the degeneracy \( 4(s + 1)^2 \). The typical representation always encompasses an equal number of bosons and fermions. The wave function \( \Omega^{(0,\ell)} \) corresponding to the atypical case has the degeneracy \( (2s + 1)(s + 1) \), with \( (s + 1)^2 \) bosons and \( s(s + 1) \) fermions.

As instructive examples, let us consider superwave functions of the simplest atypical and typical representations of \( SU(2)[1] \). The atypical superwave function \( \Omega^{(0,1)} \) consists of \((1/2, 1/2, 0)\) bosonic and \((1/2, 0, 0)\) fermionic states given by
\[ \nabla^{ia} [0], \quad \bar{\xi}_a [0]. \] (3.55)

The simplest typical superwave function \( \Omega^{(1,0)} \) has the 4-fold degeneracy:
\[ \nabla^{ia} \nabla_{ia} [0], \quad \bar{\xi}_a \nabla^{ia} [0], \quad \bar{\xi}^2 [0]. \] (3.56)

Here, bosonic states belong to \((0, 0) \oplus (0, 0)\), while fermionic states belong to \((0, 1/2)\).

4. The ‘mirror’ multiplet \((4, 4, 0)\)

The standard multiplets \((n, n, 4 - n)\) of the flat \( N = 4, d = 1 \) supersymmetry have their ‘mirror’ (or ‘twisted’) cousins which possess the same field contents but for which two commuting \( SU(2) \) automorphism algebras of the \( N = 4, d = 1 \) superalgebra switch their roles. Since these automorphism algebras enter the game in the entirely symmetric way, the difference between two mutually mirror multiplets manifests itself only in those SQM models where they are present simultaneously. In the \( SU(2)[1] \) deformed case the symmetry between the two former automorphism \( SU(2) \) groups of the flat superalgebra proves to be broken: one of these \( SU(2) \) becomes \( SU(2)_{\text{mir}} \), while only one \( U(1) \) generator \( F \) from the second \( SU(2) \) (which we denoted \( SU'(2) \)) is inherited by the \( SU(2)[1] \) superalgebra. So one can expect an essential difference between the \( SU(2)[1] \) multiplets and their possible mirror counterparts.

Here we construct the mirror version of the \( SU(2)[1] \) multiplet \((4, 4, 0)\) and show that the relevant SQM models indeed reveal a few serious distinctions from those considered in the previous sections. In particular, in the mirror case one can define the \( SU(2)[1] \) invariant superfield \( W \) and \( Z \).

Let us consider the mirror \((4, 4, 0)\) multiplet \([13, 23]\) in the framework of the harmonic \( SU(2)[1] \) superspace. The basic superfield is \((Y^A)^I = \bar{Y}_A, A = 1, 2\), satisfying the constraints\(^7\)
\[ \hat{D}^+ Y^A = \hat{D}^+ \bar{Y}^A = 0, \quad \hat{D}^+ Y^A = \hat{D}^+ \bar{Y}^A, \quad \hat{D}^+ Y^A = \hat{D}^+ \bar{Y}^A = 0. \] (4.1)

With taking into account the action of the generator \( \hat{F} \) on the spinor covariant derivatives (equation (2.22)), the constraints (4.1) uniquely fix the \( \hat{F} \) charges of the superfields \( Y^A, \bar{Y}^A \) as
\[ m \hat{F} \bar{Y}^A = -\frac{m}{2} Y^A, \quad m \hat{F} Y^A = \frac{m}{2} Y^A. \] (4.2)

The \( SU'(2)_{\text{PG}} \) doublets \( Y^A, \bar{Y}^A \) can be combined into a doublet \( Y^{I, A}, Y^{Z, A} \) of \( SU'(2) \).

\(^7\) The harmonic constraints in (4.1) necessarily follow from the Grassmann analyticity ones as the integrability conditions for the latter, in close similarity with the \( q^+ \) multiplet.
The constraints (4.1) are solved in terms of the superfields defined on the analytic three-theta superspaces (2.35):

\[
Y^A \left( \frac{\zeta}{\zeta^{(1)}} \right) = y^A - \theta^+ \psi^{JA} \phi^{-} + \theta^- \psi^{JA} \phi^{+} - 2i \theta^- \bar{\psi}^A - 2i \theta^+ \bar{\psi}^{JA} \phi^{-} \phi^{+} y^A + m \theta^- \bar{\psi}^A + m \theta^+ \bar{\psi}^{JA} \phi^{-} y^A,
\]

(4.3)

\[
P^A \left( \frac{\zeta}{\zeta^{(1)}} \right) = \bar{y}^A - \bar{\theta}^+ \psi^{JA} \phi^{-} + \bar{\theta}^- \psi^{JA} \phi^{+} - 2i \bar{\theta}^+ \bar{\psi}^A - 2i \bar{\theta}^- \bar{\psi}^{JA} \phi^{-} \phi^{+} \bar{y}^A - m \bar{\theta}^- \bar{\psi}^A - m \bar{\theta}^+ \bar{\psi}^{JA} \phi^{-} \phi^{+} \bar{y}^A,
\]

(4.3)

where

\[
(\psi^A)^+ = \bar{\psi}_A, \quad (\psi^{JA})^+ = \bar{\psi}_{JA}.
\]

(4.4)

We observe that the field content of \(Y^A\) is just (4, 4, 0), but the \(SU(2)\) assignment of the involved fields is different from that of the previous (4, 4, 0) multiplet. The bosonic fields are organized into a complex \(SU'(2)_{PG}\) doublet \(Y^A\) which is a singlet of \(SU(2)_{int}\) (though it is still rotated by the \(U(1)_{int}\) generator \(F\)); the fermionic fields are combined into doublets of both \(SU'(2)_{PG}\) and \(SU(2)_{int}\), but they are singlets of \(U(1)_{int}\).

According to the general transformation law (2.25), the superfields \(Y^A, \bar{Y}^A\) transform as

\[
\delta Y^A = -m \left( \bar{\theta}^+ e^{-} - \theta^+ \bar{e}^{-} \right) Y^A, \quad \delta \bar{Y}^A = m \left( \bar{\theta}^+ e^{-} - \theta^+ \bar{e}^{-} \right) \bar{Y}^A.
\]

(4.5)

The corresponding component field transformations are

\[
\delta y^A = -\epsilon_i \psi^{iA}, \quad \delta \bar{y}^A = -\bar{\epsilon}_i \psi^{iA}, \quad \delta \psi^{iA} = \bar{\epsilon}^i \left( 2i \bar{y}^A - my^A \right) - \epsilon^i \left( 2i \psi^{iA} + my^A \right).
\]

(4.6)

4.1. The \(\sigma\)-model action

One can write the general action in terms of the function \(\tilde{L}\) as

\[
\tilde{S}(Y, \bar{Y}) = \int dt \tilde{L} = \int d\zeta_H \tilde{L}(Y, \bar{Y}).
\]

(4.7)

It can be checked to respect \(SU(2)_{11}\) invariance only with the following condition

\[
m \tilde{F} \tilde{L}(Y, \bar{Y}) = 0 \quad \Rightarrow \quad m \left( y^B \partial_B - \bar{y}^B \partial_B \right) \tilde{L}(y, \bar{y}) = 0.
\]

(4.8)

This condition is non-trivial only for \(m \neq 0\), when \(F\) appears as an internal generator in the superalgebra (2.4). In virtue of the constraint (4.8),

\[
\tilde{L}(Y, \bar{Y}) = \tilde{L}(\dot{U}_B^A), \quad \dot{U}_B^A = Y^A \bar{Y}_B.
\]

(4.9)
The general component Lagrangian is
\[ \mathcal{L} = \left[ 2 y^A \ddot{\psi}_A^i + \frac{i}{2} \psi^{ik} \dot{\psi}_{ia} - \frac{i}{2} \psi^{ik} \psi_{ia} \left( y^c \partial_a + \dot{y}^c \partial_\dot{a} \right) + \frac{1}{48} \psi^{ik} \psi_{ia} \psi_{jB} \Delta, \right] \tilde{G} \]

\[ - \text{im}(y^A \ddot{\psi}_A^i - y^A \dot{\psi}_A^i)\tilde{G} + 2\text{im}(y^A \partial_a \dot{L} - \ddot{y}^A \partial_\dot{a} \dot{L}) - m \psi^{ik} \psi_{ia} \partial_\dot{a} \partial_\dot{b} \tilde{L} \]

\[ + \frac{m}{4} \psi^{ik} \psi_{ia} \left( y^c \partial_a \tilde{G} - \ddot{y}^c \partial_\dot{a} \tilde{G} \right) + \frac{m^2}{2} y^A \tilde{G} - m^2 (y^A \partial_a L + \ddot{y}^A \partial_\dot{a} \dot{L}). \]

(4.10)

where
\[ \tilde{G} \equiv \Delta_\gamma \tilde{L}, \quad \Delta_\gamma = -2 \epsilon^{AB} \partial_a \partial_B, \quad \partial_a = \frac{\partial}{\partial y^A}, \quad \partial_B = \frac{\partial}{\partial \dot{y}^B}. \]

(4.11)

The simplest action is the free action
\[ S_{\text{free}}(Y, \dot{Y}) = \frac{1}{4} \int d\zeta_H Y^A \ddot{\psi}_A^i. \]

(4.12)

The corresponding Lagrangian is
\[ \mathcal{L}_{\text{free}} = 2 y^A \ddot{\psi}_A^i + \frac{i}{2} \psi^{ik} \dot{\psi}_{ia} - \frac{i}{2} m \left( y^A \ddot{\psi}_A^i - y^A \dot{\psi}_A^i \right). \]

(4.13)

In contrast to the flat case, the Lagrangian (4.10) is not vanishing for \( \tilde{G} = 0 \). In the latter case, the remaining Lagrangian
\[ \mathcal{L}_{\tilde{G}=0} = 2 \text{im} \left( y^A \partial_a \dot{L} - \ddot{y}^A \partial_\dot{a} \dot{L} \right) - m \psi^{ik} \psi_{ia} \partial_\dot{a} \partial_\dot{b} \dot{L} - m^2 \left( y^A \partial_a \dot{L} + \ddot{y}^A \partial_\dot{a} \dot{L} \right) \]

(4.14)

is SU(2|1) invariant, and it can be considered as a WZ Lagrangian vanishing at \( m = 0 \). The condition \( \tilde{G} = 0 \) is equivalent to the dim 4 Laplace equation for \( \tilde{L} \)
\[ \Delta_\gamma \tilde{L} = 0. \]

(4.15)

The simplest non-trivial solution of (4.15) is
\[ \tilde{L} = c_{(AB)} Y^A \ddot{\psi}_B^i, \]

(4.16)

where \( c_{(AB)} \) is an arbitrary constant triplet which breaks SU(2)PG symmetry. Another, SU(2)PG invariant solution is
\[ \tilde{L} \sim \frac{1}{\gamma^2}, \quad \gamma^2 = y^A \tilde{\psi}_A. \]

(4.17)

The singularity at \( y^A = 0 \) can be avoided by assuming that the fields \( y^A \) have a non-trivial constant vacuum background. Other possible solutions of (4.15) are famous multi-center solutions breaking SU(2)PG invariance.

4.2. Wess–Zumino action

For the mirror (4, 4, 0) multiplet, the WZ term can be obtained not only as a special limit of the SU(2|1) invariant \( \sigma \)-model term, but can be also constructed independently. The superfield WZ action can be written as an integral over the analytic superspace
\[ S_{WZ}(Y, \dot{Y}) = -\gamma \int d\zeta (\bar{\theta}^{+} \dot{D}^{+} + \theta^{+} D^{+}) f(Y, \dot{Y}). \]

(4.18)

Since we integrate over the analytic superspace, we need to impose the analyticity condition (2.32) on the superfield Lagrangian. It gives the condition
One can define the background 4-vector \( \tilde{A}_{\tilde{B}} \) as
\[
\tilde{A}_{\tilde{B}} = i\partial_{\tilde{B}}f, \quad \tilde{A}_{\tilde{A}} = -i\partial_{\tilde{A}}f, \quad y^A = y^{\tilde{A}}, \quad y^{\tilde{A}} = y^A.
\] (4.20)
Then equation (4.19) implies the self-duality condition for \( \tilde{A}_{\tilde{A}} \)
\[
\partial_{\tilde{A}'}\tilde{A}_{\tilde{B}} - \partial_{\tilde{B}'}\tilde{A}_{\tilde{A}} = \epsilon_{\tilde{A}'}\tilde{B}_{\tilde{A}\tilde{B}}, \quad \tilde{B}_{\tilde{A}\tilde{B}} = -2i\partial_{\tilde{A}}\partial_{\tilde{B}}f,
\] (4.21)
as well as the transversal gauge condition
\[
\partial_{\tilde{A}'}\tilde{A}_{\tilde{A}} = 0.
\] (4.22)
In addition, the requirement of \( SU(2|1) \) invariance gives rise to a new constraint for \( f \) at \( m = 0 \) (cf (4.8)):
\[
m\tilde{F}(Y, \tilde{Y}) = m(y^{\tilde{B}}\partial_{\tilde{B}} - \tilde{y}^{\tilde{B}}\partial_{\tilde{B}})f(y, \tilde{y}) = 0.
\] (4.23)
This condition amounts to the invariance of (4.18) under the internal \( U(1) \) symmetry
\[
f(Y^A, \tilde{Y}_B) = f(U_B^{\dagger}), \quad U_B^\dagger := Y^A\tilde{Y}_B.
\] (4.24)
In the limit \( m = 0 \), the matrix generator \( \tilde{F} \) becomes an external automorphism generator and the condition (4.23) is satisfied trivially, without imposing any constraints on \( f(Y^A, \tilde{Y}_B) \).

The component Lagrangian corresponding to the action (4.18) reads
\[
\tilde{L}_{WZ} = 2\gamma \left\{ i\left(y^A\partial \tilde{A} - \tilde{y}^A\partial \tilde{A} - \frac{m}{2}\left(y^A\partial f + \tilde{y}^A\partial \tilde{f}\right) - \frac{1}{2}\psi^A\psi^{\tilde{A}}\partial \tilde{f}\right) \right\}.
\] (4.25)
Employing the conditions (4.19), (4.23), one can directly check that this Lagrangian is invariant under the supersymmetry transformations (4.6). The first term in (4.25) can be concisely rewritten through the external gauge field as
\[
i\left( y^A\partial \tilde{A} - \tilde{y}^A\partial \tilde{A}\right) = y^A\tilde{A}_{\tilde{A}}.
\] (4.26)
Note that the \( SU'(2) \) symmetry is broken in the full Lagrangian (4.25).

Clearly, the Lagrangian (4.25) can be identified with the Lagrangian (4.14), where
\[
\tilde{G} = \Delta_{\tilde{G}}L(y, \tilde{y}) = 0, \quad L \equiv f, \quad m \sim \gamma.
\] (4.27)
The constraint (4.23) amounts to the condition (4.8).

### 4.3. Hamiltonian formalism and quantum supercharges

We start with the total Lagrangian \( \tilde{L} + \tilde{L}_{WZ} \). The corresponding canonical Hamiltonian reads
\[
H = -\frac{1}{2}e^{AB}\tilde{G}^{-1}\left[ p_A - 2i\left(m\partial_\tilde{B}L + \gamma\partial_\tilde{B}f\right) + \frac{i}{2}\psi^c\psi^A\partial_\tilde{B}\tilde{G}\right]
\times\left[ p_\tilde{B} + 2i\left(m\partial_\tilde{A}L + \gamma\partial_\tilde{A}f\right) + \frac{i}{2}\psi^c\psi^A\partial_\tilde{B}\tilde{G}\right] - \frac{1}{48}\psi^c\psi^A\psi^B\psi^\tilde{B}\Delta_{\tilde{G}}\tilde{G}
+ \psi^A\psi^\tilde{A}\left(m\partial_\tilde{A}L + \gamma\partial_\tilde{A}f\right) - \frac{1}{2}m(y^A\partial_\tilde{A} - \tilde{y}^A\partial_\tilde{A}).
\] (4.28)
The relevant supercharges and the rest of the bosonic generators are given by

\[ Q^i = \psi^{IA} \left[ p_A - 2i( m \partial_a \hat{L} + \gamma \partial_d f) + \frac{i}{6} \psi^{IC} \psi_A \partial_c \hat{G} \right], \]

\[ \tilde{Q}_i = \psi^{IA} \left[ \tilde{p}_A + 2i( m \partial_a \hat{L} + \gamma \partial_d f) + \frac{i}{6} \psi^{IC} \psi_A \partial_c \hat{G} \right], \quad (4.29) \]

\[ F = -\frac{i}{2} \left( y^A \partial_A - \tilde{y}^A \tilde{\partial}_A \right), \quad I_k = \frac{1}{2} \psi^{IA} \psi_{kA} \hat{G}. \quad (4.30) \]

We impose the Poisson brackets and the Dirac brackets as

\[ \{ p_A, y^B \} = -\delta^B_A, \quad \{ \tilde{p}_A, y^B \} = -\delta^B_A, \quad \{ \psi^{IA}, \psi_{kA} \} = -i \hat{G}^{-1} \delta^A_B \epsilon^i_k, \]

\[ \{ p_A, \psi_{kA} \} = \frac{1}{2} \psi_{kB} \hat{G}^{-1} \partial_k \hat{G}, \quad \{ \tilde{p}_A, \psi_{kA} \} = \frac{1}{2} \psi_{kB} \hat{G}^{-1} \partial_k \hat{G}. \quad (4.31) \]

After redefining

\[ \psi^{IA} = \tilde{G}^{-\frac{1}{2}} \xi^A, \quad (4.32) \]

the brackets are simplified to

\[ \{ p_A, y^B \} = -\delta^B_A, \quad \{ \tilde{p}_A, \tilde{y}^B \} = -\delta^B_A, \quad \{ \xi^A, \xi_{kA} \} = -i \delta^A_B \epsilon^i_k. \quad (4.33) \]

We quantize these brackets in the standard way

\[ p_A = -i \partial_A, \quad \tilde{p}_A = -i \tilde{\partial}_A, \quad \xi_{kA} = \partial_j \xi^{JA}, \]

\[ [p_A, y^B] = -i \delta^B_A, \quad [\tilde{p}_A, \tilde{y}^B] = -i \delta^B_A, \quad [\xi^A, \xi_{kA}] = \delta^A_B \epsilon^i_k. \quad (4.34) \]

The quantum supercharges obtained from the classical ones by the general procedure of [21] are given by

\[ Q^{(\text{cov})}_i = -i \tilde{G}^{-\frac{1}{2}} \tilde{\xi}^{IA} \left[ \partial_A + 2( m \partial_a \tilde{L} + \gamma \partial_d f) - \frac{1}{6} \epsilon^{EC} \xi_{kA} \hat{G}^{-1} \partial_c \hat{G} \right]/\tilde{G}, \]

\[ \tilde{Q}^{(\text{cov})}_i = -i \tilde{G}^{-\frac{1}{2}} \tilde{\xi}^{IA} \left[ \tilde{\partial}_A - 2( m \partial_B \tilde{L} + \gamma \partial_d f) - \frac{1}{6} \epsilon^{EC} \tilde{\xi}_{kA} \hat{G}^{-1} \partial_c \hat{G} \right]/\tilde{G}. \quad (4.35) \]

Their anticommutator yields the quantum Hamiltonian

\[ H^{(\text{cov})} = \frac{1}{4} \epsilon^{AB} \tilde{G}^{-1} \left[ \partial_A + 2( m \partial_a \tilde{L} + \gamma \partial_d f) + \frac{1}{2} \xi_{kA} \epsilon^{IC} \hat{G}^{-1} \partial_c \hat{G} \right] \]

\[ \times \left[ \tilde{\partial}_B - 2( m \partial_B \tilde{L} + \gamma \partial_d f) + \frac{1}{2} \xi_{kB} \epsilon^{IC} \tilde{G}^{-1} \partial_c \hat{G} \right] \]

\[ + \frac{1}{48} \epsilon^{IA} \epsilon^{BC} \epsilon^{kA} \xi_{kB} + 2 \tilde{G}^{-2} \Delta \tilde{G} + \xi^{IA} \epsilon^{kA} \tilde{G}^{-1} \left( m \partial_A \partial_B \tilde{L} + \gamma \partial_d f \right) \]

\[ - \frac{m}{2} \left( y^A \partial_A - \tilde{y}^A \tilde{\partial}_A \right) - \frac{m}{2}, \quad (4.36) \]

as well as the remaining bosonic generators:

\[ F = -\frac{1}{2} \left( y^A \partial_A - \tilde{y}^A \tilde{\partial}_A \right), \quad I_k = \frac{1}{2} \xi^{IA} \xi_{kA}. \quad (4.37) \]
In analogy with (3.39), we can represent the supercharges as
\[ Q_{(cov)i} = e^{-2 m L} Q^{(m=0)}_{(cov)i} e^{2 m L}, \quad \tilde{Q}_{(cov)i} = e^{2 m L} \tilde{Q}^{(m=0)}_{(cov)i} e^{-2 m L}, \]
thereby relating them to the particular case of the undeformed supercharges of the ‘flat’ mirror (4, 4, 0) multiplet.

### 4.4. The free model

As an instructive example, here we consider the free model with $\tilde{G} = 1$ and the simplest WZ term (4.25) in which the function $f$ has been chosen as (recall (4.16))
\[ f = \frac{1}{4} c_{AB} y^A y^B, \quad c_{AB} = c_{BA}, \quad (c_{AB}) = -c_{BA}. \]

The corresponding total component Lagrangian reads:
\[ \mathcal{L}_{\text{free}} = 2 y^A \xi^B + \frac{i}{2} \xi^A \xi^B - \frac{i}{2} m \left( y^A \gamma^B - y^A \gamma^B \right) + \frac{\gamma}{2} c_{AB} \left\{ i (y^A \gamma^B - \gamma^A y^B) - \frac{m}{2} (y^A \gamma^B + \gamma^A y^B) - \frac{1}{2} \xi^A \xi^B \right\}. \]

One can choose the $SU(2)_{PG}$ frame in which the triplet $c_{AB}$ is reduced to $c_{12} = c_{21} = 1, \quad c_{11} = c_{22} = 0.$

For further use, we define the operators
\[ \nabla_A^+ = \partial_A \pm \frac{1}{2} (m \epsilon_{AC} + \gamma c_{AC}) y^C, \quad \nabla_B^- = \partial_B \pm \frac{1}{2} (m \epsilon_{BD} - \gamma c_{BD}) y^D, \]
which form the following algebra
\[ \left[ \nabla_A^+, \nabla_B^- \right] = \mp (m \epsilon_{AB} + \gamma c_{AB}). \]

The quantum Hamiltonian reads
\[ H = \frac{1}{2} \epsilon^{AB} \nabla_B^- \nabla_A^+ + \frac{\gamma}{2} c_{AB} \xi^A \xi^B - \frac{m}{2} (y^A \partial_A - \gamma \partial_A) + \frac{m}{2} \partial^2, \]
and the remaining $SU(2|1)$ generators are written as
\[ Q^i = -i \xi^i \nabla^-_A, \quad \tilde{Q}_i = -i \xi^i \nabla^+_B, \quad F = \frac{1}{2} \left( y^A \partial_A - \gamma \partial_A \right), \quad I_3 = \frac{1}{2} \xi^A \xi^A. \]

We construct the bosonic wave functions in terms of the operators $\nabla_{\pm}^A, \tilde{\nabla}_{\pm}^A$ satisfying
\[ \left[ H, \nabla_A^\pm \right] = \frac{\gamma}{2} \epsilon^B_A \nabla_B^\pm, \quad \left[ H, \tilde{\nabla}_A^\pm \right] = \frac{\gamma}{2} \epsilon^B_A \tilde{\nabla}_B^\pm, \]
\[ \left[ H, \nabla_A^- \right] = \frac{m}{2} \nabla_A^+, \quad \left[ H, \tilde{\nabla}_A^- \right] = -\frac{m}{2} \tilde{\nabla}_A^+, \quad \left[ H, \tilde{\nabla}_A^- \right] = \frac{m}{2} \nabla_A^- . \]

Imposing the following physical conditions
\[ \nabla_1^+ |0\rangle = \nabla_1^- |0\rangle = \tilde{\nabla}_A^- |0\rangle = 0, \quad \xi^i |0\rangle = 0. \]
we make sure that the spectrum of the Hamiltonian (4.44) is bounded from below for $\gamma > 0$ and $m > 0$, so that the ground state $\langle 0 \rangle$ is the lowest level. From these conditions, we find that $m = \gamma$ and the ground state wave function is obtained as
The conditions (4.47) firmly imply that the ground state $|0\rangle$ is annihilated by both supercharges defined in (4.45):

$$Q^i |0\rangle = \hat{Q}_i |0\rangle = 0.$$  \tag{4.49}

The ground state acquires the minimal energy value $E = 0$ and the Casimir $C_2$ is vanishing on it.

For the choice $\gamma = m$ the Hamiltonian (4.44) takes the form

$$H = \frac{1}{2} \epsilon^{AB} \nabla^+_B \bar{\nabla}^+_A + \frac{m}{2} \xi^i \xi^i - \frac{m}{2} \left( y^A \partial_A - \bar{y}^A \bar{\partial}_A \right),$$  \tag{4.50}

where

$$\nabla^+_1 = \partial_1 \pm m \gamma_1, \quad \nabla^+_2 = \partial_2, \quad \nabla^+_2 = \partial_2 + m \gamma_2.$$  \tag{4.51}

The generators (4.45) can also be written through these operators. The higher-order bosonic states must be constructed in terms of the operators $\nabla^+_1$, $\nabla^+_2$ the commutators of which with the Hamiltonian are as follows

$$\left[ H, \nabla^+_2 \right] = \frac{m}{2} \nabla^+_2, \quad \left[ H, \nabla^+_2 \right] = \frac{m}{2} \nabla^+_2.$$  \tag{4.52}

The bosonic state $|\ell; n\rangle$ is defined as

$$|\ell; n\rangle = (\nabla^+_1)^\ell \left( \nabla^+_2 \right)^n |0\rangle.$$  \tag{4.53}

Superwave function $\Omega^{(\ell,n)}$ is obtained as a sum of the relevant fermionic descendants of $|\ell; n\rangle$ produced by action of the supercharges as

$$Q^i |\ell; n\rangle = 2\epsilon \ell m \xi^i |\ell - 1; n\rangle, \quad Q^2 |\ell; n\rangle = 4\ell (\ell - 1) m^2 \xi^i \xi^i |\ell - 2; n\rangle, \quad \hat{Q}_i |\ell; n\rangle = 0.$$  \tag{4.54}

Here, $\Omega^{(0,0)} = |0\rangle$. One can see that the functions $\Omega^{(0,n)}$ and $\Omega^{(1,n)}$ form singlet and triplet states, respectively.

Then, the eigenvalues of (4.44) are

$$H \Omega^{(\ell,n)} = \frac{m}{2} (\ell + n) \Omega^{(\ell,n)}, \quad m > 0.$$  \tag{4.55}

The eigenvalues (C.3) of Casimir operators are given by

$$\beta = \frac{\ell}{2}, \quad \lambda = \frac{1}{2}, \quad \text{for} \ \ell \neq 0,$$

$$\beta = \lambda = 0, \quad \text{for} \ \ell = 0.$$  \tag{4.56}

Casimir operators take zero eigenvalues for $\ell = 0$ and $\ell = 1$. Hence, the functions $\Omega^{(0,n)}$ and $\Omega^{(1,n)}$ correspond to atypical representations of $SU(2|1)$ with non-equal number of bosonic and fermionic states. The functions $\Omega^{(\ell,n)}$ with $\ell \geq 2$ correspond to the typical 4-fold representations of $SU(2|1)$. The same degeneracies with respect to $\ell$ were observed in [4].

In fact, this quantum free model can be identified with one of the $SU(2|1)$ invariant models on a complex plane considered in [4].

One can exclude the operators $\nabla^+_2$ and $\nabla^+_2$ from the sets (4.50) and (4.45), since they annihilate the ground state $|0\rangle$ and all other states. Then, if we redefine the system as
we obtain exactly the space of states of the model on a complex plane with $\kappa = 1/4$ constructed by the authors in [4] for the $SU(2|1)$ multiplet (2, 4, 2).

This correspondence can be understood from the viewpoint of the Hamiltonian reduction with respect to some shift isometry of the Lagrangian (4.40) with $\gamma = m$. Indeed, in the frame (4.41) it can be written as

$$\mathcal{L}_{\text{free}} = 2\gamma_{i}^{a}\gamma_{i}^{a} + \frac{i}{2} \xi_{i}^{a} \xi_{i}^{a} - \frac{m}{2} \xi_{i}^{a} \xi_{i}^{a} - \frac{m^{2}}{2} y_{i}^{a} y_{i}^{a} + \frac{2}{2} \left( y_{j}^{2} - \frac{i}{2} m y_{j}^{2} \right) \left( y_{j}^{2} + \frac{i}{2} m y_{j}^{2} \right),$$

and it is invariant under the transformations $\delta y_{j}^{2} = \beta e^{i\eta_{j}}, \delta y_{j}^{2} = \beta^{-i\eta_{j}}, \beta$ being a constant complex parameter.

Finally, let us briefly discuss the interesting case with $m_{0}, g_{0} = 0$. In this case the Hamiltonian becomes purely bosonic and commutes with $\xi^{a}$ and $\nabla_{a}^{\pm}, \bar{\nabla}_{a}^{\pm}$, which can be treated as the generators of the 'magnetic supertranslations' in the target space (the Lagrangian (4.40) at $\gamma = 0$ is invariant under the independent shifts of the variables $\xi^{a}$ and $y^{a}$). We rewrite the Hamiltonian (4.44) as

$$H = \frac{1}{2} e^{AB} \nabla_{A} \phi_{B}^{\pm} + \frac{m}{2}.$$

By imposing the physical condition $\nabla_{i} \psi_{0} = 0$, we define the superwave function $\psi_{0}$ with the lowest energy $m/2$:

$$\psi_{0} = \left[ g_{0}(y^{A}) + g_{i}(y^{A}) \xi_{i}^{a} + g_{i}(y^{A}) \xi_{i}^{a} \xi_{i}^{a} \right] |0\rangle, \quad |0\rangle = e^{-\frac{1}{2} m y^{a}}, \quad \xi_{i}^{a} |0\rangle = 0. \quad (4.60)$$

Here $g_{0}, g_{i}, g_{i}$ are arbitrary holomorphic polynomials of $y^{A}$ which can all be generated through action of $\nabla_{i}^{\pm}$ on $|0\rangle$, taking into account that $\nabla_{i}^{\pm} |0\rangle = 0$. The infinite degeneracy of the ground state $\psi_{0}$ is just due to the above-mentioned symmetry of (4.40) at $\gamma = 0$ under the 'magnetic supertranslation' group.

Casimir operators (C.1), (C.2) take zero eigenvalues only on the following states:

$$\Omega_{0} = \left( \nabla_{i}^{\pm} - \frac{1}{2} \xi_{i}^{a} \xi_{i}^{a} \nabla_{i}^{\pm} \right) |0\rangle, \quad \Omega^{\pm} = \xi_{i}^{a} |0\rangle. \quad (4.61)$$

These one bosonic and two fermionic states form the fundamental atypical representation of $SU(2|1)$, and the supercharges (4.45) act on them as follows

$$Q^{i} \Omega_{0} = im \Omega^{i}, \quad \bar{Q}_{i} \Omega_{0} = 0,$$

$$Q^{i} \Omega^{i} = 0, \quad \bar{Q}_{i} \Omega^{i} = -i \delta_{i}^{j} \Omega_{0}. \quad (4.62)$$

All other states correspond to the 4-fold typical representations of $SU(2|1)$. For instance, the state $|0\rangle$ is a component of the following $SU(2|1)$ supermultiplet

$$|0\rangle, \quad \xi_{i}^{a} \nabla_{i}^{\pm} |0\rangle, \quad \frac{1}{2} \xi_{i}^{a} \xi_{i}^{b} \nabla_{i}^{\pm} \nabla_{i}^{\pm} |0\rangle. \quad (4.63)$$

Thus the ground state superwave function $\psi_{0}$ is represented as an infinite sum of non-singlet states of $SU(2|1)$. In other words, among states from which $\psi_{0}$ is composed there is no state which would be simultaneously annihilated by both supercharges, $Q^{i}$ and $\bar{Q}_{i}$. So $SU(2|1)$
supersymmetry is spontaneously broken at $\gamma = 0$, in contrast to the case of $\gamma = m \neq 0$. The option with $\gamma = m$ also features the spontaneous breaking of $SU(2|1)$.

5. Superconformal models

In this section we consider trigonometric superconformal models built on the $(4, 4, 0)SU(2|1)$ superfields. Their construction and basic features closely follow the pattern of superconformal models for the $SU(2|1)$ multiplets $(1, 4, 3)$ and $(2, 4, 2)$ discussed in [6].

5.1. Generalities

The most general $\mathcal{N} = 4, d = 1$ superconformal algebra is $D(2, 1; \alpha)$ [24, 25], $\alpha$ being a real parameter. The proper embedding of the superalgebra $su(2|1)$ into $D(2, 1; \alpha)$ is achieved through the following redefinition of the $U(1)$ generator $\mathcal{H}$ in (2.1):

$$\mathcal{H} = \mathcal{H} + (1 + \alpha)\mu F, \quad m = -\alpha \mu. \quad (5.1)$$

Correspondingly, the $su(2|1)$ superalgebra (2.1) is redefined as

$$\{Q^i, \bar{Q}_j\} = -2\alpha \mu t^i_j + 2\delta^i_j[\mathcal{H} + (1 + \alpha)\mu F],$$
$$\left[ I^i_j, \bar{Q}_j \right] = \frac{1}{2}\delta^i_j \bar{Q}_i - \delta^i_j \bar{Q}_j, \quad \left[ I^i_j, Q^j \right] = \delta^i_j Q^j - \frac{1}{2}\delta^i_j Q^k,$$
$$\left[ F, \bar{Q}_j \right] = -\frac{1}{2} \bar{Q}_j, \quad \left[ F, Q^k \right] = \frac{1}{2} Q^k,$$
$$\left[ \mathcal{H}, \bar{Q}_j \right] = \frac{\mu}{2} \bar{Q}_j, \quad \left[ \mathcal{H}, Q^k \right] = -\frac{\mu}{2} Q^k. \quad (5.2)$$

The particular choice $\alpha = -1$ in (5.2) yields (2.1) at $m = \mu$, with $F$ being decoupled and becoming an external automorphism generator. At $\alpha = 0$, the superalgebra (5.2) converts into some $U(1)$-deformed ‘Poincaré’ $\mathcal{N} = 4, d = 1$ superalgebra,

$$\{Q^i, \bar{Q}_j\} = 2\delta^i_j (\mathcal{H} + \mu F), \quad (5.3)$$

with $I^i_j$ becoming some external $SU(2)$ automorphism generators (the remaining non-zero commutation relations are the same as in (5.2)).

The basic virtue of rewriting the superalgebra $su(2|1)$ in the form (5.2) is that the closure of (5.2) with its $-\mu$ counterpart in the properly defined basis of the $SU(2|1)$ superspace is just $\mathcal{N} = 4$ superconformal algebra $D(2, 1; \alpha)$ in the so-called trigonometric realization [6]. This remarkable property implies the simple criterion for one or another $SU(2|1)$ invariant action to be superconformal: it should be an even function of the parameter $\mu$. The same property applies to the case $\alpha = 0$ too. Once again, $D(2, 1; 0)$ is contained in a closure of (5.3) with its $-\mu$ counterpart. The generator $\mathcal{H}$ in (5.2), (5.3) is embedded into the $d = 1$ conformal algebra so$(2, 1)$ in the standard basis as [6]

$$\mathcal{H} = \mathcal{H} + \frac{i \mu}{4} \vec{k}. \quad (5.4)$$

To ensure the analogous property in application to the harmonic and analytic $SU(2|1)$ superspaces, we need to appropriately modify the original definitions of these superspaces. We again pass to the notations (2.5) and rewrite the superalgebra (5.2) in terms of the generators $\{Q^\pm, \bar{Q}^\pm, I^{++}, I^{--}, I^0\}$. Then we extend this superalgebra by the $SU(2)_{\text{ext}}$

8 This is the basis, in which $[\mathcal{D}, \mathcal{H}] = -i \mathcal{H}$, $[\mathcal{D}, \mathcal{K}] = i \mathcal{K}$, $[\mathcal{H}, \mathcal{K}] = 2i \mathcal{D}$. 

\[ \text{Class. Quantum Grav. 33 (2016) 055001} \]
generators \(\{T^0, T^{++}, T^-\}\) and define the \(\alpha \neq 0\) analogs of the harmonic \(SU(2|1)\) super-space \((2.9)\) in the analytic basis as the coset superspaces
\[
\begin{align*}
\{\mathcal{H}, Q^\pm, \bar{Q}^\pm, F, I^\pm, I^0, T^{+\pm}, T^0\}, & \quad \alpha = 0, -1, \\
\{F, I^{++}, I^0, I^{--} - T^-, T^0\}, & \quad \alpha = -1.
\end{align*}
\]
\[(5.5)\]

The relevant \(\epsilon, \bar{\epsilon}\) coordinate transformations are obtained from \((2.10)\) via the substitution \(m \rightarrow -\alpha\mu\) and the redefinition \((B.16)\) of the Grassmann coordinates, such that the generator \(\mathcal{H}\) becomes the purely time-translation one, \(\mathcal{H} = i\partial / \partial t.A\) [6]. These transformations together with their \(-\mu\) counterparts produce, as a closure, some realization of \(D(2, 1; \alpha)\) on the harmonic superspace coordinates. This realization preserves the \(SU(2|1)\) harmonic analyticity, but only in the \(\mu = 0\) limit it reproduces the realization given in [12]. The main difference between the \(\mu \neq 0\) and \(\mu = 0\) options is that the former yields the trigonometric realization of the \(d = 1\) conformal \(SO(2, 1)\) transformations [26], while the latter gives rise to their standard parabolic realization.

For the special case \(\alpha = 0\), the appropriate coset superspace is defined as
\[
\begin{align*}
\{\mathcal{H}, Q^\pm, \bar{Q}^\pm, F, T^{+\pm}, T^0\}, & \quad \alpha = 0.
\end{align*}
\]
\[(5.6)\]

Despite the fact that the coordinate transformations \((2.10)\) at \(\alpha = 0\) take the form corresponding to the standard flat harmonic \(\mathcal{N} = 4, d = 1\) superspace [12] (because \(m = -\alpha\mu\), superconformal transformations appearing as a closure of these \(\alpha = 0\) transformations (in some \(\mu\)-dependent superspace basis) with their \(-\mu\) counterparts still constitute the trigonometric realization of \(D(2, 1; 0)\). Note that the harmonic variables \(w_i^\pm\) in the \(\alpha = 0\) case do not transform under the \(\epsilon, \bar{\epsilon}\) transformations at all (as well as under the full set of the \(D(2, 1; 0)\) transformations, including those of the relevant \(SU(2)_\text{int}\), which act only on the Grassmann coordinates).

For the generic \(\alpha\), the trigonometric superconformal transformations of the deformed \(\mathcal{N} = 4, d = 1\) superspace coordinates were given in [6]. In order to find the trigonometric realization in the analytic basis of harmonic superspace, one should use the relations \((B.4)\) with \(m = -\alpha\mu\) or the relations \((B.17)\). The basic relations of the CR structure \((2.24)\) are rewritten as
\[
\{D^+, \bar{D}^+\} = 2\alpha\mu D^{++}, \quad [D^{++}, D^+] = [\bar{D}^{++}, \bar{D}^+] = 0,
\]
\[(5.7)\]

where the covariant derivatives \(D^+, \bar{D}^+\) and \(D^{++}\) (in the original coordinates) are given by the following expressions valid for any \(\alpha\):
\[
\begin{align*}
D^+ & = e^{-i\mu t.A} \left[ \frac{\partial}{\partial \theta^-} - \alpha\mu \theta^- \bar{D}^{++} \right], \quad \bar{D}^+ = e^{i\mu t.A} \left[ -\frac{\partial}{\partial \bar{\theta}^-} + \alpha\mu \theta^- \bar{D}^{++} \right], \\
D^{++} & = \left(1 - \alpha\mu \bar{\theta}^+ \theta^- + \alpha\mu \theta^+ \bar{\theta}^- \right)^{-1} \partial^{++} + 2i \theta^+ \bar{\theta}^+ \partial(\Lambda) + 2(1 + \alpha)\mu \theta^+ \bar{\theta}^+ F \\
& + \theta^+ \frac{\partial}{\partial \theta^-} + \bar{\theta}^+ \frac{\partial}{\partial \bar{\theta}^-}.
\end{align*}
\]
\[(5.8)\]

The factors \(e^{\pm i\mu t.A}\) appear in \((5.8)\) due to the property that the elements of the supercosets \((5.5), (5.6)\) are related to an element of \((2.9)\) (and to that of the flat \(\mathcal{N} = 4, d = 1\) harmonic
supercoset in the case \((5.6)\) through multiplication of the latter element from the right by the exponential \(\exp(-iyF)\) (cf analogous relations in the non-harmonic case \([6]\)).

To define the \((4, 4, 0)\) superfields adapted to the supercosets \((5.5)\) and \((5.6)\), one should use the constraints \((3.1), (4.1)\) in which the corresponding covariant derivatives are replaced by the expressions \((5.8)\). Note that at any \(\alpha \neq 0\) the constraints \((2.31)\) give rise to \((2.32)\). At \(\alpha = 0\), equation \((2.32)\) should be imposed independently.

The superfield \(\epsilon, \bar{\epsilon}\) transformation law generalizing equation \((2.25)\) to the harmonic supercosets \((5.5)\) and \((5.6)\) is as follows (in the original coordinates)

\[
\delta \Phi = \mu \left[ 2(1 + \alpha) \left( \theta^+ \epsilon^- - \theta^+ \bar{\epsilon}^- \right) \bar{F} + \alpha \left( \bar{\theta}^+ \epsilon^- + \bar{\theta}^+ \bar{\epsilon}^- \right) \mathcal{D}^0 + \alpha \left( \theta^{-} \epsilon^+ + \theta^{-} \bar{\epsilon}^+ \right) \mathcal{D}^{++} \right] \Phi.
\]

\[\text{(5.9)}\]

### 5.2. Superconformal Lagrangians for \(q^{+a}\)

Skipping details, one can show that all component results for the \(SU(2|1)\) superfields \(q^{+a}\) defined on the supercosets \((5.5), (5.6)\) can be obtained from those for \(q^{-a}\) on the supercoset \((2.9)\) by substituting \(m \rightarrow -\alpha \mu\) and redefining the fermionic fields as

\[
\psi^a \rightarrow \psi^a e^{\bar{\epsilon} \mu}, \quad \bar{\psi}^a \rightarrow \bar{\psi}^d e^{-\bar{\epsilon} \mu}.
\]

\[\text{(5.10)}\]

This redefinition ensures that the \(U(1)\) generator \(F\) acts only on the fermionic fields and \(H\) is reduced to the pure time derivative \(i\partial_t\) on all fields.

For any \(\alpha\), the transformations \((3.6)\) are modified as

\[
\begin{align*}
\delta x^{ia} &= -\epsilon^i \psi^a 
\delta \psi^a &= \left( 2i \epsilon^i \bar{x}^{\bar{a}} + \alpha \mu \epsilon_i x_a \right) e^{\bar{\epsilon} \mu}, \\
\delta \bar{\psi}^a &= \left( 2i \bar{\epsilon}^a \bar{x}^\alpha - \alpha \mu \bar{\epsilon}^a x^\alpha \right) e^{-\bar{\epsilon} \mu}.
\end{align*}
\]

\[\text{(5.11)}\]

The transformations with \(-\mu\) amount to the \(-\mu\) version of the superalgebra \((5.2)\). The two sets of transformations produce as their closure the superconformal group \(D(2, 1; \alpha)\).

At \(\alpha = 0\) the superfields \(q^{+a}\) live on the supercoset \((5.6)\) corresponding to the \(U(1)\)-deformed Poincaré superalgebra \((5.3)\). The relevant component field transformations are the \(\alpha = 0\) version of \((5.11)\). Together with their \(-\mu\) counterparts (forming the \(-\mu\) analog of the algebra \((5.3)\)) they close on

\[D(2, 1; 0) \cong PSU(1, 1|2) \times SU(2).\]

\[\text{(5.12)}\]

As was shown in \([6]\), the trigonometric superconformal models for the multiplets \((1, 4, 3)\) and \((2, 4, 2)\) possess the notable common property that their superconformal Lagrangians are functions of \(\mu^2\) and so they are invariant under the reflection \(\mu \rightarrow -\mu\). The same feature proves to be inherent to the supermultiplets \((4, 4, 0)\) too.

After making the substitution \((5.10)\) in the Lagrangian \((3.19)\), the latter proves to contain no terms linear in \(\mu\) and so is superconformal for the choice \(G = (x^a x_a)^{\frac{1}{2\bar{\alpha}}}\) only. The relevant trigonometric superconformal Lagrangian is a deformation of the parabolic superconformal Lagrangian by the oscillator term \([17]\)

\[
\mathcal{L}_{\text{osc}}^{(\alpha)} = \left[ x^{ia} \partial_a x_{ia} + \frac{i}{2} \left( \bar{\psi}_a \psi^a - \bar{\psi}^a \psi_a \right) + \frac{i}{2} \left( \bar{\psi}_a \psi^a + \psi^a \bar{\psi}_a \right) x^{ia} x_{ia} \right] \frac{x^{ia} x_{ia}}{4}.
\]

\[\text{(5.13)}\]

The free Lagrangian corresponds to the choice \(\alpha = 1\).
In the particular case $\alpha = 0$ the expression (5.13) becomes singular and, in order to construct the meaningful superconformal action, one is led to redefine the field $x^{ia}$ as

$$x^{ia} \to x^{ia} + \frac{\rho^{ia}}{\alpha},$$

and then send $\alpha \to 0$. This gives rise to the inhomogeneous $\rho$ dependent transformations for the $\alpha = 0$ case

$$\delta x^{ia} = - \epsilon^i_{\lambda j} \bar{\psi}^{j\mu} e^{-\tilde{\delta} / \mu},$$
$$\delta \bar{\psi}^a = (2i\epsilon_k \bar{x}^a_k + \mu \epsilon_k \rho^{ka}) \epsilon^{j\mu}, \quad \delta \psi^a = (2i\epsilon^k \bar{x}^a_k - \mu \epsilon^k \rho^{ka}) e^{-\tilde{\delta} / \mu}.$$

The superconformal Lagrangian for $\alpha = 0$ is obtained as the limit

$$L^{(\alpha = 0)}_{\text{SC}} = \lim_{\alpha \to 0} L^{(\alpha, \rho)}_{\text{SC}},$$

with detaching some singular overall factor in the end. The Lagrangian is given by

$$L^{(\alpha = 0)}_{\text{SC}} = \left[ x^{ia} \bar{x}_{ia} + \frac{1}{2} \left( \bar{\psi}^a \psi^a - \frac{\tilde{\delta}}{\mu} \psi^a \right) - \frac{1}{2} \bar{\psi}^a \left( \bar{\psi}^a \bar{\psi}^a + \psi^a \psi^a \right) \right] \partial \psi^a - \frac{1}{16} \left( \bar{\psi} \right)^2 (\psi)^2 \Delta \psi - \frac{\mu^2 \rho^2}{4} \exp \left\{ \frac{2 \rho^{ia} \bar{\psi}^a}{\rho^2} \right\}.$$

Here, both the Pauli–Gürsey symmetry $SU(2)_{\text{PG}}$ acting on the indices $a$ and the automorphism $SU(2)$ symmetry with generators $I^a_k$ are broken down to the diagonal subgroup $SU(2)_{\text{diag}}$. The internal subgroup $SU(2)_{\text{int}} \subset PSU(1, 1)$ acts only on the fermionic fields. So, the Lagrangian (5.17) is invariant under the superconformal symmetry $PSU(1, 1) \rtimes SU(2)_{\text{diag}}$. The parabolic analog of this Lagrangian was considered in [10].

Note that the WZ Lagrangian (3.25) is $\alpha = 0$ superconformal, because it is invariant under both the transformations (5.11) with $\alpha = 0$ and their $-\mu$ counterparts. Indeed, it is invariant under the redefinition (5.10) and the second term in (3.29) disappears, when $m = -\alpha \mu$ and $\alpha = 0$. No additional parameters $\rho$ are required in this case.

### 5.3. Superconformal Lagrangians for $Y^{+A}$

Next, we consider the mirror multiplet $(4, 4, 0)$, proceeding from the superspaces (5.5), (5.6).

Passing to the superspaces (5.5), (5.6) should be accompanied by the following field redefinitions (cf (5.10)):

$$Y^A \to Y^A e^{-\tilde{\delta} / \mu}, \quad Y^{\bar{A}} \to Y^{\bar{A}} e^{\tilde{\delta} / \mu},$$
$$y^A \to y^A e^{-\tilde{\delta} / \mu}, \quad y^{\bar{A}} \to y^{\bar{A}} e^{\tilde{\delta} / \mu}, \quad m = -\alpha \mu.$$

The constraints (4.1) are now imposed with the derivatives (5.8). The constraints (4.2) become

$$(1 + \alpha) \mu \tilde{F} \tilde{Y} \bar{A} = - \frac{(1 + \alpha)}{2} \mu \tilde{Y} \bar{A}, \quad (1 + \alpha) \mu \tilde{F} Y^A = \frac{(1 + \alpha)}{2} \mu Y^A.$$

The redefinitions (5.18) bring the transformations (4.6) to the form:

$$\delta y^A = - \epsilon^i_{\lambda j} \bar{\psi}^{j\mu} e^{-\tilde{\delta} / \mu}, \quad \delta y^{\bar{A}} = - \bar{\epsilon}^i_{\lambda j} \psi^{j\mu} e^{-\tilde{\delta} / \mu},$$
$$\delta \bar{\psi}^a = \bar{\epsilon}^i [2i \bar{y}^A + (1 + \alpha) \mu y^A] e^{-\tilde{\delta} / \mu} - \epsilon^i [2i \bar{y}^{\bar{A}} - (1 + \alpha) \mu y^{\bar{A}}] \epsilon^{j\mu}.$$
These transformations, together with those in which the replacement $m \leftrightarrow -m$ is made, generate superconformal $D(2, 1; \alpha)$ transformations.

Requiring the superconformal invariance of the $\sigma$-model Lagrangian obtained from (4.10) through the changes (5.18) (i.e. just demanding the terms $\sim \mu$ to cancel), we find that

$$G = (y^A \psi_A)^{-\frac{1}{2m}}.$$  \hspace{1cm} (5.21)

The resulting component Lagrangian reads

$$\mathcal{L}^{(\alpha)}_{sc} = \left[ 2y^A \psi_A + \frac{i}{2} \psi^{[A} \phi_{C]} - \frac{i}{2} \psi^{[A} \phi_{C]} \left( y^C \partial_A + \bar{y}^C \bar{\partial}_A \right) \right] + \frac{1}{48} \psi^{[A} \phi_{C]} \psi^{B]} \psi_{[D} \bar{\psi}_{D]} \Delta_y - \frac{(1 - \alpha)^2}{2} y^A \psi_A \left( y^A \psi_A \right)^{-\frac{1}{2m}}. \hspace{1cm} (5.22)$$

It is equivalent to the Lagrangian (5.13) up to the substitution $\alpha \to -(1 + \alpha)$. The free system with $G = G = 1$ corresponds to the choice $\alpha = -2$ in (5.22).

At $\alpha = -1$ the Lagrangian (5.22) is singular. To construct the superconformal $\sigma$-model term in this case, one introduces an arbitrary inhomogeneity parameter $\rho^A$ as

$$y^A \to y^A + \frac{\rho^A}{1 + \alpha}, \quad \bar{y}^A \to \bar{y}^A + \frac{\bar{\rho}^A}{1 + \alpha}, \quad \rho^2 = \rho A \bar{\rho}_A \hspace{1cm} (5.23)$$

and then sends $\alpha \to -1$. At $\alpha = -1$, the transformations (5.20) become

$$\delta y^A = e^2 \left[ 2i\bar{y}^A + \mu \rho^A \right] e^{2i\lambda^A} - e^2 \left[ 2i\bar{y}^A - \mu \rho^A \right] e^{2i\lambda^A}. \hspace{1cm} (5.24)$$

Respectively, the $\alpha = -1$ superconformal $\sigma$-model Lagrangian is written as (up to some divergent overall factor which can be thrown out)

$$\mathcal{L}^{(\alpha = -1)}_{sc} = \left[ 2y^A \psi_A + \frac{i}{2} \psi^{[A} \phi_{C]} - \frac{i}{2} \psi^{[A} \phi_{C]} \left( y^C \partial_A + \bar{y}^C \bar{\partial}_A \right) \right] + \frac{1}{48} \psi^{[A} \phi_{C]} \psi^{B]} \psi_{[D} \bar{\psi}_{D]} \Delta_y - \frac{\mu^2 \rho^2}{4} \exp \left\{-y^A \rho_A + \rho^A \bar{\rho}_A \right\}. \hspace{1cm} (5.25)$$

This Lagrangian can be brought into the same form as the previous $\alpha = 0$ Lagrangian (5.17) by a simple relabeling of the involved fields. The superconformal group is reduced to $D(2, 1; -1) \cong SU(1, 1) \rtimes SU(2)_{\text{diag}}$, where $SU(2)_{\text{diag}}$ is the diagonal subgroup in the product of the external $SU(2)$ (to which the decoupled generator $F$ belongs) and $SU(2)_{\text{PG}}$ (which acts on the indices $A$).

The redefinitions (5.18) bring the WZ Lagrangian (4.25) to the form

$$\mathcal{L}^{(\alpha)}_{WZ} = 2 \gamma \left\{ i \left( y^A \partial_A - \bar{y}^A \bar{\partial}_A \right) - (1 + \alpha) \frac{\mu}{2} \left( y^A \partial_A + \bar{y}^A \bar{\partial}_A \right) \right\}. \hspace{1cm} (5.26)$$

We observe that the trigonometric superconformal WZ Lagrangian can be defined only at $\alpha = -1$, when the term $\sim \mu$ drops out. This Lagrangian is written as

$$\mathcal{L}^{(\alpha = -1)}_{scWZ} = 2 \gamma \left\{ i \left( y^A \partial_A - \bar{y}^A \bar{\partial}_A \right) - \frac{1}{2} \psi^{[A} \phi_{C]} \partial_A \bar{\partial}_B \right\}. \hspace{1cm} (5.27)$$
and it is invariant (up to a total derivative) under the homogeneous transformations (5.20) with \( \alpha = -1 \) and their \( \mu \rightarrow -\mu \) counterparts. Thus the corresponding action reveals invariance under their closure \( PSU(1, 1|2) \).

The function \( f \) satisfies the self-duality conditions (4.19), while the condition (4.23) is modified as

\[
(1 + \alpha) \mu \bar{f}(Y, \bar{Y}) = 0 \quad \Rightarrow \quad (1 + \alpha) \mu \left( y^B \partial_B - \bar{y}^B \partial_B \right) f(y, \bar{y}) = 0.
\]

At \( \alpha = -1 \) this condition is automatically satisfied and so does not impose any restrictions on the function \( f \) in (5.27).

As we briefly discussed in the previous subsection, WZ term (3.25) is superconformal at \( \alpha = 0 \). Using the definition (4.20), it is easy to show the equivalence of (5.27) and (3.25), as well as the equivalence of the relevant superconformal transformations\(^9\). In the next subsection we consider infinite-dimensional symmetries inherent to these WZ terms Note that the sum of the Lagrangians (5.27) and (5.25) is not superconformal, since the supergroup \( PSU(1, 1|2) \) is realized in these Lagrangians by transformations of the different kinds (these are inhomogeneous for (5.25) and homogeneous for (5.27)).

5.4. The centerless \( \mathcal{N} = 4 \) super Virasoro algebra

The superconformal WZ Lagrangian (5.27) is not deformed by \( \mu \). Hence, it is simultaneously invariant under the following transformations\(^10\)

\[
\delta y^A = -\eta^i \psi^A, \quad \delta \bar{y}^A = -\bar{\eta}^i \psi^A, \quad \delta \psi^A = 2i\eta^i \psi^A - 2i\bar{\eta}^i \bar{\psi}^A,
\]

which generate the standard ‘Poincaré’ \( \mathcal{N} = 4, d = 1 \) supersymmetry. The Lie brackets of these additional transformations with both (5.20) and the \(-\mu \) counterparts of (5.20) produce the new bosonic generators

\[
\sim \epsilon^{\pm \mu} i \partial_t.
\]

On the other hand, generators of the conformal algebra so\((2, 1) \subset \text{psu}(1, 1|2)\) have the following trigonometric realization [6]:

\[
\mathcal{H} = i\partial_t, \quad T = e^{-i\mu t} i \partial_t, \quad \bar{T} = e^{i\mu t} i \partial_t.
\]

Commuting the new bosonic generators with these ones, we find that the bosonic subalgebra extends to an infinite-dimensional Virasoro algebra [27]

\[
\left[ L_k, L_n \right] = (k - n)L_{k+n}, \quad L_n = \frac{2i}{\mu} \epsilon^{\pm \mu} \partial_t, \quad k, n \in \mathbb{Z}.
\]

Computing further their commutators with supercharges, we finally find, as the full symmetry of (5.27), the Ramond version of the centerless (small) \( \mathcal{N} = 4 \) super Virasoro algebra\(^11\).

The isomorphic super Virasoro algebra extending the finite-dimensional \( \alpha = 0 \) superconformal algebra \( \text{psu}(1, 1|2) \) is a symmetry of the WZ component Lagrangian (3.25).

As was noticed in [17], with regard to the \((4, 4, 0)\) multiplet the centerless super Virasoro group can possess only homogeneous \((\rho = 0)\) realizations with the scaling dimension \( \lambda_0 = 0 \). This is consistent with the absence of the superconformal WZ action for \( q^{\pm \mu} \) at

\(^9\) Note that this equivalence gets broken if we simultaneously consider WZ terms for both types of the \((4, 4, 0)\) multiplet. This sum cannot be made superconformal by any choice of \( \alpha \).

\(^10\) They coincide with the \( \mu = 0 \) case of the transformations (5.20).

\(^11\) The realization of this \( \mathcal{N} = 4 \) super Virasoro on the fields of the multiplet \((4, 4, 0)\) was found earlier in [17]. The novelty of our consideration is its derivation as a symmetry of the WZ Lagrangians (5.27), (3.25).
\[ \alpha = 0, \text{ since } \alpha \text{ is identified as } \alpha = -2\lambda_D \text{ [17, 28, 29]. The same conclusion is valid for the superfields } Y^A, \bar{Y}^\dot{A}, \text{ with } \alpha = 2\lambda_D - 1. \]

6. Concluding remarks and outlook

In this paper we generalized the \( d = 1 \) harmonic superspace approach [12] to the case of worldline realizations of the supergroup \( SU(2|1) \) as a deformation of the flat \( \mathcal{N} = 4, d = 1 \) supersymmetry and constructed the general superfield and component actions for the \( SU(2|1) \) analogs of the ‘root’ \( \mathcal{N} = 4, d = 1 \) multiplet \((4, 4, 0)\) and its ‘mirror’ version. We also selected the superconformal subclass of general \( SU(2|1) \) invariant actions of these multiplets.

In contrast to the flat harmonic superspace, there is no direct equivalence between the two types of the \((4, 4, 0)\) supermultiplet. In the case of the standard supermultiplet \((4, 4, 0)\), the \( SU(2)_{int} \) subgroup of \( SU(2|1) \) is realized only on the bosonic fields \( \chi^{\mu} \) of \( q^{\pm \mu} \). For the mirror multiplet \((4, 4, 0)\), this group is realized on the fermionic fields \( \psi^{iA} \) of \( Y^A, \bar{Y}^\dot{A} \). One of the manifestations of the non-equivalence just mentioned is the non-existence of the \( SU(2|1) \) invariant WZ action for \( q^{\pm \mu} \) and the existence of such an action for the mirror multiplet. On the other hand, the trigonometric (as well as the parabolic) superconformal models of both \((4, 4, 0)\) multiplets are equivalent to each other, up to the substitutions

\[
\chi^{\mu} \leftrightarrow \psi^{iA}, \quad \psi^{i\alpha} \leftrightarrow \psi^{i\dot{A}} \quad \alpha \leftrightarrow -(1 + \alpha),
\]

As is known, the interchange \( \alpha \leftrightarrow -(1 + \alpha) \) amounts to permuting the \( SU(2) \) and \( SU'(2) \) generators in \( D(2, 1; \alpha) \) and so is an automorphism of this superalgebra.

The scaling dimension parameter \( \lambda_D \) is identified with \( \alpha = -2\lambda_D \) for the standard multiplet \((4, 4, 0)\) and with \( \alpha = 2\lambda_D - 1 \) for the mirror multiplet \((4, 4, 0)\). The superconformal group \( D(2, 1; \alpha) \) with \( \lambda_D = 0 \) can be reduced to the \( PSU(1, 1|2) \times SU(2) \) supergroup for both supermultiplets, at \( \alpha = 0 \) for \( q^{\pm \mu} \) and \( \alpha = -1 \) for \( Y^A, \bar{Y}^\dot{A} \).

For the reader’s convenience, in the table 1 we summarize the results concerning symmetries of the \( \sigma \)-model and WZ actions. An important role is played by the inhomogeneity parameters \( \rho \) allowing one to construct superconformal \( \sigma \)-model actions for the special case \( \lambda_D = 0 \), so that they are invariant under the \( \rho \)-modified inhomogeneous transformations \((5.15), (5.24)\). On the other hand, superconformal WZ actions at \( \lambda_D = 0 \) are invariant under the homogeneous superconformal transformations given by \((5.11)\) and \((5.20)\) (with \( \alpha = 0 \) and \( \alpha = -1 \), respectively).

In the case of the mirror multiplet, we can consider \( SU(2|1)\sigma \)-model actions combined with WZ actions (the Lagrangian \((4.40)\) provides an example of such a system). The combined actions, being \( SU(2|1) \) invariant, cannot be made superconformal.

A few comments are in order concerning the third column in the table 1. For the nonconformal models, the descriptions in the standard flat \( \mathcal{N} = 4, d = 1 \) harmonic superspace [12] and its \( U(1) \)-deformed analog \((5.6)\) are completely equivalent. Seemingly the only difference is that in the case of \((5.6)\) the supersymmetry transformations produce not only the standard shift of \( t_A \), (associated as a coset coordinate with the generator \( \mathcal{H} \)) but also some induced \( U(1) \) transformation with the stability subgroup generator \( F \). However, this difference can be leveled just by redefining all the \( U(1) \) charged fields on the pattern of redefinitions \((5.10)\) or \((5.18)\). After that the canonical time-translation generator becomes \( \mathcal{H} + \mu F \) in \((5.3)\) and one recovers the standard \( \mathcal{N} = 4, d = 1 \) picture. On the other hand, in the superconformal case it is the description in \((5.6)\) that allows one to define two sets of the \( \mu \)-deformed \( \mathcal{N} = 4, d = 1 \) supersymmetry transformations related via the change \( \mu \rightarrow -\mu \) and to construct the superconformal models with the trigonometrical realization of the supergroup.
as a closure of these two sets. In the standard undeformed
harmonic superspace only parabolic realizations of superconformal symmetries are achievable.

As for further developments, here we would like to mention the construction of SQM models for the $SU(2|1)$ analog of the multiplet $(3, 4, 1)$ which in the flat case has a natural description in the analytic harmonic superspace, the construction of the multi-particle SQM models with various types of the $SU(2|1)$ multiplets taken into account, and generalizations of the harmonic superspace approach to higher rank deformed $d = 1$ supersymmetries, e.g. $SU(2|2)$, which can be regarded as a deformation of the flat $\mathcal{N} = 8$, $d = 1$ supersymmetry [4]. Besides this, it seems of interest to explicitly solve some more complicated examples of the SQM $SU(2|1)$ models, not only the simplest ones treated here and in [4]. Another problem to be clarified is whether the $SU(2|1)$ analogs of the nonlinear $(4, 4, 0)$ multiplets [13] exist and what could be the $m$ deformation of the target geometries associated with such multiplets. It would also be interesting to define and study $SU(2|1)$ analogs of various useful concepts of the flat $\mathcal{N} = 4$, $d = 1$ supersymmetry, such as the semi-dynamical spin multiplets [30], the gauging procedure in the $\mathcal{N} = 4$ SQM models [9–11], etc. All these methods and concepts essentially use the notions of the $d = 1$ harmonic superspace approach. There also remains the problem of recovering various $SU(2|1)$ SQM models through the direct dimensional reduction from the higher-dimensional theories with the curved analogs of the Poincaré supersymmetry. Recently, a variant of the deformed SQM was applied to compute the vacuum (Casimir) energy in some conformal field theories [31]. It is interesting to establish possible links of this construction with the deformations of the $\mathcal{N} = 4$, $d = 1$ supersymmetry (and its $\mathcal{N} = 2$ reductions) considered in [4–6] and in the present paper.

Acknowledgments

We are grateful to Armen Nersessian and Andrei Smilga for their interest in various aspects of $SU(2|1)$ SQM models, useful suggestions and comments. This work was partially supported by the RFBR grant no. 15–02-06670 and a grant of the Heisenberg–Landau Program.

Appendix A. Transformations of the covariant derivatives

Here we collect the $\epsilon, \bar{\epsilon}$ transformations of the covariant derivatives constituting the CR structure (2.24), as well as of the derivative $D^{\alpha\beta}$.
These variations are obtained just by making use of the coordinate transformations (2.10). Note that the redefined harmonic derivative
\[ D^{++} := (1 + m \bar{\theta}^+ \theta^- - m \bar{\theta}^- \theta^+) D^{++} = \partial^{++} + 2 \bar{\theta}^+ \bar{\theta}^+ \partial_\lambda - 2 m \theta^+ \bar{\theta}^+ \bar{\theta}, \]
has a simpler transformation law
\[ \delta D^{++} = m \left( \bar{\theta}^+ \epsilon^+ + \theta^+ \bar{\epsilon}^+ \right) D^0 + 2 m \left( \bar{\theta}^+ \epsilon^+ - \theta^+ \bar{\epsilon}^+ \right) \bar{F}. \] (A.3)

Despite the rather involved form of (A.1), the objects \( D^{++} \), \( D^+ \) and \( \bar{D}^+ \) transform according to the simple universal transformation law (2.25), e.g.
\[ \delta (D^{++} \Phi) = - m \left[ 2 \left( \bar{\theta}^+ \epsilon^+ - \theta^+ \bar{\epsilon}^+ \right) \bar{F} + \left( \bar{\theta}^+ \epsilon^+ + \theta^+ \bar{\epsilon}^+ \right) D^0 \right] \bar{D}^+ \Phi, \]
i.e. these are \( SU(2|1) \) superfields. The rest of covariant derivatives, i.e. \( D^- \Phi \) and \( \bar{D}^- \Phi \), can be obtained by the action of \( D^- \) on the basis ones forming CR structure, so they are also transformed by the law (2.25).

Appendix B. Relation to the central basis of \( SU(2|1) \) superspace

The \( SU(2|1) \) superspace in the central basis amounts to the coordinate set [4–6]
\[ \zeta = (t, \theta_i, \bar{\theta}^j). \] (B.1)
Extending these coordinates by the harmonic coordinates \( w_i^\pm \), we arrive at the central basis of the harmonic \( SU(2|1) \) superspace:
\[ \zeta_c = (t, \theta_i, \bar{\theta}^j, w_i^\pm). \] (B.2)
The odd $SU(21)$ transformations of these coordinates are
\[
\delta \theta_i = \epsilon_i + 2m \bar{z}^k z_k \theta_i, \quad \delta \bar{\theta}^j = \bar{\epsilon}^i - 2m \epsilon_k \bar{\theta}^k \theta^j, \quad \delta t = i \left( \epsilon_k \bar{\theta}^k + \bar{\epsilon}^k z_k \right), \quad \delta \omega_i^+ = \lambda^+ w_i^-, \quad \lambda^+ = -m \left( 1 - m \bar{\theta}^i \theta^j \right) \left( \bar{\theta}^j \epsilon^j + \theta^k \bar{\epsilon}^k \right) w_i^+, \quad \delta w_i^- = 0. \tag{B.3}
\]

The relation with the analytic basis coordinates (2.9) is given by
\[
\theta w_i^- = \bar{\theta}^i w_i^-, \quad \theta w_i^+ = \theta^i \left( 1 + m \bar{\theta}^i \theta^j - m \bar{\theta}^j \theta^j \right) + \bar{\theta}^i \theta^i \theta^j, \quad \bar{\theta}^i w_i^+ = \bar{\theta}^i \left( 1 + m \bar{\theta}^i \theta^j - m \bar{\theta}^j \theta^i \right) + \bar{\theta}^i \theta^i \theta^j. \tag{B.4}
\]

It is direct to show that the transformations (B.3) in the central basis yield for the analytic basis coordinates just the transformations (2.10).

Using the above correspondence, the supermultiplets $(4, 4, 0)$ can also be described in the central basis. The corresponding superfield $q_{ia}$ does not depend on harmonics and obeys the $SU(21)$ covariant constraints
\[
\mathcal{D}^{kq_{ia}} = \bar{\mathcal{D}}^{kq_{ia}} = 0, \quad \bar{F} q_{ia} = 0, \quad (q_{ia})^\dagger = q_{ia}. \tag{B.5}
\]

The expressions for $\mathcal{D}^k, \bar{\mathcal{D}}^k$ were given in [4]. Solving these constraints, we find
\[
q_{ia}(\zeta) = \left[ 1 + \frac{m}{2} \bar{\theta}^i \theta^i - \frac{5m^2}{16} \left( \theta^i \theta^j \right)^2 + \left( 1 - \frac{m}{2} \bar{\theta}^i \theta^i \right) \left( \theta^i \psi^a + \bar{\theta}^i \bar{\psi}^a \right) - i \epsilon_{kl} \left( \bar{\theta}^i \theta^l + \theta^l \bar{\theta}^i \right) \bar{\theta}^i \bar{\theta}^i \right] \theta^i \psi^a + \bar{\theta}^i \bar{\psi}^a + \frac{1}{4} \left( \bar{\theta}^i \theta^i \right)^2 \bar{\psi}^a. \tag{B.6}
\]

In contrast to the harmonic analytic superfield (3.2), this superfield explicitly involves the deformation parameter $m$. The superfields $q_{ia}^+$ and $q_{ia}^-$ are related to each other in the following way:
\[
q_{ia}^+ = q_{ia}^+ \left( 1 + m \bar{\theta}^i \theta^j - m \bar{\theta}^j \theta^i \right)^{\frac{1}{2}}, \quad q_{ia}^- = q_{ia}^- \left( 1 + m \bar{\theta}^i \theta^j - m \bar{\theta}^j \theta^i \right)^{\frac{1}{2}}, \quad q_{ia} = q_{ia}^+ \left( 1 + m \bar{\theta}^i \theta^j - m \bar{\theta}^j \theta^i \right)^{-\frac{1}{2}} - q_{ia}^- \left( 1 + m \bar{\theta}^i \theta^j - m \bar{\theta}^j \theta^i \right)^{-\frac{1}{2}}. \tag{B.7}
\]

One can also rewrite the general $q$ action as an integral over the superspace (B.1)
\[
S(q_{ia}) = - \int d\zeta \int_{\mathcal{L}} \left( q^2 \right), \quad q^2 = q_{ia}^* q_{ia}, \tag{B.9}
\]
where
\[
d\zeta = dt d^2 \theta d^2 \bar{\theta} \left( 1 + 2m \bar{\theta}^i \theta^i \right) \tag{B.10}
\]
is the $SU(21)$ invariant integration measure. Note that the analytic basis measure (2.16) is related to (B.10) as
\[
d\zeta_H = dw d\zeta. \tag{B.11}
\]
It is not $SU(21)$ invariant because the harmonic measure $dw$ is not.

In the superspace (B.1), the $SU(21)$ constraints (4.1) defining the mirror $(4, 4, 0)$ multiplet are rewritten as
Their solution reads
\[
Y^A(\zeta) = \left(1 + 2m \bar{\partial}^A \partial_k \right) \frac{i}{4} \left\{ Y^A + \theta_k \psi^A + i \bar{\partial}^A \bar{\psi}^A + \bar{\theta}_k \bar{\theta}_k \left( i \bar{y}^A + \frac{m}{2} \bar{y}^A \right) \right\} \\
+ \frac{1}{4} (\bar{\partial})^2 \left( \bar{y}^A + 2 \text{Im} \bar{y}^A \right),
\]
\[
\bar{Y}^A(\zeta) = \left(1 + 2m \partial^A \bar{\partial}_k \right) \frac{i}{4} \left\{ \bar{Y}^A + \bar{\theta}_k \bar{\psi}^A - i \bar{\partial}^A \bar{\psi}^A + \bar{\theta}_k \bar{\theta}_k \left( i \bar{y}^A - \frac{m}{2} \bar{y}^A \right) \right\} \\
- \frac{1}{4} (\partial)^2 \left( \bar{y}^A - 2 \text{Im} \bar{y}^A \right).
\]

The precise relation with the analytic basis solution \(Y^A(\zeta)\) given in (4.3) is
\[
Y^A(\zeta) = e^{2m(t_\alpha - t)} Y^A(\zeta). \tag{B.14}
\]
The general \(Y, \bar{Y}\) action can also be written as an integral over (B.1)
\[
\mathcal{S}(Y, \bar{Y}) = \int d\zeta \mathcal{L}(Y, \bar{Y}). \tag{B.15}
\]

Finally, we note the existence of another analytic basis in the \(SU(2|1)\) harmonic superspace, \(\tilde{\zeta}_H = (t_{\alpha}, \bar{\theta}^+, \bar{\theta}^-, \bar{w}_0^+)\), where
\[
\bar{\theta}^+ = \bar{\rho}^+ e^{2i\mu t_{\alpha}}, \quad \bar{\theta}^- = \bar{\rho}^- e^{-2i\mu t_{\alpha}} \left[1 - (1 + \alpha)\mu \bar{\theta}^+ \bar{\theta}^-\right], \quad m = -\alpha \mu,
\]
\[
\bar{w}_0^+ = \bar{w}_{0+}^+, \quad \bar{\theta}^+ = \bar{\theta}^+ e^{-2i\mu t_{\alpha}} \left[1 + (1 + \alpha)\mu \bar{\theta}^+ \bar{\theta}^-\right]. \tag{B.16}
\]
The \(SU(2|1)\) transformations and the non-degenerate case in this basis are easily extended to the trigonometric superconformal ones by making the reflection \(\mu \rightarrow -\mu\) and considering a closure of two such sets of transformations.

The coordinates \((t, \bar{\theta}, \bar{w}_0)\) introduced in [6] and ensuring a similar superconformal closure in the central basis, are related to (B.16) by the following simple redefinitions:
\[
\bar{\theta}^\pm = \bar{\theta}^\pm \bar{w}_0^+, \quad \bar{\theta}^\pm = \bar{\theta}^\pm \bar{w}_0^+, \quad t_{\alpha} = t - i \left( \bar{\theta}^- \bar{\theta}^+ + \bar{\theta}^+ \bar{\theta}^- \right). \tag{B.17}
\]

It is straightforward to check that the analytic subspace \((t_{\alpha}, \bar{\theta}^+, \bar{\theta}^-, \bar{w}_0^+)\) is closed under both sets of the \(SU(2|1)\) transformations and hence under their superconformal closure. The same is true for the \(\alpha = 0\) case.

Appendix C. \(SU(2|1)\) representations

The finite-dimensional \(SU(2|1)\) representations [32] are characterized by two parameters \(\lambda\) and \(\beta\). The number \(\lambda\) (‘highest weight’) is a positive half-integer or integer, and an arbitrary additional real number \(\beta\) is related to the eigenvalues of the \(U(1)\) generator \(\hat{H}\) of (2.1). Casimir operators of the \(su(2|1)\) superalgebra (2.1) are written in the model-independent way as follows:
\[
m^2 C_2 = \hat{H}^2 - \frac{m^2}{2} I_i^i I_j^j + \frac{m}{4} [Q^i, Q^j]. \tag{C.1}
\]
\[ m^2 C_0 = \frac{m^2}{2} (1 + 2C_2) \hat{H} - \frac{m}{8} \left[ m I_k^i - \delta^i_k \hat{H} \right] \left[ Q^k, \tilde{Q}^i \right]. \tag{C.2} \]

The eigenvalues of these Casimir operators can be written as

\[ C_2 = \beta^2 - \lambda^2, \quad C_3 = \beta C_2. \tag{C.3} \]

Non-zero values of Casimir operators define the typical representations of \( SU(2|1) \). In the case \(|\beta| = \lambda\), Casimir operators have zero eigenvalues and the relevant \( SU(2|1) \) representations are atypical. Typical and atypical representations have the dimensions \( 8\lambda \) and \( 4\lambda + 1 \), respectively. The fundamental \( SU(2|1) \) representation is atypical and it corresponds to the choice \(|\beta| = \lambda = 1/2\).

References

[1] Festuccia G and Seiberg N 2011 Rigid supersymmetric theories in curved superspace J. High Energy Phys. JHEP06(2011)114
[2] Dumitrescu T T, Festuccia G and Seiberg N 2012 Exploring curved superspace J. High Energy Phys. JHEP08(2012)141
[3] Samsonov I B and Sorokin D 2014 Superfield theories on \( S^1 \) and their localization J. High Energy Phys. JHEP04(2014)102
Samsonov I B and Sorokin D 2014 Gauge and matter superfield theories on \( S^2 \) J. High Energy Phys. JHEP09(2014)097
[4] Ivanov E and Sidorov S 2014 Deformed supersymmetric mechanics Class. Quantum Grav. 31 075013
[5] Ivanov E and Sidorov S 2014 Super Kähler oscillator from \( SU(2|1) \) superspace J. Phys. A: Math. Theor. 47 292002
[6] Ivanov E, Sidorov S and Toppan F 2015 Superconformal mechanics in \( SU(2|1) \) superspace Phys. Rev. D 91 085032
[7] Smilga A V 2004 Weak supersymmetry Phys. Lett. B 585 173
[8] Bellucci S, Krivonos S, Marrani A and Orazi E 2006 Root action for \( N = 4 \) supersymmetric mechanics theories Phys. Rev. D 73 025011
[9] Delduc F and Ivanov E 2006 Gauging \( N=4 \) supersymmetric mechanics Nucl. Phys. B 753 211
[10] Delduc F and Ivanov E 2007 Gauging \( N = 4 \) supersymmetric mechanics II: (1,4,3) models from the \((4,4,0)\) ones Nucl. Phys. B 770 179
[11] Delduc F and Ivanov E 2007 The common origin of linear and nonlinear chiral multiplets in \( N = 4 \) mechanics Nucl. Phys. B 787 176
[12] Ivanov E and Lechtenfeld O 2003 \( N = 4 \) supersymmetric mechanics in harmonic superspace J. High Energy Phys. JHEP09(2003)073
[13] Delduc F and Ivanov E 2012 \( N = 4 \) mechanics of general \((4,4,0)\) multiplets Nucl. Phys. B 855 815
[14] Galperin A, Ivanov E, Kalitzin S, Ogievetsky V and Sokatchev E 1984 Unconstrained \( N = 2 \) matter, Yang–Mills and supergravity theories in harmonic superspace Class. Quantum Grav. 1 469
[15] Galperin A S, Ivanov E A, Ogievetsky V I and Sokatchev E S 2001 Harmonic Superspace (Cambridge: Cambridge University Press) p 306
[16] Kuzenko S M and Sorokin D 2014 Superconformal structures on the three-sphere J. High Energy Phys. JHEP10(2014)80
[17] Holanda N L and Toppan F 2014 Four types of (super)conformal mechanics: D-module reps and invariant actions J. Math. Phys. 55 061703
[18] Galperin A, Ivanov E and Ogievetsky O 1994 Harmonic space and quaternionic manifolds Ann. Phys. 230 201
[19] Howe P S and Hartwell G G 1995 A superspace survey Class. Quantum Grav. 12 1823
Hartwell G G and Howe P S 1993 (\( n, p, q \)) harmonic superspace Int. J. Mod. Phys. A 10 3901
Heslop P and Howe P S 2000 On harmonic superspaces and superconformal fields in four-dimensions Class. Quantum Grav. 17 3743
[20] Ivanov E A and Niederle J 2009 Biharmonic superspace for $N = 4$ mechanics Phys. Rev. D \textbf{80} 065027
[21] Smilga A V 1987 How to quantize supersymmetric theories Nucl. Phys. B \textbf{292} 363
[22] Smilga A V 2013 Taming the zoo of supersymmetric quantum mechanical models J. High Energy Phys. JHEP05(2013)119
[23] Fedoruk S A, Ivanov E A and Smilga A V 2014 $N = 4$ mechanics with diverse $(4, 4, 0)$ multiplets: Explicit examples of HKT, CKT and OKT geometries J. Math. Phys. \textbf{55} 052302
[24] Fedoruk S, Ivanov E and Lechtenfeld O 2012 Superconformal mechanics J. Phys. A: Math. Theor. \textbf{45} 173001
[25] Frappat L, Sciarrino A and Sorba P 2000 \textit{Dictionary on Lie Algebras and Superalgebras} (New York: Academic)
[26] Papadopoulos G 2013 New potentials for conformal mechanics Class. Quantum Grav. \textbf{30} 075018
[27] Howe P S and Townsend P K 1990 Chern–Simons quantum mechanics Class. Quantum Grav. \textbf{7} 1655
[28] Kuznetsova Z and Toppan F 2012 D-module representations of $N = 2, 4, 8$ superconformal algebras and their superconformal mechanics J. Math. Phys. \textbf{53} 043513
[29] Khodaei S and Toppan F 2012 Critical scaling dimension of D-module representations of $N = 4, 7, 8$ superconformal algebras and constraints on superconformal mechanics J. Math. Phys. \textbf{53} 103518
[30] Fedoruk S, Ivanov E and Lechtenfeld O 2009 Supersymmetric Calogero models by gauging Phys. Rev. D \textbf{79} 105015
[31] Assel B, Cassani D, Di Pietro L, Komargodski Z, Lorenzen J and Martelli D 2015 The Casimir energy in curved space and its supersymmetric counterpart J. High Energy Phys. JHEP07 (2015)043
[32] Scheunert M, Nahm W and Rittenberg V 1977 Irreducible representations of the osp(2,1) and spl (2,1) graded Lie algebras J. Math. Phys. \textbf{18} 155