von Neumann-Morgenstern and Savage Theorems for Causal Decision Making

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Abstract

Decision making under uncertain conditions has been well studied when uncertainty can only be considered at the associative level of information. The classical Theorems of von Neumann-Morgenstern and Savage provide a formal criterion for rationally making choices using associative information. We provide here a previous result from Pearl and show that it can be considered as a causal version of the von Neumann-Morgenstern Theorem; furthermore, we consider the case when the true causal mechanism that controls the environment is unknown to the decision maker and propose a causal version of the Savage Theorem. As applications, we argue how previous optimal action learning methods for causal environments fit within the Causal Savage Theorem we present thus showing the utility of our result in the justification and design of learning algorithms; furthermore, we define a Causal Nash Equilibria for a strategic game in a causal environment in terms of the preferences induced by our Causal Decision Making Theorem.

1 Introduction

Causal reasoning is a constant element in our lives as it is human nature to constantly ask why. Looking for causes is an everyday task and, in fact, causal reasoning is to be found at the very core of our minds (Waldmann and Hagmayer (2013), Danks (2014)). It has been argued that the brain is a causal inference machine which uses effects to figure out causes in order to engage with the world (Friston (2010), Clark (2013)). Acting in the world is conceived by human beings as intervening the world and in fact humans are able to learn and use causal relations while making choices (Tversky and Kahneman (1980), Garcia-Retamero and Hoffrage (2006), Lagnado et al. (2007), Hagmayer and Medel (2008), Hagmayer and Sloman (2009), Hagmayer and Fernbach (2017)).
An important aspect of acting in the world is being able to make decisions under uncertain conditions (Danks (2014), Lake et al. (2017)). Von Neumann and Morgenstern (1944) gave an answer for how to make choices if rational preferences are assumed and the decision maker knows the stochastic relation between actions and outcomes: maximize expected utility. If no such relation is known, then Savage (1954) showed that a rational decision maker must choose as if she is maximizing expected utility using a subjective probability distribution. Such theorems provide formal criteria for decision making if rationality is assumed. This criterion is the basis for many of the techniques used in Artificial Intelligence; for example, Reinforcement Learning algorithms learn optimal policies in such a way that any action prescribed by the optimal policy achieves the maximum expected utility (Sutton and Barto (1998), Puterman (1994)). Algorithms which rely on the von Neumann-Morgenstern or Savage Theorems are based on associative relations which are expressed using correlations or probability distributions. It is a natural question how to formalize rational decision making when causal information is present. Such question has been previously considered by Nozick (1969), Lewis (1981), Joyce (1999) with out an explicit optimality criterion for decision making and by Pearl (2009) who provides an optimality criterion for decision making under causal-controlled uncertainty when the causal mechanism which controls the environment is known.

In this work we extend such criterion to the case where a decision maker does not know the causal mechanism so she holds beliefs about possible causal model and uses such beliefs as if such beliefs were true, as prescribed by Joyce (1999) in order to attempt to make a good choice given her beliefs. We provide an explicit way of making a good choice given what the decision maker believes about the causal structure of her environment, thus providing a causal version of Savage’s Theorem. We take a normative point of view to study how causal relations should be used by a rational agent when making decisions with uncertain consequences.

Our goal is to establish the foundations for decision making algorithms which rely on the existence of causal relations even though such relations are not previously known or fully observable to a decision maker such as the algorithms found in [Bareinboim et al. (2015), Lattimore et al. (2016), Sen et al. (2017), Gonzalez-Soto et al. (2018)].

2 The Classical Decision Making Theorems

A Decision problem under uncertainty is a situation in which an agent must choose one out of many available actions with uncertain consequences which depend on different, possibly unknown, factors. Such consequences are ordered in terms of the satisfaction that they produce in the decision maker, and such ordering is represented by a preference relation denoted by $\succeq$, where $a \succeq b$ is read as $a$ being preferred to $b$.

The most well-known theories for Decision Making are those from Von Neumann and Morgenstern (1944) and Savage (1954). In the former theory it is assumed that the decision maker knows the stochastic relation between actions and consequences, which is also known as decision under risk (Binmore (2008), Peterson (2017)), and in that case the theory guarantees that the decision maker behaves as if she maximizes the expected value of a utility function. If the decision maker doesn’t know the probabilities of observing an outcome given a chosen action, then Savage’s theory guarantees that the decision maker behaves as if she has in mind a subjective probability distribution and a utility function and chooses the action which maximizes the expected utility with respect to that subjective probability distribution and the utility function.

Other Decision Making theories exist, such as Prospect Theory (Kahneman and Tversky (1979)), Case-Based Decision Theory (Gilboa and Schmeidler (1995)), among others that are out of the scope of this work. We will later talk about another theory: Causal Decision Making (Joyce (1999)). For further details on classical decision making, see [Bernardo and Smith (2000), Gilboa (2009), Wakker (2010), Speekenbrink and Shanks (2013)].

2.1 von Neumann-Morgenstern

Von Neumann and Morgenstern (1944) had the objective of justifying strategies in which players in a game maximized expected utility. This theorem considers a scenario of decision under risk and rational preferences; this is, choosing between uncertain outcomes with known probabilities.
Formally, we consider a set $X$ of alternatives. Let $L$ be the set of lotteries (with finite support) over $X$. The actual object of choice are the elements $l \in L$, which are known to the decision maker; we represent the decision maker's preferences by a preference relation $\succeq \subseteq L \times L$ which satisfies being complete, transitive, continuous and a technical condition called independence. This family of conditions is called von Neumann-Morgenstern rationality axioms (Gilboa (2009)).

**Theorem 1.** A preference relation $\succeq \subseteq L \times L$ where $L$ is a set of lotteries with finite support over a set $X$ satisfies the von Neumann-Morgenstern rationality axioms if and only if there exists a function $u : X \to \mathbb{R}$ such that for every $P, Q \in L$ we have that

$$P \succeq Q \text{ if and only if } \sum_{x \in X} P(x)u(x) \geq \sum_{x \in X} Q(x)u(x). \quad (1)$$

The theorem states that if a rational decision maker knows the probabilities of obtaining a certain outcome, then she must choose as if maximizing the expected value of some function $u$ whose existence is guaranteed by Theorem 1. See Gilboa (2009) for details on the proof.

### 2.2 Savage’s Theorem

If a rational decision maker does not know the probabilities of obtaining certain outcomes and does not have a precise quantification of her preferences (utility function), then it is Savage (1954) who gives a formal choosing criterion. Savage’s Theorem extends von Neumann-Morgenstern Theorem since it considers the case in which a rational decision maker does not know neither her utility function nor the probabilities to be used in order to obtain the expected values required for making choices according to the von Neumann-Morgenstern Theorem.

The notation used here is the one used by Bernardo and Smith (2000). Consider a set $\Omega$. Consider a countable set $A$ of available actions; for each action $a_i \in A$, a partition $E_i$ of $\Omega$ and a set of consequences $C_i$. Let $\mathcal{E}$ the union of every $E_i$, we assume $\mathcal{E}$ to be an algebra of events, and $C$ the union of all $C_i$.

**Definition 2.** An uncertain environment is the tuple $(\Omega, A, C, \mathcal{E})$. Where $A$ is a non-empty set of available actions, $C$ a set of consequences and $\mathcal{E}$ an algebra of events over $\Omega$.

When we consider the preferences of some decision maker over the set of consequences of some uncertain environment we have a Decision Problem under Uncertainty:

**Definition 3.** A Decision Problem under Uncertainty is an uncertain environment $(\Omega, A, C, \mathcal{E})$ plus a preference relation $\succeq$ defined over $C$

In Bernardo and Smith (2000) the existence of a subjective probability measure is derived from a set of rationality axioms, where a decision maker has some mechanism of quantifying uncertainty in terms of real numbers within the $[0, 1]$ interval, and then classical Kolmogorov axioms are derived from such construction, and therefore all of the known machinery of Probability Theory. We now state Savage’s Theorem. Details in Kreps (1988) and Bernardo and Smith (2000).

**Theorem 4.** In a Decision Problem under Uncertainty $(\Omega, C, \mathcal{E}, \succeq)$, the preference relation $\succeq$ satisfies the Savage rationality axioms if and only if there exists:

- A probability measure $P$, called a subjective probability, that associates with each uncertain event $E \in \mathcal{E}$ a real number $P(E)$.
- A utility function $u : C \to \mathbb{R}$ such that it associates each outcome with a real number $u(c)$.

Such that for $c_1$ and $c_2$ consequences in $C$,

$$c_1 \succeq c_2 \text{ if and only if } \mathbb{E}_P[u(c_1)] \geq \mathbb{E}_P[u(c_2)]. \quad (2)$$

This theorem states that if a rational decision maker does not know the precise probabilities of outcomes given that an action is taken, then she chooses as if having in mind a probability assignment to the uncertainties in her environment and using such probabilities to calculate the expected utility with respect to a subjective utility function that represents her preferences. This result also gives a precise definition of subjective probability as a quantification of uncertainty which is used to make good decisions. See Hens (1992), Gilboa (2009) for further details. See Ellsberg (1961) and Binmore (2008) for critiques of the Savage axioms.
3 Causal Decision Problems

3.1 On causality

The concept of Causality deals with regularities found in a given environment which are stronger than probabilistic (or associative) relations in the sense that a causal relation allows for evaluating a change in the consequence given that a change in the cause is performed. The manipulationist interpretation of Causality (Woodward (2003)) is adopted here. The main paradigm is clearly expressed by Campbell and Cook (1979) as manipulation of a cause will result in a manipulation of the effect. Consider the following example from Woodward (2003): manually forcing a barometer to go down won’t cause a storm, whereas the occurrence of a storm will cause the barometer to go down.

We adopt the formal definition of Causality given by Spirtes et al. (2000) as a stochastic relation between events which is irreflexive, antisymmetric and transitive. Similar descriptions of the manipulationist approach can be found in Holland (1986) and Freedman (1997). Causal inference tools, such as Pearl’s do-calculus (Pearl (2009)) allows to find the effect of an intervention in terms of probabilistic information when certain conditions are met. For what remains, we assume the causal axioms found in Spirtes et al. (2000) with the condition known as causal sufficiency.

3.2 Causal Environments and Causal Decision Problems

We define a Causal Environment to be an uncertain environment for which there exists a Causal Graphical Model (CGM) $G$ (Koller and Friedman (2009)) which controls the environment.

Definition 5. A Causal Environment is a tuple $(\Omega, A, G, C, E)$ where $(\Omega, A, C, E)$ is an uncertain environment and $G$ is a CGM such that the set of variables of $G$ correspond to the uncertain events in $E$.

Definition 6. We define a Causal Decision Problem (CDP) as a tuple $(A, G, E, C, \succeq)$ where $(A, G, E, C)$ is a Causal Environment and $\succeq$ is a preference relation.

For the CGM in a CDP we will distinguish two particular variables: one corresponding to the available actions, and one corresponding to the produced (caused) outcome. We are considering that only one variable can be intervened upon and that the values of such variable represent the actions available to the decision maker; i.e., the value forced upon such variable under an intervention represents the action taken by the decision maker. The intuition behind the definition of a Causal Decision Problem is this: a decision maker chooses an action $a \in A$, which is automatically inputed into the model $G$, which outputs the causal outcome $c \in C$. We say a CDP is finite if the set $A$ is finite.

3.3 Related work: Causal Decision Making and Decision-Theoretic foundations of Causal Inference

A previous attempt to formalize Decision Theory in the presence of Causal Information is given in Lewis (1981), Joyce (1999). According to such formulation, a decision maker must choose whatever action is more likely to (causally) produce desired outcomes while keeping any beliefs about causal relations fixed (Peterson (2017)). This is stated by the Stalnaker (Stalnaker (1968)) equation

$$u(a) = \sum_x P(a \square \rightarrow x)u(x),$$

where $a \square \rightarrow x$ is to be read as if the decision maker does $a$ then $x$ would be the case (Gibbard and Harper (1978), Kleinberg (2013)). Lewis’ and Joyce’s work captured the intuition that causal relations may be used to control the environment and to predict what is caused by the actions of a decision maker. In Section 3.4 we refine the $\square \rightarrow$ operator by an explicit way of calculating the probability of causing an outcome by doing a certain action in terms of Pearl’s do-calculus.

Heckerman and Shachter (1995) provides a framework for defining the notions of cause and effect in terms of decision theoretical concepts, such as states and outcomes and gives a theoretical basis for graphical description of causes and effects, such as causal influence diagrams (Dawid (2002)). Heckerman gave an elegant definition of causality, but did not addressed how to actually make choices using causal information.

In Dawid (2012) we are presented with a decision-theoretic approach to causal inference in which a decision maker must take into account how alternatives compare against the other in terms of
the average causal effect, such approach uses the well-known influence diagrams (Dawid (2002), Dawid (2003)) in order to derive formulas that allow an explicit calculation of the average causal effect. Influence diagrams have the ability of expressing both intervention variables and chance variables into a single graphical structure in such a way that the standard techniques for probabilistic D AG’s still apply. In Dawid and Didelez (2008) an optimality criterion for sequential interventions is obtained by maximizing the expectation of outcomes.

3.4 A von Neumann-Morgenstern type theorem for causal environments

Consider a rational decision maker who faces a causal environment in which she knows the causal model controlling the relation between actions and outcomes. She can use the known causal model in order to find the probabilities of causing a desired outcome given she takes a certain action. The following theorem is found in Pearl (2009), but the intuitions that lie behind can be traced back to Lewis (1981) and Joyce (1999).

Consider a Causal Model \( G \) and its associated distribution \( P_G \) and let \( C \) be the set of consequences of interest for a decision maker. Then,

**Theorem 7.** If a rational decision maker faces a Causal Environment and if the causal model is known, then the preference relation \( \succeq \) satisfies the von Neumann-Morgenstern rationality axioms if and only if:

\[
a \succeq b \text{ if and only if } \sum_{c \in C} P(c|\text{do}(a))u(c) \geq \sum_{c \in C} P(c|\text{do}(b))u(c). \tag{4}
\]

Equivalently, the action that must be chosen is

\[
a^* = \arg\max_{a \in A} \sum_{c \in C} P(c|\text{do}(a))u(c).
\]

We argue that Pearl’s result can be considered as a causal version of the von Neumann-Morgenstern result since it assumes that a causal model is known. If the causal model that controls an environment is known to a decision maker, then this is equivalent of being able to know the probabilities of outcomes given actions. As stated in Section 4.1 of Pearl (2009), the utility function \( u \) is considered as given, even though the Theorem guarantees its existence. Pearl argues that decision making by maximizing such function is commonsensical without appealing to the original results in rational decision making. We avoid entering in the long-standing debate between causal and evidential decision making. We only note that both von Neumann-Morgenstern’s and Savage’s Theorems have no a priori causal interpretation and therefore rely on associative information.

3.5 A Savage type theorem for causal environments

When the decision maker does not know the causal model which controls her environment we argue that the decision maker is facing a particular case of Savage’s Theorem. The difference here being that the subjective probability must make use of the causal nature of the environment. The idea we will follow is that Savage’s Theorem gives the decision maker a subjective quantification of her uncertainty about the environment and the associated probabilities which must be used as if they were true.

In this case, where a Causal Graphical Model controls the relation between actions and outcomes, such subjective information about the environment must consider such causal structures. For this reason, we assert that the probability distribution that the decision maker has in mind is in fact a distribution over causal structures where the decision maker uses each structure as if it were the true one in order to choose the best action within each structure by using Pearl’s result. We assume a finite set of actions and outcomes. Formally:

**Theorem 8.** In a finite Causal Decision Problem \( (A, G, \mathcal{E}, C, \succeq) \), where \( G \) is a Causal Graphical Model, we have that the preferences \( \succeq \) of a decision maker are Savage-rational if and only if there exists a probability distribution \( P_C \) over a family \( \mathcal{F} \) of causal structures such that for \( a, b \in A \):

\[
a \succeq b \text{ if and only if } \sum_{c \in C} u(c) \left( \sum_{g \in \mathcal{F}} P_g(c|\text{do}(a))P_C(g) \right) \geq \sum_{c \in C} u(c) \left( \sum_{g \in \mathcal{F}} P_g(c|\text{do}(b))P_C(g) \right), \tag{5}
\]

where \( P_g \) a the probability distribution associated with the causal structure \( g \).
We have shown what is the expected utility for some action $a$. In order for the decision maker to find the probability of causing some consequence $c_j$ given that the intervention $do(a)$ is performed; in this way, the optimal action $a^*$ is given by:

$$a^* = \arg\max_{a \in A} \sum_{c \in C} u(c) \left( \sum_{g \in G} P_g(c|do(a))P_C(g) \right).$$

We note that $a^*$ is obtained by taking into account the utility obtained by every possible consequences weighted using both the probability of causing such action within a specific causal model $g$ and the probability that the decision maker assign to such $g \in G$.

We are considering a normative interpretation for Theorem 8 according to which a decision maker must use any causal information in order to obtain the best possible action. Such action must be obtained by considering the beliefs of the decision maker about the causal relations that hold in her environment (the distribution $P_C$), how such relations could produce the best action when considered as if they were true (distribution $P_g$), and the satisfaction (utility $u$) produced by the consequences of actions.

Several studies have considered how human beings use causal information when making choices. Humans tend to ignore pure probabilistic information over causal information (Tversky and Kahneman 1980). Humans are able to learn, and use, causal models in sequential decision making processes, even though such learning is not perfect (Sloman and Hagmayer 2006, Nichols and Danks 2007, Lagnado et al. 2007, Meder et al. 2010, Hagmayer and Meder 2013, Wellen and Danks 2012, Danks 2014, Rottman and Hastie 2014).
4 Application: Causal Games and Nash Equilibrium

In this section, we consider a strategic game between $N$ rational players who are situated in a causal environment. A game is a model of a situation in which several players must take an action and afterwards they will be affected both by the outcome of their own action as well as the actions of the other players. In a strategic game it is assumed that no player knows the action taken by any other players; we also assume that the causal mechanism, which represented by a Causal Graphical Model $G$, remains fixed and it is unknown for each player.

In this game, players ignore the actions taken by any other player, and since the causal model which controls the environment is unknown for every player, then players also ignore the information that players will use in order to take their respective actions: strategic games of this type are called Bayesian Game, introduced in [Harsanyi (1967), Harsanyi (1968b), Harsanyi (1968a)]. In the games we will consider, the uncertainty of every player consists of two levels: on a first level, the true causal model $G$; on a second level, what an action $do(a)$ causes if a certain CGM $ω$ is considered to be the causal model.

**Definition 9.** A Bayesian strategic game ([Osborne and Rubinstein (1994)]), consists of:

- A finite set $N$ of players.
- A finite set $Ω$ of states of nature.
- For each player, a nonempty set $A_i$ of actions.
- For each player, a finite set $T_i$ and a function $τ_i : ω \mapsto T_i$ the signal function of the player
- For each player, a probability measure $p_i$ over $Ω$ such that $p_i(τ_i^{-1}(t_i)) > 0$ for all $t_i \in T_i$.
- A preference relation $\succeq_i$ defined over the set of probability measures over $A_i \times Ω$ where $A_i = A_1 \times \cdots \times A_n$.

We will consider $Ω$ to be a family of possible causal models; in this way, $ω \in Ω$ being the true state of nature fixes a causal model which controls the environment in which the players make their choices. In classical Bayesian games, once $ω \in Ω$ is realized as the true state, then each player receives a signal $t_i = τ_i(ω)$ and the posterior belief $p_i(ω|τ_i^{-1}(t_i))$ given by $p_i(ω)/p_i(τ_i^{-1}(t_i))$ if $ω \in τ_i^{-1}(t_i)$. In the case for causal bayesian games, we must consider both the probability $p_i$ of $ω$ being the true state as well as the probability $p_i^c$ of observing a certain consequence when doing some action $a_i$ if $ω$ is the true model.

Following [Osborne and Rubinstein (1994)], we define a new game $G^*$ in which its players are all of the possible combinations $(i, t_i) \in N \times T_i$, where the possible actions for $(i, T_i)$ is $A_i$. We see that fixing a player $i \in N$, the posterior probability $p(ω|τ_i^{-1}(t_i))$ induces a lottery over the pairs $(a^*(j, τ_i(ω)))_j$, $ω$ for some other $j \in N$. This lottery assigns to $(a^*(j, τ_i(ω)))_j$, $ω$ the probability $p_i(ω)/p_i(τ_i^{-1}(t_i))$ if $ω \in τ_i^{-1}(t_i)$. The classical Bayesian game will simply call a Nash equilibrium for the game $G^*$ a Nash equilibrium of the original game; but we have the second level of uncertainty: the consequences caused by some action $a$ through a causal model $ω$. We notice that the posterior probability itself induces a probability distribution defined over actions for each player once a desired consequence is fixed, this distribution, according to Theorem 8 is given by $p_i^c(c|do(a_i^*), a_{-i}^*)p_i(ω|τ_i^{-1}(t_i))$. This motivates the following definition of a Causal Nash equilibrium.

4.1 Causal Nash Equilibrium

For each player $i \in N$ in the strategic game, we define the following probability distribution over consequences:

$$p_i^a(c) = p_i^c(c|do(a_i), a_{-i})p_i(ω) \quad \text{for } a \in A = A_1 \times \cdots \times A_N.$$  \hspace{1cm} (9)

where $p_i^c$ is the probability of causing a certain consequence within a causal structure $ω$ and $p_i$ are the player’s posterior beliefs about the causal structure that controls the environment, and $do()$ is the well known intervention operator from [Pearl (2009)]. We now define:

$$u_i^c(a) = \sum_{c \in C} u_i(c)p_i^a(c) \quad \text{for } a \in A = A_1 \times \cdots \times A_N.$$  \hspace{1cm} (10)
Notice that $u_C^i$ evaluates an action profile $a \in A$ in terms of: The knowledge about the causal structure of each player represented by $p_i$, which allows each player to evaluate the probability of causing outcomes in terms of actions by using the $do$ operator as well as the other actions taken by the other players, given by $a_{-i}$ and the preferences of each player $u_i$. Using this new function, we define the equilibria for a strategic game with causal information and Bayesian players as:

**Definition 10.** A Nash equilibrium for this causal strategic game is an action profile $a^* \in A$ if and only if

$$u_C^i(a^*) \geq u_C^i(a_i, a_{-i}^*)$$

for any other $a_i \in A_i$.

This is, an action profile is a Nash equilibrium if and only if each player uses her current knowledge about the causal structure of the environment in order to (causally) produce the best possible outcome given the actions taken by the other players. The existence of the Causal Nash Equilibrium is guaranteed if every $A_i$ is a nonempty compact convex set in some $\mathbb{R}^n$ and if the preference relation induced by $u_C^i$ is continuous and quasi-concave.

5 Limitations

We are working within the classical rationality assumption. Rationality can be ultimately though of as a consistent or coherent way of making choices, but the precise definition has been a subject of debate. See Ellsberg (1961), Binmore (2008), Gilboa (2009) and Machina and Siniscalchi (2014) for critiques of the Savage Rationality Axioms. Another line of decision making, from a descriptive point of view, has been developed by psychologists and economists; see Tversky and Kahneman (1974), Kahneman and Tversky (1979), Kahneman et al. (1982), Kahneman and Tversky (2000). It is an interesting line of research to find an adequate definition of rationality in a way that best allows for causal interpretations and from there develop the formal machinery of causal decision making.

We are considering Causal Graphical Models as the representation of causal relations in the environment, and we are considering stochastic causal relations according to the manipulationist interpretation, which is one of many. We have favoured the Causal Graphical Models over other alternatives since it has been argued that several cognitive processes, such as causal reasoning, can be best represented as graphical models (Chater and Oaksford (2013), Danks (2014), Sloman and Lagnado (2015), Hagmayer (2016)). As alternative frameworks for causality we have: Topological Causality for Dynamical Systems (Harnack et al. (2017)), Lamport’s Causality (Lamport (1978)), Granger’s Causality for Time Series (Granger (1969)), Suppes’ Causality (Suppes (1970)). A review of several theories of Causality can be found in Holland et al. (1985) and Frosini (2006).

6 Conclusion

We have defined a Causal Decision Problem in terms of a classical Decision Problem under Uncertainty provided of a Causal Mechanism which controls the relation between actions and outcomes. In the case in which a rational decision maker knows such relations, Pearl (2009) provides a causal version of Theorem 1 for rational decision making. On the other hand, when a decision maker does not know the causal mechanism, in Theorem 8 we have provided a causal version of Savage’s Theorem; our result explicitly states how a decision maker uses any subjective beliefs, encoded as a probability distribution over causal models, as well as the causal inference machinery within any causal structure considered in order to find an optimal action. Such subjective probabilities over causal models can be updated using any causal evidence provided after a decision has been made. Learning algorithms for causal environments, such as those stated in Lattimore et al. (2016), Sen et al. (2017), Gonzalez-Soto et al. (2018) fit within the machinery stated in Theorem 8 so we are confident that further implementations of such result will show that Causality is a fundamental concept for Machine Learning and Artificial Intelligence as proposed by Lake et al. (2017) and Pearl and Mackenzie (2018).
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