Entropy for the Interior of a Schwarzschild Black Hole while the Mass is Increasing With Time

Prasanta Choudhury\textsuperscript{1}, Ritabrata Biswas\textsuperscript{2}

Department of Mathematics, The University of Burdwan, Golapbag Academic Complex, Burdwan-713104, India.

Abstract

Black hole thermodynamics is the area of study that seeks to reconcile the laws of thermodynamics with the existence of black hole event horizons. Here we calculate the entropy corresponding to the interior of a Schwarzschild black hole for massless modes, assuming the mass of the black hole increasing with time. We find that the entropy is proportional to the Bekenstein-Hawking expression. Also, we can see that the evaluated entropy satisfies the second law of thermodynamics. Using the thermodynamic law we found a relation between temperature and inverse temperature.

Keywords: Black hole physics; Thermodynamics; Thermodynamic processes; Entropy; Dynamic Black Hole

1 Introduction

There exist many theories regarding the compact objects like Black holes. Even the present day gravitational wave detections\textsuperscript{1} of black hole mergers can be treated as strong evidence of their existence. For the first time, theoretically Hawking has shown that black holes can evaporate and leave thermal radiation\textsuperscript{2, 3}. This concept attracts people and different aspects of black hole thermodynamics\textsuperscript{4} has been discussed. Many thermodynamic quantities were calculated. The study was analogous to the previous studies of classical thermodynamics. Entropy for a black hole was treated to be a quantity which is proportional to its ever increasing area of event horizon. Bekenstein-Hawking’s entropy form\textsuperscript{5} is usually expressed as

\[ S_{BH} = \frac{A}{4\hbar G} \]

where, \( A \) is the area of the black hole’s event horizon. The second law of thermodynamics states that the entropy increases with time. Another natural notion of entropy is the Von Neumann’s entropy given by

\[ S[\rho] = -\text{tr}(\rho \ln \rho) \]

where, \( \rho \) is the density matrix of a quantum system. In classical level, we know that black hole mechanics follows the laws similar to the ordinary laws of thermodynamics\textsuperscript{6}. A well known formula for the entropy of black hole in terms of Noether charge was later executed in the 90’s\textsuperscript{7}. Yet, a question has arrived as unanswered in the area of classical thermodynamics of black holes: what is the source of the entropy at the classical level? More generally, what are the classical microstates that corresponds to the entropy macrostates? So, we found a long-standing issue in literature by this question, which has recently renewed: do extremal black holes have zero or non-zero entropy\textsuperscript{8}? Recently, a new path of defining the inside volume of the Schwarzhild black hole has been introduced\textsuperscript{9} by Christodoulou and Rovelli(known as CR volume). This calculation was done for Kerr\textsuperscript{9}. Now introducing the following inputs which taken in the paper\textsuperscript{10} without very clear description: (a) In expanding the Klein-Gordon equation under the time dependent background, the scalar field was chosen as

\[ \exp\{-iET\} \exp\{iI(\lambda, \theta, \phi)\} \]

where \( E \) is identified as the energy. (b) From the free energy we obtain the entropy, the derivative with respect to the inverse temperature was taken. (c) Finally, we use the flux relation(Hawking expression)\textsuperscript{10}.

Here we tried to get a clean knowledge about the above inputs. Express the interior volume as integral form\textsuperscript{10} and defined an effective metric. Then we evaluate the Hamiltonian for a particle moving in this background. Finally, we use the Gibbs’s free energy for a massless particle and find the entropy as time dependent which is a very small impact on entropy changes.

The structure of the paper is as follows. Firstly, we write a brief review of the results, given in\textsuperscript{10}. In section 3 we calculate the energy for a massless particle within the interior of the black hole. In section 4 we calculate the energy and finally we conclude in section 5.

\textsuperscript{1}prasantachoudhury98@gmail.com
\textsuperscript{2}biswas.ritabrata@gmail.com
2 Review of Previous Works

First we write the well known Schwarzschild metric in Eddington-Finkelstein coordinates

\[ ds^2 = -f dt^2 + 2dv dr + r^2 d\Omega^2 , \]

where the function \( f = f(r, t) = 1 - \frac{2M(1+\varepsilon t)}{r} \) (where, \( \varepsilon > 0 \) is a very small real number i.e. \( \varepsilon \) tends to zero) and \( v \) is advanced time defined as \( v = t + \int \frac{dt}{v} = t + r + 2M(1+\varepsilon t) \ln |r - 2M(1+\varepsilon t)| \). The units are taken here : \( G = c = \hbar = \kappa_B = 1 \). Now, considering an transformation \( v \rightarrow v(T, \lambda) \) and \( r \rightarrow r(T, \lambda) \), we have

\[ ds^2 = \left\{ -f \left( \frac{\partial v}{\partial T} \right)^2 + 2 \frac{\partial v}{\partial T} \frac{\partial r}{\partial T} \right\} dT^2 + \left\{ -f \left( \frac{\partial v}{\partial \lambda} \right)^2 + 2 \frac{\partial v}{\partial \lambda} \frac{\partial r}{\partial \lambda} \right\} d\lambda^2 + r^2 d\Omega^2 , \]

taking the cross term as zero by taking the transformation properly. Assuming that the condition \( -f \left( \frac{\partial v}{\partial T} \right)^2 + 2 \frac{\partial v}{\partial T} \frac{\partial r}{\partial T} = -1 \) is enforced, if the spherically symmetric hypersurface is considered as the direct product of a 2-sphere and an arbitrary curve parametrized by \( \lambda \) in \( v - r \) plane then we are able to find the hypersurface \( \Sigma : T = \text{constant} \) \[10, 11\] where we have

\[ ds^2_{\Sigma} = -dT^2 + \left\{ -f \dot{v}^2 + 2\dot{v} \dot{r} \right\} d\lambda^2 + r^2 d\Omega^2 . \]

The interior volume within the horizon can be written by the surface \( \Sigma \equiv \gamma \times S^2 \) on which metric can be written as

\[ ds^2 = \{ -f \dot{v}^2 + 2\dot{v} \dot{r} \} d\lambda^2 + r^2 d\Omega^2 , \]

where, \( r = r(\lambda) \) and \( v = v(\lambda) \), \( \lambda \) being an arbitrary parameter\[11\].

The volume can be written as

\[ V_\Sigma = 4\pi \int d\lambda \sqrt{r^4 \{ -f \dot{v}^2 + 2\dot{v} \dot{r} \}} . \] (2)

Now, the metric takes the form of an integrand of Lagrangian,

\[ dS^2_{eff} = r^4 \left\{ -f \dot{v}^2 + 2\dot{v} \dot{r} \right\} d\lambda^2 . \] (3)

For this Lagrangian, the coordinates are \( (r, v) \) and the momenta are \( (P_r, P_v) \) respectively. Now, \( \int dr dv dP_r dP_v \) represent the phase space volume. In this paper we evaluate the entropy inside this phase space volume. In the process of evaluation of entropy in a quantum statistical way, firstly we have to calculate the Hamiltonian of a particle, restricted inside the volume.

3 Calculating the Hamiltonian of a Particle

Considering \( m \) as the mass of a particle which moves in a space time with a background metric given by

\[ ds^2_{\text{ansatz}} = g_{ab} dx^a dx^b = -dt^2 + r^4 \left( -f(r, t) dv^2 + 2dv dr \right) , \] (4)

the action (which supposed to have the reparametrization symmetry) seems to be

\[ S = m \int_1^2 dS_{\text{ansatz}} = m \int_1^2 \left( g_{ab} dx^a dx^b \right)^{\frac{1}{2}} . \] (5)

The velocities of the particle is given by \( u^a = \frac{dx^a}{d\tau} \), where, \( \tau \) is an arbitrary parameter and \( x^a = x^a(\tau) \). The path of the particle is then given by

\[ S = \int_1^2 \mathcal{L} d\tau = m \int_1^2 \left( g_{ab} \frac{dx^a}{d\tau} \frac{dx^b}{d\tau} \right)^{\frac{1}{2}} d\tau . \] (6)

Comparison of both sides clear states the Lagrangian to have the form \( \mathcal{L} = m \left( g_{ab} \frac{dx^a}{d\tau} \frac{dx^b}{d\tau} \right)^{\frac{1}{2}} \). By using the Euler Lagrange equation, we can easily find the equation of the motion of the system as

\[ \frac{d^2 x^a}{d\tau^2} + \Gamma^a_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = 0 \Rightarrow \frac{du^a}{d\tau} + \Gamma^a_{bc} u^b u^c = 0 \] ,
where $\Gamma^a_{bc}$ is the Christoffel symbol. Also the above geodesic equation is true for any space time. To evaluate the Hamiltonian which describes the whole dynamical system, firstly, we have to calculate the momenta of the system given by

$$ P_a = \frac{\partial L}{\partial \dot{x}^a} = \frac{m^2}{\mathcal{L}} g_{ab} \frac{dx^b}{dt}. $$

Therefore, the canonical Hamiltonian is

$$ H_c = P_a \frac{dx^a}{dt} - L = 0. $$

In reparametrization invariant theory, canonical Hamiltonian was zero which is a typical signature of this theory; for Minkowski space time the same thing can be observed. For the present article, we have to analyse the Hamiltonian of the system given by (4). For chronological gauge [12], in flat space time the analysis is done and for proper time gauge in [13]. For calculation of entropy, we present the constraint which are independent.

Since the momenta are not independent,

$$ P^2 = g^{ab} P_a P_b = m^2. $$

Then we get primary constraint

$$ \Phi = P^2 - m^2 \approx 0 $$

Using Dirac’s algorithm [14], the primary constraint and the Hamiltonian are proportional to each other, therefore

$$ H_T = \xi(\tau) \Phi = \xi(\tau) \left( P^2 - m^2 \right), $$

where $\xi$ depends on $\tau$ and is known as proportionality constant.

Hence

$$ x^a = \frac{dx^a}{d\tau} = u^a = \{x^a, H_T\} = 2\xi P^a $$

and

$$ H_T = \xi(\tau) \Phi = \xi(\tau) \left( P^2 - m^2 \right), $$

also using (7) and (12) we get, $u^a = 2\xi P^a = 2\xi \frac{m^2}{\mathcal{L}} u^a$. So we have

$$ \xi = \frac{\mathcal{L}}{2m^2}. $$

Therefore, the total Hamiltonian is

$$ H_T = \frac{\mathcal{L}}{2m^2} \left( P^2 - m^2 \right). $$

So far, we have considered one constraint only (given by (10)). This characterises the system as of first class and hence it has gauge freedom [15]. Imposing some conditions on the arbitrary parameter $\tau$ which we are going to interpret as proper time, the gauge freedom can be removed. A proper time gauge will be imposed throughout the subsequent analysis.

Therefore we are now with

$$ \psi_2 = \frac{P^0}{m} \tau - x^0 \approx 0 $$

and for primary constraint (11) we have,

$$ \psi_1 = P^2 - m^2 \approx 0, $$

this makes the system as second class.

Therefore,

$$ \dot{\psi}_2 = \frac{\partial \psi_2}{\partial \tau} + \{\psi_2, H_T\} = 0 \Rightarrow \frac{P^0}{m} - \{x^0, H_T\} + \frac{\tau}{m} \{P^0, H_T\} = 0. $$

Also

$$ \{x^0, H_T\} = \{x^0, \xi P^2\} = 2\xi g^{ab} \{x^0, P_a\} P_b = 2\xi P^0 $$

and

$$ \{P^0, H_T\} = \xi \left( 2\frac{P^0}{m} \frac{\partial g^{0b}}{\partial x^a} P_b - g^{0a} \frac{\partial g^{bc}}{\partial x^a} P_b P_c \right). $$

For the metric (4), we get

$$ \{P^0, H_T\} = 0. $$
Therefore equation (18) and (19) gives
\[ \xi = \frac{1}{2m} . \] (21)

From equations (14) and (21) we can obtain \( \mathcal{L} = m \).
Also from equation (11) and (12) we get,
\[ \dot{x}_a = \frac{1}{2m}\{}x^a, P^2\{} = \frac{P_a}{m} . \] (22)

and
\[ \dot{P}_a = \frac{1}{2m}\{}P_a, P^2\{} = \frac{1}{2m}\left( \frac{\partial g^{ab}}{\partial x^c}g^{cd} - \frac{\partial g^{db}}{\partial x^c}g^{ca} \right) P_dP_b . \] (23)

Here we use the identity
\[ \partial_x g^{ab} = -g^{ad}g_{bc}\partial_x g^{de} \]
for two dynamical variables, calculating the Dirac bracket [9, 16]
\[ \{f_1, f_2\}^* \]
is
\[ \{f_1, f_2\}^* = \{f_1, f_2\} + \frac{1}{2P_0}\left( \{f_1, P^2\}\{\psi_2, f_2\} - \{f_1, \psi_2\}\{P^2, f_2\} \right) . \] (25)

For a dynamical variable, the equation of motion can be obtained by the following relation [14]
\[ \dot{f}_1 = \{f_1, H\}^* . \] (26)

In fact
\[ H = P^0 \]
serves our motive.
\[ \dot{x}^a = \{x^a, P^0\}^* = g^{a0} + \frac{1}{P_0}\left( P^a + \frac{\tau}{m}g^{a0}\Gamma^{bc}_0 P^bP^c \right) \] and \[ \dot{P}^a = \{P^a, P^0\}^* = \frac{\partial g^{ab}}{\partial x^c}g^{0c}P_b - \frac{1}{P_0}\Gamma^{bc}_0 P^bP^c + g^{0a}\Gamma^{bc}_0 P^bP^c . \] (27)

By the gauge fixing constraint
\[ a = 0 \]
has been eliminated, then the equation of motion for the space component of
\[ a(a = \mu) \]
is
\[ \dot{x}^\mu = \frac{P^\mu}{P_0} \] and \[ \dot{P}^\mu = \frac{1}{P_0}\Gamma^{\mu\nu}_ab\mathcal{P}^aP^b . \] (28)

From the above two equation, we can eliminate \( P^0 \) and get the desired geodesic equation. Now,
\[ g^{ab}P_aP_b = -P^0)^2 + 2r^{-4}P_rP_v - f r^{-4}P_r^2 = m^2 \] (29)

The energy of a particle, i.e., the Hamiltonian is given by
\[ \epsilon = P^0 = \left( \frac{-fP_r^2}{r^4} + \frac{2P_rP_v}{r^4} - m^2 \right)^{\frac{1}{2}} \] (30)

We are calculating the energy for a massless particle (like photon). So we can consider \( m \to 0 \), then the equation (30) can be written as
\[ \epsilon = \left( \frac{-fP_r^2}{r^4} + \frac{2P_rP_v}{r^4} \right)^{\frac{1}{2}} \] (31)
4 Entropy Calculation

Now, we will construct the expression for entropy w.r.t. the energy for the massless particle. This entropy is defined inside the black hole. Also, no chemical potential terms are present, hence Gibb’s free energy is 

\[ G = -\frac{1}{\beta} \ln Z = \frac{1}{\beta} \sum_c \ln(1 - \exp(-\beta \varepsilon_c)) \]

where, \( Z \) is the grand canonical partition function and \( \beta \) is the inverse temperature. In horizon, for maximum volume of the interior, we get the fixed value of the radial coordinate, i.e., \( \dot{r} = 0 \), where \( r \) can be obtained from the equation (28) which implies \( \dot{r} = r^{-4} (P_v + f P_r) = 0 \) \( \Rightarrow P_v + f P_r = 0 \). Again \( r \) and ingoing null coordinates are depending on \( \lambda \) (a parameter)[11], i.e., \( r = r(\lambda) \) and \( v = v(\lambda) \), \( v = F(r) \), a function of \( r \). Therefore the Gibb’s free energy

\[ G = \frac{1}{\beta} \int \frac{dP_v dP_r dr dv}{h^2} \ln (1 - \exp(-\beta \varepsilon)) \times \delta (P_v + f P_r) \delta(v - F(r)) . \]

Using Dirac-Delta functions

\[ G = \frac{1}{h^2 \beta} \int dP_r \ln \left[ 1 - \exp \left\{ -\beta \left( -\frac{f P_r^2}{r^4} \right) \right\} \right] , \]

where, \( P_r \) varies from 0 to \( \infty \).

So

\[ G = \frac{1}{h^2 \beta} \int dr \int_0^\infty \ln \left[ 1 - \exp \left\{ -\beta \left( -\frac{f P_r^2}{r^4} \right) \right\} \right] \]

incorporating \( \frac{\beta P_r}{r^2 \sqrt{-y}} = x \) we have

\[ G = -\frac{\pi^2}{6h^2 \beta^2} \int \frac{r^2}{\sqrt{-y}} dr \]

Let us consider as time goes the radius of the black hole increases, then the radial coordinate varies from \( 2M(1 + \varepsilon t) \) to zero, where \( \varepsilon(> 0) \) is a very small real number, i.e., \( \varepsilon \to 0 \). So, we have

\[ G = -\frac{\pi^2}{6h^2 \beta^2} \int_{2M(1+\varepsilon t)}^0 \frac{r^2}{\sqrt{2M(1+\varepsilon t) - 1}} dr . \]

Incorporating \( y = \frac{r}{2M(1+\varepsilon t)} \), we have

\[ G = \frac{5\pi^3}{12h^2 \beta^2} M^3 (1 + \varepsilon t)^3 \]

Also, the inverse temperature is given by \( \beta = \frac{8\pi M}{\hbar} \) for the Schwarzschild black hole.

Therefore, the Gibb’s free energy

\[ G = \frac{5}{6144} \hbar \beta (1 + \varepsilon t)^3 \]

Hence, the entropy is

\[ S = \beta^2 \frac{\partial G}{\partial \beta} = \frac{5\hbar \beta^2}{6144} (1 + 3\varepsilon t) , \]

neglecting the higher power of \( \varepsilon t \).

Hence, we can say that the entropy is monotonically increasing as time increases but the increment of entropy depends on time which is very small.

Some important things we have to mention here : firstly, we have considered that the massless modes inside the horizon are at a temperature whose value is same as that of the event horizon. We should remember that \( r \) varies from 0 to \( 2M \) but our consideration was slightly different than this. We have assumed the mass of the black hole is increasing as time increases because matters get accreted towards a black hole with time. But the average density of the universe is very small. So the increment of the mass of the black hole is very slow. For this we assumed that the radial coordinate’s increment also is very minute. This lead us to the consideration that the mass of the black hole is \( M(1 + \varepsilon t) \) after a time \( t \) while \( M = M(t = t_0 = 0) \) and \( \varepsilon(> 0) \) is a very small number. We have also assumed that \( r \) varies from \( 2M(1 + \varepsilon t) \) to zero.

Also, Hawking showed that black holes emit thermal Hawking radiation[5][17] corresponding to a certain temperature(Hawking temperature)[2][18]. Using the thermodynamic relationship between energy, temperature and entropy, Hawking was able to confirm Bekenstein’s conjecture and fix the constant of proportionality at \( 1/4\)[19]:
$S_{BH} = \frac{A}{4\pi}$ is the entropy on the horizon of the black hole, where $A = 16\pi M^2$ is the area of the event horizon. Hence, the relation between $S$ and $S_{BH}$ is

$$S = \frac{5\pi}{384}(1 + 3\varepsilon t)S_{BH}$$

Fig.1

Relation between $S, t$ and $S_{BH}$

Thus we can see that the entropy on the horizon is greater than the entropy inside the black hole i.e. $S_{BH} > S$. We visualize from the Fig.1 that the entropy inside the black hole is proportional to the time $(t)$ i.e. if we increase the time then the entropy $(S)$ inside the black hole increase.

The first law of thermodynamics provides the basic definition of internal energy, associated with all thermodynamics systems, and states the rule of conservation of energy. The second law is concerned with the direction of natural processes. It asserts that a natural process runs only in one sense, and is not reversible. Also, we can see from the above

$$\frac{dS}{dt} > 0$$

Hence the total entropy can never decrease over time and the process is irreversible. According to the first law of thermodynamics

$$dM = TdS$$

$$\Rightarrow M\varepsilon dt = T\frac{5h\beta^2}{6144}3\varepsilon dt$$

$$\Rightarrow \beta = \frac{768}{15\pi T}$$

where, $\beta$ is the reciprocal of the thermodynamic temperature of the system [20, 21]. Thermodynamic beta is essentially the connection between the information of a physical system through its entropy and the thermodynamics associated with its energy.
5 Conclusion

A black hole is usually formed from the collapse of a quantity of matter or radiation, both of which carry entropy. However, the black hole’s interior and contents are veiled to an exterior observer. Thus a thermodynamic description of the collapse from that observer’s viewpoint cannot be based on the entropy of that matter or radiation because these are unobservable. A stationary black hole is parametrized by just a few numbers (Ruffini and Wheeler 1971): its mass, electric charge and angular momentum (and magnetic monopole charge, except its actual existence in nature has not been demonstrated yet). For any specific choice of these parameters one can imagine many scenarios for the black hole’s formation. Thus there are many possible internal states corresponding to that black hole. In thermodynamics one meets a similar situation: many internal microstates of a system are all compatible with the one observed (macro)state. Thermodynamic entropy quantifies the said multiplicity. Thus by analogy one needs to associate entropy with a black hole.

In this paper, we calculated the entropy for the Schwarzchild black hole contained by the CR volume for massless modes. The approach are considered as statistical one. We first introduced the integrand of the expression of interior volume as an effective metric. Then easily identified the energy of the modes. To handle the situation, one can use the method of constraint analysis, since the canonical Hamiltonian vanishes. The exact calculation of the Hamiltonian is itself new. By using the Gibb’s free energy we can calculate the entropy. The effect is very interesting. As the mass of the black hole increasing as time increases, the average density of the universe is very small. So, the increment of the mass of the black hole is very slow. So we consider the mass of the black hole is $M(1 + \varepsilon t)$ after a time $t$, where $\varepsilon(>0)$ is a very small number. Then we assumed $r$ varies from $2M(1 + \varepsilon t)$ to zero. Finally we found that the entropy is monotonically increasing as time increases but the increment of entropy depends on time which is very small. Also we can see that the entropy on the horizon is greater than the entropy inside the black hole i.e. $S_{BH} > S$. The entropy inside the black hole is proportional to the time($t$) i.e. if we increase the time then the entropy($S$) inside the black hole increase. Hope this paper will give a imagination knowledge about entropy.

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