Bridging the gap: symplecticity and low regularity on the example of the KdV equation

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Abstract

Recent years have seen an increasing amount of research devoted to the development of so-called resonance-based methods for dispersive nonlinear partial differential equations. In many situations, this new class of methods allows for approximations in a much more general setting (e.g. for rough data) than, for instance, classical splitting or exponential integrator methods. However, they lack one important property: the preservation of geometric structures. This is particularly drastic in the case of the Korteweg-de Vries (KdV) equation which is a fundamental model in the broad field of dispersive equations that is completely integrable, possessing infinitely many conserved quantities, an important property which we wish to capture - at least up to some degree - also on the discrete level. A revolutionary step in this direction was set by the theory of geometric numerical integration resulting in the development of a wide range of structure-preserving algorithms for Hamiltonian systems. However, in general, these methods rely heavily on highly regular solutions. State-of-the-art low-regularity integrators, on the other hand, poorly preserve the geometric structure of the underlying PDE. This work makes a first step towards bridging the gap between low regularity and structure preservation. We introduce a novel symplectic (in the Hamiltonian picture) resonance-based method on the example of the KdV equation that allows for low-regularity approximations to the solution while preserving the underlying geometric structure of the continuous problem on the discrete level.

Keywords: Geometric numerical integration · resonances · low regularity · symplecticity

Mathematics Subject Classification: 65M12 · 65M70 · 35Q53

1 Introduction

Let us consider as a model problem the periodic Korteweg-de Vries (KdV) equation [9]

\[
\begin{aligned}
\partial_t u(t, x) + \partial_x^3 u(t, x) &= \frac{1}{2} \partial_x (u(t, x))^2, & t \in \mathbb{R}_+, & x \in \mathbb{T} = [-\pi, \pi), \\
u(0, x) &= u_0(x), & x \in \mathbb{T},
\end{aligned}
\]

(1)

and its numerical solution from two major perspectives: We aim to

(I) approximate the time dynamics of the partial differential equation under low regularity assumptions, i.e., allowing for rough data, and at the same time,
(II) preserve the underlying geometric structure of the continuous problem.

The numerical solution of nonlinear dispersive equations with low-regularity data is thereby an ongoing challenge of its own right: Classical numerical time-integrators are developed with analytic solutions in mind [25]. For this reason, classical integrators require a significant amount of regularity of the solution to converge reliably. The necessity for smooth solutions is not just a theoretical technicality: The severe order reduction of classical methods in the low-regularity setting is indeed observed in practice [6, 19, 31] (see also Section 6) leading to instability, loss of convergence and huge computational costs. Over the recent decade, this challenge has motivated the idea of so-called low-regularity integrators which are able to provide reliable convergence rates in a more general setting allowing for, e.g., rough initial data. A particular class of integrators which has proven successful in a range of applications are so-called resonance-based methods [2, 6, 31, 34, 35].

While in many situations this new class of integrators allows for approximations for much rougher data, than for instance classical splitting methods [20, 21], previous resonance-based approaches lack one important property: the preservation of geometric structures. This is particularly drastic in case of the KdV equation which is completely integrable, possessing infinitely many conserved quantities [10, 22], an important property which we wish to capture - at least up to some degree - also on the level of the discretisation. A revolutionary step in this direction was taken by the theory of geometric numerical integration [11, 17, 28, 36] resulting in the development of a wide range of structure-preserving algorithms firstly for dynamical systems and later also for partial differential equations with conservation laws [3, 7, 12, 33, 32]. However, in general, these methods rely heavily on the treatment of highly regular solutions to achieve guaranteed convergence. State-of-the-art low-regularity integrators, whilst allowing for approximations for rougher data, on the other hand, come with the major drawback of poor preservation of geometric structure of the underlying PDE (cf. [6, 31] and also Section 6).

Generally speaking, until now structure preservation seemed out of reach for low-regularity integrators, and the low regularity regime was out of reach for structure-preserving algorithms. With this work, we aim to make a first step towards bridging this gap. We introduce a novel symplectic (in the Hamiltonian picture) low-regularity integrator on the example of the KdV equation that preserves the underlying geometric structure of the continuous problem whilst at the same time allowing for low-regularity approximations. Our key idea thereby lies in embedding the dominant parts within the oscillatory integrals exactly into our numerical discretisation (in the spirit of resonance-based schemes [2, 6, 19]), while approximating the non-oscillatory parts in a symplectic way. The latter is much more involved than previous resonance-based approaches (which are based on simple Taylor series expansions) and the key novelty to obtaining structure preservation at low regularity. This is achieved by switching to the so-called interaction picture of KdV, \( v(t, x) = \exp(t\partial_x^3)u(t, x) \), revisiting the constructions of some of the first works in resonance-based methods [19, 31]. In this case, the dominant-frequency interactions in the solution are encoded directly in Duhamel’s formula which in case of KdV Eq. (1) casts into the form

\[
   v(t) = v(0) + \frac{1}{2} \int_0^t e^{s\partial_x^3} \partial_x \left( e^{-s\partial_x^3} v(s) \right)^2 ds.
\]

In order to resolve the underlying oscillations triggered by the nonlinear frequency interactions of \( e^{s\partial_x^3} \) and \( e^{-s\partial_x^3} \) we consider Duhamel’s formula in Fourier space

\[
   \hat{v}_m(t) = \hat{v}_m(0) + \sum_{a+b=m} \frac{im}{2} \int_0^t e^{-is3\omega_0^2 t} \hat{v}_a(s) \hat{v}_b(s) ds
\]

via the expansion \( v(t, x) = \sum_{m \in \mathbb{Z}} \hat{v}_m(t)e^{imx} \) (see Section 3.1 for further details). The central task then
lies in finding a suitable numerical approximation to the oscillatory integrals

\[
\int_0^T e^{-is3mab} \hat{v}_a(s) \hat{v}_b(s) ds.
\] (2)

For an extensive overview on highly oscillatory problems, we also refer to [8, 23, 24, 26, 30] and the references therein.

The central idea behind classical resonance-based schemes [6, 19, 31] lies in embedding the dominant frequency interactions within the oscillatory integral Eq. (2) (that is \(e^{-is3mab}\) in case of KdV) exactly into the numerical discretisation, while approximating the non-oscillatory terms \(\hat{v}_\sigma(s)\) \((\sigma = a, b)\) via a simple Taylor series expansion

\[
\hat{v}_\sigma(s) = \hat{v}_\sigma(0) + O(s^{\hat{v}_\sigma'}).
\] (3)

Although this approach yields, in the spirit of (1), a low-regularity approximation with a local error at order \(O(t^2 u^2)\) (compared to the local error structure \(O(t^2u^4)\) for classical exponential integrators), it brutally destroys the symplectic structure of the KdV flow

\[
v(0) \mapsto \Phi^t(v(0)) = v(0) + \frac{1}{2} \int_0^t e^{s\partial^3_x} \partial_x \left( e^{-s\partial^3_x} \Phi^s(v(0)) \right)^2 ds.
\]

In order to overcome this, we propose a novel class of resonance-based methods for the discretisation of oscillatory integrals of type Eq. (2) which do preserve the geometric structure of the underlying flow. Our key idea thereby lies in stepping away from the simple Taylor series expansion Eq. (3) towards a suitable symplectic approximation of the non-oscillatory terms. This allows us to introduce a resonance-based method which preserves exactly the symplectic form

\[
\omega = \sum_{a=1}^{\infty} dp_a \wedge dq_a
\]

associated to the Hamiltonian description of the KdV equation (see Section 2.1)

\[
\mathcal{H}_1(p, q) = -i \sum_c c^2 p_c q_c - \frac{i}{2} \sum_{a+b-c=0} (p_a q_b q_c + q_a p_b p_c).
\]

In contrast to classical resonance-based methods [6, 31, 34, 35], which are all explicit, our new symmetric and symplectic resonance-based scheme is implicit. This pursuit of the implicit approach within the construction of our new method is motivated by the fundamental result that symmetric Runge–Kutta methods are necessarily implicit (cf. [27]) and indeed, in general, our conjecture is that a similar statement holds true also for resonance-based methods:

**Conjecture 1.1.** Every symmetric resonance-based scheme is necessarily implicit.

**Outline of the paper.** The structure of this manuscript is as follows. In Section 2 we recall the central geometric properties of the KdV equation. This is followed, in Section 3, by an overview of recent progress on numerical methods for the KdV equation and the construction of our new symplectic resonance-based method. We then analyse its implicit nature and geometric properties in Section 3.2 and Section 4 and the convergence properties of our method are proven in Section 5. Our theoretical findings are underlined in computational experiments which are described in Section 6.
2 The KdV equation and its geometric structure

We consider the initial value problem for the Korteweg–de Vries (KdV) equation with periodic boundary conditions as introduced in Eq. (1), where $\mathbb{T} = [-\pi, \pi]$ denotes the one-dimensional torus. Note that the mass $\int_{\mathbb{T}} u(t, x) dx$ is conserved in this equation, and that we may (by considering $u_0 \mapsto u_0 - \int_{\mathbb{T}} u_0 dx$) therefore impose without loss of generality the following assumption.

**Assumption 1.** We assume throughout that our solution has zero mass, i.e. that
$$\int_{\mathbb{T}} u_0(x) dx = 0.$$ 

In the construction of our numerical scheme it will be helpful to consider the twisted variable
$$v(t, x) = \exp(\partial^3_x t) u(t, x).$$

This change of variable is widely known to provide a useful tool both for the analysis of dispersive nonlinear equations [4, 37] and the construction of tailored numerical schemes. The twisted variable $v$ satisfies the following initial value problem which is equivalent to Eq. (1):

$$\begin{cases}
\partial_t v(t, x) = \frac{1}{2} \partial^2_x \partial_x \left( e^{-\partial^3_x t} v(t, x) \right)^2, & (t, x) \in \mathbb{R}_+ \times \mathbb{T}, \\
v(0, x) = u_0(x), & x \in \mathbb{T}.
\end{cases}$$

(4)

It is easy to see that under Assumption 1 the twisted variable will also satisfy
$$\int_{\mathbb{T}} v(t, x) dx = 0$$
for all times $t \geq 0$.

**Remark 2.1.** We note that for any $s \in \mathbb{N}$, the norm $\| \cdot \|_{H^s}$ on the quotient space $H^s/\langle 1 \rangle$ is equivalent to $\| \partial_x^s \|_{H^s}$ and to the definition in terms of Fourier modes
$$\|g\|_{H^s} := \left( \sum_{m \in \mathbb{Z}\{0\}} |m|^{2s} |\hat{g}_m|^2 \right)^{1/2}.$$ 

Therefore we will use these three notations interchangeably throughout the paper.

2.1 First integrals and Hamiltonian formulation of the KdV equation

The KdV equation is completely integrable and has an infinite set of first integrals [10, Section 3.1]. In the present work we mainly focus on the momentum $I_0[u]$ and energy $I_1[u]$ which are given by

$$I_0[u] = \int_{\mathbb{T}} u^2 dx, \quad I_1[u] = -\frac{1}{2} \int_{\mathbb{T}} 3u_x^2 + u^3 dx.$$ 

(5)

In order to study the structure preservation properties it will be convenient to look at the following infinite-dimensional Hamiltonian formulation of the KdV equation [15, Section 1]. Let us write

$$\dot{u}_n(t) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-inx} u(t, x) dx,$$

for the Fourier modes of the solution $u$ such that $u(t, x) = \sum_{n \in \mathbb{Z}} \hat{u}_n e^{inx}$, and define

$$p_a(t) := \hat{u}_a(at), \quad q_a(t) := \hat{u}_{-a}(at) \quad \text{for} \quad a \geq 1.$$ 

Then the KdV equation Eq. (1) is equivalent to the following infinite dimensional Hamiltonian system

$$\frac{dp_a}{dt} = -\frac{\partial}{\partial q_a} \mathcal{H}_1(p, q), \quad \frac{dq_a}{dt} = \frac{\partial}{\partial p_a} \mathcal{H}_1(p, q).$$
where the \textit{time-independent} Hamiltonian \( H_1 \) is given by

\[
H_1(p,q) = -i \sum c^2 p_c q_c - i \sum_{a+b+c=0} (p_a p_b q_c + q_a q_b p_c).
\]

The corresponding symplectic form is given by

\[
\omega = \sum_{a=1}^{\infty} dp_a \wedge dq_a.
\]

In practice we will exploit the equivalent form

\[
\omega(y, \tilde{y}) = \sum_{m \in \mathbb{Z}} \text{sgn}(m) y_m \tilde{y}_m \quad \text{for } y, \tilde{y} \in l^2(\mathbb{Z} \setminus \{0\}).
\]

In order to understand the convergence properties of these (and similar) sums it will be convenient to introduce the following Hilbert spaces:

**Definition 2.2.** For \( s \in \mathbb{R}_+ \), we define for a sequence \( v = (v_m)_{m \in \mathbb{Z} \setminus \{0\}} \) the norm

\[
\|v\|_{l^2_s} := \left( \sum_{m \in \mathbb{Z} \setminus \{0\}} |m|^{2s} |v_m|^2 \right)^{1/2}
\]

and we define the space \( l^2_s \) by

\[
l^2_s := \{ v \in l^2(\mathbb{Z} \setminus \{0\}) \mid \|v\|_{l^2_s} < \infty \}.
\]

The above spaces are isometries of the classical Sobolev spaces \( H^s \), \( s > 0 \). With the aid of the Cauchy-Schwarz inequality one can in particular show that for whenever \((p,q) \in l^2_s\), i.e. whenever \( u \in H^s, s > 1/2 \), all of the above sums converge absolutely and the Hamiltonians and the symplectic form as defined above are well-defined.

For further details on the Hamiltonian structure of the KdV equation and the local well-posedness of the KdV equation in low-regularity regimes we refer the reader to [4, 16, 22, 37].

### 3 A resonance-based symplectic integrator for the KdV equation

Before introducing our new scheme, let us briefly outline recent progress on numerical integrators for the KdV equation. Symmetric exponential integrators for a general class of semilinear evolution equations were designed in [7]. Due to the loss of derivative in Burger’s type nonlinearity their error analysis does, however, not apply to the KdV equation. Numerical experiments (see Section 6) in particular suggest that symmetric exponential integrators [7] suffer severely if no Courant–Friedrichs–Lewy (CFL) condition is imposed in case of the KdV equation and, indeed, become unstable in case of low-regularity data. Important contributions to symplectic integrators for KdV were made in [3, 5] and symplectic finite difference schemes were designed and extensively tested numerically for smooth solutions. However, crucially the inherent nature of finite difference schemes requires that a CFL condition be imposed to ensure stability (see also [34]), and to the best of our knowledge, a rigorous error analysis is still lacking even for highly regular solutions.
For Strang splitting for the KdV equation it was recently shown [20, 21] that the method converges at first order in $H^r$ for some $r \geq 1$ if we require the solution to be at least in $H^{r+3}$, where $H^r$ is the periodic Sobolev of order $r$. To our knowledge, this was the first rigorous convergence result for the KdV equation in the literature.

The desire to construct symplectic numerical methods is partially motivated by the well-known result that in the case of finite dimensional Hamiltonian systems such symplectic methods lead to long-time approximate preservation of the Hamiltonian [17]. More recently it was found that, subject to some additional challenges arising from the existence of certain resonant time-steps, a similar statement holds true also for specific examples of infinite dimensional Hamiltonian systems, see for instance [12, 13, 14] in case of cubic Schrödinger.

Hofmanová & Schratz [19] constructed a resonance-based integrator for the KdV equation which achieves first-order convergence in $H^1$ with initial data in $H^3$ (and an integrator which requires data in $H^5$ for second order convergence in $H^3$). A further improvement on these results was given by [38] who developed an embedded exponential-type low-regularity integrator which can, for any $\gamma > 1/2$, achieve first order convergence in $H^{\gamma}$ for initial data in $H^{\gamma+1}$ and second order convergence for initial data in $H^{\gamma+3}$.

While the Strang splitting method is naturally symmetric and symplectic, the resonance-based schemes proposed for KdV so far are neither symmetric nor symplectic. The main reason why such structure preservation properties could not previously be embedded for low-regularity integrators is because previous work had focussed on the construction of explicit methods (cf. Conjecture 1.1). Thus, in the present work, we take a novel point-of-view by describing, for the first time, an implicit resonance-based low-regularity integrator. The construction is motivated by the algorithms described in [19], which ensures that our new method can achieve first and second order convergence for similar low-regularity assumptions as in [19]. A particular strength of both the present work and [19] is that no CFL condition is required to ensure stability, meaning these methods can be applied to compute true low-regularity solutions. In addition, through our new choice of discretisation we are able to show that the resulting method is symmetric and symplectic (in the sense made precise in Section 4.1) and preserves the momentum exactly (cf. Section 4.2). These are fundamental properties of the continuous problem (1) and hence we wish to also preserve them on the discrete level.

### 3.1 Derivation of the symplectic resonance-based scheme

In order to derive our symplectic resonance-based method we consider Duhamel’s formula for the twisted system Eq. (4) where, for geornotational simplicity, we suppress the $x$-dependence of the unknown functions in the following notation

$$v(t_n + \tau) = v(t_n) + \frac{1}{2} \int_0^\tau e^{i(t_n + s)\partial_x} \left( e^{-(t_n + s)\partial_x^3} v(t_n + s) \right)^2 ds. \tag{6}$$

We can reformulate Eq. (6) in terms of the Fourier coefficients of $v$ as follows:

$$\hat{v}_m(t_n + \tau) = \hat{v}_m(t_n) + \sum_{a+b=m} \frac{im}{2} e^{-i(a^3 - b^3) \tau} \int_0^\tau e^{-i(s^3 - a^3 - b^3)} \hat{v}_a(t_n + s) \hat{v}_b(t_n + s) ds \leq \hat{v}_m(t_n) + \sum_{a+b=m} \frac{im}{2} e^{-i(3mab) \tau} I^\tau_{\alpha, \beta}[v], \tag{7}$$

where we defined the oscillatory integral

$$I^\tau_{\alpha, \beta}[v] := \int_0^\tau e^{-i3mab \tau} \hat{v}_a(t_n + s) \hat{v}_b(t_n + s) ds \tag{8}$$
and used the algebraic relation \((a + b)^3 - a^3 - b^3 = 3(a + b)ab\). The central observation is that the nonlinear frequency interactions in the KdV system are now captured by the oscillatory terms
\[
\exp(-is3mab).
\] (9)

In the discretisation of our oscillatory integral Eq. (8) we wish to treat these central oscillations Eq. (9) exactly. Henceforth, we will only approximate numerically the corresponding non-oscillatory parts
\[
\hat{v}_\sigma(t_n + s) \quad \text{and} \quad \hat{v}_\sigma(t_n + s)
\] (10)
in Eq. (8). This will allow for approximations at low regularity. Note that \(\hat{v}_\sigma(t_n + s) (\sigma = a, b)\) are indeed slowly varying as thanks to Eq. (4) we have for any \(s > d/2\)
\[
\|\partial_t v\|_s \leq c \|\partial_x v\|_s \|v\|_s
\] (11)
for some constant \(c > 0\) independent of \(v, u_0\). Here we relied on the bilinear estimates from Lemma 3.4 and the fact that \(v \mapsto \exp(\pm t\partial_x^3)v\) is an isometry on any \(H^s\), for all \(s, t \geq 0\).

In [19] the central idea in the discretisation of the oscillatory integral Eq. (8) lies in a simple Taylor series expansion of the non-oscillatory parts Eq. (10) in the spirit of
\[
\hat{v}_\sigma(t_n + s) \approx \hat{v}_\sigma(t_n) \quad \text{for} \quad \sigma = a, b.
\] (12)
Together with the observation in Eq. (11) this leads to a local error structure at low regularity of the form
\[
O(s\partial_t v) = O(s\partial_x v^2).
\] (13)
We call the above error of \textit{low regularity} as a classical KdV approximation would introduce a local error at order
\[
O(s\partial_t u) = O(s\partial_x^2 u)
\]
which involves higher derivatives (and thus higher regularity assumptions on the solution) than Eq. (13).

All resonance-based schemes proposed in the literature so far follow exactly this construction \([6, 19, 31]\). Due to the favourable local error structure of this approach, in general one obtains better approximations at low regularity than classical numerical schemes (e.g., splitting, exponential integrator or Lawson-type methods). However, as outlined in the introduction, a major drawback lies in the fact that the quite brutal approximation Eq. (12) destroys the symplectic structure of the KdV flow
\[
v(0) \mapsto \phi_{0,t}(v(0)) = v(0) + \frac{1}{2} \int_0^t e^{s\partial_x^3} \left(e^{-s\partial_x^3} \phi_{0,s}(v(0))\right)^2 ds.
\]

In order to overcome this, our new idea lies in the fact that \textit{qualitatively} the size of the local error in the numerical scheme would remain the same if we used for \(\sigma = a, b\) the implicit approximation
\[
\hat{v}_\sigma(t_n + s) \approx \hat{v}_\sigma(t_n + \tau)
\]
or indeed if we took the average of both approximations leading to the symmetric form
\[
\hat{v}_\sigma(t_n + s) \approx \frac{1}{2} (\hat{v}_\sigma(t_n) + \hat{v}_\sigma(t_n + \tau))
\].
Motivated by the structure preservation properties of the classical midpoint rule (cf. \([17, 25]\)) we will base our new schemes on the latter approximation. More precisely, our idea is to apply the midpoint rule
to the slowly varying terms Eq. (10) in the oscillatory integral Eq. (8). This gives rise to the following implicit resonance-based discretisation of \( \tilde{v}_m \):

\[
\tilde{v}_m^{n+1} = \tilde{v}_m^n + \sum_{a+b=m} \frac{im}{8} e^{-it_a3ab} \int_0^\infty e^{-ix3ab} (\tilde{v}_a^{n+1} + \tilde{v}_a^n) (\tilde{v}_b^{n+1} + \tilde{v}_b^n) \, ds. \tag{14}
\]

We can now perform the remaining integral exactly to find

\[
\tilde{v}_m^{n+1} = \tilde{v}_m^n + \sum_{a+b=m} -\frac{1}{24ab} e^{-it_a3ab} (e^{-it3ab} - 1) (\tilde{v}_a^{n+1} + \tilde{v}_a^n) (\tilde{v}_b^{n+1} + \tilde{v}_b^n).
\]

This expression can be transformed back into physical space in the following steps

\[
\tilde{v}_m^{n+1} = \tilde{v}_m^n + \sum_{a+b=m} \frac{1}{24} e^{-it_a3ab} (e^{-it3ab} - 1) \left( \left( \partial_x^{-1}_a v^n \right)_a + \left( \partial_x^{-1}_b v^n \right)_b \right) \left( \left( \partial_x^{-1}_a v^{n+1} \right)_a + \left( \partial_x^{-1}_b v^{n+1} \right)_b \right).
\]

which in physical space takes the form

\[
v^{n+1} = v^n + \frac{1}{24} e^{it(t_3x)} \partial_x^3 \left( e^{-it(t_3x)} \partial_x^3 \partial_x^{-1} (v^{n+1} + v^n) \right) - \frac{1}{24} e^{-it(t_3x)} \partial_x^3 \left( e^{-it(t_3x)} \partial_x^3 \partial_x^{-1} (v^{n+1} + v^n) \right)^2. \tag{15}
\]

By the definition of the twisted variable, \( v(t) = \exp((t_3x)u(t)) \), we then find our new symplectic resonance-based method for the approximation of the original KdV equation Eq. (1) in the following form:

\[
u^{n+1} = e^{-\tau \partial_x^3 u^n} + \frac{1}{24} \left( \partial_x^{-1} u^{n+1} + e^{-\tau \partial_x^3 \partial_x^{-1} u^n} \right)^2 - \frac{1}{24} e^{-\tau \partial_x^3 \partial_x^{-1} u^{n+1} + \partial_x^{-1} u^n}^2. \tag{16}
\]

**Remark 3.1.** Let the numerical scheme be denoted by \( u^{n+1} = \hat{\Phi}_\tau(u^n) \) then it is easy to see that the method is symmetric, in the sense that \( \hat{\Phi}_\tau = \hat{\Phi}_\tau^{-1} \). The remaining structure preservation properties will be discussed in further detail in Section 4.

Before analysing the geometric properties of our new symplectic scheme Eq. (16) let us first make some remarks on its implicit nature.

### 3.2 Remarks on the implicit nature of the symplectic resonance-based scheme

In contrast to classical resonance-based methods [2, 6, 31, 34, 35], which are all explicit, our new symplectic resonance-based scheme is implicit. In general, our conjecture is that any symmetric resonance-based scheme must be implicit (cf. Conjecture 1.1).

The implicit nature of the method brings about novel challenges, such as the question of solution of a nonlinear equation at every time step and the stability analysis of the method. Through rigorous and careful analysis we are able to prove that fixed-point iterations yield a satisfactory means for solving the nonlinear system and derive stability and convergence results of the implicit method. A particular strength of this approach is that no CFL condition needs to be imposed on the time-step and spatial discretisation. This results in a method that is truly able to resolve low-regularity solutions, unlike Runge–Kutta methods and even exponential integrators, the latter of which for the KdV equation due to its Burgers’ type nonlinearity typically rely on the weaker CFL condition \( \Delta t \lesssim \Delta x \). In particular, to the
best of our knowledge the present work is, in fact, the first structure-preserving integrator for the KdV equation which does not require a CFL condition (cf. [3, 7]).

In this section we show how one may efficiently solve the implicit equation in our scheme Eq. (16) with fixed-point iterations at every time step. In practical implementation it is found (cf. Section 6) that even for moderate timesteps only a small number of fixed-point iterations is required for convergence. For the analysis let us define the following map:

\[
S_1(\tilde{v}) := v^n + \frac{1}{24} e^{(t_n+\tau)\partial_x^3} \left( e^{-((t_n+\tau)\partial_x^3)\partial_x^{-1}(\tilde{v} + v^n)} \right)^2 - \frac{1}{24} e^{t_n\partial_x^3} \left( e^{-(t_n\partial_x^3)\partial_x^{-1}(\tilde{v} + v^n)} \right)^2.
\]

The main result concerning the solution of Eq. (15) is then the following.

**Theorem 3.2.** Fix \( l \in \{1, 2, 3\} \) and \( R > 0 \). Then there is a \( \tau_R > 0 \) such that for all \( \tau \in [0, \tau_R) \) and any \( v^n \in B_R(H^l) := \{ \tilde{v} \in H^l \mid \|\tilde{v}\|_{H^l} < R \} \) we have \( v^{n+1} \) the exact solution of Eq. (15) is given by the following limit in \( H^l \):

\[
v^{n+1} = \lim_{j \to \infty} S_1^{(j)}(v^n), \quad \text{where} \quad S_1^{(j)}(v^n) = S_1 \circ \cdots \circ S_1(v^n).
\]

Moreover, we have the estimate

\[
\|v^{n+1} - v^n\|_{H^l} \leq \tau^\frac{l}{2} \tilde{C}_R,
\]

and, if additionally \( v^n \in B_R(H^{l+1}) \),

\[
\|v^{n+1} - v^n\|_{H^l} \leq \tau \tilde{C}_R,
\]

for some \( \tilde{C}_R > 0 \) which depends only on \( R \).

Note an analogous result holds true for the numerical method in \( u \), Eq. (16), and is proven in Cor. 3.6. In order to prove Thm. 3.2 we will rely on the following two lemmas.

**Lemma 3.3.** Let us introduce the notation

\[
F(t_n, \tau, \tilde{v}) := \frac{1}{6} e^{(t_n+\tau)\partial_x^3} \left( e^{-(t_n+\tau)\partial_x^3} e^{-((t_n+\tau)\partial_x^3)\partial_x^{-1}(\tilde{v})} \right)^2 - \frac{1}{6} e^{t_n\partial_x^3} \left( e^{-(t_n\partial_x^3)\partial_x^{-1}(\tilde{v})} \right)^2.
\]

Then, for \( l = 1, 2, 3 \), there is a continuous function \( M_l : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) such that

\[
\|F(t_n, \tau, f) - F(t_n, \tau, g)\|_{H^l} \leq \tau^\frac{l}{2} M_l(\|f\|_{H^l}, \|g\|_{H^l}) \|f - g\|_{H^l}.
\]

**Proof of Lemma 3.3.** For the case \( l = 1 \) see the proof of [19, Eq. (38)] and for the case \( l = 2 \) see the proof of [19, Lemma 2.4]. The case \( l = 3 \) follows in a similar way, and for completeness we have included the proof in Appendix A.

Throughout this and the following section the following well-known bilinear estimates will prove to be a useful tool:

**Lemma 3.4.** For any \( r > 1/2 \) there is a constant \( C_r > 0 \) such that for all \( f, g \in H^r \) we have

\[
\|fg\|_{H^r} \leq C_r \|f\|_{H^r} \|g\|_{H^r}.
\]

For further details and a proof of Lemma 3.4 see for instance [1, Eqs. (10)-(11)].
Proof of Thm. 3.2. Our goal is to apply a contraction mapping argument for \( \tau \in (0, \tau_R) \), with \( \tau_R > 0 \) sufficiently small.

**Claim 3.5.** For any \( R > 0 \) there is a constant \( \tau_R > 0 \) such that for all \( \tau \in (0, \tau_R) \) the following is true. If \( v^n, f, g \in H^1 \) are such that \( \|f\|_{H^1}, \|g\|_{H^1} < 2R, \|v^n\|_{H^1} < R \) then

\[
\|S_1(f) - S_1(g)\|_{H^1} \leq \frac{1}{2} \|f - g\|_{H^1}.
\]

Proof of Claim 3.5. By Lemma 3.3 we have

\[
\|S_1(f) - S_1(g)\|_{H^1} = \left\| F\left(\tau, \frac{f + v^n}{2}\right) - F\left(\tau, \frac{g + v^n}{2}\right) \right\|_{H^1}.
\]

Thus \( M \|v^n\|_{H^1} \leq \|f - g\|_{H^1} \). Since \( n \rightarrow H^1 \) for all \( n \in \mathbb{N} \), we have \( S_1(v^n) \rightarrow v^n \) in \( H^1 \), and so if we let \( \tau R = \tilde{M}_i^2 \) we find by induction

\[
\|S_1(v^n)\|_{H^1} \leq (1 + \tau^{1/2} \tilde{M}_i) \|v^n\|_{H^1},
\]

and

\[
\|v^n - v^n\|_{H^1} \leq \|S_1(v^n) - v^n\|_{H^1} \leq \|v^n - v^n\|_{H^1} + 2^{-j} \|S_1(v^n) - v^n\|_{H^1}.
\]

Thus \( \|S_1(v^n)\|_{H^1} \leq 2 \|v^n - v^n\|_{H^1} \), and so if we let \( \tau R = \tilde{M}_i^{-2} \) we find by induction

\[
\|S_1(v^n) - v^n\|_{H^1} \leq \|v^n - v^n\|_{H^1} + 2^{-j} \|S_1(v^n) - v^n\|_{H^1}.
\]

Thus \( \|S_1(v^n)\|_{H^1} \leq (1 + \tau^{1/2} \tilde{M}_i^{-1}) \|v^n\|_{H^1} \), and so if we let \( \tau R = \tilde{M}_i^{-2} \) we find by induction

\[
\|S_1(v^n) - v^n\|_{H^1} \leq \|v^n - v^n\|_{H^1} + 2^{-j} \|S_1(v^n) - v^n\|_{H^1}.
\]

Thus \( \langle S_1(v^n) \rangle \rightarrow v^n \) is a Cauchy sequence and its limit in \( H^1 \) is a fixed point of \( S_1 \), hence Eq. (17) follows.

Taking \( J \rightarrow \infty \) in Eq. (22) we find

\[
\|v^{n+1} - v^n\|_{H^1} \leq \|S_1(v^n) - v^n\|_{H^1}.
\]

The estimate Eq. (18) follows then by combining Eq. (21) and Eq. (23). We now have by the construction of the resonance-based method Eq. (15):

\[
\|S_1(v^n) - v^n\|_{H^1} = \frac{1}{2} \left\| \int_0^r e^{(t_k + s)\partial_x^2} \partial_x \left( e^{-(t_k + s)\partial_x^2} v^n \right)^2 \right\|_{H^1} \leq \frac{1}{2} \left\| \int_0^r \left( e^{-(t_k + s)\partial_x^2} v^n \right)^2 \right\|_{H^1+1} \leq C_1 \|v^n\|_{H^1+1}^2,
\]

for some constant \( C_1 > 0 \) independent of \( v \), where in the final line we made use of the bilinear estimates Lemma 3.4. Combining Eq. (24) and Eq. (23) implies Eq. (19).
The results of Thm. 3.2 extend directly to the solution \( u^{n+1} \) of Eq. (16). For this let us introduce the map
\[
S_2(\tilde{u}) := e^{-\tau \partial_x^3} u^n + \frac{1}{24} \left( \partial_x^{-1} \tilde{u} + e^{-\tau \partial_x^3} \partial_x^{-1} u^n \right)^2 - \frac{1}{24} e^{-\tau \partial_x^3} \left( e^{\tau \partial_x^3} \partial_x^{-1} \tilde{u} + \partial_x^{-1} u^n \right)^2.
\]

Corollary 3.6. Fix \( l \in \{1, 2, 3\} \) and \( R > 0 \). Then there is a \( \tilde{\tau}_R > 0 \) such that for all \( \tau \in [0, \tilde{\tau}_R) \) and any \( u^n \in B_R(H^l) := \{ \tilde{u} \in H^l \mid \|\tilde{u}\|_{H^l} < R \} \) we have \( u^{n+1} \) the exact solution of Eq. (16) is given by the following limit in \( H^l \):
\[
u^{n+1} = \lim_{j \to \infty} S_2^{(j)}(u^n), \quad \text{where} \quad S_2^{(j)}(u^n) = S_2 \circ \cdots \circ S_2(u^n).
\]
Moreover we have the estimate
\[
\|u^{n+1} - u^n\|_{H^l} \leq \tau \frac{2}{2} \tilde{C}_R,
\]
and, if additionally \( u^n \in B_R(H^{l+1}) \),
\[
\|u^{n+1} - u^n\|_{H^l} \leq \tau \tilde{C}_R,
\]
for some \( \tilde{C}_R > 0 \) which depends on \( R \).

Proof. Let us define
\[
\tilde{F}(t_n, \tau, \tilde{u}) := e^{-(t_n + \tau) \partial_x^3} \tilde{F}(t_n, \tau, e^{t_n \partial_x^3} \tilde{u}).
\]
Then, by recalling \( u^n = \exp(-t_n \partial_x^3)v^n \), we can express \( S_2 \) as
\[
S_2(\tilde{u}) = e^{-\tau \partial_x^3} u^n + \tilde{F} \left( \tau, \frac{e^{\tau \partial_x^3} \tilde{u} + u^n}{2} \right).
\]
Now, since \( v \mapsto \exp(\pm \partial_x^3)v \) is an isometry on \( H^l \) we have from Lemma 3.3 that
\[
\|\tilde{F}(t_n, \tau, f) - \tilde{F}(t_n, \tau, g)\|_{H^l} \leq \tau \frac{2}{2} M_l (\|f\|_{H^l}, \|g\|_{H^l}) \|f - g\|_{H^l}.
\]
Hence the result follows by taking exactly the same steps as in the proof of Thm. 3.2 but by replacing the use of Lemma 3.3 with the estimate Eq. (26).

4 Structure preservation properties

Following the introduction of our novel scheme (16) and discussion of its implicit nature let us now turn to its main novelty: structure preservation.

4.1 Symplectic nature of the method

The symplectic form of the KdV equation defined in Section 2.1 is given by
\[
\omega(y, \tilde{y}) = \sum_{j=1}^{\infty} y_j \tilde{y}_{-j} - y_{-j} \tilde{y}_j = \sum_{m \in \mathbb{Z}} \text{sgn}(m) y_m \tilde{y}_{-m}
\]
for any two vectors \( y, \tilde{y} \in l^2(\mathbb{Z}) \). We shall now show that our numerical method Eq. (16) preserves this symplectic structure exactly.
Thus it follows that for any vectors \( v, w \in H^3 \) and assume that the assumptions of Thm. 5.1 are satisfied. Then our method preserves the symplectic form \( \omega \) exactly, i.e. if \( u^k, w^k \) are the iterates of our numerical method Eq. (16) for \( k = 1, \ldots, n \), then
\[
\omega(u^k, w^k) = \omega(u^0, w^0), \quad \forall k = 1, \ldots, n.
\]

**Proof.** We begin by expressing the symplectic form in terms of the twisted variables \( v(t) = \exp(t \partial_2^2) u(t), z(t) = \exp(t \partial_2^2) w(t) \). Then we have, for all \( k = 1, \ldots, n \),
\[
\omega(u^k, w^k) = \sum_{m \in \mathbb{Z}} \text{sgn}(m) u_m^k w_{-m}^k = \sum_{m \in \mathbb{Z}} \text{sgn}(m) \left( e^{-im^2 \frac{3}{4} \tau} y_m \right) \left( e^{im^2 \frac{3}{4} \tau} w_{-m}^k \right) = \omega(u^k, w^k).
\]

Thus it is sufficient to prove that the symplectic form is preserved over a single time step in the twisted variables \( v, z \). Since the continuous flow, i.e. the exact solution of Eq. (4), preserves the symplectic form we have, writing \( \hat{v}_m(t), \hat{z}_m(t) \) for the Fourier coefficients of the continuous solution to Eq. (4) with initial conditions \( \hat{v}_m = \hat{v}_0, \hat{z}_m = \hat{z}_0 \) respectively, that
\[
\sum_{m \in \mathbb{Z}} \text{sgn}(m) \sum_{m=a+b} \frac{im}{2} e^{-i2mb \frac{3}{4} \tau} v_a \hat{v}_b \hat{z}_{-m} + \text{sgn}(m) \hat{v}_m \sum_{-m=a+b} \frac{(-i)m}{2} e^{i2mb \frac{3}{4} \tau} \hat{z}_a \hat{z}_{-m} = \sum_{m \in \mathbb{Z}} \text{sgn}(m) \frac{d^\tau}{dt^\tau} \hat{v}_m \hat{z}_{-m} + \text{sgn}(m) \frac{d^\tau}{dt^\tau} \hat{z}_m \hat{v}_{-m} = \frac{d}{dt} \omega(v, z) = 0.
\]

Thus it follows that for any vectors \( v, z \in l^2_I \) and any \( t \geq 0 \) we have
\[
\sum_{m \in \mathbb{Z}} \text{sgn}(v_m \mathcal{G}_m(t, z) + z_m \mathcal{G}_m(t, v)) = 0,
\]
where we defined the function \( \mathcal{G} \) by
\[
\mathcal{G} : \quad v = (v_m)_{m \in \mathbb{Z}} \mapsto \left( \sum_{m=a+b} \frac{im}{2} e^{-i2mb \frac{3}{4} \tau} v_a \right)_{m \in \mathbb{Z}}.
\]

We can now use \( \mathcal{G} \) to express our numerical scheme Eq. (15) as follows
\[
\hat{v}_m^{k+1} = \hat{v}_m^k + \int_0^\tau \mathcal{G}_m \left( t_k + s, \frac{\hat{v}_m^k + \hat{v}_m^{k+1}}{2} \right) ds.
\]

Thus we have
\[
\omega(\hat{v}_m^{k+1}, \hat{z}_m^{k+1}) - \omega(\hat{v}_m^k, \hat{z}_m^k) = \sum_{m \in \mathbb{Z}} \text{sgn}(m) \hat{v}_m \int_0^\tau \mathcal{G}_m \left( t_k + s, \frac{\hat{v}_m + \hat{v}_m^{k+1}}{2} \right) ds
\]
\[
+ \sum_{m \in \mathbb{Z}} \text{sgn}(m) \hat{z}_m \int_0^\tau \mathcal{G}_m \left( t_k + s, \frac{\hat{v}_m^k + \hat{v}_m^{k+1}}{2} \right) ds
\]
\[
+ \sum_{m \in \mathbb{Z}} \text{sgn}(m) \int_0^\tau \mathcal{G}_m \left( t_k + s, \frac{v_m^{k+1} + v_m^k}{2} \right) ds \int_0^\tau \mathcal{G}_m \left( t_k + s, \frac{z_m^{k+1} + z_m^0}{2} \right) ds.
\]
Because all sums converge absolutely we can exchange summation and integration in the above expression to find, using Eq. (27),

\[
\omega(\hat{\mathbf{v}}^{k+1}, \hat{\mathbf{z}}^{k+1}) - \omega(\hat{\mathbf{v}}^k, \hat{\mathbf{z}}^k) = \int_0^\tau \sum_{m \in \mathbb{Z}} \text{sgn}(m) \frac{\hat{v}_m^{k+1} - \hat{v}_m^k}{2} G_{-m} \left( t_k + s, \frac{\hat{z}_m^{k+1} + \hat{z}_m^k}{2} \right) ds
\]

\[
+ \int_0^\tau \sum_{m \in \mathbb{Z}} \text{sgn}(m) \frac{\hat{z}_m^k - \hat{z}_m^{k+1}}{2} G_m \left( t_k + s, \frac{\hat{v}_m^k + \hat{v}_m^{k+1}}{2} \right) ds
\]

\[
+ \sum_{m \in \mathbb{Z}} \text{sgn}(m) \int_0^\tau G_m \left( t_k + s, \frac{\hat{\mathbf{v}}^k + \hat{\mathbf{v}}^{k+1}}{2} \right) ds \int_0^\tau G_{-m} \left( t_k + s, \frac{\hat{\mathbf{z}}^k + \hat{\mathbf{z}}^{k+1}}{2} \right) ds
\]

\[
- \frac{1}{4} \int_0^\tau \sum_{m \in \mathbb{Z}} \text{sgn}(m) \int_0^\tau G_{-m} \left( t_k + s, \frac{\hat{\mathbf{v}}^k + \hat{\mathbf{v}}^{k+1}}{2} \right) ds \int_0^\tau G_m \left( t_k + s, \frac{\hat{\mathbf{z}}^k + \hat{\mathbf{z}}^{k+1}}{2} \right) ds
\]

\[
+ \sum_{m \in \mathbb{Z}} \text{sgn}(m) \int_0^\tau G_m \left( t_k + s, \frac{\hat{\mathbf{v}}^k + \hat{\mathbf{v}}^{k+1}}{2} \right) ds \int_0^\tau G_{-m} \left( t_k + s, \frac{\hat{\mathbf{z}}^k + \hat{\mathbf{z}}^{k+1}}{2} \right) ds.
\]

The terms on the right hand side of the equation all cancel and we thus find

\[
\omega(\hat{\mathbf{v}}^{k+1}, \hat{\mathbf{z}}^{k+1}) - \omega(\hat{\mathbf{v}}^k, \hat{\mathbf{z}}^k) = 0.
\]

4.2 Conservation of momentum

As a further result, it turns out that the semi-discrete method Eq. (16) preserves the momentum exactly:

**Theorem 4.2.** Suppose \( u \in H^3 \), then the numerical method Eq. (16) preserves the momentum of the equation exactly, i.e. if the solution \( u^k \) of the implicit equation Eq. (16) exists for \( k = 1, \ldots, n \) then

\[
\int_\tau (u^k(x))^2 dx = \int_\tau (u^0(x))^2 dx, \quad k = 1, 2, \ldots, n.
\]

**Proof.** The proof of this statement is closely related to the proof of Thm. 4.1, but the two results are not equivalent. We begin by expressing the momentum in Fourier modes. Note due to the regularity assumptions on \( u \) and the convergence result Thm. 5.1 all of the following sums converge absolutely.

\[
\int_\tau (u^k(x))^2 dx = \sum_{m \in \mathbb{Z}} \hat{u}_m^k \hat{u}_m^k = \sum_{m \in \mathbb{Z}} (\exp(-im^3 t_k) \hat{u}_m^k) (\exp(im^3 t_k) \hat{u}_m^k) = \sum_{m \in \mathbb{Z}} \hat{u}_m^k \hat{u}_m^k.
\]

Thus it suffices to prove that the momentum is conserved exactly over a single timestep in \( v \). Because momentum is preserved in the exact flow of Eq. (4) we have for any initial data and any value of \( t \geq 0 \)

\[
\sum_{m \in \mathbb{Z}} \hat{v}_m \sum_{-m = a + b} (-1)^m e^{it m^3 a b} \hat{\alpha}_a \hat{\beta}_b = 2 \int_\tau v(t, x) v'(t, x) dx = \frac{d}{dt} \int_\tau v(t, x)^2 dx = 0.
\]

Therefore it follows that for any \( v \in l^2_\tau, t \geq 0 \)

\[
\sum_{m \in \mathbb{Z}} v_m G_{-m}(t, v) = 0,
\]

(29)
where \( G \) is as introduced in Eq. (28). We can now perform the following simple calculation:

\[
\begin{align*}
\sum_{m \in \mathbb{Z}} \hat{v}^{k+1}_m \hat{v}^{k+1}_{-m} - \sum_{m \in \mathbb{Z}} \hat{v}^k_m \hat{v}^k_{-m} &= \sum_{m \in \mathbb{Z}} \hat{v}^k_m \int_0^\tau G_{-m} \left( t_k + s, \frac{\hat{\vartheta}^k + \hat{\vartheta}^{k+1}}{2} \right) ds \\
&+ \sum_{m \in \mathbb{Z}} \hat{v}^k_{-m} \int_0^\tau G_m \left( t_k + s, \frac{\hat{\vartheta}^k + \hat{\vartheta}^{k+1}}{2} \right) ds \\
&+ \sum_{m \in \mathbb{Z}} \int_0^\tau G_m \left( t_k + s, \frac{\hat{\vartheta}^k + \hat{\vartheta}^{k+1}}{2} \right) ds \int_0^\tau G_{-m} \left( t_k + s, \frac{\hat{\vartheta}^k + \hat{\vartheta}^{k+1}}{2} \right) ds.
\end{align*}
\]

Because all sums converge absolutely we can exchange summation and integration in the above expression to find using Eq. (29)

\[
\begin{align*}
\sum_{m \in \mathbb{Z}} \hat{v}^{k+1}_m \hat{v}^{k+1}_{-m} - \sum_{m \in \mathbb{Z}} \hat{v}^k_m \hat{v}^k_{-m} &= \sum_{m \in \mathbb{Z}} \hat{v}^k_m \int_0^\tau G_{-m} \left( t_k + s, \frac{\hat{\vartheta}^k + \hat{\vartheta}^{k+1}}{2} \right) ds \\
&+ \sum_{m \in \mathbb{Z}} \hat{v}^k_{-m} \int_0^\tau G_m \left( t_k + s, \frac{\hat{\vartheta}^k + \hat{\vartheta}^{k+1}}{2} \right) ds \\
&+ \sum_{m \in \mathbb{Z}} \int_0^\tau G_m \left( t_k + s, \frac{\hat{\vartheta}^k + \hat{\vartheta}^{k+1}}{2} \right) ds \int_0^\tau G_{-m} \left( t_k + s, \frac{\hat{\vartheta}^k + \hat{\vartheta}^{k+1}}{2} \right) ds \\
&= \sum_{m \in \mathbb{Z}} \hat{v}^k_m \int_0^\tau G_{-m} \left( t_k + s, \frac{\hat{\vartheta}^k + \hat{\vartheta}^{k+1}}{2} \right) ds \\
&+ \sum_{m \in \mathbb{Z}} \hat{v}^k_{-m} \int_0^\tau G_m \left( t_k + s, \frac{\hat{\vartheta}^k + \hat{\vartheta}^{k+1}}{2} \right) ds \\
&+ \sum_{m \in \mathbb{Z}} \int_0^\tau G_m \left( t_k + s, \frac{\hat{\vartheta}^k + \hat{\vartheta}^{k+1}}{2} \right) ds \int_0^\tau G_{-m} \left( t_k + s, \frac{\hat{\vartheta}^k + \hat{\vartheta}^{k+1}}{2} \right) ds.
\end{align*}
\]

The terms on the right hand side of the equation all cancel and we find

\[
\sum_{m \in \mathbb{Z}} \hat{v}^{k+1}_m \hat{v}^{k+1}_{-m} - \sum_{m \in \mathbb{Z}} \hat{v}^k_m \hat{v}^k_{-m} = 0.
\]

\[\square\]

5 Error analysis

In this section we carry out a rigorous convergence analysis of our symplectic resonance-based scheme. The main results in this section are the following:

**Theorem 5.1.** Let us denote by \( v(t) \) the exact solution to Eq. (4), let \( v^k, k = 0, \ldots, n \), be the iterates in the numerical method Eq. (15), and let \( t_n = n \tau \). Given \( R > 0 \) there is a \( \tau_R > 0 \) such that for all \( \tau \in [0, \tau_R) \), and as long as \( \| v(t) \|_{H^2} < R/2 \), we have

\[
\| v(t_n) - \tilde{v}^n \|_{H^1} \leq c_{R, t_n} \| v(t) \|_{H^1}
\]

for some constant \( c_{R, t_n} > 0 \) depending on \( R, t_n \), but which may be chosen independently of \( \tau \).
Theorem 5.2. Let $v(t), v^k, k = 0, \ldots, n$ and $t_n = n\tau$ be as in Thm. 5.1. Then we have that given $R > 0$ there is a $\tau_R > 0$ such that for all $\tau \in [0, \tau_R)$, and as long as $\sup_{t \in [0, t_n]} \|v(t)\|_{H^2} < R/2$, we have
\[
\|v(t_n) - v^n\|_{H^2} \leq \tau^2 c_{R, t_n}
\]
for some constant $c_{R, t_n} > 0$ depending on $R, t_n$, but which may be chosen independently of $\tau$.

Of course, the isomorphism properties of the twisting map $u(t, x) \mapsto v(t, x) = \exp(t\partial_x^2)u(t, x)$ imply that the analogous results hold true also for the variable $u$ and the method Eq. (16):

Corollary 5.3. Let $u(t)$ be the exact solution to Eq. (1) and be $u^k, k = 0, \ldots, n$, the iterates in the numerical method Eq. (16), and $t_n = n\tau$. Given $R > 0$ there is a $\tau_R > 0$ such that for all $\tau \in [0, \tau_R)$:

- If $\sup_{t \in [0, t_n]} \|u(t)\|_{H^3} < R/2$, we have
  \[
  \|u(t_n) - u^n\|_{H^1} \leq \tau c_{R, t_n},
  \]

- if $\sup_{t \in [0, t_n]} \|u(t)\|_{H^2} < R/2$, we have
  \[
  \|u(t_n) - u^n\|_{H^1} \leq \tau^2 c_{R, t_n},
  \]

for some constants $C_{R, t_n}, c_{R, t_n}$ depending on $R, t_n$, but which may be chosen independently of $\tau$.

Parts of our error analysis will follow the ideas in [19]. However, we need to account for the implicit nature of our method and make use of novel estimates for certain integrals arising from Duhamel’s formula to understand the higher order convergence properties of this scheme.

Remark 5.4. As discussed in Remark 3.1 the method is symmetric. Thus we would expect the method to be of even order (cf. [17, Section IX.2]) as is indeed confirmed in Thm. 5.2. The estimate in Thm. 5.1 guarantees in addition that the method reliably converges at first order when applied to solutions which are in $H^3$.

We will present the proof of Thms. 5.1 & 5.2 in Section 5.2.3, as a result of the lemmas introduced and proved in the following sections.

5.1 Error analysis in $H^2$

As a first step in our proof we need to establish the boundedness of our numerical solution in $H^2$. In this section we will prove convergence and hence boundedness of the numerical solution in $H^2$ for initial data that lies in $H^3$. The stability of our numerical scheme is proved in Section 5.1.1, the local error bound is given in Section 5.1.2 and the global error bound, Thm. 5.8, is given in Section 5.1.3.

5.1.1 Stability

Let us denote by $\Phi_{t_n, \tau} : H^1 \to H^1$ the nonlinear solution map of Eq. (15), i.e. let $\Phi_{t_n, \tau}$ be such that
\[
v^{n+1} = \Phi_{t_n, \tau}(v^n).
\]
We can then show the following stability estimate:

Lemma 5.5. Fix $R > 0$. Then there is a $\tau_R > 0$ such that for all $\tau \in [0, \tau_R)$ and any $f \in B_R(H^2), g \in B_R(H^3)$ we have
\[
\|\Phi_{t_n, \tau}(f) - \Phi_{t_n, \tau}(g)\|_{H^2} \leq \exp(\tau \hat{C}_R)\|f - g\|_{H^2},
\]
where $\hat{C}_R > 0$ depends only on $R$.
In order to prove this result we rely on the following bound from [19]:

**Lemma 5.6.** Let $\mathcal{F}$ be defined as in Eq. (20). Then, there is a continuous function $L : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for any $f \in H^2$, $g \in H^2$ and any $t_n \geq 0$ we have

$$\left| \langle \partial_x^2 \mathcal{F}(t_n, \tau, f) - \mathcal{F}(t_n, \tau, g), \partial_x^2 (f - g) \rangle \right| \leq \tau L(\|f\|_{H^2}, \|g\|_{H^2}) \|f - g\|_{H^2}^2,$$

where by $\langle \cdot, \cdot \rangle$ we denoted the usual $L^2$-inner product.

**Proof.** See Lemma 2.3 from [19].

We can now proceed to prove the stability estimate Lemma 5.5.

**Proof of Lemma 5.5.** By the definition of the numerical method Eq. (15) we have for any function $f$:

$$\Phi_{t_n, \tau}(f) = f + \mathcal{F}\left(t_n, \tau, \frac{f + \Phi_{t_n, \tau}(f)}{2}\right).$$

(30)

Thus, we have

$$\|\Phi_{t_n, \tau}(f) - \Phi_{t_n, \tau}(g)\|_{H^2}$$

$$\leq \|f - g\|_{H^2}^2 + 2 \left| \partial_x^2 \left( \mathcal{F}\left(t_n, \tau, \frac{f + \Phi_{t_n, \tau}(f)}{2}\right) - \mathcal{F}\left(t_n, \tau, \frac{g + \Phi_{t_n, \tau}(g)}{2}\right) \right), \partial_x^2 (f - g) \right|$$

$$+ \left\| \mathcal{F}\left(t_n, \tau, \frac{f + \Phi_{t_n, \tau}(f)}{2}\right) - \mathcal{F}\left(t_n, \tau, \frac{g + \Phi_{t_n, \tau}(g)}{2}\right) \right\|^2_{L^2}.$$

We can then estimate the term $A$ as follows:

$$|A| \leq 2 \left| \partial_x^2 \left( \mathcal{F}\left(t_n, \tau, \frac{f + \Phi_{t_n, \tau}(f)}{2}\right) - \mathcal{F}\left(t_n, \tau, \frac{g + \Phi_{t_n, \tau}(g)}{2}\right) \right), \partial_x^2 \left( \frac{f + \Phi_{t_n, \tau}(f)}{2} - \frac{g + \Phi_{t_n, \tau}(g)}{2} \right) \right|$$

$$+ \left\| \partial_x^2 \left( \mathcal{F}\left(t_n, \tau, \frac{f + \Phi_{t_n, \tau}(f)}{2}\right) - \mathcal{F}\left(t_n, \tau, \frac{g + \Phi_{t_n, \tau}(g)}{2}\right) \right) \right\|^2_{L^2}$$

$$\leq \tau L \left( \frac{1}{2} \|f + \Phi_{t_n, \tau}(f)\|_{H^2}, \frac{1}{2} \|g + \Phi_{t_n, \tau}(g)\|_{H^2} \right) \frac{1}{2} \|f - g + \Phi_{t_n, \tau}(f) - \Phi_{t_n, \tau}(g)\|_{H^2}^2 + B,$$

where we used Eq. (30) and Lemma 5.6. For the term $B$ we have by Lemma 3.3

$$B \leq \tau M_2 \left( \frac{1}{2} \|f + \Phi_{t_n, \tau}(f)\|_{H^2}, \frac{1}{2} \|g + \Phi_{t_n, \tau}(g)\|_{H^2} \right)^2 \frac{1}{2} \|f - g + \Phi_{t_n, \tau}(f) - \Phi_{t_n, \tau}(g)\|_{H^2}^2.$$

Now we have from Eq. (18) that there is a $\tilde{\tau}_R > 0$ such that, for all $\tau \in [0, \tilde{\tau}_R)$,

$$\|\Phi_{t_n, \tau}(f)\|_{H^2} \leq \|f\|_{H^2} + \|f - \Phi_{t_n, \tau}(f)\|_{H^2} \leq R + R = 2R,$$

$$\|\Phi_{t_n, \tau}(g)\|_{H^2} \leq \|g\|_{H^2} + \|g - \Phi_{t_n, \tau}(g)\|_{H^2} \leq R + R = 2R.$$

Therefore, by the continuity of the functions $L, M_2$, there is a constant $\tilde{C}_R > 0$ such that for all $\tau \in [0, \tilde{\tau}_R)$

$$\|\Phi_{t_n, \tau}(f) - \Phi_{t_n, \tau}(g)\|_{H^2}^2 \leq \|f - g\|_{H^2}^2 + \tilde{C}_R \left( \|f - g\|_{H^2}^2 + \|\Phi_{t_n, \tau}(f) - \Phi_{t_n, \tau}(g)\|_{H^2}^2 \right).$$

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Equivalently for all $\tau \in [0, \tilde{\tau}_R)$

$$||\Phi_{t_n, \tau}(f) - \Phi_{t_n, \tau}(g)||_{H^2} \leq \frac{1 + \tilde{\tau} \tilde{C}_R}{1 - \tilde{\tau} \tilde{C}_R} ||f - g||_{H^2}.$$  

We now recall that

$$\frac{1 + x/2}{1 - x/2} \leq \exp(x), \quad \forall x < 1$$

Thus the result follows immediately by taking $\tau_R = \min\{\tilde{\tau}_R, 1/\tilde{C}_R\}$. □

5.1.2 Local error in $H^2$

We can now proceed to estimate the local error of a single time step. We note at this point that a local error of $O(\tau^{3/2})$ is sufficient to guarantee convergence and hence boundedness of the numerical method in $H^2$. We will describe estimates that provide faster convergence rates in $H^1$ in Section 5.2.

**Lemma 5.7.** Let us denote by $\Phi_{t_n, \tau}(v(t_n)) = v(t_n + \tau)$ the solution to Eq. (4) with initial condition $\phi_{t_n, 0}(v(t_n)) = v(t_n)$. Fix $R > 0$, then there is a $\tau_R > 0$ such that for all $\tau \in [0, \tau_R)$ and any $v(t_n) \in B_R(H^3)$ such that $\sup_{t \in [0, \tau]} ||\phi_{t_n, \tau}(v(t_n))||_{H^3} < R$ we have

$$||\Phi_{t_n, \tau}(v(t_n)) - \Phi_{t_n, \tau}(v(t_n))||_{H^2} \leq c_R \tau^{\frac{3}{2}}$$

for some constant $c_R > 0$ depending only on $R > 0$.

**Proof.** The proof of this statement is closely inspired by the proof of Lemma 2.5 in [19]. According to Duhamel’s formula Eq. (7) and the construction of our resonance-based scheme Eq. (14) we have

$$||\Phi_{t_n, \tau}(v(t_n)) - \Phi_{t_n, \tau}(v(t_n))||_{H^2}$$

\begin{align*}
\leq & \left\| \frac{1}{2} \int_0^\tau e^{(t_n + s) \partial_x^3} (e^{-(t_n + s) \partial_x^3} v(t_n + s)) \partial_x^2 (e^{-(t_n + s) \partial_x^3} v(t_n)) \right\|_{H^2} \\
= & A_1 + \left\| \frac{1}{2} \int_0^\tau e^{(t_n + s) \partial_x^3} (e^{-(t_n + s) \partial_x^3} v(t_n)) \partial_x^2 \left( e^{-(t_n + s) \partial_x^3} v(t_n) + \Phi_{t_n, \tau}(v(t_n)) \right) \right\|_{H^2}
\end{align*}$$

where $c_{1,R} > 0$ is a constant depending on $R$. For the second term $A_2$ we note

$$A_2 = \left\| \frac{1}{2} \int_0^\tau e^{(t_n + s) \partial_x^3} \partial_x^2 \left( e^{-(t_n + s) \partial_x^3} v(t_n) \right) \right\|_{H^2}$$

\begin{align*}
\leq & \left\| \frac{1}{2} \int_0^\tau e^{(t_n + s) \partial_x^3} \partial_x^2 \left( e^{-(t_n + s) \partial_x^3} v(t_n) + \Phi_{t_n, \tau}(v(t_n)) \right) \right\|_{H^2} \\
= & A_{2,1} + \left\| \frac{1}{2} \int_0^\tau e^{(t_n + s) \partial_x^3} \partial_x^2 \left( e^{-(t_n + s) \partial_x^3} v(t_n) - \Phi_{t_n, \tau}(v(t_n)) \right) \right\|_{H^2}
\end{align*}$$

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We can again estimate those contributions individually. Firstly, we have by Thm. 3.2 (specifically Eq. (18)) and by the usual bilinear estimate Lemma 3.4 that under the assumptions on $\tau, v(t_n)$,

\[ A_{2,1} \leq \tilde{c}_2 \int_0^\tau \|v(t_n)\|_{H^3} \|v(t_n) - \Phi(t_\tau,v(t_n))\|_{H^3} \, ds \leq c_{2,R} R^{\frac{2}{3}}, \]  

where $c_{2,R} > 0$ depends on $R$. Similarly, we have

\[ A_{2,2} \leq \tilde{c}_3 \int_0^\tau \|v(t_n) - \Phi(t_\tau,v(t_n))\|^2_{H^3} \, ds \leq \tau^2 c_{3,R}. \]

where $c_{3,R} > 0$ depends on $R$. Combining Eqs. (31)-(34) yields the desired estimate. \hfill \square

### 5.1.3 Global error in $H^2$

We can now combine the estimates from Lemma 5.5 and Lemma 5.7 to prove the following global error estimate.

**Theorem 5.8.** Let us again denote by $v(t)$ the exact solution to Eq. (4) and be $v^k, k = 0, \ldots, n$, the iterates in the numerical method Eq. (15), and $t_n = n\tau$. Given $R > 0$ there is a $\tau_R > 0$ such that for all $\tau \in [0, \tau_R)$ and as long as $\sup_{t \in [0,t_n]} \|v(t)\|_{H^3} < R/2$ we have

\[ \|v(t_n) - v^n\|_{H^2} \leq \tau^\frac{4}{3} c_{R,t_n}, \]

for some constant $c_{R,t_n} > 0$ depending on $R,t_n$, but which may be chosen independently of $\tau$.

**Proof.** By the triangle inequality we have

\[ \|v(t_n) - v^n\|_{H^2} \leq \|\Phi(t_{n-1},\tau(v(t_{n-1})) - \Phi(t_{n-1},\tau(v^{n-1}))\|_{H^2} + \|\phi(t_{n-1},\tau(v(t_{n-1}) - \Phi(t_{n-1},\tau(v(t_{n-1}))))\|_{H^2}. \]

Iterating the estimate we have, so long as $v^k \in B_R(H^2)$ for $0 \leq k \leq n$ and $\tau \in [0, \tilde{\tau}_R)$ where $\tilde{\tau}_R$ is as given in Lemmas 5.5 & 5.7, that

\[ \|v(t_n) - v^n\|_{H^2} \leq e^{\tau C_R} \|v(t_{n-1}) - v^{n-1}\|_{H^2} + c_R \tau^\frac{2}{3} \leq e^{2\tau C_R} \|v(t_{n-2}) - v^{n-2}\|_{H^2} + e^{\tau C_R} c_R \tau^\frac{2}{3} + c_R \tau^\frac{2}{3} \]

\[ \leq c_R \tau^\frac{2}{3} \sum_{k=0}^{n-1} e^{k\tau C_R} \leq c_R \tau^\frac{2}{3} \sum_{k=0}^{n-1} e^{k\tau C_R}. \]

Thus in particular if we choose $\tau < \tau_R$ where $\tau_R = \min\{\tilde{\tau}_R, R^2 \exp(-2t_n C_R)/(4t_n^2 c_R^2)\}$, we ensure that $v^n \in B_R(H^2)$ and the result follows by induction. \hfill \square

### 5.2 Error analysis in $H^1$

Having proved the boundedness of our numerical approximation in $H^2$ (see Thm. 5.8) we can proceed to study its convergence properties in $H^1$.

#### 5.2.1 Stability

As in Section 5.1 we begin by proving the crucial stability estimate, based on the following estimate from [19]:

\[ \text{Preprint 2022} \quad 18 \quad G. \text{Maierhofer & K. Schratz} \]
Lemma 5.9. Let \( F \) be defined as in Eq. (20). Then, there is a continuous function \( L : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) such that for any \( f \in H^2 \), \( g \in H^2 \) and any \( t_n \geq 0 \) we have

\[
\| \partial_x (F(t_n, \tau, f) - F(t_n, \tau, g), \partial_x (f - g)) \| \leq \tau L(\|f\|_{H^2}, \|g\|_{H^2})\|f - g\|_{H^1}.
\]

Proof. See Eq. (36) from [19].

The stability estimate for \( H^1 \) can now be deduced analogously to the proof of Lemma 5.5:

Lemma 5.10. Fix \( R > 0 \). Then there is a \( \tau_R > 0 \) such that for all \( \tau \in [0, \tau_R) \) and any \( f, g \in B_R(H^2) \) we have

\[
\| \Phi_{t_n, \tau}(f) - \Phi_{t_n, \tau}(g) \|_{H^1} \leq \exp(\tau \tilde{C}_R)\|f - g\|_{H^1},
\]

where \( \tilde{C}_R > 0 \) depends on \( R \).

Proof. This statement can be proved analogously to Lemma 5.5 at each point replacing the use of Lemma 5.6 by the \( H^1 \)-estimate Lemma 5.9. In the interest of brevity the arguments are not repeated here.

5.2.2 Local error in \( H^1 \)

We may now proceed to prove the crucial local error estimates on the numerical scheme in \( H^1 \). In the following we will show two central results: in Lemma 5.11 we show that the method incurs a local error of size \( O(\tau^2) \) provided the solution \( v(t) \) remains uniformly bounded in \( H^3 \) over the time-interval of interest \( [t_n, t_n + \tau] \). However, we know that symmetric methods are, for sufficiently regular data, of even global order. This is indeed confirmed in Lemma 5.12 where we demonstrate the method incurs a local error of size \( O(\tau^3) \) provided the solution remains uniformly bounded in \( H^5 \) over the corresponding time-interval.

Lemma 5.11. As above let us denote by \( \phi_{t_n, \tau}(v(t_n)) = v(t_n + \tau) \) the solution to Eq. (4) with initial condition \( \phi_{t_n, \tau}(v(t_n)) = v(t_n) \). Fix \( R > 0 \), then there is a \( \tau_R > 0 \) such that for all \( \tau \in [0, \tau_R) \) and any \( v(t_n) \in B_R(H^3) \) such that sup\(_{t \in [0, \tau]} \| \phi_{t_n, \tau}(v(t_n)) \|_{H^3} < R \) we have

\[
\| \phi_{t_n, \tau}(v(t_n)) - \Phi_{t_n, \tau}(v(t_n)) \|_{H^1} \leq c_R \tau^2
\]

for some constant \( c_R > 0 \) depending on \( R > 0 \).

Proof. By construction of our numerical scheme Eq. (15) we have

\[
\phi^*(v(t_n)) - \Phi_{t_n, \tau}^*(v(t_n)) = \frac{1}{2} \int_0^\tau e^{\tau(x+s)} \partial_x^2 \left[ e^{-(t_n+s)} \partial_x^2 v(t_n + s) \right]^2 - \left( e^{-(t_n+s)} \partial_x^2 \frac{1}{2} v(t_n) + \Phi_{t_n, \tau}^*(v(t_n)) \right)^2 \] ds
\]

Thus, using the usual bilinear estimate Lemma 3.4, we find for some constant \( c > 0 \)

\[
\| \phi^*(v(t_n)) - \Phi_{t_n, \tau}^*(v(t_n)) \|_{H^1} \leq \frac{c}{2} \int_0^\tau \left( \| v(t_n + s) - \frac{1}{2} v(t_n) + \Phi_{t_n, \tau}^*(v(t_n)) \|_{H^2} + \frac{1}{2} \| v(t_n) + \Phi_{t_n, \tau}^*(v(t_n)) \|_{H^2} \right) ds
\]

\[
\leq \frac{c}{2} \sup_{s \in [0, \tau]} \left( \| v(t_n + s) - \frac{1}{2} v(t_n) + \Phi_{t_n, \tau}^*(v(t_n)) \|_{H^2} + \frac{1}{2} \| v(t_n) + \Phi_{t_n, \tau}^*(v(t_n)) \|_{H^2} \right)
\]

(35)
Note that by Thm. 3.2 (specifically Eq. (19)) we have for some $\tilde{\tau}_R > 0$ and all $\tau \in [0, \tilde{\tau}_R)$
\[
\frac{1}{2} \| v(t_n) + \Phi_{t_n, \tau}(v(t_n)) \|_{H^2} \leq R + \tau \tilde{C}_R \leq \tilde{C}_R,
\]
for some $\tilde{C}_R, \tilde{\tau}_R > 0$ which depend on $R$. It remains to bound the term
\[
\left\| v(t_n + s) - \frac{1}{2} (v(t_n) + \Phi_{t_n, \tau}(v(t_n))) \right\|_{H^2}.
\]
This can be done as follows:
\[
\left\| v(t_n + s) - \frac{1}{2} (v(t_n) + \Phi_{t_n, \tau}(v(t_n))) \right\|_{H^2} \leq \| v(t_n + s) - v(t_n + \tau) \|_{H^2} + \| v(t_n + \tau) - \Phi_{t_n, \tau}(v(t_n)) \|_{H^2} + \frac{1}{2} \| v(t_n) - \Phi_{t_n, \tau}(v(t_n)) \|_{H^2}.
\]
We then have by Lemma 5.7 for some $\tilde{\tau}_R > 0$ and all $\tau \in [0, \tilde{\tau}_R)$
\[
\| v(t_n + \tau) - \Phi_{t_n, \tau}(v(t_n)) \|_{H^2} \leq c_R \tau^{2},
\]
and, by Thm. 3.2,
\[
\frac{1}{2} \| v(t_n) - \Phi_{t_n, \tau}(v(t_n)) \|_{H^2} \leq \tau \tilde{C}_R.
\]
Finally, we can estimate using Duhamel’s formula Eq. (6) and the bilinear estimate Lemma 3.4:
\[
\| v(t_n + s) - v(t_n + \tau) \|_{H^2} = \left\| \frac{1}{2} \int_{s}^{\tau} e^{(t_n+s)\partial_{\delta}} \partial_{\tau} \left( e^{(t_n+s)\partial_{\delta}} v(t_n + \tilde{s}) \right)^2 \right\|_{H^2} \leq c \frac{1}{2} \int_{s}^{\tau} \| v(t_n + \tilde{s}) \|_{H^2}^2 d\tilde{s} \leq \tau \tilde{c}_R
\]
for some constant $\tilde{c}_R$. Thus if we choose $\tau_R = \min\{\tilde{\tau}_R, \tilde{\tau}_R, R/\tilde{C}_R\}$ and combine Eq. (35)-(40), the result follows.

Although Lemma 5.11 is already sufficient to guarantee convergence of our method in $H^1$ we can show that faster rates of $H^1$-convergence can be obtained if we allow for slightly more regular initial data:

**Lemma 5.12.** Let $\phi_{t_n, \tau}(v(t_n)), v(t_n + \tau)$ be as above. Fix $R > 0$, then there is a $	au_R > 0$ such that for all $\tau \in [0, \tau_R)$ and any $v(t_n) \in B_R(H^5)$ such that $\sup_{\tau \in [0, \tau]} \| \phi_{t_n, \tau}(v(t_n)) \|_{H^2} < R$ we have
\[
\| \phi_{t_n, \tau}(v(t_n)) - \Phi_{t_n, \tau}(v(t_n)) \|_{H^1} \leq c_R \tau^3
\]
for some constant $c_R > 0$ depending on $R > 0$.

In order to prove this statement we have to rely on the following crucial estimate:

**Lemma 5.13.** For any $j, l \in \mathbb{N}$ such that $j + l \geq 1$ there is a constant $c > 0$ such that for all $f, g \in H^{j+l}$ and any $F \in L^2$ whose Fourier coefficients satisfy
\[
\tilde{F}_m \leq \sum_{m = a + b} |m|^j |\hat{f}_a| |\hat{g}_b|, \quad \forall m \in \mathbb{Z},
\]
we have
\[
\| F \|_{H^j} \leq c \| f \|_{H^{j+l}} \| g \|_{H^{j+l}}.
\]
Proof. The proof is given in Appendix B. \qed \\

Proof of Lemma 5.12. Our starting point is again Duhamel’s formula Eq. (6) and the definition of our numerical scheme Eq. (15) which yields:

$$
\phi_{t_n, \tau}(v(t_n)) - \Phi_{t_n, \tau}(v(t_n)) = \frac{1}{2} \int_0^\tau e^{(t_n+s)\partial_x^2} \partial_x \left[ \left( e^{-(t_n+s)\partial_x^2} v(t_n + s) \right)^2 - \left( e^{-(t_n+s)\partial_x^2} \frac{1}{2} (v(t_n) + \Phi_{t_n, \tau}(v(t_n))) \right)^2 \right] ds 
\tag{41}
$$

In order to understand higher order convergence properties of the numerical method we have to iterate Eq. (6) to obtain the following expression:

$$
\left( e^{-(t_n+s)\partial_x^2} v(t_n + s) \right)^2 = \left( e^{-(t_n+s)\partial_x^2} v(t_n + \tau/2) + \frac{1}{2} e^{-(t_n+s)\partial_x^2} \int_{\tau/2}^\tau e^{(t_n+s)\partial_x^2} \partial_x \left( e^{-(t_n+s)\partial_x^2} v(t_n + \tilde{s}) \right)^2 d\tilde{s} \right)^2 
= \left( e^{-(t_n+s)\partial_x^2} v(t_n + \tau/2) \right)^2 
+ \frac{1}{4} e^{-(t_n+s)\partial_x^2} \int_{\tau/2}^\tau e^{(t_n+s)\partial_x^2} \partial_x \left( e^{-(t_n+s)\partial_x^2} v(t_n + \tilde{s}) \right)^2 d\tilde{s} 
\tag{42}
$$

We also have

$$
\frac{1}{2} \left( v(t_n) + \Phi_{t_n, \tau}(v(t_n)) \right) - \phi_{t_n, \tau/2}(v(t_n)) = \frac{1}{4} \int_0^{\tau/2} e^{(t_n+s)\partial_x^2} \partial_x \left[ \left( e^{-(t_n+s)\partial_x^2} \frac{1}{2} (v(t_n) + \Phi_{t_n, \tau}(v(t_n))) \right)^2 \right] ds 
- \frac{1}{2} \int_{\tau/2}^{\tau} e^{(t_n+s)\partial_x^2} \partial_x \left[ \left( e^{-(t_n+s)\partial_x^2} v(t_n + s) \right)^2 \right] ds 
=: D_1
$$

Thus

$$
\left( e^{-(t_n+s)\partial_x^2} \frac{1}{2} (v(t_n) + \Phi_{t_n, \tau}(v(t_n))) \right)^2 
= \left( e^{-(t_n+s)\partial_x^2} v(t_n + \tau/2) + e^{-(t_n+s)\partial_x^2} \left( \frac{1}{2} (v(t_n) + \Phi_{t_n, \tau}(v(t_n))) - v(t_n + \tau/2) \right) \right)^2 
= \left( e^{-(t_n+s)\partial_x^2} v(t_n + \tau/2) \right)^2 
+ \left( e^{-(t_n+s)\partial_x^2} v(t_n + \tau/2) \right) \left( e^{-(t_n+s)\partial_x^2} D_1 \right) + \left( e^{-(t_n+s)\partial_x^2} D_1 \right)^2 
\tag{42}
$$

Thus we have from Eq. (41)

$$
\| \phi_{t_n, \tau}(v(t_n)) - \Phi_{t_n, \tau}(v(t_n)) \|_{H^1} \leq c \int_0^\tau \| e^{(t_n+s)\partial_x^2} \partial_x A_1 \|_{H^1} ds 
+ c \int_0^\tau \| A_2 \|_{H^2} + \| B_1 \|_{H^2} + \left( e^{-(t_n+s)\partial_x^2} D_1 \right)^2 ds. 
\tag{43}
$$
Let us begin by estimating the contributions from $A_2$ and from $\left( e^{-(t_n+s)\partial_x^3} D_1 \right)^2$. For $A_2$ we have

$$\int_0^\tau \| A_2 \|_{H^2} \, ds \leq c \int_0^\tau \left\| \int_{\tau/2}^\tau e^{(t_n+s)\partial_x^3} \partial_x \left( e^{-(t_n+s)\partial_x^3} v(t_n + s) \right) \, ds \right\|_{H^2}^2 \, ds$$

$$\leq \bar{c} \int_0^\tau \left( \int_{\tau/2}^\tau \| v(t_n + s) \|_{H^2}^4 \right) \, ds \sup_{s \in [0,\tau]} \| v(t_n + s) \|_{H^2}^4 \leq \bar{c} \tau^3 \sup_{s \in [0,\tau]} \| v(t_n + s) \|_{H^2}^4,$$

(44)

for some constants $c, \bar{c} > 0$ independent of $v$. Similarly we find (for potentially different values of $c, \bar{c} > 0$) using Thm. 3.2 under the assumption that $\tau \in [0, \tau_R]$ as given in the statement of the theorem,

$$\int_0^\tau \left\| \left( e^{-(t_n+s)\partial_x^3} D_1 \right)^2 \right\|_{H^2}^2 \, ds \leq c \int_0^\tau \left\| \int_{\tau/2}^\tau e^{(t_n+s)\partial_x^3} \partial_x \left( e^{-(t_n+s)\partial_x^3} \frac{1}{2} (v(t_n) + \Phi_{t_n, \tau}(v(t_n))) \right) \, ds \right\|_{H^2}^2 \, ds$$

$$+ c \int_0^\tau \left\| \int_{\tau/2}^\tau e^{(t_n+s)\partial_x^3} \partial_x \left( e^{-(t_n+s)\partial_x^3} v(t_n + s) \right) \, ds \right\|_{H^2}^2 \, ds$$

$$\leq \bar{c} \tau^3 \sup_{s \in [0,\tau]} \| v(t_n + s) \|_{H^2}^4 + \| \Phi_{t_n, \tau}(v(t_n)) - v(t_n) \|_{H^2}^4$$

$$\leq C_R \tau^3 (1 + \tau),$$

where $C_R > 0$ depends on $R$. Since $\tau < \tau_R$ we thus have the estimate

$$\int_0^\tau \left\| \left( e^{-(t_n+s)\partial_x^3} D_1 \right)^2 \right\|_{H^2}^2 \, ds \leq \bar{C} \tau^3,$$

(45)

Now we aim to estimate the contribution from $B_1$:

$$\int_0^\tau \| B_1 \|_{H^2} \, ds.$$

To achieve a suitable estimate let us express $D_1$ in the following way

$$D_1 = \frac{1}{4} \int_0^\tau e^{(t_n+s)\partial_x^3} \partial_x \left[ e^{-(t_n+s)\partial_x^3} v(t_n) \right] \, ds - \frac{1}{2} \int_0^\tau e^{(t_n+s)\partial_x^3} \partial_x \left[ e^{-(t_n+s)\partial_x^3} v(t_n) \right] \, ds$$

$$=: D_{1,1}$$

$$+ \frac{1}{4} \int_0^\tau e^{(t_n+s)\partial_x^3} \partial_x \left[ e^{-(t_n+s)\partial_x^3} \frac{1}{2} (v(t_n) + \Phi_{t_n, \tau}(v(t_n))) \right] \, ds$$

$$- \frac{1}{2} \int_0^\tau e^{(t_n+s)\partial_x^3} \partial_x \left[ e^{-(t_n+s)\partial_x^3} v(t_n) \right] \, ds$$

$$=: D_{1,2}$$

$$- \frac{1}{2} \int_0^\tau e^{(t_n+s)\partial_x^3} \partial_x \left[ e^{-(t_n+s)\partial_x^3} v(t_n) \right] \, ds$$

$$=: D_{1,3}.$$
Noting that $|\exp(ix) - 1/|x| \leq 1$ for $x \in \mathbb{R}$ we therefore have that

$$\left| \frac{\partial^2}{\partial x^2} D_{1,m} \right| \leq \tau^2 \sum_{m=a+b} |m| |a||b||\tilde{v}_a||\tilde{v}_b|.$$  

Thus, by Lemma 5.13, it immediately follows that

$$\|D_{1,1}\|_{H^2} \leq c\tau^2 \|v\|_{H^2}^2,$$

for some constant $c > 0$, independent of $R$. We also have, using Eq. (6),

$$\|D_{1,3}\|_{H^2} \leq \tilde{c} \int_0^{\tau/2} \|v(t_n) + v(t_n + \epsilon)\|_{H^3} \|v(t_n) - v(t_n + \epsilon)\|_{H^2} d\epsilon \leq \tau^2 \tilde{C}_R,$$

where $\tilde{C}_R > 0$ depends on $R$, and a similar estimate can be derived analogously for $D_{1,2}$ for any $\tau \in [0, \tau_R)$ as defined in the assumptions of Thm. 3.2:

$$\|D_{1,2}\|_{H^2} \leq \tau^2 \tilde{C}_R.$$

From Eq. (46)&(42) we thus have for any $\tau \in [0, \tau_R)$ and some $\tilde{C}_R > 0$ depending only on $R$

$$\int_0^\tau \|B_1\|_{H^2} d\epsilon \leq \tau^3 \tilde{C}_R. \quad (47)$$

It remains to estimate the contribution from $A_1$. For this it is helpful to write

$$A_1 = \left( e^{-(t_n+s)\partial^2_x} v(t_n + \tau/2) \right) e^{-(t_n+s)\partial^2_x} \int_{\tau/2}^\epsilon v'(t_n + \tilde{s})d\tilde{s}
= \left( e^{-(t_n+s)\partial^2_x} v(t_n + \tau/2) \right) e^{-(t_n+s)\partial^2_x} (s - \tau/2)v'(t_n + \tau/2)
= A_{1,1}
+ \left( e^{-(t_n+s)\partial^2_x} v(t_n + \tau/2) \right) e^{-(t_n+s)\partial^2_x} \int_{\tau/2}^\epsilon (v'(t_n + \tilde{s}) - v'(t_n + \tau/2)) d\tilde{s}.
= A_{1,2}$$

Now we observe that

$$\left\| \int_0^\tau e^{(t_n+s)\partial^2_x} \partial_x A_1 d\epsilon \right\|_{H^2} \leq \left\| \int_0^\tau e^{(t_n+s)\partial^2_x} \partial_x A_{1,1} d\epsilon \right\|_{H^2} + \int_0^\tau \|A_{1,2}\|_{H^2} d\epsilon. \quad (48)$$

And we have by Lemma 3.4

$$\int_0^\tau \|A_{1,2}\|_{H^2} d\epsilon \leq c\tau^2 \sup_{s \in [0,\tau]} \|v(t_n + s)\|_{H^2} \sup_{s \in [0,\tau]} \|v'(t_n + s) - v'(t_n + \tau/2)\|_{H^2}.$$
Now by Eq. (4) we have in terms of Fourier coefficients

\[
v'(t_n + s) - v'(t_n, \tau/2) = \int_{\tau/2}^{s} v''(t_n + \tilde{s}) d\tilde{s}
\]

\[
= \sum_{m \in \mathbb{Z}} e^{imx} \sum_{m = a + b} \int_{\tau/2}^{s} \frac{1}{2} e^{-i(t_n + \tilde{s}) \lambda_{abm}} (-3iabm) im \tilde{v}_a(t_n + \tilde{s}) \tilde{v}_b(t_n + \tilde{s}) d\tilde{s}
\]

\[
+ \sum_{m \in \mathbb{Z}} e^{imx} \sum_{m = a + b} \int_{\tau/2}^{s} e^{-i(t_n + \tilde{s}) \lambda_{abm}} im \tilde{v}_a(t_n + \tilde{s}) \tilde{v}_b(t_n + \tilde{s}) d\tilde{s}
\]

\[
= \frac{1}{2} \int_{\tau/2}^{s} e^{i(t_n + \tilde{s})} \partial_x^2 \left( e^{-i(t_n + \tilde{s})} \partial_x^2 v(t_n + \tilde{s}) \right)^2 d\tilde{s}
\]

\[
+ \frac{1}{2} \int_{\tau/2}^{s} e^{i(t_n + \tilde{s})} \partial_x \left( e^{-i(t_n + \tilde{s})} \partial_x^2 v(t_n + \tilde{s}) \right) \partial_x \left( e^{-i(t_n + \tilde{s})} \partial_x^2 v(t_n + \tilde{s}) \right) d\tilde{s}
\]

Thus, similarly to the derivation of Eq. (44), we have the estimate

\[
\int_0^\tau \|A_{1,2}\|_{H^s} ds \leq c\tau^3 \left( \sup_{s \in [0,\tau]} \|v(t_n + s)\|_{H^s} + \sup_{s \in [0,\tau]} \|v(t_n + s)\|_{H^{s+1}} \right).
\]  

(49)

To estimate \(A_{1,1}\) let us look at the Fourier coefficients of \(A := \int_0^\tau e^{i(t_n + s)} \partial_x^2 \partial_x A_{1,1} ds\). We have

\[
|A_{1,1}| = \left| \sum_{m = a + b} \int_0^\tau e^{-im^2 - m^2 - b} e^{i(t_n + s)} (s - \tau/2) ds \tilde{v}_a(t_n + \tau/2) \tilde{v}_b(t_n + \tau/2) \right|
\]

\[
= \left| \sum_{m = a + b} \int_0^\tau e^{-im^2 - m^2 - b} e^{i(t_n + s)} (s - \tau/2) ds \tilde{v}_a(t_n + \tau/2) \tilde{v}_b(t_n + \tau/2) \right|
\]

\[
\leq \sum_{m = a + b} \left| 3mab \cos \left( \frac{3mab}{2} \right) - 2 \sin \left( \frac{3mab}{2} \right) \right| |m| \|\tilde{v}_a(t_n + \tau/2)\|_{H^s} \|\tilde{v}_b(t_n + \tau/2)\|_{H^s}
\]

Using Lemma 5.13 and observing that \(\frac{x \cos(x/2) - 2 \sin(x/2)}{x^2} \leq \frac{1}{12}\) for all \(x \in \mathbb{R}\), we have

\[
\left\| \int_0^\tau e^{i(t_n + s)} \partial_x^2 A_{1,1} ds \right\|_{H^s} = \|A\|_{H^s} \leq c\tau^3 \|v(t_n + \tau/2)\|_{H^s} \|v'(t_n + \tau/2)\|_{H^s}
\]

\[
\leq c\tau^3 \|v(t_n + \tau/2)\|_{H^s} \|v(t_n + \tau/2)\|_{H^{s+1}}
\]  

(50)

for some constants \(c, \tilde{c} > 0\).

We can now easily conclude the result by combining the estimates Eqs. (43), (44), (45), (47), (48), (49) & (50).

\[\square\]

**Remark 5.14.** A slightly more detailed (and tedious) analysis of the error in terms of Fourier coefficients would allow us to show the following slightly more resolved result which implies Lemmas 5.11 & 5.12: Fix \(R > 0\), then for any \(\gamma \in [1,2]\) there is a \(\tau_{R,\gamma}, c_{R,\gamma} > 0\) such that for all \(\tau \in [0,\tau_{R,\gamma})\) and any \(v(t_n) \in B_R(H^{1+2\gamma})\) such that \(\sup_{t \in [0,\tau]} \|\Phi_{t_n,\tau}(v(t_n))\|_{H^{1+2\gamma}} < R\) we have

\[
\|\Phi_{t_n,\tau}(v(t_n)) - \Phi_{t_n,\tau}(v(t_n))\|_{H^s} \leq c_{R,\gamma} \tau^3.
\]
5.2.3 Global error in \( H^1 \)

We can now prove Thms. 5.1 & 5.2 in similar vein to our proof of Thm. 5.8.

**Proof of Thm. 5.1.** By the triangle inequality we have

\[
\|v(t_n) - v^n\|_{H^1} \leq \|\Phi_{t_{n-1},\tau}(v(t_{n-1})) - \Phi_{t_{n-1},\tau}(v^{n-1})\|_{H^1} + \|\Phi_{t_{n-1},\tau}(v(t_{n-1}) - \Phi_{t_{n-1},\tau}(v(t_{n-1}))\|_{H^1} .
\]

(51)

We note that by Thm. 5.8 we may choose \( \tau_R \) such that, whenever \( \tau \in [0, \tau_R) \) we have \( v^k \in B_{\tau R}(H^2) \) for all \( 0 \leq k \leq n \). Thus we may iterate above estimate Eq. (51) and find that, using Lemmas 5.10 & 5.11, 

\[
\|v(t_n) - v^n\|_{H^1} \leq e^{\tau CR}\|v(t_{n-1}) - v^{n-1}\|_{H^1} + cR \tau^2 \\
\leq e^{2\tau CR}\|v(t_{n-2}) - v^{n-2}\|_{H^1} + e^{\tau CR}cR \tau^2 + cR \tau^2 \\
\leq cR \tau^2 \sum_{k=0}^{n-1} e^{\tau CR} \leq cR \tau_n e^{\tau_n CR}.
\]

which completes the proof.

**Proof of Thm. 5.2.** This result follows analogously by replacing Lemma 5.11 with Lemma 5.12 in the above proof.

6 Numerical experiments

In this section we test our symplectic resonance-based scheme (16) numerically, and exhibit its favourable properties in practice. We will study its convergence and preservation of structure in comparison with two numerical methods proposed in the literature for smooth and non-smooth initial data: A first-order resonance-based low-regularity integrator from [19, Eq. (10)] which we denote by ‘Hofmanová & Schratz 2017’ and a symmetric Lawson method (an exponential integrator) from [7, Example 3.2] which we denote by ‘Celledoni et al. 2008’.

6.1 A note on the fully discrete method and aliasing

In our numerical implementation we choose a spectral collocation method as the spatial discretisation. Here we will briefly outline the resulting fully discrete method and highlight that aliasing does not have a significant impact on the numerical performance. We follow the description of a similar discretisation for the nonlinear Schrödinger equation in [12, Chapter 6] (see also [19, Remark 3.1]).

Our spatial discretisation is a spectral collocation method defined in the following way: for \( M \in 2\mathbb{N} \) find a trigonometric polynomial

\[
u_{(M)}(t, x) = \sum_{m=-M/2+1}^{M/2} u_{(M),m}(t)e^{imx},
\]

(52)

which satisfies for all collocation points, \( x_m = 2\pi m/M, m = -M/2 + 1, M/2 \), the KdV equation Eq. (1) i.e.

\[
\begin{align*}
\partial_t u_M(t, x_m) + \partial_x^2 u_M(t, x_m) &= \frac{1}{2}\partial_x(u_M(t, x_m))^2, \quad t \in \mathbb{R}_+, \\
u_M(0, x_m) &= u_0(x_m).
\end{align*}
\]

(53)
Taking the discrete Fourier transform $F_M$, with the convention

$$(F_M z)_l = \frac{1}{M} \sum_{m=-M/2+1}^{M/2} e^{\frac{2\pi i ml}{M}} z_m,$$

of Eq. (53) we find the equation is equivalent to the following equation for the Fourier coefficients of Eq. (52):

$$\partial_t u_{(M),m}(t) = im^3 u_{(M),m}(t) + \sum_{l \in \mathbb{Z}} \sum_{m=a+b+1M}^{a+b+M/2} \frac{i(a+b)}{2} u_{(M),a}(t) u_{(M),b}(t).$$

We notice quickly that in Eq. (54) the only possible values for $l$ yielding a non-trivial contribution to the sum are $l = -1, 0, 1$. The presence of non-zero $l$ is called aliasing, but because only a finite number of non-trivial contributions appear, the time discretisation can be constructed analogously to Eq. (16) and its convergence and stability analysis can be performed similarly to Section 5.

### 6.2 Convergence properties

We will now examine the convergence properties of the proposed scheme Eq. (16) when compared against the methods ‘Hofmanová & Schratz 2017’ [19] which was shown to achieve first order convergence in $H^1$ for data in $H^3$ and the Lawson method ‘Celledoni et al. 2008’ [7] which is classically of second order for data in $C^\infty(T)$. In the following numerical experiments we consider two types of initial conditions. Firstly, we take

$$u_{0(M),m} = \begin{cases} m^{-\vartheta} U_{M,m}/10, & m \neq 0, \\ 0, & m = 0. \end{cases}$$

where the rescaled Fourier coefficients of our initial condition, $U_{M,M}$, are chosen as a single sample (using $\text{rand}(M,1)+i*\text{rand}(M,1)$ in Matlab) of a uniform random distribution $U_{M} \sim U([-1,1]^M+i[-1,1]^M)$.

This is such that $u_{0(M)} \in H^{3-\vartheta/2-\varepsilon}$ for all $\varepsilon > 0$. The second type of initial conditions we consider are $u_0 \in C^\infty(T)$ with

$$u_0(x) = \frac{1}{10(2 + \sin(x))} - \frac{1}{2\pi} \int_\mathbb{T} \frac{1}{10(2 + \sin(x))} dx.$$
6 Numerical experiments

on the particular implementation and computational architecture used, but the figure nevertheless serves as a broad indication of the relative computational cost of these methods if a comparable implementation is used. In Fig. 1b we observe in particular that the implicit nature of our method does not adversely affect the overall computational cost, as the improved convergence properties make up for the extra cost of solving the implicit system Eq. (16) at each time step. Here we chose, at each time step, to iterate Eq. (25) until the updates incurred an error less than $\tau^4$ to ensure the local error of the method is indeed realised. In practice it was found that around 10-20 fixed-point iterations sufficed to guarantee convergence of the fixed-point iterations close to machine accuracy, even for moderate values of $\tau$.

In Fig. 2 we display a similar experiment with initial data that is slightly more regular, in particular

Figure 1: Convergence properties of the methods for data in $H^{3-\epsilon}, \forall \epsilon > 0$.

Figure 2: Convergence properties of the methods for data in $H^{5-\epsilon}, \forall \epsilon > 0$. 

(a) Order plot measured in $H^1$.

(b) CPU time versus $H^1$-error.

(a) Order plot measured in $H^1$.

(b) CPU time versus $H^1$-error.
we take initial data in $H^{5-\epsilon}$, $\forall \epsilon > 0$, by choosing $\vartheta = 5.5$ in Eq. (55). As predicted in Thm. 5.2 we observe our new method converges at second order for this initial data. The Lawson method ‘Celledoni et al. 2008’ [7] shows some improved convergence properties over the previous experiment in Fig. 1, but still suffers from a highly oscillatory error with unreliable convergence. The first order integrator from [19] still converges with first order, as expected.

Finally, in Fig. 3 we see the analogous experiment for smooth initial data as defined in Eq. (56). In this case, as we might expect, the second order convergence of the Lawson method ‘Celledoni et al. 2008’ [7] is indeed fully realised, but our low-regularity integrator remains competitive even for highly regular solutions.

6.3 Structure preservation properties

Having verified the convergence properties of our proposed numerical scheme, we now study its structure preservation properties. As before we will compare our new symplectic resonance-based scheme against ‘Hofmanová & Schratz 2018’ [19] which is designed as an explicit resonance-based scheme that exhibits good convergence properties in the low-regularity regime. However it is not specifically designed to preserve geometric structure of the underlying flow and we shall see that in practice unsurprisingly it is unable to preserve momentum or Hamiltonian well.

In contrast our second method of comparison, ‘Celledoni et al. 2008’ [7] is a symmetric exponential integrator which was designed to preserve the momentum of the equation (as a quadratic first integral) exactly (cf. [7, Proposition 3.1]). We already saw in Section 6.2 that the method itself becomes unreliable in the low-regularity regime, and we shall see that this translates also to the conservation of first integrals in the low-regularity regime.

6.3.1 Momentum

As a first step let us look at how well the momentum, $\int_T u^2 \, dx$, is preserved under the numerical scheme. In the following we see two figures examining this for initial conditions of specific regularity. In all of the following experiments we took the number of Fourier modes $M = 2^9$ and the time step $\tau = 0.05$.

Figure 3: Convergence properties of the methods for data in $C^\infty(\mathbb{T})$. 

(a) Order plot measured in $H^1$. (b) CPU time versus $H^1$-error.
The figures plot the absolute error in the momentum approximation (using the notation introduced in Eq. (5)):

$$\left| I_0[u^N_{(M)}(x)] - I_0[u^0_{(M)}(x)] \right| = \left| \int_\tau (u^N_{(M)}(x))^2 \, dx - \int_\tau (u^0_{(M)}(x))^2 \, dx \right|$$

for the range $\tau N \in [0, 2500]$, where $T = N\tau$.

![Graph showing error in momentum approximation.](image)

(b) Zoom in on subinterval to observe numerical instability of ‘Celledoni et al. 2008’ [7].

Figure 4: Error in the momentum of the numerical solution, for $u^0 \in H^{3-\epsilon}$, $\forall \epsilon > 0$.

In Fig. 4 we observe this error for initial data $u_0 \in H^{3-\epsilon}$ for all $\epsilon > 0$, i.e. taking $\vartheta = 3.5$ in Eq. (55). As expected for our new method the momentum is preserved even for such low-regularity initial data, and the error remains close to double-precision machine accuracy throughout. Perhaps unsurprisingly the method ‘Hofmanová & Schratz’ [19] incurs an error which grows algebraically in $T$, in line with its global first order convergence. This is to be expected as this method was not designed for structure preservation. However, what is perhaps somewhat unexpected is the poor performance of the Lawson method, ‘Celledoni et al. 2008’ [7] which completely loses its momentum conservation property because
the method diverges after a small amount of time in this low-regularity regime (this instability is most apparent in the zoomed-in graph in Fig. 4b). Of course, this is justified by the fact that the Lawson method was not developed for low-regularity data and we see below that its performance is significantly improved in the smooth setting in Fig. 5.

Indeed once we choose smooth initial data, i.e. Eq. (56), we see in Fig. 5 that the Lawson method recovers its structure preservation properties and indeed achieves a perfect conservation of the momentum. Our new method preserves the momentum equally well in this smooth regime.

![Figure 5: Error in the momentum of the numerical solution, for $u_0 \in C^\infty(T)$.](image)

### 6.3.2 Hamiltonian

In the case of finite dimensional Hamiltonian systems it is well-known that symplectic methods preserve the Hamiltonian over long times (cf. [17, Section IX.8]). We can therefore also study the error in the Hamiltonian of the numerical solution, i.e. the quantity

$$I_1[u] = \frac{1}{2} \int_T 3u_x^2 + u^3 dx.$$

In Figs. 6-7 we observe the error

$$\left| I_1[u^{N,M}_0(x)] - I_1[u^0_{M}(x)] \right|$$

for initial conditions of various degrees of regularity as in the previous section. The numerical results were obtained with the same parameter settings as in Section 6.3.1, $M = 2^9$, $\tau = 0.05$. We observe that our symplectic numerical scheme appears to preserve the Hamiltonian well across the regularity regimes considered, whereas the Lawson method ‘Celledoni et al. 2008’ [7] achieves comparable preservation only in the smooth case, and the resonance-based integrator ‘Hofmanová & Schratz 2017’ incurs a similar algebraic drift as observed in Section 6.3.1 in the momentum. The development of the instability of ‘Celledoni et al. 2008’ can be seen in Fig. 6b.
Numerical experiments

(b) Zoom in on subinterval to observe numerical instability of ‘Celledoni et al. 2008’ [7].

Figure 6: Error in the Hamiltonian of the numerical solution for \( u_0 \in H^{3-\epsilon}, \forall \epsilon > 0 \).

Figure 7: Error in the Hamiltonian of the numerical solution for \( u_0 \in C^\infty(T) \).
6.3.3 Resonant time steps in the long time preservation of the Hamiltonian

Although it has been shown that in the finite dimensional case the symplectic nature of a numerical method for a Hamiltonian system is closely linked to the long-time preservation of the Hamiltonian [17, Chapter IX], this is no longer necessarily the case for infinite dimensional systems. For instance, for splitting methods (which are naturally symplectic) for the nonlinear Schrödinger equation it was shown by [12] that so-called resonant timesteps appear at which the Hamiltonian is no longer well-preserved (cf. also [13, 14]). In numerical experiments we found that a similar phenomenon appears for our proposed numerical scheme. In this example we consider initial data according to Eq. (55) with $\vartheta = 3.5$, take $M = 2^9$ and consider the maximum error in the Hamiltonian over a fixed time range

$$\max_{0 \leq n \tau \leq 100} |I_1[u^n_{(M)}(x)] - I_1[u^0_{(M)}(x)]|.$$

![Figure 8: Resonant time steps in the preservation of Hamiltonian.](image)

Fig. 8 shows this quantity as a function of the time step $\tau$. Given the results from Section 6.2 we might expect the graph to show a fairly smooth curve following the second order convergence as indicated in the red dashed line in the figure. However we notice the appearance of spikes at a discrete set of time steps at which the Hamiltonian is no longer preserved so well. This is very similar to the phenomena observed for splitting methods for the nonlinear Schrödinger equation and a theoretical explanation of this will form part of future research on the topic. Here we present the result as a warning for the reader to exercise caution when applying our method with the goal of preserving the Hamiltonian.

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[37] T. Tao, *Nonlinear Dispersive Equations: Local and Global Analysis*, Conference Board of the Mathematical Sciences. Regional Conference Series in Mathematics, American Mathematical Society, 2006.
A Proof of Lemma 3.3

For completeness we recall the statement of the lemma.

**Lemma A.1.** Let us introduce the notation

\[
F(t_n, \tau, \tilde{v}) := \frac{1}{6} e^{(\tau^2 + \tau^2)} \left( e^{-\tau^2} \partial_x^{-1} \tilde{v} \right)^2 - \frac{1}{6} e^{\tau^2} \left( e^{-\tau^2} \partial_x^{-1} \tilde{v} \right)^2
\]

Then, for \( l = 1, 2, 3 \), there is a continuous function \( M_l : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) such that

\[
\| F(t_n, \tau, f) - F(t_n, \tau, g) \|_{\mathcal{H}} \leq \tau^2 M_l (\| f \|_{\mathcal{H}}, \| g \|_{\mathcal{H}}). \]

**Proof.** As highlighted in Section 3.2 the cases \( l = 1, 2 \) are treated in [19, Eq. (38) & Lemma 2.4]. It remains to prove the case \( l = 3 \). For this we proceed similarly to [19, Lemma 2.4] and write

\[
\| F(t_n, \tau, f) - F(t_n, \tau, g) \|_{\mathcal{H}^2}^2 = \frac{1}{36} \partial_x^2 e^{-\tau^2} \left[ \| F(t_n, \tau, f) - F(t_n, \tau, g) \|_{\mathcal{H}}^2 \right]^2
\]

\[
= \frac{1}{36} \partial_x^2 \left[ \left( e^{-\tau^2} \partial_x^{-1} \tilde{f} \right)^2 - \left( e^{-\tau^2} \partial_x^{-1} \tilde{g} \right)^2 \right] - \partial_x^2 e^{-\tau^2} \partial_x^{-1} \left[ \left( \partial_x^{-1} \tilde{f} \right)^2 - \left( \partial_x^{-1} \tilde{g} \right)^2 \right] - \partial_x^2 e^{-\tau^2} \partial_x^{-1} \left[ \left( \partial_x^{-1} \tilde{f} \right)^2 - \left( \partial_x^{-1} \tilde{g} \right)^2 \right]
\]

\[
= A_1 + A_2
\]

where we denoted by \( \tilde{f} = e^{-\tau^2} f \) and \( \tilde{g} = e^{-\tau^2} g \). Now let us estimate \( A_2 \) first. Letting \( F := \partial_x^2 e^{-\tau^2} \left[ \partial_x^{-2} \tilde{f}^2 - \partial_x^{-1} \tilde{g}^2 \right] \), we have

\[
| A_2 | \leq \sum_{m = a + b} \left| \frac{(-im)^3}{a + b} \right| \left| \frac{F_{-m}}{F_{m}} \right| \left| 1 - e^{\tau a (a + b)} \right| \left| \tilde{f}_a \tilde{f}_b - \tilde{g}_a \tilde{g}_b \right|
\]

\[
\leq 3\tau \sum_{m = a + b} \left| \frac{a + b}{|a||b|} \right| \left| \tilde{f}_a \tilde{f}_b - \tilde{g}_a \tilde{g}_b \right|
\]

\[
\leq 3\tau \sum_{m = a + b} \left| \frac{(-im)}{a + b} \right| \left| (\tilde{f}_a - \tilde{g}_a) \tilde{f}_b + \tilde{g}_a (\tilde{f}_b - \tilde{g}_b) \right|
\]

\[
\leq 3\tau \sum_{m = a + b} \left| (\tilde{f}_a - \tilde{g}_a) \tilde{f}_b + \tilde{g}_a (\tilde{f}_b - \tilde{g}_b) \right|
\]

\[
\leq 3\tau \sum_{m = a + b} \left| (\tilde{f}_a - \tilde{g}_a) \tilde{f}_b + \tilde{g}_a (\tilde{f}_b - \tilde{g}_b) \right|
\]

[38] Y. Wu and X. Zhao, *Embedded exponential-type low-regularity integrators for KdV equation under rough data*, BIT Numerical Mathematics, (2021), pp. 1–42.
Thus by Cauchy–Schwarz we have, for some $c_1 > 0$,

$$|A_2| \leq c_1 \tau \|\partial_x F\|_{L^2} \left( \sum_{j=0}^{3} \left( \|\partial_x^j \tilde{g} \ast \partial_x^{3-j} (\tilde{f} - \tilde{g})\|_{L^2} + \|\partial_x^j \tilde{f} \ast \partial_x^{3-j} (\tilde{f} - \tilde{g})\|_{L^2} \right) \right). \quad (57)$$

We can now make use of the following observation:

**Claim A.2.** Let $j \in \{0, 1, 2, 3\}$ then there is a constant $c_2 > 0$ such that for any functions $f, g \in H^3$ we have

$$\|\partial_x^j \tilde{f} \ast \partial_x^{3-j} \tilde{g}\|_{L^2} \leq c_2 \|f\|_{H^3}\|g\|_{H^3}. \quad (58)$$

**Proof of Claim.** Let us take without loss of generality $j > 0$, then for $f^{(j)}(k) := |k|^j \tilde{f}_k, k \in \mathbb{Z} \setminus \{0\}$. We have by Young's inequality

$$\|\partial_x^j \tilde{f} \ast \partial_x^{3-j} \tilde{g}\|_{L^2} = \|f^{(j)} \ast g^{(3-j)}\|_{L^2} \leq \|f^{(j)}\|_{H^3}\|g^{(3-j)}\|_{H^3}.$$ 

Now using the Cauchy–Schwarz inequality we have $\|f^{(j)}\|_{H^3} \leq c_2 \|f^{(3)}\|_{H^3}$ for some constant $c_2 > 0$ independent of $f$ which immediately implies the bound Eq. (58).

Thus, combining Eq. (57) and Eq. (58) we conclude:

$$|A_2| \leq \tau \tilde{c}_3 \|\partial_x F\|_{L^2} \|f - g\|_{H^3} \leq \tau \tilde{c}_3 \|\tilde{F} - \tilde{g}\|_{H^3},$$

where $c_3, \tilde{c}_3 > 0$ are constants which depends continuously on $\|f\|_{H^3}, \|g\|_{H^3}$. We can now introduce $\tilde{F} := \partial_x^2 \left( \left( e^{-\tau \partial_x^2} \tilde{f} \right)^2 - \left( e^{-\tau \partial_x^2} \tilde{g} \right)^2 \right)$ and follow exactly the same estimates to show

$$|A_1| \leq \tau \tilde{c}_4 \|\tilde{F} - \tilde{g}\|_{H^3},$$

where $c_4 > 0$ is a constant which depends continuously on $\|f\|_{H^3}, \|g\|_{H^3}$. Therefore the result follows.

**B Proof of Lemma 5.13**

For completeness we recall the statement of the Lemma.

**Lemma B.1.** For any $j, l \in \mathbb{N}$ such that $j + l \geq 1$ there is a constant $c > 0$ such that for all $f, g \in H^{3+l}$ and any $F \in L^2$ whose Fourier coefficients satisfy

$$\hat{F}_m \leq \sum_{m = a + b} |m|^l |\hat{f}_a| |\hat{g}_b|, \quad \forall m \in \mathbb{Z},$$

we have

$$\|F\|_{H^3} \leq c \|f\|_{H^{3+l}} \|g\|_{H^{3+l}}.$$

**Proof.** The proof follows the arguments from [29, Appendix A] closely. We begin by noting that for any $r \geq 0$ there is a constant $C_r > 0$ such that for all $a, b \in \mathbb{Z}$ we have

$$|(a + b)^r| \leq C_r (|a|^r + |b|^r).$$
Thus we can estimate

\[ \|F\|_{H^j}^2 = \sum_{m \in \mathbb{Z} \setminus \{0\}} |m|^{2j} |\hat{F}_m|^2 \]

\[ \leq \sum_{m \in \mathbb{Z} \setminus \{0\}} \left( \sum_{m = a + b} |m|^{l+j} |\hat{f}_a| |\hat{g}_b| \right)^2 \]

\[ = \sum_{a \in \mathbb{Z} \setminus \{0\}} \left( \sum_{b \in \mathbb{Z} \setminus \{0\}} |a + b|^{l+j} |\hat{f}_a| |\hat{g}_b| \right)^2 \]

\[ \leq C \sum_{a \in \mathbb{Z} \setminus \{0\}} \left( \sum_{b \in \mathbb{Z} \setminus \{0\}} |a|^{l+j} |\hat{f}_a| |\hat{g}_b| \right)^2 + \sum_{a \in \mathbb{Z} \setminus \{0\}} \left( \sum_{b \in \mathbb{Z} \setminus \{0\}} |b|^{l+j} |\hat{f}_a| |\hat{g}_b| \right)^2. \quad (59) \]

Now by the discrete Minkowski inequality we have

\[ \sum_{a \in \mathbb{Z} \setminus \{0\}} \left( \sum_{b \in \mathbb{Z} \setminus \{0\}} |a|^{l+j} |\hat{f}_a| |\hat{g}_b| \right)^2 \leq \left( \sum_{b \in \mathbb{Z} \setminus \{0\}} \left( \sum_{a \in \mathbb{Z} \setminus \{0\}} |a|^{2(l+j)} |\hat{f}_a|^2 |\hat{g}_b|^2 \right)^{1/2} \right)^2 \]

\[ \leq \|f\|_{H^{l+j}}^2 \left( \sum_{b \in \mathbb{Z} \setminus \{0\}} |\hat{g}_b| \right)^2 \leq \left( \sum_{b \in \mathbb{Z} \setminus \{0\}} |b|^{-2(j+1)} \right) \|f\|_{H^{l+j}} \|g\|_{H^{l+j}} \]

where in the final line we used the Cauchy–Schwarz inequality. The final sum converges because \( j + l \geq 1 \), thus applying a similar estimate to the second term in Eq. (59) concludes the proof. \( \square \)