A NOTE ON A SPECTRAL CONSTANT ASSOCIATED WITH AN ANNULUS

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Abstract. Fix $R > 1$ and let $A_R = \{1/R \leq |z| \leq R\}$ be an annulus. Also, let $K(R)$ denote the smallest constant such that $A_R$ is a $K(R)$-spectral set for the bounded linear operator $T \in \mathcal{B}(H)$ whenever $\|T\| \leq R$ and $\|T^{-1}\| \leq R$. We show that $K(R) \geq 2$, for all $R > 1$. This improves on previous results by Badea, Beckermann and Crouzeix.

1. BACKGROUND

Let $X$ be a closed set in the complex plane and let $\mathcal{R}(X)$ denote the algebra of complex-valued bounded rational functions on $X$, equipped with the supremum norm $\|f\|_X = \sup\{|f(x)| : x \in X\}$.

Suppose that $T$ is a bounded linear operator acting on the (complex) Hilbert space $H$. Suppose also that the spectrum $\sigma(T)$ of $T$ is contained in the closed set $X$. Let $f = p/q \in \mathcal{R}(X)$. As the poles of the rational function $f$ are outside of $X$, the operator $f(T)$ is naturally defined as $f(T) = p(T)/q(T)$ or, equivalently, by the Riesz-Dunford functional calculus (see e.g. [4] for a treatment of this topic).

Recall that for a fixed constant $K > 0$, the set $X$ is said to be a $K$-spectral set for $T$ if $\sigma(T) \subseteq X$ and the inequality $\|f(T)\| \leq K\|f\|_X$ holds for every $f \in \mathcal{R}(X)$. The set $X$ is a spectral set for $T$ if it is a $K$-spectral set with $K = 1$. Spectral sets were introduced and studied by von Neumann in [8], where he proved the celebrated result that an operator $T$ is a contraction if and only if the closed unit disk is a spectral set for $T$ (we refer the reader to the book [9] and the survey [2] for more detailed presentations and more information on $K$-spectral sets).

We will be concerned with the case where $X = A_R := \{1/R \leq |z| \leq R\}$ ($R > 1$) is a closed annulus, the intersection of the two closed disks $D_1 = \{|z| \leq R\}$ and $D_2 = \{|z| \geq 1/R\}$. Now, the intersection of two spectral sets is not necessarily a spectral set; counterexamples for the annulus were presented in [7], [11] and [12]. However, the same question for $K$-spectral sets remains open (the counterexamples for spectral sets show that the same constant cannot be used for the intersection). Regarding the annulus in particular, Shields proved in [11] that, given an invertible operator $T \in \mathcal{B}(H)$ with $\|T\| \leq R$ and $\|T^{-1}\| \leq R$, $A_R$ is a $K$-spectral set for $T$ with $K = 2 + \sqrt{R^2 + 1}/(R^2 - 1)$. This bound is large if $R$ is close to $1$. In

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this context, Shields raised the question of finding the smallest constant $K = K(R)$ such that $A_R$ is $K(R)$-spectral. In particular, he asked whether this optimal constant $K(R)$ would remain bounded.

This question was answered positively by Badea, Beckermann and Crouzeix in [3], where they obtained that (for every $R > 1$)

$$\frac{4}{3} < \gamma(R) := 2(1 - R^{-2}) \prod_{n=1}^{\infty} \left( \frac{1 - R^{-8n}}{1 - R^{-4n}} \right)^2 \leq K(R) \leq 2 + \frac{R+1}{\sqrt{R^2 + R+1}} \leq 2 + \frac{2}{\sqrt{3}}.$$ 

It should be noted that the quantity $\gamma(R)$ was numerically shown to be greater than or equal to $\pi/2$ (leading to the universal lower bound $\pi/2$ for $K(R)$) and it also tends to 2 as $R$ tends to infinity.

Two subsequent improvements were made to the upper bound for $K(R)$: the first one in [5] by Crouzeix and the most recent one in [6] by Crouzeix and Greenbaum, where it was proved that

$$K(R) \leq 1 + \sqrt{2}, \quad \forall R > 1.$$ 

As for the lower bound, Badea obtained in [1] the statement

$$\frac{3}{2} < \frac{1 + R^2 + R}{1 + R^2 + 2R} \leq K(R), \quad \forall R > 1,$$

where the quantity $2(1 + R^2 + R)/(1 + R^2 + 2R)$ again tends to 2 as $R$ tends to infinity.

We improve the aforementioned estimates by showing that 2 is actually a universal lower bound for $K(R)$:

**Theorem 1.1.** Put $A_R = \{1/R \leq |z| \leq R\}$, for any $R > 1$. Let $K(R)$ denote the smallest positive constant such that $A_R$ is a $K(R)$-spectral set for the bounded linear operator $T \in B(H)$ whenever $||T|| \leq R$ and $||T^{-1}|| \leq R$. Then,

$$K(R) \geq 2, \quad \forall R > 1.$$ 

2. PROOF OF THEOREM

*Proof.* Fix $R > 1$. For every $n \geq 2$, define

$$g_n(z) = \frac{1}{R^n} \left( \frac{1}{z^n} + z^n \right) \in \mathcal{R}(A_R).$$

It is easy to see that

(2.1) $$||g_n||_{A_R} = g_n(R) = 1 + \frac{1}{R^{2n}}.$$ 

To achieve the stated improvement, we will apply $g_n$ to a bilateral shift operator $S$ acting on a particular weighted sequence space $L^2(\beta)$. First, define the sequence $\{\beta(k)\}_{k \in \mathbb{Z}}$ of positive numbers (weights) as follows:

$$\beta(2ln + q) = R^l, \quad \forall q \in \{0, 1, \ldots, n\}, \forall l \in \mathbb{Z};$$

$$\beta((2l+1)n + q) = R^{n-q}, \quad \forall q \in \{0, 1, \ldots, n\}, \forall l \in \mathbb{Z}.$$ 

Consider now the space of sequences $f = \{\hat{f}(k)\}_{k \in \mathbb{Z}}$ such that

$$\|f\|^2_\beta := \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2 |\beta(k)|^2 < \infty.$$
We shall use the notation $f(z) = \sum_{k \in \mathbb{Z}} \hat{f}(k) z^k$ (formal Laurent series), whether or not the series converges for any (complex) values of $z$. Our weighted sequence space will be denoted by

$$L^2(\beta) := \{ f = \{ \hat{f}(k) \}_{k \in \mathbb{Z}} : \| f \|_\beta^2 < \infty \}.$$ 

This is a Hilbert space with the inner product

$$\langle f, g \rangle_\beta := \sum_{k \in \mathbb{Z}} \hat{f}(k) \overline{\hat{g}(k)} [\beta(k)]^2.$$

Consider also the linear transformation (bilateral shift) $S$ of multiplication by $z$ on $L^2(\beta)$:

$$(Sf)(z) = \sum_{k \in \mathbb{Z}} \hat{f}(n) z^{n+1}.$$ 

In other words, we have

$$(Sf)(n) = \hat{f}(n-1), \ \forall n \in \mathbb{Z}.$$ 

Observe that

$$\| S \| = \sup_{k \in \mathbb{Z}} \frac{\beta(k+1)}{\beta(k)} = R$$ 

and

$$\| S^{-1} \| = \sup_{k \in \mathbb{Z}} \frac{\beta(k)}{\beta(k+1)} = R.$$ 

Now, let $m \geq 3$ and define $h = \{ \hat{h}(k) \}_{k \in \mathbb{Z}} \in L^2(\beta)$ by putting:

$$\hat{h}(2ln) = \frac{1}{m}, \ \forall l \in \{0, 1, 2 \ldots, m^2\};$$

$$\hat{h}(k) = 0, \ \text{in all other cases}.$$ 

We calculate

$$\| h \|_\beta^2 = \sum_{l=0}^{m^2} \frac{1}{m^2} [\beta(2ln)]^2 = \sum_{l=0}^{m^2} \frac{1}{m^2} \cdot 1^2 = \frac{m^2 + 1}{m^2},$$ 

hence

$$\| h \|_\beta = \sqrt{\frac{m^2 + 1}{m}}.$$ 

Also, put $f = (S^{-n} + S^n)h$ and notice that

$$\| (S^{-n} + S^n)h \|_\beta^2 = \| f \|_\beta^2$$

$$\geq \sum_{l=1}^{m^2} |\hat{f}((2l-1)n)|^2 [\beta((2l-1)n)]^2$$

$$= \sum_{l=1}^{m^2} \left( \frac{2m}{m} \right)^2 R^{2n}$$

$$= 4R^{2n}.$$ 

Thus,

$$\| (S^{-n} + S^n)h \|_\beta \geq 2R^n.$$
Using (2.1), (2.2) and (2.3), we can now write
\[
K(R) \geq \frac{||g_n(S)||}{||g_n||_{A_R}} \geq \frac{1}{R^n} \cdot \frac{||S^{-n} + S^n||}{1 + R^{-2n}} \geq \frac{1}{R^n + R^{-n}} \cdot \frac{||h||_{\beta}}{\beta} \geq \frac{2R^n}{\sqrt{m^2+1}}.
\]

Letting \(m \to \infty\), we obtain
\[
K(R) \geq \frac{1}{R^n + R^{-n}} \cdot \frac{2R^n}{1} = \frac{2R^n}{R^n + R^{-n}} \xrightarrow{n \to \infty} 2, \quad \text{as } R > 1.
\]

The proof is complete.

\[\square\]

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