OPERATOR ALGEBRAS WITH CONTRACTIVE APPROXIMATE IDENTITIES, IV: A LARGE OPERATOR ALGEBRA IN $c_0$

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Abstract. We exhibit a singly generated, semisimple commutative operator algebra with a contractive approximate identity, such that the spectrum of the generator is a null sequence and zero, but the algebra is not the closed linear span of the idempotents associated with the null sequence and obtained from the analytic functional calculus. Moreover the multiplication on the algebra is neither compact nor weakly compact. Thus we construct a ‘large’ operator algebra of orthogonal idempotents, which may be viewed as a dense subalgebra of $c_0$.

1. Introduction

There is an extensive history and theory of operator algebras on a Hilbert (or Banach) space that are generated by a family of idempotent operators which are orthogonal (that is, the product of any two of which is zero); and using such families in ‘spectral resolutions’ of operators. Related to this, it is well known that there exist exotic Banach algebras whose elements are sequences of scalars, with the multiplication being the obvious pointwise one. That is, there can be quite complicated Banach algebra norms on subalgebras of the $C^*$-algebra $c_0$ of null sequences. However examples of both of these kinds of algebras, of a certain interesting type described below, and which have an approximate identity, seem to be missing from the Banach algebra and operator theory literature. Our main goal here is to provide an explicit, yet in some sense universal, example of this kind. We first discuss our goal from the operator-theoretic angle, and later in the introduction we will mention the Banach algebra viewpoint.

Henceforth, by an operator algebra, we will mean a norm closed algebra of operators on a Hilbert space. There is a large literature on operator algebras generated by a family of mutually orthogonal idempotents (see e.g. [8, 11, 13, 16, 19] and references therein). Such a family arises naturally when one considers for an operator $T$ on a Hilbert space $H$ with $\text{Sp}(T)$ countable and having no nonzero limit points, the spectral idempotents obtained by the analytic functional calculus from the isolated points. These idempotents will be called minimal spectral idempotents, and they are nonzero by the Shilov idempotent theorem. In this case it is standard to try to use this family to analyze the structure of $T$ (often with an eye to decomposing $T$ in terms of these idempotents). We will henceforth assume that the norm closed algebra $B_T$ generated by $T$ in $B(H)$ is semisimple, which implies that these minimal spectral idempotents $e$ are minimal in the sense that $eB_T = \mathbb{C}e$. It seems that certain specific questions about the algebra $B_T$ that arise in this setting, are...
essentially unaddressed in the literature, to the best of our knowledge. It is a simple exercise in matrix theory (and using the blanket assumption that $B_T$ is semisimple), that if $H$ above were finite dimensional then $T$ is the sum of the minimal spectral idempotents, each multiplied by the corresponding eigenvalue. So it is natural to ask if, for $T$ as above, is $B_T$ generated by the minimal spectral idempotents of $T$? (Saying that $B$ is ‘generated’ by a subset, will for us always mean, unless stated to the contrary, that $B$ is the smallest norm closed subalgebra of $B$ containing the subset.) Such questions become quite difficult if one adds the assumption that the operator algebras involved have a contractive approximate identity (or cai). We will give a counterexample to this. In this example, $B_T$ is ‘large’, and in particular is not weakly compact. We will define, for the purposes of this paper, a commutative Banach algebra $A$ to be weakly compact (resp. compact) if multiplication on $A$ by $a$ is weakly compact (resp. compact), for every $a \in A$ (this is not the usual definition, but it is equivalent to it for algebras with a cai). Indeed it is the case that for an operator $T$ with $\text{Sp}(T)$ countable and having no nonzero limit points, $B_T$ is generated (in the norm topology) by the minimal spectral idempotents of $T$ if $B_T$ is semisimple, weakly compact, and its socle has an approximate identity (see the end of this Introduction for this and some related results, which we show there to be ‘best possible’ in some sense).

In algebraic language, $B_T$ being generated by the minimal spectral idempotents, is equivalent to $B_T$ having dense socle (or being Tauberian [16]); such algebras can be considered to be not ‘big’. However the point is that the examples in the literature of operator algebras generated by an operator and an associated sequence of mutually orthogonal minimal idempotents, tend to either be ‘small’ in this sense, or to fall within the case where this sequence is uniformly bounded, or to not have approximate identities, and do not speak to the question we have mentioned. We remark that in joint work with Le Merdy, the first author studied some natural Banach sequence algebras which were shown to be operator algebras (see [5, Chapter 5]), but again these were not ‘big’ in the sense above, and had no approximate identity.

We now discuss our goal from the Banach algebraic perspective, which goes back to Kaplansky (e.g. [14]). We recall that a natural Banach sequence algebra on $\mathbb{N}$ is a Banach algebra $A$ of scalar sequences, which contains the space $c_{00}$ of finitely supported sequences, and whose characters (i.e. nontrivial multiplicative linear functionals) are precisely the obvious ones: $\chi_n(\vec{a}) = a_n$ for $\vec{a} \in A$ (see [10, Section 4.1]). In our case the sequences in $A$ will converge to 0, so that $c_{00} \subset A \subset c_0$ (we are not assuming of course that the norm on $A$ is the $c_0$ norm). Natural Banach sequence algebras have been studied by many researchers (see e.g. [10, 16] and references therein), however we are not aware of any such algebras in the literature which are ‘big’ in the previous sense, namely that the socle of $A$, which in this case is $c_{00}$, is not dense in $A$ (that is, $A$ is not Tauberian), or, more generally, that $A$ is not weakly compact, and which also have a bounded approximate identity (or bai). In fact there is only one natural Banach sequence algebra example that we know of in the literature which does not have dense socle, an example due to Joel Feinstein in [10, Section 4.1], however it has no bai. We will construct here a ‘big’ example, which is a singly generated operator algebra on a Hilbert space, and which has a cai. To relate the Banach sequence algebra setting to that of the previous paragraphs, note that if $T$ is an operator on a Hilbert space $H$ with $\text{Sp}(T)$ countable
and having no nonzero limit points, and if the closed algebra $B_T$ generated by $T$ is semisimple, then the Gelfand transform makes $B_T$ into a (semisimple) natural Banach sequence algebra in $c_0$ (by basic Gelfand theory, e.g. the standard ideas in the proof of Corollary 1.3 below).

In this paper we exhibit an operator algebra example with the desired features discussed above. In hopes of obtaining a tool useful for solving other questions in this area in the literature, we have deliberately built our example to be as ‘large as possible’, and this has probably added to the difficulty and complexity of our proofs. In particular we show:

**Theorem 1.1.** There exists a semisimple operator algebra $A$ which has a cai and a single generator $g$ (and hence $A$ is separable), with the following properties. The spectrum of the generator of $A$ is a null sequence and zero, but $A$ is strictly larger than $A_{00}$, the norm closed linear span of the minimal spectral idempotents associated with this null sequence and obtained from the analytic functional calculus. Also, multiplication by the generator $g$ on $A$ is not a weakly compact operator (equivalently, $A$ is not an ideal in $A^{**}$ with the Arens product, see e.g. [1, Lemma 5.1]). The algebra can be chosen further with $A_{00}$ having a cai too; and with either $A$ contained in the strong operator closure of $A_{00}$, or not, as desired.

In the next section we turn to the construction of our algebras $A$ and $A_{00}$ described above. The development will become increasingly technical as the paper proceeds. However in Section 5 we will prove our main theorem with one lemma taken on faith, and in Section 6 we will pause and describe many properties that our algebras $A$ and $A_{00}$ possess. The material following Section 6 consists of the lengthy proof of the lemma just referred to.

We end this introduction with some general positive results on the topics above; namely sufficient conditions for when $B_T$ is generated by the minimal spectral idempotents of $T$. We remark that in [3, Proposition 1.1] it is shown that any closed algebra on a separable Hilbert space generated by a family of ‘mutually orthogonal’ idempotents, is topologically singly generated.

**Proposition 1.2.** Suppose that $D$ is a Banach algebra, which is an essential ideal in a commutative Arens regular Banach algebra $A$ (‘essential’ means that the canonical map $A \to B(D)$ is one-to-one). Assume that $A$ is weakly compact and $D$ has a bai. Then $A = D$.

**Proof.** By e.g. [1, Lemma 5.1] we have $A^{**} A \subset A$. Let $e \in D^\perp$ be the ‘support projection’ in $A^{**}$ of $D$, an identity for $D^\perp$, and write $1$ for the identity of the unitization of $A$. Then $(1 - e)D = 0$, and for any $a \in A$ we have $a(1 - e)D = 0$. Since $eA \subset A$, and $D$ is an essential ideal in $A$, we see that $a(1 - e) = 0$. Thus $e$ acts as an identity on $A$ and therefore also on $A^{**}$, so that $A^{**} = eA^{**} \subset D^\perp$. Hence $A = D$. 

**Corollary 1.3.** Let $T$ be an operator on a Hilbert space whose spectrum is countable and has no nonzero limit points, and let $B_T$ (resp. $B$) be the closed algebra generated by $T$ (resp. by the minimal spectral idempotents of $T$). If $B_T$ is semisimple and weakly compact, and if $B$ has a bai, then $B = B_T$.

**Proof.** Suppose that $\text{Sp}(T) \setminus \{0\} = \{\lambda_n\}$. By basic Gelfand theory, the set of characters of the unitization $B_T^1$ is $\{\chi_n : n \in \mathbb{N}_0\}$, where $\chi_0$ annihilates $B_T$, and...
\( \chi_n(T) = \lambda_n \) for \( n \in \mathbb{N} \). By the functional calculus \( \chi_m(e_n) = \delta_{nm} \) for \( n \in \mathbb{N}, m \in \mathbb{N}_0 \), where \( e_n \) is the spectral idempotent in \( B_T^1 \) corresponding to \( \lambda_n \). Hence \( e_n \in B_T \), and \( e_n T - \lambda_n e_n = \cap_m \ker(\chi_m) = \{0\} \). Thus \( e_n B_T = \mathbb{C} e_n \) for all \( n \), and \( \chi_n(a) e_n = a e_n \) for all \( a \in B_T \). Hence \( B B_T \subset B \), that is \( B \) is an ideal in \( B_T \). Indeed \( B \) is an essential ideal in \( B_T \), since the latter is semisimple (if \( a e_n = 0 \) for all \( n \) then \( \chi(a) = 0 \) for all characters \( \chi \)). Thus \( B = B_T \) by Theorem 1.2.

For operators on a Hilbert space whose spectrum is countable and has no nonzero limit points, the last result is sharp in the following sense:

**Theorem 1.4.** In the last result no one of the following three hypotheses can simply be removed, in general: \( B_T \) is semisimple; or \( B_T \) is weakly compact; or \( B \) has a bai.

**Proof.** To see that the semisimplicity condition cannot be removed, consider the long example in [6, Section 5] (or one could consider the Volterra operator, or the direct sum of the Volterra operator and a generator for \( c_0 \)). Theorem 1.1 in the present paper shows that the weak compactness condition cannot be removed, even if in addition \( B \) and \( B_T \) have cai.

Finally, we will show that the approximate identity condition cannot be removed, even if the algebra is ‘compact’. Our algebra \( A \) will be the space \( c \) of convergent sequences with product \( \vec{x} \cdot \vec{y} = (\frac{1}{n} x_n y_n) \) (an example also mentioned briefly by Mirikel in a Banach algebra context). Clearly \( A \) is generated by \( T = (1, 1, 1, \cdots) \) and \( c_{00} \). Since \( \vec{x} \cdot \vec{y} \) is the usual product of \( \vec{x}, \vec{y} \), and \( (\frac{1}{n^2}) \), \( A \) is a commutative operator algebra by [9, Remark 2 on p. 194]. The vectors \( 2^n \vec{e}_n \) are minimal idempotents in \( A \), inducing characters \( \chi_n \) on \( A \), and it is clear now that \( A \) is semisimple. Conversely, since any character on \( c_{00} \) must be induced by a sequence in \( \ell^1 \), it is easy to see that such a character is the restriction of one of the \( \chi_n \). It is also then easy that any character on \( A \) is one of the \( \chi_n \). We leave it as an exercise that \( T \) generates \( c_{00} \). Hence the spectrum of \( T \) is \( \{ \frac{1}{n} \} \cup \{0\} \) and the spectrum of \( A \) is homeomorphic to \( \mathbb{N} \). If \( E_n \) is the minimal spectral idempotent of \( T \) corresponding to \( \frac{1}{n} \) in the spectrum, then \( E_n \in A \) by an argument in Corollary 1.3. Thus these minimal spectral idempotents are exactly the \( 2^n \vec{e}_n \) above, by e.g. 1.4 Theorem 1.2. So \( A = B_T \) has discrete spectrum, but it is clearly not Tauberian.

To see that \( A \) is compact suppose that \( (\vec{x}(n))_n \) is a bounded sequence in \( c \). Then \( T \cdot \vec{x}(n) = (x(n)m / 2^m) \), which is the product of a fixed sequence in \( c_{00} \) with a bounded sequence in \( c_0 \). Since \( c_0 \) is compact, there is a convergent subsequence as desired.

**Corollary 1.5.** A semisimple, topologically singly generated operator algebra, whose socle has a bai, has a dense socle if and only if it is compact.

**Proof.** We first prove that, more generally, a semisimple, weakly compact, operator algebra with a topological single generator, whose socle has a bai, and which has discrete spectrum, has a dense socle (equivalently, is Tauberian). This follows from Corollary 1.3. Indeed if \( T \) is a topological single generator for such an algebra \( A \), then the spectrum of \( g \) (minus 0) is homeomorphic to the discrete spectrum of \( A \), so has no nonzero limit points. The minimal idempotents in \( A \), whose span defines the socle, define characters of \( A \), so, as in the proof of Corollary 1.3, they must be the minimal spectral idempotents of \( T \).
That a Banach algebra with dense socle is ‘compact’ is obvious (or see e.g. [17 Proposition 8.7.7]). Conversely, semisimple compact Banach algebras have discrete spectrum (see e.g. [17, Chapter 8]). Thus the first result follows from the last paragraph.

\[\square\]

Remark. We point out an error in [1] that momentarily led us astray early in this work. Namely, in the last assertion of [1, Theorem 5.10 (4)], to get a correct statement the characters there are not allowed to vanish on \(A\). This led to a mistaken comment at the end of the first paragraph of the Remark after Proposition 5.6 there, concerning the spectrum of \(A^{**}\). Fortunately these results have not been used elsewhere.

2. THE GENERAL CONSTRUCTION

Let \(A_0\) be the dense subalgebra of \(c_0\) generated by the unit vectors \(e_i\) and the vector \(g = \sum_{i=1}^{\infty} 2^{-i} e_i = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)\). We seek to renorm \(A_0\) so that its completion \(A\) is an operator algebra such that

1. \(A\) has a cai, and is topologically generated by \(g\).
2. the spectrum of \(g\) is \(\{2^{-n} : n \in \mathbb{N}\}\),
3. \(g \notin \text{lin}\{e_i : i \in \mathbb{N}\}\), and
4. \(A\) is semisimple.

Note that what forces us to change (increase!) the usual norm on \(c_0\) is condition (3). In fact the norm will be increased in such a way that the unit vectors \(e_i\), which will be the spectral idempotents for the generator \(g\), are unbounded. For \(n \in \mathbb{N}_0\), write \(P_n = \sum_{i=1}^{n} e_i\), and let

\[g_n = 2^n g \wedge 1 = (1, 1, 1, \ldots, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots) = \sum_{i=1}^{n} e_i + \sum_{i=1}^{\infty} 2^{-i} e_{i+n}.\]

Note that \(g_n \in 2^n g + \text{lin}\{e_1, \ldots, e_n\} \in A_0\).

Lemma 2.1. Let \(\| \cdot \|\) be any algebra norm on the algebra \(A_0\). Suppose that for some strictly increasing sequence \((\alpha_i)_{i=1}^{\infty} \subset \mathbb{N}\), we have

\[\| g^{\alpha_i}_n \| \leq 1 + \frac{1}{n}, \quad \text{and} \quad \| g^{\alpha_i}_n : g - g \| \leq \frac{1}{n}.\]

Then the vectors \(x_n = \frac{1}{\alpha_i+n} g^{\alpha_i}_n (I - P_{\alpha_i})\) are a cai for \((A_0, \| \cdot \|)\).

Proof. The conditions we are given ensure that \(\|x_n\| \leq 1\) and \(x_n g \rightarrow g\). But \(e_m = 2^m g e_m\), and so \(x_n e_m \rightarrow e_m\) also. The vectors \(e_m\) and \(g\) generate \(A_0\), and so \((x_n)_{n=1}^{\infty}\) is a cai. \(\square\)

If \(A\) is the completion of \(A_0\) in some algebra norm, we write \(\overline{A_0}\) for the closed ideal \(\overline{\text{lin}\{e_i : i \in \mathbb{N}\}}\) in \(A\).

Lemma 2.2. Once again, let \(\| \cdot \|\) be any algebra norm on the algebra \(A_0\), and let \(A\) denote the completion of \((A_0, \| \cdot \|)\). Suppose that for some strictly increasing sequence \((\alpha_i)_{i=1}^{\infty} \subset \mathbb{N}\), we have

\[\| g^{\alpha_i}_n (I - P_{\alpha_i}) \| \leq n^{-\alpha_i}.\]

Then the spectrum of \(g \in A\) is precisely \(\{2^{-n} : n \in \mathbb{N}\} \cup \{0\}\), and the spectral idempotents for the eigenvalues \(2^{-n}\), obtained from the analytic functional calculus for \(g\), are precisely the unit vectors \(e_n\).
Proof. For every \( x \in A_0 \) and \( n \in \mathbb{N} \) we have \( xe_n = \chi_n(x)e_n \) for a unique complex number \( \chi_n(x) \). Even in the completion \( A \) this will be true, because for \( y \in A \) the product \( ye_n \) is a limit of scalar multiples of \( e_n \), and so is a multiple of \( e_n \). We now see that \( \chi_n \) is a character on \( A \) with \( \chi_n(g) = 2^{-n} \). Thus \( 2^{-n} \in \text{Sp}(g) \).

Conversely, we claim that \( \text{Sp}(g) \) does not contain any \( \lambda \neq 0 \) that is not a negative integer power of 2. If it does, it contains such a \( \lambda \in \partial \text{Sp}(g) \), so \( \lambda \) is in the approximate point spectrum of \( g \). Pick \( n \) so large that \( 1/n < |\lambda| \). For \( y \in (I - P_{an})A \) with \( \|y\| = 1 \) we have by hypothesis that \( \|g^{an}y\| \leq n^{-an} < |\lambda|^{an} \). Thus \( \lambda \) is not in the spectrum of the operator of multiplication by \( g \) on \( (I - P_{an})A \). There is therefore an \( \eta > 0 \) such that \( \|gy - \lambda y\| \geq \eta \|y\| \) for all \( y \in (I - P_{an})A \). In the subalgebra

\[ P_mA = (e_1 + e_2 + \ldots e_m)A = \text{lin}\{e_j : j \leq m \} \]

the spectrum of \( g \) is \( \{2^{-j} : j \leq m \} \), so there is an \( \eta' \) such that \( \|g y' - \lambda y'\| \geq \eta' \|y'\| \) for \( y' \in P_mA \). Write \( m = a_n \) and let \( z \in A \). Then

\[ \|gz - \lambda z\| \geq \frac{1}{\|P_m\|} \cdot \|gP_m z - \lambda P_m z\| \geq \eta' \|P_m z\| / \|P_m\| \]

and also

\[ \|gz - \lambda z\| \geq \frac{1}{1 + \|P_m\|} \cdot \|g(1 - P_m)z - \lambda(1 - P_m)z\| \geq \eta' \|(1 - P_m)z\| / (1 + \|P_m\|) \].

Also \( \|gz - \lambda z\| \geq C\|P_m z + (z - P_m z)\| = C\|z\| \), for some positive constant \( C \). Therefore \( \lambda \) is not in the approximate point spectrum of \( g \). This contradiction proves the claim; that is, \( \text{Sp}(g) = \{0\} \cup \{2^{-n} : n \in \mathbb{N} \} \).

To identify the spectral idempotent for the point \( 2^{-n} \), we decompose the unitization \( A^1 \) as a direct sum of ideals \( P_{an} A^1 \oplus (I - P_{an}) A^1 \), where \( m \) is much larger than \( 2^n \). There is a corresponding decomposition of the spectral idempotent as a sum of the spectral idempotent in \( P_{an} A^1 \), which is easy to see is \( e_n \), and the spectral idempotent in \( (I - P_{an}) A^1 \). The latter is zero since by hypothesis, \( \|g^{an}(I - P_{an})\| \leq m^{-an} \), so that the spectral radius of \( g(I - P_{an}) \) is much smaller than \( 2^{-n} \). (One may also use this in conjunction with the criteria in [10, Theorem 1.2]). \( \square \)

Corollary 2.3. If the conditions of Lemma 2.2 hold, then \( A \) is singly generated by \( g \), and the space of characters of \( A \) is \( \{\chi_n : n \in \mathbb{N}\} \), with \( \chi_n \) as defined above.

Proof. The spectral idempotents \( e_n \) are, by the functional calculus, in the closed algebra \( \text{oa}(g) \) generated by \( g \) (note that \( e_n \in \text{oa}(1, g) \) by e.g. [10, Theorem 2.4.4 (ii)]), but \( e_n = 2^n ge_n \), so that \( e_n \) is in \( \text{oa}(g) \). Together with \( g \) itself, these idempotents generate \( A_0 \) algebraically; so \( A \) is singly generated by \( g \). As in the proof of Corollary 1.3 the characters \( \chi_n \) constitute the character space of \( A \). \( \square \)

Remark. It is easy to find operator algebra norms so that (1)-(4) at the start of Section 2 hold with the exception of \( A \) having a cai. The last example in the proof of Theorem 1.4 is as such. Also the operator algebra norm in [2, Example 4.30] can easily seen to work (with the help of the last two results).

3. Maximal norms

Definition 3.1. Let a strictly increasing sequence \( (a_n)_{n=1}^\infty \subset \mathbb{N} \) be given. Set \( a_0 = 1 \), and define a subset \( S_0 \subset c_0 \) as follows:
Also, let $S$ be the collection of all finite products of elements of $S_0$.

Note: later on, we are going to impose growth conditions on this underlying sequence. Our main results will happen provided the underlying sequence $(a_n)$ increases sufficiently rapidly. But for now, we note that $S_0$ includes both $g$ and nonzero multiples of the unit vectors $e_n$, so $S_0$ generates $A_0$ algebraically; indeed the linear span of $S$ is $A_0$. Thus, we get a finite seminorm on $A_0$ if we define

$$
(3.2) \quad \|x\|_{\text{max}} = \inf \{ \sum_{i=1}^{n} |\lambda_i| : x = \sum_{i=1}^{n} \lambda_i s_i : n \in \mathbb{N}, \lambda_i \in \mathbb{C}, s_i \in S \}.
$$

It will eventually transpire that, given growth conditions on the $a_n$, the completion of $A_0$ in the norm $\| \cdot \|_{\text{max}}$ gives a Banach algebra satisfying (1)–(3) at the start of Section 2, and that the quotient of this by the radical satisfies (1)–(4) there.

**Lemma 3.2.** Let $S$, $S_0$ be defined as in Definition 3.1. Then the seminorm $\| \cdot \|_{\text{max}}$ in (3.2) is a norm greater than or equal to the $c_0$-norm $\| \cdot \|_0$. Indeed $\| \cdot \|_{\text{max}}$ is the largest seminorm on $A_0$ such that $\|s\|_{\text{max}} \leq 1$ for all $s \in S$; and $\| \cdot \|_{\text{max}}$ is equal to the largest algebra seminorm on $A_0$ such that $\|s\|_{\text{max}} \leq 1$ for all $s \in S_0$.

If $\| \cdot \|$ is any algebra norm on $A_0$ with $\| x \|_0 \leq \| \cdot \| \leq \| \cdot \|_{\text{max}}$, then writing $A$ for the completion of $(A_0, \| \cdot \|)$, A has a cai, the spectrum of $g \in A$ is precisely $\{2^{-n} : n \in \mathbb{N}\} \cup \{0\}$, and the spectral idempotents for the eigenvalues $2^{-n}$, obtained from the analytic functional calculus for $g$, are the unit vectors $e_n$. Finally, $A$ is singly generated by $g$.

**Proof.** Expression (3.2) is at least the $c_0$-norm if and only if every $x \in S$ has $c_0$-norm at most 1, if and only if every $x \in S_0$ has $c_0$-norm at most 1.

Looking at the definition of $S_0$, it is obvious that the vectors $g$ and $a_n^{-1}e_n$ and $\frac{n}{n+1}g^{a_n}_n$ have $c_0$-norm at most 1; also the $\ell^\infty$-norm of $g^{a_n}_n - 1$ is 1, but its $j$th entry is zero for $j < a_n$, hence the $c_0$-norm of $(g^{a_n}_n \cdot g - g)$ is at most $\|g(1 - P_{a_n})\|_0 = 2^{-(1+a_n)} < \frac{1}{n}$. So $\|n(g^{a_n}_n \cdot g - g)\|_0 < 1$ also. Finally the $c_0$-norm of $g^{a_n}(1 - P_{a_n})$ is $2^{-a_n(1+a_n)}$ so since $a_n \geq n$, the $c_0$-norm of $n^{a_n}g^{a_n}(1 - P_{a_n})$ is at most 1 also. Thus $\|x\|_0 \leq 1$ for every $x \in S_0$, and $\|x\| \geq \|x\|_0$ for every $x \in A_0$.

Let $\| \cdot \|$ be any algebra seminorm on $A_0$ such that $\|s\| \leq 1$ for all $s \in S_0$. Then plainly we have $\|s\| \leq 1$ for all $s \in S$. And given $\| \cdot \|$ is a norm such that $\|s\| \leq 1$ for all $s \in S$, plainly we must have $\|s\| \leq \|s\|_{\text{max}}$ as given in (3.2).

Every element in the set $S$ has $\| \cdot \|_{\text{max}}$ norm at most 1, so $\| \cdot \|_{\text{max}}$ is indeed the maximal seminorm with this property, as claimed in the lemma. Also, if $\| \cdot \|$ is any algebra norm with $\| x \|_0 \leq \| \cdot \| \leq \| \cdot \|_{\text{max}}$, we have

$$
(3.3) \quad \|g\| \leq 1, \quad \text{and} \quad \|e_n\| \leq a_n
$$

for each $n \in \mathbb{N}$; also

$$
\|g^{a_n}_n\| \leq 1 + \frac{1}{n}, \quad \|g^{a_n}_n \cdot g - g\| \leq \frac{1}{n}, \quad \text{and} \quad \|g(1 - P_{a_n})\| \leq n^{-a_n}.
$$

By Lemma 2.1 the vectors $x_n = \frac{n}{n+1}g^{a_n}_n$ are a cai for $A$. By Lemma 2.2 the spectrum of $g \in A$ is precisely $\{2^{-n} : n \in \mathbb{N}\} \cup \{0\}$, and the spectral idempotents for the eigenvalues $2^{-n}$, obtained from the analytic functional calculus for $g$, are the unit vectors $e_n$. By Corollary 2.3 $A$ is topologically generated by $g$. \qed
Corollary 3.3. Let $\| \cdot \|$ be an algebra norm on $A_0$ with $\| \cdot \|_0 \leq \| \cdot \| \leq \| \cdot \|_{\text{max}}$, and let $A$ be the completion of $(A_0, \| \cdot \|)$, and $J$ the Jacobson radical of $A$. Let $q : A \rightarrow A/J$ be the quotient map. Then the norm $\| \cdot \|$ on $A_0$ with $\| x \| = \| q(x) \|$ also satisfies $\| \cdot \|_0 \leq \| \cdot \| \leq \| \cdot \|_{\text{max}}$, and the conclusions of Lemma 3.2 are also satisfied when $A$ is replaced by the semisimple Banach algebra $A/J$.

**Proof.** Since $A$ satisfies the conditions of the previous result, and therefore of Corollary 2.3 too, the characters on $A$ are the $\chi_n$ mentioned there. Of course $J$ is the intersection of the kernels of the characters of $A$, hence $A_0 \cap J = \{0\}$. Therefore the quantity $\| x \| = \| q(x) \|$ is a norm on $A_0$, and it is dominated by $\| \cdot \|_{\text{max}}$ clearly.

It is only necessary to show that $\| x \| \geq \| x \|_{0}$; for then $A/J$ is the completion of $(A_0, \| \cdot \|)$ and the conclusions of Lemma 3.2 will follow for the norm $\| \cdot \|$. But $\| x \|$ dominates $\sup \{ |\chi(x)| : \chi$ is a character of $A \} = \| x \|_{0}$. □

4. A general (Banach algebraic) theorem

Write $\Delta_n = P_{a_{n+1}} - P_{a_n}$, and

$H_n = \lim \{ e_j : a_n < j \leq a_{n+1} \} = \Delta_n A_0$.

A basis for the dual of this vector space is the characters $\{ \chi_j : j = a_n + 1, \ldots, a_{n+1} \}$. For these $j$, we have $\chi_j = \chi_j \circ \Delta_n$.

We have the following general theorem, which is part of what we need, but it does not necessarily give an operator algebra, merely a Banach algebra. However we will use it later to get an operator algebra.

**Theorem 4.1.** Let a strictly increasing sequence $(a_n)$ be given, and let $A$ be the completion of $A_0$ in an algebra norm $\| \cdot \|$, where $\| \cdot \|_0 \leq \| \cdot \| \leq \| \cdot \|_{\text{max}}$, with $\| \cdot \|_0$ being the $c_0$-norm, and $\| \cdot \|_{\text{max}}$ the maximal norm as defined in 3.2. Suppose in addition that there is a bounded sequence $(\psi_n)_{n=1}^\infty \in A^*$ such that $\psi_n(g) = 1$ and $\psi_n = \psi_n \circ \Delta_n$ for each $n$. Let $B = A/\text{rad}A$. Then $B$ has the following properties.

1. $B$ is a Banach algebra with cai, topologically generated by $g$.
2. $g$ is a contraction with spectrum $\{ 2^{-n} : n \in \mathbb{N} \}$.
3. $g \notin \overline{\text{lin}} \{ e_i : i \in \mathbb{N} \}$ and
4. $B$ is semisimple.

**Proof.** That $B$ has a cai, and the spectrum of $g$ is as stated, and that $B$ is topologically generated by $g$, follows from Corollary 3.3. Certainly $B = A/\text{rad}A$ is semisimple, so it remains to show that $g \notin \overline{\text{lin}} \{ e_i : i \in \mathbb{N} \}$. To this end, let $\psi \in A^*$ be any weak-* accumulation point of the (by hypothesis bounded) sequence $\psi_n$. Since $\psi_n = \psi_n \circ \Delta_n$, we have $\psi_n \in \overline{\text{lin}} \{ \chi_j : a_n < j \leq a_{n+1} \}$. Hence each $\psi_n$ annihilates $\text{rad}A$, so $\psi$ annihilates $\text{rad}A$, and yields a well defined element of $B^*$. Since $\psi_n(e_j) = 0$ for $j \leq a_n$ and $a_n \rightarrow \infty$, we have $\psi(e_j) = 0$ for all $j \in \mathbb{N}$. But $\psi_n(g) = 1$ for all $n$, so $\psi(g) = 1$. Therefore $g \notin \overline{\text{lin}} \{ e_i : i \in \mathbb{N} \}$. □

5. Representations on Hilbert space

It is time to construct norms which will presently turn $A_0$ into a nontrivial operator algebra. These norms will be somewhat universal, in the sense that they are defined so as to encode abstractly the critical hypotheses in the Lemmas in Section 2 which ensure that conditions (1) and (2) at the start of that Section hold (after which we will proceed to prove that (3) and (4) also hold).
For \( k \in \mathbb{N}_0 \), write \( \gamma_k^{(n)} = \sum_{j=1}^{g_n+1} 2^{-jk} e_j \). In the rest of our paper we will very often silently use the relation
\[
\gamma_k^{(n)} \gamma_{k+1}^{(n)} = \gamma_k^{(n)}.
\]
The reader can check the following relations: We have \( \Delta_n g = \gamma_1^{(n)} \), and \( \Delta_n g_{a_j} = 2^{a_j^2} \gamma_j^{(n)} \) if \( j \leq n \), while \( \Delta_n g_{a_j} = \gamma_0^{(n)} \) if \( j > n \). So \( \Delta_n g_{a_j} \cdot g - g = 2^{a_j^2} \gamma_j^{(n)} - \gamma_1^{(n)} \) if \( j \leq n \), and is zero if \( j > n \). Similarly, \( \Delta_n g_{a_j} (1 - P_n) = \gamma_0^{(n)} \) if \( j \leq n \), and is zero if \( j > n \). Let us also write
\[
\Lambda_n = \{ \sum_{i=1}^{n} t_i a_i : t_i \in \mathbb{N}_0, t_i \leq a_i \text{ for } 1 = i, \ldots, n \},
\]
and
\[
\xi_n = \max \Lambda_n = \sum_{i=1}^{n} a_i^2.
\]
We shall assume that the sequence \( (a_n) \) increases sufficiently fast that these sums are distinct for distinct sequences \((t_i)\), and that they appear in “lexicographic order”. So, we assume that, for \( t_i \in \mathbb{Z}, \ |t_i| \leq 2a_i \), we have \( \sum_{i=1}^{n} t_i a_i > 0 \) if and only if \( t_r > 0 \), where \( r = \max\{ j : t_j \neq 0 \} \). This condition, slightly stronger than our immediate need, will also ensure that elements of the set \( \Lambda_n - \Lambda_n = \{ \sum_{i=1}^{n} t_i a_i : -a_i \leq t_i \leq a_i \} \) also appear in “lexicographic” order.

We note that any collection of \( m \) of the vectors \( \gamma_k^{(n)} \) is linearly independent, provided \( m \leq a_{n+1} - a_n \). So a linear functional \( \phi \in \text{lin}\{ \varepsilon_j : a_n < j \leq a_{n+1} \} \) is specified uniquely by its action on \( \{ \gamma_k^{(n)} : 0 \leq k < a_{n+1} - a_n \} \). Let \( \phi_n \) be the unique such functional such that for all \( 0 \leq k < a_{n+1} - a_n \),
\[
(5.1) \quad \phi_n(\gamma_k^{(n)}) = \begin{cases} \prod_{i=1}^{n} 2^{-i a_i^2} (1 - t_i/a_i) & \text{if } k = 1 + \sum_{i=1}^{n} t_i a_i \in 1 + \Lambda_n, \\ 0 & \text{otherwise}. \end{cases}
\]
Here the \( t_i \in \mathbb{N}_0 \) with \( t_i \leq a_i \) of course, and \( \phi_n(\gamma_1^{(n)}) = 1 \). We may view \( \phi_n \) as a functional on \( A \) satisfying \( \phi_n = \phi_n \circ \Delta_n \). Therefore from (5.1) we have that
\[
\phi_n(g) = \phi_n(\Delta_n g) = \phi_n(\gamma_1^{(n)}) = 1.
\]
However, for \( j \leq a_n \) we have \( \phi_n(\varepsilon_j) = 0 \). We could go on from here and prove directly that the \( \phi_n \) are uniformly \( \| \cdot \|_{\max} \)-bounded, whereupon we could apply Theorem 4.1 but this would not get us an operator algebra. Instead we proceed as follows.

We have established that \( \Delta_n g = \gamma_1^{(n)} \), and \( \Delta_n g_{a_j} = 2^{a_j^2} \gamma_j^{(n)} \) if \( j \leq n \), while \( \Delta_n g_{a_j} = \gamma_0^{(n)} \) if \( j > n \). Also, \( \Delta_n \varepsilon_j = \varepsilon_j \) if \( a_n < j \leq a_{n+1} \), and is zero otherwise. Now \( \gamma_0^{(n)} \) is the identity of \( \Delta_n A_0 \), so referring to (3.1), and removing from \( \Delta_n S_0 \) positive scalar multiples of the identity or of other elements of \( \Delta_n S_0 \), we are left with the set
\[
(5.2) \quad \{ \gamma_1^{(n)}, a_i^{-1} e_i, j + \frac{1}{j} 2^{a_j^2} \gamma_j^{(n)}, j(2^{a_j^2} \gamma_j^{(n)} - \gamma_1^{(n)}) : a_n < i \leq a_{n+1}, 1 \leq j \leq n \}.
\]

**Definition 5.1.** The set given by (5.2) will be called \( S_0^{(n)} \). Let \( I^{(n)} \) denote the set of all “index functions” \( i : S_0^{(n)} \to \mathbb{N}_0 \). For \( i \in I^{(n)} \), write \( s^i \) for the product
\(\prod_{s_i \in S_0^{(n)}} s_i^{(s)}\). Equip \(H_n\) with a Euclidean seminorm \(\| \cdot \|_2^{(n)}\) as follows. For \(x \in H_n\), we define

\[
\|x\|_2^{(n)} = \left( \sum_{i \in X^{(n)}} |\phi_n(s_i^i x)|^2 \right)^{1/2}.
\]

We shall establish that \(\|x\|_2^{(n)}\) is finite, so that we do indeed have a Euclidean seminorm. Letting \(H_n\) denote the associated Euclidean space, we shall represent each \(T \in A_0\) in \(B(H_n, \| \cdot \|_2^{(n)})\) by its compression \(\Delta_n T\). This representation will be called \(\rho_n : A_0 \to B(H_n, \| \cdot \|_2^{(n)})\), and the operator norm \(\|\rho_n(T)\|\) will be called \(\|T\|_{\text{op}}^{(n)}\).

The basic Lemma we shall prove is as follows:

**Lemma 5.2.** For every \(x \in H_n\), we have \(\|x\|_2^{(n)} < \infty\), provided our underlying sequences satisfy certain growth conditions. The representation \(\rho_n\) above on \(H_n\) is well defined, and the operator norm \(\|T\|_{\text{op}}^{(n)} \leq 1\) for every \(T \in S_0\).

**Proof.** Consider \(H_n\) as an algebra with pointwise product. The spectral radius here is the \(c_0\)-norm, and for the various elements of \(S_0^{(n)}\) it is as follows:

\[
\left\| \gamma_1^{(n)} \right\| = 2^{-1-a_n}, \quad \left\| a_i^{-1} e_i \right\| = a_i^{-1}, \quad \left\| \frac{j}{j+1} 2^{a_j a_n} \gamma_j^{(n)} \right\| = \frac{j}{j+1} 2^{a_j (a_j - a_n - 1)},
\]

and

\[
\left\| j(2^{a_j a_n} \gamma_1^{(n)} - \gamma_1^{(n)}) \right\|_0 \leq j(2^{a_j (a_j - a_n)} + 2^{-1-a_n}).
\]

A mild growth condition on the \((a_j)\) will ensure that for all \(n\),

\[
2^{-1-a_n} + \sum_{i=a_n+1}^{a_{n+1}} a_i^{-1} + \sum_{j=1}^{n} (j+1)(2^{a_j (a_j - a_n)} + 2^{-1-a_n}) < 1.
\]

In particular, the spectral radii of the elements of \(S_0^{(n)}\) are then all strictly less than 1. It is a nice exercise that on any commutative algebra with a finite set of generators of spectral radius < 1, we can pick an algebra norm such that \(|\|s\|| < 1\) for all generators \(s\) simultaneously (Hint: let \(G_0\) be the set of generators, each multiplied by \(1 + \epsilon\), such that the semigroup \(G\) generated by \(G_0\) is bounded. Renorm \(A\) in the usual way so that \(G \subset \text{Ball}(A)\), see e.g. [17, Proposition 1.1.9]). With respect to such a norm on \(H_n\), the square of the sum in (5.3) is at most

\[
\sum_i \prod_{s \in S_0^{(n)}} |\|s\|^{2i(s)} \cdot ||\phi_n||^2 = ||\phi_n||^2 \prod_{s} (1 - ||s||^2)^{-1} < \infty.
\]

So the expression \(\|x\|_2^{(n)}\) in (5.3) is finite.

It is clear from (5.3) that for \(s \in S_0^{(n)}\) and \(x \in H_n\), the terms in the sum defining \(\|x\|_2^{(n)}\) include all those in the sum defining \(\|sx\|_2^{(n)}\) (plus certain extra terms, namely \(|\phi_n(s_i^i x)|^2\) for index functions \(i\) such that \(i(s) = 0\)). Therefore \(\|sx\|_2^{(n)} \leq \|x\|_2^{(n)}\), and \(|s||^{(n)}_{\text{op}} \leq 1\) for \(s \in S_0^{(n)}\). And for \(s \in S_0\), we have \(|s||^{(n)}_{\text{op}} = \|\Delta_n s||^{(n)}_{\text{op}}\), which is either a multiple of the identity \(\gamma^{(n)}_0\) of magnitude less than 1, or an element of \(S_0^{(n)}\) (again possibly multiplied by a positive scalar of magnitude less than 1). Thus \(|s||^{(n)}_{\text{op}} \leq 1\) for all \(s \in S\), as required. The last few lines also show that \(\rho_n\) above is well defined.
It will be much harder to prove the next result. In fact this proof will take up almost all of the rest of our paper.

Lemma 5.3. Given growth conditions on our underlying sequences, we have \( 0 < \| \gamma_1^{(n)} \|_2 \leq 3 \cdot \| \gamma_0^{(n)} \|_2 \) for all \( n \). We have \( \| g \|_{\text{op}}^{(n)} \geq \frac{1}{3} \).

Taking this lemma on faith for now, the rest of our assertions follow rather easily:

Theorem 5.4. Given growth conditions on the \( a_n \), the operator algebra norm defined in (5.4) is at most \( \| \cdot \|_{\text{max}} \). The completion \( A \) of \( A_0 \) in this norm is an operator algebra satisfying all of the conditions (1)–(4) at the start of Section 2.

**Proof.** We can impose on \( A_0 \) the norm
\[
(5.4) \quad \| T \| = \sup_{n \in \mathbb{N}_0} \| T \|_{\text{op}}^{(n)},
\]
where \( \| T \|_{\text{op}}^{(n)} = \| T \|_n \). The completion \( A \) of \( (A_0, \| \cdot \|) \) will be an operator algebra whose norm lies between \( \| \cdot \|_0 \) and \( \| \cdot \|_{\text{max}} \) as in Lemma 3.2 (for \( \| \cdot \|_{\text{max}} \) is the largest norm on \( A_0 \) such that \( \| s \|_{\text{max}} \leq 1 \) for all \( s \in S_0 \)). Furthermore, since \( \| g \|_{\text{op}}^{(n)} \geq \frac{1}{3} \) for each \( n \) by Lemma 5.3 there is a linear functional \( \psi_n \in (A_0, \| \cdot \|_{\text{op}}^{(n)})^* \) such that \( \| \psi_n \| \leq 3 \) and \( \psi_n(g) = 1 \). Finally, \( A \) is semisimple (there is no need to quotient out by \( \text{rad}A \)), because the operator norm \( \| T \|_{\text{op}}^{(n)} \) is zero unless one of the characters \( \chi_j(T) \neq 0 \) for some \( j \) with \( \alpha_n < j \leq \alpha_{n+1} \). Thus \( (A, \| \cdot \|) \) satisfies the requirements of Theorem 4.1 but it is clearly an operator algebra.

□

Corollary 5.5. If \( A_{\text{max}} \) is the completion of \( A \) in the maximal norm defined in (3.2), and given growth conditions on the \( a_n \), then \( A_{\text{max}} \) (resp. \( A_{\text{max}}/\text{rad}A_{\text{max}} \)) is a Banach algebra satisfying the desired conditions (1)–(3) (resp. (1)–(4)) at the start of Section 2.

**Proof.** By Lemma 3.2 and Corollary 5.3 \( A_{\text{max}} \) (resp. \( A_{\text{max}}/\text{rad}A_{\text{max}} \)) satisfy (1)–(2) (resp. (1)–(2) and (4)). If \( \psi_n \) is as in the proof of Theorem 5.4 and if \( i : A_{\text{max}} \to A \) is the canonical contraction due to \( \| \cdot \|_{\text{max}} \) being a larger norm, then \( \psi_n \circ i \) satisfies the conditions of Theorem 4.1 with \( A \) replaced by \( A_{\text{max}} \). Then Theorem 4.1 or its proof implies that (3) also holds.

□

**Remark.** In a previous draft Corollary 5.5 was proved directly, but for reasons of space this (very lengthy) computation has been omitted.

6. ADDITIONAL PROPERTIES OF OUR ALGEBRAS \( A \) AND \( A_{00} \)

In this section we take Lemma 5.3 on faith, so that our main theorems above hold for our algebra \( A \).

Let \( H = \ell^2 \oplus (\oplus_{n \in \mathbb{N}} H_n) \), and write \( \rho \) for the representation of \( A \) that we have already constructed. Namely
\[
\rho(a) = \Gamma(a) \oplus (\oplus_{n \in \mathbb{N}} \rho_n(a)).
\]
Here \( \Gamma \) is the Gelfand transform, mapping into \( c_0 \), but with \( c_0 \) viewed as ‘diagonal’ operators on \( \ell^2 \) in the usual way. Note that \( \rho \) is initially defined on \( A_0 \), and is easily seen to be a nondegenerate representation of \( A_0 \). The norm on \( A \) was defined in such a way that \( \rho \) extends to a (completely) isometric representation of \( A \), which we will continue to write as \( \rho \), and which is still nondegenerate.
Lemma 6.1. The algebra \( \overline{A}_{00} \) has a contractive approximate identity. Also \( A \) is a subalgebra of the closure of \( \overline{A}_{00} \) in the strong operator topology of \( B(H) \) for \( H \) as above.

Proof. Indeed \( (\rho(P_{a_k+1})) \) is a cai for \( A_{00} \), and hence also for \( \overline{A}_{00} \), since \( \rho_n(P_{a_k+1}) = I_{H_k} \) if \( k \geq n \), and is zero for \( k < n \).

Clearly \( \rho(P_{a_k+1}) \to I \) strongly on \( H \), and hence \( \rho(gP_{a_k+1}) = \rho(g)\rho(P_{a_k+1}) \to \rho(g) \) strongly.

Hence \( \overline{A}_{00} \) is a (complete) \( M \)-ideal in its bidual by \([3\) Theorem 4.8.5], with all the consequences that this brings (see e.g. \([12\) Chapter 3] and \([1\) Theorem 5.10]). For example, every character of \( \overline{A}_{00}^{**} \) is weak* continuous by \([1\) Theorem 5.10 (4)]. Note that the explicit representation \( \rho \) of \( \overline{A}_{00} \) given above shows that \( \overline{A}_{00} \) is a subalgebra of the compact operators on \( H \) (since each \( \rho(e_n) \) is compact on \( H \)). The characters on \( \overline{A}_{00} \) are again the \( \chi_n \) above of course, since for any such character \( \chi \) we must have \( \chi(e_n) \neq 0 \) for some \( n \), and then \( \chi(e_m) = 0 \) for all other \( m \in \mathbb{N} \), so that \( \chi = \chi_n \). The spectrum of \( \overline{A}_{00} \) is thus the same as the spectrum of \( A \) (and equals the spectrum of \( \overline{A}_{00}^{**} \) by a point above). Note that the \( (\rho(P_{a_k+1})) \) above is a (contractive) spectral resolution of the identity.

Corollary 6.2. The operator algebra \( A \) constructed in Section 5 is not compact or weakly compact. Thus \( A \) is not an ideal in its bidual.

Proof. See e.g. \([1\) Lemma 5.1] for the well known equivalence between being weakly compact and being an ideal in the bidual.

The rest follows from results at the end of the Introduction, but we give a direct proof. Assume by way of contradiction that multiplication by \( g \) on \( A \) (or, for that matter, on \( \overline{A}_{00} \)) is weakly compact. Suppose that \( (f_k) \) is a bai for \( \overline{A}_{00} \). Since \( \overline{A}_{00} \) is weakly closed, there is a subsequence \( g_{f_k} \to a \in \overline{A}_{00} \) weakly, say. Thus \( g_{f_k} e_m \to a e_m \) in norm for each \( m \in \mathbb{N} \). However \( g_{f_k} e_m \to ge_m \) in norm since \( (f_k) \) is a bai. Thus \( (g-a)e_m = 0 \) for every \( m \), yielding the contradiction \( g = a \in \overline{A}_{00} \).

The proof of Theorem 5.3 is now complete, except for Lemma 6.3 and the very last assertion, about obtaining \( g \notin \overline{A}_{00}^{SOT} \). To see the latter, we will change the Hilbert space \( A \) acts on. Suppose that \( A \) generates a \( C^* \)-algebra \( B \). Then \( \overline{A}_{00} \) generates a proper \( C^* \)-subalgebra \( B_0 \) of \( B \) (since if \( B_0 = B \) then the cai of \( \overline{A}_{00} \) would be a cai for \( A \) by \([3\) Lemma 2.1.7 (2)], and this is false). Let \( B \subset B(K) \) be the universal representation, so that \( B^{**} \) may be represented as a von Neumann algebra \( M \) on \( K \). Then \( M = \overline{B}^{w*} = \overline{B}^{SOT} \) by von Neumann’s double commutant theorem. If \( A \subset \overline{B}^{SOT} \), then

\[
A \subset \overline{B}_{0}^{SOT} = \overline{B}_{0}^{w*},
\]

so that \( B^{**} \cong \overline{B}^{w*} \subset \overline{B}_{0}^{w*} \). This implies that \( B_{0}^{\perp} = B^{**} \), and we obtain the contradiction \( B = B_{0} \).

Remark. Probably a modification of our construction in Section 5 would produce a representation in which we would explicitly have \( g \notin \overline{A}_{00}^{SOT} \). We had a more complicated construction for Theorem 5.3 in a previous draft for which this perhaps may have been true.
We recall that for a commutative semisimple Banach algebra the following are equivalent: (i) $A$ is a modular annihilator algebra; (ii) the Gelfand spectrum of $A$ is discrete; (iii) no element of $A$ has a nonzero limit point in its spectrum; and (iv) for every $a \in A$, multiplication on $A$ by $a$ is a Riesz operator (see [17, Theorem 8.6.4 and Proposition 8.7.8] and [16, p. 400]). Thus $A$ and $\overline{A_{00}}$ are modular annihilator algebras. It follows that our algebra $A$ is a commutative solution to a problem raised in [7]: is every semisimple modular annihilator algebra with a cai, weakly compact?

In [8] we found a much simpler (but still deep) noncommutative counterexample to the latter question, an example with some interesting noncommutative features.

**Theorem 6.3.** The operator algebras $A$ and $\overline{A_{00}}$ constructed above have the following additional properties:

(a) Every maximal ideal in $\overline{A_{00}}$, and every maximal modular ideal in $A$, has a bounded approximate identity.

(b) $A$ and $\overline{A_{00}}$ are regular natural Banach function algebra (in the sense of [10, Section 4.1]) on $N$ or on $\{\frac{1}{n} : n \in \mathbb{N}\}$.

(c) $A$ is not Tauberian, nor is strongly regular or Ditkin, nor satisfies spectral synthesis (see [10,16] for definitions). On the other hand, $A_{00}$ does have all these properties, indeed it is a strong Ditkin algebra.

(d) $A$ is a semisimple modular annihilator algebra, while $\overline{A_{00}}$ is a dual algebra in the sense of Kaplansky (see e.g. [17, Chapter 8]).

(e) The closure of the socle of $A$ (or of $\overline{A_{00}}$) is $\overline{A_{00}}$, and $A$ is not an annihilator algebra in the sense of [14, Chapter 8].

(f) $A$ is not nc-discrete in the sense of [2]: indeed the support projection of $\overline{A_{00}}$ in $A^{* *}$ is open but not closed.

(g) $A$, and its multiplier algebra $M(A)$, may be identified completely isometrically with subalgebras of the multiplier algebra $M(\overline{A_{00}})$.

**Proof.** We only prove some of these assertions, leaving the others as exercises.

(a) The maximal ideals are the annihilators of the $e_m$, which have as a bai $(x_n - x_n e_m)$, where $(x_n)$ is a cai for $A$ or $A_0$.

(b) These follow easily from the definitions, and the identification of the characters of these algebras.

(c) First, $A$ is not Tauberian in the sense of e.g. [16, Definition 4.7.9]), because $\overline{A_{00}} \neq A$. This implies failure of spectral synthesis by e.g. [16, p. 385]. Similar arguments show the other assertions for $A$. The statements for $A_{00}$ are easy, or follow from [10, p. 419].

(d) We have already observed this for $A$. For $\overline{A_{00}}$ this follows from [10, Proposition 4.1.35], or from the observation whose proof we omit that for a natural Banach sequence algebra being ‘dual’ is equivalent to spectral synthesis holding, or to having ‘approximate units’ [10, Definition 2.9.10].

(e) The assertion for $A_{00}$ is clear. If $f$ is a minimal idempotent in $A \setminus A_{00}$, then $f e_n = 0$ for all $n \in \mathbb{N}$ (since $f e_n \in C f \cap C e = (0)$), and so $f = 0$.

(f) This is almost identical to the proof of [8, Corollary 7.13], except that we work with $e_n$ as opposed to the $e_{n i}^n$ there. Note that $\hat{\rho}(1 - p)e_n \neq 0$ for some $n$ because otherwise the (faithful) Gelfand transform of $\hat{\rho}(1 - p)a$ would be zero for all $a \in A$.

(g) In the explicit nondegenerate representation $\rho$ of $A$ given in the proof of Corollary 6.2 it is clear that $\rho(A)\rho(A_{00}) \subset \rho(\overline{A_{00}})$. More generally, if $T \rho(A) \subset \rho(A)$
that for every $n$ is true: for all $A$ our product (7.1) will be at least half of the product of $(j/a_j)^2$. Given growth conditions on our underlying sequences, the following assertion holds if $\rho$ characterizes $A$. To see this, let us choose positive $h_i$ such an $i$ where $\rho_i$ is very large indeed compared to $j$. We claim that growth conditions will ensure that this quantity will be, for each $n \in \mathbb{N}$, the product for some $\gamma_i$. For each such $s_j$, we have $i(s_j) < a_j$. For such an $i$, the product

$$s_i \gamma_i^{(n)} = \prod_{j=1}^{n} \left( \frac{j}{j+1} \right)^{a_i^2} \gamma_i^{(n)} \sum_{j=1}^{n} a_j i_j,$$

where $i_j = i(s_j)$. The value of $\phi_n(\gamma_i^{(n)} \sum_{j=1}^{n} a_j i_j)$ is given by (5.1), and it is $\prod_{j=1}^{n} 2^{-a_i^2} (1 - i_j / a_j)$. So the sum, for these $i \in I^{(n)}$, of $|\phi_n(s_i \gamma_i^{(n)})|^2$, is precisely

$$\sum_{i=0}^{a_j} \prod_{j=1}^{n} \left( \frac{j}{j+1} \right)^{i_j} (1 - i_j / a_j))^2 = \prod_{j=1}^{n} \sum_{i=0}^{a_j} \left( \frac{j}{j+1} \right)^{2i} (1 - i_j / a_j)^2.\tag{7.1}$$

We may assume that $a_j$ is very large indeed compared to $j$. We claim that growth conditions will ensure that this quantity will be, for each $n$, at least half of the sum

$$\prod_{j=1}^{n} \sum_{i=0}^{a_j} \left( \frac{j}{j+1} \right)^{2i} = \prod_{j=1}^{n} \sum_{i=0}^{a_j} \left( \frac{j}{j+1} \right)^{2i} \sum_{j=1}^{n} \left( \frac{j}{j+1} \right)^{2i} = \prod_{j=1}^{n} \left( \frac{j}{j+1} \right)^{2j+1}.$$

To see this, let us choose positive $h_k < 1$ such that $h_k > 1/2$ for all $m \in \mathbb{N}$. Our product (7.1) will be at least half of the product of $(j+1)^2/(2j+1)$, provided that for every $j = 1, \ldots, n$, we have

$$\sum_{i=0}^{a_j} \left( \frac{j}{j+1} \right)^{2i} (1 - i_j / a_j)^2 \geq h_j \cdot \sum_{i=0}^{a_j} \left( \frac{j}{j+1} \right)^{2i}.$$

The sum $\sum_{i=0}^{a_j} (j/j+1)^{2i}$ converges, and we may think of it as an integral with respect to counting measure. If $f_a(i) = 1 - a/i$ for $i < a$, and is zero for $i \geq a$, then $\int f_a$
is uniformly bounded and converges to 1 pointwise. Therefore, by the Lebesgue dominated convergence theorem,
\[ \lim_{a \to \infty} \sum_{i} (j/j + 1)^{2i} f_{a}(i)^{2} = \sum_{i} (j/j + 1)^{2i}. \]
If each \( a_{j} \) is chosen large enough we therefore have for \( j = 1, \ldots, n \) that
\[ \sum_{i} (j/j + 1)^{2i} f_{a_{j}}(i)^{2} = \sum_{i} (j/j + 1)^{2i}(1 - 1/a_{j})^{2} \geq (1 - h_{j}) \sum_{i} (j/j + 1)^{2i}. \]
This proves the claim. Finally, \( \|\gamma_{1}(n)\|_{2}^{2} \geq \frac{1}{2} \cdot \prod_{j=1}^{n} \frac{(j+1)^{2}}{2j+1} \), as desired. \( \Box \)

8. Strategy for an upper estimate for \( \|\gamma_{0}(n)\|_{2}^{(n)} \)

In the remainder of our paper we strive for an upper estimate for \( \|\gamma_{0}(n)\|_{2}^{(n)} \). Now
\[ \begin{align*}
\|\gamma_{0}(n)\|_{2}^{(n)}^{2} &= \|\gamma_{1}(n)\|_{2}^{(n)}^{2} + \sum_{i \in I_{0}^{(n)}} |\phi_{n}(s)|^{2}, \\
\end{align*} \]
where \( I_{0}^{(n)} = \{ i \in I^{(n)} : i(\gamma_{1}(n)) = 0 \} \). Let us write
\[ I_{0}^{(n)} = I_{1}^{(n)} \cup I_{2}^{(n)} \cup I_{3}^{(n)}, \]
with
\[ I_{1}^{(n)} = \{ i \in I^{(n)} : i(\gamma_{1}(n)) = i(a_{i}^{-1} e_{i}) = 0 \text{ for all } i, \text{ and } |i| = \sum_{s} i(s) < \sqrt{a_{n+1}} \}, \]
and
\[ I_{2}^{(n)} = \{ i \in I^{(n)} : i(\gamma_{1}(n)) = i(a_{i}^{-1} e_{i}) = 0 \text{ for all } i, \text{ but } |i| \geq \sqrt{a_{n+1}} \}, \]
and
\[ I_{3}^{(n)} = \{ i \in I^{(n)} : i(\gamma_{1}(n)) = 0, \text{ but } i(a_{i}^{-1} e_{i}) > 0 \text{ for some } i \}. \]

The main contribution towards the sum \( (8.1) \) that we must investigate, is from the sum over \( i \in I_{1}^{(n)} \). We will estimate this in the lengthy Section 9. In the much easier Sections 10 and 11 we estimate the contribution from \( I_{2}^{(n)} \) and \( I_{3}^{(n)} \) respectively, and in the final Section 12 we summarize why this proves our main result.

9. Bound on \( \sum_{i \in I_{1}^{(n)}} |\phi_{n}(s)|^{2} \)

Let \( i \in I_{1}^{(n)} \). Write
\[ E(i) = \{ j \in [1, n] : i(j) = 0, \text{ and } j(\gamma_{a_{j}}^{(n)}) > a_{j} \}. \]
Let \( \eta(i) \) be the set whose elements are of form
\[ \sum_{j \in E(i)} (\lambda_{j} a_{j}) + \sum_{j=1}^{n} (\mu_{j} + \nu_{j} a_{j}), \]
for integers
\[ \lambda_j = \frac{j}{j+1} \gamma_{(n)}^{(1+a_j)}, \quad 0 \leq \nu_j \leq \mu_j = \frac{i(j(2^{j+1} \gamma_{(1+a_j)} - \gamma_{(1)}))}{j}, \nu_j \in \mathbb{N}_0. \]

**Lemma 9.1.** Let \( i \in Z_{A}^{(1)} \). If \( \phi_n (s^i) \neq 0 \), then the set \( \eta(i) \) defined above must contain a positive element of the set \( 1 + \Lambda_n - \Lambda_n \).

**Proof.** For when \( 0 \leq k < a_{n+1} - a_n \), we have \( \phi_n (\gamma_{(n)}^{(k)}) = 0 \) unless \( k \in 1 + \Lambda_n \) by [5.1] (of course things are more complicated for larger \( k \)). Writing \( \lambda_j = \frac{i(j(2^{j} \gamma_{(1+a_j)} - \gamma_{(1)}))}{j+1} \), the product \( \prod_{j \notin E(i)} (\frac{j}{j+1} \gamma_{(1+a_j)} - \gamma_{(1)})^{\lambda_j} \) is a multiple of \( \gamma_{(n)} \), where \( k = \sum_{j \notin E(i)} \mu_j a_j \in \Lambda_n \). The full product \( s_i \) is equal to
\[ s^i = \lambda \cdot \gamma_{(n)} \cdot \prod_{j \in E(i)} (\frac{j}{j+1} \gamma_{(1+a_j)} - \gamma_{(1)})^{\lambda_j} \cdot \prod_{j=1}^{n} (j(2^{j} \gamma_{(1+a_j)} - \gamma_{(1)}))^{\mu_j}, \]
where again, \( \mu_j = \frac{i(j(2^{j} \gamma_{(1+a_j)} - \gamma_{(1)}))}{j} \). This is a linear combination of vectors \( \gamma_{(m)}^{(n)} \) for \( m \in k + \eta(i) \). Furthermore, since \( |i| < \sqrt{a_{n+1}} \), given a growth condition asserting \( a_{n+1} \) large compared to \( a_n \), there is never any vector \( \gamma_{(m)}^{(n)} \) involved when \( m \geq a_{n+1} - a_n \). So the set \( k + \eta(i) \) must meet the set \( 1 + \Lambda_n \), hence the result. \( \square \)

Having got Lemma 9.1 we want to separate out the cases when \( 1 \in \eta(i) \), from the cases when \( \eta(i) \) only contains larger elements of the set
\[ 1 + \Lambda_n - \Lambda_n = \{ 1 + \sum_{i=1}^{n} t_i a_i : -a_i \leq t_i \leq a_i \}. \]

Let us write \( m_0 (i) = \min(\eta(i) \cap (1 + \Lambda_n - \Lambda_n)) \), and let us begin with the more challenging case when \( m_0 = m_0 (i) > 1 \). We can then write \( m_0 = 1 + \sum_{i=1}^{r} t_i a_j \) with \( t_r > 0 \) and \( -a_j \leq t_j \leq a_j \) for all \( j \). In particular, \( m_0 \leq 1 + \xi_r \). With \( \lambda_j \) and \( \mu_j \) as above, we can write
\[ m_0 = \sum_{j \in E(i)} a_j \lambda_j + \sum_{j=1}^{n} (\mu_j + a_j \nu_j) \]
with \( 0 \leq \nu_j \leq \mu_j \). But then, we must have \( \nu_j = 0 \) for \( j > r \) otherwise the value of \( m_0 \) will be too big. Indeed \( \nu_r = 0 \) too, or we can get a smaller element of \( \eta(i) \cap (1 + \Lambda_n - \Lambda_n) \) by considering \( m_0 - a_r \). Again, we cannot have \( j \in E(i) \) for any \( j \geq r \) otherwise the value of \( m_0 \) is again too big (these are \( j \) such that \( \lambda_j > a_j \)). So, \( E(i) \subset \{ 1, r \} \) and
\[ m_0 = \sum_{j \in E(i) \subset \{ 1, r \}} \lambda_j a_j + \sum_{j=1}^{r-1} (\mu_j + a_j \nu_j) + \sum_{j=r}^{n} \mu_j. \]

Let us consider the vector
\[ x = (\gamma_{(1)}^{(n)})^{\sum_{j=r}^{n} \mu_j} \cdot \prod_{j=1}^{r-1} \left( \frac{j}{j+1} \gamma_{(1+a_j)}^{(n)} \right)^{\lambda_j} \cdot \prod_{j=1}^{n} (j(2^{j} \gamma_{(1+a_j)} - \gamma_{(1)}))^{\mu_j}. \]
Given a growth condition, we can certainly assume that \( m_0 \in ([t_{r-\frac{1}{2}} a_r, t_{r+\frac{1}{2}} a_r]) \).

For which \( i \in [0, a_{n+1} - a_n) \) does \( x \) have a nonzero coefficient for \( \gamma_{(n)}^{(i)} \)? (We may refer to the set of such \( i \) as “the \( \gamma \)-support of \( x \)”). Where does the \( \gamma \)-support of
\( x \) lie? From (9.2), the generic element of that support is a sum 
\( m = \sum_{j=1}^{n}(\nu_j + \nu'_j a_j) + \sum_{j=1}^{r-1} \lambda_j a_j \), where \( 0 \leq \nu'_j \leq \mu_j \). The difference between \( m_0 \) (as given by (9.1)) and this expression, is a sum \( \sum_{j=1}^{r-1}(\nu'_j - \nu_j) a_j \) (with \( 0 \leq \nu'_j \leq \mu_j \)), plus the sum \( \sum_{j \in [1,r) \setminus E(i)} \lambda_j a_j \). The second sum cannot be negative, but in the worst case might be as large as \( \xi_{r-1} \). Write \( m'_0 = \sum_{j=1}^{n}(\mu_j + \nu'_j a_j) + \sum_{j \in E(i)} \lambda_j a_j \). The ratio \( m'_0/m_0 \) is in \([1/(1+a_{r-1}), 1+a_{r-1}]\), and we have \( m'_0 \leq m \leq m'_0 + \xi_{r-1} \). So the \( \gamma \)-support of \( x \) is contained in
\[
[m_0/(1+a_{r-1}), m_0(1+a_{r-1}) + \xi_{r-1}) \subset (a_r/2a_{r-1}, 2a_r^2 a_{r-1})
\]
given a growth condition.

Let us write
\[
\tau = \sum_{i=1}^{r-1} \lambda_i a_i + \sum_{i=1}^{n} \mu_i
\]
noting that \( \tau \) is the minimum of the \( \gamma \)-support of \( x \).

Given that the vector \( x \) is \( \gamma \)-supported well to the right of zero, let us introduce a Banach algebra norm \( \| \cdot \|_{\gamma} \) to make use of this. The \( l_1 \) version of this is
\[
\left\| \sum_{i=0}^{a_{n+1}a_{n-1}} y_i \gamma_i^{(n)} \right\|_{\gamma} = \sum_{i=0}^{a_{n+1}a_{n-1}} |y_i|.
\]
We have
\[
\| x \|_{\gamma} \leq \prod_{j=1}^{r-1} \left( \frac{j}{j+1} 2^{a_j^2} \right)^{\lambda_j} (j(2^{a_j^2} + 1))^{\mu_j}.
\]
For a reasonable bound on this, let us write \( \sum_{j=1}^{r-1} \lambda_j + \mu_j = L \); so the sum of all the indices \( \lambda_j, \mu_j \) involved in the last product is \( L \). Since the largest possible power is \((r-1)(2^{a_{r-1}} + 1)\), and this is a Banach algebra norm, we get
\[
\| x \|_{\gamma} \leq ((r-1)(2^{a_{r-1}} + 1))^L.
\]
The vector \( s^i \) is equal to
\[
\prod_{j=1}^{n} \left( \frac{j}{j+1} 2^{a_j^2} \gamma_j \gamma_j^{(n)} \right)^{\lambda_j} (j(2^{a_j^2} + 1))^{\mu_j} = x \cdot \prod_{j=r+1}^{n} \left( \frac{j}{j+1} 2^{a_j^2} \gamma_j \gamma_j^{(n)} \right)^{\lambda_j} (j(2^{a_j^2} + 1))^{\mu_j},
\]
\[
= x' \cdot \prod_{j=r+1}^{n} \left( \frac{j}{j+1} 2^{a_j^2} \gamma_j \gamma_j^{(n)} \right)^{\lambda_j} (j(2^{a_j^2} + 1))^{\mu_j},
\]
where
\[
x' = x \cdot \left( \frac{r}{r+1} 2^{a_r^2} \gamma_r \gamma_r^{(n)} \right)^{\lambda_r} (r(2^{a_r^2} + 1))^{\mu_r}.
\]
Now \( \sum_{j=1}^{n} \mu_j \leq m_0 \leq 1 + \xi_r \) from (9.1) and the definitions of \( r \) and \( \xi_r \). So for each \( j = r+1, \ldots, n \) we have \( \mu_j \leq 1 + \xi_r \) and \( \lambda_j + \mu_j \leq a_j + 1 + \xi_r < 2a_j \), since \( j \notin E(i) \). When \( j = r \) we have \( \lambda_j \leq a_j \), again because \( r \notin E(i) \); but we must be content with the estimate \( \mu_r \leq 1 + \xi_r \) above. So \( \lambda_r + \mu_r \leq 1 + a_r + \xi_r \leq 2a_r^2 \), given a growth condition.
Lemma 9.2. Given growth conditions, the following is true. For \( r \) as above, and any nonnegative integer \( \lambda \leq 2^a \), and any nonnegative integer \( \lambda \leq \lambda_{r-1} \), we have

\[
x \in \text{lin}\{ \gamma_{j}^{(n)} : \lambda a_r + a_r/2a_{r-1} < j \leq (\lambda + 1)a_r + a_r/2a_{r-1} \}
\]

we have

\[
(9.7) \quad |\phi_n(x)| \leq 2^{-|s+1|} \|x\|.
\]

Furthermore, if \( \lambda > 1 + a_r \), then \( \phi_n(x) = 0 \).

Proof. Equation (9.7) is equivalent to

\[
|\phi_n(\gamma_{j}^{(n)})| \leq 2^{-(\lambda+1)a^2}, \quad i \in (\lambda a_r + a_r/2a_{r-1}, (\lambda + 1)a_r + a_r/2a_{r-1}).
\]

By (5.1), the left hand side is zero unless \( i \in 1 + \Lambda_n \). But, given a growth condition, we can assume \( a_r/2a_{r-1} > \xi_1 = \max \Lambda_{r-1} \), so the element of \( 1 + \Lambda_n \) involved must be at least \( 1 + (\lambda + 1)a_r \). Equation (5.1) then gives \( |\phi_n(\gamma_{i}^{(n)})| \leq 2^{-(\lambda+1)a^2} \) as required. If \( \lambda > 1 + a_r \), we have \( i > 1 + \xi_r \), so the least element of \( 1 + \Lambda_n \) available would be \( 1 + a_{r+1} \). But given a growth condition, we can certainly assume that the absolute upper bound \( i \leq (2a_r + a_r/2a_{r-1}) \) is less than \( a_{r+1} \), so for \( \lambda > 1 + a_r \) we have \( \phi_n(x) = 0 \).

We now use the lemma to estimate \( |\phi_n(x')| \), where \( x' \) is as in (9.6). Define \( t \) to be the nonnegative integer with

\[
(9.8) \quad \tau \in [ta_r, (t+1)a_r),
\]

where \( \tau \) is the minimum of the support of \( x \) as in (9.3) and (9.2). We will have \( 0 \leq t \leq a_r \) because from (5.1),

\[
\tau \leq m_0 + \sum_{i \in [1,r) \setminus E(i)} \lambda_i a_r \leq m_0 + \xi_{r-1},
\]

and then

\[
m_0 \leq 1 + \xi_r = 1 + \xi_{r-1} + a_r^2.
\]

so \( \tau \leq 1 + 2\xi_{r-1} + a_r^2 < (a_r + 1)a_r \), given a growth condition.

The vector \( x \) given by (9.2) is \( \gamma \)-supported on \( (\max(a_r/2a_{r-1}, ta_r), 2^a a_r) \), so by applying (9.7) for various \( \lambda \) and summing the results, we find that

\[
(9.9) \quad |\phi_n(x)| \leq 2^{-a^2} \gamma_t \|x\|_\gamma \leq 2^{-a^2} \text{max}(1,t) \cdot ((r-1)(2^{a^2-1} + 1))L,
\]

where \( L = \sum_{i=1}^{t-1} \lambda_i + \mu_i \leq (t+1)a_r \), by (9.4). For all \( \lambda \leq a_r^2 \) it is easy to argue that

\[
(9.10) \quad |\phi_n(\gamma_{\lambda a_r} \cdot x)| \leq 2^{-a^2} \gamma_t \lambda \text{max}(1,t) \cdot ((r-1)(2^{a^2-1} + 1))L.
\]

Given a growth condition, we may replace the bounds on the right of (9.9) and (9.10) by \( 2^{-a^2} \text{max}(1/2, t-1/2) \) and \( 2^{-a^2} \text{max}(1/2, t-1/2) \) respectively. Accordingly the vector \( x' = x \cdot (x, 2^{-a^2} \gamma_{a_r}^{(n)}(r(2^{-a^2} \gamma_{a_r}^{(n)} - 1))^\mu_r \) from (9.10) satisfies

\[
|\phi_n(x')| \leq 2^{-a^2} \text{max}(1/2, t-1/2) \cdot (\frac{r}{r+1})^{\lambda_r} \cdot (2r)^{\mu_r},
\]

Crudey, we may estimate \( \mu_r \leq \tau \leq a_r (t+1) \) (from (9.3) and the definition of \( \tau \)), and then we have

\[
2^{-a^2} \text{max}(1/2, t-1/2) \cdot (2r)^{\mu_r} \leq 2^M,
\]

where

\[
M = (1 + \log_2 r) a_r (t+1) - a_r^2 \text{max}(1/2, t-1/2).
\]
Now a growth condition on the sequence \((a_r)\) will ensure that \((1 + \log_2 r) a_r < a_r^2 / 12\) for every \(r\), and since \((t + 1) \leq 4 \max(1/2, t - 1/2)\) for any \(t \in \mathbb{N}_0\), we will then have
\[
(1 + \log_2 r) a_r (t + 1) - a_r^2 \max(1/2, t - 1/2) \leq -\frac{2}{3} a_r^2 \max(1/2, t - 1/2).
\]
But this equals
\[
-a_r^2 \max(1/3, (2t - 1)/3) \leq -a_r^2 \max(1/3, t/3).
\]
So we have
\[
|\phi_n(x')| \leq 2^{-a_r^2 \max(1/3, t/3)} \cdot \left(\frac{r}{r + 1}\right)\lambda r.  \tag{9.11}
\]

**Lemma 9.3.** If \(w \in \text{lin}\{\gamma_i^{(n)} : 0 \leq i < a_{r+1}\}\), and \(\rho_j \in \mathbb{N}_0\), for \(0 \leq \rho_j \leq 2a_j\) for \(j = r + 1, \ldots, n\), then writing \(y = w \cdot \prod_{j=r+1}^n (\gamma_{a_j})^{\rho_j} \), we also have
\[
\phi_n(y) = \phi_n(w) \cdot \prod_{j=r+1}^n 2^{-\rho_j a_j^2} (1 - \rho_j / a_j), \tag{9.12}
\]
where as usual, \(t_+\) denotes the maximum \(\max(t, 0)\).

**Proof.** Since \(w \in \text{lin}\{\gamma_i^{(n)} : 0 < i < a_{r+1}\}\), we refer to (5.1) to find \(\phi_n(\gamma_i^{(n)} + \sum_{j=r+1}^n \rho_j a_j)\) for such \(i\) (and \(\rho_j \leq 2a_j\)), and it is \(\phi_n(\gamma_i^{(n)}) \cdot \prod_{j=r+1}^n 2^{-a_j^2 \rho_j} (1 - \rho_j / a_j)\), given the usual growth condition which ensures that, for \(\rho_j \leq 2a_j\), \(i + \sum_{j=r+1}^n \rho_j a_j\) is not in \(1 + \Lambda_n\) unless \(i \in \Lambda_n\) and all the \(\rho_j \leq a_j\). Equation (9.12) follows. \(\square\)

Equation (9.12) can also be written as
\[
\phi_n(w \cdot \prod_{j=r+1}^n (2a_j^2 \gamma_{a_j})^{\rho_j}) = \phi_n(w) \cdot \prod_{j=r+1}^n (1 - \rho_j / a_j). 
\]
So if \(\lambda_j, \mu_j \leq a_j\), and we write
\[
y' = w \cdot \prod_{j=r+1}^n (2a_j^2 \gamma_{a_j} - \gamma_0^{(n)})^{\mu_j} (2a_j^2 \gamma_{a_j})^{\lambda_j},
\]
we will therefore have
\[
\phi_n(y') = \sum_{\alpha_j = 0, \ldots, a_j} \phi_n(w \cdot \prod_{j=r+1}^n (2a_j^2 \gamma_{a_j})^{\lambda_j + \alpha_j} (-1)^{\mu_j - \alpha_j} \left(\frac{\mu_j}{\alpha_j}\right)) \tag{9.13}
\]
\[
= \sum_{\alpha_j = 0, \ldots, a_j} \phi_n(w) \cdot \prod_{j=r+1}^n (-1)^{\mu_j - \alpha_j} \left(\frac{\mu_j}{\alpha_j}\right) (1 - \frac{\alpha_j + \lambda_j}{a_j})_+. 
\]

We next note that, given a growth condition, the fact that \(x\) (as in (9.2)) is \(\gamma\)-supported on \([0, 2a_r a_{r-1}]\) tells us that \(x'\) (as in (9.6)) is supported on \([0, a_{r+1}]\). We also recall that \(m_0 \leq 1 + \xi_r\), where \(m_0\) is as in (9.1); so
\[
\mu_j \leq m_0 \leq 1 + \xi_r < a_j, \quad j > r,
\]
given a growth condition. Let us therefore apply (9.13) to obtain \( \phi_n(s^i) \) as a multiple of \( \phi_n(x') \), using equation (9.5):

\[
\phi_n(s^i) = \phi_n \left( x' \cdot \prod_{j=r+1}^{n} \left( j(2^{a_j^2 \gamma(n)} - \gamma(n)) \right)^{\mu_j} \left( \frac{j}{j+1} 2^{a_j^2 \gamma(n)} \right)^{\lambda_j} \right).
\]

But this equals

\[
\sum_{\eta \in \mathcal{I}_1(n) \setminus \{i\}} \phi_n(x') \cdot \prod_{j=r+1}^{n} (-1)^{\mu_j} \left( \frac{\mu_j}{\alpha_j} \right) \left( 1 - \frac{\alpha_j + \lambda_j}{\alpha_j} \right)^+, \]

which in turn equals

\[
\phi_n(x') \cdot \prod_{j=r+1}^{n} j^{\mu_j} \left( \frac{j}{j+1} \right)^{\lambda_j} \eta_j, \quad \text{where } \eta_j = \sum_{\alpha=0}^{\mu_j} (-1)^{\mu_j-\alpha} \left( \frac{\mu_j}{\alpha} \right) \left( 1 - \frac{\alpha + \lambda_j}{\alpha_j} \right)^+.
\]

Putting our estimate (9.11) into this equation, we have

\[
(9.14) \quad |\phi_n(s^i)| \leq 2^{-a_1 r \max(1/3, t/3)} \cdot \left( \frac{r}{r+1} \right)^{\lambda r} \prod_{j=r+1}^{n} j^{\mu_j} \left( \frac{j}{j+1} \right)^{\lambda_j} |\eta_j|.
\]

We can estimate \( |\eta_j| \) as follows. If \( \lambda_j + \mu_j \leq a_j \), there is no need to estimate: we have \( \eta_j = \sum_{\alpha=0}^{\mu_j} (-1)^{\mu_j-\alpha} \left( \frac{\mu_j}{\alpha} \right) \left( 1 - \frac{\alpha + \lambda_j}{\alpha_j} \right)^+ \). This is \( 1 - \lambda_j/a_j \) if \( \mu_j = 0 \), and is \(-1/a_j \) if \( \mu_j = 1 \). It is zero if \( \mu_j > 1 \), by a binomial series argument, or because second and higher differences of the sequence \( (1 - (\alpha + \lambda_j)/a_j)^{\infty} \) are zero. If \( \lambda_j + \mu_j > a_j \), we estimate as follows: we must have \( \lambda_j \geq a_j - \xi_r \) because \( \mu_j \leq 1 + \xi_r \); so \( (1 - \alpha + \lambda_j)/a_j \leq \xi_r/a_j \) for all \( \alpha \geq 0 \); so \( |\eta_j| \leq 2^{\mu_j} \xi_r/a_j \leq 2^{1+\xi_r} \xi_r/a_j \leq a_j^{-2/3} \), given a growth condition. We are now in a position to prove:

**Lemma 9.4.** Given growth conditions, one has

\[
\sum_{i \in \mathcal{I}_1(n) \setminus \{0\}} |\phi_n(s^i)|^2 \leq \prod_{j=1}^{n} \left( \frac{j+1}{2j+1} \right)^{\lambda_j},
\]

for all \( n \in \mathbb{N} \).

**Proof.** The full sum over \( i \in \mathcal{I}_1(n) \), \( m_0(\bar{i}) > 1 \) is a sum, from \( r = 1 \) to \( n \), of contributions involving \( i \) with \( m_0(\bar{i}) = 1 + \sum_{j=1}^{r} t_j a_j \), and \( t_r > 0 \). For a fixed \( r \), we further consider contributions for fixed \( \lambda_j (j = 1, \ldots, n) \), \( \mu_j (j = 1, \ldots, r) \) and fixed \( s = \sum_{j=r+1}^{n} \mu_j \) (where as usual, \( \lambda_j = i(\frac{i}{j} + 2^{a_j^2 \gamma(n)}) \) and \( \mu_j = i(j(2^{a_j^2 \gamma(n)} - \gamma(n))) \)). So let us write \( \mathcal{I}_1^{n,r,\lambda_1,\ldots,\lambda_n,\mu_1,\ldots,\mu_r,s}(n) \) for the set of \( i \in \mathcal{I}_1(n) \) with \( m_0(i) = 1 + \sum_{j=1}^{r} t_j a_j \), and \( t_r > 0 \), and the given values \( \lambda_j (j = 1, \ldots, n) \), \( \mu_j (j = 1, \ldots, r) \) and \( s = \sum_{j=r+1}^{n} \mu_j \). Once these are all fixed, we know the vectors \( x' \) and \( x'' \) as in (9.2) and (9.6), and also the constants \( \tau, t \) and \( L \) as in (9.3) and (9.9). For each \( j > r \), the values \( |\eta_j| \) are determined by \( \lambda_j \) and \( \mu_j \), as described below (9.14). Given \( \lambda_j \), the product \( j^{\mu_j} |\eta_j| \) can take the value \( 1 - \lambda_j/a_j \) once (when \( \mu_j = 0 \)), and the value \( j/a_j \) once (when \( \mu_j = 1 \)), and values up to \( j^{\mu_j} a_j^{-2/3} \) for any \( \mu_j = 1, \ldots, 1 + \xi_r \). The sum of the squares of all such values is at most

\[
1 + (j/a_j)^2 + (1 + \xi_r)^2 2^{2\xi_r} a_j^{-4/3} \leq 1 + a_j^{-1}
\]

for all \( j > r \), given a growth condition. So the sum of the products \( \prod_{j=r+1}^{n} \left( \frac{j}{j+1} \right)^{2\lambda_j} |\eta_j|^2 \), for various \( \mu_j (j = r+1, \ldots, n) \) with \( \sum_{j=r+1}^{n} \mu_j = s \), is at most \( \prod_{j=r+1}^{n} \left( \frac{j}{j+1} \right)^{2\lambda_j} (1 +
Writing $I_1^{(i)}$ for $I_1^{(n,r,\lambda_1,\ldots,\lambda_n,\mu_1,\ldots,\mu_\gamma,s)}$, we then get
\[
\sum_{i \in I_1^{(i)}} |\phi_n(s^i)|^2 \leq 2^{-a_j^2 \max(2/3,2t/3)} \cdot \frac{(r + 1)^2}{2r + 1} \cdot \prod_{j=r+1}^n (j + 1)^2 (1 + a_j^{-1}).
\]
from (9.14). We can sum this over all possible $\lambda_j \ (j = r, \ldots, n)$; writing $I_1^{(i)} = I_1^{(n,r,\lambda_1,\ldots,\lambda_{r-1},\mu_1,\ldots,\mu_\gamma,s)}$ for the union of all sets $I_1^{(n,r,\lambda_1,\ldots,\lambda_n,\mu_1,\ldots,\mu_\gamma,s)}$ as $\lambda_j$ varies for $j = r, \ldots, n$, we have
\[
\sum_{i \in I_1^{(n,r,t)}} |\phi_n(s^i)|^2 \leq 2^{-a_j^2 \max(2/3,2t/3)} \cdot \frac{(r + 1)^2}{2r + 1} \cdot \prod_{j=r+1}^n (j + 1)^2 (1 + a_j^{-1}).
\]
The number of ways we can choose $\lambda_1,\ldots,\lambda_{r-1},\mu_1,\ldots,\mu_\gamma, s$ in order to get a nonempty set $I_1^{(n,r,\lambda_1,\ldots,\lambda_{r-1},\mu_1,\ldots,\mu_\gamma,s)$ associated with the given value $t$, is less than the number of ways we can pick $2r$ nonnegative integers adding up to an answer $r \in [t a_r, (t + 1) a_r]$ as in (9.3). Very crudely, this number is no bigger than $((t+1)a_r)^{2r}$. So writing $I_1^{(n,r,t)}$ for the union of all sets $I_1^{(n,r,\lambda_1,\ldots,\lambda_{r-1},\mu_1,\ldots,\mu_\gamma,s)}$ such that (9.3) holds, we have
\[
\sum_{i \in I_1^{(n,r,t)}} |\phi_n(s^i)|^2 \leq ((t+1)a_r)^{2a_r} 2^{-a_j^2 \max(2/3,2t/3)} \cdot \frac{(r + 1)^2}{2r + 1} \cdot \prod_{j=r+1}^n (j + 1)^2 (1 + a_j^{-1}).
\]
This is dominated by
\[
2^{-a_j^2 \max(1/3,t/3)} \cdot \prod_{j=r+1}^n \frac{(j + 1)^2}{2j + 1} (1 + a_j^{-1}),
\]
given another growth condition. We can of course assume that $\prod_{j=1}^n (1 + a_j^{-1}) \leq 2$. So
\[
\sum_{i \in I_1^{(n,r,t)}} |\phi_n(s^i)|^2 \leq 2^{\frac{1}{a_j^2} \max(1/3,t/3)} \cdot \prod_{j=r+1}^n \frac{(j + 1)^2}{2j + 1}.
\]
Summing over all $t \in \mathbb{N}_0$ and $r \in [1,n]$ we get
\[
\sum_{i \in I_1^{(n)}} |\phi_n(s^i)|^2 \leq \sum_{r=1}^n \sum_{t=0}^\infty 2^{\frac{1}{a_j^2} \max(1/3,t/3)} \cdot \prod_{j=r+1}^n \frac{(j + 1)^2}{2j + 1} \leq \prod_{j=1}^n \frac{(j + 1)^2}{2j + 1},
\]
given another growth condition. Thus the lemma is proved. 

We can now finish this section by polishing off the case when $i \in I_1^{(n)}$ but $m_0(i) = 1$. In that case, since $\sum_{j \in E(i)} \lambda_j a_j + \sum_{j \in 0} \mu_j \leq m_0$, we must have $E(i) = \emptyset$ and at most one $\mu_j = 1$. In fact we must have exactly one $\mu_j = 1$, since $0 \notin 1 + \Lambda$. So $s^i = \prod_{j=1}^n (\frac{a_j^2}{a_r} \gamma_j^{(n)} \lambda_j r(2^{a_j^2} \gamma_j^{(n)} - \gamma_j^{(n)})$ for some $r \leq n$, and so, when $\lambda_j < a_r$, the reader can check that (5.1) tells us that
\[
\phi_n(s^i) = -\frac{r}{a_r} \prod_{j=1}^n \frac{(j + 1)^2}{j + 1} \lambda_j \prod_{j=1}^n (1 - \lambda_j/a_j) \lambda_j (\prod_{j=1}^n (1 - \lambda_j/a_j)).
\]
If \( \lambda_r = a_r \), we can safely assume that \( \phi_n(\gamma_1+(a_r+1)a_r+\sum_j \lambda_j a_j) = 0 \), and then \( \phi_n(s^j) = 0 \) in this case. So we have

\[
\sum_{i \in I(n)} |\phi_n(s^j)|^2 \leq \sum_{r=1}^n \sum_{\lambda_j=0}^{n(i)} (r/a_r)^2 \prod_{j=1}^n \frac{j}{j+1}^{\lambda_j},
\]

which is dominated by

\[
\sum_{r=1}^n (r/a_r)^2 \prod_{j=1}^n \frac{j}{j+1}^{2\lambda_j} = \sum_{r=1}^n (r/a_r)^2 \prod_{j=1}^n \frac{(j+1)^2}{2j+1}.
\]

Now \( \sum_{r=1}^\infty (r/a_r)^2 < 1 \), given a mild growth condition, so

\[
\sum_{i \in I(n)} |\phi_n(s^j)|^2 \leq \prod_{j=1}^n \frac{(j+1)^2}{2j+1}, \quad n \in \mathbb{N}.
\]

Combining this equation with the previous lemma, we have the result:

**Theorem 9.5.** Given growth conditions, we have \( \sum_{i \in I(n)} |\phi_n(s^j)|^2 < 2 \prod_{j=1}^n \frac{(j+1)^2}{2j+1} \), for all \( n \in \mathbb{N} \).

10. **Bound on \( \sum_{i \in I_2(n)} |\phi_n(s^i)|^2 \)**

To get this bound, we need to have a good estimate of \( |\phi_n(\gamma_j(n))| \) in cases when \( \gamma_j(n) \) is not one of the basis vectors for \( H_n \) with respect to which \( \phi_n \) is defined directly in (5.1). That is, we need to know about \( \phi_n(\gamma_j(n)) \) when \( j \geq a_{n+1} - a_n \).

Let \( e_j^* \) \((a_n < j \leq a_{n+1})\) denote the linear functional on \( H_n \) with \( \langle e_i, e_j^* \rangle = \delta_{i,j} \). A general linear functional \( \psi = \sum_{j=a_{n+1}}^{a_n} \lambda_j e_j^* \) will have

\[
(10.1) \quad \psi(\gamma_k(n)) = \sum_{j=a_{n+1}}^{a_n} \lambda_j 2^{-j} = p(2^{-k}),
\]

where \( p(t) = \sum_{j=1}^{a_n} \lambda_j t^j \) is a polynomial of degree at most \( a_{n+1} \), with \( t^{1+a_n} \) a factor of \( p(t) \). We will have \( p(2^{-k}) = \delta_{i,k} \) when \( 0 \leq k < a_{n+1} - a_n \), if we choose \( p = p_{n,i} \), where

\[
(10.2) \quad p_{n,i}(t) = (2t)^{1+a_n} \prod_{0 \leq j < a_{n+1} - a_n} \frac{t - 2^{-j}}{2^{-i} - 2^{-j}}.
\]

If we write \( \psi_{n,i} \) for the corresponding linear functional, we have

\[
(10.3) \quad x = \sum_{i=0}^{a_{n+1} - a_n - 1} \langle x, \psi_{n,i} \rangle \gamma_i^{(n)}
\]

for every \( x \in H_n \). This may be seen by checking it on the basis \( \langle \gamma_i^{(n)} \rangle_{i=0}^{a_{n+1} - a_n - 1} \), using (10.1) and the fact above (10.2). Indeed,

\[
(10.4) \quad \gamma_k^{(n)} = \sum_{i=0}^{a_{n+1} - a_n - 1} p_{n,i}(2^{-k}) \gamma_i^{(n)}.
\]
Lemma 10.1. Given a suitable growth conditions on our underlying sequences, we will have $|\phi_n(\gamma_l)| \leq 2^{-l(1+a_n)} - a^2_{n+1}/3$ for all $l \in \mathbb{N}$, $l \geq 1 + \xi_n$.

Proof. It is enough to show this for $l \geq a_{n+1} - a_n$, since $\phi_n(\gamma_l^{(n)}) = 0$ for $l \in (1 + \xi_n, a_{n+1} - a_n)$ by (5.1). Note that $\phi_n$ is $\gamma$-supported on $[0, 1 + \xi_n]$. When $k \leq 1 + \xi_n$, we have by (10.2) that

$$p_{n,k}(2^{-l}) = 2^{(k-1)(1+a_n)} \prod_{0 \leq j < a_{n+1}-a_n} \frac{2^{-l} - 2^{-j}}{2^{-k} - 2^{-j}}.$$  

When $j < k$ the factor $|2^{-l} - 2^{-j}|$ is in $(1, 2]$. When $j > k$ we have $|2^{-l} - 2^{-j}| \leq 2^{k-j+1}$. Thus

$$|p_{n,k}(2^{-l})| \leq 2^{(k-1)(1+a_n)} \cdot 2^k \cdot \prod_{j=k+1}^{a_{n+1}-a_n-1} 2^{k-j+1}$$

(10.5) \hspace{1cm} 2^{(k-1)(1+a_n)} \cdot 2^k \cdot 2^{-\frac{1}{2}(a_{n+1}-a_n-k-2)(a_{n+1}-a_n-k-1)} \leq 2^{-1-l(1+a_n)} - a^2_{n+1}/3,$$

given a suitable growth condition. Now the nonzero coefficients $\phi_n(\gamma_k)$ in (5.1) are positive numbers at most $2^{-\sum_j t_j a_j^2}$, where $k = 1 + \sum_j t_j a_j$, $0 \leq t_j < a_j$. These are distinct nonnegative powers of 2, so the sum of all the coefficients is at most 2. So (5.1), (10.3), and (10.5) give us $|\phi_n(\gamma_l)| \leq 2^{-l(1+a_n)} - a^2_{n+1}/3$. \hfill \square

Theorem 10.2. Given growth conditions, we have $\sum_{i \in \mathbb{I}^n} |\phi_n(\gamma_i^{(n)})|^2 \leq 2^{-2a^2_{n+1}/3}$ for all $n \in \mathbb{N}$.

Proof. If we impose the convolution multiplication on $c_00(\mathbb{N}_0)$, the norm $\|\sum_{i=0}^N \beta_i e_i\| = \sum_{i=0}^N 2^{-i(1+a_n)} |\beta_i|$ is an algebra norm. One can define an algebra homomorphism $\theta$ from $c_00(\mathbb{N}_0)$ into $H_n$ with $\theta(e_i) = \gamma_i^{(n)}$, and Lemma 10.1 can then be rephrased as follows: if $z \in c_00$ with $z \in \{e_j : j > \xi_n\}$, then $|\phi_n(\theta(z))| \leq 2^{-a^2_{n+1}/3} \|z\|$.

We note that, among the elements of $S_0^{(n)} = \sum_{j=1}^n 2a_j^{2} \gamma_1^{(n)} = \theta(u_j)$ with

$$\|u_j\| \leq 2^{a_j(1-a_j)} \leq 2^{-a_j},$$

and $j(2a_j^2 \gamma_1^{(n)} - \gamma_1^{(n)}) = \theta(u_j)$ where

$$\|v_j\| \leq 2j \cdot 2^{-a_j} \leq 2^{-a_j}/2, \quad j \leq n,$$

given a growth condition. If $i \in \mathbb{I}^{(n)}_2$ (so $|i| \geq \sqrt{a_{n+1}}$), we write (as usual) $\lambda_j = i(\sum_{j=1}^n 2a_j^2 \gamma_1^{(n)} + 2a_j + \mu_j = i(j(2a_j^2 \gamma_1^{(n)} - \gamma_1^{(n)})).$ Let $w = \prod_{j=1}^n v_j^{\lambda_j}$, then $\|w\| = \theta(w)$, and $\|w\| \leq 2^{-\sum_{j=1}^n (\lambda_j a_j + \mu_j a_n/2)}$. Also,

$$w \in \{e_i : i \geq \sqrt{a_{n+1}}\} \subset \{e_i : i > 1 + \xi_n\}$$

given a growth condition. So Lemma 10.1 applies and tells us (in its “rephrased” form) that

$$|\phi_n(s^i)| = |\phi_n(\theta(w))| \leq 2^{-a^2_{n+1}/3} \|w\| \leq 2^{-a^2_{n+1}/3 - \sum_{j=1}^n (\lambda_j a_j + \mu_j a_n/2)}.$$
So
\[ \sum_{i \in I_2^{(n)}} |\phi_n(s_i^{(n)})|^2 \leq 2^{-2a_{n+1}^2/3}. \]

A mild growth condition ensures that the right hand sum is at most 1 for any \( n \), so we have the required result.

\[ \square \]

11. An estimate for \( \sum_{i \in I_3^{(n)}} |\phi_n(s_i)|^2 \).

We now turn our attention to the set \( I_3^{(n)} \), which involves index functions \( i \) for which \( i(a_k^{-1}e_k) \neq 0 \) for some \( k \in \langle a_n, a_{n+1} \rangle \). Since the product of distinct \( e_k \) is zero, we get \( s_i^1 = 0 \) if \( i(a_k^{-1}e_k) > 0 \) for two distinct \( k \). If there is one such \( k \), and if the index \( i(a_k^{-1}e_k) = m > 0 \), we get
\[ s_i^1 = a_k^{-m}e_k \cdot s_i' = a_k^{-m}e_k(s_i')e_k, \]
where \( i' \) is an element of \( I_1^{(n)} \cup I_2^{(n)} \). Accordingly we get
\[ \sum_{i \in I_3^{(n)}} |\phi_n(s_i)|^2 = \sum_{k=m+1}^{a_n+1} \sum_{i' \in I_1^{(n)} \cup I_2^{(n)}} a_k^{-2m}|\phi_n(s_i')\phi_n(e_k)|^2. \]

This is equal to
\[ \sum_{k=m+1}^{a_n+1} (a_k^2 - 1)^{-1} \sum_{i' \in I_1^{(n)} \cup I_2^{(n)}} |\phi_n(s_i')\phi_n(e_k)|^2, \]
which is dominated (since the \( e_k^* \) are characters, hence contractive) by
\[ \sum_{k=m+1}^{a_n+1} (a_k^2 - 1)^{-1} \sum_{i' \in I_1^{(n)} \cup I_2^{(n)}} \|s_i'\|^2_0 |\phi_n(e_k)|^2. \]

The \( c_0 \) norms of the elements of \( S_0^{(n)} \) are listed in the first paragraph of the proof of Theorem 5.4. Writing
\[ \varepsilon_j = \left\| j \frac{\hat{\gamma}}{j+1} 2^{a_j^2} \gamma_{a_j} \right\|_0 \leq 2^{-a_j}, \quad \text{and} \quad \varepsilon'_j = \left\| j(2^{a_j^2} \gamma_{a_j+1} - \gamma_{a_j}) \right\|_0, \]
we have that
\[ \varepsilon'_j \leq j(2^{-a_j} + 2^{-1-a_n}) \leq 2 \varepsilon_j \leq 2^{-a_j/2}, \]
given a growth condition. Now \( \{s_i' : i' \in I_1^{(n)} \cup I_2^{(n)}\} \) is equal to
\[ \left\{ \prod_{j=1}^n \left( \frac{j}{j+1} 2^{a_j^2} \gamma_{a_j} \right)^{\lambda_j} (j(2^{a_j^2} \gamma_{a_j+1} - \gamma_{a_j}))^{\mu_j} : \lambda, \mu_j, j = 1, \ldots, n \right\}, \]
and so
\[ \sum_{i' \in I_1^{(n)} \cup I_2^{(n)}} \|s_i'\|^2_0 \leq \sum_{\lambda_1, \mu_1, \ldots, \lambda_n, \mu_n \geq 0} \prod_{j=1}^n 2^{\lambda_j} (\varepsilon'_j)^{2\mu_j} \leq \prod_{j=1}^n (1 - \varepsilon_j^2)^{-1}(1 - \varepsilon_j)^{-1}. \]

By the estimates for \( \varepsilon_j, \varepsilon'_j \) above, the latter is dominated by
\[ \prod_{j=1}^n (1 - 2^{-2a_j})^{-1}(1 - 2^{-a_j})^{-1} < 2, \]
given a growth condition. So,

\begin{equation}
|\phi_n(s^j)|^2 \leq \sum_{k=1+a_n}^{a_{n+1}} 2(a_k^2 - 1)^{-1}|\phi_n(e_k)|^2.
\end{equation}

It is easy to argue from (11.1) that \(\sum_j |\phi_n(\gamma_j^{(n)})| \leq 2\). Each \(e_k = \sum_{i=1+a_n}^{a_{n+1}} \beta_k,i \gamma_i^{(n)}\), where \(\beta_k,i = \langle e_k, \psi_{n,i} \rangle\) as in [10.3]. Now \(\langle e_k, \psi_{n,i} \rangle\) is the coefficient of \(t^k\) in the polynomial \(p_{n,k}(t)\) as in [10.2]; a crude estimate is that no coefficient of this polynomial exceeds

\[2^k(1+a_n) \cdot \prod_{\substack{0 \leq j < a_{n+1} - a_n \atop j \neq k}} \frac{2}{|2^k - 2^{-j}|} \leq 2^k(1+a_n) \cdot \prod_{j=0}^{a_{n+1} - a_n} 2^{j/2} \leq 2^{k(1+a_n) + \frac{a_{n+1} - a_n - 1}{2}}.
\]

But this is dominated by \(2^{a_{n+1}^2}\), given a mild growth condition. Putting this estimate in (11.1), we have

\[\sum_{i \in I_3^{(n)}} |\phi_n(s^i)|^2 \leq \sum_{k=1+a_n}^{a_{n+1}} 2(a_k^2 - 1)^{-1}2^{a_{n+1}^2}.
\]

**Lemma 11.1.** Given growth conditions, we have \(\sum_{i \in I_3^{(n)}} |\phi_n(s^i)|^2 \leq 1\) for every \(n \in \mathbb{N}\).

**Proof.** Given the last inequality, all we need to do is demand the growth conditions \(a_n \geq n + 1\) so that \(a_k^2 > 1 + 2^{2^r + a_k^2 - 1}\) for all \(k\). For then, since the \(a_k\) are strictly increasing, we have \(a_k^2 > 1 + 2^{r+1+a_k^2} - r\) for all \(0 < r < k\). This implies that

\[\sum_{k=1+a_n}^{a_{n+1}} 2(a_k^2 - 1)^{-1}2^{a_{n+1}^2} \leq \sum_{k=2+n}^{\infty} 2(a_k^2 - 1)^{-1}2^{a_{n+1}^2} \leq \sum_{r=1}^{\infty} 2 \cdot 2^{-r+1} \cdot 2^{a_{n+1}^2},
\]

where \(r = k - n - 1\). But the last quantity equals \(\sum_{r=1}^{\infty} 2^{1-r} = 1\).

12. Conclusions

**Theorem 12.1.** Given growth conditions on the underlying sequence \((a_k)_{k=1}^\infty\), we have \(\sum_{i \in I_3^{(n)}} |\phi_n(s^i)|^2 \leq 2 + 2 \cdot \prod_{j=1}^{n} \frac{(j+1)^2}{2j+1}\) for all \(n \in \mathbb{N}\). Lemma 5.3 holds: one has \(\|\gamma_0^{(n)}\|^2_2 \leq 3 \cdot \|\gamma_1^{(n)}\|^2_2\) for all \(n\). The operator norm \(\|g^{(n)}\|_{op} \geq \frac{1}{3}\). Theorem 7.4 is true, as is Theorem 4.1 for these choices of the underlying sequence.

**Proof.** The first estimate is obtained by summing the estimates given in Theorem 9.1 Theorem 10.2 and Lemma 11.1. Substituting in (5.1), we have

\[\|\gamma_0^{(n)}\|^2_2 < \|\gamma_1^{(n)}\|^2_2 + 2 + 2 \cdot \prod_{j=1}^{n} \frac{(j+1)^2}{2j+1}.
\]

Applying the lower estimate Lemma 7.1 for \(\|\gamma_1^{(n)}\|^2_2\), we see that \((\|\gamma_0^{(n)}\|^2_2)^2 < 9(\|\gamma_1^{(n)}\|^2_2)^2\), and the second estimate follows. Of course the operator norm

\[\|g\|_{op} \geq \frac{1}{3}\]
Theorem 4.1 and Theorem 5.4 now follow by the argument after Lemma 5.3.

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