DISTORTION OF MAPPINGS AND $L_{q,p}$-COHOMOLOGY

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ABSTRACT. We study some relation between some geometrically defined classes of diffeomorphisms between manifolds and the $L_{q,p}$-cohomology of these manifolds. Some applications to vanishing and non vanishing results in $L_{q,p}$-cohomology are given.

1. Introduction

The $L_{q,p}$-cohomology is an invariant of Riemannian manifolds defined to be the quotient of the space of $p$-integrable closed differential $k$-forms on the manifold modulo the exact forms having a $q$-integrable primitive:

$$H^k_{q,p}(M) = \{ \omega \mid \omega \text{ is a } k\text{-form, } |\omega| \in L^p(M) \text{ and } d\omega = 0 \}/\{d\theta \mid |\theta| \in L^q(M)\}.$$  

This invariant has been first defined for the special case $p = q = 2$ in the 1970’s and has been intensively studied since then, we refer to the book [22] for an overview of $L_2$-cohomology. The $L_{q,p}$-cohomology has been introduced in the early 1980’s as an invariant of the Lipschitz structure of manifolds, see [4]. During the next two decades, the main interest was focused on the case $p = q$, i.e on $L_p$-cohomology, and the last chapter of the book [15] by M. Gromov is devoted to this subject; see also [4, 15, 28, 29, 30, 31] for more geometrical applications of $L_p$-cohomology. Although the $L_{q,p}$-cohomology with $q \neq p$ has attracted less attention, it poses a richer structure. The subject is also motivated by its connections with Sobolev type inequalities [11] and quasiconformal geometry [12]. See also [15, 10, 20, 21] for other results on $L_{q,p}$-cohomology.

When an invariant of a geometric object has been defined, it is important to investigate its functorial properties, i.e. its behavior under various classes of mappings. It is one of our goal in the present paper to describe a natural class of maps which induces morphisms at the level of $L_{q,p}$-cohomology. Our answer is restricted to the case of diffeomorphisms and is given in Theorem 6.1(C) below.

A diffeomorphisms will behave functorially for $L_{q,p}$-cohomology, if its distortion is controlled in some specific way. To explain what is meant by the distortion, consider a diffeomorphism $f : M \to \tilde{M}$ between two Riemannian manifolds. On then define for any $k$ the principal invariant of $f$ as

$$\sigma_k(f, x) = \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1}(x)\lambda_{i_2}(x)\cdots\lambda_{i_k}(x),$$

where $\lambda_{i_k}(x)$ are the eigenvalues of the Hessian matrix of $f$ at $x$.
where the $\lambda_i$'s are the principal invariants of $df_x$, i.e. the eigenvalues of $\sqrt{(df_x)^*(df_x)}$.

One then say that $f$ has bounded $(s,t)$-distortion in degree $k$, and we write $f \in BD_{k,(s,t)}(M,\tilde{M})$, if

$$(\sigma_k(f,x))^s \cdot J_f^{-1}(x) \in L^t(M)$$

where $J_f$ is the Jacobian of $f$.

The class $BD_{n,\infty}$ (where $n$ is the dimension of $M$) is exactly the class of quasi-conformal diffeomorphisms (also called mappings with bounded distortion), which has been introduced by Y. Reshetnyak in the early 1960’s and has been intensively studied since then. The classes $BD_{1,\infty}$ have been studied by different authors and under various names, see [1 2 25 26 27 32 35 37 39]. The class $BD_{s,\infty}$ also appears in [35], where some obstructions are given.

As a preliminary step to the study of functoriality in $L_{q,p}$-cohomology, we study diffeomorphisms $f : M \to \tilde{M}$ that induces bounded operator between the Banach spaces of $\tilde{p}$-integrable differential $k$-forms. The result is formulated in Proposition [4.1]: it states that a diffeomorphism $f \in BD_{\tilde{p},1}(M,\tilde{M})$ induces a bounded operator $f^* : L^{\tilde{p}}(\tilde{M},\Lambda^k) \to L^p(M,\Lambda^k)$ if $p \leq \tilde{p} < \infty$ and $t = \frac{p}{p-\tilde{p}}$. Let us note that finer information are available in the case $k = 1$, see [7 9 38 39].

To obtain a functoriality in $L_{q,p}$-cohomology, we need to control the distortion of the map $f$ both on $k$-forms and on $(k-1)$-forms. This is formulated in Theorem [6.1(C)], which states in particular that a diffeomorphism $f \in BD_{\tilde{q},r}^{n-k+1}(M,\tilde{M}) \cap BD_{\tilde{p},1}(M,\tilde{M})$ induces a well defined linear map $f^* : H_{\tilde{q},p}(\tilde{M}) \to H_{q,p}(M)$ if $p \leq \tilde{p}$, $q \leq \tilde{q}$, $t = \frac{p}{p-\tilde{p}}$, $u = \frac{q}{q-\tilde{q}}$ and $\tilde{q}' = \frac{\tilde{q}}{\tilde{q}-r}$.

This is a quite technical result, and it would be nice to be able to give conditions under which the map $f^*$ is injective at the level of $L_{q,p}$-cohomology. But unfortunately, the results we give in section 5 strongly suggest that it will be hard or impossible to find conditions for injectivity, except for the special cases of quasiconformal or bilipschitz maps. However we have the following result (theorem [6.1(B)]), which allows us to prove some vanishing results in $L_{q,p}$-cohomology without requiring the functoriality: If there exists a diffeomorphism $f \in BD_{n-k+1}(M,\tilde{M}) \cap BD_{\tilde{p},1}(M,\tilde{M})$ with $p \leq \tilde{p}$, $q \leq \tilde{q}$, $t = \frac{p}{p-\tilde{p}}$, $r = \frac{q}{q-\tilde{q}}$, and $\tilde{q}' = \frac{\tilde{q}}{\tilde{q}-r}$, then $H_{q,p}^k(M) = 0$ implies $H_{\tilde{q},p}^k(\tilde{M}) = 0$. We give two concrete examples showing how this result can be used to prove vanishing and non vanishing results in $L_{q,p}$-cohomology.

The paper is organized as follows. In section 2, we recall the definition of $L_{q,p}$-cohomology and some known facts about the distortion of linear maps. In section 3, we discuss the effect of a diffeomorphism $f$ at the level of $L_{q,p}$-cohomology, assuming that the map $f$ induces bounded operators at the level of some Banach spaces of integrable differential forms; these are abstract results. In section 4, we introduce the class of diffeomorphisms with bounded $(s,t)$-distortion and in section 5, we relate these diffeomorphisms with quasiconformal and bilipschitz maps. Section 6 contains our main results, which relates the distortion of diffeomorphisms to $L_{q,p}$-cohomology and in section 7, we give two concrete applications of these results. In the last section, we shortly discuss our smoothness restrictions.
2. Preliminary notions

2.1. $L_{q,p}$-cohomology. We shortly recall the definition of $L_{q,p}$-cohomology, referring to the paper [11] for more details. Let $M$ be an oriented Riemannian manifold, we denote by $C_c^\infty(M, \Lambda^k)$ the vector space of smooth differential forms of degree $k$ with compact support on $M$ and by $L^1_{loc}(M, \Lambda^k)$ the space of differential $k$-forms whose coefficients (in any local coordinate system) are locally integrable. The form $\theta \in L^1_{loc}(M, \Lambda^k)$ is said to be the weak exterior differential of $\phi \in L^1_{loc}(M, \Lambda^{k-1})$, and one writes $d\phi = \theta$, if for each $\omega \in C_c^\infty(M, \Lambda^{n-k})$, one has

$$\int_M \theta \wedge \omega = (-1)^k \int_M \phi \wedge d\omega.$$ 

Let $L^p(M, \Lambda^k)$ be the Banach space of differential forms in $L^1_{loc}(M, \Lambda^k)$ such that

$$\|\theta\|_p := \left(\int_M |\theta|^p dx\right)^{\frac{1}{p}} < \infty.$$ 

We denote by $Z^k_p(M)$ the space of weakly closed forms in $L^p(M, \Lambda^k)$, i.e. $Z^k_p(M) = L^p(M, \Lambda^k) \cap \ker d$. It is a closed subspace. We also define

$$B^k_{q,p}(M) := d(L^q(M, \Lambda^{k-1})) \cap L^p(M, \Lambda^k),$$

this is the space of exact forms in $L^p$ having a primitive in $L^q$ and we have $B^k_{q,p}(M) \subset Z^k_p(M)$, because $d \circ d = 0$.

**Definition 2.1.** The $L_{q,p}$-cohomology of $(M, g)$ (where $1 \leq p, q \leq \infty$) is defined to be the quotient

$$H^k_{q,p}(M) := \frac{Z^k_p(M)}{B^k_{q,p}(M)},$$

and the reduced $L_{q,p}$-cohomology of $(M, g)$ is

$$\overline{H}^k_{q,p}(M) := \frac{Z^k_p(M)}{\overline{B}^k_{q,p}(M)},$$

where $\overline{B}^k_{q,p}(M)$ is the closure of $B^k_{q,p}(M)$.

The reduced cohomology is naturally a Banach space. When $p = q$, we simply speak of $L_p$-cohomology and write $H_p^k(M)$ and $\overline{H}_p^k(M)$.

2.2. Linear map between Euclidean spaces. Recall that an Euclidean vector space $(E, g)$ is a finite dimensional real vector space equipped with a scalar product. Two linear mappings $A, B \in L(E_1; E_2)$ between two Euclidean vector spaces $(E_1, g_1)$ of dimension $n$ and $m$ are said to be orthogonally equivalent if there exist orthogonal transformations $Q_1 \in O(E_1)$ and $Q_2 \in O(E_2)$ such that $B = Q_2^{-1}AQ_1$, i.e. the diagram

$$\begin{array}{ccc}
E_1 & \xrightarrow{A} & E_2 \\
Q_1 \uparrow & & \uparrow Q_2 \\
E_2 & \xrightarrow{B} & E_2
\end{array}$$

commutes. Given a linear mapping $A : (E_1, g_1) \to (E_2, g_2)$, its (right) Cauchy-Green tensor $c$ is the symmetric bilinear form on $E_1$ defined by $c(x, y) = g_2(Ax, Ay)$. The adjoint of $A$ is the linear map $A^\# : E_2 \to E_1$ satisfying

$$g_2(x, Ay) = g_1(A^\#x, y)$$



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for all $x \in E_1$ and $y \in E_2$. The Cauchy-Green tensor and the adjoint are related by
\[ c(x, y) = g_2(A x, A y) = g_1(A^\# A x, y). \]
Let us denote the eigenvalues of $A^\# A$ by $\mu_1, \mu_2, \ldots, \mu_n$. Then $\mu_i \in [0, \infty)$, for all $i$, and there exists orthonormal basis $e_1, e_2, \ldots, e_n$ of $E_1$ and $e'_1, e'_2, \ldots, e'_m$ of $E_2$ such that $A e_i = \sqrt{\mu_i} e'_i$ for all $i$. The matrix of $A^\# A$ with respect to an orthonormal basis $e_1, e_2, \ldots, e_n$ of $E_1$ coincides with the matrix $C$ of the Cauchy-Green tensor $c$ in the same basis.

**Definition 2.2.** The numbers $\lambda_i = \sqrt{\mu_i}$ are called the principal distortion coefficients of $A$ or the singular values of $A$.

The principal distortion coefficients can be computed from the distortion polynomial which is defined as follows:

**Definition 2.3.** Given an arbitrary basis $e_1, e_2, \ldots, e_n$ of $E_1$, we associate to $g_1$ and $c$, the $n \times n$ matrices $G = (g_1(e_i, e_j))$ and $C = (c(e_i, e_j))$. The distortion polynomial of $A$ is the polynomial
\[ P_A(t) := \frac{\det(C - t G)}{\det G}. \]

The distortion polynomial $P_A(t)$ is independent of the choice of the basis $\{e_i\}$, it coincides with the characteristic polynomial of $AA^\#$ and has nonnegative roots. In particular, the roots of $P_A$ are the eigenvalues $\mu_i$ of $AA^\#$ and the $\lambda_i = \sqrt{\mu_i}$ are the principal distortion coefficients of $A$ and the distortion polynomial can thus be written in terms of the principal distortion coefficients as
\[ P_A(t) = \prod_i (t - \lambda_i^2). \]

The following notion is also useful:

**Definition 2.4.** The principal invariants of $A$ are the elementary symmetric polynomials in the $\lambda_i$’s, i.e. they are defined by $\sigma_0(A) = 1$ and
\[ \sigma_k(A) = \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k} \]
for $k = 1, \ldots, 2, \ldots, n$.

The following result is well known, it can be found e.g. in ([33], page 57)

**Proposition 2.1.** Two linear mappings $A, B \in L(E_1; E_2)$ are orthogonally equivalent if and only if they have the same principal invariants: $\sigma_k(A) = \sigma_k(B)$ for $k = 1, 2, \ldots, n$.

The principal invariants of $A$ are related to the action of $A \in L(E_1; E_2)$ on the exterior algebras: Recall that if $E$ is an Euclidean vector space, then the exterior algebra $\Lambda E$ is equipped with a canonical scalar product. If $e_1, e_2, \ldots, e_n$ is an orthonormal basis of $E_1$, then the \( \binom{n}{k} \) multi-vectors $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}\}$ $(i_1 < i_2 < \cdots < i_k)$ form an orthonormal basis of $\Lambda^k E$.

To any linear map $A \in L(E_1; E_2)$ we associate a linear map $\Lambda^k A \in L(\Lambda^k E_1; \Lambda^k E_2)$, and we have
\begin{equation}
\frac{1}{\binom{n}{k}} \sigma_k \leq \| \Lambda^k A \| \leq \sigma_k.
\end{equation}
Indeed, suppose that \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \) are the principal distortion coefficients of \( A \), then we have
\[
\|A^k\| = \lambda_{n-k+1} \lambda_{n-k+2} \cdots \lambda_n
\]
and
\[
\sigma_k : = \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.
\]

If \( E_1 = E_2 = \mathbb{R}^n \) and \( A \) is a diagonal matrix with nonnegative entries, then we have
\[
\sigma_k = \text{Trace}(A^k).
\]

The principal distortion coefficients also have the following geometric interpretation:

- If \( S \subset E_1 \) is the unit ball, then \( A(S) \subset E_2 \) is an ellipsoid contained in \( \text{Im}A \) and whose principal axis are the non vanishing \( \lambda_i \).
- Suppose \( \dim(E_1) = \dim(E_2) = n \). The Jacobian \( J_A := \sigma_n = \lambda_1 \lambda_2 \cdots \lambda_n \) measures the volume distortion.
- If \( \dim(E_1) = \dim(E_2) = n \) and \( A \) is invertible, then the principal distortion coefficients of \( A^{-1} \) are the inverse of the principal distortion coefficients of \( A \).
- The norm of \( A \) as a linear operator is \( \|A\| = \max_{v \neq 0} \frac{\|Av\|}{\|v\|} = \max_i \lambda_i \).

**Lemma 2.2.** If \( \dim(E_1) = \dim(E_2) = n \) and \( A \) is invertible, then for any \( 0 \leq m \leq n \), we have
\[
\sigma_m(A^{-1}) = \frac{\sigma_{n-m}(A)}{J_A}.
\]

**Proof** Use the fact the principal distortion coefficients of \( A^{-1} \) are the inverse of the principal distortion coefficients of \( A \), and compute. \( \square \)

### 3. Diffeomorphism and \( L_{q,p} \)-cohomology

Let \((M, g)\) and \((\tilde{M}, \tilde{g})\) be two smooth oriented \( n \)-dimensional Riemannian manifolds and \( f : M \to \tilde{M} \) be a diffeomorphism such that the induced operator
\[
f^* : L^\hat{p}(\tilde{M}, \Lambda^k) \to L^p(M, \Lambda^k)
\]
is bounded for some specified \( p, \hat{p} \in (0, \infty) \). Then the condition \( f^*d = df^* \) implies that
\[
f^* : Z^k_\hat{p}(\tilde{M}) \to Z^k_p(M)
\]
is a well defined bounded operator. In the framework of \( L_{q,p} \)-cohomology there are two natural questions which then arise:

i.) Suppose that \( \omega \in B^k_{q,\hat{p}}(\tilde{M}) \). Under what conditions does this imply that \( f^*\omega \in B^k_{q,p}(M) \), i.e. that
\[
f^*(B^k_{q,\hat{p}}(\tilde{M})) \subset B^k_{q,p}(M) ?
\]

ii.) Suppose that \( f^*\omega \in B^k_{q,p}(M) \). Under what conditions can we conclude that \( \omega \in B^k_{q,\hat{p}}(M) \), i.e. that
\[
(f^{-1})^*(B^k_{q,p}(M)) \subset B^k_{q,\hat{p}}(\tilde{M}) ?
\]
A positive answer to the first question gives us a well defined linear map
\[ f^* : H^k_{\tilde{q}, \tilde{p}}(\tilde{M}) \to H^k_{q,p}(M), \]
and a positive answer to both questions implies the injectivity of this linear map.

In this section we give an answer to these questions in terms of boundedness of the operators \( f^* \), and \( f_\ast := (f^{-1})^* \). We begin with the second question.

**Theorem 3.1.** Let \( f : M \to \tilde{M} \) be a diffeomorphism, \( 1 \leq p \leq \tilde{p} < \infty \) and \( 1 \leq \tilde{q} \leq q < \infty \). Assume that both operators
\[
 f^* : L^\tilde{p}(\tilde{M}, \Lambda^k) \to L^p(M, \Lambda^k), \quad \text{and} \quad f_\ast : L^\tilde{q}(M, \Lambda^{k-1}) \to L^q(\tilde{M}, \Lambda^{k-1})
\]
are bounded. Then for any \( \omega \in Z^\tilde{p}_p(\tilde{M}) \), we have \( f^* \omega \in Z^p_p(M) \). Furthermore, if \( [f^* \omega] = 0 \) in \( H^\tilde{p}_{q,p}(M) \) then \( [\omega] = 0 \) in \( H^p_{\tilde{q},p}(\tilde{M}) \) (thus \( H^k_{\tilde{p},p}(M) = 0 \Rightarrow H^k_{\tilde{q},p}(\tilde{M}) = 0 \)).

**Remarks** We should not conclude that \( f^* : H^k_{\tilde{q},\tilde{p}}(\tilde{M}) \to H^k_{q,p}(M) \) is an injective map, because this map is a priori not even well defined.

**Proof.** Choose \( \omega \in Z^\tilde{p}_p(\tilde{M}) \). Because \( f^* : L^\tilde{p}(\tilde{M}, \Lambda^k) \to L^p(M, \Lambda^k) \) is a bounded operator, \( f^* \omega \in L^p(M, \Lambda^k) \), and since and \( d(f^* \omega) = f^* d\omega = 0 \) we have \( f^* \omega \in Z^p_p(M) \). Suppose now that \( [f^* \omega] = 0 \) in \( H^\tilde{p}_{q,p}(M) \), then \( f^* \omega \in B^\tilde{p}_{q,p}(M) \), and there exists \( \theta \in L^q(M, \Lambda^{k-1}) \) such that \( d\theta = f^* \omega \). But by the second hypothesis the operator \( f_\ast : L^\tilde{q}(M, \Lambda^k) \to L^q(\tilde{M}, \Lambda^k) \) is bounded and therefore \( f_\ast \theta \in L^q(\tilde{M}, \Lambda^k) \).

We then have
\[
\omega = f_\ast (f^* \omega) = f_\ast d\theta = d(f_\ast \theta) \in B^\tilde{q}_{q,p}(\tilde{M})
\]
Therefore \( [\omega] = 0 \) in \( H^\tilde{p}_{q,p}(\tilde{M}) \). \( \square \)

The argument of the previous proof is illustrated in the following commutative diagrams:
\[
\begin{array}{ccc}
Z^\tilde{p}_p(\tilde{M}) & \xrightarrow{f^*} & Z^p_p(M) \\
\downarrow{d} & & \uparrow{d} \\
L^\tilde{q}(\tilde{M}, \Lambda^{k-1}) & \xrightarrow{f_\ast} & L^q(M, \Lambda^{k-1})
\end{array}
\]

The next result gives us sufficient conditions for a diffeomorphism to behave functorially at the \( L_{q,p} \)-cohomology level.

**Theorem 3.2.** Let \( f : M \to \tilde{M} \) be a diffeomorphism and 1 \( \leq p \leq \tilde{p} < \infty \) and \( 1 \leq q \leq \tilde{q} < \infty \). Assume that
\[
 f^* : L^\tilde{p}(\tilde{M}, \Lambda^k) \to L^p(M, \Lambda^k), \quad \text{and} \quad f^* : L^\tilde{q}(\tilde{M}, \Lambda^{k-1}) \to L^q(M, \Lambda^{k-1})
\]
are bounded operators. Then

a.) \( f^* : \Omega^k_{\tilde{q},\tilde{p}}(M) \to \Omega^k_{q,p}(M) \) is a bounded operator,

b.) \( f^* : H^k_{\tilde{q},\tilde{p}}(\tilde{M}) \to H^k_{q,p}(M) \) is a well defined linear map,

c.) \( f^* : \overline{H}^k_{\tilde{q},\tilde{p}}(\tilde{M}) \to \overline{H}^k_{q,p}(M) \) is a well defined bounded operator,
Proof. a) By definition $\omega \in \Omega^{k-1}_{q,p}(\tilde{M})$ if $\omega \in L^q(\tilde{M}, \Lambda^{k-1})$ and $d\omega \in L^p(\tilde{M}, \Lambda^k)$. Because both operators $f^* : L^p(\tilde{M}, \Lambda^k) \to L^p(M, \Lambda^k)$, $f^* : L^q(\tilde{M}, \Lambda^{k-1}) \to L^q(M, \Lambda^{k-1})$ are bounded and $f^* d\omega = df^* \omega$ we obtain that $f^* \omega \in \Omega^{k-1}_{q,p}(M)$. The operator $f^* : \Omega^{k-1}_{q,p}(\tilde{M}) \to \Omega^{k-1}_{q,p}(M)$ is clearly bounded.

b) The condition $f^* d = df^*$ and the boundedness of the operators $f^* : L^p(\tilde{M}, \Lambda^k) \to L^p(M, \Lambda^k)$ implies that $f^* \left( Z^k_{L^q}(\tilde{M}) \right) \subset Z^k_{L^q}(M)$. Using the boundedness of the operator $f^* : \Omega^{k}_{q,p}(\tilde{M}) \to \Omega^{k}_{q,p}(M)$ and the condition $f^* d = df^*$ we see that

$$f^* \left( B^k_{q,p}(\tilde{M}) \right) = f^* \left( d\Omega^{k-1}_{q,p}(\tilde{M}) \right) = df^* \left( \Omega^{k-1}_{q,p}(\tilde{M}) \right) \subset d \left( \Omega^{k-1}_{q,p}(M) \right) = B^k_{q,p}(M).$$

The inclusions

$$f^* \left( Z^k_{L^q}(\tilde{M}) \right) \subset Z^k_p(M), \quad f^* \left( B^k_{q,p}(\tilde{M}) \right) \subset B^k_{q,p}(M)$$

imply that the linear map

$$f^* : H^k_{q,p}(\tilde{M}) = Z^k_{L^q}(\tilde{M})/B^k_{q,p}(\tilde{M}) \to Z^k_{L^q}(M)/B^k_{q,p}(M) = H^k_{q,p}(M)$$

is well defined.

c) Using the inclusions (3.1) and the continuity of the operator $f^* : \Omega^{k}_{q,p}(\tilde{M}) \to \Omega^{k}_{q,p}(M)$, we have

$$f^* \left( B^{k}_{q,p}(\tilde{M}) \right) \subset f^* \left( B^{k}_{q,p}(\tilde{M}) \right) \subset B^{k}_{q,p}(M).$$

Therefore the operator

$$f^* : \overline{H}_{q,p}(\tilde{M}) = Z^k_{L^q}(\tilde{M})/\overline{B}^k_{q,p}(\tilde{M}) \to Z^k_{L^q}(M)/\overline{B}^k_{q,p}(M) = \overline{H}_{q,p}(M)$$

is well defined and bounded.

Using the two previous theorems, we have the following result:

**Theorem 3.3.** Let $f : M \to \tilde{M}$ be a diffeomorphism and $1 \leq p \leq p \leq \infty$ and $1 \leq q = q \leq \infty$. Assume that the operator $f^* : L^p(\tilde{M}, \Lambda^k) \to L^p(M, \Lambda^k)$ is bounded and that $f^* : L^q(\tilde{M}, \Lambda^{k-1}) \to L^q(M, \Lambda^{k-1})$ is an isomorphism of Banach spaces. Then the linear map

$$f^* : H^k_{q,p}(\tilde{M}) \to H^k_{q,p}(M)$$

is well defined and injective.

The proof is immediate.

**Corollary 3.4.** Let $f : M \to \tilde{M}$ satisfying the hypothesis of the previous theorem. If $T^k_{q,p}(M) = 0$ then $T^k_{q,p}(\tilde{M}) = 0$.

**Proof** Since $T^k_{q,p}(M) = 0$, we have $B_{q,p}(M) = B^k_{q,p}(M)$. The hypothesis of Theorem 3.2 are satisfied, thus the inclusions (3.2) holds and we thus have

$$f^* \left( B_{q,p}^{k}(M) \right) \subset B_{q,p}^{k}(\tilde{M}) = B_{q,p}^{k}(M).$$
Choose now an arbitrary element $\omega \in B^{k}_{q,\bar{p}}(M)$. We have $f^*\omega \in B^{k}_{q,\bar{p}}(\hat{M})$ by the previous inclusion, this means that $[f^*\omega] = 0 \in H^{k}_{q,\bar{p}}(\hat{M})$, but $f^* : H^{k}_{q,\bar{p}}(\hat{M}) \to H^{k}_{q,\bar{p}}(M)$ is injective by the previous theorem and therefore $[\omega] = 0 \in H^{k}_{q,\bar{p}}(\hat{M})$, that is $\omega \in B^{k}_{q,\bar{p}}(M)$. Since $\omega$ was arbitrary, we have shown that $B^{k}_{q,\bar{p}}(\hat{M}) = B^{k}_{q,\bar{p}}(M)$, i.e. $T^{k}_{q,\bar{p}}(M) = 0$. □

**Remark** The hypothesis in Theorem 3.3 seem to be very restrictive, the results of section 5 suggest that it will be difficult to find diffeomorphisms satisfying these hypothesis and which aren’t bilipschitz or quasiconformal. See the discussion at the end of section 5.

### 4. Diffeomorphisms with controlled distortion.

Let $(M, g)$ and $(\hat{M}, \hat{g})$ be two smooth oriented Riemannian manifolds. In this section we study classes of diffeomorphisms $f : M \to \hat{M}$ with bounded distortion of an integral type that induce bounded operators $f^* : L^{\hat{p}}(\hat{M}, \Lambda^k) \to L^p(M, \Lambda^k)$ for $1 \leq p \leq \hat{p} \leq \infty$. To define these classes we use the notation 

$$\sigma_k(f, x) = \sigma_k(d f_x)$$

for the $k$-th principal invariant of the differential $d f_x$. We also write $\sigma_k(f)$ when there is no risk of confusion, observe that $\sigma_n(f) = J_f$, where $J_f$ is the Jacobian of $f$.

**Definition 4.1.** A diffeomorphism $f : M \to \hat{M}$ is said to be of bounded $(s, t)$-distortion in degree $k$, and we write $f \in BD_{(s, t)}^k(M, \hat{M})$, if

$$(\sigma_k(f))^s J_f^{-1} \in L^t(M).$$

It is assumed that $1 \leq s < \infty$ and $0 < t \leq \infty$.

It is convenient to introduce the quantity

$$K_{s,t,k}(f) = \left\| (\sigma_k(f))^s \right\|_{L^t(M)}^{-1}(J_f(x)),$$

the mapping $f$ belongs then to $BD_{(s, t)}^k(M, \hat{M})$, if and only if $K_{s,t,k}(f) < \infty$.

**Proposition 4.1.** Let $f : M \to \hat{M}$ be a diffeomorphism. Suppose $p \leq \hat{p} < \infty$ and for any $\omega \in L^{\hat{p}}(\hat{M}, \Lambda^k)$ we have

$$\|f^*\omega\|_{L^p(M, \Lambda^k)} \leq (K_{\hat{p},t,k}(f))^{1/\hat{p}} \|\omega\|_{L^{\hat{p}}(\hat{M}, \Lambda^k)}$$

where $t = \frac{p}{\hat{p} - p}$. In particular if $f \in BD_{(\hat{p}, t)}^k(M, \hat{M})$, then the operator

$$f^* : L^{\hat{p}}(\hat{M}, \Lambda^k) \to L^p(M, \Lambda^k)$$

is bounded.
Proof. Without loss of generality we can suppose that \( J_f(x) > 0 \). Using the fact that \(|(f^*\omega)_x| \leq |\sigma_k(f, x) \cdot [\omega_{J_f(x)}]|\), we have

\[
\|f^*\omega\|_{L^p(M, \Lambda^k)}^p \leq \int_M |(f^*\omega)_x|^p \, dx \leq \int_M (\sigma_k(f, x))^p \cdot |\omega_{J_f(x)}|^p \, dx \\
\leq \int_M \left\{ (\sigma_k(f, x) J_f^{-1/\beta}(x))^p + (|\omega_{J_f(x)}| J_f^{1/\beta}(x))^p \right\} \, dx.
\]

Using Hölder’s inequality for \( s = \frac{\beta}{p(1 - \beta)} \) and \( s' = \frac{\beta}{p} \) (so that \( \frac{1}{s} + \frac{1}{s'} = 1 \)), and the change of variable formula, we obtain

\[
\|f^*\omega\|_{L^p(M, \Lambda^k)}^p \leq \left( \int_M (\sigma_k^p f, x) J_f^{-1}(x) \right)^{\frac{p}{p - \beta}} \cdot \left( \int_M (|\omega_{J_f(x)}|^p J_f(x)) \, dx \right)^{\frac{\beta}{p - \beta}} \\
\leq (K_{p, t, k}(f))^{\frac{\beta}{p}} \left( \int_M |\omega|^p \, dy \right)^{\frac{\beta}{p}},
\]

that is

\[
\|f^*\omega\|_{L^p(M, \Lambda^k)} \leq (K_{p, t, k}(f))^{1/\beta} \|\omega\|_{L^p(M, \Lambda^k)}.
\]

\[
\square
\]

Remark Every diffeomorphism belongs to the class \( BD_{1, \infty}^n \), i.e. \( BD_{1, \infty}^n(M, \tilde{M}) = \text{Diff}(M, \tilde{M}) \). The previous proposition states in particular the well known fact that the condition for an \( n \)-form to be integrable is invariant under diffeomorphism and therefore independent of the choice of a Riemannian metric.

The next proposition describes the inverse of diffeomorphisms in \( BD_{s, t}^k \).

Proposition 4.2. Let \( f : M \to \tilde{M} \) be a diffeomorphism, \( 0 \leq m \leq n \). Let \( 1 \leq \alpha < \infty \) and \( 0 < \beta \leq \infty \) with \( \beta(\alpha - 1) > 1 \). Then the equivalence

\[
f^{-1} \in BD_{(\alpha, \beta)}^m(\tilde{M}, M) \iff f \in BD_{(s, t)}^{n-m}(M, \tilde{M})
\]

holds if and only if

\[
s = \frac{\alpha}{\alpha - 1 - \frac{1}{\beta}} \quad \text{and} \quad t = \beta(\alpha - 1) - 1.
\]

Proof. Without loss of generality we can suppose that \( J(f, x) > 0 \).

Assume first that \( \beta < \infty \), then the condition \( f^{-1} \in BD_{(\alpha, \beta)}^m(\tilde{M}, M) \) means that

\[
\int_{\tilde{M}} \left\{ \sigma_m^\alpha(f^{-1}, y) J_{f^{-1}}^{-1}(y) \right\}^\beta \, dy < \infty.
\]

By the lemma 2.2 we have

\[
\sigma_m(f^{-1}, f(x)) = \frac{\sigma_{n-m}(f, x)}{J_f(x)}
\]

at \( y = f(x) \) and for any \( 0 \leq m \leq n \). Using the relations 1.1, which can be rewritten as

\[
\alpha \beta = st = t + \beta + 1,
\]
Thus (4.3) holds if and only if $\sigma_n^\alpha m(f, x) < \infty$, using the relation $s = \frac{\alpha}{q-1}$, the equation \(12\) and $J_f^{-1} = J_f^{-1}$, we have

$$\left(\sigma_m(f^{-1})^\alpha J_f^{-1}\right) \left(\sigma_m(f^{-1})^{-\alpha} J_f\right) = \left(\sigma_m(f)^\alpha J_f^{-1}\right) \left(\sigma_m(f^{-1})^\alpha J_f^{-1}\right) \left\{\sigma_n^\alpha m(f, x)\right\}^{-1}$$

Thus \(13\) holds if and only if $\sigma_n^\alpha m(f) J_f^{-1}$ is bounded, i.e. $f \in \text{BD}^{\alpha,\infty}(M, \tilde{M})$.

## Corollary 4.3

If $\tilde{q} \leq q$ and the diffeomorphism $f$ belongs to $\text{BD}^{\alpha,\infty}(M, \tilde{M})$ with

$$s = \frac{\tilde{q}}{\tilde{q}-1}, \quad t = \frac{q(\tilde{q}-1)}{q-\tilde{q}}$$

then the operator

$$f_* : L^q(M, \Lambda^m) \to L^\tilde{q}(\tilde{M}, \tilde{\Lambda}^m)$$

is bounded.

**Proof.** This follows immediately from Proposition 4.1 and the previous proposition with $\alpha = q$ and $\beta = \frac{\tilde{q}}{q-\tilde{q}}$.

## Corollary 4.4

If the diffeomorphism $f : M \to \tilde{M}$ satisfies $f \in \text{BD}^{\alpha,\infty}(M, \tilde{M}) \cap \text{BD}^{\alpha,\infty}(M, \tilde{M})$ with $q' = \frac{\alpha}{\alpha-1}$ then $f_* : L^q(\tilde{M}, \Lambda^k) \to L^{q'}(\tilde{M}, \Lambda^k)$ is an isomorphism.

**Proof.** It follows at once from the Propositions 4.2 and 4.1.

## 5. Relation with Quasiconformal and Bilipschitz Diffeomorphisms

Recall that an orientation preserving diffeomorphism\(^1\) $f : (M, g) \to (\tilde{M}, \tilde{g})$, between two oriented $n$-dimensional Riemannian manifolds is said to be *quasiconformal* if

$$\frac{|df|^n}{J_f} \in L^\infty(M).$$

\(^1\)It is usual, and important, to consider not only diffeomorphisms, but more generally homeomorphisms in $W^{1,n}_{loc}$ when defining quasiconformal maps. In our present context, diffeomorphisms are sufficient, see however the discussion in section 8.
Lemma 5.1. For the diffeomorphism \( f : (M, g) \to (\tilde{M}, \tilde{g}) \), the following properties are equivalent

(i.) \( f \) is quasiconformal;
(ii.) \( f^{-1} \) is quasi-conformal;
(iii.) If \( \lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x) \) are the principal dilation coefficients of \( df_x \), then

\[
\sup_{x \in M} \frac{\max\{\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x)\}}{\min\{\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x)\}} < \infty.
\]

The proof of this lemma is standard and easy.

Let us denote by \( QC(M, \tilde{M}) \) the class of all quasiconformal diffeomorphisms, it is clear that \( QC(M, \tilde{M}) = BD_{n,\infty}^1(M, \tilde{M}) \), but, more generally:

**Proposition 5.2.** We have

\[
QC(M, \tilde{M}) = BD_{k,\infty}^k(M, \tilde{M})
\]

for any \( 1 \leq k \leq n - 1 \).

**Proof** Suppose that \( f : (M, g) \to (\tilde{M}, \tilde{g}) \) is quasiconformal. Let us assume that \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \), then by condition (iii) of the previous lemma, there exists a constant \( C \) such that

\[
\sigma_k(f, x) \leq C \cdot (\lambda_1(x))^k.
\]

Since \( J_f = \lambda_1 \cdot \lambda_2 \cdots \lambda_n \), we have

\[
\frac{(\sigma_k(f))^{n/k}}{J_f} \leq C \cdot \left(\frac{\lambda_1^k}{J_f}\right)^{n/k} \leq \frac{C \cdot (\lambda_1)^n}{(\lambda_1 \cdot \lambda_2 \cdots \lambda_n)} \leq C,
\]

i.e. \( f \in BD_{k,\infty}^k(M, \tilde{M}) \). We have thus shown that \( QC(M, \tilde{M}) \subseteq BD_{k,\infty}^k(M, \tilde{M}) \).

To prove the converse inclusion, we distinguish three cases: \( k = \frac{n}{2}, \ 1 \leq k < \frac{n}{2} \) and \( \frac{n}{2} < k < n \).

Let us first assume that \( k = \frac{n}{2} \), then we have

\[
\frac{\lambda_n}{\lambda_1} \leq \frac{(\lambda_{n-k+1} \cdots \lambda_n)}{(\lambda_1 \cdots \lambda_k)} \leq \frac{(\lambda_{n-k+1} \cdots \lambda_n)^2}{(\lambda_1 \cdots \lambda_k)(\lambda_{n-k+1} \cdots \lambda_n)} \leq \frac{(\sigma_k(f))^2}{J_f},
\]

which implies that \( BD_{n,\infty}^{n/2}(M, \tilde{M}) \subseteq QC(M, \tilde{M}) \).

Assume now that \( 1 \leq k < \frac{n}{2} \), i.e. \( k + 1 \leq n - k \). Observe that

\[
(\lambda_{k+1} \cdots \lambda_{n-k}) \leq (\lambda_{n-k})^{n-2k} \leq (\lambda_{n-k+1} \cdots \lambda_n)^{(n-2k)/k},
\]

therefore

\[
J_f = (\lambda_1 \cdot \lambda_2 \cdots \lambda_n)
\]

\[
= (\lambda_1 \cdots \lambda_k)(\lambda_{k+1} \cdots \lambda_{n-k})(\lambda_{n-k+1} \cdots \lambda_n)
\]

\[
\leq (\lambda_1 \cdots \lambda_k)(\lambda_{n-k+1} \cdots \lambda_n)^{n-2k+1}
\]

\[
= (\lambda_1 \cdots \lambda_k)(\lambda_{n-k+1} \cdots \lambda_n)^{n-k}.
\]

Because \( \sigma_k \geq \lambda_{n-k+1} \cdots \lambda_n \), we have from the previous inequality

\[
\frac{(\sigma_k(f))^{n/k}}{J_f} \geq \frac{(\lambda_{n-k+1} \cdots \lambda_n)^{\frac{k}{2}}}{J_f} \geq \frac{(\lambda_{n-k+1} \cdots \lambda_n)^{n-k}}{(\lambda_1 \cdots \lambda_k)}.
\]
Since
\[
\frac{\lambda_{n-k+1}}{\lambda_k}, \frac{\lambda_{n-k+2}}{\lambda_{k-1}}, \ldots, \frac{\lambda_{n-1}}{\lambda_2} \geq 1,
\]
we finally have
\[
\frac{\lambda_n}{\lambda_1} \leq \frac{(\lambda_n \cdots \lambda_{n-k+1})}{(\lambda_1 \cdots \lambda_k)} \leq \frac{(\sigma_k(f))^{n/k}}{J_f},
\]
from which follows that \( BD_{q,\infty}^k(M, \tilde{M}) \subset QC(M, \tilde{M}) \).

If \( k > \frac{n}{q} \), then \( n - k < \frac{n}{q} \) and we have from the previous argument and Proposition 4.2
\[
f^{-1} \in BD_{n-k,\infty}^q(M, M) \subset QC(\tilde{M}, M),
\]
and we deduce from lemma 5.1 that \( f \in QC(M, \tilde{M}) \).

The next result relates our class of maps to bilipschitz ones.

**Proposition 5.3.** If \( f \in BD_{q,\infty}^k(M, \tilde{M}) \cap BD_{n-k,\infty}^{q'}(M, \tilde{M}) \) with \( q' = \frac{q}{q-k} \), then \( f \) is quasiconformal. Furthermore if \( q \neq \frac{n}{k} \), then \( f \) is bilipschitz.

**Proof** Using the same notations and convention as in the previous proof, we have
\[
\frac{\lambda_n}{\lambda_1} \leq \frac{(\lambda_n \cdots \lambda_{n-k+1})}{(\lambda_1 \cdots \lambda_k)} \leq \frac{(\sigma_k(f))^{n/k}}{J_f},
\]
because \( \frac{1}{q} + \frac{1}{q'} = 1 \). It follows from this computation that any map \( f \in BD_{q,\infty}^k(M, \tilde{M}) \cap BD_{n-k,\infty}^{q'}(M, \tilde{M}) \) is quasiconformal.

We now prove that \( f \) is bilipschitz if \( q \neq \frac{n}{k} \). Because \( f \) is quasiconformal, there exists a constant \( c \) such that \( \lambda_n \leq c \cdot \lambda_1 \). Since \( \lambda_1 \leq \sigma_k(f) \) and \( J_f \leq \lambda_n \), we have
\[
|df|^{q-k-n} = \lambda_n^{q-k-n} \leq \frac{(e\lambda_1)^{kq}}{\lambda_n^q} \leq e^{kq} \cdot \frac{(\sigma_k(f))^q}{J_f},
\]
this implies that any quasiconformal map in \( BD_{q,\infty}^k(M, \tilde{M}) \) is lipschitz if \( qk > n \).

If \( qk < n \), then \( q'(n-k) < n \) and the same argument shows that any quasiconformal map in \( BD_{n-k,\infty}^{q'}(M, \tilde{M}) \) is lipschitz. Thus any \( f \in BD_{q,\infty}^k(M, \tilde{M}) \cap BD_{n-k,\infty}^{q'}(M, \tilde{M}) \) with \( q \neq \frac{n}{k} \) is lipschitz. But Proposition 4.2 implies that \( f^{-1} \in BD_{q,\infty}^k(M, M) \cap BD_{n-k,\infty}^{q'}(M, M) \), hence \( f^{-1} \) is also a lipschitz map if \( q \neq \frac{n}{k} \).

**An open question.** The previous result and the Corollary 4.4 suggest the following question: *Suppose a diffeomorphism \( f : M \to \tilde{M} \) induces an isomorphism \( f^* : L^q(M, \Lambda^k) \to L^q(M, \Lambda^k) \). Can we conclude that \( f \) is quasiconformal for \( q = \frac{n}{k} \) and bilipschitz otherwise?*

If \( k = 1 \), the answer to the above question is positive, see [7, 8, 38, 39].
For a more complete discussion of quasiconformal maps in the context of differential forms, we refer to [12].
6. $L_{q,p}$-cohomology and BD-diffeomorphisms.

Combining the results of the two previous sections, we obtain the following theorem.

**Theorem 6.1.** Suppose $p \leq \tilde{p} < \infty$, and let $f : M \to \tilde{M}$ be a diffeomorphism of the class $BD_{(\tilde{p},t)}(M, \tilde{M})$ where $t = \frac{p}{p-\tilde{p}}$. Then the following holds:

A.) $f^* : \Omega^{k-1}_{q,p}(\tilde{M}) \to \Omega^{k-1}_{q,p}(M)$ is a bounded operator and $f^*(Z^k_p(\tilde{M})) \subset Z^k_p(M)$.

B.) If $q \geq \tilde{q} > 1$ and $f \in BD_{(\tilde{p},t)}(M, \tilde{M}) \cap BD_{(\tilde{p},t)}(\tilde{M}, \tilde{M})$ with $\tilde{q}' = \frac{\tilde{q}}{q-1}, r = \frac{q(\tilde{q}'-1)}{q-\tilde{q}}$, then $f^* \omega = 0$ in $H^k_{q,p}(M)$ implies $[\omega] = 0$ in $H^k_{q,p}(\tilde{M})$ (thus $H^k_{q,p}(M) = 0 \Rightarrow H^k_{q,p}(\tilde{M}) = 0$).

C.) If $q \leq \tilde{q}$ and $f \in BD_{(\tilde{p},t)}(M, \tilde{M}) \cap BD_{(\tilde{p},t)}(\tilde{M}, \tilde{M})$ where $u = \frac{q}{q-\tilde{q}}$ and $t = \frac{p}{p-\tilde{p}}$, then

a.) $f^* : \Omega^{k-1}_{q,p}(\tilde{M}) \to \Omega^{k-1}_{q,p}(M)$ is a bounded operator,

b.) $f^* : \Omega^k_{q,p}(\tilde{M}) \to \Omega^k_{q,p}(M)$ is a well defined linear map,

c.) $f^* : \Omega^k_{q,p}(\tilde{M}) \to \Omega^k_{q,p}(M)$ is a bounded operator.

**Proof.** The statement (A) follows immediately from Proposition 4.1 and the fact that $df^* \omega = f^* d\omega$, whereas the assertion (B) follows from Proposition 4.1 Proposition 4.2 and Theorem 3.1. Finally, the property (C) follows from Proposition 4.1 and Theorem 3.2.

Part (C) of the Theorem gives us sufficient conditions on a map $f$ to have a functorial behavior in $L_{q,p}$-cohomology.

7. Some examples

In this section, we show how Theorem 6.1 can be used to produce vanishing and non vanishing results for the $L_{q,p}$-cohomology of some specific manifolds. The calculations can be quite delicate, even for familiar Riemannian manifolds, and here we only give two simple examples, without trying to obtain optimal results.

7.1. A manifold with a cusp. Let us consider the Riemannian manifold $(\tilde{M}, \tilde{g})$ such that $\tilde{M}$ is diffeomorphic to $\mathbb{R}^n$ and $\tilde{g}$ is a Riemannian metric such that in polar coordinates, we have

$$\tilde{g} = dr^2 + e^{-2r} \cdot h$$

for large enough $r$, where $h$ denotes the standard metric on the sphere $S^{n-1}$. Let us also consider the identity map $f : \mathbb{R}^n \to \tilde{M}$, where $\mathbb{R}^n$ is given its standard euclidean metric, which writes in polar coordinates as

$$ds^2 = dr^2 + r^2 \cdot h.$$

**Proposition 7.1.** If $s > \frac{n-1}{m-1}$, then the above map $f : \mathbb{R}^n \to \tilde{M}$ belongs to the class $BD_{s,t}(\mathbb{R}^n, \tilde{M})$ for any $0 < t \leq \infty$. 
Proof For \( r \) large enough, we have the following principal dilatation coefficients for \( f \):

\[
\lambda_1 = 1, \quad \lambda_2 = \lambda_3 = \cdots = \lambda_n = \frac{e^{-r}}{r}.
\]

In particular \( J_f = \left( \frac{e^{-r}}{r} \right)^{n-1} \) and

\[
\sigma_m(f) = \left( \frac{e^{-r}}{r} \right)^m + \left( \frac{n-1}{m-1} \right) \left( \frac{e^{-r}}{r} \right)^{m-1} \leq C_1 \left( \frac{e^{-r}}{r} \right)^{m-1}.
\]

and thus

\[
\frac{(\sigma_m(f))^s}{J_f} \leq C_2 \left( \frac{e^{-r}}{r} \right)^{s(m-1)-(n-1)}
\]

outside a compact set in \( \mathbb{R}^n \). Therefore \( \int_{\mathbb{R}^n} \left( \frac{\sigma_m(f)}{J_f} \right)^t \, dx < \infty \) if and only if

\[
\int_{1}^{\infty} \left( \frac{e^{-r}}{r} \right)^{t(s(m-1)-(n-1))} \cdot r^{n-1} \, dr < \infty
\]

which is the case when \( s \geq \frac{n-1}{k-1} \). This implies that \( f \in \text{BD}^m_{s,t}(\mathbb{R}^n, \tilde{M}) \) for any \( 0 < t < \infty \).

It is also clear that \( f \in \text{BD}^m_{s,\infty}(\mathbb{R}^n, \tilde{M}) \), since \( \frac{\sigma_m(f)}{J_f} \) is bounded when \( s \geq \frac{n-1}{m-1} \).

Corollary 7.2. If \( \tilde{q} < \frac{n-1}{k-1} < \tilde{p} \), then \( H^k_{\tilde{q},\tilde{p}}(\tilde{M}) = 0 \).

Proof

We will use Theorem 6.1(B) with the previous Proposition. We have \( f \in \text{BD}^k_{\tilde{p},1}(\mathbb{R}^n, \tilde{M}) \) for any \( t > 0 \), since we have \( \tilde{p} > \frac{n-1}{k-1} \) by hypothesis. We also have \( f \in \text{BD}^n_{\tilde{q}, \tilde{r}}(\mathbb{R}^n, \tilde{M}) \) if \( \tilde{q} > \frac{n}{n-k} \). But this inequality is equivalent to

\[
\tilde{q} = \frac{\tilde{q}}{\tilde{q} - 1} < \frac{n-1}{k-1},
\]

and this also holds by hypothesis. We thus have \( f \in \text{BD}^n_{\frac{\tilde{q}}{\tilde{q} - 1}, \tilde{r}}(\mathbb{R}^n, \tilde{M}) \cap \text{BD}^k_{\tilde{p},1}(\mathbb{R}^n, \tilde{M}) \) for any \( \tilde{q} < \frac{n}{k-1} < \tilde{p} \).

Let us now set \( p = \frac{n}{k} \) and \( q = \frac{n}{k-1} \), and observe that \( p \leq \frac{n-1}{k-1} \), hence \( p \leq \tilde{p} \) and \( q \geq \frac{n-1}{k-1} \), hence \( \tilde{q} \geq q \).

In [30], it is proved that \( H^k_{p,q}(\mathbb{R}^n) \neq 0 \) if \( p = \frac{n}{k} \) and \( q = \frac{n}{k-1} \). Therefore by Theorem 6.1 we have \( H^k_{\tilde{q},\tilde{p}}(\tilde{M}) = 0 \) for any \( \tilde{q} < \frac{n-1}{k-1} < \tilde{p} \).

\( \square \)

7.2. The hyperbolic space. Let us denote by \( \mathbb{H}^n \) the hyperbolic space of dimension \( n \). Recall that \( \mathbb{H}^n \) can be described in polar coordinate as follow:

\[
\mathbb{H}^n = [0, \infty) \times S^{n-1}/\{0\} \times S^{n-1},
\]

with the Riemannian metric

\[
g = dr^2 + \sinh(r)^2 h,
\]
where \( h \) is the standard metric on the sphere \( S^{n-1} \). Likewise, the euclidean space \( \mathbb{R}^n \) is given by \( \mathbb{R}^n = [0, \infty) \times S^{n-1}/\{0\} \times S^{n-1} \), with the Riemannian metric \( ds^2 = dr^2 + r^2 h \).

Let us consider the identity map \( f : \mathbb{H}^n \to \mathbb{R}^n \) (which is, from an intrinsic viewpoint, the inverse of the exponential map \( \exp : T_x \mathbb{H}^n = \mathbb{R}^n \to \mathbb{H}^n \)).

**Proposition 7.3.** The above map \( f : \mathbb{H}^n \to \mathbb{R}^n \) belongs to the class \( BD_{s,t}(\mathbb{H}^n, \mathbb{R}^n) \) for \( 1 \leq s < \infty, \ 0 < t \leq \infty \) if and only if

\[
(7.1)\quad s > \frac{n-1}{m-1} \left( 1 + \frac{1}{t} \right)
\]

and belongs \( BD_{s,\infty}(\mathbb{H}^n, \mathbb{R}^n) \) if and only if

\[
(7.2)\quad s \geq \frac{n-1}{m-1}.
\]

**Proof** We clearly have the following principal dilatation coefficients for \( f \):

\[
\lambda_1 = 1, \quad \lambda_2 = \lambda_2 = \cdots = \lambda_n = \frac{r}{\sinh(r)}.
\]

Therefore

\[
\sigma_m(f) = \left( \frac{r}{\sinh(r)} \right)^m + \left( \frac{r}{\sinh(r)} \right)^{m-1} \leq \text{const.} \left( \frac{r}{\sinh(r)} \right)^{m-1},
\]

and thus

\[
\frac{(\sigma_m(f))^s}{J_f} \leq C \left( \frac{r}{\sinh(r)} \right)^{s(m-1)-(n-1)}.
\]

It follows that \( f \in BD_{s,\infty}(\mathbb{H}^n, \mathbb{R}^n) \) if and only if \( s \geq \frac{n-1}{m-1} \). Likewise, \( f \in BD_{s,t}(\mathbb{H}^n, \mathbb{R}^n) \) for some \( t < \infty \) when the integral

\[
\int_{\mathbb{H}^n} \left( \frac{(\sigma_m(f))^s}{J_f} \right)^t \leq \text{const.} \int_0^\infty \left( \frac{r}{\sinh(r)} \right)^{t(s(m-1)-(n-1))} \cdot (\sinh(r))^{-n-1} dr
\]

is finite. This is the case if and only if

\[
t(s(m-1)-(n-1)) > (n-1).
\]

And this inequality is equivalent to (7.1).

\[\square\]

**Corollary 7.4.** If \( q < \frac{n-1}{k-1} < p \), then \( H_{q,p}^k(\mathbb{H}^n) \neq 0 \).

**Proof** We will use Theorem 6.1 with the previous Proposition. We have \( f \in BD_{(\tilde{p},\tilde{q})}^k(\mathbb{H}^n, \mathbb{R}^n) \) with \( \tilde{p} = \frac{p}{\overline{p}} \) if and only if

\[
\tilde{p} > \frac{n-1}{k-1} \left( 1 + \frac{1}{t} \right) = \frac{n-1}{k-1} \left( 1 + \frac{\tilde{p} - p}{p} \right) = \frac{n-1}{k-1} \cdot \tilde{p}.
\]

i.e.

\[
p > \frac{n-1}{k-1}.
\]

Likewise, \( f \in BD_{(\tilde{q},\tilde{r})}^{n-k+1}(\mathbb{H}^n, \mathbb{R}^n) \) with \( \tilde{q} = \frac{\tilde{q}}{\overline{q}}, \tilde{r} = \frac{\tilde{q}(\tilde{q}-1)}{\overline{q}-\overline{r}} \) if and only if

\[
\tilde{q} > \frac{\tilde{q}}{\overline{q}-1} > \frac{n-1}{n-k} \left( 1 + \frac{1}{r} \right) = \frac{n-1}{n-k} \left( 1 + \frac{q - \tilde{q}}{q - \tilde{q}} \right).
\]
This inequality is equivalent to
\[ \tilde{q} > \frac{n - 1}{n - k} \left( (\tilde{q} - 1) + \frac{q - \tilde{q}}{q} \right) = \frac{n - 1}{n - k} \left( 1 - \frac{1}{q} \right) \tilde{q}, \]
or, finally
\[ q < \frac{n - 1}{k - 1}. \]

We proved that \( f \in \text{BD}^{n-k+1}(\tilde{q}', r) \cap \text{BD}^k(\tilde{p}, t) \) whenever
\[ (7.3) \quad q < \frac{n - 1}{k - 1} < p, \quad p \leq \tilde{p}, \quad q \geq \tilde{q}, \quad t = \frac{p}{p - p}, \quad r = \frac{q(\tilde{q} - 1)}{q - \tilde{q}}. \]

In [36], it is proved that \( H^k_{\tilde{q}, \tilde{p}}(\mathbb{R}^n) \neq 0 \) unless \( \frac{\tilde{q} - 1}{\tilde{p}} = \frac{1}{n} \). Therefore one can choose some values of \( \tilde{p}, \tilde{q} \) compatible with the conditions (7.3) and use Theorem 6.1 to conclude that \( H^k_{q, p}(\mathbb{H}^n) \neq 0 \) for any \( q < \frac{n - 1}{k - 1} < p \).

The result given in the previous Theorem is not optimal and we shall discuss the \( L_{q,p} \)-cohomology of the hyperbolic space and other manifolds with negative curvature in a another paper.

8. Non-smooth mappings

We have formulated our results for diffeomorphisms, but it is clear that the Definition 4.1 makes sense for wider classes of maps such as Sobolev maps in \( W^{1,1}_{loc} \) or maps which are approximately differentiable almost everywhere, we can thus consider the class of \( W^{1,1}_{loc} \) homeomorphisms with bounded mean distortion. It is then natural and important to wonder whether our results still hold in this wider context.

Unfortunately, there is no elementary answer to this question. A careful look at our arguments show that we have used the following properties of diffeomorphisms:

i.) The change of variables formula in integrals: \( \int_M u(f(x)) J_f(x) dx = \int_{\tilde{M}} u(y) dy \) in Proposition 4.1.

ii.) The change of variables formula for the inverse map: \( \int_M u(f(x)) J_f(x) dx = \int_{\tilde{M}} u(y) dy \), this is implicitly used in Corollary 4.3.

iii.) The naturality of the exterior differential \( df^*\omega = f^*d\omega \) is used everywhere.

The change of variables formula in integrals holds for a homeomorphism \( f \) in \( W^{1,1}_{loc} \) provided we assume the Luzin \( (N) \) condition to hold. This condition states that a subset of zero measure in \( M \) is mapped by \( f \) onto a set of zero measure in \( \tilde{M} \). The map change of variables formula for the inverse map \( f^{-1} \) holds if the Luzin \( (N^{-1}) \) condition holds, that is the inverse image of subset of zero measure also has zero measure. The Luzin condition is widely studied in the literature (see, for example, [38, 16, 18, 19]). Concerning the naturality of the exterior differential, we refer to [13].

Let finally mention that for the special case of quasiconformal mapping, all these properties hold. The relation between the theory of quasiconformal mappings and \( L_{qp} \)-cohomology is studied in [12].
References

[1] Lelong-Ferrand L. Etude d’une classe d’applications liées à des homomorphismes d’algèbres de fonctions et généralisant les quasi-conformes. Duke Math. J. 40, (1973) 163-186.

[2] Gafaïti K. Algèbre de Royden et Homomorphismes à p-dilatation bornée entre espaces métriques mesurés. Thèse, EPFL Lausanne (2001).

[3] Gol’dshein V.M., Reshetnyak Yu.G., Quasiconformal Mappings and Sobolev Spaces, Kluwer Academic Publishers, Dordrecht/Boston/London, 1990.

[4] Gol’dshein V.M., Kuz’minov V.I., Sobolev Inequality for Differential forms on Lipschitz Manifolds, Siberian Math. Journal, 23, No 2 (1982), 16-30. English translation in: Siberian Math. J. 23, No 2 (1982), 151-161.

[5] Gol’dshein V.M., Kuz’minov V.I., Shvedov I.A., Lp,q-cohomology of warped cylinder, Siberian Math. Journal, 31, No 6 (1990), 55-63. English translation in: Siberian Math. J. 31, No 6 (1990), 716-727.

[6] V. M. Gol’dshein, V.I. Kuz’m inov, I.A.Shvedov Dual spaces of Spaces of Differential Forms Siberian Math. Journal, (1986), 54, No 1, 35-43.

[7] Gol’dshein V., Gurov L., Romanov A., Homeomorphisms that induce Monomorphisms of Sobolev Spaces, Israel Journal of Math. (1995), 91, No 1, 31-60.

[8] Gol’dshein V., Gurov L., Applications of change of variable operators for exact embedding theorems, Integral equations and operator theory. (1994), 19, No 1, 1-24.

[9] Gol’dshein V.M., Romanov A.S., Transformations that preserve Sobolev spaces, (1984), 25, No 3, 382-388.

[10] Gol’dshein V. and Troyanov M. The Lpq-cohomology of SOL. Annales de la Faculté des Sciences de Toulouse. Vol. VII, No 4, 1998.

[11] Gol’dshein V. and Troyanov M., Sobolev Inequality for Differential forms and Lp,q-cohomology, Journal of Geom. Anal.(2006), 16, No 4, 597-631.

[12] Gol’dshein V. and Troyanov M., A conformal de Rham complex. arXiv:0711.1286

[13] Gol’dshein V. and Troyanov M., On the naturality of exterior differential forms on Lipschitz Manifolds, Siberian Math. Journal, 31, No 6 (1990), 716-727.

[14] Gol’dshein V. and Troyanov M. The Lpq-cohomology of SOL. Annales de la Faculté des Sciences de Toulouse. Vol. VII, No 4, 1998.

[15] Gol’dshein V. and Troyanov M., Sobolev Inequality for Differential forms and Lp,q-cohomology, Journal of Geom. Anal.(2006), 16, No 4, 597-631.

[16] Gol’dshein V. and Troyanov M., A conformal de Rham complex. arXiv:0711.1286

[17] Gol’dshein V. and Troyanov M., On the naturality of exterior differential forms on Lipschitz Manifolds, Siberian Math. Journal, 31, No 6 (1990), 716-727.

[18] Gol’dshein V. and Troyanov M. The Lpq-cohomology of SOL. Annales de la Faculté des Sciences de Toulouse. Vol. VII, No 4, 1998.

[19] Gol’dshein V. and Troyanov M., Sobolev Inequality for Differential forms and Lp,q-cohomology, Journal of Geom. Anal.(2006), 16, No 4, 597-631.

[20] Gol’dshein V. and Troyanov M., A conformal de Rham complex. arXiv:0711.1286

[21] Gol’dshein V. and Troyanov M., On the naturality of exterior differential forms on Lipschitz Manifolds, Siberian Math. Journal, 31, No 6 (1990), 716-727.
[31] Pansu P. Cohomologie $L^p$ et pincement. Comment. Math. Helvetici. (to appear)
[32] Reiman M. Über harmonische Kapazität und quasikonforme Abbildungen in Raum. Comm. Math. Helv. 44 (1969) 284–307.
[33] Reshetnyak Yu.G. Space Mappings with Bounded distortion, Translations of Mathematical Monographs, (1985), 73, American Mathematical Society.
[34] Rickman S. Quasiregular mapping Springer-Verlag, Berlin-Heidelberg-New York, 1993.
[35] Troyanov M. and Vodop'yanov S.K. Liouville type theorem for mappings with bounded co-distortion. Annales de l’Institute Fourier. (2002), 52, No 6, 1754-1783.
[36] Troyanov M. On the Hodge decomposition in $\mathbb{R}^n$. arXiv:0710.5414
[37] Vodop'yanov S.K. Topological and geometrical properties of mappings with an integrable Jacobian in Sobolev classes, Siberian math. J., (2000), 41, no 4., 19-39.
[38] Vodop'yanov S.K., Gol’dshtein V.M., Quasiconformal mappings and spaces of mappings with generalized first derivatives, Siberian math. J., (1976), 17, no 3, 515-531.
[39] Vodop'yanov S.K., Ukhlov A.D. Sobolev spaces and $(p,q)$-quasiconformal mappings of Carnot groups. Siberian Math. J. (1998) 39, No 4, 776–795.

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