Some differentiation formulas for Legendre polynomials

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October 25, 2009

Abstract

In a series of recent works, we have provided a number of explicit expressions for the derivative of the associated Legendre function of the first kind with respect to its degree,

\[
\frac{\partial P_{m}^{\nu}(z)}{\partial \nu}\bigg|_{\nu=n}, \quad \text{with } m,n \in \mathbb{N}.
\]

In this communication, we use some of those expressions to obtain several, we believe new, explicit formulas for the derivatives \(d^{m}\left[P_{n}(z) \ln(z \pm 1)\right]/dz^{m}\), where \(P_{n}(z)\) is the Legendre polynomial.

KEY WORDS: Legendre polynomials; Legendre functions; special functions

MSC2010: 33C45, 33C55

1 The problem and the method

Recently, Brychkov has published a monumental reference work [1] containing, among others, a large number of explicit expressions for derivatives of various special functions with respect to their arguments and parameters. It is the purpose of the present communication to supplement the handbook [1] with several closed-form formulas for the derivatives \(d^{m}\left[P_{n}(z) \ln(z \pm 1)\right]/dz^{m}\), where \(P_{n}(z)\) is the Legendre polynomial of degree \(n\). We shall arrive at these formulas exploiting results of recent papers [2–5], in which we have extensively investigated the derivatives \(\frac{\partial P_{\nu}^{\nu}(z)}{\partial \nu}\bigg|_{\nu=n}\) and \(\frac{\partial P_{m}^{\nu}(z)}{\partial \nu}\bigg|_{\nu=n}\), where \(P_{\nu}(z)\) and \(P_{m}^{\nu}(z)\) are the Legendre function of the first kind and the associated Legendre function of the first kind, respectively.

To make the functions which appear in the following considerations single-valued, we cut the complex \(z\)-plane along the real axis from \(-\infty\) to \(+1\). Then, it follows in particular that

\[
-z - 1 = e^{\pi i}(z + 1), \quad -z + 1 = e^{\pi i}(z - 1) \quad \text{(arg}(z) \gtrless 0). \tag{1.1}
\]

Throughout the paper, it is assumed that \(\nu \in \mathbb{C}\) and \(m,n \in \mathbb{N}\).

We begin with recalling the following formulas [2–5]:

\[
\frac{\partial P_{\nu}(z)}{\partial \nu}\bigg|_{\nu=n} = P_{n}(z) \ln \frac{z + 1}{2} + R_{n}(z) \tag{1.2}
\]

and

\[
\frac{\partial P_{m}^{\nu}(z)}{\partial \nu}\bigg|_{\nu=n} = P_{n}^{m}(z) \ln \frac{z + 1}{2} + R_{n}^{m}(z) \tag{1.3}
\]

for the derivatives of \(P_{\nu}(z)\) and \(P_{m}^{\nu}(z)\) with respect to their degrees \(\nu\), the two functions being related through

\[
P_{\nu}^{m}(z) = (z^{2} - 1)^{m/2} \frac{d^{m}P_{\nu}(z)}{dz^{m}}. \tag{1.4}
\]
The relationship analogous to that in Eq. (1.4) holds also between the function \( P_n^m(z) \) and the polynomial \( R_n(z) \). In Eqs. (1.2) and (1.3), \( R_n(z) \) is a known polynomial in \( z \) of degree \( n \) (the Bromwich polynomial) and \( R_n^m(z) \) is a known function (see Section 3 below). It holds that

\[
R_n^m(z) = R_n(z),
\]

but it must be emphasized that in general \( R_n^m(z) \) is not related to \( R_n(z) \) through a formula analogous to that in Eq. (1.4).

Differentiating Eq. (1.2) \( m \) times with respect to \( z \) gives

\[
\frac{d^m}{dz^m} \left( \frac{\partial P_n(z)}{\partial \nu} \right)_{\nu=n} = \frac{d^m}{dz^m} \left( P_n(z) \ln \frac{z+1}{2} \right) + \frac{d^m R_n(z)}{dz^m}. \tag{1.6}
\]

On the other hand, from Eqs. (1.3) and (1.4) it follows that

\[
\frac{d^m}{dz^m} \left( \frac{\partial P_n(z)}{\partial \nu} \right)_{\nu=n} = \frac{d^m P_n(z)}{dz^m} \ln \frac{z+1}{2} + (z^2 - 1)^{-m/2} R_n^m(z). \tag{1.7}
\]

Combining Eqs. (1.6) and (1.7) gives

\[
\frac{d^m}{dz^m} [P_n(z) \ln(z+1)] = \frac{d^m P_n(z)}{dz^m} \ln(z+1) + (z^2 - 1)^{-m/2} R_n^m(z) - \frac{d^m R_n(z)}{dz^m}. \tag{1.8}
\]

If \( m > n \), the two derivatives on the right-hand side of Eq. (1.8) vanish and one simply has

\[
\frac{d^m}{dz^m} [P_n(z) \ln(z+1)] = (z^2 - 1)^{-m/2} R_n^m(z) \quad (m > n). \tag{1.9}
\]

Replacement of \( z \) by \(-z\) in Eqs. (1.8) and (1.9), followed by the use of Eq. (1.1) and the well-known property

\[
P_n(-z) = (-)^n P_n(z),
\]

implies the counterpart relationships

\[
\frac{d^m}{dz^m} [P_n(z) \ln(z-1)] = \frac{d^m P_n(z)}{dz^m} \ln(z-1) + (-)^n (z^2 - 1)^{-m/2} R_n^m(-z) - (-)^n \frac{d^m R_n(z)}{dz^m} \quad (m > n). \tag{1.11}
\]

and

\[
\frac{d^m}{dz^m} [P_n(z) \ln(z-1)] = (-)^n (z^2 - 1)^{-m/2} R_n^m(-z) \quad (m > n). \tag{1.12}
\]

Thus, from Eqs. (1.8), (1.9), (1.11) and (1.12) we see that once the function \( R_n^m(z) \) and the polynomial \( R_n(z) \) are known, the derivatives \( d^m [P_n(z) \ln(z \pm 1)]/dz^m \) may be evaluated.

2 Explicit representations of the polynomial \( R_n(z) \) and the function \( R_n^m(z) \)

The following three representations of the Bromwich polynomial \( R_n(z) \) have been derived in Refs. [2, 3] (in the first of these papers, the reader will find coordinates of earlier publications of Schelkunoff and Bromwich, in which the expressions (2.1) and (2.3) were found differently than in Ref. [2]):

\[
R_n(z) = -2 \psi(n+1) P_n(z) + 2 \sum_{k=0}^{n} \frac{(k+n)! \psi(k+n+1)}{(k)!^2 (n-k)!} \left( \frac{z-1}{2} \right)^k, \tag{2.1}
\]

\[
R_n(z) = 2 \sum_{k=0}^{n} (-)^{k+n} \frac{(k+n)!}{(k)!^2 (n-k)!} [\psi(k+n+1) - \psi(k+1)] \left( \frac{z+1}{2} \right)^k, \tag{2.2}
\]
\[ R_n(z) = 2[\psi(2n+1) - \psi(n+1)]P_n(z) + 2 \sum_{k=0}^{n-1} (-1)^{k+n} \frac{2k+1}{(n-k)(k+n+1)} P_k(z), \]  

(2.3)

where

\[ \psi(\zeta) = \frac{1}{\Gamma(\zeta)} \frac{d\Gamma(\zeta)}{d\zeta} \]  

(2.4)

is the digamma function. Next, in Ref. [4] it has been proved that the function \( R_n^m(z) \) may be written as

\[
R_n^m(z) = -[\psi(n+1) + \psi(n-m+1)]P_n^m(z) + \left( \frac{z^2-1}{4} \right)^{m/2} \sum_{k=0}^{n-m} \frac{(k+n+m)!}{k!(k+m)!(n-m-k)!} \left( \frac{z-1}{2} \right)^k
\]

\[
+ \frac{(n+m)!}{(n-m)!} \left( \frac{z-1}{2} \right)^{m/2} \sum_{k=0}^{n} \frac{(k+n)!}{k!(k+m)!(n-k)!} \left( \frac{z-1}{2} \right)^k
\]

(0 \leq m \leq n),  

(2.5)

while in Ref. [5] the following two alternative expressions:

\[
R_n^m(z) = -[\psi(n+m+1) + \psi(n-m+1)]P_n^m(z) + \left( \frac{z^2-1}{4} \right)^{m/2} \sum_{k=0}^{n-m} \frac{(k+n+m)!}{k!(k+m)!(n-m-k)!} \left( \frac{z-1}{2} \right)^k
\]

\[
\times [2\psi(k+n+m+1) - \psi(k+m+1)] \left( \frac{z-1}{2} \right)^k
\]

\[
+ \frac{(n+m)!}{(n-m)!} \left( \frac{z-1}{2} \right)^{m/2} \sum_{k=0}^{n} \frac{(k+n)!}{k!(k+m)!(n-k)!} \left( \frac{z-1}{2} \right)^k
\]

(0 \leq m \leq n)  

(2.6)

and

\[
R_n^m(z) = [\psi(n+m+1) - 2\psi(n+1) - \psi(n-m+1)]P_n^m(z) + \left( \frac{z^2-1}{4} \right)^{m/2} \sum_{k=0}^{n-m} \frac{(k+n+m)!}{k!(k+m)!(n-m-k)!} \left( \frac{z-1}{2} \right)^k
\]

\[
+ \frac{(n+m)!}{(n-m)!} \left( \frac{z-1}{2} \right)^{m/2} \sum_{k=0}^{n} \frac{(k+n)!}{k!(k+m)!(n-k)!} \left( \frac{z-1}{2} \right)^k
\]

\[
\times [2\psi(k+n+1) - \psi(k+m+1)] \left( \frac{z-1}{2} \right)^k
\]

(0 \leq m \leq n)  

(2.7)

have been provided. Furthermore, in Ref. [5] we have found the formula

\[
R_n^m(z) = -(-1)^n \sum_{k=0}^{n-m} (-1)^{k+n} \frac{(k+n)!}{k!(n-k)!} \left( \frac{z+1}{2} \right)^k
\]

\[
+ (-1)^n \sum_{k=0}^{n-m} (-1)^{k+n} \frac{(k+n)!}{k!(n-k)!} \left( \frac{z+1}{2} \right)^k
\]

\[
\times [2\psi(k+n+m+1) - \psi(k+m+1) - \psi(k+1)] \left( \frac{z+1}{2} \right)^k
\]

(0 \leq m \leq n),  

(2.8)
while in Ref. [4] we have arrived at

\[ R_n^m(z) = [2\psi(2n + 1) - \psi(n + 1) - \psi(n - m + 1)]P_n^m(z) \]

\[ + (-1)^m \sum_{k=0}^{n-m-1} \frac{2k + 1}{(n-k)(k+n+1)} P_k^m(z) \]

\[ + (-)^{n+m} \sum_{k=0}^{n-m-1} \frac{2k + 2m + 1}{(n-k)(k+n+m+1)} \times \left[ 1 + \frac{k!(n+m)!}{(k+2m)!(n-m)!} \right] P_{k+m}^m(z) \quad (0 \leq m \leq n). \] (2.9)

Finally, in Ref. [4] we have also obtained the remarkably simple representation

\[ R_n^m(z) = (-1)^{n+m+1}(n + m)!(m - n - 1)!P_n^{-m}(z) \quad (m > n). \] (2.10)

It is easy to see that for \( m = 0 \) the triple of equations (2.5)–(2.7) reduces to Eq. (2.1), while Eqs. (2.8) and (2.9) go over into Eqs. (2.2) and (2.3), respectively, in accordance with Eq. (1.5).

3 Closed-form expressions for the derivatives

\[ d^m[P_n(z) \ln(z \pm 1)]/dz^m \]

Inserting the expansions (2.1)–(2.3) and (2.5)–(2.10) into Eqs. (1.8), (1.9), (1.11) and (1.12), and using the relation [6, Eq. (8.936.2)]

\[ \frac{d^mP_n(z)}{dz^m} = \frac{(2m)!}{2^m m!} C_{n-m}^{(m+1/2)}(z), \] (3.1)

where \( C_n^{(\alpha)}(z) \) is the Gegenbauer polynomial, yields the following representations of the derivatives \( d^m[P_n(z) \ln(z \pm 1)]/dz^m \):

\[ \frac{d^m}{dz^m}[P_n(z) \ln(z \pm 1)] = \frac{(2m)!}{2^m m!} C_{n-m}^{(m+1/2)}(z) \ln(z \pm 1) \]

\[ + \frac{(2m)!}{2^m m!} [\psi(n + 1) - \psi(n - m + 1)] C_{n-m}^{(m+1/2)}(z) \]

\[ - (-)^n \sum_{k=0}^{n-m} \frac{C_{n-m}(z)}{k!(k+n)(n-m-k)!} \frac{(k + n + m)!\psi(k + n + m + 1)}{(n-k)(n-m-k)!} \left( \frac{z \mp 1}{2} \right)^k \]

\[ + \frac{(n+m)!}{(n-m)!} \left( \sum_{k=0}^{n-m} \frac{(k + n)!\psi(k + n + 1)}{k!(k+n)(n-k)!} \left( \frac{z \pm 1}{2} \right)^k \right) \]

\[ (0 \leq m \leq n), \] (3.2)

\[ \frac{d^m}{dz^m}[P_n(z) \ln(z \pm 1)] = \frac{(2m)!}{2^m m!} C_{n-m}^{(m+1/2)}(z) \ln(z \pm 1) \]

\[ - \frac{(2m)!}{2^m m!} [\psi(n + m + 1) - 2\psi(n + 1) + \psi(n - m + 1)] C_{n-m}^{(m+1/2)}(z) \]

\[ - (-)^n \sum_{k=0}^{n-m} \frac{C_{n-m}(z)}{k!(k+n)(n-m-k)!} \frac{(k + n + m)!\psi(k + m + 1)}{(n-k)(n-m-k)!} \left( \frac{z \mp 1}{2} \right)^k \]

\[ + \frac{(n+m)!}{(n-m)!} \left( \sum_{k=0}^{n-m} \frac{(k + n)!\psi(k + m + 1)}{k!(k+n)(n-k)!} \left( \frac{z \mp 1}{2} \right)^k \right) \]

\[ (0 \leq m \leq n), \] (3.3)
\[ \frac{d^m}{dz^m}[P_n(z) \ln(z \pm 1)] = \frac{(2m)!}{2^m m!} C_{n-m}^{(m+1/2)}(z) \ln(z \pm 1) \]

\[ + \frac{(2m)!}{2^m m!} [\psi(n + m + 1) - \psi(n - m + 1)] C_{n-m}^{(m+1/2)}(z) \]

\[ - \frac{(\mp)^{n+m}}{2^m} \sum_{k=0}^{n-m} (\pm)^k \frac{(k + n + m)!}{k!(k+m)!(n-m-k)!} \]

\[ \times [2\psi(k + n + m + 1) - \psi(k + m + 1)] \left( \frac{z + 1}{2} \right)^k \]

\[ + (\pm)^n (n + m)! (z^2 - 1)^{-m/2} \sum_{k=0}^{m-1} (-1)^k \frac{2k + 1}{(n-k)(k+m+1)} P_k^m(\pm z) \]

\[ - (\mp)^{n+m} \frac{(2m)!}{2^m m!} \sum_{k=0}^{n-m} (\pm)^k \frac{2k + 1}{(n-m-k)(k+n+m+1)} \]

\[ \times \left[ 1 - \frac{k!(n+m)!}{(k+2m)!(n-m)!} \right] C_k^{(m+1/2)}(z) \quad (0 \leq m \leq n), \quad (3.4) \]

\[ \frac{d^m}{dz^m}[P_n(z) \ln(z \pm 1)] = \frac{(2m)!}{2^m m!} C_{n-m}^{(m+1/2)}(z) \ln(z \pm 1) \]

\[ + \frac{(2m)!}{2^m m!} [\psi(n + m + 1) - \psi(n - m + 1)] C_{n-m}^{(m+1/2)}(z) \]

\[ - \frac{(\mp)^{n+m}}{2^m} \sum_{k=0}^{n-m} (\pm)^k \frac{(k + n + m)!}{k!(k+m)!(n-m-k)!} \]

\[ \times [2\psi(k + n + m + 1) - \psi(k + m + 1)] \left( \frac{z + 1}{2} \right)^k \]

\[ + (\pm)^n (n + m)! (z^2 - 1)^{-m/2} \sum_{k=0}^{m-1} (-1)^k \frac{2k + 1}{(n-k)(k+m+1)} P_k^m(\pm z) \]

\[ - (\mp)^{n+m} \frac{(2m)!}{2^m m!} \sum_{k=0}^{n-m} (\pm)^k \frac{2k + 1}{(n-m-k)(k+n+m+1)} \]

\[ \times \left[ 1 - \frac{k!(n+m)!}{(k+2m)!(n-m)!} \right] C_k^{(m+1/2)}(z) \quad (0 \leq m \leq n), \quad (3.5) \]

\[ \frac{d^m}{dz^m}[P_n(z) \ln(z \pm 1)] = \frac{(2m)!}{2^m m!} C_{n-m}^{(m+1/2)}(z) \ln(z \pm 1) \]

\[ + \frac{(2m)!}{2^m m!} [\psi(n + m + 1) - \psi(n - m + 1)] C_{n-m}^{(m+1/2)}(z) \]

\[ - \frac{(\mp)^{n+m}}{2^m} \sum_{k=0}^{n-m} (\pm)^k \frac{(k + n + m)!}{k!(k+m)!(n-m-k)!} \]

\[ \times [2\psi(k + n + m + 1) - \psi(k + m + 1)] \left( \frac{z + 1}{2} \right)^k \]

\[ + (\pm)^n (n + m)! (z^2 - 1)^{-m/2} \sum_{k=0}^{m-1} (-1)^k \frac{2k + 1}{(n-k)(k+m+1)} P_k^m(\pm z) \]

\[ - (\mp)^{n+m} \frac{(2m)!}{2^m m!} \sum_{k=0}^{n-m} (\pm)^k \frac{2k + 1}{(n-m-k)(k+n+m+1)} \]

\[ \times \left[ 1 - \frac{k!(n+m)!}{(k+2m)!(n-m)!} \right] C_k^{(m+1/2)}(z) \quad (0 \leq m \leq n), \quad (3.6) \]

and

\[ \frac{d^m}{dz^m}[P_n(z) \ln(z \pm 1)] = (\mp)^n (n + m)! (m - n + 1)! (z^2 - 1)^{-m/2} P_m^{m-1}(\pm z) \quad (m > n). \quad (3.7) \]

The sextet of formulas (3.2)–(3.7) constitutes the result of this paper.

### 4 Concluding remarks

In some applications, it might be necessary to have explicit representations of \( \frac{d^m}{dx^m}[P_n(x) \ln(1 \pm x)] / dx^m \) with \(-1 \leq x \leq 1\). Such explicit formulas may be derived from Eqs. (3.2)–(3.7) with the aid of the relationships

\[ x + 1 \pm i0 = 1 + x, \quad x - 1 \pm i0 = e^{\pm i\pi}(1 - x) \quad (-1 \leq x \leq 1) \quad (4.1) \]
and
\[
P_n^{\pm m}(x) = e^{\pm i\pi m/2}P_n^{\pm m}(x + i0) = e^{\mp i\pi m/2}P_n^{\pm m}(x - i0) \\
= \frac{1}{2} \left[ e^{\pm i\pi m/2}P_n^{\pm m}(x + i0) + e^{\mp i\pi m/2}P_n^{\pm m}(x - i0) \right] \\
(-1 \leq x \leq 1). \tag{4.2}
\]

Since the procedure is straightforward and does not offer any difficulty, we do not list the resulting expressions here.

References

[1] Yu. A. Brychkov, Handbook of Special Functions. Derivatives, Integrals, Series and Other Formulas, Chapman & Hall/CRC, Boca Raton, FL, 2008

[2] R. Szmytkowski, On the derivative of the Legendre function of the first kind with respect to its degree, J. Phys. A 39 (2006) 15147 [corrigendum: 40 (2007) 7819]

[3] R. Szmytkowski, Addendum to ‘On the derivative of the Legendre function of the first kind with respect to its degree,’ J. Phys. A 40 (2007) 14887. We note parenthetically that the simplest way to arrive at Eq. (5) in that paper is to differentiate both sides of the identity \(d^k z^\alpha / dz^k = [\Gamma(\alpha + 1)/\Gamma(\alpha - k + 1)]z^{\alpha - k}\) with respect to \(\alpha\).

[4] R. Szmytkowski, On the derivative of the associated Legendre function of the first kind of integer order with respect to its degree (with applications to the construction of the associated Legendre function of the second kind of integer degree and order), preprint arXiv:0907.3217

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