NO ELLIPTIC POINTS FROM FIXED PRIME ENDS

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Abstract. We consider area preserving maps of surfaces and extend Mather’s result on the equality of the closure of the four branches of saddles. He assumed elliptic fixed points to be Moser stable, while we require only that the derivative at this points to be a rotation by an angle different from zero. There are many results in the literature which require the hypothesis that elliptic periodic points be Moser stable that now can be extended to the case that the derivative at these points be an irrational rotation. The key point is to give more information on Cartwright and Littlewood’s fixed point theorem, to show that the fixed point obtained by a fixed prime end can not be elliptic. Hypotheses then became easier to verify: non degeneracy of fixed points and nonexistence of saddle connections. As an application we show that the result immediately implies that for the standard map family, for all values of the parameter, except one, the principal hyperbolic fixed point has homoclinic points. We also extend results to surfaces with boundary in order to be applicable to return maps to surfaces of section and broken book decompositions.

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1. Introduction.

Let $S$ be a surface provided with a finite Borel measure $\mu$ which is positive on non-empty open subsets. Let $f : S \to S$ be a homeomorphism. We say that $f$ is area preserving if $f_* \mu = \mu$.

Let $p$ be a fixed point of $f$. We say that $p$ is of saddle type if in a neighborhood $V$ of $p$ there exist continuous coordinates with $p$ at the origin and in which $f(x, y) = (\lambda x, \lambda^{-1} y)$ with $|\lambda| > 1$. The stable and unstable invariant manifolds of $p$ are defined as:

$$W^s_p = \{ x \in S : \lim_{n \to +\infty} f^n x = p \}$$
$$W^u_p = \{ x \in S : \lim_{n \to -\infty} f^n x = p \}$$

respectively.

The sets $W^s_p$ and $W^u_p$ are injectively immersed connected curves. The components of $W^s_p \cap V$ and $W^u_p \cap V$ which contain $p$ are called the local invariant manifolds of $p$ (with respect to $V$). The local branches of $p$ are the components of the complement of $p$ in the local invariant manifolds.

The set $\{(x, y) \in V \mid x \neq 0 \text{ and } y \neq 0\}$ has four connected components that contain $p$ in their closures. We call them sectors of $p$. If $\Sigma$ is one of these sectors and $\Sigma'$ is a sector of $p$ defined by means of another neighborhood $V'$ of $p$ then either $\Sigma \cap \Sigma' = \emptyset$ or $\Sigma$ and $\Sigma'$ define the same germ at $p$. We say that the set $A$ contains a sector $\Sigma$ if $A$ contains a set $\Sigma'$ germ equivalent to $\Sigma$ at $p$. We would like to say that a set $B$ accumulates on $p$ with points coming through a sector. We say that a set $B$ accumulates on a sector $\Sigma$ of $p$ if the closure of $B \cap \Sigma$ contains $p$. These definitions do not depend on the choice of $V$ neither on the choice of the linear map $(x, y) \mapsto (\lambda x, \lambda^{-1} y)$. We could have taken for the coordinates any pair of expanding and contracting homeomorphisms of $\mathbb{R}$ which fix 0. We say that a branch $L$ and a sector $\Sigma$ are adjacent if a local branch of $L$ is contained in the closure of $\Sigma$ in $S$. Two branches are adjacent if they are adjacent to a single sector.

By a connection we mean a branch which is contained in the intersection of two invariant manifolds (possibly of two different fixed points).

Let $L$ be an invariant branch and let $x \in L$. Denote by $D$ the closed arc from $x$ to $f(x)$ inside $L$. We call the set

$$\omega(L) := \{ y \in S \mid y = \lim_{n \to \infty} f^{n_i}(x_i) \text{ where } n_i \searrow \infty \text{ and } x_i \in D \}$$
the limit set of $L$. The limit set $\omega(L)$ is non empty, connected, compact, invariant and the closure of $L$ in $S$ is the union of $L$ and $\omega(L)$.

We say that a fixed point $p$ is elliptic if in a neighborhood $V$ of $p$ there exist continuous coordinates in which $f$ is differentiable at $p$ and $df_p$ is a rotation by an angle different from zero. If $p$ is of saddle type or elliptic we say that it is non degenerate. We say that $p$ is irrationally elliptic if $df_p$ is a rotation by an angle $\theta$ where $\frac{\theta}{2\pi}$ is irrational.

The fixed point $p$ is Moser stable if there is a fundamental system of neighborhoods of $p$ made of disks whose frontiers are minimal invariant sets.

If $p$ is a periodic point and $\tau = \inf\{ n \geq 1 \mid f^n(p) = p \}$ is its period, we use $f^\tau$ to define these concepts for $p$.

Now we would like to state a result of Mather [6]:

1.1. Theorem (Mather).

Let $S$ be a compact connected orientable surface provided with a finite measure $\mu$ which is positive on open sets. Let $f : S \to S$ be an orientation preserving, area preserving homeomorphism of $S$. Assume that all fixed points of $f$ are of saddle type or Moser stable and that $f$ has no invariant connections.

If $p$ is a fixed point of saddle type whose branches are invariant sets, then the four branches of $p$ have the same closure.

We would like to remark that the above result is false without the assumption that the branches of the fixed point be invariant (see the example after theorem 5.2 in [6]). The proof is based on his extension of the classical theory of prime ends to open connected subsets $U$ of $S$ with finitely many ideal boundary points (or topological ends). This is done by adding to $U$ a circle of prime ends corresponding to each non trivial ideal boundary point. Let $\tilde{U}$ be the prime ends compactification of $U$. Then $\tilde{U}$ is a compact surface and if $U$ is invariant then the restriction of $f$ to $U$ extends to a homeomorphism $\tilde{f} : \tilde{U} \to \tilde{U}$. If $S$ is orientable and $f$ is orientation preserving then so is $\tilde{f}$.

Cartwright and Littlewood [2] proved that if there exists a prime end $e$ fixed by $\tilde{f}$, then every point in the principal set of $e$ is fixed by $f$.

If $A \subset B$, we use the notation $\text{int}_B(A)$, $\text{cl}_B(A)$ and $\text{fr}_B(A)$ for the interior, closure and the frontier of $A$ in $B$, respectively. The boundary of a manifold $M$ will be denoted by $\partial M$. By a domain we mean a connected open subset of $S$.

Our result is the following:

1.2. Theorem.

Let $S$ be a compact connected orientable surface provided with a finite measure $\mu$ which is positive on open sets and let $f : S \to S$ be an orientation preserving, area preserving homeomorphism of $S$.

(1) Suppose that $U$ is an invariant domain of $S$ with finitely many ideal boundary points. Assume that all fixed points of $f$ in $\text{fr}_S U$ are non degenerate. If $e$ is a fixed regular prime end and $p$ is a principal point of $e$ (which is fixed by Cartwright-Littlewood’s theorem), then $p$ can not be elliptic. Furthermore, one of the branches of $p$ is an invariant connection contained in $\text{fr}_S U$. 
(2) Suppose that \( L \) is an invariant branch of \( f \) and that all fixed points of \( f \) contained in \( \text{cl}_S L \) are non degenerate. Then either \( L \) is a connection or \( L \) accumulates on both adjacent branches through the adjacent sectors. In the latter alternative \( L \subset \omega(L) \).

(3) Let \( p \) be a fixed point of \( f \) of saddle type and let \( L_1 \) and \( L_2 \) be adjacent branches of \( p \) that are not connections. If \( L_1 \) and \( L_2 \) are invariant and all fixed points of \( f \) contained in \( \text{cl}_S (L_1 \cup L_2) \) are non degenerate, then \( \text{cl}_S L_1 = \text{cl}_S L_2 \).

(4) Let \( p \) be a fixed point of \( f \) of saddle type. Assume that all fixed points of \( f \) contained in \( \text{cl}_S (W^u_p \cup W^s_p) \) are non degenerate. If the branches of \( p \) are invariant and none of them is a connection, then all the branches of \( p \) have the same closure in \( S \).

Allowing the existence of some degenerate fixed points is useful in applications. Item (2) gives a strong dichotomy between recurrent and non recurrent behaviour of \( L \).

As we shall see, each boundary component of \( \hat{U} \) is a circle periodic under \( \hat{f} \). Now consider the case when \( U \) is periodic of period \( n \). Let \( C \) be a boundary component of \( \hat{U} \) and \( k \) the smallest positive number such that \( f^{nk}(C) = C \). We define the rotation number of \( U \) at \( C \) as the rotation number of \( (\hat{f}^n)^k \) restricted to \( C \). We call these the boundary rotation numbers of \( U \).

Mather was interested in the differentiable setting.

Assume \( S \) and \( \mu \) to be smooth. For \( 1 \leq r \leq \infty \) let \( D^r_\mu(S) \) be the space of \( C^r \) area preserving diffeomorphisms \( f \) of \( S \) with the \( C^r \) topology.

For \( f \in D^1_\mu(S) \) consider the following properties:

(5) Every periodic point of \( f \) of period \( \tau \) is non degenerate (i.e. \( 1 \) is not an eigenvalue of \( d(f^\tau)_p \)).

(6) Every elliptic periodic point is irrationally elliptic.

(7) Every elliptic periodic point of \( f \) is Moser stable.

(8) \( f \) has no periodic connections.

It is well know that properties (5), (6) and (8) are generic for \( 1 \leq r \leq \infty \). Property (7) is generic for \( r \geq 16 \) (see remark just before Theorem 6.4 of [4]).

From Theorem 1.1 and standard category arguments, we have that if a diffeomorphism satisfies (5), (7) and (8), then the branches of every hyperbolic periodic point have the same closure. This holds generically when \( r \geq 16 \). This was Mather’s original genericity result. (He also proved that generically the boundary rotation numbers of every periodic open connected subset with finitely many ideal boundary points are irrational).

From Theorem 1.2 we have that if a diffeomorphism satisfies (5), (6) and (8), then the branches of every hyperbolic periodic point have the same closure and the rotation numbers of every periodic open connected subset with finitely many ideal boundary points are irrational. So we have the following:

### 1.3. Theorem.

Let \( S \) be a compact connected orientable smooth surface provided with a finite smooth measure \( \mu \). For \( 1 \leq r \leq \infty \) let \( D^r_\mu(S) \) be the space of \( C^r \) area preserving diffeomorphisms \( f \) of \( S \) with the \( C^r \) topology.
Let $R$ be the set of $f \in D^r_\mu(S)$ that satisfy (5), (6) and (8). Then $R$ is a residual subset of $D^r_\mu(S)$ and every $f \in R$ satisfies the following:

9. The branches of each hyperbolic periodic point have the same closure.
10. The boundary rotation numbers of every periodic domain with finitely many ideal boundary points are irrational.

For fixed points we have the following:

1.4. Theorem.

Let $S$ be a compact connected orientable smooth surface provided with a finite smooth measure.

Let $A$ be the set of $f \in D^r_\mu(S)$ such that every fixed point of $f$ is non degenerate and $f$ has no invariant connections. The set $A$ is $C^1$ open and $C^r$ dense for every $r$.

If $f \in A$ then the branches of each hyperbolic fixed point have the same closure.

In section 2 we present the facts about prime ends we need. In section 3 we prove Theorem 1.2. In section 4 we extend results to surfaces with boundary and present some results about the existence of homoclinic points. In section 5 we give applications to the standard map family.

In section 4 we prove the results about homoclinic points for partially defined area preserving maps of surfaces with boundary in order to be applicable to Poincaré maps of surfaces of section of Reeb flows and to holonomy maps of broken book decompositions. In [3] we use these results to prove that for a Kupka-Smale riemannian metric on a closed surface every hyperbolic geodesic has homoclinics in all its branches.

2. Prime ends.

In sections 2 and 3, $S$ will be a compact connected orientable surface without boundary. Now we are going to describe the theory of prime ends of Mather [6] and [7].

2.1. The ideal boundary.

We will be interested in the connected components of the complement of the closure of invariant manifolds and branches. If $K$ is a compact connected subset of $S$, a residual domain of $K$ is a component of $S - K$.

Let $U$ be a connected surface without boundary. We are going to describe a compactification of $U$ by the addition of its topological ends or boundary components. See [5], [10].

A boundary component of $U$ is a decreasing sequence $P_1 \supset P_2 \supset \cdots$ of open connected subsets of $U$ such that:

11. Every $P_n$ is not relatively compact.
12. Every $fr_U(P_n)$ is compact.
13. If $K$ is a compact subset of $U$, then there is $n_0$ such that $K \cap P_n = \emptyset$ for $n \geq n_0$.

Two boundary components $P_1 \supset P_2 \supset \cdots$ and $P'_1 \supset P'_2 \supset \cdots$ are equivalent if for every $n$ there is $m$ such that $P_m \subset P'_n$ and vice versa. An ideal boundary point is an
equivalence class of boundary components. The set of ideal boundary points \( b_1 U \) is called the ideal boundary of \( U \) and the disjoint union \( U^* = U \cup b_1 U \) is called the ideal completion of \( U \). We have that \( b_1 U = \emptyset \) if and only if \( U \) is compact. Let \( A \) be an open subset of \( U \) with \( f_{\partial U}(A) \) compact and let \( A' \) be the set of ideal boundary points whose representing boundary components \( (P_n) \) satisfy \( P_n \subset A \) for \( n \) larger than some \( n_0 \). The collection of sets \( A \cup A' \) where \( A \) is an open subset of \( U \) with \( f_{\partial U}(A) \) compact forms a basis for a topology of \( U^* \).

The space \( U^* \) is a compactification of \( U \) characterized by the facts that \( U^* \) is locally connected and \( b_1 U \) is totally disconnected and nonseparating on \( U \) (i.e. for any open connected subset \( V \) of \( U^* \), \( V - b_1 U \) is connected), (cf. [1, §36–37, ch.1]).

If \( U \) is a domain of \( S \) and \( b_1 U \) is finite, then \( U^* \) is a connected compact orientable surface without boundary (Proposition 2.1 in [6] or Prop. 3.12 in [9]). This happens for residual domains (Lemma 2.3 of [6]).

For \( b \in b_1 U \) let
\[
Z(b) := \bigcap_i \text{cl}_S(\mathcal{N}_i \cap U),
\]
where \( (\mathcal{N}_i) \) is a fundamental system of neighborhoods of \( b \) in \( U^* \). Then \( Z(b) \) is just the set of limit points in \( S \) of sequences in \( U \) that converge to \( b \) in \( U^* \). The set \( Z(b) \) is non-empty, connected, compact and \( f_{\partial S}(U) = \bigcup_{b \in b_1 U} Z(b) \). If \( Z(b) \) contains more than one point we say that \( b \) is a regular ideal boundary point of \( U \).

2.2. The prime ends compactification.

Let \( U \) be a domain of \( S \) such that \( b_1 U \) is finite. Now we describe another compactification of \( U \) by replacing each regular ideal boundary point by a circle.

A chain is a sequence \( V_1 \supset V_2 \supset \cdots \) of open connected subsets of \( U \) such that
\[
\begin{align*}
(14) & \ f_{\partial U}(V_i) \text{ non-empty and connected for every } i \geq 1, \text{ and} \\
(15) & \ \text{cl}_S(f_{\partial U}(V_i)) \cap \text{cl}_S(f_{\partial U}(V_j)) = \emptyset \text{ for } i \neq j.
\end{align*}
\]

A chain \( (W_j) \) divides \((V_i)\) if for every \( i \) there exists \( j \) such that \( W_j \subset V_i \). Two chains are equivalent of each one divides the other. A chain is prime if any chain which divides it is equivalent to it. A sufficient condition for a chain \((V_i)\) to be prime is that there exists \( p \in S \) such that \( f_{\partial U}(V_i) \to p \), that is, every neighborhood of \( p \) contains all but finitely many of the sets \( f_{\partial U}(V_i) \).

A prime point is an equivalence class of prime chains. Let \( x \in U \) and consider a family of closed disks \( D_1 \supset D_2 \supset \cdots \) in \( U \), such that \( D_{i+1} \subset \text{int}_U(D_i) \) and \( \cap_i D_i = \{x\} \). Then \( (\text{int}_U(D_i)) \) is a prime chain that defines a prime point denoted by \( \omega(x) \). A prime end is a prime point which is not of the form \( \omega(x) \) for any \( x \in U \).

The set of prime points of \( U \) is denoted by \( \hat{U} \). We consider a topology on \( \hat{U} \) defined as follows. Let \( V \) be an open subset of \( U \) and denote by \( V' \) the set of prime points of \( U \) whose representing chains are eventually contained in \( V \). The collection of sets \( V' \subset \hat{U} \) such that \( V \) is an open subset of \( U \) is a basis for the topology on \( \hat{U} \). The function \( \omega : U \to \hat{U} \) is a homeomorphism from \( U \) onto an open subset of \( \hat{U} \), and we identify \( U \) with \( \omega(U) \).

If \( e \in \hat{U} \), let
\[
\alpha(e) := \bigcap_i \text{cl}_{U^*}(V_i),
\]
where \((V_i)\) is a chain representing \(e\). Then \(\alpha(e)\) consists of one point and \(\alpha : \hat{U} \to U^*\) is a continuous function whose restriction to \(U\) is the inclusion \((\alpha \circ \omega)\) is the identity on \(U\).

A prime end \(e \in \hat{U} - U\) is said regular if \(Z(\alpha(e))\) contains more than one point.

The set \(\hat{U}\) is a compact connected surface with boundary. This is Theorem 10 of [7]. \(\hat{U} - U\) is the set of prime ends of \(U\), the boundary \(\partial \hat{U}\) is the set of regular prime ends of \(U\) and the boundary components of \(\hat{U}\) are the sets \(\alpha^{-1}(b)\) where \(b\) varies on the set of regular ideal boundary points of \(U\). The set \(\alpha^{-1}(b)\) is homeomorphic to a circle called the Carathéodory circle associated with \(b\). We denote it by \(C(b)\).

Let \(e \in \hat{U}\) and let \((V_i)\) be a chain representing \(e\). The set

\[ Y(e) = \bigcap_i cl_S(V_i) \]

is called the **impression of** \(e\). The definition does not depend on the representing chain, and \(Y(e)\) is a compact, connected, non-empty subset of \(S\).

We say that \(p \in S\) is a **principal point** of \(e\) if there is a chain \((V_i)\) which represents \(e\) and for which \(fr_U(V_i) \to p\) (given a neighborhood \(V\) of \(p\) there exists \(n_0\) such that \(fr_U(V_i) \subset V\) for all \(n \geq n_0\)). The set of principal points of \(e\) is called the principal set of \(e\) and is denoted by \(X(e)\). It is a non-empty, compact, connected subset of \(S\), and

\[ X(e) \subset Y(e) \subset Z(\alpha(e)). \]

If \(X(e) = \{p\}\) for some \(p \in fr_S(U)\), we say that \(e\) is an accessible prime end. The point \(e \in \hat{U}\) is an accessible prime end if and only if there exists a path \(\beta : [0, 1] \to U\) such that \(\lim_{t \to 0} \beta(t) = p\) in \(S\) and \(\lim_{t \to 0} \beta(t) = e\) in \(\hat{U}\).

**2.1. Lemma.**

*Let \(U\) be an open connected set. Let \(A_1\) and \(A_2\) be open connected subsets of \(U\), both non-empty and different from \(U\). Suppose that*

\[
\begin{align*}
(16) & \ A_1 \cap A_2 \neq \emptyset. \\
(17) & \ A_1 \cap (U - A_2) \neq \emptyset. \\
(18) & \ A_2 \cap (U - A_1) \neq \emptyset. \\
(19) & \ fr_U(A_1) \text{ and } fr_U(A_2) \text{ are connected and disjoint.}
\end{align*}
\]

*Then we have the following facts:*

\[
\begin{align*}
(20) & \ fr_U(A_1) \subset A_2 \text{ and } fr_U(A_2) \subset A_1. \\
(21) & \ U = A_1 \cup A_2.
\end{align*}
\]

**Proof:** Since \(fr_U(A_1)\) is connected and does not intersect \(fr_U(A_2)\), we must have either \(fr_U(A_1) \subset A_2\) or \(fr_U(A_2) \subset U - A_2\). But \(A_2\) is a connected set that intersects both \(A_1\) and \(U - A_1\), and therefore we have \(A_2 \cap fr_U(A_1) \neq \emptyset\). This implies that \(fr_U(A_1) \subset A_2\). Similarly, \(fr_U(A_2) \subset A_1\), which proves part (20).

In order to prove item (21), we are going to show that \(A_1 \cup A_2\) is open and closed in \(U\). It is obviously open. Let \((a_n)\) be a sequence in \(A_1 \cap A_2\) converging to a point \(a\). We may assume that \(a_n \in A_1\) for infinitely many values of \(n\), and therefore \(a \in cl_U(A_1)\). If \(a \in A_1\) we are done. If not, we have that \(a \in fr_U(A_1)\), and from item (20), \(a \in A_2\). 

\[ \square \]
2.3. An useful lemma.

We would like to show that accessible prime ends can be represented by chains whose frontiers are contained in arcs of circles.

For this we are going to consider $\mathbb{R}^2$ equipped with a norm $\| \|$ . We denote by $B_r$, $B_r^c$ and $C_r$ the closed ball, the open ball and the circle with center at $(0,0)$ and radius $r$, respectively. By a closed disk $D$ we mean a set homeomorphic to a closed disk in $\mathbb{R}^2$. We write $\partial D$ for its boundary and $D^o$ for its interior $D - \partial D$.

Let $e$ be an accessible prime end and let $p$ be its principal point. In a neighborhood of $p$ we consider continuous coordinates with $p$ at the origin.

2.2. Lemma.

Let $U$ be a domain of $S$ such that by $U$ is finite. Let $e$ be an accessible prime end of $U$ and $p$ its principal point. Then we have the following:

(22) There exists $\delta > 0$ such that for any decreasing sequence $(r_n)_{n \geq 1}$ contained in $(0, \delta)$ with $\lim_{n \to \infty} r_n = 0$ there exists a chain $(V_n)$ representing $e$ such that $fr_U V_n \subset C_{r_n}$.

(23) For every $n \geq 1$ if $\rho \in ]r_{n+1}, r_n]$ then there exists an open subset $W$ of $U$ such that $fr_U W$ is connected, contained in $C_b(\rho)$ and $(W_i)$ defined by $W_i = V_i$ if $i \neq n + 1$ and $W_{n+1} = W$ is also a chain representing $e$.

Proof:

Let $\beta : [0,1] \to U$ be a path such that $\lim_{t \to 0} \beta(t) = p$ in $S$ and $\lim_{t \to 0} \beta(t) = e$ in $\hat{U}$.

If $b = \alpha(e)$ then $\lim_{t \to 0} \beta(t) = b$ in $U^*$. In fact, $\cap_{t>0} cl_{U^*} \beta([0,\epsilon])$ is a non-empty, compact, connected subset of $U^*$ which is disjoint from $U$. Since $U^* - U$ is finite, $\cap_{t>0} cl_{U^*} \beta([0,\epsilon])$ is a single point equal to $\lim_{t \to 0} \beta(t)$ in $U^*$. By the definition of $\alpha$ this point must be $\alpha(e)$.

Let $D$ be a closed disk in $U^*$ such that $b \in D^o$, $D \cap (U^* - U) = \{b\}$ and $\beta([0,1]) \cap \partial D \neq \emptyset$. The subset $\partial D \subset U$ is compact. Therefore there exists $\delta > 0$ such that $B_{\delta} \cap \partial D = \emptyset$ if $r < \delta$. Let $t_0 = \inf\{ t \in [0,1] \mid \beta(t) \in \partial D \}$. Then $\beta(t_0) \in \partial D$ and $\beta([0,t_0]) \subset D^o$.

24. Claim: For any $r \in [0,\delta[$, if $\beta([0,c]) \subset B_r^c$ and $\beta(d) \notin B_r$ with $d \leq t_0$, then there exists a connected component $\xi$ of $C_r \cap U$ such that $U - \xi$ is the disjoint union of two open connected subsets of $U$ with $\beta([0,c])$ contained in one and $\{\beta(d), \beta(t_0)\}$ contained in the other.

Proof of the claim.

First note that the components of $C_r \cup U$ form an open cover of $\beta([0,d]) \cap C_r$ which is compact. Therefore $\beta([0,d])$ intersects only finitely many components of $C_r \cup U$, say $\lambda_1, \ldots, \lambda_n$.

Since $\lim_{t \to 0} \beta(t) = b \in D^o$ and $\beta([0,t_0]) \cap \partial D = \emptyset$, we have that $\beta([0,t_0]) \subset D$. Then $\beta([0,d]) \subset D - \{b\}$, and hence $\lambda_i \cap (D - \{b\}) \neq \emptyset$. On the other hand, since $\lambda_i \subset C_{r_i}$, we have that $\lambda_i \cap \partial D = \emptyset$. Therefore $\lambda_i \subset D - \{b\}$. We have that $\lambda_i$ is an open arc in $D \subset U^*$ with end points at $b$. This implies that $U - \lambda_i$ has two components.
We are going to assume that $\beta([0,c])$ and $\beta(d)$ belong to the same component of $U - \lambda_i$ for $1 \leq i \leq n$ and obtain a contradiction.

Let $\lambda_0$ be the union of the components of $C_r \cap U$ different from $\lambda_1, \ldots, \lambda_n$.

For $0 \leq i \leq n - 1$ assume that there is a path $\alpha : [a,d] \to U - (\lambda_0 \cup \cdots \cup \lambda_i)$ from $\beta(a)$ to $\beta(d)$ where $a$ is any number in $[0,c]$. We are going to show that there is another path $\gamma : [a,d] \to U - (\lambda_0 \cup \cdots \cup \lambda_i \cup \lambda_{i+1})$ from $\beta(a)$ to $\beta(d)$.

If $\alpha$ does not intersect $\lambda_{i+1}$ then we just take $\gamma = \alpha$. When $\alpha \cap \lambda_{i+1} \neq \emptyset$ let $u = \inf\{t \mid \alpha(t) \in \lambda_{i+1}\}$ and $v = \sup\{t \mid \alpha(t) \in \lambda_{i+1}\}$. Since $\beta(a)$ and $\beta(d)$ are in the same component of $U - \lambda_{i+1}$, if $u' < u$ and $v' > v$ then $\alpha([a,u'])$ and $\alpha([v',d])$ are contained in the same component of $U - \lambda_{i+1}$. If $u'$ and $v'$ are close enough to $u$ and $v$ then the arc $\eta$ of the circle $C_{r'}$ with end points at $\alpha(u')$ and $\alpha(v')$ does not intersect $\lambda_0 \cup \cdots \cup \lambda_i \cup \lambda_{i+1}$. If we define $\gamma$ by just replacing the part of $\alpha$ between $\alpha(u')$ and $\alpha(v')$ by $\eta$, we obtain the desired path $\gamma : [a,d] \to U - (\lambda_0 \cup \cdots \cup \lambda_i \cup \lambda_{i+1})$ from $\beta(a)$ to $\beta(d)$.

We have that $\beta([0,d]) \cap \lambda_0 = \emptyset$. Therefore there exists a path $\gamma_1 : [a,d] \to U - (\lambda_0 \cup \lambda_1)$ from $\beta(a)$ to $\beta(d)$. By induction there exists a path $\gamma_n : [a,d] \to U - (\lambda_0 \cup \cdots \cup \lambda_n)$ from $\beta(a)$ to $\beta(d)$. This is a contradiction, since any path in $U$ from $\beta(a)$ to $\beta(d)$ must intersect $C_r \cap U$. Therefore there is some $k$ such that $\beta([0,c])$ and $\beta(d)$ belong to different components of $U - \lambda_k$.

Each $\lambda_i$ is an open arc in $D^\circ \subset U^*$ with end points at $b$. The curve $\lambda_0^\ast = \lambda_1 \cup \{b\}$ separates $U$ into two components $V_i$ and $W_i$, where $W_i$ contains $\partial D$ and $fr_{U^*}V_i = \lambda_i^\ast$. Since $\beta(d) \notin B_r$, there is a path in $D - B_r$ joining $\beta(d)$ to a point in $\partial D$. In particular it does not intersect $\lambda_i$. This implies that $\beta(d) \in W_i$. Also $\beta(t_0) \in \partial D \subset W_i$.

For $i = k$, since $\beta(d) \in W_k$, we have that $\beta([0,c]) \subset V_k$, and $\{\beta(d), \beta(t_0)\} \subset W_k$.

We construct the chain in the following way. For $i \geq 1$ let

$$t_i = \inf\{t \in [0,1] \mid \beta(t) \in C_{r_i}\}.$$ 

Then

$$\beta(t_i) \in C_{r_i} \quad \text{and} \quad \beta([0,t_i]) \subset B_{r_i}^\circ.$$ 

The set $V_1$ is defined in the following way. By hypothesis $r_1 < \delta$, then $B_{r_1} \cap \partial D = \emptyset$. We have that $\beta(t_0) \notin B_{r_1}$ and $\beta([0,t_1]) \subset B_{r_1}^\circ$. Using Claim 24, let $\xi_1$ be a component of $C_{r_1} \cap U$ for which $U - \xi_1$ is the disjoint union $V_1 \cup W_1$ of two open connected subsets of $U$ with $\beta([0,t_1]) \subset V_1$ and $\beta(t_0) \in W_1$. Also $\beta(t_0) \notin cl_U V_1$ because $\beta(t_0) \notin B_{r_1} \supset C_{r_1} \supset fr_{U^*}V_1$ and $\beta(t_0) \notin V_1$.

Suppose now that we have defined $V_1 \supset V_2 \supset \cdots \supset V_n$ with the following properties for $1 \leq i \leq n$:

$$\beta([0,t_i]) \subset V_i,$$

$$fr_{U^*}V_i = \xi_i$$

is a connected component of $C_{r_i} \cap U$.

$$\beta(t_{i-1}) \notin cl_U V_i.$$ 

$$\beta(t_0) \notin V_i.$$ 

The previous paragraph shows that (26), (27), (28), (29) hold for $i = 1$. 

Then
By (25) we can apply the claim to \( \beta([0,t_{n+1}]) \) and \( \beta(t_n) \). Let \( \xi_{n+1} \) be a component of \( C_{r_{n+1}} \cap U \) for which \( U - \xi_{n+1} \) is the disjoint union \( V_{n+1} \cup W_{n+1} \) of two open connected subsets of \( U \) with \( \beta([0,t_{n+1}]) \subset V_{n+1} \) and \( \{\beta(t_n), \beta(t_0)\} \subset W_{n+1} \). Observe that the choice of \( V_{n+1} \) and \( W_{n+1} \) implies that \( fr_U V_{n+1} = fr_U W_{n+1} = \xi_{n+1} \). Then \( \beta(t_n) \notin V_{n+1} \) and \( \beta(t_n) \notin fr_U V_{n+1} \) because \( fr_U V_{n+1} = \xi_{n+1} \subset C_{r_{n+1}} \) and \( \beta(t_n) \in C_{r_n} \). Therefore \( \beta(t_n) \notin cl_U V_{n+1} \) and \( V_{n+1} \) satisfies (26), (27), (28), (29) above.

We show now that \( V_{n+1} \subset V_n \). We have that \( \beta([0,t_{n+1}]) \subset V_n \cap V_{n+1} \). Therefore \( V_n \) and \( V_{n+1} \) satisfy hypothesis (16) of lemma 2.1. Also \( \xi_n \cap \xi_{n+1} = \emptyset \) and then hypothesis (19) also holds. Since \( \beta(t_n) \notin cl_U V_{n+1} \) we have that \( \beta([t_n - \delta, t_n]) \cap V_{n+1} = \emptyset \) for \( \delta \) small enough. Therefore \( \beta([t_n - \delta, t_n]) \subset V_n - V_{n+1} \) and (17) holds. By (29) we have that \( \beta(t_0) \notin V_n \cup V_{n+1} \) and then conclusion (21) of Lemma 2.1 does not hold. Therefore hypothesis (18) does not hold and then \( V_{n+1} \subset V_n \).

Thus the chain \( (V_n) \) is well defined. Since \( fr_U V_n \subset C_{r_n} \), we have that \( fr_U V_n \to p \) and \( (V_n) \) is prime. Let \( e' \) be the prime end it defines. Since \( \beta([0,t_n]) \subset V_n \) for every \( n \) we have that \( \lim_{t \to 0} \beta(t) = e' \) in \( \hat{U} \).

Hence \( e = (V_n) \). This proves part (22) of the lemma.

As for part (23), if we perform the previous construction with the new sequence \( s_i = r_i \) if \( i \neq n+1 \) and \( s_{n+1} = \rho \), then \( (t_i), (\xi_i) \) and \( (V_i) \) remain the same except for \( i = n+1 \), when we obtain an open subset \( W \) of \( U \) such that \( fr_U W \) is an arc of \( C_\rho \cap U \) and \( V_{n+2} \subset W \subset V_n \).

2.4. Cartwright-Littlewood’s Theorem.

Let \( f : S \to S \) be an area preserving orientation preserving homeomorphism and \( U \) an invariant domain with \( b_f U \) finite. If we denote by \( f_U \) the restriction of \( f \) to \( U \), then \( f_U \) has a unique extension \( f^* : U^* \to U^* \), which is also an orientation preserving homeomorphism. Points in \( b_f U \) are periodic. Since \( f_U \) maps irreducible chains to irreducible chains, \( f_U \) also extends to an orientation preserving homeomorphism \( \hat{f} : \hat{U} \to \hat{U} \) and Caratheodory circles are permuted in the same way as their associated regular ideal boundary points.

Suppose now that \( U \) is periodic of period \( n \). Let \( C \) be a Caratheodory circle and let \( k \) be the the smallest positive number such that \( f^{nk} C = C \). Define the rotation number of \( U \) at \( C \) as the rotation number of \( (f^u)^k \) restricted to \( C \).

2.3. Lemma.

Let \( e \) be a fixed prime end of \( U \), \( p \) a principal point of \( e \) and \( (V_i) \) a chain defining \( e \) such that \( fr_U (V_i) \to p \). Let \( i_0 = \inf \{ i \geq 1 \mid fV_i \subset V_i \} \). Then

\[
\forall i \geq i_0, \quad fr_U (V_i) \cap fr_U (fV_i) \neq \emptyset.
\]

Proof:

We have that \( (fV_i) \) is a chain that represents \( \hat{f}(e) \). Since \( e \) is a fixed point of \( \hat{f} \), we have that \( (fV_i) \) and \( (V_i) \) are equivalent.

Let \( i_0 \) be such that \( fV_i \subset V_i \) for \( i \geq i_0 \). We are going to assume that \( fr_U (V_i) \cap fr_U (fV_i) = \emptyset \) for some \( i \geq i_0 \) and apply Lemma 2.1 to \( V_i \) and \( fV_i \) to obtain a contradiction.
So we are assuming that hypothesis (19) of Lemma 2.1 holds. The equivalence of \((V_i)\) and \((fV_i)\) implies that \(V_i \cap fV_i \neq \emptyset\) for every \(i \geq 1\) and hypothesis \((16)\) also holds. We have that \(V_i \subset V_1\) for \(i \geq 1\) and therefore \(V_i \cap fV_i \subset V_1\) for \(i \geq i_0\). Since \(V_1\) is strictly contained in \(U\), we have that conclusion \((21)\) does not hold.

This implies that hypotheses \((17)\) and \((18)\) can not both hold. So we have that \(V_i \subset fV_i\) or \(fV_i \subset V_i\). We are going to assume that \(fV_i \subset V_i\) and show that \(\mu(fV_i) < \mu(V_i)\). The other case is analogous.

Since \(fr_U(fV_i)\) is connected and
\[
(30) \quad fr_U(fV_i) \cap fr_U V_i = \emptyset,
\]
we have that \(fr_U(fV_i) \subset V_i\) or \(fr_U(fV_i) \subset U - V_i\). We claim that the case \(fr_U(fV_i) \subset U - V_i\) can not happen. If so, by equality \((30)\) we would have that \(fr_U(fV_i) \subset U - cl_U V_i\). If \(x \in fr_U(fV_i)\) then every neighborhood \(W\) of \(x\) would contain points of \(fV_i \subset V_i\). On the other hand we could choose \(W\) disjoint from \(cl_U V_i\), a contradiction.

Therefore \(fr_U(fV_i) \subset V_i\), implying that \(cl_U(fV_i) \subset V_i\). Since \(fr_U(V_i) \to p\), \(V_i \neq U\). If we had \(cl_U(fV_i) = V_i\) then \(V_i\) would be open and closed at the same time, contradicting the connectivity of \(U\). Hence \(V_i - cl_U(fV_i) \neq \emptyset\). Since \(\mu\) is positive on open sets, we have that \(\mu(cl_U(fV_i)) < \mu(V_i)\). A contradiction.

\[\square\]

Next we present Cartwright and Littlewood’s fixed point theorem.

2.4. Theorem (Cartwright and Littlewood [2]).

Let \(f : S \to S\) be an area preserving orientation preserving homeomorphism and \(U\) an invariant domain with \(b_U\) finite. Let \(e\) be a fixed prime end of \(U\) and \(p\) a principal point of \(e\). Then \(f(p) = p\).

\textbf{Proof:} Let \((V_i)\) be a chain defining \(e\) such that \(fr_U(V_i) \to p\). From Lemma 2.3 there exists \(i_0\) such that for \(i \geq i_0\) there exists a point \(x_i \in fr_U(V_i)\) such that \(f(x_i)\) also belongs to \(fr_U(V_i)\). Since \(fr_U(V_i) \to p\), we have that \(x_i \to p\) and \(f(x_i) \to p\), implying that \(f(p) = p\).

\[\square\]

3. No elliptic points from periodic prime ends.

Now we start the proof of item \((1)\) of Theorem 1.2. We have that \(b = \alpha(e) \in b_U\) is a regular ideal boundary point, \(e\) is a prime end that belongs to the Caratheodory circle associated with \(b\) and \(f(e) = e\).

Let \(p\) be a principal point of \(e\). By Theorem 2.4, \(f(p) = p\). Our hypothesis that fixed points of \(f\) contained in \(fr_SU\) are non degenerate implies that they are isolated. Therefore the principal set of \(e\) reduces to \(\{p\}\) and hence
\[
(31) \quad e \text{ is accessible}.
\]
3.1. The fixed point $p$ can not be elliptic.

We are going to assume that $p$ is elliptic and obtain a contradiction.

Let $V$ be a neighborhood of $p$ where there exist continuous coordinates with $p$ at the origin and in which $f$ is differentiable at $p$ and $df_p$ is a rotation by an angle $\alpha$ different from zero.

We write the coordinates in the plane as complex numbers. By hypothesis we have that $f'(0) = e^{i\alpha}$ with $0 < \alpha < 2\pi$.

Since $Z(b)$ is connected and contains more than one point, there exists a closed ball $B$ centered at 0 such that $Z(b)$ is not contained in $B$.

We are going to construct a closed curve $\xi$ satisfying three conditions:

- $\xi \subset U$,
- $\xi \subset B^o$,
- If $B'$ is the component of $S - \xi$ which contains $p$, then $B'$ is homeomorphic to an open disk whose closure is contained in $B^o$ and $frSB' \subset \xi$.

Since $p \in X(e)$ and $X(e) \subset Z(b)$, we have that $Z(b) \cap B' \neq \emptyset$. On the other hand, since $B' \subset B$ and $Z(b)$ is not contained in $B$, we have that $Z(b) \cap (S - B') \neq \emptyset$. Being connected $Z(b)$ must intersect $frSB'$, a contradiction since $Z(b) \subset frS(U)$ and $frSB' \subset \xi \subset U$.

Let $B_0^2$ be the open ball of radius $\delta$ centered at the origin with $\delta$ small enough so that $f(B_0^2) \subset V$. For points $z$ close to 0 we will need to estimate the distance between iterates of $z$ under $f$ and $f'(0)$.

3.1. Lemma.

Let $w$ and $w' \in \mathbb{C}$ and $\epsilon < 1$ be such that $|w' - w| < \epsilon |w|$. If $w = re^{i\eta}$ and $r' = |w'|$ then there exists a unique real number $\eta'$ such that

\[ w' = r' e^{i\eta'}, \]
\[ |r' - r| < \epsilon r, \]
\[ |\eta' - \eta| < \frac{\pi}{2} \epsilon. \]

Proof:

Let $B_r$ be the closed ball of radius $\epsilon r$ with center at $w$. Then $w'$ belongs to the interior of $B_r$ and $0 \notin B_r$.

Let $\nu$ be the angle between the ray starting at 0 through $w$ and one of the two rays starting at 0 and tangent to $B_r$. Since $0 \notin B_r$, we have that $\nu < \frac{\pi}{2}$. Then there is a unique real number $\eta'$ such that $w' = r' e^{i\eta'}$ and $|\eta' - \eta| < \nu$. For $0 < \nu < \frac{\pi}{2}$ we have that $\frac{2}{\pi}\nu < \sin \nu$. But $\sin \nu = \epsilon$. Therefore $|\eta' - \eta| < \frac{\pi}{2} \epsilon$ which proves (35) and (37). Item (36) is just the triangle inequality.

For $\epsilon \in ]0,1[$ let $\delta > 0$ be such that if $|z| < \delta$ then $|f(z) - f'(0)z| < \epsilon |z|$.

3.2. Lemma.

Let $z = re^{i\theta}$ with $r < \delta$ and let $r' = |f(z)|$. Then there exists a unique real number $\theta'$ such that
(38) $f(z) = r' e^{\theta'}$,
(39) $|r' - r| < \epsilon r$,
(40) $|\theta' - (\theta + \alpha)| < \frac{\pi}{2} \epsilon$.

**Proof:**

Let $w' = f(z)$ and $w = f'(0)z$. Then $w = r e^{i(\theta + \alpha)}$ and $|w' - w| < \epsilon |w|$. Let $\eta = \theta + \alpha$. Lemma 3.1 provides a unique number $\theta' = \eta'$ for which (38), (39) and (40) are satisfied.

By Lemma 3.2, we have that $F : \mathbb{R} \times [0, \delta] \rightarrow \mathbb{R} \times [0, +\infty]$ defined by $F(\theta, r) = (\theta', r')$ is a lifting of $f : B_\delta - \{0\} \rightarrow \mathbb{C}$ with the property that $|r - r'| < \epsilon r$ and $|\theta' - (\theta + \alpha)| < \frac{\pi}{2} \epsilon$ for every $(\theta, r) \in \mathbb{R} \times [0, \delta]$. We are going to use $(\theta_j, r_j) = F^j(\theta_0, r_0)$ to denote iterates of a point $(\theta_0, r_0)$.

Now we start the construction of $\xi$.

Let $B$ be a closed ball centered at 0 such that $Z(b)$ is not contained in $B$.

Let $n$ be such that $n \alpha > 2\pi$ and $n(2\pi - \alpha) > 2\pi$.

Let $\epsilon \in [0, 1]$ be such that

\begin{align*}
(41) & \quad n \frac{\pi}{2} \epsilon < n \alpha - 2\pi, \\
(42) & \quad n \frac{\pi}{2} \epsilon < n(2\pi - \alpha) - 2\pi, \\
(43) & \quad \alpha + \frac{\pi}{2} \epsilon < 2\pi \quad \text{and} \quad \alpha - \frac{\pi}{2} \epsilon > 0.
\end{align*}

Let $\delta > 0$ be such that $B_\delta \subset B$, $f(B_\delta^3) \subset V$ and if $|z| < \delta$ then $|f(z) - f'(0)z| < \epsilon |z|$.

If $r_0$ satisfies $r_0(1 + \epsilon)^n < \delta$, then from (39) of Lemma 3.2 we have that

$$r_j < r_0 (1 + \epsilon)^j < \delta$$

and $F^j : \mathbb{R} \times [0, \delta] \rightarrow \mathbb{R} \times [0, +\infty]$ is well defined for $1 \leq j \leq n$. From (40) of Lemma 3.2 we have that

$$|\theta_j - (\theta_0 + j\alpha)| < j \frac{\pi}{2} \epsilon \quad \text{for} \quad 1 \leq j \leq n.$$

By Lemma 2.2 there exists a chain $(V_i)$ representing $e$ such that $fr_u V_i \rightarrow p$ and the sets $fr_u V_i$ are all contained in circles with center at 0. For $i$ large enough let $\beta = fr_u V_i$ be an arc of circle of radius $r_0$ such that $r_0(1 + \epsilon)^n < \delta$.

Observe that $e$ is a fixed prime end of both $f$ and $f^n$. From Lemma 2.3 we may take $i$ large enough so that $f(\beta) \cap \beta \neq \emptyset$ and $f^n(\beta) \cap \beta \neq \emptyset$.

We have that $f^{-1}(\beta) \cap f^i(\beta) \neq \emptyset$ for $1 \leq i \leq n$ and $f^n(\beta) \cap \beta \neq \emptyset$. We are going to show that these arcs turn around $p$ at least once to close a curve with $p$ inside.

Let $\hat{\beta} := \{a, b\} \times \{r_0\}$ be a lifting of $\beta$ with $0 \leq a < 2\pi$. We have that $b - a < 2\pi$.

**3.3. Lemma.** $F\hat{\beta} \cap \hat{\beta} \neq \emptyset$ or $F\hat{\beta} \cap (\hat{\beta} + (2\pi, 0)) \neq \emptyset$.

**Proof:** Since $f \beta \cap \beta \neq \emptyset$, we have that $F\hat{\beta} \cap (\hat{\beta} + (2k\pi, 0)) \neq \emptyset$ for some $k \in \mathbb{Z}$. We are going to show that $k = 0$ or 1.
Let \((\theta_1, r_1) \in F\beta \cap (\beta + (2k\pi, 0))\). Then \((\theta_1, r_1) = F(\theta_0, r_0)\) and \(\theta_1 = \theta_0 + 2k\pi\), where both \(\theta_0\) and \(\theta_0 + 2k\pi\) belong to \([a, b][\). By (45) we have that \(|\theta_1 - (\theta_0 + \alpha)| < \frac{n\pi}{2}\pi\). Therefore

\[
|\theta_0 + 2k\pi - (\theta_0 + \alpha)| < \frac{n\pi}{2}\pi, \quad |2k\pi - \alpha| < |\theta_0 - \theta_0| + \frac{\pi}{2}\pi < 2\pi + \frac{\pi}{2}\pi, \quad \text{and} \quad -2\pi + \alpha - \frac{\pi}{2}\pi < 2k\pi < 2\pi + \alpha + \frac{\pi}{2}\pi.
\]

By (43) we have that \(-2\pi < 2k\pi < 4\pi\) and \(k = 0\) or 1.

\[\square\]

3.4. Lemma. \(F^n\beta \cap (\beta + (2k\pi, 0)) \neq \emptyset\) for some \(k \geq 1\).

Proof:
Since \(F^n\beta \cap \beta \neq \emptyset\) we have that \(F^n\beta \cap (\beta + (2k\pi, 0)) \neq \emptyset\) for some \(k \in \mathbb{Z}\).

Let \((\theta_0, r_0) \in F^n\beta \cap (\beta + (2k\pi, 0))\). Then \((\theta_0, r_0) = F^n(\theta_0, r_0)\) and \(\theta_0 = \theta_0 + 2k\pi\) where both \(\theta_0\) and \(\theta_0 + 2k\pi\) belong to \([a, b][\). By (45), we have that \(|\theta_0 - (\theta_0 + 2k\pi)| < n\pi\). Therefore

\[
|\theta_0 + 2k\pi - (\theta_0 + 2k\pi)| < \frac{n\pi}{2}\pi, \quad |2k\pi - 2k\pi| < |\theta_0 - \theta_0| + \frac{\pi}{2}\pi < 2\pi + \frac{\pi}{2}\pi, \quad \text{and} \quad -2\pi + 2k\pi - \theta_0 > 0 \quad \text{by (41)}.
\]

Hence \(k \geq 1\).

\[\square\]

In the case when \(F\beta \cap \beta \neq \emptyset\) we construct \(\xi\) in the following way.

We have \(F^{j+1}\beta \cap F^j\beta \neq \emptyset\) for \(1 \leq j \leq n - 1\) and by Lemma 3.4, \(F^n\beta \cap (\beta + (2k\pi, 0)) \neq \emptyset\) for some \(k \geq 1\). This implies that the union \(\beta \cup F\beta \cup \cdots \cup F^n\beta \cup (\beta + (2k\pi, 0))\) is path connected and therefore this union contains a path connecting a point \((\theta_0, r_0) \in \beta\) to \((\theta_0 + 2k\pi, r_0)\) with \(k \geq 1\). By (44) this path projects down to a closed path \(\xi\) contained in \(B^\circ \setminus \{0\} \subset B^\circ \setminus \{0\}\) which is not non trivial path of \(\pi_1(B^\circ \setminus \{0\})\). Since \(\beta \subset U\) and \(\xi \subset \bigcup_{j=0}^{n-1} F^j\beta\), we have that \(\xi \subset U\). We have that \(p\) and \(\partial B\) lie in different components of \(S - \xi\). Let \(B'\) be the component of \(S - \xi\) that contains \(p\). Then \(f_{rs}B' \subset \xi\). The set \(Z(b)\) intersects both \(B'\) and \(S - B'\) and therefore \(Z(b)\) intersects \(\xi\). But this is a contradiction since \(Z(b) \subset f_{rs}U\) and \(\xi \subset U\).

When \(F\beta \cap (\beta + (2\pi, 0)) \neq \emptyset\) we work with a different lifting of \(f\),

\[H(\theta, r) := F(\theta, r) - (2\pi, 0)\]

3.5. Lemma. \(H\) satisfies \(H\beta \cap \beta \neq \emptyset\) and \(H^n\beta \cap (\beta + (2k\pi, 0)) \neq \emptyset\) for some \(k \leq -1\).

Proof: The fact \(H\beta \cap \beta \neq \emptyset\) follows immediately from the definition of \(H\) and the fact that \(F\beta \cap (\beta + (2\pi, 0)) \neq \emptyset\).

Inequality \(f^n\beta \cap \beta \neq \emptyset\) implies that \(H^n\beta \cap (\beta + (2k\pi, 0)) \neq \emptyset\) for some \(k \in \mathbb{Z}\).

Let \(q \in H^n\beta \cap (\beta + (2k\pi, 0))\). Since \(H^n(\theta_0, r_0) = F^n(\theta_0, r_0) - (2n\pi, 0)\), we have that \(q = F^n(\theta_0, r_0) - (2n\pi, 0) = (\theta_n, r_n) - (2n\pi, 0)\) and \(q = (\theta_0 + 2k\pi, r_0)\) where both \(\theta_0\) and
\( \theta_0 \) belong to \([a, b]\). By (45) we have that \(|\theta_n - (\theta_0 + n\alpha)| < n\frac{\pi}{2}\epsilon\). Therefore

\begin{align*}
\left| \theta_0 + 2k\pi + 2n\pi - \theta_0 - n\alpha \right| &< n\frac{\pi}{2}\epsilon \\
2k\pi + \theta_0 - \theta_0 + n(2\pi - \alpha) &< n\frac{\pi}{2}\epsilon \\
2k\pi < \theta_0 - \theta_0 - n(2\pi - \alpha) + n\frac{\pi}{2}\epsilon < 2\pi - n(2\pi - \alpha) + n\frac{\pi}{2}\epsilon < 0
\end{align*}

by (42), implying that \( k \leq -1 \).

\[ \square \]

Now we have \( H_\beta \cap \beta \neq \emptyset \) and \( H^n_\beta \cap (\beta + (2k\pi, 0)) \neq \emptyset \) for some \( k \leq -1 \). The construction of \( \xi \) is done as in the previous case.

Now we prove that one of the branches of \( p \) is a connection contained in the frontier of \( U \).

### 3.2. One of the branches of \( p \) is a connection contained in the frontier of \( U \).

The arguments in this subsection are due to Mather [6] and our presentation is based on Franks, Le Calvez [4], where they work with area preserving diffeomorphisms of the two dimensional sphere and take powers of the map to assume that the branches of the fixed point are invariant.

Firstly a well known result.

#### 3.6. Lemma.

Let \( K \) be a compact connected invariant set and \( L \) a branch. If \( L \cap K \neq \emptyset \) then \( L \subset K \).

For a proof see Corollary 8.3 of Mather [6] or in \( S^2 \) Lemma 6.1 of Franks, Le Calvez [4], or [9] for surfaces with boundary and partially defined area preserving maps.

The domain \( U \) has finitely many ideal boundary points and therefore \( frSU \) has finitely many connected components. Since \( frSU \) is invariant, its components are periodic. Let \( L \) be a branch of \( p \). We claim that

\[ (46) \quad L \cap U \neq \emptyset \quad \implies \quad L \subset U. \]

In fact, if \( L \) were not contained in \( U \) then \( L \) would intersect a component of \( frSU \) and by Lemma 3.6 applied to a power of \( f \) this component would contain \( L \). We have a little more:

#### 3.7. Lemma.

Let \( L \) be a branch of \( p \). Then we have the following:

\[ (47) \quad \text{If } U \text{ is an open invariant set and } L \cap U \neq \emptyset \text{ then } U \text{ contains } L \text{ and the two sectors of } p \text{ adjacent to } L. \]

\[ (48) \quad \text{If } K \text{ is a compact connected invariant set then either } L \subset K \text{ or } L \text{ and its adjacent sectors are contained in one component of } S - K. \]

**Proof:**

The branch \( L \) is invariant or of period two. By (46) we have that \( L \subset U \). Let \( x \in L \) and let \( W \subset U \) be a neighborhood of the arc from \( x \) to \( f^2(x) \) in \( L \). Then \( \cup_{n \in \mathbb{Z}} f^{2n}(W) \)
contains the sectors of \( p \) adjacent to \( L \). Item (48) easily follows from item (47) letting \( U \) be a component of \( S - K \) which intersects \( L \).

\[ \square \]

Let \( b = \alpha(e) \) and \( C(b) \) be the circle of prime ends that contains \( e \).

Let \((x, y)\) be continuous coordinates in a neighborhood \( V \) of \( p \) with \( p \) at the origin where \( f(x, y) = (\lambda x, \lambda^{-1} y) \) with \( |\lambda| > 1 \). We may assume that \( V \) is the open ball \( B^\circ_\delta \) of radius \( \delta \) and center at \((0, 0)\).

This implies that \( W \) must be contained in one of the sectors of \( f \) and contains the sectors of \( S \) and \( T \). By (31), there exists a path \( \beta : [0, 1] \to U \) such that \( \lim_{t \to 0} \beta(t) = p \) in \( S \) and \( \lim_{t \to 0} \beta(t) = e \) in \( U \). We may assume that \( \delta \) is small enough so that \( \beta(1) \not\in B^\circ_\delta \) and we can apply Lemma 2.2. We may also assume that \( \beta([0, 1]) \subset V \).

For every decreasing sequence \( (r_n) \) contained in \([0, \delta]\) with \( \lim_{n \to \infty} r_n = 0 \) there exists a chain \((V_n)\) representing \( e \) such that \( \xi_n = f(r_n) \cap V_n \subset C_{r_n} \). Notice that \( \xi_n \cap \beta \neq \emptyset \) for every \( n \). Since \( \beta([0, 1]) \cup (\cup_n \xi_n) \) is connected and contained in both \( U \) and \( V \), all the arcs \( \xi_n \) must be contained in one connected component of \( U \cap V \). Denote this component by \( W \).

3.8. Lemma.

There exists an open arc of prime ends \([a, c] \subset C(b) \) such that the function \( \phi : [a, c] \to f \circ \rho \) defined by \( \phi(e') = X(e') \) is continuous, \( \phi(0, e) \) and \( \phi(e, c) \) are local branches of \( p \) and \( \phi \circ f = f \circ \phi \).

Proof:

First we consider the case when the arcs \( \xi_n \) do not intersect the local branches of \( p \). This implies that \( W \) must be contained in one of the sectors of \( p \) defined by \( B^\circ_\delta \), say \( S_1 \).

By Lemma 2.3, if \( n_0 = \inf \{ n \mid f(V_n) \subset V_1 \} \), then \( f(\xi_n) \cap \xi_n \neq \emptyset \) for \( n \geq n_0 \). This implies that \( \lambda > 1 \). Let \( \rho \) be any number in \([0, r_{n_0}] \). By (23) of Lemma 2.2 we may assume that \( \xi_m \subset C_{\rho} \) for some \( m > n_0 \).

Let us consider the sup norm on \( \mathbb{R}^2 \). In this case \( B_\rho \) is the square with vertices at \((\pm \delta, \pm \delta)\), and if \( \Gamma_\rho = ([0, \rho] \times \{\rho\}) \cup (\{\rho\} \times [0, \rho]) \) then \( \xi_m \subset \Gamma_\rho \). Since \( f(\Gamma_\rho) \cap \Gamma_\rho = \{(\rho, \lambda^{-1} \rho)\} \), we have that \( f(\xi_m) \cap \xi_m = \{(\rho, \lambda^{-1} \rho)\} \) as well. Therefore \( (\rho, \lambda^{-1} \rho) \) and \( (\lambda^{-1} \rho, \rho) \) belong to \( \xi_m \) and the arc \( \Lambda_\rho \) from \( (\rho, \lambda^{-1} \rho) \) to \( (\lambda^{-1} \rho, \rho) \) inside \( \Gamma_\rho \) is contained in \( \xi_m \subset U \). This holds for every \( \rho \in [0, r_{n_0}] \). It follows that \( \cup_{k \in \mathbb{Z}} J_k(\cup_{\rho \in [0, r_{n_0}]} \Lambda_\rho) \) is contained in \( U \) and contains \( S_1 \cap B^\circ_\delta \) for some small number \( \varepsilon < \delta \).

Therefore \( S_1 \cap B^\circ_\delta \subset W \) and \( (L_1 \cup L_2) \cap U = \emptyset \). It follows that for \( n \) sufficiently large we have that \( V_n \subset S_1 \cap B^\circ_\delta \) and the arcs \( \xi_n \) must have one end point at \( \{0\} \times [0, \varepsilon] \) and another at \([0, \varepsilon] \times \{0\} \). For any \( x \in \{\{0\} \times [0, \varepsilon] \} \cup ([0, \varepsilon] \times \{0\}) \) if \( B^\circ_\delta_{1/n}(x) \) is the open ball of radius \( \frac{1}{n} \) with center at \( x \), then \( V_n = B^\circ_{1/n}(x) \cap W \) defines a prime chain \( (V_n) = e' \) such that \( X(e') = \{x\} \). Then there exists an open arc of prime ends \([a, c] \subset C(b) \) such
that if \( e' \in [a, c] \) then \( X(e') = \{ x \} \) for some \( x = x(e') \in (\{ 0 \} \times [0, \varepsilon]) \cup ([0, \varepsilon] \times \{ 0 \}) \). If we define \( \phi : [a, c] \to S \) by \( \phi(e') = x(e') \), then it is easy to check that \( \phi \) is continuous, \( \phi(e) = p \), \( \phi([a, e]) = \{ 0 \} \times [0, \varepsilon] \subset L_2 \) and \( \phi([e_2, c]) = [0, \varepsilon] \times \{ 0 \} \subset L_1 \). Since \( X(\hat{f}(e')) = f(X(e')) \) for any prime end \( e' \) we have that \( \phi \circ f = f \circ \phi \).

Now consider the case when the arcs \( \xi_n \) intersect the local branches \( L_i \). The family \( \{ \xi_n \} \) cannot intersect the four branches, otherwise by Lemma 3.7.47, \( W \) would contain the four sectors of \( p \) implying that \( B^0_{\varepsilon_n} \cap f_{rs} U = \{ p \} \) for small \( \varepsilon \), contradicting the fact that \( b \) is regular.

So \( \{ \xi_n \} \) intersects at least one of the local branches and at least one is disjoint from \( \{ \xi_n \} \). We may assume that \( L_1 \cap \{ \xi_n \} = \emptyset \) and \( L_2 \cap \{ \xi_n \} \neq \emptyset \).

Let \( k = \sup \{ j \in \{ 2, 3, 4 \} \mid L_j \cap \{ \xi_n \} \neq \emptyset \} \). Then by Lemma 3.7.47 we have that \( L_1 \cap W = \emptyset \), \( L_{k+1} \cap W = \emptyset \) \((L_5 = L_1)\) and
\[
(S_1 \cup \cup_{i=2}^k (L_i \cup S_i)) \cap B^0_{\varepsilon} \subset W \subset S_1 \cup \cup_{i=2}^k (L_i \cup S_i) \quad \text{for some} \quad \varepsilon \in [0, \delta].
\]

It follows that for \( n \) sufficiently large we have that \( V_n \subset (S_1 \cup \cup_{i=2}^k (L_i \cup S_i)) \cap B^0_{\varepsilon} \). The sets \( L_1 \cap B^0_{\varepsilon} \) and \( L_{k+1} \cap B^0_{\varepsilon} \) are contained in \( f_{rs} W \) and the arcs \( \xi_n \) must have one end point at \( L_1 \cap B^0_{\varepsilon} \) and another at \( L_{k+1} \cap B^0_{\varepsilon} \).

As in the precious case, if \( x \in (L_1 \cup L_{k+1}) \cap B^0_{\varepsilon} \) and \( V_n = B^0_{1/n}(x) \cap W \) then \( \phi((V_n)) = x \) is the desired map. In case \( k = 4 \), \( \phi([a, c]) \subset \{ p \} \cup L_1 \).

Let \([e_1, e_2]\) be an arc of prime ends with \( \hat{f}(e_i) = e_i \) and
\[
\lim_{n \to \infty} \hat{f}^n(e_i) = e_1 \quad \text{and} \quad \lim_{n \to -\infty} \hat{f}^n(e_i) = e_2 \quad \text{for any} \quad e \in [e_1, e_2].
\]
(possibly with \( e_1 = e_2 \)). From Lemma 3.8 we know that there exists a function \( \phi_1 \) that maps an arc \([e_1, e_2]\) onto a local branch of \( p_1 = \phi_1(e_1) \) which is part of an unstable branch \( L_1 \) of \( p_1 \). For any \( e \in [e_1, e_2]\) there exist \( n \in \mathbb{Z} \) and \( e' \in [e_1, e_2] \) such that \( e = \hat{f}^n(e') \). Since \( X(\hat{f}^n(e')) = \hat{f}^n(X(e')) \) we have that \( e \) is accessible and \( \phi_1(e) = X(e) \) extends \( \phi_1 \) to \([e_1, e_2]\) with the same properties. We have that \( L_1 = \phi_1([e_1, e_2]) \).

The same happens with \( e_2 \). There exists a function \( \phi_2 \) that maps and arc \([a, e_2]\) onto a local branch of \( p_2 = \phi_2(e_2) \) which is part of a stable branch \( L_2 \) of \( p_2 \). The map \( \phi_2(e) = X(e) \) extends to \([e_1, e_2]\) in the same way and \( L_2 = \phi_2([e_1, e_2]) \). Therefore \( L_1 = \phi_1([e_1, e_2]) = \phi_2([e_1, e_2]) = L_2 \) is a connection contained in \( f_{rs} U \).

So the proof of item (1) of Theorem 1.2 is complete.

\[ \square \]

3.3. Proof of items (2), (3) and (4) of Theorem 1.2.

Let \( L \) be an invariant branch of a fixed point \( p \) of saddle type and assume that all fixed points of \( f \) contained in \( cl S L \) are non degenerate. We are going to assume that \( L \) does not accumulate on one of its adjacent branches through the adjacent sector and show that \( L \) must be a connection.

As in the previous subsection we are going to consider a system \((x, y)\) of continuous coordinates in a neighborhood \( V \) of \( p \) with \( p \) at the origin where \( f(x, y) = (\lambda x, \lambda^{-1} y) \) with \( \lambda > 1 \). We may assume that \( V \) is the open ball \( B^0_{\delta} \) of radius \( \delta \) and center at \((0,0)\).
We are going to assume that $L$ is the branch that contains $\{ (x, y) \in B_3^0 \mid x > 0, y = 0 \}$ and that $L$ is disjoint from $S_1 = \{ (x, y) \in B_0^0 \mid x > 0, y > 0 \}$. Let $U$ be the connected component of $S - cl_S L$ that contains $S_1$. Then $f(U) = U$ and by Lemma 2.3 in [6] or Proposition 3.15 in [9], $U$ has finitely many ideal boundary points (in fact $U$ has at most $g + 1$ ideal boundary points where $g$ is the genus of $S$, but we do not need this here). We also have that $fr_S U \subset cl_S L$ and thus $fr_S U$ contains no degenerate fixed point of $f$. Therefore $U$ satisfies the hypothesis of item (1).

We claim that $\{ p \}$ is the principal set of a fixed prime end $e$. When $L_2 \cap U = \emptyset$ we see this by defining the chain $V_n = B_{1/n}^0 \cap S_1$. When $L_2 \subset U$ and $L_3 \cap U = \emptyset$ we take $V_n$ as $B_{1/n}^0 \cap (S_1 \cup L_2 \cup S_2)$, when $L_3 \subset U$ and $L_4 \cap U = \emptyset$ we take $V_n$ as $B_{1/n}^0 - (L_4 \cup S_4 \cup L_1)$ and when $L_4 \subset U$ we take $V_n$ as $B_{1/n}^0 - L_1$.

From Lemma 3.8 there exists an open arc of prime ends $[a, c]$ in $C(b)$ containing $e$ such that the function $\phi : [a, c] \rightarrow fr_S U$, defined by $\phi(e') = X(e')$, is continuous and $\phi([a, e])$ and $\phi([e, c])$ are local branches of $p$. The local branch of $L$ is equal to either $\phi([a, e])$ or $\phi([e, c])$ or both. In the proof of item (1) we showed that both $\phi([a, e])$ and $\phi([e, c])$ are part of connections. Therefore $L$ must be a connection.

We proved that $L$ accumulates on both adjacent sectors. From Lemma 3.7.(48) we have that if $L$ accumulates on one adjacent sector then $L \subset \omega(L)$. This completes the proof of item (2).

Assume that $L_1$ and $L_2$ are invariant adjacent branches that are not connections and that all fixed points of $f$ contained in $cl_S (L_1 \cup L_2)$ are non degenerate. From Lemma 3.7.(48) we have that $L_2 \subset cl_S L_1$ and $L_1 \subset cl_S L_2$ which proves item (3).

Item (4) follows from item (3).

4. Surfaces with boundary and homoclinic points.

In this section we extend previous results to compact connected orientable surfaces $S$ with boundary.

Let $f : S \rightarrow S$ be an orientation preserving area preserving homeomorphism of $S$. Let $p \in \partial S$ be a periodic point of $f$ of period $\tau$. We say that $p$ is of saddle type if there exist a neighborhood $V$ of $p$ and continuous coordinates $(x, y)$, $y \geq 0$, in $V$ with $p$ at the origin and $f^\tau (x, y) = (\lambda x, \lambda^{-1} y)$ with $\lambda > 0$ and $\lambda \neq 1$. The point $p$ has two sectors and three branches, two of which are connections contained in $\partial S$ and the other is contained in $S - \partial S$.

Let $C$ be a connected component of $\partial S$. Suppose that all periodic points in $C$ are of saddle type. Then there are only finitely many periodic points in $C$ and, as we move around $C$, periodic points must be alternatively attracting and repelling in $C$ and their branches that are contained in $S - \partial S$ also alternate from stable to unstable.

4.1. Theorem.

Let $S$ be a compact connected orientable surface with boundary provided with a finite measure $\mu$ which is positive on open sets and $f : S \rightarrow S$ be an orientation preserving area preserving homeomorphism of $S$. 
(50) Suppose that $L$ is an invariant branch of $f$ and that all fixed points of $f$ contained in $\text{cl}_S L$ are non degenerate. Then either $L$ is a connection or $L$ accumulates on both adjacent sectors. In the later alternative $L \subset \omega(L)$.

(51) Let $p \in S - \partial S$ be a fixed point of $f$ of saddle type and let $L_1$ and $L_2$ be adjacent branches of $p$ that are not connections. If $L_1$ and $L_2$ are invariant and all fixed points of $f$ contained in $\text{cl}_S (L_1 \cup L_2)$ are non degenerate then $\text{cl}_S L_1 = \text{cl}_S L_2$. If in addition $S$ has genus 0 then $L_1 \cap L_2 \neq \emptyset$.

(52) Let $p \in S - \partial S$ be a fixed point of $f$ of saddle type. Assume that all fixed points of $f$ contained in $\text{cl}_S (W^u_p \cup W^s_p)$ are non degenerate. If the branches of $p$ are invariant and none of them is a connection, then all branches have the same closure in $S$. If in addition the genus of $S$ is 0 or 1, then the four branches of $p$ have homoclinic points.

(53) Let $C$ be a connected component of $\partial S$ and suppose that all fixed points $p_1, \ldots, p_{2n}$ of $f$ in $C$ are of saddle type. Let $L_i$ be the branch of $p_i$ contained in $S - \partial S$. Assume that for every $i$ all fixed points of $f$ contained in $\text{cl}_S L_i$ are non degenerate and that $L_i$ is not a connection. Then $\text{cl}_S L_i = \text{cl}_S L_j$ for any pair $(i, j)$.

If in addition $S$ has genus 0 then any pair $(L_i, L_j)$ of stable and unstable branches intersect. The same happens when the genus of $S$ is 1 provided there are at least 4 fixed points in $C$.

4.2. Remark.

The conclusion about the existence of homoclinic and heteroclinic points above is false if the genus of $S$ is greater than one. Examples could be time one maps of area preserving flows on surfaces with finitely many singularities and every other orbit dense. In item (51) the conclusion about homoclinic points is false if the genus of $S$ is one. In the torus two branches could close a connection and the other two spin around the torus like a line of irrational slope without intersecting. For the same reason, when the genus of $S$ is one in item (53), we need at least four recurrent branches in a boundary component to have heteroclinic points.

For applications we need to state a theorem obtaining homoclines for area preserving maps defined on an open set of a surface with boundary. So we split Theorem 4.1 into two theorems, one containing the results in accumulation of invariant manifolds and another containing the results on homoclines.

4.3. Theorem.

Let $S$ be a compact connected orientable surface with boundary provided with a finite measure $\mu$ which is positive on open sets and $f : S \to S$ be an orientation preserving area preserving homeomorphism of $S$.

(54) Suppose that $L$ is an invariant branch of $f$ and that all fixed points of $f$ contained in $\text{cl}_S L$ are non degenerate. Then either $L$ is a connection or $L$ accumulates on both adjacent sectors. In the later alternative $L \subset \omega(L)$.

(55) Let $p \in S - \partial S$ be a fixed point of $f$ of saddle type and let $L_1$ and $L_2$ be adjacent branches of $p$ that are not connections. If $L_1$ and $L_2$ are invariant and all fixed points of $f$ contained in $\text{cl}_S (L_1 \cup L_2)$ are non degenerate then $\text{cl}_S L_1 = \text{cl}_S L_2$. 

(56) Let \( p \in S - \partial S \) be a fixed point of \( f \) of saddle type. Assume that all fixed points of \( f \) contained in \( \text{cl}_S(W^u_p \cup W^s_p) \) are non degenerate. If the branches of \( p \) are invariant and none of them is a connection, then all branches have the same closure in \( S \).

(57) Let \( C \) be a connected component of \( \partial S \) and suppose that all fixed points \( p_1, \ldots, p_{2n} \) of \( f \) in \( C \) are of saddle type. Let \( L_i \) be the branch of \( p_i \) contained in \( S - \partial S \). Assume that for every \( i \) all fixed points of \( f \) contained in \( \text{cl}_S L_i \) are non degenerate and that \( L_i \) is not a connection. Then \( \text{cl}_S L_i = \text{cl}_S L_j \) for any pair \((i, j)\).

4.4. Theorem.

Let \( S \) be a compact connected orientable surface with boundary. Let \( S_0 \subset S \) be a submanifold with compact boundary \( \partial S_0 \subset \partial S \) and let \( f, f^{-1} : S_0 \to S \) be an orientation preserving and area preserving homeomorphism of \( S_0 \) onto open subsets \( fS_0, f^{-1}S_0 \) of \( S \) with \( f(\partial S_0) \subset \partial S_0 \).

(58) Let \( p \in S_0 - \partial S \) be a fixed point of \( f \) of saddle type and let \( L_1, L_2 \) be adjacent and invariant branches of \( p \) such that \( \text{cl}_S L_i \subset S_0 \), \( i = 1, 2 \) and that both branches accumulate on the sector bounded by them. If in addition \( S \) has genus 0, then \( L_1 \cap L_2 \neq \emptyset \).

(59) Let \( p \in S_0 - \partial S \) be a fixed point of \( f \) of saddle type. Assume that the branches of \( p \) are invariant, with closure in \( S \) included in \( S_0 \). Assume also that each branch of \( p \) is not a connection, accumulates of its adjacent sectors and that all the branches of \( p \) have the same closure in \( S \). If in addition \( S \) has genus 0 or 1, then the four branches of \( p \) have homoclinic points.

(60) Let \( C \) be a connected component of \( \partial S_0 \) and suppose that all fixed points \( p_1, \ldots, p_{2n} \) of \( f \) in \( C \) are of saddle type. Let \( L_i \) be the branch of \( p_i \) contained in \( S - \partial S \). Assume that for every \( i \), \( L_i \) is not a connection, \( L_i \) accumulates on both of its adjacent sectors and that \( \text{cl}_S L_i = \text{cl}_S L_j \subset S_0 \) for every pair \((i, j)\).

If in addition \( S \) has genus 0, then every pair \((L_i, L_j)\) of stable and unstable branches intersect. The same happens if the genus of \( S \) is 1 provided that there are at least 4 fixed points in \( C \).

4.1. The double of a surface.

In order to prove Theorem 4.1 we would like to use the previous results for surfaces without boundary. For this we are going to work with the double of \( S \).

Let \( \sim \) be the equivalence relation defined by the partition of \( S \times \{0, 1\} \) into one point sets \( \{(p, i)\} \) if \( p \notin \partial S \), \( i \in \{0, 1\} \) and two point sets \( \{(p, 0), (p, 1)\} \) if \( p \in \partial S \). Let \( S' = S \times \{0, 1\}/ \sim \) equipped with the quotient topology. The surface \( S' \) is obtained by gluing two copies of \( S \) together along their common boundary and it is called the double of \( S \). The quotient map \( S \times \{0, 1\} \to S' \) is closed implying that \( S' \) is a Hausdorff space. If \( S \) is compact, connected and orientable, then so is \( S' \). The mapping \( \iota : S \to S' \) defined by \( \iota(p) = [(p, 0)] \) is a homeomorphism from \( S \) onto a closed subset \( \iota(S) \) of \( S' \). So we just think of \( S \) as a subset of \( S' \). Since \( \iota(S) \) is closed in \( S \) we have that if \( A \subset S \) then \( \text{cl}_S A = \text{cl}_{S'} A \).

The measure \( \mu \) can be “doubled” to a measure \( \mu' \) on \( S' \) by taking the measure \( \nu(\{0\}) = \nu(\{1\}) = 1 \) on \( \{0, 1\} \), the product measure \( \mu \times \nu \) on \( S \times \{0, 1\} \) and the pushforward by
the quotient map. Homeomorphisms of $S$ extend naturally to homeomorphisms of $S'$ by defining $f'[(x, i)] = [(f(x), i)]$. If $f$ is orientation preserving and area preserving, the so is $f'$.

The set $S'$ has the structure of a surface without boundary. Charts for neighborhoods of points $[(p, i)] \in S'$ with $p \in \partial S$ may be obtained in the following way. Let $B$ be an open ball with center at $(0,0)$, $B^+ = \{(x,y) \in B \mid y \geq 0\}$ and $\phi : B^+ \to S$ a chart for a point $p = \phi(0,0) \in \partial S$. Define $\phi' : B \to S'$ by $\phi'(x,y) = [(\phi(x,y), 0)]$ if $y \geq 0$ and $\phi'(x,y) = [\phi(x,-y),1]$ if $y \leq 0$. It follows from our definitions that if $p$ is a saddle in $\partial S$ then it is a saddle in $S'$.

4.2. Proof of Theorem 4.3: the closure of invariant manifolds.

4.3. Proof of item (54): auto accumulation of invariant manifolds.

Let $L$ be an invariant branch of $f$ such that all fixed points of $f$ contained in $cl_S L$ are non degenerate. Let $S'$ be the double of $S$ and $f'$ the extension of $f$ to $S'$. As a branch of $f'$, $L$ is invariant by $f'$ and all fixed points of $f'$ contained in $cl_{S'} L$ are non degenerate. By item (2) of Theorem 1.2, item (54) holds.

4.4. Proof of items (55) and (56): equal closure of invariant manifolds.

Now let $p \in S - \partial S$ be a fixed point of $f$ of saddle type and let $L_1$ and $L_2$ be invariant adjacent branches of $p$ that are not connections and assume that all fixed points of $f$ contained in $cl_S (L_1 \cup L_2)$ are non degenerate. By item (3) of Theorem 1.2 applied to $f'$ we have that $cl_{S'} L_1 = cl_{S'} L_2$.

That all the branches in item (56) have the same closure follows immediately from item (55).

4.5. Proof of item (57): the case of boundary fixed points.

Let $C$ be a connected component of $\partial S$ and suppose that all fixed points of $f$ in $C$ are of saddle type. Let $p_1, \ldots, p_{2n}$ be these fixed points and $L_1, \ldots, L_{2n}$ their branches contained in $S - \partial S$. We are assuming that for every $i$ all fixed points of $f$ contained in $cl_S L_i$ are non degenerate and that $L_i$ is not a connection.

We need to show that all these branches have the same closure, and for this it is enough to show that this happens for any pair of branches whose corresponding fixed points have a connection contained in $C$ as a common branch.

4.5. Lemma. Let $L_1$ and $L_2$ be any branches of $f$ possibly from different periodic points in $S$. If $L_1$ accumulates on a sector adjacent to $L_2$ then $L_2 \subseteq cl_S L_1$ (for surfaces with or without boundary).

Proof:

Just take $K = cl_{S'} L_1$ and $L = L_2$ in part (48) in Lemma 3.7.

Let $p_1$ and $p_2$ be fixed points of $f$ in $C$, and let $L_1$ and $L_2$ be their branches contained in $S - \partial S$. It suffices to consider the case when there is a connection $C_{12}$ from $p_1$ to $p_2$. 

By item (54) of Theorem 4.3, the branch $L_1$ accumulates on both sectors of $p_1$ adjacent to itself and therefore by Lemma 4.5, we have that $C_{12} \subset cl_S L_1$. From this we have that $L_1$ accumulates on the sector of $p_2$ that has $C_{12}$ and $L_2$ as adjacent branches. By Lemma 4.5 again we have that $L_2 \subset cl_S L_1$. It follows that $cl_S L_1 = cl_S L_2$.

4.6. Proof of Theorem 4.4: homoclinics.

4.7. Proof of item (58): homoclinics for specified branches in genus 0.

Assume now that $S$ has genus 0. Replacing $f$ by $f^2$ if necessary, we can assume that the branches $L_1$ and $L_2$ are fixed.

Consider a system of coordinates in a neighborhood $V$ of $p$ with $p$ at the origin and in which $f(x, y) = (\lambda^{-1}x, \lambda y)$ with $0 < \lambda < 1$, $y \geq 0$. We assume that $L_1$ is the unstable branch that contains \{(x, y) \in V | y = 0, x > 0\} and $L_2$ is the stable branch that contains \{(x, y) \in V | x = 0, y > 0\}. Since the branches $L_1$, $L_2$ are invariant and contained in $S_0$, we have that all their iterates by $f$ and $f^{-1}$ are defined, i.e.

$$\forall n \in \mathbb{Z}, \forall i \in \{1, 2\} : f^n(L_i) \subset S_0.$$ 

By hypothesis the branches $L_1$ and $L_2$ accumulate on the sector $\{(x, y) \in V | x > 0, y > 0\}$. Let

$$\Sigma = \{(x, y) \in V | 0 < xy \leq \lambda^2, 0 < x \leq 1, 0 < y \leq 1\},$$

$$En(\Sigma) = \{(x, y) \in \Sigma | \lambda < y \leq 1\},$$

$$Ex(\Sigma) = \{(x, y) \in \Sigma | \lambda < x \leq 1\}. $$

Note that $En(\Sigma)$ and $Ex(\Sigma)$ are disjoint. From the dynamics of $f$ in $\Sigma$ we see that every orbit of $f$ that visits $\Sigma$ must enter $\Sigma$ through $En(\Sigma)$, meaning that if $x \notin \Sigma$ and $f^n(x) \in \Sigma$ for some positive $n$, then for the smallest $n$ with this property $f^n(x) \in En(\Sigma)$. In the same way every orbit of $f^{-1}$ that visits $\Sigma$ must enter $\Sigma$ through $Ex(\Sigma)$.

We start at $p$ and move along the unstable branch $L_1$, let $q_1$ be the first point of $L_1$ to intersect $\Sigma$. Since $L_1$ is unstable we have that $q_1 \in En(\Sigma)$. Join $q_1$ and $p$ by a line segment $\gamma_1 \subset int_S \Sigma$ and consider $\Gamma_1 = L_1[p, q_1] \cup \gamma_1$, where $L_1[p, q_1]$ is the segment of $L_1$ between $p$ and $q_1$. Observe that $\Gamma_1$ is a simple closed curve that separates $S$ into two connected open sets whose common frontier in $S$ is $\Gamma_1$.

If we start at $p$ and move along the stable branch $L_2$, let $q_2$ be the first point of $L_2$ to intersect $\Sigma$. Since $L_2$ is stable we have that $q_2 \in Ex(\Sigma)$. Join $q_2$ and $p$ by a line segment $\gamma_2 \subset int_S \Sigma$ and consider $\Gamma_2 = L_2[p, q_2] \cup \gamma_2$, where $L_2[p, q_2]$ is the segment of $L_2$ between $p$ and $q_2$. Then $\Gamma_2$ is also a simple closed curve in $S$. Note that the local branch of $L_2$ and $\gamma_2$ are in different components of $S - \Gamma_1$. From this we conclude that $\Gamma_1$ and $\Gamma_2$ must intersect again at a point $q$ different from $p$. Since $\Gamma_1 \cap \gamma_2 = \emptyset$ and $\Gamma_2 \cap \gamma_1 = \emptyset$ we have that $q \in L_1[p, q_1] \cap L_2[p, q_2]$.

4.8. Proof of item (59): homoclinics in genus 0 and 1.

The existence of homoclinic points if the genus of $S$ is 0 follows immediately from item (58). Consider now the case when $S$ has genus 1.
We may assume that $S$ is the two dimensional torus $\mathbb{T}^2$ minus a finite union of (sets homeomorphic to) open disks. By restricting the canonical 3-fold covering
\begin{equation}
\mathbb{R}^2/(3\mathbb{Z} \times \mathbb{Z}) \to \mathbb{R}^2/\mathbb{Z}^2
\end{equation}
we obtain a 3-fold covering $\pi : E \to S$ whose group of deck transformations is generated by the rotation $T(x, y) = (x + 1, y)$. Let $F : E \to E$ be a lifting of $f$ such that the three points in $\pi^{-1}\{p\}$ are fixed.

Let $\pi^{-1}(p) = \{p_1, p_2, p_3\}$. If $L$ is a branch of $p$ then $\pi^{-1}(L) = L_1 \cup L_2 \cup L_3$ where every $L_i$ is a branch of $p_i$, $L_2 = TL_1$ and $L_3 = TL_2$. In order to prove that $p$ has homoclinic points, it is enough to prove that for some pair $(i, j)$ an unstable branch of $p_i$ intersects a stable branch of $p_j$. The branches of $p$ have the same closure in $S$. We prove in Lemma 4.7 that the same happens with the points $p_i$.

Let $K_i$ be the closure in $E$ of the branches of each $p_i$.

The following version of the accumulation lemma is proven in Theorem 4.3 of [9].

4.6. Lemma. [9, Th. 4.3]

Let $S$ be a connected surface with compact boundary provided with a Borel measure $\mu$ such that open non-empty subsets have positive measure and compact subsets have finite measure. Let $S_0 \subset S$ be an open subset with $f|_{S_0}$ compact.

Let $f, f^{-1} : S_0 \to S$ be an area preserving homeomorphism of $S_0$ onto an open subsets $f(S_0)$, $f^{-1}(S_0)$ of $S$. Let $K \subset S_0$ be a compact connected invariant subset of $S_0$.

If $L \subset S_0$ is a branch of $f$ and $L \cap K \neq \emptyset$, then $L \subset K$.

4.7. Lemma.

All the branches of each $p_i$ have the same closure $K_i$. There are two disjoint alternatives:

(63) For every pair $(i, j)$ all branches of $p_i$ accumulate on all sectors of $p_j$ and $K_i = K_j$.

(64) There exist pairwise disjoint neighborhoods $V_i$ of $p_i$ such that $K_i \cap V_j = \emptyset$ if $i \neq j$.

Alternative (63) holds when there exists a pair $(i, j)$ with $i \neq j$ for which a branch of $p_i$ accumulates on a sector of $p_j$.

In the case (64) it might happen that $K_i \cap K_j \neq \emptyset$, but we do not need this here.

Proof:

Let $L^i_\alpha$, $1 \leq \alpha \leq 4$ be the branches of $p_i$ and let $S^i_\alpha$ be the sectors of $p_i$ with $(L^i_\alpha \cup L^i_{\alpha+1}) \cap B_\delta \subset cl_E(S^i_\alpha)$ as in (49).

We first prove that in alternative (64) all the branches of $p_i$ have the same closure. We use alternative (64) with $K_i$ being the closure of $\cup_{\alpha=1}^4 L^i_\alpha$. By hypothesis (59), in $S$ all the branches of $p$ have the same closure. If $1 \leq \alpha, \beta \leq 4$ the branch $L^i_\alpha$ accumulates on some branch $L^j_\beta$, i.e. $L^i_\alpha \cap cl_E(L^j_\beta) \neq \emptyset$. The statement of alternative (64) implies that $j = i$.

By Lemma 4.6, $L^i_\beta \subset cl_E(L^i_\alpha)$. Similarly $L^i_\alpha \subset cl_E(L^i_\beta)$. Thus $cl(L^i_\alpha) = cl(E(L^i_\beta) = K_i$.

65. Claim:

If there is $\alpha$ and $i \neq j$ such that $L^i_\alpha$ accumulates on $L^i_\beta$ then alternative (63) holds.
For simplicity assume that $\alpha = 1$, $(i, j) = (1, 2)$. We have that $L_1^2 \cap cl_E L_1^1 \neq \emptyset$. By Lemma 4.6, $L_1^2 \subset cl_E L_1^1$. Then $L_1^3 = TL_1^2 \subset Tcl_E(L_1^1) = cl_E(L_1^3)$. Similarly

$$L_1^1 \subset cl_E(L_1^3) \subset cl_E(L_1^2) \subset cl_E(L_1^1) =: K,$$

and the inclusions in (66) are equalities. Now let $\beta \neq 1$ and $k \in \{1, 2, 3\}$. We have that some $L_1^1$ accumulates on $L_1^k$ and hence $L_1^k \subset cl_E(L_1^j) = K$. Also $L_1^k$ accumulates on some $L_1^j$ and then $K = cl_E(L_1^j) \subset L_1^k$. This proves the claim.

The hypothesis in (59) implies that any branch of $p$ accumulates on all the sectors of $p$. By Lemma 4.6 if a branch $L_1^\alpha$ accumulates on a sector $S_\beta^j$ then $L_1^\beta \cup L_{\beta+1}^j \subset cl_E(L_1^\alpha)$.

Suppose that some $L_1^\alpha$ accumulates on the sector $S_\beta^j$ with $j \neq i$. Alternative (64) is just the negation of this assumption. So it remains to prove that alternative (63) holds. Assume by contradiction that alternative (63) does not hold. For simplicity $(i, \alpha) = (1, 1)$. Then

$$L_1^j \cup L_{\beta+1}^j \subset cl_E(L_1^1).$$

For some $k$ the branch $L_1^j$ accumulates on $L_1^k$ and then $L_1^k \subset cl_E(L_1^j) \subset cl_E(L_1^1)$. If $k \neq 1$ claim 65 implies that alternative (63) holds. Therefore $k = 1$. Hence $cl_E(L_1^j) = cl_E(L_1^1)$.

Similarly

$$cl_E(L_{\beta+1}^j) = cl_E(L_1^j) = cl_E(L_1^1).$$

The branch $L_{\beta+1}^j$ accumulates on some sector $S_{\beta+1}^k$ and hence

$$L_{\beta+1}^k \cup L_{\beta+2}^k \subset cl_E(L_{\beta+1}^1) = cl_E(L_1^1).$$

By claim 65 applied to $\alpha = \beta + 1$ we have that $k = j$. The same argument as above using (68) instead of (67) gives

$$cl_E(L_{\beta+2}^j) = cl_E(L_{\beta+1}^j) = cl_E(L_1^j) = cl_E(L_1^1).$$

Repeating the argument once more we get that $cl_E(L_1^j) = cl_E(L_1^1)$ for all $\beta$. In particular for $\beta = 1$, $cl_E(L_1^1) = cl_E(L_1^1)$. Claim 65 for $\alpha = 1$ implies that alternative (63) holds. A contradiction.

Assume that alternative (63) holds.

Choose a sector $\Sigma$ of some $p_i$, say $p_1$, and continuous coordinates about $p_1$ with $p_1$ at $(0, 0)$ such that $f(x, y) = (\lambda^{-1}x, \lambda y)$, $0 < \lambda < 1$. As before we will write $\Sigma$ as

$$\Sigma = \{(x, y) \in V \mid 0 < xy \leq \lambda^2, \ 0 < x \leq 1, \ 0 < y \leq 1\}$$

and consider the sets $En(\Sigma)$ and $Ex(\Sigma)$ defined in (61). We are going to denote the unstable branches of $p_i$ by $W^u_i(p_i)$ and $W^u(p_i)$ and the stable ones by $W^s_i(p_i)$ and $W^s(p_i)$.

All branches of all points $p_i$ accumulate on $\Sigma$.

Let $q_+^u(p_i) \in W^u_i(p_i)$ and $q_-^u(p_i) \in W^u(p_i)$ be the first point of each unstable branch to intersect $\Sigma$ as we move along the corresponding branch starting from $p_i$. Both $q_+^u(p_i)$ and $q_-^u(p_i)$ belong to $En(\Sigma)$. Join $q_-^u(p_i)$ and $q_+^u(p_i)$ by a small arc $\gamma_i^u$ contained in $En(\Sigma)$. Let $W^u[q_-^u(p_i), q_+^u(p_i)]$ be the segment inside $W^u(p_i)$ from $q_-^u(p_i)$ to $q_+^u(p_i)$. Then we have that $\Gamma_i^u = W^u[q_-^u(p_i), q_+^u(p_i)] \cup \gamma_i^u$ is a simple closed curve that contains $p_i$. 

In the same way let \( q^+_i(p_i) \in W^s_1(p_i) \) and \( q^-_i(p_i) \in W^u_1(p_i) \) be the first point of each stable branch to intersect \( \Sigma \) as we move along the corresponding branch starting from \( p_i \). Both \( q^+_i(p_i) \) and \( q^-_i(p_i) \) belong to \( Ex(\Sigma) \). Join \( q^+_i(p_i) \) and \( q^-_i(p_i) \) by a small arc \( \gamma_i^s \) contained in \( E_n(\Sigma) \). Let \( W^s[q^-_i(p_i), q^+_i(p_i)] \) be the segment inside \( W^s(p_i) \) from \( q^-_i(p_i) \) to \( q^+_i(p_i) \). Then we have that \( \Gamma_i^s = W^s[q^-_i(p_i), q^+_i(p_i)] \cup \gamma_i^s \) is a simple closed curve that contains \( p_i \).

We are going to assume that \( p \) has no homoclinic points and derive a contradiction.

We have that \( \gamma_i^s \cap \gamma_j^s = \emptyset \) for all pairs \((i,j)\). Therefore \( \Gamma_i^s \cap \Gamma_j^s = \emptyset \) if \( i \neq j \) and \( \Gamma_i^s \cap \Gamma_i^s = \{p_i\} \).

If \( \alpha \) and \( \beta \) are oriented closed curves in \( \mathbb{T}^2 \) let \( \#(\alpha, \beta) \) be the oriented intersection number of \( \alpha \) and \( \beta \). Then \( \# \) depends only on the homology classes of \( \alpha \) and \( \beta \) and is a non degenerate skew symmetric bilinear form over the integers defined on \( H_1(\mathbb{T}^2) = \mathbb{Z}^2 \).

We have that \( E \subset \mathbb{T}^2 \) and if we look at \( \Gamma_i^u \) and \( \Gamma_i^s \) as elements of \( H_1(\mathbb{T}^2) \), then

\[
\text{(69)} \quad \#(\Gamma_i^u, \Gamma_j^s) = \begin{cases} 1 & \text{if } i = j, \\
0 & \text{if } i \neq j. \end{cases}
\]

The rank\(^1 \) of \( H_1(\mathbb{T}^2) \) is 2 and therefore \( \{\Gamma_1^u, \Gamma_2^s, \Gamma_3^s\} \) is linearly dependent. So one of them is a linear combination of the other two, say \( \Gamma_i^u = a\Gamma_2^s + b\Gamma_3^s \). It follows that

\[
\#(\Gamma_i^u, \Gamma_1^s) \leq |a|\#(\Gamma_1^u, \Gamma_2^s) + |b|\#(\Gamma_1^u, \Gamma_3^s) = 0,
\]

which contradicts (69).

This proves the existence of a homoclinic point if alternative (63) holds.

Suppose now that there exist pairwise disjoint neighborhoods \( V_i \) of \( p_i \) such that \( K_i \cap V_j = \emptyset \) if \( i \neq j \). The procedure is almost the same. For each \( i \) we consider a sector \( \Sigma_i \) of \( p_i \). The branches of one \( p_i \) do not intersect \( \Sigma_j \) if \( i \neq j \). The curves \( \Gamma_i^u \) and \( \Gamma_i^s \) are constructed by considering the first point of each branch of \( p_i \) to intersect \( \Sigma_i \). They also satisfy (69), and we derive a contradiction in the same way.

In order ot complete the proof of item (52), we need to show that once \( p \) has a homoclinic point then the four branches of \( p \) all have homoclinics points. We say that two curves \( \gamma_1, \gamma_2 \) cross if they intersect at a point \( q \in \gamma_1 \cap \gamma_2 \) and there is a contractible neighborhood \( U \) of \( q \) such that \( U - \gamma_1 \) is disconnected and \( U \cap \gamma_2 \) intersects different connected components of \( U - \gamma_1 \).

4.8. Lemma. If two branches of \( p \) cross, then all branches of \( p \) cross.

Proof: Let \( q \) be a homoclinic point of \( p \) and let \( \gamma \) be the closed curve obtained by the union of the segments of the branches that contain \( q \) from \( q \) to \( f(q) \). If \( \Gamma = \gamma \cup f(\gamma) \cup f^2(\gamma) \), then \( S - \Gamma \) is disconnected. Let \( A \) be the component of \( S - \Gamma \) that contains \( p \) and its local branches and let \( B \) be any other component. Since \( f \) is area preserving, there exists \( n \) such that \( f^n B \cap \text{int}_S B \neq \emptyset \). Therefore \( \text{int}_S B \) contains points of \( \text{fr}_S(f^n B) \) which is made of invariant manifolds. But the four branches have the same closure in \( S \), implying that

\(^1\)In other surfaces their cyclic covers do not have the same Betti numbers as in the base and then the following argument does not apply.
each one intersects different components of $S - \Gamma$. It follows that each branch intersects $fr_S(S - \Gamma) = \Gamma$. 

4.9. **Proof of item (60): the case of boundary fixed points.**

**Case 1: $S$ has genus 0.**

Suppose now that $S$ has genus zero.

Let $p_1$ and $p_2$ be any fixed points of $f$ in $C$, let $L_1$ and $L_2$ be their branches contained in $S - \partial S$ and assume that $L_1$ is unstable and $L_2$ is stable. Since $cl_S L_1 = cl_S L_2$ and $L_2$ accumulates on both sectors of $p_2$ adjacent to itself, we have that $L_1$ accumulates on both sectors of $p_2$ adjacent to $L_2$.

Consider a system of coordinates in a neighborhood $V$ of $p_2$ with $p_2$ at the origin in which $f(x, y) = (\lambda^{-1}x, \lambda y)$ with $0 < \lambda < 1$, $y \geq 0$. We have that $L_2$ is the branch that contains $\{(x, y) \in V : x = 0, y > 0\}$. Consider the sector

$$\Sigma = \{(x, y) \in V : 0 < xy \leq \lambda^2, 0 < x \leq 1, 0 < y \leq 1\}$$

of $p_2$ and the sets $En(\Sigma)$ and $Ex(\Sigma)$ as defined in (61). The branch $L_1$ intersects $\Sigma$ and if we move along $L_1$ starting from $p_1$ the first point $q_1$ of $L_1$ to intersect $\Sigma$ belongs to $En(\Sigma)$. Join $q_1$ to $p_2$ with a line segment $\gamma_1 \subset int_S \Sigma$ and consider $\Gamma_1 = L_1[q_1, p_1] \cup \gamma_1 \cup C_{21}$, where $C_{21}$ is the component of $C - \{p_1, p_2\}$ that contains the branch of $p_2$ that contains $\{(x, y) \in V : x > 0, y = 0\}$. $\Gamma_1$ is a simple closed curve.

The branch $L_2$ intersects $\Sigma$ and as we move along $L_2$ starting from $p_2$, the first point $q_2$ of $L_2$ to intersect $\Sigma$ belongs to $Ex(\Sigma)$. Join $q_2$ to $p_2$ with a line segment $\gamma_2 \subset int_S \Sigma$ and consider the simple closed curve $\Gamma_2 = L_2[p_2, q_2] \cup \gamma_2$. $\Gamma_1$ and $\Gamma_2$ intersect at $p_2$. The local branch of $L_2$ and the curve $\gamma_2$ belong to different components of $S - \Gamma_1$. From this we conclude that $\Gamma_1$ and $\Gamma_2$ must intersect again at a point $q$ different from $p_2$. Since $\Gamma_1 \cap \gamma_2 = \{p_2\}$ and $\Gamma_2 \cap (\gamma_1 \cup C_{21}) = \{p_2\}$, we have that $q \in L_1[q_1, p_1] \cap L_2[p_2, q_2]$. This proves the case of genus zero.

**Case 2: $S$ has genus 1 and $C$ has at least 4 fixed points:**

Suppose now that $S$ has genus one.

Let $p_1, p_2, p_3, p_4$ be fixed points of $f$ such that $p_i$ and $p_{i+1}$ are end points of connections contained in $C$ for $1 \leq i \leq 3$. Let $L_i$ be the branch of each $p_i$ contained in $S - \partial S$ and assume that $L_1$ and $L_3$ are stable and $L_2$ and $L_4$ are unstable. By hypothesis $cl_S L_i = cl_S L_j$ for any $i, j$. Let $\pi : E \to S$ be a 3-fold covering of $S$ defined after (62) and let $F : E \to \tilde{E}$ be a lifting of $f$.

Let $\pi^{-1}(C) = C^1 \cup C^2 \cup C^3$ be the lifting of the boundary component; $\pi^{-1}(p_i) = \{p^1_i, p^2_i, p^3_i\}$, $1 \leq i \leq 4$, the lifting of the fixed points $p_i$ and $\pi^{-1}(L_i) = \{L^1_i, L^2_i, L^3_i\}$, $1 \leq i \leq 4$, the lifting of the corresponding branches.

To show that any pair $(L_i, L_j)$ of stable and unstable branches intersect, it is enough to show that this happens for one pair. This follows from the fact that they all have the same closure in $S$ and an argument similar to Lemma 4.8. To show that a pair $(L_i, L_j)$ of stable and unstable branches intersect, it is enough to show that this happens for a pair $(L^m_i, L^n_j)$ of stable and unstable branches in $E$.

We are going to assume that this does not happen and derive a contradiction.
We would like to construct simple closed curves such as $\Gamma^m_1$ and $\Gamma^p_2$ in the proof of item (59) in § 4.8, with the same intersecting properties. The same proof as in Lemma 4.7 shows that for each $n \in \{1, 2, 3\}$ the branches $L^n_1, L^n_2, L^n_3, L^n_4$ have the same closure $K^n$ in $E$ and that there are two alternatives:

(70) There exist a pair $(m, n)$ with $m \neq n$ and a pair $(i, j)$ such that a branch $L^m_i$ accumulates on a sector of $p^j_n$. In this case we have that all branches $L^m_i$ accumulate on all sectors of the points $p^n_j$ and $K^1 = K^2 = K^3$. This follows from the action of deck transformations on fibers.

(71) For every $i, j$ and $m \neq n$, the branches $L^m_i$ do not accumulate on any of the sectors of points $p^n_j$. In this case there exist neighborhoods $V^n_i$ of $p^n_i$ such that $K^n \cap V^n_i = \emptyset$ if $m \neq n$.

Suppose that alternative (70) holds.

For $n = 1, 2, 3$ let $C^n_{ij}$ be the closed arc of $C^n$ that has $p^n_i$ and $p^n_j$ as end points and contains $p^n_2, p^n_3$ and the three connections between them. For $1 \leq i < j \leq 4$, let $C^n_{ij}$ be the closed arc contained in $C^n_{ij}$ that has $p^n_i$ and $p^n_j$ as end points.

Let $\Sigma$ be the sector of $p^n_1$ that has $L^n_1$ and the connection contained in $C^n_{12}$ as adjacent branches:

$$\Sigma = \{(x, y) \in V : 0 < xy \leq \lambda^2, 0 < x \leq 1, 0 < y \leq 1\}$$

in local linearizing coordinates.

All branches $L^n_i$ intersect $\Sigma$. Let $q^n_i$ be the first point of $L^n_i$ to intersect $\Sigma$. If $i \in \{1, 3\}$ then $L^n_i$ is stable and $q^n_i \in \text{Ex}(\Sigma)$. If $i \in \{2, 4\}$ then $L^n_i$ is unstable and $q^n_i \in \text{En}(\Sigma)$. Let $L^n_i(p^n_i, q^n_i)$ be the open arc inside $L^n_i$ between $p^n_i$ and $q^n_i$.

If we define $\Gamma^n_{13}$ as $L^n_1(p^n_1, q^n_1) \cup C^n_{13} \cup L^n_3(p^n_3, q^n_3) \cup \gamma^n_{13}$ where $\gamma^n_{13}$ is a small arc from $q^n_1$ to $q^n_3$ contained in $\text{Ex}(\Sigma)$, then $\Gamma^n_{13}$ is a simple closed curve made of parts of stable branches, an arc of $C^n$ and a small arc contained in $\text{Ex}(\Sigma)$.

Similarly, let $\Gamma^n_{24} = L^n_2(p^n_2, q^n_2) \cup C^n_{24} \cup L^n_4(p^n_3, q^n_4) \cup \gamma^n_{24}$, where $\gamma^n_{24}$ is an arc from $q^n_2$ to $q^n_4$ contained in $\text{En}(\Sigma)$. Then $\Gamma^n_{24}$ is a simple closed curve made of parts of unstable branches, an arc of $C^n$ and a small arc contained in $\text{En}(\Sigma)$.

From the assumption that different branches $L^n_i$ and $L^n_j$ do not intersect, we have that $\Gamma^n_{13} \cap \Gamma^n_{24} = \emptyset$ if $n \neq m$ and $\Gamma^n_{13} \cap \Gamma^n_{24} = \text{C^n}_{24}$ for $n = 1, 2, 3$.

We have that $E \subset \mathbb{T}^2$ and if we look at the curves $\Gamma^n_{ij}$ as subsets of $\mathbb{T}^2$, then $\{\Gamma^n_{13}, \Gamma^n_{24}, \Gamma^n_{13}\}$ and $\{\Gamma^n_{24}, \Gamma^n_{24}, \Gamma^n_{24}\}$ are linearly dependent subsets of $\mathcal{H}_1(\mathbb{T}^2)$. We have that $\#(\Gamma^n_{13}, \Gamma^n_{24}) = 0$ if $n \neq m$.

We claim that $\#(\Gamma^n_{13}, \Gamma^n_{24}) = 1$ for $n = 1, 2, 3$.

In order to see why this happens we are going to homotopically modify each $\Gamma^n_{24}$ so that $\Gamma^n_{13}$ and $\Gamma^n_{24}$ intersect only at $p^n_3$ and this point is a crossing, meaning that there exists a neighborhood $V$ of $p^n_3$ such that the complement in $V$ of each curve has two components, with the other curve intersecting both.

Each boundary component $C^n$ has a collar neighborhood $W$ homeomorphic to $(\mathbb{R}/5\mathbb{Z}) \times [0, 1]$ where we take coordinates $(\theta, r)$. Points in $C^n$ have coordinate $r = 0$. The following refers to the coordinates $(\theta, r)$. 
We may assume that $p^n_0 = (i, 0)$. Therefore $\Gamma^n_{13} \cap \Gamma^n_{24} = [2, 3] \times \{0\}$.

For any $\delta_1 > 0$ there exists $\delta_2 > 0$ such that $R = (|2 - \delta_1, 3 - \delta_1| \times [0, \delta_2])$ is disjoint from $\Gamma^n_{13}$ and contains the local branch of $L^n_2$. Let $q_0 \in R$ be a point of the local branch of $L^n_2$ inside $R$. It follows that the straight line closed segment $\alpha_1$ from $q_0$ to $q_1 := (3 - \delta_1, 0)$ is contained in $R \cup \{q_1\}$. Let $\alpha_0$ be the closed segment inside $\Gamma^n_{24}$ from $q_0$ to $q_1$ that contains $p^n_2$. The arc $\alpha_0$ is the union of a segment of the local branch of $L^n_2$ with $[2, 3 - \delta_1] \times \{0\}$. Let $\alpha : [0, 1]^2 \to S$ be the straight line path homotopy $\alpha(s, t) = (1 - t)\alpha_0(s) + t\alpha_1(s)$ from $\alpha_0$ to $\alpha_1$.

Now let us consider a neighborhood $V$ of $p^n_0$ and coordinates in $V$ with $p^n_3 = (0, 0)$ and $f(x, y) = (\lambda^{-1} x, \lambda y)$, $y \geq 0$ and $\lambda < 1$. We may assume that $V$ is an open ball with center $(0, 0)$ in these coordinates. If $\delta_1$ is small enough, we have that $q_1 \in V$. Therefore for $a < 1$ and close enough to 1, we have that $\beta_a(t) := \alpha(a, t) \in V$ for all $t \in [0, 1]$. Let $\beta_1(t) = p^n_3$ and let $\beta : [a, 1] \times [0, 1] \to S$ be the straight line path homotopy from $\beta_a$ to $\beta_1$ given by

$$\beta(s, t) = \frac{1 - a}{1 - a} \beta_a(t) + \frac{a}{1 - a} \beta_1(t).$$

We have that $\alpha(a, t) = \beta(a, t)$ and therefore $\alpha$ and $\beta$ define a continuous map $\gamma : [0, 1]^2 \to S$ by $\gamma(s, t) = \alpha(s, 0)$ if $s \leq a$ and $\gamma(s, t) = \beta(s, t)$ if $s \geq a$.

Then $\gamma$ is a path homotopy between $\gamma_0(s) = \gamma(s, 0)$ and $\gamma_1(s) = \gamma(s, 1)$, where

$$\gamma_0(s) = \begin{cases} 
\alpha_0(s) & \text{if } s \leq a, \\
\frac{1 - a}{1 - a} \alpha_0(a) + \frac{a}{1 - a} p^n_3 & \text{if } s \geq a
\end{cases}$$

is the arc of $\Gamma^n_{24}$ from $q_0$ to $p^n_3$ that contains $p^n_2$, and

$$\gamma_1(s) = \begin{cases} 
\alpha_1(s) & \text{if } s \leq a, \\
\frac{1 - a}{1 - a} \alpha_1(a) + \frac{a}{1 - a} p^n_3 & \text{if } s \geq a.
\end{cases}$$

We have that $\gamma_1(0) = q_0$, $\gamma_1(1) = p^n_3$ and since $\alpha_1$ is a straight line in the coordinates of $R$ we have that $\gamma_1([0, a]) \subset R \subset S - \Gamma^n_{13}$. We also have that $\gamma_1(1) = p^n_3$ and that $\gamma_1(a)$ belongs to the sector $\Sigma = \{(x, y) \in V : x < 0, y > 0\}$. Since $\gamma_1$ is a straight line in the coordinates of $V$ we have that $\gamma_1([a, 1]) \subset \Sigma \subset S - \Gamma^n_{13}$.

Therefore $\gamma^*([0, 1]) \cap \Gamma^n_{13} = \{p^n_3\}$. Let us replace the arc of $\Gamma^n_{24}$ from $q_0$ to $p^n_3$ that contains $p^n_2$ by $\gamma_1([0, 1])$ to obtain a new simple closed curve $\Lambda$.

Then $\Lambda$ and $\Gamma^n_{24}$ are homotopic and $\Lambda \cap \Gamma^n_{13} = \{p^n_3\}$.

In the coordinates given by $V$ we have that

$$\Gamma^n_{13} \cap V = \{(x, y) \in V : x \leq 0, y = 0\} \cup \{(x, y) \in V : x = 0, y \geq 0\}$$

and the complement of $\Gamma^n_{13}$ in $V$ has two components $\Sigma = \{(x, y) \in V : x < 0, y > 0\}$ and $\Sigma' = \{(x, y) \in V : x > 0, y \geq 0\}$.

On the other hand $\Lambda \cap V$ is the disjoint union of $\gamma_1([0, 1]) \cap \Sigma$, $\{p^n_3\}$ and $\{(x, y) \in V : x > 0, y = 0\}$. Therefore $\Lambda$ intersects both $\Sigma$ and $\Sigma'$.

From this we conclude that $\#(\Lambda, \Gamma^n_{13})$ is 1 or $-1$ which proves the claim.
Now the contradiction comes easily from the fact that \( \Sigma_{n=1}^{3} a_n \Gamma_{13}^n = 0 \), with \( a_m \neq 0 \) for some \( m \); implying that

\[ 0 = \#(\Sigma_{n=1}^{3} a_n \Gamma_{13}^n, \Gamma_{24}^m) = a_m \#(\Gamma_{13}^m, \Gamma_{24}^m). \]

A contradiction.

This proves item (60) in the case \( S \) has genus one and alternative (70) holds.

When alternative (71) holds, we consider a sector \( \Sigma_n \) of \( p_i^m \) contained in \( V_1^n \) for \( 1 \leq n \leq 3 \). For each \( n \) the four branches \( L_i^m \) intersect \( \Sigma_n \) and are disjoint from \( \Sigma_m \) if \( n \neq m \). Let \( q_i^n \) be the first point of \( L_i^m \) to intersect \( \Sigma_n \).

For each \( n = 1, 2, 3 \) we define a pair of simple closed curves \( \{ \Gamma_{13}^n, \Gamma_{24}^n \} \) as in alternative (70). The only difference is that the arcs \( \gamma_{13}^n \) and \( \gamma_{24}^n \) will be contained in different sectors \( \Sigma_n \). If there are no homoclinic points then using (70) we still have that \( \Gamma_{13}^n \cap \Gamma_{24}^m = \emptyset \) if \( n \neq m \) and \( \Gamma_{13}^n \cap \Gamma_{24}^n = C_{23}^n \) for \( n = 1, 2, 3 \), and a contradiction comes up when we compute their oriented intersection number. This ends the proof of Theorem 4.4.

\( \square \)

The following corollary extends Theorem 4.1 to periodic points. It follows immediately by taking powers of \( f \) and adding the hypothesis of elliptic periodic being irrationally elliptic.

4.9. Corollary.

Let \( S \) be a compact connected orientable surface with boundary provided with a finite measure \( \mu \) which is positive on open sets and let \( f : S \to S \) be an orientation preserving area preserving homeomorphism of \( S \).

(72) Suppose that \( L \) is a (periodic) branch of \( f \) and that all periodic points of \( f \) contained in \( \text{cl}_S L \) are of saddle type or irrationally elliptic. Then either \( L \) is a connection or \( L \) accumulates on both adjacent sectors. In the later alternative \( L \subset \omega(L) \).

(73) Let \( p \in S - \partial S \) be a periodic point of \( f \) of saddle type and let \( L_1 \) and \( L_2 \) be adjacent branches of \( p \) that are not connections. If all the periodic points of \( f \) contained in \( \text{cl}_S(L_1 \cup L_2) \) are of saddle type or irrationally elliptic, then \( \text{cl}_S L_1 = \text{cl}_S L_2 \).

(74) Suppose that \( p \in S - \partial S \) is a periodic point of \( f \) of saddle type. Assume that all the periodic points contained in \( \text{cl}_S(W_p^s \cup W_p^u) \) are of saddle type or irrationally elliptic and \( p \) has no connections. Then the branches of \( p \) have the same closure and each branch of \( p \) accumulates on all the sectors of \( p \).

If in addition \( S \) has genus zero or one, then the four branches of \( p \) have homoclinic points.

(75) Let \( C \) be a connected component of \( \partial S \) and suppose that all the periodic points \( p_1, \ldots, p_m \) of \( f \) in \( C \) are of saddle type. Let \( L_i \) be the branch of \( p_i \) contained in \( S - \partial S \). Assume that for every \( i \) all the periodic points of \( f \) contained in \( \text{cl}_S L_i \) are of saddle type or irrationally elliptic and that \( L_i \) is not a connection. Then for every pair \( (i, j) \) the branch \( L_i \) accumulates on all the sectors \( p_j \) and \( \text{cl}_S L_i = \text{cl}_S L_j \).

If in addition \( S \) has genus zero then any pair \( (L_i, L_j) \) of stable and unstable branches intersect. The same happens if the genus of \( S \) is 1 provided that there are at least 4 periodic points in \( C \).
The version of the Corollary for maps defined on an open subset is the following:

4.10. Corollary.

Let $S$ be a compact connected orientable surface with boundary. Let $S_0 \subset S$ be a submanifold with compact boundary $\partial S_0 \subset \partial S$ and let $f, f^{-1}: S_0 \rightarrow S$ be an orientation preserving and area preserving homeomorphism of $S_0$ onto open subsets $fS_0, f^{-1}S_0$ of $S$ with $f(\partial S_0) \subset \partial S_0$.

(76) Let $p \in S_0 - \partial S$ be a periodic point of $f$ of saddle type. Assume that the branches of $p$ have closure included in $S_0$. Assume also that each branch of $p$ accumulates on both of its adjacent sectors and that all the branches of $p$ have the same closure in $S$. If in addition $S$ has genus 0 or 1, then the four branches of $p$ have homoclinic points.

(77) Let $C$ be a connected component of $\partial S_0$ and suppose that all the periodic points $p_1, \ldots, p_{2n}$ of $f$ in $C$ are of saddle type. Let $L_i$ be the branch of $p_i$ contained in $S - \partial S$. Assume that for every $i$, $L_i$ is not a connection and $\text{cl}_S L_i = \text{cl}_S L_j \subset S_0$ for every pair $(i, j)$.

If in addition $S$ has genus 0, then every pair $L_i, L_j$ of stable and unstable branches intersect. The same happens if the genus of $S$ is 1 provided that there are at least 4 periodic points in $C$.

5. The standard map.

The standard map is a one parameter family of area preserving diffeomorphisms of the two dimensional torus $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ given by

$$f_\lambda(x, y) = \left(x + y + \frac{\lambda}{2\pi} \sin(2\pi x), \ y + \frac{\lambda}{2\pi} \sin(2\pi x) \right), \ \lambda \in \mathbb{R}.$$ 

The map $f_0$ is just a twist. The map $\varphi(x, y) = (x + \frac{1}{2}, y)$ is a conjugacy between $f_\lambda$ and $f_{-\lambda}$, so we consider only parameters $\lambda > 0$.

For $\lambda \neq 0$ there are two fixed points, $p = (0, 0)$ and $q = (\frac{1}{2}, 0)$. For $\lambda > 0$, $p$ is always a saddle with positive eigenvalues, and $q$ is elliptic if $0 < \lambda < 4$ and a saddle with negative eigenvalues if $\lambda > 4$.

5.1. Theorem. If $\lambda \neq 4, \lambda > 0$, then the four branches of $p$ have homoclinic points.

Proof: For $\lambda \neq 4$ if $L$ is a branch of $p$ then $L$ is invariant and all fixed points of $f$ contained in $\text{cl}_S L$ are non-degenerate. The identity $f(-x, -y) = -f(x, y)$ implies that the invariant manifolds of $p$ are symmetric with respect to $(0, 0)$. Therefore if $L$ is a connection then so is $-L$.

There can not be a connection between $p$ and $q$. This is obvious when $q$ is elliptic, and when $q$ is a saddle this can not happen because the branches of $p$ are invariant and those of $q$ have period two.

Therefore if one of the branches of $p$ is a connection, then this connection equals two branches of $p$. By the symmetry shown above the invariant manifolds of $p$ are made of two connections of homoclinic points. So in this case the four branches have homoclinic points.
If no branch of \( p \) is a connection, then by (52) of Theorem 4.1, the four branches of \( p \) have homoclinic points.

\[ \square \]

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