THE ESSENTIAL NORM OF A WEIGHTED COMPOSITION OPERATOR ON BMOA

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Abstract. We provide an estimate for the essential norm of a weighted composition operator \( W_{\psi,\varphi} : f \mapsto \psi(f \circ \varphi) \) acting on the space \( BMOA \) in terms of the weight function \( \psi \) and the \( n \)-th power \( \varphi^n \) of the analytic self-map \( \varphi \) of the open unit disc \( D \). We also provide a new estimate for the norm of the weighted composition operator on \( BMOA \).

1. Introduction

Let \( D \) be the open unit disc in the complex plane and denote by \( H(D) \) the space of all analytic functions \( D \to \mathbb{C} \). Throughout the paper, \( \psi \) will denote a function in \( H(D) \), and \( \varphi \) will be an analytic self-map of \( D \), \( \varphi(D) \subset D \). These maps induce a linear weighted composition operator \( W_{\psi,\varphi} \) which is defined on \( H(D) \) by

\[ (W_{\psi,\varphi} f)(z) = (M_{\psi} C_{\varphi} f)(z) = \psi(z) f(\varphi(z)), \quad z \in D, \]

where \( M_{\psi} \) is the operator of pointwise multiplication by \( \psi \), and \( C_{\varphi} \) is the composition operator \( f \mapsto f \circ \varphi \). The main aim of this paper is to complement the literature on weighted composition operators by providing a function-theoretic estimate for the essential norm of a weighted composition operator \( W_{\psi,\varphi} \) on the space \( BMOA \), which consists of the analytic functions on \( D \) that are of bounded mean oscillation on the unit circle \( \partial D \).

Let \( H^p \) \((1 \leq p < \infty)\) be the classical Hardy space of functions \( f \in H(D) \) that satisfy

\[ \|f\|_p = \lim_{r \to 1} \left( \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} \]

and let \( H^\infty \) by the space of bounded analytic functions on \( D \) endowed with the sup norm. Then \( f \in H^2 \) belongs to the space \( BMOA \) provided that the \( BMOA \) seminorm

\[ \|f\|_* = \sup_{a \in \overline{D}} \|T[f, a]\|_2 \]

is finite. Here \( T[f, a] = f \circ \sigma_a - f(a) \), where \( \sigma_a(z) = (a - z)/(1 - \overline{a}z) \) is the conformal automorphism of \( D \) that exchanges the points 0 and \( a \). The quantity \( \|f\| = |f(0)| + \|f\|_* \) is a complete norm on \( BMOA \). The closed subspace \( VMOA \) consists of the analytic functions having vanishing mean oscillation on \( \partial D \), or equivalently, of the functions \( f \in BMOA \) such that

\[ \lim_{|a| \to 1} \|T[f, a]\|_2 = 0. \]

We refer to [5], [6] and [19] for the basic properties of the spaces \( BMOA \) and \( VMOA \).

Boundedness and compactness properties of pointwise multipliers \( M_{\psi} \) and composition operators \( C_{\varphi} \) on \( BMOA \) and \( VMOA \) have been intensively investigated in

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Theorem 1.2. Suppose that $W_{\psi, \varphi}$ is bounded on $BMOA$. Then
\[\|W_{\psi, \varphi}\|_{L(BMOA)} \asymp |\psi(0)|L(\varphi(0)) + \sup_{n\in\mathbb{D}} \|\psi\varphi^n\|_* + \sup_{a\in\mathbb{D}} \beta(a).\]

Theorem 1.1. Suppose that $W_{\psi, \varphi}$ is bounded on $BMOA$. Then
\[\|W_{\psi, \varphi}\|_e,BMOA \asymp \limsup_{n\to\infty} \|\psi\varphi^n\|_* + \limsup_{|\varphi(a)|\to1} \beta(a).\]

Very recently Pablo Galindo and the authors of this paper [4] provided an estimate for the essential norm of a composition operator on $BMOA$ in terms of the $BMOA$ norm of $\varphi^n$. This motivates the natural question of whether such an estimate could be extended for weighted composition operators. We shall give a positive answer to this question by extending the approach from [4] and combining it with earlier works [10] and [2] on weighted composition operators.

Recall that the essential norm $\|T\|_{e,X}$ of a bounded operator $T: X \to X$ is defined as the distance from $T$ to the space of compact operators on $X$. For two quantities $A$ and $B$ which may depend on $\varphi$ and $\psi$, we use the abbreviation $A \lesssim B$ whenever there is a positive constant $c$ (independent of $\varphi$ and $\psi$) such that $A \leq cB$. We write $A \simeq B$, if $A \lesssim B \lesssim A$. For basic results about composition operators on classical spaces of analytic functions, we refer the reader to the monographs [13] and [3].

We next formulate our main results. Theorem 1.1 provides a new estimate for the norm of a weighted composition operator on $BMOA$. Theorem 1.2 provides an estimate for the essential norm $\|W_{\psi, \varphi}\|_{e,BMOA}$.

Theorem 1.1. Suppose that $W_{\psi, \varphi}$ is bounded on $BMOA$. Then
\[\|W_{\psi, \varphi}\|_{L(BMOA)} \asymp |\psi(0)|L(\varphi(0)) + \sup_{n\in\mathbb{D}} \|\psi\varphi^n\|_* + \sup_{a\in\mathbb{D}} \beta(a).\]

Theorem 1.2. Suppose that $W_{\psi, \varphi}: BMOA \to BMOA$ is bounded. Then
\[\|W_{\psi, \varphi}\|_{e,BMOA} \asymp \limsup_{n\to\infty} \|\psi\varphi^n\|_* + \limsup_{|\varphi(a)|\to1} \beta(a).\]

We next formulate two corollaries to these results. The first one is a direct consequence of Theorem 1.2. Here the estimate for pointwise multipliers appears to be new; the estimate for composition operators was earlier given in [4].
Corollary 1.3. Suppose that $M_\psi$ is bounded on $BMOA$. Then
\[
\|M_\psi\|_{e, BMOA} \asymp \limsup_{n \to \infty} \|\psi z^n\|_* + \limsup_{|a| \to 1} L(a) \|T[\psi, a]\|_2
\]
and
\[
\|C_\varphi\|_{e, BMOA} \asymp \limsup_{n \to \infty} \|\varphi^n\|_*.
\]

The following result is a variant of formula (2), and it is proved in Section 4.

Corollary 1.4. If $W_{\psi, \varphi}$ is bounded on $VMOA$, then
\[
\|W_{\psi, \varphi}\|_{e, BMOA} \asymp \|W_{\psi, \varphi}\|_{e, VMOA} \asymp \limsup_{|a| \to 1} \alpha(a) + \limsup_{|a| \to 1} \beta(a).
\]

The rest of the paper is organized as follows. Section 2 contains preparatory material. In Sections 3 and 4 we establish the proofs of Theorems 1.1 and 1.2.

2. Preliminary results

In this section we collect several preliminary results needed for the proofs of our main theorems. We start with known properties of $BMOA$ functions. First, we need a pointwise estimate, which follows by integrating the well-known inequality $|f'(z)| \lesssim \|f\|(1 - |z|^2)^{-1}$, which holds for all $f \in BMOA$; see, for example, [6] or [19].

Lemma 2.1. For $f \in BMOA$ and $z \in \mathbb{D}$,
\[
|f(z)| \lesssim L(z)\|f\|,
\]
where $L(z) = \log \frac{2}{1 - |z|^2}$.

For $1 \leq p < \infty$, we define the $BMOA$ $p$-seminorm by $\|f\|_{*, p} = \sup_a \|T[f, a]\|_p$. The following reverse Hölder inequality is a consequence of the John-Nirenberg lemma; see [5], [6] or [19].

Lemma 2.2. For every $1 \leq p < \infty$,
\[
\|f\|_* \asymp \|f\|_{*, p}.
\]

The next lemma collects two inequalities which were proved in [10] Prop. 2.3 and (3.19). The first one of them is a version of the Littlewood inequality ([13], [3]). It implies, in particular, the well-known result that all composition operators are bounded on $BMOA$; see, for example, [4].

Lemma 2.3 ([10]). Let $g \in H^2$ be such that $g(0) = 0$. Then for $\varphi$ such that $\varphi(0) = 0$, we have
\[
\|g \circ \varphi\|_2 \lesssim \|\varphi\|_2 \|g\|_2
\]
and for $\frac{1}{2} \leq t < 1$ and $|z| \leq t$, we have
\[
|g(z)| \leq 2|z| \max_{|w| \leq t} |g(w)|.
\]

The next result contains four basic estimates about weighted composition operators which will be applied a number of times in the sequel. These estimates were essentially proved in [10] (Proof of Theorem 2.1 and Lemma 3.4), but were not collected there as independent results. We also formulate these estimates slightly differently from [10] and hence present short proofs.
Lemma 2.4 ([10]). (i) Let $f_a = \sigma_{\varphi(a)} - \varphi(a)$. For all $a \in \mathbb{D}$,
\[
\alpha(a) \leq \beta(a)/L(\varphi(a)) + ||W_{\psi,\varphi} f_a||_+. \tag{10}
\]
(ii) Let $g_a = h_a^2/h_a(\varphi(a))$, where $h(z) = \log(2/(1-\varphi(a)z))$. For all $a \in \mathbb{D}$,
\[
\beta(a) \leq ||T[\psi, a]g_a \circ \varphi, a||_2 + ||W_{\psi,\varphi} g_a||_+ + \alpha(a). \tag{11}
\]
(iii) For all $f \in BMOA$ and $a \in \mathbb{D}$,
\[
||T[W_{\psi,\varphi} f, a]||_2 \leq ||T[\psi, a]f \circ \varphi, a||_2 + (\alpha(a) + \beta(a)) ||f||_. \tag{12}
\]
(iv) For all $f \in BMOA$ and $a \in \mathbb{D}$,
\[
||T[\psi, a]f \circ \varphi, a||_2 \leq ||f||_+ \min_{w \in \mathbb{D}} \left\{ \sup_{w \in \mathbb{D}} \beta(w), \frac{||W_{\psi,\varphi}||_{L(BMOA)}}{\sqrt{L(\varphi(a))}} \right\}. \tag{13}
\]

Proof. (i) Since $||f_a \circ \varphi \circ \sigma_a||_{\infty} \leq 2$, we have
\[
\alpha(a) = ||\psi(a)||T[f_a \circ \varphi, a]||_2
= ||T[\psi, a] \cdot f_a \circ \varphi \circ \sigma_a - T[W_{\psi,\varphi} f_a, a]||_2
\leq 2 ||T[\psi, a]||_2 + ||W_{\psi,\varphi} f_a||_+
= 2\beta(a)/L(\varphi(a)) + ||W_{\psi,\varphi} f_a||_.
\]
(ii) By the triangle inequality and because $g_a(\varphi(a)) = L(\varphi(a))$, we get
\[
\beta(a) = ||g_a(\varphi(a))T[\psi, a]||_2
= ||T[g_a \circ \varphi, a]T[\psi, a] + \psi(a)T[g_a \circ \varphi, a] - T[W_{\psi,\varphi} g_a, a]||_2
\leq ||T[g_a \circ \varphi, a]T[\psi, a]||_2 + ||\psi(a)||T[g_a \circ \varphi, a]||_2 + ||W_{\psi,\varphi} g_a||_+.
\]
Note that $T[g_a \circ \varphi, a] = g_a \circ \sigma_{\varphi(a)} \circ (\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a) - g_a(\varphi(a))$, where $(g_a \circ \sigma_{\varphi(a)} - g_a(\varphi(a)))(0) = (\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a)(0) = 0$. Hence by Lemma 2.3, we have
\[
\psi(a)||T[g_a \circ \varphi, a]||_2 \leq \alpha(a)||g_a||_+.
\]
Because $\sup_{a \in \mathbb{D}} ||g_a||_+ < \infty$, this yields (ii).
(iii) Again, by the triangle inequality,
\[
||T[W_{\psi,\varphi} f, a]||_2 = ||T[\psi, a]T[f \circ \varphi, a] + \psi(a)T[f \circ \varphi, a] + T[\psi, a]f(\varphi(a))||_2
\leq ||T[\psi, a]T[f \circ \varphi, a]||_2 + ||\psi(a)||T[f \circ \varphi, a]||_2 + ||f(\varphi(a))||T[\psi, a]||_2,
\]
so, by Lemmas 2.3 and 2.1
\[
||\psi(a)||T[f \circ \varphi, a]||_2 + ||f(\varphi(a))||T[\psi, a]||_2 \leq \alpha(a)||f||_+ + L(\varphi(a))||T[\psi, a]||_2||f||_.
\]
(iv) By applying twice both Hölder’s inequality and Lemma 2.2, we get
\[
||T[\psi, a]T[f \circ \varphi, a]||_2^2 = ||T[\psi, a]^2T[f \circ \varphi, a]^2||_1
\leq ||T[\psi, a]^2||_2||T[\psi, a]^4||_4||T[f \circ \varphi, a]||_8^5
\leq ||T[\psi, a]||_2||\psi||_4||f \circ \varphi||_8^5
\leq \beta(a)||\psi||_+||f \circ \varphi||_8^5/L(\varphi(a)).
\]
By applying the estimate $\log 2 \leq L(\varphi(a))$ and the norm estimate (11), we have
\[
||\psi||_+ \leq \sup_{a \in \mathbb{D}} \beta(w) \leq ||W_{\psi,\varphi}||_{L(BMOA)}.
\]
Hence
\[
\beta(a)||\psi||_+||f \circ \varphi||_8^5/L(\varphi(a)) \leq (\sup_{w \in \mathbb{D}} \beta(w))^2 ||f||_8^5/L(\varphi(a)) \leq 2 \left\{ \sup_{w \in \mathbb{D}} \beta(w), \frac{||W_{\psi,\varphi}||_{L(BMOA)}}{\sqrt{L(\varphi(a))}} \right\}^2.
\]
\qed
Finally, we present a weighted version of [4, Lemma 2.3]. The idea of the proof is essentially contained also in [17] or [2, pp. 187-188].

**Lemma 2.5.** Suppose that $W_{\psi, \varphi}$ is bounded on $BMOA$ and let $f_a = \sigma_{\varphi(a)} - \varphi(a)$. Then

$$\sup_{a \in \mathbb{D}} \|W_{\psi, \varphi} f_a\|_* \leq 2 \sup_{n \geq 0} \|\psi \varphi^n\|_*$$

and

$$\limsup_{|\varphi(a)| \to 1} \|W_{\psi, \varphi} f_a\|_* \leq 2 \limsup_{n \to \infty} \|\psi \varphi^n\|_*.$$

**Proof.** We only prove the second inequality, because the first inequality follows with a similar argument. Because $W_{\psi, \varphi} f_a = (|\varphi(a)|^2 - 1) \sum_{n=0}^{\infty} \varphi(a)^n \psi \varphi^{n+1}$, we have

$$\|W_{\psi, \varphi} f_a\|_* \leq (1 - |\varphi(a)|^2) \sum_{n=0}^{\infty} |\varphi(a)|^n \|\psi \varphi^{n+1}\|_*,$$

and for each $N$, that

$$\|W_{\psi, \varphi} f_a\|_* \leq (1 - |\varphi(a)|^2) \left( \sum_{n=0}^{N} |\varphi(a)|^n \|\psi \varphi^{n+1}\|_* + \sum_{n=N+1}^{\infty} |\varphi(a)|^n \|\psi \varphi^{n+1}\|_* \right)$$

$$\leq (1 - |\varphi(a)|^2) \left( \sum_{n=0}^{N} |\varphi(a)|^n \|\psi \varphi^{n+1}\|_* + \sup_{n \geq N+1} \|\psi \varphi^{n+1}\|_* \sum_{n=0}^{\infty} \frac{|\varphi(a)|^n}{1 - |\varphi(a)|^2} \right).$$

Therefore

$$\limsup_{|\varphi(a)| \to 1} \|W_{\psi, \varphi} f_a\|_* \leq 2 \sup_{n \geq N+1} \|\psi \varphi^{n+1}\|_*,$$

and we complete the proof by letting $N \to \infty$. □

3. PROOF OF THEOREM 1.1

The proof of Theorem 1.1 follows immediately from the next two lemmas.

**Lemma 3.1.** If $W_{\psi, \varphi}$ is bounded on $BMOA$, then

$$\|W_{\psi, \varphi}\|_{L(BMOA)} \lesssim \sup_{a \in \mathbb{D}} \|\psi \varphi^n\|_* + \sup_{a \in \mathbb{D}} \beta(a) + |\psi(0)|L(\varphi(0)).$$

**Proof.** By Lemma 2.4 (iii) and (iv), we have

$$\|W_{\psi, \varphi} f\|_* \lesssim \sup_{a \in \mathbb{D}} (\alpha(a) + \beta(a)) \|f\|_*,$$

for all $f \in BMOA$. Hence, by Lemma 2.4 (i) and Lemma 2.5

$$\alpha(a) \lesssim \beta(a) + \sup_{n \geq 0} \|\psi \varphi^n\|_*,$$

because $L(\varphi(a))$ is bounded from below. Thus

$$\|W_{\psi, \varphi} f\|_* \lesssim \left( \sup_{n \geq 0} \|\psi \varphi^n\|_* + \sup_{a \in \mathbb{D}} \beta(a) \right) \|f\|_*.$$

We further have $|W_{\psi, \varphi} f(0)| = |\psi(0)| \|f(\varphi(0))\| \lesssim |\psi(0)|L(\varphi(0)) \|f\|$, by Lemma 2.4 which completes the proof. □

**Lemma 3.2.** Suppose that $W_{\psi, \varphi} : BMOA \to BMOA$ is bounded. Then

$$\|W_{\psi, \varphi}\|_{L(BMOA)} \gtrsim \sup_{n \geq 0} \|\psi \varphi^n\|_* + \sup_{a \in \mathbb{D}} \beta(a) + |\psi(0)|L(\varphi(0)).$$

**Proof.** Combine the norm estimate (1) and the fact that the functions $z \mapsto z^n$ belong uniformly to $BMOA$ for $n \geq 0$. □
4. Proof of Theorem 1.2

In this section we prove Theorem 1.2. We split the proof into three lemmas of which Theorem 1.2 is a direct consequence.

**Lemma 4.1.** Suppose that $W_{\psi, \varphi}$ is bounded on BMOA. Then
\[
\|W_{\psi, \varphi}\|_{e, \text{BMOA}} \gtrsim \limsup_{n \to \infty} \|\psi \varphi^n\|_* + \limsup_{|\varphi(a)| \to 1} |\beta(a)|.
\]

**Proof.** A standard argument shows that for any $f_n \in \text{VMOA}$ such that $\sup_n \|f_n\| < \infty$ and $f_n \to 0$ weakly in BMOA as $n \to \infty$, we have
\[
\|W_{\psi, \varphi}\|_{e, \text{BMOA}} \gtrsim \limsup_{n \to \infty} \|W_{\psi, \varphi}f_n\|,
\]
see, for example, [4]. We next apply this basic result to three different sequences of test functions. First, by setting $f_n(z) = z^n$, we get
\[
(3) \quad \|W_{\psi, \varphi}\|_{e, \text{BMOA}} \gtrsim \limsup_{n \to \infty} \|W_{\psi, \varphi}z^n\|_* = \limsup_{n \to \infty} \|\psi \varphi^n\|_*.
\]
If now $\|\varphi\|_\infty < 1$, then $\limsup_{|\varphi(a)| \to 1} \beta(a) = 0$ and the proof is complete. Otherwise, there are points $a_n \in \mathbb{D}$ such that $|\varphi(a_n)| \to 1$ and
\[
\lim_{n} \alpha(a_n) = \limsup_{|\varphi(a)| \to 1} \alpha(a).
\]
Define $f_n(z) = \sigma_{\varphi(a_n)}(z) - \varphi(a_n)$. Then, by Lemma 2.4 (iii),
\[
(4) \quad \|W_{\psi, \varphi}\|_{e, \text{BMOA}} \gtrsim \limsup_{n \to \infty} \|W_{\psi, \varphi}f_n\|_* \gtrsim \limsup_{n \to \infty} \left( \alpha(a_n) - \|W_{\psi, \varphi}\|_{L(\text{BMOA})/L(\varphi(a_n))} \right) = \limsup_{|\varphi(a)| \to 1} \alpha(a).
\]
Finally, choose $a_n \in \mathbb{D}$ such that $|\varphi(a_n)| \to 1$ and
\[
\lim_{n \to \infty} \beta(a_n) = \limsup_{|\varphi(a)| \to 1} \beta(a).
\]
Let $g_{a_n}$ be as in Lemma 2.4 (ii). It was shown in [10, p. 35] that the functions $g_{a_n}$ are uniformly bounded in BMOA, $g_{a_n} \in \text{VMOA}$ and $g_{a_n}$ converges weakly to zero in BMOA. By these facts and the estimates (4), we get
\[
\|W_{\psi, \varphi}\|_{e, \text{BMOA}} \gtrsim \limsup_{n \to \infty} \left( \|W_{\psi, \varphi}g_{a_n}\|_* + \alpha(a_n) \right).
\]
By Lemma 2.4 (ii) and (iv),
\[
(5) \quad \|W_{\psi, \varphi}\|_{e, \text{BMOA}} \gtrsim \limsup_{n \to \infty} \left( \beta(a_n) - \|g_{a_n}\|_* \right) \|W_{\psi, \varphi}\|_{L(\text{BMOA})/\sqrt{L(\varphi(a_n))}} = \limsup_{n \to \infty} \beta(a_n).
\]
Combining above estimates (3) and (5) completes the proof. 

In next lemmas we will next use the well-known fact that every $f \in H^p$ has the almost everywhere on $\partial \mathbb{D}$ existing radial limit function (also denoted by $f$) and that the $H^p$ norm of $f$ coincides with the $L^p(d\theta/2\pi)$ norm of $f$ on $\partial \mathbb{D}$. For $t \in (0, 1)$ we define the sets
\[
E(\varphi, a, t) = \{ \xi \in \partial \mathbb{D} : |(\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a)(\xi)| > t \} \subset \partial \mathbb{D}.
\]

**Lemma 4.2.** We have that
\[
\lim_{t \to 1^-} \limsup_{t \to 1} \sup_{|\varphi(a)| \leq r} \left( \int_{E(\varphi, a, t)} |\psi(\varphi(e^{i\theta}))|^t \frac{d\theta}{2\pi} \right)^{1/4} \lesssim \limsup_{n \to \infty} \|\psi \varphi^n\|_*.
\]
**Proof.** The triangle inequality yields that
\[
\| (\psi \circ \sigma_n) (\varphi^n \circ \sigma - \varphi^n (a)) \|_4 \\
\leq \| (\psi \circ \sigma_n) (\varphi^n \circ \sigma - \psi (a)) \|_4 + \| (\psi \circ \sigma_n - \psi (a)) \|_4.
\]
Now fix \( r \in (0, 1) \). Then for every \( |\varphi (a)| \leq r \), we obtain that
\[
\| (\psi \circ \sigma_n) (\varphi^n \circ \sigma - \varphi^n (a)) \|_4 \leq \| \psi \varphi^n \|_{\ast, 4} + r^n \| \psi \|_{\ast, 4}.
\]
For \( t \in (r, 1) \) we get
\[
\limsup_{t \to 1} \sup_{|\varphi(a)| \leq r} \frac{\int_{E(\varphi, a, t)} |\psi(\sigma_n(e^{i\theta}))| \frac{d\theta}{2\pi}}{t^n} = \limsup_{t \to 1} \sup_{|\varphi(a)| \leq r} \frac{\int_{E(\varphi, a, t)} |\psi(\sigma_n(e^{i\theta}))| \frac{d\theta}{2\pi}}{t^n} \\
\leq \limsup_{t \to 1} \sup_{|\varphi(a)| \leq r} \frac{\int_{E(\varphi, a, t)} |(\psi \circ \sigma_n)(e^{i\theta})| (|\varphi^n \circ \sigma_n| - |\varphi^n (a)|) \frac{d\theta}{2\pi}}{t^n} \\
\leq \limsup_{t \to 1} \sup_{|\varphi(a)| \leq r} \frac{\int_{E(\varphi, a, t)} |(\psi \circ \sigma_n)(\varphi^n \circ \sigma - \varphi^n (a))| \frac{d\theta}{2\pi}}{t^n},
\]
where \( \hat{E}(\varphi, a, t) = \{ \xi \in \partial \mathbb{D} : |\varphi \circ \sigma_n (\xi)| > t \} \). Therefore
\[
\limsup_{t \to 1} \sup_{|\varphi(a)| \leq r} \frac{\int_{E(\varphi, a, t)} |\psi(\sigma_n(e^{i\theta}))| \frac{d\theta}{2\pi}}{t^n} \leq \limsup_{t \to 1} \sup_{|\varphi(a)| \leq r} \frac{\int_{E(\varphi, a, t)} |\psi(\sigma_n(e^{i\theta}))| \frac{d\theta}{2\pi}}{t^n}.
\]
By [10] Remark 3.3, we have
\[
s(r)^{-1} (1 - |\varphi \circ \sigma_n (\xi)|) \leq 1 - |\sigma_n(\varphi) \circ \varphi \circ \sigma_n (\xi)| \leq s(r) (1 - |\varphi \circ \sigma_n (\xi)|),
\]
for \( |\varphi (a)| \leq r \) and \( \xi \in \partial \mathbb{D} \), where \( s(r) = 2(1 + r)/(1 - r) \). In particular,
\[
\hat{E}(\varphi, a, 1 - s(r)^{-1} (1 - t)) \subset E(\varphi, a, 1 - t) \subset \hat{E}(\varphi, a, 1 - s(r)(1 - t)),
\]
for \( 1 - s(r)^{-1} < t < 1 \). In view of these estimates, it follows that
\[
\limsup_{t \to 1} \sup_{|\varphi(a)| \leq r} \frac{\int_{E(\varphi, a, t)} |\psi(\sigma_n(e^{i\theta}))| \frac{d\theta}{2\pi}}{t^n} = \limsup_{t \to 1} \sup_{|\varphi(a)| \leq r} \frac{\int_{E(\varphi, a, t)} |\psi(\sigma_n(e^{i\theta}))| \frac{d\theta}{2\pi}}{t^n} \\
\leq \limsup_{\varphi(a)| \to 1} \frac{\int_{E(\varphi, a, t)} |\psi(\sigma_n(e^{i\theta}))| \frac{d\theta}{2\pi}}{t^n},
\]
We use the equivalence of the seminorms \( \| \cdot \|_\ast \) and \( \| \cdot \|_{\ast, 4} \) (Lemma 2.2) to complete the proof. \( \square \)

For \( 0 < r < 1 \) and \( 0 < t < 1 \), define
\[
Q(r, t) = r \overline{\mathbb{D}} \cup \{ \sigma_b(z) \in \mathbb{D} : b \in r \overline{\mathbb{D}}, z \in t \overline{\mathbb{D}} \}.
\]
Thus \( Q(r, t) \) is a compact subset of \( \mathbb{D} \).

**Lemma 4.3.** Suppose that \( W_{\psi, \varphi} \) is bounded on \( BMOA \). Then
\[
\| W_{\psi, \varphi} \|_{\ast, BMOA} \leq \limsup_{n \to \infty} \| \psi \varphi^n \|_{\ast} + \limsup_{|\varphi(a)| \to 1} \| \beta (a) \|_.
\]

**Proof.** For \( n \geq 0 \), let \( r_n = n/(1 + n) \) and define the linear operator \( K_n \) on \( BMOA \) by \( (K_n f)(z) = f(r_n z) \). Then standard arguments show that \( \sup_n \| K_n \| \leq 2 \) and every \( K_n \) is a compact operator. Moreover, \( \sup_{z \in K} \sup_{\| f \| \leq 1} \| (S_n f)(z) \| \) converges
to zero for compact subsets $K \subset \mathbb{D}$, where $S_nf = f - K_nf$; see, for example, [4]. We now have

$$
\|W_{\psi,\varphi}\|_{\text{BMOA}} \leq \liminf_{n \to \infty} \|W_{\psi,\varphi}S_n\|
= \liminf_{n \to \infty} \sup_{\|f\| \leq 1} |\psi(0)(S_nf)(\varphi(0))|
+ \liminf_{n \to \infty} \sup_{\|f\| \leq 1} \|W_{\psi,\varphi}S_nf\|_*
= \liminf_{n \to \infty} \sup_{\|f\| \leq 1} \|W_{\psi,\varphi}S_nf\|_*.
$$

(6)

Fix $n \geq 0$, $f \in \text{BMOA}$ with $\|f\| \leq 1$, $r \in (0,1)$ and $t \in (\frac{1}{2},1)$. Then

$$
\|W_{\psi,\varphi}S_nf\|_* \leq \sup_{|\varphi(a)| \leq r} \|T[W_{\psi,\varphi}S_nf,a]\|_2 + \sup_{|\varphi(a)| > r} \|T[W_{\psi,\varphi}S_nf,a]\|_2.
$$

By Lemma 2.1 (iii) and (iv), we get

$$
\sup_{|\varphi(a)| > r} \|T[W_{\psi,\varphi}S_nf,a]\|_2
\lesssim \|S_nf\| \cdot \sup_{|\varphi(a)| > r} \left( \alpha(a) + \beta(a) + \|W_{\psi,\varphi}\|_{L(\text{BMOA})}/\sqrt{L(\varphi(a))} \right).
$$

To estimate the other term, we use the triangle inequality and get

$$
\sup_{|\varphi(a)| \leq r} \|T[W_{\psi,\varphi}S_nf,a]\|_2 \leq \sup_{|\varphi(a)| \leq r} \|(S_nf)(\varphi(a))\| \cdot \|T[\psi,a]\|_2
+ \sup_{|\varphi(a)| \leq r} \|\psi \circ \sigma_a \cdot T[S_nf \circ \varphi,a]\|_2
\leq \|\psi\|_* \max_{w \in Q(r,t)} |(S_nf)(w)| + I_1^{1/2} + I_2^{1/2},
$$

where

$$
I_1 = \sup_{|\varphi(a)| \leq r} \int_{\partial \mathbb{D} \setminus E(\varphi,a,t)} |(\psi \circ \sigma_a)(\theta)| |T[S_nf \circ \varphi,a]\|_{\frac{2}{e}} \frac{d\theta}{2\pi};
$$

$$
I_2 = \sup_{|\varphi(a)| \leq r} \int_{E(\varphi,a,t)} |(\psi \circ \sigma_a)(\theta)| |T[S_nf \circ \varphi,a]\|_{\frac{2}{e}} \frac{d\theta}{2\pi}.
$$

Abbreviate $\varphi_a = \sigma_{\varphi(a)} \circ \varphi \circ \sigma_a$ and note that $T[S_nf \circ \varphi,a] = (S_nf) \circ \sigma_{\varphi(a)} \circ \varphi_a - (S_nf \circ \varphi)(a)$, so that

$$
|T[S_nf \circ \varphi,a]\| \leq 2|\varphi_a(\xi)| \sup_{|w| \leq t} |(S_nf) \circ \sigma_{\varphi(a)}(w) - (S_nf)(\varphi(a))|,
$$

for $\xi \in \partial \mathbb{D} \setminus E(\varphi,a,t)$ by Lemma 2.3. Hence

$$
I_1 \leq 4 \sup_{|\varphi(a)| \leq r} \sup_{|w| \leq t} \|((S_nf) \circ \sigma_{\varphi(a)})(w) - (S_nf)(\varphi(a))\|^2 \|\psi \circ \sigma_a \cdot \varphi_a\|^2_2
\leq 16 \sup_{z \in Q(r,t)} \|(S_nf)(z)\|^2 \|\psi \circ \sigma_a \cdot \varphi_a\|^2_2,
$$

where

$$
\|\psi \circ \sigma_a \cdot \varphi_a\|_2 \leq \|T[\psi,a]\|_2 |\varphi_a|_\infty + |\varphi(a)||\varphi_a|_2
\lesssim \|\psi\|_* + \alpha(a) \lesssim \|W_{\psi,\varphi}\|_{L(\text{BMOA})}.
$$

For $I_2$ we use Hölder’s inequality;

$$
I_2 \leq \sup_{|\varphi(a)| \leq r} \left( \int_{E(\varphi,a,t)} |\psi(\sigma_a(\theta))|^4 \frac{d\theta}{2\pi} \right)^{1/2} \sup_{|\varphi(a)| \leq r} \|T[S_nf \circ \varphi,a]\|_4^2,
$$

where

$$
\|T[S_nf \circ \varphi,a]\|_4^2 \leq \|S_nf \circ \varphi\|^2_4 \lesssim \|f\|^2 \leq 1,
$$

and

$$
\|\psi(\sigma_a(\theta))|^4 \frac{d\theta}{2\pi} \leq \|T[S_nf \circ \varphi,a]\|_4^2.
$$

and

$$
\|T[S_nf \circ \varphi,a]\|_4^2 \leq \|S_nf \circ \varphi\|^2_4 \lesssim \|f\|^2 \leq 1.
$$
by Lemma 2.2. By combining the above estimates with (7), we have for \( r \in (0, 1) \) and \( t \in \left( \frac{1}{2}, 1 \right) \) that
\[
\|W_{\psi, \varphi}S_nf\|_r \lesssim \sup_{|\varphi(a)| > r} \left( \alpha(a) + \beta(a) + \|W_{\psi, \varphi}\|_{L(BMOA)/\sqrt{L(\varphi(a))}} \right)
\]
\[
+ \sup_{|\varphi(a)| \leq r} \left( \int_{E(\varphi, a, t)} |\psi(\sigma_a(e^{i\theta}))|^{4} \frac{d\theta}{2\pi} \right)^{1/4}
\]
\[
+ \sup_{z \in Q(r, t)} |(S_nf)(z)|^2 \|W_{\psi, \varphi}\|_{L(BMOA)}.
\]
By taking the supremum over \( \|f\| \leq 1 \), letting \( n \to \infty \) and applying (6), we get
\[
\|W_{\psi, \varphi}\|_{e,BMOA} \lesssim \sup_{|\varphi(a)| > r} \left( \alpha(a) + \beta(a) + \|W_{\psi, \varphi}\|_{L(BMOA)/\sqrt{L(\varphi(a))}} \right)
\]
\[
+ \sup_{|\varphi(a)| \leq r} \left( \int_{E(\varphi, a, t)} |\psi(\sigma_a(e^{i\theta}))|^{4} \frac{d\theta}{2\pi} \right)^{1/4}.
\]
From Lemma 2.4 (i) and Lemma 2.3 we conclude that
\[
\limsup_{|\varphi(a)| \to 1} \alpha(a) \lesssim \limsup_{n \to \infty} \|\psi \varphi^n\|_*.
\]
Therefore, by combining the above estimates with Lemma 4.2 we see that
\[
\|W_{\psi, \varphi}\|_{e,BMOA} \lesssim \limsup_{|\varphi(a)| \to 1} \beta(a) + \limsup_{n \to \infty} \|\psi \varphi^n\|_*,
\]
and the proof is complete.

We finally prove Corollary 4.4.

Proof of Corollary 4.4. First we note that \( VMOA^{**} = BMOA \) and \( W_{\psi, \varphi} \) on \( BMOA \) is the second adjoint of \( W_{\psi, \varphi} \) on \( VMOA \). Moreover, \( (VMOA)^* = H^1 \) has the metric approximation property, so Theorem 3 in [1] gives us that the essential norms \( \|W_{\psi, \varphi}\|_{e,BMOA} \) and \( \|W_{\psi, \varphi}\|_{e,VMOA} \) are equivalent. Since \( \psi \) and \( \varphi \varphi \) belong to \( VMOA \), we conclude (see Proposition 4.1 in [10]) that
\[
\limsup_{|a| \to 1} \|\psi \circ \sigma_a - \psi(a)\|_2 = 0 \quad \text{and} \quad \limsup_{|a| \to 1} \|\varphi \circ \sigma_a - \varphi(a)\|_2 = 0.
\]

Let \( r \in (0, 1) \) be fixed. Since
\[
\|\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a\|_2 = \left\| \varphi(a) - \frac{\varphi \circ \sigma_a}{1 - \varphi(a) \varphi \circ \sigma_a} \right\|_2 \leq \frac{\|\varphi \circ \sigma_a - \varphi(a)\|_2}{1 - |\varphi(a)|},
\]
we get
\[
\lim_{s \to 1} \sup_{|\varphi(a)| \leq r, |a| > s} \|\psi(a)\| \|\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a\|_2
\]
\[
\leq (1 - r)^{-1} \limsup_{|a| \to 1} \|\varphi \circ \sigma_a - \varphi(a)\|_2 = 0
\]
and
\[
\lim_{s \to 1} \sup_{|\varphi(a)| \leq r, |a| > s} \left( \log \frac{2}{1 - |\varphi(a)|^2} \right) \|\psi \circ \sigma_a - \psi(a)\|_2
\]
\[
\leq \left( \log \frac{2}{1 - r^2} \right) \limsup_{|a| \to 1} \|\psi \circ \sigma_a - \psi(a)\|_2 = 0.
\]
Hence,
\[
\limsup_{|a| \to 1} \alpha(a) \leq \lim_{s \to 1} \left( \sup_{|\varphi(a)| \leq r, |a| > s} \alpha(a) + \sup_{|\varphi(a)| > r, |a| > s} \alpha(a) \right) \leq \sup \alpha(a)
\]
and, similarly,
\[
\limsup_{|a| \to 1} \beta(a) \leq \sup_{|\varphi(a)| > r} \beta(a).
\]
Therefore we can use the essential norm estimate (2) for $VMOA$ to conclude that
\[
\|W_{\psi, \varphi}\|_{e, BMOA} \lesssim \limsup_{|\varphi(a)| \to 1} \alpha(a) + \limsup_{|\varphi(a)| \to 1} \beta(a).
\]
On the other hand,
\[
\limsup_{n \to \infty} \|\psi \varphi^n\|_{*} \lesssim \limsup_{|\varphi(a)| \to 1} \alpha(a)
\]
so Theorem 1.2 finishes the proof.

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