Noncommutative Scalar Solitons at Finite $\theta$

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We investigate the behavior of the noncommutative scalar soliton solutions at finite noncommutative scale $\theta$. A detailed analysis of the equation of the motion indicates that fewer and fewer soliton solutions exist as $\theta$ is decreased and thus the solitonic sector of the theory exhibits an overall hierarchy structure. If the potential is bounded below, there is a finite $\theta_c$ below which all the solitons cease to exist even though the noncommutativity is still present. If the potential is not bounded below, for any nonzero $\theta$ there is always a soliton solution, which becomes singular only at $\theta = 0$. The $\phi^4$ potential is studied in detail and it is found the critical $(\theta m^2)_c = 13.92$ ($m^2$ is the coefficient of the quadratic term in the potential) is universal for all the symmetric $\phi^4$ potential.
1. Introduction

Recently, it was found that noncommutative geometry arises naturally in string theory with a constant $B$ field background \[1,2\]. In particular, in the large $B$ limit, the string field algebra factors into a direct product with the noncommutative algebra being an independent subalgebra\[3\], resulting in a noncommutative field theory as a decoupled low energy effective description of string theory in this limit. It is highly nonlocal and contains infinitely high order derivative terms, but in a controlled and self-consistent way. Compared to the commutative case, the renormalizability is improved but remains an open question, while nontrivial behavior such as UV/IR mixing adds to the difficulties \[4,5,6,7,8,9\]. The phase structure of the noncommutative scalar field theory is analyzed in the $\phi^4$ case \[10\] in which an unusual phase structure is uncovered.

The soliton sector of the noncommutative scalar field theory also exhibits an intriguingly rich structure. In the limit of large noncommutative parameter $\theta$ and ignoring the kinetic term, the solitons can be explicitly constructed via an isomorphism between the noncommutative fields and the operators on a single particle Hilbert space \[11\], and it was found that there are infinitely many solutions as long as the potential has more than one extremum. In spacetime, the soliton interpolates between the pseudo-vacuum at the core and the true vacuum at spatial infinity. This construction has been applied in the description of the unstable D-branes and of tachyonic condensation \[12,13\], based on the idea that D-branes can be constructed as solitons or lumps in the open string field theory \[14,15,16,17,18,19,20\]. These solitons are expected to disappear in the commutative limit $\theta = 0$ because Derrick’s theorem states that there is no soliton solution in more than 1+1 dimensions. Precisely how this happens, as $\theta$ changes from infinity to zero, will be investigated in this paper.

The general picture we found is as follows. First, all the solutions at $\theta = \infty$ can be extrapolated into the finite $\theta$ region, when the contribution from the kinetic term is taken into account. As we will show later, the noncommutative parameter $\theta$ becomes an overall multiplicative factor of the scalar potential, after a simple scaling. As $\theta$ decreases, the potential seen by the soliton scales down, which finally makes the soliton solution impossible. Each soliton at $\theta = \infty$ has its own critical point in $\theta$, and when $\theta$ gets smaller this particular solution disappears. In general different solutions at $\theta = \infty$ have different critical points for their own existence, so there is a hierarchy structure controlled by $\theta$, and we will see fewer and fewer soliton solutions as $\theta$ is decreased.
Depending on whether the potential is bounded below we have qualitatively different results. If the polynomial potential is bounded below and assumed to have its global minimum at the origin, there exists a lowest critical point $\theta_c$ which is nonzero. The bound for this critical point can be estimated for each particular potential, in some cases it can even be found precisely, such as in $\phi^4$ theory, which will be studied in detail in this paper. If instead the potential is not bounded below, such as for the cubic potential, there always exists a certain soliton solution at any nonzero $\theta$ and it becomes singular only in the commutative limit $\theta = 0$. Thus the noncommutative soliton owes its existence not only to the noncommutativity, but also to the dynamics.

The paper is organized as follows. In section 2 we review the construction of the noncommutative scalar soliton at $\theta = \infty$. In particular the isomorphism between the noncommutative algebra and the operator algebra on a single particle Hilbert space is discussed in detail. Ignoring the kinetic energy, the soliton in this limit is easily obtained by solving an algebraic equation. In section 3, we consider the solitons at finite $\theta$ for a general polynomial potential, and provide a proof of our result through a qualitative study of the full equation of motion expanded in the projection operator basis. In section 4, we provide a detailed numerical study of the soliton solution at finite $\theta$, for the $\phi^4$ theory. We use a more natural dimensionless parameter $\theta m^2$ instead of $\theta$, where $m^2 = V'(0)$. The approximation method in [11] gives a good estimation of $(\theta m^2)_c$. Using a simple scaling argument, we find that the dimensionless critical parameter $(\theta m^2)_c$ is the same for all $\phi^4$ potential with two degenerate vacua. In section 5, we discuss string theoretic implications of our results and give our conclusions.

2. Review of Noncommutative Scalar Solitons

The basis for noncommutative geometry lies in the deformation of the usual commutative product of the smooth functions on a flat space $\mathbb{R}^n$, into the noncommutative star product. The simplest realization of this deformation is Weyl quantization, when a Poisson bracket structure over the space of smooth functions $C^\infty(\mathbb{R}^n)$ is determined by a constant bivector field $\theta^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j}$, and the product assumes the form

$$f(x) \ast g(x) = e^{-i\theta^{ij} \frac{\partial}{\partial a^i} \frac{\partial}{\partial b^j} f(x + a)g(x + b)|_{a=b=0}}. \quad (2.1)$$

Actually this is the familiar quantization procedure in quantum mechanics, as the Poisson bracket is lifted up to the canonical commutation relations, and functions on the
phase space become operators on the quantum Hilbert space. So after the deformation quantization, the noncommutative star algebra \((C^\infty(\mathbb{R}^n), \ast)\) is naturally isomorphic to the operator algebra over a quantum Hilbert space corresponding to a finite number of particles. Under the Weyl prescription, the algebraic isomorphism is

\[
f(x) = \frac{1}{(2\pi)^n} \int d^{2n}k \tilde{f}(k) e^{-i(kx)}, \quad (2.2)
\]

\[
\hat{f}(\hat{x}) = \frac{1}{(2\pi)^n} \int d^{2n}k \tilde{f}(k) e^{-i(k\hat{x})}.
\]

The star product is mapped to the operator product, and the integration of the function over the phase space is equal to the trace of the corresponding operator in the Hilbert space \(\mathcal{H}\)

\[
f \ast g \longleftrightarrow \hat{f} \hat{g},
\]

\[
\frac{1}{(2\pi)^n} \int d^{2n}x f(x) = Tr_{\mathcal{H}} \hat{f}.
\]

As Hilbert space is separable and naturally equipped with a positive definite inner product, it always has a complete set of orthonormal basis of vectors \(\{|n\rangle, n = 0, 1, 2, \ldots\}\). Then the bounded linear operators on the Hilbert space have a corresponding basis composed of operators \(\{|m\langle n|, m, n = 0, 1, 2, \ldots\}\). Conversely, the above isomorphism, (2.2) and (2.3), allows us to find the smooth function corresponding to each operator.

We will give the detailed construction in the case that the space is two dimensional, and the generalization to the higher dimensional case is easy. \(\mathcal{H}\) now is a single particle Hilbert space. Using (2.1), the complex variable \(\bar{z}\) and \(z\) are mapped to the creation operator \(a^\dagger\) and the annihilation operator \(a\) respectively. We choose the simple harmonic oscillator basis, which are eigenstates of the operator \(a^\dagger a \sim \frac{r^2}{2}\)

\[
|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle, \quad a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle.
\]

A real function \(\phi^\dagger(x) = \phi(x)\) corresponds to an hermitian operator, and so can be diagonalized using a unitary operator \(U\) \((U^\dagger U = UU^\dagger = I)\)

\[
\hat{\phi} = U \left( \sum_{n=0}^{\infty} \lambda_n |n\rangle \langle n| \right) U^\dagger, \quad \lambda_n \in \mathbb{R}.
\]
The projection operator is easily expressed in the normal ordered form, $|n\rangle \langle n| = \frac{1}{n!} (a^\dagger)^n e^{-a^\dagger a} a^n$: which is proportional to the n-th Laguerre polynomial $L_n$ in momentum space

$$\phi_n^{(n)}(k) = e^{-\frac{k^2}{4}} L_n\left(\frac{k^2}{2}\right). \quad (2.8)$$

Actually Weyl ordering and normal ordering are two equivalent isomorphisms from the star algebra to the operators on Hilbert space (Weyl quantization uses the symmetric ordering), and in particular they differ by an integration kernel in momentum space

$$\tilde{f}_W(k) = \tilde{f}_N(k) e^{\frac{k^2}{4}}. \quad (2.9)$$

Now, using the relation (2.3), (2.8) and (2.9), we can easily find that the projection operator corresponds to a radially symmetric function

$$\phi_n(r^2) = \frac{1}{2\pi} \int d^2k e^{-\frac{k^2}{4}} L_n\left(\frac{k^2}{2}\right)e^{-ik.x} \quad (2.10)$$

The functions $\phi_n(x)$ have the same properties under the star product as the corresponding projection operators. This will greatly facilitate the construction of the noncommutative solitons.

Consider noncommutative scalar field theory in 2+1 dimensions with noncommutativity only in the spatial dimensions. The soliton is the classical extremum of the energy functional

$$E[\phi] = \int d^2x [\partial_\mu \phi \ast \partial^\mu \phi + V(\phi)]. \quad (2.11)$$

Upon changing variables to $(z, \bar{z})$, and performing a rescaling $z \rightarrow \frac{z}{\sqrt{\theta}}$, the star product will be independent of $\theta$, and the sole effect of the noncommutative parameter $\theta$ will appear as an overall scale factor for the potential

$$E = \int d^2z \left(\frac{1}{2} \partial \phi \ast \bar{\partial} \phi + \theta V(\phi)\right). \quad (2.12)$$

In the operator representation, the energy functional becomes

$$E(\hat{\phi}) = K(\hat{\phi}) + U(\hat{\phi}), \quad K(\hat{\phi}) = Tr[a, \hat{\phi}][\phi, a^\dagger], \quad U(\hat{\phi}) = \theta TrV(\phi). \quad (2.13)$$

At $\theta = \infty$, the kinetic energy is much smaller than the potential energy and so can be ignored. The potential $U(\hat{\phi})$ has a $U(\infty)$ symmetry, and the scalar field can be diagonalized
as in (2.1). In the operator representation, the potential is a function of the coefficient series \( \{\lambda_n, n = 0, 1, \ldots\} \), where the \( \lambda_n \)'s are decoupled from each other

\[
E(\phi) = \theta V(\phi) = \theta \sum_{n=0}^{+\infty} V(\lambda_n).
\]  

(2.14)

The classical equation of motion \( E'(\phi) = 0 \) is a set of independent algebraic equations, \( V'(\lambda_n) = 0 \). So the solution at \( \theta = \infty \) is a sequence of components \( (\lambda_0, \lambda_1, \lambda_2, \ldots) \) in the projection operator basis, with each \( \lambda_n \) being an extremum of the potential, and \( \lambda_n = 0 \) as \( n \to \infty \), which is the finite energy requirement for the soliton solution. Here we assume the potential has zero vacuum energy at the origin.

Therefore there are infinitely many soliton solutions at \( \theta = \infty \), as long as the potential has more than one extremum. Each solution can be regarded as a map from the positive integer to the extrema of the potential. A general solution will spontaneously break the \( U(\infty) \) group down to a finite unitary subgroup, depending on how many \( \lambda_n \) are the same in that particular solution. This has been interpreted as describing the decay of the unstable D-brane into multiple lower dimensional D-branes in the string theory[13].

3. Noncommutative solitons at finite \( \theta m^2 \): general analysis

In this section we will discuss the noncommutative soliton solutions at finite \( \theta \) in 2+1 dimensions. We assume the potential has a true vacuum at the origin with value zero. This can always be satisfied by a constant shift of the scalar field, if the highest power term is even with positive coefficient. So for example, \( \phi^3 \) potential does not satisfy this condition while \( \phi^4 \) potential does. This trivial looking assumption turns out to be essential in understanding the existence of the critical point for the solitons. We will comment on the case when the potential has odd highest power at the end of the section.

The kinetic energy has to be taken into account. It breaks the \( U(\infty) \) symmetry, so the energy functional contains the unitary matrix \( U \). Let \( U_{mn} = \langle m|U|n \rangle \) be the matrix element of \( U \) in SHO basis, the energy functional is

\[
E(\{\lambda_n\}, \{U_{mn}\}) = \sum_{n=0}^{\infty} \lambda_n^2 (1 + 2 \sum_{m=0}^{\infty} m|U_{mn}|^2) - 2 \sum_{m,n=0}^{\infty} \lambda_m \lambda_n |A_{mn}|^2 + \theta \sum_{n=0}^{\infty} V(\lambda_n),
\]

(3.1)

where

\[
A_{mn} = \sum_{k=1}^{\infty} \sqrt{k} U_{kn} U_{k-1,m}^*.
\]

(3.2)
Only the radially symmetric soliton solutions will be considered, which means the scalar field in the operator representation is diagonalized, \( U = I \). Actually adding a noncommutative \( U(1) \) gauge field into the action can restore the \( U(\infty) \) symmetry, while the scalar field lies in the adjoint representation of this \( U(\infty) \) group. Then by a proper \( U(\infty) \) transformation, the radially symmetric form of the scalar field can always be assumed.

Under such an assumption, the energy functional simplifies greatly,

\[
E(\{\lambda_n\}) = \sum_{n=0}^{\infty} [(2n + 1)\lambda_n^2 - 2(n + 1)\lambda_{n+1}\lambda_n + \theta V(\lambda_n)].
\]  

(3.3)

The classical equation of motion \( \partial E/\partial \lambda_n = 0 \) becomes a set of infinite number of coupled equations

\[
(n + 1)\lambda_{n+1} - (2n + 1)\lambda_n + n\lambda_{n-1} = \frac{1}{2}\theta V'(\lambda_n), \quad n \geq 1
\]

\[
\lambda_1 - \lambda_0 = \frac{1}{2}\theta V'(\lambda_0).
\]  

(3.4)

In addition we impose the asymptotic boundary condition required by the finiteness of the total energy

\[
\lambda_n \to 0, \quad \text{as} \quad n \to \infty
\]  

(3.5)

This is a second order difference equation for which it is hard to find a closed form solution. In general, the difference equation allows more solutions than its corresponding differential equation. In this section we first try a qualitative analysis to find the effect of \( \theta \) on the solution. In the next section, we will use both numerical and analytical methods to analyze the \( \phi^4 \) potential in detail.

Add the set of equations (3.4) up to the \( N \)-th to get an equation that is “integrated” once

\[
\lambda_{N+1} - \lambda_N = \frac{\theta}{2N + 1} \sum_{n=0}^{N} V'(\lambda_n).
\]  

(3.6)

Take the \( N \to \infty \) limit and use asymptotic condition (3.3), we obtain the necessary condition

\[
\sum_{n=0}^{\infty} V'(\lambda_n) < \infty.
\]  

(3.7)

If we regard \( n \) as the discrete time, the above equation describes a non-autonomous dynamic system. The noncommutative soliton solution is like a particle starting from a large nonzero \( \lambda_0 \) and approaches zero as time \( n \) goes to infinity. In general it is possible for this particle to go back and forth, but it should ultimately approach zero monotonically.
as dictated by the asymptotic condition. We will only study this part below, and previous “motion” only shows up as an initial condition of \( \frac{1}{N} \sum_{n=0}^{N-1} V'(\lambda_n) \) which is bounded for \( \lambda > 0 \).

First let us observe whether the particle approaches zero from below or above. Assume this imaginary particle starts from positive \( \lambda \) and decreases. If at some time \( N \), it jumps to \( \lambda_N < 0 \) close to zero, then \( \sum_{n=0}^{N-1} V'(\lambda_n) < 0 \). As the origin is a true vacuum, \( V' \) is a line with positive slope near the origin so \( V'(\lambda_N) < 0 \). Then necessarily \( \lambda_{N+1} < \lambda_N < 0 \) from (3.6) and \( \lambda_n \) would not converge to zero. Similarly, if the particle starts from the negative point, it should approach the origin from below. So we need only consider \( \lambda_n \) positive only. Then for large \( n \), \( \lambda_n \) approach zero from above monotonically, which by (3.6)gives a sharper constraint than (3.7)

\[
\sum_{n=0}^{\infty} V'(\lambda_n) \leq 0.
\]  

(3.8)

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**Fig. 1:** The general \( \phi^4 \) potential \( V(\phi) \) and its derivative \( V'(\phi) \). Notice the absolute value of the minima (the valley) of \( V'(\phi) \) for \( \phi > 0 \) is smaller than the maxima of \( V'(\phi) \) (the hump) in this case.
To see what happens as $\theta$ decreases, let us take a general $\phi^4$ potential as an example, as shown in fig. 1, and study the extrapolation of the solution corresponding to the particle’s starting from a nonzero $\lambda_0 > 0$ and jumping to $\lambda_1 << 1$ and then approach zero afterwards. This soliton solution has the lowest $\theta_c$, as will be proved in the next section. (3.6) requires $V'(\lambda_0) < 0$ and $V'(\lambda_1) + V'(\lambda_0) < 0$. But the minimum of $V'$ is bounded for $\lambda > 0$, so no matter how $\lambda_0$ changes, $\lambda_1 = \lambda_0 + \theta V'(\lambda_0)/2$ would finally increase with $\theta$ decreasing when $\theta$ becomes small enough, and the particle climbs up the hump at $\lambda_1$. In other words, $\theta$ controls how far the imaginary particle can go at each step. In this $\phi^4$ potential, it can be easily proved that the requirement $V(x) \geq V(0)$ always makes the hump larger than the absolute value of the valley. So as $\theta$ becomes small enough, it’s impossible for $\lambda_1$ to be close enough to the origin to satisfy the necessary condition $V'(\lambda_1) + V'(\lambda_0) < 0$. Then $\lambda_2 > \lambda_1$ and $\lambda_n$ wouldn’t converge to zero. Thus this solution can’t exist for such a $\theta$.

It seems that the above argument depends on the special property of the $\phi^4$ potential and presumably would not hold for a more general polynomial potential. But there is a more general argument to establish the existence of the finite critical point, although this may not reflect the actual situation of how the solution disappears. Notice that there is a bound on $\Delta \lambda_N \equiv \lambda_N - \lambda_{N-1}$

$$|\Delta \lambda_N| = \left| \frac{\theta}{2N} \sum_{n=0}^{N-1} V'(\lambda_n) \right| < \frac{\theta}{2} \inf_{x>0} V'(x).$$  \hspace{1cm} (3.9)

When $\theta$ becomes very small, $\Delta \lambda$ becomes very small, and the sum can be approximated by an integral from $\lambda_0$ to $\lambda_N$. This is exactly the same as using the limit process to find Riemann integral of a function. But this integral $V(\lambda_0) = \int_0^{\lambda_0} V'(x)dx > 0$ because zero is the true vacuum of the potential at the origin and $\lambda_0$ can not be a minimum, so there must be a critical point $\theta_c$ such that the sum becomes zero or even positive at some large $N$. Then the convergent process stops and the solution disappears.

It is interesting to observe how the difference equation allows a solution to avoid the constraint $\int_0^{\lambda_0} V'(x)dx > 0$ in the commutative limit and satisfies the constraint \((3.8)\). It is comparable to the definition of the Riemann integral through the limit of a finite sum by using a particular partition of the coordinate region determined by $\lambda_0 > \lambda_1 > \lambda_2 > ... > 0$ and letting the partition be smaller and smaller. When $\theta$ is big enough, the partition is coarse and it is possible for the sum to be different from the continuous limit and remain always negative. Decreasing $\theta$ is the same as taking the continuous limit to calculate the
Riemann integral and it will finally make the solution impossible at some finite \( \theta_c \). In this sense, the noncommutative scale \( \theta \) really controls the continuum limit.

For the polynomial potential whose highest power is odd, the above proof breaks down as either \( V'(\phi) \to -\infty \) as \( \phi \to \infty \) or \( V'(\phi) \to \infty \) as \( \phi \to -\infty \). So in principle, it is always possible to find a soliton solution satisfying the required constrains at any nonzero \( \theta \). Certainly the commutative limit \( \theta = 0 \) invalidates the scaling transformation of \( x \to x/\sqrt{\theta} \) and so it is a singular critical point of all the soliton solutions in this case.

Our qualitative analysis is also valid for non-polynomial potentials, such as the periodic cosine shaped potential, as long as it satisfies the bounded-below condition.

4. Noncommutative Solitons at finite \( \theta \): \( \phi^4 \) potential

In this section we discuss the \( \phi^4 \) potential in detail. First we will use the numerical method to explicit construct these solutions at finite \( \theta \) and see clearly how the solutions varies with \( \theta \). Second we will see that the method in [11] can be extended to find the lowest critical point \( (\theta m^2)_c \) explicitly to a very good approximation. Finally using a simple scaling argument, we find that \( (\theta m^2)_c \) is the same for all the symmetric \( \phi^4 \) potentials. Note, we use the dimensionless paremeter \( \theta m^2 \) in this section instead of \( \theta \), where \( m^2 = V'(0) \).

4.1. Numerical Results

First we explain the numerical method briefly. We use the relaxation method normally applied in solving differential equation with two point boundary conditions. The two boundary conditions for (3.4) are

\[
\lambda_{-1} = \lambda_{-2} = 0, \lambda_n \to 0 \text{ as } n \to \infty. \tag{4.1}
\]

An initial guess for the solution is required as an input. We can estimate the asympotic value of \( \lambda_n \) by going to the continuum limit and convert the difference equation into a differential equation, and ignore the nonlinear terms

\[
\lambda(u) = \frac{2}{\theta m^2} u \frac{d^2 \lambda(u)}{du^2}. \tag{4.2}
\]

The solution satisfying the asympotic boundary condition is [11]

\[
\lambda(n) = An^{\frac{4}{3}} e^{-\sqrt{\frac{2n}{\theta m^2}}} \tag{4.3}
\]
which can be used as the initial input to recursively find the true solution of the difference equation.

We use as an illustrative $\phi^4$ potential

$$V(\phi) = \frac{1}{4}\phi^4 - \frac{5}{3}\phi^3 + 3\phi^2$$

which is shown in fig. 1. It has a local maximum at $\phi = 2$ and a local minimum at $\phi = 3$ in addition to the global minimum at $\phi = 0$.

The results from the numerical analysis are as follows:

1) At $\theta = \infty$ and ignoring the kinetic term, $\phi(x) = \sum_n \lambda_n \phi_n(x)$ with $V'(\lambda_n) = 0$ is the general soliton solution. There are several changes to the solution after including the kinetic term at finite $\theta$. First, those $\lambda_n$ which are zero at $\theta = \infty$ become nonzero. They increase when $\theta$ decreases, but never become appreciably large. Second, those $\lambda_i$ which are nonzero at $\theta = \infty$ will also change such that $V'(\lambda_n)$ is negative and decreasing when $\theta$ decreases. So if $\lambda_i$ at $\theta = \infty$ is a local minimum/local maximum, it will decrease/increase when $\theta$ decreases. A typical example is the extrapolation of the solution $\phi(x) = 3\phi_0$ to $\theta m^2 = 600$, $\phi(x) = 2.89\phi_0 + 0.047\phi_1 + 0.0015\phi_2 + 0.00006\phi_3 + ...$. Here $\lambda_0$ decreases with $\theta$ because $\lambda = 3$ is the local minimum of the illustrative potential given above. Finally, while several $\lambda_n$ can take the same nonzero value at $\theta = \infty$, this is not possible at finite $\theta$. For example, the solution $\phi(x) = 3\phi_1(x) + 3\phi_2(x)$ at $\theta = \infty$, extrapolated to $\theta m^2 = 78$, is $\phi(x) = 0.038\phi_0 + 2.90\phi_1 + 2.70\phi_2 + 0.10\phi_3 + 0.004\phi_4...$, which clearly shows that $\lambda_1 \neq \lambda_2$.

2) The existence of the critical point $\theta_c$ can be seen as follows. The solution at $\theta = \infty$ is characterized by those nonzero $\lambda_n$’s. If one of them becomes zero (approximately), then this particular solution can be regarded as being nonexistent. A nonzero $\lambda_n$ at $\theta = \infty$ always changes in the direction such that $V'(\lambda_i)$ decreases when $\theta$ decreases. Because $V'(\lambda)$ is bounded below for $\lambda \leq 0$, $\lambda_n$ will reach a critical value at a finite $\theta_c$. With further decrease in $\theta$, this particular $\lambda_n$ will jump to a small value which is approximately zero, and this solution will cease to exist.

3) When $\theta$ decreases, the nonzero $\lambda_n$ with the largest $n$ reaches the critical value first. This can be explained considering the argument of section 3. The quantity $\Delta\lambda_N \equiv \lambda_N - \lambda_{N-1}$ should be negatively large enough to make the solution possible. As it is proportional to $1/N$, the $\lambda_N$ with the larger $N$ will fail this criterion first. So the solution $\phi = \lambda_0\phi_0$ gives the lowest critical point $\theta_c$.

4) We emphasize that at the critical point the solitonic solution has no singular behavior because $\theta_c$ is finite. The solution looks almost the same as the example given in 1). The only sign of the criticality is the discontinuity of the $\lambda_n$, which changes from a finite nonzero value to zero.
4.2. Determination of $(\theta m^2)_c$

In this section we will find the method of \[11\] useful to explicitly express the critical point $(\theta m^2)_c$ within a good approximation. In particular for all the symmetric $\phi^4$ potential, $(\theta m^2)_c$ is the same, which can be proved using a scaling argument.

The lowest critical point $(\theta m^2)_c$ corresponds to the extrapolation of the solution $\phi(x) = \lambda_0 \phi_0(x)$. The numerical solution indicates that $\lambda_n$ is extremely small for $n \neq 0$, so ignoring the nonliner terms is a good approximation

$$(n + 1)\lambda_{n+1} - (2n + 1)\lambda_n - n\lambda_{n-1} = \frac{1}{2}\theta m^2 \phi. \quad (4.5)$$

Going back to the coordinate space representation, and noticing that $\lambda_0 >> 0$ is equivalent to adding a source term proportional to $\phi_0(x)$

$$(-\frac{1}{\theta m^2} \partial^2 + 1)\phi(x) = A\phi_0(x), \quad (4.6)$$
except that it has a different boundary condition at $n = 0$

$$\lambda_0 = \frac{2}{\theta m^2}(\lambda_1 - \lambda_0) + A. \quad (4.7)$$

Compatibility with the boundary condition of the original equation of motion determines $A$.

Equation (4.6) is solved using properties of Laguerre polynomials $\phi_n$,

$$\lambda_n = e \int_0^{+\infty} \frac{e^{-x}}{1 + \frac{2}{\theta m^2}x} L_n(x). \quad (4.8)$$

Define the function $F$ by

$$F(a) = \int_0^{+\infty} \frac{e^{-x}}{1 + 2ax}, \quad (4.9)$$
as shown in fig.2, and $\lambda_0 = F(\frac{1}{\theta m^2})$. The two boundary conditions (4.7) and (4.1) should agree, which gives an equation for the scale factor $A$

$$m^2 A(F(\frac{1}{\theta m^2}) - 1) = V'(AF) = m^2(AF) - a(AF)^2 + b(AF)^3, \quad (4.10)$$

where we assume a general form of the $\phi^4$ potential, $V'(\phi) = m^2 \phi - a\phi^2 + b\phi^3$. The existence of a real solution for $A$ requires $\left(\frac{1}{\theta m^2}\right) \geq \frac{4m^2b}{a^2}$. $F$ is a monotonic function, so the equality determines the critical point $(\theta m^2)_c$

$$F(\frac{1}{(\theta m^2)_c}) = \frac{4m^2b}{a^2}. \quad (4.11)$$

At this critical point, $\lambda_0$ is equal to

$$\lambda_c = (eF)_c = \frac{a}{2b}. \quad (4.12)$$

These two expressions agree with the numerical results.
We have assumed that the potential has a global minimum at the origin, so $V(\phi) \geq 0$. It sets a lower bound $4m^2 b/a^2 \geq 8/9$. By (4.11), it sets a lower bound $(\theta m^2)_c \geq 14.374$. It is saturated exactly by the symmetric $\phi^4$ potential. Numerical analysis gives the exact result to be $(\theta m^2)_c = 13.92$.

Let’s study the case of the symmetric $\phi^4$ potential in more detail. The effect of the potential on $(\theta m^2)_c$ enters through its derivative $V'$, as seen in the equation of motion in the projection operator basis, equation (3.4). Assuming one of the degenerate vacuum is at the origin, the symmetric $\phi^4$ potential is characterized by the zeros of its derivative, assumed to be at 0, 1/a, 2/a. So in general $V'(\lambda) = m^2 \lambda(a\lambda - 1)(a\lambda - 2)$, and the variation of $a$ and $m^2$ gives all the symmetric $\phi^4$ potential. Writing out (3.4) explicitly as

$$(n + 1)\lambda_{n+1} - (2n + 1)\lambda_n + n\lambda_{n-1} = \frac{1}{2}\theta \frac{m^2}{2}\lambda_n (a\lambda_n - 1)(a\lambda_n - 2). \quad (4.13)$$

Under scaling transformation $\lambda \rightarrow b\lambda$, it becomes

$$(n + 1)\lambda_{n+1} - (2n + 1)\lambda_n + n\lambda_{n-1} = \frac{1}{2}\theta \frac{m^2}{2}\lambda_n (ab\lambda_n - 1)(ab\lambda_n - 2). \quad (4.14)$$

Effectively it transforms the moduli $a$ by the scaling factor $b$, which can be absorbed into $m^2$. This scaling of the variable will not affect the existence of the solution, and $\theta m^2$ remains invariant under the transformation, so the critical point $(\theta m^2)_c = 13.92$ for the existence of the nontrivial solution is the same for all the symmetric $\phi^4$ potential.
5. Discussion

From the analysis in this paper, we find that noncommutative soliton sector exhibits a nontrivial hierarchy structure controlled by the noncommutative scale $\theta$. For quite general potentials which have a global minimum, the $\theta_c$ is finite and determined by the details of the potential. This indicates that the noncommutative geometry should not be considered as merely a passive kinematic background, but may have the similar dynamic content comparable to the potential, as shown by its effect on the existence of the soliton solutions.

The noncommutative soliton solutions can be interpreted as describing the tachyon condensation on the unstable brane at large B field background \[12,13\]. It results in co-dimension-two branes obtained from the decay of the original brane. A general noncommutative soliton solution would disappear at finite $\theta$. But notice that the string field theory algebra factors into a direct product only in the limit $B \rightarrow \infty$, and those string degrees of freedom other than those involving only the center of mass coordinates have to be taken into account at finite $\theta$. This may change the picture in the string theory content.

The soliton solution at finite $\theta$ in general breaks the $U(\infty)$ symmetry completely. At $\theta = \infty$ and ignoring the kinetic energy, the level N soliton solution, such as $\phi = \sum_{n=0}^{N} \lambda_n \phi_n$ with all $\lambda_n$ equal, break the $U(\infty)$ down to $U(N) \times U(\infty - N) \[13\]$. This soliton solution describes N coincident D(n-2)-branes with $U(N)$ symmetry. Inclusion of the kinetic energy term brings $1/\theta^{-1}$ corrections and all the $\lambda_n$ are different. It breaks the residual $U(N)$ symmetry completely into $U(1)^N$. So the $U(N)$ symmetry for this solution is at most approximate, and it seems that it should be interpreted as describing N branes that are not coincident. But this conclusion may not be true if we consider the additional terms in the action coming from the open string field theory, because at finite $\theta$ it is necessary to include those string degree of freedom other than those accounted for by the noncommutative field theory. In particular, noncommutative Yang-Mills theory is not enough and the analysis of Dirac-Born-Infeld action may give a different result which will be worth exploring.

Also notice that the open superstring tachyon potential for type II superstring theory is exactly of the symmetric $\phi^4$ shape, which follows from the reflection symmetry of the potential, although the details of the potential are unknown. But we find the critical point $(\theta m^2)_c = 13.92$ for the existence of the noncommutative soliton is ignorant of the exact form of the $\phi^4$ potential, and so is characteristic of type-II in the B-field background. We haste to add that this conclusion may be changed after inclusion of the open string field theory degrees of freedom.
It would be interesting to study the stability of these soliton solutions, and their effects on the quantum structure of the whole theory. We hope to explore these issues in the future.

Note added:

We notice that the recent two papers [21,22] discussed the noncommutative solitons in the context of noncommutative scalar theory coupled with the noncommutative gauge field. These new solutions involve nontrivial gauge field configuration, which may answer some of the puzzles put forward in the last section of the paper concerning the finite $\theta$ behavior and are worth exploring.

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