On Reciprocity of Twisted Alexander Invariants

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Abstract

Given a knot and an $\text{SL}_n \mathbb{C}$ representation of its group that is conjugate to its dual, the representation that replaces each matrix with its inverse-transpose, the associated twisted Reidemeister torsion is reciprocal. An example is given of a knot group and $\text{SL}_3 \mathbb{Z}$ representation that is not conjugate to its dual for which the twisted Reidemeister torsion is not reciprocal.

Keywords: Knot, twisted Reidemeister torsion, twisted Alexander polynomial

1 Introduction

The Alexander polynomial $\Delta(t)$ of a knot $k$ can be computed from a diagram of $k$ or from a presentation of the knot group (see [5], for example). It is an integral Laurent polynomial, well defined up to multiplication by units $\pm t^i \in \mathbb{Z}[t^\pm 1]$, and it is usually normalized to be a polynomial with nonzero constant coefficient.

It is well known that $\Delta(t)$ is reciprocal in the sense that

$$\Delta(t^{-1}) = \Delta(t),$$

(1.1)
where \( \doteq \) indicates equality up to multiplication by units. This is a consequence of Poincaré duality of the knot exterior (see [14] for an alternative approach based on duality in the knot group).

In 1990 X.S. Lin introduced a more sensitive invariant using information from nonabelian representations of the knot group [9]. Later, refinements were described by M. Wada [15] and others including P. Kirk and C. Livingston [6], J. Cha [1], and others. These twisted Alexander invariants have proven to be useful for a variety of questions about knots including questions about concordance [6], knot symmetry [4] and fibrations [3]. See [2] for a survey.

We briefly review the definition of perhaps the best-known twisted Alexander invariant. Let \( k \) be a knot with exterior \( X \), endowed with the structure of a CW complex. We fix a Wirtinger presentation \( \langle x_0, x_1, \ldots, x_k \mid r_1, \ldots, r_k \rangle \) for the knot group \( \pi = \pi_1(X) \). Let \( \phi : F_k \to \pi \) be the associated projection of the free group \( F_k = \langle x_0, x_1, \ldots, x_k \mid \rangle \) to \( \pi \). It induces a ring homomorphism \( \tilde{\phi} : \mathbb{Z}[F_k] \to \mathbb{Z}[\pi] \).

Let \( \epsilon : \pi \to \mathbb{H}_1(X; \mathbb{Z}) \cong \langle t \mid \rangle \) be the abelianization mapping each \( x_i \) to \( t \). It induces a ring homomorphism \( \tilde{\epsilon} : \mathbb{Z}[\pi] \to \mathbb{Z}[t^{\pm 1}] \).

Assume that \( \gamma : \pi \to \text{SL}_n \mathbb{C} \) is a linear representation. Let \( \tilde{\gamma} : \mathbb{Z}[\pi] \to M_n(\mathbb{C}) \) be the associated ring homomorphism to the algebra of \( n \times n \) matrices over \( \mathbb{C} \). We obtain a homomorphism

\[
\tilde{\gamma} \otimes \tilde{\epsilon} : \mathbb{Z}[\pi] \to M_n(\mathbb{C}[t^{\pm 1}]),
\]

mapping \( g \) to \( t^{\epsilon(g)} \gamma(g) \), that we denote more simply by \( \Phi \).

Let \( M_{\gamma \otimes \epsilon} \) denote the \( k \times (k + 1) \) matrix with \((i, j)\)-component equal to the \( n \times n \) matrix \( \Phi(\partial r_{ij}) \in M_n(\mathbb{C}[t^{\pm 1}]) \). Here \( \partial r_{ij} \) denotes Fox partial derivative. Let \( M^0_{\gamma \otimes \epsilon} \) denote the \( k \times k \) matrix obtained by deleting the column corresponding to \( x_0 \). We regard \( M^0_{\gamma \otimes \epsilon} \) as a \( kn \times kn \) matrix with coefficients in \( \mathbb{C}[t^{\pm 1}] \).

**Definition 1.1.** The Wada invariant \( W_\gamma(t) \) is

\[
\frac{\det M^0_{\gamma \otimes \epsilon}}{\det \Phi(x_0 - 1)}.
\]

When \( \gamma \) is the trivial 1-dimensional representation, \( M^0_{\gamma \otimes \epsilon} \) is a matrix \( M(t) \) that we call the Alexander matrix of \( k \). (This terminology is used, for example, in [12], but it is not standard.) The determinant of \( M(t) \) is the (untwisted) Alexander polynomial \( \Delta(t) \) of \( k \).
Remark 1.2. The rational function $W_\gamma(t)$ need not be a polynomial. See [15].

The matrix $M_\gamma \otimes \epsilon$ represents a boundary homomorphism for a twisted chain complex
\[ C_*(X; V[t^\pm 1]) = (\mathbb{C}[t^\pm 1] \otimes_{\mathbb{C}} V) \otimes_\gamma C_*(\tilde{X}). \] (1.3)
Here $V = \mathbb{C}^n$ is a vector space on which $\pi$ acts via $\gamma$, while $C_*(\tilde{X})$ denotes the cellular chain complex of the universal cover $\tilde{X}$ with the structure of a CW complex that is lifted from $X$. The group ring $\mathbb{Z}[\pi]$ acts on the left via deck transformations. On the other hand, $C_*(t^\pm 1) \otimes_{\mathbb{C}} V$ has the structure of a right $\mathbb{Z}[\pi]$-module via
\[(p \otimes v) \cdot g = (pt^e(g)) \otimes (v\gamma(g)), \quad \text{for } \gamma \in \pi.\]

Remark 1.3. The homology group $H_1(X; V[t^\pm 1])$ of the chain complex (1.3) is a finitely generated $\mathbb{C}[t^\pm 1]$-module. Its 0th elementary divisor, $\Delta_\gamma(t)$, lately competes with $W_\gamma(t)$ for the name “twisted Alexander polynomial.” In many cases they are equal; generally, $\Delta_\gamma(t)$ is $\det M_0 \otimes \epsilon$ divided by a factor of $\det \Phi(x_0 - 1)$. See [6] or [13] for details.

Let $\mathbb{C}(t)$ denote the field of rational functions. When $\det M_0 \otimes \epsilon \neq 0$, the chain complex
\[ C_*(X; V(t)) = (\mathbb{C}(t) \otimes_{\mathbb{C}} V) \otimes_\gamma C_*(\tilde{X}) \] (1.4)
is acyclic [7], and hence the (Reidemeister) torsion $\tau_\gamma(t)$ is defined. In [6] it is shown that $\tau_\gamma(t)$ coincides with the Wada invariant $W_\gamma(t)$.

Remark 1.4. Conjugating the representation $\gamma$ corresponds to a change of basis for $V$. It is well known that the invariants $W_\gamma(t), \Delta_\gamma(t)$ and $\tau_\gamma(t)$ are unchanged.

T. Kitano used Poincaré duality to prove in [7] that for orthogonal representations $\gamma : \pi \to \text{SO}_n(\mathbb{R})$, the torsion $\tau_\gamma(t)$ is reciprocal; that is, $\tau_\gamma(t^{-1}) = \tau_\gamma(t)$. He asked whether reciprocity holds for general representations $\gamma : \pi \to \text{SL}_n(\mathbb{C})$. The question appeared more recently in [2].

Several years later, Kirk and Livingston showed in [6] that reciprocity holds whenever $\gamma$ is unitary. In particular, it holds for all representations with finite image.

It is not difficult to find representations $\gamma : \pi \to \text{GL}_n\mathbb{C}$ such that $\tau_\gamma(t)$ is non-reciprocal. For example, consider the Wirtinger presentation
\( \langle x_0, x_1, x_2 | x_0x_1 = x_2x_0, x_1x_2 = x_0x_1 \rangle \) of the trefoil knot group \( \pi \). The assignment \( x_i \mapsto X_i \in \text{GL}_2 \mathbb{C} \), such that

\[
X_0 = \begin{pmatrix} a & 0 \\ 1 & 1 \end{pmatrix}, \quad X_1 = \begin{pmatrix} a & -(a^2 - a + 1) \\ 0 & 1 \end{pmatrix}, \quad X_2 = X_1^{-1}X_0X_1
\]
yields \( \tau_\gamma(t) = at^2 + 1 \). The question of reciprocality for representations in \( \text{SL}_n \mathbb{C} \) is more subtle.

In Section 2 we show that reciprocality need not hold for general representations in \( \text{SL}_n \mathbb{C} \). The representations \( \gamma \) that we consider have the property that the dual representation \( \overline{\gamma} \), obtained by replacing each matrix \( \gamma(g), g \in \pi \), by its inverse-transpose, is not conjugate to \( \gamma \). We wish to thank Walter Neumann for suggesting to us that such a representation might yield non-reciprocal torsion.

In Section 3 we prove that if a representation \( \gamma : \pi \to \text{SL}_n \mathbb{C} \) is conjugate to its dual, then the torsion \( \tau_\gamma(t) \) is reciprocal.

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### 2 Examples

Any reciprocal even-degree integral polynomial \( \Delta(t) \) such that \( \Delta(1) = \pm 1 \) arises as the Alexander polynomial of a knot (see [5], for example). Let \( f(t) \) be any monic integral polynomial with constant coefficient \(-1\) and \( f(1) = \pm 1 \). Choose a knot \( k \) with Alexander polynomial \( \Delta(t) = f(t)f(t^{-1}) \).

Let \( C \) be the companion matrix of \( (t-1)f(t) \). Then \( C \in \text{SL}_n \mathbb{Z} \), where \( \deg f = n - 1 \). Consider the cyclic representation \( \gamma : \pi \to \text{SL}_n \mathbb{Z} \) sending each generator \( x_0, x_1, \ldots, x_k \) of a Wirtinger presentation of \( \pi \) to \( C \). We have

\[
\tau_\gamma(t) \overset{\text{def}}{=} \frac{\det M^0_\overline{\gamma} \otimes \epsilon}{\det \Phi(x_0 - 1)} \overset{\text{def}}{=} \frac{\det M^0_\gamma \otimes \epsilon}{f(t^{-1})(t - 1)}. \tag{2.1}
\]

The matrix \( M^0_\gamma \otimes \epsilon \) can be obtained from the \((k \times k)\) Alexander matrix \( M(t) \) by replacing each polynomial entry \( \sum a_i t^i \) with the \((n \times n)\) block matrix \( \sum a_i (tC)^i \). Since the \( n \times n \) blocks commute,

\[
\det M^0_\gamma \otimes \epsilon = \prod_\lambda \det M(t\lambda),
\]

where \( \lambda \) ranges over the eigenvalues of \( C \), that is, the roots of \((t-1)f(t)\) (see [8] for details). Hence

\[
\det M^0_\gamma \otimes \epsilon = \prod_\lambda \Delta(t\lambda) = \Delta(t) \prod_{\lambda : f(\lambda) = 0} f(t\lambda)f(t^{-1}\lambda^{-1}).
\]
Since $\Delta(t)$ and $\det M_0^{\gamma\otimes \epsilon}(t)$ are integral polynomials, so is

$$g(t) = \prod_{\lambda: f(\lambda)\neq 0} f(t\lambda)f(t^{-1}\lambda^{-1}).$$

**Lemma 2.1.** If $\deg f = 2$, then $g(t)$ is reciprocal.

**Proof.** Our assumptions about $f(t)$ imply that its roots have the form $\lambda, -\lambda^{-1}$, for some $\lambda \in \mathbb{C} \setminus \{0\}$. Then $g(t) = f(t\lambda)f(t^{-1}\lambda^{-1})f(-t\lambda^{-1})f(-t^{-1}\lambda)$ while $g(t^{-1}) = f(t^{-1}\lambda)f(t\lambda^{-1})f(-t^{-1}\lambda^{-1})f(-t\lambda)$. Observe that $g(t)$ and $g(t^{-1})$ have the same roots:

- $f(t\lambda)$ and $f(-t^{-1}\lambda^{-1})$ have roots: $t = 1, -\lambda^{-2}$;
- $f(t^{-1}\lambda^{-1})$ and $f(-t\lambda)$ have roots: $t = -1, \lambda^{-2}$;
- $f(-t\lambda^{-1})$ and $f(t^{-1}\lambda)$ have roots: $t = 1, -\lambda^{2}$;
- $f(-t^{-1}\lambda)$ and $f(t\lambda^{-1})$ have roots: $t = -1, \lambda^{2}$.

It follows that $g(t^{-1}) = \alpha g(t)$, for some $\alpha \in \mathbb{C} \setminus \{0\}$. Letting $t = 1$, we see that $\alpha = 1$. Hence $g(t^{-1}) = g(t)$.

**Remark 2.2.** The numerator $\det M_0^{\gamma\otimes \epsilon}$ of (1.1) is a polynomial invariant $D_\gamma(t)$ of $k$ (see [13]). Since $\Delta(t)$ is reciprocal, Lemma 2.1 implies that $D_\gamma(t)$ is reciprocal whenever $\deg f = 2$. Example 2.5 below shows that this conclusion need not hold when $\deg f > 2$.

**Proposition 2.3.** Let $f(t)$ be a polynomial as above with degree 2. If $f(t)$ is non-reciprocal, then $\tau_\gamma(t)$ is a non-reciprocal integral polynomial of the form $(t - 1)h(t)$.

**Proof.** From equation (2.1)

$$\tau_\gamma(t) = \frac{f(t)f(t^{-1})g(t)}{f(t^{-1})(t-1)} = \frac{f(t)g(t)}{t-1}. \quad (2.2)$$

Since $g(t)$ and $t - 1$ are reciprocal but $f(t)$ is not, $\tau_\gamma(t)$ is non-reciprocal. To see that $\tau_\gamma(t)$ has the desired form, note that $(t - 1)^2$ divides $g(t)$ since both factors $f(t\lambda), f(-t\lambda^{-1})$ of $g(t)$ vanish when $t = 1$.

\[\square\]
Example 2.4. Let $f(t) = t^2 - t - 1$. Then

\[
C = \begin{pmatrix}
0 & 0 & -1 \\
1 & 0 & 0 \\
0 & 1 & 2
\end{pmatrix}.
\]

Computation shows that $g(t) = (t - 1)^2(t + 1)^2(t^2 - 3t + 1)(t^2 - 3t + 1)$. By equation (2.2),

\[
\tau_\gamma(t) = (t^2 - t + 1)(t - 1)(t + 1)^2(t^2 - 3t + 1)(t^2 + 3t + 1),
\]

which is non-reciprocal.

Example 2.5. Let $f(t) = t^3 - t - 1$. Then

\[
C = \begin{pmatrix}
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix}.
\]

Computation shows that $g(t) = (t - 1)^3(t^3 - t - 1)^2(t^3 - t^2 + 2t - 1)(t^6 + 3t^5 + 5t^4 + 5t^3 + 5t^2 + 3t + 1)$. The polynomial $f(t)f(t^{-1})g(t)$ is the numerator $D_\gamma(t)$ of Wada’s invariant (1.1). It is non-reciprocal.

It is not difficult to see that for any cyclic representation, $D_\gamma(t) \cong \Delta_\gamma(t)$ (see Section 3 of [13]). Hence this example shows that $\Delta_\gamma(t)$ can also be non-reciprocal.

3 Sufficient condition for reciprocality

If $\gamma : G \to \text{GL}_n\mathbb{F}$ is a linear representation, then the dual (or contragredient) representation $\bar{\gamma}$ is defined by

\[
\bar{\gamma}(g) = t\gamma(g)^{-1},
\]

where $t$ denotes transpose.

The following elementary lemma is included for the reader’s convenience.

Lemma 3.1. A representation $\gamma : G \to \text{GL}_n\mathbb{F}$ is conjugate to its dual if and only if there exists a nondegenerate bilinear form $(v, w) \mapsto \{v, w\} \in \mathbb{F}$ on $V$ such that $\{v \cdot g, w \cdot g\} = \{v, w\}$ for all $v, w \in V$ and $g \in G$. 

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**Proof.** Assume that \( \bar{\gamma} \) is conjugate to \( \gamma \). Then there exists a matrix \( A \in \text{GL}_n\mathbb{F} \) such that \( A^{-1}\gamma(g)A = t\gamma(g)^{-1} \), for all \( g \in G \). Define \( \{v, w\} = vA^{-1}w \). Since \( A \) is invertible, the bilinear form is nondegenerate. It is easy to check that \( \{v \cdot g, w \cdot g\} = \{v, w\} \) for all \( v, w \in V \).

Conversely, assume that \( \gamma \) preserves a nondegenerate bilinear form \( (v, w) \mapsto \{v, w\} \). There exists an invertible matrix \( A \in \text{GL}_n\mathbb{F} \) such that \( \{v, w\} = vA^{-1}w \). Since \( \gamma \) preserves the form, we have \( v\gamma(g)A^{-1}\gamma(g)^{-1}w = \{v \cdot g, w \cdot g\} = \{v, w\} = vA^{-1}w \), for all \( v, w \in V, g \in G \). It follows that \( \gamma(g)A^{-1}\gamma(g)^{-1}A = \gamma(g)A^{-1}A \), for all \( g \in G \). Hence \( A^{-1}\gamma(g)A = t\gamma(g)^{-1} \), and so \( \bar{\gamma} \) is conjugate to \( \gamma \).

\[\blacksquare\]

As before, let \( k \) be a knot with group \( \pi \). Assume that \( \gamma : \pi \to \text{SL}_n\mathbb{F} \) is a representation, where \( \mathbb{F} \) is an arbitrary field. As above, \( V = \mathbb{F}^n \) is a right \( \mathbb{Z}[\pi] \)-module via \( v \cdot g = v\gamma(g) \), for all \( v \in V \) and \( \gamma \in \pi \). Let \( W = \mathbb{F}^n \) with the dual \( \mathbb{Z}[\pi] \)-module structure given by \( w \cdot g = w^\ast \gamma(t)^{-1} \).

**Theorem 3.2.** Assume that \( \det M^0_{\gamma \otimes t} \neq 0 \). If \( \gamma \) is conjugate to its dual representation \( \bar{\gamma} \), then the torsion \( t_\gamma(t) \) is reciprocal.

**Proof.** The following argument is similar to those of [7] and [6].

Recall that \( X \) is the exterior of \( k \), endowed with a CW cell structure. Let \( X' \) be the same space but with the dual cell structure. Let \( \bar{\gamma} : \mathbb{F}(t) \to \mathbb{F}(t) \) be the involution induced by \( t \mapsto t^{-1} \).

Assume that \( \gamma : \pi \to \text{SL}_n\mathbb{F} \) is a representation that is conjugate to its dual. By Lemma 3.1 there exists a nondegenerate bilinear form \( (v, w) \mapsto \{v \cdot g, w \cdot g\} \) such that \( \{v \cdot g, w \cdot g\} = \{v, w\} \) for all \( v, w \in V, g \in \pi \). Consider the twisted chain complexes

\[ C_\ast = (\mathbb{F}(t) \otimes V) \otimes C_\ast(\tilde{X}), \quad D_\ast = (\mathbb{F}(t) \otimes W) \otimes C_\ast(\tilde{X}', \partial \tilde{X}'), \]

where \( \tilde{X} \) and \( \tilde{X}' \) denote universal covering spaces of \( X \) and \( X' \), respectively. We abbreviate these by \( V_{\gamma \otimes t} \otimes C_\ast(\tilde{X}) \) and \( V_{\gamma \otimes t} \otimes C_\ast(\tilde{X}) \), respectively.

Define a bilinear pairing \( C_q \times D_{3-q} \to F(t) \) by

\[ \langle p \otimes v \otimes z_1, q \otimes w \otimes z_2 \rangle = \sum_{g \in \pi} (z_1 \cdot g z_2) pq \{v \cdot g, w\}, \quad (3.1) \]

where \( z_1 \cdot g z_2 \) is the algebraic intersection number in \( \mathbb{Z} \) of cells \( z_1 \) and \( g z_2 \). We extend linearly.

The pairing induces a \( \mathbb{F}(t) \)-module isomorphism \( D_{3-q} \to \text{Hom}(C_q, \mathbb{F}(t)) \), where \( \text{Hom} \) denotes the dual space with \( (q \cdot h)(z) = \bar{q}(h(z)) \), for all \( q \in \mathbb{F}(t), z \in C_q \). Consequently, there exists a nondegenerate pairing \( H_q(\tilde{X}; V(t)) \times \)
Choose a basis \( \{ v_i \} \) over \( \mathbb{F} \) for \( V \) and lifts to \( \tilde{X} \) of simplices of \( X \). In this way, we obtain a preferred \( \mathbb{F}(t) \)-basis for \( C_* \). Basis members have the form \( 1 \otimes v_i \otimes z_j \). We get a natural basis over \( \mathbb{F}(t) \) for \( D_* \) by picking a basis for \( W \) that is dual to the basis for \( V \) with respect to \( \{ , \} \), and choosing dual cells in \( \tilde{X}' \) of the fixed lifts of simplices of \( X \). As observed in [6], the bases for \( C_* \) and \( D_* \) that we build are dual with respect to the bilinear form (3.1).

Let \( \tau(X; V_{\gamma \otimes \epsilon}) \) denote the torsion of \( C_* \). Similarly, let \( \tau(X', \partial X'; V_{\bar{\gamma} \otimes \epsilon}) \) denote the torsion of \( D_* \). Then \( \tau(X; V_{\gamma \otimes \epsilon}) = \tau(X', \partial X'; V_{\bar{\gamma} \otimes \epsilon}) \) by Theorem 1' of [10]. Furthermore,

\[
\tau(X', \partial X'; V_{\bar{\gamma} \otimes \epsilon}) = \tau(X, \partial X; V_{\bar{\gamma} \otimes \epsilon}) \quad \text{(by subdivision)}
\]

\[
= \tau(X, \partial X; V_{\bar{\gamma} \otimes \epsilon}) \quad \text{(since } \gamma \text{ is conjugate to } \bar{\gamma})
\]

\[
= \tau(X, \partial X; V_{\gamma \otimes \epsilon})
\]

\[
= \tau(X; V_{\gamma \otimes \epsilon}).
\]

The last equality is a result of Lemma 2 of [11] and the fact that \( \tau(\partial X; V_{\gamma \otimes \epsilon}) = 1 \) (see [6]). Hence

\[
\tau_\gamma(t) = \tau(X; V_{\gamma \otimes \epsilon}) = \bar{\tau}(X; V_{\gamma \otimes \epsilon}) = \bar{\tau}_\gamma(t).
\]

Remark 3.3. If \( \mathbb{F} = \mathbb{R} \), and the bilinear form in Lemma 3.1 is positive-definite, then by considering a basis for \( V \) that is orthonormal with respect to the form, we see that \( A \) is the identity matrix. In this case, \( \gamma(g) = {}^t\gamma(g)^{-1} \) for all \( g \in G \), and hence \( \gamma \) is conjugate to an orthogonal representation. Similarly, if \( \mathbb{F} = \mathbb{C} \) and the bilinear form is hermitian and positive-definite, \( \gamma \) is conjugate to a unitary representation.

Corollary 3.4. If \( \gamma: \pi \to \text{Sp}_{2n}\mathbb{C} \) is a symplectic representation, then \( \tau_\gamma(t) \) is reciprocal.

Proof. The representation preserves the bilinear form given by \( A = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} \).

Since \( \text{Sp}_2\mathbb{C} = \text{SL}_2\mathbb{C} \), the following is immediate.

Corollary 3.5. If \( \gamma \) is any representation of \( \pi \) in \( \text{SL}_2\mathbb{C} \), then \( \tau_\gamma(t) \) is reciprocal.

Corollary [3.5] shows that Example 2.4 is, in a sense, the simplest possible.
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