Semi-invariant lightlike submanifolds of golden semi-Riemannian manifolds

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Abstract

In this study, we introduce semi-invariant lightlike submanifolds of golden semi-Riemannian manifolds. We find some conditions for integrability of distributions. Moreover, we investigate totally geodesic and mixed geodesic distributions of such a submanifold. The paper also contains an example.

Keywords: Lightlike submanifold, Golden semi-Riemann manifold, semi-invariant submanifold.

Altın semi-Riemann manifoldların semi-invaryant lightlike altmanifoldları

Öz

Bu makalede altın semi-Riemann manifoldların semi-invaryant lightlike altmanifoldlarını çalıştıktı. Distribüsyonların integrelinebilirliği için bazı koşullar bulundu. Ayrıca, böyle bir altmanifoldun tamamen geodezik ve mixed geodezik distribüsyonlarını inceledik. Makale, bir adet örnek içermektedir.

Anahtar kelimeler: Lightlike altmanifold, Altın semi-Riemann manifold, semi-invaryant altmanifold.

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1. Introduction

The applications in general relativity of degenerate submanifolds of semi-Riemannian manifolds have made such submanifolds one of the most important fields of study in differential geometry. In contrast to the Riemannian and semi-Riemannian submanifolds, the intersection of the normal bundle and the tangent bundle in lightlike submanifolds is not zero. Thus, studying the geometry of lightlike submanifolds is more difficult and complicated. The first studies on lightlike submanifolds are belong to Duggal-Bejancu [7] and Kupeli [17]. Later, Duggal and Şahin introduced a book on lightlike geometry [10]. Many authors have studied on lightlike submanifolds in various spaces, for example [2, 3, 4, 5, 8, 9, 16].

The geometries of some manifolds with differentiable geometric structures, which are quite suitable tools for the differential geometry, are interesting. These manifolds and the transformations between them have been studied extensively in differential geometry. One of these structures is $f$–structure which is described by Yano in [20]. A tensor field $\phi$ on manifold $\mathcal{N}$ is an $f$–structure if it satisfies that $\phi^2 + \phi = 0$. The almost complex and contact structures are obvious examples of $f$–structures. Goldberg and Yano introduced polynomial structures of degree $d$ in [14] by generalizing $f$–structures. A polynomial structure satisfies the equation $\theta(f) = f_d + a_1 f_{d-1} + \ldots + a_d f + a_0 I = 0$, where $a_1, a_2, \ldots, a_d$ are real numbers and $I$ is the identity tensor of type $(1,1)$.

Golden proportion $\phi$ is the real positive root of the equation $x^2 - x - 1 = 0$ (thus, $\phi = \frac{1 + \sqrt{5}}{2} = 1.618 \ldots$). Inspired by the Golden proportion, Crasmareanu and Hretcanu defined $\mathcal{P}$ golden structure which is a tensor field satisfying $\mathcal{P}^2 - \mathcal{P} - I = 0$ on $\mathcal{N}$ [6]. In the same paper, it is shown that $\mathcal{P}$ is an automorphism of $T\mathcal{N}$ and its eigenvalues are $\phi$ and $1-\phi$. We note that for golden structures, $\mathcal{P} \neq \phi I$. If $\mathcal{P} = \phi I$, then its minimal polynomial is $X - \phi$. But, the minimal polynomial of the golden structure $\mathcal{P}$ is $X^2 - X - 1$ and it is a contraction.

A Riemannian manifold $\mathcal{N}$ with a $\mathcal{P}$ golden structure is named as a golden Riemannian manifold in [6, 15] and invariant submanifolds of such manifolds were studied in [15]. Later, Erdoğan and Yıldırım researched totally umbilical semi-invariant manifolds of golden manifold in [12]. Many authors investigated the properties of golden structures and golden manifolds in Riemannian [13, 18] and semi-Riemannian geometry. Submanifolds of golden manifolds in degenerate geometry were firstly studied by Poyraz and Yaşar in [19]. In the study, they introduced lightlike hypersurfaces of a golden semi-Riemannian manifold and investigated geometry of such hypersurfaces in detail. Acet studied lightlike hypersurfaces of metallic semi-Riemannian manifolds in [1] and Erdoğan researched transversal lightlike submanifolds of metallic semi-Riemannian manifolds in [11].

In this study, we introduce semi-invariant lightlike submanifolds of golden semi-Riemannian manifolds. We find some conditions for integrability of distributions. Moreover, we investigate totally geodesic and mixed geodesic distributions of such a submanifold. The paper also contains an example.
2. Preliminaries

A golden structure $\tilde{P}$ on a differentiable manifold $\tilde{N}$ is defined as

$$\tilde{P}^2 = \tilde{P} + I$$  \hspace{1cm} (2.1)

and if

$$\bar{g}(PX,Y) = \bar{g}(X,PY)$$  \hspace{1cm} (2.2)

then $(\tilde{N},\bar{g},\tilde{P})$ is called a golden semi-Riemannian manifold [18].

Let $(\tilde{N},\bar{g},\tilde{P})$ be a semi-Riemannian golden manifold. Then for any $X,Y \in \Gamma(T\tilde{N})$, equation (2.2) can be written as

$$\bar{g}(\tilde{P}X,\tilde{P}Y) = \bar{g}(\tilde{P}X,Y) + \bar{g}(X,Y).$$  \hspace{1cm} (2.3)

We denote real space forms with constant sectional curvatures $c_p$ and $c_q$, by $N'_p$ and $N'_q$, respectively. Then similar calculations of semi-Riemannian product real space form (see [21]), the Riemannian curvature tensor $\bar{R}$ of a locally Golden product space form $(\tilde{N}=N'_p(c_p)\times N'_q(c_q),\bar{g},\tilde{P})$ is calculated as the following

$$\bar{R}(X,Y,Z) = \left(\begin{array}{c} \bar{g}(Y,Z)X - \bar{g}(X,Z)Y + \bar{g}(PY,Z)PX - \bar{g}(PX,Z)PY \\ + \bar{g}(PY,Z)X - \bar{g}(PX,Z)Y + \bar{g}(Y,Z)PX - \bar{g}(X,Z)PY \end{array}\right).$$  \hspace{1cm} (2.4)

Let $(\tilde{N},\bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold with index $q$, such that $m,n \geq 1$, $1 \leq q \leq m+n+1$ and $(N,g)$ be a $m$-dimensional submanifold of $\tilde{N}$, where $g$ is the induced metric of $\bar{g}$ on $N$. If $g$ is degenerate on the tangent bundle $T^\perp N$ of $N$ the $N$ is named a lightlike submanifold of $\tilde{N}$. For a degenerate metric $g$ on $N$

$$T^\perp N = \bigcup \{ u \in T_x\tilde{N} : \bar{g}(u,v) = 0, \forall v \in T_xN, x \in N \}$$  \hspace{1cm} (2.5)

is a degenerate $n$-dimensional subspace of $T\tilde{N}$. Since, both $T_xN^\perp$ and $T_xN^{\perp\perp}$ are degenerate orthonormal distributions, there exists a radical (null) space $Rad(T_xN)$ such that $Rad(T_xN) = T_xN \cap T_xN^{\perp\perp}$. If $\text{rank}(Rad(T_xN)) = r > 0$, then $N$ is an $r$-lightlike submanifold of $\tilde{N}$.

Denote the complementary distributions of $Rad(TN')$ in $TN'$ and $TN^\perp$ by $S(TN')$ and $S(TN^\perp)$, respectively. Also, let $tr(TN')$ and $ltr(TN')$ be complementary (but not orthogonal) vector bundles to $TN'$ in $TN^\perp$. Thus, we can write
\[ T \dot{N} = S(T \dot{N}) \perp \text{Rad}(T \dot{N}) \]  
\[ t\text{r}(T \dot{N}) = l\text{tr}(T \dot{N}) \perp S(T \dot{N}^\perp) , \]  
\[ T \dot{N} \mid \gamma = T \dot{N} \oplus t\text{r}(T \dot{N}) = \left( \text{Rad}(T \dot{N}) \oplus l\text{tr}(T \dot{N}) \right) \perp S(T \dot{N}) \perp S(T \dot{N}^\perp) , \]  

where \( S(T \dot{N}) \), \( S(T \dot{N}^\perp) \), \( t\text{r}(T \dot{N}) \) and \( l\text{tr}(T \dot{N}) \) are called as screen bundle, screen transversal bundle, transversal bundle and lightlike transversal bundle, respectively.

**Theorem 2.1.** Let \( (\dot{N}, g, S(T \dot{N}), S(T \dot{N}^\perp)) \) be an \( r \)-lightlike submanifold of a semi-Riemannian manifold \( (\bar{N}, \bar{g}) \). Assume that \( U \) is a coordinate neighbourhood of \( \dot{N} \) and \( E_i, \ i \in \{1, \ldots, r\} \) is a basis of \( \Gamma(\text{Rad}(T \dot{N})) \). Then, there exist a complementary vector subbundle \( l\text{tr}(T \dot{N}) \) of \( \text{Rad}(T \dot{N}) \) in \( S(T \dot{N}^\perp) \) and a basis \( \{ N_i \}, \ i \in \{1, \ldots, r\} \) of \( \Gamma(l\text{tr}(T \dot{N})) \) such that

\[
g(N_i, E_j) = \delta_{ij}, \ g(N_i, N_j) = 0, \]  

for any \( i, j \in \{1, \ldots, r\} \).

We say that a submanifold \( (\dot{N}, g, S(T \dot{N}), S(T \dot{N}^\perp)) \) of \( \bar{N} \) is

**Case 1:** \( r \)-lightlike if \( r < \min\{m, n\} \),

**Case 2:** Coisotropic if \( r = n < m \), \( S(T \dot{N}^\perp) = \{0\} \),

**Case 3:** Isotropic if \( r = m < n \), \( S(T \dot{N}) = \{0\} \),

**Case 4:** Totally lightlike if \( r = m = n \), \( S(T \dot{N}) = \{0\} = S(T \dot{N}^\perp) \).

Let \( \nabla \) be the Levi-Civita connection on \( \dot{N} \). The Gauss and Weingarten formulas are given by

\[
\nabla_X Y = \nabla_X Y + h(X, Y), \]  
\[
\nabla_X U = -A_X X + \nabla_X U , \]  

for any \( X, Y \in \Gamma(T \dot{N}) \) and \( U \in \text{tr}(T \dot{N}) \), where \( \{ \nabla_X Y, A_X X \} \) and \( \{ h(X, Y), \nabla_X U \} \) belong to \( \Gamma(T \dot{N}) \) and \( \Gamma(\text{tr}(T \dot{N})) \), respectively. \( \nabla \) and \( \nabla' \) are linear connections on \( \dot{N} \) and on the vector bundle \( \text{tr}(T \dot{N}) \), respectively. Using (2.8), considering the projection morphisms \( L \) and \( S \) of \( \text{tr}(T \dot{N}) \) on \( l\text{tr}(T \dot{N}) \) and \( S(T \dot{N}^\perp) \), respectively, from (2.10) and (2.11), we get

\[
\nabla_X Y = \nabla_X Y + h'(X, Y) + h^s(X, Y), \]  
\[
\nabla_X N = -A_X X + \nabla_X N + D^s(X, N), \]  
\[
\nabla_X L = -A_X X + \nabla_X L + D^s(X, L), \]  

for any \( X, Y \in \Gamma(T \dot{N}) \), \( N \in \Gamma(l\text{tr}(T \dot{N})) \) and \( L \in \Gamma(S(T \dot{N}^\perp)) \), where \( h'(X, Y) = Lh(X, Y) \), \( h^s(X, Y) = Sh(X, Y) \), \( \nabla_X N, D^s(X, L) \in \Gamma(l\text{tr}(T \dot{N})) \), \( \nabla_X L, D^s(X, N) \in \Gamma(S(T \dot{N}^\perp)) \) and \( \nabla_X Y, A_X X, A_X X \in \Gamma(T \dot{N}) \) [8]. Then, using (2.12)-(2.14) and \( \nabla \) metric connection, we derive
\[ \bar{g}(h'(X,Y),E) + \bar{g}(Y,D'(X,L)) = g(A_gX,Y), \quad (2.15) \]
\[ \bar{g}(D'(X,N),L) = \bar{g}(A_gX,N). \quad (2.16) \]

Let \( J \) be a projection of \( TN' \) on \( S(TN') \). From (2.6) we derive

\[ \nabla_x JY = \nabla_x JY + h'(X,JY), \quad (2.17) \]
\[ \nabla_x E = -A_x^tX + \nabla_x^tE, \quad (2.18) \]

for any \( X,Y \in \Gamma(TN') \) and \( E \in \Gamma(Rad(TN')) \), where \( \{ \nabla_x JY, A_x^tX \} \) and \( \{ h'(X,JY), \nabla_x^tE \} \) belong to \( \Gamma(S(TN')) \) and \( \Gamma(Rad(TN')) \), respectively.

By using above equations we obtain

\[ \bar{g}(h'(X,JY),E) = g(A_g^tX,JY), \quad (2.19) \]
\[ \bar{g}(h'(X,JY),N) = g(A_gX,JY), \quad (2.20) \]
\[ \bar{g}(h'(X,E),E) = 0, \quad A_g^tE = 0. \quad (2.21) \]

Furthermore, Gauss equation is given by

\[ \bar{R}(X,Y)Z = R(X,Y)Z + A_{h'(x,z)}Y - A_{h'(y,z)}X + A_{h'(x,z)}Y - A_{h'(x,z)}X \]
\[ + (\nabla_x h')'(Y,Z) - (\nabla_y h')'(X,Z) + D'(X,h'(Y,Z)) - D'(Y,h'(X,Z)) \]
\[ + (\nabla_x h')'(Y,Z) - (\nabla_y h')'(X,Z) + D'(X,h'(Y,Z)) - D'(Y,h'(X,Z)). \quad (2.22) \]

for any \( X,Y,Z \in \Gamma(TN') \).

3. Semi-invariant lightlike submanifolds of semi-Riemannian golden manifolds

Let \( (N',\bar{g},S(TN'),S(TN'^\perp)) \) be a lightlike submanifold of a golden semi-Riemannian manifold \( (\bar{N},\bar{g},\bar{P}) \). Then, for any \( X \in \Gamma(TN') \), we have

\[ PX = PX + wX, \quad (3.1) \]

where \( PX \) and \( wX \) are the tangential and transversal components of \( PX \), respectively. Similarly, for any \( U \in tr(TN') \), we have

\[ PU = BU + CU \quad (3.2) \]

where \( BU \) and \( CU \) are the tangential and transversal components of \( PU \), respectively.

**Definition 3.1.** Let \( N' \) be a lightlike submanifold of a golden semi-Riemannian manifold \( (\bar{N},\bar{g},\bar{P}) \). If \( \bar{P}(Rad(TN')) \subset S(TN') \), \( \bar{P}(ltr(TN')) \subset S(TN') \) and \( \bar{P}(S(TN'^\perp)) \subset S(TN') \) then we call that \( N' \) is a semi-invariant lightlike submanifold.

If we set \( D_1 = \bar{P}(Rad(TN')) \), \( D_2 = \bar{P}(ltr(TN')) \) and \( D_3 = \bar{P}(S(TN'^\perp)) \) then we have
Thus we derive

\[ S(TN^\perp) = D_0 \perp \{D_1 \oplus D_2\} \perp D_3, \quad (3.3) \]

According to this definition we can write

\[ D = D_0 \perp D_1 \perp \text{Rad}(TN^\perp), \quad (3.4) \]

and

\[ D^+ = D_2 \perp D_3. \quad (3.7) \]

Thus we have

\[ TN^\perp = D \oplus D^+ \quad (3.8) \]

For Case (2), we know that \( S(TN^\perp) = \{0\} \). Then we derive

\[ S(TN^\perp) = \{D_1 \oplus D_2\} \perp D_0, \quad (3.9) \]

\[ TN^\perp = \{D_1 \oplus D_2\} \perp D_0 \perp \text{Rad}(TN^\perp), \quad (3.10) \]

\[ TN = \{D_1 \oplus D_2\} \perp D_0 \perp \{\text{Rad}(TN^\perp) \oplus \text{ltr}(TN^\perp)\}, \quad (3.11) \]

\[ TN^\perp = D \oplus D_2. \quad (3.12) \]

**Proposition 3.2.** The distribution \( D_0 \) are \( D \) are invariant distributions with respect to \( \bar{P} \).

**Lemma 3.3.** Let \( N \perp \) be a semi-invariant lightlike submanifold of a golden semi-Riemannian manifold \( (\tilde{N}, \tilde{g}, \tilde{P}) \) with \( \tilde{\nabla}P = 0 \). Then, we have

\[
(\nabla_X P)Y = A_{x,y} X + Bh(X,Y), \quad (3.13)
\]
\[
w\nabla_X Y = \nabla^i_x wY + h(X, PY), \quad (3.14)
\]
\[
\nabla_X BU = -PA_{x} X + B\nabla^i_x U, \quad (3.15)
\]
\[
h(X, BU) = -wA_{x} X, \quad (3.16)
\]

for any \( X, Y \in \Gamma(TN^\perp) \) and \( U \in \Gamma(\text{tr}(TN^\perp)) \).

Throughout this paper, we assume \( \tilde{\nabla}P = 0 \).

**Lemma 3.4.** Let \( N \perp \) be a semi-invariant lightlike submanifold of a golden semi-Riemannian manifold \( (\tilde{N}, \tilde{g}, \tilde{P}) \). Then, we have

\[
P^2 X = PX + X - BwX, \quad (3.17)
\]
\[
wPX = wX, \quad PBU = BU, \quad wBU = U, \quad (3.18)
\]
for any $X,Y \in \Gamma(TN)$ and $U \in \Gamma(tr(TN))$.

**Lemma 3.5.** Let $N'$ be a lightlike submanifold of a golden semi-Riemannian manifold $(\bar{N},\bar{g},\bar{P})$. Then $P$ is golden structure on $N'$ iff $wX = 0$.

**Example 3.6.** Let $(\bar{N} = \mathbb{R}^7_2, \bar{g})$ be a 7-dimensional semi-Euclidean space with signature $(-,-,+,+,+,+,+)$ and $(x_1,x_2,x_3,x_4,x_5,x_6,x_7)$ be the standard coordinate system of $\mathbb{R}^7_2$. If we define a mapping $\bar{P}$ by

$$
\bar{P}(x_1,x_2,x_3,x_4,x_5,x_6,x_7) = ((1-\phi)x_1,\phi x_2,\phi x_3,(1-\phi)x_4,\phi x_5,(1-\phi)x_6,\phi x_7)
$$

then $\bar{P}^2 = \bar{P} + I$ and $\bar{P}$ is a golden structure on $\mathbb{R}^7_2$. Let $N'$ be a submanifolds of $\bar{N}$ given by

$$
x_i = \phi u_i + \frac{1}{2(2+\phi)} u_2 - u_3, \quad x_2 = u_1 - \frac{\phi}{2(2+\phi)} u_2 + \phi u_3,
$$

$$
x_3 = u_1 + \frac{\phi}{2(2+\phi)} u_2 + \phi u_3, \quad x_4 = \phi u_1 - \frac{1}{2(2+\phi)} u_2 - u_3,
$$

$$
x_5 = -u_4, \quad x_6 = -u_4 - u_5, \quad x_7 = \phi u_4 + \phi u_5,
$$

where $u_i$, $1 \leq i \leq 5$, are real parameters. Thus $TN' = Sp\{U_1,U_2,U_3,U_4,U_5\}$ where,

$$
U_1 = \phi \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}, \quad U_2 = \frac{1}{2(2+\phi)} (\frac{\partial}{\partial x_1} - \phi \frac{\partial}{\partial x_2} + \phi \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}),
$$

$$
U_3 = \frac{\partial}{\partial x_1} + \phi \frac{\partial}{\partial x_2} + \phi \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}, \quad U_4 = -\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \phi \frac{\partial}{\partial x_4}, \quad U_5 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}.
$$

Then $N'$ is a 1-lightlike submanifold. We have $Rad(TN') = Sp\{E = U_1\}$ and $S(TN') = Sp\{U_2,U_3,U_4,U_5\}$. Therefore, we derive

$$
ltr(TN') = Sp\left\{ N = -\frac{1}{2(2+\phi)} \left( \phi \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} - \phi \frac{\partial}{\partial x_4} \right) \right\}
$$

And

$$
S(TN'^\perp) = Sp\left\{ L = \phi \frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_7} \right\}.
$$

Furthermore, we get

$$
PE = U_3, \quad PN = U_2, \quad PL = U_4.
$$
If we set $D_0 = Sp[U_1]$, $D_1 = Sp[U_2]$, $D_2 = Sp[U_2]$, $D_3 = Sp[U_4]$, then $N$ is a semi-invariant lightlike submanifold of $\tilde{N}$.

**Theorem 3.7.** Let $N$ be a semi-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then $D$ is integrable iff

\[ h'(\tilde{P}X, \tilde{P}Y) = h'(\tilde{P}X, Y) + h'(\tilde{P}X, \tilde{P}Y) + h'(X, Y) \]  

(3.21)

for any $X, Y \in \Gamma(D)$.

**Proof.** For any $X, Y \in \Gamma(D), \ E \in \Gamma(Rad(TN))$, $L \in \Gamma(S(TN^\perp))$ and $N \in \Gamma(ltr(TN^\perp))$ the distribution $D$ is integrable iff

\[ \tilde{g}\left(\left[\tilde{P}X, Y\right], \tilde{P}E\right) = \tilde{g}\left(\left[\tilde{P}X, Y\right], \tilde{P}L\right) = 0. \]

Then, from (2.1) and (2.12) we obtain

\[ \tilde{g}\left(\left[\tilde{P}X, Y\right], \tilde{P}E\right) = \tilde{g}(h'(\tilde{P}X, \tilde{P}Y) - h'(Y, \tilde{P}X) - h'(Y, X), E), \]  

(3.22)

\[ \tilde{g}\left(\left[\tilde{P}X, Y\right], \tilde{P}L\right) = \tilde{g}(h'(\tilde{P}X, \tilde{P}Y) - h'(Y, \tilde{P}X) - h'(Y, X), L) \]  

(3.23)

which completes the proof.

**Theorem 3.8.** Let $N$ be a semi-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. If $D$ is integrable, then the leaves of $D$ have a golden structure.

**Proof.** Let $N$ be a semi-invariant lightlike submanifold and $M'$ a leaf of $D$. Since $D$ is integrable, then for any $p \in M'$, we have $T_pM' = (D)_p$.

Letting $P' = P_p'$, we say that $P'$ defines an (1,1)-tensor field on $M'$ because $D$ is $\tilde{P}$-invariant.

If $X \in \Gamma(D)$, then $wX = 0$. Thus we derive

\[ P'^2X = P'^2X = P^2X + X = PX + X = P'X + X \]

which proves our assertion.

**Theorem 3.9.** Let $N$ be a semi-invariant lightlike submanifolds of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then the distribution $D^\perp$ is integrable iff

(i) $\tilde{g}(h'(X, Y) - h'(Y, X), N) = \tilde{g}(A_{\tilde{v}_1}X - A_{\tilde{v}_1}Y, N)$,

(ii) $\tilde{g}(A_{\tilde{v}_1}Y, \tilde{P}N) = \tilde{g}(A_{\tilde{v}_1}X, \tilde{P}N)$,

(iii) $\tilde{g}(A_{\tilde{v}_1}Y, \tilde{P}Z) = \tilde{g}(A_{\tilde{v}_1}X, \tilde{P}Z)$,
for any $X,Y \in \Gamma(D^\perp)$, $Z \in \Gamma(D_0)$, $N \in \Gamma(ltr(TN'))$ and $U_1,U_2 \in \Gamma(tr(TN'))$.

**Proof.** For any $X,Y \in \Gamma(D^\perp)$, $Z \in \Gamma(D_0)$ and $N \in \Gamma(ltr(TN'))$ the distribution $D^\perp$ is integrable iff

$$g([X,Y],PN) = g([X,Y],N) = g([X,Y],Z) = 0$$

Choosing $X,Y \in \Gamma(D^\perp)$, there is a vector field $U_1,U_2 \in \Gamma(tr(TN'))$ such that $X = \bar{PU}_i$ and $Y = \bar{PU}_j$. Then, from (2.1), (2.11), (2.12) and (2.17) we derive

$$g([X,Y],PN) = g(\bar{\nabla}_X Y - \bar{\nabla}_Y X,PN) = g(\bar{\nabla}_X PU_2 - \bar{\nabla}_Y PU_1,PN) = g(\bar{\nabla}_X PU_2 - \bar{\nabla}_Y PU_1,N) + g(\bar{\nabla}_X U_2 - \bar{\nabla}_Y U_1,N)$$

$$= g(h'(X,\bar{PU}_2) - h'(Y,\bar{PU}_1),N) - g(A_{ij} X - A_{ij} Y,N),$$

$$g([X,Y],N) = g(\bar{\nabla}_X Y - \bar{\nabla}_Y X,N) = g(\bar{\nabla}_X PU_2 - \bar{\nabla}_Y PU_1,N) = g(\bar{\nabla}_X U_2 - \bar{\nabla}_Y U_1,PN) = g(A_{ij} Y - A_{ij} X,PN),$$

$$g([X,Y],Z) = g(\bar{\nabla}_X Y - \bar{\nabla}_Y X,Z) = g(\bar{\nabla}_X PU_2 - \bar{\nabla}_Y PU_1,Z) = g(\bar{\nabla}_X U_2 - \bar{\nabla}_Y U_1,PZ) = g(A_{ij} Y - A_{ij} X,PZ).$$

This completes our proof.

**Theorem 3.10.** Let $N'$ be a semi-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\bar{N},\bar{g},\bar{P})$. Then $Rad(TN')$ is integrable iff

(i) $g(h'(E,PE),N) = g(h'(E,PE),N),$ 
(ii) $g(h'(E,\bar{PE}),E_i) = g(h'(E,PE),E_i),$ 
(iii) $g(h'(E,PE),L) = g(h'(E,\bar{PE}),L),$ 
(iv) $g(A'_i',E,X) = g(A'_i',E,X),$

for any $X \in \Gamma(D_0), E,E_i \in \Gamma(Rad(TN'))$, $N \in \Gamma(ltr(TN'))$, $L \in \Gamma(S(TN^+))$.

**Proof.** $Rad(TN')$ is integrable iff

$$g([E,E],PN) = g([E,E],PE_i) = g([E,E],PL) = g([E,E],X) = 0$$

for any $X \in \Gamma(D_0), E,E_i \in \Gamma(Rad(TN'))$, $N \in \Gamma(ltr(TN'))$, $L \in \Gamma(S(TN^+))$. Then, from (2.12), (2.17) and (2.18) we get

$$g([E,E],PN) = g(\bar{\nabla}_E E - \bar{\nabla}_E E,PN) = g(\bar{\nabla}_E PE - \bar{\nabla}_E PE,N) = g(h'(E,PE') - h'(E',PE),N),$$

$$g([E,E],PE_i) = g(\bar{\nabla}_E E - \bar{\nabla}_E E,PE_i) = g(\bar{\nabla}_E PE - \bar{\nabla}_E PE,E_i) = g(h'(E,\bar{PE}) - h'(E',\bar{PE}),E_i).$$
\( g\left[ E, E, PL \right] = g(\nabla_E E - \nabla_E E, PL) = g(\nabla_E \hat{P}E - \nabla_E \hat{P}E, L) \)

\( = g(h'(E, \hat{P}E') - h'(E', \hat{P}E), L), \)

\( g\left[ E, E, X \right] = g(\nabla_E E - \nabla_E E, X) = g(A_E E - A_E E, X) = 0. \) \hspace{1cm} (3.29) \hspace{1cm} (3.30)

This completes the proof.

**Theorem 3.11.** Let \( N' \) be a semi-invariant lightlike submanifold of a golden semi-Riemannian manifold \((N, g, P)\). Then, \( P(Rad(TN')) \) is integrable iff

(i) \( g(h'(PE_1, E_2), E) = g(h'(PE_1, E_1), E), \)

(ii) \( g(h'(PE_1, E_2), L) = g(h'(PE_1, E_1), L) \)

(iii) \( g(A_h PE_1, PE_2) = g(A_h PE_1, PE_1) \)

(iv) \( g(A_h PE_2, PX) = g(A_h PE_1, PX) \)

for any \( X \in \Gamma(D_0), E, E_1, E_2 \in \Gamma(Rad(TN')), N \in \Gamma(Itr(TN')), L \in \Gamma(S(TN'^{-})). \)

**Proof.** \( \bar{P}Rad(TN') \) is integrable iff

\( g\left[ \bar{PE}_1, \bar{PE}_2 \right], \bar{P}E = g\left[ \bar{PE}_1, \bar{PE}_2 \right], \bar{P}L = g\left[ \bar{PE}_1, \bar{PE}_2 \right], N = g\left[ \bar{PE}_1, \bar{PE}_2 \right], X = 0. \)

for any \( X \in \Gamma(D_0), E, E_1, E_2 \in \Gamma(Rad(TN')), N \in \Gamma(Itr(TN')), L \in \Gamma(S(TN'^{-})). \) Since \( \bar{\nabla} \) is a metric connection, from (2.1), (2.12), (2.13) and (2.18) we obtain

\( g\left[ \bar{PE}_1, \bar{PE}_2 \right], \bar{P}E = g(\nabla_{\bar{P}E_1} \bar{P}E_2 - \nabla_{\bar{P}E_2} \bar{P}E_1, \bar{P}E) \)

\( = g(\nabla_{\bar{P}E_1} \bar{P}E_2, \bar{P}E) - g(\nabla_{\bar{P}E_2} \bar{P}E_1, \bar{P}E) \)

\( = g(h'(PE_1, E_2), E) + g(\nabla_{\bar{P}E_1} \bar{P}E_2, E) - g(\nabla_{\bar{P}E_2} \bar{P}E_1, E) \)

\( = g(h'(PE_1, E_2), E) + g(h'(PE_2, E_2), E) \)

\( - g(h'(PE_1, E_1), E) - g(h'(PE_2, E_1), E) \)

\( = g(h'(PE_1, E_2), E) - g(h'(PE_2, E_2), E), \) \hspace{1cm} (3.31)

\( g\left[ \bar{PE}_1, \bar{PE}_2 \right], \bar{P}L = g(\nabla_{\bar{P}E_1} \bar{P}E_2 - \nabla_{\bar{P}E_2} \bar{P}E_1, PL) \)

\( = g(\nabla_{\bar{P}E_1} \bar{P}E_2, PL) - g(\nabla_{\bar{P}E_2} \bar{P}E_1, PL) \)

\( = g(\nabla_{\bar{P}E_1} \bar{P}E_2, L) + g(\nabla_{\bar{P}E_1} \bar{P}E_2, L) - g(\nabla_{\bar{P}E_2} \bar{P}E_1, L) \)

\( = g(h'(PE_1, E_2), L) + g(h'(PE_1, E_2), L) \)

\( - g(h'(PE_2, E_2), L) - g(h'(PE_2, E_2), L) \)

\( = g(h'(PE_1, E_2), L) - g(h'(PE_2, E_2), L), \) \hspace{1cm} (3.32)

\( g\left[ \bar{PE}_1, \bar{PE}_2 \right], N = g(\nabla_{\bar{P}E_1} \bar{P}E_2 - \nabla_{\bar{P}E_2} \bar{P}E_1, N) = g(\nabla_{\bar{P}E_1} \bar{P}E_2, N) - g(\nabla_{\bar{P}E_2} \bar{P}E_1, N) \)

\( = -g(PE_2, \nabla_{\bar{P}E_1} N) + g(\bar{P}E_1, \nabla_{\bar{P}E_2} N) \)

\( = g(A_h PE_2, PE_1) - g(A_h PE_2, PE_1), \) \hspace{1cm} (3.33)
Thus the proof is completed.

**Theorem 3.12.** Let \( N' \) be a semi invariant lightlike submanifold of a golden semi-Riemannian manifold \((\bar{N}, \bar{g}, \bar{P})\). Then, each leaf of radical distribution is totally geodesic on \( N' \) iff

1. \( A_{E_1}^* E_2 \in \Gamma(D_i \perp D_j) \),
2. \( \bar{g}(h^*(E_1, \bar{P}E_2), N) = 0 \),
3. \( \bar{g}(h^*(E_1, \bar{P}E_2), L) = 0 \),

for any \( E_1, E_2 \in \Gamma(Rad(TN')) \), \( N \in \Gamma(ltr(TN')) \), \( L \in \Gamma(S(TN'^\perp)) \).

**Proof.** Radical distribution is totally geodesic iff

\[
g(\nabla_{E_1}^* E_2, \bar{P}E) = g(\nabla_{E_1}^* E_2, X) = g(\nabla_{E_1}^* E_2, \bar{P}N) = g(\nabla_{E_1}^* E_2, \bar{P}L) = 0,
\]

for any \( X \in \Gamma(D_i) \), \( E_1, E_2 \in \Gamma(Rad(TN')) \), \( N \in \Gamma(ltr(TN')) \), \( L \in \Gamma(S(TN'^\perp)) \). Using (2.12), (2.17) and (2.18), we have

\[
g(\nabla_{E_1}^* E_2, \bar{P}E) = \bar{g}(\bar{\nabla}_{E_1}^* E_2, \bar{P}E) = -g(A_{E_1}^* E_2, \bar{P}E),
\]
\[
g(\nabla_{E_1}^* E_2, X) = \bar{g}(\bar{\nabla}_{E_1}^* E_2, X) = -g(A_{E_1}^* E_2, X),
\]
\[
g(\nabla_{E_1}^* E_2, \bar{P}N) = \bar{g}(\bar{\nabla}_{E_1}^* E_2, \bar{P}N) = \bar{g}(h^*(E_1, \bar{P}E_2), N),
\]
\[
g(\nabla_{E_1}^* E_2, \bar{P}L) = \bar{g}(\bar{\nabla}_{E_1}^* E_2, \bar{P}L) = \bar{g}(h^*(E_1, \bar{P}E_2), L).
\]

Hence, from (3.35)-(3.38) we complete the proof.

**Definition 3.13.** Let \( N' \) be a proper semi-invariant \( r \)-lightlike submanifold of a golden semi-Riemannian manifold \( \bar{N} \). If

\( h(X, Y) = 0 \), \( \forall X \in \Gamma(D) \) and \( Y \in \Gamma(D^\perp) \)

then, \( N' \) is called as mixed-geodesic submanifold.

**Theorem 3.14.** Let \( N' \) be a semi invariant lightlike submanifold of a golden semi-Riemannian manifold \((\bar{N}, \bar{g}, \bar{P})\). Then the followings are equivalent:

i) \( N' \) is mixed geodesic,
ii) \( A_{E}X \) has only component in \( \Gamma(D) \),
iii) \( A_{E_1}^* X \) and \( A_{E_2}X \) have no components in \( D_i \) and \( D_j \).
iv) \( \nabla_{E}^* P, \nabla_{E}^* P \in \Gamma(D_0 \perp D_2) \)

for any \( X \in \Gamma(D), Y \in \Gamma(D^\perp), E \in \Gamma(Rad(TN')), U \in \Gamma(ltr(TN')), L \in \Gamma(S(TN'^\perp)) \).
Proof. $N$ is mixed geodesic iff

$$g(h(X,Y),E) = 0 \text{ and } g(h(X,Y),L) = 0$$

for any $X \in \Gamma(D), \ Y \in \Gamma(D^\perp), \ E \in \Gamma(Rad(TN))$, $L \in \Gamma(S(TN^\perp))$. Choosing $Y \in \Gamma(D^\perp)$, there is a vector field $U \in \Gamma(tr(TM))$ such that $Y = PU$. Using (2.10) and (2.11) we have

$$g(h(X,Y),E) = g(\nabla_X Y, E) = g(\nabla_X PU, E) = g(\nabla_X U, PE) = -g(A_y X, PE),$$

$$g(h(X,Y),L) = g(\nabla_X Y, L) = g(\nabla_X PU, L) = g(\nabla_X U, PL) = -g(A_y X, PL).$$

Thus we derive $(i) \iff (ii)$. From (2.10), (2.14) and (2.18) we get

$$g(h(X,Y),E) = g(Y, A_y X), \quad (3.39)$$

$$g(h(X,Y),L) = g(Y, A_y X - D^r(X,L)). \quad (3.40)$$

Hence we obtain $(i) \iff (iii)$. Since $\nabla$ is a metric connection, from (2.10) we derive

$$g(h(\bar{P}X,Y),E) = g(h(Y,\bar{P}X),E) = g(\nabla_{\bar{y}} \bar{P}X, E) = -g(\bar{P}X, \nabla_{\bar{y}} E)$$

$$= -g(X, \nabla_{\bar{y}} PE) = -g(X, \nabla_{\bar{y}} PE),$$

$$g(h(\bar{P}X,Y),L) = g(h(Y,\bar{P}X),L) = g(\nabla_{\bar{y}} \bar{P}X, L) = -g(\bar{P}X, \nabla_{\bar{y}} L)$$

$$= -g(X, \nabla_{\bar{y}} PL) = -g(X, \nabla_{\bar{y}} PL).$$

Thus we derive $(i) \iff (iv)$.

**Definition 3.15.** A semi-invariant submanifold $N$ of a golden semi-Riemannian manifold $N$ is named as $D$-totally geodesic (resp. $D^\perp$-totally geodesic) if its the second fundamental form $h$ satisfies $h(X,Y) = 0$ (resp. $h(Z,L) = 0$), for any $X,Y \in \Gamma(D)$, $(Z,L \in \Gamma(D^\perp))$.

**Theorem 3.16.** Let $N$ be a semi-invariant lightlike submanifold of a golden semi-Riemannian manifold $(N, g, \bar{P})$. Then the followings are equivalent:

(i) $N$ is $D$-geodesic,

(ii) $A_y X \in \Gamma(D_1 \perp D_3)$ and $g(Y, A_y X) = g(Y, D^r(X,L))$,

(iii) $\nabla^r_{\bar{y}} Y$ has no components in $D_2$ and $D_3$,

(iv) $\nabla^r_{\bar{y}} PE, \nabla^r_{\bar{y}} PL \in \Gamma(D_1 \perp D_3)$,

for any $X,Y \in \Gamma(D)$, $E \in \Gamma(Rad(TN))$, $L \in \Gamma(S(TN^\perp))$.

Proof. For any $X,Y \in \Gamma(D)$, $E \in \Gamma(Rad(TN))$ and $L \in \Gamma(S(TN^\perp))$, using (2.10), (2.14) and (2.18) we obtain
\( \bar{g}(h(X,Y),E) = g(Y, A_\kappa^* X) , \) \hspace{1cm} (3.43)

And

\( \bar{g}(h(X,Y),L) = \bar{g}(Y, A_\kappa X - D'(X,L)) . \) \hspace{1cm} (3.44)

Hence we derive \((i) \Rightarrow (ii)\). Since \( \bar{\nabla} \) is a metric connection, from (2.10), (2.14) and (2.18) we get

\[
\begin{align*}
    g(\nabla_x^* Y, PE) &= g(\nabla_x^* Y, PY, E) \\
    &= -g(PY, \nabla_x E) = g(PY, A_\kappa^* X). \\
    g(\nabla_x^* Y, PL) &= g(\nabla_x^* Y, PY, L) \\
    &= -g(PY, \nabla_x L) = g(PY, A_\kappa X - D'(X,L)).
\end{align*}
\]

This is \((ii) \Rightarrow (iii)\). Similarly, since \( \bar{\nabla} \) is a metric connection, from (2.10) and (2.17) we have

\[
\begin{align*}
    g(\nabla_x^* PE, Y) &= \bar{g}(\nabla_x^* PE, Y) = -\bar{g}(PE, \nabla_x Y) = -\bar{g}(PE, \nabla_x^* Y), \\
    g(\nabla_x^* PL, Y) &= \bar{g}(\nabla_x^* PL, Y) = -\bar{g}(PL, \nabla_x Y) = -\bar{g}(PL, \nabla_x^* Y).
\end{align*}
\]

Hence we get \((iii) \Rightarrow (iv)\). Since \( \bar{\nabla} \) is a metric connection, using (2.1), (2.10) and (2.17) we derive

\[
\begin{align*}
    \bar{g}(h(X,Y),E) &= \bar{g}(\nabla_x^* Y, E) = \bar{g}(\nabla_x^* PY, PE) - \bar{g}(\nabla_x^* Y, PE) \\
    &= -\bar{g}(PY, \nabla_x E) + g(Y, \nabla_x^* PE) + \bar{g}(Y, \nabla_x^* PE). \\
    \bar{g}(h(X,Y),L) &= \bar{g}(\nabla_x^* Y, L) = \bar{g}(\nabla_x^* PY, PL) - \bar{g}(\nabla_x^* Y, PL) \\
    &= -\bar{g}(PY, \nabla_x L) + g(Y, \nabla_x^* PL) - \bar{g}(Y, \nabla_x^* PL).
\end{align*}
\]

Thus we get \((iv) \Rightarrow (i)\).

**Theorem 3.17.** Let \( N' \) be a semi-invariant lightlike submanifold of a golden semi-Riemannian \( (\bar{N}, \bar{g}, \bar{P}) \). Then the following statements are equivalent:

(i) \( N' \) is \( D^+ \) - geodesic,

(ii) \( A_\kappa^* X \) and \( A_\kappa X \) have no components in \( D_j \) and \( D_j \),

(iii) \( A_\kappa X \) has no components in \( D_j \) and \( D_j \),

for any \( X, Y \in \Gamma(D^+), \ E \in \Gamma(Rad(TN')), \ U \in \Gamma(tr(TN')), \ L \in \Gamma(S(TN^+)) \).

**Proof.** \( N' \) is \( D^+ \) - geodesic iff

\( \bar{g}(h(X,Y),E) = 0 \) and \( \bar{g}(h(X,Y),L) = 0 \),

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for any \( X, Y \in \Gamma(D^+) \), \( E \in \Gamma(Rad(TN')) \), \( L \in \Gamma(S(TN'^-)) \). From (2.10), (2.14) and (2.18) we get

\[
\bar{g}(h(X,Y),E) = g(Y, A'_\nu X),
\]

and

\[
\bar{g}(h(X,Y),L) = g(Y, A'_\nu X - D'(X,L)).
\]

Thus we derive \((i) \Leftrightarrow (ii)\). Choosing \( Y \in \Gamma(D^+) \), there is a vector field \( U \in \Gamma(tr(TN')) \) such that \( Y = PU \). Similarly from (2.10) and (2.11) we get

\[
\bar{g}(h(X,Y),E) = \bar{g}(\bar{\nabla}_X Y, E) = \bar{g}(\bar{\nabla}_X PU, E)
\]

\[
= \bar{g}(\bar{\nabla}_X U, PE) = -\bar{g}(A'_\nu X, PE),
\]

\[
\bar{g}(h(X,Y),L) = \bar{g}(\bar{\nabla}_X Y, L) = \bar{g}(\bar{\nabla}_X PU, L)
\]

\[
= \bar{g}(\bar{\nabla}_X U, PL) = -\bar{g}(A'_\nu X, PL).
\]

This is \((i) \Leftrightarrow (iii)\).

**Theorem 3.18.** Let \( N' \) be a semi-invariant lightlike submanifold of a golden semi-Riemannian manifold \((\bar{N}, \bar{g}, P)\). Then, following statements are equivalent.

(i) \( D \) is parallel,
(ii) \( N' \) is \( D^- \)-geodesic,
(iii) \( P \) is parallel on \( N' \).

**Proof.** Using (3.4) we have, \( D \) is parallel iff \( \bar{g}(\bar{\nabla}_X Y, \bar{P}E) = \bar{g}(\bar{\nabla}_X Y, \bar{P}L) = 0 \), for any \( X, Y \in \Gamma(D) \), \( E \in \Gamma(Rad(TN')) \), \( L \in \Gamma(S(TN'^-)) \). From (2.12) we derive

\[
\bar{g}(\bar{\nabla}_X Y, \bar{P}E) = \bar{g}(h'(X, \bar{P}Y), E),
\]

\[
\bar{g}(\bar{\nabla}_X Y, \bar{P}L) = \bar{g}(h'(X, \bar{P}Y), L).
\]

Hence we derive \((i) \Leftrightarrow (ii)\). From (3.13) for any \( X, Y \in \Gamma(D) \) we have \( (\bar{\nabla}_X P)Y = Bh(X,Y) \). Since \( N' \) is \( D^- \)-geodesic, \( (\bar{\nabla}_X P)Y = 0 \). Thus we get \((ii) \Leftrightarrow (iii)\).

Now, by using the equation (2.4) and (3.1), we derive

\[
\bar{R}(X,Y)Z = \frac{1}{1+\varphi} \left\{ \bar{g}(Y, Z)X - \bar{g}(X, Z)Y + \bar{g}(\bar{P}Y, Z)PX - \bar{g}(\bar{P}X, Z)PY \right\}
\]

\[
- \bar{g}(\bar{P}X, Z)wY
\]

\[
\bar{R}(X,Y)Z = \frac{1}{1+\varphi} \left\{ \bar{g}(\bar{P}Y, Z)X + \bar{g}(Y, Z)PX + \bar{g}(Y, Z)wX - \bar{g}(\bar{P}X, Z)wY \right\}
\]

for any \( X, Y, Z \in \Gamma(TN') \). From (2.22) and (3.57), the equations of Gauss and Codazzi for the submanifold \( N' \), respectively, can be written as
\[
R(X,Y)Z = \left( -\frac{(1-\varphi)c_p - \varphi c_q}{2\sqrt{5}} \right) \{ \bar{g}(Y,Z)X - \bar{g}(X,Z)Y + \bar{g}(\bar{P}Y,Z)PX - \bar{g}(\bar{P}X,Z)PY \} \\
\]
\[
\quad + \left( -\frac{(1-\varphi)c_p + \varphi c_q}{4} \right) \{ \bar{g}(\bar{P}Y,Z)X - \bar{g}(\bar{P}X,Z)Y \} \\
\quad + \left( -\frac{(1-\varphi)c_p - \varphi c_q}{4} \right) \{ \bar{g}(Y,Z)X - \bar{g}(X,Z)Y \}.
\] (3.58)

and
\[
(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = \left( -\frac{(1-\varphi)c_p - \varphi c_q}{2\sqrt{5}} \right) \{ \bar{g}(\bar{P}Y,Z)wX - \bar{g}(\bar{P}X,Z)wY \} \\
\quad + \left( -\frac{(1-\varphi)c_p + \varphi c_q}{4} \right) \{ \bar{g}(Y,Z)wX - \bar{g}(X,Z)wY \}.
\] (3.59)

for any \( X,Y,Z \in \Gamma(TN) \).

**Definition 3.19.** Let \( N' \) be an \( r \)-lightlike submanifold of any semi-Riemannian manifold \( N \). \( N' \) is called as curvature-invariant lightlike submanifold if for \( \forall X,Y,Z \in \Gamma(TN) \),
\[
(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = 0.
\] (3.60)

**Theorem 3.20.** There is no curvature-invariant semi-invariant lightlike submanifolds in any semi-Riemannian locally golden product space form \( (\tilde{N} = N'_p(c_p) \times N'_q(c_q), \bar{g}, \bar{P}) \) with \( c_p,c_q \neq 0 \).

**Proof.** Suppose that \( N' \) is a semi curvature-invariant lightlike submanifold of a semi-Riemannian golden product space form \( (\tilde{N} = N'_p(c_p) \times N'_q(c_q), \bar{g}, \bar{P}) \) with \( c_p,c_q \neq 0 \). Then from (3.59) we have
\[
\left( -\frac{(1-\varphi)c_p - \varphi c_q}{2\sqrt{5}} \right) \{ \bar{g}(\bar{P}Y,Z)wX - \bar{g}(\bar{P}X,Z)wY \} \\
\quad + \left( -\frac{(1-\varphi)c_p + \varphi c_q}{4} \right) \{ \bar{g}(Y,Z)wX - \bar{g}(X,Z)wY \} = 0.
\] (3.61)

If we take \( X \in \Gamma(D_3) \), \( Y \in \Gamma(Rad(TN')) \) and \( Z \in \Gamma(D_2) \) in (3.61), then we get
\[
\left( -\frac{(1-\varphi)c_p - \varphi c_q}{2\sqrt{5}} \right)_{\omega}X = 0.
\] (3.62)

Similarly, if we take \( X \in \Gamma(D_3) \), \( Y \in \Gamma(D_1) \) and \( Z \in \Gamma(D_2) \) in (3.61), then we get
\[
\left( -\frac{(1-\varphi)c_p - \varphi c_q}{2\sqrt{5}} \right)_{\omega}X + \left( -\frac{(1-\varphi)c_p + \varphi c_q}{4} \right)_{\omega}Y = 0.
\] (3.63)

Using the equations (3.62) and (3.63), we get \( c_p,c_q = 0 \).
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