Average Reward Adjusted Discounted Reinforcement Learning

Near-Blackwell-Optimal Policies for Real-World Applications

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Abstract Although in recent years reinforcement learning has become very popular the number of successful applications to different kinds of operations research problems is rather scarce. Reinforcement learning is based on the well-studied dynamic programming technique and thus also aims at finding the best stationary policy for a given Markov Decision Process, but in contrast does not require any model knowledge. The policy is assessed solely on consecutive states (or state-action pairs), which are observed while an agent explores the solution space. The contributions of this paper are manifold. First we provide deep theoretical insights to the widely applied standard discounted reinforcement learning framework, which give rise to the understanding of why these algorithms are inappropriate when permanently provided with non-zero rewards, such as costs or profit. Second, we establish a novel near-Blackwell-optimal reinforcement learning algorithm. In contrary to former method it assesses the average reward per step separately and thus prevents the incautious combination of different types of state values. Thereby, the Laurent Series expansion of the discounted state values forms the foundation for this development and also provides the connection between the two approaches. Finally, we prove the viability of our algorithm on a challenging problem set, which includes a well-studied M/M/1 admission control queuing system. In contrast to standard discounted reinforcement learning our algorithm infers the optimal policy on all tested problems. The insights are that in the operations research domain machine learning techniques have to be adapted and advanced to successfully apply these methods in our settings.

Keywords machine learning, reinforcement learning, average reward, operations research, admission control.

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1 Introduction

Artificial Intelligence, or more precisely machine learning (ML), has been rising over the last decades and became a very popular topic in science and industry. It is the field that tries to mimic human learning capabilities using computer algorithms. This is done by mathematically modeling the learning process by finding a function that matches the desired outcome according to a given input. Machine learning itself can be divided into the categories of supervised learning, unsupervised learning and reinforcement learning [36, p.1ff], where we focus on reinforcement learning in this paper. Despite astonishing results in the application of reinforcement learning to game-like domains, for instance chess [35], Go [34], and various Atari games [24], no such success was reported for non-episodic operations research problems yet. We conjecture that this lack of reinforcement learning applications is due to the reason that the widely applied standard discounted reinforcement learning framework is inappropriate. To overcome this issue, we present a discounted reinforcement learning variant which is able to deduce near-(Blackwell-)optimal policies, i.e. it performs better for real-world economic problem structures, such as ones often found in the area of operations research.

Reinforcement Learning (RL), which is mathematically based on dynamic programming, is similar to supervised learning, but differs in the way that input-output pairs are actually never presented to the algorithm, but an oracle function rewards actions taken by an agent [36, p.2]. In contrast to dynamic programming, RL has the advantages that (i) the problem space is explored by an agent and thus only expectantly interesting parts need to be assessed and (ii) a comprehensive knowledge of the underlying model becomes unnecessary as the states are evaluated by consecutively observed states solely. Like in operations research the goal is to optimise some measure, termed reward in RL, by finding the best function for the underlying problem. The basic idea of RL is quite simple. An agent explores the solution space by observing the current system state and taking an action which makes the system traverse to the next state. Every time the agent chooses an action a reward is generated by the (oracle) reward function. The agent tries to learn what actions to take to maximise the reward over time, i.e. not only to maximise over the actions for the current state, but it also respects possible stochastic transitions and reward values of future states. RL is most often applied to control problems, games and other sequential decision making tasks [36], where in almost all steps the reward function returns 0. However, in real-world (economic) applications, where success is usually measured in terms of profit or costs, the function intuitively returns a non-zero reward in almost all steps as it continuously reports the actions results. Therefore, we focus on problems that produce an average reward per step that cannot be approximated by 0. Application areas of such a model could be an agent that periodically decides on buying and selling instruments at the stock market, an agent performing daily replenishment decisions or many other decision problems that arise in hierarchical supply chain and production planning systems, which are strongly capacity-oriented [42] and with aggregated feedback (e.g., see [30,12]).

Only a very limited number of operations research optimisation papers that are using RL and imposing this structure are available. Schneckenreither and Haußler [31] presented an RL algorithm which optimises order release decisions in production control environments. They use sophisticated additions to allow the agent to link the actions to the actual rewards. The results were compared to static order release measures only, which were outperformed. Gijsbrechts et al. [10] apply RL on a dual sourcing problem for replenishment of a distribution center using rail or road, and compare their results to optimal solutions if tractable or otherwise established heuristics. They found that hyperparameter\(^1\) tuning is effort-intensive and the resulting policies are often not optimal, especially

\(^1\) In machine learning parameters to be set before the start of the experiment are called hyperparameters.
for larger problems. Balaji et al. [1] use out-of-the-box RL algorithms with simple 2 layer neural networks to tackle stochastic bin packing, newsvendor and vehicle routing problems (VRP). The VRP is a generalised travelling salesman problem (TSP) where one or more vehicles have to visit nodes in a graph. They report to sometimes beat the benchmarks and find sensible solutions. The capacitated VRP is also tackled with RL by Nazari et al. [26]. They minimise the total route length and compare the results to optimal solutions for small instances. Although the optimum is not reached, one instance, which keeps track of the most probable paths, performs better than well established heuristics. Also for larger problem sizes this technique seems to outperform the other tested methods. Vera and Abad [39] extend this model to a multi-agent algorithm and thus tackle the capacitated multi-vehicle routing problem with a fixed fleet size. They also report better results compared to the heuristics, especially for large problem sizes, but in contrast to [26] are outperformed by Google’s OR-Tools\textsuperscript{2}. These applications, like other TSP applications [2,14, e.g.], of RL to the VRP, except for [1], however, calculate the reward according to the length to the finished route. Thus, the average reward for long routes can be approximated with 0 as the decision problem is episodic and therefore perform well. Finally there are applications to the beer distribution game [7,28], however the problem size and therefore its complexity imposed by the beer game is rather small. All of the above cited papers use highly sophisticated methods to tackle rather small problem sizes or result in far-from-optimal solutions. One reason for this is that they utilise algorithms of the widely applied standard discounted RL framework, which are usually evaluated on games. However, games are structured to have a terminal state describing victory or defeat, which is reported as positive or negative reward to the agent, while any other preceding decision returns reward 0. This does not comply with most of the above cited non-episodic optimisation problems.

Although some of these applications tackle intractable problems, in this paper we are concerned with smaller problem sizes to be able to investigate the underlying mathematical issues. A similar approach was taken by Mahadevan. In a series of papers the authors investigate average reward RL. In average reward RL no discount factor is used but rather nested constraint problems are approximated iteratively. In [17] they establish mainly foundations and examine R-Learning, an average reward RL algorithm. R-learning was proposed by Schwartz in [32] and is similar to our approach, but less sophisticated and therefore unable to produce near-optimal-policies. Then in [18] RL optimality criteria are thoroughly discussed, while in the papers [16,19] they present model-based and model-free bias-optimal algorithms. Similarly, Tadepalli and Ok [37] present an average reward RL algorithm called H-Learning, which under certain assumptions finds bias-optimal values. Furthermore, they apply it to simulated automatic guided vehicle scheduling. Later Mahadevan et al. [20] lift the model-free average reward RL algorithm to continuous-time semi-Markov Decision Processes (semi-MDPs), which handle non-periodic decision problems, and apply it to an inventory problem consisting of a single machine. Then the algorithm of [20], called SMART, was applied on the optimisation of transfer lines using a hierarchical approach [21] and preventive maintenance of a production inventory system [8]. Although the results are promising, the adaption to continuous-time problems eases the complexity for most applications. However, in practise usually decision have to be made on a daily basis [9]. Therefore, we refrain from this adaption and concentrate on standard MDPs only. Furthermore, continuous-time problems can be converted through uniformisation into equivalent discrete time instances (see [29,4]).

The reason for the small number of RL applications to non-episodic operations research problems and the fact that none of the available applications report tremendous success, as in game-like areas, can be explained by the average reward per step that the agent receives. The average reward per

\textsuperscript{2} See https://developers.google.com/optimization.
step is usually by far the greatest part when the discounted state values\(^3\) are decomposed into its sub-components. This results from the fact, that in contrast to the other parts, the average reward contributes to the learned state-values in an exponentially up-scaled fashion. However, for most applications the average reward is equal among all states and thus the agent has issues in choosing the best action. Additionally, iteratively shifting all state values to the base imposed by the exponentially up-scaled average reward easily causes the learning process to fail, finding itself in sub-optimal maxima and with that complicating the hyperparameterisation process tremendously. On the opposite average reward RL is cumbersome and computationally expensive, as nested constraints have to be solved. Therefore, we contribute to the research streams of discounted and average reward RL, by combining the best things of both worlds, that is, the stability and simplicity of standard discounted RL and the idea of assessing the average reward separately and aiming for broader optimality criteria of average reward RL with this work. Additionally, we contribute to the operations research domain, which was the origin of the motivation for this work, by providing a new machine learning algorithm and applying it to a well studied M/M/1 admission control queuing system. In summary the contribution is as follows.

- First we analyse and illustrate why standard discounted reinforcement learning is inappropriate for real-world applications, which usually impose an average reward that cannot be approximated with 0,
- Second, we establish a novel near-Blackwell-optimal reinforcement learning algorithm and analytically prove its optimality, and
- Third, show the viability of the algorithms by experimentally applying them to three problem specifications, one of which is a well-studied M/M/1 admission control queuing system.

The rest of the paper is structured as follows. The next section provides an overview of the proposed method. Section 3 introduces discounted RL and average reward RL, and provides the linkage between the two frameworks via the Laurent Series Expansion of discounted state values. The insights gained of the expansion form the foundations for the developed average reward adjusted discounted RL method established in Section 4. Section 5 proves the viability of the algorithm, while Section 6 concludes the paper.

2 Overview of Average Reward Adjusted Discounted Reinforcement Learning

Before establishing the mathematical details of the method in the next sections, in this section we give a high-level overview of the newly developed algorithm by providing the main concepts and ideas using an example.

Although very convenient the evaluation of newly developed RL algorithms on game-like environments, for instance chess or Atari games, brings major issues once these algorithms are applied for optimisation in structurally different real-world scenarios as found in operations research. The most intuitive goal in such economic optimisations problems is to maximise for expected profit (minimize expected costs) at every decision, that is, always taking the actions that expectantly result in the greatest profit. I.e. when modelled as reinforcement learning process the reward function must return the accumulated profit between consecutive decisions. However, reward functions

\(^3\) Note that in contrary to economics where discounting is often motivated by interest rates in discounted RL the motivation of discounting future rewards origins from the fact that the “infinite sum has a finite value as long as the reward sequence […] is bounded” [36, p.59].
of games are usually designed to return a non-zero reward only after the agent reached the state describing victory or defeat. Therefore, currently newly designed RL algorithms are evaluated with problems that impose an average reward per step of approximately 0. This leads to the fact, that when discounted RL techniques are applied to real-world scenarios, they often perform poorly. As we will see the average reward per step plays a crucial role in the performance of reinforcement learning algorithms.

Therefore, our approach in this work is to separately learn the average reward value and rather use average reward adjusted discounted state values to be able to selectively choose the best actions. By using a sufficiently large discount factor the agent reduces the set of best actions to the ones that are (bias-)optimal. Additionally, we approximate the average reward adjusted discounted state values with a smaller discount factor, which allows us to choose actions near-(Blackwell-)optimal.

To clarify consider the Markov Decision Process (MDP) in Figure 1 which consists of two possible definite policies with the only action choice in state 1. Both policies $\pi_A$ (going left in 1) and $\pi_B$ (going right) result in the same average reward $\rho^{\pi_A} = \rho^{\pi_B} = 1$. However, only the A-loop is (bias-)optimal, as the actions with non-zero rewards are selected earlier. E.g. consider starting in state 1. Under the policy $\pi_A$ which takes the A-loop the reward sequence is $(2, 0, 2, 0, \ldots)$, while for the other policy $\pi_B$ it is $(0, 2, 0, 2, \ldots)$. Our algorithm infers i) the average reward based on the state values of consecutively observed states and the returned reward, ii) approximates so called bias values, and iii) uses a smaller discount factor to infer the greatest error term values which only exist as the discount factor is strictly less than 1. Intuitively bias values are the additional reward received when starting in a specific state, and the error term incorporates the number of steps until the reward is collected.

With a discount factor of 0.999 our implementation automatically infers policy $\pi_A$ and thus produces average reward adjusted discounted state values $X^{\pi_A}_{0.9999}(A, l) = 0.493$ for going left (l) in state A and $X^{\pi_A}_{0.9999}(A, r) = 0.492$ for going right (r) in under 10k iterations. Here we use state-action pairs as function parameters. The actual bias values, analytically inferred, are $V^{\pi_A}(\langle A, l \rangle) = V^{\pi_A}(\langle A, r \rangle) = 0.5$, where the differences to the estimated values are due to the error term. Note that for policy $\pi_B$ the state values are $V^{\pi_B}(\langle A, l \rangle) = V^{\pi_B}(\langle A, r \rangle) = -0.5$. Therefore, $\pi_A$ is preferable. However, after the decision for $\pi_A$ is made, doing loop B once becomes attractive as well, as the same amount of rewards are collected. This makes the bias values being equal under the given policy. Therefore, our algorithm has a further decision layer using an approximation of average adjusted discounted state values with a smaller discount factor. As the error term is increased when the discount factor decreases, the agent chooses the action that maximises the error term. The inferred values for a discount factor of 0.8 are $X^{\pi_A}_{0.8}(\langle A, l \rangle) = 0.555$ and $X^{\pi_A}_{0.8}(\langle A, r \rangle) = 0.145$, and thus action $l$ is preferred over action $r$.

In the sequel we establish the method, provide the algorithm and present an optimality analysis thereof. I.e. we show that the presented algorithm is capable of inferring, so called, Blackwell-optimal policies [5]. Blackwell-optimal policies are policies that maximise the average reward, the bias value
and the error term, in this specific order. Near-Blackwell-optimal algorithms infer bias-optimal policies for any MDP and for given MDPs can be configured to infer Blackwell-optimal policies, where inaccuracies due to the limitations of floating point representations of modern computer systems are neglected.

3 The Laurent Series Expansion of Discounted State Values

This section briefly introduces the needed formalism, then investigates the discounted RL framework and provides the Laurent series expansion of its state values. The Laurent series expansion plays a crucial role in the development of the newly established algorithm.

Like Miller and Veinott [22] we are considering problems that are observed in a sequence of points in time labeled 1, 2, . . . and can be modelled using a finite set of states $S$, labelled 1, 2, . . . , $|S|$, where the size $|S|$ is the number of elements in $S$. At each point $t$ in time the system is in a state $s_t \in S$. Further, by choosing an action $a_t$ of a finite set of possible actions $A_s$ the system returns a reward $r_t = r(s_t, a_t)$ and transitions to another state $s_{t+1} \in S$ at time $t + 1$ with conditional probability $p(s_{t+1}, r_t \mid s_t, a_t)$. That is, we assume that reaching state $s_{t+1}$ from state $s_t$ with reward $r_t$ depends solely on the previous state $s_t$ and chosen action $a_t$. In other words, we expect the system to possess the Markov property [36, p.63]. RL processes that possess the Markov property are referred to as Markov decision processes (MDPs) [36, p.66]. A MDP is called episodic if it includes terminal states, or non-episodic otherwise [36, p.58]. Terminal states are absorbing states and are followed by a reset of the system, which starts a new episode.

Thus, the action space is defined as $F = \times_{s=1}^{|S|} A_s$, where $A_s$ is a finite set of possible actions. A policy is a sequence $\pi = (f_1, f_2, \ldots)$ of elements $f_t \in F$. Using the policy $\pi$ means that if the system is in state $s$ at time $t$ the action $f_t(s)$, i.e. the $s$-th component of $f_t$, is chosen. A stationary policy $\pi = (f, f, \ldots)$ does not depend on time. In the sequel we are concerned with stationary policies only. An ergodic MDP consists of a single set of recurrent states under all stationary policies, that is, all states are revisited with probability 1 [17]. A MDP is termed unichain if under all stationary policies the transition matrix contains a single set of recurrent states and a possible empty set of transient states. A MDP is multichain if there exists a policy with at least two recurrent classes [17]. Finally, a state is termed periodic if the greatest common divisor of all path lengths to itself is greater than 1. Otherwise, it is called aperiodic. The goal in RL is to find the optimal stationary policy $\pi^\star$ for the underlying MDP problem defined by the state and action spaces, as well as the reward function.

3.1 Discounted Reinforcement Learning

In the standard discounted framework the value of a state $V_\pi^\gamma(s)$ is defined as the expected discounted sum of rewards under the stationary policy $\pi_\gamma$ when starting in state $s$. That is,

$$V_\pi^\gamma(s) = \lim_{N \to \infty} E \left[ \sum_{t=0}^{N-1} \gamma^t R_t^\pi(s) \right] ,$$

where $0 \leq \gamma < 1$ is the discount factor and $R_t^\pi(s) = E_{\pi_\gamma}[r(s_t, a_t) \mid s_t = s, a_t = a]$ the reward received at time $t$ upon starting in state $s$ by following policy $\pi_\gamma$ [17, e.g.]. The aim is to find an
optimal policy $\pi^*_\gamma$, which when followed, maximises the state value for all states $s$ as compared to any other policy $\pi_\gamma$:

$$V^{\pi^*_\gamma}_\gamma - V^{\pi_\gamma}_\gamma \geq 0.$$ 

This criteria is usually referred to as discounted-optimality (or $\gamma$-optimality) as the discount factor $\gamma$ is fixed [18]. Note that most works omit the index $\gamma$ in the policies $\pi_\gamma, \pi^*_\gamma$ and thus incorrectly indicate that $\pi^*_\gamma = \pi^*$, where $\pi^*$ is the optimal policy for the underlying problem. For the rest of the paper we follow this convention and drop the index $\gamma$ of the policy for the discounted state value and the reward, thus $V^{\pi_\gamma}_{\pi_\gamma}(s) = V^{\pi}_{\pi}(s)$, and $R^{\pi_\gamma}_{\pi_\gamma}(s) = R^{\pi}_{\pi}(s)$.

This also means that the actual value set for $\gamma$ defines the policy which is optimised for. For instance, Figure 2 depicts a MDP with a single action choice in state 1. Rewards of 5 or 10 respectively, are received once upon traversing from state 5 or 10' to state 1. In all other cases the reward is 0. The only action choice is in state 1, in which the agent can choose between doing the printer-loop or the mail-loop. Observe that the average reward received per step equals 1 for the printer-loop and 2 for the mail-loop. Thus, the Blackwell-optimal policy is to choose the mail-loop, as it maximises the returned reward. However, if $\gamma < 3^{-\frac{4}{5}} \approx 0.8027$ an agent using standard discounted reinforcement learning selects the printer loop. Schartz [32] shows that for arbitrary large $\gamma < 1$ it is possible that standard discounted RL fails in finding the optimal policy, while Zhang and Dietterich [41] as well as Boyan [6] use RL on combinatorial optimization problems for which the choice of discount factor is crucial.

The idea behind the discount factor in standard discounted reinforcement learning is to prevent infinite state values [36, p.59]. However, Mahadevan [18,17] refers to the discounting approach as unsafe, as it encourages the agent to aim for short-term gains over long-term benefits. This results from the fact that the impact of an action choice with long-term reward decreases exponentially with time [27]. Besides bounding the state values another main idea behind the $\gamma$-parameter is to be able to specify a balance between short-term (low $\gamma$-values) and long-term (high $\gamma$-values) reward objectives [32]. But what seems to be an advantage rather becomes a disadvantage, as in almost all decision problems the aim actually is to perform well over time, i.e. to collect as much reward as possible over a presumably infinite time horizon. In terms of reward this means to seek for average reward maximising policies before more selectively choosing actions, cf. the example in Figure 2. Therefore, in almost all RL studies the discount factor $\gamma$ is set to a value very close to (but strictly

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4 In [18] they claim that only for $\gamma < 0.75$ the policy is sub-optimal.
Fig. 3: A gridworld MDP with 4 states and a set of 5 actions and the reward function

less than) 1, for instance to 0.99 [25,23,15, e.g.]. However, the issue is that the state values increase exponentially as the average reward is multiplied by \(1/(1-\gamma)\), while the bias value and error term are not. We will refer to the portion induced by the exponentially scaled average reward as \textit{base level imposed by the average reward.}

**Example 1** To clarify, consider the gridworld MDP given in Figure 3 with states \(S = \{(x, y) \mid x, y \in \{0, 1\}\}\), where (0, 0) is the goal state after which the agent is placed uniformly on the grid by using the only available action \textit{random} (see left side). In all other states the agent can choose among the actions \textit{up}, \textit{right}, \textit{down}, \textit{left}, which move the agent in the expected direction, except when moving out of the grid. In such cases the state is unchanged. The reward function is given on the right. Action \textit{random} in state (0, 0) returns reward 10, while taking any other action provides a uniformly distributed reward of \(U(0, 8)\), where moving out of the grid adds a punishment of \(-1\). It is obvious that reaching state (0, 0) with as little steps as possible is optimal. As by definition the average reward for unichain MDPs is equal among all states [17]. Thus the average reward for all states of the optimal (and greedy) policy is 7, which already imposes a base level imposed by the average reward of 700 in case of \(\gamma = 0.99\).

For episodic MDPs standard discounted RL is unable to evaluate the average reward of terminal states and its predecessors correctly, as there is no consecutive state at the end of an episode, i.e. for the gridworld example state (0, 0) is evaluated with value 10, when a new episode starts upon reaching the goal.

Furthermore, for both, episodic and non-episodic MDPs, there is another major issue. As each state is assessed separately, so is the average reward of that state. Thus, states that are visited more often in the iterative process of shifting all state values to the base level imposed by the average reward are evaluated with a higher average reward, which however, increases the likeliness that the agent will visit the state again. This behaviour forms clusters with cyclic paths of states (e.g. going back and fourth between states \((1, 0)\) and \((1, 1)\)), which are visited more and more likely. Even setting a very high exploration rate is usually no remedy by the same argument. Additionally, recall that the average reward increases once the policy gets better through decreasing the exploration rate. This information then needs to be traversed to all states, which however, requires high exploration. This implies that finding the correct hyperparameter setting (e.g. exploration decay) needs a huge amount of effort and experience, which is exactly what was reported in [10]. Regardless of that, this behaviour increases the number of required learning steps tremendously, as all states have to be adapted every time a policy change imposes a change in the average reward (see also [18,17,32]). Unfortunately this easily leads to aforementioned clusters.
It is obvious that optimal policies are very hard to obtain with all these difficulties, especially as all these effects complicate the parameterisation tremendously. However, recall that in operations research and many other real-world applications, success is constantly measured. It is easy to see that this corresponds directly to the problem structure of the gridworld example given in Figure 3. Unfortunately this also means that all these issues directly exist in such applications, which also explains the small number of works combining RL and operations research problems.

Therefore, in the sequel we present a more refined RL approach for operations research, which overcomes these issues by separately assessing the average reward. The relation of the computed values of these two approaches, that is, the aforementioned discussed standard discounted RL technique and the in this paper developed average reward adjusted discounted RL (ARA-DRL) method, are described by following established Laurent series expansion.

3.2 The Laurent Series Expansion of Discounted State Values

The Laurent series expansion of the discounted state values [22,29] provide important insights by giving rise to basically three addends. For a given discount factor $\gamma$ the first addend is solely determined by the average reward $\rho^\pi$, the second is the bias value and the third one, actually consisting of infinitely many sub-terms, is the error term. In the sequel we present the definitions of the average reward and bias value, before providing the Laurent series expansion.

**Definition 1** Due to Howard [13] for an aperiodic MDP the gain or average reward $\rho^\pi(s)$ of a policy $\pi$ and a starting state $s$ is defined as

$$\rho^\pi(s) = \lim_{N \to \infty} \frac{E[\sum_{t=0}^{N-1} R^\pi_t(s) - \rho^\pi(s)]}{N},$$

where $R^\pi_t(s) = E_\pi[r(s_t, a_t) | s_t = s, a_t = a]$ is the reward received at time $t$, starting in state $s$ and following policy $\pi$.

Clearly $\rho^\pi(s)$ expresses the expected average reward received per action taken when starting in state $s$ and following policy $\pi$. In the common case of unichain MDPs, in which only a single set of recurrent states exists, the average reward $\rho^\pi(s)$ is equal for all states $s$ [17,29]. Thus, in the sequel we may simply refer to it as $\rho^\pi$.

**Definition 2** For an aperiodic MDP problem the average adjusted sum of rewards or bias value is defined as

$$V^\pi(s) = \lim_{N \to \infty} E[\sum_{t=0}^{N-1} (R^\pi_t(s) - \rho^\pi(s))],$$

where again $R^\pi_t(s)$ is the reward received at time $t$, starting in state $s$ and following policy $\pi$.

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5 Please note that we assume a positive average reward and the objective to maximise the returned reward throughout this work.

6 In the periodic case the Cesaro limit of degree 1 is required to ensure stationary state transition probabilities and thus stationary values [29]. Therefore to ease readability we concentrate on unichain and aperiodic MDPs. However, the theory directly applies to period unichain MDPs by replacing the limits accordingly.
Fig. 4: The Blackwell-optimal policy $\pi_A$ of the MDP of Figure 1 as model-based (left) and model-free (right) version

Note that the bias values are bounded due to the subtraction of the average reward. Thus the bias value can be seen as the rewards that additionally sum up in case the process starts in state $s$.

Finally, a state value $V^\pi_\gamma(s)$ in standard discounted reinforcement learning can be decomposed into its average reward, the bias value and an additional error term, which actually consists of infinitely many subterms.

**Definition 3** Due to Miller and Veinott [22] the Laurent Series Expansion of the discounted state value for a state $s$, a discount factor $\gamma$ and a policy $\pi$ is given by

$$V^\pi_\gamma(s) = \frac{\rho^\pi(s)}{1-\gamma} + V^\pi(s) + e^\pi_\gamma(s), \quad (1)$$

where Puterman [29, p.341] shows that $\lim_{\gamma \to 1} e^\pi_\gamma(s) = 0$.

The error term $e^\pi_\gamma(s)$ incorporates what amount of reward is collected in combination with the number of steps until it is collected. The higher the discount factor the more long-sighted the agent is. Note how the first term depending on the average reward $\rho^\pi(s)$ converges to infinity as $\gamma$ increases towards 1 and that the second addend does not depend on the discount factor. If the average reward is non-zero and for large $\gamma$-values we can assume $\rho^\pi(s)/(1-\gamma) \gg V^\pi(s) + e^\pi_\gamma(s)$ which explains the behaviour of the standard discounted RL agent, cf. the example of Figure 3.

Regardless of the quality of the chosen actions all state-values need to iteratively increase from the starting state (usually 0) to the base level imposed by the average reward $\rho^\pi(s)/(1-\gamma)$ offset by $V^\pi(s) + e^\pi_\gamma(s)$. As usually there are more actions which are sub-optimal in comparison to the number of optimal ones the agent will more likely choose such an action in usually applied tabula rasa learning. Thus there is a high chance that cycles form.

**Example 2** Reconsider the task depicted in Figure 1. The (Blackwell-)optimal MDP $\pi_A$, that is, the one that chooses the A-loop, is shown on the left side of Figure 4. The right side of Figure 4 shows the same MDP for the model-free version. In model-free reinforcement learning state-action pairs as opposed to state values are estimated. The dashed line indicates that these states are connected and thus the agent has to choose among them. In this case the bias values are $V^\pi_A((0, r)) = -0.5$, $V^\pi_A((1, l)) = V^\pi_A((1, r)) = 0.5$, and $V^\pi_A((2, l)) = 1.5$.

When using standard discounted reinforcement learning, that is, estimating $V^\pi_\gamma(s)$ the average reward of 1 scales the state-action values. E.g. for a discount factor of $\gamma = 0.99$ the inferred values for a converged system are $V^\pi_{0.99}(0, r) = 99.497$, $V^\pi_{0.99}(1, l) = 100.502$, $V^\pi_{0.99}(1, r) = 100.482$, and $V^\pi_{0.99}(2, l) = 101.497$. For all states the discounted state-value consists of the scaled average reward $1/0.01 = 100$ and the bias value plus the error term. The duration of the process of learning the state values is unintentionally increased as the scaled average reward has to be learned in an
iterative manner. Furthermore, the values itself are hard to interpret and the marginal difference between optimal and non-optimal action as compared to their actual values increase the likelihood of choosing sub-optimal actions, especially when function approximation is used to represent $V_\gamma(s)$. In this example the difference of the state values for choosing action $l$ over action $r$ in state 1 is given by 0.02 as compared to their mean state value of 100.492.

4 Average Reward Adjusted Discounted Reinforcement Learning

This section establishes the average reward adjusted discounted reinforcement learning algorithm. Furthermore, we provide the Bellman Equations and an optimality discussion of the presented algorithm. For the rest of the paper we assume unichain MDPs, that is, we restrict our method to MDPs that possess a scalar average reward value $\rho^\pi$. In case of multichain MDPs we refer to the companion paper Near-Blackwell-Optimal Average Reward Reinforcement Learning.\footnote{Working paper.}

Starting from the Bellman Equations we derive the average reward adjusted reinforcement learning equation using the Laurent series expansion provided in Equation 1. However, let us first introduce a notion for the state values which are adjusted of the average reward.

**Definition 4** We define the average reward adjusted discounted state value $X_\gamma^\pi(s)$ of a state $s$ under policy $\pi$ and with discount factor $0 \leq \gamma \leq 1$ as

$$X_\gamma^\pi(s) := V_\gamma^\pi(s) + c_\gamma^\pi(s).$$

This can be reformulated to

$$X_\gamma^\pi(s) = V_\gamma^\pi(s) - \frac{\rho^\pi}{1-\gamma} = \lim_{N \to \infty} E[\sum_{t=0}^{N-1} \gamma^t R_t^\pi(s)] - \frac{\rho^\pi}{1-\gamma},$$

thus our definition is a reformulation of the average-adjusted reward values of Schwartz [32,33].

A major problem occurring at average reward RL is that the bias values are not uniquely defined without solving the first set of constraints defined by the error term addends (see [29,19, p.346]). We could overcome this issue by simply requiring $\gamma$ to be strictly less than 1. However, actually our algorithm does not require the exact solution for $V_\gamma^\pi(s)$, but a solution which is offset suffices. Clearly this observation reduces the required iteration steps tremendously as finding the exact solution, especially for large discount factors, is tedious. Therefore, we allow to set $\gamma = 1$, which induces $X_1^\pi(s) = V_1^\pi(s) + u$, where $u$ is for unichain MDPs a scalar value independent of $s$, i.e. equivalent for all states of the MDP [29, p.346]. If we are interested in correct bias values, i.e. $\gamma$ is close but strictly less than 1, our approach is a tremendous advantage over average reward RL as it reduces the number of iterative learning steps by requiring only a single constraint per state plus one for the scalar average reward value. That is, for an MDP with $N$ states only one more constraint $(N+1)$ has to be solved in ARA-DRL as compared to (at least) $2N+1$ nested constraints for average reward RL. Therefore, it is cheap to compute $X_1^\pi(s)$, while it is rather expensive to find the correct values of $V_\gamma^\pi(s)$ directly, especially in an iterative manner as RL is.

4.1 Bellman Equations

Using the above notion we are able to derive the average reward adjusted discount reinforcement learning state balance equation. To do so we make use of an equivalence found in the derivation of
the Bellman equation for $V_\gamma^\pi(s)$ (see e.g. [36, p.70]) and transform it to our needs. As in [36, p.66] we use the notation $R_{s,s'}^a = \mathbb{E}[r_t | s_t = s, a_t = a, s_{t+1} = s']$ for the expected reward to receive when traversing from any current state $s$ to state $s'$ using action $a$ and $\pi(a | s)$ the probability of taking action $a$ in state $s$ as given by policy $\pi$.

$$V_\gamma^\pi(s) = \mathbb{E}_\pi[r_t + \gamma V_\gamma^\pi(s_{t+1}) | s_t = s]$$

$$\frac{\rho^\pi}{1-\gamma} + V_\gamma^\pi(s) + e_\gamma^\pi(s) = \mathbb{E}_\pi[r_t + \gamma(\frac{\rho^\pi}{1-\gamma} + V_\gamma^\pi(s_{t+1}) + e_\gamma^\pi(s_{t+1})) | s_t = s]$$

$$X_\gamma^\pi(s) = \mathbb{E}_\pi[r_t + \gamma X_\gamma^\pi(s_{t+1}) - \rho^\pi | s_t = s]$$

$$X_\gamma^\pi(s) = \sum_a \pi(a | s) \sum_{s'} p(s' | s_t = s, a_t = a)[R_{s,s'}^a + \gamma X_\gamma^\pi(s') - \rho^\pi]$$

Thus, we can compute the average reward adjusted discounted state value $X_\gamma^\pi(s)$ of a state $s$ by the returned reward, the adjusted discounted value state $X_\gamma^\pi(s_{t+1})$ of the next state $s_{t+1}$ and the average reward $\rho^\pi$. This is very similar to the Bellman equation of standard discounted RL, cf. the first line. Further, note that in line three we use the equivalence $\rho^\pi(s) = \mathbb{E}_\pi[\rho^\pi(s)]$ described by the first addend of the Laurent series expansion (see [22]).

In the same manner we derive the Bellman optimality equation for average reward adjusted discounted reinforcement learning, cf. [36, p.76].

$$V_\gamma^{\pi*}(s) = \max_a \mathbb{E}_\pi[r_t + \gamma V_\gamma^{\pi*}(s_{t+1}) | s_t = s]$$

$$\frac{\rho^{\pi*}}{1-\gamma} + V_\gamma^{\pi*}(s) + e_\gamma^{\pi*}(s) = \max_a \mathbb{E}_\pi[r_t + \gamma(\frac{\rho^{\pi*}}{1-\gamma} + V_\gamma^{\pi*}(s) + e_\gamma^{\pi*}(s)) | s_t = s]$$

$$X_\gamma^{\pi*}(s) = \max_a \mathbb{E}_\pi[r_t + \gamma X_\gamma^{\pi*}(s) - \rho^{\pi*} | s_t = s]$$

$$X_\gamma^{\pi*}(s) = \max_a \sum_{a'} p(s' | s_t = s, a_t = a)[R_{s,s'}^{a} + \gamma X_\gamma^{\pi*}(s') - \rho^{\pi*}]$$

Again like in standard discounted RL, also in ARA-DRL the value for the optimal policy equals the expected value of the best action from that state [36, p.76], where the average reward is subtracted accordingly. This is very pleasing as we can use the same ideas in terms of exploration/exploitation and state value acquisition as known from standard discounted RL.

4.2 Near-Blackwell-Optimal Algorithm

The reinforcement learning algorithm is depicted in Algorithm 1. In model-free methods, which is what we aim for, state-action pairs are computed instead of state values only. Therefore, the algorithm operates on state-action tuples, where for simplicity we write $X_\gamma^{\pi}(s,a)$ instead of $X_\gamma^{\pi}((s,a))$ and the Bellman optimality equation is adopted as in [36, p.76], s.t. the agent is able to selectively choose actions among multiple states, cf. Figure 4. After initialising all values the agent enters the loop in which the first task is to choose an action (step 3). In this action selection process we utilise an $\epsilon$-sensitive lexicographic order $\preceq_\epsilon$ defined as $(a_1, \ldots, a_n) = a \preceq_\epsilon b = (b_1, \ldots, b_n)$ if and only if $|a_j - b_j| \leq \epsilon$ for all $j < i$ and $|a_i - b_i| > \epsilon$. Note that the resulting sets of actions may not be disjoint. Although this is an unusual order in programming, finding the set of maximizing values as
Algorithm 1 Model-free tabular near-Blackwell-optimal RL algorithm for unichain MDPs

1: Initialize state $s_0$, $\rho = 0$, $X^\pi(\cdot, \cdot) = 0$, set an exploration rate $0 \leq p_{exp} \leq 1$, exponential smoothing learning rates $0.5 \leq \gamma_0 < \gamma_1 \leq 1$, where $\gamma_1 = 1$ is usually a good choice.

2: while the stopping criterion is not fulfilled do

3: With probability $p_{exp}$ choose a random action and probability $1 - p_{exp}$ one that fulfills $\max_a < \epsilon \left( X^\pi_{\gamma_1}(s_t, a), X^\pi_{\gamma_0}(s_t, a) \right)$.

4: Carry out action $a_t$, observe reward $r_t$ and resulting state $s_{t+1}$.

5: if a non-random action was chosen then

6: Update the average reward adjusted discounted state-values.

7: Set $s \leftarrow s'$, $t \leftarrow t + 1$ and decay parameters.

in our algorithm is straightforward and thus cheap to compute. In case the resulting set of actions contains more than one element a random action of this set of actions is chosen.

As usual in reinforcement learning the state-values are exponentially smoothed using parameters $\alpha$ and $\gamma$. In step 4 the chosen action is carried out, while in step 5 the average reward estimate is calculated. As we aim for an estimate of $\rho^\pi$ we only update the value in case the greedy action was chosen. The formula used is a reformulation of the second addend of the Laurent series expansion (see [22] or [29, p.346] for details): $\rho^\pi(s) + V^\pi(s) - E[V^\pi(s)] = R_t(s)$. Like [37] we also observed that the average reward has to be updated by this formula and not by exponentially smoothing the actual observed rewards as done in the algorithms of Mahadevan [16,19], which likely leads to sub-optimal policies. Furthermore, we adapted the above given algorithm by adding an exponentially smoothed bound from below for the average reward value. The idea is that, once a policy was established for some time, it does not make sense to aim for policies with smaller average reward.

Finally the state-values are updated according to the average reward adjusted Bellman optimality equation derived above (step 6) and the environment is updated to the next state and time period (step 7). In the next subsection we will introduce further optimality criteria of RL and prove that for a given MDP the discount factors $\gamma_0$ and $\gamma_1$ and the comparison measure $\epsilon$ can be chosen s.t. under the assumption of correctly approximated values the algorithm produces Blackwell-optimal policies.

4.3 Optimality Criteria

In terms of optimality, we consider the notion of $n$-discount optimality as it is the broadest approach of optimality criteria in reinforcement learning, and further for a sufficiently large $n$ it is known that there always exists a policy which is optimal [5,38]. For a comprehensive discussion of optimality criteria in reinforcement learning we refer to [18].

---

With rate $\frac{1}{10}$ and update of 97.5% of the current reward in every period.
Definition 5 Due to Veinott [38] for MDPs a policy \( \pi^* \) is \( n \)-discount-optimal for \( n = -1, 0, 1, \ldots \) for all states \( s \in S \) with discount factor \( \gamma \) if and only if
\[
\lim_{\gamma \to 1} (1 - \gamma)^{-n} (V_{\pi^*}^\gamma (s) - V_\gamma^\pi (s)) \geq 0.
\]

As a policy can only be \( m \)-discount optimal if it is \( n \)-discount-optimal for all \( n < m \) [29,38], this leads to the component-wise comparison when greedily choosing actions, cf. the action selection process of the algorithm.

Definition 6 If a policy is \( \infty \)-discount-optimal then it is said to be Blackwell-optimal [5].

That is, Blackwell-optimal policies are the in the sense of \( n \)-discount-optimality the best achievable policies that first optimise for the highest gain, as we have for \( n = -1 \) a measure for gain-optimality [17], then for \( n = 0 \) for bias-optimality [17], and as we will see for \( n \geq 1 \) it maximises for the greatest error term. For an agent that either expects to have infinitely many time to collect rewards, or one that is unaware when the system will halt, this is the most sensible approach.

There are two known possibilities to incorporate the expected reward to be collected in the future. The first one is to use a single discounted value, while the other approach in general incorporates solving of infinitely many constraints. The following definition separates these kinds of algorithms.

Definition 7 If an algorithm infers for any MDP bias-optimal policies and for a given MDP can in theory be configured to infer \( \infty \)-discount-optimal policies, but in practise this ability is naturally limited due to the finite accuracy of floating-point representation of modern computer systems, it is said to be near-Blackwell-optimal under the given computer system. An according to a near-Blackwell-optimal algorithm inferred Blackwell-optimal policy is called near-Blackwell-optimal.

This definition is of practical relevance, as it defines a group of algorithms that are by far less computationally expensive in comparison to ones that solve infinitely many constraints, but are able to deduce sufficiently optimal policies. To the best of our knowledge there is no Blackwell-optimal algorithm that neither requires infinitely many constraints to be solved nor is restricted by the floating point precision.

4.4 Optimality Analysis

In this section we will analyse the procedure of Algorithm 1 by the above definitions of \( n \)-discount-optimality, cf. Definition 5. Thus, we will start with \( n = -1 \) and then proceed from there.

4.4.1 \((-1)\)-Discount-Optimality

For the case of \( n = -1 \) this leads to gain-optimality [17], defined as \( \rho^\pi^* (s) - \rho^\pi (s) \geq 0 \) for all policies \( \pi \) and states \( s \in S \). This means that a \((-1)\)-discount-optimal agent puts its highest priority to maximising for the greatest average reward. However, recall that in unichain MDPs the average rewards is equal among all states. Thus, all bias values are assessed with the same average reward, i.e. the agent automatically maximises for the greatest gain. Note that by definition, in case of a possibly falsely predetermined and fixed average reward value, the bias values are estimated according to the given policy induced by the average reward. Furthermore, if the average reward is fixed to a wrong value the bias values are similarly shifted as in the standard discounted framework. Therefore, we highly recommend to infer the average reward automatically as specified in the algorithm.
Example 3 Reconsider the MDP of Figure 2 and the discount factor $\gamma = 0.99$. If we fix the average reward to $\rho_n^\pi = 1$ as opposed to the correct value of $\rho^\pi = 2$, the algorithm infers values $X_{n,\gamma}^\pi(s) = \frac{\rho^n - \rho_n^\pi}{1 - \gamma} + X_\gamma^\pi(s) = 100 + X_\gamma^\pi(s)$ instead.

4.4.2 0-Discount-Optimality

In the case of $n = 0$ the criteria of $n$-discount-optimality describes bias-optimality with $V^{\pi^*}(s) - V^\pi(s) \geq 0$ for all policies $\pi$ and states $s \in S$ [17]. Thus for the algorithm the first decision level is to maximise for the policy yielding the highest bias values. As we have $\lim_{\gamma \to 1} e_{\gamma}^\pi(s) = 0$, we know that according to the chosen $\epsilon$ for sufficiently large $\gamma_1$ the agent selects the set of actions that maximise $V^\pi(s)$.

**Theorem 1** For a sufficiently large $\gamma_1 < 1$, where sufficiently large means that for all states $s$ we have $|e_{\gamma_1}^\pi(s)| \leq \epsilon$, i.e. it depends on the parameter $\epsilon$, a 0-discount-optimal agent chooses an action among the set of possible actions that maximise $X_{\gamma_1}^\pi(s)$.

**Proof** Recall that $\lim_{\gamma \to 1} e_\gamma^\pi(s) = 0$ and $\rho_\pi^* = \rho^\pi$. We have

$$\lim_{\gamma \to 1}(1 - \gamma)^0(V^\pi^*(s) - V_\gamma^\pi(s)) \geq 0$$

$$\lim_{\gamma \to 1}(\rho_\pi^* - \rho^\pi) + V^\pi^*(s) - V^\pi(s) + e_{\gamma}^\pi(s) - e_\gamma^\pi(s) \geq 0$$

$$V^\pi^*(s) - V_\gamma^\pi(s) \geq 0$$

meaning that a 0-discount-optimal policy $\pi$ has to maximise the bias values $V^\pi(s)$ for all states $s$. By definition of the $\epsilon$-sensitive lexicographic order ($a \preceq _\epsilon b$ if and only if $|a - b| \leq \epsilon$), we have for a sufficiently large $\gamma_1$-value $|e_{\gamma_1}^\pi(s)| \leq \epsilon$ and thus $|V^\pi(s) - X_{\gamma_1}^\pi(s)| \leq \epsilon$ for all states $s$. Thus the claim follows. \qed

4.4.3 $\infty$-Discount-Optimality

Finally, in case $n \geq 1$, the agent has to choose actions that satisfy $e_{\gamma_1}^\pi(s) - e_\gamma^\pi(s) \geq 0$ once $\gamma \to 1$. That is, the agent must maximise the error term, which means for $n \geq 1$ we analyse the case where $\gamma < 1$ is used to incorporate short term rewards into the discounted state value, i.e. how long-sighted the agent shall be. Therefore, the number of actions to reach a desired goal or path is taken into account, as well as when and how much rewards are collected. However, as the error term depends on infinitely many sub-terms simply estimating these and summing up does not work.

The RL algorithm depicted in Algorithm 1 is unable to generally deduce Blackwell-optimal policies. The cause is illustrated in Figure 5, where we assume the error terms to be polynomials of the same degree. Let the policies $\pi_1$, $\pi_2$ choose the actions $a_1$ and $a_2$ resp. of the set of 0-discount-optimal stationary actions in state $s$ and then follow policy $\pi$. Note the different slopes of the two approximated values of $X_{\gamma}^\pi(s)$. Therefore, by interpolation we know that for very high values of $\gamma$ we have $X_{\gamma^2}^\pi(s) > X_{\gamma^1}^\pi(s)$. However, the agent will choose action $a_1$ as it maximises $X_{\gamma_0}^\pi(s)$. Therefore, this means that the parameter $\gamma_0$ defines how long-sighted the agent is. The MDP of Figure 6 explains the idea. The only action choice is in state $S$, where the agent either decides to do the top- or bottom-loop. For both loops the same amount of rewards are collected, thus the average reward
\(\gamma\) approaches 1

Fig. 5: Visualisation of the strictly monotonically decreasing error terms \(e_\gamma^{0}(s)\) and \(e_\gamma^{2}(s)\) as \(\gamma\) approaches 1

\[
X_\gamma^{0}(s) = V^{\pi}(s) + e_\gamma^{0}(s) \\
X_\gamma^{2}(s) = V^{\pi}(s) + e_\gamma^{2}(s)
\]

Fig. 6: An example MDPs for which the discount factor \(\gamma_0\) can be used to balance short- and long-sightedness for near-Blackwell-optimal algorithm

(0.75) and bias values (0.25 for action up, 0.375 for action down) are equal, regardless of the chosen policy. But only going down is Blackwell-optimal, as the full amount of rewards is collected sooner. This also manifests in a higher bias value. However, recall that the agent is unable to separate the actions by the bias values in case the same amount of rewards are collected. If we set \(\gamma_0 = 0.50\) the agent deduces state-values \(V^{\pi}_{\gamma_0}(S) = -0.480\) and \(V^{\pi}_{\gamma_0}(S) = -0.746\) and thus like for any other value \(\gamma_0 < 0.84837\) chooses the top-loop.

This shows how for our algorithm \(\gamma_0\) functions exactly as the discount factor in standard RL is supposed to do, due to the fact that the state values are adjusted: It can be used to balance expected short-term and long-term rewards, without changing the main optimisation objective, i.e. maximise for the highest average reward and bias values, before taking path lengths into account. Especially for highly volatile systems, e.g. stochastic production and control systems, being able to set the long-sightedness can be an advantage over Blackwell-optimal agents. Nonetheless, setting very high values for \(\gamma_0\) and using the \(\epsilon \approx 0\) for the comparison of \(X_\gamma^{0}(s)\) values, could be an approach in finding Blackwell-optimal policies for many MDPs. But recall that for any arbitrary large \(\gamma_0 < 1\) it is possible to construct MDPs which lead to non-optimal policies [32].
Theorem 2 A \( n \)-discount-optimal agent for \( n \geq 1 \) has to maximise the error term \( e^\pi_\gamma(s) \) once \( \gamma \to 1 \).

\[ \lim_{\gamma \to 1} (1 - \gamma)^{-n}(V^\pi_\gamma(s) - V^\pi_\gamma(s)) \geq 0 \]
\[ \lim_{\gamma \to 1} (1 - \gamma)^{-n}(\rho^\pi_\gamma(s) \frac{1}{1 - \gamma} + V^\pi_\gamma(s) + e^\pi_\gamma(s)) - \frac{\rho^\pi_\gamma(s)}{1 - \gamma} - V^\pi(s) - e^\pi_\gamma(s)) \geq 0 \]
\[ \lim_{\gamma \to 1} (1 - \gamma)^{-n}(e^\pi_\gamma(s) - e^\pi_\gamma(s)) \geq 0 \]

which completes the proof as the error term does not depend on \( n \). However, note that as \( e^\pi_\gamma(s) \) approaches 0 as \( \gamma \to 1 \), we are interested in the cases where \( \gamma < 1 \). That is, the error term incorporates the amount and number of steps until rewards are collected.

In other works, the error term is often split into its subterms by \( e^\pi_\gamma(s) := \sum_{m=1}^{\infty} (\frac{1 - \gamma}{1 - \gamma})^m \cdot y^m_\gamma \) (for the definition see [22]). Then for any \( n \geq 1 \) the terms evaluate to maximising \( y^m_\gamma \), as for all subterms < \( n \) the values are equal due to \((n - 1)\)-discount-optimality and for \( n > 1 \) the terms evaluate to 0. This leads to the approaches of average reward dynamic programming, where \( n \)-nested sets of constraints are solved [29, e.g. p.511ff]. Clearly, this straightforward approach is computationally very expensive and infinite. Therefore, and also due to the imposed infinite polynomial structure of the error term formula we refrain on adapting this strategy and rather let the user choose an appropriate \( \gamma_0 \) value for the provided situation.

Thus, for a given MDP and under the assumption of correct approximations and wisely selected discount-factors \( \gamma_0 \) and \( \gamma_1 \) in combination of the chosen \( \epsilon \)-value our algorithm is able to infer Blackwell-optimality policies. Nonetheless due to the accuracy of floating-point representation of modern computer systems the previous statement is naturally bounded. Therefore, the in this paper established reinforcement learning algorithm ARA-DRL is near-Blackwell-optimal.

5 Experimental Evaluation

In this section we prove the viability of the algorithm with three examples. We compare the algorithm to standard discounted RL, where we choose the widely applied Q-Learning [36,40] technique as appropriate model-free comparison method. The Q-Learning algorithm is given in Algorithm 2. We have adapted the parameter names accordingly to match the ARA-DRL algorithm from above.

5.1 Printer-Mail

Reconsider Figure 2 discussed above. The agent chooses either the printer loop or the mail loop, whereas the mail loop returns a reward of 20 every tenth step and the printer loop 5 every fifth step. As there is no stochastic in the reward function, nor the transition function, the problem can be easily solved until convergence. To do so we set the learning rate \( \gamma = 0.01 \) and for ARA-DRL
Algorithm 2 Watkins Q-Learning algorithm [40]. Adapted from the version by Sutton [36, p.149].

1: Initialize state $s_0$, $Q_{\pi_1}^\pi(\cdot) = 0$, set an exploration rate $0 \leq \rho_{\text{exp}} \leq 1$ and $0 < \gamma, \gamma_1 < 1$.
2: while the stopping criterion is not fulfilled do
3: With probability $p_{\text{exp}}$ choose a random action and probability $1-p_{\text{exp}}$ one that fulfills $\max_a Q_{\pi_1}^\pi(s_t, a)$ at the current state $s_t$.
4: Carry out action $a_t$, observe reward $r_t$ and resulting state $s_{t+1}$.
5: Update the discounted state-values.

\[
Q_{\pi_1}^\pi(s_t, a_t) \leftarrow (1-\gamma)Q_{\pi_1}^\pi(s_t, a_t) + \gamma[r_t + \gamma_1 \max_{a'} Q_{\pi_1}^\pi(s_{t+1}, a')] \]

6: Set $s \leftarrow s'$, $t \leftarrow t + 1$ and decay parameters

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Measure & ARA-DRL & $\gamma_1 = 0.99$ & $\gamma_1 = 0.80$ & $\gamma_1 = 0.50$ \\
\hline
State (1, left) & -13.349 & 186.509 & 3.046 & 0.323 \\
State (1, right) & -8.787 & 191.070 & 3.011 & 0.039 \\
Steps in $10^6$ & & 1.0 & 10.3 & 0.5 & 0.4 \\
\hline
\end{tabular}
\caption{The state-values for state 1 of the printer-mail MDP and the number of steps until convergence, where ARA-DRL inferred an average reward of $\rho^\pi = 1.999$.}
\end{table}

$\alpha = 0.01$ exponentially decayed with rate 0.25 in 100k steps, where the minimum is set to $10^{-6}$, and the discount factors $\gamma_0 = 0.8$, $\gamma_1 = 0.99$. For Q-Learning we repeated the experiment with discount factors $\gamma_1 = 0.99$, $\gamma_1 = 0.8$ and $\gamma_1 = 0.50$.

The results are depicted in Figure 1, where the state-action values for $(1, \text{left})$ and $(1, \text{right})$ are shown. The table reports the values of $X_{\pi_1}^\pi(\cdot)$ for ARA-DRL, and $Q_{\pi_1}^\pi(\cdot)$ in case of Q-Learning. The reported number of steps are measured until convergence, which we defined as no state-value change in 100k steps. As ARA-DRL, as well as Q-Learning with $\gamma_1 = 0.99$, report a greater value for going right than taking action left, both infer the optimal policy, while Q-Learning with $\gamma_1 = 0.80$ and $\gamma_1 = 0.50$ are unable to find it. This shows clearly that it is crucial in standard discounted RL to use a discount factor close to 1. When setting $\gamma_1 = 0.8027$ the discounted values are $V_{\pi_1}^\pi(1, \text{left}) = V_{\pi_1}^\pi(1, \text{right}) = 3.113$. The average reward learned by ARA-DRL is 1.999 and thus very close to the actual value of 2. Note that as all states are assessed with the same average reward it is not necessarily to estimate it perfectly, and thus small deviations have no impact on the policy of ARA-DRL. The reported steps to convergence for Q-Learning increase exponentially when increasing the discount factor, which is due to the requirement of very high discount factors a rather unsatisfactory behaviour. Especially as ARA-DRL converges in $10^6$ steps, while Q-Learning $\gamma_1 = 0.99$ requires a more than ten times longer learning phase.

5.2 Gridworld

We use a scaled up version of the MDP given in Figure 3 by increasing the state space to a $5 \times 5$ grid to make the optimisation task a little bit more challenging. The reward function is adapted accordingly, where moving out of the grid is punished and otherwise a stochastic reward of $U(0, 8)$ is returned. The goal state is indifferent. Again, it is obvious that the optimal policy is to traverse to $(0, 0)$ as fast as possible. Further, the system is symmetric, where states $(m, n)$ and $(n, m)$ are
Table 2: The results of the gridworld example, where all ARA-DRL instances inferred an average reward of $p^\pi = 5.215$

| Algorithm | Sum Reward Mean | StdDev | Ave. Steps to Goal Mean | StdDev |
|-----------|-----------------|--------|-------------------------|--------|
| ARA-DRL $\gamma_1 = 0.99$ | 51894.094 | 234.260 | 5.039 | 0.047 |
| ARA-DRL $\gamma_1 = 0.999$ | 51787.069 | 282.335 | 5.063 | 0.054 |
| ARA-DRL $\gamma_1 = 1.00$ | 51856.529 | 242.953 | 5.055 | 0.039 |
| Q-Learning $\gamma_1 = 0.99$ | 34409.464 | 3391.818 | 7661.833 | 3518.656 |
| Q-Learning $\gamma_1 = 0.999$ | 33931.917 | 3076.401 | 7379.155 | 3745.914 |
| Q-Learning $\gamma_1 = 0.50$ | 30171.837 | 328.591 | 9999.000 | 0.000 |

equal when also swapping the actions left with up and right with down respectively. For the optimal policy the average number to reach the goal state is 4 steps, with a reward of on average 4 and then there is the random action which produces a reward of 10. Thus, the average reward for the optimal policy is 5.2 and the average number of steps to reach the goal state should be 5. However, note that we use no episodes, i.e. there is no terminal state, as otherwise the Q-Learning algorithm fails completely by ignoring the average reward in the terminal state, and thus producing policies with the lowest estimate in the goal state.

We initialise the learning rates $\alpha = \gamma = 0.01$, the discount factor $\gamma_0 = 0.80$ and set $\epsilon = 0.25$, $p_{\exp} = 1.00$. The learning rates and exploration are exponentially decayed as follows: $\alpha$ with a rate of 0.50 in 50k steps and a minimum of $10^{-5}$, $\gamma$ with rate of 0.50 in 150k steps and $10^{-3}$, and finally the exploration with rate 0.50 in 100k steps and a minimum of 0.01. We execute 500k learning steps before doing an evaluation run of 10k steps for which exploration and learning is disabled. The experiment, including the learning process, is repeated 40 times with the same random number streams over the different setups.

Table 2 presents the summary of the results for the gridworld experiment. We have evaluated both methods with three different discount factors, namely 0.99, 0.999 and 1.00 for ARA-DRL and 0.50, 0.99 and 0.999 for Q-Learning. We used the Friedman test with a significance level of $p = 0.05$ for a statistical analysis of the mean sum of rewards and the mean average steps until the goal step. As expected the omnibus null hypotheses (all samples are from the same distribution) are rejected (with $3.223e-33$ for the mean reward and $1.736e-34$ for the mean avg. steps to goal). Therefore, we conducted pairwise Conover post-hoc tests adjusted by the Benjamini-Hochberg FDR method [3] to reduce liberality (see the Appendix for the detailed results). Measures highlighted with the same shade of grey are not statistically distinguishable from each other. It can be seen that all ARA-DRL instance perform very well, outperforming Q-Learning in terms of amount of collected rewards, as well as finding the shortest path to the goal state. More precisely all ARA-DRL instances collect on average over all 40 experiments a reward of 5.19 per step, which is only 0.01 less than the optimum of 5.2, while the best Q-Learning variant receives 3.44 which means that Q-Learning not even learns to avoid to steer of the grid. The standard deviations, especially of the average number of steps to reach the goal state, undermine the great performance of ARA-DRL as it shows how stable the algorithm works while the Q-Learning results are rather unstable. ARA-DRL $\gamma_1 = 0.99$ performs significantly better in terms of mean average number of steps to the goal state than the other setups, while for the mean sum of reward all ARA-DRL instances are significantly indifferent. This is due to the very small deviations of the former measure. Further the average steps show Q-Learning is
Fig. 7: This Figure shows estimated values $Q^\pi_{0.99}(s, a)$ of Q-Learning after 1 million steps

|       | 0,0 | 1,0 | 2,0 | 3,0 | 4,0 |
|-------|-----|-----|-----|-----|-----|
| Value | 0.260 | 0.283 | 0.392 | 0.515 | 0.610|
| Value | 101.260 | 153.653 | 351.907 | 359.539 | 399.539 |

Fig. 8: This diagram illustrates a simple M/M/1 admission control queuing system (adapted from Mahadevan [16,19])

unable to find even close-to near-optimal policies. The estimated average reward by the ARA-DRL algorithm is $\rho^\pi = 5.215$ for all setups, and thus almost perfectly matches the analytically inferred $5.20$. An explanation, why Q-Learning performs that poorly can be seen in Figure 7. It provides discounted state-values as estimated by the Q-Learning variant after 1 Million steps of learning. Here we can see a cluster forming at the state-actions pairs of state $(0, 4)$, which have with almost 400 a far higher evaluation than the goal state $(0, 0)$ of around 100. Clearly, the surrounding states adapt accordingly worsening the situation.

5.3 Admission Control Queuing System

Finally, like Mahadevan [16,19] we evaluate the algorithm on a simple M/M/1 admission control queuing system. That is, we assume one server that processes jobs which arrive by an exponential (Markov) interarrival time distribution. Furthermore, the processing duration is assumed to be exponentially distributed. The arrival and service rate are modeled by parameter $\lambda$ and $\mu$ respectively. On each new arrival the agent has to decide whether to accept the job and thus add it to the queue, or reject the job. In case of acceptance an immediate reward is received, which however, also incurs a holding cost depending on the current queue size. The goal is to maximise the reward by balancing the admission allowance reward and the holding costs. The MDP is depicted in Figure 8 and was
observed through uniformisation from a continuous time problem to a discrete time specification (see [29, 4] for a description of uniformisation). The set of states consists of elements \((l, \text{Arr})\) with queue length \(l \in \mathbb{N}\) and a Boolean variable \(\text{Arr} \in \{T, F\}\) where \(\text{Arr}\) symbolises an arrival (T), or no arrival (F). The edges are labelled with the corresponding action, that is, accept and reject in case of a new arrival, or continue for continuation when no new job arrived, and the corresponding transition probability. We define the reward function \(r\) as in [16, 19] by

\[
\begin{align*}
    r((0, F), \text{continue}) &= r((0, T), \text{reject}) = 0 & \text{if } s = 0, \\
    r((l, F), \text{continue}) &= -f(l + 1)(\lambda + \mu) & \text{if } l \geq 1, \\
    r((l, T), \text{reject}) &= -f(l + 1)(\lambda + \mu) & \text{if } l \geq 1, \\
    r((l, T), \text{accept}) &= [R - f(l + 1)](\lambda + \mu)
\end{align*}
\]

where the factor \(\lambda + \mu\) is an artifact of the uniformisation of the continuous time problem to the discrete time MDP.

Haviv and Puterman [11] show that if the cost function has the shape \(f(l) = c \cdot l\), there are at most two gain-optimal control limit policies. Namely to admit \(L\) or \(L + 1\) jobs. However, only the policy that admits \(L + 1\) jobs is also bias-optimal as the extra reward received offsets the additional cost of the extra job. Furthermore, note that in such cases the reward function can be simplified by removing the conditions and the first line. We use exactly this cost function in our experiment.

Further, we choose the challenging problem setup with \(\lambda = 5, \mu = 5, R = 12, c = 1, \) and a maximum queue length of 20 as also selected by Mahadevan in [16, 19]. To allow a comparison to the optimal solution we implemented the constraints imposed by constraint formulation of the addends of the Laurent series expansion [22, 29, p.346] using mixed integer linear programming (MILP). The MILP result shows that \(L = 2\), i.e. both policies of admitting 2 or 3 jobs to the queue are gain-optimal imposing average reward of \(\rho^2 = 30\). However, only admitting 3 jobs is also bias-optimal and with that Blackwell-optimal, as it’s the only gain-optimal policy that is left. This makes sense, as \(R\) is only collected when an order is accepted, which is for admitting 3 jobs immediate in contrast to the policy of admitting 2. Note that the inferred average reward of about 27.5 in [19] is sub-optimal\(^9\). The correct queue lengths for admitting 2 jobs is 0.67, while for 3 it is 1.12.

For the algorithm setup we use the same values as in the gridworld experiment, except that we changed the \(\epsilon\)-Parameter to constantly be 5 as the returned reward is significantly higher as in the previous example. As the problem is more complex than the previous ones we decided to perform \(10^6\) learning steps before evaluating for 100k periods.

The results are depicted in Table 5. The Friedman test rejected the null hypotheses (with 3.399e−32 for the mean reward and 2.223e−35 for the mean queue length). We performed the pairwise Conover post-hoc tests adjusted by the Benjaminyi-Hochberg FDR method [3] and highlighted measures that are not statistically distinguishable from each other with the a grey background (see the Appendix for detailed results). The ARA-DRL variants with \(\gamma_1 = 1.0\) and \(\gamma_1 = 0.999\) infer the Blackwell-optimal policy of admitting 3 jobs and thus accumulate a reward of about 29.88 and 29.77 per step over all evaluations of the 40 replications. This clearly shows the stability of the ARA-DRL algorithm, especially when \(\gamma_1 = 1.0\). However, in this example ARA \(\gamma_1 = 0.99\) is unable to find the optimal policy in 12 replications and therefore performs worse as compared

\(^9\) It is possible that the reported results in [19] do not coincide with the given setup, as the queue lengths do not match our results either. Our algorithm infers the policy of admitting 2 and queue length 0.68 if we fix the average reward to 27.5 on the above specified setup.
to the other ARA-DRL instances. Statistically the optimal queue length of 1.12 is matched by ARA-DRL $\gamma_1 = 0.999$ and ARA-DRL $\gamma_1 = 1.0$. In contrast all Q-Learning setups are unable to find even close-to-optimal policies, resulting in rather small amount of collected rewards and very short mean queue lengths. Furthermore, we investigated how to get Q-Learning $\gamma_1 = 0.99$ to find better solutions. Using $\epsilon$-sensitive comparison, instead of the max-operator, for the action selection process we could infer the gain-optimal policy which admits 2 jobs. However, we were unable to infer the Blackwell-optimal policy of admitting 3 jobs and thus collects rewards as soon as possible with Q-Learning. Similarly by hyperparameter tuning of ARA-DRL $\gamma_1 = 1.0$, namely increasing the decay rates of $\alpha$ and $\gamma$ to 0.8 and omitting the minimum values, we were able to find more stable state-action values such that setting $\epsilon \leq 1$ is possible while still finding the optimal policy and the average reward stabilizes even more. We found that the stability of the average reward is of major importance for the stability our ARA-DRL algorithm.

### 6 Conclusion

This paper introduces deep theoretical insights to reinforcement learning and explains why standard discounted reinforcement learning is inappropriate for tasks that are presented with rewards in non-terminal steps as also as they produce an average reward per step that cannot be approximated with 0. This kind of problem structure is easily obtained in real-world problem specifications, for instance in the field of operations research, where companies constantly aim for profit-optimal decisions. Furthermore, we established a novel average reward adjusted discounted reinforcement learning algorithm ARA-DRL, which is computationally cheap and deduces near-Blackwell-optimal policies. Additionally, we implemented the algorithm and prove its viability by testing it on three different decision problems. The results experimentally expose the superiority of ARA-DRL over standard discounted reinforcement learning. In the future we plan to use neural networks for function approximation to be able to apply ARA-DRL to bigger sized problems. Adding to this we are planning to develop an actor-critic version of ARA-DRL.

Therefore, in conclusion although machine learning has tremendously advanced and solved many problems in many different areas the methods may not be directly applicable to operations research. Thus, as a researcher it is important to have broad background knowledge of the selected method instead of handling it as a black box, as we showed that sometimes the problem structures require an adaption of the machine learning technique for successful applications. Nonetheless, machine
learning and in particular reinforcement learning have been producing astonishing results over the past decades, which when handled wisely can be adapted to our field of research as we demonstrated by establishing ARA-DRL. Therefore, machine learning methods will likely play an important role in advancing the field of operations research and its techniques over the next years, especially when used in combination with theoretical insights of our field.

References

1. Balaji, B., Bell-Masterson, J., Bilgin, E., Damianou, A., Garcia, P.M., Jain, A., Luo, R., Maggiar, A., Narayanaswamy, B., Ye, C.: Orl: Reinforcement learning benchmarks for online stochastic optimization problems. arXiv preprint arXiv:1911.10641 (2019)
2. Bello, I., Pham, H., Le, Q.V., Norouzi, M., Bengio, S.: Neural combinatorial optimization with reinforcement learning. arXiv preprint arXiv:1611.09940 (2016)
3. Benjamini, Y., Hochberg, Y.: Controlling the false discovery rate: a practical and powerful approach to multiple testing. Journal of the Royal statistical society: series B (Methodological) 57(1), 289–300 (1995)
4. Bertsekas, D.P., Bertsekas, D.P., Bertsekas, D.P., Bertsekas, D.P.: Dynamic programming and optimal control, vol. 1. Athena scientific Belmont, MA (1995)
5. Blackwell, D.: Discrete dynamic programming. The Annals of Mathematical Statistics 344, 719–726 (1962). DOI 016/j.cam.2018.05.030
6. Boyan, J.A., Moore, A.W.: Generalization in reinforcement learning: Safely approximating the value function. In: Advances in neural information processing systems, pp. 369–376 (1995)
7. Chaharsooghi, S.K., Heydari, J., Zegordi, S.H.: A reinforcement learning model for supply chain ordering management: An application to the beer game. Decision Support Systems 45(4), 949–959 (2008)
8. Das, T.K., Gosavi, A., Mahadevan, S., Marchalleck, N.: Solving semi-markov decision problems using average reward reinforcement learning. Management Science 45(4), 560–574 (1999)
9. Enns, S.T., Suwanruji, P.: Work load responsive adjustment of planned lead times. Journal of Manufacturing Technology Management 15(1), 90–100 (2004)
10. Gijsbrechts, J., Boute, R.N., Van Mieghem, J.A., Zhang, D.: Can deep reinforcement learning improve inventory management? performance and implementation of dual sourcing-mode problems. Performance and Implementation of Dual Sourcing-Mode Problems (December 17, 2018) (2018)
11. Haviv, M., Puterman, M.L.: Bias optimality in controlled queueing systems. Journal of Applied Probability 35(1), 136–150 (1998)
12. Hax, A.C., Meal, H.C.: Hierarchical integration of production planning and scheduling. Report, DTIC Document (1973)
13. Howard, R.A.: Dynamic programming and markov processes. John Wiley (1960)
14. Kool, W., van Hoof, H., Welling, M.: Attention, learn to solve routing problems! arXiv preprint arXiv:1803.08475 (2018)
15. Lillicrap, T.P., Hunt, J.J., Pritzel, A., Heess, N., Erez, T., Tassa, Y., Silver, D., Wierstra, D.: Continuous control with deep reinforcement learning. arXiv preprint arXiv:1509.02971 (2015)
16. Mahadevan, S.: An average-reward reinforcement learning algorithm for computing bias-optimal policies. In: AAAI/IAAI, Vol. 1, pp. 875–880 (1996)
17. Mahadevan, S.: Average reward reinforcement learning: Foundations, algorithms, and empirical results. Machine Learning 22, 159–195 (1996)
18. Mahadevan, S.: Optimality criteria in reinforcement learning. In: Proceedings of the AAAI Fall Symposium on Learning Complex Behaviors in Adaptive Intelligent Systems (1996)
19. Mahadevan, S.: Sensitive discount optimality: Unifying discounted and average reward reinforcement learning. In: ICML, pp. 328–336 (1996)
20. Mahadevan, S., Marchalleck, N., Das, T.K., Gosavi, A.: Self-improving factory simulation using continuous-time average-reward reinforcement learning. In: Machine Learning-International Workshop Then Conference-, pp. 202–210. Morgan Kaufmann Publishers, Inc. (1997)
21. Mahadevan, S., Theocharous, G.: Optimizing production manufacturing using reinforcement learning. In: FLAIRS Conference, pp. 372–377 (1998)
22. Miller, B.L., Veinott, A.F.: Discrete dynamic programming with a small interest rate. The Annals of Mathematical Statistics 40(2), 366–370 (1969). URL http://www.jstor.org/stable/2239451
23. Mnih, V., Badia, A.P., Mirza, M., Graves, A., Lillicrap, T., Harley, T., Silver, D., Kavukcuoglu, K.: Asynchronous methods for deep reinforcement learning. In: International conference on machine learning, pp. 1928–1937 (2016)
24. Mnih, V., Kavukcuoglu, K., Silver, D., Rusu, A.A., Veness, J., Bellemare, M.G., Graves, A., Riedmiller, M., Fidjeland, A.K., Ostrovski, G., et al.: Human-level control through deep reinforcement learning. Nature 518(7540), 529 (2015)

25. Mnih, V., Kavukcuoglu, K., Silver, D., Rusu, A.A., Veness, J., Bellemare, M.G., Graves, A., Riedmiller, M., Fidjeland, A.K., Ostrovski, G., et al.: Human-level control through deep reinforcement learning. Nature 518(7540), 529 (2015)

26. Nazari, M., Oroojlooy, A., Snyder, L., Takáč, M.: Reinforcement learning for solving the vehicle routing problem. In: Advances in Neural Information Processing Systems, pp. 9839–9849 (2018)

27. Ok, D., Tadepalli, P.: Auto-exploratory average reward reinforcement learning. In: AAAI/IAAI, Vol. 1, pp. 881–887 (1996)

28. Oroojlooyjadid, A., Nazari, M., Snyder, L., Takáč, M.: A deep q-network for the beer game: A reinforcement learning algorithm to solve inventory optimization problems. arXiv preprint arXiv:1708.05924 (2017)

29. Puterman, M.L.: Markov decision processes. J. Wiley and Sons (1994)

30. Rohde, J.: Hierarchical supply chain planning using artificial neural networks to anticipate base-level outcomes. OR Spectrum 26(4), 471–492 (2004)

31. Schneckenreither, M., Haeussler, S.: Reinforcement learning methods for operations research applications: The order release problem. In: International Conference on Machine Learning, Optimization, and Data Science, pp. 545–559. Springer (2018)

32. Schwartz, A.: A reinforcement learning method for maximizing undiscounted rewards. In: Proceedings of the tenth international conference on machine learning, vol. 298, pp. 298–305 (1993)

33. Schwartz, A.: Thinking locally to act globally: A novel approach to reinforcement learning. In: Proceedings of the fifteenth annual conference of the cognitive science society, pp. 906–911. Lawrence Erlbaum Associates Hillsdale, NJ (1993)

34. Silver, D., Huang, A., Maddison, C.J., Guez, A., Sifre, L., Van Den Driessche, G., Schrittwieser, J., Antonoglou, I., Panneershelvam, V., Lanctot, M., et al.: Mastering the game of go with deep neural networks and tree search. nature 529(7587), 484 (2016)

35. Silver, D., Hubert, T., Schrittwieser, J., Antonoglou, I., Lai, M., Guez, A., Lanctot, M., Sifre, L., Kumaran, D., Graepel, T., et al.: Mastering chess and shogi by self-play with a general reinforcement learning algorithm. arXiv preprint arXiv:1712.01815 (2017)

36. Sutton, R.S., Barto, A.G., et al.: Introduction to reinforcement learning, vol. 2. MIT press Cambridge (1998)

37. Tadepalli, P., Ok, D.: Model-based average reward reinforcement learning. Artificial intelligence 100(1-2), 177–224 (1998)

38. Veinott, A.F.: Discrete dynamic programming with sensitive discount optimality criteria. The Annals of Mathematical Statistics 40(5), 1635–1660 (1969). DOI 10.1214/aoms/1177697379

39. Vera, J.M., Abad, A.G.: Deep reinforcement learning for routing a heterogeneous fleet of vehicles. arXiv preprint arXiv:1912.03341 (2019)

40. Watkins, C.J.C.H.: Learning from delayed rewards. Ph.D. thesis, King’s College (1989)

41. Zhang, W., Dietterich, T.G.: A reinforcement learning approach to job-shop scheduling. In: IJCAI, vol. 95, pp. 1114–1120. Citeseer (1995)

42. Zijm, W.H.: Towards intelligent manufacturing planning and control systems. OR-Spektrum 22(3), 313–345 (2000)
A Statistical Results

This section provides the detailed results of the pairwise Benjaminyi-Hochberg FDR adjusted Conover statistical analysis for the gridworld and the admission control queuing system examples.

A.1 Statistical Significance for the Gridworld Example

|                | ARA-DRL |                  | Q-Learning |                  |
|----------------|---------|------------------|------------|------------------|
|                | γ₁ = 0.99 | γ₁ = 0.999 | γ₁ = 1.0  | γ₁ = 0.99 | γ₁ = 0.999 |
| ARA-DRL γ₁ = 0.999 | 6.723e-01 |              |            |            |           |
| ARA-DRL γ₁ = 1.0  | 7.834e-01 | 7.834e-01      |            |            |           |
| Q-Learning γ₁ = 0.99 | 2.048e-71 | 3.365e-70      | 8.198e-71 |            |           |
| Q-Learning γ₁ = 0.999 | 1.094e-75 | 1.361e-74      | 3.764e-75 | 6.936e-02 |           |
| Q-Learning γ₁ = 0.50 | 3.893e-96 | 1.392e-95      | 6.353e-96 | 2.333e-25 | 1.148e-19 |
| ARA-DRL γ₁ = 0.999 | 2.465e-06 |              |            |            |           |
| ARA-DRL γ₁ = 1.0  | 5.249e-04 | 1.957e-01      |            |            |           |
| Q-Learning γ₁ = 0.99 | 3.083e-89 | 3.696e-79      | 5.691e-82 |            |           |
| Q-Learning γ₁ = 0.999 | 7.900e-88 | 1.531e-77      | 2.041e-80 | 4.583e-01 |           |
| Q-Learning γ₁ = 0.50 | 2.242e-99 | 1.352e-90      | 4.932e-93 | 7.775e-08 | 1.787e-09 |

Table 4: The Benjaminyi-Hochberg FDR adjusted Conover p-values for the mean sum of rewards (top) and the mean average number of steps to the goal state (bottom)

A.2 Statistical Significance for the Admission Control Queue System Example

25
\begin{table}
\centering
\begin{tabular}{lcccc}
\hline
 & \multicolumn{3}{c}{ARA-DRL} & Q-Learning \\
 & $\gamma_1 = 1.0$ & $\gamma_1 = 0.999$ & $\gamma_1 = 0.99$ & $\gamma_1 = 0.999$ \\
\hline
ARA-DRL $\gamma_1 = 0.999$ & 5.947e-01 & 6.556e-01 & 6.406e-01 & 1.590e-05 \\
ARA-DRL $\gamma_1 = 0.99$ & 7.602e-11 & 6.269e-12 & 3.152e-12 & 1.554e-05 \\
Q-Learning $\gamma_1 = 0.99$ & 6.035e-64 & 1.509e-83 & 5.459e-94 & 6.556e-01  \\
Q-Learning $\gamma_1 = 0.999$ & 3.056e-65 & 1.498e-84 & 1.061e-94 & 6.269e-12 \\
Q-Learning $\gamma_1 = 0.5$ & 5.232e-45 & 7.776e-68 & 3.485e-80 & 3.152e-12 \\
\hline
\end{tabular}
\caption{The Benjaminyi-Hochberg FDR adjusted Conover p-values for the mean sum reward (top) and mean queue length (bottom)}
\end{table}