Tensor perturbations of Palatini $f(\mathcal{R})$-branes

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Abstract: We investigate the thick brane model in Palatini $f(\mathcal{R})$ gravity. The brane is generated by a real scalar field with a scalar potential. We solve the system analytically and obtain a series of thick brane solutions for the $f(\mathcal{R}) = \mathcal{R} + \alpha \mathcal{R}^2$-brane model. It is shown that tensor perturbations of the metric are stable and the gravitational zero mode can be localized on the brane, which indicates that the four-dimensional gravity can be recovered on the brane. Mass spectra of gravitational KK modes are also discussed.

Keywords: Palatini formalism, brane world
1 Introduction

The problem of singularity has puzzled physicists for a long time. However, this is not all, recent observations indicate that to explain the motions of galaxy and the cosmic speed-up, we need to introduce dark matter and dark energy, and they sum up 96% of the total energy content. What’s more mysterious is the nature of this dark sector. In fact, general relativity has passed all tests at solar system scales only, but not at all length scales. So considering alternative theories of gravity is a reasonable choice.

Actually, Einstein has considered a new approach to variation when he found general relativity, nowadays known as Palatini variational principle. Unlike the conventional metric approach, Palatini approach assumes that both metric and connection are independent variables, and thus abandons the priori of metric. Under this variational principle, one gets two field equations. For an Einstein-Hilbert action, these two methods are proved to be equivalent [1]. However, the results will be very different if corrections to Einstein-Hilbert action are considered. It is well known that a metric formalism $f(R)$ theory leads to fourth-order field equations, while one will get second-order field equations when Palatini variational principle is used. Therefore, a Palatini formalism $f(R)$ theory avoids the matter sector instability in a metric $f(R)$ theory [2]. In recent years Palatini $f(R)$ theories of gravity have attracted much interest since they are expected to have a good description about the phenomenology of our universe [3–14]. In ref. [15], it is showed that $1/R$ correction to Einstein-Hilbert action in Palatini formalism offers an alternative explanation for late time acceleration. The models with the addition of both positive and negative powers of scalar curvature were considered [16]. It was showed that this model of modified gravity may account for both early time inflation and late time accelerated expansion.

On the other hand, higher-dimensional gravity theories have attracted much attentions since the concept of brane world [17–23] was proposed. In the framework of general relativity, it has been shown that higher-dimensional gravity theories may address some open problems in particle physics and phenomenology such as the hierarchy problem and
cosmological constant problem. In this paper, we are interested in the smoothed version of Randall-Sundrum brane model (RS-II) [23] in Palatini gravity theory. Unlike RS-II model, the brane in this scenario has a thickness, so it is known as thick brane model. As in RSII, one of the issues of a thick brane model is the localization of the gravitational zero mode, which is related to the recovery of four-dimensional gravity. Besides, the stability problem of the system is also very important. Though the massive KK modes of gravity are suppressed on the brane, they do contribute to the Newtonian potential with a correction, which provides an approach to detect to extra dimensions.

Usually, the structure of a brane world model is determined by the gravity model and the way how matter fields coupled to gravity. As is well known, the \( f(R) \) gravity theory in metric formalism modifies the gravitational sector of the Einstein equations. For the brane world models constructed in metric formalism \( f(R) \) gravity, see refs. [24–29]. We will show in this paper that, Palatini \( f(R) \) gravity is equivalent to general relativity with a modified source, thus it provides a special coupling of matter fields and gravity. It will be interesting to explore the thick brane world model in this gravity. We expect that the solutions of \( f(R) \) gravity in Palatini formalism will be different from those in metric formalism.

In this paper, we study thick brane model in Palatini \( f(R) \) gravity. We try to get analytic solutions of thick brane world model for general \( f(R) \) with constant curvature, and exact analytic solutions of the model with \( f(R) = R + \alpha R^2 \).

In particular, we try to get thick brane solutions of the Palatini \( f(R) = R + \alpha R^2 \) gravity. In Sec. 2, we first review Palatini \( f(R) \) gravity and set up our Palatini \( f(R) \)-brane model. Then we derive the second-order field equations in five dimensions for our model and show how to solve the field equations analytically. In Sec. 3, we study the gravitational fluctuations and stability problems. The localization of gravitational KK mode modes is also discussed. The discussion and conclusions are given in Sec. 4.

2 The model

In this paper, we consider the general \( f(R) \) model in five-dimensional spacetime in Palatini formalism. The action takes the form

\[
S_{\text{Pal}} = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g_5} f(R(\Gamma)) + S_M(g_{MN}, \Psi),
\]

where \( \kappa_5^2 = 8\pi G \) with \( G \) the five-dimensional fundamental gravitational constant, \( g_5 \) is the determinant of the metric, and \( S_M(g_{MN}, \Psi) = \int d^5x \sqrt{-g_5} L_M(g_{MN}, \Psi) \) is the action for ordinary matter that only couples to the metric. In this paper, capital Latin letters \( M, N, \cdots \) denote the five-dimensional coordinate indices 0, 1, 2, 3, 5 and Greek letters \( \mu, \nu, \cdots \) denote the four-dimensional brane coordinate indices 0, 1, 2, 3. \( R(\Gamma) = g^{MN} R_{MN} \) is the Ricci scalar constructed with the independent connection \( \Gamma \). In Palatini \( f(R) \) formalism, the main feature is that both metric and connection are assumed to be independent variables. It is very different from general relativity and other metric theories. This set-up will lead to special physics. Varying with respect to the metric and connection, respectively, one
gets the following two field equations:

\[ f_R R_{MN} - \frac{1}{2} f g_{MN} = \kappa^2 T_{MN}, \quad (2.2) \]
\[ \tilde{\nabla}_A \left[ \sqrt{-g} f_R g^{MN} \right] = 0, \quad (2.3) \]

where \( f_R \equiv \frac{df}{dR} \), \( T_{MN} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{MN}} \) is the energy-momentum tensor, and \( \tilde{\nabla} \) is the covariant derivative defined with the connection \( \Gamma \). Note that \( \tilde{\nabla} \) is not compatible with the metric, which implies that \( \tilde{\nabla} g_{MN} \neq 0 \). Actually, eq. (2.3) defines the auxiliary metric in Palatini \( f(R) \) gravity. If we define the auxiliary metric

\[ \sqrt{-q q^{MN}} \equiv \sqrt{-g} f_R g^{MN}, \quad (2.4) \]

then we have \( \tilde{\nabla}_A (\sqrt{-q q^{MN}}) = 0 \). It is similar to the equation \( \nabla_A (\sqrt{-g} g^{MN}) = 0 \) (it is also equivalent to \( \nabla_A g^{MN} = 0 \) in general relativity. At this point, we can say \( \tilde{\nabla} \) is compatible with the auxiliary metric \( q^{MN} \). According to this definition, we obtain

\[ q^{MN} = f_R^{-\frac{2}{3}} g^{MN}, \quad q_{MN} = f_R^\frac{2}{3} g_{MN}. \quad (2.5) \]

Obviously, \( q_{MN} \) is just the conformally transformed metric. With this metric, one can express the independent connection as

\[ \Gamma^A_{\ MN} = \frac{1}{2} q^{AB} (\partial_M q_{NB} + \partial_N q_{MB} - \partial_B q_{MN}) = \{A_{MN}\} + C^A_{MN}, \quad (2.6) \]

where \( \{A_{MN}\} \) is the Christoffel symbols and \( C^A_{MN} \) is a well-defined tensor. When \( f(R) = R \), we have \( C^A_{MN} = 0 \), and the theory will reproduce general relativity.

The expression (2.6) indicates that we are able to eliminate the independent connection \( \Gamma \) from the field equations. If this is done, then we will get one field equation which only relies on metric dynamically. With the following relations

\[ R_{MN} = R_{MN}(g) - \frac{1}{3 f_R} (3 \nabla_M \nabla_N f_R + g_{MN} \nabla_A \nabla^A f_R) + \frac{4}{3 f_R^2} \nabla_M f_R \nabla_N f_R, \quad (2.7) \]
\[ R = R - \frac{8}{3 f_R} \nabla_A \nabla^A f_R + \frac{4}{3 f_R^2} \nabla_A f_R \nabla^A f_R, \quad (2.8) \]

where \( R_{MN}(g) \) and \( R \) are the Ricci tensor and Ricci scalar constructed from the spacetime metric \( g_{MN} \), respectively, we can transform eq. (2.2) into the following one

\[ G_{MN} = \frac{\kappa^2 T_{MN}}{f_R} - \frac{1}{2} g_{MN} \left( R - \frac{f}{f_R} \right) + \frac{1}{f_R} (\nabla_M \nabla_N - g_{MN} \nabla_A \nabla^A) f_R \]
\[ - \frac{4}{3 f_R^2} \left( \nabla_M f_R \nabla_N f_R - \frac{1}{2} g_{MN} \nabla_A f_R \nabla^A f_R \right), \quad (2.9) \]

where \( G_{MN} = R_{MN} - \frac{1}{2} R g_{MN} \) is the Einstein tensor. Furthermore, from eq. (2.2), we have

\[ f_R R - \frac{5}{2} f = \kappa^2 T, \quad (2.10) \]
which shows that $R$ is related with the matter field $T$ algebraically. Thus, all of the quantities such as $R$, $f(R)$, and $f_R$ can be expressed by $T$. At this point, we have successfully eliminated the auxiliary metric $q^{MN}$ or the connection $\Gamma$ from the field equations, and the dynamical variable of the field equations (2.9) is the spacetime metric $g_{MN}$. The implication of eq. (2.9) is clear so far: it is the Einstein equation with a modified source, and the effective energy-momentum tensor is defined by the right hand side of eq. (2.9). It can be seen that Palatini $f(R)$ gravity is equivalent to a metric theory with a modified source. For more details about Palatini $f(R)$ gravity, see Refs. [12, 30, 31].

In this paper we consider the $f(R)$ brane model with a scalar field presented in the five-dimensional background spacetime. The background metric with four-dimensional Poincaré symmetry is assumed as

$$ds^2 = a^2(y)\eta_{\mu\nu}dx^\mu dx^\nu + dy^2,$$

where $a(y)$ is the warp factor. The Lagrangian of the scalar field is assumed as $L_\phi = -\frac{1}{2}\partial_M \phi \partial^M \phi - V(\phi)$. The corresponding energy-momentum tensor and equation of motion of the scalar field are

$$T_{MN} = \partial_M \phi \partial_N \phi - g_{MN} \left( \frac{1}{2} \partial_A \phi \partial^A \phi + V(\phi) \right),$$

$$\Box \phi = V(\phi).$$

For static brane solution, the scalar field is only a function of $y$, $\phi = \phi(y)$. Then, with the metric (2.11), the explicit forms of the above two equations are given by

$$T_{\mu\nu} = -a^2 \left( \frac{1}{2} \phi'^2 + V \right) \eta_{\mu\nu},$$

$$T_{55} = \frac{1}{2} \phi'^2 - V,$$

$$V' = \phi'' \phi' + 4 \frac{a'}{a} \phi'^2,$$

where the prime represents the derivative with respect to the extra dimension coordinate $y$. In this section, we expect to solve the system (2.9) and (2.16). Usually, we can obtain topologically nontrivial solutions by introducing a superpotential [32–34] or giving a scalar potential such as the $\phi^4$ or other models. However, in our case this does not work because of the complex expression of the right hand side of (2.9). To solve this system, we consider eqs. (2.2) and (2.4) instead of eq. (2.9). With the relation (2.4) between $q_{MN}$ and $g_{MN}$, it is convenient to assume the auxiliary metric as

$$ds^2 = u^2(y)\eta_{\mu\nu}dX^\mu dX^\nu + \frac{u^2(y)}{a^2(y)}dY^2.$$

Then eqs. (2.2) and (2.4) are reduced to

$$\left( \frac{6u'^2}{u^2} - 3 \frac{a' u'}{u} - 3 \frac{u''}{u} \right) f_R = \kappa^2 \phi'^2,$$

$$5f_R \left( \frac{a' u'}{u} + \frac{u''}{u} \right) - 2f_R \frac{u'^2}{u^2} + f(R) = 2\kappa^2 V.$$
and
\[ f_R = \left( \frac{u}{a} \right)^3, \]  

(2.20)

respectively.

Now we have four equations (2.13), (2.18), (2.19), and (2.20). However, eqs. (2.13), (2.18), and (2.19) are not independent because of the conservation of \(T_{MN}\). Therefore, this system has a degree of freedom. To solve it we need a constraint. Obviously, different constraints lead to different results. We first consider the case in which \(R(\Gamma)\) is a constant. According to eq. (2.5), it is straightforward to conclude that \(R(\Gamma)\) is also constant. Thus, the solutions are the same as in metric formalism \(f(R)\) gravity with constant \(R\) [25]. The solutions are listed as follows.

- For \(AdS_5\), \(R(\Gamma) = R(g) = -20\gamma^2 (\gamma > 0)\) and \(f_R < 0\), we have
  \[ a(y) = \cosh \left( \frac{5\gamma y}{2} \right), \]
  \[ \phi(y) = \pm 2 \sqrt{\frac{6|f_R|}{5\kappa^2}} \text{arctan} \left( \text{tanh} \left( \frac{5\gamma y}{4} \right) \right), \]
  \[ V(y) = V_0 + \frac{9\gamma^2 |f_R|}{4\kappa^2} \sin^2 \left( \sqrt{\frac{5\kappa^2}{6|f_R|}} \phi \right), \]
  \[ \text{where } V_0 = \frac{2f - 25\gamma^2 |f_R|}{4\kappa^2}. \]

- For \(dS_5\), \(R(\Gamma) = R(g) = 20\gamma^2\) and \(f_R > 0\),
  \[ a(y) = \cos \left( \frac{5\gamma y}{2} \right), \]
  \[ \phi(y) = \pm \sqrt{\frac{6f_R}{5\kappa^2}} \text{arctanh} \left( \sin \left( \frac{5\gamma y}{2} \right) \right), \]
  \[ V(y) = V_0 - \frac{9\gamma^2 f_R}{4\kappa^2} \sinh^2 \left( \sqrt{\frac{5\kappa^2}{6f_R}} \phi \right), \]
  \[ \text{where } V_0 = \frac{2f - 25\gamma^2 f_R}{4\kappa^2}. \]

The \(AdS_5\) solution supports a warp factor which diverges at infinity. However, it can be checked that for an observer located at \(y = 0\), photons coming from infinity cost finite time to reach \(y = 0\). Therefore, there are no event horizons here. For the \(dS_5\) solution, the extra dimension should be restricted to the interval \(-\frac{\pi}{5\gamma} < y < \frac{\pi}{5\gamma}\) and there are also no event horizons.

For nonconstant \(R(\Gamma)\), the system becomes complex. For metric formalism \(f(R)\) gravity, it is the metric that involves high derivatives, so the brane world solutions can be obtained by assuming the solution of the warp factor [29]. However, this is not a good method for our case.

Here, we consider the \(f(R) = R + \alpha R^2\) model, for which eq. (2.20) becomes
\[ 1 - 8\alpha \left( \frac{a'}{a} \frac{u'}{u} + 2 \frac{u''}{u} + \frac{a'}{a} \right) = \left( \frac{u}{a} \right)^3. \]

(2.23)
Note that the system will be greatly simplified if one impose a good relation between \( u(y) \) and \( a(y) \). For this purpose, we assume \( u(y) = c_1 a^n(y) \) with \( n \neq 0 \). Then eq. (2.23) reduces to
\[
1 - 24n^2 \frac{a^2}{a''} - 16n \frac{a''}{a} - c_1^3 a^{3(n-1)} = 0. \tag{2.24}
\]
When \( n = 1 \), we get the above \( \text{AdS}_5 \) and \( dS_5 \) solutions (2.21) and (2.22), which are constant curvature solutions. So we set \( n \neq 1 \), for which eq. (2.24) supports the following solution of the warp factor:
\[
a(y) = \text{sech}^{\frac{3}{3n-1}}(ky) \tag{2.25}
\]
with \( k = \frac{3(n-1)}{32n(3n+2)\alpha} \), but \( c_1 \) in (2.24) is fixed as \( c_1 = \left(\frac{6n-1}{6n+2}\right)^{1/3} \). Thus the potential \( V(y) \), scalar field \( \phi(y) \), and energy density \( \rho(y) \) are given by
\[
V(y) = \frac{(3n+1)(6n-1)}{32(3n+2)^2\kappa^2\alpha} \text{sech}^4(ky) + \frac{(3n+5)(6n-1)}{16(3n+2)^2\kappa^2\alpha} \text{sech}^2(ky) + \Lambda_5, \tag{2.26}
\]
\[
\phi(y) = \sqrt{\frac{2n(6n-1)}{3(3n+2)(n-1)\kappa^2}} \left( \sqrt{3} E\left(iky, \frac{2}{3}\right) - \sqrt{3} F\left(iky, \frac{2}{3}\right) + \sqrt{2 + \cosh(2ky)\tanh(ky)} \right), \tag{2.27}
\]
\[
\rho(y) = \frac{(3n-1)(6n-1)}{16(3n+2)^2\kappa^2\alpha} \text{sech}^4(ky) + \frac{(3n+1)(6n-1)}{8(3n+2)^2\kappa^2\alpha} \text{sech}^2(ky), \tag{2.28}
\]
where \( \Lambda_5 = -\frac{1}{8\kappa^2\alpha} \), \( F(y,m) \) and \( E(y,m) \) are the incomplete elliptic integrals of the first and second kinds, respectively.

The scalar field involves Elliptic integrals, and we fail to give the expression of the potential \( V(\phi) \). It can be seen that our solution is determined by two parameters \( n \) and \( \alpha \). In order to get an asymptotic \( \text{AdS}_5 \) solution, we require \( \alpha > 0 \). The solution of the scalar field indicates that \( n \) should be restricted to the interval \( n < -\frac{5}{3} \) or \( 0 < n < \frac{1}{6} \) or \( n > 1 \). Besides, any values in the interval \( 0 < n < \frac{1}{6} \) will lead to negative energy density, so we exclude these values. On the other hand, for an observer located at the origin of the extra dimension, photons coming from infinity cost finite time to reach the origin for the solution (2.25) with \( n < 1 \). We are interested in the brane model with event horizons located at boundaries of the extra dimension. So we only consider the case of \( n > 1 \). For convenience, we set \( n = \frac{5}{3} \), for which the solution reads
\[
a(y) = \text{sech}(ky), \tag{2.29}
\]
\[
\phi(y) = \sqrt{\frac{15}{7\kappa^2}} \left( \sqrt{3} E\left(iky, \frac{2}{3}\right) - \sqrt{3} F\left(iky, \frac{2}{3}\right) + \sqrt{2 + \cosh(2ky)\tanh(ky)} \right), \tag{2.30}
\]
\[
V(y) = \frac{5k^2}{\kappa^2} \left( \frac{9}{14} \text{sech}^4(ky) + \frac{15}{7} \text{sech}^2(ky) - \frac{7}{3} \right). \tag{2.31}
\]
where \( k = \sqrt{\frac{3}{280n}} \). It is clear that \( a(\pm \infty) \to 0 \), thus \( |\phi(\infty)| \to \text{constant} \), and \( V(\pm \infty) \to \Lambda_5 \). We show the plots of \( a(y) \), \( \phi(y) \), and \( V(y) \) in fig. 1. The thickness of the brane, \( 1/k \), is
determined by the parameter \( \alpha \). Another feature of our solution is that the cosmological constant \( \Lambda = V(y \to \infty) = -\frac{35k^2}{3\kappa^2} = -\frac{1}{8\pi \alpha} \) is independent of \( n \). Equation (2.28) implies that the energy density peaks at \( y = 0 \), and it does not dissipate with time. Furthermore, the Ricci scalar \( R(g) \) is given by

\[
R(g) = 4k^2 \left( 7\text{sech}^2(ky) - 5 \right). \tag{2.32}
\]

As \( y \to \pm\infty \), \( R(g) \to -20k^2 < 0 \). This is consistent with the fact that the space-time far away from the brane is asymptotically \( AdS_5 \).

![Figure 1. The shapes of the \( a(y) \), \( \kappa \phi(y) \), \( \kappa^2 V(y)/k^2 \), and \( \kappa^2 \rho(y)/k^2 \) with respect to \( ky \) for \( n = \frac{3}{4} \).](image)

### 3 Gravitational fluctuations

The fluctuations \( \delta g_{MN} \) of the background metric (2.11) can be decomposed as the transverse-traceless (TT) tensor mode, transverse vector modes, and scalar modes. It can be shown that the transverse vector modes and scalar modes are decoupled with the transverse-traceless tensor modes. In this paper we would like to investigate stability and localization of the TT tensor fluctuations of the background metric (2.11), whose KK modes are related to the four-dimensional gravitons and Newtonian potential. In our case, the perturbed metric is

\[
ds^2 = a^2(y)(\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu + dy^2, \tag{3.1}
\]
where the tensor fluctuations $h_{\mu\nu}$ satisfy the TT condition
\[ \partial_{\mu}h^{\mu\nu} = 0, \quad h \equiv \eta^{\mu\nu}h_{\mu\nu} = 0. \] 
(3.2)
Thus we have
\[ \delta g_{\mu\nu} = a^2(y)h_{\mu\nu}, \quad \delta g_{55} = \delta g_{\mu5} = 0. \] 
(3.3)
With the perturbed metric (3.1), to linear order, we get the perturbations of Ricci tensor and Ricci scalar
\[ \delta R_{\mu\nu} = \frac{1}{2} (\partial_{\nu}\partial_{\rho}h^{\rho}_{\mu} + \partial_{\rho}\partial_{\mu}h^{\rho}_{\nu} - \square^{(4)}h_{\mu\nu} - \partial_{\mu}\partial_{\nu}h) 
- 3a^2h_{\mu\nu} - a\delta''h_{\mu\nu} - 2a\delta'h_{\mu\nu}' - \frac{1}{2}a^2h_{\mu\nu}^{\prime\prime} - \frac{a\delta'h'}{2}\eta^{\mu\nu}, \] 
(3.4)
\[ \delta R = \frac{1}{a^2}(\partial_{\mu}\partial_{\nu}h_{\mu\nu} - \square^{(4)}h) - h'' - \frac{5a'}{a}h', \] 
(3.5)
where $\square^{(4)} = \eta^{\mu\nu}\partial_{\mu}\partial_{\nu}$ denotes the four-dimensional d’Alembert operator. Considering the TT condition (3.2), the perturbation of Ricci scalar vanishes, and
\[ \delta R_{\mu\nu} = -\frac{1}{2} \square^{(4)} h_{\mu\nu} - 3a^2 h_{\mu\nu} - a\delta''h_{\mu\nu} - 2a\delta'h_{\mu\nu}' - \frac{1}{2}a^2h_{\mu\nu}^{\prime\prime}. \] 
(3.6)
We immediately obtain the perturbed $\mu\nu$-components of Einstein tensor
\[ \delta G_{\mu\nu} = \delta (R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) 
= -\frac{1}{2} \square^{(4)} h_{\mu\nu} + 3a^2 h_{\mu\nu} + 3a\delta''h_{\mu\nu} - 2a\delta'h_{\mu\nu}' - \frac{1}{2}a^2h_{\mu\nu}^{\prime\prime}. \] 
(3.7)
On the other hand, the perturbations of the $\mu\nu$-components of the right hand side of field equation (2.9) reads
\[ \delta G_{\mu\nu} = \left( -\frac{\kappa(\frac{1}{2}\phi'^2 + V(\phi))}{f_R} - \frac{1}{2}\left(\frac{\kappa^2 T}{f_R} + \frac{3f}{2f_R}\right) - \frac{\partial_{\rho}\partial_{\nu}f_R}{f_R} 
- \frac{4a'\partial_{\nu}f_R}{af_R} + \frac{2}{3}\left(\frac{\partial_{\nu}f_R}{f_R}\right)^2 \right)h_{\mu\nu} + \frac{\partial_{\nu}f_R}{2f_R} h_{\mu\nu}' \] 
(3.8)
With eqs. (3.7) and (3.8) we get the following perturbed equation
\[ -\frac{1}{2} \square^{(4)} h_{\mu\nu} + 3a^2 h_{\mu\nu} + 3a\delta''h_{\mu\nu} - 2a\delta'h_{\mu\nu}' - \frac{1}{2}a^2h_{\mu\nu}^{\prime\prime} 
= \left( -\frac{\kappa(\frac{1}{2}\phi'^2 + V(\phi))}{f_R} - \frac{1}{2}\left(\frac{\kappa^2 T}{f_R} + \frac{3f}{2f_R}\right) - \frac{\partial_{\rho}\partial_{\nu}f_R}{f_R} 
- \frac{4a'\partial_{\nu}f_R}{af_R} + \frac{2}{3}\left(\frac{\partial_{\nu}f_R}{f_R}\right)^2 \right)h_{\mu\nu} + \frac{\partial_{\nu}f_R}{2f_R} h_{\mu\nu}' \] 
(3.9)
Next, we simplify the above equation. First, from eq. (2.9), we have
\[ \frac{g^{\alpha\beta}G_{\alpha\beta}}{4} = \left( -\frac{\kappa(\frac{1}{2}\phi'^2 + V(\phi))}{f_R} - \frac{1}{2}\left(\frac{\kappa^2 T}{f_R} + \frac{3f}{2f_R}\right) - \frac{\partial_{\rho}\partial_{\nu}f_R}{f_R} 
- \frac{4a'\partial_{\nu}f_R}{af_R} + \frac{2}{3}\left(\frac{\partial_{\nu}f_R}{f_R}\right)^2 + \frac{a'\partial_{\nu}f_R}{af_R} \right). \] 
(3.10)
Now, it is straightforward to get the following simplified perturbed equation by substituting eq. (3.10) into eq. (3.9):

\[
\frac{1}{2} \square^{(4)} h_{\mu\nu} + 2 a a' h'_{\mu\nu} - \frac{1}{2} a^2 h''_{\mu\nu} = - \frac{\partial_y f_R}{2 f_R} h'_{\mu\nu} + \frac{a' \partial_y f_R}{a f_R} h_{\mu\nu},
\]

where we have used the result $1/4 g^{\alpha\beta} G_{\alpha\beta} = 3(a^2 + aa'')$. The above perturbed equation is our main equation. For convenience, we will transform it to a Schrödinger-like equation. However, the third term on the left hand side contains a factor $a^2$, which destroys the formalism of the Schrödinger-like equation. To eliminate it, we introduce a coordinate transformation

\[
dy = adz.
\]

Under this transformation, the background metric (2.11) turns to be a conformally flat one. As a consequence,

\[
\partial_y = \frac{\partial_z}{a}, \quad a' = \partial_y a = \frac{\partial_z a}{a}.
\]

Then eq. (3.11) becomes

\[
\left[ \square^{(4)} - \partial_z^2 + \left( 3 \frac{\partial_z a}{a} + \frac{\partial_z f_R}{f_R} \right) \partial_z \right] h_{\mu\nu} = 0.
\]

By defining $\tilde{h}_{\mu\nu} = B(z) h_{\mu\nu}$ with $B(z) = a^2 f_R^2$, the equation of $\tilde{h}_{\mu\nu}$ reads

\[
\left( \square^{(4)} - \partial_z^2 + \frac{\partial_z^2 B}{B} \right) \tilde{h}_{\mu\nu} = 0.
\]

Now we introduce the KK decomposition $\tilde{h}_{\mu\nu}(x^\sigma, z) = \varepsilon_{\mu\nu}(x^\sigma) \Psi(z)$. Then we will get two equations:

\[
\begin{align*}
\left( \square^{(4)} + m^2 \right) \varepsilon_{\mu\nu}(x^\sigma) &= 0, \\
\left( - \partial_z^2 + \frac{\partial_z^2 B}{B} \right) \Psi(z) &= m^2 \Psi(z).
\end{align*}
\]

It is clear that the equation for the function $\Psi(z)$ is a Schrödinger-like equation with the effective potential given by

\[
W(z) = \frac{\partial_z^2 B}{B} = \frac{3 \partial_z^2 a}{2a} + \frac{\partial_z^2 f_R}{2f_R} + \frac{3}{4} \left( \frac{\partial_z a}{a} \right)^2 - \left( \frac{\partial_z f_R}{2f_R} \right)^2 - \frac{3 \partial_z a \partial_z f_R}{2a f_R}.
\]

It is easy to show that eq. (3.17) can be factorized as

\[
\left( \partial_z + \frac{A}{2} \right) \left( - \partial_z + \frac{A}{2} \right) \Psi(z) = m^2 \Psi(z),
\]

where $A = 3 \frac{\partial_z a}{a} + \frac{\partial_z f_R}{f_R}$. The above equation has the form of $Q^\dagger Q \Psi(z) = m^2 \Psi(z)$, which ensures that the eigenvalues are nonnegative, i.e., $m^2 \geq 0$. Thus, there are no gravitational tachyon modes and so the system is stable under tensor perturbations.
Now we analyze the localization of the gravitational zero mode $\Psi_0(z)$, for which $m = 0$ and the equation is reduced to $(-\partial_z^2 + \frac{2}{z^2})\Psi_0(z) = 0$. The solution of the zero mode is

$$\Psi_0(z) \propto a \frac{z}{f_R^2}.$$  \hfill (3.20)

It is clear that, for the first constant curvature solution (2.21), the zero mode cannot be localized on the brane. For the second constant curvature solution (2.22), the zero mode is localized on the brane. For the $f(R) = R + \alpha R^2$ brane model with $u = c_1 a^n$, the warp factors are given by $a(y) = \text{sech}^{\frac{2}{n-1}}(ky)$ and $u(y) = c_1 \text{sech}^{\frac{2n}{n-1}}(ky)$ with $n > 1$. Then, from eq. (2.20), the zero mode (3.20) is given by

$$\Psi_0(z(y)) \propto a^{\frac{3}{2}}(u/a)^{\frac{3}{2}} \propto a^{\frac{3n}{2}} = \text{sech}^{\frac{n}{n-1}}(ky).$$  \hfill (3.21)

The normalization condition for the zero mode is

$$\int_{-\infty}^{+\infty} \Psi_0^2 dz = \int_{-\infty}^{+\infty} \Psi_0^2 a^{-1} dy \propto \int_{-\infty}^{+\infty} \text{sech}^{ \frac{2(n-1)}{n-1}}(ky) dy < \infty,$$  \hfill (3.22)

which can be guaranteed for $n > 1$. So the zero mode can be localized on the brane and the four-dimensional gravity can be recovered on the brane.

For massive KK modes, we need to analyze the properties of the effective potential $W(z)$ given in eq. (2.2). For the $f(R) = R + \alpha R^2$ brane model with $u = c_1 a^n$, the effective potential is reduced to

$$W(z) = \frac{3n}{2} \frac{\partial_z^2 a}{a} + \frac{3n(3n-2)}{4} \left( \frac{\partial_z a}{a} \right)^2,$$  \hfill (3.23)

or

$$W(z(y)) = \frac{3n}{4} \left( 2a \partial_y^2 a + 3n(\partial_y a)^2 \right) = \frac{nk^2}{3n-1} \text{sech}^{\frac{4}{3n-1}}(ky) \left( 3n + 2 - (6n-1) \text{sech}^2(ky) \right).$$  \hfill (3.24)

From the above equation, we can see that

$$W(0) = -\frac{nk^2}{n-1},$$  \hfill (3.25)

$$W(|y| \to \infty) \to \frac{n(3n+2)k^2}{3(n-1)^2} e^{-\frac{4k|y|}{3(n-1)}} \to 0,$$  \hfill (3.26)

which shows that the effective potential has a trapping well around the brane and tends to vanish from above at infinity. The asymptotic behavior of the effective potential implies that only the zero mode is a bound state and any massive mode cannot be localized on the brane. For $n = \frac{5}{3}$, the explicit expression of the effective potential in the conformally flat coordinate $z$ can be obtained:

$$W(z) = \frac{5k^2(7k^2 z^2 - 2)}{4 ((kz)^2 + 1)^2}.$$  \hfill (3.27)
We show the plots of $W(z)$ in fig. 2.

As can be seen from eq. (3.26), the effective potential allows a series of continuous massive KK modes $\Psi_m$. They are not localized on the brane. As claimed in ref. [35], if the effective potential $W(z) \sim \frac{\alpha^{(\alpha+1)}}{z^2}$ as $|z| \to \infty$, then the Newtonian potential is corrected by $\Delta U \sim 1/r^{2\alpha}$. So in the case of $n = \frac{5}{3}$, the Newtonian potential is corrected by $\Delta U \sim 1/r^5$.

![Figure 2](image.png)

**Figure 2.** The shapes of the effective potential $W(z)/k^2$ for the gravitational KK modes with respect to $kz$ for $n = \frac{5}{3}$

### 4 Discussions and conclusions

In this work we investigated thick brane configuration generated by a background scalar field in Palatini $f(R)$ gravity. It is showed that Palatini $f(R)$ gravity contains higher derivatives of matter fields, whereas it is the metric that contains higher derivatives in metric $f(R)$ gravity. This leads to the difference of strategies to solve differential equations. In Palatini theories like Einddington inspired Born-Infeld gravity [36–38] and Platini $f(R)$ gravity, it would be sensible to introduce an auxiliary metric. By assuming the relations between the background metric and auxiliary metric, we obtained thick brane solutions of this system. The scalar field solution is a kink, which connects two vacua of the scalar potential, and it describes a domain wall. Besides, the thickness of the brane is determined by the coefficient of the $R^2$ term.

Furthermore, we analyzed the gravitational fluctuations of the brane system. For the transverse-traceless tensor perturbations, a Schrödinger-like equation is obtained. It was shown that Palatini $f(R)$-brane system with any function $f(R)$ but $f_R > 0$ is stable under tensor perturbations. The asymptotic behavior of effective potential implies that the gravitational zero mode is localized on the brane, while massive gravitons are suppressed on the brane. The four-dimensional Newtonian gravity on the brane will be recovered but with a small correction.
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