Phase Transition Threshold and Stability of Magnetic Skyrmions

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Abstract: We examine the stability of vortex-like configuration of magnetization in magnetic materials, so-called the chiral magnetic skyrmion. These correspond to critical points of the Landau–Lifshitz energy with the Dzyaloshinskii–Moriya (DM) interactions. In an earlier work of Döring and Melcher, it is known that the skyrmion is a ground state when the coefficient of the DM term is small. In this paper, we prove that there is an explicit critical value of the coefficient above which the skyrmion becomes unstable, while stable below this threshold. Moreover, we show that in the unstable regime, the infimum of energy is not bounded from below, by giving an explicit counterexample with a sort of helical configuration. It is worth noticing that this instability is different from the well-known elliptic instability of skyrmions. Our results mathematically explains the occurrence of phase transition observed in some experiments.

1. Introduction

We consider the Landau–Lifshitz energy functional of the form

\[ E_p[\mathbf{n}] = D[\mathbf{n}] + rH[\mathbf{n}] + V_p[\mathbf{n}], \quad \mathbf{n} : \mathbb{R}^2 \to S^2 \]  \hspace{1cm} (1.1)

where \( r > 0, \ p > 0 \) are constants, and

\[ D[\mathbf{n}] := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \mathbf{n}|^2 dx; \quad \text{Dirichlet energy}, \]
\[ H[\mathbf{n}] := \int_{\mathbb{R}^2} (\mathbf{n} - \mathbf{e}_3) \cdot \nabla \times \mathbf{n} dx; \quad \text{helicity}, \]
\[ V_p[\mathbf{n}] = \int_{\mathbb{R}^2} (1 - n_3) dx - \frac{p}{2} \int_{\mathbb{R}^2} (1 - n_3^2) dx; \quad \text{potential energy}. \]
First we point out that $\nabla \times \mathbf{n}$ can be defined as

$$\nabla \times \mathbf{n} = \begin{pmatrix} \partial_1 \\ \partial_2 \\ 0 \end{pmatrix} \times \mathbf{n},$$

which is the three-dimensional curl acting on maps depending only on $x_1$ and $x_2$. The Landau–Lifshitz energy arises in micromagnetics, where $\mathbf{n}$ represents magnetization vector in a magnetic material. The equilibrium state of magnetization is characterized by the stable critical points of $E_p$ (see [14] for the general theory). In the energy $E_p$, $D$ represents the exchange interaction, while $V_p$ arises from the combination of the external field (also known as the Zeeman potential) and the magneto-crystalline anisotropy perpendicular to the planar material. $H[\mathbf{n}]$ stands for the Dzyaloshinskii–Moriya interaction of magnetization that emerges in some particular crystalline structure [4,5]. When this kind of magnetic material is analyzed under the effect of strong external field, then localized vortex-like configuration of magnetization, called the chiral magnetic skyrmion, appears. This was first proposed theoretically in [4], experimentally observed (see for example in [18,23]), and mathematically proved in [8,12,16,17]. On the other hand, when the effect of the external field is weak, then skyrmions are expected to become unstable, and a sort of helical shapes emerge. This suggests the occurrence of a phase transition of the equilibrium state. Phase transition was introduced in [6,7] with numerical simulation evidence, and experimentally observed in [18,23]. There is however no rigorous analysis of such results justifying the above phenomena so far. The purpose of this paper is to establish the rigorous proof of phase transition through the analysis of the Landau–Lifshitz energy.

In $E_p$, the parameter $r$ measures the strength of external fields; larger values of $r$ correspond to weaker field strength. Before going any further, let us explain that putting different weights on $H$ and $V$, the energy can be formulated through two parameters as follows:

$$D[\mathbf{n}] + \kappa H[\mathbf{n}] + \mu V_p[\mathbf{n}].$$

However, one can always normalize the coefficient of $V_p$ using the rescaling $\mathbf{n}(x) \mapsto \mathbf{n}(\sqrt{\mu} x)$, which reduces the problem to (1.1) with $r = \kappa \mu^{-1/2}$.

In the present paper, we are interested in the particular choice of coefficient: $p = 1$, and we denote $E_c := E_1$, $V_c := V_1$. Then, we can write

$$V_c[\mathbf{n}] = \frac{1}{8} \int_{\mathbb{R}^2} |\mathbf{n} - \mathbf{e}_3|^4 dx = \frac{1}{2} \int_{\mathbb{R}^2} (1 - n_3)^2 dx.$$

The great advantage of this special choice is the nice factorization of the corresponding energy (Bogomol’nyi trick) that makes it mathematically favorable to study critical points. For instance, see (1.4). Define the function space of maps $\mathbf{n}$ by

$$\mathcal{M} := \{ \mathbf{n} : \mathbb{R}^2 \to \mathbb{S}^2 \mid D[\mathbf{n}] + V_c[\mathbf{n}] < \infty \}$$

endowed with the metric $d(\mathbf{n}, \mathbf{m}) := \| \mathbf{n} - \mathbf{m} \|_{L^4(\mathbb{R}^2)} + \| \nabla (\mathbf{n} - \mathbf{m}) \|_{L^2(\mathbb{R}^2)}$ for $\mathbf{n}, \mathbf{m} \in \mathcal{M}$. Recall that the energy functional $E_c$ is well-defined on $\mathcal{M}$ (see [8, Page 7] for the proof). In addition, thanks to the well-known inequality of Wente [22], the topological degree

$$Q[\mathbf{n}] := \frac{1}{4\pi} \int_{\mathbb{R}^2} \mathbf{n} \cdot \partial_1 \mathbf{n} \times \partial_2 \mathbf{n} dx$$
is also a well-defined, integer-valued functional on $\mathcal{M}$. $Q[n]$ represents the total number of skyrmions, and its sign gives an idea about their directions of rotation. In this paper, we will restrict ourselves to the case $Q = -1$ corresponding to a single skyrmion, which is known to be the most stable homotopy class (see [17]). Moreover, solutions in this class are known to enjoy additional properties.

The critical points of $E_c$ satisfy the Euler-Lagrange equation

$$-\Delta n + 2r \nabla \times n - (1 - n_3)e_3 - \Lambda(n)n = 0 \quad (1.2)$$

where

$$\Lambda(n) := |\nabla n|^2 + 2rn \cdot (\nabla \times n) - (1 - n_3)n_3.$$

Then (1.2) has an explicit solution:

$$h^{2r}(x) := h\left(\frac{x}{2r}\right), \quad h(x) := \left(\frac{-2x_2}{1 + |x|^2}, \frac{2x_1}{1 + |x|^2}, -\frac{1 - |x|^2}{1 + |x|^2}\right). \quad (1.3)$$

Note that $h^{2r}$ is a harmonic map, with $h^{2r} \in \mathcal{M}$ and $Q[h^{2r}] = -1$. In the work of Döring and Melcher [8], it is shown that when $0 < r \leq 1$, $h^{2r}$ is a global minimizer of the energy;

$$\min_{n \in \mathcal{M}} E_c[n] = E_c[h^{2r}] = 4\pi (1 - 2r^2).$$

In particular, their result explains the formation of skyrmion since the map $h^{2r}$ has the desired configuration. The essence of this result is that $E_c$ can be factorized as

$$E_c[n] - 4\pi r^2 Q[n] = \frac{r^2}{2} \int_{\mathbb{R}^2} |D^f_j n + n \times D^f_2 n|^2 dx + (1 - r^2)D[n] \quad (1.4)$$

where the helical derivatives $D^f_j$ are defined by

$$D^f_j := \partial_j - \frac{1}{r} e_j \times \cdot.$$

When $r \leq 1$, then the minimality immediately follows from (1.4) because $h^{2r}$ is a minimizer of $D$ with $Q = -1$, and satisfies $D^f_1 h^{2r} + n \times D^f_2 h^{2r} = 0$. However, this argument clearly breaks down when $r > 1$, where even the stability of $h^{2r}$ has not been understood so far.

Our main result is the instability of $E_c$ around $h^{2r}$ when $r > 1$.

**Theorem 1.** If $r > 1$, then the critical point $h^{2r}$ of the energy $E_c$ is unstable, in the sense that for any neighborhood of $h^{2r}$ there exists $n \in \mathcal{M}$ in that neighborhood such that $E_c[n] - E_c[h^{2r}] < 0$.

Theorem 1 rigorously explains why the phase transition occurs; the stability of skyrmions ceases to hold when the external field is weak. Moreover, Theorem 1, together with Döring-Melcher’s results, explicitly quantifies the threshold of phase transition at $r = 1$.

For the proof of Theorem 1, we follow the framework of [16]; We rewrite the quadratic form of the Hessian in terms of the coordinates of moving frame, and then decompose it with respect to Fourier modes of argument variable. To unveil unstable factors, we apply
a rescaling argument to derive a large radius asymptotic by following the strategy of [15]. Then we can find negative directions of the Hessian in the Fourier modes higher than or equal to 2. It is worth noticing that with this approach, the criticality of \( r = 1 \) manifests through the third Fourier mode; namely the unstable mode can only be observed via the Hessian at the third Fourier mode when \( r \) is close to 1+. We also observe that the Hessian at 0-th and first Fourier modes is always positive definite, regardless of \( r \). The latter mechanism has already been observed in that of the Ginzburg-Landau energy [11,15].

Next, it is natural to ask about the existence of a minimizer of \( E_4 \) in the unstable regime. We show that if \( r > 1 \), then the energy is unbounded from below.

**Theorem 2.** If \( r > 1 \), then

\[
\min_{\mathcal{M}} \min_{Q[n] = -1} E_c[n] = -\infty.
\]

This theorem suggests that (1.1) on \( \mathcal{M} \) is not well-suited to characterize equilibrium states in the regime of weak field. Nevertheless, it is worth mentioning that our example of the unbounded sequence has the same helical structure as observed in experiments of [23]. Moreover, our construction is by stretching a skyrmion in one direction, which roughly gives us the information of how the instability of skyrmions occurs. Note that this cannot happen when the domain is bounded, as minimizers may exist as expected by experiments. This is beyond the scope of this paper.

**Remark 1.** As already mentioned, our potential energy can only be seen as a particular combination of Zeeman and anisotropy term which does not seem to be generic in nature. As numerically suggested in [6,7], a similar result is expected to be true for wider range of potentials (i.e. different combinations).

We conclude this introduction by mentioning some known related studies. The minimizing problem of \( E_p \) is first addressed by [17] when \( p = 0 \), and the analysis is extended to various settings in [8,12,16]. Recently, the geometric interpretation of the integrand of the first term in the left hand side of (1.4) is given in [1,21], which yields a family of formal solutions to the corresponding Bogomol’nyi-type equation. The dynamical equation corresponding to the energy related to (1.1) is also investigated with the Gilbert damping by [8], or without damping by the second author [20]. It is worth noting that in the latter case, the equation is closely related to the nonlinear Schrödinger equation, and in fact, when the energy only consists of \( D[n] \), then a sort of dispersive properties are observed [2,3,9,10].

The organization of this paper is as follows. In Sect. 2, we first derive the Hessian of \( E_c \), then reduce the problem to its analysis. The main part is Sect. 3 where we construct an unstable direction of the Hessian, which concludes Theorem 1. In Sect. 4, we prove Theorem 2 by constructing a sequence which gives infinitely negative energy. In Appendix, we prove the technical lemmas used in the main argument.

### 2. Hessian

First of all, we observe that the difference of energy from \( h^{2r} \) can be written as a quadratic form.
**Lemma 1.** Let $n \in \mathcal{M}$ and $\xi := n - h^{2r}$. Then,

$$E_c[n] - E_c[h^{2r}] = \frac{1}{2} \langle \mathcal{L} \xi, \xi \rangle_{L^2}$$

(2.1)

where

$$\mathcal{L} \xi := -\Delta \xi + 2r \nabla \times \xi + \xi_3 e_3 - \Lambda (h^{2r}) \xi.$$

**Proof.** By the criticality of $h^{2r}$ for $E_c$, we have

$$E_c[n] - E_c[h^{2r}] = \int \Lambda (h^{2r}) h^{2r} \cdot \xi - \frac{1}{2} \int \Delta \xi \cdot \xi + r \int \nabla \times \xi \cdot \xi + \frac{1}{2} \int \xi^2_3.$$

The constraint $|n| = |h^{2r}| = 1$ yields

$$2\xi \cdot h^{2r} + |\xi|^2 = 0.$$

Thus

$$\int \Lambda (h^{2r}) h^{2r} \cdot \xi = -\frac{1}{2} \int \Lambda (h^{2r}) |\xi|^2,$$

which completes the proof. □

Let us focus on the quadratic form defined by the right hand side of (2.1). Now we claim that the perturbation $\phi$ may be linearized into the tangent space.

**Proposition 1** (Reduction of the theorem). Suppose that there is $\phi \in H^1(\mathbb{R}^2 : \mathbb{R}^3)$ with $\phi \cdot h^{2r} = 0$ such that $\langle \mathcal{L} \phi, \phi \rangle < 0$, then for any neighborhood of $h^{2r}$, there exists $n \in \mathcal{M}$ in that neighborhood such that $E_c[n] - E_c[h^{2r}] < 0$.

**Proof.** For $t > 0$, let $n_t := \frac{h^{2r} + t \phi}{|h^{2r} + t \phi|}$. Since $|h^{2r} + t \phi| = \sqrt{1 + t^2 |\phi|^2}$, $n_t$ is well-defined. By calculation, we have

$$n_t - h^{2r} = \frac{t \phi}{\sqrt{1 + t^2 |\phi|^2}} - \frac{t^2 |\phi|^2 h^{2r}}{\sqrt{1 + t^2 |\phi|^2} \left(1 + \sqrt{1 + t^2 |\phi|^2}\right)}.$$

$$\partial_j (n_t - h^{2r}) = \frac{t \partial_j \phi}{\sqrt{1 + t^2 |\phi|^2}} - \frac{t^2 |\phi|^2 \partial_j h^{2r}}{\sqrt{1 + t^2 |\phi|^2} \left(1 + \sqrt{1 + t^2 |\phi|^2}\right)} - \frac{t^3 (\phi \cdot \partial_j \phi) \phi}{(1 + t^2 |\phi|^2)^{3/2}},$$

which implies $n_t \to h^{2r}$ in $\mathcal{M}$ as $t \to 0+$. Moreover, (2.1) yields

$$E_c[n_t] - E_c[h^{2r}] = \langle \mathcal{L} (n_t - h^{2r}), n_t - h^{2r} \rangle = t^2 \langle \mathcal{L} \phi, \phi \rangle + o(t^2),$$

which is negative if $t$ is sufficiently small. This completes the proof. □

### 3. Proof of Theorem 1

In this section, we show Theorem 1. By Proposition 1, it suffices to find $\phi \in H^1(\mathbb{R}^2 : \mathbb{R}^3)$ with

$$\phi \cdot h^{2r} = 0, \quad \text{and} \quad \langle \mathcal{L} \phi, \phi \rangle < 0.$$

Following [16], one can rewrite the Hessian via several steps, using moving frame, Fourier expansion, and the Hardy decomposition.
3.1. Rescaling. For $\phi \in H^1(\mathbb{R}^2 : \mathbb{R}^3)$, we consider the $H^1(\mathbb{R}^2)$ rescaling $\phi^{2r}(\cdot) := \phi\left(\frac{\cdot}{2r}\right)$. Then a simple calculation shows that the rescaled Hessian $\mathcal{H}_r(\phi) := (L\phi^{2r}, \phi^{2r})$ can be written as

$$
\mathcal{H}_r(\phi) = \|\nabla \phi\|_{L^2}^2 + 4r^2(\nabla \times \phi, \phi)_{L^2} + 4r^2 \|\phi_3\|_{L^2}^2 - \int_{\mathbb{R}^2} \Lambda_r(h)|\phi|^2 \, dx \tag{3.1}
$$

with

$$
\Lambda_r(h) := |\nabla h|^2 + 4r^2 h \cdot \nabla h - 4r^2 (1-h_3)h_3.
$$

Note that the coefficients become balanced, and the dependence of the Hessian on $r$ gets more explicit. Our goal is now to find $\phi \in H^1(\mathbb{R}^2 : \mathbb{R}^3)$ with

$$
\phi \cdot h = 0, \quad \text{and} \quad \mathcal{H}_r(\phi) < 0. \tag{3.2}
$$

3.2. Moving frame. Let $(\rho, \psi)$ be the polar coordinates in $\mathbb{R}^2$. Then by (1.3), we can write

$$
h = \begin{pmatrix}
-\sin \psi \\
\sin \theta(\rho) \\
\cos \theta(\rho)
\end{pmatrix}
$$

where $\theta : [0, \infty) \rightarrow \mathbb{R}$ is the non-decreasing function defined using

$$
\sin \theta(\rho) = \frac{2\rho}{\rho^2 + 1}, \quad \theta(0) = \pi, \quad \theta(\infty) = 0.
$$

In particular, $\theta$ satisfies the following relations:

$$
\cos \theta = \frac{\rho^2 - 1}{\rho^2 + 1}, \quad \sin \theta - \theta(1 - \cos \theta) = 0, \quad \theta' = -\frac{2}{\rho^2 + 1} = -\frac{\sin \theta}{\rho}, \quad \theta'' + \theta' \rho - \frac{\sin \theta \cos \theta}{\rho^2} = 0. \tag{3.4}
$$

Based on (3.3), we introduce the moving frame in the tangent space at $h^{2r}$ as

$$
J_1 := \begin{pmatrix}
\cos \psi \\
\sin \psi \\
0
\end{pmatrix}, \quad J_2 := \begin{pmatrix}
-\sin \psi \cos \theta(\rho) \\
\cos \psi \cos \theta(\rho) \\
-\sin \theta(\rho)
\end{pmatrix}.
$$

For $\phi \in T_hS^2$, one can write

$$
\phi = u_1J_1 + u_2J_2, \quad \text{and then define} \quad u = ^t(u_1, u_2).
$$

Let us rewrite $\mathcal{H}_r(\phi)$ in terms of $u_1$ and $u_2$ following [16]. First note that

$$
\partial_\rho h = \theta'J_2, \quad \partial_\psi h = -\sin \theta)J_1, \quad \partial_\rho J_1 = 0,
$$

$$
\partial_\psi J_1 = (\cos \theta)J_2 + (\sin \theta)h, \quad \partial_\rho J_2 = -\theta'J_2, \quad \partial_\psi J_2 = -(\cos \theta)J_1.
$$
Hence, each component of the integrand in (3.1) can be reexpressed as

\[ |\nabla \phi|^2 = |\partial_\rho \phi|^2 + \frac{1}{\rho^2} |\partial_\phi \phi|^2 \]

\[ = |\nabla u|^2 + \frac{2 \cos \theta}{\rho^2} u \times \partial_\psi u + \frac{u_1^2}{\rho^2} + \left( \left( \partial' \right)^2 + \frac{\cos^2 \theta}{\rho^2} \right) u_2^2, \]

\[ \phi \cdot \nabla \times \phi = \phi \cdot \left[ J_1 \times \partial_\rho \phi + \frac{1}{\rho} \left( (\cos \theta) J_2 + (\sin \theta) \mathbf{h} \right) \times \partial_\phi \phi \right] \]

\[ = -\frac{\sin \theta}{\rho} u \times \partial_\psi u + \left( \theta' - \frac{\sin \theta \cos \theta}{\rho} \right) u_2^2, \]

\[ \phi_3^2 = u_2^2 \sin^2 \theta, \]

\[ \Lambda_r(\mathbf{h}) = \left( \theta' \right)^2 + \frac{\sin^2 \theta}{\rho^2} + 4r^2 \left( \theta' + \frac{\sin \theta \cos \theta}{\rho} \right) - 4r^2 (1 - \cos \theta) \cos \theta. \]

Hence by (3.4), we have

\[ \mathcal{H}_r[\phi] = \int_{\mathbb{R}^2} |\nabla u|^2 + \left( \frac{2 \cos \theta}{\rho^2} - \frac{4r^2 \sin \theta}{\rho} \right) u \times \partial_\psi u \]

\[ + \left( -\left( \theta' \right)^2 + \frac{\cos^2 \theta}{\rho^2} + \frac{4r^2 \sin \theta}{\rho} \right) (u_1^2 + u_2^2) dx \quad (3.5) \]

3.3. Fourier splitting. Next we apply Fourier expansion of \( u_j \) with respect to \( \psi \):

\[ u_j(\rho, \psi) = \alpha_j^{(0)}(\rho) + \sum_{k=1}^{\infty} \left( \alpha_j^{(k)}(\rho) \cos(k\psi) + \beta_j^{(k)}(\rho) \sin(k\psi) \right), \quad j = 1, 2. \]

Then \( \mathcal{H}_r[\phi] \) can be split in the following way:

\[ \mathcal{H}_r[\phi] = 2\pi \mathcal{H}_r'[\alpha_1^{(0)}, \alpha_2^{(0)}] + \pi \sum_{k=1}^{\infty} \left( \mathcal{H}_k'[\alpha_1^{(k)}, \beta_2^{(k)}] + \mathcal{H}_k'[\beta_1^{(k)}, -\alpha_2^{(k)}] \right) \quad (3.6) \]

where

\[ \mathcal{H}_k'[\alpha, \beta] := \int_0^\infty \left[ (\alpha')^2 + (\beta')^2 + \left( \frac{k^2}{\rho^2} - (\theta')^2 + \frac{\cos^2 \theta}{\rho^2} + \frac{4r^2 \sin \theta}{\rho} \right) (\alpha^2 + \beta^2) \right. \]

\[ + 4k \left( \frac{\cos \theta}{\rho^2} - \frac{2r^2 \sin \theta}{\rho} \right) \alpha \beta \] \quad \rho d\rho. \quad (3.7) \]

In order to find \( \phi \in H^1(\mathbb{R}^2 : \mathbb{R}^3) \) satisfying (3.2), it suffices to show that one of \( \mathcal{H}_k' \) can take negative value. In fact, it can be shown that \( \mathcal{H}_0', \mathcal{H}_1' \) are always non-negative definite for all \( r > 0 \). (See Appendix for the proof.) Thus we need to focus only on \( \mathcal{H}_k' \) with \( k \geq 2. \)
3.4. Instability at higher mode. We show the following:

**Proposition 2 (Instability at higher mode).** For \( k \geq 2 \), there exists \( r_{k,c} \geq 1 \) such that the following holds: If \( r > r_{k,c} \), then there exist \( \alpha, \beta \in C^\infty_0 (0, \infty) \) such that \( H^r_\lambda [\alpha, \beta] < 0 \). Moreover, if \( k = 3 \), then we can take \( r_{3,c} = 1 \).

For the proof, we change variables in the Hessian following the idea of [16]. We use the following lemma:

**Lemma 2.** Let \( A : (0, \infty) \to \mathbb{R} \) be nonnegative \( C^1 \) function, let \( V \in L^1_{\text{loc}} ((0, \infty) : \mathbb{R}) \), and let \( L = -\frac{d}{dp} A(p) \frac{d}{dp} + V \). Let \( f, g \in C^\infty_0 (0, \infty) \) be functions satisfying \( f = \psi g \) with some positive smooth function \( \psi : (0, \infty) \to (0, \infty) \). Then,

\[
\int_0^\infty (L f) f \, dp = \int_0^\infty \psi^2 A(g')^2 + \int_0^\infty L(\psi) \psi g^2 \, dp.
\]

Lemma 2 plays a role of simplification of quadratic forms \( \int_0^\infty (L f) f \, dp \), especially when \( L \) has a kernel as the ground state. Indeed, this is the case when \( k = 1 \), and applying Lemma 2 immediately concludes that \( H^r_\lambda \) is positive definite (see the proof of Proposition 3 in Appendix). Although \( H^r_\lambda \) for \( k \geq 2 \) does not have such kernel, we will apply Lemma 2 with \( \psi \) being the kernel of \( H^r_\lambda \), which enables us to find the unstable factors.

**Proof of Proposition 2.** First, let us set \( \alpha = \beta \). Then we have

\[
H^r_\lambda [\alpha, \alpha] = 2 \int_0^\infty \left[ \rho (\alpha')^2 + \left( \frac{(k+\cos \theta)^2 - \rho (\theta')^2 + 4r^2 (1-k) \sin \theta}{\rho} \right) \alpha^2 \right] \, dp
\]

\[
= 2 \int_0^\infty \left[ (L_1 \alpha) + \left( \frac{k^2 - 1}{\rho} + \frac{2(k-1) \cos \theta}{\rho} + 4r^2 (1-k) \sin \theta \right) \right] \alpha^2 \, dp
\]

where \( L_1 := -\frac{d}{dp} \rho \frac{d}{dp} + \frac{(1+\cos \theta)^2}{\rho} - \rho (\theta')^2 \). Noting that \( L_1 \left( \frac{\sin \theta}{\rho} \right) = 0 \), we transform \( \alpha = \frac{\sin \theta}{\rho} \xi \). Applying Lemma 2 with \( A = \rho, \psi = \frac{\sin \theta}{\rho}, V = \frac{(1+\cos \theta)^2}{\rho} - \rho (\theta')^2 \), we have

\[
H^r_\lambda \left[ \frac{\sin \theta}{\rho}, \xi \right] = \int_0^\infty \left[ \frac{2 \sin^2 \theta}{\rho} (\xi')^2 + f_k^r (\rho) \xi^2 \right] \, dp
\]

where \( f_k^r (\rho) := 2(k^2 - 1) \frac{\sin^2 \theta}{\rho^3} + 4(k-1) \frac{\sin^2 \theta \cos \theta}{\rho^3} + 8(1-k)r^2 \frac{\sin^3 \theta}{\rho^2} \).

Since

\[
\sin \theta = \frac{2}{\rho} + o(\rho^{-1}), \quad \cos \theta = 1 + o(\rho^{-1})
\]

as \( \rho \to \infty \), we have

\[
f_k^r (\rho) = -8(k-1)((8r^2 - k - 3)\rho^{-5} + o(\rho^{-5}).
\]

Now we consider the rescaling

\[
\xi_\lambda (\rho) := \frac{1}{\lambda^2} \xi (\lambda \rho), \quad \lambda > 0.
\]
Then for $\xi \in C_0^\infty (0, \infty)$, we have

$$H_k \left[ \begin{array}{l} \sin \theta / \rho \xi, \\ \sin \theta / \rho \xi \end{array} \right] = \int_0^\infty \left[ \frac{8}{\rho^3} (\xi')^2 - \frac{8(k - 1)(8r^2 - k - 3)}{\rho^5} \xi^2 \right] d\rho + o(1).$$

as $\lambda \to 0^+$. Thus we obtain

$$\lim_{\lambda \to 0^+} H_k \left[ \begin{array}{l} \sin \theta / \rho \xi \lambda, \\ \sin \theta / \rho \xi \lambda \end{array} \right] = \int_0^\infty \left[ \frac{8}{\rho^3} (\xi')^2 - \frac{8(k - 1)(8r^2 - k - 3)}{\rho^5} \xi^2 \right] d\rho,$$

which we denote $I_k[\xi]$. Hence for Proposition 2, it suffices to show that $I_k[\xi] < 0$ for some $\xi \in C_0^\infty (0, \infty)$. This problem is concerned with the optimization of the constant of the Hardy-type inequality:

$$C_H := \inf \left\{ C \left| \int_0^\infty \frac{\xi^2}{\rho^2} d\rho \leq C \int_0^\infty \frac{(\xi')^2}{\rho^3} d\rho \quad \text{for all} \quad \xi \in C_0^\infty (0, \infty) \right. \}.$$

In fact, it is known that $C_H = \frac{1}{4}$, and thus for any $\varepsilon > 0$, there exists $\xi_\varepsilon \in C_0^\infty (0, \infty) \setminus \{0\}$ such that

$$\int_0^\infty \frac{\xi_\varepsilon^2}{\rho^2} d\rho > \left( \frac{1}{4 + \varepsilon} \right) \int_0^\infty \frac{(\xi_\varepsilon')^2}{\rho^3} d\rho.$$

This fact is shown in [13] (see also [19]), while in Appendix we will reproduce the proof for reader’s convenience. Using $\xi_\varepsilon$, we have

$$I_k[\xi_\varepsilon] < \int_0^\infty \left[ \frac{8(4 + \varepsilon)}{\rho^3} \xi_\varepsilon^2 - \frac{8(k - 1)(8r^2 - k - 3)}{\rho^5} \xi_\varepsilon^2 \right] d\rho$$

$$= 8[4 + \varepsilon - (k - 1)(8r^2 - k - 3)] \int_0^\infty \frac{1}{\rho^5} \xi_\varepsilon^2 d\rho.$$

If $k \geq 2$, then the right hand side is negative for sufficiently large $r$. Especially when $k = 3$, it holds that

$$I_k[\xi_\varepsilon] < 128 \left( 1 - r^2 + \frac{\varepsilon}{16} \right) \int_0^\infty \frac{1}{\rho^5} \xi_\varepsilon^2 d\rho$$

which is negative when $r > 1$ if $\varepsilon$ is sufficiently small. Thus the proof of Proposition 2 is complete. □

Proof of Theorem 1. According to Proposition 2, if $r > 1$, then there exist $\alpha, \beta \in C_0^\infty (0, \infty)$ such that $H_3^\alpha[\alpha, \beta] < 0$. Now define

$$u_1(\rho, \psi) := \alpha(\rho) \cos (3\psi), \quad u_2(\rho, \psi) := -\beta(\rho) \sin (3\psi).$$

Then by (3.5) and (3.6), $\phi := u_1 J_1 + u_2 J_2$ satisfies $H_r[\phi] < 0$, which completes the proof by Proposition 1.
Remark 2. As noted in the proof of Lemma 2, $\xi(\rho) = \rho^2$ is an almost optimizer of (3.9). Notice that it is invariant under the rescaling (3.8), and one can obtain its corresponding map $\phi$ by taking $\alpha(\rho) = \beta(\rho) = \frac{\sin(\theta(\rho))}{\rho} \xi(\rho)$. This yields

$$\phi = \frac{2\rho^2}{\rho^2 + 1} (\cos(3\psi) J_1 - \sin(3\psi) J_2),$$

which is an example almost\(^1\) satisfying (3.2). The corresponding perturbation of $h^{2r}$ is

$$n_\varepsilon := \begin{pmatrix} - \sin\left(\psi - \frac{\varepsilon}{2r} \rho \cos 3\psi\right) \sin(\theta(\frac{\rho}{r^2}) - \varepsilon \frac{2\rho^2}{\rho^2 + 4r^2} \sin 3\psi) \\ \cos\left(\psi - \frac{\varepsilon}{2r} \rho \cos 3\psi\right) \sin(\theta(\frac{\rho}{r^2}) - \varepsilon \frac{2\rho^2}{\rho^2 + 4r^2} \sin 3\psi) \\ \cos(\theta(\frac{\rho}{r^2}) - \varepsilon \frac{2\rho^2}{\rho^2 + 4r^2} \sin 3\psi) \end{pmatrix}, \quad \varepsilon \in \mathbb{R}, \quad (3.12)$$

which is considered as energetically unstable deformation.

Figure 1 illustrates a few level sets of $n_3$. This shows a new kind of instability that is different than the well known elliptic instability (see [6]).

4. Proof of Theorem 2

Let $r > 1$. In this section we construct a sequence $\{n_\nu\}_{\nu=1}^{\infty}$ with

$$n_\nu \in \mathcal{M}, \quad Q[n_\nu] = -1, \quad \text{and} \quad \lim_{n \to \infty} E_c[n_\nu] = -\infty. \quad (4.1)$$

The key ingredient is the specific map defined as

$$b(x) := \begin{pmatrix} 0, & \frac{2r x_1}{r^2(x_1)^2 + 1}, & \frac{r^2(x_1)^2 - 1}{r^2(x_1)^2 + 1} \end{pmatrix}. \quad (4.2)$$

\(^1\) One needs to smoothly cut-off $\xi(\rho)$ to make it an example.
This is known as the line defect as defined in [1]. Note that \( \mathbf{b}(x) = \mathbf{h}^{1/r}(x_1, 0) \) where \( \mathbf{h} \) is as in (1.3), and \( \mathbf{b} \) satisfies an equation similar to the Beltrami field:

\[
\nabla \times \mathbf{b} = -\frac{b_2}{x_1} \mathbf{b}.
\]

If we calculate the integrand of \( E_c \), then

\[
\frac{1}{2} |\nabla \mathbf{b}|^2 + r(\mathbf{b} - \mathbf{e}_3) \cdot \nabla \times \mathbf{b} + \frac{1}{2} (b_3 - 1)^2 = \frac{2(1 - r^2)}{(r^2 x_1^2 + 1)^2}.
\]

Thus the energy of \( \mathbf{b} \) is \(-\infty \) if \( r > 1 \). Our construction of \( \{\mathbf{n}_r\} \) with (4.1) is based on the cut-off of \( \mathbf{b} \).

Define

\[
\mathbf{n}_L(x) := \begin{cases} 
\mathbf{b}(x) & \text{if } |x_2| \leq L, \\
\mathbf{h}^{1/r}(x_1, x_2 - L) & \text{if } x_2 > L, \\
\mathbf{h}^{1/r}(x_1, x_2 + L) & \text{if } x_2 < -L.
\end{cases}
\]

Then \( \mathbf{n}_L \in \mathcal{M} \cap C(\mathbb{R}^2) \), and we have \( Q[\mathbf{n}_L] = -1 \) since \( \mathbf{n}_L \) is homotopic to \( \mathbf{h}' \) by the homotopy with \( L \) shrinking to 0. Moreover, (4.2) gives

\[
E_c[\mathbf{n}_L] = \int_{|x_2| \leq L} \left( \frac{1}{2} |\nabla \mathbf{b}|^2 + r(\mathbf{b} - \mathbf{e}_3) \cdot \nabla \times \mathbf{b} + \frac{1}{2} (b_3 - 1)^2 \right) dx
\]

\[
+ \int_{|x_2| > L} \left[ \frac{1}{2} |\nabla \mathbf{h}^{1/r}|^2 + r(\mathbf{h}^{1/r} - \mathbf{e}_3) \cdot \nabla \times \mathbf{h}^{1/r} + \frac{1}{2} (h_3^{1/r} - 1)^2 \right] dx
\]

\[
+ \int_{|x_2| < -L} \left[ \frac{1}{2} |\nabla \mathbf{h}^{1/r}|^2 + r(\mathbf{h}^{1/r} - \mathbf{e}_3) \cdot \nabla \times \mathbf{h}^{1/r} + \frac{1}{2} (h_3^{1/r} - 1)^2 \right] dx
\]

\[
= \int_{|x_2| \leq L} \frac{2(1 - r^2)}{(r^2 x_1^2 + 1)^2} dx + E_c[\mathbf{h}^{1/r}] = (1 - r^2)C_r L + E_c[\mathbf{h}^{1/r}]
\]

where \( C_r \) is positive constant independent of \( L \). Hence we have

\[
\lim_{L \to \infty} E_c[\mathbf{n}_L] = -\infty
\]

which concludes the proof of Theorem 2.

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Declarations

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A. Appendix

A.1. Positivity of the Hessian at lower modes. We show that $\mathcal{H}_k^r$ defined as (3.7) is positive definite.

**Proposition 3.** For all $r \geq 0$, we have $\mathcal{H}_0^r[\alpha, \beta] \geq 0$, $\mathcal{H}_1^r[\alpha, \beta] \geq 0$ for all $\alpha, \beta \in C_0^\infty(0, \infty)$.

**Proof.** Our proof essentially follows [16]. We first see the case $k = 0$:

$$\mathcal{H}_0^r[\alpha, \beta] = \int_0^\infty \left[ (\alpha')^2 + (\beta')^2 + \left( - (\theta')^2 + \frac{\cos^2\theta}{\rho^2} \right) (\alpha^2 + \beta^2) + 4r^2 \frac{\sin \theta}{\rho} (\alpha^2 + \beta^2) \right] \rho d\rho.$$  

Clearly it suffices to show the case $r = 0$. Now we can write

$$\mathcal{H}_0^0[\alpha, \beta] = \int_0^\infty [(L_0 \alpha) \alpha + (L_0 \beta) \beta] d\rho$$

with $L_0 := -\frac{d}{d\rho} \rho \frac{d}{d\rho} - \rho (\theta')^2 + \frac{\cos^2 \theta}{\rho}$. Noting that $L_0(\sin \theta) = 0$, we transform

$$\alpha = (\sin \theta) \xi, \beta = (\sin \theta) \eta, \ (\xi, \eta \in C_0^\infty(0, \infty)).$$

Applying Lemma 2 with $A = \rho$, $\psi = \sin \theta$, $V = -\rho (\theta')^2 + \frac{\cos^2 \theta}{\rho}$, we obtain

$$\mathcal{H}_0^0[(\sin \theta) \xi, (\sin \theta) \eta] = \int_0^\infty \sin^2 \theta [(\xi')^2 + (\eta')^2] \rho d\rho \geq 0.$$  

Next we consider the case $k = 1$. We can write

$$\mathcal{H}_1^r[\alpha, \beta] = \int_0^\infty \left[ (\alpha')^2 + (\beta')^2 + \left( \frac{1}{\rho^2} - (\theta')^2 + \frac{\cos^2 \theta}{\rho^2} \right) (\alpha^2 + \beta^2) + \frac{4 \cos \theta}{\rho^2} \alpha \beta ight.$$

$$\left. + \frac{4r^2 \sin \theta}{\rho} (\alpha - \beta)^2 \right] \rho d\rho.$$  

Thus it also suffices to show the case $r = 0$. Then we can write

$$\mathcal{H}_1^0[\alpha, \beta] = \int_0^\infty \left[ (L_1 \alpha) \alpha + (L_1 \beta) \beta - \frac{2 \cos \theta}{\rho} (\alpha - \beta)^2 \right] d\rho$$  

with $L_1 := -\frac{d}{d\rho} \rho \frac{d}{d\rho} + \frac{(1+\cos \theta)^2}{\rho} - \rho (\theta')^2$. Noting that $L_1(\sin \theta) = 0$, we transform

$$\alpha = \frac{\sin \theta}{\rho} \xi, \beta = \frac{\sin \theta}{\rho} \eta, \ (\xi, \eta \in C_0^\infty(0, \infty)).$$

Applying Lemma 2 with $A = \rho$, $\psi = \sin \theta$, $V = \frac{(1+\cos \theta)^2}{\rho} - \rho (\theta')^2$, and using (3.4), we have

$$\mathcal{H}_1^0[(\sin \theta) \xi, (\sin \theta) \eta] = \int_0^\infty \frac{\sin^2 \theta}{\rho} (\xi'^2 + \eta'^2) + \frac{2 \sin \theta \cos \theta (\theta')}{\rho^2} (\xi - \eta)^2 d\rho$$

$$= \int_0^\infty \frac{\sin^2 \theta}{\rho} (\xi'^2 + \eta'^2) + \frac{(\sin^2 \theta)'}{\rho^2} (\xi - \eta)^2 d\rho$$

$$= \int_0^\infty \frac{\sin^2 \theta}{\rho} (\xi'^2 + \eta'^2) + \sin^2 \theta \left( \frac{2(\xi - \eta)(\xi' - \eta')}{\rho^2} - \frac{2(\xi - \eta)^2}{\rho^3} \right) d\rho$$

$$= \int_0^\infty \frac{\sin^2 \theta}{\rho} \left[ (\xi' - \frac{\xi - \eta}{\rho})^2 + (\eta' + \frac{\xi - \eta}{\rho})^2 \right] \geq 0.$$  

Hence the proof is complete. □
A.2. The optimality of the Hardy-type inequality. In this section we give a proof of the optimality $C_H = \frac{1}{4}$. (For the proof of $C_H \geq \frac{1}{4}$, see [13].) More precisely, we show the following:

**Lemma 3.** For any $\epsilon > 0$ there exists $\xi_\epsilon \in C_0^\infty(0, \infty) \setminus \{0\}$ such that

$$
\left( \frac{1}{4 + \epsilon} \right) \int_0^\infty \frac{(\xi_\epsilon')^2}{\rho^3} d\rho < \int_0^\infty \frac{\xi_\epsilon^2}{\rho^5} d\rho.
$$

**Proof.** As given in [13], $C_H$ is formally optimized by $\xi = \rho^2$. To seek the compactness of support, we take a cut-off of this function. Given $A > 1$, let $\chi = \chi_A \in C_0^\infty(0, \infty)$ be a function with $\chi(\rho) = 1$ if $\rho \in [1, A]$, $\chi(\rho) = 0$ if $\rho \notin \left( \frac{1}{2}, 2A \right)$, $|\chi'(\rho)| \leq \frac{2}{A}$ for $\rho \in [A, 2A]$, and $0 \leq \chi(\rho) \leq 1$ for all $\rho \in (0, \infty)$. Then define

$$
\xi_A(\rho) := \rho^2 \chi_A(\rho)
$$

for $\epsilon > 0$. Then calculation gives

$$
\int_0^\infty \frac{\xi_A^2}{\rho^5} d\rho = \int_0^\infty \frac{1}{\rho} \chi_A^2(\rho) d\rho,
$$

$$
\int_0^\infty \frac{(\xi_A')^2}{\rho^3} d\rho = 4 \int_0^\infty \frac{1}{\rho} \chi_A^2 d\rho + 4 \int_0^\infty \chi_A \chi'_A d\rho + \int_0^\infty \rho (\chi_A')^2 d\rho
$$

Thus it suffices to show that given $\epsilon > 0$, there exists $A > 0$ such that

$$
\int_0^\infty \rho (\chi'_A)^2 d\rho \leq \epsilon \int_0^\infty \frac{1}{\rho} \chi_A^2 d\rho.
$$

(A.1)

For (A.1), we can estimate as

$$
\int_0^\infty \frac{1}{\rho} \chi_A^2 d\rho \geq \int_1^A \frac{1}{\rho} = \log A,
$$

$$
\int_0^\infty \rho (\chi'_A)^2 d\rho \leq C \int_1^2 \rho d\rho + \frac{4}{A^2} \int_A^{2A} \rho d\rho \leq C
$$

where $C$ is independent of $A$. Thus we have

$$
\left[ \int_0^\infty \frac{1}{\rho} \chi_A^2 d\rho \right]^{-1} \int_0^\infty \rho (\chi'_A)^2 d\rho \xrightarrow{A \to \infty} 0
$$

which implies (A.1) for sufficiently large $A$. ⊓⊔
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