The product midpoint rule for Abel-type Volterra integral equations of the first kind with perturbed data

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Abstract

In the present paper we consider the regularizing properties of the product midpoint rule for the stable solution of Abel-type Volterra integral equations of the first kind with perturbed right-hand sides. The Hölder continuity of the solution and its derivative is carefully taken into account, and correction weights are considered to get rid of initial conditions. The proof of the inverse stability of the quadrature weights relies on Banach algebra techniques. Finally, numerical results are presented.

Key words. Weakly singular Volterra integral equation of the first kind; Abel integral operator; quadrature method; product integration; midpoint rule; Wiener’s theorem; Banach algebra; inverse-closed; noisy data; parameter choice strategy.

1 Introduction

1.1 Preliminary remarks

In this paper we consider linear Abel-type Volterra integral equations of the following form,

\[(Au)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - y)^{\alpha - 1} k(x, y) u(y) \, dy = f(x) \quad \text{for} \ 0 \leq x \leq a,\]

\[(1.1)\]

with \(0 < \alpha < 1\) and \(a > 0\), and with a sufficiently smooth kernel function \(k : \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq x \leq a\} \rightarrow \mathbb{R}\), and \(\Gamma\) denotes Euler’s gamma function. Moreover, the function \(f : [0, a] \rightarrow \mathbb{R}\) is supposed to be approximately given, and a function \(u : [0, a] \rightarrow \mathbb{R}\) satisfying equation (1.1) is to be determined.

In the sequel we suppose that the kernel function does not vanish on the diagonal \(0 \leq x = y \leq a\), and without loss of generality we may assume that

\[k(x, x) = 1 \quad \text{for} \ 0 \leq x \leq a\]

\[(1.2)\]

holds.

For the approximate solution of equation (1.1) with an exactly given right-hand side \(f\), there exist many quadrature methods, see e.g., Brunner/van der Houwen [3], Linz [17], and Hackbusch [12]. One of these methods is the product midpoint rule which is considered in detail, e.g., in Weiss and Anderssen [29] and in Eggermont [7], see also [17, Section 10.4].

In the present paper we investigate, for perturbed right-hand sides in equation (1.1), the regularizing properties of the product midpoint rule. The smoothness of the solution is classified in terms of Hölder continuity of the function and its derivative is considered. We also give a new proof of the inverse stability of the quadrature weights which relies on Banach algebra techniques and may be of independent interest. Finally, some numerical illustrations are presented.

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1.2 The Abel integral operator

As a first step we consider in (1.3) the special situation \( k \equiv 1 \). On the other hand, for technical reasons we allow arbitrary intervals \([0, b]\) with \( 0 < b \leq a \) instead of the fixed interval \([0, a]\) which allows to extend the obtained results for arbitrary kernels \( k \).

The resulting integral operator is the Abel integral operator

\[
(\mathcal{A}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x - y)^{\alpha - 1} \varphi(y) \, dy \quad \text{for} \quad 0 \leq x \leq b,
\]

where \( \varphi : [0, b] \to \mathbb{R} \) is supposed to be a piecewise continuous function. One of the basic properties of the Abel integral operator is as follows,

\[
(\mathcal{A}^{\alpha}y^q)(x) = \frac{\Gamma(q+1)}{\Gamma(q+1+\alpha)} x^{q+\alpha} \quad \text{for} \quad x \geq 0 \quad (q \geq 0),
\]

where \( y^q \) is short notation for the mapping \( y \mapsto y^q \). In the sequel, frequently we make use of the following elementary estimate:

\[
\sup_{0 \leq x \leq b} |(\mathcal{A}^{\alpha}\varphi)(x)| \leq \frac{b^\alpha}{\Gamma(\alpha+1)} \sup_{0 \leq x \leq b} |\varphi(y)| \quad (\varphi : [0, b] \to \mathbb{R} \text{ piecewise continuous}).
\]

Other basic properties of the Abel integral operator can be found e.g., in Gorenflo and Vessella [11] or Hackbusch [12].

2 The product midpoint rule for Abel integrals

2.1 The method

For the numerical approximation of the Abel integral operator (1.3) we introduce equidistant grid points

\[
x_n = nh, \quad n = k/2, \quad k = 0, 1, \ldots, 2N, \quad \text{with} \quad h = \frac{a}{N},
\]

where \( N \) is a positive integer. For a given continuous function \( \varphi : [0, x_n] \to \mathbb{R} \) \((n \in \{1, 2, \ldots, N\})\), the product midpoint rule for the numerical approximation of the Abel integral \((\mathcal{A}^{\alpha}\varphi)(x_n)\) is obtained by replacing the function \( \varphi \) on each subinterval \([x_{j-1}, x_j]\), \( j = 1, 2, \ldots, n \), by the constant term \( \varphi(x_{j-1/2}) \), respectively:

\[
(\mathcal{A}^{\alpha}\varphi)(x_n) \approx \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{n} \left\{ \int_{x_{j-1}}^{x_j} (x_n - y)^{\alpha - 1} \, dy \right\} \varphi(x_{j-1/2})
\]

\[
= \frac{1}{\Gamma(\alpha+1)} \sum_{j=1}^{n} \left\{ (x_n - x_{j-1})^{\alpha} - (x_n - x_j)^{\alpha} \right\} \varphi(x_{j-1/2})
\]

\[
= \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=1}^{n} \left\{ (n - j + 1)^{\alpha} - (n - j)^{\alpha} \right\} \varphi(x_{j-1/2})
\]

\[
= h^{\alpha} \sum_{j=1}^{n} \omega_{n-j} \varphi(x_{j-1/2}) =: (\Omega_{n}^{\alpha}\varphi)(x_n),
\]

where the quadrature weights \( \omega_0, \omega_1, \ldots \) are given by

\[
\omega_s = \frac{1}{\Gamma(\alpha+1)} \left\{ (s + 1)^{\alpha} - s^{\alpha} \right\} \quad \text{for} \quad s = 0, 1, \ldots.
\]

The weights have the asymptotic behavior \( \omega_n = \frac{1}{\Gamma(\alpha)} n^{\alpha - 1} + O(n^{\alpha - 2}) \) as \( n \to \infty \).
2.2 The integration error – preparations

In the sequel, we consider the integration error

\[(E_n^a \varphi)(x_n) = (V^a \varphi)(x_n) - (\Omega_n^a \varphi)(x_n)\]  

(2.5)

under different smoothness assumptions on the function \( \varphi \). As a preparation, for \( c < d, L \geq 0, m = 0, 1, \ldots \)
and \( 0 < \beta \leq 1 \), we introduce the space \( F^{m+\beta}_{L}[c, d] \) of all functions \( \varphi : [c, d] \rightarrow \mathbb{R} \) that are continuously

differentiable up to order \( m \), and the derivative \( (\phi^{(m)}) \) of order \( m \) is Hölder continuous of order \( \beta \) with Hölder

contant \( L \geq 0 \), i.e.,

\[F^{m+\beta}_{L}[c, d] = \{ \varphi \in C^m[c, d] \mid |(\varphi^{(m)}(x) - \varphi^{(m)}(y))| \leq L|x - y|^\beta \quad \text{for} \quad x, y \in [c, d] \}.\]  

(2.6)

The space of Hölder continuous functions of order \( m + \beta \) on the interval \([c, d]\) is then given by

\[F^{m+\beta}[c, d] = \{ \varphi : [c, d] \rightarrow \mathbb{R} \mid \varphi \in F_{L}^{m+\beta}[c, d] \text{ for some constant } L \geq 0 \}.\]

Other notations for those spaces are quite common, e.g., \( C^{m,\beta}[c, d] \), cf. [2] section 2.

As a preparation, for \( n \in \{1, 2, \ldots, N\} \) and \( \varphi : [0, x_n] \rightarrow \mathbb{R} \) we introduce the corresponding piecewise constant interpolating spline \( p_n \varphi : [0, x_n] \rightarrow \mathbb{R} \), i.e.,

\[(p_n \varphi)(y) = \varphi(x_{j-1}/2) \quad \text{for} \quad x_{j-1} \leq y < x_j \quad (j = 1, 2, \ldots, n),\]  

(2.7)

and in the latter case \( j = n \), this setting is also valid for \( y = x_n \). For \( \varphi \in F^{\gamma}[0, x_n] \) with \( 0 < \gamma \leq 1 \), it follows from zero order Taylor expansions at the grid points that

\[\varphi(y) = (p_n \varphi)(y) + O(h^\gamma), \quad 0 \leq y \leq x_n,\]  

(2.8)

uniformly both on \([0, x_n]\) and for \( \varphi \in F_{L}^{\gamma}[0, x_n] \), with any arbitrary but fixed constant \( L \geq 0 \), and also uniformly for \( n = 1, 2, \ldots, N \).

We consider the smooth case \( \varphi \in C^1[0, x_n], n \in \{1, 2, \ldots, N\} \), next. Let \( q_n \varphi : [0, x_n] \rightarrow \mathbb{R} \) be given by

\[(q_n \varphi)(y) = \varphi(x_{j-1}/2) + (y - x_{j-1}/2) \varphi'(x_{j-1}/2) \quad \text{for} \quad x_{j-1} \leq y < x_j \quad (j = 1, 2, \ldots, n),\]  

(2.9)

and in the latter case \( j = n \), this definition is extended to the case \( y = x_n \). For \( \varphi \in F^{\gamma}[0, x_n] \) with \( 1 < \gamma \leq 2 \), first order Taylor expansions at the grid points yield

\[\varphi(y) = (q_n \varphi)(y) + O(h^\gamma), \quad 0 \leq y \leq x_n,\]  

(2.10)

uniformly in the same manner as for (2.8).

2.3 The integration error

We are now in a position to consider, under different smoothness conditions on the function \( \varphi \), representations for the integration errors \( (E_n^a \varphi)(x_n) \) introduced in (2.5).

Lemma 2.1 Let \( n \in \{1, 2, \ldots, N\} \), and moreover let \( \varphi : [0, x_n] \rightarrow \mathbb{R} \) be a continuous function. We have the following representations for the quadrature error \( (E_n^a \varphi)(x_n) \) introduced in (2.5):

(a) We have

\[(E_n^a \varphi)(x_n) = (V^a (\varphi - p_n \varphi))(x_n).\]  

(2.11)

(b) For \( \varphi \in C^1[0, x_n] \) we have

\[(E_n^a \varphi)(x_n) = h^{a+1} \sum_{j=1}^{n} \tau_{n-j} \varphi'(x_{j-1}/2) + (V^a (\varphi - q_n \varphi))(x_n),\]  

(2.12)

where

\[\tau_n = \frac{1}{\Gamma(a+2)} ( (n+1)^{a+1} - n^{a+1} ) - \frac{1}{2\Gamma(a+1)} ( (n+1)^{a} + n^{a} ) \text{ for } n = 0, 1, \ldots .\]  

(2.13)
The grid points \(x_n\) are approximated by the product midpoint rule, respectively, see (2.3) with \(k\) has Lipschitz continuous partial derivatives up to the order 2.

Assumption 3.1 (a) There exists a solution \(u : [0, a] \to \mathbb{R}\) to the integral equation (1.1) with \(u \in F^\gamma [0, a]\), where \(c_\alpha := \min \{\alpha, 1 - \alpha\} < \gamma \leq 2\).

(b) There holds \(k(x, x) = 1\) for each \(0 \leq x \leq a\).

(c) The kernel function \(k\) has Lipschitz continuous partial derivatives up to the order 2.

(d) The grid points \(x_n\) are given by (2.1).

(e) The values of the right-hand side of equation (1.1) are approximately given at the grid points, cf. (3.1).
3.2 Formal power series

As a preparation for the proof of the main stability result of the present paper, cf. Theorem 3.3, we next consider power series. In what follows, we identify sequences \((b_n)_{n \geq 0}\) of complex numbers with their (formal) power series \(b(\xi) = \sum_{n=0}^{\infty} b_n \xi^n\), with \(\xi \in \mathbb{C}\). Pointwise multiplication of two power series

\[
\left( \sum_{\ell=0}^{\infty} b_\ell \xi^\ell \right) \cdot \left( \sum_{j=0}^{\infty} c_j \xi^j \right) = \sum_{n=0}^{\infty} d_n \xi^n, \quad \text{with} \quad d_n := \sum_{\ell=0}^{n} b_\ell c_{n-\ell} \quad \text{for} \quad n = 0, 1, \ldots
\]

makes the set of power series into a complex commutative algebra with unit element \(1 + 0 \cdot \xi + 0 \cdot \xi^2 + \cdots\). For any power series \(b(\xi) = \sum_{n=0}^{\infty} b_n \xi^n\) with \(b_0 \neq 0\), there exists a power series which inverts the power series \(b\) with respect to pointwise multiplication, and it is denoted by \(1/b(\xi)\) or by \([b(\xi)]^{-1}\). For a thorough introduction to formal power series see, e.g., Henrici [14].

In what follows, we consider the inverse

\[
[\omega(\xi)]^{-1} = \sum_{n=0}^{\infty} \omega_n^{-1} \xi^n
\]

of the generating function \(\omega(\xi) = \sum_{n=0}^{\infty} \omega_n \xi^n\), with \(\omega_n\) as in (2.4).

**Lemma 3.2** The coefficients in (3.3) have the following properties:

\[
\omega_0^{-1} > 0, \quad \omega_n^{-1} < 0 \quad \text{for} \quad n = 1, 2, \ldots, \quad (3.4)
\]

\[
\omega_0^{-1} = \Gamma(\alpha + 1) = \sum_{n=1}^{\infty} |\omega_n^{-1}|, \quad (3.5)
\]

\[
\omega_n^{-1} = \mathcal{O}(n^{-\alpha-1}) \quad \text{as} \quad n \to \infty. \quad (3.6)
\]

Estimate (3.6) can be found in [7]. Another proof of (3.6) which uses Banach algebra theory and may be of independent interest is given in section 6 of the present paper. Section 6 also contains proofs of the other statements in Lemma 3.2.

Lemma 3.2 is needed in the proof of our main result, cf. Theorem 3.3 below and section 7. We state the lemma here in explicit form since it is fundamental in the stability estimates.

3.3 The main result

We next present the first main result of this paper, cf. the following theorem, where different situations on the smoothness of the solution \(u\) are considered. For comments on the estimates presented in the theorem, see Remark 3.4 below.

**Theorem 3.3** Let the conditions of Assumption 3.1 be satisfied, and consider the approximations \(u_{1/2}^\delta, u_{1/2}^\delta, \ldots, u_{N-1/2}^\delta\) determined by scheme (3.2). Let \(c_\alpha := \min\{\alpha, 1 - \alpha\}\).

(a) If \(c_\alpha < \gamma \leq 1 + c_\alpha\), then we have

\[
\max_{n=1, 2, \ldots, N} |u_{n-1/2}^\delta - u(x_{n-1/2})| = \mathcal{O}(h^{\gamma-c_\alpha + \frac{\delta}{h^{\alpha}}} \quad \text{as} \quad (h, \delta) \to 0. \quad (3.7)
\]

(b) Let \(2 - \alpha < \gamma \leq 2\), and in addition let \(u(0) = u'(0) = 0\) be satisfied. Then

\[
\max_{n=1, 2, \ldots, N} |u_{n-1/2}^\delta - u(x_{n-1/2})| = \mathcal{O}(h^{\gamma-1+\alpha} + \frac{\delta}{h^{\alpha}} \quad \text{as} \quad (h, \delta) \to 0. \quad (3.8)
\]

The proof of Theorem 3.3 is given in section 7. Below we give some comments on Theorem 3.3.
Remark 3.4 (a) In the case $\alpha \leq \frac{1}{2}$ we have the following estimates:

$$\max_{n=1,2,\ldots,N} |u_n^\delta - u(x_{n-1/2})| = \begin{cases} O(h^{\gamma - \alpha} + \frac{\delta}{h^\alpha}), & \text{if } \alpha < \gamma \leq \alpha + 1, \\ O(h^{\gamma - 1 + \alpha} + \frac{\delta}{h^\alpha}), & \text{if } 2 - \alpha < \gamma \leq 2, \ u(0) = u'(0) = 0. \end{cases}$$

(b) In the case we $\alpha \geq \frac{1}{2}$ the following estimates hold:

$$\max_{n=1,2,\ldots,N} |u_n^\delta - u(x_{n-1/2})| = O(h^{\gamma - 1 + \alpha} + \frac{\delta}{h^\alpha}),$$

if $1 - \alpha < \gamma \leq 2 - \alpha$, or if $2 - \alpha < \gamma \leq 2$, $u(0) = u'(0) = 0$.

(c) The noise-free rates, obtained for $\gamma = 1$ and $\gamma = 2$, basically coincide with those given in the papers by Weiss and Anderssen [29] and by Eggermont [8].

d) The maximal rate in the noise-free case $\delta = 0$ is $O(h)$ without initial conditions, and it is obtained for $\gamma = 1 + c_n$. This rate is indeed maximal, which can be seen by considering the error at the first grid point $x_{1/2}$, obtained for the function $u(y) = y$, cf. Weiss and Anderssen [29]. Under the additional assumption $u(0) = u'(0) = 0$, the maximal rate is $O(h^{\alpha + 1})$, obtained for $\gamma = 2$.

(e) It is not clear if the presented rates are optimal. \(\square\)

In what follows, for step sizes $h = a/N$ we write, with a slight abuse of notation, $h \sim \delta^\beta$ as $\delta \to 0$, if there exist real constants $c_2 \geq c_1 > 0$ such that $c_1 h \leq \delta^\beta \leq c_2 h$ holds for $\delta \to 0$. As an immediate consequence of Theorem 3.3 we obtain the following main result of this paper.

Corollary 3.5 Let Assumption [27] be satisfied.

- Let $\alpha \leq 1/2$ and $\alpha < \gamma \leq \alpha + 1$. For $h = h(\delta) \sim \delta^{1/\gamma}$ we have

$$\max_{n=1,2,\ldots,N} |u_n^\delta - u(x_{n-1/2})| = O(\delta^{1-\alpha/\gamma}) \quad \text{as } \delta \to 0.$$  

- Let one of the following two conditions be satisfied: (a) $\alpha \geq 1/2$, $1 - \alpha < \gamma \leq 2 - \alpha$, or (b) $\gamma > 2 - \alpha$, $u(0) = u'(0) = 0$. Then for $h = h(\delta) \sim \delta^{1/(\gamma-1+2\alpha)}$ we have

$$\max_{n=1,2,\ldots,N} |u_n^\delta - u(x_{n-1/2})| = O\left(\delta^{1-\frac{\alpha}{\gamma-1+2\alpha}}\right) \quad \text{as } \delta \to 0.$$  

Note that in the case $\alpha < \frac{1}{2}$ for the class of functions satisfying the initial conditions $u(0) = u'(0) = 0$, there is a gap for $\alpha + 1 < \gamma \leq 2 - \alpha$ where no improvement in the rates is obtained, i.e., we have piecewise saturation $O(\delta^{1-\alpha/\gamma})$ for this range of $\gamma$. This is due to the different techniques used in the proof of Theorem 3.3.

We conclude this section with some more remarks.

Remark 3.6 (a) We mention some results on other quadrature schemes for the approximate solution of Abel-type integral equations of the first kind. The product trapezoidal method is considered, e.g., in Weiss [28], Eggermont [8], and in [21]. Fractional multistep methods are treated in Lubich [18, 19] and in [20]. Backward difference product integration methods are considered in Cameron and McKee [5, 6].

Galerkin methods for Abel-type integral equations are considered, e.g., in Eggermont [8] and in Vogeli, Nedaial and Sauter [27]. Some general references are already given in the beginning of this paper.

(b) For other special regularization methods for the approximate solution of Volterra integral equations of the first kind with perturbed right-hand sides and with possibly algebraic-type weakly singular kernels, see e.g., Bughgeim [4], Gorenflo and Vessella [11], and the references therein.
Remark 3.7 The results of Theorem 3.3 and Corollary 3.5 can be extended to linear Volterra integral equations of the first kind with smooth kernels, that is, for $\alpha = 1$. The resulting method is in fact the classical midpoint rule, and the main error estimate is as follows: if $0 < \gamma \leq 2$, then we have

$$
\max_{n=1,2,\ldots,N} |u_{n-1/2}^\delta - u(x_{n-1/2})| = \mathcal{O}(h^\gamma + \delta) \quad \text{as} \quad (h, \delta) \to 0,
$$

and initial conditions are not required anymore then. The choice $h = h(\delta) \sim \delta^{1/(\gamma+1)}$ then gives

$$
\max_{n=1,2,\ldots,N} |u_{n-1/2}^\delta - u(x_{n-1/2})| = \mathcal{O}(\delta^{\gamma/(\gamma+1)}) \quad \text{as} \quad \delta \to 0.
$$

The proof follows the lines used in this paper, with a lot of simplifications then. In particular, the inverse stability results derived in section 6 can be discarded in this case. We leave the details to the reader and indicate the basic ingredients only: we have $\omega = 1$ and $\tau_n = 0$ for $n = 0, 1, \ldots$, then, and in addition, $\omega_0(-1) = 1, \omega_1(-1) = -1$, and $\omega_n(-1) = 0$ for $n = 2, 3, \ldots$ holds. For other results on the regularizing properties of the midpoint rule for solving linear Volterra integral equations of the first kind, see [22] and Kaltenbacher [15].

4 Modified starting weights

For the product midpoint rule (2.3), applied to a continuous function $\varphi : [0, a] \to \mathbb{R}$, and with grid points as in (2.1), with $1 \leq n \leq N$ and $N \geq 2$, we now would like to overcome the conditions $\varphi(0) = \varphi'(0) = 0$. For this purpose we consider the modification

$$
(\tilde{\Omega}_h \varphi)(x_n) := h^\alpha \sum_{j=1}^n \omega_{n-j} \varphi(x_{j-1/2}) + h^\alpha \sum_{j=1}^2 w_{nj} \varphi(x_{j-1/2})
$$

(4.1)

as approximation to the fractional integral $(\mathcal{V}^\alpha \varphi)(x_n)$ at the considered grid points $x_n$, respectively. See Lubich [18, 19] and [20] for a similar approach for fractional multistep methods. In (4.1), $w_{n1}$ and $w_{n2}$ are correction weights for the starting values that are specified in the following. In fact, for each $n = 1, 2, \ldots, N$ the correction weights are chosen such that the modified product midpoint rule (4.1) is exact at $x_n = nh$ for polynomials of degree $\leq 1$, i.e.,

$$
(\tilde{\Omega}_h y^q)(x_n) = (\mathcal{V}^\alpha y^q)(x_n) \quad \text{for} \quad q = 0, 1.
$$

4.1 Computation of the correction weights

For each $n = 1, 2, \ldots, N$, a reformulation of (4.2) gives the following linear system of two equations for the starting weights $w_{nj}$, $j = 1, 2$:

$$
h^\alpha (w_{n1} + w_{n2}) = (E^\alpha_h 1)(x_n), \quad h^{\alpha+1}(w_{n1} + \frac{2}{3}w_{n2}) = (E^\alpha_h y)(x_n),
$$

cf. (2.5) for the introduction of $E^\alpha_h$. On the other hand we have

$$
(E^\alpha_h 1)(x_n) = 0, \quad (E^\alpha_h y)(x_n) = h^{\alpha+1} \sum_{j=0}^{n-1} \tau_j.
$$

Those identities follow from representations (2.11) and (2.12), respectively. From this we obtain

$$
-w_{n1} = w_{n2} = \sum_{j=0}^{n-1} \tau_j.
$$

(4.3)

This in particular means that the correction weights are independent of $h$. We finally note that the asymptotic behavior of the coefficients $\tau_j$, cf. (2.15), implies

$$
w_{nj} = \mathcal{O}(1) \quad \text{as} \quad n \to \infty \quad \text{for} \quad j = 1, 2.
$$

(4.4)
4.2 Integration error of the modified quadrature method

We now consider, for each \( n = 1, 2, \ldots, N \), the error of the modified product midpoint rule,

\[
(\tilde{E}_n^h \varphi)(x_n) = (V^n \varphi)(x_n) - (\tilde{\Omega}_h \varphi)(x_n),
\]

where \( \varphi : [0, a] \to \mathbb{R} \) denotes a continuous function.

**Lemma 4.1** Let \( n \in \{1, 2, \ldots, N\} \), and moreover let \( \varphi \in F^\gamma[0, a] \), with \( 0 < \gamma \leq 2 \). We have the following representations of the modified quadrature error \( (\tilde{E}_n^h \varphi)(x_n) \) introduced in (4.5):

(a) In the case \( 0 < \gamma \leq 1 \) we have \( (\tilde{E}_n^h \varphi)(x_n) = (E_n^h \varphi)(x_n) + O(h^{\gamma+\alpha}) \) as \( h \to 0 \).

(b) In the case \( 1 < \gamma \leq 2 \) we have, with \( \tilde{\varphi}(y) := \varphi(y) - \varphi(0) - \varphi'(0)y \) for \( 0 \leq y \leq a \),

\[
(\tilde{E}_n^h \varphi)(x_n) = (E_n^h \varphi)(x_n) + O(h^{\gamma+\alpha}) \quad \text{as} \quad h \to 0.
\]

Both statements hold uniformly for \( n = 1, 2, \ldots, N \), and for \( \varphi \in F^\gamma_L[0, a] \), with any constant \( L \geq 0 \).

**Proof.**

(a) This follows immediately from (4.1) and (4.3)–(4.5):

\[
(\tilde{E}_n^h \varphi)(x_n) = (E_n^h \varphi)(x_n) + h^\alpha w_{n1} (\varphi(x_{3/2}) - \varphi(x_{1/2})) = (E_n^h \varphi)(x_n) + O(h^{\gamma+\alpha}) \quad \text{as} \quad h \to 0.
\]

(b) Using the notation \( p(y) := \varphi(0) + \varphi'(0)y \), we have \( \varphi = \tilde{\varphi} + p \), and the linearity of the modified error functional gives

\[
(\tilde{E}_n^h \varphi)(x_n) = (\tilde{E}_n^h \tilde{\varphi})(x_n) + (\tilde{E}_n^h p)(x_n) = (E_n^h \tilde{\varphi})(x_n) - h^\alpha \sum_{j=1}^2 w_{nj} \tilde{\varphi}(x_{j-1/2})
\]

where \( \tilde{\varphi}(y) = O(y^\gamma) \) as \( y \to 0 \) has been used, and the boundedness of the correction weights, cf. (4.4), is also taken into account. \( \Box \)

4.3 Application to the Abel-type first kind integral equation

In what follows, the modified product midpoint rule (4.1) is applied to numerically solve the algebraic-type weakly singular integral equation (1.1), with noisy data as in (5.1). In order to make the starting procedure applicable, in the sequel we assume that the kernel \( k \) can be smoothly extended beyond the triangle \( \{0 \leq y \leq x \leq a\} \). For simplicity we assume that the kernel is defined on the whole square.

**Assumption 4.2** The kernel function \( k \) has Lipschitz continuous partial derivatives up to the order 2 on \([0, a] \times [0, a]\).

For each \( n = 1, 2, \ldots, N \), we consider the modified product midpoint rule (4.1) with \( \varphi(y) = k(x_n, y)u(y) \) for \( 0 \leq y \leq a \), \( n = 1, 2, \ldots, N \). This results in the following modified scheme:

\[
h^\alpha \sum_{j=1}^n \omega_{n-j} k(x_n, x_{j-1/2}) \bar{u}_{j-1/2} \delta_{\bar{u}_{j-1/2}} + h^\alpha \sum_{j=1}^2 w_{nj} k(x_n, x_{j-1/2}) \bar{u}_{j-1/2} = f_n, \quad n = 1, 2, \ldots, N. \tag{4.6}
\]

This scheme can be realized by first solving a linear system of two equations for the approximations \( \bar{u}_{n-1/2} \approx u(x_{n-1/2}) \), \( n = 1, 2 \). The approximations \( \bar{u}_{n-1/2} \approx u(x_{n-1/2}) \) for \( n = 3, 4, \ldots, N \) can be determined recursively by using scheme (4.6) then.
4.4 Uniqueness, existence and approximation properties of the starting values

We next consider uniqueness, existence and the approximation properties of the two starting values $\tilde{u}_{1/2}^\delta$ and $\tilde{u}_{3/2}^\delta$. They in fact satisfy the linear system of equations

$$h^\alpha \sum_{j=1}^2 (\omega_{n-j} + w_{nj}) k(x_n, x_{j-1/2}) \tilde{u}_{n-1/2}^\delta = f_n^\delta \quad \text{for } n = 1, 2, \quad (4.7)$$

with the notation $\omega_{-1} = 0$. In matrix notation this linear system of equations can be written as

$$h^\alpha \begin{pmatrix} \omega_{11} k(x_1, x_{1/2}) & \omega_{12} k(x_1, x_{3/2}) \\ \omega_{21} k(x_2, x_{1/2}) & \omega_{22} k(x_2, x_{3/2}) \end{pmatrix} \begin{pmatrix} \tilde{u}_{1/2}^\delta \\ \tilde{u}_{3/2}^\delta \end{pmatrix} = \begin{pmatrix} f_1^\delta \\ f_2^\delta \end{pmatrix}. \quad (4.8)$$

**Lemma 4.3** The matrix $S_h \in \mathbb{R}^{2 \times 2}$ in (4.8) is regular for sufficiently small values of $h$, and $\|S_h^{-1}\|_\infty = O(1)$ as $h \to 0$, where $\| \cdot \|_\infty$ denotes the matrix norm induced by the maximum vector norm on $\mathbb{R}^2$.

**Proof.** We first consider the special case $k \equiv 1$ and denote the matrix $S_h$ by $T$ in this special case. From (1.4) and (4.2) it follows

$$\omega_{11} + \omega_{22} = \frac{n^\alpha}{\Gamma(\alpha + 1)} \quad \frac{1}{2} \omega_{11} + \frac{1}{2} \omega_{22} = \frac{n^{\alpha+1}}{\Gamma(\alpha + 2)}, \quad n = 1, 2.$$

Hence the matrix $T$ is regular and does not depend on $h$.

We next consider the general case for $k$. Since $k(x, x) = 1$, we have $k(x_n, x_m) \to 1$ as $h \to 0$ uniformly for the four values of $k$ considered in the matrix $S_h$. This shows $S_h = T + \Delta$ with $\|\Delta\|_\infty \to 0$ as $h \to 0$ so that the matrix $S_h$ is regular for sufficiently small values $h$, with $\|S_h^{-1}\|_\infty$ being bounded as $h \to 0$. This completes the proof of the lemma. \(\square\)

We next consider the error of the modified product midpoint rule at the first two grid points $x_{1/2}$ and $x_{3/2}$.

**Proposition 4.4** Let the conditions of Assumption 3.1 and Assumption 4.2 be satisfied. Consider the approximations $\tilde{u}_{1/2}^\delta$ and $\tilde{u}_{3/2}^\delta$ determined by scheme (4.6) for $n = 1, 2$. Then we have

$$\max_{n=1,2} |\tilde{u}_{n-1/2}^\delta - u(x_{n-1/2})| = O(h^\gamma + \frac{\delta}{h^\nu}) \quad \text{as } (h, \delta) \to 0.$$

**Proof.** From (4.1), (4.5) and Lemma 4.1 applied with $\varphi_n(y) = k(x_n, y)u(y)$ for $0 \leq y \leq a$, we obtain the representation

$$h^\alpha \sum_{j=1}^2 \omega_{nj} k(x_n, x_{j-1/2}) \tilde{e}_{j-1/2}^\delta = (\tilde{E}_h \varphi_n)(x_n) + f_n^\delta - f(x_n) = O(h^{\gamma + \alpha} + \delta) \quad \text{for } n = 1, 2,$$

as $(h, \delta) \to 0$, where $\tilde{e}_{j-1/2}^\delta = \tilde{u}_{j-1/2}^\delta - u(x_{j-1/2})$, $j = 1, 2$, and the weights $\omega_{nj}$ are introduced in (4.7). Note that Lemma 2.1 and Lemma 4.1 imply, for the two integers $n = 1, 2$, that $(\tilde{E}_h \varphi_n)(x_n) = O(h^{\gamma + \alpha})$ as $h \to 0$. The proposition now follows from Lemma 4.3. \(\square\)
4.5 The regularizing properties of the modified scheme

Theorem 4.5 Let the conditions of Assumption 3.1 and Assumption 4.2 be satisfied.

(a) In the case $\alpha \leq 1/2$ we have

$$\max_{n=1,2,\ldots,N} |\tilde{u}_{n-1/2}^{\delta} - u(x_{n-1/2})| = \begin{cases} \mathcal{O}(h^{\gamma-\alpha} + \frac{\delta}{h^\alpha}) & \text{if } \alpha < \gamma \leq \alpha + 1, \\ \mathcal{O}(h^{\gamma-1+\alpha} + \frac{\delta}{h^\alpha}) & \text{if } 2 - \alpha < \gamma \leq 2. \end{cases}$$

(b) In the case $\alpha \geq 1/2$, $1 - \alpha < \gamma \leq 2$ we have

$$\max_{n=1,2,\ldots,N} |\tilde{u}_{n-1/2}^{\delta} - u(x_{n-1/2})| = \mathcal{O}(h^{\gamma-1+\alpha} + \frac{\delta}{h^\alpha}) \text{ as } (h, \delta) \to 0.$$ 

Proof. Let $\tilde{u}_{j-1/2}^{\delta} = \tilde{u}_{j-1/2}^{\delta} - u(x_{j-1/2})$ for $j = 1, 2, \ldots, N$. From (4.1), (4.4), (4.5), Lemma 4.1 and Proposition 4.4, we obtain the representation

$$h^{\alpha} \sum_{j=1}^{n} \omega_{n-j} k(x_{n}, x_{j-1/2}) \tilde{u}_{j-1/2}^{\delta} = (E_{h}^{\delta} v_{n})(x_{n}) + f(x_{n}) - f^{\delta} - h^{\alpha} \sum_{j=1}^{2} w_{nj} k(x_{n}, x_{j-1/2}) \tilde{u}_{j-1/2}^{\delta}$$

$$= (E_{h}^{\delta} v_{n})(x_{n}) + \mathcal{O}(h^{\gamma+\alpha} + \delta) = (E_{h}^{\delta} v_{n})(x_{n}) + \mathcal{O}(h^{\gamma+\alpha} + \delta)$$

as $(h, \delta) \to 0$, uniformly for $n = 1, 2, \ldots, N$, where $\tilde{v}_{n} = v_{n}$, if $\gamma \leq 1$, and $\tilde{v}_{n}(y) = v_{n}(y) - v_{n}(0) - v_{n}'(0)y$ for $\gamma > 1$. The theorem now follows by performing the same steps as in the proof of Theorem 3.3.

As an immediate consequence of Theorem 4.5, we can derive regularizing properties of the modified scheme.

Corollary 4.6 Let both Assumption 3.1 and Assumption 4.2 be satisfied.

- If $\alpha \leq 1/2$ and $\alpha < \gamma \leq \alpha + 1$, then choose $h = h(\alpha) \sim \delta^{1/\gamma}$. The resulting error estimate is

$$\max_{n=1,2,\ldots,N} |\tilde{u}_{n-1/2}^{\delta} - u(x_{n-1/2})| = \mathcal{O}(\delta^{1-\alpha/\gamma}) \text{ as } \delta \to 0.$$ 

- Let one of the following two conditions be satisfied: (a) $\alpha \geq 1/2$, $1 - \alpha < \gamma \leq 2 - \alpha$, or (b) $2 - \alpha < \gamma \leq 2$. For $h = h(\delta) \sim \delta^{1/(\gamma-1+2\alpha)}$ we then have

$$\max_{n=1,2,\ldots,N} |\tilde{u}_{n-1/2}^{\delta} - u(x_{n-1/2})| = \mathcal{O}(\delta^{1-\frac{\alpha}{\gamma-1+2\alpha}}) \text{ as } \delta \to 0.$$ 

5 Numerical experiments

We next present results of some numerical experiments with the linear Abel-type Volterra integral equation of the first kind (1.1). The following example is considered (for different values of $0 < \alpha < 1$ and $0 < q \leq 2$):

$$k(x, y) = \frac{1 + xy}{1 + x^2}, \quad f(x) = \frac{1}{1(q + 2 + \alpha)} \frac{x^{q+\alpha}}{1 + x^2}(q + 1 + \alpha + (q + 1)x^2) \quad \text{for } 0 \leq x, y \leq 1,$$

with exact solution (cf. (1.4))

$$u(y) = \frac{1}{1(q + 1)} y^q \quad \text{for } 0 \leq y \leq 1,$$

so that the conditions in (a) or (c) of Assumption 3.1 are satisfied with $\gamma = q$. We present experiments for different values of $\alpha$ and $q$, sometimes with corrections weights, sometimes without, in order to cover all variants in Corollaries 3.5 and 4.6. Here are additional remarks on the numerical tests.
The presented theory for the product midpoint rule without correction weights suggests that we have in (a)–(c) of Assumption 3.1 are satisfied with any Example 5.4

We first consider the situation (5.1)–(5.2), with (a)–(c) of Assumption 3.1 are satisfied with Example 5.2

• Experiments are employed using the program system OCTAVE (http://www.octave.org).

Example 5.1 We first consider the situation (5.1)–(5.2), with \( \alpha = \frac{1}{2} \) and \( q = 2 \). The conditions in (a)(c) of Assumption 3.1 are satisfied with \( \gamma = 2 \) (also for any \( \gamma > 2 \) in fact, but then we have saturation). We have \( u(0) = u'(0) = 0 \), so correction weights are not required here. The expected error estimate, with the choice of \( \delta = \delta(h) \) considered in the beginning of this section, is \( \max_n |u_n^\delta - u(x_n)| = O(h^{5/4}) = O(h^{5/2}) \). The numerical results are shown in Table 1.

Example 5.2 We next consider the situation (5.1)–(5.2), with \( \alpha = 0.9 \) and \( q = 0.4 \). The conditions in (a)(c) of Assumption 3.1 are satisfied with \( \gamma = 0.4 \). Since \( \gamma \leq 1 \), correction weights are not needed here. The expected error estimate, with \( \delta = \delta(h) \) as in the beginning of this section, is \( \max_n |u_n^\delta - u(x_n)| = O(h^{5/4}) = O(h^{5/2}) \). The numerical results are shown in Table 2.

Example 5.3 We next consider the situation (5.1)–(5.2), with \( \alpha = 0.2 \) and \( q = 0.5 \). The conditions in (a)(c) of Assumption 3.1 are satisfied with \( \gamma = 0.5 \) then, and the expected error estimate is \( \max_n |u_n^\delta - u(x_n)| = O(h^{5/4}) = O(h^{5/2}) \). The numerical results are shown in Table 3.

Example 5.4 We finally consider the situation (5.1)–(5.2), with \( \alpha = 0.5 \) and \( q = 1 \). Then the conditions in (a)(c) of Assumption 3.1 are satisfied with any \( \gamma > 0 \), and initial conditions are not satisfied in this case. The presented theory for the product midpoint rule without correction weights suggests that we have \( \max_n |u_n^\delta - u(x_n)| = O(h^2) = O(h) \). The corresponding numerical results are shown in Table 4.
We start with the consideration of some sequence spaces in a Banach algebra framework. For an introduction to Banach algebra theory see, e.g., Rudin [25]. The following results can be found in Rogozin [23, 24], and for completeness they are recalled here.

### 6.1 Special sequence spaces and Banach algebra theory

We now present a proof of (3.6) for the coefficients of the inverse of the considered generating power series \( \sum_{n=0}^{\infty} a_n \xi^n \) which differs from that given by Eggermont [8]. Our proof uses Banach algebra theory and may be of independent interest.

### 6 Appendix A: Proof of Lemma 3.2

We also consider the modified product midpoint rule for the same problem, i.e., correction weights are used this time. The presented theory then yields \( \max_n |u_n^i - u(x_n)| = O(\delta^{5/3}) = O(h^{5/2}) \). The related numerical results are shown in Table 5.

The last column in each table shows that the theory is confirmed in each of the five numerical experiments.

| \( N \) | \( \delta \) | \( 100 \cdot \delta / \| f \|_\infty \) | \( \max_n |u_n^i - u(x_n)| \) | \( \max_n |u_n^i - u(x_n)| / \delta^{5/6} \) |
|---|---|---|---|---|
| 32 | 5.3 \cdot 10^{-2} | 5.12 \cdot 10^0 | 1.18 \cdot 10^{-1} | 0.69 |
| 64 | 3.8 \cdot 10^{-2} | 3.62 \cdot 10^0 | 8.52 \cdot 10^{-2} | 0.61 |
| 128 | 2.7 \cdot 10^{-2} | 2.56 \cdot 10^0 | 7.86 \cdot 10^{-2} | 0.69 |
| 256 | 1.9 \cdot 10^{-2} | 1.81 \cdot 10^0 | 5.89 \cdot 10^{-2} | 0.64 |
| 512 | 1.3 \cdot 10^{-2} | 1.28 \cdot 10^0 | 5.19 \cdot 10^{-2} | 0.69 |
| 1024 | 9.4 \cdot 10^{-3} | 9.05 \cdot 10^{-1} | 4.20 \cdot 10^{-2} | 0.69 |
| 2048 | 6.6 \cdot 10^{-3} | 6.40 \cdot 10^{-1} | 3.33 \cdot 10^{-2} | 0.68 |

Table 3: Numerical results for Example 5.3

| \( N \) | \( \delta \) | \( 100 \cdot \delta / \| f \|_\infty \) | \( \max_n |u_n^i - u(x_n)| \) | \( \max_n |u_n^i - u(x_n)| / \delta^{2/3} \) |
|---|---|---|---|---|
| 32 | 1.7 \cdot 10^{-3} | 2.20 \cdot 10^{-4} | 1.26 \cdot 10^{-3} | 0.90 |
| 64 | 5.9 \cdot 10^{-4} | 7.79 \cdot 10^{-2} | 6.47 \cdot 10^{-3} | 0.92 |
| 128 | 2.1 \cdot 10^{-4} | 2.75 \cdot 10^{-2} | 3.27 \cdot 10^{-3} | 0.94 |
| 256 | 7.3 \cdot 10^{-5} | 9.74 \cdot 10^{-3} | 1.57 \cdot 10^{-3} | 0.89 |
| 512 | 2.6 \cdot 10^{-5} | 3.44 \cdot 10^{-3} | 7.72 \cdot 10^{-4} | 0.88 |
| 1024 | 9.2 \cdot 10^{-6} | 1.22 \cdot 10^{-3} | 3.95 \cdot 10^{-4} | 0.90 |
| 2048 | 3.2 \cdot 10^{-6} | 4.30 \cdot 10^{-4} | 2.06 \cdot 10^{-4} | 0.94 |

Table 4: Numerical results for Example 5.4 without correction weights

| \( N \) | \( \delta \) | \( 100 \cdot \delta / \| f \|_\infty \) | \( \max_n |u_n^i - u(x_n)| \) | \( \max_n |u_n^i - u(x_n)| / \delta^{5/4} \) |
|---|---|---|---|---|
| 32 | 2.9 \cdot 10^{-4} | 3.89 \cdot 10^{-4} | 2.10 \cdot 10^{-4} | 0.94 |
| 64 | 7.3 \cdot 10^{-5} | 9.74 \cdot 10^{-5} | 6.56 \cdot 10^{-4} | 0.83 |
| 128 | 1.8 \cdot 10^{-5} | 2.43 \cdot 10^{-5} | 2.88 \cdot 10^{-4} | 1.03 |
| 256 | 4.6 \cdot 10^{-6} | 6.09 \cdot 10^{-6} | 8.66 \cdot 10^{-5} | 0.87 |
| 512 | 1.1 \cdot 10^{-6} | 1.52 \cdot 10^{-6} | 3.46 \cdot 10^{-5} | 0.99 |
| 1024 | 2.9 \cdot 10^{-7} | 3.80 \cdot 10^{-7} | 1.22 \cdot 10^{-5} | 0.99 |
| 2048 | 7.2 \cdot 10^{-8} | 9.51 \cdot 10^{-8} | 4.31 \cdot 10^{-6} | 0.99 |

Table 5: Numerical results for Example 5.4 with correction weights
For a sequence of positive real weights \( (\sigma_n)_{n \geq 0} \), consider the following norms,

\[
\|a\|_{\infty, \sigma} = \sup_{m \geq 0} |a_m| \sigma_m + \sum_{n=0}^{\infty} |a_n|, \quad \|a\|_1 = \sum_{n=0}^{\infty} |a_n|, \quad a = (a_n)_{n \geq 0} \subset \mathbb{C},
\]

and the spaces

\[
\ell^1 = \{ a = (a_n)_{n \geq 0} \subset \mathbb{C} \mid \|a\|_1 < \infty \},
\]

\[
\ell^\infty = \{ a = (a_n)_{n \geq 0} \subset \mathbb{C} \mid \|a\|_{\infty, \sigma} < \infty \},
\]

\[
e_0 = \{ a \in \ell^\infty \mid a_n \sigma_n \to 0 \text{ as } n \to \infty \}.
\]

We obviously have \( e_0 \subset \ell^\infty \subset \ell^1 \). By using the canonical identification \( a(\xi) = \sum_{n=0}^{\infty} a_n \xi^n \), the spaces \( e_0, \ell^\infty \) and \( \ell^1 \) can be considered as function algebras on

\[
D = \{ \xi \in \mathbb{C} \mid |\xi| \leq 1 \},
\]

the closed unit disc with center 0 and radius 1. We are mainly interested in positive weights \( (\sigma_n)_{n \geq 0} \) which satisfy \( \sum_{n=0}^{\infty} \sigma_n^{-1} < \infty \). In that case, \( \sup_{n \geq 0} |a_m| \sigma_m \) for \( (a_n)_{n \geq 0} \in \ell^\infty \) defines a norm on \( \ell^\infty \) which is equivalent to the given norm \( \| \cdot \|_{\infty, \sigma} \). In particular, in the case \( \sigma_0 = 1 \) and \( \sigma_n = n^\beta \) for \( n = 1, 2, \ldots (\beta > 1) \), then \( \ell^\infty_\sigma \) is the space of sequences \( (a_n)_{n \geq 0} \) satisfying \( a_n = \mathcal{O}(n^{-\beta}) \) as \( n \to \infty \). In the sequel we assume that

\[
\sigma_n \leq c \sigma_j, \quad \frac{n}{2} \leq j \leq n, \quad n \geq 0,
\]

(6.1)

holds for some finite constant \( c > 0 \). We state without proof the following elementary result (cf. [25] for part (a) of the proposition, and [23, 24] for parts (b) and (c)).

**Proposition 6.1** Let \( \sigma_0, \sigma_1, \ldots \) be positive weights satisfying (6.1).

(a) The space \( \ell^1 \), equipped with convolution \( (ab)_n = \sum_{j=0}^{n} a_{n-j} b_j, n \geq 0 \), for \( a, b \in \ell^1 \), is a commutative complex Banach algebra, with unit \( e = (1, 0, 0, \ldots) \).

(b) The space \( \ell^\infty_\sigma \) is a subalgebra of \( \ell^1 \), i.e., it is closed with respect to addition, scalar multiplication and convolution. The norm \( \| \cdot \|_{\infty, \sigma} \) complete on \( \ell^\infty_\sigma \) and satisfies

\[
\|a * b\|_{\infty, \sigma} \leq (2c + 1)\|a\|_{\infty, \sigma} \cdot \|b\|_{\infty, \sigma}, \quad a, b \in \ell^\infty_\sigma,
\]

where \( c \) is taken from estimate (6.1).

(c) The statements of (b) are also valid for the space \( e_0^\infty \) (instead of \( \ell^\infty_\sigma \)), supplied with the norm \( \| \cdot \|_{\infty, \sigma} \).

The following proposition is based on the fact that the subalgebra generated by \( a(\xi) = \xi = (0, 1, 0, 0, \ldots) \) is dense in the space \( \ell^1 \) and in \( e_0^\infty \) as well, i.e., both spaces are single-generated in fact.

**Proposition 6.2** (Rogozin [23]) Let \( \sigma_0, \sigma_1, \ldots \) be positive weights satisfying (6.1). The spaces \( \ell^1 \) and \( e_0^\infty \) are inverse-closed, i.e., for each \( a \in \ell^1 \) with \( a(\xi) \neq 0, \xi \in D \), one has \( [a(\xi)]^{-1} \in \ell^1 \), and for each \( a \in e_0^\infty \) with \( a(\xi) \neq 0, \xi \in D \), one has \( [a(\xi)]^{-1} \in e_0^\infty \).

For the \( \ell^1 \)-case, this is Wiener’s theorem, cf., e.g., Rudin [25]. The space \( \ell^\infty_\sigma \) is not single-generated but still inverse-closed which will be used in the following. The proof is taken from Rogozin [24] and is stated here for completeness.

**Proposition 6.3** (Rogozin [24]) For positive weights \( (\sigma_n)_{n \geq 0} \) satisfying (6.1), the space \( \ell^\infty_\sigma \) is inverse-closed, i.e., for each \( a \in \ell^\infty_\sigma \) with \( a(\xi) \neq 0 \) for \( \xi \in D \) one has \( [a(\xi)]^{-1} \in \ell^\infty_\sigma \).
PROOF. Consider \(x(\xi) = \sum_{n=0}^{\infty} a_n \xi^n \in \ell_\sigma^\infty \) with \(x(\xi) \neq 0 \) for \(\xi \in \mathcal{D} \). Then \(x\) is invertible in \(\ell^1\) (cf. Proposition 6.2), i.e., \(1/x(\xi) = \sum_{n=0}^{\infty} a_n^{(-1)} \xi^n \in \ell^1\). Let us assume contradictory that \(1/x(\xi) \notin \ell_\sigma^\infty\). This means that \(\limsup_{n \to \infty} |a_n^{(-1)}| \sigma_n = \infty \) and then
\[
\kappa_n = \max_{0 \leq m \leq n} |a_m^{(-1)}| \sigma_m \to \infty \text{ as } n \to \infty, \tag{6.3}
\]
and \(\kappa_{n+1} \geq \kappa_n > 0 \) for \(n = 0, 1, \ldots\). Let \(\bar{\sigma}_n = \sigma_n/\kappa_n \) for \(n = 0, 1, \ldots\). We have
\[
0 < \bar{\sigma}_n = \frac{\sigma_n}{\kappa_n} \leq \frac{\sigma_n}{\kappa_j} = c\bar{\sigma}_j, \quad \frac{n}{2} \leq j \leq n,
\]
so the space \(c_0^\sigma = \{ a \in \ell_\sigma^\infty \mid a_n \bar{\sigma}_n \to 0 \text{ as } n \to \infty \} \) with \(\bar{\sigma} = (\bar{\sigma}_n)_{n \geq 0}\) is a Banach algebra which is inverse-closed (cf. Propositions 6.1 and 6.2).

By assumption \(\sup_{n \geq 0} |a_n| \sigma_n < \infty\) and then \(a_n \bar{\sigma}_n \to 0 \) as \(n \to \infty\). From Proposition 6.2 it then follows
\[
|a_n^{(-1)}| \bar{\sigma}_n \to 0 \text{ as } n \to \infty. \tag{6.4}
\]
However, it follows from (6.3) that for some infinite subset \(N \subset \mathbb{N}\) we have
\[
\kappa_n = |a_n^{(-1)}| \sigma_n \quad \text{for } n \in N. \tag{6.5}
\]
Otherwise there would exist an \(n_1 \geq 1\) with \(\kappa_n = \max_{0 \leq m \leq n} |a_m^{(-1)}| \sigma_m > |a_n^{(-1)}| \sigma_n\) for \(n = n_1, n_1 + 1, \ldots\), which in fact means \(\kappa_{n-1} = \max_{0 \leq m \leq n-1} |a_m^{(-1)}| \sigma_m > |a_n^{(-1)}| \sigma_n\), and then \(\kappa_n = \kappa_{n-1}\) for \(n = n_1, n_1 + 1, \ldots\), a contradiction to (6.3). From (6.5) we then get
\[
|a_n^{(-1)}| \bar{\sigma}_n = |a_n^{(-1)}| \sigma_n/\kappa_n = 1, \quad n \in \mathbb{N},
\]
a contradiction to (6.4). \(\Box\)

6.2 The power series \(\sum_{n=0}^{\infty} (n+1)^\alpha \xi^n\)

Our analysis continues with a special representation of the power series \(\sum_{n=0}^{\infty} (n+1)^\alpha \xi^n\), and we will make use of the binomial expansion
\[
(1 - \xi)^\beta = \sum_{n=0}^{\infty} \binom{\beta}{n} (-1)^n \xi^n \quad \text{for } \xi \in \mathbb{C}, \; |\xi| < 1 \quad (\beta \in \mathbb{R}), \tag{6.6}
\]
\[
(-1)^n \binom{\beta}{n} = \sum_{s=0}^{m-1} d_{\beta,s} n^{\beta-1-s} + \mathcal{O}(n^{\beta-1-m}) \quad \text{as } n \to \infty, \tag{6.7}
\]
with certain real coefficients \(d_{\beta,s}\) for \(s = 0, 1, \ldots, m-1, m = 0, 1, \ldots\), where \(d_{\beta,0} = 1/\Gamma(-\beta), \beta \neq 0, 1, \ldots\), cf. e.g., equation (6.1.47) in Abramowitz and Stegun [1]. We need the following result.

Lemma 6.4 For \(0 < \alpha < 1\) we have, with some coefficients \(r_0, r_1, \ldots\),
\[
\frac{1}{\Gamma(\alpha+1)} \sum_{n=0}^{\infty} (n+1)^\alpha \xi^n = (1 - \xi)^{-\alpha-1} r(\xi) \quad \text{for } \xi \in \mathbb{C}, \; |\xi| < 1, \tag{6.8}
\]
with \(r(\xi) = \sum_{n=0}^{\infty} r_n \xi^n, \quad r(1) = 1, \quad r_n = \mathcal{O}(n^{\alpha-2}) \) as \(n \to \infty\). \(\tag{6.9}\)
PROOF. We first observe that, for each $m \geq 0$, there exist real coefficients $c_0, c_1, \ldots, c_{m-1}$ with

$$\frac{1}{\Gamma(\alpha+1)} \sum_{n=0}^{\infty} (n+1)^{\alpha} \xi^n = \sum_{j=0}^{m-1} c_j (1-\xi)^{-\alpha-1+j} + s(\xi) \quad \text{for } \xi \in \mathbb{C}, \ |\xi| < 1, \quad (6.10)$$

with $s(\xi) = \sum_{n=0}^{\infty} s_n \xi^n$, where $s_n = \mathcal{O}(n^{\alpha-m})$ as $n \to \infty$, and we have $c_0 = 1$. This follows by comparing the coefficients in the Taylor expansion

$$\frac{1}{\Gamma(\alpha+1)} \sum_{n=0}^{\infty} (n+1)^{\alpha} \xi^n = (1-\xi)^{-\alpha} \sum_{j=0}^{m-1} c_j (1-\xi)^j + \mathcal{O}(n^{\alpha-1})$$

for $\xi \in \mathbb{C}, \ |\xi| < 1$.

A reformulation of (6.10) gives, with $m = 4$,

$$\frac{1}{\Gamma(\alpha+1)} \sum_{n=0}^{\infty} (n+1)^{\alpha} \xi^n = (1-\xi)^{-\alpha-1} \left( \sum_{j=0}^{3} c_j (1-\xi)^j + (1-\xi)^{\alpha+1} s(\xi) \right) \quad \text{for } \xi \in \mathbb{C}, \ |\xi| < 1,$n

with $s(\xi) = \sum_{n=0}^{\infty} s_n \xi^n$, $s_n = \mathcal{O}(n^{\alpha-4})$ as $n \to \infty$.

The statement of the lemma now follows from statement (b) of Proposition 6.1 applied with $\sigma_0 = 1$ and $\sigma_n = n^{\alpha+2}$ for $n = 1, 2, \ldots$, and from (6.6), (6.7) applied with $\beta = \alpha+1, m = 0$. \hfill \Box

6.3 The main results

As a consequence of Lemma 6.4 we obtain the following representation.

Corollary 6.5 For the quadrature weights $\omega_0, \omega_1, \ldots$ considered in (2.4) we have, with the power series $r$ from (6.8), (6.9),

$$\omega(\xi) = \sum_{n=0}^{\infty} \omega_n \xi^n = (1-\xi)^{-\alpha} r(\xi) \quad \text{for } \xi \in \mathbb{C}, \ |\xi| < 1. \quad (6.11)$$

PROOF. The two power series $\sum_{n=0}^{\infty} (n+1)^{\alpha} \xi^n$ and $\omega(\xi) = \sum_{n=0}^{\infty} \omega_n \xi^n$ with coefficients as in (2.4) are obviously related as follows,

$$\sum_{n=0}^{\infty} \omega_n \xi^n = \frac{1}{1(\alpha+1)} \sum_{n=0}^{\infty} (n+1)^{\alpha} \xi^n.$$n

The representation (6.8) now implies the statement of the corollary. \hfill \Box

Inverting (6.11) immediately gives the power series representation

$$\sum_{n=0}^{\infty} \omega_n (-1)^n = (1-\xi)^{\alpha} r(\xi)^{-1}, \quad (6.12)$$

where $\omega_n^{-1}$ denote the coefficients of the inverse of the power series $\omega(\xi) = \sum_{n=0}^{\infty} \omega_n \xi^n$, cf. (3.3).

In the sequel we examine the asymptotic behavior of the coefficients in the power series

$$[r(\xi)]^{-1} = \sum_{n=0}^{\infty} r_n (-1)^n \xi^n. \quad (6.13)$$

Lemma 6.6 We have $r_n^{-1} = \mathcal{O}(n^{-\alpha-2})$ as $n \to \infty$. 

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PROOF. It follows from (6.9) that the power series \( r \) considered in (6.8) satisfies \( r \in \ell^{\infty}_r \) for the specific choice \( \sigma_0 = 1 \) and \( \sigma_n = n^{\alpha + 2} \) for \( n \geq 1 \). In addition we have \( r(\xi) \neq 0 \) for \( \xi \in \mathbb{C}, |\xi| \leq 1 \) (a proof is given below). From Proposition 6.3 we then obtain \( r_n^{-1} = O(n^{-\alpha - 2}) \) as \( n \to \infty \).

It remains to show that \( r(\xi) \neq 0 \) holds for \( \xi \in \mathbb{C}, |\xi| \leq 1 \). For this purpose we consider a reformulation of (6.11).

\[
    r(\xi) = (1 - \xi)^\alpha \sum_{n=0}^{\infty} \omega_n \xi^n \quad \text{for} \quad \xi \in \mathbb{C}, |\xi| < 1.
\]

We have

\[
    |\sum_{n=0}^{\infty} \omega_n \xi^n| \geq \frac{1}{2^{\beta}} \quad \text{for} \quad \xi \in \mathbb{C}, |\xi| < 1, \tag{6.14}
\]

a proof of (6.14) is presented in the next section. Since \( r(1) \neq 0 \) and \( r \) is continuous on \( \{ \xi \in \mathbb{C} | |\xi| \leq 1 \} \), (6.14) then implies \( r(\xi) \neq 0 \) for \( \xi \in \mathbb{C}, |\xi| \leq 1 \) as desired, and thus the statement of the lemma is proved.

We are now in a position to continue with the verification of the asymptotical behavior (3.6) for the coefficients of the power series \( \{\omega(\xi)\}^{-1} \). From the representation (6.6), (6.7) with \( \beta = \alpha \) it follows that the coefficients in the expansion \( (1 - \xi)^\alpha = \sum_{n=0}^{\infty} (-1)^n {\alpha \choose n} \xi^n \) satisfy \( (-1)^n {\alpha \choose n} = O(n^{-\alpha - 1}) \) as \( n \to \infty \).

This and Lemma 6.6 (which in particular means \( r_n^{-1} = O(n^{-\alpha - 1}) \)) and part (b) of Proposition 6.1 applied with \( \sigma_0 = 1 \) and \( \sigma_n = n^{\alpha + 1} \) for \( n \geq 1 \), finally results in the desired estimate (3.6) for the coefficients of the power series \( \{\omega(\xi)\}^{-1} \).

### 6.4 The proof of the lower bound (6.14)

To complete our proof of (3.6), we need to show that (6.14) holds. We start with a useful lemma.

**Lemma 6.7** The quadrature weights \( \omega_0, \omega_1, \ldots \) in (2.4) are positive and satisfy \( \sum_{n=0}^{\infty} \omega_n = \infty \). In addition we have

\[
    \frac{\omega_{n+1}}{\omega_n} \geq \frac{\omega_n}{\omega_{n-1}} \quad \text{for} \quad n = 1, 2, \ldots \tag{6.15}
\]

**PROOF.** It follows immediately from the definition that the coefficients \( \omega_0, \omega_1, \ldots \) are positive. The identity \( \sum_{n=0}^{\infty} \omega_n = \infty \) is obvious, and we next present a proof of the inequality (6.15). Using the notation

\[
    f(x) = x^\alpha \quad \text{for} \quad x \geq 0
\]

we obtain the following,

\[
    \frac{\omega_n}{\omega_{n-1}} = \frac{f(n+1) - f(n)}{f(n) - f(n-1)} = \frac{f'(t_n)}{f(t_n) - f(t_n-1)} = (1 - \frac{1}{t_n})^{1-\alpha} =: h(t_n) \quad \text{for} \quad n = 1, 2, \ldots ,
\]

with some real number \( n < t_n < n + 1 \). Here, the identity \((*)\) follows from the generalized mean value theorem. The function \( h(s) \) is monotonically increasing for \( s > 0 \) which yields estimate (6.15). This completes the proof of the lemma. \( \square \)

For results similar to those in Lemma 6.7 see Egggernort [7] [9] and Linz [17] Section 10.4]. It follows from Lemma 6.7 that the conditions of the following lemma are satisfied for \( p_n = c\omega_n, n = 0, 1, \ldots \), with \( c > 0 \) arbitrary but fixed.
Lemma 6.8 (cf. Kaluza [16]; see also Szegö [26], Hardy [13], and Linz [17]) Let \( p_0, p_1, \ldots \) be real numbers satisfying

\[
p_n > 0 \quad \text{for} \quad n = 0, 1, \ldots, \quad \frac{p_{n+1}}{p_n} > \frac{p_n}{p_{n-1}} \quad \text{for} \quad n = 1, 2, \ldots.
\]

(6.16)

Then the inverse \([p(\xi)]^{-1}\) of the power series \( p(\xi) = \sum_{n=0}^{\infty} p_n \xi^n \) can be written as follows,

\[
[p(\xi)]^{-1} = c_0 - \sum_{n=1}^{\infty} c_n \xi^n,
\]

(6.17)

with coefficients \( c_0, c_1, \ldots \) satisfying \( c_n > 0 \) for \( n = 0, 1, \ldots \). If moreover \( \sum_{n=0}^{\infty} p_n = \infty \) holds and the power series \( p(\xi) = \sum_{n=0}^{\infty} p_n \xi^n \) has convergence radius \( 1 \), then we have \( \sum_{n=1}^{\infty} c_n = c_0 \).

**Proof.** Lemma 6.8 is Theorem 22 on page 68 of Hardy [13]. The proof of \( c_n > 0 \) for \( n = 0, 1, \ldots \) is presented there in full detail, and we do not repeat the steps here. However, the proof of \( \sum_{n=1}^{\infty} c_n = c_0 \) is omitted there, so in the sequel we present some details of this proof. Condition (6.16) and the assumption on the convergence radius of the power series \( p(\xi) \) means \( p_{n+1}/p_n \to 1 \) as \( n \to \infty \). The second condition in (6.16) then implies \( 0 < p_{n+1} < p_n \) for \( n = 0, 1, \ldots \). From \( c_n \geq 0 \) for \( n = 0, 1, \ldots \), we obtain \( p_{n-1} \sum_{j=1}^{n} c_j \leq \sum_{j=1}^{n} p_{n-j} c_j = p_n c_0 \) for \( n = 1, 2, \ldots \). The latter identity follows from the representation (6.17). Thus

\[
\sum_{j=1}^{n} c_j \leq \frac{p_n}{p_{n-1}} c_0 \leq c_0 \quad \text{for} \quad n = 1, 2, \ldots.
\]

The latter inequality means that \( c(\xi) = c_0 - \sum_{j=1}^{\infty} c_j \xi^j \) is absolutely convergent on the closed unit disc \( \{ \xi \in \mathbb{C} \mid |\xi| \leq 1 \} \) and hence is continuous on this set. This finally gives

\[
0 = \lim_{0 < x \to 1} \frac{1}{\sum_{j=0}^{\infty} p_j x^j} = c_0 - \lim_{0 < x \to 1} \sum_{j=1}^{\infty} c_j x^j = c_0 - \sum_{j=1}^{\infty} c_j.
\]

This completes the proof of the lemma.

The following lemma is closely related to results in Erdős, Feller and Pollard [10]. A detailed proof can be found in [21].

**Lemma 6.9** Let \( c_1, c_2, \ldots \) be a sequence of real numbers satisfying \( c_n > 0 \) for \( n = 1, 2, \ldots \), and \( \sum_{n=1}^{\infty} c_n = \frac{1}{2} \). Then the power series \( q(\xi) = \frac{1}{2} - \sum_{n=1}^{\infty} c_n \xi^n \) satisfies \( |q(\xi)| < 1 \) for each complex number \( \xi \) with \( |\xi| \leq 1 \).

We are now in a position to present a proof of the lower bound (6.14). In fact, from Lemma 6.7 it follows that the coefficients of the power series \( p(\xi) = 2\Gamma(\alpha + 1)\omega(\xi) \) with \( \omega(\xi) \) as in (6.11) satisfy the conditions of Lemma 6.8 and in addition \( p_0 = 2 \) holds. This implies that the coefficients of the power series

\[
\frac{1}{2\Gamma(\alpha + 1)\omega(\xi)} = c_0 - \sum_{n=1}^{\infty} c_n \xi^n
\]

satisfy \( c_n > 0 \) for \( n = 0, 1, \ldots \) and \( \sum_{n=1}^{\infty} c_n = c_0 = 1/2 \). Lemma 6.9 then implies that \( 2\Gamma(\alpha + 1)\omega(\xi) \geq 1 \) and thus \( |\omega(\xi)| \geq \frac{1}{2\Gamma(\alpha + 1)} \) for \( \xi \in \mathbb{C}, |\xi| < 1 \). This is the desired estimate (6.14) needed in the proof of Lemma 6.6.

**7 Appendix B: Proof of Theorem 3.3**

1. We apply the representations (2.3) and (2.5) with \( \varphi = \varphi_n \), where

\[
\varphi_n(y) = k(x_n, y)u(y), \quad 0 \leq y \leq x_n,
\]
and scheme (3.2) imply the following,

\[ h^n \sum_{j=1}^{n} \omega_{n-j} k(x_n, x_{j-1/2}) e^{\delta}_{j-1/2} = (E_h^\alpha \varphi_n)(x_n) + f^\delta_n - f(x_n) \quad \text{for } n = 1, 2, \ldots, N, \quad (7.1) \]

where

\[ e^{\delta}_{j-1/2} = u^{\delta}_{j-1/2} - u(x_{j-1/2}), \quad j = 1, 2, \ldots, N. \]

2. We next consider a matrix-vector formulation of (7.1). As a preparation we consider the matrix \( A_h \in \mathbb{R}^{N \times N} \) is given by

\[
A_h = \begin{pmatrix}
\omega_0 k_{1,1/2} & 0 & \cdots & \cdots & 0 \\
\omega_1 k_{2,1/2} & \omega_0 k_{2,3/2} & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
\omega_{N-1} k_{N,1} & \cdots & \cdots & \omega_1 k_{N,N-3/2} & \omega_0 k_{N,N-1/2}
\end{pmatrix}
\]

with the notation

\[ k_{n,j-1/2} = k(x_n, x_{j-1/2}) \quad \text{for } 1 \leq j \leq n \leq N. \]

Additionally we consider the vectors

\[ \Delta^\delta_h = (e^{\delta}_{j-1/2})_{j=1,2,\ldots,N}, \quad R_h = ((E_h^\alpha \varphi_n)(x_n))_{n=1,2,\ldots,N}, \quad F^\delta_h = (f^\delta_n - f(x_n))_{n=1,2,\ldots,N}. \quad (7.2) \]

Using these notations, the linear system of equations (7.1) can be written as

\[ h^n A_h \Delta^\delta_h = R_h + F^\delta_h \quad \text{with } \|F^\delta_h\|_\infty \leq \delta, \quad (7.3) \]

where \( \| \cdot \|_\infty \) denotes the maximum norm on \( \mathbb{R}^N \). In addition, occasionally we consider a modified error equation which can easily be derived from (7.3) by applying the matrix \( D_h \) to both sides of that equation:

\[ h^n D_h A_h \Delta^\delta_h = D_h R_h + D_h F^\delta_h. \quad (7.4) \]

This technique is a discrete analogue of fractional differentiation.

3. For a further treatment of the identity (7.3) and its variant (7.4), we now show

\[ \|D_h\|_\infty = \mathcal{O}(1), \quad \|(D_h A_h)^{-1}\|_\infty = \mathcal{O}(1), \quad \|A_h^{-1}\|_\infty = \mathcal{O}(1) \quad \text{as } h \to 0, \quad (7.5) \]

where the matrix \( D_h \in \mathbb{R}^{N \times N} \) given by

\[
D_h = \begin{pmatrix}
\omega_0^{(-1)} & 0 & \cdots & \cdots & 0 \\
\omega_1^{(-1)} & \omega_0^{(-1)} & 0 & 0 & \cdots \cdots \cdots \cdots \\
\omega_2^{(-1)} & \cdots & \cdots & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots & 0 \\
\omega_{N-1}^{(-1)} & \cdots & \cdots & \omega_1^{(-1)} & \omega_0^{(-1)}
\end{pmatrix}, \quad (7.6)
\]
and \( \| \cdot \|_\infty \) denotes the matrix norm induced by the maximum vector norm on \( \mathbb{R}^N \). In fact, the estimate \( \| D_h \|_\infty = O(1) \) as \( h \to 0 \) follows immediately from the decay of the coefficients of the inverse of the generating function \( \omega \), cf. estimate (3.6). For the proof of the second statement in (7.5) we use the fact that the matrix \( D_h \) can be written in the form \( D_h A_h = I_h + K_h \), where \( I_h \in \mathbb{R}^{N \times N} \) denotes the identity matrix, and \( K_h = (k_{h,n,j}) \in \mathbb{R}^{N \times N} \) denotes some lower triangular matrix which satisfies \( \max_{1 \leq j \leq n \leq N} |k_{h,n,j}| = O(h) \) as \( h \to 0 \), cf. the proof of Lemma 4.2 in Eggermont \[8\] for more details.

We only note that here it is taken into account that the kernel function is uniformly Lipschitz continuous with respect to the first variable, cf. part (c) of Assumption 3.1. This representation of \( D_h A_h \) and the discrete version of Gronwall’s inequality now yields \( \| (D_h A_h)^{-1} \|_\infty = O(1) \) as \( h \to 0 \). The third estimate in (7.5) follows immediately from the other two estimates considered in (7.5).

4. In view of (7.3)–(7.5), it remains to take a closer look at the representations of the quadrature error considered in Lemma 2.1. We consider different situations for \( \gamma \) and constantly make use of the fact that, for some finite constant \( L \geq 0 \), we have
\[
\varphi_n \in F^\gamma_0[0, x_n] \quad \text{for} \quad n = 1, 2, \ldots, N, \tag{7.7}
\]
cf. Assumption 3.1

(i) In the case \( \gamma \leq 1 \) we proceed in two different ways. The first one turns out to be useful for the case \( \alpha \leq \frac{1}{2} \), while the other one uses partial summation and is useful for the case \( \alpha \geq \frac{1}{2} \).

- Our first approach proceeds with (7.3), and we assume \( \alpha < \gamma \leq 1 \) in this case. We then easily obtain, cf. (2.11), (7.7),
\[
\| R_h \|_\infty = \max_{1 \leq n \leq N} |(E^\alpha_h \varphi_n)(x_n)| = O(h^\gamma) \quad \text{as} \quad h \to 0,
\]
and then, cf. (7.3) and (7.5), \( \| \Delta^\alpha_h \|_\infty = O(h^{-\alpha}(h^\gamma + \delta)) = O(h^{\gamma - \alpha} + \frac{\delta}{Nh}) \).

- In our second approach we would like to proceed with (7.4), and we need to consider the vector \( D_h R_h \in \mathbb{R}^N \) in more detail. For this purpose we assume that \( 1 - \alpha < \gamma \leq 1 \) holds, and we introduce the notation
\[
r_n = (E^\alpha_h \varphi_n)(x_n), \quad n = 1, 2, \ldots, N.
\]
Partial summation, applied to the \( n \)th entry of \( D_h R_h \), gives
\[
(D_h R_h)_n = \sum_{j=1}^{n} \omega^{(-1)}_{n-j} r_j = \beta_n r_1 + \sum_{\ell=1}^{n-1} \beta_{n-\ell} (r_{\ell+1} - r_\ell), \quad (7.8)
\]
where
\[
\beta_n = \sum_{\ell=0}^{n-1} \omega^{(-1)}_{\ell} \quad \text{for} \quad n = 1, 2, \ldots. \tag{7.9}
\]
From Lemma 3.2 it easily follows that \( \beta_n \geq 0 \) for \( n = 1, 2, \ldots \). In addition, we have
\[
\beta_n = O(n^{-\alpha}) \quad \text{as} \quad n \to \infty, \quad (7.10)
\]
and thus
\[
\sum_{\ell=1}^{n-1} \beta_{\ell} = O(N^{1-\alpha}) = O(h^{\alpha-1}) \quad \text{as} \quad h \to 0 \tag{7.11}
\]
uniformly for \( n = 1, 2, \ldots, N \). Hölder continuity (7.7) implies
\[
|r_1| = |(E^\alpha_h \varphi_1)(x_1)| \leq \frac{Lh^\gamma}{\Gamma(\alpha)} \int_0^h (h - y)^{\alpha-1} dy = O(h^{\gamma + \alpha}),
\]

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and we next consider the differences \( r_{\ell+1} - r_{\ell} \) in more detail. For this purpose we introduce short notation for the interpolation error,

\[
\chi_n(y) = \varphi_n(y) - p_h \varphi_n(y) \quad \text{for} \quad 0 \leq y \leq x_n, \quad n = 1, 2, \ldots, N.
\]

We then have

\[
\begin{aligned}
r_{\ell+1} - r_{\ell} &= \frac{1}{\Gamma(\alpha)} \left( \int_0^{x_{\ell+1}} (x_{\ell+1} - y)^{\alpha-1} \chi_{\ell+1}(y) \, dy - \int_0^{x_{\ell}} (x_{\ell} - y)^{\alpha-1} \chi_{\ell}(y) \, dy \right) \\
&= \frac{1}{\Gamma(\alpha)} \int_{x_{\ell}}^{x_{\ell+1}} (x_{\ell+1} - y)^{\alpha-1} \chi_{\ell+1}(y) \, dy + \frac{1}{\Gamma(\alpha)} \int_0^{x_{\ell}} (x_{\ell} - y)^{\alpha-1} (\chi_{\ell+1} - \chi_{\ell})(y) \, dy \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^{x_{\ell}} ((x_{\ell+1} - y)^{\alpha-1} - (x_{\ell} - y)^{\alpha-1}) \chi_{\ell}(y) \, dy =: s_1 + s_2 + s_3.
\end{aligned}
\]

We have \( s_1 = O(h^{\gamma + \alpha}) \) which easily follows from \( \sup_{0 \leq y \leq x_{\ell+1}} |\chi_{\ell+1}(y)| = O(h^{\gamma}) \). Moreover, a first order Taylor expansions of the kernel \( k \) with respect to the first variable at the grid point \( x_{j-1/2} \) (\( 1 \leq j \leq \ell + 1 \)) gives, for \( x_{j-1} \leq y \leq x_j \),

\[
(\chi_{\ell+1} - \chi_{\ell})(y) = k(x_{\ell+1}, y)u(y) - k(x_{\ell+1}, x_{j-1/2})u(x_{j-1/2}) \\
- \left\{ k(x_{\ell}, y)u(y) - k(x_{\ell}, x_{j-1/2})u(x_{j-1/2}) \right\} \\
= (\frac{\partial k}{\partial x}(x_{\ell}, y)h + O(h^2))u(y) - (\frac{\partial k}{\partial x}(x_{\ell}, x_{j-1/2})h + O(h^2))u(x_{j-1/2}) \\
= h(\frac{\partial k}{\partial x}(x_{\ell}, y)u(y) - \frac{\partial k}{\partial x}(x_{\ell}, x_{j-1/2})u(x_{j-1/2})) + O(h^2) = O(h^{\gamma + 1}),
\]

and this implies \( s_2 = O(h^{\gamma + 1}) \). Finally,

\[
|s_3| \leq \frac{L}{\Gamma(\alpha)} h^\gamma \int_0^{x_{\ell}} (x_{\ell} - y)^{\alpha-1} - (x_{\ell+1} - y)^{\alpha-1} \, dy = \frac{L}{\Gamma(\alpha + 1)} h^{\gamma + \alpha} (1 + \ell^\alpha - (\ell + 1)^\alpha) = O(h^{\gamma + \alpha}).
\]

Summation gives \( s_1 + s_2 + s_3 = O(h^{\gamma + \alpha}) \), and (7.8) finally results in (see also (7.11))

Check

\[
(D_h R_h)_{n} = O(h^{\gamma + \alpha} + h^{\alpha-1} h^{\gamma + \alpha}) = O(h^{\gamma + 2\alpha - 1})
\]

uniformly for \( n = 1, 2, \ldots, N \). We note that this estimate is useful for \( \alpha \geq \frac{1}{2} \) only. We are now in a position to proceed with (7.9):

\[
\|\Delta_h^{\alpha}\|_\infty = O(h^{-\alpha}\|D_h R_h\|_\infty + \frac{\delta}{\Gamma(\alpha)}) = O(h^{\gamma + \alpha - 1} + \frac{\delta}{\Gamma(\alpha)}) \quad \text{as} \quad (h, \delta) \to 0,
\]

where also (7.5) has been used. This gives the desired result.

(ii) We now proceed with the case \( 1 < \gamma \leq 2 \). Preparatory results are given in the present item (iii) and in item (iii) the final steps will be done. Representation (2.12) of the integration error gives

\[
(E_h^{\alpha} \varphi_{n})(x_n) = h^{\alpha+1} s_n + t_n, \quad \text{with} \quad s_n = \sum_{j=1}^{n} \tau_{n-j} \varphi_{n}(x_{j-1/2}), \quad t_n = (\gamma^\alpha(\varphi_{n} - q_h \varphi_{n}))(x_n),
\]

for \( n = 1, 2, \ldots, N \), or, in vector notation (for the definition of \( R_h \) see (7.2))

\[
R_h = h^{\alpha+1} S_h + T_h, \quad \text{with} \quad S_h = (s_n)_{n=1,2,\ldots,N}, \quad T_h = (t_n)_{n=1,2,\ldots,N}. \quad (7.12)
\]

In view of (7.3) and (7.4), we need to consider the four vectors \( S_h, D_h S_h, T_h \) and \( D_h T_h \in \mathbb{R}^N \) in more detail.

- From the summability of the coefficients \( \tau_n \), cf. (2.15), it immediately follows that \( \|S_h\|_\infty = O(1) \) as \( h \to 0 \).
\begin{itemize}

\item In the case $\gamma > 2 - \alpha$ and $u(0) = u'(0) = 0$, it turns out to be useful to consider the vector $D_h S_h$. Partial summation related to the $n$th entry of $D_h S_h$ gives

\[
(D_h S_h)_n = \sum_{\ell=1}^{n} \omega_{n-\ell}^{(-1)} s_{\ell} = \beta_n s_1 + \sum_{\ell=1}^{n-1} \beta_{n-\ell}(s_{\ell+1} - s_{\ell}),
\]

with $\beta_n$ given by (7.9). The smoothness property (7.7), the assumption $u(0) = u'(0) = 0$ and the boundedness $\beta_n = O(1)$, cf. (7.10), imply that $\beta_n s_1 = \beta_n \tau_0 \varphi'_1(x_{1/2}) = O(h^{\gamma-1})$. In addition,

\[
s_{\ell+1} - s_{\ell} = \sum_{j=1}^{\ell} \tau_{\ell+1-j}\varphi'_{\ell+1}(x_{j-1/2}) - \sum_{j=1}^{\ell} \tau_{\ell-j}\varphi'_{\ell}(x_{j-1/2})
\]

\[
= \tau_{\ell}\varphi'_{\ell+1}(x_{1/2}) + \sum_{j=1}^{\ell} \tau_{\ell-j}(\varphi'_{\ell+1}(x_{j+1/2}) - \varphi'_{\ell}(x_{j-1/2})) = O(h^{\gamma-1})
\]

uniformly for $\ell = 1, 2, \ldots, N - 1$. The considered partial summation (7.13) thus finally results in (see also (7.11))

\[
\|D_h S_h\|_\infty = O(h^{\gamma-1}) + O(h^{\alpha-1+\gamma-1}) = O(h^{\gamma+\alpha-2}).
\] (7.14)

\item It follows from (2.10) that $\|T_h\|_\infty = O(h^\gamma)$ as $h \to 0$. This estimate will be useful in the case $\alpha \leq \frac{1}{2}$.

\item We next consider the vector $D_h T_h$ in more detail. Partial summation applied to the $n$th entry of $D_h T_h$ gives

\[
(D_h T_h)_n = \sum_{\ell=1}^{n} \omega_{n-\ell}^{(-1)} t_{\ell} = \beta_n t_1 + \sum_{\ell=1}^{n-1} \beta_{n-\ell}(t_{\ell+1} - t_{\ell}).
\] (7.15)

We have

\[
t_1 = O(h^{\gamma+\alpha}), \quad t_{\ell+1} - t_{\ell} = O(h^{\gamma+\alpha}),
\]

uniformly for $\ell = 1, 2, \ldots, N - 1$. This in fact is verified similarly as in the second item of part 4(ii) of this proof, this time with second order Taylor expansions of the kernel $k$ as well as first order Taylor expansions of $\frac{\partial k}{\partial y}$ with respect to the first variable, respectively. We omit the simple but tedious computations. This gives

\[
\|D_h T_h\|_\infty = O(h^{\gamma+\alpha}) + O(h^{\alpha-1+\gamma+\alpha}) = O(h^{\gamma+2\alpha-1}).
\] (7.16)

This estimate will be useful in the case $\alpha \geq \frac{1}{2}$.

\end{itemize}

It should be noticed that the second of the four considered items is the only one where the initial condition $u(0) = u'(0) = 0$ is needed.

(iii) We continue with the consideration of the case $1 < \gamma \leq 2$. The results from (ii) allow us to proceed with (7.3), (7.4).

\begin{itemize}

\item We first consider the case $\alpha \leq \frac{1}{2}$, $1 < \gamma \leq \alpha + 1$. The consistency error representations in item (ii) of the present proof yield $\|R_h\|_\infty = \max_{1 \leq n \leq N} |(E^h_n \varphi_n)(x_n)| = O(h^{\alpha+1} + ||S_h||_\infty + ||T_h||_\infty) = O(h^{\alpha+1} + h^{\gamma}) = O(h^\gamma)$. From the error equation (7.4) it then follows $\|\Delta_h\|_\infty = O(h^{-\alpha}(h^\gamma + \delta)) = O(h^{-\alpha} + \delta/h^\alpha)$.

\item We next consider the case $\alpha \geq \frac{1}{2}$, $1 < \gamma \leq 2 - \alpha$. The integration error estimates obtained in item (ii) yield $\|D_h R_h\|_\infty = O(h^{\alpha+1} + ||S_h||_\infty + ||D_h T_h||_\infty) = O(h^{\alpha+1} + h^{\gamma+2\alpha-1}) = O(h^{\gamma+2\alpha-1})$, where the first identity in (7.3) has been applied. From the error equation (7.4) it then follows $\|\Delta_h^\delta\|_\infty = O(h^{-\alpha}(h^{\gamma+2\alpha-1} + \delta)) = O(h^{-\alpha} + \delta/h^\alpha)$. Finally we consider the case $2 - \alpha < \gamma \leq 2$ and $u(0) = u'(0) = 0$. The consistency error estimates in item (ii) yield $\|D_h R_h\|_\infty = O(h^{\alpha+1} + ||D_h S_h||_\infty + ||D_h T_h||_\infty) = O(h^{\gamma+2\alpha-1})$. From the error equation (7.4) it then follows $\|\Delta_h^\delta\|_\infty = O(h^{-\alpha}(h^{\gamma+2\alpha-1} + \delta)) = O(h^{-\alpha} + \delta/h^\alpha)$.

\end{itemize}

This completes the proof of the theorem.
8 Conclusions

In the present paper we have considered the product midpoint rule for the regularization of weakly singular Volterra integral equations of the first kind with perturbed given right-hand sides. The applied techniques are closely related to those used in Eggermont [7]. The presented results include intermediate smoothness degrees of the solution of the integral equation in terms of Hölder continuity. In addition we have given a new proof of the stability estimate for the inverse of the generating sequence, cf. (3.6), which may be of independent interest. Another topic is the use of correction starting weights to get rid of initial conditions on the solution. Results of some numerical experiments are also given.

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