A Slightly Supercritical Condition of Regularity of Axisymmetric Solutions to the Navier-Stokes Equations

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Abstract

In the note, a new regularity condition for axisymmetric solutions to the non-stationary 3D Navier-Stokes equations is proven. It is slightly supercritical.

Keywords Navier-Stokes equations, axisymmetric solutions, local regularity

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1 Introduction

In this note, we continue to analyze potential singularities of axisymmetric solutions to the non-stationary Navier-Stokes equations. In the previous paper [23], it has been shown that an axially symmetric solution is smooth provided a certain scale-invariant energy quantity of the velocity field is bounded. By definition, a potential singularity with bounded scale-invariant energy quantities is called the Type I blowup. It is important to notice that the above result does not follow from the so-called $\varepsilon$-regularity theory developed in [2], [15], and [10], where regularity is coming out due to smallness of those scale-invariant energy quantities.

We consider the 3D Navier-Stokes system

$$\frac{\partial v}{\partial t} + v \cdot \nabla v - \Delta v = -\nabla q, \quad \text{div} v = 0 \quad (1.1)$$

in the parabolic cylinder $Q = C \times ]1, 0[, \quad \text{where} \quad C = \{ x = (x_1, x_2, x_3) : x_1^2 + x_2^2 < 1, \quad -1 < x_3 < 1 \}$. A solution $v$ and $q$ is supposed to be a suitable weak one, which means the following:

Definition 1.1. Let $\omega \subset \mathbb{R}^3$ and $T_2 > T_1$. The pair $w$ and $r$ is a suitable weak solution to the Navier-Stokes system in $Q_* = \omega \times ]T_1, T_2[$ if:

1. $w \in L_{2,\infty}(Q_*), \quad \nabla w \in L_2(Q_*), \quad r \in L_{2,2}^3(Q_*);$

2. $w$ and $r$ satisfy the Navier-Stokes equations in $Q_*$ in the sense of distributions;

3. for a.a. $t \in [T_1, T_2]$, the local energy inequality

$$\int_{\omega} \varphi(x, t)|w(x, t)|^2dx + 2 \int_{T_1}^t \int_{\omega} \varphi|\nabla w|^2dxdt' \leq \int_{T_1}^t \int_{\omega} [||w||^2(\partial_t \varphi + \Delta \varphi) +$$

$$+w \cdot \nabla \varphi(|v|^2 + 2r)]dxdt'$$

holds for all non-negative $\varphi \in C_0^1(\omega \times ]T_1, T_2 + (T_2 - T_1)/2[)$.

In our standing assumption, it is supposed that a suitable weak solution $v$ and $q$ to the Navier-Stokes equations in $Q = C \times ]1, 0[$ is axially symmetric with respect to the axis $x_3$. The latter means the following: if we introduce the corresponding cylindrical coordinates $(\rho, \varphi, x_3)$ and use the corresponding representation $v = v_\rho e_\rho + v_\varphi e_\varphi + v_3 e_3$, then $v_\rho, v_\varphi = v_3, \varphi = q, \varphi = 0.$

There are many papers on regularity of axially symmetric solutions. We cannot pretend to cite all good works in this direction. For example, let us
mention papers: [9], [28], [13], [18], [20], [3], [26], [5], [25], [11], [19], [12], [4], [27], and [29].

Actually, our note is inspired by the paper [19], where the regularity of solutions has been proved under a slightly supercritical assumption. We would like to consider a different supercritical assumption, to give a different proof and to get a better result.

To state our supercritical assumption, additional notation is needed. Given \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \), denote \( x' = (x_1, x_2, 0) \). Next, different types of cylinders will be denoted as \( C(r) = \{x : |x'| < r, |x_3| < r\} \), \( C(x_0, r) = C(r) + x_0 \), \( Q^{\lambda, \mu}(r) = C(\lambda r) \times [-\mu R^2, 0] \), \( Q^{1, 1}(r) = Q(r) \), \( Q^{\lambda, \mu}(z_0, r) = C(x_0, \lambda r) \times [t_0 - \mu R^2, t_0] \). And, finally, we let

\[
f(R) := \frac{1}{\sqrt{R}} \left( \int_{-R^2}^{0} \left( \int_{C(R)} |v|^3 \, dx \right)^{\frac{3}{2}} \, dt \right)^{\frac{2}{3}}
\]

and

\[
M(R) := \frac{1}{\sqrt{R}} \left( \int_{Q(R)} |v|^{\frac{10}{3}} \, dz \right)^{\frac{3}{10}}
\]

for any \( 0 < R \leq 1 \) and assume that:

\[
f(R) + M(R) \leq g(R) := c_* \ln^\alpha \ln^{\frac{3}{2}}(1/R) \tag{1.2}
\]

for all \( 0 < R \leq 2/3 \), where \( c_* \) and \( \alpha \) are positive constants and \( \alpha \) obeys the condition:

\[
0 < \alpha \leq \frac{1}{224}. \tag{1.3}
\]

Without loss of generality, one may assume that \( g(R) \geq 1 \) for \( 0 < R \leq \frac{2}{3} \). To ensure the above condition, it is enough to increase the constant \( c_* \) if necessary.

Our aim could be the following completely local statement.

**Theorem 1.2.** Assume that a pair \( v \) and \( q \) is axially symmetric suitable weak solution to the Navier-Stokes equations in \( Q \) and conditions (1.2) and (1.3) hold. Then the origin \( z = 0 \) is a regular point of \( v \).

However, in this paper, we shall prove a weaker result leaving Theorem 1.2 as a plausible conjecture. We shall return to a proof of Theorem 1.2 elsewhere. In the present paper, the following fact is going to be justified.
Theorem 1.3. Let \( v \) be an axially symmetric solution to the Cauchy problem for the Xavier-Stokes equations \( (\text{[L]}) \) in \( \mathbb{R}^3 \times ]0, T[ \) with initial divergence free field \( v_0 \) from the Sobolev space \( H^2 = W^2_2(\mathbb{R}^3) \) such that

\[
\sup_{0 < t < T - \delta} \| \nabla v(\cdot, t) \|_{L^2(\mathbb{R}^3)} \leq C(\delta) < \infty
\]

for all \( 0 < \delta < T \). Assume further that

\[
\Sigma_0 = \sup_{x \in \mathbb{R}^3} |v_{02}(x)x_1 - v_{01}(x)x_2| < \infty \tag{1.4}
\]

and

\[
\sup_{0 < R \leq 2/3} \sup_{-\infty < h < \infty} f(R; (0, h, T)) + M(R; (0, h, T)) \leq g(R) \tag{1.5}
\]

with some positive constants \( c_* \) and \( \alpha \), satisfying \( (\text{[L]}) \), where

\[
f(R; z_0) := \frac{1}{\sqrt{R}} \left( \int_{t_0 - R^2}^{t_0} \left( \int_{C(z_0, R)} |v|^3 \, dx \right)^{\frac{4}{3}} \, dt \right)^{\frac{3}{4}}
\]

and

\[
M(R; z_0) := \frac{1}{\sqrt{R}} \left( \int_{Q(z_0, R)} |v|^{10} \, dz \right)^{\frac{3}{10}}
\]

Then \( v \) is a strong solution to the above Cauchy problem in \( \mathbb{R}^3 \times ]0, T[ \), i.e.,

\[
\sup_{0 < t < T} \| \nabla v(\cdot, t) \|_{L^2(\mathbb{R}^3)} < \infty.
\]

Our proof is based on the analysis of the following scalar equation

\[
\partial_t \sigma + \left( v + 2x' \frac{x'}{|x'|^2} \right) \cdot \nabla \sigma - \Delta \sigma = 0 \tag{1.6}
\]

in \( Q \setminus \{ x' = 0 \times [1, 0] \} \), where \( \sigma := \rho v_{x'} = v_2x_1 - v_1x_2 \).

Let us list some differentiability properties of \( \sigma \). Some of them follows from partial regularity theory developed by Caffarelli-Kohn-Nirenberg.

Indeed, since \( v \) and \( q \) are an axially symmetric suitable weak solution, there exists a closed set \( S^\sigma \) in \( Q \), whose 1D-parabolic measure in \( \mathbb{R}^3 \times \mathbb{R} \) is equal to zero and \( x' = 0 \) for any \( z = (x, t) \in S^\sigma \), such that any spatial derivative of \( v \) (and thus of \( \sigma \)) is Hölder continuous in \( Q \setminus S^\sigma \).
Next, we observe that
\[
|\partial_t \sigma(z) - \Delta \sigma(z)| \leq \left( \sup_{z=(x,t) \in P(\delta,R;R) \times [-R^2,0]} |v(z)| + 2/\delta \right) |\nabla \sigma(z)|
\]
for any \(0 < \delta < R < 1\), where \(P(a,b;h) = \{x: a < |x'| < b, |x_3| < h\}\). Since \(v\) is axially symmetric, the first factor on the right hand side is finite. This fact, by iteration, yields
\[
\sigma \in W^{2,1}_p(P(\delta,R;R) \times [-R^2,0])
\]
for any \(0 < \delta < R < 1\) and for any finite \(p \geq 2\).

It follows from the above partial regularity theory that, for any \(-1 < t < 0\),
\[
\sigma(x',x_3,t) \to 0 \quad \text{as} \quad |x'| \to 0 \quad (1.7)
\]
for all \(x_3 \in ]-1,1[ \cup S^\sigma_t\).

In the same way, as it has been done in [25] and [23], one can show that \(\sigma \in L_\infty(Q(R))\) for any \(0 < R < 1\).

The main part of the proof of Theorem 1.3 is the following fact.

**Proposition 1.4.** Let \(\sigma = gv_\varphi\), then
\[
\text{osc}_{z \in Q(r)} \sigma \leq C_1(c_*) \left( \frac{r}{2R} \right)^{C_2(c_*)} \text{osc}_{z \in Q(2R)} \sigma(z), \quad (1.8)
\]
where \(C_1\) and \(C_2\) are positive constants and \(0 < r < R \leq R_*(c_*,\alpha) \leq 1/6\). Here, \(\text{osc}_{z \in Q(r)} \sigma(z) = M_r - m_r\) and
\[
M_r = \sup_{z \in Q(r)} \sigma(z), \quad m_r = \inf_{z \in Q(r)} \sigma(z).
\]

The above statement is an improvement of the result in [19], where the bound for oscillations of \(\sigma\) contains a logarithmic factor only.

The proof of Proposition 1.4 is based on a technique developed in [17], see also references there. We also would like to mention interesting results for the heat equation with a divergence free drift, see [6], [7], [24], and [1].

2 Auxiliary Facts

Define the class \(V\) of functions \(\pi: Q \to \mathbb{R}\) possessing the properties:
(i) there exists a closed set $S^\pi$ in $Q$, whose 1D-parabolic measure $\mathbb{R}^3 \times \mathbb{R}$ is equal to zero and $x' = 0$ for any $z = (x', x_3, t) \in S^\pi$, such that any spatial derivative is Hölder continuous in $Q \setminus S^\pi$;

(ii) $\pi \in W^{2,1}_2(P(\delta, R; R) \times] - R^2, 0[) \cap L_\infty(Q(R))$

for any $0 < \delta < R < 1$.

We are going to use the following subclass $V_0$ of the class $V$, saying that $\pi \in V_0$ if and only if $\pi \in V$ and

$$\partial_t \pi + \left( u + 2 \frac{x'}{|x'|^2} \right) \cdot \nabla \pi - \Delta \pi = 0 \quad (2.1)$$

in $C \setminus \{x' = 0\} \times] - 1, 0[$.

We shall also say that $\pi \in V_0$ has the property $(B_R)$ in $Q(2R)$ if there exists a number $k_R > 0$ such that $\pi(0, x, t) \geq k_R$ for $-(2R)^2 \leq t \leq 0, x_3 \in] - 2R, 2R[ \setminus S^\pi_t$, where $S^\pi_t = \{ x \in C : (x, t) \in S^\pi \}$.

**Remark 2.1.** Let $0 < r \leq R$ and $\pi \in V_0$ have the property $(B_R)$ in $Q(2R)$. Then $\pi$ has the property $(B_r)$ in $Q(2r)$ with any constant less or equal to $k_R$.

In what follows, we always suppose that $0 < R \leq 1/6$.

**Proposition 2.2.** Let $\pi \in V_0$ have the property $(B_R)$. Then, for any $0 < k \leq k_R$, for any $0 < \tau_1 < \tau < 2$, and for any $0 < \gamma_1 < \gamma < 4$, the following inequality holds:

$$\sigma(z) \leq c_1(\tau_1, \tau, \gamma_1, \gamma, M(2R)) \left( \frac{1}{|Q^{\tau, \gamma}(R)|} \int_{Q^{\tau, \gamma}(R)} \sigma^{\frac{10}{3}}(R) dz \right)^\frac{3}{10},$$

where $\sigma = (k - \pi)_+$,

$$c_1(\tau_1, \tau, \gamma_1, \gamma, M(2R)) = \frac{c}{(\tau - \tau_1)^\frac{10}{3}} \left( 1 + \frac{\tau - \tau_1}{\sqrt{\gamma - \gamma_1}} + \left( \frac{1}{\gamma_1 \tau_1^2} \right)^\frac{1}{3} M(2R) \right)^3,$$

and $Q^{\tau, \gamma}(R) = C(\tau R) \times] - \gamma R^2, 0[.$

**Proof.** Repeating arguments in [23], we can get the following estimate of $h = \sigma^m$:

$$\left( \int_0^t \int_{c(r^2)} |h|^{\frac{10}{3}} dz \right)^\frac{3}{10} \leq$$
\[
\leq c \left( \int_{t_1}^{0} \int_{C(r_1)} |h|^2 \, dz \right)^\frac{2}{m} \frac{(r_1^3|t_1|)^\frac{1}{m}}{r_1 - r_2} \left( 1 + \frac{r_1 - r_2}{\sqrt{t_2 - t_1}} + \frac{r_1^3|t_1|^\frac{1}{m}}{(r_1 - r_2)^\frac{5}{m}} \right)
\]

(2.3)

for any \(0 < r_2 < r_1 < 2R\) and \(-4R^2 < t_1 < t_2 < 0\), where

\[
\overline{M}(r_1, t_1) = \left( \frac{1}{|t_1|r_1^2} \right)^\frac{1}{m} \left( \int_{t_1}^{0} \int_{C(r_1)} |\sigma|^\frac{4}{3} \, dz \right)^\frac{3}{m}.
\]

Next, we wish to iterate (2.3). To this end, let \(m = m_i = \left(\frac{4}{3}\right)^i\),

\[
r_1 = r_i = \tau_1 R + (\tau - \tau_1) R 2^{-i+1}, \quad r_2 = r_{i+1},
\]

\[
t_1 = t_i = -\gamma_1 R^2 - (\gamma - \gamma_1) R^2 4^{-i+1}, \quad t_2 = t_{i+1},
\]

where \(i = 1, 2, \ldots\). Then, we can derive from (2.3) the following inequality

\[
G_{i+1} \leq \left( \frac{c2^{i+1}}{\tau - \tau_1} \right)^\frac{1}{m_i} \left( 1 + \frac{\tau - \tau_1}{\sqrt{\gamma - \gamma_1}} + \overline{M}(r_i, t_i) + \frac{2^{(i+1)\frac{2}{5}}}{(\tau - \tau_1)^\frac{5}{2}} \right)^\frac{1}{m_i} G_i,
\]

(2.4)

where

\[
G_i = \left( \frac{1}{|t_i|r_i^2} \right)^\frac{1}{m_i} \left( \int_{t_i}^{0} \int_{C(r_i)} |\sigma|^\frac{5m_i}{3} \, dz \right)^\frac{3}{5m_i}.
\]

Noticing that

\[
\overline{M}(r_i, t_i) \leq c \left( \frac{1}{\gamma_1 \tau_1^2} \right)^\frac{1}{m} M(2R),
\]

let us make use of (2.4) to obtain the estimate

\[
G_{i+1} \leq \left( \frac{c2^{i+1}}{\tau - \tau_1} \right)^\frac{1}{m_i} \left( 1 + \frac{\tau - \tau_1}{\sqrt{\gamma - \gamma_1}} + \frac{2^{(i+1)\frac{2}{5}}}{(\tau - \tau_1)^\frac{5}{2}} + \left( \frac{1}{\gamma_1 \tau_1^2} \right)^\frac{1}{m} M(2R) \right)^\frac{1}{m_i} G_i,
\]

(2.5)

which, after iterations, gives the following

\[
G_{i+1} \leq \xi_i G_1,
\]

(2.6)
where
\[
\xi_i = \prod_{k=1}^{i} \left( \frac{c^{2k+1}}{\tau - \tau_1} \right)^{\frac{1}{m_k}} \left( 1 + \frac{\tau - \tau_1}{\sqrt{\gamma - \gamma_1}} + \frac{2^{(k+1)\frac{7}{\sigma}}}{(\tau - \tau_1)^{\frac{7}{\sigma}}} \right) \left( \frac{1}{\gamma_1 \tau_1^2} \right)^{\frac{1}{10}} M(2R) \right)^{\frac{1}{m_k}}.
\]

Obviously,
\[
\xi_i \leq \prod_{k=1}^{i} \left( \frac{c^{2k+1}}{\tau - \tau_1} \right)^{\frac{1}{m_k}} \left( 1 + \frac{2^{(k+1)\frac{7}{\sigma}}}{(\tau - \tau_1)^{\frac{7}{\sigma}}} \right)^{\frac{1}{m_k}} \left( 1 + \frac{\tau - \tau_1}{\sqrt{\gamma - \gamma_1}} + \frac{1}{\gamma_1 \tau_1^2} M(2R) \right)^{\frac{1}{m_k}}.
\]

Next,
\[
\ln \xi_i \leq A_1 + A_2 + A_3,
\]
where
\[
A_1 = \sum_{k=1}^{i} \frac{1}{m_k} \left( \ln c + (k+1) \ln 2 - \ln (\tau - \tau_1) \right) \leq \ln c - 3 \ln (\tau - \tau_1),
\]
\[
A_2 = \sum_{k=1}^{i} \frac{1}{m_k} \ln \left( 1 + \frac{2^{(k+1)\frac{7}{\sigma}}}{(\tau - \tau_1)^{\frac{7}{\sigma}}} \right) = \sum_{k=1}^{i} \frac{1}{m_k} \ln \left( \frac{2^{(k+1)\frac{7}{\sigma}}}{(\tau - \tau_1)^{\frac{7}{\sigma}}} \right) + \frac{1}{m_k} \ln \left( 1 + \frac{(\tau - \tau_1)^{\frac{7}{\sigma}}}{2^{(k+1)\frac{7}{\sigma}}} \right) \leq \ln \frac{c}{(\tau - \tau_1)^{\frac{7}{\sigma}}} + (\tau - \tau_1)^{\frac{7}{\sigma}} \sum_{k=1}^{i} \frac{1}{m_k} \frac{1}{2^{(k+1)\frac{7}{\sigma}}} \leq \ln \frac{c}{(\tau - \tau_1)^{\frac{7}{\sigma}}},
\]
and
\[
A_3 = \ln \left( 1 + \frac{\tau - \tau_1}{\sqrt{\gamma - \gamma_1}} + \frac{1}{\gamma_1 \tau_1^2} M(2R) \right) \sum_{k=1}^{i} \frac{1}{m_k} \leq \ln \left( 1 + \frac{\tau - \tau_1}{\sqrt{\gamma - \gamma_1}} + \frac{1}{\gamma_1 \tau_1^2} M(2R) \right)^{\frac{1}{10}}.
\]
So,
\[
\xi_i \leq \frac{c}{(\tau - \tau_1)^{\frac{7}{\sigma}}} \left( 1 + \frac{\tau - \tau_1}{\sqrt{\gamma - \gamma_1}} + \frac{1}{\gamma_1 \tau_1^2} M(2R) \right)^{\frac{1}{m_k}}.
\]
Passing to the limit as \(i \to \infty\) in (2.6), we complete the proof the Proposition. \(\square\)
Remark 2.3. If we additionally assume that \( \pi(\cdot, -\theta R^2) \geq k \) in \( B \) for some \( 0 < \theta \leq 1 \), then we do not need to use a cut-off in \( t \). So, for \( 0 < \lambda < 1 \), we have
\[
\sup_{Q^{\lambda, \theta}(R)} \sigma \leq c'_1(\lambda, M(2R)) \left( \frac{1}{|Q^{1, \theta}(R)|} \int_{Q^{1, \theta}(R)} \sigma^{\frac{10}{3}} \, dz \right)^{\frac{3}{10}},
\]
where
\[
c'_1(\lambda, M(2R)) = \frac{c}{(1 - \lambda)^{10}} \left( 1 + \left( \frac{1}{\theta \lambda^3} \right)^{\frac{1}{10}} M(2R) \right)^{3}.
\]

Corollary 2.4. Let a non-negative function \( \pi \in \mathcal{V}_0 \) have the property \((\mathcal{B}_R)\) in \( Q(2R) \) and let \( 0 < \lambda_1 < \lambda < 2 \) and \( 0 < \theta \leq 1 \). Suppose that
\[
|\{ \pi < k \} \cap Q^{\lambda, \theta}((0, t_0), R)| < \mu |\mathcal{C}(2R)|
\]
for some \( t_0 > -4R^2 \), for some \( 0 < k \leq k_R \), and for some
\[
0 < \mu \leq \mu_* = \left( \frac{1}{2c_1(\lambda_1, \lambda, \theta/2, \theta, M(2R))} \right)^{\frac{10}{3}}.
\]
Then \( \pi \geq \frac{k}{2} \) in \( Q^{\lambda_1, \theta/2}((0, t_0), R) \).

If, in addition, \( \pi(\cdot, t_0 - \theta R^2) > k \) in \( \mathcal{C}(\lambda R) \), then \( \pi \geq \frac{k}{2} \) in \( Q^{\lambda_1, \theta}((0, t_0), R) \).

Proof. The first statement can be proved ad absurdum with the help of inequality (2.2) and a suitable choice of the number \( \mu_* \). The second statement is proved in the same way but with the help of the inequality of Remark 2.3. Number \( \mu_* \) is defined by the constant \( c'_1 \) instead of \( c_1 \).

The two lemmas below are obvious modifications of the corresponding statements in the paper \([17]\).

Lemma 2.5. Let \( 0 \leq \pi \in \mathcal{V}_0 \) have the property \((\mathcal{B}_R)\) in \( Q(2R) \). Given \( \delta_0 \in [0, 1] \), there exists a positive number \( \theta_0(\delta_0, f(2R)) \leq 1 \) such that if, for \( 0 < \theta \leq \theta_0 \), \( 0 < k_0 \leq k_R \), there holds
\[
|\{ \pi(\cdot, t_0 - \theta R^2) \geq k_0 \} \cap \mathcal{C}(R)| > \delta_0 |\mathcal{C}(R)|,
\]
then
\[
|\{ \pi(\cdot, t) \geq \frac{\delta_0}{3} k_0 \} \cap \mathcal{C}(R)| > \frac{\delta_0}{3} |\mathcal{C}(R)|
\]
for all \( t \in [t_0 - \theta R^2, t_0] \).
Remark 2.6. There is a formula for $\theta_0$:

$$\theta_0 = \left( \frac{c_0 \delta_0}{1 + \delta_0 f(2R)} \right)^{\frac{4}{3}}.$$

Lemma 2.7. Let $0 \leq \pi \in V_0$ have the property $(B_R)$ in $Q(2R)$. Let, for any $t \in [t_0 - \theta_1 R^2, t_0]$, $\delta_0$:

$$\left\{ \pi(\cdot, t) \geq k_1 \cap C(R) \right\} \geq \delta_1 |C(R)|$$

for some $0 < k_1 \leq k_R$ and for some $0 < \delta_1 \leq 1$ and $0 < \theta_1 \leq 1$.

Then, for any $\mu_1 \in ]0, 1[$, the following inequality is valid:

$$|\{ \pi < 2^{-s} k_1 \cap Q^{1,\theta_1}((0, t_0), R) \} | \leq \mu_1 |Q^{1,\theta_1}(R)|$$

with the integer number $s$ defined as

$$s = \text{entier} \left( \frac{c}{\delta_1^2 \mu_1 \theta_1} (1 + f(2R)) \right) + 1.$$

Given $\theta \in ]0, 1[$, we can find an number $0 < R_\ast(c, \alpha, \theta) \leq 1$ so that

$$\left( \frac{1}{\pi g(2r)} \right)^{\frac{4}{3}} \leq \theta \text{ for all } 0 < r \leq R_\ast.$$

Corollary 2.8. Let $0 \leq \pi \in V_0$ have the property $(B_R)$ in $Q(2R)$. If $\pi(\cdot, t) \geq k_2$ in $C(R)$, then, for any $\sigma \in ]0, 1[$, the inequality $\pi \geq \beta_2 k_2$ holds in $Q^{\sigma, \theta_0}((0, t_0), R)$, where

$$\beta_2 = \frac{1}{6} 2^{-c(1-\sigma)} - 40 \sigma - 6g^2(2R)$$

provided $R \leq R_\ast$.

Proof. We apply Lemma 2.5 with $\delta_0 = \delta_2 = 1$ and $k_0 = k_2$. Then, for $\sigma = 4/(27c)$, we calculate

$$\theta_0 = \left( \frac{4 \pi \sigma}{1 + f(2R)} \right)^{\frac{4}{3}} \geq \left( \frac{c}{g(2R)} \right)^{\frac{4}{3}}$$

and state that the following inequality holds:

$$|\{ \pi(\cdot, t) \geq \frac{k_0}{3} \cap C(R) \} | \geq \frac{1}{3} |C(R)|$$

and state that the following inequality holds:
for any $t \in [t_0 - \theta_0 R^2, t_0]$, where $t_0 = \bar{t} + \theta_0 R^2$. In what follows, we are going to use the quantity $(c/(g(2R)))^{\frac{3}{2}}$ as a new number $\theta_0$ instead of $\theta_0(1, f(2R))$.

Now, we are going to apply Lemma 2.7 with another set of parameters $k_1 = \frac{1}{3} k_2$, $\theta_1 = \theta_0$, $\delta_1 = \frac{1}{3}$, and

$$\mu_1 = \mu_* = \left( \frac{1}{2c'_1} \right)^{\frac{10}{3}}, \quad c'_1 = \frac{c}{(1 - \sigma)^{\frac{10}{3}}} \left( 1 + \left( \frac{1}{\theta_0 \sigma^3} \right)^{\frac{1}{10}} M(2R) \right)^{\frac{3}{5}} \leq \frac{1}{(1 - \alpha)^{\frac{10}{3}}} \left( \frac{1}{\theta_0 \sigma^3} \right)^{\frac{3}{10}} g^3(2R).$$

Lemma (2.7) gives us:

$$|\{ \pi < 2^{-s} k_1 \} \cap Q^{1, \theta_1}((0, t_0), R)| < \mu_1 |Q^{1, \theta_1}(R)|,$$

where

$$s = \text{entier} \left( \frac{c}{\delta_1^2 \mu_1 \theta_1} (1 + f(2R)) \right) + 1.$$

But we know that

$$\pi(\cdot, t_0 - \theta_0 R^2) \geq k_2 > 2^{-s} k_1 = 2^{-s} \frac{k_2}{3}.$$ 

Then, from Corollary 2.4 it follows that $\pi > \frac{1}{2} 2^{-s} k_1 = \beta_2 k_2$ with $\beta_2 = \frac{1}{2} 2^{-s} \frac{1}{3}$ in $Q^{\sigma, \theta_0}((0, t_0), R)$.

\[ \square \]

**Lemma 2.9.** Let $0 \leq \pi \in \mathcal{V}_0$ have the property $(\mathcal{B}_R)$ in $Q(2R)$, assuming that $R \leq R_*(c_*, \alpha, \theta)$ for some $0 < \theta \leq 1$. Suppose further that, for some $0 < k \leq k_R$ and for some $-R^2 \leq \bar{t} \leq -\theta R^2$, there holds $\pi(\cdot, \bar{t}) \geq k$ in $C(R)$. Then $\pi \geq \beta_0 k$ in $\hat{Q} := C(\frac{2}{3} R) \times [\bar{t}, 0]$, where

$$\beta_0 \geq \ln^{-\frac{1}{2}}(1/R)$$

for $R \leq R_*(c_*, \alpha, \theta)$.

**Proof.** Let

$$N = \text{entier} \left( \frac{9}{8 \theta_0 R^2} |\bar{t}| \right) + 1,$$
where $\tilde{\theta}_0 = (c/g(\frac{2}{3}2R))^\frac{1}{4} \leq \theta$. Next, we introduce

$$\hat{\theta}_0 = \frac{\overline{t}}{\left(\frac{8N}{9} + \frac{1}{2N}\right)R^2} \leq \tilde{\theta}_0.$$

**Step 1.** By Corollary 2.8, the inequality $\pi \geq \beta_2^{(1)}k$ holds at least in $C((1 - \frac{1}{3N})R) \times [\overline{t}_1, \overline{t}_1 + \hat{\theta}_0R^2]$, where $\overline{t}_1 = \overline{t}$, $\overline{t}_2 = \overline{t}_1 + \hat{\theta}_0R^2$, $\sigma = 1 - 1/(3N) \geq 2/3$, $1 - \sigma = 1/(3N)$, and

$$\ln \beta_2^{(1)} = -\ln 6 - cN^{40}g^{25}(2R).$$

**Step 2.** Here, we are going to use Corollary 2.8 with $R(1 - 1/(3N))$ instead of $R$ and with $\sigma = (1 - 2(3N))/(1 - 1/(3N))$. As a result, we have the estimate

$$\pi \geq \beta_2^{(2)}\beta_2^{(1)}k$$

at least in $C((1 - 2/(3N))R) \times [\overline{t}_2, \overline{t}_2 + \hat{\theta}_0(1 - 1/(3N))^2R^2]$, $\overline{t}_3 = \overline{t}_2 + \hat{\theta}_0(1 - 1/(3N))^2R^2$, and

$$\ln \beta_2^{(2)} = -\ln 6 - cN^{40}g^{25}(2 - 1/(3N))R).$$

So, $\pi \geq \beta_2^{(2)}\beta_2^{(1)}k$ in $C((1 - 2(3N))R) \times [\overline{t}, \overline{t}_3]$. After $N$ steps, we shall have $\overline{t}_N = 0$ and

$$\pi \geq \beta_2^{(N)}...\beta_2^{(1)}k = \beta_0(R)k$$

in $C(\frac{2}{3}R) \times [0, 0]$, where

$$\ln \beta_2^{(i+1)} = -\ln 6 - cN^{40}g^{25}(2(1 - i/(3N))R)$$

for $i = 0, 1, ..., N - 1$.

Next, according to assumption (1.2), we can have

$$\ln \beta_0 \geq -N \ln 6 - cN^{40} \sum_{k=1}^{N-1} c_+^{25} \ln^\gamma \ln^\frac{1}{\gamma} \left(\frac{1}{2(1 - i/(3N))R}\right),$$

where $25\alpha < 1$. Since

$$\ln \frac{1}{1 - x} \leq 2x$$

provided $0 \leq x \leq 1/2$, we find, assuming that $R \leq 1/6$, the following:

$$\ln^\gamma \ln^\frac{1}{\gamma} \left(\frac{1}{2(1 - i/(3N))R}\right) \leq \ln^\gamma \left(\ln \frac{1}{2R} + \frac{i}{N}\right)^\frac{1}{\gamma}. $$
\[
\leq \ln \gamma \left( \ln \frac{\frac{1}{2R}}{\ln N} + \left( \frac{i}{N} \right)^{\frac{1}{2}} \right) = \ln \gamma \left( \ln \frac{\frac{1}{2R}}{\ln N} + \left( \frac{i}{N} \ln \frac{2}{2R} \right) \right) \leq \\
\leq \ln \gamma \left( \ln \frac{\frac{1}{2}}{2R} \left( 1 + \left( \frac{i}{N} \right)^{\frac{1}{2}} \right) = \left( \ln \left( \ln \frac{\frac{1}{2}}{2R} \right) + \ln \left( 1 + \left( \frac{i}{N} \right)^{\frac{1}{2}} \right) \right) \right) \gamma \leq \\
\leq \left( \ln \left( \ln \frac{\frac{1}{2}}{2R} \right) + \left( \frac{i}{N} \right)^{\frac{1}{2}} \right) \gamma \leq \ln \gamma \left( \ln \frac{\frac{1}{2}}{2R} \right) + \left( \frac{i}{N} \right)^{\frac{1}{2}}. 
\]

From the latter inequality, one can deduce the bound

\[
\ln \beta_0 \geq -N \ln 6 - cc^2_\gamma N^{40} \left( N \ln \gamma \ln \frac{\frac{1}{2}}{2R} + \sum_{i=0}^{N-1} \left( \frac{i}{N} \right)^{\frac{1}{2}} \right) \geq \\
\geq -N \ln 6 - cc^2_\gamma N^{41} \ln \gamma \ln \frac{\frac{1}{2}}{2R},
\]

which is valid for \( 0 < R \leq R_{s3}(\alpha) \leq 1/6 \). Taking into account that \( N \leq c(g(2R))^4 \), we conclude

\[
\ln \beta_0 \geq -c_1(c_\gamma) \frac{239 \alpha}{3} \sqrt{\ln \frac{1}{R}}.
\]

It remains to find \( R_{s4}(c_s, \alpha) \leq 1 \) such that

\[
c_1(c_\gamma) \frac{239 \alpha}{3} \sqrt{\ln \frac{1}{R}} \leq 1
\]

for all \( 0 < R \leq R_{s4} \). So, we have the required inequality provided \( 0 < R \leq R_{s2} = \min\{R_{s1}, R_{s3}, R_{s4}\} \). \( \square \)

### 3 Proof of Proposition 1.4

Now, we can state an analog of Lemma 4.2 of [17] for the class \( \mathcal{V} \).

**Lemma 3.1.** Let \( 0 \leq \pi \in \mathcal{V}_0 \) possess the property \( (B_R) \) in \( Q(2R) \).

Suppose further that

\[
\pi \leq M_0 k_R \tag{3.1}
\]

in \( Q(2R) \) for some \( M_0 \geq 1 \). Then, there exists \( \bar{t} \in [-R^2, -\frac{3}{4}R^2] \) such that

\[
|e_{\pi_0}(\bar{t})| \geq \delta_0 |B(R)| \tag{3.2}
\]
Here, \( \kappa_0 = \kappa_0(f(2R)) = c/(1 + f(2R)) \), \( e_\kappa(t) := \{ x \in C(R) : \pi(x, t) \geq \kappa k_R \} \), and
\[
\delta_0(M_0, f(2R)) = \left( \frac{c}{M_0(1 + f(2R))} \right)^{\frac{3}{2}}.
\]

Proof. Here, we follow arguments of the paper [17]. They are based on the identity:
\[
\int_Q \left( -\pi \partial_t \eta - \pi \Delta \eta - (v + 2x'/|x'|^2) \cdot \nabla \eta \pi \right) dx dt =
\]
\[
= 4\pi_0 \int_{-1}^1 \int_{-1}^1 \pi(0, x_3, t) \eta(0, x_3, t) dx_3 dt,
\]
which is valid for any non-negative test function \( \eta \) supported in \( Q \). Here, \( \pi_0 = 3.14... \). Although a similar statement has been proven in [17] under the assumption that \( \pi \) is Lipschitz, it remains to be true for functions \( \pi \) from the class \( V_0 \) as well. Indeed, take a smooth cut-off function \( \psi = \psi(x') \) so that \( \psi(x') = \Psi(|x'|), 0 \leq \psi \leq 1, \psi(x') = 0 \) if \( |x'| \leq \varepsilon/2, \psi(x') = 1 \) if \( |x'| \geq \varepsilon, \Psi'(\varrho) \leq c/\varrho \) and \( \Psi''(\varrho) \leq c/\varrho^2 \) for some positive constant \( c \). Then, it follows from (2.1) that:
\[
\int_Q \left( \pi \partial_t(\eta \psi) + \pi(u + b) \cdot \nabla(\eta \psi) + \pi \Delta(\eta \psi) \right) dz = 0.
\]

There are two difficult terms for passing to the limit as \( \varepsilon \to 0 \). The first one is as follows:
\[
I_1 := \int_Q \pi \eta \Delta \psi dx dt = J_1 + J_2,
\]
where
\[
J_1 := \int_Q (\pi \eta - (\pi \eta)|_{x'=0}) \Delta \psi dx dt,
\]

For \( J_2 \), we find
\[
J_2 := \int_Q (\pi \eta)|_{x'=0} \Delta \psi dx dt = \int_{-1}^1 \int_{-1}^1 (\pi \eta)|_{x'=0} dx_3 dt \int_{|x'|<1} \Delta \psi(x') dx'
\]
and
\[ \int_{|x'|<1} \Delta \psi(x') \, dx' = 2\pi_0 \int_\Omega \frac{1}{\varrho} \frac{\partial}{\partial \varrho} \left( \varrho \Psi'(\varrho) \right) \, d\varrho = 2\pi_0 \varrho \Psi'(\varrho) \bigg|^{\varrho=\frac{\varepsilon}{2}} = 0. \]

Now, we wish to show that
\[ J_1 := \int_\Omega \xi \Delta \psi \, dx \, dt \to 0 \]
as \( \varepsilon \to 0 \), where, \( \xi := \pi \eta - (\pi \eta)|_{x'=0} \). To this end, let us introduce the function
\[ H_\varepsilon(x_3, t) := \int_{\frac{\varepsilon}{2} < \varrho < \varepsilon} \xi \Delta \psi \, dx'. \]

It can be bounded from above and from below
\[ |H_\varepsilon(x_3, t)| \leq c \sup_{spt \eta} \pi \sup_{|x'|<1} (\pi \eta)(x', x_3, t) \frac{1}{\varepsilon^2} \int_{\frac{\varepsilon}{2}}^{\varepsilon} \varrho \, d\varrho =: h(x_3, t) \]
provided \( \varepsilon < 1 \). The function \( h \) is supported in \( ]-1, 1[ \times ]-1, 0[ \) and thus
\[ \int_{-1}^{1} \int_{-1}^{0} h(x_3, t) \, dx_3 \, dt < \infty. \]

Now, let \( (0, x_3, t) \) be a regular point of \( \pi \), i.e., \( (0, x_3, t) \notin S^\pi \). Then, \( \xi(x', x_3, t) \to 0 \) as \( |x'| \to 0 \) and thus for any \( \delta > 0 \) there exists a number \( \tau(x_3, t) > 0 \) such that \( |\xi(x', x_3, t)| < \delta \) provided \( |x'| < \tau \). So,
\[ |H_\varepsilon(x_3, t)| < c \frac{\delta}{\varepsilon^2} \int_{\frac{\varepsilon}{2}}^{\varepsilon} \varrho \, d\varrho = c \frac{\delta}{2} \]
provided \( \varepsilon < \tau \). Therefore, \( H_\varepsilon(x_3, t) \to 0 \) as \( \varepsilon \to 0 \) and by the Lebesgue theorem on dominated convergence, we find that
\[ J_1 = \int_{-1}^{1} \int_{-1}^{0} H_\varepsilon(x_3, t) \, dx_3 \, dt \to 0 \]
as $\varepsilon \to 0$.

Similar arguments work for the second difficult term:

$$I := \int_Q \pi \eta b \cdot \nabla \psi dz = J_1 + J_2,$$

where

$$J_1 = \int_Q \xi b \cdot \nabla \psi dz$$

and

$$J_2 := \int_Q (\pi \eta)_{x_3=0} b \cdot \nabla \psi dxdt = \int_{-1}^{0} \int_{-1}^{1} (\pi \eta)_{x_3=0} dx_3 dt \pi_0 \int_{\frac{\varepsilon}{2}}^{e} \frac{2}{q} \Psi'(q) q dq =$$

$$= 4 \pi_0 \int_{-1}^{0} \int_{-1}^{1} (\pi \eta)_{x_3=0} dx_3 dt.$$

The fact that $J_1 \to 0$ as $\varepsilon \to 0$ can be justified in the same way as above, replacing $H_\varepsilon$ with the function

$$G_\varepsilon(x_3, t) := \int_{\frac{\varepsilon}{2} < |x'| < \varepsilon} \xi b \cdot \nabla \psi dx'.$$

Other terms can be treated in a similar way and even easier. So, the required identity (3.3) has been proven.

Now, let us select the test function $\eta$ in (3.3), using the following notation

$$Q^{\lambda, \theta}(z_0, R) := C(x_0, \lambda R) \times]t_0 - \theta R^2, t_0[,$$

so that $\eta = 1$ in $Q^{\frac{1}{4}, \frac{1}{2}}((0, -\frac{12}{10} R^2), R)$, $\eta = 0$ out of $Q^{1, \frac{4}{3}}((0, -\frac{3}{10} R^2), R)$ and $|\partial \eta| + |\nabla \eta|^2 + |\nabla^2 \eta| \leq c/R^2$. Taking into account that $\pi$ has the property (B.R), we find

$$\frac{\pi_0}{2} k_R R^2 \leq \frac{c}{R^2} \int_{Q^{1, \frac{4}{3}}(z_R, R)} \pi dz + \frac{c}{R} \int_{Q^{1, \frac{4}{3}}(z_R, R)} \pi |v| dz + \frac{c}{R} \int_{Q^{1, \frac{4}{3}}(z_R, R)} \pi \frac{1}{|x'|} dz.$$
where $z = (0, -\frac{3}{4}R^2)$.

Setting $E_\kappa = \{(x, t) : t \in [-R^2, -\frac{3}{4}R^2], x \in e_\kappa(t)\}$, we can deduce from the latter inequality

$$\frac{\pi_0}{2} k_R R^3 \leq \frac{c}{R^2} \int_{Q^{1,\frac{\kappa}{2}(z, R)\setminus E_\kappa}} \pi dz + \frac{c}{R^2} \int_{Q^{1,\frac{\kappa}{2}(z, R)\setminus E_\kappa}} \pi |v| dz + \frac{c}{R^2} \int_{Q^{1,\frac{\kappa}{2}(z, R)\setminus E_\kappa}} \frac{\pi |x|}{|x'|} dz\]

+ \frac{c}{R^2} \int_{Q^{1,\frac{\kappa}{2}(z, R)\cap E_\kappa}} \pi dz + \frac{c}{R^2} \int_{Q^{1,\frac{\kappa}{2}(z, R)\cap E_\kappa}} \pi |v| dz + \frac{c}{R^2} \int_{Q^{1,\frac{\kappa}{2}(z, R)\cap E_\kappa}} \frac{\pi |x|}{|x'|} dz.

Applying (3.1) and recalling definitions of the sets $e_\kappa(t)$ and $E_\kappa$, we can get

$$\frac{\pi_0}{2} k_R R^3 \leq \frac{c\kappa k_R}{R^2} \left\{ |Q^{1,\frac{\kappa}{2}}(R)| + R \int_{Q^{1,\frac{\kappa}{2}(z, R)\setminus E_\kappa}} |v| dz + R \int_{Q^{1,\frac{\kappa}{2}(z, R)\setminus E_\kappa}} \frac{1}{|x'|} dz \right\} +

\frac{cM_0 k_R}{R^2} \left\{ |E_\kappa| + R \int_{Q^{1,\frac{\kappa}{2}(z, R)\cap E_\kappa}} |v| dz + R \int_{Q^{1,\frac{\kappa}{2}(z, R)\cap E_\kappa}} \frac{1}{|x'|} dz \right\}.

We need to estimate integrals in the above inequality. First, for integrals, containing $v$, Holder inequality gives

$$\int_{Q^{1,\frac{\kappa}{2}(z, R)\setminus E_\kappa}} |v| dx \leq \|v\|_{\frac{3}{2}, \frac{4}{\kappa}, Q^{1,\frac{\kappa}{2}}(R)} \left( \int_{-R^2}^{0} \left( \int_{C(R)} |v|^3 dx \right)^{\frac{4}{3}} dt \right)^{\frac{1}{4}} \leq

f(2R) R^{\frac{1}{2}} \|v\|_{\frac{3}{2}, \frac{4}{\kappa}, Q^{1,\frac{\kappa}{2}}(R)} \leq f(2R) R^{4}

and similarly

$$\int_{Q^{1,\frac{\kappa}{2}(z, R)\cap E_\kappa}} |v| dz \leq f(2R) R^{\frac{1}{2}} \|v\|_{\frac{3}{2}, \frac{4}{\kappa}, E_\kappa}.

To evaluate the last two integrals, let us take into account the fact:

$$\frac{1}{|x'|} \in L^{2, \infty}(Q^{1,\frac{\kappa}{2}}(z, R)).$$
Then,
\[ \int_{Q^{1/4}(z_R,R) \setminus E_\kappa} \frac{1}{|x'|} \, dz \leq \| \frac{1}{|x'|} \|_{L_\infty} \| Q^{1/4}(z_R,R) \|_{L_{3/4},1,1} \| \kappa \|_{L_{3/4},1,1,1} \leq \]
\[ \leq c R^{3/4} \| \kappa \|_{L_{3/4},1,1,1} \]
\[ \leq c R^4 \]
and
\[ \int_{Q^{1/4}(z_R,R) \cap E_\kappa} \frac{1}{|x|} \, dz \leq c R^{3/4} \| \kappa \|_{L_{3/4},1,E_\kappa}. \]

Hence, we have
\[ \frac{\pi_0}{2} k_R R^3 \leq c \kappa k_R R^3 (1 + f(2R)) + \]
\[ + c M_0 k_R \left[ |E_\kappa| + f(2R) R^{3/4} \| \kappa \|_{L_{3/4},1,E_\kappa} + R^{5/4} \| \kappa \|_{L_{3/4},1,E_\kappa} \right]. \]
So,
\[ \frac{\pi_0}{2} \leq c \kappa (1 + f(2R)) + \frac{c M_0}{R^5} \left[ |E_\kappa| + f(2R) R^{3/4} \| \kappa \|_{L_{3/4},1,E_\kappa} + R^{5/4} \| \kappa \|_{L_{3/4},1,E_\kappa} \right]. \]

Now, one can find \( \kappa = \kappa_0(f(2R)) = c/(1 + f(2R)) \) such that
\[ \frac{c M_0}{R^5} \left[ |E_{\kappa_0}| + f(2R) R^{3/4} \| \kappa \|_{L_{3/4},1,E_{\kappa_0}} + R^{5/4} \| \kappa \|_{L_{3/4},1,E_{\kappa_0}} \right] \geq 1. \]

It remains to estimate two integrals on the left hand side of the latter inequality:
\[ \| \kappa \|_{L_{3/4},1,E_{\kappa_0}} = \left( \int_{-R^2}^{R^2} |\kappa(t)|^{3/4} \, dt \right)^{\frac{4}{3}} \leq c |E_{\kappa_0}|^{\frac{3}{4}} R^{\frac{3}{4}} \]
and
\[ \| \kappa \|_{L_{3/4},1,E_\kappa} \leq c |E_{\kappa_0}|^{\frac{3}{4}} R^{\frac{10}{3}}. \]

Letting \( A = |E_{\kappa_0}|/R^5 \), we arrive at the following inequality
\[ f(A) := A + A^{3/4} + f(2R) A^{3/2} \geq \frac{1}{c M_0}. \]
Since $f'(A) > 0$ for $A > 0$, we can state that the last inequality implies
\[
\frac{|E_{\kappa_0}|}{|C(R)|\frac{1}{4}R^2} \geq \delta_0 = \left(\frac{c}{M_0(1 + f(2R))}\right)^{\frac{2}{3}}.
\]

It is not so difficult to show the existence of $\overline{t} \in [-R^2, -\frac{3}{4}R^2]$ with the property:
\[
|e_{\kappa_0}(\overline{t})|\frac{1}{4}R^2 \geq |E_{\kappa_0}|.
\]
So, it is proven that there exists $\overline{t} \in [-R^2, -\frac{3}{4}R^2]$ such that
\[
|\{x \in C(R) : \pi(x, \overline{t}) > \kappa_0k_R\}| \geq \delta_0|C(R)|, \tag{3.4}
\]
which completes the proof of the lemma. \(\square\)

Now, we are able to prove Proposition 1.4.

Assume that the function $\pi$ meets all the conditions of Lemma 3.1 and according to it, we can claim that:
\[
|e_{\kappa_0}(\overline{t})| = |\{x \in C(R) : \pi(x, \overline{t}) \geq \kappa_0k_R\}| \geq \delta_0|C(R)|
\]
for some $\overline{t} \in [-R^2, -\frac{3}{4}R^2]$, $\kappa_0 = c/g(2R)$, and $\delta_0 = c(M_0)/g^{\frac{2}{3}}(2R)$. Now, we can calculate
\[
\theta(\delta_0(M_0, f(2R)), f(2R)) \geq c\left(\frac{\delta_0}{1 + \frac{\delta_0^2}{\delta_0}f(2R)}\right)^{\frac{4}{3}} \geq c(M_0)\left(\frac{1}{g(2R)}\right)^{18},
\]
apply Lemma 2.5, and find
\[
|\{\pi(\cdot, t) \geq \delta_0\kappa_0k_R/3\} \cap C(R)| > \delta_0|C(R)|
\]
for all $t \in [\overline{t}, t_0]$ with $t_0 = \overline{t} + \theta_0R^2$ and $\theta_0 = c(M_0)(g(2R))^{-18}$.

Next, it follows from Lemma 2.7 that:
\[
|\{\pi < 2^{-s}\delta_0\kappa_0k_R/3\} \cap Q_{1, \theta_0}^{1, \theta_0}(0, t_0, R)| \leq \mu_*|Q_{1, \theta_0}^{1, \theta_0}(R)|,
\]
where
\[
s = \text{entier}\left(\frac{c}{\delta_0^2\mu_*\theta_0}(1 + f(2R))\right) + 1
\]

19
and $\mu_*$ is the number that appears in Corollary 2.4, see also Proposition 2.2.

In our case,

$$\mu_* = \left( \frac{1}{2c_1(3/4, 1, \theta_0/2, M(2R))} \right)^{10/3}$$

and, moreover

$$c_1(3/4, 1, \theta_0/2, M(2R)) \leq c\theta_0^{-\frac{3}{2}} g^3(2R) \leq c(M_0)(g(2R))^{30}.$$  

Then, Corollary 2.4 implies the bound

$$\pi \geq 2^{-s} \delta_0 \kappa_0 k_R/6 = \hat{\beta}_2 \kappa_0 k_R$$

in $Q^{\frac{1}{2}, \theta_0}(0, t_0, R)$. So, combining previous estimates, we find the following:

$$\hat{\beta}_2 = \frac{1}{6} 2^{-s} \delta_0 \geq e^{-s\ln 2 - \ln 6} \delta_0 \geq e^{-cs} \delta_0,$$

where

$$s \leq \frac{2g(2R)}{\delta_0^2 \mu_*^2 \theta_0} \leq c(M_0)g(2R)(g(2R))^{\frac{2}{7}} (g(2R))^{18} c_1^{\frac{20}{3}} \leq c(M_0)(g(2R))^{\frac{24}{7}} (g(2R))^{30} \leq c(M_0)(g(2R))^{224}.$$  

So,

$$\hat{\beta}_2 \geq e^{-c(M_0)(g(2R))^{224}} c(M_0)(g(2R))^{-\frac{9}{7}} \geq e^{-2c(M_0)(g(2R))^{224}} \geq e^{-c(M_0, c_*) \ln^{24\alpha} \sqrt{\ln 2}},$$

Obviously, there exists a number $0 < R_5(M_0, c_*, \alpha) \leq \min\{1/6, R_2\}$ such that

$$2c(M_0, c_*) \ln^{24\alpha - 1} \sqrt{\ln \frac{1}{R}} \leq 1$$

and

$$c(M_0, c_*) \ln^{24\alpha} \sqrt{\ln \frac{1}{R}} \geq \ln \ln \alpha \sqrt{\ln 1}$$

for $0 < R \leq R_5(M_0, c_*, \alpha)$ and thus

$$-c(M_0, c_*) \ln^{24\alpha} \sqrt{\ln \frac{1}{R}} = -2c(M_0, c_*) \ln^{24\alpha} \sqrt{\ln 1} +$$

$$+ c(M_0, c_*) \ln^{24\alpha} \sqrt{\ln \frac{1}{R}} \geq -\ln \sqrt{\ln \frac{1}{R}} + \ln \ln \alpha \sqrt{\ln 1}.$$
Now, the number $\hat{\beta}_2$ is estimated as follows:

$$\hat{\beta}_2 \geq \left( \ln \frac{1}{R} \right)^{-\frac{1}{4}} \ln \ln \sqrt{\ln \frac{1}{R}}$$ \hspace{1cm} (3.5)

for $0 < R \leq R_{s5}(M_0, c_*, \alpha)$.

Since

$$-R^2 \leq \bar{t} + \theta_0/2R^2 = t_0 - \theta_0/2R^2 < t_0 = \bar{t} + \theta_0R^2 \leq -\frac{3}{4}R^2 + \frac{1}{4}R^2 = -\frac{1}{2}R^2,$$

there is $\bar{t}_1 \in [-R^2, -\frac{1}{2}R^2]$ such that

$$\pi(\cdot, \bar{t}_1) > \hat{\beta}_2\kappa_0 k_R$$

in $C(\frac{3}{4}R)$. It allows us to apply Lemma 2.9 with $\theta = 1/2$, with $\frac{3}{4}R$ instead of $R$, with $\bar{t}_1$ instead of $\bar{t}$, and with $\hat{\beta}_2\kappa_0 k_R$ instead of $k$. According to Lemma 2.9 the inequality

$$\pi \geq \beta_0 \hat{\beta}_2\kappa_0 k_R$$

holds in $Q(R/2)$. It follows from Lemma 2.9 and from (3.5) that

$$\pi \geq \frac{c(c_*)k_R}{\ln(1/R)} = \beta(2R)k_R$$

in $Q(R/2)$.

By our assumption imposed on function $\sigma$, we can put $k_R = \frac{1}{2}\text{osc}_{z \in Q(2R)}\sigma(z)$. Then, either $\pi = \sigma - m_{2R}$ or $\pi = M_{2R} - \sigma(z)$ satisfies all the conditions of the proposition with $M_0 = 2$. Simple arguments show that

$$\text{osc}_{z \in Q(R/2)}\sigma(z) \leq \left(1 - \frac{1}{2}\beta(2R)\right)\text{osc}_{z \in Q(2R)}\sigma(z).$$

Now, after iterations of the latter inequality, we arrive at the following bound

$$\text{osc}_{z \in Q(R/2^{2k+1})} \leq \prod_{i=0}^{k} \left(1 - \frac{1}{2}\beta(R/2^{2k+1})\right)\text{osc}_{z \in Q(2R)}\sigma(z) =$$

$$= \eta_k \text{osc}_{z \in Q(2R)}\sigma(z)$$

being valid for any natural number $k$. 

21
In order to evaluate $\eta_k$, take ln of it. As a result,

$$\ln \eta_k = \sum_{i=0}^{k} \ln \left(1 - \frac{1}{2}\beta\left(R/2^{2k+1}\right)\right) \leq - \sum_{i=0}^{k} \frac{1}{2}\beta\left(R/2^{2k+1}\right) =$$

$$= -c(c_*) \sum_{i=0}^{k} (\ln(2^k/R))^{-1} = -c(c_*) \sum_{i=0}^{k} \frac{1}{k \ln 2 + \ln 1/R} \leq$$

$$\leq -c(c_*) \int_{0}^{k+1} \frac{dx}{x \ln 2 + \ln 1/R} =$$

$$= -c(c_*) \left(\ln(2^{k+1}/R) - \ln(1/R)\right) = -c(c_*) \left(\ln(2^{k+1})\right).$$

So, (1.8) follows. The proof of Proposition 1.4 is complete.

4 Proof of Theorem 1.3

By the maximum principle, we have $|\sigma| = |\rho v_\rho| \leq \Sigma_0$ in $\mathbb{R}^3 \times [0,T]$. From Proposition 1.4, it follows that

$$|\sigma(\rho, x_3, t)| \leq C_1(c_*) \left(\frac{\rho}{2R_*}\right)^{C_2(c_*)} 2\Sigma_0$$

for all $0 < \rho \leq R_*(c_*, \alpha)$, for all $x_3 \in \mathbb{R}$, and for $t \in [T - R_*^2, T]$. For $\rho > R_*$, we simply have

$$|\sigma(\rho, x_3, t)| \leq \Sigma_0 \left(\frac{\rho}{R_*}\right)^{C_2(c_*)}.$$

It remains to notice that $v(\cdot, T - R_*^2) \in H^2$. Therefore, one can use the main result of the paper [4], see also [10] and [12], for the Cauchy problem for the Navier-Stokes system (1.1) in $\mathbb{R}^3 \times [T - R_*^2, T]$ and conclude that $v$ is a strong solution in the interval $[0, T]$.

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