We define the generalized-Euler-constant function \( \gamma(z) \) as
\[
\gamma(z) = \sum_{n=1}^{\infty} z^{n-1} \left( \frac{1}{n} - \log \frac{n+1}{n} \right)
\]
when \(|z| \leq 1\). Its values include both Euler’s constant \( \gamma = \gamma(1) \) and the “alternating Euler constant” \( \log \frac{4}{\pi} = \gamma(-1) \). We extend Euler’s two zeta-function series for \( \gamma \) to polylogarithm series for \( \gamma(z) \). Integrals for \( \gamma(z) \) provide its analytic continuation to \( \mathbb{C} - [1, \infty) \). We prove several other formulas for \( \gamma(z) \), including two functional equations; one is an inversion relation between \( \gamma(z) \) and \( \gamma(1/z) \). We generalize Somos’s quadratic recurrence constant and sequence to cubic and other degrees, give asymptotic estimates, and show relations to \( \gamma(z) \) and to an infinite nested radical due to Ramanujan. We calculate \( \gamma(z) \) and \( \gamma'(z) \) at roots of unity; in particular, \( \gamma'(-1) \) involves the Glaisher-Kinkelin constant \( A \). Several related series, infinite products, and double integrals are evaluated. The methods used involve the Kinkelin-Bendersky hyperfactorial \( K \) function, the Weierstrass products for the gamma and Barnes \( G \) functions, and Jonquières’s relation for the polylogarithm.
1. Introduction

In this paper we introduce and study the generalized-Euler-constant function \( \gamma(z) \), defined by

\[
\gamma(z) = \sum_{n=1}^{\infty} z^{n-1} \left( \frac{1}{n} - \log \frac{n+1}{n} \right) = \int_{0}^{1} \int_{0}^{1} \frac{1-x}{(1-xyz)(-\log xy)} \, dx \, dy,
\]

where the series converges when \(|z| \leq 1\), and the integral gives the analytic continuation for \( z \in \mathbb{C} - [1, \infty) \). The function \( \gamma(z) \) generalizes both Euler’s constant \( \gamma = \gamma(1) \) and the “alternating Euler constant” \( \log \frac{4}{\pi} = \gamma(-1) \) \cite{16, 18}, where \( \gamma \) is defined by the limit

\[
\gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \right) = 0.57721566 \ldots .
\]  (1)

In Section 2, we extend Euler’s two zeta-function series for \( \gamma \) to polylogarithm series for \( \gamma(z) \) (Theorem 1); we also show another way in which the function \( \gamma(z) \) is related to the extended polylogarithm (Theorem 4). We give a variant of the series definition of \( \gamma(z) \) which is valid on a different domain (Theorem 5). We prove two functional equations for \( \gamma(z) \) (Theorems 6 and 7); the first is an inversion formula relating \( \gamma(z) \) and \( \gamma(1/z) \), and is proved using Jonquières’s relation for the polylogarithm.

In Section 3, we generalize one of Somos’s constants, and show relations with the function \( \gamma(z) \) (Theorem 8) and with an infinite nested radical due to Ramanujan (Corollary 2). We also generalize Somos’s quadratic recurrence sequence to cubic and other degrees (Theorem 9) and provide asymptotic estimates (Lemma 1 and Theorem 10).

In Section 4, we calculate the value of \( \gamma(z) \) at any root of unity (Theorem 13); the proof uses a result based on the Weierstrass product for the gamma function (Theorem 12). Using the Kinkelin-Bendersky hyperfactorial \( K \) function, in Section 5 we compute the derivative \( \gamma'(\omega) \) at a root of unity \( \omega \neq 1 \) (Theorem 16); the proof involves a summation formula derived using the Barnes \( G \) function (Theorem 15). In particular, we show that \( \gamma'(-1) \) involves the Glaisher-Kinkelin constant \( A \) (Corollary 4), and that it is related to an infinite product (Example 11) essentially due to Borwein and Dykshoorn \cite{3}.

Other infinite products occur in Corollary 1, Example 4 (where we generalize an accelerated product for pi \cite{17}), Section 3, and Example 7. We evaluate some new double integrals in Examples 8 and 10, Theorem 14, and Corollary 4.
2. The Generalized-Euler-Constant Function \( \gamma(z) \)

Denote by \( \mathbb{Z}^+ \), \( \mathbb{R} \), and \( \mathbb{C} \) the sets of positive integers, real numbers, and complex numbers, respectively. If \( z \in \mathbb{C} \) and \( z \neq 0 \), define

\[
\log z = \ln |z| + i \arg z \quad (-\pi < \arg z \leq \pi).
\]

The limit definition (11) of Euler’s constant is equivalent to the series formula

\[
\gamma = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \log \frac{n+1}{n} \right) \tag{2}
\]

(see [16]). The corresponding alternating series gives the “alternating Euler constant” [16], [18] (see also [7])

\[
\log \frac{4}{\pi} = \sum_{n=1}^{\infty} (-1)^{n-1} \left( \frac{1}{n} - \log \frac{n+1}{n} \right) = 0.24156447 \ldots. \tag{3}
\]

The main subject of this paper is the following function \( \gamma(z) \), which generalizes (2) and (3).

**Definition 1** The generalized-Euler-constant function \( \gamma(z) \) is defined when \( |z| \leq 1 \) by the power series

\[
\gamma(z) = \sum_{n=1}^{\infty} z^{n-1} \left( \frac{1}{n} - \log \frac{n+1}{n} \right), \tag{4}
\]

which converges by comparison to series (2) for \( \gamma \).

**Example 1** In addition to \( \gamma(1) = \gamma \) and \( \gamma(-1) = \log \frac{4}{\pi} \), Definition 1 gives \( \gamma(0) = 1 - \log 2 \). At \( z = 1/2 \), the function takes the value

\[
\gamma \left( \frac{1}{2} \right) = 2 \log \frac{2}{\sigma}, \tag{5}
\]

where

\[
\sigma = \sqrt{1 \sqrt{2 \sqrt{3} \ldots}} = 1^{1/2} 2^{1/4} 3^{1/8} \ldots = \prod_{n=1}^{\infty} n^{1/2^n} = 1.66168794 \ldots
\]

is one of Somos’s quadratic recurrence constants [13], [13, Sequence A112302] (see also [21] and [3, p. 446], where the notation \( \gamma \) is used instead of \( \sigma \)). To see this, write (14) with \( z = 1/2 \) as

\[
\gamma \left( \frac{1}{2} \right) = 2 \sum_{n=1}^{\infty} \left( \frac{1}{2^n n} - \frac{2 \log(n+1)}{2n+1} + \frac{\log n}{2^n} \right) \\
= 2 \left( \log 2 - 2 \log \sigma + \log \sigma \right) = 2 \log \frac{2}{\sigma}.
\]
Euler gave two zeta-function series for his constant \( \gamma \) (see, for example, [20, equations 3.4(23) and 3.4(151)]),

\[
\gamma = \sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k)}{k} = 1 - \sum_{k=2}^{\infty} \frac{\zeta(k) - 1}{k}.
\] (6)

We generalize them to polylogarithm series for the function \( \gamma(z) \).

**Theorem 1** If \( |z| \leq 1 \) and if \( \text{Li}_k(z) \) denotes the polylogarithm [4, Section 1.11], [7], [17], defined for \( k = 2, 3, \ldots \) by the convergent series

\[
\text{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k},
\]

then

\[
z \gamma(z) = \sum_{k=2}^{\infty} (-1)^k \frac{\text{Li}_k(z)}{k}.
\] (7)

If in addition \( z \neq 1 \), then

\[
z^2 \gamma(z) = z + (1 - z) \log(1 - z) - \sum_{k=2}^{\infty} \frac{\text{Li}_k(z) - z}{k}.
\] (8)

**Proof.** If \( |z| \leq 1 \), then Definition [4] and the expansion

\[
\log(1 - w) = -\sum_{m=1}^{\infty} \frac{w^m}{m} \quad (|w| \leq 1, w \neq 1)
\] (9)

give

\[
z \gamma(z) = z(1 - \log 2) + \sum_{n=2}^{\infty} z^n \left[ \frac{1}{n} - \log \left( 1 + \frac{1}{n} \right) \right] = z \sum_{k=2}^{\infty} \frac{(-1)^k}{k} + \sum_{n=2}^{\infty} \sum_{k=2}^{\infty} (-1)^k \frac{z^n}{kn^k}.
\]

It is easy to see that the double series converges absolutely, so we may reverse the order of summation, obtaining

\[
z \gamma(z) = \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \left( z + \sum_{n=2}^{\infty} \frac{z^n}{n^k} \right) = \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \text{Li}_k(z).
\]

This proves (7).

Now take \( z \neq 1 \) with \( |z| \leq 1 \). Definition [4] and formula (9) imply that

\[
z^2 \gamma(z) = z \sum_{n=1}^{\infty} \frac{z^n}{n} - \sum_{n=1}^{\infty} z^{n+1} \log \frac{n+1}{n} = -z \log(1 - z) - \sum_{n=2}^{\infty} z^n \log \frac{n}{n-1},
\]
where we have re-indexed the last series. Thus, using (9) again, we can write
\[
  z^2 \gamma(z) = -z \log(1 - z) + \log(1 - z) + \sum_{n=1}^{\infty} \frac{z^n}{n} + \sum_{n=2}^{\infty} z^n \log \frac{n-1}{n}
\]
\[
  = (1 - z) \log(1 - z) + z + \sum_{n=2}^{\infty} z^n \left[ \frac{1}{n} + \log \left(1 - \frac{1}{n}\right) \right].
\]
The last series is equal to the absolutely convergent double series
\[
-\sum_{n=2}^{\infty} \sum_{k=2}^{\infty} \frac{z^n}{k n^k} = -\sum_{k=2}^{\infty} \frac{1}{k} \sum_{n=2}^{\infty} \frac{z^n}{n^k} = -\sum_{k=2}^{\infty} \frac{\text{Li}_k(z) - z}{k},
\]
and the proof of (8) is complete. \qed

**Example 2** Let \( z = 1 \) in (7), and let \( z \) tend to \( 1^- \) in (8). Since \( \gamma(1) = \gamma \) and \( \text{Li}_k(1) = \zeta(k) \), we recover the two zeta series (6) for Euler’s constant.

Now take \( z = -1 \), and substitute \( \gamma(-1) = \log 4/\pi \) and \( \text{Li}_k(-1) = (2^{1-k} - 1)\zeta(k) \) (see [10, Section 9.522], [7], [17], [19, Chapter 3]). Formula (7) and the first equality in (6) give the zeta series (compare [16, equation (4)] and [20, equation 3.4(25)])
\[
  \log 4/\pi = \sum_{k=2}^{\infty} (-1)^k \frac{1 - 2^{1-k}}{k} \zeta(k) = \gamma - \sum_{k=2}^{\infty} (-1)^k \frac{k \zeta(k)}{2^k k}.
\]
Simplification in formula (8) yields a zeta series for the logarithm of pi:
\[
  \log \pi = 1 + \sum_{k=2}^{\infty} \frac{1 - (1 - 2^{1-k}) \zeta(k)}{k}.
\]

Finally, set \( z = 1/2 \). Using formula (5) for the Somos constant \( \sigma \), and defining \( \text{Li}_1(z) = -\log(1 - z) \), we can write formulas (7) and (8) as
\[
  \log \sigma = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\text{Li}_k(1/2)}{k} = \sum_{k=1}^{\infty} \frac{2\text{Li}_k(1/2) - 1}{k}.
\]

**Theorem 2** The function \( \gamma(z) \) is continuous on the closed unit disk
\[
  D = \{ z \in \mathbb{C} : |z| \leq 1 \},
\]
and holomorphic on the interior of \( D \). However, the left-hand derivative at \( z = 1 \) does not exist; more precisely,
\[
  \lim_{t \to 1^-} \frac{\gamma(1) - \gamma(t)}{1 - t} = +\infty.
\]
In particular, the Taylor series expansion of \( \gamma(z) \) at \( z = 0 \) has radius of convergence 1.
Proof. For $z \in D$, series (4) is majorized by series (2) for Euler’s constant. Therefore, series (4) converges to $\gamma(z)$ uniformly on $D$. It follows that $\gamma(z)$ is continuous on $D$, and holomorphic on the interior of $D$.

If $0 < t < 1$, then

$$
\gamma(1) - \gamma(t) = \sum_{n=1}^{\infty} (1 - t^{n-1}) \left[ \frac{1}{n} - \log \left( 1 + \frac{1}{n} \right) \right]
= \sum_{n=1}^{\infty} (1 - t^{n-1}) \left( \frac{1}{2n^2} - \frac{1}{3n^3} + \frac{1}{4n^4} - \cdots \right)
\geq \sum_{n=1}^{N} (1 - t^{n-1}) \left( \frac{1}{2n^2} - \frac{1}{3n^3} \right)
$$

for $N = 1, 2, 3, \ldots$. Multiplying by $(1 - t)^{-1}$, and letting $t$ tend to $1^-$, we obtain

$$
\liminf_{t \to 1^-} \frac{\gamma(1) - \gamma(t)}{1 - t} \geq \sum_{n=1}^{N} (n - 1) \left( \frac{1}{2n^2} - \frac{1}{3n^3} \right) \geq \sum_{n=1}^{N} \left( \frac{1}{2n} - \frac{5}{6n^2} \right).
$$

The last sum tends to infinity with $N$, and the theorem follows.

The image of the unit circle under the function $\gamma(z)$ is shown in Figure 1.

In [15, 16, 17], the first author represented Euler’s constant $\gamma = \gamma(+1)$ and the alternating Euler constant $\log \frac{4}{\pi} = \gamma(-1)$ by the double integrals

$$
\gamma(\pm 1) = \int_0^1 \int_0^1 \frac{1 - x}{(1 + xy)(-\log xy)} \, dx \, dy.
$$

We extend this to integrals for the function $\gamma(z)$, and obtain its analytic continuation to the domain $\mathbb{C} - [1, \infty)$. 

\begin{figure}[h]
\centering
\includegraphics[width=\linewidth]{unit_circle}
\caption{The image of the unit circle under the function $\gamma(z)$}
\end{figure}
Theorem 3 If \(|z| \leq 1\), then
\[
\gamma(z) = \int_0^1 \int_0^1 \frac{1-x}{(1-xyz)(-\log xy)} \, dx \, dy = \int_0^1 \frac{1-x + \log x}{(1-xz) \log x} \, dx. \tag{11}
\]

The integrals converge for all \(z \in \mathbb{C} - (1, \infty)\), and provide the analytic continuation of the generalized-Euler-constant function \(\gamma(z)\) for \(z \in \mathbb{C} - [1, \infty)\).

Proof. By [7, Theorem 4.1], if \(z \in \mathbb{C} - (1, \infty)\) with \(z \neq 0\), and if \(\Re(s) > -2\) with \(s \neq -1\), then for \(u > 0\)
\[
\int_0^1 \int_0^1 \frac{(1-x)(xy)^{n-1}}{(1-xyz)(-\log xy)^s} \, dx \, dy = \Gamma(s+2) \left[ \Phi(z, s+2, u) + \frac{(1-z)\Phi(z, s+1, u) - u^{-s-1}}{z(s+1)} \right]. \tag{12}
\]

Here \(\Phi\) is the Lerch transcendent [3 Section 1.11], [7 Section 2], the analytic continuation of the series
\[
\Phi(z, s, u) = \sum_{n=0}^{\infty} \frac{z^n}{(n+u)^s}, \tag{13}
\]
which converges for all complex \(s\) and all \(u > 0\) when \(|z| < 1\). In particular, \(\Phi(z, 0, u) = 1 + z + z^2 + \cdots = (1-z)^{-1}\). It follows, letting \(s\) tend to \(-1\) in (12), that
\[
\int_0^1 \int_0^1 \frac{(1-x)(xy)^{n-1}}{(1-xyz)(-\log xy)} \, dx \, dy = \Phi(z, 1, u) + \frac{1-z}{z} \frac{\partial \Phi}{\partial s}(z, 0, u) + \frac{\log u}{z}. \tag{14}
\]

Now set \(u = 1\) and let \(I(z)\) denote the double integral in (11). Using (13), and re-indexing, we obtain, when \(0 < |z| < 1\),
\[
I(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n} - (z^{-1} - 1) \sum_{n=1}^{\infty} z^{n-1} \log n \tag{15}
\]
\[
= \sum_{n=1}^{\infty} \frac{z^{n-1}}{n} - \sum_{n=1}^{\infty} z^{n-1} \log(n+1) + \sum_{n=1}^{\infty} z^{n-1} \log n = \gamma(z),
\]
by Definition [11]

This proves that \(\gamma(z) = I(z)\) when \(0 < |z| < 1\). It follows, since the functions \(\gamma(z)\) and \(I(z)\) are continuous on \(D - \{1\}\) (in fact, on \(D\)), that \(\gamma(z) = I(z)\) also when \(|z| = 1 \neq z\) and when \(z = 0\). Finally, \(\gamma(1) = \gamma = I(1)\), from [16].

It remains to show that \(I(z)\) is equal to the single integral in (11). To see this, make the change of variables \(x = X/Y, y = Y\), and integrate with respect to \(Y\).

The graph of the function \(\gamma(z)\) for \(z \in (-\infty, 1]\) is shown in Figure 2. Its properties are easily verified from (11) and (10). Namely, for \(z\) real, the graph of \(\gamma(z)\) is positive, increasing, and concave upward; it is asymptotic to the negative real axis, that is, \(\lim_{z \to -\infty} \gamma(-z) = 0\); and the tangent line at the point \((1, \gamma)\) is vertical.
Example 3 Using Theorem 3 and formula (5) for Somos’s constant $\sigma$, we recover the evaluations from [7]

$$\int_0^1 \int_0^1 \frac{x}{(2-xy)(-\log xy)} \, dx \, dy = \int_0^1 \frac{1-x}{(2-x)(-\log x)} \, dx = \log \sigma.$$ 

To see this, set $z = 1/2$ in (11), and replace $\gamma(1/2)$ with $2 \log(2/\sigma)$. Now multiply by 1/2, and subtract the result from the equations

$$\int_0^1 \int_0^1 \frac{1}{(2-xy)(-\log xy)} \, dx \, dy = \int_0^1 \frac{1}{2-x} \, dx = \log 2,$$

which are easily verified.

The function $\gamma(z)$ is also related to the polylogarithm in a different way from that in Theorem 1.

Theorem 4 For all $z \in \mathbb{C} - [1, \infty)$ the relation

$$z^2 \gamma(z) = (1-z) \text{Li}'_0(z) - z \log(1-z)$$

holds. Here the prime $'$ denotes $\partial/\partial s$, and $\text{Li}_s(z)$ is the extended polylogarithm, the analytic continuation [4, Section 1.11], [7, Section 2], [17, Section 5] of the series

$$\text{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}.$$
If \(|z| < 1\), then the series converges for any \(s \in \mathbb{C}\), and the relation can be written

\[
z\gamma(z) = -\log(1 - z) - (1 - z) \sum_{n=1}^{\infty} z^{n-1} \log n. \tag{18}
\]

We give two proofs.

**Proof 1.** Multiply the last two equations in (15) by \(z\), with \(|z| < 1\). Using (9), formula (18) follows. Now multiply (18) by \(z\). Using (17), we obtain relation (16), and the theorem follows by analytic continuation. \(\square\)

**Proof 2.** Setting \(u = 1\) in (14), we may replace the integral with \(\gamma(z)\), by Theorem 3. Now multiply the equation by \(z^2\). Using the formulas

\[
-\log(1 - z) = z \Phi(z, 1, 1) \quad \text{(from (9) and (13))}
\]

and

\[
\text{Li}_s(z) = z \Phi(z, s, 1) \quad \text{(19)}
\]

(from (17) and (13)), relation (16) follows. If \(|z| < 1\), substituting (17) in (16) yields (18). \(\square\)

An application of Theorem 4 is a formula for \(\gamma(z)\) involving a series which converges if \(\Re(z) < 1/2\), when \(|-z| < 1\).

**Theorem 5** If \(\Re(z) < 1/2\), then

\[
z\gamma(z) = -\log(1 - z) + \sum_{n=1}^{\infty} \left(\frac{-z}{1 - z}\right)^n \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \log(k + 1). \tag{20}
\]

Equivalently, if \(|w| < 1\), then

\[
w\gamma\left(\frac{-w}{1 - w}\right) = -(1 - w) \log(1 - w) + (1 - w) \sum_{n=1}^{\infty} w^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \log(k + 1). \tag{21}
\]

**Proof.** By a special case of [7, Corollary 5.1], if \(\Re(z) < 1/2\), then the double sum in (20) is equal to the product \((1 - z)^{\frac{\partial}{\partial s}}(z, 0, 1)\). Using (13) and (16), the first statement follows. To prove the second, note that if \(|w| < 1\) and \(z = \frac{-w}{1 - w}\), then \(\Re(z) < 1/2\), and (20) implies (21). \(\square\)

**Corollary 1** For \(m = 1, 2, 3, \ldots\), define the infinite product

\[
P_m = \prod_{n=1}^{\infty} \left( \prod_{k=0}^{n} (k + 1)^{(-1)^{k+1} \binom{n}{k}} \right)^{\left(\frac{m}{m+1}\right)^n}.
\]

Then the product converges, and

\[
\gamma(-m) = \frac{1}{m} \log \frac{m + 1}{P_m}.
\]
Proof. When \( z = -m \), the series in (20) is equal to \( \log P_m \).

\[ \square \]

**Example 4** Let \( m = 1 \). The product \( P_1 \) is the acceleration of Wallis’s product for \( \pi/2 \) in [17] (see also [7]):

\[
P_1 = \left( \frac{2}{1} \right)^{1/2} \left( \frac{2^2}{1 \cdot 3} \right)^{1/4} \left( \frac{2^3 \cdot 4}{1 \cdot 3^3} \right)^{1/8} \left( \frac{2^4 \cdot 4^4}{1 \cdot 3^6 \cdot 5} \right)^{1/16} \cdots = \frac{\pi}{2}.
\]

Thus

\[
\gamma(-1) = \log \frac{2}{P_1} = \log \frac{4}{\pi},
\]

confirming the value of \( \gamma(-1) \) in Example [1].

With \( m = 2 \), we get

\[
\gamma(-2) = \frac{1}{2} \log \frac{3}{P_2},
\]

where

\[
P_2 = \left( \frac{2}{1} \right)^{2/3} \left( \frac{2^2}{1 \cdot 3} \right)^{4/9} \left( \frac{2^3 \cdot 4}{1 \cdot 3^3} \right)^{8/27} \left( \frac{2^4 \cdot 4^4}{1 \cdot 3^6 \cdot 5} \right)^{16/81} \cdots.
\]

We now prove a functional equation for the generalized-Euler-constant function which expresses \( \gamma(1/z) \) in terms of \( \gamma(z) \).

**Theorem 6** The following inversion formula holds for all \( z \in \mathbb{C} - [0, \infty) \) with \( \Im(z) \geq 0 \):

\[
\gamma \left( \frac{1}{z} \right) = z^3 \gamma(z) - \pi iz + z^2 \log(1 - z) - z \log \frac{z}{1} + z(1 - z) \left[ \gamma + \log 2 \pi + \frac{\pi i}{2} + \psi \left( \frac{\log z}{2\pi i} \right) \right],
\]

where \( \psi(z) \) is the digamma function

\[
\psi(z) = \frac{d[\log \Gamma(z)]}{dz} = \frac{\Gamma'(z)}{\Gamma(z)}.
\]

**Proof.** The extended polylogarithm \( \text{Li}_s(z) \) (see Theorem [4] satisfies Jonquière’s relation [4] Section 1.11), [3]

\[
\text{Li}_s(z) + e^{\pi is} \text{Li}_s(1/z) = \frac{(2\pi)^s e^{\pi is/2}}{\Gamma(s)} \zeta \left( 1 - s, \frac{\log z}{2\pi i} \right), \tag{22}
\]

where \( \zeta(s, w) \) is the Hurwitz (or generalized) zeta function [4] Section 1.10], the analytic continuation of the series

\[
\zeta(s, w) = \sum_{n=0}^{\infty} \frac{1}{(n + w)^s}.
\]
which converges if $\Re(s) > 1$ and $w \in \mathbb{C} - \{0, -1, -2, \ldots\}$. If $|z| < 1$ and $s = 0$, the sum of the series (17) for $\Li_s(z)$ is $\Li_0(z) = z(1 - z)^{-1}$, which by analytic continuation holds for all complex $z \neq 1$. It follows that at $s = 0$ the left side of (22) is equal to $-1$. Therefore, the derivative of (22) with respect to $s$ at $s = 0$ is

$$\Li'_0(z) + \Li'_0(1/z) + \frac{\pi i}{z - 1} = \lim_{s \to 0} \frac{1}{s} \left[ \frac{(2\pi)^s e^{\pi is/2}}{\Gamma(s)} \zeta \left( 1 - s, \frac{\log z}{2\pi i} \right) - (-1) \right].$$

To compute the limit, we use the Taylor series for

$$(2\pi)^s e^{\pi is/2} = \exp \left[ \left( \log 2\pi + \frac{\pi i}{2} \right) s \right]$$

together with the estimates [19, Equation 43:6:1]

$$\frac{1}{\Gamma(s)} = s + \gamma s^2 + O(s^3)$$

and [4, Section 1.11]

$$\zeta(1 - s, w) = -\frac{1}{s} - \psi(w) + O(s) \quad (\Re(w) > 0),$$

which are valid for $s$ tending to 0. If $\Im(z) > 0$, the result is

$$\Li'_0(z) + \Li'_0(1/z) + \frac{\pi i}{z - 1} = -\gamma - \log 2\pi - \frac{\pi i}{2} - \psi \left( \frac{\log z}{2\pi i} \right).$$

For $z \in \mathbb{C} - [0, \infty)$, we may use (16) to replace $\Li'_0(z)$ and $\Li'_0(1/z)$ with expressions for them involving $\gamma(z)$ and $\gamma(1/z)$, respectively. Solving for $\gamma(1/z)$, we arrive at the inversion formula. This proves it when $\Im(z) > 0$. Since each term in the formula is continuous on the set $\{ z \in \mathbb{C} | \Re(z) < 0, \Im(z) \geq 0 \}$, the theorem follows. \qed

**Example 5** Take $z = -1$. Using the values $\log(-1) = \pi i$ and $\psi(1/2) = -\gamma - \log 4$ (from Gauss’s formula [20, equation 1.2(47)]

$$\psi \left( \frac{j}{q} \right) = -\gamma - \frac{\pi}{2} \cot \frac{j\pi}{q} - \log q + \sum_{k=1}^{q-1} \cos \frac{2kj\pi}{q} \log \left( 2\sin \frac{k\pi}{q} \right), \quad (23)$$

where $0 < j < q$), we again obtain $\gamma(-1) = \log \frac{4}{\pi}$. For a related application of Theorem 6, see the proof of Corollary 4.

**Remark 1** Setting $z = 1/w$ in Theorem 6 we obtain an inversion formula valid for $w \in \mathbb{C} - [0, \infty)$ with $\Im(w) \leq 0$.  

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The next theorem gives a second functional equation for the function \( \gamma(z) \). The equation relates the quantities \( \gamma(z) \), \( \gamma(-z) \), and \( \gamma(z^2) \).

**Theorem 7** The following reflection formula holds for all \( z \in \mathbb{C} - ((-\infty, -1] \cup [1, \infty)) \):

\[
z(1 + z)\gamma(z) + z(1 - z)\gamma(-z) = 2z^3\gamma(z^2) - 2z \log 2 + (1 + z) \log(1 + z) - (1 - z) \log(1 - z).
\]

**Proof.** If \( |z| < 1 \), then using (17) we see that

\[
\frac{1}{2} (\text{Li}'_0(z) + \text{Li}'_0(-z)) = -\sum_{n=1}^{\infty} z^{2n} \log 2n = \frac{z^2 \log 2}{z^2 - 1} + \text{Li}'_0(z^2),
\]

where the prime \( ' \) denotes \( \partial/\partial s \). The relation between \( \gamma(z) \) and \( \text{Li}'_0(z) \) in Theorem 4 then yields the desired formula, and the result follows by analytic continuation. \( \square \)

**Remark 2** Theorem 7 can be generalized, as follows. Given an integer \( q > 1 \), we choose a \( q \)th root of unity \( \omega \neq 1 \), and obtain the average

\[
\frac{1}{q} \sum_{j=0}^{q-1} \text{Li}'_0(\omega^j z) = \frac{z^q \log q}{z^q - 1} + \text{Li}'_0(z^q).
\]

Theorem 4 then translates this into a formula relating \( \gamma(z) \), \( \gamma(\omega z), \ldots, \gamma(\omega^{q-1} z) \) and \( \gamma(z^q) \), valid for all \( z \in \mathbb{C} \) such that \( \omega^j z \not\in [1, \infty) \) for \( j = 0, 1, \ldots, q - 1 \).

### 3. A Generalization of Somos’s Quadratic Recurrence Constant

We begin this section by generalizing both Somos’s quadratic recurrence constant \( \sigma \) and its relation (3) with the function \( \gamma(z) \). By convention, if \( a \geq 0 \) and \( t > 0 \), we assume that \( \sqrt[a]{t} = a^{1/t} \geq 0 \).

**Definition 2** For \( t > 1 \), the *generalized Somos constant* \( \sigma_t \) is given by

\[
\sigma_t = \sqrt[1/t]{1 \sqrt[2/t]{2 \sqrt[3/t]{3}} \cdots} = 1^{1/t} 2^{1/t^2} 3^{1/t^3} \cdots = \prod_{n=1}^{\infty} n^{1/t^n}.
\]

The convergence of the infinite product for \( \sigma_t \) follows from the convergence of the series

\[
\log \sigma_t = \sum_{n=1}^{\infty} \frac{\log n}{t^n} = -\text{Li}'_0 \left( \frac{1}{t} \right) = -\frac{1}{t} \frac{\partial \Phi}{\partial s} \left( \frac{1}{t}, 0, 1 \right),
\]

(24) where \( \Phi \) is the Lerch transcendent (13). Note that for \( t = 2 \) we get Somos’s constant \( \sigma = \sigma_2 \).

The following result relates the generalized Somos constant \( \sigma_t \) to the function \( \gamma(z) \), essentially generalizing Example 1.
Theorem 8 For $t > 1$, the generalized Euler constant $\gamma(1/t)$ and the generalized Somos constant $\sigma_t$ satisfy the relation

$$\gamma\left(\frac{1}{t}\right) = t \log \frac{t}{(t-1)\sigma_t^{t-1}}. \quad (25)$$

In particular,

$$\lim_{t \to 0^+} t\sigma_t^{t+1} = e^{-\gamma}. \quad (26)$$

Proof. To prove (25), set $z = 1/t$ in (18) and use (24). (The case $t = 2$ is proved in Example 1.) To prove (26), replace $t$ with $t+1$ in (25), let $t$ tend to $0^+$, and use $\lim_{x \to 1^-} \gamma(x) = \gamma(1) = \gamma$. \(\square\)

Remark 3 From Theorem 8 and the reflection formula in Theorem 7, one can also express $\gamma(-1/t)$ in terms of $\sigma_t$ and $\sigma_{t^2}$.

The next theorem generalizes a result due to Somos [14] (see [5, p. 446] and [21]).

Theorem 9 Fix $t > 1$, and define the sequence $(g_{n,t})_{n \geq 0}$ via the generalized Somos recurrence

$$g_{0,t} = 1, \quad g_{n,t} = ng_{n-1,t} \quad (n \geq 1). \quad (27)$$

Then we have the explicit solution

$$g_{n,t} = \sigma_t^n \exp \left[ \frac{1}{t} \frac{\partial}{\partial s} \left( \frac{1}{t}, 0, n+1 \right) \right] = \sigma_t^n \prod_{m=1}^{\infty} (m+n)^{-1/t^m}. \quad (28)$$

Proof. This follows by induction on $n$. \(\square\)

For $t = 2$, equation (27) is Somos’s quadratic recurrence $g_n = ng_{n-1}^2$. Somos [14], [13, Sequence A116603] (see also [5] and [21]) gave the following asymptotic formula for $g_n = g_{n,2}$ as $n$ tends to infinity:

$$g_n \sim \sigma_{2n}^n (n+2-n^{-1}+4n^{-2}-21n^{-3}+138n^{-4}-1091n^{-5}+\ldots) - 1. \quad (29)$$

We extend this to an asymptotic formula for $g_{n,t}$ given any fixed $t > 1$.

Lemma 1 For $t > 1$, let $(g_{n,t})_{n \geq 0}$ be the sequence defined by the recurrence (27). For $x \in [0, \infty)$, define $f_t(x) \in [1, \infty)$ by the infinite product

$$f_t(x) = \prod_{m=1}^{\infty} (1+mx)^{1/t^m}.$$ 

Then for each $N \geq 2$ and $n \geq 1$, there exists a positive number $\mu = \mu(N,n,t)$ such that

$$g_{n,t} = \sigma_t^n n^{-1/(t-1)} \left[ 1 + \sum_{k=1}^{N-1} \frac{1}{n^k k!} \frac{\partial^k f_t}{\partial x^k}(0) + \frac{1}{n^N N!} \frac{\partial^N f_t}{\partial x^N} (\mu) \right]^{-1}. \quad (30)$$

Moreover, for fixed $N \geq 2$ and fixed $t > 1$,

$$\lim_{n \to \infty} \mu(N,n,t) = 0, \quad \lim_{n \to \infty} \frac{\partial^N f_t}{\partial x^N} (\mu(N,n,t)) = \frac{\partial^N f_t}{\partial x^N}(0).$$
Proof. Since $t > 1$ and $x \geq 0$, the product for $f_t(x)$ converges. It follows from Theorem \ref{10} and the identity $\sum_{m=1}^{\infty} t^{-m} = 1/(t - 1)$ that

$$g_{n,t} = \sigma_t^n n^{-1/(t-1)} f_t \left( \frac{1}{n} \right)^{-1}.$$ 

Thus, by Taylor’s theorem with remainder, it suffices to show that $f_t(x)$ is infinitely differentiable with respect to $x$ for $x \in [0, \infty)$, the derivatives at $x = 0$ being right-sided. That in turn follows (see, for example, [11, p. 342, Theorem 4]) by noting that for $k \geq 1$ the $k$th termwise derivative of the series for $\log f_t(x)$, namely,

$$(-1)^{k-1}(k-1)! \sum_{m=1}^{\infty} \frac{m^k}{t^{m}(1 + mx)^k},$$

is uniformly convergent on $[0, \infty)$, as it is majorized by $(k-1)! \sum_{m=1}^{\infty} m^k t^{-m}$. \hfill \Box

**Theorem 10** For fixed $t > 1$, the sequence $(g_{n,t})_{n \geq 0}$ satisfies the following asymptotic condition as $n$ tends to infinity:

$$g_{n,t} = \sigma_t^n n^{-1/(t-1)} \left[ 1 + \frac{t}{(1-t)^2 n} - \frac{t(t^2 - t - 1)}{2(1-t)^4 n^2} + \frac{t^3 - 11t^2 + 7t + 2}{6(1-t)^6 n^3} + O \left( \frac{1}{n^4} \right) \right]^{-1}.$$ 

**Proof.** By Lemma \ref{11} we only have to compute $\frac{\partial^k f_t}{\partial x^k}(0)$. For $k = 1, 2, \ldots$ define $\phi_k(t)$ by

$$\phi_k(t) = \frac{\partial^k \log f_t}{\partial x^k}(0) = (-1)^{k-1}(k-1)! \sum_{m=1}^{\infty} \frac{m^k}{t^m} = (-1)^{k-1}(k-1)! \text{Li}_{-k} \left( \frac{1}{t} \right).$$

Using the identity

$$\phi_k(t) = (k-1)t \frac{\partial \phi_{k-1}}{\partial t}(t) \quad (k > 1),$$

we compute that

$$\phi_1(t) = \frac{t}{(1-t)^2}, \quad \phi_2(t) = \frac{t(1+t)}{(1-t)^3}, \quad \phi_3(t) = \frac{2t(t^2 + 4t + 1)}{(1-t)^4}.$$ 

Since $f_t = e^{\log f_t}$ and $f_t(0) = 1$, Faá di Bruno’s formula for the $k$th derivative of the composition of two functions (see, for example, [13, p. 21]) yields

$$\frac{\partial^k f_t}{\partial x^k}(0) = \sum_{m_1! \cdots m_k!} \frac{k!}{m_1! \cdots m_k!} \prod_{j=1}^{k} \left( \frac{\phi_j(t)}{j!} \right)^{m_j},$$

where the sum is over all $k$-tuples $(m_1, \ldots, m_k)$ of non-negative integers such that $m_1 + 2m_2 + 3m_3 + \ldots + km_k = k$.  

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In particular,
\[
\frac{\partial f_t}{\partial x}(0) = \phi_1(t), \quad \frac{\partial^2 f_t}{\partial x^2}(0) = \phi_1(t)^2 + \phi_2(t), \quad \frac{\partial^3 f_t}{\partial x^3}(0) = \phi_1(t)^3 + 3\phi_1(t)\phi_2(t) + \phi_3(t).
\]
Applying (29) with \(N = 4\), the theorem follows. \(\square\)

**Example 6** Taking \(t = 2\) gives the first four terms of the asymptotic formula (28) for Somos’s quadratic recurrence sequence \(g_n = g_{n,2} = ng_{n-1,2} = 1, 2, 12, 576, 1658880, \ldots\) [13, Sequence A052129]. With \(t = 3\) we get an asymptotic formula [13, Sequences A123853 and A123854] for the cubic recurrence sequence \(g_{n,3} = ng_{n-1,3} = 1, 2, 24, 55296, 845378412871680, \ldots\) [13, Sequence A123851], namely,

\[
g_{n,3} = \sigma_3^n n^{-1/2} \left[ 1 + \frac{3}{4n} - \frac{15}{32n^2} + \frac{113}{128n^3} + O\left(\frac{1}{n^4}\right) \right]^{-1} \quad (n \to \infty),
\]

where \(\sigma_3\) is the cubic recurrence constant [13, Sequence A123852]

\[
\sigma_3 = 3 \sqrt{1 \sqrt{2 \sqrt{3} \cdots}} = 1^{1/3}2^{1/9}3^{1/27} \cdots = 1.15636268 \ldots.
\]

The generalized Somos constants \(\sigma_t\) are connected with some of Ramanujan’s infinite nested radicals and with the Vijayaraghavan-Herschfeld convergence criterion for them, as follows. Fix \(t > 1\). Given \(a_n = a_n(t) \geq 0\) for \(n = 1, 2, \ldots\), define \(b_n = b_n(t)\) by

\[
b_n = \sqrt[t]{a_1 + \sqrt[t]{a_2 + \sqrt[t]{a_3 + \cdots + \sqrt[t]{a_{n-1} + \sqrt[t]{a_n}}}}}. \tag{30}
\]

If the sequence \((b_n)_{n \geq 1}\) converges, we call the limit \(b_\infty = b_\infty(t)\), and we define

\[
\sqrt[t]{a_1 + \sqrt[t]{a_2 + \sqrt[t]{a_3 + \cdots}}} = \lim_{n \to \infty} b_n = b_\infty. \tag{31}
\]

The following theorem is a special case of a result stated by Herschfeld [8, Theorem III]. For \(t = 2\), it was first proved by Vijayaraghavan (in Appendix I of [12, p. 348]), and independently by Herschfeld. (The proof for \(t > 1\) is similar to that for \(t = 2\), and thus is omitted by both authors.)

**Theorem 11 (Vijayaraghavan-Herschfeld)** Fix \(t > 1\), and suppose that \(a_n = a_n(t) \geq 0\) for \(n \geq 1\). The sequence \((b_n)_{n \geq 1}\) defined by (30) converges if and only if

\[
\limsup_{n \to \infty} \frac{a_n}{n^{1/3}} < \infty.
\]

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Note that this inequality is equivalent to the sequence \((a_n^{1/t^n})_{n\geq 1}\) being bounded.

We now give the promised connection between the generalized Somos constant \(\sigma_t\) and Ramanujan’s infinite radical. The latter is defined by

\[
\sqrt[3]{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots}}} = \lim_{n \to \infty} B_n(t) = B_\infty(t),
\]  

(32)

where

\[
B_n(t) = \sqrt[3]{1 + 2\sqrt[3]{1 + 3\sqrt[3]{1 + \cdots} + (n-1)\sqrt[3]{1 + n\sqrt{1}}}}.
\]

**Corollary 2** Fix \(t > 1\). If

\[a_n = a_n(t) = 1^{t^n} \cdot 2^{t^{n-1}} \cdot 3^{t^{n-2}} \cdots n^{t} \]

for \(n = 1, 2, \ldots\), then

\[
\lim_{n \to \infty} a_n^{1/t^n} = \sigma_t^t < \infty,
\]

and the infinite nested radicals \(b_\infty(t)\) and \(B_\infty(t)\) in (31) and (32) converge and are equal.

**Proof.** The equality (33) follows immediately from Definition 2. Since \(\sigma_t^t < \infty\), Theorem 11 implies that the infinite nested radical

\[
\sqrt[3]{a_1 + \sqrt[3]{a_2 + \sqrt[3]{a_3 + \cdots}}} = \sqrt[3]{1^{t^1} + \sqrt[3]{1^{t^2} \cdot 2^{t^1} + \sqrt[3]{1^{t^3} \cdot 2^{t^2} \cdot 3^{t^1} + \cdots}}} = \lim_{n \to \infty} b_n(t) = b_\infty(t)
\]

converges. Now notice that \(b_n(t) = B_n(t)\) for \(n = 1, 2, \ldots\). Therefore, the infinite nested radical \(B_\infty(t)\) also converges, and \(b_\infty(t) = B_\infty(t)\). This proves the corollary. \(\square\)

**Remark 4** The only known value of \(B_\infty(t)\) is \(B_\infty(2) = 3\), that is,

\[
\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots}}} = 3.
\]

This formula was discovered by Ramanujan [12, Question 289 and Solution, p. 323], but his proof is incomplete. Vijayaraghavan [12, p. 348] and Herschfeld [8, pp. 420-421] each completed Ramanujan’s proof. The result also appeared as Putnam problem A6 in 1966 (see [1, pp. 5 and 52]).

**4. Calculation of \(\gamma(z)\) at Roots of Unity**

The purpose of this section is to calculate the value of \(\gamma(z)\) at roots of unity. We use the following summation formula, which has some resemblance to Problem 24 in [20, p. 71].
Theorem 12 Let \((\theta_n)_{n \geq 1}\) be a periodic sequence in \(\mathbb{C}\), and let \(q\) be a positive multiple of its period. If \(\sum_{j=1}^{q} \theta_j = 0\), then

\[
\sum_{n=1}^{\infty} \theta_n \log \frac{n+1}{n} = \sum_{j=1}^{q} \theta_j \log \frac{\Gamma\left(\frac{j}{q}\right)}{\Gamma\left(\frac{j+1}{q}\right)}.
\]

Proof. The Weierstrass product for the gamma function is [6, Section 8.322]

\[
\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \frac{e^{\frac{z}{n}}}{1 + \frac{z}{n}}
\]

for \(z \in \mathbb{C} - \{0, -1, -2, \ldots\}\). It follows that for \(j = 1, \ldots, q\),

\[
\log \Gamma\left(\frac{j}{q}\right) - \log \Gamma\left(\frac{j+1}{q}\right) = \frac{\gamma}{q} + \log \frac{j+1}{j} + \sum_{n=1}^{\infty} \left( -\frac{1}{qn} + \log \frac{qn+j+1}{qn+j} \right).
\]

Now multiply by \(\theta_j\), and sum from \(j = 1\) to \(q\). Using \(\sum_{j=1}^{q} \theta_j = 0\) and \(\theta_j = \theta_{qn+j}\), we obtain

\[
\sum_{j=1}^{q} \theta_j \log \frac{\Gamma\left(\frac{j}{q}\right)}{\Gamma\left(\frac{j+1}{q}\right)} = \sum_{j=1}^{q} \theta_j \log \frac{j+1}{j} + \sum_{n=1}^{\infty} \sum_{j=1}^{q} \theta_j \log \frac{qn+j+1}{qn+j} = \sum_{n=1}^{\infty} \theta_n \log \frac{n+1}{n},
\]

where the last series converges by Dirichlet’s test. This proves the theorem. \(\square\)

Example 7 Take \(\theta_n = (-1)^{n-1}\) and \(q = 2\), and exponentiate the series. Using the identities (valid for \(z \in \mathbb{C} - \{0, -1, -2, \ldots\}\)

\[
\Gamma(z+1) = z\Gamma(z)
\]

and

\[
\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}
\]

to compute \(\Gamma(1/2)\Gamma(3/2) = \pi/2\), we recover Wallis’s product for pi, which can be written

\[
\prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1} = \frac{\pi}{2}.
\]

Now take \(\theta_n = i^{n-1}\) and \(q = 4\). Exponentiating the real and imaginary parts of the series, and putting \(z = 1/4\) in (36), we get the pair of products

\[
\prod_{n=1}^{\infty} \left(\frac{2n}{2n-1}\right)^{(-1)^{n-1}} = \frac{\Gamma(1/4)^2}{\pi^{3/2} \sqrt{2}}, \quad \prod_{n=1}^{\infty} \left(\frac{2n+1}{2n}\right)^{(-1)^{n-1}} = \frac{\Gamma(1/4)^2}{4\sqrt{2\pi}}.
\]
Multiplying them together, or dividing the second by the first, we obtain the formulas
\[
\prod_{n=1}^{\infty} \left( \frac{2n+1}{2n-1} \right)^{(-1)^{n-1}} = \frac{\Gamma(1/4)^4}{8\pi^2}, \quad \prod_{n=1}^{\infty} \left( \frac{4n^2-1}{4n^2} \right)^{(-1)^{n-1}} = \frac{\pi}{4}.
\]

The last may be a new product for \( \pi \). If we multiply it by Wallis’s product, the factors with odd \( n \) cancel, and taking the square root gives
\[
\prod_{n=1}^{\infty} \left( \frac{16n^2}{16n^2-1} \right) = \frac{\pi}{2\sqrt{2}}.
\]

These products are perhaps known.

The next result gives two formulas for the value of \( \gamma(z) \) at any root of unity \( \omega \neq 1 \).

**Theorem 13** Let \( p \) and \( q \) be relatively prime positive integers. If \( \omega = e^{i\pi p/q} \neq 1 \), then
\[
\gamma(\omega) + \log(1 - \omega) = \sum_{j=1}^{q} \omega^{-j} E_{j,p,q} = \sum_{j=1}^{2q} \omega^{-j} \log \frac{\Gamma \left( \frac{j+1}{2q} \right)}{\Gamma \left( \frac{j}{2q} \right)},
\]

where
\[
E_{j,p,q} = \begin{cases} 
\log \frac{\Gamma \left( \frac{j+1}{q} \right)}{\Gamma \left( \frac{j}{q} \right)} & \text{if } p \text{ is even}, \\
\log \frac{\Gamma \left( \frac{j+1}{2q} \right) \Gamma \left( \frac{j+q}{2q} \right)}{\Gamma \left( \frac{j}{2q} \right) \Gamma \left( \frac{j+q+1}{2q} \right)} & \text{if } p \text{ is odd}.
\end{cases}
\]

**Proof.** By Definition 1 and formula (9), we only need to prove that
\[
-\sum_{n=1}^{\infty} \omega^{n-1} \log \frac{n+1}{n} = \sum_{j=1}^{q} \omega^{-j} E_{j,p,q} = \sum_{j=1}^{2q} \omega^{-j} \log \frac{\Gamma \left( \frac{j+1}{2q} \right)}{\Gamma \left( \frac{j}{2q} \right)}.
\]

If \( p \) is even, \( (\omega^{j-1})_{j \geq 1} \) is a periodic sequence of period \( q \) with \( \sum_{j=1}^{q} \omega^{j-1} = 0 \). In this case, the first equality in (38) follows immediately from Theorem 12.

If \( p \) is odd, \( (\omega^{j-1})_{j \geq 1} \) is a periodic sequence of period \( 2q \) with \( \sum_{j=1}^{2q} \omega^{j-1} = 0 \). Hence in Theorem 12 we may replace \( q \) with \( 2q \), and take \( \theta_j = \omega^{-j} \). Since \( \omega^{j+q} = -\omega^j \) for \( j = 1, 2, \ldots, q \), the first equality in (38) holds with \( E_{j,p,q} \) given by (37).

To show the second equality in (38), use Theorem 12 and the fact that \( \omega \) is a \( 2q \)th root of unity. \( \square \)
Example 8 Take $p/q = 1/2$ and $2/3$. Using Theorem 3 and identities (35) and (36), we get

\[
\gamma(i) = \int_0^1 \int_0^1 \frac{1-x}{(1-ixy)(-\log xy)} \, dx \, dy = \frac{\pi}{4} - \log \frac{\Gamma(1/4)^2}{\pi \sqrt{2\pi}} + i \log \frac{8\sqrt{\pi}}{\Gamma(1/4)^2},
\]

\[
\gamma(e^{2\pi i/3}) = \int_0^1 \int_0^1 \int_0^1 \frac{1-x}{(1-e^{2\pi i/3}xy)(-\log xy)} \, dx \, dy = \frac{\pi}{4\sqrt{3}} - \frac{3}{2} \log \frac{2\pi}{3\Gamma(2/3)^2} + i \left( \frac{\sqrt{3}}{2} \log \frac{9}{2\pi} - \frac{\pi}{12} \right).
\]

The following corollary gives an additional formula for the value of $\gamma(z)$ at any point $\omega$ of the unit circle with argument $\pi p/q$, where $p$ and $q$ are integers with $p$ odd and $q > 0$. For such a point $\omega$, the formula expresses $\gamma(\omega)$ as a finite linear combination of powers of $\omega$ with real coefficients.

Corollary 3 Let $p$ and $q$ be integers, with $p$ odd and $q$ positive. If $\omega = e^{i\pi p/q}$, then

\[
\gamma(\omega) = \sum_{j=1}^q \omega^{j-1}(D_{j,q} + E_{j,p,q}),
\]

where $E_{j,p,q}$ is given by (37), and

\[
D_{j,q} = \frac{1}{2q} \left[ \psi\left(\frac{1}{2} + \frac{j}{2q}\right) - \psi\left(\frac{j}{2q}\right) \right].
\]

Proof. By Theorem 13 and formula (49), we only need to show that

\[
\sum_{n=1}^{\infty} \frac{\omega^{n-1}}{n} = \sum_{j=1}^q \omega^{j-1} D_{j,q}.
\]

Let $s_1, s_2, \ldots$ be the partial sums of the series. Then, using $\omega^{2q} = 1$,

\[
\lim_{k \to \infty} s_k = \lim_{k \to \infty} s_{2qk} = \lim_{k \to \infty} \sum_{j=1}^{2q} \omega^{j-1} \sum_{m=0}^{k-1} \frac{1}{2qm + j}.
\]

Since $p$ is odd, $\omega^{j+q} = -\omega^j$ for $j = 1, 2, \ldots, q$. Therefore,

\[
\lim_{k \to \infty} s_k = \sum_{j=1}^q \omega^{j-1} \frac{1}{4q} \sum_{m=0}^{\infty} \frac{1}{(m + \frac{j}{2q}) (m + \frac{j+q}{2q})}.
\]

Using the formula [6, Section 8.363, Equation 3], [19, Equation 44:5:8]

\[
\sum_{m=0}^{\infty} \frac{1}{(m + x)(m + y)} = \frac{\psi(x) - \psi(y)}{x - y},
\]

which is valid for $x, y \in \mathbb{C} - \{0, -1, -2, \ldots\}$ with $x \neq y$, equation (40) follows. \qed
Remark 5 In (39) the quantity $\psi(j/q)$ can be calculated from Gauss’s formula (23) when $0 < j < q$, together with the value $\psi(1) = -\gamma$.

Example 9 Setting $p = 1$ and $q = 3$ in Corollary 3, we multiply by 6 and obtain

$$6\gamma(e^{i\pi/3}) = \psi\left(\frac{2}{3}\right) - \psi\left(\frac{1}{6}\right) + 6\log \frac{\Gamma(1/3) \Gamma(2/3)}{\Gamma(1/6) \Gamma(5/6)}$$

$$+ e^{i\pi/3} \left[\psi\left(\frac{5}{6}\right) - \psi\left(\frac{1}{3}\right) + 6\log \frac{\Gamma(1/2) \Gamma(5/6)}{\Gamma(1/3)}\right]$$

$$+ e^{2i\pi/3} \left[-\gamma - \psi\left(\frac{1}{2}\right) + 6\log \frac{6\Gamma(2/3)}{\Gamma(1/2) \Gamma(1/6)}\right]$$

$$= \pi\sqrt{3} - 3\log \frac{6\sqrt{3}}{\pi} + i \left[\pi - 3\sqrt{3}\log \frac{\Gamma(1/2) \Gamma(5/6)}{\Gamma(1/3)^2 \sqrt{3}}\right].$$

The next theorem gives a formula for the average value of the function $\gamma(z)$ at the vertices of a regular polygon inscribed in the unit circle with the positive real axis a perpendicular bisector of one side.

Theorem 14 Given $q \in \mathbb{Z}^+$, let $\omega = e^{i\pi/q}$. Then

$$\frac{1}{q} \sum_{k=0}^{q-1} \gamma(\omega^{2k+1}) = \int_0^1 \int_0^1 \frac{1-x}{(1+xy)(-\log xy)} \, dx \, dy$$

$$= \frac{1}{2q} \left[\psi\left(\frac{q+1}{2q}\right) - \psi\left(\frac{1}{2q}\right)\right] - \log \frac{\Gamma\left(\frac{1}{2q}\right) \Gamma\left(\frac{q+2}{2q}\right)}{\Gamma\left(\frac{1}{q}\right) \Gamma\left(\frac{q+1}{2q}\right)}.$$ 

We give two proofs.

Proof 1. By the method of partial fractions, for $\zeta \in \mathbb{C} - \{\omega^{-1}, \omega^{-3}, \ldots, \omega^{-(2q-1)}\}$,

$$\sum_{k=0}^{q-1} \frac{1}{1 - \zeta \omega^{2k+1}} = \frac{q}{1 + \zeta^q}.$$

Letting $\zeta = xy$ and using (11), we obtain

$$\sum_{k=0}^{q-1} \gamma(\omega^{2k+1}) = q \int_0^1 \int_0^1 \frac{1-x}{(1+xy)(-\log xy)} \, dx \, dy. \quad (41)$$

Making the change of variables $X = x^q, Y = y^q$ gives

$$\sum_{k=0}^{q-1} \gamma(\omega^{2k+1}) = \int_0^1 \int_0^1 \frac{(XY)^{\frac{q}{q-1}} - X^{\frac{q}{q-1}} Y^{\frac{q}{q-1}}}{(1 + XY)(-\log XY)} \, dX \, dY.$$
Now use the following evaluations [7, Corollary 3.4], which hold for $u, v > 0$:

\[
\int_0^1 \int_0^1 (XY)^{u-1} (1 + XY)(-\log XY) \, dX \, dY = \frac{1}{2} \left[ \psi \left( \frac{u+1}{2} \right) - \psi \left( \frac{u}{2} \right) \right],
\]

\[
\int_0^1 \int_0^1 X^{u-1} Y^{v-1} \, dX \, dY = \frac{1}{u-v} \log \frac{\Gamma \left( \frac{u+1}{2} \right) \Gamma \left( \frac{v+1}{2} \right)}{\Gamma \left( \frac{u}{2} \right) \Gamma \left( \frac{v}{2} \right)}. \quad \Box
\]

**Proof 2.** Using Corollary 3 and the equality $E_{j,2k+1,q} = E_{j,1,q}$ from (37), we obtain

\[
\sum_{k=0}^{q-1} \gamma(\omega^{2k+1}) = \sum_{k=0}^{q-1} \sum_{j=1}^{q} (\omega^{2k+1})^{j-1} (D_{j,q} + E_{j,2k+1,q}) = \sum_{j=1}^{q} (D_{j,q} + E_{j,1,q}) \sum_{k=0}^{q-1} (\omega^{j-1})^{2k+1}.
\]

Since $\omega = e^{i\pi/q}$, if $j \in \{1, \ldots, q\}$, then $\omega^{j-1} \neq -1$, and $\omega^{j-1} = 1$ if and only if $j = 1$. It follows that $\sum_{k=0}^{q-1} (\omega^{j-1})^{2k+1} = 0$ when $j \in \{2, \ldots, q\}$, hence

\[
\sum_{k=0}^{q-1} \gamma(\omega^{2k+1}) = q(D_{1,q} + E_{1,1,q}).
\]

The theorem now follows using (37), (39), and (41). \quad \Box

**Example 10** Take $q = 3$. Using (23) and (36), we get

\[
\gamma(e^{i\pi/3}) + \gamma(e^{i\pi}) + \gamma(e^{5i\pi/3}) = 3 \int_0^1 \int_0^1 \frac{1-x}{(1+x^3y^3)(-\log xy)} \, dx \, dy = \frac{\pi}{\sqrt{3}} - \log \frac{3\sqrt{3}}{2}.
\]

### 5. The Hyperfactorial $K$ Function and the Derivative of $\gamma(z)$

In this section we study the derivative of $\gamma(z)$ at roots of unity, using the following function.

**Definition 3** The Kinkelin-Bendersky [10, 2] hyperfactorial $K$ function is the real-valued function $K(x)$ defined for $x \geq 0$ by the relation

\[
\log K(x) = \frac{x^2 - x}{2} - \frac{x}{2} \log 2\pi + \int_0^x \log \Gamma(y) \, dy. \quad (42)
\]

(Bendersky [2] uses the notation $\Gamma_1$, but most authors today use $K$ to denote the function defined by [12]; see, for example, [5, p. 135].)

The $K$ function satisfies [2, p. 279] \( K(0) = K(1) = K(2) = 1 \) and, for $x > 0$,

\[
K(x + 1) = x^2 K(x). \quad (43)
\]
Thus, by induction on \( n \in \mathbb{Z}^+ \), we have \( K(n + 1) = 1^1 2^2 \cdots n^n \).

There is an analog for the \( K \) function [2, p. 281] of Gauss’s multiplication formula for the gamma function. A special case is

\[
\prod_{j=1}^{n-1} K \left( \frac{j}{n} \right) = \frac{A_{n-1}^n}{n^{1/2} e^{-2n}},
\]

for \( n \geq 2 \), where [2, pp. 263-264]

\[
A = \lim_{x \to \infty} \frac{K(x + 1)}{x^{2\pi^2 / 12 + 1/4} e^{-x / 2}} = \lim_{n \to \infty} \frac{1^1 2^2 \cdots n^n}{n^{\pi^2 / 6} e^{-n / 4}} = 1.28242712\ldots
\]

is the Glaisher-Kinkelin constant (see [5, p. 135]). In particular,

\[
K \left( \frac{1}{2} \right) = \frac{A^{3/2}}{2^{1/2} e^{1/8}}.
\]

The following summation formula is needed in the proof of Theorem 16.

**Theorem 15** Let \( (\theta_n)_{n \geq 1} \) be a periodic sequence in \( \mathbb{C} \), and let \( q \) be a positive multiple of its period. If

\[
\sum_{j=1}^{q} \theta_j = 0,
\]

then

\[
\sum_{n=1}^{\infty} \theta_n \left( 1 - n \log \frac{n+1}{n} \right) = \sum_{j=1}^{q} \theta_j \log \frac{K \left( \frac{j+1}{q} \right)^q e^{-j/q}}{K \left( \frac{j}{q} \right)^q \Gamma \left( \frac{j+1}{q} \right)}.
\]

**Proof.** Dirichlet’s test implies that the series converges. The proof of the formula for its sum is similar to that of Theorem 12. The new ingredient is the Barnes \( G \) function, which is defined by the infinite product

\[
G(z+1) = \frac{(2\pi)^{z/2} e^{-\frac{z^2}{4}}}{\prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right)^n e^{-\frac{z^2}{2n^2}}}
\]

for \( z \in \mathbb{C} \), and which is related to the \( K \) function by \( K(x+1)G(x+1) = \Gamma(x+1)^x \) for \( x \geq 0 \) (see [5, Section 2.15], [6, Section 6.441]).

From (43) and (45), we can write the relation as \( K(x) = G(x+1)^{-1} \Gamma(x)^x \), so that

\[
\frac{K \left( \frac{j+1}{q} \right)^q}{K \left( \frac{j}{q} \right)^q \Gamma \left( \frac{j+1}{q} \right)} = \left( \frac{G \left( \frac{j}{q} + 1 \right)}{G \left( \frac{j+1}{q} + 1 \right)} \right)^q \left( \frac{\Gamma \left( \frac{j+1}{q} \right)}{\Gamma \left( \frac{j}{q} \right)} \right)^j.
\]
for \( j = 1, 2, \ldots, q \). Now take the logarithm, multiply by \( \theta_j \), and sum from \( j = 1 \) to \( q \). We compute the result in two parts. On the one hand, a calculation using (47) and (46) yields

\[
\sum_{j=1}^{q} \theta_j q \log \frac{G \left( \frac{j}{q} + 1 \right)}{G \left( \frac{j+1}{q} + 1 \right)} = \frac{1 + \gamma}{q} \sum_{j=1}^{q} \theta_j j - \sum_{n=1}^{\infty} \sum_{j=1}^{q} \theta_j \left( \frac{j}{qn} + qn \log \frac{qn + j + 1}{qn + j} \right).
\]

On the other hand, from (34) we have

\[
\sum_{j=1}^{q} \theta_j j \log \frac{\Gamma \left( \frac{j+1}{q} \right)}{\Gamma \left( \frac{j}{q} \right)} = \sum_{n=0}^{\infty} \sum_{j=1}^{q} \theta_j (qn + j) \log \frac{qn + j + 1}{qn + j}.
\]

Therefore, by addition

\[
\sum_{j=1}^{q} \theta_j j \log \frac{K \left( \frac{j+1}{q} \right) q e^{-j/q}}{K \left( \frac{j}{q} \right) \Gamma \left( \frac{j+1}{q} \right) \Gamma \left( \frac{j}{q} \right)} = \sum_{n=0}^{\infty} \sum_{j=1}^{q} \theta_j (qn + j) \log \frac{qn + j + 1}{qn + j}.
\]

Substituting \( \theta_j = \theta_{qn+j} \) on the right, and using (46) again, we deduce the desired formula. \( \square \)

**Example 11** Take \( \theta_n = (-1)^{n-1} \) and \( q = 2 \). Exponentiating the series, and using (43) and (45), we obtain the infinite product

\[
\prod_{n=1}^{\infty} \left[ \frac{e}{(1 + \frac{1}{n})^n} \right]^{(-1)^{n-1}} = \frac{2^{1/6} e \sqrt{\pi}}{A^6},
\]

where \( A \) is the Glaisher-Kinkelin constant (41).

** Remark 6** The last equality, in the form of the limits

\[
\lim_{N \to \infty} \prod_{n=1}^{2N+1} \left( 1 + \frac{1}{n} \right)^{(-1)^n+1} = e \cdot \lim_{N \to \infty} \prod_{n=1}^{2N} \left( 1 + \frac{1}{n} \right)^{(-1)^n+1} = \frac{A^6}{2^{1/6} \sqrt{\pi}},
\]

is due to Borwein and Dykshoorn [3].

Our final theorem is the main result of the section. It gives the value of the derivative of \( \gamma(z) \) at any point \( \omega \neq 1 \) of the unit circle with argument a rational multiple of \( \pi \).

**Theorem 16** Let \( p \) and \( q \) be relatively prime positive integers. If \( \omega = e^{i\pi p/q} \neq 1 \), then the derivative of the function \( \gamma(z) \) at \( z = \omega \) is

\[
\gamma'(\omega) = \frac{1}{\omega(1-\omega)} + \frac{\log(1-\omega)}{\omega^2} + \sum_{j=1}^{2q} \omega^{j-2} \log \frac{\Gamma \left( \frac{j}{2q} \right) K \left( \frac{j+1}{2q} \right)^{2q}}{\Gamma \left( \frac{j+1}{2q} \right)^2 K \left( \frac{j}{2q} \right)^{2q}}.
\]

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Proof. Using Definition 11 the Taylor series of the derivative of the product \( z\gamma(z) \) when \( |z| < 1 \) is

\[
\gamma(z) + z\gamma'(z) = \sum_{n=1}^{\infty} z^{n-1} \left( 1 - n \log \frac{n+1}{n} \right).
\] (48)

For any fixed \( z \neq 1 \) with \( |z| = 1 \), the partial sums of the series \( \sum_{n=1}^{\infty} z^{n-1} \) are bounded. It follows, using Dirichlet’s test, that the series in (48) converges for such \( z \). Since the function \( \gamma(z) + z\gamma'(z) \) is continuous at such \( z \), Abel’s limit theorem implies that (48) also holds when \( |z| = 1 \neq z \).

In particular, we may take \( z = \omega = e^{i\pi p/q} \neq 1 \). Then the sequence \( (\omega^{n-1})_{n \geq 1} \) is periodic, and \( 2q \) is a positive multiple of its period. Since \( \sum_{n=1}^{2q} \omega^{n-1} = 0 \), in Theorem 15 we may replace \( q \) with \( 2q \), and take \( \theta_n = \omega^{n-1} \). Using Theorem 13, the result follows. \( \square \)

Compare the following double integral formulas with those involving \( A \) in [7] and [16].

**Corollary 4** The first and second derivatives of the function \( \gamma(z) \) at \( z = -1 \) are

\[
\gamma'(-1) = \int_0^1 \int_0^1 \frac{xy(1-x)}{(1+xy)^2(-\log xy)} \, dx \, dy = \log \frac{2^{11/6} A^6}{\pi^{3/2} e}
\]

and

\[
\gamma''(-1) = \int_0^1 \int_0^1 \frac{2x^2y^2(1-x)}{(1+xy)^3(-\log xy)} \, dx \, dy = \log \frac{2^{10/3} A^{24}}{\pi^4 e^{13/4}} - \frac{7\zeta(3)}{2\pi^2}.
\]

Proof. In both cases the first equality follows using Theorem 3.

To obtain the value of \( \gamma'(-1) \) use Theorem 16 together with formulas (43) and (45). Alternatively, use (48), Example 11, and the value \( \gamma(-1) = \log \frac{1}{2} \).

To evaluate \( \gamma''(-1) \), differentiate the inversion formula in Theorem 4 twice at \( z = -1 + it \), let \( t \) tend to \( 0^+ \), and re-arrange terms, to get

\[
\gamma''(-1) = 4\gamma'(-1) - 3\gamma(-1) + \frac{3}{4} - \gamma - \log \pi - \psi\left(\frac{1}{2}\right) + \frac{1}{4\pi^2} \psi''\left(\frac{1}{2}\right) + i \left[ \frac{1}{\pi} \psi'\left(\frac{1}{2}\right) - \frac{\pi}{2} \right].
\]

Using the values [19] Sections 44:8 and 44:12

\[
\psi\left(\frac{1}{2}\right) = -\gamma - \ln 4, \quad \psi'\left(\frac{1}{2}\right) = \frac{\pi^2}{2}, \quad \psi''\left(\frac{1}{2}\right) = -14\zeta(3),
\]

the desired formula follows. \( \square \)

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