This article collects a few observations concerning Hitchin’s generalized Calabi–Yau structures in dimension four. I became interested in these while thinking about the moduli space of K3 surfaces (with metric and B-field) and its relation to the moduli space of $N = (2, 2)$ SCFT.

Roughly, a generalized Calabi–Yau structure is a very special even non-degenerate complex form, which usually will be called $\varphi$. The main examples are $\varphi = \sigma$, where $\sigma$ is the holomorphic two-form on a K3 surface, and $\varphi = \exp(i\omega)$, where $\omega$ is an arbitrary symplectic form. A generalized K3 surface consists of a pair $(\varphi, \varphi')$ of generalized Calabi–Yau structures satisfying certain orthogonality conditions which are modeled on the relation between the holomorphic two-form $\sigma$ on a K3 surface and a Ricci-flat Kähler form $\omega$.

As was explained by Aspinwall and Morrison (cf. [2, 11, 17]), the moduli space $\mathcal{M}_{(2, 2)}$ of $N = (2, 2)$ SCFT fibers over the moduli space $\mathcal{M}_{(4, 4)}$ of $N = (4, 4)$ SCFT. The fibre of the projection $\mathcal{M}_{(2, 2)} \to \mathcal{M}_{(4, 4)}$ is isomorphic to $S^2 \times S^2$. Using the period map, the moduli space of B-field shifts of hyperkähler metrics $\mathcal{M}_{\text{HK}}$ can be identified with an open dense subset of $\mathcal{M}_{(4, 4)}$. For any chosen hyperkähler metric $g \in \mathcal{M}_{\text{HK}}$ there is an $S^2$ worth of complex structures making this metric a Kähler metric. Thus, the moduli space $\mathcal{M}_{K3}$ of B-field shifts of complex K3 surfaces endowed with a metric fibers over $\mathcal{M}_{\text{HK}}$ and the fibre of $\mathcal{M}_{K3} \to \mathcal{M}_{\text{HK}}$ is isomorphic to $S^2$. Any point in $\mathcal{M}_{K3}$ gives rise to an $N = (2, 2)$ SCFT and the induced inclusion $\mathcal{M}_{K3} \subset \mathcal{M}_{(2, 2)}$ is compatible with the two projections. Mirror symmetry is realized as a certain discrete group action on $\mathcal{M}_{(2, 2)}$ or $\mathcal{M}_{(4, 4)}$.

Due to the fact that $\mathcal{M}_{K3} \to \mathcal{M}_{\text{HK}}$ is only an $S^2$-fibration and not an $S^2 \times S^2$-fibration as is $\mathcal{M}_{(2, 2)} \to \mathcal{M}_{(4, 4)}$, one soon realizes that points in $\mathcal{M}_{K3}$ might be mirror symmetric to points that are no longer in $\mathcal{M}_{K3}$. We will explain that Hitchin’s generalized Calabi–Yau structures allow to give a geometric meaning also to those points.

From a slightly different point of view, one could think of generalized Calabi–Yau structures as geometric realizations of points in the extended period domain which is obtained by passing from the period domain $Q \subset \mathbb{P}(H^2(M, \mathbb{C}))$, an open subset of a smooth quadric, to the analogous object $\tilde{Q} \subset \mathbb{P}(H^*(M, \mathbb{C}))$. The latter is defined in terms of the Mukai pairing on $H^*(M, \mathbb{Z})$. Recall that due to results of Siu, Todorov, and others, the period domain $Q$ is essentially the moduli space of marked K3 surfaces. The
larger moduli space corresponding to $\tilde{Q}$ contains the $B$-field shifts of those as a hyperplane section. Its complement is the open subset of $B$-field shifts of symplectic structures on a K3 surface. Thus, complex structures and symplectic structures are parametrized by the same moduli space and the discrete group $O(H^*(M,\mathbb{Z}))$ acting on $\tilde{Q}$ frequently interchanges these two.

In particular, we will prove the following result:

**Theorem 0.1.** The period map $\mathcal{P}_{\text{gen}} : \mathcal{N}_{\text{gen}} \to \tilde{Q}$ from the moduli space of generalized Calabi–Yau structures $\mathcal{N}_{\text{gen}}$ on a K3 surface $M$ to the extended period domain $\tilde{Q} \subset \mathbb{P}(H^*(M,\mathbb{C}))$ is surjective. Moreover, $\mathcal{N}_{\text{gen}}$ admits a natural symplectic structure $\Omega$ with respect to which the moduli space of symplectic structures $\text{Sympl}(M) \subset \mathcal{N}_{\text{gen}}$ is Lagrangian.

It might be worth pointing out that the B-field, from a mathematical point of view a slightly mysterious object, is indispensable when we want to view complex and symplectic structures as special instances of a more general notion. I certainly hope and expect that this unified treatment of symplectic and complex structures on K3 surfaces leads to a better understanding of both.

Here is the plan of the paper. In the first section we recall the notion of generalized Calabi–Yau structures, which is due to Hitchin, and discuss the two main examples (and their $B$-field transforms) alluded to above. In Section 2 after introducing the notion of generalized Calabi–Yau structures of (hyper)kähler type, we prove a Global Torelli theorem for generalized Calabi–Yau structures on K3 surfaces. We also discuss generalizations of the existence theorems of Siu and Yau. Moduli spaces of generalized Calabi–Yau structures are treated in Section 3. We define various period maps and show how they can be used to relate the moduli space of generalized K3 surfaces to the moduli space of $N = (2,2)$ SCFT. In Section 4 we argue that these new moduli spaces are well suited to interpret Orlov’s criterion on the equivalence of derived categories of algebraic K3 surfaces. In order to treat the twisted, still conjectural version of it, we introduce the Picard group and the transcendental lattice of a generalized Calabi–Yau structure. In the last section a natural symplectic (hermitian) structure on the moduli space of generalized Calabi–Yau structures is defined. It turns out that the part of the moduli space that parametrizes generalized Calabi–Yau structures of the form $\exp(i\omega)$, with $\omega$ a symplectic form, is Lagrangian.

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1. Hitchin’s generalized Calabi–Yau structures

Throughout this paper we will assume that $M$ is the differentiable manifold underlying a K3 surface. E.g. we could think of $M$ as the differentiable
fourfold defined by $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$ in $\mathbb{P}^3$. (Due to a result of Kodaira one knows that any K3 surface is diffeomorphic to $M$.) We will also fix the natural orientation induced by the complex structure. This will enable us to speak about positivity and negativity of four-forms on $M$.

We take the liberty to change some of Hitchin’s original conventions in order to make the theory compatible with the standard theory of K3 surfaces.

For two even complex forms $\varphi, \psi \in A^2_\mathbb{C}(M)$ one defines
\[
\langle \varphi, \psi \rangle := -\varphi_0 \wedge \psi_4 + \varphi_2 \wedge \psi_2 - \varphi_4 \wedge \psi_0 \in A^4_\mathbb{C}(M),
\]
where $\varphi_i$ and $\psi_i$ denote the parts of degree $i$ of $\varphi$ and $\psi$, respectively. This is the Mukai pairing on the level of forms.

**Definition 1.1.** A generalized Calabi–Yau structure on the four-dimensional manifold $M$ is a closed even form $\varphi \in A^2_\mathbb{C}(M)$ such that
\[
\langle \varphi, \varphi \rangle = 0 \quad \text{and} \quad \langle \varphi, \varphi' \rangle > 0.
\]

Note that such a $\varphi$ is not necessarily homogeneous and that its degree zero term is constant.

**Remark 1.2.** Hitchin defines also odd generalized Calabi–Yau structures, but they are of no importance for our purposes, as in dimension four they only exist on manifolds with non-trivial first cohomology.

The notion of generalized Calabi–Yau structures embraces symplectic and complex structures:

**Example 1.3.** i) Every symplectic structure $\omega$ on $M$ induces a generalized Calabi–Yau structure $\varphi = \exp(i\omega) = 1 + i\omega - (1/2) \cdot \omega^2$. In order to see that any symplectic structure on $M$ defines the same orientation, i.e. that $\omega^2 > 0$, one can use Seiberg-Witten theory. For our purpose we might as well just restrict to those.

ii) Let $X$ be a K3 surface. Thus, $X$ is given by a complex structure $I$ on $M$. The holomorphic two-form $\sigma$, which is unique up to scaling, defines a generalized Calabi–Yau structure $\varphi = \sigma$.

These two examples are very different from each other due to the fact that in i) the constant term is non-trivial, and after scaling we might even assume that $\varphi_0 = 1$, whereas the second example $\varphi = \sigma$ has trivial constant term. In most of the arguments that will follow, one has to distinguish between these two cases.

If $B$ is a two-form, then $\exp(B)$ acts on $A^*_\mathbb{C}(M)$ by exterior product, i.e.
\[
\exp(B) \cdot \varphi = (1 + B + (1/2) \cdot B \wedge B) \wedge \varphi.
\]

It is easy to see that multiplication with $\exp(B)$ is orthogonal with respect to the pairing $\langle \cdot, \cdot \rangle$, i.e.
\[
\langle \exp(B) \cdot \varphi, \exp(B) \cdot \varphi' \rangle = \langle \varphi, \varphi' \rangle \in A^4_\mathbb{C}(M)
\]
for all forms $\varphi, \varphi'$. This immediately yields the following observation due to Hitchin.

**Lemma 1.4.** For any generalized Calabi–Yau structure $\varphi$ and any real closed two-form $B$, the form $\exp(B) \cdot \varphi$ is again a generalized Calabi–Yau structure. \hfill $\square$

The generalized Calabi–Yau structure $\exp(B) \cdot \varphi$ is called the $B$-field transform of $\varphi$. Note that $\exp(B) \cdot \exp(i\omega) = \exp(B + i\omega)$.

The following proposition shows that any generalized Calabi–Yau structure is actually a $B$-field transform of one of the two fundamental examples.

**Proposition 1.5.** (Hitchin) Let $\varphi$ be a generalized Calabi–Yau structure.

i) If $\varphi_0 \neq 0$, then $\varphi = \varphi_0 \cdot \exp(B + i\omega)$, with $\omega$ a symplectic form and $B$ a closed real two-form.

ii) If $\varphi_0 = 0$, then $\varphi = \exp(B) \cdot \sigma = \sigma + \sigma \wedge B$, where $\sigma$ is a holomorphic two-form with respect to some complex structure on $M$ and $B$ is a closed real two-form.

**Proof.** i) More explicitly one finds in this case $\varphi^{-1} \cdot \varphi = \exp\left(\text{Re}(\varphi_0^{-1} \cdot \varphi_2)\right) \cdot \exp\left(i \cdot \text{Im}(\varphi_0^{-1} \cdot \varphi_2)\right)$.

Using $\langle \varphi, \overline{\varphi} \rangle > 0$ we obtain $\varphi_0 \overline{\varphi}_0 ((\overline{\varphi}_2 / \varphi_0) - (\varphi_2 / \varphi_0))^2 < 0$. Hence, $\text{Im}(\varphi_0^{-1} \cdot \varphi_2)$ is symplectic. The claimed equality is checked easily.

ii) Let $\varphi$ be a generalized Calabi–Yau structure with $\varphi_0 = 0$. In this case $\varphi_2 \wedge \varphi_2 = 0$ and $\varphi_2 \wedge \overline{\varphi}_2 > 0$. Due to an observation of Andreotti, there exists a unique complex structure on $M$ such that $\sigma := \varphi_2$ is a holomorphic two-form. By definition, the bundle of $(1,0)$-forms is the kernel of $\varphi_2 : \mathcal{A}_2^1 \to \mathcal{A}_2^2$. The integrability of the induced almost complex structure is equivalent to $d\varphi_2 = 0$.

Let us first assume that $\varphi_4$ is exact. Any exact four-form can be written as $\partial \gamma = d\gamma$ for a $(2,1)$-form $\gamma$ (one way to see this is to use Hodge-decomposition for $\partial$ and the fact that a four-form is exact if and only if its $d$-harmonic part is trivial if and only if its $\partial$-harmonic part is trivial). Since $\sigma$ is non-degenerate, there exists a $(0,1)$-form $\delta$ such that $\sigma \wedge \delta = \gamma$. Clearly, for degree reasons one also knows $\sigma \wedge \delta = 0$. Then with $B := d(\delta + \bar{\delta})$ one has $B \wedge \sigma = \varphi_4$.

In general, $\varphi_4$ can be written as $\varphi_4 = (\varphi_4 - \lambda \sigma \overline{\sigma}) + \lambda \sigma \overline{\sigma}$ with $\lambda \in \mathbb{C}$ such that $\varphi_4 - \lambda \sigma \overline{\sigma}$ is exact. Then choose a closed form $B'$ with $B' \wedge \sigma = \varphi_4 - \lambda \sigma \overline{\sigma}$ as before and set $B = B' + \lambda \overline{\sigma} + \lambda \sigma$. \hfill $\square$

For the notion of isomorphic generalized Calabi–Yau structures we shall consider the group $\text{Diff}_*(M)$ of all diffeomorphisms $f$ of $M$ such that the induced action $f^* : H^2(M, \mathbb{R}) \to H^2(M, \mathbb{R})$ is trivial. It seems unknown whether $\text{Diff}_*(M)$ coincides with the identity component $\text{Diff}_0(M)$ of the
diffeomorphism group. Allowing only \( \text{Diff}_\ast(M) \) and not the full diffeomorphism group \( \text{Diff}(M) \) might look not very natural, but for the moduli space considerations it is useful to divide by \( \text{Diff}(M)/\text{Diff}_\ast(M) = \text{O}(M) \) only later (see [3, 7, 11]).

**Definition 1.6.** Two generalized Calabi–Yau structures \( \varphi \) and \( \varphi' \) are called isomorphic if and only if there exists an exact B-field \( B \) and a diffeomorphism \( f \in \text{Diff}_\ast(M) \) such that \( \varphi = \exp(B) \cdot f^* \varphi' \).

Clearly, if \( \varphi \) and \( \varphi' \) are isomorphic generalized Calabi–Yau structures then \( \varphi_0 = 0 \) if and only if \( \varphi'_0 = 0 \). For the two principal examples, i.e. \( \varphi \) of the form \( \sigma \) or \( \exp(i\omega) \), this reduces to the (only slightly modified) standard definition of isomorphisms. Thus, \( \sigma \) and \( \sigma' \) are isomorphic as generalized Calabi–Yau structures if and only if there exists a diffeomorphism \( f \in \text{Diff}_\ast(M) \) such that \( f^* \sigma' = \sigma \). Similarly, \( \exp(i\omega) \) and \( \exp(i\omega') \) are isomorphic if and only if there exists a diffeomorphism \( f \in \text{Diff}_\ast(M) \) with \( \omega = f^* \omega' \).

**Remark 1.7.** If one wants to build an analogy between symplectic structures \( \omega \) and holomorphic forms \( \sigma \), one soon realizes that the natural isotropy groups \( \text{Sympl}(M, \omega) \) and \( \{ f \in \text{Diff}(M) \mid f^* \sigma = \sigma \} \) are quite different in nature. The group of symplectomorphisms is always infinite-dimensional, whereas \( \{ f \in \text{Diff}(M) \mid f^* \sigma = \sigma \} \subset \text{Aut}(M, I) \) is discrete. Only when both structures, \( \omega \) or rather \( \exp(i\omega) \) and \( \sigma \), are considered as generalized Calabi–Yau structures, the analogy emerges: Namely,

\[
\text{Aut}(\varphi = \exp(i\omega)) = \{ (f, B) \mid \exp(B) \cdot f^* \exp(i\omega) = \exp(i\omega) \} = \text{Sympl}(M, \omega)
\]

and

\[
\text{Aut}(\varphi = \sigma) = \{ (f, B) \mid \exp(B) \cdot f^* \sigma = \sigma \},
\]

the isotropy groups of the generalized Calabi–Yau structures \( \exp(i\omega) \) respectively \( \varphi = \sigma \), are both infinite-dimensional. In fact we see that \( \text{Aut}(\varphi = \sigma) \) is the set of all exact B-fields of type \((1, 1)\), which can be identified with the space of all functions modulo scalars due to the \( \partial\bar{\partial} \)-lemma.

In what follows, we will use the following notation:

**Definition 1.8.** Let \( \varphi \) be a generalized Calabi–Yau structure. Then \( P_\varphi \subset A^\ast(M) \) denotes the real vector space spanned by the real and imaginary part of \( \varphi \). Analogously, \( P_{[\varphi]} \subset H^\ast(M, \mathbb{R}) \) is the plane generated by the real and imaginary parts of the associated cohomology class.

Thus, \( P_\varphi \) with respect to \( \langle , \rangle \) is positive at every point and \( P_{[\varphi]} \subset H^\ast(M, \mathbb{R}) \) is a positive plane with respect to the Mukai pairing. Moreover, \( P_\varphi \) comes along with a natural (pointwise) orientation. Conversely, the oriented plane \( P_\varphi \subset A^\ast(M) \) determines \( \varphi \) uniquely up to non-trivial complex scalars (use \( d\varphi = 0 \)). Also note that \( P_{\exp(B)\varphi} = \exp(B) \cdot P_\varphi \).

Recall that in general there is a natural isomorphism between the (open subset of a) quadric \( Q_V := \{ x \mid x^2 = 0, x \cdot \bar{x} > 0 \} \subset \mathbb{P}(V_\mathbb{C}) \) and the
Grassmannian of oriented positive planes $\text{Gr}^o_2(V)$, where $V$ is a real vector space endowed with a non-degenerate quadratic form.

2. Global Torelli for generalized CY structures of HK type

Recall that any complex structure on $M$ defines a K3 surface (cf. [8]) and therefore is Kähler, due to Siu’s results [21]. Using the existence of Ricci-flat Kähler structures proved by Yau in [23], this also shows that any complex structure on $M$ admits a hyperkähler structure. In fact, any Kähler class is represented by a unique hyperkähler form.

In this section we shall discuss the analogous notions for generalized Calabi–Yau structures. First note that a symplectic two form $\omega$ is of type $(1,1)$ with respect to a complex structure $I$ if and only if $\sigma \wedge \omega = 0$, where $\sigma$ is the holomorphic two-form on $(M,I)$. In this case, $\omega$ or $-\omega$ is a Kähler form. (As before, we use $\omega^2 > 0$.) Thus, if $\omega$ is a symplectic form, then one of the two forms $\omega$ or $-\omega$ is a Kähler form with respect to $\sigma$ if and only if $P_\sigma$ and $P_{\exp(i\omega)}$ are pointwise orthogonal. This can be generalized as follows:

**Definition 2.1.** Let $\varphi$ be a generalized Calabi–Yau structure on $M$. We say that $\varphi$ is Kähler (or of Kähler type) if there exists another generalized Calabi–Yau structure $\varphi'$ orthogonal to $\varphi$, i.e. such that $P_\varphi$ and $P_{\varphi'}$ are pointwise orthogonal. In this case, $\varphi'$ is called a Kähler structure for $\varphi$.

Note that the orthogonality of two planes $P_\varphi$ and $P_{\varphi'}$ is in general a stronger condition than just $\langle \varphi, \varphi' \rangle \equiv 0$.

Does Siu’s existence result of Kähler structures on K3 surfaces extend to generalized Calabi–Yau structures? An affirmative answer can be given for generalized Calabi–Yau structures $\varphi$ with $\varphi_0 = 0$ (cf. Lemma 2.6).

**Example 2.2.** i) Let $\varphi = \sigma$. If $\varphi'$ is a Kähler structure for $\varphi$, then $\varphi'_0 \neq 0$, as $\wedge^2$ has only three positive eigenvalues at every point. Thus, we may assume $\varphi' = \exp(B + i\omega)$. The orthogonality of $P_\varphi$ and $P_{\varphi'}$ is equivalent to $\sigma \wedge B = \sigma \wedge \omega = 0$. Thus, $\varphi'$ is a Kähler structure for $\varphi$ if and only if $\varphi' = \exp(B + i\omega)$ (up to scalar factors) with $B$ a closed real $(1,1)$-form and $\pm \omega$ a Kähler form (both with respect to the complex structure defined by $\sigma$).

ii) Let $\varphi = \exp(i\omega)$, where $\omega$ is a symplectic form, and let $\varphi'$ be a Kähler structure for $\varphi$. There are two possible cases: Either $\varphi'_0 = 0$, then $\varphi' = \sigma$ and $\pm \omega$ is a Kähler form with respect to the complex structure defined by $\sigma$ or $\varphi'_0 \neq 0$. In the latter case $\varphi' = \exp(B' + i\omega')$ (up to scalars). The orthogonality is equivalent to the four equations $B' \wedge \omega = 0$, $B' \wedge \omega' = 0$, $\omega \wedge \omega' = 0$, and $B'^2 = \omega^2 + \omega'^2$. In particular, $\omega$, $\omega'$, and $B'$ are three pairwise pointwise orthogonal symplectic forms.

Recall that a Kähler form $\omega$ on a K3 surface is a hyperkähler form if $\omega \wedge \omega$ is a scalar multiple of the canonical volume form $\sigma \wedge \bar{\sigma}$. Scaling $\sigma$, which does not change the complex structure, makes it natural to assume that $2\omega \wedge \omega = \sigma \wedge \bar{\sigma}$. 


Definition 2.3. A generalized Calabi–Yau structure $\varphi$ is hyperkähler if there exists another generalized Calabi–Yau structure $\varphi'$ such that $\varphi$ and $\varphi'$ are orthogonal and $\langle \varphi, \bar{\varphi} \rangle = \langle \varphi', \bar{\varphi}' \rangle$. We say that $\varphi'$ is a hyperkähler structure for $\varphi$.

Example 2.4. i) A hyperkähler structure for $\varphi = \sigma$ is a generalized Calabi–Yau structure $\varphi'$ of the form $\lambda \exp(B + i\omega)$, where $0 \neq \lambda \in \mathbb{C}$, $B$ is a closed real $(1,1)$-form, and $\pm \omega$ is a hyperkähler form such that $2|\lambda|^2 \omega \wedge \omega = \sigma \wedge \bar{\sigma}$.

ii) A hyperkähler structure for $\varphi = \exp(i\omega)$ is either a holomorphic two-form $\varphi'$ = $\sigma$ with respect to which $\pm \omega$ is a hyperkähler form or it is of the form $\varphi' = \exp(B' + i\omega')$ (up to scalar factors which we omit) as in ii) of Example 2.2 with the additional condition $\omega \wedge \omega = \omega' \wedge \omega'$. This shows that $\sigma := (1/\sqrt{2})B' + i\omega'$ defines a complex structure with respect to which $\pm \omega$ is a hyperkähler form.

Remark 2.5. i) Clearly, both definitions are symmetric in $\varphi$ and $\varphi'$, i.e. if $\varphi'$ is a (hyper)kähler structure for $\varphi$ then $\varphi$ is a (hyper)kähler structure for $\varphi'$.

ii) Let $\varphi$ be a generalized Calabi–Yau structure and $\varphi'$ a (hyper)kähler structure for it. Then $\exp(B') \cdot \varphi'$ is a (hyper)kähler structure for the $B$-field transform $\exp(B) \cdot \varphi$.

Obviously, any generalized Calabi–Yau structure which is hyperkähler is also Kähler. The following lemma thus settles the existence question for both structures in the case $\varphi_0 = 0$.

Lemma 2.6. Any generalized Calabi–Yau structure $\varphi$ with $\varphi_0 = 0$ is hyperkähler.

Proof. As we have seen, a generalized Calabi–Yau structure $\varphi$ with $\varphi_0 = 0$ is of the form $\sigma + \sigma \wedge B$, where $\sigma$ is a holomorphic two-form with respect to a certain complex structure $I$ on $M$. Using the results of Siu and Yau we find a hyperkähler form $\omega$ on $(M, I)$. Thus, $\exp(i\omega)$ defines a hyperkähler structure for $\sigma$. Using the above remark, we find that $\varphi' = \exp(B + i\omega)$ is a hyperkähler structure for $\varphi = \exp(B) \cdot \sigma$.

Recall that Yau’s existence result says that for any complex structure defined by a complex two-form $\sigma$ and any Kähler form $\omega$ there exists a hyperkähler form $\omega'$ cohomologous to $\omega$. Using Moser’s result [16], which shows that $\omega$ and $\omega'$ are related by a diffeomorphism $f \in \text{Diff}_o(M)$, we find that Yau’s result is equivalent to saying that any Kähler structure for the generalized Calabi–Yau structure $\sigma$ is isomorphic to a hyperkähler structure.

Proposition 2.7. Let $\varphi'$ be a Kähler structure for a generalized Calabi–Yau structure $\varphi$. If $\varphi_0 \cdot \varphi_0 = 0$ then $\varphi'$ is isomorphic to a hyperkähler structure for $\varphi$.

Proof. Let us first assume $\varphi_0 = 0$. Then $\varphi'$ is necessarily (up to scaling) of the form $\exp(B + i\omega)$. The assertion is invariant under shifting both
structures by a $B$-field. Thus, we may assume that $\varphi = \sigma$. Yau's result immediately proves the existence of a diffeomorphism $f \in \text{Diff}_S(M)$ such that $f^*\omega$ is a hyperkähler form for $\sigma$. Moreover, $f^*B$ and $B$ differ by an exact $B$-field $B'$, i.e. $B - f^*B = B'$. Therefore, $\exp(B') \cdot f^* \exp(B + i\omega)$ is a hyperkähler structure (up to scalar factors) for the generalized Calabi–Yau structure $\sigma$.

Next assume $\varphi_0 \neq 0$. After rescaling we have $\varphi = \exp(B + i\omega)$, where $\omega$ is a symplectic form. Clearly, $\varphi$ is (hyper)kähler if and only if $\exp(i\omega)$ is (hyper)kähler. Thus, we may assume $\varphi = \exp(i\omega)$. By assumption $\varphi'$ is a Kähler structure for $\varphi$. A priori, we have to distinguish the two cases $\varphi_0' = 0$ and $\varphi_0' \neq 0$, but the second case is excluded by assumption.

If $\varphi_0' = 0$, then $\varphi' = \sigma + \sigma \wedge B$. The orthogonality of $P_\varphi$ and $P_{\varphi'}$ yields $\sigma \wedge B = 0$, i.e. $\varphi' = \sigma$. Thus, $\pm \omega$ is a Kähler structure with respect to the complex structure defined by $\sigma$. Hence, there exists a (unique) hyperkähler form $\omega'$ cohomologous to $\omega$ which can in fact be written as $\omega' = f^*\omega$ for some $f \in \text{Diff}_S(M)$ due to the result of Moser. But then $\pm \omega$ is a hyperkähler form with respect to the complex structure defined by $(f^{-1})^*\sigma$ (up to sign). Hence, $(f^{-1})^*\sigma$ is a hyperkähler structure for $\exp(i\omega)$ which is isomorphic to $\varphi'$ via $f$.

**Remark 2.8.** I certainly believe that the hypothesis $\varphi_0 \cdot \varphi_0' = 0$ is superfluous. The problem is a situation where $\varphi = \exp(i\omega)$ and $\varphi' = \exp(B' + i\omega')$. The Kähler condition is equivalent to $B' \wedge \omega = \omega' \wedge \omega = B' \wedge \omega' = 0$ and $B'^2 = \omega^2 + \omega'^2$. The problem one has to solve in this case seems very similar to the original existence question for Ricci-flat Kähler forms.

In order to see this analogy, we suppose for simplicity that $\int \omega^2 = \int \omega'^2$. Then consider the complex two-form $\sigma := \omega + i\omega'$ which is clearly orthogonal to $B'$. Moreover, $B'$ satisfies $B'^2 = \sigma \overline{\sigma}$. If we can change $B'$ by an exact form, such that the new $B'$ is still orthogonal to $\sigma$ and $B'^2 = 2\text{Im}(\sigma)^2$, then we are done. Indeed, then $\sigma' := B' + i\sqrt{2}\omega'$ would be a Calabi–Yau form orthogonal to $\omega$. The latter could be made a hyperkähler form by applying a diffeomorphism $f \in \text{Diff}_S(M)$ due to Yau's theorem. Note that $\sigma$ itself is a priori not a Calabi–Yau form, since, $\sigma^2 \neq 0$ a priori but the "$(1,1)$"-form $B$ satisfies already the condition $B'^2 = \sigma \overline{\sigma}$.

**Remark 2.9.** It is an open question whether any symplectic form on $M$ is in fact (hyper)kähler with respect to some complex structure. One expects an affirmative answer to this and a possible approach has recently been suggested by Donaldson [6]. The last proposition extended to the case $\varphi_0 \cdot \varphi_0' \neq 0$ would show in particular that if the generalized Calabi–Yau structure $\exp(i\omega)$ associated to a symplectic form $\omega$ is Kähler (as a generalized Calabi–Yau structure), then $\omega$ is actually a hyperkähler form with respect to a certain complex structure on $M$. Thus, together with an analogue of Siu's existence result, which would claim that any generalized Calabi–Yau structure is of Kähler type, the more general version of the above proposition would in particular show that any symplectic form is hyperkähler.
Remark 2.10. When conjecturing the existence of Ricci-flat metrics, Calabi gave a simple proof of the unicity, i.e., any Kähler class is represented by at most one hyperkähler form. Equivalently, if \( f \in \text{Diff}_s(M) \) such that \( f^* \sigma = \sigma \), then also \( f^* \omega = \omega \) for any hyperkähler form \( \omega \) on the complex K3 surface determined by \( \sigma \). Thus, \( f \) is an isometry and hence of finite order. It is known that this in fact yields \( f = \text{id} \). The unicity is no longer true when the role of \( \sigma \) and \( \omega \) are interchanged, i.e., for a given \( \omega \) there may exist several complex structures realizing the same period and making \( \omega \) a hyperkähler form. Indeed, if \( \omega \) is a hyperkähler form with respect to \( \sigma \) and \( \text{id} \neq f \in \text{Symp}(\omega) \cap \text{Diff}_s(M) \), then \( f^* \sigma \neq \sigma \) by the above argument. Hence, \( \sigma \) and \( f^* \sigma \) are two different hyperkähler structures for \( \exp(i\omega) \).

Again, the different behaviour of \( \omega \) and \( \sigma \) can be explained if both are considered as generalized Calabi–Yau structures. Indeed, for \( \varphi = \sigma \) there exist many different hyperkähler structures \( \exp(B + i\omega) \) in the same cohomology class. In fact, if \( \omega \) is hyperkähler for \( \sigma \), then \( \exp(B + i\omega) \) is a hyperkähler generalized Calabi–Yau structure for \( \varphi = \sigma \) whenever \( B \) is an exact \((1,1)\)-form.

The arguments used in the proof of the following result show in particular that two hyperkähler structures for a given generalized Calabi–Yau structure are always isomorphic.

Proposition 2.11. (Global Torelli theorem) Let \( \varphi \) and \( \psi \) be two generalized Calabi–Yau structures on \( M \) and suppose they are both hyperkähler. If \( P_{[\varphi]} = P_{[\psi]} \subset H^*(M, \mathbb{R}) \), then there exists a real exact \( B \)-field \( B \) and a diffeomorphism \( f \) such that \( P_{\varphi} = \exp(B) \cdot P_{f^* \psi} \), i.e., up to rescaling \( \varphi = \exp(B) \cdot f^* \psi \).

If \( \varphi_0 \neq 0 \), then \( f \) can be chosen in \( \text{Diff}_s(M) \), i.e., \( \varphi \) and \( \psi \) are isomorphic generalized Calabi–Yau structures.

Proof. First suppose that \( \varphi_0 = 0 \). Then also \( \psi_0 = 0 \) and after rescaling we may assume \([\varphi_4] = [\psi_4] \). We have to find a real exact two-form \( B \) and a diffeomorphism \( f \) such that \( \varphi_2 = f^* \psi_2 \) and \( \varphi_4 = B \wedge f^* \psi_2 + f^* \psi_4 \). As has been explained before, the assumption that \( \varphi \) and \( \psi \) are generalized Calabi–Yau structures implies that \( \varphi_2 \) and \( \psi_2 \) are holomorphic two-forms with respect to uniquely determined complex structures. Invoking the classical Global Torelli theorem for K3 surfaces we find a diffeomorphism \( f \) such that \( \varphi_2 = f^* \psi_2 \). Thus, we may assume \( \varphi_2 = \psi_2 \) already and try to find an real exact two-form \( B \) such that \( \varphi_4 - \psi_4 = B \wedge \psi_2 \). This follows directly from the argument given in the proof of Proposition \ref{prop:unicity}, because \( \varphi_4 - \psi_4 \) is exact and \( \psi_2 \) is a non-degenerate holomorphic two-form. (Note that \( f \) can be chosen in \( \text{Diff}_s(M) \) if \( \varphi \) and \( \psi \) admit hyperkähler structures \( \varphi' \) respectively \( \psi' \) with \([\varphi'] = [\psi'] \).)

If \( \varphi_0 \neq 0 \) then also \( \psi_0 \neq 0 \) and after rescaling we might assume \( \varphi_0 = \psi_0 = 1 \). Thus, we have \( \varphi = \exp(B + i\omega) \) and \( \psi = \exp(B' + i\omega') \), where \( \omega \) and \( \omega' \) are symplectic forms and \( B \) and \( B' \) are cohomologous real closed
two-forms. In particular, $B$ and $B'$ differ by an exact $B$-field. Thus, we may reduce to the case $\varphi = \exp(i\omega)$ and $\psi = \exp(i\omega')$ with $[\omega] = [\omega']$.

The assumption ensures that $\omega$ and $\omega'$ are hyperkähler forms with respect to certain holomorphic two-forms $\sigma$ and $\sigma'$, respectively.

As the moduli space of K3 surfaces (with metric) is connected, there exists a deformation $(\omega_t, \sigma_t)$ of $(\omega, \sigma)$ such that $([\omega_1], [\sigma_1]) = ([\omega'], [\sigma'])$. Moreover, since $[\omega] = [\omega']$, we may assume that $[\omega_t] = [\omega']$, where $[\omega_t]$ is constant. Using Moser’s result we then find a continuous family of diffeomorphisms $f_t$ such that $f_t^* \omega_t = \omega$. Applying $f$ also to $\sigma_t$ shows that we can in fact assume that $(\omega_t, \sigma_t) \equiv (\omega, \sigma_t)$. The upshot of all this is that whenever $\varphi = \exp(i\omega)$ and $\psi = \exp(i\omega')$ are two hyperkählerian generalized Calabi–Yau structures, then we can choose $\sigma$ and $\sigma'$ such that $[\sigma] = [\sigma']$. The standard Global Torelli theorem then yields the existence of a diffeomorphism $f \in \text{Diff}(M)$ such that $f^* \omega' = \omega$. □

**Remark 2.12.** i) We can slightly improve the above statement. Assume $\varphi$ and $\psi$ are two generalized Calabi–Yau structures. Suppose that there exists an automorphism $F$ of the K3 lattice $H^2(M, \mathbb{Z})$ such that $F[\psi] = [\varphi]$. (Here, we extend $F$ by the identity to the full cohomology.) Then there exists a diffeomorphism $f \in \text{Diff}(M)$ and an exact two-form $B$ such that $\varphi = (\exp(B) \circ \pm \text{id}_{H^2}) \circ f^*(\psi)$ and $f^* = F$. Indeed, due to a result of Borcea and Matumoto there exists a diffeomorphism $F$ such that $f^* = \pm F$. Then we may apply the proposition to $\varphi$ and $f^* \psi$.

ii) A Global Torelli theorem for generalized Calabi–Yau structures $\varphi$ and $\psi$ which are not necessarily hyperkähler would in particular show that for any two cohomologous symplectic structures $\omega$ and $\omega'$ on $M$ there exists a diffeomorphism $f$ such that $\omega = f^* \omega'$. This could in turn be used to show that every symplectic structure on $M$ is hyperkähler (cf. Remark 2.9).

### 3. Generalized K3 surfaces and moduli spaces

Marked K3 surfaces endowed with a Kähler structure and a $B$-field form a moduli space that can be described via the period map. It turns out that the period map injects this moduli space into the physics moduli space of $N = (2, 2)$ SCFT. However, not every $N = (2, 2)$ SCFT parametrized by the latter comes from a classical K3 surface (with a $B$-field); the geometric moduli space is of real codimension two. In fact, the $N = (2, 2)$ SCFT moduli space fibers over the $N = (4, 4)$ SCFT moduli space with fibre $S^2 \times S^2$. The standard K3 moduli space fibers as well over the $N = (4, 4)$ SCFT moduli space (which is interpreted as the moduli space of hyperkähler metrics), but the fibre is only $S^2$, the twistor line.

In this section we shall indicate how Hitchin’s generalized Calabi–Yau structures (or rather generalized K3 surfaces) fit nicely in this picture. We will see that the $N = (2, 2)$ SCFT moduli space can be interpreted as the moduli space of generalized K3 surfaces. For details of the moduli space
construction we refer the reader to [11] or the original articles [2] [17]. In particular, we will use the notations introduced there. Also note that we will actually not work with the SCFT moduli spaces, but rather with certain period domains, which have been shown to contain the corresponding moduli spaces (cf. [2]).

So far, by a K3 surface we meant a compact complex surface $X$ with trivial canonical bundle $K_X$ and $b_1(X) = 0$. Any K3 surface in this sense is determined by a complex structure $I$ or by a Calabi–Yau structure $\sigma \mathbb{C}$ on $M$. From now on we will reserve the name K3 surface for a complex surface already endowed with a hyperkähler form. More precisely, we have

**Definition 3.1.** A K3 structure on the differentiable manifold $M$ consist of a closed complex two-form $\sigma \in \mathcal{A}^2(M)$ with $\sigma \wedge \sigma = 0$ and a symplectic form $\omega \in \mathcal{A}^2(M)$ such that i) $\omega \wedge \sigma = 0$ and ii) $\sigma \wedge \bar{\sigma} = 2\omega \wedge \omega > 0$.

As was explained before, the complex two-form $\sigma$ defines a unique complex structure. The orthogonality condition $\sigma \wedge \omega = 0$ is equivalent to $\omega$ being a $(1,1)$-form with respect to this complex structure. Eventually, $\sigma \wedge \bar{\sigma} = 2\omega \wedge \omega$ ensures that $\pm \omega$ is a hyperkähler form. Thus, $M$ endowed with such a K3 structure is just a K3 surface with a chosen hyperkähler structure. Using the convention of the last section we give the following

**Definition 3.2.** A generalized K3 structure on $M$ consists of a pair $(\varphi, \varphi')$ of generalized Calabi–Yau structures $\varphi$ and $\varphi'$ such that $\varphi$ is a hyperkähler structure for $\varphi'$.

Two generalized K3 structures $(\varphi, \varphi')$ and $(\psi, \psi')$ on $M$ are called isomorphic if there exists a diffeomorphism $f \in \text{Diff}_s(M)$ and an exact real two-form $B \in \mathcal{A}^2(M)$ such that $(\varphi, \varphi') = \exp(B) \cdot f^*(\psi, \psi')$.

Clearly, the B-field transform $(\exp(B) \cdot \varphi, \exp(B) \cdot \varphi')$ of any generalized K3 structure $(\varphi, \varphi')$ is again a generalized K3 structure.

**Definition 3.3.** To any generalized K3 structure $(\varphi, \varphi')$ on $M$ we associate the (pointwise) oriented positive four-space $\Pi(\varphi,\varphi') \subseteq \mathcal{A}^2^*(M)$ spanned by $P_\varphi$ and $P_\varphi'$. Analogously, one defines an oriented positive four-space $\Pi([\varphi],[\varphi']) \subseteq H^*(M,\mathbb{R})$ spanned by $P_{[\varphi]}$ and $P_{[\varphi']}$.

**Remark 3.4.** The set $T_\Pi$ of all generalized K3 structures $(\varphi, \varphi')$ with fixed positive four-space $\Pi$ is naturally isomorphic to the Grassmannian of oriented planes $\text{Gr}_2^+(\Pi) = S^2 \times S^2 = \mathbb{P}^1 \times \mathbb{P}^1$. Indeed, $T_\Pi = Q_\Pi$, which is a quadric in $\mathbb{P}^3 = \mathbb{P}^3(\Pi_\mathbb{C})$. We call $T_\Pi$ or $T_{(\varphi,\varphi')}$ the (generalized) twistor space (or, more precisely, the base of it).

**Example 3.5.** If $(\varphi = \sigma, \varphi' = \exp(i\omega))$ is a classical K3 structure on $M$, then $\Pi$ is spanned by the oriented base $\text{Re}(\sigma), \text{Im}(\sigma), 1 - (1/2) \cdot \omega^2 = 1 - (1/4) \sigma \bar{\sigma}, \omega$. In other words, if we write $\sigma = \omega_J + i\omega_K$, where $J, K = IJ$ are the two other natural complex structures induced by the hyperkähler form, then $\Pi = \langle 1 - (1/2)\omega^2, \omega_I = \omega, \omega_J, \omega_K \rangle$. The classical twistor deformations
$S^2 = \{aI + bJ + cK \mid a^2 + b^2 + c^2 = 1\}$ form a $\mathbb{P}^1$ which is contained in $T_{(\sigma, \exp(\iota \omega))}$ as the hyperplane section $Q_\Pi \cap \mathbb{P}(\langle \omega_I, \omega_J, \omega_K \rangle \mathbb{C})$. In particular, it is not one of the two components of $\mathbb{P}^1 \times \mathbb{P}^1$.

Note that the other generalized K3 structures parametrized by $\mathbb{P}^1 \times \mathbb{P}^1 \setminus S^2$ are not obtained as $B$-field transforms of points in $S^2$.

**Proposition 3.6.** Let $(\varphi, \varphi')$ be a generalized K3 structure. Then there exists a classical K3 structure $(\sigma, \exp(\iota \omega))$ and a closed $B$-field $B$ with

$$\Pi_{(\varphi, \varphi')} = \exp(B) \cdot \Pi_{(\sigma, \exp(\iota \omega))}.$$  

**Proof.** Since $\Pi_{(\varphi, \varphi')} \subset H^*(M, \mathbb{R})$ is a positive four-space and $H^0 \oplus H^4$ is only two-dimensional, there exists a positive plane $H \subset \Pi_{(\varphi, \varphi')} \cap H^2(M, \mathbb{R})$.

Using the isomorphism $\Pi_{(\varphi, \varphi')} \cong \Pi_{(\varphi, \varphi')}$, we may choose a generalized Calabi–Yau structure $\psi \in \mathcal{A}_2^0(M)$ such that $\langle \text{Re}(\psi), \text{Im}(\psi) \rangle \subset \Pi_{(\varphi, \varphi')}$ corresponds to $H$. Hence, $\psi$ is of the form $\exp(B) \cdot \sigma$ with $B$ exact (cf. Prop. 1.3).

The orthogonal plane $H^\perp \subset \Pi_{(\varphi, \varphi')}$ is spanned by real and imaginary part of a form of type $\exp(B' + i\omega)$ for some closed $B$-field $B'$ and a symplectic structure $\omega$. (We use here that the Mukai pairing on $H^2(M, \mathbb{R})$ has only three positive eigenvalues.) Then, $(\psi_2, \exp(\iota \omega))$ is automatically a classical K3 structure and, moreover, $\psi_2$ and $B' - B$ are orthogonal. Thus, $\Pi_{(\varphi, \varphi')} = \Pi_{(\exp(B), \psi_2, \exp(B' + i\omega))} = \exp(B') \cdot \Pi_{(\psi_2, \exp(\iota \omega))}$. \hfill $\square$

**Remark 3.7.** The relation between generalized Kähler structures and so-called bi-hermitian structures is explained in detail in [9, Sect.6]. Gualtieri’s notion of generalized Kähler structures is formulated in terms of generalized complex structures (and not generalized Calabi–Yau structures as in this text) and is in fact slightly stronger than ours.

**Definition 3.8.** Let

$$\mathcal{M}_{\text{genK3}} = \{(\varphi, \psi)\}_{\simeq}$$

be the moduli space of all generalized K3 structures modulo isomorphism as defined in [8].

As a consequence of the above proposition 3.6 we obtain

**Corollary 3.9.** There exists a natural $S^2 \times S^2$-fibration

$$\mathcal{M}_{\text{genK3}} \longrightarrow \mathcal{M}_{\text{HK}} := \left(\text{Met}^\text{HK}(M)/\text{Diff}_s(M)\right) \times H^2(M, \mathbb{R})$$

onto the space of all $B$-field shifts of hyperkähler metrics on $M$. \hfill $\square$

**Definition 3.10.** The period of a generalized K3 structure $(\varphi, \psi)$ is the orthogonal pair of positive oriented planes $(P_{[\varphi]}, P_{[\psi]}) \in \text{Gr}^{\mathbb{P}_0}_{2, 2}(H^*(M, \mathbb{R}))$. The period map

$$\mathcal{P}_{\text{genK3}} : \mathcal{M}_{\text{genK3}} \longrightarrow \text{Gr}^{\mathbb{P}_0}_{2, 2}(H^*(M, \mathbb{R}))$$


is the map that associates to a generalized K3 structure \((\varphi, \psi)\) its period \((P_{[\varphi]}, P_{[\psi]})\).

Similarly, one defines the period map

\[
\mathcal{M}_{\text{genK3}} \xrightarrow{\mathcal{P}_{\text{genK3}}} \text{Gr}^\text{po}_2(H^*(M, \mathbb{R}))
\]

which maps the B-field shift of a hyperkähler metric given by a generalized K3 structure \((\varphi, \psi)\) to \(\Pi([\varphi],[\psi])\). We obtain a commutative diagram, where both vertical arrows are \(S^2 \times S^2\)-fibrations:

\[
\begin{array}{ccc}
\mathcal{M}_{\text{genK3}} & \xrightarrow{\mathcal{P}_{\text{genK3}}} & \text{Gr}^\text{po}_2(H^*(M, \mathbb{R})) \\
\downarrow & & \downarrow \\
\mathcal{M}_{\text{HK}} & \xrightarrow{\mathcal{P}_{\text{HK}}} & \text{Gr}^\text{po}_4(H^*(M, \mathbb{R}))
\end{array}
\]

Classically, one considers the space \(\mathcal{M}_{\text{K3}}\) of pairs \((I, \omega)\), where \(I\) is a complex structure and \(\omega\) is a hyperkähler form modulo the group \(\text{Diff}^*_+(M)\). This is equivalent to giving the holomorphic two-form \(\sigma\) (up to scaling) and the Kähler form \(\omega\). Adding the B-field one obtains the moduli space \(\mathcal{M}_{\text{K3}} \times H^2(X, \mathbb{R})\). The fibre of the natural map \(\mathcal{M}_{\text{K3}} \times H^2(X, \mathbb{R}) \rightarrow \mathcal{M}_{\text{HK}}\), which associates to \(((\sigma, \omega), B)\) the underlying hyperkähler metric \((g, B)\) (everything shifted by \(B\)), is a natural \(S^2\)-bundle, where every fibre parametrizes all complex structures associated with one hyperkähler metric.

Using the exponential map we obtain a canonical injection

\[
\mathcal{M}_{\text{K3}} \times H^2(X, \mathbb{R}) \hookrightarrow \mathcal{M}_{\text{genK3}}, \quad (\sigma, \omega, B) \mapsto (\exp(B) \cdot \sigma, \exp(B + i\omega)).
\]

Both sides are fibred over \(\mathcal{M}_{\text{HK}}\) with fibre \(S^2\) and \(S^2 \times S^2\), respectively.

Proposition 3.11. The period map

\[
\mathcal{P}_{\text{genK3}} : \mathcal{M}_{\text{genK3}} \longrightarrow \text{Gr}^\text{po}_2(H^*(M, \mathbb{R}))
\]

is an immersion with dense image.

Proof. This is essentially a consequence of the known results for classical K3 surfaces. One knows that \(\mathcal{P}_{\text{HK}} : \mathcal{M}_{\text{HK}} \rightarrow \text{Gr}^\text{po}_4\) is a dense immersion. Thus, the assertion follows from the fact that both maps \(\mathcal{P}_{\text{genK3}} : \mathcal{M}_{\text{genK3}} \rightarrow \mathcal{M}_{\text{HK}}\) and \(\text{Gr}^\text{po}_2 \rightarrow \text{Gr}^\text{po}_4\) are \(S^2 \times S^2\)-fibrations, whose fibres over points of the image of \(\mathcal{P}_{\text{HK}}\) are naturally identified via \(\mathcal{P}_{\text{genK3}}\). \(\square\)

The non-surjectivity is a phenomenon already encountered on the level of hyperkähler metrics. The period map \(\mathcal{M}_{\text{HK}} \rightarrow \text{Gr}^\text{po}_4(H^*(M, \mathbb{R}))\) is not surjective, but its image is dense and points in the complement can be interpreted as degenerate hyperkähler metrics. (In fact, only positive four-spaces which are not orthogonal to any \((-2)\)-class should be interpreted in this way and should be considered as points defining a \(N = (4, 4)\) SCFT.)
So far our discussion of the relation between generalized K3 surfaces and $N = (2, 2)$ SCFT was purely on the level of moduli spaces. Leaving aside the conformal aspects of the theory, which in any case are not mathematically established for K3 surfaces, and considering only the supersymmetric part, one can in fact show that generalized K3 surfaces are indeed related to $N = (2, 2)$ supersymmetry.

Let us briefly recall the general setting for supersymmetry in (hyper)Kähler geometry. If $X$ is a compact Kähler manifold endowed with a Kähler form $\omega$, then the Lefschetz operator $L$ and its dual $\Lambda$ generate an $s\ell(2, \mathbb{C})$ subalgebra of the algebra of endomorphisms of $\mathcal{A}^\cdot_\omega(X)$. Together with the differential operators $d$ and $d^\ast$ they generate a Lie algebra which in physics jargon is called the $N = (2, 2)$ supersymmetry algebra associated to $(X, \omega)$. Note that this Lie algebra is finite dimensional and that as a vector space it is generated by $\partial, \bar{\partial}, \partial^\ast, \bar{\partial}^\ast, \Delta, L, \Lambda, H$. (The Lie algebra ‘closes.’) The Kähler condition is crucial at this point.

If $X$ is a K3 surface and $\omega$ is Ricci-flat one can find an even bigger Lie algebra. Firstly, as Verbitsky [22] has shown more generally for hyperKähler manifolds, the Lefschetz operators $L_I, L_J, L_K$ and their dual $\Lambda_I, \Lambda_J, \Lambda_K$ associated to the three complex structures $I, J, K = IJ$ generate a Lie subalgebra isomorphic to $so(5, \mathbb{C})$. The $N = (4, 4)$ supersymmetry algebra associated with $(X, \omega)$ is the (finite dimensional!) Lie algebra generated by this $so(5, \mathbb{C})$ and $d$ and $d^\ast$. As before, it also contains the differential operators $\partial_I, \bar{\partial}_J, \bar{\partial}_K$, etc. Note that the $N = (4, 4)$ supersymmetry algebra is naturally induced by the underlying hyperKähler metric.

For K3 surfaces, the description of the Lie algebra generated by the Lefschetz operators and their dual is rather straightforward. Consider the five-dimensional vector space $V$ spanned by $1 \in \mathcal{A}^0(X), \omega_I, \omega_J, \omega_K \in \mathcal{A}^2(X), \omega_I^2 \in \mathcal{A}^4(X)$ endowed with the Mukai pairing. Clearly, all six Lefschetz operators preserve this space and it can be checked that they are all in $s\ell(V)$. Note that singling out a specific complex structure $\lambda = aI + bJ + cK$ naturally yields an inclusion $s\ell(2, \mathbb{C}) \subset so(5, \mathbb{C})$. In particular, to any such $\lambda$ the $N = (2, 2)$ supersymmetry algebra associated with $\lambda$ and $\omega_\lambda$ is naturally contained in the $N = (4, 4)$ supersymmetry algebra associated with the hyperKähler metric.

More generally, each point in the fibre of $\mathcal{M}_{genK3} \rightarrow \mathcal{M}_{HK}$ naturally parametrizes $N = (2, 2)$ supersymmetry algebras within the $N = (4, 4)$ supersymmetry algebra determined by a point in $\mathcal{M}_{(4,4)}$. The following proposition shows that also the fibre of $\mathcal{M}_{genK3} \rightarrow \mathcal{M}_{HK}$ naturally parametrizes $N = (2, 2)$ supersymmetry algebras within the $N = (4, 4)$ supersymmetry algebra given by a hyperKähler metric $g \in \mathcal{M}_{HK}$.
Let \( g \) be a hyperkähler metric. Denote by \( \Pi_g \subset A^2(M) \) and \( \mathfrak{g}_g \cong \mathfrak{so}(5, \mathbb{C}) \subset \text{End}(\Lambda^2(M)) \) the positive four-space respectively the Lie subalgebra naturally associated with \( g \). Recall that giving a generalized Calabi–Yau structure \( \varphi \) with \( P_{\varphi} \subset \Pi_g \) is equivalent to giving a generalized K3 surface \((\varphi, \varphi')\) realizing \( \Pi_g \).

**Proposition 3.12.** Any generalized Calabi–Yau structure \( \varphi \) on \( M \) with \( P_{\varphi} \subset \Pi_g \) determines a Lie subalgebra of \( \mathfrak{g}_g \cong \mathfrak{so}(5, \mathbb{C}) \) naturally isomorphic to \( \mathfrak{sl}(2, \mathbb{C}) \).

**Proof.** We know that \( \Pi_g = \langle 1 - \omega I/2, \omega I, \omega J, \omega K \rangle_{\mathbb{R}} \) and \( \mathfrak{g}_g = \mathfrak{so}(V_{\mathbb{C}}) \) where \( V = \langle 1, \omega I, \omega J, \omega K, \omega I^2 \rangle_{\mathbb{R}} \). The standard isomorphism \( \mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{so}(3, \mathbb{C}) \) can be interpreted in our context as \( \mathfrak{sl}(P_{\varphi} \subset \Pi_g) \cong \mathfrak{so}(P_{\varphi} \subset \Pi_g) \). Since \( V = P_{\varphi} \oplus \Pi_g^\perp \), where \( P_{\varphi} ^\perp \) is the orthogonal complement of \( P_{\varphi} \subset \Pi_g \), we obtain a natural inclusion \( \mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{sl}(P_{\varphi} \subset \Pi_g) \subset \mathfrak{so}(V_{\mathbb{C}}) \cong \mathfrak{so}(5, \mathbb{C}) \).

We leave it to the reader to verify that for \( \varphi = \exp(i \omega \lambda) \) the inclusion \( \mathfrak{sl}(2, \mathbb{C}) \subset \mathfrak{so}(5, \mathbb{C}) \) thus described is the one that is given by the Lefschetz operators \( L \lambda \) and \( \Lambda \lambda \).

Let us conclude this section with a discussion of the moduli space of K3 surfaces without metrics and generalized Calabi–Yau structures. They are studied in terms of the following two period domains:

\[
Q := \{ x \mid x^2 = 0, \ x \bar{x} > 0 \} \subset \mathbb{P}(H^2(M, \mathbb{C}))
\]

and

\[
\tilde{Q} := \{ x \mid \langle x, x \rangle = 0, \ \langle x, \bar{x} \rangle > 0 \} \subset \mathbb{P}(H^*(M, \mathbb{C})),
\]

where the latter involves the Mukai pairing \( \langle , , \rangle \). Clearly, \( Q \) is naturally contained in \( \tilde{Q} \).

**Definition 3.13.** We introduce the moduli space

\[
\mathfrak{M}_{\text{gen}} = \{ \varphi \cdot \mathbb{C} \} / \cong
\]

of all isomorphism classes of generalized Calabi–Yau structures \( \varphi \cdot \mathbb{C} \) of hyperkähler type on \( M \) (cf. Definition 3.2).

By definition, \( \mathfrak{M}_{\text{gen}} \) is the quotient of the set of generalized Calabi–Yau structures of hyperkähler type on \( M \) by the action of exact \( B \)-fields and of the groups \( \mathbb{C}^* \) and \( \text{Diff}_*(M) \). In contrast to the moduli spaces considered before, \( \mathfrak{M}_{\text{gen}} \) is not separated.

The classical counterpart is the moduli space

\[
\mathfrak{M} := \{ \sigma \cdot \mathbb{C} \} / \text{Diff}_*(M)
\]

of complex structures on \( M \) or, equivalently, the moduli space of marked K3 surfaces (or rather, one of the two connected components of it).
The classical period map \( P: \mathfrak{M} \to Q, \sigma \cdot \mathfrak{C} \mapsto [\sigma] \cdot \mathfrak{C} \) extends naturally to \( P_{\text{gen}}: \mathfrak{M}_{\text{gen}} \to \tilde{Q}, \varphi \cdot \mathfrak{C} \mapsto [\varphi] \cdot \mathfrak{C} \). This yields a commutative diagram

\[
\begin{array}{ccc}
\mathfrak{M} & \xrightarrow{P} & \mathbb{P}(\mathbb{C}) \\
\mathfrak{M}_{\text{gen}} & \xrightarrow{P_{\text{gen}}} & \mathbb{P}(\mathbb{C}) \\
\end{array}
\]

The surjectivity of the period map \( P: \mathfrak{M} \to Q \subset \mathbb{P}(\mathbb{C}) \), a result due to Todorov, Looijenga, Siu (cf. \[\text{(1)}\]), is one of the fundamental results in the theory of K3 surfaces. It easily generalizes to our situation:

**Proposition 3.14.** The period map \( P_{\text{gen}} \) is étale and surjective. Moreover, \( P_{\text{gen}} \) is bijective over the complement of the hyperplane section \( \mathbb{P}(\mathbb{C}) \cap \tilde{Q} \).

**Proof.** The argument for the surjectivity follows the classical proof. We define an equivalence relation on \( \tilde{Q} \) using positive four-spaces. Two positive planes \( P, P' \in \tilde{Q} \) are called equivalent if they generate a positive four-space. Obviously, this equivalence relation is open and, since \( \tilde{Q} \) is connected, there exists only one equivalence class. In particular, it suffices to show that for two planes \( P, P' \in \tilde{Q} \) generating a positive four-space \( \Pi \) one has \( P \in \operatorname{Im}(P_{\text{gen}}) \) if and only if \( P' \in \operatorname{Im}(P_{\text{gen}}) \). Since \( \mathbb{P}(\mathbb{C}) \cap \tilde{Q} \), we may assume that \( P \in Q \) and that \( \Pi \) is generated by \( P \) and the plane \( P_\varphi \) with \( \varphi = \exp([B] + i[\omega]) \) (cf. the proof of Prop. 3.6), where \([\omega]\) is a class in the positive cone. After changing \( P \) a little in \( Q \), which is allowed as the positivity of \( \langle P, P' \rangle \) is preserved, we may assume that the Picard group of the corresponding K3 surface is trivial. Hence, every element in the positive cone is actually a Kähler class. This shows that the intersection \( \mathbb{P}(\mathbb{C}) \cap \tilde{Q} \) is the image of the generalized twistor space and thus contained in the image of \( P \).

The last assertion follows from the Global Torelli theorem for generalized Calabi–Yau structures \( \varphi \) with \( \varphi_0 \neq 0 \) (see the proof of Proposition 2.11). □
It is noteworthy that by incorporating $B$-fields it is now possible to deform a K3 surface, i.e. a complex structure on $M$, continuously to a symplectic form.

One should think of $\mathbb{P}(H^2(M, \mathbb{C}) \oplus H^4(M, \mathbb{C})) \cap \tilde{Q}$ as the period domain for the moduli space of all $B$-field shifts of $Q$. Thus, the latter has to be considered as a hyperplane section of $\mathbb{N}_{\text{gen}}$. Note that the complement of this hyperplane section is complex 22-dimensional and parametrizes $B$-field shifts of hyperkähler symplectic forms, which for themselves form a real 22-dimensional subspace.

4. Generalized Calabi–Yau structures and Derived Categories

The modest aim of the present section is to illustrate that generalized Calabi–Yau structures and generalized K3 surfaces seemingly provide a natural framework for certain results and conjectures on equivalences of derived categories of coherent sheaves on algebraic K3 surfaces. More details can be found in [13].

Let us begin with Orlov’s result [19] on the equivalence of derived categories of K3 surfaces. In the following let $X$ and $X'$ be two algebraic K3 surfaces given by Calabi–Yau structures $\varphi = \sigma$ and $\varphi' = \sigma'$, respectively. Furthermore, let $D := D^b(\text{Coh}(X))$ and $D' := D^b(\text{Coh}(X'))$, respectively, be their derived categories of coherent sheaves. Then Orlov’s result can be reformulated as follows:

**Theorem 4.1. (Orlov)** There exists an exact equivalence between the triangulated categories $D$ and $D'$ if and only if one of the following two equivalent conditions is satisfied:

i) There exists an Hodge isometry $T(X) \cong T(X')$ between their transcendental lattices.

ii) The periods of $\varphi$ and $\varphi'$ are contained in the same $O(H^2(M, \mathbb{Z}))$-orbit of the natural action of the orthogonal group of the Mukai lattice on the period domain. \[ \square \]

Orlov’s result thus generalizes beautifully the Global Torelli theorem saying that $X$ and $X'$ (not necessarily algebraic) are isomorphic if and only if their periods are contained in the same $O(H^2(M, \mathbb{Z}))$-orbit. Thus, passing from the period domain $Q$ to the generalized period domain $\tilde{Q}$, or from $O(H^2(M, \mathbb{Z}))$ to $O(H^*(M, \mathbb{Z}))$, corresponds in geometrical terms to the passage from isomorphism classes of K3 surfaces to equivalence classes of their derived categories.

Note however that, for the time being, a completely satisfactory geometrical interpretation of the action of $O(H^*(M, \mathbb{Z}))$ is missing. For the smaller orthogonal group $O(H^2(M, \mathbb{Z}))$ it is provided by the surjectivity of the natural representation $\text{Diff}(M) \to O_+(H^2(M, \mathbb{Z}))$ due to Borcea and Donaldson, where $O_+ \subset O$ is the subgroup of orthogonal transformations that preserve the orientation of the positive directions.
What about generalized Calabi–Yau structures of the form \( \exp(B)\sigma \)? Here the situation is less clear, but a conjecture treating this case has been proposed by A. Căldăruș in [4]. Again, we assume that \( X \) and \( X' \) are two algebraic K3 surfaces. Moreover, we choose two torsion classes \( \alpha \in H^2(X, O_X^*) \) and \( \alpha' \in H^2(X', O_{X'}^*) \). Lifting these classes to elements in \( H^2(X, \mathbb{Q}) \) respectively in \( H^2(X', \mathbb{Q}) \) one defines \( T(X, \alpha) := \text{Ker}(\alpha : T(X) \rightarrow \mathbb{Q}/\mathbb{Z}) \) and similarly \( T(X', \alpha') \). (See below for more details.)

**Conjecture 4.2.** (Căldăruș) There exists an exact equivalence between the derived categories \( D^b(\text{Coh}_\alpha(X)) \) and \( D^b(\text{Coh}_{\alpha'}(X')) \) of twisted coherent sheaves if and only if there exists a Hodge isometry \( T(X, \alpha) \sim T(X', \alpha') \).

This conjecture has been verified in a few special cases by Căldăruș himself [4] and, more recently, by Donagi and Pantev [5]. One way to define the derived category \( D^b(\text{Coh}_\alpha(X)) \) is to view \( \alpha \) as an Azumaya algebra \( A \) (up to equivalence) and to derive the abelian category of coherent \( A \)-modules. (That the Azumaya algebra \( A \) exists is due to Grothendieck, at least in the algebraic situation considered here. The analytic analogue was recently established in [13].)

In the rest of this section we indicate how to define the transcendental lattice of a generalized Calabi–Yau structure and how to rephrase Căldăruș’s conjecture in terms of periods and the \( O(H^*(M, \mathbb{Z})) \)-action.

**Definition 4.3.** The Picard group of a generalized Calabi–Yau structure \( \varphi \) is the orthogonal complement of its cohomology class:

\[
\text{Pic}(\varphi) := \{ \delta \mid \langle \delta, \varphi \rangle = 0 \} \subset H^*(M, \mathbb{Z}).
\]

**Example 4.4.**

i) If \( \varphi \) is an ordinary Calabi–Yau structure \( \sigma \), i.e. \( \sigma \) is the holomorphic two form with respect to a complex structure on \( M \) defining a K3 surface \( X \), then

\[
\text{Pic}(\varphi) = H^0(M, \mathbb{Z}) \oplus \text{Pic}(X) \oplus H^4(M, \mathbb{Z}).
\]

So, the Picard group of a K3 surface \( X \) and the Picard group of the Calabi–Yau structure naturally defined by it differ just by the hyperbolic plane \( H^0 \oplus H^4 \).

ii) If \( \omega \) is a symplectic structure and \( \varphi = \exp(i\omega) \), then

\[
\text{Pic}(\varphi) = H^2(M, \mathbb{Z})_\omega \oplus \{ \delta_0 + \delta_4 \in (H^0 \oplus H^4)(M, \mathbb{Z}) \mid \delta_0 \int_M \omega^2 = 2 \int_M \delta_4 \}.
\]

Here, \( H^2(M, \mathbb{Z})_\omega \) is the group of \( \omega \)-primitive classes. Note that for \( \omega \) very general the Picard group \( \text{Pic}(\exp(i\omega)) \) is trivial, as \( \int_M \omega^2 \) will be irrational and any integral class orthogonal to \( \omega \) will be trivial.

iii) Let us twist an ordinary Calabi–Yau structure \( \sigma \) by a B-field \( B \in H^2(M, \mathbb{R}) \), i.e. we consider \( \varphi := \exp(B)\sigma = \sigma + B^0 \wedge \sigma \). Then

\[
\text{Pic}(\varphi) = H^4(M, \mathbb{Z}) \oplus \{ \delta_0 + \delta_2 \in (H^0 \oplus H^2)(M, \mathbb{Z}) \mid \int_M \delta_2 \wedge \sigma = \delta_0 \int_M B \wedge \sigma \}.
\]
Clearly, $H^4(M, \mathbb{Z}) \oplus \text{Pic}(X) \subset \text{Pic}(\exp(B)\sigma)$, where $X$ is the K3 surface defined by $\sigma$. In general, this inclusion will be strict. Note that $\text{Pic}(\varphi)$ depends only on $\sigma$ and the $(0,2)$-part of $B$.

If $X$ is a classical K3 surface then the Hodge decomposition $H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$ induces a weight two Hodge structure of $H^*(M, \mathbb{Z})$ whose $(1,1)$-part is $H^0(X, \mathbb{C}) \oplus H^{1,1}(X) \oplus H^4(X, \mathbb{C})$ and its $(2,0)$-part is spanned by the Calabi–Yau form $\sigma$.

In the same vain, any generalized Calabi–Yau structure $\varphi$ defines a weight two Hodge structure on $H^*(M, \mathbb{Z})$ whose $(2,0)$-part is spanned by the cohomology class of $\varphi$. This determines the other parts by requiring that the Hodge decomposition is orthogonal with respect to the Mukai pairing. Clearly, the Picard group $\text{Pic}(\varphi)$ is of pure type $(1,1)$.

**Definition 4.5.** The transcendental lattice $T(\varphi)$ of a generalized Calabi–Yau structure $\varphi$ on $M$ is the orthogonal complement of $\text{Pic}(\varphi)$ in $H^*(M, \mathbb{Z})$.

More precisely,

$$T(\varphi) := \{ \gamma \in H^*(M, \mathbb{Z}) \mid \langle \gamma, \delta \rangle = 0 \text{ for all } \delta \in \text{Pic}(\varphi) \}. $$

Via the intersection pairing, the transcendental lattice $T(\varphi)$ can be identified with the dual of $H^*(M, \mathbb{Z})/\text{Pic}(\varphi)$. Also note that the transcendental lattice $T(\varphi)$ is almost never pure.

**Example 4.6.** i) Let us recall that the transcendental lattice $T(X)$ of a classical K3 surface $X$ is the orthogonal complement of $\text{Pic}(X)$ inside $H^2(X, \mathbb{Z})$. One easily sees that $T(X) = T(\sigma)$ for a Calabi–Yau form $\sigma$ on $X$.

ii) For $\varphi = \exp(B)\sigma$ we find that $\gamma_0 + \gamma_2 + \gamma_4 \in T(\varphi)$ implies $\gamma_0 = 0$ and $\gamma_2 \in T(X)$.

Now we shall explain how to identify the transcendental lattice $T(\varphi)$ in the case $\varphi = \exp(B)\sigma$ with $T(X, \alpha_B)$ defined before. Here, $B \in H^2(X, \mathbb{R})$ is a B-field whose $(0,2)$-component $B^{0,2} \in H^{0,2}(X) = H^2(X, \mathcal{O})$ induces a torsion element $\alpha_B$ in $H^2(X, \mathcal{O}^*)$ via the exponential map $H^2(X, \mathcal{O}) \to H^2(X, \mathcal{O}^*)$.

As before, $X$ is the K3 surface defined by $\sigma$.

Let us first recall some basic facts concerning the B-field and its associated Brauer class $\alpha_B$:

i) Since $B$ is real, $B^{0,2} \in H^2(X, \mathcal{O})$ is trivial if and only if $B \in H^{1,1}(X, \mathbb{R})$.

ii) The $(0,2)$-part of $B$ is contained in the image of $H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O})$ if and only if $B \in H^2(X, \mathbb{Z}) + H^{1,1}(X, \mathbb{R})$.

iii) Using the exponential sequence $H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}) \to H^2(X, \mathcal{O}^*)$, one finds that the B-field $B \in H^2(X, \mathbb{R})$ defines an $r$-torsion class $\alpha_B \in H^2(X, \mathcal{O}^*)$ if and only if $rB \in H^2(X, \mathbb{Z}) + H^{1,1}(X, \mathbb{R})$.

Let us now assume that $\alpha_B \in H^2(X, \mathcal{O}^*)$ is $r$-torsion. Then we can write $B = \beta + \delta$ with $\beta \in (1/r)H^2(X, \mathbb{Z})$ and $\delta \in H^{1,1}(X, \mathbb{R})$. This decomposition is not unique. If, however, $B \in H^2(M, \mathbb{Q})$ then $\delta \in \text{Pic}(X)_{\mathbb{Q}}$ and, hence, the induced homomorphism $B : T(X) \to \mathbb{Q}$, $\gamma \mapsto \int_M \gamma \wedge B$ can be described in terms of $\beta$ as $\gamma \mapsto \int_M \gamma \wedge \beta$. Clearly, the image is contained $(1/r)\mathbb{Z} \subset \mathbb{Q}$.
and one has
\[ T(X, \alpha_B) = \text{Ker}(B : T(X) \to \mathbb{Q}/\mathbb{Z}) = \{ \gamma \in T(X) \mid \gamma \beta \in \mathbb{Z} \}. \]

Note that if \( B \) is not rational, even if the induced \( \alpha_B \) is torsion, then the map \( \gamma \mapsto \int_M \gamma \wedge B \) takes values in \( \mathbb{R} \) and usually \( T(X, \alpha_B) \) defined as the kernel of the induced map to \( \mathbb{R}/\mathbb{Z} \) will be too small to be interesting. But if one starts out with a torsion class \( \alpha \in H^2(X, \mathcal{O}^*) \) one always finds a rational B-field \( B \) with \( \alpha = \alpha_B \). So, we will continue to assume that \( B \) is rational.

The sublattice \( T(X, \alpha_B) \subset T(X) \) will be viewed as a sublattice of \( H^*(M, \mathbb{Z}) \) via the injection

\[ \eta : T(X, \alpha_B) \to (H^2 \oplus H^4)(M, \mathbb{Z}), \ \gamma \mapsto \gamma + \gamma \wedge B. \]

Note that \( \eta \) is the restriction of the isometry (with respect to the Mukai pairing)

\[ \exp(B) : H^*(X, \mathbb{Q}) \xrightarrow{\sim} H^*(X, \mathbb{Q}). \]

**Proposition 4.7.** For a rational B-field \( B \) the map \( \eta \) defines an Hodge isometry

\[ T(X, \alpha_B) \cong T(\varphi), \]

where \( \varphi \) is the generalized Calabi–Yau structure \( \sigma + B \wedge \sigma \).

**Proof.** Firstly, it is clear that \( \eta \) is compatible with the quadratic forms provided by the intersection and the Mukai pairing, respectively.

Secondly, the Hodge structures are respected. Indeed, the \( (2,0) \)-part of \( T(X, \alpha_B) \) is generated by \( \sigma \) and \( \eta(\sigma) = \varphi \) spans the \( (2,0) \)-part of the Hodge structure on \( H^*(M, \mathbb{Z}) \) defined by \( \varphi \). (In fact, \( \eta \) is nothing but \( \exp(B) \), so it is clearly compatible with the Hodge structures.)

Thus, it suffices to show that \( \eta \) is indeed bijective. Let us first verify the inclusion \( T(\varphi) \subset \eta(T(X, \alpha_B)) \): Let \( \gamma = \gamma_0 + \gamma_2 + \gamma_4 \in T(\varphi) \). As was observed above, one has \( \gamma_0 = 0 \) and \( \gamma_2 \in T(X) \). As \( T(X, \alpha_B) \) and \( T(\varphi) \) are independent of the \( (1,1) \)-part of \( B \), we may assume \( B = \beta \in (1/\rho)H^2(M, \mathbb{Z}) \).

Then \( r + rB + 0 \in \text{Pic}(\varphi) \), for \( \int_M (rB) \wedge \sigma = r \int_M B \wedge \sigma \). Thus, \( \gamma \) is orthogonal to \( r + rB \) and, therefore, \( \int_M \gamma_2 \wedge (rB) = r \int_M \gamma_4 \). Hence, \( \gamma \in \text{Im}(\eta) \).

For the other inclusion one has to check that \( \gamma := \gamma_2 + \gamma_2 \wedge B \) is orthogonal to \( \text{Pic}(\varphi) \) for any \( \gamma_2 \in T(X, \alpha_B) \). Clearly, any such \( \gamma \) is orthogonal to \( H^1(M, \mathbb{Z}) \oplus \text{Pic}(X) \) and to the special element \( r + rB \in \text{Pic}(\varphi) \). Let \( \delta_0 + \delta_2 \in \text{Pic}(\varphi) \). Then \( \delta_0 \int_M B \wedge \sigma = \int_M \delta_2 \wedge \sigma \). Writing \( B^{0,2} = \mu \sigma \) and \( \delta_0^{0,2} = \lambda \sigma \), this condition becomes \( \delta_0 \mu = \lambda \). Hence, \( \delta_2 = \delta_0 B + \delta_2^0 \) with \( \delta_2^0 \in H^{1,1}(X, \mathbb{Q}) \).

Here, the rationality of \( \delta_2^0 \) follows from the rationality of the other terms. For \( \gamma_2 \in T(X) \) this yields \( \gamma_2 \wedge \delta_2 = \delta_0 \gamma_2 \wedge B + \gamma_2 \wedge \delta_2^0 = \delta_0 \gamma_2 \wedge B \), i.e. \( \eta(\gamma_2) \) is orthogonal to \( \delta_0 + \delta_2^0 \).

**Corollary 4.8.** There exists an Hodge isometry \( T(X, \alpha_B) \cong T(X', \alpha_{B'}) \) if the periods of the two generalized Calabi–Yau structures \( \varphi = \sigma + B \wedge \sigma \) and \( \varphi' = \sigma' + B' \wedge \sigma' \) are contained in the same orbit of the natural action of \( O(H^*(M, \mathbb{Z})) \).
Proof. This is obvious, as $T(X, \alpha_B)$ and $T(\varphi)$ are Hodge isometric and $T(\varphi)$ is defined in terms of the period. □

In many cases the corollary can be improved to an “if and only if” statement. In general, however, this is not true. In the untwisted case any Hodge isometry $T(X) \cong T(X')$ can be extended to an Hodge isometry $\tilde{H}(X, \mathbb{Z}) \cong \tilde{H}(X', \mathbb{Z})$ due to results of Nikulin [18]. The existence of the hyperbolic plane $H^0 \oplus H^4$ in the orthogonal complement $T^\perp$ is crucial for this. In the twisted case, $T(\varphi)^\perp$ does not necessarily contain such an hyperbolic plane. In fact, in [13] we give an explicit example of an Hodge isometry $T(\varphi) \cong T(\varphi')$ which cannot be extended simply due to the fact that the Picard groups are not isometric. We furthermore put forward a version of Căldăraru’s conjecture that takes into account not only the transcendental lattice $T(X, \alpha_B) \cong T(\varphi)$, but the full Hodge structure defined by the generalized Calabi–Yau structure $\varphi = \sigma + B \wedge \sigma$.

Clearly, the action of the discrete group $\text{O}(H^*(M, \mathbb{Z}))$ studied in this section is intimately related to mirror symmetry for K3 surfaces. For instance one can observe that $\text{O}(H^*(M, \mathbb{Z}))$ frequently interchanges the periods of honest K3 surfaces with those of symplectic structures. Although mirror symmetry phenomena on the level of moduli spaces are much simpler for K3 surfaces than for arbitrary Calabi–Yau manifolds, not much is known about it on a deeper level, e.g. mirror symmetry in its homological version (but see the recent paper [20]) or mirror symmetry as a duality for conformal field theories, vertex algebras as in [14] for tori.

5. THE MODULI SPACE OF SYMPLECTIC STRUCTURES AS A LAGRANGIAN

This section is devoted to a canonical symplectic form on the moduli space $\mathfrak{M}_{\text{gen}}$ of generalized Calabi–Yau structures on $M$. We will show that it coincides with the pull-back (via the period map) of the curvature of the tautological bundle with respect to a certain hermitian structure. Moreover, the subset of generalized Calabi–Yau structures of the form $\exp(i\omega)$ is Lagrangian.

As before, $\langle \cdot, \cdot \rangle$ denotes the Mukai pairing on even forms.

Lemma 5.1. The pairing $H(\varphi, \psi) = \int \langle \varphi, \overline{\psi} \rangle$ defines a non-degenerate indefinite hermitian product on $\mathcal{A}^{2\ast}(M)$.

Proof. E.g. if $\varphi_2 \neq 0$, then $\int \langle \varphi, * \varphi_2 \rangle = ||\varphi_2||^2 > 0$. Together with similar calculations in the other cases this shows that $H$ is non-degenerate. Moreover, $H$ is indefinite, for $H(\omega, \omega) > 0$ for any symplectic structure $\omega$ and $H(\alpha, \alpha) < 0$ for any primitive $(1, 1)$-form $\alpha$ (with respect to an arbitrary Calabi–Yau structure and a Kähler form).

□

Corollary 5.2. The constant two-form $\Omega := \text{Im}(H)$ defines a symplectic structure on $\mathcal{A}^{2\ast}_\mathbb{C}(M)$.

□
Since \( \int \langle , \rangle \) is invariant under \( B \)-field transformations and diffeomorphisms, we obtain this way a hermitian structure \( H \) and a two-form \( \Omega \) on \( \mathcal{N}_{\text{gen}} \). That \( \Omega \) is indeed a symplectic structure can be seen by the following explicit description.

**Lemma 5.3.** Let \( \varphi \) be a generalized Calabi–Yau structure on \( M \). Then the tangent space \( T_\varphi \mathcal{N}_{\text{gen}} \) can be naturally identified with

\[
\{ \alpha \in H^*(M, \mathbb{C}) \mid \langle \alpha, [\varphi] \rangle = \langle \alpha, [\overline{\varphi}] \rangle = 0 \}.
\]

The form \( \Omega \), which is given by \( \Omega(\alpha, \beta) = \text{Im} \langle \alpha, \overline{\beta} \rangle \), defines a symplectic structure, i.e. it is non-degenerate and closed.

**Proof.** One way to prove this, is to note that the period map \( \mathcal{P}_{\text{gen}} \) is a local isomorphism (this is the analogue of the Local Torelli theorem). Hence, it suffices to compute the tangent space of \( \tilde{Q} \) at \([\varphi]\). Now, \( \langle [\varphi] + \varepsilon \alpha, [\varphi] + \varepsilon \alpha \rangle = 0 \) yields \( \langle [\varphi], \alpha \rangle = 0 \). Thus, the tangent space \( T_{[\varphi]} \tilde{Q} \) is canonically isomorphic to \( [\varphi] \perp / \mathbb{C} \cdot [\varphi] \), which can be identified with the subspace given above, due to \( \langle [\varphi], [\overline{\varphi}] \rangle \neq 0 \).

The Mukai pairing \( \langle , \rangle \) is non-degenerate on the orthogonal complement of \( P_\varphi \), i.e. on the tangent space \( T_\varphi \mathcal{N}_{\text{gen}} \). Hence, \( \Omega \) is a non-degenerate two-form on \( \mathcal{N}_{\text{gen}} \).

The closedness of \( \Omega \) follows from the construction: Its pull-back is a restriction of a constant form on \( \mathcal{A}^2(M)_{\mathbb{C}} \). \( \square \)

Note that since \( H \) is indefinite, the symplectic structure \( \Omega \) is not Kähler with respect to the natural complex structure on \( \mathcal{N}_{\text{gen}} \).

Let \( \mathcal{O}(-1) \) be the tautological line bundle on \( \mathbb{P}(H^*(M, \mathbb{C})) \). The hermitian structure on \( H^*(M, \mathbb{C}) \) defined by the Mukai pairing \( \langle \alpha, \beta \rangle \) can be viewed as a constant hermitian structure on the constant bundle \( H^*(M, \mathbb{C}) \otimes \mathcal{O} \) on \( \mathbb{P}(H^*(M, \mathbb{C})) \). Consider the Euler sequence and, in particular, the natural inclusion \( \mathcal{O}(-1) \subset H^*(M, \mathbb{C}) \otimes \mathcal{O} \), which provides a natural hermitian structure on \( \mathcal{O}(-1) \). Then the constant connection on \( H^*(M, \mathbb{C}) \otimes \mathcal{O} \) induces the Chern connection \( \nabla \) on \( \mathcal{O}(-1) \). A standard calculation, well-known in the positive definite case, relates the curvature \( F_\nabla \) to the above defined symplectic form \( \Omega \):

**Proposition 5.4.** \( \mathcal{P}^*(iF_\nabla) = -\Omega \). \( \square \)

As remarked above, we cannot expect, due to the indefiniteness of the chosen hermitian structure on \( \mathbb{P}(H^*(M, \mathbb{C})) \), that the curvature satisfies any positivity condition.

Let \( \text{Sympl}(M) \) be the moduli space of symplectic structures on \( M \), which is identified with a submanifold of \( \mathcal{N}_{\text{gen}} \) via the the natural inclusion \( \omega \mapsto \exp(i\omega) \). (For simplicity we assume here that any symplectic structure on \( M \) is Kähler. Otherwise, one has to work with the moduli space of all generalized Calabi–Yau structures and not only of those of hyperkähler type.)
An easy dimension count shows that $\text{Sympl}(M)$ is a real 22-dimensional submanifold of the complex 22-dimensional complex manifold $\mathcal{N}_{\text{gen}}$.

**Proposition 5.5.** The submanifold $\text{Sympl}(M) \subset \mathcal{N}_{\text{gen}}$ is a Lagrangian submanifold with respect to $\Omega$.

**Proof.** The tangent space $T_\omega \text{Sympl}(M)$ is canonically identified with $H^2(M, \mathbb{R})$ and the tangent map $d\exp : T_\omega \text{Sympl}(M) \to T_{\exp(i\omega)}$ is given by

$$H^2(M, \mathbb{R}) \to T_{\exp(i\omega)} \mathcal{N}_{\text{gen}} \subset H^*(M, \mathbb{C})$$

$$\alpha \mapsto i\alpha - \alpha \omega$$

Clearly, $\Omega(i\alpha - \alpha \omega, i\beta - \beta \omega) = \text{Im}(i\alpha, -i\beta) = 0$, whenever $\alpha, \beta$ are both real cohomology classes. $\square$

**Remark 5.6.** By construction the curvature $F_\nabla$ is invariant under the action of the orthogonal group $O(H^*(M, \mathbb{R})) = O(4, 20)$. Thus, translating $\text{Sympl}(M)$ by elements in $O(4, 20)$ yields new Lagrangian submanifolds.

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