ELECTROMAGNETIC FORM FACTORS OF THE TRANSITION
\[ \pi^++\gamma^* \to A_1. \]

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Abstract

Corrections due to gluon condensates to electromagnetic form factors of the transition \( \pi^++\gamma^* \to A_1 \) are calculated by standard QCD sum rules technique[1,2]. The obtained results are compared to the corresponding light-cone QCD sum rules[3].

1

The way of deriving sum rules in the framework of QCD is well-known. To obtain sum rules for electromagnetic form factors of \( \pi^++\gamma^* \to A_1 \) transition one should consider the three-point function:

\[ \Gamma_{\mu\nu;\lambda}(p, p'; q) = -\int dxdye^{ip'x - iqy} <0|T\{j_{A\mu}(x)j_{\lambda}^{el}(y)j_{A\mu}(0)\}|0> \]

where \( j_{A\mu} \) is the axial vector light quark current, \( j_{A\mu} = \bar{u}(x)\gamma_\mu\gamma_5d(x) \); \( j_{\lambda}^{el}(x) = e_u\bar{u}(x)\gamma_\mu u(x) + e_d\bar{d}(x)\gamma_\mu d(x) \) is the electromagnetic current and \( q = p' - p \).

In the euclidean region \( p^2, p'^2, q^2 < 0 \), when the virtualities are large enough, the amplitude can be obtained in the framework of perturbative QCD, the corresponding triangle graphs are represented in Fig. 1. If virtualities are not so large then correction to the lowest order appears. There are two types of these corrections: pertubative corrections and non-pertubative ones arising from non-trivial interactions with vacuum fields. In the following the virtuality region \( |p^2| \sim |p'^2| \sim |q^2| \sim 1 \text{ GeV}^2 \) will be important, where \( \alpha_s/\pi \sim 0.1 \), pertubative corrections are small and can be disregarded.

On the contrary, non-pertubative corrections which decrease as a power law are very important here. The systematic treatment of these corrections can be carried out in the operator product expansion. It is clear from dimensional considerations that the corrections due to gluonic \( <0|G^{a}_{\mu\nu}G^{a}_{\mu\nu}|0> \) and quark condensates \( <0|\bar{\psi}\psi|0>^2 \) give main contributions.

The amplitude has a complicate tensor structure and it is convenient to represent it in the form: \( \Gamma_{\mu\nu;\lambda}(p, p'; q) = \sum f_i(p, p'; q)e^{i}_{\mu\nu;\lambda} \) where \( e^{i}_{\mu\nu;\lambda} \) are all structures which can be built from \( p_\alpha, p'_\alpha, g_{\alpha\beta} \) and \( f_i(p, p'; q) \) are the corresponding coefficients.
The following structures: $\rho'_{\mu}g_{\nu\lambda} + p_{\nu}g_{\mu\lambda}, p_{\mu}g_{\nu\lambda} + p'_{\nu}g_{\mu\lambda}$, $q_\lambda(p_{\mu}p_{\nu} - p'_{\mu}p'_{\nu})$ will be used to determine form factors.

Let us consider the coefficient of some structure and write down the dispersion relation over axial currents virtualities $p^2, p'^2$, holding $Q^2 = -q^2$ fixed:

$$f_i(p^2, p'^2; q^2) = \int_0^\infty ds \int_0^\infty ds' \frac{\rho_i(s, s', q^2)}{(s - p^2)(s' - p'^2)} + \text{subtr. terms} \tag{2}$$

Quantity $\rho_i(p^2, p'^2, q^2)$ equals the double discontinuity of the amplitude $f_i(p^2, p'^2, q^2)$ on the cuts $0 \leq p^2, p'^2 \leq \infty$ divided by $-4\pi^2$. To get rid of the unknown subtraction terms, which are polynomial in one of variables $p^2$ or $p'^2$ but arbitrary functions in the other two, it is convenient to use the Borel transformation in variables $p^2$ and $p'^2$ simultaneously:

$$B_M^2 B_{M'}^2 f_i(p^2, p'^2, Q^2) = \int_0^\infty ds \int_0^\infty ds' e^{-\frac{s+s'}{2M^2}} \rho_i(s, s', Q^2) \tag{3}$$

Parameters $M^2$ and $M'^2$ are chosen equal. The value of $M^2$ should be optimized so the power corrections and contribution of higher states are small (say less than 30%).

Spectral function $\rho_{\mu\nu\lambda}(p^2, p'^2, Q^2)$ can be represented as a sum over physical hadronic states. The value of $M^2$ will turn out to be $1 \text{ GeV}^2$ and it means that the main contributions to Borel-transformed amplitude will give $\pi$ and $A_1$ mesons while the contributions of higher states will be exponentially suppressed. The contributions of $\pi$ and $A_1$ mesons will be explicitly accounted for and for the contribution of continuum the following model of spectral function will be adopted [2]:

$$\rho^{\text{cont}}_{\mu\nu\lambda}(p^2, p'^2, Q^2) = [1 - \theta(s_0 - s - s')] \rho^{\text{quark}}_{\mu\nu\lambda}(p^2, p'^2, Q^2) \tag{4}$$

where $\rho^{\text{quark}}$ is the spectral function calculated in the perturbative QCD, $s = p^2$, $s' = p'^2$ and $s_0$ is the continuum threshold.

2

To obtain explicitly the sum rules for the electromagnetic form factors one should match Borel-transformed phenomenological part, consisting of the contributions of $\pi$ and $A_1$ mesons, and theoretical part, consisting of unity operator (i.e. pertubative part), gluon $<0 | G^a_{\mu\nu} G^a_{\mu\nu} | 0>$ and fermion $<0 | \bar{\psi} \psi | 0>^2$ condensates.

Using the following formulae:

$$< 0 | j_{A_\mu}(0) | \pi^+(p) > = i p^\mu f_\pi \tag{5}$$

$$< 0 | j_{A_\mu}(0) | A_1^+(p, \varepsilon) > = \sqrt{2} m_A^2 \varepsilon^\mu \tag{6}$$

$$< A_1^+(p', \varepsilon') | j_{A_\mu}^{(l)}(0) | A_1^+(p, \varepsilon) > = \varepsilon'_\alpha \varepsilon^\rho \left\{ \left[ \sigma_{\lambda\rho} - p'_\rho g_{\lambda\sigma} - p_{\sigma} g_{\lambda\rho} \right] F_1^A(Q^2) + \left[ \gamma_{\lambda\rho} q_\sigma - g_{\lambda\sigma} q_\rho \right] F_2^A(Q^2) + \frac{1}{m_A^2} \left[ p^p p^\rho \sigma P_\lambda F_3^A(Q^2) \right] \right\} \tag{7}$$

2
\[ < \pi^+(p') | j_{\pi}^\mu(0) | A_1^+(p, \varepsilon) > = -\frac{1}{m_A} \left\{ (\mathcal{P}q)q_{\lambda \sigma} - \mathcal{P}_\lambda q_{\sigma} \right\} G_1(Q^2) + \]
\[ + \frac{1}{m_A} \left\{ (\mathcal{P}q)q_{\lambda} - q^2\mathcal{P}_\lambda \right\} p'_\sigma G_2(Q^2) \varepsilon_{\sigma}; \quad \mathcal{P}_\lambda = p'_\lambda + p_\lambda; \quad q'_\lambda = p'_\lambda - p_\lambda \]

one can explicitly obtain the contributions of \( \pi \) and \( A_1 \) mesons in the structures under study:

\[ p'_\mu g_{\nu \lambda} + p_\nu g_{\mu \lambda}: \]
\[ B_{M^2} B_{M^2} f_1(p^2, p'^2; Q^2) | \pi \gamma \rightarrow \pi \rangle = 0 \]
\[ B_{M^2} B_{M^2} f_1(p^2, p'^2; Q^2) | \pi \gamma \rightarrow A_1 \rangle = 0 \]
\[ B_{M^2} B_{M^2} f_1(p^2, p'^2; Q^2) | A_1 \gamma \rightarrow A_1 \rangle = \frac{2m^4}{g_A^2} e^{-2m^2/\sqrt{2g}} \left( F_1^A(Q^2) + F_2^A(Q^2) \right) \]

\[ q_\lambda (p_\mu p_\nu - p'_\mu p'_\nu): \]
\[ B_{M^2} B_{M^2} f_3(p^2, p'^2; Q^2) | \pi \gamma \rightarrow \pi \rangle = 0 \]
\[ B_{M^2} B_{M^2} f_3(p^2, p'^2; Q^2) | \pi \gamma \rightarrow A_1 \rangle = f_\pi \sqrt{2/9} e^{-m^2/\sqrt{2g}} G_2(Q^2) \]
\[ B_{M^2} B_{M^2} f_3(p^2, p'^2; Q^2) | A_1 \gamma \rightarrow A_1 \rangle = m^2 e^{-2m^2/\sqrt{2g}} \left( F_1^A(Q^2) + F_2^A(Q^2) \right) \]

The contribution of unity operator and fermion condensate was calculated in [2]:

\[ p'_\mu g_{\nu \lambda} + p_\nu g_{\mu \lambda}: \]
\[ B_{M^2} B_{M^2} f_1(p^2, p'^2; Q^2) | I \rangle = \frac{M^4}{32\pi^2} \int_0^\infty dx e^{-x} \left\{ -3k \ln \left( 1 + \frac{2x}{k} \right) + \frac{x(16x^2 + 18xk + 6k^2)}{(2x + k)^2} \right\} \]
\[ B_{M^2} B_{M^2} f_1(p^2, p'^2; Q^2) | < \bar{\psi} \psi > \rangle = \frac{4\pi \alpha_s}{81} < \bar{\psi} \psi >^2 \frac{1}{M^2} [-20 - 8k] \]

\[ p_\mu g_{\nu \lambda} + p'_\mu g_{\nu \lambda}: \]
\[ B_{M^2} B_{M^2} f_2(p^2, p'^2; Q^2) | I \rangle = -\frac{M^4}{8\pi^2} \int_0^\infty dx e^{-x} \frac{5x + 3k}{(2x + k)^2} x^2 \]
\[ B_{M^2} B_{M^2} f_2(p^2, p'^2; Q^2) | < \bar{\psi} \psi > \rangle = \frac{4\pi \alpha_s}{81} < \bar{\psi} \psi >^2 \frac{1}{M^2} [64 + 5k - 2k^2] \]
The contribution proportional $<G^2>$ into amplitude $\Gamma_{\mu
u;\lambda}(p^2, p'^2; Q^2)$ could be obtained after taking into account the interaction of quarks, running through triangle loop in Fig. 1, with external vacuum gluon field. In the first order of $\alpha_s$ there are 12 diagrams and 6 of them are represented in Fig. 2. The other 6 diagrams correspond to the case when electromagnetic current enters the d-quark line.

In fixed-point gauge $x_\mu A^a_\mu(x) = 0$ the potential $A^a_\mu(x)$ is expressed through the field strength and its covariant derivatives at the origin (black circle in the left lower angle):

$$A^a_\mu(x) = -\frac{i}{2} x_\mu G^a_{\mu\nu}(0) - \frac{i}{2} x_\nu x_\alpha (D_\alpha G^a_{\mu\nu}) \delta(0) + \ldots$$

Graphs in which there is only one external gluon line disappear after averaging over vacuum gluonic fields, because v.e.v of $G^a_{\mu\nu}$, $D_\lambda G^a_{\mu\nu}$ equals zero. Non-zero contributions appear from the graphs with two external gluon lines and vacuum averaging is performed according to the rule:

$$<0|G^a_{\mu\nu}G^b_{\lambda\rho}|0> = \frac{1}{96}\delta^{ab} <0|G^a_{\mu\nu}G^a_{\mu\nu}|0> (g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda})$$

In fixed-point gauge the diagrams in Figs. 2 e,f are equal to zero[4]. Expressions for contributions of the other diagrams in $\Gamma_{\mu\nu;\lambda}(p^2, p'^2; Q^2)$ can be obtained if one consider the equation for the quark propagator in the external gluonic field:

$$i\gamma_\mu \left( \partial_\mu + ig\frac{\lambda^a}{2} A^a_\mu(x) \right) \cdot S(x, z) = i\delta^4(x - z)$$

Solving this equation up to the second order of $A$ we get:

$$S(x, z) = i \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-z)} \left\{ \frac{k}{k^2} - \frac{1}{4} g \lambda^a G^a_{\alpha\beta} \cdot \varepsilon_{\alpha\beta\gamma\rho} \gamma_\rho \frac{k^4}{k^2} + \right.$$  

$$\left. + \frac{i}{4} g \lambda^a G^a_{\alpha\beta} z_\beta (\gamma_\alpha k^2 - 2k_\alpha k_\beta) \cdot \frac{1}{k^2} + \frac{1}{96} g^2 <0|G^2_{\mu\nu}|0> \cdot \left[ -\frac{2k^2}{k^2} - 4 \frac{k_z}{k} (kz)^2 + 2z^2 \frac{k_z}{k} \right] \right\}$$

The power corrections proportional to $<0|G^a_{\mu\nu}G^a_{\mu\nu}|0>$ can be calculated if one substitutes propagator $S(x, z)$ (17) instead of each fermionic line in Fig. 1, selects the terms proportional to $<G^2>$ averages over vacuum fluctuations of the gluonic field according
The final sum rules for electromagnetic form factors of $\pi + \gamma^* \rightarrow A_1$ transition are the following:

$$\Gamma^{\text{Fig.2}(a+b+c)}_{\mu \nu ; \lambda}(p^2, p'^2; Q^2) = \frac{ig^2}{24} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2(p-k)^2(p'-k)^2} \times$$

$$\times \left( S p \left\{ k \gamma_\nu [2 \gamma_\lambda(p - k, p' - k) + (\hat{p} - \hat{k}) \gamma_\lambda(\hat{p}' - \hat{k})] \gamma_\mu \right\} \frac{1}{k^2(p-k)^2(p'-k)^2} + 
\right.$$

$$+ S p \left\{ [2 \gamma_\nu(\hat{p}' - \hat{k}) \gamma_\lambda(pk - k^2) + (\hat{p} - \hat{k}) \gamma_\nu(\hat{p}' - \hat{k}) \gamma_\lambda \hat{k}] \gamma_\mu \right\} \frac{1}{k^2(p-k)^2(p'-k)^2} + 
$$

$$+ S p \left\{ [2 \gamma_\nu(p'k - k^2) + (\hat{p}' - \hat{k}) \gamma_\nu \hat{k}] \gamma_\lambda(\hat{p} - \hat{k}) \gamma_\mu \right\} \frac{1}{k^2(p-k)^2(p'-k)^2} \right) \times$$

(18)

The contribution of these diagrams structures of interest are the following:

$$p'_\mu g_\nu \lambda + p_\nu g_\mu \lambda:$$

$$B_{M^2}B_{M'2}^f(p^2, p'^2; Q^2)[< G^2 >] = \frac{g^2 < 0 | G_{\mu \nu}^2 | 0 >}{96\pi^2} \int_0^\infty dx e^{-x} \frac{3x^2 - xk - 2k^2}{(x + k)^2(2x + k)^2x}$$

(19)

$$p_\mu g_\nu \lambda + p'_\nu g_\mu \lambda:$$

$$B_{M^2}B_{M'2}^f(p^2, p'^2; Q^2)[< G^2 >] = -\frac{g^2 < 0 | G_{\mu \nu}^2 | 0 >}{48\pi^2} \int_0^\infty dx e^{-x} \frac{2k^3 + 4k^2x + 2x^2k + x^3}{(x + k)^2(2x + k)^2} \cdot \frac{x}{k}$$

(20)

$$q_\lambda(p_\mu p_\nu - p'_\nu p'_\mu):$$

$$B_{M^2}B_{M'2}^f(p^2, p'^2; Q^2)[< G^2 >] = \frac{g^2 < 0 | G_{\mu \nu}^2 | 0 >}{96\pi^2} \int_0^\infty dx e^{-x} \frac{4k^3 - 5k^2x + 4x^2k + x^3}{(x + k)^4} \cdot \frac{x}{k^2}$$

(21)

The analytical expression for Fig.2d turns out to be

$$\Gamma^{(d)}_{\mu \nu ; \lambda}(p^2, p'^2; Q^2) = \frac{ig^2 < 0 | G_{\mu \nu}^2 | 0 >}{192} \left[ \frac{\partial^4}{\partial p_\alpha \partial p'_\alpha \partial p_\beta \partial p'_\beta} - \frac{\partial^2}{\partial p_\alpha \partial p_\beta} \cdot \frac{\partial^2}{\partial p'_\alpha \partial p'_\beta} \right] \times$$

$$\times \int \frac{d^4 k}{(2\pi)^4} Sp \left\{ \gamma_\mu(\lambda - \lambda)(k - p) \gamma_\nu (k - p') \hat{\gamma} \right\}$$

(22)

The explicit calculations of the contribution of this diagram to the interested structures were performed using program ”MAPLE” and result is rather surprising one: this diagram exactly cancels the others, i.e. gluon condensate gives no contribution to the above mentioned structures.

4

The final sum rules for electromagnetic form factors of $\pi + \gamma^* \rightarrow A_1$ transition are the following:
\[ F_1^A + F_2^A + \sqrt{2} L e g_{e q} \frac{m^2}{m_A} G_2 = \frac{g_A^2}{m_A^2} e^{2m^2} \left[ 3M^2 \frac{\chi_0}{\chi_0} \int_0^\infty e^{-x} dx \left\{ \ln \left( 1 + \frac{2x}{k} \right) - \frac{2x}{2x+k} \right\} + \right. \]
\[ \left. + \frac{4\pi \alpha}{81} < \bar{\psi} \psi > ^2 \cdot \frac{1}{M^2} \left[ 2k + 5 - \frac{18}{k} + \frac{18}{k^2} \right] \right] \]
\[ F_1^A + F_2^A + \sqrt{2} L e g_{e q} \frac{m^2}{m_A} e^{2m^2} G_1 = \frac{g_A^2}{m_A^2} \cdot e^{2m^2} \left[ \frac{M^4}{8\pi^2} \frac{\chi_0}{0} \int_0^\infty dx e^{-x} x \frac{2(5x+3k)}{(2x+k)^2} \right] \]
\[ - \frac{4\pi \alpha}{81} < \bar{\psi} \psi > ^2 \frac{1}{M^2} \left\{ 64 + 5k - 2k^2 \right\} \]
\[ F_1^A + F_2^A = \frac{g_A^2}{2m_A^2} e^{2m^2} \left[ \frac{M^4}{32\pi^2} \int_0^\infty dx e^{-x} \left\{ -3k \ln \left( 1 + \frac{2x}{k} \right) + \frac{x(16x^2+18xk+6k^2)}{(2x+k)^2} \right\} \right] + \]
\[ \left. + \frac{4\pi \alpha}{81} < \bar{\psi} \psi > ^2 \frac{1}{M^2} \left\{ -20 - 8k \right\} \right) , \]

where \( \chi_0 = s_0/M^2 \).

These first and second sum rules coincide with the ones obtained in [2] but the third one is different.

The sum rules for \( G_1(Q^2) \) and \( G_2(Q^2) \) are stable for 0.9 GeV\(^2 < M^2 < 1.2 \) GeV\(^2 \). Figs. 3,4 show dependence of \( G_1(Q^2) \) and \( G_2(Q^2) \) upon \( M^2 \) at the following values of parameters: \( \alpha_s < 0 | \bar{\psi} \psi | 0 >^2 = 0.8 \times 10^{-4} \) GeV\(^6 \), \( g_A = 8.9 \), \( m_A = 1.26 \) GeV, \( s_0 = 3 \) GeV. Parameter \( g_A \) was determined in [5] using QCD sum rule. The left-hand vertical bars mark the \( M^2 \) value above which the ratio of power corrections to the sum of terms does not exceed 30%. To the left of right-handed vertical bars the contribution of the continuum does not exceed 30%. Choosing \( M^2 = 1.05 \) GeV\(^2 \) we get acceptable region of \( Q^2 \) for \( G_1(Q^2) \) and \( G_2(Q^2) \): 0.4 GeV\(^2 < Q^2 < 2.2 \) GeV\(^2 \) and 0.7 GeV\(^2 < Q^2 < 3.2 \) GeV\(^2 \), respectively.

There is another calculation method of form factors based on light-cone expansion [6,7,8]. Obtained results can be compared to the ones derived from the sum rules on light-cone[3] (Figs. 5,6). These sum rules are model-dependent and different lines correspond to the different wave functions of pion. Three types of wave function are used: asymptotical [9], Chernyak-Zhitnitsky [10] and Braun-Filyanov [11] ones. For \( G_1(Q^2) \) there is a discrepancy between the two methods but, probably, situation can be improved after accounting for the operator of a higher dimension. For \( G_2(Q^2) \) the agreement is sensible and the best choice of the wave function is Braun-Filyanov function.

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Figure Captions

Fig.1 - The diagrams corresponding to pertubative QCD in the zero order of $\alpha_s$.

Fig.2 - The diagrams correspondig to gluon contribution in the first order of $\alpha_s$.

Fig.3 - The $M^2$-dependence of $G_1$ at fixed $Q^2 = 0.5\text{GeV}^2$ (solid line) and $Q^2 = 1.\text{GeV}^2$ (dashed line).

Fig.4 - The $M^2$-dependence of $G_2$ at fixed $Q^2 = 2.\text{GeV}^2$ (solid line) and $Q^2 = 2.5\text{GeV}^2$ (dashed line).

Fig.5 - The $Q^2$-dependence of obtained $G_1$ in comparision to $G_1$ corresponding to light-cone sume rules.

Fig.6 - The $Q^2$-dependence of obtained $G_2$ in comparision to $G_1$ corresponding to light-cone sume rules.
3-point sum rules

CZ
BF
asympt.
3-point sum rules
CZ
BF
asympt.

$C_{2\ell}(Q^2)$

$Q^2 (GeV^2)$