SOME RESULTS CONCERNING THE $p$-ROYDEN AND $p$-HARMONIC BOUNDARIES OF A GRAPH OF BOUNDED DEGREE

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Abstract. Let $p$ be a real number greater than one and let $\Gamma$ be a connected graph of bounded degree. We show that the $p$-Royden boundary of $\Gamma$ with the $p$-harmonic boundary removed is an $F_\sigma$-set. We also characterize the $p$-harmonic boundary of $\Gamma$ in terms of the intersection of the extreme points of a certain subset of one-sided infinite paths in $\Gamma$.

1. Introduction

Let $\Gamma$ be a graph with vertex set $V_\Gamma$ and edge set $E_\Gamma$. We will write $V$ for $V_\Gamma$ and $E$ for $E_\Gamma$ if it is clear what graph $\Gamma$ we are working with. For $x \in V$, $\deg(x)$ will denote the number of neighbors of $x$ and $N_x$ will be the set of neighbors of $x$. A graph $\Gamma$ is said to be of bounded degree if there exists a positive integer $k$ such that $\deg(x) \leq k$ for every $x \in V$. A path $\gamma$ in $\Gamma$ is a sequence of vertices $x_1, x_2, \ldots, x_n$ where $x_{i+1} \in N_{x_i}$ for $1 \leq i \leq n - 1$ and $x_i \neq x_j$ if $i \neq j$. Assume throughout this paper that all infinite paths have no self-intersections. A graph is connected if any two given vertices of the graph are joined by a path. All graphs considered in this paper will be connected, of bounded degree with no self-loops and have countably infinite number of vertices. We shall say that a subset $S$ of $V$ is connected if the subgraph of $\Gamma$ induced by $S$ is connected. The Cayley graph of a finitely generated group is an example of the type of graph that we study in this paper. By assigning length one to each edge of $\Gamma$, $V$ becomes a metric space with respect to the shortest path metric. We will denote this metric by $d(x, y)$, where $x$ and $y$ are vertices of $\Gamma$. Thus $d(x, y)$ gives the length of the shortest path joining the vertices $x$ and $y$. Finally, if $x \in V$ and $n \in \mathbb{N}$, then $B_n(x)$ will denote the metric ball that contains all elements of $V$ that have distance less than $n$ from $x$.

Let $p$ be a real number greater than one. In Section 2 we will define the $p$-Royden boundary of $\Gamma$, which we will indicate by $R_p(\Gamma)$. We will also define the $p$-harmonic boundary of $\Gamma$, which is a subset of $R_p(\Gamma)$. We will use $\partial_p(\Gamma)$ to denote the $p$-harmonic boundary. Our motivation for investigating the $p$-harmonic boundary of a graph is its connection to the vanishing of the first reduced $\ell^p$-cohomology space of a finitely generated group. More specifically, this space vanishes if and only if the $p$-harmonic boundary of the group is empty or contains exactly one element, see [5, Section 7] for the details of this fact. Gromov conjectured in [1] page 150] that the first reduced $\ell^p$-cohomology space of a finitely generated amenable group

Date: July 7, 2011.

2000 Mathematics Subject Classification. Primary: 60J50; Secondary: 43A15, 31C45.

Key words and phrases. $p$-Royden boundary, $p$-harmonic boundary, $p$-harmonic function, $F_\sigma$-set, extreme points of a path, $p$-extremal length of paths.
vanishes. Thus, a better understanding of the p-harmonic boundary could be helpful in resolving Gromov’s conjecture.

Recall that in a topological space a set is said to be $F_p$ if it is a countable union of closed sets. In this paper we will prove that $R_p(\Gamma) \setminus \partial_p(\Gamma)$ is $F_p$. For each infinite path in $\Gamma$ we can associate a set of extreme points, which is roughly the “points at infinity” of the path with respect to the $p$-Royden boundary. Our other main result in this paper is that the p-harmonic boundary is precisely the intersection of the extreme points of a certain subset of one-sided infinite paths in $\Gamma$.

The research for this paper was partially supported by PSC-CUNY grant 63873-00 41 and I would like to thank them for their support.

2. The $p$-Royden and p-harmonic boundaries

Let $1 < p \in \mathbb{R}$. In this section we construct the p-Royden and p-harmonic boundaries of $\Gamma$. For a more detailed discussion about this construction see Section 2.1 of [5]. Before we can give these definitions we need to define the space of $p$-Dirichlet finite functions on $V$. For any $S \subset V$, the outer boundary $\partial S$ of $S$ is the set of vertices in $V \setminus S$ with at least one neighbor in $S$. For a real-valued function $f$ on $S \cup \partial S$ we define the $p$-th power of the gradient, the $p$-Dirichlet sum, and the $p$-Laplacian of $x \in S$ by

$$|Df(x)|^p = \sum_{y \in N_x} |f(y) - f(x)|^p,$$

$$I_p(f, S) = \sum_{x \in S} |Df(x)|^p,$$

$$\Delta_p f(x) = \sum_{y \in N_x} |f(y) - f(x)|^{p-2}(f(y) - f(x)).$$

In the case $1 < p < 2$, we make the convention that $|f(y) - f(x)|^{p-2}(f(y) - f(x)) = 0$ if $f(y) = f(x)$. Let $S \subset V$. A function $f$ is said to be $p$-harmonic on $S$ if $\Delta_p f(x) = 0$ for all $x \in S$. We shall say that $f$ is $p$-Dirichlet finite if $I_p(f, V) < \infty$. The set of all $p$-Dirichlet finite functions on $\Gamma$ will be denoted by $D_p(\Gamma)$. With respect to the following norm $D_p(\Gamma)$ is a reflexive Banach space,

$$\| f \|_{D_p} = (I_p(f, V) + |f(o)|^p)^{1/p},$$

where $o$ is a fixed vertex of $\Gamma$ and $f \in D_p(\Gamma)$. We use $\text{HD}_p(\Gamma)$ to represent the set of $p$-harmonic functions on $V$ that are contained in $D_p(\Gamma)$. Let $\ell^\infty(\Gamma)$ denote the set of bounded functions on $V$ and let $\| f \|_\infty = \sup_{v \in V} |f|$ for $f \in \ell^\infty(\Gamma)$. Set $BD_p(\Gamma) = D_p(\Gamma) \cap \ell^\infty(\Gamma)$. The set $BD_p(\Gamma)$ is a Banach space under the norm

$$\| f \|_{BD_p} = (I_p(f, V))^{1/p} + \| f \|_\infty,$$

where $f \in BD_p(\Gamma)$. Let $\text{BHD}_p(\Gamma)$ be the set of bounded $p$-harmonic functions contained in $D_p(\Gamma)$. The space $BD_p(\Gamma)$ is also closed under the usual operations of scalar multiplication, addition and pointwise multiplication. Furthermore, $\| fg \|_{BD_p} \leq \| f \|_{BD_p} \| g \|_{BD_p}$ for $f, g \in BD_p(\Gamma)$. Thus $BD_p(\Gamma)$ is a commutative Banach algebra. Let $C_c(\Gamma)$ be the set of functions on $V$ with finite support. Indicate the closure of $C_c(\Gamma)$ in $D_p(\Gamma)$ by $C_c(\Gamma)_{D_p}$. Set $B(C_c(\Gamma)_{D_p}) = C_c(\Gamma)_{D_p} \cap \ell^\infty(\Gamma)$. Using the fact that the inequality $(a + b)^{1/p} \leq a^{1/p} + b^{1/p}$ is true when $a, b \geq 0$
and $1 < p \in \mathbb{R}$, we see immediately that $\| f \|_{D_{p}} \leq \| f \|_{BD_{p}}$. It now follows that $B(C_{c}(\Gamma)_{D_{p}})$ is closed in $BD_{p}(\Gamma)$.

Let $Sp(BD_{p}(\Gamma))$ denote the set of complex-valued characters on $BD_{p}(\Gamma)$, that is the nonzero ring homomorphisms from $BD_{p}(\Gamma)$ to $\mathbb{C}$. Then with respect to the weak $*$-topology, $Sp(BD_{p}(\Gamma))$ is a compact Hausdorff space. Given a topological space $X$, let $C(X)$ denote the ring of continuous functions on $X$ endowed with the sup-norm. The Gelfand transform defined by $\hat{f}(\chi) = \chi(f)$ yields a monomorphism of Banach algebras from $BD_{p}(\Gamma)$ into $C(Sp(BD_{p}(\Gamma)))$ with dense image. Furthermore the map $i: V \to Sp(BD_{p}(\Gamma))$ given by $(i(x))(f) = f(x)$ is an injection, and $i(V)$ is an open dense subset of $Sp(BD_{p}(\Gamma))$. For the rest of this paper, we shall write $f$ for $\hat{f}$, where $f \in BD_{p}(\Gamma)$. The $p$-Royden boundary of $\Gamma$, which we shall denote by $R_{p}(\Gamma)$, is the compact set $Sp(BD_{p}(\Gamma)) \setminus i(V)$. The $p$-harmonic boundary of $\Gamma$ is the following subset of $R_{p}(\Gamma)$:

$$\partial_{p}(\Gamma): = \{ \chi \in R_{p}(\Gamma) \mid \hat{f}(\chi) = 0 \text{ for all } f \in B(C_{c}(\Gamma)_{D_{p}}) \}.$$ 

Let $S$ be an infinite subset of $V$ and let $A$ and $B$ be disjoint nonempty subsets of $S \cup \partial S$. The $p$-capacity of the condenser $(A, B, S)$ is defined by

$$cap_{p}(A, B, S) = \inf_{u} I_{p}(u),$$

where the infimum is taken over all functions $u \in D_{p}(\Gamma)$ with $u = 0$ on $A$ and $u = 1$ on $B$. Such a function is called admissible. Set $cap_{p}(A, B, S) = \infty$ if the set of admissible functions is empty.

Let $A$ be a finite subset of $S \cup \partial S$ and let $(U_{n})$ be an exhaustion of $V$ by finite connected subsets such that $A \subseteq U_{1}$. We now define

$$cap_{p}(A, \infty, S) = \lim_{n \to \infty} cap_{p}(A, (\partial S \cup \setminus U_{n}, S).$$

Since $cap_{p}(A, (\partial S \cup \setminus U_{n}, S) \geq cap_{p}(A, (\partial S \cup \setminus U_{n+1}, S)$, the above limit exists. We shall say that $S$ is $p$-hyperbolic if there exists a finite subset $A$ of $S \cup \partial S$ that satisfies $cap_{p}(A, \infty, S) > 0$. If $S$ is not $p$-hyperbolic, then it is said to be $p$-parabolic. An equivalent definition of $p$-parabolic is that $S$ is $p$-parabolic if and only if $1_{S} \in C_{c}(\Gamma)_{D_{p}}$, where $1_{S}$ is the constant function 1 on $S$ and $\Gamma_{S}$ the subgraph of $\Gamma$ induced by $S$. [3] Theorem 3.1]. We will define a graph $\Gamma$ to be $p$-hyperbolic ($p$-parabolic) if its vertex set $V$ is $p$-hyperbolic ($p$-parabolic). It was shown in [3] Proposition 4.2] that $\Gamma$ is $p$-parabolic if and only if $\partial_{p}(\Gamma) = \emptyset$. A useful property of $p$-hyperbolic graphs that we will use throughout this paper is the following $p$-Royden decomposition, see [3] Theorem 4.6] for a proof.

**Theorem 2.1.** ($p$-Royden decomposition) Let $1 < p \in \mathbb{R}$ and suppose $f \in BD_{p}(\Gamma)$. Then there exists a unique $u \in B(C_{c}(\Gamma)_{D_{p}})$ and a unique $h \in BD_{p}(\Gamma)$ such that $f = u + h$.

Let $G$ be a finitely generated group. The Cayley graph of $G$ is an example of the type of graph we study in this paper. As was mentioned in Section 1 the first reduced $\ell^{p}$-cohomology space of $G$ vanishes if and only if the cardinality of the $p$-harmonic boundary of $G$ is one or zero. The reason for this is that the first reduced $\ell^{p}$-cohomology space of $G$ vanishes if and only if the only $p$-harmonic functions on $G$ that are contained in $D_{p}(\Gamma)$ are the constants, for a proof of this see the remark after Theorem 3.5 in [3]. Furthermore, [3] Theorem 2.5] tells us that there are nonconstant $p$-harmonic functions with finite $p$-Diriclet sum on a graph of bounded
degree if and only if the cardinality of the \(p\)-harmonic boundary of the graph is greater than one. In section 7 of [5] the \(p\)-harmonic boundary is computed for several groups.

3. Statement of main results

In this section we will state our main results. In section 4 we will prove

**Theorem 3.1.** Let \(1 < p \in \mathbb{R}\) and let \(\Gamma\) be a graph of bounded degree. The set \(R_p(\Gamma) \setminus \partial_p(\Gamma)\) is \(F_\sigma\).

Before we state our other main result we need to define the set of extreme points of a path in \(\Gamma\). Let \(P\) be the set of all one-sided infinite paths with no self-intersections in \(\Gamma\). For a real-valued function \(f\) on \(V\) and a path \(\gamma \in P\), the limit of \(f(\gamma)\) as we follow \(\gamma\) to infinity is given by \(\lim_{n \to \infty} f(x_n)\), where \(x_0, x_1, \ldots, x_n, \ldots\) is the vertex representation of the path \(\gamma\). Sometimes we write \(f(\gamma) = \lim_{n \to \infty} f(x_n)\) to indicate this limit. Let \(\gamma \in P\) and denote by \(V(\gamma)\) the set of vertices on \(\gamma\).

The closure of \(i(V(\gamma))\) in \(S(pBD_p(\Gamma))\) will be indicated by \(V(\gamma)\). Recall that \(S(pBD_p(\Gamma))\) is endowed with the weak \(\ast\)-topology. Thus \(\chi \in V(\gamma)\) if and only if there exists a subsequence \((x_{n_k})\) of \((x_n)\) such that \(\lim_{k \to \infty} f(x_{n_k}) = \chi(f)\) for all \(f \in BD_p(\Gamma)\). The extreme points of a path \(\gamma\) is defined to be

\[ E(\gamma) = \overline{V(\gamma) \cap R_p(\Gamma)}. \]

Let \(f \in B(Cc(\Gamma)_{DP})\) and set \(A_f = \{\gamma \in P \mid f(\gamma) \neq 0\}\). Set

\[ E_f = \{\bigcup_{\gamma \in P \setminus A_f} E(\gamma) \mid \gamma \in P \setminus A_f\}. \]

In Section 5 we shall prove

**Theorem 3.2.** Let \(1 < p \in \mathbb{R}\) and let \(\Gamma\) be a graph of bounded degree. Then

\[ \partial_p(\Gamma) = \bigcap_{f \in B(Cc(\Gamma)_{DP})} E_f. \]

Let \(1 < p \in \mathbb{R}\). If \(\Gamma\) is \(p\)-parabolic, then \(\partial_p(\Gamma) = \emptyset\) and Theorem 3.1 is true. Also for the \(p\)-parabolic case, \(1_V \in B(Cc(\Gamma)_{DP})\) by [9, Theorem 3.2], where \(1_V\) is the constant function one on \(V\). Then \(E_{1_V} = \emptyset\) and Theorem 3.2 follows. Thus for the rest of the paper we will assume \(\Gamma\) is \(p\)-hyperbolic.

4. Proof of Theorem 3.1

In this section we will prove Theorem 3.1. We will start by giving some needed definitions and proving a comparison principle. A comparison principle for finite subsets of \(V\) was proved in [2, Theorem 3.14]. Our proof follows theirs in spirit.

Let \(f\) and \(h\) be elements of \(BD_p(\Gamma)\) and let \(1 < p \in \mathbb{R}\). Define

\[ \langle \Delta_p h, f \rangle : = \sum_{x \in V} \sum_{y \in N_x} |h(y) - h(x)|^{p-2} (h(y) - h(x))(f(y) - f(x)). \]

The sum exists since

\[ \sum_{x \in V} \sum_{y \in N_x} |h(y) - h(x)|^{p-2} (h(y) - h(x)) < \infty, \]

...
where \( \frac{1}{p} + \frac{1}{q} = 1 \). For notational convenience let
\[
T(h, f, x, y) = |h(y) - h(x)|^{p-2}(h(y) - h(x))(f(y) - f(x)).
\]

In order to prove Theorem 3.1, we will need the following:

**Lemma 4.1.** (Comparison principle) Let \( h_1, h_2 \) be elements of \( BD_p(\Gamma) \) and suppose \( h_1(x) \leq h_2(x) \) for all \( x \in \partial_p(\Gamma) \). Then \( h_1 \leq h_2 \) on \( V \).

**Proof.** Define a function \( f \) on \( V \) by \( f = \min\{h_2 - h_1, 0\} \). Theorem 4.8 of [3] says \( f \in B(\overline{C_p(\Gamma)}_{D_p}) \) since \( f = 0 \) on \( \partial_p(\Gamma) \). By Lemma 4.6 of [3] we have \( \langle \Delta_p h_1, f \rangle = 0 \) and \( \langle \Delta_p h_2, f \rangle = 0 \), which implies \( \langle \Delta_p h_1 - \Delta_p h_2, f \rangle = 0 \). Now set
\[
A = \{x \in V \mid h_1(x) \leq h_2(x)\},
\]
\[
B = \{x \in V \mid h_2(x) < h_1(x)\},
\]
and for \( a \in V \) let
\[
C_a = \{y \in V \mid y \in N_a \text{ and } h_1(y) \leq h_2(y)\},
\]
\[
D_a = \{y \in V \mid y \in N_a \text{ and } h_2(y) < h_1(y)\}.
\]

Now
\[
0 = \sum_{x \in V} \sum_{y \in N_x} (T(h_1, f, x, y) - T(h_2, f, x, y)) = T_1 + T_2 + T_3
\]
where
\[
T_1 = \sum_{x \in A} \sum_{y \in C_x} (T(h_1, f, x, y) - T(h_2, f, x, y)),
\]
\[
T_2 = \left( \sum_{x \in A} \sum_{y \in D_x} + \sum_{x \in B} \sum_{y \in C_x} \right) (T(h_1, f, x, y) - T(h_2, f, x, y)),
\]
and
\[
T_3 = \sum_{x \in B} \sum_{y \in D_x} (T(h_1, f, x, y) - T(h_2, f, x, y)).
\]
Since \( f(x) = f(y) = 0 \) for \( x \in A \) and \( y \in C_x \), it follows that \( T_1 = 0 \). We now claim that \( T_3 \leq 0 \). To see the claim let \( a \) and \( b \) be real numbers such that \( a \neq b \). It follows from the inequality
\[
|a|^{p-2}a(a - b) > |b|^{p-2}b(a - b)
\]
that
\[
T(h_1, h_1 - h_2, x, y) \geq T(h_2, h_1 - h_2, x, y).
\]
Equality occurs if and only if \( (h_1 - h_2)(x) = (h_1 - h_2)(y) \). Now if \( x \in B \) and \( y \in D_x \), then \( f(y) - f(x) = (h_2 - h_1)(y) - (h_2 - h_1)(x) \). Combining (4.2) with the fact \( T(h_k, h_1 - h_2, x, y) = -T(h_k, h_2 - h_1, x, y) \), where \( k = 1 \) or \( k = 2 \), we obtain \( T_3 \leq 0 \), which is our claim.

We now proceed to show that if there is a pair of vertices \( x \) and \( y \) that satisfy \( x \in A, y \in D_x \) or \( x \in B, y \in C_x \), then \( T_2 < 0 \). Suppose \( x \in A \) and \( y \in D_x \). Then \( f(y) - f(x) = h_2(y) - h_1(y) < 0 \) and
\[
T(h_1, f, x, y) - T(h_2, f, x, y) = (h_2(y) - h_1(y)) \times \left( |h_1(y) - h_1(x)|^{p-2}(h_1(y) - h_1(x)) - |h_2(y) - h_2(x)|^{p-2}(h_2(y) - h_2(x)) \right).
\]
Also $h_1(y) - h_1(x) > h_2(y) - h_2(x)$ because $h_1(y) - h_2(y) > 0 \geq h_1(x) - h_2(x)$. So
if $h_2(y) \geq h_2(x)$ we see that $T(h_1, f, x, y) - T(h_2, f, x, y) < 0$ since $h_1(y) - h_1(x) > h_2(y) - h_2(x)$. On the other hand if $h_2(y) < h_2(x)$ and $h_1(y) > h_1(x)$ we obtain
\[
T(h_1, f, x, y) - T(h_2, f, x, y) = (h_2(y) - h_1(y))(h_1(y) - h_1(x)^{p-1} + |h_2(y) - h_2(x)|^{p-1}) < 0
\]
since $|h_2(y) - h_2(x)| = -(h_2(y) - h_2(x))$. The only other possibility is $h_2(y) < h_2(x)$ and $h_1(y) \leq h_1(x)$. If this is the case then $h_2(y) < h_1(y) \leq h_1(x) \leq h_2(x)$ due to $x \in A$ and $y \in D_x$. Consequently, $h_2(y) - h_2(x) < h_1(y) - h_1(x)$ and $h_1(x) - h_1(y) \leq h_2(x) - h_2(y)$; hence, $|h_1(y) - h_1(x)| < |h_2(y) - h_2(x)|$. It now follows that
\[
T(h_1, f, x, y) - T(h_2, f, x, y) = (h_2(y) - h_1(y))(h_2(y) - h_2(x)^{p-1} - |h_1(y) - h_1(x)|^{p-1}) < 0.
\]
A similar argument can be used to show that $T(h_1, f, x, y) - T(h_2, f, x, y) < 0$ for each $x \in B$ and $y \in C_x$. Hence, if $x \in A$, $y \in D_x$ or $x \in B$, $y \in C_x$, then $T_2 < 0$. Since $T_1 = 0$ and $T_3 \leq 0$, it follows from (4.4) that it must be the case $T_2 = 0$. Thus it is impossible to have a pair of vertices $x$ and $y$ with $x \in A$, $y \in D_x$ or $x \in B$, $y \in C_x$.

Now assume that $h_1(z) > h_2(z)$ for some $z \in V$. We claim that there exists vertices $x_0, y_0$ in $V$ for which $y_0 \in N_{x_0}, h_1(x_0) > h_2(x_0)$ and $h_1(y_0) \leq h_2(y_0)$. To see the claim suppose $h_1 = h_2$ on $\partial_p(\Gamma)$, then $h_1 = h_2$ on $V$ by [3] Corollary 4.9. So there exists an $x \in \partial_p(\Gamma)$ that satisfies $h_1(x) < h_2(x)$. Let $(x_n) \to x$ where $(x_n)$ is a sequence in $V$. Now there exists a term $x_m$ in this sequence such that $h_1(x_m) < h_2(x_m)$. Since $\Gamma$ is connected there is a path from $z$ to $x_m$. Thus there are vertices $x_0$ and $y_0$ on this path with $y_0 \in N_{x_0}, h_1(x_0) > h_2(x_0)$, and $h_1(y_0) < h_2(y_0)$ because $h_1(z) > h_2(z)$ and $h_1(x_m) < h_2(x_m)$. Thus $x_0 \in B$ and $y_0 \in C_{x_0}$, a contradiction. Therefore, $h_1(z) \leq h_2(z)$ for all $z \in V$. \hfill \Box

Proof of Theorem 7.7 Let $1 < p \in \mathbb{R}$. Since $Sp(BD_p(\Gamma))$ is a normal space, there exists for each $x \in R_p(\Gamma)$ a sequence $(U_j(x))$ of open sets containing $x$ such that $\bigcup_{j=1}^{\infty} U_j(x) \subseteq U_{j+1}(x)$. For each $x \in N_{x_0}$ there exists a finite number of points $x_{j,k}, 1 \leq k \leq N_j$ such that $U_j(x_{j,k})$ cover $R_p(\Gamma)$. For notational simplicity we will denote $U_j(x_{j,k})$ by $U_{j,k}$. Using Urysohn’s lemma, we can construct a continuous function $\phi_{j,k}$ with $\phi_{j,k} = 2$ on $U_{j,k}$ and $\phi_{j,k} = -1$ on $Sp(BD_p(\Gamma)) \setminus U_{j-1,k}$. By the density of $BD_p(\Gamma)$ in $C(\partial_p(\Gamma))$ there exists a $g \in BD_p(\Gamma)$ such that $|\phi_{j,k} - g| < \frac{1}{2}$. Set $f_{j,k} = \max(\min(1, g), 0)$, so $f_{j,k} \in BD_p(\Gamma), 0 \leq f_{j,k} \leq 1, f_{j,k} = 1$ on $U_{j,k}$ and $f_{j,k} = 0$ on $Sp(BD_p(\Gamma)) \setminus U_{j-1,k}$. The p-Royden decomposition of $BD_p(\Gamma)$ yields a unique $p$-harmonic function $h_{j,k} \in BHD(\Gamma)$ and a unique $u_{j,k} \in B(C_c(\Gamma)D_p)$ such that $f_{j,k} = u_{j,k} + h_{j,k}$. Because $u_{j,k} = 0$ on $\partial_p(\Gamma)$ by [3] Theorem 4.8, we see that $f_{j,k} = h_{j,k}$ on $\partial_p(\Gamma)$. Now define
\[
R_{j,k} = \{x \in R_p(\Gamma) \cap U_{j,k} | \lim_{x_n \to x} h_{j,k}(x_n) < f_{j,k}(x) = 1\},
\]
where $(x_n)$ is a sequence in $V$. Observe that if $R_{j,k}$ is nonempty, then it only contains elements of $R_p(\Gamma) \setminus \partial_p(\Gamma)$.

Let $x \in R_p(\Gamma) \setminus \partial_p(\Gamma)$. We will now show that there exists $j, k \in \mathbb{N}$ such that $x \in R_{j,k}$. Since $x \notin \partial_p(\Gamma)$ there exists a $u \in B(C_c(\Gamma)D_p)$ such that $u(x) 
eq 0$. Since $B(C_c(\Gamma)D_p)$ is an ideal we may assume that $u \geq 0$ on $V$ and $u(x) > 0$. Replacing $u$ by $u^{-1}(x)u$ if necessary we may assume that $u(x) = 1$. Let $h \in BHD_p(\Gamma)$ that
satisfies $h \geq 1$ on $V$. Set $f = u + h$, so $f \in BD_p(\Gamma)$ and $f = h$ on $\partial_p(\Gamma)$. Let $(x_n)$ be a sequence in $V$ that converges to $x$. Now $\lim_{n \to \infty} h(x_n) < f(x)$. Because $f$ is continuous we can find an open set $U_{j,k}$ that contains $x$ and satisfies
\[ m = \inf_{U_{j-1,k} \cap R_p(\Gamma)} f > \lim_{n \to \infty} h(x_n). \]
It now follows
\[ f_{j,k} \leq \frac{f}{m} \quad \text{on } R_p(\Gamma), \]
which implies that $h_{j,k} \leq \frac{h}{m}$ on $\partial_p(\Gamma)$. An appeal to the comparison principle gives us
\[ \lim_{n \to \infty} h_{j,k}(x_n) \leq \frac{1}{m} \lim_{n \to \infty} h(x_n) < 1 = f_{j,k}(x), \]
hence $x \in R_{j,k}$. Furthermore,
\[ R_{j,k} = \bigcup_{i=1}^{\infty} \left( R_p(\Gamma) \cap \overline{U_{j,k}} \cap \{ y \in V \mid h_{j,k}(y) < 1 - \frac{1}{i} \} \right). \]
Thus $R_{j,k}$ is a countable union of compact sets. Theorem 3.1 now follows because
\[ R_p(\Gamma) \setminus \partial_p(\Gamma) = \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{N_j} R_{j,k}. \]

5. Proof of Theorem 3.2

Before we prove Theorem 3.2 we need to state some definitions and prove several preliminary results.

Fix a real number $p > 1$. Recall that $E$ denotes the edge set of a graph $\Gamma$. Denote by $F(E)$ the set of all real-valued functions on $E$ and let $F^+(E)$ be the subset of $F(E)$ that consists of all nonnegative functions. For $\omega \in F(E)$ set
\[ E_p(\omega) = \sum_{e \in E} |\omega(e)|^p. \]
The edge set of a path $\gamma$ in $\Gamma$ will be denoted by $Ed(\gamma)$, remember $E(\gamma)$ represents the extreme points of $\gamma$. Let $Q$ be a set of paths in $\Gamma$, denote by $A(Q)$ the set of all $\omega \in F^+(E)$ that satisfy $E_p(\omega) < \infty$ and $\sum_{e \in Ed(\gamma)} \omega(e) \geq 1$ for each $\gamma \in Q$. The extremal length of order $p$ for $Q$ is defined by
\[ \lambda_p(Q)^{-1} = \inf \{ E_p(\omega) \mid \omega \in A(Q) \}. \]
A variation of the next lemma was proved for the case $p = 2$ in [7, Lemma 6.13]. In the $p = 2$ case the conclusion of the lemma is stronger in that $g$ belongs to $C_c(\Gamma)_{D_2}$ instead of the larger space $D_2(\Gamma)$.

**Lemma 5.1.** Let $K$ be a compact subset of $R_p(\Gamma)$ with $K \cap \partial_p(\Gamma) = \emptyset$. Then there exists a function $g \in D_p(\Gamma)$ that satisfies $g = \infty$ on $K$ and $g = 0$ on $\partial_p(\Gamma)$.

**Proof.** By Urysohn’s lemma there exists an $f \in C(Sp(BD_p(\Gamma)))$ that satisfies the following: $0 \leq f \leq 1$, $f = 1$ on $K$ and $f = 0$ on $\partial_p(\Gamma)$. Using the argument from the first paragraph of the proof of Theorem 3.1 we may and do assume $f \in BD_p(\Gamma)$.

Let $(U_n)$ be an exhaustion of $V$ by finite connected subsets. Applying Theorem 3.1 of [2] yields a function $h_n$ that is $p$-harmonic on $U_n$ and equals $f$ on $V \setminus U_n$. It follows from the minimizer property of $p$-harmonic functions on $U_n$ that $I_p(h_n, V) \leq I_p(f, V)$. Hence, $h_n \in BD_p(\Gamma)$ for each $n \in \mathbb{N}$. Also, $h_n = 0$ on $\partial_p(\Gamma)$, $h_n = 1$ on
$K$ and $0 \leq h_n \leq 1$ for each $n$. By passing to a subsequence if necessary, we may assume that $(h_n)$ converges pointwise to a function $h$ because $\{h_n(x) \mid n \in \mathbb{N}\}$ is compact for each $x \in V$. By Lemma 3.2 of [2], $h$ is $p$-harmonic on $V$. Since the sequence $(I_p(h_n, V))$ is bounded, Theorem 1.6 on page 177 of [8] says that by passing to a subsequence if necessary, we may assume that $(h_n)$ converges weakly to a function $\overline{h} \in D_p(\Gamma)$. Because evaluation by $x \in V$ is a continuous linear functional on $D_p(\Gamma)$, we have that $h_n(x) \to \overline{h}(x)$ for each $x \in V$. Thus $h = \overline{h}$ and $h \in BD_p(\Gamma)$. It follows from [5 Corollary 4.9] that $h = 0$ on $V$, due to that fact $h = 0$ on $\partial_p(\Gamma)$.

Since $h_n \to h$ pointwise on $U_k$ for $k \in \mathbb{N}$, it follows $I_p(h_n, U_k) \to I_p(h, U_k) = 0$ for each $k$. Consequently, $I_p(h_n, V) \to 0$. By taking a subsequence if necessary, we may assume that $\|h_n\|_{D_p} \leq 2^{-n}$. Let $\epsilon > 0$ be given and for $m \in \mathbb{N}$, let $g_m = \sum_{k=1}^m h_k$. There exists $N \in \mathbb{N}$ such that $2^{-N} < \epsilon$. For $m, n \in \mathbb{N}$ with $m > n \geq N$ we see that

$$\|g_m - g_n\|_{D_p} = \|\sum_{k=n+1}^m h_k\|_{D_p} \leq \sum_{k=n+1}^m 2^{-k} < 2^{-n} < \epsilon.$$  

Hence, the Cauchy sequence $(g_m)$ converges to $g = \sum_{k=1}^\infty h_k$ in the $D_p$-norm. Thus $g \in D_p(\Gamma)$. For $x \in K, g_m(x) = m$, so $g(x) = \infty$; also $g = g_m = 0$ on $\partial_p(\Gamma)$. The proof of the lemma is complete.

The next result was shown to be true for the case $p = 2$ in [7 Theorem 6.16]. Our proof is essentially the same, and we include it for completeness.

**Lemma 5.2.** Let $P$ be a family of one-sided infinite paths in $\Gamma$ and let

$$K = \cup_{\gamma \in P} E(\gamma).$$

If $K$ is disjoint from $\partial_p(\Gamma)$, then $\lambda_p(P) = \infty$.

**Proof.** By Lemma 5.1 there exists a $g \in D_p(\Gamma)$ such that $g = \infty$ on $K$ and $g = 0$ on $\partial_p(\Gamma)$. Let $\gamma \in P$ and let $x_1, x_2, x_3 \ldots$ be the vertex representation of $\gamma$. Since $E(\gamma) \subseteq K$ we have that $g(\gamma) = \lim_{k \to \infty} g(x_k) = \infty$. Thus

$$\sum_{k=1}^\infty |g(x_k) - g(x_{k+1})| \geq \lim_{k \to \infty} (g(x_k) - g(x_1)) = \infty.$$  

By [3 Lemma 2.3] we obtain $\lambda_p(P) = \infty$.  

A connected infinite subset $D$ of $V$ with $\partial D \neq \emptyset$ is defined to be $D_p$-massive if there exists a $p$-harmonic function $u$ on $D$ that satisfies the following: $0 \leq u \leq 1$, $u = 0$ on $\partial D$, $\sup_D u = 1$ and $I_p(u, D) < \infty$. The function $u$ is known as an inner potential of $D$.

**Proposition 5.3.** Let $D$ be a $D_p$-massive subset, with inner potential $u$, of $V$. Denote by $P_D$ the set of all one-sided infinite paths contained in $D \cup \partial D$. Then $\lambda_p(P_D) < \infty$.

**Proof.** Let $a \in D$ and let $P_a$ be the set of all paths in $P_D$ with initial point $a$. If $\lambda_p(P_a) < \infty$, then $\lambda_p(D) < \infty$ by [3 Lemma 2.1]. Let $(B_n)$ be an exhaustion of $V$ by finite connected subsets of $V$ such that $B_1 \cap \partial D \neq \emptyset$. Pick an $a \in B_1 \cap \partial D$. By combining Theorem 2.1 and Theorem 2.4 of [4] we see that $\lambda_p(P_a) < \infty$ if and only if $\text{cap}_p(\{a\}, \infty, D) > 0$. Thus to finish the proof we need to show $\text{cap}_p(\{a\}, \infty, D) > 0$, which we now proceed to do.
Choose admissible functions \( \omega_k, k \geq 2 \), for condensers \((\{a\}, (D \cup \partial D) \setminus B_k, D)\) such that

\[
I_p(\omega_k, D \cap B_k) \leq \text{cap}_p(\{a\}, (D \cup \partial D) \setminus B_k, D) + \frac{1}{k}.
\]

Replacing all values of \( \omega_k(x) \) on \( D \cap B_k \) for which \( \omega_k(x) < 0 \) by 0 and replacing all values of \( \omega_k(x) \) on \( D \cap B_k \) for which \( \omega_k(x) > 1 \) by 1 decreases the value of \( I_p(\omega_k, D \cap B_k) \). Thus we may and do assume \( 0 \leq \omega_k \leq 1 \) on \( D \cap B_k \). Theorem 3.11 of [2] tells us that there exists a unique \( p \)-harmonic function \( v_2 \) on \( D \cap B_2 \) such that \( v_2 = \omega_2 \) on \( \partial(D \cap B_2) \). Extend \( v_2 \) to all of \( D \) by setting \( v_2 = 1 \) on \( D \setminus B_2 \). By the minimizing property of \( p \)-harmonic functions,

\[
I_p(v_2, D \cap B_2) \leq I_p(\omega_2, D \cap B_2).
\]

Since \( u \) is \( p \)-harmonic on \( D \) and \( u(x) \leq v_2(x) \) for all \( x \in \partial(D \cap B_2) \), \( u \leq v_2 \) on \( D \cap B_2 \) by [2] Theorem 3.14]. Pick \( \omega_3 \). The set \( A = \{x \in D \mid \omega_3(x) > v_2(x)\} \) is a subset of \( D \cap B_2 \). If \( A \neq \emptyset \), redefine \( \omega_3 \) by setting \( \omega_3 = v_2 \) on \( A \). The redefined \( \omega_3 \) decreases \( I_p(\omega_3, D \cap B_1) \), so (5.1) remains true. By continuing as above, we obtain a decreasing sequence of functions \( (v_k) \) such that \( v_k \) is \( p \)-harmonic on \( B \cap B_k, v_k \geq u \), and

\[
I_p(v_k, D \cap B_k) \leq I_p(\omega_k, D \cap B_k).
\]

Now assume that \( \text{cap}_p(\{a\}, (D \cup \partial D) \setminus B_k, D) \to 0 \). Then \( I_p(v_k, D \cap B_k) \to 0 \). Since \( v_k \geq u \) and \( \sup_D u = 1 \), it must be the case that \( (v_k) \to 1_D \), the constant function 1 on \( D \). This is a contradiction because \( (v_k) \) is a decreasing sequence of functions, \( 0 \leq v_2 \leq 1 \) and \( v_2 \neq 1 \). Thus, \( \text{cap}_p(\{a\}, \infty, D) > 0 \) and the proof of the proposition is complete. \( \square \)

Our next result is [7] Theorem 6.18 for the case \( p = 2 \). We give a different proof of the result.

**Lemma 5.4.** Let \( P \) be the family of all one-sided infinite paths in \( \Gamma \) and let \( P_\infty \subseteq P \) be any subfamily with \( \lambda_p(P_\infty) = \infty \). Then

\[
\partial_p(\Gamma) \subseteq \left( \bigcup_{\gamma \in P \setminus P_\infty} E(\gamma) \bigg\} \gamma \in P \setminus P_\infty \right).
\]

**Proof.** Set \( K = \left( \bigcup_{\gamma \in P \setminus P_\infty} E(\gamma) \right) \gamma \in P \setminus P_\infty \right) \). Since our standing assumption is that \( \Gamma \) is \( p \)-hyperbolic, it follows from [3] theorem 2.1] that \( \lambda_p(P) < \infty \). By [3] Lemma 2.2, \( \lambda_p(P \setminus P_\infty) < \infty \). Lemma 5.2 tells us \( K \cap \partial_p(\Gamma) \neq \emptyset \). For purposes of contradiction, assume that there exists a \( \chi \in \partial_p(\Gamma) \) for which \( \chi \notin K \). By Urysohn’s lemma there exists a continuous function \( f \) on \( \text{Sp}(BD_p(\Gamma)) \) that satisfies the following: \( 0 \leq f \leq 1 \), \( f(\chi) = 1 \) and \( f = 0 \) on \( K \cap \partial_p(\Gamma) \). By density of \( BD_p(\Gamma) \) in \( C(\text{Sp}(BD_p(\Gamma))) \) we assume \( f \in BD_p(\Gamma) \). The p-Royden decomposition for \( BD_p(\Gamma) \) yields a unique \( p \)-harmonic function \( h \) on \( V \) and a unique \( g \in B(C_c(\Gamma)_{D_p}) \) such that \( f = g + h \). Theorem 4.8 of [5] shows that \( g = 0 \) on \( \partial_p(\Gamma) \). Combining this fact with the maximum principle [5] Theorem 4.7] it follows that \( 0 < h < 1 \) on \( V \), \( h(\chi) = 1 \) and \( h = 0 \) on \( \partial_p(\Gamma) \cap K \). Let

\[
A = \{x \in V \mid h(x) > 1 - \epsilon\}.
\]

where \( 0 < \epsilon < 1 \). Let \( B \) be a component of \( A \). The set \( B \) is \( D_p \)-massive, see the proof of [5] Proposition 4.12] for a proof of this fact. Let \( P_A \) be the family of all one-sided infinite paths in \( A \), and let \( P_B \) consist of all one-sided infinite paths in
B. Since $B$ is a $D_p$-massive set, $\lambda_p(P_B) < \infty$ by Proposition \[5.3\]. It now follows from \[3\] Lemma 2.1 that $\lambda_p(P_A) < \infty$. Set

$$K_1 = \{ \cup_a E(\gamma) \mid \gamma \in P_A \setminus P_\infty \}.$$  

Another appeal to Lemma \[5.2\] shows $K_1 \cap \partial_p(\Gamma) \neq \emptyset$, because $\lambda_p(P_A \setminus P_\infty) < \infty$. Furthermore, $h = 0$ on $K_1 \cap \partial_p(\Gamma)$ since $K_1 \cap \partial_p(\Gamma) \subseteq K \cap \partial_p(\Gamma)$. However, $h(\gamma) \geq 1 - \epsilon$ for all $\gamma \in P_A$. Thus we obtain the contradiction $h(x) \geq 1 - \epsilon$ for all $x \in K_1$. Therefore, $\partial_p(\Gamma) \subseteq K$, as desired.

**Proof of Theorem \[3.3\]** Let $f \in B(C_c(\Gamma)_{D_p})$ and let $a \in V$. Denote by $P_a$ the set of all one-sided infinite paths in $\Gamma$ with initial point $a$. Set

$$A_{a,f} = \{ \gamma \in P_a \mid f(\gamma) \neq 0 \}.$$  

By \[3\] Theorem 3.3, $\lambda_p(A_{a,f}) = \infty$. Also, \[3\] Lemma 2.2 tells us $\lambda_p(A_f) = \lambda_p(\cup_{a \in V} A_{a,f}) = \infty$. The definition of $A_f$ above and $E_f$ below were given in Section \[3\]. Now Proposition \[5.4\] says that

$$\partial_p(\Gamma) \subseteq E_f.$$  

For notational convenience set $F = \cap_f E_f$, where $f$ runs through $B(C_c(\Gamma)_{D_p})$. Thus, $\partial_p(\Gamma) \subseteq F$. We now proceed to prove the reverse inclusion. Suppose there exists $\chi \in F$ for which $\chi \notin \partial_p(\Gamma)$. By \[5\] Theorem 4.8 we obtain an $f \in B(C_c(\Gamma)_{D_p})$ for which $\chi(f) \neq 0$. Let $\alpha = x_0, x_1, \ldots, x_n, \ldots$ be a one-sided path with $\chi \in \overline{\nabla}(\alpha)$. Because $\chi(f) \neq 0$, there is a subsequence $(x_{n_k})$ of $(x_n)$ that satisfies $\lim_{k \to \infty} f(x_{n_k}) \neq 0$. Thus $f(\alpha) \neq 0$ and has a result $\alpha \in A_f$. Hence $\chi \notin \{ \cup_a E(\gamma) \mid \gamma \in P \setminus A_f \}$. We are assuming $\chi \in E_f$, so it must be the case that there is a sequence $(\chi_n)$ in $\{ \cup_a E(\gamma) \mid \gamma \in P \setminus A_f \}$ with $(\chi_n) \to \chi$. Since $f(\gamma) = 0$ for each $\gamma \in P \setminus A_f$ it follows immediately that $\chi_n(f) = 0$ for each $n \in \mathbb{N}$. This implies $\chi(f) = 0$, contradicting our assumption $\chi(f) \neq 0$. Therefore, $F \subseteq \partial_p(\Gamma)$.

The proof of Theorem \[3.3\] is now complete.

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