POGORELOV TYPE $C^2$ ESTIMATES FOR SUM HESSIAN EQUATIONS AND A RIGIDITY THEOREM

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Abstract. We mainly study Pogorelov type $C^2$ estimates for solutions to the Dirichlet problem of Sum Hessian equations. We establish respectively Pogorelov type $C^2$ estimates for $k$-convex solutions and admissible solutions under some conditions. Furthermore, we apply such estimates to obtain a rigidity theorem for $k$-convex solutions of Sum Hessian equations in Euclidean space.

1. Introduction

In this paper, we consider the following Dirichlet problem of Sum Hessian equations,

\begin{align}
\sigma_k(D^2 u) + \alpha \sigma_{k-1}(D^2 u) &= f(x, u, Du), \quad \text{in } \Omega,
\sigma_k(D^2 u) &= f(x, u, Du), \quad \text{on } \partial \Omega.
\end{align}

Here, $u$ is an unknown function defined on $\Omega$, $\alpha$ is a positive constant. Denote by $Du$ and $D^2 u$ the gradient and the Hessian of $u$. We also require $f > 0$ and smooth enough with respect to every variables. Let $\sigma_k(D^2 u) = \sigma_k(\lambda(D^2 u))$ denotes the $k$-th elementary symmetric function of the eigenvalues of the Hessian matrix $D^2 u$. Namely, for $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$,

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}.$$ 

As we all know, the following classic $k$-Hessian equation

\begin{align}
\sigma_k(D^2 u) &= f(x, u, Du), \quad \text{in } \Omega
\end{align}

is an important research content in the field of fully nonlinear partial differential equations and geometric analysis, which is closely related to many important geometric problems, such as [1, 8, 9, 11, 17, 18, 26, 28, 33]. The equation (1.1) is a natural extension of (1.2), which is a Hessian type equation formed by linear combination of the $k$ Hessian operators. These Hessian type equations have been studied and applied in geometry. Harvey and Lawson [21] considered the special Lagrangian equations which is one of this category. Krylov [23] and Dong [14] also considered equations close to this type and obtained the curvature estimates using the concavity of the operators. In [20], Guan and Zhang studied the curvature estimates for equations of such type with the right hand side not depending...
on gradient term but with coefficients depending on the position of the hypersurfaces. The geometrical problems in the hyperbolic space also reduce to equations of this type [15].

An important problem in the study of Hessian equations (1.2) is how to obtain $C^2$ estimates. For the relative work, we refer the readers to [7, 8, 13, 16, 17, 18, 19]. Especially when the right hand side function depends on the gradient, how to obtain $C^2$ estimates for equations (1.2) is a longstanding problem [17].

In this paper, we are interested in Pogorelov type $C^2$ estimates for the Dirichlet problem (1.1). Pogorelov type $C^2$ estimates is a type of interior $C^2$ estimates with boundary information. This type of $C^2$ estimates were established at first by Pogorelov [28] for Monge-Ampère equations, which play an important role in studying the regularity for fully nonlinear equations. For example, Pogorelov type $C^2$ estimates were established in order to study the regularity for degenerate Monge-Ampère equations [4, 31]. For $k$-Hessian equations (1.2), if the right hand side function is $f = f(x, u)$, Chou and Wang [13] established Pogorelov $C^2$ estimates of $k$ convex solutions:

$$(-u)^{1+\delta} \Delta u \leq C.$$  

Sheng-Ubas-Wang [32] established Pogorelov type curvature estimates for the curvature equations of the graphic hypersurface including the Hessian equations. If the right hand side function depends on the gradient, Li-Wang and the last author [25] established Pogorelov type $C^2$ estimates for $k + 1$ convex solutions for equation (1.2):

$$(-u) \Delta u \leq C.$$  

In particular, for 2 convex solutions of the case $k = 2$, the following Pogorelov type $C^2$ estimates hold:

$$(-u)^{3} \Delta u \leq C.$$  

Here, positive constants $\beta$ and $C$ depend on the domain $\Omega$, $f$, $\sup_{\Omega} |u|$ and $\sup_{\Omega} |Du|$.

A natural question is whether Pogorelov type $C^2$ estimates are still valid for the Dirichlet problem (1.1)?

First, we give the definition of $k$ convex solution [8]:

**Definition 1.1.** For a domain $\Omega \subset \mathbb{R}^n$, a function $v \in C^2(\Omega)$ is called $k$ convex if the eigenvalues $\lambda(x) = (\lambda_1(x), \cdots, \lambda_n(x))$ of the Hessian $\nabla^2 v(x)$ is in $\Gamma_k$ for all $x \in \Omega$, where $\Gamma_k$ is the Garding’s cone

$$\Gamma_k = \{ \lambda \in \mathbb{R}^n \mid \sigma_m(\lambda) > 0, \ m = 1, \cdots, k \}.$$  

The main results are as follows:

**Theorem 1.2.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, and let $u \in C^4(\Omega) \cap C^{0,1}(\overline{\Omega})$ be a $k - 1$ convex solution for the Dirichlet problem (1.1), where $f(x, u, p) \in C^{1,1}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ is a positive function. Then we have

(a) For $k = 2, k = 3$, there are some positive constants $\beta$ and $C$, such that

$$(-u)^{3} \Delta u \leq C.$$  

(b) For $1 < k \leq n$, if $f^\frac{1}{k}(x,u,p)$ is a convex function with respect to $p$, there are some positive constants $\delta$ and $C$, such that

\begin{equation}
(-u)^{1+\delta} \Delta u \leq C.
\end{equation}

Here, $\delta > 0$ can be arbitrarily small, the positive constants $\beta$ and $C$ depend on the domain $\Omega$, $\alpha$, $f$, $\sup \Omega |u|$ and $\sup \Omega |Du|$.

**Remark 1.3.** In [24], Li-Wang and the last author proved that Sum Hessian operator $\sigma_k + \alpha \sigma_{k-1}$ is elliptic operator in the set

$$\tilde{\Gamma}_k = \Gamma_{k-1} \cap \{ \lambda | \sigma_k(\lambda) + \alpha \sigma_{k-1}(\lambda) > 0 \},$$

i.e., $\tilde{\Gamma}_k$ is the admissible set of the equations (1.1). Here, in addition to the requirements for $k - 1$ convex solutions, the right hand side function $f > 0$ ensures the ellipticity of the equations (1.1).

**Remark 1.4.** Similar to the result in [24], because $f$ depends on the gradient, the constant $\beta$ is large in theorem 1.2, and we cannot improve $\beta$ to 1 or 1 + $\delta$.

**Theorem 1.5.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, and let $u \in C^4(\Omega) \cap C^{0,1}(\overline{\Omega})$ be a $k$ convex solution for the Dirichlet problem (1.1), where $f(x,u,p) \in C^{1,1}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ is a positive function. Then we have

\begin{equation}
(-u) \Delta u \leq C.
\end{equation}

Here, positive constant $C$ depends on the domain $\Omega$, $\alpha$, $f$, $\sup \Omega |u|$ and $\sup \Omega |Du|$.

An application of the interior $C^2$ estimates is to prove rigidity theorems for equations. For the following $k$-Hessian equations in $n$-dimensional Euclidean spaces,

\begin{equation}
\sigma_k(D^2 u) = 1,
\end{equation}

Chang and Yuan [11] proposed a problem that: Are the entire solutions of (1.5) with lower bound only quadratic polynomials?

For $k = 1$, equation (1.5) is a linear equation. It is a obvious result coming from the Liouville property of the harmonic functions. For $k = n$, equation (1.5) is Monge-Ampère equation, which is a well known theorem. The work of Jörgens [22], Calabi [10], Pogorelov [27, 28] proved that every entire strictly convex solution is a quadratic polynomial. Then, Cheng and Yau [12] gave another more geometry proof. In 2003, Caffarelli and Li [6] extended the theorem of Jörgens, Calabi and Pogorelov.

For general $1 < k < n$, there are few related results. For $k = 2$, Chang and Yuan [11] have proved that, if

$$D^2 u \geq \delta - \sqrt{\frac{2n}{n-1}},$$

for any $\delta > 0$, then the entire solution of the equation (1.5) only has quadratic polynomials. For general $k$, Bao-Chen-Guan-Ji [3] proved that strictly entire convex solutions
of equation (1.5) satisfying a quadratic growth are quadratic polynomials. Here, the quadratic growth means that: there are some positive constants $c, b$ and sufficiently large $R$, such that
\begin{equation}
(1.6) \quad u(x) \geq c|x|^2 - b, \quad |x| \geq R.
\end{equation}

[25] relaxed the condition of strictly convex solutions in [3] to $k + 1$ convex solutions, and proved that the entire $k + 1$ convex solutions of the equations (1.5) with quadratic growth are quadratic polynomials. In 2016, Warren [34] proved that equation (1.5) has non-polynomial entire $k$ convex solutions for $n \geq 2k - 1$ by constructing examples.

In this paper, we apply Pogorelov type $C^2$ estimates to prove the rigidity theorem of the following Sum Hessian equations in Euclidean space,
\begin{equation}
(1.7) \quad \sigma_k(D^2u) + \alpha\sigma_{k-1}(D^2u) = 1.
\end{equation}

**Theorem 1.6.** The entire $k$ convex solutions of the equations (1.7) defined in $\mathbb{R}^n$ with quadratic growth (1.6) are quadratic polynomials.

At last, using the idea of Warren [34], we can give an example of non-polynomial entire $k - 1$ convex solution.

**Example 1.7.** When $n = 3$, the 1-convex function
\begin{equation*}
u(x, y, t) = e^{4t} - \frac{1}{4}(x^2 + y^2) + \frac{1}{16}(7e^{-4t} - e^{4t} - 4t^2)
\end{equation*}
solves
\begin{equation*}
\sigma_2(D^2u) + \sigma_1(D^2u) = 1.
\end{equation*}

The organization of our paper is as follows: In section 2, we give the preliminary knowledge. The proofs of Theorem 1.2 and Theorem 1.5 are respectively given in section 3 and section 4. In the last section, we prove the rigidity theorem of the equations (1.7).

2. Preliminary

In this section, we give the preliminary knowledge related to our theorems and their proofs. To make it convenient, we denote
\begin{equation*}
S_k(\lambda) := \sigma_k(\lambda) + \alpha\sigma_{k-1}(\lambda).
\end{equation*}

For the related notation of $\sigma_k$, we use the denotation in [24, 29]. Let $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$, we have
\begin{enumerate}
\item[(i)] $S_k^{pp}(\lambda) := \frac{\partial S_k(\lambda)}{\partial \lambda_p} = \sigma_{k-1}(\lambda)p + \alpha\sigma_{k-2}(\lambda)p = S_{k-1}(\lambda)p, \quad p = 1, 2, \cdots, n$;
\item[(ii)] $S_k^{pp,qq}(\lambda) := \frac{\partial^2 S_k(\lambda)}{\partial \lambda_p \partial \lambda_q} = S_{k-2}(\lambda)pq, \quad p, q = 1, 2, \cdots, n$, and $S_k^{pp,pp}(\lambda) = 0$;
\item[(iii)] $S_k(\lambda) = \lambda_i S_{k-1}(\lambda)|i| + S_k(\lambda)|i|, \quad i = 1, \cdots, n$;
\item[(iv)] $\sum_{i=1}^{n} S_k(\lambda|i) = (n - k)S_k(\lambda) + \alpha\sigma_{k-1}(\lambda)$;
\end{enumerate}
In the joint work with Li, Wang and the last author \cite{24} proved that the admissible solution set of Sum Hessian operator $S_k(\lambda)$ is

$$\tilde{\Gamma}_k = \Gamma_{k-1} \cap \{\lambda|S_k > 0\},$$

and $S_k(\lambda)$ and $(\frac{S_k}{S_l})^{1/2}$ are concave functions on $\tilde{\Gamma}_k$. Using the method in \cite{19}, we can get the following lemma.

\textbf{Lemma 2.1.} Assume that $k > l$, for $\vartheta = \frac{1}{k-l}$, then for $\lambda \in \tilde{\Gamma}_k$, we have

\begin{equation}
-\frac{S_{pp,qq}}{S_k} u_{pph} u_{qqh} + \frac{S_{pp,qq}}{S_l} u_{pph} u_{qqh} \\
\geq \left(\frac{(S_k)_h}{S_k} - \frac{(S_l)_h}{S_l}\right) \left((\vartheta - 1)\frac{(S_k)_h}{S_k} - (\vartheta + 1)\frac{(S_l)_h}{S_l}\right).
\end{equation}

Furthermore, for sufficiently small $\delta > 0$, we have

\begin{equation}
-\frac{S_{pp,qq}}{S_k} u_{pph} u_{qqh} + \left(1 - \vartheta + \frac{\vartheta}{\delta}\right)\frac{(S_k)_h}{S_k} \\
\geq S_k(\vartheta + 1 - \delta \vartheta) \left[\frac{(S_l)_h}{S_l}\right]^2 - \frac{S_k S_{pp,qq}}{S_l} u_{pph} u_{qqh}.
\end{equation}

The following Lemma comes from \cite{2}.

\textbf{Lemma 2.2.} Denote by $\text{Sym}(n)$ the set of all $n \times n$ symmetric matrices. Let $F$ be a $C^2$ symmetric function defined in some open subset $\Psi \subset \text{Sym}(n)$. At any diagonal matrix $A \in \Psi$ with distinct eigenvalues, let $\tilde{\Gamma}(B,B)$ be the second derivative of $C^2$ symmetric function $F$ in direction $B \in \text{Sym}(n)$, then

$$\tilde{\Gamma}(B,B) = \sum_{j,k=1}^{n} \tilde{f}^{jk} B_{jj} B_{kk} + 2 \sum_{j<k} \frac{\tilde{f}^{jk} - \frac{j}{k} \frac{j}{k}}{\kappa_j - \kappa_k} B_{jk}^2.$$

\textbf{Lemma 2.3.} Assume $\lambda = (\lambda_1, \cdots, \lambda_n) \in \tilde{\Gamma}_k$. If $S_{k+1}(\lambda) > 0$, then $\lambda \in \tilde{\Gamma}_{k+1}$.

\textit{Proof.} By the definition of $\tilde{\Gamma}_{k+1}$, we only need to prove $\sigma_k(\lambda) > 0$. If $\sigma_k(\lambda) \leq 0$, since $S_k(\lambda) > 0, S_{k+1}(\lambda) > 0$, then

$$\alpha \sigma_{k-1} > |\sigma_k|, \quad \sigma_{k+1} > |\alpha \sigma_k|,$$

namely, $\alpha \sigma_{k-1} \sigma_{k+1} > \alpha \sigma_k^2$, this contradicts Newton’s inequality. \hfill \Box

\textbf{Remark 2.4.} Using Lemma 2.3, we can define $\tilde{\Gamma}_k$ as

$$\tilde{\Gamma}_k = \{\lambda \in \mathbb{R}^n|S_m(\lambda) > 0, \quad m = 1, 2, \cdots, k\}.$$
Lemma 2.5. Assume that \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \tilde{\Gamma}_k, 1 \leq k \leq n, \lambda_1 \geq \cdots \geq \lambda_n, \) then
(a) For any \( 1 \leq s < k, \) we have
\[
S_s(\lambda) > \lambda_1 \cdots \lambda_s + \alpha \lambda_1 \cdots \lambda_{s-1};
\]
(b) For any \( 1 \leq j \leq k-1, \) there exists a positive constant \( \theta \) depending on \( n, k, \) such that
\[
S^j_k(\lambda) \geq \frac{\theta S_k(\lambda)}{\lambda_j}.
\]

Proof. (a) Using \( \lambda \in \tilde{\Gamma}_k \subset \Gamma_s, \) we have
\[
\lambda_1 > 0, \quad \lambda_2 > 0, \ldots, \quad \lambda_s > 0,
\]
and
\[
S_s(\lambda|1) > 0, \quad S_{s-1}(\lambda|12) > 0, \ldots, \quad S_1(\lambda|12 \cdots s) > 0.
\]
Using the above inequalities, we get
\[
S_s(\lambda) = \lambda_1 S_{s-1}(\lambda|1) + S_s(\lambda|1)
\]
\[
> \lambda_1 S_{s-1}(\lambda|1) = \lambda_1 \lambda_2 S_{s-2}(\lambda|12) + \lambda_1 S_{s-1}(\lambda|12)
\]
\[
> \lambda_1 \lambda_2 S_{s-2}(\lambda|12) = \cdots
\]
\[
> \lambda_1 \lambda_2 \cdots \lambda_{s-1}(\lambda_s + \alpha).
\]
(b) If \( S_k(\lambda|j) \leq 0, \) obviously we have
\[
S^j_k(\lambda) = \frac{S_k(\lambda) - S_k(\lambda|j)}{\lambda_j} \geq \frac{S_k(\lambda)}{\lambda_j}.
\]
If \( S_k(\lambda|j) > 0, \) then \( (\lambda|j) \in \tilde{\Gamma}_k \) by Lemma 2.3. Using (2.3), we obtain
\[
S^j_k(\lambda) > \frac{\lambda_1 \cdots \lambda_k + \alpha \lambda_1 \cdots \lambda_{k-1}}{\lambda_j}.
\]
On the other hand, \( (\lambda|j) \in \tilde{\Gamma}_k \) implies \( \lambda_k > 0 \) and
\[
\lambda_k + \cdots + \lambda_n + \alpha > 0.
\]
So there exists some constant \( \theta \) depending on \( n, k, \) such that
\[
\theta S_k(\lambda) \leq \lambda_1 \cdots \lambda_k + \alpha \lambda_1 \cdots \lambda_{k-1}.
\]
Then we obtain (2.4). \( \square \)

Lemma 2.6. For \( \lambda \in \tilde{\Gamma}_k, \) we have \( S_k^2(\lambda) - S_{k-1}(\lambda)S_{k+1}(\lambda) \geq 0. \)

Proof. Since \( \lambda \in \tilde{\Gamma}_k, \) \( S_k(\lambda) > 0, S_{k-1}(\lambda) > 0. \) If \( S_{k+1}(\lambda) \leq 0, \) the Lemma obviously holds.
If \( S_{k+1}(\lambda) > 0, \) Lemma 2.3 implies \( \lambda \in \tilde{\Gamma}_{k+1} \subset \Gamma_k. \) Thus
\[
S_k^2 - S_{k-1}S_{k+1} = (\sigma_k + \alpha \sigma_{k-1})^2 - (\sigma_{k-1} + \alpha \sigma_{k-2})(\sigma_{k+1} + \alpha \sigma_{k})
\]
\[
= \sigma_k^2 + \alpha^2 \sigma_{k-1}^2 + \alpha \sigma_k \sigma_{k-1} - (\sigma_{k-1} \sigma_{k+1} + \alpha^2 \sigma_{k-2} \sigma_k + \alpha \sigma_{k-2} \sigma_{k+1})
\]
\[
\geq 0.
\]
In the above inequality, we used Newton’s inequality and the following inequality which obtained from Newton’s inequality
\[ \sigma_k \sigma_{k-1} \geq \sigma_{k-2} \sigma_{k+1}. \]

\[ \square \]

**Lemma 2.7.** Assume that \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_k, 1 \leq k \leq n, \lambda_1 \geq \cdots \geq \lambda_n, \) and \( S_k \leq N_0 \) for some positive constant \( N_0 > 0. \) Then

(a) There exists some positive constant \( K_0, \) such that
\[ \lambda_{k-1} \leq \left( \frac{N_0}{\alpha} \right)^{\frac{1}{k-1}}, \lambda_n > -K_0; \]

(b) Denote \( \kappa_i = \lambda_i + K_0. \) For any \( 1 \leq i \leq n, \) we have
\[ 2\kappa_i^{k+2} S_k^{(1)}(\lambda) \geq \kappa_i^{k+2} S_k^{(i)}(\lambda), \]
if \( \lambda_1 \) is sufficiently large;

(c) For any \( \varepsilon_0 > 0, \) we have
\[ S_k(\lambda) > (1 - \varepsilon_0)\lambda_1 S_k^{(1)}(\lambda), \]
if \( \lambda_1 \) is sufficiently large;

(d) For any \( 1 \leq i \leq n, \) there exists some positive constant \( C_0, \) such that
\[ C_0 S_k(\lambda) \geq \lambda_i S_k^{(i)}(\lambda). \]

**Proof.** (a) Since \( \lambda \in \Gamma_k, \) by Lemma 11 in [30], we have
\[ N_0 \geq S_k = \sigma_k + \alpha \sigma_{k-1} > \alpha \sigma_{k-1} \geq \alpha \lambda_1 \cdots \lambda_{k-1} > \alpha \lambda_1^{k-1}. \]
Thus, \( \lambda_{k-1} \leq \left( \frac{N_0}{\alpha} \right)^{\frac{1}{k-1}}. \) Using \( \lambda_k \geq 0, \) we have
\[ \lambda_n > -(n - k)\lambda_k \geq -(n - k)\lambda_{k-1} > -K_0. \]

(b) From (2.5), we get \( \kappa_i > 0. \) We divide into two cases to prove (2.6).

**Case 1:** \( \lambda_i \geq \lambda_1^{1/(k+2)} \). Since
\[ 2\kappa_1 S_k^{(1)} - \kappa_i^2 S_k^{(i)} = \kappa_1 S_k^{(1)}(\lambda + \alpha S_k^{(1)}(\lambda | 1i) + S_k-1(\lambda | 1i)] - \kappa_i^2[\kappa_i S_k^{(1)}(\lambda | 1i) + S_k-1(\lambda | 1i)] \geq \kappa_1^2 S_k^{(1)}(\lambda | 1i) + \kappa_i^2(\alpha S_k^{(1)}(\lambda | 1i) + \lambda_{k-1}(\lambda | 1i)] \geq \kappa_1^2 S_k^{(1)}(\lambda | 1i) + \lambda_{k-1}(\lambda | 1i)], \]
Using \( \lambda \in \Gamma_k, \) we have
\[ S_k-1(\lambda | 1i) = \sigma_k-1(\lambda | 1i) + \alpha \sigma_k-2(\lambda | 1i) \geq \sigma_k-1(\lambda | 1i) > \sigma_k-1 (\lambda | 1i) > C_{n-2}^{k-1} \frac{\lambda_2 \cdots \lambda_k \lambda_n}{\lambda_i}, \]
and
\[ S_k^{(1)} = \sigma_k-1(\lambda | 1) + \alpha \sigma_k-2(\lambda | 1) \geq \alpha \lambda_2 \cdots \lambda_{k-1}. \]
In view of (2.5), we have $|\lambda_k\lambda_n| < K^2_0$. Using the above two formulas, we have

$$(2.11) \quad S^{11}_k + S_{k-1}(\lambda|1i) > \lambda_2 \cdots \lambda_{k-1}(\alpha + C^{k-1}_{n-2}\frac{\lambda_k\lambda_n}{\lambda_i}) > 0,$$

if $\lambda_1$ is sufficiently large. Combining (2.11) with (2.9), we can get (2.6).

Case 2: $\lambda_i < \lambda_1^{1/(k+2)}$. In this case, $2\kappa_1 > \kappa_i^{k+2}$, using (2.4), we have

$$\kappa_1 S^{11}_k \geq \kappa_1 \lambda_1 S_k \geq \kappa_1 c_0 S_k \geq \kappa_1 c_0 \lambda_1 \lambda_n \lambda_i > 0,$$

when $\lambda_1$ is sufficiently large. Then we get

$$2\kappa_1^{k+2} S^{11}_k \geq \kappa_i^{k+2} S^{11}_i.$$

(c) Using (2.10) and $\sigma_k(\lambda|1) > C^{k-1}_n \lambda_2 \cdots \lambda_k \lambda_n$, we get

$$S_k - (1 - \varepsilon_0)\lambda_1 S^{11}_k = \varepsilon_0 \lambda_1 S^{11}_k + S_k(\lambda|1)$$

$$> \varepsilon_0 \lambda_1 S^{11}_k + \sigma_k(\lambda|1)$$

$$\geq \lambda_2 \cdots \lambda_{k-1}(\varepsilon_0 \lambda_1 \alpha + C^{k-1}_n \lambda_1 \lambda_n).$$

By (2.5), we know $|\lambda_k\lambda_n| < K^2_0$. Thus when $\lambda_1$ is sufficiently large, the above formula is non-negative.

(d) Since $\lambda \in \Gamma_k$, we have

$$S_k = \sigma_k + \alpha \sigma_{k-1} \geq \alpha \lambda_1 \cdots \lambda_{k-1},$$

and

$$\sigma_k(\lambda|i) > C^{k}_n \lambda_1 \cdots \lambda_{k-1} \lambda_n,$$

which implies

$$C_0 S_k - \lambda_i S^{ii}_k = (C_0 - 1)S_k + S_k(\lambda|i)$$

$$= (C_0 - 1)S_k + \sigma_k(\lambda|i) + \alpha \sigma_{k-1}(\lambda|i)$$

$$> \lambda_1 \cdots \lambda_{k-1}(C_0 - 1)\alpha + C^{k}_n \lambda_n].$$

By (2.5), when $C_0 > 1 + \frac{K_0 C^k}{\alpha}$, the above formula is non-negative. \( \square \)

3. Pogorelov type $C^2$ estimates for $k-1$ convex solutions

In this section, we prove theorem 1.2. First, for the case of $k = 2, 3$, we establish Pogorelov type $C^2$ estimates for $k-1$ convex solutions of the Dirichlet problem (1.1), and then we give a proof for the case that $f^k_i(x,u,p)$ is convex with respect to $p$.

Proof of (1.3): Let $\lambda_1 = \lambda_1(x)$ denote the biggest eigenvalue of $D^2u$. Consider the test function:

$$\phi(x) = \lambda_1(-u)^\beta \exp\{\frac{\varepsilon}{2}|Du|^2 + \frac{a}{2}|x|^2\},$$
where $\beta, \varepsilon, a$ are undetermined positive constants. Using the trick of Brendle-Choi-Daskalopoulos [5, 35], assume test function $\phi(x)$ obtains its maximum at point $x_0$. Define a function $\psi(x)$:

$$
\psi(x) = \phi(x_0)(-u)^{-\beta} \exp\left\{-\frac{\varepsilon}{2} |Du|^2 - \frac{a}{2} |x|^2\right\}.
$$

Since

$$
\phi(x_0) = \psi(x)(-u)^{-\beta} \exp\left\{\frac{\varepsilon}{2} |Du|^2 + \frac{a}{2} |x|^2\right\}
\geq \phi(x) = \lambda_1(-u)^{-\beta} \exp\left\{\frac{\varepsilon}{2} |Du|^2 + \frac{a}{2} |x|^2\right\},
$$

we have $\psi(x) \geq \lambda_1(x)$ and $\psi(x_0) = \lambda_1(x_0)$. By rotating the coordinate, we assume that $(u_{ij})$ is a diagonal matrix at $x_0$, and $u_{ii} = \lambda_i$, $\lambda_1 \geq \cdots \geq \lambda_n$.

Let $\mu$ denote the multiplicity of the biggest eigenvalue at $x_0$, namely, $\lambda_1 = \cdots = \lambda_{\mu} > \lambda_{\mu+1} \geq \cdots \geq \lambda_n$. Then from Brendle-Choi-Daskalopoulos result (Lemma 5 in [3]), we have the followings exist at $x_0$:

$$
\psi = \lambda_1 = u_{11},
$$

$$
u_{kji} = \psi_i \delta_{kj}, \quad 1 \leq k, j \leq \mu,
$$

$$
\psi_{ii} \geq u_{11ii} + 2 \sum_{j>\mu} \frac{1}{\lambda_i - \lambda_j} u_{1ji}^2,
$$

Now we restrict all calculations at point $x_0$. Since the function

$$
\tilde{\psi} = \psi(x)(-u)^{-\beta} \exp\left\{\frac{\varepsilon}{2} |Du|^2 + \frac{a}{2} |x|^2\right\} = \phi(x_0)
$$

has constant value, at $x_0$, $(\ln \tilde{\psi})_i = 0$, $(\ln \tilde{\psi})_{ii} = 0$, using the above formulas of $\psi$, we obtain

$$
(3.1) \quad \frac{\psi_i}{\psi} + \frac{\beta u_i}{u} + \varepsilon u_i u_{ii} + ax_i = \frac{u_{11i}}{u_{11}} + \frac{\beta u_i}{u} + \varepsilon u_i u_{ii} + ax_i = 0,
$$

and

$$
0 = \frac{\psi_{ii}}{\psi} - \frac{\psi_i^2}{\psi^2} + \frac{\beta u_{ii}}{u} - \frac{\beta u_i^2}{u^2} + \sum_s \varepsilon u_s u_{ssi} + \varepsilon u_{ii}^2 + a
$$

$$
\geq \frac{\beta u_{ii}}{u} - \frac{\beta u_i^2}{u^2} + \frac{u_{11ii}}{u_{11}} + 2 \sum_{j>\mu} \frac{u_{1ji}^2}{\lambda_1 (\lambda_1 - \lambda_j)} - \frac{u_{1ii}^2}{u_{11}} + \sum_s \varepsilon u_s u_{ssi} + \varepsilon u_{ii}^2 + a.
$$

In the above inequality, contracting with $S_k^{ii}$, we have

$$
(3.2) \quad 0 \geq \frac{\beta S_k^{ii} u_{ii}}{u} - \frac{\beta S_k^{ii} u_i^2}{u^2} + \frac{S_k^{ii} u_{11ii}}{u_{11}} + 2 \sum_{j>\mu} \frac{S_k^{ii} u_{1ji}^2}{\lambda_1 (\lambda_1 - \lambda_j)} - \frac{S_k^{ii} u_{1ii}^2}{u_{11}} + \sum_s \varepsilon u_s S_k^{ii} u_{ssi} + \varepsilon S_k^{ii} u_{ii}^2 + a \sum_i S_k^{ii}.
$$

At $x_0$, differentiating equation (3.1) twice, we have

$$
(3.3) \quad S_k^{ii} u_{1ii} = f_j + f_u u_j + f_p u_{jj},
$$
and

\[(3.4) \quad S_{ii} u_{i j j} + S_{p q, r s}^{p q, r s} u_{p q j} u_{r s j} \geq -C - C u_{11} + f_{p_1 p_1} u_{11}^2 + \sum_s f_s u_{s j j}.\]

Substituting (3.4) into (3.2), we have

\[(3.5) \quad 0 \geq \beta S_{ii} u_{ii} - \beta S_{ii} u_{ii}^2 + \frac{1}{u_{11}} [-C - C u_{11} + f_{p_1 p_1} u_{11}^2 + \sum_s f_s u_{s 11} - K(S_k)^2] + K(S_k)^2 - S_{pp, qq}^{pp, qq} u_{pp} u_{qq} + 2 \sum_{j > \mu} S_{11, jj}^2 - S_{ii}^2 u_{ii}^2 + a \sum_i S_{ii}^2 - C.\]

where \(K\) is a positive constant which will be determined later.

Combining (3.1) with (3.3), we obtain

\[
\begin{align*}
0 &\geq \beta S_{ii} u_{ii} - \beta S_{ii} u_{ii}^2 + \frac{1}{u_{11}} [-C - C u_{11} + f_{p_1 p_1} u_{11}^2 + \sum_s f_s u_{s 11} - K(S_k)^2] + K(S_k)^2 - S_{pp, qq}^{pp, qq} u_{pp} u_{qq} + 2 \sum_{j > \mu} S_{11, jj}^2 - S_{ii}^2 u_{ii}^2 + a \sum_i S_{ii}^2 - C.
\end{align*}
\]

Next, we divide into two cases to deal with (3.6).

**Case A:** \(\lambda_1 \geq -\frac{\lambda_2}{2}\). For index \(i > \mu\), using (3.1), we have

\[(3.7) \quad -\frac{\beta u_{ii}^2}{u^2} \geq -\frac{2 u_{11}^2}{\beta u_{11}^2} - \frac{4}{\beta} (\varepsilon u_{ii} u_{ii})^2 - \frac{4}{\beta} (ax_i)^2.
\]
For any \( j > \mu \) and \( \beta > 6 \), we have

\[
\frac{2}{u_{11}} S_{k}^{11,ij} u_{11j}^2 + 2 \frac{S_{k}^{11}}{\lambda_1 (\lambda_1 - \lambda_j)} u_{11j}^2 - \frac{2 + \beta S_{k}^{ij}}{u_{11j}^2} u_{11j}^2 = \frac{(\beta - 2) \lambda_1 + (\beta + 2) \lambda_j S_{k}^{ij} u_{11j}^2}{\beta (\lambda_1 - \lambda_j)} \geq 0.
\]

Substituting (3.7), (3.8) into (3.6), we get

\[
(3.9) \ - \frac{C}{u} \geq - \sum_{i=1}^{\mu} \frac{\beta S_{k}^{ii} u_{11i}^2}{u^2} - \sum_{i=1}^{\mu} \frac{S_{k}^{ii} u_{11i}^2}{u_{11i}^2} + \varepsilon S_{k}^{ii} u_{11i}^2 - \frac{4 \beta}{\beta} \sum_{i > \mu} S_{k}^{ii} (\varepsilon u_{i i i})^2
\]

\[
- \frac{4 \beta}{\beta} \sum_{i > \mu} S_{k}^{ii} (ax_i)^2 + a \sum_{i} S_{k}^{ii} + \left( f_{p_1 p_1} - \frac{(k - 1) f_{p_1}^2}{k f} \right) u_{11} - C.
\]

For index \( i \leq \mu \), using (3.1), we have

\[
(3.10) \ - \frac{u_{11i}^2}{u_{11}^2} \geq -2 (\frac{\beta u_{i}}{u})^2 - 4 (\varepsilon u_{i i i})^2 - 4 (ax_i)^2.
\]

Substituting (3.10) into (3.9), we get

\[
(3.11) \ - \frac{C}{u} + \sum_{i=1}^{\mu} (\beta + 2 \beta^2) S_{k}^{ii} u_{11i}^2
\]

\[
\geq \varepsilon S_{k}^{ii} u_{11i}^2 - 4 \sum_{i=1}^{\mu} S_{k}^{ii} (\varepsilon u_{i i i})^2 - 4 \sum_{i=1}^{\mu} S_{k}^{ii} (ax_i)^2 - \frac{4 \beta}{\beta} \sum_{i > \mu} S_{k}^{ii} (\varepsilon u_{i i i})^2
\]

\[
- \frac{4 \beta}{\beta} \sum_{i > \mu} S_{k}^{ii} (ax_i)^2 + a \sum_{i} S_{k}^{ii} + \left( f_{p_1 p_1} - \frac{(k - 1) f_{p_1}^2}{k f} \right) u_{11} - C.
\]

We choose \( \varepsilon \), such that

\[
\varepsilon > 8 \varepsilon^2 \max |Du|^2.
\]

Note that \( i \leq \mu, \ u_{i i i} = u_{11} = \lambda_1 \), thus for the above selected \( \varepsilon \),

\[
(3.12) \ \frac{3 \varepsilon S_{k}^{ii} u_{11i}^2}{4} - 4 \sum_{i=1}^{\mu} S_{k}^{ii} (\varepsilon u_{i i i})^2 - 4 \sum_{i=1}^{\mu} S_{k}^{ii} (ax_i)^2 - \frac{4 \beta}{\beta} \sum_{i > \mu} S_{k}^{ii} (\varepsilon u_{i i i})^2 > 0,
\]

when \( \lambda_1 \) is sufficiently large. By (iv) in section 2, for \( k = 2, 3 \), when \( \lambda \in \tilde{\Gamma}_k \),

\[
\sum_{i=1}^{n} S_{2}^{ii} = (n - 1) \sigma_1 + n \alpha > (n - 1) \sigma_1,
\]

\[
(3.13) \ \sum_{i=1}^{n} S_{3}^{ii} = (n - 2) \sigma_2 + (n - 1) \alpha \sigma_1 > (n - 1) \alpha \sigma_1.
\]

Note that, when \( \lambda \in \tilde{\Gamma}_2 \),

\[
S_{2}^{11} = \lambda_2 + \cdots + \lambda_n + \alpha > 0,
\]
which implies $\sigma_1 > \lambda_1 - \alpha$. Taking $a$ sufficiently large, we have

\[(3.14)\quad \frac{a}{3} \sum_{i=1}^{n} S_{k}^{ii} + \left(f_{p_1p_1} - \frac{(k-1)f_{p_1}^2}{kf}ight)u_{11} - C > 0.\]

Choose $\beta > a^2$, such that

\[(3.15)\quad -\frac{4}{\beta} \sum_{i=\mu}^{\mu} S_{k}^{ii}(ax_i)^2 + \frac{a}{3} \sum_{i=\mu}^{\mu} S_{k}^{ii} > 0.\]

Substituting (3.12), (3.14), (3.15) into (3.11), we get

\[-C \frac{u}{u} + \sum_{i=1}^{\mu} \frac{(\beta + 2\beta^2)S_{k}^{ii}u_i^2}{u^2} \geq \frac{\varepsilon}{4} \sum_{i=1}^{\mu} S_{k}^{ii}u_i^2 + \frac{a}{3} \sum_{i} S_{k}^{ii}.\]

When

\[-\frac{C}{u} \geq \sum_{i=1}^{\mu} \frac{(\beta + 2\beta^2)S_{k}^{ii}u_i^2}{u^2},\]

we get

\[-\frac{2C}{u} \geq \frac{a}{3} \sum_{i} S_{k}^{ii}.\]

Using (3.13), we obtain (1.3).

When

\[-\frac{C}{u} < \sum_{i=1}^{\mu} \frac{(\beta + 2\beta^2)S_{k}^{ii}u_i^2}{u^2},\]

we get

\[2\sum_{i=1}^{\mu} \frac{(\beta + 2\beta^2)S_{k}^{ii}u_i^2}{u^2} \geq \frac{\varepsilon}{4} \sum_{i=1}^{\mu} S_{k}^{ii}u_i^2.\]

Since $u_{11} = \cdots = u_{\mu\mu}$, then $S_{k}^{11} = \cdots = S_{k}^{\mu\mu}$, and then we get

\[(-u)^2 u_{11}^2 \leq C\]

by the above inequality.

**Case B:** $\lambda_n < -\frac{\lambda_1}{2}$. Using (3.6) and (3.10), we get

\[(3.16)\quad -\frac{C}{u} + \frac{(\beta + 2\beta^2)S_{k}^{ii}u_i^2}{u^2} \geq \varepsilon S_{k}^{ii}u_i^2 - 4S_{k}^{ii}(\varepsilon u_iu_{ii})^2 - 4S_{k}^{ii}(ax_i)^2 + a \sum_{i} S_{k}^{ii} + \left(f_{p_1p_1} - \frac{(k-1)f_{p_1}^2}{kf}\right)u_{11} - C.\]
Note that $S^n_k \geq \cdots \geq S^n_{k1}$ and $\lambda^n_k > \frac{\lambda^2}{4}$. We choose $\varepsilon > 8\varepsilon^2 \max_{\Omega} |Du|^2$, then

$$
\frac{3}{4} \varepsilon S^{ii}_{k1} u^2_{ii} - 4S^{ii}_{k1} (\varepsilon u_i u_{ii})^2 - 4S^{ii}_{k1} (ax_j)^2 > 0,
$$
if $\lambda_1$ is sufficiently large. Let $a$ be sufficiently large satisfying (3.14). Substituting (3.14), (3.17) into (3.16), we get

$$
-C \frac{u_i}{u} + (\beta + 2\beta^2) S^{ii}_{k1} u^2_{ii} \geq \frac{\varepsilon}{4} S^{ii}_{k1} u^2_{ii} + 2 \frac{a}{3} \sum_i S^{ii}_{k1}.
$$

Using again (3.13), we obtain (1.3).

**Proof of (1.4):** Let $\lambda_1 = \lambda_1(x)$ denote the biggest eigenvalue of $D^2 u$. We consider the following test function:

$$
\phi(x) = \lambda_1 (-u)^\beta \exp\{\frac{\varepsilon}{2} |Du|^2\},
$$
where $\beta = 1 + \delta' = 1 + 2\delta$. We may assume that $0 < \delta < \frac{1}{3}$.

Suppose that function $\phi$ achieves its maximum value in $\Omega$ at some point $x_0$, and similar to the process of obtaining (3.1) and (3.6), let $a = 0$ and use the convexity of $f^k(x, u, p)$, we obtain

$$
-C \frac{u}{u} + (\beta + 2\beta^2) S^{ii}_{k1} u^2_{ii} \geq \left(1 + \frac{1}{\delta} \right) \varepsilon S^{ii}_{k1} u^2_{ii} + 2 \frac{a}{3} \sum_i S^{ii}_{k1}.
$$

Next, we divide into two cases to deal with (3.19).

**Case A:** $\lambda_1 \geq -\frac{\delta \lambda_1}{3}$. For index $i > \mu$, using (3.18), we have

$$
\frac{\beta u_i}{u} + S^{ii}_{k1} u^2_{ii} = 0,
$$

$$
-C \frac{u}{u} \geq - \frac{(1 + \delta) u^2_{11i}}{\beta u^2_{1i}} - \frac{1 + 1/\delta}{\beta} (\varepsilon u_i u_{ii})^2.
$$
For any $j > \mu$, we have

$$
2 S^{jj}_{k1} u^2_{11j} + 2 \frac{S^{jj}_{k1}}{\lambda_1 (\lambda_1 - \lambda_j)} u^2_{11j} \geq \frac{1 + \delta + \beta S^{jj}_{k1} u^2_{11j}}{\beta (\lambda_1 - \lambda_j)} \geq 0.
$$

Substituting (3.20), (3.21) into (3.19), we obtain

$$
-C \frac{u}{u} \geq - \sum_{i=1}^\mu \frac{\beta S^{ii}_{k1} u^2_{ii}}{u^2} - \sum_{i=1}^\mu \frac{S^{ii}_{k1} u^2_{1i}}{u^2_{1i}} + \varepsilon S^{ii}_{k1} u^2_{ii} - \frac{1 + 1/\delta}{\beta} \sum_{i>\mu} S^{ii}_{k1} (\varepsilon u_i u_{ii})^2 - C.
$$
For index $i \leq \mu$, using (3.18), we have

\begin{equation}
-\frac{u_{11}^2}{u_{11}^2} \geq -2\left(\frac{\beta u_i}{u}\right)^2 - 2(\varepsilon u_i u_{ii})^2.
\end{equation}

(3.23)

Substituting (3.23) into (3.22), we obtain

\begin{equation}
-C + \frac{\mu}{u} \left(\beta + 2\beta^2\right) S^{ii}_K u_i^2 \\
\geq \varepsilon S^{ii}_K u_{ii} - 2 \sum_{i=1}^{\mu} S^{ii}_K (\varepsilon u_i u_{ii})^2 - \frac{1 + 1/\delta}{\beta} \sum_{i > \mu} S^{ii}_K (\varepsilon u_i u_{ii})^2 - C.
\end{equation}

(3.24)

We take $\varepsilon$ sufficiently small, such that

\[ \varepsilon > \max\{4, \frac{2 + 2/\delta}{\beta}\}\varepsilon^2 \max_{\Omega} |Du|^2. \]

Then we have

\begin{equation}
\frac{1}{2} \varepsilon S^{ii}_K u_{ii}^2 - 2 \sum_{i=1}^{\mu} S^{ii}_K (\varepsilon u_i u_{ii})^2 - \frac{1 + 1/\delta}{\beta} \sum_{i > \mu} S^{ii}_K (\varepsilon u_i u_{ii})^2 > 0.
\end{equation}

(3.25)

Using (3.24), (3.25) and $\lambda_1 S^{ii}_K \geq \theta S_k$, we obtain

\[ -C + \frac{\sum_{i=1}^{\mu} (\beta + 2\beta^2) S^{ii}_K u_i^2}{u^2} \geq \varepsilon \frac{S^{ii}_K u_{ii}^2}{3} \geq \varepsilon \frac{\sum_{i=1}^{\mu} S^{ii}_K u_{ii}^2}{6} + \frac{\varepsilon \theta f_{u}}{6} u_{11}. \]

Hence, we get (1.4).

**Case B:** $\lambda_n < -\frac{\delta \lambda_1}{3}$, then $\lambda_n^2 > \frac{\delta^2 \lambda_1^2}{9}$. Using (3.19) and (3.23), we obtain

\begin{equation}
-C + \frac{(\beta + 2\beta^2) S^{ii}_K u_i^2}{u^2} \geq \varepsilon S^{ii}_K u_{ii}^2 - 2 S^{ii}_K (\varepsilon u_i u_{ii})^2 - C.
\end{equation}

(3.26)

We choose $\varepsilon > 4\varepsilon^2 \max_{\Omega} |Du|^2$. It then follows that

\[ -C + \frac{(\beta + 2\beta^2) S^{ii}_K u_i^2}{u^2} \geq \varepsilon \frac{S^{ii}_K u_{ii}^2}{3}. \]

Hence, we complete the proof of Theorem 1.2.

4. **Pogorelov Type $C^2$ Estimates for $k$-Convex Solutions**

In this section, we will prove Theorem 1.5. Similar to the method in [25], we consider the following test function.

\[ \phi = m \log(-u) + \log P_m + \frac{mN}{2} |Du|^2, \]

where

\[ P_m = \sum_{j} \kappa_{j}^{m}, \kappa_{j} = \lambda_{j} + K_{0}, \]
where $m$ and $N$ are some sufficiently large undetermined constants. The $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of the Hessian matrix $D^2u$. Let $K_0 = n \left( \frac{\sup_{\Omega} f}{\alpha} \right)^{\frac{1}{m}}$. By (2.5), $\kappa_1, \kappa_2, \ldots, \kappa_n$ are nonnegative. Suppose that function $\phi$ achieves its maximum value in $\Omega$ at some point $x_0$. By rotating the coordinates, we assume that $(u_{ij})$ is a diagonal matrix at $x_0$, and $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n$.

Differentiating test function twice and using Lemma 2.2, at $x_0$, we have

\begin{equation}
\sum_j \kappa_j^{m-1} u_{jjj} \frac{1}{P_m} + Nu_i u_{ii} + \frac{u_i}{u} = 0,
\end{equation}

and

\begin{equation}
0 \geq \frac{1}{P_m} \left[ \sum_j \kappa_j^{m-1} u_{jjj} + (m - 1) \sum_j \kappa_j^{m-2} u_{jjj}^2 + \sum_{p \neq q} \frac{\kappa_p^{m-1} - \kappa_q^{m-1}}{\kappa_p - \kappa_q} u_{ppq}^2 \right] - \frac{m}{P_m} \left( \sum_j \kappa_j^{m-1} u_{jjj} \right)^2 + \sum_s N u_s u_{sii} + Nu_i^2 + \frac{u_i}{u} - \frac{u_i^2}{u^2}.
\end{equation}

At $x_0$, differentiating the equation (1.1) twice, we have

\begin{equation}
S_k^{ii} u_{iii} = f_j + f_u u_j + f_{j_j} u_{jjj}
\end{equation}

and

\begin{equation}
S_k^{ii} u_{iij} + S_k^{pq,rs} u_{pqj} u_{rsj} \geq -C - Cu_1^2 + \sum_s f_p u_{sij}.
\end{equation}

Here, $C$ is a constant depending on $f$, the diameter of the domain $\Omega$, $\sup_{\Omega} |u|$ and $\sup_{\Omega} |Du|$. Contacting $S_k^{ii}$ in both sides of (4.2), using (4.4), we have

\begin{equation}
0 \geq \frac{1}{P_m} \left[ \sum_j \kappa_j^{m-1} \left( -C - Cu_1^2 + \sum_s f_p u_{sij} - K(S_k)^2 j + K(S_k)^2 j \right) \right. \left. - S_k^{pq,rs} u_{pqj} u_{rsj} \right] + (m - 1) S_k^{ii} \sum_j \kappa_j^{m-2} u_{jjj}^2 + S_k^{ii} \sum_{p \neq q} \frac{\kappa_p^{m-1} - \kappa_q^{m-1}}{\kappa_p - \kappa_q} u_{ppq}^2 \right] - \frac{mS_k^{ii}}{P_m^2} \left( \sum_j \kappa_j^{m-1} u_{jjj} \right)^2 + \sum_s N u_s S_k^{ii} u_{sii} + Nu_i^2 u_{ii} + \frac{u_i}{u} - \frac{u_i^2}{u^2}.
\end{equation}

Using (4.1) and (4.3), we have

\begin{equation}
\frac{1}{P_m} \sum_j \sum_s \kappa_j^{m-1} f_p u_{sij} + \sum_s N u_s S_k^{ii} u_{sii} \geq - \sum_s f_p u_s u - C.
\end{equation}
By Lemma 2.2

\[
-S_{pq,rs}^k u_{pqj} u_{rsj} = -S_{pp,qq}^k u_{ppj} u_{qqj} + \sum_{p \neq q} S_{pp,qq}^k u_{pqj}^2 \\
\geq -S_{pp,qq}^k u_{ppj} u_{qqj} + 2 \sum_{j \neq i} S_{jj,ii}^k u_{jji}^2.
\]

(4.7)

For the given index \(1 \leq i, j \leq n\), we denote

\[
A_i = \frac{\kappa_{ij}^{m-1} m}{P_m} \left[ K(S_k)_i^2 - \sum_{p,q} S_{pp,qq}^k u_{ppi} u_{qqi} \right], \quad B_i = \frac{2\kappa_{ij}^{m-1}}{P_m} \sum_j S_{jj,ii}^k u_{jji}^2,
\]

\[
C_i = \frac{m-1}{P_m} S_{ii}^k \sum_j \kappa_j^{m-2} u_{jji}^2, \quad D_i = \frac{2S_{jj}^k}{P_m} \sum_{j \neq i} \frac{\kappa_j^{m-1} - \kappa_i^{m-1}}{\kappa_j - \kappa_i} u_{jji}^2,
\]

\[
E_i = \frac{mS_{ii}^k}{P_m^2} \left( \sum_j \kappa_j^{m-1} u_{jji} \right)^2.
\]

Using (4.6), (4.7) and the above definitions, (4.5) becomes

\[
0 \geq \sum_i (A_i + B_i + C_i + D_i - E_i) + NS_{ii}^k u_{ii}^2 + \frac{S_{ii}^k u_{ii}^2}{u} - \frac{S_{ii}^k u_{ii}^2}{u^2} \\
- \sum_s f_{ps} \frac{u_s}{u} - C(K) u_{11}.
\]

(4.8)

Let’s deal with the third derivatives. We divide two cases \(i \neq 1\) and \(i = 1\). The proof is the same as in [25].

**Lemma 4.1.** For any \(i \neq 1\), we have

\[
A_i + B_i + C_i + D_i - (1 + \frac{1}{m})E_i \geq 0,
\]

for sufficiently large \(m\) and \(\lambda_1\).

**Proof.** First, let \(l = 1\) in the formula (2.2) of Lemma 2.1 then we have \(A_i \geq 0\) for sufficiently large \(K\).

Next, it is the same as (3.13) in [25]. Using

\[
\kappa_j S_{ii,ii}^k + S_{ii}^k = (\lambda_j + K_0)S_{ij,ii}^k + S_{ij}^k = K_0 S_{ij,ii}^k + S_{ij}^k - S_{k-1} = \lambda_i S_{k-2} + S_{k-1} \lambda_i
\]

\[
= (\lambda_i + K_0)S_{ii,ij}^k + S_{ii}^k \geq S_{ii}^k,
\]
and Cauchy-Schwarz inequalities we obtain

\begin{equation}
P_m^2 \left[ B_i + C_i + D_i - (1 + \frac{1}{m})E_i \right] \\
\geq \sum_{j \neq i} (m + 1)\kappa_i^m \kappa_j^{m-2} S_k^{ii} + 2\kappa_i^m S_k^{ij} \sum_{l=0}^{m-3} \kappa_i^{m-2-l} \kappa_j^l u_{jji} \\
+ \left[(m - 1)(P_m - \kappa_i^m) - 2\kappa_i^m \right] S_k^{ii} \kappa_i^{m-2} u_{iii} \\
- 2(m + 1) S_k^{ii} \kappa_i^{m-1} u_{iii} \sum_{j \neq i} \kappa_j^{m-1} u_{jji}.
\end{equation}

By \(2.6\), we have

\[2\kappa_i^m S_k^{ij} \sum_{l=0}^{m-3} \kappa_i^{m-2-l} \kappa_j^l \geq 7\kappa_i^m \kappa_j^m \]

for \(m \geq \max\{k + 2, 10\}\). Using the above formula and \(4.9\),

\[P_m^2 \left[ B_i + C_i + D_i - (1 + \frac{1}{m})E_i \right] \\
\geq \sum_{j \neq i} (m + 8)\kappa_i^m \kappa_j^{m-2} S_k^{ii} u_{jji} + [(m - 1)(P_m - \kappa_i^m) - 2\kappa_i^m \kappa_i^{m-2} u_{iii} \\
- 2(m + 1) S_k^{ii} \kappa_i^{m-1} u_{iii} \sum_{j \neq i} \kappa_j^{m-1} u_{jji} \\
\geq (m + 8)\kappa_i^m \kappa_i^{m-2} S_k^{ii} u_{11i} + [(m - 1)\kappa_i^m - 2\kappa_i^m \kappa_i^{m-2} u_{iii} \\
- 2(m + 1) S_k^{ii} \kappa_i^{m-1} u_{11i} + (m - 3)\kappa_i^m S_k^{ii} \kappa_i^{m-2} u_{11i} \\
- 2(m + 1) S_k^{ii} \kappa_i^{m-1} u_{11i} \kappa_i^{m-1} u_{11i} \\
\geq 0.
\]

In the second inequality we have used, for \(m \geq 10\)

\[\sum_{j \neq i, 1} (m + 8)\kappa_i^m \kappa_j^{m-2} S_k^{ii} u_{jji} + (m - 1) \sum_{j \neq i, 1} \kappa_j^{m} \kappa_i^{m-2} S_k^{ii} u_{jji} \\
- 2(m + 1) S_k^{ii} \kappa_i^{m-1} u_{iii} \sum_{j \neq i, 1} \kappa_j^{m-1} u_{jji} \geq 0.
\]

In the forth inequality we have used

\[(m + 8)(m - 3) \geq (m + 1)^2.
\]

Take \(m = \max\{k + 2, 10\}\). Then we obtain Lemma 4.1. \qed

**Lemma 4.2.** For \(\mu = 1, \cdots, k - 2\), if there exists some positive constant \(\delta \leq 1\), such that

\[\frac{\lambda_\mu}{\lambda_1} \geq \delta,
\]

then there exist two sufficiently small positive constants \(\eta, \delta'\) depending on \(\delta\), such
that, if \( \frac{\lambda_{\mu+1}}{\lambda_1} \leq \delta' \), we have

\[
A_1 + B_1 + C_1 + D_1 - (1 + \frac{\eta}{m})E_1 \geq 0.
\]

**Proof.** First, it is the same as (3.19) in [25]. For \( m \geq 5 \), we have

\[
(4.10) \quad P_m^2 [B_1 + C_1 + D_1 - (1 + \frac{\eta}{m})E_1] \geq - (1 + \eta)S_k^{11} \kappa_1^{2m-2}u_{111}^2 + 2P_m \kappa_1^{m-2} \sum_{j \neq 1} S_j^2 u_{jj1}^2.
\]

Next, we will replace \(- (1 + \eta)S_k^{11} \kappa_1^{2m-2}u_{111}^2\) from \( A_1 \). We divide two cases \( \mu = 1 \) and \( 2 \leq \mu \leq k - 2 \).

For \( \mu = 1 \), since \( S_1 = \sigma_1 + \alpha \) and \( S_1^{aa,bb} = 0 \), by Lemma 2.1 for sufficiently large \( K \), we have

\[
A_1 = \frac{\kappa_1^{m-1}}{P_m} [K(S_k)_1^2 - \sum_{p,q} S_k^{pp,qq}u_{pp1}u_{qq1}] \\
\geq \frac{\kappa_1^{m-1}}{P_m} S_k(1 + \frac{\theta}{2})(S_1)_1^2.
\]

Using \( S_1^{aa} = 1 \), we have

\[
(4.11) \quad \frac{\theta}{2}(S_1)_1^2 = (1 + \frac{\theta}{2})(\sum_a u_{aa1})^2 \\
= (1 + \frac{\theta}{2})u_{111}^2 + 2(1 + \frac{\theta}{2}) \sum_{a \neq 1} u_{aa1} u_{111} + (1 + \frac{\theta}{2}) \left( \sum_{a \neq 1} u_{aa1} \right)^2 \\
\geq (1 + \frac{\theta}{4})u_{111}^2 - C_\theta \sum_{a \neq 1} u_{aa1}.
\]

Combining (4.11), (4.12) and (2.7), we obtain

\[
(4.13) \quad P_m^2 A_1 \geq \frac{P_m \kappa_1^{m-1} S_k}{S_1^2} (1 + \frac{\theta}{4})u_{111}^2 - \frac{P_m \kappa_1^{m-1} C_\theta}{S_1^2} \sum_{a \neq 1} u_{aa1}^2 \\
\geq \frac{P_m \kappa_1^{m-2} S_k^{11} (1 - \delta_0)}{(1 + \sum_{j \neq 1} \frac{\lambda_j}{\lambda_1} + \frac{\alpha}{\lambda_1})^2} (1 + \frac{\theta}{4})u_{111}^2 - \frac{P_m \kappa_1^{m-1} C_\theta}{S_1^2} \sum_{a \neq 1} u_{aa1}^2 \\
\geq (1 + \eta) \kappa_1^{2m-2} S_k^{11} u_{111}^2 - P_m \kappa_1^{m-3} C_\theta \sum_{a \neq 1} u_{aa1}^2.
\]

The last inequality comes from choosing positive constants \( \delta', \varepsilon_0, \eta \) such that

\[
\frac{\lambda_1}{\lambda_1} \leq \delta', \quad (1 + \frac{\theta}{4})(1 - \varepsilon_0) \geq (1 + \eta)(1 + \eta \delta')^2.
\]
For $2 \leq \mu \leq k-2$, by Lemma 2.5, we have
\begin{equation}
A_1 \geq \frac{k_1^{m-1}}{P_m} \left[ S_k \frac{(S_\mu)^2}{S_\mu} - \frac{S_k}{S_\mu} \sum_{p,q} u_{pp1} u_{qq1} \right] \\
= \frac{k_1^{m-1}}{P_m S_\mu^2} \left[ \sum_a (S_\mu^{aa} u_{aa1})^2 + \sum_{a \neq b} (S_\mu^{aa} S_\mu^{bb} - S_\mu S_\mu^{aab,bb}) u_{aa1} u_{bb1} \right].
\end{equation}

Next, we split $\sum_{a \neq b} (S_\mu^{aa} S_\mu^{bb} - S_\mu S_\mu^{aab,bb}) u_{aa1} u_{bb1}$ into three terms to deal with:
\begin{equation}
\sum_{a \neq b} = \sum_{a \neq b, a \leq \mu} + 2 \sum_{a \leq \mu, b > \mu} + \sum_{a \neq b, a > \mu} := T_1 + T_2 + T_3.
\end{equation}

First, let’s deal with $T_1$. By Lemma 2.6 for $a \neq b$, we have
\begin{equation}
S_\mu^{aa} S_\mu^{bb} - S_\mu S_\mu^{aab,bb} = S_\mu^2 (\lambda |ab|) - S_\mu (\lambda |ab|) S_\mu (\lambda |ab|) \geq 0.
\end{equation}
Note that $\lambda \in \Gamma_k, \mu \leq k-2$, which implies $S_\mu (\lambda |ab|) > 0$. So (4.15) implies
\begin{equation}
T_1 = \sum_{a \neq b, a \leq \mu} (S_\mu^{aa} S_\mu^{bb} - S_\mu S_\mu^{aab,bb}) u_{aa1} u_{bb1}
\geq - \sum_{a \neq b, a \leq \mu} \left[ S_\mu^2 (\lambda |ab|) - S_\mu (\lambda |ab|) S_\mu (\lambda |ab|) \right] u_{aa1}^2
\geq - \sum_{a \neq b, a \leq \mu} S_\mu^2 (\lambda |ab|) u_{aa1}^2.
\end{equation}

By Lemma 2.5 for any $a, b \leq \mu$,
\begin{equation}
S_\mu^{aa} = \sigma_{\mu - 1}(\lambda |a|) + \alpha \sigma_{\mu - 2}(\lambda |a|) \geq \frac{\lambda_1 \cdots \lambda_\mu}{\lambda_a},
\end{equation}
and
\begin{equation}
S_{\mu - 1}(\lambda |ab|) \leq C \left( \frac{\lambda_1 \cdots \lambda_{\mu+1}}{\lambda_a \lambda_b} + \alpha \frac{\lambda_1 \cdots \lambda_\mu}{\lambda_a \lambda_b} \right) \leq C \left( \frac{\alpha + \lambda_{\mu+1}}{\lambda_b} \right) S_\mu^{aa}.
\end{equation}

Using (4.16), (4.17), (4.18), we obtain
\begin{equation}
T_1 \geq - \sum_{a \neq b, a \leq \mu} C_1 \frac{(\alpha + \lambda_{\mu+1})^2}{\lambda_b^2} (S_\mu^{aa} u_{aa1})^2
\geq - \frac{C_1 (\alpha + \delta' \lambda_1)^2}{\lambda_b^2} \sum_{a \leq \mu} (S_\mu^{aa} u_{aa1})^2
\geq - \varepsilon \sum_{a \leq \mu} (S_\mu^{aa} u_{aa1})^2.
\end{equation}

Here, we choose positive constants $\delta', \varepsilon$ sufficiently small and $\lambda_1$ sufficiently large, satisfying
\[ \frac{4C_1 \delta^2}{\delta^2} < \varepsilon, \quad \alpha < \delta' \lambda_1. \]
Next, let’s deal with $T_2$ and $T_3$. Using Cauchy-Schwarz inequalities we get

\[
T_2 \geq -2 \sum_{a \leq \mu; b > \mu} S_{\mu}^{aa} S_{\mu}^{bb} |u_{aa1} u_{bb1}| \\
\geq -\varepsilon \sum_{a \leq \mu; b > \mu} (S_{\mu}^{aa} u_{aa1})^2 - \frac{1}{\varepsilon} \sum_{a \leq \mu; b > \mu} (S_{\mu}^{bb} u_{bb1})^2,
\]

(4.20)

\[
T_3 \geq -\sum_{a \neq b; a, b > \mu} S_{\mu}^{aa} S_{\mu}^{bb} |u_{aa1} u_{bb1}| \geq - \sum_{a \neq b; a, b > \mu} (S_{\mu}^{aa} u_{aa1})^2.
\]

(4.21)

Hence, combining (4.14), (4.19), (4.20) and (4.21), we obtain

\[
A_1 \geq \lambda_1 \sum_{a \leq \mu} (S_{\mu}^{aa} u_{aa1})^2 - C_\varepsilon \sum_{a > \mu} (S_{\mu}^{aa} u_{aa1})^2.
\]

(4.22)

For $a > \mu$, we have

\[
S_{\mu}^{aa} \leq C_2 \lambda_1 \cdots \lambda_{\mu-1}, \quad S_{\mu} \geq \lambda_1 \cdots \lambda_\mu.
\]

So

\[
-\frac{1}{S_{\mu}^2} \sum_{a > \mu} (S_{\mu}^{aa} u_{aa1})^2 \geq -\frac{C_2^2}{\lambda_1^2} \sum_{a > \mu} u_{aa1}^2 \geq -\frac{C_3}{\kappa_1^2 \delta_1^2} \sum_{a > \mu} u_{aa1}^2.
\]

(4.24)

For $a \leq \mu$, we have

\[
S_{\mu}(\lambda|a) \leq C_4 \left(\frac{\lambda_1 \cdots \lambda_{\mu+1}}{\lambda_a} + \alpha \frac{\lambda_1 \cdots \lambda_\mu}{\lambda_a}\right).
\]

(4.25)

Using (4.23), (4.25) and (2.7), we have

\[
\frac{S_k}{S_{\mu}^2} \sum_{a \leq \mu} (S_{\mu}^{aa} u_{aa1})^2 \geq \frac{\lambda_1 S_{k}^{11}}{S_{\mu}^2} (1 - \varepsilon_0) \sum_{a \leq \mu} (S_{\mu}^{aa} u_{aa1})^2 \\
\geq \frac{S_{k}^{11}}{\lambda_1} (1 - \varepsilon_0) \sum_{a \leq \mu} \left(\frac{\lambda_a S_{\mu}^{aa}}{S_{\mu}}\right)^2 u_{aa1}^2 \\
\geq \frac{S_{k}^{11}}{\kappa_1} (1 - \varepsilon_0) \sum_{a \leq \mu} \left(1 - \frac{S_{\mu}(\lambda|a)}{S_{\mu}}\right)^2 u_{aa1}^2 \\
\geq \frac{S_{k}^{11}}{\kappa_1} (1 - \varepsilon_0) \left(1 - \frac{C_5 (\alpha + \delta' \lambda_1)}{\delta \lambda_1}\right)^2 \sum_{a \leq \mu} u_{aa1}^2 \\
\geq \frac{S_{k}^{11}}{\kappa_1} (1 - \varepsilon_0) \left(1 - \frac{C_6 \delta'}{\delta}\right)^2 \sum_{a \leq \mu} u_{aa1}^2.
\]

(4.26)
Combining (4.22), (4.24), (4.26) and $P_m > (1 + \delta^m)\kappa_1^m$, we obtain

\begin{equation}
(4.27) \quad P_m^2 A_1 \geq (1 + \delta^m)\kappa_1^2m - 2S_k^{11}(1 - \varepsilon_0)(1 - 2\varepsilon)\left(1 - \frac{C_6\delta'}{\delta}\right)^2 \sum_{a \leq \mu} u_{aa}^2
\end{equation}

\[ - \frac{P_m\kappa_1^{m-3}C'_eS_k}{\delta^2} \sum_{a > \mu} u_{aa}^2 \]

\[ \geq (1 + \eta)\kappa_1^{2m - 2}S_k^{11} \sum_{a \leq \mu} u_{aa}^2 - \frac{P_m\kappa_1^{m-3}C'_eS_k}{\delta^2} \sum_{a > \mu} u_{aa}^2. \]

Here, the last inequality comes from choosing positive constants $\varepsilon_0, \varepsilon, \delta', \eta$ sufficiently small such that

\[ \frac{C_6\delta'}{\delta} \leq 2\varepsilon, \quad (1 - \varepsilon_0)(1 - 2\varepsilon)^3(1 + \delta^m) \geq 1 + \eta. \]

Using (4.10), (4.13) or (4.27), we have

\begin{equation}
(4.28) \quad P_m^2 \left[ A_1 + B_1 + C_1 + D_1 - (1 + \frac{\eta}{m})E_1 \right]
\end{equation}

\[ \geq 2 \sum_{j \neq 1} S_{kj}^{jj} u_{jj}^2 - \frac{P_m\kappa_1^{m-3}C'_eS_k}{\delta^2} \sum_{j > \mu} u_{jj}^2. \]

Now, for $k - 1 \geq j > \mu$, by (2.24), we have

\[ \kappa_1 S_{kj}^{jj} = \frac{\kappa_1}{\lambda_j} \lambda_j S_{kj}^{jj} \geq \frac{\kappa_1}{\lambda_j} \theta S_k \geq \frac{\theta S_k}{\delta'}. \]

For $j \geq k$, we have

\[ \kappa_1 S_{kj}^{jj} \geq \kappa_1 S_k^{k-1,k-1} \geq \frac{\kappa_1}{\lambda_{k-1}} \theta S_k \geq \frac{\theta S_k}{\delta'}. \]

Hence, choose $\delta'$ small enough such that

\[ \delta' < \frac{\theta \delta^2}{C_\varepsilon}, \]

then (4.28) is nonnegative. We complete the proof of the lemma. \hfill \Box

Hence, a direct corollary of Lemma 4.2 is the following.

**Corollary 4.3.** There exist two finite sequences of positive numbers $\{\delta_i\}_{i=1}^{k-1}$ and $\{\eta_i\}_{i=1}^{k-1}$, such that, if the following inequality holds for some index $1 \leq r \leq k - 2$,

\[ \frac{\lambda_r}{\lambda_1} \geq \delta_r, \quad \frac{\lambda_{r+1}}{\lambda_1} \leq \delta_{r+1}, \]

then, for sufficiently large $K$ and $\lambda_1$, we have

\begin{equation}
(4.29) \quad A_1 + B_1 + C_1 + D_1 - (1 + \frac{\eta_r}{m})E_1 \geq 0.
\end{equation}
Now, we can prove Theorem 1.5. By Corollary 4.3, there exists some sequence \( \{\delta_i\}_{i=1}^{k-1} \). We consider the following two cases.

Case 1: If \( r = k - 1, \lambda_{k-1} \geq \delta_{k-1} \lambda_1 \), obviously we have

\[
f = \sigma_k + \alpha \sigma_{k-1} > \alpha \lambda_1 \cdots \lambda_{k-1} \geq \delta_{k-1}^{k-2} \lambda_1^{k-1},
\]

which implies \( \lambda_1 \leq C \). Hence, we complete the proof of Theorem 1.5.

Case 2: There exists some index \( 1 \leq r \leq k - 2 \), such that \( \lambda_r \lambda_1 \geq \delta_r \), \( \lambda_{r+1} \lambda_1 \leq \delta_{r+1} \).

Using Lemma 4.1 and Corollary 4.3, we have

\[
\sum_i (A_i + B_i + C_i + D_i) - E_1 - (1 + \frac{1}{m}) \sum_{i=2}^n E_i \geq 0.
\]

Combining (4.8) with (4.30), we have

\[
0 \geq \sum_{i=2}^n \left( \frac{S_{ii}^k}{m} (\sum_j \kappa_j^{m-1} u_{jjj})^2 + Nu_i^2 S_{ii}^k + \frac{S_{ii}^k u_{ii}}{u} - \frac{S_{ii}^k u_{ii}^2}{u^2} - \sum_s f_s \frac{u_s}{u} - Cu_{11} \right).
\]

By (4.1), for any fixed \( i \geq 2 \), we have

\[
-\frac{S_{ii}^k u_{ii}^2}{u^2} = -\frac{S_{ii}^k}{m} (\sum_j \kappa_j^{m-1} u_{jjj})^2 + S_{ii}^k N u_i^2 u_{ii}^2 + \frac{2N S_{ii}^k u_{ii}^2}{u} \geq -\frac{S_{ii}^k}{m} (\sum_j \kappa_j^{m-1} u_{jjj})^2 + 2NC_0 S_{ii}^k u_{ii}^2.
\]

In the above inequality, we have used (2.8). Using the above formula and (v) in section 2, (4.31) becomes

\[
-\frac{C}{u} \geq Nu_{11} S_{11}^k - \frac{u_{11}^2 S_{11}^k}{u^2} - Cu_{11}.
\]

Since \( \lambda \in \Gamma_k \), we know \( u_{11} S_{11}^k \geq \theta S_k \) by (2.4). Choose \( N \) sufficiently large in the above formula, such that

\[
\frac{N}{2} u_{11}^2 S_{11}^k - Cu_{11} > 0,
\]

we obtain

\[
-\frac{C}{u} + \frac{C S_{11}^k}{u^2} \geq \frac{N}{4} S_{11}^k \lambda_1^2 + \frac{N \theta S_k}{4} \lambda_1.
\]

Hence, we complete the proof of Theorem 1.5.
5. A Rigidity Theorem for \( k \) Convex Solutions

In this section, we prove Theorem 1.6. First, we have the following lemma.

**Lemma 5.1.** For the following Dirichlet problem of Sum Hessian equations,

\[
\begin{aligned}
S_k(D^2u) &= f(x), & \text{in} & & \Omega, \\
u &= 0, & \text{on} & & \partial\Omega.
\end{aligned}
\]

Assume that \( f > 0 \) is a smooth function in some domain \( \Omega \), \( u \) is a \( k \) convex solution of the above problem. Then there exists some sufficiently large constant \( \beta > 0 \), such that

\[
(-u)^\beta \Delta u \leq C.
\]

Here, the constants \( C \) and \( \beta \) only depend on \( n, k \) and the diameters of the domain \( \Omega \).

**Proof.** Obviously, for sufficiently large \( a \) and \( b \), the function \( w = a^2 |x|^2 - b \) can control \( u \) by comparison principal [8], namely

\[
w \leq u \leq 0.
\]

Here \( a \) and \( b \) depend on the diameter of the domain \( \Omega \). Hence, in the following proof, the constant \( \beta, C \) in (5.2) can contain sup \( \Omega |u| \).

Since \( u \) is \( k \) convex solution, by (2.5), there is some constant \( K_0 > 0 \), such that \( D^2u + K_0 I \geq 0 \). We consider the test function,

\[
\phi = m \beta \log(-u) + \log P_m + \frac{m}{2} |x|^2,
\]

where

\[
P_m = \sum_j \kappa_j^m, \quad \kappa_j = \lambda_j + K_0.
\]

Suppose \( \phi \) achieves its maximum value at \( x_0 \) in \( \Omega \). By rotating the coordinate, we assume that \( (u_{ij}) \) is a diagonal matrix at \( x_0 \), so \( u_{ii} = \lambda_i \) and \( \kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n \).

Differentiating test function twice at \( x_0 \), we have

\[
\begin{aligned}
\sum_j \frac{\kappa_j^{m-1} u_{jjij}}{P_m} + x_i + \frac{\beta u_i}{u} &= 0
\end{aligned}
\]

and

\[
\begin{aligned}
0 \geq & \frac{1}{P_m} \left[ \sum_j \kappa_j^{m-1} u_{jjij} + (m - 1) \sum_j \kappa_j^{m-2} u_{jjji}^2 + \sum_{p \neq q} \frac{\kappa_p^{m-1} - \kappa_q^{m-1}}{\kappa_p - \kappa_q} u_{pqi}^2 \right] \\
& - \frac{m}{P_m^2} \left( \sum_j \kappa_j^{m-1} u_{jjji} \right)^2 + \frac{\beta u_{ii}}{u} - \frac{\beta u_i^2}{u^2} + 1.
\end{aligned}
\]

At \( x_0 \), differentiating the equation (5.1) twice, we have

\[
S_k^{ij} u_{ij} = f_j
\]
and

\[(5.6) \quad S_k^{ii} u_{ijj} + S_k^{pq,r_s} u_{pqj} u_{rsj} = f_{jj}.\]

Using (5.3), we have

\[(5.7) \quad -\frac{\beta u^2}{u^2} \geq \frac{2}{\beta} \left( \frac{\left( \sum_j \kappa_j^{m-1} u_{jji} \right)^2}{P_m^2} \right) - \frac{2x^2_i}{\beta}.\]

Contacting \(S_k^{ii}\) in both sides of (5.4), using (5.6) and (5.7), we obtain

\[(5.8) \quad 0 \geq \frac{1}{P_m} \left[ \sum_j \kappa_j^{m-1} (f_{jj} - S_k^{pq,r_s} u_{pqj} u_{rsj}) + (m - 1) S_k^{ii} \sum_j \kappa_j^{m-2} u_{jji}^2 \right.
\quad + \left. S_k^{ii} \sum_{p \neq q} \left( \frac{\kappa_p - \kappa_q}{\kappa_{pq}} u_{pq} \right)^2 \right] - \frac{(m + 2) S_k^{ii}}{P_m^2} \left( \sum_j \kappa_j^{m-1} u_{jji} \right)^2
\quad + \frac{\beta S_k^{ii} u_{ii}}{u} - \frac{2S_k^{ii} x_i^2}{\beta} + \sum_i S_k^{ii}.\]

Using the definitions of \(A_i, B_i, C_i, D_i, E_i\) and (5.8), we have

\[(5.9) \quad 0 \geq \sum_{i=1}^n \left[ A_i + B_i + C_i + D_i - (1 + \frac{2}{m\beta}) E_i \right] + \frac{\beta S_k^{ii} u_{ii}}{u}
\quad - \frac{2S_k^{ii} x_i^2}{\beta} + \sum_i S_k^{ii} - \frac{C}{\kappa_1}.\]

For sequence \(\{\delta_i\}_{i=1}^{k-1}\) and \(\{\eta_i\}_{i=1}^{k-1}\) appearing in Corollary 4.3, we choose \(\beta\) sufficiently large such that

\[\frac{2}{\beta} < \min \{\frac{1}{2} \eta_1, \ldots, \eta_{k-1}\}.\]

If \(r = k - 1\), \(\lambda_{k-1} \geq \delta_{k-1} \lambda_1\), obviously we have

\[f = \sigma_k + \alpha \sigma_{k-1} > \lambda_1 \cdots \lambda_{k-1} \geq \delta_{k-1} \lambda_1^{k-1},\]

which implies \(\lambda_1 \leq C\). Thus, we obtain Lemma 5.1.

If there exists some index \(1 \leq r \leq k - 2\), such that

\[\frac{\lambda_r}{\lambda_1} \geq \delta_r, \quad \frac{\lambda_{r+1}}{\lambda_1} \leq \delta_{r+1},\]

then (5.9) becomes

\[(5.10) \quad 0 \geq \frac{\beta S_k^{ii} u_{ii}}{u} - \frac{2S_k^{ii} x_i^2}{\beta} + \sum_i S_k^{ii} - \frac{C}{\kappa_1}\]

by Lemma 4.1 and Corollary 4.3. We choose \(\beta\) sufficiently large, such that

\[-\frac{2S_k^{ii} x_i^2}{\beta} + \frac{1}{2} \sum_i S_k^{ii} > 0.\]
By Newton-Maclaurin inequality, we have
\[ \sigma_{k-1} \geq \frac{1}{\sigma_1^{k-1}} \sigma_k^{k-2}, \quad \sigma_{k-2} \geq \frac{1}{\sigma_1^{k-2}} \sigma_k^{k-3}, \]
which implies
\[ \sum_i S_{ki}^{ii} = (n-k+1)\sigma_{k-1} + \alpha(n-k+2)\sigma_{k-2} \geq c_0\sigma_1^{\frac{1}{k-1}}. \]

By (v) in section 2,
\[ \frac{S_{ki}^{ii}u_{ii}}{u} = \frac{kS_k - \alpha\sigma_{k-1}}{u} \geq \frac{kS_k}{u}. \]

Substituting the above inequalities into (5.10), we obtain
\[ -\frac{\beta C}{u} \geq \frac{1}{2} \sum_i S_{ki}^{ii} - \frac{C}{\kappa_1} \geq \frac{c_0}{4} \sigma_1^{\frac{1}{k-1}}. \]

Thus, we obtain Lemma 5.1. \( \square \)

**Proof of Theorem 1.3:** The proof is classical as in [33, 25]. For the convenience of the readers, the complete proof is given here. Suppose \( u \) is an entire solution of the equation (1.7). For arbitrary positive constant \( R > 1 \), we consider the set
\[ \Omega_R = \{ y \in \mathbb{R}^n; u(Ry) \leq R^2 \}. \]

Let
\[ v(y) = \frac{u(Ry) - R^2}{R^2}. \]

We consider the following Dirichlet problem
\[
\begin{cases}
\sigma_k(D^2v) + \alpha\sigma_{k-1}(D^2v) = 1, & \text{in } \Omega_R, \\
v = 0, & \text{on } \partial \Omega_R.
\end{cases}
\]

By Lemma 5.1, we have the following estimates
\[ (-v)^3 \Delta v \leq C. \]

Here the constants \( C \) and \( \beta \) depend on \( k \) and the diameter of the domain \( \Omega_R \). Now using the quadratic growth condition appears in Theorem 1.6, we have
\[ c|Ry|^2 - b \leq u(Ry) \leq R^2. \]

Namely
\[ |y|^2 \leq \frac{1 + b}{c}. \]

Hence, \( \Omega_R \) is bounded. Thus, the constant \( C \) and \( \beta \) become two absolutely constants.

Consider the domain
\[ \Omega'_R = \{ y; u(Ry) \leq \frac{R^2}{2} \} \subset \Omega_R. \]

In \( \Omega'_R \), we have
\[ v(y) \leq -\frac{1}{2}. \]
Hence, (5.12) implies that in $\Omega'_R$, we have
\[ \Delta v \leq 2^\beta C. \]

Note that
\[ \nabla^2_y v = \nabla^2_x u. \]

Thus, using the above two formulas, in $\Omega'_R = \{ x; u(x) \leq \frac{R^2}{2} \}$, we have
\[ \Delta u \leq C, \]

where $C$ is an absolutely constant. Since $R$ is arbitrary, we have the above inequality in whole $\mathbb{R}^n$. Using Evan-Krylov theory \[10\], we have
\[ |D^2 u|_{C^0(B_R)} \leq C \frac{|D^2 u|_{C^0(0)}}{R^\alpha} \leq \frac{C}{R^\alpha}. \]

Hence, when $R \to +\infty$, we obtain Theorem 1.3.

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