SUBSTRUCTURED TWO-LEVEL AND MULTILEVEL DOMAIN DECOMPOSITION METHODS

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Abstract. Two-level domain decomposition methods are very powerful techniques for the efficient numerical solution of partial differential equations (PDEs). A two-level domain decomposition method requires two main components: a one-level preconditioner (or its corresponding smoothing iterative method), which is based on domain decomposition techniques, and a coarse correction step, which relies on a coarse space. The coarse space must properly represent the error components that the chosen one-level method is not capable to deal with. In the literature most of the works introduced efficient coarse spaces obtained as the span of functions defined on the entire space domain of the considered PDE. Therefore, the corresponding two-level preconditioners and iterative methods are defined in volume.

In this paper, a new class of substructured two-level methods is introduced, for which both domain decomposition smoothers and coarse correction steps are defined on the interfaces. This approach has several advantages. On the one hand, the required computational effort is cheaper than the one required by classical volumetric two-level methods. On the other hand, it allows one to use some of the well-known efficient coarse spaces proposed in the literature. Moreover, our new substructured framework can be efficiently extended to a multi-level framework, which is always desirable when the high dimension or the scarce quality of the coarse space prevent the efficient numerical solution. Numerical experiments demonstrate the effectiveness of the proposed new numerical framework.

Key words. domain decomposition methods; Schwarz methods; substructured methods; two-level methods; coarse correction; multigrid methods; Laplace equation.

AMS subject classifications.

1. Introduction. Consider a linear problem of the form $Au = f$, which we assume well posed in a vector space $V$. To define a two-level method for the solution to this problem, a one-level method and a coarse-correction step are required.

One-level methods are generally based on a splitting technique: the operator $A : V \to V$ is decomposed as $A = M - N$, where $M : V \to V$ is assumed invertible. This splitting leads to a stationary iteration, namely $u^{k+1} = M^{-1}Nu^k + M^{-1}f$, for $k = 0, 1, \ldots$, and to a preconditioned system $M^{-1}Au = M^{-1}f$. These are strongly related, since the stationary iteration, if it converges, produces the solution of the preconditioned system; see, e.g., [7] and references therein. Notice that we have tacitly used the term “method” with different meanings. On the one hand, a stationary method is a fixed-point iteration method whose goal is to obtain the solution $u$. On the other hand, a preconditioner is a transformation method that aims at transforming the considered system to a new better conditioned one. Indeed, when talking about preconditioning, it is always implicitly assumed that the preconditioned system is solved by a Krylov iteration. Similarly, a Krylov method can be used to accelerate a stationary iteration method. For one-level methods (based on the same operator $M$), a precise relation makes these two solution strategies equivalent. However, a significant difference appears when considering two-level methods. Notice that one-level domain decomposition (DD) methods can be generally obtained by a splitting $A = M - N$. Hence, they can be used as stationary iterations or preconditioners; see, e.g., [43, 44, 18, 45, 49, 11, 25]. Unfortunately, DD methods are in general not scalable and a coarse correction step is often desirable. See, e.g., [4, 8, 9, 10, 14, 15] for exceptions and detailed scalability and non-scalability analyses.

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A two-level method is characterized by the combination of a classical one-level method, defined on $V$, and a coarse correction step, performed on a coarse space $V_c$. The coarse space $V_c$ is finite-dimensional and it must satisfy the condition $\dim V_c \ll \dim V$. The mappings between $V$ and $V_c$ are realized by a restriction operator $R : V \rightarrow V_c$ and a prolongation operator $P : V_c \rightarrow V$. In general, the restriction of $A : V \rightarrow V$ on $V_c$ is defined as $A_c = RAP$, which is assumed to be an invertible matrix. Now, we distinguish two cases: a two-level stationary method and a two-level preconditioning method. In the first case, a stationary method is used as first-level method. After each stationary iteration, which produces an approximation $u_{\text{app}}$, the residual $r = f - Au_{\text{app}}$ is mapped from $V$ to $V_c$, the coarse problem $A_c e = Rr$ is solved to get $e \in V_c$, and the coarse correction step is defined as $u_{\text{new}} = u_{\text{app}} + Pe$. This correction provides the new approximation $u_{\text{new}}$. By repeating these operations iteratively, one gets a two-level stationary method. Notice that this idea is very much related to two-grid methods. In the second case, the first-level method is purely a preconditioner $M^{-1}$. The corresponding two-level preconditioning method is

$$M_{2L} := M^{-1} + PA_c^{-1}R,$$

which is clearly obtained in an additive way: the one-level preconditioner $M^{-1}$ is added to the coarse correction matrix $PA_c^{-1}R$.

A simple calculation reveals that a two-level stationary method leads to a preconditioner $M_{s,2L}$ given by

$$M_{s,2L} = M^{-1} + PA_c^{-1}R + PA_c^{-1}RAM^{-1}.$$  

When used with appropriate implementations, the two preconditioners $M_{2L}$ and $M_{s,2L}$ require about the same computational effort per Krylov iteration. However, their different structures can lead to different performances of Krylov methods.

The literature about two-level DD methods is very rich. See, e.g., [5, 6, 12, 20, 27, 28, 29, 31, 32, 33], for references that are based on one-level DD stationary methods, and, e.g., [1, 2, 17, 19, 21, 22, 23, 30, 41, 40, 46, 47, 50], for references that are based on one-level DD preconditioners. See also general classical references as [18, 45, 49] and [36, 38].

For any given one-level DD method (stationary or preconditioning), the choices of $V_c$, $P$ and $R$ influence very strongly the convergence behavior of the corresponding two-level method. For this reason, the main focus of all the references mentioned above is the definition of different coarse spaces and new strategies to build coarse space functions, leading to efficient two-level DD stationary and preconditioning methods. Despite the mentioned references consider several one-level DD methods and different partial differential equation (PDE) problems, it is still possible to classify them in two main groups. These depend on the idea governing the definition of the coarse space. To explain it, let us consider a DD iterative method (e.g., RAS) applied to a well-posed PDE problem. Errors and residuals of the DD iterative procedure have generally very special forms. The errors are harmonic, in the sense of the underlying PDE operator, in the interior of the subdomains (excluding the interfaces). Moreover, the errors are predominant in the overlaps. The residuals are predominant on the interfaces and zero outside the overlap. For examples and more details, see, e.g., [29, 13, 12]. This difference motivated, sometimes implicitly, the construction of different coarse spaces. On the one hand, many references use different techniques to define coarse functions in the overlap (where the error is predominant), and then extending them on the remaining part of the neighboring subdomains; see, e.g., [17, 19, 21, 22, 23,
On the other hand, in other works the coarse space is created by first defining basis function on the interfaces (where the residual is non-zero), and then extending them (in different ways) on the portions of the neighboring subdomains; see, e.g., [1, 2, 5, 6, 12, 27, 30, 29, 31, 32, 40, 33]. For a good, compact and complete overview of several of the different coarse spaces present in the literature, we refer to [40, Section 5]. For other different techniques and other related discussions, see, e.g., [18, 20, 27, 28, 35, 50].

The scenario is actually even more complicated, because different one-level DD methods are used (e.g., overlapping and non-overlapping methods) and different PDEs are considered. However, the classifications we used so far are sufficiently accurate to allow us to give a precise description of the novelties of our work. We introduce for the first time so-called two-level DD substructured methods. These are two-level stationary iterative methods and the term “substructured” indicates that iterations and coarse spaces are defined on the interfaces. With this respect, they are defined in the same spirit as two-level methods whose coarse spaces are extensions in volume of interfaces basis functions. Moreover, they share some similarities with the two-level methods designed in [12] for the solution of PDEs on perforated domains.

We distinguish two different two-level substructured DD methods. The first one is the Spectral 2-level Substructured (S2S) method, for which the coarse space is obtained as the span of certain interface functions. A good choice would be to use the eigenfunctions of the one-level iteration operator corresponding to the highest in modulus eigenvalues. However, the S2S framework allows one to choose arbitrarily the coarse space functions, as, e.g., the ones proposed in several papers as [30, 29, 31, 40]. The S2S method is discussed and analyzed in Section 3. The S2S framework has several advantages if compared to a classical two-level DD method defined in volume. Since the coarse space functions are defined on the interfaces, less memory storage is required. For a three-dimensional problem with mesh size $h$, a discrete interface coarse function is an array of size $O(1/h^2)$. This is much smaller than $O(1/h^3)$, which is the size of an array corresponding to a coarse function in volume. For this reason the resulting interface restriction and prolongation operators are much smaller matrices. It is then clear that also the corresponding interpolation operations are much cheaper to be performed and possibly easier to be implemented. Therefore, assuming that the one-level stationary iteration step and the dimension of the coarse space are the same for S2S and a method in volume, each S2S iteration is generally computationally less expensive. In terms of iteration number, our S2S method performs similarly or faster than other two-level methods that use a DD smoother. Notice also, that the pre-computation part, that consists mainly in constructing the coarse space $V_c$ and assembling the operators $P$, $R$ and $A_c$ requires the same computational effort of a method in volume. As for other two-level DD methods, the main drawback of our S2S method is related to the solution of the coarse problem. As the dimension of the coarse space increases, the computational effort required for the solution of the coarse problem can dominate the other costs. This happens also if a good choice of coarse functions is not available or expensive to compute. This reason motivated our second new method. Inspired by the S2S method, we introduced the so-called Geometric 2-level Substructured (G2S) method, for which the coarse space is not explicitly constructed. The G2S method is essentially a two-grid interface method, for which the coarse correction is performed on a coarser interface grids. It is clear that the G2S framework does not require the explicit knowledge of coarse space functions. The G2S method has the same advantages of the S2S method. In addition, it does not require the explicit construction of a coarse space, and it allows a multilevel extension,
which is desirable when the dimension of the coarse space becomes too large. The G2S method is discussed and analyzed in Section 4.

This paper is organized as follows. In Section 2, we formulate the classical parallel Schwarz method in a substructured form. This is done at the continuous level and represents the starting point for the S2S method introduced in Section 3. In particular, the S2S method is described in Section 3.1. A convergence analysis is given in Section 3.2. Section 4 is devoted to the G2S method, which is described in Section 4.1. We prove convergence results and relations between G2S and S2S in Section 4.2. Section 5 focuses on the theoretical relation between substructured two-level DD methods and two-level DD methods defined in volume. In Section 6, we discuss implementation details and multilevel extensions of the G2S method. Numerical experiments are shown and discussed in Section 7. Finally, we present our conclusions in Section 8.

2. Substructured domain decomposition methods. Consider a bounded Lipschitz domain \( \Omega \subset \mathbb{R}^d \) for \( d \in \{2, 3\} \), a general second-order linear elliptic operator \( \mathcal{L} \) and a function \( f \in L^2(\Omega) \). Our goal is to introduce new domain-decomposition based methods for the efficient numerical solution of the general linear elliptic problem

\[
\mathcal{L} u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,
\]

which we assume to be uniquely solved by \( u \in H^1_0(\Omega) \).

To formulate our methods, an overlapping decomposition \( \Omega = \Omega_1 \cup \Omega_2 \) is considered, where \( \Omega_1 \) and \( \Omega_2 \) are two Lipschitz domains. The sets \( \Gamma_1 := \partial \Omega_1 \setminus \partial \Omega \) and \( \Gamma_2 := \partial \Omega_2 \setminus \partial \Omega \) are the two interfaces. An example is given in Figure 2.1. A classical result in domain decomposition theory is that (2.1) is equivalent to the system

\[
\begin{align*}
\mathcal{L} u_1 &= f \quad \text{in } \Omega_1, \quad u_1 = u_2 \quad \text{on } \Gamma_1, \\
\mathcal{L} u_2 &= f \quad \text{in } \Omega_2, \quad u_2 = u_1 \quad \text{on } \Gamma_2,
\end{align*}
\]

in the sense that the unique solution \( (u_1, u_2) \in H^1(\Omega_1) \times H^1(\Omega_2) \) of (2.2) satisfies \( u_1 = u_{|\Omega_1} \) and \( u_2 = u_{|\Omega_2} \); see, e.g., [45, 49, 25] and references therein. We wish to formulate (2.2) in a substructured form. To do so, let us recall the famous Lions-Magenes space. Let \( v \in H^{1/2}(\Gamma_j) \) be any trace function defined on \( \Gamma_j \), we denote by \( \tilde{v} \) the extension by zero of \( v \) on \( \partial \Omega_j \), \( j = 1, 2 \). The spaces \( H^{1/2}_0(\Gamma_j), \ j = 1, 2 \), are then define as

\[
H^{1/2}_0(\Gamma_j) = \{ v \in H^{1/2}(\Gamma_j) : \tilde{v} \in H^{1/2}(\partial \Omega_j) \}.
\]
For more details about these space and different equivalent definition see [48, Section 33] and [42, page 66]. See also [45, page 7]. Consider the trace operators \( \tau_j : H^1(\Omega_{3-j}) \to H^{1/2}(\Gamma_j) \) for \( j = 1,2 \), see, e.g., [16], and define the extension operators \( E_j : H^{1/2}_{00}(\Gamma_j) \times L^2(\Omega_j) \to H^1(\Omega_j) \) by \( \tilde{u}_1 = E_1(v_1, f_1) \) and \( \tilde{u}_2 = E_2(v_2, f_2) \), where \( \tilde{u}_1 \) and \( \tilde{u}_2 \) solve the equations

\begin{equation}
L\tilde{u}_j = f_j \text{ in } \Omega_j, \quad \tilde{u}_j = v_j \text{ on } \Gamma_j,
\end{equation}

with \( f_j := f|_{\Omega_j} \) for \( j = 1,2 \). We introduce the maps \( \hat{G}_j : H^{1/2}_{00}(\Gamma_j) \times L^2(\Omega_j) \to H^{1/2}_{00}(\Gamma_{3-j}) \), \( j = 1,2 \), as

\begin{equation}
(v_j, f_j) \mapsto \hat{G}_j(v_j, f_j) := \tau_{3-j}(E_j(v_j, f_j)).
\end{equation}

Since the problems (2.3) are linear with respect to \( f_j \) and \( v_j \), we can write

\( \hat{G}_j(v, f) = \hat{G}_j(v, 0) + \hat{G}_j(0, f_j) \), \( j = 1,2 \). Using \( \hat{G}_1 \) and \( \hat{G}_2 \), the system (2.2) can be written as

\begin{equation}
\hat{G}_1(v_{12}, f_1) = v_{12}, \quad \hat{G}_2(v_{12}, f_2) = v_{21}.
\end{equation}

The equivalence between (2.2) and (2.4) is explained by the following theorem.

**Theorem 2.1** (Substructured problem and problem in volume). Let the pair \((u_1, u_2) \in H^1(\Omega_1) \times H^1(\Omega_2)\) solve (2.2), then the pair \((v_{12}, v_{21}) := (\tau_2(u_1), \tau_1(u_2))\) solves (2.4). Let \((v_{12}, v_{21}) \in H^{1/2}_{00}(\Gamma_2) \times H^{1/2}_{00}(\Gamma_1)\) solves (2.4), then \((u_1, u_2) := (E_1(v_{21}, f_1), E_2(v_{12}, f_2))\) solves (2.2).

**Proof.** Let \((u_1, u_2) \) solve (2.2) and \((v_{12}, v_{21}) := (\tau_2(u_1), \tau_1(u_2))\). We have

\[
\begin{aligned}
\hat{G}_1(v_{21}, f_1) &= \tau_2(E_1(v_{21}, f_1)) = \tau_2(\tau_1(u_2)) = \tau_2(u_1) = v_{12}, \\
\hat{G}_2(v_{12}, f_2) &= \tau_1(E_2(v_{12}, f_2)) = \tau_1(\tau_2(u_1)) = \tau_1(u_2) = v_{21}.
\end{aligned}
\]

Hence \((v_{12}, v_{21})\) solves (2.4). Consider the pair \((v_{12}, v_{21})\) that solves (2.4) and define \((u_1, u_2) := (E_1(v_{21}, f_1), E_2(v_{12}, f_2))\). Then \((u_1, u_2)\) solves the problems

\[
\begin{aligned}
Lu_1 &= f_1 \text{ in } \Omega_1, \quad u_1 = v_{12} \text{ on } \Gamma_1, \\
Lu_2 &= f_2 \text{ in } \Omega_2, \quad u_2 = v_{12} \text{ on } \Gamma_2.
\end{aligned}
\]

The result follows by recalling that \( f_j = f \) in \( \Omega_j \) for \( j = 1,2 \), \( u_1 = v_{21} = u_2 \) on \( \Gamma_1 \) and \( u_2 = v_{12} = u_1 \) on \( \Gamma_2 \).

Using the properties of the operators \( \hat{G}_j, j = 1,2 \), we get

\begin{equation}
\begin{aligned}
v_{12} - \hat{G}_1(v_{21}, 0) &= \hat{G}_1(0, f_1), \\
v_{21} - \hat{G}_2(v_{12}, 0) &= \hat{G}_2(0, f_2).
\end{aligned}
\end{equation}

Take any function \( w \in H^1_0(\Omega) \) that leads to the initialization \( u_1^0 := w|_{\Omega_1} \) and \( u_2^0 := w|_{\Omega_2} \). The parallel Schwarz method (PSM) is the given by

\[
\begin{aligned}
Lu_1^n &= f \text{ in } \Omega_1, \quad u_1^n = u_2^{n-1} \text{ on } \Gamma_1, \\
Lu_2^n &= f \text{ in } \Omega_2, \quad u_2^n = u_1^{n-1} \text{ on } \Gamma_2,
\end{aligned}
\]
for $n \in \mathbb{N}^+$, and has the substructured form

\begin{equation}
(2.6) \quad v_{12}^n - \hat{G}_1(v_{21}^{n-1}, 0) = \hat{G}_1(0, f_1), \quad v_{21}^n - \hat{G}_2(v_{12}^{n-1}, 0) = \hat{G}_2(0, f_2),
\end{equation}

initialized by $v_{12}^0 := \tau_2(u_{12}^0)$ and $v_{21}^0 := \tau_1(u_{21}^0)$. Equations (2.6) and (2.5) allow us to obtain the substructured PSM in error form, that is

\begin{equation}
(2.7) \quad e_{12}^n = \hat{G}_1(e_{21}^{n-1}, 0), \quad e_{21}^n = \hat{G}_2(e_{12}^{n-1}, 0),
\end{equation}

for $n \in \mathbb{N}^+$, where $e_{12}^n := v_{12} - v_{12}^n$ and $e_{21}^n := v_{21} - v_{21}^n$ for $n \in \mathbb{N}$. Equation (2.5) can be written in the matrix form $A\mathbf{v} = \mathbf{b}$, where $\mathbf{v} = [v_{12}, v_{21}]^\top$ and

\begin{equation}
(2.8) \quad A = \begin{bmatrix}
I_{d,2} & -\hat{G}_1(\cdot, 0) \\
-\hat{G}_2(\cdot, 0) & I_{d,1}
\end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix}
\hat{G}_1(0, f_1) \\
\hat{G}_2(0, f_2)
\end{bmatrix},
\end{equation}

where $I_{d,j}$ are the identity operators on $L^2(\Gamma_j)$, for $j = 1, 2$. We introduce the operators $G_j : H_{00}^{1/2}(\Gamma_j) \to H_{00}^{1/2}(\Gamma_{3-j})$ defined by $G_j := \hat{G}_j(\cdot, 0)$ for $j = 1, 2$. The equivalent matrix form of equation (2.7) is $\mathbf{e}^n = G\mathbf{e}^{n-1}$, that is

\begin{equation}
(2.9) \quad \begin{bmatrix}
e_{12}^n \\
e_{21}^n
\end{bmatrix} = \begin{bmatrix} 0 & G_1 \\ G_2 & 0 \end{bmatrix} \begin{bmatrix} e_{12}^{n-1} \\
e_{21}^{n-1}
\end{bmatrix}, \quad \text{with} \quad G := \begin{bmatrix} 0 & G_1 \\ G_2 & 0 \end{bmatrix}
\end{equation}

and $\mathbf{e}^n := [e_{12}^n, e_{21}^n]^\top$. Notice that $G = I - A$, where $I := \text{diag}(I_{d,2}, I_{d,1})$.

3. S2S: Spectral two-level substructured DD method. In this section, we present our Spectral 2-level Substructured (S2S) method, which is detailed in Section 3.1 and analyzed in Section 3.2.

3.1. Description of the S2S method. The idea of the S2S method is to use a coarse space $V_c$ defined as the span of certain interfaces basis functions. The ideal choice would be to consider the span of some of the eigenfunctions of the smoothing operators $G_j$. We will show in Section 3.2 that this choice leads to a very efficient method. However, the eigenfunctions of $G_j$ are known only in very special cases and their numerical calculation could be quite expensive. To overcome this problem one could define $V_c$ as the span of some Fourier basis functions, that could be obtained by solving a Laplace-Beltrami eigenvalue problem on each interface.

Once the coarse space $V_c$ is constructed, the choice of restriction and prolongation operator follows naturally. Consider the two spaces $\mathcal{H}_1 := H^{1/2}(\Gamma_1)$ and $\mathcal{H}_2 := H^{1/2}(\Gamma_2)$ and define $\mathcal{H} := \mathcal{H}_2 \times \mathcal{H}_1$. Let $\{\psi_k^1\}_{k \in \mathbb{N}}$ be a basis for $\mathcal{H}_1$ and $\{\psi_k^2\}_{k \in \mathbb{N}}$ a basis for $\mathcal{H}_2$. Let us introduce an inner product $\langle \cdot, \cdot \rangle_1$ for $\mathcal{H}_1$, an inner product $\langle \cdot, \cdot \rangle_2$ for $\mathcal{H}_2$, and define $\langle (a, b), (c, d) \rangle := \langle a, c \rangle_2 + \langle b, d \rangle_1$ for all $(a, b), (c, d) \in \mathcal{H}$. Assume that the coarse space $V_c \subset \mathcal{H}$ is the span of the basis functions $(\psi_1^2, 0), \ldots, (\psi_m^2, 0)$ and $(0, \psi_1^1), \ldots, (0, \psi_m^1)$, for a finite $m > 0$, that are orthonormal with respect to $\langle \cdot, \cdot \rangle$. The operators $P : \mathbb{R}^{2m} \to \mathcal{H}$ and $R : \mathcal{H} \to \mathbb{R}^{2m}$ are then defined as

\begin{equation}
(3.1) \quad P \begin{bmatrix} \mathbf{v} \\
\mathbf{w}
\end{bmatrix} := \begin{bmatrix} \sum_{k=1}^m (\mathbf{v})_k \psi_k^2 \\
\sum_{k=1}^m (\mathbf{w})_k \psi_k^1
\end{bmatrix}^\top, \quad R \begin{bmatrix} f \\
g
\end{bmatrix} := \begin{bmatrix} (\psi_1^2, f)_2, \ldots, (\psi_m^2, f)_2, (\psi_1^1, g)_1, \ldots, (\psi_m^1, g)_1
\end{bmatrix}^\top,
\end{equation}

for any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^m$ and any $(f, g) \in \mathcal{H}$. The restriction of $A$ on $V_c$ is the operator $A_c : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ given by $A_c = RA^T.$
LEMMA 3.1 (Invertibility of a coarse operator $A_c$). Let $X_1$ and $X_2$ be inner-product spaces, $X := X_2 \times X_1$, $V_c$ be a finite-dimensional subspace of $X$ given by the span the basis functions $(\psi_1^2,0), \ldots, (\psi_m^2,0)$ and $(0,\psi_1^1), \ldots, (0,\psi_m^1)$, for a finite $m > 0$. Let $P_{V_c}$ be the orthogonal projection operator onto $V_c$. Consider an invertible operator $A : X \to X$ and the matrix $A_c = RAP \in \mathbb{R}^{2m \times 2m}$, where $P$ and $R$ are operators defined as in (3.1). Then $A_c$ has full rank if and only if $P_{V_c}(Av) \neq 0$ for any $v \in V_c \setminus \{0\}$.

Proof. We first show that if $P_{V_c}(Av) \neq 0$ for any $v \in V_c \setminus \{0\}$, then $A_c = RAP$ has full rank. This result follows from the rank-nullity theorem, if we show that the only element in the kernel of $A_c$ is the zero vector. To do so, we recall the definitions of $P$ and $R$ given in (3.1). Clearly, $Pz = 0$ if and only if $z = 0$. For any $z \in \mathbb{R}^{2m}$ the function $Pz$ is in $V_c$. Since $A$ is invertible, then $APz = 0$ if and only if $z = 0$. Moreover, it holds that $P_{V_c}(APz) \neq 0$. Now, we notice that $Rw = 0$ for all $w \in V_c \setminus \{0\}$, and $RPz = 0$ for all $w \in V_c \perp$, where $V_c \perp$ denotes the orthogonal complement of $V_c$ in $\mathcal{H}$ with respect to $\langle \cdot, \cdot \rangle$. Since $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is an inner-product space, we have $APz = P_{V_c}(APz) + (I - P_{V_c})(APz)$ with $(I - P_{V_c})(APz) \in V_c \perp$. Hence, $RAPz = RRP_{V_c}(APz) \neq 0$ for any non-zero $z \in \mathbb{R}^{2m}$.

Now we show that, if $A_c = RAP$ has full rank, then $P_{V_c}(Av) \neq 0$ for any $v \in V_c \setminus \{0\}$. We proceed by contraposition and prove that if there exists a $v \in V_c \setminus \{0\}$ such that $Av \in V_c \perp$, then $A_c = RAP$ is not full rank. Assume that there is a $v \in V_c \setminus \{0\}$ such that $Av \in V_c \perp$. Since $v$ is in $V_c$, there exists a nonzero vector $z \in \mathbb{R}^{2m}$ such that $v = Pz$. Hence $APz \in V_c \perp$. We can now write that $A_cz = R(APz) = 0$, which implies that $A_c$ is not full rank.

The following simple example shows that the invertibility of $A$ does not necessarily implies the invertibility of $A_c$.

EXAMPLE 1. Consider the invertible matrix $A := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Let us denote by $e_1$ and $e_2$ the canonical vectors in $\mathbb{R}^2$, define $V_c := \text{span}\{e_1\}$, and consider the classical scalar product for $\mathbb{R}^2$. This gives $V_c \perp := \text{span}\{e_2\}$. The prolongation and restriction operators are $P = [e_1]$ and $R = P^\top$. Clearly, we have that $Ae_1 = e_2$, which implies that $P_{V_c}(Av) = 0$ for all $v \in V_c$. Moreover, in this case we get $A_c = RAP = 0$, which shows that $A_c$ is not invertible.

Notice that, if $A(V_c) \subseteq V_c$, then it holds that $P_{V_c}(Av) \neq 0 \forall v \in V_c \setminus \{0\}$, and $A_c$ is invertible. The condition $A(V_c) \subseteq V_c$ is satisfied for $A$ defined in (2.8) if the functions $\psi_k$ are eigenfunctions of $G_j$. However, the condition $A(V_c) \subseteq V_c$ is only sufficient for the invertibility of $A_c$. As the following example shows, there exist invertible operators $A$ that do not satisfy this condition, but lead to invertible $A_c$.

EXAMPLE 2. Consider the invertible matrix $A := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Let us denote by $e_1$, $e_2$ and $e_3$ the three canonical vectors in $\mathbb{R}^3$, define $V_c := \text{span}\{e_1, e_2\}$, and consider the classical scalar product for $\mathbb{R}^3$. This gives $V_c \perp := \text{span}\{e_3\}$. The prolongation and restriction operators are $P = [e_1, e_2]$ and $R = P^\top$, and we get $A_c = RAP = I$, where $I$ is the $2 \times 2$ identity matrix. Now, we notice that $Ae_2 = e_2 + e_3$, which implies $P_{V_c}(Ae_2) \neq 0$ and $P_{V_c}(Ae_2) \neq 0$. Hence $V_c$ is not invariant under $A$, but $A_c$ is invertible.

With the operators $P$, $R$ and $A_c$ in hands, our two-level method is defined as a classical two-level strategy applied to the substructured problem (2.5) and using the
Algorithm 3.1 Two-level substructured domain decomposition method

Require: \( u^0 \) (initial guess)
1: \( u^n = G u^{n-1} + b, \ n = 1, \ldots, n_1 \) (dd pre-smoothing steps)
2: \( r = b - A u^{n_1} \) (compute the residual)
3: Solve \( A_c u_c = \tilde{R} r \) (solve the coarse problem)
4: \( u^n = u^{n_1} + P u_c \) (coarse correction)
5: \( u^n = G u^{n-1} + b, \ n = 1, \ldots, n_2 \) (dd post-smoothing steps)
6: Set \( u^0 = u^{n_2} \) (update)
7: Repeat from 1 to 6 until convergence

The well posedness of Algorithm 3.1 is proved in the following lemma.

Lemma 3.2 (Well posedness of S2S). Let \( V_c \subset H_{00}^{1/2} (\Gamma_2) \times H_{00}^{1/2} (\Gamma_1) \) be the span of basis functions \( (\psi_1^2, 0), \ldots, (\psi_m^2, 0) \) and \( (0, \psi_1^1), \ldots, (0, \psi_m^1) \), where \( \psi_k^j \) are eigenfunctions of \( G_j \). If the initialization vector \( u^0 \) is chosen in \( H_{00}^{1/2} (\Gamma_2) \times H_{00}^{1/2} (\Gamma_1) \), then \( u^n \in H_{00}^{1/2} (\Gamma_2) \times H_{00}^{1/2} (\Gamma_1) \) for \( n = 0, 1, 2, \ldots \).

Proof. It is sufficient to show that for a given \( u^0 \in H_{00}^{1/2} (\Gamma_2) \times H_{00}^{1/2} (\Gamma_1) \) all the steps of Algorithm 3.1 are well posed. It is clear that Step 1 and Step 2 produce \( u^{n_1} \) and \( r \) in \( H_{00}^{1/2} (\Gamma_2) \times H_{00}^{1/2} (\Gamma_1) \). Since \( V_c \) is the span of eigenfunctions of \( G_j \), then \( A(\psi) \subset V_c \) holds and Lemma 3.1 guarantees that \( A_c \) is invertible. Hence Step 3 is well posed. Since \( V_c \) is a subset of \( H_{00}^{1/2} (\Gamma_2) \times H_{00}^{1/2} (\Gamma_1) \), then \( P u_c \) and \( u^0 \) is Step 4 lie in \( H_{00}^{1/2} (\Gamma_2) \times H_{00}^{1/2} (\Gamma_1) \). Clearly, the element \( u^{n_2} \) produced by Step 5 is also in \( H_{00}^{1/2} (\Gamma_2) \times H_{00}^{1/2} (\Gamma_1) \). Therefore, by induction we obtain that Algorithm 3.1 is well posed in \( H_{00}^{1/2} (\Gamma_2) \times H_{00}^{1/2} (\Gamma_1) \).

A direct calculation reveals that one iteration our two-level method can be written in the form of a stationary method:

\[
(3.2) \quad u^{new} = G^{n_2} (I - PA_c^{-1} R A) G^{n_1} u^{old} + \tilde{M} b,
\]

where \( \tilde{M} \) is an operator which acts on the right-hand side vector \( b \) and which can be regarded as the preconditioner corresponding to our two-level method.

3.2. Convergence analysis of the S2S method. In error form, the iteration (3.2) becomes

\[
(3.3) \quad e^{new} = T e^{old} \quad \text{with} \quad T := G^{n_2} (I - PA_c^{-1} R A) G^{n_1},
\]

where \( e^{new} := u - u^{new} \) and \( e^{old} := u - u^{old} \). Hence, to prove convergence of the S2S method we must study of the operator \( T \). We use the operator norm

\[
\|S\|_{op} := \sup_{\|v\|_{2,\infty} = 1} \|Sv\|_{2,\infty} \quad \text{for any} \quad S \in \mathcal{L} (\mathcal{H}),
\]

where \( \mathcal{L} (\mathcal{H}) \) is the space of linear operators on \( \mathcal{H} \) and \( \|v\|_{2,\infty} := \max\{\|v_2\|_{H_2}, \|v_1\|_{H_1}\} \) with \( \|v_j\|_{H_j} := (v_j, v_j)^{1/2} \) for \( j = 1, 2 \) and any \( v = (v_2, v_1) \in \mathcal{H} \). Moreover, we also consider the contraction factor \( \rho(T) := \lim_{n \to \infty} \|T^n\|_{op}^{1/n} \).
Let us suppose that the interfaces $\Gamma_1$ and $\Gamma_2$ can be mapped one to the other by simple rotation, translation and scaling. This hypothesis allows us to identify the sets $\mathcal{H}_1$ and $\mathcal{H}_2$. Hence, we write $\mathcal{H}_0 := \mathcal{H}_1 = \mathcal{H}_2$ and $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_1 = \langle \cdot, \cdot \rangle_2$. Further, we assume that there exists a set of basis functions $\{\psi_1, \psi_2, \psi_3, \ldots\} \subset \mathcal{H}_0$, orthonormal with respect to the inner product $\langle \cdot \rangle$, that diagonalizes the operators $G_j$:

$$
(3.4) \quad G \begin{bmatrix} \psi_k \\ \psi_k \end{bmatrix} = \begin{bmatrix} 0 & G_1 \\ G_2 & 0 \end{bmatrix} \begin{bmatrix} \psi_k \\ \psi_k \end{bmatrix} = \begin{bmatrix} \rho_1(k)\psi_k \\ \rho_2(k)\psi_k \end{bmatrix},
$$

where $\rho_j(k)$ are the eigenvalues of $G_j$, for $j = 1, 2$.

**Example 3.** Consider a rectangular domain $\Omega := (-L_1, L_2) \times (0, \bar{L})$, $\bar{L}, L_1, L_2 > 0$ that is decomposed as $\Omega = \Omega_1 \cup \Omega_2$ by two overlapping subdomains $\Omega_1 := (-L_1, \delta) \times (0, \bar{L})$ and $\Omega_2 := (-\delta, L_2) \times (0, \bar{L})$ for some $0 < \delta < \min(L_1, L_2)$. The two interfaces are $\Gamma_1 := \{\delta\} \times (0, \bar{L})$ and $\Gamma_2 := \{-\delta\} \times (0, \bar{L})$. If $L = -\Delta$, then the Schwarz operators $G_1$ and $G_2$ are diagonalized by the sine-Fourier functions $\psi_k(y) = \sin(kx\pi/\bar{L})$, for $k = 1, 2, \ldots$. The corresponding eigenvalues of $G_j$ are $\rho_j(k) = \sinh\left(\frac{k\pi}{\bar{L}}\right)$, for $j = 1, 2$; see, e.g., [26, 7].

**Example 4.** Consider a disc $\Omega$ of radius $r$ and centered in the origin. One can decompose $\Omega$ as the union of two overlapping subdomains $\Omega_1$ and $\Omega_2$, where $\Omega_1$ is a disc of radius $r_1 < r$ and centered in the origin, and $\Omega_2$ is an annulus of external radius equal to $r$ and internal radius $r_2 \in (r_1, r)$. If $L = -\Delta + \eta$ with $\eta > 0$, then the two Schwarz operators $G_1$ and $G_2$ are diagonalized by periodic Fourier functions defined on circles; see, e.g., [34].

Our coarse space is defined as $V_c = (\text{span}\{\psi_1, \psi_2, \ldots, \psi_m\})^2$. Prolongation and restriction operators are (as in (3.1)) given by

$$
(3.5) \quad P \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} := \begin{bmatrix} \sum_{j=1}^{m} (\mathbf{v})_j \psi_j, \sum_{j=1}^{m} (\mathbf{w})_j \psi_j \end{bmatrix}^\top, \quad R \begin{bmatrix} f \\ g \end{bmatrix} := [(\psi_1, f), \ldots, (\psi_m, f), (\psi_1, g), \ldots, (\psi_m, g)]^\top.
$$

The restriction of $A$ on the coarse space $V_c$ is $A_c = RAP$. Notice that, since in this case $A(V_c) \subseteq V_c$, the operator $A_c$ is invertible according to Lemma 3.1. In order to analyze the convergence behavior of (3.3), we expand the error as $e_0 = \sum_{j=1}^{\infty} (\mathbf{v})_j^0 \psi_j, \sum_{j=1}^{\infty} (\mathbf{w})_j^0 \psi_j)^\top$ and study the spectral properties of the operator $T$.

**Theorem 3.3 (Convergence of the S2S method).** Consider the coarse space $V_c = (\text{span}\{\psi_1, \psi_2, \ldots, \psi_m\})^2$ and the operators $P$ and $R$ defined in (3.5). The S2S applied to the model problem (2.5) is a direct method for all the error components $(\psi_k, \psi_\ell)$ with $k, \ell \leq m$, that is $T[\psi_k, \psi_\ell]^\top = 0$ for all $k, \ell \leq m$. Moreover, if the eigenvalues $\rho_j(k)$, $j = 1, 2$, are in absolute value decreasing functions of $k$,* the contraction factor of the S2S, defined as $\rho_{S2S}(T) := \lim_{n \to \infty} (\|T^n\|_{op})^\frac{1}{n}$, is given by

$$
\rho_{S2S}(T) = \begin{cases} 
\rho_1(m+1)\rho_2(m+1)\frac{1+n_1}{2}, & \text{if } n_1, n_2 \text{ are both even or odd,} \\
\rho_1(m+1)\rho_2(m+1)\frac{1+n_2}{2}, & \text{if } n_1, n_2 \text{ are both odd or even,} \\
\max\{|\rho_1(m+1)|,|\rho_2(m+1)|\}, & \text{otherwise.}
\end{cases}
$$

*The assumption about the monotone decreasing of the absolute value of the eigenvalues is in general satisfied in case of the classical Schwarz method; see, e.g., [4, 8, 24].
Proof. Let us suppose that both \( n_1 \) and \( n_2 \) are even. The other cases follow the same calculation. For \( n_1 \) even we define \( \pi^{n_1}(k) := \rho^I_1(k) \rho^{\dagger}_2(k) \), and study the action of the operator \( T \) on a vector \([\psi_k, \psi_\ell]^\top\):
\[
T \begin{bmatrix} \psi_k \\ \psi_\ell \end{bmatrix} = G^{n_2}(I - PA_c^{-1} RA)G^{n_1} \begin{bmatrix} \psi_k \\ \psi_\ell \end{bmatrix}.
\]
We begin with the case \( k \leq m \) and \( \ell \leq m \). First, let us compute the action of the operator \( RAG^{n_1} \) on \([\psi_k, \psi_\ell]^\top\). Since the operators \( G_j \) are diagonalized by the basis \( \{\psi_k\}_k \) using (3.4) one obtains \( G^{n_1} \begin{bmatrix} \psi_k \\ \psi_\ell \end{bmatrix} = \begin{bmatrix} \pi^{n_1}(k)\psi_k \\ \pi^{n_1}(\ell)\psi_\ell \end{bmatrix} \). The action of \( A \) on \([\pi^{n_1}(k)\psi_k, \pi^{n_1}(\ell)\psi_\ell]^\top \) is
\[
A \begin{bmatrix} \pi^{n_1}(k)\psi_k \\ \pi^{n_1}(\ell)\psi_\ell \end{bmatrix} = \begin{bmatrix} I_d & -G_1 \\ -G_2 & I_d \end{bmatrix} \begin{bmatrix} \pi^{n_1}(k)\psi_k \\ \pi^{n_1}(\ell)\psi_\ell \end{bmatrix} - \begin{bmatrix} \pi^{n_1}(\ell)\rho_1(\ell)\psi_\ell \\ \pi^{n_1}(k)\rho_2(k)\psi_k \end{bmatrix}.
\]
Since \( A \) is invertible and has the form \( A = I - G \), the eigenvalues \( \rho_j(k) \) must different from one. Hence, the product \( A[\pi^{n_1}(k)\psi_k, \pi^{n_1}(\ell)\psi_\ell]^\top \neq 0 \). Now, the application of the restriction operator \( R \) on \( A \begin{bmatrix} \pi^{n_1}(k)\psi_k \\ \pi^{n_1}(\ell)\psi_\ell \end{bmatrix} \) gives us
\[
RA \begin{bmatrix} \pi^{n_1}(k)\psi_k \\ \pi^{n_1}(\ell)\psi_\ell \end{bmatrix} = \begin{bmatrix} \pi^{n_1}(k)e_k \\ \pi^{n_1}(\ell)e_\ell \end{bmatrix} - \begin{bmatrix} \pi^{n_1}(\ell)\rho_1(\ell)e_\ell \\ \pi^{n_1}(k)\rho_2(k)e_k \end{bmatrix} = A \begin{bmatrix} \pi^{n_1}(k)e_k \\ \pi^{n_1}(\ell)e_\ell \end{bmatrix},
\]
where \( e_k \) and \( e_\ell \) are canonical vectors in \( \mathbb{R}^m \) and \( \Lambda := \begin{bmatrix} I & -\rho_1(\ell)I \\ -\rho_2(k)I & I \end{bmatrix} \), with \( I \) the \( m \times m \) identity matrix. We have then obtained
\[
(3.6) \quad RA G^{n_1} \begin{bmatrix} \psi_k \\ \psi_\ell \end{bmatrix} = \Lambda \begin{bmatrix} \pi^{n_1}(k)e_k \\ \pi^{n_1}(\ell)e_\ell \end{bmatrix}.
\]
Now, by computing
\[
A_c \begin{bmatrix} \pi^{n_1}(k)e_k \\ \pi^{n_1}(\ell)e_\ell \end{bmatrix} = RA \begin{bmatrix} I_d & -G_1 \\ -G_2 & I_d \end{bmatrix} \begin{bmatrix} \pi^{n_1}(k)e_k \\ \pi^{n_1}(\ell)e_\ell \end{bmatrix} = A \begin{bmatrix} \pi^{n_1}(k)e_k \\ \pi^{n_1}(\ell)e_\ell \end{bmatrix},
\]
one obtains the action of \( A_c^{-1} \) on \( \Lambda \begin{bmatrix} \pi^{n_1}(k)e_k \\ \pi^{n_1}(\ell)e_\ell \end{bmatrix} \), that is
\[
(3.7) \quad \begin{bmatrix} \pi^{n_1}(k)e_k \\ \pi^{n_1}(\ell)e_\ell \end{bmatrix} = A_c^{-1} \begin{bmatrix} \pi^{n_1}(k)e_k \\ \pi^{n_1}(\ell)e_\ell \end{bmatrix}.
\]
Using (3.6) and (3.7) we have
\[
(3.8) \quad (I - PA_c^{-1} RA) G^{n_1} \begin{bmatrix} \psi_k \\ \psi_\ell \end{bmatrix} = \begin{bmatrix} \pi^{n_1}(k)\psi_k \\ \pi^{n_1}(\ell)\psi_\ell \end{bmatrix} - PA_c^{-1} \begin{bmatrix} \pi^{n_1}(k)e_k \\ \pi^{n_1}(\ell)e_\ell \end{bmatrix} = \begin{bmatrix} \pi^{n_1}(k)\psi_k \\ \pi^{n_1}(\ell)\psi_\ell \end{bmatrix} - \begin{bmatrix} \pi^{n_1}(k)\psi_k \\ \pi^{n_1}(\ell)\psi_\ell \end{bmatrix} = 0.
\]
This means that the S2S method is a direct method for all the pairs \( (\psi_k, \psi_\ell) \) with \( k \leq m \) and \( \ell \leq m \). The result for \( n_1 \) odd follows by similar calculations.
Similarly as before, we compute

\[ \text{RA}^{n_1} [\begin{bmatrix} 0 \\ \psi_k \\ \psi_\ell \end{bmatrix}] = R \left( \begin{bmatrix} \pi^{n_1}(k) \psi_k \\ \pi^{n_1}(\ell) \psi_\ell \\ \pi^{n_1}(k) \rho_2(k) \psi_\ell \end{bmatrix} \right) - \begin{bmatrix} \pi^{n_1}(\ell) \rho_1(\ell) \psi_\ell \\ \pi^{n_1}(\ell) \rho_1(\ell) \psi_\ell \\ \pi^{n_1}(\ell) \rho_1(\ell) \psi_\ell \end{bmatrix} \left[ \begin{bmatrix} 0 \\ -\rho_1(\ell) I \\ 0 \end{bmatrix} \right] \begin{bmatrix} 0 \\ I \\ \pi^{n_1}(\ell) e_\ell \end{bmatrix}. \]

Similarly as before, we compute

\[ \begin{bmatrix} 0 \\ \psi_k \\ \psi_\ell \end{bmatrix} = \begin{bmatrix} 0 \\ -\rho_1(\ell) I \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ I \\ \pi^{n_1}(\ell) e_\ell \end{bmatrix}. \]

Thus, we have

\[ T \begin{bmatrix} \psi_k \\ \psi_\ell \end{bmatrix} = G^n A c^{n_1} \begin{bmatrix} 0 \\ \psi_k \\ \psi_\ell \end{bmatrix} = \begin{bmatrix} 0 \\ -\rho_1(\ell) I \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ I \\ \pi^{n_1}(\ell) e_\ell \end{bmatrix}. \]

Hence for any pair \((\psi_k, \psi_\ell)\) with \(k > m\) and \(\ell \leq m\), the S2S is a direct method only for the \(\ell^{th}\) error component, which belongs to the coarse space. The component \(k\) is instead transparent to the coarse correction and only affected by the smoothing steps.

For the remaining case \(k > m\) and \(\ell > m\), the same arguments as before imply that

\[ \text{RA}^{n_1} [\begin{bmatrix} 0 \\ \psi_k \\ \psi_\ell \end{bmatrix}] = \begin{bmatrix} 0 \\ -\rho_1(\ell) I \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ I \\ \pi^{n_1}(\ell) e_\ell \end{bmatrix}. \]

Thus, we have

\[ T \begin{bmatrix} \psi_k \\ \psi_\ell \end{bmatrix} = G^n A c^{n_1} \begin{bmatrix} 0 \\ \psi_k \\ \psi_\ell \end{bmatrix} = \begin{bmatrix} 0 \\ -\rho_1(\ell) I \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ I \\ \pi^{n_1}(\ell) e_\ell \end{bmatrix}. \]

We can now study the norm of \(T\). Since \(\{\psi_k, \psi_\ell\}_{k, \ell}\) form a basis of \(\mathcal{H}\), we have

\[ \| T \|_{op} = \| T v \|_{2, \infty} = \sup_{k, \ell} \| T \psi_k \|_{2, \infty} = \sup_{k, \ell} \| T \psi_\ell \|_{2, \infty}, \]

where the last equality follows from (3.8), (3.9) and (3.10). Using (3.10) we obtain

\[ \| T \|_{op} = \sup_{k, \ell} \| T \psi_k \|_{2, \infty} = \sup_{k, \ell} \| G^n A c^{n_1} \psi_k \|_{2, \infty} = \sup_{k, \ell} \max \left( \| \pi^{n_1+n_2}(k) \psi_k \|_{2}, \| \pi^{n_1+n_2}(\ell) \psi_\ell \|_{2} \right) = \sup_{k, \ell} \max \left( |\pi^{n_1+n_2}(k)|, |\pi^{n_1+n_2}(\ell)| \right) = |\pi^{n_1+n_2}(m+1)|, \]

where in the last equality we used that \(|\rho_1(k)|\) and \(|\rho_2(k)|\) are in absolute value non-increasing functions of \(k\). Finally, a direct calculation leads to

\[ \| T^m \|_{op} = \sup_{k, \ell} \max \left( \| \pi^{n_1+n_2}(k) \|, |\pi^{n_1+n_2}(\ell)| \right) = |\pi^{n_1+n_2}(m+1)|, \]

which implies that \(\rho_{S2S}(T) = \lim_{n \to \infty} (\| T^n \|_{op})^{1/n} = |\pi^{n_1+n_2}(m+1)|. \)
Theorem 3.3 shows that the choice of the basis functions $\psi^i_k$ can affect drastically the convergence of the method. On the one hand, an inappropriate choice of $V_c$ can lead to a two-level method that performs as the corresponding one-level method. On the other hand, a good choice of $V_c$ can even make convergent a non-converging stationary method.

The result of Theorem 3.3 is intuitively easy. The corresponding proof for a case of the coarse space functions that are eigenfunctions of $G$ is much easier. We report it for completeness in the Appendix (see Theorem 9.1). However, the settings of Theorem 3.3 are more general, since the coarse space functions are not eigenfunctions of $G$.

### 3.3. A PCA approach for an automatic coarse space generation

The construction of a good coarse space $V_c$ of our S2S method is not an easy task. Indeed, one can consider any coarse space constructed by solving generalized eigenvalue problems on the interfaces; see, e.g., [2, 29, 31, 40]; see also [6, 12]. What to do if no “good” information about $V_c$ is available? Is there any other possibility to create an adequate coarse space in the off-line pre-computation phase?

The idea that we present in this section is to construct an approximation of the image of the smoother $G$ using a principal component analysis (PCA), also known as proper orthogonal decomposition (POD); see, e.g., [37] and references therein. We consider the following procedure.

- Consider a set of $q$ linearly independent randomly generated vectors 
  $\{s_k\}_{k=1}^q \subseteq \mathbb{R}^{N^s}$, where $N^s$ is the number of degrees of freedom on the two interfaces, and define the matrix $S = [s_1 \cdots s_q]$. Here, $q \approx 2m$ and $2m$ is the desired dimension of the coarse space.
- Use these as initial vectors and perform $r$ smoothing steps to create the matrix $W = GS$. This computation can be performed in parallel and we assume that $r$ is “small”.
- Compute the SVD of $W$: $W = U\Sigma V^T$. This is cheap ($O(q(N^s)^2)$) because $W \in \mathbb{R}^{N^s \times q}$ is “small” since $q$ is “small” and the vectors $v_k$ are interface vectors.
- Define $V_c := \text{span}\{u_{2k}\}_{k=1}^{2m}$ and $P := [u_1, \ldots, u_{2m}]$.

To qualitatively describe the obtained coarse space, we prove the following bound.

**Lemma 3.4 (Approximation of the random generated coarse space).** Consider a full rank orthogonal matrix $X \in \mathbb{R}^{N^s \times N^s}$ and its decomposition $X = [S, \tilde{S}]$. Let $W = G^*[S, 0]$ and $P_\ell = U_\ell \Sigma_\ell V_\ell^T$ be the rank-$\ell$ SVD of $W$ ($\ell \leq m$), where $(\Sigma_\ell)_{jj} = \sigma_j$, $j = 1, \ldots, \ell$ are the singular values of $W$. Then, it holds that

$$\|P_\ell - G^*(X)\|_2 \leq \sigma_{\ell+1} + \|G^*\|_2 \sqrt{\min(1, N^s - q)}.$$

**Proof.** Using the triangle inequality, we get

$$\|P_\ell - G^*(X)\|_2 \leq \|P_\ell - G^*([S, 0])\|_2 + \|G^*([S, 0]) - G^*(X)\|_2.$$

The first term on the right-hand side is equal to $\sigma_{\ell+1}$ by the best approximation properties of the SVD. The second term can be bounded as $\|G^*([S, 0]) - G^*(X)\|_2 \leq \|G^*\|_2 \|([S, 0] - X)\|_2$, and a direct calculation of $\|([S, 0] - X)\|_2$ leads to the result.

Despite its very simple proof, the result given by Lemma 3.4 allows us to describe the quality of the created coarse space. Larger values of $q$ and $\ell$ lead to a smaller error in the approximation of the image of $G$. Moreover, a smoother $G$ with good contraction properties, namely $\|G\|_2 \ll 1$ leads to a better approximation. Clearly, one...
can improve the approximation by enlarging \( r \) at the cost of extra subdomain solves. Notice also that if \( r = 1 \), then our procedure has approximatively the same computational cost of other strategies based on interface eigenvalue problems. The dominant costs of these procedures are the extension in volume of the interface functions. This extensions have about the same cost of constructing \( W \).

A numerical study of the described procedure is given in Section 7.1.

4. **G2S: Geometric two-level substructured dd method.** One of the main drawbacks of the S2S method is that it requires a set of “good” basis functions on each interface to define the coarse space \( V_c \). These functions are not always available or not easy to be computed, since their calculation could require the solution of eigen-problems defined on (possibly complicated) interfaces. Moreover, an extension to a multi-level framework seems to be quite involved. If the dimension of the coarse space becomes too large (if for example the basis functions are not “good enough”), then one would need to recursively repeat Algorithm 3.1 in a multi-level fashion. To do so, a new coarse space on each level is required, but to the best of our knowledge, it is not clear how to properly define a third-level coarse space. Finally, in the case that the eigenfunctions of the smoothers \( G_j \) are not available, the use of Fourier functions on the interfaces is not necessarily the best choice. These comments lead to some questions: Is it possible to avoid the explicit construction of a coarse space? Is there any practical way to implicitly define a coarse space which is not simply the span of the first \( m \) Fourier basis functions? Can one define a framework in which an extension of the two-level method to a multi-level framework is possible and easy?

In this section, we answer the above questions by introducing the so-called Geometric 2-level Substructured (G2S) method and relate it to the S2S method. The G2S method is detailed in Section 4.1. The relations between G2S and S2S are discussed in Section 4.2.1. Eventually, we provide a convergence analysis in Section 4.2.2.

4.1. **Description of the G2S method.** Let us consider a discretization of the interfaces \( \Gamma_1 \) and \( \Gamma_2 \) by two properly chosen grids \( \Gamma_{h,1} \) and \( \Gamma_{h,2} \). Here \( h \) is the grid size. We denote by \( N_1 \) the number of grid points in \( \Gamma_{h,1} \), \( N_2 \) the number of grid points in \( \Gamma_{h,2} \) and \( N^s := N_1 + N_2 \). The corresponding finite-dimensional discretizations of the operators \( G_1 : H_1 \rightarrow H_2 \) and \( G_2 : H_2 \rightarrow H_1 \) are denoted by \( G_{h,1} : \mathbb{R}^{N_1} \rightarrow \mathbb{R}^{N_2} \) and \( G_{h,2} : \mathbb{R}^{N_2} \rightarrow \mathbb{R}^{N_1} \), and the substructured problem (2.5) becomes

\[
(4.1) \quad v_{12} - G_{h,1}(v_{21}) = b_{h,1}, \quad v_{21} - G_{h,2}(v_{12}) = b_{h,2},
\]

where \( b_{h,j} \) are the discretizations of \( \hat{G}_{j}(0,f_j) \), for \( j = 1, 2 \). Notice that (4.1) can be written as \( A_h v = b_h \), where

\[
(4.2) \quad A_h = \begin{bmatrix} I_{h,2} & -G_{h,1} \\ -G_{h,2} & I_{h,1} \end{bmatrix} \quad \text{and} \quad b_h = \begin{bmatrix} b_{h,1} \\ b_{h,2} \end{bmatrix},
\]

where \( I_{h,j} \) are identity matrices acting on vectors defined on the discrete interfaces \( \Gamma_{h,j} \), for \( j = 1, 2 \). We define \( G_h := \begin{bmatrix} 0 & G_{h,1} \\ G_{h,2} & 0 \end{bmatrix} \) and write \( A_h = I_h - G_h \), where \( I_h = \text{diag}(I_{h,2},I_{h,1}) \) is the \( N^s \times N^s \) identity matrix. Clearly, if the discrete domain decomposition method converges, then \( \rho(G_h) < 1 \). Hence, the matrix \( A_h \) is invertible.

Next, we consider two coarser grids \( \Gamma_{1,2h} \) and \( \Gamma_{2,2h} \) of size \( 2h \) and \( M_1 \) and \( M_2 \) points, respectively, with \( M^s := M_1 + M_2 \). Let us denote by \( P_{2h} \in \mathbb{R}^{N^s \times M^s} \) and \( R_{2h} \in \mathbb{R}^{M^s \times N^s} \) prolongation and restriction matrices. These could be classical interpolation operators used in multigrid methods. For example, if \( \Gamma_1 \) and \( \Gamma_2 \) are one-dimensional
intervals, then $P_{2h}^h = \text{diag}(P_2, P_1)$, where $P_1 \in \mathbb{R}^{N_1 \times M_1}$ and $P_2 \in \mathbb{R}^{N_2 \times M_2}$ are linear interpolation matrices of the form

$$
\begin{bmatrix}
\frac{1}{2} \\
1 \\
\frac{1}{2} \\
\frac{1}{2} \\
\vdots \\
1 \\
\frac{1}{2}
\end{bmatrix}.
$$

The restriction operator can then be chosen to be the full weighting restriction matrix $R_{2h}^h = \text{diag}(R_2, R_1)$ with $R_j := \frac{1}{2} P_j^\top$. The restriction of $A_h$ on the coarse level can be then defined as $A_{2h} := R_{2h}^h A_h P_{2h}^h$.

The G2S procedure is defined by Algorithm 3.1, upon replacement of the operators $A, A_c, G, R$ and $P$, by $A_h, A_{2h}, G_h, R_{2h}^h$, and $P_{2h}^h$, respectively. We insist on the fact that the G2S does not require the explicit construction of a coarse space $V_c$, but it exploits directly a discretization of the interfaces. Moreover, it is clear that a simple recursion allows us to embed our G2S in a multi-grid framework.

As for the S2S method, one iteration of our G2S method can be written as

$$
\tilde{u}^{\text{new}} = G_{n_2}^{n_2}(I_n - P_{2h}^h A_{2h}^{-1} P_{2h}^h A_h) G_h^{n_1} u^{\text{old}} + \tilde{M}_b b_h,
$$

where $I_n = \text{diag}(I_{h,2}, I_{h,1})$. In error form, the iteration (4.4) becomes

$$
e^{\text{new}} = T_h e^{\text{old}} \text{ with } T_h := G_h^{n_2}(I_h - P_{2h}^h A_{2h}^{-1} P_{2h}^h A_h) G_h^{n_1},$$

where $e^{\text{new}} := u - \tilde{u}^{\text{new}}$ and $e^{\text{old}} := u - u^{\text{old}}$.

### 4.2. Analysis of the G2S method.

In this section, we study the convergence behavior of the G2S method. To do so, we consider a Laplace equation $\mathcal{L} = -\Delta$ defined on a rectangular domain $\Omega := (-L_1, L_2) \times (0, \bar{L})$, $L_1, L_2, \bar{L} > 0$, which is decomposed into two overlapping subdomains $\Omega_1 := (-L_1, \delta) \times (0, \bar{L})$ and $\Omega_2 := (-\delta, L_2) \times (0, \bar{L})$ for some $0 < \delta < \min(L_1, L_2)$. The interfaces are $\Gamma_1 := \{\delta\} \times (0, \bar{L})$ and $\Gamma_2 := \{-\delta\} \times (0, \bar{L})$.

For a given $\ell \in \mathbb{N}^+$, $\ell \geq 2$, we discretize the model problem (2.1) using a uniform grid of $N_h = 2^\ell - 1$ points on each interface so that the grid size is $h = \frac{\bar{L}}{2^{\ell-1}+1}$. Notice that $N_h = N_1 = N_2$, where $N_1$ and $N_2$ are used in Section 4.1 to denote the number of discretization points of the two interfaces. We also introduce a coarser mesh of $N_c = 2^\ell - 1$ points on each interface and mesh size $h_c = \frac{1}{2^{\ell+1}}$. We define the geometric prolongation operator $P_{2h}^h \in \mathbb{R}^{2N_c \times 2N_c}$ as $P_{2h}^h := \text{diag}(\tilde{P}, \tilde{P})$, where $\tilde{P}$ is the interpolation matrix given in (4.3). The operator $R_{2h}^h \in \mathbb{R}^{2N_c \times 2N_c}$ is defined as $R_{2h}^h := \text{diag}(\tilde{R}, \tilde{R})$, where $\tilde{R}$ is the full weighting restriction matrix $\tilde{R} := \frac{1}{2} \tilde{P}^\top$.

We suppose that the operators $G_{h,1}$ and $G_{h,2}$ have eigenvectors $\psi_k$ with eigenvalues $\rho_j(k), k = 1, \ldots, N_h, j = 1, 2$. Here, $\psi_k$ are discrete Fourier modes given by $(\psi_k)_j = \sin(k \pi j h)$, for $j, k = 1, \ldots, N_h$. Notice that $\psi_k^\top \psi_k = \delta_{\ell,k} \frac{N_c+1}{2}$, with $\delta_{\ell,k}$ the Kronecker delta.

It is well-known that the actions of $\tilde{R}$ and $\tilde{P}$ on the combination of a low-frequency mode $\psi_k$ with its high-frequency companion $\psi_{\bar{k}}$, with $\bar{k} = N_h - k + 1$, are
\[ \tilde{R} [\psi_k | \psi_k] = \phi_k [c_k^2 - s_k^2], \quad \tilde{P} \phi_k = (c_k^2 \psi_k - s_k^2 \psi_k) = [\psi_k | \psi_k] \left[ \begin{array}{c} c_k^2 \\ -s_k^2 \end{array} \right], \]

where \( c_k = \cos(k\pi \theta) \), \( s_k = \sin(k\pi \theta) \) for \( k = 1, \ldots, N_c \) and \( (\phi_k)_j = \sin(k\pi 2j) \), for \( k = 1, \ldots, N_{h+1} \) and \( j = 0, \ldots, N_{h+1} \); see, e.g., [38, 7]. These are the Fourier modes on the coarse grid. As before, the coarse matrix is \( A_{2h} = R_{2h}^h A_h P_{2h}^h \), and the G2S iteration operator is \( T_h := G_{h}^{m+1} (I - \tilde{P}_h^h A_{2h}^h R_{2h}^h A_h) G_{h}^{m} \).

So far, we tacitly assumed the invertibility of \( A_{2h} = R_{2h}^h A_h P_{2h}^h \). By Lemma 3.1 this property clearly depends on the structure of the matrix \( A_h \) (hence on the iteration matrices \( G_{h,1} \) and \( G_{h,2} \)) and on the coarse space \( V_c \), which we did not need explicitly to introduce our G2S method. Nevertheless, in Lemma 4.1 of the next section, we prove that the matrix \( A_{2h} \) is invertible (at least in the specific settings considered in Section 4.2).

**4.2.1. Relation between S2S and G2S.** The goal of this section is to discuss the relation between S2S and G2S. In particular, we show that, in the framework of Section 4.2, the G2S method corresponds to a S2S method using the coarse space.

\[ V_c = (\text{span}_{k=1, \ldots, N_c} \{ \tilde{P} \phi_k \})^2 = (\text{span}_{k=1, \ldots, N_c} \{ c_k^2 \psi_k - s_k^2 \psi_k \})^2 \subset \mathbb{R}^{2N_c}, \]

where the relation \((4.5)\) is used. To do so, we consider any invertible matrix \( U \in \mathbb{R}^{2N_c \times 2N_c} \) and compute

\[
T_h = G_{h}^{m+1} (I - P_{2h}^h A_{2h}^h R_{2h}^h A_h) G_{h}^{m} = G_{h}^{m+1} (I - P_{2h}^h U U^{-1} (R_{2h}^h A_h P_{2h}^h)^{-1} U^T U^{-1} (R_{2h}^h A_h) G_{h}^{m},
\]

\[ = G_{h}^{m+1} (I - P_{2h}^h U U^{-1} (U^{-1} R_{2h}^h A_h P_{2h}^h U)^{-1} U^T U^{-1} (R_{2h}^h A_h) G_{h}^{m},
\]

\[ = G_{h}^{m+1} (I - \tilde{P}_{2h}^h U U^{-1} (U^{-1} R_{2h}^h A_h P_{2h}^h U)^{-1} U^T U^{-1} (R_{2h}^h A_h) G_{h}^{m},
\]

\[ = G_{h}^{m+1} (I - \tilde{P}_{2h}^h U U^{-1} (R_{2h}^h A_h P_{2h}^h U)^{-1} U^T U^{-1} (R_{2h}^h A_h) G_{h}^{m},
\]

where \( \tilde{P}_{2h}^h := P_{2h}^h U \), \( \tilde{R}_{2h}^h := U^{-1} R_{2h}^h \), and \( \tilde{A}_{2h} := \tilde{R}_{2h}^h A_h \tilde{P}_{2h}^h \). This means that, using any invertible matrix \( U \), the G2S method can be written as a two-level method characterized by an iteration operator \( \tilde{T}_h \) defined via the prolongation and restriction operators \( \tilde{P}_{2h}^h \) and \( \tilde{R}_{2h}^h \).

Let us define the orthogonal matrices \( \Phi = \frac{1}{\sqrt{N_{h+1}}} [\phi_1, \ldots, \phi_{N_{h+1}}] \) and \( U := \text{diag}(\Phi, \Phi) \), and the operators \( \tilde{P}_{2h}^h := P_{2h}^h U \), \( \tilde{R}_{2h}^h := U^T R_{2h}^h \) and \( \tilde{A}_{2h} := \tilde{R}_{2h}^h A_h \tilde{P}_{2h}^h \). Notice that the columns of the matrix \( \tilde{P}_{2h}^h := P_{2h}^h U \) are the functions that span the coarse space \( V_c \) defined in \( (4.6) \). This means that the G2S method is equivalent to a S2S method defined using this coarse space. This equivalence leads to some important remarks. In a two-level setting, a two-grid method defines a coarse space whose dimension could still be large. However:

1. The equivalence between G2S and the S2S allows one to keep the geometric structure of a two-grid framework and at the same time to reduce the dimension of the coarse space by choosing less basis functions \( \tilde{P} \phi_k \). This possibility is studied in Section 7 by direct numerical experiments.
2. Thanks to the geometric interpretation, it is possible to introduce other coarser levels based on coarser interface grids (as in a multi-grid framework).

\[ \text{Notice that } (\tilde{P}_{2h}^h)^T = U^T (P_{2h}^h)^T = 2U^T R_{2h}^h = 2\tilde{R}_{2h}^h. \]
We conclude this section by turning our attention to the matrix $A_{2h}$, whose invertibility is proved in the following lemma.

**Lemma 4.1 (Invertibility of $A_{2h}$).** Assume that $\rho_1(k), \rho_2(k) \in [0, 1]$ for all $k$ and that $\rho_1(k) \geq \rho_1(\bar{k})$ and $\rho_2(k) \geq \rho_2(\bar{k})$ for any $k = 1, \ldots, N_c$ and $\bar{k} = N_h - k + 1$. The matrix $A_{2h} := R_{2h}^h A_h P_{2h}^h \in \mathbb{R}^{2N_h \times 2N_c}$ has full rank.

**Proof.** Since $A_{2h} = U^T \tilde{A}_{2h} U$, it is enough to show that $\tilde{A}_{2h}$ is invertible. To do so, we recall that $A_{2h} = R_{2h}^h A_h P_{2h}^h$ and we wish to prove that for any $z \in V_c \setminus \{0\}$ (with $V_c$ defined in (4.6)) it holds $\mathbb{P}_{V_c}(A_h z) \neq 0$ and then invoke Lemma 3.1. Here the orthogonality is understood with respect to the classical scalar product for $\mathbb{R}^{2N_h}$.

First, it is possible to show that the orthogonal complement of $V_c$ is

$$V_c^\perp = (\text{span}_{k=1}^{N_c} \{ c_k^{-2} \psi_k + s_k^{-2} \psi_k, \psi_{(N_h+1)/2} \})^2.$$ Notice that $\dim(V_c) = 2N_c$, $\dim(V_c^\perp) = 2(N_c + 1)$, and $\dim(V_c) + \dim(V_c^\perp) = 2N_h$.

We use (4.5) and compute

$$A_h \begin{bmatrix} \bar{P}\phi_k \\ \bar{P}\phi_\ell \end{bmatrix} = A_h \begin{bmatrix} c_k \hat{\psi}_k - s_k^2 \psi_k \\ c_\ell \hat{\psi}_\ell - s_\ell^2 \psi_\ell \end{bmatrix} = \begin{bmatrix} c_k^2 \hat{\psi}_k - s_k^2 \psi_k - (\rho_2(\ell) c_\ell^2 \hat{\psi}_\ell - \rho_2(\bar{\ell}) s_\ell^2 \psi_\ell) \\ c_\ell^2 \hat{\psi}_\ell - s_\ell^2 \psi_\ell - (\rho_1(k) c_k^2 \hat{\psi}_k - \rho_1(\bar{k}) s_k^2 \psi_k) \end{bmatrix},$$

(4.8)

for any $k, \ell = 1, \ldots, N_c$, where $\bar{k} = N_h - k + 1$ and $\bar{\ell} = N_h - \ell + 1$, for $k, \ell = 1, \ldots, N_c$.

Now, a direct calculation shows that

$$s_k^2 \psi_k = -\frac{s_k^2}{s_k^4 + c_k^4} \bar{P}\phi_k + \frac{1}{s_k^4 + c_k^4} \left( c_k^{-2} \psi_k + s_k^{-2} \psi_k \right),$$

$$s_\ell^2 \psi_\ell = -\frac{s_\ell^2}{s_\ell^4 + c_\ell^4} \bar{P}\phi_\ell + \frac{1}{s_\ell^4 + c_\ell^4} \left( c_\ell^{-2} \psi_\ell + s_\ell^{-2} \psi_\ell \right).$$

Inserting these equalities into (4.8), we obtain for $k \neq \ell$ that

$$\begin{bmatrix} \bar{P}\phi_k \\ \bar{P}\phi_\ell \end{bmatrix}^T A_h \begin{bmatrix} \bar{P}\phi_k \\ \bar{P}\phi_\ell \end{bmatrix} = \| \bar{P}\phi_k \|_2^2 + \| \bar{P}\phi_\ell \|_2^2 \neq 0.$$ Similarly, for $k = \ell$ we obtain that

$$\begin{bmatrix} \bar{P}\phi_k \\ \bar{P}\phi_\ell \end{bmatrix}^T A_h \begin{bmatrix} \bar{P}\phi_k \\ \bar{P}\phi_\ell \end{bmatrix} = \left( 2 - (\rho_1(k) - \rho_2(k)) + \frac{s_k^4 (\rho_2(k) - \rho_2(\bar{k}) + \rho_1(k) - \rho_1(\bar{k}))}{s_k^4 + c_k^4} \right) \| \bar{P}\phi_k \|_2^2.$$ A direct calculation using the assumptions on $\rho_j(k)$ shows that this is nonzero. \qed

**4.2.2. Convergence of the G2S method.** In the previous section, we have shown that the G2S method is equivalent to a S2S method. This allowed us also to prove Lemma 4.1, which guarantees that $A_{2h}$ is invertible and that the G2S method is well posed. We turn then our attention to the analysis of its convergence behavior.
Lemma 4.2. Consider the G2S iteration matrix $T_h := G_h^{n_2}(I - P_{2h}^h A_{2h}^{-1} P_{2h}^h A_h) G_h^{n_1}$. The action of $T_h$ on \[
athan{\begin{bmatrix}
\psi_k & \psi_k & 0 & 0 \\
0 & 0 & \psi_k & \psi_k \\
0 & 0 & \psi_k & \psi_k \\
0 & 0 & \psi_k & \psi_k
\end{bmatrix}}\] is given by,

\begin{equation}
T_h \begin{bmatrix}
\psi_k & \psi_k & 0 & 0 \\
0 & 0 & \psi_k & \psi_k \\
0 & 0 & \psi_k & \psi_k \\
0 & 0 & \psi_k & \psi_k
\end{bmatrix} = \begin{bmatrix}
\psi_k & \psi_k & 0 & 0 \\
0 & 0 & \psi_k & \psi_k \\
0 & 0 & \psi_k & \psi_k \\
0 & 0 & \psi_k & \psi_k
\end{bmatrix} \tilde{G}_k
\end{equation}

where $\tilde{G}_k := D_{n_2}(k)(D_{n_1}(k) - V(k)\Lambda_2^{-1}(k)\Lambda_1(k))$ with

\[
\Lambda_1(k) := V(k) H(k) D_{n_1}(k), \quad \Lambda_2(k) := V(k) H(k) V(k),
\]

\[
V(k) := \begin{bmatrix}
c_k^2 & 0 & \cdots & 0 \\
0 & c_k^2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & c_k^2
\end{bmatrix}, \quad H(k) := \begin{bmatrix}
1 & 0 & -\rho_1(k) & 0 \\
0 & 1 & 0 & -\rho_1(\tilde{k}) \\
-\rho_2(k) & 0 & 1 & 0 \\
0 & -\rho_2(\tilde{k}) & 0 & 1
\end{bmatrix},
\]

and $D_{n}(k)$ is given by

\[
D_{n_1}(k) := \begin{bmatrix}
\pi(k)^n & 0 & 0 & 0 \\
0 & \pi(\tilde{k})^n & 0 & 0 \\
0 & 0 & \pi(k)^n & 0 \\
0 & 0 & 0 & \pi(\tilde{k})^n
\end{bmatrix}, \quad D_{n}(k) := \begin{bmatrix}
0 & 0 & \pi_{21}(k,n) & 0 \\
0 & 0 & 0 & \pi_{21}(\tilde{k},n) \\
\pi_{12}(k,n) & 0 & 0 & 0 \\
0 & \pi_{12}(\tilde{k},n) & 0 & 0
\end{bmatrix}
\]

for $n$ even and for $n$ odd, respectively, whose entries are $\pi(k) := (\rho_1(k)\rho_2(k))^{1/2}$, $\pi_{12}(k,n) := \rho_1(k)^{(n-1)/2}\rho_2(k)^{(n+1)/2}$, and $\pi_{21}(k,n) := \rho_1(k)^{(n+1)/2}\rho_2(k)^{(n-1)/2}$.

Proof. We consider the case in which both $n_1$ and $n_2$ are even. The other cases can be obtained by similar arguments. Since $n_1$ is even, we have that

\[
G_h^{n_1} = \begin{bmatrix}
(G_{h,1} G_{h,2})^{n_1/2} & 0 \\
0 & (G_{h,2} G_{h,1})^{n_1/2}
\end{bmatrix}.
\]

Because of the relation $(G_{h,1} G_{h,2})^{n_1/2} \psi_k = (G_{h,2} G_{h,1})^{n_1/2} \psi_k = \pi^{n_1}(k) \psi_k$, where $\pi(k) := (\rho_1(k)\rho_2(k))^{1/2}$, we get

\[
G_h^{n_1} \begin{bmatrix}
\psi_k & \psi_k & 0 & 0 \\
0 & 0 & \psi_k & \psi_k \\
0 & 0 & \psi_k & \psi_k \\
0 & 0 & \psi_k & \psi_k
\end{bmatrix} = \begin{bmatrix}
\psi_k & \psi_k & 0 & 0 \\
0 & 0 & \psi_k & \psi_k \\
0 & 0 & \psi_k & \psi_k \\
0 & 0 & \psi_k & \psi_k
\end{bmatrix} D_{n_1}(k).
\]

Similarly, we obtain that $G_h^{n_2} \begin{bmatrix}
\psi_k & \psi_k & 0 & 0 \\
0 & 0 & \psi_k & \psi_k \\
0 & 0 & \psi_k & \psi_k \\
0 & 0 & \psi_k & \psi_k
\end{bmatrix} = \begin{bmatrix}
\psi_k & \psi_k & 0 & 0 \\
0 & 0 & \psi_k & \psi_k \\
0 & 0 & \psi_k & \psi_k \\
0 & 0 & \psi_k & \psi_k
\end{bmatrix} D_{n_2}(k)$. Moreover, direct calculations reveal that

\begin{equation}
A_h \begin{bmatrix}
\psi_k & \psi_k & 0 & 0 \\
0 & 0 & \psi_k & \psi_k \\
0 & 0 & \psi_k & \psi_k \\
0 & 0 & \psi_k & \psi_k
\end{bmatrix} = \begin{bmatrix}
\psi_k & \psi_k & 0 & 0 \\
0 & 0 & \psi_k & \psi_k \\
0 & 0 & \psi_k & \psi_k \\
0 & 0 & \psi_k & \psi_k
\end{bmatrix} D_{n_2}(k). \quad \text{Moreover,}
\end{equation}

and

\begin{equation}
P_{2h}^h \begin{bmatrix}
\psi_k & \psi_k & 0 & 0 \\
0 & 0 & \psi_k & \psi_k \\
0 & 0 & \psi_k & \psi_k \\
0 & 0 & \psi_k & \psi_k
\end{bmatrix} = \begin{bmatrix}
\phi_k & 0 & 0 & 0 \\
0 & \phi_k & 0 & 0 \\
0 & 0 & \phi_k & 0 \\
0 & 0 & 0 & \phi_k
\end{bmatrix} H(k),
\end{equation}

\begin{equation}
P_{2h}^h \begin{bmatrix}
\psi_k & \psi_k & 0 & 0 \\
0 & 0 & \psi_k & \psi_k \\
0 & 0 & \psi_k & \psi_k \\
0 & 0 & \psi_k & \psi_k
\end{bmatrix} = \begin{bmatrix}
\phi_k & 0 & 0 & 0 \\
0 & \phi_k & 0 & 0 \\
0 & 0 & \phi_k & 0 \\
0 & 0 & 0 & \phi_k
\end{bmatrix} V(k)^\top,
\end{equation}
where we used (4.5). Therefore, we obtain $R_{2h}^h A_h G_h \begin{bmatrix} \psi_k & \psi_k & 0 & 0 \\ 0 & \psi_k & 0 & 0 \end{bmatrix} = \begin{bmatrix} \phi_k & 0 \\ 0 & \phi_k \end{bmatrix} \Lambda_1(k)$.

We now study the action of the coarse matrix $A_{2h}$ on $\begin{bmatrix} \phi_k \\ 0 \end{bmatrix}$. We use (4.5), (4.10) and (4.11) to write

\[
A_{2h} \begin{bmatrix} \phi_k \\ 0 \end{bmatrix} = R_{2h}^h A_h P_{2h}^h \begin{bmatrix} \phi_k \\ 0 \end{bmatrix} = R_{2h}^h A_h \begin{bmatrix} \psi_k \\ 0 \\ \psi_k \end{bmatrix} V(k) = R_{2h}^h \begin{bmatrix} \psi_k \\ 0 \\ \psi_k \end{bmatrix} H(k) V(k) = \begin{bmatrix} \phi_k \\ 0 \\ \phi_k \end{bmatrix} V(k)^\top H(k) V(k).
\]

Thus, we have $A_{2h} \begin{bmatrix} \phi_k \\ 0 \end{bmatrix} = \begin{bmatrix} \phi_k \\ 0 \end{bmatrix} \Lambda_2(k)$. Hence, recalling Lemma 4.1 we get

\[
(4.12) \quad \begin{bmatrix} \phi_k \\ 0 \end{bmatrix} = A^{-1}_{h,c} \begin{bmatrix} \phi_k \\ 0 \end{bmatrix} \Lambda_2(k).
\]

A direct calculation reveals that the eigenvalues of $\Lambda_2(k)$ are $\lambda_{1,2} = c_k^2 + s_k^2 \pm \sqrt{(c_k^4 \rho_1(k) + s_k^4 \rho_1(k))(c_k^4 \rho_2(k) + s_k^4 \rho_2(k))}$ and they are nonzero for $k = 1, \ldots, N_c$. Hence, $\Lambda_2(k)$ is invertible and, using (4.12), we get

\[
A^{-1}_{h,c} \begin{bmatrix} \phi_k \\ 0 \end{bmatrix} \Lambda_1(k) = A^{-1}_{h,c} \begin{bmatrix} \phi_k \\ 0 \end{bmatrix} \Lambda_2(k) \Lambda_2^{-1}(k) \Lambda_1(k) = \begin{bmatrix} \phi_k \\ 0 \end{bmatrix} \Lambda_2^{-1}(k) \Lambda_1(k).
\]

Summarizing our results and using the definition of $T_h$, we conclude that

\[
T_h \begin{bmatrix} \psi_k & \psi_k & 0 & 0 \\ 0 & \psi_k & 0 & 0 \end{bmatrix} = \begin{bmatrix} \psi_k & \psi_k & 0 & 0 \\ 0 & \psi_k & 0 & 0 \end{bmatrix} D_{n_2}(k) \begin{bmatrix} c_k^2 & 0 \\ -s_k^2 & 0 \\ 0 & c_k^2 \\ 0 & -s_k^2 \end{bmatrix} \Lambda_2^{-1}(k) \Lambda_1(k)
\]

and our claim follows.

**Theorem 4.3** (Factorization of the iteration matrix $T_h$). There exists an invertible matrix $Q$ such that $T_h = Q \tilde{G} Q^{-1}$, where the G2S iteration matrix $T_h$ is defined in Lemma 4.2 and

\[
\tilde{G} = \begin{bmatrix} \tilde{G}_1 \\ \vdots \\ \tilde{G}_{N_c} \\ \gamma_1 \left(\frac{N_h+1}{2}\right) \\ \gamma_2 \left(\frac{N_h+1}{2}\right) \end{bmatrix},
\]

where the matrices $\tilde{G}_k \in \mathbb{R}^{4 \times 4}$ are defined in Lemma 4.2 and $\gamma_j \left(\frac{N_h+1}{2}\right)$ depend on $n_1$, $n_2$ and the eigenvalues $\rho_j \left(\frac{N_h+1}{2}\right)$ of $G_{h,j}$, for $h = 1, 2$.

**Proof.** We define the invertible matrix

\[
Q = \begin{bmatrix} \psi_1 & \psi_{N_h} & 0 & 0 & \ldots & \psi_{N_c} & \psi_{N_c+2} & 0 & 0 & \psi_{N_h} & \psi_{N_h+1} & 0 & 0 \\ 0 & 0 & \psi_1 & \psi_{N_h} & \ldots & 0 & 0 & \psi_{N_c} & \psi_{N_c+2} & 0 & 0 & \psi_{N_h} & \psi_{N_h+1} \end{bmatrix}.
\]
Moreover, we observe that regardless of the values of \( \rho \) (pre- or post-) smoothing step is necessary for the G2S method to converge. In case of Schwarz methods. Notice also that Theorem 4.4 guarantees that only one spectral radii of \( \tilde{G} \) an optimal result would be a direct calculation of the spectral radii of the matrices \( T \) where the expressions of \( \gamma \) are mapped to zero by the restriction operator, \( R_{2h}^{R} = G_{h}^{-1} G_{h}^{n_1} = 0 \), and we get

\[
T_{h} \begin{bmatrix} \psi_{N_{h}+1}^{k} & 0 \\ \psi_{N_{h}+1}^{k} & 0 \\ \psi_{N_{h}+1}^{k} & 0 \end{bmatrix} = G_{h}^{n_2} G_{h}^{n_1} \begin{bmatrix} \psi_{N_{h}+1}^{k} & 0 \\ \psi_{N_{h}+1}^{k} & 0 \\ \psi_{N_{h}+1}^{k} & 0 \end{bmatrix} = \begin{bmatrix} \gamma_{1}(N_{h}+1) \psi_{N_{h}+1}^{k} & 0 \\ 0 & \gamma_{2}(N_{h}+1) \psi_{N_{h}+1}^{k} \end{bmatrix},
\]

where the expressions of \( \gamma_{1}(N_{h}+1) \) and \( \gamma_{2}(N_{h}+1) \) depend on \( n_1 \) and \( n_2 \). For instance if \( n_1 + n_2 \) is an even number, then \( \gamma_{1}(N_{h}+1) = \gamma_{2}(N_{h}+1) = (\rho_{1}(N_{h}+1)\rho_{2}(N_{h}+1))^{n_1+n_2} \). Hence, we conclude that \( T_{h} Q = Q G \) and our claim follows.

Theorem 4.3 allows one to obtain convergence results of a G2S method. Clearly, an optimal result would be a direct calculation of the spectral radii of the matrices \( G_{h} \). However, this is in general a difficult task that requires cumbersome calculations. Nevertheless, in Theorem 4.4 we are capable to obtain an explicit expression for the spectral radii of \( G_{h} \) under some reasonable assumptions that are in general satisfied in case of Schwarz methods. Notice also that Theorem 4.4 guarantees that only one (pre- or post-) smoothing step is necessary for the G2S method to converge.

**Theorem 4.4.** Assume that \( 1 > \rho_{1}(k) = \rho_{2}(k) = \rho(k) \geq 0 \) for any \( k \) and that \( \rho(k) \) is a decreasing function of \( k \). The convergence factor of the G2S method is

\[
\rho_{G2S}(T_{h}) = \max_{k \in \{1, \ldots, N_{c}, \frac{N_{h}}{2} - 1\}} \left( \frac{c_{k}^{4}(1 - \rho(k)) \rho(\tilde{k})^{n_1+n_2} + s_{k}^{4}(1 - \rho(\tilde{k})) \rho(k)^{n_1+n_2}}{c_{k}^{4}(1 - \rho(k)) + s_{k}^{4}(1 - \rho(k))} \right) < 1.
\]

**Proof.** The convergence factor of the G2S is given by the spectral radius of the iteration matrix \( T_{h} \). Theorem 4.3 implies that

\[
\rho_{G2S}(T_{h}) = \max_{k \in \{1, \ldots, N_{c}\}} \rho(\tilde{G}_{k}), \gamma_{1} \left( \frac{N_{h} + 1}{2} \right), \gamma_{2} \left( \frac{N_{h} + 1}{2} \right).
\]

Regardless of the values of \( n_1 \) and \( n_2 \), direct calculations show that the matrices \( \tilde{G}_{k} \) have four eigenvalues:

\[
\lambda_{1}(k) = \lambda_{2}(k) = 0,
\]

\[
|\lambda_{3}(k)| = \frac{c_{k}^{4}(1 - \rho(k)) \rho(\tilde{k})^{n_1+n_2} + s_{k}^{4}(1 - \rho(\tilde{k})) \rho(k)^{n_1+n_2}}{c_{k}^{4}(1 - \rho(k)) + s_{k}^{4}(1 - \rho(k))},
\]

\[
|\lambda_{4}(k)| = \frac{c_{k}^{4}(1 + \rho(k)) \rho(\tilde{k})^{n_1+n_2} + s_{k}^{4}(1 + \rho(\tilde{k})) \rho(k)^{n_1+n_2}}{c_{k}^{4}(1 + \rho(k)) + s_{k}^{4}(1 + \rho(\tilde{k}))}.
\]

Moreover, we observe that

\[
|\lambda_{3}(k)| - |\lambda_{4}(k)| = \frac{2c_{k}^{4}s_{k}^{4}(\rho(k) - \rho(\tilde{k}))((\rho(k) - \rho(\tilde{k}))^{n_1+n_2} - (\rho(k)^{n_1+n_2} - \rho(\tilde{k})^{n_1+n_2})}{(\rho(k) + 1)c_{k}^{4} + s_{k}^{4}(\rho(k) + 1))((1 - \rho(k))c_{k}^{4} + s_{k}^{4}(1 - \rho(k))} \geq 0,
\]

where we used the monotonicity of \( \rho(k) \). On the other hand, since \( \rho_{1}(k) = \rho_{2}(k) = \rho(k) \), we have \( \gamma_{1}(N_{h}+1) = \gamma_{2}(N_{h}+1) = \rho(\frac{N_{h}+1}{2})^{n_1+n_2} \). Therefore we have that
and the result follows by observing that \( \lambda_3 \left( \frac{N_h+1}{2} \right) = \rho \left( \frac{N_h+1}{2} \right)^{n_1+n_2} \), since \( \rho(k) = \rho(k) \) for \( k = \frac{N_h+1}{2} \).

5. Two-level substructured and volumetric methods. In the previous sections, we have introduced our new two-level DD methods and showed that they are intimately related, since the G2S method can be regarded as an S2S method. It is fair at this point to pose the following questions. Is there any relation between these two methods and two-level DD methods in volume? How are these two methods related to other classical two-level/multi-level method? The answer to these questions is the content of this section.

Let \( A_v \) be a discretization of our problem (2.1). In particular, \( A_v \in \mathbb{R}^{N^v \times N^v} \) is the discretization of the elliptic operator \( L \), while \( u \in \mathbb{R}^{N^v} \) and \( f \in \mathbb{R}^{N^v} \) are the discrete counterparts of the solution \( u \) and the right-hand side function \( f \). Consider the following splittings of the matrix \( A_v \):

\[
A_v = \begin{bmatrix} A_1 & E_1 \tilde{R}_1 \\ \times & \times \end{bmatrix} = \begin{bmatrix} \times & \times \\ E_2 \tilde{R}_2 & A_2 \end{bmatrix},
\]

where \( A_j \in \mathbb{R}^{N_j^v \times N_j^v} \) for \( j = 1, 2 \). We assume that \( A_v \), \( A_1 \) and \( A_2 \) are invertible. The matrices \( \tilde{R}_1 \in \mathbb{R}^{N_1^v \times (N^v - N_1^v)} \) and \( \tilde{R}_2 \in \mathbb{R}^{N_2^v \times (N^v - N_2^v)} \) are restriction operators that take as input vectors of sizes \( N^v - N_1^v \) and \( N^v - N_2^v \) and returns as output interface vectors of sizes \( N_1 \) (interface \( \Gamma_1 \)) and \( N_2 \) (interface \( \Gamma_2 \)). The two matrices \( E_1 \in \mathbb{R}^{N_1^v \times N_1} \) and \( E_2 \in \mathbb{R}^{N_2^v \times N_2} \) are extension by zero operators. In order to obtain a discrete substructured problem, we define first the so-called augmented system in volume as

\[
A_a u_a = f_a,
\]

where \( A_a = \begin{bmatrix} A_1 & E_1 R_1 \\ E_2 R_2 & A_2 \end{bmatrix} \), \( u_a = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \), and \( f_a = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \), with \( A_j \in \mathbb{R}^{N_j^v \times N_j^v} \) and \( u_j, f_j \in \mathbb{R}^{N_j} \) for \( j = 1, 2 \). The matrices \( R_1 \in \mathbb{R}^{N_1 \times N_2} \) and \( R_2 \in \mathbb{R}^{N_2 \times N_1} \) are restriction operators that take as input volume vectors of sizes \( N_2 \) (second subdomain) and \( N_1 \) (first subdomain) and return as output interface vectors of size \( N_j \) (interface \( \Gamma_j \)) and \( N_2 \) (interface \( \Gamma_2 \)). Notice that \( R_j R_j^T = I_{N_j} \), the identity of size \( N_j \), for \( j = 1, 2 \). Moreover, we define \( N^{s} := N_1 + N_2 \) and \( N^{o} := N_1^o + N_2^o \).

The interface vectors \( v_{21} := R_1 u_2 \) and \( v_{12} := R_2 u_1 \) solve the discrete substructured system

\[
A_s \begin{bmatrix} v_{12} \\ v_{21} \end{bmatrix} = \begin{bmatrix} R_2 A_1^{-1} f_1 \\ R_1 A_2^{-1} f_2 \end{bmatrix},
\]

where \( A_s = \begin{bmatrix} I_{N_2} & R_2 A_1^{-1} E_1 \\ R_1 A_2^{-1} E_2 & I_{N_1} \end{bmatrix} \), which was denoted by \( A_h \) in (4.2) in Section 4.

The vectors \( v_{12} \) and \( v_{21} \) are the restrictions on the interfaces \( \Gamma_2 \) and \( \Gamma_1 \) of the solution vectors \( u_1 \) and \( u_2 \), and (5.2) is the substructured form of (5.1). Notice that (5.2) is the discrete counterpart of the substructured problem (2.6).
The block-Jacobi method applied to (5.1) and (5.2) leads to the iteration matrices

\[ G_a = \begin{bmatrix} 0 & -A_1^{-1}E_1R_1 \\ -A_2^{-1}E_2R_2 & 0 \end{bmatrix} \quad \text{and} \quad G_s = \begin{bmatrix} 0 & -R_2A_1^{-1}E_1 \\ -R_1A_2^{-1}E_2 & 0 \end{bmatrix}, \]

where \( G_s \) is the discretization of \( G \) defined in (2.9), which was denoted by \( G_h \) in Section 4.

Let us now introduce the matrices

\[ D := \begin{bmatrix} A_1^{-1} & 0 \\ 0 & A_2^{-1} \end{bmatrix}, \quad \tilde{T} := \begin{bmatrix} R_2 & 0 \\ 0 & R_1 \end{bmatrix} \quad \text{and} \quad \tilde{E} := \begin{bmatrix} 0 & E_1 \\ E_2 & 0 \end{bmatrix}. \]

It is easy to verify the relations

\[ \tilde{T}\tilde{T}^\top = I_{N^s}, \quad A_s\tilde{T} = \tilde{T}D\tilde{T}, \quad G_a = \tilde{T}\tilde{E}\tilde{T} \quad \text{and} \quad G_s\tilde{T} = \tilde{T}G_a. \]

In particular, the relation \( \tilde{T}\tilde{T}^\top = I_{N^s} \) is trivial, and \( A_s\tilde{T} = \tilde{T}D\tilde{T} \) can be obtained by calculating

\[ \tilde{T}D\tilde{T} = \begin{bmatrix} R_2 & 0 \\ 0 & R_1 \end{bmatrix} \begin{bmatrix} A_1^{-1} & 0 \\ 0 & A_2^{-1} \end{bmatrix} \left[ \begin{array}{cc} E_1R_1 & 0 \\ 0 & R_1 \end{array} \right] = \begin{bmatrix} I_{N_s} & A_1^{-1}E_1R_1 \\ 0 & R_1 \end{bmatrix}. \]

A similar calculation allows us to obtain that \( G_s\tilde{T} = \tilde{T}G_a \).

Since the matrices \( G_s \) and \( G_a \) are different representations of the PSM, one expects that their spectra coincide. This is shown in the next lemma.

**Lemma 5.1.** The matrices \( G_s \in \mathbb{R}^{N^s \times N^s} \) and \( G_a \in \mathbb{R}^{N^a \times N^a} \) have the same non-zero eigenvalues, that is \( \sigma(G_s) = \sigma(G_a) \setminus \{0\} \).

**Proof.** Recalling the structure of \( G_a \), one can clearly see that \( \text{rank}(G_a) = N^a \), because the matrices \( E_jR_j \) have rank \( N_j \) for \( j = 1, 2 \). Hence \( G_a \) has \( N^a \) nonzero eigenvalues. Take any eigenvector \( v \in \mathbb{R}^{N^a} \) of \( G_a \) with eigenvalue \( \lambda \neq 0 \). We note that \( \tilde{T}v \neq 0 \), otherwise we would have \( G_a v = \tilde{T}\tilde{E}\tilde{T}v = 0 \), which contradicts the hypothesis \( \lambda \neq 0 \). Using the last relation in (5.3), we write \( G_s\tilde{T}v = \tilde{T}G_a v = \lambda\tilde{T}v \). Hence \( (\tilde{T}v, \lambda) \) is an eigenpair of \( G_s \). Since this holds for any eigenpair \( (v, \lambda) \) of \( G_a \), the result follows. \( \square \)

Let us now consider arbitrary restriction and prolongation operators \( R_s \) and \( P_s \) (with \( R_s = P_s^\top \)), which can correspond to the ones used for the S2S or G2S methods. Our discrete substructured two-level iteration matrix is then given by

\[ G_s^{2L} := [I_{N^s} - P_s(R_sA_sP_s)^{-1}R_sA_s]G_s. \]

The goal is to find a volumetric two-level iteration operator \( G_a^{2L} \) that has the same spectrum of \( G_s^{2L} \). This volumetric operator must be formulated for the augmented system (5.1) and based on the iteration matrix \( G_a \). To do so, we recall the relations
and compute

\[
G_s^{2L} = [I_{N^2} - P_s(R_sA_sP_s)^{-1}R_sA_s]G_a \tilde{T} \\
= [I_{N^2} - P_s(R_sA_sP_s)^{-1}R_sA_s] \tilde{T} G_a \\
= [\tilde{T} - P_s(R_sA_sP_s)^{-1}R_sA_s \tilde{T}] G_a \\
= \tilde{T} [I_{N^2} - \tilde{T}^T P_s(R_sA_sP_s)^{-1}R_sA_s \tilde{T}] G_a \\
= \tilde{T} [I_{N^2} - \tilde{T}^T P_s(R_sA_s \tilde{T}^T P_s)^{-1}R_s \tilde{T} DA_a] G_a \\
= \tilde{T} [I_{N^2} - \tilde{T}^T P_s(R_a \tilde{T} DA_a \tilde{T}^T P_s)^{-1}R_a \tilde{T} DA_a] G_a \\
= \tilde{T} [I_{N^2} - \tilde{T}^T P_a(R_a DA_a P_a)^{-1}R_a DA_a] G_a = \tilde{T} G_a^{2L},
\]

where we defined \( P_a := \tilde{T}^T P_s, R_a := R_a \tilde{T} = P_a^T \) and

\[
G_a^{2L} := [I_{N^2} - P_a(R_a DA_a P_a)^{-1}R_a DA_a] G_a.
\]

We obtained that \( G_s^{2L} \tilde{T} = \tilde{T} G_a^{2L} \). Moreover, an argument similar to the one used to prove Lemma 5.1 allows us to show that the spectra of \( G_s^{2L} \) and \( G_a^{2L} \) coincide in the sense that \( \sigma(G_s^{2L}) = \sigma(G_a^{2L}) \setminus \{0\} \). This means that we have found a two-level volumetric iteration operator that is spectrally equivalent to our substructured two-level operator. Moreover, for any invertible matrix \( U \in \mathbb{R}^{N^2 \times N^2} \) we can repeat the calculations done in (4.7), to obtain

\[
G_a^{2L} = [I_{N^2} - \tilde{P}_a(R_a DA_a \tilde{P}_a)^{-1} \tilde{R}_a DA_a] G_a,
\]

where \( \tilde{P}_a = P_a U \) and \( \tilde{R}_a = U^{-1} R_a \) (with \( \tilde{R}_a = \tilde{P}_a^T \) if \( U \) is orthogonal). This means that there exists many two-level DD methods in volume that are equivalent to our substructured two-level methods.

We can summarize the obtained result in the following theorem.

**Theorem 5.2 (Volumetric formulation of substructured methods).** Consider the substructured two-level iteration operator \( G_a^{2L} \) given in (5.4) and denote its spectrum by \( \sigma(G_a^{2L}) \). For any invertible matrix \( U \in \mathbb{R}^{N^2 \times N^2} \), the spectrum of the matrix \( G_a^{2L} \) given in (5.6) satisfies the relation \( \sigma(G_a^{2L}) = \sigma(G_s^{2L}) \setminus \{0\} \).

The matrix \( G_a^{2L} \) has a special structure. Since \( D \) represents the block-Jacobi preconditioner for the augmented system (5.1), one can say that \( G_a^{2L} \) correspond to a two-level method applied to the preconditioned system \( DA_u = Du \), in a similar spirit of the smoothed aggregation method defined in [3, Section 2].

Let us now consider the question: what is the relation between our G2S method and a two-grid (volumetric) method that uses the same smoother (PSM)? A two-grid method in volume applied to the augmented system (5.1), would correspond to an iteration operator \( \tilde{G}_a^{2L} \) of the form

\[
\tilde{G}_a^{2L} = [I_{N^2} - \tilde{P}_a(R_a A_a \tilde{P}_a)^{-1} \tilde{R}_a A_a] G_a.
\]

Natural choices for \( \tilde{P}_a \) and \( \tilde{R}_a \) are the usual (volumetric) restriction and prolongation operators. For example, for a one-dimensional problem a natural choice is the prolongation matrix \( \tilde{P}_a \) given in (4.3) and \( \tilde{R}_a = \frac{1}{2} P_a^T \). On the other hand, our prolongation
The operator $P_a := \tilde{T}^T P_s$ is an extension by zero of a coarse interface vector to a fine volumetric vector. Moreover, $R_a := R_s \tilde{T}$ restricts a fine volumetric vector $v$ to a coarse interface vector by only interpolating the components of $v$ belonging to the (fine) interfaces. Another crucial difference is that $G_{2L}^a$ is constructed on $DA_a$, while $\hat{G}_{2L}^a$ is obtained using the matrix $A_a$. Therefore, $G_{2L}^a$ is constructed on the original augmented system $A_a u_a = f_a$, while $\hat{G}_{2L}^a$ is defined over the preconditioned system $DA_a u_a = \hat{D} f_a$.

These facts indicate clearly that our method is by far distant from a classical volumetric two-grid method that uses the PSM as smoother. This is also confirmed by the numerical results shown in Figure 5.1, where the spectral radii of three different two-level iteration matrices are depicted. In particular, we consider exactly the Laplace problem used in Section 4.2 with $\tilde{L} = 1$ and $L = \frac{1}{2}$ (hence $\Omega$ is a unit square). The problem is discretized using a classical second-order finite-difference scheme with a uniform grid of size $h = \frac{1}{N_h + 1}$, where $N_h = 2^\ell - 1$. The length of the overlap is $\delta = (N_{ov} + 1)h$, for some positive odd integer $N_{ov}$. We consider three different iterations matrices $G_{2L}^s$, $G_{2L}^a$ and $G_{2L}^{RAS}$. The first one $G_{2L}^s$ is the iteration matrix corresponding to our G2S method. The second one $G_{2L}^a$ is the iteration matrix of a two-level method applied on the augmented volumetric system (5.1). In both cases, the same classical Schwarz method is used as smoother. The third matrix $G_{2L}^{RAS}$ is the iteration operator of a classical two-grid method applied to the volumetric system $A_a u_a = f_a$ and using as smoother the RAS method. In all cases, restriction and prolongation operators correspond to linear interpolation matrices (as in (4.3)) and to the full weighting restriction matrices, respectively. Indeed, for our G2S method these are one-dimensional operators, while for the other two methods they are two-dimensional operators. In particular, for the augmented system these interpolation and restriction operators take into account the non-zero values of the discrete functions on the interfaces. For the two-level RAS method, they are obtained by a two-dimensional extension of (4.3).

In Figure 5.1, we show the spectral radii of $G_{2L}^s$, $G_{2L}^a$ and $G_{2L}^{RAS}$, obtained by a direct numerical computation, as a function of $N_{ov}$, hence the size of the overlap. The two figures correspond to two different discretizations. It is clear that our G2S method outperforms the other two methods, which have also very small contraction factors. Moreover, it is clear that the coarse correction makes all the methods very robust with respect to the number of discretization points.
6. Implementation details and multilevel algorithm. In this section, we discuss some important remarks about the implementation of our substructured two-level methods. In Section 6.1, after discussing pro and contra of substructured and volume two-level methods, we present equivalent and computationally more efficient forms of Algorithm 3.1. This is essential to make our framework computationally equal or more efficient than other existing strategies. In Section 6.2, we explain how to extend our G2S method to a multi-grid strategy.

6.1. A practical form of two-level substructured methods. One of the advantages of our new substructured framework is that most of the computations are performed with objects (vectors, matrices, arrays, etc.) that are defined on the interfaces and hence having a much smaller dimension. This is clear if one carefully studies Algorithm 3.1, where for example the products $R r$ and $P u$ are performed on interface vectors. In volume two-level methods, the same prolongation and restriction operators involve volume entities, thus their application is more costly and they might be generally more difficult to implement due to the higher dimensions. For the same reasons, less memory storage is required. Moreover substructured two-level methods do not need to extend in volume the coarse space functions that, in volume methods, are defined either on the overlap or on the interfaces. We remark that there is not a unique way to extend these functions and we refer to [40, Section 5] for an overview.

We now compare the computational costs of the S2S and of a spectral 2-level method in volume per iteration. Let $N^v$ be the size of the volume matrix $A_v = M - N$ and $N^s$ the size of the substructured matrix $A (N^s \ll N^v)$. The size of each subdomain is $N_{sub}$ and $2m$ is the dimension of the coarse space. The restriction and prolongation operators are $R_v, R_s, P_v, P_s$. For simplicity we assume $n_1 = 1, n_2 = 0$.

We now discuss the cost of the off-line and on-line computation phases. The off-line pre-computation phases for the substructured and volume coarse matrices have the same cost. For the substructured case, the bottleneck consists in the application of $A$ which consists in the solution of subdomains problems. These can be performed in parallel, and thus the cost is $N_{sub}^3$. In the volume case, the application of $A_v$ is just a standard matrix-vector multiplication, but to build prolongation and restriction operators, we need to extend the coarse space functions defined either on the overlap or on the interfaces and this costs again $N_{sub}^3$. The on-line costs reported in Table 6.1 show that one iteration of the S2S may be more expensive than a spectral two-level method in volume, since the computation of the residual requires the solution of subdomain problems in the substructured case.

To avoid this extra cost per iteration, we use the special form of the matrix $A = I - G$. We propose two new versions of Algorithm 3.1, which we call S2S-B1 and S2S-B2 (respectively with G2S). These are Algorithms 6.1 and 6.2. Notice
that Algorithm 6.1 requires for the first iteration two applications of the smoothing operator $G$, namely two subdomains solves. The next iterations, which are represented by Steps 6-10, need only one application of the smoothing operator $G$. Theorem 6.1 (a) shows that Algorithm 6.1 is equivalent to Algorithm 3.1. This means that each iteration after the first one of Algorithm 6.1 is computationally less expensive than one iteration of a volume two-level DD method. Since two-level DD methods perform generally few iterations, it could be important to get rid of the expensive first iteration.

For this reason, we introduce Algorithm 6.2, which overcome the problem of the first iteration. Theorem 6.1 (b) shows that Algorithm 6.2 is exactly an S2S method with no pre-smoothing and one post-smoothing step. Moreover, it has the same convergence behavior of Algorithm 6.1.

**Theorem 6.1 (Equivalence between S2S, S2S-B2 and S2S-B1).**

(a) Algorithm 6.1 generates the same iterates of Algorithm 3.1.

(b) Algorithm 6.2 corresponds to the stationary iterative method

$$
\mathbf{u}^n = G(\mathbb{I} - PA_c^{-1}RA)\mathbf{u}^{n-1} + \tilde{M}\mathbf{b},
$$

where $G(\mathbb{I} - PA_c^{-1}RA)$ is the iteration matrix and $\tilde{M}$ the relative preconditioner. Moreover, Algorithm 6.2 and Algorithm 6.1 have the same convergence behavior.

**Proof.** For simplicity, we suppose to work with the error equation and thus $\mathbf{b} = 0$. We call $\tilde{u}^0$ the output of the first five steps of Algorithm 6.1 and with $\hat{u}^0$ the output of Algorithm 3.1. Then given an initial guess $\mathbf{u}^0$, we have

$$
\tilde{u}^0 = \mathbf{u}^1 + Pd = \mathbf{u}^1 + PA_c^{-1}R(-\mathbf{u}^1 + \mathbf{v}) = Gu^0 + PA_c^{-1}R(-AGu^0) = (\mathbb{I} - PA_c^{-1}RA)Gu^0 = \hat{u}^0.
$$

Similar calculations show that also steps 6-10 of S2S-B1 are equivalent to an iteration of 3.1. For the second part of the Theorem, we write the iteration matrix for Algorithm 6.2 as

$$
\mathbf{u}^1 = \mathbf{v} + \tilde{P}d = Gu^0 + GPA_c^{-1}R(-A\mathbf{u}^0) = G(\mathbb{I} - PA_c^{-1}RA)\mathbf{u}^0.
$$

Hence, Algorithm 6.2 performs a post-smoothing step instead of a pre-smoothing step.
as Algorithm 6.1 does. The method still has the same convergence behavior since the matrices \( G(\mathbb{I} - PA^{-1}R) \) and \( (\mathbb{I} - PA^{-1}R)G \) have the same eigenvalues\(^4\).

Notice that, the reformulations S2S-B1 and S2S-B2 require to store the matrix \( \tilde{P} := GP \), which is anyway needed in the assembly phase of the coarse matrix, hence no extra cost is required, if compared to a volume two-level DD method. Finally, we stress that these implementation tricks can be readily generalized to a general number of pre- and post-smoothing steps.

Concerning the specific implementation details for the G2S, we remark that one can lighten the off-line assembly of the matrix \( A_{2h} = R_{2h}^h A_h P_{2h}^h \), using instead the matrix

\[
\tilde{A}_{2h} := \begin{bmatrix}
I_{2h,2} & -G_{2h,1} \\
-G_{2h,2} & I_{2h,1}
\end{bmatrix},
\]

which corresponds to a direct discretization of (2.5) on the coarse mesh, see also (4.2). Moreover, since our two-level method works directly on the interfaces, we have more freedom in the discretization of the smoothing operators \( G_j \), \( j = 1, 2 \) on each level. For instance, on each level, we could keep the corresponding volume mesh in a neighborhood of the interfaces, while away from them we could consider a coarser grid. In Table 7.2, we show a comparison of computational times between the three different implementations of the G2S method and a two-level RAS method.

**6.2. Extension to multilevel framework.** Classical two-level grid methods in volume are, sometimes, still not very efficient for the solution of large problems. This is mainly due to the dimension of the coarse space, which is too large in volume, namely about half of the full dimension. In our substructured framework, the size of the substructured coarse matrix corresponds to the number of degrees of freedom on the coarse interfaces, and thus it is already much smaller with respect to the volume case (see Section 7.1 for a comparison of their sizes in a model problem). However, they might be problems for which the direct solution of the coarse problem is inconvenient also in the substructured framework. For instance, if we considered multiple subdomains, then we would have several interfaces and therefore the size of the substructured coarse matrix increases.

The G2S is suitable to a multilevel generalization following a classical multi-grid strategy [38]. Given a sequence of grids on the two interfaces labeled from the coarsest to the finest by \( \{\ell_{\min}, \ell_{\min} + 1, \ldots, \ell_{\max}\} \), we denote with \( P_{\ell_{-1}}^\ell \) and \( R_{\ell_{-1}}^\ell \) the interpolation and restriction operators between grids \( \ell \) and \( \ell - 1 \). To build the substructured matrices on the different grids we have two possible choices. The first one corresponds to the standard Galerkin projection. Being \( A_{\ell_{\max}} \) the substructured matrix on the finest grid, we can define for \( \ell \in \{\ell_{\min}, \ell_{\min} + 1, \ldots, \ell_{\max} - 1\} \),

\[
A_\ell := R_{\ell+1}^{\ell+1} A_{\ell+1} R_{\ell}^{\ell+1},
\]

the second choice consists to define \( A_\ell \) directly as the discretization of (2.8) on the grid labeled by \( \ell \). It corresponds exactly to (6.1) for the two-grid case. The two choices are not equivalent. On the one hand, the Galerkin approach leads to a faster method in terms of iteration number. However, the Galerkin matrices \( A_\ell \) do not have the block structure as in (2.8). For instance, \( A_{\ell_{\max} - 1} = R_{\ell_{\max} - 1}^{\ell_{\max} - 1} A_{\ell_{\max}} P_{\ell_{\max} - 1}^{\ell_{\max} - 1} = R_{\ell_{\max} - 1}^{\ell_{\max} - 1} P_{\ell_{\max} - 1}^{\ell_{\max} - 1} - R_{\ell_{\max} - 1}^{\ell_{\max} - 1} G_{\ell_{\max}} P_{\ell_{\max} - 1}^{\ell_{\max} - 1} \). Thus, the identity matrix is replaced by the sparse matrix \( R_{\ell_{\max} - 1}^{\ell_{\max} - 1} P_{\ell_{\max} - 1}^{\ell_{\max} - 1} \). On the other hand, defining \( A_\ell \) directly on the current grid \( \ell \) as in (6.1) leads to a minimum increase of the iteration number but permits to conserve the original block-diagonal structure.

---

\(^4\)Given two matrices \( A \) and \( B \), \( AB \) and \( BA \) share the same non-zero eigenvalues.
Algorithm 6.3 Geometric multilevel substructured domain decomposition method function GMLS($u^0, b, \ell$)

if $\ell = \ell_{\text{min}}$, then
  set $u^0 = A_{\ell_{\text{min}}}^{-1} b$. (direct solver)
else
  $u^n = G u^{n-1} + b$, $n = 1, \ldots, n_1$ (dd pre-smoothing steps)
  $r = b - A u^{n_1}$ (compute the residual)
  $u_c = \text{GMLS}(0, R_{\ell-1}^\ell r, \ell - 1)$. (recursive call)
  $u^0 = u^{n_1} + P_{\ell-1}^{\ell} u_c$ (coarse correction)
  $u^n = G u^{n-1} + b$, $n = 1, \ldots, n_2$ (dd post-smoothing steps)
  Set $u^0 = u^{n_2}$ (update)
end if
return $u^0$.

In spite of the choice for $A_\ell$, we can define the geometric multilevel substructured dd method (GMLS) function described by Algorithm 6.3, which implements the classical V-cycle.

7. Numerical experiments. In this section, we test numerically our new computational frameworks and our purpose is twofold. On the one hand, we wish to compare our new methods with other classical existing methods, like a two-grid method in volume using RAS as smoother, or a two level DD method in volume based on the SHEM coarse space functions. This is done in Section 7.1, where we consider two simple cases of a Poisson equation defined on two-dimensional and three-dimensional boxes and we study convergence rates and computational times.

On the other hand, we wish to show the effectiveness of our new methods in solving both classical test problems, like the Laplace equation defined on a rectangle, and more complicated problems, like advection-diffusion equations with strong advection and possibly jumping diffusion coefficients. This is done in Sections 7.2 and 7.3. In particular, Section 7.2 studies the convergence behavior of the S2S and G2S methods for the solution of an advection-diffusion equation defined on a two-dimensional non-convex and non-simply connected “smiling domain”. Finally, in Section 7.3, we apply our methods to solve a diffusion equation with highly varying diffusion coefficients.

7.1. Laplace equation on 2D and 3D boxes. Consider the Poisson equation $-\Delta u = f$ in a rectangle $\Omega = (-1,1) \times (0,1)$ with homogeneous Dirichlet boundary condition. The domain $\Omega$ is decomposed into two overlapping rectangles $\Omega_1 = (-1, \delta) \times (0,1)$ and $\Omega_2 = (-\delta, 1) \times (0,1)$, where $2\delta$ is the length of the overlap. We discretize the problem using a standard second-order finite difference scheme based on a uniform grid of $N_y = 2^\ell - 1$ interior points in direction $y$ and $N_x = 2N_y + 1$ interior points in direction $x$. Here, $\ell$ is a positive integer. The grid size is denoted by $h$. The overlap is assumed to be $2\delta = h(N_{ov} + 1)$, where $N_{ov}$ represents the number of interior points in the overlap in direction $x$. The results of our numerical experiments are shown in Figures 7.1 and 7.2, where we solve the problem using the classical parallel Schwarz method (PSM), our S2S and G2S methods, a classical two-grid method using RAS as smoother (“2L-RAS” in the figures), and a classical two-level DD method in volume using SHEM coarse functions ("SHEM(m)" in the figures).

For the G2S method we use the one-dimensional interpolation operator $P_{2h}^h$ obtained by (4.3) and $R_{2h}^h = \frac{1}{2}(P_{2h}^h)^\top$ (as explained in Section 4.1). For the S2S method
and the classical two-level DD method, we use \(2m\) coarse functions, that are the first \(m\) sine Fourier functions on each interface. By extending these interface functions on each subdomain we obtain the \(2m\) SHEM coarse functions. Finally, we also use the S2S method together with \(2m\) coarse functions generated randomly by the procedure defined in Section 3.3 (with \(r = 3\) and \(\ell = q = m\)). This is denoted by “Rand\((m)\)” in the figures.

The figures show the decay of the relative errors with the respect to the number of iterations. All the methods are stopped if the relative error is smaller than \(10^{-12}\). In all the tests, the G2S and the two-grid RAS methods outperform the other methods and the PSM is obviously much slower than all the others. The two-grid RAS method performs as the G2S for \(N_{ov} = 2\). However, the G2S is faster for \(N_{ov} = 4\). Notice that while the G2S coarse space has dimension about \(N_y\), the one corresponding to the two-grid RAS method has dimension about \(N_xN_y/4 \approx N_y^2/2 > N_y\). Notice that the two curves corresponding to S2S and SHEM are always parallel. For \(m = 5\) these methods perform already very well, since with less than 10 iterations they can achieve an error of about \(10^{-6}\). The slowest two-level method (but still much faster than the PSM) is the S2S provided with random generated coarse functions. Increasing the dimension \(m\), the speed of convergence of S2S and SHEM drastically increases. For \(m = 20\), all the two-level methods show the same performance. Notice that if \(m = 20\), the dimension of the coarse spaces for S2S and SHEM is 40, while the dimension of the coarse spaces of G2S and 2L-RAS are about 60 and 1900, respectively. By doubling the value of the overlap, all the methods converge faster. In particular, the improvement is more significant for S2S and SHEM, which for \(m = 20\) also outperform 2L-RAS. Is it remarkable, that for a larger overlap the convergence of the S2S with random functions is almost the same as S2S with Fourier functions and SHEM. This is in

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Fig. 7.1: Convergence curves for \(\ell = 6\), \(N_{ov} = 2\), and \(m = 5\) (left), \(m = 10\) (middle), \(m = 20\) (right).

Fig. 7.2: Convergence curves for \(\ell = 6\), \(N_{ov} = 4\), and \(m = 5\) (left), \(m = 10\) (middle), \(m = 20\) (right).
agreement with the results of Lemma 3.4. The slower performance of 2L-RAS with respect to G2S can be traced back to the interpolation step. This operation breaks the harmonicity of the obtained correction, which therefore does not lie anymore in the space where the errors lie; see, e.g.,[31]. One could use interpolators which extend harmonically the correction inside the overlapping subdomains although this would increase significantly the computational cost of each iteration. We refer also to [33] for a similar observation.

Next, we repeat the same experiments on a three-dimensional box \( \Omega = (-1, 1) \times (0, 1) \times (0, 1) \) decomposed into two overlapping subdomains \( \Omega_1 = (-1, \delta) \times (0, 1) \times (0, 1) \) and \( \Omega_2 = (-\delta, 1) \times (0, 1) \times (0, 1) \). Since we are interested in computational times, we solve the problem (up to a tolerance of \( 10^{-10} \) on the relative error) using the G2S method, its equivalent forms G2S-B1 and G2S-B2, introduced in Section 6.1, and 2L-RAS. The results are shown in Tables 7.1 and 7.2. It is clear that the G2S methods outperform 2L-RAS, in terms of iteration numbers and computational times. In particular, G2S-B1 and G2S-B2 require per iteration about half of the computational time that 2L-RAS requires. The experiments have been performed on a workstation with 8 processors Intel Core i7-6700 CPU @ 3.40GHz and with 32 GB di RAM.

### 7.2. Advection-diffusion problem on smiling domain.

We consider the advection-diffusion equation \(-\Delta u + a \cdot \nabla u = f\) in \( \Omega \) with homogeneous Dirichlet boundary conditions. The domain \( \Omega \) is the “smiling face” depicted in Figure 7.3 (left), where the external disc has unit radius. The right-hand side function and the advection coefficients are \( f(x, y) = 10 \sin(2\pi x) \sin(2\pi y) \sin(2\pi xy) \) and \( a(x, y) = [10x^3, -20 - 30y^2]^T \). The corresponding solution is show in Figure 7.3 (right). The domain \( \Omega \) is decomposed into two overlapping subdomains as shown in Figure 7.3 (left). The overlap is denoted by \( \delta \).

We solve the problem using the parallel Schwarz method, and G2S and S2S methods. For the G2S method we use the one-dimensional interpolation operator obtained using (4.3) as described in Section 4.1. For the S2S method, we use the first \( 2m \) functions of the coarse space \( V_c \) defined in (4.6), which is the coarse space corresponding to the G2S method (if \( 2m = \dim V_c \)). In this way, on the one hand, we can study

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**Table 7.1:** Number of iterations performed by the different methods and for different number of degrees of freedom.

| # (volume) | G2S | G2S-B1 | G2S-B2 | 2L-RAS |
|------------|-----|--------|--------|--------|
| 539        | 4   | 4      | 4      | 6      |
| 6075       | 5   | 5      | 4      | 6      |
| 56699      | 4   | 4      | 4      | 6      |
| 488187     | 4   | 4      | 4      | 6      |

**Table 7.2:** Computational times performed by the different methods. In parentheses we indicate the computational time per iteration.

| # (volume) | G2S         | G2S-B1     | G2S-B2    | 2L-RAS     |
|------------|-------------|------------|-----------|------------|
| 539        | 0.023 (0.005) | 0.010 (0.003) | 0.010 (0.003) | 0.039 (0.06) |
| 6075       | 0.143 (0.028) | 0.102 (0.024) | 0.070 (0.017) | 0.190 (0.03) |
| 56699      | 2.700 (0.675) | 1.598 (0.399) | 1.280 (0.320) | 4.128 (0.688) |
| 488187     | 126.0980 (31.524) | 78.363 (19.591) | 63.131 (15.783) | 189.162 (31.527) |
the effectiveness of our methods for an advection-diffusion problem on a complicate domain, and on the other hand, we can observe the behavior of the G2S method when some basis functions are removed from its coarse space.

The numerical experiments are performed using Freefem++ [39] and the corresponding results are shown in Figure 7.4.

It is clear that the G2S method outperforms all the other methods. The benefit of a coarse correction is more evident for smaller values of the overlap. Moreover, when $m$ increases the convergence rate of the S2S method approaches the one of the S2S method. In particular, for $N = 396$ and $m = 40$ (which means a dimension of the coarse space of 80) the performance of the S2S method is already very similar to the one of the G2S method. This suggests that the entire coarse space is $V_c$ of the G2S is not always needed to achieve extremely good performances.
7.3. Diffusion problem with jumping diffusion coefficients. In this section, we test our methods for the solution of a diffusion equation $-\text{div}(\alpha \nabla u) = f$ defined in a domain $\Omega$ with homogeneous Dirichlet boundary conditions. The right-hand side function is $f(x,y) = \sin(x^2\pi)\sin(y^2\pi)\sin(xy^2\pi)$; see Figure 7.5 (bottom row - left). The domain $\Omega$ is the rectangle depicted in Figure 7.5 (top row - left), having as in the the example of Section 7.1 height equal to 1 and width equal to 2. In the domain $\Omega$ two horizontal channels are present; see Figure 7.5 (top row - right). Outside the channels the diffusion coefficient is $\alpha = 1$. Inside the channels we consider three different values of $\alpha$, namely $10^2$, $10^4$ and $10^6$, in order to study the robustness of the G2S algorithm with respect to the amplitude of the jump across the channels. Figure 7.6 (bottom row - right) shows the computed solution for $\alpha = 10^6$. The convergence results of our experiments are depicted in Figure 7.6. It is clear that our G2S method performs very well by reaching an error of order of $10^{-10}$ in less than 10 iterations. Direct numerical computations show that $G_1$ and $G_2$ have exactly two eigenvalues close to one. As soon as the coarse space contains both the slow eigenvectors, S2S converges very fast.

8. Conclusions. In this work we introduced a new framework of two-level substructured DD framework. In particular, two new class of methods are introduced: the S2S method, based on interface function coarse space, and the G2S, which is an interface two-grid method. The latter can be easily extended to a multilevel framework. Under certain reasonable hypotheses, we proved that these methods are well posed and convergent, and we also provide sharp estimates of their convergence factors. Moreover, the relations between S2S, G2S and volumetric two-level DD methods is also discussed. Finally, the effectiveness of our new methods is confirmed by extensive numerical experiments.

9. Appendix. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Consider a linear system $Ax = b$, where $b \in H$ and $A : H \to H$ an invertible operator of the form $A = I - G$, with $G$ the interface operator.
with \( I \) the identity. The operator \( G \) is the one-level iteration operator that corresponds to the stationary method \( x^{k+1} = Gx^k + M^{-1}b \). Consider a coarse space \( V_c := \text{span}\{\psi_1, \ldots, \psi_m\} \), where \( \psi_j \) are eigenfunctions of \( G \) and \( m > 0 \) is a finite integer. We denote by \( \lambda_j \) the corresponding eigenvalues and we assume that \( |\lambda_{m+1}| = \sup_{k \geq m+1} |\lambda_k| \). The two-level iteration operator is then given by \( G_{2L} := (I - PA_c^{-1}RA)G \), where \( P \psi := \sum_{k=1}^m v_j \psi_k \) and \( Rf := [\langle \psi_1, f \rangle, \ldots, \langle \psi_m, f \rangle]^\top \), for any \( \psi \in \mathcal{R}^m \) and \( f \in \mathcal{H} \). The coarse matrix is \( A_c = RAP \).

**Theorem 9.1** (Convergence of a general two-level method). 

The convergence factor of \( G_{2L} \) is \( \rho(G_{2L}) = |\lambda_{m+1}| \).

**Proof.** The matrix \( A_c \) is invertible by Lemma 3.1, hence the iteration is well-defined. Now, the proof is divided into two parts. First, we show that the coarse space functions \( \{\psi_k\}_{k=1}^m \) are in the kernel of \( G_{2L} \). Second, we show that \( \lambda_j \), for \( j \geq m+1 \), are eigenvalues of \( G_{2L} \).

To prove the first part, we consider a \( \psi_j \) with \( j \leq m \) and compute
\[
G_{2L} \psi_j = (I - PA_c^{-1}RA)G \psi_j = \lambda_j (\psi_j - (1 - \lambda_j)PA_c^{-1}R \psi_j).
\]

Since \( A \) is invertible, \( 1 - \lambda_j \neq 0 \) and, similarly as in the proof of Theorem 3.3, we observe that \( A_c((1 - \lambda_j)^{-1}v_j) = RAP((1 - \lambda_j)^{-1}v_j) = R \psi_j \). Inserting this equality in (9.1), we get
\[
G_{2L} \psi_j = \lambda_j (\psi_j - (1 - \lambda_j)PA_c^{-1}R \psi_j) = \lambda_j (\psi_j - P e_j) = \lambda_j (\psi_j - \psi_j) = 0,
\]
which is the first claim.

To prove the second claim, we take any \( \psi_j \) with \( j \geq m + 1 \) and define \( \phi_j := A^{-1}(\psi_j - P_{V_c} \psi_j) \), where \( P_{V_c} \) is the orthogonal projection operator onto \( V_c \). Notice that \( \phi_j = (1 - \lambda_j)^{-1}(\psi_j - w) \), where \( w = A^{-1}P_{V_c} \psi_j \in V_c \). Since we proved that \( V_c \) is included in the kernel of \( G_{2L} \), we have
\[
G_{2L} \phi_j = (1 - \lambda_j)^{-1}G_{2L} \psi_j = \lambda_j ((1 - \lambda_j)^{-1} \psi_j - PA_c^{-1}R \psi_j).
\]

If the \( \psi_j \) were orthonormal, then we are done. In a more general case, we proceed as follows. Now, we notice that the operator \( RP \) is invertible and \( P_{V_c} = P(RP)^{-1}R \).
Therefore, we compute
\[ PA_c^{-1} R \psi_j = PA_c^{-1}[RP(RP)^{-1}]R \psi_j = PA_c^{-1} R P V_c \psi_j = \sum_{\ell=1}^m \gamma_\ell PA_c^{-1} R \psi_\ell, \]
for some coefficients \( \gamma_\ell \). Now, we recall that \( R \psi_j = A_c((1 - \lambda_j)^{-1} e_j) \) for all \( j = 1, \ldots, m \) and write
\[ PA_c^{-1} R \psi_j = (1 - \lambda_j)^{-1} \sum_{\ell=1}^m \gamma_\ell \psi_\ell = \sum_{\ell=1}^m \gamma_\ell A^{-1} \psi_\ell = A^{-1} P V_c \psi_j. \]
Replacing this equality into (9.2), we obtain that \( G_{2L} \phi_j = \lambda_j \phi_j \).

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