A NOTE ON J-POSITIVE BLOCK OPERATOR MATRICES

ALEKSEY KOSTENKO

Abstract. We study basic spectral properties of \( J \)-self-adjoint \( 2 \times 2 \) block operator matrices. Using the linear resolvent growth condition, we obtain simple necessary conditions for the regularity of the critical point \( \infty \). In particular, we present simple examples of operators having the singular critical point \( \infty \). Also, we apply our results to the linearized operator arising in the study of soliton type solutions to the nonlinear relativistic Ginzburg–Landau equation.

1. Introduction

Let \( \mathcal{H} \) be a complex separable Hilbert space. Consider the following operators defined in \( \tilde{\mathcal{H}} = \mathcal{H} \times \mathcal{H} \) by the block operator matrices

\[
\mathcal{L} = \begin{pmatrix} iC & iB \\ -iA & -iC^* \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} A & C^* \\ C & B \end{pmatrix}.
\]

Note that \( \mathcal{L} = \mathcal{J} \mathcal{A} \), where

\[
\mathcal{J} = \mathcal{J}^* = \mathcal{J}^{-1} = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}
\]

is a fundamental symmetry on \( \mathcal{H} \times \mathcal{H} \). Also, \( \mathcal{J} \) stands for the identity operator on \( \mathcal{H} \). The operators \( \mathcal{A}, \mathcal{B}, \) and \( \mathcal{C} \) are not assumed to be bounded. In order to define the operators \( \mathcal{A} \) and \( \mathcal{L} \) correctly we shall assume the following.

Hypothesis 1.1.

(i) \( \mathcal{A} \) is closed with \( 0 \in \rho(\mathcal{A}) \) and \( \kappa_-(\mathcal{A}) < \infty \),
(ii) \( \mathcal{C} \) is closed, \( \text{dom}(\mathcal{A}) \subset \text{dom}(\mathcal{C}) \) and \( \text{dom}(\mathcal{A}) \subset \text{dom}(\mathcal{C}^*) \),
(iii) \( \mathcal{B} = \mathcal{B}^* \),
(iv) \( \text{dom}(\mathcal{S}_0) := \text{dom}(\mathcal{C}^*) \cap \text{dom}(\mathcal{B}) \) is dense in \( \mathcal{H} \) and the operator

\[
\mathcal{S}_0 := \mathcal{B} - \mathcal{C} \mathcal{A}^{-1} \mathcal{C}^*
\]

is essentially self-adjoint on \( \text{dom}(\mathcal{S}_0) \) with \( \kappa_-(\mathcal{S}_0) < \infty \).

Here \( \kappa_-(\mathcal{T}) = \dim \text{ran}_\chi(-\infty,0)(\mathcal{T}) \). Note that \( \kappa_-(\mathcal{T}) \) is the number of negative eigenvalues of \( \mathcal{T} \) if \( \kappa_-(\mathcal{T}) < \infty \).

Under the assumptions of Hypothesis 1.1 the operator \( \mathcal{A}_0 \) defined on \( \text{dom}(\mathcal{A}_0) = \text{dom}(\mathcal{A}) \times \text{dom}(\mathcal{S}_0) \) is essentially self-adjoint (see Theorem 2.1.1). If additionally the operators \( \mathcal{A} \) and \( \mathcal{S}_0 \) are positive, then so is the operator \( \mathcal{A}_0 \). Moreover, the operator \( \mathcal{L}_0 \) defined by \( \mathcal{L}_0 = \mathcal{J} \mathcal{A}_0 \) on \( \text{dom}(\mathcal{L}_0) = \text{dom}(\mathcal{A}_0) \) is closable and essentially \( \mathcal{J} \)-self-adjoint. Let us denote by \( \mathcal{A} \) and \( \mathcal{L} \) the closures of \( \mathcal{A}_0 \) and \( \mathcal{L}_0 \), respectively.

2010 Mathematics Subject Classification. Primary 47B50; Secondary 47A40; 47B15; 34L10.

Key words and phrases. Block operator matrix, \( J \)-self-adjoint operator, \( J \)-positive operator, eigenfunction expansion.

Research supported by the Austrian Science Fund (FWF) under Grant No. P26060.
Operators \( \mathcal{A} \) and \( \mathcal{L} \) arise in various areas of mathematical physics and hydrodynamics. In particular, the spectral properties of \( \mathcal{A} \) has been studied in \[12\], \[13\] (see also references therein). Note also that our choice of the fundamental symmetry \((1.2)\) is motivated by applications to the study of asymptotic stability of solutions of nonlinear wave equations. More precisely, in \[8\], \[9\], \[10\], the operator \( \mathcal{L} \) defined on \( \mathcal{H} = L^2(\mathbb{R}) \times L^2(\mathbb{R}) \) by \( (1.1) \) with \( (1.4) \)
\[
A = -\frac{d^2}{dx^2} + m^2 + V(x), \quad C = \nu \frac{d}{dx}, \quad B = I,
\]
was studied in connection with the problem of asymptotic stability of solutions of the nonlinear relativistic Ginzburg–Landau equation. The authors of \[10\] were interested in the eigenfunction expansion properties for \( \mathcal{L} \), which was used in \[1\], \[9\] for the calculation of the Fermi Golden Rule (this condition ensures a strong coupling of discrete and continuous spectral components of solutions, which provides the energy radiation to infinity and results in the asymptotic stability of solitary waves). If \( |\nu| \in [0, 1), V \to 0 \) as \( x \to \infty \) and under certain positivity assumptions on \( A \) and \( S_0 \), it was shown in \[10\] that the operator \( \mathcal{L} \) is positive and the eigenfunction expansion was constructed for all functions from the energy space \( \mathcal{H}_A \). However, the question on the eigenfunction expansion properties in the initial Hilbert space \( \mathcal{H} = L^2(\mathbb{R}) \times L^2(\mathbb{R}) \) was left to be open. It is one of our main aims to investigate this problem.

On the other hand, under the assumptions of Hypothesis \(1.1\), the operator \( \mathcal{L} \) defined by \( (1.1) \) and \( (1.4) \) is definitizable (see Theorem 4.8). Therefore (see \[11\]), the problem on the eigenfunction expansion properties is equivalent to the regularity of critical points of the operator \( \mathcal{L} \). It turns out that the operator \( \mathcal{L} \) with coefficients \( (1.4) \) has a singular critical point \( \infty \) (Theorem 4.8). First of all, this result shows that the results obtained in \[10\] are optimal in a certain sense. On the other hand, studying the spectral properties of the block operator matrix \( (1.1) \), we are able to construct a class of \( J \)-positive operators with the singular critical point \( \infty \) (see Example 3.7). The special case when all coefficients \( A, B \) and \( C \) are functions of a self-adjoint operator \( T \) on \( \mathcal{H} \) (and hence they are commutative) was studied in \[6\], \[7\].

Let us now briefly describe the content of the paper. In Section 2, we recall basic facts from \[12\] and \[13\] on spectral properties of the operator \( \mathcal{A} \). Section 3 deals with the spectral properties of the \( J \)-self-adjoint operator \( \mathcal{L} \). We describe the spectrum of \( \mathcal{L} \), provide sufficient conditions for its definitizability and obtain a necessary condition for the similarity of \( \mathcal{L} \) to a self-adjoint operator. We demonstrate our findings by examples. For instance, we present a class of \( 2 \times 2 \) block operator matrices with the singular critical point \( \infty \). In the final Section 4, we study the spectral properties of the operator \( \mathcal{L} \) defined by \( (1.1) \) and \( (1.4) \). The main result of this section, Theorem 4.8 states that the operator \( \mathcal{L} \) is definitizable and \( \infty \) is a singular critical point if the potential \( V \) satisfies \( (4.3) \).

2. Self-adjointness of the operator matrix \( \mathcal{A} \)

In this section we collect some information on basic spectral properties of the operator \( \mathcal{A} \) defined by \( (1.1) \). We begin with the following result from \[12\] (see also \[13\], Chapter II.2).
Theorem 2.1 ([12]). Assume that the operators $A$, $B$, $C$ satisfy the assumptions of Hypothesis [11]. Then the operator $A_0 : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$,

\begin{equation}
A_0 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} := \begin{pmatrix} Af_1 + C^* f_2 \\ C f_1 + B f_2 \end{pmatrix}, \quad f \in \text{dom}(A_0) := \text{dom}(A) \times \text{dom}(S_0),
\end{equation}

is essentially self-adjoint.

Proof. We shall give a proof because our further considerations rely on this construction. The proof is based on the Frobenius–Schur factorization.

\begin{equation}
A_0 - z = \begin{pmatrix} I & 0 \\ C(A-z)^{-1} & I \end{pmatrix} \begin{pmatrix} A - z & 0 \\ 0 & S(z) - z \end{pmatrix} \begin{pmatrix} I & (A-z)^{-1} C^* \\ 0 & I \end{pmatrix},
\end{equation}

where

\begin{equation}
S(z) = B - C(A-z)^{-1} C^*, \quad z \in \rho(A); \quad \text{dom}(S(z)) = \text{dom}(S_0).
\end{equation}

Assumption (ii) implies that the operators

\[ F(z) := C(A-z)^{-1} \quad \text{and} \quad G(z) := (A-z)^{-1} C^* \]

are bounded in $\mathcal{H}$ whenever $z \in \rho(A)$. Moreover, $\text{dom}(F) = \mathcal{H}$ and the closure of $G$ is a bounded operator on $\mathcal{H}$. Therefore, the operators

\begin{equation}
\mathcal{F}(z) = \begin{pmatrix} I & 0 \\ C(A-z)^{-1} & I \end{pmatrix}, \quad \mathcal{G}(z) = \begin{pmatrix} I & (A-z)^{-1} C^* \\ 0 & I \end{pmatrix},
\end{equation}

are bounded and boundedly invertible on $\mathcal{H} \times \mathcal{H}$. Noting also that $\mathcal{G}(z)^* = \mathcal{F}(z)^*$ and $\mathcal{G}(z) \subset \mathcal{F}(z)^*$, we conclude that the operator $A_0$ is essentially self-adjoint if and only if so is $S(0) = S_0$. It remains to exploit the assumption (iv). \qed

Using (2.2), we can describe the closure of $A_0$.

Corollary 2.2 ([12]). Assume the conditions of Theorem 2.1. Then the closure $\mathcal{A}$ of the operator $A_0$ is given by

\begin{equation}
\mathcal{A} = \begin{pmatrix} I & 0 \\ F(0) & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & (F(0))^* \\ 0 & I \end{pmatrix},
\end{equation}

and

\begin{equation}
\text{dom}(\mathcal{A}) = \{ f = (f_1, f_2)^T : f_1 + (F(0))^* f_2 \in \text{dom}(A), \ f_2 \in \text{dom}(S_0) \}.
\end{equation}

We also need the following description of the spectrum of $\mathcal{A}$. In what follows we shall use the following notation:

\begin{equation}
\sigma(\overline{S}) := \{ z \in \mathbb{C} : z \in \sigma(\overline{S}(z)) \},
\end{equation}

\begin{equation}
\sigma_i(\overline{S}) := \{ z \in \mathbb{C} : z \in \sigma_i(\overline{S}(z)) \}, \quad i \in \{ p, c, \text{ess} \}.
\end{equation}

Corollary 2.3. Assume the conditions of Theorem 2.1. Let also $\mathcal{A}$ and $\overline{S}(z)$ be the closures of $A_0$ and $S(z)$, respectively. Then

\begin{equation}
\sigma(\mathcal{A}) \setminus \sigma(\overline{S}) = \sigma(\overline{S}), \quad \sigma_i(\mathcal{A}) \setminus \sigma(\overline{S}) = \sigma_i(\overline{S}), \quad i \in \{ p, c \}.
\end{equation}

In particular, the operator $\mathcal{A}$ is (uniformly) positive if and only if so are $A$ and $S_0$. Moreover,

\begin{equation}
\kappa_-(\mathcal{A}) = \kappa_-(A) + \kappa_-(\overline{S}_0).
\end{equation}
Proof. For the proof of equality (2.8) we refer to [13] Theorem 2.3.3. The second claim is obvious since the operators $\mathcal{F}$ and $\mathcal{F}$ are bounded, boundedly invertible and $\mathcal{F}(z)^* = \overline{\mathcal{F}(z^*)}$. \hfill $\square$

**Remark 2.4.** Further assumptions on coefficients of $\mathcal{A}$ are required in order to extend (2.8) to the case of essential spectra. For instance, 

$$\sigma_{ess}(\mathcal{A}) \setminus \sigma(\mathcal{A}) = \sigma_{ess}(\mathcal{F})$$

if the operator $BA^{-1}$ is bounded on $\mathcal{H}$. For further details and results we refer to [12], [13] Chapter II.4.

3. ON THE REGULARITY OF CRITICAL POINTS OF BLOCK OPERATOR-MATRICES

Assume Hypothesis [1.1] Since $\mathcal{L}_0 = \mathcal{J}\mathcal{A}_0$, the operator $\mathcal{L}_0$ is essentially $\mathcal{J}$-self-adjoint if conditions (i)-(iv) of Hypothesis [1.1] are satisfied. Moreover, its closure $\mathcal{L}$ is given by $\mathcal{L} = \mathcal{J}\mathcal{A}$, where $\mathcal{A} = \mathcal{A}^* = \mathcal{A}_0^*$. We can also describe the closure using the Frobenius–Schur factorization. To this end, for all $z \in \mathbb{C}$ define the operator

$$\mathcal{L}_0 - z = \begin{pmatrix} I & - (C + iz)A^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & iT(z) \\ -iA & 0 \end{pmatrix} \begin{pmatrix} I & A^{-1}(C^* - iz) \\ 0 & I \end{pmatrix}$$

for all $f \in \text{dom}(A) \times \text{dom}(S_0)$. This representation enables us to find the closure of $\mathcal{L}_0$ and also to describe its spectrum (see, e.g., [13] Chapter II.3 and Theorem 2.4.16).

**Theorem 3.1.** Assume Hypothesis [1.1] The closure $\mathcal{L}$ of $\mathcal{L}_0$ is given by

$$\mathcal{L} = \begin{pmatrix} I & -CA^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & iT(0) \\ -iA & 0 \end{pmatrix} \begin{pmatrix} I & A^{-1}C^* \\ 0 & I \end{pmatrix}$$

and $\text{dom}(\mathcal{L}) = \text{dom}(\mathcal{A})$. Moreover,

$$\sigma(\mathcal{L}) = \sigma(\overline{T}), \quad \sigma_i(\mathcal{L}) = \sigma_i(\overline{T}), \quad i \in \{p, c, \text{ess}\},$$

where

$$\sigma(\overline{T}) = \{z \in \mathbb{C} : 0 \in \sigma(\overline{T}(z))\}, \quad \sigma_i(\overline{T}(z)) = \{z \in \mathbb{C} : 0 \in \sigma_i(\overline{T}(z))\}.$$ 

The next result is important for our further considerations.

**Corollary 3.2.** Assume Hypothesis [1.1] Then the operator $\mathcal{L}$ is definitizable if and only if there is $z \in \mathbb{C}$ such that $0 \in \rho(T(z))$.

**Proof.** By [23], the form $\langle f, g \rangle := \langle \mathcal{J}\mathcal{L}f, g \rangle = \langle Af, f \rangle$, $f \in \text{dom}(\mathcal{L})$, has finitely many negative squares. Therefore, by [11] p.11, Example (c) [see also Corollary II.2.1 in [11]], the operator $\mathcal{L}$ is definitizable if and only if $\rho(\mathcal{L}) \neq \emptyset$. It remains to apply Theorem 3.1. \hfill $\square$

**Corollary 3.3.** Assume Hypothesis [1.1] Then $\sigma(\mathcal{L})$ is symmetric with respect to the real line.

If additionally $\sigma(\mathcal{L}) \subseteq \mathbb{C}$, then the non-real spectrum $\sigma(\mathcal{L}) \setminus \mathbb{R}$ of $\mathcal{L}$ consists of a finite number of pairs $\lambda, \lambda^*$. Moreover, total algebraic multiplicity of non-real eigenvalues is at most $2\kappa_-(\mathcal{A})$. In particular, $\sigma(\mathcal{L}) \subseteq \mathbb{R}$ if $\kappa_-(\mathcal{A}) = 0$, i.e., the operator $\mathcal{A}$ is positive.
Proof. The proof follows from [11] Proposition II.2.1 (see also [3] Proposition 1.6). □

Note that in practice the condition $\sigma(T) = \{z \in \mathbb{C} : 0 \in \sigma(\overline{T(z)})\} \neq \mathbb{C}$ is difficult to check. Let us present two examples.

**Example 3.4.** Let $A$ be an unbounded self-adjoint uniformly positive operator in $\mathcal{H}$, i.e., $A = A^* > 0$ and $0 \in \rho(A)$. Let also $C = 0$ and $B = A^{-1}$, that is,

$$A = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}, \quad L = \begin{pmatrix} 0 & iA^{-1} \\ -iA & 0 \end{pmatrix}. \quad (3.6)$$

Clearly, all the assumptions of Hypothesis [7] are satisfied. By Theorem [3.1],

$$\sigma(L) = \sigma(T), \quad T(z) = A^{-1} - z^2A^{-1} = (1 - z^2)A^{-1}. \quad (3.7)$$

However, $T(\pm 1) = 0$ and $T(z)^{-1} = (1 - z^2)^{-1}A$ for all $z \neq \pm 1$. Since $A$ is unbounded, $\sigma(L) = \mathbb{C}$ and hence the operator $L$ is not definitizable, however, it is $\mathcal{J}$-self-adjoint and $\mathcal{J}$-positive.

In particular, a very simple example of a $\mathcal{J}$-self-adjoint operator $L$ with $\sigma(L) = \mathbb{C}$ is given by

$$L = \bigoplus_{n \in \mathbb{N}} \begin{pmatrix} 0 & i/n \\ -in & 0 \end{pmatrix}, \quad \mathcal{J} = l^2(\mathbb{N}; \mathbb{C}^2). \quad (3.8)$$

**Example 3.5.** Let $a$, $b$ and $c : \mathbb{R} \to \mathbb{C}$ be locally integrable functions. Assume also that $a = a^* > 0$, $b = b^* \geq 0$ a.e. on $\mathbb{R}$ and $1/a$, $c/a \in L^\infty(\mathbb{R})$. Denote by $M_a$, $M_b$ and $M_c$ the multiplication operators in $L^2(\mathbb{R})$ by $a$, $b$ and $c$, respectively, and set $A = M_a$, $B = M_b$ and $C = M_c$. Hence $A_0$ and $L_0$ are the operators on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ defined by

$$A_0 = \begin{pmatrix} M_a & M_c \\ M_b & M_b \end{pmatrix}, \quad L_0 = \begin{pmatrix} iM_c & iM_b \\ -iM_a & -iM_c \end{pmatrix}. \quad (3.9)$$

Clearly, the operator $A = \overline{A_0}$ is self-adjoint and hence $L = L_0 = \mathcal{J}A$ is $\mathcal{J}$-self-adjoint. Let us also assume that

$$a(x)b(x) - |c(x)|^2 \geq 0 \quad \text{for a.a.} \quad x \in \mathbb{R}. \quad (3.10)$$

The latter means that the operator $A$ is positive and $L$ is $\mathcal{J}$-positive.

It is easy to see that under the assumptions on the coefficients $a$, $b$ and $c$, Hypothesis [7] is satisfied. By Theorem [3.1], the resolvent set of $L$ is given by

$$z \in \rho(L) \iff \frac{a}{ab - (c + iz)(c^* - iz)} \in L^\infty(\mathbb{R}). \quad (3.11)$$

Moreover, in view of the positivity assumption (3.10), the operator $L$ is definitizable and $\sigma(L) \subseteq \mathbb{R}$ if and only if $i \in \rho(L)$, that is,

$$\frac{a}{ab - (c - 1)(c^* + 1)} \in L^\infty(\mathbb{R}). \quad (3.12)$$

Our main interest is the similarity of the operator $L$ to a self-adjoint operator.

**Lemma 3.6.** Assume Hypothesis [7] Let also $\sigma(L) \subseteq \mathbb{R}$. If the operator $L$ is similar to a self-adjoint operator, then there is a positive constant $K > 0$ such that

$$\|T(z)^{-1}\|_{\mathcal{B}} \leq \frac{K}{|z||\text{Im} \ z|}, \quad \|A^{-1}(T(z))^{-1}\|_{\mathcal{B}} \leq \frac{K}{|z||\text{Im} \ z|} \quad (3.13)$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$.\]
Proof. Using the Frobenius–Schur factorization \[3.2\], after straightforward calculations we find that the resolvent of \(\mathcal{L}\) is given by
\[
(\mathcal{L}-z)^{-1} = \begin{pmatrix} A^{-1}(C^*-iz)(T(z))^{-1} -i(A^{-1}(C^*-iz)(T(z)))^{-1}(C+iz)A^{-1} + A^{-1} \\ -i(T(z))^{-1}(C+iz)A^{-1} \end{pmatrix}.
\]
Note that, by Theorem \[3.1\] \((T(z))^{-1}\) is a bounded operator for each \(z \in \mathbb{C} \setminus \mathbb{R}\) since \(\sigma(\mathcal{L}) \subseteq \mathbb{R}\). It remains to apply the resolvent growth condition (LRG), which states that
\[
\| (\mathcal{L}-z)^{-1} \| \leq \frac{K}{|\text{Im} z|}, \quad z \in \mathbb{C} \setminus \mathbb{R},
\]
if \(\mathcal{L}\) is similar to a self-adjoint operator. \(\square\)

Lemma \[3.6\] enables us to construct a very simple example of a \(J\)-positive operator with the singular critical point infinity.

Example 3.7. Let \(A\) be a uniformly positive unbounded self-adjoint operator in \(\mathcal{H}\), \(A = A^* \geq \varepsilon^2 I > 0\). Let also \(B = I\) and \(C = 0\), that is, the operator \(\mathcal{L}\) is given by
\[
\mathcal{L} = \begin{pmatrix} 0 & iI \\ -iA & 0 \end{pmatrix}, \quad \text{dom}(\mathcal{L}) = \text{dom}(A) \times \mathcal{H}.
\]
Note that \(\mathcal{L}\) is \(J\)-self-adjoint and \(J\)-positive in \(\mathcal{H} = \mathcal{H} \times \mathcal{H}\). Moreover,
\[
T(z) = I - z^2 A^{-1} = (A - z^2)A^{-1}, \quad z \in \mathbb{C}.
\]
Therefore,
\[
\sigma(\mathcal{L}) = \{ \lambda \in \mathbb{R} : \lambda^2 \in \sigma(A) \} \subseteq \mathbb{R} \setminus (-\varepsilon, \varepsilon).
\]
Notice that \(\infty\) is a critical point of \(\mathcal{L}\) since \(A\) is unbounded. Moreover, we immediately find that
\[
\| T(z)^{-1} \| = \| A(A-z^2)^{-1} \| \geq 1
\]
for all \(z \in \rho(\mathcal{L})\). By Lemma \[3.6\], the operator \(\mathcal{L}\) is not similar to a self-adjoint operator (since it does not satisfy the LRG condition). Moreover, \(\infty\) is a singular critical point of \(\mathcal{L}\).

Remark 3.8. The results of Example \[3.7\] can be deduced from \[0\], where the norms of spectral projections are computed in terms of coefficients of \(\mathcal{L}\) (see \[4\] Satz 2.1.3).

Remark 3.9. Note that the operator \[3.10\] provides a very simple example of a \(J\)-positive operator with the singular critical point \(\infty\). For instance, it suffices to take \(\mathcal{H} = L^2(\mathbb{R}_+; d\mu)\), where \(d\mu\) is a positive Borel measure on \(\mathbb{R}_+ = (0, +\infty)\). Let also \(A\) be the usual multiplication operator in \(L^2(\mathbb{R}_+, d\mu)\)
\[
(Af)(x) = (x+1)f(x), \quad x \in \mathbb{R}_+.
\]
If \(\mu\) is a discrete measure, say \(\mu = \sum_{n \in \mathbb{N}} \delta(x-n)\), then \(L^2(\mathbb{R}_+)\) is equivalent to \(l^2(\mathbb{N})\) and the operator \(A\) is simply the orthogonal sum of \(2 \times 2\) matrices
\[
\mathcal{L} = \bigoplus_{n \in \mathbb{N}} \begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix}, \quad \mathcal{H} = l^2(\mathbb{N}; \mathbb{C}^2).
\]
Other simple examples of operators with singular critical points can be found in \[3\] pp. 92–93, \[5\] Example 2.11,
Example 3.10. Let us continue with Example 3.5. Assume additionally that the coefficients $a$, $b$ and $c$ satisfy (3.12). Then the operator $L$ is $J$-positive and $\sigma(L) \subseteq \mathbb{R}$. Clearly, the resolvent of $L$ is given by

$$
(L - z)^{-1} = \begin{pmatrix}
\frac{-ic - z}{(c + iz)(c^* - iz) - ab} & \frac{-ib}{ic - z} \\
\frac{ic - iz}{(c + iz)(c - iz) - ab} & \frac{ic - z}{(c + iz)(c^* - iz) - ab}
\end{pmatrix}.
$$

If the operator $L$ is similar to a self-adjoint operator, then it satisfies the LRG condition (3.15). Clearly, the latter is equivalent to the following inequality

$$
\frac{|a| + |b| + |c|}{ab - (c - iz)(c^* + iz)} \leq \frac{K}{|\Im z|}, \quad z \in \mathbb{C} \setminus \mathbb{R}.
$$

Here $K > 0$ is a positive constant independent on $z$.

For a detailed discussion of spectral properties of these operators we refer to [6] and [7].

4. Block matrices with differential operators

Let $V : \mathbb{R} \rightarrow \mathbb{R}$ be a locally integrable function, $V \in L^1_{loc}(\mathbb{R})$. The following operator arises in the study of stability of solitons for the 1-D relativistic Ginzburg–Landau equation (see [3, 9, 10]):

$$
\mathcal{L}_0 = \begin{pmatrix}
-\frac{d^2}{dx^2} + m^2 + V(x) & iV \\
iV & i\frac{d}{dx}
\end{pmatrix}, \quad \text{dom}(\mathcal{L}_0) = \mathcal{D}(H_V) \times W^{1,2}(\mathbb{R}).
$$

Here $\mathcal{D}(H_V)$ is the maximal domain of the operator $H_V = -\frac{d^2}{dx^2} + m^2 + V(x)$

$$
\mathcal{D}(H_V) = \{ f \in L^2(\mathbb{R}) : f, f' \in AC_{loc}(\mathbb{R}), -f'' + Vf \in L^2(\mathbb{R}) \}.
$$

We shall assume (cf. [3, 9, 10]) that $\nu \in (-1, 1)$, $m > 0$ and

$$
\lim_{z \to \infty} \int_x^{x+1} |V(t)| \, dt = 0.
$$

Note that condition (1.3) implies that the potential $V$ is a relatively compact perturbation (in the sense of forms) of $H_0 = -\frac{d^2}{dx^2} + m^2$ (cf. [4] Chapter III.43) and hence

$$
\sigma_c(H_V) = \sigma_{ess}(H_V) = [m^2, +\infty), \quad \kappa_-(H_V) = N < \infty.
$$

Assume for simplicity that $z = 0$ is not an eigenvalue of $H_V$. Then all conditions (i)–(iv) of Hypothesis 1.1 are satisfied and hence we can apply the results from the previous sections.

Remark 4.1. If $\nu = 0$, then the operator $L$ is a particular case of the operator considered in Example 3.7. In this case the operator $L$ does satisfy the LRG condition (3.15) and hence is not similar to a self-adjoint operator. We exclude this case from our further considerations.

Let $\psi_+(z, x)$ and $\psi_-(z, x)$ be the Weyl solutions of $-y'' + (m^2 + V(x))y = zy$ normalized such that $W(\psi_+, \psi_-)(z) = \psi_+(z, x)\psi'_-(z, x) - \psi'_+(z, x)\psi_-(z, x) = 1$. Then the resolvent of the 1-D Schrödinger operator is given by

$$
(H_V - z)^{-1}f = \int_{\mathbb{R}} G(z; x, y) f(y) \, dy, \quad G(z; x, y) = \begin{cases}
\psi_+(x)\psi_-(y), & y \leq x, \\
\psi_+(y)\psi_-(x), & y > x,
\end{cases}
$$

where $\psi_+ = \psi_+$ and $\psi_- = \psi_-$. The following conditions hold:

(i) $\psi_+ = \psi_+$ and $\psi_- = \psi_-$. The following conditions hold:

$$
\psi_{\alpha}(x) = \psi_{\alpha}(x), \quad \psi_{\alpha}(x) = \psi_{\alpha}(x),
$$

where $\alpha = +, -$. The following conditions hold:

$$
\psi_{\alpha}(x) = \psi_{\alpha}(x), \quad \psi_{\alpha}(x) = \psi_{\alpha}(x).
$$
Denote \( D = \frac{d}{dx} \), \( \text{dom}(D) = W^{1,2}(\mathbb{R}) \), and assume that \( 0 \in \rho(H_V) \). Then using \( (4.4) \), integration by parts shows that
\[
DH_V^{-1}f = \int_{\mathbb{R}} G_x(0; x, y)f(y)dy, \quad \overline{H_V^{-1}D}f = -\int_{\mathbb{R}} G_y(0; x, y)f(y)dy, \quad f \in L^2(\mathbb{R}),
\]
and
\[
\overline{S(0)}f = (1 + \nu^2)f - \nu^2 \int_{\mathbb{R}} G_{xy}(0; x, y)f(y)dy, \quad f \in L^2(\mathbb{R}).
\]
Here the subscript denotes the partial derivative. Since \( \overline{S(0)} \) is a bounded operator, the ranges of \( DH_V^{-1} \) and \( \overline{H_V^{-1}D} \) are contained in \( W^{1,2}(\mathbb{R}) \).

Firstly, let us describe the spectral properties of the operator \( A_0 = JL_0 \) and its closure \( A \).

**Lemma 4.2.** Let \( m > 0 \), \( V \) satisfy \( (4.3) \) and \( 0 \in \rho(H_V) \). Then the operator \( A_0 \) is essentially self-adjoint and its closure is given by
\[
(4.5) \quad A = \begin{pmatrix} I & 0 \\ \nu DH_V^{-1} & I \end{pmatrix} \left( \begin{array}{cc} H_V & 0 \\ 0 & \overline{S(0)} \end{array} \right) \begin{pmatrix} I & \nu H_V^{-1}D \\ 0 & I \end{pmatrix}
\]
on the domain
\[
(4.6) \quad \text{dom}(A) = \{ f = (f_1, f_2)^T : f_1 - \nu H_V^{-1}Df_2 \in \mathcal{D}(H_V), f_2 \in L^2(\mathbb{R}) \}.
\]

The form domain of the operator \( A \) is given by
\[
(4.7) \quad \text{dom}(A^{1/2}) = \{ f = (f_1, f_2)^T : f_1 \in W^{1,2}(\mathbb{R}), f_2 \in L^2(\mathbb{R}) \}.
\]

**Proof.** The first claim immediately follows from Corollary \( 2.2 \). To prove \( (4.7) \) it suffices to note that
\[
(4.8) \quad \text{dom}(A^{1/2}) = \{ f = (f_1, f_2)^T : f_1 - \nu H_V^{-1}Df_2 \in W^{1,2}(\mathbb{R}), f_2 \in L^2(\mathbb{R}) \}.
\]

However, \( H_V^{-1}Df_2 \in W^{1,2}(\mathbb{R}) \) whenever \( f_2 \in L^2(\mathbb{R}) \).

The next result describes the essential spectrum of \( A \) (cf. \cite[Lemma A.1]{10}).

**Corollary 4.3.** Assume the conditions of Lemma \( 4.2 \). Then
\[
(4.9) \quad \sigma_{\text{ess}}(A) = \begin{cases} [1 - \nu^2, 1] \cup [m^2, +\infty), & m \geq 1, \\ [1 - \nu^2, m^2] \cup [1, +\infty), & 0 \leq 1 - \nu^2 \leq m^2 < 1, \\ [m^2, 1 - \nu^2] \cup [1, +\infty), & 0 < m^2 < 1 - \nu^2 \leq 1. \end{cases}
\]

**Proof.** It follows from \( (4.7) \) and \( (4.3) \) that the operator \( V = V \oplus 0 \) is a relatively compact perturbation (in the sense of forms) of the operator \( A \) with \( V \equiv 0 \). Therefore, by the version of Weyl’s theorem for relatively compact perturbations, \( \sigma_{\text{ess}} \) does not depend on \( V \) and hence we can set \( V \equiv 0 \).

To find the essential spectrum of the operator \( A \) with \( V \equiv 0 \) let us apply the Fourier transform. Then the operator \( A \) is equivalent to the multiplication operator \( \hat{A} \) in \( L^2(\mathbb{R}) \times L^2(\mathbb{R}) \) defined by \( \hat{f}(\lambda) \to \hat{A}(\lambda)\hat{f}(\lambda) \), where
\[
(4.10) \quad \hat{A}(\lambda) = \begin{pmatrix} \frac{1}{\lambda^2 + m^2} & 0 \\ \frac{0}{\lambda^2 + m^2} & 1 \end{pmatrix} \begin{pmatrix} \lambda^2 + m^2 & 0 \\ 0 & 1 - \frac{\nu^2 \lambda^2}{\lambda^2 + m^2} \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda^2 + m^2} & 0 \\ \frac{0}{\lambda^2 + m^2} & 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.
\]
Since the function \( \hat{A}(\cdot) \) is continuous on \( \mathbb{R} \), we conclude that \( \sigma(A) = \sigma(\hat{A}) = \sigma_{\text{ess}}(\hat{A}) \). Straightforward calculations show that

\[
(\hat{A}(\lambda) - z)^{-1} = \frac{1}{\text{det}(\hat{A}(\lambda) - z)} \begin{pmatrix}
1 & i\nu \lambda \\
-(-i\nu \lambda) & \lambda^2 + m^2 \\
\end{pmatrix}, \quad \lambda \in \mathbb{R},
\]

\[
\text{det}(\hat{A}(\lambda) - z) = \lambda^2(1 - z - \nu^2) + (m^2 - z)(1 - z).
\]

Therefore, \( z \in \sigma(\hat{A}) \) if and only if either \( z = 1 - \nu^2 \) or \( \text{det}(\hat{A}(\lambda) - z) = 0 \) for some \( \lambda \in \mathbb{R} \). Clearly, this equation has real solutions if and only if

\[
\frac{(z - m^2)(z - 1)}{z - (1 - \nu^2)} \geq 0.
\]

This completes the proof of (4.9). \( \square \)

**Corollary 4.4.** Assume that \( 0 \notin \sigma(H_V) \). Then

(4.11) \hspace{1cm} \kappa_-(A) = \kappa_-(H_V) + \kappa_-(\overline{S(0)}) < \infty.

In particular, \( A \) is positive if and only if so are \( H_V \) and \( \overline{S(0)} \).

**Proof.** The first equality in (4.11) follows from Corollary 2.3. Moreover, due to (4.3), \( \kappa_-(H_V) = N < \infty \). It remains to show that \( \kappa_-(\overline{S(0)}) < \infty \). Denote by \( S_0(0) \) the operator \( S(0) \) with \( V \equiv 0 \). Note that \( \sigma(\overline{S_0(0)}) = \sigma_{\text{ess}}(\overline{S_0(0)}) = [1 - \nu^2, 1] \) (immediately follows by applying the Fourier transform). Moreover,

\[
S_0(0) - S(0) = \nu^2 DH_0^{-1} V H_V^{-1} D.
\]

Note that the closure of this operator is compact on \( L^2(\mathbb{R}) \) since \( V \) satisfies (4.3). Therefore, \( \sigma_{\text{ess}}(\overline{S_0(0)}) = \sigma(\overline{S_0(0)}) = [1 - \nu^2, 1] \subset (0, 1] \) since \( |\nu| \in (0, 1) \). This implies the desired inequality. \( \square \)

**Corollary 4.5.** The operator \( A \) is nonnegative if and only if so are the operators \( H_V \) and

(4.12) \hspace{1cm} H_{\nu,V} := -(1 - \nu^2) \frac{d^2}{dx^2} + m^2 + V(x), \quad \text{dom}(H_{\nu,V}) = D(H_{\nu,V}).

**Proof.** By the previous corollary, it remains to show that \( \overline{S(0)} \geq 0 \) if and only if so is \( H_{\nu,V} \). Next, the operator is positive if and only if

\[
t_S[f] = \overline{(S(0)f, f)}_{L^2} = (f, f)_{L^2} - \nu^2 (H_V^{-1} Df, Df)_{L^2} > 0
\]

for all \( f \in C_c^\infty(\mathbb{R}) \) (since this linear subspace is dense in \( L^2(\mathbb{R}) \)). Setting \( g(x) = f'(x) \) and integrating by parts once again, we finally get

\[
t_S[f] = (H_0^{-1} g, g)_{L^2} - \nu^2 (H_V^{-1} g, g)_{L^2} > 0, \quad g \in C_c^\infty(\mathbb{R}).
\]

Here \( H_0 = -\frac{d^2}{dx^2} \) is the free Hamiltonian on \( L^2(\mathbb{R}) \). The latter is equivalent to the positivity of the operator \( H_{\nu,V} \). \( \square \)

Now let us describe the spectral properties of the operator \( \mathcal{L} = \mathcal{J}A \). We begin with the description of the closure of \( \mathcal{L}_0 \).

**Lemma 4.6.** Assume the conditions of Lemma 4.2. Then the operator \( \mathcal{L}_0 \) is essentially \( \mathcal{J} \)-self-adjoint and its closure is given by

(4.13) \hspace{1cm} \mathcal{L} = \begin{pmatrix}
I & -\nu DH_V^{-1} \\
0 & I
\end{pmatrix} \begin{pmatrix}
0 & i\overline{S(0)} \\
-iH_V & 0
\end{pmatrix} \begin{pmatrix}
I & \nu H_V^{-1} D \\
0 & I
\end{pmatrix}
on the domain
\begin{equation}
\text{dom}(L) = \{ f = (f_1, f_2)^T : f_1 + \nu H^{-1} D f_2 \in W^{2,2}(\mathbb{R}), \ f_2 \in L^2(\mathbb{R}) \}.
\end{equation}

\textbf{Proof.} Note that $T(0) = S(0)$ and $\text{dom}(L) = \text{dom}(A)$. The rest of the proof follows from Theorem 3.1 and Lemma 4.2. \hfill \Box

As an immediate corollary of Theorem 3.1 we obtain the following description of $\sigma(L)$.

\textbf{Corollary 4.7.} Assume the conditions of Theorem 3.1 and set
\begin{equation}
T(z) = I + (\nu D - iz)H^{-1}_0(\nu D + iz).
\end{equation}
Then
\begin{equation}
z \in \sigma(L) \iff 0 \in \sigma(T(z)) \quad (z \in \sigma(L) \iff 0 \in \sigma(T(z)), \ i \in \{p, c, \text{ess}\}).
\end{equation}

\textbf{Theorem 4.8.} Assume the conditions of Lemma 4.2. Then the operator $L$ is definitizable and $\infty$ is its singular critical point.

\textbf{Proof.} By Lemma 1.2 and Corollary 1.4 the operator $L$ is $J$-self-adjoint and the form $(\langle J L \cdot, \cdot \rangle = \langle A \cdot, \cdot \rangle$ has finitely many negative squares. Therefore, $L$ is definitizable if $\rho(L) \neq \emptyset$. By Corollary 1.4 we need to show that there is $z \in \mathbb{C}$ such that the operator $T(z)$ is boundedly invertible. Set $z = iy$ with $y > 0$. The operators $\nu D + y$ and $\nu D - y$ are boundedly invertible in $L^2(\mathbb{R})$. Therefore, we get
\begin{equation}
(\nu D + y)^{-1}T(iy)(\nu D - y)^{-1} = (\nu^2 D^2 - y^2)^{-1} + H^{-1}_0.
\end{equation}
Since $-\nu^2 D^2 + y^2 \geq y^2 I$, we get $\|(\nu^2 D^2 - y^2)^{-1}\| \leq 1/y^2$. Therefore, the left-hand side in (4.17) is a boundedly invertible operator for $y > 0$ sufficiently large since $0 \in \rho(H^0)$. It remains to note that
\begin{equation}
(\overline{T(iy)})^{-1} = (\nu D + y)^{-1}[(\nu^2 D^2 - y^2)^{-1} + H^{-1}_0]^{-1}(\nu D - y)^{-1}.
\end{equation}
Therefore, $iy \in \rho(T)$ for all sufficiently large $y > 0$.

By [2, Theorem 4.1], $\infty$ is a singular critical point of $L$ if and only if $\infty$ is a singular critical point of $L$ with $V \equiv 0$. That is, it suffices to show that $\infty$ is a singular critical point of the operator $L = \overline{L_0}$, where $L_0$ is given in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ by
\begin{equation}
L_0 = \begin{pmatrix}
\nu D & iI \\
-iH_0 & i\nu D
\end{pmatrix} = \begin{pmatrix}
\nu \lambda f_1(\lambda) + i f_2(\lambda) \\
-i(\nu^2 + m^2) f_1(\lambda) + \nu f_2(\lambda)
\end{pmatrix}.
\end{equation}
Now using the Fourier transform we see that $L_0$ is unitarily equivalent to the multiplication operator acting in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and defined by
\begin{equation}
(\hat{L}_0 f)(\lambda) = \begin{pmatrix}
\nu \lambda \\
-i(\nu^2 + m^2)
\end{pmatrix}
\begin{pmatrix}
f_1(\lambda) \\
f_2(\lambda)
\end{pmatrix} = \begin{pmatrix}
\nu \lambda f_1(\lambda) + i f_2(\lambda) \\
-i(\nu^2 + m^2) f_1(\lambda) + \nu f_2(\lambda)
\end{pmatrix}.
\end{equation}
The operator $\hat{L}_0$ is a particular case of the operator considered in Example 3.5 with $a(\lambda) = \lambda^2 + m^2$, $b(\lambda) = 1$ and $c(\lambda) = \nu \lambda$. Clearly, setting $z = iy$ in (3.20), we get
\begin{equation}
\left\| \frac{a}{ab - (c + y)(\bar{c} - y)} \right\|_{L^\infty} = \left\| \frac{\lambda^2 + m^2}{(1 - \nu^2)\lambda^2 + m^2 + y^2 - 2i\nu \lambda} \right\|_{L^\infty} \geq \frac{1}{1 - \nu^2}.
\end{equation}
Therefore, there is no $K > 0$ such that (3.20) holds true. Hence the LRG test (3.15) for the operator $L$ fails and hence $L$ is not similar to a self-adjoint operator. It remains to note that $L$ is $J$-positive with $0 \in \rho(L)$ if $V \equiv 0$. Therefore, $\infty$ is
the only critical point of $\mathcal{L}$. Since $\mathcal{L}$ is not similar to a self-adjoint operator, $\infty$ is a singular critical point of $\mathcal{L}$.

Acknowledgements. I am deeply grateful to Branko Ćurgus, Andreas Fleige, Alexander Komech and Elena Kopylova for hints with respect to the literature and fruitful discussions.

References

[1] V. S. Buslaev and C. Sulem, On asymptotic stability of solitary waves for nonlinear Schrödinger equations, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 20 (2003), 419–475.
[2] B. Ćurgus, On the regularity of the critical point infinity of definitizable operators, Int. Equat. Oper. Theory 8 (1985), no. 4, 462–488.
[3] B. Ćurgus and B. Najman, Quasi-uniformly positive operators in Krein space, in: “Operator theory and boundary eigenvalue problems” (Vienna, 1993), Oper. Theory: Adv. Appl. 80 (1995), 90–99.
[4] I. M. Glazman, Direct Methods for Qualitative Spectral Analysis of Singular Differential Operators, Fizmagiz, Moscow, 1963.
[5] L. Grubišić, V. Kostrykin, K. Makarov, and K. Veselić, Representation theorems for indefinite quadratic forms revisited, Mathematika 59 (2013), 169–189.
[6] P. Jonas, Zur existenz von eigenspektralfunktionen für J-positive operatoren, I, Math. Nachr. 82 (1978), 241–254.
[7] P. Jonas, Zur existenz von eigenspektralfunktionen für J-positive operatoren, II, Math. Nachr. 83 (1978), 197–207.
[8] A. I. Komech and E. A. Kopylova, On asymptotic stability of moving kink for relativistic Ginzburg–Landau equation, Comm. Math. Phys. 302 (2011), no.1, 225–252. (arXiv:0910.5538)
[9] A. I. Komech and E. A. Kopylova, On asymptotic stability of kink for relativistic Ginzburg–Landau equation, Arch. Ration. Mech. Anal. 202 (2011), 213–245. (arXiv:0910.5539)
[10] A. I. Komech and E. A. Kopylova, On eigenfunction expansion of solutions to the Hamilton equations, J. Stat. Phys. 154 (2014), 503–521. (arXiv:1308.0485)
[11] H. Langer, Spectral functions of definitizable operators in Krein spaces, Lect. Notes. Math. 948 (1984), 1–46.
[12] A. A. Shkalikov, On the essential spectrum of matrix operators, Math. Notes 58 (1995), 945–949.
[13] C. Tretter, Spectral Theory of Block Operator Matrices and Applications, Imperial College Press, 2008.