A GAUGE FIXING PROCEDURE FOR CAUSAL FERMION SYSTEMS

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Abstract. Causal fermion systems incorporate local gauge symmetry in the sense that the Lagrangian and all inherent structures are invariant under local phase transformations of the physical wave functions. In the present paper it is explained and worked out in detail that, despite this local gauge freedom, the structures of a causal fermion system give rise to distinguished gauges where the local gauge freedom is fixed completely up to global gauge transformations. The main method is to use spectral and polar decompositions of operators on Hilbert spaces and on indefinite inner product spaces. We also introduce and make use of a Riemannian metric which is induced on the manifold of all regular correlation operators by the Hilbert-Schmidt scalar product. Gaussian coordinate systems corresponding to this Riemannian metric are constructed. Moreover, we work with so-called wave charts where the physical wave functions are used as coordinates. Our constructions and results are illustrated in the example of Dirac sea configurations in finite and infinite spatial volume.

Contents

1. Introduction 2
2. Preliminaries on Causal Fermion Systems 2
   2.1. Basic Definition 2
   2.2. A Few Inherent Structures 3
   2.3. Restriction to Regular Causal Fermion Systems 4
   2.4. Local Gauge Invariance and Gauge Transformations 5
3. A Smooth Manifold Structure of $F$ 6
4. A Riemannian Metric on $F$ 7
5. Gaussian Charts 7
6. Gauges and Gauge Fixing 8
   6.1. Symmetric Wave Charts 8
   6.2. Gaussian Wave Charts 13
7. Example: Dirac Systems 15
   7.1. Dirac Systems in Finite Spatial Volume 15
   7.2. The Kernel of the Fermionic Projector in Finite Volume 17
   7.3. Connection to the Kernels in Infinite Volume 18
   7.4. Gauge Fixing of Wave Functions in Space-Time 19
   7.5. Gauge Fixing of the Perturbation Expansion 22
References 25
1. Introduction

The local gauge freedom of electrodynamics is based on the observation that transforming the electromagnetic potential $A$ in Minkowski space by the derivative of a real-valued function $\Lambda$,

$$A_j(x) \rightarrow A_j(x) + \partial_j \Lambda(x),$$  \hspace{1cm} (1.1)

does not change the electromagnetic field tensor and thus has no effect on any observable quantities (see for example [21, Section I.2]). In quantum mechanics, the gauge transformation (1.1) must be complemented by a local phase transformation of the wave function $\psi$ (see [20, Section XV.111], [23, Section 4.1] or [24, Section 2.6]),

$$\psi(x) \rightarrow e^{i\Lambda(x)} \psi(x).$$  \hspace{1cm} (1.2)

The connection between these two transformation laws can be understood most easily if the electromagnetic potential is combined with the partial derivatives to gauge-covariant derivatives $D_j$ by

$$D_j := \partial_j - iA_j,$$

because then (1.1) and (1.2) give rise to the simple transformation law

$$D_j \psi(x) \rightarrow e^{i\Lambda(x)} D_j \psi(x).$$

The local gauge principle states that local gauge transformations lead to equivalent formulations of the physical system. In its generalization to non-Abelian gauge theories, the local gauge principle is one of the cornerstones of modern physics.

In most applications and calculations, the local gauge freedom is inconvenient because of the resulting non-uniqueness of the gauge potential and the gauge phases. Therefore, it is often desirable to fix the gauge, for example by choosing the Lorenz, Coulomb or radiation gauges. The general strategy of a gauge-fixing procedure is to use the local gauge freedom in order to arrange that the gauge potential has a particularly simple or convenient form. Many gauge-fixing procedures do not fix the gauge completely, but only partially up to a remaining residual gauge freedom. In particular, the residual gauge freedom typically includes the global gauge transformations (i.e. transformations of the form (1.1) and (1.2) with $\Lambda$ a constant).

Causal fermion systems are a recent approach to fundamental physics (see the basics in Section 2.1, the reviews [15, 11, 13] or the textbook [8]). This theory incorporates a local gauge principle (for details see Section 2.4). This raises the question if and to what extent the structures of a causal fermion systems make it possible to fix the gauge. In preparation for attacking this question, in [10, Section 6] it was noted that in the setting of causal fermion systems, gauge freedom corresponds to the freedom in choosing charts on $\mathcal{F}$. Based on this observation, in [11, Section 5.3] a gauge-fixing procedure was proposed for causal fermion systems. In the present paper we work out this procedure in detail and clarify how it is related to the gauges and gauge fixing in electrodynamics.

2. Preliminaries on Causal Fermion Systems

2.1. Basic Definition. We begin with the abstract definition (for more details see for example [8, Section 1.1]).

Definition 2.1. (causal fermion system) Given a separable complex Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot, \cdot \rangle_\mathcal{H}$ and a parameter $n \in \mathbb{N}$ (the “spin dimension”), we let $\mathcal{F} \subset \text{L}(\mathcal{H})$ be the set of all self-adjoint operators on $\mathcal{H}$ of finite rank, which (counting
multiplicities) have at most \( n \) positive and at most \( n \) negative eigenvalues. On \( \mathcal{F} \) we are given a positive measure \( \rho \) (defined on a \( \sigma \)-algebra of subsets of \( \mathcal{F} \)), the so-called universal measure. We refer to \( (\mathcal{H}, \mathcal{F}, \rho) \) as a causal fermion system.

The physical equations are formulated via a variational principle, the causal action principle. This variational principle will not be needed in this paper. Therefore, we do not introduce it here, but refer instead again to [8, Section 1.1]. But we need to recall a few additional structures of a causal fermion system, which are inherent in the sense that they only use information already encoded in the causal fermion system.

2.2. A Few Inherent Structures. We define space-time \( M \) as the support of the universal measure,

\[
M := \text{supp } \rho \subset \mathcal{F}
\]

(where the support is defined as the complement of the largest open set of measure zero). For every \( x \in \mathcal{F} \) we define the spin space

\[
S_x := x(\mathcal{H})
\]

it is a subspace of \( \mathcal{H} \) of dimension at most \( 2n \). On \( S_x \) we choose the inner product

\[
\langle \cdot, \cdot \rangle_x : S_x \times S_x \to \mathbb{C}, \quad \langle u | v \rangle_x = -\langle u | xv \rangle_{\mathcal{H}},
\]

(2.2)

referred to as the spin scalar product. It is an indefinite inner product of signature \((p, q)\) with \( p, q \leq n \).

A wave function \( \psi \) is defined as a function which to every space-time point \( x \in M \) associates a vector of the corresponding spin space,

\[
\psi : M \to \mathcal{H} \quad \text{with} \quad \psi(x) \in S_x \quad \text{for all} \quad x \in M.
\]

Every vector \( u \in \mathcal{H} \) gives rise to a corresponding wave function, referred to as the physical wave function \( \psi^u \). It is defined by

\[
\psi^u(x) = \pi_x u \in S_x,
\]

where \( \pi_x : \mathcal{H} \to S_x \) is the orthogonal projection in \( \mathcal{H} \) to the subspace \( S_x \subset \mathcal{H} \). Finally, it is convenient to combine all the physical wave functions to an operator, the so-called wave evaluation operator \( \Psi \) defined for any \( x \in \mathcal{F} \) by

\[
\Psi(x) : \mathcal{H} \to S_x, \quad u \mapsto \pi_x u.
\]

(2.3)

Then clearly, for every space-time point \( x \in M \) and every \( u \in \mathcal{H} \),

\[
\Psi(x) u = \psi^u(x).
\]

In what follows, we shall often take adjoints of the above operators. When doing so, one must be careful to work with the correct corresponding inner products. In order to avoid confusion, we now explain in detail how this works. The adjoint of \( \Psi(x) \) is defined formally by

\[
\Psi(x)^* : S_x \to \mathcal{H}, \quad \langle \phi | \Psi(x) u \rangle_x = \langle \Psi(x)^* \phi | u \rangle_{\mathcal{H}} \quad \text{for all} \quad u \in \mathcal{H} \text{ and } \phi \in S_x.
\]

Note that on the left side of this equation the spin scalar product appears. Using its definition \((2.2)\), we obtain the relation

\[
-\langle \phi | X \Psi(x) u \rangle_{\mathcal{H}} = \langle \Psi(x)^* \phi | u \rangle_{\mathcal{H}},
\]

(2.4)

where we introduced the short notation

\[
X := x|_{S_x} : S_x \to S_x.
\]

(2.5)
Now we can take adjoints purely with respect to the Hilbert space scalar product. Denoting those adjoints for clarity by a dagger, we can rewrite (2.4) as
\[ -\langle \Psi(x)^{\dagger} X \phi | u \rangle_{\mathcal{H}} = \langle \Psi(x)^* \phi | u \rangle_{\mathcal{H}}, \] (2.6)
implying that
\[ \Psi(x)^* = -\Psi(x)^{\dagger} X. \] (2.7)
Adjoint operators can be computed similarly. For a linear operator \( A \in L(S_x) \), for example, the adjoint is defined by
\[ \langle \phi | A \tilde{\phi} \rangle_x = \langle A^* \phi | \tilde{\phi} \rangle_x \quad \text{for all} \ \phi, \tilde{\phi} \in S_x. \]
Using again the definition of the spin scalar product (2.2), we can rewrite this equation as
\[ -\langle \phi | X A \tilde{\phi} \rangle_{\mathcal{H}} = -\langle A^* \phi | X \tilde{\phi} \rangle_{\mathcal{H}}, \]
and taking adjoints in the Hilbert space \( \mathcal{H} \) gives
\[ -\langle X^{-1} A^{\dagger} X \phi | X \tilde{\phi} \rangle_{\mathcal{H}} = -\langle A^* \phi | X \tilde{\phi} \rangle_{\mathcal{H}} \]
(note that the operator \( X \) is invertible because \( S_x \) is by definition its image (2.1)). We thus obtain
\[ A^* = X^{-1} A^{\dagger} X. \] (2.8)

We now derive an identity which will be important later on (for an alternative derivation see [8, Lemma 1.1.3]).

**Lemma 2.2.** For all \( x \in \mathcal{F} \),
\[ x = -\Psi(x)^* \Psi(x) \] (2.9)
\[ = -\Psi(x)^{\dagger} X \Psi(x). \] (2.10)

**Proof.** Combining (2.7) and (2.8), we obtain
\[ \Psi(x)^* \Psi(x) = -\Psi(x)^{\dagger} X \Psi(x) = -\pi_x^{\dagger} X \pi_x = -\pi_x X \pi_x, \]
where in the last step we used that orthogonal projections are symmetric operators on \( \mathcal{H} \). Using (2.5) gives (2.9). Rewriting this relation with the help of (2.7) gives (2.10). \( \square \)

2.3. **Restriction to Regular Causal Fermion Systems.** In the definition of causal fermion systems, the number of positive or negative eigenvalues of the operators in \( \mathcal{F} \) can be strictly smaller than \( n \). This is important because it makes \( \mathcal{F} \) a closed subspace of \( L(\mathcal{H}) \) (with respect to the sup-norm topology), which in turn is crucial for the general existence results for minimizers of the causal action principle (see [7] or [16]). However, in all physical examples in Minkowski space or in a Lorentzian space-time, all the operators in \( M \) do have exactly \( n \) positive and exactly \( n \) negative eigenvalues. This motivates the following definition (see also [8, Definition 1.1.5]).

**Definition 2.3.** An operator \( x \in \mathcal{F} \) is said to be **regular** if it has the maximal possible rank, i.e. \( \dim x(\mathcal{H}) = 2n \). Otherwise, the operator is called **singular**. A causal fermion system is **regular** if all its space-time points are regular.
In what follows, we restrict attention to regular causal fermion systems. Moreover, it is convenient to also restrict attention to all those operators in $\mathcal{F}$ which are regular,

$$
\mathcal{F}^{\text{reg}} := \{ x \in \mathcal{F} \mid x \text{ is regular} \} .
$$

(2.11)

$\mathcal{F}^{\text{reg}}$ is a dense open subset of $\mathcal{F}$ (again with respect to the sup-norm topology on $L(\mathcal{F}))$.

For notational convenience, in omit the superscript “reg” from now on. Thus by $\mathcal{F}$ we always mean $\mathcal{F}^{\text{reg}}$.

(2.12)

2.4. Local Gauge Invariance and Gauge Transformations. The setting of causal fermion systems is gauge invariant in the following sense (see also [8, Section 1.3]): In order to represent the wave functions in components, one must work with basis representations of the spin spaces. To this end, we choose a pseudo-orthonormal basis $(\epsilon_\alpha(x))_{\alpha=1,\ldots,2n}$ of every spin space $(S_x, \langle \cdot | \cdot \rangle_x)$, i.e.

$$
\langle \epsilon_\alpha(x) | \epsilon_\beta(x) \rangle_x = s_\alpha \delta^\alpha_\beta
$$

with $s_1 = \ldots = s_n = 1$ and $s_{n+1} = \ldots = s_{2n} = -1$. Then a wave function $\psi$ can be represented as

$$
\psi(x) = \sum_{\alpha=1}^{2n} \psi^{\alpha}(x) \epsilon_\alpha(x)
$$

(2.13)

with component functions $\psi^1, \ldots, \psi^{2n}$. The freedom in choosing the basis $(\epsilon_\alpha)$ is described by the group $U(n,n)$ of unitary transformations with respect to an inner product of signature $(n,n)$. This gives rise to the transformations

$$
\epsilon_\alpha(x) \rightarrow \sum_{\beta=1}^{2n} U^{-1} \epsilon_\beta(x) \quad \text{and} \quad \psi^{\alpha}(x) \rightarrow \sum_{\beta=1}^{2n} U(\epsilon_\beta(x) \psi^{\beta}(x)
$$

(2.14)

with $U \in U(n,n)$. As the basis $(\epsilon_\alpha)$ can be chosen independently at each space-time point, one obtains local gauge transformations of the wave functions, where the gauge group is determined to be the isometry group of the spin scalar product. The causal action is gauge invariant in the sense that it does not depend on the choice of spinor bases. We finally remark that the concept of identifying local gauge freedom with the freedom of choosing pseudo-orthonormal bases of the spinor spaces goes back to [2].

3. A Smooth Manifold Structure of $\mathcal{F}$

We now assume for technical simplicity that the Hilbert space $\mathcal{H}$ is finite dimensional,

$$
\dim \mathcal{H} =: f < \infty .
$$

For the sake of larger generality, instead of $\mathcal{F}$ we consider operators with different numbers of positive and negative eigenvalues. These operators are of importance in view of topological and Riemannian fermion systems as introduced and analyzed in [14].

Definition 3.1. We let $\mathcal{F}^{p,q}$ be the set of all symmetric linear operators on $\mathcal{H}$ of rank $p+q$, which (counting multiplicities) have $p$ positive and $q$ negative eigenvalues.

Clearly, setting $p = q = n$, we obtain the set $\mathcal{F}$ (or, more precisely, the set $\mathcal{F}^{\text{reg}}$; see (2.11) and (2.12)).
Theorem 3.2. The set $\mathcal{F}^{p,q}$ is a smooth manifold of dimension
$$\dim \mathcal{F}^{p,q} = 2f(p + q) - (p + q)^2.$$ 

Proof. Let $x \in \mathcal{F}^{p,q}$. We denote its image by $I \subset \mathcal{H}$ and set $J = I^\perp$ (where the orthogonal complement is taken with respect to the scalar product on $\mathcal{H}$). Using a block matrix representation in $\mathcal{H} = I \oplus J$, the operator $x$ has the representation
$$x = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}.$$ 

We now let $A$ be symmetric linear operator on $I$. By choosing its norm sufficiently small, we can arrange that the operator $X + A$ has again $p$ positive and $q$ negative eigenvalues. In particular, this operator is invertible. Next, we choose a linear operator $B : J \to I$. We form the operator
$$M := \begin{pmatrix} \mathbf{1} & 0 \\ (B^\dagger (X + A)^{-1})^\dagger & \mathbf{1} \end{pmatrix} \begin{pmatrix} X + A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1} & (X + A)^{-1}B \\ 0 & \mathbf{1} \end{pmatrix}$$ 
(3.1)

$$= \begin{pmatrix} X + A & B \\ B^\dagger & B^\dagger (X + A)^{-1}B \end{pmatrix}$$ 
(3.2)

(where for clarity the dagger again denotes the adjoint with respect to the scalar product induced from $\langle \cdot, \cdot \rangle_\mathcal{H}$; see (2.6)). This operator is symmetric and has again $p$ positive and $q$ negative eigenvalues. Thus for sufficiently small $\varepsilon$ we obtain the mapping
$$\Lambda : (\text{Symm}(I) \oplus \text{L}(I, J)) \cap B_\varepsilon(0) \to \mathcal{F}^{p,q}, \quad (A, B) \mapsto M$$

(3.2)

(where Symm($I$) denotes the linear operators on $I$ which are symmetric with respect to the induced scalar product $\langle \cdot, \cdot \rangle_\mathcal{H}|_{I \times I}$). Let us verify that (again for sufficiently small $\varepsilon$) this mapping is a homeomorphism to an open neighborhood of $x \in \mathcal{F}^{p,q}$. It is obvious from (3.1) that $\Lambda$ is injective. In order to verify that it maps to an open neighborhood of $x$, we let $y \in F^{p,q}$ with $\|x - y\| < \delta$ (with $\delta > 0$ to be specified below). Diagonalizing $y$ with a unitary operator $U$, we obtain the block matrix representation
$$y = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} X + C & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_{11}^\dagger & U_{21}^\dagger \\ U_{12}^\dagger & U_{22}^\dagger \end{pmatrix},$$

where $C$ is a symmetric linear operator on $I$. In the limit $y \to x$, the image of $y$ converges to the image of $x$, implying that the operator $U_{11}$ becomes unitary. Therefore, for sufficiently small $\delta > 0$, the operator $U_{11}$ is invertible, giving rise to the representation
$$y = \begin{pmatrix} \mathbf{1} & 0 \\ U_{21} U_{11}^{-1} & \mathbf{1} \end{pmatrix} \begin{pmatrix} U_{11} (X + C) U_{11}^\dagger & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1} & (U_{11})^{-1} U_{21}^\dagger \\ 0 & \mathbf{1} \end{pmatrix}.$$ 

This is indeed of the form (3.1), and one can even read off $A$ and $B$,
$$A = U_{11} (X + C) U_{11}^\dagger - X$$
$$B = (U_{11} (X + C) U_{11}^\dagger) (U_{11})^{-1} U_{21}^\dagger.$$ 

We conclude that $\Lambda$ is a bijection to an open neighborhood of $x \in \mathcal{F}^{p,q}$. The continuity of $\Lambda$ and of its inverse are obvious. We have thus constructed a chart on $\mathcal{F}^{p,q}$ around $x$.

Performing the above construction around every point of $\mathcal{F}^{p,q}$ gives an atlas. By direct computation one verifies that the transition maps are smooth. We conclude that, with this atlas, $\mathcal{F}^{p,q}$ is indeed a smooth manifold.
We finally determine the dimension of $\mathcal{F}^{p,q}$. The linear operator $B$ is represented by a $(p+q) \times (f-p-q)$-matrix, giving rise to $2(p+q)(f-p-q)$ real degrees of freedom. The symmetric linear operator $A$, on the other hand, is represented by a Hermitian $(p+q) \times (p+q)$-matrix, described by $(p+q)^2$ real parameters. Adding these dimensions concludes the proof. □

From now on, we always restrict attention to the case $p=q=n$ of causal fermion systems.

4. A Riemannian Metric on $\mathcal{F}$

We finally introduce another inherent structure which has not been used so far and which seems useful in the context of gauge fixing: a Riemannian metric on $\mathcal{F}$. As in the previous section, we assume for technical simplicity that $\mathcal{H}$ is finite-dimensional. Then on $\mathcal{F}$ the Hilbert-Schmidt norm gives rise to a distance function

$$d : \mathcal{F} \times \mathcal{F} \to \mathbb{R}^+_0, \quad d(x,y) = \|x - y\|_{\text{HS}} := \sqrt{\text{tr}((x - y)^2)}$$

(4.1)

(note that the existence of the trace would not be an issue even in the infinite-dimensional setting because all operators in $\mathcal{F}$ have finite rank). The square of this distance function is smooth. Moreover, its first derivative vanishes on the diagonal, i.e. $D(d(x,.)^2)|_x = 0$. Therefore, taking its quadratic Taylor expansion about a point $x \in M$ gives a scalar product on $T_x\mathcal{F}$, i.e.

$$h_x : T_x\mathcal{F} \times T_x\mathcal{F} \to \mathbb{R}, \quad h_x(u,v) = \text{tr}(uv).$$

(4.2)

Clearly, this mapping depends smoothly on $x$ and thus defines a Riemannian metric on $\mathcal{F}$.

5. Gaussian Charts

Specializing to the case $p=q=n$, in the proof of Theorem 3.2 we constructed a local parametrization of $\mathcal{F}$ given by

$$\Lambda : (\text{Symm}(I) \oplus L(I,J)) \cap B_\epsilon(0) \to \mathcal{F},$$

$$(A,B) \mapsto M = \begin{pmatrix} X + A & B \\ B^\dagger & B^\dagger(X + A)^{-1}B \end{pmatrix}.$$  

(5.1)

The image of this mapping is an open neighborhood of $x \in \mathcal{F}$ which we denote by $U$. Then the inverse of $\Lambda$ defines a chart

$$\phi_x := \Lambda^{-1} : U \subset \mathcal{F} \to \text{Symm}(I) \oplus L(I,J).$$

(5.2)

**Theorem 5.1.** The chart $(\phi_x, U)$ in (5.2) is a Gaussian coordinate system about the point $x \in U$ with respect to the Riemannian metric $h$ on $\mathcal{F}$ (see (4.2)).

**Proof.** In the chart $\phi_x$, we describe points of $\mathcal{F}$ by pairs

$$(A,B) \in (\text{Symm}(I) \oplus L(I,J)) \cap B_\epsilon(0).$$

Expanding the mapping $\Lambda$ in a Taylor series about the origin, there is a nonlinearity only in the lower right block matrix entry,

$$\Lambda(tA,tB) = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} + t \begin{pmatrix} A & B \\ B^\dagger & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}(t^2).$$
Hence the distance function \((4.1)\) has the expansion
\[
d((tA,tB), (\hat{t}A, \hat{t}B))^2 = \text{tr} \left\{ [\Lambda(tA,tB) - \Lambda(\hat{t}A, \hat{t}B)]^2 \right\}
\]
\[
= \text{tr} \left\{ \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} \right\}^2
\]
\[
= t^2 \text{tr} \left\{ \begin{pmatrix} A - \hat{A} & B - \hat{B} \\ B^\dagger - \hat{B}^\dagger & 0 \end{pmatrix} \right\}^2 + \mathcal{O}(t^4),
\]
where in the last step we made use of the cubic term in \(t\) is trace-free, because
\[
\text{tr} \left\{ \begin{pmatrix} A - \hat{A} & B - \hat{B} \\ B^\dagger - \hat{B}^\dagger & 0 \end{pmatrix} \right\}^2 = \text{tr} \left\{ \begin{pmatrix} 0 & (B - \hat{B})(*) \\ 0 & 0 \end{pmatrix} \right\} = 0
\]
(5.3)
where the star stands for an arbitrary block matrix entry).

The formula (5.3) shows that, in our coordinates, the Riemannian metric is constant up to contributions of order \(\mathcal{O}(t^2)\). Therefore, the coordinates are indeed Gaussian. □

6. Gauges and Gauge Fixing

We saw in Section 2.4 that the vectors in \(\mathcal{H}\) can be represented by \((2n)\)-component wave functions in space-time (2.13), unique up to local gauge transformations (2.14).

In order to clarify the mathematical structures, it is useful to choose an inner product space \((V, \langle \cdot | \cdot \rangle)\) of signature \((n,n)\) with pseudo-orthonormal basis \((f_1, \ldots, f_{2n})\). Then one can regard the \(\psi^\alpha(x)\) in (2.13) as component functions of vectors in \(V\),
\[
\sum_{\alpha=1}^{2n} \psi^\alpha(x) f_\alpha \in V,
\]
We thus obtain a representation of \(\mathcal{H}\) as \(V\)-valued functions in space-time. The only condition to fulfill is that at each space-time point \(x\), the resulting local correlation operator must coincide with the operator \(x \in \mathcal{F}\). This leads us to the following notion:

**Definition 6.1.** Let \((V, \langle \cdot | \cdot \rangle)\) be an indefinite inner product space of signature \((n,n)\). Moreover, let \(\Omega \subset \mathcal{F}\) be an open space-time region. A mapping
\[
\Psi^\Omega_V : \Omega \to \text{L}(\mathcal{H}, V)
\]
is called a **gauge** in \(\Omega\) if
\[
x = - (\Psi^\Omega_V(x))^* (\Psi^\Omega_V(x)) \quad \text{for all } x \in \Omega.
\]
Here the adjoint is to be taken with respect to the corresponding inner products, i.e.
\[
\langle \phi | \Psi^\Omega_V(x) u \rangle = \langle (\Psi^\Omega_V(x))^* \phi | u \rangle_{\mathcal{H}} \quad \text{for all } \phi \in V \text{ and } u \in \mathcal{H}.
\]
We remark that the concept of defining a gauge as a representation of Hilbert space vectors as wave functions goes back to [3, Definition 2.1].

In order to see that gauges exist, one can proceed as follows. Given \(\Omega\), for every \(y \in \Omega\) one chooses a unitary mapping
\[
U_y : S_y \to V
\]
such a unitary mapping exists because \(V\) and \(S_y\) have the same signature). Then the mapping
\[
\Psi^\Omega_V(y) := U_y \Psi(y)
\]
(6.2)
is indeed a gauge because
\[-(\Psi_V^\Omega(y))^* (\Psi_V^\Omega(y)) = -(\Psi(y))^* (\Psi(y)) = y\]
(where in the first step we used that $U_y$ is unitary, and in the second step we applied (2.9)). In this construction, the local gauge freedom corresponds to the freedom in choosing the isomorphisms (6.1) between the spin spaces and $V$.

Fixing the gauge amounts to constructing distinguished gauges. Our general procedure is outlined as follows. Given $x \in \mathcal{F}$, we want to construct a distinguished gauge in an open neighborhood $\Omega \subset \mathcal{F}$ of $x$. To this end, we want to construct a distinguished mapping
\[
\phi : \Omega \to L(\mathcal{H}, S_x)
\] (6.3)
with the property that the local correlation operator corresponding to $\phi(y)$ agrees with $y$, i.e.
\[y = -\phi(y)^* \phi(y) \quad \text{for all } y \in \Omega.
\]
Next, we choose a unitary operator $U_x$ from $S_x$ to $V$. Then the mapping
\[
\Psi_V^\Omega : \Omega \to L(\mathcal{H}, V), \quad \Psi_V^\Omega(y) := U_x \phi(y)
\] (6.4)
is a gauge. This construction is illustrated in the following diagram:

\[
\begin{array}{ccc}
y \in \Omega \subset \mathcal{F} & \xrightarrow{\phi} & \phi(y) \in L(\mathcal{H}, S_x) \\
& \downarrow \Psi_V^\Omega & \\
& U_x \phi(y) \in L(\mathcal{H}, V)
\end{array}
\]

Note that, in contrast to the construction (6.2), which involves the freedom in choosing a unitary operator $U_y$ at every $y \in \Omega$, the gauge (6.4) involves only one unitary operator $U_x$. In this way, the local gauge freedom has been fixed up to global gauge transformations.

The mapping $\phi$ in (6.3) has a simple interpretation as “using wave functions as coordinates.” Indeed, given $u \in \mathcal{H}$, the vector $\phi(y)u \in S_x$ can be regarded as the physical wave function at the space-time point $y$, however in a gauge where all the spin spaces are identified with $S_x$. This idea will become clearer in the next sections, when we use the wave evaluation operator $\Psi$ for the construction of $\phi$. Due to the local gauge freedom, the idea of “using wave functions as coordinates” can be realized only after invoking a gauge fixing procedure. We first introduce this gauge fixing by hand (Section 6.1) and justify it afterward by analyzing the Gaussian charts of Section 5 (Section 6.2).

6.1. **Symmetric Wave Charts.** By varying the wave evaluation operator, we obtain a mapping
\[
R : W \subset L(\mathcal{H}, S_x) \to \mathcal{F}, \quad \psi \mapsto -\psi^* \psi, \quad (6.5)
\]
where $W$ is an open neighborhood of $\Psi(x)$ which is chosen so small that all the operators in the image of $R$ have $n$ positive and $n$ negative eigenvalues (a similar construction which in addition arranges a constant trace is considered in see [10, Section 6.2]). Since every operator in $\mathcal{F}$ can be realized as its own local correlation operator (2.10) and all the spin spaces are isomorphic, it is obvious that the image of $R$ contains an open neighborhood of $x \in \mathcal{F}$. However, the operator $R$ has a kernel. In order to describe
this kernel systematically, it is convenient to again decompose the Hilbert space into the direct sum

$$\mathcal{H} = I \oplus J$$

with $I := S_x$, $J := (S_x)\perp$, $W \ni \psi = \psi_I + \psi_J$. \hfill (6.6)

For clarity, we point out that $I$ always denotes the Hilbert space with the induced scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}|I \times I}$. Thus $I$ and $S_x$ coincide as complex vector spaces. However, the spin space $S_x$ is not a Hilbert space but an indefinite inner product space, endowed with the spin scalar product $\langle \cdot, \cdot \rangle_{x} := \langle \cdot, \cdot \rangle_{\mathcal{H}}$.

The direct sum decomposition (6.6) gives rise to a corresponding decomposition of the linear operators,

$$L(\mathcal{H}, S_x) = L(I, S_x) \oplus L(J, S_x).$$

We again point out that the space $L(I, S_x)$ coincides as a vector space with $L(I)$, and $L(J, S_x)$ coincides with $L(J, I)$. However, when taking adjoints, one must be careful to take the correct inner products. Possibly by choosing $W$ smaller, we can arrange that the operators $\psi_I \in L(I, I)$ are all invertible, so that $R$ becomes a mapping

$$R : W \subset GL(S_x) \oplus L(J, S_x) \to \mathcal{F}$$

(where we canonically identified $L(I, S_x)$ with $L(S_x)$). The gauge freedom becomes manifest in the fact that the mapping $R$ has a non-trivial kernel:

**Lemma 6.2.** $R$ is injective up to gauge transformations in $U(S_x)$, meaning that

$$R(\psi_I, \psi_J) = R(\tilde{\psi}_I, \tilde{\psi}_J) \iff \exists U \in U(S_x) \text{ with } \tilde{\psi}_I = U\psi_I \text{ and } \tilde{\psi}_J = U\psi_J \quad (6.7)$$

(where $U(S_x)$ are the unitary operators with respect to the spin scalar product).

**Proof.** Unitarity with respect to the spin scalar product is defined by

$$\langle U\phi | U\tilde{\phi} \rangle_x = \langle \phi | \tilde{\phi} \rangle_x \quad \text{for all } \phi, \tilde{\phi} \in S_x.$$ Using (2.8), unitarity can be written more explicitly as the conditions

$$X^{-1} U^\dagger X = U^{-1} \quad \text{or} \quad U^\dagger X = X U^{-1}. \quad (6.8)$$

Using a block matrix notation in the direct sum decomposition (6.6), we have

$$R(\psi_I, \psi_J) = -\begin{pmatrix} \psi_I^\dagger X \psi_I & \psi_I^\dagger X \psi_J \\ \psi_J^\dagger X \psi_I & \psi_J^\dagger X \psi_J \end{pmatrix}.$$ Hence the condition $R(\psi_I, \psi_J) = R(\tilde{\psi}_I, \tilde{\psi}_J)$ is equivalent to the three equations

$$\psi_I^\dagger X \psi_I = \tilde{\psi}_I^\dagger X \tilde{\psi}_I \quad (6.9)$$

$$\psi_I^\dagger X \psi_J = \tilde{\psi}_I^\dagger X \tilde{\psi}_J \quad (6.10)$$

$$\psi_J^\dagger X \psi_J = \tilde{\psi}_J^\dagger X \tilde{\psi}_J. \quad (6.11)$$

Since $\psi_I$ and $\tilde{\psi}_I$ are invertible operators, we can write

$$\tilde{\psi}_I = U\psi_I \quad (6.12)$$

with an invertible operator $U$ on $I$. Multiplying (6.9) from the left by the inverse of $\psi_I^\dagger$ and from the right by the inverse of $\psi_I$, we obtain the condition $X = U^\dagger X U$. Comparing with (6.8), we conclude that $U \in U(S_x)$ is unitary with respect to the spin.
scalar product. Substituting (6.12) in (6.10) and using that $U$ is unitary with respect to the spin scalar product, we obtain the equivalent condition
\[ \psi_I^\dagger X_{\psi_J} = \psi_I^\dagger U^\dagger X_{\tilde{\psi}_J} = \psi_I^\dagger X U^{-1}_{\tilde{\psi}_J}, \]
and multiplying from the left by $UXU^{-1}$ gives the last identity in (6.7). If the relations on the right side of (6.7) hold, then the condition (6.11) is also satisfied. This concludes the proof. □

After these preparations, we can explain our method for fixing the gauge. The gauge freedom (6.7) means that both $\psi_I$ and $\psi_J$ can be multiplied by an arbitrary unitary operator $U \in U(S_x)$. Using this freedom, we can arrange that $\psi_I \in \text{Symm}(S_x)$ (6.13) becomes a symmetric operator. This method indeed fixes the local gauge freedom completely, as we shall now work out. Before beginning, we remark that, at present, our procedure is motivated only by the fact that it works and is canonical. A deeper justification will be given in connection with the Gaussian charts in Section 6.2 below.

We begin with a preparatory lemma.

Lemma 6.3. (unique polar decomposition) Let $(V, \prec, \succ)$ be a (finite-dimensional) indefinite inner product space. Then there is an open neighborhood $W$ of $1 \in L(V)$ such that every operator $A \in W$ has a unique polar decomposition
\[ A = U S \quad \text{with} \quad U \in U(V) \text{ and } S \in \text{Symm}(V) \cap W. \quad (6.14) \]

Proof. Writing $A = 1 + \Delta A$, it follows that
\[ B := A^* A = (1 + \Delta A)^*(1 + \Delta A) = 1 + \Delta B \]
with $\Delta B = (\Delta A)^* + (\Delta A) + (\Delta A)^*(\Delta A)$. If the neighborhood of $W$ is chosen sufficiently small, the spectral calculus for $B$ is well-defined as a power expansion in $\Delta B$. In particular, the series
\[ B^{\frac{1}{2}} := 1 + \frac{\Delta B}{2} - \sum_{n=2}^{\infty} \frac{1}{2^n} \frac{(2n - 3)!}{n!(2n - 4)!!} (\Delta B)^n, \]
\[ B^{-\frac{1}{2}} := 1 + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{(2n - 1)!}{n!(2n - 2)!!} (\Delta B)^n \]
converge absolutely. This makes it possible to form the polar decomposition (6.14) with the standard formulas
\[ U = A B^{-\frac{1}{2}} \quad \text{and} \quad S = B^{\frac{1}{2}}. \]
Here the operator $U$ is indeed unitary because
\[ UU^* = (AB^{-\frac{1}{2}})(B^{-\frac{1}{2}} A^*) = AB^{-1} A^* = A (A^* A)^{-1} A^* = AA^{-1} (A^*)^{-1} A^* = 1. \]

It remains to show uniqueness. To this end, we consider two polar decompositions,
\[ A = U S = \tilde{U} \tilde{S}. \quad (6.15) \]
Using that $U, \tilde{U}$ are unitary and $S, \tilde{S}$ are symmetric, we obtain
\[ A^* A = S^2 = \tilde{S}^2. \quad (6.16) \]
Choosing $W$ sufficiently small, the square root is again well-defined as a power series, i.e. 
\[ S = \sqrt{S^2} \quad \text{and} \quad \tilde{S} = \sqrt{\tilde{S}^2}. \]
Hence (6.16) implies that $S = \tilde{S}$. As a consequence, it follows from (6.15) that $\tilde{U} = \tilde{U}$, completing the proof. □

We now apply the previous lemma to the mapping $\psi_I$:

**Lemma 6.4.** There is an open neighborhood $W$ of $\Psi(x) \in L(\mathcal{H}, S_x)$ such that for every $\psi \in \Omega$, the operator $\psi_I : S_x \to S_x$ has a unique polar decomposition of the form 
\[ \psi_I = US \quad \text{with} \quad U \in U(S_x) \quad \text{and} \quad S \in \text{Symm}(S_x) \cap \pi_x \Omega|_{S_x}. \] (6.17)

**Proof.** Since $\Psi(x)_I = 11$, $\psi_J = 0$, we can write $\psi \in \Omega$ as 
\[
\psi_I = 11 + \Delta \psi_I \quad \psi_J = \Delta \psi_J,
\]
where $\Delta \psi$ is sufficiently small. Regarding the operator $\psi_I$ as an endomorphism of $S_x$, we can apply Lemma 6.3 with $V = S_x$ to conclude that this operator has a unique polar decomposition of the form (6.17). □

We now use the unitary operator $U$ in (6.17) to perform a gauge transformation (6.14). Due to the uniqueness of the construction, we thus obtain a chart. We have thus proved the following theorem:

**Theorem 6.5.** For every $x \in F$ there is an open neighborhood $W$ of $(11, 0) \in \text{Symm}(S_x) \oplus L(J, S_x)$ such that
\[
R : W \subset \text{Symm}(S_x) \oplus L(J, S_x) \to F, \quad \psi \mapsto -\psi^* \psi
\]
is a local parametrization of $F$ around $x$. Its inverse
\[
\phi := (R|_{\Omega})^{-1} : \Omega \subset F \to \text{Symm}(S_x) \oplus L(J, S_x)
\] (6.18)
with $\Omega := R(W) \subset F$ is a local chart of $F$.

The chart $(\phi, \Omega)$ is referred to as the symmetric wave chart about the point $x \in M$.

We finally bring the symmetric wave chart into a more explicit form:

**Proposition 6.6.** Choosing the open set $\Omega$ sufficiently small, the symmetric wave chart $\phi$ in (6.18) takes the form
\[
\phi(y) = (P(x,x)^{-1} A_{xy} P(x,x)^{-1})^{-\frac{1}{2}} P(x,x)^{-1} P(x,y) \Psi(y),
\]
where $A_{xy} := P(x,y) P(y,x)$ is the closed chain.

**Proof.** The operator $R$ introduced in (6.5) has the property that
\[-\Psi(y)^* \Psi(y) = y \overset{!}{=} R(\psi) = -\psi^* \psi.
\] As a consequence, $\psi$ differs from $\Psi(y)$ by a unitary mapping, i.e.
\[
\psi = U_{x,y} \Psi(y) : \mathcal{H} \to S_x \quad \text{with} \quad U_{x,y} \in U(S_y, S_x).
\]
We thus obtain the ansatz
\[
\phi(y) = U_{x,y} \Psi(y), \quad (6.19)
\]
where we must choose $U_{x,y}$ such that the restriction $\phi(y)|_{S_x} : S_x \to S_x$ is symmetric. Let us evaluate what this condition means: First, it is convenient to express the operators in terms of the kernel of the fermionic projector,

$$\Psi(y)|_{S_x} = \pi_y \pi_x|_{S_x} = \pi_y x X^{-1} \pi_x|_{S_x} = P(y, x) P(x, x)^{-1}|_{S_x}$$

(where we used again the notation (2.5)). Next, we form a polar decomposition of the obtained operator

$$B := P(y, x) P(x, x)^{-1}|_{S_x} : S_x \to S_y.$$ 

This gives

$$\phi(y)|_{S_x} = U_{x,y} \Psi(y)|_{S_x} = U_{x,y} (B (B^* B)^{-\frac{1}{2}}) (B^* B)^{\frac{1}{2}}|_{S_x}.$$ 

Therefore, we can choose $U_{x,y}$ as

$$U_{x,y} = (B (B^* B)^{-\frac{1}{2}})^{-1} = (B^* B)^{\frac{1}{2}} B^{-1} = (B^* B)^{\frac{1}{2}} (B^* B)^{-1} B = (B^* B)^{-\frac{1}{2}} B^*$$

This operator is indeed unitary because

$$U_{x,y} (U_{x,y}^*) = (P(x, x)^{-1} A_{xy} P(x, x)^{-1})^{-\frac{1}{2}} P(x, x)^{-1} P(x, y)$$

$$\times P(y, x) (P(x, x)^{-1} A_{xy} P(x, x)^{-1})^{-\frac{1}{2}}$$

$$= (P(x, x)^{-1} A_{xy} P(x, x)^{-1})^{-\frac{1}{2}}$$

$$\times (P(x, x)^{-1} A_{xy} P(x, x)^{-1}) (P(x, x)^{-1} A_{xy} P(x, x)^{-1})^{-\frac{1}{2}} = 1_{S_x}.$$ 

The uniqueness of $U_{x,y}$ follows from Lemma 6.3. Using (6.20) in (6.19) gives the result. \hfill \square

### 6.2. Gaussian Wave Charts.

We now analyze whether the Gaussian charts constructed in Section 5 also give rise to wave charts. Our starting point is the local parametrization $\Lambda$ in (5.1). Our strategy is to construct a mapping $\psi \in \text{L}(I, S_x) \oplus \text{L}(J, S_x)$ such that $M = R(\psi)$ (where $R$ is again the mapping (6.5)). In other words, in block matrix notation, we want to find $\psi$ such that

$$M = \left(\begin{array}{c}
\psi_I^T \\
\psi_J^T
\end{array}\right) X \left(\begin{array}{cc}
\psi_I & \psi_J
\end{array}\right).$$

(6.21)

Considering the upper left block matrix entry of $M$ and comparing with (5.1), one finds that $\psi_I$ must satisfy the equation

$$\psi_I^T X \psi_I = X + A.$$ 

(6.22)

This equation can be solved with the spectral calculus: The first step is to set

$$X + A = X (1 + X^{-1} A) = X \sqrt{1 + X^{-1} A} \sqrt{1 + X^{-1} A},$$

where the square root is defined as a power series in $X^{-1} A$. Using that for any $p \in \mathbb{N}$,

$$X (X^{-1} A)^p = (AX^{-1})^p X,$$

it follows that

$$X \sqrt{1 + X^{-1} A} = \sqrt{1 + AX^{-1}} X = \left(\sqrt{1 + X^{-1} A}\right)^\dagger X.$$
We conclude that 

$$X + A = \left(\sqrt{1 + X^{-1}A}\right)^{\dagger} X \sqrt{1 + X^{-1}A},$$  

(6.23)

giving the explicit solution of (6.22)

$$\psi_I = \sqrt{1 + X^{-1}A}.\quad (6.24)$$

Using (6.23) in (3.1), we can read off that \(\psi_J\) is given explicitly by

$$\psi_J = \sqrt{1 + X^{-1}A} (X + A)^{-1} B$$

$$= \sqrt{1 + X^{-1}A} (1 + X^{-1}A)^{-1} X^{-1} B$$

$$= (1 + X^{-1}A)^{-\frac{1}{2}} X^{-1} B.\quad (6.25)$$

With (6.24) and (6.25) we have found an explicit solution \(\psi = \psi_I + \psi_J\) of (6.21). Our findings are summarized as follows:

**Proposition 6.7.** In the Gaussian parametrization \(\Lambda\) in (5.1), the spectral calculus gives rise to a canonical mapping

\[ W : (\text{Symm}(I) \oplus L(I, J)) \cap B_{\varepsilon}(0) \to L(I, S_x) \oplus L(J, S_x) \]

\((A, B) \mapsto \left(\sqrt{1 + X^{-1}A}, (1 + X^{-1}A)^{-\frac{1}{2}} X^{-1} B\right)\)

(6.26)

with the property that

\[ \Lambda(A, B) = R(W(A, B)) \]

for all \((A, B) \in (\text{Symm}(I) \oplus L(I, J)) \cap B_{\varepsilon}(0)\. The mapping

\[ \phi := W \circ \Lambda^{-1} : \Omega \subset \mathcal{F} \to L(I, S_x) \oplus L(J, S_x) \]

with \(\Omega := \Lambda(B_{\varepsilon}(0))\) is a local chart of \(\mathcal{F}\).

The chart \(\phi, \Omega\) is referred to as the Gaussian wave chart about the point \(x \in M\). Our construction is summarized by

\[ y \in \Omega \subset \mathcal{F} \xrightarrow{\Lambda^{-1}} \Lambda^{-1}(y) \in \text{Symm}(I) \oplus L(I, J) \xrightarrow{W} \phi(y) \in L(\mathcal{K}, S_x). \]

A more detailed analysis can be found in [19].

Let us analyze what this result means. Note that the operators \(A\) and \(B\) in Proposition 6.7 map subspaces of the Hilbert space \(\mathcal{K}\) into each other; thus no indefinite inner product spaces appear. Nevertheless, the spin scalar product is important for understanding the formula in (6.26). The main observation is the following simple lemma:

**Lemma 6.8.** The operator \(\sqrt{1 + X^{-1}A}\), regarded as an endomorphism of the spin space \(S_x\), is symmetric,

\[ \sqrt{1 + X^{-1}A} \in \text{Symm}(S_x) \].

**Proof.** Using the formula (2.8), we obtain

\[ (X^{-1}A)^{\dagger} = X^{-1} (X^{-1}A)^{\dagger} X = X^{-1} (AX^{-1}) X = X^{-1} A \].

Hence all powers of \(X^{-1}A\) are also in \(\text{Symm}(S_x)\). Since the square root is defined by a power series, the result follows.  \(\square\)
As a consequence of this lemma, the operator $W$ maps to $\text{Symm}(S_x) \oplus L(J,S_x)$. Comparing with (6.18), we conclude that the Gaussian wave chart satisfies the symmetry condition (6.13) used for the construction of the symmetric wave charts. Using the uniqueness of the latter construction, we come to the following conclusion:

**Proposition 6.9.** The symmetric wave chart and the Gaussian wave chart about the point $x \in M$ coincide.

This result gives a better understanding of the above constructions. First of all, the fact that different constructions give the same wave charts shows that our wave charts are canonical. More technically, the symmetry condition which in (6.13) was introduced ad hoc, gets a more convincing justification by Proposition 6.7 and Lemma 6.8, where the condition (6.13) follows simply by rewriting the parametrization of the Gaussian chart in terms of a wave chart.

### 7. Example: Dirac Systems

We now want to illustrate our results in concrete examples. Knowing that symmetric wave charts and Gaussian wave charts coincide (see Proposition 6.9), it suffices to consider the symmetric wave chart as computed in Proposition 6.6. According to (6.4), the corresponding gauge $\Psi^\Omega_V$ (see Definition 6.1) is obtained by composing with a unitary operator $U_x : S_x \rightarrow V$.

#### 7.1. Dirac Systems in Finite Spatial Volume

We consider a system of non-interacting Dirac particles in finite spatial volume (for basics on the Dirac equation we refer to [26] or standard textbooks like [1, 25, 23]). More precisely, let $\mathcal{M}$ be the subset of Minkowski space

$$\mathcal{M} := \mathbb{R} \times [-L, L]^3 \subset \mathbb{R}^{1,3}$$

with periodic boundary conditions. The four-component Dirac spinors $\psi(x)$ in Minkowski space take values in the spinor space, which we denote by $S_\mathcal{M} \simeq \mathbb{C}^4$. The spinor space is endowed with an inner product $\langle \cdot , \cdot \rangle$ of signature $(2,2)$ (which in physics is usually written as $\langle \psi | \phi \rangle = \bar{\psi} \phi$ with the adjoint spinors $\bar{\psi} = \psi^\dagger \gamma^0$), which we refer to as the spin scalar product. For convenience, we extend the Dirac wave functions to periodic functions in all of $\mathbb{R}^{1,3}$, i.e.

$$\psi(t, \vec{x}) = \psi(t, \vec{x} + \vec{v}) \quad \text{for all} \ t \in \mathbb{R}, \vec{x} \in \mathbb{R}^3 \text{ and } \vec{v} \in (2L\mathbb{Z})^3.$$ 

The scalar product on the Dirac solutions takes the usual form

$$\langle \psi | \phi \rangle := 2\pi \int_{[-L,L]^3} \langle \psi(t, \vec{x}) | \gamma^0 \psi(t, \vec{x}) \rangle \ d^3x ,$$

Next, we make the plane-wave ansatz

$$\psi_{k\alpha s}(t, \vec{x}) = c e^{-ikx} \chi_{\tilde{k}\alpha s} \quad \text{with} \ \tilde{k} \in \left( \frac{\pi}{L} \mathbb{Z} \right)^3, \ a \in \{1,2\} \text{ and } s \in \{\pm1\} ,$$

where $c$ is a non-zero normalization constant to be determined below. Here $kx$ is the Minkowski inner product of the space-time point $x = (t, \vec{x})$ with four-momentum $k$ on the mass shell,

$$k := (s \omega(\tilde{k}), \tilde{k}) \quad \text{with} \ \omega(\tilde{k}) := \sqrt{|\tilde{k}|^2 + m^2}.$$
Moreover, the spinors $\chi_{kas}$ are solutions of the Dirac equation in momentum space

$$ (\vec{k} - m) \chi_{kas} = 0, \quad (7.4) $$

which we choose to be pseudo-orthonormal with respect to the spin scalar product, i.e.

$$ \langle \chi_{kas} | \chi_{k'a's} \rangle = s \delta_{aa'} . \quad (7.5) $$

As a consequence, for fixed $\vec{k}$ and $s$, the integrand in (7.2) is computed by

$$ \langle \chi_{kas} | \gamma^0 \chi_{k'a's} \rangle = \frac{1}{2m} \left( \langle \vec{k} \chi_{kas} | \gamma^0 \chi_{k'a's} \rangle + \langle \chi_{kas} | \gamma^0 \vec{k} \chi_{k'a's} \rangle \right) $$

$$ = \frac{2k^0}{2m} \langle \chi_{kas} | \chi_{k'a's} \rangle = \frac{s \omega(\vec{k})}{m} s \delta_{aa'} = \frac{\omega(\vec{k})}{m} \delta_{aa'} . $$

Moreover, this scalar product vanishes for fixed $\vec{k}$ if the frequencies of the waves have opposite signs,

$$ \langle \chi_{ka} | \gamma^0 \chi_{ka'} \rangle = \frac{1}{2m} \left( \langle (\omega(\vec{k}) \gamma^0 - \vec{k}\gamma^0) \chi_{ka} | \gamma^0 \chi_{ka'} \rangle + \langle \chi_{ka} | (\omega(\vec{k}) \gamma^0 - \vec{k}\gamma^0) \chi_{ka'} \rangle \right) = 0 $$

(where in the last line we used that $[\gamma^0, \gamma^0] = 0 = \{\gamma^\alpha, \gamma^0\}$ for $\alpha \in \{1, 2, 3\}$). Using these formulas in (7.2), one concludes that the plane waves (7.3) are orthogonal. Moreover, the calculation

$$ \langle \psi_{k}\bar{\psi}_{kas} \rangle := 2\pi |c|^2 (2L)^3 \langle \chi_{k} | \gamma^0 \chi_{k} \rangle = 16 \pi |c|^2 L^3 \frac{\omega(\vec{k})}{m} $$

shows that choosing the normalization constant as

$$ c = \sqrt{\frac{m}{\pi \omega(\vec{k})}} \frac{1}{4L^2}, $$

we obtain unit vectors. Our findings are summarized as follows:

**Lemma 7.1.** In a three-dimensional box (7.1) with periodic boundary conditions, the Dirac wave functions

$$ \psi_{kas}(t, \vec{x}) = \sqrt{\frac{m}{\pi \omega(\vec{k})}} \frac{1}{4L^2} e^{-ikx} \chi_{kas} \quad (7.6) $$

with $\vec{k} \in (\pi \mathbb{Z}/L)^3$, $a \in \{1, 2\}$ and $s \in \{\pm 1\}$, form an orthonormal basis of the Hilbert space of all Dirac solutions, endowed with the scalar product (7.2). Here $\chi_{kas}$ are pseudo-orthonormal solutions of the Dirac equation in momentum space (7.4) and (7.5).

We now choose the Hilbert space $\mathcal{H}$ as the subspace of the solution space of all negative-energy solutions whose energy is above $-1/\varepsilon$, i.e.

$$ \mathcal{H} := \text{span} \left\{ \psi_{ka}(t, \vec{x}) \mid a \in \{1, 2\} \text{ and } \omega(\vec{k}) < \frac{1}{\varepsilon} \right\}. \quad (7.7) $$

**Lemma 7.2.** The Hilbert space $\mathcal{H}$ is finite-dimensional. Its dimension has the following asymptotics for small $\varepsilon$,

$$ f := \dim \mathcal{H} = \frac{8}{3\pi^2} \left( \frac{L}{\varepsilon} \right)^3 \left( 1 + \mathcal{O} \left( \frac{\varepsilon}{L} \right) + \mathcal{O}(\varepsilon m) \right). $$
Proof. According to Lemma 7.1, two Dirac states of negative energy occupy a volume of \((\pi/L)^3\) in momentum space. As a consequence of the energy cutoff, we must occupy a sphere of radius \(\sqrt{\varepsilon-\varepsilon^2-m^2}\) in momentum space. Hence the number of states is counted by

\[
f = \frac{4\pi}{3} \frac{1}{\varepsilon^3} 2 \left( \frac{L}{\pi} \right)^3 \left( 1 + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon m) \right),
\]

giving the result. \(\Box\)

Having a finite-dimensional Hilbert space consisting of smooth wave functions, we can define the local correlation operators without regularization operators, i.e.

\[
F : \mathcal{M} \to \mathcal{F}, \quad (\psi | F(x) \phi) = \langle \psi(x) | \phi(x) \rangle \quad \forall \psi, \phi \in \mathcal{H}.
\]

Finally, we define the universal measure as the push-forward of the Lebesgue measure on \(\mathcal{M}\),

\[
d\rho := F_* (d\mu) \quad \text{with} \quad d\mu := d^4x.
\]

We thus obtain a causal fermion system \((\mathcal{H}, \mathcal{F}, \rho)\) of spin dimension \(n = 2\).

7.2. The Kernel of the Fermionic Projector in Finite Volume. For the computations, it is favorable to identify the spin space \(S_x\) with the space \(S_x \mathcal{M}\) of Dirac spinors at the point \(x\) of Minkowski space \(\mathcal{M}\). To this end, we introduce the evaluation operator \(e_x\) by

\[
e_x : \mathcal{H} \to S_x \mathcal{M}, \quad \psi \mapsto \psi(x) \quad (7.8)
\]

(here we use the fact that, according to (7.7), the vectors of \(\mathcal{H}\) are not merely abstract vectors but linear combinations of plane wave solutions of the Dirac equation, which can be evaluated at \(x \in \mathcal{M}\)). In [8, Section 1.2.4] it is show that if \(e_x\) is surjective, then the space-time point \(x\) is regular (see [8, eq. 1.2.15]). Using this result, we now prove that our causal fermion system is regular if the dimension of the Hilbert space is sufficiently large:

**Proposition 7.3.** If \(\dim \mathcal{H} \geq 4\), then the causal fermion system \((\mathcal{H}, \mathcal{F}, \rho)\) is regular.

**Proof.** Assume that \(\dim \mathcal{H} \geq 4\). Then, since every momentum \(\vec{k}\) gives rise to two Dirac solutions, at least two different momenta are occupied. According to (7.4) and (7.5), for given \(\vec{k}\) the two spinors \(\chi_{\vec{k}a-}\) with \(a = 1, 2\) span the image of the matrix \(\vec{k} + m\).

By direct computation, one sees that for two different momenta \(\vec{k}\) and \(\vec{k}'\), the span of the images of the operators \(\vec{k} + m\) and \(\vec{k}' + m\) is four-dimensional. As a consequence, the corresponding four plane wave solutions \(\psi_{\vec{k}a-}(x)\) and \(\psi_{\vec{k}'a-}(x)\) evaluated at \(x\) are linearly independent. This implies that the evaluation operator (7.8) has rank four, giving the result. \(\Box\)

From now on, we always assume that \(\dim \mathcal{H} \geq 4\), so that our causal fermion system is regular. Restricting the evaluation operator to the subspace \(S_x \subset \mathcal{H}\), we obtain the mapping

\[
e_x|_{S_x} : S_x \to S_x \mathcal{M}. \quad (7.9)
\]

This mapping is indeed an isomorphism from the spin space to the spinor space (for details see [8, Proposition 1.2.6]), making it possible to identify \(S_x\) and \(S_x \mathcal{M}\) as indefinite inner product spaces. This identification is useful for bringing the objects of the causal fermion system into a more explicit form, as we now explain in two examples.
Proposition 7.4. Using the identification (7.9) of the spin spaces with the spinor spaces, the wave evaluation operator (2.3) coincides with the evaluation operator (7.8),

\[ \Psi(x) : \mathcal{H} \rightarrow S_x \mathcal{A}, \quad u \mapsto e_x u = u(x) . \]

For the proof see [8, Proposition 1.2.6], choosing the regularization operator as the identity.

In the calculations it is most convenient to work with the kernel of the fermionic projector, which for clarity we denote with indices \( \varepsilon \) and \( L \),

\[ P_{\varepsilon,L}(x,y) = \pi_x y |_{S_y} : S_y \rightarrow S_x . \quad (7.10) \]

Proposition 7.5. Using the identification (7.9) of the spin spaces with the spinor spaces, the kernel of the fermionic projector (7.10) takes the form

\[ P_{\varepsilon,L}(x,y) = \frac{1}{(2L)^3} \sum_{k,a} \frac{1}{4\pi \omega(k)} e^{-ik(x-y)} (\vec{k} + m) \bigg|_{k = -\omega(k) \vec{k}} \cdot (7.11) \]

Proof. According to [8, Proposition 1.2.7], under the identification (7.9) the kernel of the fermionic projector takes the form

\[ P_{\varepsilon,L}(x,y) = -\sum_{k,a} |\psi^{\bar{\varepsilon}}_{ka-}(x)\rangle \langle \psi^{\bar{\varepsilon}}_{ka-}(y)| \quad (7.12) \]

(where we used a bra/ket notation and made use of the fact that in our example, there is no regularization operator). Using the explicit form of the plane wave solutions (7.6), we obtain

\[ P_{\varepsilon,L}(x,y) = -\frac{1}{(2L)^3} \sum_{k,a} \frac{m}{2\pi \omega(k)} e^{-ik(x-y)} |\chi^\varepsilon_{ka-}\rangle \langle \chi^\varepsilon_{ka-}| . \]

The bra/ket combination of the spinors \( \chi^\varepsilon_{ka-} \) can be calculated further. Indeed, using that these spinors form a pseudo-orthonormal basis of the solution space of the Dirac equation space (see (7.4) and (7.5)), it is clear that the operator

\[ -\sum_{a=1}^2 |\chi^\varepsilon_{ka-}\rangle \langle \chi^\varepsilon_{ka-}| \]

is an idempotent symmetric operator (w.r.to the spin scalar product) whose image coincides with that of the operator \( \vec{k} + m \). As a consequence,

\[ -\sum_{a=1}^2 |\chi^\varepsilon_{ka-}\rangle \langle \chi^\varepsilon_{ka-}| = \frac{1}{2m} (\vec{k} + m) . \]

This concludes the proof. \( \square \)

7.3. Connection to the Kernels in Infinite Volume. In order to bring the kernel of the fermionic projector in (7.11) into a more explicit form, it is useful to compare it with the corresponding kernel in infinite volume. The unregularized kernel is the integral over the lower mass shell (see [8 Section 1.2.5]),

\[ P(x,y) := \int \frac{d^4k}{(2\pi)^4} (\vec{k} + m) \delta(k^2 - m^2) \Theta(-k^0) e^{-ik(x-y)} . \]
The simplest method for the regularization is to insert a momentum cutoff (see also [8, Section 3.8.6 (B)]),

\[
P^\varepsilon(x, y) := \int \frac{d^4k}{(2\pi)^4} (\hat{k} + m) \delta(k^2 - m^2) \Theta(-k^0) \Theta(1 + \varepsilon k^0) e^{-ik(x-y)}.
\]

**Proposition 7.6.** The unregularized kernel, the regularized kernel and the kernel in finite volume are related to each other by

\[
P^\varepsilon(x, y) = \int_{-\infty}^{\infty} P(x, y + (t, \vec{0})) \frac{1}{\pi t} \sin \left(\frac{t}{\varepsilon}\right) dt \quad (7.13)
\]

\[
P^\varepsilon,L(x, y) = \sum_{\vec{z} \in (2LZ)^3} P^\varepsilon(x, y + (0, \vec{z})) \quad (7.14)
\]

**Proof.** The momentum cutoff is realized by multiplying in momentum space with the characteristic function

\[
\chi_{[-\varepsilon^{-1}, \varepsilon^{-1}]}(\omega) \quad \text{with} \quad \omega = k^0.
\]

Multiplication in momentum space corresponds to convolution in momentum space with the kernel

\[
\hat{\chi}_{[-\varepsilon^{-1}, \varepsilon^{-1}]}(t) := \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{d\omega}{2\pi} e^{-i\omega t} = \frac{1}{2\pi t} \left(e^{-\frac{\omega}{2\pi}} - e^{\frac{\omega}{2\pi}}\right) = \frac{1}{\pi t} \sin \left(\frac{t}{\varepsilon}\right).
\]

This proves (7.13).

In order to derive (7.14), we rewrite (7.11) as

\[
P^\varepsilon,L(x, y) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{(2L)^3} \sum_{\vec{k} \in (\pi\mathbb{Z}/L)^3, \omega(\vec{k}) < \varepsilon^{-1}} \delta(k^2 - m^2) \Theta(-k^0) e^{-ik(x-y)} (\hat{k} + m)
\]

\[
= \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \left(\frac{\pi}{L} \right)^3 \sum_{\vec{q} \in (\pi\mathbb{Z}/L)^3} \delta^3(\vec{k} - \vec{q}) \times \left((\hat{k} + m) \delta(k^2 - m^2) \Theta(-k^0) \Theta(1 + \varepsilon k^0)\right).
\]

Again using that multiplication in momentum space corresponds to convolution in position space, one sees that \(P^\varepsilon,L\) is obtained from \(P^\varepsilon\) by convolution with the spatial kernel

\[
h(\vec{x}) = \left(\frac{\pi}{L} \right)^3 \int \frac{d^3q}{(2\pi)^3} \sum_{\vec{k} \in (\pi\mathbb{Z}/L)^3} e^{i\vec{k}\vec{x}}
\]

\[
= \frac{1}{(2L)^3} \sum_{\vec{q} \in (\pi\mathbb{Z}/L)^3} e^{i\vec{q}\vec{z}} = \sum_{\vec{z} \in (2LZ)^3} \delta^3(\vec{x} - \vec{z})
\]

where in the last step we used the completeness relation for plane waves on the torus.

For clarity we remark that the sum in (7.14) makes \(P^\varepsilon,L\) periodic in space with period \(2L\).

**7.4. Gauge Fixing of Wave Functions in Space-Time.** We now compute the gauge (6.4) for the symmetric wave chart \(\phi\) of Proposition 6.6 more explicitly for our Dirac systems. Although this gauge was derived under the assumption that \(\mathcal{H}\) is finite-dimensional, all the formulas expressed in terms of the kernel of the fermionic projector can be used in the infinite-dimensional setting of Section 7.3 just as well. With this in
mind, the following results apply to the kernels with regularization in Proposition 7.6, both in finite and infinite spatial volume. For ease in notation, from now on we omit the indices $\varepsilon$ and $L$.

We begin with the massless case $m = 0$. Then by symmetry it follows that
\[ P(x,x) = \alpha \gamma^0 \quad \text{with} \quad \alpha \in \mathbb{R} \quad (7.15) \]
and thus
\[ U_{x,y} = (\gamma^0 A_{xy} \gamma^0)^{-\frac{1}{2}} \gamma^0 P(x,y) \, . \]
Using that $(\gamma^0)^2 = 1$, we obtain
\[ (\gamma^0 A_{xy} \gamma^0)^p = \gamma^0 A_{xy}^p \gamma^0 \]
for any $p \in \mathbb{N}$. The spectral calculus yields that this relation holds also for any real $p$. Hence
\[ U_{x,y} = (\gamma^0 A_{xy}^{-\frac{1}{2}} \gamma^0) \gamma^0 P(x,y) = \gamma^0 A_{xy}^{-\frac{1}{2}} P(x,y) \, . \]

Hence
\[ \phi(y) = U_{x,y} \Psi(y) = \gamma^0 A_{xy}^{-\frac{1}{2}} P(x,y) \, \Psi(y) \, . \]

The resulting symmetric wave gauge is
\[ \Psi^0_V(y) = U_x \, (\gamma^0 A_{xy}^{-\frac{1}{2}} P(x,y) \, \Psi(y)) \, . \quad (7.16) \]

Before going on, we point out that the combination $A_{xy}^{-\frac{1}{2}} P(x,y)$ is reminiscent of the spin connection in [12]. Indeed, the spin connection has the form (see [12, eq. (3.42)])
\[ D_{x,y} = e^{i \varphi_{xy} v_{xy}} A_{xy}^{-\frac{1}{2}} P(x,y) \, . \quad (7.17) \]
where $v_{xy}$ is the directional sign operator (see [12, Definition 3.15]). In simple terms, the factor $e^{i \varphi_{xy} v_{xy}}$ introduces generalized $\text{SU}(2)$-phases which are absent in (7.16). These phases are important for the geometric constructions in [12]. The drawback is that the spin connection (7.17) is not defined for all space-time points $y$ in an open neighborhood of $x$, but only for a more restrictive class of space-time points which satisfy the conditions subsumed in the notion of spin connectability (see [12, Definition 3.17]). With this in mind, the spin connection (7.17) cannot be used for constructing charts. The factor $\gamma^0 A_{xy}^{-\frac{1}{2}} P(x,y)$ in (7.16) can be understood as a simplified version of a spin connection, which is insufficient for describing the geometry of space-time, but which can nevertheless be used for constructing distinguished gauges.

In the massless case, the kernel of the fermionic projector $P(x,y)$ has only a vector component (see (7.11) or the similar formulas in infinite volume). Therefore, we can make the general ansatz
\[ P(x,y) = \psi(x,y) + i \zeta(x,y) \quad (7.18) \]
with two Minkowski vectors $u$ and $\zeta$. In view of (7.15),
\[ \psi(x,x) = \alpha \gamma^0 \quad \text{and} \quad \zeta(x,x) = 0 \, . \]
Moreover, we know that
\[ P(y,x) = P(x,y)^* = \psi(x,y) - i \zeta(x,y) \, . \]
Hence, omitting the arguments $x$ and $y$, we obtain for the closed chain
\[ A_{xy} = u^2 + \zeta^2 - i [\psi, \zeta] \, . \quad (7.19) \]
In the next lemma we compute the factor $A_{xy}^{-\frac{1}{2}} P(x,y)$ in (7.16).
Lemma 7.7. For the kernel of the fermionic given by \((7.18)\),
\[
A_{xy}^{-rac{1}{2}} P(x,y) = \frac{1}{2} \left( \frac{\sqrt{\lambda_+} + \sqrt{\lambda_-}}{\sqrt{\lambda_+} + \sqrt{\lambda_-}} \right) \sqrt{\lambda_+ + \sqrt{\lambda_-}} - \frac{\zeta^2 - i(u\zeta)}{\sqrt{u^2\zeta^2 - 2(u\zeta)^2}} \sqrt{\lambda_+ + \sqrt{\lambda_-}} \right) \gamma_1 \\
+ \frac{i}{2} \left( \frac{\sqrt{\lambda_+} + \sqrt{\lambda_-}}{\sqrt{\lambda_+} + \sqrt{\lambda_-}} \right) \sqrt{\lambda_+ + \sqrt{\lambda_-}} - \frac{u^2 - i(u\zeta)}{\sqrt{u^2\zeta^2 - 2(u\zeta)^2}} \sqrt{\lambda_+ + \sqrt{\lambda_-}} \right) \gamma_0,
\]
where
\[
\lambda_{\pm} = u^2 + \zeta^2 \pm 2\sqrt{u^2\zeta^2 - (u\zeta)^2}. \tag{7.20}
\]

Proof. The calculation
\[
(A_{xy} - u^2 - \zeta^2)^2 = -[\gamma_1, \gamma_0]^2 = -\gamma_1 \gamma_0 \gamma_1 \gamma_0 - \gamma_1 \gamma_0 \gamma_1 \gamma_0 + 2u^2 \zeta^2 \\
= -(2(u\zeta) \gamma_1 \gamma_0 - u^2\zeta^2) - (2(u\zeta) \gamma_1 \gamma_0 - u^2\zeta^2) + 2u^2 \zeta^2 = -4(u\zeta)^2 + 4u^2 \zeta^2
\]
shows that the matrix \(A_{xy}\) has the eigenvalues \(\lambda_{\pm}\) as given in (7.20). The corresponding spectral projection operators are given by
\[
E_{\pm} = \frac{1}{2} \left( 1 \pm \frac{A_{xy} - \lambda_{\pm}}{\lambda_{\pm} - \lambda_{\mp}} \right) = \frac{1}{2} \left( 1 \mp \frac{i [\gamma_1, \gamma_0]}{2\sqrt{u^2\zeta^2 - (u\zeta)^2}} \right),
\]
where in the last step we used (7.19) and (7.20). The spectral calculus gives
\[
A_{xy}^{-\frac{1}{2}} P(x,y) = \sum_{s=\pm} \lambda_{s}^{-\frac{1}{2}} E_{s} P(x,y).
\]
Substituting (7.18) and applying the relations
\[
[\gamma_1, \gamma_0] \gamma_1 = \gamma_1 \gamma_0 \gamma_1 - u^2 \gamma_0 = (2(u\zeta) \gamma_1 \gamma_0 - u^2 \gamma_0) - u^2 \gamma_0 = 2(u\zeta) \gamma_1 \gamma_0 - 2u^2 \gamma_0 \\
[\gamma_1, \gamma_0] \gamma_0 = -2(u\zeta) \gamma_0 + 2\zeta^2 \gamma_0
\]
gives the result. \(\square\)

We next analyze this result in an expansion near the diagonal \(x = y\). To this end, we make the ansatz
\[
u = \alpha \gamma^0 + \tau \nu_1 + O(\tau^2), \quad \zeta = \tau \zeta_1 + O(\tau^2) \tag{7.21}
\]
with a real expansion parameter \(\tau\). A straightforward computation (which we carried out with the help of Mathematica) gives the following result:

Proposition 7.8. For \(P(x,y)\) as in (7.18) with \(u\) and \(\zeta\) according to (7.21),
\[
\gamma^0 A_{xy}^{-\frac{1}{2}} P(x,y) = 1 - \tau \gamma^0 (\bar{u} \gamma_0 + \tau \frac{\zeta_1}{\alpha} \gamma_0) \gamma_0 + \tau \frac{\zeta_1}{\alpha} \gamma_0 \gamma_0 + O(\tau^2). \tag{7.22}
\]

Let us explain the above results. We begin with Proposition 7.8. Writing the matrix in (7.22) in the form \(1 + \tau A + O(\tau^2)\), the fact that \(A\) is antisymmetric (with respect to the spin scalar product) shows that this matrix is unitary. Next, one sees that only the spatial component of \(u\) and only the time component of \(\zeta\) enter (7.22). More precisely, the time component of \(\zeta\) gives a phase factor, whereas the spatial component of \(u\) gives a bilinear contribution. These contributions clearly depend on the regularization scale \(\epsilon\). The expansion in Proposition 7.8 is justified only if the difference vector \(y - x\) is as small as the regularization scale. On larger scales, one must work instead with the formulas of Lemma 7.7. In general terms, the matrix in (7.22) is a unitary mapping
from the spinor spaces at $y$ to $x$, which depends on the difference vector $y - x$ and on the regularization.

In the resulting symmetric wave gauge $\Psi^\Omega_V$ in (7.16), this matrix is multiplied by $\Psi(y)$, which is composed of the plane-wave solutions of the Dirac equation at the space-time point $y$ (see Proposition 7.4). The point of interest is that gauge phases drop out of $\Psi^\Omega_V$. This can be seen explicitly from the transformation law under gauge transformations (1.1) and (1.2), which implies that

$$\Psi(y) \rightarrow e^{i\Lambda(y)} \Psi(y)$$
$$P(x,y) \rightarrow e^{i\Lambda(x) - i\Lambda(y)} P(x,y)$$
$$A_{xy} \rightarrow A_{xy}$$
$$\Psi^\Omega_V(y) \rightarrow \Psi^\Omega_V(y).$$

Thus the local gauge freedom of electrodynamics is completely fixed.

Due to the phases depending on $y - x$ in (7.22), the symmetric wave gauge cannot be identified with any of the usual gauges of electrodynamics (like the Lorenz, Coulomb or general $R_\xi$ gauges). Instead, the local phases are determined by the detailed form of the regularization.

We finally explain how the above findings generalize to the massive case $m > 0$. In this case, the regularized kernel $P(x,y)$ also involves a scalar component, making all the formulas more complicated. However, for $y - x$ on the Planck scale, the scalar component is smaller than the vector component by a scaling factor of $\varepsilon m$. Therefore, the result of Proposition 7.8 is still valid, up to small correction terms. With this in mind, all our qualitative results remain valid, but the detailed form of the gauge fixing is more involved.

### 7.5. Gauge Fixing of the Perturbation Expansion

We now consider the situation that the Dirac wave functions are perturbed by an external electromagnetic potential $A$. It is most convenient to begin with the perturbation of the wave evaluation operator $\Psi$. Always denoting the perturbed objects by a tilde, to first order we obtain

$$\tilde{\Psi}(x) = \Psi(x) - (s_mA\Psi)(x) = \Psi(x) - \int s_m(x,y)A(y)\Psi(y)\,d^4y,$$

where $s_m$ is a Dirac Green’s operator. To higher order, one has similar formulas involving several Green’s operators (for a systematic treatment see for example [17]). Here we do not need to enter the details of the perturbation expansion. It suffices to note that the perturbation expansion respects the gauge symmetry in the sense that a pure gauge potential $\tilde{A}(x) = \phi\Lambda(x)$ gives rise to a local phase transformation,

$$\tilde{\Psi}(x) = e^{i\Lambda(x)} \Psi(x).$$

To first order, this can be verified directly from (7.23) using the computation

$$\tilde{\Psi}(x) = \Psi(x) - (s_m(\phi\Lambda)\Psi)(x) = \Psi(x) + i(s_m[i\phi - m, \Lambda]\Psi)(x)$$
$$= \Psi(x) + i\Lambda(x) \Psi(x) = e^{i\Lambda(x)} \Psi(x) + \mathcal{O}(\Lambda^2).$$

Once we know $\tilde{\Psi}$, all the other relevant objects can computed in a straightforward way. In particular, the perturbed local correlation operator and the kernel of the
fermionic projector are given by (for details see [8, Lemma 1.1.3])
\[ \tilde{F}(x) = -\tilde{\Psi}(x)^*\tilde{\Psi}(x) \quad \text{and} \quad \tilde{P}(x, y) = -\tilde{\Psi}(x)\tilde{\Psi}(y)^*. \]

We can also perturb only one of the factors in the kernel of the fermionic projector. We use the notation
\[ P(x, \tilde{F}(y)) := -\Psi(x)\tilde{\Psi}(y)^*. \]

In order to fix the gauge in the perturbation expansion, one should note that \( \tilde{F}(x) \) is again an operator in \( \mathcal{F} \). Therefore, we can work again with \( \phi \) in Proposition 6.6 choosing \( y = \tilde{F}(x) \), i.e.
\[ \tilde{\phi}(x) := \left( P(x, x)^{-1} A_x \tilde{F}(x) P(x, x)^{-1} \right)^{-1/2} P(x, x)^{-1} P(x, \tilde{F}(x)) \tilde{\Psi}(x). \]

Using again that in our Dirac examples, \( P(x, x) \) has the form (7.15), we can simplify this formula according to (7.16) to obtain the perturbation expansion in the symmetric wave gauge
\[ \tilde{\Psi}_\Omega(x) = U_x \gamma^0 A_x^{-1/2} \tilde{F}(x) P(x, \tilde{F}(x)) \tilde{\Psi}(x). \quad (7.25) \]

In order to understand what this formula means, it is useful to choose an orthonormal basis \( u_1, \ldots, u_4 \) of the subspace \( S_x \subset \mathcal{H} \) (orthonormal with respect to the scalar product \( \langle \cdot | \cdot \rangle_{\mathcal{H}} \)). Then for any \( y \in \mathcal{F} \),
\[ P(y, x) = \pi_y x |_{S_x} = \frac{4}{a=1} \pi_y u_a (u_a | x) |_{S_x} = -\frac{4}{a=1} |u_a(y)\rangle\langle u_a(x)|, \quad (7.26) \]
where in the last step as in (7.12) we again applied [8, Proposition 1.2.7] and used the identification (7.9). Choosing \( y = \tilde{F}(x) \), we obtain the simple formulas
\[ P(x, \tilde{F}(x)) = -\sum_{a=1}^4 |u_a(x)\rangle\langle u_a(x)| \quad (7.27) \]
\[ A_x \tilde{F}(x) = -\sum_{a,b=1}^4 |u_a(x)\rangle\langle u_a(x)|\tilde{u}_b(x)\rangle\langle u_a(x)|. \quad (7.28) \]

This shows that the formula (7.25) can be expressed purely in terms of the unperturbed and perturbed wave functions \( u_a \) and \( \tilde{u}_a \), all evaluated at the space-time point \( x \).

This raises the question how the wave functions \( u_a \) and \( \tilde{u}_a \) look like. Indeed, this can be read off from (7.26):

**Lemma 7.9.** For Dirac systems in Minkowski space,
\[ u_a(y) = P(y, x) \chi_a \quad (7.29) \]
\[ \tilde{u}_a(x) = P(\tilde{F}(x), x) \chi_a, \quad (7.30) \]
where \( \chi_a \in S_x \mathcal{M} \) are the spinors
\[ \chi_a = \frac{1}{\alpha} \gamma^0 u_a(x), \quad a = 1, \ldots, 4. \quad (7.31) \]

**Proof.** Multiplying (7.26) by a spinor \( \chi \in S_x \mathcal{M} \), we obtain
\[ \sum_{b=1}^4 c_b u_b(y) = P(y, x) \chi \quad (7.32) \]
with coefficients \( c_b = -\langle u_b(x)|\chi\rangle \). Hence the Dirac wave functions \( u_a(y) \) are obtained by multiplying \( P(y, x) \) with suitable spinors. In order to prove (7.30), it remains to verify that choosing \( \chi = \chi_a \) according to (7.31), the linear combination on the left of (7.32) gives the wave function \( u_a \). To this end, it suffices to evaluate (7.32) for \( y = x \),

\[
\sum_{b=1}^{4} c_b u_b(x) = P(x, x) \chi = \alpha \gamma^0 \chi,
\]

where in the last step we applied (7.15). Using (7.31) gives

\[
\sum_{b=1}^{4} c_b u_b(x) = u_a(x),
\]

concluding the proof of (7.29).

The identity (7.30) follows from (7.29) by comparing (7.26) with (7.27).

We now compute the wave functions corresponding to the vectors \( u_a \) in the symmetric wave gauge.

**Proposition 7.10.** In the gauge (7.25), the vectors \( u_1, \ldots, u_4 \) which form an orthonormal basis of \( S_x \) have the form

\[
\tilde{\Psi}^\Omega_V(x) u_a = U_x \gamma^0 A^{\frac{1}{2}}_{x, \tilde{F}(x)} \chi_a.
\]

**Proof.** Using (7.30) in (7.25) gives

\[
\tilde{\Psi}^\Omega_V(x) u_a = U_x \gamma^0 A^{\frac{1}{2}}_{x, \tilde{F}(x)} P(x, \tilde{F}(x)) \tilde{u}_a(x)
\]

\[
= U_x \gamma^0 A^{\frac{1}{2}}_{x, \tilde{F}(x)} P(x, \tilde{F}(x)) P(\tilde{F}(x), x) \chi_a
\]

\[
= U_x \gamma^0 A^{\frac{1}{2}}_{x, \tilde{F}(x)} A_{x, \tilde{F}(x)} \chi_a,
\]

giving the result. □

Our gauge-fixing procedure can be understood directly by comparing the wave functions of the vectors \( u_a \) without gauge fixing (7.30) with those in the symmetric wave gauge (7.33). In (7.30), the wave functions are modified by the electromagnetic potential. In particular, for a gauge transformation, this gives rise to the local phase in (7.24). The formula (7.33), on the other hand, involves instead of \( P(\tilde{F}(x), x) \) the matrix \( A^{\frac{1}{2}}_{x, \tilde{F}(x)} \). This matrix does not involve gauge phases, because the close chain is gauge invariant according to (7.28). The matrix \( \gamma^0 \) is needed in order to get agreement of (7.30) and (7.33) in the case when no electromagnetic potential is present.

In this way, our gauge-fixing procedure brings the wave functions corresponding to the vectors \( u_a \) into a canonical form. The point is that by doing so, the \( U(2, 2) \)-gauge freedom at the space-time point \( x \) is exhausted completely. Therefore, the wave functions corresponding to all other vectors in \( H \) at \( x \) are also determined uniquely.

We conclude by giving an intuitive picture of how the Dirac waves \( u_a \) look like in position space and outline the methods for analyzing their perturbations. According to (7.29), the space-time dependence of these waves is the same as that of the kernel of the fermionic projector \( P(y, x) \) for fixed \( x \). The unregularized kernel \( P(y, x) \) has **singularities** if \( y \) lies on the light cone centered at \( x \) (for details see for example [8].
Due to the regularization, these singularities are mollified on the scale $\varepsilon$. This means that, for small $\varepsilon$, the Dirac waves $u_\alpha(y)$ are peaked near the light cone centered at $y$. Qualitatively speaking, these waves can be regarded as wave packets of negative frequency which are as far as possible localized at time $t = x^0$ at the spatial point $\tilde{x}$. Clearly, in view of Hegerfeldt’s theorem [18] (see also [26, Section 1.8.3]), wave packets of negative frequency cannot be localized in space. This is also apparent here because, similar to the Feynman propagator, the distribution $P(y,x)$ does have a contribution if $x$ and $y$ are spatially separated, but this contribution decays exponentially in the spatial distance. More details on the waves $u_\alpha$ and related results on Dirac systems in Minkowski space can be found in [22].

The light-cone expansion is a powerful computational tool for analyzing the kernel $\tilde{P}(x,y)$ in position space (see [4, 5] or the introduction in [8, Section 2.2]). The resulting formulas show that the electromagnetic potential changes $P(x,y)$ by gauge phases and also by contributions involving the field tensor and its derivatives. More precisely, the unregularized distribution $\tilde{P}(x,y)$ can be expressed by an infinite sum of distributions which have singularities on the light cone, each multiplied by an integral over potentials or fields along the line segment $\overline{xy}$. The regularized kernel is then obtained by mollification (for details see [8, Appendix F]). The light-cone expansion of $P(x,\tilde{F}(y))$ is more involved because it typically involves unbounded line integral along the straight line joining the points $x$ and $y$. This is worked out in [6, Appendix F]; see also [9, Lemma 5.1]. However, these results give information on $\tilde{P}(x,\tilde{F}(x))$ only if $x \neq y$. Therefore, these results unfortunately do not apply to the regularized kernel $P(x,\tilde{F}(x))$ as needed for the symmetric wave gauge (7.25). At present, the only rigorous result is that a pure gauge potential $\tilde{A} = \partial \Lambda$ gives rise to a gauge phase,

$$P(x,\tilde{F}(x)) = e^{-i\Lambda(x)} P(x,x).$$

This suggests that the leading order in $\varepsilon/l_{\text{macro}}$ (where $l_{\text{macro}}$ denotes the macroscopic length scale determined by the Compton scale and typical wave lengths of the electromagnetic field) should also simply give a gauge phase. It seems a promising strategy for computing the higher orders in an expansion $\varepsilon/l_{\text{macro}}$ to work in momentum space (similar to [4, Section 3]) and to integrate over both the incoming and outgoing momenta. However, the detailed computations are somewhat technical and go beyond the scope of the present paper.

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