On Type 2 Degenerate Poly-Frobenius-Genocchi Polynomials and Numbers

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Abstract

In this paper, we consider a class of new generating function for the Frobenius-Genocchi polynomials, called the type 2 degenerate poly-Frobenius-Genocchi polynomials, by means of the polyexponential function. Then, we investigate diverse explicit expressions and some identities for those polynomials.

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1. Introduction

Special polynomials have their origin in the solution of the differential equations (or partial differential equations) under some conditions. Special polynomials can be defined in a various ways such as by generating functions, by recurrence relations, by $p$-adic integrals in the sense of fermionic and bosonic, by degenerate versions, etc.

Kim-Kim have introduced polyexponential function in \[18\] and its degenerate version in \[20],[21]. By making use of aforementioned function, they have introduced a new class of some special polynomials. This idea provides a powerful tool in order to define special numbers and polynomials by making use of polyexponential function. One may see that the notion of polyexponential function form a special class of polynomials because of their great applicability, \textit{cf.} \[12, 18-22, 26, 27, 29, 31\]. The importance of these polynomials would be to find applications in analytic number theory, applications in classical analysis and statistics, \textit{cf.} \[1-34\].

Throughout of the paper we make use of the following notations: $\mathbb{N} := \{1, 2, 3, \cdots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Here, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The classical Bernoulli $B_n(x)$, Euler $E_n(x)$ and Genocchi $G_n(x)$ polynomials and the degenerate Bernoulli $B_{n,\lambda}(x)$, Euler $E_{n,\lambda}(x)$ and Genocchi $G_{n,\lambda}(x)$ polynomials are given as follows (\textit{cf.} \[5, 8, 10, 11, 14, 16, 18-20, 22, 23, 26-32\)):

\begin{align}
\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} &= \frac{t}{e^t - 1} \quad \text{and} \quad \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!} = \frac{t}{e_{\lambda}(t) - 1} e_{\lambda}(t) \\
\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} &= \frac{2}{e^t + 1} \quad \text{and} \quad \sum_{n=0}^{\infty} E_{n,\lambda}(x) \frac{t^n}{n!} = \frac{2}{e_{\lambda}(t) + 1} e_{\lambda}(t)
\end{align}

(1.1)
\[ \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = \frac{2t}{e^t + 1} \quad \text{and} \quad \sum_{n=0}^{\infty} G_{n,\lambda}(x) \frac{t^n}{n!} = \frac{2t}{e_{\lambda}(t) + 1} e_{\lambda}(t). \] (1.3)

One may look at the references [1, 4-13, 15, 17-19, 21, 22, 25-31] to see the various applications of Bernoulli, Euler and Genocchi polynomials.

Frobenius studied the polynomials \( F_n(x | u) \) given by (cf. [2, 3])

\[ \frac{1-u}{e^t-u} e^{xt} = \sum_{n=0}^{\infty} F_n(x | u) \frac{t^n}{n!} \quad (u \in \mathbb{C}\setminus\{1\}). \] (1.4)

Upon setting \( u = -1 \), it becomes

\[ F_n(x | -1) = E_n(x). \]

Owing to relationship with the Euler polynomials as well as their important properties, and in the honor of Frobenius, the aforementioned polynomials denoted by \( F_n(x | u) \) are called the Frobenius-Euler polynomials, cf. [2, 3].

Parallel to (1.4), Ya¸sar and Özarslan [34] introduced the Frobenius-Genocchi polynomials \( G^F_n(x; u) \) given by

\[ \frac{(1-u)t}{e^t-u} e^{xt} = \sum_{n=0}^{\infty} G^F_n(x; u) \frac{t^n}{n!}, \] (1.5)

since

\[ G^F_n(x; -1) = G_n(x). \]

The case \( x = 0 \) in (1.5), \( G^F_0(0; u) := G^F_n(u) \) stands for the Frobenius-Genocchi numbers. Several recurrence relations and differential equations are also investigated in [34].

Khan and Srivastava [17] introduced a new class of the generalized Apostol type Frobenius-Genocchi polynomials and investigated some properties and relations including implicit summation formulae and various symmetric identities. Moreover a relation in between Array-type polynomials, Apostol-Bernoulli polynomials and investigated some properties and relations including implicit summation formulae and operational rule for these polynomials.

The Bernoulli polynomials of the second kind are defined by means of the following generating function

\[ \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!} = \frac{t}{\log (1+t)} (1+t)^x. \] (1.6)

When \( x = 0 \), \( b_n(0) := b_n \) are called the Bernoulli numbers of the second kind, cf. [20].

It is well-known from (1.6) that

\[ \left( \frac{t}{\log (1+t)} \right)^r (1+t)^{x-1} = \sum_{n=0}^{\infty} B^{(n-r+1)}_n(x) \frac{t^n}{n!}, \] (1.7)

where \( B^{(r)}_n(x) \) are the Bernoulli polynomials of order \( r \), see [20].

For \( \lambda \in \mathbb{C} \), the \( \lambda \)-falling factorial \( (x)_{n,\lambda} \) is defined by (see [10, 11, 20-22, 24-27, 29-31])

\[ (x)_{n,\lambda} = \begin{cases} x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), & n = 1, 2, \ldots \\ 1 & n = 0. \end{cases} \] (1.8)

In the case \( \lambda = 1 \), the \( \lambda \)-falling factorial reduces to the familiar falling factorial as follows

\[ (x)_{n,1} := (x)_n = x(x-1)\cdots(x-n+1) \text{ and } (x)_0 = 1. \]

The \( \Delta_\lambda \) difference operator is defined by (see [10, 11])

\[ \Delta_\lambda f(x) = \frac{1}{\lambda} (f(x+\lambda) - f(x)), \quad \lambda \neq 0. \] (1.9)
The degenerate exponential function $e^x_\lambda (t)$ is defined as follows

$$e^x_\lambda (t) = \left( 1 + \lambda t \right)^{\frac{x}{\lambda}} \text{ and } e^1_\lambda (t) = e_\lambda (t).$$

(1.10)

It is readily seen that $\lim_{\lambda \to 0} e^x_\lambda (t) = e^{xt}$ (cf. [10, 11, 20-22, 24-27, 29-31]). From (1.8) and (1.10), we obtain the following relation

$$e^x_\lambda (t) = \sum_{n=0}^{\infty} \left( \frac{x}{\lambda} \right)^n \frac{t^n}{n!},$$

(1.11)

which satisfies the following difference rule

$$\Delta_\lambda e^x_\lambda (t) = te^x_\lambda (t).$$

(1.12)

The Stirling numbers of the first kind $S_1 (n, k)$ and the Stirling numbers of the second kind $S_2 (n, k)$ are defined (cf. [2, 4, 5, 12]) by means of the following generating functions:

$$\left( \log (1 + t) \right)^k \frac{k!}{k!} = \sum_{n=0}^{\infty} S_1 (n, k) \frac{t^n}{n!} \text{ and } \left( e^t - 1 \right)^k \frac{k!}{k!} = \sum_{n=0}^{\infty} S_2 (n, k) \frac{t^n}{n!}.$$ \hspace{1cm} (1.13)

From (1.13), we get the following relations for $n \geq 0$:

$$(x)_n = \sum_{k=0}^{n} S_1 (n, k) x^k \text{ and } x^n = \sum_{k=0}^{n} S_1 (n, k) (x)_k.$$ \hspace{1cm} (1.14)

Very recently, Kim-Kim [22] performed to generalize the degenerate Bernoulli polynomials by using poly-exponential function

$$Ei_k (t) = \sum_{n=1}^{\infty} \frac{t^n}{(n-1)!} n^k,$$

(1.15)

as inverse to the polylogarithm function

$$Li_k (t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k}, \quad (|t| < 1; k \in \mathbb{Z})$$

(1.16)

given by

$$Ei_k \left( \log (1 + t) \right) - e^x_\lambda (t) = \sum_{n=0}^{\infty} \beta^{(k)}_{n, \lambda} (x) \frac{t^n}{n!}.$$ \hspace{1cm} (1.17)

Upon setting $x = 0$ in (1.17), $\beta^{(k)}_{n, \lambda} (0) := \beta^{(k)}_{n, \lambda}$ are called the degenerate poly-Bernoulli numbers. Kim et al. [22] studied the degenerate poly-Bernoulli polynomials and also gave some explicit expressions and several formulas for those polynomials.

For $k \in \mathbb{Z}$, the type 2 degenerate poly-Euler polynomials $E^{(k)}_{n, \lambda} (x)$ are defined, cf. [29], as follows:

$$Ei_k \left( \log (1 + 2t) \right) - e^x_\lambda (t) = \sum_{n=0}^{\infty} \mathcal{E}^{(k)}_{n, \lambda} (x) \frac{t^n}{n!}.$$ \hspace{1cm} (1.18)

When $x = 0$, $\mathcal{E}^{(k)}_{n, \lambda} (0) := \mathcal{E}^{(k)}_{n, \lambda}$ are called the type 2 degenerate poly-Euler numbers. Lee et al. [29] studied the type 2 degenerate poly-Euler polynomials and provided multifarious explicit formulas and identities.

Since $Ei_1 (t) = e^t - 1$, it is worthy to note that

$$\beta^{(1)}_{n, \lambda} (x) := B_{n, \lambda} (x) \text{ and } \mathcal{E}^{(1)}_{n, \lambda} (x) := E_{n, \lambda} (x).$$
2. The type 2 Degenerate Poly-Frobenius-Genocchi Polynomials

Now, we consider the following Definition 1 by means of the polyexponential function.

**Definition 1.** Let $k \in \mathbb{Z}$. The type 2 degenerate poly-Frobenius-Genocchi polynomials are defined via the following exponential generating function (in a suitable neighbourhood of $t = 0$) including the polyexponential function as given below:

$$E_{\lambda} \left( \log \left( 1 + (1 - u) t \right) \right) \frac{e^{\lambda t}}{e_{\lambda}(t) - u} = \sum_{n=0}^{\infty} G_{n,\lambda}^{(F,k)}(x; u) \frac{t^n}{n!}. \tag{2.1}$$

At the value $x = 0$ in (2.1), $G_{n,\lambda}^{(F,k)}(0; u) := G_{n,\lambda}^{(F,k)}(u)$ will be called type 2 degenerate poly-Frobenius-Genocchi numbers.

**Remark 1.** Taking $k = 1$ in (2.1) yields $G_{n,\lambda}^{(F,1)}(x; u) := G_{n,\lambda}^{F}(x; u)$ are the degenerate Frobenius-Genocchi polynomials $G_{n,\lambda}^{F}(x; u)$ (cf. [15]) as follows

$$\frac{(1 - u) t}{e_{\lambda}(t) - u} e^{\lambda t}(t) = \sum_{n=0}^{\infty} G_{n,\lambda}^{F}(x; u) \frac{t^n}{n!}.$$  

**Remark 2.** Upon setting $\lambda \to 0$ in (2.1) gives $\lim_{\lambda \to 0} G_{n,\lambda}^{(F,k)}(x; u) := G_{n}^{(F,k)}(x; u)$ are type 2 poly-Frobenius-Genocchi polynomials $G_{n}^{(F,k)}(x; u)$ (cf. [12]) as follows

$$E_{\lambda} \left( \log \left( 1 + (1 - u) t \right) \right) e^{\lambda t} \left( e^{\lambda t} \right)^{-1} = \sum_{n=0}^{\infty} G_{n}^{(F,k)}(x; u) \frac{t^n}{n!}.$$  

**Remark 3.** Taking $k = 1$ and $\lambda \to 0$ in (2.1) yields $G_{n,\lambda}^{(F,1)}(x; -1) := G_{n,\lambda}(x)$ are the Frobenius-Genocchi polynomials in (1.5).

A difference operator rule of type 2 degenerate poly-Frobenius-Genocchi polynomials is given as follows

$$\Delta_{\lambda} G_{n,\lambda}^{(F,k)}(x; u) = G_{n-1,\lambda}^{(F,k)}(x; u).$$

Now, we give the following theorem.

**Theorem 1.** The following relation

$$G_{n,\lambda}^{(F,k)}(x; u) = \sum_{l=0}^{n} \binom{n}{l} G_{n-l,\lambda}^{(F,k)}(u)(x)_{l,\lambda}$$  

is valid for $k \in \mathbb{Z}$ and $n \geq 0$.

**Proof.** By Definition 1, we consider that

$$\sum_{n=0}^{\infty} G_{n,\lambda}^{(F,k)}(x; u) \frac{t^n}{n!} = \frac{E_{\lambda} \left( \log \left( 1 + (1 - u) t \right) \right) e^{\lambda t}}{e_{\lambda}(t) - u} e^{\lambda t}(t)$$

$$= \left( \sum_{n=0}^{\infty} G_{n,\lambda}^{(F,k)}(u) \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!} \right)$$

$$= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} G_{n-l,\lambda}^{(F,k)}(u)(x)_{l,\lambda} \right) \frac{t^n}{n!},$$

which implies the asserted result in (2.2).
Theorem 2. The following relation
\[
\frac{d}{dx} G_{n,\lambda}^{(F,k)} (x; u) = n! \sum_{u=1}^{\infty} G_{n-u,\lambda}^{(F,k)} (x; u) \frac{(-1)^{u+1}}{(n-u)!} \lambda^{u-1} \tag{2.3}
\]
is valid for \( k \in \mathbb{Z} \) and \( n \geq 0 \).

Proof. By Definition 1, we consider that
\[
\sum_{n=0}^{\infty} \frac{d}{dx} G_{n,\lambda}^{(F,k)} (x; u) \frac{t^n}{n!} = \frac{\text{Ei}_k (\log (1 + (1 - u) t))}{e_\lambda (t) - u} \frac{d}{dx} e_\lambda^k (t)
\]
\[
= \sum_{n=0}^{\infty} G_{n,\lambda}^{(F,k)} (x; u) \frac{t^n}{n!} \ln (1 + \lambda t)
\]
\[
= \left( \sum_{n=0}^{\infty} G_{n,\lambda}^{(F,k)} (x; u) \frac{t^n}{n!} \right) \sum_{u=1}^{\infty} \frac{(-1)^{u+1}}{u} \lambda^{u-1} t^u
\]
\[
= \sum_{n=0}^{\infty} \sum_{u=1}^{\infty} G_{n,\lambda}^{(F,k)} (x; u) \frac{(-1)^{u+1}}{u} \lambda^{u-1} \frac{t^{u+n}}{n!},
\]
which implies the asserted result in (2.2). \( \square \)

A relation between the type 2 degenerate poly-Frobenius-Genocchi polynomials and the degenerate Frobenius-Genocchi polynomials is stated in the following theorem.

Theorem 3. For \( k \in \mathbb{Z} \) and \( n \geq 0 \), we have
\[
G_{n,\lambda}^{(F,k)} (x; u) = \sum_{m=0}^{n} \sum_{l=1}^{m+1} \binom{n}{m} \frac{S_1 (m+1, l)}{l^{k-1} (m+1)} G_{n-m,\lambda}^{F} (x; u). \tag{2.4}
\]

Proof. From (1.15), we observe that
\[
\text{Ei}_k (\log (1 + (1 - u) t)) = \sum_{l=1}^{\infty} \frac{(\log (1 + (1 - u) t))^l}{(l-1)! l^k}
\]
\[
= \sum_{l=1}^{\infty} \frac{1}{l^{k-1}} \sum_{m=l}^{\infty} S_1 (m, l) (1 - u)^m \frac{t^m}{m!}
\]
\[
= \sum_{m=0}^{\infty} \sum_{l=1}^{m+1} \frac{S_1 (m+1, l)}{l^{k-1} (m+1)} (1 - u)^m \frac{t^{m+1}}{m!}. \tag{2.5}
\]

Then, by (2.1), we get
\[
\frac{t (1 - u)}{e_\lambda (t) - u} \frac{1}{t (1 - u)} E_k (\log (1 + (1 - u) t)) = \sum_{n=0}^{\infty} G_{n,\lambda}^{(F,k)} (x; u) \frac{t^n}{n!}
\]
\[
\times \sum_{m=0}^{\infty} \sum_{l=1}^{m+1} \frac{S_1 (m+1, l)}{l^{k-1} (m+1)} (1 - u)^m \frac{t^m}{m!}
\]
\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=1}^{m+1} \binom{n}{m} \frac{S_1 (m+1, l)}{l^{k-1} (m+1)} G_{n-m,\lambda}^{F} (x; u) \frac{t^n}{n!},
\]
which means the asserted result in (2.4). \( \square \)

The immediate results of the Theorem 3 are stated below.
Corollary 1. For $k \in \mathbb{Z}$ and $n \geq 0$, we have
\[
G_{n,\lambda}^{(F,k)}(u) = \sum_{m=0}^{n} \sum_{l=1}^{m+1} \binom{n}{m} \frac{S_1(m+1,l) (1-u)^m}{l^k (m+1)} G_{n-m,\lambda}^{F}(u).
\] (2.6)

Corollary 2. Taking $k = 1$ in Theorem 3 gives
\[
G_{n,\lambda}^{(F,1)}(x;u) = \sum_{m=0}^{n} \sum_{l=1}^{m+1} \binom{n}{m} \frac{S_1(m+1,l) (1-u)^m}{(m+1)} G_{n-m,\lambda}(x;u).
\]

Corollary 3. Taking $k = 1$ and $u = -1$ in Theorem 3 reduces
\[
G_{n,\lambda}(x) = \sum_{m=0}^{n} \sum_{l=1}^{m+1} \binom{n}{m} \frac{S_1(m+1,l)(1-u)^m}{(m+1)} G_{n-m,\lambda}(x).
\]

Here, we give the following lemma.

Lemma 1. For $k \in \mathbb{Z}$ and $n \geq 0$, we have
\[
\frac{d}{dx} \operatorname{Ei}_k(\log(1+(1-u)x)) = \frac{1-u}{(1+(1-u)x) \log(1+(1-u)x)} \operatorname{Ei}_{k-1}(\log(1+(1-u)x)).
\] (2.7)

Proof. From (1.15), we observe that
\[
\frac{d}{dx} \operatorname{Ei}_k(\log(1+(1-u)x)) = \frac{d}{dx} \sum_{l=1}^{\infty} \frac{(\log(1+(1-u)x))^l}{(l-1)! l^k} = \frac{1-u}{(1+(1-u)x) \log(1+(1-u)x)} \sum_{l=1}^{\infty} \frac{(\log(1+(1-u)x))^l}{(l-1)! l^{k-1}} = \frac{1-u}{(1+(1-u)x) \log(1+(1-u)x)} \operatorname{Ei}_{k-1}(\log(1+(1-u)x)),
\]
which is the claimed result in (2.7).

Theorem 4. Let $k \geq 2$. We have
\[
G_{n,\lambda}^{(F,k)}(u) = \sum_{m=0}^{n} \binom{n}{m} \sum_{m_1+m_2+\cdots+m_{k-1}=m} \binom{m}{m_1, m_2, \ldots, m_{k-1}} (1-u)^{m_1+m_2+\cdots+m_{k-1}}
\times G_{n-m,\lambda}^{F}(u) B_{m_1}^{(m_1)}(0) P_{m_2}^{(m_2)}(0) \cdots P_{m_{k-1}}^{(m_{k-1})}(0) M_{m_1+1} M_{m_2+1} \cdots M_{m_{k-1}+1}.
\]

Proof. By (2.7), we consider
\[
\operatorname{Ei}_k(\log(1+(1-u)x)) = \int_{0}^{x} \frac{1-u}{(1+(1-u)x) \log(1+(1-u)x)} \operatorname{Ei}_{k-1}(\log(1+(1-u)x))dt
\]
\[
= \int_{0}^{x} \frac{1-u}{(1+(1-u)x) \log(1+(1-u)x)}
\times \int_{0}^{t} \frac{1-u}{(1+(1-u)x) \log(1+(1-u)x)} \cdots \int_{0}^{t} \frac{(1-u)^2 t}{(1+(1-u)x) \log(1+(1-u)x)} dt dt \cdots dt.
\]
Then, we obtain

\[
\sum_{n=0}^{\infty} G^{(F,k)}_{n,\lambda}(u) \frac{t^n}{n!} = \frac{\text{Ei}_k \left( \log \left( 1 + (1 - u) t \right) \right)}{e_\lambda(t) - u} \\
= \frac{1}{e_\lambda(t) - u} \int_0^x \frac{1 - u}{(1 + (1 - u) x) \log (1 + (1 - u) x)} \left( \int_0^t \frac{(1 - u)^2 t}{(1 + (1 - u) x) \log (1 + (1 - u) x)} dt \right)^{(k-2)} \, dt \\
= \frac{(1 - u) x}{e_\lambda(t) - u} \sum_{m=0}^{\infty} \sum_{m_1+m_2+\cdots+m_{k-1}=m} \left( \begin{array}{c} m \\ m_1, m_2, \ldots, m_{k-1} \end{array} \right) (1 - u)^{m_1+m_2+\cdots+m_{k-1}} \\
\times B_{m_1}(0) B_{m_2}(0) \cdots B_{m_{k-1}}(0) x^m \\
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \begin{array}{c} n \\ m \end{array} \right) \left( \begin{array}{c} m \\ m_1, m_2, \ldots, m_{k-1} \end{array} \right) (1 - u)^{m_1+m_2+\cdots+m_{k-1}} \\
\times G_{n-m,\lambda}^{F}(u) B_{m_1}(0) B_{m_2}(0) \cdots B_{m_{k-1}}(0) x^n \\
\times \frac{1}{m_1 + 1} \frac{1}{m_1 + m_2 + 1} \cdots \frac{1}{m_1 + m_2 + \cdots + m_{k-1} + 1} \frac{1}{n!}.
\]

This finalizes the proof of the theorem.

Now, we give the following theorem.

**Theorem 5.** For \( n \in \mathbb{N}_0 \), we have

\[
\sum_{m=0}^{n} \frac{S_2(n, m)}{(1 - u)^m} G^{(F,k)}_{m,\lambda}(u) = \sum_{m=0}^{n} \sum_{k=0}^{m} \left( \begin{array}{c} n \\ m \end{array} \right) \left( \begin{array}{c} m \\ k \end{array} \right) G^{(F)}_{j,\lambda}(u) S_2(k, j) \frac{B_{m-k}}{(1 - u)^j (n - m + 1)^k}.
\]

**Proof.** Replacing \( t \) by \( \frac{e^t-1}{1-u} \) in (2.1), we attain

\[
\frac{\text{Ei}_k \left( \log \left( \frac{e^t-1}{1-u} \right) \right)}{e_\lambda \left( \frac{e^t-1}{1-u} \right) - u} = \sum_{m=0}^{\infty} (1 - u)^{-m} G^{(F,k)}_{m,\lambda}(u) \left( \frac{e^t-1}{1-u} \right)^m \\
= \sum_{m=0}^{\infty} (1 - u)^{-m} G^{(F,k)}_{m,\lambda}(u) \sum_{n=0}^{\infty} S_2(n, m) \frac{t^n}{n!} \\
= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{S_2(n, m)}{(1 - u)^m} G^{(F,k)}_{m,\lambda}(u) \frac{t^n}{n!}.
\]
Also, we investigate

\[
\frac{(1-u)\left(e^{t-u}-1\right)}{e^u-1} \frac{1}{1-u} \sum_{l=1}^{\infty} \frac{t^l}{(l-1)!l^k} = \frac{(1-u)\left(e^{t-u}-1\right)}{e^u-1} \frac{1}{1-u} \sum_{l=0}^{\infty} \frac{t^l}{l! (l+1)^k}
\]

\[
= \sum_{j=0}^{k} (1-u)^{-j} G_{j,\lambda}^{(F)}(u) \frac{(e^t-1)^j}{j!} \sum_{l=0}^{\infty} B_l \frac{t^l}{l!} \sum_{l=0}^{\infty} \frac{t^l}{l! (l+1)^k}
\]

\[
= \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{(1-u)^{-j} G_{j,\lambda}^{(F)}(u) S_2(k,j)}{(1-u)^j} B_{m-k} \frac{t^m}{m!} \sum_{l=0}^{\infty} \frac{t^l}{l! (l+1)^k}
\]

\[
= \sum_{n=m=0}^{\infty} \sum_{m=0}^{k} \sum_{j=0}^{k} \left( \begin{array}{c} m \\ k \end{array} \right) \frac{(m)}{(n)} \frac{G_{j,\lambda}^{(F)}(u) S_2(k,j)}{(1-u)^j} B_{m-k} \frac{t^m}{m!} \frac{t^n}{n!} (n-m+1)^{k} n!.
\]

This completes the proof of the theorem. \(\square\)

**Theorem 6.** For \(k \in \mathbb{Z}\) and \(n \geq 0\), we have

\[
G_{n,\lambda}^{(F,k)}(x+1;u) - uG_{n,\lambda}^{(F,k)}(x;u) = \sum_{m=0}^{n} \sum_{l=1}^{m+1} \left( \begin{array}{c} m \\ l \end{array} \right) \frac{S_1(m+1,l)(1-u)^{m+1}}{l^{k-1}(m+1)} (x)_{n-m,\lambda}.
\]

**Proof.** By Definition 1 and formula (2.5), we see that

\[
\sum_{n=0}^{\infty} \left( G_{n,\lambda}^{(F,k)}(x+1;u) - uG_{n,\lambda}^{(F,k)}(x;u) \right) \frac{t^n}{n!} = \frac{E_i \left( \log(1+(1-u)t) \right)}{e^\lambda(t)-u} e^\lambda(t) (e^\lambda(t)-u)
\]

\[
= \frac{E_i \left( \log(1+(1-u)t) \right)}{e^\lambda(t)-u} e^\lambda(t)
\]

\[
= \sum_{m=0}^{\infty} \sum_{l=1}^{m+1} \frac{S_1(m+1,l)(1-u)^{m+1}}{l^{k-1}(m+1)} \frac{t^m}{m!} \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{l=1}^{m+1} \left( \begin{array}{c} m \\ l \end{array} \right) \frac{S_1(m+1,l)(1-u)^{m+1}}{l^{k-1}(m+1)} (x)_{n-m,\lambda} \frac{t^n}{n!},
\]

which gives the asserted result in (2.8). \(\square\)

3. Conclusion

In the present paper, we have considered type 2 degenerate poly-Frobenius-Genocchi polynomials and numbers by means of the polylogarithm function. Then, we have investigated diverse explicit expressions and some identities for those numbers and polynomials.

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On type 2 Degenerate Poly-Frobenius-Genocchi Polynomials

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