Universal Equivariant Multilayer Perceptrons

Siamak Ravanbakhsh
School of Computer Science, McGill University, Montreal, Canada, Mila - Quebec AI Institute.

Group invariant and equivariant Multilayer Perceptrons (MLP), also known as Equivariant Networks, have achieved remarkable success in learning on a variety of data structures, such as sequences, images, sets, and graphs. Using tools from group theory, this paper proves the universality of a broad class of equivariant MLPs with a single hidden layer. In particular, it is shown that having a hidden layer on which the group acts regularly is sufficient for universal equivariance. Next, Burnside’s table of marks is used to decompose product spaces. It is shown that product of two $G$-sets always contains an orbit larger than the input orbits. Therefore high order hidden layers inevitably contain a regular orbit, leading to universality of the corresponding MLP. It is shown that with an order larger than the logarithm of the size of stabilizer group, a high-order equivariant MLP is equivariant universal.

I. INTRODUCTION

Invariance and equivariance properties constrain the output of a function under various transformations of its input. This constraint serves as a strong learning bias that has proven useful in sample efficient learning for a wide range of structured data. In this work, we are interested in universality results for Multilayer Perceptrons (MLPs) that are constrained to be equivariant or invariant. This type of result guarantees that the model can approximate any continuous equivariant (invariant) function with an arbitrary precision, in the same way an unconstrained MLP can approximate an arbitrary continuous function [2, 12, 18].

Study of invariance in neural networks goes back to the book of Perceptron [27], where the necessity of parameter-sharing for invariance was used to prove the limitation of a single layer Perceptron. The follow-up work showed how parameter symmetries can be used to achieve invariance to finite and infinite groups [36, 37, 40, 41]. These fundamental early works went unnoticed during the resurgence of neural network research and renewed attention to symmetry [2, 7, 11, 13, 16, 19, 24]. When equivariance constraints are imposed on feed-forward layers in an MLP, the linear maps in each layer is constrained to use tied parameters [31, 41]. This model that we call an equivariant MLP appears in deep learning with sets [29, 44], exchangeable tensors [15], graphs [25], and relational data [14]. Universality results for some of these models exists [20, 35, 44]. Broader results for high order invariant MLPs appears in [26].

A parallel line of work in equivariant deep learning studies linear action of a group beyond permutations. The resulting equivariant linear layers can be written using convolution operations [8, 22]. When limited to permutation groups, group convolution is simply another expression of parameter-sharing [22, 31]; see also Section [II C]. However, in working with linear representations, one may move beyond finite groups [5]; see also [41]. Some applications include equivariance to isometries of the Euclidean space [33, 42], and sphere [4]. Extension of this view to manifolds is proposed in [6]. Finally, a third line of work in equivariant deep learning that involves a specialized architecture and learning procedure is that of Capsule networks [17, 33]; see [23] for a group theoretic generalization.

A. Summary of Results

This paper proves universality of equivariant MLPs for finite groups in two settings: First, Section [III] shows that any equivariant MLP with a single regular hidden layer is universal equivariant (invariant). Next, Section [IV] shows that a general universality result (that subsumes existing universality results for high order networks) can be derived and attributed to the existence of regular orbits in product spaces. The main tool in our analysis involving decomposition of product spaces is Burnside’s table of marks. Using the table of marks, Section [V] proves that the product of two $G$-sets always creates at least one orbit larger than the orbits of the input $G$-sets. Therefore, repeated product in high-order hidden layers inevitably leads to creation of a regular orbit, which we show is sufficient for universality. A lower-bound on the order of a high order hidden layer that is sufficient for universality is $\log(|\mathcal{H}|)$, where $\mathcal{H}$ is the stabilizer group. Using the largest possible stabilizer on a set of size $N$, this leads to a bound smaller than $N \log_2(N)$ for universal equivariance to arbitrary permutation group. This bound is an improvement over the previous bound $\frac{1}{4} N(N - 1)$ that was shown to guarantee universal invariance [26].

II. PRELIMINARIES

Let $G = \{g\}$ be a finite group with its action defined on the finite set $\mathbb{N}$. Formally, this action $\alpha : G \to \mathcal{S}_{\mathbb{N}}$ is a homomorphism into the group of all permutations of $\mathbb{N}$. The image of this map is a permutation group $\text{Im}(\alpha) = G_{\mathbb{N}} \leq \mathcal{S}_\mathbb{N}$. We use the notation $g \cdot n = g^{-1}n$ to denote this action for
\( g \in G \) and \( n \in \mathbb{N} \). Let \( \mathbb{M} \) be another \( G \)-set, where the corresponding permutation action \( G_M = \text{Im}(b) \leq \mathcal{S}_M \) is defined by \( b : G \to \mathcal{S}_M \), \( G \)-action on \( \mathbb{M} \) naturally extends to \( x \in \mathbb{R}^n \) by \( [g \cdot x](n) \equiv x(g \cdot n) \forall g \in G_n \). We also write this action as \( A_g \cdot x \), where \( A_g \) is the permutation matrix form of \( a(g, \cdot) : \mathbb{N} \to \mathbb{N} \).

A. Invariant and Equivariant Linear Maps

Let the real matrix \( W \in \mathbb{R}^{N \times M} \) denote a linear map \( W : \mathbb{R}^N \to \mathbb{R}^M \). We say this map is \( G \)-equivariant if:

\[
B_g W x = W A_g x \quad \forall x \in \mathbb{R}^N, g \in G.
\]

where similar to \( A_g \), the permutation matrix \( A_g \) is defined based on the action \( b(g, \cdot) : \mathbb{M} \to \mathbb{M} \). In this definition, we assume that the group action on the input is \textit{faithful} – that is \( a \) is injective, or \( G_N \cong G \). If the action on the output index set \( \mathbb{M} \) is not faithful, then the \textit{kernel} of this action is a non-trivial normal subgroup of \( G \), \( \text{ker}(b) \triangleleft G \). In this case \( G_M \cong G/\text{ker}(b) \) is a quotient group, and it is more accurate to say that \( W \) is equivariant to \( G/\text{ker}(b) \) and equivariant to \( G \). Using this convention \( G \)-equivariance and \( G \)-equivariance correspond to extreme cases of \( \text{ker}(b) = G \) and \( \text{ker}(b) = \{ e \} \). Moreover, composition of such invariant-equivariant functions preserves this property, motivating design of deep networks by stacking equivariant layers.

B. Orbits and Homogeneous Spaces

\( G_N \) partitions \( \mathbb{N} \) into orbits \( \mathbb{N}_1, \ldots, \mathbb{N}_O \), where \( G_N \) is \textit{transitive} on each orbit, meaning that for each pair \( n_1, n_2 \in \mathbb{N}_o \), there is at least one \( g \in G_N \) such that \( g \cdot n_1 = n_2 \). If \( G_N \) has a single orbit, it is transitive, and \( \mathbb{N} \) is called a \textit{homogeneous space} for \( G \). If moreover the choice of \( g \in G_N \) with \( g \cdot n_1 = n_2 \) is unique, then \( G_N \) is called \textit{regular}.

Given a subgroup \( H \leq G \) and \( g \in G \), the right coset of \( H \) in \( G \) defined as \( Hg = \{ hg, h \in H \} \) is a subset of \( G \). For a fixed \( H \leq G \), the set of these right-cosets, \( \mathcal{H} \triangleleft G = \{ Hg, g \in G \} \), form a partition of \( G \). \( G \) naturally acts on the right coset space, where \( g' \cdot (Hg) = H(gg') \) sends one coset to another. The significant of this action is that “any” transitive \( G \)-action is isomorphic to \( G \)-action on some right coset space. To see why, note that in this action any \( h \in H \) stabilizes the coset \( Hc \), because \( h \cdot Hc = Hc \). Therefore in any action the stabilizer identifies the coset space.

C. Parameter-Sharing and Group Convolution View

Consider the equivariance condition of (1). Since the equality holds for all \( x \in \mathbb{R}^n \), and using the fact that the inverse of a permutation matrix is its transpose, the equivariance constraint reduces to

\[
B_g W A_g^T = W \quad \forall g \in G.
\]

The equation above ties the parameters within the orbits of \( G \)-action on rows and columns of \( W \):

\[
W(m, n) = W(g \cdot m, g \cdot n) \quad \forall g \in G, n, m \in \mathbb{N} \times \mathbb{M}
\]

where \( W(g \cdot m, g \cdot n) = W_{g \cdot m, g \cdot n} \) is an element of the matrix as a linear map. This type of group action on Cartesian product space is sometimes called the \textit{diagonal action}. In this case, the action is on the Cartesian product of rows and columns of \( W \).

We saw that any homogenous \( G \)-space is isomorphic to a coset space. Using \( N \cong H \backslash G \) and \( \mathbb{M} \cong \mathcal{K} \backslash \mathbb{R} \), the parameter-sharing of (2) becomes

\[
W(Hg, Hg') = W(g^{-1} \cdot Hg, g^{-1} \cdot Hg')
\]

This rewriting also enables us to express the matrix vector multiplication of the linear map \( W \) in the form of cross-correlation of input and a kernel \( w \)

\[
|Wx|(n) = |Wx|(Hg)
\]

\[
= \sum_{Hg' \in \mathcal{K} \cap G} W(Hg, Hg') x(Hg')
\]

\[
= \sum_{Hg' \in \mathcal{K} \cap G} w(Hg' g^{-1}) x(Hg')
\]

This relates the parameter-sharing view of equivariant maps \( \mathcal{W} \) to the convolution view \( \mathcal{K} \). Therefore, the universality results in the following extends to group convolution layers \( \mathcal{W} \), for finite groups.

\[\textbf{a. Equivariant Affine Maps} \quad \text{We may extend our definition, and consider affine} \ G \text{-maps} \ Wx + b, \text{by allowing an “invariant” bias parameter} \ b \in \mathbb{R}^M \text{ satisfying}
\]

\[ B_g b = b. \]

This implies a parameter sharing constraint \( b(m) = b(g \cdot m) \). For homogeneous \( \mathbb{M} \), this constraint enforces a \textit{scalar} bias. Beyond homogeneous spaces, the number of free parameters in \( b \) grows with the number of orbits.

D. Invariant and Equivariant MLPs

One may stack multiple layers of equivariant affine maps with multiple channels, followed by a non-linearity, so as to
build an equivariant MLP. One layer of this equivariant MLP a.k.a. equivariant network is given by:

\[ x_{c}^{(l)} = \sigma \left( \sum_{c'=1}^{C^{(l-1)}} W_{c,c'} x_{c'}^{(l-1)} + b_{c}^{(l)} \right), \]

where \( 1 \leq c' \leq C^{(l-1)} \) and \( 1 \leq c \leq C^{(l)} \) index the input and output channels respectively, \( x^{(l)} \) is the output of layer \( 1 \leq l \leq L \), with \( x^{(0)} = x \) denoting the original input. Here, we assume that \( G \) faithfully acts on all \( x_{c}^{(l)} \in \mathbb{R}^{\ell^{(l)}}, \forall c, l \), with \( \mathbb{H}^{(0)} = \mathbb{N} \) and \( \mathbb{H}^{(L)} = \mathbb{M} \). The parameter matrices \( W_{c,(i),c'}^{(l)} \in \mathbb{R}^{\ell^{(l-1)} \times \ell^{(l)}}, \) and the bias vector \( b_{c}^{(l)} \in \mathbb{R}^{\ell^{(l)}} \) are constrained by the parameter-sharing conditions (2) and (9) respectively.

In an invariant MLP the faithfulness condition for \( G \)-action on the hidden and output layers are lifted. In practice, it is common to construct invariant networks by first constructing an equivariant network followed by pooling over \( \mathbb{H}^{(L)} \).

### III. UNIVERSALITY RESULTS FOR REGULAR ACTION

This section presents two simple new results on universality of both invariant and equivariant networks with a single hidden layer \((L = 2)\). Formally, we can claim that a \( G \)-equivariant continuous function \( \psi : \mathbb{R}^{N} \to \mathbb{R}^{M} \) is a universal \( G \)-equivariant approximator, if for any \( G \)-equivariant continuous function \( \psi : \mathbb{R}^{N} \to \mathbb{R}^{M}, \) any compact set \( \mathbb{K} \subseteq \mathbb{R}^{N}, \) and \( \epsilon > 0, \) there exists a choice of parameters, and number of channels such that \( |\|\psi(x) - \psi(x)\| \leq \epsilon \forall x \in \mathbb{K} \).

**Theorem III.1.** A \( G \)-invariant network

\[ \hat{\psi}(x) = \sum_{c=1}^{C} w_{c}^{T} \sigma (W_{c} x + b_{c}). \]  

with a single hidden layer, on which \( G \) acts regularly is a universal \( G \)-invariant approximator. Here, \( 1 = \begin{bmatrix} 1, \ldots, 1 \end{bmatrix}^{T} \), and \( b_{c}, w_{c}^{T} \in \mathbb{R}, |G| \)

**Proof.** The first step follows the symmetrization argument \([33]\), which in its general form is widely used in invariant theory \([33]\). Since MLP is a universal approximator, for any compact set \( \mathbb{K} \subseteq \mathbb{R}^{N}, \) we can find \( \psi_{\text{MLP}} \) such that for any \( \epsilon > 0, \) \( |\psi(x) - \psi_{\text{MLP}}(x)\| \leq \epsilon \) for \( x \in \mathbb{K} \). Let \( \mathbb{K}_{\text{sym}} = \bigcup_{g \in G} \mathbb{A}_{g} x \in \mathbb{K} \) denote the symmetrized \( \mathbb{K} \), which is again a compact subset of \( \mathbb{R}^{N} \) for finite \( G \). Let \( \hat{\psi}_{\text{MLP}+} \) approximate \( \psi \) on the symmetrized compact set \( \mathbb{K}_{\text{sym}} \). It is then easy to show that for \( G \)-invariant \( \psi \), the symmetrized MLP \( \psi_{\text{sym}}(x) = \frac{1}{|G|} \sum_{g \in G} \hat{\psi}_{\text{MLP}+}(\mathbb{A}_{g} x) \) also approximates \( \psi \)

\[ |\psi(x) - \psi_{\text{sym}}(x)| = |\psi(x) - \frac{1}{|G|} \sum_{g \in G} \psi_{\text{MLP}+}(x)| \leq \epsilon. \]

Our next step, is to show that \( \psi_{\text{sym}} \) is equal to \( \hat{\psi} \) of \((10)\), for some parameters \( W_{c} \in \mathbb{R}^{\ell^{(l)} \times N} \) constrained so that \( H_{g} W_{c} = W_{c} A_{g} \forall g \in G \), where \( A_{g} \) and \( H_{g} \) are the permutation representation of \( G \) action on the input and the hidden layer respectively.

\[ \psi_{\text{sym}}(x) = \frac{1}{|G|} \sum_{g \in G} \sum_{c=1}^{C} w_{c}^{T} \sigma (w_{c}^{T} A_{g} x) \]

\[ = \sum_{c=1}^{C} \frac{w_{c}^{T}}{|G|} \sum_{g \in G} \sigma (w_{c}^{T} A_{g} x) \]

\[ = \sum_{c=1}^{C} \frac{w_{c}^{T}}{|G|} \sigma (\begin{bmatrix} -w_{c}^{T} A_{g_{1}} & \ldots & \ldots & \ldots & -w_{c}^{T} A_{g_{|G|}} \end{bmatrix} x) \] 

where in the last step we put the summation terms into rows of the matrix \( W_{c} \), and performed the summation using multiplication by \( 1^{T} \). \( \tilde{w}_{c} \) is the rescaled \( w_{c} \).

**Proof.** The first step follows the symmetrization argument \([33]\), which in its general form is widely used in invariant theory \([33]\). Since MLP is a universal approximator, for any compact set \( \mathbb{K} \subseteq \mathbb{R}^{N}, \) we can find \( \psi_{\text{MLP}} \) such that for any \( \epsilon > 0, \) \( |\psi(x) - \psi_{\text{MLP}}(x)\| \leq \epsilon \) for \( x \in \mathbb{K} \). Let \( \mathbb{K}_{\text{sym}} = \bigcup_{g \in G} \mathbb{A}_{g} \) denote the symmetrized \( \mathbb{K} \), which is again a compact subset of \( \mathbb{R}^{N} \) for finite \( G \). Let \( \hat{\psi}_{\text{MLP}+} \) approximate \( \psi \) on the symmetrized compact set \( \mathbb{K}_{\text{sym}} \). It is then easy to show that for \( G \)-invariant \( \psi \), the symmetrized MLP \( \psi_{\text{sym}}(x) = \frac{1}{|G|} \sum_{g \in G} \hat{\psi}_{\text{MLP}+}(\mathbb{A}_{g} x) \) also approximates \( \psi \)

\[ |\psi(x) - \psi_{\text{sym}}(x)| = |\psi(x) - \frac{1}{|G|} \sum_{g \in G} \psi_{\text{MLP}+}(x)| \leq \epsilon. \]

This shows that a \( G \)-invariant network with a single hidden layer on which \( G \) acts regularly is equivalent to a symmetrized MLP, and therefore for some number of channels, it is a universal approximator of \( G \)-invariant functions. Note that the number of channels corresponds to the number of hidden units in the symmetrized MLP.

Next, we extend this to equivariant MLPs.
**Theorem III.2.** A $G$-equivariant MLP

$$\hat{\psi}(x) = \sum_{c=1}^{C} W_c^c \sigma(W_c^c x + b_c). \quad (16)$$

with a single regular hidden layer is a universal $G$-equivariant approximator.

**Proof.** In this setting, symmetricization, using the so-called Reynolds operator, for the universal MLP is given by

$$\psi_{sym}(x) = \frac{1}{|G|} \sum_{g \in G} B_g^{-1} \sum_{c=1}^{C} w'^c_g \sigma(w'^c_g A_g x + b_c) \quad (17)$$

where $w_c \in \mathbb{R}^N$ and $w'_c \in \mathbb{R}^M$ are the weight vectors in the first and second layer associated with hidden unit $c$. Our objective is to show that this symmetrized MLP is equivalent to the equivariant network of (16), in which $W_c^c \in \mathbb{R}^{M \times H}$, and $W_c \in \mathbb{R}^{H \times N}$ use parameter-sharing to satisfy

$$H_g W_c = W_c A_g \text{ and } B_g W'_c = W'_c H_g \quad \forall g \in G. \quad (18)$$

Here, $A_g$, $B_g$ and $H_g$ are the permutation representations of $G$ action on the input, the output, and the hidden layer respectively.

First, rewrite the symmetrized MLP as

$$\psi_{sym}(x) = \sum_{c=1}^{C} \sum_{g \in G} B_g^{-1} w'^c_g \sigma(A_g x + b_c)$$

$$= \sum_{c=1}^{C} W'_c \sigma(W_c x)$$

where

$$W'_c = \begin{bmatrix} B_{g_1}^c w'^c_c & \ldots & B_{g_n}^c w'^c_c \end{bmatrix}$$

$$W_c = \begin{bmatrix} -w_c A_{g_1} & \ldots & -w_c A_{g_n} \end{bmatrix}^{-1}$$

and the $\frac{1}{|G|}$ factor is absorbed in one of the weights. It remains to show that the two matrices above satisfy the equivariance condition

$$H_g W_c = W_c A_g, \quad \text{and} \quad B_g W'_c = W'_c H_g.$$ 

The proof for $W_c$ is identical to the invariant network case. For $W'_c$, we use a similar approach.

$$B_g W'_c H_g^{-1} = \begin{bmatrix} B_{g_1} B_{g_1}^{-1} w'_c & \ldots & B_{g_n} B_{g_n}^{-1} w'_c \end{bmatrix}$$

$$= \begin{bmatrix} B_{g_1} w'_c & \ldots & B_{g_n} w'_c \end{bmatrix} = W'_c.$$

In the first step, since $H_g^{-1} = H_{g^{-1}}$ is acting on the right, it moves the column indexed by $g^{-1}$ to $g^{-1}$. This means that the column currently at $g^{-1}$ is $g^{-1}$. The second step uses the following:

$$B_g B_{g^{-1}} g = B_g (g^{-1} g) = B_{g^{-1}} g^{-1} g = B_{g^{-1}}.$$ 

This proves the equality of the symmetrize MLP (17) to the equivariant MLP of (16). However, a similar argument to the proof of invariant case, shows the universality of $\psi_{sym}$. Putting these together, completes the proof of Theorem III.2. □

In the case where $G$ is an Abelian group, any faithful transitive action is regular, meaning that the hidden layer in a $G$-equivariant neural network is necessarily regular. Combined with Theorem III.2 this leads to a universality result for Abelian groups.

**Corollary 1.** For Abelian group $G$, a $G$-equivariant (invariant) neural network with a single hidden layer is a universal approximator of continuous $G$-equivariant (invariant) functions on compact subsets of $\mathbb{R}^N$.

### IV. DECOMPOSITION OF PRODUCT $G$-SETS

A prerequisite to analysis of product $G$-sets is their classification, which also leads to classification of all $G$-maps based on their input/output $G$-sets.

#### A. Classification of $G$-Sets and $G$-Maps

Recall that any transitive $G$-set $\mathcal{N}$ is isomorphic to a right-coset space $\mathcal{H} \setminus \mathcal{G}$. However, the right cosets $\mathcal{H} \setminus \mathcal{G}$ and $\{g^{-1} \mathcal{H} g \mid g \in \mathcal{G}\}$ are themselves isomorphic. This also means what we care about is conjugacy classes of subgroups $\mathcal{H} = \{g^{-1} \mathcal{H} g \mid g \in \mathcal{G}\}$, which classifies right-cosets spaces up to conjugacy $\mathcal{H} \setminus \mathcal{G} = \{g^{-1} \mathcal{H} g \mathcal{G} \mid g \in \mathcal{G}\}$. We used the bracket to identify the conjugacy class. In this notation, for $\mathcal{H} \leq \mathcal{G}$, we say $\mathcal{H} < \mathcal{G}^\prime$, iff $g^{-1} \mathcal{H} g < \mathcal{H}^\prime$, for some $g \in \mathcal{G}$.

#### B. Classification of $G$-sets

A $G$-set is transitive on each of its orbits, and we can identify each orbit with its stabilizer subgroup. Therefore a list

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5 The stabilizer subgroups of two points in a homogeneous space are conjugate, and therefore $G$-sets resulting from conjugate choice of right-cosets are isomorphic. To see why stabilizers are conjugate, assume $n = a^{-1} \cdot n$, and $h \in G_n$, then $aha^{-1} n = a^{-1} n = n$. Therefore, $a^{-1} ha \in G_n$. Since conjugation is a bijection, this means $G_n = a^{-1} G_n a$. 


of these subgroups along with their multiplicities completely defines a $G$-set up to an isomorphism [22]:

$$\mathbb{N} \cong \bigcup_{[\mathcal{H}] \leq G} p_i[\mathcal{H} \setminus G],$$

(19)

where $p_1, \ldots, p_l \in \mathbb{Z}_{\geq 0}$ denotes the multiplicity of a right-coset space, and $\mathbb{N}$ has $\sum_{i=1}^l p_i$ orbits.

To ensure a faithful $G$-action on $\mathbb{N}$, a necessary and sufficient condition is for the point-stabilizers $G_n, \forall n \in \mathbb{N}$ to have a trivial intersection. The point-stabilizers within each orbit are conjugate to each other and their intersection which is the largest normal subgroup of $G$ contained in $[\mathcal{H}]$, is called the core of $G$-action on $[\mathcal{H}]$:

$$Core_G([\mathcal{H}]) = \bigcap_{g \in G} g^{-1}[\mathcal{H}],$$

(20)

C. Classification of $G$-Maps

Next, we extend the classification of $G$-sets to $G$-equivariant maps, a.k.a. $G$-maps $W : \mathbb{R}^N \rightarrow \mathbb{R}^M$, by jointly classifying the input and the output index sets $\mathbb{N}$ and $\mathbb{M}$. We may consider a similar expression to [19] for the output index set $M = \bigcup_{[\mathcal{H}] \leq G} q_j[\mathcal{H} \setminus G]$. The linear $G$-map $W : \mathbb{R}^N \rightarrow \mathbb{R}^M$ is then equivariant to $G/\mathcal{K}$ and invariant to $\mathcal{K} \triangleleft G$ iff

$$\bigcap_{p_i > 0} Core_G([\mathcal{H}_i]) = \{e\} \text{ and } \bigcap_{q_j > 0} Core_G([\mathcal{H}_j]) = \mathcal{K},$$

(21)

where the second condition translates to $\mathcal{K}$ invariance of $G$-action on $\mathbb{M}$. Note that the first condition is simply ensuring the faithfulness of $G$-action on $\mathbb{N}$. This result means that the multiplicities $(p_1, \ldots, p_l)$ and $(q_1, \ldots, q_j)$ completely identify a (linear) $G$-map $W : \mathbb{R}^N \rightarrow \mathbb{R}^M$ that equivariant to $G/\mathcal{K}$ and invariant to $\mathcal{K} \triangleleft G$, up to an isomorphism.

D. Cartesian Product of $G$-sets

Previously, we classified all $G$-sets as the disjoint union of homogeneous spaces $\bigcup_{i=1}^N p_i[\mathcal{G}_i \setminus G]$, where $\mathcal{G}$ acts transitively on each orbit. However, $\mathcal{G}$ also naturally acts on the Cartesian product of homogeneous $G$-sets:

$$\mathbb{N}_1 \times \ldots \times \mathbb{N}_D = ([G_1 \setminus G] \times \ldots \times [G_D \setminus G])$$

where the action is defined

$$g \cdot (G_1 h_1, \ldots, G_D h_D) = (G_1 h_1 g, \ldots, G_D h_D g).$$

a. High Order Spaces  A special case is when we consider the repeated self-product of the same homogeneous space $H \cong [\mathcal{H}]$:

$$H^D \cong [\mathcal{H}] \times [\mathcal{H}] \times \ldots \times [\mathcal{H}]$$

(22)

We call this an order $D$ product space. Product spaces are used in building high-order layers in $G$-equivariant networks in several recent works [1, 21, 23]. (author?) [26] show that for

$$D \geq \frac{1}{2} ||\mathcal{H}|| (||\mathcal{H}|| - 1),$$

(23)

such MLPs with multiple hidden layers of order $D$ become universal $G$-invariant approximators. We show that better bounds for $D$ that guarantees universal invariance and equivariance follows from the universality results of Theorems [11] and [11] and the decomposition of product spaces. This means that such high order produce spaces are universal simply because they contain a regular $G$-set.

E. Burnside Ring and Decomposition of $G$-sets

Since any $G$-set can be written as a disjoint union of homogeneous spaces [19], we expect a decomposition of the product $G$-space in the form

$$[G_i \setminus G] \times [G_j \setminus G] = \bigcup_{[\mathcal{G}] \leq G} \delta_{ij}[G_i \setminus G]$$

(24)

Indeed, this decomposition exists, and the multiplicities $\delta_{ij} \in \mathbb{Z}_{\geq 0}$, are called the structure coefficient of the Burnside Ring. The (commutative semi)ring structure is due to the fact that the set of non-isomorphic $G$-sets $\Omega(G) = \{\bigcup_{[\mathcal{G}] \leq G} p_i[G_i \setminus G] \mid p_i \in \mathbb{Z}_{>0}\}$, is equipped with: 1) a commutative product operation that is the Cartesian product of $G$-spaces, and 2) a summation operation that is the disjoint union of $G$-spaces [10]. A key to analysis of product $G$-spaces is finding the structure coefficients in [23].

Example 1 (Product of Sets). The symmetric group $S_n$ acts faithfully on $\mathbb{N}$, where the stabilizer is $S_n = S_{n-\{n\}}$ — that is the stabilizer of $n \in \mathbb{N}$ is the set of all permutations of the remaining items $\mathbb{N} \setminus \{n\}$. This means $\mathbb{N} \cong [S_{n-\{n\}} \setminus S_n]$.

The diagonal $S_n$ action on the product space $\mathbb{N}^D$, decomposes into $\sum_{i=1}^D p_i = \text{Bell}(D)$ orbits, where the Bell number is the number of different partitions of a set of $D$ labelled objects [23]. One may further refine these orbits by their type in the form of [23]:

$$[S_{n-\{n\}} \setminus S_n]^D = \bigcup_{d=1}^D S(D, d)[S_{n-\{n_1, \ldots, n_d\}} \setminus S_n]$$

(25)

where the “structure coefficient” $S(D, d)$ is the Stirling number of the second kind, and it counts the number of ways $D$ could be partitioned into $d$ non-empty sets. For example, when $D = 2$, one may think of the index set $\mathbb{N} \times \mathbb{N}$ as indexing some $[N] \times [N]$ matrix. This matrix decomposes into one $S(2, 1) = 1$ diagonal $[S_{n-\{n\}} \setminus S_n]$ and one $S(2, 2) = 1$ set of off-diagonals $[S_{n-\{n_1, n_2\}} \setminus S_n]$. This decomposition is presented in
From (24) in the example above it follows that an order $D = |N| - 1$ product of sets contains a regular orbit. The following is a corollary that combines this with the universality results of Theorems III.1 and III.2.

**Corollary 2.** [Universality for Product of Sets] a $S_N$ equivariant network with a hidden layer of order $|N| - 1$, is a universal approximator of $\delta_{N^2}$-equivariant (invariant) functions, where the input and output layer may be of any order.

A universality result for the invariant case only, using a quadratic order appears in [24], where the MLP is called a hyper-graph network. (author?) [20] prove universality for the equivariant case, without giving a bound on the order of the hidden layer, and assuming an output $M = H^1$ of degree $D = 1$. In comparison, Corollary 2 uses a linear bound and applies to a much more general setting of arbitrary orders for the input and output product sets. In fact, the universality result is true for arbitrary input-output $S_N$-sets.

**a. Linear $G$-Map as a Product Space** For finite groups, the linear $G$-map $W : \mathbb{R}^N \rightarrow \mathbb{R}^M$ is indexed by $M \times N$, and therefore it is a product space. In fact the parameter-sharing of $W$ ties all the parameters $W(m,n)$ that are in the same orbit. Therefore, the decomposition (23) also identifies parameter-sharing pattern of $W$.

**Example 2 (Equivariant Maps Between Set Products).** Equation (24) gives a closed form for the decomposition of $\mathbb{N}^D$ into orbits. Assuming a similar decomposition for $M^D$, the equivariant map $W : \mathbb{R}^{\mathbb{N}^D} \rightarrow \mathbb{R}^{M^D}$ is decomposed into $\oplus_{G^j \in G}$ linear maps corresponding to the orbits of $M^D$ in $\mathbb{N}^D$. $I$ show that each orbit “type” is a form of pooling-broadcasting from/to hyper-diagonals of the corresponding tensors.

1. **Burnside’s Table of Marks**

Burnside’s table of marks simplifies working with the multiplication operation of the Burnside ring, and enables the analysis of $G$-action on product spaces [3, 28]. The mark of $\mathcal{H} \leq G$ on a finite $G$-set $\mathbb{N}$, is defined as the number of points in $\mathbb{N}$ fixed by all $h \in \mathcal{H}$:

$$m_\mathcal{H}(N) = |\{n \in \mathbb{N} | h \cdot n = n \ \forall h \in \mathcal{H}\}|. \quad (25)$$

The interesting quality of the number of fixed points is that the total number of fixed points adds up when we add two spaces $\mathbb{N}_1 \cup \mathbb{N}_2$. Also, when considering product spaces $\mathbb{N}_1 \times \mathbb{N}_2$, any combination of points fixed in both spaces will be fixed by $\mathcal{H}$. This means

$$m_{\mathbb{N}_1 \cup \mathbb{N}_2}(G_i) = m_{\mathbb{N}_1}(G_i) + m_{\mathbb{N}_2}(G_i) \quad (26)$$

$$m_{\mathbb{N}_1 \times \mathbb{N}_2}(G_i) = m_{\mathbb{N}_1}(G_i) m_{\mathbb{N}_2}(G_i). \quad (27)$$

Now define the vector of marks $m_\mathcal{H} : \Omega(G) \rightarrow \mathbb{Z}^n$ as

$$m_\mathcal{H} = [m_{\mathbb{N}_1}(G_1), \ldots, m_{\mathbb{N}_n}(G_1)]$$

where $I$ is the number of conjugacy classes of subgroups of $G$, and we have assume a fixed order on $[G_i] \leq G$. Due to Eqs. (26) and (27), given $G$-sets $\mathbb{N}_1, \ldots, \mathbb{N}_I$, we can perform elementwise addition and multiplication on the vector of integers $m_{\mathbb{N}_1}, \ldots, m_{\mathbb{N}_n}$, to obtain the mark of union and product $G$-sets respectively. Moreover, the special quality of marks, makes this vector an injective homeomorphism: we can work backward from the resulting vector of marks and decompose the union/product space into homogeneous spaces. To facilitate calculation of this vector, for any $G$-set $\mathbb{N}$, one may use the table of marks.

The **table of marks** for a group $G$, is the square matrix of marks of all subgroups on all right-coset space $\mathbb{N}$ — that is the element $i, j$ of this matrix is:

$$M_G(i,j) = m_{\mathbb{N}_i}(G_j) \quad \text{or} \quad M_G = \begin{bmatrix} m_{\mathcal{H}_i \setminus G} & \cdots & m_{\mathcal{H}_i \setminus G} \\ \vdots & \ddots & \vdots \\ m_{\mathcal{H}_i \setminus G} & \cdots & m_{\mathcal{H}_i \setminus G} \end{bmatrix}. \quad (28)$$

The matrix $M_G$, has valuable information about the subgroup structure of $G$. For example, $G_j$’s action on $G_i$ will have a fixed point, iff $[G_j] \leq [G_i]$. Therefore, the sparsity pattern in

| \{e\} | $G_i$ | $G_j$ | $G_m$ |
|---|---|---|---|
| $G_i$ | ... | $G_i$ | $G_{i,m}$ |
| $G_j$ | ... | $G_j$ | $G_{j,m}$ |
| $G_m$ | ... | $G_m$ | $G_{m,m}$ |

| $G_i \setminus G$ | $G_j \setminus G$ | $G_m \setminus G$ |
|---|---|---|
| $G_i$ | ... | $G_{i,m}$ |
| $G_j$ | ... | $G_{j,m}$ |
| $G_m$ | ... | $G_{m,m}$ |

When $\mathbb{N}$ and $M$ are homogeneous spaces, another characterization the orbits of the product space $[G_n \setminus G] \times [G_m \setminus G]$ is by showing their one-to-one correspondence with double-cosets $G_n \setminus G \setminus G_m = \{G \in G_n | g \in G \}$.  

\[ m_{G_i \setminus G_j}(G_i) = m_{G_i \setminus G_j}(G_i g^{-1}) \text{, and } m_{G_i \setminus G_j}(G_j) = m_{G_i \setminus G_j}(G_i g^{-1}) \forall g \in G. \quad (29) \]

Therefore, the table of marks’ characterization is up to conjugacy.
the table of marks, reflects the subgroup lattice structure of $G$, up to conjugacy.

A useful property of $M_G$ is that we can use it to find the marks $m_H$ on any $G$-set $H = \sum_{\ell} p_{\ell} [G_{\ell}] \notin \Omega(G)$ using the expression $m_H = [p_1, \ldots, p_I] \cdot M_G$. Moreover, the structural constants of $M_G$ can be recovered from the table of Marks:

$$\delta_{ij}^\ell = \sum_l M_G(i, l)M_G(j, l)M_G^{-1}(l, \ell) \quad (29)$$

V. UNIVERSALITY OF $G$-MAPS ON PRODUCT SPACES

Using the tools discussed in the previous section, in this section we prove some properties of product spaces that are consequential in design of equivariant maps. Previously we saw that product spaces decompose into orbits, identified by $\delta_{ij}^\ell > 0$ in $(23)$. The following theorem states that such product spaces always have orbits that are at least as large as the largest of the input orbits, and at least one of these product orbits is strictly larger than both inputs. For simplicity, this theorem is stated in terms of the stabilizers, rather than the orbits, where by the orbit-stabilizer theorem, larger stabilizers correspond to smaller orbits. Also, while the following theorem is stated for the product of homogeneous $G$-sets, it trivially extends to product of $G$-sets with multiple orbits.

**Theorem V.1.** Let $[G_i \setminus G]$ and $[G_j \setminus G]$ be transitive $G$-sets, with $\{e\} \subset G_i, G_j \subset G$. Their product $G$-set decomposes into orbits $[G_i \setminus G] \times [G_j \setminus G] = \bigcup_{\ell} \delta_{ij}^\ell [G_{\ell} \setminus G]$, such that:

- (i) $[G_i] \subseteq [G_{\ell}], [G_j] \subseteq [G_{\ell}]$ for all the resulting orbits.
- (ii) if $G_i \not\subseteq \text{Core}_G(G_j)$ and $G_j \not\subseteq \text{Core}_G(G_i)$, then $[G_{\ell}] < [G_i], [G_j]$ for at least one of the resulting orbit.

**Proof.** The proof is by analysis of the table of Marks $M_G$. The vector of mark for the product space is the element-wise product of vector of marks of the input:

$$m_{[G_i \setminus G] \times [G_j \setminus G]} = m_{[G_i \setminus G]} \odot m_{[G_j \setminus G]}.$$

The same vector, can be written as a linear combination of rows of $M_G$, with non-negative integer coefficients:

$$m_{[G_i \setminus G]} \odot m_{[G_j \setminus G]} = \sum_{\ell} \delta_{ij}^\ell m_{[G_{\ell} \setminus G]}.$$

For convenience we assume a topological ordering of the conjugacy class of subgroups $\{e\} = G_1, \ldots, G_t = G$ consistent with their partial order – that is $[G_i] \not< [G_j] \forall j > i$. This means that $M_G$ is lower-triangular, with nonzero diagonals; see Table I. Three important properties of this table are $(23)$:

1. the sparsity pattern in $M_G$ reflects the subgroup relation: $m_{[G_i \setminus G]}(\ell) > 0$ iff $G_{\ell} \subseteq [G_i]$. 2. the first column is the index of $G_i$ in $G$: $m_{[G_i \setminus G]}(1) = |G : G_i| \forall i$. 3. the diagonal element is the index of the normalizer: $m_{[G_i \setminus G]}(i) = |G : N_G(G_i)| \forall i$, where the normalizer of $H$ in $G_i$ is defined as the largest intermediate subgroup of $G_i$ in which $H$ is normal: $N_G(H) = \{ g \in G_i \mid g H g^{-1} = H \}$.

(i) From (1) it follows that the non-zeros of the product $m_{[G_i \setminus G]} \odot m_{[G_j \setminus G]}(\ell) > 0$ correspond to $G_{\ell} \leq [G_i]$ and $G_{\ell} \leq [G_j]$. Since the only rows of $M_G$ with such non-zero elements are $m_{[G_i \setminus G]}$ for $G_{\ell} \leq [G_i] \cap [G_j]$, all the resulting orbits have such stabilizers. This finishes the proof of the first claim.

(ii) If $[G_i] \not< [G_j]$ and $[G_j] \not< [G_i]$, then $[G_i]$ which is a subgroup of both groups is strictly smaller than both, which means one of the resulting orbits must be larger than both input orbits.

Next, w.l.o.g., assume $[G_i] \subseteq [G_j]$. Consider proof by contradiction: suppose the product does not have a strictly larger orbit, then from (i) it follows that $m_{[G_i \setminus G]} \odot m_{[G_j \setminus G]} = \delta_{ij}^\ell m_{[G_{\ell} \setminus G]}$ for some $\delta_{ij}^\ell > 0$. Consider the first and $i^{th}$ element of the elementwise product above:

$$|G : G_j| \times |G : G_i| = \delta_{ij}^\ell |G : G_i| \mid m_{[G_i \setminus G]}(i) \times |G : N_G(G_i)| = \delta_{ij}^\ell |G : N_G(G_i)|.$$

Substituting $\delta_{ij}^\ell = |G : G_j|$ from the first equation into the second equation and simplifying we get $m_{[G_i \setminus G]}(i) = |G : G_j|$. This means the action of $G_j$ on $[G_j \setminus G]$ fixes all points, and therefore $G_j \subseteq \text{Core}_G(G_j)$ as defined in $(20)$. This contradicts the assumption of (ii). $\square$

A sufficient condition for (ii) in Theorem V.1 is for the $G$-action on input $G$-sets to be faithful. Note that in this case the the core is trivial; see Section [IV.A] An implication of

**Table II.** Table of marks for the alternating group $A_5$.

| $\{e\}$ | $C_2$ | $C_3$ | $K_4$ | $C_5$ | $\Delta$ | $D_{10}$ | $A_4$ | $A_5$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $A_5$ | 60    | 30    | 15    | 12    | 10    | 6     | 5     | 1     |
| $C_2 \setminus A_5$ | 2         |         |         |         |         |         |         |         |
| $C_3 \setminus A_5$ | 2         |         |         |         |         |         |         |         |
| $K_4 \setminus A_5$ | 3         |         |         |         |         |         |         |         |
| $C_5 \setminus A_5$ | 2         |         |         |         |         |         |         |         |
| $\Delta \setminus A_5$ | 1         |         |         |         |         |         |         |         |
| $D_{10} \setminus A_5$ | 1         |         |         |         |         |         |         |         |
| $A_4 \setminus A_5$ | 1         |         |         |         |         |         |         |         |
| $A_5 \setminus A_5$ | 1         |         |         |         |         |         |         |         |
this theorem is that repeated self-product \([\mathcal{H} \setminus G]^D\) is bound to produce a regular orbit. The following theorem gives a lower-bound on \(D\).

**Theorem V.2.** Let \(G\) act faithfully on \(N \cong [\mathcal{H} \setminus G]\). Then \(N^D\) has a regular orbit for any

\[
D \geq \log_2(|\mathcal{H}|)
\]

**Proof.** Since \(G\) acts faithfully on \(N\), \(\text{Core}_G(\mathcal{H}) = \{e\}\). From Theorem V.1 it follows that each time we calculate a product by \(N\), a strictly smaller stabilizer is produced so that

\[
H = H^{(t=0)} > H^{(1)} > \ldots > H^{(D)} = \{e\},
\]

where \(H^{(d)}\) is the smallest stabilizer at time-step \(d\). From Lagrange theorem, the size of a proper subgroup is at most half the size of its overgroup in this sequence of stabilizers. It follows that for any \(D \geq \log_2|\mathcal{H}|, [\mathcal{H} \setminus G]^D\) has an orbit with \(H^{(t=D)} = \{e\}\) as its stabilizer.

Since the largest stabilizer for any action on \(N\) is \(S_{|N|-1}\), we can use a lower-bound for \(D\), in Theorem V.2 that is independent of the stabilizer group \(H\). The following bound follows from the Sterlings’ approximation \(N! < N^{N+\frac{1}{2}}e^{-N+1}\) to the size of the largest possible stabilizer \(S_{|N|-1} = (|N| - 1)!\).

**Corollary 3.** The \(G\)-set \(N^D\), with \(N = |N|\) has a regular orbit for

\[
D \geq [(N - \frac{1}{2})\log_2(N - 1) - (N - 2)\log_2(e)]
\]

We may then combine these results with the universality results of Theorem III.2 (Theorem III.1), to get sufficient conditions for universality of higher order equivariant (invariant) networks with a single hidden layer. Note that our bound of Corollary 3 that does not use the size of stabilizer is tighter than the bound of 22 derived in [26], which was derived for universal invariance.

**Corollary 4.** A \(G\)-equivariant (invariant) network with a high order hidden layer \(H^D = [\mathcal{H} \setminus G]^D\) is a universal equivariant (invariant) approximator, if either \(D > \log_2(|\mathcal{H}|)\) or \(D > |\mathcal{H}| \log_2(|\mathcal{H}|)\).

In general, the best value for \(D\) is obtained through the analysis of the table of Marks, as demonstrated in the following example.

**Example 3 (Universal Approximation for \(\mathcal{A}_5\)).**

The alternating group \(\mathcal{A}_5\) is the group even permutations of 5 objects. One way to create a universal approximator for this group to have a regular layer (see Theorem III.2).

A more convenient alternative is to consider the canonical action of this group on a set \(N\) of size 5, and use an order \(D\) layer to ensure universality. Using Corollary 3 we get

\[
D \geq 5 = [\frac{1}{2} \log_2(4) - 4 \log_2(e)]
\]

The natural action of \(\mathcal{A}_5\) on \(N = [5]\) is isomorphic to \([\mathcal{A}_4 \setminus \mathcal{A}_5]\) – i.e., \(\mathcal{A}_4\) is a stabilizer. Using this stabilizer in Theorem V.2 we get the same bound

\[
D \geq 5 = [\log_2(|\mathcal{A}_4|)]
\]

However, using the table of marks we can show that \(D = 3\) already produces a regular orbit in this case. The table of marks for the alternating group \(\mathcal{A}_5\) is shown in Table II. Our objective is to find the decomposition of \([\mathcal{A}_4 \setminus \mathcal{A}_5]^3\). We do this in steps, first showing

\[
[\mathcal{A}_4 \setminus \mathcal{A}_5]^2 = [\mathcal{A}_4 \setminus \mathcal{A}_5] \cup [C_3 \setminus \mathcal{A}_5]
\]

To see this, note that the element-wise product of the vector of marks \(m_{[\mathcal{A}_4 \setminus \mathcal{A}_5]}\) (which is next to last row in Table II) with itself is equal to \(m_{[\mathcal{A}_4 \setminus \mathcal{A}_5]} \times m_{[C_3 \setminus \mathcal{A}_5]}\). Since the vector of marks is an injective homomorphism, this implies (30). Applying the same idea one more time, gives

\[
[\mathcal{A}_4 \setminus \mathcal{A}_5]^3 = ([\mathcal{A}_4 \setminus \mathcal{A}_5] \cup [C_3 \setminus \mathcal{A}_5]) \times [\mathcal{A}_4 \setminus \mathcal{A}_5]
\]

This shows that \([\mathcal{A}_4 \setminus \mathcal{A}_5]^3\) contains a regular orbit \([\{e\} \setminus \mathcal{A}_5]\). Therefore, using an order \(D = 3\) hidden layer \(N^3\) on which \(\mathcal{A}_5\) acts using even permutations, also produces a universal equivariant (invariant) approximator.
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