Weighted Hardy’s inequalities and the variational problem with compact perturbations

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Abstract
Let Ω be a bounded domain of $\mathbb{R}^N$ ($N \geq 1$) whose boundary $\partial \Omega$ is a $C^2$ compact manifolds. In the present paper we shall study a variational problem relating the weighted Hardy inequalities established in [4]. As weights we adopt powers of the distance function $\delta(x)$ to the boundary $\partial \Omega$.

1. Introduction
Let Ω be a bounded domain of $\mathbb{R}^N$ ($N \geq 1$) whose boundary $\partial \Omega$ is a $C^2$ compact manifolds, for short, a bounded domain of class $C^2$ in $\mathbb{R}^N$. In [4] we have established $N$ dimensional weighted Hardy’s inequalities with weight function being powers of the distance function $\delta(x) = \text{dist}(x, \partial \Omega)$ to the boundary $\partial \Omega$. In this paper we shall study a variational problem relating to these new inequalities.

We prepare more notations to describe our results. Let $1 < p < \infty$ and $\alpha < 1 - 1/p$. By $L^p(\Omega, \delta^{\alpha p})$ we denote the space of Lebesgue measurable functions with weight $\delta^{\alpha p}$, for which

$$\|u\|_{L^p(\Omega, \delta^{\alpha p})} = \left( \int_{\Omega} |u|^p \delta^{\alpha p} \, dx \right)^{1/p} < \infty. \quad (1.1)$$

$W^{1,p}_{\alpha,0}(\Omega)$ is given by the completion of $C^\infty_c(\Omega)$ with respect to the norm defined by

$$\|u\|_{W^{1,p}_{\alpha,0}(\Omega)} = \|\nabla u\|_{L^p(\Omega, \delta^{\alpha p})}. \quad (1.2)$$

Then $W^{1,p}_{\alpha,0}(\Omega)$ becomes a Banach space with the norm $\| \cdot \|_{W^{1,p}_{\alpha,0}(\Omega)}$. Under these preparation we recall the noncritical weighted Hardy inequality in [4]. In particular,
we have the simplest one:

\[
\int_{\Omega} |\nabla u|^p \delta \, dx \geq \mu \int_{\Omega} |u|^p \delta^{(\alpha-1)p} \, dx \quad \text{for } u \in W^{1,p}_{\alpha,0}(\Omega),
\]

(1.3)

where \( \mu \) is a positive constant independent of \( u \). If \( \alpha = 0 \) and \( p = 2 \), then (1.3) is a well-known Hardy’s inequality and valid for a bounded domain \( \Omega \) of \( \mathbb{R}^N \) with Lipschitz boundary (cf. [6], [8], [11], [13]). If \( \Omega \) is convex and \( \alpha = 0 \), then (1.3) with \( \mu = (1 - 1/p)^p \) holds for arbitrary \( 1 < p < \infty \) (see [13], [14]).

The best possible \( \mu \) in (1.3) is given by the quantity

\[
\inf_{u \in W^{1,p}_{\alpha,0}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^p \delta \, dx}{\int_{\Omega} |u|^{p(\alpha-1)p} \, dx},
\]

(1.4)

which depends on \( p, \alpha \) and \( \Omega \).

In this paper we consider the following variational problem

\[
J_\lambda^\alpha = \inf_{u \in W^{1,p}_{\alpha,0}(\Omega) \setminus \{0\}} \chi_\lambda^\alpha(u)
\]

(1.5)

where \( \lambda \in \mathbb{R} \) and

\[
\chi_\lambda^\alpha(u) = \frac{\int_{\Omega} |\nabla u|^p \delta \, dx - \lambda \int_{\Omega} |u|^p \delta^{\alpha p} \, dx}{\int_{\Omega} |u|^{p(\alpha-1)p} \, dx}.
\]

(1.6)

Note that \( J_0^\alpha \) gives the best constant in (1.3). Clearly, the function \( \lambda \mapsto J_\lambda^\alpha \) is non-increasing on \( \mathbb{R} \) and \( J_\lambda^\alpha \to -\infty \) as \( \lambda \to \infty \).

**Remark 1.1.** It is worthy to remark that (1.3) is never valid in the critical case that \( \alpha \geq 1 - 1/p \). Nevertheless, we have established in this case a variant of weighted Hardy’s inequalities in [4] (cf. [10]). In a coming paper [3], we shall treat general weighted Hardy’s inequalities with compact perturbations and study relating variational problems including the critical case that \( \alpha \geq 1 - 1/p \).

This paper is organized in the following way: The main result is described in Section 2. Section 3 is devoted to the proof of main result.

## 2. Main results

Our main result is the following.

**Theorem 2.1.** Assume that \( \Omega \) is a bounded domain of class \( C^2 \) in \( \mathbb{R}^N \). Assume that \( 1 < p < \infty \) and \( \alpha < 1 - 1/p \). Then there exists a constant \( \lambda^\ast \in \mathbb{R} \) such that:

1. If \( \lambda \leq \lambda^\ast \), then \( J_\lambda^\alpha = \Lambda_{\alpha,p} \). If \( \lambda > \lambda^\ast \), then \( J_\lambda^\alpha < \Lambda_{\alpha,p} \). Here

\[
\Lambda_{\alpha,p} = \left( 1 - \alpha - \frac{1}{p} \right)^p.
\]

Moreover, it holds that:

...
2. If \( \lambda < \lambda^* \), then the infimum \( J_\lambda^\alpha \) in (1.5) is not attained.

3. If \( \lambda > \lambda^* \), then the infimum \( J_\lambda^\alpha \) in (1.5) is attained.

**Remark 2.1.**

1. In Theorem 2.1, it remains for \( \lambda = \lambda^* \) of the open problem whether the infimum \( J_\lambda^\alpha \) in (1.5) is attained or not.

2. For the case of \( \alpha = 0 \) and \( p = 2 \), it is shown that the infimum \( J_\lambda^0 \) in (1.5) is attained if and only if \( \lambda > \lambda^* \). See [6].

3. For the case of \( \alpha = 0 \) and \( \lambda = 0 \), the value of the infimum \( J_0^0 \) in (1.5) and its attainability are studied in [13].

4. In the assertion 3 of Theorem 2.1, if \( \lambda > \lambda^* \), then the minimizer \( u \) for the variational problem (1.5) is a non-trivial weak solution of the following Euler-Lagrange equation:

\[
-\text{div}(\delta^{\alpha p}|\nabla u|^{p-2}\nabla u) - \lambda \delta^{\alpha p}|u|^{p-2}u = J_\lambda^\alpha \delta^{(\alpha-1)p}|u|^{p-2}u \quad \text{in} \quad \mathcal{D}'(\Omega).
\]

**Corollary 2.1.** Under the same assumptions as in Theorem 2.1, there exists a constant \( \lambda \in \mathbb{R} \) such that for \( u \in W^{1,p}_{\alpha,0}(\Omega) \)

\[
\int_\Omega |\nabla u|^{p\delta^{\alpha p}}dx \geq \Lambda_{\alpha,p} \int_\Omega |u|^{p\delta^{(\alpha-1)p}}dx + \lambda \int_\Omega |u|^{p\delta^{\alpha p}}dx.
\] (2.2)

For each small \( \eta > 0 \), by \( \Omega_\eta \) we denote a tubular neighborhood of \( \partial \Omega \):

\[
\Omega_\eta = \{ x \in \Omega : \delta(x) = \text{dist}(x, \partial \Omega) < \eta \}.
\] (2.3)

Then we have the following inequality of Hardy type which is crucial in the proof of Theorem 2.1.

**Theorem 2.2.** Assume that \( \Omega \) is a bounded domain of class \( C^2 \) in \( \mathbb{R}^N \). Assume that \( 1 < p < \infty \) and \( \alpha < 1 - 1/p \). Assume that \( \eta \) is a sufficiently small positive number. Then we have that for \( u \in W^{1,p}_{\alpha,0}(\Omega) \)

\[
\int_{\Omega_\eta} |\nabla u|^{p\delta^{\alpha p}}dx \geq \Lambda_{\alpha,p} \int_{\Omega_\eta} |u|^{p\delta^{(\alpha-1)p}}dx,
\] (2.4)

where \( \Lambda_{\alpha,p} \) is defined by (2.1).

In [4] we have more precise estimate than (2.4).

**Corollary 2.2.** Under the same assumptions as in Theorem 2.2, there exists a positive constant \( \gamma \) such that for \( u \in W^{1,p}_{\alpha,0}(\Omega) \)

\[
\int_\Omega |\nabla u|^{p\delta^{\alpha p}}dx \geq \gamma \int_\Omega |u|^{p\delta^{(\alpha-1)p}}dx.
\] (2.5)
For any bounded domain $\Omega \subset \mathbb{R}^N$ we can prove the following:

**Theorem 2.3.** Assume that $\Omega$ is a bounded domain of $\mathbb{R}^N$. Assume that $1 < p < \infty$ and $\alpha < 1 - 1/p$. Then the followings are equivalent with each other.

1. There exists a positive number $\gamma$ such that the inequality (2.5) is valid for every $u \in W^{1,p}_0(\Omega)$.

2. For a sufficiently small positive number $\eta$, there exists a positive number $\kappa$ such that the inequality (2.4) with $\Lambda_{\alpha,p}$ replaced by $\kappa$ is valid for every $u \in W^{1,p}_0(\Omega)$.

For the proofs of Theorem 2.2, Corollary 2.2 and Theorem 2.3, see in [4].

3. **Proof of Theorem 2.1**
   
   In this section, we give the proof of Theorem 2.1.

3.1. **Upper bound of $J^\alpha_\lambda$**

   First, we prove the assertion 1 of Theorem 2.1.

   **Lemma 3.1.** Let $1 < p < \infty$ and $\alpha < 1 - 1/p$. For any $\varepsilon > 0$ and any $\eta > 0$ there exists a function $h \in W^{1,p}_{\alpha,0}(0, \eta)$ such that
   
   $$
   \int_0^\eta |h'(t)|^p t^\alpha dt \leq (\Lambda_{\alpha,p} + \varepsilon) \int_0^\eta |h(t)|^p t^{(\alpha - 1)p} dt,
   $$
   (3.1)
   
   where $\Lambda_{\alpha,p}$ is defined by (2.1).

   **Proof.** Since the inequality (3.1) is invariant with respect to scaling, we may assume that $\eta = 2$. Put
   
   $$
   h(t) = \begin{cases} 
   t^\beta & \text{if } t \in (0,1), \\
   2 - t & \text{if } t \in [1,2)
   \end{cases}
   $$
   
   with $\beta > 1 - \alpha - 1/p$. Then we see that $h \in W^{1,p}_{\alpha,0}(0, 2)$,
   
   $$
   \int_0^2 |h'(t)|^p t^\alpha dt = \frac{\beta^p}{p(\beta - 1 + \alpha + 1/p)} + C_{\alpha,p}
   $$
   (3.2)
   
   and
   
   $$
   \int_0^2 |h(t)|^p t^{(\alpha - 1)p} dt = \frac{1}{p(\beta - 1 + \alpha + 1/p)} + D_{\alpha,p},
   $$
   (3.3)
   
   where
   
   $$
   C_{\alpha,p} = \int_1^2 t^\alpha dt \quad \text{and} \quad D_{\alpha,p} = \int_1^2 (2 - t)^{\alpha - 1/p} dt
   $$
   
   are constants independent of $\beta$. It follows from (3.2) and (3.3) that
   
   $$
   \frac{\int_0^2 |h'(t)|^p t^\alpha dt}{\int_0^2 |h(t)|^p t^{(\alpha - 1)p} dt} \rightarrow \Lambda_{\alpha,p} \quad \text{as} \quad \beta \rightarrow 1 - \alpha - \frac{1}{p} + 0,
   $$
   
   which implies (3.1) with $\eta = 2$. Therefore we obtain the desired conclusion. \qed
Lemma 3.2. Let $\Omega$ be a bounded domain of class $C^2$ in $\mathbb{R}^N$. Let $1 < p < \infty$ and $\alpha < 1 - 1/p$. Then it holds that
\[
J_\lambda^\alpha \leq \Lambda_{\alpha, p}
\] (3.4)
for all $\lambda \in \mathbb{R}$.

Proof. Since the boundary $\partial \Omega$ is of class $C^2$, there exists an $\eta_0 > 0$ such that for any $\eta \in (0, \eta_0)$ and every $x \in \Omega_\eta$ we have a unique point $\sigma(x) \in \partial \Omega$ satisfying $\delta(x) = |x - \sigma(x)|$. The mapping
\[
\Omega_\eta \ni x \mapsto (\delta(x), \sigma(x)) = (t, \sigma) \in (0, \eta) \times \partial \Omega
\]
is a $C^2$ diffeomorphism, and its inverse is given by
\[
(0, \eta) \times \partial \Omega \ni (t, \sigma) \mapsto x(t, \sigma) = \sigma + t \cdot n(\sigma) \in \Omega_\eta,
\]
where $n(\sigma)$ is the inward unit normal to $\partial \Omega$ at $\sigma \in \partial \Omega$. For each $t \in (0, \eta)$, the mapping
\[
\partial \Omega \ni \sigma \mapsto \sigma_t(\sigma) = x(t, \sigma) \in \Sigma_t = \{x \in \Omega : \delta(x) = t\}
\]
is also a $C^2$ diffeomorphism of $\partial \Omega$ onto $\Sigma_t$, and its Jacobian satisfies
\[
|\text{Jac} \sigma_t(\sigma) - 1| \leq c t
\] (3.5)
for any $\sigma \in \partial \Omega$, where $c$ is a positive constant depending only on $\eta_0$, $\partial \Omega$ and the choice of local coordinates. Since $n(\sigma)$ is orthogonal to $\Sigma_t$ at $\sigma_t(\sigma) = \sigma + t \cdot n(\sigma)$, it follows that for every integrable function $v$ in $\Omega_\eta$
\[
\int_{\Omega_\eta} v(x)dx = \int_0^\eta dt \int_{\Sigma_t} v(\sigma_t)d\sigma_t
= \int_0^\eta dt \int_{\partial \Omega} v(x(t, \sigma))|\text{Jac} \sigma_t(\sigma)|d\sigma,
\] (3.6)
where $d\sigma$ and $d\sigma_t$ denote surface elements on $\partial \Omega$ and $\Sigma_t$, respectively. Hence (3.6) together with (3.5) implies that for every integrable function $v$ in $\Omega_\eta$
\[
\int_0^\eta (1 - ct)dt \int_{\partial \Omega} |v(x(t, \sigma))|d\sigma \leq \int_{\Omega_\eta} |v(x)|dx
\] (3.7)
\[
\leq \int_0^\eta (1 + ct)dt \int_{\partial \Omega} |v(x(t, \sigma))|d\sigma.
\] (3.8)

Let $\epsilon > 0$, and let $\eta \in (0, \eta_0)$. Take $h \in W^{1,p}_{\alpha, \Omega}((0, \eta))$ be a function satisfying (3.1). Put
\[
u(x) = \begin{cases} 
 h(\delta(x)) & \text{if } x \in \Omega_\eta, \\
 0 & \text{if } x \in \Omega \setminus \Omega_\eta. 
\end{cases}
\] (3.9)
Since $|\nabla \nu(x)| = |h'(\delta(x))|$ for $x \in \Omega_\eta$ by $|\nabla \delta(x)| = 1$, it follows from (3.8) that
\[
\int_{\Omega_\eta} |\nabla \nu|^p \delta^{\alpha p}dx \leq (1 + c\eta)|\partial \Omega| \int_0^\eta |h'(t)|^p t^{\alpha p}dt,
\] (3.10)
which implies $u \in W^{1,p}_{\alpha,\delta}(\Omega)$ by $\{u \neq 0\} \subset \Omega_\eta$. On the other hand, by (3.7) and (3.9) we have that
\[
\int_{\Omega_\eta} |u|^p \delta^{(\alpha-1)p} dx \geq (1 - c\eta) |\partial \Omega| \int_0^\eta |h(t)|^p \delta^{(\alpha-1)p} dt. \tag{3.11}
\]
Since $\{u \neq 0\} \subset \Omega_\eta$, by combining (3.10), (3.11) and the estimate
\[
\int_{\Omega_\eta} |u|^p \delta^{\alpha p} dx \leq \eta^p \int_{\Omega_\eta} |u|^p \delta^{(\alpha-1)p} dx,
\]
we obtain that
\[
\chi_\lambda^\alpha(u) \leq 1 + c\eta \int_0^\eta |h'(t)|^p \delta^{\alpha p} dt + \lambda |\eta|^p.
\]
This together with (3.1) implies that
\[
J_\lambda^\alpha \leq 1 + c\eta \int_0^\eta |h'(t)|^p \delta^{\alpha p} dt + |\lambda| |\eta|^p. \tag{3.12}
\]
Letting $\eta \to +0$ and $\varepsilon \to +0$ in (3.12), (3.4) follows. Therefore it concludes the proof.

Lemma 3.3. Let $\Omega$ be a bounded domain of class $C^2$ in $\mathbb{R}^N$. Let $1 < p < \infty$ and $\alpha < 1 - 1/p$. Then there exists a $\lambda \in \mathbb{R}$ such that $J_\lambda^\alpha = \Lambda_{\alpha, p}$.

Proof. Let $\eta > 0$ be a sufficiently small number as in Theorem 2.2. For any $u \in W^{1,p}_{\alpha,\delta}(\Omega) \setminus \{0\}$, by using Hardy’s inequality (2.4) and the estimate
\[
\int_{\Omega \setminus \Omega_\eta} |u|^p \delta^{(\alpha-1)p} dx \leq \eta^{-p} \int_{\Omega} |u|^p \delta^{\alpha p} dx,
\]
we have that
\[
\Lambda_{\alpha, p} \int_{\Omega} |u|^p \delta^{(\alpha-1)p} dx = \Lambda_{\alpha, p} \int_{\Omega_\eta} |u|^p \delta^{(\alpha-1)p} dx + \Lambda_{\alpha, p} \int_{\Omega \setminus \Omega_\eta} |u|^p \delta^{(\alpha-1)p} dx \leq \int_{\Omega} |\nabla u|^p \delta^{\alpha p} dx + \Lambda_{\alpha, p} \eta^{-p} \int_{\Omega} |u|^p \delta^{\alpha p} dx,
\]
which implies that
\[
\chi_\lambda^\alpha(u) \geq \Lambda_{\alpha, p}
\]
for $\lambda \leq -\Lambda_{\alpha, p} \eta^{-p}$. Consequently, it holds that $J_\lambda^\alpha \geq \Lambda_{\alpha, p}$ for $\lambda \leq -\Lambda_{\alpha, p} \eta^{-p}$. This together with (3.4) implies the desired conclusion.

Lemma 3.4. Let $\Omega$ be a bounded domain of class $C^2$ in $\mathbb{R}^N$. Let $1 < p < \infty$ and $\alpha < 1 - 1/p$. Then the function $\lambda \mapsto J_\lambda^\alpha$ is Lipschitz continuous on $\mathbb{R}$.
Proof. Let $\lambda, \bar{\lambda} \in \mathbb{R}$. Then it holds that for any $u \in W^{1,p}_{\alpha,0}(\Omega) \setminus \{0\}$

$$|\chi_\lambda^\alpha(u) - \chi_{\bar{\lambda}}^\alpha(u)| = |\lambda - \bar{\lambda}| \frac{\int_{\Omega} |u|^p \delta^{\alpha p} dx}{\int_{\Omega} |u|^p \delta^{(\alpha - 1)p} dx} \leq M^p |\lambda - \bar{\lambda}|,$$

where $M = \sup_{x \in \Omega} \delta(x)$ is a positive constant depending only on $\Omega$. Hence we see that

$$|J_\lambda^\alpha - J_{\bar{\lambda}}^\alpha| \leq M^p |\lambda - \bar{\lambda}|$$

for $\lambda, \bar{\lambda} \in \mathbb{R}$. It completes the proof. \(\square\)

**Proof of the assertion 1 of Theorem 2.1.** By Lemma 3.3 and $\lim_{\lambda \to \infty} J_\lambda^\alpha = -\infty$, the set $\{ \lambda \in \mathbb{R} : J_\lambda^\alpha = \Lambda_{\alpha,p} \}$ is non-empty and upper bounded. Hence the sup$\{ \lambda \in \mathbb{R} : J_\lambda^\alpha = \Lambda_{\alpha,p} \}$ exists finitely. Put

$$\lambda^* = \sup \{ \lambda \in \mathbb{R} : J_\lambda^\alpha = \Lambda_{\alpha,p} \}. \quad (3.13)$$

Since the function $\lambda \mapsto J_\lambda^\alpha$ is non-increasing on $\mathbb{R}$, it follows from Lemma 3.2 and Lemma 3.3 that $J_\lambda^\alpha = \Lambda_{\alpha,p}$ for $\lambda < \lambda^*$, and $J_\lambda^\alpha < \Lambda_{\alpha,p}$ for $\lambda > \lambda^*$. Further, by Lemma 3.4 we have the equality $J_{\lambda^*}^\alpha = \Lambda_{\alpha,p}$. Therefore the assertion 1 of Theorem 2.1 is valid. \(\square\)

3.2. $J_\lambda^\alpha$ is not attained when $\lambda < \lambda^*$

Next, we prove the assertion 2 of Theorem 2.1.

**Proof of the assertion 2 of Theorem 2.1.** Suppose that for some $\lambda < \lambda^*$ the infimum $J_\lambda^\alpha$ in (1.5) is attained at an element $u \in W^{1,p}_{\alpha,0}(\Omega) \setminus \{0\}$. Then, by the assertion 1 of Theorem 2.1, we have that

$$\chi_\lambda^\alpha(u) = J_\lambda^\alpha = \Lambda_{\alpha,p} \quad (3.14)$$

and for $\lambda < \tilde{\lambda} < \lambda^*$

$$\chi_{\tilde{\lambda}}^\alpha(u) \geq J_{\tilde{\lambda}}^\alpha = \Lambda_{\alpha,p}. \quad (3.15)$$

From (3.14) and (3.15) it follows that

$$(\tilde{\lambda} - \lambda) \int_{\Omega} |u|^p \delta^{\alpha p} dx \leq 0.$$ 

Since $\tilde{\lambda} - \lambda > 0$, we conclude that

$$\int_{\Omega} |u|^p \delta^{\alpha p} dx = 0,$$

which contradicts $u \neq 0$ in $W^{1,p}_{\alpha,0}(\Omega)$. Therefore it completes the proof. \(\square\)

3.3. Attainability of $J_\lambda^\alpha$ when $\lambda > \lambda^*$

At last, we prove the assertion 3 of Theorem 2.1.
Let \( \{u_k\} \) be a minimizing sequence for the variational problem (1.5) normalized so that
\[
\int_\Omega |u_k|^{p\delta(\alpha-1)p} dx = 1 \quad \text{for all } k.
\]
(3.16)

Since \( \{u_k\} \) is bounded in \( W^{1,p}_{\alpha,\delta}(\Omega) \), by taking a suitable subsequence, we may assume that there exists a \( u \in W^{1,p}_{\alpha,\delta}(\Omega) \) such that
\[
\nabla u_k \rightharpoonup \nabla u \quad \text{in } (L^p(\Omega, \delta^{\alpha p}))^N,
\]
(3.17)
and
\[
u_k \rightharpoonup u \quad \text{in } L^p(\Omega, \delta((\alpha-1)p))
\]
(3.18)

and
\[
u_k \to u \quad \text{in } L^p_{\text{loc}}(\Omega)
\]
(3.19)
by Hardy’s inequality (2.5) and the compact embedding \( W^{1,p}_{\alpha,\delta}(\Omega) \hookrightarrow L^p(\Omega, \delta^{\alpha p}) \).

Under these preparation we establish the properties of concentration and compactness for the minimizing sequence, respectively.

**Proposition 3.1.** Let \( \Omega \) be a bounded domain of class \( C^2 \) in \( \mathbb{R}^N \). Let \( 1 < p < \infty \) and \( \alpha < 1 - 1/p \). Let \( \lambda \in \mathbb{R} \). Let \( \{u_k\} \) be a minimizing sequence for (1.5) satisfying (3.16), (3.17), (3.18) and (3.19) with \( u = 0 \). Then it holds that
\[
\nabla u_k \to 0 \quad \text{in } (L^p_{\text{loc}}(\Omega))^N
\]
(3.20)
and
\[
J_\lambda = \Lambda_{\alpha,p}.
\]
(3.21)

**Proof.** Let \( \eta > 0 \) be a sufficiently small number as in Theorem 2.2. By Hardy’s inequality (2.4) and (3.16) we have that
\[
\int_{\Omega_\eta} |\nabla u_k|^{p\delta\alpha p} dx \geq \Lambda_{\alpha,p} \int_{\Omega_\eta} |u_k|^{p\delta(\alpha-1)p} dx
\]
\[
= \Lambda_{\alpha,p} \left( 1 - \int_{\Omega \setminus \Omega_\eta} |u_k|^{p\delta(\alpha-1)p} dx \right),
\]
and so
\[
\chi_\lambda(u_k) \geq \Lambda_{\alpha,p} \left( 1 - \int_{\Omega \setminus \Omega_\eta} |u_k|^{p\delta(\alpha-1)p} dx \right)
\]
\[
+ \int_{\Omega \setminus \Omega_\eta} |\nabla u_k|^{p\delta\alpha p} dx - \lambda \int_{\Omega} |u_k|^{p\delta\alpha p} dx.
\]
(3.22)

Since
\[
\int_{\Omega \setminus \Omega_\eta} |u_k|^{p\delta(\alpha-1)p} dx \leq \eta^{-p} \int_{\Omega} |u_k|^{p\delta\alpha p} dx,
\]
\[
\int_{\Omega_\eta \setminus \Omega} |u_k|^{p\delta(\alpha-1)p} dx = \int_{\Omega_\eta} |u_k|^{p\delta(\alpha-1)p} dx.
\]
it follows from (3.19) with \( u = 0 \) that
\[
\lim_{k \to \infty} \int_{\Omega \setminus \Omega_n} |u_k|^p \delta^{(\alpha-1)p} dx = 0. \tag{3.23}
\]
Hence, by (3.22), (3.23) and (3.19) with \( u = 0 \), we obtain that
\[
0 \leq \limsup_{k \to \infty} \int_{\Omega \setminus \Omega_n} |\nabla u_k|^p \delta^{\alpha p} dx \leq J_\lambda^\alpha - \Lambda_{\alpha, p}.
\]
Since \( J_\lambda^\alpha - \Lambda_{\alpha, p} \leq 0 \) by Lemma 3.2, we conclude that (3.21) and
\[
\lim_{k \to \infty} \int_{\Omega \setminus \Omega_n} |\nabla u_k|^p \delta^{\alpha p} dx = 0. \tag{3.24}
\]
By the estimate
\[
\int_{\Omega \setminus \Omega_n} |\nabla u_k|^p \delta^{\alpha p} dx \geq \min\{\eta^{\alpha p}, M^{\alpha p}\} \int_{\Omega \setminus \Omega_n} |\nabla u_k|^p dx
\]
with \( M = \sup_{x \in \Omega} \delta(x) \), it follows from (3.24) that
\[
\lim_{k \to \infty} \int_{\Omega \setminus \Omega_n} |\nabla u_k|^p dx = 0,
\]
which shows (3.20). Consequently it completes the proof. \( \square \)

**Proposition 3.2.** Let \( \Omega \) be a bounded domain of class \( C^2 \) in \( \mathbb{R}^N \). Let \( 1 < p < \infty \) and \( \alpha < 1 - 1/p \). Let \( \lambda \in \mathbb{R} \). Let \( \{u_k\} \) be a minimizing sequence for (1.5) satisfying (3.16), (3.17), (3.18) and (3.19) with \( u \neq 0 \). Then it holds that
\[
J_\lambda^\alpha = \min\{\Lambda_{\alpha, p}, \chi_\lambda^\alpha(u)\}. \tag{3.25}
\]
In addition, if \( J_\lambda^\alpha < \Lambda_{\alpha, p} \), then it holds that
\[
J_\lambda^\alpha = \chi_\lambda^\alpha(u), \tag{3.26}
\]
namely \( u \) is a minimizer for (1.5), and
\[
\text{lim}_{k \to \infty} u_k \to u \quad \text{in} \quad W^{1,p}_{0, \delta}(\Omega). \tag{3.27}
\]
**Proof.** Let \( \eta > 0 \) be a sufficiently small number as in Theorem 2.2. Then we have (3.22) by the same arguments as in the proof of Proposition 3.1. By the estimate
\[
\int_{\Omega \setminus \Omega_n} |u_k - u|^p \delta^{(\alpha-1)p} dx \leq \eta^{-p} \int_{\Omega} |u_k - u|^p \delta^{\alpha p} dx,
\]
(3.19) implies that
\[
\lim_{k \to \infty} \int_{\Omega \setminus \Omega_n} |u_k|^p \delta^{(\alpha-1)p} dx = \int_{\Omega \setminus \Omega_n} |u|^p \delta^{(\alpha-1)p} dx. \tag{3.28}
\]
Since it follows from (3.17) that \( \nabla u_k \to \nabla u \) weakly in \( (L^p(\Omega \setminus \Omega_\eta))^N \), by weakly lower semi-continuity of the \( L^p \)-norm, we see that

\[
\liminf_{k \to \infty} \int_{\Omega \setminus \Omega_\eta} |\nabla u_k|^p \delta^{o_p} dx \geq \left( \liminf_{k \to \infty} \|\nabla u_k\|_{L^p(\Omega \setminus \Omega_\eta, \delta^{o_p})} \right)^p \\
\geq \|\nabla u\|_{L^p(\Omega \setminus \Omega_\eta, \delta^{o_p})}^p \\
= \int_{\Omega \setminus \Omega_\eta} |\nabla u|^p \delta^{o_p} dx.
\]

(3.29)

Hence, by letting \( k \to \infty \) in (3.22), from (3.19), (3.28) and (3.29) it follows that

\[
J_\lambda^\alpha \geq \Lambda_{\alpha,p} \left( 1 - \int_{\Omega \setminus \Omega_\eta} |u|^p \delta^{(1-1)p} dx \right) \\
+ \int_{\Omega \setminus \Omega_\eta} |\nabla u|^p \delta^{o_p} dx - \lambda \int_{\Omega} |u|^p \delta^{o_p} dx.
\]

(3.30)

Letting \( \eta \to +0 \) in (3.30), we obtain that

\[
J_\lambda^\alpha \geq \Lambda_{\alpha,p} \left( 1 - \int_{\Omega} |u|^p \delta^{(1-1)p} dx \right) \\
+ \int_{\Omega} |\nabla u|^p \delta^{o_p} dx - \lambda \int_{\Omega} |u|^p \delta^{o_p} dx.
\]

(3.31)

Since it holds that

\[
0 < \int_{\Omega} |u|^p \delta^{(1-1)p} dx \leq \liminf_{k \to \infty} \int_{\Omega} |u_k|^p \delta^{(1-1)p} dx = 1
\]

(3.32)

by \( u \neq 0 \), (3.16), (3.18) and weakly lower semi-continuity of the \( L^p \)-norm, we have from (3.31) and (3.32) that

\[
J_\lambda^\alpha \geq \Lambda_{\alpha,p} \left( 1 - \int_{\Omega} |u|^p \delta^{(1-1)p} dx \right) + \chi_\lambda^\alpha(u) \int_{\Omega} |u|^p \delta^{(1-1)p} dx \\
\geq \min\{\Lambda_{\alpha,p}, \chi_\lambda^\alpha(u)\}.
\]

(3.33)

This together with Lemma 3.2 implies (3.25). Moreover, by (3.25) and (3.33), we conclude that

\[
J_\lambda^\alpha = \Lambda_{\alpha,p} \left( 1 - \int_{\Omega} |u|^p \delta^{(1-1)p} dx \right) + \chi_\lambda^\alpha(u) \int_{\Omega} |u|^p \delta^{(1-1)p} dx.
\]

(3.34)

In addition, if \( J_\lambda^\alpha < \Lambda_{\alpha,p} \), then we have (3.26) by (3.25), and so, it follows from (3.34) and (3.16) that

\[
\int_{\Omega} |u|^p \delta^{(1-1)p} dx = 1 = \lim_{k \to \infty} \int_{\Omega} |u_k|^p \delta^{(1-1)p} dx.
\]

(3.35)
(3.18) and (3.35) imply that
\[ u_k \rightarrow u \quad \text{in} \quad L^p(\Omega, \delta^{(\alpha-1)p}). \]  
Further, by (3.16), (3.19), (3.26) and (3.35), we obtain that
\[ \int_\Omega |\nabla u_k|^p \delta^{\alpha p} \, dx = \chi_\lambda^\alpha(u_k) + \lambda \int_\Omega |u_k|^p \delta^{\alpha p} \, dx \]
\[ \rightarrow \chi_\lambda^\alpha(u) + \lambda \int_\Omega |u|^p \delta^{\alpha p} \, dx = \int_\Omega |\nabla u|^p \delta^{\alpha p} \, dx. \]
This together with (3.17) implies that
\[ \nabla u_k \rightarrow \nabla u \quad \text{in} \quad (L^p(\Omega, \delta^{\alpha p}))^N, \]  
which shows (3.27). Consequently it completes the proof.

Proof of the assertion 3 of Theorem 2.1. Let \( \lambda > \lambda^* \). Then \( J_\lambda^\alpha < \Lambda_{\alpha,p} \) by the assertion 1 of Theorem 2.1. Let \( \{ u_k \} \) be a minimizing sequence for (1.5) satisfying (3.16), (3.17), (3.18) and (3.19). Then we see that \( u \neq 0 \) by Proposition 3.1. Therefore, by applying Proposition 3.2, we conclude that \( \chi_\lambda^\alpha(u) = J_\lambda^\alpha \), namely \( u \) is a minimizer for (1.5). It finishes the proof.

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