ON THE CONSTRUCTION OF SMALL SUBSETS CONTAINING SPECIAL ELEMENTS IN A FINITE FIELD

JIYOU LI

Abstract. In this note we construct a series of small subsets containing a non-d-th power element in a finite field by applying certain bounds on incomplete character sums.

Precisely, let $h = \lfloor q^\delta \rfloor > 1$ and $d \mid q^h - 1$. Let $r$ be a prime divisor of $q - 1$ such that the largest prime power part of $q - 1$ has the form $r^t$. Then there is a constant $0 < \epsilon < 1$ such that for a ratio at least $q^{-\epsilon h}$ of $\alpha \in F_{q^h} \setminus F_q$, the set $S = \{ \alpha - x^t, x \in F_q \}$ of cardinality $1 + q^{-1} M(h)$ contains a non-d-th power in $F_{q^h}$, where $t$ is the largest power of $r$ such that $t < \sqrt{q/h}$ and $M(h)$ is defined as

$$M(h) = \max_{r \mid (q-1)} r^{\min\{v_r(q-1), \lfloor \log_q (q/2-\log_q h) \rfloor \}}.$$ 

Here $r$ runs through prime divisors and $v_r(x)$ is the $r$-adic order of $x$. For odd $q$, the choice of $\delta = \frac{1}{2} - d = o(1) > 0$ shows that there exists an explicit subset of cardinality $q^{1-d} = O(\log^{2+\epsilon'} (q^h))$ containing a non-quadratic element in the field $F_{q^h}$. On the other hand, the choice of $h = 2$ shows that for any odd prime power $q$, there is an explicit subset of cardinality $1 + q^{-1} M(2)$ containing a non-quadratic element in $F_{q^2}$. This improves a $q - 1$ construction by Coulter and Kosick [6] since $\lfloor \log_q (q - 1) \rfloor \leq M(2) < \sqrt{q}$.

In addition, we obtain a similar construction for small sets containing a primitive element. The construction works well provided $\phi(q^h - 1)$ is very small, where $\phi$ is the Euler’s totient function.

1. Introduction

For an odd prime $p$, it is a historical and hard problem to determine $q_p$, the least quadratic non-residue or generally, the least $d$-th power non-residue of $p$. Clearly, this problem is essentially reduced to finding a nontrivial upper bound for the sum

$$\sum_{1 \leq i \leq x} \chi(x),$$

where $\chi$ is a non-principal multiplicative character modulo $p$.

There are two well-known and important upper bounds for this quantity. The first, discovered independently by Polya and Vinogradov one century ago, asserts

$$| \sum_{1 \leq i \leq x} \chi(x) | \ll \sqrt{p} \log p$$

and this implies there is a constant $C$ such that

$$q_p \leq C \sqrt{p} \log p.$$
It is believed that the Polya-Vinogradov bound can be improved to
\[ |\sum_{1 \leq i \leq x} \chi(x)| \ll \sqrt{p} \log \log p, \]
which was proved by Montgomery and Vaughan under Generalized Riemann Hypothesis (GRH).

The second, the work of Burgess and Hildebrand (1962) [12] extended the range of \( x \). It was shown that for all primitive real quadratic characters \( \chi \mod p \),
\[ |\sum_{1 \leq i \leq x} \chi(x)| = o(x) \]
provided \( x > p^{1/4+\epsilon} \). This implies that for every \( \epsilon > 0 \) there is a constant \( C \) such that
\[ q_p \leq C p^{1/4+\epsilon}. \]
Note that the index \( \frac{1}{4} \) can be replaced by \( \frac{1}{4} \sqrt{\log p} \) by a sieving argument for quadratic character.

It is also believed that Burgess bound can be improved to
\[ |\sum_{1 \leq i \leq x} \chi(x)| = o(x) \]
provided \( x > p^x \), which was also known to be true under GRH. Actually Linnik showed that under GRH, for any \( \epsilon > 0 \),
\[ q_p = o(p^\epsilon). \]

The best bound assuming GRH was given by Ankeny [1]:
\[ q_p = O(\log^2 p). \]

In the same paper the author claimed the same bound for \( d \)-th power non-residue and the least prime \( q \) which is quadratic residue mod \( p \).

On the other hand, Chowla and Turan showed that the bound \( q_p = O(\log^2 p) \) is not far from the best possible, namely, there is a positive constant \( C \) such that for infinitely many primes \( p \) one has
\[ q_p > C \log^2 p. \]

Few explicit bounds were known. Since an upper bound for primitive roots always works for quadratic non-residues, Grosswald’s work [7] shows that if \( p > e^{e^{24}} \approx 10^{10^{30}} \), then \( q_p < p^{0.499} \). And the work of Cohen, Oliveira e Silva and Trudgian [3] shows that if \( p > 4 \cdot 10^{71} \), then \( q_p < p^{1/2} - 2 \). A unconditional bound was given by Cohen and Trudgian [4] that for any \( p \),
\[ q_p < p^{0.96}. \]

In computational number theory and theoretical computer sciences, it is a basic problem to deterministically generate a quadratic non-residue efficiently. For instance, generating a quadratic non-residue plays a crucial role in computing a square root of an element mod a prime number.

The above \( \log^2 p \) type bound immediately gives a polynomial deterministic algorithm to return a quadratic non-residue under GRH. And thus it is an interesting problem to find a relatively small set containing desired elements. In fact, the deterministic output of a small subset containing special elements such as primitive elements and quadratic non-residues was widely studied.
In this paper, thanks to the Weil’s bound on function fields, we establish several constructions in finite fields by using tools from algebraic function fields. We construct a series of explicit small sets containing a quadratic non-residue, or generally a non-$d$-th power.

Let us start from a problem raised by Coulter and Kosick [6]:

**Problem 1.1** (Coulter and Kosick). Let $A$ be a subset of $F_q^*$ of order $O(q)$. What is the minimum order of a subset $S$ of $F_q^*$ so that for all $a \in A$, the set $\{a-s, s \in S\}$ contains both a square and non-square element of $F_q$?

The authors constructed such subsets $S$ of cardinality roughly equal to $\sqrt{q}$, by using methods from algebraic combinatorics. Precisely they proved the following result.

**Theorem 1.2** (Coulter and Kosick). Assume $q \geq 7$ and let $A = \alpha \in F_{q^2} - F_q$. Then for any $\alpha \in A$, the set $S = \{\alpha - a^2, a \in F_q^*\}$ contains both a square and a non-square in $F_{q^2}$.

Since finding a square is a trivial task, we may ask the following essentially equivalent question:

**Problem 1.3.** Let $A$ be a subset of $F_q^*$ of order $O(q)$. Construct small subsets $S$ of $F_q^*$ so that for all $a \in A$, the set $\{a-s, s \in S\}$ contains a non-square in $F_q$.

When the base field is special and the field extension is large enough, one can expect stronger constructions. For instance, in [8] Heath-Brown and Micheli proved the following result.

**Theorem 1.4** (Heath-Brown and Micheli). Suppose $p$ is an odd prime and $h(x) = x^p - x - a \in F_p[x]$, $F_{p^e}$ is the degree $p$ extension of $F_p$. If $a$ is not a square in $F_p$, then for any root $\alpha$ of $h(x)$, each element in the set $\alpha + F_p$ is a non-square in $F_{p^e}$.

In general, they proved the following theorem, which was used to construct large family of "dynamically irreducible" quadratic polynomials.

**Theorem 1.5** (Heath-Brown and Micheli). Suppose $p$ is an odd prime with $p \equiv 1(\text{mod } 4)$. If $q > p^e \sqrt{2 \log p}$, then there is an element $\alpha$ in $F_q \setminus F_p$ such that all of the elements in the set $\alpha + F_p$ are non-squares in $F_q$.

In this paper we give a class of explicit constructions of such kind of small subsets, in which the cardinality is much smaller than previous constructions for the same extension degree. Some constructions achieve the $\log^c q$ type bound. Precisely, we have the following general result.

**Theorem 1.6.** Let $h > 1$ and $d \mid q^h - 1$. For any $\alpha \in F_{q^h} \setminus F_q$, let $e$ be the (multiplicative) order of $\alpha$. Suppose $h, t$ satisfy the following conditions:

1. $(t, \frac{q^h-1}{e}) = 1$;
2. Each prime factor of $t$ divides $e$;
3. $q^h \equiv 1(\text{mod } 4)$ if $t \equiv 0(\text{mod } 4)$;
4. $th \leq \sqrt{q}$.

Then the set $S = V(\alpha - x^t) = \{\alpha - x^t, x \in F_q\}$ contains a non-$d$-th power in $F_{q^h}$. 
Some conditions above can be simplified in many cases. For instance, if $\omega$ is primitive in $\mathbb{F}_{q^h}$, then condition 1 can be dropped and if $q$ is odd one can always choose $t = 2^k$. To get the best construction, we need to take a prime divisor $r$ of $q - 1$ such that the largest prime power part of $q - 1$ equals $r^s$ for some $s$. In particular we have the following corollaries.

**Corollary 1.7.** Let $h = \lfloor q^{\delta} \rfloor > 1$ and $d | q^h - 1$. Let $r$ be a prime divisor of $q - 1$ such that the largest prime power part of $q - 1$ has the form $r^s$. Then there is a constant $0 < \epsilon < 1$ such that for a ratio at least $q^{-\epsilon t}$ of $\alpha \in \mathbb{F}_{q^h} \setminus \mathbb{F}_{q^t}$, the set $S = \{\alpha - x^t, x \in \mathbb{F}_q\}$ of cardinality $1 + \frac{q - 1}{M(h)}$ contains a non-$d$-th power in $\mathbb{F}_{q^h}$, where $t$ is the largest power of $r$ such that $t < \sqrt{q}/h$ and $M(h)$ is defined as

$$M(h) = \max_{r \mid (q-1)} r^{\min\{\nu_r(q-1), \lfloor \log_q q/2 - \log_q h \rfloor\}}.$$

Here $r$ runs through prime divisors and $\nu_r(x)$ is the $r$-adic order of $x$.

The case $\delta = \frac{1}{4}$ shows the construction is almost optimal, since $q^{3/4} = O(\log q^h)^3$. And for $\delta = \frac{1}{4} - \epsilon$, one can show that there is at least one $t$ such that $|S| = q^{1-\epsilon} = O(\log^2 q^t)^{2+\epsilon'}$, achieving the $O(\log^2 p)$ type bound for the prime field $\mathbb{F}_p$ case under GRH. In particular we have the following corollaries considering special cases.

**Corollary 1.8.** For any $q, 1 \leq h \leq \lfloor \sqrt{q} \rfloor$, $d | q^h - 1$, the set $S = V(\alpha - x) = \{\alpha - x, x \in \mathbb{F}_q\}$ contains a non-$d$-th power in $\mathbb{F}_{q^h}$. In particular, there is a set of cardinality $q$ in $\mathbb{F}_{q^1}$ containing a non-$d$-th power.

For another extreme case, we have

**Corollary 1.9.** Let $h = 2$ and $d | q - 1$. If both $q$ and $\frac{q^2 - 1}{e}$ are odd, then there is a set $S$ of cardinality $1 + \frac{q^2 - 1}{M(2)}$ in $\mathbb{F}_{q^2}$ containing a non-$d$-th power, where

$$M(2) = \max_{r \mid (q-1)} r^{\min\{\nu_r(q-1), \lfloor \log_q q/2 - \log_q 2 \rfloor\}}.$$

Clearly the above construction holds when $\alpha$ is primitive. Let us return to the Coulter and Kosick question. Note that $|A| = O(q^2)$ and the cardinality of $S$ equals $1 + \frac{q^2 - 1}{M(2)}$. Since $|\log_2 (q - 1)| \leq M(2) < \sqrt{q}$, our construction is necessarily better than Coulter and Kosick’s construction, which has size $q - 1$. In particular, when $M(2) \sim \sqrt{q}$, our construction set has cardinality $O(\sqrt{q})$ in $\mathbb{F}_{q^2}$.

Similarly, one may consider the constructions for small sets containing a primitive element by using approaches from [2].

**Theorem 1.10.** Let $\tau(x)$ denote the number of divisors of $x$. Suppose $t$ satisfies the following conditions

1. $(t, \frac{q^n - 1}{e}) = 1$;
2. Each prime factor of $t$ divides $e$;
3. $q^n \equiv 1(\text{mod } 4)$ if $t \equiv 0(\text{mod } 4)$;
4. $nt \leq \sqrt{q}$.

If $\tau(q^n - 1) < \frac{\sqrt{q}}{nt} + 1$, then the set

$$S = V(\alpha - x^t) = \{\alpha - x^t, x \in \mathbb{F}_q\}$$

of cardinality $O(q/t)$ in $\mathbb{F}_{q^n}$ contains a primitive element.
In particular, if \( \tau(q^h - 1) < \sqrt{q^h} + 1 \), then the set
\[
S = V(\alpha - x) = \{ \alpha - x, x \in \mathbb{F}_q \}
\]
of cardinality \( q \) in \( \mathbb{F}_{q^h} \) contains a primitive element.

Note that the above construction performs well if \( \phi(q^h - 1) \) is small.

Our main tool is a bound on incomplete character sums over finite fields.

### 2. Main Results

To prove the main result, the main tool is a key lemma on incomplete character sums over finite fields deduced from the celebrating Weil theorem. In this paper we always let \( \mathbb{F}_q \) be a finite field of cardinality \( q \) and let \( \mathbb{F}_{q^m} \) be an degree \( m \) extension field. Let \( f(x) \) be a nonconstant polynomial defined over the extension field \( \mathbb{F}_{q^m} \).

Let \( \chi \) be a non-trivial multiplicative character defined on \( \mathbb{F}_{q^m} \). Define the following incomplete character sum by
\[
S_d(\chi) = \sum_{a \in \mathbb{F}_q} \chi(f(a)).
\]

Before stating the main lemma, here is a toy example.

**Proposition 2.1.** Suppose \( q \) is odd. Let \( \mathbb{F}_q[\alpha] = \mathbb{F}_q^2 \). Assume \( \alpha \) is not a square in \( \mathbb{F}_q^2 \), \( f(x) = x^2 - \alpha \in \mathbb{F}_q^2[x] \). Let \( \beta \) be a simple root of \( f(x) \). Then
\[
| \sum_{a \in \mathbb{F}_q} \chi(a^2 - \alpha) | \leq 3\sqrt{q}.
\]

**Theorem 2.2.** \(^{13}\) Let \( f(x) \) be a nonconstant polynomial defined over \( \mathbb{F}_{q^m} \). Suppose that the largest squarefree divisor of \( f(x) \) has degree \( D \). If there is a root \( \zeta \) of multiplicity \( t \) of \( f(x) \) such that the character \( \chi^t \) is non-trivial on the image set \( \text{Norm}_{q^m}[\zeta]/\mathbb{F}_{q^m} \mathbb{F}_q[\zeta] \). Here \( \text{Norm}_{q^m}[\zeta]/\mathbb{F}_{q^m} \mathbb{F}_q[\zeta] \) is the norm map from \( \mathbb{F}_{q^m}[\zeta] \) to \( \mathbb{F}_{q^m} \). Then we have the estimate
\[
|S_d(\chi)| \leq (mD - 1)\sqrt{q}.
\]

In particular, if \( \mathbb{F}_{q^m}[\zeta] = \mathbb{F}_{q^m}[\zeta] \), then \( \chi^t \) is non-trivial on \( \mathbb{F}_{q^m} \), thus the above bound always holds.

For the proof and the details, please refer to \(^{13}\) for a nice exploration. It follows from the above lemma and the basic theory of finite fields extensions that

**Proposition 2.3.** Let \( g(x) \) be a polynomial of degree \( D \) defined over \( \mathbb{F}_q \). For an \( \omega \in \mathbb{F}_{q^m} \mathbb{F}_q \), let \( f(x, \omega) = \omega - g(x) \) be defined over \( \mathbb{F}_{q^m} \) and suppose \( f(x, \omega) \) is irreducible over \( \mathbb{F}_{q^m} \). If \( d | q^m - 1 \) and \( mD < \sqrt{q} + 1 \), then the image set
\[
S = \{ \omega - g(a), a \in \mathbb{F}_q \}
\]
contains a non-\( d \)-th power.

**Proof.** Let \( \zeta \) be a simple root of \( f(x) \). Clearly \( \mathbb{F}_q[\zeta] = \mathbb{F}_q[\zeta, \omega] = \mathbb{F}_{q^m}[\zeta] \) and the conclusion follows from Theorem 2.2 since \( \chi \) is nontrivial on \( \mathbb{F}_{q^m} \). \( \square \)

There are still two key steps. The first step is to determine if a polynomial of the above type \( f(x, \omega) \) is irreducible. This problem seems in general nontrivial. Even the existence of such kind of irreducible polynomials is still open. Precisely, Munemasa and Nakamura \(^{12}\) made the following conjecture:
Conjecture 2.4 (Munemasa and Nakamura). Let $q$ be a prime power (or equivalently, a prime), and let $k, l$ be positive integers. Then there exists a monic irreducible polynomial $f(x) \in \mathbb{F}_q[x]$ of degree $l$ such that $f(x) - f(0) \in \mathbb{F}_q[x]$ and $f(0)$ does not belong to any proper subfield of $\mathbb{F}_q$.

Some partial results were obtained. The conjecture clearly holds when $k = 1$ or $l = 1$. It follows from the irreducibility of binomials (see Lemma 2.6) that the conjecture holds for specified $l$’s. For instance, the conjecture holds for $l = 2$ when $p$ is odd.

The second step, the determination of the cardinality of the image of $g(x)$ over $\mathbb{F}_q$, is also hard in general. This problem is usually called the value set problem over finite fields, which aims to determine the cardinality of the image set of a polynomial map and to understand its algebraic or combinatorial structure. It has a wide variety of applications in number theory, algebraic geometry, coding theory and cryptography. For details of this problem, please refer to [9]. Note that if we denote

$$V(f) = \{f(x), x \in \mathbb{F}_q\},$$

then clearly $S = V(f, \omega)$.

Many interesting bounds were established, and a trivial one gives

$$\left\lceil \frac{q}{d} \right\rceil \leq |V(f)| \leq q.$$

When the upper bound is achieved, $f$ is called a permutation polynomial. The theory of permutation polynomials was extensively studied and has many applications in coding theory and cryptography. On the other hand, when the lower bound is achieved, $f$ is called having the minimal value set property.

In this paper we are certainly interested in the minimal value set property. Among them, the class of monomials, or generally, the composition of a permutation polynomial and a monomial, are the simplest but most important classes. We then first state a useful lemma on the irreducibility of the composition of an irreducible polynomial and a monomial.

Lemma 2.5. [10, 11] Suppose $f(x)$ is an irreducible polynomial of degree $n$ over $\mathbb{F}_q$ and a root of $f(x)$ has order $e$. If $t$ satisfies the following conditions, then $f(x^t)$ is also irreducible.

1. $(t, \frac{q^n-1}{e}) = 1$;
2. Each prime factor of $t$ divides $e$;
3. $q^n \equiv 1(\text{mod } 4)$ if $t \equiv 0(\text{mod } 4)$.

By this lemma, it is an interesting problem to classify polynomials are both permutating and irreducible. For simplicity, in this paper we focus on the simplest classes –linear polynomials, which are clearly both permutation polynomials and irreducible polynomials.

Composing the monomials into linear polynomials, it then suffices to consider the irreducible binomials. Fortunately, the classification of irreducible binomials were established and it will lead many good constructions of small desired subsets in finite fields. As a special case of Lemma 2.5, we have

Lemma 2.6. [9, 11] Let $t \geq 2$ be an integer and $\alpha \in \mathbb{F}_q$. Let $e$ be the order of $\alpha$ in $\mathbb{F}_q$. Then the binomial $x^t - a$ is irreducible if and only if the following conditions are satisfied:
1. \( (t, \frac{q^h - 1}{e}) = 1 \);
2. Each prime factor of \( t \) divides \( e \);
3. \( q \equiv 1 \pmod{4} \) if \( t \equiv 0 \pmod{4} \).

**Problem 2.7.** From the viewpoint of elementary number theory, we are interested in the choices of \( t \). We may ask how to compute the density of \( t \) for given \( q, h \) and \( e \) satisfying the above conditions.

Combining Theorem 2.3 and Lemma 2.6 we then have

**Theorem 2.8.** Let \( h > 1 \). For any \( \alpha \in F_{q^h} \setminus F_q \), let \( e \) be the (multiplicative) order of \( \alpha \). Suppose \( \alpha - f(x) \) is an irreducible polynomial of degree \( n \) over \( F_q \). Suppose \( h, t \) satisfy the following conditions:

1. \( (t, \frac{q^h - 1}{e}) = 1 \);
2. Each prime factor of \( t \) divides \( e \);
3. \( q^{nh} \equiv 1 \pmod{4} \) if \( t \equiv 0 \pmod{4} \);
4. \( \sqrt{n} \leq \sqrt{q} \).

Then the set \( S = V(\alpha - f(x)) = \{ \alpha - f(x), x \in F_q \} \) contains a non-\( d \)-th power in \( F_{q^h} \). In particular, choosing \( f(x) = \alpha - x \) we obtain the set \( S = V(\alpha - x^t) = \{ \alpha - x^t, x \in F_q \} \) containing a non-\( d \)-th power in \( F_{q^h} \).

Since one can always take \( t = 1 \), we have the following result.

**Corollary 2.9.** For any \( q, 1 \leq h \leq \sqrt{q} \), \( d \mid q^h - 1 \), the set \( S = V(\alpha - x) = \{ \alpha - x, x \in F_q \} \) contains a non-\( d \)-th power in \( F_{q^h} \). In particular, there is a set of cardinality \( q \) in \( F_{q^h} \) containing a non-\( d \)-th power.

We now give more constructions based on a rough classification of these parameters. Please note that they are not complete and the interested readers may give their own constructions.

**Case 1:** Assume both \( q \) and \( \frac{q^h - 1}{e} \) are odd. Since \( h > 1 \), in this case \( q^h - 1 \) is always divisible by \( 4 \), and thus one can always choose \( t = 2^k \), where \( 1 \leq k \leq \lceil \log_2 \sqrt{q} \rceil \) and one checks that \( t \) satisfies the first three conditions above. Thus we obtain our main constructions below:

**Corollary 2.10.** Let \( h > 1 \) and \( d \mid q^h - 1 \). If both \( q \) and \( \frac{q^h - 1}{e} \) are odd, then for any \( t = 2^k \), any integer \( h \) satisfying \( h \leq \sqrt{q}/2^k \), the set \( S = V(\alpha - x^{2^k}) = \{ \alpha - x^{2^k}, x \in F_q \} \) contains a non-\( d \)-th power in \( F_{q^h} \). Let \( h = \lfloor q^h \rfloor \), then there is a set of cardinality \( O(q^{\frac{1}{2} + \delta_1}) \) in \( F_{q^{\lfloor q^h \rfloor}} \) containing a non-\( d \)-th power. Here \( 0 < \delta_1 < 1 \) is determined by \( (t, q - 1) \).

For interested readers, we list one more specified subcase. Choosing \( h = 2 \) and \( t = 2^{\lceil \log_2 \sqrt{q} \rceil - 1} \approx \sqrt{q}/2 \), we have

**Corollary 2.11.** Let \( h = 2 \) and \( d \mid q^2 - 1 \). If both \( q \) and \( \frac{q^2 - 1}{e} \) are odd, then there is a set of cardinality \( O(\frac{1}{M}) \) in \( F_{q^2} \) containing a non-square, where

\[
M = \max_{p \mid q - 1} \max_{1 \leq t \leq \lceil \log_2 q \rceil/2} \gcd(p^t, q - 1).
\]

If we denote \( \nu_p(x) \) to be the \( p \)-adic order of \( x \), then \( M = \max_{p \mid q - 1} p^\min\{\nu_p(q - 1), \lceil \log_2 q \rceil/2\} \).

Thus the construction is pretty good if there is a prime divisor \( p \) of \( q - 1 \) such that \( \nu_p(q - 1) \) is large. It would be very interesting to construct smaller subsets containing a non square in \( F_{q^2} \).
Note that the above constructions hold when $\alpha$ is primitive.

**Case 2.** Assume $q$ is even and $3 \mid q^h - 1$. $(3, q^h - 1) = 1$. In this case we can always choose $t = 3^k$, where $1 \leq k \leq \lfloor \log_3 \sqrt{q} \rfloor$ and one checks that $t$ satisfies the first three conditions above. Thus we obtain another construction below:

**Corollary 2.12.** Let $q$ be even, $h > 1$ and $d \mid q^h - 1$. If $(3, q^h - 1) = 1$ and $3 \mid q - 1$, then for any $t = 3^k$, any integer $h$ satisfying $h \leq \sqrt{q}/3^k$, the set $S = V(\alpha - x^{3^k}) = \{\alpha - x^{3^k}, x \in \mathbb{F}_q\}$ contains a non-$d$-th power in $\mathbb{F}_{q^h}$. Let $h = \lfloor q^\delta \rfloor$, then there is a set of cardinality $O(q^{\frac{2}{3} + \delta})$ in $\mathbb{F}_{q^{\delta}}$ containing a non-$d$-th power.

**Remark 1.** One can also take another prime divisor of $q - 1$ other than 3. In fact, the construction works better if $q - 1$ has a small prime divisor $r$ and the $r$-adic order $v_r(q - 1)$ is large. Since in $\mathbb{F}_{q^h}$ we have at least $\phi(q^h - 1) = (q^h)^{1 - \epsilon}$ primitive elements, we obtain the following general result.

**Corollary 2.13.** Let $h = \lfloor q^\delta \rfloor > 1$ and $d \mid q^h - 1$. Let $r$ be a prime divisor of $q - 1$ such that the largest prime power part of $q - 1$ has the form $r^s$. Then there is a constant $0 < \epsilon < 1$ such that for a ratio at least $q^{-\epsilon}$ of $\alpha \in \mathbb{F}_{q^h} \setminus \mathbb{F}_q$, the set $S = \{\alpha - x^{r^s}, x \in \mathbb{F}_q\}$ of cardinality $1 + \frac{1}{\phi(q^h)}$ contains a non-$d$-th power in $\mathbb{F}_{q^{\delta}}$, where $t$ is the largest power of $r$ such that $t < \sqrt{q}/h$ and $M(h)$ is defined as

$$M(h) = \max_{(r, (q - 1)) = 1} r^{\min\{v_r(q - 1), |\log_r q - 2\log_r h|\}}.$$ 

Here $r$ runs through prime divisors and $v_r(x)$ is the $r$-adic order of $x$.

**Remark 2.** Based on the constructions, we are particularly interested in the classification of polynomials other than linear polynomials and binomials which are both permutational and irreducible. One may also consider the general interested irreducible polynomials, whose corresponding value set problem has nice estimate.

3. **ON SMALL SUBSETS CONTAINING A PRIMITIVE ELEMENT**

**Definition 3.1.** Let $r > 1$ be a positive integer and suppose $r \mid q - 1$. We call $\alpha \in \mathbb{F}_{q^r}$ an $r$-free element if $\gcd(r, (q - 1)/\phi(\alpha)) = 1$.

**Lemma 3.2 ([2]).** Let $\alpha \in \mathbb{F}_{q^r}$, and $r > 1$ be a positive integer such that $r \mid q - 1$. Then

$$\sum_{d \mid r} \frac{\mu(d)}{\phi(d)} \sum_{\text{ord}(\chi) = d} \chi(\alpha) = \left\{ \begin{array}{ll}
\frac{r^m}{\phi(r)}, & \text{if } \chi \text{ is } r\text{-free}, \\
0, & \text{otherwise}.
\end{array} \right.$$ 

where $\phi$ is the Euler totient function, $\mu$ is the Möbius function and $\text{ord}(\chi)$ is the order of the multiplicative character $\chi$ of $\mathbb{F}_q$.

Let $\alpha \in \mathbb{F}_{q^n}$ and $d$ be a positive integer. Define

$$P(d, \alpha) = \frac{\mu(d)}{\phi(d)} \sum_{\chi_d} \chi_d(\alpha), \quad P(\alpha) = \frac{\phi(q^n - 1)}{q^n - 1} \sum_{d \mid q^n - 1} P(d, \alpha),$$

where $\sum_{\chi_d}$ ranges over all multiplicative characters of $\mathbb{F}_{q^n}$ of order $d$. Thus (2.1) implies

$$P(\alpha) = \left\{ \begin{array}{ll} 
1, & \text{if } \alpha \text{ is a primitive element,} \\
0, & \text{otherwise.}
\end{array} \right.$$
Theorem 3.3. Let $\tau(x)$ denote the number of divisors of $x$. If $\tau(q^n - 1) < \frac{\sqrt{q}}{n-1} + 1$, then the set

$$S = V(\alpha - x) = \{\alpha - x, x \in \mathbb{F}_q\}$$

of cardinality $q$ in $\mathbb{F}_{q^n}$ contains a primitive element.

Proof. Let $N$ be the number of primitive elements in $\alpha + \mathbb{F}_q$. Then

$$N = \sum_{a \in \mathbb{F}_q} \frac{\phi(q^n - 1)}{q^n - 1} \sum_{d|q^n - 1} P(d, \alpha - a)$$

$$= \sum_{a \in \mathbb{F}_q} \frac{\phi(q^n - 1)}{q^n - 1} \sum_{d|q^n - 1} \frac{\mu(d)}{\phi(d)} \sum_{ord(\chi) = d} \chi(\alpha - a)$$

$$= \frac{\phi(q^n - 1)}{q^n - 1} \left( q - \sum_{d|q^n - 1, d > 1} \frac{1}{\phi(d)} \sum_{ord(\chi) = d} (n - 1)\sqrt{q} \right)$$

$$= \frac{\phi(q^n - 1)}{q^n - 1} \left( q - \sum_{d|q^n - 1, d > 1} (n - 1)\sqrt{q} \right)$$

$$= \frac{\phi(q^n - 1)}{q^n - 1} \left( q - (\tau(q^n - 1) - 1)(n - 1)\sqrt{q} \right)$$

Similarly by applying Theorem 2.2 we have

Theorem 3.4. Let $\tau(x)$ denote the number of divisors of $x$. Suppose $t$ satisfies the following conditions

1. $(t, \frac{q^n - 1}{q^n - 1}) = 1$;
2. Each prime factor of $t$ divides $e$;
3. $q^n \equiv 1 \pmod{4}$ if $t \equiv 0 \pmod{4}$;
4. $nt \leq \sqrt{q}$.

If $\tau(q^n - 1) < \frac{\sqrt{q}}{nt - 1} + 1$, then the set

$$S = V(\alpha - x^t) = \{\alpha - x^t, x \in \mathbb{F}_q\}$$

of cardinality $O(\frac{q}{\gcd(t, q^n - 1)})$ in $\mathbb{F}_{q^n}$ contains a primitive element.

Acknowledgements. We thank Professor Daqing Wan for his lectures on incomplete character sum and Dr. Robert Coulter for his help comments.
References

[1] N.C. Ankeny, The least quadratic non residue, Ann. of Math. (2) 55, (1952). 65-72.
[2] S.D. Cohen, Primitive roots in the quadratic extension of a finite field, J. London Math Soc., (1983)27(2): 221-228.
[3] S.D. Cohen, T. Oliveira e Silva, T.S. Trudgian, On Grosswald’s conjecture on primitive roots, Acta Arith. 172 (2016), 263-270.
[4] S.D. Cohen and T. Trudgian, On the least square-free primitive root modulo p, J. Number Theory 170 (2017), 10-16.
[5] T.H. Cormen, C.E. Leiserson, R.L. Rivest and C. Stein, Introduction to Algorithms, MIT Press and McGraw-Hill, 2001.
[6] R.S. Coulter and P. Kosic, On expressing elements as a sum of squares, where one square is restricted to a subfield, Finite Fields & Applications, 26 (2004), 116–122.
[7] E. Grosswald, On Burgess bound for primitive roots modulo primes and an application to $\Gamma(p)$, Amer. J. Math., 103(6):1171-1183, 1981.
[8] D.R. Heath-Brown and G. Michel, Irreducible polynomials over finite fields produced by composition of quadratics, Preprint, 2016 (available from http://arxiv.org/abs/1701.05031).
[9] R. Lidl and H. Niederreiter, Finite Fields, volume 20 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, second edition, 1997.
[10] A. Menezes, I. Blake, X.-H. Gao, R. Mullin, S. Vanstone, and T. Yaghoobian, Applications of Finite Fields, The Springer International Series in Engineering and Computer Science, Vol. 199, Springer, 1993.
[11] G.L. Mullen and D. Panario, Handbook of finite fields, Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, 2013.
[12] A. Munemasa and H. Nakamura, A Note on the Brawley-Carlitz Theorem on Irreducibility of Composed Products of Polynomials over Finite Fields, International Workshop on the Arithmetic of Finite Fields, WAIFI 2016: Arithmetic of Finite Fields pp 84-92.
[13] D. Wan, Generators and irreducible polynomials over finite fields, Math. Comp. 66 (1997) 1195-1212.