Two explicit divisor sums

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Abstract
We give explicit bounds on sums of $d(n)^2$ and $d_4(n)$, where $d(n)$ is the number of divisors of $n$ and $d_4(n)$ is the number of ways of writing $n$ as a product of four positive integers. In doing so, we make a slight improvement on the upper bound for class numbers of quartic number fields.

Keywords Divisor function · Class numbers · Explicit results

Mathematics Subject Classification 11N37 · 11R29

1 Introduction

Let $d(n)$ denote the number of divisors of $n$, and let $d_k(n)$ denote the number of ways to write $n$ as a product of $k$ positive integers, where $k \geq 2$. Using Perron’s formula with the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{d(n)^2}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)} \text{ and } \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} = \zeta^k(s)$$

(both converging in $\Re(s) > 1$), we have (see, e.g., [18, pp. 312–313])

$$\sum_{n \leq x} d_k(n) \sim \frac{1}{(k-1)!} x (\log x)^{k-1}.$$
and
\[ \sum_{n \leq x} d(n)^2 \sim \frac{1}{\pi^2} x (\log x)^3. \] (3)

The purpose of this article is to give explicit bounds for the sum in (3) by way of an explicit version of (2) in the case \( k = 4 \). The rolled-gold example for this kind of problem is \( \sum_{n \leq x} d(n) \), which corresponds to (2) with \( k = 2 \). Berkane, Bordellès and Ramaré [1, Thm. 1.1] recently gave several pairs of values \((\alpha, x_0)\) such that, for \( x \geq x_0 \), we have
\[ \sum_{n \leq x} d(n) = x (\log x + 2\gamma - 1) + \Delta(x), \] (4)
where \( \gamma \) is Euler’s constant and
\[ |\Delta(x)| \leq \alpha x^{1/2}. \] (5)

One such pair given, which we shall use frequently, is \( \alpha = 0.397 \) and \( x_0 = 5560 \).

The best-known error term in (4) is \( \Delta(x) = O(x^{31/416+\epsilon}) \) from Huxley [8] (see also the improvement announced by Bourgain and Watt [5]). It seems difficult to give an implied constant for this estimate. Therefore, asymptotically weaker explicit bounds,\(^1\) such as those in [1], are still very useful for applications.

To bound sums of \( d_k(n) \), Bordellès [4] showed that, for any \( x \geq 1 \),
\[ \sum_{n \leq x} d_k(n) \leq x \left( \log x + \gamma + \frac{1}{x} \right)^{k-1}. \] (6)

This only misses the asymptotic approximation in (2) by a factor of \( 1/(k-1)! \). Nicolas and Tenenbaum (see [4, p. 2]) were able to meet the asymptotic bound in (2) for any \( x \geq 1 \), with
\[ \sum_{n \leq x} d_k(n) \leq \frac{x}{(k-1)!} (\log x + k - 1)^{k-1}. \]

The case of \( k \geq 3 \) was improved by Bordellès [3], to
\[ \sum_{n \leq x} d_k(n) \leq \frac{x}{(k-1)!} (\log x + k - 2)^{k-1}, \] (7)
for any \( x \geq 13 \).

We improve these results for the case \( k = 4 \) in our first theorem. Note that here, and hereafter, we use \( \vartheta \) to represent a number with absolute value at most one.

**Theorem 1** For \( x \geq 2 \) we have
\[ \sum_{n \leq x} d_4(n) = C_1 x \log^3 x + C_2 x \log^2 x + C_3 x \log x + C_4 x + \vartheta (4.48 x^{3/4} \log x), \] (8)
\(^1\) It is worth noting that Theorem 1.2 in [1] has \( |\Delta(x)| \leq 0.764 x^{1/3} \log x \) for \( x \geq 9995 \). Although this is a smaller error term, it would only improve our results for very large \( x \).
where

\[ C_1 = \frac{1}{6}, \quad C_2 = 0.654 \ldots, \quad C_3 = 0.981 \ldots, \quad C_4 = 0.272 \ldots, \]

are exact constants given in Sect. (2). Furthermore, when \( x \geq 193 \) we have

\[ \sum_{n \leq x} d_4(n) \leq \frac{1}{3} x \log^3 x. \] (9)

The bound in (8) is sharper than (6) and (7) for all \( x \geq 2 \). This is primarily because the main terms are exactly those given by Perron’s formula. We are not aware of any other explicit estimate that also matches the main terms. We note that other explicit bounds are possible: we selected (9) as it is nice and neat, and valid when \( x \) is not too large.

Bounds on \( d_k(n) \) can be used to obtain bounds on class numbers of number fields. Let \( K \) be a number field of degree \( n_K = [K : \mathbb{Q}] \) and discriminant \( d_K \). Also, let \( r_1 \) (resp. \( r_2 \)) denote the number of real (resp. complex) embeddings in \( K \), so that \( n_K = r_1 + 2r_2 \). Finally, let

\[ b = b_K = \left( \frac{n_K!}{n_K^{n_K}} \right) \left( \frac{4}{\pi} \right)^{r_2} |d_K|^{1/2}, \]

denote the Minkowski bound, and let \( h_K \) denote the class number. Lenstra [11, Sect. 6]—see also Bordellès [4, Lem. 1]—proved that

\[ h_K \leq \sum_{m \leq b} d_{n_K}(m). \]

Only upper bounds on (2) are needed to bound the class number. Bordellès used (6) in its weaker form

\[ \sum_{n \leq x} d_k(n) \leq 2x(\log x)^{k-1}, \] (10)

for \( x \geq 6 \), to this end. As an aside, we note that the bound on \( h_K \) obtained with Bordellès estimate is not the sharpest possible for all degrees. For example, much more work has been done on quadratic extensions (see [10,12,16]).

Using Theorem 1 we have the following improvement to the bound on the class number of quartic number fields.

Corollary 1 Let \( K \) be a quartic number field with class number \( h_K \) and Minkowski bound \( b \). Then, if \( b \geq 193 \) we have

\[ h_K \leq \frac{1}{3} b \log^3 b. \]

This improves on the bound \( 2b \log^3 b \) obtained by Bordellès using (10). Corollary 1 can also be used to improve the leading constant of the bound in Debaene’s Lemma.
in 13 [6] for the case $n = 4$. This relates to bounding sums of norms of integral ideals—see [6, Sect. 3.3, pp. 894–899] for more details.

The sum in (3) has attracted less attention: for an interesting analogue in a finite field setting, see [2]. Ramanujan [14] showed that

$$\sum_{n \leq x} d(n)^2 - x(A \log^3 x + B \log^2 x + C \log x + D) \ll x^{3/5+\epsilon},$$  \hspace{1cm} (11)

where the constants $A, B, C, D$ can (these days) be obtained via Perron’s formula. The best asymptotic improvement to date of (11) is $x^{1/2+\epsilon}$ from Wilson [20]. We are not aware of any explicit version of this estimate. Of note is an elementary result by Gowers [7], namely, that

$$\sum_{n \leq x} d(n)^2 \leq x(\log x + 1)^3 \leq 2x(\log x)^3 \quad (x \geq 1).$$  \hspace{1cm} (12)

This is used by Kadiri et al. [9] in their work on zero-density estimates for the zeta-function. Although one would expect some lower-order terms, the bound in (12) is a factor of $2\pi^2 \approx 19.7$ times the asymptotic bound in (3), whence one should be optimistic about obtaining a saving. We obtain such a saving in our second main result.

**Theorem 2** For $x \geq 2$ we have

$$\sum_{n \leq x} d(n)^2 = D_1 x \log^3 x + D_2 x \log^2 x + D_3 x \log x + D_4 x + \vartheta \left( 9.73 x^{3/4} \log x \right),$$

where

$$D_1 = \frac{1}{\pi^2}, \quad D_2 = 0.745 \ldots, \quad D_3 = 0.824 \ldots, \quad D_4 = 0.461 \ldots,$$

are exact constants given in (18). Furthermore, for $x \geq x_j$ we have

$$\sum_{n \leq x} d(n)^2 \leq K x \log^3 x,$$

where one may take $\{K, x_j\}$ to be, among others, $\{\frac{1}{4}, 433\}$ or $\{1, 7\}$.

We proceed first with a proof of Theorem 1, followed by discussion on generalising the method for $\sum_{n \leq x} d_k(n)$. We then prove Theorem 2 in Sect. 3.
2 Bounding sums of $d_4(n)$

Since $d_4(n) = (d*d)(n)$, where $*$ denotes Dirichlet convolution, the hyperbola method gives us

$$\sum_{n \leq x} d_4(n) = 2 \sum_{a \leq \sqrt{x}} d(a) \sum_{n \leq x/a} d(n) - \left( \sum_{n \leq \sqrt{x}} d(n) \right)^2.$$  

Using (5), we arrive at

$$\sum_{n \leq x} d_4(n) = 2x \left[ (\log x + 2\gamma - 1) S_1(\sqrt{x}) - S_2(\sqrt{x}) \right] + 2 \sum_{a \leq \sqrt{x}} d(a) \Delta \left( \frac{x}{a} \right)$$  

$$- \left\{ \sqrt{x} \left( \frac{1}{2} \log x + 2\gamma - 1 \right) + \Delta(\sqrt{x}) \right\}^2,$$

where

$$S_1(x) = \sum_{n \leq x} \frac{d(n)}{n}, \quad S_2(x) = \sum_{n \leq x} \frac{d(n) \log n}{n}.$$

The absolute value of the sum over $a \leq \sqrt{x}$ in (13) can also be bounded above by

$$2ax^{1/2}S_3(\sqrt{x}),$$

where $\alpha$ is defined in (5), and

$$S_3(x) = \sum_{n \leq x} \frac{d(n)}{\sqrt{n}}.$$

We can approximate $S_1$, $S_2$, and $S_3$ with partial summation and the bound in (5). Also, note that the following only need be valid for $x \geq 2$, as (13) does not hold for $x = 1$.

Berkane et al. [1, Cor. 2.2] give a bound for $S_1(x)$. As noted by Platt and Trudgian in [13, Sect. 2.1], their constant 1.16 should be replaced by 1.641 as in Riesel and Vaughan [17, Lem. 1]. To obtain an error term in Theorem 1 of size $x^{3/4} \log x$, we should like an error term in $S_1(x)$ of size $x^{-1/2}$, which is right at the limit of what is achievable. We follow the method used by Riesel and Vaughan ([17, pp. 48–50]) to write

$$S_1(x) = \frac{1}{2} \log^2 x + 2\gamma \log x + \gamma^2 - 2\gamma_1 + \vartheta(cx^{-1/2}),$$

with $c = 1.001$ for $x \geq 6 \cdot 10^5$. We ran a simple computation in Mathematica to verify that this also holds for $2 \leq x < 6 \cdot 10^5$. Choosing a larger value than $6 \cdot 10^5$ would reduce the constant $c$, but not to anything less than unity.

To evaluate $S_2$ and $S_3$ we use the second Dirichlet series in (1), and its derivative. For $S_2$, we have

$$S_2(x) = \frac{1}{3} \log^3 x + \gamma \log^2 x + 2\gamma \gamma_1 - \gamma_2 + E_2(x),$$

with $E_2(x)$ a main term.
where
\[ E_2(x) = \frac{\Delta(x) \log x}{x} - \int_x^\infty \frac{(\log t - 1) \Delta(t)}{t^2} \, dt, \]
and \( \gamma_1 \) and \( \gamma_2 \) are the first two Stieltjes constants—see, e.g., [17, pp. 45–46]. Using (5) with \( \alpha = 0.397 \) and \( x_0 = 5560 \) gives, for \( x \geq x_0 \),
\[ |E_2(x)| \leq \alpha \left( 3 + \frac{2}{\log x_0} \right) x^{-1/2} \log x. \]

Lastly, for \( S_3 \) we have
\[ S_3(x) = 2x^{1/2} \log x + 4(\gamma - 1)x^{1/2} + E_3(x), \] (16)
where
\[ E_3(x) = 3 - 2\gamma + \frac{\Delta(x)}{x^{1/2}} + \frac{1}{2} \int_1^x \frac{\Delta(t)}{t^{3/2}} \, dt. \] (17)

For the integral in (17) to converge, we would need \( \Delta(t) \ll t^{1/2-\delta} \). Theorem 1.2 in [1] does give an explicit bound of this order. However, as mentioned in Sect. 1, it would only improve our result for very large \( x \). Instead, since \( S_3(x) \) has a relatively small contribution to the total error, we can afford a slightly larger bound on \( E_3(x) \). We can apply the bound in (5) and the triangle inequality to get
\[ |E_3(x)| \leq 3 - 2\gamma + \alpha + \frac{1}{2} \alpha \log x \leq \left( \frac{\alpha}{2} + \frac{3 - 2\gamma + \alpha}{\log x_0} \right) \log x := \beta \log x, \]
for all \( x \geq x_0 \).

Thus, the bounds in (14), (15), and (16) can be used in (13) to show that
\[ \sum_{n \leq x} d_4(n) = C_1 x \log^3 x + C_2 x \log^2 x + C_3 x \log x + C_4 x + E(x), \]
where
\[ |E(x)| \leq F_1 x^{3/2} \log x, \]
and, with \( c = 1.001 \) and \( \alpha = 0.397 \), we have
\[ C_1 = \frac{1}{6}, \quad C_2 = 2\gamma - \frac{1}{2}, \quad C_3 = 6\gamma^2 - 4\gamma - 4\gamma_1 + 1, \]
\[ C_4 = 4\gamma^3 - 6\gamma^2 + 4\gamma - 12\gamma \gamma_1 + 4\gamma_1 + 2\gamma_2 - 1, \quad \text{and} \quad F_1 = 2c + 6\alpha + \frac{2\alpha}{\log x_0}. \]

As \( x_0 = 5560 \), we have only verified (8) for \( x \geq 5560^2 \). By calculating the partial sums of \( d_4(n) \) in Mathematica, we confirmed that (8) also holds for \( 2 \leq x < 5560^2 \).
It appears possible to bound the partial sums of $d_k(n)$ for other values of $k$ by generalising the previous method. When $k$ is even, one can use $d_k(n) = (d_{k-2} * d)(n)$; when $k$ is odd, one can use $d_k(n) = (d_{k-1} * 1)(n)$. Although we have not pursued this here, it is likely to give decent bounds for small values of $k$. For large values of $k$ we expect the error term to increase rapidly with repeated application of this process. Alternatively, one might consider a 'hyperboloid' method, involving the estimation of $d_k(n) = (1 * 1 * \cdots * 1)(n)$, with a $k$-fold convolution of the unity function.

**3 Bounding sums of $d(n)^2$**

We define

$$H(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s} = \frac{1}{\zeta(2s)},$$

and

$$H^*(s) = \sum_{n=1}^{\infty} \frac{|h(n)|}{n^s} = \prod_p \left(1 + \frac{1}{p^{2s}}\right) = \frac{\zeta(2s)}{\zeta(4s)},$$

whence both $H(s)$ and $H^*(s)$ converge for $\Re(s) > \frac{1}{2}$. This, combined with (1), implies that $d(n)^2 = (d_4 * h)(n)$, so we can write

$$\sum_{n \leq x} d(n)^2 = \sum_{a \leq x} h(a) \sum_{b \leq \frac{x}{a}} d_4(b).$$

Expanding both sums with the estimate in (8), we get

$$\sum_{n \leq x} d(n)^2 = D_1 x \log^3 x + D_2 x \log^2 x + D_3 x \log x + D_4 x + \sum_{a \leq x} h(a) E \left(\frac{x}{a}\right),$$

where

$$D_1 = C_1 H(1), \quad D_2 = C_2 H(1) + 3C_1 H'(1),$$
$$D_3 = C_3 H(1) + 2C_2 H'(1) + 3C_1 H''(1),$$
$$D_4 = C_4 H(1) + C_3 H'(1) + C_2 H''(1) + C_1 H^{(3)}(1),$$

and the error term is

$$\left| \sum_{a \leq x} h(a) E \left(\frac{x}{a}\right) \right| \leq F_1 H^* \left(\frac{3}{4}\right) x^{\frac{3}{2}} \log x.$$
We note here the following values for ease of replication:

\[ H(1) = \frac{6}{\pi^2}, \quad H'(1) = -\frac{72\zeta'(2)}{\pi^4}, \quad H''(1) = \frac{1728\zeta'(2)^2}{\pi^6} - \frac{144\zeta''(2)}{\pi^4}, \]
\[ H^{(3)}(1) = -\frac{62208\zeta'(2)^3}{\pi^8} + \frac{10368\zeta'(2)\zeta''(2)}{\pi^6} - \frac{288\zeta^{(3)}(2)}{\pi^4}. \]

As an aside, Theorem 2 could have been proved with Ramaré’s general result in [15, Lem. 3.2], as modified in [19, Lem. 14] (with the constants repaired as in [17]). However, some further generalisation would have been necessary; hence we chose the preceding route.

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