Prediction for Modified Topp Leone-Chen Distribution Based on Progressive Type-II Censoring Scheme

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This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract
Prediction of future events on the basis of the past and present information is a fundamental problem of statistics, arising in many contexts and producing varied solutions. The predictor can be either a point or an interval predictor. This paper focuses on predicting the future observations from the modified Topp-Leone Chen distribution based on progressive Type-II censored scheme. The two-sample prediction is applied to obtain the maximum likelihood, Bayesian and E-Bayesian prediction (point and interval) for future order statistics. The Bayesian and E-Bayesian predictors are considered based on two different loss functions, the balanced squared error loss function; as a symmetric loss function and balanced linear exponential loss function; as an asymmetric loss function. The predictors are obtained based on conjugate gamma prior and uniform hyperprior distributions. A numerical example is provided to illustrate the theoretical results and an application using real data sets are used to demonstrate how the results can be used in practice.

Keywords: Modified Topp-Leone Chen distribution; progressive Type-II censored samples; balanced squared error loss function; balanced linear exponential loss function; two-sample prediction; maximum likelihood; Bayesian and E-Bayesian prediction; Markov Chain Monte Carlo method.
1 Introduction

AL-Sayed et al. [1] introduced the modified Topp-Leone Chen (MTLCh) distribution as a composite distribution. They obtained some statistical properties of the proposed distribution such as reliability function (rf), some stress-strength models, hazard rate function (hrf), reversed hazard rate function, quantile function, mean residual life, mean past lifetime, order statistics and Renyi entropy. They derived the maximum likelihood (ML) estimators; under progressive Type-II censored samples, for the parameters, rf and hrf. They gave a numerical example to illustrate the theoretical results and used two real data sets to demonstrate how the results can be used in practice.

The probability density function (pdf) and cumulative distribution function (cdf) of the MTLCh distribution are, respectively, given by

\[
f(x; \lambda, \alpha, \beta) = 2\lambda\alpha\beta e^{\beta x + 2\lambda(1 - e^{\beta x})} \left[1 - \exp \left(2\lambda(1 - e^{\beta x})\right)\right]^{\alpha-1}, \quad x > 0; (\lambda, \alpha, \beta > 0),
\]

and

\[
F(x; \lambda, \alpha, \beta) = \left[1 - \exp \left(2\lambda(1 - e^{\beta x})\right)\right]^\alpha, \quad x > 0; (\lambda, \alpha, \beta > 0),
\]

where \(\lambda, \alpha\) are shape parameters and \(\beta\) is a scale parameter.

The rf and hrf of the MTLCh distribution are, respectively, given by

\[
R(x; \lambda, \alpha, \beta) = 1 - \left[1 - \exp \left(2\lambda(1 - e^{\beta x})\right)\right]^\alpha, \quad x > 0; (\lambda, \alpha, \beta > 0),
\]

and

\[
h(x; \lambda, \alpha, \beta) = \frac{2\lambda\alpha\beta e^{\beta x + 2\lambda(1 - e^{\beta x})} \left[1 - \exp \left(2\lambda(1 - e^{\beta x})\right)\right]^{\alpha-1}}{1 - \left[1 - \exp \left(2\lambda(1 - e^{\beta x})\right)\right]^\alpha}, \quad x > 0; (\lambda, \alpha, \beta > 0).
\]

The MTLCh distribution is related to a wide range of well-known distributions by considering special values of the parameters \((\lambda, \alpha, \beta)\), such as exponentiated Chen, Chen and standard Chen distributions or through variables transformation of the variable \(X\) such as TL-beta Type II, Kumaraswamy Weibull, Kumaraswamy exponential, Kumaraswamy Rayleigh, exponentiated Weibull, exponentiated exponential, Weibull, Burr Type X, TL-left truncated exponential, left truncated exponential, Kumaraswamy power function, exponentiated power function, power function, TL-Weibull, TL-exponential, TL-Rayleigh, Rayleigh, TL-F, F, log-logistic, TL-Dagum, Dagum, TL-Burr Type III and Burr Type III distributions. For more details, see AL-Sayed et al. [1].

In many life testing experiments, it is desired to withdraw live units from the experiment at time points other than the final termination point of the test. The progressive censoring possesses such flexibility and thus allows in between removals of units as well. Different progressive censoring schemes have been introduced in the literature. The most popular one is known as the progressive Type-II censoring scheme and it can be briefly described as given below.

Considering \(n\) identical units are put to test and the lifetime distribution of the \(n\) units are denoted by \(X_1, X_2, \ldots, X_n\). The integer \(m(< n)\) is fixed at the beginning of the experiment and \(R_1, R_2, \ldots, R_m\) are \(m\) pre-fixed integers satisfying \(R_1 + R_2 + \cdots + R_m + m = n\). At the time of the first failure \(X_{1(m, m)}\), \(R_1\) units are chosen randomly from the remaining \(n - 1\) units and they are removed from the experiment. Similarly at the time of the second failure \(X_{2(m, m)}\), \(R_2\) of the remaining \(n - R_1 - 2\) units are removed from the test and so on. Finally, when the \(m\)-th failure is observed the experiment is terminated and the remaining surviving units \(R_m\) with
\( R_m = n - R_1 - R_2 - \cdots - R_{m-1} - m \) are removed. Here \((R_1, R_2, \ldots, R_m)\) is known as the censoring scheme and it is prefixed before the experiment starts.

Lifetime distributions under progressive Type-II censored scheme have been attracting great interest due to their wide application in the fields of science, engineering, social sciences and medicine [see, El-Sagheer and Ahsanullah [2], Dey et al. [3], Almetwaly and Almongy [4], Karakoca and Pekgör [5], Mondal and Kundu [6], Xiuyun et al. [7], Li and Gui [8] and others].

In the Bayesian approach, the unknown parameters are treated as random variables and one depends on the prior information about the parameters under study. The Bayes estimators of the parameters are obtained based on loss functions.

Ahmadi et al. [9] suggested the use of the balanced loss function (BLF), which was originated by Zellner [10], to be of the form

\[
L'(\theta, \theta^*) = \omega \, l(\theta_0, \theta^*) + (1 - \omega) \, l(\theta, \theta^*),
\]

where \(l(\theta, \theta^*)\) is an arbitrary loss function, \(\theta_0\) is a chosen target estimator of \(\theta^*\) and the weight \(\omega \in [0, 1]\). The BLF specializes to various choices of loss functions such as the absolute error loss, entropy, linear exponential (LINEX) and squared error loss (SEL) functions. Moreover, the estimator of a function using BLF is a mixture of the ML estimator, least squares estimators or any other estimator and the Bayes estimator using any loss function.

The Bayes estimator of \(\theta\), using the balanced square error loss (BSEL) function is given by

\[
\tilde{\theta}_{BSE} = \omega \, \tilde{\theta}_{ML} + (1 - \omega) \, \tilde{\theta}_{SE},
\]

where \(\tilde{\theta}_{ML}\) is the ML estimator of \(\theta\) and \(\tilde{\theta}_{SE}\) is its Bayes estimator using SEL function. Also, the Bayes estimator using the balanced LINEX loss (BLL) function of \(\theta\) is obtained as follows:

\[
\tilde{\theta}_{BL} = \frac{-1}{v} \ln\{\omega \exp(-v\tilde{\theta}_{ML}) + (1 - \omega) \, E(\exp(-v\theta) | x)\},
\]

where \(v \neq 0\) is the shape parameter of the BLL function. In this paper the Bayes and E-Bayes predictors are obtained using the BSEL and BLL functions.

Han [11] introduced the expected Bayesian (E-Bayesian) estimation method which is very simple and it’s a special Bayesian method used in the area related for the life testing of products with high reliability, small sample size or censored data. Many researchers applied the E-Bayesian method to many distributions, such as, Okasha [12], Azimi et al. [13], Okasha [14], Reyad and Ahmed [15], Reyad and Ahmed [16], Nasiri and Esfandyarifar [17], Reyad et al. [18], EL-Sagheer [19], Shawky and Al-Aboud [20], Reyad et al. [21], Reyad and Othman [22], Han [23], Okasha [24], Rabie and Li [25], Algarni et al. [26], Han [27], Piriaei et al. [28] and Rabie and Li [29].

The general problem of prediction may be described as that of inferring the values of unknown observables (future observations; known as future sample), or functions of such variables, from current available observations; known as informative sample. Prediction has been applied in a variety of disciplines such as medicine (medical prognosis, antibiotic assays and pre-operative medical diagnosis), engineering (mechanical...
tool replacements, quality control and maximization of the yield of an industrial process), business (determining the difference in future mean performance of a specified number of systems), economic and other areas as well.

Prediction for order statistics of future observables from certain distributions has been studied by several authors, such as, Valiollahi et al. [30] who obtained the ML and Bayesian prediction (point and interval) of a future observation based on Type-I, Type-II and hybrid censored samples when the lifetime distribution of the experimental units is assumed to be a generalized exponential random variable. The one-sample, two-sample prediction and intervals of the future samples under Bayesian paradigm of a weighted exponential distribution under Type-II progressive censoring were introduced by Dey et al. [3]. Also, Faizan and Sana [31] considered prediction intervals for future observations of the two unknown parameters of Chen distribution based on upper record value. The one-sample Bayesian prediction and intervals of the generalized half-normal distribution under progressive Type-II censoring were studied by Abd El-Raheem [32]. Arshad and Jamal [33] predicted future record values using Bayesian approach of the TL family of distributions. Recently, Okasha et al. [34] derived the Bayesian and E-Bayesian prediction (point and interval) for a future observation based on two samples from parameter Burr XII model based on Type-II censored data. Moreover, they obtained the predictors under symmetric and asymmetric loss functions assuming gamma conjugate prior density.

The rest of this paper is organized as follows: in Section 2, the ML, Bayesian and E-Bayesian prediction (point and interval) for a future observation of the MTLCh distribution based on two-sample prediction are derived. In Section 3, a numerical example is given to illustrate the theoretical results and an application using real data sets are used to demonstrate how the results can be used in practice. Finally, general conclusion is presented in Section 4.

2 Prediction for Modified Topp-Leone Chen Distribution

In this section, the ML, Bayesian and E-Bayesian prediction (point and interval) for a future observation $Y_{(n)}$ of the MTLCh distribution based on two-sample prediction technique are derived.

Let $X_{1:m,n}, X_{2:m,n}, \ldots, X_{m:m,n}$ denote a progressive Type-II censored sample obtained from MTLCh $(\lambda, \alpha, \beta)$ distribution. The likelihood function (LF) is given by

$$L(\theta|x) = C(n, m - 1) \prod_{i=1}^{m} f(x(i); \theta) \left[1 - F(x(i); \theta)\right]^{R_i},$$

(8)

where $\theta = (\lambda, \alpha, \beta)$, $x = (x_{1:m,n}, x_{2:m,n}, \ldots, x_{m:m,n})$ denotes an observed value of

$$X = (X_{1:m,n}, X_{2:m,n}, \ldots, X_{m:m,n}) \text{ and } C(n, m - 1) = n(n - R_1 - 1)(n - R_1 - R_2 - 2) \ldots (n - R_1 - \cdots - R_{m-1} - m + 1),$$

Then substituting (1) and (2) in (8) yields

$$L(\theta|x) \propto (\lambda \alpha \beta)^n \exp[\beta \sum_{i=1}^{m} x(i) + \sum_{i=1}^{m} \ln z(i)] \times \exp[(\alpha - 1) \sum_{i=1}^{m} \ln(1 - z(i)) + \sum_{i=1}^{m} R_i \ln\{1 - [1 - z(i)]^\alpha\}].$$

(10)

where

$$z(i) = \exp[2\lambda(1 - e^{\beta x(i)})].$$

(11)
Considering the prior knowledge of the vector of parameters, \( \theta = (\lambda, \alpha, \beta)' \), is adequately represented by a conjugate prior which is a gamma distribution with parameters \( a_j \) and \( b_j \) and pdf as follows:

\[
\pi(\theta_j; a_j, b_j) = \frac{b_j^{a_j}}{\Gamma(a_j)} \theta_j^{a_j-1} \exp(-b_j \theta_j), \quad \theta_j > 0; \ (a_j, b_j > 0), \ j = 1,2,3, \tag{12}
\]

where \( \theta_1 = \lambda, \theta_2 = \alpha \) and \( \theta_3 = \beta, a_j \) and \( b_j \) are the hyper-parameters of the prior distribution.

Assuming that the parameters \( \theta = (\lambda, \alpha, \beta)' \), are unknown and independent. Then the joint prior distribution of all the unknown parameters has a joint pdf given by

\[
\pi(\theta; a, b) \propto \lambda^{a_1-1} \alpha^{a_2-1} \beta^{a_3-1} \exp[-(b_1 \lambda + b_2 \alpha + b_3 \beta)], \quad \theta > 0; \ (a, b > 0). \tag{13}
\]

Combining the LF in (10) with the joint prior distribution given by (13), then the joint posterior distribution of the parameters, \( \theta = (\lambda, \alpha, \beta)' \), can be obtained as follows:

\[
\pi(\theta|x) = K \ L(\theta|x) \pi(\theta; a, b) = K \ \lambda^{m+a_1-1} \alpha^{m+a_2-1} \beta^{m+a_3-1} \exp[-(b_1 \lambda + b_2 \alpha + b_3 \beta)]
\times \exp[(\alpha - 1) \sum_{i=1}^{m} \ln(1 - z(i)) + \sum_{i=1}^{m} R_i \ln[1 - (1 - z(i))^\alpha]]
\times \exp[-(b_1 \lambda + b_2 \alpha + b_3 \beta)] \ d\theta, \tag{14}
\]

where \( K \) is a normalizing constant,

\[
K^{-1} = \int_{a_1}^{\infty} \int_{a_2}^{\infty} \int_{a_3}^{\infty} \lambda^{m+a_1-1} \alpha^{m+a_2-1} \beta^{m+a_3-1} \exp[-(b_1 \lambda + b_2 \alpha + b_3 \beta)]
\times \exp[(\alpha - 1) \sum_{i=1}^{m} \ln(1 - z(i)) + \sum_{i=1}^{m} R_i \ln[1 - (1 - z(i))^\alpha]]
\times \exp[-(b_1 \lambda + b_2 \alpha + b_3 \beta)] \ d\theta, \tag{15}
\]

where

\[
\int_{a_1}^{\infty} \int_{a_2}^{\infty} \int_{a_3}^{\infty} \ d\theta = d\beta \ d\alpha \ d\lambda. \tag{16}
\]

Considering that \( X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)} \) are the first \( r \) ordered life times in a random sample of \( n \) components (progressive Type-II censoring) whose failure times are identically distributed as a random variable \( X \) having the MTLCh \((\lambda, \alpha, \beta)\) distribution; informative sample, and \( Y_{(1)}, Y_{(2)}, \ldots, Y_{(r)} \) is a future independent random sample (of size \( \text{j} \)) from the same distribution. Our aim is to predict a statistic in the future sample based on the informative sample.

For the future sample of size \( \text{j} \), let \( Y_{(s)} \) denotes the \( s^{th} \) order statistic, \( 1 \leq s \leq j \). The conditional density function of \( Y_{(s)} \), given the vector of the parameters \( \theta \), is given by

\[
h(y_{(s)}|\theta) = D(s)[F(y_{(s)})]^{s-1}[1 - F(y_{(s)})]^{j-s} f(y_{(s)}|\theta), \quad y_{(s)} > 0, \tag{17}
\]

where \( s \) is the order statistic of the predicted future observation in the future sample,

\[
D(s) = s(s) = \frac{j!}{(s-1)!(j-s)!} = \frac{1}{\lambda(s-j+1)} \ \text{and} \ s = 1, 2, 3, \ldots . \tag{18}
\]

Using the binomial expansion theorem for \([1 - F(y_{(s)})]^{j-s}\), yields
\[
\begin{align*}
  h(y_3; \theta) &= D(s) f(y_3; \theta) \sum_{l_1=0}^{j-x} (-1)^{l_1} \binom{j-x}{l_1} \left[ F(y_3) \right]^{s+l_1-1} \\
  &= D(s) 2 \lambda \alpha \beta \exp\left[ \beta y_{(3)} + 2(1-e^{\beta y_{(3)}}) \right] \left( \sum_{l_3=0}^{j-x} (-1)^{l_3} \frac{(j-x)!}{l_3!} \right)^{-1} \\
  &\times \left[ 1 - \exp\left( 2\lambda (1 - e^{\beta y_{(3)}}) \right) \right]^{s+1-l_3-1}.
\end{align*}
\] 

(19)

For real value of \( \alpha \), using the binomial expansion

\[
(1+x)^a = \sum_{j=0}^{\infty} \binom{a}{j} x^j = \sum_{j=0}^{\infty} \frac{\Gamma(a+1)}{j! \Gamma(a-j+1)} x^j,
\]

(20)

then, the \( h(y_3; \theta) \) in (19) is expressed as an infinite series which is given by

\[
\begin{align*}
  h(y_3; \theta) &= D(s) 2 \lambda \alpha \beta \exp\left[ \beta y_{(3)} + 2(1-e^{\beta y_{(3)}}) \right] \left( \sum_{l_3=0}^{j-x} (-1)^{l_3} \frac{(j-x)!}{l_3!} \right)^{-1} \\
  &\times \left[ 1 - \exp\left( 2\lambda (1 - e^{\beta y_{(3)}}) \right) \right]^{s+1-l_3-1} \\
  &= D(s) 2 \lambda \alpha \beta \exp\left[ \beta y_{(3)} + 2(1-e^{\beta y_{(3)}}) \right] \left( \sum_{l_3=0}^{j-x} (-1)^{l_3} \frac{(j-x)!}{l_3!} \right)^{-1} \\
  &\times \left[ 1 - \exp\left( 2\lambda (1 - e^{\beta y_{(3)}}) \right) \right]^{s+1-l_3-1} \\
  &= D(s) 2 \lambda \alpha \beta \exp\left[ \beta y_{(3)} + 2(1-e^{\beta y_{(3)}}) \right] \left( \sum_{l_3=0}^{j-x} (-1)^{l_3} \frac{(j-x)!}{l_3!} \right)^{-1}.
\end{align*}
\]

(21)

Applying the power series for the exponential function, hence

\[
\exp\left( -2\lambda (l_2 + 1) e^{\beta y_{(3)}} \right) = \sum_{l_3=0}^{\infty} \frac{(-1)^{l_3} (2\lambda (l_2 + 1))^{l_3}}{l_3!} e^{\beta l_3 y_{(3)}},
\]

(22)

substituting (22) in (21) yields

\[
\begin{align*}
  h(y_3; \theta) &= D(s) \sum_{l_1=0}^{j-x} \sum_{l_3=0}^{j-x} \varphi_{l_1,l_2,l_3}(\lambda, \alpha, \beta) e^{\beta l_3 y_{(3)}} \\
  y_{(3)} &> 0; (\theta > 0),
\end{align*}
\]

(23)

where

\[
\varphi_{l_1,l_2,l_3}(\lambda, \alpha, \beta) = \alpha \beta (2\lambda)^{rac{l_3+1}{2}} \frac{(l_2+1)^{l_3}}{l_3!} (-1)^{l_1+l_2+l_3} \frac{(j-x)!}{l_1! l_2!}(j-x)^{rac{a(l_1 l_2)}{2}} e^{2\lambda (l_2 + 1)}.
\]

(24)

### 2.1 Maximum likelihood prediction

In this subsection, the ML prediction (point and interval) for a future observation \( Y_{(3)} \), of the MTLCh distribution based on two-sample prediction are derived.

Assuming that the parameters \( \theta \) are unknown and independent, then the ML prediction density (MLPD) of \( Y_{(3)} \) given \( \hat{\theta}_{ML} \) can be obtained using the conditional pdf of the \( s^{th} \) order statistic which is given by (23) after replacing the vector of parameters \( \theta \) by their ML estimators \( \hat{\theta}_{ML} \) as follows:

\[
\begin{align*}
  h_{1}(y_{(3)}; \hat{\theta}_{ML}) &= D(s) \sum_{l_1=0}^{j-x} \sum_{l_2=0}^{j-x} \varphi_{l_1,l_2,l_3}(\hat{\lambda}, \hat{\alpha}, \hat{\beta}) e^{\beta l_3 y_{(3)}} \\
  y_{(3)} &> 0; (\hat{\theta}_{ML} > 0).
\end{align*}
\]

(25)

#### 2.1.1 Point prediction

The ML predictor (MLP) for the future observation \( Y_{(3)} \), based on progressive Type-II censoring can be derived using (25) as follows:

\[
\begin{align*}
  \hat{Y}_{(3)}(MLP) &= E \left( y_{(3)}; \hat{\theta}_{ML} \right) \\
  &= \int y_{(3)} h_{1}(y_{(3)}; \hat{\theta}_{ML}) dy_{(3)} \\
  &= D(s) \sum_{l_1=0}^{j-x} \sum_{l_2=0}^{j-x} \varphi_{l_1,l_2,l_3}(\hat{\lambda}, \hat{\alpha}, \hat{\beta}) \int_{0}^{\infty} y_{(3)} e^{\beta l_3 y_{(3)}} dy_{(3)} \\
  &= \frac{1}{\left[ \beta (l_3 + 1) \right]^s}.
\end{align*}
\]

(26)
where \( D(s) \) is defined in (18) and \( \varphi_{l_1,l_2,l_3}(\lambda, \alpha, \beta) \) is given by (24).

### 2.1.2 Interval prediction

A 100(1-\( \tau \))\% ML predictive bounds (MLPB) for the future observation \( Y(s) \), such that \( P(L(s)(x) < Y(s) < U(s)(x) | x) = 1 - \tau \), are

\[
P(Y(s) > L(s)(x) | x) = \int_{L(s)(x)}^{\infty} h_1(y(s); \tilde{\theta}_{ML}) \, dy(s) = 1 - \frac{\tau}{2},
\]

and

\[
P(Y(s) > U(s)(x) | x) = \int_{U(s)(x)}^{\infty} h_1(y(s); \tilde{\theta}_{ML}) \, dy(s) = \frac{\tau}{2},
\]

Substituting (25) in (27) and (28), then the MLPB are obtained as follows:

\[
P(Y(s) > L(s)(x) | x) = D(s) \sum_{l_1=0}^{l_\infty} \sum_{l_2,l_3} \varphi_{l_1,l_2,l_3}(\lambda, \alpha, \beta)
\times \int_{L(s)(x)}^{\infty} e^{R(l_3+1)y(s)} \, dy(s) = 1 - \frac{\tau}{2},
\]

and

\[
P(Y(s) > U(s)(x) | x) = D(s) \sum_{l_1=0}^{l_\infty} \sum_{l_2,l_3} \varphi_{l_1,l_2,l_3}(\lambda, \alpha, \beta)
\times \int_{U(s)(x)}^{\infty} e^{R(l_3+1)y(s)} \, dy(s) = \frac{\tau}{2},
\]

where \( s = 1, 2, 3, \ldots, J \), \( D(s) \) is defined in (18) and \( \varphi_{l_1,l_2,l_3}(\lambda, \alpha, \beta) \) is given by (24).

### 2.2 Bayesian prediction

In this subsection, Bayesian prediction (point and interval) for a future observation \( Y(s) \) of the MTLCh distribution based on two-sample prediction are considered.

Assuming that the parameters \( \theta = (\lambda, \alpha, \beta)' \) are unknown and independent, then the Bayesian prediction density (BPD) of \( Y(s) \) given \( x \) based on informative prior can be obtained as follows:

\[
h_2(y(s) | x) = \int_{\theta} h(y(s) | \theta) \, \pi(\theta | x) \, d\theta, \quad y(s) > 0; \quad (\theta > 0),
\]

where \( \pi(\theta | x) \) is given by (14), \( h(y(s) | \theta) \) is defined in (23), \( \int_{\theta} \) and \( d\theta \) are given by (16).

Substituting (14) and (23) into (31), then the BPD of \( Y(s) \) given \( x \) is

\[
h_2(y(s) | x) = D(s)K \sum_{l_1=0}^{l_\infty} \sum_{l_2,l_3} \int_{\theta} \varphi_{l_1,l_2,l_3}(\lambda, \alpha, \beta) \exp[\sum_{i=1}^{m} \ln z(i)]
\times \exp[(\alpha - 1) \sum_{i=1}^{m} \ln(1 - z(i)) + \sum_{i=1}^{m} \beta \ln(1 - [1 - z(i)]^n)]
\times \exp[\beta \sum_{i=1}^{m} x(i) - (b_1 \lambda + b_2 \alpha + b_3 \beta) + \beta(l_3 + 1)y(s)] \, d\theta
\]

where \( z(i) \) is given by (11), \( K^{-1} \) is given by (15), \( \int_{\theta} \) and \( d\theta \) are given by (16) and

\[
\varphi_{l_1,l_2,l_3}(\lambda, \alpha, \beta) = 2^{(l_3+1)} \alpha^{(m+\alpha_2)} \beta^{(m+\alpha_3)} \frac{(\alpha+\alpha_1+l_3)}{l_3+1} \frac{(l_2+1) s}{l_1} \frac{(l_3+1)}{l_1+1+l_3} \frac{(l_2+1)^3}{l_3+1} \frac{(-1)^{l_1+1+l_3}}{(l_1+l_2+l_3)^2} \frac{\Gamma(l_1+l_2+l_3)}{\Gamma(l_1) \Gamma(l_2) \Gamma(l_3)}
\]

\[
\times \left( \frac{l_3}{l_1} \right) \left( \frac{l_2}{l_1} \right) e^{2(\alpha+\alpha_1+l_3)},
\]

(33)
### 2.2.1 Point prediction

Based on progressive Type-II censoring, the Bayesian prediction is considered under two types of loss functions, the BSEL function; as a symmetric loss function and BLL function; as an asymmetric loss function.

#### I. Balanced squared error loss function

The *Bayes predictor* (BP) for the future observation \( Y(x) \), under BSEL function can be derived using (6) and (32) as given below

\[
\hat{y}(x)_{BSE} = \omega \hat{y}(x)_{ML} + (1 - \omega) \int_{\mathcal{Y}(x)} \hat{y}(x) h_2(y(x) | \hat{\Sigma}) \, dy(x)
\]

\[
= \omega \hat{y}(x)_{ML} + (1 - \omega) D(s) K \sum_{i=0}^{m-1} \sum_{i_2, i_3 = 0}^{m-1} \int_{\mathcal{Y}(x)} \hat{y}(x) \varphi_{1,2,3}^*(\lambda, \alpha, \beta)
\]

\[
\times \exp\left(\beta \sum_{i=1}^m x(i) + \sum_{i=1}^m \ln z(i) + (\alpha - 1) \sum_{i=1}^m \ln(1 - z(i))\right)
\]

\[
\times \exp\left[\sum_{i=1}^m R_i \ln(1 - \left(1 - z(i)\right)^\alpha) - (b_1 \lambda + b_2 \alpha + b_3 \beta)\right]
\]

\[
\times \exp\left[\beta(l_3 + 1) y(x)\right] d\theta.
\]

(34)

where \( \hat{y}(x)_{ML} \) is the ML prediction for the future observation of \( y(x) \), \( z(i) \) is given by (11), \( K^{-1} \) is defined in (15), \( D(s) \) is given by (18), \( \varphi_{1,2,3}^*(\lambda, \alpha, \beta) \) is defined in (33), \( \int_{\hat{\Sigma}} \) and \( d\theta \) are given by (35).

#### II. Balanced linear exponential loss function

The BP for the future observation \( Y(x) \), under BLL function can be derived using (7) and (32) as follows:

\[
\hat{y}(x)_{BLL} = \frac{-1}{v} \ln \left\{ \omega \exp(-v \hat{y}(x)_{ML}) + (1 - \omega) \int_{\mathcal{Y}(x)} \exp(-v y(x)) h_2(y(x) | \hat{\Sigma}) \, dy(x) \right\}
\]

\[
= \frac{-1}{v} \ln \omega \exp(-v \hat{y}(x)_{ML}) + (1 - \omega) D(s) K \sum_{i=0}^{m-1} \sum_{i_2, i_3 = 0}^{m-1} \int_{\mathcal{Y}(x)} \varphi_{1,2,3}^*(\lambda, \alpha, \beta)
\]

\[
\times \exp\left(\beta \sum_{i=1}^m x(i) + \sum_{i=1}^m \ln z(i) + (\alpha - 1) \sum_{i=1}^m \ln(1 - z(i))\right)
\]

\[
\times \exp\left[\sum_{i=1}^m R_i \ln(1 - \left(1 - z(i)\right)^\alpha) - (b_1 \lambda + b_2 \alpha + b_3 \beta)\right]
\]

\[
\times \exp\left[-v y(x) + \beta(l_3 + 1) y(x)\right] d\theta.
\]

(36)

where \( \hat{y}(x)_{ML} \) is the ML prediction for the future observation of \( y(x) \), \( z(i) \) is given by (11), \( K^{-1} \) is defined in (15), \( D(s) \) is given by (18), \( \varphi_{1,2,3}^*(\lambda, \alpha, \beta) \) is defined in (33), \( \int_{\hat{\Sigma}} \) and \( d\theta \) are given by (35).

### 2.2.2 Interval prediction

A 100(1-\(\tau\))% Bayesian prediction bounds (BPB) for the future observation \( Y(x) \), so that \( P(L(x) < Y(x) < U(x) | \hat{x}) \) = 1 - \(\tau\), can be obtained from (32) as given below

\[
P(Y(x) > L(x) | \hat{x}) = \int_{L(x)(\hat{x})}^{\infty} h_2(y(x) | \hat{x}) \, dy(x) = 1 - \frac{\tau}{2}
\]

(37)

and

\[
P(Y(x) > U(x) | \hat{x}) = \int_{U(x)(\hat{x})}^{\infty} h_2(y(x) | \hat{x}) \, dy(x) = \frac{\tau}{2}
\]

(38)

Substituting (32) in (37) and (38), then the BPB are obtained as follows:

\[
P(Y(x) > L(x) | \hat{x}) = D(s) \int_{L(x)(\hat{x})}^{\infty} K \sum_{i=0}^{m-1} \sum_{i_2, i_3 = 0}^{m-1} \varphi_{1,2,3}^*(\lambda, \alpha, \beta)
\]

\[
\times \exp\left(\beta \sum_{i=1}^m x(i) + \sum_{i=1}^m \ln z(i) + (\alpha - 1) \sum_{i=1}^m \ln(1 - z(i))\right)
\]

\[
\times \exp\left[\sum_{i=1}^m R_i \ln(1 - \left(1 - z(i)\right)^\alpha) - (b_1 \lambda + b_2 \alpha + b_3 \beta)\right]
\]

\[
\times \exp\left[\beta(l_3 + 1) y(x)\right] d\theta \, dy(x) = 1 - \frac{\tau}{2}
\]

(39)
and

\[
P(Y_{(s)} > U_{(s)}(\mathcal{X}) | \mathcal{X}) = D(s) \int_{U_{(s)}(\mathcal{X})} \mathcal{K}_{(l_2, l_3)}(\lambda, \alpha, \beta) \times \exp\left[\beta \sum_{i=1}^{m} x_{(i)} + \sum_{i=1}^{m} \ln z_{(i)} + (\alpha - 1) \sum_{i=1}^{m} \ln(1 - z_{(i)})\right] \times \exp\left[\sum_{i=1}^{m} R_i \ln\left(1 - \left(1 - z_{(i)}\right)^{\nu}\right) - \left(b_1 \lambda + b_2 \alpha + b_3 \beta\right)\right] \times \exp(\beta(l_1 + 1) y_{(s)}) d\theta d y_{(s)} = \frac{1}{\gamma^2},
\]

where \(s = 1, 2, 3, \ldots, \gamma\), \(z_{(i)}\) is given by (11), \(K^{-1}\) is given by (15), \(\varphi_1^{\ast, l_2, l_3}(\lambda, \alpha, \beta)\) is defined in (33), \(\int_{\theta_{(s)}}\) and \(d\theta\) are given by (35).

### 2.3 E-Bayesian prediction

In this subsection, the E-Bayesian prediction (point and interval) for a future observation \(Y_{(s)}\), based on two-sample prediction technique are obtained.

According to Han [11], the hyper-parameters \(a_j\) and \(b_j\) should be selected to guarantee that \(\pi(\theta_j; a_j, b_j)\) are decreasing functions of \(\theta_j\), \((j = 1, 2, 3)\).

The derivative of \(\pi(\theta_j; a_j, b_j)\) with respect to \(\theta_j\) is

\[
\frac{d \pi(\theta_j; a_j, b_j)}{d \theta_j} = \frac{b_j}{\gamma(\alpha_j)} \theta_j^{a_j - 2} \exp\left(-b_j \theta_j\right) \left[a_j - 1 - b_j \theta_j\right], j = 1, 2, 3,
\]

for \(0 < a_j < 1\) and \(b_j > 0\), then \(\frac{d \pi(\theta_j; a_j, b_j)}{d \theta_j} < 0\), which means that \(\pi(\theta_j; a_j, b_j)\) can be decreasing functions of \(\theta_j\).

The E-Bayesian estimators of the parameters are obtained based on three different distributions of the hyper-parameters \(a_j\) and \(b_j\). These distributions are used to investigate the effect of different prior distributions on the E-Bayesian estimation of \(\theta_j\).

Assuming that the hyper-parameters \(a_j\) and \(b_j\) are independent with bivariate density functions

\[
\pi_h(a_j, b_j) = \pi_h(a_j) \pi_h(b_j), j = 1, 2, 3, h = 1, 2, \ldots, 9.
\]

Then, the bivariate uniform hyperprior distributions are:

\[
\pi_h(a_j, b_j) = \frac{2(c_j - b_j)}{c_j^2}, \quad 0 < a_j < 1, 0 < b_j < c_j,
\]

\[
\pi_h(a_j, b_j) = \frac{1}{c_j}, \quad 0 < a_j < 1, \quad 0 < b_j < c_j,
\]

\[
\pi_h(a_j, b_j) = \frac{2b_j}{c_j^2}, \quad 0 < a_j < 1, \quad 0 < b_j < c_j,
\]

The E-Bayesian estimators of \(\theta_j\) (expectation of the Bayes estimators of \(\theta_j\)) can be derived as follows:

\[
\bar{\theta}_{EB} = E_{\pi_h}\left(\bar{\theta}_{EB}(a_j, b_j)\right) = \int_{0}^{c_j} \int_{0}^{1} \bar{\theta}_{EB}(a_j, b_j) \pi_h(a_j, b_j) da_j db_j,
\]

\[
E_{\pi_h}(h = 1, 2, \ldots, 9) \text{ stands for the expectation of the bivariate hyperprior distributions and } \bar{\theta}_{EB}(a_j, b_j) \text{ are the Bayes estimators of the parameters } \theta_j, j = 1, 2, 3 \text{ based on BSEL and BLL functions.} 
\]
2.3.1 Point prediction

Based on progressive Type-II censoring, the E-Bayesian prediction is considered under two types of loss functions, the BSEL function; as a symmetric loss function and BLL function; as an asymmetric loss function.

I. Balanced squared error loss function

The three \( E-Bayes \) predictors (EBPs) for the future observation \( Y_{(s)} \), under BSEL function can be obtained by substituting (34) and (43)-(45) in (46) as given below

\[
\hat{y}_{(s)EBBS} = E_{\pi}(\hat{y}_{(s)BBSE}) = \int_0^c \int_0^1 \hat{y}_{(s)BBSE} \pi_\alpha(a, b) \, da \, db, \quad h = 1,2,3. \quad (47)
\]

II. Balanced linear exponential loss function

The EBPs for the future observation \( Y_{(s)} \), under BLL function can be derived by substituting (36) and (43)-(45) in (46) as follows:

\[
\hat{y}_{(s)EBBL} = E_{\pi}(\hat{y}_{(s)BBL}) = \int_0^c \int_0^1 \hat{y}_{(s)BBL} \pi_\alpha(a, b) \, da \, db, \quad h = 1,2,3. \quad (48)
\]

2.3.2 Interval prediction

A \( 100(1-\tau)\% \) \( E-Bayesian \) prediction bounds (EBPB) for the future observation \( Y_{(s)} \), such that \( P(L_{(s)}(X) < Y_{(s)} < U_{(s)}(X)|X) = 1 - \tau \), can be obtained by substituting (39) and (43)-(45) in (46), (40) and (43)-(45) in (46), respectively.

Remark:

- If \( s = 1 \), in (34), (36), (47) and (48), one can predict the minimum observable, \( Y_{(1)} \), which represents the first failure time in a future sample of size \( j \).
- If \( s = j \), in (34), (36), (47) and (48), one can predict the maximum observable, \( Y_{(j)} \), which represents the largest failure time in a future sample of size \( j \).
- If \( s = \frac{j+1}{2} \), in (34), (36), (47) and (48), one can predict the median observable if \( j \) is odd, \( Y_{(\frac{j+1}{2})} \), which represents the median failure time in a future sample of size \( j \).

3 Numerical Illustration

This section aims to investigate the precision of the theoretical results of prediction on the basis of simulated and real data sets.

3.1 Simulation study

In this subsection, the ML, Bayes and E-Bayes predictors (point and interval) for a future observation from the MTLCh \( (\lambda, \alpha, \beta) \) distribution based on progressive Type-II censored data are computed. All simulation studies are performed using Mathematica 9, Mathcad 14 and R programming language.

3.2 Simulation algorithm

3.2.1 Maximum likelihood prediction

Applying the algorithm given by Balakrishnan and Sandhu [35], the following steps are used to generate a progressive Type-II censored sample from the MTLCh distribution.

Step 1: Generate \( m \) independent \( U(0,1) \) random variables \( U_1, U_2, \ldots, U_m \).
Step 2: For given values of the progressive censoring scheme $R_1, R_2, ..., R_m$, 
\[ Y_i = U_1^{1/(i+\sum_{j=m-i+1}^{m} R_j)}, \quad \text{for } i = 1, 2, ..., m. \]

Step 3: Set $U P_i = 1 - (Y_m Y_{m-1} Y_{m-2} ... Y_{m-i+1}), i = 1, 2, ..., m$. Hence, $U P_1, U P_2, ..., U P_m$ are progressive Type-II censored sample of size $m$ from $U(0,1)$ distribution.

Step 4: A random $m$ progressive Type-II censored sample from MTLCh, $x_1, x_2, ..., x_m$ is generated using the following transformation, for given values of the parameters $\lambda, \alpha$ and $\beta$
\[ x_i = \frac{1}{\beta} \ln \left[ 1 - \frac{1}{2\lambda} \ln \left( 1 - U P_i^{1/\alpha} \right) \right], \quad i = 1, 2, ..., m, \]
and the generated sample is ordered.

Step 5: The ML estimates for the parameters $\lambda, \alpha$ and $\beta$ are computed based on progressive Type-II censored scheme.

Step 6: Substituting the ML estimates of the parameters in the equation of $\hat{Y}_{(s)(ML)}$ and for given values for $s$, the MLP for the future observation $Y_{(s)}$ can be computed under progressive Type-II censored sample.

Step 7: Using the ML estimates for the parameters and a certain value of $s$, the MLPB for the future observation $Y_{(s)}$, can be computed under progressive Type-II censored sample.

Step 8: Repeat all the previous steps $N=2000$ times.

3.2.2 Bayesian and E-Bayesian prediction

Step 1: Generate $a_j$ and $b_j$ from the bivariate uniform hyperprior distributions; $\pi_h(a_j, b_j), \quad j = 1, 2, 3,$
\[ h = 1, 2, ..., 9, \] which are given in (43)-(45).

Step 2: For given values of $a_j$ and $b_j$, generate $\lambda, \alpha$ and $\beta$ from the gamma prior distributions.

Step 3: Applying the previous generation steps, progressive Type-II censored sample can be generated from the MTLCh distribution.

Step 4: Calculate the joint posterior distribution for the parameters based on progressive Type-II censored sample from the MTLCh distribution.

Step 5: The BPD of the future observation $Y_{(s)}$, can be obtained.

Step 6: The BP is calculated based on BSEL and BLL functions. Also, the BPB is evaluated.

Step 7: Using the BP, the EBPs for a future observation from the MTLCh distribution based on BSEL and BLL functions are calculated. Similarly, using the BPB, the EBPBs are evaluated.

Step 8: Repeat all the previous steps $N=10000$ times.

For each sample size, three different samples schemes are considered, as given below

Scheme I: $R_1 = n - m$ and $R_2 = R_3 = \cdots = R_m = 0$. It is Type-II censoring.

Scheme II: $R_1 = R_2 = \cdots = R_{m-1} = 1$ and $R_m = n - 2m + 1$. 

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Scheme III: $0, 0, ..., 0, \frac{n-m}{3}, 0, 0, ..., 0, \frac{n-m}{3}, 0, 0, ..., 0, \frac{n-m}{3}$.

The ML two-sample predictors are presented in Table 1. Also, the Bayes and E-Bayes two-sample predictors are presented in Tables 3 and 4 based on BSEL and BLL functions.

Table 1. ML predictor and bounds of the future observation based on progressive Type-II censoring under two-sample prediction and different sample schemes ($N=2000$, $n = 100$, $m = 50$, $j = 21$, $\lambda = 0.63$, $\alpha = 0.48$ and $\beta = 0.37$)

| Scheme | $s$ | $\hat{y}_{(s)(ML)}$ | LL  | UL  | Length |
|--------|-----|----------------------|-----|-----|--------|
| Scheme I | 3   | 0.0438               | 0.0015 | 0.1763 | 0.1747 |
|        | 11  | 0.5425               | 0.1711 | 1.0936 | 0.9225 |
|        | 21  | 3.0072               | 1.7121 | 4.5284 | 2.8163 |
| Scheme II | 3   | 0.0447               | 0.0016 | 0.1785 | 0.1769 |
|        | 11  | 0.5392               | 0.1698 | 1.0876 | 0.9178 |
|        | 21  | 3.0341               | 1.7270 | 4.5703 | 2.8433 |
| Scheme III | 3   | 0.0442               | 0.0015 | 0.1778 | 0.1763 |
|        | 11  | 0.5406               | 0.1713 | 1.0874 | 0.9160 |
|        | 21  | 3.0309               | 1.7257 | 4.5634 | 2.8377 |

3.3 Some applications

The main aim of this subsection is to demonstrate how the proposed methods can be used in practice. Three real lifetime data sets are used for this purpose. The MTLCh distribution is fitted to the three real data using Kolmogorov-Smirnov goodness of fit test through Mathematica 9.

Application 1:

In lung transplant recipients, respiratory tract infections are associated with faster progression through stages of bronchiolitis obliterans syndrome and mortality. Common causative pathogens for respiratory tract infections (RTIs) include non-fermenting gram-negative bacilli (NFGNB). Data to guide optimal treatment durations for limited NFGNB RTIs in this population. This was a single-center, retrospective; cohort study of adult lung transplant recipients who received systemic antibiotic treatment for RTIs caused by NFNGB and had at least 28 days of post-treatment follow-up.

The data set was taken from web portal of Indiana University Health, and it refers to the duration of therapy (days) for NFGNB RTI treatment (DOT) for 60 lung transplant recipients:

The data set is: 9, 10, 14, 29, 16, 18, 16, 10, 7, 15, 11, 10, 21, 14, 13, 10, 7, 12, 18, 18, 19, 13, 12, 10, 13, 17, 15, 10, 18, 14, 15, 7, 14, 12, 12, 19, 26, 15, 5, 9, 10, 10, 18, 17, 12, 10, 7, 14, 14, 13, 15, 5, 19, 13, 9, 7.

Application 2:

The second application is given by Murthy et al. [36]. The data refers to the time between failures for a repairable item: 1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86 and 1.17.

Application 3:

The third data set is given by Nelson [37]. The data refers to the time to breakdown of an insulating fluid between electrodes at a voltage of 34 k.v. (minutes). The 19 times to breakdown are: 0.96, 4.15, 0.19, 0.78, 8.01, 31.75, 7.35, 6.50, 8.27, 33.91, 32.52, 3.16, 4.85, 2.78, 4.67, 1.31, 12.06, 36.71 and 72.89.

The Kolmogorov-Smirnov goodness of fit test is applied to check the validity of the introduced fitted model. The $p$ values are given, respectively, 0.772, 0.9525 and 0.8101. The $p$ value given in each case showed that the proposed model fits the data very well.

Table 2 presents the ML two-sample predictors of the real data sets. Also, Tables 5-10 display the Bayes and E-Bayes two-sample predictors of the real data sets based on BSEL and BLL functions.
Table 2. ML predictor and bounds of the future observation for real data sets based on progressive Type-II censoring under two-sample prediction and different sample schemes

| Scheme | Application I | Application II | Application III |
|--------|---------------|----------------|-----------------|
|        | $\hat{y}_{(s)}^{(ML)}$ | LL | UL | Length | $\hat{y}_{(s)}^{(ML)}$ | LL | UL | Length | $\hat{y}_{(s)}^{(ML)}$ | LL | UL | Length |
|        | $s$ | | | | | | | | | | | | |
| Scheme I | | | | | | | | | | | | | |
| 10 | 18.0415 | 15.6873 | 20.5195 | 4.8322 | 1 | 0.1249 | 0.0106 | 0.3466 | 0.3360 | 1 | 0.1466 | 0.0186 | 0.3672 | 0.3485 |
| 25 | 24.5672 | 21.7552 | 27.6472 | 5.8920 | 15 | 1.1884 | 0.7779 | 1.6704 | 0.8925 | 8 | 0.7360 | 0.4253 | 1.1200 | 0.6947 |
| 35 | 31.2683 | 27.1513 | 36.0638 | 8.9125 | 27 | 2.7489 | 0.0100 | 3.6955 | 3.6855 | 15 | 2.0600 | 1.2319 | 3.2255 | 1.9936 |
| Scheme II | | | | | | | | | | | | | |
| 10 | 18.4950 | 16.1719 | 20.9375 | 4.7656 | 1 | 0.0953 | 0.0121 | 0.2362 | 0.2240 | 1 | 0.2479 | 0.0301 | 0.6267 | 0.5965 |
| 25 | 24.9225 | 22.1545 | 27.9539 | 5.7994 | 15 | 0.7055 | 0.4817 | 0.9653 | 0.4836 | 8 | 1.2536 | 0.7263 | 1.8969 | 1.1706 |
| 35 | 31.5183 | 27.4659 | 36.2401 | 8.7742 | 27 | 1.5447 | 1.1169 | 2.0573 | 0.9404 | 15 | 3.4084 | 2.0816 | 5.2200 | 3.1385 |
| Scheme III | | | | | | | | | | | | | |
| 10 | 18.8501 | 16.4779 | 21.3441 | 4.8662 | 1 | 0.0834 | 0.0089 | 0.2176 | 0.2087 | 1 | 0.5068 | 0.1101 | 1.0730 | 0.9629 |
| 25 | 25.4122 | 22.5866 | 28.5060 | 5.9194 | 15 | 0.6938 | 0.4638 | 0.9630 | 0.4992 | 8 | 1.8737 | 1.2063 | 2.6602 | 1.4539 |
| 35 | 32.1426 | 28.0080 | 36.9589 | 8.9509 | 27 | 1.5698 | 1.1212 | 2.1086 | 0.9874 | 12 | 2.8097 | 1.9090 | 3.8880 | 1.9790 |
Table 3. Bayes, E-Bayes predictor and bounds of the future observation under balanced squared error loss function based on progressive Type-II censoring under two-sample prediction and different sample schemes (N=10000, n = 100, m = 50, j = 21, λ = 2.3, α = 1.5, β = 1.1 and ω = 0.3)

| Scheme | s   | \(\hat{y}_{(s)}(BB)\) | LL   | UL   | Length | \(\hat{y}_{(s)}(EB)\) | LL   | UL   | Length |
|--------|-----|------------------------|------|------|--------|------------------------|------|------|--------|
|        |     | Bayesian               |      |      | E-Bayesian         |           |      |      |        |
|        | 3   | 0.0243                 | 0.0168 | 0.0314 | 0.0145 | 0.0227                 | 0.0178 | 0.0256 | 0.0078 |
| Scheme I | 11 | 0.3303                 | 0.2692 | 0.3747 | 0.1055 | 0.3152                 | 0.2935 | 0.3312 | 0.0377 |
|         | 21 | 1.5869                 | 1.0801 | 2.3034 | 1.2233 | 1.4357                 | 0.9355 | 1.7344 | 0.7989 |
|         | 3  | 0.0323                 | 0.0198 | 0.0396 | 0.0198 | 0.0274                 | 0.0202 | 0.0326 | 0.0124 |
| Scheme II | 11 | 0.3629                 | 0.2630 | 0.4418 | 0.1788 | 0.2892                 | 0.2032 | 0.3702 | 0.1669 |
|         | 21 | 2.3428                 | 1.6816 | 2.7397 | 1.0582 | 2.2906                 | 2.0803 | 2.4583 | 0.3780 |
|         | 3  | 0.0227                 | 0.0169 | 0.0295 | 0.0127 | 0.0220                 | 0.0189 | 0.0263 | 0.0074 |
| Scheme III | 11 | 0.3319                 | 0.2705 | 0.4068 | 0.1363 | 0.2918                 | 0.2707 | 0.3203 | 0.0497 |
|         | 21 | 1.7248                 | 1.1476 | 2.2335 | 1.0859 | 1.7132                 | 1.4122 | 1.9590 | 0.5469 |
|         | 21 | 1.5707                 | 1.3760 | 1.7991 | 0.4230 | 1.2294                 | 0.9623 | 1.5545 | 0.5923 |
Table 4. Bayes, E-Bayes predictor and bounds of the future observation under balanced linear exponential loss function based on progressive Type-II censoring under two-sample prediction and different sample schemes (\(N=10000, n = 100, m = 50, j = 21, \lambda = 2.3, \alpha = 1.5, \beta = 1.1, \nu = -2\) and \(\omega = 0.3\))

| Scheme | s   | \(\hat{y}_s^{(BBL)}\) | LL   | UL   | Length | \(\hat{y}_s^{(EBBL)}\) | LL   | UL   | Length |
|--------|-----|-----------------------|------|------|--------|-----------------------|------|------|--------|
|        | 3   | 0.0208                | 0.0151| 0.0241| 0.0089 | 0.0176                | 0.0142| 0.0219| 0.0077 |
|        |     |                       |      |       |        | 0.0188                | 0.0161| 0.0216| 0.0055 |
|        |     |                       |      |       |        | 0.0194                | 0.0168| 0.0211| 0.0043 |
| Scheme I | 11  | 0.3427                | 0.2732| 0.4293| 0.1560 | 0.3164                | 0.2746| 0.3412| 0.0665 |
|        |     |                       |      |       |        | 0.3289                | 0.3054| 0.3531| 0.0476 |
|        |     |                       |      |       |        | 0.3276                | 0.3075| 0.3470| 0.0395 |
|        | 21  | 1.4918                | 0.7356| 2.0169| 1.2813 | 1.2111                | 0.9962| 1.4001| 0.4039 |
|        |     |                       |      |       |        | 1.0935                | 0.7276| 1.4361| 0.7086 |
|        |     |                       |      |       |        | 1.2032                | 0.8637| 1.4419| 0.5781 |
|        | 3   | 0.0253                | 0.0190| 0.0319| 0.0128 | 0.0219                | 0.0173| 0.0256| 0.0082 |
|        |     |                       |      |       |        | 0.0222                | 0.0200| 0.0244| 0.0045 |
|        |     |                       |      |       |        | 0.0253                | 0.0237| 0.0265| 0.0027 |
| Scheme II | 11  | 0.3277                | 0.2718| 0.3783| 0.1065 | 0.2937                | 0.2484| 0.3303| 0.0819 |
|        |     |                       |      |       |        | 0.2961                | 0.2730| 0.3185| 0.0455 |
|        |     |                       |      |       |        | 0.3262                | 0.3106| 0.3379| 0.0273 |
|        | 21  | 1.7029                | 1.0076| 2.4021| 1.3945 | 1.6740                | 1.5441| 1.7419| 0.1978 |
|        |     |                       |      |       |        | 1.5766                | 1.3979| 1.6849| 0.2870 |
|        |     |                       |      |       |        | 1.6860                | 1.3759| 1.8302| 0.4543 |
|        | 3   | 0.0213                | 0.0153| 0.0248| 0.0095 | 0.0194                | 0.0172| 0.0223| 0.0051 |
|        |     |                       |      |       |        | 0.0203                | 0.0182| 0.0218| 0.0036 |
|        |     |                       |      |       |        | 0.0213                | 0.0195| 0.0232| 0.0037 |
| Scheme III | 11  | 0.2168                | 0.1623| 0.2811| 0.1187 | 0.1892                | 0.1737| 0.2068| 0.0331 |
|        |     |                       |      |       |        | 0.2135                | 0.2038| 0.2203| 0.0166 |
|        |     |                       |      |       |        | 0.1787                | 0.1507| 0.2118| 0.0611 |
|        | 21  | 1.6276                | 1.0470| 2.3050| 1.2580 | 1.3931                | 1.2017| 1.6397| 0.4380 |
|        |     |                       |      |       |        | 1.4056                | 1.1448| 1.6131| 0.4683 |
|        |     |                       |      |       |        | 1.6207                | 1.4370| 1.7974| 0.3605 |
Table 5. Bayes, E-Bayes predictor and bounds of the future observation for real data sets (Application 1) under balanced squared error loss function based on progressive Type-II censoring under two-sample prediction and different sample schemes

| Scheme | s  | $\hat{y}_{(s)BBL}$ | LL      | UL      | Length | $\hat{y}_{(s)EBBL}$ | LL      | UL      | Length |
|--------|----|---------------------|---------|---------|--------|---------------------|---------|---------|--------|
|        | 10 | 16.8100             | 15.8100 | 17.5104 | 1.7004 | 16.7590             | 16.4858 | 17.0707 | 0.5849 |
| Scheme I | 25 | 21.2858             | 20.1496 | 22.3494 | 2.1998 | 21.1209             | 20.7122 | 21.5033 | 0.7911 |
|        | 35 | 30.2700             | 28.3837 | 32.8416 | 4.4579 | 31.4362             | 30.2859 | 32.2228 | 1.9368 |
|        | 10 | 15.9275             | 14.3915 | 17.0691 | 2.6776 | 15.5194             | 14.9879 | 15.9253 | 0.9374 |
| Scheme II | 25 | 21.5545             | 19.4060 | 23.2723 | 3.8662 | 21.5178             | 20.9278 | 22.0287 | 1.1008 |
|        | 35 | 30.1841             | 27.0818 | 33.6530 | 6.5711 | 27.1328             | 25.6184 | 29.6811 | 4.0627 |
|        | 10 | 15.1821             | 14.3983 | 15.8700 | 1.4717 | 14.8021             | 14.1955 | 15.4478 | 1.2523 |
| Scheme III | 25 | 21.7577             | 20.4352 | 22.9358 | 2.5006 | 20.9814             | 19.9088 | 21.6578 | 1.7490 |
|        | 35 | 28.8131             | 25.3474 | 31.9388 | 6.5913 | 28.0883             | 27.1876 | 29.0320 | 1.8444 |
|        |    |                     |         |         |        | 28.7484             | 28.1643 | 29.2862 | 1.1219 |
|        |    |                     |         |         |        | 28.5269             | 27.5749 | 29.0052 | 1.4303 |
Table 6. Bayes, E-Bayes predictor and bounds of the future observation for real data sets (Application 1) under balanced linear exponential loss function based on progressive Type-II censoring under two-sample prediction and different sample schemes

| Scheme   | s    | $\hat{y}_{(s)}(BBL)$ | LL   | UL   | Length |
|----------|------|-----------------------|------|------|--------|
|          | 10   | 16.1400               | 15.6063 | 16.9161 | 1.3099 |
|          | 25   | 21.6143               | 20.5652 | 22.1800 | 1.6149 |
|          | 35   | 28.7235               | 27.1428 | 29.9223 | 2.7795 |
| Scheme I |      |                       |       |       |        |
|          | 10   | 15.3654               | 14.2504 | 16.2937 | 2.0433 |
|          | 25   | 20.3305               | 18.0564 | 21.8102 | 3.7538 |
|          | 35   | 27.8597               | 25.4294 | 30.0043 | 4.5749 |
| Scheme II|      |                       |       |       |        |
|          | 10   | 14.5728               | 13.7993 | 15.0640 | 1.2648 |
|          | 25   | 20.7166               | 19.7065 | 22.0608 | 2.3543 |
|          | 35   | 28.3319               | 26.1759 | 29.6041 | 3.4282 |
| Scheme III|     |                       |       |       |        |

|          | 15.1358 | 14.7387 | 15.5000 | 0.7613 |
|          | 15.2300 | 14.8274 | 15.4436 | 0.6162 |
|          | 15.3267 | 14.9765 | 15.6666 | 0.6901 |
|          | 19.8054 | 19.1702 | 20.5443 | 1.3741 |
|          | 19.8028 | 19.2258 | 20.3766 | 1.1507 |
|          | 20.0403 | 19.1870 | 20.5841 | 1.3971 |
|          | 27.7094 | 26.6138 | 28.3795 | 1.7657 |
|          | 27.0038 | 26.5820 | 27.7591 | 1.1771 |
|          | 27.3220 | 26.4048 | 28.1413 | 1.7364 |
|          | 14.5231 | 14.2747 | 14.7664 | 0.4917 |
|          | 14.3509 | 13.8781 | 14.7298 | 0.8517 |
|          | 14.3816 | 14.0860 | 14.6307 | 0.5447 |
|          | 20.4752 | 19.9890 | 20.8650 | 0.8761 |
|          | 20.6496 | 20.1399 | 21.0612 | 0.9212 |
|          | 20.5906 | 20.1593 | 20.9690 | 0.8097 |
|          | 27.7395 | 26.8832 | 28.3146 | 1.4314 |
|          | 28.4384 | 27.7779 | 28.9987 | 1.2208 |
|          | 27.7349 | 27.0225 | 28.3119 | 1.2894 |
Table 7. Bayes, E-Bayes predictor and bounds of the future observation for real data sets (Application 2) under balanced squared error loss function based on progressive Type-II censoring under two-sample prediction and different sample schemes

| Scheme | \(s\) | \(\hat{y}_{(S)}(RBS)\) | LL | UL | Length | \(\hat{y}_{(S)}(EBBS)\) | LL | UL | Length |
|--------|------|-----------------|----|---|-------|-----------------|----|---|-------|
| I      | 1    | 0.0351          | 0.0260 | 0.0420 | 0.0159 | 0.0341          | 0.0303 | 0.0368 | 0.0065 |
|        | 15   | 1.1107          | 1.0204 | 1.1832 | 0.1628 | 1.0876          | 1.0405 | 1.1248 | 0.0843 |
|        | 27   | 1.5808          | 0.3829 | 2.2594 | 1.8765 | 1.3451          | 0.6051 | 1.7804 | 1.1753 |
|        | 1    | 0.0604          | 0.0492 | 0.0687 | 0.0195 | 0.0581          | 0.0534 | 0.0618 | 0.0084 |
|        | 15   | 1.1010          | 1.0063 | 1.1711 | 0.1647 | 1.0933          | 1.0425 | 1.1369 | 0.0944 |
|        | 27   | 1.7855          | 0.4419 | 2.3624 | 1.9205 | 1.7576          | 1.5011 | 1.9541 | 0.4530 |
|        | 1    | 0.0604          | 0.0492 | 0.0687 | 0.0195 | 0.0585          | 0.0546 | 0.0616 | 0.0070 |
|        | 15   | 1.0856          | 0.9360 | 1.1415 | 0.2054 | 1.0667          | 1.0346 | 1.1130 | 0.0784 |
|        | 27   | 1.7975          | 1.1444 | 2.3382 | 1.1938 | 1.7203          | 1.5029 | 1.8712 | 0.3683 |
| II     | 1    | 0.0604          | 0.0492 | 0.0687 | 0.0195 | 0.0586          | 0.0544 | 0.0623 | 0.0079 |
|        | 15   | 1.0856          | 0.9360 | 1.1415 | 0.2054 | 1.0667          | 1.0346 | 1.1130 | 0.0784 |
|        | 27   | 1.7975          | 1.1444 | 2.3382 | 1.1938 | 1.7203          | 1.5029 | 1.8712 | 0.3683 |
| III    | 1    | 0.0604          | 0.0492 | 0.0687 | 0.0195 | 0.0586          | 0.0544 | 0.0623 | 0.0079 |
|        | 15   | 1.0856          | 0.9360 | 1.1415 | 0.2054 | 1.0667          | 1.0346 | 1.1130 | 0.0784 |
|        | 27   | 1.7975          | 1.1444 | 2.3382 | 1.1938 | 1.7203          | 1.5029 | 1.8712 | 0.3683 |
Table 8. Bayes, E-Bayes predictor and bounds of the future observation for real data sets (Application 2) under balanced linear exponential loss function based on progressive Type-II censoring under two-sample prediction and different sample schemes

| Scheme | s   | $\tilde{y}_{(s)(BBL)}$ | LL    | UL    | Length | $\tilde{y}_{(s)(EBBL)}$ | LL    | UL    | Length |
|--------|-----|------------------------|-------|-------|--------|------------------------|-------|-------|--------|
|        | 1   | 0.0327                 | 0.0277| 0.0356| 0.0079 | 0.0303                 | 0.0267| 0.0333| 0.0066 |
|        |     |                        |       |       |        | 0.0326                 | 0.0309| 0.0337| 0.0028 |
|        |     |                        |       |       |        | 0.0305                 | 0.0282| 0.0321| 0.0039 |
| Scheme I | 15  | 0.9951                 | 0.9526| 1.0220| 0.0694 | 0.9819                 | 0.9594| 0.9935| 0.0341 |
|        |     |                        |       |       |        | 0.9943                 | 0.9553| 1.0200| 0.0647 |
|        |     |                        |       |       |        | 0.9904                 | 0.9701| 1.0056| 0.0354 |
|        | 27  | 1.4937                 | 0.7827| 2.3211| 1.5384 | 1.2915                 | 1.1621| 1.4665| 0.3043 |
|        |     |                        |       |       |        | 1.3632                 | 1.2114| 1.4784| 0.2669 |
|        |     |                        |       |       |        | 1.3030                 | 1.1190| 1.4157| 0.2967 |
| Scheme II | 1   | 0.0493                 | 0.0451| 0.0531| 0.0080 | 0.0462                 | 0.0430| 0.0495| 0.0065 |
|        |     |                        |       |       |        | 0.0485                 | 0.0462| 0.0503| 0.0041 |
|        |     |                        |       |       |        | 0.0456                 | 0.0415| 0.0495| 0.0080 |
|        | 15  | 0.9761                 | 0.9233| 1.0247| 0.1014 | 0.9726                 | 0.9575| 0.9930| 0.0355 |
|        |     |                        |       |       |        | 0.9648                 | 0.9460| 0.9797| 0.0338 |
|        |     |                        |       |       |        | 0.9741                 | 0.9519| 0.9922| 0.0402 |
|        | 27  | 1.4683                 | 0.9865| 1.8510| 0.8645 | 1.1994                 | 0.9229| 1.4977| 0.5747 |
|        |     |                        |       |       |        | 1.3675                 | 1.1276| 1.5318| 0.4042 |
|        |     |                        |       |       |        | 1.1468                 | 0.7042| 1.4977| 0.7935 |
| Scheme III | 1   | 0.0506                 | 0.0451| 0.0538| 0.0088 | 0.0506                 | 0.0489| 0.0520| 0.0031 |
|        |     |                        |       |       |        | 0.0490                 | 0.0441| 0.0526| 0.0085 |
|        |     |                        |       |       |        | 0.0509                 | 0.0487| 0.0527| 0.0039 |
|        | 15  | 1.0437                 | 1.0166| 1.0699| 0.0533 | 1.0307                 | 1.0163| 1.0431| 0.0267 |
|        |     |                        |       |       |        | 1.0260                 | 1.0067| 1.0417| 0.0350 |
|        |     |                        |       |       |        | 1.0104                 | 0.9947| 1.0285| 0.0338 |
|        | 27  | 1.6791                 | 1.2670| 1.9730| 0.7060 | 1.5842                 | 1.1663| 1.8208| 0.6544 |
|        |     |                        |       |       |        | 1.6500                 | 1.5216| 1.7312| 0.2096 |
|        |     |                        |       |       |        | 1.4369                 | 1.1346| 1.7334| 0.5988 |
Table 9. Bayes, E-Bayes predictor and bounds of the future observation for real data sets (Application 3) under balanced squared error loss function based on progressive Type-II censoring under two-sample prediction and different sample schemes

| Scheme | s  | $\hat{y}_{(\text{BB})}$ | LL   | UL   | Length | $\hat{y}_{(\text{EBB})}$ | LL   | UL   | Length |
|--------|----|-------------------------|-------|-------|--------|-------------------------|-------|-------|--------|
|        | 1  | 0.0179                  | 0.0100| 0.0226| 0.0126 | 0.0128                  | 0.0071| 0.0181| 0.0109 |
|        | 8  | 0.1925                  | 0.1004| 0.2516| 0.1512 | 0.1809                  | 0.1537| 0.2025| 0.0488 |
|        | 15 | 0.7027                  | 0.3223| 1.0775| 0.7551 | 0.7020                  | 0.4660| 0.8479| 0.3820 |
| Scheme I |
|        | 1  | 0.0218                  | 0.0098| 0.0269| 0.0170 | 0.0160                  | 0.0111| 0.0220| 0.0110 |
|        | 8  | 0.1782                  | 0.0982| 0.2338| 0.1355 | 0.1736                  | 0.1185| 0.2100| 0.0914 |
|        | 15 | 0.8520                  | 0.3084| 1.2698| 0.9614 | 0.7450                  | 0.3485| 0.9211| 0.5727 |
| Scheme II |
|        | 1  | 0.0179                  | 0.0100| 0.0226| 0.0126 | 0.0139                  | 0.0081| 0.0182| 0.0102 |
|        | 8  | 0.1695                  | 0.1000| 0.2126| 0.1126 | 0.1296                  | 0.0716| 0.1727| 0.1011 |
|        | 15 | 0.9098                  | 0.7822| 0.9821| 0.1999 | 0.9021                  | 0.8766| 0.9263| 0.0497 |
| Scheme III |
Table 10. Bayes, E-Bayes predictor and bounds of the future observation for real data sets (Application 3) under balanced linear exponential loss function based on progressive Type-II censoring under two-sample prediction and different sample schemes

| Scheme   | s   | $\hat{y}_{(o)(BBL)}$ | LL  | UL  | Length | $\hat{y}_{(o)(EBBL)}$ | LL  | UL  | Length |
|----------|-----|----------------------|-----|-----|--------|----------------------|-----|-----|--------|
|          |     |                      |     |     |        |                      |     |     |        |
| I        | 1   | 0.0113               | 0.0080 | 0.0136 | 0.0056 | 0.0094               | 0.0076 | 0.0109 | 0.0033 |
|          | 8   | 0.1576               | 0.0782 | 0.2259 | 0.1477 | 0.1406               | 0.1149 | 0.1578 | 0.0429 |
|          | 15  | 0.6140               | 0.2443 | 0.9396 | 0.6953 | 0.5086               | 0.3372 | 0.6250 | 0.2878 |
| II       | 1   | 0.0126               | 0.0070 | 0.0196 | 0.0127 | 0.0125               | 0.0107 | 0.0136 | 0.0029 |
|          | 8   | 0.1286               | 0.0632 | 0.1766 | 0.1135 | 0.1188               | 0.0983 | 0.1426 | 0.0443 |
|          | 15  | 0.7017               | 0.2786 | 0.9859 | 0.7073 | 0.6570               | 0.4827 | 0.7820 | 0.2993 |
| III      | 1   | 0.0156               | 0.0086 | 0.0215 | 0.0129 | 0.0117               | 0.0079 | 0.0166 | 0.0087 |
|          | 8   | 0.1463               | 0.0799 | 0.2031 | 0.1232 | 0.1070               | 0.0688 | 0.1554 | 0.0866 |
|          | 15  | 0.7949               | 0.7416 | 0.8311 | 0.0894 | 0.7942               | 0.7644 | 0.8098 | 0.0453 |
3.4 Concluding remarks

- The results in Tables 1-10 indicate that the length of the interval of the first future order statistic is smaller than the length of the interval of the last future order statistic.
- The ML, Bayes and E-Bayes intervals include the predictive values (between the lower limit (LL) and upper limit (UL)).
- The lengths of the Bayesian and E-Bayesian predictive intervals are shorter than the corresponding ML predictive ones.
- The lengths of the intervals of the E-Bayes predictors are less than the lengths of the intervals of the Bayes predictors, so the E-Bayesian prediction technique is better than the Bayesian prediction technique.
- In all cases, the lengths of the intervals of E-Bayes predictors under BLL function are less than the lengths of the intervals of the E-Bayes predictors under BSEL function.
- The lengths of the intervals of the Bayes predictors under BLL function in most cases are less than the lengths of the intervals of the Bayes predictors under BSEL function.
- The lengths of the intervals of the ML, Bayes and E-Bayes predictors increase when \( s \) increases.

4 Conclusion

In this research, the two-sample prediction technique is applied to obtain the ML, Bayesian and E-Bayesian prediction (point and interval) for future order statistics of the MTLCh distribution based on progressive Type-II censored samples. The predictors are considered under two different loss functions, the BSEL function; as a symmetric loss function and BLL function; as an asymmetric loss function. The predictors are obtained based on conjugate gamma prior and uniform hyperprior distributions. A numerical example is given to illustrate the theoretical results and three applications using real data sets are used to demonstrate how the results can be used in practice. In general, numerical computations showed that the length of the interval of the first future order statistic is smaller than the length of the interval of the last future order statistic, in Table 1 and scheme I as an example, the length of the interval of the first future order statistic is 0.1747 and the length of the interval of the last future order statistic is 2.8163. The ML, Bayes and E-Bayes intervals include the predictive values, in Table 3 as an example, at \( s = 11 \) in scheme II, the intervals (0.2630, 0.4418) include the Bayes predictive value \( \hat{y}_{(11)}(\text{BB}) = 0.3629 \), similarly, the intervals (0.2064,0.3619) include the E-Bayes predictive value \( \hat{y}_{(11)}(\text{EBB}) = 0.3024 \). Also, the lengths of the interval of the E-Bayes predictors are less than the lengths of the interval of the Bayes predictors, in Table 4 as an example, at \( s = 21 \) in scheme III, the length of the interval of the Bayes predictors is 1.2580 but the lengths of the intervals of the E-Bayes predictors are 0.4380, 0.4683 and 0.3605, so the E-Bayes prediction technique is better than the Bayes prediction technique. The Bayesian and E-Bayesian prediction (point and interval) for future order statistics of the MTLCh distribution under different type of loss functions such as general entropy and Precautionary loss functions would be useful as a basis for further researches in distribution theory.

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Competing Interests

Authors have declared that no competing interests exist.

References

[1] AL-Sayed NT, Swielum EM, AL-Dayian GR, EL-Helbawy AA. Modified Topp-Leone Chen distribution: Properties and estimation based on progressive Type-II censoring scheme. The 54th Annual
Conference on Statistics, Computer Sciences and Operation Research 9-11 Dec. Cairo University, Faculty of Graduate Studies for Statistical Research. 2019; 50-75.

[2] El-Sagheer RM, Ahsanullah M. Statistical inference for a step-stress partially accelerated life test model based on progressively Type-II censored data from Lomax distribution. Journal of Applied Statistical Science. 2015; 21(4):307–323. Available:https://www.researchgate.net/publication/289497650.

[3] Dey S, Kayal T, Tripathi YM. Statistical inference for the weighted exponential distribution under progressive Type-II censoring with binomial removal. American Journal of Mathematical and Management Sciences. 2018; 37(2):188-208. DOI: https://doi.org/10.1080/01966324.2017.1395375

[4] Almetwaly EM, Almongy HM. Estimation of the generalized Power Weibull distribution parameters using progressive censoring schemes. International Journal of Probability and Statistics. 2018; 7(2):51-61. DOI: https://doi.org/10.5923/j.ijps.20180702.03

[5] Karakoca A, Pekgör A. Maximum likelihood estimation of the parameters of progressively Type-II censored samples from Weibull distribution using Genetic Algorithm. Academic Platform Journal of Engineering and Science. 2019; 7(2):1-11. DOI: https://doi.org/10.21541/apjes.452564

[6] Mondal S, Kundu D. A new two sample Type-II progressive censoring scheme. Communications in Statistics - Theory and Methods. 2019; 48(10):2602-2618. DOI: https://doi.org/10.1080/03610926.2018.1472781

[7] Xiuyn P, Yan X, Zaizai Y. Reliability analysis of Birnbaum–Saunders model based on progressive Type-II censoring. Journal of Statistical Computation and Simulation. 2019; 89(3):461-477. DOI: https://doi.org/10.1080/00949655.2018.1555251

[8] Li S, Gui W. Bayesian survival analysis for generalized Pareto distribution under progressively Type-II censored data. International Journal of Reliability, Quality and Safety Engineering. 2020; 27(1). DOI: https://doi.org/10.1142/S0218539320500011

[9] Ahmadi J, Jozani MJ, Marchand E, Parsian A. Bayes estimation based on k-record data from a general class of distributions under balanced type loss functions. Journal of Statistical Planning and Inference. 2009; 139(3):1180-1189. DOI: https://doi.org/10.1016/j.jspi.2008.07.008

[10] Zellner A. Bayesian and non-Bayesian estimation using balanced loss functions. In: S. S. Gupta and J. O. Burger, Eds. Statistical Decision Theory and Related Topics, Springer, New York. 1994; 377-390.

[11] Han M. E-Bayesian estimation of failure probability and its application. Mathematical and Computer Modelling. 2007; 45(9-10):1272–1279. DOI: https://doi.org/10.1016/j.mcm.2006.11.007

[12] Okasha H. E-Bayesian estimation of system reliability with Weibull distribution of components based on Type-II censoring. Journal of Advanced Research in Scientific Computing. 2012; 4(4):33-45.

[13] Azimi R, Yaghmaei F, Fasih B. E-Bayesian estimation based on generalized half logistic progressive Type-II censored data. International Journal of Advanced Mathematical Sciences. 2013; 1(2):56-63. DOI: https://doi.org/10.14419/ijams.v1i2.759

[14] Okasha HM. E-Bayesian estimation for the Lomax distribution based on Type-II censored data. Journal of the Egyptian Mathematical Society. 2014; 22(3):489–495. DOI: http://dx.doi.org/10.1016/j.joems.2013.12.009
[15] Reyad HM, Ahmed SO. E-Bayesian analysis of the Gumbel Type-II distribution under Type-II censored scheme. International Journal of Advanced Mathematical Sciences. 2015; 3(2):108-120. DOI: https://doi.org/10.14419/ijams.v3i2.5093

[16] Reyad HM, Ahmed SO. Bayesian and E-Bayesian estimation for the Kumaraswamy distribution based on Type-II censoring. International Journal of Advanced Mathematical Sciences. 2016; 4(1):10-17. DOI: https://doi.org/10.14419/ijams.v4i1.5750

[17] Nasiri P, Esfandyarifar H. E-Bayesian estimation of the parameter of the logarithmic series distribution. Journal of Modern Applied Statistical Methods. 2016; 15(2):643-655. DOI: https://doi.org/10.22237/jmasm/1478003700

[18] Reyad HM, Younis AM, Alkhedir AA. Comparison of estimates using censored samples from Gompertz model: Bayesian, E-Bayesian, hierarchical Bayesian and empirical Bayesian schemes. International Journal of Advanced Statistics and Probability. 2016; 4(1):47-61. DOI: https://doi.org/10.14419/ijasp.v4i1.5914

[19] EL-Sagheer RM. E-Bayesian estimation for Rayleigh model using Progressive Type-II censoring data. Journal of Statistical Theory and Applications. 2017; 16(2):239–247. DOI: https://doi.org/10.2991/jsta.2017.16.2.8

[20] Shawky AI, Al-Aboud FM. Bayesian and E-Bayesian estimations for the compound Rayleigh distribution based on upper record values. Journal of Scientific and Engineering Research. 2017; 4(7):299-308.

[21] Reyad HM, Younis AM, Othman SA. E-Bayesian and Hierarchical Bayesian estimations based on dual generalized order statistics from the inverse Weibull model. Journal of Advances in Mathematics and Computer Science. 2017; 23(1):1-29. DOI: https://doi.org/10.9734/JAMCS/2017/34540

[22] Reyad HM, Othman SA. E-Bayesian estimation of two-component mixture of inverse Lomax distribution based on Type-I censoring scheme. Journal of Advances in Mathematics and Computer Science. 2018; 26(2):1-22. DOI: https://doi.org/10.9734/JAMCS/2018/39087

[23] Han M. E-Bayesian estimation of the exponentiated distribution family parameter under LINEX loss function. Communications in Statistics - Theory and Methods. 2019; 48(3):648-659. DOI: https://doi.org/10.1080/03610926.2017.1417432

[24] Okasha HM. E-Bayesian estimation for the exponential model based on record statistics. Journal of Statistical Theory and Applications. 2019; 18(3):236-243. DOI: https://doi.org/10.2991/jsta.d.190820.001

[25] Rabie A, Li J. E-Bayesian estimation based on Burr-X generalized Type-II hybrid censored data. Symmetry. 2019; 11(5):1-14. DOI: https://doi.org/10.3390/sym11050626

[26] Algarni A, Almarashi AM, Okasha H, Ng HKT. E-Bayesian estimation of Chen distribution based on Type-I censoring scheme. Entropy. 2020; 22(6):1-14. DOI: https://doi.org/10.3390/e22060636

[27] Han M. E-Bayesian estimation and its E-posterior risk of the exponential distribution parameter based on complete and Type-I censored samples. Communications in Statistics - Theory and Methods. 2020; 49(8):1858-1872. DOI: https://doi.org/10.1080/03610926.2019.1565837
[28] Piriaei H, Yari G, Farnoosh R. E-Bayesian estimations for the cumulative hazard rate and mean residual life based on exponential distribution and record data. Journal of Statistical Computation and Simulation. 2020; 90(2):271-290. DOI: https://doi.org/10.1080/00949655.2019.1678623

[29] Rabie A, Li J. E-Bayesian estimation for Burr-X distribution based on generalized Type-I hybrid censoring scheme. American Journal of Mathematical and Management Sciences. 2020; 39(1):41-55. DOI: https://doi.org/10.1080/01966324.2019.1579123

[30] Valiollahi R, Asgharzadeh A, Kundu D. Prediction of future failures for generalized exponential distribution under Type-I or Type-II hybrid censoring. Brazilian Journal of Probability and Statistics. 2017; 31(1):41-61. DOI: https://doi.org/10.1214/15-BJPS302

[31] Faizan M, Sana. Bayesian estimation and prediction for Chen distribution based on upper record values. Journal of Mathematics and Statistical Science. 2018; 1(6):235-243.

[32] Abd El-Raheem AM. Inference and optimal design of multiple constant-stress testing for generalized half-normal distribution under Type-II progressive censoring. Journal of Statistical Computation and Simulation. 2019; 89(2):1-30. DOI: https://doi.org/10.1080/00949655.2019.1656722

[33] Arshad M, Jamal QA. Statistical inference for Topp–Leone-generated family of distributions based on records. Journal of Statistical Theory and Applications. 2019; 18(1):65–78. DOI: https://doi.org/10.2991/jsta.d.190306.008

[34] Okasha HM, Wang C, Wang J. E-Bayesian prediction for the Burr XII model based on Type-II censored data with two samples. Advances in Mathematical Physics. 2020; 5:1:13. DOI: https://doi.org/10.1155/2020/3510673

[35] Balakrishnan N, Sandhu RA. A simple simulational algorithm for generating progressive Type-II censored samples. The American Statistician. 1995; 49(2):229-230.

[36] Murthy DNP, Xie M, Jiang R. Weibull models. John Wiley and Sons, Inc., Hoboken, New Jersey; 2004.

[37] Nelson W. Applied life data analysis. John Wiley and Sons, Inc., New York; 1982.

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