NON-BOLTZMANN EQUILIBRIUM
PROBABILITY DENSITIES FOR NON-LINEAR LÉVY OSCILLATOR

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Abstract

We study, both analytically and by numerical modeling the equilibrium probability density function for an non-linear Lévy oscillator with the Lévy index $\alpha$, $1 \leq \alpha \leq 2$, and the potential energy $x^4$. In particular, we show that the equilibrium PDF is bimodal and has power law asymptotics with the exponent $-(\alpha + 3)$.

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1 Starting equations

Recently, kinetic equations with fractional derivatives have attracted attention as a possible tool for the description of anomalous diffusion and relaxation phenomena, see, e.g., recent review \cite{1}, semi-review papers \cite{2,3,4} and references on earlier studies therein. It was also recognized \cite{5,6} that the fractional kinetic equations may be viewed as “hydrodynamic” (that is, long-time and long-space) limits of the CTRW (Continuous Time Random Walk) scheme \cite{7}, which was successfully applied to the description of anomalous diffusion phenomena in many areas, e.g., turbulence
disordered medium, intermittent chaotic systems, etc. However, the kinetic equations have two advantages over a random walk approach: firstly, they allow one to explore various boundary conditions (e.g., reflecting and/or absorbing) and, secondly, to study diffusion and/or relaxation phenomena in external fields, both possibilities are difficult to realize in the framework of CTRW (we point, however, to the paper, in which a fractional kinetic equation was obtained from generalized CTRW). Fractional kinetic equations can be divided into three classes: the first one, describing Markovian processes, contains equations with fractional space or velocity derivatives and the first time derivative, the second one, describing non-Markovian processes, contains equations with fractional time derivative, and the third class, naturally, contains both fractional space and time derivatives, as well. In this paper we deal with a one-dimensional kinetic equation belonging to the first class, namely, with the Fractional Symmetric Einstein-Smoluchowski Equation (FSESE), which, from one hand, is a natural generalization of the diffusion-like equation with the symmetric fractional space derivative \[2, 11\] and, from the other hand, is a Markovian generalization of the Einstein-Smoluchowski kinetic equation, which describes a motion of a particle subjected to a white Gaussian noise in a strong friction limit, see, e.g., \[13\]. From this point of view, the FSESE describes a motion of a particle subjected to a white Lévy noise, also in a strong friction limit \[14\].

In dimensionless units the one-dimensional FSESE has the form

\[
\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x} \left( F f \right) + \frac{\partial^\alpha f}{\partial |x|^\alpha}, \quad t > 0, \quad x \in \mathbb{R}, \quad f(x, 0) = \delta(x),
\]

(1.1)

where \(f(x, t)\) is the probability density function, \(F\) is the deterministic external force, \(\alpha\) is the Lévy index, \(0 < \alpha \leq 2\), \(\partial^\alpha / \partial |x|\) is the symmetric fractional space derivative \[2, 11\], which is defined, for a “sufficiently well-behaved function \(\phi(x)\)” through its Fourier transform \(\hat{\phi}(k)\) as

\[
\frac{d^\alpha}{d |x|^\alpha} \phi(x) = -|k|^\alpha \hat{\phi}(k),
\]

(1.2)

or in terms of the Riemann-Liouville derivatives as

\[
\frac{d^\alpha}{d |x|^\alpha} \phi(x) = -\frac{1}{2\cos(\pi \alpha/2)} \left[ D_+^\alpha \phi(x) + D_-^\alpha \phi(x) \right],
\]

(1.3)

where \(\alpha > 0\), \(\alpha \neq 1, 3, \ldots\),

\[
D_+^\alpha \phi(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^\infty \frac{\phi(x')dx'}{(x-x')^\alpha}, \quad D_-^\alpha \phi(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^\infty \frac{\phi(x')dx'}{(x'-x)^\alpha}
\]

(1.4)

for \(0 < \alpha < 1\). For \(\alpha \geq 1\)

\[
D_+^\alpha \phi(x) = \frac{(\pm 1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^\infty \xi^{n-\alpha-1} \phi(x \mp \xi) d\xi,
\]

(1.5)

\(n - 1 < \alpha \leq n \in \mathbb{N}\). The derivatives (1.4) and (1.5) are characterized in their Fourier representation by

\[
D_+^\alpha \phi(x) \div (\mp ik)^\alpha \hat{\phi}(k),
\]

(1.6)
\[ (\mp ik)^\alpha = |k|^{\alpha} \exp \left( \mp \frac{\text{sgn} k}{\alpha} \right) \cdot \frac{\pi}{2}. \]

One can easily recover Eq. (1.2) by combining Eq. (1.3) with Eq. (1.6). A detailed theory of the Riemann - Liouville and other forms of fractional derivatives is presented in \[15\].

The solution of Eq. (1.1) in the force - free case, \( F = 0 \), is known to be a probability density function (PDF) for an \( \alpha \) - stable symmetric process. The solution of Eq. (1.1) was also obtained for the constant field, \( F = \text{const} \) on the whole axis, \( -\infty < x < \infty \), and for a linear Lévy oscillator \[16\]. As far as the authors know, Eq. (1.1) was not considered for a more complicated potential fields. In the present paper we study the equilibrium solution of the FSESE for non - linear Lévy oscillator with the potential energy

\[ U(x) = \frac{x^4}{4}, \quad F = -\frac{dU}{dx}. \quad (1.7) \]

The models with the potential energy (1.7) play an important role in the theory of dynamical chaos \[17\] and in the theory of Brownian motion in an open auto - oscillation systems \[18\], and have various applications \[19\]. One may expect that the models of non - linear Lévy oscillator will also possess an important place in the theory of the systems influenced by non - Gaussian Lévy noises and obeying fractional kinetic equations. For our purposes we pass to the equation for the characteristic function \( \hat{f}(k) \),

\[ \hat{f}(k) = \int_{-\infty}^{\infty} dx f(x) \exp(ikx). \quad (1.8) \]

For the equilibrium state \( \hat{f}(k) \) obeys the equation, which follows from Eqs. (1.1) and (1.7):

\[ \frac{d^3 \hat{f}_{eq}}{dk^3} = \text{sgn}(k) |k|^{\alpha-1} \hat{f}_{eq}(k), \quad (1.9) \]

where the index “eq.” denotes equilibrium solution. Below in the paper we omit subscript “eq” for brevity. The characteristic function obeys the following conditions.

1. \( \hat{f}(k) = \hat{f}^*(k) = \hat{f}(-k) \), where the asterisk implies complex conjugate. The first equality is a consequence of the Khintchine theorem about reality of the characteristic function for the symmetric PDF, whereas the second equality is the consequence of the Bochner - Khintchine theorem about positive definiteness of the characteristic function. In fact, one can see that the solution of Eq. (1.9) obeys the condition 1.

2. The boundary conditions for \( \hat{f}(k) \) are as follows:

\[ \hat{f}(0) = 1, \quad \hat{f}(\pm \infty) = 0, \quad \frac{d\hat{f}(0)}{dk} = 0. \quad (1.10) \]

The first property stems from normalization of the PDF, \( \int_{-\infty}^{\infty} f(x)dx = 1. \)

The second property stems from the existence of the PDF.

As to the third property, we remind that the integer moments of the PDF (if exist) are connected with the derivatives of the characteristic function at \( k = 0 \) as

\[ \langle x^p \rangle = \frac{1}{i^p} \frac{d\hat{f}^{(p)}(0)}{dk^p}, \quad p = 1, 2, ... \]

The third property is the consequence of this theorem:

\[ \frac{d\hat{f}^{(p)}(0)}{dk^p} = 0, \quad p = 1, 3, 5, ... \]
because the PDF is a symmetric function, and hence, all odd moments are equal zero. The last equality is valid for those odd $p$, for which the $p$-th moments of the PDF exist.

To make results more transparent, the rest part of the paper will be organized as follows. In Sect.2 we recall the results for non-linear Brownian oscillator, $\alpha = 2$. In Sect.3 we get an equilibrium solution for an non-linear Cauchy oscillator, $\alpha = 1$. In Sect.4, which is the main one, we present analytical and numerical results on equilibrium solutions for non-linear Lévy oscillators with the Lévy indexes lying between 1 and 2, Brownian and Cauchy oscillators being their particular cases. Finally, in Sect.5 we present the discussion and summarize the results. Some subsidiary studies, which support the results obtained in Sec.4, are presented in Appendix A. An integral which is necessary for estimating asymptotics of the PDF is calculated in Appendix B.

2 Equilibrium solution for a non-linear Brownian oscillator

In this Section, which is presented here mainly for the methodical purposes, we remind the equilibrium solution for non-linear Brownian oscillator, $\alpha = 2$. It is well-known, that in this case there is no need to pass to the equation (1.9); on the contrary, the starting point is the stationary equation, which follows from Eqs. (1.1) and (1.7):

$$\frac{df(x)}{dx} = -\frac{dU}{dx}f(x). \tag{2.1}$$

It has the Boltzmann solution

$$f(x) = C \exp(-U(x)), \tag{2.2}$$

where the constant $C$ is determined from normalization condition. For a non-linear quartic Brownian oscillator

$$C = \frac{\sqrt{2}}{\Gamma(1/4)}. \tag{2.3}$$

We are also in position to get the characteristic function by making Fourier transform of equilibrium solution (2.2), (2.3) and, then expanding $\exp(ikx)$ into the power series and integrating over $x$ each term separately:

$$\hat{f}(k) = \sum_{j=0}^{\infty} c_j k^{2j}, \tag{2.4}$$

where

$$c_j = \frac{(-1)^j 2^j \Gamma \left(\frac{2j+1}{4}\right)}{(2j)! \Gamma(1/4)}. \tag{2.5}$$

In Sect. 4 we show that this solution is a particular case of that for a non-linear Lévy oscillator.
3 Equilibrium solution for non-linear Cauchy oscillator

For this case we start from Eq. (1.9) with $\alpha = 1$. The solution is

$$\hat{f}(k) = \frac{2}{\sqrt{3}} \exp\left(-\frac{|k|}{2}\right) \cos\left(\frac{\sqrt{3}|k|}{2} - \frac{\pi}{6}\right)$$ (3.1)

This solution can be expanded into the power series of $|k|$ as

$$\hat{f}(k) = \sum_{j=0}^{\infty} c_j |k|^j,$$ (3.2)

where

$$c_j = \frac{1}{2j!} \left[ \left(1 - \frac{i}{\sqrt{3}}\right) \exp\left(\frac{2\pi j}{3}\right) + \left(1 + \frac{i}{\sqrt{3}}\right) \exp\left(\frac{4\pi j}{3}\right)\right].$$ (3.3)

Making an inverse Fourier transform of Eq. (3.1), we get equilibrium PDF for the Cauchy oscillator,

$$f(x) = \frac{1}{\pi(1 - x^2 + x^4)}.$$ (3.4)

Clearly, $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$, so $f$ is a PDF.

Equation (3.4) clearly indicates strongly non-Boltzmann character of the equilibrium PDF for non-linear Cauchy oscillator. In Fig.1 both PDFs for Brownian and Cauchy oscillators are shown in a linear (at the top) and in a semi-logarithmic (at the bottom) scales. The two important distinctive features of the equilibrium PDF for the Cauchy oscillator are (i) the bimodality, and (ii) power law asymptotic at $x \to \pm \infty$. The latter property is clearly visualized in a linear scale, whereas the former is better shown in a semi-logarithmic scale. It appears that both features are inherent not only to a non-linear Cauchy oscillator, but also to equilibrium PDFs of non-linear Lévy oscillators with the Lévy indexes $\alpha$, such that $1 \leq \alpha < 2$. In the next Section we get the solution for an arbitrary $\alpha$ between 1 and 2 and clearly demonstrate this fact.

4 Equilibrium solution for non-linear Lévy oscillator, $1 \leq \alpha \leq 2$

We turn to the solution of Eq. (1.9) with the boundary conditions (1.10). One can convince himself that the solution of Eq. (1.9) with the first and the third conditions from Eqs. (1.10) being taken into account can be represented as

$$\hat{f}(k) = \Sigma_1 + ak^2 \Sigma_2,$$ (4.1)

where

$$\Sigma_1 = 1 + \sum_{j=1}^{\infty} a_j |k|^j^{(\alpha+2)},$$ (4.2)
\[ \sum_2 = 1 + \sum_{j=1}^{\infty} b_j |k|^j(\alpha+2). \]  

(4.3)

and the coefficients \(a_j, b_j\) are obtained by inserting Eqs. (4.2) and (4.3) into Eq. (1.9) and equating the terms of the same powers of \(k\) in the right - and left - hand sides:

\[ a_j j(\alpha + 2)(j\alpha + 2j - 1)(j\alpha + 2j - 2) = a_{j-1}, \]  

(4.4)

\[ b_j j(\alpha + 2)(j\alpha + 2j + 1)(j\alpha + 2j + 2) = b_{j-1}, \]  

(4.5)

\(j \geq 1,\ a_0 = b_0 = 1.\)

Obviously, the \(a_j\) as well as \(b_j\) tend to zero extremely fast.

The terms \(\Sigma_1\) and \(k^2\Sigma_2\) are, in fact, two independent particular solutions of Eq. (1.9). Since the condition at the infinity from Eq. (1.10) has not been employed yet, the general solution thus depends on an arbitrary constant \(a\). We define it numerically by demanding

\[ \hat{f}(k \to \infty) \to 0, \]  

(4.6)

that is,

\[ a = - \lim_{k \to \infty} \frac{\sum_1}{k^2 \sum_2}. \]  

(4.7)

Our numerical simulations show that with \(k\) increasing the value of \(a\) rapidly reaches the constant value, which, of course, depends on \(\alpha\). Further increase of \(k\) allows us to get \(a\) with higher accuracy, that is, with more significant digits. It is also worthwhile to note that the radius \(R\) of convergence for both power sets in numerator and denominator of Eq. (4.7) is infinite. This fact can be easily shown with the help of the Cauchy - Hadamard theorem (see, e.g., [20], p. 300), according to which, e.g., for \(\Sigma_1\)

\[ \frac{1}{R} = \lim_{j \to \infty} |a_j|^{1/j}, \]

where the bar denotes the largest limit for the sequence \(\{|a_j|^{1/j}\}\).

In Fig. 2 the obtained solution for \(\hat{f}(k)\) is shown in a linear (at the top) and in a semi - logarithmic (in the bottom) scales, \(\alpha = 1.7\). One can see an oscillatory character of the solution at large \(k\); this property can not be visualized directly from the power series expansion. Therefore, we also get a large \(k\)--asymptotic of the solution to Eq. (1.9). The derivation of it is presented in Appendix A in detail. There we demonstrate that (i) the asymptotics has an oscillatory character and an exponentially decreasing amplitude, and (ii) the period of oscillations coincide with high accuracy with that obtained numerically from power series expansion, see Fig.2. Thus, with the help of Appendix A we have an independent evidence of the correctness of our procedure and of the solution presented in this Section above.

Another evidence of the correctness of the obtained results stems from the comparison of the general expressions (4.1) - (4.5) with those for the two particular cases \(\alpha = 2\) and 1, presented in Sect.. 2, see Eqs. (2.4), (2.5), and in Sect.. 3, see Eqs. (3.2), (3.3), respectively. Indeed, let us consider Brownian oscillator at first. By comparing expansion (4.1) at \(\alpha = 2\) with that given by Eq. (2.4) one can see that the odd
coefficients $c_{2j}$ and $c_{2j+2}$ given by Eq. (2.5) have the same recurrent relation as the coefficients $a_j$ and $a_{j+1}$ given by Eq. (4.4) have, whereas the even coefficients $c_{2j+1}$ and $c_{2j+3}$ given by Eq. (2.5) have the same recurrent relation as the coefficients $b_j$ and $b_{j+1}$ given by Eq. (4.5) have. Now, let us compare general expressions with those for the Cauchy oscillator. By comparing expansion (4.1) at $\alpha = 1$ with that given by Eq. (3.2) one can see that the coefficients $c_{3j}$ and $c_{3j+3}$ in Eq. (3.3) obey recurrent relation between the coefficients $a_j$ and $a_{j+1}$, see Eq. (4.4), whereas the coefficients $c_{3j-1}$ and $c_{3j+2}$ obey recurrent relation for the coefficients $b_j$ and $b_{j+1}$, see Eq. (4.5). At last, $c_{3j+1} = 0, j \geq 1$, which is also in agreement with general expansion, if we set $\alpha = 1$ in Eq. (4.1). Thus, we may conclude that the general expansion (4.1) - (4.5) is in agreement with the particular cases of Brownian and Cauchy oscillators. However, we have to take in mind that for the Brownian and Cauchy cases we are able to evaluate coefficient $a$ analytically. On the contrary, for an arbitrary we have to estimate this coefficient numerically with the use of Eq. (4.7). Of course, the results of numerical and analytical estimates coincide for $\alpha = 1$ and 2.

Now we consider two important properties which have been already discussed for the particular case of the Cauchy oscillator in Sect. 3, namely, power law tails and bimodality. Consider power law tails at $x \to \pm \infty$ at first. These asymptotics are determined by the first non-analytical term in the power series expansion (4.1), that is, the term $a_1 |k|^{\alpha+2}$. By making an inverse Fourier transform of this term, we get

$$f(x) \approx a_1 \int_{-\infty}^{\infty} \frac{dk}{2\pi} |k|^{\alpha+2} \exp(-ikx), \quad x \to \pm \infty,$$

where $a_1 = [\alpha(\alpha + 1)(\alpha + 2)]^{-1}$. This integral is calculated by passing to the complex plane, the detailed derivation is presented in Appendix B. The result is

$$f(x) \approx \frac{\sin(\pi \alpha/2) \Gamma(\alpha)}{\pi |x|^{\alpha+3}}, \quad x \to \pm \infty. \quad (4.9)$$

It follows from Eq. (4.8) that the equilibrium PDF has a power law tail, $f(x) \propto |x|^{-(\alpha+3)}$, and, thus the integer moments of the order greater than 3 diverge. This behavior is strikingly different from that of a non-linear Brownian oscillator. The "long tails" can be explained qualitatively, if we turn to the Langevin description of the Lévy oscillator, The Langevin approach relevant to the FSESE [13] implies that the non-linear overdamped oscillator is influenced by "white Lévy noise" $\xi(t)$, whose PDF behaves as $|\xi|^{1-\alpha}$ at $|\xi| \to \infty$. These "long tails" imply that the large absolute values of the noise occur frequently, which, in turn, lead to large increments of the coordinate. However, it is also clear that the PDF of the coordinate $x$ must fall off more rapidly at $x \to \infty$ than the PDF of the noise $\xi$, because of the presence of the potential well, which prevents $x$ from "escaping" rather far from the origin.

In Fig. 3 equilibrium PDF is shown by solid lines in a linear (at the top) and semi-logarithmic (at the bottom) scales. The PDF is obtained by an inverse Fourier transform of the characteristic function shown in Fig. 2, $\alpha = 1.5$. The dashed lines indicate asymptotic (4.9). One can see, especially from the semi-logarithmic plot, that the asymptotics is a good approximation beginning from $k$ equal nearly 2. In this figure the second important property, namely, bimodality is clearly seen in a linear scale. In Fig. 4 the profiles of equilibrium PDF's (obtained by an inverse Fourier transform of corresponding power series in $k$-space) are shown for the different Lévy indexes from
\( \alpha = 1 \) at the top of the figure till \( \alpha = 2 \) at the bottom. It is seen that the bimodality is most strongly expressed for \( \alpha = 1 \). With the Lévy index increasing the bimodal profile “smooths” and, finally, it turns to a unimodal one at \( \alpha = 2 \), that is, for the Boltzmann distribution.

We also perform numerical simulation based on numerical solution of the Langevin equation

\[
\frac{dx}{dt} = -x^3 + \xi, \tag{4.10}
\]

where \( \xi(t) \) is a white Gaussian noise or a white Lévy noise, whose generator is described in detail in our previous papers devoted to the studies of self-affine properties of ordinary and fractional Lévy motions [21],[22]. A discussion on equivalency of the description of a stochastic system with the help of the Langevin equation (21) and fractional kinetic equation (1) is presented in Ref. [14]; for the Brownian motion this problem is discussed, e.g., in Ref. [13], in detail.

In Fig. 5 the results of numerical modelling are presented for the Brownian oscillator, \( \alpha = 2 \) (above) and the Lévy oscillator, \( \alpha = 1.1 \) (below). At the left the trajectories \( x(N) \) are presented, where \( N = t/\Delta t \) is the number of time steps in numerical modeling, \( t \) is the length of a single step, \( \Delta t << 1 \). The result of numerical solution of the Langevin equation must not depend on \( \Delta t \); for the Brownian oscillator this requirement is fulfilled at \( \Delta t \leq 10^{-2} \), whereas for the Lévy oscillator \( \Delta t \leq 10^{-3} \). We also studied the time-dependence of the second moments and fix when the moments become constant, thus indicating equilibrium state. From the left figures a clear difference between trajectories of Brownian (above) and Lévy (below) oscillators is seen: there are large “jumps” on the figure below, which are due to existence of large “pushes” from an external Lévy noise \( \xi \) or, equivalently, due to power law asymptotic of the PDF of the Lévy noise \( \xi \). Now we turn to the right figures. The designations are as follows. Thick solid line 1 and dotted line 2 show the potential well and its curvature, respectively, in conventional units. Thin solid line 3 shows the Boltzmann distribution (2.2). The black points depict the PDFs obtained in numerical simulations by statistical averaging over 50 trajectories each of 20,000 steps. Finally, the power law asymptotic (4.9) are depicted by a thin solid line 4. It is seen from the figure above that the PDF obtained in numerical simulations agrees quantitatively with the Boltzmann PDF, whereas the figure below demonstrates drastic difference between the Boltzmann PDF and numerical PDF. The latter has long power asymptotic, which start far away from the maximum of the potential well curvature, and whose exponent is close to that obtained theoretically. This conclusion is confirmed by the results of numerical modeling for the Lévy oscillators with different Lévy indexes, see Fig. 6. In this figure the black points depict the values \( \gamma \) of the exponents of the asymptotic \( |x|^{-\gamma} \) of the equilibrium PDFs for \( \alpha = 1.1, 1.3, 1.5, 1.7, 1.9 \). The values \( \gamma \) of are estimated as a tangent of a slope angle of a rectilinear asymptotic in a double logarithmic scale. The statistical averaging is over 50 trajectories, each consisting of 20,000 time steps, \( \Delta t = 10^{-3} \). Dotted line depicts theoretical dependence \( \gamma = \alpha + 3 \) of the asymptotic exponent versus \( \alpha \). The results of numerical simulations coincide with analytical estimates within the error limits.

In Fig. 7 the results of numerical modeling are presented in more detail for small values of \( x \), when power law asymptotic is inadequate. Similarly to Fig. 5, thick solid line 1 and dotted line 2 show the potential well and its curvature, respectively. Solid line 3 indicates PDF obtained by an inverse Fourier transform of the power series in \( k \)-space. The black points depict PDF obtained in numerical simulations by statistical...
averaging over 20 trajectories, each of $10^{-5}$ steps. The Lévy index is 1.2. It is clearly seen that both PDFs are in qualitative agreement and the both are bimodal. Therefore, from Figs. 5 - 7 we may conclude that the results of simulations based on numerical solution of the Langevin equation confirm qualitatively and even quantitatively the results based on analytical solution of Eq. (1.9) for the characteristic function.

5 Discussion and Results

In this paper we study, both analytically and numerically, the properties of equilibrium PDF of a non-linear ($x^4$) Lévy oscillator, that is, the oscillator which is subjected to a white Lévy noise obeying a Lévy stable probability law. We restrict ourselves to the case of the Lévy indexes such that $1 \leq \alpha \leq 2$. It is known that the Lévy stable distributions (as the Gaussian one, which corresponds to $\alpha$ equal 2) appear in problems, whose result is determined by the sum of a great number of independent identical factors. Since the Brownian oscillator is subjected to a white Gaussian noise, the Lévy oscillator is a natural generalization of a Brownian one.

For the analytical studies the starting equation is the so-called fractional symmetric Einstein - Smoluchowski equation, which contains fractional symmetric space derivative and is a natural generalization of the kinetic Einstein - Smoluchowski equation used in the theory of Brownian motion.

The main results are as follows:

1. We get analytically the characteristic function of the equilibrium PDF in the form of a power series. Its inverse Fourier transform, realized numerically, allows us to obtain the PDF. The two main distinctive features of the equilibrium PDFs for non-linear Lévy oscillator with $\alpha \neq 2$ are (i) the power law asymptotic at large $x$, and (ii) bimodality. Both features imply that the PDFs for non-linear Lévy oscillators are strikingly different from Boltzmann distribution, which is equilibrium PDF for the Brownian oscillator.

2. The power law asymptotics is determined by the first non-analytical term in a power series expansion of the characteristic function. We find that the asymptotics behaves as $|x|^{-(\alpha+3)}$.

4. The bimodal profile of the equilibrium PDF is clearly visualized after an inverse Fourier transform of the power series expansion for the characteristic function. Two maxima are most strongly expressed for $\alpha = 1$ (Cauchy oscillator). With the Lévy index increasing the profile “smooths” and at $\alpha = 2$ (Brownian oscillator) turns to profile with a single maximum.

5. We make three independent verifications of the solution for the characteristic function.

First: we find that the power series expansion for the characteristic function of the Brownian oscillator ($\alpha = 2$) is in agreement with the solution for $1 \leq \alpha \leq 2$, if we set $\alpha = 2$ in the latter.

Second: the particular case of the Cauchy oscillator ($\alpha = 1$) admits complete analytical study, which allows us also to check the agreement between power series and to demonstrate power law asymptotics and bimodality of the PDF.

Third: we obtain the solution of the equation for the characteristic function at large values of the argument $k$ and show that the periods of oscillations of this asymptotic solution in $k$-space coincide with high accuracy with those obtained from the power series.
Therefore, all three particular verifications testify to correctness of our approach in the general case, \(1 \leq \alpha \leq 2\).

6. We also perform numerical simulation based on numerical solution of the Langevin equation for a non-linear oscillator subjected to a white Gaussian noise. The equilibrium PDFs obtained in simulation show quantitative agreement with PDFs obtained analytically.

At the end we note that the general case of a non-linear Lévy oscillator \((x^{2n}, n = 1, 2, ...)\) can be treated in a similar way. In particular, it can be shown that in this case the power law asymptotics have an exponent \(- (\alpha + 2n - 1)\).

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APPENDIX A. Asymptotics of equilibrium characteristic function at large \(k\)

We start from Eq. (1.9) for \(k > 0\). Using the transformation

\[
\eta(\xi) = k^{b/3} \hat{f}(k), \quad \xi = ck^{1+b/3},
\]

where \(b = \alpha - 1\), and \(c\) is an arbitrary positive (for definiteness) parameter, we get

\[
\xi^3 \eta''' + (1 - \nu^2) \xi \eta' + \left( \nu^2 - 1 - \frac{\alpha \nu^3}{c^3} \xi^3 \right) \eta = 0, \quad \nu = \frac{3}{\alpha + 2}. \tag{A.2}
\]

At large Eq. (A.2) reduces to

\[
\eta'' - \left( \frac{\nu}{c} \right)^3 \eta = 0,
\]

whose solution, which tends to zero at large is

\[
\eta(\xi) = C_1 \exp \left[ \frac{\nu}{c} \xi \left( -\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) \right] + C_2 \exp \left[ \frac{\nu}{c} \xi \left( -\frac{1}{2} - i \frac{\sqrt{3}}{2} \right) \right]. \tag{A.3}
\]

where \(C_1\) and \(C_2\) are arbitrary constants. Returning to \(\hat{f}(k)\), we get at large \(k\)

\[
\hat{f}(k) = C k^{-(\alpha-1)/3} \exp \left[ -\frac{3}{2(\alpha + 2)} k^{(\alpha+2)/3} \right] \times \cos \left[ \frac{3\sqrt{3}}{2(\alpha + 2)} k^{(\alpha+2)/3} - \theta \right]. \tag{A.4}
\]

It depends on two unknown constants, \(C\) and \(\theta\), since we use the condition at infinity only. We can check this formula for two particular cases, \(\alpha = 1\) and 2.

1. Cauchy oscillator, \(\alpha = 1\).
   By comparing Eqs. (A.4) and (3.1) we get \(C = 2/\sqrt{3}\), \(\theta = \pi/6\).
2. Brownian oscillator, \(\alpha = 2\).

We explore asymptotics of the following integral at large \(k\), which can be obtained with the help of the saddle-point method, see, e.g., [24]:

\[
\int_{-\infty}^{\infty} dx \exp(-x^{2m} + ikx) = \frac{2\sqrt{\pi}}{\sqrt{m(2m-1)}} (2m)^{(m-1)/(2m-1)} k^{-(m-1)/(2m-1)} \times
\]
\[
\exp \left( k^{2m/(2m-1)} c_m \cos \frac{\pi m}{2m-1} \right) \cos \left( k^{2m/(2m-1)} c_m \sin \frac{\pi m}{2m-1} - \frac{\pi}{2 \cdot 2m-1} \right),
\]
where \( m = 1, 2, \ldots \),
\[
c_m = \frac{2m - 1}{2m} (2m)^{-1/(2m-1)}.
\]

With the help of Eq. (A.5) we get, also using Eqs. (2.2), (2.3),
\[
\hat{f}(k) = \int \frac{dx f(x) \exp(ikx)}{2\pi} = \frac{\sqrt{2}}{\Gamma(1/4)} \int dx \exp \left( -\frac{x^4}{4} + i k x \right) \approx \sqrt{2} \sqrt{\frac{\pi}{3}} \Gamma(1/4) \approx \sqrt{\frac{\pi}{3}} \Gamma(1/4), \quad \theta = \pi/6.
\]
at \(|k| >> 1\). By comparing Eqs. (A.6) and (A.4) we get \( C = 2\sqrt{\pi}/(\sqrt{3} \Gamma(1/4)) \), \( \theta = \pi/6 \).

We also make a comparison between the period of the asymptotics (A.4) and the period of oscillations of the solution (4.1) - (4.7). Firstly, we are able to get with high accuracy the values \( k_j \), \( j = 1, 2, 3, 4, 5 \), at which the solution given by Eqs. (4.1) - (4.7) is equal zero, see also Fig. 2. Then, we insert \( k_j \) into the cosine in Eq. (A.4) and estimate
\[
1 - \frac{3 \sqrt{3}}{2(\alpha + 2)} \frac{k_j^{(\alpha+2)/3} - k_j^{(\alpha+2)/3}}{\pi} = \delta_{j+1,j}.
\]

It is seen, that \( \delta_{j+1,j} \) can serve as a measure of difference between zeros of \( \hat{f}(k) \) estimated from Eqs. (4.1) - (4.7) and those estimated from Eq. (A.4). For example, for the characteristic function with \( \alpha = 1.7 \), which is shown in Fig. 2, we get \( \delta_{2,1} = 56 \cdot 10^{-4} \), \( \delta_{3,2} = 17 \cdot 10^{-4} \), \( \delta_{4,3} = 8 \cdot 10^{-4} \), \( \delta_{5,4} = 5 \cdot 10^{-4} \). These results demonstrate the smallness of \( \delta_{j+1,j} \) and serve as one more confirmation of the correctness of our approach.

**APPENDIX B. Power law asymptotics of equilibrium PDF**

In this Appendix we evaluate the main value of the integral
\[
I = \int_{-\infty}^{\infty} \frac{dk}{2\pi} |k|^\alpha \exp(-ikx),
\]
which gives the asymptotics of the equilibrium PDF, see Eq. (4.8). We present (B1) as
\[
I = 2 \Re I_1,
\]
where
\[
I_1 = \int_{0}^{\infty} \frac{dk}{2\pi} k^{\alpha+2} \exp(-ikx).
\]
In order to evaluate $I_1$, we pass to the complex plane. Since the integral over the closed contour shown in Fig. 8 is equal to zero, we get

$$I_1 = -\int_{-\infty}^{0} \frac{dk}{2\pi k^{\alpha+2}} \exp(-ikx) = i \exp(-i\alpha\pi/2) \int_{0}^{\infty} \frac{dk}{2\pi k^{\alpha+2}} \exp(-kx) =$$

$$= \frac{i \exp(-i\alpha\pi/2)}{2\pi x^{\alpha+3}} \Gamma(\alpha + 3)$$  \hspace{1cm} (B.4)

and

$$I = \frac{\sin\left(\frac{\pi\alpha}{2}\right) \Gamma(\alpha + 3)}{\pi x^{\alpha+3}}.$$  \hspace{1cm} (B.5)

With the use of Eqs. (4.8) and (B.5) we get power law asymptotics of the equilibrium PDF, see Eq. (4.9).

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FIGURE CAPTIONS

Fig.1. Equilibrium PDFs for non - linear Brownian and Cauchy oscillators in the linear (above) and semi - logarithmic (below) scales.

Fig.2. Equilibrium characteristic function for non - linear Lévy oscillator in the linear (above) and semi - logarithmic (below) scales, $\alpha = 1.5$. Characteristic function (4.1) is evaluated by the use of Eqs. (4.1) - (4.7).

Fig. 3. Equilibrium PDF and its asymptotics in the linear (above) and semi - logarithmic (below) scales, $\alpha = 1.5$. The PDF obtained by inverse Fourier transform of a power series (4.1) is indicated by solid lines. Dashed lines indicate power law asymptotics (4.9).

Fig. 4. The profiles of equilibrium PDFs for different values of the Lévy indexes.

Fig.5. Comparison of analytical estimates with the results of numerical simulation based on numerical integration of the Langevin equations for the Brownian (above) and the Lévy (below) oscillators. The Lévy index is 1.1. At the left: typical trajectories, $N$ is the number of steps. At the right the semi - logarithmic scale is used. The designations are as follows. Thick solid line 1 and dotted line 2 show the potential well and its curvature, respectively, in conventional units. Boltzmann distribution is indicated by solid line 3, the PDF obtained in numerical simulations is shown by black points, solid line 4 shows the power law asymptotics (4.9).

Fig.6. Comparison of analytical estimates with the results of numerical simulation based on numerical integration of the Langevin equations. The black points indicate the slope $\gamma$ of the power law asymptotics of equilibrium PDFs obtained in numerical simulations for different Lévy indexes. Dotted line depicts theoretical value $\gamma = \alpha + 3$.

Fig.7. Comparison of analytical estimates with the results of numerical simulation based on numerical integration of the Langevin equations. The Lévy index is 1.2. Thick solid line 1 and dotted line 2 show the potential well and its curvature, respectively. Solid line 3 indicates PDF obtained by inverse Fourier transform from Eq. (4.1). Black points indicate PDF obtained in numerical simulation.

Fig. 8. Contour of integration for evaluation $I_1$, see Eq. (B.3).
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