Susy $CP^{N-1}$ model and surfaces in $\mathbb{R}^{N^2-1}$

V. Hussin

Centre de recherches mathématiques et Département de mathématiques et de statistique,
Université de Montréal, C.P. 6128, succ. Centre-ville,
Montréal (Québec), H3C 3J7 Canada.

W. J. Zakrzewski

Department of Mathematical Sciences,
University of Durham, Durham DH1 3LE, United Kingdom

(Dated: July 13, 2021)

We describe surfaces in $\mathbb{R}^{N^2-1}$ generated by the holomorphic solutions of the supersymmetric $CP^{N-1}$ model. We show that these surfaces are described by the fundamental projector constructed out of the solutions of this model and that in the $CP^{N-1}$ case the corresponding surface is a sphere. Although the coordinates of the sphere are superfields the sphere’s curvature is constant. We show that for $N > 2$ the corresponding surfaces can also be constructed from the similar projector.

PACS numbers: 03.65.Fd, 02.20.–a, 42.50Ar

I. INTRODUCTION

The subject of Weierstrass representations of surfaces immersed in multidimensional spaces was introduced few years ago by Konopelchenko et al.\cite{1,2}. This has generated interest in looking at the properties of these surfaces and relating them to the solutions of the $CP^{N-1}$ model. Recently one of us (WJZ), together with Grundland\cite{5} presented a general procedure for the construction of such surfaces from the harmonic $CP^{N-1}$ maps. This approach involved writing the equation for the harmonic map as a conservation law and then observing that in this construction a special operator played a key role. This operator, related to the fundamental projector of the harmonic map was then used in the construction of the surface.

The $CP^{N-1}$ model has been supersymmetrised thus giving us supersymmetric harmonic maps. The question then arises what surfaces these supersymmetric maps correspond to and what properties they have. This is the problem that is studied in this paper.

In the next section we briefly review the supersymmetric $CP^{N-1}$ harmonic maps (using the formalism as given in \cite{7}. We then construct the operators which are the supersymmetric generalisation of the operators of the purely bosonic maps. We also construct the Weierstrass surfaces and show that, like in the purely bosonic model, the surfaces are described by the projector of the harmonic map. This allows us to show that in the $CP^1$ case, like in the corresponding bosonic case, the resultant surface is the surface of a sphere. In the final sections of the paper we discuss its properties and present a short discussion of the corresponding surfaces for $N > 2$. 

\footnote{Electronic address: hussin@dms.umontreal.ca}

\footnote{Electronic address: w.j.zakrzewski@durham.ac.uk}
II. FORMALISM

A. The supersymmetric $CP^{N-1}$ model

We are interested here in the supersymmetric (SUSY) $CP^{N-1}$ model which is constructed on the two-dimensional superspace $(x, y, \theta_1, \theta_2)$ where the anticommuting quantities $\theta_1$ and $\theta_2$ denote two components of a majorana spinor $\theta$ and can be thought of as being real. For our considerations, a better choice of coordinates will be the complexified superspace $(x^+, x^-, \theta^+, \theta^-)$ where

$$x_{\pm} = x \pm iy, \quad \theta_{\pm} = \theta_1 \pm i\theta_2.$$  \hfill (II.1)

We consider in particular a complex bosonic superfield which is a $N$-column vector defined as

$$\Phi(x^+, x^-, \theta^+, \theta^-) = z(x^+, x^-) + i\theta^+_+ \chi^+_+(x^+, x^-) + i\theta^-_- \chi^-_-(x^+, x^-) - \frac{1}{2}\theta^+_+ \theta^-_- F(x^+, x^-),$$  \hfill (II.2)

where $z$, $F$ are $N$-component bosonic fields and $\chi^+_+$, $\chi^-_-$ are $N$-component fermionic fields. Since the fermionic fields $\chi^+_+$, $\chi^-_-$ anticommute with each other and with $\theta^+_+$, $\theta^-_-$, the hermitian conjugate of the superfield $\Phi$ is given by

$$\Phi^\dagger(x^+, x^-, \theta^+, \theta^-) = z^\dagger(x^+, x^-) + i\theta^-_- \chi^+_+(x^+, x^-) + i\theta^+_+ \chi^-_-(x^+, x^-) - \frac{1}{2}\theta^+_+ \theta^-_- F^\dagger(x^+, x^-).$$  \hfill (II.3)

In the SUSY $CP^{N-1}$ model, $\Phi$ satisfies $\Phi^\dagger \Phi = 1$. In terms of $z$, $\chi^+_+$, $\chi^-_-$ and $F$, this condition writes

1) if $\Phi$ is a bosonic superfield, we have

$$z^\dagger z = 1, \quad \chi^+_+ z + z^\dagger \chi^-_- = 0, \quad F^\dagger z + z^\dagger F = 2(\chi^+_+ \chi^-_- - \chi^+_+ \chi^-_+).$$  \hfill (II.4-6)

2) if $\Psi$ is a fermionic superfield, we have

$$\hat{\partial}_{\mp} = -i\partial_{\mp} + \theta_{\mp} \partial_{\pm}.$$  \hfill (II.7)

They are fermionic and satisfy anticommuting properties that we have to take into account in the calculations. For example, the following relations will be useful later:

3) In general, we have

$$\hat{\partial}_{\mp} \hat{\partial}_{\mp} = -i\partial_{\mp}.$$  \hfill (II.10)

Let us recall that we are considering SUSY models. This means that the corresponding Lagrangian density and equations of motion must be expressed in terms of the superfields...
\( \Phi, \Phi^\dagger \) and the associated supercovariant derivatives. A definition of these supercovariant derivatives has thus to be given. Let us note that they will be dependant on the superfields \( \Phi \) and \( \Phi^\dagger \) and will be defined as acting on bosonic as well as fermionic superfields. We get

\[
\hat{D}_\pm \Lambda = \hat{\partial}_\pm \Lambda - \Lambda A_\pm, \quad A_\pm = \Phi^\dagger \hat{\partial}_\pm \Phi, \tag{II.11}
\]

where \( \Lambda \) is an arbitrary homogeneous (bosonic or fermionic) superfield. In our SUSY \( CP^{N-1} \) model, the quantities \( A_\pm \) are scalar fermionic superfields. In particular, we have

\[
(\hat{D}_\pm \Phi)^\dagger = \hat{\partial}_\mp \Phi^\dagger (\mathbb{I} - \mathbb{P}). \tag{II.12}
\]

We can now write both the Lagrangian density and the equations of motion of our model as:

\[
\mathcal{L} = 2(|\hat{D}_+ \Phi|^2 - |\hat{D}_- \Phi|^2) \tag{II.14}
\]

and

\[
\hat{D}_+ \hat{D}_- \Phi + |\hat{D}_- \Phi|^2 \Phi = 0. \tag{II.15}
\]

Similarly to the case of the non-SUSY \( CP^{N-1} \) model, we can introduce the following spectral equations \((\lambda \in \mathbb{R})\)

\[
\hat{\partial}_+ \Lambda = \frac{2}{1 + \lambda} \mathbb{K}^\dagger \Lambda, \quad \hat{\partial}_- \Lambda = \frac{2}{1 - \lambda} \mathbb{K} \Lambda, \tag{II.16}
\]

where

\[
\mathbb{K} = [\hat{\partial}_- \mathbb{P}, \mathbb{P}], \quad \mathbb{K}^\dagger = [\hat{\partial}_+ \mathbb{P}, \mathbb{P}]. \tag{II.17}
\]

So the equation of motion (II.15) is a compatibility condition for these spectral equations that could be written as a superconservation law:

\[
\hat{\partial}_+ \mathbb{K} + \hat{\partial}_- \mathbb{K}^\dagger = 0. \tag{II.18}
\]

Let us now show that \( \mathbb{K} = \mathbb{M} + \mathbb{L} \) is in fact a linear combination of two distinct conserved quantities. Indeed, since we have \( \Phi^\dagger \Phi = 1 \), we can set

\[
\Phi = |w|^{-1} w \tag{II.19}
\]

and

\[
\mathbb{P} = \Phi \Phi^\dagger = |w|^{-2} w w^\dagger. \tag{II.20}
\]

Let us recall that we thus get

\[
tr\mathbb{P} = 1, \quad \text{and} \quad \det \mathbb{P} = 0. \tag{II.21}
\]

Now \( \mathbb{K} = \mathbb{M} + \mathbb{L} \) takes the form

\[
\mathbb{K} = [\hat{\partial}_- \mathbb{P}, \mathbb{P}] = |w|^{-2}(\hat{\partial}_- w w^\dagger - w \hat{\partial}_- w^\dagger) + |w|^{-4}(\hat{\partial}_- w^\dagger w - w^\dagger \hat{\partial}_- w)w w^\dagger. \tag{II.22}
\]
Setting
\[ \mathcal{M} = (\mathbb{I} - \mathcal{P}) \frac{\partial_- w }{|w|^2} \quad \mathcal{L} = - \frac{w}{|w|^2} \frac{\partial_- w |w|^2}{|w|^2} (\mathbb{I} - \mathcal{P}), \]  
we easily get \( \mathbb{K} = \mathcal{M} + \mathcal{L} \). Since we also have
\[ \mathcal{M} - \mathcal{L} = \frac{\partial_- |w|}{|w|} - \frac{1}{|w|} \frac{\partial_- |w|^2}{|w|^2}, \]
(II.23)
\[ \mathbb{L} \quad \text{and} \quad \mathbb{M} \] are conserved.

Incidently the equations (II.15) when written in terms of \( w \), take the form:
\[ \frac{1}{|w|^2} \frac{\partial_- w }{w} - \frac{1}{|w|^2} \frac{\partial_- w |w|^2}{|w|^2} + \frac{2}{|w|^2} \frac{\partial_- w }{w} \frac{w}{|w|^2} = 0. \]
(II.25)

**B. Special solutions of the equations of motion**

Let us now take \( \Phi \) as in (II.19) where we assume that
\[ \frac{\partial_- \Phi = (\partial_- |w|^{-1})w + |w|^{-1} \partial_- w = (\partial_- |w|^{-1})w} \]
and
\[ \mathcal{P} \frac{\partial_- \Phi = \partial_- \Phi}, \]
we get
\[ \hat{\mathcal{D}} \Phi = 0. \]
(II.28)
Thus we see that
\[ w = w(x_+, \theta_+) \]
solves the equation of motion (II.15). In analogy with the purely bosonic case we shall call such a solution ‘holomorphic’.

In this case we also have \( \mathcal{M} = 0 \) and
\[ \mathbb{K} = \mathbb{L} = - \frac{\partial_- \mathcal{P} = |w|^{-2}(-w \partial_- w^\dagger) + |w|^{-4}(\partial_- w^\dagger w)ww^\dagger}. \]
(II.30)

Let us now define the bosonic quantity \( \mathcal{L} = -i \partial_- \mathcal{L} \) and the hermitian conjugate which, from (II.9), is given by \( \mathcal{L}^\dagger = i(\partial_- \mathcal{L})^\dagger = -i \partial_+ \mathcal{L}^\dagger \). From (II.30), we get
\[ \mathcal{L} = \partial_- \mathcal{P}, \quad \mathcal{L}^\dagger = \partial_+ \mathcal{P}, \]
(II.31)
so that \( \mathcal{L} \) is conserved in the usual sense, i.e.
\[ \partial_+ \mathcal{L} + \partial_- \mathcal{L}^\dagger = 0. \]
(II.32)
Similarly as in the non-SUSY case, we can construct
\[ X = \int_\gamma \mathcal{L} \, dx_- + \int_\gamma \mathcal{L}^\dagger \, dx_+, \]
(II.33)
which is independent on the contour of integration and we see that

\[ X = \mathbb{P}. \]  

(II.34)

So, our surface is described by the projector \( \mathbb{P} \).

As \( tr \mathbb{P} = 1 \) not all components of \( \mathbb{P} \) are independent, so if we want to think of a vector \( X \) describing our surface - we can take it as a vector with \( N \times N - 1 \) components constructed from the entries of \( \mathbb{P} \).

III. **THE CP\(^1\) CASE.**

A. **Explicit form of X**

Now we look at the case of \( CP^1 \). In this case all our original vectors have only two components. Thus \( \mathbb{P} \) is a \( 2 \times 2 \) matrix which can be written as

\[ \mathbb{P} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \frac{1}{2}(\mathbb{I} + X_i \sigma_i), \tag{III.1} \]

where

\[ X_1 = P_{12} + P_{21}, \quad X_2 = i(P_{12} - P_{21}), \quad X_3 = P_{11} - P_{22}. \tag{III.2} \]

Then using (II.21) we easily get

\[ X_1^2 + X_2^2 + X_3^2 = 1. \tag{III.3} \]

This suggests that we take for our 3 component vector \( X \) the vector whose components are given by the quantities given above. Given this choice we see that our surface is the surface of a sphere or radius 1.

To get the explicit form of \( X_i \) we can proceed as follows:

First, using the overall gauge freedom, we can choose

\[ w = \begin{pmatrix} 1 \\ W \end{pmatrix}, \tag{III.4} \]

and so we see that, effectively we are dealing with a bosonic superfunction \( W \). Of course now \( \mathbb{P} \) is given by

\[ \mathbb{P} = \frac{1}{1 + |W|^2} \begin{pmatrix} 1 & W^\dagger \\ W & |W|^2 \end{pmatrix}, \tag{III.5} \]

and the components of the vector \( X \) are given by

\[ X_1 = \frac{W + W^\dagger}{1 + |W|^2}, \quad X_2 = \frac{i(W^\dagger - W)}{1 + |W|^2}, \quad X_3 = \frac{1 - |W|^2}{1 + |W|^2}. \tag{III.6} \]

However, these fields are just the fields of the alternative \( (S^2) \) description of the \( CP^1 \) model. The relation between them is given by

\[ X_i = \Phi^\dagger \sigma_i \Phi. \tag{III.7} \]

Thus the situation is the same as in the purely bosonic case.
In that case we also knew that for holomorphic solutions of the \( CP^1 \) model the generated surface corresponded to a sphere.

Our result showing that this surface is described by the projector \( \mathbb{P} \), and then the surface vector \( X \) which is constructed from this projector, in fact, corresponds to the alternative formulation of the model, is not altered by the supersymmetrisation of the model.

For the solutions of the equations of motion (III.15) we have

\[
W = f + i \theta^+ g, \quad \text{(III.8)}
\]

where \( f \) and \( g \) are, respectively, bosonic and fermionic functions of \( x_+ \).

Putting all the expressions in (III.6) we see that the explicit form of the vector \( X \) is given by

\[
X_1 = \frac{(f + \bar{f})}{1 + |f|^2} + i \theta^+ \bar{g}(1 - f^2) \left( \frac{1}{1 + |f|^2} \right) + i \theta^+ g(1 - \bar{f}^2) \left( \frac{1}{1 + |f|^2} \right) + 2 \theta^+ \theta^- \bar{g}g(f + \bar{f}) \left( \frac{1}{1 + |f|^2} \right)^2,
\]

\[
X_2 = i \left( \frac{f - \bar{f}}{1 + |f|^2} - \theta^+ \bar{g}(1 + f^2) \left( \frac{1}{1 + |f|^2} \right) - \theta^+ g(\bar{f}^2 + 1) \left( \frac{1}{1 + |f|^2} \right) + 2 i \theta^+ \theta^- \bar{g}g(f - \bar{f}) \left( \frac{1}{1 + |f|^2} \right)^2, \quad \text{(III.9)}
\]

\[
X_3 = \frac{(1 - |f|^2)}{1 + |f|^2} - 2 i \theta^+ \bar{g}f \left( \frac{1}{1 + |f|^2} \right)^2 - 2 i \theta^+ g(\bar{f}) \left( \frac{1}{1 + |f|^2} \right)^2 + 2 \theta^+ \theta^- \bar{g}g(1 - |f|^2) \left( \frac{1}{1 + |f|^2} \right)^3.
\]

We note that although the components of \( X \) satisfy (III.3) they are, in fact, superfields - ie they have fermionic parts.

**B. Metric**

Next we look at the metric induced on the surface and its curvature.

We introduce the metric by putting

\[
g_{ij} = \partial_i X_k \partial_j X_k, \quad \text{(III.10)}
\]

where the sum goes over all the components of \( X \).

However, it is more convenient to change variable to the holomorphic basis and so introduce \( g_{\pm \pm} \), where the indices +(-) denote the \( x_+ \)(\( x_- \)) components of the metric. Then, as we shall see below, only the \( g_{+-} = g_{-+} \) components are nonzero.

Note that as our vector \( X \) is constructed from the components of \( \mathbb{P} \) we have

\[
g_{\pm \pm} = tr \partial_{\pm} \mathbb{P} \partial_{\pm} \mathbb{P}. \quad \text{(III.11)}
\]

Then as

\[
\partial_+ \mathbb{P} = -(\mathbb{I} - \mathbb{P}) \frac{\partial_+ w w^+}{|w|^2} \quad \text{(III.12)}
\]

we see that

\[
\partial_+ \mathbb{P} \partial_+ \mathbb{P} = 0, \quad \text{(III.13)}
\]

and so we see that \( g_{++} = 0 \).

Of course \( g_{--} \) also vanishes as it is given by \( g_{--} = \bar{g}_{++} \).
However \( g_{+-} \) is nonzero. To calculate it we note that its is given by
\[
\frac{\partial_+ W \partial_- \bar{W}}{[1 + |W|^2]^2}. \tag{III.14}
\]
Note that this expression, superficially, is similar to the energy density. It would have been it had the derivatives been \( \bar{\partial} \) and not \( \partial s \). As \( W \) is a superfield \( g_{+-} \) is a superfield too. So what are its components?

Clearly, the bosonic part, which comes from putting \( \theta_{\pm} = 0 \) in (III.14) is given by
\[
\frac{\partial_+ f \partial_- \bar{f}}{[1 + |f|^2]^2}. \tag{III.15}
\]
It is the bosonic energy density \( ie \) the term that we get in a nonsupersymmetric version of the problem. Calculating the other parts of \( g_{+-} \) we obtain the complete result as
\[
g_{+-} = \frac{\partial_+ f \partial_- \bar{f}}{[1 + |f|^2]^2} + i\theta_+ \partial_+ \left( \frac{\bar{g} \partial_- \bar{f}}{[1 + |f|^2]^2} \right) + i\theta_- \partial_- \left( \frac{\bar{g} \partial_+ f}{[1 + |f|^2]^2} \right) - \theta_+ \theta_- \partial_+ \partial_- \left( \frac{\bar{g} g}{[1 + |f|^2]^2} \right). \tag{III.16}
\]
Hence we see that the metric does have fermionic corrections but, as they are total derivatives, they average to zero (\( ie \) vanish after integration over \( x_+ \) and \( x_- \)).

C. Curvature

Next we calculate the curvature of our metric. As the metric has only the \( g_{+-} \) component the curvature is given by
\[
K = -2 \frac{1}{F} \partial_+ \partial_- \ln F, \tag{III.17}
\]
where \( F = \frac{1}{2} g_{+-} \).

To perform the calculation we note that
\[
\ln \left( \frac{\partial_+ W \partial_- \bar{W}}{[1 + |W|^2]^2} \right) = \ln (\partial_+ W) + \ln (\partial_- \bar{W}) - 2 \ln ([1 + |W|^2]). \tag{III.18}
\]
However as \( W = W(x_+, \theta_+) \) only the first two terms in (III.18) vanish when one applies to them \( \partial_+ \partial_- \) and so we get
\[
K = -\frac{[1 + |W|^2]^2}{\partial_+ W \partial_- \bar{W}} (-2) \partial_+ \partial_- (\ln [1 + |W|^2]) = 2. \tag{III.19}
\]
Thus the curvature is purely bosonic and, as expected, is 2.

In a way this may be not unexpected as our surface is a surface of a sphere. However, it is interesting that although the coordinates of this surface are superfields and the induced metric is also described by a superfield all the fermionic effects cancel and the curvature is just \( K = 2 \). Hence the fermionic modification does not alter the curvature of the surface.

IV. WEIERSTRASS SYSTEM FOR \( CP^1 \)

Let us recall the regular Weierstrass problem for the nonsupersymmetric \( CP^1 \) system. In this case one considers two complex functions \( \psi, \phi \) of \( x_+ \) and \( x_- \) which satisfy the equations
\[
\partial_+ \psi = (|\psi|^2 + |\phi|^2) \phi, \quad \partial_- \phi = -(|\psi|^2 + |\phi|^2) \psi. \tag{IV.1}
\]
Then to find a solution of these two equations one can put
\[ V = \frac{\psi}{\phi} \]  
and eliminate \( \psi \). Then one rewrites (IV.1) as
\[ \partial_+ V = \phi^2 (1 + |V|^2)^2, \quad \partial_- \phi^2 = -2|\phi|^4 V (1 + |V|^2). \]  
Thus
\[ \phi^2 = \frac{\partial_+ V}{(1 + |V|^2)^2} \]  
and we see that \( V \) satisfies
\[ \partial_- \partial_+ V = 2 \frac{\tilde{V} \partial_+ V \partial_- V}{1 + |V|^2}, \]  
\textit{i.e.} the equation of the \( CP^1 \) model.

What is the supersymmetric version of this problem? As we know, in the supersymmetric case, \( V \) becomes \( W \) as in (III.4). Its equation of motion can be deduced easily from (III.4) and (II.25) and it is
\[ \tilde{\partial}_+ \tilde{\partial}_- W = 2 \tilde{V} \tilde{\partial}_+ W \tilde{\partial}_- W \frac{1}{1 + |W|^2}. \]  
Having \( W \) for \( V \), we take \( Z^2 \) as the supersymmetric analogue of \( \phi^2 \) defined in (IV.4). We require that \( W \) and \( Z^2 \) satisfy
\[ \tilde{\partial}_+ \partial_- W = (1 + |W|^2)^2 Z^2, \quad \tilde{\partial}_+ Z^2 = -2 W Z^2 Z^2 (1 + |W|^2). \]  
Note that \( W \) is bosonic while \( Z^2 \) is fermionic. We have
\[ Z^2 = \frac{\tilde{\partial}_+ W}{(1 + |W|^2)^2} \]  
and, as is easy to check, \( W \) solves the equation (IV.6).

Can one take the nonsupersymmetric limit of this problem? This is difficult as \( Z^2 \) is fermionic. However, we can put
\[ \phi^2 = \tilde{\partial}_+ Z^2. \]  
Then, as
\[ \tilde{\partial}_+ \tilde{W} = 0 \]  
we see that
\[ \tilde{\partial}_+ Z^2 = \frac{\tilde{\partial}_+ \tilde{\partial}_+ W}{1 + |W|^2}. \]  
Note that due to (II.10) we see that (up to an over factor \(-i\)) this is the correct expression for \( \phi^2 \) after we have set all \( \theta_\pm = 0 \).
V. GENERALISATION TO $CP^{N-1}$

Some of our results generalise easily to the $CP^{N-1}$ case. This is the case in particular with the projector $\mathbb{P}$ which gives us a surface in $\mathbb{R}^{N^2-1}$ for the $CP^{N-1}$ model.

So our surface is defined in terms of $\mathbb{P}$. How should we then define our vector $X$? A little thought shows that, like in the nonsupersymmetric case, we should take $X$ in such a form that an analogue of (III.1) holds, i.e., $\partial_- X_i \partial_+ X_i$ is proportional to $tr(\partial_- \mathbb{P} \partial_+ \mathbb{P})$ as then $g_{++} = g_{--} = 0$.

This requires that we take off-diagonal entries of matrix $\mathbb{P}$, say $\mathbb{P}_{ij}$ and form from them components $\mathbb{P}_{ij} + \mathbb{P}_{ji}$ and $i(\mathbb{P}_{ij} - \mathbb{P}_{ji})$. For the diagonal entries we have some choice. We want the $N-1$ vector components $X_i$ to be such that

$$\sum_{i=1}^{N-1} \partial_+ X_i \partial_- X_i = 2 \sum_{i=0}^{N} \partial_+ \mathbb{P}_{ii} \partial_- \mathbb{P}_{ii}. \quad (V.1)$$

In the $CP^1$ case this tells us that we should take, as shown in (III.1), $X_3 = \mathbb{P}_{11} - \mathbb{P}_{22}$. For larger $N$ we have more choices; thus for $CP^2$ we can take (this choice is based on Gell Mann’s $SU(3)$ λ matrices)

$$X_1 = \mathbb{P}_{11} - \mathbb{P}_{22}, \quad X_2 = \sqrt{3}(\mathbb{P}_{11} + \mathbb{P}_{22}). \quad (V.2)$$

or we could make another choice. In general, for $CP^2$ we could take

$$\mathbb{P}_{11} = \frac{1}{3} + aX_1 + bX_2, \quad \mathbb{P}_{22} = \frac{1}{3} + cX_1 + dX_2. \quad (V.3)$$

Then we choose $a$, $b$, $c$ and $d$ so that

$$\partial_+ X_1 \partial_- X_1 + \partial_+ X_2 \partial_- X_2 \quad (V.4)$$

give the same expression as

$$\partial_+ \mathbb{P}_{11} \partial_- \mathbb{P}_{11} + \partial_+ \mathbb{P}_{22} \partial_- \mathbb{P}_{22} + \partial_+ \mathbb{P}_{33} \partial_- \mathbb{P}_{33} \quad (V.5)$$

in which we can eliminate $\mathbb{P}_{33}$ by $\mathbb{P}_{33} = 1 - \mathbb{P}_{11} - \mathbb{P}_{22}$.

This guarantees that only $g_{+-}$ is nonzero. A simple calculation shows that we have a one-parameter family of solutions

$$a = \frac{2}{\sqrt{3}} \cos \alpha, \quad b = \frac{2}{\sqrt{3}} \sin \alpha, \quad (V.6)$$

$$c = \mp \sin \alpha - \frac{1}{\sqrt{3}} \cos \alpha, \quad d = -\frac{1}{\sqrt{3}} \sin \alpha \pm \cos \alpha.$$}

For $N > 2$ the solutions are even more nonunique.

Note also that with all these choices we always have

$$g_{+-} = tr(\partial_+ \mathbb{P} \partial_- \mathbb{P}). \quad (V.7)$$

Moreover, the other components of the metric vanish. Thus the metric has a nontrivial dependence on the fermionic degrees of freedom. A simple calculation shows that we can rewrite (V.7) as

$$g_{+-} = (\partial_+ \Phi^\dagger (I - \mathbb{P}) \partial_- \Phi + (\partial_- \Phi^\dagger (I - \mathbb{P}) \partial_+ \Phi). \quad (V.8)$$
This is closely related to the energy density of the original map - in the nonsupersymmetric case it is proportional to this density; this is not the case here as \( \nu \) involves \( \partial \) derivatives and not \( \bar{\partial} \)!

It is easy to see that the fermionic contributions to both the metric and the curvature do not cancel. We have looked at these corrections in the \( CP^2 \) case. Then the vector \( w \) has three components which can be taken in the form

\[
w = \begin{pmatrix}
1 \\
W_1 \\
W_2
\end{pmatrix},
\]

(V.9)

The detailed calculations then show that \( g_{++} \) is again given by the same expression as the energy density of the nonsupersymmetric model with, however, superfields in place of bosonic fields. Thus

\[
g_{++} = \frac{\partial^2 W_1^2 + \partial^2 W_2^2 + |W_2 \partial^2 W_1 - W_1 \partial^2 W_2|^2}{1 + |W_1|^2 + |W_2|^2}. \tag{V.10}
\]

We can now expand this expression in powers of \( \theta \). However, it is easy to check that as, say, the \( \theta_+ \) corrections involve expressions that are not total derivatives. The same is true for the calculation of the curvature. In the \( CP^1 \) case we had the nice factorisation of the terms in \( g_{++} \) leading to the fact that the derivative terms did not contribute to \( \partial_+ \partial_- \ln(g_{++}) \). This was essential for the cancellation of various factors leading to \( K = 2 \). This time the numerator in (V.10) contains 3 terms and it does contribute to \( \partial_+ \partial_- \ln(g_{++}) \).

In consequence \( K \) is not very simple and the fermionic contributions to it do not cancel. We have checked this explicitly but as the obtained expression is quite complicated we do not present it here. Hence, the simple results of the \( CP^1 \) case do not hold any more; both the metric and its curvature are given by full superfields.

VI. CONCLUSIONS

In this paper we have discussed the supersymmetrisation of the Weierstrass problem and extended to the supersymmetric case the work of Grundland et al. Our results have shown that with small modifications the extension has not lead to results which are significantly different from the purely bosonic case. In the \( CP^1 \) case we have again obtained a sphere. Its coordinates are given by real bosonic superfields and, as such, this sphere, is deformed but its fermionic structure. However, these fermionic fields do not play a role in the description of some of its properties; eg in the calculation of the curvature all the fermionic contributions cancel and, as in purely bosonic case, we get \( K = 2 \). They do play a role in the metric - but as they are given by total derivatives, they cancel when we integrate over \( x_+ \) and \( x_- \).

When taking larger \( N \) we have found that, for the holomorphic \( CP^{N-1} \) fields, the projector \( \mathbb{P} \) still describes the surfaces in \( R^{N^2-1} \). This time, however, the curvature is not constant and, furthermore, it contains fermionic corrections.

The more general solutions of the supersymmetric \( CP^{N-1} \) model, for \( N > 2 \), are given by fields which are neither holomorphic nor antiholomorphic. Their description is somewhat complicated due to the constraints of the model. The corresponding surfaces are expected to be more complicated. They have not been studied yet due to these constraint problems which still have to be resolved. This work is currently under consideration.
Acknowledgments

The work reported in this paper was started when V. Hussin visited the University of Durham in Michaelmas term 2005. She would like to thank the University of Durham for the award of a Grey College Fellowship and the Department of Mathematical Sciences for its hospitality.

The research of V. Hussin is partially supported by research grants from NSERC of Canada.

This paper was finished when W. J. Zakrzewski visited the University of Montreal in March 2006. He wishes to thank the University of Montreal for hospitality and the LMS for its travel grant.

1 B. Konopelchenko and I. Taimanov, Constant mean curvature surfaces via an integrable dynamical system, *J. Phys. A* **29**, 1261-1265 (1996).

2 R. Carroll and B. Konopelchenko, Generalised Weierstrass-Enneper inducing conformal immersions and gravity, *Int. J. Mod. Phys. A* **11**, (7), 1183-1216 (1996).

3 B. Konopelchenko and G. Landolfi, Generalised Weierstrass representation for surfaces in multi-dimensional Riemannian spaces, *Stud. Appl. Maths.* **104**, 129-169 (1999) and references therein.

4 P. Bracken and A.M. Grundland, Symmetry properties and explicit solutions of the generalised Weierstrass system, *J. Math. Phys.* **42**, 1250-1282 (2001) and references therein.

5 A.M. Grundland, and W.J. Zakrzewski, $CP^{N-1}$ harmonic maps and the Weierstrass problem, *J. Math. Phys.* **44**, 3370-3382 (2003).

6 A. D’Adda, M. Luscher and P. Di Vecchia, Confinement and chiral symmetry breaking in $CP^{N-1}$ models with quarks, *Nucl. Phys., B* **152**, 125-144, (1979).

7 see eg W.J. Zakrzewski, *Low Dimensional Sigma Models* (Hilger, Bristol, 1989).