Solutions of the Painlevé VI Equation from Reduction of Integrable Hierarchy in a Grassmannian Approach

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Abstract

We present an explicit method to perform similarity reduction of a Riemann-Hilbert factorization problem for a homogeneous $\hat{GL}(N, \mathbb{C})$ loop group and use our results to find solutions to the Painlevé VI equation for $N = 3$. The tau function of the reduced hierarchy is shown to satisfy the $\sigma$-form of the Painlevé VI equation. A class of tau functions of the reduced integrable hierarchy is constructed by means of a Grassmannian formulation. These solutions provide rational solutions of the Painlevé VI equation.

1 Introduction

One of main challenges in the study of integrable models is derivation of a tau function providing solutions of the nonlinear partial differential hierarchy equations. In this paper, we shall give a systematic and explicit construction of a self-similarity reduction of an infinite-dimensional integrable $\hat{GL}(N, \mathbb{C})$ hierarchy [1, 3], which in the case of $N = 3$ describes a three-component Kadomtsev-Petviashvili (KP) hierarchy [13]. The self-similarity reduction yields the $\sigma$-form of the Painlevé VI equation for which we explicitly construct a class of tau function solutions. We adopt Grassmannian approach to derive tau functions in terms of determinants obtained from expectation values of certain Fermi operators constructed using boson-fermion correspondence [4]. In particular, we obtain a description of rational solutions of the Painlevé VI equation in terms of Schur polynomials.

In the context of soliton partial differential equations a self-similarity reduction is closely related to the scaling behavior of their solutions. To illustrate this relation let us examine an example of the modified Korteweg-de Vries (mKdV) equation :

$$u_t - 6u^2u_x + u_{xxx} = 0$$

which has a self-similarity reduction [10] :

$$u(x, t) = (3t)^{-1/3} f(z), \quad z = x(3t)^{-1/3}. \quad (1.1)$$
Substitution of such function $u(x, t)$ into the mKdV equation yields the following ordinary differential equation

$$f'' = 2f^3 + zf - \alpha, \quad \alpha = \text{constant},$$

where $'$ denotes derivative with respect to the appropriate argument. We recognize in the above relation the second Painlevé equation. It is of interest to point out that the single similarity variable $z$ combines the two variables $x$ and $t$ in such a way as to ensure a scaling property:

$$u(\lambda x, \lambda^3 t) = \lambda^{-1} u(x, t), \quad (1.2)$$

which characterizes function $u(x, t)$ in the self-similarity limit. Alternatively one can reformulate the above scaling property as a linear condition $(x \partial_x + 3t \partial_t) u(x, t) = -u(x, t)$ being a special case of the so-called $L_{-1}$ Virasoro constraint.

A common feature of soliton equations is that they are members of integrable hierarchies, each hierarchy forms an infinite sequence of evolution equations, labeled by their order. In the example considered above the mKdV equation is the first member of the mKdV hierarchy while the second Painlevé equation is the first member of the the second Painlevé hierarchy.

The Riemann-Hilbert factorization problem serves the purpose of generating all evolution equations of the underlying hierarchy. For the purpose of this paper we will use the $\hat{G}L(N, \mathbb{C})$ Riemann-Hilbert factorization problem of the form:

$$\exp \left( \sum_{j=1}^{N} \sum_{n=1}^{\infty} z^n E_{jj} n^{(j)} \right) g(z) = \Theta^{-1}(u, z) \Pi(u, z). \quad (1.3)$$

A self-similarity reduction (to which we will also refer as a Painlevé reduction) is implemented by restricting $g : S^1 \rightarrow GL(N)$ to be of the special form:

$$g(z) = z^{\mu} g_0 z^{-\nu}, \quad (1.4)$$

where $z^{\mu} = \text{diag}(z^{\mu_1}, \ldots, z^{\mu_N})$ and $z^{-\nu} = \text{diag}(z^{-\nu_1}, \ldots, z^{-\nu_N})$. The reduction imposed by condition (1.4) yields an integrable model parametrized by a set of conformal (scaling) dimensions $\mu_i, \nu_i, i = 1, \ldots, N$. The notation used in eq. (1.3) and definitions of the dressing matrices $\Theta(u, z)$ and $\Pi(u, z)$ will be explained in the next section. Here we only note that $E_{ij}$ is a unit matrix with matrix elements given by $(E_{ij})_{kl} = \delta_{ik} \delta_{jl}$ and that the homogeneous gradation defined by the gradation operator

$$d = z \frac{d}{dz}$$

is assumed.

As seen below equation (1.2) generators of the conformal symmetry naturally enter discussion of the self-similarity reduction process. As indicated by (1.2) an alternative mechanism to arrive at result of equation (1.1) involves restricting $u(x, t)$ to be in the space of stationary solutions of an appropriate Virasoro symmetry
generator [10, 20] of the original hierarchy. In a related method, the same result is obtained by imposing constraints on the Lax and Orlov-Schulman operators entering string equation of the KdV hierarchy [22], which when reformulated in terms of the wave-function can again be expressed as a linear constraint involving action of the Virasoro symmetry generator. This shows equivalence of introducing a self-similarity limit by either imposing a certain scaling behavior or a certain Virasoro condition. Below, in equation (2.27), we will show that the appropriate Virasoro condition $L_{-1} \Psi = \nu \Psi$ given in terms of the Virasoro operator $L_{-1}$, characterizes in an alternative way the reduction carried out in this paper.

The idea of imposing a well-defined scaling behavior on dressing matrices to perform the self-similarity reduction of the $\hat{GL}(3, \mathbb{C})$ integrable hierarchy was already used recently by Kakei and Kikuchi [15] who established connection to $3 \times 3$ Fuchs–Garnier system [11, 19] (see also [16]) and subsequently to $2 \times 2$ Schlesinger system of isomonodromy deformation flows known [12, 21] to be associated with the Painlevé VI equation. In a related development, a one-dimensional reduction to the generic Painlevé VI equation of the three-wave resonant system was constructed in [6]. Here we provide a direct and explicit construction relating the tau function of the reduced $\hat{GL}(3, \mathbb{C})$ hierarchy to the solution of the $\sigma$-form of the Painlevé VI equation (see eq. (1.9) below).

To set the work in context let us point out that the construction in this paper generalizes a simpler case of the three-dimensional Frobenius manifold of the two-dimensional conformal field theory. In that special case it was shown in [3] that the above construction with a much simpler choice of $\mu = \text{diag}(-\mu, 0, \mu)$ and $\nu = 0$ leads to the Painlevé VI equation:

$$\frac{d^2 y}{dt^2} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) \left( \frac{dy}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) \frac{dy}{dt} + \frac{y(y-1)(y-t)}{t(t-1)^2} \left\{ \alpha + \beta \frac{t}{y^2} + \gamma \frac{t-1}{(y-1)^2} + \delta \frac{t(t-1)}{(y-t)^2} \right\},$$

with the Painlevé parameters $(\alpha, \beta, \gamma, \delta)$ fully determined by a single conformal scaling parameter $\mu$ through:

$$\alpha = \frac{(1 + \mu)^2}{2}, \quad \beta = -\frac{\mu^2}{2}, \quad \gamma = \frac{\mu^2}{2}, \quad \delta = \frac{1 - \mu^2}{2}. \quad (1.6)$$

Note that, in an equivalent parametrization of [9, 3], the Painlevé parameters $(\alpha, \beta, \gamma, \delta)$ take the following form:

$$\alpha = \frac{(1 \pm 2\mu)^2}{2}, \quad \beta = 0, \quad \gamma = 0, \quad \delta = \frac{1}{2}. \quad (1.7)$$

The above restriction on the allowed values of the Painlevé parameters $(\alpha, \beta, \gamma, \delta)$ no longer holds in a more general setting defined by (1.4) and, as shown below, we find relations of the type:

$$\beta = -\frac{(\nu_2 - \mu_3)^2}{2}, \quad \gamma = \frac{(\mu_1 - \mu_2)^2}{2}, \quad \alpha = \frac{(\nu_3 - \mu_3)^2}{2}, \quad \delta = \frac{1}{2} \left( 1 - (1 + \nu - \mu_3)^2 \right).$$
For \( g(z) \) as in (1.4) and \( N = 3 \) we will show that certain dressing matrix can be transformed by a matrix similarity transformation in such a way that it is determined by six variables \( \omega_i, \bar{\omega}_i, i = 1, 2, 3 \), which only depend on one single variable

\[
t = \frac{u_1^{(2)} - u_1^{(1)}}{u_1^{(3)} - u_1^{(1)}}.
\]

These six variables satisfy generalizations the familiar Euler top equations:

\[
\begin{align*}
\frac{\partial}{\partial t} \omega_3 &= \frac{\bar{\omega}_1 \omega_2}{1 - t}, \\
\frac{\partial}{\partial t} \omega_2 &= \frac{\omega_1 \omega_3}{t(t-1)} - \frac{\omega_2}{t}(\nu_3 - \nu_1), \\
\frac{\partial}{\partial t} \omega_1 &= \frac{\omega_2 \omega_3}{t} + \frac{\omega_1}{t(t-1)}(\nu_3 - \nu_2),
\end{align*}
\]

and are constrained by a quadratic algebraic relation \( \sum_i \omega_i \bar{\omega}_i = -R^2 \) with a constant \( R \) defined in (2.43) as well as a cubic algebraic relation given in relation (2.52).

Equivalence is established between on the one hand the above system of equations (1.8) and the algebraic constraints for arbitrary values of scaling parameters \( \mu_i, \nu_i, i = 1, 2, 3 \) and on the other hand the Painlevé VI equation (1.5) with the complete set of the Painlevé parameters \( (\alpha, \beta, \gamma, \delta) \). Explicit construction shows that the system consisting of relations

\[
\omega_2 \bar{\omega}_2 = f', \quad \omega_1 \bar{\omega}_1 = -f + (t-1)f',
\]

between \( \omega \)’s and the tau function \( \tau \) represented by \( f(t) = t(t-1) d\log \tau / dt \) and the algebraic constraints for \( \omega \)’s is equivalent to the so-called \( \sigma \)-form of the Painlevé VI equation [12]:

\[
\frac{d\sigma}{dt} \left( t(t-1) \frac{d^2 \sigma}{dt^2} \right)^2 + \left( \frac{d\sigma}{dt} \left[ 2\sigma - (2t-1) \frac{d\sigma}{dt} \right] + v_1 v_2 v_3 v_4 \right)^2 = \prod_{k=1}^4 \left( \frac{d\sigma}{dt} + v_k^2 \right),
\]

where, for certain constants \( a, b \),

\[
\sigma = t(t-1) \frac{d\log \tau}{dt} - at - b.
\]

Note, that a similar relation between the tau function underlying the Painlevé VI system and \( \sigma \) appeared in [21] with different values of \( a \) and \( b \), which implies that our KP derived tau function differs from the tau function used in [21].

The \( \sigma \)-form of the Painlevé VI equation exhibits a \( D_4 \) root system symmetry in the parameters \( v_1, \ldots, v_4 \), which are related to the general Painlevé parameters \( (\alpha, \beta, \gamma, \delta) \) through

\[
v_1 + v_2 = \sqrt{-2\beta}, \quad v_1 - v_2 = \sqrt{2\gamma}, \quad v_3 + v_4 + 1 = \sqrt{1 - 2\delta}, \quad v_3 - v_4 = \sqrt{2\alpha}.
\]
Below, we will find the realization of the above parameters in terms of the scaling dimensions \( \mu_i, \nu_i, i = 1, \ldots, 3 \).

In the 3-component Clifford algebra setting the tau-function is explicitly realized as

\[
\tau(\nu_1, \nu_2, \nu_3; \mathbf{u}) = \langle 0| Q_3^{-\nu_3} Q_2^{-\nu_2} Q_1^{-\nu_1} e^{\sum_{i,k} \alpha(i) u_k(i)} G |0\rangle,
\]

where, as explained below, the element \( Q_3^{-\nu_3} Q_2^{-\nu_2} Q_1^{-\nu_1} \) is related to \( z^\nu, e^{\sum_{i,k} \alpha(i) u_k(i)} \) to \( e^{\sum_{i,k} z^i E_{kk} u_k(i)} \) and \( G |0\rangle \) corresponds to \( g_0 z^{-\mu} \) (cf. (1.4)) acting on the vacuum \( |0\rangle = e_3 \wedge e_2 \wedge e_1 \wedge z e_3 \wedge z e_2 \wedge z e_1 \wedge z^2 e_3 \wedge z^2 e_2 \wedge z^2 e_1 \wedge z^3 e_3 \wedge z^3 e_2 \wedge z^3 e_1 \wedge z^4 e_3 \wedge \cdots \), where \( \{e_i\}_{1 \leq i \leq 3} \) is a basis of \( \mathbb{C}^3 \).

The paper is organized as follows. The main subject of Section 2 is a \( \hat{GL}(N, \mathbb{C}) \) hierarchy in the self-similarity limit characterized by the scaling parameters \( \mu \) and \( \nu \). In the end of this section the \( \sigma \)-form of the Painlevé VI equation is obtained for the underlying tau function and the parameters of the Painlevé VI equation are related to the scaling parameters \( \mu \) and \( \nu \). Section 3 develops an explicit Grassmannian realization of Riemann-Hilbert factorization problem leading to the tau function given as an expectation value within the 3-component Clifford algebra setting. Some background material about semi-infinite wedge space and Clifford algebra is also given here. Section 4 derives conditions satisfied by the tau function constructed in the previous section. In the next section, Section 5, the tau function is cast in the determinant form. The technical details of this Section are relocated to Appendix A. The wave matrix is calculated in Section 7. In Section 6 we describe an explicit example, which produces e.g. the solution

\[
y = \frac{-D_2^2 t^5 + 3 D_2^2 t^4 + 2 D_1 D_2 t^3 + 2 D_1 D_2 t^2 + 3 D_1^2 t - D_1^2}{D_1^2 + 4 D_2 D_1 t - 6 D_1 D_2 t^2 + 4 D_1 D_2 t^3 + D_2^2 t^4}
\]

of the Painlevé VI equation (1.5) for arbitrary constants \( D_1, D_2 \in \mathbb{C} \) and the parameters

\[
\alpha = \frac{1}{2}, \quad \beta = -2, \quad \gamma = 2, \quad \text{and} \quad \delta = -\frac{3}{2}.
\]

## 2 \( \hat{GL}(N, \mathbb{C}) \) Hierarchy in the Self-Similarity Limit

Let the Riemann-Hilbert factorization problem for \( \hat{GL}(N, \mathbb{C}) \) be defined as in equations (1.3) and (1.4) with all higher flows \( u_n^{(j)}, n > 1 \) set to zero. For notational convenience we set \( u_j = u_1^{(j)}, j = 1, \ldots, N \) and use the multi-time notation with \( u = (u_1, \ldots, u_N) \) to denote all \( N \) \( u_j \)-flows. For brevity we sometimes denote \( \partial_j = \partial/\partial u_j \) for \( j = 1, \ldots, N \). (Later on we will also use the notation \( \mathbf{u} \) for all \( u_i^{(k)}, i = 1, 2, 3, \ldots, k = 1, 2, 3, \) when we do not put all higher flows \( u_i^{(k)}, i > 1 \) to zero.)

Equation (1.3) simplifies to

\[
\exp \left( \sum_{j=1}^N z E_{jj} u_j \right) g(z) = \Theta^{-1}(u, z) \Pi(u, z),
\]

(2.1)
The dressing matrices $\Theta, \Pi$ in equation (2.1) have the following expansions into positive and negative modes with respect to the gradeation operator:

$$
\Theta(u, z) = 1 + \theta^{(-1)}(u)z^{-1} + \theta^{(-2)}(u)z^{-2} + \ldots \quad (2.2)
$$

$$
\Pi(u, z) = M(u) \left(1 + \pi^{(1)}(u)z + \pi^{(2)}(u)z^2 + \ldots \right) . \quad (2.3)
$$

One derives from (2.1) the following expressions for the symmetry $u_j$-flows:

$$
\frac{\partial}{\partial u_j} \Theta(u, z) = - (\Theta z E_{jj} \Theta^{-1})_+ \Theta(u, z) \quad (2.4)
$$

$$
\frac{\partial}{\partial u_j} \Pi(u, z) = (\Theta z E_{jj} \Theta^{-1})_+ \Pi(u, z) \quad (2.5)
$$

where $(\ldots)_\pm$ denote the projections onto the negative and positive powers of $z$, respectively.

We now choose $g : S^1 \to GL(N)$ to be given by relation (1.4) so that the Riemann-Hilbert factorization problem becomes

$$
e^z u z \nu g_0 z^{-\mu} = \Theta^{-1}(u, z) \Pi(u, z) \quad (2.6)
$$

where $g_0$ is a grade zero element. The Riemann-Hilbert factorization problem (2.6) leads to the $u_j$-flows of the same functional form as in (2.4) and (2.5) as they are not affected of the functional form of $g(z)$.

Applying the grading operator $d$ on both sides of eq. (2.6) one finds for the negative and strictly positive grades:

$$
d\Theta = - (\Theta z u \Theta^{-1})_\Theta - (\Theta z u \Theta^{-1})_\Theta = - (\Theta z u \Theta^{-1})_\Theta - \Theta \nu + \nu \Theta
$$

$$
d\Pi = (\Theta z u \Theta^{-1}) >_0 \Pi - (\Pi \mu \Pi^{-1}) >_0 \Pi \quad (2.7)
$$

For the grade zero one obtains from equation (2.6) a consistency condition:

$$
(\Theta z u \Theta^{-1})_0 + (\Theta z u \Theta^{-1})_0 - (\Pi \mu \Pi^{-1})_0 = (\Theta z u \Theta^{-1})_0 + \nu - M \mu M^{-1} = 0 \quad (2.8)
$$

It is convenient to define the unity $\hat{I}$ and the Euler $\hat{E}$ vector fields as:

$$
\hat{I} = \sum_{j=1}^{N} \frac{\partial}{\partial u_j}, \quad \hat{E} = \sum_{j=1}^{N} u_j \frac{\partial}{\partial u_j} \quad (2.9)
$$

Note that from (2.7) and (2.4) one finds

$$
\left( z \frac{\partial}{\partial z} - \hat{E} \right) \Theta(u, z) = - (\Theta z u \Theta^{-1})_\Theta = [\nu, \Theta], \quad \hat{I}(\Theta) = 0 . \quad (2.10)
$$

Relation (2.10) implies the following scaling law for $\Theta$:

$$
\Theta(u, \lambda z) = \lambda^{\nu} \Theta(u z, \lambda) \lambda^{-\nu} . \quad (2.11)
$$
From eq. (2.4) it follows quite generally for the diagonal elements of \( \theta^{(-1)} \):

\[
\partial_j (\theta^{(-1)})_{ii} = - (\Theta z E_{jj} \Theta^{-1})_{-1} ii = (\theta^{(-1)} E_{jj} \theta^{(-1)})_{ii} = \beta_{ij} \beta_{ji} \quad i \neq j = 1, \ldots, N
\]

where we introduced the “rotation coefficients” \( \beta_{ij} \) with \( i \neq j \) as the off-diagonal elements of the \( \theta^{(-1)} \) matrix:

\[
\beta_{ij}(u) = (\theta^{(-1)})_{ij}(u), \quad i \neq j = 1, \ldots, N.
\] (2.12)

Thus,

\[
\partial_j (\theta^{(-1)})_{ii} - \partial_i (\theta^{(-1)})_{jj} = 0
\]

and accordingly we can express the diagonal elements of the \( \theta^{(-1)} \) matrix as a derivative of a logarithm of a tau-function:

\[
(\theta^{(-1)})_{ii} = - \partial_i \log \tau.
\] (2.13)

It follows that

\[
\partial_j \partial_i \log \tau = - \beta_{ij} \beta_{ji} \quad i \neq j = 1, \ldots, N
\] (2.14)

For \( i = j \) in the above equation we get from eq. (2.4):

\[
\partial_i (\theta^{(-1)})_{ii} = - (\Theta z E_{ii} \Theta^{-1})_{-1} ii = (\theta^{(-1)} E_{ii} \theta^{(-1)})_{ii} - (\theta^{(-1)})^2_{ii} = - \sum_{k \neq i} \beta_{ik} \beta_{ki}
\]

To summarize, we have found that

\[
\partial_i \partial_j \log \tau = \begin{cases} 
- \beta_{ij} \beta_{ji} & i \neq j \\
\sum_{k \neq i} \beta_{ik} \beta_{ki} & i = j.
\end{cases}
\] (2.15)

From eq. (2.11) we derive

\[
\theta^{(-1)}(u) = \lambda^{1+\nu} \theta^{(-1)}(\lambda u) \lambda^{-\nu}
\] (2.16)

or for the matrix-elements of \( \theta^{(-1)}(u) \):

\[
\tilde{E} (\theta^{(-1)}_{ij}) = -(1 + \nu_i - \nu_j) \theta^{(-1)}_{ij} \quad \text{or} \quad \tilde{E} (\beta_{ij}) = -(1 + \nu_i - \nu_j) \beta_{ij}
\] (2.17)

The above results also follow from eq. (2.10). In particular, for the diagonal elements of the dressing matrix one gets:

\[
\tilde{E} (\theta^{(-1)}_{ii}) = - \theta^{(-1)}_{ii} \quad \rightarrow \quad \tilde{E} (\partial_i \log \tau) = - \partial_i \log \tau \quad \rightarrow \quad \partial_i \tilde{E} (\log \tau) = 0
\]

which amounts to

\[
\tilde{E} (\log \tau) = \text{const}.
\]
Using eqs. (2.14) and (2.15) we obtain

\[ \hat{E} \left( \partial_i \log \tau \right) = \sum_{j=1}^{N} u_j \partial_j \partial_i \log \tau = - \sum_{j \neq i}^{N} u_j \beta_{ij} \beta_{ji} + u_i \sum_{k \neq i}^{N} \beta_{ik} \beta_{ki} \]
\[ = - \sum_{j=1}^{N} (u_j - u_i) \beta_{ij} \beta_{ji} \]

It follows from \( 0 = \partial_i \hat{E} \left( \log \tau \right) = \partial_i \log \tau + \hat{E} \left( \partial_i \log \tau \right) \) that

\[ \partial_i \log \tau = - \theta_i^{(-1)} = \sum_{j=1}^{N} (u_j - u_i) \beta_{ij} \beta_{ji} . \]

Thus

\[ \hat{I} \left( \log \tau \right) = \sum_{i,j=1}^{N} (u_j - u_i) \beta_{ij} \beta_{ji} = 0 \]

and

\[ \hat{E} \left( \log \tau \right) = \sum_{i=1}^{N} u_i \partial_i \log \tau = \sum_{i,j=1}^{N} u_i (u_j - u_i) \beta_{ij} \beta_{ji} \]
\[ = - \beta_{12} \beta_{21} (u_1 - u_2)^2 - \beta_{13} \beta_{31} (u_1 - u_3)^2 - \beta_{32} \beta_{23} (u_3 - u_2)^2 \]
\[ = V_{12} V_{21} + V_{13} V_{31} + V_{32} V_{23} = \frac{1}{2} \text{Tr} \left( V^2 \right) \quad \text{for} \ N = 3 , \]

where we have introduced

\[ V \equiv [\theta^{(-1)}, u] = (\Theta z u \Theta^{-1})_0 . \]

Matrix \( V \) reads in components:

\[ V_{ij} = (u_j - u_i) \theta_{ij}^{(-1)} = (u_j - u_i) \beta_{ij}, \quad i, j = 1, \ldots, N . \]

Thus we conclude that \( \text{Tr} \left( V^2 \right) \) is a constant. Below we will find that \( \text{Tr} \left( V^2 \right) = \sum_i \mu_i^2 - \sum_i \nu_i^2 . \)

Applying the scaling law (2.11) to expressions (2.12), (2.13) we obtain scaling rules

\[ \tau(\lambda u) = \tau(u), \quad \beta_{ij}(\lambda u) = \lambda^{-1+\nu_i-\nu_j} \beta_{ij}(u), \]
which are consistent with relations (2.17).

The similar scaling law for \( \Pi \) can be obtained from second of eqs. (2.7). First using relation (2.8) we obtain

\[ d \Pi = (\Theta z u \Theta^{-1})_+ \Pi + (\Theta u \Theta^{-1})_0 - (\Pi u \Pi^{-1})_+ \Pi , \]

which leads to

\[ \left( d - \hat{E} \right) \Pi(u, z) = \nu \Pi(u, z) - \Pi(u, z) \mu \]
\[ (2.22) \]
consistent with the following scaling law

\[ \Pi(u, \lambda z) = \lambda^\frac{\nu}{\mu} \Pi(\lambda u, z) \lambda^{-\frac{\nu}{\mu}}. \]  

(2.23)

From eqs. (2.22) and (2.23) it clearly follows that

\[ \hat{E}(M)(u) = M(u)\mu - \nu M(u), \quad M(u) = \lambda^\frac{\nu}{\mu} M(\lambda u) \lambda^{-\frac{\nu}{\mu}}. \]  

(2.24)

Define a wave matrix \( \Psi \) as :

\[ \Psi(u, z) = \Theta(u, z) e^{zu^2 z^2}. \]  

(2.25)

Then

\[ d \Psi = ((\Theta(z u)\Theta^{-1})_+ + \nu) \Psi \]  

(2.26)

and

\[ \frac{\partial}{\partial u_i} \Psi(z, t) = (\Theta z E_{ii} \Theta^{-1})_+ \Psi(z, t) \]

Thus

\[ \left(d - \hat{E}\right)\Psi(u, z) = \nu \Psi(u, z), \]  

(2.27)

which ensures the scaling behavior \( \Psi(u, \lambda z) = \lambda^\frac{\nu}{\mu} \Psi(\lambda u, z) \).

The operator on the right hand side was identified with the Virasoro operator \( \mathcal{L}_{-1} \) in [2], where the corresponding left hand side of the Virasoro condition vanished.

### 2.1 Reduction. \( \widehat{GL}(3, \mathbb{C}) \) Hierarchy.

From now on we set \( N = 3 \).

The scaling law for the matrix elements of \( V \) reads

\[ V_{ij}(u) = \lambda^{\nu_i - \nu_j} V_{ij}(\lambda u) \]

In terms of the matrix \( V \) condition (2.8) becomes :

\[ V + \nu - M \mu M^{-1} = 0 \]  

(2.28)

or

\[ VM = M\mu - \nu M = \hat{E}(M). \]

Eq. (2.26) can be written as :

\[ d \Psi = (zu + V + \nu) \Psi \]  

(2.29)

Furthermore

\[ \frac{\partial \Psi}{\partial u_i} = (zE_{ij} + V_j)\Psi. \]  

(2.30)

and

\[ \frac{\partial}{\partial u_j} M = V_j M. \]  

(2.31)
where we introduced a matrix

\[ V_j \equiv [\theta^{(-1)}, E_{jj}], \quad (V_j)_{kl} = (\delta_{ij} - \delta_{kj}) \theta^{(-1)}_{kl}, \tag{2.32} \]

which scales as in (2.16).

From compatibility of the above equations it follows that

\[
\begin{align*}
\partial_j V &= [V_j, V + \nu] \\
\partial_j V_i &= \partial_i V_j + [V_j, V_i] \\
[V, E_{jj}] &= [V_j, u]
\end{align*}
\tag{2.33}
\]

Also, since diagonal elements of \( V = [\theta^{(-1)}, u] \) are zero, it follows from (2.28) that

\[
\nu_i = (M \mu M^{-1})_{ii}, \quad i = 1, \ldots, N
\tag{2.34}
\]

Taking trace on both sides we obtain a trace condition:

\[
\nu_1 + \nu_2 + \nu_3 = \mu_1 + \mu_2 + \mu_3
\tag{2.35}
\]

The special case of \( \mu = -\mu, 0, \mu \) and \( \nu = 0 \) was already considered in [3].

It holds that

\[
\hat{I}(V) = 0, \quad \hat{E}(V) = [V, \nu], \quad \hat{E}(M) = VM.
\tag{2.36}
\]

The identity \( \hat{I}(V) = 0 \), shows that \( V \) is a function of two variables. Those can be identified with \( t \) and \( h \) given by:

\[
t = \frac{u_2 - u_1}{u_3 - u_1}, \quad h = u_2 - u_1
\tag{2.37}
\]

and thus \( V(u) = V(t, h) \). Making use of technical identities:

\[
\frac{\partial t}{\partial u_1} = \frac{1}{h} (t - 1) t, \quad \frac{\partial t}{\partial u_2} = \frac{1}{h} t, \quad \frac{\partial t}{\partial u_3} = -\frac{1}{h} t^2,
\]

one easily derives

\[
\frac{\partial}{\partial u_1} = \frac{t(t-1)}{h} \frac{\partial}{\partial t} - \frac{\partial}{\partial h}, \quad \frac{\partial}{\partial u_2} = \frac{t}{h} \frac{\partial}{\partial t} + \frac{\partial}{\partial h}, \quad \frac{\partial}{\partial u_3} = -\frac{t^2}{h} \frac{\partial}{\partial t},
\]

from which

\[
\hat{E} = h \frac{\partial}{\partial h}
\]

follows.

Now, define

\[
\tilde{V} = e^{\nu \log h} V(t, h) e^{-\nu \log h} = [\tilde{\theta}^{(-1)}, u],
\tag{2.38}
\]

where

\[
\tilde{\theta}^{(-1)} = e^{\nu \log h} \theta^{(-1)} e^{-\nu \log h}
\tag{2.39}
\]
has scaling dimension one.

\( \vec{V} \) satisfies

\[ \hat{E} (\vec{V}) = \hbar \frac{\partial \vec{V}}{\partial \hbar} = [\nu, \vec{V}] + e^{\nu \log \hbar} [V, \nu] e^{-\nu \log \hbar} = 0 \]

Because of \( \hat{I}(\hbar) = 0 \) it also follows that \( \hat{I}(\vec{V}) = 0 \) and, thus, the matrix \( \vec{V} \) is a function of only one variable \( t \): \( \vec{V} = \vec{V}(t) \).

Similarly define,

\[ \vec{M} = e^{\nu \log \hbar} M e^{-\nu \log \hbar} \quad (2.40) \]

Then

\[ \vec{V} + \nu = \vec{M} \mu \vec{M}^{-1} = e^{\nu \log \hbar} M \mu M^{-1} e^{-\nu \log \hbar} \]
\[ \hat{E} (\vec{M}) = (\vec{V} + \nu) \vec{M} - \vec{M} \mu = 0 \quad (2.41) \]

Thus, the matrix \( \vec{M} \) is also a function of only one variable \( t \) (since it follows from (2.31) that \( \hat{I}(\vec{M}) = 0 \)) and we set \( \vec{M} = \vec{M}(t) \).

We will use the following parametrization of matrices from eq. (2.41)

\[ \vec{V} + \nu = \vec{M} \mu \vec{M}^{-1} = \begin{bmatrix} \nu_1 & \omega_3 & -\omega_2 \\ -\omega_3 & \nu_2 & \omega_1 \\ \omega_2 & -\omega_1 & \nu_1 \end{bmatrix} \quad (2.42) \]

From the first of equations (2.33) we find

\[ \frac{\partial}{\partial t} \vec{V} = -\frac{\hbar}{t^2} [\vec{V}, \vec{V} + \nu], \quad \hat{V}_j = \left[ e^{\nu \log \hbar} \theta(1) e^{-\nu \log \hbar}, E_{jj} \right], \]

which leads to equations (1.8) for the matrix elements from (2.42). The basic observation to make in connection with equations (1.8) is that the limit \( \omega_i = \bar{\omega}_i \) is not consistent with these equations unless \( \nu_3 = \nu_2 = \nu_1 \). In such limit \( V \) is skew-symmetric and \( M \) is orthogonal.

Next, consider \( \text{Tr}(V^2) = \text{Tr}(\vec{V}^2) = \sum_i \omega_i \bar{\omega}_i \). From equations (1.8) it follows that

\[ \frac{\partial}{\partial t} \sum_i \omega_i \bar{\omega}_i = 0, \]

in agreement with relation (2.18). A direct calculation gives from eq. (2.28)

\[ \text{Tr} ((V + \nu)^2) = \text{Tr} (\mu^2) \]

or

\[ \text{Tr} (V^2) = \text{Tr}(\vec{V}^2) = \text{Tr} (\mu^2) - \text{Tr} (\nu^2) \]

which indeed is a constant.

In addition from the trace relation \( \text{Tr} ((V + \nu)^2) = \sum_i (\nu_i^2 - 2\omega_i \bar{\omega}_i) \) we have a condition:

\[ \omega_1 \bar{\omega}_1 + \omega_2 \bar{\omega}_2 + \omega_3 \bar{\omega}_3 = -\frac{1}{2} (\mu_1^2 + \mu_2^2 + \mu_3^2 - \nu_1^2 - \nu_2^2 - \nu_3^2) = -R^2, \quad (2.43) \]

where for convenience we have introduced on the right hand side a new constant \( R \). Since the trace condition (2.35) holds, one finds for \( c \in \mathbb{C} \) that

\[ R^2 = \frac{1}{2} \left( (\mu_1 - c)^2 + (\mu_2 - c)^2 + (\mu_3 - c)^2 - (\nu_1 - c)^2 - (\nu_2 - c)^2 - (\nu_3 - c)^2 \right). \]
2.2 The tau function and the $\sigma$-form of the Painlevé VI equation

We can use matrices $V$ and $V_j$ to find an alternative expressions for derivatives of the tau function as:

\[
\partial_j \log \tau = \frac{1}{2} \text{Tr}(V_j \tilde{V}), \quad \partial_i \partial_j \log \tau = -\frac{1}{2} \text{Tr}(\tilde{V}_i \tilde{V}_j), \quad j = 1, 2, 3, \quad (2.44)
\]

These identities allow us to express the right hand side of (2.15) by $\omega_i$, $\bar{\omega}_i$. First note that from eq. (2.20) rewritten in a “barred” version it follows that

\[
\bar{\beta}_{12} = \frac{\omega_3}{h}, \quad \bar{\beta}_{21} = \frac{\bar{\omega}_3}{h},
\]
\[
\bar{\beta}_{13} = -\frac{t\omega_2}{h}, \quad \bar{\beta}_{31} = -\frac{t\bar{\omega}_2}{h},
\]
\[
\bar{\beta}_{23} = \frac{t\omega_1}{h(1-t)}, \quad \bar{\beta}_{32} = \frac{t\bar{\omega}_1}{h(1-t)}.
\]

Furthermore from

\[
\frac{\partial^2}{\partial u_1 \partial u_3} = \frac{t^2}{h^2} \left[ t(t-1) \frac{\partial^2}{\partial t^2} + (2t-1) \frac{\partial}{\partial t} - h \frac{\partial}{\partial h} \frac{\partial}{\partial t} \right]
\]
\[
\frac{\partial^2}{\partial u_1 \partial u_2} = \frac{1}{h^2} \left[ t^2(t-1) \frac{\partial^2}{\partial t^2} + t^2 \frac{\partial}{\partial t} + h(t^2-2t) \frac{\partial}{\partial h} \frac{\partial}{\partial t} - h^2 \frac{\partial^2}{\partial h^2} \right]
\]
\[
\frac{\partial^2}{\partial u_3 \partial u_2} = \frac{t^2}{h^2(1-t^2)} \left[ -t(t-1)^2 \frac{\partial^2}{\partial t^2} - (1-t)^2 \frac{\partial}{\partial t} - h(1-t)^2 \frac{\partial}{\partial h} \frac{\partial}{\partial t} \right]
\]

follows that

\[
\left[ t(t-1) \frac{\partial^2}{\partial t^2} + (2t-1) \frac{\partial}{\partial t} - h \frac{\partial}{\partial h} \frac{\partial}{\partial t} \right] \log \tau = \omega_2 \bar{\omega}_2
\]
\[
\left[ t^2(t-1) \frac{\partial^2}{\partial t^2} + t^2 \frac{\partial}{\partial t} + h(t^2-2t) \frac{\partial}{\partial h} \frac{\partial}{\partial t} - h^2 \frac{\partial^2}{\partial h^2} \right] \log \tau = -\omega_3 \bar{\omega}_3 \quad (2.45)
\]
\[
\left[ -t(t-1)^2 \frac{\partial^2}{\partial t^2} - (1-t)^2 \frac{\partial}{\partial t} - h(1-t)^2 \frac{\partial}{\partial h} \frac{\partial}{\partial t} \right] \log \tau = -\omega_1 \bar{\omega}_1.
\]

Summing the right hand sides of (2.45) and using (2.43) we find

\[
\omega_1 \bar{\omega}_1 + \omega_2 \bar{\omega}_2 + \omega_3 \bar{\omega}_3 = h^2 \frac{\partial^2}{\partial h^2} \log \tau = -R^2, \quad (2.46)
\]

which leads to a separation of variables formula

\[
\log \tau = R^2 \log h + \log \tau_0(t), \quad (2.47)
\]

where on the right hand side we have introduced a single isomonodromic tau function $\tau_0$, which solely depends on single variable $t$. Let,

\[
f = t(t-1) \frac{d \log \tau_0}{dt}.
\]
Substituting relation (2.47) into equation (2.45) we arrive at a following parametrization of \( \omega_i \)'s in terms of \( \tau_0 \) or rather \( f \):

\[ \omega_2 \bar{\omega}_2 = f', \quad \omega_3 \bar{\omega}_3 = f - tf' - R^2, \quad \omega_1 \bar{\omega}_1 = -f + (t - 1)f'. \quad (2.48) \]

Taking a derivative of (2.48) yields

\[ t \frac{d(\omega_1 \bar{\omega}_1)}{dt} = t(t - 1) \frac{d(\omega_2 \bar{\omega}_2)}{dt} = -(t - 1) \frac{d(\omega_3 \bar{\omega}_3)}{dt} = t(t - 1)f''. \quad (2.49) \]

On the other hand from the generalized Euler top eqs. (1.8) it follows that

\[ t \frac{d(\omega_1 \bar{\omega}_1)}{dt} = t(t - 1) \frac{d(\omega_2 \bar{\omega}_2)}{dt} = -(t - 1) \frac{d(\omega_3 \bar{\omega}_3)}{dt} = \bar{\omega}_1 \omega_2 \bar{\omega}_3 + \omega_1 \bar{\omega}_2 \omega_3. \quad (2.50) \]

Thus

\[ t(t - 1)f'' = \bar{\omega}_1 \omega_2 \bar{\omega}_3 + \omega_1 \bar{\omega}_2 \omega_3. \quad (2.51) \]

Take now the determinant of \( \bar{V} + \nu \):

\[ \text{Det} \left( \bar{V} + \nu \right) = \nu_1 \nu_2 \nu_3 + \omega_1 \omega_2 \omega_3 - \omega_1 \bar{\omega}_2 \omega_3 + \sum_{i=1}^{3} \nu_i \omega_i \bar{\omega}_i = \text{Det} (\mu) = \mu_1 \mu_2 \mu_3 \]

or

\[ \omega_1 \omega_2 \omega_3 - \omega_1 \bar{\omega}_2 \omega_3 = \prod_{i=1}^{3} \mu_i - \prod_{i=1}^{3} \nu_i - \sum_{i=1}^{3} \nu_i \omega_i \bar{\omega}_i. \quad (2.52) \]

By squaring eqs. (2.51) and (2.52) we obtain

\[ t^2(t - 1)^2 (f'')^2 = 4 \prod_{i=1}^{3} \omega_i \bar{\omega}_i + \prod_{i=1}^{3} \mu_i^2 + \prod_{i=1}^{3} \nu_i^2 - 2 \prod_{i=1}^{3} \mu_i \nu_i + (\sum_{i=1}^{3} \nu_i \omega_i \bar{\omega}_i)^2 \]

\[ - 2 \left( \prod_{i=1}^{3} \mu_i - \prod_{i=1}^{3} \nu_i \right) \sum_{i=1}^{3} \nu_i \omega_i \bar{\omega}_i \]

\[ = 4(-f + (t - 1)f')f'(-R^2 + f - tf') \]

\[ + (\nu_1(-f + (t - 1)f') + \nu_2 f' + \nu_3(-R^2 + f - tf'))^2 \]

\[ - 2 \left( \prod_{i=1}^{3} \mu_i - \prod_{i=1}^{3} \nu_i \right) \left( \nu_1(-f + (t - 1)f') + \nu_2 f' + \nu_3(-R^2 + f - tf') \right) \]

\[ + \left( \prod_{i=1}^{3} \mu_i - \prod_{i=1}^{3} \nu_i \right)^2 \]

\[ = -4 \left[ f' (tf' - f)^2 - (f')^2 (tf' - f) + c_5 (tf' - f)^2 + c_6 f' (tf' - f) + c_7 (f')^2 + c_8 (tf' - f) + c_9 f' + c_{10} \right]. \quad (2.53) \]
where coefficients:

\[
c_5 = \left(-\frac{1}{4}\right) (\nu_1 - \nu_3)^2
\]

\[
c_6 = \left(-\frac{1}{4}\right) \left(-4 R^2 + 2 (\nu_2 - \nu_1) (\nu_1 - \nu_3)\right)
\]

\[
c_7 = \left(-\frac{1}{4}\right) \left(4 R^2 + (\nu_1 - \nu_2)^2\right)
\]

\[
c_8 = \left(-\frac{1}{4}\right) \left(2 (\nu_3 - \nu_1) \left(\mu_1 \mu_2 \mu_3 - \nu_1 \nu_2 \nu_3 + \nu_3 R^2\right)\right)
\]

\[
c_9 = \left(-\frac{1}{4}\right) \left(2 (\nu_1 - \nu_2) \left(\mu_1 \mu_2 \mu_3 - \nu_1 \nu_2 \nu_3 + \nu_3 R^2\right)\right)
\]

\[
c_{10} = \left(-\frac{1}{4}\right) \left(\mu_1 \mu_2 \mu_3 - \nu_1 \nu_2 \nu_3 + \nu_3 R^2\right)^2
\]

were introduced to make it more convenient to compare with the technical steps taken in [7] (see also [6]). We perform a change variable \(f \rightarrow \rho\) as follows. We set

\[
f = t(t-1) \frac{d \log \tau_0}{dt} = \rho + at + b
\]

Thus \(f'' = \rho''\), \(f' = \rho' + a\) and \(t f' - f = t \rho' - \rho - b\). Further we set

\[
a = -c_5, \quad b = c_5 + \frac{1}{2} c_6
\]

in order to eliminate terms with \(c_5, c_6\) in equation (2.53). This way we obtain for \(\rho\):

\[
[t(t-1)]^2 (\rho'')^2 = -4 \left[\rho' (t \rho' - \rho)^2 - (\rho')^2 (t \rho' - \rho)\right] + A_1 (\rho')^2 + A_2 (t \rho' - \rho) + A_3 \rho' + A_4,
\]

where

\[
A_1 = (-4) \left(c_7 + c_5 + \frac{1}{2} c_6\right)
\]

\[
A_2 = (-4) \left(c_8 - c_5^2 - c_5 c_6\right)
\]

\[
A_3 = (-4) \left(c_9 - c_5^2 - \frac{1}{4} c_6^2 - c_5 c_6 - 2 c_5 c_7\right)
\]

\[
A_4 = (-4) \left(c_{10} + \frac{3}{2} c_5^2 c_6 + \frac{1}{2} c_5 c_6^2 + c_5^3 + c_5^2 c_7 - c_5 c_8 - \frac{1}{2} c_6 c_8 - c_5 c_9\right).
\]

Now we compare eq. (2.53) to the Jimbo-Miwa-Okamoto \(\sigma\)-form of the Painlevé VI equation (1.9) with

\[
v_1 = \frac{1}{2} \left(\sqrt{-2\beta} + \sqrt{2\gamma}\right), \quad v_2 = \frac{1}{2} \left(\sqrt{-2\beta} - \sqrt{2\gamma}\right),
\]

\[
v_3 = \frac{1}{2} \left(\sqrt{2\alpha} + \sqrt{1 - 2\delta} - 1\right), \quad v_4 = \frac{1}{2} \left(-\sqrt{2\alpha} + \sqrt{1 - 2\delta} - 1\right).
\]
Expanding the products in (1.9) and dividing by $\sigma'$ gives

$$t^2(t-1)^2(\sigma')^2 + 4 \left[ \sigma' (t\sigma' - \sigma)^2 - (\sigma')^2 (t\sigma' - \sigma) \right] - 4v_1v_2v_3v_4 (t\sigma' - \sigma)$$

$$= (\sigma')^2 \left( \sum_{k=1}^{4} v_k^2 \right) + \sigma' \left( \sum_{i \neq j}^{4} v_i^2v_j^2 - 2v_1v_2v_3v_4 \right) + \sum_{i \neq j \neq k} v_i^2v_j^2v_k^2 \tag{2.59}$$

Comparing with eq. (2.55) we see that the equations will agree for $\rho = \sigma$ and:

$$A_1 = \sum_{k=1}^{4} v_k^2, \quad A_2 = 4v_1v_2v_3v_4$$

$$A_3 = \left( \sum_{i \neq j}^{4} v_i^2v_j^2 - 2v_1v_2v_3v_4 \right), \quad A_4 = \sum_{i \neq j \neq k} v_i^2v_j^2v_k^2,$$

with $v_k^2, k = 1, 2, 3, 4$ being roots of the fourth-order polynomial $x^4 - A_1x^3 + (A_3 + A_2/2)x^2 - A_4x + A_3^2/16$. Plugging the values of $A_i$'s from equation (2.56) into the above fourth-order polynomial leads to the following generic solution for its roots:

$$v_i^2 = \left( \frac{\nu_1 + \nu_3}{2} - \mu_i \right)^2, \quad i = 1, 2, 3, \quad v_4^2 = \left( \frac{\nu_1 - \nu_3}{2} \right)^2. \tag{2.60}$$

To satisfy condition $A_2 = 4v_1v_2v_3v_4$ we choose

$$v_i = \frac{\nu_1 + \nu_3}{2} - \mu_i, \quad i = 1, 2, 3, \quad v_4 = \frac{\nu_1 - \nu_3}{2} \tag{2.61}$$

and comparing with relation (1.10) we get (c.f. [5]):

$$\beta = -\frac{(\nu_2 - \mu_i)^2}{2}, \quad \gamma = \frac{(\mu_j - \mu_k)^2}{2}$$

$$\alpha = \frac{(\nu_3 - \mu_i)^2}{2}, \quad \delta = \frac{1}{2} \left( 1 + (1 + \nu_1 - \mu_i)^2 \right), \tag{2.62}$$

with $(i, j, k) = (3, 1, 2)$. Due to manifest $D_4$ symmetry of the Painlevé VI equation (1.9) any permutation of $v_1, v_2, v_3, v_4$ as well as change of signs in front of even number of $v_i$'s in equation (1.10) will lead to other equivalent solutions.

For example for solution

$$v_i = (-1)^i \left( \frac{\nu_1 + \nu_3}{2} - \mu_i \right), \quad i = 1, 2, 3, \quad v_4 = \frac{\nu_1 - \nu_3}{2}, \tag{2.63}$$

obtained from (2.61) by simultaneously changing signs in front of $v_1$ and $v_3$ we obtain

$$\beta = -\frac{(\mu_1 - \mu_2)^2}{2}, \quad \gamma = \frac{(\nu_2 - \mu_3)^2}{2}$$

$$\alpha = \frac{(\nu_1 - \mu_3)^2}{2}, \quad \delta = \frac{1}{2} \left[ 1 - (1 + \mu_3 - \nu_3)^2 \right]. \tag{2.64}$$
Furthermore, exchanging $v_1 \leftrightarrow v_3$ in equation (2.61) and next exchanging $v_2 \leftrightarrow v_3$ in equation (2.61) we get relation (2.62) with $(i, j, k) = (1, 2, 3)$ and $(i, j, k) = (2, 1, 3)$, respectively.

The relation (2.62) compares well with the result found in [15] after inserting $\mu_3 = -1$, needed in [15] in order to establish equivalence with the system of $2 \times 2$ Schlesinger equations.

Note, that the relation (2.6) is invariant under simultaneous transformations $\nu \rightarrow \nu + cI$ and $\mu \rightarrow \mu + cI$, with a constant $c$. It is indeed easy to confirm by checking all the above expressions for the coefficients $v_i$, $i = 1, \ldots, 4$ that they are left invariant under $\nu_i \rightarrow \nu_i + c$, $\mu_i \rightarrow \mu_i + c$ for $i = 1, 2, 3$. This fact will be used later below eq. (3.7).

$A_i$, $i = 1, \ldots, 4$ can also be expressed directly in terms of the Painlevé parameters $\alpha, \beta, \gamma, \delta$ via relation (1.10) as follows

\begin{align*}
A_1 &= -\beta + \gamma + \alpha - \delta - \sqrt{1 - 2\delta} + 1 \\
A_2 &= (\beta + \gamma) \left(\alpha + \delta + \sqrt{1 - 2\delta} - 1\right) \\
A_3 &= (\beta - \gamma) \left(-\alpha + \delta + \sqrt{1 - 2\delta} - 1\right) + \frac{1}{4} \left(-\alpha - \delta - \sqrt{1 - 2\delta} + \beta + \gamma + 1\right)^2 \\
A_4 &= -\frac{1}{4} (\beta - \gamma) \left(\alpha + \delta + \sqrt{1 - 2\delta} - 1\right)^2 + \frac{1}{4} (\beta + \gamma)^2 \left(\alpha - \delta - \sqrt{1 - 2\delta} + 1\right). \\
\end{align*}

According to [21] the solution $y$ to the Painlevé VI equation (1.5) is obtained from $\sigma$ via relation:

\begin{equation}
y = \frac{1}{2A} \left((v_3 + v_4)B + \left(\frac{d\sigma}{dt} - v_3 v_4\right)C\right),
\end{equation}

where

\begin{equation}
A = \left(\frac{d\sigma}{dt} + v_3^2\right) \left(\frac{d\sigma}{dt} + v_4^2\right),
\end{equation}

\begin{equation}
B = t(t-1)\frac{d^2\sigma}{dt^2} + (v_1 + v_2 + v_3 + v_4)\frac{d\sigma}{dt} - (v_1 v_2 v_3 + v_1 v_2 v_4 + v_1 v_3 v_4 + v_2 v_3 v_4),
\end{equation}

and

\begin{equation}
C = 2 \left(t \frac{d\sigma}{dt} - \sigma\right) - (v_1 v_2 + v_1 v_3 + v_1 v_4 + v_2 v_3 + v_2 v_4 + v_3 v_4).
\end{equation}

### 3 Construction of the tau function

In this section we will construct the tau function discussed above. We will follow [13], and introduce a semi-infinite wedge space $F = \Lambda^{1 \infty} C^\infty$ as the vector space with a basis consisting of all semi-infinite monomials of the form $v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \ldots$, with $i_j \in \frac{1}{2} + Z$, where $i_1 > i_2 > i_3 > \ldots$ and $i_{\ell+1} = i_\ell - 1$ for $\ell >> 0$. Define the
wedging and contracting operators $\psi_j^+$ and $\psi_j^-$ ($j \in \mathbb{Z} + \frac{1}{2}$) on $F$ by

$$
\psi_j^+(v_i \wedge v_{i+1} \wedge \cdots) =
\begin{cases}
0 & \text{if } -j = i_s \text{ for some } s \\
(-1)^s v_i \wedge v_{i+1} \wedge \cdots & \text{if } i_s = -j > i_{s+1}
\end{cases}
$$

$$
\psi_j^-(v_i \wedge v_{i+1} \wedge \cdots) =
\begin{cases}
0 & \text{if } j \neq i_s \text{ for all } s \\
(-1)^{s+1} v_i \wedge v_{i+1} \wedge \cdots & \text{if } j = i_s.
\end{cases}
$$

These operators satisfy the following relations ($i, j \in \mathbb{Z} + \frac{1}{2}, \lambda, \mu = +, -$):

$$
\psi_i^\lambda \psi_j^\mu + \psi_j^\mu \psi_i^\lambda = \delta_{\lambda,-\mu} \delta_{i,-j},
$$

(3.1)

hence they generate a Clifford algebra, which we denote by $\mathcal{C}_\ell$.

Introduce the following elements of $F$ ($m \in \mathbb{Z}$):

$$
|m\rangle = v_{m-\frac{1}{2}} \wedge v_{m-\frac{3}{2}} \wedge v_{m-\frac{5}{2}} \wedge \cdots.
$$

It is clear that $F$ is an irreducible $\mathcal{C}_\ell$-module such that

$$
\psi_j^\pm |0\rangle = 0 \text{ for } j > 0.
$$

Think of the adjoint module $F^*$ in the following way, it is the vector space with a basis consisting of all semi-infinite monomials of the form $v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots$, where $i_1 < i_2 < i_3 < \ldots$ and $i_{\ell+1} = i_\ell + 1$ for $\ell >> 0$. The operators $\psi_j^+$ and $\psi_j^-$ ($j \in \mathbb{Z} + \frac{1}{2}$) also act on $F^*$ by contracting and wedging, but in a different way, viz.,

$$
(\cdots v_i) \psi_j^+ =
\begin{cases}
0 & \text{if } j \neq i_s \text{ for all } s \\
(-1)^{s+1} \cdots v_{i_{s+1}} \wedge v_{i_{s-1}} \wedge \cdots & \text{if } i_s = j
\end{cases}
$$

$$
(\cdots v_i) \psi_j^- =
\begin{cases}
0 & \text{if } -j = i_s \text{ for some } s \\
(-1)^s \cdots v_{i_{s+1}} \wedge v_i \wedge v_{i_{s-1}} \wedge \cdots & \text{if } i_s < -j < i_{s+1}.
\end{cases}
$$

We introduce the element $\langle m |$, by

$$
\langle m | = \cdots \wedge v_{m+\frac{1}{2}} \wedge v_{m+\frac{3}{2}} \wedge v_{m+\frac{5}{2}},
$$

such that

$$
\langle 0 | \psi_j^\pm = 0 \text{ for } j < 0.
$$

We define the vacuum expectation value by

$$
\langle 0 | 0 \rangle = 1, \quad \text{and denote } \langle A \rangle = \langle 0 | A | 0 \rangle.
$$

We next identify $\mathbb{C}^\infty$ with the space $\mathbb{C}[z, z^{-1}]^N$ as follows. Let $\{e_i\}_{1 \leq i \leq N}$ be a basis of $\mathbb{C}^N$, we relabel the basis vectors $v_i$

$$
v^{(j)}_i = z^{-\frac{1}{2}-\frac{j}{2}} e_j = v_{N\ell-\frac{1}{2}(N-2j+1)},
$$

(3.2)
and with them the corresponding fermionic operators (the wedging and contracting operators):

\[ \psi^{\pm(j)}_{\ell} = \psi^{\pm}_{N\ell + \frac{j}{2}(N-2j+1)}. \]

Note that with this relabeling we have:

\[ \psi^{\pm(j)}_{\ell} |0\rangle = 0 \text{ for } \ell > 0 \]

and

\[ |0\rangle = z^0 e_N \land z^0 e_{N-1} \land \cdots \land z^0 e_1 \land ze_N \land ze_{N-1} \land \cdots \land ze_1 \land z^2 e_N \land z^2 e_{N-1} \land \cdots. \]

Introduce the fermionic fields \((0 \neq \lambda \in \mathbb{C})\):

\[ \psi^{\pm(j)}(\lambda) = \sum_{\ell \in \mathbb{Z} + \frac{1}{2}} \psi^{\pm(j)}_{\ell} \lambda^{-\ell - \frac{1}{2}}. \]

Next, we introduce bosonic fields \((1 \leq i \leq n)\):

\[ \alpha^{(i)}(\lambda) \equiv \sum_{k \in \mathbb{Z}} \alpha^{(i)}_{k} \lambda^{-k-1} =: \psi^{+(i)}(\lambda) \psi^{-(i)}(\lambda) :, \]

where \(:\) stands for the normal ordered product defined in the usual way \((\lambda, \mu = + \text{ or } -)\):

\[ : \psi^{\lambda(i)}_{\ell} \psi^{\mu(j)}_{\ell} : = \left\{ \begin{array}{ll}
\psi^{\lambda(i)}_{\ell} \psi^{\mu(j)}_{\ell} & \text{if } \ell > 0 \\
-\psi^{\mu(j)}_{\ell} \psi^{\lambda(i)}_{\ell} & \text{if } \ell < 0.
\end{array} \right. \]

One checks (using e.g. the Wick formula) that the operators \(\alpha^{(i)}_{k}\) satisfy the canonical commutation relation of the associative oscillator algebra, which we denote by \(\mathfrak{a}\):

\[ [\alpha^{(i)}_{k}, \alpha^{(j)}_{\ell}] = k \delta_{ij} \delta_{k,-\ell}, \]

and one has

\[ \alpha^{(i)}_{k} |m\rangle = 0 \text{ for } k > 0, \quad \langle m | \alpha^{(i)}_{k} = 0 \text{ for } k < 0. \]

In order to express the fermionic fields \(\psi^{\pm(i)}(\lambda)\) in terms of the bosonic fields \(\alpha^{(i)}(\lambda)\), we need some additional operators \(Q_{i}, \ i = 1, 2, \ldots, N\), on \(F\). These operators are uniquely defined by the following conditions:

\[ Q_{i} |0\rangle = \psi^{+(i)}_{-\frac{1}{2}} |0\rangle, \quad Q_{i} \psi^{\pm(j)}_{\ell} = (-1)^{\delta_{ij}+1} \psi^{\pm(j)}_{\ell+\delta_{ij}} Q_{i}. \]

They satisfy the following commutation relations:

\[ Q_{i} Q_{j} = -Q_{j} Q_{i} \text{ if } i \neq j, \quad [\alpha^{(i)}_{k}, Q_{j}] = \delta_{ij} \delta_{k0} Q_{j}. \]

We shall use below the following notation

\[ |k_{1}, k_{2}, \ldots, k_{N}\rangle = Q_{1}^{k_{1}} Q_{2}^{k_{2}} \cdots Q_{N}^{k_{N}} |0\rangle, \quad \langle k_{1}, k_{2}, \ldots, k_{N}| = \langle 0| Q_{N}^{-k_{N}} \cdots Q_{1}^{-k_{1}}. \]
such that
\[ \langle k_1, k_2, \ldots, k_N | k_1, k_2, \ldots, k_N \rangle = \langle 0 | 0 \rangle = 1 . \]

One easily checks the following relations:
\[ [a_k^{(i)}, \psi_m^{(j)}] = \pm \delta_{ij} \psi_k^{(m)} \]
and
\[ Q_i^{\pm 1} | k_1, k_2, \ldots, k_N \rangle = (-)^{k_1 + k_2 + \cdots + k_{i-1}} | k_1, k_2, \ldots, k_{i-1}, k_i \pm 1, k_{i+1}, \ldots, k_N \rangle , \]
\[ \langle k_1, k_2, \ldots, k_N | Q_i^{\pm 1} = (-)^{k_1 + k_2 + \cdots + k_{i-1}} \langle k_1, k_2, \ldots, k_{i-1}, k_i \mp 1, k_{i+1}, \ldots, k_N | . \]

These formulas lead to the following vertex operator expression for \( \psi^{(i)}(\lambda) \). Given any sequence \( s = (s_1, s_2, \ldots) \), define
\[ \Gamma^{(j)}_\pm(s) = \exp \left( \sum_{k=1}^{\infty} s_k a_k^{(j)} \right) , \]
then (see e.g. [8], [17])
\[ \psi^{(i)}(\lambda) = Q_i^{\pm 1} \lambda^{\pm \alpha_0^{(i)}} \exp(\mp \sum_{k<0} \frac{1}{k} a_k^{(i)} \lambda^{-k}) \exp(\mp \sum_{k>0} \frac{1}{k} a_k^{(i)} \lambda^{-k}) \]
\[ = Q_i^{\pm 1} \lambda^{\pm \alpha_0^{(i)}} \Gamma^{(i)}_\pm(\pm[\lambda]) \Gamma^{(i)}_\mp(\mp[\lambda^{-1}]) , \]
where \([\lambda] = (\lambda, \frac{\lambda^2}{2}, \frac{\lambda^3}{3}, \ldots)\). Note, that
\[ \Gamma^{(j)}_\pm(s) | k_1, k_2, \ldots, k_n \rangle = | k_1, k_2, \ldots, k_n \rangle , \]
\[ \langle k_1, k_2, \ldots, k_n | \Gamma^{(j)}_\pm(s) = \langle k_1, k_2, \ldots, k_n | . \]

Also observe that
\[ \Gamma^{(j)}_+(s) \Gamma^{(k)}_+(s') = \gamma(s, s') \delta_{jk} \Gamma^{(k)}_-(s') \Gamma^{(j)}_+(s) , \]
where
\[ \gamma(s, s') = e^{\sum n s_n s_n'} . \]

We have
\[ \Gamma^{(j)}_\pm(s) \psi^{+(k)}(\lambda) = \gamma(s, [\lambda^{\pm 1}]) \delta_{jk} \psi^{+(k)}(\lambda) \Gamma^{(j)}_\pm(s) \]
\[ \Gamma^{(j)}_\pm(s) \psi^{-(k)}(\lambda) = \gamma(s, -[\lambda^{\pm 1}]) \delta_{jk} \psi^{-(k)}(\lambda) \Gamma^{(j)}_\pm(s) . \]

Note that
\[ \gamma(t, [\lambda]) = \exp \left( \sum_{n \geq 1} t_n \lambda^n \right) . \]

Assume from now on that \( N = 3 \). In the 3-component setting we will construct the element \( g_0 z^{-\bar{\alpha}} \) (cf. (1.4) acting on the vacuum \( |0\rangle \)). It is obvious, see e.g. [13],

19
that this element in the orbit will satisfy the 3-component KP-hierarchy. To obtain
the most general construction, we do not assume any order in the size of the $\mu_i$. But
we do the following. Since (2.6) is invariant under simultaneous transformations
$\nu \rightarrow \nu + cI$ and $\mu \rightarrow \mu + cI$, with a constant $c$, we subtract such a constant $c$ from
all $\mu_i$ and $\nu_i$ such that all $\mu_i$ become non-positive. Define
\[ m_1 = \max\{-\mu_i\} \]
and choose $m_2$ and $m_3$ in such a way that
\[ m_1 \geq m_2 \geq m_3 \geq 0 \quad \text{and} \quad \{m_1, m_2, m_3\} = \{-\mu_1, -\mu_2, -\mu_3\}. \]
In fact one can choose $c$ in such a way, without loss of generality, that $m_3 = 0$, however we will not do that.

We start with an example. Take $-\mu_1 = m_1 = 3$, $-\mu_3 = m_2 = 2$ and $-\mu_2 = m_3 = 0$ and note that the vacuum is equal to
\[ |0\rangle = e_3 \wedge e_2 \wedge e_1 \wedge z e_3 \wedge z e_2 \wedge z e_1 \wedge z^2 e_3 \wedge z^2 e_2 \wedge z^2 e_1 \wedge z^3 e_3 \wedge z^3 e_2 \wedge z^3 e_1 \wedge z^4 e_3 \wedge \cdots \]
Now let $G$ act on this. Set
\[ g_0(e_1) = w^{(1)}, \quad g_0(e_2) = w^{(3)}, \quad g_0(e_3) = w^{(2)}, \]
thus $G = g_0 z^{-\mu}$ acting on the vacuum gives
\[ G|0\rangle = z^2 w^{(2)} \wedge w^{(3)} \wedge z^3 w^{(1)} \wedge z^3 w^{(2)} \wedge z^3 w^{(3)} \wedge z^4 w^{(1)} \wedge z^4 w^{(2)} \wedge z^4 w^{(3)} \wedge z^5 w^{(1)} \wedge z^5 w^{(2)} \wedge z^5 w^{(3)} \wedge z^6 w^{(1)} \wedge \cdots \]
Now permute the elements infinitely many times, this gives up to a sign,
\[ G|0\rangle = \pm w^{(3)} \wedge z w^{(3)} \wedge z^2 w^{(2)} \wedge z^2 w^{(3)} \wedge z^3 w^{(1)} \wedge z^3 w^{(2)} \wedge z^3 w^{(3)} \wedge z^4 w^{(1)} \wedge z^4 w^{(2)} \wedge z^4 w^{(3)} \wedge \cdots \]
Up to some multiplicative scalar $K$, which might be an infinite constant, this is equal to
\[ G|0\rangle = Kw^{(3)} \wedge z w^{(3)} \wedge z^2 w^{(3)} \wedge z^2 w^{(2)} \wedge z^3 e_3 \wedge z^3 e_2 \wedge z^3 e_1 \wedge z^4 e_3 \wedge z^4 e_2 \wedge z^4 e_1 \wedge \cdots \]
\[ = L\phi^{(3)}_{\frac{1}{2}} \phi^{(3)}_{-\frac{1}{2}} \phi^{(3)}_{-\frac{1}{2}} \phi^{(3)}_{-\frac{1}{2}} (z^3 e_3 \wedge z^3 e_2 \wedge z^3 e_1 \wedge z^4 e_3 \wedge z^4 e_2 \wedge z^4 e_1 \wedge \cdots) \]
\[ = M\phi^{(3)}_{\frac{1}{2}} \phi^{(3)}_{\frac{1}{2}} \phi^{(3)}_{\frac{1}{2}} \phi^{(3)}_{\frac{1}{2}} \phi^{(3)}_{\frac{1}{2}} | -3, -3, -3 \rangle, \]
where
\[ \phi^{(i)}_{\frac{1}{2}} = w^{(1)}_{\frac{1}{2}} \psi^{(1)}_{\frac{1}{2}} + w^{(2)}_{\frac{1}{2}} \psi^{(2)}_{\frac{1}{2}} + w^{(3)}_{\frac{1}{2}} \psi^{(3)}_{\frac{1}{2}}, \quad \frac{1}{2} \in \mathbb{Z} \]
and
\[ w^{(i)} = (w^{(1)}_{i}, w^{(2)}_{i}, w^{(3)}_{i}), \quad i = 1, 2, 3. \]
Since we want to avoid infinite constants, we do not consider the element $g_0 z^{-\mu}|0\rangle$, but rather
\[ G|0\rangle = \phi^{(3)}_{m_3 + \frac{1}{2}} \phi^{(3)}_{m_3 + \frac{1}{2}} \cdots \phi^{(3)}_{m_1 + \frac{1}{2}} \phi^{(3)}_{m_2 + \frac{1}{2}} \phi^{(3)}_{m_2 + \frac{1}{2}} \cdots \phi^{(2)}_{m_1 - \frac{1}{2}} | -m_1, -m_1, -m_1 \rangle, \quad (3.8) \]
20
Let \((\nu_1, \nu_2, \nu_3) \in \mathbb{Z}^3\), we decompose the element \(G|0\) as
\[
G|0\rangle = \sum_{\nu_1, \nu_2, \nu_3 \in \mathbb{Z}} g(\nu_1, \nu_2, \nu_3)|\nu_1, \nu_2, \nu_3\rangle ,
\] (3.9)
then
\[
g(\nu_1, \nu_2, \nu_3) = \langle \nu_1, \nu_2, \nu_3|G|0\rangle .
\]
Using the boson-fermion correspondence \(\sigma\) as in e.g. [14], [13] or [18], we can express
\(g(\nu_1, \nu_2, \nu_3)\) in terms of of the \(u_i^{(k)}\), \(i = 1, 2, 3, \ldots, k = 1, 2, 3\) viz.
\[
\sigma(G|0\rangle) = \sum_{\nu_1, \nu_2, \nu_3 \in \mathbb{Z}} \tau(\nu_1, \nu_2, \nu_3; \mathbf{u})|\nu_1, \nu_2, \nu_3\rangle .
\] (3.10)

We use the notation \(u_i^{(k)}\) for all \(u_i^{(k)}\), \(i = 1, 2, 3, \ldots,\) and \(\mathbf{u}\) for all \(u_i^{(k)}\), \(i = 1, 2, 3, \ldots,\) 
\(k = 1, 2, 3\). Note, that we do not put all higher flows \(u_i^{(k)}\), \(i > 1\) to zero yet. Now \(\tau(\nu_1, \nu_2, \nu_3; \mathbf{u})\), with all higher times \(u_i^{(k)}\), \(i > 1\) set to zero, will turn out to be a tau function of the Painlevé VI equation.

One has
\[
\tau(\nu_1, \nu_2, \nu_3; \mathbf{u}) = \langle \nu_1, \nu_2, \nu_3|e^{\sum_{i,k} \alpha^{(i)} u_i^{(k)} G}|0\rangle = \langle 0|Q_3^{-\nu_3} Q_2^{-\nu_2} Q_1^{-\nu_1} e^{\sum_{i,k} \alpha^{(i)} u_i^{(k)}} G|0\rangle .
\]
In this 3-component Clifford algebra setting the element \(Q_3^{-\nu_3} Q_2^{-\nu_2} Q_1^{-\nu_1}\) is related to \(z^L\) and \(e^{\sum_{i,k} \alpha^{(i)} u_i^{(k)}}\) to \(e^{\sum_{i,k} e^i E_{kk} u_i^{(k)}}\).

First note that
\[
e^{\sum_{i,k} \alpha^{(i)} u_i^{(k)} I_3^{-\nu_3} Q_2^{-\nu_2} Q_1^{-\nu_1} e^{\sum_{i,k} \alpha^{(i)} u_i^{(k)}}} = Q_3^{-\nu_3} Q_2^{-\nu_2} Q_1^{-\nu_1} e^{\sum_{i,k} \alpha^{(i)} u_i^{(k)}}
\]
and that
\[
e^{\sum_{i,k} \alpha^{(i)} u_i^{(k)}} \psi_k^{+}\left(e^{-\sum_{i,k} \alpha^{(i)} u_i^{(k)}} = \sum_{j=0}^{\infty} \psi_k^{+} S_j(u^{(i)})
\]
here the \(S_j(t)\) are the elementary Schur functions. Also
\[
e^{\sum_{i,k} \alpha^{(i)} u_i^{(k)}} - m_1, -m_1, -m_1\rangle = | - m_1, -m_1, -m_1\rangle .
\]
Next, let \(e^{\sum_{i,k} \alpha^{(i)} u_i^{(k)}}\) act on the element (3.8). This gives
\[
\tilde{G}(\mathbf{u})|0\rangle :=
\]
\[
e^{\sum_{i,k} \alpha^{(i)} u_i^{(k)}} \phi_{m_1^{(3)}}^{+}\left(u^{(3)}\right) \cdots \phi_{m_1^{(3)}}^{+}\left(u^{(3)}\right) \phi_{m_2^{(2)}}^{+}\left(u^{(2)}\right) \cdots \phi_{m_1^{(1)}}^{+}\left(u^{(2)}\right) - m_1, -m_1, -m_1\rangle =
\]
\[
\phi_{m_1^{(3)}}^{+}\left(u^{(3)}\right) \cdots \phi_{m_1^{(3)}}^{+}\left(u^{(3)}\right) \phi_{m_2^{(2)}}^{+}\left(u^{(2)}\right) \cdots \phi_{m_1^{(1)}}^{+}\left(u^{(2)}\right) - m_1, -m_1, -m_1\rangle ,
\] (3.11)
where
\[
\phi_k^{+}(u) = \sum_{a=1}^{3} \sum_{j=k}^{m_1^{(-1)}} u_a^{(i)} \psi_j^{+}(a) S_{j-k}(u^{(a)}) ,
\] (3.12)
Next we let $Q_3^{-\nu_3}Q_2^{-\nu_2}Q_1^{-\nu_1}$ act on the element (3.11). This gives

\[
\hat{G}(\nu; u)|0\rangle := Q_3^{-\nu_3}Q_2^{-\nu_2}Q_1^{-\nu_1} \phi^{+ (3)}_{m_3 + \frac{1}{2}}(u) \phi^{+ (3)}_{m_3 + \frac{1}{2}}(u) \cdots \\
\phi^{+ (3)}_{m_1 - \frac{1}{2}}(u) \phi^{+ (2)}_{m_2 + \frac{1}{2}}(u) \phi^{+ (2)}_{m_2 + \frac{1}{2}}(u) \cdots \phi^{+ (2)}_{m_1 - \frac{1}{2}}(u)| - m_1, - m_1, - m_1 \rangle 
\]

Then

\[
\tau(\nu; u) = \tau(\nu_1, \nu_2, \nu_3; u) = \langle 0| \hat{G}(\nu; u)|0\rangle = \langle 0| Q_3^{-\nu_3}Q_2^{-\nu_2}Q_1^{-\nu_1} \times \\
\phi^{+ (3)}_{m_3 + \frac{1}{2}}(u) \phi^{+ (3)}_{m_3 + \frac{1}{2}}(u) \cdots \phi^{+ (3)}_{m_1 - \frac{1}{2}}(u) \phi^{+ (2)}_{m_2 + \frac{1}{2}}(u) \phi^{+ (2)}_{m_2 + \frac{1}{2}}(u) \cdots \phi^{+ (2)}_{m_1 - \frac{1}{2}}(u) | - m_1, - m_1, - m_1 \rangle ,
\]

Define

\[
f^{(b)}_a(\nu) = (-)^{\nu_1 + \nu_2 + \nu_3 + \nu_a} w^{(b)}_a
\]

then

\[
\hat{\phi}^{+ (b)}_\ell(\nu; u) := Q_3^{-\nu_3}Q_2^{-\nu_2}Q_1^{-\nu_1} \phi^{+ (b)}_\ell(u) \eta^{\nu_3 \nu_2 \nu_1}_1 \\
= \sum_{a=1}^{3} \sum_{j=\ell}^{m_1-\frac{1}{2}} (-)^{\nu_1 + \nu_2 + \nu_a} w^{(b)}_a \psi^{+ (a)}_{j+\nu_a} S_{j-\ell}(u^{(a)})
\]

\[
= \sum_{a=1}^{3} \sum_{j=\ell}^{m_1-\frac{1}{2}} f^{(b)}_a(\nu) \psi^{+ (a)}_{j+\nu_a} S_{j-\ell}(u^{(a)})
\]

\[
= \sum_{a=1}^{3} \sum_{j=\ell+\nu_a}^{m_1+\nu_a-\frac{1}{2}} f^{(b)}_a(\nu) \psi^{+ (a)}_{j-\nu_a} S_{j-\ell}(u^{(a)})
\]

and

\[
Q_3^{-\nu_3}Q_2^{-\nu_2}Q_1^{-\nu_1}| - m_1, - m_1, - m_1 \rangle = (-)^{m_1 \nu_2 + \nu_1 \nu_2 + \nu_1 \nu_3 + \nu_2 \nu_3} | - m_1 - \nu_1, - m_1 - \nu_2, - m_1 - \nu_3 \rangle .
\]

Thus

\[
\hat{G}(\nu; u)|0\rangle = (-)^{m_1 \nu_2 + \nu_1 \nu_2 + \nu_1 \nu_3 + \nu_2 \nu_3} \hat{\phi}^{+ (3)}_{m_3 + \frac{1}{2}}(\nu; u) \hat{\phi}^{+ (3)}_{m_3 + \frac{1}{2}}(\nu; u) \cdots \hat{\phi}^{+ (3)}_{m_1 - \frac{1}{2}}(\nu; u) \times \\
\hat{\phi}^{+ (2)}_{m_2 + \frac{1}{2}}(\nu; u) \hat{\phi}^{+ (2)}_{m_2 + \frac{1}{2}}(\nu; u) \cdots \hat{\phi}^{+ (2)}_{m_1 - \frac{1}{2}}(\nu; u)| - m_1 - \nu_1, - m_1 - \nu_2, - m_1 - \nu_3 \rangle
\]

and

\[
\tau(\nu; u) = (-)^{m_1 \nu_2 + \nu_1 \nu_2 + \nu_1 \nu_3 + \nu_2 \nu_3} |0\rangle \hat{\phi}^{+ (3)}_{m_3 + \frac{1}{2}}(\nu; u) \hat{\phi}^{+ (3)}_{m_3 + \frac{1}{2}}(\nu; u) \cdots \hat{\phi}^{+ (3)}_{m_1 - \frac{1}{2}}(\nu; u) \times \\
\hat{\phi}^{+ (2)}_{m_2 + \frac{1}{2}}(\nu; u) \hat{\phi}^{+ (2)}_{m_2 + \frac{1}{2}}(\nu; u) \cdots \hat{\phi}^{+ (2)}_{m_1 - \frac{1}{2}}(\nu; u)| - m_1 - \nu_1, - m_1 - \nu_2, - m_1 - \nu_3 \rangle.
\]
Let \( p = \max\{ m_1 + \nu_i \} \), then it is a straightforward calculation to check that
\[
\hat{G}(\nu; u)|0\rangle = (-)^{\frac{1}{2}p(p+1)+m_1\nu_2+m_3p+\nu_1\nu_2+\nu_1\nu_3+m_2\nu_2\nu_3} \hat{\phi}^{(+3)}_{m_3+\frac{1}{2}}(\nu; u) \hat{\phi}^{(+3)}_{m_3+\frac{1}{2}}(\nu; u) \cdots \\
\cdots \hat{\phi}^{(+2)}_{m_1+\nu_1+\frac{1}{2}}(\nu; u) \hat{\phi}^{(+2)}_{m_2+\frac{1}{2}}(\nu; u) \cdots \\
\cdots \hat{\psi}^{+(i)}_{\nu_1+\frac{1}{2}}(\nu; u) \hat{\psi}^{+(i)}_{\nu_1+\frac{1}{2}}(\nu; u) \cdots \\
\cdots \hat{\psi}^{+(i)}_{p-\frac{1}{2}}(\nu; u) \hat{\psi}^{+(i)}_{p-\frac{1}{2}}(\nu; u) \cdots \\
| - 3p \rangle = (Q_1^{-1}Q_2^{-1}Q_3^{-1})^p|0\rangle \\
= v_{-3p-\frac{1}{2}} \wedge v_{-3p-\frac{1}{2}} \wedge v_{-3p-\frac{1}{2}} \wedge \cdots .
\]

Hence, it is possible to express the tau function as a \( 3p \times 3p \)-determinant.

4 Conditions satisfied by the constructed tau function

First we observe by using (3.14) and (3.16) that
\[
\tau(\nu; u) = 0 \quad \text{if} \quad \nu_1 + \nu_2 + \nu_3 \neq -m_1 - m_2 - m_3 = \mu_1 + \mu_2 + \mu_3 , \\
or one \quad \nu_i < -m_1 , \\
or one \quad \nu_i > -m_3 .
\]

Note that these equations (4.1) define the complement of a convex polygon in the plane \( x_1 + x_2 + x_3 = \mu_1 + \mu_2 + \mu_3 \). In other words, if one defines supp \( \tau \), the support of \( \tau \), those \( \nu \in \mathbb{Z}^3 \) for which \( \tau(\nu; u) \neq 0 \), then supp \( \tau \) is within a convex polygon (see also [13]).

From (3.14) we deduce that
\[
\sum_{a=1}^{3} \frac{\partial \hat{\phi}^{+(b)}(\nu; u)}{\partial u_1^{(a)}} = \sum_{a=1}^{3} \sum_{j=\nu_1}^{m_1+\nu_2-\frac{1}{2}} f_a^{(b)}(\nu) \hat{\psi}^{+(a)}_{\nu_1} \frac{\partial S_{j-\nu_1}(u^{(a)})}{\partial u_1^{(a)}} \\
= \sum_{a=1}^{3} \sum_{j=\nu_1}^{m_1+\nu_2-\frac{1}{2}} f_a^{(b)}(\nu) \hat{\psi}^{+(a)}_{\nu_1} S_{j-\nu_1-1}(u^{(a)}) \\
= \hat{\phi}^{+(b)}_{\nu_1}(\nu; u) .
\]
Also observe in (3.14) that \( \hat{\varphi}^{(b)}_{m_1 + \frac{1}{2}}(\nu; u) = 0 \). Hence from the explicit form of (3.16), we conclude that

\[
\hat{I}(\tau(\nu; u)) = \sum_{a=1}^{3} \frac{\partial \tau(\nu; u)}{\partial u_{1}^{(a)}} = 0. \tag{4.2}
\]

There exists a natural gradation on this Spin module with times \( u_k^{(i)} \), which we denote by \( \deg \). This gradation is the sum of two separate gradations, a \( u \)-gradation \( \deg_u \), which is given by \( \deg_u(u_{1}^{(a)}) = i \) and a fermionic gradation \( \deg_f \), which is defined by \( \deg_f(\psi_{r}^{(a)}) = -k \). So

\[
\deg = \deg_u + \deg_f.
\]

Now,

\[
\deg_u(S_j(u^{(a)})) = j
\]

and the \( u \)-gradation is related to the Euler vector field \( \hat{E} \), viz. for a homogeneous function \( f \) one has

\[
\deg_u(f(u))f(u) = \sum_{a=1}^{3} \sum_{i=1}^{\infty} iu_{1}^{(a)} \frac{\partial f(u)}{\partial u_{1}^{(a)}}.
\]

Since, \( \deg|m_1, m_2, m_3| = \frac{1}{2}(m_1^2 + m_2^2 + m_3^2) \) and \( \deg \psi_{k}^{(a)} = -k \), it is straightforward to show that \( \deg G|0| = \frac{1}{2}(m_1^2 + m_2^2 + m_3^2) \), (see (3.8)). Thus, we see from (3.9) that

\[
\deg g(\nu_1, \nu_2, \nu_3) = \deg G|0| - \deg|\nu_1, \nu_2, \nu_3| = \frac{1}{2}\left(m_1^2 + m_2^2 + m_3^2 - \nu_1^2 - \nu_2^2 - \nu_3^2\right),
\]

which by (2.43) is equal to \( R^2 \). Hence, using (3.10), we find that

\[
\deg_u(\tau(\nu; u)) = R^2. \tag{4.3}
\]

Hence we can conclude that

\[
\sum_{a=1}^{3} \sum_{i=1}^{\infty} iu_{1}^{(a)} \frac{\partial \tau(\nu; u)}{\partial u_{1}^{(a)}} = R^2 \tau(\nu; u). \tag{4.4}
\]

If one then puts all \( u_{1}^{(a)} \), for all \( i > 1 \), in \( \tau(\nu; u) \) equal to zero and writes \( \tau(\nu; u) \) for this tau function, equation (4.4) reduces to

\[
\hat{E}(\tau(\nu; u)) = \sum_{a=1}^{3} u_{1}^{(a)} \frac{\partial \tau(\nu; u)}{\partial u_{1}^{(a)}} = R^2 \tau(\nu; u). \tag{4.5}
\]
5 The tau function as a determinant

Recall from (3.2) that $\tau_j^{(a)} = \nu_{3j-2+a}$. In analogy with (3.14) we define

$$
\tau_i^{(b)}(\nu; u) = \sum_{a=1}^{3} \sum_{j=\nu_a}^{\nu_a - \frac{1}{2}} f_a^{(b)}(\nu) S_{j-\nu_a}(u^{(a)}) \nu_j^{(a)}
$$

(5.1)

Hence, from (3.17) we obtain that

$$
\hat{G}(\nu; u)|0\rangle = (-)^{\frac{1}{2}(p+1)+m_1\nu_2+m_1 p+m_1 \nu_2+m_1 \nu_2+m_1 \nu_3+m_1 \nu_3+1} E_{m_1}^{m_1+\frac{1}{2}}(\nu; u) \wedge \tau^{(3)}(\nu; u) \wedge \cdots
$$

$$
\cdots \wedge \nu^{(3)}_{m_1-\frac{1}{2}}(\nu; u) \wedge \nu^{(2)}_{m_2+\frac{1}{2}}(\nu; u) \wedge \nu^{(2)}_{m_2+\frac{1}{2}}(\nu; u) \cdots \wedge \nu^{(1)}_{m_3-\frac{1}{2}}(\nu; u) \wedge \nu^{(1)}_{m_3-\frac{1}{2}}(\nu; u) \wedge \cdots
$$

(5.2)

Using a formula from [14], viz. (4.48), we find that

$$
\tau(\nu; u) = (-)^{\frac{1}{2}p(p+1)+m_1 \nu_2+m_1 p+m_1 \nu_2+m_1 \nu_3+m_1 \nu_3+1} \det(A),
$$

where $A$ is the following $3p \times 3p$ matrix:

$$
\begin{align*}
A &= \sum_{i=1}^{m_1-m_3} \sum_{k=1}^{3} f_k^{(2)}(\nu) \sum_{j=0}^{m_1+\nu_k-1} S_{j-m_3-\nu_k-i+1}(u^{(k)}) E_{-3j+k-\frac{7}{2},-i+\frac{1}{2}}^{m_3+m_2-m_2 \nu_k-2m_1+\frac{1}{2}} \\
&+ \sum_{i=1}^{m_1-m_2} \sum_{k=1}^{3} f_k^{(2)}(\nu) \sum_{j=0}^{m_1+\nu_k-1} S_{j-m_2-\nu_k-i+1}(u^{(k)}) E_{-3j+k-\frac{7}{2},m_3-m_1-i+\frac{1}{2}}^{m_3+m_2-2m_1+\frac{1}{2}} \\
&+ \sum_{i=1}^{p-m_1-\nu_1} E_{-3(m_1+\nu_1+i)+\frac{1}{2},m_3+m_2-2m_1+\frac{1}{2}-i}^{m_3+m_2-m_2 \nu_k-2m_1+\frac{1}{2}} \\
&+ \sum_{i=1}^{p-m_1-\nu_2} E_{-3(m_1+\nu_2+i)+\frac{1}{2},m_3+m_2-2m_1+\frac{1}{2}-i}^{m_3+m_2-m_2 \nu_k-2m_1+\frac{1}{2}} \\
&+ \sum_{i=1}^{p-m_1-\nu_3} E_{-3(m_1+\nu_3+i)+\frac{1}{2},m_3+m_2-2m_1+\frac{1}{2}-i}^{m_3+m_2-2m_1+\frac{1}{2}-i}.
\end{align*}
$$

(5.3)

Now replacing the indices such that $E_{-i,-j}$ becomes $E_{i+\frac{1}{2},j+\frac{1}{2}}$ and using some elementary matrix operations, see appendix A for the explicit calculations, we obtain that

$$
\tau(\nu; u) = (-)^{m_1 \nu_2+m_1 \nu_2+m_1 \nu_3+m_1 \nu_3+1} \det(E),
$$

25
where

\[ E = w_1^{(3)} \sum_{j=1}^{m_1+\nu_1} E_{2m_2-m_3-j+1,m_1-m_3-j+1} + \]

\[ \sum_{i=1}^{m_1-m_3} \left( w_2^{(3)} \sum_{j=1}^{m_1+\nu_2} S_{j-m_3-\nu_2-i}(u^{(2)} - u^{(1)})E_{m_1+\nu_3+j,i} + \right. \]

\[ w_3^{(3)} \sum_{j=1}^{m_1+\nu_3} S_{j-m_3-\nu_3-i}(u^{(3)} - u^{(1)})E_{j,i} \right) + \]

\[ w_1^{(2)} \sum_{j=1}^{\min\{m_1+\nu_1,m_1-m_2\}} E_{2m_2-m_3-j+1,2m_1-m_2-m_3-j+1} + \]

\[ \sum_{i=1}^{m_1-m_2} \left( w_2^{(2)} \sum_{j=1}^{m_1+\nu_2} S_{j-m_2-\nu_2-i}(u^{(2)} - u^{(1)})E_{m_1+\nu_3+j,-m_3+m_1+i} + \right. \]

\[ w_3^{(2)} \sum_{j=1}^{m_1+\nu_3} S_{j-m_2-\nu_3-i}(u^{(3)} - u^{(1)})E_{j,-m_3+m_1+i} \right) . \]

To make the connection with the Painlevé VI tau-function, we put all higher times \( u_n^{(a)} \) for \( n > 0 \) to zero. Then \( S_k(u^{(a)} - u^{(b)}) \) becomes \( \frac{(u_a-u_b)^k}{k!} \). Using (2.37) we can express this tau function in \( t \) and \( h \), viz. we substitute

\[ u_2 - u_1 = h, \quad u_3 - u_1 = \frac{h}{t} . \]

Since \( \tau(\nu; u) \) is homogeneous of \( u \)-degree \( R^2 \) (see (4.3), we find that

\[ \tau(\nu; t, h) = h^{R_2} \tau_0(\nu; t) \]

and

\[ \tau_0(\nu; t) = \frac{(-)^{m_1\nu_2+\nu_1\nu_2+\nu_1\nu_3+\nu_2\nu_3}}{t^{R_2}} \det(T_\nu) . \]

26
where

$$T_\nu = w_1^{(3)} \sum_{j=1}^{m_1+\nu_1} E_{2m_1-m_2-m_3-j+1,m_1-m_3-j+1} +$$

$$\sum_{i=1}^{m_1-m_3} \left( w_2^{(3)} \sum_{j=1}^{m_1+\nu_2} t^{(j-m_3-\nu_2-i)} E_{m_1+\nu_2+j,i} + w_3^{(3)} \sum_{j=1}^{m_1+\nu_3} 1^{(j-m_3-\nu_3-i)} E_{j,i} \right) +$$

$$w_1^{(2)} \min\{m_1+\nu_1,m_1-m_2\} \sum_{j=1}^{m_1-m_3-m_2-j+1,2m_1-m_2-m_3-j+1} E_{2m_1-m_2-m_3-j+1,2m_1-m_2-m_3-j+1} +$$

$$\sum_{i=1}^{m_1-m_2} \left( w_2^{(2)} \sum_{j=1}^{m_1+\nu_2} t^{(j-m_2-\nu_2-i)} E_{m_1+\nu_2+j,-m_3+m_1+i} +
\right.$$  

$$w_2^{(2)} \sum_{j=1}^{m_1+\nu_3} 1^{(j-m_2-\nu_3-i)} E_{j,-m_3+m_1+i} \right) .$$

(5.5)

Here $t^{(n)}$ stands for the divided power

$$t^{(n)} = \frac{t^n}{n!}.$$  

Note that we use the convention that $n! = \infty$ for $n < 0$, such that $1^{(n)} = \frac{1}{n!} = t^{(n)} = \frac{t^n}{n!} = 0$.

6 An example

Now assume that $\mu_1 = -4$, $\mu_2 = -2$, $\mu_3 = 0$, $\nu_1 = -3$, $\nu_2 = -2$ and $\nu_3 = -1$. Then $R^2 = 3$. From (2.61) we find

$$v_1 = 2, \quad v_2 = 0, \quad v_3 = -2 \quad \text{and} \quad v_4 = -1 .$$

(6.1)

and equation (2.62) with $(i,j,k) = (3,2,1)$ yields for the PVI parameters:

$$\alpha = \frac{1}{2}, \quad \beta = -2, \quad \gamma = 2, \quad \text{and} \quad \delta = -\frac{3}{2}.$$  

27
Then
\[ \tau = -(h/t)^3 \det \begin{pmatrix} w_3^{(3)} & w_3^{(3)} & 0 & 0 & 0 & 0 \\ w_3^{(3)} & w_3^{(3)} & w_3^{(3)} & 0 & w_3^{(2)} & 0 \\ w_3^{(3)} & w_3^{(3)} & w_3^{(3)} & w_3^{(3)} & w_3^{(2)} & w_3^{(2)} \\ t^2 w_2^{(3)} & tw_2^{(3)} & w_2^{(3)} & 0 & w_2^{(2)} & 0 \\ t^2 w_2^{(3)} & t^2 w_2^{(3)} & tw_2^{(3)} & tw_2^{(3)} & tw_2^{(2)} & tw_2^{(2)} \\ 0 & 0 & 0 & w_1^{(3)} & 0 & w_1^{(2)} \end{pmatrix}. \]

Thus
\[ \tau_0 = \frac{D}{6t^3} (D_1 + 3D_2 t^2 - 2D_2 t^3) , \]
where
\[ D = w_3^{(3)} (w_2^{(2)} w_3^{(3)} - w_2^{(3)} w_3^{(2)}) \]
and
\[ D_1 = w_3^{(3)} (w_1^{(2)} w_2^{(3)} - w_1^{(3)} w_2^{(2)}) \quad D_2 = w_2^{(3)} (w_1^{(3)} w_3^{(2)} - w_1^{(2)} w_3^{(3)}). \]

Then
\[ \sigma = 2 \frac{D_1 - 2D_1 t - 2D_2 t^3 + D_2 t^4}{D_1 + 3D_2 t^2 - 2D_2 t^3} \]
and according to (2.66) this gives a Painlevé IV solution (1.11), presented in the introduction.

From (2.63) and (2.64) we get
\[ v_1 = -2, \quad v_2 = 0, \quad v_3 = 2 \quad \text{and} \quad v_4 = -1. \]

together with
\[ \alpha = \frac{9}{2}, \quad \beta = -2, \quad \gamma = 2, \quad \text{and} \quad \delta = -\frac{3}{2}. \]

This time the solution of the Painlevé VI equation (1.5) takes the form
\[
y = \frac{1}{3} \left( -D_1 + 2 D_1 t + 2 D_2 t^3 - D_2 t^4 \right)^{-1} \\
\left( D_1^2 - 6 D_1 D_2 t^2 + 4 D_1 D_2 t^3 + D_2^2 t^4 + 4 D_2 t D_1 \right)^{-1} \\
\left( -D_2^3 t^6 + 3 D_2^2 t^5 - 6 D_1 D_2^2 t^7 \\
+ 42 D_1 D_2^2 t^6 - 57 D_1 D_2^2 t^5 + 27 D_1^2 D_2 t^5 + 27 D_1 D_2^2 t^4 \\
- 57 D_1^2 D_2 t^4 + 42 D_1^2 D_2 t^3 - 6 D_1^2 D_2 t^2 + 3 t D_1^3 - D_1^3 \right)
\]
as dictated by relation (2.66).

Interchanging \( v_2 \) with \( v_3 \) in equation (6.1), which can be done by choosing \( \mu_2 = 0 \)
and \( \mu_3 = -2 \) (this does not change the tau-function), yields
\[ \alpha = \frac{1}{2}, \quad \beta = 0, \quad \gamma = 8, \quad \text{and} \quad \delta = \frac{1}{2}. \]
according to equation (2.62) with \((i, j, k) = (2, 1, 3)\). The corresponding solution of the Painlevé VI equation (1.5) reads:

\[
\begin{align*}
y = \left( D_1 + D_2 t^3 - 3 D_2 t^2 + 3 D_2 t \right)^{-1} & \left( D_1^2 - 6 D_1 D_2 t^2 \right) \\
+ 4 D_1 D_2 t^3 + D_2^2 t^4 + 4 D_2 D_1 t \right)^{-1} (D_2 t) \\
& \left(-15 t^2 D_1^2 + 7 t D_1^2 + 9 t^3 D_1^2 - D_1^2 + 6 D_1 D_2 t^2 - 26 D_1 D_2 t^3 \right) \\
-6 D_1 D_2 t^5 + 26 D_1 D_2 t^4 - 9 D_2^2 t^4 - 7 D_2^2 t^6 + 15 D_2^2 t^5 + D_2^2 t^7
\end{align*}
\]

7 Wave matrix

We want to calculate the wave matrix corresponding to \(\nu\). For this we let the fermionic field \(\psi^{(j)}(z)\) act on \(G|0\rangle\). Then

\[
\sigma \left( \psi^{(j)}(z) G|0\rangle \right) = \sigma \left( \psi^{(j)}(z) \sum_{\nu \in \text{supp } \tau} g(\nu) |\nu\rangle \right)
\]

\[
= \sum_{\nu \in \text{supp } \tau} (-)^{\nu_1 \ldots \nu_{i-1} - \nu_{i} \nu_{j} - \nu_{k}} e^{\sum_k u_k^{(j)} z^j} e^{-\sum_{k=1}^{\infty} \frac{\partial}{\partial u_k^{(j)}} z^k} \tau(\nu; u) |\nu_1 + \delta_1, \nu_2 + \delta_2, \nu_3 + \delta_3\rangle
\]

Now let \(\Phi_{ij}(\nu; u)\) be the coefficient of \(|\nu_1 + \delta_1, \nu_2 + \delta_2, \nu_3 + \delta_3\rangle\) of this expression, i.e., this term corresponds to

\[
\langle \nu_1 + \delta_1, \nu_2 + \delta_2, \nu_3 + \delta_3 | \psi^{(j)}(z) G|0\rangle,
\]

then

\[
\Phi_{ij}(\nu; u) = \text{sign}(i - j) (-)^{\nu_1 \ldots \nu_{i-1} - \nu_{i} \nu_{j} - \nu_{k}} e^{\sum_k u_k^{(j)} z^j} \\
\times e^{-\sum_{k=1}^{\infty} \frac{\partial}{\partial u_k^{(j)}} z^k} \tau(\nu_1 + \delta_1, \nu_2 + \delta_2 - \delta_{j_2}, \nu_3 + \delta_{i_3} - \delta_{j_3}; u)
\]

For every \(\nu \in \text{supp } \tau\) we define a wave matrix \(\Psi(\nu; u, z)\) as follows

\[
\Psi(\nu; u, z) = \Theta(\nu; u, z) \exp \left( \sum_{j=1}^{3} \sum_{n=1}^{\infty} z^n E_{jj} u_n^{(j)} \right) z^\nu,
\]

\[
\Psi(\nu; u, z) = \left( \Phi_{ij}(\nu; u, z) \right)_{1 \leq ij \leq 3},
\]

\[
\Theta(\nu; u, z) = (\Theta(\nu; u, z))_{1 \leq ij \leq 3}
\]

with

\[
\tau(\nu; u) \Theta_{ij}(\nu; u, z) = \text{sign}(i - j) z^{\delta_{ij} - 1} \exp \left( \sum_{k=1}^{\infty} \frac{\partial}{\partial u_k^{(j)}} z^{-k} \right) \tau(\nu_1 + \delta_1 - \delta_{j_1}, \nu_2 + \delta_{i_2} - \delta_{j_2}, \nu_3 + \delta_{i_3} - \delta_{j_3}; u).
\]

(7.1)
The coefficient of $z^{-1}$ of $\Theta(\nu; u, z)$ is the matrix $\theta^{(-1)}(\nu; u)$, see (2.2). The off-diagonal elements are the rotation coefficients $\beta_{ij}(\nu; u)$ (see (2.12)). Now

$$
\beta_{ij}(\nu; u) = \text{sign}(i - j) \frac{\tau(\nu_1 + \delta_{i1} - \delta_{j1}, \nu_2 + \delta_{i2} - \delta_{j2}, \nu_3 + \delta_{i3} - \delta_{j3}; u)}{\tau(\nu_1, \nu_2, \nu_3; u)}
$$

(7.2)

Using (4.3) it is easy to deduce that

$$
\deg_{u}(\beta_{ij}(\nu; u)) = -1 - \nu_i + \nu_j.
$$

If we then put in (7.2) all higher times $u^{(n)} = 0$ for $n > 0$ and make the substitution (2.37), then for $k \neq i, j$ we find that

$$
\beta_{ij}(\nu; t, h) = \\
- \text{sign}(i - j)(-)^{m_1(\delta_{i2} + \delta_{j2}) + \nu_1 + \nu_2 + \nu_3 + \nu_k} \left( \frac{h}{t} \right)^{\nu_j - \nu_i - 1} \frac{\det(T_{(\nu_1 + \delta_{i1} - \delta_{j1}, \nu_2 + \delta_{i2} - \delta_{j2}, \nu_3 + \delta_{i3} - \delta_{j3})})}{\det(T_{(\nu_1, \nu_2, \nu_3)})}
$$

(7.3)

Looking at (3.8) it is clear that for $j = 1, 2, 3$:

$$
\phi^{(j)}_k G|0\rangle = \begin{cases} 0 & \text{for } k > m_j, \\ \neq 0 & \text{for } k < m_j, \end{cases}
$$

or in other words

$$
z^{m_j} \sum_{\ell=1}^{3} w_\ell^{(j)} \psi^{\ell}(z) G|0\rangle
$$

is a non-negative power series in $z$. It is easy to see that in fact for $i = 1, 2, 3$:

$$
\langle \nu_1 + \delta_{i1}, \nu_2 + \delta_{i2}, \nu_3 + \delta_{i3} | z^{m_j} \sum_{\ell=1}^{3} w_\ell^{(j)} \psi^{\ell}(z) G|0\rangle
$$

is a non-negative power series in $z$, with nonzero constant term. Using the boson-fermion correspondence this term corresponds to

$$
z^{m_j} \sum_{\ell=1}^{3} w_\ell^{(j)} \Phi_{id}(\nu; u).
$$

Thus we have shown that

$$
z^{m_j} \Psi(\nu; u, z) \begin{pmatrix} w_1^{(j)} \\ w_2^{(j)} \\ w_3^{(j)} \end{pmatrix}
$$

contains no negative powers of $z$. Stated in another way this means that

$$
\Pi(\nu; u, z) := \Psi(\nu; u, z) g_0 z^{-\underline{u}}
$$

has no negative powers of $z$ and the constant coefficient is an invertible matrix $M(\nu; u)$.
8 Conclusion/Outlook

In summary we have shown how the general Painlevé VI equation and its class of solutions are obtained from a self-similarity reduction of a 3-component KP hierarchy. We have shown how reduction is imposed through restricting group elements $g$ in the Riemann-Hilbert factorization to be of special form determined by five independent scaling parameters. These parameters specify the scaling laws of the underlying dressing matrices and values of Painlevé VI coefficients through simple relations.

The particular advantage of our construction is that it allows for an explicit construction of the tau function solutions of the sigma-form of the Painlevé VI equation in terms of elementary Schur functions using Grassmannian techniques.

The approach outlined in this paper offers powerful techniques to study symmetries of the Painlevé VI equation which originate in the formalism of Bäcklund transformations within the setup of a 3-component KP hierarchy. The work is in progress to develop detailed and exhaustive results within this conceptual framework.

A Appendix: Calculation of the tau function

Consider the matrix $A$ given in (5.3), replace the indices such that $E_{-i,-j}$ becomes $E_{i+\frac{1}{2}j+\frac{1}{2}}$ then $A$ has the form

\[
A = \sum_{i=1}^{m_1-m_3} \sum_{k=1}^{3} f_k^{(3)}(\nu) \sum_{j=0}^{m_1+\nu_k-1} S_{j-m_3-\nu_k-i+1}(u^{(k)}) E_{3j+4-k,i} + \sum_{i=1}^{m_1-m_2} \sum_{k=1}^{3} f_k^{(2)}(\nu) \sum_{j=0}^{m_1+\nu_k-1} S_{j-m_2-\nu_k-i+1}(u^{(k)}) E_{3j+4-k,-m_3+m_1+i} \\
+ \sum_{i=1}^{p-m_1-\nu_1} E_{3(m_1+\nu_1+i),-m_3-m_2+2m_1+i} + \sum_{i=1}^{p-m_1-\nu_2} E_{3(m_1+\nu_2+i)-1,-m_3-m_2+m_1+p-\nu_1+i} \\
+ \sum_{i=1}^{p-m_1-\nu_3} E_{3(m_1+\nu_3+i)-2,-m_3-m_2+2p-\nu_1-\nu_2+i}.
\]

(A.1)

Next permute the rows of this matrix, then we obtain that

\[
\tau(\nu;u) = (-)^{p+m_1\nu_2+m_1p+p\nu_2+p\nu_1\nu_2+p\nu_1\nu_3+p\nu_2\nu_3} \det(B),
\]
where
\[
B = \sum_{i=1}^{m_1-m_3} \sum_{k=1}^{3} f_k^{(3)}(\nu) \sum_{j=0}^{m_1+\nu_k-1} S_{j-m_3-\nu_k-i+1}(u^{(k)}) E_{(3-k)p+j+1,i} + \sum_{i=1}^{m_1-m_2} \sum_{k=1}^{3} f_k^{(2)}(\nu) \sum_{j=0}^{m_1+\nu_k-1} S_{j-m_2-\nu_k-i+1}(u^{(k)}) E_{(3-k)p+j+1,-m_3+m_1+i}
\]
\[
+ \sum_{i=1}^{m_1-m_1-\nu_1} E_{2p+m_1+\nu_1+i,-m_3-m_2+2m_1+i} + \sum_{i=1}^{m_1-m_2-\nu_2} E_{p+m_1+\nu_2+i,-m_3-m_2+m_1+p-\nu_1+i} + \sum_{i=1}^{m_1-\nu_3} E_{m_1+\nu_3+i,-m_3-m_2+2p-\nu_1-\nu_2+i}.
\]

Now developing the determinant to the last columns we obtain that
\[
\tau(\nu; u) = (-)^{m_1+\nu_1+\nu_2+\nu_3+\nu_3} \det (C),
\]
where
\[
C = \sum_{i=1}^{m_1-m_3} \sum_{k=1}^{3} f_k^{(3)}(\nu) \sum_{j=1}^{m_1+\nu_k} S_{j-m_3-\nu_k-i}(u^{(k)}) E_{(3-k)m_1+\sum p>\nu_j+p,i} + \sum_{i=1}^{m_1-m_2} \sum_{k=1}^{3} f_k^{(2)}(\nu) \sum_{j=1}^{m_1+\nu_k} S_{j-m_2-\nu_k-i}(u^{(k)}) E_{(3-k)m_1+\sum p>\nu_j+p,-m_3+m_1+i}.
\]

Now note that \( \det (C) = \det (D) \),
\[
D = \sum_{i=1}^{m_1-m_3} \sum_{k=1}^{3} w_k^{(3)} \sum_{j=1}^{m_1+\nu_k} S_{j-m_3-\nu_k-i}(u^{(k)}) E_{(3-k)m_1+\sum p>\nu_j+p,i} + \sum_{i=1}^{m_1-m_2} \sum_{k=1}^{3} w_k^{(2)} \sum_{j=1}^{m_1+\nu_k} S_{j-m_2-\nu_k-i}(u^{(k)}) E_{(3-k)m_1+\sum p>\nu_j+p,-m_3+m_1+i}.
\]

Next multiply \( D \) from the right with the matrix (with determinant 1):
\[
\sum_{i,j=1}^{m_1-m_3} S_{i-j}(-u^{(1)}) E_{ij} + \sum_{i,j=1}^{m_1-m_2} S_{i-j}(-u^{(1)}) E_{m_1-m_3+i,m_1-m_3+j}
\]

Thus \( \det (D) = \det (E) \), for \( E \) as in (5.4). Here we have used that \( 3m_1+\nu_1+\nu_2+\nu_3 = 2m_1-m_2-m_3 \).

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