Self-organization in trees and motifs of two-dimensional chaotic maps with time delay

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Abstract. We study two-dimensional chaotic standard maps coupled along the edges of scale-free trees and tree-like subgraphs (4-star) with a non-symplectic coupling and time delay between the nodes. Apart from the chaotic and regular 2-periodic motion, the coupled map system exhibits a variety of dynamical effects in a wide range of coupling strengths. This includes dynamical localization, emergent periodicity and the appearance of strange non-chaotic attractors. Near the strange attractors we find long-range correlations in the intervals of return times to specified parts of the phase space. We substantiate the analysis with the finite-time Lyapunov stability. We also give some quantitative evidence of how the small-scale dynamics at 4-star motifs participates in the genesis of the collective behavior at the whole network.

Keywords: self-organized criticality (theory), network dynamics, new applications of statistical mechanics
1. Introduction

Network dynamics, diffusion and chaos. Complex dynamical systems can be efficiently modeled by network topologies where the edges represent the interactions between the dynamical variables attached to the network nodes [1]–[3]. Various dynamical processes on the networks have been studied in which the temporal fluctuations in the dynamical variables constrained by the network structure are subject to the inter-node interactions and/or driving by the external noise. The interplay between the network topology and emergent dynamical behavior can be readily demonstrated in diffusive processes on scale-free networks [3]. Of particular importance for the dynamics are the networks with well-developed modular structures or topological subgraphs, which determine their own timescales [4]. Such structures are often present in biological systems, in particular, in gene interaction networks, where structural modules, or dynamical motifs [5], represent gene functional units mutually connected through the network. Although the cooperativity in gene functions can be easily demonstrated, e.g. with the appropriate statistical analysis [6] of the empirical gene-expression data, both the dynamics of gene motifs and the structure of gene networks remains a challenging problem. Recently, an application of discrete-time dynamics has been suggested for modeling gene interactions [7]–[9]. This approach brings a new prospective, in particular with respect to the advanced stability analysis and an intriguing possibility of a chaotic behavior.

The coupled chaotic maps on regular structures [10]–[14], structured small graphs [15]–[17], random [18] and scale-free networks [19]–[23] are currently the subject of an intensive study. Many of the known one-dimensional chaotic maps were studied with the emphasis on the effects of coupling strengths and of the time delay [24]–[26] between the units. The central points in these studies are the phenomena related with the synchronization of spatially extended coupled maps [12, 14, 17, 19, 22], [24]–[26], nature of the transition to the synchronized state [11] and coexistence of different types of attractors [15].

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Generally, coupled chaotic maps are interesting dynamical systems in which the
dynamic stability can be examined with respect to variations in:

- initial conditions (the trajectory divergence);
- parameters (coupling at a node, coupling between the units, geometry, time delay);
- drivings (external frequencies, endogenous driving).

In this respect, two-dimensional coupled chaotic maps, which are much less studied so
far [12,16,23], leave more room for different types of coupling and, consequently, new
dynamical effects. In particular, in a system of standard maps with symplectic coupling
on a full graph trapping of the trajectories and anomalous diffusion was found [16]. In the
coupling with time delay on scale-free trees the standard maps lead to characteristic
dynamical patterns with robust statistical properties [23]. The type of the coupling
between two degrees of freedom for a general one-dimensional map coupled along a linear
chain was shown [12] to affect the transition to the synchronized state.

In this work we study the discrete versions of two-dimensional periodically kicked
oscillators, usually referred to as the Chirikov standard maps [27], coupled along their
phase variables on large scale-free trees and their typical subgraph, the 4-star motif. As
described in detail below, we impose a fixed time delay between coupled neighboring units
(maps), thus rendering the coupled maps system non-symplectic. Our main purpose in
this paper is to characterize the dynamic behavior that emerges as a consequence of the
inter-node coupling both at small-scale structure—the 4-star motif—and at the large-
scale structure—the tree with \( N = 1000 \) nodes. As will be explained in detail later, we
find three dynamical regimes with different behaviors and determine their characteristic
statistical properties and dynamical stability of the coupled maps.

This paper is organized as follows. After defining our coupled map system in section 2,
we report the qualitative and statistical features of the coupled maps in section 3. In
section 4 we investigate the dynamic stability and describe behavior near the fractal
attractors. Finally in section 5 we give a short summary and discussion of the results.

2. The graphs of coupled maps

We consider the standard map [27] in its original unbounded version, with \( x \) being the
phase (angle) and \( y \) being the action (momentum) variable defined on each node \([i]\) of a
network with \( N \) nodes:

\[
\begin{pmatrix}
x'\[i]\[i]
y'\[i]\[i]
\end{pmatrix} = \begin{pmatrix}
x\[i] + y\[i] + \varepsilon \sin(2\pi x\[i]) \mod 1
y\[i] + \varepsilon \sin(2\pi x\[i])
\end{pmatrix}.
\]

(1)

This map has been extensively studied as an isolated system and its properties are well
known, in particular the ones regarding the phase space diffusion for \( \varepsilon \) values above the
chaotic transition threshold (\( \varepsilon_c \sim 0.155 \)). This is a symplectic map (a discrete version
of a Hamiltonian system) which has a very intricate phase space structure with invariant
KAM-torai persisting below the \( \varepsilon_c \) value [28].

We define our coupled map system (CMS) in accordance with the usually adopted
scheme [20] as: \((1 - \mu) \times \) (standardmap-update) + \( \mu \times \) (coupling). Also in analogy with
the previous works on 1D maps \([13,20,29]\), we consider the diffusive phase-coupling in
the $x$ variable of equation (1). A one-step time delay between the neighboring nodes is imposed, so that our CMS is

$$
\begin{pmatrix}
  x'[i]_{n+1} \\
  y'[i]_{n+1}
\end{pmatrix}
= (1 - \mu)
\begin{pmatrix}
  x'[i]_n \\
  y'[i]_n
\end{pmatrix}
+ \frac{\mu}{k_i}
\begin{pmatrix}
  \sum_{j \in \mathcal{K}_i} (x[j]_n - x'[j]_n) \\
  0
\end{pmatrix}.
$$

Here, $'$ denotes the next iterate of equation (1), $n$ is the global discrete time, $\mu$ is the coupling strength, $[i]$ indexes the nodes on the graph, $k_i$ is the node degree and $\mathcal{K}_i$ denotes the neighborhood of the node $[i]$. The update of each node is the sum of a contribution given by its standard map update (the $'$ part) plus a coupling contribution given by the sum of differences between the node’s phase value and the phase values of neighboring nodes in the previous iteration, normalized by the node’s degree. The couplings on the 4-star are schematically shown in figure 1(a).

In contrast to the well-studied phase-coupled 1D oscillators [13, 20, 29], our CMS defined by equations (1)–(2) is essentially a network of time-delayed phase-coupled 1D oscillators with an additional coupling to the local momentum variable at each node. From a continuous-time viewpoint, seeing $x'(i, t) \equiv x[i]'(t)$ as a continuous variable over a discrete space–time defined by the network, this coupling functional form can be seen as

$$
x'(i, t + 1) = (1 - \mu)x[i]' + \frac{\mu}{k_i} \left[ \sum_{j \in \mathcal{K}_i} (x[j]' - x'[j]) + \sum_{j \in \mathcal{K}_i} (x[j] - x[j]') \right] + \frac{\mu}{k_i} \left[ \nabla^2 x[i] - \sum_{j \in \mathcal{K}_i} \partial_t x[j]' \right] \quad \approx \quad \left( 1 - \mu \right) x[i]' + \frac{\mu}{k_i} (\nabla^2 - k_i \partial_t) x'(i, t)
$$

$$
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$$
which represents a sort of diffusive phase-coupling on the network. In comparison to
the diffusive CMS that are typically studied on chains (e.g. [12]), our model involves
2D elements on a network with a time delay, causing additional dynamical effects.
Note also that, in contrast to the study in [16], our coupling scheme defined by equa-
tion (2) is non-symplectic, i.e. for each pair of coupled nodes whenever \( \mu > 0 \) we have
\[ |Df(x, y)| = (1 - \mu)^3(1 - 3\mu) < 1, \]
and in addition it includes a fixed time delay. This type of coupling is motivated by a phenomenological model of the node interaction com-
ing from the oscillatory origin of the standard map. Another important feature of the
CMS defined by equation (2) is its systematic inhibition of the diffusion along the action
coordinate, which is made possible by the factor \( (1 - \mu) \) in front of \( y[i]' \) (see section 3.),
thus allowing more coherent interactions among the coupled nodes.

Geometries. The coupled maps of equation (2) interact along the links of a network
determined by its adjacency matrix \( C_{ij} \). In this work we primarily focus on the coupling
strength effects. Therefore, we consider tree-like structures in order to avoid possible
loop effects on the time-delayed interactions. In particular, we study CMS defined by
equation (2) on a large scale-free tree with \( N = 1000 \) nodes and on a typical 4-star
motif, both shown in figure 1. Our previous study on trees [23] suggests that the chaotic
maps with the coupling in equation (2) stabilize through the localization of orbits into
well-defined bands in the action coordinate. These bands determine different clusters of
periodically synchronized nodes, which make a statistically stable pattern on the tree with
the characteristic distance \( d = 2 \) between two synchronized nodes [23]. Note that the 4-
star is a typical motif which captures both the characteristic distance of synchronization
and the tree-like nature of the topology. Here we systematically compare the emergent
dynamic behavior of a 4-star and the whole tree in an extended range of couplings, where
the CMS exhibits new dynamical phenomena, as we describe below.

In figure 1(b) we show a typical situation of the tree at the threshold of the momentum
localization as the coupling \( \mu \) approaches a critical value \( \mu_c \approx 0.021 \) from below. In colors
are shown nodes which first achieve periodic orbits, in contrast to black nodes, at which
the maps still remain chaotic. The figure shows that the localization of orbits on the tree
starts from the nodes with the least number of links.

In the following two sections we describe different types of dynamical behaviors
obtained at varied coupling strengths. Our particular emphasis is on the region \( 0.02 < \mu < 0.08 \), i.e. between the above-mentioned initial localization of orbits with many
momentum bands at weaker couplings, and the global synchronization of nodes at strong
couplings. We focus on the statistical properties of the collective motion as our primary
goal. Throughout this work we fix the coupling \( \epsilon = 0.9 \) between two components \( (x[i], y[i]) \)
at each node and the time delay \( \tau = 1 \) between coupled nodes, and vary the inter-node
coupling \( \mu \). The results are collected after an initial transient, typically \( 10^5 \) steps for each
trajectory, and averaged over many (typically 1000) initial conditions. In all runs we pick
the initial condition for all the nodes of the network randomly with a uniform probability
from the phase space subset \( (x, y) \in [0, 1] \times [-1, 1] \).

3. Orbits structure and dynamical regimes

To study CMS on networks, the following types of orbits (after transient time, denoted
by \( n_0 \)) can be considered:

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• Orbits of each individual node, coupled to the rest of the network:

\[
(x[i]_n, y[i]_n)_{n>n_0}, \quad i = 1, \ldots, N.
\]  

(4)

For a selected node on the network we can in this way investigate in detail 2D discrete dynamics using well-known study techniques developed for (uncoupled) dynamical systems.

• Network-averaged orbit (n.a.o.), defined as

\[
(\hat{x}_n, \hat{y}_n)_{n>n_0} = \frac{1}{N} \sum_{i=1}^{N} (x[i]_n, y[i]_n)_{n>n_0},
\]  

(5)

is another useful measure of the collective dynamics of the entire network, in particular in the context of the synchronization and stability of the CMS.

• Time-averaged orbit (t.a.o.), defined for each node on the tree as

\[
(\bar{x}[i], \bar{y}[i]) = \lim_{n \to \infty} \frac{1}{n - n_0} \sum_{k=n_0}^{n} (x[i]_k, y[i]_k),
\]  

(6)

reduces the whole orbit of a node to a single point in the phase space, that qualitatively captures the emergent motion.

In the following we always refer to a particular type of orbit used.

Structure of emergent orbits of coupled maps. In figure 2 we show examples of three typical orbits obtained for different \(\mu\)-values for a node on the 4-star motif. In particular, at weak coupling between nodes, \(\mu = 0.005\), we have a chaotic motion with orbits wandering in the phase space in an irregular fashion (figures 2(a) and (d)), resembling the orbits of the uncoupled standard map equation (1). However, above a threshold coupling \(\mu \sim 0.02\) the CMS exhibits a regular motion. Orbits of different periodicity can be found in this region, such as the one in figure 2(b). As a rule, the 2-periodic orbits are the ones which survive at large couplings \(\mu \geq 0.08\). In the region between \(0.02 < \mu < 0.08\), depending on the initial conditions, we also find a non-periodic motion with the fractal attractors, as the example shown in figure 2(c). The time dependence of the momentum coordinate in the respective three dynamical regimes is also illustrated in figures 2(d)–(f). Clearly, our CMS of equation (2) exhibits a variety of emergent orbits which are not known in the isolated standard map. In the appearance of these orbits both the sensitivity to coupling strength and to the initial conditions play a role. The appearance of the fractal attractors, similar to the one in figure 2(c), with possibly non-chaotic behavior is a remarkable new feature of the collective dynamics of our CMS, to which we devote the section 4.2. Here we describe the differences between these three dynamical regimes in a quantitative manner.

Dynamical localization. In order to understand the nature of our CMS, we first analyze the chaotic orbit occurring for arbitrarily small but non-zero couplings. Although these orbits are chaotic (see also the stability analysis in the sections 4.2), in the presence of coupling they tend to localize in the action \(y\) coordinate, as shown in figures 2(a) and (d). Specifically, as opposed to the chaotic diffusion in the \(y\) coordinate, which is well known for the uncoupled standard map equation (1), see, e.g., [28], the interaction between the nodes in equation (2) inhibits the diffusion. Quantitatively, the mean-square distance \(<y^2(n)\) as

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Figure 2. Three typical emergent orbits ((a)–(c)) and the respective time dependence of the $y$ coordinate ((d)–(f)) for a node on the 4-star motif: ((a), (d)) chaotic orbit at $\mu = 0.005$, and ((b), (e)) periodic orbit at $\mu = 0.02$, both for the central node, and ((c), (f)) strange attractor of a branch node at $\mu = 0.049$.

a function of time $n$ for different coupling strengths $\mu$ is shown in figure 3(a). Clearly, the normal diffusion ($\gamma \simeq 1$) observed for the uncoupled maps $\mu = 0$ is replaced by the asymptotically localized orbits (finite m.s.r.) when $\mu > 0$. A similar effect was demonstrated in the spreading of the bandwidths occupied by the orbit, figure 3(b). The bandwidth in $y$ coordinate increases as a power of time for the uncoupled standard maps, whereas the stretching rate decreases with a non-zero coupling. Apart from quantitative differences, such localization occurs both on large trees and on small motifs, see figures 4(a) and (b) for all $\mu$ values. Note that for this range of coupling strengths each orbit of a node localizes at a given momentum, defined by $\bar{y}$, as shown in figure 4. The observed dynamical localization is a clear argument that the collective effects due to node interaction are occurring. As we shall see in the following paragraphs, further decrease in the stretching rate of the bandwidths occupied by an orbit may trigger the appearance of a regular motion.

Generalized synchronization and periodic orbits. The appearance of periodic orbits is one of the central collective feature exhibited by the CMS equation (2). The trajectories, similar to the one in figures 2(b) and (e), oscillate in the phase space between two localized groups of points in a way to maintain a constant average $\bar{y}$. For further quantitative study we use the time-averaged orbit $(\bar{x}, \bar{y})[i]$ defined in equation (6). As t.a.o. is represented
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Figure 3. Dynamical localization of orbits for various coupling strengths: $y$ coordinate mean-square distance (a) and average stretching of the bandwidth (b). Plots obtained by averaging over an ensemble of orbits of the 4-star branch node. Transients were also included.

Figure 4. 2D distribution histogram of time-averaged emergent orbits $\bar{y}$ against coupling $\mu$ for (a) one node of the 4-star branch with many initial conditions and (b) all nodes of the tree, but each with one initial condition. Color code: logarithm of the fraction of orbits corresponding to a given $\bar{y}$.

by a point in 2D node’s phase space, here we plot only its $\bar{y}$ coordinate as a function of the coupling strength $\mu$, as shown in figure 4(a). The color code represents (on log-scale) the number of orbits that end up with a given value of $\bar{y}$.

The Gaussian distribution of $\bar{y}$ values for small $\mu$ reduces to an organized set of clusters as the coupling strength is increased above $\mu \simeq 0.01$. With the disappearance of the chaotic e.o., classes of periodic orbits appear. In this region, depending on the initial conditions, an orbit ends up with a given $\bar{y}$ value, i.e. at one of the horizontal lines in figure 4(a). The orbit performs an oscillatory motion around a fixed set of phase space points, which are located symmetrically around the respective average value $\bar{y}$. The
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Figure 5. (a) Distribution of return times to a selected part of the phase space in three dynamical regimes corresponding to three different coupling strengths \( \mu \) for the star’s branch node. Phase space grid of \( 1000 \times 100,000 \) covering \( (x, y) \in [0, 1] \times [-50, 50] \) was used and distributions averaged over an ensemble of initial conditions. (b) Distribution of periods of the network-averaged orbits on the tree for \( \mu = 0.08 \). Inset: ranking statistics of periods for the 4-star branch node and an outer node on the tree, for \( \mu = 0.021 \).

emergent periodicity is studied in more detail later. A similar structure of orbits is found for each node on the tree, with robust patterns of clusters of nodes having the same \( \bar{y} \) value, see figure 4(b). Spatial patterns of such clusters of nodes depend on the local structure of the graph [23]. On the scale-free tree graph the number of possible groups gradually decreases with increased coupling strength and eventually reduces to a single group centered around \( \bar{y} = 0 \) for \( \mu \approx 0.07 \).

Appearance of non-periodic orbits. Furthermore, in the same region of couplings we find a fraction of initial conditions leading to non-periodic orbits. Such orbits are discussed in more details in section 4.

3.1. Statistical features of emergent orbits

As the above results and, in particular, figure 4 suggest, the 2D coupled chaotic maps defined by equations (1) and (2) on tree graphs exhibit new types of collective dynamic behaviors, apart from synchronized regular motion, which dominates at large couplings. In particular, we find (i) weakly localized chaos, (ii) emergent periodicity and (iii) non-periodic orbits with possibly non-chaotic features (see later). In this section we further differentiate between these dynamical regimes using the appropriate statistical analysis. The analysis of the dynamical stability will be given in section 4.

Return times statistics. The first-passage (or return-time) statistics represents an interesting statistical measure of correlations in the dynamical systems. In this context, we monitor return of the trajectories to a marked small area in the phase space. The results, integrated over the phase space subset \( (x, y) \in [0, 1] \times [-50, 50] \) and averaged over many initial conditions for three representative values of the coupling \( \mu \), are shown in figure 5(a). Clearly, the distributions of return times have different tails for the chaotic
motion (the case with \( \mu = 0.005 \)), periodic (for \( \mu = 0.021 \)) and non-periodic (the case with \( \mu = 0.048 \)) behavior. In particular, the cooperative behavior leading to the power-law tail of the distribution in the case for \( \mu = 0.048 \), suggests that the observed non-periodic orbits in this regime are non-chaotic. This is in contrast to the chaotic behavior, where all return times appear with equal probability (see middle curve in figure 5(a)). It is interesting that the differences between these dynamical behaviors are also expressed in the transient regime (approximately the first 10\(^3\) iterations in figure 5(a)).

*Distribution of periods.* Another measure of the cooperative behavior in the CMS is the emergent periodicity of the orbits, measured at the level of an individual node coupled to the system and in the system-averaged trajectory. In figure 4(b) we show the distribution of periods \( \pi \) of the periodic orbits for the tree-average trajectory and, in the inset, ranking statistics of the periods observed at an outer node coupled on a tree and on the 4-star. It is interesting that at the tree-like topologies the odd and even periodicities appear to have different statistics, which is especially pronounced at lower values of \( \pi \) and at weaker couplings. Although the large periods can be found at nodes on small graphs, our findings suggest that the large-scale structures are more efficient in building-up the power-law dependences in the distribution of periodicity. The network-averaged trajectories, defined by equation (5), exhibit new periodicities with the power-law distribution shown in figure 5(b). Without discussing a possible origin of the distribution, here we mention that the curve can be fitted well by the \( q \)-exponential form [30]:

\[
P(X) = B_q \left( 1 - (1 - q) \frac{X}{X_0} \right)^{1/q-1},
\]

with the characteristic period \( X_0 = 63 \) above which the power-law tail appears and the slope is 1.66. The fit is also shown in figure 5(b).

4. Dynamic stability and evidence of SNA

The analysis in previous sections revealed that our CMS maintains different types of cooperative dynamical behaviors, depending on the coupling strength. In order to further substantiate the nature of the emergent dynamics, in particular the occurrence of the non-periodic orbits, here we discuss the appropriate dynamic stability of the system.

We analyze the trajectory divergence [31] within small neighborhoods of the orbit. As was shown above, for the non-symplectic CMS of equation (2) the trajectories at long times tend to localize at a finite distance in the action coordinate. This localization implies that only limited divergence between different trajectories can be observed at \( t \to \infty \), see figure 6. Thus, the appropriate stability measure is the initial divergence rate, which lasts for a finite-time interval and determines the finite-time Lyapunov exponent [31].

4.1. Finite-time Lyapunov stability

The maximal Lyapunov exponents (MLE) for a phase space point is generally defined as the maximum rate of divergence among all the trajectories starting in the vicinity of the considered point. In figure 6(a) we show several examples of different initial divergence rates for different trajectories. As expected, all curves reach a saturation (slope zero) at long times. However, the trajectory separation increases over a certain number of
initial iterations $\tilde{n}$ with a well-defined slope, before the saturation occurs. These slopes determine the finite-time MLE, denoted as $\Lambda^t_{\text{max}}$. More precisely, the finite-time Lyapunov exponent $\Lambda^t_{\text{max}}(x_0)$ associated with the point $x_0 \equiv (x_0, y_0)$ on a trajectory is defined by

$$\Lambda^t_{\text{max}}(x_0) = \max_{x \in \mathcal{N}} \left\{ \text{initial slope} \left[ \frac{1}{\tilde{n}} \ln \frac{d(U_n x, U_n x_0)}{d(x, x_0)} \right] \right\},$$

where $\mathcal{N}$ stays for a small neighborhood around the point $x_0$, $d(\cdot, \cdot)$ denotes distance and $U_n$ stands for the discrete time-evolution dynamics given by CMS equations (1) and (2). Note that $\tilde{n}$ should be appropriately selected for each trajectory separately. As before, we consider trajectories of a selected node coupled on the 4-star and on the tree.

The procedure employed for computing the $\Lambda^t_{\text{max}}(x_0)$ is the following:

1. Take an initial condition $x_0$ (for all the nodes) and compute their orbits;
2. Focus on the final state (point) of a selected node and consider a random point in its close neighborhood, with the distance $d$ defined as the ‘Manhattan distance’ $d((x, y), (x_0, y_0)) = |x - x_0| + |y - y_0|$;
3. Iterate the dynamics for both points by systematically recording the rate that their distance ratio in equation (8) changes in time and determine the optimal $\tilde{n}$; determine the slope over the region $(1, \tilde{n})$.
4. Repeat the same procedure for a few other points in the same neighborhood and determine the maximal slope obtained in this way.

The exponent $\Lambda^t_{\text{max}}(x_0)$ corresponds to only one point on the orbit. Starting with different points a spectrum of $\Lambda^t_{\text{max}}$-s can be determined to characterize the entire orbit. The average value $\lambda^t_{\text{max}}$ over the distribution of $\Lambda^t_{\text{max}}(x_0)$ for a given orbit is defined as

$$\lambda^t_{\text{max}} = \langle \Lambda^t_{\text{max}}(x_0) \rangle_{x_0 \in \text{orbit}}.$$
We compute the average $\lambda_{t\text{max}}$ for a branch node on the 4-star for many trajectories and different coupling strengths $\mu$. The results are shown in figure 6(b) in the form of a 2D histogram for $\mu$ in the interval $[0.0, 0.08]$. Not surprisingly, fairly large positive values of $\lambda_{t\text{max}}$ at $\mu < 0.01$ are in full accordance with the chaotic motion of weakly coupled nodes. More interesting is the appearance of the orbits with $\lambda_{t\text{max}} > 0$ in the range of couplings $0.02 < \mu < 0.08$, where most of the trajectories are localized (cf. figures 4(a) and (b)). Note, however, that the measured values of $\lambda_{t\text{max}}$ in this region are far smaller compared to the chaotic orbits at small $\mu$. This indicates that the nature of these orbits might be fundamentally different. It should also be noted that, in this region of couplings we can find both positive and negative $\lambda_{t\text{max}}$ for different initial conditions (or different parts of the trajectory), as discussed later. A qualitatively similar picture is found for the trajectories of an outer node on the large tree.

Next we compare the occurrence of the trajectories with $\lambda_{t\text{max}} > 0$ and the non-periodic orbits, mentioned before. Note that the periodicity is usually determined after long times when the trajectory is localized, whereas the value of $\lambda_{t\text{max}}$ is the average computed along the entire trajectory. Nevertheless, as shown in figures 7(a) and (b), we find a significant overlap in the fraction of the non-periodic orbits and the orbits with a positive finite-time Lyapunov exponent $\lambda_{t\text{max}}$. In the case of a 4-star’s node the overlap is nearly complete, whereas in the case of a tree’s node the qualitative similarities are clear.

4.2. Appearance of strange attractors

The observed excessive number of non-periodic orbits, as compared to the orbits with an average $\lambda_{t\text{max}} > 0$, suggests that we take a close-up look at the properties along the non-periodic orbits. The two examples of such orbits found for the branch node on the 4-star for $\mu = 0.048$ end up at the strange attractors, parts of which are shown in figures 8(a) and (b). A systematic study of the exponent $\Lambda_{t\text{max}}(x_0)$, defined in equation (8), at different points $x_0$ along the trajectory, gives the spectrum of $\Lambda_{t\text{max}}$ values. For the two trajectories in figures 8(a) and (b) the respective spectra are given in figures 8(c) and (d). Note that the
\[ \lambda_{\text{max}}^t \]

reported above, roughly corresponds to the peak value of the spectrum. The fractal attractors, as for those shown in figures 8(a) and (b), have a common feature: the spectra have tails on both positive and negative values of \( \Lambda_{\text{max}}^t(x_0) \). Specifically, the attractor in figure 8(b) has a negative distribution-average value \( \lambda_{\text{max}}^t \) close to zero. Attractors with the atypical presence of a negative Lyapunov exponent which is accompanied with the fractal structure are good candidates for strange non-chaotic attractors SNA [31].

Following the classical references of the SNA [31]–[34], here we present additional analysis of the attractor in figure 8(b) in order to demonstrate its strangeness. First, we note that the attractor was found for a branch node of the 4-star, which is coupled to the hub node only, as shown schematically in figure 1(a). At every time step \( n \) the phase variable \( x_n[\text{branch}] \) of the branch node can be seen as ‘driven’ with the phase of the hub \( x_n[\text{hub}] \), which in this particular case follows a non-periodic orbit. One should stress that, in contrast to well-known examples with externally forced systems, the coupled maps on the 4-star in figure 1(a) mutually drive each other, however, with a time delay of one time step. One can easily realize that a regular 2-periodic or 4-periodic orbit becomes two or four separate clouds of points which make our fractal attractor. We find that the trajectory still regularly visits these separate parts of the attractor: however, when it returns to the same cloud, the successive points are scattered, as shown in figure 9(a).
Having the time series for the phase variables of all nodes of the 4-star, in figure 9(b) we plot the $x_n[\text{branch}]$ against $x_n[\text{hub}]$ at equal time steps, which clearly indicates the absence of a smooth curve. Furthermore, following the approach demonstrated in [32], we can demonstrate nondifferentiability of the respective time series, $F^n_h \equiv \frac{\partial x_n[\text{branch}]}{\partial x_n[\text{hub}]}$. In particular, one can define the sum

$$S_m = \sum_{k=1}^{k=m} F_h^{k-1} \times \exp\{\Lambda^t_{\text{max}}(m - k)\},$$

which takes into account the non-trivial effects of the positive tail in the distribution of the Lyapunov exponents $\Lambda^t_{\text{max}}$ on the attractor. The derivative $F^n_h$ is computed numerically using the functional dependence between the two phases shown in figure 9(b). The sum for different values of $m$ is shown in figure 9(c). The inset to the same figure shows that its maximal value $\max |S_m|$ is systematically increasing with the number of terms $m$. This indicates that the sum is unbounded, corresponding to a non-smooth attractor. Another feature that the studied attractor shares with the well-known SNA in non-periodically driven maps [32] is the saturation of the maximal distance between points on...
the attractor, independently of the time gap between the points. In figure 9(d) we show the maximal distance between points as a function of time gap $\Delta$. Two lines correspond to two separated parts of the attractor.

We find a large number of fractal attractors in our CMS. An incomplete search for fractal attractors gives the distribution of the peak values $\lambda_{\text{max}}^t$ of the respective spectra shown in figure 10(a), indicating roughly four different groups of such attractors in our CMS.

A further interesting question is how the appearance of such attractors affects the collective motion of the CMS, which was our central point in this paper. Here we present the distributions of the return times to a given portion of the phase space, $P(\Delta t)$, defined above. For the two attractors in figures 8(a) and (b) the results for these distributions are shown in figure 10(b). In both cases the distributions of return times exhibit a power-law tail, which can be fitted with the $q$-exponential function in equation (7), suggesting a non-ergodic dynamics near these attractors. Although both attractors in figures 8(a) and (b) have quantitatively similar fractal dimensions, close to 1.5, obviously their other features affect the return-time statistics. In particular, the larger slope and smaller characteristic scale $X_0 = 80$ was found in the case of the attractor with SNA features, figure 8(b), whereas, at the attractor figure 8(a) with a positive $\lambda_{\text{max}}^t$, we find $X_0 = 8000$ and an increased probability of long return times.

5. Conclusion

We have demonstrated that a variety of new dynamical effects occur in a two-dimensional standard map with a non-symplectic coupling and time delay between coupled units on tree-like graphs. By focusing on a single node coupled to the tree, we were able to apply the established methods of the discrete-time dynamics, and thus closely follow how a coupled unit adapts to the evolution of the extended dynamical system. In particular, as a consequence of the coupling we find the dynamical localization of orbits, emergent
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periodicity, non-periodic orbits and the occurrence of fractal attractors in a wide range of coupling strengths. The self-organized dynamical behavior of mutually coupled maps in our CMS is characterized by the power-law dependences, specifically in the emergent (tree averaged) periodicity and in the return-time intervals near the strange attractors.

We have shown that some of the strange attractors in our CMS have non-positive Lyapunov exponents, similarly to the strange non-chaotic attractors [31]–[34] found in the quasi-periodically forced one-dimensional maps. It should be stressed that in our two-dimensional coupled maps no external driving was imposed. Contrary to the forced isolated maps, in our CMS different coupled units on the 4-star or on the scale-free tree provide the dynamical input to each other. For an example of such an attractor, we presented quantitative arguments, summarized in figures 8 and 9, which show nondifferentiability and finite maximal distance between points on the attractor, in full analogy to the classical SNA in quasi-periodically enforced systems [32]. Full understanding of the routes to the strange non-chaotic attractors in this coupled map system requires additional study. We speculate that both the observed localization of orbits and time delay between coupled units, which prevents full synchronization of neighboring nodes, play an important role.

With the parallel analysis of the dynamics at the 4-star motif and on the large tree, in this work we obtained certain quantitative arguments, which imply how the self-organized dynamics on the large-scale networks occurs. Specifically, the 4-star appears as a basic dynamical unit on which the non-periodic orbits and strange attractors appear and spread on the entire tree, cf. figure 4. The role of cycles on the graph as well as the effects of different/variable delays between units are left for a separate work. Further interesting subjects in the context of our CMS, as the appearance of partial synchronization patterns (see also [23]) and the problem of coexisting attractors on the graph, are left for future study.

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