Analytic calculation of quasi-normal modes*

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Abstract

We discuss the analytic calculation of quasi-normal modes of various types of perturbations of black holes both in asymptotically flat and anti-de Sitter spaces. We obtain asymptotic expressions and also show how corrections can be calculated perturbatively. We pay special attention to low-frequency modes in anti-de Sitter space because they govern the hydrodynamic properties of a gauge theory fluid according to the AdS/CFT correspondence. The latter may have experimental consequences for the quark-gluon plasma formed in heavy ion collisions.

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1 Introduction

To many practitioners of quantum gravity the black hole plays the role of a soliton, a non-perturbative field configuration that is added to the spectrum of particle-like objects only after the basic equations of their theory have been put down, much like what is done in gauge theories of elementary particles, where Yang-Mills equations with small coupling constants determine the small-distance structure, and solitons and instantons govern the large-distance behavior.

Such an attitude however is probably not correct in quantum gravity. The coupling constant increases with decreasing distance scale which implies that the smaller the distance scale, the stronger the influences of “solitons”. At the Planck scale it may well be impossible to disentangle black holes from elementary particles.

Quasi-normal modes (QNMs) describe small perturbations of a black hole which is a thermodynamical system whose (Hawking) temperature and entropy are given in terms of its global characteristics (total mass, charge and angular momentum). They are obtained by solving a wave equation for small fluctuations subject to the conditions that the flux be ingoing at the horizon and outgoing at asymptotic infinity. These boundary conditions in general lead to a discrete spectrum of complex frequencies whose imaginary part determines the decay time of the small fluctuations

\[ \Im \omega = \frac{1}{\tau} \]

There is a vast literature on quasi-normal modes and we make no attempt to review it. Instead, we concentrate on obtaining analytic expressions for quasi-normal modes of various black hole perturbations of interest. One can rarely obtain analytic expressions in closed form. Instead, we discuss techniques which allow one to calculate the spectrum perturbatively starting with an asymptotic regime (e.g., high or low overtones). In asymptotically flat space, we discuss the cases of four-dimensional Schwarzschild and Kerr black holes. Generalization to higher-dimensional spacetimes does not present substantially new calculational challenges. However, we should point out that the case of a rotating black hole is considerably harder than the Schwarzschild case.

We also discuss asymptotically AdS spaces and obtain the spectrum as a perturbative expansion around high overtones. At leading order the frequencies are proportional to the radius of the horizon. When expanding around low overtones, one in general obtains an additional frequency which is inversely proportional to the horizon radius. Thus for large black holes there is a gap between the lowest frequency and the rest of the spectrum of quasi-normal modes. We pay special attention to the lowest frequencies because they govern the behavior of the gauge theory fluid on the boundary per the AdS/CFT correspondence. The latter may have experimental consequences pertaining to the formation of the quark-gluon plasma in heavy ion collisions.

2 Flat spacetime

We start with a study of QNMs in asymptotically flat space-times. We discuss scalar perturbations of Schwarzschild and Kerr black holes in four dimensions.
2.1 Schwarzschild black holes

The metric of a Schwarzschild black hole in four dimensions is

\[ ds^2 = -f(r) \, dt^2 + \frac{dr^2}{f(r)} + r^2 \, d\Omega^2, \quad f(r) = 1 - \frac{2GM}{r} \quad (2) \]

The Hawking temperature is

\[ T_H = \frac{1}{8\pi GM} = \frac{1}{4\pi r_0} \quad (3) \]

where \( r_0 = 2GM \) is the radius of the horizon.

A spin-\( j \) perturbation of frequency \( \omega \) is governed by the radial equation

\[ -f(r) \frac{d}{dr} \left( f(r) \frac{d\Psi}{dr} \right) + V(r) \Psi = \omega^2 \Psi \quad (4) \]

where \( V(r) \) is the “Regge-Wheeler” potential

\[ V(r) = f(r) \left( \frac{\ell(\ell+1)}{r^2} + \frac{(1-j^2)r_0}{r^3} \right) \quad (5) \]

The spin is \( j = 0, 1, 2 \) for scalar, electromagnetic and gravitational perturbations, respectively. It is advantageous to avoid integer values of \( j \) throughout the discussion and only take the limit \( j \to \text{integer} \) at the end of the calculation.

By defining the “tortoise coordinate”

\[ r_* = \int \frac{dr}{f(r)} = r + r_0 \ln \left( \frac{r}{r_0} - 1 \right) \quad (6) \]

the wave equation may be brought into a Schrödinger-like form,

\[ -\frac{d^2\Psi}{dr_*^2} + V(r(r_*)) \Psi = \omega^2 \Psi \quad (7) \]

to be solved along the entire real \( r_* \)-axis. At both ends the potential vanishes (\( V \to 0 \) as \( r_* \to \pm\infty \)) therefore the solutions behave as \( \Psi \sim e^{\pm i\omega r_*} \). For QNMs, we demand

\[ \Psi \sim e^{\pm i\omega r_*}, \quad r_* \to \pm\infty \quad (8) \]

assuming \( \Re \omega > 0 \).

2.1.1 Limit \( \ell \to \infty \)

In this case it suffices to consider the potential near its maximum. Expanding around the maximum of the potential (\( V'_0(r_{\text{max}}) = 0 \) [1],

\[ r_{\text{max}} = \frac{3}{2} r_0 + \mathcal{O}(1/\ell), \quad (9) \]

we obtain

\[ V_0[r(r_*)] \approx \alpha^2 - \beta^2 (r_* - r_*(r_{\text{max}}))^2, \quad (10) \]
where
\[ \alpha^2 = \frac{4}{27} \left( \ell + \frac{1}{2} \right) r_0^2 + \mathcal{O}(1/\ell) , \quad \beta^2 = \frac{16}{729} \left( \ell + \frac{1}{2} \right) + \mathcal{O}(1/\ell) . \] (11)

The solutions to the wave equation are
\[ \Psi_n = H_n(\sqrt{\beta}x)e^{\beta x^2/2} , \quad n = 0, 1, 2, \ldots \] (12)
where \( H_n \) are Hermite polynomials. The corresponding eigenvalues are
\[ \omega_n = \frac{2}{3\sqrt{3}r_0} \left\{ \ell + 1 + i(n + 1/2) \right\} + \mathcal{O}(1/\ell) \] (13)

This result is in agreement with the standard WKB approach [2].

**2.1.2 Limit \( n \to \infty \)**

The asymptotic form of QNMs for large \( n \) is
\[ \frac{\omega_n}{T_H} = (2n + 1)\pi i + \ln 3 \] (14)

independent of the angular momentum quantum number \( \ell \). This form was first derived numerically [3–7] and subsequently confirmed analytically [8]. The large imaginary part of the frequency \( \Im(\omega_n) \) makes the numerical analysis cumbersome but is easy to understand because the spacing of frequencies is \( 2\pi iT_H \) which is the same as the spacing of poles of a thermal Green function on the Schwarzschild black hole background. On the other hand the real part \( \Re(\omega_n) \) is small. Its analytical value was first proposed by Hod [9].

The analytical derivation of the asymptotic form (14) of QNMs by Motl and Neitzke [8] offered a new surprise because it heavily relied on the black hole singularity. It is intriguing that the unobservable region beyond the horizon influences the behavior of physical quantities.

We shall calculate the asymptotic formula for QNMs including first-order corrections [10] by solving the wave equation perturbatively for arbitrary spin of the wave. We shall obtain agreement with results from numerical analysis for gravitational and scalar waves [5, 11] and WKB analysis for gravitational waves [12].

Let
\[ \Psi = e^{-i\omega r_*} f(r_*) . \] (15)

We have \( f(r_*) \sim 1 \) as \( r_* \to +\infty \) and near the horizon, \( f(r_*) \sim e^{2i\omega r_0} \) (as \( r_* \to -\infty \)). Let us continue \( r \) analytically into the complex plane and define the boundary condition at the horizon in terms of the monodromy of \( f(r_*(r)) \) around the singular point \( r = r_0 \),
\[ \mathcal{M}(r_0) = e^{-4\pi i\omega r_0} \] (16)
along a contour running counterclockwise. We may deform the contour in the complex \( r \)-plane so that it either lies beyond the horizon \( (\Re r < r_0) \) or at infinity \( (r \to \infty) \). The monodromy only gets a contribution from the segment lying beyond the horizon.

It is convenient to change variables to
\[ z = \omega(r_* - i\pi r_0) = \omega(r + r_0 \ln(1 - r/r_0)) \] (17)
(where we chose a branch such that $z \to 0$ as $r \to 0$). The potential can be written as a series in $\sqrt{z}$,

$$V(z) = -\frac{\omega^2}{4z^2} \left( 1 - j^2 + \frac{3\ell(\ell + 1) + 1 - j^2}{3} \sqrt{-\frac{2z}{\omega r_0} + \ldots} \right) \tag{18}$$

which is a formal expansion in $1/\sqrt{\omega}$.

Now deform the contour defining the monodromy so that it gets mapped onto the real axis in the $z$-plane. Near the singularity $z = 0$,

$$z \approx -\frac{\omega}{2r_0} r^2 \tag{19}$$

Choose a contour in the $r$-plane so that near $r = 0$, the positive and negative real axes in the $z$-plane are mapped onto

$$\arg r = \pi - \frac{\omega}{2}, \quad \arg r = \frac{3\pi}{2} - \frac{\omega}{2} \tag{20}$$

in the $r$-plane, respectively. These segments form a $\pi/2$ angle (independent of $\arg \omega$).

To avoid the $r = 0$ singularity, go around an arc of angle $3\pi/2$ which corresponds to an angle of $3\pi$ around $z = 0$ in the $z$-plane.

Considering the black hole singularity ($r = 0$), we note that there are two solutions,

$$f_{\pm}(r) = r^{1 \pm j} Z_{\pm}(r) \tag{21}$$

where $Z_{\pm}$ are analytic functions of $r$. Going around an arc of angle of $3\pi/2$, we obtain

$$f_{\pm}(e^{3\pi i/2} r) = e^{3\pi(1 \pm j)i/2} f_{\pm}(r) \tag{22}$$

which is an exact result.

To proceed further, we need to relate the behavior of the wavefunction near the black hole singularity to its behavior at large $r$ in the complex $r$-plane. To this end, we shall solve the wave equation perturbatively, thus writing the wavefunction as a perturbation series in $1/\sqrt{\omega}$.

At zeroth order, the wave equation reads

$$\frac{d^2 \Psi^{(0)}}{dz^2} + \left( \frac{1 - j^2}{4z^2} + 1 \right) \Psi^{(0)} = 0 \tag{23}$$

Two linearly independent solutions are

$$f_{\pm}^{(0)}(z) = e^{iz} \psi_{\pm}^{(0)} = e^{iz} \sqrt{\frac{\pi z}{2}} J_{\pm j/2}(z) \tag{24}$$

in terms of Bessel functions. We deduce the behavior at infinity ($z \to \infty$)

$$f_{\pm}^{(0)}(z) \sim e^{iz} \cos(z - \pi(1 \pm j)/4) \tag{25}$$

The boundary conditions imply $f(z) \sim \text{const.}$ as $z \to \infty$ along the positive real axis in the $z$-plane. Therefore, we ought to adopt the linear combination

$$f^{(0)} = f_{+}^{(0)} - e^{-\pi j i/2} f_{-}^{(0)} \sim e^{iz} \sqrt{\pi} J_{\pm j/2}(z) \tag{26}$$

(in terms of a Hankel function). As $z \to \infty$, we obtain

$$f^{(0)}(z) \sim -e^{-\pi(1 + j)i/4} \sin(\pi j/2) \tag{27}$$
a constant, as desired.

Going along the $3\pi$ arc around $z = 0$ in the $z$-plane, we have

$$f^{(0)}(e^{3\pi i z}) = e^{3\pi(1+j)i/2} \left( f^{(0)}_-(z) - e^{-7\pi i/2} f^{(0)}_+(z) \right)$$

(28)

As $z \to \infty$,

$$f^{(0)}(z) \sim e^{-\pi(1+j)i/4} \sin(3\pi j/2) + e^{\pi(1-j)i/4} \sin(2\pi j) e^{2iz}$$

(29)

The monodromy to zeroth order is

$$\mathcal{M}(r_0) = -\sin\left(\frac{3\pi j}{2}\right)\sin\left(\frac{\pi j}{2}\right) = -(1+2\cos(\pi j))$$

(30)

leading to a discrete set of complex frequencies (QNMs) [8]

$$\frac{\omega_0}{T_H} = (2n+1)\pi i + \ln(1+2\cos(\pi j)) + \mathcal{O}(1/\sqrt{n})$$

(31)

Next, we calculate the first-order correction to the above expression [10]. Expanding the wavefunction in $1/\sqrt{\omega}$,

$$\Psi = \Psi^{(0)} + \frac{1}{\sqrt{-\omega r_0}} \Psi^{(1)} + \mathcal{O}(1/\omega)$$

(32)

the first-order correction obeys

$$\frac{d^2\Psi^{(1)}}{dz^2} + \left( \frac{1-f^2}{4z^2} + 1 \right) \Psi^{(1)} = \sqrt{-\omega r_0} \delta V \Psi^{(0)}$$

(33)

where

$$\delta V(z) = \frac{1-f^2}{4z^2} + \frac{1}{\omega z^2} V[r(z)]$$

(34)

Two linearly independent solutions are

$$\Psi^{(1)}_\pm(z) = \mathcal{C} \Psi^{(0)}_\pm(z) \int_z^\infty \Psi^{(0)}_\mp \delta V \Psi^{(0)}_\pm - \mathcal{C} \Psi^{(0)}_\mp(z) \int_z^\infty \Psi^{(0)}_\pm \delta V \Psi^{(0)}_\pm$$

(35)

where $\mathcal{C} = \sqrt{-\omega r_0 \sin(\pi j/2)}$ and the integral is along the positive real axis on the $z$-plane ($z > 0$). We obtain the large-$z$ behavior

$$\Psi^{(1)}_\pm(z) \sim c_{\pm} \cos(z - \pi(1+j)/4) - c_{\pm} \cos(z - \pi(1-j)/4)$$

(36)

where

$$c_{\pm} = \mathcal{C} \int_0^\infty \Psi^{(0)}_\pm \delta V \Psi^{(0)}_\pm$$

(37)

To obtain the small-$z$ behavior, expand

$$\delta V(z) = -\frac{3\ell(\ell+1) + 1-f^2}{6\sqrt{-2\omega r_0}} z^{-3/2} + \mathcal{O}(1/\omega)$$

(38)

It follows that

$$\Psi^{(1)}_\pm = z^{1\pm j/2} G_\pm(z) + \mathcal{O}(1/\omega)$$

(39)

where $G_\pm$ are even analytic functions of $z$. 

6
For the desired behavior as $z \to \infty$, define

$$
\Psi = \Psi_+^{(0)} + \frac{1}{\sqrt{-\omega r_0}} \left\{ \Psi_+^{(1)} - e^{-\pi j/2} \Psi_+^{(1)} + e^{-\pi j/2} \xi \Psi_-^{(1)} \right\} + \ldots
$$

(40)

where $\xi \sim \mathcal{O}(1)$ and dots represent terms of order higher than $\mathcal{O}(1/\sqrt{\omega})$. By demanding $\Psi \sim e^{-iz}$ as $z \to +\infty$, we fix

$$
\xi = \xi_+ + \xi_- , \quad \xi_+ = c_+ e^{\pi j/2} - c_- , \quad \xi_- = c_- e^{-\pi j/2} - c_+
$$

(41)

Then the requirement $f(z) = e^{iz} \Psi(z) \sim \text{const.}$ as $z \to \infty$ yields

$$
f(z) \sim -e^{-\pi (1+j)/4} \sin(\pi j/2) \left\{ 1 - \frac{\xi_-}{\sqrt{-\omega r_0}} \right\}
$$

(42)

In the neighborhood of the black hole singularity (around $z = 0$), going around a $3\pi$ arc, we obtain

$$
\Psi_+^{(1)}(e^{3\pi i}z) = e^{3\pi(2+j)i/2} \Psi_+^{(1)}(z)
$$

(43)

therefore

$$
\Psi(e^{3\pi i}z) = \Psi^{(0)}(e^{3\pi i}z)
$$

$$
- e^{3\pi j/2} \frac{1}{\sqrt{-\omega r_0}} \left\{ \Psi_+^{(1)}(z) - e^{-7\pi j/(2\Psi_+^{(1)}(z))} - i\xi \Psi_-^{(1)}(z) \right\}
$$

(44)

As $z \to \infty$ along the real axis,

$$
f(z) \sim e^{-\pi (1+j)/4} \sin(3\pi j/2) \left\{ 1 - \frac{1}{\sqrt{-\omega r_0}} A \right\}
$$

$$
+ e^{\pi(1-j)/4} \sin(2\pi j) \left\{ 1 - \frac{1}{\sqrt{-\omega r_0}} B \right\} e^{2iz}
$$

where

$$
A = \frac{i - 1}{2} e^{\pi j/2} (\xi_+ + i\xi_- - \xi \cot(3\pi j/2))
$$

(45)

and $B$ is not needed for our purposes. The monodromy to this order reads

$$
\mathcal{M}(r_0) = -\frac{\sin(3\pi j/2)}{\sin(\pi j/2)} \left\{ 1 + \frac{i - 1}{2\sqrt{-\omega r_0}} e^{\pi j/2} (\xi_- - \xi_+ + \xi \cot(3\pi j/2)) \right\}
$$

(46)

leading to the QNM frequencies [10]

$$
\frac{\omega_n}{T_H} = (2n + 1)\pi i + \ln(1 + 2\cos(\pi j)) + \frac{e^{\pi j/2}}{\sqrt{n + 1/2}} (\xi_- - \xi_+ + \xi \cot(3\pi j/2))
$$

$$
+ \mathcal{O}(1/n)
$$

(47)

which includes the $\mathcal{O}(1/\sqrt{n})$ correction to the $\mathcal{O}(1)$ asymptotic expression (31).

For an explicit expression, use

$$
\mathcal{J}(\nu, \mu) \equiv \int_0^\infty dz z^{-1/2} J_\nu(z) J_\mu(z) = \frac{\sqrt{\pi/2} \Gamma(\frac{\nu+\mu+1/2}{2})}{\Gamma(\frac{\nu+\mu+3/2}{2}) \Gamma(\frac{\nu-\mu+3/2}{2})}
$$

(48)
We obtain
\[ c_{\pm} = \frac{\pi}{12\sqrt{2} \sin(\pi j/2)} J(\pm j/2, \pm j/2) \] (49)
therefore
\[ \xi_- - \xi_+ + \xi \cot(3\pi j/2) = (1 - i) \frac{\ell(\ell + 1) + 1 - j^2}{24\sqrt{2}\pi^{3/2}} \sin(3\pi j/2) \]
\[ \times \Gamma^2(1/4) \Gamma(1/4 + j/2) \Gamma(1/4 - j/2) \] (50)
where we also used the identity \[ \Gamma(y)\Gamma(1 - y) = \frac{\pi}{\sin(\pi y)}. \] This expression has a well-defined finite limit as \( j \to \) integer.

For scalar waves, let \( j \to 0^+ \). We obtain
\[ \frac{\omega_n}{T_H} = (2n + 1)\pi i + \ln 3 + \frac{1 - i}{\sqrt{n + 1/2}} \frac{\ell(\ell + 1) + 1/3}{6\sqrt{2}\pi^{3/2}} \Gamma^4(1/4) + O(1/n) \] (51)
which is in agreement with numerical results [11].

For gravitational waves, we let \( j \to 2 \) and obtain
\[ \frac{\omega_n}{T_H} = (2n + 1)\pi i + \ln 3 + \frac{1 - i}{\sqrt{n + 1/2}} \frac{\ell(\ell + 1) - 1}{18\sqrt{2}\pi^{3/2}} \Gamma^4(1/4) + O(1/n) \] (52)
which is in agreement with the results from a WKB analysis [12] as well as numerical analysis [5].

### 2.2 Kerr black holes

Extending the above discussion to rotating (Kerr) black holes is not straightforward. Bohr’s correspondence principle
\[ \delta M = \hbar \Re \omega \] (53)
and the first law of black hole mechanics
\[ \delta M = T_H \delta S_{BH} + \Omega \delta J \] (54)
imply the asymptotic expression[9]
\[ \Re \omega = T_H \ln 3 + m\Omega \] (55)
where \( m \) is the azimuthal eigenvalue of the wave and \( \Omega \) is the angular velocity of horizon. In deriving the above, we identified \( \delta S_{BH} \equiv \ln 3 \) [13]. Even though the above result has the correct limit as \( \Omega \to 0 \) (in agreement with the Schwarzschild expression (14)), it is in conflict with numerical results [14] indicating \( \Re \omega \approx m\Omega \).

To resolve the above contradiction, we shall obtain an analytic solution to the wave (Teukolsky [15]) equation which will be valid for asymptotic modes bounded from above by \( 1/a \), where
\[ a = \frac{J}{M} \] (56)
with \( J \) being the angular momentum and \( M \) the mass of the Kerr black hole. The calculation will be valid for \( a \ll 1 \) which includes the Schwarzschild case \( (a = 0) \) [16]. Our results will confirm Hod’s expression (55) and not necessarily contradict numerical results (the latter may still be valid in the
asymptotic regime $1/a \lesssim \omega$). In the Schwarzschild limit ($a \to 0$) the range of frequencies extends to infinity and our expression reduces to the expected form (14). The metric of a Kerr black hole is

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 + \frac{4Mar^2}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \sin^2 \theta \left(r^2 + a^2 + \frac{2Ma^2r}{\Sigma}\right) d\phi^2$$

(57)

where $\Sigma = r^2 + a^2 \cos^2 \theta$, $\Delta = r^2 - 2Mr + a^2 = (r - r_-)(r - r_+)$ and we have set Newton's constant $G = 1$. The angular velocity of the horizon and Hawking temperature, respectively, are

$$\Omega = \frac{a}{2Mr_+}, \quad T_H = \frac{1 - r_-/r_+}{8\pi M}$$

(58)

2.2.1 Massless perturbations

Massless perturbations are governed by the Teukolsky wave equation [15]

$$\left(\frac{r^2 + a^2}{\Delta} - a^2 \sin^2 \theta\right) \frac{\partial^2 \Psi}{\partial t^2} + 4Mar^2 \frac{\partial^2 \Psi}{\partial t \partial \phi} + \left(\frac{a^2}{\Delta} - \frac{1}{\sin^2 \theta}\right) \frac{\partial^2 \Psi}{\partial \phi^2}$$

\[ - \frac{1}{\Delta} \frac{\partial}{\partial r} \left(\Delta^{s+1} \frac{\partial \Psi}{\partial r}\right) - 2s \left(\frac{M(r^2 - a^2)}{\Delta} - r - iax \cos \theta\right) \frac{\partial \Psi}{\partial t} \]

\[ - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Psi}{\partial \theta}\right) - 2s \left(\frac{a(r-M)}{\Delta} + \frac{ic \cos \theta}{\sin^2 \theta}\right) \frac{\partial \Psi}{\partial \phi} + (s^2 \cot^2 \theta - s) \Psi = 0 \]

(59)

where $s = 0, -1, -2$ for scalar, electromagnetic and gravitational perturbations, respectively. Writing the wavefunction in the form

$$\Psi = e^{-i\omega t}e^{im\phi}S(\theta)f(r)$$

(60)

we obtain the angular equation

$$\frac{1}{\sin \theta}(\sin \theta S')' + \left(a^2 \omega^2 \cos^2 \theta - \frac{m^2}{\sin^2 \theta} - 2a\omega \cos \theta - \frac{2ms \cos \theta}{\sin^2 \theta} - s^2 \cot^2 \theta\right) S$$

$$= -(A+s)S$$

(61)

where $A$ is the separation constant (eigenvalue) and the radial equation

$$\frac{1}{\Delta^s}(\Delta^{s+1}f')' + V(r)f = (A + a^2 \omega^2)f$$

(62)

where the potential is given by

$$V(r) = \frac{(r^2 + a^2)^2 \omega^2 - 4aMr \omega m + a^2 m^2 + 2ia(r - M)m + 2imr^2 - 2M(r^2 - a^2)\omega s}{\Delta} + 2i\omega s$$

(63)

Let us simplify the notation by placing the horizon at $r = 1$, i.e., by setting

$$2M = 1 + a^2, \quad r_- = a^2, \quad r_+ = 1$$

(64)
and solve the two wave equations by expanding in \(a\). We shall keep terms up to \(O(a)\) assuming \(\omega\) is large but bounded from above by \(1/a, (1 \lesssim \omega \lesssim 1/a)\). Thus \(\omega\) is in an intermediate range which becomes asymptotic in the Schwarzschild limit \(a \to 0\).

The solutions to the angular equation to lowest order are spin-weighted spherical harmonics with eigenvalue

\[
A = \ell(\ell + 1) - s(s + 1) + O(a) \tag{65}
\]

Near the horizon \((r \to 1)\),

\[
f(r) \sim (r - 1)\lambda, \quad \lambda = i(\omega - am) + O(1/\omega) \tag{66}
\]

At infinity \((r \to \infty)\), \(f(r) \sim e^{i\omega r}\). Introducing the “tortoise coordinate”

\[
z = \omega r + (\omega - am) \ln(r - 1) \tag{67}
\]

the boundary conditions read

\[
f(z) \sim e^{\pm iz}, \quad z \to \pm \infty \tag{68}
\]

From the boundary condition at the horizon we deduce the monodromy for the function \(F(z) \equiv e^{iz}f(z)\) (notice that \(F \sim \text{const. as } z \to +\infty\) around the singular point \(r = 1\),

\[
\mathcal{M}(1) = e^{4\pi(\omega - am)} + O(a^2) \tag{69}
\]

To express the radial equation in terms of the tortoise coordinate, define

\[
f(r) = \Delta_0^{-1/2} \frac{R(r)}{\sqrt{r(\omega r - am)}} \tag{70}
\]

\(\Delta_0 = r(r - 1)\) (note \(\Delta = \Delta_0 + O(a^2)\)). Inverting \(z = z(r)\),

\[
r = \sqrt{-\frac{2z}{\omega}} + O(1/\omega) \tag{71}
\]

the radial equation to lowest order in \(1/\sqrt{\omega}\) in terms of \(R\) reads

\[
d^2R \over dz^2 + \left\{1 + \frac{3is}{2z} + \frac{4 - s^2 - 4i\omega m}{16z^2}\right\} R = 0 \tag{72}
\]

to be solved along the entire real axis. This is Whittaker’s equation. The solutions may be written as

\[
M_{\kappa, \pm \mu}(x) = e^{-x/2}x^{\pm \mu + 1/2}M\left(\frac{1}{2} \pm \mu - \kappa, 1 \pm 2\mu, x\right) \tag{73}
\]

where \(\kappa = \frac{3s}{4}, \mu^2 = \frac{s(s+4iam)}{16}\), \(M_{\kappa, \pm \mu}\) is Kummer’s function (also called \(\Phi\)) and we set \(x = 2iz\). We need to introduce Whittaker’s function

\[
W_{\kappa, \mu}(x) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2} - \mu - \kappa)} M_{\kappa, \mu}(x) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2} + \mu - \kappa)} M_{\kappa, -\mu}(x) \tag{74}
\]

due to its clean asymptotic behavior,

\[
W_{\kappa, \mu}(x) \sim e^{-x/2}x^\kappa\left(1 + O(1/x)\right), \quad |x| \to \infty. \tag{75}
\]
We may compute the monodromy by deforming the contour as before. Going around an arc of angle $3\pi$, we have

$$M_{\kappa,\pm\mu}(e^{3\pi i}x) = -ie^{3\pi i\mu}M_{-\kappa,\pm\mu}(x)$$

where we used $M(a,b,-x) = e^{-x}M(b-a,b,x)$, therefore

$$W_{\kappa,\mu}(e^{3\pi i}x) = -ie^{3\pi i\mu} \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2} - \mu - \kappa)} M_{-\kappa,\mu}(x) - ie^{-3\pi i\mu} \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2} + \mu - \kappa)} M_{-\kappa,-\mu}(x)$$

(77)

To find the asymptotic behavior, we need

$$M_{-\kappa,\mu}(x) = \frac{\Gamma(1+2\mu)}{\Gamma(\frac{1}{2} + \mu + \kappa)} e^{-i\pi\kappa} W_{\kappa,\mu}(e^{i\pi}x) + \frac{\Gamma(1+2\mu)}{\Gamma(\frac{1}{2} + \mu - \kappa)} e^{-i\pi(\frac{1}{2}+\mu+\kappa)} W_{-\kappa,\mu}(x)$$

(78)

As $|x| \to \infty$, we obtain

$$W_{\kappa,\mu}(e^{3\pi i}x) \sim Ae^{x/2}x^{\kappa} + Be^{-x/2}x^{-\kappa}$$

(79)

where

$$A = -ie^{3\pi i\mu} \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2} - \mu - \kappa)} \frac{\Gamma(1+2\mu)}{\Gamma(\frac{1}{2} + \mu + \kappa)} e^{-i\pi\kappa} + (\mu \to -\mu)$$

(80)

and $B$ is not needed for our purposes. After some algebra, we deduce

$$A = -(1+2\cos\pi\kappa) + O(a^2)$$

(81)

where we used the identities $\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin\pi x}$, $\Gamma(\frac{1}{2} + x)\Gamma(\frac{1}{2} - x) = \frac{\pi}{\cos\pi x}$. The monodromy around $r = 1$ is

$$\mathcal{M}(1) = e^{4\pi(i\omega - ma)} = A$$

(82)

therefore [16]

$$\Re\omega = \frac{1}{4\pi} \ln(1+2\cos\pi\kappa) + ma + O(a^2)$$

(83)

in agreement with Hod’s formula for gravitational waves ($s = -2$) in the small-$a$ limit (in which $\Omega \approx a$, $T_H \approx \frac{1}{4\pi}$). However, it should be emphasized that these are not asymptotic values of QNMs but bounded from above by $1/a$.

### 2.2.2 Massive perturbations

The case of massive perturbations is interesting because it reveals instabilities. As is well-known, the Schwarzschild spacetime is stable against all kinds of perturbations, massive or massless which makes the Schwarzschild geometry appropriate to study astrophysical objects. On the other hand, Kerr spacetime is stable against massless perturbations but not against massive bosonic fields [17]. The instability timescale is much larger than the age of the Universe so the problem is not expected to have observable consequences. Nevertheless, the study of instabilities is an important subject and QNMs provide an indispensable tool.

For a massive scalar of mass $\mu$, the radial wave equation reads

$$\frac{d}{dr} \left( \Delta \frac{dR}{dr} \right) + \left\{ \frac{\omega^2(r^2 + a^2)^2 - 4\mu M m \omega r + m^2 a^2}{\Delta} - \mu^2 r^2 - a^2 \omega^2 - \ell(\ell + 1) \right\} R = 0,$$

(84)

We are interested in solving this equation for a small mass and low frequencies ($\mu, \omega \ll 1/M$) [17].
Away from the horizon ($r \gg M$), we may approximate by
\[
\frac{d^2}{dr^2}(rR) + \left[ -k^2 + \frac{2M\mu^2}{r} - \frac{\ell(\ell+1)}{r^2} \right] rR = 0 \quad , \quad k^2 = \mu^2 - \omega^2 \tag{85}
\]
The solution to this equation is given in terms of a confluent hypergeometric function,
\[
R(r) = (2kr)^{\ell}e^{-kr} U(\ell + 1 - M\mu^2/k, 2(\ell+1), 2kr) \tag{86}
\]
Near the horizon ($r \ll \ell/|k|$), we may approximate by
\[
z(z+1) \frac{d}{dz} \left[ z(z+1) \frac{dR}{dz} \right] + \left[ P^2 - \ell(\ell+1)z(z+1) \right] R = 0 \tag{87}
\]
where $P = \frac{am-2Mr_+\omega}{r_+ - r_-}$, $z = \frac{r-r_-}{r_+ - r_-}$. The solution to this equation is given in terms of a hypergeometric function,
\[
R(z) = \left( \frac{z}{z+1} \right)^{ip} F(-\ell, \ell + 1; 1 - 2iP; z+1) \tag{88}
\]
Matching the two expressions in the overlap region ($M \ll r \ll \ell/|k|$), we obtain the frequencies
\[
\omega_n \approx \mu + i\gamma_n \quad , \quad n \in \mathbb{N} \tag{89}
\]
where
\[
\gamma_n = C_{\ell n} \mu (\mu M)^{4(\ell+1)-\ell} \frac{am}{M - 2\mu r_+} \prod_{j=1}^{\ell} \left[ 2j \left( 1 - \frac{a^2}{M^2} \right) + (\frac{am}{M} - 2\mu r_+) \right] \tag{90}
\]
and $C_{\ell n} = \frac{2^{3(2\ell+1)}(2\ell+1+1)!^2(\ell)!^2}{(\ell+1+n)!^2(2\ell+1)!^2n!(2\ell+1)!^2}$. For $m > 0$, we have $\gamma_0 > 0$ yielding an instability. For the fastest growing mode (with $\ell = 1$, $m = 1$, $n = 2$ (2p state)) we have
\[
\tau = \frac{1}{\gamma} = \frac{24}{a\mu^2(\mu M)^2} \tag{91}
\]
which is generally large.

Notice that there is no instability in the Schwarzschild limit ($a \rightarrow 0$) and for massless perturbations ($\mu \rightarrow 0$); in both cases, $\gamma \rightarrow 0$ and therefore the lifetime $\tau \rightarrow \infty$.

### 2.3 Half-integer spin

In the case of a perturbation of half-integer spin we need to solve the Teukolsky equation [18] with potential
\[
V(r) = f(r) \left( \frac{\ell(\ell+1)}{r^2} + \frac{1}{r^3} \right) + \frac{2i\omega j}{r} - \frac{3i\omega j}{r^2} + \frac{j^2}{4r^4} \tag{92}
\]
where $j$ is the spin of the perturbing field (e.g., $j = 1/2$ for Dirac fermion). We shall set $r_0 = 1$, for simplicity, so $f(r) = 1 - \frac{1}{r}$.

Expanding around the black hole singularity $z = \omega r_+ = 0$,
\[
\frac{1}{\omega^2} V(z) = \frac{3ij}{2z} - \frac{4 - j^2}{16z^2} + \frac{\omega^3}{\omega^{1/2}z^{3/2}} + O(1/\omega) \quad , \quad \omega^3 = \frac{\ell(\ell+1) + \frac{1-j^2}{3}}{2\sqrt{2}} \tag{93}
\]
we obtain the zeroth-order wave equation

\[ \frac{d^2 \Psi}{dz^2} + \left[ 1 - 3 i j \frac{z}{2z} - 4 - j^2 \right] \Psi = 0 \quad (94) \]

whose solutions are the Whittaker functions

\[ \Psi^{(0)}(z) = M_{\lambda, \pm \mu}(-2iz), \quad \lambda = \frac{3j}{4}, \quad \mu = \frac{j}{4} \quad (95) \]

The calculation of the monodromy as before leads to the modes \[18\]

\[ \frac{\omega_n}{T_H} = -(2n + 1) \pi i + \ln(1 + 2 \cos \pi j) + \mathcal{O}(1/\sqrt{n}) \quad (96) \]

in agreement with the result for integer spin (which came from the Regge-Wheeler equation). For a Dirac fermion, \( j = 1/2 \), so asymptotically, the real part vanishes.

The first-order correction may also be calculated as before \[19\]. The result is

\[ \frac{\omega_n}{T_H} = -(2n + 1) \pi i + \ln(1 + 2 \cos \pi j) - \frac{2i}{\sqrt{-ln/2}} \sin 4\pi \mu \frac{\tilde{b}_+ A_+ B_- + \tilde{b}_- A_- B_+}{e^{-4\pi \mu} A_+ B_- - e^{4\pi \mu} A_- B_+} + \mathcal{O}(1/n) \quad (97) \]

where

\[ \tilde{b}_\pm = \frac{\omega}{4\mu} \int_0^\infty \frac{dz}{z^{3/2}} M_{\lambda, \pm \mu}(-2iz) M_{\lambda, \pm \mu}(-2iz) \quad (98) \]

and \( A_\pm = \frac{\Gamma(1 \pm 2\mu)}{\Gamma(\frac{1}{2} \pm \mu + \lambda)} e^{ip\frac{1}{2} \pm \mu - \lambda}, B_\pm = \frac{\Gamma(1 \pm 2\mu)}{\Gamma(\frac{1}{2} \pm \mu - \lambda)} e^{-ip\lambda} \). This result appears to be a complicated function of \( j \), so let us look at specific cases.

For \( j = 1/2 \) (Dirac fermions), we obtain

\[ \frac{\omega_n}{T_H} = -(2n + 1) \pi i + \frac{1 + i}{2\sqrt{n}} \left( \ell + \frac{1}{2} \right)^2 \Gamma^2 \left( \frac{1}{4} \right) + \mathcal{O}(1/n) \quad (99) \]

which is in good agreement with numerical data \[19\].

For \( j = 3/2 \), we find

\[ \frac{\omega_n}{T_H} = -(2n + 1) \pi i + \mathcal{O}(1/n) \quad (100) \]

so there are no first-order corrections to the spectrum.

For \( j = 5/2 \), we have

\[ \frac{\omega_n}{T_H} = -(2n + 1) \pi i + \frac{1 + i}{\sqrt{2n}} \frac{\omega}{\sqrt{2} \mu} \Gamma^2 \left( \frac{1}{4} \right) + \mathcal{O}(1/n) \quad (101) \]

etc.

All of the above spectra agree with the general expression we obtained for integer spin using the Regge-Wheeler equation. The relation of the latter to the Teukolsky equation is worth exploring further.
3 Anti-de Sitter spacetime

According to the AdS/CFT correspondence, QNMs of AdS black holes are expected to correspond to perturbations of the dual Conformal Field Theory (CFT) on the boundary. The establishment of such a correspondence is hindered by difficulties in solving the wave equation governing the various types of perturbation. In three dimensions one obtains a hypergeometric equation which leads to explicit analytic expressions for the QNMs \[20, 21\]. In five dimensions one obtains a Heun equation and a derivation of analytic expressions for QNMs is no longer possible. On the other hand, numerical results exist in four, five and seven dimensions \[22–24\].

3.1 Scalar perturbations

To find the asymptotic form of QNMs, we need to find an approximation to the wave equation valid in the high frequency regime. In three dimensions the resulting wave equation will be an exact equation (hypergeometric equation). In five dimensions, we shall turn the Heun equation into a hypergeometric equation which will lead to an analytic expression for the asymptotic form of QNM frequencies in agreement with numerical results.

3.1.1 AdS\(_3\)

In three dimensions the wave equation for a massless scalar field is

\[
\frac{1}{R^2} r \frac{\partial}{\partial r} \left( r^3 \left( 1 - \frac{r_0^2}{r^2} \right) \partial_r \Phi \right) - \frac{R^2}{r^2 - r_0^2} \partial_r^2 \Phi + \frac{1}{r^2} \partial_x^2 \Phi = 0 \tag{102}
\]

Writing the wavefunction in the form

\[
\Phi = e^{i(\hat{\omega} t - px)} \Psi(y), \quad y = \frac{r_0^2}{r^2} \tag{103}
\]

the wave function becomes

\[
y^2(y - 1) \left( (y - 1)\Psi' \right)' + \hat{\omega}^2 y \Psi + \hat{p}^2 y(y - 1) \Psi = 0 \tag{104}
\]

to be solved in the interval \(0 < y < 1\), where

\[
\hat{\omega} = \frac{\omega R^2}{2r_0} = \frac{\omega}{4\pi T_H}, \quad \hat{p} = \frac{pR}{2r_0} = \frac{p}{4\pi R T_H}. \tag{105}
\]

For QNMs, we are interested in the solution

\[
\Psi(y) = y(1 - y) F_1 \left( \frac{i\hat{\omega}}{2}, \frac{1}{2}; 1 + i(\hat{\omega} + \hat{p}), 1 + i(\hat{\omega} - \hat{p}); 2; y \right) \tag{106}
\]

which vanishes at the boundary \((y \to 0)\). Near the horizon \((y \to 1)\), we obtain a mixture of ingoing and outgoing waves,

\[
\Psi \sim A_+(1 - y)^{-i\hat{\omega}} + A_-(1 - y)^{+i\hat{\omega}} \quad A_\pm = \frac{\Gamma(\pm 2i\hat{\omega})}{\Gamma(1 \pm i(\hat{\omega} + \hat{p})) \Gamma(1 \pm i(\hat{\omega} - \hat{p}))}
\]

Setting \(A_- = 0\), we deduce the quasi-normal frequencies

\[
\hat{\omega} = \pm \hat{p} - in \quad , \quad n = 1, 2, \ldots \tag{107}
\]

which form a discrete spectrum of complex frequencies with \(\Im \hat{\omega} < 0\).
3.1.2 AdS$_5$

Restricting attention to the case of a large black hole, the massless scalar wave equation reads

$$\frac{1}{r^2} \partial_r (r^2 f(r) \partial_r \Phi) - \frac{R^4}{r^2 f(r)} \partial_y^2 \Phi - \frac{R^2}{r^2} \bar{\nabla}^2 \Phi = 0 \ , \ f(r) = 1 - \frac{r_0^4}{r^4} \quad (108)$$

Writing the solution in the form

$$\Phi = e^{i\omega r - \beta y} \Psi(y) \ , \ y = \frac{r^2}{r_0^2} \quad (109)$$

the radial wave equation becomes

$$(y^2 - 1) \left(y(y^2 - 1)\Psi'\right)' + \left(\frac{\hat{\omega}^2}{4} y^2 - \frac{\hat{\beta}^2}{4} (y^2 - 1)\right) \Psi = 0 \quad (110)$$

For QNMs, we are interested in the analytic solution which vanishes at the boundary and behaves as an ingoing wave at the horizon. The wave equation contains an additional (unphysical) singularity at $y = -1$, at which the wavefunction behaves as $\Psi \sim (y + 1)^{\pm \hat{\omega}/4}$. Isolating the behavior of the wavefunction near the singularities $y = \pm 1$,

$$\Psi(y) = (y - 1)^{-i\hat{\omega}/4} (y + 1)^{\pm \hat{\omega}/4} F_\pm(y) \quad (111)$$

we shall obtain two sets of modes with the same $\Im \hat{\omega}$, but opposite $\Re \hat{\omega}$.

$F_\pm(y)$ satisfies the Heun equation

$$y(y^2 - 1)F''_\pm + \left\{\left(3 - i\frac{1 \pm 1}{2} \hat{\omega}\right) y^2 - i\frac{1 \pm 1}{2} \hat{\omega} y - 1\right\} F'_\pm + \left\{\frac{\hat{\omega}}{2} \left(\pm i\hat{\omega} \mp 1 - i\right) y - (i \mp 1)\frac{\hat{\omega}}{4} \frac{\hat{\beta}^2}{4}\right\} F_\pm = 0 \quad (112)$$

to be solved in a region in the complex $y$-plane containing $|y| \geq 1$ which includes the physical regime $r > r_h$.

For large $\hat{\omega}$, the constant terms in the polynomial coefficients of $F'$ and $F$ are small compared with the other terms, therefore they may be dropped. The wave equation may then be approximated by a hypergeometric equation

$$(y^2 - 1)F''_\pm + \left\{\left(3 - i\frac{1 \pm 1}{2} \hat{\omega}\right) y - i\frac{1 \pm 1}{2} \hat{\omega}\right\} F'_\pm + \frac{\hat{\omega}}{2} \left(\pm i\hat{\omega} \mp 1 - i\right) F_\pm = 0 \quad (113)$$

in the asymptotic limit of large frequencies $\hat{\omega}$. The acceptable solution is

$$F_0(x) = 2F_1(a_+, a_-; c; (y + 1)/2) \ , \ a_\pm = 1 - \frac{i + 1}{4} \hat{\omega} \pm 1 \ , \ c = \frac{3}{2} \pm \frac{1}{2} \hat{\omega} \quad (114)$$

For proper behavior at the boundary ($y \to \infty$), we demand that $F$ be a polynomial, which leads to the condition

$$a_+ = -n \ , \ n = 1, 2, \ldots \quad (115)$$

Indeed, it implies that $F$ is a polynomial of order $n$, so as $y \to \infty$, $F \sim y^n \sim y^{-a_+}$ and $\Psi \sim y^{-i\hat{\omega}/4}y^{\pm \hat{\omega}/4}y^{-a_+} \sim y^{-2}$, as expected.

15
We deduce the quasi-normal frequencies \[25\]
\[ \hat{\omega} = \frac{\omega}{4\pi T_H} = 2n(\pm 1 - i) \] (116)
in agreement with numerical results.

It is perhaps worth mentioning that these frequencies may also be deduced by a simple monodromy argument [25]. Considering the monodromies around the singularities, if the wavefunction has no singularities other than \( y = \pm 1 \), the contour around \( y = +1 \) may be unobstructedly deformed into the contour around \( y = -1 \), which yields

\[ \mathcal{M}(1)\mathcal{M}(-1) = 1 \] (117)

Since the respective monodromies are \( \mathcal{M}(1) = e^{\pi \hat{\omega}/2} \) and \( \mathcal{M}(-1) = e^{\mp i\pi \hat{\omega}/2} \), using \( \Im \hat{\omega} < 0 \), we deduce \( \hat{\omega} = 2n(\pm 1 - i) \), in agreement with our result above.

### 3.2 Gravitational perturbations

Next we consider gravitational perturbations of AdS Schwarzschild black holes of arbitrary size in \( d \) dimensions. We shall derive analytic expressions for the asymptotic spectrum [26] including first-order corrections [27]. Our results will be in good agreement with numerical results.

The metric is

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega_{d-2}^2, \quad f(r) = \frac{r^2}{R^2} + 1 - \frac{2\mu}{r^{d-3}}. \] (118)

The radial wave equation can be cast into a Schrödinger-like form,

\[ -\frac{d^2\Psi}{dr_*^2} + V[r(r_*)]\Psi = \omega^2\Psi, \] (119)
in terms of the tortoise coordinate defined by

\[ \frac{dr_*}{dr} = \frac{1}{f(r)}. \] (120)

The potential \( V \) for the various types of perturbation has been found by Ishibashi and Kodama [28]. For tensor, vector and scalar perturbations, we obtain, respectively,

\[ V_T(r) = f(r) \left\{ \frac{\ell(\ell + d - 3)}{r^2} + \frac{(d - 2)(d - 4)f(r)}{4r^2} + \frac{(d - 2)f'(r)}{2r} \right\} \] (121)

\[ V_V(r) = f(r) \left\{ \frac{\ell(\ell + d - 3)}{r^2} + \frac{(d - 2)(d - 4)f(r)}{4r^2} - \frac{rf'''(r)}{2(d - 3)} \right\} \] (122)
where
\[ j \]
Near the black hole singularity \( (r \sim 0) \),
\[
V_T = -\frac{1}{4r_s^2} + \frac{\mathcal{A}_T}{[-2(d-2)\mu]^{\frac{d-4}{2}}} r_s^{-\frac{d-4}{2}} + \ldots, \quad \mathcal{A}_T = \frac{(d-3)^2}{2(2d-5)} + \frac{\ell(\ell+d-3)}{d-2},
\]
\[
V_V = \frac{3}{4r_s^2} + \frac{\mathcal{A}_V}{[-2(d-2)\mu]^{\frac{d-4}{2}}} r_s^{-\frac{d-4}{2}} + \ldots, \quad \mathcal{A}_V = \frac{d^2 - 8d + 13}{2(2d-15)} + \frac{\ell(\ell+d-3)}{d-2}
\]
and
\[
V_S = -\frac{1}{4r_s^2} + \frac{\mathcal{A}_S}{[-2(d-2)\mu]^{\frac{d-4}{2}}} r_s^{-\frac{d-4}{2}} + \ldots,
\]
where
\[
\mathcal{A}_S = \frac{2d^3 - 24d^2 + 94d - 116}{4(2d-5)(d-2)} + \frac{(d^2 - 7d + 14)[\ell(\ell+d-3) - (d-2)]}{(d-1)(d-2)^2}
\]
We have included only the terms which contribute to the order we are interested in. We may summarize the behavior of the potential near the origin by
\[
V = \frac{j^2 - 1}{4r_s^2} + \mathcal{A} r_s^{-\frac{d-4}{2}} + \ldots
\]
where \( j = 0 \) (2) for scalar and tensor (vector) perturbations.

On the other hand, near the boundary (large \( r \)),
\[
V = \frac{j_m^2 - 1}{4(r_s - r_s)^2} + \ldots, \quad \bar{r}_s = \int_0^{r_m} \frac{dr}{f(r)}
\]
where \( j_m = d - 1, d - 3 \) and \( d - 5 \) for tensor, vector and scalar perturbations, respectively.

After rescaling the tortoise coordinate \( (z = \omega r_s) \), the wave equation to first order becomes
\[
\left( \mathcal{H}_0 + \omega^{-\frac{d-4}{2}} \mathcal{H}_1 \right) \Psi = 0,
\]
where
\[
\mathcal{H}_0 = \frac{d^2}{dz^2} - \left[ \frac{j^2 - 1}{4z^2} - 1 \right], \quad \mathcal{H}_1 = -\mathcal{A} z^{-\frac{d-4}{2}}.
\]
By treating $\mathcal{H}_1$ as a perturbation, we may expand the wave function

$$\Psi(z) = \Psi_0(z) + \omega^{-\frac{d-1}{2}} \Psi_1(z) + \ldots$$

(131)

and solve the wave equation perturbatively.

The zeroth-order wave equation,

$$\mathcal{H}_0 \Psi_0(z) = 0,$$

(132)

may be solved in terms of Bessel functions,

$$\Psi_0(z) = A_1 \sqrt{z} J_{\nu}(z) + A_2 \sqrt{z} N_{\nu}(z).$$

(133)

For large $z$, it behaves as

$$\Psi_0(z) \sim \sqrt{\frac{2}{\pi}} [A_1 \cos(z - \alpha_+) + A_2 \sin(z - \alpha_+)]$$

$$= \frac{1}{\sqrt{2\pi}} (A_1 - iA_2) e^{-i\alpha_+} e^{iz} + \frac{1}{\sqrt{2\pi}} (A_1 + iA_2)e^{+i\alpha_+} e^{-iz}$$

where $\alpha_+ = \frac{\pi}{2} (1 \pm j)$.

At the boundary ($r \to \infty$), the wavefunction ought to vanish, therefore the acceptable solution is

$$\Psi_0(r_o) = B \sqrt{\omega (r_o - \bar{r}_o)} J_{\nu}(\omega (r_o - \bar{r}_o))$$

(134)

Indeed, $\Psi \to 0$ as $r_o \to \bar{r}_o$, as desired.

Asymptotically (large $z$), it behaves as

$$\Psi(r_o) \sim \sqrt{\frac{2}{\pi}} B \cos[\omega (r_o - \bar{r}_o) + \beta], \quad \beta = \frac{\pi}{4} (1 + j_\infty)$$

(135)

We ought to match this to the asymptotic form of the wavefunction in the vicinity of the black-hole singularity along the Stokes line $\Im z = \Im (\omega r_o) = 0$. This leads to a constraint on the coefficients $A_1, A_2$,

$$A_1 \tan(\omega \bar{r}_o - \beta - \alpha_+) - A_2 = 0.$$

(136)

By imposing the boundary condition at the horizon

$$\Psi(z) \sim e^{i\omega}, \quad z \to -\infty,$$

(137)

we obtain a second constraint. To find it, we need to analytically continue the wavefunction near the black hole singularity ($z = 0$) to negative values of $z$. A rotation of $z$ by $-\pi$ corresponds to a rotation by $-\frac{\pi}{d-1}$ near the origin in the complex $r$-plane. Using the known behavior of Bessel functions

$$J_{\nu}(e^{-i\pi z}) = e^{-i\pi \nu} J_{\nu}(z), \quad N_{\nu}(e^{-i\pi z}) = e^{i\pi \nu} N_{\nu}(z) - 2i \cos(\pi \nu) J_{\nu}(z)$$

(138)

for $z < 0$ the wavefunction changes to

$$\Psi_0(z) = e^{-i\pi (j+1)/2} \sqrt{-z} \left\{ \left[ A_1 - i(1 + e^{i\pi j})A_2 \right] J_{\nu}(-z) + A_2 e^{i\pi j} N_{\nu}(-z) \right\}$$

(139)

whose asymptotic behavior is given by

$$\Psi \sim \frac{e^{-i\pi (j+1)/2}}{\sqrt{2\pi}} \left[ A_1 - i(1 + 2e^{i\pi j})A_2 \right] e^{-iz} + \frac{e^{-i\pi (j+1)/2}}{\sqrt{2\pi}} \left[ A_1 - iA_2 \right] e^{iz}$$

(140)
Therefore we obtain a second constraint
\[ A_1 - i(1 + 2e^{i\pi})A_2 = 0 \] (141)
The two constraints are compatible provided
\[
\begin{vmatrix}
1 & -i(1 + 2e^{i\pi}) \\
\tan(\omega \bar{r}_s - \beta - \alpha_+) & -1
\end{vmatrix} = 0
\] (142)
which yields the quasi-normal frequencies [26]
\[
\omega \bar{r}_s = \frac{\pi}{4} (2 + j + j_\infty) - \tan^{-1} \frac{i}{1 + 2e^{i\pi}} + n\pi
\] (143)
The first-order correction to the above asymptotic expression may be found by standard perturbation theory [27]. To first order, the wave equation becomes
\[ \mathcal{H}_0 \Psi_1 + \mathcal{H}_1 \Psi_0 = 0 \] (144)
The solution is
\[
\Psi_1(z) = \sqrt{2}N_\frac{j}{2}(z) \int_0^z dz' \sqrt{z'} J_{\frac{j}{2}}(z') \mathcal{H}_1 \Psi_0(z') - \sqrt{2} \frac{J_{\frac{j}{2}}(z) \mathcal{H}_1 \Psi_0(z)}{W}
\] (145)
where \( W = 2/\pi \) is the Wronskian.
The wavefunction to first order reads
\[
\Psi(z) = \{A_1[1 - b(z)] - A_2a_2(z)\} \sqrt{2}J_{\frac{j}{2}}(z) + \{A_2[1 + b(z)] + A_1a_1(z)\} \sqrt{2}N_\frac{j}{2}(z)
\] (146)
where
\[
\begin{align*}
a_1(z) &= \frac{\pi \alpha'}{2} \omega \frac{z^2}{\pi^2} \int_0^z dz' z' \frac{1}{z^2} J_{\frac{j}{2}}(z') J_{\frac{j}{2}}(z') \\
a_2(z) &= \frac{\pi \alpha'}{2} \omega \frac{z^2}{\pi^2} \int_0^z dz' z' \frac{1}{z^2} N_{\frac{j}{2}}(z') N_{\frac{j}{2}}(z') \\
b(z) &= \frac{\pi \alpha'}{2} \omega \frac{z^2}{\pi^2} \int_0^z dz' z' \frac{1}{z^2} J_{\frac{j}{2}}(z') N_{\frac{j}{2}}(z')
\end{align*}
\] and \( \alpha' \) depends on the type of perturbation.
Asymptotically, it behaves as
\[
\Psi(z) \sim \sqrt{\frac{2}{\pi}} \left[ A'_1 \cos(z - \alpha_+) + A'_2 \sin(z - \alpha_+) \right]
\] (147)
where
\[
A'_1 = [1 - \bar{b}]A_1 - \bar{a}_2A_2 , \quad A'_2 = [1 + \bar{b}]A_2 + \bar{a}_1A_1
\] (148)
and we introduced the notation
\[
\bar{a}_1 = a_1(\infty) , \quad \bar{a}_2 = a_2(\infty) , \quad \bar{b} = b(\infty).
\] (149)
The first constraint is modified to
\[
A'_1 \tan(\omega \bar{r}_s - \beta - \alpha_+) - A'_2 = 0
\] (150)
Explicitly,

\[(1 - \vec{b}) \tan(\omega \vec{r}_s - \beta - \alpha_+) - \vec{a}_1]A_1 - [1 + \vec{b} + \vec{a}_2 \tan(\omega \vec{r}_s - \beta - \alpha_+)]A_2 = 0 \tag{151}\]

To find the second constraint to first order, we need to approach the horizon. This entails a rotation by \(-\pi\) in the \(z\)-plane. Using

\[a_1(e^{-i\pi z}) = e^{-i\pi \frac{d}{2(d-2)}} e^{-i\pi j} a_1(z),\]
\[a_2(e^{-i\pi z}) = e^{-i\pi \frac{d}{2(d-2)}} \left[ e^{i\pi j} a_2(z) - 4 \cos^2 \frac{\pi j}{2} a_1(z) - 2i(1 + e^{i\pi j}) b(z) \right],\]
\[b(e^{-i\pi z}) = e^{-i\pi \frac{d}{2(d-2)}} \left[ b(z) - i(1 + e^{-i\pi j}) a_1(z) \right]\]

in the limit \(z \to -\infty\) we obtain

\[\Psi(z) \sim -ie^{-i\pi/2} B_1 \cos(-z - \alpha_+) - ie^{i\pi/2} B_2 \sin(-z - \alpha_+) \tag{152}\]

where

\[B_1 = A_1 - A_1 e^{-i\pi \frac{d}{2(d-2)}} [\vec{b} - i(1 + e^{-i\pi j}) \vec{a}_1] - A_2 e^{-i\pi \frac{d}{2(d-2)}} \left[ e^{i\pi j} \vec{a}_2 - 4 \cos^2 \frac{\pi j}{2} \vec{a}_1 - 2i(1 + e^{i\pi j}) \vec{b} \right] - i(1 + e^{i\pi j}) [A_2 + A_2 e^{-i\pi \frac{d}{2(d-2)}} [\vec{b} - i(1 + e^{-i\pi j}) \vec{a}_1] + A_2 e^{-i\pi \frac{d}{2(d-2)}} e^{-i\pi j} \vec{a}_1] - A_2 + A_2 e^{-i\pi \frac{d}{2(d-2)}} [\vec{b} - i(1 + e^{-i\pi j}) \vec{a}_1] + A_2 e^{-i\pi \frac{d}{2(d-2)}} e^{-i\pi j} \vec{a}_1\]

Therefore the second constraint to first order reads

\[1 + \vec{b} + \vec{a}_2 \tan(\omega \vec{r}_s - \beta - \alpha_+) \quad \vec{b} - i(1 + 2e^{i\pi j}) + e^{-i\pi \frac{d}{2(d-2)}} ((1 + e^{i\pi j}) \vec{a}_1 + e^{i\pi j} \vec{a}_2 - i\vec{b})]A_2 = 0 \tag{153}\]

Compatibility of the two first-order constraints yields

\[\left| \begin{array}{c}
1 + \vec{b} + \vec{a}_2 \tan(\omega \vec{r}_s - \beta - \alpha_+) \\
(1 - \vec{b}) \tan(\omega \vec{r}_s - \beta - \alpha_+) - \vec{a}_1 \\
\end{array} \right| = 0 \tag{154}\]

leading to the first-order expression for quasi-normal frequencies,

\[\omega \vec{r}_s = \frac{\pi}{4} (2 + j + j_\infty) + \frac{1}{2t} \ln 2 + n\pi - \frac{1}{8} \left\{ 6i\vec{b} - 2ie^{-i\pi \frac{d}{2(d-2)}} \vec{b} - 9\vec{a}_1 + e^{i\pi \frac{d}{2(d-2)}} \vec{a}_1 + \vec{a}_2 - e^{-i\pi \frac{d}{2(d-2)}} \vec{a}_2 \right\} \]

where

\[\vec{a}_1 = \frac{\pi \omega}{4} \left( \frac{n\pi}{2\vec{r}_s} \right)^{-\frac{d}{2(d-2)}} \frac{\Gamma(\frac{1}{2} - \frac{j}{2}) \Gamma(\frac{1}{2} + \frac{d-3}{2(d-2)})}{\Gamma(\frac{1}{2} - \frac{j}{2}) \Gamma(\frac{1}{2} + \frac{d-1}{2(d-2)})} \]
\[\vec{a}_2 = \left[ 1 + 2\cot \frac{\pi(d-3)}{2(d-2)} \cot \frac{\pi}{2} \left( -j + \frac{d-3}{d-2} \right) \right] \vec{a}_1\]
\[\vec{b} = -\cot \frac{\pi(d-3)}{2(d-2)} \vec{a}_1\]

Thus the first-order correction is \(~\vec{b} (n^{-d/2})\).

The above analytic results are in good agreement with numerical results [29] (see ref. [27] for a detailed comparison).
3.3 Electromagnetic perturbations

The electromagnetic potential in four dimensions is

\[ V_{EM} = \frac{\ell (\ell + 1)}{r^2} f(r). \]  

(155)

Near the origin,

\[ V_{EM} = j^2 - \frac{1}{4} + \frac{\ell (\ell + 1)}{2\sqrt{-4\mu}} + \ldots, \]  

(156)

where \( j = 1 \). Therefore we have a vanishing potential to zeroth order. To calculate the QNM spectrum we need to include first-order corrections from the outset. Working as with gravitational perturbations, we obtain the QNMs

\[ \omega \mathcal{r}_s = n\pi - \frac{i}{4} \ln n + \frac{1}{2i} \ln \left( 2(1 + i)\mathcal{A} \sqrt{\mathcal{r}_s} \right), \quad \mathcal{A} = \frac{\ell (\ell + 1)}{2\sqrt{-4\mu}} \]  

(157)

Notice that the first-order correction behaves as \( \ln n \), a fact which may be associated with gauge invariance.

As with gravitational perturbations, the above analytic results are in good agreement with numerical results [29] (see ref. [27] for a detailed comparison).

4 AdS/CFT correspondence and hydrodynamics

A second unexpected connection comes from studies carried out using the Relativistic Heavy Ion Collider, a particle accelerator at Brookhaven National Laboratory. This machine smashes together nuclei at high energy to produce a hot, strongly interacting plasma. Physicists have found that some of the properties of this plasma are better modeled (via duality) as a tiny black hole in a space with extra dimensions than as the expected clump of elementary particles in the usual four dimensions of spacetime. The prediction here is again not a sharp one, as the string model works much better than expected. String-theory skeptics could take the point of view that it is just a mathematical spinoff. However, one of the repeated lessons of physics is unity - nature uses a small number of principles in diverse ways. And so the quantum gravity that is manifesting itself in dual form at Brookhaven is likely to be the same one that operates everywhere else in the universe.

– Joe Polchinski

There is a correspondence between \( \mathcal{N} = 4 \) Super Yang-Mills (SYM) theory in the large \( N \) limit and type-IIB string theory in AdS\(_5 \times S^5 \) (AdS/CFT correspondence). In the low energy limit, string theory is reduced to classical supergravity and the AdS/CFT correspondence allows one to calculate all gauge field-theory correlation functions in the strong coupling limit leading to non-trivial predictions on the behavior of gauge theory fluids. For example, the entropy of \( \mathcal{N} = 4 \) SYM theory in the limit of large \( \text{'t Hooft} \) coupling is precisely \( 3/4 \) its value in the zero coupling limit.

The long-distance (low-frequency) behavior of any interacting theory at finite temperature must be described by fluid mechanics (hydrodynamics). This leads to a universality in physical properties because hydrodynamics implies very precise constraints on correlation functions of conserved currents and the stress-energy tensor. Their correlators are fixed once a few transport coefficients are known.
4.1 Hydrodynamics

To study hydrodynamics of the gauge theory fluid, suppose it possesses a conserved current $j^\mu$. For simplicity, let us set the chemical potential $\mu = 0$, so that in thermal equilibrium the charge density $\langle j^0 \rangle = 0$. The retarded thermal Green function is given by

$$
G^R_{\mu\nu}(\omega, q) = -i \int d^4 x e^{-i q \cdot x} \theta(t) \langle [j_\mu(x), j_\nu(0)] \rangle,
$$

(158)

where $q = (\omega, \vec{q})$, $x = (t, \vec{x})$. It determines the response of the system to a small external source coupled to the current. For small $\omega$ and $|\vec{q}|$, the external perturbation varies slowly in space and time. Then a macroscopic hydrodynamic description for its evolution is possible [30].

For a charged density obeying the diffusion equation

$$
\partial_0 j^0 = D \nabla^2 j^0,
$$

(159)

where $D$ is the diffusion constant with dimension of length, we obtain an overdamped mode with dispersion relation

$$
\omega = -i D \vec{q}^2,
$$

(160)

The corresponding retarded Green function has a pole at $\omega = -i D \vec{q}^2$ in the complex $\omega$-plane.

Another important conserved current is the stress-energy tensor $T^{\mu\nu}$. Its conservation law may be written as

$$
\partial_0 \tilde{T}^{00} + \partial_i \tilde{T}^{0i} = 0,
\partial_0 T^{0i} + \partial_j \tilde{T}^{ij} = 0,
$$

(161)

where

$$
\tilde{T}^{00} = T^{00} - \rho, \quad \rho = \langle T^{00} \rangle,
\tilde{T}^{ij} = T^{ij} - p \delta^{ij} = -\frac{1}{\rho + p} \left[ \eta \left( \partial_i T^{0j} + \partial_j T^{0i} - \frac{2}{3} \delta^{ij} \partial_k T^{0k} \right) + \zeta \delta^{ij} \partial_k T^{0k} \right],
$$

(162)

and $\rho$ ($p$) is the energy density (pressure) of the fluid, $\eta$ ($\zeta$) is its shear (bulk) viscosity.

One obtains two types of eigenmodes, the shear modes which consist of transverse fluctuations of the momentum density $T^{0i}$, with a purely imaginary eigenvalue

$$
\omega = -i D \vec{q}^2, \quad D = \frac{\eta}{\rho + p},
$$

(163)

and a sound wave due to simultaneous fluctuations of the energy density $T^{00}$ and the longitudinal component of momentum density $T^{0i}$, with dispersion relation

$$
\omega = u_s q - \frac{i}{2} \frac{1}{\rho + p} \left( \zeta + \frac{4}{3} \eta \right) q^2, \quad u_s^2 = \frac{\partial p}{\partial \rho}.
$$

(164)

In a conformal field theory, the stress-energy tensor is traceless, so

$$
\rho = 3p, \quad \zeta = 0, \quad u_s = \frac{1}{\sqrt{3}}
$$

(165)
4.2 Branes

To understand the gravitational side of the AdS/CFT correspondence, consider a non-extremal 3-brane which is a solution of type-IIB low energy equations of motion. In the near-horizon limit $r \ll R$ where $R$ is the AdS radius, the metric becomes

$$ds_{10}^2 = \frac{(\pi T R)^2}{u} (-f(u) dt^2 + dx^2 + dy^2 + dz^2) + \frac{R^2}{4u^2 f(u)} du^2 + R^2 d\Omega_2^2,$$

(166)

where $T = \frac{r_0}{\pi R}$ is the Hawking temperature, and we have defined $u = \frac{r^2}{r_0^2}$, $f(u) = 1 - u^2$. The horizon corresponds to $u = 1$ whereas spatial infinity is at $u = 0$.

According to the gauge theory/gravity correspondence, the above background metric with non-extremality parameter $r_0$ is dual to $\mathcal{N} = 4$ SU($N$) SYM at finite temperature $T$ in the limit of $N \to \infty$, $g_{YM}^2 N \to \infty$. For the retarded Green function

$$G_{\mu\nu,\lambda\rho}(\omega, \vec{q}) = -i \int d^4x e^{-i\omega t} \theta(t) \langle [T_{\mu\nu}(x), T_{\lambda\rho}(0)] \rangle,$$

(167)

we deduce by considering an appropriate perturbation of the background metric [30],

$$G_{xy,xy}(\omega, \vec{q}) = -\frac{N^2 T^2}{16} (i2\pi T \omega + \vec{q}^2).$$

(168)

leading to the shear viscosity of strongly coupled $\mathcal{N} = 4$ SYM plasma (Kubo formula)

$$\eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int dt d\vec{x} e^{i\omega t} \theta(t) \langle [T_{xy}(x), T_{xy}(0)] \rangle = \frac{\pi}{8} N^2 T^3.$$

(169)

Other correlators may also be found by different perturbations of the metric. One obtains

$$G_{xx,xx}(\omega, \vec{q}) = \frac{N^2 \pi T^3 q^2}{8(i\omega - \mathcal{D} q^2)} + \ldots,$$

$$G_{xx,xc}(\omega, \vec{q}) = \frac{N^2 \pi T^3 \omega q}{8(i\omega - \mathcal{D} q^2)} + \ldots,$$

$$G_{xc,xc}(\omega, \vec{q}) = \frac{N^2 \pi T^3 \omega^2}{8(i\omega - \mathcal{D} q^2)} + \ldots,$$

(170a)

where $\mathcal{D} = \frac{1}{4\pi T}$.

From the above results, one may deduce the viscosity $\eta$. Indeed, recall from hydrodynamics $\mathcal{D} = \frac{\eta}{\rho + p}$. Using the entropy

$$s = \frac{3}{4} s_0 = \frac{\pi^2}{2} N^2 T^3,$$

(171)

where $s_0$ is the entropy at zero coupling, and the thermodynamic equations $s = \frac{2p}{\rho T}$, $\rho = 3p$, we deduce $\rho + p = \frac{9}{4} N^2 T^4$, therefore

$$\eta = \frac{\pi}{8} N^2 T^3, \quad \frac{\eta}{s} = \frac{1}{4\pi}$$

(172)

which agrees with the Kubo formula. It should be pointed out that there is no agreement unless $s = \frac{3}{4} s_0$, a fact which is still poorly understood.
The above result for the viscosity is based on the gravity dual of the gauge theory fluid and should correspond to its strong coupling regime. At weak coupling, one obtains by a direct calculation
\[
\frac{\eta}{s} \gg \frac{1}{4\pi}
\]
Thus the viscosity coefficient \(\eta\) varies as a function of the ‘t Hooft coupling,
\[
\eta = f_\eta (g_{\text{YM}}^2 N^2 T^3)
\]
(174)
where \(f_\eta (x) \sim \frac{1}{-x^2 \ln x}\) for \(x \ll 1\) and \(f_\eta (x) = \frac{\pi}{8}\) for \(x \gg 1\).

4.3 Schwarzschild black holes

In the metric considered above, the horizon was flat. This corresponds to the limit of a large black hole. For a black hole of finite size, the horizon is generally a sphere. Then the boundary of spacetime is \(S^3 \times \mathbb{R}\). This may be conformally mapped onto a flat Minkowski space. Then by holographic renormalization, the AdS\(_5\)-Schwarzschild black hole is dual to a spherical shell of plasma on the four-dimensional Minkowski space which first contracts and then expands (conformal soliton flow) [31].

Quasi-normal modes govern the properties of this plasma with long-lived modes (i.e., of small \(\Im \omega\)) having the most influence. For example, one obtains the ratio
\[
\frac{v^2}{\delta} = \frac{1}{6\pi} \frac{\omega^4 - 40\omega^2 + 72}{\omega^3 - 4\omega} \sin \frac{\pi \omega}{2}
\]
(175)
where \(v_2 = \langle \cos 2\phi \rangle\) evaluated at \(\theta = \frac{\pi}{2}\) (mid-rapidity) and averaged with respect to the energy density at late times; \(\delta = \frac{\langle y^2 - x^2 \rangle}{\langle y^2 + x^2 \rangle}\) is the eccentricity at time \(t = 0\). Numerically, \(\frac{v^2}{\delta} = 0.37\), which compares well with the result from RHIC data, \(\frac{v^2}{\delta} \approx 0.323\) [32].

Another observable is the thermalization time which is found to be
\[
\tau = \frac{1}{2|\Im \omega|} \approx \frac{1}{8.6T_{\text{peak}}} \approx 0.08 \text{ fm/c} \quad , \quad T_{\text{peak}} = 300 \text{ MeV}
\]
(176)
not in agreement with the RHIC result \(\tau \sim 0.6 \text{ fm/c}\) [33], but still encouragingly small. For comparison, the corresponding result from perturbative QCD is \(\tau \gtrsim 2.5 \text{ fm/c}\) [34, 35].

The above results motivate the calculation of low-lying QNMs. Earlier, we calculated analytically the asymptotic form of QNMs for large black holes. We obtained frequencies which were proportional to the horizon radius \(r_0\). We found an infinite spectrum, however we missed the lowest frequencies which are inversely proportional to \(r_0\). The latter are important in the understanding of the hydrodynamic behavior of the gauge theory fluid via the AdS/CFT correspondence.

4.3.1 Vector perturbations

We start with vector perturbations and work in a \(d\)-dimensional Schwarzschild background. It is convenient to introduce the coordinate [36]
\[
u = \left(\frac{r_0}{r}\right)^{d-3}
\]
(177)
The wave equation becomes
\[
-(d - 3)^2 u^{d-3} \hat{f}(u) \left(u^{d-3} \hat{f}(u)\Psi\right)' + \hat{V}(u)\Psi = \hat{\omega}^2 \Psi \quad , \quad \hat{\omega} = \frac{\omega}{r_0}
\]
(178)
where prime denotes differentiation with respect to \( u \) and we have defined

\[
\hat{f}(u) \equiv \frac{f(r)}{r^2} = 1 - u^{\frac{d-3}{2}} \left( u - \frac{1 - u}{r_0^2} \right)
\]  

(179)

\[
\hat{V}_V(u) \equiv \frac{V}{r_0^2} = \hat{f}(u) \left\{ \mathcal{L}^2 + \frac{(d-2)(d-4)}{4} u^{-\frac{d-3}{2}} \hat{f}(u) - \frac{(d-1)(d-2)}{2} \left( 1 + \frac{1}{r_0^2} \right) \right\}
\]  

(180)

where \( \mathcal{L}^2 = \frac{\ell(\ell + d-3)}{r_0^2} \).

First let us consider the large black hole limit \( r_0 \to \infty \) keeping \( \omega \) and \( \mathcal{L} \) fixed (small). Factoring out the behavior at the horizon \((u = 1)\)

\[
\Psi = (1 - u)^{-i\omega \mathcal{L}} F(u)
\]  

(181)

the wave equation simplifies to

\[
\mathcal{A} F'' + \mathcal{B}_\omega F' + \mathcal{C}_\omega \mathcal{L} F = 0
\]  

(182)

where

\[
\mathcal{A} = -(d - 3)^2 u^{\frac{d-3}{2}} (1 - u^{\frac{d-1}{2}})
\]

\[
\mathcal{B}_\omega = -(d - 3)[d - 4 - (2d - 5)u^{\frac{d-3}{2}}]u^{\frac{d-3}{2}} - 2(d - 3)^2 \frac{i\omega}{d - 1} \frac{u^{\frac{d-3}{2}} (1 - u^{\frac{d-1}{2}})}{1 - u}
\]

\[
\mathcal{C}_\omega \mathcal{L} = \mathcal{L}^2 + \frac{(d-2)[d - 4 - 3(d - 2)u^{\frac{d-3}{2}}]}{4} u^{-\frac{d-3}{2}}
\]

\[
- \frac{\omega^2}{1 - u^{\frac{d-3}{2}}} + (d - 3)^2 \frac{\omega^2}{(d-1)^2} \frac{u^{\frac{d-3}{2}} (1 - u^{\frac{d-1}{2}})}{(1 - u)^2}
\]

\[
-(d - 3) \frac{i\omega}{d - 1} \frac{[d - 4 - (2d - 5)u^{\frac{d-3}{2}}]u^{\frac{d-3}{2}}}{1 - u} - (d - 3)^2 \frac{i\omega}{d - 1} \frac{u^{\frac{d-3}{2}} (1 - u^{\frac{d-1}{2}})}{(1 - u)^2}
\]

We may solve this equation perturbatively by separating

\[
(\mathcal{H}_0 + \mathcal{H}_1) F = 0
\]  

(183)

where

\[
\mathcal{H}_0 F \equiv \mathcal{A} F'' + \mathcal{B}_0 F' + \mathcal{C}_{0,0} F
\]

\[
\mathcal{H}_1 F \equiv (\mathcal{B}_\omega - \mathcal{B}_0) F' + (\mathcal{C}_{\omega \mathcal{L}} - \mathcal{C}_{0,0}) F
\]

Expanding the wavefunction perturbatively,

\[
F = F_0 + F_1 + \ldots
\]  

(184)

at zeroth order the wave equation reads

\[
\mathcal{H}_0 F_0 = 0
\]  

(185)

whose acceptable solution is

\[
F_0 = u^{\frac{d-3}{2(d-3)}}
\]  

(186)
being regular at both the horizon \((u = 1)\) and the boundary \((u = 0, \text{ or } \Psi \sim r^{-\frac{d-3}{2}} \to 0 \text{ as } r \to \infty)\). The Wronskian is
\[
W = \frac{1}{u^{\frac{d-3}{2}}(1 - u^{\frac{d-3}{2}})}
\]
and another linearly independent solution is
\[
\tilde{F}_0 = F_0 \int \frac{\mathcal{H}_1 F_0}{\mathcal{W}}
\]
which is unacceptable because it diverges at both the horizon \((\tilde{F}_0 \sim \ln(1 - u) \text{ for } u \approx 1)\) and the boundary \((\tilde{F}_0 \sim u^{-\frac{d-4}{2(d-3)}} \text{ for } u \approx 0, \text{ or } \Psi \sim r^{-\frac{d-4}{2}} \to \infty \text{ as } r \to \infty)\).

At first order the wave equation reads
\[
\mathcal{H}_0 F_1 = -\mathcal{H}_1 F_0
\]
whose solution may be written as
\[
F_1 = F_0 \int \frac{W}{F_0^2} \int \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{W}}
\]
The limits of the inner integral may be adjusted at will because this amounts to adding an arbitrary amount of the unacceptable solution. To ensure regularity at the horizon, choose one of the limits of integration at \(u = 1\) rendering the integrand regular at the horizon. Then at the boundary \((u = 0)\),
\[
F_1 = \tilde{F}_0 \int_0^1 F_0 \mathcal{H}_1 F_0 \frac{1}{\mathcal{W}} + \text{regular terms}
\]
The coefficient of the singularity ought to vanish,
\[
\int_0^1 \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{W}} = 0
\]
which yields a constraint on the parameters (dispersion relation)
\[
a_0 \hat{L}^2 - i a_1 \hat{\omega} - a_2 \hat{\omega}^2 = 0
\]
After some algebra, we arrive at
\[
a_0 = \frac{d-3}{d-1}, \quad a_1 = d-3
\]
The coefficient \(a_2\) may also be found explicitly for each dimension \(d\), but it cannot be written as a function of \(d\) in closed form. It does not contribute to the dispersion relation at lowest order. E.g., for \(d = 4, 5\), we obtain, respectively
\[
a_2 = \frac{65}{108} - \frac{1}{3} \ln 3, \quad \frac{5}{6} - \frac{1}{2} \ln 2
\]
Eq. (193) is quadratic in \(\hat{\omega}\) and has two solutions,
\[
\hat{\omega}_0 \approx -i \frac{L^2}{d-1}, \quad \hat{\omega}_1 \approx -i \frac{d-3}{a_2} + i \frac{L^2}{d-1}
\]
In terms of the frequency $\omega$ and the quantum number $\ell$,
\[
\omega_0 \approx -\frac{i\ell (\ell + d - 3)}{(d-1)r_0}, \quad \frac{\omega_1}{r_0} \approx -\frac{i}{a_2} + \frac{i\ell (\ell + d - 3)}{(d-1)r_0^2} \tag{197}
\]

The smaller of the two, $\omega_0$, is inversely proportional to the radius of the horizon and is not included in the asymptotic spectrum. The other solution, $\omega_1$, is a crude estimate of the first overtone in the asymptotic spectrum, nevertheless it shares two important features with the asymptotic spectrum: it is proportional to $r_0$ and its dependence on $\ell$ is $O(1/r_0^2)$. The approximation may be improved by including higher-order terms. This increases the degree of the polynomial in the dispersion relation (193) whose roots then yield approximate values of more QNMs. This method reproduces the asymptotic spectrum derived earlier albeit not in an efficient way.

To include finite size effects, we shall use perturbation theory (assuming $1/r_0$ is small) and replace $H_1$ by
\[
H_1' = H_1 + \frac{1}{r_0} H_H \tag{198}
\]

where
\[
H_H F \equiv A_H F'' + B_H F' + C_H F \tag{199}
\]

The coefficients may be easily deduced by collecting $O(1/r_0^2)$ terms in the exact wave equation. We obtain
\[
A_H = -2(d - 3)^2 u^2 (1 - u) \\
B_H = -(d - 3)u \left[ (d - 3)(2 - 3u) - (d - 1) \frac{1 - u}{1 - u^{d - 2}} u^{d - 4} \right] \\
C_H = \frac{d - 2}{2} \left[ d - 4 - (2d - 5)u - (d - 1) \frac{1 - u}{1 - u^{d - 2}} u^{d - 4} \right]
\]

Interestingly, the zeroth order wavefunction $F_0$ is an eigenfunction of $H_H$,
\[
H_H F_0 = -(d - 2)F_0 \tag{200}
\]

therefore the first-order finite-size effect is a simple shift of the angular momentum operator
\[
\hat{L}^2 \to \hat{L}^2 - \frac{d - 2}{r_0^2} \tag{201}
\]

The QNMs of lowest frequency are modified to
\[
\omega_0 = -i\frac{\ell(\ell + d - 3) - (d - 2)}{(d-1)r_0} + O(1/r_0^2) \tag{202}
\]

For $d = 4, 5$, we have respectively,
\[
\omega_0 = -i\frac{(\ell - 1)(\ell + 2)}{3r_0}, \quad -i\frac{(\ell + 1)^2 - 4}{4r_0} \tag{203}
\]

in agreement with numerical results [29, 31].

According to the AdS/CFT correspondence, dual to the AdS Schwarzschild black hole is a gauge theory fluid on the boundary of AdS ($S^{d-2} \times \mathbb{R}$). Consider the fluid dynamics ansatz
\[
u_i = \mathcal{H} e^{-i\Omega \tau} \psi_i \tag{204}
\]
where \( u_i \) is the (small) velocity of a point in the fluid, and \( \nabla_i \) a vector harmonic on \( S^{d-2} \). Demanding that this ansatz satisfy the standard equations of linearized hydrodynamics, one arrives at a constraint on the frequency of the perturbation \( \Omega \) which yields [37]

\[
\Omega = -i \frac{\ell (\ell + d - 3) - (d - 2)}{(d-1)r_0} + o(1/r_0^3)
\]

in perfect agreement with its dual counterpart.

### 4.3.2 Scalar perturbations

Next we consider scalar perturbations which are calculationally more involved but phenomenologically more important because their spectrum contains the lowest frequencies. For a scalar perturbation we ought to replace the potential \( \hat{V}_Y \) by

\[
\hat{V}_S(u) = \frac{\hat{f}(u)}{4} \left[ \hat{m} + \left(1 + \frac{1}{r_0^2}\right) u \right]^{-2} 
\]

\[
\times \left\{ d(d-2) \left(1 + \frac{1}{r_0^2}\right)^2 u^{\frac{d - 3}{d-1}} - 6(d-2)(d-4)\hat{m} \left(1 + \frac{1}{r_0^2}\right) u^{\frac{d - 5}{d-3}} 
+ (d-4)(d-6)\hat{m}^2 u^{\frac{d - 3}{d-1}} + (d-2)^2 \left(1 + \frac{1}{r_0^2}\right)^3 u^3 
+ 2(2d^2 - 11d + 18)\hat{m} \left(1 + \frac{1}{r_0^2}\right)^2 u^2 
+ \frac{(d-4)(d-6) \left(1 + \frac{1}{r_0^2}\right)^2}{r_0^2} u^2 - 3(d-2)(d-6)\hat{m}^2 \left(1 + \frac{1}{r_0^2}\right) u 
- \frac{6(d-2)(d-4)\hat{m} \left(1 + \frac{1}{r_0^2}\right)}{r_0^2} u + 2(d-1)(d-2)\hat{m}^2 + d(d-2)\hat{m}^2 \right\}
\]

where \( \hat{m} = \frac{2\ell (\ell + d - 3) - (d - 2)}{(d-1)(d-2)\hat{m}^2} = \frac{2(\ell + d - 2)(\ell - 1)}{(d-1)(d-2)r_0^2} \).

In the large black hole limit \( r_0 \to \infty \) with \( \hat{m} \) fixed (small), the potential simplifies to

\[
\hat{V}_S^{(0)}(u) = \frac{1 - u^{\frac{d - 1}{d - 3}}}{4(\hat{m} + u)^2} \left\{ d(d-2)u^{\frac{d - 3}{d-3}} - 6(d-2)(d-4)\hat{m}u^{\frac{d - 5}{d-3}} 
+ (d-4)(d-6)\hat{m}^2 u^{\frac{d - 3}{d-1}} + (d-2)^2 u^3 
+ 2(2d^2 - 11d + 18)\hat{m}u^2 - 3(d-2)(d-6)\hat{m}^2 u + 2(d-1)(d-2)\hat{m}^3 \right\}
\]

The wave equation has an additional singularity due to the double pole of the scalar potential at \( u = -\hat{m} \). It is desirable to factor out the behavior not only at the horizon, but also at the boundary and the pole of the scalar potential,

\[
\Psi = (1 - u)^{-\frac{d-1}{d-3}} \frac{u^{\frac{d - 1}{d-3}}}{\hat{m} + u} F(u)
\]
Then the wave equation reads

$$\mathcal{A} F'' + \mathcal{B}_\omega F' + \mathcal{C}_\omega F = 0$$  \hspace{1cm} (209)

where

$$\mathcal{A} = -(d - 3)^2 u^{\frac{d-4}{d-3}} (1 - \frac{4}{d-3})$$

$$\mathcal{B}_\omega = -(d - 3) u^{\frac{d-4}{d-3}} (1 - \frac{4}{d-3}) \left[ \frac{d-4}{u} - \frac{2(d - 3)}{\hat{m} + u} \right]$$

$$\mathcal{C}_\omega = -u^{\frac{d-4}{d-3}} (1 - \frac{4}{d-3}) \left[ \frac{(d-4)(d-4) - 2(d-3)^2 i\hat{\omega}}{d-1} - \frac{2(d-3)^2}{(\hat{m} + u)^2} \right]$$

$$= -(d - 3) [d - 4 - (2d - 5) u^{\frac{4}{d-3}}] u^{\frac{d-4}{d-3}} - 2(d - 3)^2 \frac{i\hat{\omega}}{d-1} \frac{u^{\frac{d-4}{d-3}} (1 - \frac{4}{d-3})}{1 - u}$$

$$+ \left\{ d - 4 - (2d - 5) u^{\frac{4}{d-3}} \right\} u^{\frac{d-4}{d-3}} + 2(d - 3) \frac{i\hat{\omega}}{d-1} \frac{u^{\frac{d-4}{d-3}} (1 - \frac{4}{d-3})}{1 - u}$$

$$- (d - 3)^2 \left( \frac{i\hat{\omega}}{d-1} \frac{[d - 4 - (2d - 5) u^{\frac{4}{d-3}}] u^{\frac{d-4}{d-3}}}{1 - u} - (d - 3)^2 \frac{i\hat{\omega}}{d-1} \frac{u^{\frac{d-4}{d-3}} (1 - \frac{4}{d-3})}{1 - u} \right)$$

$$+ \hat{\Psi}^{(0)}(u) - \hat{\omega}^2 + (d - 3)^2 \frac{\hat{\omega}^2 - u^{\frac{d-4}{d-3}} (1 - \frac{4}{d-3})}{(d-1)^2}$$

We shall define zeroth-order wave equation as $\mathcal{H}_0 F_0 = 0$, where

$$\mathcal{H}_0 F \equiv \mathcal{A} F'' + \mathcal{B}_0 F'$$  \hspace{1cm} (210)

The acceptable zeroth-order solution is

$$F_0(u) = 1$$  \hspace{1cm} (211)

which is plainly regular at all singular points ($u = 0, 1, -\hat{m}$). It corresponds to a wavefunction vanishing at the boundary ($\Psi \sim r^{-\frac{d-4}{2}}$ as $r \to \infty$).

The Wronskian is

$$\mathcal{W} = \frac{(\hat{m} + u)^2}{u^{\frac{d-4}{d-3}} (1 - \frac{4}{d-3})}$$  \hspace{1cm} (212)

and an unacceptable solution is $\tilde{F}_0 = \int \mathcal{W}$. It can be written in terms of hypergeometric functions.

For $d \geq 6$, it has a singularity at the boundary, $\tilde{F}_0 \sim u^{-\frac{d-4}{2}}$ for $u \approx 0$, or $\Psi \sim r^{-\frac{d-6}{2}} \to \infty$ as $r \to \infty$. For $d = 5$, the acceptable wavefunction behaves as $r^{-1/2}$ whereas the unacceptable one behaves as $r^{-1/2} \ln r$. For $d = 4$, the roles of $F_0$ and $\tilde{F}_0$ are reversed, however the results still valid because the correct boundary condition at the boundary is a Robin boundary condition $[36, 37]$. Finally, we note that $\tilde{F}_0$ is also singular (logarithmically) at the horizon ($u = 1$).

Working as in the case of vector modes, we arrive at the first-order constraint

$$\int_0^1 \frac{\mathcal{C}_\omega}{\mathcal{A} \mathcal{W}} = 0$$  \hspace{1cm} (213)

because $\mathcal{H}_1 F_0 \equiv (\mathcal{B}_\omega - \mathcal{B}_0) F_0' + \mathcal{C}_\omega F_0 = \mathcal{C}_\omega$. This leads to the dispersion relation

$$a_0 - a_1 i\hat{\omega} - a_2 \hat{\omega}^2 = 0$$  \hspace{1cm} (214)

After some algebra, we obtain

$$a_0 = \frac{d - 1}{2} \frac{1 + (d - 2)\hat{m}}{(1 + \hat{m})^2} , \hspace{0.5cm} a_1 = \frac{d - 3}{(1 + \hat{m})^2} , \hspace{0.5cm} a_2 = \frac{1}{\hat{m}} \{ 1 + O(\hat{m}) \}$$  \hspace{1cm} (215)
For small $\hat{m}$, the quadratic equation has solutions

$$\hat{\omega}_0^\pm \approx -i \frac{d-3}{2} \hat{m} \pm \sqrt{\frac{d-1}{2} \hat{m}}$$

related to each other by $\hat{\omega}_0^+ = -\hat{\omega}_0^-$, which is a general symmetry of the spectrum.

Finite size effects at first order amount to a shift of the coefficient $a_0$ in the dispersion relation

$$a_0 \to a_0 + \frac{1}{r_0^2} a_H$$

After some tedious but straightforward algebra, we obtain

$$a_H = \frac{1}{\hat{m}} \{ 1 + O(\hat{m}) \}$$

The modified dispersion relation yields the modes

$$\hat{\omega}_0^\pm \approx -i \frac{d-3}{2} \hat{m} \pm \sqrt{\frac{d-1}{2} \hat{m} + 1}$$

In terms of the quantum number $\ell$,

$$\omega_0^\pm \approx -i(d-3) \frac{\ell(\ell+d-3) - (d-2)}{(d-1)(d-2)r_0} \pm \sqrt{\frac{\ell(\ell+d-3)}{d-2}}$$

in agreement with numerical results [31].

Notice that the imaginary part is inversely proportional to $r_0$, as in vector case. In the scalar case, we also obtained a finite real part independent of $r_0$. It yields the speed of sound $v_s = \frac{1}{\sqrt{d-2}}$, which is the correct value in the presence of conformal invariance.

Turning to the implications of the above results for the AdS/CFT correspondence, we may perturb the gauge theory fluid on the boundary of AdS ($S^{d-2} \times \mathbb{R}$) using the ansatz

$$u_i = \mathcal{H} e^{-i\Omega \tau} \nabla_i S, \quad \delta p = \mathcal{H}' e^{-i\Omega \tau} S$$

where $u_i$ is the (small) velocity of a point in the fluid and $\delta p$ is a pressure perturbation. They are both given in terms of $S$, a scalar harmonic on $S^{d-2}$. Demanding that this ansatz satisfy the equations of linearized hydrodynamics, one obtains a frequency of perturbation $\Omega$ in perfect agreement with our analytic result [36, 37].

### 4.3.3 Tensor perturbations

Finally, for completeness we discuss the case of tensor perturbations. Unlike the other two cases, the asymptotic spectrum of tensor perturbations is the entire spectrum. To see this, note that in the large black hole limit, the wave equation reads

$$-(d-3)^2 (\frac{d+3}{4} u - u^3) \Psi'' - (d-3)(d-4)u \frac{d-3}{4} - (2d-5)u^2 \Psi'$$

$$+ \left\{ \hat{L}^2 + \frac{d(d-2)}{4} u^2 - \frac{(d-2)^2}{4} u^2 - \frac{\hat{\omega}^2}{1 - u^\frac{d-3}{2}} \right\} \Psi = 0$$
For the zeroth-order equation, we may set $\hat{L} = 0 = \hat{\omega}$. The resulting equation may be solved exactly. Two linearly independent solutions are ($\Psi = F_0$ at zeroth order)

$$F_0(u) = u^{\frac{d-2}{d-3}}, \quad \tilde{F}_0(u) = u^{-\frac{d-2}{2(d-3)}} \ln \left(1 - u^{\frac{d-4}{d-3}}\right)$$  \hspace{1cm} (222)

Neither behaves nicely at both ends ($u = 0, 1$). Therefore both are unacceptable which makes it impossible to build a perturbation theory to calculate small frequencies which are inversely proportional to $r_0$. This negative result is in agreement with numerical results [29, 31] and in accordance with the AdS/CFT correspondence. Indeed, there is no ansatz that can be built from tensor spherical harmonics $T_{ij}$ satisfying the linearized hydrodynamic equations, because of the conservation and tracelessness properties of $T_{ij}$.

5 Conclusion

We discussed the calculation of analytic asymptotic expressions for quasi-normal modes of various perturbations of black holes in asymptotically flat as well as anti-de Sitter spaces. We also showed how perturbative corrections to the asymptotic expressions can be systematically calculated.

In view of the AdS/CFT correspondence, in AdS spaces we concentrated on low frequency modes because they govern the hydrodynamic behavior of the gauge theory fluid which is dual to the black hole. Thus, these modes provide a powerful tool in understanding the hydrodynamics of a gauge theory at strong coupling. They may lead to experimental consequences pertaining to the quark-gluon plasma produced in heavy ion collisions at RHIC and the LHC.

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