Tractability of Multivariate Approximation Defined over Hilbert Spaces with Exponential Weights

Christian Irrgeher*, Peter Kritzer†, Friedrich Pillichshammer‡, Henryk Woźniakowski§

Abstract

We study multivariate approximation defined over tensor product Hilbert spaces. The domain space is a weighted tensor product Hilbert space with exponential weights which depend on two sequences \( a = \{a_j\}_{j \in \mathbb{N}} \) and \( b = \{b_j\}_{j \in \mathbb{N}} \) of positive numbers, and on a bounded sequence of positive integers \( m = \{m_j\}_{j \in \mathbb{N}} \). The sequence \( a \) is non-decreasing and the sequence \( b \) is bounded from below by a positive number. We find necessary and sufficient conditions on \( a, b \) and \( m \) to achieve the standard and new notions of tractability in the worst case setting.

Keywords: Multivariate Approximation, Tractability, Hilbert Spaces with Exponential Weights

2010 MSC: 41A25, 41A63, 65D15, 65Y20

1 Introduction

We approximate \( s \)-variate problems by algorithms that use finitely many linear functionals. The information complexity \( n(\varepsilon, s) \) is defined as the minimal number of linear functionals which are needed to find an approximation to within an error threshold \( \varepsilon \).

The standard notions of tractability deal with the characterization of \( s \)-variate problems for which the information complexity \( n(\varepsilon, s) \) is not exponential in \( \varepsilon^{-1} \) and \( s \). Since there are many different ways of measuring the lack of the exponential dependence we have various notions of tractability. For instance, weak tractability (WT) means that

\[
\log n(\varepsilon, s)/(s + \varepsilon^{-1}) \text{ goes to zero as } s + \varepsilon^{-1} \text{ approaches infinity,}
\]

whereas quasi-polynomial tractability (QPT) means that \( n(\varepsilon, s) \) can be bounded for all \( s \in \mathbb{N} \) and all \( \varepsilon \in (0, 1] \) by

\[
C \exp(t (1 + \log s)(1 + \log \varepsilon^{-1}))
\]

for some \( C \) and \( t \) independent of both \( \varepsilon^{-1} \) and \( s \). Analogously, we have polynomial tractability (PT) if \( n(\varepsilon, s) \) can be bounded by a polynomial in \( \varepsilon^{-1} \) and \( s \), and strong polynomial tractability (SPT) if \( n(\varepsilon, s) \) can be bounded by a
positive integers, the Hilbert space $H$ is a space of functions or that it is a reproducing kernel Hilbert space. Therefore we can given positive numbers $a, b, \omega$ by assuming that $(\varepsilon_1, \varepsilon_2)$ goes to zero as $s + \varepsilon^{-1}$ approaches infinity for some positive $t_1$ and $t_2$. Uniform weak tractability (UWT) was defined in [17] by assuming that $(t_1, t_2)$-WT holds for all $t_1, t_2 \in (0, 1]$. It is easy to check that for $t_1, t_2 \in (0, 1]$ we have the following hierarchy

$$\text{SPT} \Rightarrow \text{PT} \Rightarrow \text{QPT} \Rightarrow \text{UWT} \Rightarrow (t_1, t_2)$$.WT \Rightarrow \text{WT}.$$  

All these standard notions are appropriate for $s$-variate problems for which the minimal errors are polynomially decaying. That is, for any $n \in \mathbb{N}$ we can find $n$ linear functionals and an algorithm using these $n$ linear functionals whose error decays like $O(n^{-p})$ for some positive $p$ and with the factor in the big $O$ notation that may depend on $s$.

There is a stream of work with new notions of tractability which is relevant for $s$-variate problems for which the minimal errors are exponentially decaying, see [2, 3, 8, 10, 11, 15]. The new notions of tractability correspond to the standard notions of tractability but for the pair $(s, 1 + \log \varepsilon^{-1})$ instead of the pair $(s, \varepsilon^{-1})$. For instance the new notion of strong polynomial tractability means that we can bound $n(\varepsilon, s)$ by a polynomial in $1 + \log \varepsilon^{-1}$ for all $s \in \mathbb{N}$. Obviously, the new notions of tractability are more demanding than the standard ones. To distinguish them from the standard notions we add the prefix EC (exponential convergence) and we have EC-WT, EC-UWT, EC-(t_1, t_2)-WT, EC-QPT, EC-PT, and EC-SPT. For $t_1, t_2 \in (0, 1]$, we obviously have

$$\text{EC-SPT} \Rightarrow \text{EC-PT} \Rightarrow \text{EC-QPT} \Rightarrow \text{EC-UWT} \Rightarrow \text{EC-(t_1, t_2)}-\text{WT} \Rightarrow \text{EC-WT}.$$  

We study $(t_1, t_2)$-WT and EC-(t_1, t_2)-WT for general positive $t_1$ and $t_2$, i.e., dropping the assumption that they are from $(0, 1]$. Obviously, if $t_1 > 1$ we do not have an exponential dependence on $s^{t_1}$ but we may have the exponential dependence on $s^\tau$ for $\tau < t_1$. For $\tau = 1$, we may have an exponential dependence on $s$ which is usually called the curse of dimensionality. Nevertheless, the parameters $t_1$ and $t_2$ control the level of exponential behaviour with respect to $s$ and $\varepsilon^{-1}$, and it seems to be an interesting problem to find the minimal, say, $t_1$ for which we have $(t_1, t_2)$-WT or EC-(t_1, t_2)-WT.

In this paper we study all these standard and new notions of tractability. This is done for general multivariate approximation defined over tensor product Hilbert spaces in the worst case setting. The construction of our problem is roughly as follows. For $s = 1$, we take a separable Hilbert space $H$ of infinite dimension with an orthonormal basis $\{e_k\}_{k \in \mathbb{N}_0}$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, and inner product $\langle \cdot, \cdot \rangle_H$. In general, we do not assume that $H$ is a space of functions or that it is a reproducing kernel Hilbert space. Therefore we can only consider linear functionals as information used by algorithms.

From the space $H$, we construct a weighted Hilbert space in the following way. For given positive numbers $a, b, \omega$ with $\omega \in (0, 1)$, and a bounded sequence $m = \{m_k\}_{k \in \mathbb{N}_0}$ of positive integers, the Hilbert space $H_{a,b}$ is a subspace of $H$ for which $f \in H_{a,b}$ iff

$$\|f\|_{H_{a,b}}^2 := \left( \sum_{j=0}^{m_0-1} |\langle f, e_j \rangle_H|^2 + \sum_{k=1}^{\infty} \omega^{-ak^b} \sum_{j=0}^{m_k-1} |\langle f, e_{m_0+\ldots+m_{k-1}+j} \rangle_H|^2 \right)^{1/2} < \infty.$$  

Note that $\omega^{-ak^b}$ goes exponentially fast to infinity with $k$. Therefore, $\|f\|_{H_{a,b}} < \infty$ means that the sum of $|\langle f, e_{m_0+\ldots+m_k+j} \rangle_H|^2$ for $j = 0, 1, \ldots, m_k - 1$ must decay exponentially fast with $k$.  

2
The univariate approximation problem \( APP_1 : H_{a,b} \rightarrow H \) is defined as the embedding operator \( APP_1 f = f \). The \( s \)-variate approximation problem

\[
APP_s : H_{s,a,b} \rightarrow H_s := \bigotimes_{j=1}^{s} H
\]

is the embedding operator \( APP_s f = f \), where

\[
H_{s,a,b} := H_{a_1,b_1} \otimes H_{a_2,b_2} \otimes \cdots \otimes H_{a_s,b_s}
\]

is the \( s \)-fold tensor product of the weighted spaces \( H_{a_j,b_j} \). Here \( a = \{a_j\}_{j \in \mathbb{N}} \) and \( b = \{b_j\}_{j \in \mathbb{N}} \). We assume that \( a_1 > 0 \), the \( a_j \)'s are nondecreasing, and \( \inf_j b_j > 0 \).

The space \( H_{s,a,b} \) is a subset of \( H_s \) with exponentially decaying coefficients in the basis of \( H_s \). The speed of the decay depends on the parameters \( a, b, m \) and \( \omega \) of the problem.

Special instances of the spaces \( H_{s,a,b} \) are weighted Hermite and Korobov spaces which were already analyzed in the papers mentioned before. In fact, similarity in the analysis before does not have to be a space of functions. But even if we assume that \( H_{s,a,b} \) is the embedding operator \( APP \)

functions then the functions \( e \) is important. That is why the results for the space \( H_{s,a,b} \) are similar to the results for the weighted Hermite and Korobov spaces.

We now briefly summarize the main results obtained in this paper. We first study when exponential convergence (EXP) and uniform exponential convergence (UEXP) hold. EXP holds if there is \( q \in (0, 1) \) such that for all \( s \in \mathbb{N} \) we can find positive \( C_s, M_s, p_s \) for which the \( n \)th minimal worst case error for approximating \( APP_s \), see Section 3, is bounded by

\[
C_s q^{(n/M_s)^{p_s}} \quad \text{for all} \quad n \in \mathbb{N}.
\]

The supremum of such \( p_s \) is called the exponent of EXP and denoted by \( p^* \). UEXP holds if we can take \( p_s = p > 0 \) for all \( s \in \mathbb{N} \), and the supremum of such \( p \) is called the exponent of UEXP and denoted by \( p^* \).

We prove that EXP holds always with no extra conditions on the parameters \( a, b, m, \omega \), and \( p^* = 1/\sum_{j=1}^{s} b_j^{-1} \), whereas UEXP holds iff \( B := \sum_{j=1}^{\infty} b_j^{-1} < \infty \) and then \( p^* = 1/B \). Hence, UEXP only requires that \( b_j^{-1} \)'s are summable and there are no extra conditions on the rest of the parameters.

We now turn to tractability. We obtain necessary and sufficient conditions on standard and new notions of tractability in terms of the parameters \( a, b, m \) and \( \omega \) of the problems. Such conditions were not known before even for weighted Hermite or Korobov spaces. More precisely, UWT, QPT, EC-UWT, EC-QPT as well as \((t_1,t_2)\)-WT and EC-\((t_1,t_2)\)-WT were not studied before for the weighted Korobov spaces, and approximation has not been studied at all for Hermite spaces before.
To stress that we approximate APPs, we denote the information complexity $n(\varepsilon, s)$ by $n(\varepsilon, \text{APP}_s)$, see again Section 3. In this paper we present specific lower and upper bounds on $n(\varepsilon, \text{APP}_s)$ from which we conclude various notions of standard and new tractability. We also present estimates of the tractability exponents. They are defined as the infimum of $t$ for QPT and EC-QPT, or the infimum of the degree of polynomials in $\varepsilon^{-1}$ for SPT and in $1 + \log \varepsilon^{-1}$ for EC-SPT which bound the information complexity $n(\varepsilon, \text{APP}_s)$. We usually do not have the exact values of these exponents but only lower and upper bounds. It would be of interest to improve these bounds. In this section, we only mention when various tractability notions hold. We prove:

- $(t_1, t_2)$-WT holds for the parameters $a, b, m$ and $\omega$ iff $t_1 > 1$ or $m_0 = 1$.
- EC-$(t_1, t_2)$-WT holds for the parameters $a, b, m$ and $\omega$ iff $t_1 > 1$, or $t_2 > 1$ and $m_0 = 1$.
- WT holds iff $m_0 = 1$, whereas EC-WT holds iff $m_0 = 1$ and $\lim_{j \to \infty} a_j = \infty$.
- UWT holds iff $m_0 = 1$, whereas EC-UWT holds iff $m_0 = 1$ and $\lim_{j \to \infty} \frac{\log a_j}{\log j} = \infty$.
- QPT holds iff $m_0 = 1$, whereas EC-QPT holds iff $m_0 = 1$, $\sum_{j=1}^{s} \frac{b_j^{-1}}{1 + \log s} < \infty$, and $\liminf_{j \to \infty} (1 + \log j) \frac{\log a_j}{j} > 0$.
- PT holds iff SPT holds iff $m_0 = 1$ and $\lim_{j \to \infty} \frac{a_j}{\log j} > 0$.
- EC-PT holds iff EC-SPT holds iff $m_0 = 1$, $\sum_{j=1}^{\infty} b_j^{-1} < \infty$, and $\liminf_{j \to \infty} \frac{\log a_j}{j} > 0$.

Observe that for $m_0 > 1$, only $(t_1, t_2)$-WT and EC-$(t_1, t_2)$-WT with $t_1 > 1$ hold. The reason is that

$$n(\varepsilon, \text{APP}_s) \geq m_0^s$$

for all $\varepsilon \in (0, 1)$, and we have the curse of dimensionality. This also shows that the condition $t_1 > 1$ is sharp. So we have to assume that $m_0 = 1$ to obtain other notions of tractability in terms of the conditions on $a$ and $b$. Interestingly enough there are no conditions on $m_k$ for $k > 0$ and on $\omega$. However, the exponents of tractability as well as constants depend on $m_k$ for $k > 0$ and on $\omega$. We illustrate the necessary and sufficient conditions on various notions of tractability for $m_0 = 1$ and for

$$a_j = j^{v_1} \exp(v_2 j) \text{ and } b_j = j^{v_3} \text{ for } j \geq 1$$

for some non-negative $v_1, v_2$ and $v_3$. Then

\footnote{Under a simplifying assumption that the limit of $a_j / \log j$ exists.}
• EXP, \((t_1, t_2)\)-WT, EC-\((t_1, t_2)\)-WT with \(t_1 > 1\), WT and QPT hold for all \(v_1, v_2, v_3\),
• UEXP holds iff \(v_3 > 1\),
• EC-WT, PT and SPT hold iff \(v_1^2 + v_2^2 > 0\),
• EC-PT and EC-SPT hold iff \(v_2 > 0\) and \(v_3 > 1\).

The remaining sections of this paper are structured in the following way. We provide detailed information on the Hilbert spaces which are studied in the paper in Section 2. We outline the setting of the approximation problem in Section 3. The results on exponential and uniform exponential convergence are shown in Section 4. In Section 5 we prove the results on the various notions of tractability. A table which summarizes all conditions is presented in Section 6.

2 Weighted Hilbert Spaces

Let \(H\) be a separable Hilbert space over the real or complex field. To omit special cases, we also assume that \(H\) has infinite dimension. Let \(\{e_k\}_{k \in \mathbb{N}_0}\) be its orthonormal basis, \(\langle e_k, e_j \rangle_H = \delta_{k,j}\) for all \(k, j \in \mathbb{N}_0\). Hence, \(f \in H\) iff

\[
f = \sum_{k=0}^{\infty} \langle f, e_k \rangle_H e_k \quad \text{with} \quad \sum_{k=0}^{\infty} |\langle f, e_k \rangle_H|^2 < \infty.
\]

For \(s \in \mathbb{N} := \{1, 2, \ldots\}\), by \(H_s = H \otimes H \otimes \cdots \otimes H\) we mean the \(s\)-fold tensor product of \(H\). For \(k = [k_1, k_2, \ldots, k_s] \in \mathbb{N}_0^s\), let \(e_k = e_{k_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_s}\). Clearly, \(\{e_k\}_{k \in \mathbb{N}_0^s}\) is an orthonormal basis of \(H_s\) and \(f \in H_s\) iff

\[
f = \sum_{k \in \mathbb{N}_0^s} \langle f, e_k \rangle_{H_s} e_k \quad \text{with} \quad \sum_{k \in \mathbb{N}_0^s} |\langle f, e_k \rangle_{H_s}|^2 < \infty.
\]

We now define a weighted Hilbert space which will depend on a number of parameters. Some of these parameters will be fixed while others will be varying. The fixed parameters are: a number \(\omega \in (0, 1)\) and a bounded sequence \(m = \{m_k\}_{k \in \mathbb{N}_0}\) of positive integers. With the sequence \(m\) we associate a sequence \(r = \{r_k\}_{k \in \mathbb{N}_0}\) given by

\[
\begin{align*}
    r_0 &= 0, \\
    r_k &= m_0 + m_1 + \cdots + m_{k-1} \quad \text{for all} \quad k \in \mathbb{N}.
\end{align*}
\]

Clearly, \(r_{k+1} = r_k + m_k \geq r_k + 1\). Furthermore,

\[
\mathbb{N}_0 = \bigcup_{k=0}^{\infty} \{r_k, r_k + 1, \ldots, r_{k+1} - 1\}
\]

and the sets \(\{r_k, r_k + 1, \ldots, r_{k+1} - 1\}\) are disjoint.

The varying parameters are positive real numbers \(a\) and \(b\). The weighted Hilbert space will be therefore denoted by \(H_{a,b}\) and is defined as

\[
H_{a,b} = \left\{ f \in H : \|f\|_{H_{a,b}} := \left( \sum_{k=0}^{\infty} \omega^{-ak^b} \sum_{j=r_k}^{r_{k+1}-1} |\langle f, e_j \rangle_H|^2 \right)^{1/2} < \infty \right\}.
\]
As an example, consider \( m_k \equiv 1 \). Then \( r_k = k \) and

\[
\|f\|_{H_{a,b}} := \left( \sum_{k=0}^{\infty} \omega^{-ak^b} |\langle f, e_k \rangle_H|^2 \right)^{1/2}.
\]

For a general \( m \), note that \( \omega^{-ak^b} \) goes exponentially fast to infinity with \( k \). Therefore \( \|f\|_{H_{a,b}} < \infty \) means that \( \sum_{j=r_k}^{r_{k+1}-1} |\langle f, e_j \rangle_H|^2 \) must decay exponentially fast to zero as \( k \) goes to infinity.

The inner product in \( H_{a,b} \) is given for \( f, g \in H_{a,b} \) by

\[
\langle f, g \rangle_{H_{a,b}} = \sum_{k=0}^{\infty} \omega^{-ak^b} \sum_{j=r_k}^{r_{k+1}-1} \langle f, e_j \rangle_H \langle g, e_j \rangle_H.
\]

Since \( \omega^{-ak^b} \geq 1 \), we have

\[
\|f\|_H \leq \|f\|_{H_{a,b}} \quad \text{for all } f \in H_{a,b}.
\]  \hfill (1)

We now find an orthonormal basis \( \{e_{n,a,b}\}_{n \in \mathbb{N}_0} \) of \( H_{a,b} \). For \( n \in \mathbb{N}_0 \), there is a unique \( k = k(n) \) such that \( n \in \{r_{k(n)}, r_{k(n)}+1, \ldots, r_{k(n)+1} - 1\} \). Then we set

\[
e_{n,a,b} = \omega^{a[k(n)]^{b/2}} e_n.
\]

We now verify that the sequence \( \{e_{n,a,b}\}_{n \in \mathbb{N}_0} \) is orthonormal in \( H_{a,b} \). Indeed, take \( n_1, n_2 \in \mathbb{N}_0 \). Then

\[
\langle e_{n_1,a,b}, e_{n_2,a,b} \rangle_{H_{a,b}} = \sum_{k=0}^{\infty} \omega^{-ak^b} \sum_{j=r_k}^{r_{k+1}-1} \langle e_{n_1,a,b}, e_j \rangle_H \langle e_{n_2,a,b}, e_j \rangle_H = \sum_{k=0}^{\infty} \omega^{-ak^b+a[k(n_1)]^{b/2}} \sum_{j=r_k}^{r_{k+1}-1} \langle e_{n_1}, e_j \rangle_H \langle e_{n_2}, e_j \rangle_H.
\]

Suppose that \( n_1 \neq n_2 \). Then the last sum over \( j \) is zero for all \( k \in \mathbb{N}_0 \) due to the orthonormality of \( \{e_j\}_{j \in \mathbb{N}_0} \). Suppose now that \( n_1 = n_2 \). Then the only non-zero term is for \( k = k(n_1) \) and \( j = n_1 \), so that the sum is 1. Hence, \( \langle e_{n_1,a,b}, e_{n_2,a,b} \rangle_{H_{a,b}} = \delta_{n_1,n_2} \).

Finally, note that \( H_{a,b} \subseteq H = \text{span}(e_1, e_2, \ldots) = \text{span}(e_{1,a,b}, e_{2,a,b}, \ldots) \), which means that \( \{e_{n,a,b}\}_{n \in \mathbb{N}_0} \) is an orthonormal basis of \( H_{a,b} \), as claimed.

The norm in \( H_{a,b} \) can now also be written as

\[
\|f\|_{H_{a,b}} = \left( \sum_{n=0}^{\infty} |\langle f, e_{n,a,b} \rangle_{H_{a,b}}|^2 \right)^{1/2}.
\]

We remark that \( k(n) = 0 \) for \( n \in \{0, 1, \ldots, m_0 - 1\} \) and therefore, \( e_{n,a,b} = e_n \) and

\[
\|e_{n,a,b}\|_{H_{a,b}} = \|e_n\|_H = 1 \quad \text{for all } n \in \{0, 1, \ldots, m_0 - 1\}.
\]

The last equality holds for \( m_0 \) elements, and \( m_0 \geq 1 \). This and (1) imply

\[
\sup_{\|f\|_{H_{a,b}} \leq 1} \|f\|_H = 1.
\]  \hfill (2)
Similarly as for the space $H_s$, we take the $s$-fold tensor products of the weighted space $H_{a_j,b_j}$ with possibly different $a_j$ and $b_j$ such that

$$0 < a_1 \leq a_2 \leq \cdots \quad \text{and} \quad \inf_{j \in \mathbb{N}} b_j > 0. \quad (3)$$

That is,

$$H_{s,a,b} = H_{a_1,b_1} \otimes H_{a_2,b_2} \otimes \cdots \otimes H_{a_s,b_s}.$$ 

For $n = [n_1,n_2,\ldots,n_s] \in \mathbb{N}_0^s$, define

$$e_{n,a,b} = e_{n_1,a_1,b_1} \otimes e_{n_2,a_2,b_2} \otimes \cdots \otimes e_{n_s,a_s,b_s}.$$ 

Then $\{e_{n,a,b}\}_{n \in \mathbb{N}_0^s}$ is an orthonormal basis of $H_{s,a,b}$ and $f \in H_{s,a,b}$ iff

$$f = \sum_{n \in \mathbb{N}_0^s} \langle f, e_{n,a,b} \rangle_{H_{s,a,b}} e_{n,a,b} \quad \text{with} \quad ||f||_{H_{s,a,b}} := \left( \sum_{n \in \mathbb{N}_0^s} | \langle f, e_{n,a,b} \rangle_{H_{s,a,b}} |^2 \right)^{1/2} < \infty.$$ 

We now show that

$$||f||_{H_s} \leq ||f||_{H_{s,a,b}} \quad \text{for all} \quad f \in H_{s,a,b}. \quad (4)$$

Indeed, for $f \in H_{s,a,b}$ we have $f = \sum_{n \in \mathbb{N}_0^s} \alpha_n e_{n,a,b}$ with $||f||_{H_{s,a,b}}^2 = \sum_{n \in \mathbb{N}_0^s} |\alpha_n|^2 < \infty$. For any $n_j \in \mathbb{N}_0$ there is a unique $k(n_j) \in \mathbb{N}_0$ such that $n_j \in \{r_k(n_j), r_k(n_j)+1, \ldots, r_k(n_j)+1-1\}$, and $e_{n_j,a_j,b_j} = \omega^{a_j[k(n_j)]^s/2} e_{n_j}$. Therefore

$$e_{n,a,b} = \left( \prod_{j=1}^s \omega^{a_j[k(n_j)]^s/2} \right) e_n. \quad (5)$$

We have $f = \sum_{n \in \mathbb{N}_0^s} \alpha_n \left( \prod_{j=1}^s \omega^{a_j[k(n_j)]^s/2} \right) e_n$ and

$$||f||_{H_s} = \left( \sum_{n \in \mathbb{N}_0^s} |\alpha_n|^2 \prod_{j=1}^s \omega^{a_j[k(n_j)]^s} \right)^{1/2} \leq \left( \sum_{n \in \mathbb{N}_0^s} |\alpha_n|^2 \right)^{1/2} = ||f||_{H_{s,a,b}},$$

as claimed.

For $n \in \{0,1,\ldots,m_0-1\}^s$, we have $k(n_j) = 0$ for $j = 1,2,\ldots,s$. Therefore $e_{n,a,b} = e_n$ and

$$||e_{n,a,b}||_{H_{s,a,b}} = ||e_n||_{H_s} = 1 \quad \text{for all} \quad n \in \{0,1,\ldots,m_0-1\}^s.$$ 

The last equality holds for $m_0^s$ elements. This and (4) imply

$$\sup_{||f||_{H_{s,a,b}} \leq 1} ||f||_{H_s} = 1. \quad (6)$$

Note that (5) implies that $\{e_n\}_{n \in \mathbb{N}_0^s}$ is orthogonal in $H_{s,a,b}$ and

$$||e_n||_{H_{s,a,b}} = \prod_{j=1}^s \omega^{-a_j[k(n_j)]^s/2} \quad \text{for all} \quad n \in \mathbb{N}_0^s.$$


For $f \in H_s$ we have $f = \sum_{n \in \mathbb{N}_0^s} \langle f, e_n \rangle_{H_s} e_n$ with $\sum_{n \in \mathbb{N}_0^s} |\langle f, e_n \rangle_{H_s}|^2 < \infty$. Such $f$ belongs to $H_{s,a,b}$ iff

$$
\|f\|_{H_{s,a,b}} = \left( \sum_{n \in \mathbb{N}_0^s} \prod_{j=1}^s \omega^{-a_j |k(n_j)|^{b_j}} |\langle f, e_n \rangle_{H_s}|^2 \right)^{1/2} < \infty.
$$

As for the univariate case, we see that $\prod_{j=1}^s \omega^{-a_j |k(n_j)|^{b_j}}$ goes exponentially fast to infinity if one of the components of $n$ goes to infinity. Therefore $|\langle f, e_n \rangle_{H_s}|$ must decay exponentially fast to zero if one of the components of $n$ approaches infinity.

**Remark 1.**

We stress that the spaces $H$, $H_s$ and $H_{s,a,b}$ do not have to be reproducing kernel Hilbert spaces, see [1] for general facts on reproducing kernel Hilbert spaces. Indeed, the initial space $H$ does not have to be a function space. But if $H$ is a Hilbert space of real or complex valued functions defined on, say, a common domain $D$, then it is well known that $H$ is a reproducing kernel Hilbert space iff

$$
\sum_{k=0}^\infty |e_k(x)|^2 < \infty \quad \text{for all } x \in D. \tag{7}
$$

If (7) holds then

$$
K(x, y) = \sum_{k=0}^\infty e_k(x) \overline{e_k(y)} \quad \text{for all } x, y \in D
$$

is a reproducing kernel of $H$ and

$$
f(y) = \langle f, K(\cdot, y) \rangle_H \quad \text{for all } f \in H \text{ and } y \in D.
$$

If (7) holds then $H_s$ is also a reproducing kernel Hilbert space and its kernel is

$$
K_s(x, y) = \prod_{j=1}^s K(x_j, y_j) = \sum_{k \in \mathbb{N}_0^s} e_k(x) \overline{e_k(y)} \quad \text{for all } x, y \in D^s.
$$

Similarly, the weighted space $H_{a,b}$ is a reproducing kernel Hilbert space iff

$$
\sum_{k=0}^\infty \omega^{a_{k+1}-b} \sum_{j=r_k}^{r_k} |e_j(x)|^2 < \infty \quad \text{for all } x \in D. \tag{8}
$$

Clearly, the condition (8) is weaker than the condition (7). Hence, it may happen that $H$ is not a reproducing kernel Hilbert space but $H_{a,b}$ is. We shall see examples of such spaces in a moment.

If (8) holds then the reproducing kernel of $H_{a,b}$ is

$$
K_{a,b}(x, y) = \sum_{k=0}^\infty e_{k,a,b}(x) \overline{e_{k,a,b}(y)} = \sum_{k=0}^\infty \omega^{a_{k+1}-b} \sum_{j=r_k}^{r_k} e_j(x) \overline{e_j(y)} \quad \text{for all } x, y \in D.
$$
If \( \Box \) holds then \( H_{s,a,b} \) is also a reproducing kernel Hilbert space and its kernel is

\[
K_{s,a,b}(x, y) = \prod_{j=1}^{s} K_{a_j,b_j}(x_j, y_j) = \sum_{k \in \mathbb{N}_0^s} c_{k,a,b}(x) e_{k,a,b}(y) \quad \text{for all } x, y \in D^s.
\]

We illustrate the weighted Hilbert spaces \( H_{s,a,b} \) by five examples.

**Example 1. Weighted \( \ell_2 \) Space**

Let \( H = \ell_2 \) be the space of sequences in \( \mathbb{C} \) with finite quadratic norm, i.e., \( H = \ell_2 = \{ f : \mathbb{N}_0 \to \mathbb{C} : \sum_{n=0}^{\infty} |f(n)|^2 < \infty \} \). Let \( e_k \) be the \( k \)-th canonical element \( e_k(n) = \delta_{k,n} \).

Then \( H_s = \{ f : \mathbb{N}_0^s \to \mathbb{C} : \sum_{n \in \mathbb{N}_0^s} |f(n)|^2 < \infty \}. \) For \( k = [k_1, k_2, \ldots, k_s] \in \mathbb{N}_0^s \), let \( e_k(n) = \prod_{j=1}^{s} e_{k_j}(n_j) \) for all \( n = [n_1, n_2, \ldots, n_s] \in \mathbb{N}_0^s \). The inner product in \( H_s \) is \( \langle f, g \rangle_{H_s} = \sum_{k \in \mathbb{N}_0^s} f(k)g(k) \). Note that \( \langle e_{k_1}, e_{k_2} \rangle_{H_s} = \delta_{k_1,k_2} \) and \( \sum_{k \in \mathbb{N}_0^s} |e_k(n)|^2 = 1 \). Hence, \( H_s \) is a reproducing kernel Hilbert space with kernel function

\[
K_s(l, n) = \sum_{k \in \mathbb{N}_0^s} e_k(l) e_k(n) = \delta_{l,n} \quad \text{for } l, n \in \mathbb{N}_0^s.
\]

For \( H_{s,a,b} \), we take \( m_k \equiv 1 \). Then \( r_k = k \) and \( k(n) = n \). The inner product of \( H_{s,a,b} \) for \( f, g \in H_{s,a,b} \) is given by

\[
\langle f, g \rangle_{H_{s,a,b}} = \sum_{k \in \mathbb{N}_0^s} \omega^{-\frac{1}{2}} \sum_{j=1}^{s} a_j k_j^b_j f(k)g(k).
\]

Hence \( f \in H_{s,a,b} \) means that the \( |f(k)| \) of \( f \) decrease exponentially fast. \( H_{s,a,b} \) is a reproducing kernel Hilbert space with kernel

\[
K_{s,a,b}(l, n) = \sum_{k \in \mathbb{N}_0^s} \omega^{\frac{1}{2}} \sum_{j=1}^{s} a_j k_j^b_j e_k(l)e_k(n).
\]

**Example 2. Weighted Hermite Space**

Let \( H = L_2(\mathbb{R}, \rho) \) be the \( L_2 \)-space of real Lebesgue square-integrable functions defined on the real line \( \mathbb{R} \) with the Gaussian weight \( \rho(x) = (2\pi)^{-1/2} \exp(-x^2/2) \) for all \( x \in \mathbb{R} \).

Let \( \text{Her}_k \) be the Hermite polynomial of degree \( k \),

\[
\text{Her}_k(x) = \frac{(-1)^k}{\sqrt{k!}} \exp(x^2/2) \frac{d^k}{dx^k} \exp(-x^2/2) \quad \text{for all } x \in \mathbb{R}.
\]

It is known that \( \{\text{Her}_k\}_{k \in \mathbb{N}_0} \) is orthonormal. Hence, we can take \( e_k = \text{Her}_k \). Clearly, \( H \) is not a reproducing kernel Hilbert space.

Then \( H_s = L_2(\mathbb{R}^s, \rho_s) \) with

\[
\rho_s(x) = \frac{1}{(2\pi)^{s/2}} \exp\left(-\frac{1}{2} \sum_{j=1}^{s} x_j^2\right) \quad \text{for all } x = [x_1, x_2, \ldots, x_s] \in \mathbb{R}^s.
\]
For \( k \in \mathbb{N}_0 \), we take \( e_k(x) = \text{Her}_k(x) = \prod_{j=1}^{s} \text{Her}_{kj}(x_j) \) for all \( x \in \mathbb{R}^s \). Then \( \{e_k\}_{k \in \mathbb{N}_0} \) is an orthonormal basis of \( H_s \). Obviously, \( H_s \) is not a reproducing kernel Hilbert space for any \( s \in \mathbb{N} \).

The weighted Hermite space \( H_{s,a,b} \) is obtained by taking \( m_k \equiv 1 \). Then \( r_k = k \) and \( k(n) = n \). The inner product of \( H_{s,a,b} \) for \( f, g \in H_{s,a,b} \) is given by

\[
\langle f, g \rangle_{H_{s,a,b}} = \sum_{k \in \mathbb{N}_0^s} \omega^{-\sum_{j=1}^{s} a_j k_j^b_j} \hat{f}_k \hat{g}_k,
\]

where \( \hat{f}_k \) and \( \hat{g}_k \) denote the \( k \)th Hermite coefficients of \( f \) and \( g \),

\[
\hat{f}_k = \langle f, \text{Her}_k \rangle_{L_2(\mathbb{R}^s, \rho_s)} = \int_{\mathbb{R}^s} f(x) \text{Her}_k(x) \rho_s(x) \, dx \quad \text{for all } k \in \mathbb{N}_0^s.
\]

The weighted Hermite space \( H_{s,a,b} \) is a reproducing kernel Hilbert space due to Cramer's bound which states that

\[
|\text{Her}_k(x)| \leq (2\pi)^{1/4} \exp(x^2/4) \quad \text{for all } x \in \mathbb{R} \text{ and } k \in \mathbb{N}_0,
\]

see \cite{16} p. 324]. Indeed, this bound leads to

\[
\sum_{k \in \mathbb{N}_0^s} [e_{k,a,b}(x)]^2 = \prod_{j=1}^{s} \sum_{k=0}^{\infty} \omega^{a_j k_j^b_j} [\text{Her}_k(x_j)]^2 \leq \prod_{j=1}^{s} (2\pi)^{1/2} \exp(x_j^2/2) \sum_{k=0}^{\infty} \omega^{a_j k_j^b_j} < \infty
\]

since the series \( \sum_{k=0}^{\infty} \omega^{a_j k_j^b_j} < \infty \) for all positive \( a_j \) and \( b_j \). The reproducing kernel of \( H_{s,a,b} \) is

\[
K_{s,a,b}(x, y) = \sum_{k \in \mathbb{N}_0^s} \omega^{\sum_{j=1}^{s} a_j k_j^b_j} \text{Her}_k(x) \text{Her}_k(y) \quad \text{for all } x, y \in \mathbb{R}^s.
\]

More information on weighted Hermite spaces can be found in \cite{8,9}.

**Example 3. Weighted Korobov Space**

We now take \( H = L_2([0,1]) \) as the \( L_2 \)-space of complex-valued functions defined on \([0,1] \). The orthonormal basis \( \{e_k\}_{k \in \mathbb{N}_0} \) of \( H \) is taken as

\[
e_0(x) = 1, \quad e_{2k-1}(x) = \exp(2\pi i k x), \quad e_{2k}(x) = \exp(-2\pi i k x),
\]

for \( k \in \mathbb{N} \) with \( i = \sqrt{-1} \). Then \( H_s = L_2([0,1]^s) \) and \( \{e_k\}_{k \in \mathbb{N}_0} \) with

\[
e_k(x) = \prod_{j=1}^{s} e_{kj}(x_j) \quad \text{for all } x \in [0,1]^s
\]

as its orthonormal basis. Then \( f \in H_s \) iff

\[
f(x) = \sum_{h \in \mathbb{Z}^s} \hat{f}_h \exp(2\pi i h \cdot x) \quad \text{for all } x \in [0,1]^s
\]
with \( \sum_{h \in \mathbb{Z}^s} |\hat{f}_h|^2 < \infty \), where \( \mathbf{h} \cdot \mathbf{x} \) denotes the usual dot product. Here, \( \mathbb{Z} \) is the set of all integers, \( \mathbb{Z} := \{\ldots, -1, 0, 1, \ldots\} \), and

\[
    \hat{f}_h = \langle f, e_h \rangle_{L^2} = \int_{[0,1]^s} f(x) \exp(-2\pi i \mathbf{h} \cdot x) \, dx
\]

is the \( h \)th Fourier coefficient. Clearly, \( H_s \) is not a reproducing kernel Hilbert space for all \( s \in \mathbb{N} \).

The weighted Korobov space \( H_{s,a,b} \) is obtained by taking \( m_0 = 1 \) and \( m_k = 2 \) for all \( k \in \mathbb{N} \). Then \( r_0 = 0 \) and \( r_k = 2k - 1 \) for all \( k \in \mathbb{N} \). The inner product of \( H_{s,a,b} \) for \( f, g \in H_{s,a,b} \) is given by

\[
    \langle f, g \rangle_{H_{s,a,b}} = \sum_{h \in \mathbb{Z}^s} \omega^{-\sum_{j=1}^s a_j |h_j|^b} \hat{f}_h \hat{g}_h.
\]

The space \( H_{s,a,b} \) is a reproducing kernel Hilbert space and its reproducing kernel is

\[
    K_{s,a,b}(x, y) = \sum_{h \in \mathbb{Z}^s} \omega^{\sum_{j=1}^s a_j |h_j|^b} \exp(2\pi i \mathbf{h} \cdot (x - y)) \quad \text{for all } x, y \in [0,1]^s.
\]

The weighted Korobov space \( H_{s,a,b} \) is a space of periodic functions with period 1 for each variable. More information on these spaces can be found in \([2, 3, 10, 11]\).

Example 4. Weighted Cosine Space

We take \( H = L^2([0,1]) \) as the \( L^2 \)-space of real-valued functions defined on \([0,1]\). The orthonormal basis \( \{e_k\}_{k \in \mathbb{N}_0} \) of \( H \) is now taken as

\[
    e_0(x) = 1, \text{ and } e_k(x) = \sqrt{2} \cos(\pi k x) \quad \text{for } k \in \mathbb{N}.
\]

Then \( H_s = L^2([0,1]^s) \) and \( \{e_k\}_{k \in \mathbb{N}_0} \) with

\[
    e_k(x) = \prod_{j=1}^s e_{k_j}(x_j) \quad \text{for all } x \in [0,1]^s
\]

as its orthonormal basis. For \( \mathbf{h} = [h_1, h_2, \ldots, h_s] \in \mathbb{N}_0^s \) we denote by \( |\mathbf{h}|_0 \) the number of indices \( j \in \{1, 2, \ldots, s\} \) for which \( h_j \neq 0 \). Then \( f \in H_s \) iff

\[
    f(x) = \sum_{\mathbf{h} \in \mathbb{N}_0^s} \tilde{f}_h(\sqrt{2})^{|\mathbf{h}|_0} \left( \prod_{j=1}^s \cos(\pi h_j x_j) \right) \quad \text{for all } x \in [0,1]^s
\]

with \( \sum_{\mathbf{h} \in \mathbb{N}_0^s} |\tilde{f}_h|^2 < \infty \). Here

\[
    \tilde{f}_h = \langle f, e_h \rangle_{L^2} = \int_{[0,1]^s} f(x) (\sqrt{2})^{|\mathbf{h}|_0} \left( \prod_{j=1}^s \cos(\pi h_j x_j) \right) \, dx
\]

is the \( h \)th cosine coefficient. Clearly, \( H_s \) is not a reproducing kernel Hilbert space for all \( s \in \mathbb{N} \).
The weighted cosine space \( H_{s,a,b} \) is obtained by taking \( m_k \equiv 1 \). Then \( r_k = k \) and \( k(n) = n \). The inner product of \( H_{s,a,b} \) for \( f, g \in H_{s,a,b} \) is given by
\[
\langle f, g \rangle_{H_{s,a,b}} = \sum_{h \in \mathbb{N}_0^s} \omega^{-\sum_{j=1}^s a_j b_j} \tilde{f}_h \tilde{g}_h.
\]

The space \( H_{s,a,b} \) is a reproducing kernel Hilbert space and its reproducing kernel is
\[
K_{s,a,b}(x, y) = \sum_{h \in \mathbb{N}_0^s} \omega^{\sum_{j=1}^s a_j b_j} 2^{j_0} \left( \prod_{j=1}^s \cos(\pi h_j x_j) \cos(\pi h_j y_j) \right) \quad \text{for all } x, y \in [0, 1]^s.
\]

More information on cosine spaces with finite smoothness can be found in [4].

**Example 5. Weighted Walsh Space**

We once more consider \( H = L_2([0,1]) \) of complex-valued functions. For this example the orthonormal basis \( \{e_k\}_{k \in \mathbb{N}_0} \) of \( H \) is taken as
\[
e_k(x) = \text{wal}_k(x) \quad \text{for all } k \in \mathbb{N}_0,
\]
where \( \text{wal}_k \) is the \( k \)th Walsh function in some fixed integer base \( b \geq 2 \), see for example [6, Appendix A] for further details.

Then \( H_s = L_2([0,1]^s) \) and \( \{e_k\}_{k \in \mathbb{N}_0^s} \) with
\[
e_k(x) = \prod_{j=1}^s e_{k_j}(x_j) = \text{wal}_k(x) := \prod_{j=1}^s \text{wal}_{k_j}(x_j) \quad \text{for all } x \in [0,1]^s
\]
as its orthonormal basis. Then \( f \in H_s \) iff
\[
f(x) = \sum_{h \in \mathbb{N}_0^s} \hat{f}_{h,\text{wal}} \text{wal}_h(x) \quad \text{for all } x \in [0,1]^s
\]
with \( \sum_{h \in \mathbb{N}_0^s} |\hat{f}_{h,\text{wal}}|^2 < \infty \). Here,
\[
\hat{f}_{h,\text{wal}} = \langle f, \text{wal}_h \rangle_{L_2} = \int_{[0,1]^s} f(x) \text{wal}_h(x) \, dx
\]
is the \( h \)th Walsh coefficient.

The weighted Walsh space \( H_{s,a,b} \) is obtained by taking \( m_k \equiv 1 \). Then \( r_k = k \) and \( k(n) = n \). The inner product of \( H_{s,a,b} \) for \( f, g \in H_{s,a,b} \) is given by
\[
\langle f, g \rangle_{H_{s,a,b}} = \sum_{h \in \mathbb{N}_0^s} \omega^{-\sum_{j=1}^s a_j b_j} \tilde{f}_{h,\text{wal}} \tilde{g}_{h,\text{wal}}.
\]

The space \( H_{s,a,b} \) is a reproducing kernel Hilbert space and its reproducing kernel is
\[
K_{s,a,b}(x, y) = \sum_{h \in \mathbb{N}_0^s} \omega^{\sum_{j=1}^s a_j b_j} \text{wal}_h(x) \overline{\text{wal}_h(y)} \quad \text{for all } x, y \in [0,1]^s.
\]

More information on the Walsh spaces with finite smoothness can be found in [5, 6].
3 Multivariate Approximation

By multivariate approximation we mean an embedding operator \( \text{APP}_s : H_{s,a,b} \to H_s \) given by

\[
\text{APP}_s f = f \quad \text{for all } f \in H_{s,a,b}.
\]

Due to (6), the operator \( \text{APP}_s \) is well defined, and it is a continuous linear operator. Furthermore, \( \|\text{APP}_s f\|_{H_s} \leq \|f\|_{H_{s,a,b}} \) for all \( f \in H_{s,a,b} \) and

\[
\|\text{APP}_s\| = 1 \quad \text{for all } s \in \mathbb{N}.
\]

We will later show that \( \text{APP}_s \) is a compact operator.

We want to approximate \( \text{APP}_s f \) by algorithms \( A_n : H_{s,a,b} \to H_s \) that use at most \( n \) continuous linear functionals of \( f \). Without loss of generality, see e.g. [12, 19], we may restrict ourselves to linear algorithms of the form

\[
A_n f = \sum_{j=1}^{n} L_j(f) g_j \quad \text{for all } f \in H_{s,a,b}
\]

for some \( L_j \in H^*_{s,a,b} \) and \( g_j \in H_s \) for \( j = 1, 2, \ldots, n \).

We consider the worst case setting in which the error of \( A_n \) is defined as

\[
e(A_n) = \sup_{\|f\|_{H_{s,a,b}} \leq 1} \|\text{APP}_s f - A_n f\|_{H_s} = \|\text{APP}_s - A_n\|.
\]

For \( n = 0 \), we have the so-called initial error which is achieved by the zero algorithm \( A_0 = 0 \), and \( e(A_0) = \|\text{APP}_s\| = 1 \).

By the \( n \)th minimal (worst case) error we mean the minimal error among all algorithms \( A_n \),

\[
e(n, \text{APP}_s) = \inf_{A_n} e(A_n).
\]

Clearly, \( e(0, \text{APP}_s) = 1 \). In a moment an algorithm \( A^*_n \) for which the infimum is attained will be presented.

By the information complexity \( n(\varepsilon, \text{APP}_s) \) we mean the minimal \( n \) for which we can find an algorithm \( A_n \) with error at most \( \varepsilon \in (0, \infty) \),

\[
n(\varepsilon, \text{APP}_s) = \min\{n : e(n, \text{APP}_s) \leq \varepsilon\}.
\]

Clearly, \( n(\varepsilon, \text{APP}_s) = 0 \) for all \( \varepsilon \geq 1 \), and therefore the only \( \varepsilon \)'s of interest are from \( (0, 1) \).

It is well known, see again e.g., [12, 19], that the \( n \)th minimal errors \( e(n, \text{APP}_s) \) and the information complexity \( n(\varepsilon, \text{APP}_s) \) depend on the eigenvalues of the continuous and linear operator \( W_s = \text{APP}_s^* \text{APP}_s : H_{s,a,b} \to H_{s,a,b} \). The operator \( W_s \) is self-adjoint and in a moment we shall see that \( W_s \) is also compact. Let \( (\lambda_{s,j}, \eta_{s,j}) \) be the eigenpairs of \( W_s \),

\[
W_s \eta_{s,j} = \lambda_{s,j} \eta_{s,j} \quad \text{for all } j \in \mathbb{N},
\]

where the eigenvalues \( \lambda_{s,j} \) are ordered,

\[
\lambda_{s,1} \geq \lambda_{s,2} \geq \cdots \geq 0,
\]
and the eigenelements $\eta_{s,j}$ are orthonormal,
\[
\langle \eta_{s,j_1}, \eta_{s,j_2} \rangle_{H_{s,a,b}} = \delta_{j_1, j_2} \quad \text{for all} \quad j_1, j_2 \in \mathbb{N}.
\]

Then the $n$th minimal error is attained for the algorithm
\[
A_n^* f = \sum_{j=1}^{n} \langle f, \eta_{s,j} \rangle_{H_{s,a,b}} \eta_{s,j} \quad \text{for all} \quad f \in H_{s,a,b},
\]
and
\[
e(n, \text{APP}_s) = e(A_n^*) = \sqrt{\lambda_{s,n+1}} \quad \text{for all} \quad n \in \mathbb{N}_0.
\]

This implies that the information complexity is equal to
\[
n(\varepsilon, \text{APP}_s) = \min\{ n \in \mathbb{N}_0 : \lambda_{s,n+1} \leq \varepsilon^2 \}.
\]

We now find the eigenpairs of $W_s$. Using the notation and results of the previous section, we know that $\{e_{n,a,b}\}_{n \in \mathbb{N}_0}$ is an orthonormal basis of $H_{s,a,b}$. We prove that
\[
W_s e_{n,a,b} = \omega \sum_{j=1}^{s} a_j |k(n_j)| b_j \quad e_{n,a,b} \quad \text{for all} \quad n \in \mathbb{N}_0^s.
\]

Indeed, for $f, g \in H_{s,a,b}$ we have
\[
\langle \text{APP}_s f, \text{APP}_s g \rangle_{H_s} = \langle f, \text{APP}_s^* \text{APP}_s g \rangle_{H_{s,a,b}} = \langle f, W_s g \rangle_{H_{s,a,b}}.
\]

Taking $f = e_{n_1,a,b}$ and $g = e_{n_2,a,b}$ for arbitrary $n_1, n_2 \in \mathbb{N}_0^s$ we obtain from (5),
\[
\langle e_{n_1,a,b}, W_s e_{n_2,a,b} \rangle_{H_{s,a,b}} = \langle e_{n_1,a,b}, e_{n_2,a,b} \rangle_{H_s}
\]
\[
= \left( \prod_{j=1}^{s} \omega^{a_j |k(n_1_j)| b_j / 2 + a_j |k(n_2_j)| b_j / 2} \right) \langle e_{n_1}, e_{n_2} \rangle_{H_s}
\]
\[
= \omega \sum_{j=1}^{s} a_j |k(n_1_j)| b_j / 2 + a_j |k(n_2_j)| b_j / 2 \delta_{n_1, n_2}.
\]

Hence,
\[
\langle e_{n_1,a,b}, W_s e_{n_2,a,b} \rangle_{H_{s,a,b}} = 0 \quad \text{for all} \quad n_1 \neq n_2,
\]

and
\[
\langle e_{n,a,b}, W_s e_{n,a,b} \rangle_{H_{s,a,b}} = \omega \sum_{j=1}^{s} a_j |k(n_j)| b_j.
\]

This means that
\[
W_s e_{n,a,b} = \sum_{n_1 \in \mathbb{N}_0^s} \langle W_s e_{n,a,b}, e_{n_1,a,b} \rangle_{H_{s,a,b}} e_{n_1,a,b} = \omega \sum_{j=1}^{s} a_j |k(n_j)| b_j \quad e_{n,a,b},
\]
as claimed. Hence,
\[
\left( \omega \sum_{j=1}^{s} a_j |k(n_j)| b_j, e_{n,a,b} \right)_{n \in \mathbb{N}_0^s}
\]
are the eigenpairs of $W_s$.

As an example consider the weighted Hermite space or the weighted cosine space for which $m_k \equiv 1$. Then $k(n_j) = n_j$ and the eigenpairs are of the form
\[
\left( \omega \sum_{j=1}^{s} a_j n_j^{b_j}, \omega \sum_{j=1}^{s} a_j n_j^{b_j} / 2 e_n \right)_{n \in \mathbb{N}_0^s}.
\]
For the weighted Korobov space, we have $m_0 = 1$ and $m_k = 2$ for all $k \in \mathbb{N}$. Then $k(n_j) = \lceil n_j/2 \rceil$ and the eigenpairs are of the form

\[
\left( \omega \sum_{j=1}^{s} a_j [n_j/2]^{b_j}, \omega \sum_{j=1}^{s} a_j [n_j/2]^{b_j} e_n \right)_{n \in \mathbb{N}_0^s}.
\]

We turn to the general case. The eigenvalues of $W_s$ may be multiple. Indeed, for $n_j \in \mathbb{N}_0$ we obtain the same $k(n_j)$ for all $n_j \in \{ r_{k(n_j)}, r_{k(n_j)} + 1, \ldots, r_{k(n_j) + 1} - 1 \}$, i.e., for $r_{k(n_j)} + 1 - r_{k(n_j)} = m_k(n_j)$ different values of $n_j$. This means that $W_s$ has the eigenvalues

\[
\omega \sum_{j=1}^{s} a_j k_j^{b_j}
\]

of multiplicity $m_k := m_{k_1} m_{k_2} \cdots m_{k_s}$.

In particular, the largest eigenvalue $\lambda_{s,1} = 1$, obtained for $k_j = 0$ for all $j = 1, 2, \ldots, s$, has multiplicity $m_0^s$. For $m_0 = 1$ the largest eigenvalues is single, and the second largest eigenvalue $\lambda_{s,2} = \omega^{a_1}$ has multiplicity $m_1$.

Clearly, the sequence of ordered eigenvalues $\{ \lambda_{s,j} \}_{j \in \mathbb{N}}$ is the same as the sequence $\{ \omega \sum_{j=1}^{s} a_j [k(j)]^{b_j} \}_{n \in \mathbb{N}_0^s}$. Furthermore it is obvious that $\lim_{j \to \infty} \lambda_{s,j} = 0$, which implies that APPs as well as $W_s$ are compact.

We now find a more convenient formula for the information complexity $n(\varepsilon, \text{APP}_s)$. From Eq. (11) we conclude that for $\varepsilon \in (0, \infty)$ we have

\[
n(\varepsilon, \text{APP}_s) = |\{ j \in \mathbb{N}_0 : \lambda_{s,j} > \varepsilon^2 \}|,
\]

or equivalently

\[
n(\varepsilon, \text{APP}_s) = |\{ j \in \mathbb{N}_0 : \log \lambda_{s,j}^{-1} < \log \varepsilon^{-2} \}|.
\]

All eigenvalues $\lambda_{s,j}$ are of the form $\omega \sum_{j=1}^{s} a_j k_j^{b_j}$ with multiplicity $m_k$ for $k \in \mathbb{N}_0^s$. Therefore

\[
\log \omega^{\sum_{j=1}^{s} a_j k_j^{b_j}} = \left( \sum_{j=1}^{s} a_j k_j^{-b_j} \right) \log \omega^{-1}
\]

and

\[
\log \omega^{\sum_{j=1}^{s} a_j k_j^{b_j}} < \log \varepsilon^{-2} \text{ iff } \sum_{j=1}^{s} a_j k_j^{b_j} < \frac{\log \varepsilon^{-2}}{\log \omega^{-1}}.
\]

Let

\[
A(\varepsilon, s) = \left\{ k \in \mathbb{N}_0^s : \sum_{j=1}^{s} a_j k_j^{b_j} < \frac{\log \varepsilon^{-2}}{\log \omega^{-1}} \right\}.
\]

Then

\[
n(\varepsilon, \text{APP}_s) = \sum_{k \in A(\varepsilon, s)} m_{k_1} m_{k_2} \cdots m_{k_s}.
\]

(10)

Note that for $m_k \equiv 1$, as e.g. for the weighted Hermite space and the weighted cosine space, we have

\[
n(\varepsilon, \text{APP}_s) = |A(\varepsilon, s)|.
\]

For the general case, the set $A(\varepsilon, s)$ is empty for $\varepsilon \geq 1$, and then $n(\varepsilon, \text{APP}_s) = 0$ as we already remarked. Let

\[
x(t) = \frac{\log t^{-2}}{\log \omega^{-1}} \text{ for all } t \in (0, \infty).
\]

(11)
For \( s = 1 \), it is easy to check that \( A(\varepsilon, 1) = \{0, 1, \ldots, \lceil (x(\varepsilon)/a_1)^{1/b} \rceil - 1 \} \) and
\[
n(\varepsilon, \text{APP}_1) = m_0 + m_1 + \cdots + m_{\lceil (x(\varepsilon)/a_1)^{1/b} \rceil - 1}.
\] (12)

For \( s \geq 2 \), we have
\[
A(\varepsilon, s) = \bigcup_{k_\varepsilon=0}^{\infty} \left\{ k \in \mathbb{N}_0^{s-1} : \sum_{j=1}^{s-1} a_j k_j^{b_j} < x(\varepsilon) - a_s k_s^{b_s} \right\}
\]
\[
= \bigcup_{k_s=0}^{\lceil (x(\varepsilon)/a_s)^{1/b_s} \rceil - 1} \left\{ k \in \mathbb{N}_0^{s-1} : \sum_{j=1}^{s-1} a_j k_j^{b_j} < x(\varepsilon) - a_s k_s^{b_s} \right\}.
\]

Since \( x(\varepsilon) - a_s k_s^{b_s} = (\log (\varepsilon \omega^{-a_s k_s^{b_s}/2})^{-2}) / \log \omega^{-1} \), we obtain from (10)
\[
n(\varepsilon, \text{APP}_s) = \sum_{k=0}^{\lceil (x(\varepsilon)/a_s)^{1/b_s} \rceil - 1} m_k n\left( \varepsilon \omega^{-a_s k_s^{b_s}/2}, \text{APP}_{s-1} \right).
\] (13)

For \( \varepsilon_1 \leq \varepsilon_2 \) we have \( n(\varepsilon_2, \text{APP}_s) \leq n(\varepsilon_1, \text{APP}_s) \). Since \( \varepsilon \leq \varepsilon \omega^{-a_s k_s^{b_s}/2} \) for all \( k \in \mathbb{N}_0 \), we conclude that
\[
n(\varepsilon, \text{APP}_s) \leq \left( \sum_{k=0}^{\lceil (x(\varepsilon)/a_s)^{1/b_s} \rceil - 1} m_k \right) n(\varepsilon, \text{APP}_{s-1}) \quad \text{for all } s \geq 2.
\] (14)

We obtain a lower bound on \( n(\varepsilon, \text{APP}_s) \) if we consider only the term \( k = 0 \) in (13). Then
\[
n(\varepsilon, \text{APP}_s) \geq m_0 n(\varepsilon, \text{APP}_{s-1}) \quad \text{for all } s \geq 2.
\]

For \( \varepsilon \geq \omega^{a_s/2} \) we have
\[
n(\varepsilon, \text{APP}_s) = m_0 n(\varepsilon, \text{APP}_{s-1}) \quad \text{for all } s \geq 2
\]
since \( \varepsilon \omega^{-a_s k_s^{b_s}/2} \geq 1 \) for all positive \( k \) and the terms in (13) for \( k > 0 \) are zero.

For \( x(\varepsilon) > a_1 \), define
\[
j(\varepsilon) = \sup \{ j \in \mathbb{N} : x(\varepsilon) > a_j \}.
\] (15)

Obviously, \( j(\varepsilon) \geq 1 \). For \( \lim_j a_j < \infty \), we have \( j(\varepsilon) = \infty \) for small \( \varepsilon \). On the other hand, if \( \lim_j a_j = \infty \) we can replace the supremum in the definition of \( j(\varepsilon) \) by the maximum and \( j(\varepsilon) \) is finite for all \( \varepsilon \) with \( x(\varepsilon) > a_1 \). However, \( j(\varepsilon) \) tends to infinity as \( \varepsilon \) tends to zero.

If \( j(\varepsilon) \) is finite then
\[
n(\varepsilon, \text{APP}_s) = m_{0 - j(\varepsilon)} n(\varepsilon, \text{APP}_{j(\varepsilon)}) \quad \text{for all } s \geq j(\varepsilon).
\]

Indeed, for \( j \in (j(\varepsilon), s] \) we have \( x(\varepsilon) \leq a_j \) and \( x(\varepsilon) - a_j k_j^{b_j} \leq 0 \) for all \( k \geq 1 \). This implies that \( \varepsilon \omega^{-a_j k_j^{b_j}/2} \geq 1 \) for all \( k \geq 1 \), and the sum in (13) reduces to one term for
\( k = 0 \). Hence, \( n(\varepsilon, \text{APP}_s) = m_0 n(\varepsilon, \text{APP}_{s-1}) = \cdots = m_0^{s-j(\varepsilon)} n(\varepsilon, \text{APP}_{j(\varepsilon)}) \), as claimed. Therefore, if \( j(\varepsilon) < \infty \) and \( m_0 = 1 \) then \( n(\varepsilon, \text{APP}_s) \) is independent of \( s \) for large \( s \), and

\[
\lim_{s \to \infty} \frac{\log n(\varepsilon, \text{APP}_s)}{s} = 0.
\]

Recall that we assume that the sequence \( m = \{m_k\}_{k \in \mathbb{N}_0} \) of multiplicities is bounded. That is,

\[
m_{\text{max}} = \max_{k \in \mathbb{N}} m_k
\]

is well defined and \( m_{\text{max}} < \infty \). We also set

\[
m_{\text{min}} = \min_{k \in \mathbb{N}} m_k.
\]

Clearly, \( m_{\text{min}} \geq 1 \).

We are ready to prove the following lemma.

**Lemma 1.**

Let \( x(\varepsilon), j(\varepsilon), m_{\text{max}} \) and \( m_{\text{min}} \) be defined as above.

(i) For \( \varepsilon \in (0, 1) \) we have

\[ n(\varepsilon, \text{APP}_s) \geq m_0^s, \]

whereas for \( \varepsilon \in (0, 1) \) and \( x(\varepsilon) \leq a_1 \) we have

\[ n(\varepsilon, \text{APP}_s) = m_0^s. \]

(ii) For \( x(\varepsilon) > a_1 + a_2 + \cdots + a_s \) we have

\[ n(\varepsilon, \text{APP}_s) \geq (m_0 + m_1)^s. \]

(iii) For \( x(\varepsilon) > a_1 \) and \( \varepsilon \in (0, 1) \) we have

\[
 n(\varepsilon, \text{APP}_s) \leq m_0^s \prod_{j=1}^{\min(s,j(\varepsilon))} \left( 1 + \frac{m_{\text{max}}}{m_0} \left( \left( \frac{x(\varepsilon)}{a_j} \right)^{1/b_j} - 1 \right) \right).
\]

(iv) For \( x(\varepsilon) > a_1, \varepsilon \in (0, 1), \) and arbitrary \( \alpha_j \in [0, 1] \) we have

\[
 n(\varepsilon, \text{APP}_s) \geq m_0^s \prod_{j=1}^{\min(s,j(\varepsilon))} \left( 1 + \frac{m_{\text{min}}}{m_0} \left( \left( \frac{x(\varepsilon)}{a_j} (1 - \alpha_j) \prod_{k=j+1}^{s} \alpha_k \right)^{1/b_j} - 1 \right) \right).
\]

In particular, for \( \alpha_j = (j - 1)/j \) we have

\[
 n(\varepsilon, \text{APP}_s) \geq m_0^s \prod_{j=1}^{\min(s,j(\varepsilon))} \left( 1 + \frac{m_{\text{min}}}{m_0} \left( \left( \frac{x(\varepsilon)}{a_j^s} \right)^{1/b_j} - 1 \right) \right).
\]
Proof. To prove (i), observe that for \( \varepsilon \in (0, 1) \) the set \( A(\varepsilon, s) \) is nonempty since \( k = 0 \in A(\varepsilon, s) \). Therefore \((10)\) yields \( n(\varepsilon, \text{APP}_s) \geq m_0^s \). Furthermore, for \( x(\varepsilon) \leq a_1 \) the set \( A(\varepsilon) = \{0\} \) and therefore \( n(\varepsilon, \text{APP}_s) = m_0^s \), as claimed.

To prove (ii), observe that all \( k \in \{0, 1\}^s \) belong to the set \( A(\varepsilon, s) \). Therefore

\[
n(\varepsilon, \text{APP}_s) \geq \sum_{k_1, k_2, \ldots, k_s = 0}^1 m_{k_1} m_{k_2} \cdots m_{k_s} = (m_0 + m_1)^s,
\]
as claimed.

To prove (iii), we first take \( s = 1 \). Then \((12)\) yields

\[
n(\varepsilon, \text{APP}_1) \leq m_0 + m_{\text{max}} \left( \left\lceil \left( \frac{x(\varepsilon)}{a_1} \right)^{1/b_1} \right\rceil - 1 \right) = m_0 \left( 1 + \frac{m_{\text{max}}}{m_0} \left( \left\lceil \left( \frac{x(\varepsilon)}{a_1} \right)^{1/b_1} \right\rceil - 1 \right) \right),
\]
as needed. For \( s \geq 2 \) we use \((14)\) and obtain

\[
n(\varepsilon, \text{APP}_s) \leq m_0 \left( 1 + \frac{m_{\text{max}}}{m_0} \left( \left\lceil \left( \frac{x(\varepsilon)}{a_s} \right)^{1/b_s} \right\rceil - 1 \right) \right) n(\varepsilon, \text{APP}_{s-1}).
\]

This implies that

\[
n(\varepsilon, \text{APP}_s) \leq m_0 \prod_{j=1}^s \left( 1 + \frac{m_{\text{max}}}{m_0} \left( \left\lceil \left( \frac{x(\varepsilon)}{a_j} \right)^{1/b_j} \right\rceil - 1 \right) \right) n(\varepsilon, \text{APP}_{s-1}).
\]

Note that for \( j(\varepsilon) < s \) we have \( x(\varepsilon) \leq a_j \) for all \( j \in [j(\varepsilon) + 1, s] \) and therefore

\[
1 + \frac{m_{\text{max}}}{m_0} \left( \left\lceil \left( \frac{x(\varepsilon)}{a_j} \right)^{1/b_j} \right\rceil - 1 \right) = 1.
\]

This means that we can restrict the product to \( j \) up to \( j(\varepsilon) \). This completes the proof of (iii).

To prove (iv), it is enough to prove that

\[
n(\varepsilon, \text{APP}_s) \geq m_0 \prod_{j=1}^s \left( 1 + \frac{m_{\text{min}}}{m_0} \left( \left\lceil \left( \frac{x(\varepsilon)}{a_j} \right)^{1/b_j} \right\rceil - 1 \right) \right) \sum_{k=j+1}^s \alpha_k \left( \prod_{k=j+1}^s \alpha_k \right)^{1/b_j} - 1
\]
since for \( j \in (j(\varepsilon), s] \) we have \( x(\varepsilon) \leq a_j \) and the corresponding factors are one.

Take first \( s = 1 \). Then \((12)\) yields

\[
n(\varepsilon, \text{APP}_1) \geq m_0 \left( 1 + \frac{m_{\text{min}}}{m_0} \left( \left\lceil \left( \frac{x(\varepsilon)}{a_1} \right)^{1/b_1} \right\rceil - 1 \right) \right)
\]

\[
\geq m_0 \left( 1 + \frac{m_{\text{min}}}{m_0} \left( \left\lceil \left( \frac{x(\varepsilon)}{a_1} (1 - \alpha_1) \right)^{1/b_1} \right\rceil - 1 \right) \right),
\]
as needed. For \( s \geq 2 \), note that \( x(\varepsilon) - a_s x^k b_s > \alpha_s x(\varepsilon) \) for all \( k \in \mathbb{N} \) for which

\[
k \leq \left\lfloor \left( \frac{x(\varepsilon)(1 - \alpha_s)}{a_s} \right)^{1/b_s} \right\rfloor - 1.
\]
For such $k$ we have $\varepsilon \omega^{-a_s k^s/2} < \varepsilon^{\alpha_s}$ and therefore from (13) we obtain

$$n(\varepsilon, \text{APP}_s) \geq \sum_{k=0}^{[(x(\varepsilon)(1-\alpha_s)/a_s)^{1/b_s}-1]} m_k n(\varepsilon^{\alpha_s}, \text{APP}_{s-1})$$

$$\geq m_0 \left(1 + \frac{m_{\min}}{m_0} \left[\left(\frac{x(\varepsilon)(1-\alpha_s)}{a_s}\right)^{1/b_s} - 1\right]\right) n(\varepsilon^{\alpha_s}, \text{APP}_{s-1}).$$

Since $x(\varepsilon^{\alpha_s}) = \alpha_s x(\varepsilon)$, the proof is completed by applying induction on $s$. For $\alpha_j = (j-1)/j$ we have $(1-\alpha_j) \prod_{k=j+1}^s \alpha_k = 1/s$, which completes the proof. 

4 Exponential Convergence

As in [2, 3, 8, 10], by exponential convergence (EXP) we mean that the $n$th minimal errors $\epsilon(n, \text{APP}_s)$ are bounded by $n(\varepsilon, \text{APP}_s) \leq C_s q^{n/M_s} p_s$ for all $n \in \mathbb{N}$, for some positive $C_s, M_s$ and $p_s$ with $q \in (0, 1)$. The supremum of $p_s$ for which the last bound holds is denoted by $p^*_s$ and is called the exponent of EXP for the $s$-variate case. We also have the concept of uniform exponential convergence (UEXP) if we can take $p_s = p > 0$ for all $s \in \mathbb{N}$. Then the supremum of such $p$ is denoted by $p^*$ and is called the exponent of UEXP.

We want to verify when EXP and UEXP hold for the approximation problem $\text{APP} = \{\text{APP}_s\}_{s \in \mathbb{N}}$ in terms of the varying parameters $a = \{a_s\}_{s \in \mathbb{N}}$ and $b = \{b_s\}_{s \in \mathbb{N}}$, which define the domain spaces $H_{s,a,b}$ of $\text{APP}_s$ and satisfy (3).

**Theorem 1.**

Consider the approximation problem $\text{APP} = \{\text{APP}_s\}_{s \in \mathbb{N}}$ with the embedding operators $\text{APP}_s : H_{s,a,b} \to H_s$. Then

(i) EXP holds for arbitrary $a$ and $b$ with the exponent $p^*_s = 1/B_s$ and $B_s := \sum_{j=1}^s 1/b_j$.

(ii) UEXP holds iff $a$ is arbitrary and $b$ is such that $B := \sum_{j=1}^\infty 1/b_j < \infty$.

If $B < \infty$ then the exponent of UEXP is $p^* = 1/B$.

**Proof.** From (iii) and (iv) of Lemma 1 with a fixed $s$ we conclude that there are positive numbers $c_1(s)$ and $c_2(s)$ such that

$$c_1(s)[x(\varepsilon)]^{B_s} \leq n(\varepsilon, \text{APP}_s) \leq c_2(s) [x(\varepsilon)]^{B_s} \quad \text{for all } \varepsilon \in (0, 1).$$
Clearly, \( x(\varepsilon) = \Theta(\log \varepsilon^{-1}) \). Therefore
\[
n(\varepsilon, \text{APP}_s) = \Theta\left(\log \varepsilon^{-1}B_s\right).
\]
Since \( e(n, \text{APP}_s) = \sqrt{A_s \cdot n(\varepsilon, \text{APP}_s) + 1} \), we can find positive \( c_j(s) \) for \( j = 3, 4, 5, 6 \) such that
\[
c_3(s) e^{-(n/c_3(s))1/B_s} \leq e(n, \text{APP}_s) \leq c_5(s) e^{-(n/c_5(s))1/B_s}
\]
where \( e = \exp(1) \). This proves \( \text{EXP} \) with \( p^* = 1/B_s \), as claimed.

We now turn to \( \text{UEXP} \). Suppose that \( \text{UEXP} \) holds. Then \( e(n, \text{APP}_s) \leq C_s q^{(n/M_s) p} \).

This implies that
\[
n(\varepsilon, \text{APP}_s) = \Theta\left(\log \varepsilon^{-1}1/p\right).
\]
Thus, \( B_s \leq 1/p \) for all \( s \in \mathbb{N} \). Therefore \( B \leq 1/p < \infty \) and \( p^* \leq 1/B \). On the other hand, if \( B < \infty \) then we can set \( p_s = 1/B \) and obtain \( \text{UEXP} \). Hence, \( p^* \geq 1/B \), and therefore \( p^* = 1/B \). This completes the proof.

We stress that \( \text{EXP} \) and \( \text{UEXP} \) hold for arbitrary sequences \( m \) of multiplicity and the only condition is on \( b \) for \( \text{UEXP} \). This is true since the concepts of \( \text{EXP} \) and \( \text{UEXP} \) do not specify how \( C_s \) and \( M_s \) depend on \( s \). In fact, in general, it is easy to see from Lemma \ref{lem:trac} that \( c_1(s) \) and \( c_2(s) \), as well as the other \( c_j(s) \), depend exponentially on \( s \). It is especially clear if the multiplicity \( m_0 \geq 2 \). If we wish to control the dependence on \( s \) and to control the exponential dependence on \( s \) then we need to study tractability which is the subject of the next section.

## 5 Tractability

Tractability studies how the information complexity depends on both \( \varepsilon^{-1} \) and \( s \). The key point is to characterize when this dependence is not exponential in \((s^n, \varepsilon^{-t_2})\) or in \((s^t_1, (1 + \log \varepsilon^{-1})t_2)\) for some positive \( t_1 \) and \( t_2 \), and when this dependence is polynomial in \((s, \varepsilon^{-1})\) or in \((s, 1 + \log \varepsilon^{-1})\). For \( t_1 = t_2 = 1 \), the survey of tractability results for general multivariate problems and for the pair \((s, \varepsilon^{-1})\) can be found in \cite{12, 13, 14}, and for more specific multivariate problems and the pair \((s, 1 + \log \varepsilon^{-1})\) in \cite{2, 3, 8, 10}.

We will cover a number of tractability notions and verify when they hold for the approximation problem \( \text{APP} = \{\text{APP}_s\}_{s \in \mathbb{N}} \) in terms of the parameters \( a, b, m \) and \( \omega \). We will analyze the tractability notions starting from the weakest notions and continuing to the strongest notions. A table which gives an overview of the obtained tractability results is presented in Section \ref{sec:trac}.

### 5.1 Standard Notions of Tractability

By the standard notions of tractability we mean tractability notions with respect to the pair \((s, \varepsilon^{-1})\).

- \((t_1, t_2)\)-Weak Tractability

As in \cite{18}, we say that \( \text{APP} \) is \((t_1, t_2)\)-weakly tractable (shortly \((t_1, t_2)\)-WT) for positive \( t_1 \) and \( t_2 \) if
\[
\lim_{s+\varepsilon^{-1}\to\infty} \frac{\log n(\varepsilon, \text{APP}_s)}{s^{t_1} + \varepsilon^{-t_2}} = 0.
\]
This means that \( n(\varepsilon, APP_s) \) is not exponential in \( s^{t_1} \) and \( \varepsilon^{-t_2} \) but it may be exponential in \( s^{\tau_1} \) or \( \varepsilon^{-\tau_2} \) for positive \( \tau_1 < t_1 \) or \( \tau_2 < t_2 \). In particular, if \( t_1 > 1 \) we may have the exponential dependence on \( s \) which is called the curse of dimensionality.

**Theorem 2.**

APP is \((t_1, t_2)\)-WT for the parameters \( a, b, m \) and \( \omega \) iff \( t_1 > 1 \) or \( m_0 = 1 \).

**Proof.** Suppose that APP is \((t_1, t_2)\)-WT for the parameters \( a, b, m \) and \( \omega \). Then for fixed \( \varepsilon \in (0, 1) \) we obtain from (iii) of Lemma 1 that

\[
0 = \lim_{s \to \infty} \frac{n(\varepsilon, APP_s)}{s^{t_1} + \varepsilon^{-t_2}} \geq \lim_{s \to \infty} \frac{s \log m_0}{s^{t_1} + \varepsilon^{-t_2}} = \lim_{s \to \infty} s^{1-t_1} \log m_0
\]

and hence we must have \( t_1 > 1 \) or \( m_0 = 1 \).

Suppose now that \( t_1 > 1 \). We first show that the hardest case of APP is for constant \( a \) and \( b \), i.e., \( a_j = a_1 \) and \( b_j \equiv b_1 \). Indeed, the eigenvalues of \( W_s \), which define \( n(\varepsilon, APP_s) \), are \( \omega \sum_{j=1}^{s} a_j[k(n_j)]^{b_j} \). Clearly,

\[
\sum_{j=1}^{s} a_j[k(n_j)]^{b_j} \geq \sum_{j=1}^{s} a_1[k(n_j)]^{b_0},
\]

where \( b_0 = \inf_j b_j \). Due to (iii), we have \( b_0 > 0 \). Therefore

\[
\omega \sum_{j=1}^{s} a_j[k(n_j)]^{b_j} \leq \omega \sum_{j=1}^{s} a_1[k(n_j)]^{b_0},
\]

and \( n(\varepsilon, APP_s) \) is maximized for \( a_j \equiv a_1 \) and \( b_j \equiv b_0 \) (and just now \( b_0 = b_1 \)). Hence, it is enough to show \((t_1, t_2)\)-WT for constant \( a \) and \( b \). From (iii) of Lemma 1 we have

\[
\log n(\varepsilon, APP_s) \leq s \left( \log m_0 + \log \left( 1 + \frac{m_{\text{max}}^{2/b_1}}{m_0(a_1 \log \omega^{-1})^{1/b_1} \log (\varepsilon^{-1})^{1/b_1}} \right) \right).
\]

This shows that for small \( \varepsilon \) we have

\[
\log n(\varepsilon, APP_s) = \mathcal{O}(s \log \log \varepsilon^{-1})
\]

with the factor in the big \( \mathcal{O} \) notation independent of \( s \) and \( \varepsilon^{-1} \). Hence,

\[
\frac{\log n(\varepsilon, APP_s)}{s^{t_1} + \varepsilon^{-t_2}} = \mathcal{O} \left( \frac{s \log \log \varepsilon^{-1}}{s^{t_1} + \varepsilon^{-t_2}} \right).
\]

Let \( y = \max(s^{t_1}, \varepsilon^{-t_2}) \). Then \( \varepsilon^{-1} \leq y^{1/t_2}, s \leq y^{1/t_1} \) and

\[
\frac{s \log \log \varepsilon^{-1}}{s^{t_1} + \varepsilon^{-t_2}} \leq \frac{y^{1/t_1} \log \log y^{1/t_2}}{y^{1/t_2}} = \frac{\log \log y^{1/t_2}}{y^{1/t_1}}
\]

and it goes to zero as \( s + \varepsilon^{-1} \), or equivalently \( y \), approaches infinity since \( t_1 > 1 \) and \( t_2 > 0 \). This proves \((t_1, t_2)\)-WT.
Finally, suppose that $m_0 = 1$. Then the second largest eigenvalue for all $s$ is $\lambda_{1,2} = \omega^{a_1}$, which is smaller than the largest eigenvalue $\lambda_{s,1} = 1$. As above it suffices to consider APP for constant $a$ and $b$, i.e. $a_j \equiv a_1$ and $b_j \equiv b_1$. In this case, we can use an estimate for the information complexity which has been shown in [15, p. 611], and which states

$$n(\varepsilon, \text{APP}_s) \leq \frac{s!}{(s - a_s(\varepsilon))!} \prod_{j=1}^{a_s(\varepsilon)} n(\varepsilon^{1/j}, \text{APP}_1),$$

where

$$a_s(\varepsilon) = \min \left\{ s, \left\lfloor \frac{2 \log \varepsilon^{-1}}{\log \omega^{-a_1}} \right\rfloor - 1 \right\}. $$

Then we have

$$\log n(\varepsilon, \text{APP}_s) \leq \log \frac{s!}{(s - a_s(\varepsilon))!} + \sum_{j=1}^{a_s(\varepsilon)} \log n(\varepsilon^{1/j}, \text{APP}_1).$$

From (12) with the assumption $m_0 = 1$ we obtain

$$n(\varepsilon^{1/j}, \text{APP}_1) \leq 1 + m_{\max} \left( \frac{x(\varepsilon^{1/j})}{a_1} \right)^{1/b_1} \leq 2 m_{\max} \max \left( 1, \left( \frac{x(\varepsilon^{1/j})}{a_1} \right)^{1/b_1} \right).$$

Note that

$$\left( \frac{x(\varepsilon^{1/j})}{a_1} \right)^{1/b_1} \leq \left( \frac{2 \log \varepsilon^{-1}}{a_1 j \log \omega^{-1}} \right)^{1/b_1} \leq \left( \frac{2 \log \varepsilon^{-1}}{a_1 \log \omega^{-1}} \right)^{1/b_1}.$$ 

Assume that $\varepsilon \leq \omega^{a_1/2}$. Then the last right hand side is at least one and therefore

$$\log n(\varepsilon^{1/j}, \text{APP}_1) \leq \log \left( 2 m_{\max} \left( \frac{2}{a_1} \right)^{1/b_1} \right) + \frac{1}{b_1} \log \left( \frac{\log \varepsilon^{-1}}{\log \omega^{-1}} \right)$$

$$= C_1 + C_2 \log \log \varepsilon^{-1},$$

where $C_1 = \log(2 m_{\max}(\frac{2}{a_1})^{1/b_1}) - \frac{1}{b_1} \log \log \omega^{-1}$ and $C_2 = \frac{1}{b_1}$. Hence we obtain

$$\sum_{j=1}^{a_s(\varepsilon)} \log n(\varepsilon^{1/j}, \text{APP}_1) \leq a_s(\varepsilon)(C_1 + C_2 \log \log \varepsilon^{-1}) \leq C_3 \log \varepsilon^{-1} \log \log \varepsilon^{-1}$$

(18)

with a suitable $C_3 > 0$. Hence we have

$$\limsup_{s + \varepsilon^{-1} \to \infty} \frac{\log n(\varepsilon, \text{APP}_s)}{s^{l_1} + \varepsilon^{-l_2}} \leq \limsup_{s + \varepsilon^{-1} \to \infty} \frac{\log \frac{s!}{(s - a_s(\varepsilon))!}}{s^{l_1} + \varepsilon^{-l_2}}.$$

Since

$$\frac{s!}{(s - a_s(\varepsilon))!} = (s - a_s(\varepsilon) + 1)(s - a_s(\varepsilon) + 2) \cdot \cdot \cdot s \leq s^{a_s(\varepsilon)},$$

22
we have
\[
\log \frac{s!}{(s - a_s(\varepsilon))!} \leq a_s(\varepsilon) \log s = O(\log \varepsilon^{-1} \log s). \tag{19}
\]
As before, let \( y = \max(s^{t_1}, \varepsilon^{-t_2}) \). Since \( t_1 > 0 \) and \( t_2 > 0 \) we have \( \varepsilon^{-1} \leq y^{1/t_2} \), \( s \leq y^{1/t_1} \) and
\[
\log \varepsilon^{-1} \log s \leq \frac{[\log y]^2}{t_1 t_2 y}
\]
goesto zero as \( s^{t_1} + \varepsilon^{-t_2} \), or equivalently \( y \), approaches infinity. Hence
\[
\limsup_{s+\varepsilon^{-1} \to \infty} \frac{\log n(\varepsilon, \text{APP}_s)}{s^{t_1} + \varepsilon^{-t_2}} = \lim_{s+\varepsilon^{-1} \to \infty} \frac{\log n(\varepsilon, \text{APP}_s)}{s^{t_1} + \varepsilon^{-t_2}} = 0.
\]

\begin{itemize}
\item **Weak and Uniform Weak Tractability**

Weak tractability (WT) corresponds to \((t_1, t_2)\)-WT for \( t_1 = t_2 = 1 \). Uniform weak tractability (UWT) holds iff we have \((t_1, t_2)\)-WT for all \( t_1, t_2 \in (0, 1] \).

**Theorem 3.**

\[ \text{APP is WT as well as UWT \ if \ } m_0 = 1. \]

**Proof.** Since UWT implies WT, it is enough to show that WT implies \( m_0 = 1 \), and that \( m_0 = 1 \) implies UWT. Suppose then that APP is WT. From the previous proof we conclude that \( m_0 = 1 \). On the other hand, if \( m_0 = 1 \) then APP is not only UWT but it is quasi-polynomially tractable which is a stronger notion than UWT. This will be shown in a moment.

\begin{itemize}
\item **Quasi-Polynomial Tractability**

APP is quasi-polynomially tractable (QPT) iff there are positive numbers \( C \) and \( t \) such that
\[
n(\varepsilon, \text{APP}_s) \leq C \exp\left( t(1 + \log s)(1 + \log \varepsilon^{-1}) \right) \quad \text{for all} \ s \in \mathbb{N}, \ \varepsilon \in (0, 1).
\]
The infimum of \( t \) satisfying the bound above is denoted by \( t^* \), and is called the exponent of QPT. Clearly, QPT implies UWT.

**Theorem 4.**

\[ \text{APP is QPT \ if \ } m_0 = 1. \]

**Proof.** Suppose that APP is QPT. Then APP is UWT and \( m_0 = 1 \). We now show that \( m_0 = 1 \) implies QPT and \( t^* \leq \frac{2}{a_1 \log \omega^{-1}} \) and the last bound becomes an equality for constant \( a \) and \( b \).

We now show that \( m_0 = 1 \) implies QPT and \( t^* \leq 2/(a_1 \log \omega^{-1}) \). As before, it is enough to prove it for constant \( a \) and \( b \). In this case \( H_{a,b} \) is the tensor product of \( s \) copies of \( H_{a,b} \). Then the eigenvalues \( \{\lambda_{s,k}\}_{k \in \mathbb{N}} \) of \( W_s \) are products of the eigenvalues \( \{\lambda_k\}_{k \in \mathbb{N}} \) of \( W_1 \), i.e., \( \{\lambda_{s,k}\}_{k \in \mathbb{N}} = \{\lambda_1 \cdots \lambda_k\}_{k_1, k_2, \ldots, k_s \in \mathbb{N}} \) with the

23
ordered eigenvalues \( \lambda_k = \omega^{a_1(k-1)b_1} \) for \( k \in \mathbb{N} \). It is proved in [7] that APP is QPT iff \( \lambda_2 < \lambda_1 \) and \( \text{decay}_\lambda := \sup \{ r : \lim_k k^r \lambda_k = 0 \} > 0 \). If so then

\[
T^* = \max \left( \frac{2}{\text{decay}_\lambda}, \frac{2}{\log \frac{\lambda_1}{\lambda_2}} \right).
\]

In our case, \( \lambda_1 = 1 \) and \( \lambda_2 = \omega^{a_1} \) so that the assumption \( \lambda_2 < \lambda_1 \) holds. Furthermore \( \lim_k k^r \omega^{a_1(k-1)b_1} = 0 \) for all \( r > 0 \), so that \( \text{decay}_\lambda = \infty \). Hence, \( T^* = 2/(a_1 \log \omega^{-1}) \), as claimed.

- **Polynomial and Strong Polynomial Tractability**

APP is polynomially tractable (PT) iff there are positive \( C, p \) and \( q \geq 0 \) such that

\[
n(\varepsilon, \text{APP}_s) \leq C s^q \varepsilon^{-p} \quad \text{for all} \quad s \in \mathbb{N}, \varepsilon \in (0,1).
\]

APP is strongly polynomially tractable (SPT) iff the last bound holds for \( q = 0 \). Then the infimum of \( p \) in the bound above is denoted by \( p^* \), and is called the exponent of SPT. For simplicity, we assume that

\[
\alpha := \lim_{j \to \infty} \frac{a_j}{\log j}
\]

exists.

**Theorem 5.**

APP is SPT iff APP is PT iff

\[
m_0 = 1 \quad \text{and} \quad \alpha > 0.
\]

If this is the case then the exponent of SPT is \( p^* = \frac{2}{\alpha \log \omega^{-1}} \).

**Proof.** We use [12, Theorem 5.2] which states necessary and sufficient conditions on PT and SPT in terms of the eigenvalues \( \{\lambda_{s,k}\}_{k \in \mathbb{N}} \) of \( W_s \). For our problem we have \( \lambda_{s,1} = 1 \). Namely, APP is PT iff there are numbers \( q \geq 0 \) and \( \tau > 0 \) such that

\[
\sup_{s \in \mathbb{N}} \left( \sum_{k=1}^{\infty} \lambda_{s,k}^\tau \right)^{1/\tau} s^{-q} < \infty, \tag{20}
\]

and it is SPT iff the last inequality holds with \( q = 0 \). Then the exponent \( p^* \) of SPT is the infimum of \( 2\tau \) for \( \tau \) satisfying (20) with \( q = 0 \).

In our case,

\[
\sum_{k=1}^{\infty} \lambda_{s,k}^\tau = \sum_{n \in \mathbb{N}_0} \prod_{j=1}^{s} \omega^{a_j([n/j]+b_j)} = \prod_{j=1}^{s} \left( m_0 + \sum_{k=1}^{\infty} m_k \omega^{a_j [k]} b_j \right).
\]

Let \( b_0 = \inf_j b_j \) and \( C_\tau = m_{\max} \sum_{k=1}^{\infty} \omega^{\tau a_1(kb_0-1)} \). Then \( b_0 > 0 \) and \( C_\tau < \infty \) for all \( \tau > 0 \). Furthermore,

\[
m_1 \omega^{\tau a_j} \leq \sum_{k=1}^{\infty} m_k \omega^{\tau a_j[k]} \leq m_{\max} \omega^{\tau a_j} \sum_{k=1}^{\infty} \omega^{\tau a_j(kb_j-1)} \leq C_\tau \omega^{\tau a_1}.
\]

24
Therefore
\[ \prod_{j=1}^{s} (m_0 + m_1 \omega^{\tau a_j}) \leq \prod_{j=1}^{s} \left( m_0 + \sum_{k=1}^{\infty} m_k \omega^{\tau a_j k^b} \right) \leq \prod_{j=1}^{s} (m_0 + C_\tau \omega^{\tau a_j}). \quad (21) \]

Let \( \omega^{\tau a_j} = (j+1)^{-x_j} \). That is,
\[ x_j = \frac{a_j}{\log(j+1)} \tau \log \omega^{-1} \quad \text{and} \quad \lim_{j \to \infty} x_j = \alpha \tau \log \omega^{-1}. \]

Then
\[ \prod_{j=1}^{s} \left( m_0 + \frac{m_1}{(j+1)^{x_j}} \right) \leq \prod_{j=1}^{s} \left( m_0 + \sum_{k=1}^{\infty} m_k \omega^{\tau a_j k^b} \right) \leq \prod_{j=1}^{s} \left( m_0 + \frac{C_\tau}{(j+1)^{x_j}} \right). \]

Hence, (20) holds iff \( m_0 = 1 \) and \( \lim_j x_j \geq 1 \). Indeed, \( m_0 = 1 \) is clear because otherwise we have an exponential dependence on \( s \). For \( m_0 = 1 \), let \( \beta \in \{m_1, C_\tau\} \).

Then
\[ \prod_{j=1}^{s} \left( m_0 + \frac{\beta}{(j+1)^{x_j}} \right) = \exp \left( \sum_{j=1}^{s} \log(1 + \beta (j+1)^{-x_j}) \right). \]

Furthermore,
\[ \sum_{j=1}^{s} \log(1 + \beta (j+1)^{-x_j}) = \Theta \left( \sum_{j=1}^{s} (j+1)^{-x_j} \right), \]

with the factors in the big \( \Theta \) notation independent of \( s \) and \( j \).

Suppose that \( \alpha = 0 \). Then \( \lim_j x_j = 0 \) for all \( \tau \). This means for all \( \delta \in (0, 1) \) there is an integer \( j(\delta, \tau) \) such that \( x_j \leq \delta \) for all \( j \geq j(\delta, \tau) \), and \( \sum_{j=1}^{s} (j+1)^{-x_j} = \Theta(s^{1-\delta}) \).

Hence
\[ \prod_{j=1}^{s} \left( m_0 + \frac{\beta}{(j+1)^{x_j}} \right) \quad \text{as well as} \quad \left( \sum_{k=1}^{\infty} \lambda_{s,k}^\tau \right)^{1/\tau} \]

is exponential in \( s^{1-\delta} \). This means that (20) does not hold for any positive \( \tau \) and non-negative \( q \). Hence, we do not have PT.

Suppose now that \( \alpha > 0 \). Then \( \lim_j x_j > 0 \) for \( \tau > (\alpha \log \omega^{-1})^{-1} \). This implies that
\[ \sum_{j=1}^{s} (j+1)^{-x_j} \quad \text{as well as} \quad \left( \sum_{k=1}^{\infty} \lambda_{s,k}^\tau \right)^{1/\tau} \]

is uniformly bounded in \( s \). Hence, (20) holds for \( q = 0 \) and we have SPT with the exponent \( p^* \leq 2/(\alpha \log \omega^{-1}) \). For \( \tau < (\alpha \log \omega^{-1})^{-1} \), the series \( \sum_{j=1}^{s} (j+1)^{-x_j} \) is of order at least \( \log s \) and (20) may hold only for \( q > 0 \). This contradicts SPT. Hence \( p^* \geq 2/(\alpha \log \omega^{-1}) \), which completes the proof. \( \square \)
5.2 New Notions of Tractability

We now turn to new notions of tractability which correspond to the standard notions of tractability for the pair \((s, 1 + \log \epsilon^{-1})\) instead of the pair \((s, \epsilon^{-1})\). To distinguish between the standard and new notions of tractability, we add the prefix EC (exponential convergence) when we consider the new notions. As before, we study the new notions of tractability for the approximation problem \(APP = \{APP_s\}_{s \in \mathbb{N}}\) for general parameters \(a, b, m\) and \(\omega\).

- **EC-\((t_1, t_2)\)-Weak Tractability**
  
  We say that \(APP\) is **EC-\((t_1, t_2)\)-WT** iff
  \[
  \lim_{s + \epsilon^{-1} \to \infty} \frac{\log n(\epsilon, APP_s)}{s^{t_1} + \lfloor \log \epsilon^{-1} \rfloor^{t_2}} = 0.
  \]

  Obviously, **EC-\((t_1, t_2)\)-WT** implies **\((t_1, t_2)\)-WT**. For \(t_1 = 1\) and \(t_2 > 1\), this notion was introduced and studied in [15].

  **Theorem 6.**

  \(APP\) is **EC-\((t_1, t_2)\)-WT** for the parameters \(a, b, m\) and \(\omega\) iff \(t_1 > 1\), or \(t_2 > 1\) and \(m_0 = 1\).

  **Proof.** Suppose that \(APP\) is **EC-\((t_1, t_2)\)-WT** for the parameters \(a, b, m\) and \(\omega\). Then for fixed \(\epsilon \in (0, 1)\) we obtain from (i) of Lemma 1 that
  \[
  0 = \lim_{s \to \infty} \frac{n(\epsilon, APP_s)}{s^{t_1} + \lfloor \log \epsilon^{-1} \rfloor^{t_2}} \geq \lim_{s \to \infty} \frac{s \log m_0}{s^{t_1} + \lfloor \log \epsilon^{-1} \rfloor^{t_2}} = \lim_{s \to \infty} s^{1-t_1} \log m_0.
  \]

  Hence, we conclude that \(t_1 > 1\) or that \(m_0 = 1\). For \(t_1 \leq 1\) and \(m_0 = 1\), it remains to show that \(t_2 > 1\). As in [15, p. 609] we find that for \(\epsilon_s \in (\lambda^{\lfloor s/2 \rfloor+1}/2, \lambda^{\lfloor s/2 \rfloor}/2) =: L_s\), where \(\lambda := \omega^{a_1}\), we have \(n(\epsilon_s, APP_s) \geq 2^{\lfloor s/2 \rfloor}\). Then
  \[
  0 = \lim_{s \to \infty} \frac{n(\epsilon_s, APP_s)}{s^{t_1} + \lfloor \log \epsilon^{-1} \rfloor^{t_2}} \geq \lim_{s \to \infty} \frac{|s/2| \log 2}{s^{t_1} + \left(\frac{|s/2|+1}{2} \log \lambda^{-1}\right)^{t_2}}.
  \]

  This can only hold if \(t_2 > 1\).

  Suppose now that \(t_1 > 1\). From [16] we have for small \(\epsilon_s\),
  \[
  \frac{n(\epsilon_s, APP_s)}{s^{t_1} + \lfloor \log \epsilon^{-1} \rfloor^{t_2}} = O\left(\frac{s \log \log \epsilon^{-1}}{s^{t_1} + \lfloor \log \epsilon^{-1} \rfloor^{t_2}}\right).
  \]

  Let \(y = \max(s^{t_1}, \lfloor \log \epsilon^{-1} \rfloor^{t_2})\). Then
  \[
  \frac{s \log \log \epsilon^{-1}}{s^{t_1} + \lfloor \log \epsilon^{-1} \rfloor^{t_2}} \leq \frac{y^{1/t_1} \log y}{t_2 y}.
  \]

  Clearly, this goes to zero as \(s + \epsilon^{-1}\) approaches infinity since \(t_1 > 1\) and \(t_2 > 0\). Hence, we have **EC-\((t_1, t_2)\)-WT**, as claimed.
Suppose now that \( t_2 > 1 \) and \( m_0 = 1 \). From the proof of Theorem 2 (17), (18) and (19) we obtain

\[
\log n(\varepsilon, \text{APP}_s) \leq C \log^{-1} \log s + C_3 \log^{-1} \log \log \varepsilon^{-1}
\]

with suitable constants \( C, C_3 > 0 \). Since \( t_1 > 0 \) and \( t_2 > 1 \) and using the same argument for \( y = \max(s^{t_1}, [\log \varepsilon^{-1}]^{t_2}) \) as above, it follows that

\[
\lim_{s + \varepsilon^{-1} \to \infty} \frac{\log n(\varepsilon, \text{APP}_s)}{s^{t_1} + [\log \varepsilon^{-1}]^{t_2}} = 0.
\]

Hence, we have EC-(\( t_1, t_2 \))-WT. This completes the proof. \( \Box \)

- **EC-Weak and EC-Uniform Weak Tractability**

EC-weak tractability (EC-WT) corresponds to EC-(1,1)-WT. EC-uniform weak tractability (EC-UWT) means that EC-(\( t_1, t_2 \))-WT holds for all \( t_1, t_2 \in (0, 1] \). Clearly, EC-WT implies WT, and EC-UWT implies UWT.

**Theorem 7.**

- APP is EC-WT iff \( m_0 = 1 \) and \( \lim_{j \to \infty} a_j = \infty \),
- APP is EC-UWT iff \( m_0 = 1 \) and \( \lim_{j \to \infty} \frac{\log a_j}{\log j} = \infty \).

**Proof.** We first assume that EC-WT or EC-UWT holds. Since EC-WT implies WT and EC-UWT implies UWT, Theorem 3 implies that \( m_0 = 1 \). For \( t_1, t_2 \in (0, 1] \), let

\[
z_{s,t_1,t_2} = \frac{\log n(\varepsilon, \text{APP}_s)}{s^{t_1} + [\log \varepsilon^{-1}]^{t_2}}.
\]

Let \( \delta > 0 \) and take \( x(\varepsilon) = (1 + \delta)(a_1 + \cdots + a_s) \). Due to the definition (11) of \( x(\varepsilon) \) this means that

\[
\log \varepsilon^{-1} = \log \omega^{-1} \frac{2}{(1 + \delta) (a_1 + a_2 + \cdots + a_s)}.
\]

From (ii) of Lemma 1 we have

\[
z_{s,t_1,t_2} \geq \frac{s \log(1 + m_1)}{s^{t_1} + [\log \varepsilon^{-1}]^{t_2}} \geq \frac{\log 2}{s^{t_1-1} + \frac{[\frac{1}{2}(1 + \delta)}{\log \omega^{-1}]^{t_2} y_s},
\]

where

\[
y_s = \frac{(a_1 + a_2 + \cdots + a_s)^{t_2}}{s} \leq \frac{(s a_s)^{t_2}}{s} = \frac{a_s^{t_2}}{s^{1-t_2}}.
\]

Then \( \lim_s z_{s,t_1,t_2} = 0 \) implies that \( \lim_s y_s = \infty \), which in turn implies that

\[
\lim_{s \to \infty} \frac{a_s^{t_2}}{s^{1-t_2}} = \lim_{s \to \infty} \frac{a_s}{s^{t_2-1}} = \infty.
\]

If we have EC-WT then \( \lim_s z_{s,1,1} = 0 \) and \( \lim_s a_j = \infty \), as claimed. If we have EC-UWT then, in particular, \( \lim_s z_{s,1,t_2} = 0 \) for all positive \( t_2 \). Then (22) yields there is a number \( s^* = s^*(t_2) \) such that

\[
a_s \geq s^{t_2-1} \quad \text{for all} \quad s \geq s^*(t_2),
\]

27
or equivalently
\[
\frac{\log a_s}{\log s} \geq \frac{1 - t_2}{t_2}
\]
for all \( s \geq s^*(t_2) \).

Since \( t_2 \) can be arbitrarily close to zero this implies that
\[
\lim_{s \to \infty} \frac{\log a_s}{\log s} = \infty,
\]
as claimed.

We now prove that \( m_0 = 1 \) and \( \lim_j a_j = \infty \) imply EC-WT. For any positive \( \eta \) we compute the \( \eta \) powers of the eigenvalues \( \lambda_{s,k} \) of the operator \( W_s \). We have
\[
n \lambda_{s,n}^\eta \leq \prod_{k=1}^\infty \lambda_{s,k}^\eta = \prod_{j=1}^s \left( 1 + \sum_{k=1}^\infty m_k \omega^{\eta a_j k^b_j} \right) \leq \prod_{j=1}^s (1 + C_\eta \omega^{\eta a_j}),
\]
due to (21). Hence,
\[
\lambda_{s,n} \leq \frac{\prod_{j=1}^s (1 + C_\eta \omega^{\eta a_j})^{1/\eta}}{n^{1/\eta}}.
\]
Then (22) yields
\[
n(\varepsilon, \text{APP}_s) \leq \frac{\prod_{j=1}^s (1 + C_\eta \omega^{\eta a_j})}{\varepsilon^{2\eta}}.
\]
Since \( \log(1 + x) \leq x \) for positive \( x \), we conclude
\[
\log n(\varepsilon, \text{APP}_s) \leq 2\eta \log \varepsilon^{-1} + C_\eta \sum_{j=1}^s \omega^{\eta a_j}.
\]
Observe that \( \lim_j a_j = \infty \) implies \( \lim_j \omega^{\eta a_j} = 0 \) and \( \lim_s \sum_{j=1}^s \omega^{\eta a_j} / s = 0 \). Therefore
\[
\limsup_{s+\varepsilon^{-1} \to \infty} \frac{\log n(\varepsilon, \text{APP}_s)}{s + \log \varepsilon^{-1}} \leq 2\eta.
\]
Since \( \eta \) can be arbitrarily small this proves that
\[
\lim_{s+\varepsilon^{-1} \to \infty} \frac{\log n(\varepsilon, \text{APP}_s)}{s + \log \varepsilon^{-1}} = 0.
\]
Hence EC-WT holds.

Finally, we prove that \( m_0 = 1 \) and \( \lim_s(\log a_s) / \log s = \infty \) imply EC-UWT. For \( x(\varepsilon) > a_1 \), (13) of Lemma 1 yields
\[
\log n(\varepsilon, \text{APP}_s) \leq \sum_{j=1}^{j(\varepsilon)} \log \left( 1 + m_{\max} \left( \frac{x(\varepsilon)}{a_j} \right)^{1/b_j} \right).
\]
Let \( b_0 = \inf_j b_j > 0 \) and
\[
\alpha = m_{\max} \left( \frac{2}{a_1 \log \omega^{-1}} \right)^{1/b_0}.
\]
From the definition (11) of $x(\varepsilon)$ we then have
\[
m_{\max} \left( \frac{x(\varepsilon)}{a_j} \right)^{1/b_j} \leq \alpha \left[ \log \varepsilon^{-1} \right]^{1/b_0},
\]
and
\[
\log n(\varepsilon, \text{APP}_s) \leq j(\varepsilon) \log \left( 1 + \alpha \left[ \log \varepsilon^{-1} \right]^{1/b_0} \right) = O \left( j(\varepsilon) \log \log \varepsilon^{-1} \right).
\]
We now estimate $j(\varepsilon)$ using the assumption that $\lim_j (\log a_j) / \log j = \infty$. We know that for all positive $\tau$ there is a number $j_\tau$ such that
\[
a_j \geq j_\tau \quad \text{for all} \quad j \geq j_\tau.
\]
This implies that
\[
j(\varepsilon) \leq \max \left( j_\tau - 1, x(\varepsilon) \right)^{1/\tau} = O \left( \left[ \log \varepsilon^{-1} \right]^{1/\tau} \right).
\]
Therefore
\[
\log n(\varepsilon, \text{APP}_s) = O \left( \left[ \log \varepsilon^{-1} \right]^{1/\tau} \log \log \varepsilon^{-1} \right).
\]
We stress that the factors in the big $O$ notation do not depend on $s$. Then for any positive $t_1, t_2 \in (0, 1]$ we take $\tau > 1/t_2$, and conclude
\[
\lim_{s+\varepsilon^{-1} \to \infty} \frac{\log n(\varepsilon, \text{APP}_s)}{s^{t_1} + \left[ \log \varepsilon^{-1} \right]^{t_2}} = 0.
\]
This proves EC-UWT, and completes the proof. \(\square\)

If we compare Theorems 3 and 7, we see that the assumption $m_0 = 1$ is always needed. However, WT holds for all $a = \{a_j\}_{j \in \mathbb{N}}$, whereas EC-WT requires that $\lim_j a_j = \infty$. Similarly, UWT holds for all $a = \{a_j\}_{j \in \mathbb{N}}$, whereas EC-UWT requires that $\lim_j (\log a_j) / \log j = \infty$. Hence, $a_j$’s may go to infinity arbitrarily slowly for EC-WT, whereas they must go to infinity faster than polynomially to get EC-UWT. It seems interesting that WT, UWT, EC-WT and EC-UWT do not depend on $b, m$ (with $m_0 = 1$) and $\omega$.

**EC-Quasi-Polynomial Tractability**

APP is EC-QPT if there are positive $C$ and $t$ such that
\[
n(\varepsilon, \text{APP}_s) \leq C \exp \left( t \left( 1 + \log s \right) \left( 1 + \log \left( 1 + \log \varepsilon^{-1} \right) \right) \right) \quad \text{for all} \quad s \in \mathbb{N}, \varepsilon \in (0, 1).
\]
The infimum of $t$ satisfying the bound above is denoted by $t^*$, and is called the exponent of EC-QPT. Obviously, EC-QPT implies EC-WT.

Observe that
\[
\exp \left( t \left( 1 + \log s \right) \left( 1 + \log \left( 1 + \log \varepsilon^{-1} \right) \right) \right) = \left[ e \right]^{t \left( 1 + \log \left( 1 + \log \varepsilon^{-1} \right) \right)} = \left[ e \left( 1 + \log \varepsilon^{-1} \right) \right]^{t \left( 1 + \log s \right)}.
\]
We will sometimes use these equivalent formulations to establish EC-QPT.
Theorem 8.

APP is EC-QPT iff

\[ m_0 = 1, \quad B^* := \sup_{s \in \mathbb{N}} \frac{\sum_{j=1}^s b_j^{-1}}{1 + \log s} < \infty, \quad \text{and} \quad \alpha := \liminf_{j \to \infty} \frac{(1 + \log j) \log a_j}{j} > 0. \]

If this holds then the exponent of EC-QPT satisfies

\[ t^* \in \left[ \max \left( B^*, \frac{\log (1 + m_1)}{\alpha} \right), B^* + \frac{\log (1 + m_{\text{max}})}{\alpha} \right]. \]

In particular, if \( \alpha = \infty \) then \( t^* = B^* \).

Proof. We first prove that EC-QPT implies the conditions on \( m_0, B^* \) and \( \alpha \). Since EC-QPT yields EC-WT, we have \( m_0 = 1 \). To prove that \( B^* < \infty \), we relate EC-QPT to EXP. From

\[ n = n(\varepsilon, \text{APP}_s) \leq C \exp \left( t (1 + \log s)(1 + \log (1 + \log \varepsilon^{-1})) \right) \]

we conclude that

\[ e(n, \text{APP}_s) \leq \varepsilon \leq e \cdot \exp \left( -\frac{1}{e} \left( \frac{n}{\varepsilon} \right)^{(t(1+\log s))^{-1}} \right). \]

Due to Theorem \([1]\) EXP holds with the exponent \( 1/B_s = 1/\sum_{j=1}^s b_j^{-1} \). Therefore \( 1/(t(1+\log s)) \leq 1/B_s \) and

\[ \frac{B_s}{1 + \log s} \leq t \quad \text{for all} \quad s \in \mathbb{N}. \]

Hence, \( B^* < \infty \) and \( t \geq B^* \), as claimed.

To prove that \( \alpha > 0 \), we proceed similarly as for EC-WT and EC-UWT. That is, for a positive \( \delta \), we take

\[ x(\varepsilon) = (1 + \delta)(a_1 + \cdots + a_s) \leq (1 + \delta) s a_s. \]

Now \([2]\) of Lemma \([2]\) yields

\[ s \log (1 + m_1) \leq \log n(\varepsilon, \text{APP}_s) \leq \log C + t(1 + \log s)(1 + \log (1 + \log \varepsilon^{-1})), \]

where

\[ 1 + \log (1 + \log \varepsilon^{-1}) \leq 1 + \log \left( 1 + \frac{(1 + \delta) \log \omega^{-1}}{2} s a_s \right), \]

For large \( s \), this proves that

\[ \frac{t (1 + \log s) \log a_s}{s} \geq \log (1 + m_1) + o(s). \]

Hence, \( \alpha > 0 \) and \( t \geq \log(1 + m_1)/\alpha \), as claimed.

We now prove that \( m_0 = 1, \ B^* < \alpha \) and \( \alpha > 0 \) imply EC-QPT.
From $m_0 = 1$ and Lemma 1 we have $n(\varepsilon, \text{APP}_s) = 1$ for $x(\varepsilon) \leq a_1$, whereas for $x(\varepsilon) > a_1$, we have

$$n(\varepsilon, \text{APP}_s) \leq \prod_{j=1}^{\min(s,j(\varepsilon))} \left(1 + m_{\max}\left(\frac{x(\varepsilon)}{a_j}\right)^{1/b_j}\right)$$

$$\leq (1 + m_{\max})^{\min(s,j(\varepsilon))} \prod_{j=1}^{\min(s,j(\varepsilon))} \frac{x(\varepsilon)}{a_j}^{1/b_j}.$$ 

From the definition (11) of $x(\varepsilon)$, we get

$$n(\varepsilon, \text{APP}_s) \leq (1 + m_{\max})^{\min(s,j(\varepsilon))} C_{s,\varepsilon} \left[ e \log \varepsilon^{-1} \right]^{\sum_{j=1}^{\min(s,j(\varepsilon))} b_j^{-1}},$$

where

$$C_{s,\varepsilon} = \prod_{j=1}^{\min(s,j(\varepsilon))} \left(\frac{2}{a_j e \log \omega^{-1}}\right)^{1/b_j}.$$ 

Note that (23) holds for all $s \in \mathbb{N}$ and all $\varepsilon \in (0,1)$ if we take $j(\varepsilon) = 0$ for $x(\varepsilon) \leq a_1$. We now use the assumption that $\alpha > 0$. This means that for any $\delta \in (0,\alpha)$ there is an integer $j_\delta$ such that

$$a_j \geq \exp\left(\frac{\delta j}{1 + \log j}\right) \text{ for all } j \geq j_\delta.$$ 

This means that $\lim_j a_j = \infty$, and this convergence is almost exponential in $j$. We turn to $j(\varepsilon)$ defined by (15). Now $j(\varepsilon)$ goes to infinity as $\varepsilon$ approaches zero. For $\log x(\varepsilon) \geq \delta$, i.e., for $\varepsilon \leq \omega^{e^\delta/2}$, we have

$$j(\varepsilon) \leq \max(j_\delta, J(\varepsilon)),$$

where $J(\varepsilon)$ is a solution of the nonlinear equation

$$\frac{\log x(\varepsilon)}{\delta} = \frac{J(\varepsilon)}{1 + \log J(\varepsilon)}.$$ 

The solution is unique since the function $y/(1 + \log y)$ is increasing for $y \geq 1$. Let $a(\varepsilon) = (\log x(\varepsilon))/\delta$. Then we have from (24) that $J(\varepsilon) = a(\varepsilon)(1 + \log J(\varepsilon))$. Now we write $J(\varepsilon)$ in the form

$$J(\varepsilon) = (1 + f(\varepsilon))a(\varepsilon) \log a(\varepsilon),$$

where $f(\varepsilon)$ is given by

$$f(\varepsilon) = \frac{1 + \log(1 + \log J(\varepsilon))}{\log a(\varepsilon)} = \frac{1 + \frac{1}{\log(1 + \log J(\varepsilon))}}{\log J(\varepsilon)/\log(1 + \log J(\varepsilon)) - 1} = o(1) \text{ for } \varepsilon \to 0.$$ 

Hence we have

$$J(\varepsilon) = (1 + o(1))a(\varepsilon) \log a(\varepsilon)$$

31
\[ \begin{align*}
&= \frac{1 + o(1)}{\delta} \left[ \log x(\varepsilon) \right] \log \frac{\log x(\varepsilon)}{\delta} \\
&= \frac{1 + o(1)}{\delta} \left[ \log \log \varepsilon^{-1} \right] \log \log \varepsilon^{-1}. 
\end{align*} \tag{25} \]

We turn to (23). Note that \( \lim_{j} a_j = \infty \) implies that only a finite number of factors in \( C_{s,\varepsilon} \) is larger than one. Therefore

\[ C_{s,\varepsilon} \leq C_1 := \sup_{s \in \mathbb{N}, \varepsilon \in (0,1)} \prod_{j=1}^{\min(s,j(\varepsilon))} \left( \frac{2}{a_j e \log \omega^{-1}} \right)^{1/b_j} < \infty. \]

Furthermore, from the assumption \( B^* < \infty \) we have

\[ \sum_{j=1}^{\min(s,j(\varepsilon))} b_j^{-1} = \sum_{j=1}^{\min(s,j(\varepsilon))} \frac{b_j^{-1}}{1 + \log s} (1 + \log s) \leq B^* (1 + \log s). \]

Therefore we can rewrite (23) as

\[ n(\varepsilon, \text{APP}_s) \leq (1 + m_{\text{max}})^{\min(s,j(\varepsilon))} C_1 \left[ e \left( 1 + \log \varepsilon^{-1} \right) \right] B^* (1 + \log s). \tag{26} \]

We now analyze the first factor \( \beta := (1 + m_{\text{max}})^{\min(s,j(\varepsilon))} \) in (26). Let \( s^* \in \mathbb{N} \) and \( \varepsilon^* \in (0,1) \) be arbitrary. Note that for \( s \leq s^* \) or for \( \varepsilon \in [\varepsilon^*,1] \) we have

\[ \beta \leq (1 + m_{\text{max}})^{\max(s^*,j(\varepsilon^*))} =: C_2 < \infty. \]

Hence, without loss of generality we can consider

\[ s > s^* \quad \text{and} \quad \varepsilon \in (0, \varepsilon^*). \]

We now choose \( s^* \) such that \( s^* \geq j_\delta \). For any positive \( \eta \in (0,1) \) we choose a positive \( \varepsilon^* \) such that for all \( \varepsilon \in (0, \varepsilon^*) \) we have

\[ \log \log \log \varepsilon^{-1} \geq \frac{\delta}{1 - \eta}, \tag{27} \]

\[ \frac{\delta J(\varepsilon)}{\log \log \varepsilon^{-1} \log \log \log \varepsilon^{-1}} \in [1 - \eta, 1 + \eta], \tag{28} \]

\[ \frac{1 + \eta}{\delta} \leq \frac{1 + 2\eta}{\delta}. \tag{29} \]

Observe that such a positive \( \varepsilon^* \) exists since (27) clearly holds for small \( \varepsilon \), whereas (28) holds due to (25), and (29) holds since the limit of the left hand side is \( (1 + \eta)/\delta \) which is smaller than the right hand side.

We are ready to estimate

\[ \beta = [es]^{y_{s,\varepsilon}} = [e \left( 1 + \log \varepsilon^{-1} \right)]^{z_{s,\varepsilon}}, \]

where

\[ y_{s,\varepsilon} = \frac{\min(s,j(\varepsilon)) \log (1 + m_{\text{max}})}{\log (es)}, \]

72
\[ z_{s,\varepsilon} = \frac{\min(s, j(\varepsilon)) \log (1 + m_{\text{max}})}{\log (e (1 + \log \varepsilon^{-1}))}. \]

We consider two cases depending on whether \( s \) or \( J(\varepsilon) \) is larger.

**Case 1.** Assume that \( s \leq J(\varepsilon) \).

Note that the function \( y/(1 + \log y) \) is an increasing function of \( y \in [1, \infty) \). Therefore

\[ \frac{s}{1 + \log s} \leq \frac{J(\varepsilon)}{1 + \log J(\varepsilon)}. \]

Due to (28) and (29),

\[ \frac{J(\varepsilon)}{1 + \log J(\varepsilon)} \leq \frac{(1 + \eta) \log \log \varepsilon^{-1} \log \log \log \varepsilon^{-1}}{\delta (1 + \log \frac{1 + \eta}{\delta} + \log \log \varepsilon^{-1} + \log \log \log \log \varepsilon^{-1})} \leq \frac{1 + 2\eta}{\delta} \log \log \varepsilon^{-1} \leq \frac{1 + 2\eta}{\delta} (1 + \log (1 + \log \varepsilon^{-1})). \]

Hence,

\[ y_{s,\varepsilon} \leq \frac{s}{1 + \log s} \log (1 + m_{\text{max}}) \leq \frac{J(\varepsilon)}{1 + \log J(\varepsilon)} \log (1 + m_{\text{max}}) \leq \frac{(1 + 2\eta) \log(1 + m_{\text{max}})}{\delta} (1 + \log (1 + \log \varepsilon^{-1})). \]

This yields

\[ \beta \leq \lfloor e s \rfloor \delta^{-1} (1 + 2\eta) \log (1 + m_{\text{max}}) (1 + \log (1 + \log \varepsilon^{-1})) \]

which can be equivalently written as

\[ \beta \leq \exp \left( \left[ \frac{1 + 2\eta}{\delta} \log (1 + m_{\text{max}}) \right] (1 + \log s) (1 + \log (1 + \log \varepsilon^{-1})) \right). \]

This and (26) yield EC-QPT with \( t \leq B^* + \delta^{-1} (1 + 2\eta) \log (1 + m_{\text{max}}) \). Since \( \delta \) can be arbitrarily close to \( \alpha \) and \( \eta \) can be arbitrarily small, we conclude that the exponent of EC-QPT in this case satisfies

\[ t \leq B^* + \frac{\log (1 + m_{\text{max}})}{\alpha}. \]

**Case 2.** Assume that \( s > J(\varepsilon) \).

Then \( s > \delta^{-1} (1 - \eta) \log \log \varepsilon^{-1} \geq \log \log \varepsilon^{-1} \) due to (27) and (28). Hence, \( \log s \geq \log \log \log \varepsilon^{-1} \). We now estimate \( z_{s,\varepsilon} \). Assume that \( \varepsilon > 0 \) is small enough such that \( j(\varepsilon) \leq J(\varepsilon) \). Then we have

\[ z_{s,\varepsilon} \leq \frac{j(\varepsilon) \log (1 + m_{\text{max}})}{1 + \log (1 + \log \varepsilon^{-1})} \leq \frac{J(\varepsilon) \log (1 + m_{\text{max}})}{1 + \log (1 + \log \varepsilon^{-1})} \leq \frac{(1 + \eta) \log (1 + m_{\text{max}})}{\delta} \log \log \varepsilon^{-1} \leq \frac{(1 + \eta) \log (1 + m_{\text{max}})}{\delta} \log \log \log \varepsilon^{-1}. \]
\[
\leq \frac{(1 + \eta) \log (1 + m_{\text{max}})}{\delta} (1 + \log s).
\]

Hence we have
\[
\beta \leq \exp \left( \left[ \frac{1 + \eta}{\delta} \log (1 + m_{\text{max}}) \right] (1 + \log s)(1 + \log (1 + \log \varepsilon^{-1})) \right),
\]
and the rest of the proof goes like in Case 1. This completes the proof. \(\square\)

We compare Theorems 4 and 8. The assumption \(m_0 = 1\) is needed for both QPT and EC-QPT. However, QPT holds for all \(a\) and \(b\), whereas for EC-QPT we need to assume that \(B^* < \infty\) and \(\alpha > 0\). This means that \(b_j\) must go to infinity roughly at least like \(j\), and \(a_j\) must go to infinity almost exponentially fast.

- **EC-Polynomial and EC-Strong Polynomial Tractability**

APP is EC-PT iff there are positive \(C, p\) and \(q \geq 0\) such that
\[
n(\varepsilon, \text{APP}_s) \leq C s^q (1 + \log \varepsilon^{-1})^p \quad \text{for all } s \in \mathbb{N}, \varepsilon \in (0, 1).
\]

APP is EC-SPT if the last bound holds with \(q = 0\), and then the infimum of \(p\) is denoted by \(p^*\), and is called the exponent of EC-SPT.

**Theorem 9.**

APP is EC-PT iff APP is EC-SPT iff

\[
m_0 = 1, \quad B := \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty \quad \text{and} \quad \alpha^* = \liminf_{j \to \infty} \frac{\log a_j}{j} > 0.
\]

If these conditions hold then the exponent of EC-SPT satisfies
\[
p^* \in \left[ \max \left( B, \frac{\log(1 + m_1)}{\alpha^*} \right), B + \frac{\log(1 + m_{\text{max}})}{\alpha^*} \right].
\]

In particular, if \(\alpha^* = \infty\) then \(p^* = B\).

**Proof.** We prove that EC-PT implies \(m_0 = 1\), \(B < \infty\) and \(\alpha^* > 0\), and then that \(m_0 = 1\), \(B < \infty\) and \(\alpha^* > 0\) imply EC-SPT and find bounds on the exponent of EC-SPT.

EC-PT implies EC-WT and therefore \(m_0 = 1\). It is easy to show that EC-PT implies UEXP. Indeed, the bound on EC-PT yields that
\[
e(n, \text{APP}_s) \leq e \cdot e^{-(n-1)/(C s^q))^{1/p}} \quad \text{for all } n \in \mathbb{N}.
\]

Hence, UEXP holds and the exponent of UEXP is at least \(1/p\). Then Theorem 4 implies that \(B < \infty\), and \(p \geq B\).

To prove that \(\alpha^* > 0\), we proceed similarly as for EC-WT. That is, for \(\delta > 0\) we take \(x(\varepsilon) = (1 + \delta)(a_1 + \cdots + a_s)\) and then (ii) of Lemma 1 and the bound on EC-PT yield
\[
\left(1 + m_1\right)^* \leq n(\varepsilon, \text{APP}_s) \leq C s^q \left(1 + \frac{(1 + \delta) \log \omega^{-1}}{2}(a_1 + \cdots + a_s)\right)^p.
\]
Since $a_1 \leq a_2 \leq \cdots$, this implies that
\[
sa_s \geq a_1 + \cdots + a_s \geq \frac{2}{(1+\delta) \log \omega^{-1}} \left[ \left( \frac{(1+m_1)^s}{C \omega^q} \right)^{1/p} - 1 \right].
\]
Hence,
\[
\alpha^* = \liminf_{s \to \infty} \frac{\log a_s}{s} \geq \frac{\log(1+m_1)}{p} > 0,
\]
as claimed. This also shows that $p \geq \log(1+m_1)/\alpha^*$. This reasoning also holds for all $p$ for which EC-SPT holds. Therefore the exponent $p^*$ of EC-SPT is at least $p^* \geq \log(1+m_1)/\alpha^*$. Furthermore, $p^*$ cannot be smaller then the reciprocal of the exponent of UEXP, so that $p^* \geq B$. This proves the lower bound on $p^*$.

We now assume that $m_0 = 1$, $B < \infty$ and $\alpha^* > 0$. Note that $\alpha^* > 0$ means that $a_j$ are exponentially large in $j$ for large $j$. Indeed, for $\delta \in (0, \alpha^*)$ there is an integer $j^*_\delta$ such that
\[
a_j \geq \exp(\delta j) \quad \text{for all} \quad j \geq j^*_\delta.
\]
This yields that for $j(\varepsilon)$ defined by (15) we have
\[
j(\varepsilon) \leq \max \left( j^*_\delta, \frac{\log x(\varepsilon)}{\delta} \right).
\]
For $x(\varepsilon) \leq a_1$, (i) of Lemma 1 states that $n(\varepsilon, \text{APP}_s) = 1$, whereas for $x(\varepsilon) > a_1$, (iii) of Lemma 1 yields
\[
n(\varepsilon, \text{APP}_s) \leq \prod_{j=1}^{\min(s,j(\varepsilon))} \left( 1 + m_{\max} \left( \frac{x(\varepsilon)}{a_j} \right)^{1/b_j} \right)
\leq \prod_{j=1}^{\min(s,j(\varepsilon))} \left( \frac{x(\varepsilon)}{a_j} \right)^{1/b_j} \left( 1 + m_{\max} \right)^{\min(s,j(\varepsilon))}
\leq \left( \frac{x(\varepsilon)}{a_1} \right)^B \max \left( (1 + m_{\max})^{j^*_\delta}, [x(\varepsilon)]^{(\log(1+m_{\max}))/\delta} \right).
\]
Since $x(\varepsilon) = \Theta(\log \varepsilon^{-1})$ we obtain EC-SPT with $p \leq B + (\log(1+m_{\max}))/\delta$. Taking $\delta$ arbitrarily close to $\alpha^*$, we obtain that the exponent of EC-SPT is at most
\[
p^* \leq B + \frac{\log(1+m_{\max})}{\alpha^*}.
\]
This completes the proof. □

We now compare Theorems 5 and 9 for SPT and EC-SPT. In both cases, we have $m_0 = 1$. However, the conditions on $a$ and $b$ are quite different. SPT holds for all $b$, whereas for EC-SPT we must assume that $B < \infty$, i.e., $b_j$ must go to infinity at least like $j$. The conditions on $a$ are even more striking. SPT holds for $a_j$ going to infinity quite slowly like $\log j$, whereas EC-SPT requires that $a_j$ goes exponentially fast to infinity with $j$. 

35
6 Summary

In the following table we summarize the tractability results. We tabulate the various notions of tractability with their corresponding “if and only if” conditions:

| Tractability notion | iff-conditions |
|---------------------|----------------|
| $(t_1, t_2)$-WT     | $t_1 > 1$ or $m_0 = 1$ |
| WT, UWT, QPT        | $m_0 = 1$       |
| PT, SPT             | $m_0 = 1$, and $\lim_j \frac{a_j}{\log j} > 0$ |
| EC-$(t_1, t_2)$-WT  | $t_1 > 1$, or $t_2 > 1$ and $m_0 = 1$ |
| EC-WT               | $m_0 = 1$, and $\lim_j a_j = \infty$ |
| EC-UWT              | $m_0 = 1$, and $\lim_j \frac{\log a_j}{\log j} = \infty$ |
| EC-QPT              | $m_0 = 1$, $\sup_s \sum_{j=1}^s b_j^{-1} < \infty$, and $\lim \inf_j \frac{(1+\log j)\log a_j}{j} > 0$ |
| EC-PT, EC-SPT       | $m_0 = 1$, $\sum_{j=1}^\infty b_j^{-1} < \infty$, and $\lim \inf_j \frac{\log a_j}{j} > 0$ |

References

[1] N. Aronszajn: Theory of reproducing kernels. Trans. Amer. Math. Soc. 68: 337–404, 1950.

[2] J. Dick, P. Kritzer, F. Pillichshammer, and H. Woźniakowski: Approximation of analytic functions in Korobov spaces. J. Complexity 30: 2–28, 2014.

[3] J. Dick, G. Larcher, F. Pillichshammer, and H. Woźniakowski: Exponential convergence and tractability of multivariate integration for Korobov spaces. Math. Comp. 80: 905–930, 2011.

[4] J. Dick, D. Nuyens, and F. Pillichshammer: Lattice rules for nonperiodic smooth integrands. Numer. Math. 126: 259–291, 2014.

[5] J. Dick and F. Pillichshammer. Multivariate integration in weighted Hilbert spaces based on Walsh functions and weighted Sobolev spaces. J. Complexity 21: 149–195, 2005.

[6] J. Dick and F. Pillichshammer. Digital Nets and Sequences. Discrepancy Theory and Quasi-Monte Carlo Integration. Cambridge University Press, Cambridge, 2010.

[7] M. Gnewuch and H. Woźniakowski: Quasi-polynomial tractability. J. Complexity 27: 312–330, 2011.
[8] C. Irrgeher, P. Kritzer, G. Leobacher, and F. Pillichshammer: Integration in Hermite space of analytic functions. To appear in J. Complexity, 2015.

[9] C. Irrgeher and G. Leobacher: High-dimensional integration on the $\mathbb{R}^d$, weighted Hermite spaces, and orthogonal transforms. J. Complexity 31: 174–205, 2015.

[10] P. Kritzer, F. Pillichshammer, and H. Woźniakowski: Multivariate integration of infinitely many times differentiable functions in weighted Korobov spaces. Math. Comp. 83: 1189–1206, 2014.

[11] P. Kritzer, F. Pillichshammer, and H. Woźniakowski: Tractability of multivariate analytic problems. In: Uniform Distribution and Quasi-Monte Carlo Methods. Discrepancy, Integration and Applications (P. Kritzer, H. Niederreiter, F. Pillichshammer and A. Winterhof, eds.). De Gruyter, Berlin, 2014.

[12] E. Novak and H. Woźniakowski: Tractability of Multivariate Problems, Volume I: Linear Information. EMS, Zurich, 2008.

[13] E. Novak and H. Woźniakowski: Tractability of Multivariate Problems, Volume II: Standard Information for Functionals. EMS, Zurich, 2010.

[14] E. Novak and H. Woźniakowski: Tractability of Multivariate Problems, Volume III: Standard Information for Operators. EMS, Zurich, 2012.

[15] A. Papageorgiou and I. Petras: A new criterion for tractability of multivariate problems. J. Complexity 30: 604–619, 2014.

[16] G. Sansone: Orthogonal Functions. 2nd ed. John Wiley and Sons Inc, New York, 1977.

[17] P. Siedlecki: Uniform weak tractability. J. Complexity 29: 438–453, 2013.

[18] P. Siedlecki and M. Weimar: Notes on $(s,t)$-weak tractability: a refined classification of problems with (sub) exponential information complexity. Submitted for publication; arXiv:1411.3466

[19] J. F. Traub, G. W. Wasilkowski and H. Woźniakowski: Information-Based Complexity. Academic Press, New York, 1988.

Authors’ addresses:

Christian Irrgeher, Peter Kritzer, Friedrich Pillichshammer, Department of Financial Mathematics and Applied Number Theory, Johannes Kepler University Linz, Altenbergerstr. 69, 4040 Linz, Austria

Henryk Woźniakowski, Department of Computer Science, Columbia University, New York 10027, USA, and Institute of Applied Mathematics, University of Warsaw, ul. Banacha 2, 02-097 Warszawa, Poland

E-mail: christian.irrgeher@jku.at
peter.kritzer@jku.at
friedrich.pillichshammer@jku.at
henryk@cs.columbia.edu