Group Theory

Complete reducibility and Steinberg endomorphisms

Réductibilité complète et endomorphismes de Steinberg

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\textbf{A B S T R A C T}

Let $G$ be a connected reductive algebraic group defined over an algebraically closed field of positive characteristic. We study a generalization of the notion of $G$-complete reducibility in the context of Steinberg endomorphisms of $G$. Our main theorem extends a special case of a rationality result in this setting.

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\textbf{R É S U M É}

Soit $G$ un groupe algébrique réductible connexe défini sur un corps algébriquement clos de caractéristique positive. Dans cette Note on étudie une généralisation de la notion de réductibilité $G$-complète dans le contexte des endomorphismes de Steinberg de $G$. Le théorème fondamental de la Note généralise un cas particulier d’un résultat de rationalité.

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\section{1. Introduction}

Let $p$ be a prime number and let $k = \mathbb{F}_p$ be the algebraic closure of the field of $p$ elements. Let $G$ be a connected reductive linear algebraic group defined over $k$ and let $H$ be a closed subgroup of $G$. Let $\mathbb{F}_q \subseteq k' \subseteq k$ be a field extension of $\mathbb{F}_p$. Following Serre [12], we say that a $k'$-defined subgroup $H$ of $G$ is $G$-completely reducible over $k'$ provided that whenever $H$ is contained in a $k'$-defined parabolic subgroup $P$ of $G$, it is contained in a $k'$-defined Levi subgroup of $P$. If $k' = k$, then $H$ is $G$-completely reducible over $k'$ if and only if $H$ is $G$-completely reducible (or $G$-cr for short). For an overview of this concept see for instance [11] and [12].

The starting point for our discussion is the following special case of the rationality result [1, Theorem 5.8]. Let $q$ be a power of $p$ and let $\mathbb{F}_q$ be the field of $q$ elements.

\textbf{Theorem 1.1.} Suppose that both $G$ and $H$ are defined over $\mathbb{F}_q$. Then $H$ is $G$-completely reducible if and only if it is $G$-completely reducible over $\mathbb{F}_q$.

Let $\sigma : G \to G$ be a Steinberg endomorphism of $G$, i.e. a surjective endomorphism of $G$ that fixes only finitely many points, see Steinberg [14] for a detailed discussion (for this terminology, see [6, Definition 1.15.1b]). The set of all Steinberg endomorphisms of $G$ is a subset of all isogenies $G \to G$ (see [14, 7.1(a)]) that encompasses in particular all (generalized)
Frobenius endomorphisms, i.e. endomorphisms of $G$ some power of which are Frobenius endomorphisms corresponding to some $F_q$-rational structure on $G$. 

**Example 1.2.** Let $F_1$, $F_2$ be the Frobenius maps of $G = SL_2$ given by raising coefficients to the $p$th and $p^2$th powers, respectively. Then the map $\sigma = F_1 \times F_2 : G \times G \to G \times G$ is a Steinberg morphism of $G \times G$ that is not a (generalized) Frobenius morphism, cf. the remark following [6, Theorem 2.111].

If $G$ is almost simple, then $\sigma$ is a (generalized) Frobenius map (e.g. see [6, Theorem 2.111]), and the possibilities for $\sigma$ are well known ([14, §11], e.g. see [7, Theorem 1.4]): $\sigma$ is conjugate to either $\sigma_q$, $\tau \sigma_q$, $\tau' \sigma_q$ or $\tau'$, where $\sigma_q$ is a standard Frobenius morphism, $\tau$ is an automorphism of algebraic groups coming from a graph automorphism of types $A_n$, $D_n$ or $E_6$, and $\tau'$ is a bijective endomorphism coming from a graph automorphism of type $B_2$ ($p = 2$), $F_4$ ($p = 2$) or $G_2$ ($p = 3$).

**Example 1.3.** If $G$ is not simple, then a generalized Frobenius map may fail to factor into a field and a graph automorphism as stated above. For example, let $p = 2$ and let $H_1$, $H_2$ be simple, simply connected groups of type $B_n$ and $C_n$ ($n \geq 3$), respectively. Then there are special isogenies $\phi_1 : H_1 \to H_2$ and $\phi_2 : H_2 \to H_1$ whose composites $\phi_1 \circ \phi_2$ and $\phi_2 \circ \phi_1$ are standard Frobenius maps with respect to $p$ on $H_2$, respectively $H_1$, see [4, p. 5 of Expose 24]. Let $G = H_1 \times H_2$ and define $\sigma : G \to G$ by $\sigma(h_1, h_2) = (\phi_2(h_1), \phi_1(h_2))$. Then $\sigma$ is an example of such a more complicated generalized Frobenius map.

We now give an extension of Serre’s notion of G-complete reducibility in this setting of Steinberg endomorphisms: Let $\sigma$ be a Steinberg endomorphism of $G$ and let $H$ be a subgroup of $G$. We say that $H$ is $\sigma$-completely reducible (or $\sigma$-cr for short), provided that whenever $H$ lies in a $\sigma$-stable parabolic subgroup $P$ of $G$, it lies in a $\sigma$-stable Levi subgroup of $P$. This notion is motivated as follows: If $\sigma_q$ is a standard Frobenius morphism of $G$, then a subgroup $H$ of $G$ is defined over $F_q$ if and only if it is $\sigma_q$-stable and if so, $H$ is $G$-completely reducible over $F_q$ if and only if it is $\sigma_q$-completely reducible. In view of this new notion, the goal of this note is the following generalization of Theorem 1.1 to arbitrary Steinberg endomorphisms of $G$ (the special case of Theorem 1.4 when $\sigma = \sigma_q$ gives Theorem 1.1).

**Theorem 1.4.** Let $\sigma$ be a Steinberg endomorphism of $G$. Let $H$ be a $\sigma$-stable subgroup of $G$. Then $H$ is $\sigma$-completely reducible if and only if $H$ is $\sigma$-completely reducible.

Theorem 1.4 follows from Theorems 2.4 and 2.5 proved in the next section.

**Example 1.5.** Theorem 1.4 is false without the $\sigma$-stability condition on $H$. For instance, a maximal torus $T$ of $G$ is always $G$-cr, cf. [1, Lemma 2.6]. But it may happen that $T$ is contained in a $\sigma$-stable Borel subgroup of $G$, without being itself $\sigma$-stable. Then $T$ clearly fails to be $\sigma$-cr. In the other direction, $G$ may contain a maximal parabolic subgroup $P$ of $G$ that is not $\sigma$-stable. The only $\sigma$-stable parabolic subgroup of $G$ containing $P$ is $G$ itself. Then $P$ is $\sigma$-cr for trivial reasons, whereas a proper parabolic subgroup of $G$ is not $G$-cr.

**Remark 1.6.** Even if $H$ is not $\sigma$-stable, Theorem 1.4 gives some information about the notion of $\sigma$-complete reducibility, as follows. Let $\overline{H}^\sigma$ be the algebraic subgroup of $G$ generated by all translates $\sigma^i H$, $i \geq 0$. Then $\overline{H}^\sigma$ is $\sigma$-stable and contained in the same $\sigma$-stable subgroups of $G$ as $H$. In particular, $H$ is $\sigma$-cr if and only if $\overline{H}^\sigma$ is $\sigma$-cr. Thus, by Theorem 1.4, this is equivalent to $\overline{H}^\sigma$ being $G$-cr.

**2. Proof of Theorem 1.4**

In addition to the notation already fixed in the Introduction, $\sigma : G \to G$ is always a Steinberg endomorphism of $G$ and from now on the subgroup $H$ of $G$ is assumed to be $\sigma$-stable. We begin with a generalization of (a special case of) [8, Proposition 2.2 and Remark 2.4]. The proof of Proposition 2.1 consists in a reduction to the case when $H$ is finite, covered in [8, Proposition 2.2 and Remark 2.4].

**Proposition 2.1.** If $H$ is not $G$-completely reducible, then there exists a proper $\sigma$-stable parabolic subgroup of $G$ containing $H$.

**Proof.** First we assume that $G$ is almost simple. We want to reduce to the case where $H$ is a finite, $\sigma$-stable subgroup of $G$, and then apply [8, Proposition 2.2 and Remark 2.4]. Since $G$ is almost simple, we can assume that $\sigma^m = \sigma_q$ is a standard Frobenius map for some positive integer $m$. We choose a closed embedding $G \to GL_n(k)$ so that $\sigma_q$ is the restriction of the standard Frobenius map of $GL_n(k)$ that raises coefficients to the $q$th power (see [5, Proposition 4.111]). For $r \in \mathbb{Z}$, $r \geq 1$, let $H(r) = H \cap GL_n(F_{q^r})$. Then we can write $H$ as the directed union of finite subgroups $H = \bigcup_{r \geq 1} H(r)$. Note that the union is indeed directed, that is

$$\overline{H(r)} \subseteq \overline{H(r + 1)} \quad \forall r \geq 1.$$  

(2.2)
We wish to construct a similar, but $\sigma$-stable filtration of $H$. For this purpose we set $H(r) = \bigcap_{i=0}^{m-1} \sigma^i H(t)$. Then each $H(r)$ is a finite, $\sigma$-stable subgroup of $H$ (for the $\sigma$-stability, we use that each $\sigma^i H(t)$ is stable under $\sigma^m = \sigma_0$). Moreover, we claim that $H$ is the directed union $H = \bigcup_{r \geq 1} H(r)$. Indeed, if $h \in H$, then the identities $H = \sigma H$ and $H = \bigcup_{r \geq 1} H(r)$ imply that for each $i = 0, \ldots, m - 1$ we can find some $r_i$ such that $h \in \sigma^i H(t)$. Then (2.2) implies that $h \in H(r)$ for $r \geq \max\{r_0, \ldots, r_{m-1}\}$. It follows from the argument in the proof of [1, Lemma 2.10] that there is an integer $r'$ so that $H(r')$ has the following property: $H$ is contained in a parabolic subgroup $P$ of $G$ (respectively a Levi subgroup $L$ of $G$) if and only if $H(r')$ is contained in $P$ (respectively in $L$). Therefore, if $H$ is not $G$-cr, then neither is $H(r')$, and we can apply [8, Proposition 2.2 and Remark 4.4] to obtain a proper $\sigma$-stable parabolic subgroup $P$ of $G$ that contains $H(r')$. But then $P$ also contains $H$.

Next we drop the simplicity assumption on $G$. Then we can use the almost simple components of $G$ to reduce to the almost simple case: Let $\pi : G' := Z(G)^o \times G_1 \times \cdots \times G_r \to G$ be the product map, where $G_1, \ldots, G_r$ are the almost simple components of the semisimple group $[G, G]$ and let $\pi_i : G' \to G_i$ be the projection $(1 \leq i \leq r)$. Then $\pi$ is an isogeny. Let $H' = \pi^{-1}(H)$. Using [1, Lemma 2.12] and the fact that $Z(G)^o$ is a torus, we find that there is some index $i$ such that $H_i := \pi_i(H') \subseteq G_i$ is not $G_i$-cr. We can assume that $i = 1$. We are now in the situation of the first part of the proof (for $H_1 \subseteq G_1$), except that we have yet to specify a Steinberg endomorphism of $G_1$ that stabilizes $H_1$. Since $\sigma$ stabilizes $[G, G]$ and maps components to components [4, Expose 18, Proposition 3], we can assume that $\sigma$ permutes $G_1, \ldots, G_r$ cyclically for some $s \leq r$. Moreover, $\sigma$ stabilizes $Z(G)^o = R(G)$ (because $\sigma$ is an isogeny). Using the restrictions $\sigma|_{Z(G)^o}$ and $\sigma|_{[G, G]}$, we can define a Steinberg endomorphism $\sigma' : G' \to G'$ of $\sigma \sigma' = \sigma \circ \pi$. We denote by $H''$ the image (under the projection) of $H'$ in $G'' := G_1 \times \cdots \times G_r$. Now let $\tau = \sigma|_{G_1} : G_1 \to G_1$ denote the generalized Frobenius map on $G_1$ induced by $\sigma$ [6, Theorems 2.1.2(g) and 2.1.11]. Then $H_1$ is $\tau$-stable, since $H$ is $\sigma^s$-stable. We apply the first part of the proof to $H_1 \subseteq G_1$ to obtain a proper $\tau$-stable parabolic subgroup $P_1$ of $G_1$ containing $H_1$. Then $P'' := P_1 \times \sigma P_1 \times \cdots \times \sigma^{s-1}P_1 \subseteq G''$ is a proper $\sigma'^s$-stable parabolic subgroup of $G''$ [13, Corollary 6.2.8]. The bijectivity of $\sigma^s|_{H_1} : H_1 \to H_1$ for $1 \leq i \leq s$ implies that $H_1 = \sigma^s_1 H_1$ for $1 \leq i \leq s$. We get that $P''$ contains $H''$, since we have $H'' \subseteq H_1 \times H_2 \times \cdots \times H_r$ and $H_1 \subseteq P_1$. Consequently, $P' = Z(G)^o \times P'' \times G_{s+1} \times \cdots \times G_r$ is a proper $\sigma'$-stable parabolic subgroup of $G'$ containing $H''$. Finally, $P = \pi'(P')$ is a proper $\sigma$-stable parabolic subgroup of $G$ containing $H$, as desired. □

Remark 2.3. In [8, Proposition 2.2 and Remark 2.4], Liebeck, Martin and Shalev prove the following: Let $G$ be an almost simple algebraic group over $k$ as above. Let $Aut^s(G)$ denote the group of abstract automorphisms of $G$ that is generated by inner automorphisms of $G$, together with $p'$ power field morphisms ($i \geq 1$), and abstract graph automorphisms (which may include the bijective algebraic endomorphisms coming from a graph automorphism of type $B_2$ ($p = 2$), $F_4$ ($p = 2$) or $G_2$ ($p = 3$)). (Note that $Aut^s(G)$ is an extension of the group $Aut^*(G)$ from [8, p. 455].) Let $S$ be a subgroup of $Aut^s(G)$ and suppose that $G \subseteq G'$ is a finite, $S$-stable subgroup that is not $G$-cr. Then $H$ is contained in a proper $S$-invariant parabolic subgroup of $G$ (note that the notion of strongly reductive subgroups in $G$ is equivalent to the notion of $G$-completely reducible subgroups, cf. [1, Theorem 3.1]). If we take $S$ to be generated by a (generalized) Frobenius endomorphism $\sigma$ of $G$, then we get the assertion of Proposition 2.1 for $G$ almost simple and $H$ finite.

Theorem 2.4. If $H$ is $\sigma$-completely reducible, then it is $G$-completely reducible.

Proof. If $H$ is not contained in any proper $\sigma$-stable parabolic subgroup of $G$, then it is $G$-cr according to Proposition 2.1. So we can assume that there is a proper $\sigma$-stable parabolic subgroup $P$ of $G$ containing $H$. We choose $P$ minimal with these properties. Since $H$ is $\sigma$-cr, it is contained in an $\sigma$-stable Levi subgroup $L$ of $P$. Suppose there is a proper $\sigma$-stable parabolic subgroup $P_1$ of $L$ containing $H$. Then $P' := P_1 R_\alpha(P) \not\subseteq P$ is another parabolic subgroup of $G$ (see [3, Proposition 4.4(c)]) containing $H$, and $P'$ is $\sigma$-stable ($\sigma$ stabilizes $R_\alpha(P)$ as any isogeny does). But this contradicts our choice of $P$. So we can use Proposition 2.1 again to deduce that $H$ is $L$-cr, which in turn implies that $H$ is $G$-cr [1, Corollary 3.2]. □

For the converse of Theorem 2.4 we argue as in the last part of the proof of [9, Theorem 9]. But first we recall a parametrization of the parabolic and Levi subgroups of $G$ in terms of cocharacters of $G$, e.g. see [1, Lemma 2.4]: Given a parabolic subgroup $P$ of $G$ and any Levi subgroup $L$ of $P$, there exists some cocharacter $\lambda$ of $G$ such that $P$ and $L$ are of the form $P = P_\lambda = \{g \in G \mid \lim_{-\to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists}\}$ and $L = L_\lambda = C_G(\lambda(k^s))$, respectively. The unipotent radical of $P_\lambda$ is then given by $R_\alpha(P_\lambda) = \{g \in G \mid \lim_{-\to 0} \lambda(t) g \lambda(t)^{-1} = 1\}.

Theorem 2.5. If $H$ is $G$-completely reducible, then it is $\sigma$-completely reducible.

Proof. Suppose that $P$ is a $\sigma$-stable parabolic subgroup of $G$ containing $H$. Since $H$ is $G$-cr, there is some Levi subgroup $L$ of $P$ that contains $H$. Let $U = R_\alpha(P)$. Then $A = \{u L u^{-1} \mid u \in U, H \subseteq u L u^{-1}\}$ is the set of all Levi subgroups of $P$ that contain $H$. Clearly, $A$ is $\sigma$-stable, since $H$ is contained in $A$. We need to prove that $A$ contains an element fixed by $\sigma$.

If $u L u^{-1}$ is in $A$, then $u^{-1} H U \subseteq U H = H$, so that $u$ normalizes $H$. In fact, $u$ centralizes $H$, since $[N_U(H), H] \subseteq H \cap U = 1$. So the group $C = C_G(H)$ acts transitively on $A$. We claim that $C$ is connected. In order to prove this, write $P = P_\lambda$, $L = L_\lambda$ and $U = R_\alpha(P_\lambda)$ for some suitable cocharacter $\lambda$ of $G$. The torus $\lambda(k^s)$ normalizes $C_G(H)$ (because $H$ is
contained in \( L \) and \( U \), hence it normalizes \( C \). Whence, for any fixed \( c \in C \), the map \( \phi_c : k^* \to C \), given by \( t \mapsto \lambda(t)c\lambda(t)^{-1} \), is well-defined. Moreover, \( C \subseteq U \) implies that \( \phi_c \) extends to a morphism \( \hat{\phi}_c : k \to C \) that maps 0 to 1 and 1 to \( c \). Since the image of \( \phi_c \) is connected, we get \( c \in C^0 \). It follows that \( C = C^0 \). But now we can apply the Lang–Steinberg theorem (see [14, Theorem 10.1]) to conclude that \( A \) contains an element fixed by \( \sigma \).

Remark 2.6. We conclude by outlining a short alternative approach to Proposition 2.1; the latter was crucial in the proof of Theorem 2.4. This variant utilizes the so-called Centre Conjecture for spherical buildings due to J. Tits from the 1950s. This deep conjecture has recently been established by work of Leeb and Ramos-Cuevas, e.g. see [2, §2] and the references therein for further details. This conjecture states that in the building \( \Delta = \Delta(G) \) of \( G \) any convex contractible subcomplex \( \Sigma \) has a simplex which is fixed under any building automorphism of \( \Delta \) which stabilizes \( \Sigma \) as a subcomplex. Such a fixed simplex is often referred to as a 'centre' giving this conjecture its name. Here is a sketch of a building theoretic alternative to the proof of Proposition 2.1: Let \( H \) be a \( \sigma \)-stable subgroup of \( G \) which is not \( G \)-cr. Consider the subcomplex \( \Delta^H \) of \( H \)-fixed points of the building \( \Delta \), i.e., \( \Delta^H \) corresponds to the set of all parabolic subgroups of \( G \) that contain \( H \). Note that \( \Delta^H \) is always convex [12, Proposition 3.1] and since \( H \) is not \( G \)-cr, \( \Delta^H \) is also contractible [10, Theorem 2]. The Steinberg morphism \( \sigma \) of \( G \) affords a building automorphism of \( \Delta \), also denoted by \( \sigma \). Since \( H \) is \( \sigma \)-stable, so is \( \Delta^H \). Now since \( \Delta^H \) is convex and contractible, the Centre Conjecture asserts the existence of a centre of \( \Delta^H \) with respect to the action of \( \sigma \) which corresponds to a proper parabolic subgroup of \( G \) which is \( \sigma \)-stable and contains \( H \). This is precisely the conclusion of Proposition 2.1.

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