Optimal forward contract design for inventory: a value-of-waiting analysis

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Abstract. A classical inventory problem is studied from the perspective of embedded options, reducing inventory-management to the design of optimal contracts for forward delivery of stock (commodity). Financial option techniques à la Black-Scholes are invoked to value the additional ‘option to expand stock’. A simplified approach which ignores distant time effects identifies an optimal ‘time to deliver’ and an optimal ‘amount to deliver’ for a production process run in continuous time modelled by a Cobb-Douglas revenue function. Commodity prices, quoted in initial value terms, are assumed to evolve as a geometric Brownian process with positive drift. Expected revenue maximization identifies an optimal ‘strike price’ for the expansion option to be exercised, and uncovers the underlying martingale in a truncated (censored) commodity price. The paper establishes comparative statics of the censor in terms of drift and volatility, and uses asymptotic approximation for a tractable analysis of the optimal timing.

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1 Problem formulation and model

We enhance a classical inventory-management problem by studying its embedded options, reducing the problem to the design of optimal contracts for forward delivery of inventory. The approach borrows much from the Black-Scholes model for valuing financial options (see Musiela and Rutkowski [11] Chapter 5) and uncovers the underlying martingale to be a truncated (right-censored) discounted commodity price.

A production process runs continuously over a unit time interval and the manager is permitted to acquire raw input materials at two dates: initially, at time \( t = 0 \), and again at one other time \( \theta < 1 \), selected freely, but committed to at time \( t = 0 \). This framework is intended as a proxy for a multi-stage inventory management problem, since ‘proximal’ effects of forward contracting, as represented by the date \( \theta \), are more significant than any additional ‘distal’ dates for forward delivery. Distal dates for additional forward deliveries are thus neglected in this model (see the ‘Interpretation’ paragraph at the end of Section 4). Inputs are consumed in a continuous production process which creates an instantaneous revenue rate at time \( t \) equal to \( f(x_t) \) (quoted in present-value terms), where \( x_t \) is the instantaneous input rate of consumed material. To begin with, \( f(x) \) is standardly an Inada-type increasing function, viz. twice differentiable, unboundedly increasing from zero, with slope unbounded at the origin and strictly decreasing to zero at infinity; eventually \( f(x) \) is specialized to a Cobb-Douglas production function. The revenue from any interval \([a, b]\) is taken to be

\[
\int_a^b f(x_t)dt.
\]
If the manager decides to use up a proportion $\theta x$ in the period $[0, \theta]$ then, with $\theta$ fixed, the Euler-Lagrange equation implies that a constant instantaneous input rate equal to $x$ is optimal. A further quantity $(1 - \theta)y$ may similarly be consumed in the remaining time interval. If the quantity $(1 - \theta)y$ is made up from a contracted forward delivery of $(1 - \theta)u$ and from a possible supplement, purchased at time $\theta$, of a non-negative quantity $(1 - \theta)z$, the revenue from the second interval will be

$$
\int^{1}_{\theta} f(x_t) dt = (1 - \theta)f(u + z).
$$

Values here and below are quoted in discounted terms, i.e. present-value terms relative to time $t = 0$. (We side-step a discussion of the relevant discount factor. In brief, discounting would be done relative to the required rate of return on capital given the risk-class of the investment project; see Dixit and Pindyck [4, Chapter 4, Section 2].)

Whilst the model of revenue assumes a steady (deterministic) market for the output, the input prices are assumed stochastic. (We prefer this modelling choice over the more general approach of including also a stochastic output price. Indeed, what then determines optimal behaviour is the ratio of the two prices; so, in a sense, the present simpler arrangement subsumes it.) Specifically, we suppose that at time 0 the price of inputs is $b_0 = 1$, and that, as time $t$ progresses, the present value of the spot price, $b_t$, follows the stochastic differential equation:

$$
\frac{db_t}{b_t} = \bar{\mu} dt + \bar{\sigma} dw_t,
$$

with $w_t$ a standard Wiener process. It is assumed that the constant growth rate $\bar{\mu}$ is positive, so that the expected (present-value/discounted) price at time $t$ is $e^{\bar{\mu}t}$; thus the price is expected to grow above the initial price of unity. The price $b_t$ is log-normally distributed with a mean which we denote by $\nu = (\bar{\mu} - \frac{1}{2}\bar{\sigma}^2)t$ and a variance $\sigma^2 = \bar{\sigma}^2 t$. Write $q_t(\cdot) = q(\cdot; \bar{\mu}t, \bar{\sigma}\sqrt{t})$ for the density of $b_t$. Conditional on the initial choice of $\theta$, the expected future revenue consequent on the choice of $x, u$ and $z$ (with $z$ selected at time $\theta$) is

$$
\theta(f(x) - x) + (1 - \theta) \left( \int^{\infty}_{0} \{ f(z + u) - bz \} q_0(b) db - u \right) .
$$

This is a classical inventory problem but amended by the inclusion explicitly of the ‘option to expand inventory’ (choice of $z$) and of a ‘forward’ contract (choice of $u$). We will evaluate the embedded option in a framework reminiscent of Black-Scholes option-pricing. The ‘forward contract’ is construed here as a contract signed at the earlier date $t = 0$ with an agreed specified delivered quantity, $u$, a specified delivery date $t = \theta$, and a price standardized here to unity per unit delivered. The latter standardization fixes the unit of money, since, in the absence of arbitrage and storage costs, as is well-known, the forward price equals the price of inputs at the initial time of contracting, compounded up to term-value at the required rate of interest. Note that the advance purchase of $u$ has by assumption nil resale value on delivery. This makes the delivered asset a ‘non-tradeable’ commodity, so that the usual martingale valuation approach applied to a discounted security price is not immediately appropriate; our analysis makes recourse to dynamical programming, as in Eberly and Van Mieghem [5], and thereby identifies the underlyling martingale structure via an appropriately truncated (right-censored) price.

Apart from offering a real-options approach with optimal design in mind, in contrast to the classical inventory literature (see for instance Bensoussan et al. [2], or Scarf [15]), an additional contribution of the current paper is to provide information about the sensitivity in regard to model parameters of the critical ‘strike price’ for stock expansion (its comparative statics and asymptotics), an issue omitted from consideration in Eberly and Van Mieghem [5].

The current study of profit dependence on timing, drift and variance is motivated by the general discrete-time multi-period model of Gietzmann and Ostaszewski [7], but with the simplifying removal of costly liquidation of inventory. There the latter feature was necessary for a more comprehensive study.
into the dependence of a firm’s ‘future value’ on accounting data. Such themes are explored in [12] in this volume.

Our option-based analysis is simpler than [3], though similar in spirit. There the (retailer’s) inventory control problem addresses re-distribution of a storable product; one uses a (long) forward for delivery combined with an option to dispose of any excess (a put, with a lower salvage price) coupled with an option for additional supply (a call, with a penalty cost for ‘emergency supply’); for background on these ‘option’ terms see e.g. [8]. A similar approach, albeit in discrete time, is taken in [10] using at each date a continuum of puts and calls maturing at the next date taken together with a short (negative) forward.

The rest of the paper is organized as follows. In §2 we study optimality conditions, which identify a threshold price level (the price censor) above which it is not worth purchasing the input. We consider its sensitivity (comparative statics) to price drift and volatility in §3. Then in §4 we assess the expected revenue and in §5 the optimal timing. Proofs (sensitivity analysis) is spread across §6 and §7.

2 Optimality: the censor and value of waiting

From (2) the optimization problem separates into maximization of $f(x) - x$ (with solution specified by $f'(x) = 1$) and the maximization, over choice of scalar $u \geq 0$ and function $z(b)$, of the (time $t = 0$) expectation

$$E[ f(z(b) + u) - bz(b)] - u. \quad (3)$$

**Definition.** For any Inada-type, strictly concave function $f(x)$ define the ‘indirect profit’ (i.e. maximised profit) for a deterministic price $b$ by

$$h(b) = \max_{x \geq 0} [ f(x) - bx]. \quad (4)$$

Evidently $h(b) = f(I(b)) - bl(b)$, where $I$ traditionally denotes the inverse function to $f'$.

**Theorem 1 (Optimal forward delivered quantity).** In the model setting above, with time $\theta$ given, let $\tilde{b} := \tilde{b}(\mu, \sigma, \theta)$ be the scalar solving the equation

$$E[b_{\theta} \wedge \tilde{b}] = b_{\theta} = 1, \quad (5)$$

where $b_{\theta}$ denotes the random price at time $\theta$. Then the profit-optimizing level of advance purchase $u = u(\mu, \sigma, \theta)$ for (3) satisfies

$$f'(u) = \tilde{b}, \quad (6)$$

and the optimal expected profit is given by

$$g(\mu, \sigma) = E[h(b_{\theta}), b \leq \tilde{b}] + h(\tilde{b}) \cdot Pr[b > \tilde{b}]. \quad (7)$$

**Proof.** With $\beta$ arbitrary, select $u$ with $\beta = f'(u)$. Note that $h(\beta) = f(u) - \beta u$ and $h'(\beta) = -u$. Define the right-censored random variable $B_{\theta} = B_{\theta}(\beta)$ by

$$B_{\theta} = b_{\theta} \wedge \beta.$$  

For given price $b$ the quantity $z = z(b)$ which maximizes $f(u + z) - bz$, is either zero, or satisfies the first-order condition

$$f'(z + u) = b.$$  

In view of the monotonicity of $f'$ we thus have $z(b) = 0$, unless $b \leq \tilde{b}$. Given that $u$ has been purchased at a price of unity, the profit, when $b_{\theta} \leq \beta$, is $f(u + z) - (b_{\theta} z(b_{\theta}) + u) = h(b_{\theta}) + (u b_{\theta} - u)$. Otherwise it is $f(u) - u = h(\beta) + u \beta - u$. Thus the expected profit is

$$\Pi(\beta) := E[h(B_{\theta}) + uB_{\theta} - u] = E[h(B_{\theta})] + uE[B_{\theta}] - u.$$
Differentiating $\Pi$ with respect to $\beta$, and noting that
\[
dE[h(b_\theta \land \beta)]/d\beta = h'(\beta) \Pr[b_\theta \geq \beta],
\]
we obtain, after some cancellations in view of $h'(\beta) = -u$, the optimality condition $E[B_\theta] = 1$ on $\beta$. The model assumption that $\mu$ is positive ensures the existence of a solution of equation (5). With $\beta$ set equal to the solution $\tilde{b}$ of equation (5) we have $\tilde{b} = f'(u)$, i.e. (6).

**Definition.** In view of the right-censoring of the price $b$ occurring under the expectation, we call the solution of (5) the censor $\tilde{b} = \tilde{b}(\mu, \sigma, \theta)$ at time $\theta$. This definition follows Gietzmann and Ostaszewski [6]. The censored variable is thus a martingale.

**Remark.** It is clear from the proof above that the censor describes the upper limit of those prices which trigger the exercise of the option to expand stock. So evidently, $\tilde{b} > 1$. We return in the next section to a consideration of its behaviour. Whilst this threshold role makes the censor similar to the ‘optimal ISD control limit’ studied by Eberly and Van Mieghe [5], their thresholds correspond to Investing/Staying/Disinvesting and are distinct in respect of the treatment of capital depreciation.

**Proposition 1 (Value of waiting).** The expected profit $g(\mu, \sigma)$ defined in (7) obtained by optimal forward contracting is no worse than the indirect profit $h(1)$ obtained by only using purchases at initial prices, that is

\[
h(1) < g(\mu, \sigma) = E[h(b_\theta \land \tilde{b})].
\]

**Proof.** This follows from a simple application of Jensen’s inequality, as $h(b)$ is strictly convex in $b$. Indeed, we then have

\[
h(1) = h(E[b_\theta \land \tilde{b}]) < E[h(b_\theta \land \tilde{b})].
\]

Of course $-h(b)$ is the Fenchel dual of the strictly concave function $f$, so $-h(b)$ is strictly concave in $b$ (see [13] Section 12). In the specific case of $f(x)$ twice differentiable the asserted convexity follows from $h''(b) = -1/f''(I(b))$, where $I$ denotes, as before, the inverse function of $f'$.

3 Sensitivity: Censor comparative statics

Assuming an Inada-type production function, for the geometric Brownian model adopted in respect of price as in (1), the censor equation (5) which defines $\tilde{b} = \tilde{b}(\mu, \sigma)$ can be re-written as:

\[
1 = e^{\mu \Phi(W - \sigma)} + \tilde{b} \Phi(-W). \tag{8}
\]

Here $\Phi(x) = \int_{-\infty}^{x} \varphi(w)dw$, with $\varphi(w) = e^{-\frac{1}{2}w^2}/\sqrt{2\pi}$, denotes the standard normal cumulative distribution function, $W = w(\tilde{b})$, and

\[
w(b) := \frac{\ln b - \nu}{\sigma}, \quad \text{where} \quad \nu = \mu - \frac{1}{2} \sigma^2. \tag{9}
\]

This formulation leads naturally to a further definition.

**Definition.** The normal censor is the implicit function $W(\mu, \sigma)$ defined for $\mu, \sigma > 0$ as follows:

\[
e^{-\mu} = F(W, \sigma), \quad \text{where} \quad F(W, \sigma) := \Phi(W - \sigma) + e^{\sigma W - \frac{1}{2} \sigma^2} \Phi(-W). \tag{10}
\]

We note that $W$ is well defined since $\partial F/\partial W > 0$. It is helpful to be aware of the hidden connection between the function $F$ and the normal hazard rate $H(x) = \varphi(x)/\Phi(-x)$ (or its reciprocal, the Mills’ Ratio) and to use properties of this function. We refer to Kendall and Stuart [9] p.104], or Patel and Read [13] for details. From $\varphi(\sigma - W) = e^{\sigma W - \frac{1}{2} \sigma^2} \varphi(W)$,

\[
F(W, \sigma) = \varphi(\sigma - W) \left( \frac{1}{H(\sigma - W)} + \frac{1}{H(W)} \right).
\]
From (10), $W(\mu, \sigma)$ is decreasing in $\mu$, as $e^{-\mu}$ is decreasing. Less obvious is the fact that $W(\mu, \sigma)$ is increasing in $\sigma$ since in fact $\partial W/\partial \sigma > 1$. This is shown in Section 7, where we deduce the comparative statics of $\tilde{b}(\mu, \sigma)$ from corresponding properties of $W(\mu, \sigma)$. The main results proved there are cited below.

**Theorem 5.** The censor $\tilde{b}(\mu, \sigma)$ is decreasing in the drift and is increasing in the standard deviation.

These two properties together suggest the following result, obtained by setting $\mu = \bar{\mu} \theta$ and $\sigma = \bar{\sigma} \sqrt{\theta}$, and noting that (10) permits arbitrary positive $\theta$.

**Theorem 6.** The censor $\tilde{b}(\theta) = \tilde{b}(\bar{\mu} \theta, \bar{\sigma} \sqrt{\theta})$ is either unimodal or increasing on the interval $0 < \theta < \infty$, according as $\bar{\mu} \geq \frac{1}{2} \bar{\sigma}^2$ or $\bar{\mu} < \frac{1}{2} \bar{\sigma}^2$.

### 4 Cobb-Douglas revenue: asymptotic results

We now assume $f(x)$ is Cobb-Douglas, specifically $f(x) = 2\sqrt{x}$, so that the indirect profit defined by (4) is $h(b) = b^{-1}$. This choice for the power of $x$ inflicts no loss of generality, because in the presence of a log-normally distributed price any other choice of power is equivalent to a re-scaling of $\bar{\mu}, \bar{\sigma}$. Substituting into the definition (7) yields

$$g(\mu, \sigma) = e^{(\sigma^2 - \mu)}\Phi(W + \sigma) + e^{-\mu - \sigma W + \frac{1}{2} \sigma^2} \Phi(-W),$$

(11)

as the profit per unit time arising after the re-stocking date $\theta$. We also define the associated function

$$\tilde{g}(\theta) := g(\bar{\mu} \theta, \bar{\sigma} \sqrt{\theta}),$$

for $0 < \theta < \infty$ (with some re-sizing of $\bar{\mu}, \bar{\sigma}$ in mind, as in Proposition 4 of Section 5). To study these functions we are led to analyse the behaviour of first $W(\mu, \sigma)$ and then $\tilde{W}(t) = \text{def} W(\bar{\mu} t, \bar{\sigma} \sqrt{t})$. The following are derived in Section 6.

**Proposition 2.** For fixed $\mu > 0$,

$$W(\mu, \sigma) = -\frac{\mu}{\sigma} + \frac{1}{2} \sigma + o(\sigma) \quad \text{as } \sigma \to 0 + .$$

**Proposition 3.** For fixed $\mu > 0$,

$$W(\mu, \sigma) = -\tilde{\mu} - \frac{1}{\sigma - \tilde{\mu}} \{1 + o(1)\} \quad \text{as } \sigma \to \infty, \text{ with } \tilde{\mu} = -\Phi^{-1}(e^{-\mu}).$$

From (11) and standard asymptotic estimates of $\Phi(x)$ (see Abramowitz & Stegan [1] Section 7) Theorem A below is immediate. It also turns out that $\tilde{W}(t)$ behaves rather like $\pm \sqrt{t}$ (except when $\bar{\mu} = \frac{1}{2} \bar{\sigma}^2$).

**Theorem A (Asymptotic behaviour of the profit $g(\mu, \sigma)$).**

(i) $g = e^{\sigma^2 - \mu} + o(1/\sigma)$ as $\sigma \to \infty$;

(ii) $g = e^{-\mu} + (1 - e^{-\mu})\Phi(\mu/\sigma) + o(\sigma)$ as $\sigma \to 0 +$.

**Theorem B (Behaviour of the profit $\tilde{g}(\theta)$ at the origin).**

We have $\tilde{g}'(0) = \sigma^2$ so that

$$\tilde{g}(\theta) = 1 + \tilde{\sigma}^2 \theta + o(\theta).$$

**Theorem C (Asymptotic behaviour of the profit $\tilde{g}(\theta)$ at infinity).**
Figure 1: Typical graph of $g(\theta)$ in the case (i) $\sigma^2 < \bar{\mu}$

(i) If $\bar{\sigma}^2 < \bar{\mu}$ we have as $\theta \to \infty$ that

$$g(\theta) = 1 + o(1/\sqrt{\theta}) \to 1,$$

and $g(\theta)$ has a maximum whose location tends to infinity as $\bar{\sigma}^2 \to \bar{\mu}$.

(ii) If $\bar{\mu} \leq \bar{\sigma}^2 < 2\bar{\mu}$ we have as $\theta \to \infty$ that

$$g(\theta) = 1 + e^{(\bar{\sigma}^2 - \bar{\mu})\theta} + o(1/\sqrt{\theta}).$$

(iii) If $2\bar{\mu} < \bar{\sigma}^2$ we have as $\theta \to \infty$ that

$$g(\theta) = e^{(\bar{\sigma}^2 - \bar{\mu})\theta} + o(1/\sqrt{\theta}).$$

(iv) If $\bar{\sigma}^2 = 2\bar{\mu}$ we have as $\theta \to \infty$ that

$$g(\theta) = \frac{1}{4} + e^{\theta(\sqrt{2\bar{\mu}\theta} + o(1/\sqrt{\theta}) = \frac{1}{4} + e^{\theta} + o(1/\sqrt{\theta}).$$

For the proofs, see §6.

Figures 1-4 with a parameter value $\bar{\mu} = 0.05$ show the computed graphs of $g$ (bold) alongside the relevant approximation (faint); Figure 4 shows the first to the right of the two approximations given in the case (iv).

**Interpretation.** Under ‘myopic management’, i.e. in the absence of forward contracting, for a given re-stocking date $\theta$ the expected profit would be

$$E_0[h(b_\theta)] = E[1/b_\theta] = e^{(\bar{\sigma}^2 - \bar{\mu})\theta}.$$

The theorem thus implies that forward contracting advantages lose significance as variance increases, or as the re-stocking date $\theta$ advances. This ultimately is our justification for excluding any additional dates for further forward deliveries.
5 Cobb-Douglas optimal timing: estimates

Assuming as above, again without real loss of generality, that \( f(x) = 2\sqrt{x} \), we turn now to revenue optimization in respect of the time \( \theta \) to be selected freely in \([0, 1]\). Supposing there are no associated management costs in choosing \( \theta \), the optimal revenue for a selected value of \( \theta \) is, from (2), given by:

\[
R(\theta) = \theta + (1 - \theta)\bar{g}(\theta),
\]

the first term being justified by \( h(1) = 1 \). As \( \bar{g}(0) = 1 \) the optimal choice of \( \theta \), assuming such exists, is given by the following first-order condition:

\[
\frac{\bar{g}(\theta) - \bar{g}(0)}{\bar{g}'(\theta)} = 1 - \theta. \tag{12}
\]

**Proposition 4.** The first-order condition for \( R \) in (12) is satisfied for some \( \theta \) with \( 0 < \theta < 1 \). The smallest solution is a local maximum of \( R \). If \( \bar{g} \) is concave on \([0, 1]\), then the solution of (12) is unique.

**Proof.** In general, by Proposition 1 on the Value of Waiting (Section 2), \( \bar{g}(1) - \bar{g}(0) > 0 \) and so the first assertion is obvious, as the right-hand side is zero at \( \theta = 1 \) and is positive at \( \theta = 0 \); indeed,
by Theorem B above, the left-hand side has a limiting value zero as \( \theta \to 0^+ \) for \( \sigma > 0 \). If, however, \( \bar{g}(1) - \bar{g}(0) = 0 \) (i.e. \( h \) failed to be strictly convex), then since the function \( \bar{g} \) is initially increasing for \( \theta > 0 \), \( \bar{g} \) has an internal local maximum at \( \bar{\theta} \) for some \( 0 < \bar{\theta} < 1 \) (by the Mean Value Theorem). In this case the first-order condition for \( R \) is satisfied by some \( \theta < \bar{\theta} \), since the left-hand side tends to \( +\infty \) as \( \theta \to \bar{\theta} \).

Any internal solution \( \theta^* \) to equation (12) has \( \bar{g}'(\theta^*) > 0 \) and so the second assertion follows since \( R'(\theta^* -) > 0 \) and \( R'(\theta^* +) < 0 \). Observe that if \( \bar{g}''(\theta) < 0 \), then we have

\[
\frac{d}{d\theta} \left( \frac{\bar{g}(\theta) - \bar{g}(0)}{\bar{g}'(\theta)} \right) = 1 - \bar{g}''(\theta) \frac{\bar{g}(\theta) - \bar{g}(0)}{[\bar{g}'(\theta)]^2} > 0,
\]

so the third assertion is clear; indeed concavity ensures that the left-hand side of (12) is an increasing function of \( \theta \).

One would wish to improve on Proposition 4 to show in more general circumstances (beyond the concavity which can sometimes fail, as Figure 1 shows) that (12) has a unique solution, and to study dependence on the two parameters of the problem. This appears analytically intractable. For the purposes of gaining an insight we propose therefore to replace \( \bar{g}(\theta) \) by a function related to it through asymptotic analysis (as \( t \) varies), on the grounds that from numeric observation the substitute is qualitatively similar.

Examination of behaviour for large \( t \) may be justified by re-sizing the parameters \( \bar{\mu}, \bar{\sigma} \) which enables the termination date to become ‘large’. This observation then ushers in the advantages of the asymptotic viewpoint.

Guided by Theorems B and C, we are led to a considerably simpler problem obtained by making one of two ‘typical’ substitutions for \( \bar{g}(\theta) \), namely

\[
1 + A \theta e^{-\alpha \theta}, \text{ if } \bar{\sigma}^2 < \bar{\mu}, \text{ or } \quad e^{\alpha \theta}, \text{ if } \bar{\mu} < \bar{\sigma}^2,
\]

according as variance is low, or high. Here \( \alpha = |\bar{\sigma}^2 - \bar{\mu}| > 0 \). The substitution in the first of the two situations fits qualitatively with numeric observation on the form of \( \bar{g} \) (see the Figure 1); it agrees in the second situation with the general form observed in other Figures and also the asymptotic form as \( t \to \infty \).

Case (i): \( \alpha = \bar{\mu} - \bar{\sigma}^2 > 0 \). In this case the optimum time \( \theta \) is the solution of

\[
\theta/(1 - \alpha \theta) = 1 - \theta,
\]
a quadratic relation, leading to the explicit formula

\[
\theta = \theta(\alpha) := \frac{1}{2} - \frac{1}{\alpha} \left( -1 + \sqrt{1 + \frac{\alpha^2}{4}} \right),
\]
so that, as $\alpha$ increases from zero, the optimal time $\theta$ recedes from the mid-point towards the origin. That is, low volatilities bring the replenishment timing back.

**Case (ii):** $\alpha = \bar{\sigma}^2 - \bar{\mu} > 0$. The first-order condition here reduces to

$$(1 - e^{-\alpha \theta})/\alpha = 1 - \theta,$$

with a unique solution in the unit interval. Here we can use a quadratic approximation for the exponential term and solve for $\theta$ to obtain, for $\alpha < 2$, the approximation

$$\theta(\alpha) = \frac{1}{1 + \sqrt{1 - \alpha/2}},$$

so that the optimal choice of $\theta$ is close to the midpoint $\theta = 1/2$, when $\alpha$ is small, but advances, as $\alpha$ increases, towards unity (as a direct computation shows). That is, high volatilities bring the replenishment position forward (meaning that waiting longer, beyond the mid-term, is optimal for higher volatilities).

### 6 Asymptotic analysis: the proofs

In this section we give outline arguments (for further details, see the Appendix) leading to the Propositions 2 and 3 and Theorems B and C of Section 4.

**Lemma 1.** We have for fixed $\mu$

$$\lim_{\sigma \to 0^+} W(\mu, \sigma) = -\infty,$$

and

$$\lim_{\sigma \to 0^+} \sigma W(\mu, \sigma) = -\mu.$$  

This is proved directly from the definition of $W(\mu, \sigma)$. We now prove:

**Proposition 5.** For $\mu > 0$,

$$W(\mu, \sigma) = -\frac{\mu}{\sigma} + \frac{1}{2} \sigma + o(\sigma) \text{ as } \sigma \to 0^+. $$

**Proof.** For an intuition, note that for small enough $\sigma$ we have $e^{-\mu} \simeq e^{\sigma W - \frac{1}{2} \sigma^2}$ and so

$$W(\mu, \sigma) \sim -\frac{\mu}{\sigma} + \frac{1}{2} \sigma.$$

This argument can be embellished as follows. For any non-zero $\varepsilon$ let

$$W(\varepsilon) := -\frac{\mu}{\sigma} + \frac{1}{2} \sigma + \sigma \varepsilon,$$

so that $\sigma W(\varepsilon) - \frac{1}{2} \sigma^2 = -\mu + \sigma \varepsilon$ and hence

$$\sigma - W(\varepsilon) = \frac{\mu}{\sigma} + \frac{1}{2} \sigma - \sigma \varepsilon.$$

We will prove that for positive $\varepsilon$ we have, for small enough $\sigma$, that

$$W(-\varepsilon) < W(\mu, \sigma) < W(\varepsilon).$$

This is achieved by showing that for all small enough $\sigma$ the expression below has the same sign as $\varepsilon$:

$$D(\sigma) := F(W(\varepsilon), \sigma) - F(W(\mu, \sigma), \sigma) = F(W(\varepsilon), \sigma) - e^{-\mu}.$$  

This implies the Proposition. Now $D(0^+) = 0$ and, since $D(\sigma) = \Phi(W(\varepsilon) - \sigma) + e^{\sigma W(\varepsilon) - \frac{1}{2} \sigma^2} \Phi(-W(\varepsilon))$,

$$D'(\sigma) = e^{-\frac{1}{2}(W(\varepsilon) + \sigma^2)} \frac{1}{\sqrt{2\pi}} \left\{ -\frac{\mu}{\sigma^2} + \frac{1}{2} - \varepsilon \right\} + e^{-\mu + \sigma^2 \varepsilon} \{2 \sigma \varepsilon(1 + o(\sigma))}$$
\[+e^{-\mu+\sigma^2}e^{-\frac{1}{2}W(\varepsilon)^2}\left\{\frac{\mu}{\sigma^2} + \frac{1}{2} + \varepsilon\right\}.\]

Note that the first and third terms contain a factor \(\sigma \exp[-\mu^2/\sigma^2]\), which is small compared to \(\sigma\). So for small enough \(\sigma\) the derivative \(D'(\sigma)\) has the same sign as \(\varepsilon\). So the same is true for \(D(\sigma)\).

**Definitions.** Recall from (10) that \(\partial F/\partial W > 0\) and \(F(-\infty, \sigma) = 0, F(+\infty, \sigma) = 1\). Let \(m\) be fixed; for the purposes only of the current section it it convenient to define

\[\Phi(m) = 1 - \Phi(m)\]

and to introduce, also as a temporary measure, a variant form \(\hat{W}(m, \sigma)\) of \(W(m, \sigma)\) obtained by replacing \(e^{-\mu}\) in (10) by \(\hat{\Phi}(m)\) so that now

\[F(\hat{W}(m, \sigma), \sigma) = \hat{\Phi}(m) < 1. \tag{13}\]

**Claim.** For \(c\) any constant

\[\lim_{\sigma \to \infty} F(\sigma - c, \sigma) = \hat{\Phi}(c).\]

The proof is routine.

**Conclusion from claim.** Notice the consequences for the choices \(c = (1 \pm \varepsilon)m\). Since

\[\lim_{\sigma \to \infty} F(\sigma - (1 + \varepsilon)m, \sigma) = \hat{\Phi}((1 + \varepsilon)m) < \hat{\Phi}(m),\]

for large enough \(\sigma\) we have

\[F(\sigma - (1 + \varepsilon)m, \sigma) < F(W, \sigma).\]

Hence for large enough \(\sigma\) we have \(W > \sigma - (1 + \varepsilon)m\). Similarly, taking \(c = (1 - \varepsilon)m\) we obtain \(W < \sigma - (1 - \varepsilon)m\). Thus

\[W(m, \sigma) = \sigma - m\{1 + o(1)\}\]

This result can be improved by an argument similar to that of Proposition 2 by reference to

\[D(\sigma) = \Phi(\sigma - W) + e^{\sigma W - \frac{1}{2}\sigma^2} \Phi(-W) - \hat{\Phi}(m)\]

to yield the following.

**Proposition 6.** With the definition (13), for fixed \(m\)

\[\hat{W}(m, \sigma) = \sigma - m - \frac{1}{\sigma - m}\{1 + o(1)\} \quad \text{(as } \sigma \to \infty)\]

**Conclusion.** \(\hat{W}(m, \sigma) = W(\mu, \sigma)\) when \(m = \hat{\mu}\) where \(e^{-\mu} = \hat{\Phi}(m)\). Restating this equation as \(e^{-\mu} = 1 - \Phi(\hat{\mu}) = \Phi(-\mu)\), we see that \(\hat{\mu} > 0\) if and only if \(\mu > \ln 2\), since \(\hat{\mu} = -\Phi^{-1}(e^{-\mu})\); in particular for small \(\mu\) we thus have \(\hat{\mu} < 0\).

**Lemma 2.**

\[\lim_{\theta \to 0^+} \sqrt{\theta}\hat{W}(\theta) = 0 \quad \text{and} \quad \lim_{\theta \to 0^+} \hat{W}(\theta) = +\infty \quad \text{for fixed } \hat{\mu}, \sigma > 0.\]

This follows again by a routine argument starting from (10), but requires the claim below.

**Claim.**

\[L = \lim_{\theta \to 0^+} \sigma V(\theta) = 0.\]
Remark. This leaves the identification of the appropriate sign as a separate task. The proof is by con-

Proof of Theorem B. Differentiation of (10) with respect to $\theta$ gives

$$-\bar{\mu}e^{-\bar{\mu}W} = \varphi(W - \sigma)(W' - \sigma') + e^{\sigma W - \frac{1}{2}\sigma^2} \varphi(-W)(-W') + \Phi(-W)e^{\sigma W - \frac{1}{2}\sigma^2}[\frac{1}{2} \sigma^2 + (\sigma W')].$$

Now

$$\varphi(W - \sigma)\sigma' = e^{\sigma W - \frac{1}{2}\sigma^2} \varphi(W) \frac{\sigma}{2\sqrt{\theta}}$$

$$= \left(e^{\sigma W - \frac{1}{2}\sigma^2} \varphi(W) \frac{1}{W} \right) \frac{1}{\theta} \frac{W}{2} \to \bar{\mu} \cdot 0 = 0 \quad (\theta \to 0+),$$

using (10) and $\lim_{W \to +\infty} \varphi(W)/(W\Phi(-W)) = 1$ to deal with the bracketed term. Thus

$$-\bar{\mu} = \lim_{\theta \to 0+} [\Phi(-W)(\sigma W')].$$

Differentiation of (11) with respect to $\theta$ gives

$$\bar{g}' = [\bar{\sigma}^2 - \bar{\mu}]e^{(\bar{\sigma}^2 - \bar{\mu})} \Phi(W + \sigma) + e^{(\bar{\sigma}^2 - \bar{\mu})} \varphi(W + \sigma)(W' + \sigma') + e^{-\bar{\mu} - \sigma W + \frac{1}{2}\sigma^2} \varphi(-W)(-W') + e^{-\bar{\mu} - \sigma W + \frac{1}{2}\sigma^2} \Phi(-W)[\frac{1}{2} \sigma^2 - \bar{\mu} - (\sigma W')].$$

Now

$$\bar{g}' = [\bar{\sigma}^2 - \bar{\mu}] - \lim_{\theta \to 0+} \Phi(-W)(\sigma W') = \bar{\sigma}^2.$$ 

Lemma 3. If $\frac{1}{2} \sigma^2 \neq \bar{\mu}$, then

$$\lim_{\theta \to \infty} \bar{W}(\theta) = \pm \infty.$$

Remark. This leaves the identification of the appropriate sign as a separate task. The proof is by con-

Lemma 4. $\lim_{\theta \to \infty} \bar{W}(\theta) - \sigma = -\infty$.

Lemma 5. If $\frac{1}{2} \sigma^2 = \bar{\mu}$, then $\lim_{\theta \to \infty} \bar{\sigma} \sqrt{\theta} \bar{W}(\theta) = \log 2$.

Conclusion 1. If $\lim_{\theta \to \infty} \bar{W}(\theta) = -\infty$, then for $\bar{\mu} > \frac{1}{2}$

$$\bar{W}(\theta) = -\frac{\bar{\mu} - \frac{1}{2} \sigma^2}{\sigma} \sqrt{\theta} + o(\sqrt{\theta}).$$

Lemma 6. If $\sigma^2 < 2\bar{\mu}$, then

$$\lim_{\theta \to \infty} e^{-\sigma \bar{W} + \left(\frac{1}{2} \sigma^2 - \bar{\mu}\right)\theta} = 1.$$ 

This follows directly from (10) and Lemmas 3 and 4.

Proof of Theorem C. Lemma 6 establishes case (ii) of Theorem C. Next we note:

Conclusion 2. If $\lim_{\theta \to \infty} \bar{W}(\theta) = +\infty$, then for $\bar{\mu} < \frac{1}{2} \bar{\sigma}^2$

$$\bar{W}(\theta) = (\bar{\sigma} - \sqrt{2\bar{\mu}}) \sqrt{\theta} + O(1/\sqrt{\theta}).$$
Case (iii) of Theorem C follows from this estimate. Combining (ii) and (iii) gives (i). Turning to case (iv), if $\tilde{\sigma}^2 = 2\tilde{\mu}$, then as $\theta \to \infty$ we have $\sigma + \tilde{W}(\theta) \to +\infty$, by Lemma 5, so since $\lim_{\theta \to \infty} e^{\tilde{\sigma} \tilde{W}(\theta)} = 2$, and appealing to standard asymptotic estimates of $\Phi(x)$, the censor equation (10) yields

$$g(\theta) = e^{(\tilde{\sigma}^2 - \tilde{\mu})\theta} \Phi(\sigma + \tilde{W}(\theta)) + e^{-(\tilde{\sigma}^2 - \tilde{\mu})\theta} \Phi(-\tilde{W}(\theta))$$

$$= e^{\tilde{\sigma} \theta} \Phi(\sigma + \tilde{W}(\theta)) + e^{-\sigma \tilde{W}} \Phi(-\tilde{W}(\theta))$$

$$= e^{\sigma \theta} + \frac{1}{4} + o(1/\sigma).$$

7 Censor comparative statics: reprieve

This section considers the sensitivity of $\tilde{b}(\mu, \sigma)$ to $\mu$ and $\sigma$, and the dependence of $\tilde{b}(\theta) = \tilde{b}(\mu, \sigma, \sqrt{\theta})$ on $\theta$ as given in Section 3.

**Theorem 2.** The censor $\tilde{b}(\mu, \sigma)$ is decreasing in the drift $\mu$.

**Proof.** The derivative of $\tilde{b} = \exp(\sigma W + \mu - \frac{1}{2} \sigma^2)$ with respect to $\mu$ is positive iff:

$$-\sigma \frac{\partial W(\mu, \sigma)}{\partial \mu} > 1. \quad (14)$$

But differentiation of (10) and

$$\Phi(W(\mu, \sigma) - \sigma) = e^{\sigma W - \frac{1}{2} \sigma^2} \Phi(W(\mu, \sigma))$$

yield

$$1 = \tilde{b} \Phi(-W(\mu, \sigma)) \left( -\sigma \frac{\partial W}{\partial \mu} \right).$$

So (14) holds iff $\tilde{b} \Phi(-W(\mu, \sigma)) < 1$. But the latter follows from (8).

**Theorem 3.** The censor $\tilde{b}(\mu, \sigma)$ is increasing in the standard deviation $\sigma$.

**Proof.** Differentiating $\tilde{b} = \exp(\sigma W(\mu, \sigma) + \mu - \frac{1}{2} \sigma^2)$ with respect to $\sigma$ yields

$$\frac{\partial \tilde{b}}{\partial \sigma} = \tilde{b}(\mu, \sigma) \left\{ \sigma \frac{\partial W}{\partial \sigma} + W(\mu, \sigma) - \sigma \right\}.$$ 

Differentiating also the censor equation (10) with respect to $\sigma$, we obtain after some cancellations

$$\Phi(W(\mu, \sigma) - \sigma) = e^{\sigma W - \frac{1}{2} \sigma^2} \Phi(-W(\mu, \sigma)) \left\{ W(\mu, \sigma) + \sigma \frac{\partial W}{\partial \sigma} - \sigma \right\}. $$

The bracketed term appearing here and earlier is thus positive, and so

$$\frac{\partial \tilde{b}(\mu, \sigma)}{\partial \sigma} > 0.$$

Using $\Phi(\sigma - W) = e^{\sigma W - \frac{1}{2} \sigma^2} \Phi(W)$ (cf. Section 2) we note the identity

$$W(\mu, \sigma) + \sigma \frac{\partial W}{\partial \sigma} - \sigma = \frac{\Phi(W(\mu, \sigma))}{\Phi(-W(\mu, \sigma))} = H(W(\mu, \sigma)),$$

where $H(x)$ denotes the normal hazard rate ($\phi(x)/\Phi(-x)$). Since $H(x) > x$ for all $x$, equation (15) gives $\partial W/\partial \sigma > 1$ for $\sigma > 0$. Recalling from Section 2 that $\partial W/\partial \mu < 0$, we have the two results:
Theorem 4. The two functions \( \sigma W(\mu, \sigma) - \frac{1}{2} \sigma^2, \quad W(\mu, \sigma) - \sigma \) are increasing in \( \sigma \) for \( \sigma > 0 \).

Theorem 5. The normal censor \( W(\mu, \sigma) \) is increasing in standard deviation and decreasing with drift.

Our final result is the following.

Theorem 6. The censor \( \tilde{b}(\theta) = \tilde{b}(\mu \theta, \sigma \sqrt{\theta}) \) is either unimodal or increasing on the interval \( 0 < \theta < \infty \), according as \( \mu \geq \frac{1}{2} \sigma^2 \), or as \( \mu < \frac{1}{2} \sigma^2 \).

Proof. Using \( \tilde{b} \phi(W) = e^\mu \phi(W - \sigma) \) and applying the Chain Rule to \( \tilde{b}(\theta) = \tilde{b}(\mu \theta, \sigma \sqrt{\theta}) \), we obtain

\[
\theta \Phi(-W) rac{d \tilde{b}(\theta)}{d \theta} = -\mu \{ e^\mu \Phi(W - \sigma) \} + \frac{1}{2} \sigma \tilde{b} \phi(W).
\]

The stationarity condition for \( \tilde{b}(\theta) \) can be written using the normal hazard rate \( H(x) = \phi(x)/\Phi(-x) \) as

\[
\mu = \frac{1}{2} \sigma H(-W(\mu, \sigma) + \sigma), \tag{16}
\]

where \( \mu = \mu \theta \) and \( \sigma = \sigma \sqrt{\theta} \), and \( W(\mu, \sigma) \) is the normal censor as in (10).

We now regard \( \mu \) and \( \sigma \) as free variables and let \( \kappa := \mu / \sigma^2 \) be the dispersion parameter. In this setting we seek a stationary point \( \theta \) of \( \tilde{b}(\theta) \) by first finding the values \( \mu = \mu^* \) and \( \sigma = \sigma^* \) which satisfy the equation (16) simultaneously with the equation:

\[
\mu = \kappa \sigma^2. \tag{17}
\]

We will show that this is possible (uniquely) if and only if \( \kappa \geq 1/2 \) (i.e. \( \mu \geq \frac{1}{2} \sigma^2 \)). Thus for \( \sigma^2 > 2 \mu \) the function \( \tilde{b}(\theta) \) is increasing, but otherwise has a unique maximum at \( \theta = \mu^*/\mu = \sigma^{*2}/\sigma^2 \).

We begin by noting that (16) defines an implicit function \( \mu = \mu(\sigma) \) for all \( \sigma > 0 \). Indeed, elimination of \( \mu \) between (10) and (16) leads to

\[
\exp \left( -\frac{1}{2} \sigma H(-w + \sigma) \right) = F(w, \sigma), \tag{18}
\]

and then routine analysis shows that there is a unique solution \( w = \omega(\sigma) \) of (18). Since \( \partial W / \partial \mu < 0 \), we may recover \( \mu(\sigma) > 0 \), for \( \sigma > 0 \), from \( \omega(\sigma) := W(\mu(\sigma), \sigma) \).

Linearization of both sides of (18) around \( \sigma = 0 \), yields the equation

\[
H(-w) = 2(\phi(w) + w \Phi(-w)),
\]

with unique solution \( w = \omega(0) = 0 \). Hence \( \lim_{\sigma \to 0} W(\mu(\sigma), \sigma) = 0 \) and so, for small \( \sigma \), we have the approximation to (16) given by the convex function

\[
\mu(\sigma) := \frac{1}{2} \sigma H(\sigma).
\]

Numerical investigation of the positive function \( w = \omega(\sigma) \) finds its maximum to be 0.051 for \( \sigma \) approximately 2.547. To see why, rewrite (18) in the equivalent form:

\[
\exp \left( \frac{1}{2} \sigma^2 - \frac{1}{2} \sigma H(-w + \sigma) \right) = \frac{\Phi(w - \sigma)}{\varphi(\sigma) \sqrt{2\pi}} + e^{\sigma w} \Phi(-w).
\]

For fixed \( 0 \leq w \leq 1 \), and large \( \sigma \), the left-hand side is close to \( e^{\frac{1}{2}(\sigma w - 1)} \), in view of the asymptotic over-approximation \( (x + 1/x) \) for \( H(x) \) (when \( x \) is large), whereas the first term on the right is of the order of \( 1/(\sigma \sqrt{2\pi}) \). Neglecting the latter, and replacing \( \Phi(-w) \) by \( \frac{1}{2} \), the solution for \( w \) may be estimated by \( (2 \log 2 - 1)/\sigma \).
Finally, using the same asymptotic approximation for \( H(\sigma) \), we may over-approximate \( \frac{1}{2} \sigma H(\sigma - \omega(\sigma)) \) by \( \frac{1}{2} \sigma^2 + \frac{1}{2} \). From here we may conclude that, for \( \kappa > \frac{1}{2} \), the equations \([16]\) and \([17]\) have a solution with a crude over-estimate for \( \sigma^* \) given by:

\[
\sigma^2 = \frac{1}{2\kappa - 1}.
\]

The supporting line \( \mu = \frac{1}{2} H(0) \sigma \) provides the crude under-estimate \( \sigma = 1/\kappa \sqrt{2\pi} \). For the special case \( \kappa = \frac{1}{2} \) the solution to \([16]\) and \([17]\) is \( \sigma^* = 4.331 \). For \( \kappa < \frac{1}{2} \) there is no solution, since \( \frac{1}{2} \sigma H(\sigma) > \kappa \sigma^2 \) for \( \sigma > 0 \).

**Acknowledgement.** It is a pleasure to thank Alain Bensoussan for his very helpful advice and encouragement.

**Postscript.** Harold Wilson (1916-1995, prime minister 1964-70 and 1974-76) famously always emphasized the importance of keeping his options open.

**Appendix**

**Proof of Proposition 6.** For convenience put

\[
R(W, \sigma) := \sqrt{2\pi} F(W, \sigma) = \int_{-W + \sigma}^{\infty} e^{-\frac{1}{2}x^2} dx + e^{\sigma W - \frac{1}{2} \sigma^2} \int_{W}^{\infty} e^{-\frac{1}{2}x^2} dx.
\]

Consider an arbitrary non-zero \( \varepsilon \); let \( W_\varepsilon := \sigma - m - \delta \) and put

\[
\delta := \frac{1 - \varepsilon}{\sigma - m}.
\]

Now, with \( D \) as in §6,

\[
D(\sigma) = \left( \int_{-W + \sigma}^{\infty} e^{-\frac{1}{2}x^2} dx + e^{\sigma W - \frac{1}{2} \sigma^2} \int_{W}^{\infty} e^{-\frac{1}{2}x^2} dx \right) - \int_{m}^{\infty} e^{-\frac{1}{2}x^2} dx
\]

\[
= \left( \int_{m + \delta}^{\infty} e^{-\frac{1}{2}x^2} dx - \int_{m}^{\infty} e^{-\frac{1}{2}x^2} dx \right) + e^{\sigma(m - \delta)} - \frac{1}{\sqrt{\pi}} \left( 1 + O \left( \frac{1}{\sigma^2} \right) \right)
\]

\[
= -\delta e^{-\frac{1}{2}(m+\delta)^2} + O(\delta^2) + e^{\sigma(m - \delta)} - \frac{1}{\sqrt{\pi}} \left( 1 + O \left( \frac{1}{\sigma^2} \right) \right)
\]

\[
= -\delta e^{-\frac{1}{2}(m+\delta)^2} + O(\delta^2) + \frac{1}{\sigma - m - \delta} e^{-\frac{1}{2}(m+\delta)^2} \left( 1 + O \left( \frac{1}{\sigma^2} \right) \right)
\]

\[
= \left( \frac{1}{\sigma - m - \delta} - \delta \right) e^{-\frac{1}{2}(m+\delta)^2} + O(\delta^2) + \frac{1}{\sigma - m - \delta} e^{-\frac{1}{2}(m+\delta)^2} \left( 1 + O \left( \frac{1}{\sigma^2} \right) \right)
\]

\[
= \frac{1}{\sigma - m} - \frac{1 - \varepsilon}{\sigma - m} e^{-\frac{1}{2}(m+\delta)^2} + O(\delta^2) + \frac{1}{\sigma - m} e^{-\frac{1}{2}(m+\delta)^2} \left( 1 + O \left( \frac{1}{\sigma^2} \right) \right)
\]

\[
= \frac{\varepsilon(\sigma - m)^2 + (1 - \varepsilon)^2}{(\sigma - m)^2 - (1 - \varepsilon)(\sigma - m)} e^{-\frac{1}{2}(m+\delta)^2} + O(\delta^2)
\]

\[
= \frac{\varepsilon}{\sigma - m} e^{-\frac{1}{2}(m+\delta)^2} + O(\delta^2),
\]

and this has the same sign as \( \varepsilon \). Thus, for \( \varepsilon > 0 \),

\[
R(W_{-\varepsilon}, \sigma) < R(W(m, \sigma), \sigma) < R(W_\varepsilon, \sigma),
\]

14
and so, since $\partial R(W, \sigma)/\partial W > 0$,

$$W_{-\epsilon} < W(m, \sigma) < W_{\epsilon}. \quad \Box$$

**Proof of Lemma 2.** We begin with the associated Claim (§6 above), for which we need first to put

$$V := V(\theta) = \bar{W}(\theta) - \sigma \sqrt{\theta},$$

and then to note (by the definition of the normal censor in §3):

$$(e^{-\mu} - 1) - \{\Phi(-\sigma - V) - \Phi(-V)\} = [e^{\sigma V + \frac{1}{2} \sigma^2} - 1]\Phi(-\sigma - V). \quad (\ast)$$

From here, for some $V^*$ between $V$ and $V + \sigma$,

$$(e^{-\mu} - 1) - \sigma \varphi(V^*) = [e^{\sigma V + \frac{1}{2} \sigma^2} - 1]\Phi(-\sigma - V),$$

so that

$$-\mu \theta + \sigma \varphi(V^*) \sim [e^{\sigma V + \frac{1}{2} \sigma^2} - 1]\Phi(-\sigma - V).$$

**Proof of Claim.** Suppose $L = \lim_{\theta \to 0^+} \sigma V(\theta) \neq 0$ along a sequence of values of $\theta$; then

$$V(\theta) \approx L/(\sigma \sqrt{\theta}) : \quad \sigma \varphi(V^*) \sim \sigma \sqrt{\theta} \exp(-L^2/\sigma^2 \theta)/\sqrt{2\pi}$$

and so

$$-\mu \theta \{1 - (\sigma/\sqrt{\theta}) \exp(-L^2/\sigma^2 \theta)/\sqrt{2\pi} \} \sim -\mu \theta.$$

So, for small enough $\theta$,

$$[e^{\sigma V + \frac{1}{2} \sigma^2} - 1]\Phi(-\sigma - V) < 0,$$

so that $\bar{V} \leq 0$. Suppose first that $\bar{V} = -\infty$; then $L = 0$, since $\Phi(\infty) = 1$ reduces equation $(\ast)$ to

$$0 = (e^{L} - 1),$$

contradicting $L \neq 0$. Likewise, from equation $(\ast)$, the finiteness of $\bar{V}$ yields $L = 0$, a final contradiction.

$\Box$ 

Turning to Lemma 2 proper, put $\bar{V} := \lim_{\theta \to 0^+} V(\theta)$. As above

$$(e^{-\mu} - 1) + \sigma \varphi(V^*) \sim [e^{\sigma V + \frac{1}{2} \sigma^2} - 1]\Phi(-\sigma - V).$$

By the Claim, $\sigma V$ is small; so we may expand the exponential and, dividing by $\sigma = \sigma \sqrt{\theta}$, obtain

$$-\mu \theta \sqrt{\theta} + \varphi(V^*) = (V + \frac{1}{2} \sigma)\Phi(-\sigma - V).$$

If $V \to \bar{V}$ a finite limit, then the Mills ratio (hazard rate), defined by

$$H(\bar{V}) := \frac{\varphi(\bar{V})}{\Phi(-\bar{V})},$$

satisfies $H(\bar{V}) = \bar{V}$, a contradiction, since the ratio is always greater than $\bar{V}$. Thus the limit $\bar{V}$ must be infinite, hence $\varphi(\bar{V}) = 0$. So $\bar{V} = +\infty$, as otherwise $\bar{V} = -\infty$ leads to the contradiction

$$0 = \varphi(\bar{V}) = \bar{V} \cdot 1. \quad \Box$$

**Proof of Lemma 3.** As in the definition of the normal censor

$$e^{-\mu \theta} = \Phi(\bar{W}(\theta) - \sigma) + e^{\sigma W(\theta) - \frac{1}{2} \sigma^2} \Phi(-\bar{W}(\theta)),$$
or
\[ e^{-\mu - \sigma \bar{W}(\theta)} + \frac{1}{2} \sigma^2 = e^{-\sigma \bar{W}(\theta)} + \frac{1}{2} \sigma^2 \Phi(\bar{W}(\theta) - \sigma) + \Phi(-\bar{W}(\theta)), \quad (**) \]
\[ e^{-\mu - \sigma \bar{W}(\theta)} + \frac{1}{2} \sigma^2 = \Phi(-\bar{W}(\theta)) + \varphi(\bar{W}(\theta))/H(\sigma - \bar{W}(\theta)), \]
where, as above, \( H(\cdot) \) denotes the hazard rate. Assume that \( \bar{W}(\theta) \to \bar{w} \). We are to prove that \( \bar{w} \) is not finite. We argue by cases.

Case 1: \( \frac{1}{2} \sigma^2 > \mu \). The left hand side is unbounded, whereas the right-hand side is bounded for large \( \theta \) by
\[ 1 + \varphi(\bar{w})/(\sigma \sqrt{\theta - \bar{w}}). \]

Case 2: \( \frac{1}{2} \sigma^2 < \mu \). Letting \( \theta \to \infty \) gives the contradiction:
\[ 0 = \Phi(-\bar{w}) + 0. \quad \Box \]

**Proof of Lemma 4.** As before, if \( V := \bar{W}(\theta) - \sigma \), then
\[ e^{-\mu} = \Phi(V) + e^{\sigma V + \frac{1}{2} \sigma^2} \Phi(-\sigma - V). \]

Suppose \( V \to -\infty \) is false. Then either \( V \to \infty \), or \( V \to V \), a finite limit. In either case we have
\[ e^{\sigma V + \frac{1}{2} \sigma^2} \Phi(-\sigma - V) \leq e^{-\frac{1}{2} V^2} \varphi(V + \sigma)/(V + \sigma) \to 0, \]
as \( \theta \to \infty \) (since \( \sigma \to \infty \)). This implies that \( 0 = \Phi(V) \), a contradiction in either case. So \( V \to -\infty \). \( \Box \)

**Proof of Lemma 5.** As before suppose \( \bar{W}(\theta) \to \bar{w} \). If \( \bar{w} < 0 \) (possibly infinite), then we have in the limit \( \Phi(\bar{w}) = \infty \), a contradiction. If \( 0 < \bar{w} < \infty \), then by \( (**) \) above \( 0 = \Phi(-\bar{w}) \), again a contradiction. This leaves two possibilities: either \( \bar{w} = \infty \) or \( \bar{w} = 0 \).

Suppose the former. Noting that
\[ 1 = \lim_{\theta \to \infty} \left[ e^{\mu \theta} \Phi(\bar{W}(\theta) - \sigma) + e^{\sigma \bar{W}(\theta)} \Phi(-\bar{W}(\theta)) \right], \]
then \( e^{\sigma \bar{W}(\theta)} \Phi(-\bar{W}(\theta)) \) is bounded. But
\[ \lim_{\theta \to \infty} \varphi(\bar{W}(\theta)) \Phi(-\bar{W}(\theta)) = \lim_{\theta \to \infty} \frac{\varphi(\bar{W}(\theta)) e^{-\frac{1}{2} \bar{W}(\theta)^2}}{\bar{W}(\theta) \sqrt{2\pi}} = \lim_{\theta \to \infty} \frac{\varphi(\bar{W}(\theta)) e^{-\frac{1}{2} \bar{W}(\theta)^2}}{\bar{W}(\theta) \sqrt{2\pi}} = \infty, \]
by Lemma 4 and by our assumption, a contradiction.

Thus after all \( \bar{w} = 0 \). So
\[ 1 = \lim_{\theta \to \infty} \left[ e^{\mu \theta} \Phi(\bar{W}(\theta) - \sigma) + e^{\sigma \bar{W}(\theta)} \Phi(0) \right] \]
\[ = \lim_{\theta \to \infty} \frac{e^{\frac{1}{2} \sigma^2 \theta} e^{-\frac{1}{2} \bar{W}(\theta)^2} - \frac{1}{2} \sigma \bar{W}(\theta)}{\bar{W}(\theta) \sqrt{2\pi}} + e^{\sigma \bar{W}(\theta)} \Phi(0) \]
\[ = \lim_{\theta \to \infty} e^{\sigma \bar{W}(\theta)} \left[ \frac{1}{2} - \frac{1}{\sigma \sqrt{2\pi}} \right] = \frac{1}{2} \lim_{\theta \to \infty} e^{\sigma \bar{W}(\theta)}. \quad \Box \]

**Proof of Conclusion 1:** For any \( \varepsilon \), put
\[ W_\varepsilon(\theta) := \frac{\bar{\mu} - \frac{1}{2} \sigma^2 + \varepsilon \sqrt{\theta}}{\bar{\sigma}}: \quad e^{-\mu \theta + \sigma W_\varepsilon(\theta) + \frac{1}{2} \sigma^2} = e^{\varepsilon \sqrt{\theta}}. \]

For \( \varepsilon > 0 \) and large enough \( \theta \),
\[ e^{-\mu \theta - \sigma \bar{W}(\theta) + \frac{1}{2} \sigma^2} = \Phi(-\bar{W}(\theta)) + \varphi(-\bar{W}(\theta))/H(\sigma - \bar{W}(\theta)) \]

16
\[ e^\epsilon \sqrt{\sigma} = e^{-\frac{\partial}{\sigma} + \sigma W(\theta)} + \frac{1}{2} \sigma^2 : \quad -\bar{W}(\theta) < W(\theta). \]

On the other hand, for \( \epsilon < 0 \) and large enough \( \theta \)

\[ e^{-\bar{\mu} \theta - \sigma \bar{W}(\theta)} + \frac{1}{2} \sigma^2 > \Phi(-\bar{W}(\theta)) > e^{\epsilon \sqrt{\sigma}} = e^{-\bar{\mu} \theta + \sigma W(\theta)} + \frac{1}{2} \sigma^2 : \quad W(\theta) < -\bar{W}(\theta). \]

**Proof of Lemma 6.** Here, for large \( \theta \),

\[ \bar{W}(\theta) = -\left( \mu - \frac{1}{2} \sigma^2 \right) \sqrt{\theta} + o(\sqrt{\theta}). \]

So

\[ \sigma + \bar{W}(\theta) = \frac{\frac{3}{2} \sigma^2 - \bar{\mu}}{\sigma} \sqrt{\theta} + o(\sqrt{\theta}), \quad \sigma - \bar{W}(\theta) = \frac{\frac{1}{2} \sigma^2 + \bar{\mu}}{\sigma} \sqrt{\theta} + o(\sqrt{\theta}). \]

Furthermore, rewriting the normal censor equation,

\[ e^{-\bar{\mu} \theta + \frac{1}{2} \sigma^2 - \sigma \bar{W}(\theta)} = e^{-\bar{\mu} \theta + \frac{1}{2} \sigma^2} \Phi(-\bar{\sigma} + \bar{W}(\theta))/\Phi(-\bar{W}(\theta) + \sigma) + \Phi(-\bar{W}(\theta)). \]

So by Lemma 4, and since \( \bar{W}(\theta) \to -\infty \),

\[ \lim_{\theta \to \infty} e^{-\bar{\mu} \theta + \frac{1}{2} \sigma^2 - \sigma \bar{W}(\theta)} = 1. \]

In fact, we have \( e^{-\bar{\mu} \theta + \frac{1}{2} \sigma^2 - \sigma \bar{W}(\theta)} = 1 + o(1/\sqrt{\theta}). \)

**Proof of Conclusion 2.** As \( 2\bar{\mu} > \bar{\sigma}^2 \), note that

\[ (\bar{\sigma}^2 - \bar{\mu}) - \frac{1}{2} \left( \frac{3}{2} \bar{\sigma}^2 - \bar{\mu} \right) = (\bar{\sigma}^2 - \bar{\mu}) - \frac{9}{8} \bar{\sigma}^2 - \frac{1}{2} \bar{\mu}^2 + \frac{3}{2} \bar{\mu} = -\frac{1}{8} \bar{\sigma}^2 + \frac{1}{2} \bar{\mu} \bar{\sigma}^2 - \frac{1}{2} \bar{\mu}^2 < 0; \]

indeed,

\[ \bar{\mu}^2 - \bar{\mu} \bar{\sigma}^2 + \frac{1}{4} \bar{\sigma}^4 = (\bar{\mu} - \frac{1}{2} \bar{\sigma}^2)^2 > 0. \]

From §6,

\[ s(\theta) = e^{(\sigma^2 - \bar{\mu}) \theta} \Phi(\sigma + \bar{W}(\theta)) + \Phi(-\bar{W}(\theta) + o(1/\sqrt{\theta}) \]

\[ = e^{(\sigma^2 - \bar{\mu}) \theta} \Phi(\frac{3}{2} \frac{\sigma^2 - \bar{\mu}}{\bar{\sigma}^2} \sqrt{\theta}) + \Phi(\frac{\bar{\mu} - \frac{1}{2} \bar{\sigma}^2}{\bar{\sigma}^2} \sqrt{\theta}) + o(1/\sqrt{\theta}). \]

Applying the asymptotic expansion [11]

\[ \Phi(x) \sim 1 - \frac{e^{-x^2/2}}{x \sqrt{2 \pi}} \quad (\text{as } x \to +\infty), \]

yields

\[ e^{(\sigma^2 - \bar{\mu}) \theta} \Phi(\frac{3}{2} \frac{\sigma^2 - \bar{\mu}}{\bar{\sigma}^2} \sqrt{\theta}) = e^{(\sigma^2 - \bar{\mu}) \theta} + o(1/\sqrt{\theta}). \]

\[ \square \]
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