Decays of a $T_{cc}^+$ heavy-quark-spin molecular partner to $D^*D\pi$

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Starting from the hypothesis that the $T_{cc}^+$ discovered at LHCb is a $D^{*+}D^0/D^{*0}D^+$ hadronic molecule, we consider the partial width of its heavy quark spin partner, the $T_{cc}^{*+}$ as a $D^{*+}D^{*0}$ shallow bound state, decaying into the $D^*D\pi$ final states including the contributions of the $D^*D$ and $D^*\pi$ final state interaction by using a nonrelativistic effective field theory. Because of the existence of the $T_{cc}^+$ pole, the $I=0$ $D^*D$ rescattering contributes at the leading order, the same order as that of the tree diagram, while the $D^*\pi$ rescattering contribution is one magnitude smaller. The partial widths of $T_{cc}^{*+}\to D^{*+}D^0\pi^0$, $D^{*0}D^+\pi^0$, and $D^{*0}D^0\pi^+$ are about 44 keV, 20 keV, and 18 keV, respectively.

I. INTRODUCTION

Recently, a double-charm exotic candidate, the $T_{cc}^+$ with probable quantum numbers $I(J^P) = 0(1^+)$, was discovered by the LHCb Collaboration in the $D^0D^0\pi^+$ invariant mass distribution [1,2]. The difference between its mass and the $D^0D^{*+}$ threshold, $\delta m$, and its decay width, $\Gamma$, were obtained in two different models. A fit using a relativistic $P$-wave two-body Breit-Wigner function with a Blatt-Weisskopf form factor gave [1,2]

$$\delta m_{BW} = -273 \pm 61 \pm 5_{-14}^{+11} \text{keV},$$

$$\Gamma_{BW} = 410 \pm 165 \pm 43_{-38}^{+18} \text{keV} ; \tag{1}$$

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while a unitarized Breit-Wigner profile showed \cite{2}:

\[
\delta m_{\text{pole}} = -360 \pm 40^{+4}_{-50} \text{keV},
\]

\[
\Gamma_{\text{pole}} = 48 \pm 2^{+14}_{-14} \text{keV}.
\] (2)

Both results demonstrate the closeness of the \( T_{cc}^+ \) mass to the \( DD^* \) threshold, and therefore the \( T_{cc}^+ \) is an excellent candidate of a hadronic molecule, as analyzed in Refs. \cite{3,8}. Based on the assumption that the \( T_{cc}^+ \) is a \( DD^* \) molecular state with respect to the heavy quark spin symmetry (HQSS) \cite{9,11}, the \( T_{cc}^{0+} \), as a cousin of the \( T_{cc}^+ \), is predicted as a \( D^*D^0 \) hadronic molecule with the quantum numbers \( I(J^P) = 0(1^+) \) in Refs. \cite{3,12}. In particular, the mass of the \( T_{cc}^{0+} \) relative to the \( D^*D^* \) threshold is predicted to be \( \mathcal{B} = 2m_{D^*} - m_{T_{cc}} = (503 \pm 40) \text{keV} \) in Ref. \cite{3}, which is called the binding energy of the \( T_{cc}^{0+} \) in the following. As a heavy-quark-spin partner of the \( T_{cc}^+ \), the \( T_{cc}^{0+} \) is plausible to be observed in the strong decay process \( T_{cc}^+ \to D^*D\pi \), whose partial width can be calculated in a nonrelativistic effective field theory called the XEFT \cite{13,28}.

The XEFT was first constructed in Ref. \cite{13} to systematically study the properties of the exotic \( X(3872) \) \cite{29,30}, also known as \( \chi_{c1}(3872) \), including the effects of dynamic pions. With a mass coinciding with the \( D^0\bar{D}^{*0} \) threshold\cite{2} the \( X(3872) \) is assumed to be a hadronic molecule composed of \( D^0\bar{D}^{*0} + \text{c.c.} \) with an extremely small binding energy, and thus the elementary degrees of freedom, the \( D, D^*, \bar{D}, \bar{D}^* \) and \( \pi \), are all treated nonrelativistically in the XEFT. The decay width of \( X(3872) \to D^0\bar{D}^{*0}\pi^0 \) was calculated to the next-to-leading order (NLO) in Ref. \cite{13} by using the XEFT and the leading order (LO) results are consistent with those in Ref. \cite{33}, which exploits the universal behavior of the long-range \( D^0\bar{D}^{*0} + \text{c.c.} \) part of the \( X(3872) \) wave function with a small binding energy. The \( \pi^0D^0, \pi^0\bar{D}^0 \) and \( D^0\bar{D}^0 \) rescattering effects, which were neglected in Ref. \cite{13}, were shown to be significant at NLO \cite{25} in the XEFT calculation of the \( X(3872) \to D^0\bar{D}^{*0}\pi^0 \) partial width as it doubles the uncertainty of the partial width as a function of the \( X(3872) \) binding energy predicted in Ref. \cite{13}. Since the mass of the \( T_{cc}^{(s)+} \) is very close to the thresholds of \( D^{(s)}D^* \) and \( D^{(s)}D\pi \), the XEFT is also valid for the study of the \( T_{cc}^{(s)+} \) properties. The partial widths for the decays \( T_{cc}^+ \to D^0D^0\pi^+, D^+D^0\pi^0 \) and \( D^+D^0\gamma \) were calculated in Ref. \cite{34} by using the XEFT, and the total width obtained therein is close to the value given in Eq. (2) extracted from fitting the experiment data with a unitarized Breit-Wigner model \cite{2}. The calculation of the \( T_{cc}^+ \to D^*D\pi \) decay widths in the XEFT will give a measure of the probability to search the \( T_{cc}^{0+} \) in the \( D^*D\pi \) invariant mass distributions.

\footnote{An analysis of the LHCb data with the full \( DD\pi \) three-body effects taken into account gives \( \delta m_{\text{pole}} = -356^{+39}_{-38} \text{keV} \) and \( \Gamma_{\text{pole}} = (56 \pm 2) \text{keV} \) \cite{3}.}

\footnote{The recent updated difference between the \( D^0\bar{D}^{*0} \) threshold and the mass of the \( X(3872) \) is \( \delta m_{X(3872)} = m_{D^0} + m_{D^*0} - m_{X(3872)} = (0.01 \pm 0.14) \text{MeV} \) in Ref. \cite{31}, and \( \delta m_{X(3872)} = (0.12 \pm 0.13) \text{MeV} \) in Ref. \cite{32}; the values of the \( X(3872) \) mass in both measurements were determined from a Breit-Wigner fit.}
In this paper, we assume that the $T_{cc}^{*+}$ is a $D^{*+}D^{*0}$ shallow bound state with a binding energy $\mathcal{B} = (503 \pm 40)$ keV predicted in Ref. [3], and use the XEFT to calculate the partial decay widths of $T_{cc}^{*+} \rightarrow D^{*+}D^{0}\pi^0$, $D^{*0}D^{+}\pi^0$ and $D^{*0}D^{0}\pi^+$, including the corrections from the $D^*\pi$ and $D^*D$ final state interactions (FSIs), as well as the $D^*$ selfenergy. Due to the existence of the $T_{cc}^{*+}$, the $S$-wave isoscalar $D^*D$ rescattering contributes at the same order as the tree-level amplitude, and gives a significant contribution to the $T_{cc}^{*+}$ decay width at LO.

This paper is organized as follows. In Section II, we introduce the XEFT effective Lagrangian for the charmed mesons and pion, and the power countings of the Feynman diagrams in the $T_{cc}^{*+} \rightarrow D^*D\pi$ processes. The amplitudes and partial decay rates of the $T_{cc}^{*+} \rightarrow D^*D\pi$ including the corrections from the $D^*\pi$ and $D^*D$ FSIs in the XEFT are derived in Section III, and the numerical results for the partial decay widths of the $T_{cc}^{*+}$ are shown in Section IV. Finally, all the results are summarized in Section V.

II. EFFECTIVE LAGRANGIAN AND POWER COUNTING

In this section, we introduce the effective Lagrangian for the decays of the $T_{cc}^{*+}$ and the power counting rules of the diagrams in the decay processes. As a heavy-quark-spin partner of the $T_{cc}^{*+}$, the $T_{cc}^{*+}$ is predicted as an $S$-wave isoscalar $D^{*+}D^{*0}$ shallow bound state with $J^P = 1^+$ and a binding energy $\mathcal{B} = (503 \pm 40)$ keV [3]. With such a binding energy, the upper bounds of the typical momentum and velocity of the $D^*$ mesons in the $T_{cc}^{*+}$ bound state are $p_{D^*} \sim \gamma \equiv \sqrt{2\mu_{D^*}\mathcal{B}} \lesssim 33$ MeV and $v_{D^*} \simeq \sqrt{\mathcal{B}/(2\mu_{D^*})} \lesssim 0.02$, respectively, where $\mu_{D^*}$ is the reduced mass of $D^{*+}$ and $D^{*0}$, and therefore the nonrelativistic approximation is valid for the $D^*$ and $D$ mesons. The maximum kinetic energy of the emitted pion in the $T_{cc}^{*+}$ decays is

$$E_{p\pi} = \frac{m_{T_{cc}^{*+}}^2 - (m_D + m_{D^*})^2 + m_{\pi}^2}{2m_{T_{cc}^{*+}}} - m_{\pi} \simeq 3.4 \text{ MeV},$$

which leads to the upper bounds of the typical momentum and velocity of the emitted pion to be $p_\pi \simeq \sqrt{2m_\pi E_{p\pi}} \lesssim 31$ MeV and $v_\pi \simeq p_\pi/m_\pi \lesssim 0.22$. Here $m_{T_{cc}^{*+}}, m_D, m_{D^*}$ and $m_\pi$ are the masses of $T_{cc}^{*+}$, $D$, $D^*$, and $\pi$, respectively. Clearly, the pions can also be treated nonrelativistically in the $T_{cc}^{*+} \rightarrow D^*D\pi$ decays.

The elementary degrees of freedom in the effective Lagrangian are the nonrelativistic $D^*$ and $D$ mesons, which are written as isodoublets of the pseudoscalar and vector fields [13]

$$H = \begin{pmatrix} D^0 \\ D^+ \end{pmatrix}, \quad H^* = \begin{pmatrix} D^{*0} \\ D^{*+} \end{pmatrix},$$

(4)
and the pions, which are in the isospin adjoint representation,

\[ \pi = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}. \]  

(5)

The LO XEFT effective Lagrangian for the \( T_{cc}^+ \) has been given in Ref. [34]. As an analogy, the XEFT Lagrangian we use for the \( T_{cc}^+ \) as a heavy-quark-spin partner of the \( T_{cc}^+ \) reads \[13,35\]

\[
\mathcal{L} = H^+ \left( i\partial^0 + \frac{\nabla^2}{2m_H} \right) H^* + H^\dagger \left( i\partial^0 + \frac{\nabla^2}{2m_H} \right) H + \frac{1}{2} \left\{ \pi^\dagger \left( i\partial^0 + \frac{\nabla^2}{2m_\pi} + \delta \right) \pi \right\} \\
- C_0 (H_i^* T^a T H^* \pi) \dagger \left( H_i^* T^a T H^* \pi \right) - C_1 (H_i^* T^a T H^* \pi) \dagger \left( H_i^* T^a T H^* \pi \right) \\
+ \frac{g}{F_\pi \sqrt{2m_\pi}} \left( H^\dagger \partial_i \pi H^* \pi + H.c. \right) + \frac{C_{0D}}{2} (H^T T^a T H^*) \dagger (H^T T^a T H^*) + \frac{C_{1D}}{6m_\pi} (\pi \tau_1 H^*) \dagger (\pi \tau_1 H^*) \\
+ \frac{C_{2D}}{12m_\pi} \left\{ \left[ \left( \pi \tau_3 + \frac{1}{2} \langle \pi \tau_3 \rangle \right) \tau_1 H^* \right] \dagger \left[ \left( \pi \tau_3 + \frac{1}{2} \langle \pi \tau_3 \rangle \right) \tau_1 H \right]^\dagger \right\} \\
+ 3 \left( \pi \tau_3 - \frac{1}{2} \langle \pi \tau_3 \rangle \right) H^* \right\} \),  

(6)

where \( m_H, m_{H^*} \) and \( m_\pi \) are the masses of the \( H, H^* \) and \( \pi \) particles, respectively; \( \delta = \Delta - m_\pi \simeq 7 \text{ MeV} \) with \( \Delta = m_{D^*0} - m_{D^0} \) comes from the shift of the residual mass from the \( D^* \) kinetic term \[13\] and is related to a small scale \( \mu = \sqrt{\Delta^2 - m_\pi^2} \simeq \sqrt{2m_\pi}\delta \approx 45 \text{ MeV} \) appearing in the pion propagator \[13,25\]; the pion decay constant is taken as \( F_\pi = 92.2 \text{ MeV} \), and \( \tau_a \) with \( a = 1, 2, 3 \) are the Pauli matrices in the isospin space, in which the traces (\( \langle \rangle \)) act. Notice that in Eq. (6), both pions and \( D^{(*)} \) mesons are nonrelativistic particles, which means the \( \pi \) operator annihilates and the \( \pi^\dagger \) operator creates the \( \pi \) quanta, so as the \( D^{(*)} \) and \( D^{(*)} \dagger \) \[13\], and one has

\[
\pi^\dagger = \begin{pmatrix} (\pi^0)^\dagger & \sqrt{2}(\pi^-)^\dagger \\ \sqrt{2}(\pi^+)^\dagger & -(\pi^0)^\dagger \end{pmatrix}, \quad H^\dagger = \begin{pmatrix} (D^0)^\dagger \\ (D^+)^\dagger \end{pmatrix}^T, \quad H^* \dagger = \begin{pmatrix} (D^{*0})^\dagger \\ (D^{*+})^\dagger \end{pmatrix}^T. 
\]

(7)

The first line of Eq. (6) includes the kinetic terms for the charmed mesons and pions. The second line contains the contact interactions of the \( D^{*+} \) and \( D^{*0} \), where the term with \( C_0 \) mediates the \( D^* D^* \) scattering in the \( I = 0 \) channel, and the term with \( C_1 \) mediates the scattering in the \( I = 1 \) channel. The first term in the third line is the same term in Ref. [34] which couples the charmed mesons to pions derived from the heavy hadron chiral perturbation theory (HH\( \chi \)PT), and the coupling constant \( g \simeq 0.27 \) \[3\] is determined from the updated \( D^{*+} \) decay width \[30\]. The second term in the third line is the contact interaction for \( D^* D \to D^* D \) with \( I = 0 \), and the resummation effect of the coupling \( C_{0D} \) shown in Fig. \[1\] needs to be considered \[25\] due to the existence of

\[3\] Notice that \( \bar{g} \) is related to the \( g \) in Ref. [34] by \( \bar{g} = g/2 \).
the $T_{cc}^+$ shallow bound state. The resummation effect is equivalent to replacing $C_{0D}$ with the near-threshold $T$-matrix [36]

$$C_{0D} \to T_{DD^*} = -\frac{2\pi}{\mu_D} \frac{1}{1/a + ip},$$

where $\mu_D$ is the reduced mass of $D^{*+}(D^{*0})$ and $D^0(D^+)$, $p = |\vec{p}_{D^*} - \vec{p}_D|/2$ is the relative momentum between $D^{*+}(D^{*0})$ and $D^0(D^+)$ in the $D^*D$ center-of-mass frame, and the $D^{*+}D^0(D^{*0}D^+)$ scattering length $a$ is set to be $a = [-\left(6.72^{+0.36}_{-0.45}\right) - i \left(0.10^{+0.03}_{-0.03}\right)]$ fm [3]. There is no isovector state like the $T_{cc}^+$ found near the $D^*D$ threshold, so there should be no near-threshold pole singularity in the $I = 1$ scattering amplitude for $D^*D \to D^*D$; thus, the isovector $DD^*$ FSI should be much weaker than the isoscalar one and is neglected in our calculation. The last two terms with $C_{\frac{1}{2}\pi}^{1}$ and $C_{\frac{1}{2}\pi}^{2}$ are the $D^*\pi \to D^*\pi$ contact interactions for $I = \frac{1}{2}$ and $I = \frac{3}{2}$, respectively, where $C_{\frac{1}{2}\pi}^{1} = 25.2 \text{ GeV}^{-1}, C_{\frac{1}{2}\pi}^{2} = -6.8 \text{ GeV}^{-1}$ are derived by matching to the $D^*\pi$ scattering lengths given in Ref. [37] (for detailed derivations, see Appendix [B]).

The square of effective coupling between the $T_{cc}^{*+}$ hadronic molecule and the $D^{*+}D^{*0}$ components can be derived from the residue of the $D^{*+}D^{*0}$ scattering amplitude at the $T_{cc}^{*+}$ pole as [35, 38, 39]

$$g_0^2 = \frac{2\pi\gamma}{\mu_{D^*}^2},$$

and thus the effective Lagrangian for the $T_{cc}^{*+}$ coupling to $D^{*+}D^{*0}$ can be written as

$$\mathcal{L}_0 = \frac{g_0}{\sqrt{2}} \varepsilon_{ijk} T_{cc}^{*+} \gamma^i D^{*+} \gamma^j D^{*0} \gamma^k,$$

where $\varepsilon_{ijk}$ is the 3-dimensional antisymmetric Levi-Civita tensor.

With the above Lagrangians in Eqs. (6) and (10), the LO amplitude for the $T_{cc}^{*} \to D^*D\pi$ including the effects of the $D^*D$ and $D^*\pi$ FSIs and the $D^*$ selfenergy are shown in Fig. 2. Here we only show the diagrams for the decay $T_{cc}^{*+} \to D^{*+}D^0\pi^0$, and there are also similar diagrams for the $T_{cc}^{*+} \to D^{*0}D^+\pi^0$ and the $T_{cc}^{*+} \to D^{*0}D^0\pi^+$, except that no $D^{*0}D^0$ FSI diagram is included in the $T_{cc}^{*+} \to D^{*0}D^0\pi^+$ as it only contains $D^*D$ pair with $I = 1$.

![Diagram](image-url)
In the following, we will give a brief power counting to the contributions of all these diagrams. The power counting for the decays of the $X(3872)$ has been given in detail in Refs. [13, 25, 40], and the discussions here for the $T_{cc}^{*+} \rightarrow D^{*+}D^{0}\pi^{0}$ are similar. The relevant small momenta involved in the decays of the $T_{cc}^{*+}$ are $\{p_D, p_{D^*}, p_\pi, \gamma, \mu\}$, which are at the same order and denoted by $Q$ to be the power counting scale. In the decay diagrams, each pion vertex contributes at $O(Q)$, and each nonrelativistic propagator contributes at $O(Q^{-2})$. As the nonrelativistic energy counts as $O(Q^2)$, each loop integral is of $O(Q^5)$. The $C_\pi$ contact term is related to the $D\pi$ contact term in Ref. [25] via the HQSS and therefore $C_\pi$ is of the same order as the $D\pi$ vertex in Ref. [25], i.e., $O(Q^0)$. As the $I = 0$ contact interaction between the $D^*$ and $D$, the $C_{0D}$ should be replaced with $T_{DD^*}$ in Eq. (8) due to the near threshold $T_{cc}^{*+}$ pole, and contribute at $O(Q^{-1})$ [13, 35]. For the diagrams in Fig. 2, the amplitude from the diagram in Fig. 2(a) scales as $O(Q/Q^2) = O(Q^{-1})$ since there are one nonrelativistic propagator and one $P$-wave pion vertex which gives a factor of $p_\pi \sim O(Q)$. The diagrams in Figs. 2(b) and 2(c) also scale as $O(Q^{-1})$ for the decays $T_{cc}^{*+} \rightarrow D^{*+}D^0\pi^0$, $T_{cc}^{*+} \rightarrow D^*D^+\pi^0$ considering the resummation effect (with $C_{0D}$ replaced by $T_{DD^*}$ which has a
near-threshold $T_{cc}$ pole) and can contribute at LO. The amplitudes from diagrams in Figs. 2(d), 2(e) and 2(f) scale as $O(Q^0)$ and can only contribute at NLO.

## III. Differential Decay Rate of $T_{cc}^* \rightarrow D^* D\pi$

In this section, we give all the decay amplitudes of $T_{cc}^* \rightarrow D^* D\pi$ in Fig. 2, including the processes $T_{cc}^{*+} \rightarrow D^{*+} D^0 \pi^0$, $T_{cc}^{*+} \rightarrow D^{*0} D^+ \pi^0$ and $T_{cc}^{*+} \rightarrow D^{*0} D^0 \pi^+$, and give the partial differential decay rates including the effects of $D^* D$ and $D^* \pi$ rescattering. The contribution of the $D^*$ selfenergy in Fig. 2(f) is considered by including the $D^*$ width in the $D^*$ propagator, $G_{D^*}(p)$, i.e.,

$$G_{D^*}(p) = \frac{i}{p^0_{D^*} - m_{D^*} - \frac{p_{D^*}^2}{2m_{D^*}} + i\frac{\Gamma_{D^*}}{2}},$$

where $D^*$ denotes $D^{*+}$ or $D^{*0}$, $p = (p_{D^+}^0, \vec{p}_{D^+})$ is the 4-momentum of the $D^*$, $\Gamma_{D^{*+}} = 83.4$ keV, and $\Gamma_{D^{*0}} = 55.3$ keV is settled by isospin symmetry [41].

### A. Partial decay rate of $T_{cc}^{*+} \rightarrow D^{*+} D^0 \pi^0$

First, we consider the partial decay rate of $T_{cc}^{*+} \rightarrow D^{*+} D^0 \pi^0$. The LO amplitude from the tree diagram in Fig. 2(a) reads

$$i A_a = \frac{-g_0 \bar{g} \mu_{D^*}}{\sqrt{m_{\pi^0} F_\pi}} \cdot \frac{1}{p_{D^{*+}}^0} \varepsilon_{ijk} \varepsilon^i \left(T_{cc}^{*+}\right) \varepsilon^j \left(D^{*+}\right) p_{\pi^0}^k,$$

where $\vec{p}_{D^{*+}}$ is the three-momentum of the external $D^{*+}$, $\vec{p}_{\pi^0}$ is the three-momentum of the final state $\pi^0$, and the $\varepsilon^i \left(T_{cc}^{*+}\right)$ and $\varepsilon^j \left(D^{*+}\right)$ are the polarization vectors of the $T_{cc}^{*+}$ and $D^{*+}$, respectively.

The LO amplitude from the $D^{*+} D^0 / D^{*0} D^+$ rescattering diagrams in Figs. 2(b) and 2(c) reads

$$i A_{bc} = \frac{g_0 \bar{g}}{2 \sqrt{m_{\pi^0} F_\pi}} \left\{-C_{0D1} I_b(p_{\pi^0}) + C_{0D1\text{ex}} I_c(p_{\pi^0})\right\} \varepsilon_{ijk} \varepsilon^i \left(T_{cc}^{*+}\right) \varepsilon^j \left(D^{*+}\right) p_{\pi^0}^k,$$

where $C_{0D1} = +\frac{1}{2} C_{0D}$ and $C_{0D1\text{ex}} = -\frac{1}{2} C_{0D}$ are the contact interactions for the $D^{*+} D^0 \rightarrow D^{*+} D^0$ and the $D^{*0} D^+ \rightarrow D^{*+} D^0$, respectively, and $C_{0D}$ needs to be replaced by $T_{DD^*}$ considering the resummation effect due to the existence of the nearby $T_{cc}$ pole, and the exact form of the 3-point scalar loop integral $I(p)$ is given in Appendix C [25, 42], with the masses of the three particles in the loop represented by $m_1$, $m_2$ and $m_3$. Here $m_1$, $m_2$ and $m_3$ are taken to be the masses of $D^{*0}$, $D^{*+}$ and $D^0$ in the integral $I_b(p_{\pi^0})$ appearing in Fig. 2(b) and the masses of $D^{*+}$, $D^{*0}$ and $D^+$ in the integral $I_c(p_{\pi^0})$ appearing in Fig. 2(c), respectively.
The NLO amplitude from the $D^{*}\pi$ rescattering diagrams in Figs. 2(d) and 2(e) is

$$iA_{de} = \frac{g_{\bar{q}q}g}{4m_{\pi}^{3/2}F_{\pi}} \left\{ -C_{\pi 1} \left( I_{d}^{(1)}(p_D) - I_{d}(p_D) \right) + \sqrt{2}C_{\pi 1\text{ex}} \left[ I_{e}^{(1)}(p_D) + I_{e}(p_D) \right] \right\} \varepsilon_{ijk} e^{i} \left( T_{cc}^{*+} \right) e^{j} \left( D^{*+} \right) p_{D0}^{k},$$

(14)

where the couplings $C_{\pi 1} = \frac{2}{3} C_{3/2}^{\pi} + \frac{1}{3} C_{1/2}^{\pi} = 4.1 \text{ GeV}^{-1}$ and $C_{\pi 1\text{ex}} = -\frac{\sqrt{2}}{3} C_{3/2}^{\pi} + \frac{\sqrt{2}}{3} C_{1/2}^{\pi} = 15.1 \text{ GeV}^{-1}$ are the contact interactions for the $D^{*+}\pi^{0} \to D^{+}\pi^{0}$ and the $D^{*0}\pi^{+} \to D^{+}\pi^{0}$, respectively, and the 3-point vector loop integral $I^{(1)}(p)$ is given in Appendix C, with the masses of the three particles in the loop represented by $m_1$, $m_2$, and $m_3$; see Eq. (C1). Here $m_1$, $m_2$, and $m_3$ are taken to be the masses of $D^{*0}$, $D^{*+}$, and $\pi^{0}$ in the integrals $I_{d}(p_D)$ and $I_{d}^{(1)}(p_D)$ appearing in Fig. 2(d) and the masses of $D^{*+}$, $D^{*0}$, and $\pi^{+}$ in the integrals $I_{e}(p_D)$ and $I_{e}^{(1)}(p_D)$ appearing in Fig. 2(e) respectively.

The decay rate is given by

$$d\Gamma = 2M^{2}M_{1}2M_{2}2M_{3}\frac{1}{2M} \frac{1}{2j+1} \sum_{\text{spins}} |A|^{2} d\Phi_{3},$$

(15)

where the overall factor comes from the normalization of nonrelativistic particles, with $M$ being the mass of the initial particle, $M_{1}$, $M_{2}$ and $M_{3}$ being the masses of three final state particles, respectively, $j$ is the total spin of the initial particle, and there is a sum over all the polarizations of the final state particles. Here the three-body phase space

$$\int d\Phi_{3} = \int \frac{1}{32\pi^{3}} \frac{1}{4M_{1}M_{2}} |d\vec{p}_{1}|^{2} |d\vec{p}_{2}|^{2},$$

(16)

is derived in Appendix A where $\vec{p}_{1}$ and $\vec{p}_{2}$ are the three-momenta for two of the final state particles.

The NLO partial differential rate for the $T_{cc}^{*+} \to D^{*+}D^{0}\pi^{0}$ including the corrections from the $D^{*}D$, $D^{*}\pi$ rescattering, and the $D^{*}$ selfenergy reads

$$\frac{d\Gamma_{T_{cc}^{*+} \to D^{*+}D^{0}\pi^{0}}}{dp_{D0}^{2} dp_{D^{*+}}^{2}} = \frac{1}{3} \frac{m_{\pi^{0}}}{16\pi^{3}} \sum_{\text{spins}} |A_{a} + A_{bc}|^{2} + \frac{1}{3} \frac{m_{\pi^{0}}}{16\pi^{3}} 2Re \left[ \sum_{\text{spins}} (A_{a} + A_{bc}) \times A_{de}^{*} \right],$$

(17)

where the second term includes the correction of the $D^{*}\pi$ rescattering, which is the interference term between the amplitudes at LO and NLO.

### B. Partial decay rate of $T_{cc}^{*+} \to D^{*0}D^{+}\pi^{0}$

For the decay $T_{cc}^{*+} \to D^{*0}D^{+}\pi^{0}$, the LO amplitude from the tree diagram in Fig. 2(a) reads

$$iA_{a2} = \frac{g_{\bar{q}q}g_{D^{*}}}{\sqrt{m_{\pi^{0}}F_{\pi}} p_{D^{*0}}^{2}} \frac{1}{\gamma^{2}} \varepsilon_{ijk} e^{i} \left( T_{cc}^{*+} \right) e^{j} \left( D^{*0} \right) p_{D^{*0}}^{k},$$

(18)
where $\vec{p}_{D^*0}$ is the three-momentum of the external $D^{*0}$, and the $\epsilon^j(D^*)$ is the polarization vector of the $D^*$. The LO amplitude from the $D^{*+}D^0/D^{*0}D^+$ rescattering diagrams in Figs. 2(b) and 2(c) is

$$iA_{bc2} = \frac{g_\pi g}{2\sqrt{m_\pi F_\pi}} \left[ C_{0D2} I_{2b}(p_{\pi^0}) - C_{0D2ex} I_{2c}(p_{\pi^0}) \right] \varepsilon_{ijk} \epsilon^i (T_{cc}^{*+}) \epsilon^j (D^{*0}) p_{\pi^0}^k,$$

(19)

where $C_{0D2} = \frac{1}{2} C_{0D}$ and $C_{0D2ex} = -\frac{1}{2} C_{0D}$ are the contact interactions for the $D^{*0}D^+ \to D^{*0}D^+$ and the $D^{*+}D^0 \to D^{*0}D^+$, respectively, and $C_{0D}$ needs to be replaced by $T_{DD^*}$ considering the resummation effect, and the masses $m_1$, $m_2$ and $m_3$ in the loop integrals $I_{2b}(p_{\pi^0})$ appearing in Fig. 2(b) and $I_{2c}(p_{\pi^0})$ appearing in Fig. 2(c) are taken to be the masses of $D^{*+}$, $D^{*0}$ and $D^+$ and the masses of $D^{*0}$, $D^{*+}$ and $D^0$, respectively.

The NLO amplitude from the $D^{*}\pi$ rescattering diagrams in Figs. 2(d) and 2(e) is

$$iA_{de2} = \frac{g_\pi g}{4m_\pi^3 F_\pi} \left\{ C_{\pi2} \left[ I_{2d}^{(1)}(p_{D^+}) - I_{2d}(p_{D^+}) \right] + \sqrt{2} C_{\pi2ex} \left[ I_{2c}^{(1)}(p_{D^+}) + I_{2c}(p_{D^+}) \right] \right\} \times \varepsilon_{ijk} \epsilon^i (T_{cc}^{*+}) \epsilon^j (D^{*0}) p_{D^+}^k,$$

(20)

where $C_{\pi2} = \frac{2}{3} C_{\frac{2}{3}} + \frac{1}{3} C_{\frac{1}{2}} = 4.1 \text{ GeV}^{-1}$ and $C_{\pi2ex} = \frac{\sqrt{2}}{3} C_{\frac{3}{2}} - \frac{\sqrt{1}}{3} C_{\frac{1}{2}} = -15.1 \text{ GeV}^{-1}$ are the contact interactions for $D^{*0}\pi^0 \to D^{*0}\pi^0$ and $D^{*+}\pi^- \to D^{*0}\pi^0$, respectively; $m_1$, $m_2$ and $m_3$ are the masses of $D^{*+}$, $D^{*0}$ and $\pi^0$ in the loop integrals $I_{2d}(p_{D^+})$ and $I_{2d}^{(1)}(p_{D^+})$ appearing in Fig. 2(d) and are the masses of $D^{*0}$, $D^{*+}$ and $\pi^-$ in the loop integrals $I_{2c}(p_{D^+})$ and $I_{2c}^{(1)}(p_{D^+})$ appearing in Fig. 2(e).

The NLO partial differential rate for the $T_{cc}^{*+} \to D^{*0}D^+\pi^0$ including the corrections from the $D^*D$, $D^{*}\pi$ rescattering and the $D^*$ selfenergy is

$$\frac{d\Gamma_{T_{cc}^{*+} \to D^{*0}D^+\pi^0}}{dp_{D^+}^2 dp_{D^{*0}}^2} = \frac{1}{3} \frac{m_\pi}{16\pi^3} \sum_{\text{spins}} |A_{a2} + A_{bc2}|^2 + \frac{1}{3} \frac{m_\pi}{16\pi^3} 2\text{Re} \left[ \sum_{\text{spins}} (A_{a2} + A_{bc2}) \times A_{de2}^* \right],$$

(21)

where the second term includes the correction of the $D^{*}\pi$ rescattering, which is the interference term between the amplitudes at LO and NLO.

C. Partial decay rate of $T_{cc}^{*+} \to D^{*0}D^0\pi^+$

For the decay $T_{cc}^{*+} \to D^{*0}D^0\pi^+$, the LO amplitude from the tree diagram in Fig. 2(a) reads

$$iA_{a3} = -\frac{2g_\pi g \mu_{D^*}}{\sqrt{2} m_\pi F_\pi p_{D^{*0}}^2 + \gamma^2} \varepsilon_{ijk} \epsilon^i (T_{cc}^{*+}) \epsilon^j (D^{*0}) p_{\pi^+}^k,$$

(22)
where $\mathbf{p}_{\pi^+}$ is the three-momentum of the final state $\pi^+$. There are no terms with $I = 0$ in the $D^{*0} D^0$ rescattering that can be related to the $T^+_{cc}$ and thus there is no LO contribution from the $D^{*0} D^0$ rescattering.

The NLO amplitude from the $D^* \pi$ rescattering diagrams in Figs. 2(d) and 2(e) is

$$iA_{de3} = \frac{g_{d}g_{h}}{4m_{\pi}^{2}F_{\pi}} \left\{ -\sqrt{2}\sqrt{2}C_{\pi3} \left[ I_{3d}^{(1)} (p_{D^0}) - I_{3d} (p_{D^0}) \right] + C_{\pi3ex} \left[ I_{3e}^{(1)} (p_{D^0}) + I_{3e} (p_{D^0}) \right] \right\} \times \varepsilon_{ijk} \varepsilon \left( T^{*0} \right) p_{D^0}^{k}, \quad (23)$$

where $C_{\pi3} = \frac{1}{3} C_{12}^{2} + \frac{2}{3} C_{12} \frac{1}{2} = 14.4 \text{GeV}^{-1}$ and $C_{\pi3ex} = -\frac{\sqrt{2}}{3} C_{12}^{2} + \frac{\sqrt{2}}{3} C_{12} \frac{1}{2} = 15.1 \text{GeV}^{-1}$ are the contact interactions for the $D^{*0} \pi^+ \rightarrow D^{*0} \pi^+$ and the $D^{*+} \pi^0 \rightarrow D^{*0} \pi^+$, respectively, and the masses $m_1$, $m_2$ and $m_3$ are the masses of $D^{*+}$, $D^{*0}$ and $\pi^0$ in the loop integrals $I_{3d} (p_{D^0})$ and $I_{3d}^{(1)} (p_{D^0})$ appearing in Fig. 2(d) and are the masses of $D^{*0}$, $D^{*+}$ and $\pi^0$ in the loop integrals $I_{3e} (p_{D^+})$ and $I_{3e}^{(1)} (p_{D^+})$ appearing in Fig. 2(e).

The NLO partial differential rate for the $T^{*+}_{cc} \rightarrow D^{*0} D^0 \pi^+$ including the corrections from the $D^* D$, $D^* \pi$ rescattering and the $D^*$ selfenergy is

$$\frac{d\Gamma_{T^{*+}_{cc} \rightarrow D^{*0} D^0 \pi^+}}{dp_{D^0} dp_{D^0}} = \frac{1}{3} \frac{m_{\pi}}{16 \pi^{3}} \sum_{spins} \left| A_{03} \right|^{2} + \frac{1}{3} \frac{m_{\pi}}{16 \pi^{3}} \text{Re} \left[ \sum_{spins} A_{03} \times A_{de3}^{*} \right], \quad (24)$$

where again the second term includes the correction of the $D^* \pi$ rescattering, which is the interference term between the amplitudes at LO and NLO.

### IV. PARTIAL DECAY WIDTHS FOR $T^+_{cc} \rightarrow D^* D \pi$

![Graphs](https://via.placeholder.com/150)

**FIG. 3.** Partial decay widths of the $T^+_{cc} \rightarrow D^* D \pi$ versus the binding energy of the $T^+_{cc}$.

In this section, we give the partial decay widths for the decays $T^+_{cc} \rightarrow D^* D \pi$. Table [I] shows the decay widths with the binding energy of the $T^+_{cc}$ being $\mathcal{B} = (503 \pm 40)$ keV. $\Gamma_{\text{Tree}}$ is the decay
TABLE I. Partial decay widths of the $T^{*+}_{cc}$ with a binding energy $B = (503 \pm 40)$ keV. $\Gamma_{\text{Tree}}$ contains the contributions from the tree-level diagrams, $\Gamma_{LO}$ is the LO decay width which includes the contributions from the tree-level and $D^*D$ rescattering diagrams, $\Gamma_{NLO}$ is the decay width which includes the corrections from the $D^*\pi$ rescattering to the $\Gamma_{LO}$, and $\Gamma'_{NLO}$ further includes the corrections from the $D^*$ selfenergy. Since no isovector $D^*D$ rescattering is considered, $\Gamma_{LO} = \Gamma_{\text{Tree}}$ in the last row. The errors come from that of the predicted binding energy $B$.

| Decay                                      | $\Gamma_{\text{Tree}}$ [keV] | $\Gamma_{LO}$ [keV]   | $\Gamma_{NLO}$ [keV]   | $\Gamma'_{NLO}$ [keV] |
|--------------------------------------------|-------------------------------|-----------------------|------------------------|-----------------------|
| $T^{*+}_{cc} \rightarrow D^{*+} D^0 \pi^0$| $12.92^{+0.58}_{-0.34}$      | $43.53^{+1.06}_{-1.04}$| $46.85^{+1.10}_{-1.09}$| $46.87^{+1.10}_{-1.09}$|
| $T^{*+}_{cc} \rightarrow D^{*+} D^+ \pi^0$| $8.06^{+0.42}_{-0.39}$       | $19.67^{+0.65}_{-0.63}$| $21.04^{+0.68}_{-0.66}$| $21.04^{+0.68}_{-0.66}$|
| $T^{*+}_{cc} \rightarrow D^{*0} D^0 \pi^+$| $18.34^{+0.94}_{-0.87}$      | $18.34^{+0.94}_{-0.87}$| $20.30^{+0.99}_{-0.92}$| $20.34^{+0.99}_{-0.93}$|

width including only the contribution from the tree-level diagram. The $I = 0$ $D^*D$ rescattering contributes to the decay width at the same order as $\Gamma_{\text{Tree}}$, and $\Gamma_{LO}$ is thus the LO decay width including the tree-level and the $D^*D$ rescattering contributions. One sees that the isoscalar $D^*D$ rescattering which contains a $T_{cc}$ pole indeed increases the results significantly. $\Gamma_{NLO}$ is the decay width considering the NLO corrections only from the $D^*\pi$ rescattering, whose contributions are minor for all the three decays $T^{*+}_{cc} \rightarrow D^{*+} D^0 \pi^0$, $D^{*0} D^+ \pi^0$ and $D^{*0} D^0 \pi^+$. Adding the tiny correction (smaller than 1% of $\Gamma_{LO}$) from the $D^*$ selfenergy contribution to $\Gamma_{NLO}$ gives the $\Gamma'_{NLO}$ in the last column of Table I which should be regarded as the final predictions in this work.

Since the binding energy of the $T^{*+}_{cc}$ is uncertain, we further give the partial widths of $T^{*}_{cc} \rightarrow D^* D \pi$ with the binding energy varying from 0.01 MeV to 0.80 MeV in Fig. 3 where the red dashed lines show the decay widths from the tree-level diagram, the blue dotdashed lines show the LO decay width including the tree-level and the $D^*D$ rescattering contributions, and the black solid lines show the decay width including the corrections from the $D^*\pi$ rescattering to the LO results. To see the contributions of the $D^*D$ and $D^*\pi$ rescattering to the decay widths more clearly, the corrections from the $D^*D$ and $D^*\pi$ FSIs with the binding energy being 0.01 $\sim$ 0.80 MeV are demonstrated by the blue dotdashed lines and black solid lines in Fig. 4 respectively. One can see that the isoscalar $D^*D$ FSI indeed contributes at the same order as the tree-level diagram for different $T^{*}_{cc}$ binding energies, and the $D^*\pi$ FSI keeps contributing at NLO.
In this paper, we calculated the contributions of the $D^*D$ and $D^*\pi$ rescattering, as well as that of the $D^*$ selfenergy, to the partial decay widths of the $T_{cc}^{*+} \rightarrow D^{*+}D^0\pi^0$ through the XEFT assuming that the $T_{cc}^{*+}$ is a $D^{*+}D^{*0}$ shallow bound state. We found that the $I=0$ $D^{*+}D^0/D^{*0}D^+$ rescattering, which generates a $T_{cc}^{*+}$ pole just below the threshold, contributes at LO and has a significant impact on the partial widths of the $T_{cc}^{*+} \rightarrow D^{*+}D^0\pi^0$ and $T_{cc}^{*+} \rightarrow D^{*0}D^+\pi^0$. The corrections from the $D^*\pi$ rescattering to the LO result are marginal, at the level of 10%, and the corrections from the $D^*$ selfenergy are smaller than 1%. Being an isoscalar $1^+ D^{*+}D^{*0}$ molecular state, the $T_{cc}^{*+}$ should decay dominantly into the three $D^*D\pi$ channels calculated here. Summing up their partial widths gives an estimate of the $T_{cc}^{*+}$ hadronic width to be about 90 keV; thus the total width of the $T_{cc}^{*+}$ is sizably larger than that of the $T_{cc}^{*+}$. The result reported here should be useful for searching the $T_{cc}^{*+}$ state at LHCb in the future.

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Appendix A: Three-body phase space

In this section, we derive the three-body phase space in Eq. (16) in the rest frame of the initial particle. The three-body phase space can be written as

\[ \int d\Phi_3 (P; p_1, p_2, p_3) = \int (2\pi)^4 \delta^4 (P - p_1 - p_2 - p_3) \frac{d^3 \vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3 \vec{p}_2}{(2\pi)^3 2E_2} \frac{d^3 \vec{p}_3}{(2\pi)^3 2E_3} \]

\[ = \int \frac{1}{(2\pi)^5} \delta (E - E_1 - E_2 - E_3) \frac{d^3 \vec{p}_1}{2E_1} \frac{d^3 \vec{p}_2}{2E_2} \frac{d^3 \vec{p}_3}{2E_3} \]

\[ = \int \frac{1}{(2\pi)^5} \delta (E - E_1 - E_2 - E_3) \frac{|\vec{p}_1| |\vec{p}_2| |\vec{p}_3|}{4E_1} \frac{1}{2E_2} \frac{1}{2E_3}, \]

where \( P = (M, \vec{0}) \), \( d^3 \vec{p}_1 = |\vec{p}_1|^2 d\Omega_1 \) and \( d^3 \vec{p}_2 = |\vec{p}_2|^2 d\Omega_1 \). Here \( \Omega_1 \) is the solid angle between the moving directions of particle 1 and particle 2, \( d\Omega_1 = \sin \theta_1 d\theta_1 d\varphi_1 \), where \( \theta_1 \) is the angle between particles 1 and 2 and is related to the three-momenta as

\[ |\vec{p}_3|^2 = |\vec{p}_1|^2 + |\vec{p}_2|^2 - 2 \cos \theta_1 |\vec{p}_1| |\vec{p}_2|. \]  

The integration over \( \theta_1 \) in Eq. (A1) can be changed to the integration over \( E_3 \) through

\[ d|\vec{p}_3|^2 = 2|\vec{p}_1| |\vec{p}_2| \sin \theta_1 d\theta_1 dE_3 = 2E_3 dE_3, \]

and the three-body phase space reads

\[ \int d\Phi_3 (P; p_1, p_2, p_3) = \int \frac{1}{(2\pi)^5} \delta (E - E_1 - E_2 - E_3) \frac{|\vec{p}_1| |\vec{p}_2|}{16E_1 E_2} d\Omega_1 \frac{d|\vec{p}_3|^2}{2|\vec{p}_1||\vec{p}_2|} d\varphi_1 \frac{1}{2E_3} d|\vec{p}_1|^2 d|\vec{p}_2|^2 \]

\[ = \int \frac{1}{(2\pi)^5} \delta (E - E_1 - E_2 - E_3) \frac{|\vec{p}_1| |\vec{p}_2|}{16E_1 E_2} d\Omega_1 \frac{2E_3}{2|\vec{p}_1||\vec{p}_2|} d\varphi_1 \frac{1}{2E_3} d|\vec{p}_1|^2 d|\vec{p}_2|^2 \]

\[ = \int \frac{1}{32\pi^3} \frac{1}{4E_1 E_2} d|\vec{p}_1|^2 d|\vec{p}_2|^2, \]

For the final-state particles being nonrelativistic, one has \( E_1 \simeq M_1, E_2 \simeq M_2 \), and

\[ \int d\Phi_3 = \int \frac{1}{32\pi^3} \frac{1}{4M_1 M_2} d|\vec{p}_1|^2 d|\vec{p}_2|^2. \]

Appendix B: Isospin phase conventions and contact interactions

In this section, we give the isospin phase conventions in our calculation and derive the couplings \( C_{3/2}^{1/2} \) and \( C_{1/2}^{3/2} \) in Eq. (3) from the \( D^* \pi \) scattering lengths. The isospin phase conventions for \( D^* \)
and π are [25]

\[ |\pi^+\rangle = -|1, +1\rangle, \quad |\pi^0\rangle = |1, 0\rangle, \quad |\pi^-\rangle = |1, -1\rangle, \]

\[ |D^{*+}\rangle = |D^+\rangle = \left| \frac{1}{2}, +\frac{1}{2} \right\rangle, \quad |D^{*0}\rangle = |D^0\rangle = \left| \frac{1}{2}, -\frac{1}{2} \right\rangle, \quad (B1) \]

where the right-hand side represents states \(|I, I_3\rangle\) in the isospin basis with \(I\) and \(I_3\) the isospin and its third component, respectively. For the derivation of the contact interactions between the \(D^*\) and \(D\), all the couplings can be expressed in terms of two couplings, \(C_{0D}\) with \(I = 0\) and \(C_{1D}\) with \(I = 1\). The \(|D^*D\rangle\) states can be expressed in terms of the isospin basis as

\[ |D^{*+}D^0\rangle = \sqrt{\frac{1}{2}} |1,0\rangle + \sqrt{\frac{1}{2}} |0,0\rangle, \quad (B2) \]

\[ |D^{*0}D^+\rangle = \sqrt{\frac{1}{2}} |1,0\rangle - \sqrt{\frac{1}{2}} |0,0\rangle, \quad (B3) \]

\[ |D^{*0}D^0\rangle = |1, -1\rangle. \quad (B4) \]

The \(D^*D\) amplitude can be written in terms of the amplitudes with the total isospin \(I = 1\) and \(I = 0\) as

\[ \langle D^{*+}D^0|T|D^{*+}D^0\rangle = \frac{1}{2} \langle D^*D|T|D^*D\rangle_{I=1} + \frac{1}{2} \langle D^*D|T|D^*D\rangle_{I=0}, \quad (B5) \]

\[ \langle D^{*+}D^0|T|D^{*0}D^+\rangle = \frac{1}{2} \langle D^*D|T|D^*D\rangle_{I=1} - \frac{1}{2} \langle D^*D|T|D^*D\rangle_{I=0}, \quad (B6) \]

\[ \langle D^{*0}D^+|T|D^{*0}D^+\rangle = \frac{1}{2} \langle D^*D|T|D^*D\rangle_{I=1} + \frac{1}{2} \langle D^*D|T|D^*D\rangle_{I=0}, \quad (B7) \]

\[ \langle D^{*0}D^+|T|D^{*+}D^0\rangle = \frac{1}{2} \langle D^*D|T|D^*D\rangle_{I=1} - \frac{1}{2} \langle D^*D|T|D^*D\rangle_{I=0}, \quad (B8) \]

\[ \langle D^{*0}D^0|T|D^{*0}D^0\rangle = \langle D^*D|T|D^*D\rangle_{I=1}, \quad (B9) \]

which give the expressions of \(C_{0D1}, C_{0D2}, C_{D3}, C_{0D1ex}\) and \(C_{0D2ex}\) in terms of \(C_{0D}\) and \(C_{1D}\) as

\[ C_{0D1} = \frac{1}{2}C_{1D} + \frac{1}{2}C_{0D}, \quad (B10) \]

\[ C_{0D1ex} = \frac{1}{2}C_{1D} - \frac{1}{2}C_{0D}, \quad (B11) \]

\[ C_{0D2} = \frac{1}{2}C_{1D} + \frac{1}{2}C_{0D}, \quad (B12) \]

\[ C_{0D2ex} = \frac{1}{2}C_{1D} - \frac{1}{2}C_{0D}, \quad (B13) \]

\[ C_{D3} = C_{1D}. \quad (B14) \]

Here \(C_{1D}\) is the \(D^*D\) contact interaction with \(I = 1\) and is neglected in our calculation as there is no isovector exotic state like the \(T_{cc}\) near the \(D^*D\) threshold, \(C_{0D1}, C_{0D2}\) and \(C_{D3}\) are the contact
couplings for $D^* D^0 \rightarrow D^* D^0$, $D^* D^+ \rightarrow D^+ D^0$ and $D^* D^0 \rightarrow D^* D^0$, respectively, and $C_{0D1\text{ex}}$ and $C_{0D2\text{ex}}$ are the contact couplings for $D^0 D^+ \rightarrow D^* D^0$ and $D^* D^0 \rightarrow D^* D^+$, respectively.

For the derivation of the contact interactions between the $D^*$ and $\pi$, all the couplings can be expressed in terms of two couplings, $C_{\frac{3}{2}\pi}$ with $I = \frac{3}{2}$ and $C_{\frac{1}{2}\pi}$ with $I = \frac{1}{2}$, and the two couplings can be obtained by matching the $D^* \pi$ scattering amplitude at the $D^* \pi$ threshold,

$$
\sqrt{2m_D^* 2m_{D^*} 2m_D} \frac{C_{\frac{3}{2}\pi}}{2m_{D^*}} = A_{D^{*}\pi}^I \left( \sqrt{s} = m_{D^*} + m_{\pi} \right) = 8\pi (m_{D^*} + m_{\pi}) a_{D^{*}\pi}^I,
$$

where $I = \frac{3}{2}, \frac{1}{2}$, and $a_{D^{*}\pi}^I$ is the $D^* \pi$ scattering length with isospin $I$. By using the central values of the scattering lengths $a_{D^{*}\pi}^{3/2} = a_{D^{*}\pi}^{3/2} = -(0.100 \pm 0.002)$ fm and $a_{D^{*}\pi}^{1/2} = a_{D^{*}\pi}^{1/2} = 0.37^{+0.03}_{-0.02}$ fm given in Ref. [37], we have

$$
C_{\frac{3}{2}\pi} = -6.8 \text{ GeV}^{-1},
$$

$$
C_{\frac{1}{2}\pi} = 25.2 \text{ GeV}^{-1}.
$$

The $|D^* \pi\rangle$ states can be expressed in terms of the isospin basis as

$$
-D^{*0} \pi^+ = \sqrt{\frac{3}{2}} \left( \frac{1}{2}, \frac{1}{2} \right) + \sqrt{\frac{1}{2}} \left( 0, \frac{1}{2} \right),
$$

$$
|D^{*0} \pi^0\rangle = \sqrt{\frac{3}{2}} \left( \frac{1}{2}, -\frac{1}{2} \right) + \sqrt{\frac{1}{2}} \left( 0, -\frac{1}{2} \right),
$$

$$
|D^{*+} \pi^0\rangle = \sqrt{\frac{3}{2}} \left( \frac{1}{2}, 0 \right) - \sqrt{\frac{1}{2}} \left( 0, 0 \right),
$$

$$
|D^{*+} \pi^-\rangle = \sqrt{\frac{3}{2}} \left( \frac{1}{2}, -\frac{1}{2} \right) - \sqrt{\frac{1}{2}} \left( 0, -\frac{1}{2} \right).
$$

The $D^* \pi$ amplitude can be written in terms of the amplitudes with total isospin $I = \frac{3}{2}$ and $I = \frac{1}{2}$ as

$$
\langle D^{*+} \pi^0 | T | D^{*+} \pi^0 \rangle = \frac{2}{3} \langle D^{*+} \pi^0 | T | D^{*+} \pi^0 \rangle_{I = \frac{3}{2}} + \frac{1}{3} \langle D^{*+} \pi^0 | T | D^{*+} \pi^0 \rangle_{I = \frac{1}{2}},
$$

$$
\langle D^{*+} \pi^0 | T | D^{*0} \pi^+ \rangle = -\frac{\sqrt{2}}{3} \langle D^{*+} \pi^0 | T | D^{*0} \pi^+ \rangle_{I = \frac{3}{2}} + \frac{\sqrt{2}}{3} \langle D^{*+} \pi^0 | T | D^{*0} \pi^+ \rangle_{I = \frac{1}{2}},
$$

$$
\langle D^{*0} \pi^0 | T | D^{*+} \pi^0 \rangle = \frac{2}{3} \langle D^{*0} \pi^0 | T | D^{*+} \pi^0 \rangle_{I = \frac{3}{2}} + \frac{1}{3} \langle D^{*0} \pi^0 | T | D^{*+} \pi^0 \rangle_{I = \frac{1}{2}},
$$

$$
\langle D^{*0} \pi^0 | T | D^{*+} \pi^- \rangle = \frac{\sqrt{2}}{3} \langle D^{*0} \pi^0 | T | D^{*+} \pi^- \rangle_{I = \frac{3}{2}} - \frac{\sqrt{2}}{3} \langle D^{*0} \pi^0 | T | D^{*+} \pi^- \rangle_{I = \frac{1}{2}},
$$

$$
\langle D^{*+} \pi^+ | T | D^{*0} \pi^+ \rangle = \frac{1}{3} \langle D^{*+} \pi^+ | T | D^{*0} \pi^+ \rangle_{I = \frac{3}{2}} + \frac{2}{3} \langle D^{*+} \pi^+ | T | D^{*0} \pi^+ \rangle_{I = \frac{1}{2}},
$$

$$
\langle D^{*0} \pi^+ | T | D^{*+} \pi^0 \rangle = -\frac{\sqrt{2}}{3} \langle D^{*0} \pi^+ | T | D^{*+} \pi^0 \rangle_{I = \frac{3}{2}} + \frac{\sqrt{2}}{3} \langle D^{*0} \pi^+ | T | D^{*+} \pi^0 \rangle_{I = \frac{1}{2}}.
$$
which give the expressions of $C_{\pi 1}, C_{\pi 2}, C_{\pi 3}, C_{\pi 1\text{ex}}, C_{\pi 2\text{ex}}$ and $C_{\pi 3\text{ex}}$ in terms of $C_{\frac{1}{2}\pi}$ and $C_{\frac{1}{4}\pi}$ as

$$C_{\pi 1} = \frac{2}{3} C_{\frac{1}{4}\pi} + \frac{1}{3} C_{\frac{1}{2}\pi} = 4.1 \text{ GeV}^{-1},$$ (B28)

$$C_{\pi 1\text{ex}} = -\sqrt{2} \frac{2}{3} C_{\frac{1}{4}\pi} + \frac{\sqrt{2}}{3} C_{\frac{1}{2}\pi} = 15.1 \text{ GeV}^{-1},$$ (B29)

$$C_{\pi 2} = \frac{2}{3} C_{\frac{1}{4}\pi} + \frac{1}{3} C_{\frac{1}{2}\pi} = 4.1 \text{ GeV}^{-1},$$ (B30)

$$C_{\pi 2\text{ex}} = \sqrt{2} C_{\frac{1}{4}\pi} - \sqrt{2} \frac{2}{3} C_{\frac{1}{2}\pi} = -15.1 \text{ GeV}^{-1},$$ (B31)

$$C_{\pi 3} = \frac{1}{3} C_{\frac{1}{4}\pi} + \frac{3}{3} C_{\frac{1}{2}\pi} = 14.4 \text{ GeV}^{-1},$$ (B32)

$$C_{\pi 3\text{ex}} = -\sqrt{2} \frac{2}{3} C_{\frac{1}{4}\pi} + \sqrt{2} \frac{3}{3} C_{\frac{1}{2}\pi} = 15.1 \text{ GeV}^{-1}.$$ (B33)

Here $C_{\pi 1}, C_{\pi 2},$ and $C_{\pi 3}$ are the contact interactions for $D^{*+}\pi^0 \to D^{*+}\pi^0$, $D^{*0}\pi^0 \to D^{*0}\pi^0$, and $D^{*0}\pi^+ \to D^{*0}\pi^+$, respectively, and $C_{\pi 1\text{ex}}, C_{\pi 2\text{ex}}$ and $C_{\pi 3\text{ex}}$ are the contact interactions for $D^{*0}\pi^+ \to D^{*+}\pi^0$, $D^{*+}\pi^- \to D^{*0}\pi^0$, and $D^{*+}\pi^0 \to D^{*0}\pi^+$, respectively.

**Appendix C: 3-point loop integrals**

In the rest frame of the decay particle, the scalar 3-point loop integral is ultraviolet (UV) convergent and can be worked out as [42]

\[
I(q) = i \int \frac{d^dl}{(2\pi)^d} \frac{1}{(p^0 - m_1 - \frac{p^2}{2m_1} + i\epsilon)(M - p^0 - m_2 - \frac{p^2}{2m_2} + i\epsilon)} \left( \frac{p^0 - q^0 - m_3 - \frac{(i\cdot q)^2}{2m_3} + i\epsilon}{p^0 - q^0 - m_3 - \frac{(i\cdot q)^2}{2m_3} + i\epsilon} \right)
\]

\[= 4\mu_{12}\mu_{23} \int \frac{d^{d-1}l}{(2\pi)^{d-1}} \left( \frac{b_{12} + \frac{p^2}{2m_1} - i\epsilon}{b_{23} + \frac{p^2}{2m_2} + \frac{(i\cdot q)^2}{2m_3} - i\epsilon} \right)\]

\[= \frac{4\mu_{12}\mu_{23}}{(4\pi)^{(d-1)/2}} \Gamma \left( \frac{5-d}{2} \right) \int_0^1 dx \frac{1}{\left[ \frac{5-d}{2} - ax^2 + (c_2 - c_1) x + c_1 - i\epsilon \right]^{(d-5)/2}}\]

\[= \frac{\mu_{12}\mu_{23}}{2\pi} \frac{1}{\sqrt{a}} \left[ \tan^{-1} \left( \frac{c_2 - c_1}{2\sqrt{ac_1}} \right) + \tan^{-1} \left( \frac{2a + c_1 - c_2}{2\sqrt{a(c_2 - a)}} \right) \right],\] (C1)

where $\mu_{ij} = m_i m_j / (m_i + m_j)$ are the reduced masses, $b_{12} = m_1 + m_2 - M, b_{23} = m_2 + m_3 + q^0 - M,$ and

\[a = \left( \frac{\mu_{23}}{m_3} \right)^2 \frac{q^2}{2}, \quad c_1 = 2\mu_{12}b_{12}, \quad c_2 = 2\mu_{23}b_{23} + \frac{\mu_{23}q^2}{m_3} .\] (C2)
There is no pole for the spacetime dimension \( d \leq 4 \), and we have taken \( d = 4 \) in the last step of Eq. (C1).

We also need the vector integral which is defined as

\[
q^i I^{(1)}(q) = \int \frac{d^3 l}{(2\pi)^3} \left( b_{12} + \frac{\hat{p}_{12}}{2m_{12}} - i\epsilon \right) \left[ b_{23} + \frac{\hat{p}_{23}}{2m_{23}} + \frac{(l-q)^2}{2m_3} - i\epsilon \right],
\]

and \( I^{(1)}(q) \) can be expressed in terms of the scalar 2-point and 3-point loop integrals as

\[
I^{(1)}(q) = \frac{\mu_{23}}{a m_3} \left[ B(c_2 - a) - B(c_1) + \frac{1}{2} (c_2 - c_1) I(q) \right],
\]

where the two-point function \( B(c) = 2\mu_{12}\mu_{23}\Sigma(c) \), with \( \Sigma(c) \) defined in the power divergence substraction (PDS) scheme [43] as

\[
\Sigma(c) \equiv \left( \frac{\Lambda_{\text{PDS}}}{2} \right)^{4-d} \int \frac{d^{d-1} l}{(2\pi)^{d-1}} \frac{1}{l^2 + c - i\epsilon},
\]

\[
= \left( \frac{\Lambda_{\text{PDS}}}{2} \right)^{4-d} \left( 4\pi \right)^{(1-d)/2} \Gamma \left( \frac{3-d}{2} \right) (c - i\epsilon)^{(d-3)/2},
\]

\[
= \frac{1}{4\pi} \left( \Lambda_{\text{PDS}} - \sqrt{c - i\epsilon} \right),
\]

where \( \Lambda_{\text{PDS}} \) is the sharp cutoff to regulate the UV divergence in the two-point scalar loop integral, and \( I^{(1)}(q) \) is also UV convergent. When the particles in the 3-point integrals are unstable, e.g., considering the \( D^* \) selfenergy contribution shown in Eq. (11), one needs to include their widths by replacing

\[
m_k \rightarrow m_k - i\frac{\Gamma_k}{2}, \quad k = 1, 2, 3,
\]

and in the loop integrals \( I(q) \) and \( I^{(1)}(q) \), one just makes the replacements for \( c_1 \) and \( c_2 \) as

\[
c_1 \rightarrow c_1 - i\mu_{12}(\Gamma_1 + \Gamma_2), \quad c_2 \rightarrow c_1 - i\mu_{12}(\Gamma_2 + \Gamma_3).
\]

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