Vortex solutions in axial or chiral
coupled
non-relativistic spinor- Chern-Simons theory

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Abstract.

The interaction of a spin 1/2 particle (described by the non-relativistic ”Dirac” equation of Lévy-Leblond) with Chern-Simons gauge fields is studied. It is shown, that similarly to the four dimensional spinor models, there is a consistent possibility of coupling them also by axial or chiral type currents. Static self dual vortex solutions together with a vortex-lattice are found with the new couplings.

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1. Introduction

Several exact static solutions of the field equations have been found in models, describing the coupling of Chern-Simons (CS) gauge fields with various types of matter fields. Among them is the well known self-dual solution of Jackiw and Pi [1] in a model of CS gauge fields and non relativistic matter field, described by the nonlinear Schrödinger equation. Another family of solutions of this model, obeying periodic boundary conditions and forming a vortex lattice on the plane, is presented by Olesen [2]. Duval Horváthy and Palla [3] described a system of nonrelativistic spinors coupled to CS fields, where the matter field satisfies the Lévy-Leblond equations [3,4], and they also presented some static self-dual solutions.

In this paper the nonrelativistic spinor field is also assumed to satisfy the Lévy-Leblond equations, but the current, coupling it to the CS fields, is not the vector current, but is a different, ‘axial’ or ‘chiral type’ one. The ‘axial type’ current is the result of a light-like Kaluza-Klein reduction from the 3+1 dimensional axial current in a similar way as the current in [3] is the reduced form of the 3+1 dimensional vector current. From the 2+1 dimensional point of view, with this coupling, the gauge field would be coupled to the spin density of the matter. The chiral currents are linear combinations of the vector and axial type ones. Various models with these couplings are studied and it is shown, that even with the new couplings the system of field equations is explicitly solvable (at least in the self dual case) and several exact solutions are presented.

The paper is organized as follows: In Sec. 2. the axially coupled system is described, and in Sec. 3. I solve this system with the self-duality condition used in [3]. In Sec. 4. a new self-duality condition is put forward, allowing more general solutions, and is shown how this new condition leads to the vortex lattice solutions. Sec. 5. deals with the chiral coupling.

2. The model

In this paper we are interested in a model described by the Levi-Leblond and Chern-Simons field equations, like in the nonrelativistic spinor Chern-Simons theory [3], but coupled with an axial type current. The Levi-Leblond equations (the 2+1 dimensional version of the non relativistic ‘Dirac equation’), eq. (2.1), and the field current identity (FCI), eq. (2.2), can be written as:

\[
\begin{align*}
\left\{ \begin{array}{l}
(\vec{\sigma} \cdot \vec{D}) \Phi + 2m \chi = 0, \\
D_t \Phi + i(\vec{\sigma} \cdot \vec{D}) \chi = 0,
\end{array} \right.
\end{align*}
\]  

\[\kappa B \equiv \kappa \epsilon^{ij} \partial_i A^j = -e \varphi,\]  

\[\kappa E^i \equiv \kappa (-\partial_i A^0 - \partial_t A^i) = e \epsilon^{ij} J^j,\]  

\[(2.1)\]  

\[(2.2)\]
where
\[ \Phi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} \quad \text{and} \quad \chi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}, \] (2.3)
are two-component ‘Pauli’ spinors and \((\vec{\sigma} \cdot \vec{D}) = \sum_{j=1}^2 \sigma^j D_j\), with \(D_j = \partial_j - ie A_j\), \(\sigma^j\) denoting the Pauli matrices. Here \(\kappa\) stands for the CS self coupling, \(e\) is the gauge coupling and \(J^\alpha = (\rho, \vec{J})\) denote the charge and current densities that couple the LL and CS equations.

In the model of Duval, Horváth and Palla
\[ \rho = |\Phi|^2, \quad \text{and} \quad \vec{J} = i(\Phi^\dagger \vec{\sigma} \chi - \chi^\dagger \vec{\sigma} \Phi), \] (2.4)
while in the new model these quantities are given as
\[ \rho = |\Psi_+|^2 - |\Psi_-|^2, \quad \text{and} \quad \vec{J} = -i(\Phi^\dagger \vec{\sigma} \sigma_3 \chi + \chi^\dagger \vec{\sigma} \sigma_3 \Phi). \] (2.5)

The FCI is self consistent if the continuity equation holds for \(J^\alpha\). It can be seen easily that it is satisfied also for the new current, thus the system is self consistent.

Let us see why this current is of axial type! The LL equation with the 2+1 dimensional FCI can be obtained in a Kaluza-Klein type reduction procedure from the higher (i.e. 3+1) dimensional system of massless Dirac equation and 3+1 dimensional form of FCI [5]. (It is a reduction from a four dimensional Bargmann manifold \((M, g, \xi)\) in a trivializing coordinate system \((t, x, y, s)\), where the particular vector field \(\xi\) is given by \(\xi = \partial_s\), to the quotient, \(Q\), of \(M\), by the flow of \(\xi\), in coordinates: \(t\) and \(\vec{x} = (x, y)\).) The four dimensional system has the form:
\[ \begin{cases} \hat{D} \psi = 0, & D_\xi \psi = im\psi, \\ \kappa f_{\mu\nu} = e\sqrt{-g} \epsilon_{\mu\nu\rho\sigma} j^\rho j^\sigma, & \partial_{[\mu} f_{\nu\rho]} = 0. \end{cases} \] (2.6)
The four dimensional spinors \(\psi\) are related to \(\Phi\) and \(\chi\) as:
\[ \psi = e^{ims} \begin{pmatrix} \Phi \\ \chi \end{pmatrix}. \] (2.7)

In Minkowski space, using light-cone coordinates, the metric is written \(d\vec{x}^2 + 2dt ds\), and the Dirac matrices can be chosen as:
\[ \gamma^t = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \gamma^\rho = \begin{pmatrix} -i\vec{\sigma} & 0 \\ 0 & i\vec{\sigma} \end{pmatrix}, \quad \gamma^s = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}. \] (2.8)

The chirality operator \(\Gamma\) is:
\[ \Gamma \equiv \gamma^5 = -\sqrt{-g} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma, \] (2.9)
in coordinates:
\[ \Gamma = \begin{pmatrix} -i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}. \] (2.10)

In four dimensions both the vector and the axial coupling can be used in equations (2.6), because the matter field is massless. Carrying out the light-like Kaluza-Klein type reduction on the equations (2.6) coupled by the axial current:
\[ j\mu = \overline{\psi}\gamma_\mu i\Gamma\psi, \quad \overline{\psi} = \psi^\dagger G, \quad G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]
results in our 2+1 dimensional problem. Thus this model is the axial type counter part of the model considered in [3]. Also the symmetries of the two system are the same: The \( \xi \)-preserving conformal transformations
\[ L_X g = k g \quad \text{and} \quad L_X \xi = 0, \] (2.11)
( where \( k \) is smooth function ), act as symmetries on the axial coupled spinor-CS system (2.1,2,3). To show this we can use the same process employed in [5], with the same results for the transformations of the equations (2.6), and we conclude the transformations are symmetries if
\[ \delta_X (\overline{\psi}\gamma_\mu i\Gamma\psi) = \delta_X \overline{\psi}\gamma_\mu i\Gamma\psi + \overline{\psi}\gamma_\mu i\Gamma\delta_X \psi = L_X (\overline{\psi}\gamma_\mu i\Gamma\psi) + 2k \overline{\psi}\gamma_\mu i\Gamma\psi, \]
namely if the axial Dirac current transforms in the same way as the CS current does. Since this is so, we find indeed that: the \( \xi \)-preserving conformal transformations are symmetries of the system. These symmetries form a finite dimensional Lie group, called the extended ‘Schrödinger’ group [6].

Let us see another version of our model! Realising that the lower component of \( \psi \) can be expressed from (2.1) simply: \( \chi = -(1/2m)(\vec{\sigma} \cdot \vec{D})\Phi \), we find, that \( \Phi \) is the relevant component i.e. one can write a system of equations including only the \( \Phi \). Thus the equations to solve are:
\[ iD_t \Psi_\pm = \left[ -\frac{\vec{D}^2}{2m} + \lambda (\Psi_\pm^\dagger \Psi_\pm) \right] \Psi_\pm, \quad \lambda \equiv \frac{e^2}{2m\kappa}, \] (2.12)

together with the FCI. For the static case they simplify to:
\[
\begin{align*}
\left\{ \begin{array}{l}
-\frac{1}{2m}(\vec{D}^2 + eB\sigma_3) - eA_t \Phi = 0, \\
\vec{J} = -\frac{\kappa}{e} \vec{\nabla} \times A_t, \\
\kappa B = -e\varrho.
\end{array} \right.
\] (2.13)
I will use mainly this form in what follows.

We now have a system described by its equations of motion, and we know, that these equations are self consistent. There is another way to describe a model, namely describing it with an action. It is an important question whether we can find an action producing equations (2.1,2) with the current (2.5). Looking at the 2+1 dimensional action \( \int d^3 x \mathcal{L} \) with

\[
\mathcal{L} = \frac{\kappa}{4} \epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} + \left\{ [\Phi^\dagger \sigma_3 (D_t \Phi + i \sigma^i D_i \chi)] + [\chi^\dagger \sigma_3 (\sigma^i D_i \Phi + 2m \chi)] \right\},
\]

we find that the density and the current, following from this action, (the coefficient of \(-i A_0\) and \(-i \vec{A}\) in the spinor part of \(\mathcal{L}\)), is the same, as the current in eq. (2.5), and the variational equations are indeed equivalent to (2.1,2) (they give the Levi-Leblond equations multiplied by the constant matrix \(\sigma_3\)).

3. Self-dual solutions

Let us consider the solutions of this model. First (using the similarity of the model to that of in [3]) we try to find time independent self dual solutions like the ones found there. In doing so I assume the same self duality condition which is used in [3]. Notice, that for solutions in which the two-component spinor, \(\Phi\), has only one (upper or lower) nonvanishing component, \(\rho\) has a definite sign, and is given by \(\rho = \pm |\Psi|/2\) or \(\rho = -|\Psi|/2\). The self duality condition of [3] can be written:

\[
(D_1 + iD_2) \Phi = 0. \tag{3.1}
\]

As a result if \(\Phi\) satisfies (3.1) we can replace \(\vec{D}^2 = D_1^2 + D_2^2\) by \(\mp eB\) in our equations (2.13). Let’s consider the static self dual version of our system! Only the first equation of (2.13) changes its form:

\[
\left[ -\frac{1}{2m} eB(\mp 1 + \sigma_3) - e A_t \right] \Phi = 0, \tag{3.2}
\]

and the others remain as in (2.13). One can see, that there are solutions with vanishing \(A_t\), and \(\vec{J}\) by choosing \(\Phi = \Phi_+\) (or \(\Phi = \Phi_-\)), where

\[
\Phi_+ = \begin{pmatrix} \Psi_+ \\ 0 \end{pmatrix} \quad \text{and} \quad \Phi_- = \begin{pmatrix} 0 \\ \Psi_- \end{pmatrix}. \tag{3.3}
\]

It can be seen easily, that \(\chi\) is zero for this choice. These expressions solve the static system, if they solve the remaining equations of self duality and \(B = -(e \rho)/\kappa\), where \(\rho\) has the special form given above. We get from the self duality:

\[
\vec{A} = \pm \frac{1}{2e} \vec{\nabla} \times \ln |\Psi_\pm|^2. \tag{3.4}
\]
In the gauge $\Psi_+ = \rho^{1/2}$ or $\Psi_- = -\rho^{1/2}$ the remaining equations reduce to the Liouville equation
\begin{equation}
\Delta \ln |\Psi_\pm|^2 = \frac{2e^2}{\kappa} |\Psi_\pm|^2.
\end{equation}

The ± sign in the connection between $\rho$ and $\Psi_\pm$ and in the solution of the self-duality conditions combined nicely to yield a universal plus sign in (3.5). Normalizable solutions are obtained when $\kappa < 0$. The well known general solution of the Liouville equation is:
\begin{equation}
\pm \varrho = |\Psi_\pm|^2 = -(4\kappa/e^2)|f'(z)|^2(1 + |f(z)|^2)^{-2} \text{ where } f(z) \text{ is complex analytic.}
\end{equation}

As we can see these solutions of the axial type problem are very similar to the solutions of the vector one. It is easy to see the reason behind this: In this very special situation, when $\Phi$ has only one nonzero component, our generally indefinite density, $\rho$, becomes definite, and differ just in a + or − sign from the density in the vector model. We will see however, that these are not the only possible static solutions of our model.

4. The new self-duality and the “vortex lattice” solutions

Let’s consider how the LL equations change when the self-duality and the special form of $\Phi$, ($\Phi_+$ or $\Phi_-$ see in eq (17)) is used. It is easy to see, that in this case the LL simplify to:
\begin{equation}
\begin{cases}
2m \chi = 0, \\
D_t \Phi + i(\bar{\sigma} \cdot \bar{D}) \chi = 0.
\end{cases}
\end{equation}

These are the equations which can be satisfied with the static solutions, having $\chi = 0$ and $A_t = 0$. The current is zero because $\chi = 0$. The remaining equations are $B = -(e\varrho)/\kappa$ plus some conditions, simplifying LL to (4.1). If we don’t want the special one-component form of $\Phi$, but want more general $\Phi$-s, we must choose the condition, what solves the LL equations, in the new form as:
\begin{equation}
(\bar{\sigma} \cdot \bar{D}) \Phi = 0.
\end{equation}

This is our new self-duality condition (1). For $\Phi$-s having only upper (or lower) nonzero components it requires the same as the earlier condition (3.1). What are the equations we have to solve now? As we have seen, static solutions with $\chi = 0$ and $A_t = 0$ satisfy the LL equations with the new self-duality. For this kind of solutions $\kappa E^i \equiv -\partial_i A^0 - \partial_t A^i = e \epsilon^{ij} J^j$ is an identity. The remaining equations that must be solved, are the new self-duality condition (4.2), and the ‘CS Gauss law’:
\begin{equation}
B = -(e/\kappa) \varrho,
\end{equation}

\footnote{This kind of self duality equation can be used to solve the vector model as well. This condition minimalizes the effective Hamiltonian for $\Phi$ in the vector model too.}
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where \( \varrho = |\Psi_+|^2 - |\Psi_-|^2 \) in this model. In the gauge \( \Psi_+ = |\Psi_+|, \Psi_- = |\Psi_-|e^{i\alpha} \) we find from them:

\[
\vec{A} = \frac{1}{2e} \vec{\nabla} \times \ln |\Psi_+|^2 = -\frac{1}{2e} \vec{\nabla} \times \ln |\Psi_-|^2 + \frac{1}{e} \vec{\nabla} \alpha. \quad (4.3)
\]

Therefore \( \vec{\nabla} \times \ln |\Psi_+|^2 = -\vec{\nabla} \times \ln |\Psi_-|^2 + 2\vec{\nabla} \alpha \). Using \( \vec{\nabla} \times \vec{\nabla} \alpha = 0 \) it follows, that \( \Delta \ln |\Psi_-|^2 |\Psi_+|^2 = 0 \), thus \( \ln |\Psi_-|^2 |\Psi_+|^2 = 4F(x,y) \), where \( F \) is real and harmonic, that is \( \Delta F = 0 \). From the definition of \( F \) we get

\[
|\Psi_-|^2 = e^{4F} |\Psi_+|^{-2}.
\]

Using the Chern-Simons Gauss law yields

\[
\Delta \ln |\Psi_+|^2 = -\frac{2e^2}{\kappa} (|\Psi_+|^2 - e^{4F} |\Psi_+|^{-2}). \quad (4.4)
\]

It is useful to work with the complex variable \( z = x + iy \). If \( z = f(w) \) is an analytic function, then \( \Delta w = |f'|^2 \Delta z \). Introducing

\[
\sigma = \ln |\Phi_+(f(w))|^2 + \ln |f'(w)|^2,
\]

equation (4.4) can be simplified further as

\[
\Delta_w \sigma = -\left( \frac{2e^2}{\kappa} \right) \left( e^{\sigma} - (e^{F} |f'|)^4 e^{-\sigma} \right).
\]

Choosing \( f \) so that \( |f'| = e^{-F} \) we get from here:

\[
\Delta \sigma = -\left( \frac{4e^2}{\kappa} \right) \sinh(\sigma). \quad (4.5)
\]

This equation shows a manifest scaling symmetry, and after the scale transformation \( w' = \frac{2e}{\kappa^{1/2}} w \), the constant \( \frac{4e^2}{\kappa} \) is scaled to 1 and we have:

\[
\Delta \sigma = -\sinh(\sigma). \quad (4.6)
\]

This is the two dimensional sinh-Poisson equation, which has well known numerical [7,8], and analytical [9] solutions such as the solutions of a non-linear boundary value problem on a square or on a rectangle. In these papers some arguments are put forward which indicate, that there are no regular solutions on the whole plane, just on finite regions. The large structure forms a vortex lattice on the plane. The boundary conditions in \( x \) and \( y \), related to periodic solutions are: \( \sigma = 0 \) on the sides of a square. The functions solving eq. (4.6) with this boundary conditions are:

\[
\sigma = 2 \ln \left[ \frac{\theta(\vec{l} + 1/2\vec{1}, \tau)}{\theta(\vec{l}, \tau)} \right], \quad (4.7)
\]
where
\[ \theta(\vec{l}, \tau) = \sum_{m_1, m_N = -\infty}^{\infty} \exp(2i\pi \sum_{i=1}^{N} m_i l_i + i\pi \sum_{i,j=1}^{N} m_i \tau_{ij} m_j), \]
is the Riemann theta function,
\[ l_j(x, y) = k_j x + \omega_j y + l_{0j}, \quad \vec{l} = (1, \ldots, 1), \]
and \( k_j, \omega_j, \tau_{ij} \) have nontrivial dependence on a set of 2N complex parameters \( E_j \)-s (called the main spectrum) through some closed contour integrals on a two-sheeted Riemann-surface. These contours run through and around the branch cuts beginning and ending at the \( E_j \)-s. The structure of the main spectrum determines the properties of \( \sigma \). If the \( E_j \)-s have some special symmetries \( \sigma \) becomes real. These real solutions are non linear "standing waves", and their periods depend on the wave numbers \( k_j \) and \( \omega_j \). Returning to our physical problem, for the interesting quantities, \( \rho \) and \( \vec{A} \), we have the following results:
\[ \rho = e^{2F} \sinh(\sigma) = e^{2F} \left[ \frac{\theta(\vec{l} + 1/2\vec{l}, \tau)^2}{\theta(\vec{l}, \tau)^2} - \frac{\theta(\vec{l}, \tau)^2}{\theta(\vec{l} + 1/2\vec{l}, \tau)^2} \right], \quad (4.8) \]
in terms of the scaled variables. If we want periodicity and smooth transition from domain to domain, we must choose \( \exp(2F) \) to be a constant, denote it by \( c \). This means the following: the phase shift, \( \exp(i\alpha) \), between \( \Psi_+ \) and \( \Psi_- \) is an \( x \) and \( y \) independent quantity. (\( \vec{\nabla} \alpha = \vec{\nabla} \times F = 0 \)). After it we can write:
\[ \rho = c \left[ \frac{\theta(\vec{l} + 1/2\vec{l}, \tau)^2}{\theta(\vec{l}, \tau)^2} - \frac{\theta(\vec{l}, \tau)^2}{\theta(\vec{l} + 1/2\vec{l}, \tau)^2} \right], \quad (4.9) \]
and in a general gauge:
\[ \Phi = \left( \begin{array}{c} c^{1/2} \frac{\theta(\vec{l} + 1/2\vec{l}, \tau)}{\theta(\vec{l}, \tau)} e^{i\alpha} \theta(\vec{l}, \tau) \\ c^{1/2} e^{i\alpha} \frac{\theta(\vec{l}, \tau)}{\theta(\vec{l} + 1/2\vec{l}, \tau)} \end{array} \right) e^{i\beta(x)}, \quad (4.10) \]
\[ \vec{A} = \frac{1}{2e} \nabla \times \ln \left[ \frac{\theta(\vec{l} + 1/2\vec{l}, \tau)}{\theta(\vec{l}, \tau)} \right] + \frac{1}{e} \nabla \beta(x). \quad (4.11) \]
The value of the constant \( c \) is quantised by the boundary condition, and, what can be seen easily, it depends on the number density of particles with + or - spin polarity.

The stability of the solutions can be studied [7] with numerical methods, and an important result of these numerical calculations is, that the solution having the greatest size, (in which the vortex number per domain is the smallest possible), is the thermodynamically stable one.
5. Chiral coupling

Notice, that we already proved the possibility and consistency of the vector and axial vector couplings. This suggests that there can be consistent systems coupled by the (left or right) chiral currents. The chiral invariance of the four dimensional Dirac-equation is well known. It can be seen easily, that the four dimensional system, (2.6), with the current

\[ j_\mu = \frac{1}{2}(j^v_\mu \pm j^{ax}_\mu) = \frac{1}{2}\psi\gamma_\mu(1 \pm i\Gamma)\psi, \]  

is well defined. The \(\xi\)-preserving conformal transformations are symmetries of these systems too, following from the results above and from the linearity of the transformations.

The 2+1 dimensional equations which one obtains after the reduction are equations (2.1) and (2.2) with the current \(\vec{J}_{L,R} = \frac{1}{2}(\vec{J}^v \pm \vec{J}^{ax})\), where \(\vec{J}^v\) is the 2+1 dimensional vector and \(\vec{J}^{ax}\) is the axialvector current (defined in (2.4) and (2.5)).

\[ \rho_{L,R} = |\Psi_\pm|^2, \]  

\[ J^1_{L,R} = -(\psi_\pm^1 \sigma^2 \psi_\pm), \]  

\[ J^2_{L,R} = \pm(\psi_\pm^1 \sigma^1 \psi_\pm), \]

where

\[ \psi_+ = \begin{pmatrix} \Psi_+ \\ \chi_+ \end{pmatrix} \] and \[ \psi_- = \begin{pmatrix} \Psi_- \\ \chi_- \end{pmatrix}, \]

are the chiral components of the s-independent part of the Dirac spinor \(\psi\).

Note: \(J^1_L\) and \(J^2_L\) are related to the imaginary and real parts of the complex function \(\Psi_+^* \chi_+\) and we have a similar relation for \(\vec{J}_R\) and \(\Psi_-^* \chi_-\).

Using the linearity we can see, that the continuity equation is valid for the density \(\rho_{L,R}\) and the current \(\vec{J}_{L,R}\), and we can look for the static solutions of this consistent system of equations. The LL equations split into two uncoupled systems for the chiral components, but both of these have static \(A_t = 0\) solutions, with \(\vec{J}_{L,R} = 0\), if the self-duality (4.2) is valid. Notice we must solve now the same self-dual system, what we have solved in sec. 2., and after some simple calculations we end up again with the Liouville equation. (The chiral densities have the special form (3.3). The remaining equations with this kind of \(\rho\) lead to the Liouville equation, as we have seen it earlier.) There are normalisable solutions for the left chiral case when \(\kappa < 0\), and for the right chiral coupling when \(\kappa > 0\), thus the sign of \(\kappa\) has to be connected with the sign determining the chirality.

6. Discussion

In a model in which the Chern-Simons gauge field is coupled to a spinor field, not only the vector current can play the role of the coupling current, but the axial and chiral type...
currents are possible too. In this paper we studied the other possibilities, and looked for static self-dual solutions. With the axial type coupling I found a kind of vortex lattice in which alternating regions are dominated by up and down spin states. In the chiral case the problem simplified to the solution of the Liouville equation, what is well known. The required sign of the CS coupling constant $\kappa$ is connected with the "chirality".

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