THE FOURIER-JACOBI PERIODS: THE CASE OF $Mp(2m) \times Sp(2n)$

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Abstract. In this paper, we prove one direction of the Gan–Gross–Prasad conjecture on metaplectic-symplectic groups for tempered cases. Furthermore, we also prove one direction of the non-tempered GGP conjecture for residual representations with relevant $A$-parameters. As an application, we discuss the non-vanishing of the central value of quadratic twists of automorphic $L$-functions of $GL_{2n}$.

1. Introduction

In a highly influential paper [12], Gan, Gross, and Prasad formulated a profound conjecture that establishes a connection between certain automorphic periods of classical (and metaplectic) groups and the central $L$-values of specific automorphic $L$-functions. The investigation of periods of automorphic forms has become a vibrant and active research area in modern number theory due to its significant arithmetic implications.

In their work [12], the authors considered two distinct types of periods: Bessel periods and Fourier–Jacobi (FJ) periods. In this paper, our primary focus lies on the FJ periods associated with symplectic-metaplectic groups, which we now recall.

Let $F$ be a number field with the adèle ring $\mathbb{A}$. Additionally, we fix a nontrivial character $\psi$ of $F \setminus \mathbb{A}$. Consider $W_n$ (resp. $W_m$) as $2n$ (resp. $2m$)-dimensional symplectic spaces over $F$. Suppose $n \geq m$, and let $X$ and $X^*$ denote isotropic subspaces of $W_n$ such that $X + X^*$ forms the direct sum of $r$ hyperbolic planes. We define $r = n - m$, and $W_m$ represents the orthogonal complement of $X + X^*$ within $W_n$.

For each $k = n, m$, let $G_k$ denote the isometry group of $W_k$, and $\tilde{G}_k(\mathbb{A})$ represent the metaplectic double cover of $G_k(\mathbb{A})$. We consider $G_m$ as a subgroup of $G_n$ that acts trivially on the orthogonal complement of $W_m$ within $W_n$. We fix a complete filtration $\bar{X}$ of $X$, and let $N_{n,r}$ be the unipotent subgroup of the parabolic subgroup of $G_n$ defined as the stabilizer of $\bar{X}$. Notably, $G_m$ acts on $N_{n,r}$ through conjugation.

We denote the semidirect product $N_{n,r} \rtimes G_m$ by $H_{m,r}$ and regard $H_{m,r}$ as a subgroup of $G_n$ via the map $(n, g) \mapsto ng$. Furthermore, there exists a global generic Weil representation $\nu_{\psi^{-1}, W_m}$ of $(N_{n,r} \rtimes \tilde{G}_m)(\mathbb{A})$, realized on the Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{A}^m)$.

For each $f \in \mathcal{S}$, the associated theta series is defined as follows:

$$\Theta_{\psi^{-1}, W_m}(h, f) = \sum_{x \in F_m} (\nu_{\psi^{-1}, W_m}(h)f)(x), \quad h \in (N_{n,r} \rtimes \tilde{G}_m)(\mathbb{A}).$$

The space of theta series provides an $H_{m,r}(F)$-invariant genuine automorphic realization of $\nu_{\psi^{-1}, W_m}$.

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As a notational convenience, we sometimes express an automorphic representation of $G_k(A)$ as a non-genuine automorphic representation of $\widetilde{G}_k(A)$. Consider an irreducible cuspidal automorphic representation $\pi_1 \boxtimes \pi_2$ of $\widetilde{G}_n(A) \times \widetilde{G}_m(A)$, where exactly one of $\pi_1$ or $\pi_2$ is genuine. Then the $\psi$-Fourier-Jacobi periods $\mathcal{F}_{\psi}$ are defined by the following integrals:

For $\varphi_1 \in \pi_1$, $\varphi_2 \in \pi_2$, and $f \in \nu_{\psi^{-1},W_m}$, we have

$$\mathcal{F}_{\psi}(\varphi_1, \varphi_2, f) := \int_{(N_{n,r} \times G_m)^{(F)} \setminus (N_{n,r} \times G_m)(A)} \varphi_1(n\tilde{g})\varphi_2(\tilde{g})\Theta_{\psi^{-1},W_m}((n,\tilde{g}), f) \, dn \, dg,$n

where $\tilde{g}$ represents a preimage of $g$ in $\widetilde{G}_m(A)$. Since the product of two genuine automorphic forms is no longer genuine, the above integral is independent of the choice of the preimage $\tilde{g}$ of $g$.

When either $\pi_1$ or $\pi_2$ is cuspidal, the integral mentioned above converges absolutely and is well-defined. However, if both $\pi_1$ and $\pi_2$ are not cuspidal, the integral may diverge. In the work [20], the author introduced regularized Fourier–Jacobi periods. These periods can be defined not only for cusp forms but also for non-cusp forms $\varphi_1$ and $\varphi_2$, provided their exponents satisfy a certain mild restriction (see Definition 6.5 of [20]). Since this concept generalizes the original periods, we keep the same notation $\mathcal{F}_{\psi}$ for the regularized period (see [20] Remark 6.6).

The Gan–Gross–Prasad (GGP) conjecture [12 Conjecture 24.1] predicts that when $\pi_1 \boxtimes \pi_2$ is tempered, the following two conditions are equivalent:

(i) $\mathcal{F}_{\psi}$ is non-vanishing,

(ii) $\text{Hom}_{H_{m,r}(F_v)}(\pi_1, v \boxtimes \pi_2, v \otimes \nu_{\psi^{-1},W_m}, C) \neq 0$ for all places $v$, and $L(\frac{1}{2}, FL_{\psi}(\pi_1) \times FL_{\psi}(\pi_2)) \neq 0$.

Here, $FL_{\psi}$ stands for the $\psi$-weak Langlands functorial lift to general linear groups, and the $L$-function on the right-hand side is the Rankin-Selberg $L$-function. When $\pi$ is a non-genuine representation of $\widetilde{G}_k(A)$ (i.e., a representation of $G_k(A)$), $FL_{\psi}(\pi)$ is just the weak functorial lift $FL(\pi)$ of $G_k(A)$ to $GL_{2k+1}(A)$.

By the generalized Ramanujan conjecture, it is widely believed that the conjecture would also hold for irreducible globally generic cuspidal representations $\pi_1 \boxtimes \pi_2$. In 2004, even before the GGP conjecture was extended to all classical groups, Ginzburg, Jiang, Rallis [16] Theorem 5.1] (and later together with B. Liu [15]) proved the direction from (i) to (ii) of the conjecture for globally generic cuspidal representations when $FL_{\psi}(\pi)$ and $FL_{\psi}(\pi_2)$ are both cuspidal. Around 2013, some experts found a serious gap in the original proof of [16 Proposition 5.3]. Thereafter, in 2019, G–J–R, together with B. Liu, uploaded the correction of the proof of [16 Proposition 5.3] to arXiv (15).

In 2013, Arthur ([1]) established the endoscopic classification of irreducible automorphic representations in the discrete spectrum of symplectic groups and quasi-split orthogonal groups. Based upon Arthur’s work on orthogonal groups, Gan–Ichino [13] has established a similar endoscopic classification for metaplectic groups via an ingenious use of the theta correspondence between metaplectic groups and orthogonal groups.

Therefore, building upon the endoscopic classifications of automorphic representations of symplectic and metaplectic groups, we can restate the GGP conjecture using the notion of generic $A$-parameters (see Section 2.2 for the definition of generic $A$-parameters for symplectic and metaplectic groups, respectively).

The first main result of the paper is to prove the one direction of the GGP conjecture for symplectic-metaplectic groups, which includes the result of [15] [16].

**Theorem A.** Let $n, m$ be two integers such that $r = n - m$ is a non-negative integer. Let $\pi_1 \boxtimes \pi_2$ be an irreducible cuspidal automorphic representation of $\widetilde{G}_n(A) \times \widetilde{G}_m(A)$ with generic $A$-parameter. Assume that exactly one of $\pi_1$ or $\pi_2$ is genuine, and distinguish them by writing $\pi'$ for the genuine representation.
of $\tilde{G}_{k(n')}(A)$ and $\pi$ for the non-genuine one of $\tilde{G}_{k(n)}(A)$. If there are $\varphi_1 \in \pi_1$, $\varphi_2 \in \pi_2$, and $f \in \nu_{W_m}$ such that

$$\mathcal{F}\mathcal{J}_\psi(\varphi_1, \varphi_2, f) \neq 0,$$

then $L\left(\frac{1}{2}, FL_\psi(\pi') \times FL(\pi)\right) \neq 0$.

**Remark 1.1.** The reason we prefer to work with representations with generic $A$-parameters rather than tempered representations is that we can use the following six properties of $\pi_1$ and $\pi_2$ in the course of the proof of the theorem:

(i) The weak functorial lifts $FL(\pi)$ and $FL_\psi(\pi')$ exist.

(ii) $FL(\pi)$ can be decomposed as the isobaric sum $\sigma_1 \oplus \cdots \oplus \sigma_t$, where $\sigma_i$'s are irreducible cuspidal automorphic representations of the general linear groups of orthogonal type (i.e., $L(s, \sigma_i, Sym^2)$ has a pole at $s = 1$ for all $1 \leq i \leq t$).

(iii) $FL_\psi(\pi')$ can be decomposed as the isobaric sum $\sigma'_1 \oplus \cdots \oplus \sigma'_\tau$, where $\sigma'_i$'s are irreducible cuspidal automorphic representations of the general linear groups of symplectic type (i.e., $L(s, \sigma'_i, \wedge^2)$ has a pole at $s = 1$ for all $1 \leq i \leq \tau$).

(iv) The local normalized intertwining operators are holomorphic for all $z$ with $\text{Re}(z) \geq \frac{1}{2}$.

(v) If $\pi_v$ is the unique unramified subquotient of some principal series representation $\text{Ind}_{B_k^+}(\chi_1 \boxtimes \cdots \boxtimes \chi_k)$, then $\chi_i = \nu_i \cdot | \cdot |^{s_i}$, where $\nu_i$'s are unramified unitary characters and $0 \leq s_i < \frac{1}{2}$.

(vi) If $\pi'_v$ are the unique unramified subquotients of some principal series representations $\text{Ind}_{B_k^+}(\chi'_1, \psi \boxtimes \cdots \boxtimes \chi'_{k, \psi})$, then $\chi'_i = \nu'_i \cdot | \cdot |^{s_i}$, where $\nu'_i$'s are unramified unitary characters and $0 \leq s_i < \frac{1}{2}$.

(See Sect. 4 and Sect. 5 for unexplained notations.)

The works of Arthur [1] and Gan–Ichino [13] on the description of the automorphic discrete spectrum of symplectic and metaplectic groups guarantee that $\pi_1 \boxtimes \pi_2$ with generic $A$-parameter satisfies the above properties. It is known that globally generic representations of $\tilde{G}_n(A) \times \tilde{G}_m(A)$ (or $\tilde{G}_n(A) \times \tilde{G}_m(A)$) also satisfies (i)-(vi) (see [18], Theorem 11.2.) Therefore, Theorem [A] would also hold for globally generic cuspidal representations of $\tilde{G}_n(A) \times \tilde{G}_m(A)$.

When $r = 0$, Theorem [A] is nothing but Yamana’s result [48], Theorem 1.1]. Therefore, we shall prove it only for non-equal rank cases. The proof for non-equal rank cases should be treated separately depending on whether $\pi' = \pi_1$ or $\pi' = \pi_2$. In [20], the author proved an analogue of Theorem [A] for skew-hermitian unitary groups. Essentially, we follow a similar approach here, and thus, we provide a proof only when there are non-trivial differences. We give just statements if the proof is essentially the same as the corresponding results in [20].

As announced in [28] and [29], we mention that D. Jiang and L. Zhang are preparing a paper [30] which deals with the theory of twisted automorphic descents for the Fourier–Jacobi case as an analogue of [28], which deals with the Bessel case. Once completed, this paper will prove Theorem [A] under the hypothesis on the unramified computation of the local zeta integral related to the Fourier–Jacobi functionals. This is ongoing work by D. Jiang, D. Soudry, and L. Zhang ([27]). Although these two approaches bear some resemblance in that both use residual Eisenstein series, the merit of our method lies in not appealing to unpublished results in [27] to deduce Theorem [A].

It is worthwhile to mention that Theorem [A] is closely related to the non-tempered GGP conjecture. In [11], Gan, Gross, and Prasad extended the original GGP conjecture to non-tempered cases using the notion of relevant $A$-parameters (see Definition 6.1). More precisely, they conjectured that the regularized Bessel
(and Fourier–Jacobi) periods of two irreducible automorphic representations $\pi_1 \boxtimes \pi_2$ vanish if their $A$-parameters are not relevant. When their $A$-parameters are relevant, they conjectured that the regularized Bessel (and Fourier–Jacobi) periods are non-vanishing if and only if

$$\text{Hom}_{H_m,V}(\pi_{1,v} \boxtimes \pi_{2,v} \boxtimes \nu_{\psi_v^{-1}},W_m,\mathbb{C}) \neq 0$$

for all places $v$ and

$$\frac{L(s + \frac{1}{2}, \pi_1 \times \pi_2)}{L(s + 1, \pi_1, \text{Ad}) \cdot L(s + 1, \pi_2, \text{Ad})}|_{s=0} \neq 0.$$

Since arbitrary tempered cuspidal representations have relevant $A$-parameters and their adjoint $L$-function is holomorphic and nonzero at $s = 0$, the general GGP conjecture contains the original conjecture.

Our second main result in the paper is to prove one direction ($\Rightarrow$) of this conjecture for certain non-tempered representations with non-generic relevant $A$-parameters. More precisely, we prove the following.

**Theorem B.** Let $n, m$ be two integers such that $r = n - m$ is a non-negative integer. Let $\pi_1 \boxtimes \pi_2$ be certain type irreducible residual representation of $\widetilde{G}_n(\mathbb{A}) \times \widetilde{G}_m(\mathbb{A})$ with relevant $A$-parameters. Assume that exactly one of $\pi_1$ or $\pi_2$ is genuine, and distinguish them by writing $\pi'$ for the genuine representation of $G_k(\pi')(\mathbb{A})$ and $\pi$ for the non-genuine one of $G_k(\pi)(\mathbb{A})$. If there are $\varphi_1 \in \pi_1$, $\varphi_2 \in \pi_2$, and $f \in \nu_{W_m}$ such that $\mathcal{F}^{\mathcal{F}_\psi}(\varphi_1, \varphi_2, f) \neq 0$ then,

$$\frac{L(s + \frac{1}{2}, \pi_1 \times \pi_2)}{L(s + 1, \pi_1, \text{Ad}) \cdot L(s + 1, \pi_2, \text{Ad})}|_{s=0} \neq 0.$$

For the precise meaning of certain types of residual representations with relevant $A$-parameters in the theorem, please refer to Subsection 6.2. To the author’s knowledge, this is the first result that touches upon the non-tempered GGP conjecture when $\pi_1$ and $\pi_2$ are both non-tempered.

Theorem A has two non-trivial applications; one is direct, and the other is rather indirect. In [2], H. Atobe has extended the theory of Miyawaki lifting in the framework of representation theory. Moreover, he proved the non-vanishing criterion of the generalized Miyawaki lifting ([2, Theorem 5.5]) under the following three assumptions:

- (i) $\Rightarrow$ (ii) direction of the GGP conjecture for $\widetilde{G}_{n+1}(\mathbb{A}) \times \widetilde{G}_n(\mathbb{A})$
- (ii) $\Rightarrow$ (i) direction of the GGP conjecture for $\widetilde{G}_n(\mathbb{A}) \times \widetilde{G}_n(\mathbb{A})$
- the generic summand conjecture for the Fourier–Jacobi coefficients (see [2, Hypothesis 5.3 (A)]).

Theorem A sharpens [2, Theorem 5.5] by removing the above first assumption.

The second application is concerned with the quadratic twists of automorphic $L$-functions of $GL_{2n}$. In 1995, Friedberg and Hoffstein [10]. Theorem B] proved the following theorem using analytic methods:

**Theorem 1.2.** Let $\tau_0$ be an irreducible unitary cuspidal automorphic representation of $GL_{2}(\mathbb{A})$ of symplectic type such that $\epsilon(\frac{1}{2}, \pi_0 \times \eta_0) = 1$ for some quadratic character $\eta_0 : \mathbb{A}^\times / F^\times \rightarrow \mathbb{C}$. Then there are infinitely many quadratic characters $\eta : \mathbb{A}^\times / F^\times \rightarrow \mathbb{C}^\times$ such that $L(\frac{1}{2}, \tau_0 \otimes \eta) \neq 0$.

Indeed, they proved a much stronger result by imposing certain restrictions on the character $\eta$. However, extending such analytic methods to higher rank groups is considered highly non-trivial.

In the meantime, B. Liu and B. Xu [36] Theorem 6.8] have recently reproved this result using representation-theoretic methods. Inspired by their approach, we extend their result to higher rank groups as follows:

**Theorem C.** Let $\tau_0$ be an irreducible unitary cuspidal automorphic representation of $GL_{2n}(\mathbb{A})$ of symplectic type such that $\tau_0$ is ‘good’ at some place $v_0$. Then, under some reasonable hypotheses (see Hypothesis 7.2 and Hypothesis 7.6), there exists a quadratic character $\eta : \mathbb{A}^\times / F^\times \rightarrow \mathbb{C}^\times$ such that $L(\frac{1}{2}, \tau_0 \otimes \eta) \neq 0$. (For the precise definition of ‘good’, see Definition 7.1.)
This theorem is expected to hold in general without such hypotheses (see Conjecture 7.3). We will explore the possibility of removing these two hypotheses in future work.

The organization of this paper is as follows:
- In Sect. 2, we introduce preliminaries, including notations used throughout the paper.
- In Sect. 3, we discuss the Jacquet module corresponding to the Fourier-Jacobi character and prove two theorems required for Proposition 5.4.
- In Sect. 4, we explain the definition of (residual) Eisenstein series and review related lemmas involving L-functions.
- In Sect. 5, we prove a Reciprocal Non-vanishing Theorem, from which Theorem A is easily deduced.
- In Sect. 6, we introduce the non-tempered GGP conjecture and prove one direction of it for residual representations with relevant A-parameters.
- In Sect. 7, we discuss the application of Theorem A to the quadratic twists of automorphic L-functions of $GL_{2n}$.

2. Preliminary

2.1. Notations. We introduce some notation that will be used throughout the paper.

Let $F$ be a number field with adele ring $\mathbb{A}$ and $\mathbb{A}_{\text{fin}}$ the ring of finite adeles of $F$. For a place $v$ of $F$, write $F_v$ for the localization of $F$ at $v$ and $| \cdot |_v$ for the normalized absolute valuation on $F_v$. Let $q$ be the cardinality of the residue field of $F_v$. For an algebraic group $G$ over $F$ and for a $F$-algebra $R$, write $G(R)$ for the group of $R$-points of $G$. Denote $G(F) \backslash G(\mathbb{A})$ by $[G]$. Let $\psi$ be a non-trivial additive character of $F \backslash \mathbb{A}$ and for $\alpha \in F$, put $\psi_\alpha(x) := \psi(\alpha x)$ for $x \in \mathbb{A}$ and $\eta_\alpha$ the quadratic character of $\mathbb{A}^\times / F^\times$ associated to the quadratic extension of $F(\sqrt{\alpha})/F$ by the global class field theory. For $x \in \mathbb{A}^\times$, put $|x| := \prod_v |x|_v$ and for a positive integer $k$, let $\mu_k$ be the group of $k$-th root of unity in $\mathbb{C}$. Denote by $(\cdot, \cdot)_v : F_v \times F_v \rightarrow \mu_2$ the local Hilbert symbol and by $(\cdot, \cdot) := \prod_v (\cdot, \cdot)_v : \mathbb{A} \times \mathbb{A} \rightarrow \mathbb{A}^\times$ the global Hilbert symbol.

Occasionally, we view $| \cdot |$ and $| \cdot |_v$ as a character of $GL_n(\mathbb{A})$ or $GL_n(F_v)$ by composing with det, respectively. For a character $\chi$ of $GL_1(F_v)$ and $d \geq 0$, put $\chi(d) := \chi(\det(\mu_{2d}(F_v)))$ a character of $GL_d(F_v)$.

For a non-negative integer $k$, let $W_k$ be a $2k$-dimensional $F$-vector space with a symplectic structure $\langle \cdot, \cdot \rangle$ and write $G_k$ for its symplectic group. We also write $H_{W_k} := W_k \oplus F$ the Heisenberg group associated to $(W_k, \langle \cdot, \cdot \rangle)$ and view it as an algebraic group over $F$. When $(W, \langle \cdot, \cdot \rangle)$ is a symplectic space and $W'$ is a subspace of $W$, we view $W'$ is also symplectic space with the inherited symplectic form $\langle \cdot, \cdot \rangle$ of $W$.

Fix an arbitrary positive integer $k$ and maximal totally isotropic subspaces $X$ and $X^*$ of $W_k$, which are in duality with respect to $\langle \cdot, \cdot \rangle$. Fix a complete flag in $X$

$$0 = X_0 \subset X_1 \subset \cdots \subset X_k = X,$$

and choose a basis $\{e_1, e_2, \cdots, e_k\}$ of $X$ such that $\{e_1, \cdots, e_j\}$ is a basis of $X_j$ for $1 \leq j \leq k$. Let $\{f_1, f_2, \cdots, f_k\}$ be the basis of $X^*$ which is dual to the fixed basis $\{e_1, e_2, \cdots, e_k\}$ of $X$, i.e., $\langle e_i, f_j \rangle = \delta_{ij}$ for $1 \leq i, j \leq n$, where $\delta_{ij}$ denotes the Kronecker delta. We write $X^*_j$ for the subspace of $X^*$ spanned by $\{f_1, f_2, \cdots, f_j\}$ and $W_{k-j}$ for the orthogonal complement of $X_j + X_j^*$ in $W_k$. Then $(W_{k-j}, \langle \cdot, \cdot \rangle)$ is a symplectic $F$-space of dimension $2(k - j)$.

For each $1 \leq j \leq k$, denote by $P_{k,j}$ the parabolic subgroup of $G_k$ stabilizing $X_j$, by $U_{k,j}$ its unipotent radical, and by $M_{k,j}$ the Levi subgroup of $P_{k,j}$ stabilizing $X_j^*$. Then $M_{k,j} \simeq GL(X_j) \times GL_{k-j}$. (Here, we regard $GL(X_j)$ as the subgroup of $M_{k,j}$ that acts as the identity map on $W_{k-j}$.) Let $P_0$ be the minimal parabolic subgroup of $G_k$ that stabilizes the above complete flag of $X$, and $P_0 = M_0 U_0$ represents its
Levi decomposition. We call any $F$-parabolic subgroup $P = MU$ of $G_k$ standard if it contains $P_0$. Fix a maximal compact subgroup $K = \prod_v K_v$ of $G_k(\mathfrak{A})$ such that $G_k(\mathfrak{A}) = P_0(\mathfrak{A})K$, and for every standard parabolic subgroup $P = MU$ of $G_k$,

$$P(\mathfrak{A}) \cap K = (M(\mathfrak{A}) \cap K)(U(\mathfrak{A}) \cap K),$$

where $M(\mathfrak{A}) \cap K$ remains maximal compact in $M(\mathfrak{A})$ (see [32 I.1.4]).

We give the Haar measure on $U(\mathfrak{A})$ so that the volume of $[U]$ is 1 and give the Haar measure on $K$ so that the total volume of $K$ is 1. We also choose Haar measures on $M(\mathfrak{A})$ for all Levi subgroups $M$ of $G$ compatibly with respect to the Iwasawa decomposition.

We define the Haar measure on $U(\mathfrak{A})$ such that the volume of $[U]$ is 1, and we choose the Haar measure on $K$ so that the total volume of $K$ is 1. Additionally, we select Haar measures on $M(\mathfrak{A})$ for all Levi subgroups $M$ of $G$, ensuring compatibility with respect to the Iwasawa decomposition.

For each place $v$, let $\widetilde{G}_k(F_v)$ denote the metaplectic double covering of $G_k(F_v)$. Specifically,

$$\widetilde{G}_k(F_v) = G_k(F_v) \times \{\pm 1\}$$

as a set, with a group law given by

$$(g_1, \epsilon_1) \cdot (g_2, \epsilon_2) = (g_1g_2, \epsilon_1\epsilon_2 \cdot c(g_1, g_2))$$

for some 2-cocycle $c$ on $G_k(F_v)$ valued in $\{\pm 1\}$. We denote the double covering map as $pr : \widetilde{G}_k(F_v) \to G_k(F_v)$. Similarly, let $\widetilde{G}_k(\mathfrak{A})$ be the metaplectic double covering of $G_k(\mathfrak{A})$, compatible with $\widetilde{G}_k(F_v)$ for all places $v$. Again, we use the notation $pr$ for the double cover map from $\widetilde{G}_k(\mathfrak{A})$ to $G_k(\mathfrak{A})$. Notably, $U_0(\mathfrak{A})$ canonically lifts into $\widetilde{G}_k(\mathfrak{A})$ (see [32 Appendix I]), and Weil’s result [47] ensures that $G_k(F)$ lifts uniquely into $\widetilde{G}_k(\mathfrak{A})$.

When considering the unipotent subgroup $U$ of any parabolic subgroup of $G_k$ (or a subgroup $S$ of $G_k(F)$), we maintain the same notation $U$ (or $S$) to denote their image under this lifting. Furthermore, for a subgroup $J$ of $G_k$, we use $\widetilde{J}(\mathfrak{A})$ (resp. $\widetilde{J}(F_v)$) to represent $pr^{-1}(J(\mathfrak{A}))$ (resp. $pr^{-1}(J(F_v))$). When $J$ is the trivial group, $\widetilde{J}(\mathfrak{A})$ (resp. $\widetilde{J}(F_v)$) equals to $\{1, -1\}$. Lastly, a function $f$ on $\widetilde{G}_k(\mathfrak{A})$ (resp. $\widetilde{G}_k(F_v)$) is considered genuine if $f(\epsilon \cdot g) = \epsilon \cdot f(g)$ for $\epsilon \in \{1, -1\}$ and $g \in \widetilde{G}_k(\mathfrak{A})$ (resp. $\widetilde{G}_k(F_v)$).

For $0 \leq j \leq k$, write $N_{k,j}$ (resp. $N_j$) for the unipotent radical of the parabolic subgroup of $G_k$ (resp., $GL(X_j)$) stabilizing the flag $\{0\} = X_0 \subset X_1 \subset \cdots \subset X_j$. If we regard $N_j$ as a subgroup of $M_{k,j} \simeq GL(X_j) \times G_{k-j}$, it acts on $U_{k,j}$ and so $N_{k,j} = U_{k,j} \rtimes N_j$. When $j = 0$, $N_0$ denotes the trivial group.

Let $\delta_P$ denote the modulus function of $P(\mathfrak{A})$ (resp. $\widetilde{P}(\mathfrak{A})$) (see [32 I.2.17]). We use the following notation: Ind for the normalized induction functor, ind for the unnormalized induction functor, and c-ind for the compactly supported induction.

For an automorphic representation $\rho$ of $M(\mathfrak{A})$ (resp. $\widetilde{M}(\mathfrak{A})$) and $s \in \mathbb{C}$, we define $I(s, \rho)$ as the induced representation:

$$I(s, \rho) = \text{Ind}_{P(\mathfrak{A})}^{G_k(\mathfrak{A})}(\delta_P^s \cdot \rho) \quad (\text{resp. } I(s, \rho) = \text{Ind}_{\widetilde{P}(\mathfrak{A})}^{\widetilde{G}_k(\mathfrak{A})}(\delta_P^s \cdot \rho)).$$

This induced representation has a realization in the $\mathbb{C}$-valued function space as follows:

Let $\mathcal{A}(P_k)$ (resp. $\mathcal{A}(\widetilde{P}_k)$) be the space of (resp. genuine) automorphic forms on $U(\mathfrak{A})P(F)\backslash G_k(\mathfrak{A})$ (resp. $U(\mathfrak{A})P(F)\backslash \widetilde{G}_k(\mathfrak{A})$). These forms are smooth $K$-finite (resp. $\widetilde{K}$-finite) and $\mathfrak{z}$-finite functions on $U(\mathfrak{A})P(F)\backslash G_k(\mathfrak{A})$ (resp. $U(\mathfrak{A})P(F)\backslash \widetilde{G}_k(\mathfrak{A})$), where $\mathfrak{z}$ represents the center of the universal enveloping
algebra of the complexified Lie algebra $\mathfrak{g}_C$ of $G_k(F \otimes_{\mathbb{Q}} \mathbb{R})$. These spaces are modules of the groups $(G_k(\mathbb{A}_{\text{fin}}) \times (\mathfrak{g}_C, \prod_{v \mid \mathbb{Q}} K_v))$ (resp. $(\tilde{G}_k(\mathbb{A}_{\text{fin}}) \times (\mathfrak{g}_C, \prod_{v \mid \mathbb{Q}} \tilde{K}_v))$).

When $P = G$, we simply write $\mathcal{A}(G_k)$ (resp. $\mathcal{A}(\tilde{G}_k)$) to denote $\mathcal{A}_G(G_k)$ (resp. $\mathcal{A}_{\tilde{G}}(\tilde{G}_k)$).

For an (resp. genuine) automorphic representation $\rho$ of $M(\mathbb{A})$ (resp. $\tilde{M}(\mathbb{A})$) and $s \in \mathbb{C}$, we define $\mathcal{A}_P^{[\rho, \chi]}(G_k)$ as the subspace of functions $\phi \in \mathcal{A}_P(G_k)$ (resp. $\phi \in \mathcal{A}_P(\tilde{G}_k)$) such that for all $k \in K$ (resp. $\tilde{K}$), the function $m \mapsto \delta_P(m)^{-\frac{s}{2} - s} \cdot \phi(mk)$ belongs to the space of $\rho$.

2.2. Discrete global $A$-parameters. For non-negative integer $d$, denote the unique $(d+1)$-dimensional irreducible representation of $SL_2(\mathbb{C})$ by $\text{Sym}^d(\mathbb{C}^2)$. When $d = -1$, we understand $\text{Sym}^{-1}(\mathbb{C}^2)$ to be 0. A discrete global $A$-parameter for $G_k(\mathbb{A})$ (resp. $\tilde{G}_k(\mathbb{A})$) is a formal finite direct sum

$$(2.1) \quad M = \bigoplus_{i=1}^r M_i \otimes \text{Sym}^d(\mathbb{C}^2),$$

where

- $M_i$ is an irreducible cuspidal unitary automorphic representation of $GL_{n_i}(\mathbb{A})$;
- $d_i$ is an integer greater than or equal to $-1$ such that $\sum_{i=1}^r n_i(d_i + 1) = 2k + 1$ (resp. $2k$);
- if $d_i$ is even (resp. odd), then $L(s, \text{Sym}^2(M_i))$, the symmetric square $L$-function of $M_i$ has a pole at $s = 1$ (i.e., $M_i$ is orthogonal);
- if $d_i$ is odd (resp. even), then $L(s, \wedge^2(M_i))$, the exterior square $L$-function of $M_i$ has a pole at $s = 1$ (i.e., $M_i$ is symplectic);
- the central character $\omega_{M_i}$ of $M_i$ satisfies $\prod_{i=1}^r \omega_{M_i}^{d_i+1} = 1$;
- if $(d_i, M_i) = (d_j, M_j)$, then $i = j$.

If $M$ is a discrete global $A$-parameter of $G_k(\mathbb{A})$ (resp. $\tilde{G}_k(\mathbb{A})$), we call that $M$ is orthogonal (resp. symplectic). If $M$ is decomposed as in (2.1), we call that $M$ is generic if $d_i = 0$ for all $i$. In this case, we simply write $M$ as $\bigoplus_{i=1}^r M_i$.

For each place $v$ of $F$, put

$L_{F_v} = \begin{cases} 
\text{the Weil group of } F_v, & \text{if } F_v \text{ is archimedean,} \\
\text{the Weil–Deligne group of } F_v & \text{if } F_v \text{ is non-archimedean}
\end{cases}$

and let $M_v = \bigoplus_{i=1}^r M_{i,v} \otimes \text{Sym}^{d_i}(\mathbb{C}^2)$ be the localization of $M$ at $v$. Here, we regard $M_{i,v}$ as an $n_i$-dimensional representation of $L_{F_v}$ via the local Langlands correspondence [23, 22, 21, 42], which provides a bijection between irreducible smooth representations of $GL_{n_i}$ and $n_i$-dimensional representations of $L_{F_v}$. Then $M_v$ gives rise to an $A$-parameter $\phi_{M_v} : L_{F_v} \times SL_2(\mathbb{C}) \rightarrow \text{SO}_{2k+1}(\mathbb{C})$ (resp. $\text{Sp}_{2k}(\mathbb{C})$). Then we can associate to it an $L$-parameter $\varphi_{M_v} : L_{F_v} \rightarrow \text{SO}_{2k+1}(\mathbb{C})$ (resp. $\text{Sp}_{2k}(\mathbb{C})$) defined by

$$\varphi_{M_v}(w) = \phi_{M_v}((w, \begin{pmatrix} |w|^\frac{1}{2} & 0 \\
0 & |w|^{-\frac{1}{2}} \end{pmatrix}), \quad w \in L_{F_v}.$$ Let $\mathcal{A}^2(G_k)$ (resp. $\mathcal{A}^2(\tilde{G}_k)$) be the set of square-integrable automorphic forms in $A_{G_k}(G_k)$ (resp. $A_{\tilde{G}_k}(\tilde{G}_k)$). Then it is also a module of $(G_k(\mathbb{A}_{\text{fin}}) \times (\mathfrak{g}_C, \prod_{v \mid \mathbb{Q}} K_v))$ (resp. $(\tilde{G}_k(\mathbb{A}_{\text{fin}}) \times (\mathfrak{g}_C, \prod_{v \mid \mathbb{Q}} \tilde{K}_v))$). An irreducible $(G_k(\mathbb{A}_{\text{fin}}) \times (\mathfrak{g}_C, \prod_{v \mid \mathbb{Q}} K_v))$ (resp. $(\tilde{G}_k(\mathbb{A}_{\text{fin}}) \times (\mathfrak{g}_C, \prod_{v \mid \mathbb{Q}} \tilde{K}_v))$)-submodule of $\mathcal{A}^2(G_k)$ (resp. $\mathcal{A}^2(\tilde{G}_k)$) is called an automorphic representation of $G_k(\mathbb{A})$ (resp. $G_k(\mathbb{A})$).
For two automorphic representations \( \pi_1 \) and \( \pi_2 \) of \( G_k(\mathbb{A}) \) (resp. \( \widetilde{G}_k(\mathbb{A}) \)), we say that \( \pi_1 \) and \( \pi_2 \) are nearly equivalent if \( \pi_{1,v} \simeq \pi_{2,v} \) for almost all places \( v \) of \( F \). For a discrete global \( A \)-parameter \( M \) of \( G_k(\mathbb{A}) \) (resp. \( \widetilde{G}_k(\mathbb{A}) \)), there exists a set \( \Pi_M \) of nearly equivalent classes of automorphic representations \( \pi \) in \( A^2(G_k) \) (resp. \( A^2(\widetilde{G}_k) \)) such that \( \pi_v \) has local \( L \)-parameter \( \varphi_{M_v} \) for almost all places \( v \) of \( F \). Such \( \Pi_M \) is called the global \( A \)-packet associated to \( M \) and each elements in \( \Pi_M \) is said to have global \( A \)-parameter \( M \).

The following theorem shows the relation of Satake parameters of \( M \) and elements of \( \Pi_M \).

**Theorem 2.1** ([\( \Pi \) Theorem 1.5.2], [[13] Theorems 1.1]). Let \( M = \bigoplus_{i=1}^{r} M_i \otimes \text{Sym}^{d_i}(\mathbb{C}^2) \) be a discrete \( A \)-parameter for \( G_k(\mathbb{A}) \), (resp. \( \widetilde{G}_k(\mathbb{A}) \)) where \( M_i \) is an irreducible cuspidal automorphic representation of \( GL_n(\mathbb{A}) \). Let \( \pi \) be an element in \( \Pi_M \). For almost all places \( v \) of \( F \), \( \Pi_{M,v} \) is unramified and let \( \{c_{1,v}, \ldots, c_{v,v}\} \) the set of Satake parameter of \( M_{i,v} \). Then for almost all places \( v \) of \( F \), where \( \pi_v \) is unramified, the set of Satake parameter of \( \pi_v \) is

\[
\bigcup_{i=1}^{r} \bigcup_{j=1}^{t_i} \{c_{j,i,v} \cdot q^{-\frac{d_j}{2}} \cdot q^{-\frac{d_j}{2}+1}, \ldots, c_{j,i,v} \cdot q^{-\frac{d_j}{2}+1}, c_{j,i,v} \cdot q^{-\frac{d_j}{2}}\}.
\]

3. Jacquet module corresponding to Fourier-Jacobi characters

In this section, we introduce the Jacquet module corresponding to a Fourier-Jacobi character and extract some properties of it for certain cases. This play a key role in proving Proposition 5.1 which asserts the vanishing of the Fourier-Jacobi periods when it involves a residual representation.

Fix a finite place \( v \) of \( F \) and a character \( \psi_v \) of \( F_v^\times \). Sometimes, we regard \( \psi_v \) as a character of \( U_{k,k}(F_v) \) defined by \( \psi_v(u) = \psi(\langle uf_1, f_k \rangle) \). For any \( \alpha \in F_v \), define \( \psi_{\alpha,v} : F_v \rightarrow \mathbb{C}^\times \) as

\[
\psi_{\alpha,v}(x) := \psi_v(\alpha x), \quad \text{for } x \in F_v.
\]

Let \( \Omega_{\alpha,W_{k-j},v} \) represent the Weil representation of \( \mathcal{H}_{W_{k-j}}(F_v) \times \mathcal{G}_{k-j}(F_v) \) with respect to \( \psi_{\alpha,v}^{-1} \). It is worth noting that \( \Omega_{\alpha,W_{0},v} = \psi_{\alpha,v}^{-1} \).

As this section is dedicated to the local context, we will henceforth omit the subscript \( v \) and the field \( F_v \) from our notation.

Since \( N_j \subseteq N_{k,j-1} \) and \( N_{k,j-1}\backslash N_{k,j} \simeq \mathcal{H}_{W_{k-j}} \), we can pull back \( \Omega_{\alpha,W_{k-j}} \) to \( N_{k,j} \times \mathcal{G}_{k-j} \) and denote it by the same symbol \( \Omega_{\alpha,W_{k-j}} \). When \( j \geq 2 \), we define a character \( \lambda_j : N_j \rightarrow \mathbb{C}^\times \) by

\[
\lambda_j(n) = \psi(\langle ne_2, f_1 \rangle + \langle ne_3, f_2 \rangle + \cdots + \langle ne_j, f_{j-1} \rangle), \quad n \in N_j.
\]

When \( j = 0, 1 \), define \( \lambda_j \) as the trivial character. Put \( \nu_{\alpha,W_{k-j}} = \lambda_j^{-1} \otimes \Omega_{\alpha,W_{k-j}} \) and denote \( \mathcal{H}_{k,j} = N_{k,j} \times \mathcal{G}_{k-j} \). Then \( \nu_{\alpha,W_{k-j}} \) is a smooth representation of \( \mathcal{H}_{k,j} \). We shall denote by \( \omega_{\alpha,W_{k-j}} \) the restriction of \( \nu_{\alpha,W_{k-j}} \) to \( \mathcal{G}_{k-j} \).

For \( 0 \leq l \leq k-1 \), we define characters \( \psi_l \) of \( N_{k,l+1} \), which factors through the quotient map \( n : N_{k,l+1} \rightarrow U_{k,l+1}\backslash N_{k,l+1} \simeq N_{l+1} \), by setting

\[
\psi_l(u) = \lambda_{l+1}(n(u)).
\]

For an arbitrary \( \alpha \in F_v^\times \), we also define a character \( \psi_l^{(\alpha)} \) of \( N_{k,l+1} \) by \( \psi_l^{(\alpha)}(u) = \psi_l(u) \cdot \psi(\alpha \langle ue_{l+2}, f_{l+1} \rangle) \).

(Here, we regard \( e_{l+1} = f_1 \) when \( l = k - 1 \).)

For an irreducible genuine (resp. non-genuine) smooth representation \( \pi' \) of \( \mathcal{G}_k \), write \( J_{\psi^{(\alpha)} \pi'} \) for the Jacquet module of \( \pi' \) with respect to the group \( N_{k,k} \) and its character \( \psi_{k-1}^{(\alpha)} \). Then it is a genuine (resp.
non-genuine) representation of the group \( \widetilde{P}_k \). We say that \( \pi' \) is \( \psi^{(\alpha)} \)-generic if \( J_{\psi^{(\alpha)}}(\pi') \) is non-zero. If \( \pi' \) is \( \psi^{(\alpha)} \)-generic for some \( \alpha \in F^\times \), we say that \( \pi' \) is generic.

We also write \( J_{\psi^{(\alpha)}}(\pi' \otimes \Omega_{\alpha,W_{k-l-1}}) \) for the Jacquet module of \( \pi' \otimes \Omega_{\alpha,W_{k-l-1}} \) with respect to the group \( N_{k,l+1} \) and its character \( \psi' \). Then it is a non-genuine (resp. genuine) representation of the group \( \widetilde{G}_{k-l-1} \). We can write this as a two-step Jacquet module as follows (see [18, Chapter 6]).

For \( 1 \leq l \leq k \), let \( C_{l-1} \) be the subgroup of \( N_{k,l} \) corresponding to the center of the Heisenberg group \( H_{W_{k-l}} \) through the isomorphism \( N_{k,l} \setminus H_{W_{k-l}} \cong H_{W_{k-l}} \). Note that \( C_{l-1}(F) \cong F \) and \( C_{l-1} \) acts on \( \Omega_{\alpha,W_{k-l}} \) by the character \( \psi_{\alpha}^{-1} \). For \( 1 \leq l \leq k-1 \), put \( N_{k,l+1}^0 = N_{k,l}C_l \subseteq N_{k,l+1} \) and define a character \( \psi_{\alpha,l}^0 : N_{k,l+1}^0 \to \mathbb{C}^\times \) as \( \psi_{\alpha,l}(nc) = \psi_{l-1}(n) \psi_\alpha(c) \) for \( n \in N_{k,l}, c \in C_l \).

Let \( J_{\psi_{\alpha,l}^{0}}(\pi') \) be the Jacquet module of \( \pi' \) with respect to \( N_{k,l+1}^0 \) and \( \psi_{\alpha,l}^0 \). We also denote by \( J_{\mathcal{H}_{W_{k-1-l}}/C_l} \) the Jacquet functor with respect to \( W_{k-1-l}/C_l \) and the trivial character.

Then we have

\[
(3.1) \quad J_{\psi}(\pi' \otimes \Omega_{\alpha,W_{k-l-1}}) \simeq J_{\mathcal{H}_{W_{k-1-l}}/C_l}(J_{\psi_{\alpha,l}^{0}}(\pi') \otimes \Omega_{\alpha,W_{k-l-1}}).
\]

When \( \alpha = 1 \), we simply write \( \psi, J_{\psi_{0,l}^{0}}, \nu_{\alpha,W_{k-j}}, \omega_{\alpha,W_{k-j}} \) and \( \Omega_{\alpha,W_{k-l-1}} \) as \( \psi, J_{\psi_{0}}, \nu_{W_{k-j}}, \omega_{W_{k-j}} \) and \( \Omega_{W_{k-l-1}} \), respectively.

Consider the group \( \widetilde{GL}_j \cong GL_j \times \mu_2 \) with multiplication

\[
(g_1, \zeta_1) \cdot (g_2, \zeta_2) = (g_1g_2, \zeta_1\zeta_2 \cdot (\det g_1, \det g_2)).
\]

We can define a genuine character of \( \widetilde{GL}_1 \) by

\[
\gamma_\psi(\beta, \zeta) = \zeta \gamma(\beta, \psi)^{-1}
\]

with

\[
\gamma(\beta, \psi) = \gamma(\psi_\beta) / \gamma(\psi)
\]

for \( \beta \in F^\times \) and \( \zeta \in \mu_2 \), where \( \gamma(\psi) \in \mu_8 \) is a Weil constant associated with \( \psi \) (see [39]). Using this character, we obtain a bijection, which depends on the choice of the additive character \( \psi \), between the set of equivalence classes of admissible representations of \( GL_j \) and that of genuine admissible representations of \( GL_j \) via \( \sigma \mapsto \sigma_\psi \), where

\[
\sigma_\psi(\zeta, \psi) = \gamma_\psi(\det g, \zeta) \cdot \sigma_\psi(g).
\]

Let \( B_k \) and \( B_n \) denote the standard Borel subgroups of \( GL_k \) and \( G_n \), respectively. For a sequence \( \bf{a} = (a_1, a_2, \ldots, a_l) \) of positive integers whose sum is \( k \), we denote by \( P_\bf{a} = M_\bf{a} U_\bf{a} \) the standard parabolic subgroup of \( G_n \) whose Levi subgroup \( M_\bf{a} \) is isomorphic to \( GL_{a_1} \times \cdots \times GL_{a_l} \times G_{n-k} \). We also denote by \( \mathcal{P}_{k,j} \) (resp. \( \mathcal{P}_\bf{a} \)) the subgroup of \( GL_k \) (resp. the parabolic subgroup of \( GL_k \)) whose Levi subgroup is isomorphic to \( GL_{a_1} \times \cdots \times GL_{a_l} \) consisting of \( \begin{pmatrix} GL_{k-j} & x \\ 0 & N_j \end{pmatrix} \) \( x \in \text{Mat}_{k-j} \times \text{Mat}_{j} \). We write \( N_j \) in \( \mathcal{P}_{k,j} \) as \( N_{k,j} \) (i.e., \( N_{k,j} = \mathcal{P}_{k,j} \cap (\text{Id}_{GL_{k-j}} \times GL_j) \)).

Then we have the Levi decomposition \( \widetilde{P}_\bf{a} \cong \widetilde{M}_\bf{a} \cdot U_\bf{a} \), where

\[
\widetilde{M}_\bf{a} \cong GL_{a_1} \times \mu_2 \cdots \times \mu_2 GL_{a_l} \times \mu_2 G_{n-k}.
\]

For a smooth representation \( \sigma_j \) of \( GL_{a_j} \) and a genuine smooth representation \( \pi'' \) of \( \widetilde{G}_{n-k} \), we can form an induced representation \( \text{Ind}_{\widetilde{P}_\bf{a}}^{\widetilde{G}_n}(\sigma_1 \psi \boxtimes \cdots \boxtimes \sigma_l \psi \boxtimes \pi'') \) of \( \widetilde{G}_n \). Although \( \widetilde{G}_n \) is not a linear group, many
basic results concerning with the induction and Jacquet functors remain valid. For a justification of this, the reader can consult [23].

**Lemma 3.1.** Let $n, m$ positive integers such that $r = n - m > 0$. For a positive integer $a > 0$, let $E$ be a smooth representation of $\widetilde{G}_{n+a}$ of finite length and $I$ a smooth representation of $\widetilde{G}_{m+a}$ of finite length. Assume that exactly one of $E$ or $I$ is genuine. Then

\[
\text{Hom}_{\text{N}_{n+a}} \left( E \otimes \nu_{m+a} \otimes I, \mathbb{C} \right) \\
\simeq \text{Hom}_{G_{m+a}} \left( J_{\psi^{-1}} \left( E \otimes \Omega_{m+a} \right) \otimes I, \mathbb{C} \right).
\]

**Proof.** By the Frobenius reciprocity,

\[
\text{Hom}_{\text{N}_{n+a}} \left( E \otimes \nu_{m+a} \otimes I, \mathbb{C} \right) \\
= \text{Hom}_{\text{N}_{n+a}} \left( E \otimes \lambda_1 \otimes \Omega_{m+a} \otimes I, \mathbb{C} \right) \\
\simeq \text{Hom}_{G_{m+a}} \left( J_{\psi^{-1}} \left( E \otimes \Omega_{m+a} \right) \otimes I, \mathbb{C} \right).
\]

For each $2 \leq t \leq k$, we define characters $\lambda'_t : \mathcal{N}_k \to \mathbb{C}^\times$ and $\lambda''_t : \mathcal{N}_k \to \mathbb{C}^\times$ as follows:

\[
\lambda'_t(n) = \psi^{-1}(\langle ne_{k-t+2}, f_{k-t+1} \rangle + \langle ne_{k-t+3}, f_{k-t+2} \rangle + \cdots + \langle ne_k, f_k \rangle), \quad n \in \mathcal{N}_k,
\]

\[
\lambda''_t(n) = \psi^{-1}(\langle ne_{k-t+2}, f_{k-t+1} \rangle + \langle ne_{k-t+3}, f_{k-t+2} \rangle + \cdots + \langle ne_k, f_k \rangle + \langle ne_{k-t+1}, f_{k-t} \rangle), \quad n \in \mathcal{N}_k.
\]

When $t = 0$ or $t = 1$, we define $\lambda'_t$ and $\lambda''_t$ as the trivial characters. For a smooth representation $\sigma$ of $GL_k$ and $0 \leq i \leq k$, let $\sigma^{(i)}$ (resp. $\sigma^{(j)}$) be the smooth representation of $GL_{k-i}$ (resp. $\mathcal{P}_{k-i-1,1}$) defined by the Jacquet module of $\sigma$ with respect to the character $\lambda'_t$ (resp. $\lambda''_t$) of $\mathcal{N}_k$ as in [18, p.87]. Note that $\sigma^{(i)}$ is nothing but the $i$-th Bernstein-Zelevinsky derivative of $\sigma$ if we use the unnormalized Jacquet functors in the definition of Bernstein-Zelevinsky derivative as in [7]. Note that the Bernstein-Zelevinsky derivative of $\chi(k)$ is given by

\[
(\chi(k))^{(t)} = \begin{cases} 
\chi(k-t) & \text{if } t \leq 1, \\
0 & \text{if } t \geq 2.
\end{cases}
\]

**Remark 3.2.** For a partition $(m_1, \cdots, m_t)$ of $k$ and characters $\mu_i$'s, let $\sigma = \text{Ind}_{\mathcal{P}(m_1, \cdots, m_t)}^{GL_k} (\mu_1(m_1) \boxtimes \cdots \boxtimes \mu_t(m_t))$. Now we use the notation $A \overset{\text{SS}}{\simeq} B$ to mean that $A$ and $B$ are isomorphic up to semi-simplication.

Using the Leibniz rule of the Bernstein-Zelevinsky derivative, it is not so difficult to check that for every integer $j \geq t+1$, $| \cdot |^{1/t_j} \cdot \sigma^{(j)} = 0$ and for $0 \leq j \leq t$,

\[
(3.2) \quad | \cdot |^{1/t_j} \cdot \sigma^{(j)} \overset{\text{SS}}{\simeq} \bigoplus_{i_1+\cdots+i_t=j} \text{Ind}_{\mathcal{P}(m_1-i_1, \cdots, m_t-i_t)}^{GL_{k-j}} (| \cdot |^{1/t_{i_1}} \cdot \mu_1(m_1-i_1) \boxtimes \cdots \boxtimes | \cdot |^{1/t_{i_t}} \cdot \mu_t(m_t-i_t))
\]

(cf. [15] (5.34)). By [6, p. 452], $\sigma^{(j)}$ has a finite $\mathcal{P}_{k-j-1,1}$-filtration whose subquotients are (compactly) induced from derivatives of $\sigma^{(j)}$. Note that for all $e \geq 1$, $(\sigma^{(j)})^{(e)} \simeq \sigma^{(e+j)}$ as a restriction of a $\mathcal{P}_{k-j-1,1}$-module to $GL_{k-(e+j)}$. In particular, if $j \geq t$, then all subquotients of $\sigma^{(j)}$ are zero and it leads to $\sigma^{(j)} = 0$.

For two non-negative integers $t, s$, put $\delta_{t,s} = \begin{cases} 1 & \text{if } t \geq s, \\
0 & \text{if } t < s.
\end{cases}$ Choose an arbitrary $\alpha \in F$. The following theorem is a slight generalization of [15, Theorem 6.1], reflecting $\alpha$, whose proof is almost the same.
Theorem 3.3 (\cite{IS} Theorem 6.1). Let $\sigma$ be a smooth representation of $GL_j$ and $\bar{\tau}$ a smooth representation of $\widetilde{G}_n$, respectively. Choose an arbitrary non-negative integer $l$. Then
\[
J_{\psi_{a,l}}\left(\text{Ind}_{P_{n+j,j}}^{\widetilde{G}_{n+j,l}}(\sigma_\psi \boxtimes \bar{\tau})\right)_{s,t} \simeq \bigoplus_{l-n \leq t \leq l} \bigoplus_{0 \leq j \leq 1} \text{ind}_{P_{j-l}}^{G_{n+j-(l+1) \times H_{W_{n+j-(l+1)}}}} \left(\left|\cdot\right|^{-\frac{i}{2}} \cdot \sigma(t) \boxtimes J_{\psi_{a,l-t}}(\bar{\tau})\right).
\]
(For the definitions of $P_{j-l}'$ and $P_{j-l}''$, we refer the reader to \cite{IS} (6.4) and \cite{IS} (6.8).)

Using (\ref{3.3}), we obtain the following two theorems by combining special cases of Theorem 3.3 with \cite{IS} Proposition 6.6 and \cite{IS} Proposition 6.7. (Here, we reflected some typos in \cite{IS} Proposition 6.7.)

Theorem 3.4. Let $\sigma$ be a smooth representation of $GL_j$ and $\bar{\tau}$ a smooth representation of $\widetilde{G}_n$, respectively. For an integer $l \geq 0$, assume either $j \leq l$ or $\sigma(t) = 0$. Then
\[
J_{\psi_{l}}\left(\text{Ind}_{P_{n+j,j}}^{\widetilde{G}_{n+j,l}}(\sigma_\psi \boxtimes \bar{\tau}) \otimes \Omega_{\delta W_{n+j-(l+1)}}\right)_{s,t} \simeq \bigoplus_{l-n \leq t \leq l} \bigoplus_{0 \leq j \leq 1} \text{ind}_{P_{j-l}}^{G_{n+j-(l+1) \times H_{W_{n+j-(l+1)}}}} \left(\left|\cdot\right|^{-\frac{i}{2}} \cdot \sigma(t) \boxtimes J_{\psi_{l-t}}(\bar{\tau} \otimes \Omega_{\delta W_{n+j-(l+1)}})\right).
\]

Theorem 3.5. For an integer $j \geq 1$ and a character $\mu$ of $GL_1$, let $\sigma = \mu(j)$. Let $\bar{\tau}$ be a smooth representation of $\widetilde{G}_n$, respectively. Choose an arbitrary non-negative integer $l \leq n + j - 2$. Then
\[
J_{\psi_{l}}\left(\text{Ind}_{P_{n+j,j}}^{\widetilde{G}_{n+j,l}}(\sigma_\psi \boxtimes \bar{\tau}) \otimes \Omega_{W_{n+j-(l+1)}}\right)_{s,t} \simeq \bigoplus_{l+1-n \leq t \leq l} \bigoplus_{0 \leq j \leq 1} \text{ind}_{P_{j-l}}^{G_{n+j-(l+1) \times H_{W_{n+j-(l+1)}}}} \left(\left|\cdot\right|^{-\frac{i}{2}} \cdot \sigma(t) \boxtimes J_{\psi_{l-t}}(\bar{\tau} \otimes \Omega_{W_{n+j-(l+1)}})\right) \bigoplus \delta_{0,l} \cdot \text{Ind}_{P_{n+j+1,l-1}}^{G_{n+j+1-1 \times H_{W_{n+j+1-1}}}}(\sigma(1) \boxtimes (\bar{\tau} \otimes \Omega_{W_{n+j-1}})).
\]

By applying Theorem 3.5, we are able to establish two propositions, both of which will be instrumental in subsequent discussions. This two propositions do not appear in \cite{sato}. To set the stage for these, we first introduce some relevant notations and observe a particular fact.

For a character $\chi$ of $GL_1$, we denote by $e(\chi)$ a real number such that the product $\chi \cdot |\cdot|^{-e(\chi)}$ forms a unitary character. For $1 \leq a \leq n$, denote by $J_{U_{n,a}}$ (resp. $J_{n,a}^n$) the unnormalized (resp. normalized) Jacquet functor with respect to $U_{n,a}$ and the trivial character. We write $Q_{n,a} = P_{n,a} \times U_{n,n}$.

Note that there is an exact sequence
\[
\begin{equation}
0 \rightarrow c\text{-ind}_{\widetilde{Q}_{n,1}}^{\widetilde{P}_{n,n}}(\left|\cdot\right|^s \psi \boxtimes \psi^{-1}) \rightarrow \Omega_{\widetilde{P}_{n,n}} \rightarrow \left|\cdot\right|^s \psi \rightarrow 0
\end{equation}
(see the proof of \cite{Is} Theorem 16.1.)

Proposition 3.6. Let $\mu$ and $\{\chi_i\}_{1 \leq i \leq k}$ be unramified characters of $GL_1$ such that for all $1 \leq i \leq k$, $0 \leq e(\chi_i) < \frac{1}{2}$. For $z \in \mathbb{C}$, put $\mu_z = \mu \cdot |\cdot|^z$ and $\chi_{i,z} = \chi_i$ or $\chi_i \cdot |\cdot|^z$. Write $\pi_z = \text{ind}_{B_{k+1}^1}^{GL_{k+1}}(\chi_1, z \cdot \cdots \cdot \chi_k, z \cdot \mu_z)$. We define $\pi_z = \text{ind}_{B_{k+1}^1}^{GL_{k+1}^1}(\mu_z)$ when $k = 0$. Let $\mathcal{E}_{k+1}$ be an irreducible unitary subquotient of an unramified principal series representation $\text{Ind}_{B_{k+1}^1}^{GL_{k+1}^1}(\mu_1 \otimes \cdots \otimes \mu_{k+1})$ satisfying one of the following conditions:

(i) $|\mu_i| = |\cdot|$ for some $1 \leq i \leq k + 1$.

(ii) There exists a real number $0 < s_0 < \frac{1}{2}$ such that $e(\mu_i) = 1 + s_0$ and $e(\mu_j) = 1 - s_0$ for some $1 \leq i, j \leq k + 1$. In this case, $1 \leq k$. 

Then,
\[ \text{Hom}_{G^{k+1}} \left( \text{Ind}_{P^{k+1,k+1}}^{G^{k+1}} ((\pi_z)_\psi) \otimes \omega_{W^{k+1}}, \mathcal{E}^{k+1} \right) = 0 \]
for almost all \( z \).

**Proof.** By the Frobenius reciprocity in [6, Theorem 2.29], we have
\[ \text{Hom}_{G^{k+1}} \left( \text{Ind}_{P^{k+1,k+1}}^{G^{k+1}} ((\pi_z)_\psi) \otimes \omega_{W^{k+1}}, \mathcal{E}^{k+1} \right) \simeq \text{Hom}_{P^{k+1,k+1}} \left( \delta_{P^{k+1,k+1}}^{-\frac{1}{2}} \cdot (\pi_z)_\psi \otimes \omega_{W^{k+1}}, \mathcal{E}^{k+1} \right). \]

Note that
\[ (3.4) \quad \text{Hom}_{P^{k+1,k+1}} \left( \delta_{P^{k+1,k+1}}^{-\frac{1}{2}} \cdot (\pi_z)_\psi \otimes \text{c-ind}_{Q^{k+1,1}}^{P^{k+1,k+1}} \left( (| \cdot |^\frac{1}{2})_\psi \otimes \psi^{-1} \right), \mathcal{E}^{k+1} \right) \]
\[ \simeq \text{Hom}_{Q^{k+1,1}} \left( \delta_{Q^{k+1,1}}^{-1} \cdot | \cdot |^\frac{1}{2} \cdot \pi_z \otimes \psi^{-1}, \mathcal{E}^{k+1} \right) \]
because \( \delta_{P^{k+1,k+1}} = | \cdot |^2 \).

By a result of Bernstein and Zelevinsky (see [7 Section 3.5]), the restriction of \( \pi_z \) to \( P_{k+1,1} \) has a filtration
\[ \pi_z = \pi_{z,1} \supset \pi_{z,2} \supset \cdots \supset \pi_{z,k+1} \supset \pi_{z,k+2} = 0 \]
such that \( \pi_z \mid \pi_{z,i+k+1} \simeq (\Phi^+)^{-1}\psi^+((\pi_z)^{(i)}) \) for \( 1 \leq i \leq k+1 \). (For the definition of \( \Phi^+ \) and \( \Psi^+ \), the reader should consult to [7 Sect. 3].) Then for \( 1 \leq i \leq k+1 \),
\[ (\Phi^+)^{-1}\psi^+((\pi_z)^{(i)}) = \text{c-ind}_{P^{k+1,1}}^{P^{k+1,k+1}} | \cdot |^\frac{1}{2} \cdot \pi_z \otimes \psi^{-1} | \cdot |^\frac{1}{2} \cdot \pi_{z,i+k+1} \otimes \psi^{-1} \]
We view \( \psi^{-1} \) as a character of \( U_{k+1,k+1} \) and write \( J_{\psi^{-1}} \) the Jacquet functor relative to \( U_{k+1,k+1} \) and \( \psi^{-1} \).

By the Frobenius reciprocity [6 Theorem 2.29] again, for \( 1 \leq j \leq k+1 \), we have
\[ (3.5) \quad \text{Hom}_{Q^{k+1,1}} \left( \delta_{Q^{k+1,1}}^{-1} \cdot | \cdot |^\frac{1}{2} \cdot (\Phi^+)^{-1}\psi^+((\pi_z)^{(j)}) \otimes \psi^{-1}, \mathcal{E}^{k+1} \right) \]
\[ \simeq \text{Hom}_{Q^{k+1,j}} \left( (\delta_{Q^{k+1,j}}^{-1} \cdot | \cdot |^\frac{1}{2} \cdot (\pi_z)^{(j)} \otimes \lambda_{k+1}|N_{k+1,j}) \otimes \psi^{-1}, \mathcal{E}^{k+1} \right) \]
\[ \simeq \text{Hom}_{Q^{k+1,j}} \left( (\delta_{Q^{k+1,j}}^{-1} \cdot | \cdot |^\frac{1}{2} \cdot (\pi_z)^{(j)} \otimes \lambda_{k+1}|N_{k+1,j}), J_{U_{k+1,k+1-j}}(J_{\psi^{-1}}(\mathcal{E}^{k+1})) \right) \]
\[ (3.6) \quad \simeq \text{Hom}_{GL_{k+1-j}} \left( \delta_{Q^{k+1,j}}^{-1} \cdot | \cdot |^\frac{1}{2} \cdot (\pi_z)^{(j)}, J_{N_{k+1,j}}, \lambda_{k+1}|N_{k+1,j}) \right) \]
where \( U_{k+1,k+1-j} \) is the unipotent radical of \( P_{k+1,k+1-j} \) and \( J_{U_{k+1,k+1-j}} \) (resp. \( J_{N_{k+1,j}}, \lambda_{k+1}|N_{k+1,j}) \) is the (resp. twisted) Jacquet functor with respect to \( U_{k+1,k+1-j} \) and the trivial character (resp. \( N_{k+1,j} \) and \( \lambda_{k+1}|N_{k+1,j}) \).

We claim that for \( 1 \leq j < k+1 \), (3.6) = 0 for almost all \( z \). It is easy to check that \( \delta_{Q^{k+1,j}} = | \cdot |^{\frac{j+3}{2}} \). By (3.2), the constituents of \( \delta_{Q^{k+1,j}}^{-1} \cdot | \cdot |^{\frac{j+3}{2}} \cdot (\pi_z)^{(j)} \) are \( | \cdot |^{\frac{j+3}{2}} \cdot \text{Ind}_{\mathcal{B}^{k+1,j}}^{GL_{k+1-j}}(\lambda_{i_1,z} \otimes \cdots \otimes \lambda_{i_{k+1-j},z}) \)
or \( | \cdot |^{\frac{j+3}{2}} \cdot \text{Ind}_{\mathcal{B}^{k+1,j}}^{GL_{k+1-j}}(\chi_{i_1,z} \otimes \cdots \otimes \chi_{i_{k+1-j},z} \otimes \mu_{i_1,z} \otimes \cdots \otimes \chi_{i_{k+1-j},z}) \) for subsets \( \{i_1, \ldots, i_{k+1-j}\} \) of \( \{1, \ldots, k+1\} \).

Since the functor \( \text{Hom}(\cdot, X) \) is the contravariant left exact, we are enough to prove that
\[ (3.7) \quad \text{Hom}_{GL_{k+1-j}} \left( \Pi_z, J_{N_{k+1,j}}, \lambda_{k+1}|N_{k+1,j}) \right) \]
for each constituent \( \Pi_z \) of \( \delta_{Q^{k+1,j}}^{-1} \cdot | \cdot |^{\frac{j+3}{2}} \cdot (\pi_z)^{(j)} \).

Note that
\[ J_{N_{k+1,j}}, \lambda_{k+1}|N_{k+1,j}) (J_{U_{k+1,k+1-j}}(J_{\psi^{-1}}(\mathcal{E}^{k+1}))) = J_{\psi^{-1}}(J_{U_{k+1,k+1-j}}(\mathcal{E}^{k+1})). \]
As a representation of $GL_{k+1-j} \times G_j$, if $J_{U_{k+1,j}}(E_{k+1}) \overset{\sim}{=} \bigoplus_{i \in I} A_i \boxtimes B_i$, then

$$J_{\psi_{j-1}}(J_{U_{k+1,j}}(E_{k+1})) \overset{\sim}{=} \bigoplus_{i \in I} A_i \boxtimes J_{\psi_{j-1}}(B_i).$$

Put $I_0 = \{ i \in I : J_{\psi_{j-1}}(B_i) \neq 0 \}$. Since the functor Hom$(Y, \cdot)$ is the covariant left exact, proving (3.7) is reduced to prove

$$(3.8) \quad \text{Hom}_{GL_{k+1-j}}(\Pi_z, A_i) = 0 \text{ for almost all } z$$

for every $i \in I_0$.

If $i_0 \in I_0$, it forces that $B_{i_0}$ should be generic. Then by the geometric lemma and [35, Theorem 1.1], we may assume that $A_{i_0}$ is an irreducible subquotient of a principal series $\delta_{\mathcal{P}_{k+1,k+1-j}}^{\pm} \cdot \text{Ind}_{B_0}^{GL_{k+1-j}}(\mu_1' \otimes \cdots \otimes \mu_{k+1-j}')$ where $\{ \mu_i' \}_{1 \leq i \leq k+1-j}$ are unramified characters such that $e(\mu_1') \in \{ -1 \}$ in case (i) and $e(\mu_1') \in \{ 1 \}$ in case (ii).

If $\text{Hom}_{GL_{k+1-j}}(\Pi_z, \delta_{\mathcal{P}_{k+1,k+1-j}}^{\pm} \cdot \text{Ind}_{B_0}^{GL_{k+1-j}}(\mu_1' \otimes \cdots \otimes \mu_{k+1-j}')) = 0$ for some $z$, then by [7, Theorem 2.9], $\Pi_z$ and $\delta_{\mathcal{P}_{k+1,k+1-j}}^{\pm} \cdot \text{Ind}_{B_0}^{GL_{k+1-j}}(\mu_1' \otimes \cdots \otimes \mu_{k+1-j}')$ have no common constituent and hence $A_{i_0}$ is not a constituent of $\Pi_z$. Therefore, to prove (3.8), we are enough to prove

$$(3.9) \quad \text{Hom}_{GL_{k+1-j}}(\Pi_z, \delta_{\mathcal{P}_{k+1,k+1-j}}^{\pm} \cdot \text{Ind}_{B_0}^{GL_{k+1-j}}(\mu_1' \otimes \cdots \otimes \mu_{k+1-j}')) = 0 \text{ for almost all } z$$

Choose an arbitrary subset $\{ i_1, \ldots, i_{k+1-j} \}$ of $\{ 1, \ldots, k+1 \}$.

When $\Pi_z = | \cdot |^{\frac{3}{2}} \cdot \text{Ind}_{B_0}^{GL_{k+1-j}}(\chi_{i_1,z} \otimes \cdots \otimes \chi_{i_{k+1-j},z} \otimes \mu \cdot | \cdot |^j \otimes \chi_{i_{k+1-j},z} \otimes \cdots \otimes \chi_{i_{k+1-j},z})$, (3.9) is easily verified by comparing the central characters.

Next, we prove (3.9) when $\Pi_z = | \cdot |^{\frac{3}{2}} \cdot \text{Ind}_{B_0}^{GL_{k+1-j}}(\chi_{i_1,z} \otimes \cdots \otimes \chi_{i_{k+1-j},z}).$

If $e(\chi_{i_1,z}) = z + e(\chi_{i_1})$ for some $1 \leq t \leq k+1-j$, then by the same reason above, (3.9) holds. Therefore, we only consider the case $e(\chi_{i_1,z}) = e(\chi_{i_1})$ for all $1 \leq t \leq k+1-j$. Note that $\frac{3}{2} \leq e(| \cdot |^{\frac{3}{2}} \chi_{i_1}) < 2$ for all $1 \leq i \leq n$ and $\delta_{\mathcal{P}_{k+1,k+1-j}}^{\pm} = | \cdot |^{\frac{3}{2}j+2}$.

If $j \geq 3$, then $\min_{1 \leq i \leq k+1-j} e(| \cdot |^{j+1} \cdot \mu_i') > \frac{5}{2}$ and henceforth, (3.9) holds by [7, Theorem 2.9]. Therefore, we are enough to check (3.9) only when $j = 1, 2$.

When $j = 2$, suppose $e(\mu_1') \in \{ 1 \}$. Since $e(\delta_{\mathcal{P}_{k+1,k+1-j}}^{\pm} \mu_1') \geq 2, (3.9)$ holds by [7, Theorem 2.9]. Next, we suppose that $e(\mu_1') \in \{ 1, 2 \}$ and $e(\mu_2') \in \{ 1 \}$. If $e(\mu_1') = 1 + s_0$ or $e(\mu_2') = 1 + s_0$, then (3.9) holds by [7, Theorem 2.9]. Therefore, we consider the case $e(\mu_1') = 1 + s_0$ or $e(\mu_2') = 1 + s_0$. However, since $e(\delta_{\mathcal{P}_{k+1,k+1-j}}^{\pm} \mu_1') + e(\delta_{\mathcal{P}_{k+1,k+1-j}}^{\pm} \mu_2') = 4, e(\delta_{\mathcal{P}_{k+1,k+1-j}}^{\pm} \mu_1') \geq 2$ or $e(\delta_{\mathcal{P}_{k+1,k+1-j}}^{\pm} \mu_2') \geq 2$. Therefore, (3.9) holds by [7, Theorem 2.9].

When $j = 1$, suppose $e(\mu_1') \in \{ 1 \}$. Then $e(\delta_{\mathcal{P}_{k+1,k+1-j}}^{\pm} \mu_1') = 1$ or $2$ and so (3.9) holds by [7, Theorem 2.9]. Next, we suppose that $e(\mu_1') \in \{ 1 \}$ and $e(\mu_2') \in \{ 1, 2 \}$. If $e(\mu_1') = 1 + s_0$ or $e(\mu_2') = 1 + s_0$, then (3.9) holds by [7, Theorem 2.9] in this case. Next we consider the case $e(\mu_1') = 1 + s_0$ or $e(\mu_2') = 1 + s_0$. However, since $e(\delta_{\mathcal{P}_{k+1,k+1-j}}^{\pm} \mu_1') + e(\delta_{\mathcal{P}_{k+1,k+1-j}}^{\pm} \mu_2') = 2, e(\delta_{\mathcal{P}_{k+1,k+1-j}}^{\pm} \mu_1') \leq 1$ or $e(\delta_{\mathcal{P}_{k+1,k+1-j}}^{\pm} \mu_2') \leq 1$. Therefore, (3.9) holds by [7, Theorem 2.9] in this case.
Therefore, (3.8) is verified for $1 \leq j < k+1$, and henceforth, for $1 \leq j < k+1$, $\delta_{P_{k+1,k+1}}(z) = 0$ for almost all $z$.

By applying the functor $\text{Hom}(\cdot, X)$ again, one see that for almost all $z$, (3.4) is a subspace of (3.6) when $j = k+1$, in which case it is isomorphic to $J_{\psi_{k-1}}(E_{k+1})$. However, since $E_{k+1}$ is non-generic by Theorem 1.1, (3.4) is zero for almost all $z$.

Therefore, by applying the functor $\text{Hom}_{P_{k+1,k+1}}(\cdot, E_{k+1})$ to the tensoring of (3.3) with $\delta_{P_{k+1,k+1}}(z)\psi$, $\text{Hom}_{P_{k+1,k+1}}(\delta_{P_{k+1,k+1}}(z)\psi, \omega_{W_{k+1}}, E_{k+1}) \simeq \text{Hom}_{GL_{k+1}}(\delta_{P_{k+1,k+1}}(z)\psi, (|z|^{1/2}, \psi, J_{U_{k+1,k+1}}(E_{k+1}))$ for almost all $z$. By comparing the central characters, $\text{Hom}_{GL_{k+1}}(\delta_{P_{k+1,k+1}}(z)\psi, J_{U_{k+1,k+1}}(E_{k+1}))$ is zero for almost all $z$. This completes the proof.

**Proposition 3.7.** For $n \geq 1$, let $\mu$ and $\{\chi_i\}_{1 \leq i \leq n}$ be unramified characters of $GL_1$ such that for all $1 \leq i \leq n$, $0 < e(\chi_i) < \frac{1}{2}$. For $z \in \mathbb{C}$, put $\mu_z = \mu|z|$. Write $\pi_z = \text{Ind}_{G_{n+1}}^{GL_{n+1}}(\chi_1 \otimes \cdots \otimes \chi_n \otimes \mathbb{C})$. For an arbitrary non-negative integer $l \leq n-1$, let $E_{n-l}$ be an irreducible unitary subquotient of an unramified principal series representation $\text{Ind}_{B_{n-l}}^{G_{n-l}}(\mu_1 \otimes \cdots \otimes \mu_{n-l})$ satisfying one of the following conditions:

(i) $|\mu_z| = |z|$ for some $1 \leq i \leq n-l$. 

(ii) There exists a real number $0 < s_0 < \frac{1}{2}$ such that $e(\mu_z) = 1 + s_0$ and $e(\mu_z) = 1 - s_0$ for some $1 \leq i, j \leq n-l$. Furthermore, there is some $1 \leq l \leq n$ such that $\chi_{l,z} = \chi_l|z|^2$.

Then, $\text{Hom}_{G_{n-l}}(J_{\psi_1} \left( \text{Ind}_{P_{n+1,n+1}}^{G_{n+1}}((\pi_z)\psi) \otimes \omega_{W_{n-l}} \right), E_{n-l}) = 0$ for almost all $z$.

**Proof.** Write $\sigma = \text{Ind}_{G_{n}}^{GL_{n}}(\chi_1 \otimes \cdots \otimes \chi_n \otimes \mathbb{C})$ and $\tau_z = \text{Ind}_{F_{n+1,n+1}}^{GL_{n}}((\pi_z)\psi)$. Then $\text{Ind}_{P_{n+1,n+1}}^{G_{n+1}}((\pi_z)\psi) = \text{Ind}_{P_{n+1,n+1}}^{G_{n+1}}((\sigma_z)\psi)$.

We prove the proposition by induction on $n$. When $n = 1$, $l$ should be 0. By Theorem 3.5, $J_{\psi_0} \left( \text{Ind}_{P_{2,2}}^{G_{2}}((\pi_z)\psi) \otimes \omega_{W_{1}} \right) \simeq \text{Ind}_{P_{1,1}}^{GL_{1}}(|\cdot|^{1/2} \cdot \text{Ind}_{B_{1}}^{GL_{1}}(\chi_1, z)) \bigoplus \tau_z \otimes \omega_{W_1}$, where $A < B$ means that $A$ is a submodule of $B$ up to semi-simplification.

If $\chi_{1,z} = \chi_1|z|^2$, by [7, Theorem 2.9], $\text{Hom}_{G_{1}}(\text{Ind}_{P_{1,1}}^{GL_1}(|\cdot|^{1/2} \cdot \text{Ind}_{B_{1}}^{GL_1}(\chi_1, z)), \text{Ind}_{B_{1}}^{GL_1}(\mu_1)) = 0$ for almost all $z$. If $\chi_{1,z} = \chi_1$, $e(|\cdot|^{1/2} \cdot \chi_{1,z}) < 1$ and by our assumption, $e(\mu_1) = 1$. Again by [7, Theorem 2.9], $\text{Hom}_{G_{1}}(\text{Ind}_{P_{1,1}}^{GL_1}(|\cdot|^{1/2} \cdot \text{Ind}_{B_{1}}^{GL_1}(\chi_1, z)), \text{Ind}_{B_{1}}^{GL_1}(\mu_1)) = 0$. By applying Lemma 3.6 with $k = 0$, we have $\text{Hom}_{G_{1}}(\tau_z \otimes \omega_{W_1}, \text{Ind}_{B_{1}}^{GL_1}(\mu_1)) = 0$. This completes the proof for the case $n = 1$.

Now, suppose that the proposition holds when $n = k$ and we prove when $n = k+1$. Choose an arbitrary integer $0 \leq l_0 \leq k$.

Choose $\pi_z = \text{Ind}_{B_{n+1}}^{GL_{n+k+2}}(\chi_1 \otimes \cdots \otimes \chi_{k+1} \otimes \mathbb{C})$ and $E_{k+1-l_0}$ an irreducible subquotient of $\text{Ind}_{B_{k+1-l_0}}^{GL_{k+1-l_0}}(\mu_1 \otimes \cdots \otimes \mu_{k+1-l_0})$ satisfying the conditions of the proposition. Write $\sigma = \text{Ind}_{B_{n+1}}^{GL_{n+k+1}}(\chi_1 \otimes \cdots \otimes \chi_{k+1})$. If there
exists a real number $0 < s_0 < \frac{1}{2}$ such that $e(\mu_i) = 1 + s_0$ and $e(\mu_j) = 1 - s_0$ for some $1 \leq i, j \leq k + 1 - l_0$, then by the assumption, there exists some $1 \leq t_0 \leq k + 1$ satisfying $\chi_{t_0,z} = \chi_{t_0} \cdot |z|$. In this case, we put $t = t_0$. If there is no such $0 < s_0 < \frac{1}{2}$, $t$ denotes an arbitrary integer between 1 and $k + 1$.

Put $\sigma_t = \text{Ind}_{\text{B}_k}^{GL_k} (\chi_1 \otimes \cdots \otimes \chi_{t-1} \otimes \chi_{t+1} \otimes \cdots \otimes \chi_{k+1})$, $\sigma_{t,z} = \text{Ind}_{\text{B}_k}^{GL_k} (\chi_{1,z} \otimes \cdots \otimes \chi_{t-1,z} \otimes \chi_{t+1,z} \otimes \cdots \otimes \chi_{k+1,z})$ and $\pi_{t,z} = \text{Ind}_{\text{B}_{k+1}}^{\text{GL}_{k+1}} ((\sigma_t,z) \psi \otimes \pi_z)$. Note that $\text{Ind}_{\text{B}_{k+2}}^{\text{GL}_{k+2}} ((\chi_{t,z}) \psi \otimes \pi_{t,z})$.

By applying Theorem 3.5 with $j = 1$, $n = k + 1$, $\mu = \chi_{t,z}$, $\tau = \pi_{t,z}$, we have

$$J_{\psi_{t_0}} \left( \text{Ind}_{\text{B}_{k+1}}^{\text{GL}_{k+1}} ((\chi_{t,z}) \psi \otimes \Omega_{k+1-l_0}) \right) \overset{ss}{=} \text{Ind}_{\text{B}_{k+1}}^{\text{GL}_{k+1}} ((\chi_{t,z}) \psi \otimes \pi_{t,z}) \otimes \Omega_{k+1-l_0}) \overset{ss}{=}$$

$$\bigoplus_{0 \leq s \leq 1, k-l_0 \leq s \leq k-l_0} \text{Ind}_{\text{B}_{k+1-l_0}}^{\text{GL}_{k+1-l_0}} ((1 \cdot |z| \cdot \chi_{t,z}(1-s) \otimes J_{\psi_{t_0-s}}(\pi_{t,z} \otimes \Omega_{k+1-l_0}) \bigoplus \delta_{0,l_0} \cdot (\pi_{t,z} \otimes \Omega_{k+1-l_0}))$$

up to semi-simplification.

If $l_0 = 0$, by Proposition 3.4, $\text{Hom}_{G_{k+1}}(\pi_{t,z} \otimes \Omega_{k+1-l_0}, \mathcal{E}_{k+1}) = 0$ for almost all $z$. Since the functor $\text{Hom}(\cdot, \mathcal{E}_{k+1-l_0})$ is the contravariant left exact, we are enough to show that for each $0 \leq s \leq \min\{1, l_0\}$,

$$\text{Hom}_{G_{k+1-l_0}}(\text{Ind}_{\text{B}_{k+1-l_0}}^{\text{GL}_{k+1-l_0}} ((1 \cdot |z| \cdot \chi_{t,z}(1-s) \otimes J_{\psi_{t_0-s}}(\pi_{t,z} \otimes \Omega_{k+1-l_0}))), \mathcal{E}_{k+1-l_0}) = 0$$

for almost all $z$.

By the Frobenius reciprocity in [8] Theorem 2.29, for each $0 \leq s \leq \min\{1, l_0\}$,

$$\text{Hom}_{G_{k+1-l_0}}(\text{Ind}_{\text{B}_{k+1-l_0}}^{\text{GL}_{k+1-l_0}} ((1 \cdot |z| \cdot \chi_{t,z}(1-s) \otimes J_{\psi_{t_0-s}}(\pi_{t,z} \otimes \Omega_{k+1-l_0}))), \mathcal{E}_{k+1-l_0})$$

$$\simeq \text{Hom}_{\text{B}_{k+1-l_0}}((\delta_{1}^{\otimes 1} \cdot | \cdot |^{\Delta_{1}}, \mathcal{E}_{k+1-l_0} \otimes \Omega_{k+1-l_0}))) \simeq \text{Hom}_{\text{B}_{k+1-l_0}}((\delta_{1}^{\otimes 1} \cdot | \cdot |^{\Delta_{1}}, \mathcal{E}_{k+1-l_0} \otimes \Omega_{k+1-l_0})))$$

By the geometric lemma, we can decompose $J_{\psi_{t_0-s}}^{\partial}(\pi_{t,z} \otimes \Omega_{k+1-l_0}) \overset{ss}{=} \bigoplus_{i \in I} A_i \otimes B_i$, where $A_i$ and $B_i$ are an irreducible subquotient of $\text{Ind}_{\text{B}_{k+1-l_0}}^{\text{GL}_{k+1-l_0}}((\delta_{1}^{\otimes 1} \cdot | \cdot |^{\Delta_{1}}, \mathcal{E}_{k+1-l_0} \otimes \Omega_{k+1-l_0})))$ and $\text{Ind}_{\text{B}_{k+1-l_0}}^{\text{GL}_{k+1-l_0}}((\delta_{2}^{\otimes 1} \cdot | \cdot |^{\Delta_{2}}, \mathcal{E}_{k+1-l_0} \otimes \Omega_{k+1-l_0})))$, respectively. (Here, $p$ is a permutation of $\{1, 2, \cdots, n-k-l_0\}$ and $\epsilon$ is a function $\epsilon : \{1, 2, \cdots, n-k-l_0\} \mapsto \{ \pm 1 \}$.)

Since the functor $\text{Hom}(X, \cdot)$ is the covariant left exact, we are enough to show that for each $i \in I$,

$$\text{Hom}_{\text{B}_{k+1-l_0}}((\delta_{1}^{\otimes 1} \cdot | \cdot |^{\Delta_{1}}, \mathcal{E}_{k+1-l_0} \otimes \Omega_{k+1-l_0}))) \simeq \text{Hom}_{\text{B}_{k+1-l_0}}((\delta_{2}^{\otimes 1} \cdot | \cdot |^{\Delta_{2}}, \mathcal{E}_{k+1-l_0} \otimes \Omega_{k+1-l_0})))$$

(3.10) $\text{Hom}_{\text{B}_{k+1-l_0}}((\delta_{1}^{\otimes 1} \cdot | \cdot |^{\Delta_{1}}, \mathcal{E}_{k+1-l_0} \otimes \Omega_{k+1-l_0}))) = 0$ for almost all $z$ for each $0 \leq s \leq \min\{1, l_0\}$.

When $s = 1$, (3.10) holds for all $i \in I$ by the induction hypothesis with $n = k, l = l_0 - 1$.

Next we consider the case $s = 0$. If $\chi_{t,z} = \chi_{t} \cdot |z|$, then by comparing the central characters, (3.10) holds for all $i \in I$. If $\chi_{t,z} = \chi_{t}$ for all $1 \leq j \leq k + 1$, then by the condition on $\mathcal{E}_{k+1-l_0}$, there is some $1 \leq j_0 \leq k + 1 - l_0$ such that $e(\mu_{j_0}) = 1$. Fix an arbitrary $i_0 \in I$. If $A_{i_0}$ is an irreducible subquotient of $\text{Ind}_{\text{B}_{k+1-l_0}}^{\text{GL}_{k+1-l_0}}((\delta_{1}^{\otimes 1} \cdot | \cdot |^{\Delta_{1}}, \mathcal{E}_{k+1-l_0} \otimes \Omega_{k+1-l_0})))$, (3.10) holds by [7] Theorem 2.9 because $\frac{1}{2} \leq e(| \cdot |^{\Delta_{1}} \cdot \chi_{t,z}) < 1$. If $A_{i_0}$ is not an irreducible subquotient of $\text{Ind}_{\text{B}_{k+1-l_0}}^{\text{GL}_{k+1-l_0}}((\delta_{1}^{\otimes 1} \cdot | \cdot |^{\Delta_{1}}, \mathcal{E}_{k+1-l_0} \otimes \Omega_{k+1-l_0})))$ such that $p(r) = j_0$ for some $2 \leq r \leq k + 1 - l_0$. Then $l_0 \leq k - 1$.

By the induction hypothesis with $n = k, l = l_0$, (3.10) holds for $i = i_0$. This completes the proof. 

□
The following proposition is an analog of Proposition 3.7.

**Proposition 3.8.** For $n \geq 1$, let $\mu$ and $\{\chi_i\}_{1 \leq i \leq n}$ be unramified characters of $GL_1$ such that for all $1 \leq i \leq n$, $-\frac{1}{2} < e(\lambda_i) < \frac{1}{2}$. For $z \in \mathbb{C}$, put $\mu_z = \mu|z|^2$ and $\chi_{i,z} = \chi_i$ or $|\chi_i||z|^2$. Write $\sigma_z = Ind_{B_n}(\chi_1,z \ldots \chi_n,z)$ and $\tau_z = Ind_{P_n}(\mu_z)$. For an arbitrary non-negative integer $l \leq n - 1$, let $\mathcal{E}_{n-l}$ be an irreducible unitary subquotient of an unramified principal series representation $Ind_{B_{n-l}}(\mu_1, \psi \otimes \ldots \otimes \mu_{n-l}, \psi)$ satisfying the following condition:

- There exists a real number $0 \leq s_0 < \frac{1}{2}$ such that $e(\mu_i) = 1 + s_0$ and $e(\mu_j) = 1 - s_0$ for some $1 \leq i, j \leq n - l$. In this case, there is some $1 \leq t \leq n$ such that $\chi_{t,z} = |\chi_t||z|^2$.

Then, $$\text{Hom}_{\widetilde{G}_{n-l}}(J_{\psi_l} \left( \text{Ind}_{P_{n+1,n}}(\sigma_z \otimes \tau_z) \otimes \Omega_{W_{n-l}} \right), \mathcal{E}_{n-l}) = 0$$ for almost all $z$.

The proof of Proposition 3.8 closely mirrors that of Proposition 3.7; therefore, to avoid redundancy, we will highlight only the key differences. In applying the argument from Proposition 3.7 to Proposition 3.8 we encounter the absence of an analogous version of [7, Theorem 2.9] for metaplectic groups. However, one of the key ingredients of the proof of [7, Theorem 2.9] is the geometric lemma for reductive groups. Since [23, Section 3.3] supplies all the necessary components to establish the geometric lemma for metaplectic groups, the proof can be adapted almost verbatim from [7, Theorem 2.9].

4. Residual representation

In this section, we introduce the residual Eisenstein series representations and review some of their properties related to $L$-functions. Let $\pi$ and $\pi'$ be irreducible cuspidal automorphic representations of $G_k(\mathbb{A})$ and $\widetilde{G}_k(\mathbb{A})$ with generic $A$-parameters, respectively, and let $\sigma$ be an irreducible unitary cuspidal automorphic representation of $GL_a(\mathbb{A})$.

For $g \in \widetilde{G}_{k+a}(\mathbb{A})$ (resp. $G_{k+a}(\mathbb{A})$), write $g = m_ugkg$ for $m_ug \in \widetilde{M}_{k+a,a}(\mathbb{A})$ (resp. $m_ug \in M_{k+a,a}(\mathbb{A})$), $u_g \in U_{k+a,a}(\mathbb{A})$, $k_g \in K_{k+a,a}$ (resp. $k_g \in K_{k+a,a}$). Since $\widetilde{M}_{k+a,a} \simeq GL(X_a) \times \mu_2 \widetilde{G}_k$ (resp. $M_{k+a,a} \simeq GL(X_a) \times \mu_2 G_k$), we decompose $m_ug = (m_1, \epsilon) \cdot (m_2, \epsilon)$ (resp. $m_ug = m_1 \cdot m_2$), where $m_1 \in GL_a(\mathbb{A}), m_2 \in G_k(\mathbb{A}), \epsilon \in \mu_2$. Define $d(g) = |\det m_1|_\mathbb{A}$. We set $\sigma_\psi(g, \epsilon) = \epsilon \cdot \sigma(g) \prod_\psi \gamma(\det g_v, \psi_v^{-1})^{-1}$ for $g \in GL_a(\mathbb{A})$ and $\epsilon \in \mu_2$. For every $a \in \mathbb{A}$, the product $\prod_\psi \gamma(a_v, \psi_v^{-1})$ is well defined in view of [23, Proposition A.11]. Note that $\sigma_\psi \otimes \pi'$ (resp. $\sigma \otimes \pi$) is a representation of $\widetilde{M}_{k+a,a}(\mathbb{A})$ (resp. $M_{k+a,a}(\mathbb{A})$). Let $\phi$ be an element in $\sigma_\psi \otimes \pi$ (resp. $\sigma \otimes \pi$). For any $z \in \mathbb{C}$, write $\phi_z := d^z \cdot \phi$. Then $\phi_z$ belongs to $\sigma_{\psi, \pi'}(G_{k+a})$ (resp. $\sigma_{\psi, \pi}(G_{k+a})$) for all $z \in \mathbb{C}$ and call $\phi_z$ a flat section. Note that the restriction of a flat section $\phi_z$ to $K_{k+a,a}$ (resp. $K_{k+a,a}$) is the restriction of $\phi$ to $K_{k+a,a}$ (resp. $K_{k+a,a}$), which is independent of $z \in \mathbb{C}$. Furthermore, the Eisenstein series associated to $\phi_z$ is defined by

$$E(g, \phi_z, 0) = E(g, \phi, z) := \sum_{P_{k+a,a}(F) \backslash \widetilde{G}_{k+a}(F)} \phi(g) \overline{d(g)}.$$ 

This series converges absolutely when $\text{Re}(z)$ is sufficiently large and admits a meromorphic continuation to the whole complex plane. [32, IV, 1.8.]

As is well known, the analytic properties of the Eisenstein series are governed by the global intertwining operator $M_z$. Here, we review the definition of $M_z$. Let $w_0$ be a Weyl element in $G_{k+a}$ which takes $U_{k+a,a}$
to its opposite \( U_{k+a,a}^- \). Then for an arbitrary \( f \in \mathcal{A}_{k+a,a}^{f|\sigma|\psi}\otimes (\hat{G}_{k+a}) \) (resp. \( \mathcal{A}_{k+a,a}^{f|\sigma|\psi}\otimes (G_{k+a}) \)), \( M_z(f) \) is defined by

\[
M_z(f)(\cdot) := \int_{[U_{k+a,a}]} f(w_0^{-1}u)du.
\]

Then, \( M_z : I(z, \sigma \otimes \pi') \rightarrow I(-z, w_0(\sigma \otimes \pi')) \) (resp. \( M_z : I(z, \sigma \otimes \pi) \rightarrow I(-z, w_0(\sigma \otimes \pi)) \)) is a \( \hat{G}_{k+a}(\mathbb{A}) \) (resp. \( G_{k+a}(\mathbb{A}) \))-invariant map. If \( f = \otimes_v f_v \) is factorizable, then \( M_z(f) = \prod_v M_{z,v}(f_v) \), where \( M_{z,v} : \text{Ind}_{P(F_v)} G_h(F_v) \delta_P^z \cdot w_0(\sigma_{\psi,v} \otimes \pi_v') \rightarrow \text{Ind}_{P(F_v)} G_h(F_v) \delta_P^z \cdot w_0(\sigma_{\psi,v} \otimes \pi_v') \) (resp. \( M_{z,v} : \text{Ind}_{P(F_v)} G_h(F_v) \delta_P^z \cdot (\sigma_{\psi,v} \otimes \pi_v') \rightarrow \text{Ind}_{P(F_v)} G_h(F_v) \delta_P^z \cdot w_0(\sigma_{\psi,v} \otimes \pi_v')) \) is the local intertwining operator. (Note that \( M_{z,v} \) is associated to \( \sigma_v, \pi_v' \) (resp. \( \pi_v, \psi_v \) and \( w_0 \)).)

Define the local normalizing operator \( N_{z,v} \) by multiplying \( M_{z,v} \) with the normalizing factor

\[
\alpha_v := \frac{L_v(z + 1, \sigma_v \otimes \pi_v') \cdot L_v(2z + 1, \sigma_v, \text{Sym}^2) \cdot \epsilon_v(z, \sigma_v \otimes \pi_v', \psi_v) \cdot \epsilon_v(2z, \sigma_v, \text{Sym}^2, \psi_v)}{L_v(z, \sigma_v \otimes \pi_v') \cdot L_v(2z, \sigma_v, \text{Sym}^2)}
\]

(resp. \( \alpha_v := \frac{L_v(z + 1, \sigma_v \otimes \pi_v) \cdot L_v(2z + 1, \sigma_v, \text{Sym}^2) \cdot \epsilon_v(z, \sigma_v \otimes \pi_v, \psi_v) \cdot \epsilon_v(2z, \sigma_v, \text{Sym}^2, \psi_v)}{L_v(z, \sigma_v \otimes \pi_v) \cdot L_v(2z, \sigma_v, \text{Sym}^2)} \)

where the local \( L \)-factors and \( \epsilon \)-factors are defined through the localization of the global \( A \)-parameters of \( \pi' \) and \( \pi \) as described in [1] and [13]. Then

\[
(4.1) \quad M_z(f) = \frac{L_v(z, \sigma \otimes \pi') \cdot L(2z, \sigma, \text{Sym}^2)}{L_v(z + 1, \sigma \otimes \pi') \cdot L(2z + 1, \sigma, \text{Sym}^2) \cdot \epsilon_v(z, \sigma \otimes \pi', \psi) \cdot \epsilon_v(2z, \sigma, \text{Sym}^2, \psi)} \prod_v N_{z,v}(f_v)
\]

\[
(4.2) \quad \text{resp.} \quad M_z(f) = \frac{L(z, \sigma \otimes \pi) \cdot L(2z, \sigma, \text{Sym}^2)}{L(z + 1, \sigma \otimes \pi) \cdot L(2z + 1, \sigma, \text{Sym}^2) \cdot \epsilon(z, \sigma \otimes \pi, \psi) \cdot \epsilon(2z, \sigma, \text{Sym}^2, \psi)} \prod_v N_{z,v}(f_v),
\]

where \( L_v(z, \sigma \otimes \pi') \) (resp. \( L(z, \sigma \otimes \pi) \)) is the Rankin-Selberg \( L \)-function \( L(z, \sigma \otimes \text{FL}_v(\psi')) \) (resp. \( L(z, \sigma \otimes \text{FL}(\pi')) \)).

The following is Remark [11] (iv), which is an analog of [28, Theorem 5.1] and can be proved in the same way.

**Proposition 4.1** (cf. [29, Theorem 4.1]). Let \( \pi' \) (resp. \( \pi \)) be an irreducible cuspidal automorphic representation of \( \hat{G}_k(\mathbb{A}) \) (resp. \( G_k(\mathbb{A}) \)) with generic \( A \)-parameter and \( \sigma \) an irreducible unitary cuspidal automorphic representations of \( GL_a(\mathbb{A}) \). Choose an arbitrary factorizable \( \phi = \otimes_v \phi_v \in \mathcal{A}_{k+a,a}^{f|\sigma \otimes \pi'}(\hat{G}_{k+a}) \) (resp. \( \mathcal{A}_{k+a,a}^{f|\sigma \otimes \pi}(G_{k+a}) \)). Then for each place \( v \) of \( F \), \( N_{z,v}(\phi_v, z) \) is holomorphic and nonzero for \( z \geq \frac{1}{2} \).

Some properties of automorphic representations with generic \( A \)-parameters (i.e., Remark [11] (i),(ii),(iii),(iv)) play essential roles in the proofs of the following propositions.

**Proposition 4.2.** Let \( \pi \) be an irreducible automorphic representation of \( G_k(\mathbb{A}) \) with generic \( A \)-parameter and \( \sigma \) an irreducible unitary cuspidal automorphic representation of \( GL_a(\mathbb{A}) \). For \( \phi \in \mathcal{A}_{k+a,a}^{f|\sigma \otimes \pi}(G_{k+a}) \), the Eisenstein series \( E(\phi, z) \) has at most a simple pole at \( z = \frac{1}{2} \) and \( z = 1 \). Moreover, it has a pole at \( z = \frac{1}{2} \) as \( \phi \) varies if and only if \( L(s, \sigma \otimes \pi) \) is non-zero at \( s = \frac{1}{2} \) and \( L(s, \sigma, \text{Sym}^2) \) has a pole at \( z = 1 \). Furthermore, it has a pole at \( s = 1 \) if and only if \( L(s, \sigma \otimes \pi) \) has a pole at \( s = 1 \).

**Proof.** By the Langlands theory of Eisenstein series, the analytic properties of the family of \( E(\phi, z) \) are controlled by those of the family

\[
E_{P_{k+a,a}}(\phi, z) = \phi_z + M_z(\phi_z).
\]
Therefore, $E(\phi, z)$ has a pole at $z = z_0$ of order $r$ if and only if $M_z(\phi_z)$ has a pole at $z = z_0$ of order $r$. For a factorizable $\phi = \otimes_v \phi_v \in \mathcal{A}_\phi^{G, \pi}(G_{k+a})$, (4.4) and Proposition 4.1 tells that $M_z(\phi_z)$ has a pole at $z = z_0$ of order $r$ if and only if

$$L(z, \sigma \times \pi) \cdot L(2z + 1, \sigma, \wedge^2)$$

has a pole at $z = z_0$ of order $r$.

By [25, 41] and [19, Theorem 1.3], $L(z, \sigma \times \pi)$ and $L(z, \sigma, \wedge^2)$ converges absolutely and does not vanish for $\text{Re}(z) \geq 1$. Furthermore, $L(z, \sigma \times \pi)$ and $L(z, \sigma, \wedge^2)$ are holomorphic on $0 < \text{Re}(z) < 1$ and they have possible simple pole at $z = 1$. By putting all these together, the proposition follows.

**Proposition 4.3.** Assume that $\sigma$ is isomorphic to one of the isobaric summands of FL(\pi). Then there exists $\phi \in \mathcal{A}_\phi^{G, \pi}(G_{k+a})$ such that $E(\phi, z)$ has a pole at $z = 1$.

**Proof.** Note that $L(z, \sigma \times \sigma) = L(z, \sigma, \wedge^2) \cdot L(z, \sigma, \text{Sym}^2)$. It is known that neither $L(z, \sigma, \wedge^2)$ and $L(s, \sigma, \text{Sym}^2)$ has a zero at $z = 1$. By Remark 4.1 (ii), $L(s, \sigma, \text{Sym}^2)$ has a simple pole at $s = 1$. Therefore, $L(s, \sigma \times \sigma)$ has a simple pole at $s = 1$ and it implies that $\sigma = \sigma'$. Then the proposition follows from Proposition 4.2.

Since the proofs of the following propositions are almost same with the above propositions, we omit the proofs.

**Proposition 4.4.** Let $\pi'$ be an irreducible cuspidal automorphic representation of $\widetilde{G}_k(\mathbb{A})$ with generic $A$-parameter and $\sigma$ an irreducible cuspidal automorphic representation of $GL_a(\mathbb{A})$. For $\phi' \in \mathcal{A}_{\phi'}^{G, \pi'}(G_{k+a})$, the Eisenstein series $E(\phi', z)$ has at most a simple pole at $z = \frac{1}{2}$ and $z = 1$. Moreover, it has a pole at $z = \frac{1}{2}$ as $\phi'$ varies if and only if $L(\psi(s, \sigma \times \pi'))$ is non-zero at $s = \frac{1}{2}$ and $L(s, \sigma, \text{Sym}^2)$ has a pole at $z = 1$.

**Proof.** It directly follows from the computation of the constant terms of the Eisenstein series $E(\phi', z)$ in [16, Sec. 3.2].

**Proposition 4.5 ([48, Proposition 5.3]).** Assume that $\sigma$ is isomorphic to one of the isobaric summands of FL_{\psi^{-1}}(\pi')$. Then there exists $\phi' \in \mathcal{A}_{\phi'}^{G, \pi'}(G_{k+a})$ such that $E(\phi', z)$ has a pole at $z = 1$.

**Remark 4.6.** When $\pi'$ and $\pi$ are irreducible globally generic cuspidal representations of $\widetilde{G}_{k+a}(\mathbb{A})$ and $G_{k+a}(\mathbb{A})$, respectively, then Proposition 4.2–4.5 are proved in [16, Proposition 3.2], [48, Proposition 5.1, Proposition 5.3].

Thanks to Proposition 4.2 and Proposition 4.3, we can define the residues of the Eisenstein series to be the limits

$$\mathcal{E}^0(\phi) = \lim_{z \to \frac{1}{2}} (z - \frac{1}{2}) \cdot E(\phi, z), \quad \mathcal{E}^1(\phi) = \lim_{z \to 1} (z - 1) \cdot E(\phi, z), \quad \phi \in \mathcal{A}_\phi^{G, \pi}(G_{k+a}).$$

Let $\mathcal{E}^0(\sigma, \pi)$ (resp. $\mathcal{E}^1(\sigma, \pi)$) be the residual representation of $G_{k+a}(\mathbb{A})$ generated by $\mathcal{E}^0(\phi)$ (resp. $\mathcal{E}^1(\phi)$) for $\phi \in \mathcal{A}_\phi^{G, \pi}(G_{k+a})$. Thanks to Proposition 4.4 and Proposition 4.5, we can similarly define the residues of the Eisenstein series to be the limits

$$\mathcal{E}^0(\phi') = \lim_{z \to \frac{1}{2}} (z - \frac{1}{2}) \cdot E(\phi', z), \quad \mathcal{E}^1(\phi') = \lim_{z \to 1} (z - 1) \cdot E(\phi', z), \quad \phi' \in \mathcal{A}_{\phi'}^{G, \pi'}(G_{k+a}).$$
Let $\mathcal{E}^0(\sigma, \pi')$ (resp. $\mathcal{E}^1(\sigma, \pi')$) be the residual representation of $\widetilde{G}_{k+a}$ generated by $\mathcal{E}^0(\varphi')$ (resp. $\mathcal{E}^1(\varphi')$) for $\varphi' \in \mathcal{A}_{\mathcal{P}_{k+a,a}}(\widetilde{G}_{k+a})$.

**Remark 4.7.** The condition $L(s, \sigma, \wedge^2)$ has a pole at $s = 1$ implies that $a$ is even and that $\sigma$ is self-dual and has trivial character (see [26]). Furthermore, by [44 Cor 5.11], the condition $L(s, \sigma, \text{Sym}^2)$ has a pole at $s = 1$ implies that $\sigma$ is self-dual and $\omega_\sigma$, the central character of $\sigma$, is quadratic. (i.e. $\omega_\sigma^2 = 1$.)

## 5. Reciprocal Non-vanishing of the Fourier–Jacobi periods

In this section, we shall prove our main theorem. To do this, we should prove a reciprocal non-vanishing theorem of the Fourier–Jacobi periods which has its own value but also crucial to prove our main theorem.

Let $m, a$ be positive integers and $r$ non-negative integer. Write $n = m + r$ and let $(W_{n+a}, \langle \cdot, \cdot \rangle)$ be a symplectic spaces over $F$ of dimension $2(n + a)$. Fix maximal totally isotropic subspaces $X$ and $X^*$ of $W_{n+a}$, in duality, with respect to $\langle \cdot, \cdot \rangle$. Fix a complete flag in $X$

$$0 = X_0 \subset X_1 \subset \cdots \subset X_{n+a} = X,$$

and choose a basis $\{e_1, e_2, \ldots, e_{n+a}\}$ of $X$ such that $\{e_1, \ldots, e_i\}$ is a basis of $X_i$ for each $1 \leq i \leq n + a$. Let $\{f_1, f_2, \ldots, f_{n+a}\}$ be the basis of $X^*$ which is dual to the fixed basis $\{e_1, e_2, \ldots, e_{n+a}\}$ of $X$, i.e., $\langle e_i, f_j \rangle = \delta_{ij}$ for $1 \leq i, j \leq n + a$, where $\delta_{ij}$ denotes the Kronecker delta. For each $1 \leq i \leq n + a$, write $X_i^*$ for the subspace of $X^*$ spanned by $\{f_1, f_2, \ldots, f_i\}$ and write $Z_m$ (resp. $Z_m^*$) be the subspace of $X$ (resp. $X^*$) generated by $e_{r+a+1}, \ldots, e_{n+a}$ (resp. $f_{r+a+1}, \ldots, f_{n+a}$). We also write $Y_a$ (resp. $Y_a^*$) for the subspace of $X$ (resp. $X^*$) spanned by $e_{r+1}, \ldots, e_{r+a}$ (resp. $f_{r+1}, \ldots, f_{r+a}$). Then $X$ (resp. $X^*$) has the polar decomposition $X = X_r + Y_a + Z_m$ (resp. $X^* = X_r^* + Y_a^* + Z_m^*$). For a $F$-dimensional vector space $V$, let $\mathcal{S}(V)(\mathbb{A})$ be the Schwartz space on $V(\mathbb{A})$. For an arbitrary positive integer $k$, the Schodinger model gives rise to the global Weil representation $\Omega_{\psi^{-1},W_h}$ of $H_{W_h}(\mathbb{A}) \rtimes \widetilde{G}_{k}(\mathbb{A})$ on $\mathcal{S}(X_k^*)(\mathbb{A})$. As in Sec. 3, we similarly define a generic character $\lambda_r$ of $N_{r}(\mathbb{A})$. Then $\nu_{\psi^{-1},W_h} = \lambda_r^{-1} \otimes \Omega_{\psi^{-1},W_h}$ is a generic global Weil representation of $H_{k+r,r}(\mathbb{A}) := N_{k+r,r}(\mathbb{A}) \rtimes \widetilde{G}_{k}(\mathbb{A})$. Via the natural inclusion maps $N_{n,r} \hookrightarrow N_{n+r,r}$, $\tilde{G}_{m} \hookrightarrow \tilde{G}_{m+a}$, we regard elements of $N_{n,r}$ and $\tilde{G}_{m}$ as those of $N_{n+r,r}$ and $\tilde{G}_{m+a}$, respectively. For $h \in GL(Y_a)$, put $h^* \in GL(Y_a)$ satisfying $\langle hx, y \rangle = \langle x, h^*y \rangle$. Write $S = (S(Y_a^*) \otimes S(Y_a^*))(\mathbb{A})$. Since $S \simeq S(Y_a^*)(\mathbb{A}) \otimes S(Z_m^*)(\mathbb{A})$, we describe the (partial) $N_{n+r,r}(\mathbb{A}) \rtimes \tilde{G}_{m+a}(\mathbb{A})$-action of $\nu_{\psi^{-1},W_m+a}$ on $\mathcal{S}(Y_a^*)(\mathbb{A}) \otimes S(Z_m^*)(\mathbb{A})$ as follows: (see [3] pp. 58-59.)

For $f_1 \otimes f_2 \in \mathcal{S}(Y_a^*)(\mathbb{A}) \otimes S(Z_m^*)(\mathbb{A})$ and $y \in Y_a^*(\mathbb{A})$, $z \in Z_m^*(\mathbb{A})$,

(1.1)

$$\langle \nu_{\psi^{-1},W_m+a}((n,1))(f_1 \otimes f_2), (y, z) \rangle = f_1(y) \cdot \langle \nu_{\psi^{-1},W_m+a}((n,1))(f_2), (z) \rangle, \quad n \in N_{n,r}(\mathbb{A}),$$

(1.2)

$$\langle \nu_{\psi^{-1},W_m+a}((1,\tilde{g}_0))(f_1 \otimes f_2), (y, z) \rangle = f_1(y) \cdot \langle \nu_{\psi^{-1},W_m+a}((1,\tilde{g}_0))(f_2), (z) \rangle, \quad \tilde{g}_0 \in \tilde{G}_m(\mathbb{A}),$$

(1.3)

$$\langle \nu_{\psi^{-1},W_m+a}((1,(h_0,\zeta)))(f_1 \otimes f_2), (y, z) \rangle = \langle (\cdot \frac{1}{2})_{\psi^{-1}}(\det(h_0),\zeta) \cdot f_1(h_0)(y) \cdot f_2(z), (h_0,\zeta) \in GL(Y_a)(\mathbb{A}),$$

(1.4)

$$\langle \Omega_{\psi^{-1},W_m+a}((x+x',0))(f_1 \otimes f_2), (y, z) \rangle = \psi^{-1}\left(\langle y, x \rangle + \frac{1}{2} \langle x', x \rangle \right) f_1(y+x') \cdot f_2(z), \quad x \in Y_a(\mathbb{A}), x' \in Y_a^*(\mathbb{A}).$$
For \( f \in S = S(Y_a^* + Z_m^*)(\mathbb{A}) \), its associated theta function \( \Theta_{\psi^{-1},W_{m+a}}(\cdot, f) \) is defined by
\[
\Theta_{\psi^{-1},W_{m+a}}(h, f) = \sum_{y \in (Y_a^* + Z_m^*)(F)} \left( \nu_{\psi^{-1},W_{m+a}}(h)f \right)(y),
\]
where \( h = ((u, n), g) \in \left( \prod_{r \neq 1} \mathbb{A}_r \right) \times \tilde{G}_{m+a} \). When \( f \in S \) is a pure tensor \( f_1 \otimes f_2 \in S(Y_a^*)(\mathbb{A}) \otimes S(Z_m^*)(\mathbb{A}) \), then
\[
\Theta_{\psi^{-1},W_{m+a}}(h, f) = \sum_{y \in (Y_a^*)(F), z \in Z_m^*(F)} \left( \Omega_{\psi^{-1},Y_a^*+Y_m^*}(h)f_1 \right)(y) \cdot \left( \nu_{\psi^{-1},W_{m+a}}(h)f_2 \right)(z).
\]
For each \( f \in S \), \( \Theta_{\psi^{-1},W_{m+a}}(f) \) is an automorphic form and the map \( f \to \Theta_{\psi^{-1},W_{m+a}}(f) \) gives an automorphic realization of \( \nu_{\psi^{-1},W_{m+a}} \). Since we have fixed \( \psi \), we suppress \( \psi \) from the notation and write \( \nu_{W_{m+a}} \) and \( \Theta_{W_{m+a}}(f) \) by \( \nu_W \) and \( \Theta_W(f) \), respectively. We also simply write \( \Theta_{P_{m+a,a}}(f) \) for the constant term of \( \Theta_{W_{m+a}}(f) \) along \( P_{m+a,a} \).

The following is an analogue of [20, Lemma 9.1] whose proof is identically same.

Lemma 5.1 (cf. [18, Lemma 6.2]). For \( h \in H_{n+a,r}(\mathbb{A}) \), \( f \in S \),
\[
\Theta_{P_{m+a,a}}(h, f) = \sum_{z \in Z_m^*(F)} \left( \nu_{W_{m+a}}(h)f \right)(0, z).
\]

Remark 5.2. Let \( e \) be the identity element of \( H_{n+a,r}(\mathbb{A}) = N_{n+a,r}(\mathbb{A}) \times G_{m+a}(\mathbb{A}) \). If \( f \in S \) is a pure tensor \( f_1 \otimes f_2 \in S(Y_a^*)(\mathbb{A}) \otimes S(Z_m^*)(\mathbb{A}) \), then
\[
\Theta_{P_{m+a,a}}(e, f_1 \otimes f_2) = f_1(0) \cdot \left( \sum_{z \in Z_m^*(F)} f_2(z) \right).
\]

Since every Schwartz function in \( S(Z_m^*)(\mathbb{A}) \) can be obtained as evaluation at \( 0 \in Y_a^* \) of some Schwartz function in \( S(Y_a^* + Z_m^*)(\mathbb{A}) \), we can regard theta functions in \( \nu_{W_{m+a}} \) as the evaluation at \( e \) of the constant term of theta functions in \( \nu_{W_{m+a}} \) along \( P_{m+a,a} \). This observation will be used in the proof of Lemma 5.6.

Remark 5.3. From Lemma 5.1 and (5.3) we have
\[
\Theta_{P_{m+a,a}}\left((u, (h_0, \varsigma)g), f\right) = (\cdot | \frac{1}{2})_{\psi^{-1}}(\det(h_0), \varsigma) \cdot \Theta_{P_{m+a,a}}\left((u, g), f\right)
\]
for \( u \in N_{n+a,r}, (h_0, \varsigma) \in GL(Y_a(\mathbb{A})) \) and \( g \in G_{m+a} \). Thus from the the Lemma 5.1 and \([5.1], [5.2], [5.3]\), we see that the constant terms of theta functions \( \{ \Theta_{P_{m+a,a}}(\cdot, f) \}_{f \in S} \) belong to the induced representation
\[
\text{ind}_{\prod_{r \neq 1} \mathbb{A}_r \times \tilde{G}_{m+a}}^{G_{n}(\mathbb{A}) \times G_{m}(\mathbb{A})}(\cdot | \frac{1}{2})_{\psi^{-1}} \otimes \nu_{W_{m+a}}.
\]

From now on, we consider the Fourier-Jacobi periods involving a (residual) Eisenstein series. Since the (residual) Eisenstein series is not a cusp form, \( \hat{\mathcal{F}}_\psi \) here denotes the regularized Fourier-Jacobi periods defined in [20, Definition 4.5]. (Note that the regularized Fourier-Jacobi periods naturally extend the definition of the original Fourier-Jacobi period, so it justifies using the same notation. See [20, Remark 4.6].) Since we have fixed \( \psi \), we suppress it from the notation and simply write it as \( \hat{\mathcal{F}} \).

The following proposition is an analog of [20, Lemma 9.4], but its proof is much simpler due to Proposition 3.7 and Proposition 3.8.

Proposition 5.4. Let \( \sigma \) be an irreducible cuspidal automorphic representation of \( GL_a(\mathbb{A}) \). Let \( \pi_1 \boxtimes \pi_2 \) be an irreducible cuspidal automorphic representation of \( \tilde{G}_n(\mathbb{A}) \) and \( \tilde{G}_m(\mathbb{A}) \) with generic \( A \)-parameter.
Assume that exactly one of $\pi_1$ or $\pi_2$ is genuine and distinguish them by writing $\pi'$ for the genuine representation of $G_{k(\pi')}(\mathbb{A})$ and $\pi$ for the non-genuine one of $G_{k(\pi)}(\mathbb{A})$. (Here, $\{k(\pi'), k(\pi)\} = \{n, m\}$.)

Write $FL(\pi)$ as an isobaric sum $\sigma_1 + \cdots + \sigma_s$, where $\sigma_1, \ldots, \sigma_s$ are irreducible cuspidal automorphic representations of the general linear groups such that $L(s, \sigma_i, \text{Sym}^2)$ have a pole at $s = 1$. We also write $FL_{\psi^{-1}}(\pi')$ as an isobaric sum $\sigma'_1 + \cdots + \sigma'_t$, where $\sigma'_1, \ldots, \sigma'_t$ are irreducible cuspidal automorphic representations of the general linear groups such that $L(s, \sigma'_i, \Lambda^2)$ have a pole at $s = 1$.

When $\pi_1 = \pi'$, assume $\sigma \simeq \sigma_j$ for some $1 \leq j \leq s$ and when $\pi_1 = \pi$, assume $\sigma \simeq \sigma'_i$ for some $1 \leq i \leq t$. Then the following holds.

\begin{enumerate}
  \item When $\pi_1 = \pi'$ and $\pi_2 = \pi$, then for all $\varphi \in \mathcal{E}^1(\sigma, \pi), \varphi' \in \mathcal{A}_{P_{n+a,a}}^{\emptyset}(G_{n+a})$ and $f \in \nu_{W_{m+a}},$

  \[ \mathcal{FJ}(E(\varphi', z), \varphi, f) = 0. \]

  \item When $\pi_1 = \pi$ and $\pi_2 = \pi'$, then for all $\varphi' \in \mathcal{E}^1(\sigma, \pi'), \varphi \in \mathcal{A}_{P_{n+a,a}}^{\emptyset}(G_{n+a})$ and $f \in \nu_{W_{m+a}},$

  \[ \mathcal{FJ}(E(\varphi, z), \varphi', f) = 0. \]
\end{enumerate}

**Proof.** Since $G_{m+a}(\mathbb{A}) = G_{m+a}(\mathbb{A})^1$ and $\{E(\varphi', z) \mid \varphi' \in \mathcal{A}_{P_{n+a,a}}^{\emptyset}(G_{n+a})\}$ is an automorphic realization of $\text{Ind}_{P_{n+a,a}}^{G_{n+a}(\mathbb{A})} (\sigma \psi \cdot |^z \boxtimes \pi')$, we can regard the functional $\mathcal{FJ}(E(\varphi', z), \varphi, f)$ as an element of

\begin{equation}
\text{Hom}_{N_{n+a}, r(\mathbb{A}) \otimes G_{m+a}(\mathbb{A})} \left( \text{Ind}_{P_{n+a,a}}^{G_{n+a}(\mathbb{A})} (\sigma \psi \cdot |^z \boxtimes \pi') \otimes \nu_{W_{m+a}} \otimes \mathcal{E}^1(\sigma, \pi, \mathbb{C}) \right)
\end{equation}

by [20] Proposition 6.3 (iii). Similarly, the functional $\mathcal{FJ}(E(\varphi, z), \varphi', f)$ belongs to

\begin{equation}
\text{Hom}_{N_{n+a}, r(\mathbb{A}) \otimes G_{m+a}(\mathbb{A})} \left( \text{Ind}_{P_{n+a,a}}^{G_{n+a}(\mathbb{A})} (\sigma \cdot |^z \boxtimes \pi) \otimes \nu_{W_{m+a}} \otimes \mathcal{E}^1(\sigma, \pi', \mathbb{C}) \right).
\end{equation}

When $\sigma \simeq \sigma_i$, the residue $\mathcal{E}^1(\sigma, \pi)$ is non-zero by Remark 4.7. We first prove the irreducibility of $\mathcal{E}^1(\sigma, \pi)$. Since the cuspidal support of the residues in $\mathcal{E}^1(\sigma, \pi)$ consists only of $\sigma \cdot |^{-1} \boxtimes \pi$, the residues are square integrable by [32] Lemma I.4.11. Thus $\mathcal{E}^1(\sigma, \pi)$ is a unitary quotient of $\text{Ind}_{P_{n+a,a}}^{G_{n+a}(\mathbb{A})}(\sigma \cdot | \boxtimes \pi)$. Since the Langlands quotient of $\text{Ind}_{P_{n+a,a}}^{G_{n+a}(\mathbb{A})}(\sigma \cdot | \boxtimes \pi)$ is its unique semisimple quotient, it is isomorphic to $\mathcal{E}^1(\sigma, \pi)$. Similarly, one can show that $\mathcal{E}^1(\sigma, \pi')$ is also irreducible.

We fix a finite place $v$ of $F$ such that the local $v$-components of $\psi, \sigma, \pi, \pi', \mathcal{E}^1(\sigma, \pi)$ and $\mathcal{E}^1(\sigma, \pi')$ are all unramified. For the moment, we consider only the local situation and suppress the subscript $v$ and drop the field $F_v$ from the notation. Recall that an irreducible generic unramified representation of $GL_n$ is an irreducible principal series representation.

From now on, we divide the proof into two cases.

**Case (i)** $\pi_1 = \pi'$, $\pi_2 = \pi$

We claim that

\begin{equation}
\text{Hom}_{N_{n+a}, r \otimes G_{m+a}} \left( \text{Ind}_{P_{n+a,a}}^{G_{n+a}(\mathbb{A})} \left( (\sigma \cdot |^z) \otimes \pi_1 \right) \otimes \nu_{W_{m+a}} \otimes \mathcal{E}^1(\sigma, \pi_2, \mathbb{C}) \right)
\end{equation}

is zero for almost all $z$. By Lemma 3.1,

\begin{equation}
\text{Hom}_{G_{m+a}} \left( J_{\psi^{-1}} \left( \text{Ind}_{P_{n+a,a}}^{G_{n+a}(\mathbb{A})} \left( (\sigma \cdot |^z) \otimes \pi_1 \right) \otimes \Omega_{W_{m+a}} \right), \mathcal{E}^1(\sigma, \pi_2)^\vee \right).
\end{equation}
By Remark 3.1(v), we can write \( \pi_2 \) as the irreducible unramified constituent of
\[
\text{Ind}_{B_m}^{G_m}(\chi_1 \otimes \cdots \otimes \chi_m),
\]
where \( \chi_i \)'s are unramified characters of \( F^\times \) such that \( 0 \leq e(\chi_i) < \frac{1}{2} \).

Since \( \sigma \) is an isobaric summand of \( FL(\pi_2) \), by Remark 4.7 we may assume
\[
\sigma \simeq \begin{cases}
\text{Ind}_{B_m}^{GL_n}(\chi_1 \otimes \cdots \otimes \chi_b \otimes \chi_b^{-1} \otimes \cdots \otimes \chi_1^{-1}) & \text{if } a = \text{even}, \ \omega_\eta = 1 \\
\text{Ind}_{B_m}^{GL_n}(\chi_1 \otimes \cdots \otimes \chi_{b-1} \otimes 1 \otimes \chi_0 \otimes \chi_{b-1}^{-1} \otimes \cdots \otimes \chi_1^{-1}) & \text{if } a = \text{even}, \ \omega_\eta = \text{non-trivial} \\
\text{Ind}_{B_m}^{GL_n}(\chi_1 \otimes \cdots \otimes \chi_{b-1} \otimes \chi_0 \otimes \chi_{b-1}^{-1} \otimes \cdots \otimes \chi_1^{-1}) & \text{if } a = \text{odd},
\end{cases}
\]
where \( \chi_0 \) is an unramified quadratic character of \( F^\times \).

Put \( \mu = \begin{cases} \chi_1, & \text{if } a = \text{even}, \ \omega_\eta = 1 \\
\chi_0, & \text{other cases} \end{cases} \) and \( \mu_z = \mu | z \). Then \( \sigma = \text{Ind}_{P_{a,1}}^{GL_n}(\mu \otimes \sigma_1) \) for some character \( \mu \) of \( F^\times \) and an unramified principal series representation \( \sigma_1 \) of \( GL_{a-1} \). By Remark 1.1(vi), we can write \( \pi_1 \) as the irreducible unramified constituent of
\[
\text{Ind}_{B_m}^{G_m}(\chi_{1,\psi} \otimes \cdots \otimes \chi_{n,\psi}),
\]
where \( \chi_i \)'s are unramified characters of \( F^\times \) such that \( 0 \leq e(\chi_i') < \frac{1}{2} \).

For \( z \in \mathbb{C} \), write \( \pi_z = \text{Ind}_{P_{(a-1,1,\cdots,1)}}^{GL_{n+a}}(\sigma_1 \cdot | z | \otimes \chi_{1} \otimes \cdots \otimes \chi_{n} \otimes \mu_z) \). Note that
\[
\text{Ind}_{P_{n+a,a}}^{G_{n+a}}((\sigma \cdot | z |) \otimes \pi_1) \simeq \text{Ind}_{P_{n+a,n+a-1}}^{G_{n+a}}((\pi_z) \otimes \pi_1).
\]

Since the Jacquet functor \( J_{\psi_{r-1}} \) is exact, we are enough to show that
\[
\text{Hom}_{G_{m+a}}\left( J_{\psi_{r-1}} \left( \text{Ind}_{P_{n+a,n+a-1}}^{G_{n+a}}((\pi_z) \otimes \Omega_{W_{m+a}}) \right), \mathcal{E}^1(\sigma, \pi_2)^\vee \right) = 0
\]
for almost all \( z \). This follows from Proposition 3.7 with \( n = n + a - 1 \), \( l = r - 1 \), \( E_{n-l} = E^1(\sigma, \pi_2)^\vee \) because such \( \pi_z \) and \( E^1(\sigma, \pi_2)^\vee \) satisfy the conditions of the proposition.

From what we have discussed so far, we see that
\[
\mathcal{F}(E(\phi', z), \varphi, f) = 0
\]
at least when \( \text{Re}(z) \gg 0 \). Since \( z \mapsto \mathcal{F}(E(\phi', z), \varphi, f) \) is meromorphic, \( \mathcal{F}(E(\phi', z), \varphi, f) \) is identically zero.

**Case (ii)** \( \pi_1 = \pi, \ \pi_2 = \pi' \)

As we see in the case \( \pi_1 = \pi' \) and \( \pi_2 = \pi \), it is sufficient to show that
\[
(5.8) \quad \text{Hom}_{N_{n+a,r} \times G_{m+a}}\left( \text{Ind}_{P_{n+a,a}}^{G_{n+a}}(\sigma \cdot | z | \otimes \pi_1) \otimes \nu_{W_{m+a}} \otimes \mathcal{E}^1(\sigma, \pi_2), \mathbb{C} \right)
\]
is zero for almost all \( z \).

By Lemma 3.1
\[
(5.8) \simeq \text{Hom}_{G_{m+a}}\left( J_{\psi_{r-1}} \left( \text{Ind}_{P_{n+a,a}}^{G_{n+a}}(\sigma \cdot | z | \otimes \pi_1) \otimes \Omega_{W_{m+a}} \right), \mathcal{E}^1(\sigma, \pi_2)^\vee \right).
\]

By Remark 1.1(vi), we can write \( \pi_2 \) as the irreducible unramified constituent of
\[
\text{Ind}_{B_m}^{G_m}(\chi_{1,\psi} \otimes \cdots \otimes \chi_{m,\psi}),
\]
where $\chi_i$'s are unramified characters of $F^\times$ such that $0 \leq \epsilon(\chi_i) < \frac{1}{2}$. Since $\sigma$ is an isobaric summand of $FL(\pi_2)$, by Lemma 4.7, we may assume

$$\sigma \simeq \text{Ind}_{B_a(F)}^{GL_a(F)}(\chi_{1,\psi} \boxtimes \cdots \boxtimes \chi_{b,\psi} \boxtimes \chi_{b,\psi}^{-1} \boxtimes \cdots \boxtimes \chi_{1,\psi}^{-1}).$$

Now we claim that

$$(5.9) \quad \text{Hom}_{G_{m+a}} \left( J_{\psi,-1}^{G_{n+a}} \left( \text{Ind}_{P_{n+a,a}}^{G_{n+a}}(\sigma| \cdot |^2 \pi_1) \otimes \Omega_{W_{m+a}} \right), \mathcal{E}_b^1(\sigma,\pi_2)^{\vee} \right) = 0$$

for almost all $z$. This follows from Proposition 5.8 as in Case (i).

□

From now on, we simply write $P$ (resp. $P_a, M_a, K_a$) for $P_{n+a,a}$ (resp. $P_{m+a,a}, M_{m+a,a}, K_{m+a}$). Note that $N_{n+a,r} = (\text{Hom}(Y_a, X) \times \text{Hom}(Y_a^t, X)) \cdot N_{n,r}$ and $N_{n+a,r} \cap P = \text{Hom}(Y_a^t, X) \cdot N_{n,r}$. Denote $\text{Hom}(Y_a, X) \times \text{Hom}(Y_a^t, X)$ by $V$ and let $dv$ be the Haar measure of $V$ such that $dn' = d\nu/m$.

Using Proposition 5.4, we can prove the following lemma.

Lemma 5.5 (cf. [18 Lemma 6.3], [20 Lemma 9.6]). With the same notation as in Lemma 5.4, we assume $\sigma \simeq \sigma_i$ for some $1 \leq i \leq t$. Then the following holds.

(i) When $\pi_1 = \pi'$ and $\pi_2 = \pi$, then for all $\varphi \in \mathcal{E}_b^1(\sigma,\pi), \phi' \in \mathcal{A}^{\sigma_{\psi \boxtimes \pi'}}_{F_{n+a,a}}(\widehat{G_{n+a}})$ and $f \in \nu_{W_{m+a}}$,

$$FJ(\mathcal{E}_b^0(\phi'), \varphi, f) =$$

$$\int_{K_a} [V] \left( \int_{M_a(F) \setminus M_a(\mathbb{A})} \left( \int_{[N_{n,r}]} \phi'(nmvk) \cdot \Theta_{P_a}(f)((n, \bar{m}) \cdot (v, k)) \, d\nu \right) \varphi_{P_a}(mk) \, dm \right) \cdot e^{\frac{\mu_{P_a}(vk)}{2}} \, dvdk.$$

(ii) When $\pi_1 = \pi$ and $\pi_2 = \pi'$, then for all $\varphi' \in \mathcal{E}_b^1(\sigma,\pi'), \phi \in \mathcal{A}^{\sigma_{\psi \boxtimes \pi}}_{F_{n+a,a}}(G_{n+a})$ and $f \in \nu_{W_{m+a}}$,

$$FJ(\mathcal{E}_b^0(\phi), \phi', f) =$$

$$\int_{K_a} [V] \left( \int_{M_a(F) \setminus M_a(\mathbb{A})} \left( \int_{[N_{n,r}]} \phi(nmvk) \cdot \Theta_{P_a}(f)((n, \bar{m}) \cdot (v, k)) \, d\nu \right) \phi_{P_a}(mk) \, dm \right) \cdot e^{\frac{\mu_{P_a}(vk)}{2}} \, dvdk.$$

Proof. A proof is identical to that of [20 Lemma 9.6]. We omit the details. □

Lemma 5.6 ([18 Lemma 6.5], [20 Lemma 9.7]). With the same notation as in Lemma 5.4, we assume $\sigma \simeq \sigma_i$ for some $1 \leq i \leq t$.

(i) When $\pi_1 = \pi'$ and $\pi_2 = \pi$, then the following is equivalent;

(a) The Fourier–Jacobi period functional $FJ(\pi', \pi, \nu_{W_{m}})$ is nonzero

(b) There exist $\varphi \in \mathcal{E}_b^1(\sigma,\pi), \phi \in \mathcal{A}^{\sigma_{\psi \boxtimes \pi'}}_{F_{n+a,a}}(\widehat{G_{n+a}})$ and $f \in \nu_{W_{m+a}}$ such that

$$\int_{K_a} [V] \left( \int_{M_a(F) \setminus M_a(\mathbb{A})} \left( \int_{[N_{n,r}]} \phi(nmvk) \cdot \Theta_{P_a}(f)((n, \bar{m}) \cdot (v, k)) \, d\nu \right) \varphi_{P_a}(mk) \, dm \right) \cdot e^{\frac{\mu_{P_a}(vk)}{2}} \, dvdk \neq 0.$$

(ii) When $\pi_1 = \pi$ and $\pi_2 = \pi'$, then the following is equivalent;

(a) The Fourier–Jacobi period functional $FJ(\pi', \pi, \nu_{W_{m}})$ is nonzero
(b) There exist $\varphi' \in \mathcal{E}^1(\sigma, \pi')$, $\phi \in \mathcal{A}_{\nu_{P_{n+a,a}}}(G_{n+a})$ and $f \in \nu_{W_{m+a}}$ such that
\begin{equation}
(5.11)
\int_{K_a} \int_{[V]} \left( \int_{(M_a(F) \setminus M_a(\mathbb{A}))^1} \phi(nmvk) \cdot \Theta_{P_a}(f) ((n, \tilde{m}) \cdot (v, k)) \, dn \right) \varphi_a(\tilde{mk}) \, dm \cdot e^{\frac{\mu_P(vk)}{2}} \, dv dk \neq 0.
\end{equation}

Proof. Though the proof is essentially same with [20, Lemma 9.7], for its importance, we give the proof here. Since the case $\pi_1 = \pi$ and $\pi_2 = \pi'$ is similar, we only prove the case $\pi_1 = \pi'$ and $\pi_2 = \pi$. We first prove the $(a) \rightarrow (b)$ direction.

Put
\begin{equation}
\Pi' = (\sigma | \cdot |^{-1})_{\psi} \otimes \pi', \quad \Pi = (\sigma | \cdot |^{1/2} \otimes \pi, \quad \Pi'' = (\cdot | \psi^{-1} \otimes \nu_{W_m}).
\end{equation}

We define a functional on $\Pi' \boxtimes \Pi \boxtimes \Pi''$ by
\begin{equation}
l(n' \otimes n \otimes n'') = \int_{M_a(F) \setminus M_a(\mathbb{A})^1} n(m) \left( \int_{(N_{n,r}(F) \setminus N_{n,r}(\mathbb{A}))} n'(nm) n''(nm) \, dn \right) \, dm.
\end{equation}

Let $(\Pi' \boxtimes \Pi \boxtimes \Pi'')^\infty$ be the canonical Casselman-Wallach globalization of $\Pi' \boxtimes \Pi \boxtimes \Pi''$ realized in the space of smooth automorphic forms without the $K_{M_{n+a}} \times K_{M_{m+a}} \times K_{M_{m+a}}$-finiteness condition, where $K_{M_{n+a}} = K_{n+a} \cap M_{n+a,a}(\mathbb{A})$ for $i = n, m$ (cf. [8], [45, Chapter 11]). Since cusp forms are bounded, $l$ can be uniquely extended to a continuous functional on $(\Pi' \boxtimes \Pi \boxtimes \Pi'')^\infty$ and denote it by the same notation. Our assumption enables us to choose $n' \in \Pi'$, $n \in \Pi$ and $n'' \in \Pi''$ so that $l(n' \otimes n \otimes n'') \neq 0$. We may assume that $n', n$ and $n''$ are pure tensors. By [37, 43], the functional $l$ is a product of local functionals $l_v \in \text{Hom}_{M_{n,v}}((\Pi'_v \boxtimes \Pi_v \boxtimes \Pi''_v)^\infty, \mathbb{C})$, where we set $(\Pi'_v \boxtimes \Pi_v \boxtimes \Pi''_v)^\infty = \Pi'_v \boxtimes \Pi_v \boxtimes \Pi''_v$ if $v$ is finite. Then we have $l_v(n'_v \otimes n_v \otimes n''_v) \neq 0$.

Denote by $e$ the identity element of $G_{m+a}$. Choose $\varphi \in \mathcal{E}^1(\sigma, \pi)$ and a pure tensor $f = f_1 \otimes f_2 \in \nu_{W_{m+a}} = \Omega_{Y_a + Y_a^*} \otimes \nu_{W_m}$ such that
\begin{enumerate}
\item $\varphi_{P'_a} \cdot \varphi_a = \xi_v \varphi_v$, $\Theta_{P_a}(f) = \xi_v f_v = \xi_v (f_{1,v}(0) \otimes f_{2,v})$;
\item $\varphi_v(e) = n_v$, $f_v((1, e)) = n''_v$.
\end{enumerate}
(We can choose such $f$ by Remark 5.2 and Remark 5.3.)

Let $U_{n+a,a}$ the unipotent radical of the opposite parabolic subgroup $P_{n+a,a}$. For each place $v$, let $\alpha_v$ be a smooth function on $U_{n+a,a,v}$ whose compact support lies in near $e_v$. We can define a section $\phi'_v$ by requiring
\begin{equation}
\phi'_v(muu-) = \delta_{P_v}(m) \alpha_v(u-) \cdot (P'_v)^\infty(\tilde{m}) n_v, \quad m \in M_{n+2a,a,v}, \ u \in U_{n+2a,a,v}, \ u_- \in U_{n+2a,a,v}^-.
\end{equation}
Since $N_{n+a,r} \simeq \text{Hom}(Y_a, X) \times \text{Hom}(Y_a^*, X) \times N_{n,r}$, we can regard $\text{Hom}(Y_a, X)$ and $\text{Hom}(Y_a^*, X)$ as the subgroups of $N_{n+a,r}$. By (5.4), for every $p_1 \in \text{Hom}(Y_a, X)_v$, $f_v((p_1, e)) = (\Omega_{Y_a + Y_a^*}(p_1) f_{1,v})(0) \otimes f_{2,v} = f_{1,v}(0) \otimes f_{2,v} = f_v((1, e)) = n''_v$.

Using the Iwasawa decomposition of $G_{n+a,v}$ with respect to $P_{n+a,a,v}$, it is easy to check that $\text{Hom}(Y_a, X)_v$ is contained in $U_{n+a,a,v}^-$. By taking the support of each $\alpha_v$ sufficiently small, we have
\begin{equation}
\int_{\text{Hom}(Y_a, X)_v} l_v \left( \phi'_v(p_1) \otimes n_v \otimes f_v((p_1, e)) \right) \cdot e^{\frac{\mu_P(p_1)}{2}} = \int_{\text{Hom}(Y_a, X)_v} l_v \left( \phi'_v(p_1) \otimes n_v \otimes n''_v \right) \cdot e^{\frac{\mu_P(p_1)}{2}} \neq 0.
\end{equation}
On the other hand, there is a small neighborhood $N_v$ of 0 in $\text{Hom}(Y^*_a, X)_v$ such that

$$\int_{\text{Hom}(Y^*_a, X)_v} \int_{\text{Hom}(Y_a, X)_v} l_v \left( \phi'_v((p_1 p_2), \eta \boxtimes f_v((p_1 p_2, e))) \right) \cdot e^{\frac{H_F((p_1 p_2))}{2}} \cdot \chi_{N_v}(p_2)dp_1 dp_2 \neq 0.$$ 

(Here, $\chi_{N_v}$ is the characteristic function on $N_v$.)

By taking the support of the Schwartz function $f_1$ sufficiently small neighborhood of 0 $\in Y^*_a$, from (5.4), we may assume that the support of $f_1,v$ is contained in $N_v$. (Here, we regard $f_1,v$ as a function on $\text{Hom}(Y^*_a, X)_v$ defined by $f_1,v(p_2) = (\Omega_{Y_a + Y^*_a}(p_2) f_1,v) (0)$.) Then

$$\int_{\text{Hom}(Y^*_a, X)_v} \int_{\text{Hom}(Y_a, X)_v} l_v \left( \phi'_v((p_1 p_2), \eta \boxtimes f_v((p_1 p_2, e))) \right) \cdot e^{\frac{H_F((p_1 p_2))}{2}} \cdot \chi_{N_v}(p_2)dp_1 dp_2 \neq 0.$$

Now, it is sufficient to choose a $K_{n+a,v}$-finite smooth function $\phi_v$ on $\widetilde{G}_{m+a,v}$, whose values in ($\Pi'_v$) that satisfies

- $\phi'_v(\tilde{m} u \tilde{g}) = \delta_{P_v}(m)^{\frac{1}{2}} \cdot (\Pi')^\infty(v) \phi_v(g)$ for $m \in M_{a,v}$, $u \in U_{a,v}$, $g \in G_{m+a,v}$,
- $\int_{\text{Hom}(Y^*_a, X)_v} \int_{\text{Hom}(Y_a, X)_v} l_v \left( \phi'_v(p_1 p_2 k) \boxtimes \varphi_v(k) \boxtimes f_v((p_1 p_2, e))) \right) \cdot e^{\frac{H_F((p_1 p_2 k))}{2}} dp_1 dp_2 dk \neq 0$.

For $\phi_v \in \text{Ind}_{P}^{G_{m+a,v}}(\Pi')^\infty$, put

$$I(\phi_v) = \int_{\text{Hom}(Y^*_a, X)_v} \int_{\text{Hom}(Y_a, X)_v} l_v \left( \phi'_v(p_1 p_2 k) \boxtimes \varphi_v(k) \boxtimes f_v((p_1 p_2, e))) \right) \cdot e^{\frac{H_F((p_1 p_2 k))}{2}} dp_1 dp_2 dk.$$ 

and write $U^-_{a,v}$ for the unipotent radical of the parabolic subgroup opposite to $P_v$. Since $P_{a,v} \cdot U^-_{a,v}$ is an open dense subset of $G_{m+a,v}$, we can rewrite $I(\phi'_v)$ as

$$\int_{U^-_{a,v}} \int_{\text{Hom}(Y^*_a, X)_v} \int_{\text{Hom}(Y_a, X)_v} \alpha_v(u^-) \cdot l_v \left( \phi'_v(p_1 p_2) \boxtimes \varphi_v(u^-) \boxtimes f_v((p_1 p_2 u^-, e))) \right) \cdot e^{\frac{H_F((p_1 p_2 u^-))}{2}} dp_1 dp_2 du^-.$$ 

We can choose $\alpha_v$ to be supported in a small neighborhood of $e$ so that $I(\phi'_v) \neq 0$. Furthermore, since $I$ is continuous and $K_{n+a,v}$-finite vectors are dense in the induced representation $\text{Ind}_{P_v}^{G_{m+a,v}}(\Pi')^\infty$, we can choose $K_{n+a,v}$-finite function $\phi_v'$ such that $I(\phi'_v) \neq 0$. This completes the proof of the $(a) \rightarrow (b)$ direction.

The proof of $(b) \rightarrow (a)$ direction is almost immediate. From what we have seen in the above, we see that the Fourier–Jacobi period integral $\mathcal{F}J(\pi', \pi, \nu_{W_m})$ is a partial inner period integral in (5.10). Therefore, if $\mathcal{F}J(\pi', \pi, \nu_{W_m}) = 0$, the integral (5.10) is always zero. \(\square\)

By combining Lemma 5.5 with Lemma 5.6, we get the following reciprocal non-vanishing theorem.

**Theorem 5.7.** With the same notation as in Proposition 5.4, assume $\sigma \simeq \sigma_i$ for some $1 \leq i \leq t$. Then $\mathcal{F}J(\pi_1, \pi_2, \nu_{W_m}) \neq 0$ is equivalent to $\mathcal{F}J(\mathcal{E}^0(\sigma, \pi_1), \mathcal{E}^1(\sigma, \pi_2), \nu_{W_{m+2a}}) \neq 0$.

Now piecing together everything we have developed so far, we can prove our main theorem.

**Proof of Theorem A.** Since the proof of $\pi_1 = \pi$ case is similar, we only prove $\pi_1 = \pi'$ case. Write $FL(\pi)$ as an isobaric sum of the form $\sigma_1 \boxplus \cdots \boxplus \sigma_t$, where $\sigma_1, \ldots, \sigma_t$ are distinct irreducible cuspidal automorphic representations of general linear groups such that the exterior square $L$-function $L(s, \sigma_i, \Lambda)$
has a pole at \( s = 1 \). On the other hand, \( \mathcal{E}(\sigma_i, \pi') \) is nonzero by Theorem 5.7. Thus by Proposition 4.4, we have \( L \left( \frac{1}{2}, FL_\psi(\pi') \times \sigma_i \right) \neq 0 \) for all \( 1 \leq i \leq t \) and we get

\[
L \left( \frac{1}{2}, FL_\psi(\pi_1) \times FL_\psi(\pi_2) \right) = \prod_{i=1}^{t} L \left( \frac{1}{2}, FL_\psi(\pi') \times \sigma_i \right) \neq 0.
\]

\( \square \)

6. Non-tempered GGP conjecture for residual representations

In this section, we introduce the non-tempered GGP conjecture and verify one direction of it for certain important classes of non-tempered representations, namely, residual representations.

6.1. Non-tempered GGP conjecture. Let \( \pi_1 \) and \( \pi_2 \) be cuspidal automorphic representations of \( GL_n(\mathbb{A}) \) and \( GL_m(\mathbb{A}) \), respectively. Let \( d_1, d_2 \) be non-negative integers.

Define

\[
L(s, \pi_1 \boxtimes \text{Sym}^{d_1}(\mathbb{C}^2)) := \prod_{i=0}^{d_1} L(s + \frac{d_1}{2} - i, \pi_1)
\]

\[
L(s, (\pi_1 \times \pi_2) \boxtimes \text{Sym}^{d_1}(\mathbb{C}^2)) := \prod_{i=0}^{d_1} L(s + \frac{d_1}{2} - i, \pi_1 \times \pi_2)
\]

\[
L(s, (\pi_1 \boxtimes \text{Sym}^{d_1}(\mathbb{C}^2)) \times (\pi_2 \boxtimes \text{Sym}^{d_2}(\mathbb{C}^2))) := \prod_{k=0}^{\min(d_1, d_2)} L(s, (\pi_1 \times \pi_2) \boxtimes \text{Sym}^{d_1+d_2-k}(\mathbb{C}^2)).
\]

Here, \( L(s, \pi_1) \) and \( L(s, \pi_1 \times \pi_2) \) denote the completed standard automorphic and Rankin–Selberg \( L \)-functions, respectively.

For two discrete global \( A \)-parameters \( M \) and \( N \), decompose \( M \) and \( N \) as:

\[
M = \sum_{\alpha} V_\alpha, \quad N = \sum_{\beta} W_\beta
\]

where \( V_\alpha \) and \( W_\beta \)'s are irreducible \( A \)-parameters.

Then we define:

\[
L(s, M) := \prod_{\alpha} L(s, V_\alpha), \quad L(s, M \times N) := \prod_{\alpha, \beta} L(s, V_\alpha \times W_\beta).
\]

**Definition 6.1.** Given two discrete global \( A \)-parameters \( M \) and \( N \), we can write \( M \) and \( N \) as follows:

\[
M = \sum_{i=1}^{t} M_i \boxtimes \text{Sym}^{n_i}(\mathbb{C}^2)
\]

\[
N = \sum_{i=1}^{t} M_i \boxtimes \text{Sym}^{m_i}(\mathbb{C}^2),
\]

where \( M_i \) is an irreducible unitary cuspidal automorphic representation of \( GL_{n_i} \) and \( n_i, m_i \) are integers greater than or equal to \(-1\). (Note that \( \text{Sym}^{-1}(\mathbb{C}^2) = 0 \).)
We say that $M$ and $N$ are relevant if there exists a permutation $p$ of $\{1, 2, \ldots, t\}$ such that

$$M = \sum_{i=1}^{t} M_i \boxtimes \text{Sym}^{n_{p(i)}}(\mathbb{C}^2)$$

and $|n_{p(i)} - m_i| = 1$ for all $1 \leq i \leq t$. In this case, one of $M$ and $N$ is symplectic, and the other one is orthogonal.

When $M$ and $N$ are relevant discrete $A$-parameters with orthogonal $M$ and symplectic $N$, it is proved that

$$L(s, M, N) := \frac{L(s + \frac{1}{2}, M \times N)}{L(s + 1, \wedge^2(M)) \cdot L(s + 1, \text{Sym}^2(N))}$$

is holomorphic at $s = 0$ (see [11, Theorem 9.7]).

The following is the non-tempered GGP conjecture for the Fourier–Jacobi case (see [11, Conjecture 9.1] for the Bessel case.)

**Conjecture 6.2** (cf. [11, Conjecture 9.1]). For $n \geq m$, let $\pi_1 \boxtimes \pi_2$ be an irreducible automorphic representation of $\widetilde{G}_n(\mathbb{A}) \times \widetilde{G}_m(\mathbb{A})$ such that exactly one of $\pi_1$ and $\pi_2$ is genuine. Let $M \times N$ be a discrete $A$-parameters associated with $\pi_1 \boxtimes \pi_2$. Then the following hold:

(i) If $\mathcal{J}_\psi(\pi_1, \pi_2, \nu_{\psi^{-1}, W_m}) \neq 0$, then $(M, N)$ should be a relevant pair.

(ii) Assume that $(M, N)$ is a relevant pair. Then the following are equivalent:

(a) $\mathcal{J}_\psi(\pi_1, \pi_2, \nu_{\psi^{-1}, W_m}) \neq 0$

(b) $\text{Hom}_{G_n(F_v)}(\pi_{1,v} \boxtimes \pi_{2,v} \boxtimes \nu_{\psi^{-1}, W_m}, \mathbb{C}) \neq 0$ for all places $v$ and $L(s, M, N)|_{s = 0} \neq 0$

**6.2. The Fourier–Jacobi period for residual representations.** In this subsection, we establish the direction $(a) \Rightarrow (b)$ in Conjecture 6.2 (ii) for certain significant classes of automorphic representations with non-generic relevant $A$-parameters $M$ and $N$.

For $n \geq m$, let $\pi$ and $\overline{\pi}$ be cuspidal automorphic representations of $\widetilde{G}_n(\mathbb{A})$ and $\widetilde{G}_m(\mathbb{A})$ with generic $A$-parameters $M_1$ and $N_1$, respectively. Suppose that $\pi$ is non-genuine and $\overline{\pi}$ is genuine. Write $M_1 = \sigma_1 + \cdots + \sigma_{t_1}$, where $\sigma_i$ is an irreducible cuspidal automorphic representation of the general linear group such that $L(s, \sigma_i, \text{Sym}^2)$ has a pole at $s = 1$. We also write $N_1'$ as $\sigma'_1 + \cdots + \sigma'_{t_2}$, where $\sigma'_j$ is an irreducible cuspidal automorphic representation of the general linear group such that $L(s, \sigma'_j, \wedge^2)$ has a pole at $s = 1$.

Consider an irreducible cuspidal automorphic representation $\sigma$ of $GL_a(\mathbb{A})$ such that $\sigma \simeq \sigma'_{j_0}$ for some $1 \leq j_0 \leq t_1$. Assume that $L(\frac{1}{2}, \sigma \times \pi) \neq 0$, so that $\mathcal{E}(\sigma, \pi) \neq 0$ (see Proposition 4.2). Write $\pi_1 := \mathcal{E}(\sigma, \pi)$ and $\pi_2 := \mathcal{E}(\sigma, \overline{\pi})$. Define two $A$-parameters $M$ and $N$ as follows:

$$M := \sigma \boxtimes \text{Sym}^1(\mathbb{C}^2) + \sum_{i=1}^{t_1} \sigma_i \boxtimes \text{Sym}^0(\mathbb{C}^2)$$

$$N := \sigma \boxtimes \text{Sym}^2(\mathbb{C}^2) + \sum_{1 \leq j \leq t_2 \atop j \neq j_0} \sigma'_j \boxtimes \text{Sym}^0(\mathbb{C}^2).$$

It is obvious that $M$ and $N$ are relevant. Furthermore, by applying Theorem 2.1, we can easily verify that the $A$-parameters of $\pi_1$ and $\pi_2$ are $M$ and $N$, respectively.

**Theorem 6.3.** Keep the notations as above. If $\mathcal{J}_\psi(\pi_1, \pi_2, \nu_{\psi^{-1}, W_{m+a}}) \neq 0$, then $L(s, M, N)|_{s = 0} \neq 0$. 
To prove this, we need some analytic properties of automorphic L-functions. We review some of them (refer to [25], [41], and [19, Theorem 1.3]).

Let $\sigma_1$ and $\sigma_2$ be cuspidal automorphic representations of $GL_n(\mathbb{A})$ and $GL_m(\mathbb{A})$, respectively. Write $Sym^2$ for symmetric square representation and $\wedge^2$ for exterior square representation. We also put $L(s, \gamma(\sigma_1))$ for the complete symmetric or exterior square $L$-function according to $\gamma = Sym^2$ or $\wedge^2$. We review some analytic properties of $L(s, \sigma_1 \times \sigma_2)$ and $L(s, \gamma(\sigma_1))$.

- $L(s, \sigma_1 \times \sigma_2)$ and $L(s, \gamma(\sigma_1))$ have meromorphic continuations to the whole complex plane.
- $L(s, \sigma_1 \times \sigma_2)$ and $L(s, \gamma(\sigma_1))$ satisfy functional equations relating the values at $s$ and $1 - s$.
- $L(s, \sigma_1 \times \sigma_2)$ and $L(s, \gamma(\sigma_1))$ are holomorphic for all $s \in \mathbb{C} \setminus \{0, 1\}$.
- For Re$(s) \geq 1$ and Re$(s) \leq 0$, $L(s, \sigma_1 \times \sigma_2)$ and $L(\gamma(\sigma_1), s)$ are non-zero.
- $L(s, \sigma_1 \times \sigma_2)$ has a pole at $s = 0, 1$ if and only if $m = n$ and $\sigma_2 = \sigma_1^\vee$ (the contragredient dual of $\sigma_1$). In this case, the pole is a simple pole.
- If $\sigma_1$ is not self-dual, then $L(s, \gamma(\sigma_1))$ is an entire function.
- If $\sigma_1$ is self-dual, exactly one of $L(s, Sym^2(\sigma_1))$ and $L(s, \wedge^2(\sigma_1))$ has a simple pole at $s = 0, 1$.

On the other hand, in the course of the proof, we freely use the following elementary facts.

**Fact 6.4.** Let $M$ and $N$ be a relevant pair of discrete $A$-parameters which has a decomposition

$$M = \bigoplus_{\alpha} V_\alpha, \quad N = \bigoplus_{\alpha} W_\alpha$$

so that each $(V_\alpha, W_\alpha)$ is a relevant pair of irreducible $A$-parameters. Assume that $M$ is orthogonal and $N$ is symplectic. Let $V$ be a representation of $SL_2(\mathbb{C})$. Then:

- $M \times N = \bigoplus_{\alpha} (V_\alpha \otimes W_\alpha) + \bigoplus_{\alpha \neq \beta} (V_\alpha \otimes W_\beta + V_\beta \otimes W_\alpha)$
- $\wedge^2(M) = \bigoplus_{\alpha} \wedge^2(V_\alpha) + \bigoplus_{\alpha \neq \beta} V_\alpha \otimes V_\beta$
- $Sym^2(N) = \bigoplus_{\alpha} Sym^2(W_\alpha) + \bigoplus_{\alpha \neq \beta} W_\alpha \otimes W_\beta$
- $Sym^2(Sym^2(\mathbb{C}^2)) = Sym^2(\mathbb{C}^2) + Sym^{2i-4}(\mathbb{C}^2) + \cdots$
- $\wedge^2(Sym^2(\mathbb{C}^2)) = Sym^{2i-2}(\mathbb{C}^2) + Sym^{2i-6}(\mathbb{C}^2) + \cdots$
- $Sym^2(\rho \otimes V) = Sym^2(\rho) \otimes Sym^2(V) + \wedge^2(\rho) \otimes \wedge^2(V)$
- $\wedge^2(\rho \otimes V) = Sym^2(\rho) \otimes \wedge^2(V) + \wedge^2(\rho) \otimes Sym^2(V)$
- $Sym^4(\mathbb{C}^2) \otimes Sym^2(\mathbb{C}^2) = \bigoplus_{k=0}^{\min(i,j)} Sym^{i+j-2k}(\mathbb{C}^2)$

Now we prove Theorem 6.3.

**Proof of Theorem 6.3.** Let $I = \{1, 2, \cdots, t_1\}, \ J = \{1, 2, \cdots, t_2\}$ and $J_0 = J \setminus \{j_0\}$. For brevity, we write $[i]$ for $Sym^4(\mathbb{C}^2)$ and $\sigma_j$ for $Sym^2(\mathbb{C}^2)$. Using the Fact 6.4, we see that

$$M \otimes N = (\sigma \times \sigma) \otimes 1 + (\sigma \times \sigma) \otimes [2] + \sum_{i \in I} (\sigma_i \times \sigma_j) \otimes [1] + \sum_{i \in I} (\sigma_i \times \sigma_j) \otimes [2] + \sum_{i \in I, j \in J_0} (\sigma_i \times \sigma_j) \otimes [0]$$

$$\wedge^2(M) = \wedge^2(\sigma \otimes [1]) + \sum_{i \in I} \wedge^2(\sigma_i \otimes [0]) + \sum_{i \in I} (\sigma \times \sigma_i) \otimes [1] + \sum_{i,j \in I} (\sigma_i \times \sigma_j) \otimes [0]$$

$$= Sym^2(\sigma) \otimes [0] + \wedge^2(\sigma) \otimes [2] + \sum_{i \in I} \wedge^2(\sigma_i) \otimes [0] + \sum_{i \in I} (\sigma \times \sigma_i) \otimes [1] + \sum_{i,j \in I} (\sigma_i \times \sigma_j) \otimes [0]$$

$$Sym^2(N) = Sym^2(\sigma \otimes [2]) + \sum_{j \in J_0} Sym^2(\sigma_j \otimes [0]) + \sum_{j \in J_0} (\sigma \times \sigma_j) \otimes [2] + \sum_{i,j \in J_0} (\sigma_i \times \sigma_j) \otimes [0]$$
Then by applying Theorem A, we see that though we have thus far considered the scenario where natural hypotheses. To state the conjecture, we require some terminology.

Consequently, the orders of \( L(s + \frac{1}{2}, M \times N) \), \( L(s + 1, \Lambda^{2}(M)) \) and \( L(s + 1, \text{Sym}^{2}(N)) \) at \( s = 0 \) are as follows:

\[
\text{ord}_{s=0} L(s + \frac{1}{2}, M \times N) = -2 - 2 + \sum_{i \in I} \text{ord}_{s=\frac{1}{2}} L(s, \sigma_{i} \otimes \sigma) + \sum_{i \in I, j \in J_{0}} \text{ord}_{s=\frac{1}{2}} L(s, \sigma_{i} \otimes \sigma_{j}')
\]

\[
\text{ord}_{s=0} L(s + 1, \Lambda^{2}(M)) = -2 + \sum_{i \in I} \text{ord}_{s=1} L(s, \sigma \otimes \sigma_{i})
\]

\[
\text{ord}_{s=0} L(s + 1, \text{Sym}^{2}(N)) = -2.
\]

Therefore,

\[
L(s, M, N)|_{s=0} = \sum_{i \in I} \text{ord}_{s=\frac{1}{2}} L(s, \sigma_{i} \otimes \sigma) + \sum_{i \in I, j \in J_{0}} \text{ord}_{s=\frac{1}{2}} L(s, \sigma_{i} \otimes \sigma_{j}') - \sum_{i \in I} \text{ord}_{s=1} L(s, \sigma \otimes \sigma_{i})
\]

\[
= \sum_{i \in I, j \in J} \text{ord}_{s=\frac{1}{2}} L(s, \sigma_{i} \otimes \sigma_{j}') - \sum_{i \in I} \text{ord}_{s=1} L(s, \sigma \otimes \sigma_{i}) = \sum_{i \in I, j \in J} \text{ord}_{s=\frac{1}{2}} L(s, \sigma_{i} \otimes \sigma_{j}).
\]

(The last equality follows from the assumption \( L_{\psi}(\frac{1}{2}, \sigma \times \pi) \neq 0 \).)

Given that \( FJ_{\pi_{1}, \pi_{2}, \nu_{\psi}^{-1}, W_{m+2n}} \neq 0 \), we can infer from Theorem [5, 7] that \( FJ_{\psi(\pi, \bar{\pi}, \nu_{\psi}^{-1}, W_{m})} \neq 0 \). Then by applying Theorem [4] we see that \( L_{\psi}(\frac{1}{2}, \pi \times \bar{\pi}) \neq 0 \) and consequently, \( \sum_{i \in I, j \in J} \text{ord}_{s=\frac{1}{2}} L(s, \sigma_{i} \otimes \sigma_{j}') \neq 0 \). This completes the proof.

Remark 6.5. Though we have thus far considered the scenario where \( \pi \) is non-genuine and \( \bar{\pi} \) is genuine, we can also address the converse situation: when \( \pi \) is genuine and \( \bar{\pi} \) is non-genuine. In this case, the \( \lambda \)-parameters \( M \) (resp. \( N \)) of \( \pi_{1} \) (resp. \( \pi_{2} \)) is symplectic (resp. orthogonal). For brevity, we omit further discussion.

7. Non-vanishing of the quadratic twists of L-functions

In this section, we first introduce a conjecture regarding the non-vanishing of central \( L \)-values for quadratic twists of a certain type of automorphic representations of \( GL_{2n}(\mathbb{A}) \), and we prove it under some natural hypotheses. To state the conjecture, we require some terminology.

Let \( K \) be a local field of characteristic zero and let \( \phi \) be an \( L \)-parameter of \( GL_{n}(K) \). Then we say that \( \phi \) is symplectic if there exists a non-degenerate antisymmetric bilinear form \( b : \mathbb{C}^{n} \times \mathbb{C}^{n} \rightarrow \mathbb{C} \) such that \( b(\phi(w)x, \phi(w)y) = b(x, y) \) for all \( w \in L_{K} \) and \( x, y \in \mathbb{C}^{n} \). In particular, if \( \phi \) is symplectic, then \( n \) is even number.

Definition 7.1. Let \( \tau \) be an irreducible unitary cuspidal automorphic representation \( \tau = \otimes \tau_{v} \) of \( GL_{2n}(\mathbb{A}) \) of symplectic type and \( \phi \) be a generic \( \lambda \)-parameter of \( \tilde{G}_{n}(\mathbb{A}) \) associated to \( \tau \). For a place \( v \) of \( F \), if \( \phi_{v} \) has an irreducible symplectic subrepresentation, we say that \( \tau \) is good at \( v \).

Remark 7.2. When \( n = 1 \), \( \tau \) is good at \( v \) is equivalent to that \( \tau_{v} \) is square integrable. Let \( \phi \) be a generic \( \lambda \)-parameter of \( \tilde{G}_{n}(\mathbb{A}) \) associated to \( \tau \). If \( \tau \) is good at \( v \), the local component group \( S_{\phi_{v}} \) of the centralizer...
of the image of \( \phi_v \) in \( G_n(\mathbb{C}) \) is not singleton. Therefore, \( z(S_{\phi,A}) \), the number of elements in the group \( S_{\phi,A} := \prod_v S_{\phi_v} \) is greater than or equal to 2. Let \( \hat{S}_{\phi,A} \) be the group of continuous characters of \( S_{\phi,A} \). By [13] Theorem 1.4, Proposition 4.1], \( z(\Pi_\phi) \) the number of elements in global \( A \)-packet associated to \( \phi \) is \( \frac{z(\hat{S}_{\phi,A})}{2} \). Therefore, if \( \tau \) is good at some place ensures that there exists an irreducible unitary cuspidal automorphic representation \( \pi \) of \( \hat{G}_n(\mathbb{A}) \) such that the \( L \)-parameter of \( \pi_v \) is \( \phi_v \) for almost all places of \( F \). It plays a similar role as the assumption that \( \phi \) is trivial because \( \omega_\tau \) is a quadratic character of \( \mathbb{A}^\times/F^\times \) and therefore, it is

Now, we can state a conjecture which can be regarded as a (weak) generalization of Theorem 1.2.

**Conjecture 7.3.** Let \( \tau_0 \) be an irreducible unitary cuspidal automorphic representation of \( GL_{2n}(\mathbb{A}) \) of symplectic type such that the \( \tau_{0,v} \) is good at some place \( v_0 \). Then there exists a quadratic character \( \eta : \mathbb{A}^\times/F^\times \rightarrow \mathbb{C}^\times \) such that \( L(\frac{1}{2}, \tau_0 \otimes \eta) \neq 0 \).

When \( n = 1 \), this conjecture follows from the result of Friedberg and Hoffstein [10]. Recently, B. Liu and B. Xu [36, Theorem 6.8] reproved this result by combining the theory of twisted automorphic descents and the global GGP conjecture for \( \hat{G}_1(\mathbb{A}) \times \hat{G}_1(\mathbb{A}) \). By extending B. Liu and B. Xu’s arguments, we prove Conjecture 7.3 under two reasonable hypotheses. To explicate such hypotheses, we require some notations and terminologies related to the theory of twisted automorphic descents.

Let \( \pi' \) be an automorphic representation of \( \hat{G}_n(\mathbb{A}) \) and choose an automorphic form \( \phi \in \pi' \). For an integer \( 0 \leq k < n \) and \( \alpha \in F \), define the depth-\( k \) Fourier–Jacobi coefficients of \( \phi \) as

\[
FJ^{k,f}_{\psi,\alpha}(\phi)(g) := \int_{[N_{n,k+1}]} \varphi(n\tilde{g}) \Theta_{\psi,\alpha}^{-1} \cdot ((n, \tilde{g}), f) \, dn, \quad f \in \nu_{\psi,\alpha}^{-1} \cdot W_{n,k-1}.
\]

It is easy to check that \( FJ^{k,f}_{\psi,\alpha}(\phi) \) is an automorphic form on \( G_{n-k-1}(\mathbb{A}) \). We denote by \( D^k_{\psi,\alpha}(\pi') \) the automorphic representation of \( G_{n-k-1}(\mathbb{A}) \) generated by \( \{FJ^{k,f}_{\psi,\alpha}(\phi) \mid \phi \in \pi', f \in \nu_{\psi,\alpha}^{-1} \cdot W_{n,k-1} \} \). When \( \pi' \) is a residual Eisenstein series representation, \( D^k_{\psi,\alpha}(\pi') \) is called a twisted automorphic descent of \( \pi' \).

For an irreducible unitary cuspidal automorphic representation \( \tau \) of \( GL_{2n}(\mathbb{A}) \) of symplectic type, we denote by \( \phi_\tau \) the generic global \( A \)-parameter of \( \hat{G}_n(\mathbb{A}) \) associated to \( \tau \).

The following is a generalization of [36, Proposition 3.1, Proposition 4.3].

**Proposition 7.4.** Let \( \tau_0 \) be an irreducible unitary cuspidal automorphic representation of \( GL_{2n}(\mathbb{A}) \) of symplectic type. For some \( m \geq n \), let \( \tau \) be an irreducible generic isobaric sum automorphic representation

\[
\tau = \tau_1 \boxplus \cdots \boxplus \tau_l
\]

of \( GL_{2m}(\mathbb{A}) \) such that \( \tau_i \)'s are irreducible cuspidal automorphic representation of \( GL_{m_i}(\mathbb{A}) \) of orthogonal type with \( m_i \geq 1 \). For an irreducible unitary genuine cuspidal automorphic representation \( \pi' \) of \( \hat{G}_n(\mathbb{A}) \) with generic \( A \)-parameter \( \phi_{\tau_0} \), \( D^k_{\psi,\alpha}(\mathcal{E}_{\tau \otimes \pi'}) \) is zero for all \( k \geq n + m \) and \( \alpha \in F \).

Furthermore, if there is some \( \beta \in F \) such that the automorphic descent \( D^{n+m-1}_{\psi,\beta}(\mathcal{E}_{\tau \otimes \pi'}) \) is non-zero, then any irreducible constituent \( \pi_\beta \) of \( D^{n+m-1}_{\psi,\beta}(\mathcal{E}_{\tau \otimes \pi'}) \) has a generic \( A \)-parameter.

**Proof.** For \( k \geq 1 \), let \( \pi_{k,\alpha} \) be any irreducible constituent of \( D^k_{\psi,\alpha}(\mathcal{E}_{\tau \otimes \pi'}) \) and we decompose \( \pi_{k,\alpha} \simeq \prod_v \Pi_{k,v} \).

Fix a place \( v \) of \( F \) such that \( \tau_v, \pi_v, \psi_v, (\mathcal{E}_{\tau \otimes \pi'})_v \) and \( \Pi_{k,v} \) are all unramified and \( \pi'_v \) is generic. Furthermore, we may also assume that \( \omega_{\tau_v} \) is trivial because \( \omega_\tau \) is a quadratic character of \( \mathbb{A}^\times/F^\times \) and therefore, it is
trivial for infinitely many places. From now on, we suppress the subscript \( v \) and \( F_v \) from the notation. From the condition on \( \tau \), we can write

\[
\tau = \text{Ind}^{G_{L2m}}_{B_{2m}}(\mu_1 \times \cdots \times \mu_m \times \mu_m^{-1} \times \cdots \times \mu_1^{-1})
\]

for some unramified characters \( \mu_i \)'s of \( F^\times \).

Note that \( \mathcal{E}_{\tau \otimes \pi'} \) is the unramified constituent of \( \text{Ind}^{G_{2m+n-1}}_{P_{2m+n,2m}}(\tau_\psi) \otimes \pi') \). Since \( \mathcal{E}_{\tau \otimes \pi'} \) is unramified, it is again that of \( \text{Ind}^{G_{2m+n}}_{P_{2m+n,2m}}(\tau'_\psi \boxtimes \pi') \), where \( \tau' = \text{Ind}^{G_{2m}}_{\Omega_{2m}}(\mu_1(2) \times \cdots \times \mu_m(2)) \). Therefore, \( \Pi_{k,v} \) is the unramified constituent of the Jacquet module \( J_{\psi_k}(\text{Ind}^{G_{2m+n}}_{P_{2m+n,2m}}(\tau'_\psi \boxtimes \pi') \otimes \Omega_{\alpha,W_{2m+n-(k+1)}}) \). Using (3.11), we can write this as

\[
J_{HW_{2m+n-(k+1)}}/C_k(J_{\psi_{\alpha,k}}(\text{Ind}^{G_{2m+n}}_{P_{2m+n,2m}}(\tau'_\psi \boxtimes \pi')) \otimes \Omega_{\alpha,W_{2m+n-(k+1)}}).
\]

Since \( \tau'(t) \) and \( \tau'_v(t-1) \) vanish for \( t \geq m \) by Remark 3.2 when \( k \geq n + m \), if we apply Theorem 3.3 with \( j = 2m \), \( \tau = \tau' \), \( \sigma = \pi' \) and \( l = k \), then \( J_{\psi_{\alpha,k}}(\text{Ind}^{G_{2m+n}}_{P_{2m+n,2m}}(\tau'_\psi \boxtimes \pi')) \) vanishes and so \( J_{\psi_k}(\text{Ind}^{G_{2m+n}}_{P_{2m+n,2m}}(\tau'_\psi \boxtimes \pi') \otimes \Omega_{\alpha,W_{2m+n-(k+1)}}) \) also vanishes. This proves the first assertion.

Now we consider the case \( k = n + m - 1 \). In this case, by Theorem 3.3 and (3.2),

\[
J_{\psi_{\alpha,n+m-1}}(\text{Ind}^{G_{2m+n}}_{P_{2m+n,2m}}(\tau'_\psi \boxtimes \pi')) \simeq \text{ind}^{G_{M \vee H_{W_{m}}}_{P_{m}}}_{P_{m}}(\text{Ind}^{GL_m}_{B_m}(\mu_1 \psi \times \cdots \times \mu_m \psi) \otimes J_{\psi_{\alpha,n-1}}(\pi'))
\]

where \( \widetilde{P}_{m} \) is the same as being defined in [8 (6.4)].

Write \( \pi' = \text{Ind}^{G_{m}}_{P_{m-1,1}}(\tau'_\psi \boxtimes \text{Ind}^{G_{1}}_{B_1}(\mu'_n \psi)) \), where \( \pi'' = \text{Ind}^{GL_{n-1}}_{B_{n-1}}(\mu'_1 \times \cdots \times \mu'_{n-1}) \) for some unramified characters \( \mu'_i \)'s and \( \mu'_n \) of \( F^\times \). By [2 Corollary B.3] and [3 Theorem 1.1], \( \text{Ind}^{G_{1}}_{B_1}(\mu'_n \psi) \) is irreducible and generic. As explained in the previous paragraph, by applying Theorem 3.3 again, we have

\[
J_{\psi_{\alpha,n-1}}(\pi') \simeq J_{\psi_{\alpha,0}}(\text{Ind}^{G_{1}}_{B_1}(\mu'_n \psi)) = 1
\]

because \( | \cdot |^{2m} \cdot (\tau''(n-1) = 1 \) by (3.2).

Then by Theorem 3.4, we have

\[
J_{\psi_{n+m-1}}(\text{Ind}^{G_{2m+n}}_{P_{2m+n,2m}}(\tau'_\psi \boxtimes \pi') \otimes \Omega_{\alpha,W_{m}}) \simeq \text{Ind}^{G_{m}}_{B_m}(\eta_0 \cdot (\mu_1 \times \cdots \times \mu_m)),
\]

where \( \eta_0 \) comes from the Weil representation \( \Omega_{\alpha,W_{m}} \).

Therefore, there are infinitely many local components of \( \pi_{k,0} \), which are subquotients of representations of the form \( \text{Ind}^{G_{m}}_{B_m}(\eta_0 \cdot (\mu_1 \times \cdots \times \mu_m)) \). From this, we see that \( \pi_{k,0} \) has a generic global \( A \)-parameter because if it is not, then it would have non-generic local components at almost all places. \( \square \)

The following two hypotheses are what expected to hold in general. Both are crucial to prove Theorem C.

**Hypothesis 7.5.** Let \( \tau_0 \) be an irreducible unitary cuspidal automorphic representation of \( GL_{2n}(A) \) of symplectic type. Then there exists an irreducible generic isobaric sum automorphic representation

\[
\tau = \tau_1 \boxplus \cdots \boxplus \tau_t
\]

of \( GL_{2n}(A) \) such that \( m \geq n \) and \( \tau_i \)'s are irreducible cuspidal automorphic representation of \( GL_n(A) \) of orthogonal type with \( n_i > 1 \) and \( L(\frac{1}{2}, \tau \times \tau_0) \neq 0 \).
Hypothesis 7.6. Let $\tau_0$ be an irreducible unitary cuspidal automorphic representation of $GL_{2n}(\mathbb{A})$ of symplectic type. Let $\pi$ be an irreducible generic isobaric sum automorphic representation of $GL_{2n}(\mathbb{A})$ satisfying the Hypothesis 7.3. If there is an irreducible unitary genuine cuspidal automorphic representation $\pi'$ of $G_n(\mathbb{A})$ with generic $A$-parameter $\phi_{\eta_0}$, there is some $\beta \in F$ such that the twisted automorphic descent $D_{\psi_0}^{n+m-1}(E_{\tau \otimes \pi'})$ is a non-zero automorphic representation of $G_m(\mathbb{A})$. Furthermore, it satisfies the following two properties;

(i) $D_{\psi_0}^{n+m-1}(E_{\tau \otimes \pi'})$ is cuspidal 

(ii) any irreducible constituent $\pi_\beta$ of $D_{\psi_0}^{n+m-1}(E_{\tau \otimes \pi'})$ has a non-zero $\psi_\beta$-Fourier–Jacobi period with $\pi'$.

Remark 7.7.

(i) In [57 Theorem 1.2] and [49 Theorem 1.2], W. Zhang and his collaborators proved that given a cuspidal automorphic representation $\pi_{n+1}$ of $GL_{n+1}(\mathbb{A}_E)$, there exists a cuspidal automorphic representation $\pi_n$ such that $L(1/2, \pi_{n+1} \times \pi_n) \neq 0$. Here, $E/F$ denotes a quadratic extension of number fields. To prove this, they employed the (⇒) direction of the global Gan-Gross-Prasad conjecture for unitary groups and the local Gan-Gross-Prasad conjecture for general linear groups, leveraging the isomorphism between the unitary group and the general linear group at split places of $F$. However, this approach cannot be directly applied to Hypothesis 7.5 due to the absence of a similar bridge between symplectic/metaplectic groups and general linear groups. Instead, when $n = 1$, Hypothesis 7.5 is proved in the proof of [36 Theorem 5.10]. In the case of general $n$, once the non-vanishing of $D_{\psi_0}^{n+m-1}(E_{\tau \otimes \pi'})$ is proved, then (ii) will follow from a reciprocity formula [29 Theorem 3.2] (see [36 Proposition 2.1].)

Now we are ready to prove Theorem C.

Proof of Theorem C. Since $\tau_0$ is good at $v_0$, as discussed in Remark 7.2, there exists an irreducible unitary genuine cuspidal automorphic representation $\pi'$ of $G_n(\mathbb{A})$ with generic $A$-parameter $\phi_{\eta_0}$. Then by Proposition 7.4, Hypothesis 7.5 and Hypothesis 7.6, any irreducible constituent $\pi_\alpha$ of $D_{\psi_0}^{n+m-1}(E_{\tau \otimes \pi'})$ has a generic $A$-parameter $\phi_\alpha'$ of the form $\phi_\alpha' = \phi_{\tau'} \boxplus \phi_{\eta_\alpha}$, where $\tau'$ is an irreducible cuspidal automorphic representation of $GL_{2m}(\mathbb{A})$ and $\gamma \in F^\times$ such that $\omega_{\tau'} = \eta_\gamma$. Therefore, by Hypothesis 7.6 (ii) and our main theorem,

$$L_{\psi_0}(\frac{1}{2}, \pi_\alpha \times \pi') = L(\frac{1}{2}, (\tau' \boxplus \eta_\gamma) \times \tau_0 \cdot \eta_\alpha) = L(\frac{1}{2}, \tau' \times \tau_0 \cdot \eta_\alpha) \cdot L(\frac{1}{2}, \eta \times \tau_0) \neq 0,$$

where $\eta = \eta_\gamma \cdot \eta_\alpha$ (for the first identity, see [17 Sect. 3.1, page 198].) Since $\eta$ is a quadratic character and $L(\frac{1}{2}, \eta \times \tau_0) \neq 0$, it completes the proof. □

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