Fault-tolerant control under controller-driven sampling using virtual actuator strategy*

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Abstract—We present a new output feedback fault tolerant control strategy for continuous-time linear systems. The strategy combines a digital nominal controller under controller-driven (varying) sampling with virtual-actuator (VA)-based controller reconfiguration to compensate for actuator faults. In the proposed scheme, the controller controls both the plant and the sampling period, and performs controller reconfiguration by engaging in the loop the VA adapted to the diagnosed fault. The VA also operates under controller-driven sampling. Two independent objectives are considered: (a) closed-loop stability with setpoint tracking and (b) controller reconfiguration under faults. Our main contribution is to extend an existing VA-based controller reconfiguration strategy to systems under controller-driven sampling in such a way that if objective (a) is possible under controller-driven sampling (without VA) and objective (b) is possible under uniform sampling (without controller-driven sampling), then closed-loop stability and setpoint tracking will be preserved under both healthy and faulty operation for all possible sampling rate evolutions that may be selected by the controller.

I. INTRODUCTION

Active Fault-Tolerant Control (FTC) systems aim to maintain control performance levels under a number of fault scenarios, by means of a controller reconfiguration mechanism. An interesting approach to controller reconfiguration for FTC is the one based on the concept of virtual actuators (VA) (a complete reference on VA and its applications and details can be found in [24], [16], [19], [20]). The main advantage of the VA approach is that it allows the engineer to design the controller for the nominal (“healthy”) plant, without considering the possible faults. More specifically, the method uses a single nominal controller, designed for the healthy system, which is always present in the closed-loop system, and a virtual actuator, which introduces an interface between the plant and the controller taking different actions according to the evaluated fault situation of the plant. In healthy operation the virtual actuator is inactive and the whole control action is provided by the nominal controller. In faulty operation the virtual actuator generates additional signals that combine with the existing signals in specific ways in order to cancel or mitigate the effect of the fault in the closed-loop system. The advantage of this approach is that any existing nominal controller which has been designed to satisfy the desired specifications for the fault-free plant, can be kept in the loop at all times. In addition, the design of the virtual actuator (which has to adapt to each type of detected fault) is independent of the controller and is aimed at preserving specific closed-loop properties in the presence of faults as, for example, stability and setpoint tracking.

Currently, many control systems involve some kind of shared network environment with limited bandwidth. Such systems are usually referred to as Networked Control Systems (NCS) (see the special issues [1], [2]). Since the network may be shared among processes, then sampling and acting over the system while keeping a constant rate may be difficult because it introduces a trade-off between requiring too much bandwidth and hence restricting other processes from accessing the network (sampling at a high rate) or too little bandwidth and hence reducing control performance (sampling at a low rate). Thus, much research effort has focused on designing control strategies for systems under varying sampling rate (VSR).

In the present paper, we consider a type of VSR where a central controller may be in charge not only of computing feedback but also of administering access to the shared network. In this setting, the controller may perform on-line variations of the sampling rate in order to accommodate for the bandwidth requirements of the different processes being controlled. This setting is akin to that described in [3] and has been addressed in previous work by some of the authors [12], [13], [17], [18]. In the current work, we combine this specific type of VSR with an extension of the VA-based controller reconfiguration strategy of [22] to the VSR setting, yielding a control scheme as illustrated in Figure 1. In the proposed scheme, the VSR controller is designed for the nominal or fault-free plant, and hence knowledge of the fault scenario is not needed at the control design stage. The fault tolerance mechanism is responsible for achieving correct closed-loop control under faults. This mechanism requires knowledge of the sampling period command (h_k in Figure 1) issued by the controller but does not modify the sampling and hold operations. The fault tolerance mechanism considered involves a bank of VAs and a fault detection and isolation (FDI) strategy. Correct closed-loop control under faults is achieved by engaging in the loop the VA from the bank of VAs that is adapted to the fault diagnosed by the FDI unit. In this paper, we focus on the design and properties of the scheme of Figure 1 to achieve correct setpoint tracking and stability under both nominal and faulty operation, assuming that some FDI strategy is able to correctly diagnose faults. We thus concentrate on controller reconfiguration under faults and leave the integration of an FDI unit as a topic for future work.

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We extend the existing VA-based controller reconfiguration strategy of [22] to the VSR case considered, ensuring closed-loop stability and setpoint tracking under nominal and faulty operation regardless of how the VSR controller performs online variations of the sampling rate. In this context, our main contribution is to show that the difficulties in this combined scheme are not greater as those for VSR control or uniform-sampling VA-based reconfiguration taken independently.

The proposed scheme can be particularly interesting in circumstances in which a system with input redundancy is sampled V A-based reconfiguration taken independently. Scheme are not greater as those for VSR control or uniform-contribution is to show that the difficulties in this combined line variations of the sampling rate. In this context, our main loop stability and setpoint tracking under nominal and faulty strategy of [22] to the VSR case considered, ensuring closed-

We extend the existing V A-based controller reconfiguration of performance objectives can still be reached in the sequel, we explain the different components of the feedback control system considered.

\[ \dot{x} = Ax + BFu, \]
\[ y = Cx, \]
\[ v = Cv, \]

where \( x \in \mathbb{R}^n, u \in \mathbb{R}^m \) is the control input, \( y \in \mathbb{R}^n \) is the plant measured output, and \( v \in \mathbb{R}^q \) is a performance output. As in [22] we represent the operability situation of the \( N = m \) actuators by the matrix \( F \in \mathbb{R}^{m \times m} \), taking values from a finite set

\[ F \in \mathcal{F} := \{F_0, F_1, \cdots, F_N\}, \quad F_0 = I. \]

Under healthy operation, the matrix \( F \) in (1) takes the value \( F = F_0 = I \), so that \( B \) in (1) represents the “healthy” plant input matrix. The matrix \( F_j, j = 1, \cdots, N \), models the total loss of the \( j \)-th actuator. Hence, \( F_j \) is obtained by setting to 0 the \( j \)-th diagonal entry of \( F_0 = I \). We assume that the pairs \((A, BF_i)\) are stabilisable for \( i = 0, 1, \cdots, N \), \((C, A)\) is detectable, and \( A \) is invertible.

The performance output \( v \) in (3) and the different fault situations in (1) must be such that for every desired constant value \( v_{ref} \) of the performance output and every fault \( i \), there exists a constant input value \( \bar{u}_i \) so that the equilibrium state \( \bar{x}_i \) that corresponds to the constant input \( \bar{u}_i \) under fault \( i \) is such that \( C\bar{x}_i = v_{ref} \), i.e. there exist \( \bar{x}_i, \bar{u}_i \) such that

\[ \begin{bmatrix} A & BF_i \\ C & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_i \\ \bar{u}_i \end{bmatrix} = \begin{bmatrix} 0 \\ v_{ref} \end{bmatrix} \]

for all \( i \in \{0, 1, \cdots, N\} \). Condition (5) means that the plant has sufficient levels of redundancy to admit setpoint tracking in each fault scenario. Note that if for a given \( v_{ref} \) and specific fault \( i \), no \( \bar{u}_i \) exists satisfying (5), then the performance output will not converge to \( v_{ref} \) under fault \( i \), no matter how sophisticated the fault tolerance mechanism may be. On the other hand, the VA fault tolerance mechanism has to be designed so that the required equilibrium values satisfying (5) are achieved under all possible fault situations; this will be addressed in detail in Section III-B.

### B. Fault-ignorant varying-sampling-rate controller

As previously explained, knowledge of the fault scenario is not needed at the controller design stage. Consequently, controller design is independent of virtual actuator design. We consider a healthy-plant-model-based reference-tracking sampled-data controller given by

\[ u_c = -K^h(\hat{x} - x_{ref}) + u_{ref}, \]
\[ \hat{x}^+ = A^h\hat{x} + B^h u_c + L^h(y_c - C\hat{x}), \]

where \( u_c \) represents the controller-computed plant input signal, \( y_c \) is the plant output signal supplied to the controller, \( x_{ref}, u_{ref} \) are state and input constant reference signals, respectively, and \( \hat{x}, \hat{x}^+ \) are the current and successor states

![Figure 1. Considered scheme: a central controller controls both the process and the next sampling instant.](image-url)
of the observer (7). The matrices $A^h$ and $B^h$ are the discrete-time equivalents of $A$ and $B$ in (1), corresponding to a sampling period $h$,

$$A^h := e^{Ah}, \quad B^h := \int_{0}^{h} e^{At} B dt,$$  

and the reference signals satisfy

$$Ax_{ref} + Bu_{ref} = 0, \quad C_x x_{ref} = v_{ref}.$$  

The controller may perform on-line variations of the sampling period $h$, under the constraint that all possible sampling periods are taken from a finite set:

$$h \in \mathcal{H} := \{h_1, \cdots, h_n\},$$  

where every $h \in \mathcal{H}$ should be non-pathological (see [4] for further details on pathological sampling). The feedback and observer gains $K^h$ and $L^h$ employed by the controller may depend on the sampling period selected. The computation of these gains will be explained in Section III-A. If no fault tolerance mechanism were present, the plant input $u$ would equal the controller-computed plant input $u_c$ and the plant output supplied to the controller, $y_c$, would equal the true plant output, $y$, at all sampling instances. In the presence of the fault tolerance mechanism discussed in the current paper, the equalities $y = y_c$ and $u = u_c$ will be true only under nominal (healthy) conditions and provided the fault tolerance mechanism accurately detects that the plant is under healthy operation.

C. Nominal plant-controller feedback loop

As explained previously, in the considered scenario the controller does not only apply control, but also determines the sampling instances (based, for example, on the states of the processes and/or restrictions over the network). Under nominal conditions, at instant $t_k$ the controller receives the sample $y_c$ and processes it in order to compute the required feedback action. To do so, it also determines the instant $t_{k+1} = t_k + h_k$, with $h_k \in \mathcal{H}$, at which it will take the next sample and control action. The plant dynamics at the sampling instants can be written as

$$x^+ = A^h x + B^h F u,$$  

where $A^h$ and $B^h$ are defined in (8), $x = x(t_k)$, $u = u(t_k)$, and $x^+ = x(t_{k+1})$ (the successor state).

We observe that if constant reference signals $x_{ref}$ and $u_{ref}$ satisfy condition (9) for the continuous-time plant, then the same $x_{ref}$ and $u_{ref}$ satisfy

$$x_{ref} = A^h x_{ref} + B^h u_{ref}$$  

for all $h \in \mathcal{H}$. To see this, premultiply $B^h$ in (8) by $A$, producing

$$AB^h = A \int_{0}^{h} e^{At} dt B = (A^h - I)B.$$  

Since $A$ is invertible and commutes with $A^h$, then for every $h \in \mathcal{H}$

$$(I - A^h)^{-1} B^h = -A^{-1} B.$$  

Next, we write

$$x_{ref} = -A^{-1} Bu_{ref} = (I - A^h)^{-1} B^h u_{ref},$$  

from which (12) is obtained.

Since the controller may perform on-line variations of the sampling period, the discrete-time system (11), obtained by looking at the continuous-time plant only at the sampling instants, can be regarded as a Discrete-Time Switched System (DTSS). Since the sequence of sampling periods selected by the controller is arbitrary (that is, we do not require a priori knowledge on rules for this selection), we are interested in establishing closed-loop properties (such as stability and setpoint tracking) that hold irrespective of such sequence. In switched systems terminology, we are interested in establishing closed-loop properties that hold under arbitrary switching (see, e.g. [23], [15]). System design in order to achieve the required closed-loop properties under arbitrary switching will be addressed in Section III.

D. Bank of virtual actuators

As in [22], we consider a bank of VAs where each of the VAs in the bank is designed to compensate for a specific actuator fault. The VA corresponding to the $i$-th fault situation $F_i$ is given by

$$\theta_i^+ = A^h \theta_i + B^h u_c - B^h F_i u_i,$$  

$$u_i = -M_i^h \theta_i + N_i^h u_c,$$  

$$y_i = y + C \theta_i.$$  

The variable $\theta_i$ represents the $i$-th VA internal state, $u_i$ is the $i$-th VA plant input signal and $y_i$ the $i$-th VA output to be supplied to the controller. How $u_i$ and $y_i$ relate to the true plant input $u$ and the true plant output supplied to the controller $y_c$ is explained in Section III-E. The $0$-th VA corresponds to nominal operating conditions (healthy or fault-free) and has

$$M_0^h = 0 \quad \text{and} \quad N_0^h = I,$$  

for all sampling periods $h$. For $i = 1, \ldots, N$, the VAs’ internal matrices $M_i^h$ and $N_i^h$ may depend on the sampling period $h$ selected by the controller. The design of $M_i^h$ and $N_i^h$ is explained in Section III-B. For future reference, note that substitution of (17) into (16) yields

$$\theta_i^+ = A^h \theta_i + B^h (I - F_i N_i^h) u_c, \quad \text{with}$$  

$$A_i^h := A^h + B^h F_i M_i^h.$$  

that is, the dynamics of each VA is driven by the controller-computed plant input signal, $u_c$.

E. FDI and controller reconfiguration mechanism

Controller reconfiguration is achieved through a selector that, in response to the diagnosed fault situation, interconnects the appropriate virtual actuator from the bank of virtual actuators with the controller and the plant.

When an FDI mechanism (not described here) detects that the $j$-th fault has occurred, the $j$-th virtual actuator is interconnected with the controller and plant by making $u = u_j$ and $y_c = y_j$. Hence, whenever the FDI diagnoses
that the plant is under healthy operation, the selector will set $u = u_0$ and $y_c = y_0$. The reconfiguration also resets the 0-th virtual actuator state $\theta_0$ to zero whenever healthy operation is detected.

Under the above considerations, we next show that if the plant is under healthy operation and if the FDI mechanism successfully assesses the plant’s healthy condition, then the plant and controller feedback loop will operate as if the bank of virtual actuators and reconfiguration mechanism were not present. From (17) and (19), then $u_0 = u_c$. Therefore, if at time $k_0$ the FDI mechanism detects that healthy operation is restored, then $\theta_0 = 0$ according to the virtual actuator state reset condition, $y_0 = y$ from (18), and it follows that $y_c = y_0 = y$ and $u = u_0 = u_c$ at time $k_0$. If the plant continues under healthy operation and the FDI mechanism continues to successfully assess the plant’s healthy condition, then from (16) it follows that $\theta_0 = 0$ and $y_c = y_0 = y$ and $u = u_0 = u_c$ will continue to hold for time instants $k \geq k_0$ until either a fault occurs or the FDI mechanism ceases to successfully diagnose the plant’s condition.

III. CONTROLLER AND VIRTUAL ACTUATOR DESIGN

The controller and bank of virtual actuators must be designed so that the performance output $v$ [see (3)] is able to track a constant reference in closed loop, even if faults occur, and so that all closed-loop variables remain bounded.

A. Controller design

Controller design involves the appropriate selection of the matrices $K^h$ and $L^h$ in (6)–(7). In order for the desired closed-loop properties to hold irrespective of the sampling period sequence selected by the VSR controller, the matrices $K^h$ and $L^h$ should be selected so that the closed-loop matrices

$$A^{h,\text{cl}} := A^h - B^h K^h,$$

$$A^{h,\text{o}} := A^h - L^h C,$$  

make the sets \{\(A^{h,\text{cl}} : h \in \mathcal{H}\)\} and \{\(A^{h,\text{o}} : h \in \mathcal{H}\)\} stable under arbitrary switching. Stability under arbitrary switching is equivalent to the existence of a Lyapunov function common to every matrix in the corresponding sets (see, e.g., [23], [15]). In general, this common Lyapunov function may be not quadratic.

If $K^h$ and $L^h$ exist so that the closed-loop matrices (22) on the one hand, and (23) on the other, share a common quadratic Lyapunov function (CQLF) for all $h \in \mathcal{H}$, then $K^h$ and $L^h$ can be computed via linear matrix inequalities (LMIs) (see, e.g., [5], [21]).

In some cases, $K^h$ and $L^h$ can be found so that not only CQLFs exist, but also additional properties hold for the closed-loop matrices (22) and (23). One such case is when invertible $T_{\text{cl}}$ and $T_0$ exist so that $T_{\text{cl}}^{-1} A^{h,\text{cl}} T_{\text{cl}}$ and $T_0^{-1} A^{h,\text{o}} T_0$ are upper triangular for all $h \in \mathcal{H}$ (solvable Lie algebra case). Several works address the computation of $K^h$ (and $L^h$) so that this simultaneous triangularization is achieved for $A^{h,\text{cl}}$ (and $A^{h,\text{o}}$) for general switched linear systems [10], [11], [6] and specifically for cases as the current one, where $A^h$ and $B^h$ arise from sampling a single continuous-time systems at different rates [13], [17], [12], [18]. Several facts make this apparently more restrictive design criterion appealing in the current context because the chances of successful computation of $K^h$ and $L^h$ increase when either:

- the DTSS arises from sampling a single continuous-time system at different rates [13], [17], [12], [18],
- the system has many inputs [7], [8], [9].

Note that both these situations occur in the current case, the second one because input redundancy is required for successful trajectory tracking in the presence of total actuator loss, as explained in Section II-A.

B. Virtual actuator features and design

The bank of VAs, as defined in (16)–(18), jointly with the controller reconfiguration mechanism endow the feedback loop with specific features when the plant’s fault situation has been correctly diagnosed. One of these features is known as “fault hiding” because the controller variables $u_c$ and $y_c$ are related in such a way as if a plant under nominal conditions were connected to the controller. In order to see this feature, define

$$\xi_i := x + \theta_i, \quad i = 1, \ldots, N,$$  

and write, using (16) and (11),

$$\xi^+_i := A^h \xi_i + B^h (F u - F_i u_i) + B^h u_c.$$  

When the plant fault situation is correctly diagnosed, we have $F_i = F$, $u_i = u$ and $y_c = y_i$. From the latter equalities, (2), (18) and (25), it follows that

$$\xi^+_i = A^h \xi_i + B^h u_c,$$  

$$y_c = C \xi_i.$$  

Eqs. (26)–(27) show that the controller effectively sees a nominal plant, whose state is $\xi_i$ instead of $x$.

A second feature of the bank of VAs and switching mechanism is that the desired setpoint $v_{\text{ref}}$ for the performance output $v$ defined in (9) should be preserved for all fault situations and sampling period variations, provided the plant fault situation is correctly diagnosed. In closed loop, the boundedness of all variables and the tracking of the desired setpoint $v_{\text{ref}}$ are achieved by ensuring the following:

a) the controller-computed plant input $u_c$ converges to the steady state value $u_c = v_{\text{ref}}$,

b) the VA state vector $\theta_i$ converges to a constant steady-state value $\overline{\theta}_i$,

c) Under fault $i$, the plant state $x$ and input $u$ both converge to steady-state values $\overline{x}_i$ and $\overline{u}_i$ (independent of $h$) and satisfy (5).

In Section IV, we will show that items a)–c) above will be true if we select the matrices $M_i^h$ and $N_i^h$ as explained next.

The matrices $M_i^h$ should be selected so that for every $i \in \{0, 1, \ldots, N\}$, the matrices in the set \{\(A^h_i : h \in \mathcal{H}\)\}, with $A_i^h$ as in (21), are stable under arbitrary switching. The latter
can be achieved using, for example, LMI- or Lie-algebraic-solvability-based methods, as mentioned in Section III-A and implies that every $A^h$ is Schur.

Once the $M^h$ are designed, select one sampling period $h' \in \mathcal{H}$ and compute

$$N^h_{i} = [X^h_i]^\dagger C_v (I - A^h) - 1 B^h,$$

$$X^h_i := C_v (I - A^h) - 1 B^h F_i \quad \text{for all } h \in \mathcal{H},$$

where $\dagger$ denotes the Moore-Penrose generalised inverse. For every sampling period $h \in \mathcal{H}$, select the corresponding $N^h_{i}$ as follows:

$$N^h_{i} = N^h_{i} - (M^h - M^h) P^h_{i},$$

$$P^h_{i} := (I - A^h - 1 B^h (I - F_i N^h_{i})) \quad \text{for all } h \in \mathcal{H}.$$

The following result concerning the expression for $P^h_{i}$ above will be required in Section IV.

**Lemma 1:** Let $h \in \mathcal{H}$ and $i \in \{0, \ldots, N\}$. Then,

$$\begin{align*}
(I - A^h)^{-1} B^h &= -(A + B F_i M^h) - 1 B. 
\end{align*}$$

**Proof:** Using (21) and (14), it follows that

$$\begin{align*}
(I - A^h)^{-1} B^h &= (I - A^h - B^h F_i M^h)^{-1} B \\
&= [(I - A^h)^{-1} (I - A^h - B^h F_i M^h)]^{-1} B^h \\
&= [A^{-1} (A + B F_i M^h)]^{-1} (I - A^h)^{-1} B^h \\
&= (A + B F_i M^h)^{-1} A [-A^{-1} B]
\end{align*}$$

whence (32) follows. ■

In the next section, we show that if the $M^h$ and $N^h$ are selected as previously explained, then items [2]–[3] above will be ensured and the closed-loop system will successfully track the desired setpoint $v_{ref}$ under both nominal and faulty conditions, even when the VSR controller performs on-line variations of the sampling period.

**IV. CLOSED-LOOP PROPERTIES UNDER VSR**

In this section, we present the main results of the paper. These results are given below as Theorems [1]–[3]. Each theorem establishes the validity of one of the items [2]–[3] detailed in Section III-B under the design conditions and assumptions explained in Sections I and II. These results ensure the appropriate operation of the VA for the considered VSR case, by ensuring the boundedness of all closed-loop variables and the convergence of the performance output to the desired reference value under persistent faults.

**A. Control-computed plant input convergence**

To proceed with our first main result, let us define the following observer and tracking errors

$$\begin{align*}
\tilde{\xi}_i &= \xi_i - \hat{x}, \\
\zeta_i &= \xi_i - x_{ref},
\end{align*}$$

with $\tilde{\xi}_i$ as in (24), and express the controller-computed plant input $u_c$ in (6) as

$$u_c = -K^h \zeta_i + K^h \tilde{\zeta}_i + v_{ref}.$$

We next establish item [3] of Section III-B. This is done in the following Theorem.

**Theorem 1:** Consider the continuous-time plant [1]–[2] with VSR controller [3]–[9] and bank of VAs [10]–[19]. Suppose that there exist feedback matrices $K^h$ and observer-gain matrices $L^h$ as requested in III-A. If the plant’s fault condition is persistent and successfully diagnosed by the FDI unit, then

i) the combined plant-VA state $\xi_i$ in (33), where $i$ identifies the plant’s fault condition, converges to the steady-state value $\xi_i = x_{ref}$, and so does the observer state $\hat{x}$.

ii) the controller-computed plant input $u_c$ converges to the steady-state value $\bar{u}_c = u_{ref}$.

**Proof:** The equality (35) is valid for all $i \in \{0, \ldots, N\}$. Using (6), (7), (15), (24), and (27), we obtain

$$\begin{align*}
\hat{\xi}_i^+ &= A^h \bar{\xi}_i + B^h (F u - F_i u_i) - L^h (y_c - C \hat{x}) \\
\zeta_i^+ &= (A^h - B^h K^h) \zeta_i + B^h (F u - F_i u_i) + B^h K^h \hat{\xi}_i.
\end{align*}$$

By hypothesis, the FDI unit correctly diagnoses the plant’s fault condition and hence interconnects the $i$-th VA with the controller and plant $(u = u_i, F = F_i, y_c = y_i)$. The error dynamics (36)–(37) hence become

$$\begin{align*}
\hat{\xi}_i^+ &= (A^h - L^h C) \zeta_i + B^h (F u - F_i u_i) + B^h K^h \hat{\xi}_i.
\end{align*}$$

Since both $A^h - L^h C$ and $A^h - B^h K^h$ are stable under arbitrary switching, then

$$\begin{align*}
\lim_{k \to \infty} \hat{\xi}_i &= 0 \quad \text{and} \quad \lim_{k \to \infty} \zeta_i = 0,
\end{align*}$$

which establishes 2. From (35), then

$$\bar{u}_c = \lim_{k \to \infty} u_c = u_{ref},$$

which establishes 3. Note that both (40) and (41) are true for every possible evolution of the sampling periods $h \in \mathcal{H}$ (even when varied on-line). ■

**B. VA state convergence**

The convergence of the VA state, as per item [3] of Section III-B, is established in Theorem 3 below. We require the following auxiliary result.

**Lemma 2:** Consider the matrices $X^h_i$ as defined in (29). Suppose that the continuous-time system matrices $A, B$ are such that $(A, B F_i)$ are stabilisable for $i = 0, 1, \ldots, N$, and for each fault matrix $F_i$ there exist constant values $\bar{x}_i$ and $\bar{u}_i$ satisfying (5). Then, $X^h_i [X^h_i]^\dagger = I$.

**Proof:** The existence of constant values $\bar{x}_i$ and $\bar{u}_i$ satisfying (5) is equivalent to the condition that the matrix

$$\begin{bmatrix}
-A & B F_i \\
-C_v & 0
\end{bmatrix}$$

has rank $n + q$. Under non-pathological sampling $h \in \mathcal{H}$, the latter rank condition implies (see, e.g., the proof of Lemma IV.3 in [14])

$$\begin{align*}
\text{rank } \begin{bmatrix}
I_n & -A^h & B^h F_i \\
-C_v & 0
\end{bmatrix} &= n + q. 
\end{align*}$$

\footnotesize{\textsuperscript{1}}For clarity, in this proof we use a subindex to indicate the dimensions of the identity matrices, that is, $I_n$ denotes the $n \times n$ identity matrix.
Correct design of $M^h_i$ (recall Section III-B) implies that $A^h_i$ defined in (21) is Schur; then $(I_n - A^h_i)$ is invertible and we can write

$$
\begin{bmatrix}
I_n & 0 \\
C_v (I_n - A^h_i)^{-1} & 0
\end{bmatrix}
\begin{bmatrix}
I_n - A^h_i \\
0
\end{bmatrix}
\begin{bmatrix}
B^h F_i \\
-M^h_i
\end{bmatrix}
= I_q,
$$

(43)

where $X^h_i \in \mathbb{R}^{q \times m}$ is defined in (29). Since the first and third matrices on the left hand side (LHS) of (43) are invertible, it follows (using Sylvester’s inequality and properties of the matrix rank) that the rank of the second matrix on the LHS is equal to the rank of the matrix on the right hand side of (35). Using (42), we then have rank $X^h_i = q$, that is, $X^h_i$ has full row rank. Thus, its Moore-Penrose generalised inverse $[X^h_i]^\dagger = [X^h_i]^T [X^h_i X^h_i]^T^{-1}$ exists and satisfies $X^h_i [X^h_i]^\dagger = I_q$. The result then follows.

We are now ready to establish item b) of Section III-B

**Theorem 2:** Under the hypotheses of Theorem 1 consider the performance output $v$, and suppose that for each fault matrix $F_i$, $i = 0, 1, \ldots, N$, there exist constant values $\bar{x}_i$ and $\bar{u}_i$ satisfying (5), and matrices $M^h_i$ so that $\{A^h_i : h \in \mathcal{H}\}$, with $A^h_i$ as in (21), is stable under arbitrary switching. If $N^h_i$ are selected as explained in Section III-B and if the plant’s fault condition is persistent and successfully diagnosed by the FDI unit, then

$$
\lim_{k \to \infty} \theta_i = \bar{\theta}_i \quad \text{and} \quad C_v \bar{\theta}_i = 0.
$$

(44)

**Proof:** From Theorem 1(iii), we know that $\bar{u}_c = u_{ref}$. Let $\bar{\theta}_h^i$ denote the equilibrium value of the VA state $\theta_i$ if a constant sampling period $h$ were kept by the controller. Solving from (20) and using (31), we can write

$$
\bar{\theta}_h^i = P^h_i u_{ref}.
$$

(45)

We next show that $P^h_i$ is independent of $h$. Let $h' \in \mathcal{H}$ be the sampling period selected for the computation of $N^h_i$ as in (28). Using (31) and Lemma 1, we can write

$$
P^h_i = -(A + BF_i M^h_i)^{-1} [B - BF_i N^h_i]
$$

(46)

for all $h \in \mathcal{H}$. Replacing $N^h_i$ by the expression (30), adding $-AP^h_i + AP^h_i$ inside the square brackets, and operating, yields

$$
P^h_i = -(A + BF_i M^h_i)^{-1} \left[ B(I - F_i N^h_i) - (A + BF_i M^h_i)P^h_i + (A + BF_i M^h_i)P^h_i \right].
$$

(47)

Using (46), then $(A + BF_i M^h_i)P^h_i = -B(I - F_i N^h_i)$. Using the latter expression in (47) yields

$$
P^h_i = -(A + BF_i M^h_i)^{-1} \left[ -P^h_i \right] = P^h_i,
$$

which establishes that $P^h_i$ is independent of $h$. We can thus write $P^h_i = P_i$ for all $h \in \mathcal{H}$. Therefore, the steady-state value $\bar{\theta}_h^i$ also is independent of $h$, as follows from (45), and we can write $\bar{\theta}_h^i = \bar{\theta}_i$ for all $h \in \mathcal{H}$. Define the incremental variables

$$
\Delta \theta_i := \theta_i - \bar{\theta}_i,
$$

(48)

$$
\Delta u_c := u_c - \bar{u}_c = u_c - u_{ref}.
$$

(49)

Using (20), the VA dynamics in the incremental variables can be written as

$$
\Delta \theta_i^+ = A^h_i \Delta \theta_i + B^h (I - F_i N^h_i) \Delta u,
$$

where $\Delta u \to 0$ by Theorem 1 and $\{A^h_i : h \in \mathcal{H}\}$ are stable under arbitrary switching for every $i = 0, \ldots, N$. It follows that $\Delta \theta_i \to 0$ and hence $\lim_{k \to \infty} \theta_i = \bar{\theta}_i$. Using (31) and (28)-(29), we can write

$$
C_v P_i = C_v P_i^h = \left( I - X^h_i [X^h_i]^\dagger \right) C_v (I - A^h_i)^{-1} B^h.
$$

Using Lemma 2 then $C_v P_i = C_v P_i^h = 0$ for all $h \in \mathcal{H}$. Recalling (45), then $C_v \bar{\theta}_i = C_v \bar{\theta}_h^i = C_v P_i u_{ref} = 0$.

Theorem 2 shows that the virtual actuator state converges to a constant steady-state value that is independent of the sampling periods $h \in \mathcal{H}$ and, in addition, is in the null space of the performance output matrix $C_v$ [see (5)]. This property is key to achieving the correct setpoint $v_{ref}$ for the performance output $v$, a property that is established in the following section.

### C. Setpoint tracking

We next present our last result, which establishes item c) of Section III-B related to the setpoint tracking property of the VSR VA introduced.

**Theorem 3:** Under the same hypotheses as for Theorem 2 the plant state $x$ and the performance output $v$ will satisfy

$$
\lim_{k \to \infty} x = \bar{x}_i \quad \text{and} \quad \lim_{k \to \infty} v = v_{ref}.
$$

**Proof:** From Theorem 1(ii), the combined plant and VA state $\bar{x}_i$ converges to the steady-state value $x_{ref}$ and from Theorem 2 the VA state $\bar{\theta}_i$ converges to $\bar{\theta}_i$. Recalling (24), then the plant state $x$ must converge to the steady-state value $\bar{x}_i = x_{ref} - \bar{\theta}_i$. The performance output thus satisfies

$$
\lim_{k \to \infty} v = C_v \bar{x}_i = C_v x_{ref} - C_v \bar{\theta}_i = v_{ref},
$$

where we have used (2) and (44).

The above result shows that, if the correct fault situation has been diagnosed and the matching VA has been engaged in the closed-loop system, then the VA-reconfigured system will achieve the desired constant setpoint tracking, irrespective of the selected sampling periods $h \in \mathcal{H}$.

### V. Example

As an application of the proposed strategy, we revisit the two tanks example presented in [24]. The considered plant is composed of two interconnected tanks A and B, where the objective is to control the outflow of tank B, using as control input the inflow of tank A and the opening of the valve between them.

The control objective is to keep a constant referenced outflow from tank B. In the linearised model, controlling
the level of tank B is equivalent to controlling its outflow. Thus, the level of tank B will be the considered objective. The linearised plant equations are given by

\[
\dot{x} = Ax + BFu
\]
\[
y = x, \quad v = C_vx
\]

where

\[
A := \begin{pmatrix} -0.25 & 0 \\ 0.25 & -0.25 \end{pmatrix}, \quad B := \begin{pmatrix} 1 & -0.5 \\ 0 & 0.5 \end{pmatrix}
\]
\[
C_v := \begin{pmatrix} 0 & 1 \end{pmatrix}
\]

In this example, we will consider only the loss of actuator \(v_2\) (connecting valve blocked in nominal position), referred as Fault type 2 in [24]. Thus,

\[
F \in \mathcal{F} := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}.
\]

The considered sampling periods set is given by

\[
\mathcal{H} := \{0.1, 0.05, 0.025\}.
\]

Note that, while each of the considered sampling periods are multiple of \(h_3\), this is only for simplicity and not required by the proposed strategy. Using [6, Algorithm 1], we are able to compute feedback matrices such that the closed loop share a common triangularizing transformation. The algorithm uses a procedure that computes, if possible, common eigenvectors with stability. To do so, it requires the definition of two auxiliary values (in this case, selected as \(\epsilon_c = \epsilon_d = 10^{-18}\), the first associated with the stability limits and the second preventing of selecting eigenvectors in the image of the input matrix. In this example, we computed the set of feedback matrices for both, the controller \((K^h)\) and the VA \((M^h_2)\). Computation of such matrices using the corresponding representations of the matrix pairs \((A^h, B^h F_i)\) yields

\[
K^{h_1} = \begin{pmatrix} 9.99 \\ -6.14 \times 10^{-2} \end{pmatrix}, \quad K^{h_2} = \begin{pmatrix} 19.99 \\ -6.19 \times 10^{-2} \end{pmatrix}, \quad K^{h_3} = \begin{pmatrix} 39.99 \\ -6.21 \times 10^{-2} \end{pmatrix}
\]

and matrices \(M^h_2\)

\[
M^{h_1}_2 = -\begin{pmatrix} 11.23 & 107.99 \\ 0 & 0 \end{pmatrix}, \quad M^{h_2}_2 = -\begin{pmatrix} 21.34 & 233.18 \\ 0 & 0 \end{pmatrix}, \quad M^{h_3}_2 = -\begin{pmatrix} 41.39 & 485.57 \\ 0 & 0 \end{pmatrix}
\]

Note that the subindex in \(M^h_i\) and \(N^h_i\) refers to the actuator’s fault index considered. Using Eq. (28) for \(h_1\), we can compute \(N^{h_1}_2\) and \(P_2\). Then, we are able to compute matrices \(N^{h_3}_2\) and \(N^{h_4}_2\) using (30). Their computed values are as follows.

\[
N^{h_1}_2 = \begin{pmatrix} 1.00 & 0.22 \end{pmatrix}, \quad N^{h_2}_2 = \begin{pmatrix} 1.00 & 2.25 \end{pmatrix}, \quad N^{h_3}_2 = \begin{pmatrix} 1.00 & -37.86 \end{pmatrix}
\]

Since \(C = I\), we can choose \(L^h = A^h\).

In order to test the reference tracking properties, we change the reference from \(x_{2,ref} = 0\) to \(x_{2,ref} = 0.05\), and while the plant is reaching the new setpoint, we simulate the considered actuator fault; after a second setpoint change for \(x_2\) from 0.05 to 0, we simulate the actuator restitution to the healthy situation. The resulting responses are shown in Figure 2. Observe that, in both situations the controller always tracks the reference provided the fault. Also observe that, since during the fault the only active actuator is the pump, then the only way to keep the level of the second tank at 0.05 is by increasing \(x_1\), the level in the first tank, as shown by the blue curve in the top plot of Figure 2. The fault index and the selected sampling period index variation are shown in the bottom plot of Figure 2. We used [6, Algorithm 1] setting a closed loop eigenvalue at zero, and hence, the response may be aggressive and differ from the one in [24].

VI. CONCLUSIONS

In this paper we have presented a new approach for the virtual actuator technique under varying sampling rate control systems. In this approach, the controller is in charge of both providing the control action and administering the sampling periods, taken from a finite set. The considered fault scenario consists of abrupt actuator outages and we have assumed that correct fault detection and isolation is provided externally. The main results of this paper show that in steady state, the control input will achieve its desired constant reference value, the VA states will converge to a constant value (irrespective of the sampling period used, and the variations on it), and the desired constant setpoint tracking objective is ensured for the performance variable. The considered approach can be of particular interest for plants with input redundancy controlled using a NCS. Future
work will focus on the design of an automatic fault detection and isolation method for the current approach.

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