Methods of arbitrary optimal order with tetrahedral finite-element meshes forming polyhedral approximations of curved domains

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Abstract

In a recent paper [17] the author introduced a simple alternative to isoparametric finite elements of the $N$-simplex type, to enhance the accuracy of approximations of boundary value problems with Dirichlet boundary conditions, posed in smooth curved two-dimensional domains. This technique is based upon trial-functions consisting of piecewise polynomials defined on triangular meshes forming a sequence of approximating polygons, interpolating the Dirichlet boundary conditions at points of the true boundary. In contrast the test-functions are defined upon the standard degrees of freedom associated with the method in use. In this work this method is extended to problems posed in three-dimensional curved domains solved with tetrahedron-based finite element methods. Although the method is as universal as can be, for the sake of simplicity we consider as a model the Poisson equation. Optimal a priori error estimates of arbitrary order are derived for the classical Lagrangian family of finite elements. A series of numerical examples illustrates the potential of this technique.

Key words: curved domain, Dirichlet, finite elements, interpolated boundary conditions, Lagrange, tetrahedron.

1 Introduction

This work deals with a finite element method to solve boundary value problems with Dirichlet boundary conditions, posed in three-dimensional domains with a smooth curved boundary of arbitrary shape. The principle it is based upon is the same as in [17] for the analogous two-dimensional case. The latter in turn is inspired by the technique known as interpolated boundary conditions, or simply IBC, studied in [4]. In spite of being very intuitive and known since the seventies (cf. [14] and [20]) IBC has not been much used so far. This is certainly due to its difficult implementation, the lack of an extension to three-dimensional problems, and most of all, restrictions on the choice of boundary nodal points to reach optimal convergence rates. In contrast implementation of our method is straightforward in both two- and three-dimensional geometries. This is due to the fact that only polynomial algebra is necessary, while the domain is simply approximated by the polytope equal to the union of standard $N$-simplexes of a finite-element mesh. Furthermore approximations of optimal order can be obtained for non-restrictive choices of boundary nodal points.

Generally speaking our methodology is designed to handle Dirichlet conditions to be prescribed at boundary points different from mesh vertices, or yet over entire boundary edges or faces, in connection with methods of order greater than one in problem’s natural norm, for a wide spectrum of boundary value problems. For example, in principle the application of underlying ideas would avoid order erosion of the $RT_1$ mixed finite element (cf. [15]) or yet the second order modification of the $BDM_1$ mixed method considered in [6], in case fluxes are prescribed all over disjoint smooth curved portions of the boundary.

Here we confine our study of our technique taking as a model the Poisson equation solved by the classical Lagrange tetrahedron-based methods of degree greater than one. For instance, if quadratic finite elements are employed and we shift prescribed solution boundary values from the true boundary to the
mid-points of the boundary edges of the approximating polyhedron, the error of the numerical solution will be of order not greater than 1.5 in the energy norm (cf. [7]), instead of the best possible second order. Unfortunately this only happens if the true domain itself is a polyhedron, assuming of course that the solution is sufficiently smooth.

Since early days finite element users considered method’s isoparametric version, with meshes consisting of curved triangles or tetrahedra, as the ideal way to recover optimality in the case of a curved domain (cf. [22]). However, besides an elaborated description of the mesh, the isoparametric technique inevitably leads to the integration of rational functions to compute the system matrix. This raises the delicate question on what numerical quadrature formula should be used in order to avoid qualitative losses in the error estimates, or yet ill-posedness of approximate problems. In contrast, in the technique introduced in [17] and further exploited in this work, exact numerical integration can always be used for this purpose, since we only have to deal with polynomial integrands to compute element matrices. Furthermore the element geometry remains the same as in the case of polytopic domains. It is noteworthy that both advantages do not bring about any order erosion in the error estimates that hold for our method, as compared to the equivalent isoparametric one. As a matter of fact the former can be viewed as a small perturbation of standard conforming Lagrange finite elements based on meshes consisting of tetrahedra with plane faces, in connection with the usual Galerkin formulation.

An outline of the paper is as follows. In Section 2 we describe the new method to solve a model problem with Dirichlet boundary conditions in a smooth curved three-dimensional domain. In Section 3 underlying well-posedness results and a priori error estimates are given, as mere extensions of those proved in [17] for the analogous two-dimensional case. In Section 4 we illustrate the approximation properties of the method studied in the previous sections, by solving several significant test-problems with the popular $P_2$ finite element method. We conclude in Section 5 with some comments on the whole work.

2 Method description

As a model we consider the Poisson equation with Dirichlet boundary conditions in a three-dimensional smooth domain $\Omega$ with boundary $\Gamma$, namely:

$$
\begin{align*}
- \Delta u &= f \text{ in } \Omega \\
  u &= g \text{ on } \Gamma,
\end{align*}
$$

where $f$ and $g$ are given functions with suitable regularity properties, defined in $\Omega$ and on $\Gamma$, respectively. Our technique is most effective in connection with methods of order $k > 1$ in the standard energy norm $\| \text{grad}(\cdot) \|_0$, in case $u \in H^{k+1}(\Omega)$, where $\| \cdot \|_0$ equals $\int_\Omega (\cdot)^2/2$ (that is, the standard norm of $L^2(\Omega)$). Accordingly, in order to make sure that $u$ possesses the $H^{k+1}$-regularity property we shall assume that $f \in H^{k-1}(\Omega)$ and $g \in H^{k+1/2}(\Gamma)$ (cf. [1]), and moreover that $\Omega$ is sufficiently smooth. For instance, if $k = 2$ we assume at least that $\Gamma$ is of the $C^1$-class.

At this point it is important to stress the fact that, owing to the Sobolev Embedding Theorem [1] $g$ is necessarily a continuous function, since $k$ is not less than one. We also note that our regularity assumptions rule out the case where $\Gamma$ is the union of smooth curved portions which do not form a manifold of the $C^1$-class.

Now let us be given a mesh $T_h$ consisting of straight-edged tetrahedra satisfying the usual compatibility conditions (see e.g. [7]). Every element of $T_h$ is to be viewed as a closed set. Moreover this partition is assumed to fit $\Omega$ in such a way that all the vertices of the polyhedron $\cup_{T \in T_h} T$ lie on $\Gamma$. We denote the interior of this union set by $\Omega_h$. $T_h$ is assumed to belong to a uniformly regular family of partitions. Let $\Gamma_h$ be the boundary of $\Omega_h$, $h_T$ be the diameter of $T \in T_h$ and $h := \max_{T \in T_h} h_T$. In order to avoid non-essential difficulties we make the assumption that no element in $T_h$ has more than one face on $\Gamma_h$. Notice that such a condition is commonly fulfilled in practice, since otherwise there would be excessively flat tetrahedra in the mesh.
We also need some definitions regarding the set \((Ω \setminus Ω_h) \cup (Ω_h \setminus Ω)\). Let \(S_h\) be the subset of \(T_h\) consisting of tetrahedra having one face on \(Γ_h\) and by \(R_h\) the subset of \(T_h \setminus S_h\) of tetrahedra having exactly one edge on \(Γ_h\). Notice that, owing to our initial assumption, no tetrahedron in \(T_h \setminus \{S_h \cup R_h\}\) has a non-empty intersection with \(Γ_h\).

With every edge \(e\) of \(Γ_h\) we associate a plane skin \(δ_e\) containing \(e\), and delimited by \(Γ\) and \(e\) itself. Except for the fact that each skin contains an edge of \(Γ_h\), its plane can be arbitrarily chosen. In Figure 1 we illustrate one out of three such skins corresponding to the edges of a face \(F_T\) or \(F_T'\) contained in \(Γ_h\), of two tetrahedra \(T\) and \(T'\) belonging to \(S_h\). More precisely in Figure 1 we show the skin \(δ_e\), \(e\) being the edge common to \(F_T\) and \(F_T'\). Further, for every \(T \in S_h\), we define a closed set \(Δ_T\) delimited by \(Γ\), the face \(F_T\) and the three plane skins associated with the edges of \(F_T\), as illustrated in Figure 1. In this manner we can assert that, if \(Ω\) is convex, \(Ω_h\) is a proper subset of \(Ω\) and moreover \(Ω\) is the union of the disjoint sets \(Ω_h\) and \(∪_{T \in S_h} Δ_T\) (cf. Figure 1). Otherwise \(Ω_h \setminus Ω\) is a non-empty set that equals the union of certain parts of the sets \(Δ_T\) (whose volume is an \(O(h^4)\)) and tiny portions of \(R \in R_h\) (whose volume is an \(O(h^2)\)), both types of subsets corresponding to non-convex portions of \(Γ\). Whatever the case, these configurations are merely illustrative and play no role in the practical implementation of our method, as seen hereafter.

![Figure 1: Sets Δ_T, Δ_{T'}, δ_e for tetrahedra T, T' ∈ S_h with a common edge e and a tetrahedron T'' ∈ R_h](image)

Next we introduce a space \(V_h\) and a linear manifold \(W^g_h\), both associated with \(T_h\).

\(V_h\) is the standard Lagrange finite element space consisting of continuous functions \(v\) defined in \(Ω_h\) that vanish on \(Γ_h\), whose restriction to every \(T \in T_h\) is a polynomial of degree less than or equal to \(k\) for \(k ≥ 2\). For convenience we extend by zero every function \(v \in V_h\) to \(Ω \setminus Ω_h\). We recall that a function in \(V_h\) is uniquely defined by its values at the points which are vertices of the partition of each mesh tetrahedron into \(k^3\) equal tetrahedra (see e.g. [22] and [7]). Henceforth these points will be referred to as the Lagrangian nodes of order \(k\) of the mesh.

\(W^g_h\) in turn is the set of functions defined in \(Ω_h\) having the properties listed below.

1. The restriction of \(w \in W^g_h\) to every \(T \in T_h\) is a polynomial of degree less than or equal to \(k\);
2. Every \(w \in W^g_h\) is single-valued at the vertices of \(Ω_h\) and at all the inner Lagrangian nodes of the mesh, that is, all its Lagrangian nodes of order \(k\) except those located on \(Γ_h\) which are not vertices of \(Ω_h\);

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3. Except for points located in (the closure of) the skins $\delta_e$, a function $w \in W^g_h$ is also defined in $\bar{\Omega} \setminus \Omega_h$, in such a way that its polynomial expression in $T \in S_h$ also applies to points in $\Delta_T$;

4. Except for the nodes located on $\Gamma$ (see next two items), a function $w \in W^g_h$ is multi-valued in the skins $\delta_e$ where its polynomial expression may be anyone of those in a tetrahedron belonging to $S_h \cup R_h$ to which $\delta_e$ is attached;

5. A function $w \in W^g_h$ takes the value $g(S)$ at any vertex $S$ of $\Gamma_h$;

6. $\forall T \in S_h, w(P) = g(P)$ for every point $P$ among the $(k - 1)(k - 2)/2$ nearest intersections with $\Gamma$ of the line passing through the vertex $O_T$ of $T$ not belonging to $\Gamma$ and the $(k - 1)(k - 2)/2$ points $M$ not belonging to any edge of $F_T$ among the $(k + 2)(k + 1)/2$ points of $F_T$ that subdivide this face (opposite to $O_T$) into $k^2$ equal triangles (see illustration in Figure 2 for $k = 3$);

7. $\forall T \in S_h \cup R_h, w(Q) = g(Q)$ for every $Q$ among the $k - 1$ nearest intersections with $\Gamma$ of the line orthogonal to $e$ in the skin $\delta_e$, passing through the points $M \in e$ different from vertices of $T$, subdividing $e$ into $k$ equal segments, where $e$ represents a generic edge of $T$ contained in $\Gamma_h$ (see illustration in Figure 3 for $k = 3$).

**Figure 2:** Construction of node $P \in \Gamma$ of $W^g_h$ related to the Lagrangian node $M$ in the interior of $F_T \subset \Gamma_h$

**Remark 1** As a consequence of the above definition it is clear that, as a rule, a function in $W^g_h$ is not continuous in $\Omega_h$. This is because its traces from both sides of a face $F$ common to two tetrahedra in either $S_h$ or $R_h$ necessarily coincide only at a set of $(k + 2)(k + 1)/2 - (k - 1)$ Lagrangian nodes on $F$. However this by no means erodes the optimality of our method in terms of order, as seen hereafter.

**Remark 2** The construction of the nodes associated with $W^g_h$ located on $\Gamma$ advocated in items 6. and 7. is not mandatory. Notice that it differs from the intuitive construction of such nodes lying on normals to faces of $\Gamma_h$ commonly used in the isoparametric technique. The main advantage of this proposal is the determination by linearity of the coordinates of the boundary nodes $P$ in the case of item 6. Nonetheless the choice of boundary nodes ensuring our method’s optimality is absolutely very wide.

The fact that $W^g_h$ is a non-empty set is a trivial consequence of the two following lemmata, where $\mathcal{P}_k(T)$ represents the space of polynomials defined in $T \in S_h$ of degree less than or equal to $k$. 

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Figure 3: Construction of nodes $Q \in \Gamma \cap \delta e$ of $W_h^g$ related to the Lagrange nodes $M \in e \subset \Gamma_h$

Lemma 2.1 Provided $h$ is small enough $\forall T \in S_h$, given a set of $m_k$ real values $b_i$, $i = 1, \ldots, m_k$ with $m_k = k(k + 2)(k + 1)/6$, there exists a unique function $w_T \in \mathcal{P}_k(T)$ that takes the value of $g$ at the three vertices $S$ of $T$ located on $\Gamma$, at the $(k - 1)(k - 2)/2$ points $P$ of $\Gamma$ defined in accordance with item 6. and at the $3(k - 1)$ points $Q$ of $\Gamma$ defined in accordance with item 7. of the above definition of $W_h^g$, and takes the value $b_i$ respectively at the $m_k$ Lagrange nodes of $T$ not located on $\Gamma_h$.

Proof. Let us first extend the vector $\vec{b} = [b_1, b_2, \ldots, b_{m_k}]$ of $\mathbb{R}^{m_k}$ into a vector of $\mathbb{R}^{m_k}$ still denoted by $\vec{b}$, with $n_k := m_k + (k + 2)(k + 1)/2$, by adding $n_k - m_k$ components $b_i$ which are the values of $g$ at the $(k + 2)(k + 1)/2$ nodes $P$ or $Q$ of $T \cup \Delta_T$ located on $\Gamma$. If the latter nodes were replaced by the corresponding $M \in \Gamma_h \cap T$, it is clear that the result would hold true, according to the well-known properties of Lagrange finite elements. The vector $\vec{a}$ of coefficients $a_i$ for $i = 1, 2, \ldots, n_k$ of the canonical basis functions $\varphi_i$ of $\mathcal{P}_k(T)$ for $1 \leq i \leq n_k$ would be precisely $b_i$ for $1 \leq i \leq n_k$. Denoting by $M_k$ the Lagrangian nodes of $T$, $i = 1, 2, \ldots, n_k$, this means that the matrix $K$ whose entries are $k_{ij} := \varphi_j(M_i)$ is the identity matrix. Let $M_i = M_i$ if $M_i \not\in \Gamma \setminus \Gamma_h$ and $M_i$ be the node of the type $P$ or $Q$ associated with $M_i$ otherwise. The Lemma will be proved if the $n_k \times n_k$ linear system $K \vec{a} = \vec{b}$ is uniquely solvable, where $K$ is the matrix with entries $k_{ij} := \varphi_j(M_i)$. Clearly we have $K = K + E_K$, where the entries of $E_K$ are $e_{ij} := \varphi_j(M_i) - \varphi_j(M_i)$. At this point we observe that there exists a constant $C_\Omega$ depending only on $\Omega$ such that the length of the segment $M_iM_i$ is bounded above by $C_\Omega h_T^2$. It follows that $\forall i, j, k_{ij} \leq C_\Omega h_T^2 \max_{x \in T \cup \Delta_T} |\text{grad } \varphi_j(x)|$. Since $\varphi_j$ is a polynomial and $\Delta_T$ is at most a small perturbation of $T$, the maximum of $|\text{grad } \varphi_j|$ in $T \cup \Delta_T$ must be bounded by a certain mesh independent constant times $\max_{x \in T} |\text{grad } \varphi_j(x)|$. From standard arguments we know that the latter maximum is bounded above by a mesh-independent constant times $h_T^{-1}$. In short we have $|e_{ij}| \leq C_E h_T \forall i, j$, where $C_E$ is a mesh independent constant. Hence the matrix $K$ equals the identity matrix plus an $O(h_T)$ matrix $E_K$. Therefore $K$ is an invertible matrix, as long as $h$ is sufficiently small. \hfill $\blacksquare$

Lemma 2.2 Provided $h$ is small enough $\forall T \in \mathcal{R}_h$, given a set of $p_k$ real values $b_i$, $i = 1, \ldots, p_k$ with $p_k = (k + 1)(k + 2)(k + 3)/6 - (k + 1)$, there exists a unique function $w_T \in \mathcal{P}_k(T)$ that takes the value of $g$ at the two end-points $S$ of the edge $e$ of $T$ located on $\Gamma$ and at the $k - 1$ points $Q$ of $\Gamma$ defined in accordance with item 7. of the above definition of $W_h^g$, and takes the value $b_i$ respectively at the $p_k$ Lagrange nodes of $T$ not located on $\Gamma_h$.

Proof. The proof of this lemma is based on the same argument as in Lemma 2.1. \hfill $\blacksquare$
Remark 3 It is important to stress that, in contrast to its two-dimensional counterpart, the set $W^g_h$ does not necessarily consist of continuous functions. This is because of the interfaces between elements in $R_h$ and $S_h$. Indeed a function $w \in W^g_h$ is not forcibly single-valued at all the Lagrange nodes located on one such an interface, owing to the enforcement of the boundary condition at the points $Q \in \Gamma$ instead of the corresponding Lagrange node $M \in \Gamma_h$, in accordance with item 6. in the definition of $W^g_h$. On the other hand $w$ is necessarily continuous over all other faces common to two mesh tetrahedra. □

Next we set the problem associated with the space $V_h$ and the manifold $W^g_h$, whose solution is an approximation of $u$, that is, the solution of (1). With this aim we first introduce the broken gradient operator $\text{grad}_h$ defined on $\Omega_h$ for any function $w$ which is continuously differentiable in all the elements of $T_h$, given by $[\text{grad}_h w]|_T = \text{grad} w|_T \forall T \in T_h$.

Extending $f$ by zero in $\Omega_h \setminus \Omega$ and still denoting the resulting function by $f$, we wish to solve,

$$\begin{cases}
\text{Find } u_h \in W^g_h \text{ such that} \\
\text{where } a_h(w, v) := \int_{\Omega_h} \text{grad}_h w \cdot \text{grad} v \text{ and } F_h(v) := \int_{\Omega_h} f v.
\end{cases}$$

(2)

3 Analysis of the approximate problem

In this section we address the essential elements of the numerical analysis of problem (2), by means of which one can infer that the underlying method is reliable and effective.

Hereafter $\| \cdot \|_{s,D}$ (resp. $\| \cdot \|_{s,D}$) represents the standard norm (resp. semi-norm) of Sobolev space $H^s(D)$ for a non-negative real number $s$ (c.f. [1]), $D$ being any subset of $\Omega$. Whenever $D = \Omega$ the subscript $(\cdot)_{s,D}$ becomes simply $(\cdot)_s$. We also denote by $\| \cdot \|_{0,h}$ the standard norm of $L^2(\Omega_h)$.

To begin with we prove:

Proposition 3.1 Let $W^g_h$ be the space of functions equal to $W^g_h$ for $g \equiv 0$. Then provided $h$ is sufficiently small there exists a constant $\alpha > 0$ independent of $h$ such that,

$$\forall w \in W^g_h \neq 0, \sup_{v \in V_h \setminus \{0\}} \frac{a_h(w, v)}{\|\text{grad}_h w\|_{0,h} \|\text{grad} v\|_{0,h}} \geq \alpha.$$  

(3)

Proof. Given $w \in W^g_h$ let $v \in V_h$ coincide with $w$ at all Lagrangian nodes of elements $T \in T_h$ not belonging to $S_h \cup R_h$. As for an element $T \in S_h \cup R_h$ we set $v = w$ at the Lagrange nodes not belonging to $\Gamma_h$ and $v = 0$ at the Lagrangian nodes located on $\Gamma_h$. The fact that on the edges common to two mesh elements $T^-$ and $T^+$, both $v|_{T^-}$ and $v|_{T^+}$ are polynomials of degree less than or equal to $k$ in two variables coinciding at the exact number of Lagrangian nodes required to uniquely define such a function, implies that $v$ is continuous in $\Omega_h$. Moreover for the same reason $v$ vanishes all over $\Gamma_h$.

Let us denote by $\mathcal{L}_T$ the set of Lagrangian nodes of $T \in S_h \cup R_h$ that belong to $\Gamma_h$, and are different from vertices. Clearly enough we have

$$a_h(w, v) = \sum_{T \in T_h} \int_T |\text{grad} w|^2 - \sum_{T \in S_h \cup R_h} \int_T \text{grad} w \cdot \text{grad} r_T(w),$$

where $r_T(w) = \sum_{M \in \mathcal{L}_T} w(M) \varphi_M, \varphi_M$ being the canonical basis function of the space $\mathcal{P}_h(T)$ associated with Lagrangian node $M$.

Now from standard results it holds $\| \text{grad} \varphi_M \|_{0,T} \leq C_\varphi h_T^{1/2}$ where $\| \cdot \|_{0,T}$ denotes the norm of $L^2(T)$ and $C_\varphi$ is a mesh independent constant. Moreover, since $w(P) = 0$ (resp. $w(Q) = 0$), where $P$ (resp. $Q$) generically represent the point of $\Gamma$ corresponding to $M \in \Gamma_h$ in accordance with the definition of $W^g_h$, a simple Taylor expansion about $P$ (resp. $Q$) allows us to conclude that $|w(M)| \leq$
Now using arguments in all similar to those employed above, we easily conclude that

\[ \| \nabla w \|_{0,h} \leq \| \nabla h w \|_{0,2} + \| \nabla v - \nabla h w \|_{0,2} \leq (1 + C h) \| \nabla h w \|_{0,2}. \]  

Combining (6) and (7), provided \( h \leq (2C)^{-1} \) we establish (3) with \( \alpha = 1/3 \).

Next let \( u^H \in H^1(\Omega) \) be the solution of the Laplace equation \( \Delta u^H = 0 \) in \( \Omega \) fulfilling \( u^H = g \) on \( \Gamma \).

We may assume that \( u^H \in H^{k+1}(\Omega) \) with \( k > 1 \), as a trivial consequence of suitable assumptions on \( g \) and \( \Omega \). Thus we can define the interpolate \( u^H_h \) of \( u^H \) in \( W^q_h \). Moreover the simple application of standard error estimates for the interpolating function (cf. [4]) ensures the existence of a mesh independent constant \( C \) such that

\[ \| \nabla u^H - \nabla h u^H_h \|_{0,\Omega \cap \partial_h} \leq C h^k |u^H|_{k+1}. \]

Now let \( u^0_h \in W^q_h \) satisfy

\[ a_h(u^0_h, v) = F_h^0(v) \quad \forall v \in V_h \quad \text{where} \quad F_h^0(v) := F_h(v) - a_h(u^H_h, v). \]

Next we prove:

**Proposition 3.2** Provided \( h \) is sufficiently small, problem (2) has a unique solution.

**Proof.** First we note that \( F_h^0 \) is a continuous linear form on \( V_h \), and \( a_h \) is a continuous bilinear form on \( W_h^0 \times V_h \), the spaces \( V_h \) and \( W_h^0 \) being equipped with the norms \( \| \nabla \cdot \|_{0,h} \) and \( \| \nabla h \cdot \|_{0,h} \), respectively. Thus the facts that (3) holds and \( \text{dim}(V_h) = \text{dim}(W_h^0) \) imply the existence and uniqueness of \( u^0_h \) according to the theory of non-coercive approximate linear variational problems (cf. [2], [5] and [10]). Therefore \( u_h := u^0_h + u^H_h \) is a solution to (2), and its uniqueness is a direct consequence of (3). ■

Now let \( u^0 \in H^1_0(\Omega) \) be the solution of the equation \( -\Delta u^0 = f \) in \( \Omega \). Clearly enough \( u^0 + u^H \) is the solution of (1), and hence \( u^0 \) fulfills:

\[ a(u^0, v) = F^0(v) \quad \forall v \in H^1_0(\Omega), \quad \text{where} \quad F^0(v) := F(v) - a(u^H, v), \]

with

\[ a(w, v) := \int_\Omega \nabla h w \cdot \nabla h v \quad \text{and} \quad F(v) := \int_\Omega f v. \]

We shall next derive error estimates for problem (2) by comparing the solutions of (10) and (9). In order to simplify the analysis, let us first assume that \( \Omega \) is convex.

We have,

**Theorem 3.3** As long as \( h \) is sufficiently small, if \( \Omega \) is a sufficiently smooth convex domain for the solution \( u \) of (1) to belong to \( H^{k+1}(\Omega) \) when \( f \in H^{k-1}(\Omega) \) and \( g \in H^{k+1/2}(\Gamma) \), for \( k > 1 \), then for a suitable constant \( C(f, g) \) depending only on \( f \) and \( g \), the solution \( u_h \) of (2) satisfies:

\[ \| \nabla h (u - u_h) \|_{0,h} \leq C(f, g) h^k. \]
PROOF. Owing to the convexity of $\Omega$ we have $V_h \subset H^1_0(\Omega)$. Hence the variational residual $a(u^0, v) - F_0(v)$ vanishes for every $v \in V_h$. On the other hand $a_h(u^0_h, v) = a(u^0, v)$ if $v \in V_h$; it follows that the variational residual $a_h(u^0, v) - F_0^h(v)$ equals $F^0(v) - F^0_h(v) \forall v \in V_h$. Recalling [10] we thus have:

$$
\| \text{grad}_h(u^0 - u^0_h) \|_{0,h} \leq \frac{1}{\alpha} \left[ \inf_{w \in W_h^0} \| \text{grad}_h(u^0 - w) \|_{0,h} + \sup_{v \in V_h \setminus \{0\}} \| F^0(v) - F^0_h(v) \|_{\| \text{grad} v \|_{0,h}} \right].
$$

(13)

We know that $\inf_{w \in W_h^0} \| \text{grad}_h(u^0 - w) \|_{0,h} \leq C h^k |u^0|_{k+1}.

Moreover $|F^0(v) - F^0_h(v)| = |a_h(u^H_h - u^H, v)| \leq C h^k |u^H|_{k+1}$ $\| \text{grad} v \|_{0,h}$, according to (8).

Summarizing, it holds

$$
\| \text{grad}_h(u^0 - u^0_h) \|_{0,h} \leq \frac{C}{\alpha} h^k |u^0|_{k+1} + |u^H|_{k+1}.
$$

(14)

Finally using the triangle inequality we easily derive (12) with $C(f, g) = C_F C/\alpha \| f \|_{k-1} + C_G (1 + C/\alpha) \| g \|_{k+1/2, \Gamma}$, where $C_G$ and $C_F$ are constants such that $|u^0|_{k+1} \leq C_F \| f \|_0$ and $|u^H|_{k+1} \leq C_G \| g \|_{k+1/2, \Gamma}$.



Remark 4 It is noticeable that the continuity of functions in $W_h^0$ is nowhere required in the error analysis. Indeed in the generalization given in [10] of classical error bounds such as Strang’s inequalities, only the residual $a_h(u, v) - F_h(v)$ needs to be evaluated for $v \in V_h$. Thanks to the continuity of functions in $V_h$ this residual trivially vanishes. Incidentally this explains why it is not reasonable to replace $V_h$ by $W_h^0$, as one might be tempted to in order to define a symmetric approximate problem.

Next we consider the non-convex case, which is more delicate because the residual $a_h(u, v) - F_h(v)$ is not even defined for $v \in V_h$. Let us then consider a smooth domain $\Omega'$ close to $\Omega$ which strictly contains $\Omega \cup \partial \Omega_h$ for all $h$ sufficiently small. More precisely, denoting by $\Gamma'$ the boundary of $\Omega'$ we assume that $|\text{meas}(\Gamma') - \text{meas}(\Gamma)| \leq \eps$ for $\eps$ conveniently small. Henceforth we consider that $f$ was also extended to $\Omega' \setminus \Omega$. We denote the extended $f$ by $\tilde{f}$, which is arbitrarily chosen, except for the requirement that $\tilde{f} \in H^{k-1}(\Omega')$. There are different ways of achieving such a regularity and in this respect the author refers for instance to [11] or [13].

Then instead of (2) we solve:

$$
\left\{ \begin{array}{l}
\text{Find } u_h \in W_h^0 \text{ such that } \\
a_h(u_h, v) = F_h^0(v) := \int_{\partial \Omega} \tilde{f} v \forall v \in V_h.
\end{array} \right.
$$

(15)

Akin to problem (2) and thanks to (3), problem (15) has a unique solution. Notice that now the residual $a_h(u^0, v) - F_h^0(v)$ vanishes $\forall v \in V_h$. This property allows us to derive the following preliminary result:

Theorem 3.4 Assume that the solution $u$ of (1) belongs to $H^{k+1}(\Omega)$. Further assume that for $f' \in H^{k-1}(\Omega')$ there exists a function $u'$ defined in $\Omega'$ having the following properties:

- $-\Delta u' = f'$ in $\Omega'$;
- $u'_\Omega = u$;
- $u' = g$ a.e. on $\Gamma$;
- $u' \in H^{k+1}(\Omega')$.

Then for $k > 1$ and a suitable constant $C'$ independent of $h$ it holds:

$$
\| \text{grad}_h(u - u_h) \|_{0,\Omega_h} \leq C' |u'|_{k+1,\Omega'} h^k,
$$

(16)

where $\Omega_h := \Omega_h \cap \Omega$. 

8
PROOF. Here, instead of adapting the distance inequalities in ([10]) to this specific situation, we employ a more straightforward argument. First we recall (3) to note that \( \forall w \in W_h^g \) we have:

\[
\| \grad_h(u_h - w) \|_{0,h} \leq \frac{1}{\alpha} \sup_{v \in V_h \setminus \{0\}} \frac{|a_h(u_h, v) - a_h(w, v)|}{\| \grad v \|_{0,h}}.
\]

(17)

Since \( a_h(u', v) = F_h'(v) = a_h(u_h, v) \forall v \in V_h \) we can further write for every \( w \in W_h^g \):

\[
\| \grad_h(u_h - w) \|_{0,h} \leq \frac{1}{\alpha} \sup_{v \in V_h \setminus \{0\}} \frac{|a_h(u' - w, v)|}{\| \grad v \|_{0,h}} \leq \frac{1}{\alpha} \| \grad_h(u' - w) \|_{0,h}.
\]

(18)

From the triangle inequality this further yields:

\[
\| \grad_h(u_h - w) \|_{0,h} \leq \left[ 1 + \frac{1}{\alpha} \right] \| \grad_h(u' - w) \|_{0,h}.
\]

(19)

Choosing \( w \) to be the \( W_h^g \)-interpolate of \( u' \) in \( \Omega_h \), and using standard interpolation results (cf. [4]), from (19) we establish (16). 

It is noteworthy that the knowledge of a regular extension \( f' \) of the right hand side datum \( f \) associated with a regular extension \( u' \) of \( u \) is necessary to solve problem (15). Of course, except for very particular situations such as the toy problems used to illustrate the performance of our method in the next section, in most cases such an extension of \( f \) is not known. Even if we go the other around by prescribing a regular \( f' \), the existence of an associated \( u' \) satisfying the assumptions of Theorem 3.4 can also be questioned. However using some results available in the literature it is possible to identify cases where such an extension \( u' \) does exist. Let us consider for instance a simply connected domain \( \Omega \) of the \( C^\infty \)-class and a datum \( f \) infinitely differentiable in \( \Omega \). Taking an extension \( f' \in C^\infty(\tilde{\Omega}) \cap H^{k-1}(\tilde{\Omega}) \) of \( f \) to an enlarged domain \( \tilde{\Omega} \) also of the \( C^\infty \)-class, we first solve \(-\Delta u_0 = f' \) in \( \tilde{\Omega} \) and \( u_0 = 0 \) on \( \Gamma' \). According to well-known results (cf. [12]) \( u_0 \in C^\infty(\tilde{\Omega}) \) and hence the trace \( g_0 \) of \( u_0 \) on \( \Gamma \) belongs to \( C^\infty(\Gamma) \). Next we denote by \( u_H \) the harmonic function in \( \Omega \) such that \( u_H = g_0 \) on \( \Gamma \). Let \( r_0 \) be the radius of the largest (open) ball \( B \) contained in \( \tilde{\Omega} \) and \( O = (x_0, y_0, z_0) \) be its center. Assuming that \( f' \) is not too wild, so that the Taylor series of \( u_H(x, y, z_0) \) and \( \partial u_H/\partial z \) (\( x, y, z_0 \)) centered at \( O \) converge in a disk of the plane \( z = z_0 \) centered at \( O \) with radius equal to \( r_0 \sqrt{2} + \delta \) for a certain \( \delta > 0 \), according to [9] there exists a harmonic extension of \( u_H \) to \( B_{r_0}' \) centered at \( O \) with radius \( r_0 + \delta \sqrt{2} \). Clearly in this case, as long as \( \delta \) is large enough for \( B' \) to contain \( \tilde{\Omega} \), we can define \( u_\delta := u_H - u_H \) as a function in \( H^{k+1}(\tilde{\Omega}) \) that vanishes on \( \Gamma \). Now further assuming that \( g \in C^\infty(\Gamma) \) we can also define an extension of the harmonic function \( u_H \) whose value is \( g \) on \( \Gamma \) into \( u_{H'} \in H^{k+1}(\tilde{\Omega}) \) in the very same manner as \( u_H \). The extension \( u' \) of \( u \) to \( \tilde{\Omega} \) given by \( u' := u_{H'} + u_\delta \) satisfies the required properties.

In the general case however, a convenient way to bypass the uncertain existence of an extension \( u' \) satisfying the assumptions of Theorem 3.4, is to resort to numerical integration on the right hand side. Under certain conditions rather easily satisfied, this leads to the definition of an alternative approximate problem, in which only values of \( f \) (in \( \tilde{\Omega} \)) come into play. This trick is inspired by the celebrated work due to Ciarlet and Raviart on the isoparametric finite element method (cf. [8] and [7]). To be more specific, these authors employ the following argument, assuming that \( h \) is small enough: if a numerical integration formula is used, which has no integration points different from vertices on the faces of a tetrahedron, then only values of \( f \) (in \( \tilde{\Omega} \)) will be needed to compute the corresponding approximation of \( F_h'(v) \). This means that the knowledge of \( u' \), and thus of \( f' \), will not be necessary for implementation purposes. Moreover, provided the accuracy of the numerical integration formula is compatible with method’s order, the resulting modification of (15) will be a method of order \( k \) in the norm \( \| \cdot \|_{0, \tilde{\Omega}_h} \) of \( \grad u - \grad_h u_h \).

Nevertheless it is possible to get rid of the above argument based on numerical integration in the most important cases in practice, namely, the one of quadratic and cubic Lagrange finite elements. Let us see
First of all we consider that \( f \) is extended by zero in \( \Delta \Omega := \Omega' \setminus \Omega \), and resort to the extension \( u' \) of \( u \) to the same set constructed in accordance to Stein et al. [21]. This extension does not satisfy \( \Delta u' = 0 \) in \( \Delta \Omega \) but the function denoted in the same way such that \( u'|_{\Omega} = u \) does belong to \( H^{k+1}(\Omega') \).

Since \( k > 1 \) this means in particular that the traces of the functions \( u \) and \( u' \) coincide on \( \Gamma \) and that \( \partial u/\partial n = -\partial u'/\partial n' = 0 \) a.e. on \( \Gamma \) where the normal derivatives on the left and right hand side of this relation are outer normal derivatives with respect to \( \Omega \) and \( \Delta \Omega \) respectively (the trace of the laplacian of both functions also coincide on \( \Gamma \) but this is not relevant for our purposes). Based on this extension of \( u \) to \( \Omega_h \) for all such polyhedra of interest, we next prove the following results for the approximate problem (2), without assuming that \( \Omega \) is convex, and still denoting by \( f \) the function identical in \( \Omega \) to the right hand side datum of (1), that vanishes identically in \( \Delta \Omega \).

**Theorem 3.5** Let \( k = 2 \) and assume that \( u \in H^3(\Omega) \). Provided \( h \) is sufficiently small, there exists a mesh independent constant \( C_2 \) such that the unique solution \( u_h \) to (2) satisfies:

\[
\| \text{grad}_h(u - u_h) \|_{0, \Omega_h} \leq C_2 [h^2 |u|_{3, \Omega'} + h^{5/2} \| \Delta u' \|_{0, \Omega'}] \tag{20}
\]

where \( u' \in H^3(\Omega') \) is the regular extension of \( u \) to \( \Omega' \) constructed in accordance to Stein et al. [21].

**Proof.** First we recall (17), from which we obtain:

\[
\| \text{grad}_h(u_h - w) \|_{0,h} \leq \frac{1}{\alpha} \sup_{v \in V_h \setminus \{0\}} \frac{|a_h(u', v) - F_h(v)| + |a_h(u' - w, v)|}{\| \text{grad} v \|_{0,h}}. \tag{21}
\]

Thanks to the following facts the first term in the numerator of (21) is expressed as in (22): Since \( u' \in H^3(\Omega') \) we can apply First Green’s identity to \( a_h(u', v) \) thereby getting rid of integrals on portions of \( \Gamma \); next we note that \( \Delta u + f = 0 \) in every \( T \in T_h \setminus \{ S_h \cup R_h \} \); this is also true of elements \( T \) not belonging to the subset \( S_h \cup R_h \) consisting of elements \( T \) such that \( T \setminus \Omega \) is not restricted to a set of vertices of \( \Omega_h \); finally we recall that \( \Delta u' + f \) vanishes identically in the set \( T \cap \Omega \) and denote by \( \Delta_T \) the interior of the set \( T \setminus \Omega \forall T \in Q_h \). In short we can write:

\[
|a_h(u', v) - F_h(v)| = \sum_{T \in Q_h} \int_{\Delta_T} -\Delta u' v \, dx \leq \sum_{T \in Q_h} \| \Delta u' \|_{0, \Delta_T} \| v \|_{0, \Delta_T}. \tag{22}
\]

Let us first consider the case where \( T \in S_h \cap Q_h \). Since \( v = 0 \) on \( \Omega_h \) there exists a mesh-independent constant \( C_Q \) such that \( |v(x)| \leq C_Q h_T^3 \| \text{grad} v \|_{0, \infty, \Delta_T', \forall x \in \Delta_T'} \), where \( \| \cdot \|_{0, \infty, D} \) denotes the standard norm of \( L^\infty(D), D \) being a bounded open set of \( \mathbb{R}^3 \). Now from a classical inverse inequality it holds \( \| \text{grad} v \|_{0, \infty, \Delta_T'} \leq C_{Ih_T} h_T^{-3/2} \| \text{grad} v \|_{0, T} \) for a mesh-independent constant \( C_I \). Noticing that \( \Delta_T' \) is bounded by a constant depending only on \( \Omega \) times \( h_T^3 \), after straightforward calculations we obtain for a certain mesh-independent constant \( C_Q \):

\[
\| \Delta u' \|_{0, \Delta_T'} \| v \|_{0, \Delta_T'} \leq C_Q h_T^{5/2} \| \Delta u' \|_{0, \Delta_T'} \| \text{grad} v \|_{0, T} \forall T \in Q_h \cap S_h. \tag{23}
\]

Now we consider the elements \( T \) in the set \( Q_h \cap R_h \). Since in this case the measure of \( \Delta_T' \) is bounded above by a constant depending only on \( \Omega \) times \( h_T^3 \), we obtain for such elements a bound similar to (23) with \( h_T^{5/2} \) instead of \( h_T^{5/2} \). Since \( h_T << 1 \) by assumption we can assert that (23) also holds for elements in this set.

Now plugging (23) into (22) and applying the Cauchy-Schwarz inequality, we easily come up with,

\[
|a_h(u', v) - F_h(v)| \leq C_Q h_T^{5/2} \| \Delta u' \|_{0, \Omega'} \| \text{grad} v \|_{0, h}. \tag{24}
\]

Finally plugging (24) into (21) we immediately establish the validity of error estimate (20).
Theorem 3.6  Let $k = 3$ and assume that $u \in H^4(\Omega)$. Provided $h$ is sufficiently small, there exists a mesh independent constant $C_3$ such that the unique solution $u_h$ to (2) satisfies:

$$\| \text{grad}_h (u - u_h) \|_{0,\Omega_h} \leq C_3[h^3|u'|_{4,\Omega'} + h^{7/2} \| \Delta u' \|_{0,\infty,\Omega'}]$$

(25)

where $u' \in H^4(\Omega')$ is the regular extension of $u$ to $\Omega'$ constructed in accordance to Stein et al. [21].

Proof. First of all we point out that, according to the Sobolev embedding Theorem [1], $\Delta u' \in L^\infty(\Omega')$, since $u' \in H^4(\Omega')$ by assumption.

Following the same steps as in the proof of Theorem 3.5 up to equation (22), the latter becomes for a certain mesh-independent constant $C_R$,

$$|a_h(u', v) - F_h(v)| \leq C_R \sum_{T \in Q_h} h_T^2 \| \Delta u' \|_{0,\infty,\Omega'} \| v \|_{0,\infty,\Delta_T}.$$  (26)

Using the same arguments leading to (23) this yields in turn, for a constant $C_S$ equal to $C_RC_I$ times a constant depending only on $\Omega$:

$$|a_h(u', v) - F_h(v)| \leq C_S \sum_{T \in Q_h} h_T^{9/2} \| \Delta u' \|_{0,\infty,\Omega'} \| \text{grad} v \|_{0,T}.$$  (27)

Further applying the Cauchy-Schwarz inequality to the right hand side of (27) we easily obtain:

$$|a_h(u', v) - F_h(v)| \leq C_S h^{7/2} \| \Delta u' \|_{0,\infty,\Omega'} \left[ \sum_{T \in Q_h} h_T^{2} \right]^{1/2} \| \text{grad} v \|_{0,h}.$$  (28)

From the fact that the family of meshes in use is uniformly regular we conclude that there exists a constant $C_K$ such that $\left[ \sum_{T \in Q_h} h_T^{2} \right]^{1/2}$ is bounded above by $C_K \text{meas}(\Gamma)$. Plugging this into (28) and the resulting relation into (21) we immediately establish error estimate (25). $\blacksquare$

Remark 5  The author conjectures that it is possible to apply the classical Aubin-Nitsche duality argument to derive $O(h^{k+1})$ error estimates for (2) in the $L^2$-norm. Such a conjecture is strongly corroborated by the results of the numerical experiments reported in the next section. Unfortunately the proof of an error estimate in the $L^2$-norm requires some tools which appear to be yet unavailable in the literature, as pointed out in [17]. As a matter of fact, to the best of author’s knowledge to date no error estimate in $L^2$ has even been established for the isoparametric technique. $\blacksquare$

4 Numerical experiments

In this section we assess the accuracy of the method studied in Sections 2 and 3 - referred to hereafter as the new method -, by solving equation (1) in some relevant test-cases, taking $k = 2$. A comparison with the approach consisting of shifting boundary conditions from the true boundary to the boundary of the approximating polyhedron is also carried out. Hereafter the latter approach will be called the simple method. In all the examples numerical integration of the right hand side term was performed with the 15-point Gauss quadrature formula given in [22].

In order to dispel any skepticism, we first solved our model problem with a constant right hand side equal to $2(a^{-2} + b^{-2} + 1)$ in the ellipsoid centered at the origin, whose equation is $p(x, y, z) = 1$ where $p(x, y, z) = (x/a)^2 + (y/b)^2 + z^2$. Taking $d \equiv 1$, the exact solution is the quadratic function $-p$, and thus the new method is expected to reproduce it up to machine precision for any mesh (except for round-off errors). Here we used a mesh consisting of 3072 tetrahedra resulting from the transformation of a standard uniform $6 \times 8 \times 8 \times 8$ mesh of a unit cube $\Omega_0$ into tetrahedra having one edge coincident with a
diagonal parallel to the line \(x = y = z\) of a cube with edge equal to 1/8, resulting from a first subdivision of \(\Omega_0\) into 8\(^3\) equal cubes. The final tetrahedral mesh of the ellipsoid octant corresponding to positive values of \(x, y, z\), contains the same number of elements and is generated by mapping the unit cube into the latter domain through the transformation of Cartesian coordinates into spherical coordinates using a procedure described in [16].

It turns out that the absolute error in the \(H^1\)-semi-norm \(\| \text{grad}(\cdot) \|_{0,h}\) resulting from computations with \(a = 0.6\) and \(b = 0.8\), equals approximately \(0.29896592 \times 10^{-7}\), for an exact value of ca. 1.0324886.

Even though the computations were done in double precision, this numerical solution can be considered as almost exact, since there are necessarily non-negligible round-off errors due to matrix inversions in order to compute element matrices for tetrahedra in \(S_h \cup R_h\) (see Comment 3. in the next section). On the other hand the absolute error measured in the same way for the simple method is about 0.01663104, i.e., a relative error of about 1.6 percent. One might object that this is not so bad for a rather coarse mesh. However substantial gains with the new method over the simple method will be manifest in the examples that follow.

### 4.1 Test-problems in a convex domain

We next validate error estimate (12) and assess method’s accuracy in the \(L^2\)-norm of the error function \(u - u_h\) in \(\Omega_h\). With this aim we solved two test-problems with known exact solution. Corresponding results are reported below.

**Test-problem 1:** Here \(\Omega\) is the unit sphere centered at the origin. We take the exact solution \(u = \rho^2 - \rho^4\) where \(\rho^2 = x^2 + y^2 + z^2\), which means that \(d \equiv 0\) and \(f = -6 + 20\rho^2\). Owing to symmetry we consider only the octant sub-domain given by \(x > 0, y > 0\) and \(z > 0\) by prescribing Neumann boundary conditions on \(x = 0, y = 0\) and \(z = 0\). We computed with quasi-uniform meshes defined by a single integer parameter \(J\), constructed by the procedure proposed in [16] and described in main lines at the beginning of this section. Roughly speaking the mesh of the computational sub-domain is the spherical-coordinate counterpart of the standard \(J \times J \times J\) uniform mesh of the unit cube \((0,1) \times (0,1) \times (0,1)\). Each tetrahedron of the final mesh results from the transformation of the groups of six tetrahedra generated by the subdivision of each cubic cell of the partition of the unit cube using their diagonals parallel to the line \(x = y = z\). Since the mesh is symmetric with respect to the three Cartesian axes only one third of the chosen octant sub-domain was actually taken into account in the computations.

In Table 1 we display the absolute errors in the norms \(\| \text{grad}(\cdot) \|_{0,h}\) and \(\| \cdot \|_{0,h}\) for increasing values of \(J\), namely, \(J = 4, 8, 12, 16, 20\). Since the true value of \(h\) equals \(c/J\) for a suitable constant \(c\), as a reference we set \(h = 1/J\) to simplify things. As one infers from Table 1, the approximations obtained with the new method perfectly conform to the theoretical estimate (12). Indeed as \(J\) increases the errors in the gradient \(L^2\)-norm decrease roughly like \(h^2\), as predicted. The error in the \(L^2\)-norm in turn tends to decrease as an \(O(h^3)\). In Table 2 we display the same kind of results obtained with the simple method. As one can observe the error in the gradient \(L^2\)-norm decreases roughly like \(h^{1.5}\), as predicted by the mathematical theory of the finite element method, while the errors in the \(L^2\)-norm seem to behave like an \(O(h^2)\).

**Test-problem 2:** In order to make sure that the previous example presents no particularity due to the simple form of the domain, we now consider \(\Omega\) to be the ellipsoid centered at the origin with semi
prescribing Neumann boundary conditions on \( r \) and \( r \) convex, taking now a non-polynomial exact solution. More precisely (1) is solved in the torus \( \Omega \) roughly as \( J \to \infty \) to the theoretical estimate (12). Indeed as \( J \) increases in Test-problem 1, akin to Test-problem 1, the approximations obtained with the new method perfectly conform to those observed in Test-problem 1.

4.2 Test-problem in a non-convex domain

The aim of the following test-problem is to assess the behavior of the new method when \( \Omega \) is non-convex, taking now a non-polynomial exact solution. More precisely (1) is solved in the torus \( \Omega \) with minor radius \( r_m \) and major radius \( r_M \). This means that the torus’ inner radius \( r_i \) equals \( r_M - r_m \) and its outer radius \( r_e \) equals \( r_M + r_m \). Hence \( \Gamma \) is given by the equation \( (r_M - \sqrt{x^2 + y^2})^2 + z^2 = r_m^2 \).

We only consider problems with symmetry about the \( z \)-axis, and with respect to the plane \( z = 0 \). For this reason we may work with a computational domain given by \( \{ (x,y,z) \in \Omega \mid z \geq 0; \ 0 \leq \theta \leq \pi/4 \} \) with \( \theta = \tan^{-1}(y/x) \). A family of meshes of this domain depending on a single even integer parameter \( I \) containing 6\( I^3 \) tetrahedra is generated by the following procedure. First we generate a partition of the cube \( (0,1) \times (0,1) \times (0,1) \) into \( I^3/2 \) equal rectangular boxes by subdividing the edges parallel to the \( x \)-axis, the \( y \)-axis and the \( z \)-axis into \( 2I, I/2 \) and \( I/2 \) equal segments, respectively. Then

\[
\begin{array}{cccccccc}
\| \text{grad}_h (u - u_h) \|_{0,h} & \longrightarrow & 1/4 & 1/8 & 1/12 & 1/16 & 1/20 \\
& \longrightarrow & 0.257134 \ E-1 & 0.917910 \ E-2 & 0.50152682 \ E-2 & 0.326410 \ E-2 & 0.233854 \ E-2
\end{array}
\]

\[
\begin{array}{cccccccc}
\| u - u_h \|_{0,h} & \longrightarrow & 1/4 & 1/8 & 1/12 \\
& \longrightarrow & 0.454733 \ E-2 & 0.113568 E-2 & 0.502166 E-3 & 0.281468 E-3 & 0.179698 E-3
\end{array}
\]

Table 2: Errors with the simple method measured in two different manners for Test-problem 1.

\[
\begin{array}{cccccccc}
\| \text{grad}_h (u - u_h) \|_{0,h} & \longrightarrow & 1/4 & 1/8 & 1/12 \\
& \longrightarrow & 0.117716 E+0 & 0.353096 E-1 & 0.943753 E-2 & 0.427408 E-2
\end{array}
\]

\[
\begin{array}{cccccccc}
\| u - u_h \|_{0,h} & \longrightarrow & 1/4 & 1/8 & 1/12 \\
& \longrightarrow & 0.705684 E-2 & 0.956478 E-3 & 0.122026 E-3 & 0.364375 E-4
\end{array}
\]

Table 3: Errors with the new method measured in two different manners for Test-problem 2.

\[
\begin{array}{cccccccc}
\| \text{grad}_h (u - u_h) \|_{0,h} & \longrightarrow & 1/4 & 1/8 & 1/12 \\
& \longrightarrow & 0.124723 E+0 & 0.368763 E-1 & 0.104133 E-1 & 0.501084 E-2
\end{array}
\]

\[
\begin{array}{cccccccc}
\| u - u_h \|_{0,h} & \longrightarrow & 1/4 & 1/8 & 1/12 \\
& \longrightarrow & 0.807272 E-2 & 0.163738 E-2 & 0.365620 E-3 & 0.157317 E-3
\end{array}
\]

Table 4: Errors with the simple method measured in two different manners for Test-problem 2.
each box is subdivided into six tetrahedra having an edge parallel to the line $4x = y = z$. This mesh with $3I^3$ tetrahedra is transformed into the mesh of the quarter cylinder $\{(x, y, z) \mid 0 \leq x \leq 1, y \geq 0, z \geq 0, y^2 + z^2 \leq 1\}$, following the transformation of the mesh consisting of $I^2/2$ equal right triangles formed by the faces of the mesh elements contained in the unit cube’s section given by $x = j/(2I)$, for $j = 0, 1, \ldots, 2I$. The latter transformation is based on the mapping of the Cartesian coordinates $(y, z)$ into the polar coordinates $(r, \varphi)$ with $r = \sqrt{y^2 + z^2}$, using a procedure of the same nature as the one described in [16] (cf. Figure 4). Then the resulting mesh of the quarter cylinder is transformed into the mesh with $6I^3$ tetrahedra of the half cylinder $\{(x, y, z) \mid 0 \leq x \leq 1, -1 \leq y \leq 1, z \geq 0, y^2 + z^2 \leq 1\}$ by symmetry with respect to the plane $y = 0$. Finally this mesh is transformed into the computational mesh (of an eighth of half-torus) by first mapping the Cartesian coordinates $(x, y)$ into polar coordinates $(\rho, \theta)$, with $\rho = r_M + yr_m$ and $\theta = x\pi/4$, and then the latter coordinates into new Cartesian coordinates $(x, y)$ using the relations $x = \rho \cos \theta$ and $y = \rho \sin \theta$. Notice that the faces of the final tetrahedral mesh on the sections of the torus given by $\theta = j\pi/(8I)$, for $j = 0, 1, \ldots, 2I$, form a triangular mesh of a disk with radius equal to $r_m$, having the pattern illustrated in Figure 4 for a quarter disk, taking $I = 4, \theta = 0$ and $r_m = 1$ (cf. [16]).

Recalling that here $\rho = \sqrt{x^2 + y^2}$, we take $r_M = 5/6$, $r_m = 1/6$ and $f' = 6 - 5/(3\rho)$. For $d \equiv 0$ the exact solution is given by $u = 1/36 - z^2 - (5/6 - \rho)^2$. Obviously enough we take the same expression for $u'$. In Table 5 we display the absolute errors in the norm $\|\text{grad}(\cdot)\|_0,h$ and in the norm of $L^2(\Omega_h)$, for increasing values of $I$, namely $I = 2^m$ for $m = 1, 2, 3, 4$. Now we take as a reference $h = \pi/(8I)$. As one can observe from Table 5, here again the quality of the approximations obtained with the new method is in very good agreement with the theoretical result (16), for as $I$ increases the errors in the gradient $L^2$-norm decrease roughly as $1/I^2$, as predicted. On the other hand here again the errors in the $L^2$-norm tend to decrease as $1/I^3$. Table 6 in turn shows a qualitative erosion of the solution errors obtained with the simple method, similar to the case of convex domains.
\[
\begin{array}{cccccc}
\hline
h & \rightarrow & \pi/32 & \pi/64 & \pi/128 & \pi/256 \\
\parallel \text{grad}_h(u' - u_h) \parallel_{0,h} & \rightarrow & 0.786085 \times 10^{-3} & 0.205622 \times 10^{-3} & 0.522963 \times 10^{-4} & 0.131844 \times 10^{-4} \\
\parallel u' - u_h \parallel_{0,h} & \rightarrow & 0.133794 \times 10^{-4} & 0.171222 \times 10^{-5} & 0.214555 \times 10^{-6} & 0.269187 \times 10^{-7} \\
\hline
\end{array}
\]

Table 5: Errors with the new method measured in two different manners for Test-problem 3.

\[
\begin{array}{cccccc}
\hline
h & \rightarrow & \pi/32 & \pi/64 & \pi/128 & \pi/256 \\
\parallel \text{grad}_h(u' - u_h) \parallel_{0,h} & \rightarrow & 0.829181 \times 10^{-2} & 0.327176 \times 10^{-2} & 0.119077 \times 10^{-2} & 0.425739 \times 10^{-3} \\
\parallel u' - u_h \parallel_{0,h} & \rightarrow & 0.579150 \times 10^{-3} & 0.143425 \times 10^{-3} & 0.343823 \times 10^{-4} & 0.834136 \times 10^{-5} \\
\hline
\end{array}
\]

Table 6: Errors with the simple method measured in two different manners for Test-problem 3.

5 Final comments

To conclude we make some comments on the methodology studied in this work.

1. First of all a word on method’s universality is in order. The technique illustrated here in the framework of the solution of the three-dimensional Poisson equation with Dirichlet boundary conditions in curved domains with classical Lagrange finite elements provides a simple and reliable manner to overcome technical difficulties brought about by more complicated problems and interpolations. For example, Hermite finite element methods to solve second or fourth order problems in curved domains with normal derivative degrees of freedom can also be dealt with very easily by means of our new method. This was shown in author’s paper [19] with Silva Ramos, and in his conference paper [18].

2. As for the Poisson equation with homogeneous Neumann boundary conditions \( \partial u / \partial n = 0 \) on \( \Gamma \) (provided \( f \) satisfies the underlying scalar condition) our method practically coincides with the standard Lagrange finite element method. Indeed, the fact that the degrees of freedom on \( \Gamma_h \) are shifted to \( \Gamma \) is not supposed to bring about any improvement. However it is well-known that even for the standard method there is order erosion for \( k \geq 2 \), unless in the variational formulation the domain of integration is taken closer to \( \Omega \) than \( \Omega_h \). For more details the author refers to [3]. Besides this, if inhomogeneous Neumann boundary conditions are prescribed, optimality can only be recovered if the linear form \( F_h \) is modified, in such a way that boundary integrals for elements \( T \in S_h \) are shifted to a curved boundary approximation sufficiently close to \( \Gamma \). But definitively, these are issues that have nothing to do with our method, which is basically aimed at resolving those related to the prescription of degrees of freedom in the case of Dirichlet boundary conditions.

3. As we should observe our method leads to linear systems of equations with a non-symmetric matrix, even when the original problem is symmetric. Moreover in order to compute the element matrix and right side vector for an element in \( S_h \cup R_h \), the inverse of an \( n_k \times n_k \) matrix has to be computed. However this represents a rather small extra effort nowadays, in view of the significant progress already accomplished in Computational Linear Algebra.

4. The assumption made throughout the paper that meshes be sufficiently fine (also made by celebrated finite-element authors in the same context) seem to be of academic interest only. Actually in several computations with meshes having no more than 50 tetrahedra the new method behaved pretty well, by producing coherent results with respect to successively refined meshes.
$h \rightarrow 1/4 \quad 1/8 \quad 1/16 \quad 1/32 \quad 1/64$

$\| \text{grad}_h(u - u_h) \|_{0,h} \rightarrow 0.140071 \times 10^{-1} \quad 0.361685 \times 10^{-2} \quad 0.918504 \times 10^{-3} \quad 0.231512 \times 10^{-3} \quad 0.581281 \times 10^{-4}$

$\| \text{grad}(u - \tilde{u}_h) \|_{0,h} \rightarrow 0.155867 \times 10^{-1} \quad 0.383671 \times 10^{-2} \quad 0.947667 \times 10^{-3} \quad 0.235271 \times 10^{-3} \quad 0.586053 \times 10^{-4}$

$\| u - u_h \|_{0,h} \rightarrow 0.438951 \times 10^{-3} \quad 0.564603 \times 10^{-4} \quad 0.717088 \times 10^{-5} \quad 0.905923 \times 10^{-6} \quad 0.124276 \times 10^{-6}$

$\| u - \tilde{u}_h \|_{0,h} \rightarrow 0.493034 \times 10^{-3} \quad 0.604713 \times 10^{-4} \quad 0.744364 \times 10^{-5} \quad 0.924795 \times 10^{-6} \quad 0.128341 \times 10^{-6}$

Table 7: Errors with the new and the isoparametric approach for a test-problem in a disk taking $k = 2$.

Summarizing, this paper validates the finite-element methodology introduced in [17] to solve boundary value problems posed in smooth curved domains. Indeed it proved to be an advantageous alternative in many respects to more classical techniques such as isoparametric finite elements.

The author would like to stress that his method’s most outstanding features are not only universality but also simplicity, and eventually accuracy and CPU time too. However the two latter aspects were not our point from the beginning. Nevertheless we have compared our technique with the isoparametric one in both respects, by solving some two-dimensional test-problems using both approaches, as described in [17] and [7] respectively. It turned out that the new method was a little more accurate all the way. Just to illustrate this assertion we supply in Table 7 the errors in the $L^2(\Omega_h)$-norm of the solution gradient and of the solution itself, when both methods with $k = 2$ are used to solve a toy Poisson problem (1) in the unit disk for $f(x,y) := 9(x^2 + y^2)^{1/2}$ and $g \equiv 0$. The exact solution is $u(x,y) = 1 - (x^2 + y^2)^{3/2}$ and the meshes used in these computations are of the type illustrated in Figure 4.2, i.e. they depend on an integer parameter $I$ in such a way that $h = 1/I$. In Table 7 the solution obtained with isoparametric elements is denoted by $\tilde{u}_h$. Crout’s method was employed for both methods to solve the resulting linear systems. As far as CPU time is concerned our method was slightly more demanding than the isoparametric technique. But this is no surprise owing to the $6 \times 6$ matrix inversions for the computation of basis functions related to boundary elements.

As a conclusion the new method is indeed competitive in terms of both accuracy and CPU time. However from the author's point of view the most important gain in three-dimension space is in terms of simplicity, since in this case much more complex rational functions have to be handled for isoparametric elements.

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