The hydrodynamical phase-slippage geometry and a semi-classical Josephson-RCSJ formulism are combined to make an analogy between the periodically driven Josephson ac effect and the quantum Hall effect. Both of these two macroscopic quantum effects are postulated to be originated from the one-dimensional linear vortex-electron relative motion. The mechanisms of the quantized Hall resistance of IQHE and FQHE are unified under the phase-locking dynamics of a Josephson-like oscillation which is periodically driven by the electron wave of the coherent Hall current, with the time-averaged frequency ratio playing the role of the filling factor reciprocal. Supporting experimental evidences for this scenario as well as necessary future investigations are remarked.

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When the applied current is larger than a critical value, the Josephson junction develops an ac current along with a voltage drop across the barrier. This is known as the ac Josephson effect \[1\]. If the junction is further exposed to a microwave irradiation with an angular frequency \(\omega_{r,f}\), the current-voltage characteristic curve shows Shapiro voltage steps \[2\] \[3\]:

\[ V = \alpha \frac{h}{2e} \omega_{r,f}, \tag{1} \]

where \(2e\) denotes the electric charge of the Cooper pair, \(V\) represents the voltage drop across the junction, the step index \(\alpha\) can be an integer or a fractional number and the Planck constant \(h = h/2\pi\). According to such a voltage-frequency relation, the r.f-biased ac Josephson effect provides a metrological quantum standard of the voltage.

To realize the Shapiro steps, one starts from the Josephson relations \[1\] \[4\]\

\[ V = \frac{h}{2e} \frac{d\varphi}{dt}, \tag{2} \]

where \(\varphi\) represents the electronic phase difference across the junction, and

\[ I_J = I_c \sin \varphi \tag{3} \]

where \(I_J\) is the Josephson supercurrent, \(I_c\) is the critical current, and then solves the semi-classical equation describing the current conservation across the junction. A discrete time computing of the well-known, r.f biased, Resistively and Capacitively Shunted Josephson Junction (RCSJ \[3\] \[4\]) model shows \[3\] the main features of the I-V curve which includes various typical properties of a (2+1)-dimensional nonlinear dynamical system such as deterministic chaos, quasiperiodic and the phase-locking behavior. The Shapiro step index \(\alpha\) can be shown as an integral, or a fractional, winding number \[3\], namely, time averaged frequency ratio \(<\Delta\varphi/\Delta t>/\omega_{r,f}\), in the phase-locked stationary states of the dynamical system.

Serving the metrology as another macroscopic quantum standard of a classical quantity, the transverse Hall resistance \(R_H\) in the Quantum Hall effect (QHE) \[9\] \[10\] can be quantized in units of \(h/e^2\):

\[ R_H = \frac{1}{\nu} \frac{h}{e^2}, \tag{4} \]

once the longitudinal Hall current simultaneously becomes perfectly conducting. In \[9\] \(\nu\) is an integer for the integer quantum Hall effect (IQHE) \[9\] or a fractional number for the fractional quantum Hall effect (FQHE) \[10\].

Interestingly, we see from a dimensional check that the quantized Hall resistance can be expressed as \(h/e^2\) multiplied by the frequency ratio of a Josephson-like oscillation to an electron wave:

\[ R_H = \frac{V_H}{I_H} = \frac{\omega_H}{e f_c} = \frac{\omega_H}{\omega_c} \left(\frac{h}{e^2}\right), \tag{5} \]

where \(V_H\) and \(I_H\) is the Hall voltage and the Hall current respectively, \([\ldots]\) denotes dimensional equalities, therein \(\omega_H\) implies the angular frequency of a presumable single charge Josephson oscillation and \(f_c(=\omega_c/2\pi)\) represents the frequency of a coherent electron wave. In the following work, we would like to answer the inspiring question: Is it possible to connect these two effects in a unified framework which solves the IQHE and FQHE by a common dynamical theory?

Compared to the simplicity the semi-classical phase-locking picture offered in describing both the integer and fractional Shapiro steps in a unified scheme, up to date quantum mechanical approaches from IQHE to FQHE appear to be more complicated. Without loosing the basic common quantum-mechanical features of those treatments, we take the Gauge Invariance picture for example to review the geometry, based on which the boundary condition for QHE was given. If a closed integral path \(\oint dl\) is taken to be a loop of electron current flow, as

\[ \oint dl = \oint dl \tag{6} \]

...
that suggested by Laughlin’s ribbon loop \([13]\), the quantization behavior of the charge carriers in IQHE can be attributed to the single-electron gauge invariance condition
\[
\oint A \cdot dl = n\Phi_0,
\]
where \(A\) is the vector potential of the gauge field threading the looping current, \(I\) implies the position of the circulating electrons and \(\Phi_0\) denotes the gauge flux quantum \(\hbar/e\). This current loop geometry provided a periodic boundary condition by which the phase increment of the electrons during each cycle as well as crossing from one edge of the ribbon to the other was quantum-mechanically restricted to be \(2\pi\), and the factor \(\nu\) of \([1]\) for IQHE was shown to be the number of charges involved in the cycling motion. According to this imaginary geometry, the FQHE can be realized by the introduction of a fractional charge state \([14]\) for an interactive electron fluid within the many-body theoretical framework. On the other hand, as to be illustrated below, the vortex motion scenario figured by P.W. Anderson \([15]\) for the ac Josephson effect provides another possible, and even more realistic, geometric relation between the Hall current and voltage drop to show the quantized Hall resistance.

In the phase-slippage picture, a quantization condition subjected to the Josephson voltage relation \([8]\) was adopted in almost a same mathematical form compared with \([3]\):
\[
\oint A \cdot d\mathbf{s} = 2n\pi,
\]
only that the integral path \(\oint d\mathbf{s}\) was notably taken around an isolated Josephson vortex core as shown in Fig.1(a) instead of a trajectory of circulating current, herein \(\phi\) denoted the electronic phase. It should be noted that the crucial underlying physical difference between these two pictures is that the former provides a quantization criterion for the single electron particle surrounding a flux while the latter has shown by itself a quantization of the vortex existing in the electron fluid. To recall a real image of the relative motion between electron fluid and vortices, one considers a thin layer Josephson junction lying on the \(x-z\) plane with the voltage drop developing along the \(y\) direction and the vortex moving along the \(x\) direction, see Fig.1(b). If the electronic phase difference between side 1 and side 2 of the junction barrier, \(\varphi_{12}(=\phi_1-\phi_2)\), at a certain \(x\) position is measured, one finds that it gains \(2\pi\) increment as each traveling vortex passes through that \(x\) position. An analogy of quantum oscillation can thus be introduced into the quantized Hall system if we simply view QHE as the same relative motion between the flux quantum and the electric charge from another observation point, i.e., the static vortex reference frame, see Fig.1(c), instead of from the static electron reference frame for the Josephson ac effect. In other words, for a stationary electronic background as that in the zero-field Josephson ac effect mentioned above, the relative motion of charge and flux is realized by the traveling vortex while in a static magnetic background of the quantum Hall system, we assume it resulted from the propagating charge, namely, the Hall current that travels through static flux quanta.

From this common phase slippage scenario, one finds that the electron fluid flowing along the \(x\) direction in a planar Hall bar lying on the \(x-y\) plane would build up, along the \(y\) direction across the flux, a voltage drop \(V_H\) together with a time evolution of the phase difference \(\varphi_{12}\), which satisfy a Josephson-like voltage-frequency relation
\[
V_H = \left(\frac{\hbar}{e}\right) \omega_{JH},
\]
where the ”Josephson-Hall frequency” \(\omega_{JH} = d\varphi_{12}/dt\). With the spin-polarized electron playing the role of the superconducting Cooper pair, this equation of the quantum Hall system can be regarded as a single-charged version of the Josephson relation established in a self-organized Josephson-like extended junction with the ”virtual electrodes” separated by a linear channel lined up along \(x\) direction with parallel quantum fluxes. According to this picture, one first see that keeping the applied Hall current unchanged, we can increase the Hall voltage by two ways. One is to increase the density of fluxes or the applied magnetic field, the other is to decrease the charge carrier concentration or the device gate voltage. Since both of the two operations increase the number of fluxes passed, and thus the phase difference \(\varphi_{12}\) experienced, by each propagating carrier within a certain period of time. Such a continuous variation of the voltage would correspond to the ac Josephson effect without external driving. However, in addition to the current-induced phase evolution of the Josephson vortex, the coherent Hall current \(I_H = e\varphi_{12}\) itself results in another frequency \(\omega_c = 2\pi f_c\) of the traveling wave of electrons, or holes, which is also observed at any fixed \(x\) position and further provides a periodic driving for the quantum Hall oscillation at this position. The system thus turns into a periodically forced quantum oscillation and the quantized Hall resistance is able to be explained by the phase-locking behavior between the Josephson-like oscillation and the Hall current. An example with the locked frequency ratio \(2/3\), or \(\nu = 3/2\), is explicitly illustrated in Fig.2.

Although we regard the pictorial argument to be rather straightforward, the corresponding theoretical formulation turns out to be apparently unconventional. Instead of traditionally quantizing the electron motion under external field, one follows the Josephson-RCSJ approach to construct a semi-classical model for the Josephson-Hall oscillation which is modulated by electron waves. The phenomenological current conservation law in \(y\) direction at a fixed \(x\) position should thus appear like:
where \( \varphi_{1H} \) represents the "Josephson-Hall phase" which is also the electronic phase difference \( \varphi_{12} \) across the flux channel, the capacitance \( C \) and the resistance \( R \) are macroscopic parameters, \(|I_e| f(\omega_e t)\) represents the periodic driving provided by the coherent Hall current, \( I_c g(\varphi_{1H}) \) represents a nonlinear vortex current for which a simple image can be obtained if we designate \( g(\varphi_{1H}) = \sin \varphi_{1H} \) to be the \( y \) component of a coherent circularly orbiting current within the quantized vortex. The Hall resistance

\[
R_H = \frac{V_H}{I_H} = \left( \frac{\hbar}{e f_e} \right) \varphi_{1H} = \left( \frac{\omega_H}{\omega_e} \right) \frac{\hbar}{e^2}
\]

(10)
can thus be shown to be quantized by solving (9) to find a locked time-averaged frequency ratio \( \frac{\omega_H}{\omega_e} \).

From this simple dynamical approach, it has been shown that the quantum phase-locking scenario provides a unified mechanism for IQHE and FQHE. Experimental supports of this semi-classical model can be found in the reported chaotic phenomenon \[16\] and a remarkable bistability with both dissipative and non-dissipative states \[17\] in the current-induced breakdown of the QHE, these typical nonlinear dissipative dynamical features strongly suggest a "chaotic" point of view for physicists to deal with the QHE breakdown problem. Also to be noted, the striking discovery of the novel 1/2 state in the multilayer system \[13\] \[24\] can be intuitively understood within the phase-locking scenario as that the extra layers provide additional coupling to tame the previously un-locked 1/2 state.

Further study is supposed to be twofold: Experimental justification and refinements of the equation of motion (9), as well as its comparison to the existing Josephson effect counterpart can do much helps to tackle crucial open questions of QHE. For example, the sample-dependent occurrence of the even-denominator states is expected to find its dynamical explanation in the "Arnol’d tongue" pattern \[18\] of the parameter space. The current and voltage distribution, whether in the edge or the bulk region, should be treated as a synchronized Josephson junction array issue, no matter for the integer or the fractional quantized Hall states. The Farey series order of appearance of the quantized plateaux and the phase transition behaviors between the plateaux should also be compared between these two system under a unified dynamical framework. Finally, the very question remained to be answered turns out to be highly fundamental, if not been taken as philosophical. As we followed the Josephson-RCSJ method to alternatively propose an electron-perturbed semi-classical equation of motion, we have been forced only to examine this formulism as a new macroscopic quantization rule, or in a practical sense, a semi-classical dual quantum theory. In either way, we are obliged to see further extension of this temporal system to a spatial-temporal mathematical structure with a quantum theoretical point of view. Foreseeable research should first include a unified dynamical study of the quantum soliton in both system. Among other quantum mechanical features, a relativistic Lorentz Contraction \[23\] in the QHE may be discovered as the charge-flux relative velocity approaches the sample light velocity.

In summary, adopting the hydrodynamical phase-slippage geometry together with the semi-classical Josephson-RCSJ dynamical formulism, this letter proposes a pictorial analogy between the periodically driven Josephson ac effect and the quantum Hall effect. These two macroscopic quantum effects are postulated to be both originated from the linear vortex-electron relative motion, with the moving and standing quanta been altered from the flux of the former to the electron of the latter. The quantized Hall resistance of both IQHE and FQHE is suggested to be described by a unified phase-locking dynamics of a Josephson-like oscillation periodically driven by the electron wave of the coherent Hall current, with the time-averaged frequency ratio playing the role of the filling factor reciprocal.

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FIG. 1. (a) An isolated vortex with equal-phase lines. (b) For a non-propagating electron background as the ac Josephson effect, the $x$ direction relative motion between flux and charge comes from the translational motion of the vortex. During the time $t_f - t_i$, the vortex travels from $x_1$ to $x_2$ and the phase difference between side 1 and side 2 $\varphi_{12}$ experiences a $2\pi$ increment. (c) For a static magnetic background as the quantum Hall effect, the relative motion comes from the propagation of electrons. For a corresponding example, during the time $t_f - t_i$, the electron fluid flows from $x_2$ to $x_1$ and the phase difference between 1 and 2 is also increased by $2\pi$.

FIG. 2. A one dimensional coherent electron wave passing through quantized fluxes makes a periodically driven Josephson-like oscillation. Take $\beta = 3/2$ locked state as an example: During the time $\Delta t$, electron $e_1$ originally at $x_1$ passes through two fluxes $f_1$ and $f_2$ and causes a $4\pi$ increment for the phase difference between side 1 and side 2, which can be measured at any certain $x$ position including $x_2$. Meanwhile, there are three electron wave packets $e_1$, $e_2$ and $e_3$ denoted by the sine curve, passing through $x_2$. Thus a phase-locked electron-driven Josephson-like oscillation results in at $x_2$ the quantization of the Hall resistance $R_H = V_H/I_H = (h/e)(4\pi/\Delta t)/(3e/\Delta t) = (2/3)(h/e^2)$. Also can be seen in this figure, the phase-locking dynamics corresponds to an elastic deformation of the flux chain, i.e., the line density of the fluxes can be self-adjusted to maintain the locked density ratio of electrons to fluxes.