EXISTENCE AND REGULARITY OF STEADY-STATE SOLUTIONS OF THE NAVIER-STOKES EQUATIONS ARISING FROM IRREGULAR DATA

GAEL Y. DIEBOU

Abstract. We analyze the forced incompressible stationary Navier-Stokes flow in \( \mathbb{R}^n \), \( n > 2 \). Existence of a unique solution satisfying a global integrability property measured in a scale of tent spaces is established for small data in homogeneous Sobolev space with \( s = -\frac{1}{2} \) degree of smoothness. The velocity field is shown to be locally H"older continuous while the pressure belongs to \( L^p_{\text{loc}} \) for any \( p \in (1, \infty) \). Our approach is based on the analysis of the inhomogeneous Stokes system for which we derive a new solvability result involving Dirichlet data in Triebel-Lizorkin classes with negative amount of smoothness and is of independent interest.

1. Introduction

The steady state (forced) incompressible Navier-Stokes equations in a domain \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \) is the following system

\[
\begin{align*}
-\Delta u + \nabla \pi + u \cdot \nabla u &= F & \text{in } \Omega \\
\text{div } u &= 0 & \text{in } \Omega
\end{align*}
\]

(\textbf{NS})

where \( u : \Omega \to \mathbb{R}^n \) is the unknown velocity field, \( \pi : \Omega \to \mathbb{R} \) is the unknown scalar pressure and \( F : \Omega \to \mathbb{R}^n \) is a given external force. This system is supplemented by the boundary condition

\[
\begin{align*}
u &= f & \text{on } \partial \Omega
\end{align*}
\]

where \( f = (f_1, \ldots, f_n) \) is a prescribed vector field satisfying (in the case \( \Omega \) smooth bounded) the compatibility condition \( \int_{\partial \Omega} f \cdot N d\sigma = 0 \) with \( N = (N_1, \cdots, N_n) \) being the outer unit normal vector at the boundary.

Probably, the first striking result regarding the solvability of the Dirichlet problem for the Navier-Stokes equations was obtained by Leray [21]. In a smooth bounded three-dimensional domain, he showed the existence of a weak solution \( (u, \pi) \in W^{1,2}(\Omega) \times L^2(\Omega) \) provided \( f \in W^{1/2,2}(\partial \Omega) \) and \( F \in W^{-1,2}(\Omega) \). Existence of generalized weak solutions of (\textbf{NS})-(1.1) in \( \Omega \subset \mathbb{R}^3 \), those are \( (u, \pi) \in W^{1,q}(\Omega) \times L^q(\Omega) \), \( q \in (2, \infty) \) is a consequence of the important work by Cattabriga [6]. Without some extra condition on the data, the regularity of weak solutions to (\textbf{NS})-(1.1) is still far from being fully understood. Available contributions are mostly dimension dependent. Indeed, in the physically most relevant dimensions \( n = 2, 3 \); any weak solution is smooth (see e.g. [27] in the case of nonhomogeneous smooth data). In four dimensions, the \( L^p \)-regularity theory of weak solutions with zero Dirichlet data was established in [14]. When \( \Omega = \mathbb{R}^5 \) or the torus \( \mathbb{T}^5 \), it is known due to

*Date:* Thursday 15th February, 2024.

*Key words and phrases.* Stationary Navier-Stokes, existence, regularity, homogeneous Triebel-Lizorkin spaces, weighted tent spaces.

The author was partially funded by the Fields Institute for Research in Mathematical Sciences.
Struwe [31] that (NS) admits a solution \((u, \pi) \in W_{loc}^{2,p} \times W_{loc}^{1,p}\) for any \(p < \infty\) whenever the external force is smooth and compactly supported. This result was partially extended to the case \(\Omega = \mathbb{T}^n\) and very recently to \(\Omega = \mathbb{R}^n\) with \(n \in [5, 15]\) in [11] and [33], respectively. As for uniqueness of weak solutions, it seems that a smallness assumption on given data is necessary and a recent result by Luo [22] predicts that this condition cannot be dropped.

There have been growing interest in recent years in the analysis of the Navier-Stokes equations subject to low regularity data. By this, we mean that \(f\) satisfies a weaker regularity than that needed in order for generalized weak solutions to exist. In such a context, does problem (NS)-(1.1) admit a solution? In the affirmative case, what are the qualitative properties of such solutions? Prescribing boundary value with low regularity forces one to consider a notion of solution weaker than weak solution. A good candidate, roughly speaking is obtained by testing (NS) against a suitable divergence-free smooth vector field and performing two successive integration by parts taking (1.1) into account. This idea to the best of the author’s knowledge first appeared in [2] and the resulting formulation gives rise to the so-called very weak solution. When \(\Omega \subset \mathbb{R}^n, n = 2, 3\) is a \(C^2\) regular bounded domain, the author in [23] constructed such a solution in \(L^{2n/(n-1)}(\Omega)\) provided \(f \in L^2(\partial \Omega)\) (with arbitrarily large norm) and \(F \in W^{-1,2n/(n-1)}(\Omega)\). Existence, uniqueness and regularity of very weak solutions \((u, \pi)\) in the class \(L^q(\Omega) \times W^{-1,3}(\Omega)\) have been obtained in [8, 13] under certain smallness conditions on \(f \in W^{-1,2n/(n-1)}(\partial \Omega)\) and \(F \in W^{-1,r}(\Omega)\) with \(1 < r < q < \infty, 1/r \leq 1/n + 1/q\). These results were generalized in [20] wherein the author gave a complete solvability theory for very weak solutions of (NS)-(1.1). In particular, refining the definition of very weak solutions and using some ideas from the preceding references, the author showed the existence of a very weak solution \((u, \pi)\) in \(L^n(\Omega) \times W^{-1,n}(\Omega)\) for arbitrary large data \(f \in W^{-1/n,n}(\partial \Omega)\) and forcing term \(F\) for \(n = 3, 4\). In two dimensions, he proved the existence of a solution \((u, \pi) \in L^n(\Omega) \times W^{-1,q}(\Omega)\), \(2 < q < 3\). Moreover, he investigated the regularity of these solutions and also derived uniqueness results under suitable smallness assumptions. The existence theory for very weak solutions in unbounded domains (the half-space, exterior domains, etc.) seems to be more subtle. In general, similar methods as those employed for instance in [20] which rely on duality arguments and functional analytic tools cannot be carried out. We refer the reader to [10] for an interesting discussion pertaining to generalized (weak) solutions – we point out however, some recent existence results for the linear Stokes problem in half-space domain [9] and in exterior domains [19].

This paper aims at establishing the solvability theory for (NS)-(1.1) in the case \(\Omega = \mathbb{R}^n_+\) (with possible adaptation of the ideas to the case of bounded smooth domains) by means of novel ideas. The techniques employed here are new and complement those introduced in [35] for the analysis of elliptic “critical” problems subject to low regularity data. In addition, they can be invoked to study similar questions for other semilinear elliptic problems. Assuming \(\Omega = \mathbb{R}^n_+\), we seek for a velocity field of the form \(u = v + w\) where \(v\) solves the linear Stokes equation with Dirichlet data \(f\) while \(w\) solves the inhomogeneous Stokes problem with zero boundary data and source term \(F = -u \cdot \nabla u\). Odqvist [26] proved that \(v\) assumes an integral representation, it is the Stokes extension of \(f\) to \(\mathbb{R}^n_+\) (see Section 2). We then look for the data \(f\) in a large class of distributions on \(\mathbb{R}^{n-1}\) for which \(v\) is well-defined and has \(f\) as trace in a suitable sense. On one hand, (NS) is scaling (and translation) invariant with respect to the maps

\[ u_\lambda(x) = \lambda u(\lambda x), \quad \pi_\lambda(x) = \lambda^2 \pi(\lambda x), \quad \lambda > 0 \]

for appropriately rescaled external force and boundary data. On the other hand, we want to have \(u\) in the (local) Lebesgue space \(L^2_{loc}\) in order to make sense of the equation. From these observations, we are led to the consideration of \(v\) in \(\dot{T}_{s_2}^{2(n-1)/2}\), \(s_2 = -\frac{1}{4(n-1)}\) a scale
of weighted tent spaces introduced in [7, 17]. Thanks to the continuous characterization of homogeneous Triebel-Lizorkin spaces [32], we expect that \( v \) must have a distributional trace \( f \) in the homogeneous negative Sobolev space \( \dot{H}^{-1/2, 2(n-1)}(\mathbb{R}^{n-1}) \). By the same token, the pressure \( \pi \) is sought for in the weighted tent space \( T^{2(n-1), 2}_{\sigma_2} \), \( \sigma_2 = -\frac{4}{2(n-1)} \) (see below for the definition of weighted tent spaces). These expectations are actually confirmed using some special properties of the weighted tent kernel (smoothness, cancellation), see Proposition 2.1.

Tent spaces naturally arise in the analysis of linear elliptic equations and systems, see e.g. [4, 17] and references therein. We quote the recent work [35] where these spaces are used in the context of nonlinear systems.

Our main result states that there exists a unique solution of (NS)-(1.1) in a suitable framework under a smallness condition on \( f \in \dot{H}^{-1/2, 2(n-1)}(\mathbb{R}^{n-1}) \) and the external force. A more general statement involving Dirichlet data in homogeneous Triebel-Lizorkin spaces is obtained. In both cases, the constructed solution satisfies a global integrability property, expressed in terms of a weighted tent norm and it is further shown that this solution enjoys a better regularity locally. More precisely, the velocity field \( u \) belongs to \( C^{0,\alpha} \) and the pressure \( \pi \) is an element of \( L^p_{\text{loc}} \) for any \( p \in (1, \infty) \). These features are derived from the pointwise decay rate of the velocity field near the boundary. In order to derive all these results, we study the inhomogeneous Stokes problem in \( \mathbb{R}^n_+ \) (which plays a fundamental role in the analysis of (NS)-(1.1) when the flow takes place in an exterior region, a channel or a pipe) and establish key estimates of the solution for prescribed data in the homogeneous Triebel-Lizorkin class with negative amount of smoothness.

### 1.1. Tent spaces and functional settings.

Throughout, a point \( x \in \mathbb{R}^n_+ \) will typically be denoted by \( (x', x_n) \), \( x' \in \mathbb{R}^{n-1} \) and \( x_n > 0 \). For \( R > 0 \), \( B_R(x') \) is the closed ball in \( \mathbb{R}^{n-1} \) with radius \( R > 0 \) and center at \( x' \in \mathbb{R}^{n-1} \). Given \( \alpha > 0 \), define the cone (nontangential region) with vertex at \( x' \in \mathbb{R}^{n-1} \) by

\[
\Gamma_{\alpha}(x') := \{(y', y_n) \in \mathbb{R}^n_+ : |x' - y'| < \alpha y_n\}.
\]

When \( \alpha = 1 \), \( \Gamma_1(x') \) will instead be denoted as \( \Gamma(x') \). Given a ball \( B = B_{R}(x') \times (0, 2R) \) the Carleson box over \( B_R(x') \). For \( q \in [1, \infty] \), consider the functionals \( \mathcal{A}_q, \mathcal{C}_q \) defined for \( F \) measurable in \( \mathbb{R}^n_+ \) by

\[
\mathcal{A}_q F(x') = \left( \int_{\Gamma(x')} |F(y', y_n)| q y_n^{-1} dy' dy_n \right)^{1/q}, \quad \mathcal{A}_\infty F(x') = \sup_{(y', y_n) \in \Gamma(x')} |F(y', y_n)|
\]

\[
\mathcal{C}_q F(x') = \sup_{x' \in B} \left( \frac{1}{|B|} \int_{\Gamma(x')} |F(y', y_n)| q y_n^{-1} dy' dy_n \right)^{1/q}
\]

where the supremum is taken over all balls containing \( x' \). The membership of each of these functionals in Lebesgue spaces gives rise to a scale of function spaces first introduced by Coifman, Meyer and Stein [7]. The tent space \( T^{p,q} \) with \( p, q \in [1, \infty] \) collects all functions \( F \in L^q_{\text{loc}}(\mathbb{R}^n_+) \) for which \( \mathcal{A}_q F \in L^p(\mathbb{R}^{n-1}) \). We equip this space with the norm

\[
\|F\|_{T^{p,q}} := \|\mathcal{A}_q F\|_{L^p(\mathbb{R}^{n-1})}.
\]

When \( p = \infty \), the space \( T^{\infty,q} \) is defined using the Carleson functional \( \mathcal{C}_q \) by

\[
T^{\infty,q} = \{F \in L^q_{\text{loc}}(\mathbb{R}^n_+) : \mathcal{C}_q F \in L^\infty(\mathbb{R}^{n-1})\}.
\]

The space \( T^{\infty,q} \) is intrinsically linked to Carleson measures. In fact, it can alternatively be interpreted as the space of functions \( F \in L^q_{\text{loc}}(\mathbb{R}^n_+) \) for which \( d\mu(y', y_n) = |F|^q y_n^{-1} dy' dy_n \) is a Carleson measure in \( \mathbb{R}^n_+ \). Such measures have some interesting application in the study of nonlinear PDEs with critical growth in the gradient, see [35]. Note that at the endpoint
case \( p = q = \infty \), \( T^{p,q}_s \) may be identified to \( L^\infty(\mathbb{R}^n_+) \). For \( s \in \mathbb{R} \), we say that \( F : \mathbb{R}^n_+ \rightarrow \mathbb{R} \) belongs to the weighted tent space \([1, 17]\), which we denote by \( T^{p,q}_s \) if

\[
(y', y_n) \mapsto y_n^{(n-1)\alpha} F(y', y_n) \in T^{p,q}_s.
\]

It can easily be verified that \( \|F\|_{T^{p,q}_s} := \|y_n^{(1-n)\alpha} F\|_{T^{p,q}_s} \) defines a norm on \( T^{p,q}_s \) and is a Banach space. In particular, the space \( T^{p,q}_0 \), \( s_q = -\frac{1}{q(\alpha - p)} \) contains the space of functions in \( L^q(\mathbb{R}^n_+) \) with compact support as a dense subspace. This property, together with the completeness of \( T^{p,q}_{s_q} \) follows from Lemma 1.1 below.

**Lemma 1.1.** Let \( K \) be a compact set in \( \mathbb{R}^n_+ \) and assume that \( F \in T^{p,q}_{s_q} \) for \( p, q \in [1, \infty) \). Then

\[
C_1 \|1_K F\|_{T^{p,q}_{s_q}} \leq \|F\|_{L^q(K)} \leq C_2 \|F\|_{T^{p,q}_{s_q}}
\]

where the constant \( C_1, C_2 \) only depend on \( p, q, n \) and \( K \).

Out of convenience, we defer the proof of Lemma 1.1 to the Appendix.

**Remark 1.1.** The change of aperture of the cone does not modify the tent norm in the sense that if one defines the \( \alpha \)-conical functional \( A^\alpha_q \) by

\[
A^\alpha_q F(x') := \left( \int_{I_n(x')} |F(y', y_n)|^q (\alpha y_n)^{-(n-1)} dy' dy_n \right)^{1/q}, \quad \alpha > 0,
\]

then

\[
\|A^\alpha_q F\|_{L^p(\mathbb{R}^{n-1})} \approx \|A^\beta_q F\|_{L^p(\mathbb{R}^{n-1})}
\]

where the implicit constant depends on \( p, q \) and \( \alpha, \beta \in (0, \infty) \). See \([7, \text{Proposition 4, p. 309}]\) which remains valid for \( q \neq 2 \).

Moreover, for \( s_1, s_2 \in \mathbb{R} \) such that \( s_1 > s_2 \) and \( 1 \leq p_1 < p_2 \leq \infty, q \in (0, \infty) \) the following continuous embedding holds (see \([1, \text{Lemma 2.19}]\))

\[
T^{p_1,q}_{s_1} \subset T^{p_2,q}_{s_2}
\]

provided \( s_2 - s_1 = \frac{1}{p_2} - \frac{1}{p_1} \). Recall Hölder’s inequality in weighted tent spaces.

**Lemma 1.2.** Let \( p_i, q_i \in [1, \infty] \) and \( s_i \in \mathbb{R}, i \in \{0, 1, 2\} \) such that \( \sum_{i=1}^{2} 1/p_i = 1/p_0 \) and

\[
\sum_{i=1}^{2} 1/q_i = 1/q_0 \text{ with the convention } 1/\infty = 0.
\]

If \( f \in T^{p_1,q_1}_{s_1} \) and \( g \in T^{p_2,q_2}_{s_2} \), then \( fg \in T^{p_0,q_0}_{s_0} \) and it holds that

\[
\|fg\|_{T^{p_0,q_0}_{s_0}} \leq C \|f\|_{T^{p_1,q_1}_{s_1}} \|g\|_{T^{p_2,q_2}_{s_2}}
\]

provided \( s_0 = s_1 + s_2 \).

This lemma can be proved via factorization of tent spaces \([16, \text{Theorem 3.4}]\). It is long-established that there is an intrinsic connection between weighted tent spaces and Triebel-Lizorkin spaces which we now recall their definitions.

Let us denote by \( \mathcal{S}(\mathbb{R}^{n-1}) \) the class of Schwartz (smooth rapidly decreasing) functions on \( \mathbb{R}^{n-1} \) and \( \mathcal{S}'(\mathbb{R}^{n-1}) \) its topological dual space endowed with the weak-\( \star \) topology. Define the space

\[
\mathcal{S}(\mathbb{R}^{n-1}) = \{ f \in \mathcal{S}(\mathbb{R}^{n-1}) \mid \int z^\gamma f(z) dz = 0, \ \forall \ \gamma \in \mathbb{N}^n \}
\]
which inherits the topology of \( \mathcal{S}(\mathbb{R}^{n-1}) \) as subspace. This space may be identified with the space of Schwartz functions whose Fourier transforms vanish together with all their derivatives at the origin. Its dual space is denoted by \( \mathcal{S}'(\mathbb{R}^{n-1}) \). Let \( \varphi \in \mathcal{S}(\mathbb{R}^{n-1}) \) such that
\[
\varphi(\xi) = \begin{cases} 
1 & \text{if } |\xi| \leq 1 \\
0 & \text{if } |\xi| > 2.
\end{cases}
\]

Let \( \psi(\xi) = \varphi(\xi) - \varphi(2\xi) \) so that
\[
\sum_{j \in \mathbb{Z}} \psi(2^j \xi) = 1, \quad \forall \xi \in \mathbb{R}^{n-1} \setminus \{0\}.
\]

Let us denote by \( \mathcal{F} \) the Fourier transform of \( f \) on \( \mathbb{R}^{n-1} \) and by \( \hat{\Delta}_j = \mathcal{F}^{-1}(\psi(2^{-j} \cdot)) \mathcal{F} \) the homogeneous Littlewood-Paley operator. For \( s \in \mathbb{R} \) and \( p, q \in [1, \infty) \), we say that a distribution \( f \in \mathcal{S}'(\mathbb{R}^{n-1}) \) belongs to the Triebel-Lizorkin space \( \dot{F}^{s}_{p,q}(\mathbb{R}^{n-1}) \) if
\[
\|f\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^{n-1})} := \left\| \left( \sum_{j=-\infty}^{\infty} (2^{js} |\hat{\Delta}_j f|^q)^{1/q} \right)^{1/p} \right\|_{L^p(\mathbb{R}^{n-1})} < \infty.
\]

This space is complete and equivalent to the homogeneous Sobolev space \( \dot{H}^{s,p}(\mathbb{R}^{n-1}) \) whenever \( q = 2 \) and \( 1 < p < \infty \). Moreover, for \( 1 \leq q_1, q_2 \leq \infty \) and \( s_1, s_2 \in \mathbb{R} \) with \( s_2 < s_1 \) we have the continuous inclusion
\[
\dot{F}^{s_1}_{p_1, q_1}(\mathbb{R}^{n-1}) \subset \dot{F}^{s_2}_{p_2, q_2}(\mathbb{R}^{n-1})
\]
provided \( p_1, p_2 \in (1, \infty) \) with \( s_1 - s_2 = \frac{a-1}{p_1} - \frac{a-1}{p_2} > 0 \).

Remark 1.2. Let \( \phi \in \mathcal{S}(\mathbb{R}^{n-1}) \) with Fourier transform \( \hat{\phi} \) supported in \( \{1 \leq |\xi| \leq 2\} \) and \( \sum_{k \in \mathbb{Z}} (\hat{\phi}(2^{-k} \xi))^2 = 1 \) for all \( \xi \in \mathbb{R}^{n-1} \setminus \{0\} \). For each \( k \in \mathbb{Z} \), let \( \phi_k = 2^{k(n-1)} \hat{\phi}(2^k \cdot) \) be the dyadic dilation of \( \phi \). Whenever \( \phi_k \ast f, f \in \mathcal{S}'(\mathbb{R}^{n-1}) \) is a function we define the discrete Peetre maximal function
\[
(\phi_k f)(x') = \sup_{y' \in \mathbb{R}^{n-1}} |(\phi_k \ast f)(x' - y')(1 + 2^k |y'|)^{-\lambda}|, \quad x' \in \mathbb{R}^{n-1}.
\]

This object can also be used to characterize function spaces of Triebel-Lizorkin and Besov type. In particular, if \( \lambda > \max\{(n-1)/p, (n-1)/q\} \), then
\[
\left( \sum_{k \in \mathbb{Z}} (2^{ks} \phi_k^* f)^q \right)^{1/q} \left\| \right\|_{L^p(\mathbb{R}^{n-1})} \leq C \|f\|_{\dot{F}^{s}_{p,q}}
\]
where the constant \( C \) is independent of \( f \). It is interesting that in (1.10), the kernel \( \phi \) needs not be smooth, see for instance [5].

Definition 1.1. For \( n > 2, p \in (1, \infty) \) and \( q \in [1, p) \) we define \( \mathbf{X}_{p,q} \) as the space of vector fields \( u : \mathbb{R}^n_+ \to \mathbb{R}^n \) satisfying \( \|u\|_{\mathbf{X}_{p,q}} < \infty \) and \( \mathbf{Z}_{p,q} := \{ \pi : \mathbb{R}^n_+ \to \mathbb{R}^n \mid \|\pi\|_{\mathbf{Z}_{p,q}} < \infty \} \) where
\[
\|u\|_{\mathbf{X}_{p,q}} = \sup_{x_n > 0} x_n^{-\frac{1}{q} + \frac{n-1}{p}} \|u(\cdot, x_n)\|_{L^p(\mathbb{R}^{n-1})} + \|u\|_{T^{s,q}_{p,q}}, \quad s_q = -\frac{1}{q(n-1)}
\]
and
\[
\|\pi\|_{\mathbf{Z}_{p,q}} := \|\pi\|_{T^{s,q}_{p,q}}, \quad s_q = -\frac{1}{q} + \frac{1}{n-1}.
\]

Note that either \( \| \cdot \|_{\mathbf{X}_{p,q}} \) or \( \| \cdot \|_{\mathbf{Z}_{p,q}} \) defines a norm on \( \mathbf{X}_{p,q} \) and \( \mathbf{Z}_{p,q} \) respectively, they are Banach spaces. For convenience, when \( p = 2(n-1) \) and \( q = 2 \), we will specially denote the spaces \( \mathbf{X}_{p,q} \) and \( \mathbf{Z}_{p,q} \) by \( \mathbf{X} \) and \( \mathbf{Z} \), respectively.
1.2. Main results. Our first result deals with the well-posedness theory. In what follows, the dimension is assumed larger or equal to 3 unless otherwise stated.

**Theorem 1.1.** Assume that $F = 0$. Then the Dirichlet problem (NS)\textsuperscript{\textdash} (1.1) has a solution $(u, \pi)$ in $X \times Z$ which is unique in a small closed ball provided the data $f$ has a sufficiently small $\|\dot{H}^\frac{1}{2}-\frac{n}{2}(n-1)\|^{\eta}$-norm.

In presence of the forcing term, our main finding reads as follows.

**Theorem 1.2.** Let $\eta, \tau$ and $\varepsilon > 0$ be as in Theorem 1.2, take $2 < q < p < \infty$, $\eta_1 \in (1, q)$ and $\eta_1 < \tau_1 < p$. Given $f = (f_1, ..., f_n)$ in $[\dot{H}^\frac{1}{2}-\frac{n}{2}(n-1)]^n$ and $F \in X_{\tau, \eta}$, there exists $\varepsilon^* \in (0, \varepsilon)$ such that if $\|f\|_{\dot{H}^\frac{1}{2}-\frac{n}{2}(n-1)} + \|F\|_{X_{\tau, \eta}} < \varepsilon^*$ then there exists a solution $(u, \pi)$ of (NS)\textsuperscript{\textdash} (1.1) in the space $X_{p, q} \times Z_{p, q}$ which is unique in the ball

$$B_{2\varepsilon^*}(0, 0) = \{(u, \pi) \in X_{p, q} \times Z_{p, q} : \|u\|_{X_{p, q}} + \|\pi\|_{Z_{p, q}} \leq 2\varepsilon^*\}$$

provided $\frac{1}{\eta_1} + \frac{n-1}{\tau_1} = 2 + \frac{1}{q} + \frac{n-1}{p}$ and $s = -\frac{1}{2}q$.

The uniqueness of the pressure as claimed in the previous results should be understood up to an additive constant. Let us now record the following regularity result which arises as a consequence of the local boundedness property of the velocity field and the elliptic regularity theory for the Stokes system [12].

**Theorem 1.4.** If $(u, \pi) \in X \times Z$ is the solution of problem (NS)\textsuperscript{\textdash} (1.1) constructed in Theorem 1.1 or 1.2, then $(u, \pi) \in [C^{0, \alpha}_{\text{loc}}(\mathbb{R}^n_+)]^n \times L^p_{\text{loc}}(\mathbb{R}^n_+)$ for some $\alpha \in (0, 1)$ and every $p \in (1, \infty)$.

The regularity $s = -1/2$ is special in the analysis of (NS)\textsuperscript{\textdash} (1.1). In fact, the space $T^{2(n-1)}_{x_2}$ (resp. the boundary class $\dot{H}^{-1/2, 2(n-2)}(\mathbb{R}^n)$) is one scale within the one parameter family of spaces $T^{2(n-1)}_{x_2}$ (resp. the space $\dot{F}^s_{p, 2}(\mathbb{R}^n)$) with $s = \frac{n-1}{p} - 1 < 0$ which are all admissible according to the scaling symmetry. However, if $p > 2$ is fixed and $H = u \otimes u \in T^{p/2, 1}_{x_2}$, then in order to estimate the solution in $X_{p, 2}$ (the weighted $L^\infty$ norm to be more precise) by the corresponding norm of $H$, the requirement $p \leq 2(n - 1)$, is necessary, at least from our arguments (see for instance the proof of Proposition 2.2) so that $s = -1/2$ is the endpoint. This is a particular case of the natural condition $\omega(y_n) = y_n^{2s} \in \mathcal{A}_2$, $\mathcal{A}_2$ being the Muckenhoupt weight class.

**Remark 1.3.** It is important to emphasize that the solution $(u, \pi)$ found in the preceding theorems is in fact a weak solution of (NS)\textsuperscript{\textdash} (1.1) (compare with the notion of regular solution defined in [33]) in the sense that

$$u \in W^{1, p}_{\text{loc}} \quad \text{and} \quad \pi \in L^p_{\text{loc}},$$
for any $p < \infty$. Pertaining to Theorem 1.1, the construction of a solution $(u, \pi)$ in the framework $L^2(\mathbb{R}^2_n) \times L^2(\mathbb{R}^2_+, x_n dx)$, that is when $n = 2$, seems challenging – the main difficulty in the analysis comes from the critical regularity of the nonlinear term $u \cdot \nabla u$.

2. Auxiliary results

This section is devoted to the analysis of the Dirichlet problem for the following system

\[
(S) \begin{cases}
-\Delta u + \nabla \pi = F - \text{div} \, H & \text{in } \mathbb{R}^n_+ \\
\text{div} \, u = 0 & \text{in } \mathbb{R}^n_+ \\
u = f & \text{on } \partial \mathbb{R}^n_+
\end{cases}
\]

with given vector fields $f, F$ and tensor $H$. Our goal is to prove that $(S)$ admits a solution $(u, \pi)$ in the target space $\mathbf{X}_{p,q} \times \mathbf{Z}_{p,q}$ whose norm can be estimated by the norms of $f, F$ and $H$ in suitable function spaces. To this end, for better readability we simply separate the study into two parts: the homogeneous case ($f = 0$) and the inhomogeneous case ($F = 0, H = 0$).

2.1. Homogeneous Stokes system and linear estimates. Consider the Stokes operator $L_S$ acting on pair of functions $(u, \pi) \in [\mathcal{D}'(\mathbb{R}^n)]^n \times [\mathcal{D}'(\mathbb{R}^n)]^n$, $n \geq 2$ and given by

\[
L_S(u, \pi) = (-\Delta u_1 + \partial_1 \pi, \ldots, -\Delta u_n + \partial_n \pi, \sum_{i=1}^n \partial_i u_i).
\]

Let us denote by $\mathcal{M}_{n \times n}(\mathcal{S}'(\mathbb{R}^n))$ the collection of all $n \times n$ matrices with coefficients in $\mathcal{S}'(\mathbb{R}^n)$. A fundamental solution of the Stokes operator $L_S$ in $\mathbb{R}^n$ is a pair $(\mathbf{E}, \mathbf{b})$ with $\mathbf{E} = (E_{ij})_{i,j=1}^n \in \mathcal{M}_{n \times n}(\mathcal{S}'(\mathbb{R}^n))$ and $\mathbf{b} = (b_1, ..., b_n) \in [\mathcal{S}'(\mathbb{R}^n)]^n$ satisfying coordinate-wise the equations

\[
\begin{cases}
-\Delta E_{ij} + \partial_i b_j = \delta_{ij} \delta & \text{in } \mathcal{S}'(\mathbb{R}^n), \ \forall \ i, j \in \{1, ..., n\} \\
\sum_{k=1}^n \partial_k E_{kj} = 0 & \text{in } \mathcal{S}'(\mathbb{R}^n), \ \forall \ j \in \{1, ..., n\}.
\end{cases}
\]

For $n \geq 3$, one may apply the Fourier transform to both sides of each of the above equations after which one deduces the explicit expressions

\[
E_{ij}(x) = \frac{1}{2\omega_{n-1}} \left[ \frac{1}{n-2} \frac{\delta_{ij}}{|x|^{n-2}} + \frac{x_i x_j}{|x|^n} \right], \quad b_j = \frac{1}{\omega_{n-1}} \frac{x_j}{|x|^n}; \quad i, j \in \{1, ..., n\}
\]

defined for $x \in \mathbb{R}^n \setminus \{0\}$ where $\omega_{n-1}$ is the surface measure of the unit sphere in $\mathbb{R}^n$ centered at zero. Details of computations leading to (2.1) can be found in [25, Chap. 10].

On the other hand, when $n = 2$, $\mathbf{E}$ and $\mathbf{b}$ assume the following forms

\[
E_{ij}(x) = \frac{1}{4\pi} \left[ \frac{x_i x_j}{|x|^2} - \delta_{ij} \log |x| \right], \quad b_j = \frac{1}{2\pi} \frac{x_j}{|x|^2}; \quad x \in \mathbb{R}^2 \setminus \{0\}, \quad i, j \in \{1, 2\}.
\]

Now, let us consider the homogeneous Stokes system

\[
(2.3) \begin{cases}
-\Delta u + \nabla \pi = 0 & \text{in } \mathbb{R}^n_+ \\
\text{div} \, u = 0 & \text{in } \mathbb{R}^n_+ \\
u = f & \text{on } \partial \mathbb{R}^n_+
\end{cases}
\]

with the convolution being understood in a component wise sense, define

\[
\mathcal{H} f(x', x_n) = (K_{x_n} * f)(x'), \quad \mathcal{E} f(x', x_n) = (k_{x_n} * f)(x')
\]

where

\[
K_{x_n}(x') = (K_{ij}(x', x_n))_{1 \leq i, j \leq n} \quad \text{and} \quad k_{x_n}(x') = (k_1(x', x_n), ..., k_n(x', x_n))
\]
are commonly referred to as the Odqvist kernels [26] – each entry of the tensors assuming an explicit form in terms of $E$ and $b$ via the formulas

$$K_{ij}(x) = 2(\partial_i E_{ij} + \partial_j E_{in} - \delta_{jn} b_i) = -\frac{2n}{\omega_{n-1}} \frac{x_n x_i x_j}{|x|^{n+2}}$$

and

$$k_j(x) = 4 \partial_j b_n = \frac{4}{\omega_{n-1}} \partial_j \frac{x_n}{|x|^n}.$$

For the full derivation of these kernels, the interested reader may consult the articles [26, 29].

Note that if $f$ belongs to the weighted Lebesgue space $L^1(\mathbb{R}^{n-1}, \frac{dx'}{1 + |x'|^p})$, then $u = \mathcal{H}f$ and $\pi = \mathcal{E}f$ are both meaningful as absolutely convergent integrals and $(u, \pi)$ is the unique solution of Eq. (2.3) decaying at infinity. The Stokes extension of a tempered distribution may not be meaningful in general (as another tempered distribution). Harmonic Poisson extensions of Schwartz distributions have been studied by H. Triebel [32] – they characterize almost all scale of Triebel-Lizorkin spaces on $\mathbb{R}^{n-1}$ via tent spaces. In particular, the following equivalence holds true

$$\left\| A_q[x_n^{m-s} \partial_x \mathcal{P}_{x_n} f] \right\|_{L^p(\mathbb{R}^{n-1})} \sim \| f \|_{\dot{F}^s_{p,q}(\mathbb{R}^{n-1})}, \quad 1 \leq p, q < \infty$$

where $m$ is a large nonnegative integer, $m > s$ and $\mathcal{P}_{x_n} f$ denotes the convolution of $f$ with the kernel $\mathcal{P}_{x_n}(x') = \frac{2}{\omega_{n-1}} x_n(|x'|^2 + x_n^2)^{-\frac{\gamma}{2}}$. It is not straightforward to see that the convolution in (2.7) is meaningful given $f \in \dot{F}^s_{p,q}(\mathbb{R}^{n-1})$ since the considered kernel is non-smooth. Stein [30] introduced the notion of bounded distributions (i.e. $f * \phi \in L^\infty(\mathbb{R}^{n-1})$ for any $\phi \in \mathcal{S}(\mathbb{R}^{n-1}))$ and showed that if $f$ is a bounded distribution then $\mathcal{P}_{x_n} f$ is a bounded smooth function in $\mathbb{R}^n_+$. This notion was recently extended in [5] to distributions of finite growth.

**Definition 2.1.** A tempered distribution $f$ in $\mathbb{R}^{n-1}$ is said to be of finite growth $\gamma \geq 0$ if $f * \phi = O(|x'|^\gamma)$ as $|x'| \to \infty$ whenever $\phi \in \mathcal{S}(\mathbb{R}^{n-1})$.

By this, we then understand bounded distributions as those with growth $\gamma = 0$. Given a kernel $\psi$ such that $(1 + |.|)^{\gamma} \psi \in L^1(\mathbb{R}^{n-1})$, its convolution with a tempered distribution of finite growth makes sense as a tempered distribution. This idea was used in [5] to establish a non-smooth characterization of homogeneous Triebel-Lizorkin and Besov spaces via two essential arguments. Any $f \in \dot{F}^s_{p,q}(\mathbb{R}^{n-1})$ is up to an additive polynomial a distribution of growth $\gamma > s - (n - 1)/p$ and $f * \psi, \psi \in A_{A,m,r}$ is a bounded continuous function using the Calderón reproducing formula for elements in $\dot{F}^s_{p,q}(\mathbb{R}^{n-1})$, see [5, Theorem 3.1]. Here, for $A \geq 1$, $m, r \in \mathbb{R}$, $A_{A,m,r}$ is the class of functions in $L^1(\mathbb{R}^{n-1})$ such that $\psi \in C^{n+|A|}(\mathbb{R}^{n-1} \setminus \{0\})$ and satisfies for all multi-indices $\alpha$ with $|\alpha| \leq n + |A|

(2.8) $|\partial^\alpha \hat{\psi}(\xi')| = O(|\xi'|^{-|\alpha|})$ as $|\xi'| \to 0$ and $|\partial^\alpha \hat{\psi}(\xi')| = O(|\xi'|^{-n - (n-1) - m})$ as $|\xi'| \to \infty$.

From these arguments, the question regarding the meaning of the convolution in (2.7) is then settled as each of the kernels involved satisfies (2.8).

Now, we would like to investigate the validity of a one sided-estimate in (2.7) but having the Poisson kernel replaced by those in (2.5) and (2.6), respectively. Since the results of this paper deal with spaces of negative smoothness indexes and for simplicity sake, let us assume from now on that $s$ is negative. The following lemma collects some useful properties of $K_{jk}$ and $k_j$. 

Lemma 2.1. Set $\sigma_n = \frac{2n}{\omega_{n-1}}$ and consider the kernels

$$P_0(x') = \frac{-\sigma_n}{(|x'|^2 + 1)^{\frac{n+2}{2}}}, \quad P_j(x') = \frac{-\sigma_n x_j}{(|x'|^2 + 1)^{\frac{n+2}{2}}}, \quad Q_{jk} = \frac{-\sigma_n x_j x_k}{(|x'|^2 + 1)^{\frac{n+2}{2}}}, \quad j, k = 1, \ldots, n - 1.$$  

Then $P_0, P_j$ and $Q_{jk}$ belong to $\bigcap_{1 \leq p \leq \infty} L^p(\mathbb{R}^{n-1})$. Moreover, if $F_{x'}$ denotes the Fourier transform in $\mathbb{R}^{n-1}$, then

$$\langle F_{x'} P_0 \rangle (\xi') = |\xi'|^{-1/2} e^{-|\xi'|}, \quad \langle F_{x'} P_j \rangle (\xi') = i \xi_j e^{-|\xi'|}, \quad \xi' \in \mathbb{R}^{n-1}. \tag{2.9}$$

$$\langle F_{x'} Q_{jk} \rangle (\xi') = -\delta_{jk} e^{-|\xi'|} + \frac{\xi_j' \xi_k'}{|\xi'|} e^{-|\xi'|}, \quad \xi' \in \mathbb{R}^{n-1} \setminus \{0\}. \tag{2.10}$$

We know according to [5, Corollary 2.3] that after possibly subtracting a suitable polynomial to $f \in F^s_p,q(\mathbb{R}^{n-1})$, the resulting expression is a bounded distribution. Moreover, either of the kernels $P_0, P_j$ and $Q_{jk}$ belongs to $L^1(\mathbb{R}^{n-1})$ and satisfies the cancellation and smoothness condition (2.8). As a consequence and by application of [5, Theorem 3.1], the convolutions in (2.4) are not only well-defined distributions but also bounded continuous functions.

Remark 2.1. There is another simple argument showing that the distribution in (2.4) are in fact bounded functions whenever $f \in F^s_{p,q}(\mathbb{R}^{n-1})$. Assume that $K \in L^1(\mathbb{R}^{n-1})$ is such that $\hat{K}$ is rapidly decreasing and smooth except at the origin. Take $\eta \in \mathcal{S}(\mathbb{R}^{n-1})$ with $\hat{\eta} = 1$ near the origin, one can write $K_{x_n} = \eta_{x_n} * f * K_{x_n} + \chi_{x_n} * f$ where $\chi \in \mathcal{S}(\mathbb{R}^{n-1})$ is such that $\hat{\chi} = (1 - \hat{\eta})\hat{K}$ and conclude that $K_{x_n} f$ is bounded since $f$ is a bounded distribution.

Another application of Lemma 2.1 is the following

Proposition 2.1. Let $s \in \mathbb{R}, s < 0$ and $p, q \in (1, \infty)$. Take $A = \max\{n-1)/q, (n-1)/p\}$ and let $r, m \in \mathbb{R}$ with $r > s$, $m + s > \lambda$ for some $\lambda \in (A, [A] + 1)$. Assume further that $\psi \in \mathcal{A}_{A,m,r}$ and set $\psi_{x_n}(x') = x_n^{(n-1)} \psi(x'/x_n), \quad (x', x_n) \in \mathbb{R}_+^n$. If $f \in F^s_{p,q}(\mathbb{R}^{n-1})$, then there exists $C > 0$ independent of $f$ such that

$$\|\psi_{x_n} f\|_{F^s_{p,q}(\mathbb{R}^{n-1})} \leq C \|f\|_{F^s_{p,q}(\mathbb{R}^{n-1})}, \quad \psi_{x_n} f = \psi_{x_n} * f. \tag{2.11}$$

The proof of this result as well as that of Lemma 2.1 are both postponed to the Appendix. The previous proposition is interesting in that it provides a continuous characterization of Triebel-Lizorkin spaces by non-smooth kernels. In fact, $\psi$ can be either of the kernels in Lemma 2.1 - each of them belongs to the class $\mathcal{A}_{A,m,r}$ for suitable choices of parameters. Another special choice of $\psi$ is the function $\frac{2}{\omega_{n-1}}(1 + |\cdot|^2)^{-n/2}$ in $\mathbb{R}^{n-1}$, which corresponds to a particular case of the harmonic characterization in (2.7). However, we do not claim the reverse inequality in (2.11) under the prescribed assumptions on $\psi$.

As discussed above, $\mathcal{H} f$ and $\mathcal{E} f$ for any $f \in F^{1/q}_{p,q}(\mathbb{R}^{n-1})$ are bounded continuous functions in the tangential variable. Our next lemma shows more, it establishes their precise decay rates as well as their membership to some weighted tent spaces.

Lemma 2.2. Let $1 < q < p < \infty$. There exists a constant $C := C(n, p, q) > 0$ such that

$$\|\mathcal{H} f\|_{X_{p,q}} + \|\mathcal{E} f\|_{Z_{p,q}} + \sup_{x_n > 0} x_n^{1+1/q+(n-1)/p} \|\mathcal{E} f(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} \leq C \|f\|_{F^{1/q}_{p,q}(\mathbb{R}^{n-1})} \tag{2.12}$$

for all $f \in F^{1/q}_{p,q}(\mathbb{R}^{n-1})^n$ where $\mathcal{H} f$ and $\mathcal{E} f$ are defined as in (2.4).

We state two more auxiliary results which will be useful in the demonstration of Lemma 2.2.
Lemma 2.3 (Averaging Lemma). Assume that $F \in L^q(\mathbb{R}^n_+)$, $q \geq 1$. We have
\[
\int_{\mathbb{R}^n_+} \int_{\Gamma(x')} |F(y', y_n)|^q \frac{dy'dy_n}{y_n^{n-1}} dx' = c \int_{\mathbb{R}^n_+} |F(y)|^q dy
\]
where $c > 0$ only depends on $n$, the dimension.

The proof of this identity follows from a simple application of Fubini–Tonelli’s Theorem. In what follows, for a subset $S \subset \mathbb{R}^n_+$, we set $E(S) = \{x' \in \mathbb{R}^{n-1} : S \cap \Gamma(x') \neq \emptyset\}$.

Lemma 2.4. Let $K \subset \mathbb{R}^n_+$ be a compact set. Then $E(K)$ is open, its Lebesgue measure $|E(K)|$ is finite. Moreover, there exists a constant $\theta > 0$ depending on $K$ such that $|E(K)| \leq C \theta^{n-1}$ for some constant $C := C(n) > 0$.

Proof. Let $x' \in E(K)$, there exists $(y', y_n) \in \mathbb{R}^n_+$ with $(y', y_n) \in K$ and $y' \in B(y_n)(x')$. Putting $R = y_n - |x' - y'| > 0$, it plainly follows that $B(x', R) \subset E(K)$. Moving on, we remark that $E(K)$ is actually bounded. Fix $b > 1$ and observe that since $K$ is compact, one may assume without loss of generality that $K = Q \times [\ell(Q), b\ell(Q)]$ where $Q$ is a closed cube in $\mathbb{R}^{n-1}$ with side-length $\ell(Q)$. Hence, $|E(K)| \leq |B(x', b\ell(Q))| \leq C(b\ell(Q))^{n-1}, x' \in Q$. \qed

Now we are ready to prove Lemma 2.2.

Proof of Lemma 2.2. By a direct computation, we have for all $(x', x_n) \in \mathbb{R}^n_+$,
\[
k_j(x', x_n) = \begin{cases} -\frac{4n}{\omega_{n-1}} \frac{x_n x_j}{|x|^{n+2}} = 2x_n^{-n} P_j(x'/x_n) & \text{if } j = 1, \ldots, n-1 \\ 2\partial x_n P_{x_n}(x') & \text{if } j = n \end{cases}
\]
where $P_j$ is as in Lemma 2.1. Now set $(\overline{\pi}, \overline{\pi}) = (\mathcal{H}(f), \mathcal{E}(f))$ and for $x \in \mathbb{R}^n_+$ fixed, use the interior estimate for the linear Stokes problem (see e.g. [28]) together with Lemma 2.3 to write
\[
|\overline{\pi}(x)|^q \leq c |B_{x_n/2}(x)|^{-1} \int_{B_{x_n/2}(x)} |\overline{\pi}(y', y_n)|^q dy'dy_n
\]
\[
\leq c |B_{x_n/2}(x)|^{-1} \sum_{j=1}^n \int_{B_{x_n/2}(x)} |k_j(\cdot, y_n) \ast f_j|^q dy'dy_n
\]
\[
\leq c |B_{x_n/2}(x)|^{-1} \left( \int_{B_{x_n}(x') \times [x_n, 2x_n]} |\partial y_n P_{y_n} f_j|^q dy'dy_n + \sum_{j=1}^{n-1} \int_{B_{x_n}(x') \times [x_n, 2x_n]} |y_n^{-n} P_j(\cdot/y_n) \ast f_j|^q dy'dy_n \right)
\]
\[
\leq c x_n^{-n-q} \left( \int_{E(B_{x_n}(x') \times [x_n, 2x_n])} \int_{\Gamma(z')} |y_n^{1+1/q} \partial y_n P_{y_n} f_j|^q \frac{dy'dy_n}{y_n^{n}} dz' + \sum_{j=1}^{n-1} \int_{E(B_{x_n}(x') \times [x_n, 2x_n])} \int_{\Gamma(z')} |y_n^{1/q}(P_j y_n f_j)|^q \frac{dy'dy_n}{y_n^{n}} dz' \right)
\]
\[
\leq c x_n^{-\frac{n+1}{q} + \frac{n-1}{p}} \left( \|A_{y_n^{1+1/q}} \partial y_n P_{y_n} f_j\|_{L^p(\mathbb{R}^n_+)}^q + \sum_{j=1}^{n-1} \|A_{y_n^{1/q}} (P_j y_n f_j)\|_{L^p(\mathbb{R}^n_+)}^q \right)
\]
\[
(2.13) \leq c x_n^{-\frac{n+1}{q} + \frac{n-1}{p}} \|f\|_{L^p(\mathbb{R}^n_+)}^q,
\]
Observe that we have used Hölder’s inequality and Lemma 2.4 to get the penultimate estimate and (2.7) and Proposition 2.1 for the last. From the above remark on the kernel \( k_j \), the estimate on \( E f \) in \( T^{p,q}_{\pi} \) is a consequence of the extrinsic characterization (2.7) (applied with \( m = 1 \)) and Proposition 2.1. The same observation pertaining to the velocity field gives \( \| \pi \| T^{p,q}_{\pi} \leq C \| f \| _{E^{p-1/q}(\mathbb{R}^{n-1})} \). To see this, we simply write

\[
K_{jk}(\cdot, x_n) = x_n^{-(n-1)} \begin{cases} Q_{jk}(x_n^{-1} \cdot) & \text{if } j, k = 1, \ldots, n - 1 \\ P_k(x_n^{-1} \cdot) & \text{if } j = n, k = 1, \ldots, n - 1 \\ P_0(x_n^{-1} \cdot) & \text{if } j = k = n \end{cases}
\]

and apply Proposition 2.1. It then remains to establish the bound

\[
\sup_{x_n > 0} \frac{1}{x_n^{\frac{1}{p} + \frac{n-1}{p}}} \| \pi(\cdot, x_n) \| _{L^\infty(\mathbb{R}^{n-1})} \leq C \| f \| _{E^{p-1/q}(\mathbb{R}^{n-1})}.
\]

By the mean value property for the velocity field [28, Theorem 4.5] and with the same notation as above, we have

\[
|\mathbb{v}_k(x)| \leq \int_{B_{x_n/2}(x)} |\mathbb{v}_k(y)|dy + \frac{1}{2} \int_{B_{x_n/2}(x)} |\pi(z)| |z_k - x_k|dz := I + II, \quad k = 1, 2, \ldots, n.
\]

Using Lemma 2.3, Hölder’s inequality and Proposition 2.1 simultaneously we estimate \( I \) as follows. If \( k = 1, \ldots, n - 1 \), then

\[
I^q \leq C \| B_{x_n/2}(x) \|^{-1} \left( \sum_{j=1}^{n-1} \int_{B_{x_n/2}(x)} |y_n^{-(n-1)} Q_{jk}(y_n^{-1} \cdot) * f_j| |y_n^{-1} \cdot| dy \right) + \int_{B_{x_n/2}(x)} |(P_k)_{y_n} f_n| |y_n^{-1} \cdot|^q dy
\]

\[
\leq C x_n^{-n} \left( \sum_{j=1}^{n-1} \int_{E(B_{x_n}(x') \times [x_n/3, x_n])} \int_{\Gamma(z')} |y_n^{-(n-1)/q} Q_{jk}(y_n^{-1} \cdot) * f_j| |y_n^{-1} \cdot|^q dy' dy_n dz' + \right.
\]

\[
\left. \int_{E(B_{x_n}(x') \times [x_n/3, x_n])} \int_{\Gamma(z')} |y_n^{1/q} (P_k)_{y_n} f_n| |y_n^{-1} \cdot|^q dy' dy_n dz' \right)
\]

\[
\leq C x_n^{-1} \left( \sum_{j=1}^{n-1} \int_{E(B_{x_n}(x') \times [x_n/3, x_n])} \int_{\Gamma(z')} |y_n^{-(n-1)} Q_{jk}(y_n^{-1} \cdot) * f_j| |y_n^{-1} \cdot|^q dy' dy_n dz' \right)
\]

\[
\leq C x_n^{-(1/q + (n-1)/q)} \| f \| _{E^{p-1/q}(\mathbb{R}^{n-1})}^q
\]

When \( k = n \), one may repeat the above steps with the kernels \( P_j \) and \( P_0 \) respectively. In order to estimate the integral \( II := \frac{1}{2} \int_{B_{x_n/2}(x)} |\pi(z)| |z_k - x_k|dz \), we use the pressure estimate

\[
\left( \int_{B_{x_n}(x)} |\pi(y)|^q dy \right)^{1/q} \leq C x_n^{-(1+1/q + (n-1)/q)} \| f \| _{E^{p-1/q}(\mathbb{R}^{n-1})}^q, \quad \forall x \in \mathbb{R}^n_+
\]

which can be derived from the steps leading to (2.13) and the fact that if \( z \in B_{x_n/2}(x) \), then \( B_{x_n/2}(z) \subset B_{x_n}(x) \). Indeed, we have

\[
II \leq \| B_{x_n/2}(x) \|^{-1} \int_{B_{x_n/2}(x)} \left( \int_{B_{x_n/2}(z)} |\pi(y)| dy \right) |z_k - x_k|dz
\]

\[
\leq C \| B_{x_n}(x) \|^{-1} \left( \int_{B_{x_n}(x)} |\pi(y)|^q dy \right)^{1/q} \int_{B_{x_n/2}(x)} |z - x|dz
\]
Inhomogeneous Stokes system. Consider the operators $\mathcal{G}$ and $\Psi$ in $\mathbb{R}^n_+$ respectively defined by

$$
\mathcal{G}(F, H)(x) = \int_{\mathbb{R}^n_+} G(x, y) F(y) dy + \int_{\mathbb{R}^n_+} \nabla_y G(x, y) H(y) dy
$$

$$
\Psi(F, H)(x) = \int_{\mathbb{R}^n_+} g(x, y) F(y) dy + \int_{\mathbb{R}^n_+} \nabla_y g(x, y) H(y) dy
$$

whenever the integrals make sense for almost every $x \in \mathbb{R}^n_+$. The kernels $G(x, y) = (G_{ij}(x, y))_{i,j=1}^n$ and $g(x, y) = (g_{ij}(x, y))_{i,j=1}^n$, $x \neq y$ are the Green tensors for the Stokes operator in $\mathbb{R}^n_+$, that is, coordinates-wise the functions satisfying

$$
\begin{align*}
-\Delta_x G_{ij} + \partial_i g_j - \delta_x \delta_{ij} & \text{ in } \mathbb{R}^n_+ \\
\partial_i G_{ij} &= 0 \text{ in } \mathbb{R}^n_+ \\
G_{ij}(x, \cdot)|_{\{x', x_n): x_n=0\}} &= 0
\end{align*}
$$

(2.18)

in the sense of distributions where $\delta_x$ is the Dirac distribution with mass at $x \in \mathbb{R}^n_+$. Under mild assumptions on $F$ and $H$, the vector-valued functions $v = \mathcal{G}(F, H)$ and $w = \Psi(F, H)$ satisfy the system of equations

$$
\begin{align*}
-\Delta v + \nabla w &= F - \text{div } H \text{ in } \mathbb{R}^n_+ \\
\text{div } v &= 0 \text{ in } \mathbb{R}^n_+ \\
v &= 0 \text{ on } \partial\mathbb{R}^n_+.
\end{align*}
$$

(2.19)

Refined properties of Green matrices were recently obtained by the authors in [18] relying on ideas introduced earlier in the articles [24] (for $n = 2, 3$) and [12] (for the general case). For our purpose we will need the following properties which include sharp pointwise decay bounds.

**Lemma 2.5.** Let $n \geq 2$. The Green tensor $G$ is symmetric, $G_{ij}(x, y) = G_{ji}(y, x)$ for all $x, y \in \mathbb{R}^n_+$, $x \neq y$ and satisfies together with $g$ the pointwise estimates

$$
|G_{ij}(x, y)| \leq C \left( \frac{x_n y_n}{|x - y|^n} + 1_{\{n=2\}} \log(2 + y_n |x - y|^{-1}) \right)
$$

(2.20)

$$
|\nabla_x^\alpha \nabla_y^\beta G_{ij}(x, y)| \leq C_N \begin{cases}
|\frac{x_n y_n}{|x - y|^{n+1}}| & \text{if } \alpha_n = \beta_n = 0 \\
\frac{|x - y|^{-(n-2+N)}}{|x - y|^{n+1+N}} & \text{if } \alpha_n = 0
\end{cases}
$$

(2.21)

for all multi-indices $\alpha, \beta$ with $|\alpha| + |\beta| = N > 0$. Moreover,

$$
|\nabla_x^\alpha g_j(x, y)| \leq C_\alpha |x - y|^{-(n-1)-|\alpha|}, \ j = 1, ..., n
$$

(2.22)

where the constants are all independent of $x$ and $y$.

These inequalities find their applicability in our next result which deals with the mapping properties of the potentials $\mathcal{G}$ and $\Psi$. Recall the spaces $X_{p,q}$ and $Z_{p,q}$ introduced in Section 1.
Proposition 2.2. Fix \( n \geq 3 \), let \( p,q \in (1,\infty) \) with \( q < p \); let \( \sigma \in [1,q) \) and \( \eta \in (1,q) \). Take \( \tau \in (\eta,p) \) and \( \Lambda \in (\sigma,p) \) with the condition
\[
\frac{1}{\eta} + \frac{n-1}{\tau} = 1 + \frac{1}{\sigma} + \frac{n-1}{\Lambda} = 2 + \frac{1}{q} + \frac{n-1}{p}.
\]
Then for all \( F \in X_{\tau,\eta} \) and \( H \in X_{\Lambda,\sigma} \) we have \( \mathcal{G}(F,H) \in X_{p,q} \), \( \Psi(F,H) \in Z_{p,q} \) and it holds that
\[
\|\mathcal{G}(F,H)\|_{X_{p,q}} + \|\Psi(F,H)\|_{Z_{p,q}} \leq C(\|F\|_{X_{\tau,\eta}} + \|H\|_{X_{\Lambda,\sigma}})
\]
for some constant \( C := C(n,p,q) > 0 \) independent of \( F \) and \( H \).

Remark 2.2. The proof of the above result reveals that elliptic estimates of the form
\[
\sup_{x_n > 0} x_n^{1/q - [1/n]} \left\| \partial^\alpha u(\cdot,x_n) \right\|_{L^\infty(R^{n-1})} \leq C(\|F\|_{X_{\tau,\eta}} + \|H\|_{X_{\Lambda,\sigma}})
\]
are valid for each multi-index \( \alpha \) with \( \alpha_n = 0 \) where \( u \) is a solution of the Stokes system (2.19). However, it is not clear whether vertical derivatives of \( u \) enjoy this property. In fact, we are relying heavily on the second and third bound in (2.21) which seem to fail in the case \( \alpha_n \neq 0 \) or \( \beta_n \neq 0 \), see [18, Remark 2.6].

The proof of the proposition partially relies on the mapping properties in mixed Lebesgue spaces of the operator \( G_\beta \) defined for \( 0 < \beta < n \) by
\[
G_\beta F(y) = \int_{R^+_n} F(z)dz \left| y - z \right|^{n-\beta}
\]
whenever the integral exists for almost all \( y \in R^+_n \). For \( p,q \in [1,\infty) \), let us denote by \( L^p L^q(R^+_n) \) the mixed Lebesgue space of functions \( F : R^+_n \rightarrow R \) with the property that \( x' \mapsto F(x',\cdot) \in L^p(R^{n-1}) \) and \( x_n \mapsto F(\cdot,x_n) \in L^q(R^+_n) \) and equip it with the norm
\[
\|F\|_{L^p L^q(R^+_n)} := \|\|F(\cdot,x_n)\|_{L^q(R^+_n,dx_n)}\|_{L^p(R^{n-1})}.
\]

Lemma 2.6. Let \( \beta \in (0,n) \), \( \eta \in (1,\infty) \) and assume that \( p,q,\tau \in (1,\infty) \) are such that \( \tau < p < \eta \) and
\[
\frac{1}{q} - [1/n] < \beta + \frac{1}{q} - [1/n] = \frac{n-1}{\tau} + \frac{1}{\eta} - \frac{1}{q} - \beta.
\]
Then the operator \( G_\beta \) is bounded from \( L^p L^q(R^+_n) \) into \( L^p L^q(R^+_n) \).

Recall the Riesz potential \( I_\alpha \) of order \( \alpha \in (0,n-1) \), that is, the convolution operator with the kernel \( |x'|^{\alpha-(n-1)} \), \( x' \in R^{n-1} \setminus \{0\} \).

Proof. Along the lines of the proof of [34, Lemma 2.2], take \( F \in L^p L^q(R^+_n) \) and let \( \tilde{F} \) be the zero extension of \( F \) to \( R^n \). For \( 1 \leq \eta < \infty \) and \( 1 < \tau < \infty \) we have
\[
\|G_\beta F\|_{L^p L^q(R^+_n)} = \|\|G_\beta F(\cdot,y)\|_{L^q(R^+_n)}\|_{L^p(R^{n-1})}.
\]
Let \( x' \in R^{n-1} \) and set \( S(x',s) = (|x'|^2 + s^2)^{-\frac{n-\beta}{2}} \), \( s \in R \). For \( 1 \leq \theta < \infty \) such that \( \frac{1}{q} + 1 = \frac{1}{\theta} + \frac{1}{\theta} \) we use Minkowski’s inequality to arrive at
\[
\|G_\beta F(y',\cdot)\|_{L^q(R^+_n)} \leq \left\| \int_{R^+_n} \frac{|F(z',z_n)|dz'dz_n}{(y' - z'|^2 + |z_n|^2)^{\frac{n-\beta}{2}}} \|_{L^q(R^+_n)}
\]

\[
\leq \left\| \int_{R^{n-1}} |S(y' - z'| \cdot \cdot) * |\tilde{F}(z'| \cdot \cdot)\| dy' \right\|_{L^q(R^+_n,dy_n)}
\]
\[ \leq C \int_{\mathbb{R}^{n-1}} \|S(y' - z', \cdot)\|_{L^p(\mathbb{R})} \|\tilde{F}(z', \cdot)\|_{L^q(\mathbb{R})} \, dz' \]
\[ \leq C \int_{\mathbb{R}^{n-1}} \|S(y' - z', \cdot)\|_{L^p(\mathbb{R}^n_+)} \|\tilde{F}(z', \cdot)\|_{L^q(\mathbb{R}^n_+)} \, dz' \]
\[ \leq C [I_{\beta + \frac{1}{p} - 1}] |F(\cdot, y_n)|_{L^n(\mathbb{R}^{n+1}, dy_n)}(y') \text{,} y' \in \mathbb{R}^{n-1} \]

Thus, if \( \frac{n-1}{p} = \frac{n-1}{r} - (\beta + \frac{1}{p} - 1) \), then by the boundedness of \( I_\alpha \) in Lebesgue spaces, we find that
\[ \|G_\beta F\|_{L^p L^n(\mathbb{R}^{n+1})} \leq C \|I_{\beta - \frac{1}{p} - 1} F(y', \cdot)\|_{L^n(\mathbb{R}^n_+, dy')} \leq C \|F\|_{L^r L^n(\mathbb{R}^{n+1})}. \]

\[ \Box \]

**Remark 2.3.** In the sequel, we will need an analogue of Lemma 2.6 in weighted mixed Lebesgue spaces of the form
\[ \|G_\beta F(\cdot, y_n)\|_{L^p(\mathbb{R}^n_+, y_n^\beta dy_n)} \|_{L^n(\mathbb{R}^{n-1})} \leq C \|F(\cdot, y_n)\|_{L^n(\mathbb{R}^n_+, y_n^\beta dy_n)} \|_{L^r(\mathbb{R}^{n-1})} \]
for all functions \( F \) such that \((x', x_n) \mapsto x_n^a F \in L^r L^n(\mathbb{R}^{n+1}_+)\). This estimate is true under the following set of conditions
\[ \begin{cases} 
2 + \frac{1}{q} = (n - 1) \left( \frac{1}{r} - \frac{1}{p} \right) + \frac{1}{\eta} + b - (\beta - 1) \\
1 < r < p < \infty, \ b \geq 1 \\
n > \beta + 2 + \frac{1}{q} - \frac{1}{\eta} - b. 
\end{cases} \]

In fact, one may use the same strategy as before to prove (2.27). If \( a \geq 1 \) and \( \delta > 1 \) are such that
\[ \frac{1}{\delta} + a = (n - 1) \left( \frac{1}{r} - \frac{1}{p} \right) - (\beta - 1), \]
then using the weighted convolution inequality [15, Theorem 1.2] for \( n = 1 \), we obtain
\[ \|G_\beta F(y', \cdot)\|_{L^n(\mathbb{R}^n_+, y^\beta dy_n)} \leq C \int_{\mathbb{R}^{n-1}} \|S(y' - z', \cdot)\|_{L^\delta(\mathbb{R}^n_+, y^\delta dy_n)} \|F(z', \cdot)\|_{L^n(\mathbb{R}^n_+, y^\delta dy_n)} \, dz' \]
\[ \leq I_{\beta + a + \frac{1}{p} - 1} |F(\cdot, y_n)|_{L^n(\mathbb{R}^n_+, y^\beta dy_n)}(y'), \ y' \in \mathbb{R}^{n-1}. \]

This, in conjunction with (2.28) gives the desired bound after taking the \( L^p \)-norm on both sides of the inequality.

We are now ready to prove Proposition 2.2 and we divide the proof in two steps.

**Step 1.** The bound
\[ \|\mathcal{G}(F, H)\|_{X_{r,q}} \leq C (\|F\|_{X_{r,q}} + \|H\|_{X_{\Lambda,\sigma}}). \]

Let \( 1 < \eta < \infty, 1 \leq \sigma < \infty \) and \( 1 < \tau, \Lambda < \infty \) such that \( \frac{1}{\eta} + \frac{2-1}{p} = 2 + \frac{1}{\eta} + \frac{2-1}{p} = 1 + \frac{1}{\sigma} + \frac{n-1}{A}. \)

Pick \( F \) in \( X_{r,\eta} \) and \( H \in X_{\Lambda,\sigma}. \) We first prove that
\[ \sup_{x_n > 0} x_n^{1/q + (n-1)/p} \|\mathcal{G}(F, H)(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n+1})} \leq \|F\|_{X_{r,\eta}}. \]

Fix \( x' \in \mathbb{R}^{n-1} \) and \( x_n > 0 \) and write
\[ \int_{\mathbb{R}^{n+1}} G(x', x_n, y) \, F(y) \, dy = J_1 + J_2 + J_3 + J_4 \]
where

\[ J_1 = \int_{B_{x_n}(x') \cap \mathbb{R}^n} G(x, y) F(y) dy, \quad J_2 = \int_{B_{x_n}(x') \cap \mathbb{R}^n} G(x, y) F(y) dy, \]

\[ J_3 = \int_{\mathbb{R}^n \setminus B_{x_n}(x')} \int_{x_n/2} G(x, y) F(y) dy, \quad J_4 = \int_{\mathbb{R}^n \setminus 2x_n} G(x, y) F(y) dy. \]

Next, we estimate each of these integrals by means of the pointwise inequalities from Lemma 2.5. Indeed, starting with \( J_1 \) and using the summation convention, we have

\[
|J_1| \leq \int_{B_{x_n}(x')} \int_{x_n/2} |G(x', x_n, y)||F(y)| dy
\]

\[
\leq C \int_{B_{x_n}(x')} \int_{x_n/2} \frac{|F(y)|}{(|x' - y|^2 + (x_n - y_n)^2)^{n-2}} dy_n dy'
\]

\[
\leq C x_n^{-\frac{n-2}{2}} \left( \int_{B_{x_n}(x')} \int_{x_n/2} |F(y)|^\eta dy_n dy' \right)^{\frac{1}{\eta}}
\]

\[
\leq C x_n^{-\frac{n-2}{2}} \left( \int_{\mathbb{R}^n} \int_{x_n/2} |F(y)|^\eta y_n^{-(n-1)} dy_n dy' dz \right)^{\frac{1}{\eta}}
\]

\[
\leq C x_n^{-\frac{n-2}{2}} \left( \sum_{k=1}^{\infty} \int_{B_{x_n}(x') \cap \mathbb{R}^n} |F(y)|^\eta x_n^{-(n-1)} \right)^{\frac{1}{\eta}}
\]

where we have utilized Hölder’s inequality in order to derive the third and fifth bounds in the above chain of estimates and \( \frac{1}{\eta} + \frac{1}{\eta} = 1 \). On the other hand,

\[
|J_2| \leq \int_{B_{x_n}(x')} \int_{x_n/2} |G(x, y)||F(y)| dy
\]

\[
\leq C \int_{B_{x_n}(x')} \int_{x_n/2} |x - y|^{-(n-2)} |F(y)| dy
\]

\[
\leq C \sup_{y_n > 0} \left( \int_{B_{x_n}(x')} \int_{x_n/2} \frac{1}{y_n^{\frac{n-1}{2}}} \frac{1}{y_n^{\frac{n-1}{2}}} dy_n dy' \right)^{\frac{1}{\eta}}
\]

\[
\leq C x_n^{-\frac{n-1}{2}} \frac{1}{y_n^{\frac{n-1}{2}}} \left( \int_{B_{x_n}(x')} \int_{x_n/2} |x' - y'|^{-(n-2)} dy_n dy' \right)^{\frac{1}{\eta}}
\]

\[
\leq C x_n^{-\frac{n-1}{2}} \frac{1}{y_n^{\frac{n-1}{2}}} \left( \int_{B_{x_n}(x')} \int_{x_n/2} |x' - y'|^{-(n-2)} dy' \right)^{\frac{1}{\eta}}
\]

\[
\leq C x_n^{-\frac{n-1}{2}} \frac{1}{y_n^{\frac{n-1}{2}}} \left( \int_{B_{x_n}(x')} \int_{x_n/2} |x' - y'|^{-(n-2)} \right)^{\frac{1}{\eta}}
\]

Similarly as above, by Lemma 2.3 and Hölder’s inequality, we find that

\[
|J_3| \leq \int_{\mathbb{R}^n \setminus B_{x_n}(x')} \int_{y_n} |G(x, y)||F(y)| dy
\]

\[
\leq C x_n \int_{\mathbb{R}^n \setminus B_{x_n}(x')} \int_{x_n/2} |x - y|^{-(n-1)} |F(y)| dy
\]

\[
\leq C x_n \sum_{k=1}^{\infty} \int_{2^k B_{x_n}(x') \setminus 2^{k-1} B_{x_n}(x')} \int_{x_n/2} |x - y|^{-(n-1)} |F(y)| dy
\]
we arrive at the weighted gradient sup-norm estimate

\[ \sup_{x \in \mathbb{R}^n} x^{\frac{1}{q} + \frac{p}{n-1}} \left\| \nabla G(\cdot, x_n) H(y) \right\|_{L^\infty(\mathbb{R}^{n-1})} \leq C \| H \|_{X_{A,n}}. \]

Decompose the solid integral in the above estimate into four parts to get

\[ L_1 = \int_{B_{x_n}(x')} \int_{0}^{x_n/2} \nabla_y G(x, y) H(y) dy \]
\[ L_2 = \int_{B_{x_n}(x')} \int_{x_n/2}^{2x_n} \nabla_y G(x, y) H(y) dy, \]
\[ L_3 = \int_{B_{x_n}(x')} \int_{x_n/2}^{2x_n} \nabla_y G(x, y) H(y) dy, \]
\[ L_4 = \int_{\mathbb{R}^{n-1}} \int_{2x_n}^{\infty} \nabla_y G(x, y) H(y) dy. \]

Utilizing (2.21), Hölder’s inequality, Lemmas 2.3 and 2.4 we arrive at

\[ |L_1| \leq C \int_{B_{x_n}(x')} \int_{0}^{x_n/2} \left| \nabla_y G(x, y) \right| |H(y)| dy \]
\[ \leq C \int_{B_{x_n}(x')} \int_{0}^{x_n/2} \frac{|H(y', y_n)|}{(|x' - y'|^2 + (x_n - y_n)^2)^{\frac{n-1}{2}}} dy_n dy' \]
\[ \leq C x_n^{1 - \frac{\alpha}{\sigma} - \frac{n-1}{\alpha}} \| A_{\sigma}(y_n^{1/\sigma} H) \|_{L^A(\mathbb{R}^{n-1})} \]
\[ \leq C x_n^{1 - \frac{\alpha}{\sigma} - \frac{n-1}{\alpha}} \| H \|_{X_{A,n}}. \]

Next, noticing that \( |\nabla G_{ij}(x, \cdot)| \) belongs to the weak-Lebesgue space \( L^{\frac{n}{n-\alpha}}(\mathbb{R}^n_+) \) uniformly for all \( x \in \mathbb{R}^n_+ \), it follows that

\[ |L_2| \leq C \sup_{x_n > 0} x_n^{1 - \frac{\alpha}{\sigma} - \frac{n-1}{\alpha}} \| H(\cdot, x_n) \|_{L^\infty(\mathbb{R}^{n-1})} \int_{B_{x_n}(x')} \int_{x_n/2}^{2x_n} \frac{y_n^{-\frac{1}{\beta}} \| \nabla_y G(x, y) \|_{L^\infty(\mathbb{R}^{n-1})}}{y_n^{\frac{n-1}{\alpha}}} dy_n dy'. \]
\[
\leq C x_n^{-\frac{1}{\sigma} - \frac{n-1}{\sigma}} \sup_{x_n > 0} x_n^{\frac{1}{\sigma} + \frac{n-1}{\sigma}} \|H(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} \|\nabla_y G(x, \cdot)\|_{L^1(B_{x_n}(x') \times [x_n/2, x_n])}
\]
\[
\leq C x_n^{-\frac{1}{\sigma} - \frac{n-1}{\sigma}} \|H\|_{X_{\lambda, \sigma}}.
\]

Recall here that for any \( p > 1 \) the belonging of \( f \) to \( L^{p, \infty}(\mathbb{R}^{n-1}) \) is equivalent to the condition
\[
\sup_{E \subset \mathbb{R}^{n-1}} |E|^{1/p-1} \int_E |f(y)| dy < \infty
\]
where the supremum runs over all open set \( E \) of \( \mathbb{R}^{n-1} \). We argue as before to bound \( L_3 \)
\[
|L_3| \leq \int_{\mathbb{R}^{n-1} \setminus B_{x_n}(x')} \int_{0}^{2x_n} |\nabla_y G(x, y) H(y)| dy_n dy'
\]
\[
\leq \sum_{k=1}^{\infty} \int_{2kx_n(\chi') \setminus 2k-1 B_{x_n}(x')} \int_{0}^{2x_n} |\nabla_y G(x, y)| H(y)| dy_n dy'
\]
\[
\leq C \sum_{k=1}^{\infty} \int_{2kx_n(\chi') \setminus 2k-1 B_{x_n}(x')} \int_{0}^{2x_n} |x-y|^{-n+1} |H(y)| dy_n dy'
\]
\[
\leq C x_n^{-\frac{n}{\sigma}} \sum_{k=1}^{\infty} 2^{-(n-1)k + \frac{k(n-1)}{\sigma}} \left( \int_{\mathbb{R}^{n-1} \setminus \{ \Lambda(\chi') \cap [2k B_{x_n}(x') \times (0, 2x_n)] \}} |H(y)|^{\sigma} \frac{dy_n dy'}{y_n^{1/\sigma}} \right)^{1/\sigma}
\]
\[
\leq C x_n^{-\frac{n}{\sigma}} \sum_{k=1}^{\infty} 2^{-(n-1)k + \frac{k(n-1)}{\sigma} + k(n-1) \frac{\eta}{\lambda}}
\]
\[
\leq C x_n^{-\frac{1}{\sigma} - \frac{n}{\sigma}} \|H\|_{X_{\lambda, \sigma}}.
\]

Finally, observe that for \( y_n > 2x_n \), we have \( y_n - x_n > \frac{1}{2} y_n \) so that by the third bound in (2.21), we find that
\[
|L_4| \leq \int_{\mathbb{R}^{n-1}} \int_{0}^{\infty} |\nabla_y G(x, y)| H(y) dy
\]
\[
\leq C \int_{\mathbb{R}^{n-1}} \int_{2x_n}^{\infty} x_n |H(y)| dy_n dy'
\]
\[
\leq C \int_{\mathbb{R}^{n-1}} \int_{2x_n}^{\infty} x_n |H(y)| dy_n dy'
\]
\[
\leq C \sup_{x_n > 0} x_n^{\frac{1}{\sigma} - \frac{n-1}{\sigma}} \|H(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} \left( \int_{\mathbb{R}^{n-1}} \frac{dy'}{(y')^{2-\frac{2}{\sigma}}} \right) \left( \int_{2x_n}^{\infty} x_n y_n^{\frac{1}{\sigma} - \frac{n-1}{\sigma}} dy_n \right)
\]
\[
\leq C x_n^{-\frac{1}{\sigma} - \frac{n-1}{\sigma}} \|H\|_{X_{\lambda, \sigma}}.
\]

Summing up all the above inequalities, one obtains (2.31). Next, we show that
\[
\|\mathcal{G}(F, H)\|_{T^{p,q}_{\eta}} \leq C (\|F\|_{T^{r,q}_{\eta}} + \|H\|_{T^{1,p}_{\lambda, \sigma}}).
\]
Write
\[
\|\mathcal{G}(F, H)\|_{T^{p,q}_{\eta}} \leq \left\| \int_{\mathbb{R}^+} G(y, \cdot) F(y) dy \right\|_{T^{p,q}_{\eta}} + \left\| \int_{\mathbb{R}^+} \nabla_y G(y, \cdot) H(y) dy \right\|_{T^{p,q}_{\eta}}
\]
\[
:= I + I'.
\]
Claim 2.1. For all $\Sigma$ and $n$, we control $\Sigma$ such that $\int_{\Omega} |\xi|^n \leq C$ and using the following

\[ I := \Sigma_1 + \Sigma_2 + \Sigma_3, \quad \Sigma_i = \left\| \int_{\Omega} G(\cdot, y) F^i(y) dy \right\|_{L_{\nu}^{p,q}}, \quad i = 1, 2, 3. \]

We control $\Sigma_3$ using the following

**Claim 2.1.** For all $x' \in \mathbb{R}^{n-1}$ and $y_n > 0$, there exists $C > 0$ independent on $x'$, $y_n$ and $F$ such that

\[ A(x', y_n) \leq C G_2 \left( \int_{B_{y_n}(\cdot)} |F(z', \cdot)| dz' \right)(x', y_n). \]

Here,

\[ A(x', y_n) = \left( \int_{B_{y_n}(x')} \left[ \int_{\Omega} G(y, z) F^3(z) dz \right] dy \right)^{\frac{1}{q}}, \quad (x', y_n) \in \mathbb{R}^n_+. \]

**Proof.** We have

\[ A(x', y_n) \leq \left( \int_{B_{y_n}(x')} \left( \int_{\Omega} G(y, z) |F^3(z)| dz \right)^q dy \right)^{\frac{1}{q}} \]

\[ \leq C \left( \int_{B_{y_n}(x')} \left( \int_{0}^{\infty} \int_{\mathbb{R}^{n-1} \setminus B_{y_n}(x')} \frac{|F(z', z_n)| dz' dz_n}{(|y' - z'|^2 + |y_n - z_n|^2)^{\frac{n-2}{2}}} \right)^q dy \right)^{\frac{1}{q}} \]

\[ \leq C \left( \int_{B_{y_n}(x')} \left( \int_{0}^{\infty} \int_{|z'| > 4y_n} \frac{|F(z', z_n)| dz' dz_n}{(|y' - z'|^2 + |y_n - z_n|^2)^{\frac{n-2}{2}}} \int_{B_{y_n}(z')} \left( \int_{\Omega} G(y, z) F^3(z) dz \right)^q dy \right)^{\frac{1}{q}} \]

\[ \leq C \left( \int_{B_{y_n}(x')} \left( \int_{0}^{\infty} \int_{|z'| > 3y_n} \frac{(|x' - w|^2 + |y_n - z_n|^2)^{-\frac{n-2}{2}}}{|y' - z'|^2 + |y_n - z_n|^2} \left( \int_{B_{y_n}(w)} F(z', z_n) dz' dz_n \right)^q \right)^{\frac{1}{q}} \]

\[ \leq C \left( \int_{B_{y_n}(x')} \left( \int_{0}^{\infty} \int_{|z'| > 3y_n} \frac{(|x' - w|^2 + |y_n - z_n|^2)^{-\frac{n-2}{2}}}{|y' - z'|^2 + |y_n - z_n|^2} \left( \int_{B_{y_n}(w)} F(z', z_n) dz' dz_n \right)^q \right)^{\frac{1}{q}} \]

\[ \leq C \left( \int_{0}^{\infty} \int_{|z'| > 3y_n} \frac{(|x' - w|^2 + |y_n - z_n|^2)^{-\frac{n-2}{2}}}{|y' - z'|^2 + |y_n - z_n|^2} \left( \int_{B_{y_n}(w)} F(z', z_n) dz' dz_n \right)^q \right)^{\frac{1}{q}} \]

\[ \leq C G_2 \left( \int_{B_{y_n}(\cdot)} |F(z', \cdot)| dz' \right)(x', y_n). \]

\[ \square \]
Applying Lemma 2.6 and Jensen’s inequality, the above claim clearly implies that
\[
\Sigma_3 = \|A\|_{L_p L^q(R^n_+)} \leq C \left\| G_2 \left( \int_{B_{y_n}} |F'(z',\cdot)| dz' \right) \right\|_{L^p L^q(R^n_+)} \\
\leq C \left\| \int_{B_{y_n}} |F'(z',y_n)| dz' \right\|_{L^p L^q(R^n_+)} \leq C \|F\|_{T_0^{r,q}}.
\]
To bound \( \Sigma_2 \), we first observe that
\[
\left| \int_{R^n_+} G(y,z)F^2(z)dz \right| \leq C y_n \left[ A_2^4(z^{1/\eta} F) \right](x') \left( \int_{E_{y_n} \setminus B_{4y_n} (x')} \frac{z_n^{1+\eta^{-1}}}{1+n^{-1} dy} \right)^{\frac{1}{q}}
\]
(2.33)
\[
\leq C y_n^{2-\frac{1}{q}} \left[ A_2^4(z^{1/\eta} F) \right](x'), \quad x' \in B(y', y_n).
\]
On the other hand, this inequality also implies, thanks to Remark 1.1 the pointwise bound
(2.34) \[
\left| \int_{R^n_+} G(y,z)F^2(z)dz \right| \leq C y_n^{-\frac{n-1}{q} + \frac{1}{q}} \|A_2^4(z^{1/\eta} F)\|_{L^r(R^{n-1})}, \quad y' \in B(x', y_n).
\]
Let \( M > 0 \) to be determined later. Using (2.33) and (2.34), we find that
\[
\int_{\Gamma (x')} \left| \int_{R^n_+} G(y,z)F^2(z)dz \right|^{\frac{1}{q}} dy \leq \int_{0}^{M} \int_{B_{y_n}(x')} G(y,z)F^2(z)dz |y|^q dy' dy_n + \int_{M}^{\infty} \int_{B_{y_n}(x')} G(y,z)F^2(z)dz |y|^q dy' dy_n
\]
\[
\leq C M^{1+(2-\frac{1}{q})q} \left[ A_2^4(z^{1/\eta} F) \right](x') \|F\|_{T_0^{r,q}}^{\frac{q}{n-1}}.
\]
Optimizing this inequality with respect to \( M \), that is taking \( M = \left( \frac{\|F\|_{T_0^{r,q}}}{\left[ A_2^4(z^{1/\eta} F) \right](x')} \right)^{\frac{1}{n-1}} \), we arrive at
\[
\left( \int_{\Gamma (x')} \left| \int_{R^n_+} G(y,z)F^2(z)dz \right|^{\frac{1}{q}} y_n^{-\frac{n}{2}} dy dy_n \right)^\frac{q}{n} \leq C \|F\|_{T_0^{r,q}}^{1-\frac{q}{n}} \left[ A_2^4(z^{1/\eta} F) \right](x') \|F\|_{T_0^{r,q}}^{\frac{q}{n-1}}, \quad x' \in R^{n-1}.
\]
Taking the \( L^q \) norm on both sides of the inequality and using Remark 1.1, we conclude that
\[
\Sigma_2 \leq C \|F\|_{T_0^{r,q}}.
\]
Finally, estimating \( \Sigma_1 \) goes through a duality argument. Let \( \rho = p/q \) and \( \varphi \in L^\rho(R^{n-1}) \), \( \varphi \geq 0 \) and define the operator \( M_t \), \( t > 0 \) by
\[
M_t \varphi(x') = t^{-(n-1)} \int_{B_t(x')} \varphi(y') dy', \quad t > 0.
\]
If \( \langle \cdot, \cdot \rangle \) denotes the duality bracket between \( L^p(R^{n-1}) \) and its dual \( L^{p'}(R^{n-1}) \), then
\[
\langle A_2^4 y_n^{1/2} \int_{R^n_+} G(\cdot, z)F^1(z)dz, \varphi \rangle = \int_{R^{n-1}} \int_{\Gamma (x')} \left| \int_{R^n_+} G(y,z)F^1(z)dz \right|^{\frac{q}{n-1}} y_n^{-\frac{n}{2}} \varphi(x') dx' dx' \leq C \int_{R^{n-1}} \int_{0}^{\infty} y_n^{-\frac{n}{2}} \varphi(x') dx' [G_2^2 F](y', y_n) dy' dy_n \leq C \|G_2^2 F\|_{L^p L^q(R^n_+)} \|M_t \varphi\|_{L^{p'}(R^{n-1})} \leq C \|G_2^2 F\|_{L^p L^q(R^n_+)} \|M \varphi\|_{L^{p'} L^\infty(R^n_+)}.
From the proof of Claim 2.2, the following inequality is a consequence of Lebesgue’s differentiation Theorem and Fatou’s lemma:
\[
\int_0^\infty |F(y',y_n)|^\eta dy_n \leq c_n \liminf_{\eta \to 0^+} |A_n(y_n^{1/\eta} F)(y')|^\eta, \quad y' \in \mathbb{R}^{n-1}.
\]
Applying Lemma 2.6, the boundedness of the Hardy-Littlewood maximal function \( M \) in Lebesgue spaces successively, we obtain
\[
\left( \mathcal{A}_n^q \left[ \frac{1}{y_n^{\eta/\eta}} \int_{\mathbb{R}^n_+} G(\cdot,z) F^1(z) dz \right], \varphi \right) \leq C \| F \|_{T_{r_n,q}^{r_n,q}, \mathcal{M}} \| \varphi \|_{L^{r'}(\mathbb{R}^{n-1})} \quad \forall \varphi \in L^{r'}(\mathbb{R}^{n-1}),
\]
from which it plainly follows that
\[
\Sigma_1 \leq C \| F \|_{T_{r_n,q}^{r_n,q}}.
\]
We equally estimate II splitting \( H \) into three components exactly as before and follow the same procedure (details are left to the interested reader). This yields
\[
\left\| \int_{\mathbb{R}^n_+} \nabla_y G(\cdot,y) H(y) dy \right\|_{T_{r_n,q}^{r_n,q}} \leq C \| H \|_{T_{r_n,q}^{r_n,q}}.
\]
Summarizing, we see that (2.29) holds true. This finishes Step 1.

**Step 2.** The estimate
\[
(2.35) \quad \| \Psi(F,H) \|_{T_{p,q}^{p,q}} \leq C \left( \| F \|_{X_{r,q}} + \| H \|_{X_{A,\sigma}} \right)
\]
for all \( F \in X_{r,q} \) and \( H \in X_{A,\sigma} \). We have
\[
\| \Psi(F,H) \|_{T_{p,q}^{p,q}} \leq \left\| \int_{\mathbb{R}^n_+} g(\cdot,z) F(z) dz \right\|_{T_{p,q}^{p,q}} + \left\| \int_{\mathbb{R}^n_+} \nabla_y g(\cdot,z) H(z) dz \right\|_{T_{p,q}^{p,q}} := III + IV.
\]
Let \( F^1, F^2 \) and \( F^3 \) as above and write correspondingly
\[
III \leq III_1 + III_2 + III_3, \quad III_i = \left\| \int_{\mathbb{R}^n_+} g(\cdot,z) F^i(z) dz \right\|_{T_{p,q}^{p,q}}, \quad i = 1, 2, 3.
\]
From the proof of Claim 2.1, we easily obtain the estimate
\[
\left( \int_{B_{y_n}(x')} \left| \int_{\mathbb{R}^n_+} g(y,z) F^3(z) dz \right|^q dy' \right)^{\frac{1}{q}} \leq CG_1 \left( \int_{B_{y_n}(x')} |F(z',\cdot)| dz' \right)(x',y_n), \quad (x',y_n) \in \mathbb{R}^n_+.
\]
Now let \( r \in (\tau,p) \) such that \( \frac{1}{r} + \frac{1}{n+1} = \frac{1}{q} \). Invoking (2.27) with \( \beta = 1 \) and \( b = (n-1)(\frac{1}{r} - \frac{1}{q}) \) together with Jensen’s inequality we arrive at
\[
III_3 \leq C \left\| \left( \int_{B_{y_n}(x')} |F(z',\cdot)| dz' \right) \right\|_{L^r(\mathbb{R}^{n+1},dy_n)} \left\| G_1 \right\|_{L^p(\mathbb{R}^{n+1})}
\]
\[
\leq C \left( \int_{\mathbb{R}^{n-1}} \left( \int_0^\infty \int_{B_{y_n}(x')} |y_n^b F(z',y_n)|^\eta dz' dy_n \right)^{\frac{1}{\eta}} dx' \right)^{\frac{1}{b}}
\]
\[
\leq C \| \mathcal{A}_n(y_n^{1/\eta} F) \|_{L^{r_n,q}(\mathbb{R}^{n-1})} \leq C \| F \|_{T_{r_n,q}^{r_n,q}}.
\]
The last inequality follows from the embedding (1.7) (with \( s_1 = -\frac{1}{q(n-1)} \), \( s_2 = -\frac{b+1/\eta}{n-1} \), \( q = \eta \), \( p_1 = \tau \) and \( p_2 = r \)). Moving on, we use (2.22) and Hölder’s inequality to get the pointwise bound
\[
\left| \int_{\mathbb{R}^n_+} g(y,z) F^2(z) dz \right| \leq C|G_1 F^2(y)|
\]
Note that the penultimate inequality follows from Remark L. Hence, (after taking the $M$-norm on both sides of the previous inequality)

\[
\left( \int \int \frac{g(y,z)F^2(z)dz}{y_n} \right)^{\frac{q}{n+1}} \leq C\left\| F \left( z_n^{1/n} F \right) \right\|_{T_{r,n}^q}, \quad y' \in B(x', y_n).
\]

Therefore, for $M > 0$ to be determined later, we have

\[
\int \int \frac{g(y,z)F^2(z)dz}{y_n} \leq C\left\| F \left( z_n^{1/n} F \right) \right\|_{T_{r,n}^q}.
\]

The choice $M = \left( \frac{\left\| F \right\|_{T_{r,n}^q}}{\left\| A_n^{\frac{1}{4}} \left( z_n^{1/n} F \right) \right\|_{T_{r,n}^q}} \right)^{\frac{1}{n+1}}$ yields the bound

\[
\left( \int \int \frac{g(y,z)F^2(z)dz}{y_n} \right)^{\frac{q}{n+1}} \leq C\left\| F \right\|_{T_{r,n}^q} \left\| A_n^{\frac{1}{4}} \left( z_n^{1/n} F \right) \right\|_{T_{r,n}^q}.
\]

Hence, (after taking the $L^p$-norm on both sides of the previous inequality)

\[
III_1 \leq C\left\| F \right\|_{T_{r,n}^q}.
\]

We also claim that $III_2 \leq C\left\| F \right\|_{T_{r,n}^q}$. In fact, setting $VF(y', y_n) = y_n^{1+\frac{1}{q}} \int \int \frac{g(y,z)F^2(z)dz}{y_n}$, for all $\phi \in L^p(\mathbb{R}^{n-1})$ we have that

\[
\left\langle A_n^{\frac{1}{4}}(VF), \phi \right\rangle = \int \int \phi(x')dx' \leq C\left\| F \right\|_{L^p(\mathbb{R}^{n-1})} \left\| M\phi \right\|_{L^p(\mathbb{R}^{n-1})}.
\]

Note that the penultimate inequality follows from Remark 2.3 with $r \in (r, p)$ such that $\frac{1}{r} + \frac{1}{n-1} \leq \frac{1}{p}$ and $b = (n - 1)(\frac{1}{r} - \frac{1}{p})$ while the last bound comes from \((1.7)\). Collecting and summing up all the estimates on the $III_i$’s, we find that

\[
\left\| \int \frac{g(\cdot, z)F(z)dz}{T_{r,n}^q} \right\|_{T_{r,n}^q} \leq C\left\| F \right\|_{T_{r,n}^q}.
\]
The remaining estimate reads
\[ \| \int_{\mathbb{R}^+} \nabla_z g(\cdot, z)H(z)dz \|_{T_{p,q}^{\sigma,q}} \leq C \| H \|_{T_{\sigma}^{\Lambda, \sigma}}. \]

The argument we plan to use here is similar to the previous one. In fact, for \((y', y_n) \in \mathbb{R}^n_+\) we write
\[
y_n \left| \int_{\mathbb{R}^+} \nabla_z g(y, z)H(z)dz \right| \leq \sum_{k=1}^3 \Gamma_k(y', y_n),
\]
with
\[
\begin{align*}
\Gamma_1(y', y_n) &= y_n \int_{\mathbb{R}^n_+ \setminus B_{4y_n}(y')} \int_0^\infty |\nabla_z g(y, z)||H(z)|dz \\
\Gamma_2(y', y_n) &= y_n \int_{B_{4y_n}(y')} \int_{4y_n}^\infty |\nabla_z g(y, z)||H(z)|dz \\
\Gamma_3(y', y_n) &= y_n \int_{B_{4y_n}(y')} \int_0^{4y_n} |\nabla_z g(y, z)||H(z)|dz.
\end{align*}
\]

Mimicking the proof of the Claim 2.1, it is easy to see by Lemma 2.5 that for any \((x', y_n) \in \mathbb{R}^n_+\)
\[
\left( \int_{B_{y_n}(x')} |\Gamma_1(y', y_n)|^q dy' \right)^{1/q} \leq cG_1 \left( \int_{B_{y_n}(\cdot)} |H(z', \cdot)|dz' \right)(x', y_n).
\]

Then, by applying Lemma 2.6 with \(\eta = \sigma\) and \(\tau = \Lambda\), we deduce the estimate
\[
\left\| \int_{\mathbb{R}^+} \nabla_z g(\cdot, z)H^3(z)dz \right\|_{T_{p,q}^{\sigma,q}} \leq C \left\| G_1 \left( \int_{B_{y_n}(\cdot)} |H(z', \cdot)|dz' \right) \right\|_{L_p L_q(\mathbb{R}^n)} \leq C \| H \|_{T_{\sigma}^{\Lambda, \sigma}}.
\]

Next, we show that
\[ (2.38) \quad \left\| \int_{\mathbb{R}^+} \nabla_z g(\cdot, z)H^2(z)dz \right\|_{T_{p,q}^{\sigma,q}} \leq C \| H \|_{T_{\sigma}^{\Lambda, \sigma}}. \]

To achieve this, let us primarily observe that
\[
|\Gamma_2(y', y_n)| \leq Cy_n \left[ A^4_\sigma(z_{1/\sigma}^{1/\sigma} H)(x') \right] \left( \int_{B_{4y_n}(x')} \int_{4y_n}^\infty \frac{\frac{\sigma}{\sigma+1} dz_{\sigma} dz'}{\| y' - z \|^2 + |y_n - z_n|^{2(\sigma - 1)}} \right)^{\frac{1}{\sigma + 1}}
\]
\[
\leq Cy_n^{1-\frac{1}{\sigma}} \left[ A^4_\sigma(z_{1/\sigma}^{1/\sigma} H)(x') \right], \quad x' \in B(y', y_n).
\]

Taking the \(\Lambda\)-power of both sides of the last inequality and integrating with respect to the variable \(x'\) leads to
\[
|\Gamma_2(y', y_n)| \leq Cy_n^{1-\frac{1}{\sigma} - \frac{\sigma - 1}{\sigma+1}} \left[ A^4_\sigma(z_{1/\sigma}^{1/\sigma} H) \right]_{L^1(\mathbb{R}^{n-1})} \leq C \| H \|_{T_{\sigma}^{\Lambda, \sigma}}, \quad y' \in B(x', y_n).
\]

Let \(\delta > 0\) to be determined later. The preceding inequalities imply for each \(x' \in \mathbb{R}^{n-1}\)
\[
A_q \left( y_n^{1+1/q} \int_{\mathbb{R}^+} \nabla_z g(\cdot, z)H^2(z)dz \right)(x') = \left( \int_0^\infty \int_{B_{y_n}(x')} |\Gamma_2(y', y_n)|^q dy'dy_n \right)^{1/q}
\]
\[
\leq \left[ \left( \int_0^\delta + \int_\delta^\infty \right) \int_{B_{y_n}(x')} |\Gamma_2(y', y_n)|^q dy'dy_n \right]^{1/q}
\]
\[
\leq C \delta^{(n-1)(\frac{1}{\sigma+1} - \frac{1}{p})} \left[ A^4_\sigma(z_{1/\sigma}^{1/\sigma} H)(x') \right] + \delta^{-\frac{\sigma - 1}{\sigma+1}} \| H \|_{T_{\sigma}^{\Lambda, \sigma}}.
\]
Optimizing with respect to $\delta$, that is, choosing $\delta = \left( \|H\|_{T^{\Lambda,\sigma}_s} / \left( A_\sigma^1 (z_n^{1/\sigma} H) (x') \right) \right)^{\frac{1}{\alpha + 1}}$ yields
\[
\left( \int_0^\infty \int_{B_{y_n}(x')} |F_2(y', y_n)|^q dy' dy_n \right)^{\frac{1}{q}} \leq C \|H\|_{T^{\Lambda,\sigma}_s}^{\frac{1}{\alpha + 1}} \left( A_\sigma^1 (z_n^{1/\sigma} H) (x') \right)^{\frac{1}{\alpha + 1}}, \quad x' \in \mathbb{R}^{n-1}.
\]
Taking the $L^p$-norm on both sides and using Remark 1.1 gives (2.38). Finally, one claims that the $T^{\Lambda,\sigma}_s$-norm of $\Gamma_3$ is controlled from above by a constant multiple of $\|H\|_{T^{\Lambda,\sigma}_s}$. This is derived from a simple duality argument in the same fashion as before. The proof of Proposition 2.2 is now complete.

We can now summarize the findings obtained above into a single theorem establishing the well-posedness of System (S) for boundary data in the scale of Triebel-Lizorkin space with negative amount of smoothness. We say that a pair $(u, \pi)$ is a solution to (S) if $u$ and $\pi$ satisfy the relations
\[(2.39) \quad u(x) = Hf(x) + \mathcal{G}(F, H)(x), \quad \pi(x) = \mathcal{E}f(x) + \Psi(F, H)(x), \quad x \in \mathbb{R}^n.\]

**Theorem 2.1.** Assume that the numbers $\eta, \tau, \sigma, \Lambda$ and $p, q$ are as in Proposition 2.2. Then for any $f \in [F_{\tau,q}^{1/q} (\mathbb{R}^{n-1})]^n$, $F \in X_{\tau,\eta}$ and $H \in X_{\Lambda,\sigma}$, the Stokes system (S) has a solution $(u, \pi) \in X_{p,q} \times Z_{p,q}$ (in the sense made precise in (2.39)) and the following estimate holds:
\[(2.40) \quad \|u\|_{X_{p,q}} + \|\pi\|_{Z_{p,q}} \leq C(\|f\|_{F_{\tau,q}^{1/q} (\mathbb{R}^{n-1})} + \|F\|_{X_{\tau,\eta}} + \|H\|_{X_{\Lambda,\sigma}})\]
for some constant $C > 0$ independent of $f$, $F$ and $H$.

**Remark 2.4.** Practically, Theorem 2.1 can easily be extended to the case where the vector field $u$ is not necessarily solenoidal, i.e. $\text{div} \ u = \phi$ under suitable conditions on $\phi$ (integrability and compatibility) using the formulation derived in [29, formula 2.32], see also [6] so that our result gives an alternative approach to the Dirichlet problem for the Stokes system (to be compared to [3] and [9] wherein the analysis is carried out in weighted Sobolev spaces and Lebesgue spaces, respectively). These estimates of the velocity field and the pressure in weighted tent framework against boundary data in low regularity spaces are new and generalize well-known results. In fact, our boundary class $F_{\tau,q}^{1/q} (\mathbb{R}^{n-1})$ contains the homogeneous Sobolev space $H^{s,q}(\mathbb{R}^{n-1})$ for $-1/q < s < (n-1)/r$ under the conditions $1 \leq r < p$ and $(n-1)/r - s = 1/q + (n-1)/p$.

### 3. Proofs of main results

The proof of Theorem 1.1 can easily be deduced from that of Theorem 1.2. Hence, only the proofs of Theorems 1.2 and 1.3 will be exposed in details and in the process, one essentially relies on preliminary results obtained in Section 2.

**Proof of Theorem 1.2.** Let $f \in [\mathcal{H}^{-\frac{1}{2} - 2(n-1)}(\mathbb{R}^{n-1})]^n$ with $n > 2$ and assume $F \in X_{\tau,\eta}$ for $\eta \in (1, 2)$ and $\tau \in (\eta, 2(n-1))$ with $\frac{1}{\eta} + \frac{n-1}{\tau} = 3$. Equip the Banach space $X \times Z$ with the norm $\| \cdot \| := \| \cdot X + \| \cdot \| Z$ and introduce the operators $\mathcal{L}$ defined by
\[\mathcal{L}(u, \pi) = (\mathcal{H}f + \mathcal{G}[F, u \otimes u], \mathcal{E}(f) + \Psi[F, u \otimes u])\]
where $\mathcal{H}$ and $\mathcal{E}$ are given by (2.4). A solution of (NS)-(1.1) according to the definition given by (2.39) is a couple $(u, \pi)$ satisfying the fixed point equation
\[(3.1) \quad (u, \pi) = \mathcal{L}(u, \pi) \text{ in } \mathbb{R}^n.\]
Using a Banach fixed point argument, we wish to show that the latter equation admits a solution in $X \times Z$. Assume that $(u, \pi)$ and $(v, \pi')$ in $X \times Z$ satisfy (3.1) and use Proposition 2.2 with $(p, q) = (2(n - 1), 2)$, $(A, \sigma) = (n - 1, 1)$ to get
\[
\left\| \mathcal{L}(u, \pi) - \mathcal{L}(v, \pi') \right\| = \left\| \mathcal{G}(0, \frac{u \otimes u - v \otimes v}{x_n} + \frac{u \otimes u - v \otimes v}{x_n}) \right\|_{X} + \left\| \Psi(0, \frac{u \otimes u - v \otimes v}{x_n}) \right\|_{Z} \\
\leq C\left\| u \otimes u - v \otimes v \right\|_{X, n-1, 1} \\
\leq C\left(\left\| (u \otimes u - v \otimes v) \right\|_{X, n-1, 1} + \left\| (u - v) \right\|_{X, n-1, 1} \right) \\
\leq C\left( \sup_{x_n > 0} x_n^2 \left\| (u \otimes u - v \otimes v) \right\|_{L^\infty(\mathbb{R}^n-1)} + \left\| (u \otimes u - v \otimes v) \right\|_{L^{p, n-1, 1}} + \right. \\
\left. \sup_{x_n > 0} x_n^2 \left\| (u - v) \right\|_{L^\infty(\mathbb{R}^n-1)} + \left\| (u - v) \right\|_{L^{p, n-1, 1}} \right) \\
\leq C\left( \sup_{x_n > 0} x_n \left\| u(\cdot, x_n) \right\|_{L^\infty(\mathbb{R}^n-1)} \sup_{x_n > 0} x_n \left\| (u - v)(\cdot, x_n) \right\|_{L^\infty(\mathbb{R}^n-1)} + \right. \\
\left. \sup_{x_n > 0} x_n \left\| (u - v)(\cdot, x_n) \right\|_{L^\infty(\mathbb{R}^n-1)} \sup_{x_n > 0} x_n \left\| v(\cdot, x_n) \right\|_{L^\infty(\mathbb{R}^n-1)} + \right. \\
\left. \left\| u \right\|_{H^{2(n-1), 2}} \left\| u - v \right\|_{H^{2(n-1), 2}} + \left\| u - v \right\|_{H^{2(n-1), 2}} \right) \\
\leq C\left\| u - v \right\|_{X} \left(\left\| u \right\|_{X} + \left\| v \right\|_{X} \right) \\
(3.2)
\]
One can deduce from the previous estimate and in light of Lemma 2.2 and Proposition 2.2 the bound
\[
\left\| \mathcal{L}(u, \pi) \right\| \leq C\left(\left\| u \right\|_{X} + \left\| f \right\|_{H^{-\frac{1}{2}, 2(n-1), 2}(\mathbb{R}^n-1)} + \left\| F \right\|_{X_{\sigma, q}} \right).
(3.3)
\]
Now pick $\varepsilon > 0$ such that $\left\| f \right\|_{H^{-\frac{1}{2}, 2(n-1), 2}(\mathbb{R}^n-1)} + \left\| F \right\|_{X_{\sigma, q}} \leq \varepsilon$. If $\varepsilon$ is sufficiently small, $\varepsilon < \min(1/4C, 1/2)$ then it readily follows from (3.2) and (3.3) that $\mathcal{L}$ has a unique fixed point in a closed ball of $X \times Z$ centered at the origin with radius $2\varepsilon$.

Proof of Theorem 1.3. Let $2 < q < p < \infty$. Further, let $\eta_1 \in (1, q)$ and $\tau_1 \in (\eta_1, p)$ such that
\[
\frac{1}{\eta_1} + \frac{n - 1}{\tau_1} = 2 + \frac{1}{q} + \frac{n - 1}{p}.
(3.4)
\]
Assume $f \in H^{-\frac{1}{2}, 2(n-1)} \cap F_{\eta, q}^{\frac{1}{2}}(\mathbb{R}^n-1)$ and $F \in X_{\tau_1, q} \cap X_{\tau_1, q}$. We remark that the solution found above may be realized as the unique limit in $X \times Z$ of the following sequence of approximations given by
\[
\begin{align*}
\{(u_1, \pi_1) &= (\mathcal{H}(f), \mathcal{E}(f)) \\
(u_{j+1}, \pi_{j+1}) &= (\mathcal{G}[F, u_j \otimes u_j] + u_1, \Psi[F, u_j \otimes u_j] + \pi_1), j = 1, 2, ...
\end{align*}
\]
Each element of this sequence belongs to $X_{\eta, q} \times Z_{\eta, q}$. In fact, since $(u_1, \pi_1) \in X_{\eta, q} \times Z_{\eta, q}$ (see Lemma 2.2) one may proceed via an induction argument to prove the claim. Choose $(\sigma, A)$ such that $\frac{1}{\sigma} = \frac{1}{2} + \frac{1}{q}, \frac{1}{\tau} = \frac{2(n-1)}{2(n-1)} + \frac{1}{p}$ and invoke Proposition 2.2, Hölder’s inequality in tent spaces simultaneously to have for each $j$,
\[
\left\| (u_{j+1}, \pi_{j+1}) \right\|_{X_{\eta, q} \times Z_{\eta, q}} \\
= \left\| \mathcal{G}[F, u_j \otimes u_j] + u_1 \right\|_{X_{\eta, q}} + \left\| \Psi[F, u_j \otimes u_j] + \pi_1 \right\|_{Z_{\eta, q}} \\
\leq C\left(\left\| f \right\|_{F_{\eta, q, q}^{\frac{1}{2}}(\mathbb{R}^n-1)} + \left\| F \right\|_{X_{\tau_1, q}} \right) + \left\| u_j \otimes u_j \right\|_{X_{\tau_1, q}} \\
\leq C\left(\left\| f \right\|_{F_{\eta, q}^{\frac{1}{2}}(\mathbb{R}^n-1)} + \left\| F \right\|_{X_{\tau_1, q}} \right) + \left\| u_j \otimes u_j \right\|_{X_{\tau_1, q}} \\
\leq C\left(\left\| f \right\|_{F_{\eta, q}^{\frac{1}{2}}(\mathbb{R}^n-1)} + \left\| F \right\|_{X_{\tau_1, q}} \right) + \left\| u_j \otimes u_j \right\|_{X_{\tau_1, q}} \\
so that \((u_j, \pi_j) \in X_{p,q} \times Z_{p,q}\), then so is \((u_{j+1}, \pi_{j+1})\). Next, we show that the latter sequence is Cauchy in \(X_{p,q} \times Z_{p,q}\). We estimate \((w_j, q_j) = (u_{j+1} - u_j, \pi_{j+1} - \pi_j), j = 1, 2, \ldots\)

\[
\| (w_j, q_j) \|_{X_{p,q} \times Z_{p,q}} = \| \Psi[0, u_j \otimes u_j - u_{j-1} \otimes u_{j-1}] \|_{X_{p,q}} + \| \Psi[0, u_j \otimes u_j - u_{j-1} \otimes u_{j-1}] \|_{Z_{p,q}}
\]

\[
\leq C\| u_j \otimes u_j - u_{j-1} \otimes u_{j-1} \|_{X_{\lambda,\sigma}}
\]

\[
\leq C\| u_j \otimes w_{j-1} + w_j \otimes u_{j-1} \|_{X_{\lambda,\sigma}}
\]

\[
\leq C\| w_{j-1} \|_{X_{p,q}} (\| u_j \|_X + \| u_{j-1} \|_X)
\]

\[
\leq C\| (w_{j-1}, q_{j-1}) \|_{X_{p,q} \times Z_{p,q}} (\| u_j \|_X + \| u_{j-1} \|_X).
\]

Let \(\varepsilon > 0\) be as in Theorem 1.2 and take \(0 < \varepsilon_* < \varepsilon\). If \(\| f \|_{\bar{H}^{1-\frac{2}{p}}(\mathbb{R}^n)} + \| F \|_{\mathcal{L}_{q,\eta}} \leq \varepsilon_*, \) then the conclusion of Theorem 1.2 shows that \(\| u_j \|_X \leq 2\varepsilon_*\). Whence,

\[
\| (w_j, q_j) \|_{X_{p,q} \times Z_{p,q}} \leq 4C\varepsilon_*(\| w_{j-1}, q_{j-1} \|_{X_{p,q} \times Z_{p,q}}).
\]

A simple iteration of the previous inequality yields

\[
\| (w_j, q_j) \|_{X_{p,q} \times Z_{p,q}} \leq (4C\varepsilon_*)^{j-1}(\| w_1, q_1 \|_{X_{p,q} \times Z_{p,q}})
\]

thus implying the convergence of the sequence \((w_j, q_j)\) in \(X_{p,q} \times Z_{p,q}\) since \(\varepsilon_* < 1/4C\). The limit of this sequence solves (3.1) and by uniqueness, it is the same as the solution constructed in Theorem 1.2.

### 4. Appendix

Here we sketch the proof of Lemma 1.1. Let \(K \subset \mathbb{R}^n_+\) be a compact set. Then by Lemma 2.4 we know that \(E(K) = \{ x' \in \mathbb{R}^{n-1} : K \cap \Gamma(x') \neq \emptyset \}\) has a finite Lebesgue measure.

**Proof of Lemma 1.1.** Let us denote by \(1_K\) the characteristic function of the compact set \(K\). If \(p \leq q\), then via Hölder’s inequality, one obtains

\[
\| 1_K f \|_{T^{p,q}} = \left( \int_{\mathbb{R}^{n-1}} \left( \int_{\Gamma(x')} 1_K |f|^q y_n^{-(n-1)} dy' dy_n \right)^{p/q} dx' \right)^{1/p}
\]

\[
= \left( \int_{E(K)} \left( \int_{\Gamma(x')} |f|^q y_n^{-(n-1)} dy' dy_n \right)^{p/q} dx' \right)^{1/p}
\]

\[
\leq \left( \int_{E(K)} \left( \int_{\Gamma(x')} |f|^q y_n^{-(n-1)} dy' dy_n dx' \right)^{1/q} |E(K)| \right)^{1/p - \frac{1}{q}}
\]

\[
\leq |E(K)|^{\frac{1}{p} - \frac{1}{q}} \left( \int_{\mathbb{R}^{n-1}} \int_{\Gamma(x')} |f|^q y_n^{-(n-1)} dy' dy_n dx' \right)^{1/q}
\]

\[
\leq C|E(K)|^{\frac{1}{p} - \frac{1}{q}} \| f \|_{L^q(K)}.
\]

Moving on, for \(q < p\), applying Minkowski’s inequality implies

\[
\| 1_K f \|_{T^{p,q}} = \left( \int_{\mathbb{R}^{n-1}} \left( \int_{\Gamma(x')} 1_K |f|^q y_n^{-(n-1)} dy' dy_n \right)^{p/q} dx' \right)^{1/p}
\]

\[
= \left( \int_{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}^{n-1}} \int_0^\infty 1_B y_n(x') 1_K(y', y_n) |f|^q y_n^{-(n-1)} dy' dy_n dx' \right)^{p/q} dx' \right)^{1/p}
\]

\[
\leq C_K \left( \int_{\mathbb{R}^{n-1}} \int_0^\infty 1_K |f|^q dy' dy_n \right)^{1/q}
\]

\[
\leq C_K \| f \|_{L^q(K)}.
\]
Assuming that \( p \leq q \), we use Lemma 2.3 and Minkowski's inequality simultaneously to get

\[
\|f\|_{L^q(K)} = \left( \int_{\mathbb{R}^n_+} 1_K |f|^q dy'dyn \right)^{1/q}
\leq C \left( \int_{\mathbb{R}^{n-1}} \int_{\Gamma(x')} 1_K |f|^q y_n^{1-n} dy'dyn dx' \right)^{1/q}
\leq C K \left( \int_{\mathbb{R}^{n-1}} \left( \int_{\Gamma(x') E(K)} |f|^q y_n^{1-n} dy'dyn \right)^{\frac{q}{p}} dy' \right)^{1/q}
\leq C K \left( \int_{\mathbb{R}^{n-1}} \left( \int_{\Gamma(x')} |f|^q y_n^{1-n} dy'dyn \right)^{\frac{p}{q}} dx' \right)^{1/p}
\leq C K \|f\|_{T_p^q}.
\]

When \( p > q \), the desired bound follows from Hölder's inequality. Indeed, we have

\[
\|f\|_{L^q(K)} = \left( \int_{\mathbb{R}^n_+} 1_K |f|^q dy'dyn \right)^{1/q}
\leq C \left( \int_{\mathbb{R}^{n-1}} \int_{\Gamma(x')} 1_K |f|^q y_n^{1-n} dy'dyn dx' \right)^{1/q}
\leq C \left( \int_{E(K)} \left( \int_{\Gamma(x')} |f|^q y_n^{1-n} dy'dyn \right)^{1/q} dx' \right)^{1/p}
\leq C \left( E(K) \right)^{1/p} \|f\|_{T_p^q}.
\]

Next, we prove Lemma 2.1. Before we proceed, let us recall some properties of the Fourier transform. Let \( \varphi \in S(\mathbb{R}^{n-1}) \), its Fourier transform is denoted by the standard notation \( \hat{\varphi} \) and defined as \( \hat{\varphi}(\xi') = \int_{\mathbb{R}^{n-1}} e^{-iy'\cdot\xi'} \varphi(y') dy' \) for all \( \xi' \in \mathbb{R}^{n-1} \). If \( \varphi' = \varphi(-\cdot) \) and \( u \) is a tempered distribution in \( \mathbb{R}^{n-1} \), then the map \( u' \) defined by \( u' : S(\mathbb{R}^{n-1}) \rightarrow \mathbb{R} \), \( u'(\varphi) = \langle u, \varphi' \rangle \) for all \( \varphi \in S(\mathbb{R}^{n-1}) \) is a tempered distribution in \( \mathbb{R}^{n-1} \) and

\[
(4.1) \quad \mathcal{F}^{-1} u = (2\pi)^{1-n} u'.
\]

In addition to this formula, we will systematically use below the Fourier transforms of the integrable functions \( f(z) = (z^2 + b^2)^{-1}, z \in \mathbb{R}, b > 0 \) and \( p(y') = \frac{2}{\omega_{n-1}} (1 + |y'|^2)^{-n/2}, y' \in \mathbb{R}^{n-1} \) given respectively by

\[
(4.2) \quad \hat{f}(\eta) = \frac{\pi}{b} e^{-b|\eta|}, \quad \forall \eta \in \mathbb{R} \text{ and in } S'(\mathbb{R}) \text{ and } \hat{p}(\xi') = e^{-|\xi'|}, \xi' \in \mathbb{R}^{n-1} \setminus \{0\}.
\]

**Proof of Lemma 2.1.** We start with the computation of \( \widehat{P_0} \) by means of partial Fourier transform. Observe that \( \mathcal{F}_{\xi'}(K_{mn}(\cdot, x_n))(\xi') = -x_n|\xi'| e^{-x_n|\xi'|} - e^{-x_n|\xi'|} \forall \xi' \in \mathbb{R}^{n-1} \setminus \{0\} \). This is easily seen from the formula \( K_{mn}(x) = \frac{2}{\omega_{n-1}} (x_n \partial_n \frac{x_n}{|x|} - \frac{x_n}{|x|^2}) \) and (4.2). From the identity \( K_{mn}(x) = x_n^{-(n-1)} P_0(\frac{x'}{x_n}), \ x = (x', x_n) \in \mathbb{R}^n_+ \), one deduces that for each \( \xi' \in \mathbb{R}^{n-1} \)

\[
(4.3) \quad \widehat{P_0}(\xi') = -|\xi'| e^{-|\xi'|} - e^{-|\xi'|}.
\]
The Fourier transform of $Q_{jk}$, $j, k = 1, \ldots, n - 1$ can be derived from the simple identity
\begin{equation}
Q_{jk}(x') = -\delta_{jk}p(x') - \frac{2}{\omega_{n-1}(n-2)}\partial_{jj}'(|x'|^2 + 1)^{\frac{2-n}{2}}
\end{equation}
if $\frac{1}{2} + \frac{2-n}{2}$ is explicit. In this regard, note that $|x|^{-(n-2)}$, $x \in \mathbb{R}^n \setminus \{0\}$ belongs to $S'(\mathbb{R}^n)$ and for each $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$ and $\eta \in \mathbb{R}$ we have
\[
|.|^{-(n-2)}(\xi', \eta) = (n-2)\omega_{n-1}(\xi'^2 + |\eta|^2)^{-1}.
\]
Hence, using (4.1) in one dimension and (4.2) with $b = |\xi'|$ we arrive at
\[
\mathcal{F}_{x'}(|(x',x_n)|^{2-n})(\xi') = \mathcal{F}_{x'}[|.|^{-(n-2)}(\xi', \eta)](x_n)
\]
\[
= (n-2)\omega_{n-1}\mathcal{F}_{x'}\left(\frac{1}{|\xi'|^2 + |\eta|^2}\right)(x_n)
\]
\[
= (2\pi)^{-1}(n-2)\omega_{n-1}\mathcal{F}_{x'}\left(\frac{1}{|\xi'|^2 + |\eta|^2}\right)(x_n) = \frac{(n-2)\omega_{n-1}}{2}|\xi'|^{-1}e^{-|x_n\xi'|},
\]
In particular, when $x_n > 0$, \(\mathcal{F}_{x'}(|x|^{2-n})(\xi') = \frac{(n-2)\omega_{n-1}}{2}|\xi'|^{-1}e^{-|x_n|\xi'|} \) for all $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$.
As a consequence, \(\frac{1}{2} + \frac{2-n}{2} = \frac{(n-2)\omega_{n-1}}{2}|\xi'|^{-1}e^{-|\xi'|}, \xi' \in \mathbb{R}^{n-1} \setminus \{0\}\) and by (4.2), it follows from (4.4) that
\[
\widehat{Q}_{jk}(\xi') = -\delta_{jk}e^{-|\xi'|} + \frac{\xi'_j\xi'_k}{|\xi'|}e^{-|\xi'|}, \xi' \in \mathbb{R}^{n-1} \setminus \{0\}.
\]
Finally, since $P_j(x') = \partial_{x'}p(x')$, one uses (4.2) to get $\widehat{P}_j(\xi) = i\xi'_je^{-|\xi'|}, \xi' \in \mathbb{R}^{n-1}$.

We now turn to the proof of Proposition 2.1. Given $\lambda > 0$, a kernel $\psi$ defined on $\mathbb{R}^{n-1}$ and a distribution $f \in S'(\mathbb{R}^{n-1})$ such that $\psi_t* f$, $t > 0$ is continuous, the Peetre maximal function associated to $f$ is defined as
\begin{equation}
(\psi_t f)^{**}(y') = \sup_{z' \in \mathbb{R}^{n-1}}|\psi_t f(z')|(1 + (t^{-1}|z' - y'|))^{-\lambda}, y' \in \mathbb{R}^{n-1}.
\end{equation}

**Proof of Proposition 2.1.** Let $f \in \mathcal{F}_{x'}^{s,q}(\mathbb{R}^{n-1})$ with $s < 0$ and $\psi \in \mathcal{A}_{A,m,r}$ where $r > s$, $m + s > \lambda$ with $\lambda \in (A, [A]+1)$, $A = \max\{(n-1)/p, (n-1)/q\}$. Then $\psi_n f$ is a bounded continuous function. We primarily estimate the conical functional of the underlying convolution by the Littlewood-Paley functional of the Peetre maximal functional associated to $f$. Indeed,
\[
A_q(y_n^{-s}\psi_n f)(x') = \left(\int_{\mathbb{R}^{(n-1)}} \left|y_n^{-s}\psi_n f(y')\right|^q y_n^{-n}dy dy\right)^{1/q}
\]
\[
\leq 2^{\lambda/q} \left(\int_0^{\infty} \int_{B_{yn}(x')} \left|y_n^{-s}(\psi_n f)^{**}(x')\right|^q y_n^{-n}dy dy\right)^{1/q}
\]
so that
\begin{equation}
\|\psi_n f\|_{\mathcal{T}_{x,n}^{p,q}(n-1)} \leq 2^{\lambda/q} \left(\int_0^{\infty} \left|y_n^{-s}(\psi_n f)^{**}(\cdot)\right|^q dy\right)^{1/q} \left\|\frac{dy_n}{y_n}\right\|_{L^p(\mathbb{R}^{n-1})}.
\end{equation}
Now, our goal is to estimate the function
\[
\chi(y, z') = |(\psi_n f)(z')|(1 + y_n^{-1}|z' - y'|)^{-\lambda}, \quad y \in \mathbb{R}^n, z' \in \mathbb{R}^{n-1}.
\]
After possibly subtracting a suitable polynomial to $f$, one may use the Calderón reproducing formula for elements in $\tilde{F}_{p,q}^s(\mathbb{R}^{n-1})$ to obtain the pointwise identity (see [5, Theorem 3.1])

$$\psi_{y_n}f(z') = \sum_{k \in \mathbb{Z}} \phi_k \ast \phi_k \ast \psi_{y_n} \ast f(z'), \ z' \in \mathbb{R}^{n-1}$$

where $\phi_k$ is as in Remark 1.2. Let $l \in \mathbb{Z}$ and assume that $y_n \in [2^{-l-1}, 2^{-l}]$. Then, we have

$$\chi(y, z') \leq \sum_{k \in \mathbb{Z}} |\psi_{y_n} \ast \phi_k \ast \phi_k \ast f(z')|(1 + y_n^{-1}|y' - z'|)^{-\lambda}$$

$$\leq \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^{n-1}} |\psi_{y_n} \ast \phi_k(z')||\phi_k \ast f(z' - v')|dv'(1 + y_n^{-1}|y' - z'|)^{-\lambda}$$

$$\leq 2^{(n-1)l} \sum_{k \in \mathbb{Z}} \phi^*_k f(y') \int_{\mathbb{R}^{n-1}} |\psi_{y_n} \ast \phi_{k-l}(2^l v')| \frac{(1 + 2^k|y' - z' + v'|)^\lambda}{(1 + 2^l|y' - z'|)^\lambda} dv'$$

$$\leq \sum_{k \in \mathbb{Z}} \phi^*_k f(y') \int_{\mathbb{R}^{n-1}} |\psi_{y_n} \ast \phi_{k-l}(v')| \frac{(1 + 2^k|y' - z'| + v'|)^\lambda}{(1 + 2^l|y' - z'|)^\lambda} dv'.$$

This clearly implies the pointwise estimate

$$(4.7) \quad y_n^{-s}(\psi_{y_n}f)^\ast(y') \leq \sum_{k \in \mathbb{Z}} a_{l-k} 2^{ks} \phi^*_k f(y'): \ y' \in \mathbb{R}^{n-1}, \ y_n \in [2^{-l-1}, 2^{-l}]$$

where the sequence $(a_k)_{k \in \mathbb{Z}}$ reads

$$a_k = 2^{ks} \sup_{2^{-l-1} \leq \theta \leq 1} \sup_{z' \in \mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} |\psi_{\theta z'} \ast \phi_{-k}(v')| \frac{(1 + 2^k|v'|)^\lambda}{(1 + |v'|)^\lambda} dv'.$$

Hence, for any $y' \in \mathbb{R}^{n-1}$, it follows from (4.7) that

$$\left( \int_0^\infty \left[ y_n^{-s}(\psi_{y_n}f)^\ast(y') \right]^q y_n^{-1} dy_n \right)^{1/q} \leq \left( \sum_{l \in \mathbb{Z}} \left( \int_{2^{-l-1}}^{2^{-l}} \left[ y_n^{-s}(\psi_{y_n}f)^\ast(y') \right]^q y_n^{-1} dy_n \right) \right)^{1/q}$$

$$\leq \left( \sum_{l \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} a_{l-k} 2^{ks} \phi^*_k f(y') \right)^q \right)^{1/q}.$$ 

The desired estimate will now follow from the above estimate, (4.6) and Remark 1.2 as soon as $(a_k)_{k \in \ell^1}$. Observe that

$$\frac{1 + 2^{-k}|z' + v'|}{1 + |z'|} \leq \begin{cases} 2^{-k}(1 + |v'|) & \text{if } k < 0 \\ (1 + 2^{-k}|v'|) & \text{if } k \geq 0 \end{cases}$$

and using the fact that $\psi$ belongs to $A_{\Lambda, m, r}$, an application of Lemma 2.4 in [5] yields the estimate

$$(4.8) \quad |\psi \ast \phi_{-k}(v')| \leq \begin{cases} 2^{km}(1 + |v'|)^{-(n[4])} & \text{if } k < 0 \\ 2^{-k(n-1+r)}(1 + 2^{-k}|v'|)^{-(n[4])} & \text{if } k \geq 0. \end{cases}$$

Thus, by definition of $a_k$ and after rescaling, one obtains for each $k \in \mathbb{Z}$ the estimate

$$a_k \leq C \begin{cases} 2^{-k(\lambda - m - s)} & \text{if } k < 0 \\ 2^{-k(r - s)} & \text{if } k \geq 0 \end{cases}$$

from which it easily follows by the assumptions on the parameters $\Lambda, m, r$ that $(a_k)_{k \in \ell^1}$. The proof of Proposition 2.1 is now complete. \qed
References

[1] A. Amenta. Interpolation and embeddings of weighted tent spaces, *J. Fourier Anal. Appl.* 24 (2018), 108–140.

[2] H. Amann. On the strong solvability of the Navier-Stokes equations, *J. Math. Fluid Mech.* 2 (2000), 16–98.

[3] C. Amrouche, S. Nečasová and Y. Raudin. Very weak, generalized and strong solutions to the Stokes system in the half-space, *J. Diff. Equ.* 244 (2008), 887–915.

[4] P. Auscher and A. Axelsson. Weighted maximal regularity estimates and solvability of non-smooth elliptic systems I, *Invent. Math.* 184 (2011), 47–115.

[5] H-Q. Bui and T. Candy. A characterisation of the Besov-Lipschitz and Triebel-Lizorkin spaces using Poisson like kernels, arXiv:1502.06836v2 (2016).

[6] L. Cattabriga. Su un problema al contorno relativo al sistema di equazioni di Stokes, *Rend. Sem. Mat. Univ. Padova* 31 (1961), 308–340.

[7] R. R. Coifman, Y. Meyer and E. M. Stein. Some new function spaces and their applications to Harmonic analysis, *J. Funct. Anal.* 62 (1985), 304–335.

[8] R. Farwig, G.P. Galdi and H. Sohr. Very weak solutions and large uniqueness of classes of stationary Navier-Stokes equations in bounded domains in $\mathbb{R}^2$, *J. Diff. Equ.* 227 (2006), 564–580.

[9] R. Farwig and J. Sauer. Very weak solutions of the stationary Stokes equations in unbounded domains of half space type, *Math. Bohemica* 140 (2015), 81–109.

[10] R. Finn. On the steady-state solutions of the Navier-Stokes equations, III, *Acta Math.* 105 (1961), 197–244.

[11] J. Frehse and M. Růžička. Existence of regular solutions to the stationary Navier-Stokes equations, *Math. Ann.* 302 (1995), 699–717.

[12] G.P. Galdi. An introduction to the mathematical theory of the Navier-Stokes equations, *Springer Monographs in Mathematics*, 2nd Ed., New York, (2011).

[13] G.P. Galdi, C. G. Simader and H. Sohr. A class of solutions to stationary Stokes and Navier-Stokes equations with boundary data in $W^{-1/q,q}$, *Math. Ann.* 331 (2005), 41–74.

[14] C. Gerhardt. Stationary solutions to the Navier-Stokes equations in dimension four, *Math. Z.* 165 (1979), 193–197.

[15] W. Guo, D. Fan, H. Wu and G. Zhao. Sharp weighted convolution inequalities and some applications, *Studia Mat.* 241 (2018), 201–239.

[16] Y. Huang. Weighted tent spaces with Whitney averages: factorization, interpolation and duality, *Math. Z.* 282 (2016), 913–933.

[17] S. Hofmann, S. Mayboroda and A. McIntosh. Second order elliptic operators with complex bounded measurable coefficients in $L^p$, Sobolev and Hardy spaces, *Ann. Sci. Éc. Norm. Supér* 44(5) (2011), 723–800.

[18] K. Kang, H. Miura and T.P. Tsai. Green tensor of the Stokes system and asymptotics of stationary Navier-Stokes flows in the half space, *Adv. Math.* 323 (2018), 326–366.

[19] D. Kim, H. Kim and S. Park. Very weak solutions of the stationary Stokes equations on exterior domains, *Adv. Diff. Eqn.* 20 (11/12) (2015), 1119–1164.

[20] H. Kim. Existence and regularity of very weak solutions of the Navier-Stokes equations, *Arch. Ration. Mech. Anal.* 193 (2009), 117–152.

[21] J. Leray. Études de diverses équations intégrales non-linéaires et quelques problèmes que pose l’hydrodynamique, *J. Math. Pures Appl.* 12 (1933), 1–82.

[22] X. Luo. Stationary solutions and nonuniqueness of weak solutions for the Navier-Stokes equations in higher dimensions, *Arch. Ration. Mech. Anal.* 233 (2019), 701–747.

[23] E. Marušić-Paloka. Solvability of the Navier-Stokes systems with $L^2$ boundary data, *Appl. Math. Optim.* 41 (2000), 365–375.
[24] V.G. Maz’ya, B.A. Plamenevskii and L.I. Stupyalis. The three-dimensional problem of steady-state motion of a fluid with a free surface, Trans. Amer. Math. Soc. 123 (1984), 171–268.

[25] D. Mitrea. Distributions, partial differential equations, and harmonic analysis, Universitext, Springer, (2013).

[26] F.K.G. Odqvist. Über die Randwertaufgaben der Hydrodynamik zäher Flüssigkeiten, Math. Z. 32(1) (1930), 329–375.

[27] D. Serre. Equations de Navier-Stokes stationnaires avec données peu régulières, Ann. Sci. Norm. Sup. Pisa (4)10 (1983), 543–559.

[28] C.G. Simader. Mean value formulas, Weyl’s lemma and Liouville theorems for $\Delta^2$ and Stokes’ system, Results. Math. 22 (1992), 761–780.

[29] V. A. Solonnikov. Estimates for solutions of nonstationary Navier-Stokes equations, J. Math. Sci. 8 (1977), 467–529.

[30] E.M. Stein. Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton Mathematical series, Princeton University Press, Princeton New Jersey, 1993.

[31] M. Struwe. Regular solutions of the stationary Navier-Stokes equations on $\mathbb{R}^5$, Math. Ann. 302 (1995), 719–741.

[32] H. Triebel. Theory of Function Spaces II, Monographs in Mathematics, Vol. 84 Birkhäuser, Switzerland, 1992.

[33] L. Yanyan and Z. Yang. Regular solutions of the stationary Navier–Stokes equations on high dimensional Euclidean space, Commun. Math. Phys. 394 (2022), 711–734.

[34] G. D. Yomgné. On a nonlinear Laplace equation related to the boundary Yamabe problem in the upper-half-space, Comm. Pure Appl. Anal. 21(2) (2022), 517-539.

[35] G. D. Yomgné and H. Koch. The Dirichlet problem for weakly harmonic maps with rough data, Comm. Partial Differential Equations 47(7) (2022), 1504–1535.

Gael Y. Diebou, University of Toronto, Department of Mathematics, 40 St George Street, Toronto, ON, M5S 2E4, Canada

Email address: gaely.diebou@utoronto.ca