MANIFOLDS WITH INFINITE DIMENSIONAL GROUP OF
HOLOMORPHIC AUTOMORPHISMS AND THE LINEARIZATION
PROBLEM

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Abstract. We overview a number of precise notions for a holomorphic automorphism
group to be big together with their implications, in particular we give an exposition
of the notions of flexibility and of density property.
These studies have their origin in the famous result of Andersén and Lempert
from 1992 proving that the overshears generate a dense subgroup in the holomorphic
automorphism group of \( \mathbb{C}^n, n \geq 2 \). There are many applications to natural geometric
questions in complex geometry, several of which we mention here.
Also the Linearization Problem, well known since the 1950s and considered by
many authors, has had a strong influence on those studies. It asks whether a compact
subgroup in the holomorphic automorphism group of \( \mathbb{C}^n \) is necessarily conjugate to
a group of linear automorphisms. Despite many positive results, the answer in this
generality is negative as shown by Derksen and the author. We describe various
developments around that problem.

CONTENTS

1. Introduction 1
2. Complex Affine Space 3
3. Notions for the largeness of an automorphism group 7
3.1. Density property 7
3.2. Flexibility 12
4. Applications to natural geometric questions 15
5. The holomorphic linearization problem 21
5.1. Basic notions and properties 21
5.2. History of the linearization problem 23
5.3. Counterexamples 24
5.4. Relation to famous problems 26
5.5. Optimal positive results 27
References 29

1. INTRODUCTION

The holomorphic and at the same time algebraic automorphism group of the complex
line \( \mathbb{C} \) consists of invertible affine maps \( z \mapsto az + b, a \in \mathbb{C}^*, b \in \mathbb{C} \). It is a complex Lie
group generated by the translations \( z \mapsto z + b \) and the rotations \( z \mapsto az \).

It is a classical fact that \( \mathbb{C}^n \) for \( n \geq 2 \) has infinite dimensional groups of algebraic and
of holomorphic automorphisms. Indeed, maps of the form

\[(1.1) \quad (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{n-1}, z_n + f(z_1, \ldots, z_{n-1})) , \]

where \( f \in \mathcal{O}(\mathbb{C}^{n-1}) \) is an arbitrary polynomial or holomorphic function of \( n - 1 \) vari-
bles, are automorphisms. They can be viewed as time-1 maps of the vector field

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\( \theta = f(z_1, \ldots, z_{n-1}) \frac{\partial}{\partial z_n} \). In complex analysis (when \( f \) is holomorphic) such an automorphism is called a shear and such a vector field is called a shear field. If \( f \) is a polynomial the complex analysts call the automorphism a polynomial shear, whereas in affine algebraic geometry it is called an elementary automorphism. The geometric idea behind this automorphism is to look at \( \mathbb{C}^n \) as a trivial line bundle over \( \mathbb{C}^{n-1} \) and performing a translation in each of the fibers of that line bundle, that depends polynomially or holomorphically on the base point. In the same way one can use rotations depending on the base point

\[
(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{n-1}, f(z_1, \ldots, z_{n-1}) \cdot z_n),
\]

where \( f \in \mathcal{O}^*(\mathbb{C}^{n-1}) \) is a nowhere vanishing holomorphic function. By simple connectedness of \( \mathbb{C}^{n-1} \) the function \( f \) is the exponential \( f = e^\theta \) of some holomorphic \( g \in \mathcal{O}(\mathbb{C}^{n-1}) \). Again such an automorphism is the time-1 map of a complete (ly integrable) holomorphic vector field \( \theta = g(z_1, \ldots, z_{n-1})z_n \frac{\partial}{\partial z_n} \). These automorphisms are called overshears and the corresponding vector fields overshear fields. This notion is not relevant for affine algebraic geometry, since nowhere vanishing polynomials on affine space are constant. These maps together with affine automorphism are the first obvious candidates for a generating set of the group of holomorphic automorphisms \( \text{Aut}_{hol}(\mathbb{C}^n) \). We will come to this point in the next section.

The content of our article is an account of the holomorphic side of the problem of understanding the automorphism group of affine space and understanding how to detect affine space among manifolds.

The attempts of getting a better understanding and possibly some structure theorems for the infinite dimensional groups \( \text{Aut}_{hol}(\mathbb{C}^n) \) and \( \text{Aut}_{alg}(\mathbb{C}^n) \) started earlier in the algebraic case than in the holomorphic category. We present some of the the known results in the next section.

One attempt of better understanding these groups was the Linearization Problem asking about the ways a reductive (for example a finite) group can act on \( \mathbb{C}^n \). Or equivalently, how to find all subgroups of the algebraic or the holomorphic automorphism group of \( \mathbb{C}^n \) isomorphic to a given reductive group \( G \). The conjectured answer was that all such groups are conjugated into the group of linear transformations \( \text{GL}_n(\mathbb{C}) \).

Another intriguing question is whether this rich group of holomorphic or algebraic automorphisms can be used to characterize affine space \( \mathbb{C}^n \) holomorphically among Stein manifolds or algebraically among smooth affine algebraic varieties, i.e., subsets of \( \mathbb{C}^N \) given by finitely many polynomial equations. Remember that a Stein manifold is by definition a complex manifold that is holomorphically convex and holomorphically separable. Let be reminded that by the Bishop–Remmert embedding theorem any Stein manifold is a closed complex submanifold of some \( \mathbb{C}^N \) given by finitely many holomorphic equations. Thus Stein manifolds are the natural holomorphic analogue of affine algebraic manifolds.

Surprisingly both problems Linearization and characterization of \( \mathbb{C}^n \) are intimately related which is the reason why we present them both here.

It was understood since a long time that in order to solve the Linearization Problem one needs to be able to characterize \( \mathbb{C}^n \). Most easily this is seen at the example of the famous Zariski Cancellation Problem (see Problem 5.11) (the author learned this from Hanspeter Kraft, who says that several people knew it). Suppose \( X \times \mathbb{C} \) is algebraically isomorphic (resp. biholomorphic) to \( \mathbb{C}^{n+1} \) where \( X \) is not algebraically isomorphic (resp. biholomorphic) to \( \mathbb{C}^n \). Then the action of the group of 2 elements generated by \( \sigma: X \times \mathbb{C} \to X \times \mathbb{C} \ (x,t) \mapsto (x,-t) \) acts on \( \mathbb{C}^{n+1} \cong X \times \mathbb{C} \) with a fixed point set \( X \) not algebraically isomorphic (resp. biholomorphic) to \( \mathbb{C}^n \). Thus this action cannot be conjugate to a linear action since the fixed point sets of linear actions are affine spaces. Thus to answer the linearization question to the positive it is necessary to solve Zariski’s Cancellation Problem to the positive and this in turn would most likely require a certain characterization of \( \mathbb{C}^n \). More information in this direction can be
found in section 2 and subsection 5.4. When looking for a such a characterization of $\mathbb{C}^n$ a natural attempt is to use the automorphism group. It is a general phenomenon in mathematics and nature that highly symmetric objects are rare and such objects often can be characterized by their symmetries. As an instance in our subject one can name the fact that among bounded domains the unit ball $B := \{ z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |z_1|^2 + |z_2|^2 + \ldots + |z_n|^2 < 1 \}$ can be characterized among bounded strictly pseudoconvex domains by the dimension of its holomorphic automorphism group. Now the algebraic and holomorphic automorphism groups of are large, at least infinite dimensional as we saw above. What would be a natural "largeness" condition which is exclusively satisfied by the holomorphic automorphism group of $\mathbb{C}^n$. The search for structure on $\text{Aut}_{hol}(\mathbb{C}^n)$ lead to a nice guess, the density property (see the Varolin Toth Conjecture 5.12). More precisely, Andersén and Lempert proved a remarkable result which was further developed by Forstnerič and Rosay to the so called Andersén-Lempert Theorem 2.5. This result shows in a precise way how big $\text{Aut}_{hol}(\mathbb{C}^n)$ is. The crucial ingredient of the Andersén-Lempert Theorem (sometimes called the Andersén-Lempert Lemma) was then naturally generalized by Varolin to the notion of density property. Now the search for Stein manifolds with this property started and it is still not clear at the moment whether this property (together with an obvious condition having the right structure as a differentiable manifold) characterizes $\mathbb{C}^n$. The Andersén-Lempert Theorem in turn was used to construct non-straightenable holomorphic embeddings which then in turn led to the counterexamples to Holomorphic Linearization described in the last section. Thus the attempt of finding structure led to the study of largeness properties of $\text{Aut}_{hol}(\mathbb{C}^n)$ and a nice application of the theory to embedding questions finally led to the negative solution of the Linearization Problem. The details of this construction based on the ingenious trick of Asanuma are described in subsection 5.3.

Thus one concrete question pointing in the direction of characterizing affine space, was solved to the negative by another (not yet finished) attempt of charactering affine space. Welcome to the story.

The article is organized as follows. In Section 2 we give an account of the knowledge about the algebraic and holomorphic automorphism groups of affine space $\mathbb{C}^n$. The notions of density property and flexibility are explained in Section 3 together with their implications, most prominent for the density property this is the Andersén-Lempert Theorem 3.2 and for flexibility this is the Oka principle 3.2.1. In Section 4 we give a list of beautiful applications of the theory behind these notions to natural geometric problems. In Section 5 we turn to the Holomorphic Linearization Problem. After explaining basic notions like the categorical quotient in Section 5.1. we go through the history in Section 5.2.. In Section 5.3. we explain one of the most spectacular applications of number (2) in the list of applications from Section 4: The counterexamples to Holomorphic Linearization found by Derksen and the author. In Section 5.4. we relate these counterexamples to famous ZariskiCancellation Problem and other long standing problems. In Section 5.5. we explain recent positive results on Holomorphic Linearization which very heavily point to the conjecture that the method of constructing counterexamples explained in Section 5.3. is the only possible method.

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2. Complex Affine Space

From now on $n \geq 2$. As we have seen in the introduction there are plenty of polynomial automorphisms of $\mathbb{C}^n$ given by formula (1.1). Together with the affine automorphisms $\text{Aff}(\mathbb{C}^n)$ consisting of maps $Z \mapsto AZ + b$, where $A \in \text{GL}_n(\mathbb{C})$ and $b \in \mathbb{C}^n$, they generate the group $\text{Tame}(\mathbb{C}^n)$ of tame automorphisms. Remark that using conjugation with linear automorphisms permuting the coordinates we see that all automorphisms of the form (1.1) with any other variable $z_i$ playing the role of $z_n$ are contained in the group $\text{Tame}(\mathbb{C}^n)$ of tame automorphisms. The natural question arising is whether all algebraic
automorphisms are tame automorphisms, i.e., is $\text{Aut}_{\text{alg}}(\mathbb{C}^n) = \text{Tame}(\mathbb{C}^n)$? Suggesting a positive answer to that question this was known under the name ”Tame Generator Conjecture”. The conjecture was supported by the classical positive answer for $\mathbb{C}^2$.

**Theorem 2.1** (Jung 1942 [Jun42]). $\text{Tame}(\mathbb{C}^2) = \text{Aut}_{\text{alg}}(\mathbb{C}^2)$.

However already in 1972 Nagata proposed a candidate for a counterexample in $\mathbb{C}^3$, now well known under the name Nagata automorphism.

$$ (z_1, z_2, z_3) \mapsto (z_1 - 2(z_2 z_3 + z_2^3)z_2 - (z_1 z_3 + z_2^3)z_2 + (z_1 z_3 + z_2^3)z_3, z_3) $$

In order to see that this map is invertible it helps to observe that it preserves the polynomial $u = z_1 z_3 + z_2^3$.

It took until 2003 when Umirbaev and Shestakov [SU03] were able to confirm that the Nagata automorphism is not tame. The Tame Generator Conjecture is still open for $\mathbb{C}^n$.

**Theorem 2.3** (Ahern, Rudin [AR95]). The group $\text{Tame}(\mathbb{C}^n)$ is a free amalgamated product of the group of affine automorphisms $\text{Aff}(\mathbb{C}^2)$ and the group of elementary automorphisms $\text{Jonq}(\mathbb{C}^n)$ over their intersection.

Inspired by the algebraic results and the Tame Generator Conjecture W. Rudin together with P. Ahern proved the analogous structure result in the holomorphic category. Here the group generated by holomorphic shears (see equation (1.1)), overshears (see equation (1.2)) and affine automorphisms is called the overshear group $\text{Osh}(\mathbb{C}^n)$. The definition of the holomorphic de Jonquières group $\text{Jonq}_{\text{hol}}(\mathbb{C}^n)$ is clear. For convenience of the reader we write their members for $\mathbb{C}^2$

$$ (z, w) \mapsto (az + b, e^{f(z)}w + g(z)) $$

where $a \in \mathbb{C}^*$, $b \in \mathbb{C}$ are numbers and $f, g \in \mathcal{O}(\mathbb{C})$ holomorphic functions.

**Theorem 2.2** (van der Kulk 1953 [vdK53]). The group $\text{Osh}(\mathbb{C}^2)$ is a free amalgamated product of $\text{Aff}(\mathbb{C}^2)$ and $\text{Jonq}_{\text{hol}}(\mathbb{C}^2)$ over their intersection.

Similar structure results for higher dimensions $n \geq 3$ do not hold. We invite the reader to write the identity automorphism as a non-trivial product of triangular and affine automorphisms. Only for two dimensional affine manifolds similar structure results are known. For example automorphism groups of Danielewski surfaces admit structures of free amalgamated products. Danielewski surfaces are the surfaces given by $D := \{(u, v, z) \in \mathbb{C}^3 : uv = f(z)\}$, where $f$ is either a polynomial or a holomorphic function with simple zero’s only. This condition of simple zeros ensures the smoothness of the corresponding affine algebraic variety (resp. Stein manifold). In the algebraic category these structure results are due to Makar-Limanov [ML90] and in the holomorphic to Andrist, Lind and the author [AKL15], see these reference for relevant definitions of the involved groups in the amalgamated product structure.

Rudin very much promoted the analogue of the Tame Generator Conjecture in the holomorphic category. Is $\text{Osh}(\mathbb{C}^n) = \text{Aut}_{\text{hol}}(\mathbb{C}^n)$?

In their seminal paper Andersén and Lempert [AL92] gave two answers to this question, the first of them solving the problem in the negative has not been of much future importance. The group $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ is a topological group when equipped with the topology of uniform convergence on compacts, the usual c.-o.-topology. The following metric
$d$ on $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ induces this topology and makes this topological space a complete metric space, i.e., a Fréchet space ([AL92] Proposition 6.4).

For $\Psi, \Phi \in \text{Aut}_{\text{hol}}(\mathbb{C}^n)$, and $r = 1, 2, \ldots$ put

$$d_r(\Psi, \Phi) = \max \{\max_{|z| \leq r} |\Phi(z) - \Psi(z)|, \max_{|z| \leq r} |\Phi^{-1}(z) - \Psi^{-1}(z)|\},$$

(2.3) $$d(\Psi, \Phi) = \sum_{r=1}^{\infty} \min(1, d_r(\Psi, \Phi)) 2^{-r}$$

**Theorem 2.4.** For all $n \geq 2$ the overshear group $\text{Osh}(\mathbb{C}^n)$ is meagre in $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$.

Interestingly for $n \geq 3$ no concrete holomorphic automorphism is known which is not contained in $\text{Osh}(\mathbb{C}^n)$. It is for example not known whether the Nagata automorphism from formula (2.1) is contained in $\text{Osh}(\mathbb{C}^3)$. In dimension 2 it follows easily from the structure Theorem 2.3 that automorphisms of the form

$$(z_1, z_2) \mapsto (e^{f(z_1 z_2)} z_1, e^{-f(z_1 z_2)} z_2)$$

are not contained in $\text{Osh}(\mathbb{C}^2)$ ([And90], [KK90].

The second answer Andersén and Lempert gave to Rudin’s question is of great value, it has been the starting point of an enormous development leading to many beautiful results of geometric nature in complex analysis. This still extremely active area will be described in the next section.

**Theorem 2.5.** For all $n \geq 2$ the overshear group $\text{Osh}(\mathbb{C}^n)$ is dense in $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ with respect to the c.o.-topology.

This exemplifies the fact that finite compositions are not the appropriate notion for complex analysis when it comes to infinite dimensional automorphism groups. Rather limits of finite compositions are the appropriate notion.

Finally let us come to the Holomorphic Linearization Problem that was long standing and could be solved only with the methods of the area which is described in the next section and which was started by the seminal paper of Andersén and Lempert [AL92].

In what follows $G$ will be a complex reductive group. For us a complex reductive group $G$ is a complex Lie group which is the universal complexification $G = K^C$ of its maximal compact subgroup $K$. Examples of such groups are $\text{GL}_n(\mathbb{C}) = \text{U}_n^C$, $\text{SL}_n(\mathbb{C}) = \text{SU}_n^C$ or even unconnected groups like $\text{O}_n(\mathbb{C})$ which has 2 connected components. Even finite groups are most interesting (which by definition are the universal complexification of themselves). Suppose a reductive group $G$ is acting (effectively) holomorphically on $\mathbb{C}^n$, $n \geq 2$, in other words, let a holomorphic (injective) group homomorphism $G \hookrightarrow \text{Aut}_{\text{hol}}(\mathbb{C}^n)$ be given.

**Problem 2.6 (Holomorphic Linearization Problem).** [Huc90]

*Does there exist a holomorphic change of variables $\alpha \in \text{Aut}_{\text{hol}}(\mathbb{C}^n)$ such that $\alpha \circ G \circ \alpha^{-1}$ is linear?*

If the answer is positive, we say the action is linearizable. The problem can be equivalently formulated for compact (real-analytic Lie) groups instead due to the following result of the author. Using the discussion about completeness of vector fields at the beginning of section 3.1.1 one can generalize this result from $\mathbb{C}^n$ to any Stein manifold with the density property.

**Theorem 2.7.** [Kut98] If a real Lie group $H$ acts on $\mathbb{C}^n$ by holomorphic transformations, then this action extends (uniquely) to a holomorphic action of the complexification $H^C$.

Thus an action of a compact group $K$ extends to an action of the reductive group $G = K^C$ and is linearizable, if the action of $G$ is. This follows directly from the identity principle for holomorphic mappings since $K$ is totally real of maximal dimension in $G$.
As a sort of curiosity, it also follows from the above theorem that the real Lie groups \( \text{SL}_k(\mathbb{R}) \) for \( k \geq 2 \) (the universal covering of \( \text{SL}_k(\mathbb{R}) \)) cannot act effectively on \( \mathbb{C}^n \) since they are not injective into their universal complexifications \( \text{SL}_k(\mathbb{C}) \).

The linearization question in the algebraic setting is classical and has drawn much attention. We don’t want to go into these details, we just want to explain the counterexamples to the Algebraic Linearization Problem first constructed by Gerald Schwarz. They are known as \( G \)-vector bundles over representations. A representation is a \( \mathbb{C}^k \) with a linear action (representation) of \( G \), i.e., a group homomorphism \( \alpha : G \to \text{GL}_k(\mathbb{C}) \). By the Quillen-Suslin solution to the Serre question, every algebraic vector bundle (say of rank \( l \)) over \( \mathbb{C}^k \) is trivial, i.e., isomorphic to \( \mathbb{C}^k \times \mathbb{C}^l \to \mathbb{C}^k \). Now we require the linear \( G \)-action on the base \( \mathbb{C}^k \) to lift to the total space as vector bundle automorphisms. This means there is a map \( \beta : G \times \mathbb{C}^k \to \text{GL}_l(\mathbb{C}) \) such the action of \( G \) on the total space \( \mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^l \) of the bundle, which is a map \( G \times (\mathbb{C}^k \times \mathbb{C}^l) \to (\mathbb{C}^k \times \mathbb{C}^l) \), will be given by the formula

\[
(\beta g, (z, w)) \mapsto (\alpha(g) \cdot z, \beta(g, z) \cdot w)
\]

and the \( \cdot \) is multiplication of a vector by a matrix. Clearly the map \( \beta \) has to satisfy some obvious rules coming from the requirements of a group action. The only non-linearity in this sort of action is the dependence of the map \( \beta \) on the “base point” \( z \). A concrete example used by G. Schwarz is the following:

**Example 2.8.** We will construct a non-linearizable action of the group \( G = \mathbb{C}^* \times \text{O}_2(\mathbb{C}) \) on \( \mathbb{C}^4 \). For that we consider \( \mathbb{C}^4 \) as a trivial vector bundle of rank 2 over \( \mathbb{C}^2 \). First we describe the action of \( \text{O}_2(\mathbb{C}) \cong \text{SO}_2(\mathbb{C}) \times \mathbb{Z}/2\mathbb{Z} \cong \mathbb{C}^* \times \mathbb{Z}/2\mathbb{Z} \) by vector bundle automorphisms of the trivial vector bundle. For that let \( \lambda \) denote the variable in \( \text{SO}_2(\mathbb{C}) \cong \mathbb{C}^* \) and \( \sigma \) the generator of \( \mathbb{Z}/2\mathbb{Z} \). Then the action of \( \lambda \) on the trivial vector bundle \( \mathbb{C}^2 \times \mathbb{C}^2 \) with coordinates \( ((x, y), (u, v)) \) is given by

\[
(\lambda, (u, v), (x, y)) \mapsto ((\lambda^2 u, \lambda^{-2} v), (\lambda^3 x, \lambda^{-3} y))
\]

and the action of \( \sigma \) is given by

\[
((u, v), (x, y)) \mapsto ((u, v), ((1 + uv + (uv)^2)y - v^3 x, u^3 y + (1 - uv)x))
\]

The action of \( \text{SO}_2(\mathbb{C}) \) is linear however the element \( \sigma \) makes this bundle non-trivial as an \( \text{O}_2(\mathbb{C}) \)-bundle. The action of the additional factor \( \mathbb{C}^* \) in \( G \) is just the multiplication in the fibre. This has the consequence, that a \( \mathbb{C}^* \)-equivariant isomorphism is nothing than a bundle isomorphism. Since there is no linearization of the \( \text{SO}_2(\mathbb{C}) \)-action by bundle isomorphisms, we have a non-linearizable action of \( G \). The reader is invited to linearize the bundle, and thus the action, holomorphically by explicit formulas involving the exponential function.

Already Schwarz mentioned that this particular action is **holomorphically** linearizable. This is an instance of a more general result described in section 5 based on the Equivariant Oka Principle (Theorem 5.6). Let us just finish the discussion of the Algebraic Linearization Problem by saying that there are non-linearizable actions of all semi-simple groups, that no counterexamples for abelian groups are known yet, and that actions of connected reductive groups on \( \mathbb{C}^2 \) and on \( \mathbb{C}^4 \) [KR14] are known to be linearizable. For \( \mathbb{C}^2 \) this follows classically from Theorem 2.2. The most striking case was that of \( \mathbb{C}^* \)-actions on \( \mathbb{C}^3 \), a result with very indirect proof [KKMLR97]. First came the classification of all contractible affine algebraic 3-folds with \( \mathbb{C}^* \)-action. Then it remained to prove that some of those varieties, which if they were isomorphic to \( \mathbb{C}^3 \) would carry a non-linearizable action, are in fact not isomorphic to \( \mathbb{C}^3 \). The most famous example is the so called Koras-Russell cubic threefold defined by the equation

\[
M_{KR} := \{ (x, y, s, t) \in \mathbb{C}^4 : x + x^2 y + s^2 + t^3 = 0 \}
\]

It was Makar-Limanov who could distinguish this variety algebraically from \( \mathbb{C}^3 \) by inventing his famous Makar-Limanov invariant [ML96].
We will continue this discussion of the Linearization Problem in the last section, but then concentrating more on the holomorphic setting.

3. Notions for the largeness of an automorphism group

A natural way of saying that $\text{Aut}_{\text{hol}}(X)$ of a complex manifold is large is that it is infinite-dimensional. This is by definition the case if the Lie algebra (and equivalently the vector space) generated by $\mathbb{R}$-complete holomorphic vector fields on $X$ is infinite-dimensional. This condition is equivalent to the impossibility of introducing on $\text{Aut}_{\text{hol}}(X)$ a topology with respect to which it becomes a Lie transformation group (possibly, with uncountably many connected components) in the sense of Palais ([Pal57], p.p. 99, 101, 103). In particular, in this case $\text{Aut}_{\text{hol}}(X)$ is not a Lie group in the compact-open topology and not even locally compact (see [MZ74], p. 208).

The notions we will introduce in this section are much stronger (or more precise) and imply the infinite-dimensionality of the holomorphic automorphism group. For example consider the manifold $X = \mathbb{D} \times \mathbb{C}$ which is the product of the unit disc $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ with the complex line. It has an infinite-dimensional group of holomorphic automorphisms. Indeed it contains the shear-like maps $(z, w) \mapsto (z, f(z) + w)$ for all holomorphic functions $f \in \mathcal{O}(\mathbb{D})$. On the other hand this manifold does not have the notions we will introduce in this chapter, it does not have the density property and is not holomorphically flexible. The same holds true for the direct product of any Kobayashi hyperbolic manifold with $\mathbb{C}$.

Another class of examples of manifolds with infinite-dimensional holomorphic automorphism group are homogeneous Stein manifolds of dimension $\geq 2$ as proved by Huckleberry and Isaev [HI09]. In contrary to the example $\mathbb{D} \times \mathbb{C}$, it seems possible that the Stein homogeneous spaces of dimension $\geq 2$, except possibly $(\mathbb{C}^*)^n,$ all have the density property. If one assumes that the homogeneous space is affine algebraic (instead of assuming Stein) this is a recent result of Kaliman and the author, see Example (1) in the list of examples with density property from subsection 3.1.2.

3.1. Density property. Considering a question promoted by Walter Rudin, Andersén and Lempert in 1992 [AL92] proved a remarkable fact about the affine $n$-space $n \geq 2$, namely that the group generated by shears (maps of the form

$$(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{n-1}, z_n + f(z_1, \ldots, z_{n-1}))$$

where $f \in \mathcal{O}(\mathbb{C}^{n-1})$ is a holomorphic function and any linear conjugate of such a map) and overshears (maps of the form

$$(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{n-1}, z_ng(z_1, \ldots, z_{n-1}))$$

where $g \in \mathcal{O}^*(\mathbb{C}^{n-1})$ is a nowhere vanishing holomorphic function (and any linear conjugate of such a map) are dense in holomorphic automorphism group of $\mathbb{C}^n$, endowed with compact-open topology.

The main importance of their work was not the mentioned result but the proof itself which implies, as observed by Forstnerič and Rosay in [FR93] for $X = \mathbb{C}^n$, the remarkable Andersén–Lempert theorem, see below. The natural generalization from $\mathbb{C}^n$ to arbitrary manifolds $X$ was made by Varolin [Var01] who introduced the important notion of the density property.

3.1.1. Definition and main features. Recall that a holomorphic vector field $\Theta$ on a complex manifold $X$ is called complete or completely integrable if the ODE

$$\frac{d}{dt} \varphi(x, t) = \Theta(\varphi(x, t))$$

$$\varphi(x, 0) = x$$

has a solution $\varphi(x, t)$ defined for all complex times $t \in \mathbb{C}$ and all starting points $x \in X$. It gives a complex one-parameter subgroup in the holomorphic automorphism group
\text{Aut}_{hol}(X)$ (by definition this is a holomorphic group homomorphism $\mathbb{C} \to \text{Aut}_{hol}(X)$) whose elements $\phi_t = \varphi(\cdot, t) : X \to X$ we call flow maps or time-$t$-maps of the field $\Theta$.

We will denote the set of completely integrable holomorphic vector fields by $\text{CVF}_{hol}(X)$ and the smallest Lie algebra containing a subset $A$ of the Lie algebra of all vector fields on $X$ will be denoted by $\text{Lie}(A)$. We call it the Lie algebra generated by $A$.

A remark about completeness of vector fields is in order. The just defined notion is usually called $\mathbb{C}$-complete. When the above ODE has a solution for all real times $t \in \mathbb{R}$ a field is called $\mathbb{R}$-complete and when it has a solution for positive real times, the field is called $\mathbb{R}^+$-complete. On Kobayashi hyperbolic manifolds, for example the unit ball in $\mathbb{C}^n$, no $\mathbb{R}$-complete field can be $\mathbb{C}$-complete, since entire holomorphic maps to such manifolds are constant. Also on the unit ball, there exist $\mathbb{R}^+$-complete holomorphic fields which are not $\mathbb{R}$-complete, e.g., the Euler vector field contracting the ball to zero.

In contrast to this situation one could define the density property (next definition) using $\mathbb{R}$-complete holomorphic vector fields and derive the fact that on Stein manifolds with the density property any $\mathbb{R}$-complete holomorphic vector field is automatically $\mathbb{C}$-complete. The main ingredient in the proof of this fact is that on Stein manifolds with density property bounded plurisubharmonic functions are constant (see property (1) of Stein manifolds with DP at the end of this subsection). Thus by the theorem in [AFR00] even all $\mathbb{R}^+$-complete holomorphic fields on such manifolds are automatically $\mathbb{C}$-complete. This is the reason why we use the simplified notion "complete" or "completely integrable" holomorphic vector field. Following the reasoning along these lines one can prove that Theorem 2.7 is holds true when $\mathbb{C}^n$ is replaced by any Stein manifold with density property.

**Definition 3.1.** A complex manifold $X$ has the density property (for short DP) if in the compact-open topology the Lie algebra $\text{Lie}(\text{CVF}_{hol}(X))$ generated by completely integrable holomorphic vector fields on $X$ is dense in the Lie algebra $\text{VF}_{hol}(X)$ of all holomorphic vector fields on $X$.

The density property is a precise way of saying that the automorphism group of a manifold is big, in particular for a Stein manifold this is underlined by the main result of the theory (see [FR93], [Var01], a detailed proof can be found in the Appendix of [Rit13] or in [For17]).

**Theorem 3.2 (Andersén-Lempert Theorem).** Let $X$ be a Stein manifold with the density property and let $\Omega$ be an open subset of $X$. Suppose that $\Phi : [0, 1] \times \Omega \to X$ is a $C^1$-smooth map such that

1. $\Phi_t : \Omega \to X$ is holomorphic and injective for every $t \in [0, 1]$,  
2. $\Phi_0 : \Omega \to X$ is the natural embedding of $\Omega$ into $X$, and  
3. $\Phi_t(\Omega)$ is a Runge subset\(^1\) of $X$ for every $t \in [0, 1]$.

Then for each $\epsilon > 0$ and every compact subset $K \subset \Omega$ there is a continuous family,\(^2\) $\alpha : [0, 1] \to \text{Aut}_{hol}(X)$ of holomorphic automorphisms of $X$ such that $\alpha_t \in \text{CVF}_{hol}(X)$ satisfies $\alpha_t(\Omega) = \Phi_t(\Omega)$ and $|\alpha_t - \Phi_t|_K < \epsilon$ for every $t \in [0, 1]$.

Moreover, given a collection $A$ of completely integrable vector fields such that $\text{Lie}(A)$ is dense in $\text{VF}_{hol}(X)$, the automorphisms $\alpha_t$ can be chosen to be finite compositions of flow maps of vector fields from the collection $A$.

Philosophically one can think of the density property as a tool for realizing local movements by global maps (automorphisms). In some sense it is a substitute for cutoff functions which in the differentiable category are used for globalizing local movements. In the holomorphic category we of course lose control on automorphisms outside the compact set $K$. This makes constructions more complicated. Nevertheless constructing

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\(^1\)Recall that an open subset $U$ of $X$ is Runge if any holomorphic function on $U$ can be approximated by global holomorphic functions on $X$ in the compact-open topology. Actually, for $X$ Stein by Cartan’s Theorem A this definition implies more: for any coherent sheaf on $X$ its section over $U$ can be approximated by global sections.

\(^2\)A remark about completeness of vector fields is in order. The just defined notion is usually called $\mathbb{C}$-complete. When the above ODE has a solution for all real times $t \in \mathbb{R}$ a field is called $\mathbb{R}$-complete and when it has a solution for positive real times, the field is called $\mathbb{R}^+$-complete. On Kobayashi hyperbolic manifolds, for example the unit ball in $\mathbb{C}^n$, no $\mathbb{R}$-complete field can be $\mathbb{C}$-complete, since entire holomorphic maps to such manifolds are constant. Also on the unit ball, there exist $\mathbb{R}^+$-complete holomorphic fields which are not $\mathbb{R}$-complete, e.g., the Euler vector field contracting the ball to zero.
sequences of automorphisms by iterated use of the Andersén-Lempert theorem has led to remarkable applications.

Let us further remark that the implications of the density property for manifolds which are not Stein have not been explored very much yet. If the manifold is compact all (holomorphic) vector fields are completely integrable, the density property trivially holds and thus cannot give any further information on the manifold.

Remark 3.3. Andersén and Lempert proved that every algebraic vector field on \( \mathbb{C}^n \) is a finite sum of algebraic shear fields (fields of form \( p(z_1, \ldots, z_{n-1}) \frac{\partial}{\partial z_n} \) for a polynomial \( p \in \mathbb{C}[\mathbb{C}^{n-1}] \) and their linear conjugates, i.e., fields whose one-parameter subgroups consist of shears, see equation (1.1)) and overshear fields (fields of form \( p(z_1, \ldots, z_{n-1})z_n \frac{\partial}{\partial z_n} \) for a polynomial \( p \in \mathbb{C}[\mathbb{C}^{n-1}] \) and their linear conjugates, i.e., fields whose one-parameter subgroups consist of overshears, see equation (1.2)). Here an algebraic vector field is an algebraic section of the tangent bundle, for example on \( \mathbb{C}^n \) it can be written as \( \sum_{i=1}^n p_i(z_1, \ldots, z_n) \frac{\partial}{\partial z_i} \) with polynomials \( p_i \in \mathbb{C}[\mathbb{C}^n] \).

Since algebraic vector fields are dense in c.o.-topology in the holomorphic vector fields, their result shows that \( \mathbb{C}^n \) has the density property (they do not even need Lie brackets). Together with the simple fact that any holomorphic automorphism of \( \mathbb{C}^n \) can be joined to the identity by a smooth path, this shows how the Andersén-Lempert theorem implies that the group \( \text{Osh}(\mathbb{C}^n) \) is dense in the holomorphic automorphism group of \( \mathbb{C}^n \), (Theorem 2.5 above).

Definition 3.4. An affine algebraic manifold \( X \) has the algebraic density property (for short ADP) if the Lie algebra \( \text{Lie}(\text{CVF}_{\text{alg}}(X)) \) generated by completely integrable algebraic vector fields on it coincides with the Lie algebra \( \text{VF}_{\text{alg}}(X) \) of all algebraic vector fields on it.

The algebraic density property for affine algebraic manifolds can be viewed as a tool to prove the density property, whereas the ways of proving it are purely algebraic work. Since on an affine algebraic manifold the algebraic vector fields are dense in c.o.-topology in the holomorphic vector fields, the algebraic density property implies the density property: \( \text{ADP} \implies \text{DP} \).

If an algebraic vector field is completely integrable, its flow gives a one parameter subgroup in the holomorphic automorphism group not necessarily in the algebraic automorphism group. For example, a polynomial overshear field of the form \( p(z_1, \ldots, z_{n-1})z_n \frac{\partial}{\partial z_n} \) has the flow map \( \gamma(t, z) = (z_1, \ldots, z_{n-1}, \exp(tp(z_1, \ldots, z_{n-1}))z_n) \). This is the reason that the study of the algebraic density property is in the intersection of affine algebraic geometry and complex analysis. It is an algebraic notion, proven using algebraic methods, but has implications for the holomorphic automorphism group.

3.1.2. Criteria and Examples. The first example of a manifold with the density property was \( \mathbb{C}^n \) as explained above. Then Varolin himself found many examples \([\text{Var00}]\) and together with Toth he proved that semi-simple complex Lie groups and certain homogeneous spaces of them have DP. Their proofs rely on results from representation theory and are therefore not applicable in a more general context. Later Kaliman and the author found very effective criteria to prove DP, see e.g. \([\text{KK08a}], \text{[KK17]}\). We will explain one of them now. The idea has two steps.

STEP1: This crucial step is a way of finding a submodule in \( \text{VF}_{\text{hol}}(X) \) contained in the closure of \( \text{Lie}(\text{CVF}_{\text{hol}}(X)) \). That is highly non-trivial, the main idea is the following:

Definition 3.5. A compatible pair is a pair \( (\nu, \mu) \) of complete holomorphic vector fields such that the closure of the linear span of the product of the kernels \( \ker \nu \ker \mu \) contains a non-trivial ideal \( I \subset \mathcal{O}(X) \) and there is a function \( h \in \mathcal{O}(X) \) with \( \nu(h) \in \ker \nu \setminus 0 \) and \( h \in \ker \mu \). The biggest ideal \( I \) with this property will be called the ideal of the pair \( (\nu, \mu) \).
Lemma 3.6. Let $(\nu, \mu)$ be a compatible pair, and let $I$ be its ideal and $h$ its function. Then the submodule of $\text{VF}_{\text{hol}}(X)$ given by $I\nu(h)\mu$ is contained in the closure of $\text{Lie}(\text{CVF}_{\text{hol}}(X))$.

Proof. Let $f \in \ker \nu$ and $g \in \ker \mu$, then $f\nu, fh\nu, g\mu, gh\mu \in \text{CVF}_{\text{hol}}(X)$. A standard calculation shows

$$[f\nu, gh\mu] - [f\nu, g\mu] = f\nu(h)\mu \in \text{Lie}(\text{CVF}_{\text{hol}}(X)).$$

Thus an arbitrary element $\sum_{i=1}^{N}(f_i g_i)\nu(h)\mu \in \text{span}(\ker \nu \ker \mu)\nu(h)\mu$ with $f_i \in \ker \nu$ and $g_i \in \ker \mu$ is contained in $\text{Lie}(\text{CVF}_{\text{hol}}(X))$ which concludes the proof.

STEP 2: Once some submodules are found, simple transitivity creates more of them as follows:

Definition 3.7. Let $p \in X$. A set $U \subset T_pX$ is called generating set for $T_pX$ if the orbit of $U$ under the induced action of the stabilizer $\text{Aut}_{\text{hol}}(X)_p$ contains a basis of $T_pX$.

Proposition 3.8. Let $X$ be a Stein manifold such that $\text{Aut}_{\text{hol}}(X)$ acts transitively on $X$. Assume that there are holomorphic vector fields $v_1, ..., v_n \in \text{VF}_{\text{hol}}(X)$ which generate a submodule $\mathcal{O}(X)\mathcal{V}_I$ that is contained in the closure of $\text{Lie}(\text{CVF}_{\text{hol}}(X))$ and assume that there is a point $p \in X$ such that the tangent vectors $v_i(p) \in T_pX$ are a generating set for the tangent space $T_pX$. Then $X$ has DP.

Proof. The basic observation is that the change of variables does not effect the complete integrability of a vector field and commutes with the operations of forming the Lie bracket and the sum of vector fields. Thus for any holomorphic automorphism $\alpha \in \text{Aut}_{\text{hol}}(X)$ we have $\alpha^*(\text{CVF}_{\text{hol}}(X)) = \text{CVF}_{\text{hol}}(X)$ and $\alpha^*(\text{Lie}(\text{CVF}_{\text{hol}}(X))) = \text{Lie}(\text{CVF}_{\text{hol}}(X))$ (and the same for the closure). Also if $L$ is a submodule of $\text{VF}_{\text{hol}}(X)$ (as $\mathcal{O}(X)$ module), then $\alpha^*(L)$ is again a submodule.

We may assume that the vectors $v_i(p)$ contain a basis of $T_pX$. Indeed, the vectors $v_i(p)$ are a generating set of $T_pX$. Thus after adding some pullbacks of some vector fields $v_i$ by automorphisms in the stabilizer $\text{Aut}_{\text{hol}}(X)_p$ we get the desired basis of $T_pX$. Let $A \subset X$ be the analytic subset of points where the vectors $v_i(a)$ do not span the whole tangent space $T_aX$. Let $K \subset X$ be a compact set. After replacing $K$ by its $\mathcal{O}(X)$-convex hull we may assume that $K = \mathcal{O}(X)$-convex. Let $Y$ be a neighborhood of $K$ which is Stein and Runge, and moreover such that the closure of $Y$ is compact. After adding finitely many pullbacks of the $v_i$ by automorphisms to the collection of vector fields $v_1, ..., v_n$, we get that $Y \cap A = \emptyset$. Indeed, since the closure of $Y$ is compact, $Y \cap A$ is a finite union of irreducible analytic subsets. Let $A_0 \subset A$ be an irreducible component of maximal dimension. Pick any $a \in A_0$ and $\varphi \in \text{Aut}_{\text{hol}}(X)$ such that $\varphi(a) \in Y \setminus A$. Since the vectors $v_i(\varphi(a))$ span the tangent space $T_{\varphi(a)}X$ the vectors $(\varphi_1)(a)$ span the tangent space $T_aX$. Thus after adding some of the pull backs to $v_1, ..., v_n$, the component $A_0 \cap Y$ is replaced by finitely many components of lower dimension. Repeating the same procedure, inductively we get after finitely steps a list of complete vector fields $v_1, ..., v_N$ such that $A \cap Y = \emptyset$. By the Nakayama lemma they generate the stalk of the tangent sheaf at each point in $Y$. By standard results in the theory of coherent $\mathcal{O}$-module sheaves on Stein manifolds, the sum of the corresponding modules $\mathcal{O}(X)\mathcal{V}_I$ approximates every holomorphic vector field on the compact $K$. Thus by the observation in the beginning of the proof every holomorphic vector field is in the closure of $\text{Lie}(\text{CVF}_{\text{hol}}(X))$.

Summarizing we obtain our criterion

Theorem 3.9. Let $X$ be a Stein manifold such that the holomorphic automorphisms $\text{Aut}_{\text{hol}}(X)$ act transitively on $X$. If there are compatible pairs $(\nu_i, \mu_i)$ such that there is a point $p \in X$ where the vectors $\mu_i(p)$ form a generating set of $T_pX$, then $X$ has the density property.

Proof. Let $I_i$ be the ideals and $h_i$ the functions of the pairs $(\nu_i, \mu_i)$ and pick any non-trivial $f_i \in I_i \mu_i(h_i)$ for every $i$. Since the set of points $q \in X$ where the vector fields $\mu_i(q)$
are a generating set is open and non-empty, there is some \( p \in X \) where the vector fields \( f_i(p) \mu_i(p) \) are a generating set for \( T_p X \). By Lemma 3.6 the module generated by the vector fields \( f_i \mu_i \) is contained in the closure of \( \text{Lie}(\text{CVF}_{\text{had}}(X)) \) and thus by Proposition 3.8 the manifold \( X \) has the density property. \(\square\)

Before we come to the complete list of examples of Stein manifolds known to have the density property, we would like to show the power of our criterion in some examples:

**Example 3.10.** On \( \mathbb{C}^n \), \( n \geq 2 \) the pair of vector fields \( \left( \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2} \right) \) is compatible, with function \( z_1 \) and \( I = \mathcal{O}(\mathbb{C}^n) \). Since we can permute coordinates, \( \{ \frac{\partial}{\partial z_i} \} \) is a generating set for each tangent space. Thus \( \mathbb{C}^n \) has DP.

**Example 3.11.** On \( X = \text{SL}_2(\mathbb{C}) \) denote an element of \( X \) by

\[
A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}
\]

The pair \( (\delta_1, \delta_2) \) is compatible, where

\[
\delta_1 = a_1 \frac{\partial}{\partial b_1} + a_2 \frac{\partial}{\partial b_2},
\delta_2 = b_1 \frac{\partial}{\partial a_1} + b_2 \frac{\partial}{\partial a_2}.
\]

with \( I = \mathcal{O}(X) \) and function \( h = a_1 \). These fields are corresponding to adding multiples of the first row to the second row and vice versa. To see that \( \delta_2 \) is a generating set at the identity one can for example use the fact that the adjoint representation is irreducible. Thus \( \text{SL}_2(\mathbb{C}) \) has DP. The proof for \( \text{SL}_n(\mathbb{C}) \) and \( \text{GL}_n(\mathbb{C}) \), \( n \geq 3 \) goes the same way.

**List of examples of Stein manifolds known to have the density property:**

1. A homogeneous space \( X = G/H \) where \( G \) is a linear algebraic group and \( H \) is a closed algebraic subgroup such that \( X \) is affine and whose connected component is different from \( \mathbb{C} \) and from \( (\mathbb{C}^*)^n, n \geq 1 \) has DP.

It is known that if \( H \) is reductive the space \( X = G/H \) is always affine, however there is no known group-theoretic characterization which would say when \( X \) is affine. This result has a long history, it includes all examples known from the work of Andersén-Lempert and Varolin and Varolin-Toth and Kaliman-Kutzschebauch, Donzelli-Dvorsky-Kaliman, the final result is proven by Kaliman and the author in [KK17], \( \mathbb{C} \) and \( \mathbb{C}^* \) do not have DP, however the following problem is well known and seems notoriously difficult

**Open Problem:** Does \( (\mathbb{C}^*)^n, n \geq 2 \) have DP?

It is conjectured that the answer is no, more precisely one expects that all holomorphic automorphisms of \( (\mathbb{C}^*)^n, n \geq 2 \) respect the form \( \bigwedge_{i=1}^n \frac{dz_i}{z_i} \) up to sign.

2. The manifolds \( X \) given as a submanifold in \( \mathbb{C}^{n+2} \) with coordinates \( u \in \mathbb{C}, v \in \mathbb{C}, z \in \mathbb{C}^n \) by the equation \( uv = p(z) \), where the zero fiber of the polynomial \( p \in \mathbb{C}[\mathbb{C}^n] \) is smooth (otherwise \( X \) is not smooth) have ADP and thus DP [KK08a].

3. The only known non-algebraic examples with DP are firstly the holomorphic analogues of (2), namely the manifolds \( X \) given as a submanifold in \( \mathbb{C}^{n+2} \) with coordinates \( u \in \mathbb{C}, v \in \mathbb{C}, z \in \mathbb{C}^n \) by the equation \( uv = f(z) \), where the zero fiber of the holomorphic function \( f \in \mathcal{O}(\mathbb{C}^n) \) is smooth (again otherwise \( X \) is not smooth) [KK08a]. Secondly a special case of (4), namely when the Gizatullin surface can be completed by four rational curves, the Stein manifolds given by holomorphic analogues of the concrete algebraic equations have the density property [AKP17].

4. Smooth Gizatullin surfaces which admit a \( \mathbb{C} \)-fibration with at most one singular and reduced fibre. Sometimes these surfaces are called generalized Danielewski surfaces [And17b].
Recall that Gizatullin surfaces are by definition the normal affine surfaces on which the algebraic automorphism groups acts with an open orbit whose complement is a finite set of points. By the classical result of Gizatullin they can be characterized by admitting a completion with a simple normal crossing chain of rational curves at infinity. Every Gizatullin surface admits a $\mathbb{C}$-fibration with at most one singular fibre which however is not always reduced. (5) Certain hypersurfaces in $\mathbb{C}^{n+3}$ with coordinates $z = (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1}, x \in \mathbb{C}, y \in \mathbb{C}$ given by the equation $x^2y = a(z) + xb(z)$ where $\deg_{z_0} a \leq 2, \deg_{z_0} b \leq 1$ and not both degrees are zero, including the Koras-Russell threefold from equation (2.5) [Leu16].

Here is a number of consequences the density property has, the proof of each of them is a certain application of the Anderes-Lempert theorem:

If $X$ is a Stein manifold with DP, then

(1) $X$ is covered by Fatou-Bieberbach domains, i.e., each $x \in X$ has a neighborhood $\Omega_x \subset X$ biholomorphic to $\mathbb{C}^{\dim X}$ [Var00]. In particular all Eisenman measures on $X$ vanish identically and bounded plurisubharmonic functions are constant.

(2) There is $\varphi : X \to X$, injective holomorphic not surjective (biholomorphic images of $X$ in itself) [Var00].

(3) If $X$ is Stein with DP, $\dim X \geq 3$ and $Y$ is a complex manifold such that $\text{End}(X)$ and $\text{End}(Y)$ are isomorphic as abstract semigroups, then $X$ and $Y$ are biholomorphic or anti-biholomorphic, see [And11], [AW14]. We believe that the same is true if the dimension of $X$ is 2, but the known proofs do not apply.

(4) There are finitely many complete holomorphic vector fields $\theta_1, \ldots, \theta_N \in \text{CVF}_{hol}(X)$ such that $\text{span}(\theta_1(x), \ldots, \theta_N(x)) = T_xX \quad \forall x \in X$ (see [KK11]) and thus $X$ is holomorphically flexible (see Definition 3.16).

3.2. Flexibility.

3.2.1. Definition and main features. The notion of flexibility is even more recent than the density property. It was defined in [AFK+13]. First we state the algebraic version:

**Definition 3.12.** Let $X$ be an algebraic variety defined over $\mathbb{C}$ (any algebraically closed field would do). We let $\text{SAut}(X)$ denote the subgroup of $\text{Aut}_{alg}(X)$ generated by all algebraic one-parameter unipotent subgroups of $\text{Aut}_{alg}(X)$, i.e., algebraic subgroups isomorphic to the additive group $\mathbb{G}_a$ (usually denoted $\mathbb{C}^+$ in complex analysis). The group $\text{SAut}(X)$ is called the special automorphism group of $X$; this is a normal subgroup of $\text{Aut}_{alg}(X)$.

**Definition 3.13.** We say that a point $x \in X_{\text{reg}}$ is algebraically flexible if the tangent space $T_xX$ is spanned by the tangent vectors to the orbits $H \cdot x$ of one-parameter unipotent subgroups $H \subseteq \text{Aut}_{alg}(X)$. A variety $X$ is called algebraically flexible if every point $x \in X_{\text{reg}}$ is.

Clearly, $X$ is algebraically flexible if one point of $X_{\text{reg}}$ is and the group $\text{Aut}_{alg}(X)$ acts transitively on $X_{\text{reg}}$.

The main feature of algebraic flexibility is the following result from [AFK+13] (whose proof mainly relies on the Rosenlicht theorem).

**Theorem 3.14.** For an irreducible affine variety $X$ of dimension $\geq 2$, the following conditions are equivalent.

1. The group $\text{SAut}(X)$ acts transitively on $X_{\text{reg}}$.
2. The group $\text{SAut}(X)$ acts infinitely transitively on $X_{\text{reg}}$.
3. $X$ is an algebraically flexible variety.
The paper [AFK+13] also contains versions of simultaneous transitivity (where the space \(X_{\text{reg}}\) is stratified by orbits of \(\text{SAut}(X)\)) and versions with jet-interpolation. Moreover, it was recently remarked that the theorem holds for quasi-affine varieties, see Theorem 1.11 in [FKZ16].

Examples of algebraically flexible varieties are homogeneous spaces of semisimple Lie groups (or extensions of semisimple Lie groups by unipotent radicals), toric varieties without non-constant invertible regular functions, cones over flag varieties and cones over Del Pezzo surfaces of degree at least 4, normal hypersurfaces of the form \(ww = p(x)\) in \(\mathbb{C}^{n+2}\). Moreover, algebraic subsets of codimension at least 2 can be removed as recently shown by Flenner, Kaliman and Zaidenberg in [FKZ16]:

**Theorem 3.15.** Let \(X\) be a smooth quasi-affine variety of dimension \(\geq 2\) and \(Y \subset X\) a closed subvariety of codimension \(\geq 2\). If \(X\) is flexible then so is \(X \setminus Y\).

The holomorphic version of this notion for a reduced complex space \(X\) is much less explored, it is obviously implied by the algebraic version in case \(X\) is an algebraic variety.

**Definition 3.16.** We say that a point \(x \in X_{\text{reg}}\) is holomorphically flexible if the tangent space \(T_xX\) is spanned by the tangent vectors of completely integrable holomorphic vector fields, i.e. holomorphic one-parameter subgroups in \(\text{Aut}_{\text{hol}}(X)\). A reduced complex space \(X\) is called holomorphically flexible if every point \(x \in X_{\text{reg}}\) is.

Clearly, \(X\) is holomorphically flexible if one point of \(X_{\text{reg}}\) is and the group \(\text{Aut}_{\text{hol}}(X)\) acts transitively on \(X_{\text{reg}}\).

In the holomorphic category it is still open whether an analogue of Theorem 3.14 holds.

**Open Problem:** Are the three equivalences from Theorem 3.14 true for a reduced irreducible Stein space \(X\)? More precisely, if a reduced irreducible Stein space \(X\) is holomorphically flexible, does the holomorphic automorphism group \(\text{Aut}_{\text{hol}}(X)\) act infinitely transitively on \(X_{\text{reg}}\)?

It is clear that holomorphic flexibility of \(X\) implies that \(\text{Aut}_{\text{hol}}(X)\) acts transitively on \(X_{\text{reg}}\), i.e., the implication (3) \(\Rightarrow\) (1) is true. Indeed, let \(\theta_i, i = 1, 2, \ldots, n\) be completely integrable holomorphic vector fields which span the tangent space \(T_xX\) at some point \(x \in X_{\text{reg}}\), where \(n = \dim X\). If \(\psi^i : C \times X \to X, (t, x) \mapsto \psi^i_t(x)\) denote the corresponding one-parameter subgroups, then the map \(C^n \to X, (t_1, t_2, \ldots, t_n) \mapsto \psi^1_{t_1} \circ \psi^{n-1}_{t_{n-1}} \circ \cdots \circ \psi^1_{t_1}(x)\) is of full rank at \(t = 0\) and thus by the Inverse Function Theorem a local biholomorphism from a neighborhood of 0 to a neighborhood of \(x\). Thus the \(\text{Aut}_{\text{hol}}(X)\)-orbit through any point of \(X_{\text{reg}}\) is open. If all orbits are open, each orbit is also closed, being the complement of all other orbits. Since \(X_{\text{reg}}\) is connected, this implies that it consists of one orbit.

The inverse implication (1) \(\Rightarrow\) (3) is also true. For the proof we appeal to the Hermann–Nagano Theorem which states that if \(g\) is a Lie algebra of holomorphic vector fields on a manifold \(X\), then the orbit \(R_g(x)\) (which is the union of all points \(z\) over any collection of finitely many fields \(v_1, \ldots, v_N \in g\) and over all times \((t_1, \ldots, t_N)\) for which the expression \(z = \psi^1 t_1 \circ \psi^{N-1} t_{N-1} \circ \cdots \circ \psi^1 t_1(x)\) is defined) is a locally closed submanifold and its tangent space at any point \(y \in R_g(x)\) is \(T_y R_g(x) = \text{span}_{v \in g} v(y)\). We consider the Lie algebra \(g = \text{Lie}(\text{CVF}_{\text{hol}}(X))\) generated by completely integrable holomorphic vector fields. Since by the assumption the orbit is \(X_{\text{reg}}\), we conclude that Lie combinations of completely integrable holomorphic vector fields span the tangent space at each point in \(X_{\text{reg}}\). Now suppose at some point \(x_0\) the completely integrable fields do not generate \(T_{x_0} X_{\text{reg}}\), i.e., there is a proper linear subspace \(W\) of \(T_{x_0} X_{\text{reg}}\) such that \(v(x_0) \in W\) for all completely integrable holomorphic fields \(v\). Any Lie combination of completely integrable holomorphic fields is a limit (in the compact open topology) of sums of completely integrable holomorphic fields due to the formula \(\{v, w\} = \lim_{t \to 0} \frac{\phi_t^v(w) - w}{t}\), where \(\phi_t\) is the flow of \(v\), for the Lie bracket \(\phi_t^v(w)\) is a completely integrable field pulled back by
an automorphism, thus completely integrable!). Therefore all Lie combinations of completely integrable fields evaluated at \( x_0 \) are contained in \( W \subset T_{x_0}X_{\text{reg}} \), a contradiction.

In order to prove the remaining implication (3) \( \Rightarrow \) (2) in the same way as in the algebraic case, one would like to find suitable functions \( f \in \ker \theta \) for a completely integrable holomorphic vector field \( \theta \), vanishing at one point and not vanishing at some other point of \( X \). In general these functions may not exist, an orbit of \( \theta \) can be holomorphically Zariski dense in \( X \).

At this point it is worth mentioning that for a Stein manifold the density property \( \text{DP} \) implies all three conditions from Theorem 3.14. For flexibility this has been mentioned above, infinite transitivity (with jet-interpolation) is proved by Varolin in [Var00], see also Lemma 4.3 below.

The main importance of holomorphic flexibility is the fact that holomorphically flexible manifolds are sources for the Oka-Grauert-Gromov-h(omotopy)-principle in Complex Analysis.

Let us explain this more precisely. A holomorphically flexible complex manifold \( X \) is an Oka–Forstnerič manifold which means it is an appropriate (nonlinear) target for generalizing classical Oka–Weil interpolation and Runge approximation for holomorphic functions (linear target \( \mathbb{C} \)) or sections of vector bundles (linear target as well). More precisely, the following is true for an Oka–Forstnerič manifold \( X \) (see [For17] Corollary 5.4.5):

**Oka principle**

*For any Stein space \( W \), complex subspace \( W' \), compact \( \mathcal{O}(W) \)-convex subset \( K = \hat{K} \subset W \) and any \( \varphi : W \to X \) continuous, such that the restriction to \( W' \cup K \) is holomorphic, there is a homotopy of continuous maps

\[
h : [0, 1] \times W \to X
\]

from the continuous map \( h_0 = \varphi \) to a holomorphic map \( h_1 \),

with interpolation: \( h_t = \varphi \) on \( W' \) and

with approximation: \( |h_t - \varphi|_K \) arbitrary small \( \forall t \in [0, 1] \).

Moreover parametric versions are true: The inclusion of the space of holomorphic maps \( \text{Hol}(W, X) \) into the space of continuous maps \( \text{Cont}(W, X) \) is a weak homotopy equivalence.

We refer the reader to the monograph of Forstnerič for more details. Let us just remark that Gromov introduced the notion of an elliptic manifold, which by definition is a complex manifold with a dominating spray, and proved that the above conclusions are true for an elliptic manifold.

A spray for a complex manifold \( X \) is a holomorphic vector bundle \( \pi : E \to X \) together with a holomorphic (spray) map \( s : E \to X \), such that \( s \) is the identity on the zero section \( X \hookrightarrow E \). The spray is dominating if for each \( x \in X \) the induced differential map sends the fibre \( E_x = \pi^{-1}(x) \), viewed as a linear subspace of \( T_xE \) surjectively onto \( T_xX \). Gromov’s standard example for a spray is used to see that a holomorphic flexible manifold is elliptic.

**Example 3.17.** Let \( X \) be holomorphically flexible. We need an easy fact proved by the author in [Kut14] (see also the appendix of [AFK+13]), namely that there are finitely many completely integrable holomorphic vector fields \( \theta_1, \theta_2, \ldots, \theta_N \) such that at each point \( x \in X \) they span the tangent space (by definition for each point there are finitely many spanning at this point only). Let \( \psi^i : \mathbb{C} \times X \to X, (t, x) \mapsto \psi^i_t(x) \) denote the corresponding flow maps.

Then the map \( s : \mathbb{C}^N \times X \to X \) defined by \((t_1, t_2, \ldots, t_N, x) \mapsto \psi^1_{t_N} \circ \psi^{N-1}_{t_{N-1}} \circ \cdots \circ \psi^1_1(x)\) is of full rank at \( t = 0 \) for any \( x \), and thus a dominating spray map from the trivial bundle \( X \times \mathbb{C}^N \to X \).
4. Applications to natural geometric questions

All applications described below are applications of the density property DP and some of them use flexibility at the same time. Sometimes variants of the density property are used. There are versions for volume preserving automorphisms, so called Volume Density Property, relative versions, e.g., for automorphisms fixing subvarieties or fibered versions (considering automorphisms leaving invariant a fibration). We leave it to the interested reader to look up in the given references which version is used. Only the very last application does not use DP, it uses flexibility only via a stratified version of the Oka principle.

(1) A first application that we would like to mention is to the notoriously difficult question whether every open Riemann surface can be properly holomorphically embedded into $\mathbb{C}^2$. This is the only dimension for which the conjecture of Forster [For70], saying that every Stein manifold of dimension $n$ can be properly holomorphically embedded into $\mathbb{C}^N$ for $N = \lfloor \frac{n}{2} \rfloor + 1$, is still unsolved. The conjectured dimension is sharp by examples of Forster [For70] and has been proven by Eliashberg, Gromov [EG92] and Schürmann [Sch97] for all dimensions $n \geq 2$. Their methods of proof fail in dimension $n = 1$. But Forneass Wold invented a clever combination of a use of shears (nice projection property) and Theorem 3.2 which led to many new embedding theorems for open Riemann surfaces. As an example we like to mention the following two recent results of Forstnerič and Forneass Wold [FW13], [FW09] the first of them being the most general one for open subsets of the complex line:

**Theorem 4.1.** Every domain in the Riemann sphere with at least one and at most countably many boundary components, none of which are points, admits a proper holomorphic embedding into $\mathbb{C}^2$.

**Theorem 4.2.** If $\Sigma$ is a (possibly reducible) compact complex curve in $\mathbb{C}^2$ with boundary $\partial \Sigma$ of class $C^r$ for some $r > 1$, then the inclusion map $i: \Sigma = \bar{\Sigma} \setminus \partial \Sigma \to \mathbb{C}^2$ can be approximated, uniformly on compacts in $\Sigma$, by proper holomorphic embeddings $\Sigma \to \mathbb{C}^2$.

Many versions of embeddings with interpolation are also known and proven using the same methods invented by Forneass Wold in [Wol06]. In particular the Gromov–Eliashberg–Schürmann Theorem mentioned above is true with interpolation on discrete subsets [FIKP07].

(2) Another application is to construct non-straightenable holomorphic embeddings of $\mathbb{C}^k$ into $\mathbb{C}^n$ for all pairs of dimensions $0 < k < n$, a fact which is contrary to the situation in affine algebraic geometry. It is contrary to the famous Abhyankar-Moh-Suzuki theorem for $k = 1, n = 2$ and also to work of Kaliman [Kal15] for $2k + 1 < n$, whereas straightenability for the other dimension pairs is still unknown in algebraic geometry. Here non-straightenable for an embedding $\mathbb{C}^k$ into $\mathbb{C}^n$ means to be not equivalent to the standard embedding.

To give the reader an idea about the flavor of the subject, let us explain briefly how Theorem 3.2 is used to construct a non-straightenable embedding of $\mathbb{C}$ into $\mathbb{C}^n$.

The idea is to use the existence of a non-tame discrete subset $E = \{e_1, e_2, \ldots \}$ in $\mathbb{C}^n$ and to construct a proper holomorphic embedding $\varphi: \mathbb{C} \to \mathbb{C}^n$ whose image contains $E$, $\varphi(\mathbb{C}) \supset E$. The notion of a tame subset in $\mathbb{C}^n$ goes back to Rosay and Rudin [RR88] and is by definition a subset in $A \subset \mathbb{C}^n$ which can be mapped by a holomorphic automorphism $\alpha \in Aut_{ad}(\mathbb{C}^n)$ onto the subset $\{(i, 0, \ldots, 0) \in \mathbb{C}^n, i \in \mathbb{N}\}$. The only information we need is the fact that any countable discrete subset of the first coordinate line $\{(z, 0, \ldots, 0) \in \mathbb{C}^n : z \in \mathbb{C}\}$ is tame. This shows that our embedding containing a non-tame subset $E$ cannot be straightenable. Indeed, were it straightenable, the set $E$ would be mapped by the straightening automorphism into the first coordinate line and therefore be tame, a contradiction. The existence of non-tame subsets of $\mathbb{C}^n$ has been proved by Rosay and Rudin [RR88] and for the reader familiar with Eisenman measures we just mention that one can construct a discrete subset in $\mathbb{C}^n$ whose complement is
Eisenman $n$-hyperbolic (also called volume hyperbolic). Since the complement of the first coordinate line has constantly vanishing Eisenman volume such an $E$ cannot be contained in a coordinate line. The step where we embed $\mathbb{C}$ through that set is where we make use of DP applying Theorem 3.2. Let us describe this a little more detailed in order to let the reader feel the flavor of the subject. The step where we embed the line through a discrete subset is presented in the more general situation where $\mathbb{C}^n$ is replaced by any Stein manifold with DP.

We begin with an easy lemma which already demonstrates the power of the density property. Expressed in words it means that $N$ different points can be moved around independently of each other by automorphisms inside small neighborhoods, and the nearer the points are to their targets the nearer the automorphism is to the identity. This lemma easily implies that the holomorphic automorphism group of $X$ acts infinitely transitive on $X$, a result of Varolin [Var00].

**Lemma 4.3.** For any $N$-tuple of pairwise distinct points $x_1, x_2, \ldots, x_N$ in a Stein manifold $X$ of dimension $\dim X = n$ with DP, there is an open neighborhood $P$ of $0 \in \mathbb{C}^n\mathbb{N}$ together with an injective holomorphic map $\psi : P \to \text{Aut}_{hol}(X)$ satisfying $\psi(0) = \text{id}$ such that the map $P \to X^N$ defined by $p \mapsto (\psi(p)(x_1), \psi(p)(x_2), \ldots, \psi(p)(x_N))$ is a biholomorphism from $P$ to an open neighborhood of the point $(x_1, x_2, \ldots, x_N)$ in the $N$-fold product $X^N$.

**Proof.** Choose coordinate balls $B_i$ which are Runge around each point $x_i$ so small that their union $\bigcup_{i=1}^N B_i$ is Runge too. Consider the collection of $nN$ vector fields $\theta_i^j$ defined on $\bigcup_{i=1}^N B_i$ by $\theta_i^j = \frac{\partial}{\partial x^j}$ on $B_i$ and identically zero on $B_k$ for $k \neq i$. They naturally induce vector fields on $B_1 \times B_2 \times \cdots \times B_N$ which span the tangent space to $X^N$ at each point there. Using DP they can be approximated by Lie combinations, in fact sums (see the proof of (1) $\Rightarrow$ (3) after Definition 3.10) of complete vector fields. Linear algebra shows that there are $nN$ complete holomorphic vector fields $\bar{\theta}_i$ whose naturally induced fields on $X^N$ span the tangent space at the point $(x_1, x_2, \ldots, x_N)$. The map from $\mathbb{C}^n\mathbb{N}$ to $X^N$ defined by applying the flows of the fields in any (but fixed) order to the point $(x_1, x_2, \ldots, x_N)$ (as in Example 3.17) is of full rank. By the implicit function theorem it is a local biholomorphism, which finishes the proof. \(\square\)

**Proposition 4.4.** Given an analytic subset $S$ in a Stein manifold $X$ with DP and a countable discrete subset $E \subset X$. Then there is a proper holomorphic embedding $\varphi : S \hookrightarrow X$ with $E \subset \varphi(S)$.

**Proof.** The idea is to construct inductively a sequence of holomorphic automorphisms $\alpha_i \in \text{Aut}_{hol}(X)$ such that the limit $\lim_{n \to \infty} \alpha_n \circ \cdots \circ \alpha_2 \circ \alpha_1$ converges uniformly on compacts on an open subset $\Omega$ of $X$ containing $S$ to a biholomorphic map $\psi : \Omega \to X$ (thus $\Omega$ is a so called Fatou-Bieberbach domain in $X$). Moreover $E \subset \psi(S)$. For describing the inductive procedure we fix a strictly plurisubharmonic (spsh) exhaustion function $\rho : X \to [0, \infty)$ of the Stein manifold $X$. The existence of such a function is guaranteed by embedding $X$ as a closed submanifold into some affine space $\mathbb{C}^N$ of high enough dimension and restricting the function $|z|^2 = |z_1|^2 + |z_2|^2 + \ldots + |z_N|^2$ to $X$. By moving the origin by an arbitrarily small amount in $\mathbb{C}^N$ if necessary and changing the enumeration of the points in $E$, we can assume that $\rho(e_1) < \rho(e_2) < \ldots < \rho(e_k) < \rho(e_{k+1}) < \ldots$. Also we choose numbers $r_k > 0$ with $\rho(e_k) < r_k < \rho(e_{k+1})$. Since $\rho$ is spsh its sub-level sets $X_r := \rho^{-1}([0, r))$ are Runge subsets of $X$. We can restrict $\rho$ to the closed subvariety $S$ in order to obtain a spsh exhaustion function of $S$. Again the sub level sets $S_R := \rho^{-1}([0, R]) \cap S$ are holomorphically convex Runge subsets of $S$. Moreover it can be shown that for each $R > r$ there is a neighborhood basis $U_i$ of $S_R$ with the property that $X_r \cup U_i$ is a Runge subset in $x$. Finally choose $e_k < r_{k+1} - r_k$ and with $\sum_k e_k < \infty$.

We can assume that $e_1 \notin S$ by the following application of Theorem 3.2. Suppose $e_1 \notin S$, choose a point $s \in S$ and connect the points $s$ and $e_1$ by a continuous path
\( \gamma(t) \). It is an easy exercise using local charts and cutoff functions that the map \( \gamma_t \) can be extended to a neighborhood \( W \) of the point \( s \) to a map \( \Gamma : W \times [0,1] \to X \) with \( \Gamma(s,t) = \gamma(t) \) so that the maps \( \Gamma_t \) are biholomorphisms of \( W \) onto its image \( \Gamma_t(W) \) which is Runge in \( X \) and with \( \Gamma_0 = id \). For \( X = \mathbb{C}^n \) a simple translation \( \Gamma_t(z) = z + \gamma(t) \) will do the job. By Theorem 3.2 we can approximate the final map \( \Gamma_1 \) by an automorphism which will move the point \( e_1 \) arbitrarily close to the point \( s \in S \). Using Lemma 4.3 we can move \( e_1 \) by a further automorphism exactly to \( s \in S \).

Now we describe the inductive step: The automorphisms \( \alpha_k \) will satisfy

\[
\left\| \alpha_k - id \right\|_{X_{r_k} \cup S_{r_k+1}} < \epsilon_k \quad \alpha_k(e_i) = e_i \quad i = 1, 2, \ldots, k
\]

We assume \( \alpha_1, \ldots, \alpha_{k-1} \) have been constructed. To construct \( \alpha_k \) we proceed as follows: If \( \epsilon_{k+1} \in \alpha_{k-1} \circ \cdots \circ \alpha_1(S) \) we set \( \alpha_k = id \), if not we connect \( e_{k+1} \) by a continuous path \( \gamma(t) \) never intersecting \( X_{r_k} \cup S_{r_k+1} \) to a point \( s \) in \( S \setminus S_{r_k+1} \). Now the local data for application of Theorem 3.2 is identity on \( X_{r_k} \cup S_{r_k+1} \) and an extension of the path \( \gamma(t) \) to biholomorphic maps \( \Gamma_t \) of a small neighborhood \( W \) of \( s \) as above. If \( W \) is small enough the sets \( X_{r_k} \cup S_{r_k+1} \cup \Gamma_t(W) \) are Runge and an application of Theorem 3.2 gives an automorphism \( \alpha \) which satisfies the first of the three conditions above and the second and third condition are satisfied approximately. By composing \( \alpha \) with an automorphism from Lemma 4.3 (not destroying the first condition, if the points had moved/stayed nearby enough) we find our desired \( \alpha_k \). The claim that the limit \( \lim_{n \to \infty} \alpha_n \circ \cdots \circ \alpha_2 \circ \alpha_1 \) converges uniformly on compacts on an open subset \( \Omega \) of \( X \) containing \( S \) to a biholomorphic (Fatou-Bieberbach) map \( \psi : \Omega \to X \) follows standardly from \( \left\| \alpha_k - id \right\|_{X_{r_k}} < \epsilon_k \) and the restrictions on the \( \epsilon_k \) (see e.g. [ForL17] Proposition 4.4.1. and Corollary 4.4.2.). Moreover \( \left\| \alpha_k - id \right\|_{S_{r_k+1}} < \epsilon_k \) implies \( S \subset \Omega \) and since \( S \) is closed in \( \Omega \) clearly \( \psi(S) \) is closed in \( X = \psi(\Omega) \) and the last two properties of \( \alpha_k \) ensure that \( E \subset \psi(S) \).

More generally one can think of the possible ways of embedding any Stein manifold \( X \) into \( \mathbb{C}^n \).

**Definition 4.5.** Two embeddings \( \Phi, \Psi : X \hookrightarrow \mathbb{C}^n \) are equivalent if there exist automorphisms \( \varphi \in \text{Aut}(\mathbb{C}^n) \) and \( \psi \in \text{Aut}(X) \) such that \( \varphi \circ \Phi = \Psi \circ \psi \).

The best and quite striking result in this direction says that there are even holomorphic families of pairwise non-equivalent holomorphic embeddings.

**Theorem 4.6.** [KLL13, BK06]. Let \( n, l \) be natural numbers with \( n \geq l + 2 \). There exist, for \( k = n - l - 1 \), a family of holomorphic embeddings of \( \mathbb{C}^l \) into \( \mathbb{C}^n \) parametrized by \( \mathbb{C}^k \), such that for different parameters \( w_1 \neq w_2 \in \mathbb{C}^k \) the embeddings \( \psi_{w_1}, \psi_{w_2} : \mathbb{C}^l \hookrightarrow \mathbb{C}^n \) are non-equivalent. Moreover, there are uncountably many non-equivalent holomorphic embeddings of \( \mathbb{C}^{n-1} \) into \( \mathbb{C}^n \).

We will see a beautiful application of Theorem 4.6 to the holomorphic linearization problem in the last section.

It is clear that in Definition 4.5 the ambient space \( \mathbb{C}^n \) can be replaced by any other manifold, most interesting by a Stein manifold with DP, since they are all targets for embedding of Stein manifolds, exactly as affine spaces are, see the next application.

**Open Problem:** Suppose \( X \) is a Stein manifold with density property and \( Y \subset X \) is a closed submanifold. Is there always another proper holomorphic embedding \( \varphi : Y \hookrightarrow X \) which is not equivalent to the inclusion \( i : Y \hookrightarrow X \)?

We should remark that an affirmative answer to this problem is stated in [Var00], but the author apparently had another (weaker) notion of equivalence in mind.

(3) As mentioned above not only affine spaces \( \mathbb{C}^n \) are the universal targets for embedding Stein manifolds. All Stein manifolds with DP are such targets as well.
Open Problem: Does every Stein manifold admit an acyclic proper holomorphic embedding into a Stein Oka–Forstnerič manifold?

For the connection to the density property remember that Stein manifolds with DP are Oka–Forstnerič manifolds.

Theorem 4.8. Let $W$ be a Stein manifold and $X$ a Stein manifold with the density property. Let $x : W \to X^N \setminus \{(z^1, \ldots, z^N) \in X^N : z^i = z^j \text{ for some } i \neq j\}$ be a holomorphic map. Then the parametrized points $x^1(w), \ldots, x^N(w)$ are simultaneously standardizable by an automorphism lying in the path-connected component of the identity $(\text{Aut}_W(X))^0$ of $\text{Aut}_W(X)$ if and only if $x$ is null-homotopic.

Here simultaneously standardizable means that given any fixed positions $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N \in X$ there are holomorphic automorphisms $\alpha$ of $X$ depending holomorphically on $w$, i.e., an element of $\text{Aut}_W(X) = \{\alpha \in \text{Aut}(W \times X); \alpha(w, z) = (w, \alpha(w)(z))\}$, with $\alpha^N(x^1(w)) = \tilde{x}_j$ for all $w \in W$ and $j = 1, \ldots, N$.

The proof of this result uses extensively DP via Theorem 3.2 and the Oka principle.

All Fatou–Bieberbach domains arising as basins of attraction or more generally as domains of convergence of sequences of automorphisms of $\mathbb{C}^n$ are always Runge domains. Thus it is natural to ask whether all Fatou–Bieberbach domains in $\mathbb{C}^n$ have to be Runge. This problem was solved by Fornæss Wold who constructed a Fatou–Bieberbach domain in $\mathbb{C} \times \mathbb{C}^*$ which is not Runge in $\mathbb{C}^2$ (but Runge in $\mathbb{C} \times \mathbb{C}^*$) using the density property of $\mathbb{C} \times \mathbb{C}^*$ [Wo08].

One of the questions coming from complex dynamical systems is the description of the boundaries of Fatou–Bieberbach domains. Say, a surprising result of Stensønes [Ste97] provides such a domain in $\mathbb{C}^2$ with a smooth boundary which has, therefore, Hausdorff dimension $d = 3$. Furthermore, it was established by methods of complex dynamical systems that such a dimension can take any value $3 \leq d < 4$. However, the question about a Fatou–Bieberbach domain in $\mathbb{C}^2$ with a boundary of Hausdorff dimension $d = 4$ remained open until Peters and Fornæss Wold [PW05] managed to construct it using the DP.

A question posed by Siu asks whether there exists always a Fatou–Bieberbach domain contained in the complement to a closed algebraic subvariety $Z$ of $\mathbb{C}^n$ such that $\dim Z \leq n - 2$. The affirmative answer was obtained by Buzzard and Hubbard [BH00] who used some concrete construction. Another proof of this fact was given by Kaliman and the author who used a version of the density property for such complements.
More precisely for any point \( x \in \mathbb{C}^n \setminus Z \) there is a Fatou–Bieberbach (i.e. holomorphic injective) map \( f : \mathbb{C}^n \to \mathbb{C}^n \setminus Z \) with \( f(0) = x \) (actually, the existence of complete holomorphic vector fields on such complements \( \mathbb{C}^m \setminus Z \) has been observed in earlier papers of Gromov \cite{Gro89} and Winkelmann \cite{Win90}).

In particular all Eisenman measures on \( \mathbb{C}^n \setminus Z \) are trivial. It is worth mentioning that closed analytic subsets of \( \mathbb{C}^n \) of codimension \( k \) may have \( k \)-Eisenman hyperbolic complements. More precisely, it was shown in \cite{BK08} that if a complex manifold \( M \) of a compact manifold \( \mathbb{C}^n \setminus Z \) is biholomorphic to \( C \), then it has also another proper holomorphic embedding with \( (n - \dim Y) \)-Eisenman hyperbolic complement to the image (the proof is based on the DP and a generalized idea from \cite{BF99}).

(8) A beautiful combination of differential-topological methods with hard analysis (solutions of \( \partial \)-equations with exact estimates) and the Andersén-Lempert-Theorem is required for understanding of how many totally real differentiable embeddings of a real manifold \( M \) into \( \mathbb{C}^n \) can exist.

If \( f_0, f_1 : M \to \mathbb{C}^n \) are two totally real, polynomially convex real-analytic embeddings of a compact manifold \( M \) into \( \mathbb{C}^n \), we say that \( f_0 \) and \( f_1 \) are Aut\( _{\text{hol}}(\mathbb{C}^n) \)-equivalent\(^2\) if \( f_1 = F \circ f_0 \), where \( F : U \to F(U) \subset \mathbb{C}^n \) is a biholomorphism defined in a neighborhood \( U \) of \( f_0(M) \) such that \( F \) is the uniform limit in \( U \) of a sequence of elements of Aut\( _{\text{hol}}(\mathbb{C}^n) \). Conditions for Aut\( _{\text{hol}}(\mathbb{C}^n) \)-equivalence were found in \cite{FR93}, using volume-preserving automorphisms (and an approach using automorphisms preserving the holomorphic symplectic form was considered in \cite{For95}).

In the smooth case let \( E^r(M, \mathbb{C}^n) \) be the set of all totally real polynomially convex \( C^r \)-embeddings of \( M \) into \( \mathbb{C}^n \) (for \( 2 \leq r \leq \infty \)). It is proved by Forstnerič and Løw that two embeddings \( f_0, f_1 \in E^\infty(M, \mathbb{C}^n) \) belong to the same connected component (in the space of \( C^r \)-embeddings of \( M \) into \( \mathbb{C}^n \) equipped with the usual topology of uniform convergence of all derivatives up to order \( r \)) if and only if there exists a sequence \( \{ \Phi_j \} \subset \text{Aut}_{\text{hol}}(\mathbb{C}^n) \) such that \( \Phi_j \circ f_0 \to f_1 \) and \( \Phi_j^{-1} \circ f_1 \to f_0 \) in \( C^\infty(M) \) as \( j \to \infty \). Precise results in the case \( r < \infty \) were obtained in \cite{FL00}.

(9) Recall that a “long \( \mathbb{C}^n \)” is a complex manifold \( X \) that can be exhausted by open subsets \( \Omega_i \) which are all biholomorphic to \( \mathbb{C}^n \), i.e. \( X = \bigcup_{i=1}^{\infty} \Omega_i, \Omega_i \subset \Omega_{i+1}, \text{and } \Omega_i \cong \mathbb{C}^n \) for all \( i \in \mathbb{N} \). (Here, of course, \( \Omega_i \subset \Omega_{i+1} \) is not a Runge pair.) The first examples of such manifolds not biholomorphic to \( \mathbb{C}^n \) for \( n \geq 3 \) were constructed by Fornæss \cite{For76} in 1976. The case of \( n = 2 \) had been resistant until recently when, developing further the ideas from application (5), Fornæss Wold \cite{Wol10} constructed a “long \( \mathbb{C}^n \)” which is not biholomorphic to \( \mathbb{C}^2 \). Every known “long \( \mathbb{C}^n \)” (including the case of \( n = 2 \)) is non-Stein. Recently families of non-isomorphic long \( \mathbb{C}^n \)’s have been constructed using DP via Theorem 3.2 by Forstnerič and Boc Thaler \cite{BTF16}.

(10) The classical approximation theorem of Carleman states that for each continuous function \( \lambda : \mathbb{R} \to \mathbb{C} \) and a positive continuous function \( \epsilon : \mathbb{R} \to (0, \infty) \) there exists an entire function \( f \) on \( \mathbb{C} \) such that \( |f(t) - \lambda(t)| < \epsilon(t) \) for every \( t \in \mathbb{R} \).

Using the Andersén-Lempert theorem together with some explicit shears Buzzard and Forstnerič \cite{BF97} were able to prove a similar result for holomorphic embeddings into \( \mathbb{C}^n \). Namely, for any proper embedding \( \lambda : \mathbb{R} \to \mathbb{C}^n \) of class \( C^r \) (where \( n \geq 2 \) and \( r \geq 0 \)) and a positive continuous function \( \epsilon : \mathbb{R} \to (0, \infty) \) there exists a proper holomorphic embedding \( f : \mathbb{C} \to \mathbb{C}^n \) such that

\[
|f^{(s)}(t) - \lambda^{(s)}(t)| < \epsilon(t) \quad \forall \ t \in \mathbb{R}, \ 0 \leq s \leq r.
\]

Actually this fact remains valid under the additional requirement that the embedding satisfies the interpolation property as in Proposition 4.4.

(11) The spectral ball of dimension \( n \in \mathbb{N} \) is defined to be

\[
\Omega_n := \{ A \in \text{Mat}(n \times n; \mathbb{C}) : \rho(A) < 1 \}
\]

\(^2\)It is unfortunate that in the literature the term “Aut\( _{\text{hol}}(\mathbb{C}^n) \)-equivalence” is used in different meanings - see the sentence after the Open Problem in application (2).
where $\rho$ denotes the spectral radius, i.e. the modulus of the largest eigenvalue.

The study of the group of holomorphic automorphisms of the spectral ball started with the work of Ransford and White [RW91] in 1991 and was continued by various authors. A conjecture from [RW91] was disproved by Kosiński [Kos13] for $n = 2$ and moreover he described a dense subgroup of the $2 \times 2$ spectral ball $\Omega_2$. Andrist and the author generalized these results to $\Omega_n$ for $n \geq 2$ with an new approach using the fibered density property. The notion of $\text{SL}_n(\mathbb{C})$-shears and -overshears is easy to understand. One uses a one-parameter subgroup of $\text{SL}_n(\mathbb{C})$ which by conjugation acts on $\Omega_n$ and thus gives rise to a complete vector field $\theta$ on $\Omega_n$. Multiplying $\theta$ with a function $f$ in the kernel gives a shear field $f\theta$ and with a function $f$ in the second kernel ($\theta(f) \in \ker f \setminus \{0\}$) we get an overshear field $f\theta$. Shears and overshears are time $t$-maps of the corresponding fields. This is the same way as shears and overshears in $\mathbb{C}^n$ from formulas (1.1) and (1.2) arise from the complete fields $\partial / \partial z_i$.

**Theorem 4.9.** [And17a] The $\text{SL}_n(\mathbb{C})$-shears and the $\text{SL}_n(\mathbb{C})$-overshears together with matrix transposition and matrix valued Möbius transformations generate a dense subgroup (in compact-open topology) of the holomorphic automorphism group $\text{Aut} \Omega_n$.

For the convenience of the reader we recall that a matrix valued Möbius transformation is a map of the form $h : \Omega_n \to \Omega_n$

\begin{equation}
A \mapsto \gamma \cdot (A - \alpha \cdot \text{id}) \cdot (\text{id} - \overline{\alpha} A)^{-1}, \quad \alpha \in \mathbb{D}, \gamma \in \partial \mathbb{D}
\end{equation}

(12) It is standard material in a Linear Algebra course that the group $\text{SL}_m(\mathbb{C})$ is generated by elementary matrices $E + \alpha e_{ij}$, $i \neq j$, i.e., matrices with 1’s on the diagonal and all entries outside the diagonal are zero, except one entry. Equivalently, every matrix $A \in \text{SL}_m(\mathbb{C})$ can be written as a finite product of upper and lower diagonal unipotent matrices (in interchanging order). The same question for matrices in $\text{SL}_n(R)$ where $R$ is a commutative ring instead of the field $\mathbb{C}$ is much more delicate. For example, if $R$ is the ring of complex valued functions (continuous, smooth, algebraic or holomorphic) from a space $X$ the problem amounts to finding for a given map $f : X \to \text{SL}_m(\mathbb{C})$ a factorization as a product of upper and lower diagonal unipotent matrices

$$f(x) = \begin{pmatrix}
1 & 0 \\
G_1(x) & 1
\end{pmatrix}
\begin{pmatrix}
1 & G_2(x) \\
0 & 1
\end{pmatrix}
\cdots
\begin{pmatrix}
1 & G_N(x) \\
0 & 1
\end{pmatrix}$$

where the $G_i$ are maps $G_i : X \to \mathbb{C}^{m \times (m-1)/2}$.

Since any product of (upper and lower diagonal) unipotent matrices is homotopic to a constant map (multiplying each entry outside the diagonals by $t \in [0,1]$ we get a homotopy to the identity matrix), one has to assume that the given map $f : X \to \text{SL}_m(\mathbb{C})$ is homotopic to a constant map or as we will say null-homotopic. In particular this assumption holds if the space $X$ is contractible.

This very general problem has been studied in the case of polynomials of $n$ variables. For $n = 1$, i.e., $f : \mathbb{C} \to \text{SL}_m(\mathbb{C})$ a polynomial map (the ring $R$ equals $\mathbb{C}[z]$) it is an easy consequence of the fact that $\mathbb{C}[z]$ is an Euclidean ring that such $f$ factors through a product of upper and lower diagonal unipotent matrices. For $m = n = 2$ the following counterexample was found by Cohen [Coh60]: the matrix

$$\begin{pmatrix}
1 - z_1 z_2 \\
- z_2^2 \\
2 z_1 \\
1 + z_1 z_2
\end{pmatrix} \in \text{SL}_2(\mathbb{C}[z_1, z_2])$$

does not decompose as a finite product of unipotent matrices.

For $m \geq 3$ (and any $n$) it is a deep result of Suslin [Sus77] that any matrix in $\text{SL}_m(\mathbb{C}[\mathbb{C}^n])$ decomposes as a finite product of unipotent (and equivalently elementary) matrices. More results in the algebraic setting can be found in [Sus77] and [GMV94]. For a connection to the Jacobian problem on $\mathbb{C}^2$ see [Wri78].

In the case of continuous complex valued functions on a topological space $X$ the problem was studied and partially solved by Thurston and Vaserstein [TV86] and then finally solved by Vaserstein [Vas88].
It is natural to consider the problem for rings of holomorphic functions on Stein spaces, in particular on $\mathbb{C}^n$. Explicitly this problem was posed by Gromov in his groundbreaking paper [Gro89] where he extends the classical Oka-Grauert theorem from bundles with homogeneous fibers to fibrations with elliptic fibers, e.g., fibrations admitting a dominating spray (for definition see before Example 3.17). In spite of the above mentioned result of Vaserstein he calls it the Vaserstein problem:

(see [Gro89, sec 3.5.G])

Does every holomorphic map $\mathbb{C}^n \rightarrow SL_m(\mathbb{C})$ decompose into a finite product of holomorphic maps sending $\mathbb{C}^n$ into unipotent subgroups in $SL_m(\mathbb{C})$?

Gromov’s interest in this question comes from the question about s-homotopies (s for spray). In this particular example the spray on $SL_m(\mathbb{C})$ is that coming from the multiplication with unipotent matrices. Of course one cannot use the upper and lower diagonal unipotent matrices only to get a spray (there is no submersivity at the zero section!), there need to be at least one more unipotent subgroup to be used in the multiplication. Therefore the factorization in a product of upper and lower diagonal matrices seems to be a stronger condition than to find a map into the iterated spray, but since all maximal unipotent subgroups in $SL_m(\mathbb{C})$ are conjugated and the upper and lower diagonal matrices generate $SL_m(\mathbb{C})$ these two problems are in fact equivalent.

We refer the reader for more information on the subject to Gromov’s above mentioned paper.

As an application of flexibility via a stratified version of the Oka principle from [For10] Ivarsson and the author gave a complete positive solution of Gromov’s Vaserstein problem.

**Theorem 4.10.** [IK12] Let $X$ be a finite dimensional reduced Stein space and $f: X \rightarrow SL_m(\mathbb{C})$ be a holomorphic mapping that is null-homotopic. Then there exist a natural number $K$ and holomorphic mappings $G_1, \ldots, G_K: X \rightarrow \mathbb{C}^{m(m-1)/2}$ such that $f$ can be written as a product of upper and lower diagonal unipotent matrices

$$f(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & G_K(x) \\ 0 & 1 \end{pmatrix}$$

for every $x \in X$.

We leave it as an exercise to the reader to factorize the Cohn example holomorphically.

5. The holomorphic linearization problem

The holomorphic linearization problem was well known to the German Complex Analysis School of Grauert and Remmert. For example a first special case of the Luna slice theorem was proved by Kuhlmann. The problem was officially stated by Huckleberry in his chapter in the (that time still Soviet) Encyclopedia of Mathematical Sciences. [Huc90]

**Problem 5.1 (Holomorphic Linearization Problem).** Suppose a reductive group $G$ is acting holomorphically on $\mathbb{C}^n$, $n \geq 2$, Does there exist a holomorphic change of variables $\alpha \in \text{Aut}_{hol}(\mathbb{C}^n)$ such that $\alpha \circ G \circ \alpha^{-1}$ is linear?

Independently the late Walter Rudin got interested in this problem, he formulates it more specifically for a finite cyclic group $G = \mathbb{Z}/n\mathbb{Z}$ in his paper with Ahern [AR95].

5.1. Basic notions and properties. Two students of Alan T. Huckleberry, namely Dennis Snow and my adviser Peter Heinzner [Hei91] developed the theory of reductive group actions on Stein spaces guided by the results from algebraic geometry, most prominently by the slice theorem in étale topology due to Domingo Luna.

Let a reductive group $G = K^C$, the universal complexification of its maximal compact subgroup $K$, act on a Stein space $X$. Identify two points $x_1 \equiv x_2$ if $f(x_1) = f(x_2)$ for all $G$-invariant holomorphic functions $f \in O^G(X)$. The quotient space $X//G$ has a natural
complex structure such that \( \pi : X \to X//G =: Q_X \) is holomorphic. The quotient space \( Q_X \) (whose holomorphic functions are the invariant holomorphic functions on \( X \), \( \mathcal{O}(Q_X) = \mathcal{O}^G(X) \)) is a Stein space \[^{[\text{Sno82}]}\]. Each fiber of \( \pi \) contains a unique closed \( G \)-orbit which is contained in the closure of any of the other \( G \)-orbits in the same \( \pi \)-fiber. For a point \( x \) in this unique closed orbit the complexification of the \( K \)-isotropy is equal to the \( G \)-isotropy, \( (K_x)_x = (K^C)_x \).

If \( X \) is affine algebraic and \( G \) acts algebraically the categorical quotient is the same as the "Hilbert" quotient \((\text{Spec}(\mathbb{C}[X]^G)) = X//G\).

In general the fibers of \( \pi \) are affine \( G \)-varieties not necessarily reduced. This can be easily seen from the fact that locally every Stein \( G \)-manifold can be properly equivariantly embedded into a linear \( G \)-representation, i.e., a \( \mathbb{C}^N \) with linear \( G \) action, together with fact 1 below. If \( X \) has only finitely many slice types there is even a global such embedding. This is the main result of Peter Heinzner’s Ph.D. thesis \[^{[\text{Hei88}]}\].

The following facts are easy. The first fact uses the property that \( G \)-invariant holomorphic functions from a subvariety \( Y \) of a Stein space \( X \) extend to \( G \)-invariant functions on \( X \) (proved by using Cartan extension and averaging over the maximal compact subgroup \( K \) of \( G \)). The second fact is obvious.

**Fact 1:** If \( Y \) is a closed \( G \)-invariant subspace of a Stein \( G \)-space \( X \), then the restriction of the categorical quotient map \( \pi : X \to Q_X \) to \( Y \) is a categorical quotient map for \( Y \) and the categorical quotient for \( Y \) is the image \( Q_Y \cong \pi(Y) \) of \( Y \), which is a closed subspace of \( Q_X \).

**Fact 2:** If \( X \) is a Stein \( G \)-space with categorical quotient map \( \pi : X \to Q_X \) and \( Y \) is a Stein space with trivial \( G \)-action, then the categorical quotient map for \( X \times Y \) is \( \pi \times \text{Id}_Y : X \times Y \to Q_X \times Y \cong Q_{X \times Y} \).

**Example 5.2.** Let the group \( \text{SL}_n(\mathbb{C}) \) act on the vector space of all \( n \) by \( n \) matrices \( \text{Mat}(n \times n, \mathbb{C}) \) by conjugation \( \text{SL}_n(\mathbb{C}) \times \text{Mat}(n \times n, \mathbb{C}) \to \text{Mat}(n \times n, \mathbb{C}) (G, M) \to GM^{-1} \). We know that the orbits of this action correspond to Jordan normal forms. The categorical quotient is given by the coefficients of the characteristic polynomial \( \chi_M(\lambda) = \det(\lambda E - M) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0 \) or equivalently by the elementary symmetric polynomials in the eigenvalues of the matrix \( M \). This means

\[
\pi : \text{Mat}(n \times n, \mathbb{C}) \to \mathbb{C}^n, M \mapsto (a_0, a_1, \ldots, a_{n-1})
\]

is a categorical quotient map for the above action and thus \( Q_{\text{Mat}(n \times n, \mathbb{C})} = \text{Mat}(n \times n, \mathbb{C})//\text{SL}_n(\mathbb{C}) \cong \mathbb{C}^n \).

The categorical quotient carries a stratification defined as follows. We say that two points \( q, q' \in Q_X \) are in the same Luna stratum if the fibers \( \pi^{-1}(q) \) and \( \pi^{-1}(q') \) are \( G \)-biholomorphic. Remember the fibers are affine not necessarily reduced \( G \)-varieties.

**Theorem 5.3** (Luna stratification). \[^{[\text{Sno82}]}\] The Luna strata form a locally finite stratification of \( Q_X \) by locally closed smooth analytic subvarieties.

The following very easy example will play a role for our counterexamples to Holomorphic Linearization.

**Example 5.4.** Let \( G = \mathbb{C}^* \) act on \( \mathbb{C}^3 \) by the rule

\[
G \times \mathbb{C}^3 \to \mathbb{C}^3, \quad (\lambda, (u, v, w)) \mapsto (\lambda^2 u, \lambda^{-2} v, \lambda w)
\]

The categorical quotient \( Q_{\mathbb{C}^*} \) is isomorphic to \( \mathbb{C}^2 \) and the map is \( (u, v, w) \mapsto (uv, w^2 v) \). The Luna stratification of the quotient is easy: \( Q_{\mathbb{C}^*} \cong \mathbb{C}^2 \supset \mathbb{C} \times \{0\} \supset \{(0, 0)\} \). The isotropy groups in the closed orbits over points of the strata are isomorphic to \{Id\}, \{±Id\}, \mathbb{C}^* respectively.
Moreover the local structure of holomorphic $G$-actions on Stein manifolds is well understood, thanks to the following slice theorem due to Snow [Sno82].

The setting of this holomorphic slice theorem is as follows. Let the complexification $G = K^C$ of a compact Lie group $K$ act holomorphically on a Stein space $X$ with categorical quotient $\pi : X \to X/K^C$. Let $x$ be a point in the unique closed $G$-orbit in the fiber $\pi^{-1}(q)$, where $q = \pi(x)$, with stabilizer $L^C$, where $L = K_x$. Let $V = T_xX/T_xK^C x$ be the normal space to the orbit $K^C x$ at $x$. It is an $L^C$-module. With respect to the identification $K^C/L^C \cong K^C x$, the normal bundle $N$ of $K^C x$ in $X$ is isomorphic to $K^C \times L^C V$.

**Theorem 5.5** (Luna slice theorem). There is a saturated (with respect to $\pi$) Stein neighborhood $U$ of the orbit $K^C x$ in $X$, $K^C$-equivariantly biholomorphic to a subvariety $A$ of a neighborhood of the zero section of $N$. The embedding $\iota : U \to N$ maps $K^C x$ biholomorphically onto the zero section of $N$.

If $X$ is smooth, the submanifold $A$ in the Luna Slice Theorem is open. Thus the representation of $L^C$ on $V = T_xX/T_xK^C x$, called the slice representation, determines the local structure of the $K^C$-manifold $X$ on a saturated neighborhood of $x$. Therefore our Luna stratification is the same as the stratification by slice representations from [Sno82]. We will call the categorical quotient $Q_X$ equipped with the Luna stratification the Luna quotient. An isomorphism of Luna quotients $Q_X$ and $Q_Y$ (for Stein $G$-manifolds $X$ and $Y$) is by definition a biholomorphism between the Stein spaces $Q_X$ and $Q_Y$ mapping corresponding Luna strata onto each other. Clearly an equivariant biholomorphism between $X$ and $Y$ induces an isomorphism of the Luna quotients.

### 5.2. History of the linearization problem.

Let us describe the most important results of the holomorphic part of the problem

- If the quotient $\mathbb{C}^n//G$ is zero dimensional (one point) the action is linearizable. This follows from Luna’s slice theorem (Theorem 5.5).
- Holomorphic $\mathbb{C}^*$-actions on $\mathbb{C}^2$ are linearizable (M. Suzuki 1977 [Suz77]).
- Holomorphic actions with one-dimensional quotient are linearizable. This is the main result from the Ph.D. thesis of Jiang 1990. A new proof is given by Lárussson, Schwarz and the author in 2016. [KLS17c].
- The first counterexamples to the Algebraic Linearization Problem are given in 1989 by G. Schwarz: These examples are $G$-vector bundles over representation spaces, see Example 2.3 and the discussion preceding it. Based on the same method F. Knop constructs algebraically non-linearizable actions for all semisimple groups $G$ [Kno91].
- Holomorphic $G$-vector bundles over representation spaces are holomorphically trivial, in other words: the corresponding actions are linearizable. This is an application of the Equivariant Oka Principle proved by Heinzner and the author in 1995. For the definition of a Kempf-Ness set we refer the interested reader to [HK93], for our application the existence of the homotopy is not essential. The existence of the homotopy over the whole space $X$ together with a generalization to bundles with homogeneous fiber see the recent work of Lárussson, Schwarz and the author [KLSYa].

**Theorem 5.6** (Equivariant Oka Principle). (a) Every topological principal $K$-$G$-bundle on a Stein space $X$ is topologically $K$-isomorphic to a holomorphic principal $K^C$-$G$-bundle on $X$.

(b) Let $P_1$ and $P_2$ be holomorphic principal $K^C$-$G$-bundles on $X$. Let $c$ be a continuous $K$-equivariant section of $\text{Iso}(P_1, P_2)$ over $R$. Then there exists a homotopy of continuous $K$-equivariant sections $\gamma(t)$, $t \in [0, 1]$, of $\text{Iso}(P_1, P_2)$ over a Kempf-Ness set $R$ such that $\gamma(0) = c$ and $\gamma(1)$ extends to a holomorphic $K$-equivariant isomorphism from $P_1$ to $P_2$. 


Holomorphic actions of de Jonquières (triangular) type (see formula 5.22) on \( \mathbb{C}^n \) are linearizable. Also holomorphic actions in the overshear group of \( \mathbb{C}^2 \) are linearizable. This result of Kraft and the author [KK96] also makes use of the Equivariant Oka Principle (Theorem 5.6).

5.3. Counterexamples. The first counterexamples to the Holomorphic Linearization Problem were found by Derksen and the author. In contrast to the algebraic situation, where counterexamples for abelian groups are still unknown, they constructed counterexamples for any reductive group \( G \). For the case of finite groups \( G \) they had to use the classification of finite simple groups.

**Theorem 5.7.** [DK98] For all reductive groups \( G \) there exists a number \( N(G) \) such that for all \( n \geq N(G) \) there is a non-linearizable action of \( G \) on \( \mathbb{C}^n \).

The smallest dimension \( N(G) \) in which we know counterexamples is 4. These are counterexamples for \( G = \mathbb{C}^* \) or \( G = \mathbb{Z}/2\mathbb{Z} \). The exact value of \( N(G) \) is not known for a single group \( G \). Funny enough the counterexample for \( G = \mathbb{Z}/2\mathbb{Z} \) is not explicit. There are two examples: One is an action on \( \mathbb{C}^4 \), the other one on a certain 4-dimensional manifold \( Y \). If the first example is linearizable, then the manifold \( Y \) is biholomorphic to \( \mathbb{C}^4 \) and the action on it is not linearizable. We will come back to this topic, when discussing the relation to famous open problems.

Of course we want to give the reader an impression how these counterexamples are constructed. Hereby we will limit our presentation to the case of \( G = \mathbb{C}^* \). The main idea, originating from the work of Asanuma [Asa99], is to use non-straightenable holomorphic embeddings of \( \mathbb{C}^l \) into \( \mathbb{C}^n \) to produce an action on some \( \mathbb{C}^N \) which cannot be linearizable by the following reason:

*The Luna quotient of this action on \( \mathbb{C}^n \) is not isomorphic to the Luna quotient of any linear \( G \)-action on \( \mathbb{C}^N \).*

Let us describe the method from [DK98] and [DK99] (based on [Asa99]) to construct (non-linearizable) \( \mathbb{C}^* \)-actions on affine spaces out of (non-straightenable) embeddings \( \mathbb{C}^l \hookrightarrow \mathbb{C}^n \). At the same time we want to present a strengthening of the original method to a parametrized version. It will turn out that if the embeddings are holomorphically parametrized, then the resulting \( \mathbb{C}^* \)-actions depend holomorphically on the parameter. Theorem 5.6 will give us then the following quite striking result due to Lodin and the author [KL13], a whole family of counterexamples to linearization:

**Theorem 5.8.** For any \( n \geq 5 \) there is a holomorphic family of \( \mathbb{C}^* \)-actions on \( \mathbb{C}^n \) parametrized by \( \mathbb{C}^{n - 4} \)

\[
\mathbb{C}^{n - 4} \times \mathbb{C}^* \times \mathbb{C}^n \to \mathbb{C}^n \quad (w, \theta, z) \mapsto \theta_w(z)
\]

so that for different parameters \( w_1 \neq w_2 \in \mathbb{C}^{n - 4} \) there is no equivariant isomorphism between the actions \( \theta_{w_1} \) and \( \theta_{w_2} \).

Moreover for \( n \geq 5 \) there are such \( \mathbb{C}^* \)-actions on \( \mathbb{C}^n \) parametrized by \( \mathbb{C} \) with the additional property that \( \theta_0 \) is a linear action.

Let’s go through the method: For an embedding \( \varphi : \mathbb{C}^l \to \mathbb{C}^n \) take generators of the ideal \( I_{\varphi(\mathbb{C}^l)} < \mathcal{O}(\mathbb{C}^n) \) of the image manifold, say \( f_1, \ldots, f_N \in \mathcal{O}(\mathbb{C}^n) \) (in this case \( N = n - l \) would be sufficient, since \( \mathbb{C}^l \) is always a complete intersection in \( \mathbb{C}^n \) by results of Forster and Ramspott [FR66], but this is not important for the construction) and consider the manifold

\[
M := \{(z_1, \ldots, z_n, u_1, \ldots u_N, v) \in \mathbb{C}^{n+N+1} : f_i(z_1, \ldots, z_n) = u_i \quad \forall \ i = 1, \ldots, N\}
\]

which in [DK98] is called Rees space. This notion was introduced there by the authors since they were not aware of the fact that this is a well-known construction, called affine modification, going back to Oscar Zariski. Geometrically the manifold \( M \) results from
\( \mathbb{C}^{n+1} \) by blowing up along the center \( C = \varphi(C^l) \times \mathbb{C} \), and deleting the proper transform of the divisor \( D = \{ v = 0 \} \). Since our center is not algebraic but analytic, the process usually is called pseudo-affine modification.

Let’s denote the constructed manifold \( M \) by \( \text{Mod}(\mathbb{C}^{n+1}, D, C) = \text{Mod}(\mathbb{C}^{n+1}, \{ v = 0 \}, \varphi(C^l) \times \{ v = 0 \}) \). It’s clear from the geometric description that the resulting manifold does not depend on the choice of generators for the ideal \( I_C \) of the center. The important fact about the above modifications is that

\[
\text{Mod}(\mathbb{C}^{n+1}, \{ v = 0 \}, \varphi(C^l) \times \{ v = 0 \}) \times C^l
\]

is biholomorphic to

\[
\mathbb{C}^{n+l+1} \cong \text{Mod}(\mathbb{C}^{n+l+1}, \{ v = 0 \}, \varphi(C^l) \times 0_w \times 0_v).
\]

The later biholomorphism comes from the fact that there is an automorphism of \( \mathbb{C}^{n+l+1} \) leaving the divisor \( \{ v = 0 \} \) invariant and straightening the center \( \varphi(C^l) \times 0_v \) inside the divisor (see Lemma 2.5. in [DK98]). This is the so called Asanuma trick. Let’s present this important fact with holomorphic dependence on a parameter.

**Lemma 5.9.** Let \( \Phi_1 : C^k \times X \hookrightarrow C^k \times C^n, \Phi_1(w,x) = (w, \varphi(w,x)) \) and \( \Phi_2 : C^k \times X \hookrightarrow C^k \times C^n, \Phi_2(w,x) = (w, \varphi_2(w,x)) \) be two holomorphic families of proper holomorphic embeddings of a complex space \( X \) into \( C^n \) resp. \( C^m \) parametrized by \( C^k \). Then there is an automorphism \( \alpha \) of \( C^{n+m} \) parametrized by \( C^k \), i.e., \( \alpha \in \text{Aut}_{\text{hol}}(C^k \times C^{n+m}) \) with \( \alpha(w,z) = (w, \tilde{\alpha}(w,z)) \), such that \( \alpha \circ (\Phi_1 \times 0_m) = 0_n \times \Phi_2 \).

**Proof.** By an application of Theorem B the holomorphic map \( \varphi_1 : C^k \times X \to C^n \) extends to a holomorphic map \( \mu_1 \) from \( C^k \times C^m \supset \Phi_2(C^k \times X) \) to \( C^n \) (so \( \mu_1 \circ \varphi_2 = \varphi_1 \)). Likewise there is a holomorphic map \( \mu_2 : C^k \times C^n \to C^m \) with \( \mu_2 \circ \varphi_1 = \varphi_2 \). Define the parametrized automorphisms \( \alpha_1, \alpha_2 \) of \( C^k \times C^n \times C^m \) by \( \alpha_1(w,z,y) = (w,z,y + \mu_2(w,z)) \) and \( \alpha_2(w,z,y) = (w,z + \mu_1(w,y), y) \), and \( \alpha = \alpha_2^{-1} \circ \alpha_1 \) is the desired automorphism. \( \square \)

**Lemma 5.10.** Let \( \Phi : C^k \times C^l \hookrightarrow C^k \times C^n \Phi(w,\theta) = (w, \varphi(w,\theta)) \) be a holomorphic family of proper holomorphic embeddings of \( C^l \) into \( C^n \) parametrized by \( C^k \).

Then \( \text{Mod}(\mathbb{C}^{k+n+l+1}, \{ v = 0 \}, \Phi(C^k \times C^l) \times \{ v = 0 \}) \times C^l \cong \mathbb{C}^{k+n+l+1} \). Moreover there is a biholomorphism such that the restriction to each fixed parameter \( w \in C^k \) is a biholomorphism from \( \text{Mod}(\mathbb{C}^{k+n+1}, \{ v = 0 \}, \Phi(C^k \times C^l) \times \{ v = 0 \}) \times C^l \cong \mathbb{C}^{k+n+1} \).

**Proof.** Apply Lemma 5.9 to the families \( \Phi_1 = \Phi \) and \( \Phi_2 \) the trivial family \( \Phi_2 : C^k \times C^l \hookrightarrow C^k \times C^l \). Let \( \alpha \in \text{Aut}_{\text{hol}}(C^k \times C^n \times C^l) \) be the resulting parametrized automorphism which we extend to \( C^{k+n+l+1} \) by letting it act trivial on the last coordinate \( v \). Then by definition \( \text{Mod}(\mathbb{C}^{k+n+l+1}, \{ v = 0 \}, \Phi(C^k \times C^l) \times \{ v = 0 \}) \times C^l \) = \( \text{Mod}(\mathbb{C}^{k+n+l+1}, \{ v = 0 \}, \Phi(C^k \times C^l) \times \{ v = 0 \}) \times C^l \) and applying (the extended) \( \alpha \) we get that the later is biholomorphic to \( \text{Mod}(\mathbb{C}^{k+n+l+1}, \{ v = 0 \}, C^k \times 0_n \times C^l \times \{ v = 0 \}) \).

The last manifold is obviously biholomorphic to \( C^{k+n+l+1} \) since blowing up along a straight center and deleting the proper transform of a straight divisor does not change the affine space. The above constructed biholomorphism restricts to each fixed parameter as desired since \( \alpha \) is a parametrized automorphism. This can also be seen by writing down concrete formulas for the modifications using generators \( f_1(w,z), \ldots, f_N(w,z) \) of the ideal \( I_{\phi(C^k \times C^l)} \) in \( \mathcal{O}(C^{k+n}) \) and remarking that for each fixed \( w \in C^k \) the functions \( f_1(w,\cdot), \ldots, f_N(w,\cdot) \) generate the ideal \( I_{\phi(w,C^l)} \).

Now we describe the group actions:

Let \( f_1(w,z), \ldots, f_N(w,z) \) be generators of the ideal \( I_{\phi(C^k \times C^l)} \) in \( \mathcal{O}(C^{k+n}) \) and consider \( \text{Mod}(\mathbb{C}^{k+n+1}, \{ v = 0 \}, \Phi(C^k \times C^l) \times \{ v = 0 \}) \times C^l \cong \mathbb{C}^{k+n+l+1} \) as the affine manifold given by equations:

\[
\{(w,z,v,u) \in C^k \times C^n \times C \times C^N : f_i(w,z) = u_i \ v \ \forall \ i = 1, \ldots, N \} \times C^l
\]
On it we consider the action of $\mathbb{C}^*_r$ given by the restriction of the following linear action on the ambient space:

\[(5.2) \quad \mathbb{C}^* \times \mathbb{C}^k \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^N \times \mathbb{C}^l \to \mathbb{C}^k \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^N \times \mathbb{C}^l \]

\[\begin{array}{c}
(\nu, (w, z, v, u, x)) \mapsto (w, z, \nu^2 v, \nu^{-2} u_1, \ldots, \nu^{-2} u_N, \nu x_1, \ldots, \nu x_l)
\end{array}\]

This gives by Lemma 5.10 a holomorphic family of $\mathbb{C}^*$-actions on $\mathbb{C}^{n+l+1}$ parametrized by $\mathbb{C}^k$, i.e., an action $\mathbb{C}^* \times \mathbb{C}^k \times \mathbb{C}^{n+l+1} \to \mathbb{C}^k \times \mathbb{C}^{n+l+1}$ of the form $(\nu(w, z)) \mapsto (w, \nu(w, z))$. Calculating (as in [DK99]) the Luna-stratification of the categorical quotient $\mathbb{C}^{n+l+1}/\mathbb{C}^*$ for the $\mathbb{C}^*$-action for fixed $w$, in particular the inclusion of the fixed point stratum in the $\mathbb{Z}/2\mathbb{Z}$-isotropy stratum one sees that this inclusion is biholomorphic to $\Phi_w(\mathbb{C}^l) \subset \mathbb{C}^n$. The reader is invited to do this calculation using Example 5.3 and Fact 1 from section 5.1. Thus if for different parameters $w_1 \neq w_2$ there were an equivariant automorphism $\alpha \in \text{Aut}_{\text{hol}}(\mathbb{C}^n)$, the induced isomorphism of the categorical quotients would map the Luna-stratifications onto each other. Therefore the restriction of that induced isomorphism to the $\mathbb{Z}/2\mathbb{Z}$-isotropy stratum would give an automorphism $\beta$ of $\mathbb{C}^n$ with $\beta(\Phi_{w_1}(\mathbb{C}^l)) = \Phi_{w_2}(\mathbb{C}^l)$. This shows that pairwise non-equivalent embeddings lead to non-equivalent $\mathbb{C}^*$-actions. This concludes the proof of Theorem 5.8 except for the moreover part. The proof of this fact is a simple trick of contracting the parameter space. We refer the reader to [KL13]. Also it is clear from the above discussion that one can construct uncountably many non-linearizable $\mathbb{C}^*$-actions on $\mathbb{C}^4$ using the last assertion from Theorem 4.6.

### 5.4. Relation to famous problems

The Holomorphic Linearization Problem is connected to famous problems about complex affine space $\mathbb{C}^n$. The first one is the holomorphic version of the Zariski Cancellation Problem, a problem which is still open in both the algebraic category over $\mathbb{C}$ and in the holomorphic category.

**Problem 5.11** (Zariski Cancellation). *If $X$ is a complex manifold such that $X \times \mathbb{C}$ is biholomorphic to $\mathbb{C}^{n+1}$, Does it follow that $X$ is biholomorphic to $\mathbb{C}^n$?*

There is an easy connection to Linearization since if we had a counterexample $X$ to the Zariski Cancellation Problem, say $\dim X = n$ then on $\mathbb{C}^{n+1} \cong X \times \mathbb{C}$ the $\mathbb{Z}/2\mathbb{Z}$-action given by $\mathbb{Z}/2\mathbb{Z} \times (X \times \mathbb{C}) \to X \times \mathbb{C}$, $(\sigma(x, t)) \mapsto (x, -t)$ would have a fixed point set $X$ which would not be biholomorphic to an affine space. Clearly fixed point sets of linear actions are affine spaces, thus the $\mathbb{Z}/2\mathbb{Z}$-action would be non-linearizable. Another less obvious connection comes from the construction of our counterexamples, the Asanuma trick. Set $X = \text{Mod}(\mathbb{C}^{n+1}_{x,v}, \{v = 0\}, \Phi(C^l) \subset \{v = 0\})$ for a non-straightenable holomorphic embedding $\Phi : C^l \to \mathbb{C}^n$. By Lemma 5.10 $X \times C^l \cong \mathbb{C}^{n+l+1}$. If $X$ were biholomorphic to $\mathbb{C}^{n+1}$ we would have non-linearizable actions in lower dimensions. But if $X$ were not biholomorphic to $\mathbb{C}^{n+1}$ there would be a counterexample to Zariski Cancellation.

Here is a connection to another well known problem, formulated as a conjecture by Varolin and Toth.

**Problem 5.12** (Varolin-Toth-Conjecture). *If a Stein manifold $X$ is diffeomorphic to $\mathbb{R}^{2n}$ ($n \geq 2$) and has the density property, is $X$ then biholomorphic to $\mathbb{C}^n$?*

Again our $X = \text{Mod}(\mathbb{C}^{n+1}_{x,v}, \{v = 0\}, \Phi(\{C^l\} \subset \{v = 0\})$ is the candidate. From (3) in our list of examples in section 3.1.2 we see that it has the density property. Also it can be proven that $X$ is diffeomorphic to $\mathbb{R}^{2n+2}$, see [KK08b]. Were it not biholomorphic to affine space, we would have a counterexample to the Varolin-Toth-Conjecture. There are more candidates for counterexamples to this conjecture. A famous one is the Koras-Russel threefold from equation 2.5, which is well known to be diffeomorphic to $\mathbb{R}^6$ and has the density property, see (5) in our list of examples in section 3.1.2.
5.5. **Optimal positive results.** The way of constructing counterexamples to the Holomorphic Linearization Problem was providing group actions on \( \mathbb{C}^n \) whose Luna quotient is not isomorphic to the Luna quotient of a linear action. This raises the following natural question:

**Question 1:** If the Luna quotient of an action of a reductive group \( G \) on \( \mathbb{C}^n \) is birational to the quotient of a linear action, does it follow that the action is linearizable?

In general one can replace \( \mathbb{C}^n \) by arbitrary Stein manifolds.

**Question 2:** If two Stein \( G \)-manifolds have isomorphic Luna quotients, under which additional assumptions are they \( G \)-biholomorphic?

The following example shows that there is at least a topological obstruction for this to hold true.

**Example 5.13.** Take any Stein manifold \( M \) which admits 2 non-isomorphic holomorphic line bundles \( \pi_1 : L_1 \to M \) and \( \pi_2 : L_2 \to M \). Remember that by Oka principle \( H^2(M, \mathbb{C}) \) parametrizes the holomorphic line bundles on \( M \), so this cohomolgy group has to be non-trivial. Consider the action of \( \mathbb{C}^* \) on the total spaces \( X = L_1 \) and \( Y = L_2 \) of the line bundles by fibre wise multiplication. The categorical quotient maps for these actions are just the bundle projections. Both categorical quotients are just isomorphic to \( M \). The fibers of the categorical quotient maps consist of 2 orbits, a fixed point (contained in the zero section of the bundle) and the rest of the line, a free \( \mathbb{C}^* \) orbit. The Luna stratification is trivial, with one stratum \( M \). A \( \mathbb{C}^* \)-equivariant biholomorphism between \( L_1 \) and \( L_2 \) is linear in the fibers of the line bundle, thus a bundle isomorphism. But the line bundles are non-isomorphic. The obstruction to an equivariant biholomorphism under the existence of a Luna biholomorphism in this example is purely topological: \( H^2(M, \mathbb{C}) \).

We are now going to present recent results on the above two questions obtained by Lárusson, Schwarz and the author, all results are from the papers [KLS15], [KLS17c], [KLS17b]. The setting is as follows: Let \( X \) and \( Y \) be Stein manifolds on which \( G \) acts holomorphically. We have quotient mappings \( p_X : X \to Q_X \) and \( p_Y : Y \to Q_Y \) where \( Q_X \) and \( Q_Y \) are normal Stein spaces, the categorical quotients of \( X \) and \( Y \). Suppose there is a biholomorphism \( \phi : Q_X \to Q_Y \) which preserves the Luna strata, i.e., \( X_q \) is \( G \)-biholomorphic to \( Y_{\phi(q)} \) for all \( q \in Q_X \). We say that \( X \) and \( Y \) have common quotient \( Q \).

Set

\[
\text{Iso}(X, Y) = \prod_{q \in Q} \text{Iso}(X_q, Y_q)
\]

where \( \text{Iso}(X_q, Y_q) \) denotes the set of \( G \)-biholomorphisms of \( X_q \) and \( Y_q \). In general, there is no reasonable structure of complex variety on \( \text{Iso}(X, Y) \) as examples in [KLS17c] show.

Let \( \Phi : X \to Y \) be a \( G \)-diffeomorphism inducing the identity on the quotient. \( \Phi \) is called **strict** if it induces a \( G \)-biholomorphism of \( (X_q)_{red} \) with \( (Y_q)_{red} \) for all \( q \in Q \). One can think of the strict \( G \)-diffeomorphisms as “smooth sections of \( \text{Iso}(X, Y) \)” (although the later has no good structure).

**Theorem 5.14.** Let \( X \) and \( Y \) be Stein \( G \)-manifolds with common quotient \( Q \). Suppose that there is a strict \( G \)-diffeomorphism \( \Phi : X \to Y \). Then \( \Phi \) is homotopic, through strict \( G \)-diffeomorphisms, to a \( G \)-biholomorphism from \( X \) to \( Y \).

We would like to comment on this theorem whose proof has two major steps. Our aim is to lift a Luna isomorphism \( \phi \) between \( Q_X \) and \( Q_Y \) in the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & Y \\
\downarrow p_X & & \downarrow p_Y \\
Q_X & \xrightarrow{\phi} & Q_Y
\end{array}
\]
to a $G$-equivariant biholomorphism between $X$ and $Y$ (under the assumption that there is a lift to a strict $G$-diffeomorphism). The first step which may readily be overlooked is the local lifting. For each $q \in Q_X$ and $\phi(q) \in Q_Y$ there are open neighborhoods $U \subset Q_X$ and $V = \varphi(U) \subset Q_Y$ whose preimages under the categorical quotient maps are described by the Luna slice theorem Theorem 5.15. And since $\phi$ respects the Luna stratification, both $p_X^{-1}(U)$ and $p_Y^{-1}(V)$ are described by the same slice model and thus are $G$-biholomorphic. The problem is that this existing $G$-biholomorphism need not to be a lift of $\phi$, it may induce another local Luna isomorphism between $U$ and $V$. Therefore our first step is non-trivial and uses the additional information, namely the existence of a strict $G$-diffeomorphism.

The second step is then the gluing of the local lifts of the Luna isomorphism to a global lift. This is an Oka principle, which is a separate important result:

Suppose that we have a stratified biholomorphism $\phi: Q_X \to Q_Y$ where $V$ is a $G$-module. Again we identify $Q_X$ and $Q_Y$ and call the common quotient $Q$. We have quotient mappings $p : X \to Q$ and $r : V \to Q$. Assume there is an open cover $\{U_i\}_{i \in I}$ of $Q$ and $G$-equivariant biholomorphisms $\Phi_i : p^{-1}(U_i) \to r^{-1}(U_i)$ over $U_i$ (meaning that $\Phi_i$ descends to the identity map of $U_i$). We express the assumption by saying that $X$ and $V$ are locally $G$-biholomorphic over a common quotient. Equivalently, our original $\phi : Q_X \to Q_Y$ locally lifts to $G$-biholomorphisms of $X$ to $V$.

**Theorem 5.15.** Let $X$ and $Y$ be Stein $G$-manifolds which locally $G$-biholomorphic over a common quotient $Q$. Suppose that there is a strict $G$-diffeomorphism $\Phi : X \to Y$. Then $\Phi$ is homotopic, through strict $G$-diffeomorphisms, to a $G$-biholomorphism from $X$ to $Y$.

In the case of a generic action (see Definition 5.17 below) this theorem can be deduced from the Equivariant Oka Principle (Theorem 5.18). The proof in the general case is much more involved.

Together with the above described first step this proves Theorem 5.14. The additional obstruction, besides the Luna-isomorphism of the quotients, is the strict $G$-diffeomorphism. We also have a result, where the obstruction is more topological (instead of smooth). This is the existence of a strong $G$-homeomorphism, which is the right version of “continuous sections of Iso($X,Y$)”. Since the definition of strong $G$-homeomorphism is not so straightforward and the result includes an extra assumption on the common quotient we refer the interested reader to [KLS 17c] for details.

Now let us get back to the special case of linearization, i.e., one of the $G$-manifolds, say $Y$, is a linear representation of $G$, i.e., a $G$-module. This of course gives immediately more information on the Luna quotient and there is hope that the additional obstruction just disappears because of the simple topology or diffeomorphism type of $\mathbb{C}^n$. We have not been able to confirm this hope completely, but substantial results are proven. Remember that the proof of Theorem 5.14 consisted of two steps, the first step, the local $G$-diffeomorphisms are the problem, the second step is solved:

**Theorem 5.16.** Suppose that $X$ is a Stein $G$-manifold, $V$ is a $G$-module and $X$ and $V$ are locally $G$-biholomorphic over a common quotient. Then $X$ and $V$ are $G$-biholomorphic.

For the local $G$-isomorphisms we still do not know the optimal result. We have to make an additional technical assumption on the representation.

**Definition 5.17.** Assume that the set of closed orbits with trivial isotropy group is open in $X$ and that the complement, a closed subvariety of $X$, has complex codimension at least two. We say that $X$ is generic. Let $X^{(n)}$ denote the subset of $X$ whose isotropy groups have dimension $n$. We say that $X$ is large if $X$ is generic and $\text{codim} X^{(n)} \geq n + 2$ for $n \geq 1$.

For a simple group all but finitely many irreducible representations are large. A representations is large if all irreducible factors are large.
Theorem 5.18. Suppose that $X$ is a Stein $G$-manifold and $V$ is a $G$-module satisfying the following conditions.

1. There is a stratified biholomorphism $\phi$ from $Q_X$ to $Q_V$.
2. $V$ (equivalently, $X$) is large.

Then, by perhaps changing $\phi$, one can arrange that $X$ and $V$ are locally $G$-biholomorphic over $Q_X \simeq Q_V$, hence $X$ and $V$ are $G$-biholomorphic.

Some special cases where the representation is not large can be dealt with as well:

Theorem 5.19. Suppose that $X$ is a Stein $\text{SL}_2(\mathbb{C})$-manifold and $V$ is a $\text{SL}_2(\mathbb{C})$-module. If there is a stratified biholomorphism (Luna isomorphism) $\phi$ from $Q_X$ to $Q_V$, then $X$ is $\text{SL}_2(\mathbb{C})$-biholomorphic to $V$.

Also an old result of Jiang from our history of the Holomorphic Linearization Problem can be reproved by using this approach.

Theorem 5.20. Suppose that $X$ is a Stein $G$-manifold and $V$ is a $G$-module with one-dimensional categorical quotient $Q_V$. If there is a stratified biholomorphism $\phi$ from $Q_X$ to $Q_V$, then $X$ is $G$-biholomorphic to $V$.

A comment on the last three theorems is in order, namely that we do not assume from the beginning that the Stein manifold $X$ is biholomorphic to affine space. Therefore these theorems can be viewed as a characterization of $\mathbb{C}^n$. On the other hand it seems unlikely that this characterization can be applied in any interesting case to prove that some Stein manifold is $\mathbb{C}^n$. For example let’s pretend we would like to use it to prove that the Koras-Russel cubic threefold $M_{KR}$ given by equation (2.3) is biholomorphic to $\mathbb{C}^3$. The only known action of a reductive group on $M_{KR}$ is the famous $\mathbb{C}^*$-action given by the restriction to $M_{KR}$ of the linear action $\mathbb{C}^* \times \mathbb{C}^4 \to \mathbb{C}^4$

$$\lambda, (x, y, s, t)) \mapsto (\lambda^6 x, \lambda^{-6} y, \lambda^3 s, \lambda^2 t).$$

The reader is invited to calculate the Luna quotient and to observe that this Luna quotient is not isomorphic to the Luna quotient of a linear action. The theorem cannot be applied, and thus no information whether $M_{KR}$ is biholomorphic to $\mathbb{C}^3$ or not can be obtained. On the other hand, if there were a(nother) way to conclude that $M_{KR}$ is biholomorphic to $\mathbb{C}^3$, we would have found a non-linearizable holomorphic $\mathbb{C}^*$-action on $\mathbb{C}^3$. Up to now the holomorphic linearization for $\mathbb{C}^*$ is known to hold on $\mathbb{C}^2$, not to hold on $\mathbb{C}^n$ for $n \geq 4$ (see section 5.2) and is open on $\mathbb{C}^3$.

The last three theorems give already a lot of evidence that the answer to Question 1 could be positive. If so, this would in some sense give one beautiful answer to the Holomorphic Linearization Problem. On the other hand the question about minimal dimension for non-linearizable actions of a given reductive group $G$ is much more difficult to answer. Until we have an effective criterion which can distinguish affine space $\mathbb{C}^n$ among Stein manifolds with density property there is no hope for an answer. At this point it is worth mentioning another criterion for characterization of $\mathbb{C}^n$ obtained by Isaev and Kruzhilin. A Stein manifold $X$ of dimension $n$ whose holomorphic automorphism group $\text{Aut}_{hol}(X)$ is isomorphic as topological group to $\text{Aut}_{hol}(\mathbb{C}^n)$ is biholomorphic to $\mathbb{C}^n$ [Isa01]. This conclusion even holds without the Stein assumption [JK02]. Unfortunately this criterion is equally not applicable to the above problems as ours from the last three theorems. In the early 1990s J.P. Rosay asked the author whether $\text{Aut}_{hol}(\mathbb{C}^n)$ as an abstract group (without topology) could characterize $\mathbb{C}^n$. Unfortunately even a positive answer to this seemingly difficult question would not be very helpful for our problems.

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