Dynamic Games and Strategies

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Abstract

The present paper gives a mathematical formulation of intensionality and dynamics in computation in terms of games and strategies. More specifically, we give a game semantics for a prototypical programming language for a simple arithmetic that distinguishes terms with the same value but different algorithms, equipped with the hiding operation on strategies that precisely corresponds to the (small-step) operational semantics of the language. Categorically, our games and strategies give rise to a cartesian closed bicategory, and our game semantics forms an instance of a dynamic generalization of the standard interpretation of functional languages in cartesian closed categories. This work is intended to be the first step towards a mathematical foundation for intensional and dynamic aspects of computation; our approach should be applicable to a wide range of languages.

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1 Introduction

In [GTL89], Girard mentions the dichotomy between the static and dynamic viewpoints in logic and computation; the former identifies terms with their denotations (i.e., results of their computations in an ideal sense), while the latter focuses on their senses (i.e., algorithms or intensionality) and the dynamics of computation. This distinction is certainly reflected as the two mutually complementary semantics of programming languages: the denotational and operational ones [Win93, Gun92]. He points out that a mathematical formulation of the former has been relatively well-developed, but it is not the case for the latter; the treatment of senses has been based on syntactic manipulation. He then emphasizes the importance of a mathematical formulation of senses [GTL89]:

The establishment of a truly operational semantics of algorithms is perhaps the most important problem in computer science.

The present paper addresses this problem; more precisely, we establish an interpretation $⟦-⟧_D$ of a programming language $\mathcal{L}$ with a small-step operational semantics $\rightarrow$ and a syntax-independent operation $\triangleright$ that satisfy the following dynamic correspondence property (DCP): If $M_1 \rightarrow M_2$ in $\mathcal{L}$, then $⟦M_1⟧_D \neq ⟦M_2⟧_D$ and the diagram

\[
\begin{array}{c}
M_1 \rightarrow M_2 \\
\downarrow \quad \downarrow \\
⟦M_1⟧_D \triangleright [M_2]_D
\end{array}
\]

commutes. Note in particular that the interpretation $⟦-⟧_D$ is finer than the usual (sound) interpretation because $M_1 \rightarrow M_2$ implies $⟦M_1⟧_D \neq ⟦M_2⟧_D$. Thus, the interpretation $⟦-⟧_D$ and the operation $\triangleright$ capture intensionality and dynamics in computation, respectively.

Although our framework in this paper is intended to be a general approach, and it should be applicable to a wide range of computations, as a first step we focus on a simple language for intuitionistic Peano Arithmetic (PA) or Heyting Arithmetic (HA), known as System $T$ [AF98, GTL89] customized for our purpose.

1.1 Game Semantics

Our approach is based on a particular kind of semantics of programming languages, called game semantics, since it is an intensional model that captures the dynamics of computation in

\footnote{Clearly, two terms often have the same denotation but different senses.}
a natural and intuitive yet mathematically precise and syntax-independent manner \cite{A+97,Hy97}. Also, it is very flexible: A wide range of programming languages have been modeled via the unified framework of game semantics by varying constraints on morphisms \cite{AM99}; we have chosen this approach with the hope that it is also the case for intensional and dynamic aspects of computation.

In game semantics, types and terms are interpreted as games and strategies, respectively. Roughly, a game is a certain kind of forest whose branches represent possible developments or plays of the “game in the usual sense” (such as chess, poker, etc.) it represents. For our purpose, it suffices to concentrate on games played by just two participants, Player (who represents a “part of the computational system under consideration”) and Opponent (who represents an “environment”), in which Opponent always starts a play, and then they alternately make a move allowed by the rules of the game. On the other hand, a strategy on a game is what tells Player which move she should make next at each of her turns in the game. To summarize, a game semantics \llbracket \cdot \rrbracket_G interprets a type \(A\) as the game \llbracket A \rrbracket_G that specifies possible interactions between Opponent and Player in it and a term \(M : A\) as a strategy \llbracket M \rrbracket_G that describes an algorithm for Player to play in \llbracket A \rrbracket_G; an “execution” of the term \(M\) is then interpreted as a play in \llbracket A \rrbracket_G between the two participants in which Player follows \llbracket M \rrbracket_G.

Let us consider a simple example: The game \(N\) of natural numbers is the following tree (which is infinite in width):

\[
\begin{array}{c}
0 \\
1 \\
2 \\
3 \\
\vdots
\end{array}
\]

in which a play starts with the Opponent’s question \(q\) (“What is your number?”) and ends with a Player’s answer \(n \in \mathbb{N}\) (“My number is \(n\!)”). A strategy \(10\) on this game, for instance, that corresponds to the number \(10\) can be represented by the function \(q \mapsto 10\). As another, a bit more elaborate example, consider the game \(\mathbb{N} \rightarrow \mathbb{N}\) of numeric functions, where a play is of the form \(qqnm\), where \(n,m \in \mathbb{N}\), or diagrammatically:

\[
\begin{array}{c}
\mathbb{N} \rightarrow \mathbb{N} \\
q \q n \\
q n \rightarrow m
\end{array}
\]

which can be read as follows:

1. Opponent’s question \(q\) for an output (“What is your output?”)
2. Player’s question \(q\) for an input (“Wait, what is your input?”)
3. Opponent’s answer, say, \(n\) to the second \(q\) (“OK, here is the input \(n\) for you.”)
4. Player’s answer, say, \(m\) to the first \(q\) (“Alright, the output is then \(m\).”)

A strategy \(\text{succ}\) on this game that corresponds to the successor function can be represented by the function \(q \mapsto q, n \mapsto n + 1\).

Even these simple examples illustrate the intuitive nature of game semantics as well as how it represents intensionality and dynamics in computation.

\footnote{For simplicity, here we focus on terms with the empty context.}

\footnote{Note that a play is just a certain finite sequence of moves of a game. The diagram is depicted only to clarify which component game each move belongs to; it should be read just as a finite sequence, namely \(qqnm\).}
1.2 Existing Game Semantics Is Static

However, game semantics $\llbracket \cdot \rrbracket_G$ has been employed as a denotational semantics, and so in particular sound: If two terms evaluate to the same value, then their interpretations are identical. As an immediate consequence, the existing game semantics is static in the sense that if we have a reduction $M_1 \rightarrow M_2$ in syntax, then we have the equation $[M_1]_G = [M_2]_G$ between the strategies. Thus, the existing game semantics is not fully intensional or dynamic in the sense that it cannot satisfy a DCP.

In order to develop what should be called dynamic game semantics, let us see how the existing game semantics fails to be fully intensional or dynamic. In a word, it is because the “internal communication” between strategies for their composition is a priori hidden, and so the resulting strategy is always in “normal form”. For instance, the composition $\text{succ}; \text{double} : N \rightarrow N$ of the successor strategy $\text{succ} : N \rightarrow N$ and the doubling strategy $\text{double} : N \rightarrow N$

\[
\begin{array}{c}
N \xrightarrow{\text{succ}} N \\
q \\
m \\
m+1 \\
\end{array}
\quad
\begin{array}{c}
N \xrightarrow{\text{double}} N \\
q \\
n \\
2\cdot n \\
\end{array}
\]

is calculated as follows: First, by the “internal communication”, Player plays the role of Opponent in the intermediate component games $N_2, N_3$ just by “copying and pasting” her last moves, resulting in the following play:

\[
\begin{array}{cccc}
N_1 & \xrightarrow{\text{succ}} & N_2 & \xrightarrow{\text{double}} & N_3 & \rightarrow & N_4 \\
q & \quad & q & \quad & n & \quad & 2\cdot(n+1) \\
q & \quad & n & \quad & n+1 & \quad & n+1 \\
\end{array}
\]

where the subscripts 1, 2, 3, 4 are just to distinguish different copies of $N$, and the moves for the “internal communication” are marked by a square box. This play is read as follows:

1. Opponent’s question $q$ for an output in $N_1 \rightarrow N_4$ (“What is your output?”)
2. Player’s question $q$ by $\text{double}$ for an input in $N_3 \rightarrow N_4$ (“Wait, what is your input?”)
3. 2 in turn stimulates the question $q$ for an output in $N_1 \rightarrow N_2$ (“What is your output?”)
4. Player’s question $q$ by $\text{succ}$ for an input in $N_1 \rightarrow N_2$ (“Wait, what is your input?”)
5. Opponent’s answer, say, $n$ to 4 in $N_1 \rightarrow N_4$ (“OK, here is the input $n$ for you.”)
6. Player’s answer $n+1$ to 3 by $\text{succ}$ (“Alright, the output is then $n+1$."
7. 6 in turn stimulates the answer $n+1$ to 2 (“OK, here is the input $n+1$ for you.”)
8. Player’s answer $2\cdot(n+1)$ to 1 by $\text{double}$ (“Alright, the output is then $2\cdot(n+1)$."


where note that Opponent plays on the game $N_1 \rightarrow N_4$, and so he can “see” only the moves in $N_1$ or $N_4$.

Next, “hiding” means to hide or delete all the moves with the square box, resulting in the strategy for the function $n \mapsto 2 \cdot (n+1)$ as expected:

\[
\begin{array}{cccc}
N & \xrightarrow{\text{succ; double}} & N \\
q & \xrightarrow{\text{n}} & 2 \cdot (n+1)
\end{array}
\]

Moreover, let us plug in the strategy $\mathbf{5} : q \mapsto 5$ on the game $I \rightarrow N$, which is essentially $N$ because there is no possible move in the terminal game $I_4$. The composition $\mathbf{5}; \text{succ; double} : I \rightarrow N$ of $\mathbf{5}; \text{succ}$ and $\text{double}$ is computed again by the “internal communication”:

\[
\begin{array}{cccc}
I & \xrightarrow{\mathbf{5}} & N & \xrightarrow{\text{succ; double}} & N & \xrightarrow{\text{n}} & N & \xrightarrow{\text{n}} & q
\end{array}
\]

plus “hiding”:

\[
\begin{array}{cccc}
I & \xrightarrow{\mathbf{5}; \text{succ; double}} & N \\
q & \xrightarrow{\text{n}} & 12
\end{array}
\]

In syntax, on the other hand, assuming the constants $n$ for each number $n \in \mathbb{N}$, succ and double for the successor and doubling functions, respectively, equipped with the operational semantics $\text{succ} n \rightarrow n + 1$, $\text{double} n \rightarrow 2 \cdot n$ for all $n \in \mathbb{N}$, the program $p_1 \equiv \lambda x. (\lambda y. \text{double} y)((\lambda z. \text{succ} z)x)$ represents the syntactic composition succ; double. When it is applied to the numeral $\mathbf{5}$, we have the following chain of reductions:

\[
p_1 \mathbf{5} \rightarrow^* (\lambda x. \text{double} (\text{succ} x)) \mathbf{5} \\
\rightarrow^* \text{double} (\text{succ} \mathbf{5}) \\
\rightarrow^* \text{double} \mathbf{6} \\
\rightarrow^* 12.
\]

Thus, it seems that (syntactic) reduction corresponds to “hiding internal communication” in game semantics. However, as seen in the above examples, this game-theoretic normalization

\[\text{We will define it formally in a later section.}^4\]

\[\text{Composition of strategies is associative, which we shall see shortly, so the order of applying composition does not matter.}^5\]
process is *a priori* executed and thus invisible in the existing game semantics. As a result, the two programs $p_1$ and $12$ are interpreted as the same strategy, namely $12 : q \mapsto 12$. Moreover, observe that moves with the square box describe *intensionality* in computation or *step-by-step processes* to compute an output from an input, but they are invisible after the “hiding”. Thus, e.g., another program $p_2 \equiv \lambda x. (\lambda y. \text{succy})(\lambda z. \text{succz})(\lambda w. \text{doublew})(x)$, which represents the same function (as a graph) as $p_1$ but a different algorithm, is interpreted as the same as $p_1$:

$$[p_2] = [\text{double}; \text{succ}; \text{succ}] = [\text{succ}; \text{double}] = [p_1].$$

To sum up, we have observed the following:

1. **Reduction as hiding.** A reduction in syntax corresponds to “hiding intermediate moves” in game semantics.

2. **A priori normalization.** However, this process of “hiding” is *a priori* executed in the existing game semantics; thus, strategies are always in “normal form”.

3. **Intermediate moves as intensionality.** “Intermediate moves” constitute *intensionality* in computation; however, they are not captured in the existing game semantics again due to “a priori hiding”.

### 1.3 Dynamic Games and Strategies

From these observations, we define a *dynamic* variant of games and strategies, in which “intermediate moves” are not *a priori* hidden, representing intensionality in computation, and the *hiding operations* $\mathcal{H}$ on these dynamic games and strategies that hide “intermediate moves” in a step-by-step fashion, interpreting the dynamic process of reduction.

In doing so, we have tried to develop additional structures and axioms that are *conceptually natural* and *mathematically elegant*, *independently of syntax*. This is in order to inherit the natural and intuitive nature of the existing game semantics, so that the resulting interpretation would be insightful, convincing and useful. Also, mathematics often leads us to a “correct” formulation: If a definition gives rise to neat mathematical structures, then it is likely to succeed in capturing the essence of the concepts or phenomena of concern and subsume various instances. In fact, dynamic games and strategies are a natural generalization of the existing games and strategies, and they satisfy beautiful algebraic laws. Categorically, they form a *cartesian closed bicategory* (CCB) $\text{DG}$, in which 0- (resp. 1-) cells are a certain kind of dynamic games (resp. strategies) and 2-cells correspond to the *extensional equivalence* between 1-cells, and $\mathcal{H}$ induces the hiding functor $\mathcal{H}^e : \text{DG} \to \text{MC}$, where $\text{MC}$ is the cartesian closed category (CCC) of games and strategies in $\text{AM99}$, and it can be seen as a “extensionally collapsed” subbicategory of $\text{DG}$. (For this paper it suffices to know that a CCB is a generalized CCC in the sense that the axioms are required to hold only up to 2-cell isomorphisms.)

### 1.4 Dynamic Game Semantics

We then establish an interpretation $\llbracket \cdot \rrbracket_{\text{DG}}$ of our System T in $\text{DG}$ that together with $\mathcal{H}$ on 1-cells satisfies a DCP. Let us remark that it does not make much sense to ask whether full abstraction holds for our dynamic game semantics as its aim is to capture intensionality in computation.

On the other hand, we may establish (intensional) definability by an inductive construction of a subclass $T\text{DG}$ of dynamic games and strategies; however, it is rather trivial. We leave a non-inductive characterization of $T\text{DG}$ as future work.

Also, our model does not satisfy faithfulness: The semantic equation is of course finer than $\beta$-equivalence but coarser than $\alpha$-equivalence, e.g., consider the terms $(\lambda x. 0) 1$, $(\lambda x. 0) 2 : \text{N}$. 


1.5 Related Work and Our Contribution

To the best of our knowledge, the present work is the first syntax-independent characterization of dynamics of computation in the sense that it satisfies a DCP.

The work closest in spirit is Girard’s geometry of interaction (GoI) [Gir89]. However, GoI appears ad hoc as it does not follow the standard categorical approach [Pit01, Cro93]; also, it does not capture the step-by-step process of reduction.

From the opposite, “semantics-first” point of view, TDG per se can be seen as a mathematical model of computation in the same sense as Turing machines [Tur36] that captures syntactic practice. Seeing the literature, Turing machines, for instance, are the first syntax-independent model of computation, but their computational processes are too low-level to match those of programming languages; the λ-calculus [Chu36, Chu40] is the origin of functional programming, and so it is rather syntactic.

Next, the idea of exhibiting “intermediate moves” in the composition of strategies is nothing new; there are game-theoretic models [DGL05, Gre05, BO08, Ong06] that give such moves an official status. However, since their aim is rather to develop a tool for program verification, they do not study in depth mathematical structures thereof, give an intensional game semantics that follows the usual categorical semantics of type theories [Pit01, Cro93], or formulate a step-by-step hiding process.

Also, there have been several works to model dynamics of computation by 2-categories [BS10, See87, Mel05]. In these works, however, horizontal composition of 1-cells is the “normalizing” one, which is why the structure is 2-categories rather than bicategories. Also 2-cells are rewriting, not external equivalence. Note that 2-cells in a bicategory cannot interpret rewriting unless the horizontal composition is “normalizing” since associativity of “non-normalizing” composition with respect to such 2-cells does not hold.

Thus, although their motivation is similar to ours, our bicategorical model of computation seems a novel one, which interprets application of terms via “non-normalizing” composition, extensional equivalence of terms by 2-cells and rewriting by an equipped operation on 1-cells. Moreover, their framework is just categorical, while we have instantiated our bicategorical model by a game-theoretic one.

Finally, let us mention that this work has several implications from theoretical as well as practical perspectives. From the theoretical point of view, it enables us to study intensionality and dynamics in computation as purely mathematical (or semantic) concepts, just like any concepts in pure mathematics such as differential and integral in calculus, homotopy in topology, etc. Thus, we may rigorously analyze the essence of these concepts ignoring superfluous syntactic details; e.g., it may be an accurate measure for computational complexity of functional programming. On the other hand, from the practical view, it can be a useful tool for language analysis and design; e.g., our System T would not exist without the present work.

1.6 Overview

The rest of the paper proceeds as follows. In Section 2, we formulate our variant of System T and its categorical semantics that satisfies a DCP; it then remains to establish its game-theoretic instance. Next, Section 3 defines our games and strategies, and Section 4 gives their interpretation of System T. Finally, Section 5 makes a conclusion and proposes future work.

◮ Notation. In the rest of the paper, we employ the following notations:

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6Otherwise, unit law does not strictly hold.
7I.e., we cannot have a “rewriting” between 1-cells \((f; g); h\) and \(f; (g; h)\).
8In mathematics, one usually captures the essence of various concepts and phenomena in a syntax-independent manner, e.g., the notion of natural numbers is independent of representation, which may be unary, binary or decimal.
We use sans-serif letters such as Γ, a, A for syntactic objects, and write \( \equiv \) for syntactic equality.

Let \( V \) be a countably infinite set of variables, written \( x, y, z, \) etc., for which we assume the variable convention.

For a partially ordered set \( P \) and a subset \( S \subseteq P \), we write \( \text{sup}(S) \) for the supremum of \( S \).

For a function \( f : A \to B \) and a subset \( S \subseteq A \), we define \( f \upharpoonright S : S \to B \) to be the restriction of \( f \) to \( S \).

We use bold letters \( s, t, u, v, w, \) etc. to denote sequences; \( \epsilon \) denotes the empty sequence.

We write \( s \preceq t \) if \( s \) is a prefix of a sequence \( t \).

We use letters \( a, b, c, d, e, m, n, p, q, x, y, z, \) etc. to denote elements of sequences.

A concatenation of sequences is represented by a juxtaposition of them, but we usually write \( as, tb, ucv \) for sequences \((a)s, t(b), u(c)v\), respectively. For readability, we sometimes write \( st \) for the concatenation \( st \).

We write \( \text{even}(s) \) and \( \text{odd}(t) \) to mean that sequences \( s \) and \( t \) are of even-length and odd-length, respectively.

For a set \( S \) of sequences, we define \( S^{\text{even}} \overset{\text{def}}{=} \{ s \in S \mid \text{even}(s) \} \) and \( S^{\text{odd}} \overset{\text{def}}{=} \{ t \in S \mid \text{odd}(t) \} \).

For a set \( X \) of elements, we define \( X^* \) to be the set of finite sequences of elements in \( X \).

Given a sequence \( s \) and a set \( X \), we write \( s \upharpoonright X \) for the subsequence of \( s \) that consists of elements in \( X \). We often have \( s \in Z^* \) with \( Z = X + Y \) for some set \( Y \); in such a case, we abuse the notation: The operation deletes the “tags” for the disjoint union, so that \( s \upharpoonright X \in X^* \).

## 2 Categorical Semantics of Intensionality and Dynamics in Computation

This section presents an abstract, categorical description of how our games and strategies interpret intensionality and dynamics in computation, and shows that it is a refinement of the standard categorical semantics of type theories \cite{Pitts91, Coquand93, Jacobs99}.

### 2.1 Bicategories of Computation

The categorical structure that is essential for our interpretation is bicategories of computation (BiCs), which are bicategories whose 2-cells are “extensional equivalences” of 1-cells, equipped with values and evaluations satisfying certain axioms:

\( \text{Definition 2.1.1 (BiCs).} \) A \textit{bicategory of computation (BiC)} is a bicategory \( C \) equipped with a pair \( (V, E) \) such that for all \( A, B \in C \) morphisms in the category \( C(A, B) \) are the equivalence relation \( \cong \) defined below, \( V \) assigns a subset \( V(A, B) \subseteq C(A, B) \) whose elements are called values, and \( E \) assigns some unique \( E(f) \in C(A, B) \) to each \( f \in C(A, B) \) called the \textit{evaluation} of \( f \), satisfying the following axioms:

\footnote{\textit{I.e., we assume that in any terms of concern every bound variable is chosen to be different from any free variables.}}
Fixed-points. $E^{n+1}(f) = E^n(f) \Rightarrow E^n(f) \in \mathcal{V}(A, B)$

Values. $f \in \mathcal{V}(A, B) \Rightarrow E(f) = f$

Composition. $f; g \notin \mathcal{V}(A, C) \land (f \downarrow \land g \downarrow \Rightarrow f; g \downarrow)$

Identities. $id_A \in \mathcal{V}(A, A)$

2-cells. $C(A, B)(f, f') \neq \emptyset \Leftrightarrow f \cong f'$

for all $A, B, C \in \mathcal{C}$, $f, f' \in C(A, B)$, $g \in C(B, C)$, $n \in \mathbb{N}$, where we write $f \downarrow$, or more specifically $f \downarrow E^{n_0}(f)$, if $E^{n_0+1}(f) = E^{n_0}(f)$ for some $n_0 \in \mathbb{N}$ and $f \uparrow$ otherwise, and define:

$$f \cong f' \overset{\text{df}}{=} (\exists v \in \mathcal{V}(A, B). f \downarrow v \land g \downarrow v) \lor (f \uparrow \land f' \uparrow).$$

It is cartesian closed if so is $\mathcal{C}$ in the sense of [Oua97].

Notation. If $f \downarrow E^{n_0}(f)$ for some $n_0 \in \mathbb{N}$, then we call $E^{n_0}(f)$ the fixed-point (or value) of $f$ and rather write $E^n(f)$ for it.

The intuition behind the definition is as follows. In a BiC $\mathcal{C} = (\mathcal{C}, \mathcal{V}, \mathcal{E})$, 1-cells are (not necessarily "effective") computations with their domain and codomain "types" (i.e., 0-cells) specified, and values are extensional computations such as functions (as graphs). The horizontal composition of 1-cells should be considered as "concatenation" of computations (so it is "non-normalizing") and horizontal identities as "unit computations" (they are just like identity functions). Then the "execution" of a computation $f$ is achieved by evaluating it into a unique value $E^n(f)$, which corresponds to dynamics of computation. Moreover, 2-cells $\cong$ "witness" extensional equivalence between 1-cells. The five axioms given above are natural ones from this viewpoint, where the unit law holds only up to $\cong$ because we are concerned with processes.

Note that (both vertical and horizontal) compositions and identities of 2-cells as well as the natural isomorphisms for associativity and unit law in a BiC are the obvious and unique ones, and so the coherence trivially holds. It in particular implies the following:

Proposition 2.1.2 (Characterization of BiCs). A (resp. cartesian closed) BiC is equivalent to a (resp. cartesian closed) category equipped with values and evaluations satisfying the five axioms in which the composition (resp. pairing and currying) preserves $\cong$ and the required equations hold only up to $\cong$.

Proof. Let $\mathcal{C} = (\mathcal{C}, \mathcal{V}, \mathcal{E})$ a BiC. Clearly, its 0-cells (as objects), 1-cells (as morphisms), horizontal identities and composition on 1-cells (as identities and composition) form a category in which associativity and unit law hold only up to $\cong$. Note that the horizontal composition of 2-cells implies that the composition of morphisms preserves $\cong$. If $\mathcal{C}$ is cartesian closed, then it induces a cartesian closed structure of that category in which the pairing and currying preserve $\cong$ (by these operations on 2-cells) and the required equalities hold only up to $\cong$.

Conversely, given a category $\mathcal{C}'$ equipped with values $\mathcal{V}'$ and evaluations $\mathcal{E}'$ satisfying the five axioms (in Definition 2.1.1) in which the composition preserves $\cong$ and associativity and unit law hold only up to $\cong$, we may induce a BiC $\mathcal{C}' = (\mathcal{C}', \mathcal{V}', \mathcal{E}')$ as follows. Its 0-, 1-cells are the objects and morphisms in $\mathcal{C}'$, and horizontal identities and composition on 1-cells are the identities and composition in $\mathcal{C}'$. Its 2-cells are of course the induced equivalence relation $\cong$; the vertical identities as well as the vertical and horizontal compositions on 2-cells are the obvious (and unique) ones. Note that the horizontal composition on 2-cells is well-defined as the composition on morphisms preserves $\cong$. Also, note that the functoriality of the horizontal compositions...
composition trivially holds. Moreover, the natural 2-cell isomorphisms for associativity and unit law exist simply by the fact that associativity and unit law of morphisms hold up to \(\cong\), where note that the required coherence conditions trivially hold. If the category \(C'\) is cartesian closed in which pairing and currying preserve \(\cong\) and the required equations hold only up to \(\cong\), then in the same manner, it gives the corresponding cartesian closed structure on the BiC \(C'\) in the sense defined in [Oua97].

Finally, these constructions are clearly mutually inverse, completing the proof. ■

Thanks to this proposition, we do not have to care much about subtleties in the definition of CCBs such as coherence. Also, it suffices to specify a “CCC up to \(\cong\)” together with values and evaluations to give a CCBiC. Moreover, Proposition 2.1 implies that a BiC \(C = (C, V, E)\) induces the category \(V\) of values defined by:

- **Objects** are 0-cells of \(C\)
- **Morphisms** \(A \rightarrow B\) are values in \(V(A, B)\)
- **The composition of morphisms** \(v : A \rightarrow B, w : B \rightarrow C\) is \(E(\omega)(v; w) : A \rightarrow C\)
- **The identities** are those in \(C\).

Of course, \(V\) is cartesian closed if so is \(C\). Moreover, seeing \(V\) as a trivial 2-category, \(E\) induces the “evaluation 2-functor” \(E(\omega) : C \rightarrow V\) that maps \(A \mapsto A\) for 0-cells \(A\), \(f \mapsto E(\omega)(f)\) for 1-cells \(f\), \(\cong \mapsto \cong\) for 2-cells \(\cong\).

The point is that we may now decompose the usual (static) interpretation \([\cdot]_S\) of functional languages in CCCs [Pit01, Cro93, Jac99] as a more fine-grained “intensional interpretation” \([\cdot]_D\) in a cartesian closed bicategory of computation (CCBiC) \(C = (C, V, E)\) plus “semantic evaluation” \(E(\omega)\) on 1-cells, i.e., \([\cdot]_S = E(\omega)([\cdot]_D)\), and talk about intensional difference between computations: Terms \(M, M'\) are interpreted to be intensionally equal if \([M]_D = [M']_D\) and extensionally equal if \([M]_D \cong [M']_D\). Also, the evaluation \(E\) is intended to capture the operational semantics of the target language. We shall make this point precise for a simple language below.

2.2 System \(T_\vartheta\)

As a first step of our framework, we consider a simple extension of the *simply-typed \(\lambda\)-calculus* for Heyting Arithmetic (HA), known as Gödel’s *System T* [AF98, GTL89]. System T is far more expressive than the simply-typed \(\lambda\)-calculus, equipped with natural numbers and primitive recursion, but it has the *strong normalization* property, i.e., every computation terminates.

However, System T in the usual form does not match our perspective on computation; e.g., the 2\(^{nd}\)-numeral is usually the expression \(\text{succ}(\text{succ}(0))\), but it should represent the *process* of applying the successor twice to 0, not the number 2. For this point, we represent every term via PCF Böhm trees [AM00, HO00, AC98], exploiting the *strong definability* of the game semantics in [HO00, AM99]. Moreover, we design an eager operational semantics that computes “genuine values”. We call the resulting variant *System T\(_\vartheta\)*:

**Definition 2.2.1** (System \(T_\vartheta\)). *System T\(_\vartheta\)* is a functional programming language for simple arithmetic defined as follows:

11Of course, it would be possible to give a simpler dynamic game semantics for the simply-typed \(\lambda\)-calculus by the same framework.
**Types** $\mathcal{A}$ are generated by the following grammar:

$$\mathcal{A} \overset{df}{=} \mathcal{N} \mid \mathcal{A}_1 \Rightarrow \mathcal{A}_2$$

where $\mathcal{N}$ is the **natural numbers type** and $\mathcal{A}_1 \Rightarrow \mathcal{A}_2$ is the **function type** from $\mathcal{A}_1$ to $\mathcal{A}_2$ ($\Rightarrow$ is right associative). We write $\mathcal{A}, \mathcal{B}, \mathcal{C},$ etc. for types. Note that each type $\mathcal{A}$ is of the form $\mathcal{A}_1 \Rightarrow \mathcal{A}_2 \Rightarrow \cdots \Rightarrow \mathcal{A}_k \Rightarrow \mathcal{N}$, where $k \in \mathbb{N}$.

**Raw-terms** $\mathcal{M}$ are generated by the following grammar:

$$\mathcal{M} \overset{df}{=} x \mid n \cdot \text{case}(\mathcal{M}) \cdot \lambda x^{\mathcal{A}}. \mathcal{M} \cdot \mathcal{M}_1 \mathcal{M}_2$$

where $x$ ranges over variables, $\mathcal{M}'$ over countably infinite sequences of raw-terms, $\mathcal{A}$ over types, and $\mathcal{M}_n^{\mathcal{A}}$ are generated by the following grammar:

- $\mathcal{A} \equiv \mathcal{N} \mid \mathcal{A}_1 \Rightarrow \mathcal{A}_2$
- $\mathcal{M} \equiv x \mid n \cdot \text{case}(\mathcal{M}) \cdot \lambda x^{\mathcal{A}}. \mathcal{M} \cdot \mathcal{M}_1 \mathcal{M}_2$

**Contexts** $\Gamma$ are finite sequences $x_1 : \mathcal{A}_1, x_2 : \mathcal{A}_2, \ldots, x_k : \mathcal{A}_k$ of (variable : type)-pairs such that $x_i \neq x_j$ whenever $i \neq j$. We write $\Gamma, \Delta, \Theta,$ etc. for contexts.

**Terms** are judgements $\Gamma \vdash \{ \mathcal{M} \}_{\mathcal{B}} : \mathcal{B}$, where $\Gamma$ is a **context**, $\mathcal{M}$ is a raw-term, $e \in \mathbb{N}$ is the **execution number**, and $\mathcal{B}$ is a type, generated by the following **typing rules**:

1. $\frac{n \in \mathbb{N}}{\Gamma \vdash \{ \text{nat} \}_{\mathcal{B}} : \mathcal{B}}$
2. $\frac{\forall i \in \{1, 2, \ldots, k\}. \Gamma \vdash \{ \mathcal{M}_i \}_{\mathcal{B}} : \mathcal{A}_i \ \forall n \in \mathbb{N}. \Gamma \vdash \{ \mathcal{M}_n^{\mathcal{A}} \}_{\mathcal{B}} : \mathcal{N}}{\Gamma, x : \mathcal{A}, \Delta \vdash \{ \text{case}(x \mathcal{M}_1 \mathcal{M}_2 \ldots \mathcal{M}_k)_{\mathcal{B}} \}_{\mathcal{B}} : \mathcal{N}}$
3. $\frac{\Gamma \vdash \{ \mathcal{M} \}_{\mathcal{B}} : \mathcal{N} \ \forall n \in \mathbb{N}. \Gamma \vdash \{ \mathcal{M}_n^{\mathcal{A}} \}_{\mathcal{B}} : \mathcal{N}}{\Gamma \vdash \{ \text{case}(\mathcal{M})_{\mathcal{B}} \}_{\mathcal{B}} : \mathcal{N}}$
4. $\frac{\Gamma, x : \mathcal{A} \vdash \{ \mathcal{M} \}_{\mathcal{B}} : \mathcal{B} \ \Gamma \vdash \{ \text{succ} \}_{\mathcal{B}} : \mathcal{N}}{\Gamma \vdash \{ \lambda x^{\mathcal{A}}. \mathcal{M} \}_{\mathcal{B}} : \mathcal{A} \Rightarrow \mathcal{B}}$

Note that a deduction (tree) for each term is unique. A term $\Gamma \vdash \{ \mathcal{M} \}_{\mathcal{B}} : \mathcal{B}$ is **finite** if so is the length (i.e., the number of letters) of $\mathcal{M}$. We omit execution numbers and the brackets $\{ \cdot \}$ (even contexts and types) of terms whenever they are not important, and identify terms up to $\alpha$-equivalence. A **value** is a term generated by the rules nat, $\lambda$, and a **configuration** is a term constructed by the rules nat, $\lambda$, $\alpha$. A **subterm** of a term $\Gamma \vdash \mathcal{M} : \mathcal{B}$ is a term that occurs in the deduction of $\Gamma \vdash \mathcal{M} : \mathcal{B}$. **Atomic terms** are the following values:

1. $\frac{n \in \mathbb{N}}{\Gamma \vdash \{ \text{nat} \}_{\mathcal{B}} : \mathcal{B}}$
2. $\frac{\Gamma, x : \mathcal{A}, \Delta \vdash \{ \text{succ} \}_{\mathcal{B}} : \mathcal{A}}{\Gamma \vdash \{ \lambda x^{\mathcal{A}}. \mathcal{M} \}_{\mathcal{B}} : \mathcal{A} \Rightarrow \mathcal{B}}$
3. $\frac{\Gamma \vdash \{ \text{pred} \}_{\mathcal{B}} : \mathcal{N} \Rightarrow \mathcal{N}}{\Gamma, x : \mathcal{A} \vdash \{ \text{cond} \}_{\mathcal{B}} : \mathcal{N} \Rightarrow \mathcal{N} \Rightarrow \mathcal{N}}$
4. $\frac{\Gamma \vdash \{ \text{itr}^{\mathcal{A}} \}_{\mathcal{B}} : (\mathcal{A} \Rightarrow \mathcal{A}) \Rightarrow \mathcal{A} \Rightarrow \mathcal{N} \Rightarrow \mathcal{A}}{\Gamma \vdash \{ \text{itr}^{\mathcal{A}} \}_{\mathcal{B}} : (\mathcal{A} \Rightarrow \mathcal{A}) \Rightarrow \mathcal{A} \Rightarrow \mathcal{N} \Rightarrow \mathcal{A}}$

where:
\[ \begin{align*}
\triangleright \ &\blacktriangleleft A \overset{\text{df}}{=} \lambda x_1^A \ldots x_k^A. \text{case}(x_1, A_1, \ldots, A_k) [n \mapsto n], A \equiv A_1 \Rightarrow A_2 \Rightarrow \cdots \Rightarrow A_k \Rightarrow N \quad \text{(we often abbreviate \(A^k\) as \(x\))} \\
\triangleright \ &\text{succ} \overset{\text{df}}{=} \lambda x^N. \text{case}(x)[0 \mapsto 0, n + 1 \mapsto n] \\
\triangleright \ &\text{pred} \overset{\text{df}}{=} \lambda x^N. \text{case}(x)[0 \mapsto 0, n + 1 \mapsto n] \\
\triangleright \ &\text{cond} \overset{\text{df}}{=} \lambda x^N y^N z^N. \text{case}(z)[0 \mapsto x^N, n + 1 \mapsto y^N] \\
\triangleright \ &\text{itr} \overset{\text{df}}{=} \lambda f^A A^x y^N. \text{case}(y)[n \mapsto f(f(x), \ldots)), n^f \overset{\text{df}}{=} f(f(\ldots(f(x)), \ldots))] \end{align*} \]

\textit{Programs} are configurations generated from atomic terms via the rules \(\lambda, \alpha\).

\begin{itemize}
\item The \(\beta\alpha\)-\textit{reduction} \(\rightarrow_{\beta\alpha}\) on terms is the \textit{contextual closure} of the union of the rules
\[ (\lambda x. M) N \rightarrow_{\beta} M[N/x] \]
\[ \text{case}(\chi)[M] \rightarrow_{\alpha_1} M_n \]
\[ \text{case}(\text{case}(\chi) M) [P] \rightarrow_{\alpha_2} \text{case}(\text{case}(\chi) M) [n \mapsto \text{case}(P_n) [Q]] \]
\[ \text{case}(\text{case}(M) [P]) [Q] \rightarrow_{\alpha_3} \text{case}(M) [n \mapsto \text{case}(P_n) [Q]] \]
\end{itemize}

where \(M[N/x]\) denotes the \textit{capture-free substitution} of \(P\) for \(x\) in \(M\). The \textit{parallel \(\beta\alpha\)-reduction} \(\equiv_{\beta\alpha}\) on terms \(M\) is the simultaneous execution of \(\rightarrow_{\beta\alpha}\) for every \(\beta\alpha\)-redex in \(M\) (possibly infinitely many). We write \(n^f(M)\) for the \textit{normal form} of each term \(M\) with respect to \(\equiv_{\beta\alpha}\).

\begin{itemize}
\item The \textit{operational semantics} \(\rightarrow_{T_\theta}\) on terms \(M\) is the simultaneous execution of \(\equiv_{\beta\alpha}\) for all subterms of \(M\) with the execution number 1, where the execution number of every subterm of \(M\) is decreased by 1 if it is non-zero.
\end{itemize}

The \textit{theory} \(\mathit{Eq}(T_\theta)\) is the equational theory that consists of judgements \(\Gamma \vdash M =_{T_\theta} M' : B\), where \(\Gamma \vdash M : B, \Gamma \vdash M' : B\) are programs of System \(T_\theta\) such that \(n^f(M) = n^f(M')\).

Note that System \(T_\theta\) is \textit{infinitary} due to the natural numbers type. This is inevitable, however, if one aims to compute “genuine values” or \textit{completely extensional} outputs, e.g., a function as a graph; this is our perspective on computation.

In this view, e.g., a lazy language that takes any \(\lambda\)-abstraction as a value can be seen as an “intensional compromise” to gain a finitary nature. Similarly, we shall focus on \textit{programs} in System \(T_\theta\), employing the syntactic sugar for atomic terms, \(\rightarrow_{\beta\alpha}\) and \(n^f\) introduced above, so that we may regain the \textit{finitary} nature of the original System \(T\).

Thus, System \(T_\theta\) computes as follows: Given a program \(\Gamma \vdash \{M\}_x : B\), it produces a \textit{finite} chain of \textit{finitary} rewriting
\[ M \rightarrow_{T_\theta} M_1 \rightarrow_{T_\theta} M_2 \rightarrow_{T_\theta} \cdots \rightarrow_{T_\theta} M_e \]  
(1)
where \(M_1, M_2, \ldots, M_e\) are configurations, and in particular \(M_e\) is a value\(^{12}\). Note that \(M\) is a finite application of values (where \textit{currying} is possibly involved), and the computation \(\overset{13}{(1)}\) is executed in the “first-applications-first-evaluated (FAFE)” fashion; e.g., if \(M \equiv (V_1 V_2)(V_3 V_4)(V_5 V_6)\) and \(e = 3\), where \(V_1, V_2, \ldots, V_6\) are values, then \(\overset{14}{(1)}\) would be of the form
\[ (V_1 V_2)(V_3 V_4)(V_5 V_6) \rightarrow_{T_\theta} V_7 V_8 V_9 \rightarrow_{T_\theta} V_7 V_{10} \rightarrow_{T_\theta} V_{11} \]
where \(V_7 \equiv n^f(V_1 V_2), V_8 \equiv n^f(V_3 V_4), V_9 \equiv n^f(V_5 V_6), V_{10} \equiv n^f(V_6 V_9), V_{11} \equiv n^f(V_7 V_{10})\).

\(^{12}\) I.e., the closure with respect to the typing rules.

\(^{13}\) We need the typing rule \(\epsilon_2\) and the reduction \(\rightarrow_{\alpha_3}\) to handle the hidden “intermediate terms” between the configurations in \(\overset{14}{(1)}\).
Example 2.2.2. Consider a program $\Gamma \vdash \text{double} : N \Rightarrow N$ that doubles a given number, where double $\equiv \text{itr}^N\text{succ}\,0$, $\text{succ}^2 \equiv \lambda x.\text{succ}(\text{succ}(x))$. Then it computes as follows:

$$
\begin{align*}
\text{double} & \rightarrow^2 \text{itr}^N(\text{tr})_0, \text{where} \quad \text{tr} \equiv \lambda x.\text{case}(x)[m \mapsto m + 2] \\
& \rightarrow_{\tau_\tau} \lambda y.\text{case}(z)[m \mapsto nf((\text{ff}([++]/f))^y)]_0 \\
& \rightarrow_{\tau_\tau} \lambda x^N\text{case}(z)[m \mapsto nf((\lambda x^N.\text{case}(x))[m \mapsto m + 2])^y]_0 \\
& \equiv \lambda z^N.\text{case}(z)[m \mapsto nf((\lambda x^N.\text{case}(x))[m \mapsto m + 2])^y]_0 \\
& \equiv \lambda z^N.\text{case}(z)[m \mapsto 2n]
\end{align*}
$$

where the last equation is derived by induction on $n \in \mathbb{N}$. Note that here we use the finitary syntactic sugar for infinite terms and infinitary rewriting, evaluating to a “genuine value”.

Below, we show that the computation $\text{tr}_0$ of System $T_\alpha$ in fact correctly works. First, by the following Proposition 2.2.3 and Theorem 2.2.4 it makes sense that $\rightarrow_{\beta_\alpha}$ is defined on terms:

Proposition 2.2.3 (Unique typing). If $\Gamma \vdash \{M\}_\beta : B$ and $\Gamma \vdash \{M\}_\beta : B'$, then $\beta \equiv \beta'$ and $B \equiv B'$.

Proof. By induction on $\Gamma \vdash \{M\}_\beta : B$.

Lemma 2.2.4 (Free variable lemma). If $\Gamma \vdash M : B$ and $x \in \mathcal{V}$ occurs free in $M$, then $x : A \in \Gamma$ for some type $A$.

Proof. By a straightforward induction on the judgement $\Gamma \vdash M : B$.

Lemma 2.2.5 (Exchange and weakening lemma). If $\Gamma, x_1 : A_1, x_2 : A_2, \ldots, x_k : A_k \vdash \{M\}_\beta : B$, then $\Gamma, x_{\sigma(1)} : A_{\sigma(1)}, x_{\sigma(2)} : A_{\sigma(2)}, \ldots, x_{\sigma(k)} : A_{\sigma(k)} \vdash \{M\}_\beta : B$ for any permutation $\sigma$ of the set $\{1, 2, \ldots, k\}$, and $\Gamma, x_1 : A_1, x_2 : A_2, \ldots, x_k : A_k, x_{k+1} : A_{k+1} \vdash \{M\}_\beta : B$ for any $x_{k+1} \in \mathcal{V}$ such that $x_{k+1} \neq x_i$ for $i = 1, 2, \ldots, k$ and type $A_{k+1}$.

Proof. By induction on the judgement $\Gamma, x_1 : A_1, x_2 : A_2, \ldots, x_k : A_k \vdash \{M\}_\beta : B$.

Lemma 2.2.6 (Substitution lemma). If $\Gamma, x : A \vdash \{P\}_\beta : B$ and $\Gamma \vdash Q : A$, then $\Gamma \vdash \{P[Q/x]\}_\beta : B$.

Proof. By a simple induction on the length of $P$:

- If $\Gamma, x : A \vdash \{A\}_\beta : N$ and $\Gamma \vdash Q : A$, then the claim trivially holds as $\Gamma \vdash \{A\}_\beta : N$ by the rule $\text{nat}$.
- If $\Gamma, x : A \vdash \{\lambda z^P.\beta\}_\beta : C \Rightarrow B$ and $\Gamma \vdash Q : A$, then we need $\Gamma \vdash \{\lambda z^P.\beta[Q/x]\}_\beta : C \Rightarrow B$. We clearly have $\Gamma, x, z, C : A \vdash \{P\}_\beta : B$; thus, $\Gamma, z : C, x : A \vdash \{P\}_\beta : B$ by the exchange lemma. We then have $\Gamma, z : C \vdash \{P[Q/x]\}_\beta : B$ by the induction hypothesis, whence we conclude that $\Gamma \vdash \{\lambda z^P.\beta[Q/x]\}_\beta : C \Rightarrow B$ by the rule $\text{abs}$.
- If $\Gamma, x : A \vdash \{\text{case}(yM_1 \ldots M_k)[M_0][M_1][\ldots]\}_\beta : N$ and $\Gamma \vdash Q : A$ with $y \neq x$, then $y : A \in \Gamma$ by the free variable lemma, $A \equiv A_1 \Rightarrow A_2 \Rightarrow \cdots \Rightarrow A_k \Rightarrow N$ for some types $A_1, A_2, \ldots, A_k$, and $\Gamma, x : A \vdash \{M_i\}_\beta : A_i$ for $i = 1, 2, \ldots, k$, $\Gamma, x : A \vdash \{M'_n\}_\beta : N$ for all $n \in \mathbb{N}$. By the induction hypothesis, $\Gamma \vdash \{M_i[Q/x]\}_\beta : A_i$ for $i = 1, 2, \ldots, k$ and $\Gamma \vdash \{M'_n[Q/x]\}_\beta : N$ for all $n \in \mathbb{N}$. Thus, $\Gamma \vdash \{\text{case}(yM_1[Q/x]\ldots[M_k[Q/x])][M_0[Q/x][M_1[Q/x][\ldots]\}_\beta : N$ by the rule $c_1$, i.e., $\Gamma \vdash \{\text{case}(yM_1 \ldots M_k)[M_0][M_1][\ldots][Q/x]\}_\beta : N$.
Theorem 2.2.7 (Subject reduction). If $\Gamma \vdash M : B$ and $M \rightarrow_{\beta\eta} R$, then $\Gamma \vdash R : B$.

Proof. By a simple induction on the structure $M \rightarrow_{\beta\eta} R$. In the following, let us write $P : Q$ for $\alpha \rightarrow \text{case}(P_\alpha)Q$.

If $M \equiv (\lambda^A.P)Q$ and $R \equiv P[Q/x]$, then $\Gamma \vdash Q : A$ and $\Gamma, x : A \vdash P : B$. By the substitution lemma, we have $\Gamma \vdash P[Q/x] : B$.

If $M \equiv \text{case}(n)(M')$ and $R \equiv M'_n$, then $B \equiv N$. From $\Gamma \vdash \text{case}(n)(M') : N$, we may conclude that $\Gamma \vdash M'_n : N$.

If $M \equiv \text{case}(\text{case}(P)(P'))(Q')$ and $R \equiv \text{case}(P)(P')Q'$, then $B \equiv N$. Then $\Gamma \vdash \text{case}(P)(P') : N$, from which we may deduce $\Gamma \vdash P : N$ and $\Gamma \vdash P_\alpha : N$ for all $n \in N$. Also, $\Gamma \vdash Q'_n : N$ for all $n \in N$, whence $\Gamma \vdash \text{case}(P_\alpha)(Q') : N$ for all $n \in N$ by $c_2$. Hence, we conclude that $\Gamma \vdash \text{case}(P)(P'); Q' : N$ by the rule $c_2$.

If $M \equiv \text{case}(\text{case}(xM_1 \ldots M_k)(M'))(N')$ and $R \equiv \text{case}(xM_1 \ldots M_k)(M'; N')$, then it is handled in a similar way to the above case.

If $M \equiv \lambda^A.P$, $B \equiv A \Rightarrow C$, $R \equiv \lambda^A.Q$, and $P \rightarrow_{\beta\eta} Q$, then $\Gamma, x : A \vdash P : C$. By the induction hypothesis, $\Gamma, x : A \vdash C : Q$. Thus, we conclude that $\Gamma \vdash \lambda^A.Q : A \Rightarrow C$ by the rule abs.

If $M \equiv \text{case}(P)(P')$ and $R \equiv \text{case}(Q)(Q')$ with $(P \rightarrow_{\beta\eta} Q \equiv P \equiv Q) \wedge (P'_0 \rightarrow_{\beta\eta} Q'_0 \equiv Q'_0) \wedge (P'_1 \rightarrow_{\beta\eta} Q'_1 \equiv Q'_1) \wedge \ldots$ in such a way that just one $\rightarrow_{\beta\eta}$ holds, then $B \equiv N$. In either case, it follows from the induction hypothesis and the rule $c_2$ that $\Gamma \vdash \text{case}(Q)(Q') : N$.

If $M \equiv \text{case}(xT_1 \ldots T_k)(T')$ with $(M_1 \rightarrow_{\beta\eta} T_1 \equiv T_1) \wedge \ldots \wedge (M_k \rightarrow_{\beta\eta} T_k \equiv T_k) \wedge (M'_0 \equiv T'_0) \wedge (M'_1 \equiv T'_1) \wedge \ldots$ in such a way that just one $\rightarrow_{\beta\eta}$ holds, then $B \equiv N$. In either case, it immediately follows from the induction hypothesis and the rule $c_1$ that $\Gamma \vdash \text{case}(xT_1 \ldots T_k)(T') : N$.

If $M \equiv PQ$, $R \equiv TS$, and $(P \rightarrow_{\beta\eta} Q \equiv S) \wedge (Q \rightarrow_{\beta\eta} S \equiv P \equiv T)$, then $\Gamma \vdash P : A \Rightarrow B$ and $\Gamma \vdash Q : A$ for some type $A$. By the induction hypothesis, if $P \rightarrow_{\beta\eta} T$, then $\Gamma \vdash T : A \Rightarrow B$ and $\Gamma \vdash S : A$; thus, $\Gamma \vdash TS : B$ by the rule abs. The other case is similar.

We have considered all the cases for $M \rightarrow_{\beta\eta} R$. 

\[\square\]
Next, we show that the parallel $\beta\vartheta$-reduction $\Rightarrow_{\beta\vartheta}$ is well-defined (Theorems 2.2.9, 2.2.10).

**Lemma 2.2.8** (Hidley-Rosen [Han94]). Let $R_1$ and $R_2$ be any binary relations on the set $T$ of terms, and let us write $\rightarrow_{R_i}$ for the contextual closure of $R_i$ for $i = 1, 2$. If $\rightarrow_{R_1}$ and $\rightarrow_{R_2}$ are both Church-Rosser, and satisfy $\forall M, P, Q \in T. M \rightarrow_{R_1}^* P \land M \rightarrow_{R_2}^* Q \Rightarrow \exists R \in T. P \rightarrow_{R_1}^* R \land Q \rightarrow_{R_2}^* R$, then $\rightarrow_{R_1 \cup R_2}$ is Church-Rosser.

*Proof.* By a simple “diagram chase”; see [Han94] for the details. ■

**Theorem 2.2.9** (CR). The $\beta\vartheta$-reduction $\rightarrow_{\beta\vartheta}$ is Church-Rosser.

*Proof.* We follow the outline of the proof of Mitschke’s theorem [Han94] (note that we cannot simply apply the theorem as the outputs of the rules $\rightarrow_{\vartheta i}$ depend on inputs). First, it is easy to see that the $\vartheta$-reduction $\rightarrow_{\vartheta} \overset{df}{=} \bigcup_{i=1}^3 \rightarrow_{\vartheta i}$ is Church-Rosser. Also, we may show that

\[ M \rightarrow_{\beta} P \land M \rightarrow_{\vartheta} Q \Rightarrow \exists R. P \rightarrow_{\vartheta}^* R \land Q \rightarrow_{\beta}^* R \tag{2} \]

holds for all terms $M, P, Q$ by the case analysis on the relation between the $\beta$, $\vartheta$-redexes in $M$:

- If the $\beta$-redex is inside the $\vartheta$-redex, then it is easy to see that the claim (2) holds.

- If the $\vartheta$-redex is inside the body of the function subterm of the $\beta$-redex, then it suffices to show that $\rightarrow_{\vartheta}$ commutes with substitution, but it is straightforward.

- If $\vartheta$-redex is inside the argument of the $\beta$-redex, then it may be duplicated by a finite number $n$ (possibly zero). Whatever the number $n$ is, the claim (2) clearly holds.

- If the $\beta$- and $\vartheta$-redexes are disjoint, then the claim (2) trivially holds.

It then follows from the claim (2) that

\[ M \rightarrow_{\beta}^* P \land M \rightarrow_{\vartheta}^* Q \Rightarrow \exists R. P \rightarrow_{\vartheta}^* R \land Q \rightarrow_{\beta}^* R \tag{3} \]

holds for all terms $M, P, Q$. Applying the Hidley-Rosen lemma [Han94] to (3) (or equivalently by the well-known “diagram chase” argument on $\rightarrow_{\vartheta}$ and $\rightarrow_{\beta}^*$), we may conclude that the $\beta\vartheta$-reduction $\rightarrow_{\beta\vartheta} = \rightarrow_{\beta} \cup \rightarrow_{\vartheta}$ is Church-Rosser. ■

**Theorem 2.2.10** (Normalization). The parallel $\beta\vartheta$-reduction $\Rightarrow_{\beta\vartheta}$ is normalizing, i.e., there is no infinite chain of $\Rightarrow_{\beta\vartheta}$.

*Proof.* By a slight modification of the proof of strong normalization of the simply-typed $\lambda$-calculus in [Han94]. ■

Thus, the normal form $nf(M)$ of every term $M$ uniquely exists. Moreover, we have:

**Theorem 2.2.11** (Normal forms are values). The normal form $nf(M)$ of every term $M$ is a value.

*Proof.* It has been shown in [AC98] during the proof to show that PCF Böhm trees are closed under composition. ■

Therefore we have shown that the operational semantics $\rightarrow_{\beta\vartheta}$ is well-defined:

**Corollary 2.2.12** (Correctness). If $\Gamma \vdash \{M\}^e : B$ is a configuration and $e > 1$ (resp. $e = 1$), then there exists a unique configuration (resp. value) $\Gamma \vdash \{M’\}^{e-1} : B$ that satisfies $M \rightarrow_{\beta\vartheta} M’$.

*Proof.* Immediate from Theorems 2.2.7, 2.2.9, 2.2.10, 2.2.11. ■

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2.3 Dynamic Semantics for System T_0

We now give a general recipe to give a dynamic semantics for System T_0 in a CCBIC:

- **Definition 2.3.1 (Structures for System T_0).** A structure for System T_0 in a CCBIC C = (C, V, E) with countably infinite products is a triple T = (N, (ω), θ), where:
  - N ∈ C
  - (ω) assigns a value n : 1 → N in C to each n ∈ N, where 1 is a terminal object
  - θ : N × N^ω → N is a value in C, where N^ω is a countably infinite product of N.

An interpretation [T]_C of System T_0 induced by T in C assigns a 0-cell [A]_T ∈ C to each type A, a 0-cell [Γ]_T ∈ C to each context Γ and a 1-cell [M]_T : [Γ]_T → [B]_T to each configuration Γ ⊢ M : B as follows:

- **Types.** [N]_T df N, [A ⇒ B]_T df [A]_T ⇒ [B]_T
- **Contexts.** [Γ]_T df 1, [Γ, x : A]_T df [Γ]_T × [A]_T
- **Configurations.** [Γ ⊢ n : N]_T df E^n(Γ[T]_Γ), [Γ ⊢ λx.M : A ⇒ B]_T df Λ[x : A ⊢ M : B]_T, [Γ ⊢ MN : B]_T df ([Γ ⊢ M : A ⇒ B]_T, [Γ ⊢ N : A]_T), [Γ ⊢ case(xM) : N]_T df E^ω(Γ[T]_Γ, [Γ ⊢ M_1 : N]_T, [Γ ⊢ M_2 : N]_T, ..., df θ), where Λ, App are the currying and evaluation in C, respectively, and Γ ⊢ x : A[T]_Γ df [Γ]_T ⊢ [A]_T is a projection in C.

Note that the interpretation E^ω([Γ]_Γ) coincides with the standard interpretation of the operational equivalence Eq(T_0) in a CCC V [Pit01, Cro93, Jac99].

Now, we may reduce the DCP for System T_0 to the following:

- **Definition 2.3.2 (PDCP).** The interpretation [T]_C of System T_0 induced by a structure T in a CCBIC C = (C, V, E) is said to satisfy the pointwise dynamic correspondence property (PDCP) if E([Γ ⊢ (λx.A)V]W : B[T]_Γ) = [Γ ⊢ U : B]_T for every rewriting (λx.A)VW →_T U, where V, W, U are values, and E evaluates configurations in the "first-concatenations-first-evaluated (FCFE)" fashion, i.e., it evaluates the 1-cells corresponding to subterms with the execution number 1.

For instance, the evaluation (v1 ⊢ v2) ∘ ((v3 ⊢ v4) ⊢ (v5 ⊢ v6)) E v7 v8 v9 v10 v11, where v_i’s are values such that E(v1 v2) = v7, E(v3 v4) = v8, E(v5 v6) = v9, E(v8 v9) = v10, E(v7 v10) = v11, is FCFE fashion (which corresponds to the syntactic example given previously).

- **Theorem 2.3.3 (Standard semantics for System T_0).** The interpretation [T]_C of System T_0 induced by a structure T in a CCBIC C satisfies DCP for configurations if it satisfies PDCP.

Proof. Assume that [T]_C satisfies the PDCP. Then clearly, the operational semantics →_T and the evaluation E compute the corresponding applications and concatenations. Thus, it remains for the DCP to show [Γ ⊢ (λx.A)V]W : B[T]_Γ ≠ [Γ ⊢ U : B]_T for every rewriting (λx.A)VW →_T U on values. But it clearly holds as each concatenation is not a value.

14Γ ⊢ x : A is not a configuration, but we need it to interpret the application xM.
3 Dynamic Games and Strategies

The main idea of dynamic games and strategies is to introduce a distinction between internal and external moves; internal moves constitute “internal communication” between dynamic strategies, and they are to be a posteriori hidden by the hiding operation. Conceptually, internal moves are “invisible” to Opponent because they represent how Player internally calculates the next external move.

Dynamic games and strategies are based on the variant in [AM99], which we call static games and strategies (more in general, to distinguish our “dynamic concepts” from the existing ones, we usually add the word static in front of the corresponding notions in [AM99]; e.g., static arenas, static legal positions, etc.), because it combines good points of the two best-known variants: AJM-games [AJM00] and HO-games [HO00]. It interprets the linear decomposition of implication [Gir87], and it is flexible enough to model a wide range of programming features [AM99]. We have chosen this variant with the hope that our framework is also applicable to various logics and programming languages. This section introduces dynamic games and strategies.

3.1 Dynamic Arenas and Legal Positions

Just as static games [AM99], our games are based on two preliminary concepts: arenas and legal positions. An arena defines basic components of a game, which in turn induces a set of legal positions that specifies basic rules of the game.

Definition 3.1.1 (Dynamic arenas). A (dynamic) arena is a triple \( G = (M_G, \lambda_G, \triangleright_G) \), where:

- \( M_G \) is a set, whose elements are called moves
- \( \lambda_G : M_G \rightarrow \{O, P\} \times \{Q, A\} \times \mathbb{N} \), where \( O, P, Q, A \) are some fixed symbols, is a function called the labeling function that satisfies \( \sup(\{\lambda_G^N(m) | m \in M_G\}) \in \mathbb{N} \)
- \( \triangleright_G \subseteq (\{\ast\} \times M_G) \times M_G \), where \( \ast \) is an arbitrary element, is a relation, called the enabling relation, satisfying:
  - (E1) If \( \ast \triangleright_G m \), then \( \lambda_G(m) = (O, Q, 0) \) and \( n = \ast \) whenever \( n \triangleright_G m \)
  - (E2) If \( m \triangleright_G n \) and \( \lambda_G^A(n) = A \), then \( \lambda_G^A(m) = Q \) and \( \lambda_G^N(m) = \lambda_G^N(n) \)
  - (E3) If \( m \triangleright_G n \) and \( m \neq \ast \), then \( \lambda_G^{OP}(m) \neq \lambda_G^{OP}(n) \)
  - (E4) If \( m \triangleright_G n \), \( m \neq \ast \) and \( \lambda_G^{OP}(m) \neq \lambda_G^{OP}(n) \), then \( \lambda_G^{OP}(m) = O \) (and \( \lambda_G^{OP}(n) = P \))

in which \( \lambda_G^{OP} \equiv \lambda_G; \pi_1 : M_G \rightarrow \{O, P\}, \lambda_G^{OA} \equiv \lambda_G; \pi_2 : M_G \rightarrow \{Q, A\}, \lambda_G^{NI} \equiv \lambda_G; \pi_3 : M_G \rightarrow \mathbb{N} \), a move \( m \in M_G \) is called initial if \( \ast \triangleright m \), an O-move (resp. a P-move) if \( \lambda_G^{OP}(m) = O \) (resp. if \( \lambda_G^{OP}(m) = P \)), a question (resp. an answer) if \( \lambda_G^{OA}(m) = Q \) (resp. if \( \lambda_G^{OA}(m) = A \)), and internal (or more specifically \( \lambda_G^{NI}(m) \)-internal) (resp. external) if \( \lambda_G^{NI}(m) > 0 \) (resp. if \( \lambda_G^{NI}(m) = 0 \)). A sequence \( s \in M_G^* \) of moves is called d-complete \( (d \in \mathbb{N} \cup \{\omega\}) \) if it ends with an external or d'-internal move with \( d' > d \), where \( \omega \) denotes the least transfinite ordinal number.

That is, a dynamic arena is a static arena [AM99] equipped with the additional degree of internality \( \lambda_G^{NI} \) on moves and satisfying some axioms about it. We need all natural numbers for \( \lambda_G^{NI} \), not only the internal/external (I/E) distinction, to define a step-by-step execution of the hiding operation later. From the opposite angle, dynamic arenas are a generalization of static arenas: A static arena is a dynamic arena whose moves are all external.

Recall that a static arena determines possible moves of a game, each of which is an Opponent’s or Player’s question or answer, and specifies which move \( n \) can be made for each move \( m \) by the relation \( m \triangleright_G n \) (\( \ast \triangleright_G m \) means that \( m \) can initiate a play). Its axioms are the following:
E1 sets the convention that an initial move must be an Opponent’s question.

The first requirement of E2 states that an answer must be made for a question.

E3 mentions that an O-move must be made for a P-move, and vice versa.

The additional (dynamic) axioms are intuitively natural:

- The condition on the labeling function requires an upper bound of degrees of internality of the moves.
- E1 adds the equation $\lambda^N_G(m) = 0$ for all initial moves $m \in M_G$ since Opponent cannot “see” internal moves.
- The second requirement of E2 states that the degree of internality between a “QA-pair” must be the same.
- E4 determines that only Player can make a move for a previous move if they have different degrees of internality because internal moves are “invisible” to Opponent.

Convention. From now on, the word arenas refers to dynamic arenas by default.

Given an arena, we are interested in certain finite sequences of its moves equipped with a justifying relation:

Definition 3.1.2 (Justified sequences [HO00, AM99, McC98]). A justified sequence (j-sequence) in an arena $G$ is a finite sequence $s \in M_G^*$, in which each non-initial move $n$ is associated with (or points at) a unique move $m$, called the justifier of $n$ in $s$, that occurs previously in $s$ and satisfies $m \vdash_G n$. We say that $n$ is justified by $m$, or there is a pointer from $n$ to $m$.

Notation. We usually write $J_s(n)$ for the justifier of a non-initial move $n$ in a j-sequence $s$, where $J_s$ denotes the “function of pointers” in $s$, and $J_G$ for the set of j-sequences in an arena $G$.

The idea is that each non-initial move in a j-sequence must be made for a specific previous move, called its justifier.

We now consider justifiers from the “external viewpoint”:

Definition 3.1.3 (External justifiers). Let $G$ be an arena, and assume $s \in J_G$, $d \in \mathbb{N} \cup \{\omega\}$. Each non-initial move $n$ in $s$ has a unique sequence of justifiers $nn_1m_2 \ldots m_km (k \geq 0)$, i.e., $J_s(n) = m_1, J_s(m_1) = m_2, \ldots, J_s(m_{k-1}) = m_k, J_s(m_k) = m$, such that $m_1, m_2, \ldots, m_k$ are $d$-internal with $0 < d' \leq d$ but $n$ is not. We call $m$ the $d$-external justifier of $n$ in $s$.

Notation. We usually write $J_s^{\omega d}(n)$ for the $d$-external justifier of $n$ in a j-sequence $s$.

Note that $d$-external justifiers are a simple generalization of justifiers: 0-external justifiers coincide with justifiers (as there is no “0-internal” move). More generally, $d$-external justifiers are intended to be justifiers after executing the (1-) hiding operation $d$-times; this intuition is made precise below.

Definition 3.1.4 (External justified subsequences). Let $s$ be a j-sequence in an arena $G$ and $d \in \mathbb{N} \cup \{\omega\}$. The $d$-external justified (j-) subsequence $\mathcal{H}^d_G(s)$ of $s$ is obtained from $s$ by deleting $d$-internal moves, $0 < d' \leq d$, equipped with the pointers $J_s^{\omega d}$ (more precisely, $\mathcal{J}_{\mathcal{H}^d_G(s)}$ is a restriction of $J_s^{\omega d}$).

Definition 3.1.5 (Hiding operation on arenas). Let $d \in \mathbb{N} \cup \{\omega\}$. The $d$-hiding operation $\mathcal{H}^d$ on arenas is defined as follows. Given an arena $G$, the arena $\mathcal{H}^d(G)$ is given by:
Proof. Function \( \lambda_{H^d(G)} \) are well-defined. Now, let us verify the axioms for the enabling relation:

\[ m \vdash_{H^d(G)} n \text{ for all } m, n \in \mathbb{N}, m_1, m_2, \ldots, m_{2k-1}, m_{2k} \in M_G \setminus M_{H^d(G)}, m \vdash_G m_1 \wedge m_1 \vdash_G m_2 \wedge \cdots \wedge m_{2k-1} \vdash_G m_{2k} \wedge m_{2k} \vdash_G n \]

(I.e., if \( k = 0 \), then \( m \vdash_G n \).

\[ \lambda_{H^d(G)}(n) = \lambda_G(n) \] for all \( n \in \mathbb{N} \)

\[ m \vdash_{H^d(G)} n \text{ if and only if } \exists k \in \mathbb{N}, m, m_1, m_2, \ldots, m_{2k-1}, m_{2k} \in M_G \setminus M_{H^d(G)}, m \vdash_G m_1 \wedge m_1 \vdash_G m_2 \wedge \cdots \wedge m_{2k-1} \vdash_G m_{2k} \wedge m_{2k} \vdash_G n \]

Lemma 3.1.6 (Closure of arenas and j-sequences under hiding). If \( G \) is an arena, then so is \( H^d(G) \) for all \( d \in \mathbb{N} \) such that \( H^d(G) = G \) and \( H^d(s) \in J_{H^d(G)} \) for all \( s \in J_G \).

Proof. The case \( d = 0 \) is immediate, so assume \( d > 0 \). Clearly, the set of moves and the labeling function are well-defined. Now, let us verify the axioms for the enabling relation:

- (E1) Note that \( \star \vdash_{H^d(G)} m \Leftrightarrow \star \vdash_G m \) (\( \Rightarrow \) is immediate; \( \Rightarrow \) holds by E4 on \( G \) as initial moves are external). Thus, if \( \star \vdash_{H^d(G)} m \), then \( \lambda_{H^d(G)}(m) = \lambda_G(m) = (O, Q, 0) \) and \( n \vdash_{H^d(G)} m \Leftrightarrow n = \star \).

- (E2) Assume \( m \vdash_{H^d(G)} n \) and \( \lambda_{H^d(G)}(n) = A \). We proceed by a case analysis. If \( m \vdash_G n \), then \( \lambda_{H^d(G)}(n) = \lambda_{H^d(G)}(m) = Q \) and \( \lambda_{H^d(G)}(m) = \lambda_{G}(m) \circ d = \lambda_{G}(n) \circ d = \lambda_{H^d(G)}(n) \). Otherwise, i.e., there are some \( k \in \mathbb{N}^+ \) and \( m_1, m_2, \ldots, m_{2k} \in M_G \setminus M_{H^d(G)} \) such that \( m \vdash_G m_1 \wedge \vdash_G m_2 \wedge \cdots \wedge m_{2k-1} \vdash_G m_{2k} \wedge m_{2k} \vdash_G n \). Then we have

- (E3) Assume \( m \vdash_{H^d(G)} n \) and \( m \neq \star \). Again, we apply a case analysis. If \( m \vdash_G n \), then \( \lambda_{H^d(G)}(m) = \lambda_{G}(m) \neq \lambda_{G}(n) \). If \( m \vdash_G m_1 \wedge m_1 \vdash_G m_2 \wedge \cdots \wedge m_{2k-1} \vdash_G m_{2k} \wedge m_{2k} \vdash_G n \) for some \( k \in \mathbb{N}^+ \), \( m_1, m_2, \ldots, m_{2k} \in M_G \setminus M_{H^d(G)} \), then \( \lambda_{H^d(G)}(m) = \lambda_{G}(m) = \lambda_{G}(m_2) = \lambda_{G}(m_4) = \cdots = \lambda_{G}(m_{2k}) = \lambda_{H^d(G)}(n) \).

- (E4) Assume \( m \vdash_{H^d(G)} n \) and \( m \neq \star \). Then we have \( \lambda_{H^d(G)}(m) \neq \lambda_{G}(m) \). Thus, again by the same case analysis, if \( m \vdash_G n \), then it is trivial; otherwise, there are some \( k \in \mathbb{N}^+ \), \( m_1, m_2, \ldots, m_{2k} \in M_G \setminus M_{H^d(G)} \) with the same property as in E3, \( \lambda_{H^d(G)}(m) = \lambda_{G}(m) = 0 \) by E3 on \( G \) since \( \lambda_{H^d(G)}(m) \neq \lambda_{H^d(G)}(m_1) \).

Hence, we have shown that the structure \( H^d(G) \) forms a well-defined arena.

Next, let \( s \in J_G \); we have to show \( H^d(s) \in J_{H^d(G)} \). Assume that \( m \) is an occurrence of a non-initial move in \( H^d(s) \). By the definition, the justifier \( J_{H^d(G)}(m) = m_0 \) occurs in \( H^d(s) \). If \( m \) is a P-move, then the sequence of justifiers \( m_0 \vdash_G m_1 \vdash_G \cdots \vdash_G m_k \vdash \vdash_G n \) satisfies even \( (k) \) by E3 and E4 on \( G \), so that \( m_0 \vdash_{H^d(G)} m \) by the definition. If \( m \) is an O-move, then its justifier \( J_{G}(m) = n \) satisfies \( \lambda_{H^d(G)}(n) = \lambda_{H^d(G)}(m) \) by E4 on \( G \), and so \( n \vdash_{H^d(G)} m \) by the definition. Since \( m \) is arbitrary, we have shown that \( H^d(s) \in J_{H^d(G)} \), completing the proof.

Proposition 3.1.7 (Stepwise hiding on arenas). For any \( i \in \mathbb{N} \), let \( \mathcal{H}^i \) denote the i-times iteration of the 1-hiding operation \( \mathcal{H}^1 \) on arenas. Then \( \mathcal{H}^i = \mathcal{H}^i \) for all \( i \in \mathbb{N} \).
Proof. Let an arbitrary arena \( G \) be fixed; we show \( \tilde{\mathcal{H}}^i(G) = \mathcal{H}^i(G) \) for all \( i \in \mathbb{N} \) by induction on \( i \). The base case \( i = 0 \) is trivial. For the inductive step \( i + 1 \), note that \( \tilde{\mathcal{H}}^{i+1}(G) = \mathcal{H}^i(\tilde{\mathcal{H}}^i(G)) = \mathcal{H}^i(\mathcal{H}^{i-1}(G)) \) by the induction hypothesis; thus, it suffices to show \( \mathcal{H}^{i+1}(G) = \mathcal{H}^i(\mathcal{H}^i(G)) \).

For the sets of moves, we clearly have:

\[
M_{\mathcal{H}^{i+1}(G)} = \{ m \in M_G | \lambda_{G}^i(m) = \emptyset \}
\]

\[
= \{ m \in M_{\mathcal{H}^i(G)} | \lambda_{G}^i(m) = \emptyset \}
\]

\[
= \{ m \in M_{\mathcal{H}^{i+1}(G)} | \lambda_{G}^i(m) = \emptyset \}
\]

\[
= M_{\mathcal{H}^i(\mathcal{H}^i(G))}.
\]

Next, the labeling functions clearly coincide:

\[
\lambda_{\mathcal{H}^{i+1}(G)} = \lambda_{G}^i \circ \lambda_{M_{\mathcal{H}^{i+1}(G)}}
\]

\[
= \lambda_{M_{\mathcal{H}^i(G)}} \circ \lambda_{M_{\mathcal{H}^{i+1}(G)}}
\]

\[
= \lambda_{\mathcal{H}^i(\mathcal{H}^i(G))}.
\]

Finally, for the enabling relations between \( m, n \), if \( m = * \), then it is trivial: \( * \models_{\mathcal{H}^{i+1}(G)} n \iff * \models_{\mathcal{H}^i(G)} n \). Therefore assume \( m \neq * \); then we have:

\[
m \models_{\mathcal{H}^{i+1}(G)} n
\]

\[
\iff \exists k \in \mathbb{N}, m_1, m_2, \ldots, m_{2k} \in M_G \models_{\mathcal{H}^{i+1}(G)} m_1 \land m_1 \models_{\mathcal{H}^i(G)} m_2 \land \ldots \land m_{2k-1} \models_{\mathcal{H}^i(G)} m_{2k} \land m_{2k} \models_{\mathcal{H}^i(G)} n
\]

\[
m \models_{\mathcal{H}^i(G)} n
\]

\[
\iff m \models_{\mathcal{H}^i(G)} n
\]

which completes the proof.

Thus, we focus on \( \mathcal{H}^i \): From now on, we write \( \mathcal{H} \) for \( \mathcal{H}^1 \) and call it the hiding operation on arenas; \( \mathcal{H}^i \) for each \( i \in \mathbb{N} \) denotes the \( i \)-times iteration of \( \mathcal{H} \).

We may establish a similar inductive property for the hiding operation on \( j \)-sequences:

\[\textbf{Proposition 3.1.8} \text{ (Stepwise hiding on justified sequences). Let } s \text{ be a } j \text{-sequence in an arena } G. \text{ Then for all } i \in \mathbb{N} \text{ we have } \mathcal{H}^{i+1}_G(s) = \mathcal{H}^i_{\mathcal{H}^i(G)}(\mathcal{H}^i_G(s)).\]

\text{Proof.} \text{ First, note that } \mathcal{H}^{i+1}_G(s), \mathcal{H}^i_{\mathcal{H}^i(G)}(\mathcal{H}^i_G(s)) \in J_{\mathcal{H}^{i+1}(G)} \text{ by Lemma 3.1.6 and Proposition 3.1.7. We show the equation by induction on } i \in \mathbb{N}. \text{ The base case } i = 0 \text{ is trivial.}

\text{Consider the inductive step } i + 1. \text{ Note that } \mathcal{H}^{i+1}_G(s) \text{ is obtained from } s \text{ by deleting all the moves } m \text{ such that } 1 \leq \lambda^i_G(m) \leq i + 1, \text{ equipped with the pointers } J^i_s\circ(i+1). \text{ On the other hand, } \mathcal{H}^i_{\mathcal{H}^i(G)}(\mathcal{H}^i_G(s)) \text{ is obtained from } \mathcal{H}^i_G(s) \text{ by deleting all the moves } m \text{ such that } \lambda^i_{\mathcal{H}^i(G)}(m) = 1, \text{ equipped with the pointers } J^i_s\circ(i+1) = J^i_s\circ(i+1). \text{ Note that } \lambda^i_{\mathcal{H}^i(G)}(m) = 1 \Rightarrow \lambda^i_G(m) = i + 1, \text{ and } \mathcal{H}^i_G(s) \text{ is obtained from } s \text{ by deleting all the moves } m \text{ with } 1 \leq \lambda^i_G(m) \leq i. \text{ Hence, they are in fact the same } j \text{-sequence in the arena } \mathcal{H}^{i+1}(G).\]
Definition 3.1.9 (Point-wise hiding operation). Let $s$ be a justified sequence in an arena $G$. We define the point-wise hiding operation $\hat{H}_G$ on each move $m$ and the pointers to $m$ in $s$ by:

$$\hat{H}_G^s(m) = \begin{cases} e, & \text{the pointers to } m \text{ changed to pointing } J_s(m) \text{ if } m \text{ is } 1\text{-internal} \\ m, & \text{the pointers to } m \text{ unchanged otherwise.} \end{cases}$$

Furthermore, for any subsequence $t = m_1m_2 \ldots m_k$ of $s$, $\hat{H}_G^s(t)$ is defined to be the result of applying $\hat{H}_G^s$ to $m_i$ for $i = 1, 2, \ldots, k$.

Note that the point-wise hiding operation $\hat{H}_G^s$ makes sense only in the context of $s$; it affects some part of $s$. The point here is that the usual hiding operation on j-sequences can be executed in the “point-wise” fashion (in any order):

Definition 3.1.10 (Homomorphism theorem for hiding on j-sequences). For any j-sequence $s$ in an arena $G$, we have $H_G(s) = \hat{H}_G^s(s)$.

Proof. It suffices to establish, for each j-sequence $s = m_1m_2 \ldots m_k$ in $G$, the equation

$$H_G(s) = \hat{H}_G^s(m_1)\hat{H}_G^s(m_2)\ldots\hat{H}_G^s(m_k).$$

First, it is clear by the definition that $H_G(s)$ and $\hat{H}_G^s(m_1)\hat{H}_G^s(m_2)\ldots\hat{H}_G^s(m_k)$ are both the subsequence of $s$ obtained from $s$ by deleting 1-internal moves. Thus, it suffices to show that each move $m$ in $\hat{H}_G^s(m_1)\hat{H}_G^s(m_2)\ldots\hat{H}_G^s(m_k)$ points to $J_s^{\ominus 1}(m)$.

Let $m$ be any non-1-internal move in $s$. For the pointer from $m$ in $\hat{H}_G^s(m_1)\hat{H}_G^s(m_2)\ldots\hat{H}_G^s(m_k)$, it suffices to consider the subsequence $n_1n_2\ldots n_m$ of $s$, where $n_1, n_2, \ldots, n_1$ are 1-internal but $n$ is not, satisfying $J_s(n_1) = n_1, J_s(n_2) = n_1-1, \ldots, J_s(n_2) = n_1, J_s(n_3) = n$ because the operation on the other moves will not affect the pointers from $m$. Applying the operation $\hat{H}_G^s$ to $n_1, n_2, \ldots, n_1$ in any order, the resulting pointer from $m$ clearly points to $n$, which is $J_s^{\ominus 1}(m)$.

By virtue of the proposition, we may identify the operations $H_G$ and $\hat{H}_G$; and from now on, we shall not notationally distinguish them, and use only the former. As a result, what we have established is the “point-wise” procedure to execute the hiding operation on j-sequences, in which the order of moves to apply the “point-wise operation” is irrelevant. In particular, we now have $H_G^d(s) = H_G^d(s)H_G^d(t)$ for any arena $G$, $d \in \mathbb{N} \cup \{\omega\}$, $st \in J_G$, which provides a convenient framework for the rest of the paper.

Next, let us recall the notion of “relevant part” of previous moves for each move in a j-sequence, called views:

Definition 3.1.11 (Views). Given a j-sequence $s$ in an arena $G$, we define the Player view (P-view) $|s|_G$ and the Opponent view (O-view) $|s|_G$ by induction on the length of $s$ as follows:
\[ [\epsilon]_G \overset{df}{=} \epsilon \]
\[ [sm]_G \overset{df}{=} [s]_{G,m}, \text{ if } m \text{ is a P-move} \]
\[ [sm]_G \overset{df}{=} \epsilon, \text{ if } m \text{ is initial} \]
\[ [smtn]_G \overset{df}{=} [s]_{G,mm}, \text{ if } n \text{ is an O-move with } j_{smtn}(n) = m \]
\[ [\epsilon]_G \overset{df}{=} \epsilon \]
\[ [sm]_G \overset{df}{=} [s]_{G,m}, \text{ if } m \text{ is an O-move} \]
\[ [smtn]_G \overset{df}{=} [s]_{G,mm}, \text{ if } n \text{ is a P-move with } j_{smtn}(n) = m. \]

where the justifiers of the remaining non-initial moves are unchanged (it is well-defined in a legal position by visibility \cite{HO00, AM99} below). We often omit the subscript \( G \) in \([\_]_G \) and \([\_]_G \) when it is obvious.

The idea behind this definition is as follows. Given a “position” or prefix \( tm \) of a j-sequence \( s \) in an arena \( G \) such that \( m \) is a P-move (resp. an O-move), the P-view \([t]\) (resp. the O-view \([t]\)) is intended to be the currently “relevant” part of \( t \) for Player (resp. Opponent). That is, Player (resp. Opponent) is concerned only with the last O-move (resp. P-move), its justifier and that justifier’s “concern”, i.e., P-view (resp. O-view), which then recursively proceeds.

We are now ready to introduce a dynamic generalization of (static) legal positions \cite{AM99}:

\begin{definition} (Dynamic legal positions) A (dynamic) legal position in an arena \( G \) is a sequence \( s \in M_G^* \) that satisfies:
\begin{itemize}
  \item **Justification.** \( s \) is a j-sequence in \( G \).
  \item **Alternation.** If \( s = s_1mnss_2 \), then \( \lambda_G^{OP}(m) \neq \lambda_G^{OP}(n) \).
  \item **Generalized visibility.** If \( s = tmu \) with \( m \) non-initial and \( d \in \mathbb{N} \cup \{\omega\} \) satisfy \( \lambda_G^{i}(m) = 0 \lor \lambda_G^{i}(m) > d \), then \( j_{smtn}(m) \) occurs in \([H_G^d(t)]_{jH_G^d(G)}\) if \( m \) is a P-move, and it occurs in \([H_G^d(t)]_{jH_G^d(G)}\) if \( m \) is an O-move.
  \item **IE-switch.** If \( s = s_1mnss_2 \) with \( \lambda_G^{jH_G^d}(m) \neq \lambda_G^{jH_G^d}(n) \), then \( m \) is an O-move (and \( n \) is a P-move).
\end{itemize}

The set of dynamic legal positions in \( G \) is denoted by \( L_G \).

Recall that a static legal position is a finite sequence of moves that satisfies justification, alternation and visibility (i.e., generalized visibility only for \( d = 0 \)). It specifies the basic rules of a static game in the sense that every development or valid position of the game (see Definition \ref{def:validity} below) must be its legal position (but the converse does not necessarily hold):

\begin{itemize}
  \item In a play of the static game, Opponent always makes the first move by a question, and then Player and Opponent alternately play (by alternation), in which every non-initial move must be made for a specific previous move (by justification).
  \item The justifier of each non-initial move must belong to the “relevant” part of the previous moves (by visibility).
\end{itemize}

For dynamic legal positions, we add natural axioms:
Generalized visibility is a natural generalization of visibility; it requires that visibility holds after any iteration of the hiding operation.

IE-switch states that only Player can change the degree of internality during a play because internal moves are “invisible” to Opponent.

Also, note that a dynamic legal position in a static arena is automatically a static legal position.

Convention. From now on, the word legal positions refers to dynamic legal positions by default.

3.2 Dynamic Games

The last preliminary notion to define dynamic games is threads [AM99, McC98]. In a legal position, there may be several initial moves; the legal position consists of chains of justifiers initiated by such initial moves, and chains with the same initial move form a thread. Formally,

Definition 3.2.1 (Threads [AM99, McC98]). Let $G$ be an arena, and $s \in LG$. Assume that $m$ is an occurrence of a move in $s$. The chain of justifiers from $m$ is a sequence $m_0m_1 \ldots m_k \in MG_k$ such that $k \geq 0$, $m_k = m$, $J^s_k(m_k) = m_{k-1}$, $J^s_k(m_{k-1}) = m_{k-2}, \ldots, J^s_k(m_1) = m_0$, and $m_0$ is initial. In this case, we say that $m$ is hereditarily justified by $m_0$. The subsequence of $s$ consisting of the chains of justifiers in which $m_0$ occurs is called the thread of $m_0$ in $s$. An occurrence of an initial move is often called an initial occurrence.

Notation. We write $s \upharpoonright I$, where $s \in LG$, for an arena $G$ and $I$ is a set of initial occurrences in $s$, for the subsequence of $s$ consisting of threads of initial occurrences in $I$; we rather write $s \upharpoonright m$ for $s \upharpoonright \{m\}$.

We are now ready to define the notion of dynamic games:

Definition 3.2.2 (Dynamic games). A (dynamic) game is a quadruple $G = (MG, \lambda G, \cdash G, PG)$, where:

- The triple $(MG, \lambda G, \cdash G)$ forms an arena.
- $PG$ is a subset of $LG$ whose elements are called (valid) positions in $G$ that satisfies:
  
  - (V1) $PG$ is non-empty and prefix-closed (i.e., $sm \in PG \Rightarrow s \in PG$).
  - (V2) If $s \in PG$ and $I$ is a set of initial occurrences in $s$, then $s \upharpoonright I \in PG$.
  - (V3) For any $sm, s'm' \in P^{odd}_G, i \in \mathbb{N}$ such that $i < \lambda^N_G(m) = \lambda^N_G(m')$, if $H^i_G(s) = H^i_G(s')$, then $m = m'$ and $J^{s'm'}_m(m) = J^{s'm'}_m(m')$.

A play in $G$ is a (finitely or infinitely) increasing (with respect to $\preceq$) sequence $s, m_1, m_1m_2, \ldots$ of valid positions in $G$.

Note that a dynamic game is a static game [AM99] that satisfies the additional axiom V3. Or conversely, a static game is a dynamic game whose moves are all external.

V1 talks about the natural phenomenon that each non-empty “moment” of a play must have the previous “moment”, while V2 corresponds to the idea that a play consists of several “subplays” developed in parallel. In addition, V3 is to enable Player to “play alone” for the internal part of a play since Opponent cannot “see” moves there.

Convention. From now on, games refer to dynamic games by default, and a game is called static if its moves are all external.
Definition 3.2.3 (Well-opened games [AM99, McCo98]). A game $G$ is well-opened if $sm \in P_G$ with $m$ initial implies $s = \epsilon$.

Notation. Given a function $f : X \to Y$ and a game $G$ such that $M_G \subseteq X$, we write $f \mid G$, $f \mid G$ for the restrictions $f \mid M_G$, $f \mid (X \setminus M_G)$, respectively.

Definition 3.2.4 (Subgames). A subgame of a game $G$ is a game $H$ that satisfies $M_H \subseteq M_G$, $\lambda_H = \lambda_G \mid H$, $H \subseteq \tau \setminus G \mid (\{s\} + M_H) \times M_H$ and $P_H \subseteq P_G$. In this case we write $H \leq G$.

We are now ready to define the hiding operation on games:

Definition 3.2.5 (Hiding operation on games). For each $d \in \mathbb{N} \cup \{\omega\}$, the $d$-hiding operation $H^d$ on games is defined as follows. Given a game $G$, the game $H^d(G)$ is defined by:

- $(M_{H^d(G)}, \lambda_{H^d(G)}, \tau')$ is the arena $H^d(G)$
- $P_{H^d(G)}(s) = \{H^d(s) \mid s \in P_G\}$

We write $H$ for $H^1$, and call it the hiding operation on games; $H^i$ denotes the $i$-times iteration of $H$ for all $i \in \mathbb{N}$.

Theorem 3.2.6 (Closure of games under hiding). For any game $G$, $H^d(G)$ forms a well-defined game for all $d \in \mathbb{N} \cup \{\omega\}$. Moreover, if $H \leq G$, then $H^d(H) \leq H^d(G)$ for all $d \in \mathbb{N} \cup \{\omega\}$.

Proof. Based on Lemma 3.1.6 it suffices to show that every j-sequence in $P_{H^d(G)}$ satisfies alternation, generalized visibility and IE-switch, and $P_{H^d(G)}$ satisfies the axioms V1, V2, V3.

For alternation, assume $s_1 \ldots m m s_2 \in P_{H^d(G)}$; we have to show $\lambda_{H^d(G)}^\text{op}(m) \neq \lambda_{H^d(G)}^\text{op}(n)$. We have $H^d_G(t_1 \ldots m m t_2) = s_1 \ldots m m s_2$ for some $t_1 \ldots m m t_2 \in P_G$, where $\lambda_{H^d(G)}(t_1) = s_1$, $\lambda_{H^d(G)}(t_2) = s_2$, $\lambda_{H^d(G)}(m m t_2) = \epsilon$. Note that $\lambda_G^\text{op}(m), \lambda_G^\text{op}(n) > d$ and $0 < \lambda_G^\text{op}(m) \leq d$ for $i = 1, 2, \ldots, k$. By E3 and E4 on $G$, $k$ must be an even number, and so $\lambda_{H^d(G)}^\text{op}(m) = \lambda_G^\text{op}(m) = \lambda_G^\text{op}(m_2) = \lambda_G^\text{op}(m_4) = \cdots = \lambda_G^\text{op}(m_k) \neq \lambda_G^\text{op}(n) = \lambda_{H^d(G)}^\text{op}(n)$.

For generalized visibility, let $tmu \in P_{H^d(G)}$ with $m$ non-initial. We have to show, for each $e \in \mathbb{N} \cup \{\omega\}$, that if $tmu$ is $e$-complete, then:

- if $m$ is a P-move, then the justifier $(J^e_s)^{\ominus e}(m)$ occurs in $[H^d_{H^d(G)}(t)]_{H^e(H^d(G))}$
- if $m$ is an O-move, then the justifier $(J^e_s)^{\ominus e}(m)$ occurs in $[H^d_{H^d(G)}(t)]_{H^e(H^d(G))}$

Since the case $d = \omega$ is trivial, assume $d \in \mathbb{N}$. Also, since $tmu$ is finite, we may assume without loss of generality that $e \in \mathbb{N}$. Note that the condition is then equivalent to:

- if $m$ is a P-move, then the justifier $J^e_s(\omega^e(m))$ occurs in $[H^d_{H^d(G)}(t')]_{H^{d+e}(G)}$
- if $m$ is an O-move, then the justifier $J^e_s(\omega^e(m))$ occurs in $[H^d_{H^d(G)}(t')]_{H^{d+e}(G)}$

where $t'mu \in P_G$ such that $H^d_G(t'm) = tm$. This clearly holds by generalized visibility on $G$.

For IE-switch, let $s_1 \ldots m m s_2 \in P_{H^d(G)}$ such that $\lambda_{H^d(G)}^\text{op}(m) \neq \lambda_{H^d(G)}(n)$. Then there exists some $t_1 \ldots m m t_2 \in P_G$ such that $H^d_G(t_1 \ldots m m t_2) = s_1 \ldots m m s_2$, where note that $\lambda_G^\text{op}(m) \neq \lambda_G^\text{op}(n)$.

Therefore, if $u = \epsilon$, we then clearly have $\lambda_{H^d(G)}^\text{op}(m) = 0$ by IE-switch on $G$; otherwise, i.e., $u = mu'$, then we have the same conclusion as $\lambda_G^\text{op}(m) \neq \lambda_G^\text{op}(l)$.

We have established $P_{H^d(G)} \subseteq L_{H^d(G)}$. Hence, it remains to verify the axioms V1, V2, V3:
(V1) Because $c \in P_G$, we have $c = H_G^d(c) \in P_{H^d(G)}$, so $P_{H^d(G)}$ is non-empty. For prefix-closure, let $sm \in P_{H^d(G)}$; we have to show $s \in P_{H^d(G)}$. There must be some $tm \in P_G$ such that $sm = H_G^d(tm) = H_G^d(t)m$. Thus, we may conclude that $s = H_G^d(t) \in P_{H^d(G)}$.

(V2) Let $s \in P_{H^d(G)}$ and $I$ a set of initial moves occurring in $s$; we have to show $s \mid I \in P_{H^d(G)}$. There must be some $t \in P_G$ such that $s = H_G^d(t)$. Note that $t \mid I \in P_G$, and every initial move is external. Thus, we have $s \mid I = H_G^d(t) \mid I = H_G^d(t \mid I) \in P_{H^d(G)}$.

(V3) Assume $sm, s'm' \in P_{H^d(G)}^{odd}$, $i \in \mathbb{N}$ such that $i < \lambda_{H^d(G)}^s(m) = \lambda_{H^d(G)}^s(m')$ and $H_{H^d(G)}^i(s) = H_{H^d(G)}^i(s')$. There must be some $tm, t'm' \in P_G$ with $H_G^d(t) = s$, $H_G^d(t') = s'$. Then $H_G^{d+i}(t) = H_{H^d(G)}^i(H_G^d(t)) = H_{H^d(G)}^i(s) = H_{H^d(G)}^i(s') = H_{H^d(G)}^i(H_G^d(t')) = H_G^{d+i}(t')$. Hence by V3 on $G$, we conclude that $m = m'$ and $J_{sm}^{\otimes}(m) = J_{tm}^{\otimes(d+i)}(m') = J_{s'm'}^{\otimes}(m')$, which establishes V3 for $H^d(G)$.

We have shown that $H^d(G)$ forms a well-defined game. Finally, the preservation of the subgame relation $\leq$ under the $d$-hiding operation for all $d \in \mathbb{N} \cup \{\omega\}$ is clear from the definition.

We have a useful corollary:

**Corollary 3.2.7** (Hiding on legal positions). We have $\{H_G^d(s) \mid s \in L_G\} = L_{H^d(G)}$ for any arena $G$ and number $d \in \mathbb{N} \cup \{\omega\}$.

*Proof.* Since there is an upper bound on the set $\{\lambda_G^s(m) \mid m \in M_G\} \subseteq \mathbb{N}$, it suffices to consider the case $d \in \mathbb{N}$. Then by the inductive property of the hiding operations on arenas and j-sequences, we may just focus on the case $d = 1$.

The inclusion $\{H_G(s) \mid s \in L_G\} \subseteq L_{H(G)}$ is immediate by Theorem 3.2.6. For the other inclusion, let $t \in L_{H(G)}$. We shall find some $s \in L_G$ such that

1. $H_G(s) = t$;
2. 1-internal moves in $s$ occur as even-length consecutive segments $m_1m_2\ldots m_{2k}$, where $m_i$ justifies $m_{i+1}$ for $i = 1, 2, \ldots, 2k - 1$; and
3. $s$ is 1-complete.

We proceed by induction on the length of $t$. The base case $t = c$ is trivial. For the inductive step, let $tm \in L_{H(G)}$. Then $t \in L_{H(G)}$, and by the induction hypothesis there is some $s \in L_G$ that satisfies the three conditions.

If $m$ is initial, then it is straightforward to see that $sm \in L_G$, and $sm$ satisfies the three conditions. Thus, assume that $m$ is non-initial; we may write $tm = t_1nt_2m$, where $m$ is justified by $n$. We then need a case analysis:

- Assume $n \vdash_G m$. Then we take $sm$, in which $m$ points to $n$. Then, $sm \in L_G$ because:
  - **Justification.** It is immediate because $n \vdash_G m$.
  - **Alternation.** By the condition 3 for $s$, the last moves of $s$ and $t$ coincide. Thus the alternation condition holds for $sm$.
  - **Generalized visibility.** It suffices to establish the visibility for $sm$, as the other cases are included as the generalized visibility for $tm$. It is straightforward to see that, by the condition 2 for $s$, if the view of $t$ contains $n$, then so does the view of $s$. And since $tm \in L_{H(G)}$, the view of $t$ contains $n$. Hence, the view of $s$ contains $n$ as well.
3.3 Constructions on Games

Next, we show that dynamic games accommodate all the standard constructions on static games [AM99], i.e., they preserve the additional axioms for dynamic games. This result implies that our definition of games is in some sense “correct”.

▷ Notation. For brevity, we usually omit the “tags” for disjoint union, e.g., we write $a \in A + B$ if $a \in A, b \in B$. Also, given relations $R_A \subseteq A \times A, R_B \subseteq B \times B$, we write $R_A + R_B$ for the relation on $A + B$ such that $(x, y) \in R_A + R_B$ if $(x, y) \in R_A \cup (x, y) \in R_B$.

We begin with tensor product. Conceptually, the tensor product $A \otimes B$ is the game in which the component games $A, B$ are played “in parallel without communication”.

Definition 3.3.1 (Tensor product [AJ94] [AM99] [McC98]). Given games $A, B$, we define their tensor product $A \otimes B$ by $M_{A \otimes B} \overset{df}{=} M_A + M_B, \lambda_{A \otimes B} \overset{df}{=} [\lambda_A, \lambda_B], \tau_{A \otimes B} \overset{df}{=} \tau_A + \tau_B, P_{A \otimes B} \overset{df}{=} \{ s \in L_{A \otimes B} \mid s | A \in P_A, s | B \in P_B \}$, where $s | A$ (resp. $s | B$) denotes the $j$-subsequence of $s$ that consists of moves of $A$ (resp. $B$) equipped with the same justifiers as those in $s$.

Proposition 3.3.2 (Well-defined tensor product). Given games $A, B$, their tensor product $A \otimes B$ forms a well-defined game.

Proof. Since the proposition for static games has been established in the literature [AM99] [McC98], it suffices to show that $\otimes$ preserves the additional conditions for the labeling function and the axioms E1, E2, E4, V3. However, the non-trivial one is just V3, so we focus on it here. Let $slmn, s'l'm'n' \in P_{A \otimes B}^{odd}, i \in \mathbb{N}$ such that $\mathcal{H}_{A \otimes B}(slm) = \mathcal{H}_{A \otimes B}(s'l'm')$ and $i < \lambda_{A \otimes B}^N(n) = \lambda_{A \otimes B}^N(n')$. Note that $\lambda_{A \otimes B}^N(m) = \lambda_{A \otimes B}^N(n) = \lambda_{A \otimes B}^N(n') = \lambda_{A \otimes B}^N(m')$ by IE-switch.

At a first glance, it seems that $A \otimes B$ does not satisfy V3 as Opponent may choose to play either $A$ or $B$ at will (in contrast, Player cannot do so by the switching condition [A+97]). However,

15 This notation can be generalized for more than one component game in the obvious way; e.g., it should be clear what $s | A, C$ denotes, where $s \in P(A \otimes B) \otimes C$. 

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it is not the case for internal moves because \( slmn \in P_{A \otimes B}^{\text{odd}} \) with \( m \) internal implies \( m, n \in M_A \) or \( m, n \in M_B \). This property immediately follows from Table 1 which shows the possible transitions of OP- and IE-parities for a play in \( A \otimes B \), where a state \( (X^Y, Z^W) \) indicates that the next move in \( A \) (resp. \( B \)) has the OP-parity \( X \) (resp. \( Z \)) and the IE-parity \( Y \) (resp. \( W \)).

Now, note that \( m = m' \) and \( J_{slmn}(m) = J_{s't'm'(m')} \) as \( H^{\text{A} \otimes B}_{A}(slm) = H^{\text{A} \otimes B}_{A}(s'l'm') \). Therefore \( n, m, m', n' \) must belong to the same component game. If \( m, n, m', n' \in M_A \), then \( (sl \upharpoonright A)mn, (s'l' \upharpoonright A)m'n' \in P_{A}^{\text{odd}}, H^{\text{A}}_{A}((sl \upharpoonright A)m) = H^{\text{A} \otimes B}_{A}(slm) \mid A = H^{\text{A} \otimes B}_{A}(s'l'm') \mid A = H^{\text{A} \otimes B}_{A}(s'l'(m')) \) and \( i < \lambda^{Y}_{B}(n) = \lambda^{Y}_{B}(n') \); thus by V3 on \( A \), we may conclude that \( n = n' \) and \( J_{slmn}(n) = J_{s't'm'(m')} \). The other case is completely similar. ■

Next, we consider linear implication, which is the “space of linear functions”.

**Definition 3.3.3** (Linear implication [A94] [AM99] [McC98]). Given games \( A, B \) with \( A \) static, we define their linear implication \( A \Rightarrow B \) by \( M_{A \Rightarrow B} \equiv M_A + M_B, \lambda_{A \Rightarrow B} \equiv \overline{\lambda}_{A, \lambda_B} \), \( \star \vdash_{A \Rightarrow B} \)

\[
m \equiv \bullet \vdash_B m, m \vdash_{A \Rightarrow B} n (m \neq \bullet) \begin{cases} \downarrow \vdash_B (m \vdash_A n) & \\downarrow \vdash_B (m \vdash_B n) \lor (m \vdash_B n) \lor (\star \vdash_B m \land \star \vdash_A n) \end{cases} \]

\( n, m, m', n' \in \mathbb{N}^{\text{odd}} \) are moves in \( B \). Thus \( (sl \upharpoonright B)mn, (s'l' \upharpoonright B)m'n' \in P_{B}^{\text{odd}}, i \in \mathbb{N}^{\text{odd}} \) such that \( H^{\text{B}}_{B}((sl \upharpoonright B)m) = H^{\text{B} \otimes A}_{B}(slm) \mid B = H^{\text{B} \otimes A}_{B}(s'l'm') \mid B = H^{\text{B}}_{B}((s'l' \upharpoonright B)m).n') \) and \( i < \lambda^{Z}_{A}(n) = \lambda^{Z}_{A}(n') \), and so by V3 on \( B \), we may conclude that \( n = n' \) and \( J_{slmn}(n) = J_{s't'm'(m')} \). The construction of product is the categorical product in the CCC of static games and strategies [AM99] [McC98].

**Definition 3.3.5** (Product [HO00] [AM99] [McC98]). Given games \( A, B \), we define their product \( A \& B \) by \( M_{A \& B} \equiv M_A + M_B, \lambda_{A \& B} \equiv \overline{\lambda}_{A, \lambda_B} \), \( \vdash_{A \& B} \equiv \vdash_A + \vdash_B, P_{A \& B} \equiv \{ s \in L_{A \& B} \mid s \vdash_A A \in P_A, s \vdash_B B = \epsilon \} \cup \{ s \in L_{A \& B} \mid s \vdash_A A = \epsilon, s \vdash_B B \in P_B \} \).

**Proposition 3.3.6** (Well-defined product). Given games \( A, B \), their product \( A \& B \) forms a well-defined game.
Proof. By a similar but simpler argument as in the case of the tensor product $\otimes$. ■

Now, we make a straightforward generalization of product:

- **Definition 3.3.7** (Generalized product). Given games $L, R$ such that $\mathcal{H}^n(L) \subseteq C \rightarrow A$ and $\mathcal{H}^n(R) \subseteq C \rightarrow B$ for some static games $A, B, C$, we define their **generalized product** $L \& R$ by

\[
L \& R \overset{df}{=} M_C + (M_L \setminus M_C) + (M_R \setminus M_C), \quad \lambda_{L \& R} \overset{df}{=} \{C, \lambda_L | C, \lambda_R | C\}, \quad m \vdash_{L \& R} m \vdash_L n \lor m \vdash_R n, \quad P_{L \& R} \overset{df}{=} \{s \in L_{L \& R} | s \vdash_L \in P_L, s \vdash_R \in P_R\}.
\]

- **Proposition 3.3.8** (Well-defined generalized product). Given games $L, R$ with $\mathcal{H}^n(L) \subseteq C \rightarrow A$ and $\mathcal{H}^n(R) \subseteq C \rightarrow B$ for some games $A, B, C$, their generalized product $L \& R$ is a well-defined game.

**Proof.** Straightforward and similar to the case of product $\&$. ■

Intuitively, the exponential $!A$ of a game $A$ is the infinite iteration of tensor product $\otimes$ on $A$:

- **Definition 3.3.9** (Exponential [AM99, McC98]). For any game $A$, we define its **exponential** $!A$ as follows: The arena $!A$ is $A$, and $P_A \overset{df}{=} \{s \in L_A | \forall m \in \text{InitOcc}(s). s \vdash m \in P_A\}$, where $\text{InitOcc}(s)$ denotes the set of initial occurrences in $s$.

**Proposition 3.3.10** (Well-defined exponential). Given a game $A$, its exponential $!A$ forms a well-defined game.

**Proof.** It suffices to establish the preservation property on the axiom V3, but it can be done by essentially the same argument as in the case of tensor product $\otimes$. ■

Now, we introduce our new construction on games:

- **Definition 3.3.11** (Concatenation of games). Let $J, K$ be games such that $\mathcal{H}^n(J) \subseteq A \rightarrow B$, $\mathcal{H}^n(K) \subseteq B \rightarrow C$ for some static games $A, B, C$. Their **concatenation** $J \upharpoonright K$ is defined by:

\[
M_{J \upharpoonright K} \overset{df}{=} M_J + M_K
\]

\[
\lambda_{J \upharpoonright K} \overset{df}{=} [\lambda_J \upharpoonright B_1, \lambda_J \upharpoonright B_1, \lambda_K \upharpoonright B_2, \lambda_K \upharpoonright B_2], \quad \text{where } i \overset{df}{=} \sup(\{\lambda^G_j(m) | m \in M_J\} \cup \{\lambda^G_k(m) | m \in M_K\}) + 1, \lambda_{J \upharpoonright K} \overset{df}{=} (\lambda^G_{J \upharpoonright K} : \lambda^G, n \rightarrow \lambda^G_{J \upharpoonright K}(n) + i) \text{ for any game } G, \text{ and the subscripts 1, 2 are to distinguish the two copies of } B
\]

\[
\vdash_J \upharpoonright_K \quad m \overset{df}{=} \vdash_J m \quad \text{and} \quad m \vdash_{J \upharpoonright K} n \quad (m \neq *) \overset{df}{=} m \vdash_J n \lor m \vdash_K n \lor (\vdash_K \upharpoonright_B m \wedge \vdash_B n)
\]

\[
P_{J \upharpoonright K} \overset{df}{=} \{s \in M_{J \upharpoonright K} | s \vdash J \in P_J, s \vdash K \in P_K, s \vdash B_1, s \vdash B_2 \in P_B\}, \quad \text{where } s \vdash J \quad (\text{resp. } s \vdash K, \quad s \vdash B_1, B_2) \text{ denotes the i-subsequence of } s \text{ consisting of moves of } J \quad (\text{resp. } K, B) \text{ with the same justifiers as in } s \overset{df}{=} \vdash_B \quad B_1 \vdash \quad B_2\}.
\]

We shall see later that the “non-hiding composition” of strategies $\sigma: J$, $\tau: K$ forms a strategy on the game $J \upharpoonright K$.

- **Theorem 3.3.12** (Well-defined concatenation). Given games $J, K$ such that $\mathcal{H}^n(J) \subseteq A \rightarrow B$, $\mathcal{H}^n(K) \subseteq B \rightarrow C$ for some static games $A, B, C$, their concatenation $J \upharpoonright K$ forms a well-defined game.

\[\text{Note that moves } b \in M_{B_1} \text{ in } s \text{ with } s \vdash_{J} \text{ is initial in } s \vdash_{J}.\]
Proof. We first show that the arena $J \dagger K$ is well-defined. The set of moves and the labeling function are clearly well-defined, where note that the upper bound of degrees of internality of each game is for this construction. For the enabling relation, the axioms E1, E3 clearly hold.

For the axiom E2, if $m \vdash_{J\mathcal{K}} n$ and $\lambda_{J\mathcal{K}}^n(m) = A$, then $m, n \in M_K \setminus M_B$, $m, n \in M_B$, $m \in M_B$, or $m, n \in M_J \setminus M_B$. In either case, we clearly have $\lambda_{J\mathcal{K}}^m(m) = Q$ and $\lambda_{J\mathcal{K}}^n(m) = \lambda_{J\mathcal{K}}^m(n)$.

For the axiom E4, let $m \vdash_{J\mathcal{K}} n$, $m \neq \star$ and $\lambda_{J\mathcal{K}}^n(m) \neq \lambda_{J\mathcal{K}}^m(n)$. We proceed by a case analysis. If $(m \vdash_K n) \land (m, n \in M_K \setminus M_B)$, then we may just apply E4 on $K$. It is similar for the case $(m \vdash_J n) \land (m, n \in M_J \setminus M_B)$. Note that the case $\star \vdash_{J\mathcal{K}} m \land \star \vdash_{J\mathcal{K}} n$ cannot happen. Now, consider the case $m \vdash_K n \land m \in M_K \setminus M_B$. If $m$ is external, then $m \in M_C$, and so E4 for $J \dagger K$ is satisfied by the definition of $B \rightarrow C$; if $m$ is internal, then we may just apply E4 on $K$. The case $m \vdash_K n \land n \in M_K \setminus M_B \land m \in M_B$ is simpler as $n$ must be internal. The remaining cases $m \vdash_J n \land m \in M_J \setminus M_B \land n \in M_B$, $m \vdash_J n \land n \in M_J \setminus M_B \land m \in M_B$, are similar. Thus, the arena $J \dagger K$ is well-defined.

Next, we show that each element of $J \dagger K$ is a legal position in $J \dagger K$. For justification, let $sm \in P_{J\mathcal{K}}$ with $m$ non-initial and $d \in \mathbb{N}$ such that $sm$ is $d$-complete. Without loss of generality, we may assume $d \in \mathbb{N}$ as $s$ is finite. It is not hard to see that $H^d_{J\mathcal{K}}(sm) \in P_{H^d_{J\mathcal{K}}(H^d_{\mathcal{K}})}$ if $H^d(J \dagger K)$ is not static; in this case, we may just apply the (usual) visibility for $H^d(J \dagger K)$, otherwise, it is no harm to select the least $d \in \mathbb{N}$ such that $H^d(J \dagger K)$ is static; then $H^d_{J\mathcal{K}}(sm) \in P_{\left(A \rightarrow B_J \downarrow \left(B_J \rightarrow C \right)\right)}$, and so the visibility of $H^d_{J\mathcal{K}}(sm) = H^d_{\mathcal{K}}(H^d_{J\mathcal{K}}(H^d_{J\mathcal{K}}(sm)))$ can be shown completely in the same way as the proof that shows the composition of strategies is well-defined (in particular it satisfies visibility) [McC98, AM99]. As a consequence, it suffices to consider the case $d = 0$, i.e., to show the (usual) visibility.

For this, we need the following:

**Lemma 3.3.13** (Visibility lemma). Assume that $t \in P_{J\mathcal{K}}$ and $t \neq \epsilon$. Then we have:

1. If the last move of $t$ is in $M_J \setminus M_B$, then $[t \upharpoonright J]_J \leq [t]_{J\mathcal{K}} \downarrow J$ and $[t \upharpoonright J]_J \leq [t]_{J\mathcal{K}} \downarrow J$. 

Table 2: The concatenation double parity diagram
2. If the last move of $t$ is in $M_K \setminus M_{B_2}$, then $[t \upharpoonright K \upharpoonright K] \preceq [t \upharpoonright J_{1K} \upharpoonright K]$ and $[t \upharpoonright K \upharpoonright K] \preceq [t \upharpoonright J_{1K} \upharpoonright K]$.

3. If the last move of $t$ is an $O$-move in $M_{B_1} \cup M_{B_2}$, then $[t \upharpoonright B_1, B_2]m_{B_1 \rightarrow B_2} \preceq [t \upharpoonright J_{1K} \upharpoonright B_1, B_2]$.

Proof of the lemma. By a simple but lengthy case analysis; see Appendix A.

Note that we may write $sm = s_1n_2m$, where $n$ justifies $m$. If $s_2 = \epsilon$, then it is trivial; so assume $s_2 = s'\tau$. We then proceed by a case analysis on $m$:

- Assume $m \in M_J \setminus M_{B_2}$. Then $n \in M_J$, and $r \in M_J$ by Table 2. By Lemma 5.3.13, $[s \upharpoonright J] \preceq [s \upharpoonright J]$ and $[s \upharpoonright J] \preceq [s \upharpoonright J]$. Also, since $(s \upharpoonright J).m \in P_{J}$, visibility on $J$ implies:

  - $n$ occurs in $[s \upharpoonright J]$ if $m$ is a $P$-move;
  - $n$ occurs in $[s \upharpoonright J]$ if $m$ is an $O$-move.

Hence we may conclude that $n$ occurs in $[s]$ (resp. $[s]$) if $m$ is a $P$-move (resp. an $O$-move).

- Assume $m \in M_K \setminus M_{B_2}$. This case can be handled in a completely analogous way to the above case.

- Assume $m \in M_{B_1}$. If $m$ is a $P$-move, then $n, r \in M_J$ and so it can be handled in the same way as the case $m \in M_J \setminus M_{B_1}$. Thus assume that $m$ is an $O$-move; then $r \in M_{B_2}$ and it is a "copy" of $m$. Since $r$ is an $O$-move in $B_1 \rightarrow B_2$, by Lemma 5.3.13 we have $[s \upharpoonright B_1, B_2] \preceq [s \upharpoonright B_1, B_2]$. Note that $n$ is a move in $B_1$ or an initial move in $B_2$. In either case, we have $(s \upharpoonright B_1, B_2).m \in P_{B_1 \rightarrow B_2}$, so $n$ occurs in $[s \upharpoonright B_1, B_2]$. Hence we may conclude that $n$ occurs in $[s]$.

We have shown that $P_{J1K} \subseteq L_{J1K}$. It remains to verify the axioms V1, V2, V3. For V1, it is clear that $e \in P_{J1K}$. For the prefix-closure, let $s m \in P_{J1K}$. If $m \in M_J \setminus M_{B_1}$, then $(s \upharpoonright J).m = s m \upharpoonright J \in P_J$; thus $s \upharpoonright J \in P_J$, $s \upharpoonright K = s m \upharpoonright K \in P_K$ and $s \upharpoonright B_1, B_2 = s m \upharpoonright B_1, B_2 \in \text{pr}_B$, whence $s \in P_{J1K}$. The other cases may be handled similarly.

Next, for V2, let $s \in P_{J1K}, I \subseteq \text{InitOcc}(s)$. Define $I_J \overset{df}{=} \{ b \in M_{B_1} \mid * \upharpoonright J, b \land b \text{ occurs in } s \upharpoonright I \}$, $I_B \overset{df}{=} \{ b \in M_{B_2} \mid * \upharpoonright B_2, b \land b \text{ occurs in } s \upharpoonright I \}$, $I_K \overset{df}{=} I$. Note that $I_J \subseteq \text{InitOcc}(s \upharpoonright J)$, $I_B \subseteq \text{InitOcc}(s \upharpoonright B_1, B_2)$, $I_K \subseteq \text{InitOcc}(s \upharpoonright K)$. Then it is straightforward to see that:

$$(s \upharpoonright I) \upharpoonright J = (s \upharpoonright J) \upharpoonright J \in P_J$$

$$(s \upharpoonright I) \upharpoonright B_1, B_2 = (s \upharpoonright B_1, B_2) \upharpoonright \text{pr}_B$$

$$(s \upharpoonright I) \upharpoonright K = (s \upharpoonright K) \upharpoonright I_K \in P_K.$$ 

Therefore we may conclude that $s \upharpoonright I \in P_{J1K}$.

Finally for V3, let $sm, sm' \in P_{J1K}^\text{odd}$, $i \in \mathbb{N}$ such that $i < \lambda_{J1K}^N(m) = \lambda_{J1K}^N(m')$ and $\mathcal{H}_{J1K}^i(s) = \mathcal{H}_{J1K}^i(s')$. Without loss of generality, we may assume $i = 0$ and $\lambda_{J1K}^N(m) = 1 = \lambda_{J1K}^N(m')$ because if $\lambda_{J1K}^N(m) = \lambda_{J1K}^N(m') = j > 1$, then we may consider $\mathcal{H}_{J1K}^{j-1}(sm), \mathcal{H}_{J1K}^{j-1}(s'm') \in P_{\mathcal{H}_{J1K}^{j-1}(J)\mathcal{H}_{J1K}^{j-1}(K)}$ (note also that the justifiers of $m, m'$ both have the same degree of internality). Thus, $s = s' = s$ and $m, m' \in M_J \lor m, m' \in M_K$. If $m, m' \in M_J$ (resp. $m, m' \in M_K$), then $(s \upharpoonright J).m, (s' \upharpoonright J).m' \in P_{J}^\text{odd}$ (resp. $(s \upharpoonright K).m, (s' \upharpoonright K).m' \in P_{K}^\text{odd}$), and so we may just apply V3 for $J$ (resp. $K$), completing the proof.

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We now show that these constructions preserve the subgame relation:

**Lemma 3.3.14** (Preservation of subgames). Let \( \bullet_i \in I \) be a construction on games, where \( I = \{1\} \) or \( \{1, 2\} \), and \( H_i \preceq G_i \) for all \( i \) such that \( \bullet_i \in H_i \), \( \bullet_i \in G_i \) are well-defined. Then \( \bullet_i \in H_i \preceq \bullet_i \in G_i \). (If \( \bullet_i \) is \( \dagger \), then we require \( \sup(\{\lambda_H(m)\mid m \in M_{H_i}\}) = \sup(\{\lambda_G(m)\mid m \in M_{G_i}\}) \) for \( i = 1, 2 \).

**Proof.** Let us first consider tensor product. It is trivial to check the conditions on the sets of moves and the labeling functions, and so we omit them.

For the enabling relations, we have:

\[
\vdash_{H_1 \otimes H_2} = \vdash_{H_1} + \vdash_{H_2}
\]

\[
\subseteq (\vdash_{G_1} \cap ((* + M_{H_1}) \times M_{H_1})) + (\vdash_{G_2} \cap ((* + M_{H_2}) \times M_{H_2}))
\]

\[
= (\vdash_{G_1} \cap ((* + M_{H_1 \otimes H_2}) \times M_{H_1 \otimes H_2})) + (\vdash_{G_2} \cap ((* + M_{H_1 \otimes H_2}) \times M_{H_1 \otimes H_2}))
\]

\[
\vdash_{G_1 \otimes G_2} \cap (\vdash_{H_1 \otimes H_2}).
\]

For the valid positions, we have:

\[
P_{H_1 \otimes H_2} = \{ s \in L_{H_1 \otimes H_2} \mid \forall i \in \{1, 2\}. s \models H_i \} = P_{G_1 \otimes G_2}.
\]

Therefore we have shown that \( H_1 \otimes H_2 \preceq G_1 \otimes G_2 \).

Linear implication and exponential are similar, and (generalized) product is even simpler; thus we omit them. Next, let us consider concatenation. Assume that \( \mathcal{H}(H_1) \preceq A \rightarrow E \), \( \mathcal{H}(H_2) \preceq B \rightarrow C \), \( \mathcal{H}(G_1) \preceq D \rightarrow E \), \( \mathcal{H}(G_2) \preceq E \rightarrow F \) for some static games \( A, B, C, D, E, F \); without loss of generality, we assume that these static games are the least ones for the given conditions above with respect to the subgame relation \( \preceq \). Note that \( \mathcal{H}(H_1) \preceq \mathcal{H}(G_1) \preceq D \rightarrow E \) and \( \mathcal{H}(H_2) \preceq \mathcal{H}(G_2) \preceq E \rightarrow F \), which in turn implies \( A \preceq D \), \( B \preceq E \) and \( C \preceq F \). First, we clearly have \( M_{H_1 \otimes H_2} \subseteq M_{G_1 \otimes G_2} \) and \( \lambda_{G_1 \otimes G_2} [M_{H_1 \otimes H_2} = \lambda_{H_1 \otimes H_2} \), where the additional condition \( \sup(\{\lambda_H(m)\mid m \in M_{H_i}\}) = \sup(\{\lambda_G(m)\mid m \in M_{G_i}\}) \) for \( i = 1, 2 \) ensures that the degree of internality of moves in \( B \) coincide. Next, for the enabling relations, we have:

\[
\star \vdash_{H_1 \otimes H_2} m \Leftrightarrow \star \vdash_{H_2} m \Leftrightarrow \star \vdash_{G_2} m \Leftrightarrow \star \vdash_{G_1 \otimes G_2} m
\]

and

\[
m \vdash_{H_1 \otimes H_2} n \Leftrightarrow m \vdash_{H_1 \otimes H_2} n \Leftrightarrow \star \vdash_{B_2} m \Leftrightarrow \star \vdash_{B_2} n
\]

\[
\Rightarrow m \vdash_{G_1 \otimes G_2} n \Leftrightarrow \star \vdash_{E_2} m \Leftrightarrow \star \vdash_{E_2} n
\]

\[
m \vdash_{G_1 \otimes G_2} n
\]

for any \( m, n \in M_{H_1 \otimes H_2} \). Finally, we show \( P_{H_1 \otimes H_2} \subseteq P_{G_1 \otimes G_2} \):

\[
P_{H_1 \otimes H_2} = \{ s \in M^{*}_{H_1 \otimes H_2} \mid s \mid H_1 \in P_{H_1}, s \mid H_2 \in P_{H_2}, s \mid B_1, B_2 \in \text{pr}_B \}
\]

\[
\subseteq \{ s \in M^{*}_{G_1 \otimes G_2} \mid s \mid G_1 \in P_{G_1}, s \mid G_2 \in P_{G_2}, s \mid E_1, E_2 \in \text{pr}_E \}
\]

\[
= P_{G_1 \otimes G_2}
\]

which completes the proof. \( \square \)

At the end of the present section, we establish the following lemma:

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Lemma 3.3.15 (Hiding lemma on games). Let $\mathbb{A}_i \in I$ be a construction on games, where $I$ is $\{1\}$ or $\{1, 2\}$, and $G_i$ a game for all $i \in I$ such that $\mathbb{A}_i \in G_i$ is well-defined. Then for all $d \in \mathbb{N} \cup \{\omega\}$ we have:

1. $\mathbb{H}^d(\mathbb{A}_i \in G_i) \subseteq \mathbb{A}_i \in \mathbb{H}^d(G_i)$ if $\mathbb{A}_i$ is not $\omega$.

2. $\mathbb{H}^d(G_1 \upharpoonright G_2) = \mathbb{H}^d(G_1) \upharpoonright \mathbb{H}^d(G_2)$ if $\mathbb{H}^d(G_1)$ is not static, and $\mathbb{H}^d((G_1) \upharpoonright (G_2)) \subseteq A \rightarrow C$ otherwise, where $A, B, C$ are static games such that $\mathbb{H}^d(G_1) \subseteq A \rightarrow B$ and $\mathbb{H}^d(G_2) \subseteq B \rightarrow C$.

Proof. Since there is an upper bound of the degrees of internality for each game, it suffices to consider the case $d = 1$. Then, because $\mathbb{H}^{i+1} = \mathbb{H}^i \mathbb{H}^i$ for all $i \in \mathbb{N}$, we may just focus on the case $d = 1$.

For the valid positions, we have to show that $\mathbb{H}(G_1 \otimes G_2) \subseteq \mathbb{H}(G_1) \otimes \mathbb{H}(G_2)$. Their sets of moves and labeling functions clearly coincide. For the enabling relations, $\ast \vdash_{\mathbb{H}(G_1) \otimes \mathbb{H}(G_2)} m \iff \ast \vdash_{\mathbb{H}(G_1)} m \otimes \ast \vdash_{\mathbb{H}(G_2)} m \iff \ast \vdash_{\mathbb{H}(G_1)} m \otimes \ast \vdash_{\mathbb{H}(G_2)} m$, and

$$m \vdash_{\mathbb{H}(G_1) \otimes \mathbb{H}(G_2)} n \iff (m \vdash_{G_1 \otimes G_2} n) \lor \exists k \in \mathbb{N}^+, m_1, m_2, \ldots, m_{2k} \in M_{G_1 \otimes G_2} \setminus M_{H(G_1) \otimes H(G_2)}, m \vdash_{G_1 \otimes G_2} m_1 \land m_1 \vdash_{G_1 \otimes G_2} m_2$$

$$\ldots \land m_{2k-1} \vdash_{G_1 \otimes G_2} m_{2k} \land m_{2k} \vdash_{G_1 \otimes G_2} n$$

$$\lor (m \vdash_{G_1} n \lor m \vdash_{G_2} n) \lor \exists i \in \{1, 2\}, k \in \mathbb{N}^+, m_1, m_2, \ldots, m_{2k} \in M_{G_i} \setminus M_{H(G_1) \otimes H(G_2)}, m \vdash_{G_i} m_1 \land m_1 \vdash_{G_i} m_2$$

$$\ldots \land m_{2k-1} \vdash_{G_i} m_{2k} \land m_{2k} \vdash_{G_i} n$$

$$\lor \exists i \in \{1, 2\}, m \vdash_{G_i} n \lor \exists k \in \mathbb{N}^+, m_1, m_2, \ldots, m_{2k} \in M_{G_i} \setminus M_{H(G_i)}, m \vdash_{G_i} m_1 \land m_1 \vdash_{G_i} m_2$$

$$\ldots \land m_{2k-1} \vdash_{G_i} m_{2k} \land m_{2k} \vdash_{G_i} n$$

$$\iff m \vdash_{H(G_1) \otimes H(G_2)} n$$

for all $m, n \in M_{H(G_1) \otimes H(G_2)} (= M_{H(G_1) \otimes H(G_2)})$. Thus, the arenas $\mathbb{H}(G_1 \otimes G_2), \mathbb{H}(G_1) \otimes \mathbb{H}(G_2)$ coincide. For the valid positions, we have:

$$s \in P_{H(G_1) \otimes H(G_2)} \Rightarrow \exists t \in L_{G_1 \otimes G_2}, H_{G_1 \otimes G_2}(t) = s \land \forall i \in \{1, 2\}, t | G_i \in P_{G_i}$$

$$\Rightarrow \exists t \in L_{G_1 \otimes G_2}, H_{G_1 \otimes G_2}(t) = s \land \forall i \in \{1, 2\}, H_{G_i}(t | G_i) \in P_{H(G_i)}$$

$$\Rightarrow \exists t \in L_{H(G_1) \otimes H(G_2)}(t) = t \land \forall i \in \{1, 2\}, H_{G_i}(t | H(G_i)) \in P_{H(G_i)}$$

$$\Rightarrow s \in L_{H(G_1) \otimes H(G_2)} = L_{H(G_1) \otimes H(G_2)} \land \forall i \in \{1, 2\}, s | H(G_i) \in P_{H(G_i)}$$

$$\Rightarrow s \in P_{H(G_1) \otimes H(G_2)}.$$

Next, for linear implication, we show $\mathbb{H}(G_1 \rightarrow G_2) \subseteq \mathbb{H}(G_1) \rightarrow \mathbb{H}(G_2)$, where $G_1$ is static. The sets of moves, labeling function and valid positions may be handled similarly to the case of tensor product, so we just show $\vdash_{H(G_1) \rightarrow H(G_2)} = \vdash_{H(G_1)} \rightarrow \vdash_{H(G_2)}$. Let $m, n \in M_{H(G_1) \rightarrow G_2} (= M_{H(G_1) \rightarrow H(G_2)})$. We clearly have $\ast \vdash_{H(G_1 \rightarrow G_2)} m \iff \ast \vdash_{G_1 \rightarrow G_2} m \iff \ast \vdash_{G_2} m \iff \ast \vdash_{H(G_1)} m \iff \ast \vdash_{H(G_1) \rightarrow H(G_2)} m$, and

$$m \vdash_{H(G_1) \rightarrow H(G_2)} n \iff m \vdash_{G_1 \rightarrow G_2} n \lor \exists k \in \mathbb{N}^+, m_1, m_2, \ldots, m_{2k} \in M_{G_1 \rightarrow G_2} \setminus M_{H(G_1) \rightarrow G_2}, m \vdash_{G_1 \rightarrow G_2} m_1 \land m_1 \vdash_{G_1 \rightarrow G_2} m_2$$

$$\ldots \land m_{2k-1} \vdash_{G_1 \rightarrow G_2} m_{2k} \land m_{2k} \vdash_{G_1 \rightarrow G_2} n$$

$$\lor (m \vdash_{G_2} m \land \ast \vdash_{G_1} n) \lor \exists i \in \{1, 2\}, m \vdash_{G_i} n \lor \exists k \in \mathbb{N}^+, m_1, m_2, \ldots, m_{2k} \in M_{G_i} \setminus M_{H(G_i)}, m \vdash_{G_i} m_1$$

$$\ldots \land m_{2k-1} \vdash_{G_i} m_{2k} \land m_{2k} \vdash_{G_i} n$$

$$\lor (\ast \vdash_{H(G_i)} m \land \ast \vdash_{H(G_i)} n) \lor m \vdash_{H(G_i)} n \lor m \vdash_{H(G_i) \rightarrow H(G_2)} n$$

$$\iff m \vdash_{H(G_1) \rightarrow H(G_2)} n.$$
Next, (generalized) product and exponential are similar to and simpler than tensor product, so we omit them. Finally, we consider concatenation. First, assume that $\mathcal{H}(G_{1} \uparrow G_{2})$ is not static; we have to show $\mathcal{H}(G_{1} \uparrow G_{2}) \leq \mathcal{H}(G_{1}) \uparrow \mathcal{H}(G_{2})$. Clearly, their sets of moves and labeling functions coincide. For the enabling relations, for any $m, n \in M_{\mathcal{H}(G_{1} \uparrow G_{2})} (= M_{\mathcal{H}(G_{1}) \uparrow \mathcal{H}(G_{2})})$, we have $\star \vdash_{\mathcal{H}(G_{1} \uparrow G_{2})} m \iff \star \vdash_{\mathcal{H}(G_{1}) \uparrow \mathcal{H}(G_{2})} m \iff \star \vdash_{\mathcal{H}(G_{1})} m$ and

$$m \vdash_{\mathcal{H}(G_{1} \uparrow G_{2})} n \,(m \neq \ast)$$

$$\iff m \vdash_{G_{1} \uparrow G_{2}} n \,\lor \exists k \in \mathbb{N}^{+}, m_{1}, m_{2}, \ldots, m_{2k} \in M_{G_{1} \uparrow G_{2}} \setminus M_{\mathcal{H}(G_{1} \uparrow G_{2})}, m \vdash_{G_{1} \uparrow G_{2}} m_{1} \land m_{1} \vdash_{G_{1} \uparrow G_{2}} m_{2}$$

$$\land \cdots \land m_{2k-1} \vdash_{G_{1} \uparrow G_{2}} m_{2k} \land m_{2k} \vdash_{G_{1} \uparrow G_{2}} n$$

$$\iff m \vdash_{G_{1}}, m \vdash_{G_{2}}, n \,\lor (\star \vdash_{B_{1}} m \land \star \vdash_{B_{2}} n) \,\lor \exists i \in \{1, 2\}, k \in \mathbb{N}^{+}, m_{1}, m_{2}, \ldots, m_{2k} \in M_{G_{1}} \setminus M_{H(G_{1})},$$

$$m \vdash_{G_{1}}, m_{1} \land m_{1} \vdash_{G_{2}}, m_{2} \land \cdots \land m_{2k-1} \vdash_{G_{2}}, m_{2k} \land m_{2k} \vdash_{G_{2}} n \,(\text{as moves in } B \text{ are not "hidden"})$$

$$\iff (\star \vdash_{B_{1}} m \land \star \vdash_{B_{2}} n) \lor \exists i \in \{1, 2\}, m \vdash_{G_{1}}, n \,\lor k \in \mathbb{N}^{+}, m_{1}, m_{2}, \ldots, m_{2k} \in M_{G_{1}} \setminus M_{H(G_{1})}, m \vdash_{G_{1}}, m_{1}$$

$$\land m_{1} \vdash_{G_{2}}, m_{2} \land \cdots \land m_{2k-1} \vdash_{G_{2}}, m_{2k} \land m_{2k} \vdash_{G_{2}} n$$

$$\iff (\ast \vdash_{B_{1}} m \land \ast \vdash_{B_{2}} n) \lor \exists i \in \{1, 2\}, m \vdash_{H(G_{1})} n \implies m \vdash_{H(G_{1})} n.$$
3.4 Dynamic Strategies

To define dynamic strategies, we just apply the definition of static strategies [AM99, McC98] in the context of dynamic games:

Definition 3.4.1 (Dynamic strategies [AM99, McC98]). A (dynamic) strategy on a game $G$ is a subset $\sigma \subseteq P_{G}^{\text{ssm}}$ that satisfies:

- (S1) It is non-empty and even-prefix-closed (i.e., $\text{ssm} \in \sigma \Rightarrow s \in \sigma$).
- (S2) It is deterministic (i.e., $\text{ssm}, \text{ssm}' \in \sigma \Rightarrow n = n' \land \mathcal{J}_{\text{ssm}}(n) = \mathcal{J}_{\text{ssm}'}(n')$).

Convention. From now on, strategies refer to dynamic strategies by default, and a strategy is called static if its moves are all external. We write $\sigma : G$ for a strategy $\sigma$ on a game $G$.

Since internal moves are “invisible” to Opponent, a strategy $\sigma : G$ must be externally consistent: If $\text{ssm}, \text{ssm}' \in \sigma$, $\lambda_G^d(n) = \lambda_G^d(n') = 0$ and $\mathcal{H}_G^d(\text{ssm}) = \mathcal{H}_G^d(\text{ssm}')$, then $n = n'$ and $\mathcal{J}_{\text{ssm}}(n) = \mathcal{J}_{\text{ssm}'}(n')$. In fact, a more general statement holds (see Theorem 3.4.3 below).

Lemma 3.4.2 (O-determinacy). Let $\sigma : G, s, s' \in \sigma, d \in \mathbb{N} \cup \{\omega\}$, and $\text{ssm}, \text{ssm}' \in P_G$ are both $d$-complete. If $\mathcal{H}_G^d(\text{ssm}) = \mathcal{H}_G^d(\text{ssm}')$, then $\text{ssm} = \text{ssm}'$.

Proof. By induction on the length of $s$. The base case $s = \epsilon$ is trivial: For any $d \in \mathbb{N} \cup \{\omega\}$, if $\mathcal{H}_G^d(\text{ssm}) = \mathcal{H}_G^d(\text{ssm}')$, then $\mathcal{H}_G^d(\text{ssm}) = \mathcal{H}_G^d(\text{ssm}') = m$, and so $\text{ssm}' = m = \text{ssm}$.

For the induction step, let $d \in \mathbb{N} \cup \{\omega\}$ be fixed, and assume $\mathcal{H}_G^d(\text{ssm}) = \mathcal{H}_G^d(\text{ssm}')$. Without loss of generality, we may suppose that $\text{ssm} = \text{tlrm}$, where $l$ is the rightmost O-move in $s$ such that $\lambda_G^d(l) = 0 \lor \lambda_G^d(l) > d$. Then $\mathcal{H}_G^d(\text{ssm}') = \mathcal{H}_G^d(\text{ssm}) = \mathcal{H}_G^d(t).l.\mathcal{H}_G^d(\text{rm})$, and so we may write $\text{ssm}' = t'.l.t''.m'$. Now, $t, t'_l \in \sigma$, $t, t'_l \in P_G$, $\mathcal{H}_G^d(t) = \mathcal{H}_G^d(t'_l)$, and $t, t'_l$ are both $d$-complete; thus, by the induction hypothesis, $\mathcal{H}_G^d(\text{rm}) = \mathcal{H}_G^d(\text{rm}')$. Thus, $\mathcal{H}_G^d(t).l.\mathcal{H}_G^d(\text{rm}) = \mathcal{H}_G^d(\text{rm}') = \mathcal{H}_G^d(\text{rm}') = \mathcal{H}_G^d(\text{rm}') = \mathcal{H}_G^d(\text{rm}')$, whence $t'' = r.\mathcal{H}_G^d(\text{rm})$ by the determinacy of $\sigma$. Hence, $\text{ssm} = \text{tlrm}$ and $\text{ssm}' = t.\mathcal{H}_G^d(\text{rm})$. Finally, if $r$ is external, then so is $m$ by IE-switch, and so $\text{ssm}' = sm'; if r is j-internal (j > d), then so is m, and so we may apply the axiom V3 for $i = j - 1$ to $s, s'$, concluding that $\text{ssm} = \text{ssm}'$.

Theorem 3.4.3 (Consistency). Let $\sigma : G, s, s' \in \sigma$, and $d \in \mathbb{N} \cup \{\omega\}$. If $\text{ssm}, \text{ssm}' \in \sigma$ are both $d$-complete and $\mathcal{H}_G^d(\text{ssm}) = \mathcal{H}_G^d(\text{ssm}')$, then $n = n'$ and $\mathcal{J}_{\text{ssm}}(n) = \mathcal{J}_{\text{ssm}'}(n')$.

Proof. Let $\sigma : G, s, s' \in \sigma$, and $d \in \mathbb{N} \cup \{\omega\}$, and assume that $\text{ssm}, \text{ssm}' \in \sigma$ are both $d$-complete and $\mathcal{H}_G^d(\text{ssm}) = \mathcal{H}_G^d(\text{ssm}')$. By Lemma 3.4.2 we have $\text{ssm} = \text{ssm}'$; thus, by the axiom S2 on $\sigma$, we have $n = n'$ and $\mathcal{J}_{\text{ssm}}(n) = \mathcal{J}_{\text{ssm}'}(n')$ (whence $\mathcal{J}_{\text{ssm}}(n) = \mathcal{J}_{\text{ssm}'}(n')$).

Now, note that even-length position is not necessarily preserved under hiding on j-sequences. For instance, let $\text{ssm}t$ be an even-length position in a game $G$ such that $\text{ssm}$ (resp. $nt$) consists of external (resp. internal) moves only. By IE-switch on $G$, $m$ is an O-move, and so $\mathcal{H}_G^d(\text{ssm}t) = sm$ is of odd-length. Taking into account this fact, we define:

Definition 3.4.4 (Hiding operation on strategies). Let $G$ be a game. For any $s \in P_G$, $d \in \mathbb{N} \cup \{\omega\}$, we define:

$$s^d_H G = \begin{cases} H_G^d(s) & \text{if } s \text{ is } d\text{-complete} \\ t & \text{otherwise, where } H_G^d(s) = tm. \end{cases}$$

We then define the $d$-hiding operation $\mathcal{H}_G^d$ on strategies by:

$$\mathcal{H}_G^d : (\sigma : G) \mapsto \{ s^d_H G | s \in \sigma \}.$$
Next, we shall show a beautiful fact: \( \sigma : G \Rightarrow \mathcal{H}^d(\sigma) : \mathcal{H}^d(G) \) for all \( d \in \mathbb{N} \cup \{\omega\} \). For this, we need the following lemma:

**Lemma 3.4.5** (Asymmetry lemma). Let \( \sigma : G, d \in \mathbb{N} \cup \{\omega\} \). Assume that \( \text{smn} \in \mathcal{H}^d(\sigma) \), where \( \text{smn} = t\text{mun}v_2\mathcal{H}^d_G \) with \( t\text{mun}v \in \sigma \) not d-complete. Then \( \text{smn} = \mathcal{H}^d(t\text{mun}v) = \mathcal{H}^d(t)\text{mn} \).

**Proof.** Since \( t\text{mun}v \in \sigma \) is not d-complete, we may write \( v = v_1lv_2r \) with \( \lambda_G^N(l) = 0 \lor \lambda_G^N(r) > d \), \( 0 < \lambda_G^N(x) \leq d \) for all moves \( x \) in \( v_1, v_2 \). Then we have:

\[
\text{smn} = t\text{mun}v_1lv_2r\mathcal{H}^d_G = \mathcal{H}^d_G(t)\mathcal{H}^d_G(u)\mathcal{H}^d_G(t)\text{mn}.
\]

We are now ready to establish:

**Theorem 3.4.6** (Hiding theorem). If \( \sigma : G, \text{then } \mathcal{H}^d(\sigma) : \mathcal{H}^d(G) \) for all \( d \in \mathbb{N} \cup \{\omega\} \).

**Proof.** We first show \( \mathcal{H}^d(\sigma) \subseteq P_{\mathcal{H}^d(G)}^\text{even} \). Let \( s \in \mathcal{H}^d(\sigma) \), i.e., \( s = t\mathcal{H}^d_G \) for some \( t \in \sigma \). Let us write \( t = t'm \) as the case \( t = \epsilon \) is trivial.

- If \( t \) is d-complete, then \( s = t\mathcal{H}^d_G = \mathcal{H}^d_G(t) \in P_{\mathcal{H}^d(G)} \). Also, since \( s = \mathcal{H}^d_G(t')m \) and \( m \) is a P-move, \( s \) must be of even-length by alternation on \( \mathcal{H}^d(G) \).

- If \( t \) is not d-complete, then we may write \( t = t''m_0m_1 \ldots m_k \), where \( m_k = m \), \( t''m_0 \) is d-complete, and \( 0 < \lambda_G^N(m_i) \leq d \) for \( i = 1, 2, \ldots, k \). By IE-switch, \( m_0 \) is an O-move, and thus \( s = H_G^d(t'') \in P_{\mathcal{H}^d(G)} \) is of even-length.

It remains to verify the axioms S1, S2. For S1, \( \mathcal{H}^d(\sigma) \) is clearly non-empty as \( \epsilon \in \mathcal{H}^d(\sigma) \). For the even-prefix-closure, let \( \text{smn} \in \mathcal{H}^d(\sigma) \); we have to show \( s \in \mathcal{H}^d(\sigma) \). We have some \( t\text{mun}v \in \sigma \) such that \( t\text{mun}v\mathcal{H}^d_G = \text{smn} \). By Lemma 3.4.5, \( \text{smn} = \mathcal{H}^d_G(t)\text{mn} \), whence \( s = \mathcal{H}^d_G(t) \). Note that \( t\text{m} \) is d-complete, and so is \( t \) by IE-switch. Therefore \( s = \mathcal{H}^d_G(t) = t\mathcal{H}^d_G \in \mathcal{H}^d(\sigma) \).

Finally, for S2, let \( s, s' \in \mathcal{H}^d(\sigma) \); we have to show \( n = n' \) and \( J_{\text{sm}}(n) = J_{\text{sm}}(n') \). By the definition, \( n = \text{smn} = t\text{mun}v_2\mathcal{H}^d_G \), \( s' = t'\text{mun}'v'_2\mathcal{H}^d_G \) for some \( t\text{mun}v, t'\text{mun}'v' \in \sigma \). By Lemma 3.4.5, \( s = \mathcal{H}^d_G(t\text{mun}) \) and \( s' = \mathcal{H}^d_G(t'\text{mun}') \). Therefore by Theorem 3.4.3, we may conclude that \( n = n' \) and \( J_{\text{sm}}(n) = J_{\text{sm}}(n') \), completing the proof.

At the end of this section, we establish an inductive property of the d-hiding operations on strategies for \( d \in \mathbb{N} \) as in the case of games.

**Notation.** Given \( \sigma : G, d \in \mathbb{N} \cup \{\omega\} \), we define \( \sigma^d = \{ s \in \sigma \mid s \text{ d-complete} \} \) and \( \sigma^d = \sigma \setminus \sigma^d \).

**Lemma 3.4.7** (Hiding and complete plays). Let \( \sigma : G \) be a strategy. For any \( i, d \in \mathbb{N} \) with \( i \geq d \), we have \( \mathcal{H}^i(\sigma^d) = \mathcal{H}^i(\sigma^d) = \{ s \mathcal{H}^i_G | s \in \sigma^d \} \).

**Proof.** The inclusion \( \mathcal{H}^i(\sigma^d) \subseteq \mathcal{H}^i(\sigma) \) is obvious. For the opposite inclusion, let \( s \in \mathcal{H}^i(\sigma) \), i.e., \( s = t_0i\mathcal{H}^i_G \) for some \( t \in \sigma \); we have to show \( s \in \mathcal{H}^i(\sigma^d) \). If \( t \in \sigma^d \), then we are done; so assume otherwise. Also, if there is no external or \( j \)-internal move with \( j > i \) other than the first move \( m_0 \) in \( t \), then \( s = \epsilon \in \mathcal{H}^i(\sigma^d) \); so assume otherwise. As a consequence, we may write

\[
t = m_0t_1inmtn_2r
\]

where \( t_2r \) consists of only \( j \)-internal moves with \( 0 < j < i \), and \( m, n \) are P-, O-moves such that \( \lambda_G^N(m) = \lambda_G^N(n) = 0 \lor \lambda_G^N(m) = \lambda_G^N(n) > i \). Then we may take \( m_0t_1m \in \sigma^d \) that satisfies \( m_0t_1m\mathcal{H}^i_G = m_0\mathcal{H}^i_G(t_1)m = t_2\mathcal{H}^i_G = s \), whence \( s \in \mathcal{H}^i(\sigma^d) \).

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We are now ready to show:

**Proposition 3.4.8** (Stepwise hiding on strategies). Let $\sigma : G$ be a strategy. Then $\mathcal{H}^{i+1}(\sigma) = \mathcal{H}^{i}(\mathcal{H}^{i}(\sigma))$ for all $i \in \mathbb{N}$.

*Proof.* Let us first show the inclusion $\mathcal{H}^{i+1}(\sigma) \subseteq \mathcal{H}^{i}(\mathcal{H}^{i}(\sigma))$. By Lemma 3.4.7, we may write any element of the set $\mathcal{H}^{i+1}(\sigma)$ as $s^{i}_{j}H^{i+1}_{G}$ for some $s \in \sigma^{i+1}$. Then observe that:

$$s^{i}_{j}H^{i+1}_{G} = H^{i+1}_{G}(s) = H^{i}(H^{i}(s)) = (s^{i}_{j}H^{i}_{L}(G)(H^{i}_{G}(s))) \in \mathcal{H}^{i}(\mathcal{H}^{i}(\sigma)).$$

For the opposite inclusion $\mathcal{H}^{i}(\mathcal{H}^{i}(\sigma)) \subseteq \mathcal{H}^{i+1}(\sigma)$, again by Lemma 3.4.7, we may write any element of $\mathcal{H}^{i}(\mathcal{H}^{i}(\sigma))$ as $(s^{i}_{j}H^{i}_{L}(G))H^{i}_{H}(G)$ for some $s \in \sigma^{i+1}$. We have to show that $(s^{i}_{j}H^{i}_{L}(G))H^{i}_{H}(G) \in \mathcal{H}^{i+1}(\sigma)$. If $s \in \sigma^{i+1}$, then it is completely analogous to the above argument; so assume otherwise. Also, if an external or $j$-internal move with $j > i + 1$ in $s$ is only the first move $m_0$, then $(s^{i}_{j}H^{i}_{L}(G))H^{i}_{H}(G) = e \in \mathcal{H}^{i+1}(\sigma)$; thus assume otherwise. Now, we may write:

$$s = s^{i}_{mnm_{1}m_{2}...m_{2n}}$$

where $\lambda^{j}_{s}(r) = i + 1, m_{1}, m_{2},...m_{2n}$ are $j$-internal with $0 < j \leq i + 1$, and $m, n$ are external or $j$-internal $P\lhd, O$-moves with $j > i + 1$. Then we have:

$$(s^{i}_{j}H^{i}_{L}(G))H^{i}_{H}(G) = H^{i}_{G}(s)H^{i}_{H}(G) = H^{i}(H^{i}(G)(s'))m = H^{i+1}(s')m = s^{i}_{j}H^{i+1}_{G} \in \mathcal{H}^{i+1}(\sigma)$$

which completes the proof. ■

Thus, as in the case of games, we may focus on the 1-hiding operation $\mathcal{H}^{1}$ on strategies. From now on, we write $\mathcal{H}$ for $\mathcal{H}^{1}$ and call it the hiding operation on strategies, and $\mathcal{H}^{t}$ denotes the $i$-times iteration of $\mathcal{H}$ for any $i \in \mathbb{N}$.

### 3.5 Constructions on Strategies

We now consider constructions on strategies. However, since dynamic strategies are static strategies on dynamic games, there is nothing to prove for existing constructions such as copy-cat strategies $\text{cp}_A$, tensor (product) $\otimes$, composition $\cdot$, pairing $\langle \cdot, \cdot \rangle$, promotion $\langle \cdot \rangle^{1}$, derelictions $\text{der}_A$, and so here we just quickly review their definitions.

**Definition 3.5.1** (Copy-cat strategies $\text{cp}_A$). The copy-cat strategy $\text{cp}_A : A \rightarrow A$ on a game $A$ is defined by:

$$\text{cp}_A \triangleq \{ (s \in P^{\text{even}}_{A_{1} \leftarrow A_{2}}) \mid \forall t \leq s. \text{even}(t) \Rightarrow t \mid A_{1} = t \mid A_{2} \}.$$

The subscripts 1, 2 on $A$ are to distinguish the two copies of $A$. Note that $t \mid A_{1} = t \mid A_{2}$ indicates the equality of justifiers as well.

**Definition 3.5.2** (Tensor product $\otimes$). Given $\sigma : A \rightarrow C$, $\tau : B \rightarrow D$, their tensor product $\sigma \otimes \tau : A \otimes B \rightarrow C \otimes D$ is defined by:

$$\sigma \otimes \tau \triangleq \{ s \in L_{A \otimes B \rightarrow C \otimes D} \mid s \mid A, C \in \sigma, s \mid B, D \in \tau \}.$$

**Definition 3.5.3** (Pairing $\langle \cdot, \cdot \rangle$). Given $\sigma : C \rightarrow A$, $\tau : C \rightarrow B$, their pairing $\langle \sigma, \tau \rangle : C \rightarrow \text{Ak}_{B}$ is defined by:

$$\langle \sigma, \tau \rangle \triangleq \{ s \in L_{C \rightarrow \text{Ak}_{B}} \mid s \mid C, A \in \sigma, s \mid B = \epsilon \} \cup \{ s \in L_{C \rightarrow \text{Ak}_{B}} \mid C, B \in \tau, s \mid A = \epsilon \}.$$

**Definition 3.5.4** (Promotion $\sigma^{\uparrow}$). Given $\sigma : A \rightarrow B$, its promotion $\sigma^{\uparrow} : A \rightarrow \epsilon_{B}$ is defined by:

$$\sigma^{\uparrow} \triangleq \{ s \in L_{A \rightarrow \epsilon_{B}} \mid \forall m \in \text{InitOcc}(s). s \mid m \in \sigma \}.$$
Definition 3.5.5 (Derelicitions [AJM00 McC98]). Let \( A \) be a well-opened game. The \textit{derelicition} \( \text{der}_A : !A \to A \) on \( A \) is defined to be the copy-cat strategy \( cp_A \) up to the “tags” for the disjoint union of sets of moves for \( !A \).

Definition 3.5.6 (Parallel composition [A⁻⁺⁹⁷]). Given \( \sigma : A \to B, \tau : B \to C \), their \textit{parallel composition} \( \sigma \parallel \tau \) is defined by \( \sigma \parallel \tau \overset{df}{=} \{ s \in M^* | \sigma \downharpoonright s \upharpoonright A, B_1 \in \sigma, s \upharpoonright B_2, C \in \tau, s \upharpoonright B_1, B_2 \in \text{pr}_B \} \), where \( B_1 \) and \( B_2 \) are two copies of \( B \), \( M \overset{df}{=} M((A \to B_1) \to B_2) \to C \), \( s \in M^* \) is a j-sequence in the arena \( ((A \to B_1) \to B_2) \to C \), \( \text{pr}_B \overset{df}{=} \{ s \in P_{B_1} \to B_2 | \forall t \leq s . \text{even}(t) \Rightarrow t \upharpoonright B_1 = t \upharpoonright B_2 \} \), and \( s \upharpoonright A, B_1 \) is a subsequence of \( s \) consisting of moves in \( A \) and \( B_1 \) equipped with the pointer

\[
m \leftarrow n \overset{df}{=} \exists m_1, m_2, \ldots, m_k \in M \setminus M_{A \to B_1}, m \leftarrow m_1 \leftarrow m_2 \leftarrow \cdots \leftarrow m_k \leftarrow n \text{ in } s.
\]

The j-sequences \( s \upharpoonright B_2, C \) and \( s \upharpoonright B_1, B_2 \) are defined similarly.

Remark. Parallel composition is just a preliminary notion for the following official composition of the category of games and strategies [A⁻⁺⁹⁷ AM99 McC98]; it is not a well-defined operation.

Definition 3.5.7 (Composition [AJ94 A⁻⁺⁹⁷ McC98]). Given \( \sigma : A \to B, \tau : B \to C \), their \textit{composition} \( \sigma \circ \tau : A \to C \) (also written \( \tau \circ \sigma \)) is defined by \( \sigma \circ \tau \overset{df}{=} \{ s \upharpoonright A, C | s \in \sigma \parallel \tau \} \), where \( \sigma \downharpoonright A, C \) denotes the obvious j-sequence in the arena \( A \to C \) as in the case of parallel composition.

Now, recall that composition of strategies is “internal communication plus hiding”, which can be seen precisely in the above definition. We now reformulate it as follows:

Definition 3.5.8 (Concatenation and composition of strategies). Let \( \sigma : J, \tau : K \), and assume that \( H^\omega(J) \subseteq A \to B, H^\omega(K) \subseteq B \to C \) for some static games \( A, B, C \). Their \textit{concatenation} \( \sigma \circ \tau \) is defined by

\[
\sigma \circ \tau \overset{df}{=} \{ s \in M_{JK}^* | s \upharpoonright J \in \sigma, s \upharpoonright K \in \tau, s \upharpoonright B_1, B_2 \in \text{pr}_B \}
\]

and their \textit{composition} \( \sigma ; \tau \) is defined by

\[
\sigma ; \tau \overset{df}{=} H^\omega(\sigma \circ \tau).
\]

Theorem 3.5.9 (Well-defined concatenation). Assume that \( \sigma : J, \tau : K, H^\omega(J) \subseteq A \to B, H^\omega(K) \subseteq B \to C \), where \( A, B, C \) are static games. Then \( \sigma \circ \tau : J \to K \) and \( \sigma ; \tau : A \to C \).

Proof. It suffices to prove \( \sigma \circ \tau : J \to K \) since it implies \( \sigma ; \tau = H^\omega(\sigma \circ \tau) : H^\omega(J \to K) \subseteq A \to C \) by Lemma 3.5.14 14.4.6. First, we have \( \sigma \downharpoonright \subseteq P_{JK} \) as any \( s \in \sigma \parallel \tau \) satisfies \( s \in M_{JK}^* \), \( s \upharpoonright J \in \sigma \subseteq P_J, s \upharpoonright K \in \tau \subseteq P_K \) and \( s \upharpoonright B_1, B_2 \in \text{pr}_B \). It is also immediate that such an \( s \) is of even-length. It remains to verify the axioms S1, S2. For this, we need the following:

(\( \Diamond \)) Every \( s \in \sigma \circ \tau \) consists of only adjacent pairs \( m, n \) such that \( m, n \in M_J \) or \( m, n \in M_K \).

Proof of \( \Diamond \). By induction on the length of \( s \). The base case is trivial. For the inductive step, let \( s mn \in \sigma \circ \tau \). If \( m \in M_J \), then \( s \upharpoonright J, m.n \upharpoonright J \in \sigma \), where \( s \upharpoonright J \) is of even-length by the induction hypothesis. Thus, we must have \( m \in M_J \). If \( m \in M_K \), then \( n \in M_K \) by the same argument.

Now, we are ready to show the axioms.

(\( S1 \)) Clearly, \( \epsilon \in \sigma \circ \tau \), so \( \sigma \circ \tau \) is non-empty. For even-prefix-closure, assume \( s mn \in \sigma \circ \tau \). Then by the claim \( (\Diamond) \), either \( m, n \in M_J \) or \( m, n \in M_K \). In either case, it is straightforward to see that \( s \in P_{JK}^* \), \( s \upharpoonright J \in \sigma, s \upharpoonright K \in \tau \) and \( s \upharpoonright B_1, B_2 \in \text{pr}_B \), i.e., \( s \in \sigma \circ \tau \).

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Lemma 3.5.14 is helpful for strategies. We first need the following:

Proof. Therefore we have shown that \( \sigma \vdash \tau : J \vdash K \).

Note that our composition generalizes that of static strategies \([AJM00, AM99, McC98, HO00]\):

Given \( \sigma : A \to B, \tau : B \to C \), where \( A, B, C \) are static, our composition \( \sigma; \tau = H^\omega(\sigma \vdash \tau) : H^\omega((A \to B) \vdash (B \to C)) \subseteq A \to C \) clearly coincides with the existing composition of \( \sigma \) and \( \tau \).

Let us introduce another new construction:

Definition 3.5.11 (Generalized paring). Let \( \sigma : J, \tau : K \) be strategies such that \( H^\omega(J) \subseteq C \to A \), \( H^\omega(K) \subseteq C \to B \), where \( A, B, C \) are static games. Their generalized paring \( \langle \sigma, \tau \rangle \) is defined by

\[
\langle \sigma, \tau \rangle \overset{df}{=} \{ s \in L_{J\&K} \mid s \vdash J \in \sigma, s \vdash K \in \tau, s \vdash J = \epsilon \} \cup \{ s \in L_{J\&K} \mid s \vdash K \in \tau, s \vdash J = \epsilon \}.
\]

Clearly, the notion of generalized paring is a generalization of the usual paring. We often call a generalized paring just paring. Then, it is easy to establish:

Proposition 3.5.12 (Well-defined generalized paring). Given \( \sigma : J, \tau : K \) with \( H^\omega(J) \subseteq C \to A \), \( H^\omega(K) \subseteq C \to B \) for some static games \( A, B, C \), we have \( \langle \sigma, \tau \rangle : J\&K \).

Proof. Straightforward. ■

At the end of the present section, as in the case of games, we establish the hiding lemma on strategies. We first need the following:

Lemma 3.5.13 (Hiding on legal positions in the second form). For any arena \( G \) and number \( d \in \mathbb{N} \cup \{ \omega \} \), we have \( L_{H^d(G)} = \{ s^*_d H^d_G \mid s \in L_G \} \).

Proof. Observe that:

\[
\{ s^*_d H^d_G \mid s \in L_G \} = \{ s^*_d H^d_G \mid s \in L_G, s \text{ is } d\text{-complete} \}
\]

This is clear; \( \subseteq \) is by the prefix-closure of the set of legal positions.

\[
= \{ H^d_G(s) \mid s \in L_G, s \text{ is } d\text{-complete} \}
\]

\[
= \{ H^d_G(s) \mid s \in L_G \} \text{ (by the same argument as above)}
\]

\[
= L_{H^d(G)} \text{ (by Corollary 3.2.7).}
\]

Now we are ready to prove:

Lemma 3.5.14 (Hiding Lemma on strategies). Let \( \&_{i \in I} \) be a construction on strategies, where \( I \) is \( \{ 1 \} \) or \( \{ 1, 2 \} \), and \( \sigma_i : G_i \) a strategy for all \( i \in I \) such that \( \&_{i \in I} \sigma_i \) is well-defined. Then, for all \( d \in \mathbb{N} \cup \{ \omega \} \), we have:
1. $\mathcal{H}^d(\omega_{i} \in \sigma_i) = \omega_{i} \in \mathcal{H}^d(\sigma_i)$ if $\omega_{i}$ is (generalized) pairing $\langle \omega_{i} \rangle$ or promotion $\langle \omega_{i} \rangle^\dagger$.

2. $\mathcal{H}^d(\sigma_1 \upharpoonright \sigma_2) = \mathcal{H}^d(\sigma_1) \upharpoonright \mathcal{H}^d(\sigma_2)$ if $\mathcal{H}^d(\sigma_1 \upharpoonright \sigma_2)$ is not static, and $\mathcal{H}^d(\sigma_1 \upharpoonright \sigma_2) = \mathcal{H}^d(\sigma_1); \mathcal{H}^d(\sigma_2)$ otherwise.

Proof. As in the case of games, it suffices to assume $d = 1$. Also, the statement for concatenation can be established in the same manner as in the case of (positions of) games.

For pairing, let $\sigma_1 : J, \sigma_2 : K$ be strategies such that $\mathcal{H}^d(J) \subseteq C \rightarrow A_1$, $\mathcal{H}^d(K) \subseteq C \rightarrow A_2$. We first show $\mathcal{H}(\langle \sigma_1, \sigma_2 \rangle) \subseteq \langle \mathcal{H}(\sigma_1), \mathcal{H}(\sigma_2) \rangle$:

$$s \in \mathcal{H}(\langle \sigma_1, \sigma_2 \rangle) \Rightarrow \exists t \in \langle \sigma_1, \sigma_2 \rangle. t \upharpoonright \mathcal{H}^d_{1 \& K} = s$$

$$\Rightarrow \exists t \in L_{J \& K} \land t \upharpoonright \mathcal{H}^d_{1 \& K} = s \land ((t \upharpoonright J \in \sigma_1 \land t \upharpoonright K = \epsilon) \lor (t \upharpoonright K \in \sigma_2 \land t \upharpoonright J = \epsilon))$$

$$\Rightarrow s \in L_{H(J \& K)} \land (s \upharpoonright H(J) \in \mathcal{H}(\sigma_1) \land s \upharpoonright H(K) = \epsilon) \lor (s \upharpoonright H(K) \in \mathcal{H}(\sigma_2) \land s \upharpoonright H(J) = \epsilon)$$

(by Lemma 3.5.13)

$$\Rightarrow s \in \langle \mathcal{H}(\sigma_1), \mathcal{H}(\sigma_2) \rangle.$$ 

Next, we show the converse:

$$s \in \langle \mathcal{H}(\sigma_1), \mathcal{H}(\sigma_2) \rangle \Rightarrow s \in L_{H(J \& K)} \land (s \upharpoonright H(J) \in \mathcal{H}(\sigma_1) \land s \upharpoonright H(K) = \epsilon) \lor (s \upharpoonright H(K) \in \mathcal{H}(\sigma_2) \land s \upharpoonright H(J) = \epsilon)$$

$$\Rightarrow (\exists u \in \sigma_1, u \upharpoonright H^d_{1 \& K} = s \upharpoonright H(J) \land u \upharpoonright K = \epsilon) \lor (\exists v \in \sigma_2, v \upharpoonright H^d_{1 \& K} = s \upharpoonright H(K) \land v \upharpoonright H(J) = \epsilon)$$

$$\Rightarrow \exists w \in \langle \sigma_1, \sigma_2 \rangle. w \upharpoonright H^d_{1 \& K} = s$$

$$\Rightarrow s \in \mathcal{H}(\langle \sigma_1, \sigma_2 \rangle).$$

Next, for promotion, let $\psi : J$ be a strategy. Then we have:

$$\mathcal{H}(\psi^\dagger) = \{ s \upharpoonright H^d_{1 \& J} | s \in \psi^\dagger \}$$

$$= \{ s \upharpoonright H^d_{1 \& J} | s \in L_{J}, s \upharpoonright m \in \psi \text{ for all initial } m \}$$

$$\subseteq \{ s \upharpoonright H^d_{1 \& J} | s \in L_{J}, (s \upharpoonright m) \upharpoonright H^d_{1 \& J} \in \mathcal{H}(\psi) \text{ for all initial } m \}$$

$$= \{ s \upharpoonright H^d_{1 \& J} | s \in L_{J}, (s \upharpoonright m) \upharpoonright m \in \mathcal{H}(\psi) \text{ for all initial } m \}$$

$$= \{ t \in L_{H(J)} | t \upharpoonright m \in \mathcal{H}(\psi) \text{ for all initial } m \} \text{ (by Lemma 3.5.13) }$$

$$\Rightarrow \mathcal{H}(\psi^\dagger) = \mathcal{H}(\psi)^\dagger.$$ 

For the opposite inclusion, observe the following:

$$s \in \mathcal{H}(\psi^\dagger) \Rightarrow s \in L_{H(J)} \land \forall m \in \text{initOcc}(s). s \upharpoonright m \in \mathcal{H}(\psi)$$

$$\Rightarrow s \in L_{H(J)} \land \forall m \in \text{initOcc}(s). \exists t_m \in \psi. t_m \upharpoonright H^d_{1 \& J} = s \upharpoonright m$$

$$\Rightarrow \exists t \in \psi^\dagger. t \upharpoonright H^d_{1 \& J} = s$$

$$\Rightarrow s \in \mathcal{H}(\psi^\dagger).$$

This lemma, together with the hiding lemma on games (Lemma 3.3.13), will play a key role in establishing our game-theoretic CCBiC shortly.
4 Dynamic Game Semantics for System Tϑ

This section is the climax of the present paper: We first establish a game-theoretic instance of a CCBiC DG and a structure T in DG, and then show that the induced interpretation \([\cdot]_T^{DG}\) satisfies PDCP, thus DCP as well, giving a dynamic game semantics for System Tϑ.

4.1 Dynamic Game Semantics for System Tϑ

We are now ready to give our game-theoretic CCBiC DG of dynamic games and strategies. Note that by Proposition 2.1.2 it suffices to give a (generalized) CCC equipped with values and evaluations in which the equality = is replaced by the induced equivalence relation \(\cong\).

Definition 4.1.1 (The CCBiC DG). The CCBiC DG = (DG, V, H) is defined by:

- 0-cells \(A \in DG\) are well-opened static games
- 1-cells \(\sigma \in DG(A, B)\) are strategies \(\sigma : J\) such that \(H^\omega(J) \leq !A \multimap B\)
- The horizontal composition of 1-cells \(\sigma : A \to B, \tau : B \to C\) is the concatenation \(\sigma^\dagger \tau : A \to C\), and the horizontal identity \(id_A : A \to A\) is the dereliction \(der_A\)
- \(\mathcal{V}(A, B) \overset{df}{=} \{\sigma \in DG(A, B) | H^\omega(\sigma) = \sigma\}\)
- \(H\) is the hiding operation on strategies
- The terminal object is the terminal game \(I = (\emptyset, \emptyset, \emptyset, \{\epsilon\})\)
- Binary products and exponentials are defined by \(A \times B \overset{df}{=} A \& B, A \Rightarrow B \overset{df}{=} !A \multimap B\), respectively, for all \(A, B \in DG\) (clearly, DG has countably infinite products)
- Pairing \(\langle \_\_ \rangle\) is the pairing of strategies, and projections \(\pi_1, \pi_2\) are appropriate derelictions \[AM99\]
- Currying \(\Lambda\) is “adjusting tags for disjoint union”, and evaluations \(App\) are appropriate derelictions \[AM99, A^+97\].

Theorem 4.1.2 (Well-defined DG). The structure DG forms a well-defined CCBiC.

Proof. By Proposition 2.1.2 it suffices to give a (generalized) CCC equipped with values and evaluations in which the operations on morphisms (i.e., composition, pairing and currying) preserve the induced equivalence relation \(\cong\) and the required equations hold up to \(\cong\).

First, the horizontal composition \(\dagger\) on 1-cells is well-defined: Given objects \(A, B, C\) and 1-cells \(\sigma : A \to B, \tau : B \to C\), i.e., \(\sigma : J, \tau : K\) for some games \(J, K\) with \(H^\omega(J) \leq !A \multimap B, H^\omega(K) \leq !B \multimap C\), we have \(\sigma^\dagger : !J\) with \(H^\omega(!J) \leq !(H^\omega(J)) = !(A \multimap B) = !A \multimap !B\), by Lemmata 3.5.14 and 3.3.15.

Clearly, associativity up to 2-cell isomorphisms holds: Given 1-cells \(\sigma : A \to B, \tau : B \to C, \rho : C \to D\), we have \(H^\omega((\sigma^\dagger \tau)^\dagger \rho) = (H^\omega(\sigma)^\dagger ; H^\omega(\tau)^\dagger) ; H^\omega(\rho) = H^\omega(\sigma)^\dagger ; (H^\omega(\tau)^\dagger ; H^\omega(\rho)) = H^\omega(\sigma^\dagger (\tau^\dagger \rho))\) by Lemma 3.5.14. Note that in the CCC of static games \[AM99, McC98\] the composition is associative, and \((f^\dagger ; g)^\dagger = f^\dagger ; g\) holds for any composable morphisms \(f, g\).

Unit law up to \(\cong\) holds in a similar manner: Given a 1-cell \(\sigma : A \to B\), we have \(H^\omega(id_A^\dagger ; \sigma) = \sigma = H^\omega(\sigma)\) and \(H^\omega(\sigma^\dagger ; id_B) = H^\omega(\sigma)^\dagger = H^\omega(\sigma)\) again by Lemma 3.5.14 and basic properties of derelictions \[AM99, McC98\].
Also, $\mathcal{V}$ and $\mathcal{H}$ clearly satisfy the required five axioms. Thus, we have shown that $\mathcal{D}G$ is a BiC. It remains to verify the cartesian closed structure up to $\cong$.

The projections $\pi_1 : !(A\&B) \to A$, $\pi_2 : !(A\&B) \to B$ are clearly well-defined values $\pi_1 : A\&B \to A$, $\pi_2 : A\&B \to B$. Given any 1-cells $\sigma : C \to A$, $\tau : C \to B$, i.e., $\sigma : J$, $\tau : K$ such that $\mathcal{H}^\omega(J) \leq !C \to A$, $\mathcal{H}^\omega(K) \leq !C \to B$, we may obtain the pairing $\langle \sigma, \tau \rangle : J\&K$ such that $\mathcal{H}^\omega(J\&K) \leq \mathcal{H}^\omega(J)\&\mathcal{H}^\omega(K) \leq !C \to (A\&B)$ again by Lemma 3.3.15 and so $\langle \sigma, \tau \rangle$ is a well-defined 1-cell $C \to A\&B$. Also, $\mathcal{H}^\omega(\langle \sigma, \tau \rangle) = \langle \mathcal{H}^\omega(\sigma)\rangle, \mathcal{H}^\omega(\tau)\rangle; \pi_1 = \mathcal{H}^\omega(\sigma)$ by Lemma 3.5.14 and a basic property of derelictions $[AM99, McC98]$: similarly, $\pi_2 \equiv \tau$. Analogously, given a 1-cell $\phi : C \to A\&B$, we have $\mathcal{H}^\omega(\langle \phi, \pi_1 \rangle, \pi_2) = \langle \mathcal{H}^\omega(\phi), \pi_1, \mathcal{H}^\omega(\phi)\rangle; \pi_2 = \mathcal{H}^\omega(\phi)$. The evaluation $\text{App}_{B,C} : !((B \Rightarrow C)\&B) \Rightarrow C$ for any games $B, C$ is clearly a value $\text{App}_{B,C} : (B \Rightarrow C)\&B \Rightarrow C$ that satisfies

$\mathcal{H}^\omega((\Lambda_B(\psi) \times \text{id}_B) \downarrow \downarrow \text{App}_{B,C}) = (\Lambda_B(\mathcal{H}^\omega(\psi)) \times \text{der}_B) \downarrow \downarrow \text{App}_{B,C} = \mathcal{H}^\omega(\psi)$

$\mathcal{H}^\tau((\Lambda_B(\varphi) \times \text{id}_B) \downarrow \downarrow \text{App}_{B,C}) = \Lambda_B((\mathcal{H}^\omega(\varphi) \times \text{der}_B) \downarrow \downarrow \text{App}_{B,C}) = \mathcal{H}^\omega(\varphi)$

for any 1-cells $\psi : A\&B \Rightarrow C$, $\varphi : A \Rightarrow (B \Rightarrow C)$ by Lemma 3.5.14 and the cartesian closed structure of the CCC of static games in $[McC98, AM99]$. Finally, the composition $\langle \cdot \rangle \downarrow \downarrow \langle \cdot \rangle$, pairing $\langle \cdot, \cdot \rangle$ and currying $\Lambda$ preserve the equivalence relation $\cong$ by Lemma 3.5.14 completing the proof.

We proceed to give a structure for System $T_\theta$ in $\mathcal{D}G$:

**Definition 4.1.3** (Structure $T$ in $\mathcal{D}G$). The structure $T = (N, (\cdot), \vartheta)$ of dynamic games and strategies for System $T_\theta$ in $\mathcal{D}G$ is defined as follows:

- $N$ is the natural numbers game in Example 3.5.10
- $n \triangleq \text{pref}(\{qn\})$ is a strategy on $I \Rightarrow N$ for each $n \in \mathbb{N}$
- $\vartheta : N \times N^\omega \Rightarrow N$ is the standard interpretation of the case-construction in the language PCF $[HO00, AM99]$.  

**Remark.** Our interpretation of values coincides with the interpretation of PCF Böhm trees as innocent strategies in $[HO00]$ (see the proof of the strong definability result). In a sense, this interpretation is generalized here: Configurations, which are constructed from values via application, are interpreted utilizing the “non-hiding composition” $\downarrow \downarrow$, and the rule $c_2$ is interpreted in the same way as the rule $c_1$. Moreover, the rule $c_1$ itself is reformulated in terms $\vartheta, \downarrow \downarrow$ and $\mathcal{H}$.

### 4.2 Main Result

At last, we prove that our dynamic game semantics for System $T_\theta$ in fact satisfies the dynamic correspondence property (DCP) via the pointwise dynamic correspondence property (PDCP).

**Theorem 4.2.1** (Main theorem). The interpretation $[\cdot]_{\mathcal{D}G}^T$ of System $T_\theta$ satisfies PDCP.

**Proof.** Since $\mathcal{H}$ clearly computes in the FCFE-fashion, it remains to show that for any rewriting $\langle \lambda x^A. V \rangle W \Rightarrow [U], \text{ where } V, W, U$ are values, $\mathcal{H}([\langle \lambda x^A. V \rangle W])_{\mathcal{D}G} = [U]_{\mathcal{D}G}$. For this, we define the height $Ht(B) \in \mathbb{N}$ of each type $B$ by $Ht(N) \triangleq 0$, $Ht(B_1 \Rightarrow B_2) \triangleq \max(Ht(B_1)+1, Ht(B_2))$. Then the equation is shown by induction on the height of the type $A$ of $W$.

Below, given strategies $\tau : C \Rightarrow (A \Rightarrow B)$, $\sigma : C \Rightarrow A$, we define $\tau[\sigma] \triangleq (\tau, \sigma) \downarrow \downarrow \text{App}_{A,B} : C \Rightarrow B$. If $\tau : C \Rightarrow (A_1 \Rightarrow A_2 \Rightarrow \cdots \Rightarrow A_k \Rightarrow B)$ and $\sigma_i : C \Rightarrow A_i$ for $i = 1, 2, \ldots, k$ then...
we write \( \tau | \sigma_1, \sigma_2, \ldots, \sigma_k \) for \( \tau | \sigma_1 \parallel \sigma_2 \parallel \ldots \parallel \sigma_k : C \rightarrow B \). Also, we abbreviate the operational semantics \( \rightarrow_T \), and the interpretation \( \llbracket \cdot \rrbracket_T \) as \( \rightarrow \) and \( \llbracket \cdot \rrbracket \), respectively.

For the base case, assume \( Ht(A) = 0 \), i.e., \( A \equiv N \). By induction on the length of \( V \), we have:

- If \( V \equiv n \) with \( n \in \mathbb{N} \), then \( (\lambda x^n)W \rightarrow n \) and it is easy to see that \( \mathcal{H}((\llbracket (\lambda x^n)W \rrbracket)) = [n] \).
- If \( V \equiv \lambda y^C V' \), then \( (\lambda x^A y^C V')W \rightarrow \lambda y^C V' \) such that \( (\lambda x^A y^C V')W \rightarrow U' \) (as \( nf((\lambda x^A y^C V')W) \equiv nf(\lambda y^C V'[W/x]) \equiv \lambda y^C nf((\lambda x^A V')W)) \). By the induction hypothesis, we have \( \mathcal{H}((\llbracket (\lambda x^A V')W \rrbracket)) = [U'] \). Hence, we may conclude that:

\[
\mathcal{H}(\llbracket VW \rrbracket) = \mathcal{H}((\Lambda [A](\Lambda [C]([V']))) ([W])^{\uparrow} App)
\]

\[
= (\Lambda [A](\Lambda [C]([V']))) ([W])^{\uparrow} App \; (by \text{Lemma 3.5.14})
\]

\[
= \Lambda [C](\Lambda [A]([V'])) ([W])^{\uparrow} App
\]

\[
= \Lambda [C](\mathcal{H}(\Lambda [A]([V']))) ([W])^{\uparrow} App
\]

\[
= \Lambda [C](\llbracket (\lambda x^A V')W \rrbracket)
\]

\[
= \Lambda [C](\llbracket U' \rrbracket)
\]

\[
= [\lambda y^C U'].
\]

- If \( V \equiv \text{case}(y V_1 \ldots V_k) [V'_0 [V'_1] \ldots] \) with \( x \neq y \), then \( (\lambda x^A V)W \rightarrow U \), where

\[
U \equiv \text{case}(y nf(V_1[W/x]) \ldots nf(V_k[W/x]))[nf(V_0[W/x])]nf(V'_1[W/x]) \ldots].
\]

By the induction hypothesis and the interpretation of the variable \( y \), we have:

\[
\llbracket (\lambda x^A V)W \rrbracket
\]

\[
= \mathcal{H}(\Lambda [A]([y]) ([V_1], \ldots, [V_k], [V'_0], [V'_1], \ldots)^{\uparrow} \vartheta) ([W])
\]

\[
= \mathcal{H}((\Lambda [A]([y]) ([W]) [\Lambda [A]([V_1]) ([W]), \ldots, \Lambda [A]([V_k]) ([W]), \Lambda [A]([V'_0]) ([W]), \Lambda [A]([V'_1]) ([W]), \ldots)^{\uparrow} \vartheta)
\]

\[
= \mathcal{H}((\llbracket (\lambda x^A V)W \rrbracket) [\llbracket (\lambda x^A V)W \rrbracket], \ldots, \llbracket (\lambda x^A V)W \rrbracket, \llbracket (\lambda x V'_0) W \rrbracket, \llbracket (\lambda x V'_1) W \rrbracket, \ldots)^{\uparrow} \vartheta)
\]

\[
= \mathcal{H}((\llbracket y \rrbracket [nf(V_1[W/x])], \ldots, [nf(V_k[W/x])], [nf(V'_0[W/x])], [nf(V'_1[W/x])], \ldots)^{\uparrow} \vartheta)
\]

(by the induction hypothesis)

\[
= [U].
\]

- If \( V \equiv \text{case}(x) [V'_0 [V'_1] \ldots] \), then \( (\lambda x^A V)W \rightarrow U \), where

\[
U \equiv \text{case}(W)[nf(V'_0[W/x])]nf(V'_1[W/x]) \ldots].
\]

By the same reasoning as the above case, we may conclude that \( \mathcal{H}((\llbracket (\lambda x^A V)W \rrbracket)) = [U] \).

Next, consider the inductive case; assume that \( Ht(A) = h + 1 \). We may proceed in the same way as the base case, i.e., by induction on the length of \( V \), except that the last case is now generalized as \( V \equiv \text{case}(x V_1 \ldots V_k) [V'_0 [V'_1] \ldots] \) and \( A \equiv A_1 \Rightarrow A_2 \Rightarrow \cdots \Rightarrow A_k \Rightarrow N (k \geq 0) \). We have to consider the cases for \( k \geq 1 \); then we have \( (\lambda x^A V)W \rightarrow U \), where

\[
U \equiv \text{case}(nf(W(V_1[W/x]) \ldots [V_k[W/x]])[nf(V'_0[W/x])]nf(V'_1[W/x]) \ldots].
\]
We then have the following chain of equations between strategies:

\[
\mathcal{H}[\langle x \rangle_{\Sigma}]_{\zeta}
\]

\[
= \mathcal{H}(\lambda \alpha(\mathcal{H}[\langle x \rangle_{\Sigma}]_{\zeta}))_{\zeta}
\]

\[
= \mathcal{H}^E(\langle \lambda \alpha(x) \rangle_{\Sigma} [\lambda \beta]_{\zeta} [\lambda \alpha(x) \beta]_{\zeta} \ldots)_{\zeta}
\]

\[
= \mathcal{H}^E(\langle [\lambda \alpha(x)]_{\Sigma} [\lambda \beta]_{\zeta} [\lambda \alpha(x) \beta]_{\zeta} \ldots)_{\zeta}
\]

which completes the proof.

**Corollary 4.2.2 (Dynamic correspondence).** The interpretation \( [\_]_{T^D} \) of System \( T^D \) and the hiding operation \( \mathcal{H} \) satisfy the DCP with respect to the operational semantics \( \rightarrow_{T^D} \).

**Proof.** By Theorems [3.39][3.4.1].

In fact, the relation between the syntax and semantics is much tighter than this corollary: If we carve out a model of computation \( T^D \) from \( D^\alpha \) that consists of definable elements, then exploiting the correspondence between PCF Böhm trees and strategies, \([AC98][HO00]\), System \( T^D \) can be seen as a formal calculus for \( T^D \). As an illustration:

**Example 4.2.3.** We may now make the example in the introduction precise. The program \( \vdash (\lambda x.\nu^N.\text{double}(\text{succ}^N))_{\Sigma} : \text{N} \) evaluates as:

\[
(\lambda x.\text{double}(\text{succ}^N))_{\Sigma}
\]

\[\begin{align*}
\rightarrow_{T^D} & (\lambda x.\text{case}(z)[n \mapsto 2 \cdot n]\text{case}(x)[n \mapsto n + 1])_{\Sigma} \\
\rightarrow_{T^D} & (\lambda x.\text{case}(x)[n \mapsto 2 \cdot (n + 1)])_{\Sigma} \rightarrow_{T^D} 12
\end{align*}
\]

In \( T^D \), the corresponding computation proceeds as:

\[
\begin{align*}
q^5& qqq56(12) \rightarrow_{T^D} qqq5(12) \rightarrow_{T^D} q(12)
\end{align*}
\]

where we simplify the duplication of internal moves.

In contrast, the program \( \vdash (\lambda y.\nu^N.\text{succ}(\langle \lambda x.\nu^N.\text{double}(\text{succ}^N) \rangle y))_{\Sigma} : \text{N} \) evaluates as:

\[
(\lambda y.\text{succ}(\langle \lambda x.\text{double}(\text{succ}^N) \rangle y))_{\Sigma}
\]

\[\begin{align*}
\rightarrow_{T^D} & (\lambda y.\text{succ}(\langle \lambda x.\text{case}(x)[n \mapsto 2 \cdot n]\rangle y))_{\Sigma} \\
\rightarrow_{T^D} & (\lambda y.\text{succ}(\langle \lambda x.\text{case}(x)[n \mapsto (2 \cdot n) + 1]\rangle y))_{\Sigma} \\
\rightarrow_{T^D} & (\lambda y.\text{case}(y)[n \mapsto (2 \cdot n) + 1])_{\Sigma} \rightarrow_{T^D} 12
\end{align*}
\]

The corresponding computation in \( T^D \) is as follows:

\[
\begin{align*}
q^5& qqqq5(10)(11)(12) \rightarrow_{T^D} qqq5(11)(12) \rightarrow_{T^D} qqq5(12) \rightarrow_{T^D} q(12).
\end{align*}
\]
5 Conclusion and Future Work

We have presented the first mathematical (and syntax-independent) formulation of intentionality and dynamics in computation in terms of games and strategies. From the opposite angle, we have developed a game-theoretic model of computation with a convenient formal calculus.

The most immediate future work is to apply this framework to various logics and computations as in the case of static (usual) game semantics. Moreover, since the hiding operation can be further refined into the “move-wise” fashion, the present work may be applicable for finer calculi such as explicit substitution [Ros96] and the differential λ-calculus [ER03]. Also, it would be interesting to see how accurately our game-theoretic model can measure the complexity of (higher-order) programming. Finally, the notion of BiCs can be a concept of interest in its own right; for instance, it would be fruitful to develop it further in order to accommodate various models of computations in the same spirit of [LN15] but on computation, not computability.

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A Proof of Lemma 3.3.13

We proceed by induction on the length of \( t = ur \). If \( r \) is initial, then the clause 2 holds:

\[
[t]_K = [(u \downarrow K) r]_K = r = r \downarrow K = [ur]_{JIK} \downarrow K = [t]_{JIK} \downarrow K
\]

and

\[
[t \downarrow K]_K = [(u \downarrow K).r]_K \\
= [u \downarrow K].r \\
\leq ([(u \downarrow \downarrow K) \downarrow K).r \text{ (by the induction hypothesis)} \\
= ([u]_{JIK}).r \downarrow K \\
= [ur]_{JIK} \downarrow K \\
= [t]_{JIK} \downarrow K
\]

Now, we may assume that \( r \) is non-initial; so we may write \( t = ur = u_1l u_2r \), where \( r \) is justified by \( l \). We proceed by a case analysis on \( r \). Let us first assume that \( r \) is an O-move:

1. If \( r \in M_J \setminus M_{B_1} \), then \( l \in M_J \). Thus, we have:

\[
[t \downarrow J]_J = [(u_1 \downarrow J).l.(u_2 \downarrow J).r]_J \\
= [u_1 \downarrow J].l.\downarrow r \\
\leq ([(u_1 \downarrow JIK) \downarrow J).l.\downarrow r \text{ (by the induction hypothesis)} \\
= ([u_1]_{JIK}).l.\downarrow J \\
= [u_1l u_2r]_{JIK} \downarrow J \\
= [t]_{JIK} \downarrow J
\]

and

\[
[t \downarrow J]_J = [(u \downarrow J).r]_J \\
= [u \downarrow J].r \\
\leq ([(u \downarrow JIK) \downarrow J).r \text{ (by the induction hypothesis)} \\
= [u]_{JIK}.r \downarrow J \\
= [ur]_{JIK} \downarrow J \\
= [t]_{JIK} \downarrow J
\]
2. If \( r \in M_K \setminus M_{B_2} \), then \( l \in M_K \). Thus, we have:

\[
[t | K]_K = ([u_1 | K].l.(t_2 | K).r]_K \\
= [u_1 | K]_K lr \\
\leq ([u_1]_{J|K} | K).lr \text{ (by the induction hypothesis)} \\
= ([u_1[lu_2r]]_{J|K} | K \\
= [t]_{J|K} | K
\]

and

\[
[t | K]_K = ([u | K].r]_K \\
= [u | K]_K r \\
\leq ([u]_{J|K} | K).r \text{ (by the induction hypothesis)} \\
= [u]_{J|K} | K \\
= [ur]_{J|K} | K \\
= [t]_{J|K} | K.
\]

3. If \( r \in M_{B_1} \cup M_{B_2} \) and it is an O-move in \( B_1 \not\rightarrow B_2 \), then \( l \in M_{B_1} \cup M_{B_2} \). Thus, we have:

\[
[t | B_1, B_2]_{B_1 \not\rightarrow B_2} = ([u_1 | B_1, B_2].l.(u_2 | B_1, B_2).r]_{B_1 \not\rightarrow B_2} \\
= ([u_1 | B_1, B_2])_{B_1 \not\rightarrow B_2} lr \\
\leq ([u_1]_{J|K} | B_1, B_2).lr \text{ (by the induction hypothesis)} \\
= [u_1]_{J|K} | B_1, B_2 \\
= [u_1lu_2r]_{J|K} | B_1, B_2 \text{ (as } r \text{ is a P-move in } J \not\rightarrow K) \\
= [t]_{J|K} | B_1, B_2.
\]

and

\[
[t | B_1, B_2]_{B_1 \not\rightarrow B_2} = ([u | B_1, B_2].r]_{B_1 \not\rightarrow B_2} \\
= [u | B_1, B_2]_{B_1 \not\rightarrow B_2} r \\
\leq ([u]_{J|K} | B_1, B_2).r \text{ (by the induction hypothesis)} \\
= [u]_{J|K} | B_1, B_2 \\
= [ur]_{J|K} | B_1, B_2 \text{ (as } r \text{ is a P-move in } J \not\rightarrow K) \\
= [t]_{J|K} | B_1, B_2.
\]

Next, let us consider the case where \( r \) is a P-move:

1. If \( r \in M_J \setminus M_{B_1} \), then \( l \in M_J \). Thus, we have:

\[
[t | J]_J = ([u | J].r]_J \\
= [u | J]_J r \\
\leq ([u]_{J|K} | J).r \text{ (by the induction hypothesis)} \\
= [u]_{J|K} | J \\
= [ur]_{J|K} | J \\
= [t]_{J|K} | J
\]
and

\[ [t \upharpoonright J]_J = [(u_1 \upharpoonright J) . l . (u_2 \upharpoonright J) . r]_J \]
\[ = [u_1 \upharpoonright J]_J . lr \]
\[ \leq ([u_1]_{J \upharpoonright K} \upharpoonright J) . lr \] (by the induction hypothesis)
\[ = ([u_1]_{J \upharpoonright K}) . lr \upharpoonright J \]
\[ = [u_1 l u_2 r]_{J \upharpoonright K} \]
\[ = [t]_{J \upharpoonright K} \]

2. If \( r \in M_K \setminus M_B \), then \( l \in M_K \). Thus, we have:

\[ [t \upharpoonright K]_K = [(u \upharpoonright K) . r]_K \]
\[ = [u \upharpoonright K]_K . r \]
\[ \geq ([u]_{J \upharpoonright K} \upharpoonright K) . r \] (by the induction hypothesis)
\[ = [u]_{J \upharpoonright K} . r \upharpoonright K \]
\[ = [u r]_{J \upharpoonright K} \]
\[ = [t]_{J \upharpoonright K} \]

and

\[ [t \upharpoonright K]_K = [(u_1 \upharpoonright K) . l . (u_2 \upharpoonright K) . r]_K \]
\[ = [u_1 \upharpoonright K]_K . lr \]
\[ \leq ([u_1]_{J \upharpoonright K} \upharpoonright K) . lr \] (by the induction hypothesis)
\[ = ([u_1]_{J \upharpoonright K}) . lr \upharpoonright K \]
\[ = [u_1 l u_2 r]_{J \upharpoonright K} \]
\[ = [t]_{J \upharpoonright K} \]

which completes the proof of the lemma.