PLACTIC RELATIONS FOR $r$-DOMINO TABLEAUX

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Abstract. The recent work of Bonnafé et al. [2] shows through two conjectures that $r$-domino tableaux have an important role in Kazhdan-Lusztig theory of type $B$ with unequal parameters. In this paper we provide plactic relations on signed permutations which determine whether given two signed permutations have the same insertion $r$-domino tableaux in Grafinkle’s algorithm [3]. Moreover, we show that a particular extension of these relations can describe Garfinkle’s equivalence relation [3] on $r$-domino tableaux which is given through the notion of open cycles. With these results we articulate the conjectures of [2] by providing necessary tools for their proof.

1. Introduction

Let $W$ be a finite Coxeter group and let $L : W \mapsto \mathbb{Z}_{\geq 0}$ be a weight function such that

$$L(uw) = L(u) + L(w) \quad \text{if and only if} \quad l(uw) = l(u) + l(w)$$

where $l : W \mapsto \mathbb{Z}_{\geq 0}$ is the usual length function on $W$. As it is described by Lusztig in [13] every weight function determines an Iwahori-Hecke algebra and some preorderings on $W$ whose equivalence classes are called left, right and two-sided cells. Importance of these cells lies on the fact that they carry representations of $W$ and its corresponding Iwahori-Hecke algebra $\mathcal{H}$. Furthermore they have an important role in the representation theory of reductive algebraic groups over finite or $p$-adic fields [13] and in the study of rational Cherednik algebras [7] and the Calogero-Moser spaces [8].

The case $L = l$ is in fact first introduced by Kazhdan and Lusztig in [10] as a purely combinatorial tool to the theory of primitive ideals in the universal enveloping algebras of semisimple complex Lie algebras. In this case the combinatorial characterizations of cells are well known, where Knuth (or plactic) relations appears as the mediating tool. Namely, when $W$ is type $A$ then each right (left) cell corresponds to the plactic (respectively coplactic) class of some standard Young tableau whereas each two-sided cell consists of those permutations which lie in the plactic classes of tableaux of the same shape. This characterizations depends on Joseph’s classification of primitive ideals in type $A$, where Knuth (plactic) relations play a crucial role.

On the types B,C and D, on the other hand the emerging combinatorial objects are standard domino tableaux. The connection is first revealed in the work of Barbash and Vogan [1] where they provide necessary conditions for the characterizations of primitive ideals through an algorithm which uses the palindrome representations of signed permutations in order to assign to every signed permutations $\alpha$ a pair of same shape standard $r$-domino tableaux $(P^r(\alpha), Q^r(\alpha))$ bijectively, for $r = 0$ or $r = 1$. Meanwhile, an analog of Knuth relations provided by Joseph in [9] established the sufficient conditions. On the other hand Garfinkle [3, 11, 12] finalized classification problem for these types by showing through her two algorithms on domino tableaux that these two sets of relations are in fact equivalent. Her first algorithm assigns any signed permutation to a pair of same shape standard $r$-domino tableaux for $r$ equals to 0 or 1 and the second defines and equivalence relation between domino tableaux through the notion of open cycles. We want to remark that the extension of Garfinkle and Barbash-Vogan algorithm for larger $r$ is given in [12] and [2] respectively.

The case $L \neq l$ is also known as unequal parameters Kazhdan-Lusztig theory and it appears for the types $B_n, I_2(n)$ and $F_4$ where the classification problem for the later two can be dealt with computational methods.

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Theorem 1.3. For type $B_n$, the weight function is determined by two integers $a, b > 0$ such that
\[
L(s_i) = \begin{cases} 
    a & \text{if } 1 \leq i \leq n-1 \\
    b & \text{if } i = 0
\end{cases}
\]
where $s_0$ is the transposition $(-1,1)$ and $\{ s_i = (i, i+1) | 1 \leq i \leq n-1 \}$ is the type $A$ generators of $B_n$.

Recently, the role of $r$-domino tableaux in this theory is revealed in the work of Bonnafé, Geck, Iancu, and Lam [2] through two main conjectures:

- **Conjecture A:** If $ra < b < (r+1)a$ for some $r \geq 0$ then two signed permutations lie in the same Kazhdan Lusztig right (left) cell if and only if their insertion (recording) $r$-domino tableau are the same.
- **Conjecture B:** If $b = ra$ for some $r \geq 1$ then two signed permutations lie in the same Kazhdan Lusztig right (left) cell if and only if their insertion (recording) $r$-domino tableau or $(r-1)$-domino tableau are the same.

Furthermore, Pietraho [15] adopts Garfinkle’s notion of cycles to extend it for any positive integer $r$ and he shows that the set of signed permutations whose insertion $r$ or $(r+1)$-domino tableau are the same is equivalent to the set of signed permutations whose insertion $r$-domino tableaux are equivalent through the notion of open cycles.

On the other hand, in order to establish the proofs of these conjecture one definitely needs the plactic relations between signed permutations which determines when the insertion $r$-domino tableaux of two signed permutations are the same or equivalent through the notion of open cycles. Our aim here is to fill this gap.

**Definition 1.1.** For $\alpha = \alpha_1 \ldots \alpha_n \in B_n$ and $r \geq 0$ consider the following relations:

- **$D'_1$:** If $\alpha_i < \alpha_{i+2} < \alpha_{i+1}$ (or $\alpha_i < \alpha_{i-1} < \alpha_{i+1}$) for some $i$, then
  \[ \alpha = \alpha_1 \ldots \alpha_{i-1} (\alpha_i \alpha_{i+1}) \alpha_{i+2} \ldots \alpha_n \sim \alpha_1 \ldots \alpha_{i-1} (\alpha_{i+1} \alpha_i) \alpha_{i+2} \ldots \alpha_n \]

- **$D'_2$:** If there exists $0 < j \leq r$ such that $\alpha_j > 0$ and $\alpha_{j+1} < 0$ (or $\alpha_j < 0$ and $\alpha_{j+1} > 0$) and $\alpha_1 \ldots \alpha_j \alpha_{j+1}$ is a shuffle of some positive decreasing and negative increasing sequence ending with $\alpha_j$ and $\alpha_{j+1}$ (or respectively $\alpha_{j+1}$ and $\alpha_j$) then
  \[ \alpha = \alpha_1 \ldots (\alpha_j \alpha_{j+1}) \ldots \alpha_{r+2} \ldots \alpha_n \sim \alpha_1 \ldots (\alpha_{j+1} \alpha_j) \ldots \alpha_{r+2} \ldots \alpha_n \]

- **$D'_3$:** If $|\alpha_1| > |\alpha_i|$ for all $2 \leq i \leq r+2$ and $\alpha_2 \ldots \alpha_{r+2}$ is a shuffle of some positive decreasing and negative increasing sequences, then
  \[ \alpha = \alpha_1 \alpha_2 \ldots \alpha_{r+2} \ldots \alpha_n \sim (-\alpha_1) \alpha_2 \ldots \alpha_{r+2} \ldots \alpha_n \]

For $\alpha$ and $\beta$ are two signed permutations in $B_n$, we say

i. $\alpha$ and $\beta$ are $r$-plactic equivalent, $\alpha \overset{p}{\simeq} \beta$, if one of them can be obtained by applying a sequence of $D'_1$, $D'_2$ or $D'_3$ relations to the other. Moreover, we say $\alpha$ and $\beta$ are $r$-coplactic equivalent, $\alpha \overset{p^*}{\simeq} \beta$, if $\alpha^{-1} \overset{p}{\simeq} \beta^{-1}$.

ii. $\alpha$ and $\beta$ are $r$-cycle equivalent, $\alpha \overset{c}{\simeq} \beta$, if one of them can be obtained by applying a sequence of $D'_1$, $D'_2$, $D'_3$ or $D'_4^{(r-1)}$ relations to the other. We say $\alpha$ and $\beta$ are $r$-coycle equivalent, $\alpha \overset{c^*}{\simeq} \beta$, if $\alpha^{-1} \overset{c}{\simeq} \beta^{-1}$.

Now we are ready to state our results.

**Theorem 1.2.** $\alpha$ and $\beta$ in $B_n$ are $r$-plactic equivalent if and only if they have the same insertion $r$-domino tableaux.

**Theorem 1.3.** $\alpha$ and $\beta$ in $B_n$ are $r$-cycle equivalent if and only if their insertion $r$-domino tableaux are either the same or equivalent through some set of open cycles.

Then one can reformulate the conjectures of [2] in the following manner.

**Conjecture 1.4.**

- **Conjecture A:** If $ra < b < (r+1)a$ for some $r \geq 0$, then each Kazhdan Lusztig right (left) cell consist of all signed permutations which are $r$-plactic ($r$-coplactic) equivalent.
• Conjecture B: If \( b = ra \) for some \( r \geq 1 \) then each Kazhdan Lusztig right (left) cell consists of all signed permutations which are \( r \)-cycle (\( r \)-cocycle) equivalent.

Remark 1.5. A set of relations for \( r \)-domino tableaux is defined in [2], but as it is already discussed there it is far from being sufficient for the characterization. In fact the plactic relation in [2, Section 3.8] can be shown to be equivalent to the one given with \( D_1^r \) and \( D_2^r \) here.

Remark 1.6. Recall that for a signed permutation \( \alpha = \alpha_1 \ldots \alpha_n \) in \( B_n \), its palindrome representation is given by \( \alpha^1 = \overline{\alpha_1} \ldots \overline{\alpha_1} \ldots \overline{\alpha_n} \) where \( \overline{\alpha_i} = -\alpha_i \). Then \( D_1^r \) is just the usual Knuth (plactic) relation on \( \alpha^1 \) for any non negative integer \( r \), whereas \( D_2^r \) and \( D_3^r \) are also usual Knuth relation on \( \alpha^1 \) when \( r = 1 \).

Remark 1.7. In fact the Theorem [1.3] can be proven by applying Theorem 1.2 to the result of Pietraho [15]. On the other hand, the notion of open cycles is discussed in the last section where the proof to some lemmas which are essential in the proofs of our results. Section 3 is devoted to the proof of Theorem 1.3 is also given.

2. Related background

A sequence \( \lambda = (\lambda_1, \ldots, \lambda_k) \) is a partition of \( n \), denoted by \( \lambda \vdash n \), if \( \sum_{i=1}^k \lambda_i = n \) and \( \lambda_i \geq \lambda_{i+1} > 0 \). \( \lambda \vdash n \) can be represented by Ferrers diagram which consists of left justified arrows of boxes such that \( i \)-th row has \( \lambda_i \) boxes. For example

\[
\lambda = (2, 2, 1) = \begin{array}{ccc}
\circ & \circ & \\
\circ & \\
\end{array}
\]

A partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \) can be also seen as a set of integers pairs \( (i, j) \) such that \( 1 \leq i \leq k \) and \( 1 \leq j \leq \lambda_i \). Therefore for two partitions \( \lambda \) and \( \mu \), we can define usual set operations such as \( \lambda \cup \mu \), \( \lambda \cap \mu \), \( \lambda \subset \mu \), \( \lambda - \mu \) but resulting sets do not necessarily correspond to some partitions.

Definition 2.1. For two partitions \( \lambda \) and \( \mu \) satisfying \( \mu \subset \lambda \) we define \( \lambda / \mu = \lambda - \mu \) to be the skew partition determined by \( \lambda \) and \( \mu \).

Definition 2.2. Let \( \gamma \) and \( \gamma' \) be two skew shapes.
1. If \( \gamma \cap \gamma' = \emptyset \) and \( \gamma \cup \gamma' \) also corresponds a skew shape then we define \( \gamma \oplus \gamma' = \gamma \cup \gamma' \).
2. If \( \gamma' \subset \gamma \) and \( \gamma - \gamma' \) also corresponds a skew shape then we define \( \gamma \ominus \gamma' = \gamma - \gamma' \).

Definition 2.3. For \( \lambda \) is a partition consider cell \( (i, j) \):
1. If \( (i, j) \in \lambda \) and \( \lambda \oplus (i, j) \) is also a partition then \( (i, j) \) is called a corner of \( \lambda \).
2. If \( (i, j) \notin \lambda \) and \( \lambda \oplus (i, j) \) is also a partition then \( (i, j) \) is called an empty corner of \( \lambda \).

Definition 2.4. A skew tableau \( T \) is obtained by labeling the cells of \( \lambda/\mu \) with non repeating positive integers such that the numbers increase from left to right and from top to bottom. If \( \mu = \emptyset \) then \( T \) is called a Young tableau. We denote by

\[
\text{label}(T) \quad \text{and} \quad \text{shape}(T)
\]

respectively, the set of numbers labeling each box of \( T \) and the partition underlying \( T \). If the size of \( \text{shape}(T) = n \) and \( \text{label}(T) = \{1, 2, \ldots, n\} \) then \( T \) is called a standard skew or standard Young tableau according to the shape of \( T \). We denote by \( \text{SYT}_n \) the set of all standard Young tableaux of \( n \) cells.

There is an important connection, between standard Young tableaux \( \text{SYT}_n \) and the symmetric group \( S_n \), known as Robinson-Schensted correspondence (RSK), which is realized by Robinson and Schensted independently. In this correspondence, every permutation \( w \in S_n \) is assigned bijectively to a pair of same shape tableaux \((P(w), Q(w))\) in \( \text{SYT}_n \times \text{SYT}_n \) through insertion and recording algorithms. Let us explain these algorithms briefly. We denote by \((P_{i-1}, Q_{i-1})\) the tableaux obtained by insertion and recording algorithms
on the first \( i - 1 \) indices of \( w = w_1 \ldots w_n \). In order to get \( P_i \) we proceed as follows: if \( w_i \) is greater then the last number on the first row of \( P_{i-1} \), then \( w_i \) is concatenated to the first row of \( P_{i-1} \) from the right, otherwise \( w_i \) replaces the smallest number, say \( a \), among all numbers in the first row which are greater then \( w_i \) and the insertion algorithm continues with the insertion of \( a \) to the next row. Observe that after finitely many steps the insertion algorithm terminates with a new appearing cell on some row of \( P_{i-1} \). The resulting tableau is then \( P_i \) and recording tableau \( Q_i \) is found by filling this new cell in \( Q_{i-1} \) with the number \( i \). We illustrate these algorithms with the following example.

**Example 2.5.** Let \( w = 52413 \). Then,

\[
P_1 = \begin{array}{c}
5 \\
2 \\
4 \\
5 \\
\end{array}, \quad
P_2 = \begin{array}{c}
2 \\
5 \\
4 \\
\end{array}, \quad
P_3 = \begin{array}{c}
2 \\
5 \\
\end{array}, \quad
P_4 = \begin{array}{c}
1 \\
4 \\
\end{array}, \quad
P_5 = \begin{array}{c}
1 \\
3 \\
\end{array} = P(w)
\]

\[
Q_1 = \begin{array}{c}
1 \\
2 \\
3 \\
\end{array}, \quad
Q_2 = \begin{array}{c}
1 \\
2 \\
\end{array}, \quad
Q_3 = \begin{array}{c}
1 \\
2 \\
\end{array}, \quad
Q_4 = \begin{array}{c}
1 \\
3 \\
\end{array}, \quad
Q_5 = \begin{array}{c}
1 \\
3 \\
\end{array} = Q(w)
\]

There are two equivalence relations introduced by Knuth which have very important applications in the combinatorics of tableaux.

**Definition 2.6.** For \( u \in S_n \) consider the following relation: If \( u_i < u_{i+2} < u_{i+1} \) or \( u_i < u_{i-1} < u_{i+1} \) for some \( i \) then

\[
u = u_1 \ldots u_{i-1} (u_i u_{i+1}) u_{i+2} \ldots u_n \sim^K u_1 \ldots u_{i-1} (u_i u_{i+1}) u_{i+2} \ldots u_n = u'.
\]

We say \( u, w \in S_n \) are Knuth equivalent, \( u \sim^K w \), if \( w \) can be obtained from \( u \) by applying a sequence of \( \sim^K \) relations. On the other hand if \( w^{-1} \sim^K w'^{-1} \) then \( u \) and \( w \) are called dual Knuth equivalent, \( u \simw^K \).

The following theorem given by Knuth [11] is fundamental.

**Theorem 2.7.** Let \( u, w \in S_n \). Then

i) \( u \sim^K w \iff P(u) = P(w) \)

ii) \( u \sim^w w \iff Q(u) = Q(w) \).

We next illustrate the forward and backward slides of Schützenberger’s jeu de taquin [18] without the definition. We remark that jeu de taquin slides can be used to give alternative descriptions of both the Robinson-Schensted algorithm and Knuth relations. The following theorem provided by Schützenberger in [18] reveal this connection.

**Example 2.8.** Below we illustrate a forward slide on the tableau \( S \) through cell \( c_{12} \) and backward slide on the tableau \( T \) through cell \( c_{32} \).

\[
S = \begin{array}{c}
\bullet & 4 \\
2 & 5 \\
1 & 3 \\
\end{array} \rightarrow \begin{array}{c}
2 & 4 \\
\bullet & 5 \\
1 & 3 \\
\end{array} \rightarrow \begin{array}{c}
2 & 4 \\
3 & 5 \\
1 & \bullet \\
\end{array}
\]

\[
T = \begin{array}{c}
2 & 4 \\
3 & 5 \\
1 & \bullet \\
\end{array} \rightarrow \begin{array}{c}
2 & 4 \\
\bullet & 5 \\
1 & 3 \\
\end{array} \rightarrow \begin{array}{c}
\bullet & 4 \\
1 & 3 \\
\end{array}
\]

**Theorem 2.9.** If \( P \) is a partial skew tableau that is brought to a normal tableau \( P' \) by slides, then \( P' \) is unique. In fact, \( P' \) is the insertion tableau for the row word of \( P \).

**Definition 2.10.** The set of two adjacent cell \( A = \{(i, j), (i, j + 1)\} \) (or \( A = \{(i, j), (i + 1, j)\} \)) is called a horizontal (or respectively vertical) domino cell. By a labeling of domino cell \( A \) we mean a pair of positive numbers \((a, a')\) which label the boxes of \( A \) such that \( a \leq a' \) and \( a \) labels the cell of \( A \) which is smaller in the lexicographic order. When we want to indicate the domino cell \( A \) with its labeling, we use the notation

\[
[A, (a, a')]
\]

so that \( \text{shape}([A, (a, a')]) = A \) and \( \text{label}([A, (a, a')]) = (a, a') \).
Let $\lambda$ be a partition and $A$ be a domino cell. If $\lambda \oplus A$ is a partition then $A$ is called an empty domino corner of $\lambda$ whereas if $\lambda \ominus A$ is also a partition then $A$ is called a domino corner of $\lambda$. Clearly, if a partition has no domino corner then it must be a $r$-staircase shape $(r, \ldots, 2, 1)$ for some $r > 0$. On the other hand it is easy to see that any partition $\lambda$ can be reduced uniquely to a $r$-staircase shape $(r, \ldots, 2, 1)$ for some $r \geq 0$, by subsequent removal of existing domino corners one at a time. In this case we say $\lambda$ has a 2-core equivalent to $(r, \ldots, 2, 1)$ and we denote by $P(2n, r)$ the set of all such partitions of size $2n + r(r + 1)/2$.

**Definition 2.11.** A $r$-domino tableau $T$ of shape $\lambda \in P(2n, r)$ is obtained by tiling the skew subtableau $\lambda'/(r, \ldots, 2, 1)$ of $\lambda$ with labeled horizontal or vertical dominos $\{[A_1, (a_1, a_1)], \ldots, [A_n, (a_n, a_n)]\}$ such that $a_i \neq a_j$ for $i \neq j$ and the labels increase from left to right and from top to bottom. In this case

$$\text{label}(T) = \{a_1, a_2, \ldots, a_n\}.$$  

A standard $r$-domino tableau $T$ is a $r$-domino tableau which has $\text{label}(T) = \{1, \ldots, n\}$. We denote by $SDT^r(n)$ the set of all standard $r$-domino tableaux of $n$ dominos.

**Definition 2.12.** Let $T$ be a $r$-domino tableau and $\lambda = \text{shape}(T)$. For $a \in \text{label}(T)$ and $A$ a domino cell in $\lambda$ we define,

1. $\text{Dom}(T, a)$ to be the domino cell of $T$ whose both cells are labeled with $a$ in $T$.
2. $\text{dom}(T, a) = \text{shape}(\text{Dom}(T, a))$.
3. $\text{label}(T, A)$ to be the pair of integers $(a, a')$ which label the domino cell $A$ in $T$, where $a \leq a'$.

**Example 2.13.** For example the following is a 2-domino tableau in $SD^2(5)$.

$$T = \begin{array}{ccc}
1 & 1 & 5 \\
3 & 4 & 4 \\
2 & 3 & 5 \\
2 & & \end{array}$$

Here $\text{shape}(T)$ has two domino corners: $A_1 = \{(1, 5), (2, 5)\}$ and $A_2 = \{(2, 4), (2, 5)\}$ whereas $\text{label}(T, A_1) = (5, 5)$ and $\text{label}(T, A_2) = (4, 5)$. On the other hand $\text{dom}(T, 5) = A_1$ and $\text{dom}(T, 4) = \{(2, 3), (2, 4)\} \neq A_2$.

**Definition 2.14.** For two $r$-domino tableaux $S$ and $T$ satisfying $S \subset T$ we define $T/S = T - S$ to be the skew $r$-domino tableau determined by $S$ and $T$.

**Definition 2.15.** Let $R$ and $R'$ be two skew $r$-domino tableaux and let $\text{shape}(R) = \gamma$ and $\text{shape}(R') = \gamma'$.

1. If $\gamma \ominus \gamma'$ is defined and $R \cup R'$ corresponds to some skew $r$-domino tableau as a set then we define $R \oplus R' = R \cup R'$.
2. If $\gamma \ominus \gamma'$ is defined and if $R - R'$ corresponds to some skew $r$-domino tableau as a set then we define $R \ominus R' = R - R'$.

**Definition 2.16.** Let $T$ be a (skew) $r$-domino tableau and $a \in \text{label}(T)$. Then we define

1. $T_{<a}$ ($T_{\leq a}$) to be the $r$-domino tableau obtained by restricting $T$ to its dominos labeled with integers less than (and equal to) $a$.
2. $T_{>a}$ ($T_{\geq a}$) to be the skew $r$-domino tableau which obtained by restricting $T$ to its dominos labeled with integers greater than (and equal to) $a$.

2.1. **Garfinkle’s algorithm.** Recall that a signed permutation $\alpha \in B_n$ is a bijection of $[-n, +n]$ such that $\alpha^{-1} = -\alpha$. The usual presentation of $\alpha \in B_n$ is denoted as $\alpha = \alpha_1 \alpha_2 \ldots \alpha_n$ where $\alpha_i = \alpha(i)$ for $1 \leq i \leq n$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} = \{1, 2, \ldots, n\}$.

In [3] Garfinkle provide an algorithm for $r = 0, 1$ by which any signed permutation $\alpha \in B_n$ is assigned bijectively to a pair of same shape standard $r$-domino tableau ($P^r(\alpha), Q^r(\alpha)$), where $P^r(\alpha)$ is called insertion and $Q^r(\alpha)$ is called recording tableau of $\alpha$. Her algorithm is extended by van Leeuwen [12] for larger cores.
In the following we will explain how to insert an integer into a \( r \)-domino tableau according to Garfinkle’s algorithm. Let \( T \) be a \( r \)-domino tableau which does not have any domino labeled with \( a \). We denote by

\[
T^{i_α}
\]

the tableau which is obtained by insertion \( a \) into \( T \). Observe that \( T_{\leq a} = T_{\leq a} \) since \( a \notin \text{label}(T) \). We first concatenate a horizontal domino labeled with \((a, a)\) to the first row of \( T_{\leq a} \) from the right if \( a > 0 \), otherwise a vertical domino labeled with \((a, a)\) to the first column of \( T_{\leq a} \) from the bottom. Let us denote the resulting tableau by \((T_{\leq a})^{i_α}\). If the skew tableau \( T_{> a} \) is empty then we have

\[
T^{i_α} = (T_{\leq a})^{i_α}.
\]

Otherwise let \( a = a_0 \) and let \( a_1, a_2, \ldots, a_t \) be the increasing sequence of the labels of domino in \( T_{> a} \). We will find \( T^{i_α} \) in the following sequence

\[
T^{i_α} = ((T_{\leq a_0})^{i_α} \leftarrow a_1 \leftarrow a_2 \ldots \leftarrow a_t = \ldots = (T_{\leq a_i})^{i_α} \leftarrow a_{i+1} \ldots \leftarrow a_t = \ldots = (T_{\leq a_{t-1}})^{i_α} \leftarrow a_t
\]

where \((T_{\leq a_{i+1}})^{i_α} = (T_{\leq a_i})^{i_α} \leftarrow a_{i+1}\) is obtained by sliding the domino cell \( \text{Dom}(T, a_{i+1}) \) over \((T_{\leq a_i})^{i_α}\) in the following way: For \( 0 \leq i < t \), let

\[
B_i = \text{shape}((T_{\leq a_i})^{i_α}) - \text{shape}(T_{\leq a_i}).
\]

We first assume that \( A_{i+1} = \text{dom}(T, a_{i+1}) = \{(k, l), (k, l + 1)\} \) is a horizontal domino cell. Then we have the following possibilities:

\[ H_1 \] \( A_{i+1} \cap B_i = \emptyset \). Then \((T_{\leq a_{i+1}})^{i_α} = (T_{\leq a_i})^{i_α} \oplus \text{Dom}(T, a_{i+1})\).

\[ H_2 \] \( A_{i+1} = B_i \). Then a horizontal domino labeled with \((a_{i+1}, a_{i+1})\) is concatenated to the \((k + 1)\)-th row of \((T_{\leq a_i})^{i_α}\) from the right in order to find \((T_{\leq a_{i+1}})^{i_α}\).

\[ H_3 \] \( A_{i+1} \cap B_i = \{(k, l)\} \). Then \((T_{\leq a_{i+1}})^{i_α} = (T_{\leq a_i})^{i_α} \oplus \{(k, l), (k + 1, l + 1)\}, (a_{i+1}, a_{i+1})\} \).

Now we assume that \( A_{i+1} = \text{dom}(T, a_{i+1}) = \{(k, l), (k + 1, l)\} \) is a vertical domino cell. Then we have the following possibilities:

\[ V_1 \] \( A_{i+1} \cap B_i = \emptyset \). Then \((T_{\leq a_{i+1}})^{i_α} = (T_{\leq a_i})^{i_α} \oplus \text{Dom}(T, a_{i+1})\).

\[ V_2 \] \( A_{i+1} = B_i \). Then a vertical domino labeled with \((a_{i+1}, a_{i+1})\) is concatenated to the \((l + 1)\)-th column of \((T_{\leq a_i})^{i_α}\) from the bottom in order to find \((T_{\leq a_{i+1}})^{i_α}\).

\[ V_3 \] \( A_{i+1} \cap B_i = \{(k, l)\} \). Then \((T_{\leq a_{i+1}})^{i_α} = (T_{\leq a_i})^{i_α} \oplus \{(k + 1, l), (k + 1, l + 1)\}, (a_{i+1}, a_{i+1})\} \).

Then insertion and recording \( r \)-domino tableaux for any \( \alpha = \alpha_1 \ldots \alpha_n \) is found in the following way: Suppose that \( P_0 \) and \( Q_0 \) are the tableaux of shape \((r, \ldots, 2, 1)\) whose cells are all filed with \( 0 \). For \( \alpha = \alpha_1 \ldots \alpha_n \in B_n \) let \( P_{i+1} = P_{i}^{\alpha_i} \) and let \( Q_{i+1} \) be obtained from \( Q_{i} \) by filing the newly appearing the domino corner of \( P_{i+1} \) with \((i + 1, i + 1)\) in \( Q_{i+1} \). Then one can obtain \( P^\alpha(\alpha) \) and \( Q^\alpha(\alpha) \) by erasing all zeros of \( P_n \) and respectively \( Q_n \).

We also want to remark that as one of the main features of the Garfinkle’s algorithm we have \( T^{i - a} = (T^i)^{1 - a} \) where \( T^i \) is the transpose of \( T \).

**Example 2.17.** We will find \( T^{1 - 2} \) for \( T = \begin{array}{ccc}
1 & 1 & 3 \\
4 & 5 & 5 \\
4 & 6 & 6
\end{array} \) where \( T_{\leq 2} = T_{\leq 2} = \begin{array}{c}
1 & 1 \\
4 & 5 & 5 \\
4 & 6 & 6
\end{array} \).
2.1.1. Reverse insertion of domino corners. Based on the previous description of Garfinkle’s algorithm we now describe the reverse-insertion of domino corners from \( r \)-domino tableaux. Then we will provide several lemmas which are the main tool in the proof of Theorem 1.2 and Theorem 1.3.

Let \( T \) be a \( r \)-domino tableau and \( A \) be a domino corner in shape(\( T \)). We denote by

\[
T^\dagger A \quad \text{and} \quad \eta(T^\dagger A)
\]

respectively the tableau which is obtained by reverse-insertion of \( A \), and the number which is bumped out of \( T \) as a result of reverse insertion out of \( T \). In the following We assume that \( A \) is a horizontal domino corner. When a vertical domino corner is under consideration one can apply the same algorithm on the transpose of \( T \), since

\[
T^\dagger A = ((T^\dagger)^\dagger)^t \quad \text{and} \quad \eta(T^\dagger A) = -\eta((T^\dagger)^\dagger).
\]

**Case 1:** If \( A = \{(1,1), (1, l+1)\} \) lies on the first row of shape(\( T \)), then \( \text{label}(T,A) = (b,b) \) for some integer \( b \). In that case \( T^\dagger A \) is obtained by erasing \( \text{Dom}(T,b) = [A, (b,b)] \) from \( T \) and \( \eta(T^\dagger A) = b \).

**Case 2:** Let \( A = \{(k,l), (k, l+1)\} \) and \( k > 1 \). We first assume that \( \text{label}(T,A) = (b,b) \) as it is indicated by the barred letters below. Then the insertion algorithm indicates that both labels \( b \) must be bump from the \((k-1)\)th row by some horizontal domino cell say \( A' \). Therefore the reverse insertion \( A \) is obtained by first replacing labels of \( A' \) by \( (b,b) \) and continuing the reverse insertion with the domino cell \( A' \). Let \( y \) be the largest of all the numbers in the \((k-1)\)th row which are smaller than \( b \). Then we have either \( \text{label}(T,A') = (y,y) \) or \( \text{label}(T,A') = (x,y) \) for some number \( x < y \). In the following the barred letters indicate the domino cell which is pushed back during the reverse insertion algorithm.

\[
\text{It is clear that the reverse insertion of } A \text{ does not have any effect on the skew domino tableau } T_{>y}.
\]

Therefore \( \eta(T^\dagger A) = \eta((T_{\leq y})^\dagger A') \) and

\[
T^\dagger A = (T_{\leq y})^\dagger A' \oplus [A', (b,b)] \oplus T_{>y}.
\]

**Case 3:** again let \( A = \{(k,l), (k, l+1)\} \) and \( k > 1 \). This time we assume that \( \text{label}(T,A) = (y,b) \) for some \( y < b \). The insertion algorithm indicates that bottom cell of \( \text{Dom}(T,b) \) must be bump from the \((k-1)\)th row by some smaller number. Therefore in the reverse insertion, bottom label \( b \) must be rotated to the left to the top label \( b \).

Let \( A' = \{(k-1, l)(k,l)\} \). Then as in the previous case \( \eta(T^\dagger A) = \eta((T_{\leq y})^\dagger A') \) but this time

\[
T^\dagger A = (T_{\leq y})^\dagger A' \oplus \{(k-1, l), (k-1, l+1)\}, (b,b) \oplus T_{>y}.
\]

**Corollary 2.18.** Let \( T \) be an \( r \)-domino tableau and \( A \) is a domino corner. Furthermore let \( A' \) be the domino cell which is pushed back by \( A \) in reverse insertion \( T^\dagger A \). Then

i) If \( A = \{(i,j), (i, j+1)\} \) is horizontal, then either \( A' = \{(i-1,j), (i,j)\} \) or \( A' \subset \{(i-1,k) \mid k \geq j\} \).

ii) If \( A = \{(i,j), (i+1, j)\} \) is vertical, then either \( A' = \{(i,j-1), (i,j)\} \) or \( A' \subset \{(k,j-1) \mid k \geq i\} \).
Example 2.19. Let \( S, T \in SD^2(5) \) as given below. We will show that \( \eta(S^TA) = 1 \) and \( \eta(T^1B) = -2 \) where \( A = \{(3,3),(4,3)\} \) and \( B = \{(4,2),(5,3)\} \).

\[
S = \begin{array}{cccc}
1 & 1 & \downarrow & 1 \\
3 & 3 & \downarrow & 3 \\
2 & 4 & 7 & 2 \\
2 & 5 & 7 & 2 \\
\end{array}
\quad \implies \quad
\begin{array}{cccc}
1 & 1 & \downarrow & 1 \\
3 /4 & 3 /4 & \downarrow & 3 /4 \\
2 & 5 & 7 & 2 \\
2 & 5 & 7 & 2 \\
\end{array}
\quad \implies \quad
\begin{array}{cccc}
1 & 1 & \downarrow & 1 \\
3 /4 & 3 /4 & \downarrow & 3 /4 \\
2 & 5 & 7 & 2 \\
2 & 5 & 7 & 2 \\
\end{array}
\quad \implies \quad
\begin{array}{cccc}
1 & 1 & \downarrow & 1 \\
3 /4 & 3 /4 & \downarrow & 3 /4 \\
2 & 5 & 7 & 2 \\
2 & 5 & 7 & 2 \\
\end{array}
= S^TA
\]

\[
T = \begin{array}{cccc}
1 & 1 & \downarrow & 1 \\
3 & 3 & \downarrow & 3 \\
2 & 4 & 5 & 2 \\
2 & 4 & 5 & 2 \\
\end{array}
\quad \implies \quad
\begin{array}{cccc}
1 & 1 & \downarrow & 1 \\
3 & 3 & \downarrow & 3 \\
2 & 4 /5 & 5 & 2 \\
2 & 4 /5 & 5 & 2 \\
\end{array}
\quad \implies \quad
\begin{array}{cccc}
1 & 1 & \downarrow & 1 \\
3 & 3 & \downarrow & 3 \\
2 & 4 /5 & 5 & 2 \\
2 & 4 /5 & 5 & 2 \\
\end{array}
\quad \implies \quad
\begin{array}{cccc}
1 & 1 & \downarrow & 1 \\
3 & 3 & \downarrow & 3 \\
2 & 4 /5 & 5 & 2 \\
2 & 4 /5 & 5 & 2 \\
\end{array}
= T^1B
\]

Following lemma is given in [3] and it is essential in Garfinkel’s algorithm which defines a bijection between signed permutations and pair of same shape standard \( r \)-domino tableaux. One can easily see that the proof follows easily by induction on the size of the tableaux.

**Lemma 2.20.** Let \( T \) be a \( r \)-domino tableau and \( A \) be a domino corner of shape(\( T \)). Then \( T^1A \) and \( \eta(T^1A) \) are unique.

**Definition 2.21.** Let \( T \) be a \( r \)-domino tableau and \( A \) be a domino corner of shape(\( T \)) such that \( A = \{(i,j),(i,j+1)\} \) or \( A = \{(i,j),(i+1,j)\} \). We denote by \((T,A,ne)\) and \((T,A,sw)\) the regions of \( T \)

\[
(T,A,ne) := \{(k,l) \mid k < i \mbox{ and } l \geq j\}
\]

\[
(T,A,sw) := \{(k,l) \mid k \geq i \mbox{ and } l < j\}
\]

as illustrated in Figure 11.

Now we are ready to give the following lemma which is crucial in the proof of Theorem 2.2.

**Lemma 2.22.** Let \( T \) be a \( r \)-domino tableau and \( A \) be a domino corner of shape(\( T \)).

i) Suppose \( B \) is a domino corner of shape(\( T^1A \)) which lies in the portion \((T,A,sw)\). Then

\[
\eta(T^1A | B) < \eta(T^1A).
\]

ii) Suppose \( B \) is a domino corner of shape(\( T^1A \)) which lies in the portion \((T,A,ne)\). Then

\[
\eta(T^1A | B) > \eta(T^1A).
\]

**Proof.** Observe that it is enough to prove the lemma when \( A \) is a horizontal domino corner of shape(\( T \)) since \( -\eta(T^1A) = \eta((T')^1A') \). In the following we proceed by induction on the size of dominos in \( T \). Let label\((T,A) = (a',a)\).

i) Suppose that \( B \) is a domino cell in \((T,A,sw)\) such that label\((T,B) = (b',b)\). First observe \( a \) can not be equal to \( b \) since \( A \) is horizontal and \( B \) lies in \((T,A,sw)\).

We first assume that \( a > b \). Since the reverse insertion of \( A \) does not have any effect on the domino cells whose labels are greater then \( a \), we can conclude that

\[
\eta(T^1A) = \eta(T^1A_{<a}) \quad \mbox{and} \quad \eta(T^1A | B) = \eta((T_{a})^1A | B).
\]
If \( T_{\leq a} \neq \emptyset \) one can apply induction argument on \( T_{\leq a} \) in order to get desired result. If \( T_{\geq a} = \emptyset \) it means that \( T_{\geq a} = T \) and \( a \) is the largest number in label(\( T \)). Observe from Corollary 2.18 that the domino cell, say \( A' \), which is pushed back by \( A \) in the first step of reverse insertion \( T^{\uparrow A} \) must be a domino corner of \( T_{\leq a} \) and moreover \( B \) still lies in \( (T_{\leq a}, A', \text{sw}) \). Therefore \( \eta(T^{\uparrow A}) = \eta(T^{\uparrow A}_{\leq b}) \) and \( \eta(T^{\uparrow A_{1}B}) = \eta(T^{\uparrow A_{1}B}_{\leq b}) \).

On the other hand by induction hypothesis we have \( \eta((T_{\leq a})^{\uparrow A_{1}B}) < \eta((T_{\leq a})^{\uparrow A}) \) and the desired results follows.

Now we assume that \( b > a \). Since the reverse insertion of \( A \) does not have any effect on dominos whose label greater then \( a \) we have

\[
\eta(T^{\uparrow A}) = \eta(T^{\uparrow A}_{\leq b}) = \eta((T_{\leq b})^{\uparrow A}) \text{ and } \eta(T^{\uparrow A_{1}B}) = \eta(T^{\uparrow A_{1}B}_{\leq b}) = \eta((T_{\leq b})^{\uparrow A_{1}B}).
\]

If \( T_{\geq b} \neq \emptyset \), the result follows from induction argument as in the previous case. On the other hand, if \( T_{\geq b} = \emptyset \) then \( b \) is the largest label in \( T \). Therefore the domino cell, say \( B' \) of \( T^{\uparrow A} \), which is pushed back by \( B \) in the reverse insertion \( T^{\uparrow A_{1}B} \) must be a domino corner of \( (T^{\uparrow A})_{\leq b} = (T_{\leq b})^{\uparrow A} \). Then clearly \( \eta(T^{\uparrow A_{1}B}) = \eta(T^{\uparrow A_{1}B}_{\leq b}) \). On the other hand since \( b > a \), domino labeled by \( (b, b) \) must lie strictly below domino labeled by \( a \), which also implies by Corollary 2.18 that \( B' \) must lie in \( (T_{\leq b}, A, \text{sw}) \). Therefore induction argument applied on \( T_{\leq b} \) gives \( \eta((T_{\leq b})^{\uparrow A_{1}B'}) < \eta((T_{\leq b})^{\uparrow A}) \). Therefore \( \eta(T^{\uparrow A_{1}B}) < \eta(T^{\uparrow A}) \).

\[ \text{ii)} \] Now suppose that \( B \) is a domino cell in \( (T, A, \text{ne}) \) such that label\( (T, B) = (b', b) \). We first assume that \( a < b \). If \( T_{\geq b} \neq \emptyset \) then one can apply induction argument as in the previous cases, since

\[
\eta(T^{\uparrow A}) = \eta((T_{\leq b})^{\uparrow A}) = \eta((T_{\leq b})^{\uparrow A}) \text{ and } \eta(T^{\uparrow A_{1}B}) = \eta((T^{\uparrow A_{1}B}_{\leq b}) \).
\]

Otherwise let \( B' \) the domino cell of \( T^{\uparrow A} \) which is pushed back by \( B \) in the reverse insertion \( T^{\uparrow A_{1}B} \). Since \( b \) is the largest label of \( T \), one can deduce that \( B' \) must be a domino corner of \( (T_{\leq b})^{\uparrow A_{1}B} \) and moreover \( \eta(T^{\uparrow A_{1}B}) = \eta((T_{\leq b})^{\uparrow A_{1}B'}) \). On the other hand \( B' \) also lies also in \( (T_{\leq b}, A, \text{ne}) \). Then we have by induction \( \eta((T_{\leq b})^{\uparrow A_{1}B'}) > \eta((T_{\leq b})^{\uparrow A}) \). Therefore \( \eta(T^{\uparrow A_{1}B}) > \eta(T^{\uparrow A}) \).

Now we will show the desired result for \( a \geq b \). If \( T_{\geq a} \neq \emptyset \) one can apply induction argument on \( T_{\leq a} \) in order to get the desired result. Otherwise let \( A' \) be a domino cell of \( T \) which is pushed back by \( A \) in the first step of the reverse insertion \( T^{\uparrow A} \). Observe that since \( a \) is the largest label in \( T \), \( A' \) must be a domino corner of \( T_{\leq a} \) and moreover \( \eta(T^{\uparrow A}) = \eta((T_{\leq a})^{\uparrow A'}) \). On the other hand let \( B' \) be the domino cell of \( T^{\uparrow A} \) which is pushed back by \( B \) in the first step of reverse insertion \( T^{\uparrow A_{1}B} \). One can observe that \( B' \) is a domino corner of \( (T_{\leq a})^{\uparrow A'} \) lying in \( (T_{\leq a}, A', \text{ne}) \) and \( \eta(T^{\uparrow A_{1}B}) = \eta((T_{\leq a})^{\uparrow A_{1}B'}) \). Therefore again induction argument on \( T_{\leq a} \) gives the desired result.

\[ \square \]

2.2. Barbash and Vogan algorithm. We will now explain the algorithm which is provided by Barbash and Vogan in [1] to establish the bijection between signed permutations and standard \( r \)-domino tableaux for \( r = 0, 1 \) whereas its extension for larger cores is provided in [2]. We also remark that the equivalence of Barbash-Vogan algorithm to Garfinkle’s algorithm for \( r = 0, 1 \), is due to Van Leeuwen [12].
Recall that for $\alpha = \alpha_1 \alpha_2 \ldots \alpha_n$ is a signed permutation the palindrome representation of $\alpha$ is given by $\alpha^0 = \pi_{n-1} \pi_{n-2} \ldots \pi_1 \alpha_1 \alpha_2 \ldots \alpha_n$ if $\alpha$ lies in $C_n$, or $\alpha^1 = \pi_{n-1} \pi_{n-2} \ldots \pi_1 0 \alpha_1 \alpha_2 \ldots \alpha_n$ if $\alpha$ lies in $B_n$, where $\pi_i = -\alpha_i$. We will call $\alpha^0$ and $\alpha^1$ as 0-core and 1-core representation of $\alpha$ respectively. By following the approach of [2], let us describe how to extend this representation for larger cores. We first identify \{1, 2, \ldots, r(r+1)/2\} with \{0, 1, 2, \ldots, n\} together with the total ordering $-n < \ldots < -2 < -1 < 0 < 2 < \ldots < n$.

Let $w \in S_{r(r+1)/2}$ be a permutation under this identification, whose RSK insertion tableau is of shape $(r, r-1, \ldots, 1)$. Now for $\alpha \in B_n$ let $r$-core representation of $\alpha$ to be $\alpha^r = \pi_{n-1} \pi_{n-2} \ldots \pi_1 w \alpha_1 \alpha_2 \ldots \alpha_n$.

The algorithm introduced by Barbash and Vogan for $r = 0$ and $r = 1$ first applies RSK algorithm on $\alpha^0$ and respectively $\alpha^1$. Then starting from the lowest number $\bar{i}$, it vacates the negative integer $\bar{i}$ in the tableaux by jeu de taquin slides until it becomes adjacent to $i$, where the vacation is repeated for $i-1$ until $i = 1$. The following example illustrates this algorithm for $r = 1$.

**Example 2.23.** For $\alpha = 3 \ 1 \ 2 \in B_n$ let $\alpha^1 = 2 \ 1 \ 3 \ 0 \ 3 \ 1 \ 2$ be its 1-core representation. Then Barbash-Vogan algorithm yields:

\[
P(\alpha^1) = \begin{array}{ccc}
1 & 2 & 3 \\
2 & 0 & 3 \\
1 & & \\
\end{array} \quad \begin{array}{ccc}
1 & 2 & 2 \\
0 & 3 & 3 \\
1 & & \\
\end{array} \quad \begin{array}{ccc}
0 & 2 & 2 \\
1 & 3 & 3 \\
1 & & \\
\end{array} = P^1(\alpha).
\]

Similarly $Q(\alpha^1) = \begin{array}{ccc}
3 & 2 & 1 \\
1 & 0 & 3 \\
2 & & \\
\end{array} \quad \begin{array}{ccc}
1 & 1 & \\
2 & 3 & 3 \\
2 & & \\
\end{array} = Q^1(\alpha)$.

On the other hand by the result of [2], one only needs to apply the same algorithm on $\alpha^r$ in order to find $r$-domino tableaux $P^r(\alpha)$ and $Q^r(\alpha)$ for larger cores.

**Theorem 2.24 ([2], Theorem 3.3).** Signed permutations $\alpha$ and $\beta$ have the same insertion $r$-domino tableau if and only if $\alpha^r$ and $\beta^r$ have the same RSK insertion tableau.

The following proposition is a consequence of Theorem 2.23 and Theorem 2.24. Recall the definition of $D_{\bar{i}}$ from Definition 1.1.

**Proposition 2.25.** Let $\alpha$ and $\beta$ be two signed permutations which differ by a single $D_{\bar{i}}$ relation. Then $P^r(\alpha) = P^r(\beta)$, in other words $\alpha$ and $\beta$ have the same insertion $r$-domino tableau.

**2.2.1. Descents of domino tableaux and Vogan's map.** Recall that $B_n$ carries a Coxeter group structure with the generator set $S = \{s_0, s_1, \ldots, s_{n-1}\}$ where $s_i = (i, i+1) | 1 \leq i \leq n-1$ is the set of transpositions which also generates the symmetric group $S_n$ and $s_0$ corresponds to the transposition $(-1, 1)$. Let $l(\alpha)$ denote the length of $\alpha$, which is the minimum number of generators of $\alpha$ and let

\[
\text{Des}_L(\alpha) := \{ i \mid l(s_i \alpha) < l(\alpha) \text{ and } 0 \leq i \leq n-1 \} = \{ i \mid 1 \leq i \leq n-1 \text{ and } i+1 \text{ comes before } i \text{ in } \alpha^0 \} \cup \{ 0 \mid 1 \text{ comes before } -1 \text{ in } \alpha^0 \}
\]

\[
\text{Des}_R(\alpha) := \{ i \mid l(s_i \alpha) < l(\alpha) \text{ and } 0 \leq i \leq n-1 \} = \{ i \mid 1 \leq i \leq n-1 \text{ and } \alpha_i > \alpha_{i+1} \} \cup \{ 0 \mid \alpha_1 < 0 \}
\]
denote respectively the sets of left and right descents of $\alpha$.

Now we define the descent set of a $r$-domino tableau $T$ in the following way:

\[
\text{Des}(T) := \{ i \mid \text{domino labeled with } (i, i+1) \text{ lies below the one labeled with } (i, i) \} \cup \{ 0 \mid \text{domino labeled with } (1, 1) \text{ is vertical} \}
\]
It is well known property of RSK algorithm that for a permutation \( w \in S_n \), we have \( \text{Des}_L(w) = \text{Des}(P(w)) \) where the descent set of a (skew or Young) tableau \( T \) is defined by \( \text{Des}(T) = \{ i \mid i + 1 \text{ lies below } i \text{ in } T \} \). On the other hand jeu de taquin slides do not change the descent sets of tableaux, therefore the following result is a consequence of Theorem 2.24.

**Corollary 2.26.** For \( \alpha \in B_n \) we have \( \text{Des}_L(\alpha) = \text{Des}(P^r(\alpha)) \).

Observe that if \( \alpha \) and \( \beta \) differ by a single \( D_i^r \) relations in \( B_n \) then \( P^r(\alpha) = P^r(\beta) \) and we have either \( i \in \text{Des}_L(\alpha^{-1}) \) but \( i + 1 \notin \text{Des}_L(\alpha^{-1}) \) or \( i \notin \text{Des}_L(\alpha^{-1}) \) but \( i + 1 \in \text{Des}_L(\alpha^{-1}) \) and either \( \beta^{-1} = s_i \cdot \alpha^{-1} \) or \( \beta^{-1} = s_{i+1} \cdot \alpha^{-1} \)

for some \( 1 \leq i \leq n - 2 \). In the following we will follow Garfinkle’s approach in [3] in order to study the effect of a single \( D_i^r \) relations on the recording tableaux \( Q^r(\alpha) \) and \( Q^r(\beta) \).

For \( i, j \) two adjacent integers satisfying \( 1 \leq i, j \leq n - 1 \), consider the following sets:

\[
D_{i,j}(B_n) := \{ \alpha \in B_n \mid i \in \text{Des}_L(\alpha) \text{ but } j \notin \text{Des}_L(\alpha) \}
\]

\[
D_{i,j}(SDT^r(n)) := \{ T \in SDT^r(n) \mid i \in \text{Des}(T) \text{ but } j \notin \text{Des}(T) \}
\]

together with the map \( V_{i,j} : D_{i,j}(B_n) \to D_{i,j}(B_n) \) where \( V_{i,j}(\alpha) = \{ s_i \cdot \alpha, s_j \cdot \alpha \} \cap D_{i,j}(B_n) \). We also define a map

\[
V_{i,j} : D_{i,j}(SDT^r(n)) \to D_{i,j}(SDT^r(n))
\]

in the following manner: Without lost of generality we assume that \( j > i \), in other words \( j = i + 1 \). Observe that if \( i \in \text{Des}(T) \) but \( i + 1 \notin \text{Des}(T) \) then \( i + 1 \) lies strictly below \( i \) in \( T \) whereas \( i + 2 \) lies strictly right to \( i + 1 \) in \( T \). On the other hand we have two cases according to the positions of dominos labeled with \( (i, i) \) and \( (i + 2, i + 2) \) with respect to each other.

**Case 1.** We first assume that \( i + 2 \) lies strictly below \( i \) in \( T \). Since the \( i + 2 \) lies strictly to the right of \( i + 1 \) and \( i + 1 \) lies below \( i \) we have two cases to consider: If the boundaries \( \text{Dom}(T, i + 1) \) and \( \text{Dom}(T, i) \) intersect at most on a point then \( V_{i,i+1}(T) \) is obtained by interchanging the labels \( i \) and \( i + 1 \) in \( T \). Otherwise there is only one possibility which satisfies \( i + 2 \) lies below \( i \) and it lies to the right of \( i + 1 \), in which \( T \) has the subtableau \( U \) as illustrated below and \( V_{i,i+1}(T) \) is obtained by substituting \( U \) with \( U' \) in \( T \).

**Case 2.** Now we assume \( i + 2 \) lies strictly right to \( i \) in \( T \). Again if the boundaries \( \text{Dom}(T, i + 1) \) and \( \text{Dom}(T, i + 2) \) intersect at most on a point then \( V_{i,i+1}(T) \) is obtained by interchanging the labels \( i + 1 \) and \( i + 2 \) in \( T \). Otherwise there is only one possible cases where \( T \) has the subtableau \( U \) given below and \( V_{i,i+1}(T) \) is obtained by substituting \( U \) with \( U' \) in \( T \).

**Example 2.27.** We have \( T_2 = V_{5,6}(T_1), T_3 = V_{3,4}(T_2), \) and \( T_4 = V_{4,5}(T_3) = V_{6,5}(T_3) \) for the following tableaux.
Remark 2.28. The map \( V_{i,j} \) is first introduced on the symmetric group by Vogan [20], in the aim of classifying the primitive ideals in the universal enveloping algebra of complex semi simple Lie algebras. In fact when it is considered on the symmetric group the map \( V_{i,j} \) produces nothing but the dual Knuth relation on the permutations and their insertion tableaux.

Lemma 2.29. Let \( i \) and \( j \) be two consecutive integers such that \( 1 \leq i, j \leq n-1 \). Suppose \( \alpha \in D_{i,j}(B_n) \). Then \( P^r(\alpha) \in D_{i,j}(SDT^r(n)) \) and
\[
P^r(V_{i,j}(\alpha)) = V_{i,j}(P^r(\alpha)).
\]

Proof. This result is first proven by Garfinkle [4], Theorem 2.19. for \( r = 0, 1 \). On the other hand one can check that her proof does not rely on the specific value of \( r \) and it can easily be extended for any value of \( r \). We omit the proof for the sake of place.

The following result follows directly from Lemma 2.29 and it has an important role in the proof of Theorem 1.2.

Corollary 2.30. Suppose signed permutations \( \alpha \) and \( \beta \) in \( B_n \) differ by a single \( D^r \) relation. Then for some consecutive integers \( 1 \leq i, j \leq n-1 \) we have
\[
\beta^{-1} = V_{i,j}(\alpha^{-1}) \quad \text{and} \quad Q^r(\beta) = V_{i,j}(Q^r(\alpha)).
\]

3. Plactic and coplactic relations for \( r \)-domino tableaux

3.1. Proof of Theorem 1.2. In this section we will prove the main Theorem 1.2, i.e., we will show that the relations \( D_1^r, D_2^r \) and \( D_3^r \) from Definition 1.1 are sufficient and necessary to characterize plactic classes of standard \( r \)-domino tableaux.

Proof of Theorem 1.2. Let \( \alpha \) and \( \beta \) be two signed permutations which differ by a sequence of \( D_1^r, D_2^r \) or \( D_3^r \) relations. By using Garfinkle’s insertion algorithm for \( r \)-domino tableau and Proposition 2.25 it is easy to check that \( P^r(\alpha) = P^r(\alpha') \) if \( \alpha \) and \( \alpha' \) differs by a single \( D^r \) relations for \( i = 1, 2, 3 \). Therefore \( P^r(\alpha) \) must be equal to \( P^r(\beta) \).

Now we let \( \alpha = \alpha_1 \ldots \alpha_{n-1} \alpha_n \) and \( \beta = \beta_1 \ldots \beta_{n-1} \beta_n \) such that \( T = P^r(\alpha) = P^r(\beta) \). We will show by induction that \( \alpha \triangleright \beta \). Let \( r \geq 0 \) arbitrary. If \( n = 1 \) there is nothing to prove. Therefore we assume that the statement holds for all signed permutations of size \( n-1 \).

If \( P^r(\alpha_1 \ldots \alpha_{n-1}) = P^r(\beta_1 \ldots \beta_{n-1}) = T^{1A} \) for some domino corner \( A \) of shape \( T \) then \( \alpha_n = \beta_n \) by Lemma 2.20. By induction argument we can assume that \( \alpha_1 \ldots \alpha_{n-1} \triangleright \beta_1 \ldots \beta_{n-1} \). Therefore \( \alpha = \alpha_1 \ldots \alpha_{n-1} \triangleright \beta_1 \beta_{n-1} \beta_n = \beta \).

If \( P^r(\alpha_1 \ldots \alpha_{n-1}) \neq P^r(\beta_1 \ldots \beta_{n-1}) \) then there exist two different domino corners say \( A \) and \( B \) of \( T \) such that
\[
T^{1A} = P^r(\alpha_1 \ldots \alpha_{n-1}) \quad \text{and} \quad \eta(T^{1A}) = \alpha_n \n
T^{1B} = P^r(\beta_1 \ldots \beta_{n-1}) \quad \text{and} \quad \eta(T^{1B}) = \beta_n.
\]

In the following we proceed according to the orientation of \( A \) and \( B \) as illustrated in Figure 2 where in the first four pictures \( (T, A, \text{ne}) \cap (T, B, \text{sw}) \) is represented with the shaded areas.

Cases (1), (2) and (3): We will first show that \( \alpha \triangleright \beta \) for the first three cases of Figure 2. Consider the domino corner \( B \) of \( T^{1A} \) and let \( b = \eta(T^{1A}B) \). It is easy to see that there exists a domino corner, say \( C \) of \( T^{1A}B \) which lies in \( (T, A, \text{ne}) \cap (T, B, \text{sw}) \). Let \( c = \eta(T^{1A}B|C) \) and \( \tilde{u} \) be a signed word such that \( P^r(\tilde{u}) = T^{1A}B|C \). Therefore by Lemma 2.20 we have
\[
P^r(\tilde{u} \alpha \beta c a_n) = P^r(\tilde{u}) \sim \eta(T^{1A}B|C) = \eta(T^{1A}B) = T^{1A} = P^r(\alpha_1 \ldots \alpha_{n-1}).
\]
and by induction hypothesis \( \tilde{u} \alpha \beta c a_n \sim \alpha_1 \ldots \alpha_{n-1} \) since \( P^r(\tilde{u} \alpha \beta c) = T^{1A} = P^r(\alpha_1 \ldots \alpha_{n-1}) \). Therefore letting \( u \) denote the signed permutation \( \tilde{u} \alpha \beta c a_n \), we have \( \alpha \triangleright \beta u \).

Observe that since \( P^r(\tilde{u}) = T^{1A}B|C \), the recording tableau \( Q^r(\tilde{u} \alpha \beta c a_n) \) has its domino cells \( A, B \) and \( C \) labeled with \((n,n)\), \((n-1,n-1)\) and \((n-2,n-2)\) respectively.
On the other hand having $B$ in $(T, A, ne)$ and $C$ in $(T, B, sw)$ yields by Lemma 2.22 that $b = \eta(T^{\uparrow A\downarrow B}) > \eta(T^{\uparrow A}) = \alpha_n$ and $b = \eta(T^{\uparrow A\downarrow B}) > \eta(T^{\uparrow A\downarrow B\downarrow C}) = c$. Therefore we have

\begin{equation}
\text{either } b > \alpha_n > c, \text{ and hence } u = \hat{u}c\alpha_n \cong \hat{u}\alpha_n = w \text{ and } Q^r(w) = V_{n-1,n-2}(Q^r(u))
\end{equation}

or $b > c > \alpha_n$, and hence $u = \hat{u}c\alpha_n \cong \hat{u}\alpha_n b = w$ and $Q^r(w) = V_{n-2,n-1}(Q^r(u))$

The last argument implies that in both cases signed permutation $w$ has its recording tableau $Q^r(w)$ obtained by interchanging the labels $(n, n)$ of $A$ and $(n-1, n-1)$ of $B$ in $Q^r(u)$ i.e., $Q^r(w)$ had the domino corner $B$ labeled with $(n, n)$. Then by Lemma 2.20 we have

\begin{equation}
P^r(w_1 \ldots w_{n-1}) = T^\uparrow B = P^r(\beta_1 \ldots \beta_{n-1}) \text{ and } w_n = \beta_n
\end{equation}

and by induction argument $w_1 \ldots w_{n-1} \cong \beta_1 \ldots \beta_{n-1}$. Therefore $w \cong \beta$, and $\alpha \cong u \cong w \cong \beta$.

**Case (4):** For the fourth case of Figure 2 let $\alpha, \beta \in B_n$ as in 3.1. If there exists a domino corner in $(T, A, ne) \cap (T, B, sw)$ then one can follow the same argument which is used for Cases (1), (2), (3). On the other hand it may happen that $(T, A, ne) \cap (T, B, sw)$ is a staircase shape and in the following we consider several subcases illustrated in Figure 3.

Observe that, in case $T$ has the configuration of Figure 3(a), we have $n \leq r + 1$, $\alpha_n < 0$, $\beta_n > 0$ and

\begin{equation}
\eta(T^{\uparrow A\downarrow B}) = \beta_n
\end{equation}

\begin{equation}
\eta(T^{\uparrow B\downarrow A}) = \alpha_n
\end{equation}

\begin{equation}
P^r(\alpha_1 \ldots \alpha_{n-2}) = T^\uparrow A\downarrow B = T^\uparrow B\downarrow A = P^r(\beta_1 \ldots \beta_{n-2}).
\end{equation}

Let $\check{u}$ be a signed word such that $P^r(\check{u}) = T^\uparrow A\downarrow B = T^\uparrow B\downarrow A$. Clearly $\check{u}$ must be a shuffle of positive decreasing and negative increasing sequences and $P^r(\check{u}\alpha_n \beta_n) = T = P^r(\check{u}\beta_n \alpha_n)$. Therefore $\check{u}\alpha_n \beta_n \cong \check{u}\beta_n \alpha_n$. On the other hand $P^r(\check{u}\beta_n) = T^\uparrow A$ and $P^r(\check{u}\alpha_n) = T^\uparrow B$ and by induction hypothesis we have $\alpha_1 \ldots \alpha_{n-1} \cong \check{u}\beta_n$ and $\beta_1 \ldots \beta_{n-1} \cong \check{u}\alpha_n$. Hence

\begin{equation}
\alpha = \alpha_1 \ldots \alpha_{n-1} \alpha_n \cong \check{u}\beta_n \alpha_n \cong \check{u}\alpha_n \beta_n \cong \beta_1 \ldots \beta_{n-1} \beta_n = \beta.
\end{equation}

If $T$ has the configuration of Figure 3(b) or Figure 3(c), $T$ may have a domino corner, say $C$ lying in $(T, A, ne)$. Let $\sigma = \sigma_1 \ldots \sigma_n \in B_n$ such that $P^r(\sigma_1 \ldots \sigma_{n-1}) = T^\uparrow C$. Observe that there exist a domino corner in $(T, C, ne) \cap (T, A, sw)$ and $(T, C, ne) \cap (T, B, sw)$ therefore we can apply the argument which is used for Cases (1), (2) and (3) in order to get $\alpha \cong \sigma$ and $\beta \cong \sigma$ and hence $\alpha \cong \beta$. On the other hand if $C$ lies in $(T, B, sw)$ the same argument applied on $T^\uparrow B$ gives the desired result.

Now we assume that $T$ has the configuration of Figure 3(d) i.e. the corner $C$ and $A$ (or $C'$ and $B$) intersect. Again we let $\sigma_1 \ldots \sigma_n \in B_n$ such that $P^r(\sigma_1 \ldots \sigma_{n-1}) = T^\uparrow C$. Observe that there is a corner in $(T, C, ne) \cap (T, B, sw)$ therefore $\beta \cong \sigma$ follows. We only need to show $\alpha \cong \sigma$. 

**Figure 2.**
Observe that since $T$ has the configuration of Figure 2 (d) we have a domino corner $A'$ of $T \uparrow A$ and $A''$ of $T \uparrow A' \uparrow A''$ as it is illustrated in Figure 3. Let $a' = \eta(T \uparrow A' \uparrow A'')$ and $a'' = \eta(T \uparrow A' \uparrow A'' \uparrow A')$. Suppose $\tilde{u}$ be a signed word such that $P^r(\tilde{u}) = T \uparrow A' \uparrow A''$. Then the signed permutation $u = \tilde{u}a''a'\alpha_n$ has $P^r(u) = T$ whereas it is recording tableau $Q^r(u)$ must have the form shown in Figure 4.

On the other hand since $P^r(\tilde{u}a''a') = T \uparrow A = P^r(\alpha_1 \ldots \alpha_{n-1})$ we have $\tilde{u}a''a' \preceq \alpha_1 \ldots \alpha_{n-1}$ by induction hypothesis and therefore $u = \tilde{u}a''a'\alpha_n \preceq \alpha_1 \ldots \alpha_{n-1}\alpha_n = \alpha$.

Furthermore having $A'$ in $(T, A, sw)$ and $A''$ in $(T, A, ne)$ yields by Lemma 2.22 that
\[
a' = \eta(T \uparrow A' \uparrow A') < \eta(T \uparrow A') = \alpha_n \quad \text{and} \quad a'' = \eta(T \uparrow A' \uparrow A'') < \eta(T \uparrow A' \uparrow A'') = \alpha''.
\]

Therefore we have
\[
either a'' > \alpha_n > a', \quad \text{and hence} \quad u = \tilde{u}a''a'\alpha_n \overset{D^r}{\sim} \tilde{u}a''a''\alpha_n = w \quad \text{and} \quad Q^r(w) = V_{n-1, n-2}(Q^r(u))
\]
\[\text{or} \quad \alpha_n > a'' > a', \quad \text{and hence} \quad u = \tilde{u}a''a'\alpha_n \overset{D^r}{\sim} \tilde{u}a''a''a' = w \quad \text{and} \quad Q^r(w) = V_{n-2, n-1}(Q^r(u)).
\]

In both cases Corollary 2.30 yields that the recording tableau $Q^r(w)$ of $w$ has the form illustrated in Figure 4 and by Lemma 2.20 we have
\[
P^r(w_1 \ldots w_{n-1}) = T \uparrow C = P^r(\sigma_1 \ldots \sigma_{n-1}) \quad \text{and} \quad w_n = \beta_n.
\]
Then by induction hypothesis $w_1 \ldots w_{n-1} \preceq \sigma_1 \ldots \sigma_{n-1}$ and hence $w \preceq \sigma$. Hence as desired
\[
\alpha \preceq u \overset{D^r}{\sim} w \overset{D^r}{\sim} \beta.
\]

**Case (5):** Again let $\alpha, \beta \in B_n$ as in (5) and suppose that $T$ has the configuration illustrated in Figure 2 (5). We consider Figure 5 for several cases.

If $T$ has a corner say $C$ lying in $(T, A, sw)$, as it is illustrated in Figure 5 (a), let $\sigma_1 \ldots \sigma_n \in B_n$ such that $P^r(\sigma_1 \ldots \sigma_{n-1}) = T \uparrow C$. Since there is a domino corner (namely $A$) in $(T, C, ne) \cap (T, B, sw)$ we have $\beta \preceq \sigma$
as in the Cases (1),(2) and (3). If $C$ is a horizontal domino corner then by the argument of the Cases (1),(2) and (3) we have $\alpha \overset{p}{\sim} \sigma$, otherwise the argument of Case (4) gives that $\alpha \overset{p}{\sim} \sigma$. Therefore $\beta \overset{p}{\sim} \alpha$. On the other hand $T$ has a corner say $C'$ lying in $(T, B, ne)$ one can use the same argument in the transpose of $T$.

If there no domino corner in $T$ other then $A$ and $B$ there are two possibility as illustrated in Figure 5. Observe that the case given in Figure 5-(b) is just the transpose of the case illustrated in Figure 4-(d), therefore it follows directly that $\alpha \overset{p}{\sim} \beta$.

For the later case shown in Figure 5-(c), observe that the shaded area is a $r$-staircase shape and we must have either the domino corner $A$ or $B$ of $T$ labeled by $(n,n)$. Here we assume $A$ is labeled by $(n,n)$ since for the other case one can consider the transpose tableau $T^t$. So as Figure 4(c) illustrates, let $x_1 \ldots x_k$ be the labels of horizontal domino cells and $y_1 \ldots y_l$ be the vertical domino cells which are both positive decreasing sequence such that $r+1 = k+l$. Observe that $\eta(T^tA) = \eta(T^tB) = x_k > 0$ therefore $\alpha_n = \beta_n = x_k$. Let $\tilde{u}$ be a signed word which is a shuffle of $x_1 \ldots x_{k-1}$ and $-y_1 \ldots -y_l$. It is easy to see that $P^r(n\tilde{u}x_k) = T = P^r(-n\tilde{u}x_k)$, and $n\tilde{u}x_k \overset{D^r}{\sim} (-n)\tilde{u}x_k$. On the other hand $P^r(n\tilde{u}) = T^tA$ and $P^r(-n\tilde{u}) = T^tB$ and by induction hypothesis we have $n\tilde{u} \overset{p}{\sim} \alpha_1 \ldots \alpha_{n-1}$ and $(-n)\tilde{u} \overset{p}{\sim} \beta_1 \ldots \beta_{n-1}$. Hence

$$\alpha = \alpha_1 \ldots \alpha_{n-1}x_k \overset{p}{\sim} n\tilde{u}x_k \overset{D^r}{\sim} (-n)\tilde{u}x_k \overset{p}{\sim} \beta_1 \ldots \beta_{n-1}x_k = \beta.$$ 

3.2. Description of coplactic operations on $r$-domino tableaux. In this section we explain the effect of $r$-plactic relations on the recording tableaux.

We first assume that $\alpha \overset{D^r}{\sim} \beta$, i.e., $|\alpha_1| > |\alpha_i|$ for all $2 \leq i \leq r+2$ and $\alpha_1 \ldots \alpha_{r+2}$ is a shuffle of some positive decreasing and negative increasing sequences, then $\beta$ is obtained by interchanging the signed of $\alpha_1$
\( \alpha = \alpha_1 \alpha_2 \ldots \alpha_{r+2} \ldots \alpha_n \overset{D^r}{\sim} (-\alpha_1) \alpha_2 \ldots \alpha_{r+2} \ldots \alpha_n = \beta \)

W.L.O.G. we assume that \( \alpha_1 > 0 \). Then \( Q(\alpha)_{\leq r+2} \) is obtained by tiling some horizontal dominos labeled by \( \{1 = x_1, x_2, \ldots, x_k\} \) and vertical dominos labeled by \( \{y_1, \ldots, y_l\} \) around the \( r \)-staircase as illustrated in Figure 6(a). Moreover

\[
\{x_1, \ldots, x_k\} \cup \{y_1, \ldots, y_l\} = \{1, \ldots, r+1, r+2\}.
\]

Let \( A \) be the domino corner of \( Q(\alpha)_{\leq r+2} \) which is labeled by \( (r+2, r+2) \). Let \( a = \eta((Q(\alpha)_{\leq r+2})^A) \). Observe that since \( \alpha_1 > 0 \) we must have \( a = 1 \), otherwise \( a = -1 \). Moreover

\[
Q(\beta)_{\leq r+2} = (Q(\alpha)_{\leq r+2})^A \uparrow \downarrow a
\]

and \( Q(\beta) = Q(\beta)_{\leq r+2} \oplus Q(\alpha)_{> r+2} \).

Now we assume that \( \alpha \) and \( \beta \) are equivalent through single \( D^r \) i.e. for some \( j \leq r \),

\[
\alpha = \alpha_1 \ldots \alpha_j \alpha_{j+1} \ldots \alpha_{r+2} \ldots \alpha_n \overset{D^r}{\sim} \alpha_1 \ldots \alpha_j \alpha_{j+1} \alpha_j \ldots \alpha_{r+2} \ldots \alpha_n = \beta.
\]

where \( \alpha_j > 0, \alpha_{j+1} < 0 \) (or vice versa) and \( \alpha_1 \ldots \alpha_j \alpha_{j+1} \) is a shuffle of some positive decreasing and negative increasing sequence ending with \( \alpha_j \) and \( \alpha_{j+1} \) (or vice versa), respectively. Then observe that \( Q(\alpha)_{\leq j+1} \) is obtained by tiling some horizontal dominos labeled by \( \{x_1, \ldots, x_k = j\} \) and vertical dominos labeled by \( \{y_1, \ldots, y_l = j+1\} \) around the \( r \)-stair case as illustrated in Figure 6(b) such that

\[
\{x_1, \ldots, x_k\} \cup \{y_1, \ldots, y_l\} = \{1, \ldots, j, j+1\}.
\]

Then \( Q(\beta)_{\leq j+1} \) is found by interchanging the labels \( j \) and \( j+1 \) in \( Q(\alpha)_{\leq j+1} \). Therefore \( Q(\beta) = Q(\beta)_{\leq j+1} \oplus Q(\alpha)_{> j+1} \).

Lastly when \( \alpha \overset{D^r}{\sim} \beta \), the configurations given in Figure 6(c) and their transposes illustrates the results of Corollary 2.30.
4. Open cycles and $r$-cycle relations.

**Definition 4.1.** Let $\lambda \in P(2n, r)$ and $(i, j) \in \lambda/(r, r-1, \ldots, 1)$.

i) If $i + j$ and $r$ have the same parity then we say $(i, j)$ is a fixed cell. In this case if $i$ is odd then $(i, j)$ is called as $Y$ cell, otherwise it is called as $Z$ cell.

ii) If $i + j$ and $r + 1$ have the same parity then we say $(i, j)$ is a variable cell. In this case if $i$ is odd then $(i, j)$ is called as $X$ cell, otherwise it is called as $W$ cell.

In the following we suppose that for any $r$-domino tableau $T$ the cells lying in $r$-staircase are labeled with 0. The neighbor cells which lie to the north or to the west of shape$(T)$ are also labeled with 0 whereas the rest of the neighbor cells are labeled with $\infty$. We provide the following 1-domino tableau as an example:

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & * & \infty \\
0 & * & * & \infty \\
0 & \infty & \infty \\
\end{array}
\]

**Definition 4.2.** Let $A = \text{Dom}(T, a)$ for some $a \in \text{label}(T)$ and $(i, j)$ be a fixed cell of shape$(T)$.

i) If shape$(A) = \{(i, j), (i, j + 1)\}$ or shape$(A) = \{(i - 1, j), (i, j)\}$ then let $x$ be the label of the cell $(i + 1, j - 1)$ in $T$. Then

\[
\psi(A) = \begin{cases} 
\{(i, j), (i + 1, j)\}, (a, a) & \text{if } a > x \\
\{(i, j - 1), (i, j)\}, (a, a) & \text{if } a < x 
\end{cases}
\]

ii) If shape$(A) = \{(i, j - 1), (i, j)\}$ or shape$(A) = \{(i, j), (i + 1, j)\}$ then let $x$ be the label of the cell $(i - 1, j + 1)$ in $T$. Then

\[
\psi(A) = \begin{cases} 
\{(i, j), (i, j + 1)\}, (a, a) & \text{if } a > x \\
\{(i - 1, j), (i, j)\}, (a, a) & \text{if } a < x 
\end{cases}
\]

**Definition 4.3.** Let $T$ be a $r$-domino tableau. Then a cycle is a maximum sequence $C = \{A_0, \ldots, A_k\}$ of domino cells in $T$ which satisfy:

1. $A_i = \text{Dom}(T, a_i)$ for some $a_i \in \text{label}(T)$
2. $A_i \neq A_j$ when $i \neq j$
3. shape$(\psi(A_i)) \cap \text{shape}(A_{i+1}) \neq \emptyset$ for $0 \leq i < k$ and shape$(\psi(A_k))$ does not intersect with any domino cell in $T - C$

If shape$(\psi(A_k))$ and shape$(A_0)$ has empty intersection then $C$ is called an open cycle, otherwise they intersect on a variable cell and in this case $C$ is called a closed cycle.

**Example 4.4.** Let $T \in SD^1(7)$ be the tableau given below with the corresponding cell types.

\[
T = \begin{array}{cccc}
1 & 1 & 4 & 4 \\
2 & 2 & 5 & 5 \\
3 & 3 & 7 & 7 \\
6 & 6 & 7 & 7 \\
\end{array}
\]

Then $T$ has one open cycle $C_1 = \{\text{Dom}(T, 5), \text{Dom}(T, 7), \text{Dom}(T, 6)\} = C_{(2, 4), (5, 1)}$ and one closed cycle $C_2 = \{\text{Dom}(T, 2), \text{Dom}(T, 3)\}$. On the other hand observe that shape$(\psi(\text{Dom}(T, 1)))$ only intersect with $T$ in 1-staircase shape. Moreover since shape$(\psi(\text{Dom}(T, 4))) \cap \text{shape}(\text{Dom}(T, 1)) \neq \emptyset$ any cycle containing Dom$(T, 4)$ must also contain Dom$(T, 1)$. Therefore $T$ can not have a cycle which contains Dom$(T, 4)$ or Dom$(T, 1)$. 

**Definition 4.5.** Let $\mathcal{C} = \{A_0, A_1, \ldots, A_k\}$ be a cycle in $T$. We denote by $\Psi_C(T)$ the tableau which is obtained by replacing $\{A_0, A_1, \ldots, A_k\}$ with $\{\psi(A_0), \psi(A_1), \ldots, \psi(A_k)\}$ in $T$. Moreover for a set of cycles $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$ in $T$ we have

$$\Psi_{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k} = \Psi_{\mathcal{C}_1} \circ \Psi_{\mathcal{C}_2} \circ \cdots \circ \Psi_{\mathcal{C}_k}(T).$$

**Example 4.6.** Let $T$ be as in Example 4.4 together with open cycle $C_1$ and closed cycle $C_2$. Then

\[
\begin{array}{|c|c|c|c|c|}
\hline
1 & 1 & 4 & 4 & \\
2 & 2 & 5 & 5 & \\
3 & 3 & 7 & & \\
6 & 6 & 7 & & \\
\hline
\end{array}
\quad \rightarrow \quad
\begin{array}{|c|c|c|c|c|}
\hline
1 & 1 & 4 & 4 & \\
2 & 2 & 5 & & \\
3 & 3 & 5/7 & 7 & \\
6 & 6 & 7 & 7 & \\
\hline
\end{array}
\quad \rightarrow \quad
\begin{array}{|c|c|c|c|c|}
\hline
1 & 1 & 4 & 4 & \\
2 & 2 & 5 & & \\
3 & 3 & 5 & & \\
6 & 6 & 7 & & \\
\hline
\end{array}
\quad = \Psi_{C_1}(T).
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
1 & 1 & 4 & 4 & \\
2 & 2 & 5 & 5 & \\
3 & 3 & 7 & & \\
6 & 6 & 7 & & \\
\hline
\end{array}
\quad \rightarrow \quad
\begin{array}{|c|c|c|c|c|}
\hline
1 & 1 & 4 & 4 & \\
2 & 5 & 5 & & \\
3 & 3 & 7 & & \\
6 & 6 & 7 & & \\
\hline
\end{array}
\quad \rightarrow \quad
\begin{array}{|c|c|c|c|c|}
\hline
1 & 1 & 4 & 4 & \\
2 & 3 & 5 & 5 & \\
3 & 3 & 7 & & \\
6 & 6 & 7 & & \\
\hline
\end{array}
\quad = \Psi_{C_2}(T)
\]

The following two result can directly be deduced from the definition of cycle.

**Lemma 4.7.** Let $\mathcal{C} = \{A_0, A_1, \ldots, A_k\}$ be a cycle in $T$. Then $\mathcal{C}' = \{\psi(A_0), \psi(A_1), \ldots, \psi(A_k)\}$ is also a cycle in $\Psi_C(T)$ and

$$T = \Psi_{\mathcal{C}'} \circ \Psi_C(T).$$

**Lemma 4.8.** For any two cycle $C_1$ and $C_2$ in $T$ we have

$$\Psi_{C_1, C_2}(T) = \Psi_{C_2, C_1}(T).$$

**Remark 4.9.** Let $\mathcal{C} = \{A_0, A_1, \ldots, A_k\}$ is a cycle in $T$ and $T' = \Psi_C(T)$. One can observe that if $\mathcal{C}$ is a closed cycle then

$$\text{shape}(T) = \text{shape}(T').$$

On the other hand if it is an open cycle then it is easy to see that for some variable corner $c$ of $T$ and $c'$ of $T'$ we have

$$\text{shape}(T') = \text{shape}(T) \odot c \oplus c'.$$

In that case we denote the cycle $C$ by $C_{[c,c']}$ and the map $\Psi_C$ by $\Psi_{[c,c']}$. 

**Definition 4.10.** Consider the relation which is given on $r$-domino tableaux which share the same labels:

$$S \preceq T \quad \text{if and only if} \quad S = T \quad \text{or} \quad T = \Psi_{[c_1,c_1']}, \ldots, [c_k,c_k'](S)$$

for some open cycles $C_{[c_1,c_1']}, \ldots, C_{[c_k,c_k']}$ of $S$. We say $S$ and $T$ are cycle equivalent if $S \preceq T$.

**Remark 4.11.** In [15], Pietraho uses the following rule for determining the fixed and variable cells of $\lambda \in P(2n,r)$: if $i + j$ and $r$ have the same parity then $(i, j)$ is called variable cell otherwise it is called fixed cell. In fact his configurations is necessary for his result that for two signed permutation $\alpha$ and $\beta$,

$$P^r(\alpha) \succeq P^r(\beta) \quad \text{if and only if} \quad P^r(\alpha) = P^r(\beta) \quad \text{or} \quad P^{r+1}(\alpha) = P^{r+1}(\beta)$$

whereas the Conjecture B of [2] requires either $P^r(\alpha) = P^r(\beta)$ or $P^{r-1}(\alpha) = P^{r-1}(\beta)$ for being in the same right cell. On the other hand, one can apply his method for showing that with the our configuration given in Definition 4.1

$$P^r(\alpha) \succeq P^r(\beta) \quad \text{if and only if} \quad P^r(\alpha) = P^r(\beta) \quad \text{or} \quad P^{r-1}(\alpha) = P^{r-1}(\beta).$$
4.1. **Proof of Theorem 1.3** The following two lemmas which will be proven later in this section have fundamental role in the proof of Theorem 1.3.

**Lemma 4.12.** Let $S$ and $T$ be two $r$-domino tableaux sharing the same labels. Suppose that $S \sim T$ through one open cycle $C_{[c,c']}$. If there exist horizontal (or vertical) domino corners say $A$ of shape($S$) and $A'$ of shape($T$) which lie in the same row (respectively same column), then

$$
\eta(S^{1A}) = \eta(T^{1A'}) \quad \text{and} \quad S^{1A} \sim T^{1A'} \quad \text{through one open cycle.}
$$

**Lemma 4.13.** Let $S$ and $T$ be two $r$-domino tableaux such that $\text{label}(T) = \text{label}(S)$ and let $S \sim T$ through $t > 0$ open cycle. Let $a$ be an integer such that $|a| \notin \text{label}(T)$. Then $S^{1a} \sim T^{1a}$ through $t - 1$, $t$ or $t + 1$ open cycles.

**Proof of Theorem 1.3.** We first assume that $S \sim T$. If $S = T$ then the statement follows from Theorem 1.2. Observe that it is enough to consider the case where $S$ and $T$ are equivalent through one open cycle and in that case we have

$$\text{shape}(T) = \text{shape}(S) \oplus c \oplus c'$$

for some variable corner $c$ of $S$ and $c'$ of $T$.

If there exist domino corners $A$ of $S$ and $A'$ of $T$, both horizontal (or vertical) lying in the same row of $S$ and $T$, by Lemma 4.12 we have $\eta(S^{1A}) = \eta(T^{1A'}) = x$ and $S^{1A}$ and $T^{1A'}$ are equivalent through one open cycle. By induction we can find two signed words, $\tilde{u}$ and $\tilde{w}$ such that $P^r(\tilde{u}) = S^{1A}$ and $P^r(\tilde{w}) = T^{1A'}$ satisfying $\tilde{u} \sim \tilde{w}$. Then obviously $\tilde{u}x \sim \tilde{w}x$ where $\tilde{u}x$ and $\tilde{w}x$ lie in the plactic class of $S$ and $T$ respectively.

If $S$ and $T$ do not have any domino corners lying in the same row or same column, then observe that we must have $\text{dom}(S,n) = \{(i,j), (i, j + 1)\}$ and $\text{dom}(T,n) = \{(i,j), (i + 1,j)\}$ (or vice versa) for some $i, j > 0$ and the cell $(i,j)$ must be a (fixed) Y or Z cell. Moreover since $S$ and $T$ are equivalent through one open cycle we must have $S \sim T$. Therefore we have two cases:

\[
\begin{array}{|c|c|c|c|c|}
\hline
S & \cdots & \cdots & \cdots & \cdots \\
\hline
\cdots & a_1 & a_1 & \cdots & \cdots \\
\cdots & n & \cdots & \cdots & \cdots \\
\hline
\cdots & b_1 & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\hline
\end{array}
\quad \quad \quad \quad \quad
\begin{array}{|c|c|c|c|c|}
\hline
T & \cdots & \cdots & \cdots & \cdots \\
\hline
\cdots & a_1 & a_1 & \cdots & \cdots \\
\cdots & n & \cdots & \cdots & \cdots \\
\hline
\cdots & b_1 & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\hline
\end{array}
\quad \quad \quad \quad \quad
\begin{array}{|c|c|c|c|c|}
\hline
S & \cdots & \cdots & \cdots & \cdots \\
\hline
\cdots & a_1 & a_1 & \cdots & \cdots \\
\cdots & n & \cdots & \cdots & \cdots \\
\hline
\cdots & b_1 & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\hline
\end{array}
\quad \quad \quad \quad \quad
\begin{array}{|c|c|c|c|c|}
\hline
T & \cdots & \cdots & \cdots & \cdots \\
\hline
\cdots & a_1 & a_1 & \cdots & \cdots \\
\cdots & n & \cdots & \cdots & \cdots \\
\hline
\cdots & b_1 & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\hline
\end{array}
\]

where in the second case the vacated cell (which is labeled by $n$) is a fixed cell, and this contradicts to definition of the cycle.

So we just need to consider the first case. As it is illustrated above, let $a_1 \ldots a_k$ be the labels of horizontal domino cells and $b_1 \ldots b_l$ be the labels of vertical domino cells which are both positive decreasing sequence such that $r = k + l$. Observe that any shuffle $\tilde{u}$ of $a_1 \ldots a_k$ and $-b_1 \ldots -b_l$ lies in the plactic classes of $S_{\leq n}$ and $T_{\leq n}$ and moreover $P^r(n\tilde{u}) = S$ and $P^r(-n\tilde{u}) = T$. On the other hand $n\tilde{u}$ and $n\tilde{u}$ are equivalent through $D_3^{-1}$.

Now for the reverse direction it is enough to assume that $\alpha$ and $\beta$ differ by a single $D_1$, $D_2$, $D_3$ or $D_3^{-1}$ relations. By Theorem 1.2 we have $P^r(\alpha) = P^r(\beta)$ if $\alpha$ and $\beta$ differ by a single $D_1$, $D_2$, $D_3$ relations, therefore $P^r(\alpha) \sim P^r(\beta)$. One can check easily that when $\alpha$ and $\beta$ differ by a single $D_3^{-1}$ then $P^r(\alpha_1 \ldots \alpha_{r+2}) \sim P^r(\beta_1 \ldots \beta_{r+2})$ through one open cycle which consists of only the domino labeled by $(n,n)$. On the other hand since $\alpha_{r+3} \ldots \alpha_n = \beta_{r+3} \ldots \beta_n$, by Lemma 4.12 we have

$$
P^r(\alpha) = P^r(\alpha_1 \ldots \alpha_{r+2})^{1\alpha_{r+3} \ldots \alpha_n} \sim P^r(\beta_1 \ldots \beta_{r+2})^{1\beta_{r+3} \ldots \beta_n} = P^r(\beta).
$$

\[\square\]

**Remark 4.14.** If the relation $D_3^{-1}$ of Definition 1.1 is replaced by $D_2^{+1}$ then one can show, by following the same argument in the proof of Theorem 1.3 that $\alpha \sim \beta$ if and only if either $P^r(\alpha) = P^r(\beta)$ or...
$P^{r+1}(\alpha) = P^{r+1}(\beta)$. Therefore in this case $\alpha \preceq \beta$ gives $P^r(\alpha) \preceq P^r(\beta)$ according to the labeling used by Pietraho in [15].

In the following we will first prove the Lemma 4.12. Then we provide more detailed version of the Lemma 4.13, namely Lemma 4.15, together with its proof.

**Proof of Lemma 4.12.** We will prove the statement by induction and again we just consider the case where $A$ and $A'$ are both horizontal domino corners lying on the same row, since otherwise one can use the same argument on the transpose of $S$ and $T$ to get the desired result. So let $r \geq 0$ be arbitrary number. One can check easily that the statement of lemma is true when $S$ and $T$ is tiled by one labeled domino.

Let $n$ be the maximum of all numbers labeling $S$ (and so $T$) and $m$ be the maximum of the all numbers which label $A$ and $A'$.

**Case 1.** We first assume that $m < n$. In that case observe that $A$ and $A'$ are still domino corners of shape($S_{<n}$) and shape($T_{<n}$) respectively and since the reverse insertion of $A$ in $S$ (and $A'$ in $T$) does not have any effect on Dom($S,n$) (and Dom($T,n$)), we have

$$
\eta(S^{1,A}) = \eta((S_{<n})^{1,A}) \quad \text{and} \quad \eta(T^{1,A'}) = \eta((T_{<n})^{1,A'})
$$

(4.1)

$$S^{1,A} = (S_{<n})^{1,A} \oplus \text{Dom} (S,n) \quad \text{and} \quad T^{1,A'} = (T_{<n})^{1,A'} \oplus \text{Dom} (T,n).$$

We have three subcases:

**Case 1.1.** Dom$(S,n) \notin C_{[c,c']}$.

In that case we must have Dom$(S,n) = \text{Dom}(T,n)$ and therefore $S_{<n} \preceq T_{<n}$ through the open cycle $C_{[c,c']}$. By induction argument we can assume that $(S_{<n})^{1,A} \preceq (S_{<n})^{1,A}$ through one open cycle say $C_{[c,c']}$ and $\eta(S^{1,A}) = \eta(T^{1,A})$. On the other hand (4.1) yields that $\eta(S^{1,A}) = \eta(T^{1,A})$ and $S^{1,A} \preceq T^{1,A'}$ through the open cycle $C_{[c,c']}$.

**Case 1.2.** $C_{[c,c']} = \{\text{Dom}(S,n)\}$. In that case we must have and $\psi(\text{Dom}(S,n)) = \text{Dom}(T,n)$ where

dom$(S,n) = \{(i,j), (i,j+1)\}$ and dom$(T,n) = \{(i,j), (i+1,j)\}$ (or vice versa)

where $(i,j)$ is a fixed cell for some $i$ and $j$. Moreover the tableaux $S_{<n}$ and $T_{<n}$ must be equal as well as the domino cells $A$ and $A'$, therefore by Lemma 2.20 we have $(S_{<n})^{1,A} = (T_{<n})^{1,A}$ and $\eta(S^{1,A}) = \eta(T^{1,A})$.

On the other hand from (4.1) it follows that $S^{1,A} \preceq T^{1,A'}$ through the open cycle $C_{[c,c']}$.

**Case 1.3.** \{Dom$(S,n)$\} $\subseteq C_{[c,c']}$. In that case we have Dom$(T,n) = \psi(\text{Dom}(T,n))$. If dom$(S,n) = \{(i,j), (i,j+1)\}$ with the fixed cell $(i,j)$ then dom$(T,n) = \{(i,j-1), (i,j)\}$. Therefore $C_{[c,c']} = C_{[(i,j), (i,j+1), c]}$ and $S_{<n} \preceq T_{<n}$ through one open cycle $C_{[(i,j), (i,j+1), c]}$. By induction argument $\eta((S_{<n})^{1,A}) = \eta(T_{<n})^{1,A}$ and $(S_{<n})^{1,A} \preceq (T_{<n})^{1,A}$ through one open cycle $C_{[(i,j), (i,j+1), c]}$ since $A$ and $A'$ can not lie on the $i$-th row. Therefore by (4.1) we again have $(S_{<n})^{1,A} \preceq (T_{<n})^{1,A}$ through the open cycle $C_{[(i,j), (i,j+1), c]}$.

If dom$(S,n) = \{(i,j), (i,j+1)\}$ with the fixed cell $(i,j+1)$ then dom$(T,n) = \{(i-1,j+1), (i,j+1)\}$. This implies that $S_{<n} \preceq T_{<n}$ through two open cycle $C_{[(i,j), (i,j+1), c]}$ and $C_{[(i-1,j+1), c]}$. By induction argument $\eta(S^{1,A}) = \eta(S^{1,A})$ and $(S_{<n})^{1,A} \preceq (T_{<n})^{1,A}$ through two open cycles $C_{[(i,j), (i,j+1), c]}$ and $C_{[(i-1,j+1), c]}$. Then from (4.1) it follows that Dom$(S,n)$ joins these two cycles and therefore $S^{1,A} \preceq T^{1,A}$ through one open cycle $C_{[c,c']}$.

**Case 2.** So from now on we suppose that $m = n$, therefore at least one of the numbers labeling $A$ and $A'$ is $n$. We have several possibilities as illustrated with the figures below, where the barred letters represents domino cells $A$ and $A'$ in $S$ and $T$ respectively.

**Case 2.1.** label$(S,A) = (n,n)$ and label$(T,A') = (n,n)$. Since $A$ and $A'$ lie in the same row $S_{<n}$ and $T_{<n}$ must be cycle equivalent through one open cycle and the reverse insertion of $A$ and $A'$ bumps a horizontal domino corners say $B$ and $B'$ which lies in the same row of shape$(S_{<n})$ and shape$(T_{<n})$ respectively. Then clearly

$$
\eta(S^{1,A}) = \eta((S_{<n})^{1,B}) \quad \text{and} \quad S^{1,A} = (S_{<n})^{1,B} \oplus [B,(n,n)]
$$

$$\eta(T^{1,A'}) = \eta((T_{<n})^{1,B'}) \quad \text{and} \quad T^{1,A'} = (T_{<n})^{1,B'} \oplus [B',(n,n)]$$

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On the other hand, induction argument yields that
\[ \eta((S_{\leq n})^B) = \eta((T_{\leq n})^{B'}) \quad \text{and} \quad (S_{\leq n})^B \preceq (T_{\leq n})^{B'} \]
through one open cycle, say \( C_{[c_1,c_1']} \). Therefore if \( B = B' \) then \( S^{1A} \preceq T^{1A} \) through the cycle \( C_{[c_1,c_1']} \cup \{ \text{Dom}(S^{1A}, n) \} \).

**Case 2.2.** \( \text{label}(S, A) = (\ast, n) \) and \( \text{label}(T, A') = (\ast, n) \). Observe that since \( A \) and \( A' \) are horizontal domino corners lying in the same row of \( \text{shape}(S) \) and \( \text{shape}(T) \) respectively, we must have for some \( i \) and \( j \), \( \text{dom}(S, n) = \text{dom}(T, n) = \{(i - 1, j), (i, j)\} \) where \( A = A' = \{(i, j - 1), (i, j)\} \). Therefore we have only one possibility:

\[
\begin{array}{ccc}
S & T & S^{1A} & T^{1A'} \\
\ast & \ast & n & n \\
\ast & \pi & \pi & \pi \\
\pi & \ast & \pi & \pi \\
\end{array}
\]

Here \( S_{\leq n} \preceq T_{\leq n} \) through the open cycle \( C_{[c_1,c_1']} \). Let \( B \) and \( B' \) be the domino corners of \( \text{shape}(S_{\leq n}) \) and \( \text{shape}(T_{\leq n}) \) respectively such that \( B = \{(i - 1, j - 1), (i, j - 1)\} = B' \). Therefore
\[ \eta(S^{1A}) = \eta((S_{\leq n})^B) \quad \text{and} \quad S^{1A} = (S_{\leq n})^B \odot \{(i - 1, j - 1), (i, j - 1), (i, j), (n, n)\} \]
\[ \eta(T^{1A'}) = \eta((T_{\leq n})^{B'}) \quad \text{and} \quad T^{1A'} = (T_{\leq n})^{B'} \odot \{(i - 1, j - 1), (i, j - 1), (i, j), (n, n)\} \]

By induction argument \( \eta((S_{\leq n})^B) = \eta((T_{\leq n})^{B'}) \) and \( (S_{\leq n})^B \preceq (T_{\leq n})^{B'} \) through one open cycle say \( C_{[c_1,c_1']} \). Therefore \( \eta(S^{1A}) = \eta(T^{1A'}) \) and as it is illustrated above \( S^{1A} \preceq T^{1A'} \) through the same open cycle \( C_{[c_1,c_1']} \).

**Case 2.3.** \( \text{label}(S, A) = (n, n) \) and \( \text{label}(T, A') = (\ast, n) \) or vice versa. Clearly \( \psi(\text{Dom}(S, n)) = \text{Dom}(T, n) \) and from the definition of the map \( \psi \) it follows that
\[ \text{dom}(S, n) = \{(i, j), (i, j + 1)\} \quad \text{and} \quad \text{dom}(T, n) = \{(i - 1, j + 1), (i, j + 1)\} \]
where \( (i, j + 1) \) is a fixed cell. Therefore \( A = A' = \{(i, j), (i, j + 1)\} \) and there are four possibilities as illustrated below:

\[
\begin{array}{ccc}
S & T & \preceq \\
\ast & \ast & z \\
\ast & \ast & z \\
\ast & \ast & z \\
\ast & \pi & \pi \\
\ast & \pi & \pi \\
\pi & \ast & \pi \\
\pi & \ast & \pi \\
\end{array}
\]

Here we assume that \( x < z \) where the domino cell \( \{(i - 1, j), (i - 1, j + 1)\} \) is labeled with \( (z, z) \) and the cell \( (i, j - 1) \) is labeled by \( x \) in \( S \). In that case we have a closed cycle which consist of \( \text{Dom}(S, n) \) and \( \text{Dom}(S, z) \) which contradicts to the fact that \( T \) is obtained from \( S \) through one open cycle.

\[
\begin{array}{ccc}
S & T & S^{1A} & T^{1A'} \\
\ast & \ast & z & z \\
\ast & \ast & z & z \\
\ast & \ast & z & z \\
\ast & \pi & \pi & \pi \\
\ast & \pi & \pi & \pi \\
\ast & \ast & \pi & \pi \\
\ast & \ast & \pi & \pi \\
\ast & \ast & \pi & \pi \\
\ast & \ast & \pi & \pi \\
\end{array}
\]

Here \( x > z \) where the domino cell \( \{(i - 1, j), (i - 1, j + 1)\} \) is labeled with \( (z, z) \) and the cell \( (i, j - 1) \) is labeled by \( x \) in \( S \). Observe that the cell \( (i - 1, j - 1) \) in \( S \) can not be labeled by \( x \), since \( x > z \) and the cell \( (i, j) \) in \( T \) can only be labeled by \( x \). Moreover the domino cells \( \text{Dom}(S, n), \text{Dom}(S, z) \) and \( \text{Dom}(S, x) \) are all belong to \( C_{[c_1,c_1']} \). Let
\[ B = \{(i - 1, j), (i - 1, j + 1)\} \quad \text{and} \quad B' = \{(i - 1, j - 1), (i, j - 1)\} \]
be the domino cells of \( S \) and \( T \) respectively and let \( S' = S_{>x} \cap S_{\leq n} \) i.e., \( S' \) is the subtableau of \( S \) which consist of all the domino cells whose labels greater than \( x \) but less than \( n \). Similarly let \( T' = T_{>x} \cap T_{\leq n} \).
Then

\[ \eta(S^{1A}) = \eta((S_{<x})^{1B}) \quad \text{and} \quad \eta(T^{1A'}) = \eta((T_{<x})^{1B'}) \]

\[ S^{1A} = (S_{<x})^{1B} \oplus \text{Dom}(S, x) \oplus S' \oplus \{(i - 1, j), (i - 1, j + 1)\}, \{n, n\} \]

\[ T^{1A'} = (T_{<x})^{1B'} \oplus \{(i, j - 1), (i - 1, j - 1)\}, \{x, x\} \oplus T' \oplus \{(i - 1, j), (i - 1, j + 1)\}, \{n, n\} \]

By induction argument \( \eta((S_{<x})^{1B}) = \eta((T_{<x})^{1B'}) \) and \( (S_{<x})^{1B} \preceq (T_{<x})^{1B'} \) through one open cycle say \( \mathcal{C}_{[c_1, c'_1]} \). Then \( \eta(S^{1A}) = \eta(T^{1A'}) \) and \( S^{1A} \preceq T^{1A'} \) through one open cycle which is equal to

\[ \mathcal{C}_{[c_1, c'_1]} \cup (S' \cap \mathcal{C}_{[c, c']}) \cup \{\text{Dom}(S, x)\}. \]

Here we have \( y > x \) where the domino cell \( \{(i - 1, j), (i - 1, j + 1)\} \) is labeled by \((y, z)\) and the cell \((i, j - 1)\) be labeled by \(x\) in \( S \) where \( y > x \). Observe that in that case the cells \((i, j - 1)\) and \((i - 1, j)\) which are labeled by \(x\) and \(y\) above are fixed cell. Therefore the cell \((i, j)\) in \( T \) must be labeled by \(y\) and moreover domino cells \(\text{Dom}(S, n)\), \(\text{Dom}(S, z)\) and \(\text{Dom}(S, y)\) are all belong to \(\mathcal{C}_{[c, c']}\).

Let \( S' = S_{>y} \cap S_{<z} \) and \( S'' = S_{>z} \cap S_{<n} \). Similarly let \( T' = T_{>y} \cap T_{<z} \) and \( T'' = T_{>z} \cap T_{<n} \). Then

\[ \eta(T^{1A'}) = \eta((T_{>y})^{1B'}) \quad \text{and} \quad \eta(S^{1A}) = \eta((S_{>y})^{1B}) \]

\[ S^{1A} = (S_{>y})^{1B} \oplus S' \oplus \{(i - 2, j), (i - 2, j + 1)\}, \{z, z\} \oplus S'' \oplus \{(i - 1, j), (i - 1, j + 1)\}, \{n, n\} \]

\[ T^{1A'} = (T_{>y})^{1B'} \oplus T' \oplus \text{Dom}(T, z) \oplus T'' \oplus \{(i - 1, j), (i - 1, j + 1)\}, \{n, n\} \]

where \( B = \{(i - 2, j), (i - 1, j)\} \) and \( B' = \{(i - 1, j), (i, j)\} \). By induction argument \( \eta((S_{>y})^{1B}) = \eta((T_{>y})^{1B'}) \) and \( (S_{>y})^{1B} \preceq (T_{>y})^{1B'} \) through one open cycle say \( \mathcal{C}_{[c_1, c'_1]} \). Then \( \eta(S^{1A}) = \eta(T^{1A'}) \) and \( S^{1A} \preceq T^{1A'} \) through the one open cycle which is equal to

\[ \mathcal{C}_{[c_1, c'_1]} \cup (S' \cap \mathcal{C}_{[c, c']}) \cup (S'' \cap \mathcal{C}_{[c, c']}) \cup \{\text{Dom}(S, z)\}. \]

Here \( y < x \) where the domino cell \( \{(i - 1, j), (i - 1, j + 1)\} \) is labeled by \((y, z)\) and the cell \((i, j - 1)\) be labeled by \(x\) in \( S \). On the other hand one can apply the same argument of (4.4) on \( S_{>x} \) and \( T_{>x} \) in order to get the desired result.

Case 2.4. \( \text{label}(S, A) = (\ast, \ast) \) and \( \text{label}(T, A') = (\ast, n) \) (or vice versa). Observe that \( \text{Dom}(S, n) \) must be included in \( \mathcal{C}_{[c, c']} \) and for some \( i \) and \( j \) we have

\[ A = \{(i, j - 1), (i, j)\} \quad \text{and} \quad A' = \{(i, j), (i, j + 1)\} \quad \text{(or vice versa)} \]

where \((i, j)\) is a fixed cell. There are eight possibilities:
Since (4.8) \[ S \] respectively, such that

Then (4.10) assume (4.9) \[ C \]

In the case of (4.6) and (4.6), \( C_{[c,c']} = \{ \text{Dom}(S,n) \} \) since \( S \sim T \) through one open cycle. Therefore \( S_{<n} = T_{<n} \). Let \( S' = S_{>b} \cap S_{<n} \) and \( T' = T_{>b} \cap T_{<n} \) and let \( B \) and \( B' \) be the domino cells of \( S \) and \( T \) respectively, such that

\[ B = \{(i-1,j-1),(i,j-1)\} = B' \] as in (4.6) or \( B = \{(i-1,j-1),(i-1,j)\} = B' \) as in (4.7).

Then

\[ \eta(S'^A) = \eta((S_{<b})^B) \text{ and } \eta(T'^A) = \eta((T_{<b})^B) \]

\[ S'^A = (S_{<b})^B \oplus \{(i-1,j-1),(i-1,j),(b,b)\} \oplus S' \oplus \{(i-1,j+1),(i-1,j+2),(n,n)\} \]

\[ T'^A = (T_{<b})^B' \oplus \{(i-1,j-1),(i-1,j),(b,b)\} \oplus T' \oplus \{(i-1,j),(i-1,j+1),(n,n)\} \]

Since \( S_{<b} = T_{<b} \) then \( (S_{<b})^B = (T_{<b})^B' \) and \( \eta((T_{<b})^B') = \eta((T_{<b})^B) \). Therefore \( \eta(S'^A) = \eta(T'^A) \) and \( S'^A \sim T'^A \) through one open cycle

\[ \{\text{Dom}(S'^A,b),\text{Dom}(S'^A,n)\}. \]

In the case of (4.8) and (4.9), we have \( \{\text{Dom}(S,b)\} \not\subset C_{[c,c']} \), therefore \( S_{<n} \neq T_{<n} \). On the other hand we assume \( \{\text{Dom}(S,b)\} \not\subset C_{[c,c']} \), therefore similar argument to that of (4.6) and (4.7) can be applied here.

In the case of (4.10), still \( \{\text{Dom}(S,n)\} \not\subset C_{[c,c']} \) but this time we assume that \( \{\text{Dom}(S,b)\} \in C_{[c,c']} \) where \( x < a \). On the other hand \( \{\text{Dom}(S,b),\{\text{Dom}(S,a)\} \) defines a closed cycle which contradicts to the fact that \( S \sim T \) through one open cycle.
Moreover let $S$ and $T$ be the domino cells of $S$ and $T$ respectively, such that

$$B = \{(i - 1, j - 1), (i - 1, j)\} \text{ and } B' = \{(i - 1, j - 2), (i - 1, j - 1)\} \text{ as in } \text{(4.11)} \text{ and } \text{(4.12)}.$$ 

Moreover let $S' = S_{>x} \cap S_{<b}$ and $S'' = S_{>b} \cap S_{<n}$ and similarly $T' = T_{>x} \cap T_{<b}$ and $T'' = T_{>b} \cap T_{<n}$. Then

$$\eta(S^{1A}) = \eta((S_{<x})^1B) \text{ and } \eta(T^{1A'}) = \eta((T_{<x})^1B')$$

$$S^{1A} = (S_{<x})^1B \oplus \text{Dom}(S, x) \oplus S' \oplus [B, (b, b)] \oplus S'' \oplus \text{Dom}(S, n)$$

$$T^{1A'} = (T_{<x})^1B' \oplus [(i, j - 2), (i - 1, j - 2), (x, x)] \oplus T' \oplus [(i, j - 1), (i - 1, j - 1), (b, b)]$$

$$\oplus T'' \oplus [(i, j - 1), (i - 1, j + 1), (n, n)].$$

By induction argument we have $\eta((S_{<x})^1B) = \eta((T_{<x})^1B')$ and $(S_{<x})^1B \not\subseteq (T_{<x})^1B'$ through one open cycle say $C_{[e, c]}$. Then $\eta(S^{1A}) = \eta(T^{1A'})$ and $S^{1A} \not\subseteq T^{1A'}$ through one open cycle which is equal to

$$C_{[e, c]} \cup (S' \cap C_{[c, e]}) \cup (S'' \cap C_{[c, e]}) \cup \{\text{Dom}(S^{1A}, x), \text{Dom}(S^{1A}, b) \text{ Dom}(S^{1A}, n)\}.$$

In $\text{(4.13)}$ again we have $\{\text{Dom}(S, n)\} \subseteq C_{[e, c]}$ and $\{\text{Dom}(S, b)\} \subseteq C_{[e, c]}$, but this time we assume that $x < y$. Let $B$ and $B'$ be the domino cells of $S$ and $T$ respectively such that

$$B = \{(i - 1, j - 1), (i - 1, j)\} \text{ and } B' = \{(i - 1, j - 1), (i, j - 1)\} \text{ as in } \text{(4.13)}.$$ 

Moreover let $S' = S_{>a} \cap S_{<b}$, $S'' = S_{>b} \cap S_{<n}$ and similarly $T' = T_{>a} \cap T_{<b}$, $T'' = T_{>b} \cap T_{<n}$. Then

$$\eta(S^{1A}) = \eta((S_{<a})^1B) \text{ and } \eta(T^{1A'}) = \eta((T_{<a})^1B')$$

$$S^{1A} = (S_{<a})^1B \oplus [(i - 2, j - 1), (i - 2, j), (a, a)] \oplus S' \oplus [(i - 1, j - 1), (i - 1, j), (b, b)]$$

$$\oplus S'' \oplus \text{Dom}(S, n)$$

$$T^{1A'} = (T_{<a})^1B' \oplus \text{Dom}(T, a) \oplus T' \oplus [(i - 1, j - 1), (i, j - 1), (b, b)]$$

$$\oplus T'' \oplus [(i - 1, j), (i - 1, j + 1), (n, n)].$$

By induction argument we have $\eta((S_{<a})^1B) = \eta((T_{<a})^1B')$ and $(S_{<a})^1B \not\subseteq (T_{<a})^1B'$ through one open cycle say $C_{[e, c]}$. Then $\eta(S^{1A}) = \eta(T^{1A'})$ and $S^{1A} \not\subseteq T^{1A'}$ through one open cycle which is equal to

$$C_{[e, c]} \cup (S' \cap C_{[c, e]}) \cup (S'' \cap C_{[c, e]}) \cup \{\text{Dom}(S^{1A}, a), \text{Dom}(S^{1A}, b) \text{ Dom}(S^{1A}, n)\}.$$
Lemma 4.15. Let $S$ and $T$ be two r-domino tableaux such that $\text{label}(T) = \text{label}(S)$ and let $S \mathrel{\lessdot} T$ through $t > 0$ open cycle $C_{[c_1,c'_1]} \ldots C_{[c_n,c'_n]}$ for some variable corners $c_1, \ldots, c_t$ of $S$ and $c'_1, \ldots, c'_t$ of $T$ i.e., $\Psi_{[c_1,c'_1] \ldots [c_n,c'_n]}(S) = T$. Let $a$ be an integer such that $|a| \notin \text{label}(T)$ and

$$A = \text{shape}(S^{\downarrow a}) \oplus \text{shape}(S) \quad \text{and} \quad A' = \text{shape}(T^{\downarrow a}) \oplus \text{shape}(T).$$

Then for some $1 \leq s \leq t$ and fixed cell $(i,j)$, one of the following holds.

1. $A = \{(i,j), (i,j+1)\} = A'$ and $S \mathrel{\lessdot} T$ through $t$ open cycle such that $C_{[c_1,c'_1]} \ldots C_{[c_s,c'_s]} \ldots C_{[c_t,c'_t]}$

2. $A = \{(i,j), (i+1,j)\} = A'$ and $S \mathrel{\lessdot} T$ through $t$ open cycle such that $C_{[c_1,c'_1]} \ldots C_{[c_s,c'_s]} \ldots C_{[c_t,c'_t]}$

3. $A = \{(i,j), (i,j+1)\}$ and $A' = \{(i,j-1), (i,j)\}$ (or vice versa) and $S \mathrel{\lessdot} T$ through $t$ open cycle such that $C_{[c_1,c'_1]} \ldots C_{[(i+1,j),c'_i]} \ldots C_{[c_t,c'_t]} \quad \text{(or respectively $C_{[c_1,c'_1]} \ldots C_{[(i+1,j),c_i]} \ldots C_{[c_t,c'_t]}$)}$

4. $A = \{(i,j), (i+1,j)\}$ and $A' = \{(i-1,j), (i,j)\}$ (or vice versa) and $S \mathrel{\lessdot} T$ through $t$ open cycle such that $C_{[c_1,c'_1]} \ldots C_{[(i+1,j),c'_i]} \ldots C_{[c_t,c'_t]} \quad \text{(or respectively $C_{[c_1,c'_1]} \ldots C_{[(i+1,j),c_i]} \ldots C_{[c_t,c'_t]}$)}$

5. $A = \{(i,j), (i,j+1)\}$ and $A' = \{(i,j), (i+1,j)\}$ and $S \mathrel{\lessdot} T$ through $t+1$ open cycle such that $C_{[c_1,c'_1]} \ldots C_{[(i+1,j)+1],c'_i} \ldots C_{[c_t,c'_t]}$

6. $A = \{(i,j), (i+1,j)\}$ and $A' = \{(i,j), (i,j+1)\}$ and $S \mathrel{\lessdot} T$ through $t+1$ open cycle such that $C_{[c_1,c'_1]} \ldots C_{[(i+1,j),c'_i]} \ldots C_{[c_t,c'_t]}$

7. $A = \{(i,j-1), (i,j)\}$ and $A' = \{(i-1,j), (i,j)\}$ and $S \mathrel{\lessdot} T$ through $t-1$ open cycle such that $C_{[c_1,c'_1]} \ldots C_{[(i-1,j),c'_i]} \cdot C_{[c_t,c'_t]} \quad \text{where $C_{[c_1,c'_1]} = C_{[(i-1,j),c_i]} = \{\text{dom}(S,b)\}$ for some $b \in \text{label}(S)$.}$

8. $A = \{(i-1,j), (i,j)\}$ and $A' = \{(i,j-1), (i,j)\}$ and $S \mathrel{\lessdot} T$ through $t-1$ open cycle such that $C_{[c_1,c'_1]} \ldots C_{[(i-1,j)-1],c'_i} \ldots C_{[c_t,c'_t]} \quad \text{where $C_{[c_1,c'_1]} = C_{[(i-1,j),c_i]} \text{ or } C_{[(i,j-1),c_i]} \quad \text{for some} \quad b \in \text{label}(S).}$

Proof. We will apply induction method in order to prove the lemma. Let $n$ be the maximum of all the numbers labeling $S$ (and $T$). If $|a| > n$, then $S^{\downarrow a}$ (and $T^{\downarrow a}$) are obtained by concatenating horizontal domino labeled by $(a,a)$ to the first row of $(S$ (and $T))$ from the right incase $a > 0$, whereas if $a < 0$ then a vertical domino labeled by $(a,a)$ is concatenated to the first column of $(S$ (and $T))$ from the bottom. Clearly resulting tableau satisfy $(1),(2),(3)$ or $(4)$ of the hypothesis.

So in the following we will assume that $|a| < n$. In case there is a cycle which consist of only Dom$(s,n)$ then $S \mathrel{\lessdot} T \mathrel{\lessdot} T$ through either $t-1$ open cycle. Otherwise $S \mathrel{\lessdot} T \mathrel{\lessdot} T$ through either $t$ open cycle. Then by induction argument we may assume that the tableaux $(S^{\downarrow a})$ and $(T^{\downarrow a})$ satisfy one of the hypothesis of the Lemma together with the domino cells

$$B = \text{shape}((S^{\downarrow a})) - \text{shape}(S) \quad \text{and} \quad B' = \text{shape}((T^{\downarrow a})) - \text{shape}(T).$$

Recall from Garfinkle’s algorithm that

$$S^{\downarrow a} = (S^{\downarrow a}) \leftarrow \text{Dom}(S,n) \quad \text{and} \quad T^{\downarrow a} = (T^{\downarrow a}) \leftarrow \text{Dom}(T,n).$$

where sliding in both cases strongly depends respectively on the domino cell $B$ and $B'$ and in fact one of the following holds:

i. $\text{dom}(S,n) \cap B = \emptyset$

ii. $\text{dom}(S,n) = B$

iii. For some $i$ and $j$, $\text{dom}(S,n) = \{(i,j), (i,j+1)\}$ and $B = \{(i,j), (i+1,j)\}$ or vice versa.

On the other hand since $S \mathrel{\lessdot} T$ through $t$ open cycle then we have either $\text{dom}(S,n) = \text{dom}(T,n)$ or $\text{dom}(T,n) = \psi(\text{dom}(S,n))$. Therefore we have the following cases: Let

$$A = \text{shape}(S^{\downarrow a}) - \text{shape}(S) \quad \text{and} \quad A' = \text{shape}(T^{\downarrow a}) - \text{shape}(T).$$
Case 1. dom(S, n) = dom(T, n) = \{(k, l), (k, l + 1)\} for some k, l. Then \(S_{<n} \sim T_{<n}\) through \(t\) open cycles such that

\[\Psi_{[c_1, c'_1]...[c_s, c'_s]}(S_{<n}) = T_{<n}\]

and by induction hypothesis \((S_{<n})^1_{\alpha} \sim (T_{<n})^1_{\alpha}\) through \(t, t-1\) or \(t+1\) open cycles.

If \(\text{dom}(S, n) \cap B = \emptyset\) and \(\text{dom}(T, n) \cap B' = \emptyset\), then

\[S^1_{\alpha} = (S_{<n})^1_{\alpha} \oplus \text{Dom}(S, n)\]

and clearly \(A = B\) and \(A' = B'\) and \(\Psi_{[c_1, c'_1]...[c_s, c'_s]}(S^1_{\alpha}) = T^1_{\alpha}\).

In the following we assume that \(\text{dom}(S, n) \cap B\) or \(\text{dom}(T, n) \cap B'\) is non empty. One can easily check that in that case \(B\) and \(B'\) can not satisfy the hypothesis (3), (4), (7) and (8) of the Lemma.

If \(B\) and \(B'\) satisfy (1) we must have \(B = B' = \{(k, l), (k, l+1)\}\). Then \((S_{<n})^1_{\alpha}\) (and \((T_{<n})^1_{\alpha}\)) is obtained by concatenating a horizontal domino labeled by \((n, n)\) to the \(k + 1\)-th row of \((S_{<n})^1_{\alpha}\) (respectively \((T_{<n})^1_{\alpha}\)).

Therefore \(A\) and \(A'\) satisfy the hypothesis (1) or (3).

If \(B\) and \(B'\) satisfy (2) we must have \(B = B' = \{(k, l), (k+1, l)\}\). Then

\[S^1_{\alpha} = (S_{<n})^1_{\alpha} \oplus \{(k+1, l), (k+1, l+1)\}\]

where \(A = A' = \{(k+1, l), (k+1, l+1)\}\) and they satisfy the hypothesis (2).

If \(B\) and \(B'\) satisfy (5) we must have \(B = \{(k, l), (k+1, l)\}\) and \(B' = \{(k, l), (k+1, l)\}\) where \((k, l)\) is a fixed cell and \((S_{<n})^1_{\alpha} \sim T_{<n})^1_{\alpha}\) through \(t+1\) open cycles i.e., for some \(1 \leq s < t\)

\[\Psi_{[c_1, c'_1]...[c_s, (k,l+1)]...[c_t, c'_t]}((S_{<n})^1_{\alpha}) = (T_{<n})^1_{\alpha}\]

On the other hand

\[S^1_{\alpha} = (S_{<n})^1_{\alpha} \oplus \{(k+1, l), (k+1, l+1)\}\]

and

\[T^1_{\alpha} = (T_{<n})^1_{\alpha} \oplus \{(k, l + 1), (k+1, l + 1)\}\]

Observe that in this case \(A = A' = \{(k+1, l), (k+1, l+1)\}\) and the cycles \(C_{[(n, n), (k, l+1)]}\) and \(C_{[(k+1, l), c'_t]}\) is joined by the domino labeled by \(n\) Therefore \(A\) and \(A'\) satisfy the hypothesis (1).

On the other hand if \(B\) and \(B'\) satisfy (6) the proof follows similar to the previous case.

Case 2. dom(S, n) = dom(T, n) = \{(k, l), (k, l+1)\} for some k, l. This case is just the transpose of the Case 1. Therefore arguments used there definitely can be applied here.

Case 3. dom(S, n) = \{(k, l), (k, l + 1)\}, dom(T, n) = \{(k, l-1), (k, l)\} where \((k, l)\) is a fixed cell. Then one of the variable corner say \(c_s\) of \(S\) must be the cell \((k, l) + 1\) i.e.,

\[\Psi_{[c_1, c'_1]...[c_s, (k,l+1)]...[c_t, c'_t]}(S) = T\]

therefore \(\Psi_{[c_1, c'_1]...[c_s, (k,l+1)]...[c_t, c'_t]}(S_{<n}) = T_{<n}\).

By induction hypothesis we may assume that \(B\) and \(B'\) satisfy one of the hypothesis i.e., \((S_{<n})^1_{\alpha} \sim (T_{<n})^1_{\alpha}\) through \(t, t-1\) or \(t+1\) open cycles.

If \(\text{dom}(S, n) \cap B = \emptyset\) and \(\text{dom}(T, n) \cap B' = \emptyset\) then the variable corner \((k, l - 1)\) still determines an open cycle through which \((S_{<n})^1_{\alpha} \sim (T_{<n})^1_{\alpha}\) are equivalent. Moreover, since

\[S^1_{\alpha} = (S_{<n})^1_{\alpha} \oplus \text{Dom}(S, n)\]

and

\[T^1_{\alpha} = (T_{<n})^1_{\alpha} \oplus \text{Dom}(T, n)\]

then \(A = B\) and \(A' = B'\) and \(S^1_{\alpha} \sim T^1_{\alpha}\) through the same cycle structure as \((S_{<n})^1_{\alpha} \sim (T_{<n})^1_{\alpha}\) except that the variable corner cell \((k, l-1)\) must be replaced by \((k, l + 1)\).

In the following we assume that one of the intersections \(\text{dom}(S, n) \cap B\) or \(\text{dom}(T, n) \cap B'\) is non empty. One can easily check that in that case \(B\) and \(B'\) can not satisfy the hypothesis (1), (2), (5) and (6).

So we first assume that \(B\) and \(B'\) satisfy (3). In this case since both of them are horizontal they must satisfy \(B = \text{dom}(S, n)\) and \(B' = \text{dom}(T, n)\). Then

\[\Psi_{[c_1, c'_1]...[c_s, (k,l+1)]...[c_t, c'_t]}((S_{<n})^1_{\alpha}) = (T_{<n})^1_{\alpha}\]

Since \((S_{<n})^1_{\alpha}\) and \((T_{<n})^1_{\alpha}\) is obtained by concatenating horizontal domino labeled by \((n, n)\) to the \((k+1)\)-th row of \((S_{<n})^1_{\alpha}\) and \((T_{<n})^1_{\alpha}\) respectively, then \(S^1_{\alpha} \sim T^1_{\alpha}\) through the same cycle structure as \((S_{<n})^1_{\alpha} \sim (T_{<n})^1_{\alpha}\) except that if there is a variable corner say \((k+1, l')\), in the \((k + 1)\)-th row \((S_{<n})^1_{\alpha}\) which determines an open cycle then this corner should be replaced by \((k + 1, l' + 2)\).
Next we assume that $B$ and $B'$ satisfy (4). Observe that the only possible configuration which satisfy Garfinkle’s algorithm in that case is that $B = \{(k + 1, l - 1), (k + 2, l - 1)\}$ and $B' = \{(k, l - 1), (k + 1, l - 1)\}$ where $(k + 1, l - 1)$ is a fixed cell, moreover

$$\Psi_{[c_1, c'_{1}][...[k+2,l-1],c'_{1}][...[c_t,c'_{t}]}((S_{<n})^{1a}) = (T_{<n})^{1a}.$$ 

On the other hand

$$S^{1a} = (S_{<n})^{1a} \oplus \text{Dom}(S, n) = (S_{<n})^{1a} \oplus \{(k, l), (k, l + 1)\}, (n, n)\]$$

$$T^{1a} = (T_{<n})^{1a} \oplus \{(k, l), (k, l + 1)\}, (n, n)\]$$

where $(k, l)$ is a fixed cell. Therefore the cycle structure of $S^{1a} \preceq T^{1a}$ is obtained by adding to the cycle structure as $(S_{<n})^{1a} \preceq (T_{<n})^{1a}$ a single open cycle \{\text{Dom}(S^{1a}, n)\} i.e.,

$$\Psi_{[c_1, c'_{1}][...[(k+1+l),k+1+l],[k+2+l],c'_{1}][...[c_t,c'_{t}]}(S^{1a}) = T^{1a}.\]$$

Since $A = B = \{(k + 1, l - 1), (k + 2, l - 1)\}$ and $A' = \{(k + 1, l - 1), (k + 1, l)\}$ as a result $A$ and $A'$ satisfy (5).

Now we assume that $B$ and $B'$ satisfy (7). Then $(S_{<n})^{1a} \preceq (T_{<n})^{1a}$ through $t - 1$ cycles. By Garfinkle’s rule the only configuration of $B$ and $B'$ which makes one of the intersections $\text{dom}(S, n) \cap B$ or $\text{dom}(T, n) \cap B'$ non empty is the following:

$$B = \{(k + 1, l - 2), (k + 1, l - 1)\} \text{ and } B' = \{(k, l - 1), (k + 1, l - 1)\} \text{ where } (k + 1, l - 1) \text{ is a fixed cell.}$$

Therefore $\Psi_{[c_1, c'_{1}][...[(c_{-1}, c'_{-1}),(c_{+1}, c'_{+1}),(c_t, c'_{t})]}((S_{<n})^{1a}) = (T_{<n})^{1a}$ where $C_{[c_t, c'_{t}]} = C_{[(k, l - 1), (k + 1, l - 2)]}.$

On the other hand

$$S^{1a} = (S_{<n})^{1a} \oplus \text{Dom}(S, n) = (S_{<n})^{1a} \oplus \{(k, l), (k, l + 1)\}, (n, n)\]$$

$$T^{1a} = (T_{<n})^{1a} \oplus \{(k, l), (k, l + 1)\}, (n, n)\]$$

where $(k, l)$ is a fixed cell and $A = B = \{(k + 1, l - 2), (k + 1, l - 1)\}$ and $A' = \{(k + 1, l - 1), (k + 1, l)\}$. Therefore $S^{1a} \preceq T^{1a}$ is obtained by adding the new cycle $C_{[(k, l+1),(k+1,l)]} = \{\text{Dom}(S^{1a}, n)\}$ to $t - 1$ cycles of $(S_{<n})^{1a} \preceq (T_{<n})^{1a}$ i.e., $A$ and $A'$ satisfy (4) with

$$\Psi_{[c_1, c'_{1}][...[(c_{-1}, c'_{-1}),(c_{+1}, c'_{+1}),(c_t, c'_{t})]}((S_{<n})^{1a}) = (T_{<n})^{1a}.$$

Lastly if $B$ and $B'$ satisfy (8) then one can use the previous argument since the case (8) is just the transpose of (7).

Case 4. $\text{dom}(S, n) = \{(k, l), (k + 1, l)\}$, $\text{dom}(T, n) = \{(k - 1, l), (k, l)\}$ where $(k, l)$ is a fixed cell. Again this case is just the transpose of the Case 3. Therefore arguments used there definitely can be applied here.

Case 5. $\text{dom}(S, n) = \{(k, l), (k, l + 1)\}$, $\text{dom}(T, n) = \{(k, l), (k + 1, l)\}$ where $(k, l)$ is a fixed cell. In this case one of the cycles of $S \preceq T$ say $C_{[c_t, c'_{t}]}$ must be equal to $C_{[(k, l+1),(k+1,l)]} = \{\text{dom}(S, n)\}$ where $\text{dom}(T, n) = \psi(\text{dom}(S, n))$. Therefore $S_{<n} \preceq T_{<n}$ through $t - 1$ open cycles i.e.,

$$\Psi_{[c_1, c'_{1}][...[(c_{-1}, c'_{-1}),(c_{+1}, c'_{+1}),(c_t, c'_{t})]}((S_{<n})^{1a}) = (T_{<n})^{1a}.$$ 

and by induction argument $(S_{<n})^{1a} \preceq (T_{<n})^{1a}$ through $t - 2$, $t - 1$ or $t$ open cycles with respect to which $B$ and $B'$ satisfy one of the hypothesis of the Lemma.

If $\text{dom}(S, n) \cap B = \emptyset$ and $\text{dom}(T, n) \cap B' = \emptyset$ then since

$$S^{1a} = (S_{<n})^{1a} \oplus \text{Dom}(S, n) \text{ and } T^{1a} = (T_{<n})^{1a} \oplus \text{Dom}(T, n)$$

we have $A = B$ and $A' = B'$ therefore $A$ and $B$ satisfy the same hypothesis which is satisfied by $B$ and $B'$, except that the cycle structure of $S^{1a} \preceq T^{1a}$ is obtained by adding the cycle $C_{[(k, l+1),(k+1,l)]}$ to that of $(S_{<n})^{1a} \preceq (T_{<n})^{1a}$.

In the following we assume that one of the intersections $\text{dom}(S, n) \cap B$ or $\text{dom}(T, n) \cap B'$ is non empty. In this case one can observe from Garfinkle’s rule that $B$ and $B'$ can not satisfy (3),(4), (7) and (8).
We first assume that $B$ and $B'$ satisfy (1). Then we must have $B = B' = \{(k,l), (k,l+1)\}$ and
\[
\Psi_{[c_1,c'_1],\ldots,[c_{t-1},c'_{t-1}]}((S_{<n})^1) = (T_{<n})^1.
\]
Moreover since
\[
S^1 = (S_{<n})^1 \oplus \{\{(k+1,l), (k+1,l+1)\}, (n,n)\}
\]
\[
T^1 = (T_{<n})^1 \oplus \{\{(k+1,l), (k+1,l+1)\}, (n,n)\}
\]
we have $A = \{(k+1,l), (k+1,l+1)\}$ and $A' = \{(k+1,l), (k+1,l+1)\}$ and therefore hypothesis (5) is satisfied where the cycle structure of $S^1 \subseteq T^1$ is the same as that of $(S_{<n})^1 \subseteq (T_{<n})^1$.

We next assume that $B$ and $B'$ satisfy (5). Therefore in order to have at least one non empty intersection with $\text{dom}(S,n)$ or $\text{dom}(T,n)$ we must have $B = \{(k,l), (k,l+1)\}$ and $B' = \{(k,l), (k+1,l)\}$. The last argument implies that $(S_{<n})^1 \not\subseteq (T_{<n})^1$ through $t$ open cycle such that
\[
\Psi_{[c_1,c'_1],\ldots,[c_{t-1},c'_{t-1}]}((S_{<n})^1) = (T_{<n})^1.
\]
On the other hand
\[
S^1 = (S_{<n})^1 \oplus \{\{(k+1,l), (k+1,l+1)\}, (n,n)\}
\]
\[
T^1 = (T_{<n})^1 \oplus \{\{(k+1,l), (k+1,l+1)\}, (n,n)\}
\]
where $(k+1,l+1)$ is a fixed cell. Therefore $A = \{(k+1,l), (k+1,l+1)\}$ and $A' = \{(k+1,l), (k+1,l+1)\}$ therefore the cycles $C_{[c_s,(k,l+1)]}$ and $C_{[(k+1,l),c'_s]}$ is joined by the domino labeled by $(n,n)$. Hence $A$ and $A'$ satisfy (7) and $S^1 \subseteq T^1$ through $t-1$ cycle such that
\[
\Psi_{[c_1,c'_1],\ldots,[c_{t-1},c'_{t-1}]}(S^1) = T^1.
\]
Lastly if $B$ and $B'$ satisfy (2) or (6), then since these cases are the transpose of (1) and (5) respectively one can get the result directly.

**Case 6.** $\text{dom}(S,n) = \{(k,l), (k,l+1)\}$, $\text{dom}(T,n) = \{(k,l), (k+1,l)\}$ where $(k,l)$ is a fixed cell. Since this case is just the transpose of the Case 5, one can apply the same arguments here.

**Case 7.** $\text{dom}(S,n) = \{(k,l-1), (k,l)\}$, $\text{dom}(T,n) = \{(k-1,l), (k,l)\}$ where $(k,l)$ is a fixed cell. In that case one of the cycles of $S \subseteq T$ satisfy $\text{Dom}(S,n) \not\subseteq C_{[c_s,c'_s]}$ and it is easy to see that removing $\text{Dom}(S,n)$ divides $C_{[c_s,c'_s]}$ into two parts. Hence
\[
\Psi_{[c_1,c'_1],\ldots,[c_{t-1},c'_{t-1}]}(S_{<n}) = T_{<n}
\]
with $t+1$ cycles and by induction argument $(S_{<n})^1 \not\subseteq (T_{<n})^1$ through $t+1$, $t$ or $t+2$ open cycles corresponding to which $B$ and $B'$ satisfy one of the hypothesis of the Lemma.

If $\text{dom}(S,n) \cap B = \emptyset$ and $\text{dom}(T,n) \cap B' = \emptyset$ then $(S_{<n})^1 \not\subseteq (T_{<n})^1$ have still two cycles of the form $C_{[(k-1,l),c'_s]}$. Moreover since
\[
S^1 = (S_{<n})^1 \oplus \text{Dom}(S,n) \quad \text{and} \quad T^1 = (T_{<n})^1 \oplus \text{Dom}(T,n)
\]
we have $A = B$ and $A' = B'$ and the cycles $C_{[s,(k,l-1)]}$ and $C_{[(k-1,l),s]}$ are joined together by $\text{dom}(S^1,n) = \psi(\text{dom}(T^1,n))$ so that $(S)^1 \subseteq (T)^1$ through $t$, $t-1$ or $t+1$ open cycles corresponding to which $A$ and $A'$ satisfy one of the hypothesis of the Lemma.

In the following we assume that one of the intersection $\text{dom}(S,n) \cap B$ or $\text{dom}(T,n) \cap B'$ is non empty. It is easy to see that $B$ and $B'$ can not satisfy (1), (2),(5),(6). On the other hand let us explain why $B$ and $B'$ can not satisfy (7).

Observe $B$ and $B'$ satisfy (7) and one of the intersection $B \cap \text{Dom}(S,n)$ and $B' \cap \text{Dom}(T,n)$ non empty we must have $B = \{(k,l-1), (k,l)\} = \text{dom}(S,n)$ and $B' = \{(k-1,l), (k,l)\} = \text{dom}(T,n)$ where $(k,l)$ is a fixed cell. Then $(S_{<n})^1 \not\subseteq (T_{<n})^1$ through $t$ open cycle which is obtained by deleting the open cycle $C_{[(k-1,l),(k,l-1)]}$ which consist of only $\text{dom}(S,b) = \psi(\text{dom}(T,b))$ for some $b \in \text{label}(S)$. This means that removing $\text{Dom}(S,n)$ from $S$ and $\text{Dom}(T,n)$ from $T$ gives the $C_{[(k-1,l),(k,l-1)]}$, but this contradicts to fact that removing $\text{dom}(S,n)$ from $S$ and $\text{dom}(T,n)$ from $T$ produces two open cycle $C_{[(k-1,l-1),c'_s]}$ and $C_{[c_s,(k,l-1)]}$.
So we first assume that $B$ and $B'$ satisfy (3). Observe that this happens if and only if $B = \{(k-1,l+1), (k-1,l+2)\}$ and $B' = \{(k-1,l), (k-1,l+1)\}$ where $(k-1,l+1)$ is a fixed cell and $(S_{<n})_i^a \sim (T_{<n})_i^a$ through $t + 1$ open cycle such that

$$\Psi_{[c_1, c'_1], \ldots, [c_t, c'_t]}((S_{<n})_i^a) = (T_{<n})_i^a$$

On the other hand,

$$S_{i}^{1a} = (S_{<n})_i^a \oplus \operatorname{Dom}(S, n) = (S_{<n})_i^a \oplus \{(k, l), (k, l), (n, n)\}$$

$$T_{i}^{1a} = (T_{<n})_i^a \oplus \{(k, l), (k, l + 1), (n, n)\}$$

where $(k, l)$ is a fixed cell with $A = \{(k-1,l+1), (k-1,l+2)\}$ and $A' = \{(k-1,l+1), (k-1,l+2)\}$. Therefore $A$ and $A'$ satisfy (5) and $S_{i}^{1a} \sim T_{i}^{1a}$ through $t$ open cycle such that

$$\Psi_{[c_1, c'_1], \ldots, [c_t, c'_t]}((S_{<n})_i^a) = (T_{<n})_i^a$$

Now we assume that $B$ and $B'$ satisfy (8). It is easy to see that in that only possibility for $B$ and $B'$ is:

$B = \{(k-2,l+1), (k-1,l+1)\}$ and $B' = \{(k-1,l), (k-1,l+1)\}$

where $(k-1,l+1)$ is a fixed cell and the cycle structure of $(S_{<n})_i^a \sim (T_{<n})_i^a$ obtained by removing the cycle $C_{[k-1,l], [k-2,l+1]} = \{\operatorname{Dom}(S_{<n}, b)\}$, for some $b$, from the structure of $S_{<n} \sim T_{<n}$.

On the other hand,

$$S_{i}^{1a} = (S_{<n})_i^a \oplus \operatorname{Dom}(S, n) = (S_{<n})_i^a \oplus \{(k, l), (k, l), (n, n)\}$$

$$T_{i}^{1a} = (T_{<n})_i^a \oplus \{(k, l), (k, l + 1), (n, n)\}$$

where $(k, l)$ is a fixed cell with $A = \{(k-2,l+1), (k-1,l+1)\}$ and $A' = \{(k-1,l+1), (k, l+1)\}$. Therefore $A$ and $A'$ satisfy (4) and $S_{i}^{1a} \sim T_{i}^{1a}$ through $t$ open cycle such that

$$\Psi_{[c_1, c'_1], \ldots, [c_t, c'_t]}((S_{<n})_i^a) = (T_{<n})_i^a$$

**Case 8.** $\operatorname{dom}(S, n) = \{(k-1,l), (k, l)\}$, $\operatorname{dom}(T, n) = \{(k, l-1), (k, l)\}$ where $(k, l)$ is a fixed cell. Since this case is just the transpose of the Case 7, one can apply the same arguments here.  \(\square\)

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