ADJOINT REIDEMEISTER TORSIONS OF ONCE-PUNCTURED TORUS BUNDLES

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Abstract. Gang, Kim and Yoon have recently proposed a conjecture on a vanishing identity of adjoint Reidemeister torsions of hyperbolic 3-manifolds with torus boundary, from the viewpoint of wrapped M5-branes. In this paper, we provide infinitely many new supporting examples to this conjecture. These examples come from hyperbolic once-punctured torus bundles. We show that the vanishing identity holds for all hyperbolic once-punctured torus bundles with tunnel number one. We also show the vanishing identity does not hold for any torus knot exteriors.

1. Introduction

D. Gang, S. Kim and S. Yoon proposed a vanishing identity of the adjoint Reidemeister torsion in [GKY]. Their vanishing identity is based on the observation by 3D–3D correspondence between 3D supersymmetric quantum field theories and mathematics of 3-manifolds and knots.

Conjecture ([GKY Conjecture 1.1]). Let \( M \) be a compact 3-manifold with a torus boundary whose interior admits a hyperbolic structure. Suppose that the character variety of irreducible \( SL_2(\mathbb{C}) \)-representations of \( \pi_1(M) \) consists of only irreducible components of dimension 1. Then for any slope \( \gamma \in H_1(\partial M; \mathbb{Z}) \) we have

\[
\sum_{[\rho] \in \text{tr}_\gamma^{-1}(z)} \frac{1}{T_{M,\gamma}(\rho)} = 0
\]

for generic \( z \in \mathbb{C} \). Here \( \text{tr}_\gamma \) is the trace function of \( \gamma \) defined by \( \text{tr}_\gamma([\rho]) = \text{tr} \rho(\gamma) \) on the character variety and \( T_{M,\gamma}(\rho) \) is the adjoint Reidemeister torsion with respect to \( \rho \) and \( \gamma \).

It was shown in [GKY] that the vanishing identity holds for the figure eight knot exterior. Then Yoon has shown that the vanishing identity holds for all hyperbolic twist knot exteriors in [Yo1] and furthermore extended the result of [Yo1] to all hyperbolic two-bridge knots in [Yo2] with respect to a meridian slope.

The purpose of this paper is to study the vanishing identity of the adjoint Reidemeister torsions for hyperbolic 3-manifolds different from knot exteriors. We investigate the vanishing identity for hyperbolic once-punctured torus bundles with tunnel number one.

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according to the observation in [GKY]. Gang–Kim–Yoon’s observation provides the description of the adjoint Reidemeister torsion as a rational function by using defining polynomials of character varieties and proves the vanishing identity by the Jacobi’s residue theorem. We will show infinitely many new supporting examples to the conjecture by Gang, Kim and Yoon in once-punctured torus bundles over the circle.

**Theorem.** The vanishing identity of adjoint Reidemeister torsion holds for all hyperbolic once-punctured torus bundles with tunnel number one.

We also show that the inverse sum of adjoint Reidemeister torsions equals $\pm 2$ for all torus knot exteriors $M_{r,s}$ of type $(r, s)$ and all slopes $\gamma$.

**Theorem (Theorem 7.4).** For any slope $\gamma = \mu^p \lambda^q$ and a generic $c \in \mathbb{C}$, it holds that

$$\sum_{[\rho] \in H_1(c)} \frac{1}{T_{M_{r,s},\gamma}(\rho)} = \pm 2.$$ 

**Corollary (Corollary 7.5).** The vanishing identity of adjoint Reidemeister torsion does not hold for any torus knot exteriors and any slopes.

2. **Preliminaries**

2.1. **The fundamental groups of once-punctured torus bundles.** We almost follow the convention and notation used in [BP]. Let $\beta$ and $\beta'$ denote curves on the once-punctured torus $T$ transversally intersecting once and $\tau_a$ be the right-handed Dehn twist along the curve $a$. We can regard the once-punctured torus bundles with tunnel number one, up to mirror images, as a one-parameter family $\{M_n\}_{n \in \mathbb{Z}}$ of the mapping tori of the homeomorphims $\phi_n = \tau_{\beta'} \tau_{\beta}^{-n+2}$ on $T$, that is,

$$M_n = T \times [0, 1]/(x, 0) \sim (\phi_n(x), 1).$$

If $n$ satisfies $|n| > 2$, then the once-punctured torus bundle $M_n$ is hyperbolic.

By abuse of notation, we use the same letters $\beta$ and $\beta'$ for the homotopy classes in $\pi_1(T, *)$ with the base point $* \in \partial T$. According to [BP] the induced isomorphism $(\phi_n)_*$ maps $\beta$ and $\beta'$ to $\beta \beta'$ and $\beta' (\beta \beta')^{-n-2}$ respectively. Then we have the following presentation of $\pi_1(M_n)$:

$$\pi_1(M_n) = \langle \beta', \beta, \mu \mid \beta (\beta' \beta')^{-n+2} = \mu \beta' \mu^{-1}, \beta \beta' = \mu \beta \mu^{-1} \rangle$$

$$= \langle \alpha, \beta \mid \beta^{-n} = \alpha^{-1} \beta \alpha^2 \beta \alpha^{-1} \rangle$$

where $\mu$ stands for the homotopy class of an embedded curve in $\partial M_n$ transversally intersecting each fiber once and we put $\alpha = \beta^{-1} \mu$ in the second equality.

We call a curve representing $\mu = \beta \alpha$ the meridian of $M_n$. Let $\lambda$ be the homotopy class represented by the boundary of a fiber corresponding $\partial T$. Giving $\partial T$ the boundary orientation as $\partial T = \beta' \beta' \beta^{-1} \beta^{-1}$, we have a pair of meridian-longitude such that $\mu = \beta \alpha$.
\[ \beta \alpha \text{ and } \lambda = \beta' \beta'^{-1} \beta^{-1} = \alpha \beta \alpha^{-1} \beta^{-1}, \text{ which follows from } \beta' = \beta^{-1} \mu \beta \mu^{-1} = \alpha \beta \alpha^{-1} \beta^{-1}. \]

2.2. Review on the character varieties for once-punctured torus bundles. Let \( \rho \) denote a homomorphism from \( \pi_1(M_n) \) into \( \text{SL}_2(\mathbb{C}) \). We call \( \rho \) an \( \text{SL}_2(\mathbb{C}) \)-representation of \( \pi_1(M_n) \). We will observe \( \text{SL}_2(\mathbb{C}) \)-representations \( \rho \) up to conjugation. This means that we will often replace an \( \text{SL}_2(\mathbb{C}) \)-representation \( \rho \) by the composition with an inner automorphism of \( \text{SL}_2(\mathbb{C}) \), that is \( A \rho A^{-1} \) for some \( A \in \text{SL}_2(\mathbb{C}) \).

An \( \text{SL}_2(\mathbb{C}) \)-representation \( \rho \) is referred to as being irreducible if the standard action of \( \rho(\pi_1(M_n)) \) on \( \mathbb{C}^2 \) has no nontrivial invariant subspace of \( \mathbb{C}^2 \), in other words, the \( \text{SL}_2(\mathbb{C}) \)-matrices \( \rho(\alpha) \) and \( \rho(\beta) \) are not conjugate to upper triangular matrices simultaneously.

It is known that we can think of the set of conjugacy classes of irreducible \( \text{SL}_2(\mathbb{C}) \)-representations as an algebraic set called the character variety (precisely, its Zariski closure is called the character variety for irreducible representations). We will denote the character variety by \( X(\pi_1(M_n)) \). The following functions on \( X(\pi_1(M_n)) \):

\[ x = \text{tr} \rho(\alpha), \quad y = \text{tr} \rho(\beta), \quad z = \text{tr} \rho(\alpha \beta) \]

give the structure of an algebraic set in \( \mathbb{C}^3 \) when the fundamental group \( \pi_1(M_n) \) are generated by two generators \( \alpha \) and \( \beta \) (for details see \cite[Proposition 5.20]{BP}). This means that the variables \( x, y \) and \( z \) play a role as a coordinate \( (x, y, z) \) of the character variety \( X(\pi_1(M_n)) \).

We will review more details on the algebraic structure of the character varieties for once-punctured torus bundles with tunnel number one. Let \( f_n(y) \) be the Chebyshev polynomials defined by \( f_0(y) = 0, f_1(y) = 1 \) and \( f_{n+1}(y) = yf_n(y) - f_{n-1}(y) \) for \( n \in \mathbb{Z} \). Note that if \( y = b + b^{-1} \) and \( b \neq \pm 1 \), then \( f_n(y) \) can be expressed as

\[ f_n(y) = \frac{b^n - b^{-n}}{b - b^{-1}}. \]

According to \cite[Proposition 5.23]{BP} \( \rho : \pi_1(M_n) \rightarrow \text{SL}_2(\mathbb{C}) \) is an irreducible representation if and only if the above variables \( x, y \) and \( z \) satisfy

\begin{align*}
(2.1) & \quad x^2 - 1 + f_{n-1}(y) = 0, \\
(2.2) & \quad xz - y + f_n(y) = 0, \\
(2.3) & \quad x(f_{n+1}(y) - 1) - zf_n(y) = 0.
\end{align*}

There are 3 cases on the parametrization of \( X(\pi_1(M_n)) \) (for details we refer to \cite[Propositions 5.23 & 5.32 and Remark 5.33]{BP})�

(1) \( x = \varepsilon \sqrt{1 - f_{n-1}(y)}, \quad z = \varepsilon \frac{y - f_n(y)}{1 - f_{n-1}(y)} \) where \( \varepsilon = \pm 1 \) and \( f_{n-1}(y) \neq 1 \).

(2) \( x = z = 0 \) and \( y \in R_{n-2} \). Here \( R_{n-2} = \{2 \cos(2\pi k/(n - 2)) \mid k = 0, \ldots, n - 3\} \).

If we write \( m^{\pm 1} \) and \( \ell^{\pm 1} \) for the eigenvalues of \( \rho(\mu) \) and \( \rho(\lambda) \) respectively, then \( m^2 = -1 \) and \( \ell = 1 \), so \( \text{tr} \rho(\mu^p \lambda^q) \in \{\pm 2, 0\} \).
The cases (1) and (2) form a parametrization of a hyperelliptic curve in $X(\pi_1(M_n))$. We denote by $D$ this hyperelliptic curve. The points in (2) are given by $f_{n-1}(y) = 1$. We will see the details in Subsection 2.3. Moreover the coordinate of a discrete faithful representation of $\pi_1(M_n)$ appears in the case (2).

It is also shown in [BP] Theorem 5.1 that the character variety $X(\pi_1(M_n))$ consists of the unique component $D$ when $n \equiv 2 \pmod{4}$ and $X(\pi_1(M_n))$ consists of two components $D$ and $L$, where $L$ is parametrized by the case (3) when $n \equiv 2 \pmod{4}$. We observe the inverse sum of adjoint Reidemeister torsions on the components $D$ and $L$ for the conjecture by Gang–Kim–Yoon. We will call the component $D$ the geometric component and $L$ the extra component.

2.3. Factorization of Chebyshev polynomials. We review the details on the defining polynomials of $X(\pi_1(M_n))$ to investigate functions on it. This section explains factorization of Chebyshev concerning the defining polynomials of $X(\pi_1(M_n))$.

**Definition 2.1** (Definition 4.8 in [BP]). Set

\[
\begin{align*}
  h_n(u) &= \begin{cases} 
    f_{m-1}(u) & \text{if } n = 2m \\
    f_m(u) + f_{m-1}(u) & \text{if } n = 2m + 1,
  \end{cases} \\
  j_n(u) &= \begin{cases} 
    f_m(u) & \text{if } n = 2m \\
    f_{m+1}(u) + f_m(u) & \text{if } n = 2m + 1,
  \end{cases} \\
  k_n(u) &= \begin{cases} 
    f_{m+2}(u) - f_m(u) & \text{if } n = 2m \\
    f_{m+2}(u) - f_{m+1}(u) & \text{if } n = 2m + 1,
  \end{cases} \\
  \ell_n(u) &= \begin{cases} 
    f_{m+1}(u) - f_{m-1}(u) & \text{if } n = 2m \\
    f_{m+1}(u) - f_m(u) & \text{if } n = 2m + 1.
  \end{cases}
\end{align*}
\]

There are useful factorizations as follows.

**Lemma 2.2** (Lemma 4.9 in [BP]). For all $n$,

\[
\begin{align*}
  f_n(u) &= j_n(u)\ell_n(u), \\
  f_{n+1}(u) - 1 &= j_n(u)k_n(u), \\
  f_{n-1}(u) - 1 &= h_n(u)\ell_n(u), \\
  f_n(u) - u &= h_n(u)k_n(u),
\end{align*}
\]

whence $(f_{n+1}(u) - 1)(f_{n-1}(u) - 1) = f_n(u)(f_n(u) - u)$.

We also review the common factors among $h_n(u)$, $j_n(u)$, $k_n(u)$ and $\ell_n(u)$.

**Lemma 2.3** (Lemma 4.13 in [BP]). The symbol $(a, b)$ denotes the ideal generated by polynomials $a(u)$ and $b(u)$ in $\mathbb{C}[u]$. Then we have the following:

1. For all $n$, $(h_n, j_n) = (1)$. 

(2) For all \( n \), \( \gcd(k_n, \ell_n) = (1) \).

(3) If \( n \not\equiv 2 \pmod{8} \), then \( (h_n, k_n) = (1) \). Otherwise, \( (h_n, k_n) = (u^2 - 2) \).

(4) If \( n \equiv 0 \pmod{4} \), then \( (j_n, k_n) = (u) \). Otherwise, \( (j_n(u), k_n(u)) = (1) \).

(5) If \( n \equiv 2 \pmod{4} \), then \( (h_n, \ell_n) = (u) \). Otherwise, \( (h_n, \ell_n) = (1) \).

**Remark 2.4.** We modified the assumption of (3) from \( n \not\equiv 2 \) in [BP, Lemma 4.13 (3)] to \( n \not\equiv 2 \pmod{8} \).

**Remark 2.5.** When \( n \equiv 2 \pmod{8} \), it also holds that \((u \pm \sqrt{2}) \not\mid h_n(u)\) and \((u \pm \sqrt{2}) \not\mid k_n(u)\).

We can rewrite the defining equations (2.1), (2.2) and (2.3) of \( X(\pi_1(M_n)) \) as follows.

\begin{align*}
(2.4) & \quad x^2 + h_n(y)\ell_n(y) = 0, \\
(2.5) & \quad xz + h_n(y)k_n(y) = 0, \\
(2.6) & \quad j_n(y)(xk_n(y) - z\ell_n(y)) = 0.
\end{align*}

Define \( \hat{\ell}_n(y) \) and \( \hat{h}_n(y) \) such that \( \hat{\ell}_n(y) = \ell_n(y) \) and \( \hat{h}_n(y) = h_n(y) \) when \( n \not\equiv 2 \pmod{4} \), \( y\hat{\ell}_n(y) = \ell_n(y) \) and \( y\hat{h}_n(y) = h_n(y) \) when \( n \equiv 2 \pmod{4} \). The parametrization of \( X(\pi_1(M_n)) \) can be written as

\begin{align*}
(1) & \quad \{(\varepsilon \sqrt{-h_n(y)\ell_n(y)}, y, \varepsilon k_n(y)\sqrt{-\hat{h}_n(y)/\ell_n(y)}) \mid \varepsilon = \pm 1, \hat{\ell}(y) \neq 0, \hat{h}_n(y) \neq 0\} \subset D. \quad \text{Note that} \quad h_n(y)/\ell_n(y) = \hat{h}_n(y)/\hat{\ell}_n(y). \\
(2) & \quad \{(0, y, 0) \mid \hat{h}_n(y) = 0\} \subset D. \quad \text{The zeros of} \quad \hat{h}_n(y) \quad \text{form} \quad \mathcal{R}_{n-2}. \\
(3) & \quad \{(0, 0, z) \mid z \in \mathbb{C}\}, \quad \text{which gives the line} \quad L \quad \text{when} \quad n \equiv 2 \pmod{4}.
\end{align*}

Combining the above cases (1) and (2), we have the parametrization of the curve component \( D \) in [BP, Definition 5.31], that is, \( D \) is parametrized on \( \mathbb{C}^3 \) by

\begin{align*}
\{(\varepsilon \sqrt{-h_n(y)\ell_n(y)}, y, \varepsilon k_n(y)\sqrt{-\hat{h}_n(y)/\ell_n(y)}) \mid \varepsilon = \pm 1, \hat{\ell}(y) \neq 0\}
\end{align*}

We put a list of remarks needed in this paper.

**Remark 2.6.**

1. The polynomials \( \hat{\ell}_n(y) \) and \( \hat{h}_n(y) \) have no common factors and no factor \( y \), that is, \( (\hat{\ell}_n, \hat{h}_n) = \mathbb{C}[u] \), \( \hat{\ell}_n(0) \neq 0 \) and \( \hat{h}_n(0) \neq 0 \).

2. It holds that \( \hat{\ell}_n(y) \neq 0 \) on \( X(\pi_1(M_n)) \). If \( \hat{\ell}_n(y) = 0 \), then we have \( f_{n-1}(y) - 1 = h_n(y)\ell_n(y) = 0 \) and \( f_n(y) - y = k_n(y)h_n(y) \neq 0 \) since neither \( k_n(y) \) nor \( h_n(y) \) has any common roots with \( \hat{\ell}_n(y) \). The defining equations (2.1) and (2.2) do not hold simultaneously.

3. In the case of \( n \equiv 2 \pmod{4} \), the curve \( D \) and the line \( L \) in \( X(\pi_1(M_n)) \) intersect in the two points \( (0, 0, \pm \sqrt{1/2 - 1/n}) \) according to [BP, Proposition 5.35]. Namely \( X(\pi_1(M_n)) \) is connected.

4. If \( n \neq 0 \), then the degree of \( f_n(u) \) is \( |n| - 1 \).
2.4. General formula of the adjoint Reidemeister torsion for once-punctured torus bundles. We can find a general formula of the adjoint Reidemeister torsion for hyperbolic once-punctured torus bundles in [Po, Section 4.5]. We review the general formula in Porti’s book and then apply it to our situation. For the details, we refer readers to [Po, Section 4.5].

Let $M$ be a once-punctured torus bundle over the circle with monodromy $\phi$ and $T$ the fiber in $M$. The character variety $X(\pi_1(M))$ is an algebraic subset in $\mathbb{C}^3$ under the coordinate functions

$$x_1 = \text{tr } \rho(g), \quad x_2 = \text{tr } \rho(h), \quad x_3 = \text{tr } \rho(gh)$$

where $g$ and $h$ are a pair of generators for $\pi_1(T)$. This is derived from the fact that the character variety of the free group $\pi_1(T) = \langle g, h \rangle$ is isomorphic to $\mathbb{C}^3$ by corresponding the traces for $g, h, gh$ to the coordinate $(x_1, x_2, x_3) \in \mathbb{C}^3$. The restriction map induces the homomorphism $i : \pi_1(T) \to \pi_1(M)$ and then the map $r : X(\pi_1(M_n)) \to X(\pi_1(T)) \cong \mathbb{C}^3$ is induced by the pull-back by $i$. We can think of the image of $r$ as the fixed point set by the action of the monodromy $\phi$ on $X(\pi_1(T)) \cong \mathbb{C}^3$. It has been shown in [Po, Proposition 4.19] that the adjoint Reidemeister torsion is determined by the eigenvalues of the action induced by the monodromy $\phi$ on the tangent space of $X(\pi_1(T))$ at $r([\rho])$. Here $[\rho]$ is the conjugacy class of an $\text{SL}_2(\mathbb{C})$-representation $\rho$ of $\pi_1(M)$ and the tangent space of $X(\pi_1(T))$ at $r([\rho])$ is the vector space $\mathbb{C}^3$. The action of the monodromy $\phi$ on $X(\pi_1(T)) \cong \mathbb{C}^3$ is given by $2 \times 2$-matrix $A_\phi$ which represents the induced action of $\phi$ on $H_1(T; \mathbb{Z}) \cong \mathbb{Z}^2$ as follows:

- the matrix $A_\phi$ is conjugate to $\pm R^{a_1}L^{a_2} \cdots R^{a_{k-1}}L^{a_k}$ ($a_1, \ldots, a_k \in \mathbb{Z}$) where $L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
- Let $P_i$ ($i = 1, 2, 3$) be a polynomial in the variables $x_1, x_2$ and $x_3$ with coefficient $\mathbb{Z}$. The actions corresponding to $L$ and $R$ on the coordinates $(P_1, P_2, P_3)$ are defined by

\begin{align}
(2.7) & \quad (P_1, P_2, P_3)^L = (P_3, P_2, P_2P_3 - P_1), \\
(2.8) & \quad (P_1, P_2, P_3)^R = (P_1, P_3, P_1P_3 - P_2)
\end{align}

respectively.

We write $(P_1^W, P_2^W, P_3^W)$ for the action $(P_1, P_2, P_3)^W$ of a word $W$ in $L$ and $R$. The polynomials $P_i^{A_\phi}$ ($i = 1, 2, 3$) are defined by the action $(P_1, P_2, P_3)^{A_\phi}$ with the initial condition $(P_1, P_2, P_3)^{id} = (x_1, x_2, x_3)$. The action of the monodromy $\phi$ on the tangent space $T_{r([\rho])}X(\pi_1(T))$ is presented by the Jacobian matrix $[\partial P_i^{A_\phi}/\partial x_j]_{i,j}$. By [Po, Lemma 4.21], the adjoint Reidemeister torsion $T_{M,\lambda}(\rho)$ is expressed as

\begin{equation}
(2.9) \quad T_{M,\lambda}(\rho) = 3 - \text{tr} \left[ \frac{\partial P_i^{A_\phi}}{\partial x_j} \right]_{i,j}
\end{equation}
where \( x_i (i = 1, 2, 3) \) is evaluated at \( r([\rho]) \).

3. Adjoint Reidemeister torsion

We apply the general formula in Subsection 2.4 to hyperbolic once-punctured torus bundles with tunnel number one. The monodromy \( \phi_n = \tau^{n+2}_\beta \) induces the action given by \( LR^{-(n+2)} \) on \( H_1(T; \mathbb{Z}) \). Let \( k \) be a non-negative integer. By Eq. (2.8), there are recurrence relations

\[
(P^{LRk}_1, P^{LRk}_2, P^{LRk}_3) = (P^{LRk-1}_1, P^{LRk-1}_3 - P^{LRk-1}_2)
\]

and

\[
(P^{LR-k}_1, P^{LR-k}_2, P^{LR-k}_3) = (P^{LR-(k-1)}_1, P^{LR-(k-1)}_1 - P^{LR-(k-1)}_2, P^{LR-(k-1)}_3 - P^{LR-(k-1)}_2).
\]

The polynomials \( P^{LR\pm k}_1, P^{LR\pm k}_2 \) and \( P^{LR\pm k}_3 \) satisfy

\[
P^{LRk}_1 = P^L_1, \quad P^{LRk}_2 = P^{LRk-1}_3, \quad P^{LRk}_3 = P^L_1 P^{LRk-1}_3 - P^{LRk-2}_3
\]

and

\[
P^{LR-k}_1 = P^L_1, \quad P^{LR-k}_2 = P^L_2 - P^{LR-k}_1 = f_2(P^L_1)P^L_3 - f_1(P^L_1)P^L_2
\]

Both \( P^{LRk}_3 \) and \( P^{LR-k}_3 \) satisfy the same recurrence relation as the Chebyshev polynomial \( f_k(P^L_1) \). Note that any linear combination of Chebyshev polynomials also satisfies the same recurrence relation. We regard the initial conditions of \( P^{LRk}_3 \) and \( P^{LR-k}_3 \) for \( k = 0, 1 \) as

\[
P^L_3 = f_1(P^L_1)P^L_3, \quad \quad P^{LR}_3 = P^L_1 P^L_3 - P^L_2 = f_2(P^L_1)P^L_3 - f_1(P^L_1)P^L_2
\]

and

\[
P^L_2 = f_1(P^L_1)P^L_2, \quad \quad P^{LR-1}_3 = P^L_1 P^L_2 - P^L_3 = f_2(P^L_1)P^L_2 - f_1(P^L_1)P^L_3
\]

since \( f_2(y) = y, f_1(y) = 1, f_0(y) = 0 \). According to the initial conditions of \( P^{LRk}_3 \) and \( P^{LR-k}_3 \), the polynomials \( P^{LRk}_3 \) and \( P^{LR-k}_3 \) in \( P^L_3 \) can be expressed as

\[
P^{LRk}_3 = f_{k+1}(P^L_1)P^L_3 - f_k(P^L_1)P^L_2,
\]

\[
P^{LR-k}_3 = f_{k+1}(P^L_1)P^L_2 - f_k(P^L_1)P^L_3.
\]

By Eq. (2.7) the triple \( (P^L_1, P^L_2, P^L_3) \) of polynomials is given by

\[
(x_1, x_2, x_3)^L = (x_3, x_2, x_2 x_3 - x_1)
\]

and then we get

\[
P^{LRk}_3 = (x_2 x_3 - x_1) f_{k+1}(x_3) - x_2 f_k(x_3) = x_2 f_{k+2}(x_3) - x_1 f_{k+1}(x_3),
\]

\[
P^{LR-k}_3 = x_2 f_{k+1}(x_3) - (x_2 x_3 - x_1) f_k(x_3) = -x_2 f_{k-1}(x_3) + x_1 f_k(x_3).
\]

Summarizing, the action of \( LR^{\pm k} \) yields \( (P^{LR^\pm k}_1, P^{LR^\pm k}_2, P^{LR^\pm k}_3) \) as follows.
Lemma 3.1. For a non-negative integer $k$, the polynomials $P^L_{iR^{\pm k}}$ $(i = 1, 2, 3)$ are given by

$$(P^L_{1R^{\pm k}}, P^L_{2R^{\pm k}}, P^L_{3R^{\pm k}}) = (x_3, x_2 f_{\pm k+1}(x_3) - x_1 f_{\pm k}(x_3), x_2 f_{\pm k+2}(x_3) - x_1 f_{\pm k+1}(x_3)).$$

Proof. The polynomials $P^L_{iR^k}$ $(i = 1, 2, 3)$ follow by Eqs. (3.1) and (3.3). Since $f_{-n}(y) = -f_n(y)$ for any integer $n$, it follows from Eqs. (3.2) and (3.4) that

$$P^L_{iR^{-k}} = x_3,$$

$$P^L_{iR^{-k}} = -x_2 f_{k-1}(x_3) + x_1 f_k(x_3) = x_2 f_{-k+1}(x_3) - x_1 f_{-k}(x_3),$$

$$P^L_{iR^{-(k-1)}} = x_2 f_{-k+2}(x_3) - x_1 f_{-k+1}(x_3).$$

Applying the general formula (2.9) to $A_{\phi_n} = LR^{-(n+2)}$ together with Lemma 3.1, we obtain the adjoint Reidemeister torsion for the once-punctured torus bundle $M_n$ as follows.

Lemma 3.2. The adjoint Reidemeister torsion $T_{M_n, \lambda}(\rho)$ is expressed as

$$T_{M_n, \lambda}(\rho) = 3 + f_{n+1}(x_3) + x_2 \frac{df_n(x_3)}{dx_3} - x_1 \frac{df_{n+1}(x_3)}{dx_3}.$$ 

Proof. By Lemma 3.1, the polynomials $P^L_{iR^{-(n+2)}}$ $(i = 1, 2, 3)$ turn out to be

$$P^L_{1R^{-(n+2)}} = x_3,$$

$$P^L_{2R^{-(n+2)}} = x_2 f_{-n+1}(x_3) - x_1 f_{-(n+2)}(x_3) = -x_2 f_{n+1}(x_3) + x_1 f_{n+2}(x_3),$$

$$P^L_{3R^{-(n+2)}} = x_2 f_{-n}(x_3) - x_1 f_{-(n+1)}(x_3) = -x_2 f_n(x_3) + x_1 f_{n+1}(x_3).$$

It follows from the general formula (2.9) that

$$T_{M_n, \lambda}(\rho) = 3 - \frac{\partial P^L_{1R^{-(n+2)}}}{\partial x_1} - \frac{\partial P^L_{2R^{-(n+2)}}}{\partial x_2} - \frac{\partial P^L_{3R^{-(n+2)}}}{\partial x_3}$$

$$= 3 - \left( -f_{n+1}(x_3) - x_2 \frac{df_n(x_3)}{dx_3} + x_1 \frac{df_{n+1}(x_3)}{dx_3} \right)$$

$$= 3 + f_{n+1}(x_3) + x_2 \frac{df_n(x_3)}{dx_3} - x_1 \frac{df_{n+1}(x_3)}{dx_3}.$$ 

The variables $x_i$ in our situation are expressed as

$$x_1 = \text{tr} \rho(\beta) = y,$$

$$x_2 = \text{tr} \rho(\beta') = \text{tr} \rho(\beta^{-1} \mu \beta \mu^{-1}) = \text{tr} \rho(\alpha \beta \alpha^{-1} \beta^{-1}) = x^2 + y^2 + z^2 - xyz - 2,$$

$$x_3 = \text{tr} \rho(\beta \beta') = \text{tr} \rho(\mu \beta \mu^{-1}) = \text{tr} \rho(\beta) = y.$$ 

We can regard the adjoint Reidemeister torsion $T_{M_n, \lambda}(\rho)$ in Lemma 3.2 as a function in $y$ on our geometric components.
Proposition 3.3. If $x \neq 0$, then the variable $x_2$ satisfies

$$x_2 = x^2 + y^2 + z^2 - xyz - 2 = \frac{yz}{x} = y \frac{f_n(y) - y}{f_{n-1}(y) - 1}.$$ 

Hence we can express the adjoint Reidemeister torsion $T_{M_n, \lambda}(\rho)$ in Lemma 3.2 as a function in $y$ as follows:

$$T_{M_n, \lambda}(\rho) = 3 + f_{n+1}(y) + y \frac{f_n(y) - y}{f_{n-1}(y) - 1} \frac{df_n(y)}{dy} - y \frac{df_{n+1}(y)}{dy}.$$ 

Proof. By Eqs. (2.1) and (2.2) the variable $x_2 = x^2 + y^2 + z^2 - xyz - 2$ turns into

$$(3.5) \quad 1 - f_{n-1}(y) + y^2 + z^2 - y(y - f_n(y)) - 2 = y f_n(y) - f_{n-1}(y) - 1 + z^2.$$ 

By Eq. (2.3) we have $y f_n(y) - f_{n-1}(y) - 1 = z f_n(y) / x$. We can rewrite (3.5) as

$$\frac{z f_n(y)}{x} + z^2 = \frac{z(f_n(y) + xz)}{x} = \frac{yz}{x}$$

using Eq. (2.2) again. \qed

We close this section with the explicit form of adjoint Reidemeister torsion on the extra components $L$. Suppose $x = y = 0$ (this occurs only when $n = 4k + 2$, $k \in \mathbb{Z}$). Then we have the slope $\ell = m^{-4}$ and the local coordinate $x_2 = z^2 - 2 = m^2 + m^{-2}$. By Lemma 3.2 and the subsequent explanation of the variables, the adjoint Reidemeister torsion $T_{M_n, \lambda}(\rho)$ on the extra component $L$ is expressed as

$$(3.6) \quad T_{M_n, \lambda}(\rho) = 3 + f_{n+1}(y) + (m^2 + m^{-2}) \frac{df_n(y)}{dy} - y \frac{df_{n+1}(y)}{dy}$$

at $y = 0$. Moreover the derivative of $f_k(y)$ is also expressed as follows.

Lemma 3.4. We have

$$\frac{df_k(y)}{dy} = \frac{(k - 1) f_{k+1}(y) - (k + 1) f_{k-1}(y)}{y^2 - 4}.$$ 

The value of $f_n$ at $y = 0$ is expressed as follows.

Lemma 3.5 ([BP Lemma 4.3]). $f_{2k}(0) = 0$ and $f_{2k+1}(0) = (-1)^k$.

From Lemma 3.4 and $f_{n+1}(0) = f_{4k+3}(0) = -1$ the derivatives of (3.6) turn out to be

$$\left. \frac{df_n(y)}{dy} \right|_{y=0} = \frac{(n - 1) f_{n+1}(y) - (n + 1) f_{n-1}(y)}{y^2 - 4} \bigg|_{y=0} = \frac{(n - 1)(-1) - (n + 1)(1)}{-4} = \frac{n}{2},$$

$$\left. \frac{df_{n+1}(y)}{dy} \right|_{y=0} = \frac{n f_{n+2}(y) - (n + 2) f_n(y)}{y^2 - 4} \bigg|_{y=0} = 0.$$ 

Hence we can rewrite Eq. (3.6) as

$$T_{M_n, \lambda}(\rho) = 2 + (m^2 + m^{-2}) \frac{n}{2}.$$
Proposition 3.6. When \( n \equiv 2 \pmod{4} \), we can express the adjoint Reidemeister torsion \( T_{M_n, \gamma}(\rho) \) with respect to \( \gamma = \mu \rho \lambda^g \) on the extra component \( L \) as the following function in \( m \):

\[
T_{M_n, \gamma}(\rho) = \left( -\frac{p}{4} + q \right) \left( 2 + (m^2 + m^{-2}) \frac{n}{2} \right).
\]

Proof. This follows from the curve change formula (see \cite[Theorem 4.1]{Po}), which says we may assume that

\[
\lambda = \text{longitude}.
\]

The trace of \( \rho \) is irreducible, then \( \rho(\mu) = \rho(\beta \alpha) \) and \( \rho(\beta) \) are also not conjugate to upper triangular matrices simultaneously. For an irreducible \( SL_2(\mathbb{C}) \)-representation \( \rho \), up to conjugation, we may assume that

\[
\rho(\mu) = \rho(\beta \alpha) = \begin{bmatrix} m & 1 \\ 0 & 1/m \end{bmatrix} \quad \text{and} \quad \rho(\beta) = \begin{bmatrix} b & 0 \\ * & 1/b \end{bmatrix}.
\]

The trace of \( \rho(\alpha) = \rho(\beta)^{-1} \rho(\mu) \) equals to \( m/b - \ast + b/m \). Using the variable \( x = \text{tr} \rho(\alpha) \), we express \( \rho(\beta) \) as

\[
\rho(\beta) = \begin{bmatrix} b \\ -x + b/m + m/b & 1/b \end{bmatrix}.
\]

It follows from the irreducibility of \( \rho \) that \( -x + b/m + m/b \neq 0 \).

Since \( \lambda = \alpha \beta \alpha^{-1} \beta \alpha^{-1} \beta^{-1} \) and \( \alpha = \beta^{-1} \mu \), by a direct calculation we have

\[
\rho(\lambda) = \begin{bmatrix} b \ell (\ast \ast) \\ \ell (\ast \ast) \end{bmatrix}
\]

where

\[
\ast \ast = (b^4 m^2 x - b^3 m_3 x^2 - b^3 m_3 x^2 - b^3 m_3 x^2 - b^3 m + b^2 m^4 x + b^2 m^2 x^3 + 2 b^2 m^2 x + b^2 x - b m^3 x^2 - b m^3 - b m x^2 - b m + m^2 x) / (b^3 m^4).
\]

and \( \ell = -m(\ast \ast) + m^{-4} - x y m^{-3} + x^2 m^{-2} \). Since \( \rho(\mu) \) and \( \rho(\lambda) \) are commutative, the \((2,1)\)-entry of \( \rho(\lambda) \) equals to 0. It follows from \(-x + b/m + m/b \neq 0 \) that \( \ast \ast = 0 \). Thus we get \( \ell = m^{-4} - x y m^{-3} + x^2 m^{-2} \). Similarly, from the \((2,2)\)-entry of \( \rho(\lambda) \) we get \( \ell^{-1} = m^4 - x y m^3 + x^2 m^2 \).

We have seen that the variables \( x \) and \( z \) can be regarded as functions in \( y \). We also describe the eigenvalues \( m \) and \( \ell \) as functions in \( y \).
On the geometric component the functions \( x \) and \( z \) are expressed as
\[
x = \varepsilon \sqrt{1 - f_{n-1}(y)}, \quad z = \varepsilon \frac{y - f_n(y)}{\sqrt{1 - f_{n-1}(y)}},
\]
where \( \varepsilon = \pm 1 \) and \( f_{n-1}(y) \neq 1 \). We can also assume \( f_n(y) \neq y \) since we consider generic points on \( X(\pi_1(M_n)) \) (actually it is enough to assume \( k_n(y) \neq 0 \), see Subsection 5.1). Then
\[
x = z \frac{1 - f_{n-1}(y)}{y - f_n(y)} = (m + m^{-1}) \frac{1 - f_{n-1}(y)}{y - f_n(y)}.
\]
We have
\[
\ell m^2 = m^{-2} - xym^{-1} + x^2
\]
\[
= m^{-2} - (1 + m^{-2}) \frac{y - yf_{n-1}(y)}{y - f_n(y)} + 1 - f_{n-1}(y)
\]
\[
= m^{-2}(1 - \frac{y - yf_{n-1}(y)}{y - f_n(y)}) + 1 - f_{n-1}(y) - \frac{y - yf_{n-1}(y)}{y - f_n(y)}.
\]
Hence we can regard the eigenvalues \( m \) and \( \ell \) as the following functions in \( y \):
\[
\ell = u(y)m^{-2} + v(y)m^{-4}, \quad m^2 + m^{-2} = s(y)
\]
where \( u(y) \), \( v(y) \) and \( s(y) \) are
\[
u(y) = -\frac{(1 - f_{n-1}(y))f_n(y)}{y - f_n(y)}, \quad v(y) = \frac{f_{n-2}(y)}{y - f_n(y)}, \quad s(y) = \frac{(y - f_n(y))^2}{1 - f_{n-1}(y)} - 2(= z^2 - 2).
\]
We can extend \( u(y) \) and \( v(y) \) to the case of \( y = 0 \) by setting \( u(0) = 0 \) and \( v(0) = 1 \).

Similarly \( \ell^{-1} = um^2 + vm^4 \). We can rewrite \( \ell \ell^{-1} = 1 \) as \( u^2 + v^2 + suv = 1 \).

5. Relation between adjoint torsion and character variety

We evaluate the adjoint torsion \( T_{M,\gamma}(\rho) \) for any slope \( \gamma = \mu^p\lambda^q \). The adjoint torsion \( T_{M,\gamma}(\rho) \) for a slope \( \gamma = \mu^p\lambda^q \) is derived from \( T_{M,\lambda}(\rho) \) in Proposition 3.3 and the curve change formula of the adjoint Reidemeister torsion. The purpose of this section is to give an explicit form the adjoint torsion by the Jacobian determinant of functions on the geometric component \( D \) in the character variety \( X(\pi_1(M_n)) \) following [Yo1].

5.1. Defining polynomials of the character variety. We set function \( E_\gamma(m, y) \) for a slope \( \gamma = \mu^p\lambda^q \) on \( X(\pi_1(M_n)) \) as
\[
E_\gamma(m, y) := m^p\ell^q = m^p(u(y)m^{-2} + v(y)m^{-4})^q.
\]
Note that a solution of \( t^2 - (\text{tr}_\gamma)t + 1 = 0 \) is non-constant since the trace function \( \text{tr}_\gamma \) is non-constant on \( D \) which is a norm curve in \( X(\pi_1(M_n)) \).

The functions \( u(y) \) and \( v(y) \) can be rewritten as
\[
u(y) = -\frac{j_n(y)\ell^2(y)}{k_n(y)}, \quad v(y) = 1 - y \frac{\ell_n(y)}{k_n(y)}.
\]
We regard $u(y)$ and $v(y)$ as being defined under the assumption that $k_n(y) \neq 0$ on $X(\pi_1(M_n))$. We also rewrite the defining polynomials of $X(\pi_1(M_n))$ to describe $z = m + m^{-1}$ as an implicit function in $y$ under the assumption that $k_n(y) \neq 0$.

**Lemma 5.1.** We can regard $X(\pi_1(M_n)) \setminus \{k_n(y) = 0\}$ as the set of points satisfying

\begin{align}
(5.1) & \quad z^2\ell_n(y) + h_n(y)k_n^2(y) = 0, \\
(5.2) & \quad xk_n(y) - z\ell_n(y) = 0, \\
(5.3) & \quad k_n(y) \neq 0.
\end{align}

**Proof.** First we show $\hat{\ell}_n(y) \neq 0$ on the set of points satisfying (5.1), (5.2) and (5.3). Suppose $\hat{\ell}_n(y) = 0$. Then $h_n(y) = 0$ or $k_n(y) = 0$ by (5.1) which is a contradiction to that neither of $h_n(y)$ nor $k_n(y)$ has any common roots with $\ell_n(y)$ by Lemma 2.3.

We denote by $\varphi_1, \varphi_2, \varphi_3$ the defining polynomials in Eqs. (2.4), (2.5) and (2.6), respectively. Write $\varphi_3 = j_n(y)\varphi'_3$. The character variety $X(\pi_1(M_n))$ is the vanishing set of the ideal $(\varphi_1, \varphi_2, \varphi_3)$. Here the symbol $(\varphi_1, \varphi_2, \varphi_3)$ stands for the ideal generated by $\varphi_1$, $\varphi_2$ and $\varphi_3$. According to Eqs. (2.4)– (2.6), $X(\pi_1(M_n))$ is the union the vanishing set of the ideal $(\varphi_1, \varphi_2, \varphi'_3)$ and that of $(\varphi_1, \varphi_2, j_n(y))$. It was shown in [BP] Proposition 5.25] that the vanishing set of $(\varphi_1, \varphi_2, j_n(y))$ is contained in that of $(\varphi_1, \varphi_2, \varphi'_3)$. We can regard $X(\pi_1(M_n))$ as the vanishing set of the ideal $(\varphi_1, \varphi_2, \varphi'_3)$.

Set $\varphi'_1 = z\ell_n(y) + h_n(y)k_n^2(y)$. Under the assumption $k_n(y) \neq 0$, each point of $X(\pi_1(M_n))$ is contained in the vanishing set of the ideal $(\varphi'_1, \varphi'_3)$ since we can rewrite $\varphi_2 = 0$ as $\varphi'_1 = 0$ by $\varphi_3 = 0$ on $X(\pi_1(M_n))$.

On the other hand, we can rewrite $\varphi'_1 = 0$ as $\varphi_2 = 0$ on the vanishing set of the ideal $(\varphi'_1, \varphi'_3)$ under the assumption $k_n(y) \neq 0$. When $n \neq 2 (\text{mod} 4)$, we can rewrite $\ell_n(y)\varphi_1 = 0$ by $\varphi'_3 = 0$ on the vanishing set of $(\varphi'_1, \varphi'_3)$ since $\ell_n(y) = \hat{\ell}_n(y) \neq 0$. When $n \equiv 2 (\text{mod} 4)$, we can rewrite $\ell_n(y)\varphi_1 = 0$ by $\varphi'_3 = 0$ on the vanishing set of $(\varphi'_1, \varphi'_3)$ under the assumption $y \neq 0$ since $\ell_n(y) = y\hat{\ell}_n(y)$. If $y = 0$, then $\varphi'_3 = 0$ implies $x = 0$ under the assumption $k_n(y) \neq 0$. These points $(0, 0, z)$ lie in the line $L$ of $X(\pi_1(M_n))$. Hence we can regard the vanishing set of $(\varphi'_1, \varphi'_3)$ is contained in $X(\pi_1(M_n))$ under the assumption $k_n(y) \neq 0$. \hfill $\square$

**Remark 5.2.** The points satisfying $k_n(y) = 0$ on $X(\pi_1(M_n))$ are given by $(\pm \sqrt{2 - y^2}, y, 0)$ where $y = 2\cos(2k - 1)\pi/(n + 2)$ for $1 \leq k \leq n + 1$.

We define function $G(m, y)$ as

$$G(m, y) := (m + m^{-1})^2\ell_n(y) + h_n(y)k_n^2(y)$$

also set $F(m, y) = m^2 + m^{-2} - s(y)$.

**Remark 5.3.** We have the following factorization of $s(y)$ and relation between $G(m, y)$ and $F(m, y)$.
• We can rewrite \( s(y) \) as
  \[
  \frac{(y - f_n(y))^2}{1 - f_{n-1}(y)} - 2 = -\frac{\dot{h}_n(y)k_n^2(y)}{\ell_n(y)} - 2.
  \]

• The function \( G(m, y) \) satisfies that \( G(m, y) = \ell_n(y)F(m, y) \) on \( \dot{\ell}_n(y) \neq 0 \).

We also touch the values of \( k_n, \dot{h}_n \) and \( \dot{\ell}_n \) at \( y = 0 \) which are needed later. These values can be found in \([BP]\) Proof of Proposition 5.35.

**Lemma 5.4.** If \( n = 4k + 2 \), then \( k_n(0), \dot{h}_n(0) \) and \( \dot{\ell}_n(0) \) are expressed as
\[
k_n(0) = (-1)^{k+1}2, \quad \dot{h}_n(0) = (-1)^{k-1} k \quad \text{and} \quad \dot{\ell}_n(0) = (-1)^{\frac{k}{2}}.
\]

**Proof.** By Lemmas 3.4 and 3.5 it holds that

\[
\left. \frac{f_{2t}(y)}{y} \right|_{y=0} = \left. \frac{d}{dy} f_{2t}(y) \right|_{y=0} = l(-1)^{l-1}.
\]

The values of \( k_n, \dot{h}_n \) and \( \dot{\ell}_n \) at \( y = 0 \) follow from the definitions. \( \square \)

5.2. Jacobian determinant and adjoint Reidemeister torsion. We show the equality between the adjoint Reidemeister torsion for a slope \( \gamma \) and the Jacobian determinant given by the pair of functions \( G \) and \( E_\gamma \) in the variables \( m \) and \( y \).

**Proposition 5.5.** Set a slope \( \gamma = \mu^p \lambda^q \). Then it holds on \( X(\pi_1(M_n)) \setminus \{ k_n(y)\hat{h}_n(y) = 0 \} \) that

\[
(5.4) \quad \frac{\partial(G, E_\gamma)}{\partial(m, y)} = \frac{2E_\gamma}{m} k_n(y)\mathbb{T}_{M_n, \gamma}(\rho).
\]

Here \( \frac{\partial(G, E_\gamma)}{\partial(m, y)} \) denotes the Jacobian determinant \( \begin{bmatrix} \frac{\partial G}{\partial m} & \frac{\partial G}{\partial y} \\ \frac{\partial E_\gamma}{\partial m} & \frac{\partial E_\gamma}{\partial y} \end{bmatrix} \).

**Remark 5.6.** The assumption \( X(\pi_1(M_n)) \setminus \{ k_n(y)\hat{h}_n(y) = 0 \} \) means that \( D \setminus \{ f_n(y) - y = 0 \} \) when \( n \not\equiv 2 \) (mod 4) or \( D \setminus \{ f_n(y) - y = 0 \} \cup L \) when \( n \equiv 2 \) (mod 4) since \( f_n(y) - y \) equals \( k_n(y)\hat{h}_n(y) \) when \( n \not\equiv 2 \) (mod 4) or \( k_n(y)\hat{h}_n(y) \) when \( n \equiv 2 \) (mod 4).

**Proposition 5.5** follows from the next lemmas and the curve change formula (3.8) of the adjoint Reidemeister torsion (see \([Po]\) Theorem 4.1)).

**Lemma 5.7.** On \( D \setminus \{ f_n(y) - y = 0 \} \),

\[
\frac{\partial(F, E_\gamma)}{\partial(m, y)} = -\frac{2E_\gamma}{m} \left( \frac{p}{m} \frac{dm}{d\ell} + q \right) \frac{2\nu's + 2u' + v's}{v}.
\]

**Proof.** By \( F = 0 \) we have \( 0 = dF = \frac{\partial E_\gamma}{\partial m} dm + \frac{\partial E_\gamma}{\partial y} dy \). This implies that

\[
\frac{\partial(F, E_\gamma)}{\partial(m, y)} = -\frac{\partial F}{\partial y} \frac{\partial E_\gamma}{\partial m} + \frac{\partial F}{\partial m} \frac{\partial E_\gamma}{\partial y}.
\]
This implies that

\[
\frac{\partial F \partial E_{\gamma}}{\partial y \partial m} - \frac{\partial F \partial E_{\gamma}}{\partial y \partial m} \frac{\partial E_{\gamma}}{\partial y}
\]

\[
= - \frac{\partial F}{\partial y} \left( \frac{\partial E_{\gamma}}{\partial m} + \frac{\partial E_{\gamma}}{\partial y} \frac{dy}{dm} \right)
\]

by \(dE_{\gamma} = \frac{\partial E_{\gamma}}{\partial m} dm + \frac{\partial E_{\gamma}}{\partial y} dy = \frac{\partial E_{\gamma}}{\partial m} dm + \frac{\partial E_{\gamma}}{\partial y} \frac{d\ell}{m}\)

\[
= - \frac{\partial F}{\partial y} \left( \frac{\partial E_{\gamma}}{\partial m} + \frac{\partial E_{\gamma}}{\partial \ell} \frac{d\ell}{dm} \right)
\]

\[
= - \frac{\partial F}{\partial y} \left( \frac{p}{m} \frac{dm}{d\ell} + q \frac{E_{\gamma}}{\ell} \right) \frac{d\ell}{dm}
\]

\[
= - \frac{E_{\gamma}}{\ell} \left( \frac{p}{m} \frac{dm}{d\ell} + q \right) \frac{\partial F}{\partial y} \frac{d\ell}{dm}.
\]

Since \(\ell = u(y)m^{-2} + v(y)m^{-4}\) and \(d\ell = \frac{\partial \ell}{\partial m} dm + \frac{\partial \ell}{\partial y} dy\), we can rewrite \(-\frac{\partial F}{\partial y} d\ell\) as

\[
- \frac{\partial F}{\partial y} \frac{d\ell}{dm} = - \frac{\partial F}{\partial y} \left( -2m^{-3}u - 4m^{-5}v + (u'm^{-2} + v'm^{-4}) \frac{dy}{dm} \right)
\]

\[
= (2m^{-3}u + 4m^{-5}v) \frac{\partial F}{\partial y} + (u'm^{-2} + v'm^{-4}) \frac{\partial F}{\partial m}
\]

\[
= (2m^{-3}u + 4m^{-5}v)(-s') + (u'm^{-2} + v'm^{-4})(2m - 2m^{-3})
\]

\[
= 2m^{-5}(u'm^4 + v'(s-m^2))
\]

By taking the derivative of \(u^2 + v^2 + uv = 1\) we have \(2uu' + 2vv' + uv's + uv's' + uvs' = 0\). This implies that \(u's - us' + 2v' = -\frac{v}{u}(2vs' + 2u' + v')\) and so

\[
- \frac{\partial F}{\partial y} \frac{d\ell}{dm} = -2m^{-5} \frac{2vs' + 2u' + v'}{v} \frac{u'm^2}{v} = -2m^{-1} \frac{2vs' + 2u' + v'}{v}.
\]

Hence we obtain

\[
\frac{\partial (F, E_{\gamma})}{\partial (m, y)} = - \frac{E_{\gamma}}{\ell} \left( \frac{p}{m} \frac{dm}{d\ell} + q \right) \frac{\partial F}{\partial y} \frac{d\ell}{dm}
\]

\[
= -2 \frac{E_{\gamma}}{m} \left( \frac{p}{m} \frac{dm}{d\ell} + q \right) \frac{2vs' + 2u' + v'}{v}.
\]

\[
\square
\]

**Lemma 5.8.** On \(D \setminus \{f_n(y) - y = 0\}\),

\[
2vs' + 2u' + v' = \frac{f_{n-2}(y)}{1 - f_{n-1}(y)} T_{M_n, \lambda}(\rho) = \frac{v(y)k_n(y)}{\ell_n(y)} T_{M_n, \lambda}(\rho).
\]
Proof. Write $y = b + b^{-1}$. Then $f_k(y) = \frac{b^k - b^{-k}}{b - b^{-1}}$. By a direct calculation, using $\frac{d a(y)}{dy} = \frac{d a(y)}{db}/\frac{d y}{db}$, we can express $2us' + 2u' + v's$ as

$$2us' + 2u' + v's = \frac{b^{-n-1}(b^n + b^2)}{(b - 1)^2( b + 1)^2 (b^n + 1)} (-nb^{2n} - 4b^{2n} + 6b^{n+2} - nb^{n+4} - 4b^{n+4} + 6b^{2n+2} + nb^{2n+4} - 4b^{2n+4} + 2b^{3n+2} + nb^n - b^n + 2b^2),$$

$$\mathbb{T}_{M_n, \lambda}(\rho) = -\frac{b^{-n}}{(b - 1)^2( b + 1)^2 (b^n + 1)} (-nb^{2n} - 4b^{2n} + 6b^{n+2} - nb^{n+4} - 4b^{n+4} + 6b^{2n+2} + nb^{2n+4} - 4b^{2n+4} + 2b^{3n+2} + nb^n - b^n + 2b^2).$$

The lemma follows from

$$\frac{f_n - 2(y)}{1 - f_n - 1(y)} = -\frac{b^n + b^2}{b(b^n + 1)}$$

and

$$\frac{f_n - 2(y)}{1 - f_n - 1(y)} = \frac{v(y)(y - f_n(y))}{1 - f_n - 1(y)} = \frac{v(y)k_n(y)}{\ell_n(y)}.$$ 

We turn to the proof of Proposition 5.5.

Proof of Proposition 5.5. The Jacobian determinant in the left hand side turns into

$$\frac{\partial(\ell_n F, E_\gamma)}{\partial(m, y)} = \ell_n(y) \frac{\partial(F, E_\gamma)}{\partial(m, y)} - \ell_n(y)F(m, y) \frac{\partial E_\gamma}{\partial m}.$$ 

We have $F(m, y) = 0$ on $D \setminus \{ f_n(y) - y = 0 \}$. It follows from Lemmas 5.7 and 5.8 that

$$\frac{\partial(G, E_\gamma)}{\partial(m, y)} = -\frac{2E_\gamma}{\ell} \left( \ell \frac{d m}{m} + q \right) k_n(y) \mathbb{T}_{M_n, \lambda}(\rho)$$

$$-\frac{2E_\gamma k_n(y) \mathbb{T}_{M_n, \gamma}(\rho)}{m}.$$ 

Note that we use the curve change formula (3.8), that is,

$$\mathbb{T}_{M_n, \gamma}(\rho) = \frac{d \log(m^p \ell^q)}{d \log \ell} \mathbb{T}_{M_n, \lambda}(\rho) = \left( \frac{\ell}{m} \frac{d m}{d \ell} + q \right) \mathbb{T}_{M_n, \lambda}(\rho)$$

in the last equality.

Next we consider the component $L$ in the case of $n \equiv 2 \pmod{4}$. The function $\ell_n(y)$ turns into $y \hat{\ell}_n(y)$. It holds that $x = 0$, $y = 0$ and $m^4 \ell = 1$ on $L$. Together with

$$\frac{\partial E_\gamma}{\partial m} = \frac{\partial}{\partial m} m^p \ell^q = p \frac{E_\gamma}{m} + q \frac{E_\gamma \ell}{m} \frac{d m}{d \ell} = E_\gamma \left( p \frac{d m}{m} + q \right) \frac{d \ell}{d m},$$

we have

$$\frac{\partial(G, E_\gamma)}{\partial(m, y)} = -\frac{E_\gamma \ell}{\ell} \frac{d \ell}{d m} \left( p \frac{d m}{m} + q \right) \hat{\ell}_n(y) F(m, y) \bigg|_{y=0}.$$ 

It follows from Remark 5.4 that

$$\hat{\ell}_n(0) = (-1)^k \frac{n}{2}$$

and

$$-\frac{k_n^2(0) \hat{h}_n(0)}{\hat{\ell}_n(0)} = 4 \left( \frac{1}{2} - \frac{1}{n} \right).$$
where \( n = 2(2k + 1) \). Since \( m^4 \ell = 1 \) on \( L \) and \( F(m, y) = m^2 + m^{-2} - s(y) = m^2 + m^{-2} + (k_n^2(y)\hat{h}_n(y))/\hat{\ell}_n(y) + 2 \), the Jacobian determinant \( \frac{\partial(G, E_y)}{\partial(m, y)} \) is expressed as

\[
\frac{\partial(G, E_y)}{\partial(m, y)} = -\frac{E_y}{m^4}(-4m^{-5}) \left( p\frac{dm}{m} + q \right) \bigg|_{y=0} (-1)^k n^2 \left( m^2 + m^{-2} + \frac{4}{n} \right) 
\]

\[
= -\frac{2E_y}{m}(-1)^{k+2} \left( -\frac{p}{4} + q \right) \left( \frac{n}{2}(m^2 + m^{-2}) + 2 \right).
\]

By \( k_n(0) = (-1)^{k+1}2 \) from Remark 5.4 and Proposition 5.6, we have

\[
\frac{\partial(G, E_y)}{\partial(m, y)} = -\frac{2E_y}{m}k_n(0)\mathbb{T}_{M_n,\gamma}(\rho).
\]

\[
\square
\]

6. Checking conjecture

We will show the following main theorem which gives infinitely many new supporting examples to the conjecture by [GKY].

**Theorem 6.1.** The conjecture by [GKY] is true for every hyperbolic once-punctured torus bundle \( M_n \) with tunnel number one.

Since \( \mathbb{T}_{M_n,\gamma^{-1}}(\rho) = -\mathbb{T}_{M_n,\gamma}(\rho) \) we can assume that \( q \geq 0 \). The proof will be divided into the cases \( n \not\equiv 2 \) and \( n \equiv 2 \) (mod 4). Subsections 6.1 and 6.2 deal with the cases of \( n \not\equiv 2 \) and \( n \equiv 2 \) (mod 4) respectively.

6.1. \( n \not\equiv 2 \) (mod 4). The character variety has only geometric component \( D \). The function \( \ell_n(y) \) and \( h_n(y) \) satisfy \( \ell_n(y) = \ell_n(y) \) and \( h_n(y) = \hat{h}_n(y) \). We have \( m^2 + m^{-2} - s(y) = 0 \) from \( G(m, y) = 0 \) since \( G(m, y) = \ell_n(y)F(m, y) \) and \( \ell_n(y) \neq 0 \).

6.1.1. \( p \) is even. By \( m^2 + m^{-2} = s \) and \( f_k(s) = (m^{2k} - m^{-2k})/(m^2 - m^{-2}) \) we have \( m^{2k} = f_k(s)m^{2k} - f_{k-1}(s) \). Let \( \delta_k(y) = \sum_{k=0}^{q} \binom{q}{k}u^k(y)v^{q-k}(y)f_{k+r}(s(y)) \). Recall that \( s \) is a function in \( y \). Hence \( E_{\gamma}(m, y) \) turns out to be

\[
E_{\gamma}(m, y) = m^p\ell^q = m^{p-4q}(um^2 + v)^q = \sum_{k=0}^{q} \binom{q}{k}u^k v^{q-k}m^{2k+p-4q} = g(y)m^2 - h(y)
\]

where \( g(y) = \delta_{p/2-2q}(y) \) and \( h(y) = \delta_{p/2-2q-1}(y) \).

Similarly, \( m^{-p}\ell^{-q} = g(y)m^{-2} - h(y) \). For abbreviation, we use \( g \) and \( h \) instead of \( g(y) \) and \( h(y) \). We can rewrite \( (m^p\ell^q)(m^{-p}\ell^{-q}) = 1 \) as \( g^2 + h^2 - sgh = 1 \). By Proposition 5.5 and Eq. (5.5) we can see that

\[
\frac{-2E_{\gamma}}{m}k_n(y)\mathbb{T}_{M_n,\gamma}(\rho) = \frac{\partial(G(m, y), gm^2 - h)}{\partial(m, y)}
\]

\[
= \ell_n(y)\frac{\partial(F(m, y), gm^2 - h)}{\partial(m, y)}
\]
\[ Then \]
\[ -E_\gamma \frac{k_n(y)}{\ell_n(y)} T_{M_n, \gamma}(\rho) = (m^2 - m^{-2})(m^2 g' - h') + m^2 gs' \]
\[ = m^4 g' + m^2(gs' - h') - g' + m^{-2}h' \]
\[ = (sm^2 - 1)g' + m^2(gs' - h') - g' + (s - m^2)h' \]
\[ = m^2(gs' + gs' - 2h') - 2g' + sh'. \]

By taking derivative of \( g^2 + h^2 - sgh = 1 \) with respect to \( y \), we have \( 2gg' + 2hh' - (sg)'h - sgh' = 0 \). This implies that \( 2g' - sh' = \frac{1}{g}((sg)' - 2h') \) and
\[ -E_\gamma \frac{k_n(y)}{\ell_n(y)} T_{M_n, \gamma}(\rho) = \left( \frac{m^2 - h}{g} \right) ((sg)' - 2h') = \frac{E_\gamma}{g} ((sg)' - 2h'). \]

We have
\[ -\frac{k_n(y)}{\ell_n(y)} T_{M_n, \gamma}(\rho) = \frac{P'}{g} \]
by putting \( P(y) = s g - 2 h = g(m^2 + m^{-2}) - 2 h = \text{tr} \rho (\mu^x \lambda^y) \).

Since \( s = \frac{k_n(y)h_n(y)}{\ell_n(y)} - 2 \) and \( g^2 + h^2 - sgh = 1 \) it follows that
\[ P^2 - 4 = (gs - 2h)^2 - 4(g^2 + h^2 - sgh) \]
\[ = (s^2 - 4)g^2 \]
\[ = h_n(y) \left[ k_n^2(y)h_n(y) + 4\ell_n(y) \right] \left( \frac{k_n(y)}{\ell_n(y)} g \right)^2. \]

Note that any prime factor of the denominator of \( g \) is a factor of \( k_n(y) \) or \( \ell_n(y) \) since \( g \) is defined as a polynomial of a rational functions \( u(y), v(y) \) and \( s(y) \) whose denominators are expressed as \( k_n \) and \( \ell_n \).

So \( h_n(y) \left[ k_n^2(y)h_n(y) + 4\ell_n(y) \right] \) is coprime with the denominator of \( \frac{k_n(y)}{\ell_n(y)} g \). This implies that the denominator of \( \left( \frac{k_n(y)}{\ell_n(y)} g \right)^2 \) in (6.1) is exactly the denominator of \( P^2 - 4 \). Hence the denominator of \( \frac{k_n(y)}{\ell_n(y)} g \) is exactly the denominator of \( P \).

Write \( P = \frac{P_1}{P_2} \) and \( -\frac{k_n(y)}{\ell_n(y)} g = \frac{R}{P_2} \) where each pair of \( \{P_1, P_2\} \) and \( \{R, P_2\} \) is coprime. Then we can express the sum of \( (T_{M_n, \gamma}(\rho))^{-1} \) over \( P = c \) as
\[ \sum_{P=c} 1 T_{M_n, \gamma}(\rho) = \sum_{P_1 = cP_2 = 0} \left( \frac{R}{P_2} \right)^{P_1 - cP_2} = \sum_{P_1 = cP_2 = 0} \frac{R}{(P_1 - cP_2)^{P_2}} = \sum_{P_1 = cP_2 = 0} \frac{R}{(P_1 - cP_2)^r}. \]

We now apply the Jacobi’s residue theorem: if \( f \) is a non-constant polynomial with \( f(0) \neq 0 \) and no repeated zero and \( g \) is a polynomial with \( \deg g \leq \deg f - 2 \) then
\[ \sum_{f(0) = 0} \frac{g(z)}{f'(z)} = 0. \]
(We refer the reader to [GH, Chap. 5] for the details.)

For generic \( c \), \( P_1 - cP_2 \) has nonzero constant coefficient and no repeated zero. Hence we have the equality that \( \sum_{P=c} \frac{1}{\ell_{Mn,\gamma}(\rho)} = 0 \) if we can show that \( \deg R \leq \deg(P_1 - cP_2) - 2 \). This is equivalent to showing that \( \deg R/P_2 \leq \deg(P_1/P_2 - c) - 2 \), that is \( \deg \frac{\ell_{n}(y)}{\ell_{n}(y)} g \leq \deg(P - c) - 2 \).

Note that the degree of \( s \) equals \( n \) if \( n \geq 3 \) and equals \( -n - 2 \) if \( n \leq -3 \). By \( P^2 - 4 = (s^2 - 4)g^2 \), we have \( \deg sg \leq \max\{\deg P, 0\} = \deg(P - c) \) for generic \( c \). Hence it holds that

\[
\deg(P - c) - \deg \frac{k_n(y)}{\ell_n(y)} g \geq \deg s(y) - \deg \frac{k_n(y)}{\ell_n(y)} = \deg k_n(y)h_n(y) = |n| - 1 \geq 2,
\]

by \( f_n(y) - y = k_n(y)h_n(y) \). This proves that \( \sum_{P=c} \frac{1}{\ell_{Mn,\gamma}(\rho)} = 0 \).

**6.1.2. \( p \) is odd.** In this case we can express \( E_\gamma(m, y) \) as

\[
E_\gamma(m, y) = m^{p-4q}(u(y)m^2 + v(y))^q
\]

\[
= m^{-1} \sum_{k=0}^{q} \binom{q}{k} u^k(y)v^{q-k}(y)m^{2k+p-4q-1}
\]

\[
= gm - hm^{-1}
\]

where we use the same notations \( g = \delta_{(p+1)/2-2q}(y) \) and \( h = \delta_{(p-1)/2-2q-1}(y) \) as in the case of an even \( p \). Similarly \( m^{-p} \ell^{-q} = gm^{-1} - hm \). By Eq. (5.5) the right hand in Proposition 5.5 turns out to be

\[
-2E_\gamma \frac{k_n(y)}{\ell_n(y)} T_{Mn,\gamma}(\rho) = \ell_n(y) \frac{\partial(m^2 + m^{-2} - s, gm - hm^{-1})}{\partial(m, y)}
\]

\[
-2E_\gamma \frac{k_n(y)}{\ell_n(y)} T_{Mn,\gamma}(\rho) = (2m - 2m^{-3})(mg' - h'm^{-1}) + s'(g + hm^{-2})
\]

\[
= 2m^2g' + s'g - 2h' + m^{-2}(s'h - 2g') + 2m^{-4}h'
\]

\[
= 2(s - m^{-2})g' + s'g - 2h' + m^{-2}(s'h - 2g') + 2(sm^{-2} - 1)h'
\]

\[
= m^{-2}(-4g' + s'h + 2sh') + 2sg' + s'g - 4h'.
\]

(6.2)

By taking derivative of \( g^2 + h^2 - sgh = 1 \), we have \( 2gg' + 2hh' - s'gh - sg'h - sgh' = 0 \). This implies that \( 2sg' + s'g - 4h' = -\frac{g}{h}(-4g' + s'h + 2sh') \) and we can rewrite Eq. (6.2) as

\[
-2E_\gamma \frac{k_n(y)}{\ell_n(y)} T_{Mn,\gamma}(\rho) = (m^{-1} - \frac{g}{h}m)(-4g' + s'h + 2sh')
\]

\[
= \frac{E_\gamma}{h}(-4g' + s'h + 2sh').
\]

Since \( E_\gamma = m^p \ell^q \neq 0 \), we have that

\[
-\frac{k_n(y)}{\ell_n(y)} T_{Mn,\gamma}(\rho) = -\frac{-4g' + s'h + 2sh'}{2h}
\]
(6.3) 
\[ \frac{Q'}{g^2 - h^2} = \frac{2sg' + s'g - 4h'}{2g}. \]

Set \( Q = (\text{tr}(\mu p \lambda^q))^2 \). The function \( (\text{tr}(\mu p \lambda^q))^2 \) equals to \( (g - h)^2(m + m^{-1})^2 = (g - h)^2(s + 2) \). Then the fraction \( Q'/(g - h) \) is expressed as
\[
\frac{Q'}{g - h} = 2(g' - h')(s + 2) + (g - h)s' \\
= (2sg' + s'g - 4h') - (-4g' + s'h + 2sh') \\
= (2sg' + s'g - 4h')(1 + \frac{h}{g}).
\]

This implies that
\[
\frac{Q'}{g^2 - h^2} = \frac{2sg' + s'g - 4h'}{g}.
\]

Hence Eq. (6.3) turns out to be
\[
(6.4) 
- \frac{k_n(y)}{\ell_n(y)} T_{M_n,\gamma}(\rho) = \frac{Q'}{2(g^2 - h^2)}.
\]

Note that \( \text{tr}(\mu p \lambda^q) = (g - h)z \) where \( z = \varepsilon k_n(y) \sqrt{\frac{h_n(y)}{\ell_n(y)}} \) and \( \varepsilon = \pm 1 \). Hence it follows that
\[
\sum_{\text{tr}(\mu p \lambda^q) = c} \frac{1}{T_{M_n,\gamma}(\rho)} = \frac{1}{2} \sum_{Q = c^2} \frac{1}{T_{M_n,\gamma}(\rho)}
\]
for generic \( c \in \mathbb{C} \). By Eq. (6.4) we can rewrite \( \frac{1}{2} \sum_{Q = c^2} \frac{1}{T_{M_n,\gamma}(\rho)} \) as
\[
\sum_{Q = c^2} \frac{-k_n(y)}{\ell_n(y)} \cdot \frac{2(g^2 - h^2)}{Q'}
\]

since \( z = \pm \sqrt{s(y) + 2} \). By similar arguments as in the above case with \( c, P \) and \( g \) replaced by \( c^2 - 2, Q - 2 \) and \( g^2 - h^2 \) respectively, we can show
\[
\sum_{Q - 2 = d} \frac{-k_n(y)}{\ell_n(y)} \cdot \frac{2(g^2 - h^2)}{(Q - 2)'} = 0
\]
for generic \( d(= c^2 - 2) \in \mathbb{C} \). Since \( g^2 + h^2 - sgh = 1 \) and \( Q - 4 = (s + 2)(g - h)^2 - 4(g^2 + h^2 - sgh) = (s - 2)(g + h)^2 \), the function \( Q - 2 = (m^p \ell^q)^2 + (m^{-p} \ell^{-q})^2 = \text{tr}(\mu p \lambda^q)^2 \) satisfies \( (Q - 2)^2 - 4 = Q(Q - 4) = (s - 4)(g^2 - h^2)^2 \) which corresponds to (6.1).

We also have
\[
(Q - 2)^2 - 4 = h_n(y)[h_n(y)k_n^2(y) + 4\ell_n(y)]k_n^2(y)\ell_n^2(y)(g^2 - h^2)^2.
\]

This implies that the denominator of \( (Q - 2)^2 - 4 \) is exactly the denominator of \( \frac{k_n(y)}{\ell_n(y)}(g^2 - h^2)^2 \). Hence the denominator of \( Q - 2 \) is exactly the denominator of \( \frac{k_n(y)}{\ell_n(y)}(g^2 - h^2)^2 \).
Write $Q - 2 = \frac{Q_1}{Q_2}$ and $\frac{k_n(y)}{\ell_n(y)} 2(g^2 - h^2) = \frac{R}{Q_2}$ where each pair of $\{Q_1, Q_2\}$ and $\{R, Q_2\}$ is coprime. Then we can express the sum of $(\mathbb{T}_{M_n,\gamma}(\rho))^{-1}$ as

$$
\frac{1}{2} \sum_{Q_2=d} \frac{1}{\mathbb{T}_{M_n,\gamma}(\rho)} = \sum_{Q_1-dQ_2=0} \frac{R}{Q_2} \left( \frac{Q_1-dQ_2}{Q_2} \right)^y = \sum_{Q_1-dQ_2=0} \frac{R}{Q_2} \frac{(Q_1-dQ_2)^y}{(Q_1-dQ_2)^y} = \sum_{Q_1-dQ_2=0} \frac{R}{(Q_1-dQ_2)^y}.
$$

This implies that we can apply the Jacobi’s residue theorem for generic $d$. Hence we have the equality that $\sum_{\text{tr}_\rho(\mu^p\lambda^s) = c} \frac{1}{\mathbb{T}_{M_n,\gamma}(\rho)} = 0$ if we can show that $\deg R \leq \deg(Q_1 - dQ_2) - 2$ which is equivalent to $\deg(\frac{k_n(y)}{\ell_n(y)}(g^2 - h^2)) \leq \deg(Q_2 - d) - 2$.

By $(Q - 2)^2 - 4 = (s^2 - 4)(g^2 - h^2)^2$, we have $\deg s(g^2 - h^2) \leq \max\{\deg(Q - 2), 0\} = \deg(Q_2 - d)$ for generic $d$. Hence it holds that

$$
\deg(Q_2 - d) - \deg \frac{k_n(y)}{\ell_n(y)}(g^2 - h^2)
$$

$$
\geq \deg s(g^2 - h^2) - \deg \frac{k_n(y)}{\ell_n(y)}(g^2 - h^2)
$$

$$
= \deg \frac{k_n^2(y)h_n(y)}{\ell_n(y)} - \deg \frac{k_n(y)}{\ell(y)}
$$

$$
= \deg k_n(y)h_n(y) = |n| - 1 \geq 2
$$

by $k_n(y)h_n(y) = f_n(y) - y$. We have thus proved

$$
\sum_{\text{tr}_\rho(\mu^p\lambda^s) = c} \frac{1}{\mathbb{T}_{M_n,\gamma}(\rho)} = 0
$$

for generic $c \in \mathbb{C}$ in the case of any odd $p$.

6.2. $n \equiv 2 \pmod{4}$. The character variety $X(\pi_1(M_n))$ is given by $D \cup L$ in this case. We will see that the sums of $(\mathbb{T}_{M_n,\gamma}(\rho))^{-1}$ on $D$ and $L$ are cancelled. Propositions 6.3 and 6.4 prove Theorem 6.1 in the case of even $p$. Propositions 6.6 and 6.7 prove thereom 6.1 in the case of odd $p$.

6.2.1. $p$ is even. As in the case $n \not\equiv 2 \pmod{4}$, on the geometric component $D$ we have

$$
\frac{-k_n(y)}{\ell_n(y)} \mathbb{T}_{M_n,\gamma}(\rho) = \frac{P'}{g}
$$

$$
\frac{-k_n(y)}{g \ell_n(y)} \mathbb{T}_{M_n,\gamma}(\rho) = \frac{1}{\mathbb{T}_{M_n,\gamma}(\rho)}
$$

where $P = sg - 2h = g(m^2 + m^2) - 2h = \text{tr}_\rho(\mu^p\lambda^s)$. We can see the rational function $P$ and $\frac{k_n(y)}{\ell_n(y)}g$ has the same denominator as follows. It follows from $g^2 + h^2 - sgh = 1$, $h_n(y) = yh_n(y)$ and $\ell_n(y) = y\ell_n(y)$ that

$$
P^2 - 4 = (gs - 2h)^2 - 4(g^2 + h^2 - sgh)
$$

$$
= (s^2 - 4)g^2
$$
$$= h_n(y) [h_n(y)k_n^2(y) + 4\ell_n(y)] \left( \frac{k_n(y)}{\ell_n(y)} g \right)^2 \tag{6.5}$$

Write $P = \frac{P_1}{P_2}$ and $\frac{k_n(y)}{\ell_n(y)} g = \frac{R_1}{R_2}$, where each pair of $\{P_1, P_2\}$ and $\{R_1, R_2\}$ is coprime. Since any prime factor of the denominator of $g$ is a prime factor of $k_n$ or $\hat{\ell}_n$, any prime factor of the denominator $R_2$ is also a prime factor $k_n$ or $\hat{\ell}_n$. When we rewrite

$$\hat{\ell}_n(y) [\hat{\ell}_n(y)k_n^2(y) + 4\hat{\ell}_n(y)] \frac{R_1^2}{R_2^2}$$

for the right hand of Eq. (6.5), the denominator $R_2^2$ is coprime with $\hat{\ell}_n(y) [\hat{\ell}_n(y)k_n^2(y) + 4\hat{\ell}_n(y)]$. If there were any common factor of $R_2^2$ and $\hat{\ell}_n(y) [\hat{\ell}_n(y)k_n^2(y) + 4\hat{\ell}_n(y)]$, it would be $(y^2 - 2)^{2n} \alpha \geq 1$ in the case of $n \equiv 2 \pmod{8}$ since the denominator $P^2 - 4$ is $P_2^2$. This is impossible according to that $\hat{\ell}_n(y)$ is coprime with $\hat{h}_n(y)$. We can set $R_2 = P_2$ and $R_1 = R$ such that $P_2$ and $R$ are coprime. We can also see $R$ and $y$ are coprime which is equivalent to $g(0) \neq 0$ since $k_n(0) = (-1)^{k+1}2 \neq 0$ by Remark 5.4.

Since $u = 0$ and $v = 1$, that is $m^4\ell = 1$, at $y = 0$, $g(0)$ turns out to be

$$g(0) = \delta_{p/2-2q}(0) = \sum_{k=0}^{q} \binom{q}{k} u^k(0)v^{q-k}(0)f_{k+p/2-2q}(s(0)) = f_{p/2-2q}(s(0)).$$

**Lemma 6.2.** It holds that

$$s(0) = -\frac{4}{n} \quad \text{and} \quad g(0) \neq 0.$$  

*Proof.* It follows from Remark 2.6 (3) that

$$s(0) = z^2 - 2 = 4 \left( \frac{1}{2} - \frac{1}{n} \right) - 2 = -\frac{4}{n}.$$  

If $g(0) = 0$, then $s(0) = m^2 + m^{-2}$ is a root of $f_{p/2-2q}(u)$, $m \pm^2$ are $(p-4q)$-th roots of unity which means $-2 \leq s(0) \leq 2$. Since every Chebyshev polynomial is a monic polynomial with integer coefficients, the roots of $f_{p/2-2q}(u)$ are algebraic integers. It is known that algebraic integers in $\mathbb{Q}$ are integers. Hence $s(0) = -4/n$ must be $\pm 1, \pm 2$ which can not occur by $|n| > 2$ and $n \equiv 2 \pmod{4}$. \hfill \Box

We have the sum of $(\mathbb{T}_{M_n, \gamma}(\rho))^{-1}$ on the component $D$ as follows.

**Proposition 6.3.** It holds that, on the component $D$,

$$\sum_{[\rho] \in \mathfrak{g}_{\mathbb{C}}^{-1}(c)} \frac{1}{\mathbb{T}_{M_n, \gamma}(\rho)} = -\frac{8}{n} \frac{f_{p/2-2q}(-4/n)}{T_{p/2-2q}(-4/n) - c},$$

where $T_k(z)$ stands for $f_{k+1}(z) - f_{k-1}(z)$. 

Note that \( T_k(z) = a^k + a^{-k} \) if \( z = a + a^{-1} \).

**Proof.** We can express the sum of \( (\mathbb{T}_{M,\gamma}(\rho))^{-1} \) over \( P(y) = c \) as

\[
\sum_{P = c} \frac{1}{\mathbb{T}_{M,\gamma}(\rho)} = \sum_{P_1 - cP_2 = 0} \frac{-R}{yP_2} \left( \frac{P_1 - cP_2}{P_2} \right)^y = \sum_{P_1 - cP_2 = 0} \frac{-R}{yP_2} \left( \frac{P_1 - cP_2}{P_2} \right)^y
\]

As in the case \( n \not\equiv 2 \pmod{4} \), the Jacobi’s residue theorem implies that

\[
\frac{-R}{[y(P_1 - cP_2)]^y} \bigg|_{y=0} + \sum_{P_1 - cP_2 = 0} \frac{-R}{[yP_1 - cP_2]^y} = \sum_{y(P_1 - cP_2) = 0} \frac{-R}{[yP_1 - cP_2]^y} = 0.
\]

Hence it holds that

\[
\sum_{P = c} \frac{1}{\mathbb{T}_{M,\gamma}(\rho)} = \left. \frac{R}{[y(P_1 - cP_2)]^y} \right|_{y=0} = \left. \frac{R}{P_1 - cP_2} \right|_{y=0} = \left. \frac{R/P_2}{(P_1 - cP_2)/P_2} \right|_{y=0} = \left. \frac{k_n(y)}{\ell_n(y) P - c} \right|_{y=0}
\]

We are going to carry out \( P(0) = s(0)g(0) - 2h(0) \).

\( h(0) = \delta_{p/2-2q-1}(0) = \sum_{k=0}^q \left( \begin{array}{c} q \\ k \end{array} \right) u^k(y)v^{q-k}(y)f_{k+p/2-2q-1}(s(y)) \bigg|_{y=0} = f_{p/2-2q-1}(s(0)). \)

This implies that

\( P(0) = s(0)g(0) - 2h(0) \)

\( = s(0)f_{p/2-2q}(s(0)) - 2f_{p/2-2q-1}(s(0)) \)

\( = f_{p/2-2q+1}(s(0)) - f_{p/2-2q-1}(s(0)) \)

\( = T_{p/2-2q}(s(0)). \)

It follows from Remark 5.4 and Lemma 5.2 that

\[
\sum_{P(y) = c} \frac{1}{\mathbb{T}_{M,\gamma}(\rho)} = -\frac{4}{n} \frac{f_{p/2-2q}(-4/n)}{T_{p/2-2q}(-4/n) - c}.
\]

Since \( z = \pm \sqrt{2 + s(y)} \), we have the following equality

\[
\sum_{\rho \in \mathfrak{U}_{z}^{-1}(c)} \frac{1}{\mathbb{T}_{M,\gamma}(\rho)} = 2 \sum_{P(y) = c} \frac{1}{\mathbb{T}_{M,\gamma}(\rho)} = -\frac{8}{n} \frac{f_{p/2-2q}(-4/n)}{T_{p/2-2q}(-4/n) - c}.
\]
On the component $L$ giving $y = 0$, we have $\ell = m^{-4}$. We set $t = m^2 + m^{-2}(= z^2 - 2)$ since $s$ is no longer equal to $m^2 + m^{-2}$ on $L$. Then $P = \text{tr} \rho(m^{p_q}) = m^{p-4q} + m^{-p-4q} = T_{p/2-2q}(t)$. We express $T_{M_n,\gamma}(\rho)$ as

$$T_{M_n,\gamma}(\rho) = -\frac{p - 4q}{4} \left(2 + (m^2 + m^{-2})\frac{n}{2}\right) = -\frac{p - 4q}{4} \left(2 + \frac{n}{2}\right).$$

Since $z = \pm \sqrt{t + 2}$, we have

$$\sum_{[\rho] \in \text{tr}_{\gamma}^{-1}(c)} \frac{1}{T_{M_n,\gamma}(\rho)} = 2 \sum_{T_{p/2-2q}(t) = c} \frac{1}{p - 4q 2 + t^2} = \frac{-8/n}{p/2 - 2q} \sum_{T_{p/2-2q}(t) = c} \frac{1}{t + 4/n}. $$

**Proposition 6.4.** On the component $L$ it holds that

$$\sum_{[\rho] \in \text{tr}_{\gamma}^{-1}(c)} \frac{1}{T_{M_n,\gamma}(\rho)} = \frac{8}{n} \frac{f_{p/2-2q}(-4/n)}{T_{p/2-2q}(-4/n) - c}. $$

**Proof.** It follows from Lemma 6.5 that

$$\sum_{[\rho] \in \text{tr}_{\gamma}^{-1}(c)} \frac{1}{T_{M_n,\gamma}(\rho)} = \frac{-8/n}{p/2 - 2q} \sum_{T_{p/2-2q}(t) = c} \frac{1}{t + 4/n} = \frac{-8/n}{p/2 - 2q} \left(\frac{(p/2 - 2q)f_{p/2-2q}(-4/n)}{T_{p/2-2q}(-4/n) - c}\right).$$

□

**Lemma 6.5.**

$$\sum_{T_r(t) = c} \frac{1}{t - b} = -\frac{r f_r(b)}{T_r(b) - c}.$$

**Proof.** We first claim that $T'_r(t) = r f_r(t)$. If we write $t = a + a^{-1}$, then $T'_r(t) = \frac{dT_r}{da} = r f_r(t)$. By $1/r = f_r(t)/T'_r(t)$ we have

$$\frac{1}{r} \sum_{T_r(t) = c} \frac{1}{t - b} = \sum_{T_r(t) = c} \frac{f_r(t)}{t - b (T_r(t) - c)} = \sum_{T_r(t) = c} \frac{f_r(t)}{(t - b)(T_r(t) - c)}.$$

Since the degree of $f_r(t)$ is $|r| - 1$ and the degree of $(t - b)(T_r(t) - c)$ is $|r| + 1$, the Jacobi’s residue theorem says that

$$\sum_{(t-b)(T_r(t)-c)=0} \frac{f_r(t)}{[(t-b)(T_r(t)-c)]^r} = 0.$$
This implies that
\[
\sum_{T_r(t)=c} \frac{f_r(t)}{[(t-b)(T_r(t)-c)]'} = - \frac{f_r(t)}{[(t-b)(T_r(t)-c)]'} \bigg|_{t=b} \\
= - \frac{f_r(t)}{T_r(t)-c} \bigg|_{t=b} \\
= - \frac{f_r(b)}{T_r(b)-c}.
\]

Hence
\[
\sum_{T_r(t)=c} \frac{1}{t-b} = r \sum_{T_r(t)=c} \frac{f_r(t)}{[(t-b)(T_r(t)-c)]'} = - \frac{rf_r(b)}{T_r(b)-c}.
\]

Combining Propositions \ref{Proposition:6.3} and \ref{Proposition:6.4}, we obtain Theorem \ref{Theorem:6.1} in the case that \( n \equiv 2 \) (mod 4) and \( p \) is even.

6.2.2. \( p \) is odd. On the component \( D \), we adapt the discussion in the case of \( n \not\equiv 2 \) (mod 4) as like the case of even \( p \) for \( n \equiv 2 \) (mod 4). Recall that \( E_\gamma = mp\ell^q = gm-hm^{-1} \) and \( \text{tr} \rho(\mu^p\lambda^q) = (g-h)(m+m^{-1}) = (g-h)z \). When we set \( Q = (\text{tr} \rho(\mu^p\lambda^q))^2 \), we have the following as in Subsection 6.1.2,

\[
-\frac{k_n(y)}{\ell_n(y)} T_{M_n,\gamma}(\rho) = \frac{Q'}{2(g^2-h^2)}.
\]

**Proposition 6.6.** On the component \( D \), we have

\[
\sum_{\text{tr} \rho(\mu^p\lambda^q)=c} \frac{1}{T_{M_n,\gamma}(\rho)} = -\frac{8}{nT_{p-4q}(-4/n) + 2 - c^2}.
\]

**Proof.** It follows from \( z = \pm \sqrt{s(y)+2} \) that for generic \( c \in \mathbb{C} \),

\[
\sum_{\text{tr} \rho(\mu^p\lambda^q)=c} \frac{1}{T_{M_n,\gamma}(\rho)} = \sum_{Q(y)=c^2} \frac{-k_n(y)2(g^2-h^2)}{\ell_n(y)Q'}
\]

putting \( d = c^2 - 2 \)

\[
= \sum_{Q(y)-2=d} \frac{-k_n(y)2(g^2-h^2)}{\ell_n(y)(Q-2)'} ,
\]

since we can write \( Q - 2 = \frac{Q_1}{Q_2} \) and \( -\frac{k_n(y)}{\ell_n(y)}2(g^2-h^2) = \frac{R}{Q_2} \) by the same argument as in Subsection 6.2.1,

\[
= \sum_{Q_1-dQ_2=0} \left( \frac{R}{yQ_2} \right) \left( \frac{Q_1-dQ_2}{Q_2} \right) .
\]
\[= \sum_{Q_1 - dQ_2 = 0} \frac{R}{y(Q_1 - dQ_2)'}\]
\[= \sum_{Q_1 - dQ_2 = 0} \frac{R}{\lvert y(Q_1 - dQ_2)\rvert'}\]

The Jacobi's residue theorem implies that
\[\left.\frac{R}{\lvert y(Q_1 - dQ_2)\rvert'}\right|_{y=0} + \sum_{Q_1 - dQ_2 = 0} \frac{R}{\lvert y(Q_1 - dQ_2)\rvert'} = \sum_{y(Q_1 - dQ_2) = 0} \frac{R}{\lvert y(Q_1 - dQ_2)\rvert'} = 0.\]

Therefore it holds that
\[\sum \frac{1}{\text{tr} \rho(\mu^p \lambda^q)} = -\left.\frac{R}{\lvert y(Q_1 - dQ_2)\rvert'}\right|_{y=0} = -\left.\frac{R}{(Q_1 - dQ_2)'}\right|_{y=0} = -\left.\frac{R/Q_2}{(Q_1 - dQ_2)/Q_2}\right|_{y=0} = \left.\frac{-k_u(y)}{\ell_n(y)}\frac{2(g^2 - h^2)}{Q - c^2}\right|_{y=0}.\]  \hfill (6.6)

The value of \(g^2 - h^2\) at \(y = 0\) turns out to be
\[g^2(0) - h^2(0) = j_{p-4q}(s(0))\ell_{p-4q}(0) = f_{p-4q}(s(0))\]
since
\[g(0) = \delta_{(p+1)/2-2q}(0) = f_{(p+1)/2-2q}(s(0)),\]
\[h(0) = \delta_{(p-1)/2-2q-1}(0) = f_{(p-1)/2-2q}(s(0))\]
by \(u = 0\) and \(v = 1\) at \(y = 0\). By \(Q = (m^{p-4q} + m^{-(p-4q)})^2 = m^{2(p-4q)} + m^{-2(p-4q)} + 2 = T_{p-4q}(s(0)) + 2\) and \(k_u(0)/\ell_n(0) = s(0) = -4/n\), we can rewrite Eq. (6.6) as
\[\sum \frac{1}{\text{tr} \rho(\mu^p \lambda^q)} = -\frac{4}{n T_{p-4q}(-4/n)} \left.\frac{2f_{p-4q}(-4/n)}{n T_{p-4q}(-4/n) + 2 - c^2}\right|_{y=0}.\]

We can also carry out the sum of \((\text{tr} \rho(\mu^p \lambda^q))^{-1}\) on \(L\).

**Proposition 6.7.** On the component \(L\), we have
\[\sum \frac{1}{\text{tr} \rho(\mu^p \lambda^q)} = \frac{8}{n} \frac{f_{p-4q}(-4/n)}{n T_{p-4q}(-4/n) + 2 - c^2}.\]
Proof. Recall that \( z = m + m^{-1} \). We use the symbol \( t \) for \( m^2 + m^{-2} = z^2 - 2 \). The trace \( \text{tr} \rho(\mu^p) \) turns into \( m^p + m^{-p} = T_{p-4q}(z) \) by \( \ell = m^{-4} \). We have

\[
\sum_{\text{tr} \rho(\mu^p) = c} \frac{1}{\mathbb{T}_{M_n, \gamma}(\rho)} = \sum_{T_{p-4q}(z) = c} \frac{1}{\mathbb{T}_{M_n, \gamma}(\rho)}
= \frac{1}{2} \sum_{T_{p-4q}(z) = c^2} \frac{1}{\mathbb{T}_{M_n, \gamma}(\rho)}
= \frac{1}{2} \sum_{T_{2p-8q}(z) = c^2 - 2} \frac{1}{\mathbb{T}_{M_n, \gamma}(\rho)}
= \sum_{T_{p-4q}(t) = c^2 - 2} \frac{1}{\mathbb{T}_{M_n, \gamma}(\rho)}
= \sum_{T_{p-4q}(t) = c^2 - 2} \frac{-4}{p - 4q} \frac{1}{2 + t^2}
= -\frac{8}{n} \sum_{T_{p-4q}(t) = c^2 - 2} \frac{1}{p - 4q} \frac{1}{t^2 + \frac{1}{n}}
= \frac{8}{n} f_{p-4q}(-4/n) \sum_{T_{p-4q}(-4/n) = (c^2 - 2)}
\]

by Lemma 6.5.

Combining Propositions 6.6 and 6.7, we obtain Theorem 6.1 in the case that \( n \equiv 2 \) (mod 4) and \( p \) is odd.

7. Torus knot exteriors

Let \( r, s \) be coprime integers and \( T_{r,s} \) the torus knot of type \((r,s)\). Set \( M_{r,s} = S^3 \setminus \text{Int}N(T_{r,s}) \) where \( N(T_{r,s}) \) is a neighborhood of \( T_{r,s} \). We show that the sum of \( \mathbb{T}_{M_{r,s}, \gamma}(\rho) \) equals \( \pm 2 \) for all slopes \( \gamma \). For simplicity, we assume \( r > 0 \) and \( s > 0 \).

7.1. The sum for \( \gamma = \mu \). When we present \( \pi_1(M_{r,s}) \) as

\[
\pi_1(M_{r,s}) = \langle x, y | x^r = y^s \rangle,
\]

according to [Jo], the character variety \( X(\pi_1(M_{r,s})) \) consists of \((r-1)(s-1)/2\) components which are indexed by the following integers \( a \) and \( b \):

- \( 0 < a < r, 0 < b < s \);
- \( a \equiv b \) (mod 2) and;
- \( \text{tr} \rho(x) = 2\cos(\pi a/r), \text{tr} \rho(y) = 2\cos(\pi b/s) \) for any conjugacy class \([\rho]\) on its component.
Each component in $X(\pi_1(M_{r,s}))$ is a complex line $\mathbb{C}$ with local parameter $\text{tr}_\mu$ of the meridian $\mu$. Under the presentation (7.1) the longitude $\lambda$ satisfies

$$\lambda = x^r \mu^{-rs}.$$ 

The adjoint Reidemeister torsion $\mathbb{T}_{M_{r,s},\mu}(\rho)$ turns to be locally constant on $X(\pi_1(M_{r,s}))$ as follows.

**Lemma 7.1.** For an $\text{SL}_2(\mathbb{C})$-representation $\rho$ such that $[\rho]$ lies in the component with index $(a, b)$, the inverse of $\mathbb{T}_{M_{r,s},\mu}(\rho)$ is expressed as

$$(\mathbb{T}_{M_{r,s},\mu}(\rho))^{-1} = \frac{16}{rs} \sin^2 \left( \frac{\pi a}{r} \right) \sin^2 \left( \frac{\pi b}{s} \right).$$

**Proof.** The eigenvalues $m$ and $\ell$ for $\rho(\mu)$ and $\rho(\lambda)$ satisfy that $\ell = (-1)^a m^{-rs}$. By the curve change formula (3.8), we have

$$(\mathbb{T}_{M_{r,s},\mu}(\rho))^{-1} = \left( \frac{d \log m}{d \log \ell} \right)^{-1} (\mathbb{T}_{M_{r,s},\lambda}(\rho))^{-1} = \left( \frac{1}{-rs} \right)^{-1} (\mathbb{T}_{M_{r,s},\lambda}(\rho))^{-1}.$$

The lemma follows from the result in [Du, § 6.2] that

$$(\mathbb{T}_{M_{r,s},\lambda}(\rho))^{-1} = -\frac{16}{r^2 s^2} \sin^2 \left( \frac{\pi a}{r} \right) \sin^2 \left( \frac{\pi b}{s} \right).$$

Note that the torsion in [Du] turns into the inverse of our torsion in this paper according to the difference in the conventions. \hfill \square

We can now carry out the inverse sum of adjoint Reidemeister torsions for the torus knot exterior $M_{r,s}$ and the meridian $\mu$.

**Theorem 7.2.** For a generic complex number $c$, it holds that

$$(7.2) \sum_{[\rho] \in \text{tr}_{\mu}^{-1}(c)} \frac{1}{\mathbb{T}_{M_{r,s},\mu}(\rho)} = 2.$$

**Proof.** Since we can regard every component in $X(\pi_1(M_{r,s}))$ as a complex line $\mathbb{C}$ by the function $\text{tr}_\mu$, there exists a single conjugacy class $[\rho]$ such that $\text{tr} \rho(\mu) = c$ on each component. The inverse of $\mathbb{T}_{M_{r,s},\mu}(\rho)$ runs over all values once in the summation.

In the case of even $r$ and odd $s$, we can express the sum of $(\mathbb{T}_{M_{r,s},\mu}(\rho))^{-1}$ as

$$\sum_{[\rho] \in \text{tr}_{\mu}^{-1}(c)} \frac{1}{\mathbb{T}_{M_{r,s},\mu}(\rho)} = \frac{16}{rs} \left\{ \sin^2 \left( \frac{\pi}{r} \right) + \cdots + \sin^2 \left( \frac{(r - 1)\pi}{r} \right) \right\}\left( \sin^2 \left( \frac{\pi}{s} \right) + \cdots + \sin^2 \left( \frac{(s - 2)\pi}{s} \right) \right)$$
\[(7.3)\]
\[+ \left( \sin^2 \left( \frac{2\pi}{r} \right) + \cdots + \sin^2 \left( \frac{(r-2)\pi}{r} \right) \right) \left( \sin^2 \left( \frac{2\pi}{s} \right) + \cdots + \sin^2 \left( \frac{(s-1)\pi}{s} \right) \right) \] 

from Lemma 7.1. It follows from \(\sin^2(\pi/s) + \cdots + \sin^2((s-2)\pi/s) = \sin^2(\pi - \pi/s) + \cdots + \sin^2(2\pi/s)\) that

\[
\sin^2 \left( \frac{\pi}{s} \right) + \cdots + \sin^2 \left( \frac{(s-2)\pi}{s} \right) = \frac{1}{2} \sum_{k=1}^{s-1} \sin^2 \left( \frac{k\pi}{s} \right)
\]

Similarly we have \(\sum_{k=1}^{r-1} \sin^2(k\pi/r) = r/2\). The sum \(7.3\) turns to be

\[
(7.3) = \frac{16}{rs} \left( \sum_{k=1}^{r-1} \sin^2 \left( \frac{k\pi}{r} \right) \right) \frac{s}{4} = 2.
\]

In the case of odd \(r\) and odd \(s\), we can also have

\[
\sum_{[\rho] \in \tau^{-1}_{\gamma}(c)} \frac{1}{T_{M_{r,s},\gamma}(\rho)} = \frac{16}{rs} \left( \frac{r}{4} + \frac{s}{4} \right) = 2.
\]

**Remark 7.3.** In the case of \(r = 2\), we can find another approach to compute the inverse sum of \(T_{M_{2,s},\nu}(\rho)\) in [Yo2, Remark 1.5].

### 7.2. The sum for a general slope.

Finally we are going to carry out the sum of \((T_{M_{r,s},\gamma}(\rho))^{-1}\) for all slopes \(\gamma = \mu^p \lambda^q\).

**Theorem 7.4.** For any slope \(\gamma = \mu^p \lambda^q\) and a generic \(c \in \mathbb{C}\), it holds that

\[
\sum_{[\rho] \in \tau^{-1}_{\gamma}(c)} \frac{1}{T_{M_{r,s},\gamma}(\rho)} = \frac{1}{p - rsq} \frac{|p - rsq|}{2}.
\]

**Proof.** The slope \(\gamma\) satisfies that \(\gamma = \mu^p \lambda^q = x^q \mu^{p-rsq}\). The eigenvalues \(\xi^{\pm 1}\) of \(\rho(\gamma)\) satisfies \(\xi = (-1)^{aq}m^{p-rsq}\) on the component with index \((a,b)\) in \(X(\pi_1(M_{r,s}))\). By the curve change formula (3.8), we can express \((T_{M_{r,s},\gamma}(\rho))^{-1}\) as

\[
(T_{M_{r,s},\gamma}(\rho))^{-1} = \left( \frac{d \log \xi}{d \log m} \right)^{-1} (T_{M_{m,n},\mu}(\rho))^{-1} = \frac{1}{p - rsq} \left( \frac{d \log \xi}{d \log m} \right)^{-1}.
\]
Recall from Proposition 6.3 that \( T_n(z) = f_{n+1}(z) - f_{n-1}(z) \). If we put \( z = m + m^{-1} \), then \( T_n \) satisfies \( T_n(m + m^{-1}) = m^n + m^{-n} \). We can rewrite the condition \( \text{tr} \rho(\gamma) = c \) on the component with index \((a, b)\) as

\[
T_{p-rsq}(z) = (-1)^aq c
\]

for \( z = \text{tr} \rho(\mu) = m + m^{-1} \). Actually \( T_n \) is the Chebyshev polynomial of the first kind with degree \(|n|\). It is known that \( T_n \) has \(|n|\) different simple roots. Since \( z = \text{tr} \rho(\mu) \) is a local coordinate on every component in \( X(\pi_1(M_{r,s})) \), there exists \(|p - rsq|\) conjugacy classes \([\rho]\) such that \( \text{tr} \rho(\gamma) = c \) on each component. By Theorem 7.2, we obtain

\[
\sum_{[\rho] \in \text{tr}^{-1}_1(c)} \frac{1}{T_{M_{r,s},\gamma}(\rho)} = \frac{1}{p - rsq} \sum_{[\rho] \in \text{tr}^{-1}_1(c)} \frac{1}{T_{M_{r,s},\mu}(\rho)} = \frac{|p - rsq|}{p - rsq} 2.
\]

\[\square\]

**Corollary 7.5.** The vanishing identity of the adjoint Reidemeister torsion does not hold for any torus knot exteriors and any slopes.

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**References**

[BP] K. L. Baker and K. L. Petersen. *Character varieties of once-punctured torus bundles with tunnel number one*, Internat. J. Math. 24 (2013), no. 6, 1350048, 57 pp.

[Du] J. Dubois, *Non Abelian Twisted Reidemeister torsion for Fibered knots*, Canad. Math. Bull 49 (2006), 55–71.

[Jo] D. Johnson, *A geometric form of Casson’s invariant and its connection to Reidemeister torsion*, unpublished lecture notes.

[Po] J. Porti. *Torsion de Reidemeister pour les variétés hyperboliques*, Mem. Amer. Math. Soc. 128 (1997), no. 612, 139 pp.

[GKY] D. Gang, S. Kim, and S. Yoon. *Adjoint Reidemeister torsions from wrapped M5-branes*, Adv. Theor. Math. Phys. 25 (2021), no. 7, 1819–1845.

[GH] P. Griffiths, J. Harris. *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York (1978).

[Yo1] S. Yoon. *A vanishing identity on adjoint Reidemeister torsions of twist knots*, Algebr. Geom. Topol. 22 (2022), no. 1, 227–249.
[Yo2] S. Yoon. *Adjoint Reidemeister torsions of two-bridge knots*, Proc. Amer. Math. Soc 150 (2022), 4543–4556.