Unique continuation property for biharmonic hypersurfaces in spheres

Hiba Bibi1 · Eric Loubeau1 · Cezar Oniciuc2

Received: 5 September 2020 / Accepted: 26 August 2021 / Published online: 13 September 2021
© The Author(s), under exclusive licence to Springer Nature B.V. 2021

Abstract
We prove a unique continuation theorem for non-minimal biharmonic hypersurfaces of spheres, based on Aronszajn’s 1957 article. Under the right hypotheses, this result shows that, for these immersions, CMC on an open subset implies globally CMC. We then deduce new rigidity theorems to support the Conjecture that biharmonic submanifolds of Euclidean spheres must be of constant mean curvature.

Keywords Biharmonic submanifolds · Constant mean curvature · Unique continuation

Mathematics Subject Classification 58E20 · 53C43 · 31B30

1 Introduction
The study of biharmonic maps was introduced by G.-Y. Jiang in the mid-80’s, and in [1], he computed the first variational formula of the bienergy functional

This was first suggested by J. Eells and L. Lemaire in [2], where as a variation on the theme of harmonic maps, they considered polyharmonic maps, i.e. critical points of the functional

This work was supported by a grant of the Romanian Ministry of Research and Innovation, CCCDI-UEFISCDI, project number PN-III-P3-3.1-PM-RO-FR-2019-0234 / IBM / 2019, within PNCDI III and the PHC Brancusi 2019 project no 43460 TL.

✉ Hiba Bibi
Hiba.Bibi@univ-brest.fr

Eric Loubeau
Eric.Loubeau@univ-brest.fr

Cezar Oniciuc
oniciucc@uaic.ro

1 LMBA, Univ. Brest, CNRS UMR 6205, F-29238 Brest, France
2 Faculty of Mathematics, Al. I. Cuza University of Iasi, Bd. Carol I, no. 11, 700506 Iasi, Romania
so that \( F_2(\varphi) = E_2(\varphi) \). Hence, biharmonic maps come from a variational problem, generalizing the well-known harmonic maps. Using a simple Bochner formula, G.-Y. Jiang proved that biharmonic maps from a compact manifold to a non-positively curved space is harmonic, so the first interesting target manifold is the Euclidean sphere. By definition, biharmonic submanifolds are isometric immersions which are biharmonic maps.

Independently, in [3], B.-Y. Chen defined biharmonic submanifolds of the Euclidean space as isometric immersions with harmonic mean curvature vector field, or alternatively, the components of the immersion are biharmonic functions. In [4, 5], it was proved that biharmonic surfaces in \( \mathbb{R}^3 \) are minimal. This has led to Chen’s Conjecture [6]: \textit{Biharmonic submanifolds of Euclidean spaces are minimal.} Some particular sub-cases have been proved for example in [7–9].

For hypersurfaces of the unit Euclidean sphere \( \mathbb{S}^n \), it is natural and useful to split the Euler-Lagrange equation into its tangential and normal components and to rewrite the biharmonic equation as follows:

\[
\begin{align*}
\Delta f &= (m - |A|^2)f \\
A(\nabla f) &= -\frac{m}{2}f \nabla f,
\end{align*}
\]

see [3, 10, 11]. Here, \( f \) is the mean curvature, and \( A \) is the shape operator of \( M \) in \( \mathbb{S}^n \), and the convention for the Laplacian on functions is \( \Delta = -\text{trace} \nabla d \).

The first example of a non-minimal biharmonic hypersurface (due to G.-Y. Jiang in [1]) is the Generalized Clifford torus \( \mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2}) \) in \( \mathbb{S}^{m_1+m_2+1} \), with \( m_1 \neq m_2 \). Recall that, when \( m_1 = m_2 \), this Clifford torus is minimal. In [12], the authors observed that the 45-th parallel \( \mathbb{S}^{m}(1/\sqrt{2}) \) in \( \mathbb{S}^{m+1} \), which is umbilical, is another non-minimal biharmonic hypersurface. These two examples have constant mean curvature (CMC) and motivate the following Conjecture [13]: \textit{Any biharmonic submanifold in a Euclidean sphere is CMC}.

This Conjecture was settled for spheres of dimension 3 in [12] and dimension 4 in [14]. For arbitrary dimensions, Y. Fu and M.-C. Hong [15] proved the Conjecture when the scalar curvature is constant and the number of principal curvatures is at most 6, while S. Maeta and Y.-L. Ou [16] proved it for compact hypersurfaces with constant scalar curvature. In [17, 18], the authors use new Liouville-type theorems to prove special cases of the Conjecture.

This paper is a contribution to this Conjecture and gives rigidity results based on a new technique of unique continuation theorem (UCT). In [19], a similar approach was used to prove that if a biharmonic map is harmonic on an open subset, then it must be harmonic everywhere.

Inspired by this work of V. Branding and C. Oniciuc, and relying on the UCT of N. Aronszajn [20], our objective in this article is not to show minimality, but rather to prove the weaker condition of CMC. Using a gradient inequality between the norm of \( A \) and the mean curvature we show that, for proper-biharmonic hypersurfaces (i.e. non-minimal) in a sphere, locally CMC implies globally CMC.

In Sect. 4, we exploit this UCT property to prove new rigidity results, and in Theorem 4.1, use an integral condition involving both the scalar and the mean curvatures to force biharmonic hypersurfaces to be CMC. This extends the main result of S. Maeta...
and Y.-L. Ou [16] to non-constant scalar curvature, while relying on a different technique of proof.

1.1 Conventions

Manifolds will be assumed to be connected, oriented and without boundary, but unless stated explicitly, they are not assumed to be compact or complete.

Throughout this paper all manifolds, metrics and maps are taken to be of class $C^\infty$, and we adopt the Einstein summation convention.

Let $\varphi: M^m \to N^n$ between two Riemannian manifolds, the rough Laplacian acting on sections of the pullback bundle $\varphi^{-1}(TN)$ is given by

$$\Delta^\varphi = - \text{trace} \ (\nabla^\varphi)^2 = - \text{trace} \ (\nabla^\varphi \nabla^\varphi - \nabla^\varphi),$$

where $\nabla^\varphi$ is the pullback connection.

Our convention for the curvature tensor field is

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

2 Preliminaries

Let $\varphi: M^m \hookrightarrow \mathbb{S}^{m+1}$ be a hypersurface, which we assume, without loss of generality, to be oriented. Let $\eta \in C(NM)$ be a globally defined unit normal vector field and $A$ the shape operator

$$A^\eta(X) = - \nabla_{X} \eta,$$

where $X \in C(TM)$ and $\nabla_{\mathbb{S}^{m+1}}$ is the Levi-Civita connection on $\mathbb{S}^{m+1}$. Then, the mean curvature is

$$f = \frac{1}{m} \text{trace} A.$$

In general, $f$ can take positive and negative values.

Let $H = f \eta$ be the mean curvature vector field, so that $M$ is minimal when $H = 0$.

The second fundamental form $B \in C(\otimes^2 T^* M \otimes NM)$ is

$$\langle B(X, Y), \eta \rangle = \langle A(X), Y \rangle,$$

and

$$\tau(\varphi) = \text{trace} B = mf \eta.$$

It is known that [19, 21], for a proper-biharmonic map $\varphi: M \to N$, the subset $\{ p \in M : \tau(\varphi)(p) \neq 0 \}$ is open and dense in $M$. Thus,

$$\Omega = \{ p \in M : f(p) \neq 0 \}$$

is open and dense in $M$. Note that this subset can have several connected components.
Lemma 2.1 (J.-H. Chen [10]) Let \( \varphi : M^m \hookrightarrow \mathbb{S}^{m+1} \) be a proper-biharmonic hypersurface. Then, at points where \( \nabla f \neq 0 \), we have

\[
|A|^2 \geq \frac{m^2(m + 8)}{4(m - 1)} f^2.
\]

In low dimensions, since the Conjecture is settled for \( \mathbb{S}^3 \) and \( \mathbb{S}^4 \), Lemma 2.1 has the following direct consequence.

Proposition 2.2 Let \( \varphi : M^m \hookrightarrow \mathbb{S}^{m+1} \) be a proper-biharmonic hypersurface. Assume that \( m = 4 \) and \( \text{Scal}^M > m(m-1) \), then \( M \) has constant mean curvature.

Proof Assume that \( M \) does not have constant mean curvature, then there exists \( p_0 \in M \) such that \( (\nabla f)(p_0) \neq 0 \). Thus, Lemma 2.1 allows us to infer that

\[
|A(p_0)|^2 \geq \frac{m^2(m + 8)}{4(m - 1)} f^2(p_0).
\]

On the other hand, taking traces in the Gauss Equation (see for example [22]), we have on \( M \):

\[
\text{Scal}^{\mathbb{S}^{m+1}} = \text{Scal}^M + |A|^2 - m^2 f^2 + 2 \text{Ricci}^{\mathbb{S}^{m+1}}(\eta, \eta),
\]

where \( \eta \) is the unit normal vector field. Therefore, we have

\[
|A|^2 = m(m - 1) + m^2 f^2 - \text{Scal}^M,
\]

as \( \text{Scal}^M > m(m-1) \) we obtain, at \( p_0 \),

\[
\frac{m^2(m + 8)}{4(m - 1)} f^2(p_0) \leq |A(p_0)|^2 < m^2 f^2(p_0)
\]

which forces \( m \) to be at least 5, which is a contradiction. Therefore, \( \nabla f = 0 \) on \( M \). \( \square \)

The conjecture says that any proper-biharmonic hypersurface in \( \mathbb{S}^{m+1} \) has constant mean curvature. When \( M \) is compact, this conjecture was proved in several cases, under additional hypotheses. When \( M \) is not compact, and the additional hypotheses are still satisfied, we can only say that the points where \( \nabla f \neq 0 \), if they exist, cannot form a set with a simple structure. We present here only one result of this type.

Proposition 2.3 Let \( \varphi : M^m \hookrightarrow \mathbb{S}^{m+1} \) be a proper-biharmonic hypersurface. Assume that \( M \) does not have constant mean curvature. If \( |A|^2 \geq m \), or \( |A|^2 \leq m \), then \( W = \{ p \in M : (\nabla f)(p) \neq 0 \} \) cannot have a connected component \( W_0 \) with the following properties:

1. \( W_0 \) is compact;
2. the boundary of \( W_0 \) in \( M \) is a regular (not necessarily connected) hypersurface of \( M \);
3. there exists an open subset \( U \) of \( M \) such that \( \overline{W_0} \cap U \) and \( \nabla f = 0 \) on \( U \setminus W_0 \).
Proof  Assume that $W$ has a connected component $W_0$ with the above properties, and we argue by contradiction.

Since $\partial W_0$ is a regular hypersurface of $M$, we have

$$\text{int} \left( U \setminus W_0 \right) = U \setminus \overline{W_0} = U \setminus W_0 = U \setminus (W_0 \cup \partial W_0) \neq \emptyset,$$

otherwise $U = W_0 \cup \partial W_0$ is closed in $M$, so $U = M$, i.e. $M = W_0 \cup \partial W_0$ is a manifold with boundary; and

$$\text{int} \left( U \setminus W_0 \right) = U \setminus W_0.$$

On int $(U \setminus W_0)$, that may have several connected components, $\text{grad} f = 0$, $f \neq 0 |A|^2 = m$.

Assume now that $|A|^2 \leq m$. It was proved in [10], see also [23, Inequality (1.74)] that on $M$ we have

$$\frac{1}{2} \Delta (| \text{grad} f |^2 + \frac{m^2}{8} f^4 + f^2) + \frac{1}{2} \text{div} (|A|^2 \text{grad} f^2) \leq \frac{8(m - 1)}{m(m + 8)} (|A|^2 - m)|A|^2 f^2.$$

Equivalently,

$$- \text{div} Z \leq \frac{8(m - 1)}{m(m + 8)} (|A|^2 - m)|A|^2 f^2 \leq 0, \quad (2.1)$$

where

$$Z = \frac{1}{2} \text{grad} (| \text{grad} f |^2 + \frac{m^2}{8} f^4 + f^2) - \frac{1}{2} |A|^2 \text{grad} f^2.$$

Since $Z = 0$ on int $(U \setminus W_0)$, it follows that $Z = 0$ on $U \setminus W_0$ and so on $\partial W_0$. Integrating Inequality (2.1) on $\overline{W_0}$ and using the Divergence Theorem, as $Z = 0$ on $\partial W_0$, we obtain $(|A|^2 - m)|A|^2 f^2 = 0$ on $\overline{W_0}$. As in [23], we obtain $|A|^2 = m$ on $W_0$, and so on $\overline{W_0}$. By the first equation of (1.1), it follows that $\Delta f = 0$ on $\overline{W_0}$.

Furthermore, we integrate $\Delta f^2$ on $\overline{W_0}$, and since $\text{grad} f^2 = 0$ on $\partial W_0$, we obtain $\text{grad} f = 0$ on $\overline{W_0}$ which is impossible.

The case $|A|^2 \geq m$ is easy to prove as

$$\frac{1}{2} \Delta f^2 = (m - |A|^2) f^2 - | \text{grad} f |^2 \leq 0$$

on $M$, and integrating on $\overline{W_0}$ we obtain again $\text{grad} f = 0$ on $\overline{W_0}$.

$\square$

In the next section (see Corollary 3.3), we will see that under a stronger hypothesis, i.e. $|A|^2$ is constant, the points of a non-CMC proper-biharmonic hypersurface where $\text{grad} f \neq 0$ form an open dense subset of $M$.

Before stating the last result of this section, we need to recall some well-known facts about the smoothness of the principal curvatures.

Let $\varphi : M^m \hookrightarrow S^{m+1}$ be a hypersurface with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ its principal curvatures, i.e. the eigenvalue functions of the shape operator $A$. The functions $\lambda_i$ are continuous on $M$ for all $i = 1, \ldots, m$. The set of points where the numbers of distinct principal curvatures is locally constant is a set $M_1$ that is open and dense in $M$. On a non-empty connected component of $M_1$, which is open in $M$, and so in $M$, the number of distinct
principal curvatures is constant. Thus, the multiplicities of the principal curvatures are constant, and so, on that connected component, \( \lambda_i \)'s are smooth, and \( A \) is (smoothly) locally diagonalizable (see [24–26]).

**Proposition 2.4** Let \( \varphi : M^m \hookrightarrow \mathbb{S}^{m+1} \) be a proper-biharmonic hypersurface. Assume that at any point of \( M \) the multiplicity of each distinct principal curvature is at least 2. Then \( M \) has constant mean curvature.

**Proof** Assume that \( M \) is not CMC and denote

\[
W := \{ p \in M : (\nabla f)(p) \neq 0 \}.
\]

Clearly, \( W \) is a non-empty open subset of \( M \). Since \( M_A \) is dense, \( W \cap M_A \neq \emptyset \), and so \( W \) intersects a connected component of \( M_A \). On that intersection, \( \lambda_i \)'s are smooth, \( i = 1, \ldots, m \), and \( A \) is smoothly diagonalizable, i.e. \( A(E_i) = \lambda_i E_i \), \( i = 1, \ldots, m \), where \( \{ E_i \}_{i=1}^m \) is an orthonormal frame field.

Recall that by the biharmonic Eq. (1.1), on \( W \), \( -\left( \frac{m}{2} \right) f \) is an eigenvalue of the shape operator \( A \). From the hypothesis on the multiplicity of eigenvalues, we can assume for simplicity that \( \lambda_1 = \lambda_2 = -\frac{m}{2} f \) and \( E_1 = \frac{\nabla f}{|\nabla f|} \). Since \( \langle E_a , E_1 \rangle = 0 \), we have

\[
E_a f = 0, \quad a = 2, \ldots, m.
\]

Now, we use the connection equations with respect to the frame field \( \{ E_i \}_{i=1}^m \):

\[
\nabla_{E_i} E_j = \omega^k_j(E_i) E_k,
\]

and we rewrite the Codazzi equation

\[
(\nabla_{E_i} A)(E_j) = (\nabla_{E_j} A)(E_i)
\]

as

\[
(E_i \lambda_j) E_j + \sum_{k=1}^m (\lambda_j - \lambda_k) \omega^k_j(E_i) E_k = (E_j \lambda_i) E_i + \sum_{k=1}^m (\lambda_i - \lambda_k) \omega^k_i(E_j) E_k.
\]

(2.3)

For \( i = 1 \) and \( j = 2 \), we obtain

\[
(E_1 \lambda_2) E_2 + \sum_{k=1}^m (\lambda_2 - \lambda_k) \omega^k_2(E_1) E_k = (E_2 \lambda_1) E_1 + \sum_{k=1}^m (\lambda_1 - \lambda_k) \omega^k_1(E_2) E_k
\]

\[
= \sum_{k=1}^m (\lambda_1 - \lambda_k) \omega^k_1(E_2) E_k.
\]

(2.4)

Furthermore, we take the scalar product of the above relation with \( E_2 \), and we obtain

\[
E_1 \lambda_2 = E_2 \lambda_1 = 0,
\]

i.e. \( E_1 f = 0 \). Thus, from Eq. (2.2) we conclude that \( \nabla f = 0 \) which is impossible. \( \square \)
3 The unique continuation theorem

Very little is known on the local properties, in particular analytical ones, of biharmonic submanifolds in Euclidean spheres.

An essential tool in the analysis of PDE’s is a unique continuation property, which we establish in Theorem 3.1 under a global condition on the gradients of the norm of the shape operator and mean curvature.

The objective here departs from [19] as the conclusion is that the manifold has constant mean curvature, instead of the stronger condition of minimality, but the method is similar and is based on Aronszajn’s unique continuation theorem of 1957 [20].

In Corollaries 3.3 and 3.4, the main hypothesis of Theorem 3.1 is replaced by more geometrical constraints and allows to extend known results from the compact to the non-compact cases.

**Theorem 3.1** Let \( \varphi : M^m \hookrightarrow \mathbb{S}^{m+1} \) be a proper-biharmonic hypersurface. Assume that there exists a non-negative function \( h \) on \( M \) such that\[
\| \nabla |A|^2 | \leq h \| \nabla f \| \text{ on } M.
\]
If \( \nabla f \) vanishes on a non-empty open connected subset of \( M \), then \( M \) has constant mean curvature.

**Proof** Denote by \( V \) the non-empty open connected subset of \( M \) where \( \nabla f = 0 \).

Consider the subset \( A_0 := \{ p \in M : (\nabla f)(p) = 0 \} \).

It is clear that \( A_0 \) is closed, \( \text{int} A_0 \neq \emptyset \) and \( \text{int} A_0 \) may have several connected components. Indeed, \( A_0 = (\nabla f)^{-1}(\{0\}) \) where \( \{0\} \) is closed in \( TM \) as being the zero section, and \( \nabla f : M \to TM \) is continuous, thus \( A_0 \) is closed, and since \( V \) is a non-empty open subset of \( A_0 \), we obtain \( \text{int} A_0 \) is also non-empty.

Assume that \( \partial(\text{int} A_0) = \emptyset \). Then

\[
\emptyset = \partial(\text{int} A_0) = \partial(\text{int} A_0)^M \cap (M \setminus \text{int} A_0)^M = (\text{int} A_0)^M \cap (M \setminus \text{int} A_0).
\]

Now, as \( \emptyset = (\text{int} A_0)^M \cap (M \setminus \text{int} A_0) \), we obtain \( (\text{int} A_0)^M \subseteq \text{int} A_0 \) which implies that \( (\text{int} A_0)^M = \text{int} A_0 \), thus \( \text{int} A_0 \) is closed in \( M \). But \( \text{int} A_0 \) is non-empty open, and \( M \) is connected, we conclude that \( \text{int} A_0 = M \), so \( \text{int} A_0 = A_0 = M \) and \( \nabla f = 0 \) on \( M \).

Assume now that \( \partial(\text{int} A_0) \neq \emptyset \), we will obtain a contradiction. Let \( p_0 \in \partial(\text{int} A_0) \), \( \partial(\text{int} A_0) = (\text{int} A_0)^M \setminus \text{int} A_0 \), necessarily \( p_0 \notin \text{int} A_0 \). Let \( U \) be an open subset containing \( p_0 \), then \( U \cap \text{int} A_0 = \emptyset \).

On the other hand, we have

\[
p_0 \in \partial(\text{int} A_0) \subseteq \partial A_0,
\]

so

\[
p_0 \in \partial A_0 = \partial(M \setminus A_0).
\]

Since \( A_0 \) is closed in \( M \), then \( M \setminus A_0 \) is non-empty open in \( M \), and so \( p_0 \notin M \setminus A_0 \). Of course, \( p_0 \in (M \setminus A_0)^M \) implies that \( U \cap (M \setminus A_0) \neq \emptyset \).

In conclusion:
(1) \( U \cap \text{int} A_0 \) is a non-empty open subset of \( \text{int} A_0 \) that does not contain \( p_0 \), so there exists a non-empty open subset on which \( \text{grad} f = 0 \).

(2) \( U \cap (M \setminus A_0) \) is a non-empty open subset that does not contain \( p_0 \), and is included in \( M \setminus A_0 \), so there exists a non-empty open subset on which \( \text{grad} f \neq 0 \) at any point.

Let \( (U, x')_{j=1,...,n} \) be a local chart on \( M \) around \( p_0 \in \partial (\text{int} A_0) \). Consider an open connected subset \( D \) in \( M \) containing \( p_0 \), such that \( \partial D \) is compact and \( \overline{D} \subset U \). Note that \( D \) also contains a non-empty open subset where \( \text{grad} f = 0 \) everywhere, and a non-empty open subset where \( \text{grad} f \neq 0 \) at any point.

As usual, we identify \( \text{grad} f \in C(TM) \) with \( d\varphi (\text{grad} f) \in C(\varphi^{-1} T\mathbb{S}^{m+1}) \), or

\[
d(i \circ \varphi)(\text{grad} f) \in C((i \circ \varphi)^{-1} T\mathbb{R}^{m+2}),
\]

where \( i : \mathbb{S}^{m+1} \cong \mathbb{R}^{m+2} \) is the canonical inclusion. Let us write \( \text{grad} f = u^a e_a \), where \( u^a \in C^\infty (M) \), \( \forall a = 1, \ldots, m + 2 \), and \( \{ e_a \}_{a=1}^{m+2} \) is the canonical basis in \( \mathbb{R}^{m+2} \). For all \( a = 1, \ldots, m + 2 \), the function \( u^a \) vanishes on \( V \).

As \( \varphi \) is biharmonic, we have

\[
\Delta f = (m - |A|^2) f,
\]

and taking its differential, we obtain

\[
d\Delta f = (m - |A|^2) df - f d(|A|^2),
\]

hence, by the musical isomorphism:

\[
(d\Delta f)^\sharp = [(m - |A|^2) df - f d(|A|^2)]^\sharp,
\]

Since \( (df)^\sharp = \text{grad} f \) and \( d\Delta^\text{Hodge} = \Delta^\text{Hodge} d \), we can rewrite Eq. (3.1) as

\[
(\Delta^\text{Hodge} (df))^\sharp = (m - |A|^2) \text{grad} f - f \text{grad} |A|^2.
\]

On the other hand, by the Weitzenböck formula

\[
(\Delta^\text{Hodge} (df))^\sharp = -\text{trace} \nabla^2 \text{grad} f + \text{Ricci}^M (\text{grad} f),
\]

thus

\[
-\text{trace} \nabla^2 \text{grad} f = -\text{Ricci}^M (\text{grad} f) + (m - |A|^2) \text{grad} f - f \text{ grad} |A|^2.
\]

As \( \text{Ricci}^M (\text{grad} f) = \text{Ricci}^M (u^a e_a) \) and

\[
\text{grad} f = u^a e_a = u^a (e^T_a + e^T_a) = u^a e^T_a,
\]

where \( e^T_a \) and \( e^T_a \) are the normal and the tangential components to \( M \) of \( e_a \) in \( \mathbb{R}^{m+2} \), respectively, we obtain

\[
\text{Ricci}^M (\text{grad} f) = \text{Ricci}^M (u^a e^T_a) = u^a \text{Ricci}^M (e^T_a).
\]

On \( U \), we combine the second fundamental forms of \( M \) in \( \mathbb{S}^{m+1} \) and \( \mathbb{S}^{m+1} \) in \( \mathbb{R}^{m+2} \) to compute
\[ \nabla^M_a \text{grad } f = \nabla^{S^{m+1}}_a \text{grad } f - B(\frac{\partial}{\partial x^a}, \text{grad } f) \]
\[ = \nabla^{R^{m+2}}_a \text{grad } f + \left\langle \frac{\partial}{\partial x^a}, u^a e^T_a \right\rangle r - B(\frac{\partial}{\partial x^a}, \text{grad } f) \]
\[ = \nabla^{R^{m+2}}_a (u^a e_a) + \left\langle \frac{\partial}{\partial x^a}, u^a e^T_a \right\rangle r - B(\frac{\partial}{\partial x^a}, u^a e^T_a) \]
\[ = \frac{\partial u^a}{\partial x^i} e_a + u^a \nabla^{R^{m+2}}_a e_a + u^a \left\langle \frac{\partial}{\partial x^a}, e^T_a \right\rangle r - u^a B(\frac{\partial}{\partial x^a}, e^T_a) \]
\[ = \frac{\partial u^a}{\partial x^i} e_a + u^a \left\langle \frac{\partial}{\partial x^a}, e^T_a \right\rangle r - u^a B(\frac{\partial}{\partial x^a}, e^T_a) \]
\[ = \frac{\partial u^a}{\partial x^i} e^T_a, \]

where \( r \) is the position vector field on \( \mathbb{R}^{m+2} \). Put
\[ Y_i = \frac{\partial u^a}{\partial x^i} e^T_a. \]

For our purposes, it is convenient to write \( Y_i \) as
\[ Y_i = \frac{\partial u^a}{\partial x^i} e_a + u^a Z_{a,i}, \]

where
\[ Z_{a,i} = \left\langle \frac{\partial}{\partial x^a}, e^T_a \right\rangle r - B(\frac{\partial}{\partial x^a}, e^T_a) \]
is a vector field normal to \( M \) in \( \mathbb{R}^{m+2} \).

We repeat this process to obtain, on \( U \), the second derivatives of \( \text{grad } f \),
\[ \nabla^M_a \nabla^M_a \text{grad } f = \nabla^M_a Y_j \]
\[ = \nabla^{R^{m+2}}_a Y_j + \left\langle \frac{\partial}{\partial x^a}, Y_j \right\rangle r - B(\frac{\partial}{\partial x^a}, Y_j) \]
\[ = \nabla^{R^{m+2}}_a \left\{ \frac{\partial u^a}{\partial x^j} e_a + u^a Z_{a,j} \right\} + \left\langle \frac{\partial}{\partial x^a}, Y_j \right\rangle r - B(\frac{\partial}{\partial x^a}, Y_j) \]
\[ = \frac{\partial^2 u^a}{\partial x^i \partial x^j} e_a + \frac{\partial u^a}{\partial x^i} Z_{a,j} + u^a \nabla^{R^{m+2}}_a Z_{a,j} + \left\langle \frac{\partial}{\partial x^a}, Y_j \right\rangle r - B(\frac{\partial}{\partial x^a}, Y_j). \]
Thus, each term on the right-hand side of Eq. (3.9), except for the last one, contains either $\nabla^{a}$ or $\partial^{a}/\partial x^{a}$. Replacing (3.4) and (3.5) in (3.2), and using (3.3), we obtain

\[
(\Delta u^{a}) e_{a} = g^{ij} \frac{\partial u^{a}}{\partial x^{i}} Z_{a,j} + g^{ij} \left( \frac{\partial}{\partial x^{i}}, Y_{j} \right) r + g^{ij} B \left( \frac{\partial}{\partial x^{i}}, Y_{j} \right) + g^{ij} u^{a} \nabla^{R^{m+2}} Z_{a,j} + g^{ij} u^{a} B \left( \nabla^{M} \frac{\partial}{\partial x^{i}}, e_{a}^{T} \right) + g^{ij} u^{a} B \left( \nabla^{M} \frac{\partial}{\partial x^{i}}, e_{a}^{T} \right)
\]

(3.6)

so

\[
(\Delta u^{a}) e_{a} = g^{ij} \left( \frac{\partial}{\partial x^{i}}, Y_{j} \right) r + g^{ij} \left( \frac{\partial}{\partial x^{i}}, e_{a}^{T} \right) \nabla^{a} r + g^{ij} u^{a} \nabla^{R^{m+2}} Z_{a,j} + g^{ij} u^{a} B \left( \nabla^{M} \frac{\partial}{\partial x^{i}}, e_{a}^{T} \right) + g^{ij} u^{a} B \left( \nabla^{M} \frac{\partial}{\partial x^{i}}, e_{a}^{T} \right) - f \text{ grad } |A|^{2}.
\]

But

\[
g^{ij} \left( \frac{\partial}{\partial x^{i}}, Y_{j} \right) r = g^{ij} \left( \frac{\partial}{\partial x^{i}}, e_{a}^{T} \right) \nabla^{a} r,
\]

(3.7)

and

\[
g^{ij} B \left( \frac{\partial}{\partial x^{i}}, Y_{j} \right) = g^{ij} \left( \frac{\partial}{\partial x^{i}}, e_{a}^{T} \right) B \left( \frac{\partial}{\partial x^{i}}, e_{a}^{T} \right).
\]

(3.8)

so replacing (3.7) and (3.8) in (3.6), we obtain

\[
(\Delta u^{a}) e_{a} = g^{ij} \left( \frac{\partial u^{a}}{\partial x^{i}} Z_{a,j} + g^{ij} \left( \frac{\partial}{\partial x^{i}}, e_{a}^{T} \right) \nabla^{a} r + g^{ij} u^{a} \nabla^{R^{m+2}} Z_{a,j} + g^{ij} u^{a} B \left( \nabla^{M} \frac{\partial}{\partial x^{i}}, e_{a}^{T} \right) + g^{ij} u^{a} B \left( \nabla^{M} \frac{\partial}{\partial x^{i}}, e_{a}^{T} \right) - u^{a} \text{ Ricci } M \left( e_{a}^{T} \right) + (m - |A|^{2}) u^{a} e_{a}^{T} - f \text{ grad } |A|^{2}.
\]

(3.9)

Thus, each term on the right-hand side of Eq. (3.9), except for the last one, contains either $\partial^{a}/\partial x^{a}$ or $u^{a}$. By the triangle inequality

\[
\text{ Springer}
\]
and since all functions and vector fields are smooth on $U$, they are bounded on $\overline{D}$, and so, on $D$. Using the hypothesis and standard inequalities, we obtain

$$|\Delta u^0| \leq C \left( \sum_{a,i} \left| \frac{\partial u^a}{\partial x^i} \right| + \sum_a |u^a| \right)$$

on $D$. Since $u^a$ is zero on a non-empty open subset of $D$, by Aronszajn’s unique continuation principle we deduce that $u^a$ is equal to zero on $D$, and thus $\text{grad} f$ vanishes on $D$. This is impossible, hence the assumption $\partial (\text{int} A_0) \neq \emptyset$ is false. In conclusion, $\partial (\text{int} A_0) = \emptyset$, and so $\text{grad} f$ vanishes on the whole of $M$.

Theorem 3.1 can be rephrased as follows:

**Corollary 3.2** Let $\varphi : M^m \hookrightarrow \mathbb{S}^{m+1}$ be a proper-biharmonic hypersurface. Assume that there exists a non-negative function $h$ on $M$ such that $|\text{grad} |A|^2| \leq h \cdot |\text{grad} f|$ on $M$. Then, either $M$ has constant mean curvature, or the set of points where $\text{grad} f \neq 0$ is an open dense subset of $M$.

**Proof** Assume that $M$ is not CMC. Let

$$W := \{ p \in M : (\text{grad} f)(p) \neq 0 \},$$

be a non-empty open subset in $M$. Assume that $\overline{W}^M \subsetneq M$, then $V = M \setminus \overline{W}^M$ is a non-empty, open subset of $M$ and $\text{grad} f|_V = 0$, therefore $f$ is constant on a connected component $V_1$ of $V$. As $f$ is constant on $V_1$ and $|\text{grad} |A|^2| \leq h \cdot |\text{grad} f|$ over $M$, by Theorem 3.1 we deduce that $f$ is constant on $M$, which is a contradiction, therefore

$$\overline{W}^M = M.$$

The hypothesis on the existence of the function $h$ in Theorem 3.1 can be obtained under natural conditions on $|A|^2$ or the scalar curvature of $M$.

**Corollary 3.3** Let $\varphi : M^m \hookrightarrow \mathbb{S}^{m+1}$ be a proper-biharmonic hypersurface with $|A|^2$ constant. Then, either $M$ has constant mean curvature, or the set of points where $\text{grad} f \neq 0$ is an open dense subset of $M$.

**Proof** As $|A|^2$ is constant, the condition $|\text{grad} |A|^2| \leq h \cdot |\text{grad} f|$ on $M$ is automatically satisfied, thus, by Corollary 3.2 we conclude.

**Corollary 3.4** Let $\varphi : M^m \hookrightarrow \mathbb{S}^{m+1}$ be a proper-biharmonic hypersurface with constant scalar curvature. Then, either $M$ has constant mean curvature, or the set of points where $\text{grad} f \neq 0$ is an open dense subset of $M$.

**Proof** By Proposition 2.2, we have
\[ |A|^2 = m(m-1) + m^2f^2 - \text{Scal}^M, \]

which implies
\[ |\text{grad } |A|^2| = 2m^2|f| |\text{grad } f|. \]

Therefore, the condition
\[ |\text{grad } |A|^2| \leq h |\text{grad } f| \]

holds on \( M \), and we apply Corollary 3.2 to conclude. \( \square \)

**Remark 3.5**

(1) Corollaries 3.3 and 3.4 are meaningful because \( M \) is not assumed to be compact:

(a) A direct consequence of J.-H. Chen’s result is that if \( M \) is compact, and \( |A|^2 \) is constant, then \( \text{grad } f \) vanishes on the whole manifold \( M \) (see [23]).

(b) If \( M \) is compact and its scalar curvature is constant, S. Maeta and Y.-L. Ou show in [16] that \( f \) is constant.

Therefore, Corollaries 3.3 and 3.4 can be seen as extensions of results in [16] and [23], where it is shown that if \( f \) is constant on a non-empty open subset of \( M \) then \( f \) is constant on \( M \).

(2) Theorem 3.1 is meaningful even in the compact case.

(3) Consider \( \varphi : M^m \hookrightarrow S^{m+1}(c) \), \( c \leq 0 \) a proper-biharmonic hypersurface. Assume that \( \text{grad } f \) vanishes on an open subset. Then, it follows that \( f \) is constant on an open (connected) subset. But, as \( c \leq 0 \), the constant has to be zero (see [11] for a more general statement), so \( \varphi \) is harmonic on an open subset, therefore on the whole manifold \( M \).

As a direct application of Corollary 3.4, we can give the following result.

**Proposition 3.6** Let \( \varphi : M^m \hookrightarrow S^{m+1} \) be a proper-biharmonic hypersurface with constant scalar curvature. Assume that there exists a connected component of \( M_A \) where the number of distinct principal curvatures is at most six. Then \( M \) has constant mean curvature.

**Proof** Let \( U \) be a connected component of \( M_A \). The number of distinct principal curvatures is constant and at most 6.

As \( \text{Scal}^M \) is constant, by Theorem 1.1 of [15] we obtain that \( f \) is constant on \( U \). On the other hand, by Corollary 3.4, we deduce that \( f \) is constant on \( M \). \( \square \)

**Corollary 3.7** Let \( \varphi : M^m \hookrightarrow S^{m+1} \) be a proper-biharmonic hypersurface. Assume that there exists a non-negative function \( h \) such that \( |\text{grad } |A|^2| \leq h |\text{grad } f| \), and \( M \) is not CMC. Denote by \( U \) an open connected component of \( M_A \). Then, on \( U \) we have:

(1) \(-\frac{m}{\text{grad } f} \) is a principal curvature with multiplicity equal to 1;

(2) \( \frac{\text{grad } f}{|\text{grad } f|} \) is a vector field defined on an open dense subset of \( U \) and its integral curves are geodesics;
(3) the number of distinct principal curvatures is at least 3 and $|A|^2 > \frac{m^2(m+8)}{4(m-1)}f^2$ on an open dense subset of $U$(see [13]).

**Proof** Since $M$ does not have constant mean curvature, by Corollary 3.2 we deduce that the points of $U$ where $\text{grad} f \neq 0$ form an open dense subset of $U$. Now, by continuity we obtain $-m f = \lambda_{i_0}$, for some $i_0$, on $U$, and by Proposition 2.4, we obtain that the multiplicity of $\lambda_{i_0}$ is 1.

Furthermore, for simplicity, we consider $i_0 = 1$, and work on an open connected subset of $U$ where $\text{grad} f \neq 0$ at any point. We have $E_1 = \frac{\text{grad} f}{|\text{grad} f|}$ and taking the inner product of Equation (2.3) with $E_1$ for $i = 1$ and $j = a$, we obtain

$$\omega_{i_0}^i(E_1) = 0,$$

and thus $\nabla_{E_i} E_1 = 0$.

If the number of distinct principal curvatures is at most 2 then $U$ is CMC (see [13]). As J.-H. Chen’s Inequality is based on the Cauchy-Schwarz Inequality applied to the principal curvatures, we have a strict inequality. □

**Remark 3.8** We note that the distribution orthogonal to that determined by $\frac{\text{grad} f}{|\text{grad} f|}$ is completely integrable. The level hypersurfaces of the mean curvature $f$ have flat normal connection as submanifolds in $S^{m+1}$ of codimension 2 (see [27, Theorem 1.40]).

Corollary 3.2 allows the re-writing of some known results replacing their global hypothesis with local variants.

**Corollary 3.9** Let $\varphi : M^m \hookrightarrow S^{m+1}$ be a proper-biharmonic hypersurface. Assume that $\frac{|\text{grad} A|^2}{|\text{grad} f|} \leq h |\text{grad} f|$ on $M$, where $h$ is a non-negative function on $M$. If $M$ is not CMC, then J.-H. Chen’s Inequality

$$|A|^2 \geq \frac{m^2(m+8)}{4(m-1)}f^2$$

(3.10)

is valid everywhere on $M$.

**Proof** Inequality (3.10) holds on $W$, and we conclude by continuity. □

J.-H. Chen’s Inequality enables us to obtain a more geometric version of Theorem 3.1.

**Theorem 3.10** Let $\varphi : M^m \hookrightarrow S^{m+1}$ be a proper-biharmonic hypersurface. Assume that $f^2 > \frac{4(m-1)}{m(m+8)}$. If $\text{grad} f$ vanishes on a non-empty open connected subset of $M$, then $M$ has constant mean curvature.

**Proof** Let us denote

$$A_0 := \{ p \in M : (\text{grad} f)(p) = 0 \}.$$

In the proof of Theorem 3.1, we have shown that $A_0$ is a closed subset of $M$, $\text{int} A_0 \neq \emptyset$, and if $\partial(\text{int} A_0) = \emptyset$ then $\text{grad} f$ vanishes on $M$. 

 Springer
As in the proof of Theorem 3.1, assume that ∂(int A₀) ≠ ∅, to reach a contradiction. Let p₀ ∈ ∂(int A₀), it follows that there exists a sequence of points {pⁿ} n∈ℕ₀ converging to p₀, pⁿ ≠ p₀ and pⁿ ∈ int A₀ for any n ∈ ℕ₀, and there exists a sequence of points {pⁿ} n∈ℕ₀, that converges to p₀, pⁿ ≠ p₀ and (grad f)(pⁿ) ≠ 0 for any n ∈ ℕ₀.

From Lemma (2.1), we have

\[ |A|^2(pⁿ) ≥ \frac{m^2(m+8)}{4(m-1)} f^2(pⁿ), \forall n ∈ ℕ₀. \]  \hspace{1cm} (3.11)

Now, each connected component of int A₀ is open in int A₀ and so in M. Thus, on each connected component of int A₀ the function f is constant. But the constant cannot be zero as φ is not harmonic and so |A|^2 = m. In conclusion, we have |A|^2 = m on int A₀ and

\[ |A|^2(pⁿ) = m, \forall n ∈ ℕ₀. \]  \hspace{1cm} (3.12)

Passing to the limit in (3.11) and (3.12), we obtain

\[ m = |A|^2(p₀) ≥ \frac{m^2(m+8)}{4(m-1)} f^2(p₀), \]

thus

\[ f^2 ≤ \frac{4(m-1)}{m(m+8)} \]

which is impossible. \hfill \Box

**Remark 3.11** Compare the above result with [23, Proposition 1.38 and Corollary 1.40].

### 4 Rigidity results for biharmonic hypersurfaces

The unique continuation properties of Sect. 2 can be exploited to obtain new rigidity results. Theorem 4.1 relies essentially on the combination of the Bochner formula applied to the vector field grad f and the J.-H. Chen’s Inequality, made possible thanks to Corollary 3.9, while Theorem 4.2 is a more technical alternative which puts together a bound on the Ricci curvature and an averaged version of the condition of [23, Proposition 1.38].

**Theorem 4.1** Let φ : Mᵐ ↪ Sᵐ⁺¹ be a compact proper-biharmonic hypersurface. Assume that |grad |A|^2| ≤ h |grad f| on M, where h is a non-negative function on M, Scal M ≥ 0 and

\[ \int_M [m(m+8)f^2 - 4(m-1)] Scal_M f^2 dv_g ≥ 0. \]  \hspace{1cm} (4.1)

Then M has constant mean curvature.

**Proof** Assume that M does not have constant mean curvature, we will argue by contradiction.

Starting with the Bochner Formula (see for example [28]), we have

\[ \Box \] Springer
\[-\frac{1}{2} \Delta |\nabla f|^2 = |\nabla df|^2 - \langle \nabla \Delta f, \nabla f \rangle + \text{Ricci}^M(\nabla f, \nabla f).\]

Now, by the Gauss Equation, we get
\[
\text{Ricci}^\mathbb{S}^{m+1}(\nabla f, \nabla f) = \text{Ricci}^M(\nabla f, \nabla f) + |A(\nabla f)|^2 - mf\langle A(\nabla f), \nabla f \rangle + R^\mathbb{S}^{m+1}(\nabla f, \eta, \nabla f, \eta),
\]
where \(\eta\) is the unit normal vector field.

On the other hand, we have
\[
\text{Ricci}^\mathbb{S}^{m+1}(\nabla f, \nabla f) = m|\nabla f|^2.
\]

Now, since \(M\) is a biharmonic submanifold of \(\mathbb{S}^{m+1}\),
\[
A(\nabla f) = -\frac{m}{2} f \nabla f,
\]
\[
|A(\nabla f)|^2 = \frac{m^2}{4} |\nabla f|^2,
\]
\[
-mf\langle A(\nabla f), \nabla f \rangle = \frac{m^2}{2} f^2 |\nabla f|^2.
\]

Thus, using Eq. (4.2) we deduce that
\[
\text{Ricci}^M(\nabla f, \nabla f) = \left(m - 1 - \frac{3m^2}{4} f^2\right) |\nabla f|^2.
\]

Denote the scalar curvature \(\text{Scal}^M\) by \(s\). We have
\[
|A|^2 = m(m-1) + m^2 f^2 - s.
\]

As \(\varphi\) is biharmonic, we have
\[
\Delta f = (m - |A|^2) f,
\]
thus
\[
\nabla \Delta f = \nabla [(m - |A|^2) f]
= m \nabla f - f |A|^2 - |A|^2 \nabla f
= m \nabla f - f \nabla (m^2 f^2 - s) - |A|^2 \nabla f
= m \nabla f - 2m^2 f^2 \nabla f + f \nabla s - |A|^2 \nabla f
= (m - 2m^2 f^2 - |A|^2) \nabla f + f \nabla s.
\]

On the other hand, for a local orthonormal frame field \(\{e_i\}_i\),
\[ |\nabla df|^2 \geq \sum_{i,j=1}^{m} (\nabla df(e_i, e_j))^2 \]
\[ \geq \sum_{i=1}^{m} (\nabla df(e_i, e_i))^2 \]
\[ \geq \frac{1}{m} \left( \sum_{i=1}^{m} \nabla df(e_i, e_i) \right)^2 \]
\[ \geq \frac{1}{m} (\Delta f)^2. \]

Now
\[ -\frac{1}{2} \Delta |\nabla f|^2 \geq \frac{1}{m} (\Delta f)^2 - \langle (m - 2m^2 f^2 - |A|^2) \nabla f + f \nabla s, \nabla f \rangle \]
\[ + \left( m - 1 - \frac{3m^2}{4} f^2 \right) |\nabla f|^2 \]
\[ \geq \frac{1}{m} (\Delta f)^2 + \left( |A|^2 + \frac{5m^2}{4} f^2 - 1 \right) |\nabla f|^2 - f \langle \nabla s, \nabla f \rangle. \]

We have
\[ \frac{1}{m} \int_M (\Delta f)^2 \, dv_g = \frac{1}{m} \int_M (\Delta f)(\Delta f) \, dv_g \]
\[ = - \frac{1}{m} \int_M (\Delta f) \nabla (\nabla f) \, dv_g, \]

and using the Divergence Theorem, we obtain
\[ \frac{1}{m} \int_M (\Delta f)^2 \, dv_g = \frac{1}{m} \int_M \langle \nabla \Delta f, \nabla f \rangle \, dv_g \]
\[ = - \frac{1}{m} \int_M (|A|^2 + 2m^2 f^2 - m) |\nabla f|^2 \, dv_g \]
\[ + \frac{1}{m} \int_M f \langle \nabla s, \nabla f \rangle \, dv_g. \]

Integrating the Bochner Formula over \( M \) and using the Divergence Theorem, we get
\[ 0 \geq -\frac{1}{m} \int_M (|A|^2 + 2m^2 f^2 - m) |\nabla f|^2 \, dv_g + \frac{1}{m} \int_M f \langle \nabla s, \nabla f \rangle \, dv_g \]
\[ + \int_M (|A|^2 + \frac{5m^2}{4} f^2 - 1) |\nabla f|^2 \, dv_g - \int_M f \langle \nabla s, \nabla f \rangle \, dv_g \]
\[ \geq \int_M \left[ \left( 1 - \frac{1}{m} \right) |A|^2 + \left( -\frac{2}{m} + \frac{5}{4} \right) m^2 f^2 \right] |\nabla f|^2 \, dv_g \]
\[ + \left( \frac{1 - m}{m} \right) \int_M f \langle \nabla s, \nabla f \rangle \, dv_g. \]

(4.3)

To obtain a lower bound of the first term, we need Corollary 3.9, and we apply it to Eq. (4.3) to obtain
$$0 \geq \int_M \left[ \left( \frac{m-1}{m} \right) \left( \frac{m^2 (m+8)}{4(m-1)} \right) f^2 + \left( \frac{5m-8}{4m} \right) m^2 f^2 \right] |\text{grad} f|^2 \, dv_g$$

$$+ \left( \frac{1-m}{2m} \right) \int_M \langle \text{grad} s, \text{grad} f^2 \rangle \, dv_g$$

$$\geq \int_M \left[ \frac{m(m+8)}{4} f^2 + \frac{m(5m-8)}{4} f^2 \right] |\text{grad} f|^2 \, dv_g + \left( \frac{1-m}{2m} \right) \int_M s \Delta f^2 \, dv_g$$

$$\geq \frac{3m^2}{2} \int_M f^2 |\text{grad} f|^2 \, dv_g + \left( \frac{1-m}{2m} \right) \int_M \, dv_g,$$

Now, we have

$$\Delta f^2 = 2 \left( (m - |A|^2) f^2 - |\text{grad} f|^2 \right),$$

thus

$$\frac{3m^2}{2} \int_M f^2 |\text{grad} f|^2 \, dv_g + \left( \frac{1-m}{2m} \right) \int_M 2s \left[ (m - |A|^2) f^2 - |\text{grad} f|^2 \right] \, dv_g$$

$$= \frac{3m^2}{2} \int_M f^2 |\text{grad} f|^2 \, dv_g + \left( \frac{1-m}{2m} \right) \int_M s(m - |A|^2) f^2 \, dv_g$$

$$+ \left( \frac{m-1}{m} \right) \int_M s |\text{grad} f|^2 \, dv_g.$$

Using Corollary 3.9 and the fact that $s \geq 0$, we obtain

$$0 \geq \frac{3m^2}{2} \int_M f^2 |\text{grad} f|^2 \, dv_g + (1-m) \int_M sf^2 \, dv_g$$

$$+ \frac{m(m+8)}{4} \int_M sf^4 \, dv_g + \left( \frac{m-1}{m} \right) \int_M s |\text{grad} f|^2 \, dv_g. \quad (4.4)$$

Now as $s \geq 0$, we obtain

$$\int_M s |\text{grad} f|^2 \, dv_g \geq 0. \quad (4.5)$$

Then from Inequality (4.4), we have

$$0 \geq \frac{3m^2}{2} \int_M f^2 |\text{grad} f|^2 \, dv_g + (1-m) \int_M sf^2 \, dv_g + \frac{m(m+8)}{4} \int_M sf^4 \, dv_g. \quad (4.6)$$

Multiplying Inequality (4.6) by 4, we obtain

$$0 \geq 6m^2 \int_M f^2 |\text{grad} f|^2 \, dv_g + \int_M [4(1-m) + m(m+8)f^2] \, dv_g,$$

Now as $\int_M [4(1-m) + m(m+8)f^2]sf^2 \, dv_g \geq 0$ , we obtain

$$0 \geq 6m^2 \int_M f^2 |\text{grad} f|^2 \, dv_g + \int_M [4(1-m) + m(m+8)f^2] \, dv_g \geq 0,$$

by the sandwich rule, we conclude that
\[ 6m^2 \int_M f^2 |\nabla f|^2 \, dv_g + \int_M [4(1-m) + m(m + 8)f^2]s f^2 \, dv_g = 0. \]

Hence, at every point of \( M \), \( f^2 |\nabla f|^2 = 0 \) which implies that, at each point of \( M \) \( f = 0 \) or \( \nabla f = 0 \).

Let \( p \in M \) such that \( (\nabla f)(p) \neq 0 \), then \( f = 0 \) around \( p \), thus \( \nabla f = 0 \) at \( p \), which is a contradiction. Therefore, \( \nabla f = 0 \) at each point \( p \), which implies that \( f \) is constant everywhere on \( M \), and contradicts our assumption. \[\square\]

A weaker version of Theorem 4.1 can be formulated, replacing the condition on the scalar curvature by a combination of two lower bounds on the Ricci and scalar curvature, and an inequality involving the average of the mean curvature.

**Theorem 4.2** Let \( \varphi : M^m \hookrightarrow S^{m+1} \) be a compact proper-biharmonic hypersurface. Assume that there exist a non-negative function \( h \) on \( M \) such that \( |\nabla |A|^2| \leq h |\nabla f| \) on \( M \), and a real number \( a > 0 \) such that:

1. \( \text{Ricci}_M(X,X) \geq a > 0 \), for all \( X \in T_pM \), \( |X| = 1 \) and for all \( p \in M \);
2. \( \int_M [m^2(m + 8)af^2 - 4(m - 1)s]f^2 \, dv_g \geq 0. \)

Then \( M \) has constant mean curvature.

**Proof** Proceeding in the exact same way as in the proof of the previous theorem, we reach

\[ 0 \geq \frac{3m^2}{2} \int_M f^2 |\nabla f|^2 \, dv_g + (1 - m) \int_M sf^2 \, dv_g + \frac{m(m + 8)}{4} \int_M sf^4 \, dv_g + \left( \frac{m - 1}{m} \right) \int_M s |\nabla f|^2 \, dv_g. \tag{4.7} \]

To control the terms in \( f^2 \), for the Hilbert space \( L^2(M) \), we consider an orthonormal basis \( \{f_i\}_{i=0}^{\infty} \) of \( C^\infty(M) \)-eigenfunctions of the Laplacian, i.e. \( \Delta f_i = \lambda_i f_i \), where \( \lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \), and \( \int_M f_i f_j \, dv_g = \delta_{ij} \).

Let \( f \in C^\infty(M) \), then \( f = \sum_{i=0}^{\infty} \mu_i f_i \), where \( f_0 = \frac{1}{\sqrt{\text{Vol}(M)}} \) and \( \mu_0 = \frac{1}{\sqrt{\text{Vol}(M)}} \int_M f \, dv_g \).

Then

\[ \int_M f^2 \, dv_g = \sum_{i=0}^{\infty} \mu_i^2. \]

Also,

\[ \Delta f = \sum_{i=0}^{\infty} \lambda_i \mu_i f_i = \sum_{i=1}^{\infty} \mu_i \lambda_i f_i, \]

so, we have...
But
\[ \int_M f \Delta f \, dv_g = \sum_{i=1}^{\infty} \lambda_i \mu_i^2 \]
\[ \geq \lambda_1 \sum_{i=1}^{\infty} \mu_i^2 = \lambda_1 \left( \int_M f^2 \, dv_g - \mu_0^2 \right). \]

Now, by Obata [29], Ricci\(^M\)(X,X) \geq a|X|^2 > 0 implies that \( \lambda_1 \geq \frac{ma}{m-1}. \) Since \( s \geq ma, \) we have
\[ \int_M s|\nabla f|^2 \, dv_g \geq ma \int_M |\nabla f|^2 \, dv_g, \]
\[ \geq \frac{m^2 a^2}{m-1} \left[ \int_M f^2 \, dv_g - \frac{1}{\text{Vol}(M)} \left( \int_M f \, dv_g \right)^2 \right]. \] (4.8)

Then from Inequality (4.7)
\[ 0 \geq \frac{3m^2}{2} \int_M f^2 |\nabla f|^2 \, dv_g + (1 - m) \int_M sf^2 \, dv_g + \frac{m(m + 8)}{4} \int_M sf^4 \, dv_g \]
\[ + \ ma^2 \int_M f^2 \, dv_g - \frac{ma^2}{\text{Vol}(M)} \left( \int_M f \, dv_g \right)^2, \] (4.9)
and multiplying Inequality (4.9) by 4, and using \( s \geq ma, \) we obtain
\[ 0 \geq 6m^2 \int_M f^2 |\nabla f|^2 \, dv_g + 4(1 - m) \int_M sf^2 \, dv_g + m^2(m + 8)a \int_M f^4 \, dv_g \]
\[ + \ 4ma^2 \int_M f^2 \, dv_g - \frac{4ma^2}{\text{Vol}(M)} \left( \int_M f \, dv_g \right)^2 \geq 6m^2 \int_M f^2 |\nabla f|^2 \, dv_g + 4 \int_M \left[(1 - m)s + ma^2\right] f^2 \, dv_g \]
\[ + \ m^2(m + 8)a \int_M f^4 \, dv_g - \frac{4ma^2}{\text{Vol}(M)} \left( \int_M f \, dv_g \right)^2. \]

Now, using the Cauchy-Schwarz Inequality, we obtain
By Condition 2, we conclude that

\[ 0 \geq 6m^2 \int_M f^2 |\nabla f|^2 \, dv_g + \int_M \left[ 4(1 - m)s + 4ma^2 + m^2(m + 8)af^2 - 4ma^2 \right] f^2 \, dv_g. \]

By Condition 2, we conclude that

\[ 6m^2 \int_M f^2 |\nabla f|^2 \, dv_g + \int_M \left[ 4(1 - m)s + m^2(m + 8)af^2 \right] f^2 \, dv_g = 0, \]

to prove Theorem 4.2. \qed

Remark 4.3 The conditions in Theorems 4.1 and 4.2 are satisfied by the 45th-parallel \( S^m(1/\sqrt{2}) \hookrightarrow S^{m+1} (m \geq 2) \) since \( \text{Scal} = 2m(m - 1), f^2 = 1, \) and we can take \( a = 2(m - 1) \).

The generalized Clifford torus \( S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2}) \hookrightarrow S^{m+1} (m_1 < m_2) \) has \( \text{Scal} = 2m_1(m_1 - 1) + 2m_2(m_2 - 1), \) and \( f^2 = \left( \frac{m_1 - m_2}{m_1 + m_2} \right)^2, \) by straightforward calculations it satisfies Condition 4.1 in Theorem 4.1 if and only if \( m_1 \in \left[ \frac{m}{2} - \sqrt{\frac{m(m-1)}{m+8}} \right] \) \( (m = m_1 + m_2) \) which includes \( m_2 \geq 2m_1 + 2. \) For \( m \) large enough, there always exist Clifford tori satisfying the conditions of Theorem 4.2, for \( a = 2(m_1 - 1), m_1 \geq 2. \)

References

1. Jiang, G.-Y.: 2-harmonic maps and their first and second variational formulas. Chin. Ann. Math. Ser. A 7(4), 389–402 (1986)
2. Eells, J., Lemaire, L.: Selected Topics in Harmonic Maps. American Mathematical Society, Providence (1983)
3. Chen, B.-Y.: Total Mean Curvature and Submanifolds of Finite Type. Series in Pure Mathematics, World Scientific Publishing Co, Singapore (1984)
4. Chen, B.-Y., Ishikawa, S.: Biharmonic surfaces in pseudo-Euclidean spaces. Mem. Fac. Sci. Kyushu Univ. Ser. A 45, 323–347 (1991)
5. Jiang, G.-Y.: Some nonexistence theorems on 2-harmonic and isometric immersions in Euclidean space. Chin. Ann. Math. Ser. B 8(3), 377–383 (1987)
6. Chen, B.-Y.: Some open problems and conjectures on submanifolds of finite type. Soochow J. Math. 17, 169–188 (1991)
7. Akutagawa, K., Maeta, S.: Biharmonic properly immersed submanifolds in Euclidean spaces. Geom. Dedic. 164, 351–355 (2013)
8. Dimitric, I.: Submanifolds of \( \mathbb{E}^m \) with harmonic mean curvature vector. Bull. Inst. Math. Acad. Sinica 20, 53–65 (1992)
9. Hasanis, T., Valchos, T.: Hypersurfaces in \( \mathbb{E}^4 \) with harmonic mean curvature vector field. Math. Nachr. 172, 145–169 (1995)
10. Chen, J.-H.: Compact 2-harmonic hypersurfaces in \( S^{m+1}(1) \). Acta Math. Sinica 36, 49–56 (1993)
11. Oniciuc, C.: Biharmonic maps between Riemannian manifolds. An. Stiint. Univ. Al.I. Cuza Iasi Mat (N.S.) 48, 237–248 (2002)
12. Caddeo, R., Montaldo, S., Oniciuc, C.: Biharmonic submanifolds of \( S^3 \). Int. J. Math. 12, 867–876 (2001)
13. Balmuş, A., Montaldo, S., Oniciuc, C.: Classification results for biharmonic submanifolds in spheres. Isr. J. Math. 168, 201–220 (2008)
14. Balmuş, A., Montaldo, S., Oniciuc, C.: Biharmonic hypersurfaces in 4-dimensional space forms. Math.Nachr. 283, 1696–1705 (2010)
15. Fu, Y., Hong, M.-C.: Biharmonic hypersurfaces with constant scalar curvature in space forms. Pac. J. Math. 294(2), 329–350 (2018)
16. Maeta, S., Ou, Y.-L.: Some classifications of biharmonic hypersurfaces with constant scalar curvature. Pac. J. Math. 306(1), 281–290 (2020)
17. Luo, Y., Maeta, S.: Biharmonic hypersurfaces in a sphere. Proc. Amer. Math. Soc. 145(7), 3109–3116 (2017)
18. Maeta, S.: Biharmonic hypersurfaces with bounded mean curvature. Proc. Amer. Math. Soc. 145(4), 1773–1779 (2017)
19. Branding, V., Oniciuc, C.: Unique continuation theorems for biharmonic maps. Bull. Lond. Math. Soc. 51, 603–621 (2019)
20. Aronszajn, N.: A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order. J. Math. Pures Appl. 9(36), 235–249 (1957)
21. Caddeo, R., Montaldo, S., Oniciuc, C.: Biharmonic submanifolds in spheres. Isr. J. Math. 130, 109–123 (2002)
22. do Carmo, M.P.: Riemannian Geometry. Mathematics Theory & Applications. Birkhauser, Basel (1992)
23. Oniciuc, C.: Biharmonic submanifolds in space forms. Habilitation Thesis (2012), www.researchgate.net, https://doi.org/10.13140/2.1.4980.5605
24. Nomizu, K.: Characteristic roots and vectors of a differentiable family of symmetric matrices. Lin. Multilinear Algebra 1(2), 159–162 (1973)
25. Ryan, P.J.: Hypersurfaces with parallel Ricci tensor. Osaka J. Math. 8, 251–259 (1971)
26. Ryan, P.J.: Homogeneity and some curvature conditions for hypersurfaces. Tohoku Math. J. 2(21), 363–388 (1969)
27. Nistor, S.: Biharmonicity and biconservativity topics in the theory of submanifolds. PhD Thesis, Univ. Al.I. Cuza Iasi (2017)
28. Petersen, P.: Riemannian Geometry. Graduate Texts in Mathematics, Springer, New-York (2006)
29. Obata, M.: Certain conditions for a Riemannian manifold to be isometric with a sphere. J. Math. Soc. Jpn. 4(3), 333–340 (1962)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.