Integer partition manifolds and phonon damping in one dimension

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We develop a quantum model based on the correspondence between energy distribution between harmonic oscillators and the partition of an integer number. A proper choice of the interaction Hamiltonian acting within this manifold of states allows us to examine both the quantum typicality and the non-exponential relaxation in the same system. A quantitative agreement between the field-theoretical calculations and the exact diagonalization of the Hamiltonian is demonstrated.

The problem of thermalization in isolated quantum systems is almost as old as the quantum theory itself. Von Neumann pioneered in this field with his proof of the quantum ergodic theorem \cite{1} (see also the English translation \cite{2} and an extended commentary \cite{3}). Another important achievement in understanding the emergence of thermodynamics from quantum theory relies on the random matrix theory \cite{4–7}, which is often considered as the basis for the definition of quantum chaos \cite{8}. A new approach has been put forward independently by Deutsch \cite{9} and later by Srednicki \cite{10}, who named it the eigenstate thermalization hypothesis (ETH). The ETH allows for quantitative estimations of correlations and fluctuations in thermalizing quantum systems \cite{11}. The relation between von Neumann’s approach and the ETH is elucidated in Refs. \cite{12, 13}.

Numerical tests of these theoretical concepts usually employ such models as hard-core bosons \cite{14} or fermions \cite{15} on a lattice or spin chains \cite{16}. The problem is that the dimension of the Hilbert space for a half-filled lattice or a spin system with zero projection of the total spin grows exponentially with the system size. This is especially restrictive in the case of studying the Bose-Hubbard model with a finite interaction strength \cite{17, 18}. A numerical method to the calculation of correlations in integrable systems describable by the Bethe ansatz, i.e., in Heisenberg spin chains and in one-dimensional (1D) Bose gases has been developed \cite{19}; this method is an example of a tour de force in combining analytic and numerical approaches. However, the method of reducing the dimensionality of the Hilbert space to the states providing most of the contribution to the correlation functions in Ref. \cite{18} is restricted to integrable systems only.

Therefore it is always desirable to develop a model that is capable of solving experimentally relevant problems and has the dimension growing subexponentially with the number of the modes taken into account. In this Letter we present such a model. Consider a set of harmonic oscillators with the frequencies equal to an integer multiple of \(\Omega\), i.e., \(\omega_k = k\Omega\), each integer number \(k = 1, 2, 3, \ldots\) appearing once and only once. Obviously, the state with the energy \(n\hbar\Omega\) has a degeneracy equal to the number of partitions of the integer number \(n\). Indeed, a partition represents \(n\) as a sum of positive integers, the number \(k\) appearing in the sum \(N_k\) times:

\[
\sum_k kN_k = n. \tag{1}
\]

Obviously, the number of ways to distribute the excitation energy \(n\hbar\Omega\) among the oscillators with frequencies \(\omega_k\) is equal to the number \(p(n)\) of all partitions of the integer \(n\). Therefore we call a set of all states \(|\{N_k\}\rangle\) satisfying the condition (1) where \(N_k\) is the number of quanta in the \(k\)th oscillator, an integer partition manifold (IPM).

If we introduce an interaction with a typical matrix element much smaller than \(\Omega\), then — to the lowest order of the perturbation theory — we can neglect mixing of different IPMs and consider coupling of levels within each IPM separately. Each oscillator will be regarded as a bosonic mode of a system. We see that the number \(n\) characterizes the highest mode that can be occupied by pumping all the available energy into that mode. The dimension \(D_n\) of the IPM is equal to the number of partitions \(p(n)\) of the integer \(n\). An asymptotic estimation for \(p(n)\) was found by Hardy and Ramanujan 100 years ago \cite{19}

\[
p(n) \approx \frac{\exp\left(\pi \sqrt{2n/3}\right)}{4n^{3/4}}. \tag{2}
\]

This expression grows slower than an exponential function of \(n\), which makes the model suitable for exact diagonalization: we can take into account many modes without exploding the dimension of the Hilbert space.

To provide a physical example where an IPM spans the Hilbert space, we consider the damping of a phonon in a 1D quasicondensate \cite{20, 21}. The \(k\)th mode is identified with a phonon mode with the momentum \(k\hbar q_{\text{min}}\). To discretize the spectrum of phonons, as required by the construction of IPMs, we assume a finite size \(L\) of the 1D system, thus introducing the smallest possible phonon momentum \(q_{\text{min}} = 2\pi\hbar/L\). The interaction Hamiltonian
is then
\[ \hat{H}_{\text{int}} = \eta \sum_{k=1}^{n} \sum_{q=1}^{[k/2]} \sqrt{kq(k-q)} \left( \hat{b}_k^\dagger \hat{b}_q + \hat{b}_q^\dagger \hat{b}_k - \hat{b}_k^\dagger \hat{b}_q - \hat{b}_q^\dagger \hat{b}_k \right), \]
where the operator $\hat{b}_k$ ($\hat{b}_k^\dagger$) annihilates (creates) an excitation in the $k$th mode, $[k/2]$ denotes an entire part of $k/2$, and $\eta = \sqrt{9\pi \sigma_{\text{min}}^2 \hbar c/(16 \Omega_{1D})}$, where $c$ is the speed of sound and $\sigma_{\text{min}}$ is the 1D mass density of the quasicondensate. The difference is different from the decay of a Bogoliubov quasiparticle into three quasiparticles due to integrability-breaking effective three-body elastic collisions. Numerical solution of the Gross-Pitaevskii equation demonstrated the non-trivial dynamics of phonons in a 1D quasicondensate in the integrable limit. Therefore, we find it interesting to elucidate the quantum origins of phonon relaxation in 1D.

We performed exact diagonalization of the Hamiltonian for $n$ from 20 to 30, $\hat{H}_{\text{int}} |\psi_j\rangle = E_j |\psi_j\rangle$, using the states $|\{N_k\}\rangle$ as the basis. The standard matrix diagonalization package of Mathematica 8.0 has been applied. First of all, we notice that the distribution of eigenvalues of $\hat{H}_{\text{int}}$ is close to a Gaussian (the left inset in Fig. 1) with the width proportional to $n^s$ with $s \approx 1.33$. A more detailed examination reveals that the eigenvalue $E = 0$ is always degenerate. The ratio $W_0$ of the number of states with $E = 0$ (within the numerical error $\sim 10^{-14} \eta$) to the dimension $D_n$ of the IPM can be approximated as $W_0 \approx 90/n^3$ (right inset in Fig. 1).

The repulsion of levels resulting in energy intervals $\Delta E$ between eigenstates obeying the Wigner distribution $\propto \Delta E \exp[-(\Delta E/\Delta E_s)^2]$, where $\Delta E_s$ sets the typical energy scale, is a signature of quantum chaos. The main plot in Fig. 1 shows the distribution of energy intervals between the eigenstates of $\hat{H}_{\text{int}}$, the degenerate states with $E = 0$ being excluded. We see that the Wigner distribution well describes the statistics of most of the intervals. However, the actual distribution shows a tail for relatively high values $\Delta E$ that exceeds the Wigner distribution and this excess comprises about 20% of the energy intervals without a clear dependence on $n$. This tail spreads far beyond $\Delta E_s$, therefore many too high values of $\Delta E > 0.4\eta$ are not displayed in Fig. 1.

We check next the quantum typicality of the eigenstates of $\hat{H}_{\text{int}}$. The considerations based on the ETH lead to the following estimation of a matrix element of an observable $A$:
\[ \langle \psi_i | \hat{A} | \psi_j \rangle = \delta_{ij} \bar{A}(n) + \frac{f(n, E_j - E_i) R_{ij}}{\sqrt{D_n}}. \]

The diagonal matrix element $\bar{A}(n)$ depends mainly on $n$ and only weakly on the state $|\psi_j\rangle$. The off-diagonal matrix elements consist of $1/\sqrt{D_n}$ as prefactor of an “envelope function” $f$, which is a smooth function of $E_j - E_i$, and $R_{ij}$, which can be considered a random function of $i$ and $j$, its sign being positive or negative with equal probability and its absolute value being normalized by $D_n^{-1} \sum_j R_{ij}^2 = 1$. As the observable to be tested we choose the numbers of quanta $N_k$ in the modes $k = 1, \ldots, n$. Unlike the basis functions $|\{N_k\}\rangle$, the eigenstates $|\psi_j\rangle$ of the interaction Hamiltonian are not eigenstates of $\hat{N}_k = \hat{b}_k^\dagger \hat{b}_k$. The matrix elements of interest are
\[ \langle \psi_i | \hat{N}_k | \psi_j \rangle = \sum_{\{N_k\}} \langle \psi_i | \{N_k\} \rangle N_k \langle \{N_k\} | \psi_j \rangle. \]
In Fig. 2 we show the mode populations, i.e., diagonal matrix elements $\langle \psi_j | N_k | \psi_i \rangle$ for four different eigenstates $| \psi_i \rangle$. The populations are close to the repetition numbers $N_k$ of the number $k$ averaged over all possible partitions of $n$,

$$N_k = \frac{1}{p(n)} \sum_{\{N_k\}} N_k,$$

as long as they are large enough, $N_k \gtrsim 10^{-1}$. The values of $N_k$ in the same range are well described by a Bose-Einstein distribution with zero chemical potential and dimensionless “inverse temperature” $\beta$ defined by the condition $\sum_{k=1}^{n} k/\exp(\beta k) - 1 = n$ (for the connection between the Bose-Einstein statistics and partitions of integers see, e.g., Refs. 27, 28).

The properties of off-diagonal matrix elements of $N_k$ are illustrated in Fig. 3 for a state, which is expected to be typical, because its energy is neither too large nor too small. We see that $N_1$ couples the states in a narrow band of energies. In contrast, off-diagonal matrix elements for $N_{11}$ are uniformly distributed over almost the whole range of states with $j \neq i$.

Finally, we apply the results of the exact diagonalization of the interaction operator to the solution of a dynamical problem. We consider the decay of a phonon in 1D at zero temperature, i.e., we assume that initially, at the time $t = 0$ only a single mode is populated: $N_k(0) = \delta_{kn}$. In what follows, we work in the interaction representation and set $\hat{b}_k = \exp(-i\omega_k t)\hat{b}_k$. The initial quantum state is $| \Psi(0) \rangle = \hat{b}_0^\dagger | \text{vac} \rangle$, where $| \text{vac} \rangle$ denotes the vacuum state of the phonons. The further evolution is given by $| \Psi(t) \rangle = \exp(-i\hat{H}_{int}/\hbar)| \Psi(0) \rangle$, and we want to calculate the survival probability $P_n(t) = |\langle \Psi(t) | \Psi(0) \rangle|^2$ in the $n$th mode for $t > 0$.

We are aware that this problem is a very idealized toy model, since thermal fluctuations in 1D systems always play an important role at experimentally attainable temperatures [29], but this problem is a good starting point for demonstrating the agreement of the results of our model with those of the field theory and to stress their non-trivial feature (non-exponential decay law). The problem of decay of a phonon in 1D at finite temperatures corresponding to the classical limit of the Bose-Einstein statistics [20] will be a subject of our future work. It requires the use of an IPM with a very high dimension, that makes full diagonalization of the interaction operator a challenging task. Instead, we can invoke the sparsity of the matrix $\{\langle N_j \rangle \hat{H}_{int} \{N_k\}\}$ [the number of its non-zero elements is $\sim \frac{1}{2} n^2 p(n)$] and find the evolution of the system's state of the system using the modern methods for sparse systems of differential equations [30].

From the field-theoretical point of view, the survival probability

$$P_n(t) = |G_n^R(t)|^2$$

(7)
can be expressed through the retarded Green’s function $G_n^R(t) = \sum_{\omega} \int_{-\infty}^{\infty} d\omega \, e^{-i\omega t} \hat{G}_n^R(\omega)$ of a phonon in vacuum, where $\hat{G}_n^R(\omega) = [\omega - \tilde{\Sigma}_n(\omega) + i0^+]^{-1}$.

A renormalization that replaces bare Green’s functions by dressed ones in the standard perturbative expression for the self-energy leads to the expression [21]

$$\tilde{\Sigma}_n(\omega) = \left( \frac{\eta}{\hbar} \right)^2 \int_0^{n/2} dk \frac{k(n-k)n}{\omega - \tilde{\Sigma}_k(\omega) - \tilde{\Sigma}_{n-k}(\omega) + i0^+}.$$

(8)

A similar equation has been derived by Andreev in the finite-temperature case [20]. The off-shell contribution to $\tilde{\Sigma}_n$ (i.e., coupling to other IPMs in our case) is neglected in Eq. (8). We replaced in Eq. (8) summation over discrete set of mode numbers by integration, as a traditional approximation in field theory, valid for $n \gg 1$.

In the previous works [20, 21] equations like Eq. (8) were solved by setting $\omega \to 0$. In this limit, Eq. (8) can be solved exactly, yielding $\tilde{\Sigma}_n(0) = -i\zeta n^2/\hbar$ with $\zeta = \left\{ \int_{0}^{1/2} dq \, q(1-q)/q^2 + (1-q)^2 \right\}^{1/2} = [(\pi - 2)/8]^{1/2}$ and, hence, $P_n(t) = \exp(-2\zeta n^2t/\hbar)$. However, we show that this approach provides only a correct dependence of the typical decay time on $n$ and, but the dependence of $\tilde{\Sigma}_n$ on $\omega$ becomes essential and the single-pole approximation for $\hat{G}_n^R(\omega)$ breaks down, therefore the phonon relaxation is non-exponential.
The large frequencies correspond to short times. Therefore, in order to investigate the initial stage of the phonon decay, we will expand $\tilde{\Sigma}_n(\omega)$ and $G_n^R(\omega)$ in series in negative powers of $\omega$ that converge for $\omega \gtrsim n^2 \eta/\hbar$. One can see from Eq. (8) that the self-energy can be expressed as $\tilde{\Sigma}_n(\omega) = (\omega + i0^+) \Xi(\sigma_n)$, where $\Xi$ is a universal function and its argument

$$\sigma_n = \left[ \frac{\eta n^2}{-i\hbar(\omega + i0^+)} \right]^2$$

depends on $n$. By changing the integration variable to $q = k/n$ we transform Eq. (8) to

$$\Xi(\sigma_n) = \sigma_n \int_0^{1/2} dq \frac{q(1-q)}{1 + \Xi(q^4 \sigma_n) + \Xi((1-q)^4 \sigma_n)}. \quad (10)$$

We make an ansatz $\Xi(\sigma_n) = \sum_{l}^{\infty} \Xi_l \sigma_n^l$ and also expand the r.h.s. of Eq. (10) in powers of $\sigma_n$. The expansion coefficients $\Xi_l$ are determined from comparison of prefactors in front of $\sigma_n^l$ on both sides of Eq. (10). We obtain $\Xi_1 = \frac{1}{12}$ and $\Xi_l$ for $l > 1$ can be expressed through $\Xi_{l-1}$ in a recursive way.

Knowing $\Xi_l$, one can expand Green’s function in series:

$$G_n^R(\omega) = \left\{ (\omega + i0^+) \left[ 1 + \Xi \left( -\frac{\eta^2 n^4}{\hbar^2 (\omega + i0^+)^2} \right) \right] \right\}^{-1} = \frac{1}{\omega + i0^+} \left\{ 1 + \sum_{l=1}^{\infty} (-1)^l \xi_l \left( \frac{\eta n^2}{\hbar(\omega + i0^+)} \right)^{2l} \right\}. \quad (11)$$

The inverse Fourier transform yields an expression

$$G_n^R(t) = 1 + \sum_{l=1}^{\infty} \xi_l \frac{(\eta n^2 t/\hbar)^{2l}}{(2l)!} \quad (12)$$

that holds for not too large $t$ and is to be substituted into Eq. (7). The numerically observed convergence of the series allows us to restrict ourselves to a moderate (up to 20) number of terms in expansions for $\Xi$ and $G_n^R$. The first six values of $\xi_l$ are given in Table I.

| $l$ | $\xi_l$ | $l$ | $\xi_l$ | $l$ | $\xi_l$ |
|-----|---------|-----|---------|-----|---------|
| 1   | -0.083 333 33 | 3   | -0.000 993 07 | 5   | -0.000 012 80 |
| 2   | 0.008 928 57 | 4   | 0.000 112 24 | 6   | 0.000 001 47 |

TABLE I: Expansion coefficients in Eqs. (11, 12).

On the other hand, the exact diagonalization of the interaction Hamiltonian allows us to directly compute

$$P_n(t) = \left| \sum_{j=1}^{D_n} e^{-iE_j t/\hbar} \langle \psi_j | \Psi(0) \rangle \right|^2. \quad (13)$$

Fig. 4 demonstrates a good agreement between Eq. (13) and the field-theoretical approach Eqs. (11,12) for $\eta n^2 t/\hbar \lesssim 5$. $P_n$ decreases practically to 0 on this time scale, a partial revival occurs on longer times.

Our analysis of the non-exponential decay of a phonon should be distinguished from that of particle tunneling [31]. In the latter case, the Green’s function is derived by means of single-particle quantum scattering theory, where the boundary conditions for outgoing wave are specified. We try to connect the field-theoretical approach to the exact diagonalization of a Hamiltonian that models the low-energy effective theory of a many-body 1D system (dynamics of phonons). We hope that our approach to non-exponential decay via expansion like Eqs. (11,12) can be extended to a broad class of problems.

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