SPECTRAL ANALYSIS FOR SINGULARITY FORMATION OF THE TWO DIMENSIONAL KELLER-SEGEL SYSTEM

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ABSTRACT. We analyse an operator arising in the description of singular solutions to the two-dimensional Keller-Segel problem. It corresponds to the linearised operator in parabolic self-similar variables, close to a concentrated stationary state. This is a two-scale problem, with a vanishing thin transition zone near the origin. Via rigorous matched asymptotic expansions, we describe the eigenvalues and eigenfunctions precisely. We also show a stability result with respect to suitable perturbations, as well as a coercivity estimate for the non-radial part. These results are used as key arguments in a new rigorous proof of the existence and refined description of singular solutions for the Keller-Segel problem by the authors [7]. The present paper extends the result by Dejak, Lushnikov, Yu, Ovchinnikov and Sigal [11]. Two major difficulties arise in the analysis: this is a singular limit problem, and a degeneracy causes corrections not being polynomial but logarithmic with respect to the main parameter.

1. Introduction

We describe in this paper a detailed spectral analysis for the linear operator

$$L_z f = \Delta f - \nabla \cdot (f \nabla \Phi_U + U_{\nu} \nabla \Phi_f) - \beta \nabla \cdot (zf)$$

(1.1)

in the radial and non-radial settings, where

$$\Phi_f = -\frac{1}{2\pi} \log |z| * f, \quad U_{\nu}(z) = \frac{8\nu^2}{(\nu^2 + |z|^2)^2}, \quad \nabla \Phi_{U_{\nu}}(z) = -\frac{4z}{\nu^2 + |z|^2},$$

and $$\beta > 0$$ is a fixed constant and $$0 < \nu \ll 1$$ is the main parameter of the problem.

1.1. Origin of the spectral problem

The linear operator $$L_z$$ appears in the study of singularities of the following two dimensional parabolic-elliptic Keller-Segel system:

$$\begin{cases}
\partial_t u = \Delta u - \nabla \cdot (u \nabla \Phi_u), \\
\Phi_u = -\frac{1}{2\pi} \log |x| * u, & (x,t) \in \mathbb{R}^2 \times \mathbb{R}_+, \\
u(t = 0) = u_0 \geq 0,
\end{cases}$$

(1.2)

see [18], [19], [20], [23], and [16] for a survey of the problem. It is well known (see for example, [17], [5], [4], [13], [2], [3] and references therein) that the problem (1.2) exhibits finite time blowup solutions if the initial datum satisfies

$$M = \int_{\mathbb{R}^2} u_0(x) dx > 8\pi.$$ 

The threshold $$8\pi$$ is related to the family of stationary solutions $$(U_{\eta})_{\eta > 0}$$ of (1.2), where

$$U_{\eta}(x) = \frac{1}{\eta^2} U\left(\frac{x}{\eta}\right) \quad \text{with} \quad U(x) = \frac{8}{(1 + |x|^2)^2} \quad \text{and} \quad \int_{\mathbb{R}^2} U_{\eta}(x) dx = 8\pi.$$
The parameter \( \eta \) is linked to the scaling symmetry of the problem: if \( u \) is a solution to (1.2), then for any \( \eta > 0 \), \( u_\eta \) defined by
\[
    u_\eta(x, t) = \frac{1}{\eta^2} u \left( \frac{x}{\eta}, \frac{t}{\eta^2} \right)
\]
is a solution as well. As the mass \( M \) which is a conserved quantity for (1.2) is invariant under the above transformation, the problem is called critical. A key problem in understanding singular solutions is to analyse their asymptotic self-similarity. If a solution to (1.2) is both singular at \( t = 0 \) and invariant under the transformation (1.4), it would be of the form \(-t\)^{-1}w(x/\sqrt{-t})\); non-degenerate self-similarity would then refer to blowup solutions satisfying \( u \sim (T - t)^{-1}w(x/\sqrt{T - t}) \). However, one of the remarkable facts about finite time blowup solutions of (1.2) is that they present a degenerate self-similarity. Precisely, they are of type II blowup (see Theorem 8.19 in [25] and Theorem 10 in [22] for such a statement) in the following sense. A solution \( u(t) \) of (1.2) exhibits type I blowup at \( t = T \) if there exist a constant \( C > 0 \) such that
\[
\limsup_{t \to T} (T - t)\|u(t)\|_{L^\infty(\mathbb{R}^2)} \leq C,
\]
otherwise, the blowup is of type II. Equivalently, in the parabolic self-similar variables
\[
u
\]
\[
    u(x, t) = \frac{1}{\mu^2} w(z, \tau), \quad \Phi_u(x, t) = \Phi_w(y, s), \quad z = \frac{x}{\mu}, \quad \frac{d\tau}{dt} = \frac{1}{\mu^2}, \quad \mu(t) = \sqrt{T - t},
\]
where \( w(z, \tau) \) solves the equation
\[
\partial_\tau w = \nabla \cdot (\nabla w - w \nabla \Phi_w) - \beta \nabla \cdot (zw) \quad \text{with} \quad \beta = \frac{-\mu_\nu}{\mu} = \frac{1}{2},
\]
\( u \) is a type II finite time blowup solution of (1.2) if and only if \( w \) is a global but unbounded solution of (1.7). The mechanism of singularity formation then involves crucially the above family of solutions \( U_\eta \), see for example, [26, 28, 27, 12, 14, 24] and references therein. The key idea is that in equation (1.2) the time variation \( \partial_\tau u \) is asymptotically of lower order, the solution approaches the family of stationary states \( u \sim U_\eta \) and a scaling instability drives the parameter \( \eta \) to 0 as \( t \to T \). This motivates the study of a solution in the variables (1.7) having the form
\[
w(z, \tau) = U_\nu(z) + \varepsilon(z, \tau),
\]
where \( \nu = \eta/\sqrt{T - t} \) is time dependent with \( \nu(\tau) \to 0 \) as \( \tau \to \infty \), and \( \varepsilon \) is a lower order perturbation solving:
\[
\partial_\tau \varepsilon = \mathcal{L}^z \varepsilon - \nabla \cdot (\varepsilon \nabla \Phi_\varepsilon) + \left( \frac{\nu_\varepsilon}{\nu} - \beta \right) \nabla \cdot (zU_\nu).
\]
Above, \( \mathcal{L}^z \) is precisely the operator introduced in (1.1) that we aim at studying in the present paper. The importance of its study is motivated by the following. Previous works [26, 28, 27] emphasise the two scales problem \( (z \sim 1 \text{ and } \nu \sim \nu) \) and its singular limit, but remain formal. The full nonlinear problem is solved in [24] where the solution is studied in blow up variables \( y = \frac{x}{\nu} \sim 1 \) where a refined description is obtained. The description involves parameters, and their evolution (the modulation laws) is computed based on so called tail-dynamics, relying on suitable cancellations in the parabolic zone \( z \sim 1 \). The analysis of the tail-dynamics is however heavy, as it does not involve a refined understanding of the solution in self-similar variables. Our precise spectral study for the operator (1.8), however, gives a framework to control the solution accurately, on both scales simultaneously, and the temporal evolution of the parameters is easily related to the projection of the dynamics on its eigenmodes. The present paper is a key result in this new approach to the construction of singular solutions to (1.2) that is implemented in [7], and allows to obtain a refined description (see Remark 1.3).

It is remarkable that in the radial setting, the nonlocal linear operator \( \mathcal{L}^z \) reduces to a local linear one in terms of the partial mass
\[
m_f(\zeta) = \frac{1}{2\pi} \int_{B(0, \zeta)} f(z) dz, \quad \zeta = |z|,
\]
where \( B(0, \zeta) \) the ball centered at 0 of radius \( \zeta \). Indeed, if \( f \) is spherically symmetric, then we have the relation
\[
\mathcal{L}^z f(\zeta) = \frac{1}{\zeta} \partial_\zeta \left( \mathcal{A}^\zeta m_f(\zeta) \right),
\]
where \( \mathcal{A}^\zeta \) is the linear operator defined by
\[
\mathcal{A}^\zeta = \mathcal{A}_0^\zeta - \beta \zeta \partial_\zeta \quad \text{with} \quad \mathcal{A}_0^\zeta = \partial_\zeta^2 - \frac{1}{\zeta} \partial_\zeta + \frac{Q_\nu(\zeta)}{\zeta} \quad \text{and} \quad Q_\nu(\zeta) = \frac{4\zeta^2}{\zeta^2 + \nu^2}.
\]
Hence, in the radial setting \( \mathcal{L}^z \) and \( \mathcal{A}^\zeta \) share the same spectrum and if \( \varphi \) and \( \phi \) are the radial eigenfunctions of \( \mathcal{L}^z \) and \( \mathcal{A}^\zeta \) respectively, we have the relation
\[
\varphi(\zeta) = \frac{\partial_\zeta \phi(\zeta)}{\zeta}, \quad \Phi(\zeta) = -\frac{\phi(\zeta)}{\zeta}.
\]
Therefore, we are interested in the eigenproblem
\[
\mathcal{A}^\zeta \phi(\zeta) = \lambda \phi(\zeta), \quad \zeta \in [0, +\infty),
\]
in the regime
\[
\beta \sim 1, \quad 0 < \nu \ll 1.
\]
Note that the constant \( \beta \) is not necessarily close to 1, it can be any fixed positive constant.

### 1.2. Main results

The analysis on the spectrum of the operator \( \mathcal{L}^z \) in the radial setting had been done by Dejak, Lushnikov, Yu, Ovchinnikov and Sigal [11] via matched asymptotic expansions. Our approach, similar in spirit, is inspired by the work of Collot, Merle, and Raphaël [6] for the study of type II supercritical singularities of the semilinear heat equation \( u_t = \Delta u + |u|^{p-1}u \) (see also [8, 15, 21] for related problems). The strategy is to construct suitable eigenfunctions near the origin and away from the origin, and to match them rigorously to produce a full eigenfunction. Differentiating the matching condition then provides information on the dependence of the eigenfunctions on the parameters. The current work extends this approach to a critical problem, showing its robustness. Solving (1.11), though, is not just a mere adaptation the techniques of [6] because of the following points.

This critical case displays two new degeneracies that have to be handled. First, this is a singular limit problem. Indeed, from the explicit formula (1.10) for \( Q_\nu \), we note that the operator \( \mathcal{A}^\zeta \) converges to a limit operator pointwise outside the origin, namely that for any smooth function \( f \) and at any fixed \( \zeta > 0 \), we have
\[
\mathcal{A}^\zeta f(\zeta) \to \partial_\zeta^2 f(\zeta) + \frac{3}{\zeta} \partial_\zeta f(\zeta) - \beta \zeta \partial_\zeta f(\zeta) \quad \text{as} \quad \nu \to 0.
\]
The limit operator \( \partial_\zeta^2 + \frac{3}{\zeta} \partial_\zeta - \beta \zeta \partial_\zeta \) is well understood, its spectrum is \( \{0, -2\beta, -4\beta, -6\beta, \ldots\} \) and its eigenfunctions are Hermite polynomials. However, the limit \( \nu \to 0 \) for the problem (1.11) is a singular one. The problem involves two scales: one is \( \zeta \sim 1 \) and the other is \( \zeta \sim \nu \). What happens at the latter actually prevents the convergence to the aforementioned limit operator: the spectrum is shifted by the constant \( 2\beta \) at the leading order as is shown in Proposition 1.1 below. This in particular prevents the use of a bifurcation argument. Second, this problem also presents another degeneracy from which most of the technical difficulty stems from, since next order corrections, instead of being polynomial in the parameter \( \nu \), are actually polynomial in \( 1/|\log \nu| \). We then need to refine to higher order the description of both the inner solution at \( \zeta \sim \nu \) and the outer solution at \( \zeta \sim 1 \).

To state our results, we use the notation \( A \lesssim B \) to say that there exists a constant \( C > 0 \) such that \( 0 \leq A \leq CB \). Similarly, \( A \sim B \) means that there exist constants \( 0 < c < C \) such that \( cA \leq B \leq CA \). We write \( \langle r \rangle = \sqrt{1 + r^2} \), and use the notation \( D^k \) for \( k \in \mathbb{N} \) for \( k \)-th adapted derivative defined by
\[
D^{2k} = \left( \zeta \partial_\zeta \left( \frac{\partial_\zeta}{\zeta} \right) \right)^{2k}, \quad D^{2k+1} = \partial_\zeta D^{2k},
\]
and the weight functions
\[
\omega_\nu(\zeta) = \frac{\nu^2}{U_\nu(\zeta)}e^{-\frac{\zeta^2}{2}} = \frac{\nu^2}{U_\nu(\zeta)}\rho_0(\zeta), \quad \rho_0(\zeta) = e^{-\frac{\zeta^2}{2}},
\]
with corresponding weighted Lebesgue space \(L^2_\omega\) and Sobolev spaces \(H^k_\omega = \{ f : \sum_0^k \| D^k f \|_{L^2_\omega} < \infty \}\).

Our first main result is to describe in details spectral properties of \(\mathcal{A}_\nu\) in the regime (1.12).

**Proposition 1.1** (Spectral properties of \(\mathcal{A}_\nu\)). The linear operator \(\mathcal{A}_\nu : H^2_\omega \to L^2_\omega\) is essentially self-adjoint with compact resolvent. Moreover, given any \(N \in \mathbb{N}\), \(0 < \beta_\nu < \beta^*\) and \(0 < \delta_\nu < 1\), there exists a \(\nu^* > 0\) such that the following holds for all \(0 < \nu < \nu^*\) and \(\beta_\nu \leq \beta^*\).

(i) (Eigenvalues) The first \(N + 1\) eigenvalues are given by
\[
\lambda_{n,\nu} = 2\beta \left(1 - n + \tilde{\alpha}_{n,\nu}\right), \quad n = 0, 1, \ldots, N,
\]
where
\[
\tilde{\alpha}_{n,\nu} = \frac{1}{2\ln \nu} + \tilde{\alpha}_{n,\nu} \quad \text{with} \quad |\tilde{\alpha}_{n,\nu}| + |\nu \partial_\nu \tilde{\alpha}_{n,\nu}| \lesssim \frac{1}{|\ln \nu|^2}.
\]

In particular, we have the refinement of the first two eigenvalues,
\[
i = 0, 1, \quad |\tilde{\alpha}_1 - \frac{1}{2\ln \nu} - \frac{\ln 2 - \gamma - i - \ln \beta}{4|\ln \nu|^2}| \lesssim \frac{1}{|\ln \nu|^3},
\]
where \(\gamma\) is the Euler’s constant.

(ii) (Eigenfunctions) There exists associated eigenfunctions \(\phi_{n,\nu}\) satisfying the following: For all \((m, n) \in \mathbb{N}^2\),
\[
\langle \phi_{n,\nu}, \phi_{m,\nu} \rangle_{L^2_\omega} = c_n \delta_{m,n}, \quad c_0 \sim \frac{|\ln \nu|}{8}, \quad c_1 \sim \frac{|\ln \nu|^2}{4}, \quad c|\ln \nu|^2 \leq c_n \leq \frac{1}{c}|\ln \nu|^2,
\]
where \(c\) is some strictly positive constant. We also have the pointwise estimates for \(k = 0, 1, 2\),
\[
|D^k \phi_{n,\nu}(\zeta)| + |D^k \beta \partial_\beta \phi_{n,\nu}(\zeta)| + |D^k \nu \partial_\nu \phi_{n,\nu}(\zeta)| \lesssim \left(\frac{\zeta}{\nu + \zeta}\right)^{2 - (k \mod 2)} \left(\frac{\zeta}{\nu + \zeta}\right)^{2 + \delta} \left(1 + \zeta^2 \ln \frac{\zeta}{\nu}\right)^2 \{\nu \geq 1\},
\]

(iii) (Spectral gap estimate) For any \(g \in L^2_\omega\) with \(\langle g, \phi_{j,\nu} \rangle_{L^2_\omega} = 0\) for \(0 \leq j \leq N\), one has
\[
\langle g, \mathcal{A}_\nu g \rangle_{L^2_\omega} \leq \lambda_{N+1,\nu} \| g \|^2_{L^2_\omega}.
\]

**Remark 1.2.** We recover the same eigenvalues as in [11], and our proof has a similar spirit by means of matched asymptotic expansions. Here, we adopt a different approach inspired by the work of [6], yielding a more detailed information on the eigenfunctions and on the variations with respect to the parameter \(\nu\). We also mention that the matching procedure performed in [11] was formal as those analysis did not involve the matching of derivatives. To match the derivatives, we found a degeneracy that forces us to expand both inner and outer solutions to the next order, which renders the analysis much more involved. In addition, Propositions 1.6 and 1.9 are new.

**Remark 1.3.** Based on Proposition 1.1, we are able to construct for the problem (1.2) in [7] finite time blowup solutions according to the following precise asymptotic dynamics as \(t \to T\):
\[
u = \eta^2(\eta(t))U \left(\frac{x}{\eta(t)}\right),
\]
where the blowup rate is given by either
\[
\eta(t) = 2e^{-\frac{\gamma^2}{2} \sqrt{T-t}} e^{-\sqrt{\frac{|\ln(T-t)|}{\gamma^2}} (1 + o_{\gamma T}(1))},
\]
or
\[ \eta(t) \sim C(u_0)(T-t)^{\frac{\ell}{2}}|\ln(T-t)|^{-\frac{\ell+1}{2}} \quad \text{for some} \quad \ell \in \mathbb{N}, \quad \ell \geq 2. \]

It is worth saying that the rigorous analysis performed in [7] is greatly simplified thanks to Proposition 1.1 in comparison with the one of [24]. Importantly, we believe that the precise description of the spectrum of \( \mathcal{A}_\xi \) is one of the crucial steps toward the classification of all possibilities of blowup speeds for (1.2) (at least in the radial setting) which is a challenging problem in the analysis of blowup.

**Remark 1.4.** Our proof provides a more detailed description of the eigenfunction \( \phi_{n,\nu} \). More precisely, it is given by
\[
\phi_{n,\nu}(\zeta) = \sum_{j=0}^{n} c_{n,j} \beta^{j} \nu^{2j-2} T_{j}(\frac{\zeta}{\nu}) + \tilde{\phi}_{n,\nu}, \quad c_{n,j} = 2^{j} \frac{n!}{(n-j)!},
\]
where \( \tilde{\phi}_{n,\nu} \) is a lower order term, and \( T_{j} \)'s are profiles defined iteratively by
\[
T_{j+1}(r) = -\mathcal{A}_0^{-1} T_{j}(r) \quad \text{with} \quad T_{0}(r) = \frac{r^{2}}{(1+r^{2})^{2}}, \quad \mathcal{A}_0 T_{0} = 0,
\]
and admit the asymptotic behavior at infinity (see Lemma 2.2)
\[
T_{j}(r) \sim \tilde{d}_{j} r^{2j-2} \ln r, \quad r \gg 1, \quad \tilde{d}_{j+1} = -\frac{\tilde{d}_{j}}{4j(j+1)}.
\]
Here, \( \mathcal{A}_0 \) is the linearised operator near the stationary state, a rescaled version of \( \mathcal{A}_\xi \) via the change of variable \( \zeta = \nu r \), i.e.
\[
\mathcal{A}_0 = \partial_{r}^{2} - \frac{1}{r} \partial_{r} + \frac{\partial_{r}(Q \cdot)}{r} \quad \text{with} \quad Q(r) = \frac{4r^{2}}{1+r^{2}}.
\]

**Remark 1.5.** The present result deals with the critical Keller-Segel system. We believe that other critical problems can be studied with this approach, such as the harmonic heat flow and the semilinear heat equation. Related spectral studies were performed in the case of non-degenerate self-similar singularities for wave type equations, see for example [10], [9] for the study of stability of self-similar wave maps. It is an interesting direction to implement the present work to the hyperbolic setting.

Our second result aims at understanding under what kind of perturbations Proposition 1.1 is stable. This is a particular importance for the full nonlinear problem (1.2) analysed in [7], and shows the robustness of our approach. As a direct consequence of our construction, the spectral properties of \( \mathcal{A}_\xi \) stated in the previous proposition still hold true for the following perturbed operator of the form
\[
\mathcal{A}_\xi^\epsilon = \mathcal{A}_\xi + \frac{1}{\xi} \partial_\zeta (P \cdot),
\]
where the perturbation \( P \) satisfies
\[
|P(\zeta)| + |\zeta \partial_\zeta P(\zeta)| \lesssim \frac{\nu^{2}}{|\ln \nu| (\nu^{2} + \zeta^{2})^{2}}.
\]

**Proposition 1.6.** Assume the bound (1.22) and the same hypotheses as in Proposition 1.1. Then, the operator \( \mathcal{A}_\xi^\epsilon : H_{\frac{\xi}{\zeta}}^{2} \rightarrow L_{\frac{\xi}{\zeta}}^{2} \) is essentially self-adjoint with compact resolvent, where
\[
\tilde{\omega}_{n}(\zeta) = \omega_{n}(\zeta) \exp \left( \int_{0}^{\zeta} \frac{P(\zeta)}{\zeta} d\zeta \right).
\]
The first \( N + 1 \) eigenvalues \( \{\tilde{\lambda}_{n,\nu}\}_{0 \leq n \leq N} \) of \( \mathcal{A}_\xi^\epsilon \) satisfy
\[
|\tilde{\lambda}_{n,\nu} - \lambda_{n,\nu}| \leq \frac{C'}{|\log \nu|^{2}},
\]
and the associated eigenfunctions \( \{ \phi_{n,\nu} \}_{0 \leq n \leq N} \) satisfy
\[
\frac{\| \phi_{n,\nu} - \phi_{n,\nu^*} \|_{L^2(\mathbb{R}^N)}^2}{\| \phi_{n,\nu} \|_{L^2(\mathbb{R}^N)}^2} \leq C' \frac{\| \phi_{n,\nu} \|_{L^2(\mathbb{R}^N)}}{\sqrt{\log \nu}}.
\]  
\[(1.24)\]

**Remark 1.7.** Note that Proposition 1.6 is not a direct consequence of Proposition 1.1 in the sense that a standard perturbation argument does not work here. Indeed, the potential part \( \partial_\nu P/\zeta \) of the perturbation in (1.21) is of size \( \nu^{-2} \) in \( L^\infty \) (up to a logarithmic accuracy), while the eigenvalues of the unperturbed operator \( \mathcal{A}^\zeta \) are of order 1. The crucial point is that the algebraic form of the perturbation, \( \partial_\nu (P\cdot)/\zeta \), ensures its orthogonality to the resonance of the operator \( \mathcal{A}_0 \) near the origin, see Lemma 2.4 and its proof.

**Remark 1.8.** In [7], the use of Proposition 1.6 is essential to handle nonlinear terms, where the precise control of the solution near the origin involves the rescaled stationary state at a slightly different scale \( \tilde{\nu} \), and the corresponding perturbed linear operator is (1.21) with the perturbation potential
\[
P(\zeta) = \frac{Q_\nu(\zeta) - Q_\nu(\zeta)}{2}, \quad \left| \frac{\tilde{\nu}}{\nu} - 1 \right| \lesssim \frac{1}{|\log \nu|}.
\]

and the corresponding weight function
\[
\tilde{\omega}_\nu(\zeta) = \frac{\nu \tilde{\nu}}{\sqrt{U_\nu U_\nu'}} \rho(\zeta).
\]

Our third and last results concerns the decay of the linearised dynamics associated to \( \mathcal{L}^z \) for the nonradial part of the perturbation. An analogue of the radial spectral analysis of Proposition 1.1 is not straightforward. Indeed, while the operator \( \nabla \Delta^{-1} \) is an integral operator from the origin in the radial case, the integral involve the behaviour of the function at infinity on higher order spherical harmonics, see (A.6) and (A.8). In particular, it is not possible to make sense of \( \nabla \Delta^{-1} \) for nonradial functions with strong polygonal growth at infinity. To work around this problem we prove a coercivity estimate for a modified version of the linearised operator, in which the source term for the Poisson field is localised near the origin.

On the one hand, at the \( |z| \sim \nu \) scale, there is a natural scalar product for the linearised operator without scaling term, coming from the free energy. The following corresponds to [24], Lemma 2.1 and Proposition 2.3. The linearized operator at scale \( \nu \) is written as
\[
\mathcal{L}_0 u = \Delta u - \nabla \cdot (u \nabla \Phi_U) - \nabla \cdot (U \nabla \Phi_u) = \nabla \cdot (U \nabla \mathcal{M} u) \quad \text{with} \quad \mathcal{M} u = \frac{u}{U} - \Phi_u,
\]
\[(1.25)\]

The quadratic form \( \int u \mathcal{M} ud\nu \) is symmetric. There holds the estimates if \( \int u d\nu = 0 \):
\[
\int_{\mathbb{R}^2} U |\mathcal{M} u|^2 d\nu \lesssim \int_{\mathbb{R}^2} \frac{u^2}{U^2} d\nu,
\]
\[(1.26)\]

the nonnegativity \( \int u \mathcal{M} u d\nu \geq 0 \) and for some \( \delta_1 > 0 \),
\[
\int_{\mathbb{R}^2} u \mathcal{M} u d\nu \geq \delta_1 \int_{\mathbb{R}^2} \frac{u^2}{U^2} d\nu - C \left[ \langle u, \Lambda U \rangle_{L^2}^2 + \langle u, \partial_1 U \rangle_{L^2}^2 + \langle u, \partial_2 U \rangle_{L^2}^2 \right].
\]
\[(1.27)\]

For functions orthogonal to \( \Lambda U, \partial_{y_1} U, \partial_{y_2} U \) in the \( L^2 \) sense, the norms defined by \( \int \frac{u^2}{U^2} d\nu \) and \( \int u \mathcal{M} u d\nu \) are then equivalent. On the other hand, at scale \( |z| \sim 1 \), from (1.3) and as \( \partial \Phi_U = -4\zeta/(\nu^2 + \zeta^2) \) we get that \( \mathcal{L}^z \) converges pointwise to \( \Delta + 4/\zeta \partial_\nu - \beta \nabla \cdot (\cdot.) \) as \( \nu \to 0 \). This operator is self adjoint in \( L^2(\zeta^4 \rho_0) \).

We thus introduce the "mixed" scalar product
\[
\langle u, v \rangle_* := \nu^2 \int_{\mathbb{R}^2} u \sqrt{\rho_0} \mathcal{M}^z (v \sqrt{\rho_0}) d\nu, \quad \mathcal{M}^z f = \frac{f}{U_\nu} - \Phi_f
\]
\[(1.28)\]

It matches to leading order the first scalar product at scale \( \nu \) and the second at scale 1, and localises the Poisson field. It is equivalent to the \( L^2_{\omega_\nu} \) scalar product under the aforementioned orthogonality conditions. We localize the Poisson field in the linearized operator accordingly,
\[
\tilde{\mathcal{L}}^z u = \Delta u - \nabla \cdot (u \nabla \Phi_{\nu}) - \nabla \cdot (U_\nu \nabla \Phi_u) - \beta \nabla \cdot (zu), \quad \tilde{\Phi}_u = \frac{1}{\sqrt{\rho_0}} (-\Delta)^{-1}(\sqrt{\rho_0} u).
\]
We claimed that in the non-radial sector, the localized operator $\hat{L}$ is coercive for the mixed scalar product $\langle \cdot, \cdot \rangle_*$ under the natural orthogonality assumption to $\nabla U_\nu$. Its proof adapts the arguments of [24] for the above coercivity of $L_0$ to a perturbation framework $0 < \nu \ll 1$.

**Proposition 1.9.** For any $0 < \beta_* < \beta$, there exists $c, C > 0$ and $\nu^* > 0$ such that for all $\beta_* < \beta < \beta^*$ and $0 < \nu \leq \nu^*$, if $u \in \dot{H}^1_\nu$ satisfies $\int_{|z|=\zeta} udz = 0$ for almost every $\zeta$, then:

$$\langle -\hat{L}^z u, u \rangle_* \geq c \|\nabla u\|^2_{L_2^\nu} - C \nu^6 \left( \left( \int_{\mathbb{R}^2} u\partial_{z_1} U_\nu \sqrt{\rho_0} dz \right)^2 + \left( \int_{\mathbb{R}^2} u\partial_{z_2} U_\nu \sqrt{\rho_0} dz \right)^2 \right). \quad (1.29)$$

**Remark 1.10.** The above Proposition holds for $\hat{L}^z$ instead of $L^z$: a part of the Poisson field outside the origin has been neglected. However, the worst contribution to this field for the perturbation is from the origin, and the stationary states decays rapidly at infinity. The difference $L^z - \hat{L}^z$ can then be controlled from other norms, see [7].

The rest of the paper is devoted to the proof of Propositions 1.1, 1.6 and 1.9.

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**2. Proof of the spectral Proposition**

This section is devoted to the proof of Proposition 1.1, namely that we solve the eigenproblem (1.11). After a scaling change $\zeta = \nu r$, the problem (1.11) is equivalent to the following

$$(\mathcal{A}_0 - br\partial_r)\phi = \alpha \phi, \quad r \in [0, +\infty), \quad (2.1)$$

where $\mathcal{A}_0$ is introduced in (1.20) and

$$b = \beta \nu^2, \quad \alpha = \lambda \nu^2.$$  

We will solve the problem (2.1) in the regime $0 < b \ll 1$ by means of matched asymptotic expansions in the following sense. Let $\zeta_0$ and $R_0$ be fixed as

$$0 < \zeta_0 \ll 1, \quad R_0 = \frac{\zeta_0}{\sqrt{b}} \gg 1.$$  

By perturbation theory, we first solve (2.1) in the inner region $r \leq R_0$, and the solution is named by $\phi^{in}$, then in the outer region $r \geq R_0$ and the solution is named by $\phi^{ex}$. The two solutions must coincide at $r = R_0$ up to the first derivative from which we determine the value of $\alpha$ by standard arguments based on the implicit function theorem. Our technique is inspired by the work of Collot-Merle-Raphael [6] for the energy super-critical semilinear heat equation. In particular, we aim at providing more information about the eigenvalues and the eigenfunctions from which we can perform a rigorous analysis for studying singularity formation in the two dimensional Keller-Segel system (1.2), see [7]. Proposition 1.1 is a direct consequence of the following after a simple scaling change $\zeta = \nu r$ and $b = \beta \nu^2$.

**Proposition 2.1** (Spectral properties of $\mathcal{A} = \mathcal{A}_0 - br\partial_r$). Assume $0 < b \ll 1$, the linear operator $\mathcal{A}$ is self-adjoint in $L^2(\mathbb{R}^+, \omega_0 dr)$ with $\omega_0^{-1} = rUe^{\frac{b^2}{2}}$ and has purely discrete spectrum given by

$$\sigma(\mathcal{A}) = \left\{ \alpha_n = 2b \left( 1 - n + \tilde{\alpha}_n \right), \quad n \in \mathbb{N} \right\}. \quad (2.2)$$

(i) (Eigenvalues) We have

$$\tilde{\alpha}_n = \frac{1}{\ln b} + \bar{\alpha}_n \quad \text{with} \quad |\bar{\alpha}_n| + |b\partial_b \tilde{\alpha}_n| \lesssim \frac{1}{|\ln b|^2}. \quad (2.3)$$

In particular, we have a refinement of the first two eigenvalues,

$$i = 0, 1, \quad \left| \tilde{\alpha}_i - \frac{1}{\ln b} - \frac{\ln 2 - \gamma - i}{|\ln b|^2} \right| \lesssim \frac{1}{|\ln b|^3},$$
where \( \gamma \) is the Euler’s constant.

(ii) (Eigenfunctions) The eigenfunction \( \phi_n \) corresponding to \( \alpha_n \) is defined by (2.68) and the following properties hold:
- (Sign-changing) For \( n \geq 1 \), we have
  \[
  Z[\phi_n, (0, +\infty)] = n - 1,
  \]
  where \( Z[f, (a, b)] \) denotes the number of zeros of \( f \) in the interval \((a, b)\).
- (Orthogonality)
  \[
  \forall (m, n) \in \mathbb{N}^2, \quad \langle \phi_n, \phi_m \rangle_{L^2_{\overline{\mathcal{A}}}} = c_n \delta_{m,n}, \quad c_n \sim \begin{cases} 2^{-4/3} |\ln b|^{n+1} e_n |\ln b|^2 & \text{for } n = 0, 1, \\ e_n |\ln b|^2 & \text{for } n \geq 2, \end{cases}
  \]
  where \( e_n \)'s are some strictly positive constants.
- (Pointwise estimates) For \( k = 0, 1, 2 \),
  \[
  \left| \partial^k \phi,_{\overline{\mathcal{A}}} (\zeta) \right| \lesssim \frac{(\zeta)^{2n+2}(1 + |\ln b| 1_{\{n \geq 1\}})}{(\zeta + \sqrt{b})^{2+k}}, \quad \left| \partial^k \overline{\mathcal{A}} \phi,_{\overline{\mathcal{A}}} (\zeta) \right| \lesssim \frac{(\zeta)^{2n+2}}{(\zeta + \sqrt{2b})^{2+k}},
  \]
  where \( \phi,_{\overline{\mathcal{A}}} (\zeta) = \frac{1}{\sqrt{2b}} \phi_n \left( \frac{1}{\sqrt{2b}} \right) \).

(iii) (Spectral gap estimate) For any \( g \in L^2(\mathbb{R}^+, \omega_b dr) \) with \( \langle g, \phi_j \rangle_{L^2_{\overline{\mathcal{A}}}} = 0 \) for \( 0 \leq j \leq k \), one has
  \[
  \|g,_{\overline{\mathcal{A}}}\|^2_{L^2_{\overline{\mathcal{A}}}} \leq \alpha_{k+1} \|g\|^2_{L^2_{\overline{\mathcal{A}}}}.
  \]

Proof. Since the computation of \((\alpha_n, \phi_n)\) through the matching asymptotic procedure is long and technical, it is left to next subsections. Here we deal with the discreteness of \( \sigma(\mathcal{A}) \) and its uniqueness relying on classical arguments for the second order linear operator.

- Discreteness of \( \sigma(\mathcal{A}) \): The first observation is that we can rewrite the linear operator \( \mathcal{A} \) as
  \[
  \mathcal{A} f = \partial^2 f - \left( \frac{1}{r} + \Phi' + b r \right) \partial_r f + U f = \frac{1}{\omega_b} \partial_r (\omega_b \partial_r f) + U f,
  \]
  which directly gives the self-adjointness of \( \mathcal{A} \) in \( L^2(\mathbb{R}^+, \omega_b dr) \). For the discreteness of the spectrum of \( \mathcal{A} \), we let
  \[
  \rho_b(r) = \frac{1}{r^2 \sqrt{U} e^{-\frac{br}{2}}},
  \]
  and observe that
  \[
  \mathcal{A} f = \rho_b \mathcal{A} \left( \rho_b^{-1} f \right) = \frac{\rho_b}{\omega_b} \partial_r (\omega_b \partial_r (\rho_b^{-1} f)) + U f
  \]
  \[
  = \partial^2 f + \frac{\rho_b}{\omega_b} \left[ \partial_r \left( \frac{\omega_b}{\rho_b} \right) + \omega_b \partial_r (\rho_b^{-1}) \right] \partial_r f + \frac{\rho_b}{\omega_b} \partial_r (\omega_b \partial_r (\rho_b^{-1})) f + U f
  \]
  \[
  = \partial^2 f + \frac{3}{r} \partial_r f + \left[ -\frac{b^2 r^2}{4} + \frac{2b^2 r^2}{1 + r^2} \right] f + U f.
  \]
  The linear operator \( \mathcal{A} \) is of Schrödinger type and self-adjoint in \( L^2(\mathbb{R}^+, r^3 dr) \). Since \( \mathcal{A} \) has the real potential tending to infinity as \( r \to +\infty \), its spectrum is purely discrete by standard arguments. This concludes the discreteness of spectrum of \( \mathcal{A} \).

- Uniqueness of \((\alpha_n, \phi_n)\): We rely on Sturm comparison theorem to show that the eigenvalues of \( \mathcal{A} \) are only given by (2.2) for \( b \ll 1 \). We argue by contradiction and assume that there exists \( \alpha^*_n \in (\alpha_{n+1}, \alpha_n) \) for some \( n \in \mathbb{N} \) is an eigenvalue of \( \mathcal{A} \). Denote by \( \phi^*_n \) the eigenfunction corresponding to \( \alpha^*_n \), Sturm comparison theorem asserts that
  \[
  Z[\phi_{n+1}, (a, b)] \geq Z[\phi^*_n, (a, b)] \geq Z[\phi_n, (a, b)],
  \]
  from which and (2.4) we have
  \[
  n \geq Z[\phi^*_n, (0, +\infty)] \geq n - 1
  \]
The Sturm theorem again deduces that if $\mathcal{Z}[\phi_n^*, (0 + \infty)] = n$, then $\mathcal{Z}[\phi_{n+1}, (0 + \infty)] = n + 1$ or if $\mathcal{Z}[\phi_n^*, (0 + \infty)] = n - 1$, then $\mathcal{Z}[\phi_n, (0 + \infty)] = n - 2$, which is a contradiction. This concludes the proof of the uniqueness of $\alpha_n$ given in (2.2) as well as the proof of Proposition 1.1.

\section{2.1. Analysis in the inner zone $r \leq R_0$}

In this part, we solve equation (2.1) in the interval $[0, R_0]$ where we consider $-br\partial_r - \alpha$ as a small perturbation of $\mathcal{A}_0$. Let us recall some basic properties of $\mathcal{A}_0$ in the following. We introduce the norms

$$\|f\|_{X_r^a} = \sup_{r \in [0, R_0]} \frac{\langle r \rangle^{2-a}}{r^2(1 + \ln \langle r \rangle)}|f(r)|$$

for $a \in \mathbb{R}$, and the function spaces for $i = -1, 0, 1$,

$$\mathcal{T}_i^a = \left\{ f : \|f\|_{\mathcal{T}_i^a} \triangleq \|f\|_{X_r^a} + \|r \partial_r f\|_{X_r^a} + \|r^2 \partial_r^2 f\|_{X_r^a} < +\infty \right\}.$$  \hspace{1cm} (2.8)

\textbf{Lemma 2.2 (Properties of $\mathcal{A}_0$).}

(i) (Inversion) For any $f \in \mathcal{C}(\mathbb{R}^+, \mathbb{R})$, a solution to $\mathcal{A}_0 u = f$ is given by:

$$\mathcal{A}_0^{-1} f(r) = \frac{1}{2} \psi_0(r) \int_r^1 \frac{\xi^4 + 4\xi^2 \ln \xi - 1}{\xi} f(\xi) d\xi + \frac{1}{2} \tilde{\psi}_0(r) \int_0^r \xi f(\xi) d\xi,$$

where $\psi_0$ and $\tilde{\psi}_0$ are the two linearly independent solutions to $\mathcal{A}_0 \psi = 0$ given by

$$\psi_0(r) = \frac{2}{\langle r \rangle^4} \quad \text{and} \quad \tilde{\psi}_0(r) = \frac{r^4 + 4r^2 \ln r - 1}{\langle r \rangle^4}.$$  \hspace{1cm} (2.9)

(ii) (Continuity) Let $i \in \{-1, 0, 1\}$ and $a > -2$, then there holds the estimate:

$$\|\mathcal{A}_0^{-1} f\|_{\mathcal{T}_i^{a+2}} \lesssim \|f\|_{X_r^a}.$$  \hspace{1cm} (2.10)

(iv) (Iterative kernel of $\mathcal{A}_0$) There exists a family of smooth radial functions $\{T_i\}_{i \in \mathbb{N}}$ defined as

$$\mathcal{A}_0 T_{i+1} = -T_i, \quad T_0 = \psi_0,$$

which admit the asymptotic estimates

$$|r \partial_r^p T_i| = \mathcal{O}(r^3) \quad \text{as} \quad r \to 0, \quad \forall p \in \mathbb{N},$$

$$T_i = r^{2(i-1)} \left( \hat{d}_i \ln r + d_i + \hat{d}_0 \right) + \mathcal{O}(r^{2(i-2)} \ln^{i+1} r) \quad \text{as} \quad r \to +\infty,$$

$$r \partial_r T_i = r^{2(i-1)} \left[ 2(i-1) (\hat{d}_i \ln r + d_i + \hat{d}_0) + \hat{d}_i \right] + \mathcal{O}(r^{2(i-2)} \ln^{i+1} r),$$

$$|r \partial_r^p T_i| = \mathcal{O}(r^{2(i-1)} \ln r) \quad \text{as} \quad r \to +\infty, \quad \forall p \in \mathbb{N},$$

where $d_i \in \mathbb{R}$ and

$$\hat{d}_1 = \frac{1}{2}, \quad d_1 = \frac{1}{4}, \quad \hat{d}_{i+1} = -\frac{d_i}{4i(i+1)}, \quad d_{i+1} = \frac{1}{8} \left( \frac{d_i - 2i d_i}{r^2} - \frac{d_i - (2i + 2)d_i}{(i + 1)^2} \right).$$  \hspace{1cm} (2.11)

\textbf{Proof.} (i) By the scaling invariance of the problem (1.2), we have $\frac{d}{dx} [\Delta U - \nabla \cdot (\nabla U)]|_{x=1} = 0$, or $\mathcal{L}^y \nabla U = 0$. Hence $\psi_0 = \frac{1}{8} \int_0^1 \nabla U(x) dx$ is the first fundamental solution to $\mathcal{A}_0 \psi = 0$. The explicit formula of $\tilde{\psi}_0$ follows from the integration of the Wronskian relation, and the formula (2.9) is a standard way to solve linear second order ODEs.

(ii) We denote $u = \mathcal{A}_0^{-1} f$. We directly compute from (2.9) for $r \leq 1$ that for any $a, i,$

$$|u(r)| \lesssim \left| \psi_0(r) \int_r^1 \frac{\xi^4 + 4\xi^2 \ln \xi - 1}{\xi} f(\xi) d\xi + \tilde{\psi}_0(r) \int_0^r \xi f(\xi) d\xi \right| \lesssim \left( \sup_{0 \leq \xi \leq 2} \xi^{-2} |f(\xi)| \right) \left( r^2 \int_r^1 \xi d\xi + \int_0^r \xi^3 d\xi \right) \lesssim r^4 \|f\|_{X_r^a}.$$
For $1 \leq r \leq R_0$, we use again formula (2.9) to compute for $t = 0, 1$ and $a > -2$:

$$|u(r)| \lesssim |\psi_0(r)| \int_0^r \xi^3|f(\xi)|d\xi + |\tilde{\psi}_0(r)| \int_0^r \xi|f(\xi)|d\xi \lesssim r^{-2} \sup_{1 \leq \xi \leq R_0} \frac{\xi^{-a}|f(\xi)|}{(1 + \ln(\xi))^3} \int_1^\xi \xi^3 \xi^{a}(1 + \ln(\xi))^3d\xi + \sup_{0 \leq \xi \leq R_0} \frac{(\xi^{-a}|f(\xi)|}{(1 + \ln(\xi))^3} \int_0^\xi \xi^{a}(1 + \ln(\xi))^3d\xi \lesssim ||f||_{X^a} r^{a+2}(1 + \ln \langle r \rangle)^3.$$  

For $i = -1$ we first notice that the function $1 + \frac{2 \ln \langle r \rangle}{\ln b}$ is decreasing and satisfies for any $r \in [0, R_0]$:

$$\frac{1}{\ln b} \leq \left| \ln \xi \right| \leq 1 + \frac{2 \ln \langle r \rangle}{\ln b} \leq 1,$$

so that for $r \in [1, R_0]$ and $a > -1$, with constants independent on $b$:  

$$\int_0^r (\xi^a(1 + \frac{2 \ln \langle \xi \rangle}{\ln b}))d\xi \lesssim 1 + \left| \int_2^r \xi^a(1 + \frac{2 \ln \langle \xi \rangle}{\ln b})d\xi \right| \lesssim 1 + (\frac{r^{a+1}}{\ln b}) \approx (1 + \frac{2 \ln \langle r \rangle}{\ln b}).$$  

Hence for $i = -1$ and $a > -2$, computing as above:

$$|u(r)| \lesssim r^{-2} \sup_{1 \leq \xi \leq R_0} \frac{\xi^{-a}|f(\xi)|}{1 + (\frac{2 \ln \langle \xi \rangle}{\ln b})} \int_1^\xi \xi^3 \xi^{a}(1 + \frac{2 \ln \langle \xi \rangle}{\ln b})d\xi + \sup_{0 \leq \xi \leq R_0} \frac{(\xi^{-a}|f(\xi)|}{1 + (\frac{2 \ln \langle \xi \rangle}{\ln b})} \int_0^\xi \xi^{a}(1 + \frac{2 \ln \langle \xi \rangle}{\ln b})d\xi \lesssim ||f||_{X^a} r^{a+2}(1 + \frac{2 \ln \langle r \rangle}{\ln b}).$$

The estimates above imply for any $a > -2$ and $i = -1, 0, 1$, with a constant independent on $b$ and $\xi_0$:  

$$||\mathcal{A}_0^{-1} f||_{X^{a+2}} \lesssim ||f||_{X^a}.$$  

To estimate the derivatives, we notice from (2.9) that

$$\partial_r u = \frac{1}{2} \partial_r \psi_0 \int_0^1 \xi^{4} + 4 \xi^{2} \ln \frac{\xi}{\xi} \langle f(\xi) \rangle d\xi + \frac{1}{2} \partial_r \tilde{\psi}_0 \int_0^t \xi f(\xi) d\xi,$$  

Hence, with the very same estimates that we do not repeat we obtain for $i = -1, 0, 1$ and $a > -2$:

$$||r \partial_r u||_{X^{a+2}} \lesssim ||f||_{X^a}.$$  

Next, using that $\mathcal{A}_0 u = f$ and the definition of $\mathcal{A}_0$ yields

$$\partial_r^2 u = f + \left( \frac{1}{r} - \frac{4r}{\langle r \rangle^2} \right) \partial_r u - \frac{8}{\langle r \rangle^4} u,$$

so that for $i = -1, 0, 1$ and $a > -2$, using the previous estimates for $u$ and $r \partial_r u$:

$$||r^2 \partial_r u||_{X^{a+2}} \lesssim ||r^2 f||_{X^{a+2}} + \left( \frac{1}{r} - \frac{4r^3}{\langle r \rangle^2} \right) \partial_r u||_{X^{a+2}} + \frac{8r^4}{\langle r \rangle^4} ||f||_{X^{a+2}} \lesssim ||f||_{X^{a}} + ||f||_{X^{a}} + ||f||_{X^{a}} \lesssim ||f||_{X^{a}}.$$  

This concludes the proof of (2.11).

(iv) For $r \ll 1$, we compute from (2.9)

$$|T_1(r)| + |r \partial_r T_1(r)| = O \left( r^2 \int_r^1 \xi^{-1} \xi^2 d\xi + \int_0^r \xi \xi^2 d\xi \right) = O(r^2) \quad \text{as} \quad r \to 0.$$
We use $\mathcal{A}_0 T_1 = -\psi_0$ and the definition (1.20) of $\mathcal{A}_0$ to estimate for $k \in \mathbb{N}$,

$$|(r \partial_r)^{k+2}T_1(r)| = O \left( \sum_{j=0}^{k} |r^{j+1} \partial_r^{j+1} T_1| + r^{k+2} |\partial_r^k \psi_0| \right) = O(r^2) \quad \text{as} \quad r \to 0.$$ 

Hence, the estimate (2.13) holds for $i = 1$. By induction, we assume that estimate (2.13) holds for $i \geq 1$. We compute from (2.9) and the relation $T_{i+1} = -\mathcal{A}_0^{-1} T_i$,

$$|T_{i+1}| + |r \partial_r T_{i+1}| = O \left( r^2 \int_r^1 \xi^{-1} \xi^2 d\xi + \int_0^r \xi \xi^2 d\xi \right) = O(r^2),$$

as $r \to 0$. The estimate for higher derivative follows from the relation $\mathcal{A} T_{i+1} = -T_i$ and the definition (1.20) of $\mathcal{A}_0$.

For $1 \ll r \leq R_0$, we prove (2.14) by induction. For $i = 1$, we compute from (2.9) and the relation $T_1 = -\mathcal{A}_0^{-1} \psi_0$

$$T_1(r) = \frac{1}{2} \psi_0 \int_r^1 \frac{\xi^4 + 4 \xi^2 \ln \xi - 1}{\xi} \psi_0(\xi) d\xi + \frac{1}{2} \psi_0 \int_0^r \xi \psi_0(\xi) d\xi \quad \text{and the elementary identity}$$

$$\int r^k \ln r dr = \frac{r^{k+1}[(k+1) \ln r - 1]}{(k+1)^2} \quad \text{for all} \quad k \in \mathbb{N},$$

to compute

$$T_{i+1} = \frac{1}{2} \psi_0 \left( \int_r^1 \frac{\xi^4 + 4 \xi^2 \ln \xi - 1}{\xi} T_i(\xi) d\xi + \frac{1}{2} \psi_0 \int_0^r \xi T_i(\xi) d\xi \right)$$

$$= \frac{1}{2} \psi_0 \left( \int_1^2 O(\xi) d\xi + \int_r^r \frac{[\xi^4 + O(\xi \ln \xi)] \xi^{-2} + O(\xi^{-4})}{\xi} d\xi \right)$$

$$+ \frac{1}{2} \psi_0 \left( \int_0^2 O(\xi^2) d\xi + \int_r^r \xi \left[ \xi^2 (\xi^{-1}) (\xi \ln \xi + d_i) + O(\xi^2 \ln^{i+1} \xi) \right] d\xi \right)$$

$$= \left( \frac{1}{2r^2} + O(r^{-4}) \right) \left[ \frac{d_i}{2i} \ln r - \frac{d_i - (2i+2)d_i}{(2i+2)^2} + O \left( r^{-2} \ln^{i+2} r \right) \right]$$

$$- \left( \frac{1}{2} + O(r^{-2} \ln r) \right) r^{2i} \left[ \frac{d_i}{2i} \ln r - \frac{d_i - 2id_i}{4i^2} + O \left( r^{-2} \ln^{i+1} r \right) \right]$$

$$= r^{2i} \left[ \frac{-d_i}{4i(i+1)} \ln r + \frac{1}{8} \left( \frac{d_i - 2id_i}{i^2} - \frac{d_i - (2i+2)d_i}{(i+1)^2} \right) \right] + O \left( r^{2i-2} \ln^{i+2} r \right),$$

which gives

$$\hat{d}_{i+1} = -\frac{i}{4i(i+1)}, \quad d_{i+1} = \frac{1}{8} \left( \frac{d_i - 2id_i}{i^2} - \frac{d_i - (2i+2)d_i}{(i+1)^2} \right).$$

This concludes the proof of (2.14).
The proof of (2.15) follows similarly by induction. Indeed, assuming that (2.15) holds for $i \in \mathbb{N}$, we compute from (2.9), the relation $T_{i+1} = \mathcal{A}_0^{-1} T_i$ and expansion (2.15) for $1 \leq r \leq R_0$,

\[
 r \partial_r T_{i+1} = \frac{r}{2} \partial_r \psi_0 \int_0^1 \frac{\xi e^4 + 4 \xi^2 \ln \xi - 1}{\xi} T_i(\xi) d\xi + \frac{r}{2} \partial_r \psi_0 \int_0^r \xi T_i(\xi) d\xi
\]

\[
 = (-r^{2i} + O(r^{2i-2})) \left[ \frac{\hat{d}_i}{2(i+2)} \ln r - \frac{\hat{d}_i - (2i + 2) d_i}{(2i + 2)^2} + O\left( r^{-2 \ln i^2} \right) \right] + O\left( \frac{\ln^2 r}{r^{2-2i}} \right)
\]

\[
 = r^{2i} \left[ \frac{-\hat{d}_i}{2(i+1)} \ln r + \frac{d_i - 2(i+1) d_i}{4(i+1)^2} \right] + O\left( r^{2i-2 \ln i^2} r \right).
\]

Using the recursive definition of $\hat{d}_i$ and $d_i$, i.e,

\[
 \hat{d}_{i+1} = -\frac{\hat{d}_i}{4i(i+1)}, \quad d_i = -4i(i+1)d_{i+1} - 2(2i+1)d_{i+1}, \quad (2.19)
\]

we have the simplification $\frac{-\hat{d}_i}{2(i+1)} = 2i \hat{d}_{i+1}$ and

\[
 \frac{\hat{d}_i - 2(i+1) d_i}{4(i+1)^2} = \frac{\hat{d}_i}{4i^2} - \frac{d_i}{2i} - 2d_{i+1}
\]

\[
 = -\frac{i+1}{i} \hat{d}_{i+1} + \frac{2i+1}{i} \hat{d}_{i+1} + 2i \hat{d}_{i+1} = \hat{d}_{i+1} + 2i \hat{d}_{i+1}.
\]

This concludes the proof of (2.15). The estimate (2.16) follows from the definition of $\mathcal{A}_0$ and the relation $\mathcal{A}_0 T_{i+1} = -T_i$ by induction and the Leibniz rule. This completes the proof of Lemma 2.2. \hfill \Box

In the following we show that the profile $T_j$ given in Lemma 2.2 are actually the building blocks of the eigenfunction of the linear operator $\mathcal{A} = \mathcal{A}_0 - b \partial_r$ on $[0, R_0]$. In particular, we have the following.

**Lemma 2.3** (Inner eigenfunctions for the radial mode). Let $n \in \mathbb{N}$, $0 < \zeta_0 \ll 1$ and $0 < b \ll 1$ be small enough. Then for any $|\alpha| \leq \frac{1}{\ln b} - 2$ there exists a smooth function $\phi_n^{(i)} \in C^\infty([0, R_0], \mathbb{R})$ satisfying

\[
 \mathcal{A} \phi_n^{(i)} = 2b(1 - n + \hat{\alpha}) \phi_n^{(i)} \quad \text{with} \quad \hat{\alpha} = \frac{1}{\ln b} + \bar{\alpha},
\]

(2.20)

where $\phi_n^{(i)}$ is of the form

\[
 \phi_n^{(i)}(r) = \sum_{j=0}^{n} c_{n,j} b^j T_j + b \left( \frac{2}{\ln b} T_1 + \mathcal{A}_0^{-1} \Theta_0 \right) + 2\alpha \sum_{j=0}^{n} b^{j+1} \left( -c_{n,j} T_{j+1} + S_j \right) + b \mathcal{R}_n,
\]

and the constants $(c_{n,j})_{0 \leq j \leq n}$ are given by

\[
 c_{n,j} = 2\frac{j}{(n-j)!}, \quad c_{n,j+1} = 2(n-j)c_{n,j}, \quad c_{n,0} = 1.
\]

(2.22)

The corrective functions $R_n$, $S_j$ satisfy the following estimates:

\[
 \| S_j \|_{L^2_1} + \| \partial_\alpha S_j \|_{L^2_1} + \| b \partial_\alpha S_j \|_{L^2_1} \lesssim \zeta_0^2,
\]

\[
 \| R_n \|_{L^2_0} + \| b \partial_\alpha R_n \|_{L^2_0} + \| \partial_\alpha R_n \|_{L^2_0} \lesssim 1.
\]

(2.23)

(2.24)

with the following refinements for $n = 0$ and $n = 1$:

\[
 S_0 = \frac{1}{2} \sum_{i=1}^{\infty} \frac{1}{(2i)^2} b^i r^{2i} \log(r+1) + \mathcal{S}_0, \quad \| \mathcal{S}_0 \|_{L^2_0} \lesssim b, \quad \text{for } n = 0.
\]

(2.25)

\[
 \mathcal{R}_0 = -\frac{1}{2} \sum_{i=1}^{\infty} \frac{1}{(2i)^2} b^i r^{2i} \left\{ \frac{1}{\log b} [2 \ln(r+1) - \Psi(i+2) - \gamma] + 1 \right\} + \mathcal{R}_0, \quad \| \mathcal{R}_0 \|_{L^2_0} \lesssim |\log b|^{-1}.
\]

(2.26)
\[
\mathcal{R}_1 = -\frac{1}{2} \sum_{i=1}^{\infty} \frac{(1)_i}{(2)_i i! 2^i} b^{i+2i} \left\{ \frac{1}{\log \theta} \left[ 2 \log(r+1) - \frac{1}{i} - \Psi(i+2) - \gamma \right] + 1 - \frac{1}{\log \theta} \right\} + \tilde{\mathcal{R}}_1, \quad (2.27)
\]

\[
\|\tilde{\mathcal{R}}_1\|_{L^2_{x-1}} \lesssim |\log \theta|^{-1}, \quad \|\partial_\theta \tilde{\mathcal{R}}_1\|_{L^2_{x-1}} \lesssim 1.
\]

\[
\|S_0\|_{L^2_{x-1}} + \|\partial_\theta S_0\|_{L^2_{x-1}} \lesssim 1, \quad S_1 = -\frac{1}{2} \sum_{i=2}^{\infty} \frac{(1)_i}{(2)_i i! 2^i} b^{i-1} r^{i+2i} \log(r+1) + \tilde{S}_1, \quad \|\tilde{S}_1\|_{L^2_{x-1}} + \|\partial_\theta \tilde{S}_1\|_{L^2_{x-1}} \lesssim 1. \quad (2.28)
\]

\textbf{Proof.} The proof mainly relies on classical arguments based on the Banach fixed point theorem to construct the profiles \(\mathcal{R}_n\) and \(S_j\) for \(0 \leq j \leq n\). For \(j \in \mathbb{N}\), we let

\[
\Theta_j = r \partial_\theta T_j - 2(j-1)T_j, \quad (2.29)
\]

which admits the following slowly growing tail from (2.14) and (2.15),

\[
|\Theta_0(r)| = O(r^{-4}), \quad |\Theta_j(r)| = O(r^{2(j-1)}) \quad \text{for } j \geq 1, \quad \text{as } r \to +\infty. \quad (2.30)
\]

and for \(j \geq 1:

\[
\left| \Theta_j(r) + \frac{2}{\ln \theta} T_j(r) \right| \lesssim r^2 \langle r \rangle^{2(j-2)} \left( 1 + \frac{2 \ln(r+1)}{\ln \theta} \right). \quad (2.31)
\]

We compute the integral:

\[
\int_0^\infty r \Theta_0(r) dr = \frac{2}{\ln \theta} \int_0^\infty T_0(r) r dr + \lim_{R \to \infty} \int_0^R r^2 \partial_\theta T_0(r) dr
\]

\[
= 2 \int_0^\infty T_0(r) r dr + \lim_{r \to \infty} \left( R^2 T_0(R) - 2 \int_0^R r T_0(r) dr \right)
\]

\[
= \lim_{r \to \infty} R^2 T_0(R) = 1.
\]

From this and (2.9), as \(|r \partial_\theta T_0 + 2T_0| \lesssim (1 + r)^{-4}\) the following corrective term satisfies as \(r \to \infty\):

\[
\alpha_0^{-1} \Theta_0(r) = O(\psi_0(r) \log(r)) + O(r^{-2}) + \frac{1}{2} \psi_0(r) \int_0^\infty \zeta f(\zeta) d\zeta = \frac{1}{2} + O(r^{-2} \log r),
\]

and hence:

\[
\left| \frac{2}{\ln \theta} T_1 + \alpha_0^{-1} \Theta_0 \right| \lesssim r^2 \langle r \rangle^{-2} \left( 1 + \frac{2 \ln(r+1)}{\ln \theta} \right). \quad (2.32)
\]

These estimates show that

\[
\|b^{j-1} \left( \Theta_j(r) + \frac{2}{\ln \theta} T_j(r) \right) \|_{X_{a-1}^0} \lesssim \zeta_0^{2(j-1)} \|\Theta_j(r) + \frac{2}{\ln \theta} T_j(r)\|_{X_{a-1}^2} \lesssim 1,
\]

\[
\| - \frac{2}{\ln \theta} T_1 + \alpha_0^{-1} \Theta_0 \|_{X_{a-1}^0} \lesssim 1. \quad (2.33)
\]
- Equations satisfied by $S_j$ and $\mathcal{R}_n$: Plugging the decomposition (2.21) to (2.20) and using $\mathcal{A}_0T_j = -T_{j-1}$ with the convention $T_{-1} = 0$ yields

$$\left[\mathcal{A}_0 - br\partial_r - 2b(1 - n + \frac{1}{\ln b} + \bar{\alpha})\right] \sum_{j=0}^{n} c_{n,j}b^jT_j$$

$$= -\sum_{j=0}^{n-1} c_{n,j+1}b^{j+1}T_j - \sum_{j=0}^{n} c_{n,j}b^{j+1}\left[\Theta_j + 2(j-1)T_j\right]$$

$$+ 2(n - 1) \sum_{j=0}^{n} c_{n,j}b^{j+1}T_j - 2\left(\frac{1}{\ln b} + \bar{\alpha}\right)\sum_{j=0}^{n} c_{n,j}b^{j+1}T_j$$

$$= -\sum_{j=0}^{n-1} c_{n,j}b^{j+1}T_j\left[2(n - j) + 2(j-1) - 2(n - 1)\right] - \sum_{j=0}^{n} c_{n,j}b^{j+1}\Theta_j - 2\left(\frac{1}{\ln b} + \bar{\alpha}\right)\sum_{j=0}^{n} c_{n,j}b^{j+1}T_j$$

$$= -\sum_{j=1}^{n} c_{n,j}b^{j+1} \left(\Theta_j + \frac{2}{\ln b}T_j\right) - b\Theta_0 - \frac{2b}{\ln b}T_0 - 2\bar{\alpha} \sum_{j=0}^{n} c_{n,j}b^{j+1}T_j,$$

and

$$\left[\mathcal{A}_0 - br\partial_r - 2b(1 - n + \frac{1}{\ln b} + \bar{\alpha})\right] b\left(-\frac{2}{\ln b}T_1 + \mathcal{A}_0^{-1}\Theta_0\right)$$

$$= b\Theta_0 + \frac{2b}{\ln b}T_0 - b\left[r\partial_r + 2\left(1 - n + \frac{1}{\ln b} + \bar{\alpha}\right)\right] b\left(-\frac{2}{\ln b}T_1 + \mathcal{A}_0^{-1}\Theta_0\right)$$

and

$$\left[\mathcal{A}_0 - br\partial_r - 2b(1 - n + \bar{\alpha})\right] \left(2\bar{\alpha} \sum_{j=0}^{n} b^{j+1}\left[-c_{n,j}T_{j+1} + S_j\right]\right)$$

$$= 2\bar{\alpha} \sum_{j=0}^{n} b^{j+1} \left\{\mathcal{A}_0S_j - \left[br\partial_r + 2b(1 - n + \bar{\alpha})\right]\left(-c_{n,j}T_{j+1} + S_j\right)\right\} + 2\bar{\alpha} \sum_{j=0}^{n} c_{n,j}b^{j+1}T_j.$$  

We then rewrite equation (2.20) as

$$0 = \left[\mathcal{A}_0 - br\partial_r - 2b(1 - n + \bar{\alpha})\right] \phi_n^{in}$$

$$= \bar{\alpha} \sum_{j=0}^{n} b^{j+1} \left\{\mathcal{A}_0S_j - \left[br\partial_r + 2(1 - n + \bar{\alpha})\right]\left(-c_{n,j}T_{j+1} + S_j\right)\right\}$$

$$+ b\left\{\mathcal{A}_0\mathcal{R}_n - \left[br\partial_r + 2(1 - n + \bar{\alpha})\right]\mathcal{R}_n - \sum_{j=1}^{n} c_{n,j}b^{j}\left(\Theta_j + \frac{2}{\ln b}T_j\right) - \left[r\partial_r + 2\left(1 - n + \bar{\alpha}\right)\right] b\left(-\frac{2}{\ln b}T_1 + \mathcal{A}_0^{-1}\Theta_0\right)\right\}$$

$$= \left[\mathcal{A}_0 - br\partial_r - 2b(1 - n + \bar{\alpha})\right] \left(2\bar{\alpha} \sum_{j=0}^{n} b^{j+1}\left[-c_{n,j}T_{j+1} + S_j\right]\right)$$

$$= 2\bar{\alpha} \sum_{j=0}^{n} b^{j+1} \left\{\mathcal{A}_0S_j - \left[br\partial_r + 2b(1 - n + \bar{\alpha})\right]\left(-c_{n,j}T_{j+1} + S_j\right)\right\} + 2\bar{\alpha} \sum_{j=0}^{n} c_{n,j}b^{j+1}T_j.$$  

(2.34)

- Computation of $(S_j)_{0 \leq j \leq n}$: From equation (2.34), we choose $S_j$ to be the solution of the equation

$$\mathcal{A}_0S_j = b\left[r\partial_r + 2(1 - n + \bar{\alpha})\right]\left(-c_{n,j}T_{j+1} + S_j\right).$$  

(2.35)

Note from part (iii) of Lemma 2.2 that $T_{j+1} \in \mathcal{I}^{2j}_1$ for $j \geq 0$. We aim at proving that for $b$ and $\zeta_0$ small enough, there exists a unique solution $S_j \in \mathcal{I}^{2j}_1$ to equation (2.35) though a standard argument based on the Banach fixed point theorem. Let $\Gamma$ be the affine mapping acting on $f \in \mathcal{I}^{2j}_1$ defined as

$$\Gamma(f) = \mathcal{A}_0^{-1}\left[b\left(r\partial_r + 2\left(1 - n + \bar{\alpha}\right)\right)\left(-c_{n,j}T_{j+1} + f\right)\right] = \Gamma(0) + D\Gamma(f),$$
where $A_0^{-1}$ is defined as in (2.9) and
\[
\Gamma(0) = bc_{n,j}A_0^{-1} \left( \left[ r \partial_{rr} + 2(1 - n + \alpha) \right] T_{j+1} \right),
\]
\[
D \Gamma(f) = bA_0^{-1} \left( \left[ r \partial_{rr} + 2(1 - n + \alpha) \right] f \right).
\]

We estimate from (2.11),
\[
\| \Gamma(0) \| \mathcal{T}_1^{2j+1} \lesssim R_0^2 \| \Gamma(0) \| \mathcal{T}_1^{2j+2} \lesssim R_0^2 b \| T_{j+1} \| \mathcal{T}_1^{2j} \lesssim R_0^2 b \lesssim \zeta_0^2,
\]
and for all $f \in \mathcal{T}_1$ with $a = 2j$ or $a = 2j + 2$,
\[
\| D \Gamma(f) \| \mathcal{T}_1^{2j+1} \lesssim R_0^2 \| D \Gamma(f) \| \mathcal{T}_1^{2j+2} \lesssim R_0^2 b \| A_0^{-1} \left( \left[ r \partial_{rr} + 2(1 - n + \alpha) \right] f \right) \| \mathcal{T}_1^{2j+2}
\]
\[
\lesssim \zeta_0^2 \left\| \left[ r \partial_{rr} + 2(1 - n + \alpha) \right] f \right\| \mathcal{X}_1 \lesssim \zeta_0^2 \| f \| \mathcal{T}_1
\]
(2.36)

Since $0 < \zeta_0 \ll 1$ and $\Gamma$ is an affine mapping, the above estimates implies that $\Gamma$ is a contraction on $B_{\mathcal{T}_1^{2j}}(0, C\zeta_0^2)$ for some constant $C > 0$ independent of the problem. Therefore, there exists a unique fixed point $S_j = \Gamma(S_j)$ such that $\| S_j \| \mathcal{T}_1^{2j} \lesssim \zeta_0^2$ so that the first estimate in (2.23) holds. For the estimates of $\partial_\alpha S_j$ and $\partial_\alpha S_j$, we differentiate the relation $S_j = \Gamma(S_j)$ to obtain
\[
\partial_\beta S_j = D \Gamma(\partial_\beta S_j) + (\partial_\beta \Gamma)(S_j),
\]
\[
\partial_\alpha S_j = D \Gamma(\partial_\alpha S_j) + (\partial_\alpha \Gamma)(S_j),
\]
where we have the identities since $b \partial_\alpha \tilde{\alpha} = -1/(\ln b)^2$ and $\partial_\alpha \tilde{\alpha} = 1$:
\[
\partial_\beta \Gamma(f) = \mathcal{A}_0^{-1} \left[ \left( r \partial_{rr} + 2(1 - n + \alpha - \frac{1}{\ln b^2}) \right) \left( - c_{n,j} T_{j+1} + f \right) \right],
\]
(2.37)
\[
\partial_\alpha \Gamma(f) = b \mathcal{A}_0^{-1} \left[ - c_{n,j} T_{j+1} + f \right].
\]
(2.38)

From (2.36), we see that $\| D \Gamma \| \mathcal{T}_1 \rightarrow \mathcal{T}_1^{2j} \lesssim \zeta_0^2$ with $a = 2j + 2$ or $a = 2j$. Hence $Id - D \Gamma$ is invertible and:
\[
\| \partial_\beta S_j \| \mathcal{T}_1^{2j+2} = \| (Id - D \Gamma)^{-1}(\partial_\beta \Gamma)(S_j) \| \mathcal{T}_1^{2j+2} \lesssim \| (\partial_\beta \Gamma)(S_j) \| \mathcal{T}_1^{2j+2},
\]
\[
\| \partial_\alpha S_j \| \mathcal{T}_1^{2j} = \| (Id - D \Gamma)^{-1}(\partial_\alpha \Gamma)(S_j) \| \mathcal{T}_1^{2j} \lesssim \| (\partial_\alpha \Gamma)(S_j) \| \mathcal{T}_1^{2j}.
\]

We estimate from (2.11) and (2.37),
\[
\| (\partial_\beta \Gamma)(S_j) \| \mathcal{T}_1^{2j+2} \lesssim \| T_{j+1} \| \mathcal{T}_1^{2j} + \| S_{j+1} \| \mathcal{T}_1^{2j} \lesssim 1.
\]
Similarly, we estimate from (2.11) and (2.38),
\[
\| (\partial_\alpha \Gamma)(S_j) \| \mathcal{T}_1^{2j} \lesssim bR_0^2 \left( \| T_{j+1} \| \mathcal{T}_1^{2j} + \| S_{j+1} \| \mathcal{T}_1^{2j} \right) \lesssim \zeta_0^2,
\]
which concludes the proof of (2.23).

- Refinement for $n = 1$. We do not give technical details are these are the very same ones as above for the general case. For $n = 1$ the $S_0$ equation is:
\[
\mathcal{A}_0 S_0 = b \left[ r \partial_{rr} + 2 \tilde{\alpha} \right] \left( - T_1 + S_0 \right).
\]

As $\| b (r \partial_{rr} + 2 \tilde{\alpha}) T_1 \| \mathcal{T}_1 \lesssim b$ from (2.14) and (2.20) we get by the same strategy as above $\| S_0 \| \mathcal{T}_0 + \| S_0 \| \mathcal{T}_1 \lesssim \zeta_0^2 b$. The $S_1$ equation is:
\[
\mathcal{A}_0 S_1 = b \left[ r \partial_{rr} + 2 \tilde{\alpha} \right] \left( - 2 T_2 + S_1 \right).
\]

Let $\hat{S}_1 = -\frac{1}{2} \sum_{i=2}^{\infty} (\frac{1}{2}, \frac{1}{2}) b^{i+1} r^{2i} \ln r$, which produces $(\partial_{rr} + \frac{3}{r} - br \partial_r) \hat{S}_1 = -\frac{br^2 \ln r}{4}$. Looking for a solution $S_1 = \hat{S}_1 + \dot{S}_1$ produces
\[
\mathcal{A}_0 \dot{S}_1 = b \left[ r \partial_{rr} + 2 \tilde{\alpha} \right] \hat{S}_1 + b \left( r^2 \frac{\ln r}{4} - 2 r \partial_r T_2 \right) - 4 \tilde{\alpha} T_2 + b^2 \dot{S}_1 - \left( \frac{4 r}{\langle r \rangle^4} - \frac{4}{r} \right) \partial_r + \frac{8}{\langle r \rangle^4} \dot{S}_1.
\]
The source term above is of size 1 in $T_0^2$ from (2.14) and (2.20) so that from the strategy used above $\|\tilde{S}_1\|_{T_0^2} \lesssim \zeta_0^6$ and $\|\partial_\xi \tilde{S}_1\|_{T_1^2} \lesssim \zeta_0^6$.

Refinement for $n = 0$. For $n = 0$ the $S_0$ equation is:

$$\mathcal{A}_0 S_0 = b \left[ r T_1 + S_0 \right].$$

We look for a solution $S_0 = \tilde{S}_0(r+1)+\tilde{S}_0$ with $\tilde{S}_0 = \frac{1}{2} \sum_{i=1}^{\infty} \frac{1}{(2i)^2} b^j r^{2i} \log(r)$. As $(\partial_{\xi r} + \frac{3}{2} \partial_r - b(r \partial_r + 2)) \tilde{S}_0 = b \log r$, $\tilde{S}_0$ solves

$$\mathcal{A}_0 \tilde{S}_0 = -b(2T_1 + \log(r+1)) - (4r - \frac{1}{r} - \frac{3}{r+1}) \partial_r + \frac{8}{(r)^4} b \tilde{S}_0(r+1).$$

The source term above is of size $b$ in $T_0^2$ from (2.14) and (2.20) so that from the strategy used above $\|\tilde{S}_1\|_{T_0^2} \lesssim b \zeta_0^2$.

Computation of $R_n$. From (2.34), we choose $R_n$ to be the solution of the equation

$$\mathcal{A}_0 R_n = b \left[ r T_1 + \sum_{j=1}^{n} c_{n,j} b^j \left( \Theta_j + \frac{2}{\ln b} T_j \right) \right],$$

$$+ \left[ r T_1 + \sum_{j=1}^{n} c_{n,j} b^j \left( \Theta_j + \frac{2}{\ln b} T_j \right) \right] = 0$$

where $\Theta_j$ is introduced in (2.29). The computation is similar as for $S_j$. We let $\Gamma$ be the affine mapping

$$\Gamma(f) = \Gamma(0) + D \Gamma(f),$$

where

$$\Gamma(0) = -b \mathcal{A}_0^{-1} \left[ \sum_{j=1}^{n} c_{n,j} b^j \left( \Theta_j + \frac{2}{\ln b} T_j \right) \right],$$

$$+ \left[ r T_1 + \sum_{j=1}^{n} c_{n,j} b^j \left( \Theta_j + \frac{2}{\ln b} T_j \right) \right] \left( -\frac{2}{\ln b} T_1 + \mathcal{A}_0^{-1} \Theta_0 \right),$$

$$D \Gamma(f) = b \mathcal{A}_0^{-1} \left[ (r \partial_r + 2(1 - n + \alpha)) f \right].$$

From (2.33) and (2.11) we obtain:

$$\|\Gamma(0)\|_{T_2^2} \lesssim b \sum_{j=0}^{n} \|\mathcal{A}_0^{-1} b^j \left( \Theta_j + \frac{2}{\ln b} T_j \right)\|_{T_2^2} + b \|\mathcal{A}_0^{-1} \Theta\|_{T_2^2},$$

$$\lesssim b \sum_{j=0}^{n} \|b^j \left( \Theta_j + \frac{2}{\ln b} T_j \right)\|_{T_2^2} + b \|\Theta\|_{T_2^2} \lesssim b.$$

Using (2.11), we estimate for all $f \in T_2^2$,

$$\|D \Gamma(f)\|_{T_2^2} \lesssim b \left\| (r T_1 + \sum_{j=1}^{n} c_{n,j} b^j \left( \Theta_j + \frac{2}{\ln b} T_j \right) \right\|_{T_2^2} \lesssim b R_0^2 \|f\|_{T_2^2} \lesssim \zeta_0^2 \|f\|_{T_2^2}. \quad (2.39)$$

We then deduce that $\Gamma(f)$ is contraction on $B_{T_2^2}(0, bC)$ for some constant $C > 0$, hence, there exists a unique fixed point $R_n = \Gamma(R_n)$ satisfying $\|R_n\|_{T_2^1} \lesssim b$. As $\|R_n\|_{T_2^1} \lesssim b^{-1} \|R_n\|_{T_2^1} \lesssim 1$ the first estimates in (2.24) holds. For the estimates of $\partial_\xi R_n$ and $\partial_\alpha R_n$, we differentiate the relation $R_n = \Gamma(R_n)$:

$$\partial_\xi R_n = D \Gamma(\partial_\xi R_n) + (\partial_\xi \Gamma)(R_n), \quad \partial_\alpha R_n = D \Gamma(\partial_\alpha R_n) + (\partial_\alpha \Gamma)(R_n),$$

where we have the identities since $b \partial_\alpha \tilde{\alpha} = -1/(\ln b)^2$ and $\partial_\alpha \tilde{\alpha} = 1$,
\[
\partial_b \Gamma(f) = \mathcal{A}_0^{-1} \sum_{j=1}^{n} c_{n,j} b^{j-1} \left( j \left( \Theta_j + \frac{2}{\ln b} T_j \right) - \frac{2}{\ln b^2} T_j \right) - \mathcal{A}_0^{-1} \left[ r \partial_r + 2 \left( 1 - n + \tilde{\alpha} - \frac{1}{\ln b^2} \right) \right] \left( - \frac{2}{\ln b} T_1 + \mathcal{A}_0^{-1} \Theta_0 + \frac{2}{\ln b^2} T_j \right) + \mathcal{A}_0^{-1} \left[ (r \partial_r + 2 \left( 1 - n + \tilde{\alpha} - \frac{1}{\ln b^2} \right)) f \right],
\]

\[
\partial_\alpha \Gamma(f) = b \mathcal{A}_0^{-1} f + b \mathcal{A}_0^{-1} \left( - \frac{2}{\ln b} T_1 + \mathcal{A}_0^{-1} \Theta_0 \right)
\]

We have derived from (2.39) that \( \|D\Gamma\|_{L^2_1 \to L^2_1} \lesssim \zeta_0^2 \), hence, \( \text{Id} - D\Gamma \) is invertible on \( L^2_1 \). In particular, we have the estimates

\[
\|\partial_b \mathcal{R}_n\|_{L^2_1} = \|(\text{Id} - D\Gamma)^{-1} (\partial_b \Gamma)(\mathcal{R}_n)\|_{L^2_1} \lesssim \| (\partial_b \Gamma)(\mathcal{R}_n) \|_{L^2_1},
\]

\[
\|\partial_\alpha \mathcal{R}_n\|_{L^2_1} = \|(\text{Id} - D\Gamma)^{-1} (\partial_\alpha \Gamma)(\mathcal{R}_n)\|_{L^2_1} \lesssim \| (\partial_\alpha \Gamma)(\mathcal{R}_n) \|_{L^2_1}.
\]

Using (2.33) \( \| \frac{b^{j-1}}{\log b} T_j \|_{L^2_1} \lesssim 1 \), the estimate on \( \mathcal{R}_n \), we have by (2.11):

\[
\| (\partial_b \Gamma)(\mathcal{R}_n) \|_{L^2_1} \lesssim \sum_{j=1}^{n} \left( b^{j-1} \| \mathcal{A}_0^{-1} \left( \Theta_j + \frac{2}{\ln b} T_j \right) \|_{L^2_1} + \| \mathcal{A}_0^{-1} b^{j-1} \frac{1}{\ln b^2} T_j \|_{L^2_1} \right)
\]

\[
+ \| \mathcal{A}_0^{-1} \left[ r \partial_r + 2 \left( 1 - n + \tilde{\alpha} - \frac{1}{\ln b^2} \right) \right] \left( - \frac{2}{\ln b} T_1 + \mathcal{A}_0^{-1} \Theta_0 + \frac{2}{\ln b^2} T_j \right) \|_{L^2_1}
\]

\[
+ \| \mathcal{A}_0^{-1} \left[ (r \partial_r + 2 \left( 1 - n + \tilde{\alpha} - \frac{1}{\ln b^2} \right)) \mathcal{R}_n \|_{L^2_1}
\]

\[
\lesssim 1 + \| \mathcal{R}_n \|_{L^2_1} \lesssim 1 + b^{-1} \| \mathcal{R}_n \|_{L^2_1} \lesssim 1.
\]

Similarly, we have by (2.11),

\[
\| (\partial_\alpha \Gamma)(\mathcal{R}_n) \|_{L^2_1} \lesssim \| b \mathcal{A}_0^{-1} \mathcal{R}_n \|_{L^2_1} + b \| \mathcal{A}_0^{-1} \left( - \frac{2}{\ln b} T_1 + \mathcal{A}_0^{-1} \Theta_0 \right) \|_{L^2_1}
\]

\[
\lesssim \| b \mathcal{R}_n \|_{L^2_1} + b \| - \frac{2}{\ln b} T_1 + \mathcal{A}_0^{-1} \Theta_0 \|_{L^2_1} \lesssim \| \mathcal{R}_n \|_{L^2_1} + b \lesssim b.
\]

Hence \( \|\partial_\alpha \mathcal{R}_n\|_{L^2_1} \lesssim b^{-1} \|\partial_\alpha \mathcal{R}_n\|_{L^2_1} \lesssim 1 \).

- **Computation of \( \mathcal{R}_1 \):** For \( n = 1 \) a refinement is necessary. The equation for \( \mathcal{R}_1 \) is:

\[
\mathcal{A}_0 \mathcal{R}_1 = b \left[ r \partial_r + 2 \tilde{\alpha} \right] \mathcal{R}_1 + 2 b \left( r \partial_r T_1 + \frac{2}{\ln b} T_1 \right) + \left[ r \partial_r + 2 \tilde{\alpha} \right] b \left( - \frac{2}{\ln b} T_1 + \mathcal{A}_0^{-1} \Theta_0 \right).
\]

We look for a solution under the form \( \mathcal{R}_1(r) = \mathcal{R}_{1,1}(r) + \mathcal{R}_{1,2}(r + 1) + \tilde{\mathcal{R}} \) where

\[
\mathcal{R}_{1,1} = - \left( \frac{1}{2} - \frac{1}{\ln b} \right) \sum_{i=1}^{\infty} \frac{(1)i-1}{(2)i!2^i} \frac{1}{b^i}, \quad (\partial_r + \frac{3}{r} \partial_r - br \partial_r) \mathcal{R}_{1,1} = -(1 - \frac{1}{\ln b}) b.
\]

\[
\mathcal{R}_{1,2} = - \frac{1}{2} \sum_{i=1}^{\infty} \frac{(1)i-1}{(2)i!2^i} \frac{1}{b^i} \left[ 2 \ln(r) - \frac{1}{i} - \Psi(i + 2) - \gamma \right],
\]

\[
(\partial_r^2 + \frac{3}{r} \partial_r - br \partial_r) \mathcal{R}_{1,2} = - \frac{b}{\log b} (2 \log r - 1),
\]
so that
\[
\mathcal{A}_0 \tilde{\mathcal{R}}_1 - b(r \partial_r + 2\tilde{\alpha}) \tilde{\mathcal{R}}_1 = \left(1 - \frac{1}{\log b}\right) b + \frac{b}{\log b} \left(2 \log(r + 1) - 1\right) + 2b(r \partial_r T_1 + \frac{2}{\log b} T_1) + \left[r \partial_r + 2\tilde{\alpha}\right] b \left(-\frac{2}{\ln b} T_1 + \mathcal{A}_0^{-1} \Theta_0\right)
\]
\[-\left((\frac{4r}{\langle r \rangle^4} - \frac{4}{r}) \partial_r + \frac{8}{\langle r \rangle^4}\right) \mathcal{R}_{1,1} - \left((\frac{4r}{\langle r \rangle^4} - \frac{1}{r} - \frac{3}{r + 1}) \partial_r + \frac{8}{\langle r \rangle^4} + b \partial_r \right) \mathcal{R}_{1,2}(r + 1)
\]
\[+2b\tilde{\alpha}(\mathcal{R}_{1,1} + \mathcal{R}_{1,2}(r + 1))\]

Each line in the right hand side above contains cancellations as \(r \to \infty\): the first is \(O(br^{-1} \log r)\) from (2.14), so is the second from the definition of \(\mathcal{R}_{1,1}\) and \(\mathcal{R}_{1,2}\). For the last line, \(\|\mathcal{R}_{1,1} + \mathcal{R}_{1,2}\|_{\mathcal{T}_0^1} \lesssim 1\) and \(|\tilde{\alpha}| \lesssim |\log b|^{-1}\). This shows that the right hand side is of size \(|\log b|^{-1}\) in \(\mathcal{T}_0^1\). So that \(\|\tilde{\mathcal{R}}_1\|_{\mathcal{T}_0^1} \lesssim |\log b|^{-1}\) and \(\|\partial_\theta \tilde{\mathcal{R}}_1\|_{\mathcal{T}_0^1} \lesssim 1\).

- **Computation of \(\mathcal{R}_0\):** For \(n = 0\) a refinement is also necessary. The equation for \(\mathcal{R}_0\) is:
\[
\mathcal{A}_0 \mathcal{R}_0 = b\left[r \partial_r + 2(1 + \tilde{\alpha})\right] \mathcal{R}_0 + \left[r \partial_r + 2\left(1 + \tilde{\alpha}\right)\right] b \left(-\frac{2}{\ln b} T_1 + \mathcal{A}_0^{-1} \Theta_0\right).
\]
We look for a solution under the form \(\mathcal{R}_0(r) = \mathcal{R}_{0,1}(r) + \mathcal{R}_{0,2}(r + 1) + \mathcal{R}_0\) where
\[
\mathcal{R}_{0,1} = \frac{1}{2} \sum_{i=1}^{\infty} \left(\frac{1}{(2i)!}\right)^2 i^2 r^{2i}, \quad (\partial_{rr} + \frac{3}{r} \partial_r - b(r \partial_r + 2)) \mathcal{R}_{0,1} = b
\]
\[
\mathcal{R}_{0,2} = \frac{1}{2} \log b \sum_{i=1}^{\infty} \left(\frac{1}{(2i)!}\right)^2 i^2 r^{2i} [2 \log(r) - \Psi(i + 2) - \gamma], \quad (\partial_{rr} + \frac{3}{r} \partial_r - b(r \partial_r + 2)) \mathcal{R}_{0,2} = \frac{2b}{\log b} \log r
\]
so that
\[
\mathcal{A}_0 \tilde{\mathcal{R}}_0 - b(r \partial_r + 2\tilde{\alpha}) \tilde{\mathcal{R}}_0 = -b - \frac{2b}{\log b} \log(r + 1) + \left[r \partial_r + 2 + 2\tilde{\alpha}\right] b \left(-\frac{2}{\ln b} T_1 + \mathcal{A}_0^{-1} \Theta_0\right)
\]
\[-\left((\frac{4r}{\langle r \rangle^4} - \frac{4}{r}) \partial_r + \frac{8}{\langle r \rangle^4}\right) \mathcal{R}_{1,1} - \left((\frac{4r}{\langle r \rangle^4} - \frac{1}{r} - \frac{3}{r + 1}) \partial_r + \frac{8}{\langle r \rangle^4} + b \partial_r \right) \mathcal{R}_{1,2}(r + 1)
\]
\[+2b\tilde{\alpha}(\mathcal{R}_{1,1} + \mathcal{R}_{1,2}(r + 1))\]

In the right hand side, the first line is \(O(br^{-1} \log r)\) from (2.14), and so is the second from the definition of \(\mathcal{R}_{0,1}\) and \(\mathcal{R}_{0,2}\). For the last line, \(\|\mathcal{R}_{0,1} + \mathcal{R}_{0,2}\|_{\mathcal{T}_0^1} \lesssim 1\) and \(|\tilde{\alpha}| \lesssim |\log b|^{-1}\). Therefore the right hand side is of size \(|\log b|^{-1}\) in \(\mathcal{T}_0^1\), and we get \(\|\tilde{\mathcal{R}}_0\|_{\mathcal{T}_0^1} \lesssim |\log b|^{-1}\).

**The perturbation problem related to (1.21):** We look for the solution of the form \(\tilde{\phi}_n^{\text{in},V} = \phi_n^{\text{in}} + \tilde{\phi}_n^{\text{in},V}\), where the remainder satisfies the equation
\[
\mathcal{A}_0 \tilde{\phi}_n^{\text{in},V} = b(r \partial_r - 2(1 - n + \tilde{\alpha})) \phi_n^{\text{in}} + r^{-1} \partial_r (V(\phi_n^{\text{in}})) + r^{-1} \partial_r (V(\phi_n^{\text{in},V})),
\]
where \(V(r) = P(nr)\). We let \(\Gamma\) be the affine mapping
\[
\Gamma(f) = \Gamma(0) + D\Gamma(f),
\]
where
\[
\Gamma(0) = \mathcal{A}_0^{-1} \left(r^{-1} \partial_r (V(\phi_n^{\text{in}}))\right),
\]
\[
D\Gamma(f) = \mathcal{A}_0^{-1} \left[b(r \partial_r - 2(1 - n + \tilde{\alpha})) f + r^{-1} \partial_r (V f)\right].
\]
From (2.21) and the various bounds obtained in Lemma 2.3 we get that \( \| \phi_n^{\text{in}} \|_{X_0^{-2}} \lesssim 1 \). Hence,

\[
\| \Gamma(0) \|_{L_0^2} \lesssim \| V \phi_n^{\text{in}} \|_{L_0^{-4}} \lesssim \frac{1}{|\ln b|} \| \phi_n^{\text{in}} \|_{L_0^{-2}} \lesssim \frac{1}{|\ln b|}.
\]

Using (2.11), we estimate for all \( f \in L^2 \),

\[
\| D \Gamma(f) \|_{L^2} \lesssim b \left\| (r \partial_r + 2(1 - n + \bar{\alpha})) f \right\|_{L_0^{-1}} \lesssim b R_0^2 \| f \|_{L_0^2} \lesssim \zeta_0^2 \| f \|_{L_2^1}.
\]

We then deduce that \( \Gamma(f) \) is contraction on \( B_{L_0^2}(0, bC) \) for some constant \( C > 0 \), hence, there exists a unique fixed point \( R_n = \Gamma(R_n) \) satisfying \( \| R_n \|_{L_0^2} \lesssim b \). As \( \| R_n \|_{L_0^2} \lesssim b^{-1} \| R_n \|_{L_0^2} \lesssim 1 \) the first estimates in (2.24) holds.

\[\square\]

**Lemma 2.4.** Let \( V \) be a smooth function satisfying \( |\partial_r^k V| \leq |\ln b|^{-1} r^{2-k} (r)^{-4} \) for \( k = 0, 1 \). Then for any fixed \( n \), for \( \zeta_0 \) small enough, there exists \( b^* > 0 \) such that for all \( 0 < b < b^* \) and \( \bar{\alpha} = O(|\ln b|^{-1}) \), there exists a solution \( \phi_n^{\text{in}} V \) to

\[
\Delta_0 \phi_n^{\text{in}} V - b [r \partial_r + 2(1 - n + \bar{\alpha})] \phi_n^{\text{in}} V + r^{-1} \partial_r (V \phi_n^{\text{in}} V) = 0
\]

on \([0, R_0]\) which satisfies

\[
\| \phi_n^{\text{in}} V - \phi_n^{\text{in}} \|_{L_0^{-2}} \lesssim \frac{1}{|\ln b|}.
\]

**Proof.** We only treat the case \( n = 1 \). Indeed, we will show that \( \phi^{\text{in}} \) vanishes once on \([0, R_0]\), whereas for \( n = 0 \) it does not. Reintegrating the Wronskian relation is then harder in the case \( n = 1 \), and the case \( n = 0 \) can be treated with the very same ideas but simpler computations. We first state some results on the first fundamental solution \( \phi^{\text{in}} \). One has the following decomposition from Lemma 2.3,

\[
\phi^{\text{in}} = T_0(r) + 2nbT_1(r) + \phi^{\text{in}}(r), \quad \text{with} \quad |\phi^{\text{in}}(r)| + r |\partial_r \phi^{\text{in}}(r)| \leq b \zeta_0^2 r^2 (r)^{-2} |\ln r|,
\]

where the bound is valid on \([0, R_0]\) and recall from (2.13):

\[
T_0(r) + 2nbT_1(r) = \frac{1}{r^2} - bn \log(r) + O(b + r^{-3}) \quad \text{as} \quad r \to \infty,
\]

\[
\partial_r T_0(r) + 2nb \partial_r T_1(r) = -\frac{2}{r^3} - \frac{bn}{r} + O(b |\ln r| r^{-2} + r^{-4}) \quad \text{as} \quad r \to \infty.
\]

From the above identities, we obtain that \( \phi^{\text{in}} \) vanishes exactly once on \([0, R_0]\) at the point \( r_0 \),

\[
r_0 = \frac{1}{\sqrt{b \sqrt{n}} |\ln b|} (1 + O(\zeta_0^2)),
\]

and that there exists a constant \( c > 0 \) such that

\[
\frac{c(r_0 - r)}{r_0 r^2} \leq \phi^{\text{in}}(r) \leq \frac{c(r_0 - r)}{r_0 r^2} \quad \text{on} \quad [1, r_0], \quad \frac{c(r_0 - r)}{r_0 r^2} \leq \phi^{\text{in}}(r) \leq \frac{c(r_0 - r)}{r_0 r^2} \quad \text{on} \quad [r_0, R_0].
\]

**Step 1** Uniform asymptotic for the second fundamental solution. We claim that there exists \( \Gamma \) another linearly independent solution to

\[
\Delta_0 \Gamma - b (r \partial_r + 2(1 - n + \bar{\alpha})) \Gamma = 0
\]

on \([0, R_0]\) such that:

\[
|\Gamma(r)| \leq C \quad \text{and} \quad |\partial_r \Gamma(r)| \leq C |\ln r| (r)^{-2} (|\ln r|)^{-1} \quad \text{on} \quad [0, R_0]
\]

with a constant \( C \) that is independent of \( b \) and \( \bar{\alpha} \). Indeed, from standard arguments, the Wronskian

\[
W = \Gamma \phi^{\text{in}} - \Gamma \phi^{\text{in}} \quad \text{is} \quad \text{fixing the integration constant without loss of generality):
\]

\[
W = \frac{r}{(1 + r^2)^{1/2}} b \zeta_0^2
\]
so the second fundamental solution is given by, reintegrating the Wronskian relation (we again fix here an integration constant without loss of generality):

$$\Gamma(r) = \phi^{in}(r) \int_1^r \frac{W(\xi)}{|\phi^{in}(\xi)|^2} d\xi = \phi^{in}(r) \int_1^r \frac{\xi e^{b\xi}}{(1 + \xi^2)^2|\phi^{in}(\xi)|^2} d\xi.$$  

The asymptotic near the origin follows from (2.21), (2.41) and (2.13), and direct computations, so we only focus on the asymptotic of \( \Gamma \) for \( r \) large. For \( 1 \leq r \leq r_0 \) from (2.44):

$$|\Gamma(r)| \lesssim \frac{(r_0 - r)r_0}{r^2} \int_1^r \frac{\xi}{(r_0 - \xi)^2} d\xi \lesssim 1.$$  

Next, for \( r \geq r_0 \), we avoid the singularity in the integral by noticing that there exists a constant \( C \) such that

$$\Gamma(r) = C \phi^{in}(r) + \phi^{in}(r) \int_{r_0}^r \frac{W(\xi)}{|\phi^{in}(\xi)|^2} d\xi.$$  

To estimate \( C \), one computes from the first formula for \( \Gamma \) and the asymptotic (2.44) near \( r_0 \) of \( \phi^{in} \):

$$\Gamma'(r_0) = \lim_{r \to r_0} \left( \partial_r \phi^{in}(r) \int_1^r \frac{W(\xi)}{|\phi^{in}(\xi)|^2} d\xi + \frac{W(r)}{\phi^{in}(r)} \right) = O(r_0^{-1}).$$  

Similarly, we have

$$\Gamma'(r_0) = C(\phi^{in}(r_0))' + \lim_{r \to r_0} \left( \partial_r \phi^{in}(r) \int_{r_0}^r \frac{W(\xi)}{|\phi^{in}(\xi)|^2} d\xi + \frac{W(r)}{\phi^{in}(r)} \right) = C(\phi^{in}(r_0))' + O(r_0^{-1}).$$  

As \( \partial_r \phi^{in}(r_0) = -2r_0^{-3}(1 + O(1)) \) we obtain \( C = O(r_0^{-1}) = O(b^{-1}|\log b|^{-1}) \). For all \( r_0 \leq r \leq R_0 \) we find from (2.44):

$$\left| \phi^{in}(r) \int_{r_0}^r \frac{W(\xi)}{|\phi^{in}(\xi)|^2} d\xi \right| \lesssim \frac{(r - r_0)r_0^2}{r} \int_{r_0}^r \frac{d\xi}{(\xi - r_0)^2} \lesssim 1.$$  

and:

$$|C \phi^{in}(r)| \lesssim r_0^2 \frac{(r - r_0)}{r_0^2(r - r_0)^2} \lesssim 1.$$  

Hence \( |\Gamma(r)| \lesssim 1 \) for \( r_0 < r \leq R_0 \) as well. This proves (2.45) for \( \Gamma \). The proof for \( \partial_r \Gamma \) is verbatim the same so that we skip it.

**Step 2 Bound for the resolvent under orthogonality condition.** Let a solution to \( \mathcal{A}_b \partial u = r^{-1} \partial_r (V f) \) be given by

$$u(r) = \phi^{in}(r) \int_r^{R_0} \frac{\Gamma(\xi)}{W(\xi)} \xi^{-1} \partial_\xi (V f)(\xi) d\xi + \Gamma(r) \int_0^r \frac{\phi^{in}(\xi)}{W(\xi)} \xi^{-1} \partial_\xi (V f)(\xi) d\xi,$$

then we claim the resolvent bound:

$$\|u\|_{X^{-2}} \lesssim \frac{1}{|\ln b|} \left( \|f\|_{X^{-2}} + \|r \partial_r f\|_{X^{-2}} \right).$$  

We now prove this claim. From the hypothesis on \( V \), (2.41) and (2.45), the first term can be bounded by

$$\left| \phi^{in}(r) \int_r^{R_0} \frac{\Gamma(\xi)}{W(\xi)} \xi^{-1} \partial_\xi (V f)(\xi) d\xi \right| \lesssim \frac{1}{|\ln b|} (r^{-4} + b(r^{-2} |\ln r|)) \int_r^{R_0} \left( |\xi|^{-1} |f(\xi)| + |\partial_\xi f(\xi)| \right) d\xi$$  

$$\lesssim \frac{\|f\|_{X^{-2}} + \|r \partial_r f\|_{X^{-2}}}{|\ln b|} (r^{-4} + b(r^{-2} |\ln r|)) \int_r^{R_0} \xi^{-4} |\ln(\xi)| d\xi$$  

$$\lesssim \|f\|_{X^{-2}} + \|r \partial_r f\|_{X^{-2}} \frac{r^{-6} |\ln(\xi)| + b(r^{-4} (\ln r)^2)}{|\ln b|}$$  

$$\lesssim \|f\|_{X^{-2}} + \|r \partial_r f\|_{X^{-2}} \frac{r^{-4} (\ln r)^2}{|\ln b|}.$$
For the second term, we use the decomposition (2.41), the identities (2.10) and (2.46), the bound (2.45) and the bounds on $V$ to get

$$\left| \Gamma^{in}(r) \int_0^r \frac{\phi^{in}(\xi)}{W(\xi)} \xi^{-1} \partial_\xi (V f)(\xi) d\xi \right| = \left| \Gamma^{in}(r) \int_0^r \frac{\langle \xi \rangle^4 e^{-b\frac{\xi^2}{2r}}}{\xi^2} \left( \frac{\xi^2}{\langle \xi \rangle^4} + b T_1 + \tilde{\phi}^{in} \right) \partial_\xi (V f)(\xi) d\xi \right|$$

$$= \left| \Gamma^{in}(r) \left( V(r) f(r) e^{-\frac{b r^2}{2}} - \int_0^r b \xi V f e^{-\frac{b \xi^2}{2r}} d\xi + \int_0^r \frac{\langle \xi \rangle^4 e^{-b\frac{\xi^2}{2r}}}{\xi^2} (b T_1 + \tilde{\phi}^{in}) \partial_\xi (V f)(\xi) d\xi \right) \right|$$

$$\lesssim \frac{1}{|\ln b|} \left( r^{-2}(r^{-4}) |f(r)| + b \int_0^r \xi^3 |\langle \xi \rangle^4| |f| d\xi + b \int_0^r \xi^3 |\langle \xi \rangle^4| (|f| + \xi |\partial_\xi f|) d\xi \right)$$

$$\lesssim \frac{\|f\|_{X^{-2}} + \|r \partial_r f\|_{X^{-2}}}{|\ln b|} \left( r^4 (r)^{-8} + b \int_0^r \xi^3 (\langle \xi \rangle^4)^{-8} d\xi + b \int_0^r \xi^3 (\langle \xi \rangle^4)^{-6} (|f| + \xi |\partial_\xi f|) d\xi \right)$$

$$\lesssim \frac{\|f\|_{X^{-2}} + \|r \partial_r f\|_{X^{-2}}}{|\ln b|} r^2 (r)^{-4}$$

because $r \lesssim b^{-1}$. Combining the above two bounds yields the following estimate on $[0, R_0]$,

$$\|u(r)\| \lesssim \frac{\|f\|_{X^{-2}} + \|r \partial_r f\|_{X^{-2}}}{|\ln b|} r^2 (r)^{-4}.$$ 

Differentiating the identity satisfied by $u$ yields

$$\partial_r u = \partial_r \phi^{in}(r) \int_0^{R_0} \frac{\Gamma(\xi)}{W(\xi)} \xi^{-1} \partial_\xi (V f)(\xi) d\xi + \partial_r \Gamma(r) \int_0^r \frac{\phi^{in}(\xi)}{W(\xi)} \xi^{-1} \partial_\xi (V f)(\xi) d\xi.$$ 

Hence, computing the same way the integral terms as we just did, and using (2.41), (2.43) and (2.45) to get

$$\|\partial_r u(r)\| \lesssim \frac{\|f\|_{X^{-2}} + \|r \partial_r f\|_{X^{-2}}}{|\ln b|} r^2 (r)^{-4}.$$ 

Using the definition of $\mathcal{A}$, we write

$$\partial_r^2 u = \left( \frac{1}{r} - \frac{Q}{r} \right) \partial_r u - \frac{\partial_r Q}{r} u + b (r \partial_r + 2(1 - n + \tilde{\alpha})) u + r^{-1} \partial_r (V f),$$

from which and the hypotheses on $V$ and the bounds on $u$ and $\partial_r u$, we obtain

$$\|\partial_r^2 u\| \lesssim \frac{\|f\|_{X^{-2}} + \|r \partial_r f\|_{X^{-2}}}{|\ln b|} (r)^{-4}.$$ 

The bounds on $u$, $\partial_r u$ and $\partial_r^2 u$ imply (2.47).

**Step 3 Fixed point.** We look for a solution to $[\mathcal{A}_0 - b (r \partial_r + 2(1 - n + \tilde{\alpha})) - r^{-1} \partial_r (V f)] \phi^{in, V} = 0$ under the form

$$\phi^{in, V} = \phi^{in} + \tilde{\phi}^{in, V}.$$ 

Then, $\tilde{\phi}^{in, V}$ solves

$$[\mathcal{A}_0 - b r \partial_r + 2b (1 - n + \tilde{\alpha})]^{-1} \tilde{\phi}^{in, V} = r^{-1} \partial_r (V \phi^{in}) + r^{-1} \partial_r (V \tilde{\phi}^{in, V}).$$ 

We solve this using a fixed point argument in $I_0^{-2}$. As $\|\phi^{in}\|_{I^{-2}} \lesssim 1$ from Lemma 2.3, as $\|\cdot\|_{I^{-2}} \lesssim \|\cdot\|_{I_0^{-2}}$ from the very definition of these spaces, the bound (2.47) implies

$$\| \mathcal{A}_0 - b r \partial_r + 2b (1 - n + \tilde{\alpha}) \|^{-1} (r^{-1} \partial_r (V \tilde{\phi}^{in})) \| \lesssim \frac{1}{|\ln b|} (\|\phi^{in}\|_{X^{-2}} + \|r \partial_r \phi^{in}\|_{X^{-2}}) \lesssim \frac{1}{|\ln b|} \|\phi^{in}\|_{X^{-2}} \lesssim \frac{1}{|\ln b|},$$

$$\mathcal{A}_0 = b r \partial_r + 2b (1 - n + \tilde{\alpha}),$$

and

$$\|\phi^{in}\|_{X^{-2}} \lesssim \frac{1}{|\ln b|}.$$
and
\[ \left[ \mathcal{A}_0 - b r \partial_r + 2b(1 - n + \tilde{\alpha}) \right]^{-1} \left( r^{-1} \partial_r (V \tilde{\phi}^{in,V}) \right) \lesssim \frac{\| \tilde{\phi}^{in,V} \|_{L^2}}{| \ln b |} \lesssim \frac{\| \tilde{\phi}^{in,V} \|_{L^2}}{| \ln b |}. \]

Hence, the mapping which to \( \tilde{\phi}^{in,V} \) assigns
\[ \left[ \mathcal{A}_0 - b r \partial_r + 2b(1 - n + \tilde{\alpha}) \right]^{-1} \left( r^{-1} \partial_r (V \phi^{in}) + r^{-1} \partial_r (V \tilde{\phi}^{in,V}) \right) \]
is a contraction in \( B_{|r-2}(0,C | \ln b |^{-1}) \) for \( C \) large enough and then for \( b \) small enough. Its unique fixed point is the desired solution, and satisfies the conclusion of the lemma.

\( \square \)

2.2. Analysis in the outer zone \( r \geq R_0 \)

In this part we solve problem \((2.1)\) in the interval \([R_0, +\infty)\) where the potential term can be treated as a small perturbation. To this end, we rewrite equation \((2.1)\) as
\[ \partial_t^2 \phi + \frac{3}{r} \partial_r \phi - b r \partial_r \phi - \alpha \phi - \frac{4}{r(1 + r^2)} \partial_r \phi + \frac{8}{(1 + r^2)^2} \phi = 0. \quad (2.48) \]

Introducing the change of variable
\[ \phi^{ex}(r) = q(z) \quad \text{with} \quad z = \frac{b r^2}{2}, \quad (2.49) \]
yields the equation satisfied by \( q \),
\[ (K_\theta + P_0)q(z) = 0, \quad z \geq z_0 = \frac{c_0^2}{2}, \quad \theta = \frac{\alpha}{2b}, \quad (2.50) \]
where \( K_\theta \) is a Kummer type operator defined by
\[ K_\theta = z \partial_z^2 + (2 - z) \partial_z - \theta, \quad (2.51) \]
and \( P_0 \) is the potential
\[ P_0 = -\frac{2b}{(b + 2z)} \partial_z + \frac{4b}{(b + 2z)^2}. \quad (2.52) \]
We will treat the differential operator \( P_0 \) as a perturbation of \( K_\theta \) in the outer zone. We first claim the following.

**Lemma 2.5** (Properties of \( K_\theta \)).

(i) **(Inversion)** Assume that \(-\theta \not\in \mathbb{N}\), then an explicit inversion of \( K_\theta \) is given by
\[ K_{-\theta}^{-1} f = h_\theta(z) \int_{z_0}^z \tilde{h}_\theta(\xi) f(\xi) e^{-\xi} d\xi + \tilde{h}_\theta(z) \int_{z}^{+\infty} h_\theta(\xi) f(\xi) e^{-\xi} d\xi, \quad (2.53) \]
where \( h_\theta \) and \( \tilde{h}_\theta \) are the two linearly independent solutions to Kummer’s equation \( K_\theta h = 0 \):
\[ h_\theta(z) = \frac{1}{z \Gamma(\theta)} + \frac{1}{\Gamma(\theta - 1)} \sum_{i=0}^{\infty} \frac{(\theta)_i}{(2)_i i!} z^i \ln z + \Psi(\theta + i) - \Psi(1 + i) - \Psi(2 + i), \quad (2.54) \]
\[ \tilde{h}_\theta(z) = \sum_{i=0}^{\infty} \frac{(\theta)_i}{(2)_i i!} z^i, \quad (2.55) \]
where \( (a)_i = \frac{\Gamma(a+i)}{\Gamma(a)} \), \( \Gamma \) is the Gamma function, and \( \Psi = \Gamma'/\Gamma \) is the digamma function. Moreover, we have the asymptotic behavior as \( z \to +\infty \),
\[ h_\theta(z) = z^{-\theta} (1 + O(z^{-1})), \quad \tilde{h}_\theta(z) = \frac{\Gamma(2)}{\Gamma(\theta)} e^z z^{\theta - 2} (1 + O(z^{-1})). \quad (2.56) \]
and for \( z_0 \leq z \leq 2 \), for \( C \) dependent of \( z_0 \) if \( n = 0 \) and independent if \( n \geq 1 \):
\[ | h_\theta(z) | + | \tilde{h}_\theta(z) | \lesssim C \quad (2.57) \]
(ii) (Continuity) Let $a \in \mathbb{R}$, and $\mathcal{E}_0^{p,a}$ be the Banach space of functions $f : [z_0, +\infty) \to \mathbb{R}$ equipped with the norm
\[
\|f\|_{\mathcal{E}_0^a} \triangleq \sup_{z \geq z_0} \langle z \rangle^{-a} |f(z)| + z \partial_z f(z) + z^2 \partial_z^2 f(z)|.
\]
Then for any continuous function $f : [z_0, +\infty) \to \mathbb{R}$, we have the estimate for $a > -\theta$:
\[
\|\mathcal{K}_\theta^{-1} f\|_{\mathcal{E}_0^a} \leq C(z_0) \sup_{z_0 \leq z < +\infty} \langle z \rangle^{-a} |f(z)|.
\] (2.58)

(iii) Let $H_\theta = \mathcal{K}_\theta^{-1} h_\theta$, then we have the estimates:
\[
\begin{align*}
H_\theta(z) &= \frac{\Gamma(2)}{\Gamma(\nu)} z^{1-\theta} + O(z^{-\theta}) \quad \text{as } z \to +\infty, \\
H_\theta(z) &= C_0 + O(z), \quad C_0 > 0, \quad \text{as } z \to 0, \\
|z \partial_z H_\theta| + |z^2 \partial_z^2 H_\theta| &= O(z^{1-\theta}) \quad \text{as } z \to +\infty, \\
|z \partial_z H_\theta| + |z^2 \partial_z^2 H_\theta| &= O(z) \quad \text{as } z \to 0.
\end{align*}
\] (2.59) (2.60) (2.61) (2.62)

Proof. (i) See formulas 13.1.2, 13.1.6 and 13.1.22 in [1] for the definition of $h_\theta$, $\tilde{h}_\theta$ and the Wronskian $W(h_\theta, \tilde{h}_\theta)$ respectively. For the bound for $z_0 \leq z \leq 2$, notice that from the Gamma function’s recurrence relation and the bound on $\alpha$:
\[
\Gamma(\theta) = \frac{\Gamma(\theta + n)}{\theta(\theta + 1)\ldots(\theta + n - 1)} = \frac{\Gamma(1 + \tilde{\alpha})}{(1 - n + \tilde{\alpha})(2 - n + \tilde{\alpha})\ldots(-1 + \tilde{\alpha})}
\sim \frac{(-1)^n}{(n - 1)!\tilde{\alpha}} + O(1) = O(|\ln b|),
\] (2.63)
for $n \geq 1$, and $\Gamma(\theta) = \Gamma(1 + \tilde{\alpha}) = O(1)$ for $n = 0$. (ii) The proof follows from straightforward computations. Let
\[
D = \sup_{z_0 \leq z < +\infty} \langle z \rangle^{-a} |f(z)|.
\]
From (2.56), we compute for $z \geq 2$,
\[
\left| \tilde{h}_\theta(z) \int_z^{+\infty} f h_\theta \xi e^{-\xi} d\xi \right| \lesssim D z^{a-1} - 2 z \int_z^{+\infty} \xi a \xi e^{-\xi} d\xi \lesssim D z^{a-1},
\]
and from (2.54), we compute for $z \in [z_0, 2]$,
\[
\left| \tilde{h}_\theta(z) \int_z^{+\infty} f h_\theta \xi e^{-\xi} d\xi \right| \lesssim \tilde{h}_\theta(z) \int_z^{+\infty} f h_\theta \xi e^{-\xi} d\xi + \tilde{h}_\theta(z) \int_z^{+\infty} f h_\theta \xi e^{-\xi} d\xi \lesssim D \int_z^{+\infty} \xi d\xi \lesssim D.
\]
Similarly, we have for $z \geq 2$, as $a > -\theta$
\[
\left| h_\theta(z) \int_z^{+\infty} f h_\theta \xi e^{-\xi} d\xi \right| \lesssim D z^{-a} - 2 z \int_{z_0}^{z} \xi a \xi e^{-\xi} d\xi \lesssim D z^{a},
\]
and for $z_0 \leq z \leq 2$,
\[
\left| \int_{z_0}^{z} f h_\theta \xi e^{-\xi} d\xi \right| \lesssim D \int_{z_0}^{z} \xi d\xi \lesssim D.
\]
This proves the continuity bound (2.58) for $\mathcal{K}_\theta^{-1} f$. We now take derivatives. For $z \geq 2$, we estimate from (2.53), (2.56):
\[
\left| z \partial_z \mathcal{K}_\theta^{-1} f(z) \right| \lesssim \left| z \partial_z \tilde{h}_\theta \int_z^{+\infty} f h_\theta \xi e^{-\xi} d\xi \right| + \left| z \partial_z h_\theta \int_z^{+\infty} f \tilde{h}_\theta \xi e^{-\xi} d\xi \right| \lesssim D \left( e^z z^{\theta-1} z^{-a+1} e^{-z} + z^{-\theta} z^{a+\theta} \right) \lesssim D z^{a}.
\]
For $z \in [z_0, 2]$, we estimate from (2.55):

$$|z \partial_z K_\theta^{-1} f(z)| \lesssim |z \partial_z h_\theta \int_z^{+\infty} h_\theta \xi e^{-\xi} d\xi| + |z \partial_z h_\theta \int_{z_0}^z f h_\theta \xi e^{-\xi} d\xi| \lesssim D \int_z^2 \xi d\xi + \int_{z_0}^{+\infty} \xi^{a-\theta + 1} e^{-\xi} d\xi + D \int_{z_0}^z \xi d\xi \lesssim D.$$

Using $K_\theta K_\theta^{-1} f = f$ and the definition of $K_\theta$, we have the estimate for $z \geq 2$,

$$|z \partial_z^2 K_\theta^{-1} f(z)| \lesssim |z \partial_z K_\theta^{-1} f(z)| + |K_\theta^{-1} f(z)| \lesssim D z^a,$$

and for $z \in [z_0, 2]$,

$$|z^2 \partial_z^2 K_\theta^{-1} f(z)| \lesssim |z \partial_z K_\theta^{-1} f(z)| + |z K_\theta^{-1} f(z)| \lesssim D.$$

Collecting the above estimates yields the estimate (2.58).

(iii) This is a straightforward computation. For $z \leq 2$, we compute from (2.55) and (2.54),

$$\int_z^2 h_\theta^2(\xi)\xi^2 e^{-\xi} d\xi = \int_z^2 \left( \frac{1}{\xi^2 \Gamma^2(\theta)} + O(\xi^{-1} |\ln \xi|) \right) \xi^2 (1 + O(\xi)) d\xi = C_0 + O(z),$$

for some $C_0 > 0$, and

$$\int_{z_0}^z h_\theta(\xi) h_\theta(\xi) \xi^2 e^{-\xi} d\xi = O \left( \int_{z_0}^z \xi d\xi \right) = O(z^2).$$

For $z \geq 2$, we have

$$\int_z^{+\infty} h_\theta^2(\xi)\xi^2 e^{-\xi} d\xi = \int_z^{+\infty} \xi^{2-2\theta} (1 + O(\xi^{-1})) e^{-\xi} d\xi = z^{2-2\theta} e^{-z} \left( 1 + O(z^{-1}) \right),$$

and

$$\int_{z_0}^z h_\theta(\xi) h_\theta(\xi) \xi^2 e^{-\xi} d\xi = \int_{z_0}^2 h_\theta(\xi) h_\theta(\xi) \xi^2 e^{-\xi} d\xi + \int_{z_0}^z \frac{\Gamma(2)}{\Gamma(\theta)} \xi^{\theta-2} (1 + O(\xi^{-1})) \xi^{2-\theta} \xi^2 e^{-\xi} d\xi$$

$$= O(1) + \int_{z_0}^z \frac{\Gamma(2)}{\Gamma(\theta)} (1 + O(\xi^{-1})) d\xi = \frac{\Gamma(2)}{\Gamma(\theta)} z + O(\ln z).$$

From (2.53) and the above estimates, we obtain for $z \to +\infty$

$$H_\theta(z) = \frac{\Gamma(2)}{\Gamma(\theta)} e^z z^{\theta-2} \left( 1 + O(z^{-1}) \right) z^{2-2\theta} e^{-z} \left( 1 + O(z^{-1}) \right) + z^{-\theta} \left( 1 + O(z^{-1}) \right) \left( \frac{\Gamma(2)}{\Gamma(\theta)} z + O(\ln z) \right) = \frac{\Gamma(2)}{\Gamma(\theta)} z^{1-\theta} + O(z^{-\theta}),$$

and for $z \to 0$

$$H_\theta(z) = (1 + O(z)) (C_0 + O(z)) + \left[ \frac{1}{z \Gamma(\theta)} + O(\ln z) \right] O(z^2) = C_0 + O(z),$$

which concludes the proof of (2.59) and (2.60).

The estimates (2.61) and (2.62) are obtained in the same manner by using the above estimates and the formula

$$z \partial_z H_\theta(z) = z \partial_z h_\theta(z) \int_z^{+\infty} h_\theta^2(\xi)\xi^2 e^{-\xi} d\xi + z \partial_z h_\theta(z) \int_{z_0}^z h_\theta(\xi) h_\theta(\xi) \xi^2 e^{-\xi} d\xi.$$ 

We have for $z \gg 1$,

$$z \partial_z H_\theta(z) = \frac{\Gamma(2)}{\Gamma(\theta)} e^z z^{\theta-1} \left( 1 + O(z^{-1}) \right) z^{2-2\theta} e^{-z} \left( 1 + O(z^{-1}) \right)$$

$$- \theta z^{-\theta} \left( 1 + O(z^{-1}) \right) \left( \frac{\Gamma(2)}{\Gamma(\theta)} z + O(\ln z) \right) = O(z^{1-\theta}),$$
and for \( z \ll 1 \)
\[
z \partial_z H_\theta(z) = \left( \frac{z \theta}{2} + O(z^2) \right) \left( C_0 + O(z) \right) + \left( -\frac{1}{z \Gamma(\theta)} + O(1) \right) O(z^2) = O(z).
\]

For the control of \( z^2 \partial_z^2 H_\theta \), we use the definition of \( K_\theta \) and the relation \( h_\theta = K_\theta H_\theta \) to write
\[
z^2 \partial_z^2 H_\theta = zh_\theta - z(2 - z)\partial_z H_\theta + \theta z H_\theta,
\]
and then use the bounds on \( H_\theta \) and \( \partial_z H_\theta \) that we already obtained. This concludes the proof of Lemma 2.5.

\[ \square \]

We are now in the position of computing the solution \( q \) to equation (2.50) (hence, \( \phi^{ex} \)) by means of perturbation theory.

**Lemma 2.6** (Outer eigenfunctions for the radial mode). Fix \( n \in \mathbb{N} \), and \( \theta = 1 - n + 1/\ln b + \tilde{\alpha} \). For \( 0 < \zeta_0 \ll 1 \) and any small \( 0 < \delta \ll 1 \), there exist \( b^*>0 \) such that for all \( 0 < b \leq b^* \), for all \( \tilde{\alpha} = O(|\ln b|^{-2}) \) there exists a smooth solution

\[
q(b, \tilde{\alpha}, z) = \Gamma(\theta)h_\theta(z) + G(b, \tilde{\alpha}, z)
\]

to (2.50) on \([z_0, +\infty)\), where \( h_\theta \) is introduced in Lemma 2.5 and \( G \) satisfies the following estimates for some universal \( C > 0 \):

\[
\|G\|_{\mathcal{E}^{-\theta+\delta}} \lesssim b|\ln b|^C, \quad \|b \partial_b G\|_{\mathcal{E}^{-\theta+\delta}} \lesssim b|\ln b|^C, \quad \|\partial_\theta G\|_{\mathcal{E}^{-\theta+\delta}} \lesssim b|\ln b|^C.
\]

(2.65)

where the constants in the estimates depend on \( z_0 \).

**Lemma 2.7.** Assume \( P_0 \) is replaced by \( P_0(q) + \frac{1}{2} \partial_z (\tilde{V} q)/z \) where \( \tilde{V} \) satisfies \( |\tilde{V}| + |z \partial_z \tilde{V}| \lesssim b|\ln b|^{-1}z^{-1} \) on \([z_0, \infty)\). Then existence result of Lemma 2.6 of a solution \( q^V = \Gamma(\theta)h_\theta(z) + G^V(b, \tilde{\alpha}, z) \) and the first bound in (2.65) still hold true.

**Proof of Lemma 2.6.** From the bound on the Gamma function (2.63), we will simply consider a solution of the form \( q(z) = h_\theta(z) + G(b, \tilde{\alpha}, z) \) (with the abuse of notation of keeping the notation \( G \)), and prove the estimate (2.65) for \( G(b, \tilde{\alpha}, z) \), which will prove the Lemma upon multiplication by \( \Gamma(\theta) \). Note that \( P_0 \) has the form:

\[
P_0(q) = V_1q + V_2\partial_z q, \quad \text{with} \quad |V_1| + |zV_1| \lesssim bz^{-1}.
\]

(2.66)

Let us write from (2.64) the equation satisfied by \( G \),

\[
K_\theta G + P_0 G + P_0 h_\theta = 0.
\]

Let \( \Gamma \) the affine mapping defined as

\[
\Gamma(f) = -K_\theta^{-1}[P_0 f + P_0 h_\theta] \equiv D\Gamma(f) + D\Gamma(h_\theta),
\]

where

\[
D\Gamma(f) = -K_\theta^{-1}[P_0 f],
\]

and \( K_\theta^{-1} \) is given by (2.53). We estimate from the definition (2.52) of \( P_0 \), (2.60), (2.59) and (2.58),

\[
\|D\Gamma(h_\theta)\|_{\mathcal{E}^{-\theta+\delta}} \lesssim \|K_\theta^{-1} P_0 h_\theta\|_{\mathcal{E}^{-\theta+\delta}} \lesssim \sup_{z \in [z_0, +\infty)} \langle z \rangle^{\theta-\delta} |P_0 h_\theta| \lesssim b.
\]

From (2.58), we estimate for all \( f \in \mathcal{E}^{0,-\theta+\delta} \),

\[
\|D\Gamma(f)\|_{\mathcal{E}^{-\theta+\delta}} \lesssim \sup_{z \in [z_0, +\infty)} \langle z \rangle^{\theta-\delta} |P_0 f(z)| \lesssim b\|f\|_{\mathcal{E}^{-\theta+\delta}}.
\]

(2.67)

It follows that \( \Gamma \) is a contraction mapping on \( B_{\mathcal{E}^{-\theta+\delta}}(0, Mb) \) for some \( M = M(\zeta_0) > 0 \) large enough. Hence, there exists a unique fixed point \( G \) with

\[
G = \Gamma(G) \quad \text{with} \quad \|G\|_{\mathcal{E}^{-\theta+\delta}} \lesssim b.
\]

Differentiating the above fixed point relation yields:

\[
\partial_\theta G = D\Gamma(\partial_\theta G) + (\partial_\theta \Gamma)(G), \quad \partial_b G = D\Gamma(\partial_b G) + (\partial_\theta \Gamma)(G).
\]
Since $P_0$ depends on $b$ and not on $\theta$, whereas $h_\theta$, $h'_\theta$ and $K_\theta$ depend on $\theta$ and not on $b$, we have the identities:

$$(\partial_\theta \Gamma)(\mathcal{G}) = -\partial_\theta(K_\theta^{-1})(P_0(\mathcal{G} + h_\theta)) - K^{-1}(P_0\partial_\theta h_\theta), \quad (\partial_\theta \Gamma)(\mathcal{G}) = -K_\theta^{-1}(\partial_\theta P_0(\mathcal{G} + h_\theta)).$$

We compute from (2.54) that:

$$\partial_\theta h_\theta(z) = -\frac{\Psi(\theta)}{z\Gamma(\theta)} - \frac{\Psi(\theta - 1)}{\Gamma(\theta - 1)} \sum_{i=0}^{\infty} \frac{(\theta)_i}{(2i)!} z^i \left[ \ln z + \Psi(\theta + i) - \Psi(1 + i) - \Psi(2 + i) \right] + \frac{1}{\Gamma(\theta - 1)} \sum_{i=0}^{\infty} \frac{(\theta)_i}{(2i)!} z^i \left[ \ln z + \Psi'(\theta + i) - \Psi(1 + i) - \Psi(2 + i) \right] + \frac{1}{\Gamma(\theta - 1)} \sum_{i=0}^{\infty} \frac{(\theta)_i}{(2i)!} z^i \left[ \ln z + \Psi(\theta + i) - \Psi(1 + i) - \Psi(2 + i) \right].$$

Hence, we infer from $|\Psi(\theta)| + |\Psi(\theta - 1)| \lesssim |\tilde{\alpha}|^{-1} \lesssim |\ln b|$, $|\Psi'(\theta + i)| \lesssim |\tilde{\alpha}|^{-2} \lesssim |\ln b|^2$ and $|\theta|_i|\Psi(\theta + i) - \Psi(\theta)|| \lesssim 1$ the rough upper bound on $[z_0, \infty)$:

$$|\partial_\theta h_\theta(z)| \lesssim |\ln b|^2 z^{-1} \ln(z)^{-\theta},$$

which extends to derivatives. Similarly, we have from (2.55)

$$\partial_\theta \tilde{h}_\theta(z) = \sum_{i=0}^{\infty} \frac{(\theta)_i}{(2i)!} \left( \Psi(\theta + i) - \Psi(\theta) \right) z^i,$$

satisfies the rough upper bound $|\partial_\theta \tilde{h}_\theta(z)| \lesssim \ln(z)^{\theta - 2} e^z$ on $[z_0, \infty)$. We get from (2.53):

$$(\partial_\theta K_\theta^{-1}) f = (\partial_\theta h_\theta)(z) \int_{z_0}^{z} \tilde{h}_\theta(\xi) f(\xi) e^\xi d\xi + h_\theta(z) \int_{z_0}^{z} \partial_\theta \tilde{h}_\theta(\xi) f(\xi) e^\xi d\xi + \tilde{h}_\theta(z) \int_{z_0}^{z} \partial_\theta h_\theta(\xi) f(\xi) e^\xi d\xi.$$

Hence, as from the above, the bounds for $h_\theta$ and $\tilde{h}_\theta$ still hold up to a logarithmic loss in $z$ and $b$ and $\delta > 0$, using the same argument as in the proof of Lemma 2.5 we get:

$$\|\partial_\theta K_\theta^{-1}) f\|_{L^2(e^{-\theta + \delta})} \lesssim \|\ln b\| \sup_{z_0 \leq z < \infty} \langle z \rangle^{\theta - \delta} |f(z)|$$

and from (2.58):

$$\|K_\theta^{-1}(P_0\partial_\theta h_\theta)\|_{L^2(e^{-\theta + \delta})} \lesssim \|P_0\partial_\theta h_\theta\| \|b\| \ln b\|^2.$$

Thus, as $\delta$ is small, from the definition of $P_0$:

$$\|\partial_\theta (K_\theta^{-1})(P_0(\mathcal{G} + h_\theta))\|_{L^2(e^{-\theta + \delta})} \lesssim \|\ln b\|^2 \|P_0(\mathcal{G} + h_\theta)\| \|b\| \ln b\|^2 \mathcal{G} \lesssim b\|\ln b\|^2.$$

We proved above the continuity bound $\|DT\|_{C(\mathcal{G} e^{-\theta + \delta})} \lesssim b$ and the identity,

$$\partial_\theta \mathcal{G} = D\Gamma(\partial_\theta \mathcal{G}) - \partial_\theta (K_\theta^{-1})(P_0(\mathcal{G} + h_\theta)) = K^{-1}(P_0\partial_\theta h_\theta).$$

Hence one can invert the operator $I \partial_\theta - D\Gamma$ for $b$ small enough, with $\|I \partial_\theta - D\Gamma\|_{C(\mathcal{G} e^{-\theta + \delta})} \lesssim 1$ and the above identity gives:

$$\|\partial_\theta \mathcal{G}\|_{L^2(e^{-\theta + \delta})} = \|I \partial_\theta - D\Gamma\|^{-1} \|\partial_\theta (K_\theta^{-1})(P_0(\mathcal{G} + h_\theta)) + K^{-1}(P_0\partial_\theta h_\theta)\|_{L^2(e^{-\theta + \delta})} \lesssim b\|\ln b\|^2.$$

From the definition of $P_0$ and (2.58) we find:

$$\|K_\theta^{-1}(\partial_\theta P_0(\mathcal{G} + h_\theta))\|_{L^2(e^{-\theta + \delta})} \lesssim \|\partial_\theta P_0(\mathcal{G} + h_\theta)\| \|b\| \ln b\|^2 \lesssim 1.$$

Hence we obtain similarly from the relation $\partial_\theta \mathcal{G} = D\Gamma(\partial_\theta \mathcal{G}) - K_\theta^{-1}(\partial_\theta P_0(\mathcal{G} + h_\theta))$ the bound:

$$\|\partial_\theta \mathcal{G}\|_{L^2(e^{-\theta + \delta})} \lesssim \|I \partial_\theta - D\Gamma\|^{-1} \|\partial_\theta P_0(\mathcal{G} + h_\theta)\|_{L^2(e^{-\theta + \delta})} \lesssim 1.$$

This concludes the proof of the first part of Lemma 2.6. For the second part, where $P_0$ is replaced by $P_0 + \frac{1}{2} \partial_2(\tilde{V} \cdot)/z$, note that the decomposition (2.66) and the associated bounds still hold for $P_0 + \frac{1}{2} \partial_2(\tilde{V} \cdot)/z$. 

This was the only information we used on $P_0$, so the same proof applies. This shows the last part of Lemma 2.6.

**Proof of Lemma 2.7.** The decomposition (2.66) and the associated bounds still hold for $P_0 + \frac{1}{2} \partial z(\bar{V}')/z$. This was the only information used on $P_0$ in the proof of Lemma 2.6, so the very same proof applies.

### 2.3. Conclusion via matching asymptotic expansions

From Lemmas 2.3 and 2.6, we are now able to derive the full solution to the eigenproblem (2.1). In particular we claim the following.

**Lemma 2.8** (Matched eigenfunction for the radial mode). Fix $n \in \mathbb{N}$. Then there exists $C > 0$, such that for $\zeta_0$ small enough, there exists $0 < b^* \ll 1$ such that for all $0 < b \leq b^*$, there exists $|\tilde{\alpha}_n| \leq C|\ln b|^{-2}$ such that the following holds for the function

$$
\phi_n(r) = \begin{cases} 
\phi_n^{in}(r) & \text{for } r \leq R_0, \\
\beta_0 \phi_n^{ex}(r) & \text{for } r \geq R_0,
\end{cases}
$$

where

$$
\phi_n^{in} = \phi_n^{in}[b, \tilde{\alpha}] \quad \text{and} \quad \phi_n^{ex}(r) = \phi_n^{ex}[b, \tilde{\alpha}](r) = q[b, \tilde{\alpha}]
$$

are described in Lemmas 2.3 and 2.6 respectively.

(i) The function $\phi_n$ is a smooth solution to the equation

$$
(\mathcal{A}_0 - br \partial_r) \phi_n = 2b(1 - n + \frac{1}{\ln b} + \tilde{\alpha}_n) \phi_n.
$$

(ii) The estimates for $\phi_n$ and $\alpha_n$ described in Proposition 1.1 hold true.

(iii) In the cases $n = 0, 1$, we have the refinements

$$
\hat{\alpha}_0 = \frac{e_0}{|\ln b|^2} + \hat{\alpha}_0, \quad \hat{\alpha}_1 = \frac{e_1}{|\ln b|^2} + \hat{\alpha}_1,
$$

where $e_0 = \ln 2 - \gamma$, $e_1 = \ln 2 - \gamma - 1$, and $|\hat{\alpha}_0| + |\hat{\alpha}_1| \leq C|\ln b|^{-3}$.

**Corollary 2.9.** For the perturbed operator $\mathcal{A}_0 \phi_n - br \partial_r \phi_n + r^{-1} \partial_r (V \cdot)$ where $V$ satisfies $|\partial_r^k V| \lesssim |\ln b|^{-1/2 - k}(r)^{-4}$ for $k = 0, 1$, then item (i) of Lemma 2.8 holds true if the inner and outer eigenfunctions are those associated to the perturbed problems described by Lemma 2.4 and 2.7 respectively.

**Proof of Lemma 2.8.** Recall from (2.51) the relation

$$
\theta = 1 - n + \hat{\alpha}, \quad \hat{\alpha} = \frac{1}{\ln b} + \tilde{\alpha}.
$$

Since the equation (2.69) is a second order ODE with smooth coefficients outside the origin, it suffices to prove that the two functions and their first order derivatives agree on both sides of $R_0$, and (2.68) will then provide a global solution on $(0, \infty)$. From the special choice of $\beta_0$ this is equivalent to:

$$
\frac{\partial_r \phi_n^{in}(R_0)}{\partial_r \phi_n^{ex}(R_0)} = \beta_0 \iff \Theta(b, \tilde{\alpha}) = \frac{(r \partial_r) \phi_n^{in}(R_0)}{2 \phi_n^{in}(R_0)} - \frac{(z \partial_z)q(z_0)}{q(z_0)} = 0.
$$

We aim at showing that for $b$ small enough there exists $\alpha = \alpha_n(b)$ such that $\Theta(b, \tilde{\alpha}) = 0$ from a standard argument based on the implicit function theorem. The estimate for $\partial_b \alpha_n$ then follows by

$$
\partial_b \tilde{\alpha}_n = -\frac{(\partial_b \Theta)(b, \tilde{\alpha}_n)}{(\partial_{\tilde{\alpha}_n} \Theta)(b, \tilde{\alpha}_n)}.
$$

To ease the writing, we mention only the dependence in $b$ and $\tilde{\alpha}$ in few expressions in what follows.
The interior term: It’s convenient to rewrite from (2.21) the expression of \( \phi \) as

\[
\phi \in \frac{b, \alpha}(r) = F_n[b](r) + \bar{\alpha}bG_n[b, \alpha](r) + E_n[b, \bar{\alpha}](r),
\]

where \( F_n \) and \( \bar{\alpha}G_n \) are leading order terms and \( E_n \) is a remainder:

\[
F_n[b](r) = \sum_{j=0}^{n} c_{n,j} b^j T_j(r), \quad G_n[b, \alpha] = \sum_{j=0}^{n} b^j \left( -c_{n,j} T_{j+1}(r) + S_j[b, \alpha](r) \right), \tag{2.74}
\]

\[
E_n[b, \bar{\alpha}](r) = b \left( -\frac{2}{\ln b} T_1(r) + \alpha \frac{\alpha^{-1}}{T_0(r)} \right) + b R_n[b, \alpha](r)
\]

We have the following estimates from (2.14), (2.33), (2.24), and assuming \( |\alpha| \lesssim |\ln b|^{-2} \):

\[
\sum_{0 \leq k \leq 2, 0 \leq \ell + \ell' \leq 1} |(r \partial_r)^k (b \partial_b)^\ell \partial_{\alpha}^\ell E_n)[R_0]| \leq C(\zeta_0) \frac{b}{|\ln b|}, \tag{2.75}
\]

\[
F_n(R_0) = b \left( -\frac{\ln b}{2} H_n(\zeta_0) + K_n(\zeta_0) \right) + \mathcal{O}(b^{3}), \tag{2.76}
\]

\[
(r \partial_r F_n[R_0])(R_0) = b \left( -\frac{\ln b}{2} \zeta \partial_{\zeta} H_n(\zeta_0) + \zeta \partial_{\zeta} K_n(\zeta_0) \right) + \mathcal{O}(b^{3}),
\]

where \( H_n \) and \( G_n \) are defined by:

\[
H_n(\zeta_0) = \sum_{i=1}^{n} c_{n,i} \bar{\alpha}(i-1) \zeta_0, \quad K_n(\zeta_0) = \frac{1}{\zeta_0} + \sum_{i=1}^{n} c_{n,i} \zeta_0^{(i-1)} \left( \hat{d}_i \ln \zeta_0 + d_i \right) \tag{2.77}
\]

Notice for \( 0 < \zeta_0 \ll 1 \) small that \( |H_n(\zeta_0)| \neq 0 \). Gathering all these estimates and (2.23) we arrive at

\[
\phi \in (R_0) = b \left( -\frac{\ln b}{2} H_n(\zeta_0) + K_n(\zeta_0) + \bar{\alpha}G_n[R_0] + \mathcal{O}(\frac{1}{|\ln b|}) \right),
\]

\[
r \partial_r \phi \in (R_0) = b \left( -\frac{\ln b}{2} \zeta \partial_{\zeta} H_n(\zeta_0) + \zeta \partial_{\zeta} K_n(\zeta_0) + \bar{\alpha}r \partial_r G_n[R_0] + \mathcal{O}(\frac{1}{|\ln b|}) \right),
\]

\[
\partial_b \left( \frac{1}{b \ln b} \phi \in (R_0) \right) = -\frac{1}{b |\ln b|^2} (\zeta \partial_{\zeta} K_n(\zeta_0) + \bar{\alpha}r \partial_r G_n[R_0] + \mathcal{O}(\frac{1}{|\ln b|}))
\]

\[
+ \frac{1}{b \ln b} (\bar{\alpha} \partial_b G_n[R_0] + b \partial_b E[R_0]) = \mathcal{O}(\frac{1}{|\ln b|^2}),
\]

\[
\partial_b \left( \frac{1}{b \ln b} r \partial_r \phi \in (R_0) \right) = -\frac{1}{b |\ln b|^2} (\zeta \partial_{\zeta} K_n(\zeta_0) + \bar{\alpha}r \partial_r G_n[R_0] + \mathcal{O}(\frac{1}{|\ln b|}))
\]

\[
+ \frac{1}{b \ln b} (\bar{\alpha} \partial_b r \partial_r G_n[R_0] + b \partial_b r \partial_r E[R_0]) = \mathcal{O}(\frac{1}{|\ln b|^2}),
\]

\[
\partial_{b} \phi \in (R_0) = b \partial_{b} G_n[R_0] + \bar{\alpha} \partial_{b} \partial_{b} G_n[R_0] + b \partial_{b} E[R_0]
\]

\[
= b \partial_{b} G_n[R_0] + b \mathcal{O}(\ln |b|^{-1}) \mathcal{O}(\ln b) + b \mathcal{O}(\ln b |b|^{-1}) = b \left( G_n[R_0] + \mathcal{O}(\ln |b|^{-1}) \right),
\]

\[
\partial_{b} (r \partial_r \phi \in (R_0)) = b \partial_{b} \partial_{r} G_n[R_0] + \bar{\alpha} \partial_{b} \partial_{b} r \partial_r G_n[R_0] + b \partial_{b} r \partial_r E[R_0]
\]

\[
= b \partial_{b} \partial_{r} G_n[R_0] + b \mathcal{O}(\ln |b|^{-1}) \mathcal{O}(\ln b) + b \mathcal{O}(\ln b |b|^{-1}) = b \left( r \partial_r G_n[R_0] + \mathcal{O}(\ln |b|^{-1}) \right).
\]

We compute that, from (2.23):

\[
|r \partial_r G_n(R_0)| + |G_n(R_0)| \leq C(n) |\ln b|, \quad \text{with } C(n) \text{ independent of } \zeta_0.
\]
The collection of the above identities gives us the following leading order expressions for the matching quantity involving the refined asymptotics (2.27) and (2.28) gives

\[ \phi^n_1(r) = F_1(r) + \alpha bG_1(r) + E_1(r), \]

where

\[ F_1(r) = T_0(r) + 2b T_1(r) + b \left( -\frac{2}{\ln b} T_1(r) + \alpha_0^{-1} \Theta_0 \right) \]

\[ + \frac{2e_1}{\ln b} (b T_1(r) - b^2 T_2(r) - \frac{b^3}{2} \sum_{i=2}^{\infty} \frac{(1)_{i-1}}{(2i)!} b^{i+1} r^{2i} \ln(r + 1)) \]

\[ - \frac{b}{2} \sum_{i=1}^{\infty} \frac{(1)_{i-1}}{(2i)!} b^i r^2 \left( \frac{1}{\ln b} \left[ 2 \ln(r + 1) - \frac{1}{i} - \Psi(i + 2) - \gamma \right] + 1 - \frac{1}{\ln b} \right), \]

\[ G_1(r) = 2(-T_1(r) + S_0(r) - 2b T_2 + b S_1(r)), \quad E_1(r) = b \tilde{R}_1(r) + \frac{2e_1}{\ln b} (b S_0(r) + b^2 \tilde{S}_1(r)). \]
One has from (2.14), as \( \hat{d}_1 = -1/2, \, d_1 = 1/4 \) and \( \hat{d}_2 = 1/16, \, e_1 = \ln 2 - \gamma - 1 \) and \( R_0 = \zeta_0/\sqrt{b} \):

\[
F_1(R_0) = \frac{b}{\zeta_0^2} + 2b \left( \frac{\ln \zeta_0 - \ln b}{2} + \frac{1}{4} \right) + b \left( -\frac{2}{\ln b} \left( -\frac{\ln \zeta_0 - \ln b}{2} \right) + \frac{1}{2} \right) \\
+ \frac{2e_1}{|\ln b|^2} \left( -\frac{b \ln b}{4} + \frac{b \zeta_0^2 \ln b}{16} + b \ln b \sum_{i=2}^{\infty} \frac{(1)_{i-1}}{(2)_{i!^2}} \zeta_0^{2i} \right) \\
- \frac{b}{2} \sum_{i=1}^{\infty} \frac{(1)_{i-1}}{(2)_{i!^2}} \zeta_0^{2i} \left( \frac{1}{\ln b} \left[ 2 \ln \zeta_0 - \ln b - \frac{1}{i} - \Psi(i + 2) - \gamma \right] + 1 - \frac{1}{\ln b} \right) + O \left( \frac{b}{|\ln b|^2} \right) \\
= b \left\{ \frac{\ln b}{2} + \frac{1}{\zeta_0^2} - \ln \zeta_0 + \frac{1}{2} + \frac{\ln \zeta_0}{\ln b} + \frac{e_1}{2 \ln b} \left( -1 + \frac{1}{2} \sum_{i=1}^{\infty} \frac{(1)_{i-1}}{(2)_{i!^2}} \zeta_0^{2i} \right) \right\} \\
+ O \left( \frac{b}{|\ln b|^2} \right),
\]

and similarly, we have

\[
(r \partial_r F_1)(R_0) = \frac{-2b}{\zeta_0^2} - b + \frac{b}{\log b} - \frac{b}{2 \ln b} \sum_{i=1}^{\infty} \frac{(1)_{i-1}}{(2)_{i!^2}} \zeta_0^{2i} \left[ 2 \ln \zeta_0 - \Psi(i + 2) \right] + O \left( \frac{b}{|\log b|^2} \right).
\]

From (2.27) and (2.28), we obtain

\[
\sum_{0 \leq k \leq 2} |(r \partial_r)^k E_1(R_0)| \leq C(\zeta_0) \frac{b}{|\ln b|^2}.
\]

Hence, as \( G_1(R_0) = O(|\ln b|) \) and \( r \partial_r G_1(R_0) = O(|\ln b|) \), we obtain from the above identities

\[
\phi_1^{in}(R_0) = b \left[ -\frac{\ln b}{2} H_1(\zeta_0) + K_1(\zeta_0) + \frac{1}{2 \ln b} J_1(\zeta_0) + \hat{\alpha} b G_1(R_0) + O \left( \frac{1}{|\ln b|^2} \right) \right],
\]

\[
r \partial_r \phi_1^{in}(R_0) = b \left[ \zeta \partial_\zeta K_1 + \frac{1}{2 \ln b} \zeta \partial_\zeta J_1 + \hat{\alpha} r \partial_r G_1(R_0) + O \left( \frac{1}{|\ln b|^2} \right) \right],
\]

where we used (2.77), so that \( H_1(\zeta) = 1 \) and \( K_1(\zeta) = \zeta^{-2} - \ln \zeta + 1/2 \) and

\[
J_1(\zeta_0) = 2 \ln \zeta_0 - e_1 - \sum_{i=1}^{\infty} \frac{(1)_{i-1}}{(2)_{i!^2}} \zeta_0^{2i} \left[ 2 \ln \zeta_0 - \ln 2 - \frac{1}{i} - \Psi(i + 2) \right].
\]
We finally obtain
\[
\frac{r\partial_r \phi_0^{in}(R_0)}{\phi_1^{in}(R_0)} = \frac{\zeta \partial_z K_1 + \frac{1}{2\ln b} \zeta \partial_z J_1 + \hat{\alpha} r \partial_r G_1(R_0) + O \left( \frac{1}{\ln b^2} \right)}{-\frac{\ln b}{2} H_1(\zeta_0) + K_1(\zeta_0) + \frac{1}{2\ln b} J_1(\zeta_0) + \hat{\alpha} b G_1(R_0) + O(\frac{1}{\ln b^2})} = \frac{2}{\ln b} H_1(\zeta_0) - \frac{2}{\ln b} K_1(\zeta_0) - \frac{1}{\ln b} J_1(\zeta_0) - \frac{2}{\ln b} \hat{\alpha} b G_1(R_0) + O(\frac{1}{\ln b^2})
\]
\[
= -\frac{2}{\ln b} H_1(\zeta_0) + \frac{1}{\ln b} \zeta \partial_z J_1 H_1 + 2K_1 \zeta \partial_z K_1 + \hat{\alpha} \frac{1}{H_1^2} + O(\|\ln b\|^{-2})
\]
\[
= -\frac{2}{\ln b} H_1(\zeta_0) + \frac{1}{\ln b} \zeta \partial_z J_1 H_1 + 2K_1 \zeta \partial_z K_1 + \hat{\alpha} \frac{O(1)}{H_1^2} + O(\|\ln b\|^{-3}) \tag{2.83}
\]
where the constant in the $O(1)$ is independent of $\zeta_0$.

**The case $n = 0$:** We first use the refined asymptotics (2.26) and (2.25) to obtain:
\[
\phi_0^{in}(r) = F_0(r) + \hat{\alpha} b G_0(r) + E_0(r),
\]
where:
\[
F_0(r) = T_0(r) + b \left( -\frac{2}{\ln b} T_1(r) + \mathcal{A}^{-1} \Theta_0 \right) + \frac{b}{2} \sum_{i=1}^{\infty} \frac{1}{(2)^i \mathbb{Z}^2} \left\{ \frac{1}{\ln b} [2 \ln(r + 1) - \Psi(i + 2) - \gamma] + 1 \right\},
\]
\[
G_0(r) = 2 \left( -T_1(r) + \frac{1}{2} \sum_{i=1}^{\infty} \frac{1}{(2)^i \mathbb{Z}^2} b^{i+2} \log(r + 1) \right),
\]
\[
E_0(r) = b \tilde{R}(r) + 2 \hat{\alpha} b \tilde{S}(r).
\]
One has from (2.14), as $d_1 = -1/2$, $d_1 = 1/4$:
\[
F_0(R_0) = \frac{b}{\zeta_0^2} + b \left( -\frac{2}{\ln b} \left( -\frac{\ln \zeta_0 - \frac{\ln b}{2}}{4} + \frac{1}{2} \right) + \frac{1}{2} \right)
\]
\[
+ \frac{b}{2} \sum_{i=1}^{\infty} \frac{1}{(2)^i \mathbb{Z}^2} \left\{ \frac{1}{\ln b} [2 \ln \zeta_0 - \Psi(i + 2) - \gamma] \right\} + O(b^2)
\]
\[
= \frac{b}{\zeta_0^2} + \frac{b \ln \zeta_0}{\ln b} - \frac{b}{2 \ln b} + \frac{b}{2 \ln b} \sum_{i=1}^{\infty} \frac{1}{(2)^i \mathbb{Z}^2} [2 \ln \zeta_0 - \Psi(i + 2) - \gamma] + O(b^2),
\]
and similarly, we have
\[
(r \partial_r F_0)(R_0) = \frac{-2b}{\zeta_0^2} + b \left( \frac{1}{\log b} + \frac{b}{2 \ln b} \sum_{i=1}^{\infty} \frac{1}{(2)^i \mathbb{Z}^2} [2 \ln \zeta_0 + \frac{1}{i} - \Psi(i + 2) - \gamma] + O(b^2) \right)
\]
\[
\partial_b (b^{-1} F_0(R_0)) = O \left( \frac{1}{b \ln b^2} \right), \quad \partial_b (b^{-1} r \partial_r F_0(R_0)) = O \left( \frac{1}{b \ln b^2} \right).
\]
From (2.26), we obtain
\[
\sum_{0 \leq k \leq 2, \ 0 \leq \ell + \ell' \leq 1} ((b \partial_b)^{\ell} \partial_{\alpha}^{\ell'} (r \partial_r)^k E_0)(R_0) \leq C(\zeta_0) \frac{b}{\ln b^2}.
\]
One also has
\[
G_0(R_0) = 2 \left( -\frac{1}{4} \ln b - \frac{1}{4} \ln b \sum_{i=1}^{\infty} \frac{1}{(2)^i \mathbb{Z}^2} \right) + O(1) = -\frac{\ln b}{2} \tilde{G}_0(\zeta_0) + O(1), \quad \partial_b G_0(R_0) = O \left( \frac{1}{b} \right),
\]
where
\[
\tilde{G}_0(\zeta_0) = \sum_{i=0}^{\infty} \frac{1}{(2i)^2} \zeta_0^{2i},
\] (2.84)
so that
\[
r \partial_r \tilde{G}_0(R_0) = -\frac{\ln b}{2} \zeta \partial_\zeta \tilde{G}_0(\zeta_0) + \mathcal{O}(1), \quad \partial_\zeta r \partial_r \tilde{G}_0(R_0) = \mathcal{O} \left( \frac{1}{b} \right).
\]
We obtain from the above identities
\[
\phi_0^{in}(R_0) = b \left[ \frac{1}{\zeta_0^2} + \frac{1}{2 \ln b} J_0(\zeta_0) - \frac{\ln b}{2} \tilde{G}_0(\zeta_0) + \mathcal{O}(\ln b^{-2}) \right],
\]
\[
r \partial_r \phi_0^{in}(R_0) = b \left[ -\frac{2}{\zeta_0^2} + \frac{1}{2 \ln b} \zeta \partial_\zeta J_0(\zeta_0) - \frac{\ln b}{2} \tilde{G}_0(\zeta_0) + \mathcal{O}(\ln b^{-2}) \right],
\]
where
\[
J_0(\zeta_0) = 2 \ln \zeta_0 - 1 + \sum_{i=1}^{\infty} \frac{1}{(2i)^2} \zeta_0^{2i} [2 \ln \zeta_0 - \Psi(i + 2) - \gamma],
\] (2.85)
and for \( \tilde{\alpha} = \mathcal{O}(\ln b^{-2}) \),
\[
\partial_\zeta \left( b^{-1} \phi_0^{in}(R_0) \right) = \mathcal{O} \left( \frac{1}{|b| \ln b^2} \right), \quad \partial_\zeta \left( b^{-1} r \partial_r \phi_0^{in}(R_0) \right) = \mathcal{O} \left( \frac{1}{|b| \ln b^2} \right),
\]
\[
\partial_{\tilde{\alpha}} \left( \phi_0^{in}(R_0) \right) = -\frac{b \ln b}{2} \tilde{G}_0(\zeta_0) + \mathcal{O}(b), \quad \partial_{\tilde{\alpha}} \left( r \partial_r \phi_0^{in}(R_0) \right) = -\frac{b \ln b}{2} r \partial_r \tilde{G}_0(\zeta_0) + \mathcal{O}(b)
\]
We finally obtain
\[
\frac{r \partial_r \phi_0^{in}(R_0)}{\phi_0^{in}(R_0)} = \frac{-\frac{2}{\zeta_0^2} + \frac{1}{2 \ln b} \zeta \partial_\zeta J_0(\zeta_0) + \tilde{\alpha} r \partial_r G_0(R_0) + \mathcal{O}(\ln b^{-2})}{\frac{1}{\zeta_0^2} + \frac{1}{2 \ln b} J_0(\zeta_0) + \tilde{\alpha} G_0(R_0) + \mathcal{O}(\ln b^{-2})}
\]
\[
= -2 + \frac{\zeta_0^2}{2 \ln b} \zeta \partial_\zeta J_0(\zeta_0) + \frac{\zeta_0^2}{2 \ln b} \partial_\zeta \tilde{G}_0(\zeta_0) + \mathcal{O}(\ln b^{-2})
\]
\[
= -2 + \frac{1}{2 \ln b} \zeta_0^2 \left( \frac{1}{2} \zeta \partial_\zeta J_0 + J_0 \right) - \frac{\ln b}{2} \tilde{\alpha} \z_0^2 \left( \z_0 \partial_\zeta \tilde{G}_1(\zeta_0) + 2 \tilde{G}_1(\zeta_0) + \mathcal{O}(\ln b^{-1}) \right) + \mathcal{O}(\ln b^{-2}),
\] (2.86)
and
\[
\frac{r \partial_r \phi_0^{in}(R_0)}{\phi_0^{in}(R_0)} = \partial_\zeta \left( b^{-1} r \partial_r \phi_0^{in}(R_0) \right)
\]
\[
= \partial_\zeta \left( b^{-1} r \partial_r \phi_0^{in}(R_0) \right) - \frac{\partial_\zeta \phi_0^{in}(R_0)}{b^{-1} \phi_0^{in}(R_0)} \partial_\zeta \left( b^{-1} \phi_0^{in}(R_0) \right)
\]
\[
= \mathcal{O} \left( \frac{1}{|b| \ln b^2} \right).
\] (2.87)
and
\[
\partial_{\tilde{\alpha}} \left( r \partial_r \phi_0^{in}(R_0) \right) = -\frac{b \ln b}{2} \zeta_0^2 \left( \z_0 \partial_\zeta \tilde{G}_1(\zeta_0) + 2 \tilde{G}_1(\zeta_0) + \mathcal{O}(\ln b^{-1}) \right),
\] (2.88)
where the constant in the \( \mathcal{O}(\ln b^{-2}) \) is independent of \( \tilde{\alpha} \).
The exterior term: Recall the decomposition \( q[b, \tilde{b}](z) = \Gamma(\theta) h_\theta(z) + \mathcal{G}[b, \tilde{b}](z) \) from (2.64). From the estimates (2.65) the second term is of lower order and satisfies:

\[
\sum_{0 \leq k + \ell \leq 1} |(b \partial_b)^k \partial_\alpha^\ell (\mathcal{G}(z_0))| + |(b \partial_b)_0^k \partial_\alpha^\ell (z \partial_z \mathcal{G}(z_0))| \lesssim b^{\frac{3}{2}}. \tag{2.89}
\]

We now investigate the formula giving \( h_\theta \). From the recurrence relation of the Gamma function and the identity \( \partial_\theta (\theta) = (\theta)(\theta + \gamma) \):

\[
\Gamma(\theta) h_\theta(z) = \frac{\Gamma(z + \theta)}{z} \left( \ln z + \Psi(\theta + i) - \Psi(\theta + 2i) \right),
\]

and

\[
z \partial_z \Gamma(\theta) h_\theta(z) = \frac{\Gamma(z + \theta)}{z^2} \left( \ln z + \Psi(\theta + i) - \Psi(\theta + 2i) \right) \left( \frac{\theta}{z} \partial_z \theta \right)
\]

and

\[
\partial_\theta \Gamma(\theta) h_\theta(z) = \sum_{i=0}^{\infty} \frac{\Gamma(z + \theta)}{(2i)!} i^i \left( \ln z + \Psi(\theta + i) - \Psi(\theta + 2i) \right) \left( 1 + \frac{\theta}{z} \partial_z \theta \right) + \frac{\theta}{z} \partial_z \theta \Psi(\theta + i) + 1.
\]

We now decompose all above expressions into leading order and lower terms. We first collect some estimates on the coefficients. Note that for \( i < n \) one has from the recurrence relation of the Gamma function:

\[
(\theta)_i = \frac{\Gamma(\theta + i)}{\Gamma(\theta)} = (\theta)(\theta + 1) \cdots (\theta + i - 1) = (1 - n + \tilde{b})(2 - n + \tilde{b}) \cdots (i - n + \tilde{b}) = O(|\tilde{b}|) \tag{2.90}
\]

because there is some \( 0 \leq j \leq i - 1 \) such that \( 1 - n + j = 0 \). Moreover, for a large enough argument the digamma function

\[
\Psi(\theta + i) = \Psi(1 - n + i + \tilde{b}) = \Psi(1 - n + i) + O(\tilde{b}) = O(1) \quad \text{for } i \geq n \tag{2.91}
\]

is non-singular since \( 1 - n + i > 1 \). We recall the recurrence relation for the digamma function \( \Psi(z + 1) = \Psi(z) + 1/z \), with \( \Psi(1) = -\gamma \) the Euler constant. Then, if \( k \) is an integer:

\[
\Psi(k + 1) = \frac{1}{k} + \Psi(k) = \frac{1}{k} + \frac{1}{k-1} + \cdots + \frac{1}{2} + 1 - \gamma.
\]

Hence, refining (2.90) for \( i < n \), we obtain

\[
(\theta)_i = (1 - n)_i (1 + \tilde{b} \Psi(n - i) - \Psi(n)) + O(\tilde{b}^2)
\]

and

\[
\Psi(\theta + i) = -\frac{1}{\theta + i} + \Psi(\theta + i + 1) = -\frac{1}{1 - n + i + \tilde{b}} - \frac{1}{2 - n + i + \tilde{b}} - \cdots - \frac{1}{-1 + \tilde{b}} - \frac{1}{\tilde{b}} + \Psi(1 + \tilde{b})
\]

\[
= -\frac{1}{\tilde{b}} + \Psi(n - i) + O(\tilde{b}), \tag{2.92}
\]

\[
\partial_\theta \Psi(\theta + i) = \partial_\tilde{b} \Psi(\theta + i) = \frac{1}{\tilde{b}^2} + O(1) \quad \text{for } i < n. \tag{2.93}
\]
The coefficients that will appearing in the expansion are related to the inner expansion the following way. Using the recurrence relations (2.19)-(2.22) and the initial values for \( c_{n,1} \) and \( d_1 \),

\[-c_{n,i+1} \hat{d}_{i+1} = n \frac{(1-n)_i}{(2)_i i! 2^i}, \tag{2.94}\]

and similarly using the recurrence relations (2.19),

\[-\frac{2d_{i+1}}{\hat{d}_{i+1}} = 2 + \frac{2}{2} + \frac{2}{3} + \ldots + \frac{1}{i+1} = \Psi(i+2) + \Psi(i+1) + 2\gamma. \tag{2.95}\]

Hence, the strategy is the following. We first truncate the series (2.54) expressing \( h_\theta \) for \( 0 < z \ll 1 \) using (2.91) and (2.90). Then, we expand it with respect to \( \tilde{\alpha} \). Finally, we express the coefficients in function of those of the inner expansion via (2.94)-(2.95). The result of this strategy is given by

\[
\Gamma(\Theta) h_\theta(z) = \frac{1}{z} + (\theta - 1) \sum_{i=0}^{\infty} \frac{(\theta)_i}{(2)_i i!} z^i \left[ \ln z + \Psi(\theta + i) - \Psi(1+i) - \Psi(2+i) + O(|\tilde{\alpha}|) \right]
\]

\[
= \frac{1}{z} + (\theta - 1) \left[ \sum_{i=0}^{n-1} [...] + (\theta - 1) \sum_{i=n}^{\infty} [...] \right]
\]

\[
= \frac{1}{z} + (\theta - 1) \sum_{i=0}^{n-1} \frac{(\theta)_i}{(2)_i i!} z^i \left[ \ln z + \Psi(\theta + i) - \Psi(1+i) - \Psi(2+i) \right] + O(|\tilde{\alpha}|)
\]

\[
= \frac{1}{z} + (\tilde{\alpha} - n) \sum_{i=0}^{n-1} \frac{(1-n)_i}{(2)_i i!} \left( 1 + \tilde{\alpha} (\Psi(n-i) - \Psi(n)) + O(\tilde{\alpha}^2) \right) z^i
\]

\[
\times \left[ \ln z - \frac{1}{\tilde{\alpha}} + \Psi(n-i) + O(|\tilde{\alpha}|) - \Psi(1+i) - \Psi(2+i) \right] + O(|\tilde{\alpha}|)
\]

\[
= \frac{1}{z} + \sum_{i=0}^{n-1} \frac{n(1-n)_i}{(2)_i i!} z^i \left[ -\ln z - \Psi(n+1) + \Psi(i+1) + \Psi(i+2) + \frac{1}{\tilde{\alpha}} \right] + O(|\tilde{\alpha}|)
\]

\[
= \frac{1}{z} \sum_{i=0}^{n-1} \frac{(1-n)_i}{(2)_i i!} z^i \left[ -\ln z - \ln 2 + \Psi(i+1) + \Psi(i+2) + 2\gamma + \frac{1}{\tilde{\alpha}} \right] + O(|\tilde{\alpha}|),
\]

\[
= \frac{1}{z} \sum_{i=1}^{n} 2^{i-1} c_{n,i} z^{i-1} \left( \hat{d}_i \left( \ln z + \ln 2 - e_n - \frac{1}{\tilde{\alpha}} \right) + 2d_i \right) + O(|\tilde{\alpha}|). \tag{2.96}\]

Similarly, skipping the computations which are verbatim the same as the one above yields

\[
z \partial_z \Gamma(\Theta) h_\theta(z) = -\frac{1}{z} + (\theta - 1) \sum_{i=0}^{\infty} \frac{(\theta)_i}{(2)_i i!} z^i \left[ i \left( \ln z + \Psi(\theta + i) - \Psi(1+i) - \Psi(2+i) \right) + 1 \right]
\]

\[
= -\frac{1}{z} \sum_{i=1}^{n} 2^{i-1} c_{n,i} z^{i-1} \left[ (i-1) \left( \hat{d}_i \left( \ln z + \ln 2 - e_n - \frac{1}{\tilde{\alpha}} \right) + 2d_i \right) + \hat{d}_i \right] + O(|\tilde{\alpha}|) \tag{2.97}\]

Then, using (2.90), (2.91), (2.92), (2.93) and \( \partial_\theta \tilde{\alpha} = 1 \), we compute

\[
\partial_\theta (\Gamma(\Theta) h_\theta(z))
\]

\[
= \frac{1}{z^2} \sum_{i=0}^{\infty} \frac{(\theta)_i}{(2)_i i!} z^i \left[ \ln z + \Psi(\theta + i) - \Psi(1+i) - \Psi(2+i) \right] \left( 1 + (\theta-1)(\Psi(\theta+i) - \Psi(\theta)) \right) + (\theta-1) \partial_\theta \Psi(\theta+i)
\]

\[
= -\frac{1}{\tilde{\alpha}^2} \sum_{i=0}^{n-1} \frac{(1-n)_i}{(2)_i i!} z^i + O(1) = \frac{1}{\tilde{\alpha}^2} \sum_{i=1}^{n} 2^{i-1} c_{n,i} \hat{d}_i z^{i-1} + O(1)
\]
so that from (2.96):
\[
\partial_\theta (\tilde{\alpha}\Gamma(\theta) h_\theta(z)) = \Gamma(\theta) h_\theta(z) + \tilde{\alpha} \partial_\theta (\Gamma(\theta) h_\theta(z))
\]
\[
= \frac{1}{z} + \sum_{i=1}^{n} 2^{i-1} c_{n,i} z^{i-1} \left( \hat{d}_i \left( \ln z + \ln 2 - e_n - \frac{1}{\tilde{\alpha}} \right) + 2d_i \right) + O(|\tilde{\alpha}|) + \frac{1}{\alpha} \sum_{i=1}^{n} 2^{i-1} c_{n,i} \hat{d}_i z^{i-1} + O(|\tilde{\alpha}|)
\]
\[
= \frac{1}{z} + \sum_{i=1}^{n} 2^{i-1} c_{n,i} z^{i-1} \left( \hat{d}_i \left( \ln z + \ln 2 - e_n \right) + 2d_i \right) + O(|\tilde{\alpha}|),
\]
(2.98)

and similarly
\[
\partial_\theta \left( z \partial_\zeta \Gamma(\theta) h_\theta(z) \right)
\]
\[
= \sum_{i=0}^{\infty} \frac{(\theta)_i}{(2\eta)^i} i^i \left[ i \left( (\ln z + \Psi(\theta + i) - \Psi(1 + i) - \Psi(2 + i) \right) (1 + (\theta - 1)(\Psi(\theta + i) - \Psi((\theta))) + (\theta - 1) \partial_\theta \Psi(\theta + i)) + 1 \right]
\]
\[
= - \frac{n^{-1}}{\alpha^2} \sum_{i=0}^{n-1} \frac{(1-n)_i i^i}{(2\eta)^i} + O(1) = \frac{1}{\alpha^2} \sum_{i=1}^{n} 2^{i-1} (i-1) z^{i-1} c_{n,i} \hat{d}_i + O(1)
\]

so that from (2.97), we get
\[
\partial_\theta (\tilde{\alpha} z \partial_\zeta \Gamma(\theta) h_\theta(z)) = z \partial_\zeta \Gamma(\theta) h_\theta(z) + \tilde{\alpha} \partial_\theta (z \partial_\zeta \Gamma(\theta) h_\theta(z))
\]
\[
= - \frac{1}{z} + \sum_{i=1}^{n} 2^{i-1} c_{n,i} z^{i-1} \left[ (i - 1) \left( \hat{d}_i \left( \ln z + \ln 2 - e_n \right) + 2d_i \right) + \hat{d}_i \right] + O(|\tilde{\alpha}|).
\]
(2.99)

Therefore we obtain from (2.96), (2.89), as \( z = \zeta^2/2 \) and \( \tilde{\alpha} = 1/\log b + O(|\log b|-2) \):
\[
q(z_0) = \frac{2}{\zeta_0^2} + \sum_{i=1}^{n} 2^{i-1} c_{n,i} \left( \frac{\zeta_0^2}{2} \right)^{i-1} \left[ (i - 1) \left( \hat{d}_i \left( \ln \left( \frac{\zeta_0^2}{2} \right) + \ln 2 - e_n - \frac{1}{\tilde{\alpha}} \right) + 2d_i \right) + O(|\tilde{\alpha}|) + O(b^\frac{3}{2}) \right.
\]
\[
= - \frac{1}{\alpha} H_n(\zeta_0) + 2K_n(\zeta_0) - e_n H_n(\zeta_0) + O(|\tilde{\alpha}|),
\]
where \( H_n \) and \( G_n \) are given by (2.77). Similarly, we compute from (2.97) and (2.89),
\[
(z \partial_\zeta) q(z_0) = - \frac{2}{\zeta_0^2} + \sum_{i=1}^{n} 2^{i-1} c_{n,i} \left( \frac{\zeta_0^2}{2} \right)^{i-1} \left[ (i - 1) \left( \hat{d}_i \left( \ln \left( \frac{\zeta_0^2}{2} \right) + \ln 2 - e_n - \frac{1}{\tilde{\alpha}} \right) + 2d_i \right) + \hat{d}_i \right]
\]
\[
+ O(|\tilde{\alpha}|) + O(b^\frac{3}{2})
\]
\[
= - \frac{1}{2\tilde{\alpha}} \zeta \partial_\zeta H_n(\zeta_0) + \zeta \partial_\zeta K_n(\zeta_0) - \frac{e_n}{\alpha} \zeta \partial_\zeta H_n(\zeta_0) + O(|\tilde{\alpha}|).
\]
From (2.89), (2.98), (2.99), recalling that \( b \) and \( \tilde{\alpha} \) are two independent parameters for the moment, using the relations \( b \partial_\theta = -1/|\ln b|^2 = O(1/|\ln b|^2) \) and \( \partial_{\tilde{\alpha}} = \partial_{\theta} \):
\[
b \partial_b (\tilde{\alpha} q(z_0)) = O\left( \frac{1}{|\ln b|^2} \right) \partial_b (\tilde{\alpha} \Gamma(\theta) h(\theta)(z_0)) + O(b^\frac{3}{2}) = O(\tilde{\alpha}^2),
\]
(2.100)
\[
b \partial_b (\tilde{\alpha} z \partial_\zeta q(z_0)) = O\left( \frac{1}{|\ln b|^2} \right) \partial_b (\tilde{\alpha} z \partial_\zeta \Gamma(\theta) h(\theta)(z_0)) + O(b^\frac{3}{2}) = O(\tilde{\alpha}^2),
\]
(2.101)
\[
\partial_{\tilde{\alpha}} (\tilde{\alpha} q(z_0)) = \partial_{\theta} (\tilde{\alpha} q(z_0))
\]
\[
= \frac{2}{\zeta_0^2} + \sum_{i=1}^{n} 2^{i-1} c_{n,i} \left( \frac{\zeta_0^2}{2} \right)^{i-1} \left( \hat{d}_i \left( \ln \left( \frac{\zeta_0^2}{2} \right) + \ln 2 - e_n \right) + 2d_i \right) + O(|\tilde{\alpha}|) + O(b^\frac{1}{2})
\]
\[
= 2K_n(\zeta_0) - e_n H_n(\zeta_0) + O(|\tilde{\alpha}|),
\]
(2.102)
We deduce that for $n \geq 2$,

$$
\partial_{\tilde{\alpha}} (\tilde{\alpha} z \partial_z q(z_0)) = \partial_b (\tilde{\alpha} z \partial_z q(z_0)) = -2 \frac{\zeta}{\alpha} \partial_{\tilde{\alpha}} H_n(\zeta_0) - \frac{\zeta}{\alpha} \partial_{\tilde{\alpha}} K_n(\zeta_0) + O(\tilde{\alpha}^2) + O(\tilde{\alpha}^2) + O(|\tilde{\alpha}|).
$$

(2.103)

and similarly from (2.100), (2.102), (2.102) and (2.103),

$$
b \partial_b \left( \frac{z \partial_z q(z_0)}{q(z_0)} \right) = b \partial_b \left( \frac{\tilde{\alpha} z \partial_z q(z_0)}{\tilde{\alpha} q(z_0)} \right) = b \partial_b (\tilde{\alpha} z \partial_z q(z_0)) \hat{\alpha} q(z_0) - \tilde{\alpha} z \partial_z q(z_0) b \partial_b (\tilde{\alpha} q(z_0)) = \frac{O(\tilde{\alpha}^2)}{\tilde{\alpha}^2 q(z_0)^2} = O(\tilde{\alpha}^2),
$$

(2.105)

$$
\partial_{\tilde{\alpha}} \left( \frac{z \partial_z q(z_0)}{q(z_0)} \right) = \partial_{\tilde{\alpha}} \left( \frac{\tilde{\alpha} z \partial_z q(z_0)}{\tilde{\alpha} q(z_0)} \right) = \frac{\partial_{\tilde{\alpha}} (\tilde{\alpha} z \partial_z q(z_0)) \hat{\alpha} q(z_0) - \partial_{\tilde{\alpha}} (\hat{\alpha} q(z_0)) \tilde{\alpha} z \partial_z q(z_0)}{\tilde{\alpha}^2 q(z_0)^2} = \frac{K_n(\zeta_0) \zeta \partial_{\tilde{\alpha}} H_n(\zeta_0) - \zeta \partial_{\tilde{\alpha}} K_n(\zeta_0) H(\zeta_0)}{H_n^2(\zeta_0)} + O(|\tilde{\alpha}|).
$$

(2.106)

The case $n = 1$: For $n = 1$, $\theta = \tilde{\alpha}$, so we refine further $\tilde{\alpha}$ and take

$$
\tilde{\alpha} = \frac{1}{\ln b} + \frac{e_1}{|\ln b|^2} + \hat{\alpha}, \quad e_1 = \ln 2 - \gamma - 1 = \ln 2 - \Psi(2) - 2\gamma, \quad \hat{\alpha} = O(|\ln b|^{-3}).
$$
We then refine further $\Gamma(\theta)h_\theta$ by noticing that for $i \geq 1$, $(\hat{\alpha})_i = \hat{\alpha}(i) + O(\hat{\alpha}^2)$ and $\Psi(\hat{\alpha}) = -\hat{\alpha}^{-1} - \gamma + \pi^2 \hat{\alpha}/6 + O(\hat{\alpha}^2)$,

$$
\Gamma(\Theta)h_\theta(z) = \frac{1}{z} + (\hat{\alpha} - 1) \sum_{i=0}^{\infty} \frac{(\hat{\alpha})_i}{(2i)!} z^i \left[ \ln z + \Psi(\hat{\alpha} + i) - \Psi(1 + i) - \Psi(2 + i) \right] \\
= \frac{1}{z} + (\hat{\alpha} - 1) \left[ \ln z + \Psi(\hat{\alpha}) - \Psi(1) - \Psi(2) \right] \\
- \hat{\alpha} \sum_{i=1}^{\infty} \frac{\Gamma(i)}{(2i)!} z^i \left[ \ln z + \Psi(i) - \Psi(1 + i) - \Psi(2 + i) \right] + O(\hat{\alpha}^2) \\
= \frac{1}{z} + (\hat{\alpha} - 1) \left[ \ln z - \frac{1}{\alpha} + \hat{\alpha} \frac{\pi^2}{6} - \Psi(2) \right] \\
- \hat{\alpha} \sum_{i=1}^{\infty} \frac{\Gamma(i)}{(2i)!} z^i \left[ \ln z + \Psi(i) - \Psi(1 + i) - \Psi(2 + i) \right] + O(\hat{\alpha}^2) \\
= \frac{1}{\alpha} + \frac{1}{z} - [\ln z + \gamma] \\
+ \hat{\alpha} \left( \ln z - \Psi(2) - \frac{\pi^2}{6} - \sum_{i=1}^{\infty} \frac{\Gamma(i)}{(2i)!} z^i \left[ \ln z + \Psi(i) - \Psi(1 + i) - \Psi(2 + i) \right] \right) + O(\hat{\alpha}^2).
$$

With this, a further refinement of (2.90) with the same computation as above yields in this case, using (2.82),

$$
q(z_0) = -\frac{1}{\alpha} H_1(\zeta_0) + 2K_1(\zeta_0) - e_1H_1(\zeta_0) + \hat{\alpha}(J_1 - 2 - \frac{\pi^2}{6})(\zeta_0) + O(|\hat{\alpha}|^2),
$$

$$
z \partial_z q(z_0) = \zeta \partial_{\zeta} K_1(\zeta_0) + \frac{\hat{\alpha}}{2} \zeta \partial_{\zeta} J_1(\zeta_0) + O(\hat{\alpha}^2),
$$

$$
\partial_\alpha(z_\partial_z q(z_0)) = 1 - \sum_{i=1}^{\infty} \frac{\Gamma(i)}{(2i)!} \zeta^2 \left[ 2 \ln \zeta_0 - \ln 2 - \Psi(2 + i) \right] + O\left( \frac{1}{|\log b|} \right).
$$

Hence, combining these identities with the previous estimates, and using $H_1 = \log b/2 + O(1)$, we obtain

$$
\frac{z \partial_z q(z_0)}{q(z_0)} = \frac{\zeta \partial_{\zeta} K_n(\zeta_0) + \hat{\alpha} \zeta \partial_{\zeta} J_1(\zeta_0) + O(\hat{\alpha}^2)}{-\frac{1}{\alpha} H_1(\zeta_0) + 2K_1(\zeta_0) - e_1H_1(\zeta_0) + \hat{\alpha}(J_1 - 2 - \frac{\pi^2}{6})(\zeta_0) + O(|\hat{\alpha}|^2)} \\
= \hat{\alpha} \left[ \frac{-\zeta \partial_{\zeta} K_1(\zeta_0)}{H_1(\zeta_0)} + \frac{\zeta \partial_{\zeta} J_1(\zeta_0)}{H_1(\zeta_0)} \right] + \frac{1}{H_1(\zeta_0)} + \frac{2 \zeta \partial_{\zeta} K_1(\zeta_0) K_1(\zeta_0)}{H_1^2(\zeta_0)} + \frac{\zeta \partial_{\zeta} J_1(\zeta_0) H_1(\zeta_0)}{H_1^2(\zeta_0)} + O(\hat{\alpha}^3) + O(\hat{\alpha}^3).
$$

We now use the expansion $\hat{\alpha} = 1/\ln b + e_1/(\ln b)^2 + \hat{\alpha}$ to derive

$$
\frac{z \partial_z q(z_0)}{q(z_0)} = \frac{1}{\ln b} \frac{\zeta \partial_{\zeta} K_1}{H_1} \\
- \frac{1}{\ln b^2} \frac{2 \zeta \partial_{\zeta} K_1(\zeta_0) K_1(\zeta_0) + \zeta \partial_{\zeta} J_1(\zeta_0) H_1(\zeta_0)}{H_1^2(\zeta_0)} - \hat{\alpha} \frac{\zeta \partial_{\zeta} K_1}{H_1} + O(|\log b|^{-3}),
$$

and

$$
\partial_\alpha \left( \frac{z \partial_z q(z_0)}{q(z_0)} \right) = -\frac{\zeta \partial_{\zeta} K_1(\zeta_0)}{H_1(\zeta_0)} + O(\hat{\alpha}^2),
$$

and

$$
\partial_\alpha \left( \frac{z \partial_z q(z_0)}{q(z_0)} \right) = O(|\hat{\alpha}|^2).
$$
For \( n = 0, \theta = 1 + \tilde{\alpha} \). We then refine further \( \Gamma(\theta)h_\theta \) by noticing that for \( i \geq 0 \),
\[
(1 + \tilde{\alpha})_i = (1)_i + \mathcal{O}(|\tilde{\alpha}|) \quad \text{and} \quad \Psi(1 + \tilde{\alpha} + i) = \Psi(1 + i) + \mathcal{O}(|\tilde{\alpha}|),
\]
\[
\Gamma(\Theta)h_\theta(z) = \frac{1}{z} + \tilde{\alpha} \sum_{i=0}^{\infty} \frac{(1 + \tilde{\alpha})_i}{(2i)!} z^i \left[ \ln z + \Psi(1 + \tilde{\alpha} + i) - \Psi(1 + i) - \Psi(2 + i) \right]
\]
\[
= \frac{1}{z} + \tilde{\alpha} \sum_{i=0}^{\infty} \frac{1}{(2i)!} z^i \left[ \ln z - \Psi(2 + i) \right] + \mathcal{O}(|\tilde{\alpha}|^2)
\]
With this, performing the same computations as the previous ones and using \( \tilde{\alpha} = 1/\ln b + \mathcal{O}(\ln b)^{-2} \), we obtain
\[
q(z_0) = \frac{2}{\zeta_0} + \tilde{\alpha} \left( J_0(\zeta_0) + (\gamma - \ln 2)\tilde{G}_0(\zeta_0) \right) + \mathcal{O}(\tilde{\alpha}^2),
\]
\[
z\partial_z q(z_0) = -\frac{2}{\zeta_0^2} + \tilde{\alpha} \frac{1}{2} \left( \zeta \partial_\zeta J_0(\zeta_0) + (\gamma - \ln 2)\zeta \partial_\zeta \tilde{G}_0(\zeta_0) \right) + \mathcal{O}(\tilde{\alpha}^2),
\]
where \( J_0 \) and \( \tilde{G}_0 \) are defined in (2.85) and (2.84), and
\[
\partial_\tilde{\alpha}(q(z_0)) = J_0(\zeta_0) - 1 + (\gamma - \ln 2)\tilde{G}_0(\zeta_0) + \mathcal{O}(|\tilde{\alpha}|),
\]
\[
\partial_\tilde{\alpha}(z\partial_z q(z_0)) = \frac{1}{2} \zeta \partial_\zeta J_0(\zeta_0) + \frac{\gamma - \ln 2}{2} \zeta \partial_\zeta \tilde{G}_0(\zeta_0) + \mathcal{O}(|\tilde{\alpha}|^2).
\]
Hence, using \( \partial_\theta \tilde{\alpha} = -1/b|\ln b|^2 \), we obtain
\[
\frac{z\partial_z q(z_0)}{q(z_0)} = -\frac{2}{\zeta_0^2} + \tilde{\alpha} \frac{1}{2} \left( \zeta \partial_\zeta J_0(\zeta_0) + (\gamma - \ln 2)\zeta \partial_\zeta \tilde{G}_0(\zeta_0) \right) + \mathcal{O}(\tilde{\alpha}^2)
\]
\[
= -1 + \tilde{\alpha} \zeta_0^2 \left( \frac{1}{4} \zeta \partial_\zeta J_0 + \frac{\gamma - \ln 2}{4} \zeta \partial_\zeta \tilde{G}_0(\zeta_0) + \frac{1}{2} J_0 + \frac{\gamma - \ln 2}{2} \tilde{G}_1(\zeta_0) \right) + \mathcal{O}(\tilde{\alpha}^2),
\]
(2.108)
\[
\partial_\tilde{\alpha} \left( \frac{z\partial_z q(z_0)}{q(z_0)} \right) = \zeta_0^2 \left( \frac{1}{4} \zeta \partial_\zeta J_0 + \frac{\gamma - \ln 2}{4} \zeta \partial_\zeta \tilde{G}_0(\zeta_0) + \frac{1}{2} J_0 + \frac{\gamma - \ln 2}{2} \tilde{G}_1(\zeta_0) \right) + \mathcal{O}(|\tilde{\alpha}|),
\]
(2.109)
\[
\partial_\theta \left( \frac{z\partial_z q(z_0)}{q(z_0)} \right) = \mathcal{O} \left( \frac{1}{|\ln b|^2} \right).
\]
(2.110)

**Conclusion of the proof:** We first compute \( \tilde{\alpha}_n \). From (2.71), (2.78), (2.104) we arrive at the following: For \( n \geq 2 \), we have
\[
\Theta(b, \tilde{\alpha}) = \frac{\zeta \partial_\zeta H_n(\zeta_0)}{2H_n(\zeta_0)} + \frac{1}{\ln b} \frac{K_n(\zeta_0)\zeta \partial_\zeta H_n(\zeta_0) - H_n(\zeta_0)\zeta \partial_\zeta K_n(\zeta_0)}{H_n^2(\zeta_0)} + \tilde{\alpha} \frac{\mathcal{O}(1)}{H_n(\zeta_0)^2} + \mathcal{O}(\ln b)^{-2}
\]
\[
- \frac{1}{2} \zeta \partial_\zeta H_n(\zeta_0) - \frac{1}{2} \frac{K_n(\zeta_0)\zeta \partial_\zeta H_n(\zeta_0) - \zeta \partial_\zeta K_n(\zeta_0)H_n(\zeta_0)}{H_n^2(\zeta_0)} + \mathcal{O}(\tilde{\alpha}^2)
\]
\[
= \tilde{\alpha} \frac{K_n(\zeta_0)\zeta \partial_\zeta H_n(\zeta_0) - \zeta \partial_\zeta K_n(\zeta_0)H_n(\zeta_0)}{H_n^2(\zeta_0)} + \mathcal{O}(\ln b)^{-2}
\]
where the constant in the $O(1)$ is independent of $\zeta_0$, and the constant in the $O(|\ln b|^{-2})$ is independent of $\tilde{\alpha}$. We compute for \( n \geq 1 \) from (2.76) the nondegeneracy for $\zeta_0$ small enough, as $d_1 = -1/2$ and $c_{n,1} = 2n:

\begin{align*}
-K_n \zeta \partial_\zeta H_n + \zeta \partial_\zeta K_n H_n & = - \left( \frac{1}{\zeta_0} + \sum_{i=1}^{n} c_{n,i} \zeta_0^{2(i-1)} \left( \hat{d}_i \ln \zeta_0 + d_i \right) \right) \left( \sum_{i=1}^{n} 2(i-1)c_{n,i} \hat{d}_i \zeta_0^{2(i-1)} \right) \\
& \quad + \left( \frac{-2}{\zeta_0} + \sum_{i=1}^{n} c_{n,i} \zeta_0^{2(i-1)} \left( 2(i-1)(\hat{d}_i \ln \zeta_0 + d_i + \hat{d}_i) \right) \right) \left( \sum_{i=1}^{n} c_{n,i} \hat{d}_i \zeta_0^{2(i-1)} \right) \\
& = - \left( \frac{1}{\zeta_0} + O(|\ln \zeta_0|) \right) \left( O(\frac{2}{\zeta_0}) \right) + \left( \frac{-2}{\zeta_0} + O(1) \right) \left( -n + O(\zeta_0^2) \right) \\
& = \frac{2n}{\zeta_0} + O(1).
\end{align*}

(2.111)

So that, as $H_n(\zeta_0) = -n + O(\zeta_0^2)$ we arrive at:

$$\Theta(b, \tilde{\alpha}) = \tilde{\alpha} \left( \frac{2}{n \zeta_0} + O(1) \right) + O(|\ln b|^{-2}).$$

An application of the intermediate value theorem then yields that there exists at least one value $\tilde{\alpha} = \tilde{\alpha}_n = O(|\ln b|^{-2})$ (its uniqueness is proved by a standard Sturm-type oscillation argument) such that $\Theta(b, \tilde{\alpha}) = 0$.

For $n = 1$, we obtain from the refined identities (2.83) and (2.107):

$$\Theta = \frac{1}{\ln b} \frac{\zeta \partial_\zeta K_1}{H_1} - \frac{1}{\ln b^2} \frac{\zeta \partial_\zeta J_1 H_1 + 2K_1 \zeta \partial_\zeta K_1}{H_1^2} + \tilde{\alpha} \frac{O(1)}{H_1^2} + O(|\ln b|^{-3})$$

$$- \left( \frac{1}{\ln b} \frac{\zeta \partial_\zeta K_1}{H_1^2} - \frac{1}{\ln b^2} \frac{2\zeta \partial_\zeta J_1(\zeta_0) K_1(\zeta_0) + \zeta \partial_\zeta J_1(\zeta_0) H_1(\zeta_0)}{H_1^2(\zeta_0)} - \tilde{\alpha} \frac{\zeta \partial_\zeta K_1}{H_1} + O(|\ln b|^{-3}) \right)$$

$$= \tilde{\alpha} \frac{\zeta \partial_\zeta K_1 + O(1)}{(H_1)^2} + O \left( \frac{1}{|\log b|^3} \right).$$

From the nondegeneracy (2.111), an application of the intermediate value Theorem yields that there exists at least one value $\tilde{\alpha} = \tilde{\alpha}_n = O(|\ln b|^{-3})$ such that $\Theta = 0$.

For $n = 0$, we obtain from the identities (2.86) and (2.108), injecting $\tilde{\alpha} = 1/\ln b + e_0/(\ln b)^2 + \tilde{\alpha}$ with $e_0 = \ln 2 - \gamma$ and $\tilde{\alpha} = O(|\ln b|^{-3})$:

$$\Theta = -1 + \frac{1}{2\ln b} \zeta_0^2 \left( \frac{1}{4} \zeta \partial_\zeta J_0 + J_0 \right) - \frac{\ln b}{4} \zeta_0^2 \left( \zeta \partial_\zeta \tilde{G}_0(\zeta_0) + 2\tilde{G}_0(\zeta_0) + O(|\ln b|^{-1}) \right) + O(|\ln b|^{-2})$$

$$- \left( -1 + \frac{\ln b}{4} \zeta_0^2 \left( \zeta \partial_\zeta \tilde{G}_0(\zeta_0) + 2\tilde{G}_0(\zeta_0) + O(|\ln b|^{-1}) \right) + O(|\ln b|^{-2}) \right)$$

$$= - \frac{\ln b}{4} \zeta_0^2 \left( \zeta \partial_\zeta \tilde{G}_0(\zeta_0) + 2\tilde{G}_0(\zeta_0) + O(|\ln b|^{-1}) \right) + O(|\ln b|^{-2})$$

$$+ \frac{\ln 2 - \gamma}{4 \ln b} \left( \zeta \partial_\zeta \tilde{G}_0(\zeta_0) + 2\tilde{G}_0(\zeta_0) \right)$$

$$- \frac{\ln b}{4} \zeta_0^2 \left( \zeta \partial_\zeta \tilde{G}_0(\zeta_0) + 2\tilde{G}_0(\zeta_0) + O(|\ln b|^{-1}) \right) + O(|\ln b|^{-2}).$$

Therefore, as $\zeta \partial_\zeta \tilde{G}_0(\zeta_0) + 2\tilde{G}_0(\zeta_0) \neq 0$ for $\zeta_0$ small enough, an application of the implicit function Theorem gives the existence of $\tilde{\alpha} = \tilde{\alpha}_0 = O(|\ln b|^{-3})$ such that $\Theta(b, \tilde{\alpha}_0) = 0$. 

- **Estimate of $\partial_b \tilde{\alpha}_n$:** We estimate for $n \geq 1$ from (2.79), (2.79), (2.105), (2.106) and (2.111),

$$
\partial_b \Theta = \partial_b \left( \frac{r \partial_r \phi_n^\infty(R_0)}{2 \phi_n^\infty(R_0)} \right) - \partial_b \left( \frac{z \partial_z q(z_0)}{q(z_0)} \right) = \mathcal{O}(b^{-1} \ln b^{-2}),
$$

and

$$
\partial_\alpha \Theta = \partial_\alpha \left( \frac{r \partial_r \phi_n^\infty(R_0)}{2 \phi_n^\infty(R_0)} \right) - \partial_\alpha \left( \frac{z \partial_z q(z_0)}{q(z_0)} \right) = -K_n(\zeta_0)\zeta_0 \frac{b}{2} H_n(\zeta_0) + \mathcal{O}(1) = \frac{2}{n \zeta_0^2} + \mathcal{O}(1).
$$

Therefore, differentiating the fixed point relation $\Theta(b, \tilde{\alpha}(b)) = 0$ gives $\partial_b \tilde{\alpha} \partial_\alpha \Theta = -\partial_b \Theta$, so $|\partial_b \tilde{\alpha}_n| = \left| \frac{\partial_\alpha \Theta}{\partial_\alpha \Theta} \right| = \mathcal{O}(\frac{1}{b \ln b})$ which concludes the proof of (2.3) for $n \geq 1$. For $n = 0$ the very same computation yields the same estimate, using (2.87), (2.88), (2.109) and (2.110).

- **Pointwise estimate of $\phi_n$ in the self-similar variable:** Let us introduce

$$
\tilde{\phi}_{n, \sqrt{2b}}(\zeta) = \phi_{n, \sqrt{2b}}(\zeta) - \frac{1}{2b} \sum_{j=0}^{n} c_{n,j} b^j T_j(r), \quad r = \frac{\zeta}{\sqrt{2b}},
$$

where $T_j$’s are defined in part (iii) of Lemma 2.2. By (2.68), we distinguish the estimate into two zones:

- For $r \leq R_0$, i.e. $\zeta \leq 2 \sqrt{\varepsilon_0}$: We write from (2.21) and use estimates given in Lemma 2.2 and estimates (2.23)-(2.24),

$$
|\partial_\zeta^k \tilde{\phi}_{n, \sqrt{2b}}(\zeta)| \leq \frac{1}{(\sqrt{2b})^{k+2}} \partial_\zeta^k \left[ \tilde{\alpha}_n \sum_{j=0}^{n} b^{j+1} \left( -c_{n,j} T_{j+1}(r) + S_j(r) \right) + b R_n(r) \right] \lesssim \frac{1}{(\sqrt{2b})^k} \left| \tilde{\alpha}_n \sum_{j=0}^{n} b^j (r)^j \ln(r) + \epsilon_0 (r)^{-k} \right| \lesssim \frac{1}{(\sqrt{2b} + \zeta)^k},
$$

where we used $|\tilde{\alpha}_n| + \epsilon_0 \lesssim \frac{1}{\ln b}$ and $b r^2 = \frac{\zeta^2}{2} \ll 1$. We also have the estimate

$$
|\partial_\zeta^k b \partial_\zeta \tilde{\phi}_{n, \sqrt{2b}}(\zeta)| = \left| \frac{1}{(\sqrt{2b})^{k+2}} \partial_\zeta^k b \partial_\zeta \left[ \tilde{\alpha}_n \sum_{j=0}^{n} b^{j+1} \left( -c_{n,j} T_{j+1}(r) + S_j(r) \right) + b R_n(r) \right] \right| \lesssim \frac{1}{(\sqrt{2b} + \zeta)^k} \left[ \left( |\tilde{\alpha}_n| + |b \partial_\zeta \tilde{\alpha}_n| \right) \sum_{j=0}^{n} b^j (r)^j \ln(r) + \epsilon_0 (r)^{-k} \right] \lesssim \frac{1}{(\sqrt{2b} + \zeta)^k}.
$$

- For $r \geq R_0$, i.e. $\zeta \geq 2 \sqrt{\varepsilon_0}$: From (2.68) and (2.64), we rewrite

$$
\tilde{\phi}_{n, \sqrt{2b}}(\zeta) = \beta_0(h_{\theta_n} + \mathcal{G}) \left( \frac{\zeta}{\sqrt{2b}} \right)^2 - \frac{1}{2b} \sum_{j=0}^{n} c_{n,j} b^j T_j \left( \frac{\zeta}{\sqrt{2b}} \right),
$$

where we recall from (2.2) that $\theta_n = \frac{\theta_n}{\sqrt{b}} = 1 - n - \frac{1}{\ln b} + \mathcal{O}\left( \frac{1}{\ln b \ln b} \right)$, the constant $\beta_0 = \mathcal{O}(b)$, $h_{\theta_n}$ is the Kummer’s function introduced in Lemma 2.5 and $\mathcal{G}$ is described as in Lemma 2.6. We estimate by using

...
We now estimate from part \( (iii) \) of Lemma 2.2 the leading order term for \( \zeta > 0 \):

\[
\left| \partial_{\zeta}^{k} \left( \frac{1}{2b} \sum_{j=0}^{n} c_{n,j} b^{j} T_{j}(r) \right) \right| = \left| \frac{1}{(\sqrt{2b})^{k+2}} \sum_{j=0}^{n} c_{n,j} b^{j} \partial_{\zeta}^{k} T_{j}(r) \right| \lesssim \frac{1}{(\sqrt{2b})^{k+2}} \sum_{j=0}^{n} b^{j} \langle r \rangle^{2j} \lesssim \frac{\langle \zeta \rangle^{2n+2}(1 + |\ln b| \langle n \rangle_{n})}{(\sqrt{2b} + \zeta)^{2+k}},
\]

and from (2.29)-(2.30), we have

\[
\left| \partial_{\zeta}^{k} b \partial_{\bar{b}} \left( \frac{1}{2b} \sum_{j=0}^{n} c_{n,j} b^{j} T_{j}(r) \right) \right| = \left| \frac{1}{4} \partial_{\zeta}^{k} \sum_{j=0}^{n} c_{n,j} b^{j-1} \Theta_{j}(r) \right| \lesssim \frac{1}{(\sqrt{2b})^{k+2}} \sum_{j=0}^{n} b^{j} \langle r \rangle^{2j-2-k} \lesssim \frac{\langle \zeta \rangle^{2n+2}}{(\sqrt{2b} + \zeta)^{2+k}}.
\]

A collection of all above estimates yields estimate (2.6).

**Estimate the weighted \( L^{2} \) norm of \( \hat{\phi}_{n} \):** By (2.68), we write

\[
\int_{0}^{+\infty} |\phi_{n}(r)|^{2} \omega_{b} dr = \int_{0}^{R_{0}} |\phi_{n}^{m}(r)|^{2} \omega_{b} dr + \beta_{0}^{2} \int_{R_{0}}^{+\infty} |\phi_{n}^{ex}(r)|^{2} \omega_{b} dr = I_{n}^{m} + I_{n}^{ex}.
\]

We compute asymptotically from (2.54), (2.56), the relation \( \phi_{n}^{ex}(r) = q(z) \) with \( z = \frac{b r^{2}}{2}, \epsilon_{0} = \frac{b R_{0}^{2}}{2} \) and the fact that \( |\beta_{0}| = O(b) \),

\[
I_{n}^{ex} = \frac{2|\beta_{0}|^{2}}{b^{2}} \int_{0}^{+\infty} q(z)^{2}(z + b + b^{2} z^{-1}/4) e^{-z} dz = O \left( \int_{0}^{1} z^{-1} dz + \int_{1}^{+\infty} z^{1-2\theta_{b}} e^{-z} dz \right) = O(|\ln \epsilon_{0}|).
\]

From (2.21), part \( (iii) \) of Lemma 2.2 and the integral identity

\[
\forall k \in \mathbb{N}, \quad b^{k} \int_{10}^{+\infty} r^{2k-1}(\ln r)^{2} e^{-br^{2}/2} dr = 2^{k-3}(k - 1)! |\ln b|^{2} + O(|\ln b|),
\]

we compute \( I_{n}^{m} \) at the leading order,

\[
I_{n}^{m} = \int_{0}^{R_{0}} \left| \sum_{j=0}^{n} c_{n,j} b^{j} T_{j}(r) \right|^{2} \left( \frac{1 + r^{2}}{r} \right)^{2} e^{-br^{2}/2} dr
= \frac{1}{8} \int_{0}^{R_{0}} \left| \sum_{j=1}^{n} c_{n,j} b^{j} d_{j} r^{2(j-1)} \ln r \right|^{2} r^{3} e^{-br^{2}/2} dr + O(|\ln b|) \sim \tilde{e}_{n} |\ln b|^{2},
\]

from some strictly positive constant \( \tilde{e}_{n} \), with \( \tilde{e}_{n} = 2^{-4} \) for \( n = 0, 1 \). This concludes the proof of Lemma 2.8 as well as Proposition 1.1. \( \Box \)

**Proof of Corollary 2.9.** We claim that the same proof applies as for Lemma 2.8. Indeed, notice that from Lemma 2.4 and the bound (2.4), the inner solution for the perturbed problem is of the very same form as the original problem (2.73):

\[
\phi_{n}^{in,V}[b, \alpha](r) = F_{n}[b](r) + \hat{\alpha} b G_{n}[b, \alpha](r) + E_{n}^{V}[b, \alpha](r),
\]
where $E_n^V = E_n + \phi^{in,V} - \phi^{in}$ satisfies the analogue of (2.75):
\[ \sum_{0 \leq k \leq 2} |(r \partial_r)^k E_n(R_0)| \leq C(\zeta_0) \frac{b}{|\ln b|}. \]
So all computations made for the inner solution of the original problem are also valid for the perturbed problem. Notice similarly from Lemma 2.7 that the outer solution for the perturbed problem is of the form very same form as that of the original problem:
\[ q_n^V[b, \tilde{\alpha}](z) = \Gamma(\theta) h(\theta) + G_n^V[b, \tilde{\alpha}](z) \]
where $G$ satisfies the analogue of (2.89):
\[ |G_n^V(z_0)| + |(z \partial_z G_n^V(z_0))| \lesssim b^{\frac{3}{2}}. \]
So all computations made for the outer solution of the original problem are also valid for the perturbed problem. The matching procedure can thus be done verbatim the same way. The only informations that we do not get in comparison with the original problem are the estimates for the variation with respect to $\tilde{\alpha}$ and $b$, and the next order $|\ln b|^{-2}$ term in the expansion of $\tilde{\alpha}$ for $n = 0, 1$, but these informations are not required. This concludes the proof of the Corollary.

**Proof of Proposition 1.6.** The existence part and the estimates on the eigenvalues are direct consequences of Corollary 2.9. The bound (1.24) is a direct consequence of (2.4) and (2.7).

3. **Coercivity of the linearized operator $\mathcal{L}$.**

This part is devoted to prove Proposition 1.9. Our argument takes place on the stationary state variables, namely that
\[ \mathcal{L} u = \Delta u - \nabla \cdot (u \nabla \Phi_U) - \nabla \cdot (U \nabla \Phi_u) - b \nabla \cdot (yu), \quad 0 < b = \nu^2 \beta \ll 1, \quad y = \frac{z}{\sqrt{\beta \nu}}. \]
The operator $\mathcal{L}$ can be written in two differently divergence forms
\[ \mathcal{L} u = \mathcal{L}_0 u - b \nabla \cdot (yu) \quad \text{or} \quad \mathcal{L} u = \mathcal{H} u - \nabla U \cdot \nabla \Phi_u, \quad (3.1) \]
where $\mathcal{L}_0$ is defined in (1.25), and
\[ \mathcal{H} u = \frac{1}{\omega[b]} \nabla \cdot \left( \omega[b] \nabla u \right) + 2(U - b)u, \]
with the weighted function (we will forget about the $b$ subscript from now on in this section)
\[ \omega := \omega[b] = \frac{\rho[b]}{U}, \quad \rho[b] = e^{-\frac{b|y|^2}{4}}. \quad (3.2) \]
To prove Proposition 1.9 we will then show the analogue estimate in $y$ variables, namely that
\[ \langle -\mathcal{L} u, u \rangle \geq c \|\nabla u\|_{L_2}^2 - C \left( \int_{\mathbb{R}^2} u \partial_y U \sqrt{\rho} dy \right)^2 + \left( \int_{\mathbb{R}^2} u \partial_y U \sqrt{\rho} dy \right)^2, \quad (3.3) \]
where $\mathcal{L}$ is related to $\mathcal{L}$ through
\[ \mathcal{L} u = \mathcal{L} u - \nabla U \cdot \nabla (\Phi_u - \tilde{\Phi}_u), \]
and $\tilde{\Phi}_u$ is defined by
\[ \tilde{\Phi}_u = \tilde{\Phi}[b]_u = -\frac{1}{\sqrt{\beta}} \left[ \frac{1}{2\pi} \log(|y|) \ast (u \sqrt{\rho}) \right], \quad (3.4) \]
and that
\[ -\Delta \left( \Phi_u \sqrt{\rho} \right) = u \sqrt{\rho} \quad \text{and} \quad \Delta \Phi_u = -u + by \cdot \nabla \Phi_u + \left( b + \frac{b^2}{4} |y|^2 \right) \Phi_u. \]
The proof is done in two parts: In the first part, we deal with the linear operator $\mathcal{L}_0$ and derive its coercivity under some suitable orthogonality conditions. Then, we extend this coercive property to the
Part 1: Coercivity of \( \mathcal{L}_0 \). Our first result is that of coercivity at the \( H^1 \) level. While [24] proves a similar estimate at the \( H^2 \) level, we state and prove the following result for the sake of completeness.

Lemma 3.1 (Coercivity of \( \mathcal{L}_0 \), [24]). Let \( u \) be such that \( \int_{\mathbb{R}^2} u\,dy = 0 \) and \( \nabla u \in L^2(U^{-1}) \). Then, we have for some constants \( \delta_2 > 0 \) and \( C > 0 \):

\[
\int_{\mathbb{R}^2} U|\nabla(\mathcal{M}u)|^2\,dy \geq \delta_2 \int_{\mathbb{R}^2} \frac{|\nabla u|^2}{U} \,dy - C \left[ \langle u, \partial_1 U \rangle_{L^2}^2 + \langle u, \partial_2 U \rangle_{L^2}^2 \right].
\]  

(3.5)

Proof. We first prove that the projections are well-defined. This is a consequence of the following Hardy-type inequality:

\[
\int_{\mathbb{R}^2} u^2(1 + |y|^2)\,dy \lesssim \int_{\mathbb{R}^2} |\nabla u|^2(1 + |y|^4)\,dy,
\]  

(3.6)

and of the decay \( |U| \lesssim (1 + |y|)^{-4} \):

\[
\langle u, \partial_1 U \rangle_{L^2}^2 \lesssim \left( \int_{\mathbb{R}^2} |u|^2(1 + |y|)^2 \right)^{\frac{1}{2}} \lesssim \left( \int_{\mathbb{R}^2} |\nabla u|^2(1 + |y|^4) \right)^{\frac{1}{2}} \lesssim \left( \int_{\mathbb{R}^2} \frac{|\nabla u|^2}{U} \right)^{\frac{1}{2}}.
\]

Step 1: Subcoercivity estimate. We use Young’s inequality \( ab \leq a^2/4 + b^2 \) to obtain:

\[
\int_{\mathbb{R}^2} U|\nabla(\mathcal{M}u)|^2 = \int_{\mathbb{R}^2} U \left[ \left| \nabla \left( \frac{u}{U} \right) \right|^2 + 2\nabla \left( \frac{u}{U} \right) \cdot \nabla \Phi_u + |\nabla \Phi_u|^2 \right] \\
\geq \frac{1}{2} \int_{\mathbb{R}^2} U \left| \nabla \left( \frac{u}{U} \right) \right|^2 - \int_{\mathbb{R}^2} U |\nabla \Phi_u|^2.
\]

From the algebraic identity

\[
\int_{\mathbb{R}^2} U \left| \nabla \left( \frac{u}{U} \right) \right|^2 = \int_{\mathbb{R}^2} \frac{|\nabla u|^2}{U} - \int_{\mathbb{R}^2} U u^2,
\]

and the control of the Poisson field (A.9)

\[
\int_{\mathbb{R}^2} U|\nabla \Phi_u|^2 \lesssim \int u^2,
\]

and the decay \( U(y) \lesssim (1 + |y|)^{-4} \), one gets the following subcoercive estimate for some \( C > 0 \):

\[
\int_{\mathbb{R}^2} U|\nabla(\mathcal{M}u)|^2 \geq \frac{1}{2} \int_{\mathbb{R}^2} \frac{|\nabla u|^2}{U} - C \int_{\mathbb{R}^2} |u|^2.
\]

(3.7)

Step 2: Coercivity estimate. We apply a standard minimisation technique. Assume by contradiction (3.5) is false. Then there exists a sequence of functions \((u_n)_{n \in \mathbb{N}} \in H^1((1 + |y|)^4\,dy)\) without radial component such that

\[
\int_{\mathbb{R}^2} \frac{|\nabla u_n|^2}{U} = 1, \quad \int_{\mathbb{R}^2} u_n \partial_i U = 0 \quad \text{for} \quad i = 1, 2, \quad \int_{\mathbb{R}^2} U|\nabla(\mathcal{M}u_n)|^2 \to 0.
\]

(3.8)

Up to a subsequence there exists a limit \( u_\infty \) of \( u_n \) in \( H^1_{\text{loc}} \). Moreover, from the lower semi-continuity and the weak continuity, we have

\[
\int_{\mathbb{R}^2} \frac{|\nabla u_\infty|^2}{U} \leq 1, \quad \int_{\mathbb{R}^2} u_\infty \partial_i U = 0 \quad \text{for} \quad i = 1, 2.
\]

We now write

\[
\int_{\mathbb{R}^2} U|\nabla(\mathcal{M}u_n)|^2 = \int_{\mathbb{R}^2} \frac{|\nabla u_n|^2}{U} - \int_{\mathbb{R}^2} U u_n^2.
\]

Above, \( \nabla u_n \) converges weakly in \( L^2(U\,dy) \). We remark that

\[
\int_{\mathbb{R}^2} u_n^2(1 + |y|^2) \lesssim \int_{\mathbb{R}^2} \frac{|\nabla u_n|^2}{U}.
\]
From this and from the compactness of the embedding of $H^1(\Omega)$ in $L^2(\Omega)$ for $\Omega$ compact, $u_n$ converges strongly in $L^2(dy)$. Hence, from (3.8) and lower semi-continuity:

$$\int_{\mathbb{R}^2} U|\nabla (\mathcal{M} u_n)|^2 = \int_{\mathbb{R}^2} \frac{\nabla u_n}{U} \cdot \nabla u_n - \int_{\mathbb{R}^2} U u_n^2 \lesssim 0.$$ 

Therefore, $\nabla \mathcal{M} u_\infty = 0$. Since $u_\infty$ is without radial component, one obtains $\mathcal{M} u_\infty = 0$. Hence, $u_\infty$ belongs to the Kernel of $\mathcal{M}$ intersected with $L^2((1 + |y|)^2 dy)$, which is $\text{Span}(\partial_y U, \partial_y U)$. From the orthogonality condition (3.8), one gets that necessarily $u_\infty = 0$. From the subcoercivity estimate (3.7),

$$\int_{\mathbb{R}^2} |u_n|^2 \geq \frac{1}{C} \left( \frac{1}{2} \int_{\mathbb{R}^2} \frac{\nabla u_n}{U} \cdot \nabla u_n - \int_{\mathbb{R}^2} U|\nabla (\mathcal{M} u)|^2 \right),$$

and hence from (3.8):

$$\liminf \int_{\mathbb{R}^2} |u_n|^2 \geq \frac{1}{C} > 0.$$ 

As $u_n$ converges strongly in $L^2(dy)$, this implies $\int_{\mathbb{R}^2} |u_\infty|^2 \neq 0$ which contradicts $u_\infty = 0$. This concludes the proof of Lemma 3.1. □

**Part 2: Coercivity of $\hat{\mathcal{L}}$.** We are now in the position to conclude the proof of Proposition 1.9 thanks to Lemma 3.1. We first note from the self-adjointness of $\mathcal{M}$ that $(u, v)_* = (v, u)_*$ and the bound

$$\int_{\mathbb{R}^2} U|\mathcal{M}(v\sqrt{\rho})|^2 dy \lesssim \|v\|_{L^2}^2,$$

so that the quadratic form (1.28) is symmetric and continuous. From Lemma 3.1, one has the coercivity estimate

$$(u, u)_* \geq c\|u\|_{L^2}^2 - C \left( \int_{\mathbb{R}^2} u\Lambda U \sqrt{\rho} dy \right)^2 + \sum_{i=1}^2 \left( \int_{\mathbb{R}^2} u_i \partial_i U \sqrt{\rho} dy \right)^2.$$

Therefore, on $\text{Span}(\Lambda U \sqrt{\rho}, \nabla U \sqrt{\rho}, \sqrt{\rho})$, the quadratic form (1.28) is equivalent to the usual scalar product in $L^2_\rho$. This observation motivates us to consider the modified operator

$$\hat{\mathcal{L}} u = \Delta u - \nabla u \cdot \nabla \Phi_U + 2U u - \nabla U \cdot \nabla \Phi_u - b \nabla \cdot (y u),$$

so that

$$\mathcal{L} u = \hat{\mathcal{L}} u - \nabla U \cdot \nabla (\Phi_u - \tilde{\Phi}_u),$$

where $\tilde{\Phi}_u$ is defined by

$$\tilde{\Phi}_u = \tilde{\Phi}[b]_u = -\frac{1}{\sqrt{\rho}} \left[ \frac{1}{2\pi} \log(|y|) + (u\sqrt{\rho}) \right],$$

namely that

$$-\Delta \left( \tilde{\Phi}_u \sqrt{\rho} \right) = u\sqrt{\rho} \quad \text{and} \quad \Delta \tilde{\Phi}_u = -u + by \cdot \nabla \tilde{\Phi}_u \left( b + \frac{b^2}{4} |y|^2 \right) \tilde{\Phi}_u.$$

Let $\tilde{\mathcal{M}}$ be the linear operator

$$\tilde{\mathcal{M}} = \tilde{\mathcal{M}}[b] : u \mapsto \frac{u}{U} - \tilde{\Phi}_u,$$

and note that $\rho \tilde{\mathcal{M}} v \equiv \sqrt{\rho} \tilde{\mathcal{M}} \left(v\sqrt{\rho}\right)$. We rewrite the scalar product defined in (1.28) as

$$(u, v)_* = \int_{\mathbb{R}^2} u\sqrt{\rho} \mathcal{M} \left(v\sqrt{\rho}\right) dy \equiv \int_{\mathbb{R}^2} u\tilde{\mathcal{M}} v \rho dy. \quad (3.9)$$

By noting that $\Delta u - \nabla \Phi_U \cdot \nabla u + uU = \nabla \cdot \left[ U \nabla \left( \frac{u}{U} \right) \right]$ and

$$U u - \nabla U \cdot \nabla \tilde{\Phi}_u = -\nabla \left( U \nabla \tilde{\Phi}_u \right) - bU y \cdot \nabla \tilde{\Phi}_u - \left( b + \frac{b^2}{4} |y|^2 \right) U \tilde{\Phi}_u,$$
we rewrite the linear operator $\mathcal{L}$ in terms of $\mathcal{M}$ as follows:

$$
\mathcal{L}u = \nabla \cdot \left( U \nabla \mathcal{M} u - by u \right) - b U y \cdot \nabla \tilde{\Phi}_u - \left( b + \frac{b^2}{4} |y|^2 \right) U \tilde{\Phi}_u.
$$

One has the identity

$$
- \int \nabla \cdot \left( U \nabla \mathcal{M} u - by u \right) v \rho dy
= \int U \nabla \mathcal{M} u \cdot \nabla \mathcal{M} v \rho + b \int y \cdot \nabla \Phi_v u \rho \sqrt{\rho} + b \int U y \cdot \nabla \tilde{\Phi}_u \mathcal{M} v \rho dy + 2b \int u \mathcal{M} v \rho.
$$

This leads to the following almost self-adjointness of $\mathcal{L}$:

$$
\langle -\mathcal{L}u, v \rangle_* = F(u, v) + G(u, v) + 2b \langle u, v \rangle_*,
$$

where $\langle \cdot, \cdot \rangle_*$ is introduced in (3.9), $F$ is the leading order part given by

$$
F(u, v) = \int U \nabla \mathcal{M} u \cdot \nabla \mathcal{M} v \rho dy + b \int y \cdot \nabla \Phi_v u \mathcal{M} v \sqrt{\rho},
$$

and $G$ contains lower order terms,

$$
G(u, v) := \int \left( 2b U y \cdot \nabla \tilde{\Phi}_u + \left( b + \frac{b^2}{4} |y|^2 \right) U \tilde{\Phi}_u \right) \mathcal{M} v \rho dy.
$$

Using the modified bilinear form (3.10), we are now able to derive the coercivity estimate (3.3).

**Proof of (3.3).** We proceed in two steps:

**Step 1: Subcoercivity estimate.** We claim that for $u \in \dot{H}^1_\rho$,

$$
F(u, u) + G(u, u) = \|\nabla u\|_{L^2}^2 \mathcal{O} \left( \|\nabla u\|_{L^2} \right)^2 \left( \frac{u}{1 + |y|^{1/2}} \right) + \|\nabla u\|_{L^2}^2 + \frac{b^2}{4} \|\nabla u\|_{L^2}^2 + b \|\nabla u\|_{L^2}^2 + b \|\nabla u\|_{L^2}^2,
$$

where the constant in the $\mathcal{O}(\cdot)$ does not depend on $b$. Let us begin with the form $F$ by writing

$$
F(u, u) = \int U \left\| \nabla \left( \frac{u}{U} \right) \right\|_2^2 \rho + b \int y \cdot \nabla \Phi_v u \rho u^2 \rho - 2 \int U \nabla \left( \frac{u}{U} \right) \cdot \nabla \tilde{\Phi}_u \rho + \int U \|\nabla \tilde{\Phi}_u\|_2^2 \rho.
$$

The first line gathers the leading order terms at infinity. We compute

$$
\int U \left\| \nabla \left( \frac{u}{U} \right) \right\|_2^2 \rho + b \int y \cdot \nabla \Phi_v u \rho u^2 \rho
= \int \|\nabla u\|_{L^2}^2 \rho - 2 \int \frac{u}{U} \nabla u \cdot \nabla \Phi_v \rho + \int \frac{u^2 \|\nabla \Phi_v\|_2^2}{U} \rho + b \int y \cdot \nabla \Phi_v u \rho \rho + b \int \|\nabla \Phi_v u\|_2^2 \rho = \|\nabla u\|_{L^2}^2 - \int u^2 \rho.
$$

Thus, we have

$$
F(u, u) = \|\nabla u\|_{L^2}^2 - \int u^2 \rho - 2 \int U \nabla \left( \frac{u}{U} \right) \cdot \nabla \tilde{\Phi}_u \rho + \int U \|\nabla \tilde{\Phi}_u\|_2^2 \rho.
$$

From (A.9) with $\alpha = 7/4$, and (A.5) with $\alpha = 1/2$ we get:

$$
\frac{b^2}{4} \|\tilde{\Phi}_u(y)\|^2 \lesssim \rho^{-1}(1 + |y|)^{-\frac{3}{2}} |y|^2 \int_{\mathbb{R}^2} |u|^2 (1 + |y|) \frac{\sqrt{2}}{1} e^{-\frac{|y|^2}{2}} dy \lesssim \rho^{-1}(1 + |y|)^{-\frac{3}{2}} \|u\|^2_{L^2}.
$$

As $\nabla \tilde{\Phi}_u = \nabla (\rho^{-1/2} \Phi_{\rho'/2u})$, using the above inequality, and (A.9) with $\alpha = 1/2$, we obtain:

$$
\|\nabla \tilde{\Phi}_u(y)\|^2 \lesssim \rho^{-1}(1 + |y|)^{-\frac{5}{2}} (1 + |y|) \log |y| \int u^2 (1 + |y|) \rho + b \|\nabla \tilde{\Phi}_u(y)\|^2 \lesssim \rho^{-1}(1 + |y|)^{-\frac{5}{2}} \|u\|^2_{L^2}.
$$

(3.13)
We now turn to the terms in $G$. From (3.13), (3.14), (A.4) and $|U| \lesssim (1 + |y|)^4$, we get

$$
\sqrt{\rho} \left| 2bU \cdot \nabla \tilde{\Phi} + \left( b + \frac{b^2}{4} |y|^2 \right) U \tilde{\Phi} \right| \lesssim \|\nabla u\|_{L^2} \left( (1 + |y|)^{-\frac{7}{2}} + b^\frac{13}{14} (1 + |y|)^{-\frac{23}{24}} + b^\frac{2}{5} (1 + |y|)^{-\frac{17}{24}} \right). \tag{3.15}
$$

Using $|U|^{-1} \lesssim (1 + |y|)^4$ and Cauchy-Schwarz, we obtain for the two first terms below from (A.5) with $\alpha = 3/2$, and for the third with (A.5) with $\alpha = 3/4$:

$$
\int b(1 + |y|)^{-\frac{7}{2}} u \sqrt{\rho} \lesssim b^\frac{1}{2} \left( \frac{b^2}{2} \int u^2 (1 + |y|^5) \rho \right)^{\frac{1}{2}} \left( \int (1 + |y|)^{-4} \right) \lesssim b^\frac{1}{2} \|\nabla u\|_{L^2},
$$

$$
\int b^{\frac{13}{14}} (1 + |y|)^{-\frac{23}{24}} u \sqrt{\rho} \lesssim b^\frac{1}{2} \left( \frac{b^2}{2} \int u^2 (1 + |y|^5) \rho \right)^{\frac{1}{2}} \left( \int (1 + |y|)^{-\frac{23}{24}} \right) \lesssim b^\frac{1}{2} \|\nabla u\|_{L^2},
$$

$$
\int b^\frac{2}{5} (1 + |y|)^{-\frac{17}{24}} u \lesssim b^\frac{1}{2} \left( \frac{b^2}{2} \int u^2 (1 + |y|^5) \rho dy \right)^{\frac{1}{2}} \left( \int (1 + |y|)^{-\frac{17}{24}} \right) \lesssim b^\frac{1}{2} \|\nabla u\|_{L^2},
$$

from which we obtain the bound

$$
\left\| 2bU \cdot \nabla \tilde{\Phi} + \left( b + \frac{b^2}{4} |y|^2 \right) U \tilde{\Phi} \right\|_{L^2} \lesssim \frac{u}{\rho} \lesssim b^\frac{1}{2} \|\nabla u\|_{L^2}.
$$

By using the estimate (A.9) with $\alpha = 1$ and (A.4) we get:

$$
\sqrt{\rho} \|\tilde{\Phi}\| \lesssim (1 + |y|)^{\frac{1}{2}} (1 + 1_{|y| \leq 1} |\log |y||) \|\nabla u\|_{L^2},
$$

and hence from (3.15) one gets

$$
\left\| 2bU \cdot \nabla \tilde{\Phi} + \left( b + \frac{b^2}{4} |y|^2 \right) U \tilde{\Phi} \rho \right\| \lesssim b^\frac{1}{2} \|\nabla u\|_{L^2}.
$$

We then arrive at the estimate for $G$:

$$
|G(u, u)| \leq \left\| \left( 2bU \cdot \nabla \tilde{\Phi} + \left( b + \frac{b^2}{4} |y|^2 \right) U \tilde{\Phi} \right) \cdot \omega dy \right\| \lesssim b^\frac{1}{2} \|\nabla u\|_{L^2}^2. \tag{3.16}
$$

The estimates for $F$ and $G$ above yield the desired subcoercivity estimate (3.11).

**Step 2: Asymptotic problem and rigidity.** Assume by contradiction that for functions without radial component

$$
m := \liminf_{b \to 0} \inf_{u \in H^1_{x, \sqrt{\rho} \nabla U} = 0} \frac{F[b](u, u) + G[b](u, u)}{\|\nabla u\|_{L^2_{x, \sqrt{\rho} \nabla U}}} \leq 0 \text{ with } \omega[b] = \rho[b] U, \quad \rho[b] = e^{-\frac{b|y|^2}{2}}.
$$

From the subcoercivity estimate (3.11) and (A.4), we infer that $-\infty < m \leq 0$. Let $b_n \to 0$ and $u_n$ be sequences such that, without loss of generality, $\|\nabla u_n\|_{L^2_{x, \sqrt{\rho} \nabla U}} = 1$, $\langle u_n, \sqrt{\rho} \nabla Q \rangle = 0$ and

$$
F[b_n](u_n, u_n) + G[b_n](u_n, u_n) \to 0.
$$
The above limit, with (3.11) and \( \|\nabla u_n\|_{L^2(\omega_n)} = 1 \), imply that there exists \( c > 0 \) such that for all \( n \):
\[
\int u_n^2 (1 + |y|) \rho[b_n] dy \geq c.
\]
The sequence \( f_n = u_n \sqrt{\rho[b_n]} \) is then uniformly bounded in \( H^1((1 + |y|^4) dy) \) from (A.3), with:
\[
\int f_n^2 (1 + |y|) \geq c.
\]
Since also \( \int f_n^2 (1 + |y|^2) \) is uniformly bounded by (A.4), there exist \( R, c' > 0 \) such that, up to a subsequence,
\[
\int_{|y| \leq R} |f_n|^2 dy \geq c'.
\]
We pass to the limit: there exists \( f_\infty \in H^1((1 + |y|^4) dy) \) that is the weak limit in this space of \( f_n \).
Moreover, by compactness of \( H^1 \) in \( L^2 \) on bounded sets, the convergence is strong in \( L^2((1 + |y|) dy) \), so that \( f_\infty \neq 0 \) from the above inequality. Let us write
\[
\sqrt{\rho[b]} \nabla \tilde{\Phi}_u = \nabla \tilde{\Phi}_{u_0} - \frac{by}{4} \Phi_{u_0} \sqrt{\rho[b]}.
\]
From (A.9), we infer that the first term, i.e. the mapping \( \sqrt{\rho[b]} u \rightarrow \nabla \tilde{\Phi}_{u_0} \sqrt{\rho[b]} \), is continuous from \( L^2(1 + |y|) \) into \( L^2((1 + |y|)^{-4}) \). Similarly, the second term is controlled by
\[
\left\| \frac{by}{2} \Phi_{u_0} \sqrt{\rho[b]} \right\|_{L^2((1 + |y|)^{-4})} \leq \sqrt{b} \|u\|_{H^1} \rightarrow 0 \quad \text{as} \quad b \rightarrow 0.
\]
Therefore, \( \sqrt{\rho[b_n]} \nabla \tilde{\Phi}_{u_n} \) converges strongly to \( \nabla \tilde{\Phi}_{f_\infty} \) in \( L^2((1 + |y|)^{-4}) \). Consequently, one has the continuity at the limit,
\[
- \int u_n \rho[b_n] - 2 \int U \nabla \left( \frac{u_n}{U} \right) \cdot \nabla \tilde{\Phi}_{u_n} \rho[b_n] + \int U |\nabla \tilde{\Phi}_{u_n}|^2 \rho[b_n] \rightarrow - \int f_\infty^2 - 2 \int U \nabla \left( \frac{f_\infty}{U} \right) \cdot \nabla \tilde{\Phi}_{f_\infty} + \int U |\nabla \tilde{\Phi}_{f_\infty}|^2.
\]
Together with the continuity estimate for \( G \) (3.16), which implies its asymptotic vanishing, and lower-semicontinuity, we deduce
\[
0 = \lim_{n \rightarrow \infty} F[b_n](u_n, u_n) + G[b_n](u_n, u_n) \geq \int \frac{|\nabla f_\infty|^2}{U} - \int f_\infty^2 - 2 \int U \nabla \left( \frac{f_\infty}{U} \right) \cdot \nabla \tilde{\Phi}_{f_\infty} + \int U |\nabla \tilde{\Phi}_{f_\infty}|^2.
\]
However,
\[
\int \frac{|\nabla f_\infty|^2}{U} - \int f_\infty^2 - 2 \int U \nabla \left( \frac{f_\infty}{U} \right) \cdot \nabla \tilde{\Phi}_{f_\infty} + \int U |\nabla \tilde{\Phi}_{f_\infty}|^2 = \int U |\nabla \tilde{\Phi}_{f_\infty}|^2.
\]
Hence, as \( f_\infty \) is without radial component we deduce that \( \mathbb{M}_{f_\infty} = 0 \) and hence that \( f_\infty = c_1 \partial_{y^1} U + c_2 \partial_{y^2} U \), with one coefficient being non zero since \( f_\infty \neq 0 \). On the other hand, the orthogonality \( \langle u_n, \sqrt{\rho} \nabla U \rangle \) passes to the limit, yielding \( \langle f_\infty, \nabla U \rangle = 0 \) so that \( c_1 = c_2 = 0 \) which is a contradiction. This concludes the proof of Proposition 1.9.

\[\square\]

A. Estimates on the Poisson field

We first recall estimates relative to the weight \( e^{-|z|^2 / 2} \) with polynomial corrections. First, there holds the bound for any \( k \geq 0 \) for any function without radial component
\[
\int v^2 |z|^{2k} (1 + |z|^2) e^{-|z|^2 / 2} dz \lesssim \int |\nabla v|^2 |z|^{2k} e^{-|z|^2 / 2} dz.
\] (A.1)
By a scaling argument, this implies that for $0 < b \leq 1$:

$$\int b^2 (|y|^2 + |y|^6) u^2 e^{-b|y|^2} \lesssim \int (1 + |y|^4)|\nabla u|^2 e^{-b|y|^2}$$  \hspace{1cm} (A.2)

with constant independent on $b$. Therefore:

$$\int (1 + |y|^4)|\nabla (ue^{-b|y|^2})|^2 \leq C \int (1 + |y|^4)|\nabla u|^2 e^{-b|y|^2}.$$  \hspace{1cm} (A.3)

Applying (3.6) one obtains from the above inequality the Hardy-type inequality with weight $e^{-b|y|^2/2}$:

$$\int (1 + |y|^2)u^2 e^{-b|y|^2/2} \lesssim \int (1 + |y|^4)|\nabla u|^2 e^{-b|y|^2/2},$$

with constant independent on $b$. Interpolating between the above inequality and (A.2) we obtain that for any $0 \leq \alpha \leq 2$:

$$b^\alpha \int_{\mathbb{R}^2} |u|^2 (1 + |y|^{2+2\alpha}) e^{-b|y|^2} dy \lesssim \int |\nabla u|^2 (1 + |y|^4) e^{-b|y|^2} dy.$$  \hspace{1cm} (A.5)

For $u$ localised on a single spherical harmonics $Y^{(k,i)}$ with

$$Y^{(k,i)}(y) = \begin{cases} \cos^k \left( \frac{y}{|y|} \right) & \text{if } i = 1, \\ \sin^k \left( \frac{y}{|y|} \right) & \text{if } i = 2, \end{cases}$$

where we identify $y/|y|$ with its angle on the unit circle, the Laplace operator is written as

$$\Delta u(x) = \Delta^{(k)}(u^{(k,i)})(r)Y^{(k,i)} \left( \frac{y}{|y|} \right), \quad \Delta^{(k)} := \partial_{rr} + \frac{1}{r} \partial_r - \frac{k^2}{r^2}.$$  \hspace{1cm} \text{for } k \geq 1.

The fundamental solutions to $\Delta^{(k)} f = 0$ are $\log(r)$ and 1 for $k = 0$, and $r^k$ and $r^{-k}$ for $k \geq 1$, with Wronskian relations:

$$W^{(0)} = \frac{d}{dr} \log(r) = r^{-1} \quad \text{and} \quad W^{(k)} = \frac{d}{dr} (r^k) r^{-k} - \frac{r}{k} \frac{d}{dr} (r^{-k}) = 2kr^{-1} \quad \text{for } k \geq 1.$$  \hspace{1cm} \text{The solution to } -\Delta \Phi_u = u \text{ given by } \Phi_u = -(2\pi)^{-1} \log(|x|) \ast u \text{ is then given on spherical harmonics by:}

$$\Phi_u^{(0,0)}(r) = -\log(r) \int_0^r u^{(0,0)}(\bar{r}) \bar{r} d\bar{r} - \int_r^{+\infty} u^{(0,0)}(\bar{r}) \log(\bar{r}) \bar{r} d\bar{r},$$

$$\nabla \Phi_u^{(0,0)}(x) = -\frac{x}{|x|^2} \int_0^{|x|} u^{(0,0)}(\bar{r}) \bar{r} d\bar{r},$$  \hspace{1cm} (A.6)

$$\Phi_u^{(k,i)}(r) = \frac{r^k}{2k} \int_r^{+\infty} u^{(k,i)}(\bar{r}) \bar{r}^{1-k} d\bar{r} + \frac{r^{-k}}{2k} \int_0^r u^{(k,i)}(\bar{r}) \bar{r}^{1+k} d\bar{r},$$  \hspace{1cm} (A.7)

$$\partial_r \Phi_u^{(k,i)}(r) = \frac{r^{k-1}}{2} \int_r^{+\infty} u^{(k,i)}(\bar{r}) \bar{r}^{1-k} d\bar{r} - \frac{r^{-k-1}}{2} \int_0^r u^{(k,i)}(\bar{r}) \bar{r}^{1+k} d\bar{r}.$$  \hspace{1cm} (A.8)

**Lemma A.1.** If $u$ is without radial component, for any $0 < \alpha < 2$:

$$|\Phi_u|^2 + |y|^2 |\nabla \Phi_u|^2 \lesssim |y|^2 (1 + |y|)^{-2\alpha} (1 + 1_{|y| \leq 1} |\log |y||) \int_{\mathbb{R}^2} |u|^2 (1 + |y|)^{2\alpha} dy.$$  \hspace{1cm} (A.9)

**Proof.** We decompose $\Phi_u$ in spherical harmonics. Note that $\Phi_u^{(0,0)} = 0$ as $u$ has no radial component. Applying Cauchy-Schwartz inequality in both terms in (A.8) one gets for $k \geq 1$, as $0 < \alpha < 2$:

$$\left| \int_r^{+\infty} u^{(k,i)}(\bar{r}) \bar{r}^{1-k} d\bar{r} \right| \lesssim \left( \int_r^{+\infty} |u^{(k,i)}|^2 (1 + r)^{2\alpha} \bar{r} d\bar{r} \right)^{\frac{1}{2}} \left( \int_r^{+\infty} (1 + r)^{-2\alpha} \bar{r}^{1-2k} d\bar{r} \right)^{\frac{1}{2}}$$

$$\lesssim r^{1-k} (1 + r)^{-\alpha} (1 + 1_{r \leq 1} |\log r|) \left( \int_0^{+\infty} |u^{(k,i)}|^2 (1 + r)^{2\alpha} \bar{r} d\bar{r} \right)^{\frac{1}{2}},$$
\[ \left| \int_0^r u^{(k,i)}(\tilde{r})r^{1+k}d\tilde{r} \right| \lesssim \left( \int_0^r |u^{(k,i)}|^2(1+r)^{2\alpha}\tilde{r}d\tilde{r} \right)^{1/2} \left( \int_0^r (1+r)^{-2\alpha}\tilde{r}^{1+2k}d\tilde{r} \right)^{1/2} \lesssim r^{1+k}(1+r)^{-\alpha} \left( \int_0^\infty |u^{(k,i)}|^2(1+r)^{2\alpha}\tilde{r}d\tilde{r} \right)^{1/2}. \]

The two above inequalities, injected in (A.7), (A.8) produce:

\[ |\Phi^{(k,i)}_u(r)| \lesssim \frac{1}{k} r^1 (1 + 1_{r \leq 1} \log r) (1+r)^{-\alpha} \left( \int_0^\infty |u^{(k,i)}|^2(1+r)^{2\alpha}\tilde{r}d\tilde{r} \right)^{1/2}, \]

\[ |\partial_r \Phi^{(k,i)}_u(r)| \lesssim (1 + 1_{r \leq 1} \log r) (1+r)^{-\alpha} \left( \int_0^\infty |u^{(k,i)}|^2(1+r)^{2\alpha}\tilde{r}d\tilde{r} \right)^{1/2}. \]

On each spherical harmonic we thus have:

\[ |\Phi^{(k,i)}_u Y^{(k,i)}|^2 + r^2 \left| \nabla \left( \Phi^{(k,i)}_u Y^{(k,i)} \right) \right|^2 \lesssim r^2 (1+r)^{-2\alpha} (1 + 1_{r \leq 1} \log r) \int_0^\infty |u^{(k,i)}|^2(1+r)^{2\alpha}\tilde{r}d\tilde{r}. \]

The constant in the inequality above is independent on \( k, i \), so by summing we obtain (A.9).

\[ \square \]

REFERENCES

[1] M. Abramowitz and I. A. Stegun, editors. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. Dover Publications, Inc., New York, 1992. ISBN 0-486-61272-4. Reprint of the 1972 edition.

[2] A. Blanchet, J. Dolbeault, and B. Perthame. Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions. *Electron. J. Differential Equations*, pages No. 44, 32 pp. (electronic), 2006. ISSN 1072-6691.

[3] A. Blanchet, E. A. Carlen, and J. A. Carrillo. Functional inequalities, thick tails and asymptotics for the critical mass Patlak-Keller-Segel model. *J. Funct. Anal.*, 262(5):2142–2230, 2012. ISSN 0022-0396. doi: 10.1016/j.jfa.2011.12.012. URL http://dx.doi.org/10.1016/j.jfa.2011.12.012.

[4] S. Childress. Chemotaxis collapse in two dimensions. In *Modelling of patterns in space and time (Heidelberg, 1983)*, volume 55 of *Lecture Notes in Biomath.*, pages 61–66. Springer, Berlin, 1984. doi: 10.1007/978-3-642-45589-6_6. URL http://dx.doi.org/10.1007/978-3-642-45589-6_6.

[5] S. Childress and J. K. Percus. Nonlinear aspects of chemotaxis. *Math. Biosci.*, 56(3-4):217–237, 1981. ISSN 0025-5564. doi: 10.1016/0025-5564(81)90055-9. URL http://dx.doi.org/10.1016/0025-5564(81)90055-9.

[6] C. Collot, F. Merle, and P. Raphaël. On strongly anisotropic type II blow up. arXiv:1709.04941, 2017. URL https://arxiv.org/abs/1709.04941.

[7] C. Collot, T. Ghoul, N. Masmoudi, and V.T Nguyen. Refined description and stability for singular solutions of the 2D Keller-Segel system. arXiv, 2019a. URL https://arxiv.org/abs/1709.04941.

[8] C. Collot, P. Raphaël, and J. Szeftel. On the stability of type I blow up for the energy super critical heat equation. *Mem. Amer. Math. Soc.*, 260(1255):v+v-97, 2019b. ISSN 0065-9266. doi: 10.1090/memo/1255. URL https://doi.org/10.1090/memo/1255.

[9] O. Costin, R. Donninger, and X. Xia. A proof for the mode stability of a self-similar wave map. *Nonlinearity*, 29(8):2451–2473, 2016. ISSN 0951-7715. doi: 10.1088/0951-7715/29/8/2451. URL https://doi.org/10.1088/0951-7715/29/8/2451.

[10] O. Costin, R. Donninger, and I. Glodić. Mode stability of self-similar wave maps in higher dimensions. *Comm. Math. Phys.*, 351(3):959–972, 2017. ISSN 0010-3616. doi: 10.1007/s00220-016-2776-7. URL https://doi.org/10.1007/s00220-016-2776-7.

[11] S. I. Dejak, P. M. Lushnikov, Yu. N. Ovchinnikov, and I. M. Sigal. On spectra of linearized operators for Keller-Segel models of chemotaxis. *Phys. D*, 241(15):1245–1254, 2012. ISSN 0167-2789. URL https://doi.org/10.1016/j.physd.2012.04.003.

[12] Dejak, S and Egli, Daniel and Lushnikov, P and Sigal, I On blowup dynamics in the Keller–Segel model of chemotaxis, *St. Petersburg Mathematical Journal*, 25 (4):547–574, 2014

[13] J. I. Diaz, T. Nagai, and J.-M. Rakotoson. Symmetrization techniques on unbounded domains: application to a chemotaxis system on \( \mathbb{R}^N \). *J. Differential Equations*, 145(1):156–183, 1998. ISSN 0022-0396. doi: 10.1006/jdeq.1997.3389. URL http://dx.doi.org/10.1006/jdeq.1997.3389.

[14] Dyachenko, Sergey A and Lushnikov, Pavel M and Vladimirova, Natalia Logarithmic scaling of the collapse in the critical Keller–Segel equation *Nonlinearity*, 26 (11), 3011, 2013 IOP Publishing

[15] M. Hadžić and P. Raphaël. On melting and freezing for the 2D radial Stefan problem. *J. Eur. Math. Soc. (JEMS)*, 21 (11):3259–3341, 2019. ISSN 1435-9855. doi: 10.4171/JEMS/904. URL https://doi.org/10.4171/JEMS/904.
[16] D. Horstmann. From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I. Jahresber. Deutsch. Math.-Verein., 105(3):103–165, 2003. ISSN 0012-0456.

[17] W. Jäger and S. Luckhaus. On explosions of solutions to a system of partial differential equations modelling chemotaxis. Trans. Amer. Math. Soc., 329(2):819–824, 1992. ISSN 0002-9947. doi: 10.2307/2153966. URL http://dx.doi.org/10.2307/2153966.

[18] E. F. Keller and L. A. Segel. Initiation of slime mold aggregation viewed as an instability. Journal of Theoretical Biology, 26(3):399 – 415, 1970. ISSN 0022-5193. doi: http://dx.doi.org/10.1016/0022-5193(70)90092-5. URL http://www.sciencedirect.com/science/article/pii/0022519370900925.

[19] E. F. Keller and L. A. Segel. Traveling bands of chemotactic bacteria: A theoretical analysis. Journal of Theoretical Biology, 30(2):235 – 248, 1971a. ISSN 0022-5193. doi: http://dx.doi.org/10.1016/0022-5193(71)90051-8. URL http://www.sciencedirect.com/science/article/pii/0022519371900518.

[20] E. F. Keller and L. A. Segel. Model for chemotaxis. Journal of Theoretical Biology, 30(2):225 – 234, 1971b. ISSN 0022-5193. doi: http://dx.doi.org/10.1016/0022-5193(71)90050-6. URL http://www.sciencedirect.com/science/article/pii/0022519371900506.

[21] Merle, Frank and Raphael, Pierre and Szeftel, Jeremie On Strongly Anisotropic Type I Blowup International Mathematics Research Notices, 03 ISSN 1073-7928 doi: 10.1093/imrn/rny012 http://oup.prod.sis.lan/imrn/advance-article-pdf/doi/10.1093/imrn/rny012/24349812/rny012.pdf

[22] Y. Naito and T. Suzuki. Self-similarity in chemotaxis systems. Colloq. Math., 111(1):11–34, 2008. ISSN 0010-1354. doi: 10.4064/cm111-1-2. URL http://dx.doi.org/10.4064/cm111-1-2.

[23] C. S. Patlak. Random walk with persistence and external bias. Bull. Math. Biophys., 15:311–338, 1953.

[24] P. Raphaël and R. Schweyer. On the stability of critical chemotactic aggregation. Math. Ann., 359(1-2):267–377, 2014. ISSN 0025-5831. doi: 10.1007/s00208-013-1002-6. URL http://dx.doi.org/10.1007/s00208-013-1002-6.

[25] T. Suzuki and T. Senba. Applied analysis. Published by Imperial College Press, London; Distributed by World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, second edition, 2011. ISBN 978-1-84816-652-3; 1-84816-652-4. doi: 10.1142/p753. URL http://dx.doi.org/10.1142/p753. Mathematical methods in natural science.

[26] J. J. L. Velázquez. Stability of some mechanisms of chemotactic aggregation. SIAM J. Appl. Math., 62(5):1581–1633, 2002. ISSN 0036-1399. doi: 10.1137/S0036139900380049. URL http://dx.doi.org/10.1137/S0036139900380049.

[27] J. J. L. Velázquez. Point dynamics in a singular limit of the Keller-Segel model. I. Motion of the concentration regions. SIAM J. Appl. Math., 64(4):1198–1223, 2004a. ISSN 0036-1399. doi: 10.1137/S0036139903433888. URL http://dx.doi.org/10.1137/S0036139903433888.

[28] J. J. L. Velázquez. Point dynamics in a singular limit of the Keller-Segel model. II. Formation of the concentration regions. SIAM J. Appl. Math., 64(4):1224–1248, 2004b. ISSN 0036-1399. doi: 10.1137/S003613990343389X. URL http://dx.doi.org/10.1137/S003613990343389X.

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