ON THE $p$-ADIC WEIGHT-MONODROMY CONJECTURE FOR COMPLETE INTERSECTIONS IN TORIC VARIETIES

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ABSTRACT. We give a proof of the $p$-adic weight monodromy conjecture for scheme-theoretic complete intersections in projective smooth toric varieties. The strategy is based on Scholze’s proof in the $\ell$-adic setting, which we adapt using homotopical results developed in the context of rigid analytic motives.

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1. INTRODUCTION

Let $K$ be a non-archimedean local field of mixed characteristic $(0, p)$ with ring of integers $\mathcal{O}_K$ and residue field $k = \mathbb{F}_q$, $q = p^a$, and let $K_0 = W(k)[1/p]$ be the maximal unramified sub-extension of $K$ over $\mathbb{Q}_p$. We let $\varphi$ denote the lift of the arithmetic Frobenius in $\text{Gal}(K_0/\mathbb{Q}_p)$. Let $Y$ be a proper smooth scheme over $K$ with a proper flat integral model $\mathcal{Y}$ over $\mathcal{O}_K$. We let $\mathcal{Y}_s$ denote its special fiber. When $\mathcal{Y}$ is smooth over $\mathcal{O}_K$, the crystalline cohomology groups $H^i_{\text{crys}}(\mathcal{Y}_s/W(k))[1/p]$ come naturally equipped with a $\varphi$-semilinear endomorphism $\Phi$ called the Frobenius endomorphism. The Weil conjectures for crystalline cohomology proved by Katz–Messing in [KM74, Theorem 1] imply that $H^i_{\text{crys}}(\mathcal{Y}_s/W(k))[1/p]$ is pure of weight $i$ (that is, the eigenvalues of the linear operator $\Phi^a$ have complex absolute value $q^{i/2}$ via any embedding $\bar{K} \subset \mathbb{C}$).

When $\mathcal{Y}$ is semistable over $\mathcal{O}_K$, a similarly well-behaved $p$-adic cohomology theory is given by Hyodo–Kato cohomology (i.e., log crystalline cohomology due to Hyodo–Kato [HK94]), which we denote by $H^i_{\text{HK}}(\mathcal{Y}) = H^i_{\text{log-crys}}(\mathcal{Y}_s/W(k)^0)[1/p]$, where $\mathcal{Y}_s$ here refers to the special fiber of $\mathcal{Y}$ equipped with its natural log structure, and $W(k)^0$ is the scheme $\text{Spec} W(k)$ equipped with the log structure associated to the homomorphism of monoids $\mathbb{N} \to W(k)$, $1 \mapsto 0$. It comes naturally with a $\varphi$-semilinear Frobenius endomorphism $\Phi$ and a nilpotent endomorphism $N$ called the monodromy operator, which satisfy the equality $N\Phi = p\Phi N$.

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As Beilinson observed [Bei13], Hyodo–Kato cohomology can be viewed as a cohomology theory on the generic fiber, and it extends to the whole category of proper smooth algebraic varieties over $K$: he defined a cohomology group $H^i_{\text{HK}}(Y_K)$, which is a $\mathbb{Q}_p^\text{nr}$-vector space equipped with a Frobenius and a monodromy operator as above, and an action of the Galois group $\text{Gal}(\overline{K}/K)$. Moreover, he showed, as predicted by Jannsen and Fontaine [Jan89, Conjecture 6.2.1], that it agrees with $p$-adic étale cohomology via Fontaine’s functor: [Bei13, Formula 3.3.1] states that we have a canonical isomorphism of $(\varphi, N, G_K)$-modules

$$H^i_{\text{HK}}(Y_K) \otimes_{\mathbb{Q}_p} B_{\text{st}} \cong H^i_{\text{ét}}(Y_K, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{st}},$$

which reproves the $C_{\text{st}}$-conjecture in $p$-adic Hodge theory proved by Tsuji [Tsu99], Faltings [Fal02], Nizioł [Niz08].

We now recall that Deligne’s weight-monodromy conjecture [Del71] predicts that the $\ell$-adic étale cohomology group $H^i_{\text{ét}}(Y_K, \mathbb{Q}_\ell)$ is quasi-pure of weight $i$, i.e., on the $j$-th graded quotient $\text{gr}_j^M H^i(Y_K, \mathbb{Q}_\ell)$ of the monodromy filtration $M_\ast$, the action of a Frobenius-lift is pure of weight $i + j$.

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**Conjecture 1.1** ([Jan89, Page 347]). The $i$-th Hyodo–Kato cohomology group $H^i_{\text{HK}}(Y_K)$ is quasi-pure of weight $i$, i.e., the $j$-th graded quotient $\text{gr}_j^M H^i_{\text{HK}}(Y_K)$ of the monodromy filtration $M_\ast$ is Frobenius-pure of weight $i + j$.

To our knowledge, the known cases of the $p$-adic weight-monodromy conjecture are essentially the following kind: when the dimension of $Y$ is $\leq 2$ as shown by Mokrane [Mok93, Théorème 5.3, Corollaire 6.2.3] (see also [Nak05]) and when $Y$ is a $p$-adically uniformized variety, as shown independently by de Shalit [dS05] and Ito [Ito05a]. The $\ell$-adic weight-monodromy conjecture is also known in the above cases [RZ82, Ito05a]. Moreover, it has also been proved in the case where $Y$ is a smooth complete intersection in a projective smooth toric variety by Scholze in his seminal paper [Sch12], by using his theory of perfectoid spaces to reduce the problem to the case of an equal characteristic local field, which had already been proved by Deligne in [Del80] (cf. [Ito05b]).

We note that the (analogue of the) $p$-adic weight monodromy conjecture in the equal characteristic case is also known: Crew [Cre98] proved it in the globally defined case, by eventually reducing it to the Weil conjectures for crystalline cohomology using his $p$-adic analogue of Deligne’s method; and building on it, Lazda and Pal [LP16] proved the general case.

Considering this, it is natural to ask if Scholze’s strategy can be adapted also to reduce the $p$-adic weight monodromy conjecture for complete intersections to the equal characteristic case. We provide an affirmative answer in regards to this expectation:

**Theorem 1.2** (Theorem 5.6). Let $K$ be a finite extension of $\mathbb{Q}_p$ and $Y$ be a smooth scheme-theoretic complete intersection inside a projective smooth toric variety over $K$. Then the $p$-adic weight-monodromy conjecture holds for $Y$.

Though Scholze proves the $\ell$-adic case under the slightly weaker assumption that $Y$ is a scheme-theoretic complete intersection, we do need to assume that $Y$ is a scheme-theoretic complete intersection (see also Sections 1.3,1.4). To explain the outline of the proof, we briefly recall how Scholze reduced the problem to the equal characteristic case.

1.1. **Review of Scholze’s proof.** Assume that $Y$ is a smooth complete intersection in a projective smooth toric variety $X_{\Sigma,K}$ over $K$. We may and do replace $K$ with the perfectoid field $K_\infty$ obtained as the completion of the purely ramified extension $K(\varpi^{1/p^\infty})$, for a uniformizer $\varpi$ of $K$. His goal is to realize the cohomology group $H^i(Y) = H^i_{\text{ét}}(Y_K, \mathbb{Q}_\ell)$ as a direct summand of the cohomology group of some proper smooth variety $Z$ of equal characteristic. The variety $Z$ is constructed using the natural continuous map of topological spaces $\pi : |X_{\Sigma,K}^\text{an}| \to |X_{\Sigma,K}^\text{an}|$. 

obtained from the tilting equivalence. More precisely, the construction consists of the following three steps:

(A) We take, using results of Huber [Hub96], [Hub98a], a small (analytic) open neighborhood \( \tilde{Y} \) of (the analytification of) \( Y \) inside \( X_{\Sigma, K}^{an} \) such that \( H^*(Y) \cong H^*(\tilde{Y}) \).

(B) We find, by an approximation argument [Sch12, Proposition 8.7], a closed algebraic subvariety inside \( \pi^{-1}(\tilde{Y}) \) defined over \( k((\varpi^p)) \) that has the same dimension as \( Y \), and then take \( Z \) to be a smooth alteration of it.

(C) Using the tilting equivalence of the étale site [Sch12, Theorem 7.12], we obtain canonical maps

\[ H^i(Y) \cong H^i(\tilde{Y}) \rightarrow H^i(Z) \]

that are equivariant with respect to the actions of \( G_K \cong G_{K^\vee} \).

The latter morphism can be shown to be invertible on the top degree: this follows from the Chern classes formalism. Poincaré duality then provides a desired splitting.

It is tempting to follow the same blueprint in the \( p \)-adic setting. Note first that Step (B) is purely geometric and can be applied to the \( p \)-adic case verbatim. On the other hand, Steps (A) and (C) of the above recipe require some arguments that are particular to \( \ell \)-adic cohomology, and it is unclear a priori how to adapt them to the \( p \)-adic situation. Note that we cannot expect that this argument works for \( p \)-adic étale cohomology: the equal characteristic case due to Lazda–Pal is not about \( p \)-adic étale cohomology and Huber’s existence theorem on a good tubular neighborhood cannot hold (e.g., \( H^1 \) is infinite-dimensional for the rigid disk). With this in mind, we work with Hyodo–Kato cohomology on the generic fiber. Even then, we encounter the following difficulties: in Step (C), the morphism between the étale sites does not directly induce a map on the Hyodo–Kato cohomology groups; in Step (A), Huber’s theorem on the existence of a tubular neighborhood is only available for \( \ell \)-adic cohomology. The goal of our paper is to overcome these difficulties using homotopical methods.

1.2. Tilting and Hyodo–Kato cohomologies. In Sections 2 and 3, we focus on giving a \( p \)-adic version of Step (C). The key ingredient for this step is the motivic tilting equivalence established in [Vez19a], which provides a natural way to discuss relations between tilting and \( p \)-adic cohomology theories (cf. [LBV21]). It is stated in terms of the theory of motives of rigid analytic varieties \( \text{RigDA}(K) \) introduced by Ayoub [Ayo15], which can be applied to our situation since Hyodo–Kato cohomology can be extended to rigid analytic varieties thanks to the work of Colmez–Nizioł [CN19], and it can be shown to be motivic. The motivic tilting equivalence implies that any “well-behaved” cohomology theory defined for rigid analytic varieties over \( K \) can be “tilted” to a “well-behaved” cohomology theory defined for rigid analytic varieties over \( K^\varphi \) (and vice versa). In particular, it allows us to “tilt” Hyodo–Kato cohomology \( R^\Gamma_{\text{HK}} \) and obtain a cohomology theory \( R^\Gamma_{\text{HK}^\varphi} \) on the category of smooth rigid analytic varieties over \( \mathbb{C}_p \) equipped with a functorial \((\varphi, N)\)-structure.

Following Scholze’s strategy, and assuming Step (A), we can again realize \( H^i_{\text{HK}}(Y_K) \) as a direct summand of \( R^\Gamma_{\text{HK}}(Z_{\mathbb{C}_p}^\an) \) for some proper smooth variety \( Z \) of equal characteristic. Then our proof is reduced to showing that the new cohomology theory \( R^\Gamma_{\text{HK}^\varphi} \) satisfies the weight monodromy conjecture (Proposition 5.3). For this purpose, we construct, in the case of semistable reduction, a comparison isomorphism that connects \( R^\Gamma_{\text{HK}^\varphi} \) to the classical Hyodo–Kato cohomology, which satisfies the weight monodromy conjecture by the above-mentioned result of Crew and Lazda–Pal.

**Theorem 1.3** (Corollary 3.13). *Let \( Z \) be a smooth rigid analytic variety over a finite extension \( F \) of \( k((\varpi^p)) \) (inside \( \mathbb{C}_p \)) that admits a semistable formal model with log special fiber \( Z_0 \). Then*
we have a canonical quasi-isomorphism
\[ R \Gamma_{HK}^\varphi(Z_{\mathbb{C}_p}) \cong R \Gamma_{HK}(Z_0/W(\kappa)) \otimes_{W(\kappa)[1/p]} \mathbb{Q}_p^{\text{nr}}, \]
where \( \kappa \) denotes the residue field of \( F \).

We remark that the good reduction case (i.e., non-log case) has been already proved in [Vez19b]. In order to compare cohomology theories on the generic fiber and on the special fiber, the key point of loc. cit. is the construction of the “motivic Monsky–Washnitzer” functor \( \text{DA}(k) \rightarrow \text{RigDA}(K) \) from the category of algebraic motives to that of rigid analytic motives. Its definition is based on the invariance of motives under nilpotent thickenings [Ayo15, Corollaire 1.4.24], which essentially follows from the localization theorem of Morel–Voevodsky. In [Vez19b], it is shown that the motivic Monsky–Washnitzer functor is compatible with the motivic tilting equivalence. Then the good reduction case is obtained by a suitable \( p \)-adic realization on \( \text{RigDA}(K) \).

In order to generalize this strategy to the log case, we introduce the category of log (formal) motives with respect to the strict-étale topology and rational coefficients, and show the following:

**Theorem 1.4** (Theorem 2.22, Proposition 2.31). (1) Let \( \mathcal{S} \) be a quasi-coherent integral log formal scheme of finite Krull topological dimension. The special fiber functor induces an equivalence
\[ \text{logFDA}(\mathcal{S}) \rightleftharpoons \text{logDA}(\mathcal{S}_s), \]
between the category of log formal motives over \( \mathcal{S} \) and that of log motives over the special fiber \( \mathcal{S}_s \).

(2) Let \( K \) be a perfectoid field with residue \( k \). Then the diagram
\[ \begin{array}{ccc}
\text{logFDA}^v(\mathcal{O}_K^\times) & \xrightarrow{\sim} & \text{logDA}^v(k^0) \\
\xi_K & & \mapdown{\xi_K^\varphi} \\
\text{RigDA}(K) & \xrightarrow{\sim} & \text{RigDA}(K^\varphi)
\end{array} \]
is commutative up to an invertible natural transformation.

In the theorem above, we denote by \( \mathcal{O}_K^\times \) the formal scheme \( \text{Spf} \mathcal{O} \) equipped with the natural log structure, and by \( k^0 \) the log scheme \( \text{Spec} k \) equipped with the pullback log structure. The superscript \( v \) refers to the full subcategory of *vertical* log motives (which have a trivial log structure on the generic fiber) and \( \xi \) the functor induced by the rigid analytic generic fiber. By putting \( K = \mathbb{C}_p \) we obtain the comparison isomorphism in Theorem 1.3 as a realization of this commutative diagram. Thus, our task has been reduced to giving a \( p \)-adic analogue of Step (A).

1.3. **Existence of a good tubular neighborhood.** In Section 4, we give a motivic version of Huber’s theorem on the existence of a good tubular neighborhood of \( Y \). Let \( \mathbb{B}_K^N \) denote the \( n \)-dimensional rigid poly disk over a non-archimedean field \( K \).

**Theorem 1.5** (Proposition 4.1). Let \( X \rightarrow S \) be a qcqs smooth morphism of smooth rigid analytic varieties over \( K \). Let \( s \) be a \( K \)-rational point of \( S \) and let \( X_s \) denote the fiber of \( X \) over it. Then, for any sufficiently small open neighborhood \( U \) of \( s \) that is isomorphic to \( \mathbb{B}_K^N \), the natural morphism from the motive of \( X_s \) to that of \( X \times_S U \) in \( \text{RigDA}(K) \) is invertible.

We prove this as a consequence of the “spreading out” property of rigid analytic motives shown in [AGV20, Theorem 2.8.15]. In particular, we can take a tubular neighborhood that does not change the \( \ell \)-adic cohomology groups independently on \( \ell \), which reproves/generalizes,
with a completely different method, a smooth intersection case of the main result of [Ito20], in which this $\ell$-independence property is proved in an algebraizable situation but including non-smooth intersections, using the theory of nearby cycles over general bases.

1.4. Remarks on the proof of the main theorem. In Section 5 we finally put together the above ingredients to prove our main theorem. Note that, since our proof of Theorem 1.2 is motivic, it can be applied to the $\ell$-adic setting as well, in which case it essentially coincides with Scholze’s. Nonetheless, note that Theorem 1.3 requires the morphism $X \to S$ to be smooth, contrarily to what is shown by Huber. This is the reason why our methods only allow us to get the weight-monodromy conjecture for scheme-theoretic complete intersections; not for set-theoretic complete intersections as in [Sch12] (but, see Remark 4.3).

Finally, we also remark that a motivic approach allows us to define a version of Hyodo–Kato cohomology purely on the generic fiber, without making any reference to log schemes or the log-de Rham Witt complex (Appendix A). This is coherent with Fontaine’s initial expectation that the monodromy operator should be defined directly on a $\mathbb{Q}_p^{nr}$-model of the de Rham cohomology, using rigid analytic techniques [Fon94, Remarque 6.2.11]. This construction is a consequence of the description of rigid analytic motives in terms of modules over the residue field, proved in [AGV20], by which any compact motive in $\text{RigDA}(K)$ can be seen as a motive over $\mathbb{G}_{m,k}$ (up to a finite field extension). The Frobenius-action and the monodromy operator arise naturally from the motivic nearby cycle functor (paired up with the notion of weight-structures studied by Bondarko), giving a unified motivic version on the $\ell$-adic and $p$-adic Steenbrick complexes (see also [BGS97]). Such an interpretation will be further developed in a future paper.

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2. LOG MOTIVES AND RIGID MOTIVES

In this section we introduce and study categories of motives that will be relevant throughout the paper.

2.1. Preliminaries on log structures. Even though we are ultimately interested in (semistable) logarithmic schemes over the residue field of $k$, we need some general facts on motives attached to log (formal) schemes.

Our general references for log geometry are [Kat89, Ogu18] for fine log structures and [Kos20] for (not necessarily fine) log (formal) schemes. In particular, our log structures are always defined on the small étale sites, i.e., a log (formal) schemes is a pair $\mathcal{X} = (X, M_X)$ with $X$ being a (formal) scheme and $M_X$ being a sheaf of monoid on the small étale site $X^{\text{ét}}$ together with $\alpha : M_X \to O_{X^{\text{ét}}}$ that induces isomorphism $\alpha^{-1}(O^{*}_{X^{\text{ét}}}) \cong O^{*}_{X^{\text{ét}}}$. We will only consider quasi-coherent, integral log structures ([Kat89, Section 2]).

If $\mathcal{X} = (X, M_X)$ is a log formal scheme with $X$ adic of finite ideal type (see [FK18, Definitions I.1.1.14 and I.1.1.16]) the morphism of étale sites $X^{\text{ét}}_{\text{rig}} \to X^{\text{ét}}$ induces a log structure $M_{X^{\text{ét}}_{\text{rig}}} \to O_{X^{\text{ét}}_{\text{rig}}}^{*}$ on the generic fiber $X^{\text{rig}}$ of $X$ (see [FK18, Section II.3.2.9] for the definition of $X^{\text{rig}}$).
the structure sheaf). One can find further information on log rigid analytic spaces in [DLLZ19], though they will not play any role in our paper.

Definition 2.1. Let $\mathcal{G}$ be an integral quasi-coherent log (formal) scheme. We say that a morphism $f : \mathcal{X} \rightarrow \mathcal{G}$ is smooth if, étale locally on $\mathcal{X}$ and $\mathcal{G}$, there exists a fine log structure $M_0$ on $\mathcal{G}$ with a morphism $M_0 \rightarrow M_\mathcal{G}$, a log smooth morphism $(\mathcal{X}, M_{\mathcal{X}0}) \rightarrow (\mathcal{G}, M_0)$, and an isomorphism $(\mathcal{X}, M_{\mathcal{X}0}) \times (\mathcal{G}, M_0) \mathcal{G} \cong \mathcal{X}$ over $\mathcal{G}$, where the fiber product is taken in the category of integral log (formal) schemes.

We let $	ext{Sm}/\mathcal{G}$ denote the category of log smooth log (formal) schemes over $\mathcal{G}$. This category can be equipped with the strict-étale topology, which we denote by $\acute{e}t$.

Remark 2.2. By [Yao21, Lemma A.1] (see also [Kos20, Proposition A.1]) any quasi-coherent log (formal) scheme has étale locally a canonical chart given by the identity map $\Gamma(\mathcal{G}, M_\mathcal{G}) \rightarrow \Gamma(\mathcal{G}, M_\mathcal{G})$. In particular, any morphism between quasi-coherent log (formal) schemes has a chart, étale locally on the source and the target. Because of this, in the definition of smooth morphisms above, we may equivalently impose $M_0$ to be a fine sub-chart of $\Gamma(\mathcal{G}, M_\mathcal{G})$.

Remark 2.3. The above definition of smoothness recovers the definition “Kato-smooth and locally of finite presentation” of Koshikawa (see [Kos20, Definition A.11, Remark A.13.(3), and Proposition A.15]) using Remark 2.2.

Proposition 2.4. For any morphism $f : \mathcal{X} \rightarrow \mathcal{X}'$ in $	ext{Sm}/\mathcal{G}$, étale locally on $\mathcal{X}$, $\mathcal{X}'$, and $\mathcal{G}$ there is a fine log structure $M_0$ on $\mathcal{G}$ equipped with a morphism $M_0 \rightarrow M_\mathcal{G}$, and a morphism $(\mathcal{X}, M_{\mathcal{X}0}) \rightarrow (\mathcal{X}', M_{\mathcal{X}'0})$ in $\text{Sm}/(\mathcal{G}, M_0)$ such that $f$ is its pullback along $M_0 \rightarrow M_\mathcal{G}$.

Proof. We may and do suppose that $\mathcal{G} = \text{Spf} A$ is affine and that $M_\mathcal{G}$ has a chart $M_\infty \rightarrow A$. Also, we may and do suppose that $g : \mathcal{X} \rightarrow \mathcal{G}$ [resp. $g' : \mathcal{X}' \rightarrow \mathcal{G}$] has a model $(\mathcal{X}, M_{\mathcal{X}0})$ [resp. $(\mathcal{X}', M_{\mathcal{X}'0})$] over the fine log structure $M_0$ [resp. $M_{0'}$] on $\mathcal{G}$. We may even assume there is a fine chart $M_0 \rightarrow N_0$ [resp. $M_{0'} \rightarrow N_{0'}$] of this map. By considering the log structure associated to $M_0 \oplus M_{0'}$, we may suppose that $M_0 = M_{0'}$. By replacing $\mathcal{G}$ be an étale cover, we may assume that the log structure $M_\mathcal{G}$ has a chart $M_\infty$ which is the filtered colimit $\varprojlim M_i$ as $M_i$ varies among fine log structures on $\mathcal{G}$ over $M_0$ with a map to $M_\mathcal{G}$.

The monoid $M_\infty$ is the filtered colimit $\varprojlim M_i$ as $M_i$ varies among fine monoids over $M_0$ with a map to $M_\infty$. We let $N_i$ [resp. $N_i'$] be the integral pushout $N_0 \oplus M_0 M_i$ [resp. $N_0' \oplus M_0 M_{i'}$] and $N_\infty$ be the integral pushout $N_0 \oplus M_0 M_\infty$ [resp. $N_0' \oplus M_0 M_\infty$].

We are left to show that a $M_\infty$-linear morphism $f : N_\infty \rightarrow N_{i\infty}'$ has a $M_i$-linear model $N_i \rightarrow N_{i\infty}'$ for some large enough $i$. We consider the following composite morphism over $M_0$

$$N_0 \rightarrow N_\infty \rightarrow N_{i\infty}' = \varinjlim N_{i\infty}'$$

Since $N_0$ is finitely presented over the fine monoid $M_0$ in the sense of [Ogu18, Lemma I.2.1.9] there exists an index $i$ for which the morphism above factors over $M_0$ as $N_0 \xrightarrow{f_0} N_{i\infty}' \rightarrow N_{i\infty}'$. By eventually taking the base change over $M_i$ of $N_0$ and renaming the indices, we may and do assume that $i = 0$. Since $N_\infty$ [resp. $N_{i\infty}'$] is the push-out of $N_0$ [resp. $N_{i\infty}'$] along $M_0 \rightarrow M_\infty$, we also deduce that the morphism $f_\infty$ is the push-out of $f_0$ along $N_0 \rightarrow N_\infty$, as wanted. □

We will mostly restrict ourselves to vertical morphisms of log schemes:

Definition 2.5 ([Nak97, 7.3]). We say that a morphism of log (formal) schemes $f : \mathcal{X} \rightarrow \mathcal{G}$ is vertical if the cokernel of $f^{-1} M_\mathcal{G} \rightarrow M_\mathcal{X}$ is a sheaf of groups. We denote by $\text{Sm}^e/\mathcal{G}$ the full subcategory of $\text{Sm}/\mathcal{G}$ consisting vertical and log smooth (formal) schemes over $\mathcal{G}$.

Remark 2.6. (1) For a morphism $\varphi : P \rightarrow Q$ of monoids, the cokernel of $\varphi$ is a group if and only if, for every $q \in Q$, there exists $q' \in Q$ and $p \in P$ such that $q = q' + \varphi(p)$. It is also
equivalent to the ideal $\emptyset \subset Q$ being the only prime ideal $q \subset Q$ such that $\varphi^{-1}(q) = \emptyset$. In particular, it is equivalent to $P/P^* \to Q/Q^*$ satisfying the same condition.

(2) Vertical morphisms are stable under composition: if $G \to P$ is a morphism of monoids with $G$ a group such that the cokernel is also a group, then $P$ is a group.

(3) Vertical morphisms are stable under pullbacks and taking the rigid analytic fiber (when $\mathfrak{S}$ is adic and of finite ideal type): for a morphism of monoids, the condition for the cokernel to be a group is stable under pushout.

We will often use the following notation.

**Notation 2.7.** Let $K$ be a non-archimedean field with ring of integers $O_K$ and residue field $k$. Let $O_K^\times$ denote the log formal scheme whose underlying scheme is $\text{Spec} K$ and whose log structure is the one associated to the pre-log structure $O_K \setminus \{0\} \to O_K$. We will denote by $k^0$ the log scheme whose underlying scheme is $\text{Spec} k$ and whose log structure is given by the pullback along $\text{Spec} k \to \text{Spec} O_K$.

We will implicitly use the following elementary fact about the log structure on perfectoid rings:

**Lemma 2.8.** Let $K$ be a perfectoid field with tilt $K^\circ$. The multiplicative map $\sharp : O_K^\times \cong \lim_{\varphi \to \varphi'} O_K \to O_K$, or equivalently, the composite of the Teichmüller lift $O_K^\times \to A_{\text{inf}} = W(O_K^\times)$ and Fontaine’s map $\theta : A_{\text{inf}} \to O_K$, induces a chart $O_K^\times \setminus \{0\} \to O_K$ of the log scheme $O_K^\times$. In particular, the two log structures on $\text{Spec} k$ induced from $O_K$ and $O_K^\times$ are canonically identified.

**Proof.** This follows from the bijectivity of $\sharp : (O_K^\times \setminus \{0\})/O_K^\times \to (O_K \setminus \{0\})/O_K^\times$. \qed

**Definition 2.9.**

(1) Let $K$ be a discrete valuation field and let $\mathfrak{X}$ be a formal scheme over $\text{Spf} O_K$. We say that $\mathfrak{X}$ is semistable if it Zariski-locally it admits an étale $O_K$-morphism to

$$\text{Spf}(O_K(u_1, \ldots, u_n)/(u_1 \cdots u_m - \varpi)),$$

for some integers $0 \leq m \leq n$, where $\varpi$ is a uniformizer of $K$. The category of semistable formal schemes over $O_K$ will be denoted by $\text{FSch}^{ss}/O_K$.

(2) Let $C$ be the completion of an algebraic closure of a discrete valuation field $K$. We say that a formal scheme $\mathfrak{X}$ over $\text{Spf} O_C$ is $K$-semistable if it is basic semistable in the sense of [CN19, 2.2.1(b)] i.e. obtained as the pullback along $\text{Spf} O_C \to \text{Spf} O_L$ from a semistable formal scheme over $O_L$ for some finite field extension $L/K$. The category of $K$-semistable formal schemes will be denoted by $\text{FSch}^{ss}/O_C$.

(3) We say that a log formal scheme $\mathfrak{X} = (\mathfrak{X}, M_\mathfrak{X})$ over $O_K^\times$ [resp. $O_K^\times$] is semistable [resp. $O_K^\times$] if $\mathfrak{X}$ is log smooth and vertical over $O_K^\times$ and if $\mathfrak{X}$ is a semistable [resp. $K$-semistable] formal scheme over $O_K$ [resp. $O_C$]. We denote by $\text{Sm}^{ss}/O_K^\times$ [resp. $\text{logFSm}^{ss}/O_K^\times$] the full subcategory of $\text{Sm}/O_K^\times$ [resp. $\text{Sm}/O_C^\times$] consisting of semistable log formal schemes over $O_K^\times$ [resp. $O_C^\times$].

One could also introduce pluri-nodal or poly-stable versions of the definitions above (cf. Section 3.3).

**Remark 2.10.** Assume that $K$ is the completed algebraic closure of a discrete valuation field. The category $\text{FSch}^{ss}/O_K$ is canonically equivalent to $\text{Sm}^{ss}/O_K^\times$. Indeed, if $\mathfrak{X} \to O_K^\times$ is vertical and log smooth, $\mathfrak{X}$ is (log) regular by [Ogu18, IV.3.5.3] and the log structure on $\mathfrak{X}$ is the log structure associated to the pre-log structure $(O_{\mathfrak{X}}[1/p]^\times) \cap O_{\mathfrak{X}} \to O_{\mathfrak{X}}$ by [Niz06, Proposition 2.6] (or [GR18 Prop. 12.5.54], cf. [Kat94, Theorem 11.6], [Kos20 Example A.1]). Thus,
the equivalence of categories follows (see also [Ogu18 III.1.6.2]). Note that the same equivalence holds for the pluri-nodal or the poly-stable versions since both (like the property of being semistable) are properties of the underlying schemes.

We note that the log structure on the residue field $k_C$ is the one induced by the pre-log structure $\mathbb{Q}_{\geq 0} \cong |\varpi|^{\mathbb{Q}_{\geq 0}} \to C$ sending $a \in \mathbb{Q}_{>0}$ to 0. On the residue field, it then makes sense to give the following definitions.

**Definition 2.11.** Let $k$ be a field. For an integral monoid $\Gamma$, whose monoid structure we will write multiplicatively, we let $M^0_\Gamma$ denote the log structure on $\text{Spec} \ k$ associated to $\Gamma \to k$ sending $\gamma \neq 1$ to 0. For an element $\gamma \in \Gamma$, let $\iota_\gamma$ denote the morphism of log schemes $(\text{Spec} \ k, M^0_0) \to (\text{Spec} \ k, M^0_\Gamma)$ obtained by the identity on $\text{Spec} \ k$ and the map $\mathbb{N} \to \Gamma; 1 \mapsto \gamma$ of monoids.

1. Let $X = (X, M_X)$ be a log smooth log scheme over $(\text{Spec} \ k, M^0_0)$. We say that $X$ is **semistable** if Zariski-locally on $X$, the log scheme $X$ admits a (strictly) étale $(\text{Spec} \ k, M^0_\Gamma)$-morphism to $k[u_1, \ldots, u_n]/(u_1u_2 \cdots u_m)$ equipped with the standard log structure, for some $0 \leq m \leq n$ (cf. [Ogu18 III.1.8.4]). We denote by $\text{Sm}^{ss} / (\text{Spec} \ k, M^0_\Gamma)$ the full subcategory of $\text{Sm} / (\text{Spec} \ k, M^0_\Gamma)$ consisting of semistable log schemes over $(\text{Spec} \ k, M^0_\Gamma)$.

2. Let $X = (X, M_X)$ be a log smooth log scheme over $(\text{Spec} \ k, M^0_0)$. We say that $X$ is **$\gamma$-semistable** if it is isomorphic to the fiber product $X_0 \times (\text{Spec} \ k, M^0_0)_{\delta} (\text{Spec} \ k, M^0_{\Gamma})$ for some $\delta$ such that $\gamma \in \delta^N$ and some semistable log scheme $X_0$ over $(\text{Spec} \ k, M^0_0)$. We denote by $\text{Sm}^{ss} / (\text{Spec} \ k, M^0_0)$ the full subcategory of $\text{Sm} / (\text{Spec} \ k, M^0_0)$ consisting of $\gamma$-semistable log schemes over $(\text{Spec} \ k, M^0_0)$.

**Remark 2.12.** In the above definition, the only relevant case for us is when $k$ is the residue field of the completion $C$ of an algebraic closure of a discrete valued field $K$. In this case, we will always chose $\gamma = |\varpi|$ where $\varpi$ is a uniformizer of $K$. From the definitions above, we then obtain that the special fiber of a $K$-semistable log formal scheme over $\mathcal{O}_C$ is $|\varpi_K|$-semistable.

### 2.2. Logarithmic motives.

For a formal scheme $\mathfrak{S}$, we let $A^1_{\mathfrak{S}}$ [resp. $\mathbb{G}_{m, \mathfrak{S}}$] denote the formal scheme given by $\text{Spf} \ O_{\mathfrak{S}}(u)$ [resp. $\text{Spf} \ O_{\mathfrak{S}}(u^{+1})$]. For a log (formal) scheme $\mathfrak{S}$, we let $A^1_{\mathfrak{S}}$ [resp. $\mathbb{G}_{m, \mathfrak{S}}$] denote the log formal scheme $(A^1_{\mathfrak{S}}, p^* M_\mathfrak{S})$ [resp. $(\mathbb{G}_{m, \mathfrak{S}}, p^* M_\mathfrak{S})$] where $p$ is the natural projection to $\mathfrak{S}$. The following is a straightforward generalization of the classical (infinity-categorical) definition of motives, see e.g. [AGV20 Definitions 2.1.15 and 3.1.3]

**Definition 2.13.** Let $\mathfrak{S}$ be a log (formal) scheme.

1. We let $\text{logFDA}^{\text{eff}}(\mathfrak{S})$ be the full monoidal infinity-subcategory consisting of $A^1_{\mathfrak{S}}$-invariant objects in the monoidal stable infinity-category of étale hypersheaves $\text{Sh}_{\text{ét}}(\text{Sm} / \mathfrak{S}, \text{Ch} \mathbb{Q})$, where $\text{ét}$ denotes the strict-étale topology. We let $L_{A^1_{\mathfrak{S}}}$ be the localization functor (left adjoint to the natural inclusion) $\text{Sh}_{\text{ét}}(\text{Sm} / \mathfrak{S}, \text{Ch} \mathbb{Q}) \to \text{logFDA}^{\text{eff}}(\mathfrak{S})$.

2. We denote by $\mathbb{Q}(1)$ the image by $L_{A^1_{\mathfrak{S}}}$ of the cofiber of the split inclusion of representable sheaves induced by the morphism $1 : \mathfrak{S} \to \mathbb{G}_{m, \mathfrak{S}}$. We let $\text{logFDA}(\mathfrak{S})$ be its stabilization with respect to the Tate twist $\otimes \mathbb{Q}(1)$ (i.e., the formal inversion of $\mathbb{Q}(1)$ in the sense in [Rob15, Definition 2.6]).

For a log scheme $S = (S, M_S)$, we write $\text{logDA}^{\text{eff}}(S)$ instead of $\text{logFDA}^{\text{eff}}(S)$. There is a Yoneda functor from $\text{log Sm} / \mathfrak{S}$ to $\text{logFDA}(\mathfrak{S})$, which we denote by $\mathfrak{X} \mapsto \mathbb{Q}_\mathfrak{S}(\mathfrak{X})$. We use the same notation for $\text{logDA}(S)$. For $n \in \mathbb{Z}$, the $n$-th power Tate twist will be denoted by $M \mapsto M(n)$. 

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We define the category $\logFDA^v(\mathcal{S})$ of vertical log formal motives similarly, i.e., using $\text{Sm}^v/\mathcal{S}$ instead of $\text{Sm}/\mathcal{S}$. If $K$ is [the completed algebraic closure of] a discrete valuation field, we can also define the categories $\logFDA^{ss}(O_K^1)$ of semistable log formal motives, with the obvious notation, and if $k$ is its residue field, we can analogously introduce the category $\logDA^{ss}(k^0)$ of semistable log motives.

**Remark 2.14.** There is an obvious universal property of motives with respect to functors having strict-étale descent, $\mathbb{A}^1$-invariance and inverting the Tate twist: see [LBV21] Remark 2.8.

**Remark 2.15.** Sometimes it would be more natural to consider the *Kummer-étale* or the *log étale* topology on log schemes rather than the strict-étale topology. We remark that in that case one would get a further localization of the category $\logDA$ introduced above. Any log smooth vertical morphism over $O_K$ is log étale locally polystable if the group of values of the valuation ring $O_K$ is divisible (e.g., when $K$ is algebraically closed) by [ALPT18] Theorem 5.2.16. This is also true for the coarser topology divêt (see [BPØ22] Definition 3.1.5). Sometimes it would also be natural to consider the $(\mathbb{P}^1, \infty)$-localization rather than the $\mathbb{A}^1$-localization to define log motives (see [BPØ22]). We do not pursue this approach here.

**Remark 2.16.** Arguing as in [AGV20] Theorem 2.3.4, it is immediate to see that the functor $\mathcal{S} \mapsto \logFDA^v(\mathcal{S})$, $f \mapsto f^*$ where $f^*$ is induced by the pullback, has étale hyperdescent.

**Remark 2.17.** In Definition 2.13, the category of étale hypersheaves consists of those presheaves having descent with respect to étale hypercovers (see [AGV20], Definition 2.3.1 and Remark 2.3.2). If the base $\mathcal{S}$ has a finite (topological) Krull dimension (which will be the case in all the relevant situations) then this category coincides with that of *sheaves* $\text{Sh}_{et}(\text{Sm}/\mathcal{S}, \text{Ch} \mathbb{Q})$ i.e., those having descent with respect to Čech étale hypercovers (see [AGV20] Lemma 2.4.18).

**Remark 2.18.** In case $\mathcal{S}$ has a finite (topological) Krull dimension, the category $\logDA(\mathcal{S})$ is compactly generated. A class of compact generators is given by motives attached to affine log (formal) schemes that are log smooth over the base. A qcqs morphism between log formal schemes of finite Krull dimension induces a functor $f^*$ which is in $\text{Pr}^L_{\mathcal{S}}$ (see [AGV20] Proposition 2.4.22).

**Remark 2.19.** Arguing as in [AGV20] Remark 2.1.13, the categories $\logDA(\mathcal{S})$ and the functors $f^*$ introduced above are in $\text{CAlg}(\text{Pr}^L)$ i.e., they are equipped with a symmetric monoidal structure such that $Q_{\mathcal{S}}(X) \otimes Q_{\mathcal{S}}(X) \cong Q_{\mathcal{S}}(X'')$ where the fiber product $X'' = X \times_{\mathcal{S}} X'$ is taken in the category of integral quasi-coherent log (formal) schemes.

**Remark 2.20.** We warn the reader that we will denote by $\logDA^{(eff)}(\mathcal{S})$ the categories which, in the context of [AGV20], would be denoted as $\logDA^{(eff)}_{et}(\mathcal{S}, \mathbb{Q})\otimes$.

**Remark 2.21.** Note that any log scheme strict-étale over a vertical [resp. semistable] log scheme is vertical [resp. semistable]. In particular, the categories $\logFDA^v(\mathcal{S})$ [resp. $\logFDA^{ss}(O_K^1)$ and $\logDA^{ss}(k^0)$] are full subcategories of $\logDA(\mathcal{S})$ [resp. of $\logFDA(\mathcal{S})$ and $\logDA(k^0)$, respectively] (see for example [Vez19a] Proposition 3.17). The same is true for the poly-stable or the pluri-nodal versions.

### 2.3 Invariance of log motives under the special fiber.
Assume $\mathcal{S}$ is a log formal scheme. We let $\mathcal{S}_\sigma$ be the special fiber of $\mathcal{S}$ (i.e., the reduced scheme associated to $\mathcal{S}/\mathcal{J}$ where $\mathcal{J}$ is an ideal of definition) and $\mathcal{S}_\sigma = (\mathcal{S}_\sigma, M_{\mathcal{S}_\sigma})$ the log scheme induced by the closed immersion $\mathcal{S}_\sigma \subset \mathcal{S}$. The inclusion $\iota: \mathcal{S}_\sigma \rightarrow \mathcal{S}$ induces an adjunction $(\iota^*, \iota_*)$ (see also [AGV20] Notation 3.1.9))

$$
\begin{align*}
\logFDA^{(eff)}(\mathcal{S}) & \xleftarrow{\iota^*} \logDA^{(eff)}(\mathcal{S}_{\text{rig}}) \\
\logDA^{(eff)}(\mathcal{S}) & \xrightarrow{\iota_*} \logDA^{(eff)}(\mathcal{S}_{\text{rig}})
\end{align*}
$$
The following is a log-variant of [Ayo15 Corollary 1.4.29] (see also [AGV20 Theorem 3.1.10]).

**Theorem 2.22.** Let $\mathcal{G}$ is a quasi-coherent integral log formal scheme. Assume that at least one of the following holds:

1. The log structure $M_{\mathcal{G}}$ is fine;
2. $\mathcal{G}$ has finite topological Krull dimension.

Then the adjunction $(\iota^*, \iota_*)$ gives an equivalence in $\text{CAlg}(\mathbb{P}^1)$:

$$\log\text{FDA}^{(\text{eff})}(\mathcal{G}) \cong \log\text{DA}^{(\text{eff})}(\mathcal{G}_\sigma).$$

We will follow closely the proof of [Ayo15 Corollary 1.4.29] which is in turn based on the classical proof of the localization axiom for motives by Morel-Voevodsky [MV99 Theorem 2.21]. We will reproduce it, borrowing some notation from [Hoy21], adapting it to our specific setting.

**Proof.** As in the classical case, it suffices to prove the (stronger) effective version of the statement (i.e. before Tate stabilization) and, since $\iota^*$ is monoidal, we can prove the claim in $\mathbb{P}^1$. We first prove the statement assuming $M_{\mathcal{G}}$ is fine.

By the topological invariance of the étale site [EGAIV Théorème 18.1.2] and the characterization of log smooth maps [Kat89 Theorem 3.5], it is immediate to see that the functor $\iota^*$ sends a class of generators to a class of generators, and that $\iota_*$ commutes with all colimits (see [AGV20 Lemmas 2.2.4 and 2.2.5]).

In particular, in order to show that $$\text{F} \cong \iota_* \iota^* \text{F},$$ as the functors $\iota^*$ and $\iota_*$ commute with colimits, we may assume that $\text{F} = \mathcal{Q}_{\mathcal{G}}(\mathcal{X})$ for some $\mathcal{X} \in \text{Sm}/\mathcal{G}$. We remark that the functor $\iota_*$ between the categories of (complexes of $\mathcal{Q}$-linear) presheaves preserves $(\text{ét}, \mathcal{A}^1)$-equivalences. We shall then consider these objects as presheaves, and prove that this morphism is a $(\text{ét}, \mathcal{A}^1)$-local equivalence (i.e. it becomes invertible in the motivic category).

To this aim, it suffices to check that for any $\mathcal{G}'$ smooth over the base $\mathcal{G}$, and any morphism $\text{F}' := \mathcal{Q}_{\mathcal{G}}(\mathcal{G}') \to \iota_* \iota^* \text{F}$ that corresponds to a morphism $s': \mathcal{G}'_\sigma \to \mathcal{X}$, the pullback $\text{F} \times_{\iota_* \iota^* \text{F}} \text{F}' \to \text{F}'$ is an equivalence. This morphism (see [Ayo07b Corollaire 4.5.40] or [Hoy21 Corollary 5]) is the image via a functor $f'_* \mathcal{Q}$ (which preserves $(\mathcal{A}^1, \text{ét})$-equivalences) of the morphism $T_{(\mathcal{X}', s)} \to \mathcal{Q}$ between presheaves on $\text{Sm}/\mathcal{G}'$, where we put $\mathcal{X}' := \mathcal{G}' \times_{\mathcal{G}} \mathcal{X}$ and where $T_{(\mathcal{X}', s)}$ is the presheaf sending $\mathcal{P}$ to the free $\mathcal{Q}$-module on the set of morphisms $\mathcal{P} \to \mathcal{X}'$ over $\mathcal{G}'$ that factor over the morphism $s': \mathcal{G}'_\sigma \to \mathcal{X}'$ induced by $s$ on the special fiber. In particular, it suffices to show that $T_{(\mathcal{X}', s)} \to \mathcal{Q}$ is an equivalence. Up to renaming the objects, we then fix a section $s: \mathcal{G}_\sigma \to \mathcal{X}$ of $\mathcal{X} \to \mathcal{G}$ and we are left to prove that the presheaf $T_{(\mathcal{X}, s)}$ on $\text{Sm}/\mathcal{G}$ is $(\text{ét}, \mathcal{A}^1)$-equivalent to $\mathcal{Q}$.

By [Kat89 Proposition 4.10], for some strict étale cover $(\mathcal{U}_i)$ of $\mathcal{G}$, the composite $$(\mathcal{U}_i)_\sigma \to \mathcal{G}_\sigma \xrightarrow{s} \mathcal{X}$$ factors into a strict morphism $s': (\mathcal{U}_i)_\sigma \to \mathcal{X}'$ followed by a log étale morphism $e_i: \mathcal{X}'_i \to \mathcal{X}$. In particular, the section $s_i: (\mathcal{U}_i)_\sigma \to \mathcal{X} \times_{\mathcal{G}} \mathcal{U}_i$ factors as $$(\mathcal{U}_i)_\sigma \to (\mathcal{U}_i \times_{\mathcal{G}} \mathcal{U}_i)_\sigma \to \mathcal{X}'_i \times_{\mathcal{G}} \mathcal{U}_i \to \mathcal{X} \times_{\mathcal{G}} \mathcal{U}_i,$$ where the first morphism is the diagonal. Since the statement that we want to prove is étale local on $\mathcal{G}$ (see [Ayo15 Étape 1 of the proof 2.4.21]), we may and do assume that the section $s$ decomposes into a strict closed immersion $s': \mathcal{G}_\sigma \to \mathcal{X}'$, followed by a log étale morphism $e: \mathcal{X}' \to \mathcal{X}$. Since $s'$ targets the strict locus of the morphism $\mathcal{X}' \to \mathcal{G}$, which is an open of $\mathcal{X}$ (see, for example, [Ols03 Proposition 3.19]), by eventually shrinking $\mathcal{X}'$ to this open subscheme, we may and do assume that $\mathcal{X}' \to \mathcal{G}$ is strict.

We now claim that the natural morphism $T_{(\mathcal{X}', s')} \to T_{(\mathcal{X}, s)}$ is an isomorphism. We can argue as follows: let $f: \mathcal{P} \to \mathcal{G}$ be in $\text{Sm}/\mathcal{G}$ and let $h: \mathcal{P} \to \mathcal{X}$ be a section of $T_{(\mathcal{X}, s)}$ on $\mathcal{P}$, i.e. a morphism over $\mathcal{G}$ which factors over $s$ on the special fiber. Note that the following diagram

\[
\begin{array}{c}
\mathcal{P} \\
\downarrow f \\
\mathcal{G} \\
\downarrow h \\
\mathcal{X}
\end{array}
\]
commutes.

\[ \mathcal{P}_\sigma \longrightarrow \mathcal{X}_\sigma \\
\downarrow s' \quad \downarrow \\
\mathcal{S}_\sigma \longleftarrow \mathcal{X}_\sigma \]

We have to show that \( h \) determines uniquely a section of \( T(\mathcal{X}', s') \) on \( \mathcal{P} \). Indeed, let \( \mathcal{P}' = \mathcal{P} \times_\mathcal{X} \mathcal{X}' \) in the category of log formal schemes: it is log étale over \( \mathcal{P} \), and the diagram above gives a section \( \tau_\sigma \) on the special fiber of the morphism \( g : \mathcal{P}' \to \mathcal{P} \), which composed with \( \mathcal{P}'_\sigma \to \mathcal{X}'_\sigma \) agrees with \( s' \circ f_\sigma \). It is then enough to show that \( \tau_\sigma \) can be lifted uniquely to a section \( \tau \) of the log étale morphism \( g \). This is true as soon as the log étale toposes on \( \mathcal{P} \) and \( \mathcal{P}_\sigma \) are equivalent. Since the morphism \( \mathcal{P}_\sigma \to \mathcal{P} \) is a strict universal homeomorphism, we can use the main result of [Vid01] (together with [Ogu90, Corollaries I.2.2.5 and IV.3.1.11] to get rid of the saturated assumption) to conclude.

By what we have just proved, we may and do replace \( \mathcal{X} \) with \( \mathcal{X}' \) and hence assume that the morphism \( \mathcal{X} \to \mathcal{S} \) is strict, and hence, induced by a smooth morphism between the underlying schemes. In this case, we may conclude the statement of the theorem as in [Ayo15, Étape 3 of Proposition 1.4.21] or [Ayo07b, Proposition 4.5.42]. This concludes the proof under the assumption that \( M_\mathcal{S} \) is fine.

We now move to the non-fine case. Fix a quasi-coherent integral log formal scheme \( \mathcal{S} = (\mathcal{S}, M_\mathcal{S}) \) with finite Krull dimension, and note that the functor \( \logFDA((-)) \to \logDA((-)_\mathcal{S}) \) is a morphism between hypersheaves on \( \mathcal{S}_{\text{ét}} \) with values in \( \Pr^1_{\mathcal{O}_\mathcal{S}} \) (see Remarks 2.16 and 2.17). In order to prove it is an equivalence, by [AGV20, Proposition 2.8.1] it suffices to check it is invertible on stalks. To this aim, we fix a geometric point \( \bar{s} \to \mathcal{S} \) and we remark that (see the notation and the result of [AGV20, Proposition 2.5.8])

\[
\lim_{\rightarrow} \logFDA(\U) \cong \logFDA((\U)_{\mathcal{U}}),
\]

where \( \U \) runs among étale neighborhood of \( \bar{s} \) in \( \mathcal{S} \). By definition of smooth morphisms and Proposition 2.4, the étale topos on \( \lim_{\rightarrow} Sm/\U \) is equivalent to the étale topos on \( \lim_{\rightarrow} Sm/(\U, M_{\mathcal{U}}') \) where we now let \( \U \) vary among étale neighborhoods of \( \bar{s} \) and \( M_{\mathcal{U}}' \) vary among fine log structures on \( \mathcal{U} \) with a morphism to \( M_\mathcal{U} \). We then deduce that

\[
\lim_{\rightarrow} \logFDA(\U) \cong \logFDA(((\U, M_{\mathcal{U}}'))_{(\U, M_{\mathcal{U}}')}) \cong \lim_{\rightarrow} \logFDA((\U, M_{\mathcal{U}}')).
\]

As the analogous equivalences hold for \( \logDA((-)_\sigma) \), we can deduce the claim from the fine case.

By Remark 2.21 we can restrict the previous equivalence to vertical/semistable motives.

**Corollary 2.23.** Let \( \mathcal{S} \) be a quasi-coherent integral log formal scheme with a fine log structure, or with an underlying formal scheme of finite topological Krull dimension. The equivalence of Theorem 2.22 restricts to an equivalence

\[
\logFDA^v(\mathcal{S}) \cong \logDA^v(\mathcal{S}_\sigma).
\]

Analogously, if \( K \) is a complete non-archimedean field, we obtain an an equivalence

\[
\logFDA^{ss}(\mathcal{O}_K^\times) \cong \logDA^{ss}(k^0).
\]

The same is true if one restricts to poly-stable or pluri-nodal motives on both sides.  \( \Box \)
Remark 2.24. We note that, based on the proof above, one can get a general “localization triangle” (see [AGV20, Proposition 2.2.3(2)]) for (formal) log étale log motives: in this case, one can work log étale locally and hence replace \( X \) with \( X' \) (in the notation of the proof above) without invoking any special property of the closed immersion chosen.

Remark 2.25. In the proof of Theorem 2.22, we do not use \( \mathbb{Q} \)-coefficients in a crucial way. Part (1) holds for the categories of motives with coefficients in any commutative ring spectrum \( \Lambda \). The proof of part (2) needs some admissibility conditions on the base to ensure that the categories involved are in \( \text{Pr}_L^{\omega} \). We expect this condition to be superfluous.

Remark 2.26. Let \( K \) be a local field. We remark that Corollary 2.23 gives in particular natural equivalences

\[
\logFDA^{\text{ss}}(\mathbb{O}_K^\times) \sim \logDA^{\text{ss}}(k^{\log}) \sim \logFDA^{\text{ss}}(W(k)^0),
\]

where \( W(k)^0 \) is the log formal scheme structure on \( \text{Spf} W(k) \) induced by \( \mathbb{N} \to 0 \). This is the motivic interpretation of the definition of Hyodo–Kato cohomology [HK94, GK05] which can be canonically defined on the right-most category, and hence on the left-most category as well (see [GK05, Theorem 0.1]).

Corollary 2.27. Let \( (\mathcal{S}_i)_i \) be a cofiltered inverse system of quasi-compact and quasi-separated log formal schemes with affine transition morphisms, and let \( \mathcal{S} \) be its limit. Assume that each log formal scheme in \( \{\mathcal{S}_i, \mathcal{S}\} \) has a fine log structure or a finite topological Krull dimension. Then the canonical functor defines an equivalence

\[
\lim_{\to} \logFDA(\mathcal{S}_i) \cong \logFDA(\mathcal{S}).
\]

Proof. We may prove the analogous statement for the special fibers, where it follows from the equivalence of the étale toposes on \( \text{Sm}_{/\mathcal{S}} \) and on \( \lim_{\to} \text{Sm}_{/\mathcal{S}_i,\sigma} \). □

Example 2.28. If \( C \) is the completion of an algebraic extension of a local field \( K \) then

\[
\logFDA^{\text{ss}}(\mathbb{O}_C^\times) \sim \lim_{\to} \logFDA^{\text{ss}}(\mathbb{O}_{L}^\times)
\]

as \( L \) varies among finite extensions of \( K \) inside \( C \).

2.4. The log Monsky–Washnitzer functor and its compatibility with tilting. In this subsection, we show that the “log Monsky–Washnitzer functors” are compatible with the motivic tilting equivalence from [Vez19a], which is a log generalization of the main result of [Vez19b].

First of all, we introduce the categories of rigid analytic motives.

Definition 2.29. Let \( S \) be a rigid analytic space (in the sense of [FK18, Definition II.2.2.18]). We denote by \( \text{RigDA}^{(\text{eff})}(S) \) the category \( \text{RigDA}^{(\text{eff})}_\text{ét}(S, \mathbb{Q}) \) of (effective) hypercomplete rigid analytic motives over \( S \) with respect to the étale topology ([AGV20, Definition 2.1.15 and Remark 2.1.18]). It has a natural symmetric monoidal structure. For a non-archimedean field \( K \), we write \( \text{RigDA}^{(\text{eff})}(K) \) for the category \( \text{RigDA}^{(\text{eff})}((\text{Spf } \mathbb{O}_K)^{\text{rig}}) \). We denote by \( \text{RigDA}(K) \) the full subcategory of \( \text{RigDA}(K) \) of compact objects (or, equivalently, of fully dualisable objects by [Ayo15, Theorem 2.31] and [Rio05]).

Remark 2.30. In [AGV20], the category \( \text{RigDA}^{(\text{eff})}(S) \) endowed with its monoidal structure is typically denoted by \( \text{RigDA}^{(\text{eff})}_\text{ét}(S, \mathbb{Q})^{\otimes} \). In the following, we will only consider rigid analytic spaces with a finite Krull dimension, for which the non-hypercomplete version of the definition \( \text{RigDA}(S, \mathbb{Q}) \) coincides with the one given above [AGV20, Lemma 2.4.18].

Let \( \mathcal{S} = (\mathcal{S}, M_{\mathcal{S}}) \) be a log formal scheme. We assume that \( \mathcal{S} \) is adic and of finite ideal type, so that we have the rigid space \( \mathcal{S}^{\rig} \) associated to \( \mathcal{S} \) (see [FK18]). We assume, for simplicity, that the log structure \( M_{\mathcal{S}} \) is trivial on the rigid generic fiber, in which case we write \( \mathcal{S}^{\rig} \) for
Then the functor $\text{Sm}^v/\mathcal{S} \to \text{Sm}/\mathcal{S}^{\text{rig}}$: $X \mapsto X^{\text{rig}} = X^{\text{rig}}$ induces an adjunction (see also [AGV20 Notation 3.1.12])

$$\log\text{FDA}^v(\mathcal{S}) \cong \log\text{FDA}^v(\mathcal{S}^{\text{rig}}).$$

We call the following composite the log Monsky–Washnitzer functor

$$\log\text{DA}^v(\mathcal{O}_K^\times) \cong \log\text{FDA}^v(\mathcal{S}) \to \text{RigDA}(\mathcal{S}^{\text{rig}}).$$

Let $\mathcal{O}_K$ be perfectoid field with tilt $\mathcal{O}_K^\varphi$. Recall that the main theorem of [Vez19a] gives a motivic tilting equivalence $\text{RigDA}(\mathcal{O}_K^{\varphi}) \cong \text{RigDA}(\mathcal{O}_K)$ (see [BV21 Theorem 3.12]).

According to Lemma 2.8, the log schemes $\mathcal{O}_K^\times$ and $\mathcal{O}_K^{\varphi^\times}$ (see Notation 2.7) induce the same log structure on the residue field. Then the compatibility of the motivic tilting equivalence and the Monsky–Washnitzer functor can be stated as follows, which generalizes [Vez19b Theorem 3.2] (see also [LBV21 Proposition 5.11]).

**Proposition 2.31.** Let $\mathcal{O}_K$ be perfectoid field with tilt $\mathcal{O}_K^\varphi$. The following diagram commutes up to a canonical invertible natural transformation.

$$
\begin{array}{ccc}
\log\text{FDA}^v(\mathcal{O}_K^\times) & \cong & \text{RigDA}(\mathcal{O}_K^\varphi) \\
\downarrow & & \downarrow \\
\log\text{DA}^v(\mathcal{O}_K^\times) & \cong & \text{RigDA}(\mathcal{O}_K^{\varphi^\times})
\end{array}
$$

For the proof, we use some notation of [LBV21].

**Remark 2.32.** Let $\mathcal{O}_K$ be a perfectoid field. The base change along the Frobenius defines an endofunctor $\text{RigDA}(\mathcal{O}_K^\varphi) \xrightarrow{\phi^*} \text{RigDA}(\mathcal{O}_K^\varphi)$, equipped with a natural invertible transformation $\phi^* \Rightarrow \phi^*$. In particular, for every motive $M$ there is a natural equivalence $M \cong M^{\varphi^*}$, which can be used to define a natural functor $\text{RigDA}(\mathcal{O}_K^\varphi) \to \text{RigDA}(\mathcal{O}_K^{\varphi^{\text{h\varphi}}})$ to the Frobenius fixed points (see [LBV21 Section 2.3] and the notations therein). This functor is compatible with the rigid-analytic generic fiber functor $\xi$ and with the restriction functor $\iota^*$ in the sense that the diagram

$$
\begin{array}{ccc}
\log\text{DA}^v(\mathcal{O}_K^\times) & \cong & \text{RigDA}(\mathcal{O}_K^\varphi) \\
\downarrow & & \downarrow \\
\log\text{DA}^v(\mathcal{O}_K^{\varphi^\times}) & \cong & \text{RigDA}(\mathcal{O}_K^{\varphi^{\text{h\varphi}}})
\end{array}
$$

is commutative. Here the left vertical functors are the canonical one to the lax homotopy fixed points of the Frobenius computed in $\text{P}_\mathcal{T}.^L_{\omega}$. Informally the objects of $\text{C}^{\text{h\varphi-L}}$ for a compactly generated presentable $\infty$-category $\mathcal{C}$ equipped with an endofunctor $F: \mathcal{C} \to \mathcal{C}$ with a right adjoint are pairs $(X, \alpha)$, where $X$ is an object of $\mathcal{C}$ and $\alpha$ is a map $X \to FX$. Note that the homotopy fixed points are a localization of the lax fixed points using e.g. [Lur09 Proposition 5.5.3.17]. See [NS18 Definition II.1.4] for more details. The subscript $\omega$ stands for the full subcategory generated under filtered colimits by compact objects.

**Proof of Proposition 2.31.** We recall that the equivalence $\text{RigDA}(\mathcal{O}_K) \cong \text{RigDA}(\mathcal{O}_K^\varphi)$ can be obtained by passing over the Fargues-Fontaine curve as in [LBV21 Corollary 5.14]. That is, for a fixed affinoid open neighborhood $U = \text{Spa}(B_{[0,\varepsilon]}, B_{[0,\varepsilon]+})$ of $\mathcal{O}_K^\varphi = \text{Spa} \mathcal{O}_K^{\varphi}$ in $\text{Spa} \mathcal{O}_K^{\varphi}$
containing $x^\dagger = \text{Spa} K$ the tilting equivalence is given as the composition of the first line in the following diagram, where we use the notation of [LBV21, Section 5.1]:

$$
\begin{array}{c}
\text{RigDA}(K^\flat) \xrightarrow{h^\flat \varphi} \text{RigDA}(K) \\
\downarrow \quad \downarrow \\
\log FDA^\vee(O^\times_K) \xrightarrow{\sim} \log FDA^\vee((\text{A}_{\text{inf}}, M_K^\flat)) \xrightarrow{\sim} \log FDA^\vee((B^+_{[0,\varepsilon]}, M_K^\flat)) \xrightarrow{\sim} \log FDA^\vee(O^\times_K) \\
\downarrow \quad \downarrow \\
\text{logDA}^\vee(k^0) \xrightarrow{\sim} \logDA^\vee(k^0) \xrightarrow{\sim} \logDA^\vee(k^0) \xrightarrow{\sim} \text{logDA}^\vee(k^0) \\
\downarrow \quad \downarrow \\
\logFDA^\vee(O^\times_K) \xrightarrow{\sim} \logFDA^\vee(O^\times_{K^\flat}) \\
\downarrow \quad \downarrow \\
\logDA^\vee(k^\flat) \xrightarrow{\sim} \logDA^\vee(k^\flat) \\
\end{array}
$$

in which the squares commute (see Remark 2.32). We now show that the composite $\text{logDA}^\vee(k^0) \to \text{RigDA}(K)$ obtained in the diagram above is induced by the natural one. To this aim, it suffices to consider the following commutative $\varphi$-equivariant diagram

$$
\begin{array}{c}
\logFDA^\vee(O^\times_K) \xrightarrow{\sim} \logDA^\vee(k^0) \xrightarrow{\sim} \logDA^\vee(k^0) \\
\downarrow \quad \downarrow \\
\logFDA^\vee(O^\times_{C^\flat}) \\
\end{array}
$$

in which the functors on the left are invertible by means of Corollary 2.23. □

**Corollary 2.33.** Let $C$ be the completion of an algebraic closure of a discrete valuation field $K$ with uniformizer $\varpi$ and residue field $k$. Let $C^\flat$ be its tilt, which is the completion of an algebraic closure of the discrete valuation field $k((\varpi^\flat))$. The following diagram commutes, up to a canonical invertible natural transformation.

$$
\begin{array}{c}
\logFDA^\vee(O^\times_C) \xrightarrow{\sim} \logDA^\vee(k^0) \xrightarrow{\sim} \logDA^\vee(k^0) \\
\downarrow \quad \downarrow \\
\logFDA^\vee(O^\times_{C^\flat}) \\
\end{array}
$$

**Proof.** The diagram is connected to the one of Proposition 2.31 via fully faithful functors. □

The relation between rigid motives and semistable log motives is encoded by the following.

**Proposition 2.34.** Let $C$ be the completion of an algebraic closure of a discrete valuation field $K$ of residual characteristic $p > 0$. The generic fiber functor $\xi$ exhibits $\text{RigDA}(\text{eff})(C)$ [resp. $\text{RigDA}(\text{eff})(C^\flat)$] as the rig-étale localization of the category $\logFDA^\vee(O^\times_C)$ [resp. $\logFDA^\vee(O^\times_{C^\flat})$].

**Proof.** We use that $\text{RigDA}(C) \cong \varprojlim \text{RigDA}(L)$ runs over finite extensions $L/K$ inside $C$ [AGV20, Theorem 2.8.15]. Then, by [AGV20, Notation 2.5.5], $\text{RigDA}(C)$ is the category of motives over the étale topos $\varprojlim \text{RigSm}/L$. It now suffices to show that the étale topos on $\text{RigSm}/K$ is equivalent to the rig-étale topos on potentially semistable formal models over $O_K$. □
This is the content of [CN19 Proposition 2.8]. The same proof works in the equi-characteristic case.

In light of Proposition 2.34, we may restate Corollary 2.33 as follows.

**Corollary 2.35.** Let $C$ be the completion of an algebraic closure of a discrete valuation field. Via the following natural equivalence

$$
\text{logFDA}^{ss}(O_C^\infty) \xrightarrow{\sim} \text{logDA}^{ss}(\bar{k}^0) \xleftarrow{\sim} \text{logFDA}^{ss}(O_C^\infty),
$$

the rig-étale localization on $\text{logFDA}^{ss}(O_C^\infty)$ corresponds to the rig-étale localization on $\text{logFDA}^{ss}(O_C^\infty)$.

**Remark 2.36.** The equivalence displayed in Corollary 2.35 obtained from Corollary 2.23 looks like a “stronger” form of the motivic tilting equivalence (of [Vez19a]) since we are comparing $\text{RigDA}(\mathbb{C}_p)$ and $\text{RigDA}(\mathbb{C}_p^\circ)$ before the two rig-étale localizations over $\mathbb{C}_p$ and over $\mathbb{C}_p^\circ$. Nonetheless, a priori it is not clear that such localizations agree, as they may not be expressed in terms of a Grothendieck topology defined intrinsically on the special fibers.

### 3. The Motivic Hyodo–Kato Realization

The aim of this section is to recall from [CN19] the construction of (overconvergent) Hyodo–Kato cohomology on the rigid generic fiber and to prove that its tilt, which is a cohomology theory on rigid analytic varieties over $\mathbb{C}_p^\circ$, is compatible with the classical Hyodo–Kato cohomology of Große-Klönne in the semistable case (Corollary 3.13). We will give both a “geometric” realization (that is, over a local field) and an “arithmetic” realization (that is, over a local field). We note that only the former will be used in the proof of our main theorem.

#### 3.1. A motivic geometric Hyodo–Kato realization

Let $K$ be a complete discrete valuation field with a perfect residue field $k$ and $C$ be the completion of an algebraic closure $\bar{K}$ of $K$, whose residue field is denoted by $\bar{k}$. We let $K_0$ denote the field $W(k)[1/p]$ viewed as a subfield of $K$ and $K_0^{nr}$ denote the maximal unramified extension of $K_0$ inside $\bar{K}$.

We collect some definitions on Hyodo–Kato cohomology, following Große-Klönne [GK05], Beilinson [Bei13], Ertl–Yamada [EY19] and Colmez–Nizioł [CN19], reinterpreted using a motivic language. Hyodo–Kato cohomology on the rigid generic fiber defined in [CN19] can be thought of as a rigid analytic analogue of motivic statements of [DN18], which are based on [NN16].

Though there is a way to define Hyodo–Kato cohomology purely in terms of the generic fiber (see Appendix A), we follow in this section the classical approach using log structures.

**Definition 3.1** ([Bei13, 1.15]). Let $\varphi : K_0 \rightarrow K_0'$ (resp. $\varphi : K_0^{nr} \rightarrow K_0^{nr}$) denote the endomorphism induced by the absolute Frobenius $a \mapsto a^p$ on the residue field.

1. A $\varphi$-module over $K_0$ (resp. over $K_0^{nr}$) is a $K_0$-vector space (resp. a $K_0^{nr}$-vector space) $D$ equipped with a $\varphi$-semilinear endomorphism $\varphi : D \rightarrow D$.

2. A $(\varphi, N)$-module over $K_0$ (resp. over $K_0^{nr}$) is a $\varphi$-module $D$ over $K_0$ (resp. over $K_0^{nr}$) equipped with a linear endomorphism $N : D \rightarrow D$ satisfying $N\varphi = p\varphi N$. We denote by $D_{\varphi,N}(K_0)$ (resp. $D_{\varphi,N}(K_0^{nr})$) the DG-derived category of $(\varphi, N)$-modules over $K_0$ (resp. over $K_0^{nr}$).

**Remark 3.2.** The category $D_{(\varphi,N)}(K_0^{nr})$ is equivalent to the category $\lim \text{Pr}^L\text{D}_{(\varphi,N)}(W(k')[1/p])$ computed in $\text{Pr}^L$ as $k'$ runs among finite extensions of $k$ inside $\bar{k}$. More explicitly, a compact object is given by a bounded complex of finite dimensional $(\varphi, N)$-modules over $k'$ i.e. by some compact object of $D_{\varphi,N}(W(k')[1/p])$ for a sufficiently large $k'$. 

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We use Notation\[2.7\] for finite extensions of \( K \) and \( C \). In particular, for a finite extension of \( k' \) of \( k \) inside \( \bar{k} \), we let \( k'^0 \) [resp. \( \bar{k}^0 \)] denote the scheme \( \mathrm{Spec} \ k' \) [resp. \( \mathrm{Spec} \ \bar{k} \)] equipped with the log structure associated to \( \mathbb{N} \to k' \colon 1 \mapsto 0 \) [resp. \( \Gamma = (\mathcal{O}_C \setminus \{0\})/\mathcal{O}_C^* \to \bar{k} \) sending \( \gamma \neq 1 \) to 0].

**Definition 3.3.** Let \( X = (X, M_X) \) be a semistable log scheme over \( k^0 \). We denote by \( R\Gamma_{\text{HK}}(X/W(k^0)) \) the Hyodo–Kato cohomology complex \( R\Gamma_{\text{rig}}^{\text{ss}}(X, X) \) defined as in [\text{EY19}, Definition 3.17 and Remark 3.19] (see also [\text{CN19}, Remark 5.3] and [\text{EY21}, Section 1.3.3]). It is an object in \( D(X^\text{ss}_N)(K^0) \), whose cohomology groups will be denoted by \( H^i_{\text{HK}}(X) \).

**Remark 3.4.** (1) The complex above is quasi isomorphic to the log rigid cohomology complex defined by Große-Klönne in [\text{GK05}, Lemma 1.4].

(2) By [\text{GK05}, Section 3.11], if \( X \) is proper, the cohomology groups \( H^i_{\text{HK}}(X) \) are canonically isomorphic to the crystalline Hyodo–Kato cohomology groups defined in [\text{HK94}, Theorem 5.1].

For a finite extension \( k' \) of \( k \) inside \( \bar{k} \) and an integer \( d \geq 1 \), let \( k^{0(d)} \) denote the log scheme \( k^0 \) viewed as a \( k^0 \)-log scheme via the morphism \( k^0 \to k^0 \) induced by the inclusion \( k \subset k' \) and the map \( \mathbb{N} \to \mathbb{N}; 1 \mapsto d \). We always regard \( \bar{k}^0 \) as a \( k^{0(d)} \)-log scheme via the morphism \( \iota_d \colon \bar{k}^0 \to k^{0(d)} \) induced by the inclusion \( k' \subset \bar{k} \) and the map \( \mathbb{N} \cong |p| \mathbb{N} \subset \Gamma = (\mathcal{O}_C \setminus \{0\})/\mathcal{O}_C^* \).

For a semistable log scheme \( X' \) over \( k^{0(d)} \), we denote by \( R\Gamma_{\text{HK}}^{\text{ss}}(X'/W(k')) \) the complex \( R\Gamma_{\text{HK}}(X'/W(k')) \) equipped with the monodromy operator \( N(d) = \frac{1}{d} N \), where \( N \) is the usual monodromy operator (cf. [\text{NN16} and \text{EY19}, Section 3.1]). Note that, if \( X' \) comes from a semistable log scheme \( X \) over \( k^0 \), i.e., \( X' \cong X \times_{k^0} k^{0(d)} \), then we have a canonical \( (\varphi, N) \)-equivariant quasi-isomorphism

\[
R\Gamma_{\text{HK}}^{\text{ss}}(X \times_{k^0} k^{0(d)}/W(k')) \cong R\Gamma_{\text{HK}}(X/W(k^0)) \otimes_{W(k^0)[1/p]} W(k')[1/p].
\]

**Remark 3.5.** The case \( d = 1 \) of the above isomorphism is known as “unramified base change” for Hyodo–Kato cohomology [\text{CN19}, Paragraph 5.3.1(i)], whereas the case \( d > 1 \) corresponds to “ramified base change” [\text{NN16}, Section 3A].

**Definition 3.6.** Let \( X = (X, M_X) \) be a \( [p] \)-semistable log scheme over \( \bar{k}^0 \) (see Definition\[2.11\]). Let \( R\Gamma_{\text{HK}}(X) \) denote the complex defined by

\[
\lim_{\longrightarrow} R\Gamma_{\text{HK}}^{\text{ss}}(X_i/W(k_i)) \otimes_{W(k_i)[1/p]} K_0^{\text{nr}},
\]

where the colimit is taken over the filtered category of objects \( X_i/k_i^{0(d_i)} \) with \( d_i \geq 1 \) an integer, \( k_i \) a finite extension of \( k \) inside \( \bar{k} \), and \( X_i \) a semistable log scheme over \( k_i^{0(d_i)} \) together with an isomorphism \( X_i \times_{k_i^{0(d_i)}} \bar{k}^0 \cong X \). Since the transition maps in the colimit are \( (\varphi, N) \)-equivariant (and even quasi-isomorphic), the complex \( R\Gamma_{\text{HK}}(X) \) is naturally equipped with a \( (\varphi, N) \)-structure and defines a functor \( R\Gamma_{\text{HK}} \colon \text{Sm}^{\text{ss}}/\bar{k}^0 \to D_{\varphi, N}(K_0^{\text{nr}})^{\text{op}} \).

**Remark 3.7.** We note that, thanks to the existence of Hyodo–Kato comparison map (in this setting the statement of [\text{GK05}, Theorem 0.1] suffices) the functor

\[
R\Gamma_{\text{HK}} \colon \text{Sm}^{\text{ss}}/\bar{k}^0 \to D_{\varphi, N}(K_0^{\text{nr}})^{\text{op}}
\]

satisfies \( \mathbb{A}^1 \)-invariance, étale descent and Tate-stability (see also the proof of Proposition\[3.8\]) and hence produces a functor

\[
R\Gamma_{\text{HK}} \colon \text{logDA}^{\text{ss}}(\bar{k}^0) \to D_{\varphi, N}(K_0^{\text{nr}})^{\text{op}}.
\]

We can extend this functor to log motives over \( \mathcal{O}_C^X \) as follows.
Proposition 3.8. The composite functor

\[ \text{Sm}^{ss}/\mathcal{O}^\times_C \to \text{Sm}^{ss}/\bar{k}^0, \quad R\Gamma_{\text{HK}} \to \mathcal{D}_{(\varphi,N)}(K_{0}^{nr}) \]

has rig-étale descent, $\mathbb{A}^1$-invariance and Tate stability.

The proposition is essentially a formality given the comparison theorems of Colmez–Nizioł. Nonetheless, we briefly recall these constructions, in order to set some notation.

Proof. Note that the functor \( X \mapsto R\Gamma_{\text{HK}}(X) \) satisfies Zariski descent. Thus, in order to prove it has étale descent, we may take a rig-étale cover \( U \to X \) in $\text{FSch}^{ss}/\mathcal{O}_{C_p}$ underlying a rig-étale cover of affine semistable weak formal schemes (see [CN19, Proposition 2.13]) and show that \( R\Gamma_{\text{HK}}(X) \) is the limit of the Cech nerve associated to the cover \( U \).

We consider the quasi-isomorphism

\[ R\Gamma_{\text{HK}}(X) \otimes_{K_{0}^{nr}} C \sim R\Gamma_{dR}^!(X) \]

from [CN19, Formula 5.16] (note that the complex \( R\Gamma_{\text{HK}}(X) \) is a perfect complex under our assumptions, so that the tensor product of loc. cit. is merely a scalar extension).

Since \( R\Gamma_{dR}^! \) has étale descent (see [Vez18, Proposition 5.12]) we conclude that the complex on the left is the limit of the Cech nerve associated to the cover \( U \), as wanted.

Similarly, $\mathbb{A}^1$-invariance follows from the $B^1$-invariance of the overconvergent de Rham complex. Tate stability follows from the explicit computation \( H_{1,\text{HK}}(\mathbb{G}_m,\bar{k}^0) \sim K_{0}^{nr}(1) \). □

Definition 3.9. (1) By Proposition 2.34 and Proposition 3.8, the functor

\[ \text{Sm}^{ss}/\mathcal{O}^\times_C \to \text{Sm}^{ss}/\bar{k}^0, \quad R\Gamma_{\text{HK}} \to \mathcal{D}_{(\varphi,N)}(K_{0}^{nr})^{\text{op}} \]

induces a functor $\text{Rig}_{DA}(C) \to \mathcal{D}_{(\varphi,N)}(K_{0}^{nr})^{\text{op}}$, which we again denote by $R\Gamma_{\text{HK}}$. We will use the same notation for the following induced functors

\[ \text{logDA}^{ss}(\bar{k}^0) \xrightarrow{\xi} \text{RigDA}(C) \xrightarrow{R\Gamma_{\text{HK}}} \mathcal{D}_{(\varphi,N)}(K_{0}^{nr})^{\text{op}} \]

\[
\text{DA}(C) \xrightarrow{\text{An}^*} \text{RigDA}(C) \xrightarrow{R\Gamma_{\text{HK}}} \mathcal{D}_{(\varphi,N)}(K_{0}^{nr})^{\text{op}},
\]

where $\text{An}^*$ denotes the functor induced by analytification.

(2) We let $R\Gamma_{\text{HK}}^{\text{op}}$ denote the composite functor

\[ R\Gamma_{\text{HK}}^{\text{op}}: \text{RigDA}(C^\circ) \cong \text{RigDA}(C) \xrightarrow{R\Gamma_{\text{HK}}} \mathcal{D}_{(\varphi,N)}(K_{0}^{nr})^{\text{op}}, \]

where the first equivalence is the motivic tilting equivalence given in [Vez19a, Theorem 7.26]. We will use the same notation for the induced functor

\[ \text{DA}(C^\circ) \xrightarrow{\text{An}^*} \text{RigDA}(C^\circ) \xrightarrow{R\Gamma_{\text{HK}}^{\text{op}}} \mathcal{D}_{(\varphi,N)}(K_{0}^{nr})^{\text{op}}. \]

Remark 3.10. Proposition 3.8 gives us a commutative diagram of realizations:

\[ \xymatrix{ \text{logFDA}^{ss}(\mathcal{O}^\times_C) \ar[r]^\sim \ar[d]^{\xi} & \text{logDA}^{ss}(\bar{k}^0) \ar[d]^{R\Gamma_{\text{HK}}} \\ \text{RigDA}(C) \ar[r]^{R\Gamma_{\text{HK}}} & \mathcal{D}_{(\varphi,N)}(K_{0}^{nr})^{\text{op}}, } \]

where the left vertical map is the logarithmic Monsky–Washnitzer functor. Recall again from Proposition 2.34 that $\xi$ is a localization with respect to the rig-étale topology.
Remark 3.11. The above definition of Hyodo-Kato cohomology on the rigid generic fiber is essentially borrowed from [CN19, Section 5.3.1]. Their cohomology theory also has étale descent (by construction), ℂ₁-invariance, and Tate stability, and hence induces a realization functor
\[ RΓ_{\text{HK}} : \text{RigDA}(C) \cong \text{RigDA}^{+}(C) \to D_{(\varphi,N)}(\mathbb{K}_{0}^{nr})^{\text{op}}. \]
Here, \text{RigDA}^{+}(C) denotes the category of overconvergent rigid analytic motives (see [Vez18, Definition 4.18]), which is equivalent to \text{RigDA}(C) via the canonical functor sending a dagger variety to the underlying rigid variety [Vez18, Theorem 4.23].

As they agree on semi-stable models and both satisfy rig-étale descent, the functors \( RΓ_{\text{HK}} \) of Proposition 3.12 agree.

Remark 3.12. The realization \( RΓ_{\text{HK}} : \text{RigDA}(C)^{ct} \to D_{\varphi,N}(\mathbb{K}_{0}^{nr})^{\text{op}} \) is monoidal. Indeed, the first category is generated under finite colimits by motives of the form \( \mathbb{Q}_{C}(X^{an}) \) with \( X/C \) a smooth and proper algebraic variety by [Ayo15, Théorème 2.5.35], and the monoidal structure is the one extended by the Day convolution (see [AGV20, Remark 2.1.6]) \( \mathbb{Q}_{C}(X) \otimes \mathbb{Q}_{C}(X') = \mathbb{Q}_{C}(X \times X') \). In particular, monoidality of \( RΓ_{\text{HK}} \) can be tested on these motives, and in this case the formula \( RΓ_{\text{HK}}(X) \otimes RΓ_{\text{HK}}(X') \cong RΓ_{\text{HK}}(X \times X') \) is the usual Kunneth formula for the algebraic Hyodo–Kato cohomology (see e.g. [DN18, Lemma 2.21]).

Corollary 3.13. Let \( F \) be a finite extension of \( k((p^{e})) \) inside \( C^{o} \), where \( p^{e} \in C^{o} \) is the element defined by a system of \( p \)-th power roots of \( p \). Let \( \kappa \) denote the residue field of \( F \) and \( e \) the ramification index of the finite extension \( F/k((p^{e})) \). Let \( Z \) be a semistable formal scheme over \( \text{Spf} \mathcal{O}_{F} \) equipped with the natural log structure. Let \( Z_{0} \) denote the rigid generic fiber and \( Z_{0}^{\prime} \), the special fiber as a log scheme. Then we have a canonical quasi-isomorphism
\[ RΓ_{\text{HK}}^{\varphi}(Z_{0}/\mathbb{K}) \cong RΓ_{\text{HK}}^{e}(Z_{0}/W(\kappa)) \otimes_{W(\kappa)[1/p]} \mathbb{K}_{0}^{nr} \]
in \( D_{\varphi,N}(\mathbb{K}_{0}^{nr}) \), where \( RΓ_{\text{HK}}^{e} \) is as in Definition 3.6.

Proof. By Proposition 2.31 we have \( \mathbb{Q}(Z_{0})^{\varphi} \cong \xi\mathbb{Q}(Z_{0} \times_{\kappa^{(e)}} \overline{k}^{0}) \). Then, by the definitions of \( RΓ_{\text{HK}}^{\varphi} \) and \( RΓ_{\text{HK}} \) (cf. Remark 3.10), the left hand side is canonically isomorphic to \( RΓ_{\text{HK}}(Z_{0} \times_{\kappa^{(e)}} \overline{k}^{0}) \), which is then identified with the right hand side by Definition 3.6.

Remark 3.14. Note that the normalization on the monodromy operator does not alter the monodromy filtration on the cohomology groups of \( RΓ_{\text{HK}}^{e}(Z_{0}/W(\kappa)) \).

3.2. Abstract properties of realizations on algebraic motives. In this section, we recall from [CD12, DM15] that any monoidal algebraic realization is automatically equipped with some formal structures of a Weil cohomology, such as Poincaré duality and a formalism of Chern classes. For such structures on Hyodo–Kato cohomology, see also [NN16, Section 5], [EY20], [LP16].

Definition 3.15. Let \( \kappa \) be a field. We let \( \text{DA}(\kappa)^{ct} \) be the full subcategory of compact objects (or, equivalently, of fully dualisable objects, see [Ric05]) in \( \text{DA}(\kappa) = \text{DA}_{\text{ct}}(\kappa, \mathbb{Q}) = \text{DM}_{\text{ct}}(\kappa, \mathbb{Q}) \). The category \( \text{DA}(\kappa)^{ct} \) is equivalent to Voevodsky’s category of geometric motives [Voe00, [Ayo14b, Proposition 8.3, Théorème B.1].

Let \( \kappa \) be a field and \( C \) be a \( \mathbb{Q} \)-linear symmetric monoidal, compactly generated infinity-category with a complete \( t \)-structure and a \( \mathbb{Q} \)-linear \( t \)-exact conservative functor \( f : C \to \mathcal{D}(\Lambda) \) for some field extension \( \Lambda / \mathbb{Q} \). Let \( RΓ : \text{DA}(\kappa) \to \mathcal{C}^{op} \) be a (cohomological) realization functor, i.e., a \( \mathbb{Q} \)-linear functor in \( \text{Pr}^{L}_{\mathbb{Q}} \). We assume that its restriction to compact objects \( RΓ^{ct} \) is monoidal. We will denote by \( H^{i} \) the \( i \)-th cohomology of \( RΓ \).
Remark 3.16. Note that, by Remark 3.12 and the monoidality of the analyti fication functor [AGV20, Proposition 2.2.13], the algebraic Hyodo–Kato realizations \( R_{\Gamma_{HK}} : \mathcal{D}_\varphi, N(K^\nu_0) \rightarrow \mathcal{D}_{\varphi, N}(K^\nu_0) \) and \( R_{\Gamma^\vee_{HK}} : \mathcal{D}_\varphi, N(K^\nu_0) \rightarrow \mathcal{D}_{\varphi, N}(K^\nu_0) \) fit in this framework, and all the following properties can be deduced for such algebraic Hyodo–Kato (co)homologies.

**Theorem 3.17** (Poincaré duality; [Voe00, Theorem 4.3.2], [CD12, Theorem 1] [Ayo14a, Theorem 3.11]). Let \( X \) be a proper smooth \( \kappa \)-scheme purely of dimension \( d \). Then the motive \( \mathbb{Q}(X) \) has a strong dual in \( \mathcal{D}_\kappa \) given by \( \mathbb{Q}(X)(-d)[2d] \). Thus, \( R\Gamma(X) \) is strongly dualizable with dual \( R\Gamma(X)(d)[2d] \), and hence we have a perfect pairing in \( \mathcal{C} \)

\[
H^i(X) \otimes H^{2d-i}(X)(d) \rightarrow 1.
\]

Now we recall that (oriented) motivic realizations come naturally equipped with a theory of Chern classes and cycle classes (cf. [CD12, Section 2.3]).

Let \( R\Gamma^\vee \) be the composition of \( R\Gamma^\text{ct} \) with the (canonical) dual endofunctor on \( \mathcal{D}_\kappa^\text{ct} \). Since \( \mathcal{D}_\kappa = \text{Ind}(\mathcal{D}_\kappa^\text{ct}) \) [Lur17, Lemma 5.3.2.9] we can extend this realization formally to a monoidal functor that preserves colimits (i.e. to a functor in \( \mathcal{C}_{\text{Alg}}(\text{Pr}_L) \), see [Lur09, Lemma 5.3.5.8]):

\[
R\Gamma^\vee : \mathcal{D}_\kappa \rightarrow \mathcal{C}.
\]

This corresponds to the associated homological realization. Informally, if \( M \) is a colimit of compact objects \( M = \text{colim} K_i \) then its image is \( \text{colim} R\Gamma(K^\vee_i) \).

Let \( R\Gamma^\vee \) denote a right adjoint of \( R\Gamma^\vee \) and \( \mathcal{E} \) be the object \( R\Gamma^\vee_1 \). It is a \( \mathbb{E}_{\infty} \)-ring object in \( \mathcal{D}_\kappa^\text{ct} \) (see [Lur18, Proposition 2.5.5.1]) that represents the cohomology theory \( f \circ R\Gamma \) on compact objects. In particular, we can apply the construction in [DM15, 2.1] which yields the following.

**Proposition 3.18** (Cycle classes). For a quasi-compact smooth scheme \( X \) over \( \kappa \) and for each integer \( n \geq 0 \), we let

\[
\text{cyc} : \text{CH}^n(X) \rightarrow H^{2n}(X)(n).
\]

denote the map \( \sigma_\mathcal{E} \) defined in [DM15, (2.1.3.b)]. These maps are compatible with pullback and pushforward.

**Proof.** Follows from [DM15, 2.1.3 and 2.1.6.(4)] and [Voe00, Proposition 4.2.3 and Corollary 4.2.5]. \( \square \)

The following result gives a nonzero criterion which is used in the proof of our main theorem, following Scholze.

**Corollary 3.19.** Let \( h : Z \rightarrow Y \) be a morphism of projective smooth schemes over \( \kappa \) with \( Z \) geometrically irreducible of dimension \( d \). Assume that the push-forward \( h_*[Z] \) is a positive cycle class [Ful98, §12]. Then the natural map \( H^{2d}(Y) \rightarrow H^{2d}(Z) \) is a nonzero map.

**Proof.** By Proposition 3.18 we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{CH}^d(Y) & \xrightarrow{\text{cyc}} & H^{2d}(Y) \\
\downarrow h^* & & \downarrow h^* \\
\text{CH}^d(Z) & \xrightarrow{\text{cyc}} & H^{2d}(Z) \\
\downarrow \text{deg} & & \downarrow \text{tr} \\
Z & \rightarrow & \mathbb{A}.
\end{array}
\]
Since the trace map is bijective (Remark 3.21), it suffices to show that the composite \( \deg \circ h^1 \) is nonzero. Let \( L \) be an ample line bundle on \( Y \). Then, by [Ful98] Proposition 8.3.(c), we have \( \deg h^1 c_1(L)^d = \deg (c_1(L)^d \cap h_! [Z]) \). The latter is nonzero by [Ful98] Lemma 12.1.

\[ \square \]

**Remark 3.20.** Note that if \( h : Z \to Y \) is proper with a scheme theoretic image \( Z' \) of dimension \( d = \dim Z \), then \( h_! [Z] \) is \( \deg (\kappa(Z)/\kappa(Z')) \cdot [Z'] \), which is a positive cycle class.

**Remark 3.21 (Dimension axiom).** Suppose now \( \kappa \) is the completion of an algebraic closure of a discrete valuation field of residue characteristic \( p \) and assume that the trace map induces a bijection \( H^{2d}(X)(d) \cong \Lambda \) whenever \( X \) is a proper smooth variety of semistable reduction of dimension \( d \). This is the case for Hyodo–Kato cohomology theories \( R\Gamma, R\Gamma^p \) by Remark 3.4(2) and Corollary 3.13.

For a variety \( f : Z \to \text{Spec} \kappa \), let \( H^*_c(Z) \) be the cohomology groups with compact support attached to \( Z \) i.e. the cohomology groups of \( R\Gamma(f_* f^! 1) \). By the localization triangle, alterations and induction on the dimension, one deduces that the trace map \( H^{2d}_c(Z)(d) \to \Lambda \) is bijective, whenever \( Z \) is a geometrically connected variety of dimension \( d \).

3.3. **Arithmetic Hyodo–Kato cohomology via motives.** The reader who is only interested in the proof of the main theorem can safely skip this section, which is devoted to a definition of an arithmetic motivic Hyodo–Kato realization in equi-characteristic and mixed characteristic, and their comparison. To this aim, we first recall some properties of the categories of rigid analytic motives with good reduction.

**Definition 3.22.** We let \( \text{RigDA}_{\text{gr}}(K) \) be the full subcategory of \( \text{RigDA}(K) \) generated, under colimits, shifts and twists, by motives “of good reduction”, i.e. those attached to varieties \( X_\eta \) where \( X \) is a smooth formal scheme over \( \mathcal{O}_K \).

Despite the name, the category generated by motives of good reduction may contain motives of varieties which do not have good reduction: for instance, it contains all varieties with semistable reduction (see Proposition 3.29). As a matter of fact, if \( K \) is algebraically closed, then \( \text{RigDA}_{\text{gr}}(K) \cong \text{RigDA}(K) \). This is part of the following proposition, in which we sum up some results of [AGV20].

**Proposition 3.23.** Let \( L \) be the completion of an algebraic extension of \( K \) with residue field \( k_L \) and let \( C \) be the completion of an algebraic closure of \( K \).

1. We have \( \text{RigDA}_{\text{gr}}(L) \cong \lim_{\rightarrow} \text{RigDA}_{\text{gr}}(K') \) as \( K'/K \) varies among finite extensions inside \( L \).
2. If \( L/K \) is totally ramified, then the pullback induces a canonical equivalence \( \text{RigDA}_{\text{gr}}(K) \cong \text{RigDA}_{\text{gr}}(L) \). In particular we have \( \text{RigDA}_{\text{gr}}(L) \cong \text{RigDA}_{\text{gr}}(L_0) \) with \( L_0 = W(k_L)[1/p] \) being the completion of the maximal unramified extension of \( K_0 \) inside \( L \).
3. The functor \( K \mapsto \text{RigDA}_{\text{gr}}(K) \) has descent with respect to the \( \acute{e} \text{tale} \) site of unramified field extensions and
   \[ \text{RigDA}_{\text{gr}}(W(k)[1/p]) \cong \lim_{\rightarrow} \text{RigDA}_{\text{gr}}(W(k')[1/p]) \cong \text{RigDA}_{\text{gr}}(C) = \text{RigDA}(C). \]
4. Assume \( L \) is perfectoid. Then the following diagram commutes.
   \[
   \begin{array}{ccc}
   \text{RigDA}_{\text{gr}}(L) & \longrightarrow & \text{RigDA}(L) \\
   \sim & & \sim \\
   \text{RigDA}_{\text{gr}}(L^p) & \longrightarrow & \text{RigDA}(L^p)
   \end{array}
   \]
Proof. By means of [AGV20, Theorem 3.3.3(1)], the category $\text{RigDA}_{gr}(K)$ is canonically equivalent to the category of $\chi_1$-modules in $\text{DA}(k)$ with $\chi$ being the right adjoint to the Monsky–Washnitzer functor $\xi: \text{DA}(k) \rightarrow \text{RigDA}(K)$ (see for example [AGV20, Notation 3.1.12]). We then deduce property (1) from [AGV20, Theorem 3.5.3], and property (3) from [AGV20 Propositions 3.5.1 and 3.7.17]. Property (4) is already shown in [Vez19b]. We are left to show (2). In light of (1) we may assume $L$ to be a finite extension of $L_0$. The base change functor $\text{Mod}_{\chi_0}(\text{DA}(k_L)) = \text{RigDA}_{gr}(L_0) \rightarrow \text{RigDA}_{gr}(L) = \text{Mod}_{\chi_1}(\text{DA}(k_L))$ corresponds to a base change along a map of algebras $\chi_0 \rightarrow \chi_1$. We remind that a choice of a uniformizer gives rise to an equivalence $\chi_1 \cong 1 \oplus 1 (-1)[-1] \cong \chi_0$ (see [AGV20, Remark 3.8.2]). Under these identifications, the map of algebras $\chi_0 \rightarrow \chi_1$ corresponds to the map $\left( \begin{smallmatrix} 1 & 0 \\ 0 & e_L \end{smallmatrix} \right)$ with $e_L \neq 0$ as computed in [Ayo07c, 1.4], [Ayo07b, 3.4.14-3.5.12] and is therefore invertible (we use that our coefficient ring is $\mathbb{Q}$).

Remark 3.24. Recall that the cohomological motive $q_1$ of $q: \mathbb{G}_m \rightarrow k$ is canonically equivalent to $1 \oplus 1 (-1)[-1]$ (see e.g., [Ayo07a, Scholie 1.4.2, Subsection 1.5.3, and Théorème 2.3.75]). In the previous proof, we remarked in particular that a choice of a uniformizer determines an equivalence $\chi_1 \cong q_1$, and that the category $\text{RigDA}_{gr}(K)$ is equivalent to the category $\text{Mod}_{\chi_1}(\text{DA}(k)) \cong \text{Mod}_{q_1}(\text{DA}(k))$. This implies that, after the choice of a uniformizer, $\text{RigDA}_{gr}(K)$ can be alternatively thought as the subcategory $\text{UDA}(k)$ of $\text{DA}(\mathbb{G}_m,k)$, generated under colimits by motives coming from $k$ (see [Spi04, Corollaries 15.12 and 15.14]).

Remark 3.25. The equivalence

$$\text{RigDA}(C)^{ct} \cong \lim_{\rightarrow} \text{RigDA}_{gr}(W(k')[1/p])^{ct}$$

already shows that the (overconvergent) de Rham cohomology of a rigid analytic variety comes equipped with a $K_0^{nr}$-structure, and the equivalence $\text{RigDA}(C) \cong \text{Mod}_{\chi_1}(\text{DA}(k))$ shows that it also has a $\varphi$-structure. Also the monodromy can be build formally from Proposition 3.23 see Appendix A.

Remark 3.26. The coefficient $e_K$ appearing in the proof above is the ramification index of $K$. We shall see (in Remark A.3) that this is compatible with the presence of the normalization factor on the monodromy.

We now show that varieties of semistable reduction lie in $\text{RigDA}_{gr}(K)$. We start by recalling the following definition from [Ber99].

Definition 3.27. Let $L$ be a complete non-archimedean valued field with pseudo-uniformizer $\varpi$. A formal scheme $\mathfrak{X}$ over $\text{Spf } O_L$ is pluri-nodal if Zariski-locally it can be written as part of a sequence

$$\mathfrak{X} = \mathfrak{X}_d \xrightarrow{f_d} \mathfrak{X}_{d-1} \xrightarrow{f_{d-2}} \cdots \xrightarrow{f_1} \mathfrak{X}_1 \xrightarrow{f_0} \text{Spf } O_L$$

in which each transition map $f_i$ is, étale locally on source and target, given by a composition $\text{Spf } A_{i+1} = \mathfrak{X}_{i+1} \xrightarrow{\eta} \text{Spf } A_i(u,v)/(uv-a_i) \rightarrow \text{Spf } A_i$ with $e_i$ étale and $a_i$ invertible in $A_i[\varpi^{-1}]$. The category they form will be denoted $\text{FSch}^{pl}/O_L$.

Remark 3.28. We remind the reader that semistable formal schemes are pluri-nodal. The same is true for polystable formal schemes (in the sense of [Ber99]).

The following fact appears in [Ayo15].

Proposition 3.29. Let $L$ be a complete non-archimedean valued field and let $\mathfrak{X}$ be in $\text{FSch}^{pl}/O_L$. Then the motive $M(\mathfrak{X}_n)$ lies in $\text{RigDA}_{gr}(L)$. 

where semistable varieties are considered with respect to $B$. We let
\[ R \Gamma_{HK}^0 : \text{logFDA}^s_\sigma(W(k)[1/p]) \rightarrow \mathcal{D}_{\varphi,N}(K_0^{nr})^{op} \]

where semistable varieties are considered with respect to $k((T))[\text{resp. } |T|]$. We define:
\[ R \Gamma_{HK}^0 : \text{logFDA}^s_{\sigma}(O_{\sigma}^L) \cong \text{logDA}^s_{\sigma}(\bar{k}) \rightarrow \mathcal{D}_{\varphi,N}(K_0^{nr})^{op} \]

where the functor $R \Gamma_{HK}$ on the right is induced by Definition 3.3 and the one below computes the arithmetic overconvergent Hyodo–Kato cohomology of $[CN19]$, §5.2.2.

Proof. Note that $\text{RigDA}(C) = \lim R \text{rig}_{\text{gr}}(W(k)[1/p])$ by Proposition 3.23. The diagram then follows from Proposition 3.3 by unramified Galois descent and $[CN19]$ Proposition 5.13.

One may generalize the construction above to the case of local fields of equi-characteristic as follows.

Definition 3.31. We let $C$ be the completion of an algebraic closure of $k((T))$ and we define:

\[ \text{logFDA}^s_{\sigma}(O_{\sigma}^L) \cong \text{logDA}^s_{\sigma}(\bar{k}) \rightarrow \mathcal{D}_{\varphi,N}(K_0^{nr})^{op} \]

where $\text{logFDA}^s_{\sigma}$ is the log FDA over $W(k)[1/p]$ and $\text{logDA}^s_{\sigma}$ is the log DA over $W(k)$.

By definition, if $X$ is a strictly semistable formal scheme over a finite unramified extension $L$ of $\text{Spf}(k[T])$, its Hyodo–Kato cohomology is defined as $R \Gamma_{HK}(X) := R \Gamma(X_{\sigma}/W(k_L)^0)$ as $(W(k_L)[1/p], \varphi, N)$-module. In particular, it depends only on the log special fiber $X_{\sigma}$ as log scheme over $k_{L,\log}^{log}$. Note that this is exactly the same formula used in the mixed characteristic case: the precise relationship is summarized in the following corollary.

Corollary 3.32. (1) The realization $R \Gamma_{HK}^p : \text{logFDA}^s_\sigma(W(k)[1/p]) \rightarrow \mathcal{D}_{\varphi,N}(K_0^{nr})^{op}$ has rig-étale descent, and hence factors over a realization

\[ R \Gamma_{HK}^p : \text{RigDA}(C) \rightarrow \mathcal{D}_{\varphi,N}(K_0^{nr})^{op}. \]
Moreover, the following diagram commutes:

\[
\begin{array}{ccc}
\log\text{FDA}^\text{ss}(\mathcal{O}_C^\times) & \longrightarrow & \text{RigDA}(C) \\
\log\text{DA}^\text{ss}(k^0) & \overset{\text{R}_{\Gamma_{\text{HK}}}}{\longrightarrow} & \mathcal{D}_{\varphi,N}(K_0^\text{nr})^{\text{op}} \\
\log\text{FDA}^\text{ss}(\mathcal{O}_{C_\varphi}^\times) & \longrightarrow & \text{RigDA}(C^\phi)
\end{array}
\]

(2) Let \( K \) be \( k(\langle p^\phi \rangle) \). There is a commutative diagram of motivic realization functors, compatible with field extensions:

\[
\begin{array}{ccc}
\log\text{FDA}^\text{ss}(\mathcal{O}_K^\times) & \sim & \log\text{DA}^\text{ss}(k^0) \\
\text{RigDA}^\text{gr}(K) & \overset{\text{R}_{\Gamma_{\text{HK}}}}{\sim} & \mathcal{D}_{\varphi,N}(W(k)[1/p])^{\text{op}} \\
\log\text{FDA}^\text{ss}(\mathcal{O}_{\varphi,K}^\times) & \sim & \text{RigDA}^\text{gr}(k^\phi)
\end{array}
\]

where the functor \( \text{R}_{\Gamma_{\text{HK}}} \) is induced by the Hyodo–Kato cohomology of Definition 3.3.

Proof. For the first part, use Proposition 3.8 and Corollary 2.35. For the second, use once again that \( \text{RigDA}^\text{gr}(C^\phi) \cong \text{RigDA}^\text{gr}(k(\langle p^\phi \rangle)) \) and unramified Galois descent. The last claim is immediate from the definitions.

We can extend the previous definitions to an arbitrary \( K \), in accordance with [CN19, Remark 4.13].

Definition 3.33. Let \( L \) be a finite extension of \( K \) [resp. \( k(\langle T \rangle) \)] with residue \( k_L \), and let \( L_0 \) be the maximal subfield of \( L \) which is unramified over \( K_0 \) [resp. \( k(\langle T \rangle) \)]. We denote by \( \text{R}_{\Gamma_{\text{HK}}} \) the following contravariant realization functor

\[
\text{RigDA}^\text{gr}(L) \cong \text{RigDA}^\text{gr}(L_0) \rightarrow \mathcal{D}_{\varphi,N}(K_0^\text{nr})^{\text{op}}
\]

where the first equivalence is the one of Proposition 3.23. It is compatible with base change over \( L \) by definition.

The compatibility with tilting easily descends to the case of finite extensions.

Corollary 3.34. Let \( L \) be a finite extension of \( K \) with residue \( k_L \). Let \( L_\infty \) be the completion of \( L(p^{1/p_\infty}) \). The following diagram commutes

\[
\begin{array}{ccc}
\log\text{FDA}^\text{ss}(\mathcal{O}_{L_0}^\times) & \longrightarrow & \text{RigDA}^\text{gr}(L_0) \sim \text{RigDA}^\text{gr}(L_\infty) \\
\log\text{DA}^\text{ss}(k_0^L) & \overset{\text{R}_{\Gamma_{\text{HK}}}}{\longrightarrow} & \mathcal{D}_{\varphi,N}(W(k_L)[1/p])^{\text{op}} \\
\log\text{FDA}^\text{ss}(\mathcal{O}_{\varphi,K}^\times) & \sim & \text{RigDA}^\text{gr}(k_L(\langle p^\phi \rangle)) \sim \text{RigDA}^\text{gr}(L_\infty^\phi)
\end{array}
\]

Proof. In light of Corollary 2.33, Proposition 3.8(2) and Corollary 3.32(2), only the triangle in the middle must be shown to be commutative. This follows from the Galois-invariance of the tilting equivalence \( \text{RigDA}^\text{gr}(K_\infty) \cong \text{RigDA}^\text{gr}(K_\infty^\phi) \) (see e.g. [LBV21, Theorem 5.13]), and Corollary 3.32(1).

\( \square \)
We also note the realizations above can be enriched with a $G_L$-structure, and extended to the whole of $\text{RigDA}(L)$, as in the classical Hyodo–Kato setting.

**Corollary 3.35.** The functor $R\Gamma_{HK}$ can be extended to a functor $$\text{RigDA}(L)^{\text{ct}} \to \mathcal{D}^b_{\varphi,N,G_L}(K_0^{nr})^{\text{op}}$$ where the category on the right is the derived DG-category of $(\varphi, N, G_L)$-modules [DN18, Section 2.6].

**Proof.** As there is a natural transformation $\mathcal{D}^b_{\varphi,N}((-)_0) \to \mathcal{D}^b_{\varphi,N,G_L}((-)_0)$ which takes the same values on $C$, and the functor on the right has Galois descent, it suffices to apply the étale sheafification on the functor $R\Gamma_{HK}$. See [AGV20, Theorem 3.3.3(2)].

**Remark 3.36.** The restrictions to $\text{RigDA}(-)^{\text{ct}}$ of the realization functors above are monoidal. This is a formal consequence of Remark 3.12.

**Remark 3.37.** It is clear from our definition that the Hyodo–Kato realization $R\Gamma^T_{HK}$ induced on $\text{DA}(k((T)))$, with its structure as $(\varphi, N)$-module, coincides with the one considered by Lazda and Pal [LP16, Chapter 5] as they both have étale descent, excision and compare to Hyodo–Kato [LP16, Theorem 4.27, Lemma 4.36, Theorem 5.46].

The following corollary is an "arithmetic" version of Corollary 3.13.

**Corollary 3.38.** Let $K$ be a finite extension of $\mathbb{Q}_p$. Let $K_\infty$ be a totally ramified perfectoid extension of $K$, whose tilt $K_\infty^0$ is the completed perfection of a finite extension $K^0$ of $\mathbb{F}_p((T))$. If $Z$ is a smooth proper variety over $K^0$ with semi-stable reduction, then the image of $\mathbb{Q}_K(Z^{an})$ under the composition below $$\text{RigDA}_{gr}(K^0) \cong \text{RigDA}_{gr}(K_\infty^0) \cong \text{RigDA}_{gr}(K_\infty) \cong \text{RigDA}_{gr}(K) \to \mathcal{D}_{\varphi,N}(K_0)^{\text{op}}$$ coincides with the Hyodo–Kato cohomology of its special fiber.

**Proof.** Use the diagrams of Corollary 2.33, Proposition 3.8(2) and Corollary 3.32(2).

4. Motivic Existence of Tubular Neighborhoods

We now prove a motivic analogue of [Hub98a Theorem 3.6] (see also [Hub98b] and [Hub07]). We start by proving that the motive of a smooth family of rigid analytic varieties is locally constant around each rational point of the base (Proposition 4.1(1)). This follows from a "spreading-out" (or "continuity") property of rigid motives, proved in [AGV20]. From this, the existence of a tubular neighborhood that does not change the motive immediately follows. A similar trick is used in the proof of [LBV21 Theorem 4.46].

We remark that if $S$ is smooth over $K$, around any $K$-rational point there is an open neighborhood which is isomorphic to $\mathbb{B}_K^N$ ([Ber93 Theorem 2.1.5]).

**Proposition 4.1.** Let $X \to S$ be a qcqs smooth morphism of rigid analytic varieties over a complete non-archimedean field $K$. Let $s$ be a $K$-rational point of $S$ and let $X_s$ denote the fiber of $X$ over it.

1. For any sufficiently small open neighborhood $U$ of $s$, we have a canonical equivalence $\mathbb{Q}_S(X_s \times_K U) \cong \mathbb{Q}_S(X_s \times_S U)$ in $\text{RigDA}(S)$.

2. Assume that $S$ is smooth over $\text{Spa} \ K$. Then, for any sufficiently small open neighborhood $U$ of $s$ that is isomorphic to $\mathbb{B}_K^N$, the natural morphism $\mathbb{Q}_K(X_s) \to \mathbb{Q}_K(X_s \times_S U)$ in $\text{RigDA}(K)$ is invertible.
Proof. Since the statement of (1) is local on \( S \), we may assume that \( X \) is qcqs. By [AGV20, Corollary 2.4.13 and Theorem 2.8.15] the natural specialization maps \( s^* \) induce an equivalence \( \text{RigDA}(s)^{ct} = \text{RigDA}(K)^{ct} \cong \lim_{\to} \text{RigDA}(V)^{ct} \) in \( \text{P}^{1, \omega}_K \) as \( V \) runs along affinoid neighborhoods of \( s \).

We notice that the pullback \( \Pi^* \) along the structural morphism \( \Pi : S \to \text{Spa} \ K \) defines an explicit quasi-inverse of the functor above. This implies that for any compact motive \( M \) in \( \text{RigDA}(S) \), its restriction to \( U \) is canonically isomorphic to \( \Pi^* s^* M \) in \( \text{RigDA}(U) \) for \( U \) sufficiently small. If we apply this to \( M = Q_S(X) \) (which is compact by [AGV20, Corollary 2.4.13 and Proposition 2.4.22]) we deduce \( Q_U(X_s \times_K U) \cong Q_U(X \times_S U) \) or, equivalently \( Q_S(X_s \times_K U) \cong Q_S(X \times_S U) \).

To obtain (2), we note that, by applying the functor \( \Pi_U \) (see [AGV20, Proposition 2.2.1(1)] for the definition) to (1), we obtain a canonical equivalence \( Q_K(X_s \times U) \cong Q_K(X|U) \), whose composition with the equivalence \( Q_K(X_s) \cong Q_K(X_s \times_K U) \) is the morphism \( Q_K(X_s) \to Q_K(X \times_S U) \) defined by the canonical inclusion \( X_s \to X \times_S U \). Thus, the assertion follows.

\( \square \)

Remark 4.2. Note that in Proposition 4.1 we could choose \( U \) to be any rigid analytic variety which becomes contractible in \( \text{RigDA}(K) \), e.g. a poly-disc with with radius in \( \sqrt{|K|} \) [Ayo15, Proposition 1.3.4].

Remark 4.3. Note that we need the smoothness assumption contrary to Huber’s result [Hub98a, Theorem 3.6]. In the above proof, we use it to deduce that the motive \( f_1 f^1 1 \) is compact in \( \text{RigDA}(S) \) for the structural morphism \( f \); this would not be true if \( f \) had singular fibers (cf. [Hub96, Theorem 6.2.2], [Hub98b (0.2)], and [Hub98a (0.1)]). However, we expect that the conclusion of Proposition 4.1 holds in more general situations.

Remark 4.4. A version of Proposition 4.1 with torsion coefficients recovers, under the smoothness assumption, a result of K. Ito: In [It020, Theorem 1.2], it is proved that, in an algebraizable situation, we can \( \ell \)-independently take a tubular neighborhood that does not change the \( \mathbb{F}_\ell \)-coefficient étale cohomology, where \( \ell \) is prime to the residual characteristic of \( K \). There, the proof uses the theory of nearby cycles over general bases, and is completely different from ours.

We now focus on the main geometric object to which we will apply the above result. We refer to [Sch12, Section 8] for details on toric varieties. We fix a complete non-archimedean valued field \( K \), a projective smooth toric variety \( X_\Sigma \) over \( K \) and a smooth subvariety \( Y \) of codimension \( c \) which is the scheme-theoretic intersection \( Y_1 \times_{X_\Sigma} \cdots \times_{X_\Sigma} Y_c \) of \( c \) hypersurfaces \( Y_i = V(f_i) \) with \( f_i \in H^0(X_\Sigma, O(D_i)) \) a global section of some line bundle \( O(D_i) \). For \( \varepsilon \in |K^*| \), let \( Y(\varepsilon) \) denote the analytic open neighborhood of \( Y^{an} \) inside \( X_\Sigma^{an} \) introduced in the proof of [Sch12, Proposition 8.7], which is defined (locally) by the inequalities \( |f_i(x)| \leq \varepsilon \).

Corollary 4.5. Let \( K \) be a complete non-archimedean field, \( X_\Sigma \) be a projective smooth toric variety over \( K \), and \( Y \) be a smooth scheme-theoretic intersection \( Y_1 \times_{X_\Sigma} \cdots \times_{X_\Sigma} Y_c \) of \( c \) hypersurfaces \( Y_i = V(f_i) \). Then, there exists an element \( \varepsilon_0 \in |K^*| \) such that for any \( \varepsilon < \varepsilon_0 \), the natural morphism \( Q_K(Y) \to Q_K(Y(\varepsilon)) \) in \( \text{RigDA}(K) \) is invertible.

Proof. Since \( X_\Sigma \) is quasi-compact, we can prove the statement locally on \( X_\Sigma^{an} \), where we can apply Proposition 4.1(2). \( \square \)

5. The \( p \)-adic weight-monodromy for complete intersections

We can now prove our main theorem. We first make some recollections on the notation we will use.

25
Let $\overline{Q}_p$ denote an algebraic closure of $Q_p$ and $C_p$ its completion. Let $Q_p^m$ denote the maximal unramified extension of $Q_p$ inside $\overline{Q}_p$.

**Definition 5.1.** Let $w$ be an integer.

1. Let $K_0$ be a finite unramified extension of $Q_p$. We let $p^\alpha$ be the cardinality of the residue field of $K_0$. We say that a finitely dimensional $\varphi$-module $D$ is pure of weight $w$ if every eigenvalue $\alpha$ of the $K_0$-linear endomorphism $\varphi^\alpha$ is pure of weight $w$ in the sense of [Del80, Definition 1.2.1].

2. Let $D$ be a finitely dimensional $\varphi$-module over $Q_p^m$. We can take a finite unramified extension $K_0$ of $Q_p$ inside $Q_p^m$ and a $\varphi$-module $D_0$ over $K_0$ together with an isomorphism $D_0 \otimes_{K_0} Q_p^m \cong D$ of $\varphi$-modules over $K_0^m$. We say that $D$ is pure of weight $w$ if $D_0$ is pure of weight $w$. This definition does not depend on the choices.

3. Let $D$ be a finitely dimensional $(\varphi, N)$-module over $Q_p^m$. Then $N$ is a nilpotent endomorphism, and hence it has the associated monodromy filtration $M_\bullet$ defined in [Del80, Proposition 1.6.1]. We say that $D$ is quasi-pure of weight $w$ if, for every integer $j$, the $j$-th graded quotient $\text{gr}_j D$ of the monodromy filtration, which is a $\varphi$-module over $Q_p^m$, is pure of weight $w + j$.

As noted in the introduction, we may give an equivalent formulation of the weight-monodromy conjecture:

**Conjecture 5.2.** Let $X$ be a proper smooth algebraic variety over $C_p$ defined over $\overline{Q}_p$. Then the $(\varphi, N)$-module $H^i_{\text{HK}}(X^m)$ is quasi-pure of weight $i$.

Let $C_p^e$ denote the tilt of $C_p$, which is the completion of an algebraic closure of the Laurent series field $F = F_p((p^\alpha))$ ([Sch12, Theorem 3.7]). Let $\bar{F}$ denote the algebraic closure of $F$ inside $C_p^e$. Let $H^i_{\text{HK}}$ denote the tilted Hyodo–Kato cohomology defined in Section 3.1.

**Proposition 5.3.** Let $Z$ be a proper smooth variety over $C_p$ defined over $\bar{F}$. Then the $(\varphi, N)$-module $H^i_{\text{HK}}(Z^m)$ satisfies the weight-monodromy conjecture.

**Proof.** We may assume that $Z$ is the base change of a proper smooth variety $Z'$ over a finite extension $E$ of $F$ inside $\bar{F}$. Let $k$ denote the residue field of $E$. By [Jar96, Theorem 6.5.2] and Poincaré duality (Theorem 3.17 and Remark 3.21) we may and do assume that $Z'$ has semistable reduction over $O_E$ (cf. [Ked90, Lemma 7.1.2], see also [LP16, Lemma 5.36]).

Let $Z'_0$ denote the special fiber of a semistable model of $Z'$ over $O_E$ equipped with the natural log structure. Then, by Corollary 3.13, the $(\varphi, N)$-module $H_{\text{HK}}^i(Z^m)$ is isomorphic to the $(\varphi, N)$-module $H^i_{\text{HK}}(Z'_0/W(k)^0) \otimes_{W(k)[1/p]} Q_p^m$ (with twisted monodromy operator). The latter is quasi-pure of weight $i$ by [LP16, Theorem 5.33, Theorem 5.46].

As in [Sch12, Section 8], we use the notation $X_{\Sigma, \kappa}$ for a toric variety $X_{\Sigma}$ over a field $\kappa$ to clarify the ground field when necessary. For a perfectoid field $L$, we use the notations $X_{\Sigma, L}^{an}$, $X_{\Sigma, L}^{perf}$ introduced in [Sch12, Paragraph after 8.4] (there the notation $(-)^{\text{ad}}$ is used instead of $(-)^{an}$). Then as noted there, if $X_{\Sigma, L}$ is proper, then $X_{\Sigma, L}^{an}$ is identified with the analytification $X_{\Sigma, L}^{an}$ of $X_{\Sigma, L}$. Recall from [Sch12, Theorem 8.5] that the tilt of $X_{\Sigma, L}^{an}$ is canonically isomorphic to the perfectoid space $X_{\Sigma, L}^{perf}$ for the same fan $\Sigma$ defining $X_{\Sigma, L}$.

The following is a motivic analogue of [Sch12, Proposition 8.6).

**Lemma 5.4.** Let $L$ be a perfectoid field and $X_{\Sigma, L}$ be a smooth toric variety. Then we have $Q_L(X_{\Sigma, L}^{an})^\sim \cong Q_L(X_{\Sigma, L}^{perf})$, where $(-)^\sim$ denotes the equivalence $\text{RigDA}(L') \sim \text{RigDA}(L)$ from [Vez19a, Theorem 7.26].
Proof. Via the equivalence $\text{RigDA}(L^\flat) \cong \text{PerfDA}(L^\flat)$ of [Vez19a, Theorem 6.9] (see the proof of [BV21, Theorem 3.8] on how to omit the Frob-localization) the motive of $X^\an_{\Sigma,L}$ is sent to that of $X^\perf_{\Sigma,L}$. Note that $X^\an_{\Sigma,L}$ has a smooth formal model $\tilde{X}_{\Sigma,O_L}$ over $O_L$ and that $\lim_{\varphi} X^\an_{\Sigma,L}$ is (locally) a presentation of good reduction in the sense of [Vez19a, Definition 2.4]. Then, by [Vez19a, Proposition 5.4], we deduce that the projection gives an equivalence $\mathbb{Q}_L(X^\an_{\Sigma,L}) = \mathbb{Q}_L(\lim_{\varphi} X^\an_{\Sigma,L}) \cong \mathbb{Q}_L(X^\an_{\Sigma,L}).$ 

Construction 5.5. Let $L$ be a perfectoid field and $X_{\Sigma,L}$ be a smooth toric variety. Recall from [Sch12, Theorem 8.5.(iii)] that we have a natural continuous map $\pi: |X^\an_{\Sigma,L}| \to |X^\an_{\Sigma,L}|$. Let $V \subset X_{\Sigma,L}$ be an open adic subspace. We consider the inverse images $V^\perf \subset X^\perf_{\Sigma,L}$ and $\pi^{-1}(V) \subset X^\an_{\Sigma,L}$ viewed as open adic subspaces. Then the morphism $V^\perf \to V$ of adic spaces induces a natural morphism $j^*\mathbb{Q}(V^\perf) \to \iota^*\mathbb{Q}(V) \to \iota^*\mathbb{Q}(X^\an_{\Sigma,L})$ in $\text{RigDA}(L)$ (with the notation in [Vez19a Page 40]), and hence the following commutative diagram in $\text{RigDA}(L)$

$$
\begin{array}{c}
\mathbb{Q}(X^\an_{\Sigma,L})^2 \longrightarrow \mathbb{Q}(X^\an_{\Sigma,L}) \\
\downarrow \downarrow \\
\mathbb{Q}(\pi^{-1}(V))^d \longrightarrow \mathbb{Q}(V).
\end{array}
$$

We have now all the ingredients to adapt Scholze’s proof to the $p$-adic setting.

Theorem 5.6. Let $\mathbb{C}_p$ be the completion of an algebraic closure of $\mathbb{Q}_p$, and let $Y$ be a smooth variety over $\mathbb{C}_p$, which is a scheme-theoretic complete intersection inside a projective smooth toric variety $X^\an_{\Sigma,\mathbb{C}_p}$. Then the $p$-adic weight monodromy conjecture holds for $Y$, that is, the $(\varphi, N)$-module $H^i_{\text{HK}}(Y^\an)$ is quasi-pure of weight $i$.

Proof. By Corollary 4.5, there is an open neighborhood $\tilde{Y}$ of $Y^\an$ inside $X^\an_{\Sigma,\mathbb{C}_p}$ whose motive coincides with that of $Y^\an$. We let $\pi^{-1}(\tilde{Y}) \subset X^\an_{\Sigma,\mathbb{C}_p}$ be the inverse image of $\tilde{Y}$ via the continuous map $\pi: |X^\an_{\Sigma,\mathbb{C}_p}| \to |X^\an_{\Sigma,\mathbb{C}_p}|$. By [Sch12, Corollary 8.8], we can find an irreducible closed algebraic subvariety $Z$ of $X^\an_{\Sigma,\mathbb{C}_p}$ defined over the algebraic closure of $F_p((p^d))$ in $\mathbb{C}_p^\circ$ such that $\dim Z = \dim Y$ and its analytification lies in $\pi^{-1}(\tilde{Y})$. We use [J96, Theorem 4.1] to take a smooth alteration $Z'$ of $Z$.

Then Construction 5.5 gives us the following commutative square in $\text{RigDA}(\mathbb{C}_p)$

$$
\begin{array}{c}
\mathbb{Q}(X^\an_{\Sigma,\mathbb{C}_p})^2 \longrightarrow \mathbb{Q}(X^\an_{\Sigma,\mathbb{C}_p}) \\
\downarrow \downarrow \\
\mathbb{Q}(Z^\an)^d \longrightarrow \mathbb{Q}(Y^\an).
\end{array}
$$

Applying the realization $R\Gamma_{\text{HK}}$, we obtain a natural morphism

$$
\alpha: R\Gamma_{\text{HK}}(Y^\an) \to R\Gamma_{\text{HK}}^\circ(Z^\an)
$$
in $D_{\varphi,N}(\mathbb{C}_p)$. Let $d$ denote $\dim Y = \dim Z'$ and for an integer $i$, let $\alpha^i$ denote the map on cohomology $H^i_{\text{HK}}(Y^\an) \to H^i_{\text{HK}}(Z^\an)$ induced by $\alpha$. By Corollary 3.19 and Remark 3.20, the canonical map $H^d_{\text{HK}}(X^\an_{\Sigma,\mathbb{C}_p}) \cong H^d_{\text{HK}}(X^\an_{\Sigma,\mathbb{C}_p}) \to H^d_{\text{HK}}(Z^\an)$ can’t be zero. In particular, the map $\alpha^{2d}: H^{2d}_{\text{HK}}(Y^\an) \to H^{2d}_{\text{HK}}(Z^\an)$ is not zero. Since both sides of $\alpha^{2d}$ are one-dimensional (Remark 3.21) we deduce that $\alpha^{2d}$ is an isomorphism.
By Poincaré duality (Theorem 5.17) we deduce that the dual of $\alpha^{2d-i}$ with respect to $H^i_{\text{HK}}(Y^{an}) \cong H^i_{\text{HK}}(Z^{an})$ gives a $(\varphi, N)$-equivariant splitting of $\alpha^i$, and hence $H^i_{\text{HK}}(Y^{an})$ is a direct summand of $H^i_{\text{HK}}(Z^{an})$. Thus, the assertion follows from Proposition 5.3. \qed

Remark 5.7. By taking the motivic $\ell$-adic realization instead of the $p$-adic one, the proof above coincides with (a motivic version of) Scholze’s proof of the $\ell$-adic weight monodromy conjecture for scheme-theoretical complete intersections in toric varieties. For a motivic rigid analytic $\ell$-adic realization functor, one can use [BV21].

Remark 5.8. If $Y$ is a complete intersection of positive dimension in a smooth and projective toric variety $X_\Sigma$, then $Y$ is geometrically connected. This can be deduced from the Grothendieck-Lefschetz theorem [SGAI Corollaire XII.3.5] (we recall that a complete intersection of positive dimension in a smooth and projective toric variety $X_\Sigma$, this is also reflected on the (weight filtration of the) cohomology groups. This is in accordance with the formulas of [Mok93].

APPENDIX A. A LOG-FREE HYODO–KATO COHOMOLOGY

In this appendix, we discuss a motivic way to equip the $K_0$-overconvergent de Rham realization with a $(\varphi, N)$-structure which does not make any reference to the Hyodo–Kato isomorphism or log structures. This procedure can be done in the $\ell$-adic realization as well, see [Ayo14b] Section 11.

The basic idea is to note that the graded pieces with respect to the weight filtration on the de Rham realization factors over the motivic nearby cycle functor $\Psi$, giving a formula $\text{gr}_k^\omega H^i_{\text{dR}, K_0}(X) \cong \text{gr}_k^\omega H^i_{\text{rig}, k}((\Psi(X))$. Since the functor $\Psi$ is equipped with a monodromy operator $N$, this is also reflected on the (weight filtration of the) cohomology groups. This is in accordance with the formulas of [Mok93].

Let the notations be as in the beginning of Section 3.1. In particular, $K$ is a complete discrete valuation field of characteristic with perfect residue field $k$ and $K_0$ is the subfield $W(k)[1/p]$. As in Remark 3.25 we consider the $K_0$-overconvergent de Rham realization $R\Gamma_{\text{dR}, K_0}: \text{RigDA}_\text{gr}(K) \cong \text{RigDA}_\text{gr}(K_0) \to \mathcal{D}(K_0)$. We propose to define the $(\varphi, N)$-structure on it purely in terms of the generic fiber. In what follows, we will construct a Frobenius structure on the complex and a monodromy operator on the level of graded pieces of the weight filtration on each cohomology group.

A.1. Motivic nearby cycles. We first recall how the motivic nearby cycle functor $\Psi$ is defined on rigid analytic motives (of good reduction) following Ayoub. Recall that $\text{RigDA}_\text{gr}(K)$ is canonically equivalent to the category of $\chi_1$-modules in $\text{DA}(k)$ ([AGV20 Theorem 3.3.3(1)]) and that it contains the motives of varieties with pluri-nodal reduction see Proposition 3.29. We choose $p$ as a uniformiser of $K_0$ to fix an identification $\chi_1 \cong 1 \oplus 1(-1)[-1]$, i.e., via $\chi_1 \cong \chi_0$. We consider the augmentation morphism $\chi_1 \cong 1 \oplus 1(-1)[-1] \to 1$ in $\text{DA}(k)$ corresponding to $(1, 0)$, which is the only one algebra morphism. In light of [Ayo15 Scholie 1.3.26], we give the following definition.

**Definition A.1.** The motivic nearby cycle is the functor

$$\Psi: \text{RigDA}_\text{gr}(K) \cong \text{Mod}_{\text{DA}(k)}(\chi_1) \to \text{Mod}_{\text{DA}(k)}(1) \cong \text{DA}(k).$$

induced by the canonical augmentation $\chi_1 \cong 1 \oplus 1(-1)[-1] \to 1$.

**Proposition A.2.** If $M$ lies in $\text{RigDA}_\text{gr}(K)^{ct}$, the object $\Psi M$ is canonically equipped with a morphism $N: \Psi M \to \Psi M(-1)$ in $\text{DA}(k)$, i.e., the functor $\Psi$ factors as

$$\Psi: \text{RigDA}_\text{gr}(K)^{ct} \cong \text{Mod}_{\text{DA}(k)}(\chi_1)^{ct} \to \text{DA}(k)^{ct}_{\text{N}} \to \text{DA}(k)^{ct},$$

where the category $\text{DA}(k)^{ct}_{\text{N}}$ is given by maps of compact objects $N: M \to M(-1)$, i.e., compact comodules over the free coalgebra $\bigoplus \mathbb{Q}(-n)$ over $\mathbb{Q}(-1)$. 

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Proof. As in Remark 3.24 we let \( q: G_{m,k} \to \text{Spec } k \) be the structural morphism and UDA\((k)\) be the full subcategory of DA\( (G_{m,k}) \) generated under colimits by motives of the form \( q^* M \) for \( M \in \text{DA}(k) \). It can be identified with the category Mod\( \chi_{1}(\text{DA}(k)) \) and hence with RigDA\( (G_{m,k})(K) \) (see Remark 3.24). The restriction of the (monoidal) base-change functor \( 1^*: \text{DA}(G_{m}) \to \text{DA}(k) \) to UDA\((k)^{ct}\) is the functor induced by the algebra morphism \( q_{*}1 \to 1 \) in \( \text{DA}(k) \) so it coincides with the functor \( \Psi \) above, modulo the above identifications.

By the last part of the proof of [Ayo15, Scholie 1.3.26.2] we may also identify it with the functor \( \Upsilon_{id} \) (see, for example, [Ayo14b] Section 10) for the definition). The existence of the monodromy operator follows then from [Ayo14b, Théorème 11.16]. \( \square \)

Remark A.3. Had we chosen to identify \( \chi_{1} \) with \( 1 \oplus 1(-1)[-1] \) via a uniformiser of \( K \), then \( N \) would have been multiplied by the constant \( e_{K} \) (see Remark 3.26).

Remark A.4. We expect that the \( K_{0}\)-overconvergent de Rham realization \( R\Gamma_{K_{0},dR}^{rig} \) of \( \text{DA}(G_{m,k})(K) \) factors as \( R\Gamma_{\text{rig}}(-/K_{0}) \circ \Psi \), and hence the monodromy operator on the motivic nearby cycle induces a canonical monodromy operator on \( R\Gamma_{K_{0},dR}^{rig} \) (see more precisely Remark A.16). We now prove this fact on the graded pieces of the weight filtration.

A.2. Weight structures. We shall construct the monodromy operator on the graded pieces of the weight filtration of each cohomology group via a factorization over the nearby cycle functor. To make this precise, we recall the following definition, initially due to Bondarko and Pauksztello, then rephrased in the infinity-language by Sosnilo.

Definition A.5 ([Sos19, Definition 1.13, Remark 1.14.1]). Let \( C \) be a [closed symmetric monoidal] stable infinity-category. A [closed symmetric monoidal] bounded weight structure of \( C \) is a choice of a full subcategory \( \mathcal{H} \) (the heart) of \( C \) such that

1. it is closed under direct summands [and tensor products] in \( C \);
2. it generates \( C \) under finite limits (or colimits);
3. \( \text{Map}(X,Y) \) is \( 1 \)-coconnective for all \( X, Y \in \mathcal{H} \). i.e. \( \pi_{i} \text{Map}(X,Y) = 0 \) if \( i > 0 \).

A functor between categories \( C \to C' \) equipped with a [closed symmetric monoidal] bounded weight structure is called weight-exact if the image of \( \mathcal{H} \) lies in \( \mathcal{H}' \).

Example A.6. The following are examples of closed monoidal bounded weight structures.

1. For an additive category \( \mathcal{A} \), the \( \infty \)-category \( \text{Ch}^{b}(\mathcal{A}) \) of bounded chain complexes together with the full subcategory \( \mathcal{A} \) embedded as complexes concentrated in degree zero is a bounded weight structure by [Sos19, Section 1.3].
2. Let \( \kappa \) be a field. The Karoubi-closure \( \mathcal{P} \) of the category of Chow motives, i.e., objects in \( \text{DA}(\kappa) \) of the form \( \mathbb{Q}_{\kappa}(X)(i)[2] \) with \( i \in \mathbb{Z} \) and \( X/\text{Spec } \kappa \) smooth and proper is a symmetric closed monoidal bounded weight structure in \( \text{DA}(\kappa)^{ct} \) by [Bon10, Proposition 6.5.3].
3. Let \( K \) be a non-archimedian field. The Karoubi-closure \( \mathcal{P}_{K} \) of the category of motives of the form \( \xi M \) with \( M \in \mathcal{P} \) defines a symmetric closed monoidal bounded weight structure on \( \text{RigDA}(G_{m,k})(K)^{ct} \). To see this, notice that if \( A, B \) are in \( \mathcal{P} \) then \( \text{Map}(\xi A, \xi B) \cong \text{Map}(A, B) \oplus \text{Map}(A, (B(-1)[-2])[1]) \) (this follows from the identification of \( \chi_{1} \) given in the proof of Proposition A.2). The object \( B(-1)[-2] \) lies in \( \mathcal{P} \) so that the complex on the right is 0-coconnective.

Let \( (\mathcal{C}, \mathcal{H}) \) be a bounded weight structure. By [Sos19, Proposition 3.3.2] and [Sos22, Proposition 3.2.3] (see also [Aok20]), the functor \( \mathcal{H} \to \text{Ho } \mathcal{H} \) induces a [monoidal] weight-exact functor \( \omega: \mathcal{C} \to \text{Ch}^{b}(\text{Ho } \mathcal{H}) \), which is unique up to homotopy.
Definition A.7. The functor $\omega$ is called the weight-complex functor. For an object $X \in \mathcal{C}$, we write

$$\omega(X) = (\ldots \to X^{(-1)} \to X^{(0)} \to X^{(1)} \to \ldots) ,$$

where each $X^{(i)}$ is in $\mathcal{H}$.

There are situations where the weight complex is enough to recover the cohomology theory, equipping it with a weight filtration, as the following proposition by Bondarko [Bon10] shows (also relevant [Vol13] Section 3).

Proposition A.8 ([Bon10 Theorem 2.4.2 and Remark 2.4.3(1)]). Let $A$ be an abelian category, $(\mathcal{C}, \mathcal{H})$ be a bounded weight structure, and $R\Gamma: \mathcal{C} \to D(A)^{\text{op}}$ be a functor:

1. There is a functorial spectral sequence

$$E_1^{p,q} = H^q(R\Gamma(X^{-p})) \Rightarrow H^{p+q}R\Gamma(X)$$

whose differentials at page $E_1$ are defined by the differentials of the weight complex.

2. We assume that $A$ is equipped with abelian subcategories $\{A_{\omega=i} \subset A\}_{i \in \mathbb{Z}}$ that are closed under subquotients and orthogonal and that $H^i(R\Gamma(M)) \in A_{\omega=i}$ for each $M \in \mathcal{H}$. Then the above spectral sequence degenerates at $E_2$, and its induced graded pieces $\text{gr}_i^R H^n$ lie in $A_{\omega=i}$.

In particular, the bi-graded object $\bigoplus gr_i^R H^n(R\Gamma(X))$ only depends on $\omega(X)$. More precisely, the functor $gr_i^R H^n R\Gamma: \mathcal{C} \to \text{gr}_{\mathbb{Z} \times \mathbb{Z}}(A)$ factors as

$$\mathcal{C} \xrightarrow{\sim} \text{Ch}(\text{Ho} \mathcal{H}) \to \text{gr}_{\mathbb{Z} \times \mathbb{Z}}(A),$$

where the second functor is given by

$$M^* \mapsto \bigoplus_{n,i} H^{n-i}(\ldots \to H^1 R\Gamma(M^{-p}) \to H^i R\Gamma(M^{-p-1}) \to \ldots).$$

Example A.9. Let $K$ be a complete discrete valuation field with perfect residue field. The Frobenius-enrichment $F: \text{DA}(k) \to \text{DA}(k)_{\omega}^{h_\chi}$ on the category $\text{DA}(k)$ (see Remark 2.32) induces

$$\text{Mod}_{\text{chol}}(\text{DA}(k)^{ct}) \to \text{Mod}_{F(\text{chol})}(\text{DA}(k)^{ct,h_\varphi}) \to \text{Mod}_{\text{chol}}(\text{DA}(k)^{ct})^{h_\varphi} \cong \text{RigDA}_{\text{gr}}(K)_{0}^{ct,h_\varphi},$$

where now the Frobenius operator $\varphi$ on the right is by the lift of Frobenius. In particular, by applying we apply the de Rham realization (see [LBV21 Corollary 4.39]) we obtain

$$R\Gamma_{dR,K_0}: \text{RigDA}_{\text{gr}}(K)^{ct} \to \text{RigDA}_{\text{gr}}(K_{0})^{ct,h_\varphi} \to \text{QCoh}(\text{Spa} K_{0})^{ct,h_\varphi} \cong D_{\varphi}(K_{0})^{ct}.$$

When the residue field $k$ of $K$ is finite, this satisfies the hypotheses of Proposition A.8(2): if $Y$ is a smooth and proper variety over $k$, the $i$-th cohomology group $H^i_{\text{rig}}(Y/K_0)$ is $\varphi$-pure of weight $i$. Thus, for any $X \in \text{RigDA}_{\text{gr}}(K)^{ct}$, each $H^i_{dR,K_0}(X)$ comes equipped with a functorial weight filtration.

A.3. Motivic monodromy operator. Let $K$ be a non-archimedean field. The Monsky–Washnitzer functor $\xi: \text{DA}(k)^{ct} \to \text{RigDA}(K)^{ct}_{\text{gr}}$ is weight-exact by definition. More precisely, we have:

Proposition A.10. The functors $\text{Ho} \mathcal{P} \to \text{Ho} \mathcal{P}_K$ and $\text{Ho} \mathcal{P}_K \to \text{Ho} \mathcal{P}$ induced by $\xi$ and $\Psi$ are quasi-inverse to each other. In particular, we have a commutative square of monoidal functors

$$
\begin{array}{ccc}
\text{RigDA}_{\text{gr}}(K)^{ct} & \xrightarrow{\psi} & \text{Ch}^\psi(\text{Ho} \mathcal{P}_K) \\
\downarrow \phi & & \downarrow \psi \\
\text{DA}(k)^{ct} & \xrightarrow{\sim} & \text{Ch}^\psi(\text{Ho} \mathcal{P}).
\end{array}
$$
**Proof.** As already remarked in Example [A.6][5], for any \(A, B\) in \(\mathcal{P}\) there is an equivalence \(\text{Hom}(\xi A, \xi B) \cong \text{Hom}(A, B)\). This shows that \(\xi \mathcal{P}\) is Karoubi-closed and that \(\text{Ho} \mathcal{P} \cong \text{Ho} \mathcal{P}_K\) via \(\xi\). This proves the statement since \(\Psi \xi \cong \text{id}\). \(\Box\)

**Remark A.11.** If \(K\) is algebraically closed, the analytification functor \(\text{DA}(K)^{ct} \to \text{RigDA}(K)^{ct}\) is not weight exact with respect to the structures described above.

We now use Propositions [A.2][6] and [A.10][7] to endow the weight complex of an object \(X\) in \(\text{RigDA}_{gr}(K)^{ct}\) with a monodromy operator.

We define the Tate twist functor \(C^* \mapsto C^*(-1)\) on \(\text{Ch}^b(\mathcal{P})\) by \(C^*(-1) = (C(-1)[-2]^*)[2]\), more precisely, the \(p\)-th component of \(C^*(-1)\) is given by \(C_{p+2}(-1)[-2]\). We remark that if \(C\) lies in \(\mathcal{P}\), then so does \(C(-1)[-2]\). Then, for any object \(M\) in \(\text{RigDA}_{gr}(K)\), we have a functorial isomorphism \(\omega(M(-1)) \cong \omega(M)(-1)\).

**Corollary A.12.** For any object \(M\) in \(\text{RigDA}_{gr}(K)^{ct}\), its weight complex \(\omega(M)\) is canonically equipped with a monodromy operator \(\omega(M) \to \omega(M)(-1)\), i.e., the weight complex functor as

\[
\text{RigDA}_{gr}(K)^{ct} \to \text{CoMod}_N(\text{Ch}^b(\text{Ho} \mathcal{P}_K)) \to \text{Ch}^b(\text{Ho} \mathcal{P}_K).
\]

**Proof.** This follows from the fact that the left arrow in the commutative square from Proposition [A.10][8] factors over \(\text{CoMod}_N(\text{DA}(k))\). \(\Box\)

**Remark A.13.** As \(C^*\) is bounded, any monodromy operator \(C^* \to (C(-1)[-2]^*)[2]\) is nilpotent.

Finally, we apply the above observations to \(K_0\)-overconvergent de Rham cohomology \(R\Gamma_{dR,K_0} : \text{RigDA}_{gr}(K) \to \mathcal{D}_\varphi(K_0)\). By Corollary [A.12][9], the realization

\[
\text{RigDA}_{gr}(K)^{ct} \to \mathcal{D}_\varphi(K_0)^{ct} \to \text{gr}_{\mathbb{Z} \times \mathbb{Z}} \text{Mod}_\varphi(K_0)
\]

from Example [A.9][10] canonically factors through \(\text{CoMod}_N(\text{Ch}^b(\mathcal{P}))\). More concretely, for any object \(X\) in \(\text{RigDA}_{gr}(K)\), the monodromy operator \(\omega(X) \to \omega(X)(-1)\) from Corollary [A.12][11] induces a (functorial) monodromy operator on the bi-graded object \(\bigoplus_{n,i} \text{gr}_i H^n_{dR,K_0}(X)\): we obtain, by taking \((p+2)\)-th components, a morphism \(X^{(-p-2)} \to X^{(-p)}(-1)[-2]\) in \(\mathcal{P}\); by taking the \((q-2)\)-th rigid cohomology groups

\[
N : H^n_{\text{rig}}(X^{(-p)})(1) \to H^{n-2}_{\text{rig}}(X^{(-p-2)}),
\]

inducing a (nilpotent) monodromy operator \(N : \text{gr}_i \omega(H^n_{dR,K_0}(X)) \to \text{gr}_{i-2} \omega(H^{n-2}_{dR,K_0}(X))(1)\). In the semistable reduction case, the spectral sequence above is expected to be compared, in an appropriate sense, with the classical weight spectral sequence considered by Mokrane [Mok93] (see also Nakajima [Nak05] and Große-Klönne [GK05]). We will make this precise in a future work.

**Remark A.14.** The Frobenius structure on \(\text{RigDA}(K)\) can be encoded more geometrically via the equivalence \(\text{RigDA}(K) \cong \text{RigDA}(\text{Spd}(k))\) of [LBV21, Theorem 5.13].

**Remark A.15.** Note that the formalism of weight-structures gives a positive answer to the “generalized” weight-monodromy conjecture in equi-characteristic \(p\) of Lazda–Pal [LP16, Conjecture 5.55] by taking as “geometric” weight filtration the filtration on cohomology induced by the weight-complex.

**Remark A.16.** We expect that the category \(\text{RigDA}_{gr}(K) \cong \text{Mod}_{\mathbb{S}^1}(\text{DA}(k))\) is equivalent to the category of comodules \(\text{CoMod}_N(\text{DA}(k))\) with \(N\) being the free tensor coalgebra \(N := \bigoplus \mathbb{Q}(-n)\). If so, in particular, its compact objects are given by “motivic monodromy maps”
$M \to M(-1)$ in $DA(k)$ (which are necessarily nilpotent, using weight considerations). This would imply that any monoidal realization $\text{Rig}DA_{gr}(K) \to \mathcal{C}$ can be enriched with a monodromy operator $\text{Rig}DA_{gr}(K) \to \text{CoMod}_{N}(\mathcal{C})$, and that $\Psi$ corresponds to the functor “forgetting” monodromy. This is an infinity-version of the constructions above, which will be dealt with in a future work.

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