A DESCRIPTION OF THE QUANTUM SUPERALGEBRA $U_q[osp(2n+1/2m)]$
VIA GREEN GENERATORS

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1. Introduction. Outline of the results

In the present talk I’ll describe the orthosymplectic Lie superalgebra $osp(2n+1/2m)$ and also its $q$–deformed analogue $U_q[osp(2n+1/2m)]$ in terms of a new set of generators, called Green generators. These generators are very different form the well known Chevalley generators. Let me underline from the very beginning that I am not going to consider new deformation of $U_q[osp(2n+1/2m)]$. The deformation will be the known Hopf algebra deformation as given, for instance, in [1-4]. The description, however, will be given in terms of new free generators.

For me personally the interest in the construction stems from the observation that the Green generators are of a direct physical significance. In a certain representation of $osp(2n+1/2m)$ part of these generators are Bose operators, whereas the rest are Fermi operators. Considered as elements from the universal enveloping algebra, the Green generators are para-Bose and para-Fermi operators [5]. To begin with I’ll state the final result. It is contained in the following:

**Theorem.** $U_q[osp(2n+1/2m)]$ is an associative superalgebra with 1, generators $a_i^\pm$, $L_i$, $\bar{L}_i \equiv L_i^{-1}$, $i = 1, 2, \ldots, m + n = N$, relations ($\xi, \eta = \pm \text{ or } \pm 1$, $\bar{q} \equiv q^{-1}$)

\[
L_i L_i^{-1} = L_i^{-1} L_i = 1, \quad L_i L_j = L_j L_i, \quad L_i a_j^\pm = q^{\pm \delta_{ij} (-1)^i} a_j^\pm,
\]

\[
[a_i^-, a_i^+] = -2 \frac{L_i - \bar{L}_i}{\bar{q} - q}, \quad [[a_i^\eta, a_j^{-\xi}], a_j^\eta]_{q^{-\xi(-1)^{-i} \delta_{ij}}} = 2(\eta^{(j+1)} \delta_{j+1, i} L_j^{-\xi} a_i^\eta,
\]

\[[a_N^\xi, a_N^{-\xi}], a_N^\eta]_{q^{-2}} = 0.
\]

and $\mathbb{Z}_2$-grading induced from

\[
deg(L_i) = \bar{0}, \quad \deg(a_i^\pm) = \langle i \rangle \equiv \begin{cases} 1, & \text{for } i \leq m \\ 0, & \text{for } i > m. \end{cases}
\]

Here and throughout

\[
[x, y]_q = xy - qyx, \quad \langle x, y \rangle_q = xy + qyx, \quad [x, y]_q = xy - (-1)^{\deg(x)\deg(y)} qyx.
\]

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This theorem extends the results of several previous publications. The first deformation of one pair of para-Bose operators was given independently in [6] and [7]. The second paper includes also all Hopf algebra operations. This result was generalized to any number of parabosons in [8, 9], including some representations in the root of unity case [10]. A similar problem for any number of parafermions was solved in [11]. The deformation of one pair of parafermions and one pair of parabosons was carried out in [12]. Finally, the nondeformed version of the present investigation is given in [13].

The plan of the exposition will be the following. First in Sect 2 I’ll recall the definition of the orthosim-
plectic Lie superalgebra (LS) $osp(2n + 1/2m)$ in a matrix form. As next steps, a description of its universal enveloping algebra in terms of operators, called preoscillator generators (Sect. 3), and via Green generators (Sect. 4) will be given. Finally, in Sect. 5 the deformed algebra will be considered and some indications of how the proof of the Theorem goes will be mentioned.

2. Definition of $osp(2n + 1/2m)$ in a matrix form [14]

The Lie superalgebra $osp(2n + 1/2m)$ can be defined as the set of all $(2n + 2m + 1) \times (2n + 2m + 1)$ matrices of the form (T=transposition)

$$
\begin{pmatrix}
  a & b & u & x & x_1 \\
  c & -a^T & v & y & y_1 \\
 -v^T & -u^T & 0 & z & z_1 \\
 y_1^T & x_1^T & z_1^T & d & e \\
 -y^T & -x^T & -z^T & f & -d^T
\end{pmatrix},
$$

(5)

where $a$ is any $n \times n$ matrix, $b$ and $c$ are skew symmetric $n \times n$ matrices, $d$ is any $m \times m$ matrix, $e$ and $f$ are symmetric $m \times m$ matrices, $x, x_1, y, y_1$ are $n \times m$ matrices, $u$ and $v$ are $n \times 1$ columns, $z, z_1$ are $1 \times m$ rows. The even subalgebra consists of all matrices with $x = x_1 = y = y_1 = z = z_1 = 0$, namely

$$
\begin{pmatrix}
  a & b & u & 0 & 0 \\
  c & -a^T & v & 0 & 0 \\
 -v^T & -u^T & 0 & 0 & 0 \\
 0 & 0 & 0 & d & e \\
 0 & 0 & 0 & f & -d^T
\end{pmatrix},
$$

(6)

and it is isomorphic to the Lie algebra $so(2n + 1) \oplus sp(2m)$. The odd subspace is given with all matrices

$$
\begin{pmatrix}
  0 & 0 & 0 & x & x_1 \\
  0 & 0 & 0 & y & y_1 \\
  0 & 0 & 0 & z & z_1 \\
 y_1^T & x_1^T & z_1^T & 0 & 0 \\
 -y^T & -x^T & -z^T & 0 & 0
\end{pmatrix}.
$$

(7)

The product (= the supercommutator) is defined on any two homogeneous elements $a$ and $b$ as

$$
[a, b] = ab - (-1)^{\deg(a)\deg(b)}ba.
$$

(8)

Let $L(n/m)$ be the $2(n + m)$-dimensional $\mathbb{Z}_2$-graded subspace, consisting of all matrices

$$
\begin{pmatrix}
  0 & 0 & u & 0 & 0 \\
  0 & 0 & v & 0 & 0 \\
 -v^T & -u^T & 0 & z & z_1 \\
  0 & 0 & z_1^T & 0 & 0 \\
  0 & 0 & -z^T & 0 & 0
\end{pmatrix}.
$$

(9)
Label the rows and the columns with the indices $A, B = -2n, -2n + 1, \ldots, -1, 0, 1, 2, \ldots, 2m$ and let $e_{AB}$ be a matrix with 1 at the intersection of the $A$th row and the $B$th column and zero elsewhere. Then the following elements (matrices) constitute a basis in $L(n/m)$:

$$
a_i^- = B_i^- = \sqrt{2}(e_{0,i} - e_{i+m,0}), \quad a_i^+ = B_i^+ = \sqrt{2}(e_{0,i+m} + e_{i,0}), \quad i = 1, \ldots, m,$$

$$
a_{j+m}^- = F_j^- = \sqrt{2}(e_{-j,0} - e_{-j-n,0}), \quad a_{j+m}^+ = F_j^+ = \sqrt{2}(e_{-j,-j-n,0}), \quad j = 1, \ldots, n,$$

with $\deg(a_i^\pm) = \langle i \rangle$.

**Proposition 1.** The LS $osp(2n + 1/2m)$ is generated from $a_i^\pm$, $i = 1, \ldots, m + n \equiv N$.

It is straightforward to show that

$$
osp(2n + 1/2m) = \text{lin.env.}\{a_i^-, [a_j^0, a_k^\pm] | i, j, k = 1, \ldots, N, \quad \xi, \eta, \varepsilon = \pm\}.
$$

Hence any further supercommutator between $a_i^\pm$, $[a_j^0, a_k^\pm]$, $\xi, \eta, \varepsilon = \pm$, is a linear combination of the same type elements. A more precise computation gives:

$$\left[[a_i^\pm, a_j^0], a_k^\pm\right] = 2\epsilon^{(k)} \delta_{jk} \delta_{\varepsilon,-\eta} a_i^\pm - 2\epsilon^{(k)} (-1)^{(j)(k)} \delta_{ik} \delta_{\varepsilon,-\xi} a_j^0.
$$

Eqs. (12) are among the supercommutation relations of all Cartan-Weyl generators

$$
a_i^\pm, [a_j^0, a_k^\pm], i, j, k = 1, \ldots, N, \quad \xi, \eta, \varepsilon = \pm.
$$

The rest of the supercommutation relations follow from (12) and the (graded) Jacobi identity ($i, j, k, l = 1, \ldots, N, \quad \xi, \eta, \varepsilon, \varphi = \pm$):

$$\left[[a_i^\pm, a_j^0], [a_k^\pm, a_l^\pm]\right] = 2\epsilon^{(k)} \delta_{jk} \delta_{\varepsilon,-\eta} [a_i^\pm, a_l^\pm] - 2\epsilon^{(k)} (-1)^{(j)(k)} \delta_{ik} \delta_{\varepsilon,-\xi} [a_j^0, a_l^\pm]
$$

$$
- 2\varphi^{(k)} (-1)^{(j)(k)} \delta_{jl} \delta_{\varphi,-\eta} [a_i^\pm, a_k^\pm] + 2\varphi^{(k)} (-1)^{(j)(k)} \delta_{il} \delta_{\varphi,-\xi} [a_j^0, a_k^\pm].
$$

### 3. Description of $U[osp(2n + 1/2m)]$ via preoscillator generators

The relations (12) are representation independent. More precisely, the universal enveloping algebra (UEA) $U(osp(2n + 1/2m))$ of $osp(2n + 1/2m)$ is by definition the (free) associative algebra with 1 of the indeterminates $a_1^\pm, a_2^\pm, \ldots, a_{m+n}^\pm \equiv a_N^\pm$, subject to the relations (12) and (14). Since however Eqs. (14) follow from (12), we have

**Proposition 2.** (1) $U[osp(2n + 1/2m)]$ is the associative unital algebra with generators

$$a_1^+, a_2^+, \ldots, a_{m-1}^+, a_m^+, a_{m+1}^+, \ldots, a_{m+n}^+ \equiv a_N^+,$$

relations

$$[[a_i^\pm, a_j^0], a_k^\pm] = 2\epsilon^{(k)} \delta_{jk} \delta_{\varepsilon,-\eta} a_i^\pm - 2\epsilon^{(k)} (-1)^{(j)(k)} \delta_{ik} \delta_{\varepsilon,-\xi} a_j^0
$$

and $\mathbb{Z}_2$-grading induced from

$$\deg(a_i^\pm) = \langle i \rangle.
$$

(2)

$$osp(2n + 1/2m) = \text{lin.env.}\{a_i^\pm, [a_j^0, a_k^\pm] | i, j, k = 1, \ldots, N, \quad \xi, \eta, \varepsilon = \pm\},$$

with a natural supercommutator (turning every associative superalgebra into a Lie superalgebra):

$$[a, b] = ab - (-1)^{\deg(a)\deg(b)}ba.
$$
The above proposition gives a definition of $U[osp(2n+1/2m)]$ in terms of a new set of generators, which are very different from the Chevalley generators. The relevance of the generators $a_i^{\pm}$ stems from the following observation. The operators $a_i^{\pm} = B_i^{\pm}$ with $i = 1, \ldots, m$ satisfy the triple relations

$$[[B_i^\xi, B_j^n], B_k^n] = 2\varepsilon\delta_{jk}\delta_{\xi,-\eta}B_i^\xi + 2\varepsilon\delta_{ik}\delta_{\xi,-\eta}B_j^n,$$

whereas $a_{i+m}^{\pm} = F_i^{\pm}$ with $i = 1, \ldots, n$ yields:

$$[[F_i^\xi, F_j^n], F_k^n] = 2\delta_{jk}\delta_{\xi,-\eta}F_i^\xi - 2\delta_{ik}\delta_{\xi,-\eta}F_j^n.$$

The relations (20) and (21) are known in quantum field theory. They are defining relations for para-Bose (resp. the para-Fermi) operators. The Green generators (23) are the preoscillator generators of $osp(2n+1/2m)$, whereas $m$ pairs of para-Bose operators generate a Lie superalgebra $[16]$, which is isomorphic to $osp(1/2m)$ [17].

In the Fock representation the para-Bose (resp. the para-Fermi) operators become usual Bose (resp. Fermi) operators, namely oscillator generators. For this reason we call the operators (15) preoscillator (creation and annihilation) generators of $U[osp(2n+1/2m)]$ (resp. of $osp(2n+1/2m)$). The preoscillator generators give an alternative to the Chevalley description of $U[osp(2n+1/2m)]$. Observe that in this setting the para-Bose (resp. the Bose) operators are odd, whereas the para-Fermi (and the Fermi) operators are even generators.

Coming back to the defining relations (16) of the preoscillator generators we note that they define a linear map

$$L(n/m) \otimes L(n/m) \otimes L(n/m) \to L(n/m),$$

which identifies $osp(2n+1/2m)$ also as a Lie-supertriple system, an approach which was recently developed in [18].

Our purpose is to quantize $U[osp(2n+1/2m)]$ via the preoscillator creation and annihilation operators. This is however difficult to be done directly via the relations (16). Therefore in the next section we select a subset of relations from (16), which describe completely $U[osp(2n+1/2m)]$, and which are convenient for quantization.

4. Description of $U[osp(2n+1/2m)]$ via Green generators [13]

Proposition 3. $U[osp(2n+1/2m)]$ is an associative unital superalgebra with generators

$$a_i^{\pm}, a_2^{\pm}, \ldots, a_m^{\pm}, a_{m+1}^{\pm}, a_{m+2}^{\pm}, \ldots, a_{2n+1}^{\pm}, a_{2n+2}^{\pm}, \ldots, a_N^{\pm} = a_N^\pm,$$  

referred as to Green generators, relations $(\xi, \eta = \pm$ or $\pm 1)$

$$[[a_i^\eta, a_j^{-\eta}], a_k^n] = 2\eta(\delta_{jk})\delta_{\xi,-\eta}a_i^n, \quad |i-j| \leq 1, \quad \eta = \pm,$$

$$[[a_N^{-\eta}, a_k^n], a_k^n] = 0, \quad \eta = \pm,$$

and $\mathbb{Z}_2$-grading

$$\deg(a_i^{\pm}) = \langle i \rangle.$$  

The Green generators (23) are the preoscillator generators of $U[osp(2n+1/2m)]$.

In order to indicate how the proof can be done we recall the Chevalley definition of $U[osp(2n+1/2m)]$ and write down explicit relations between the Green and the Chevalley generators. Let $(\alpha_{ij}), \quad i, j = 1, \ldots, N$ be an $N \times N$ symmetric Cartan matrix chosen as:

$$(a_{ij}) = (-1)^{j}\delta_{i+1,j} + (-1)^{i}\delta_{i,j+1} - [(-1)^{j+1} + (-1)^{i}]\delta_{ij} + \delta_{i,m+n}\delta_{j,m+n}.$$
For instance the Cartan matrix of \( B(4/4) \equiv osp(9/8) \) is \( 8 \times 8 \) dimensional matrix:

\[
(a_{ij}) = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
\end{pmatrix}.
\] (27)

Then \( U[osp(2n + 1/2m)] \) is defined as an associative superalgebra with 1 in terms of a number of generators subject to a number of relations. The generators are the Chevalley generators \( h_i, e_i, f_i \), \( i = 1, \ldots, N \); the relations are the Cartan-Kac relations

\[
[h_i, h_j] = 0, [h_i, e_j] = a_{ij} e_j, [h_i, f_j] = -a_{ij} f_j, \ [e_i, f_j] = \delta_{ij} h_i, \tag{28}
\]

the \( e \)-Serre relations

\[
[e_i, e_j] = 0, \text{ for } |i - j| > 1; \quad [e_i, [e_i, e_{i \pm 1}]] = 0, \ i \neq N; \\
\{[e_{m-1}, e_m], [e_m, e_{m+1}]\} = 0; \quad [e_N, [e_N, [e_N, e_{N-1}]]] = 0; \tag{29a}
\]

and the \( f \)-Serre relations

\[
[f_i, f_j] = 0, \text{ for } |i - j| > 1; \quad [f_i, [f_i, f_{i \pm 1}]] = 0, \ i \neq N; \\
\{[f_{m-1}, f_m], [f_m, f_{m+1}]\} = 0; \quad [f_N, [f_N, [f_N, f_{N-1}]]] = 0. \tag{29b}
\]

The grading on \( U[osp(2n + 1/2m)] \) is induced from: \( \deg(e_m) = \deg(f_m) = 1, \ \deg(e_i) = \deg(f_i) = 0 \) for \( i \neq m \).

The expressions of the Green generators in terms of the Chevalley generators read \( i = 1, \ldots, N - 1 \):

\[
a_i^- = (-1)^{(m-i)}(3) \sqrt{2} [e_i, [e_{i+1}, \ldots, [e_{N-2}, [e_{N-1}, e_N] \ldots]]], \quad a_N^- = \sqrt{2} e_N, \\
a_i^+ = -\sqrt{2} [f_i, [f_{i+1}, \ldots, [f_{N-2}, [f_{N-1}, f_N] \ldots]]], \quad a_N^+ = -\sqrt{2} f_N. \tag{30}
\]

Then one proves that \( a_i^\pm \) generate \( U[osp(2n + 1/2m)] \) \( i = 1, \ldots, N - 1 \),

\[
h_i = \frac{1}{2} [a_{i+1}^+, a_{i+1}], \quad h_N = -\frac{1}{2} [a_N^+, a_N], \\
e_i = \frac{1}{2} [a_i^+, a_{i+1}], \quad e_N = \frac{1}{\sqrt{2}} a_N, \\
f_i = \frac{1}{2} [a_i^+, a_i^-], \quad f_N = -\frac{1}{\sqrt{2}} a_N, \tag{31}
\]

and that the Cartan-Kac and the Serre relations follow from (24) and (31).

5. **Description of \( U_q[osp(2n + 1/2m)] \) via deformed Green generators**

The \( q \)-deformed superalgebra \( U_q[osp(2n + 1/2m)] \), a Hopf superalgebra, is by now a classical concept. See, for instance, \([1-4]\) where all Hopf algebra operations are explicitly given. Here, following [4], we write only the algebra operations.
Proposition 4. $U_q[osp(2n + 1/2m)]$ is an associative unital algebra with Chevalley generators $e_i$, $f_i$, $k_i = q^{k_i}$, $k_i \equiv k_i^1 = q^{-k_i}$, $i = 1, \ldots, N$, which satisfy the Cartan-Kac relations

\begin{align}
&k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i, \\
&k_i e_j = q^{a_{ij}} e_j k_i, \quad k_i f_j = q^{-a_{ij}} f_j k_i, \\
&[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}},
\end{align}

(32)

the $e$–Serre relations

\begin{align}
&e_1 \quad [e_i, e_j] = 0, \quad |i - j| \neq 1, \\
&e_2 \quad [e_i, [e_i, e_{i + 1}]_q]_q = [e_i, [e_i, e_{i + 1}]_q]_q = 0, \quad i \neq m, \quad i \neq N, \\
&e_3 \quad \{[e_m, e_{m-1}]_q, [e_m, e_{m+1}]_q\} = 0, \\
&e_4 \quad [e_N, [e_N, [e_N, e_{N - 1}]_q]]_q = [e_N, [e_N, [e_N, e_{N - 1}]_q]]_q = 0,
\end{align}

(33)

and the $f$–Serre relations

\begin{align}
&f_1 \quad [f_i, f_j] = 0, \quad |i - j| \neq 1, \\
&f_2 \quad [f_i, [f_i, f_{i + 1}]_q]_q = [f_i, [f_i, f_{i + 1}]_q]_q = 0, \quad i \neq m, \quad i \neq N, \\
&f_3 \quad \{[f_m, f_{m-1}]_q, [f_m, f_{m+1}]_q\} = 0, \\
&f_4 \quad [f_N, [f_N, [f_N, f_{N - 1}]_q]]_q = [f_N, [f_N, [f_N, f_{N - 1}]_q]]_q = 0.
\end{align}

(34)

The $e_3$ and $f_3$ Serre relations are the additional Serre relations [19-21], which were initially omitted.

We are now ready to state our main result, given also in the Introduction.

Theorem. $U_q[osp(2n + 1/2m)]$ is an associative superalgebra with 1, generators $a_i^\pm$, $L_i$, $\bar{L}_i \equiv L_i^{-1}$, $i = 1, 2, \ldots, m + n = N$, relations ($\xi, \eta = \pm$ or $\pm 1$, $\tilde{q} \equiv q^{-1}$)

\begin{align}
L_i L_i^{-1} &= L_i^{-1} L_i = 1, \quad L_i L_j = L_j L_i, \\
L_i a_j^\pm &= q^{\pm \delta_{ij} (i)} a_j^\pm, \\
[a_i^-, a_i^+] &= -2 \frac{L_i - L_i^{-1}}{q - \tilde{q}}, \\
[[[a_i^\eta, a_{i+1}^{+\xi}], a_j^\eta]]_{q^{-\epsilon(i+1)} \delta_{ij}} &= 2 (\eta) (\delta) \delta_{j, i+1} L_j^{-\epsilon} a_i^\eta, \\
[[a_{N-1}^\xi, a_N^\eta], a_{N+1}^\xi]_{q} &= 0.
\end{align}

(35)

and $\mathbf{Z}_2$-grading $\deg(L_i) = \tilde{0}$, $\deg(a_i^\pm) = \langle i \rangle$.

The expressions of $a_i^\pm$ and $L_i$ via the Chevalley generators read ($i = 1, \ldots, N - 1$):

\begin{align}
L_i &= k_i k_{i+1} \cdots k_N (\text{including } i = N), \\
a_i^- &= (-1)^{(m - i)(i)} \sqrt{2} [e_i, e_{i+1}, \ldots, [e_{N-2}, [e_{N-1}, e_N]_{q_{N-1}}]_{q_{N-2}} \cdots]_{q_{i+1}} q_i, \\
a_i^+ &= (-1)^{N-i+1} \sqrt{2} [[\cdots [f_N, f_{N-1}]_{q_{N-1}}, f_{N-2} \cdots]_{q_{i+1}} f_i]_{q_i}, \\
&\quad \text{where} \quad q_i = \tilde{q}, \quad i = 1, \ldots, m - 1; \quad q_i = q, \quad i = m, \ldots, N.
\end{align}

The next result is essential for the proof of the Theorem.
Proposition 5. The following relations hold:

1. \[ [e_i, a^+_j] = -\delta_{ij}(-1)^{(i+1)}k_i a^+_{i+1}, \quad i \neq N, \]  
2. \[ [a_j, f_i] = \delta_{ij} a^-_{i+1} k_i, \quad i \neq N, \]  
3. \[ [e_i, a^-_j] = 0, \quad \text{if } i < j - 1 \text{ or } i > j, \quad i \neq N, \]  
   \[ [e_i, a^-_j]_{q_i} = (-1)^{(i+1)}a^-_i, \quad i \neq N, \]  
   \[ [e_i, a^-_j]_{q_{i-1}} = 0, \quad i \neq N, \]  
4. \[ [a^+_i, f_j] = 0, \quad \text{if } i < j - 1 \text{ or } i > j, \quad i \neq N, \]  
   \[ [a^+_{i+1}, f_j]_{q_i} = -a^+_i, \quad i \neq N. \]  
   \[ [a^+_i, f_j]_{q_{i-1}} = 0, \quad i \neq N. \]  

Also here one proves that \( a^+_i \) and \( L^+_i \) generate \( U_q[osp(2n+1/2m)] \). More precisely (\( i = 1, \ldots, N - 1 \)),

\[ k_i = L_i L_{i+1}, \quad L_N = k_N, \]  
\[ e_i = \frac{1}{2} L_i [a^-_i, a^+_i], \quad e_N = \frac{1}{\sqrt{2}} a^-_N, \]  
\[ f_i = \frac{1}{2} [a^-_i, a^+_{i+1}] L_{i+1}, \quad f_N = -\frac{1}{\sqrt{2}} a^+_N. \]  

It is a long computation to show, using only the relations (35), that the operators (41) satisfy the Cartan-Kac and the Serre relations. The proof is based on repeated use of nontrivial identities. Here is one of them.

Proposition 6. If \( B \) or \( C \) is an even element, then for any values of the parameters \( x, y, z, t, r, s \) subject to the relations

\[ x = zs, \quad y = zr, \quad t = zsr, \]  

the following identity holds:

\[ [A, [B, C]_x]_y = [[A, B]_z, C]_t + (-1)^{\deg(A)\deg(B)} z [B, [A, C]_r]_s. \]  

In particular it is nontrivial to prove that \( e^2_m = 0 \), which is one of the Serre relations, or to show that the additional Serre relations (e3) and (f3) hold.

5. Concluding remarks

The root system of the orthosymplectic Lie superalgebra \( osp(2n+1/2m) \) reads:

\[ \Delta = \{ \xi\varepsilon_i + \eta\varepsilon_j; \xi\varepsilon_i, 2\xi\varepsilon_k \}, \quad i \neq j = 1, \ldots, m + n \equiv N; \quad k = 1, \ldots, m; \quad \xi, \eta = \pm \}. \]  

The roots \( \varepsilon_1, \ldots, \varepsilon_N \) are orthogonal with respect to the Killing form on \( osp(2n+1/2m) \). The Green generators are the root vectors, corresponding to the orthogonal roots. More precisely, the correspondence reads:

\[ a^\pm_i \leftrightarrow \mp \varepsilon_i, \quad i = 1, \ldots, N. \]  

Therefore what we have done here is

(1) to describe \( U[osp(2n+1/2m)] \) in terms of a “minimal” set of relations among the positive and the negative root vectors, corresponding to the orthogonal roots.
(2) to describe $U_q[osp(2n + 1/2m)]$ entirely in terms of deformed “orthogonal” root vectors, namely deformed Green generators.

This is a good opportunity to mention that the canonical quantum statistics and its generalization, the parastatistics, is based on the representation theory of orthosymplectic Lie superalgebras. For instance the Bose operators $B_i^\pm$, $i = 1, \ldots, n$ are generators of $osp(1/2n)$ in a particular representation. Similar statement holds for $n$ pairs of Fermi creation and annihilation operators: they are generators of the Lie superalgebra $so(2n+1)$ in a particular, the Fock representation. Both $osp(1/2n)$ and $so(2n+1)$ are among the superalgebras from the class $B$ in the classification of Kac of the basic Lie superalgebras [14]. Therefore the canonical quantum statistics and its generalization, the parastatistics, could be called $B$-statistics.

One can associate a concept of creation and annihilation operators with every simple Lie algebra [22-24] and presumably also with every basic Lie superalgebra. The creation and the annihilation operators of the Lie superalgebra $sl(1/n)$ were given in [24]. Therefore, parallel to the $B$-statistics, i.e., the parastatistics, there exists $A$-statistics, $C$-statistics and $D$-statistics. The corresponding deformations, certainly, also exist. In fact the $A$-statistics belongs to the class of the exclusion statistics, recently introduced by Haldane [25] in solid state physics.

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