ON FLOW-EQUIVALENCE OF R-GRAPH SHIFTS

WOLFGANG KRIEGER

Abstract. We show that Property (A) of subshifts and the semigroup, that is associated to subshifts with Property (A), are invariants of flow equivalence. We show for certain $\mathcal{R}$-graphs that their isomorphism is implied by the flow equivalence of their $\mathcal{R}$-graph shifts.

1. Introduction

Let $\Sigma$ be a finite alphabet, and let $S_\Sigma$ be the shift on the shift space $\Sigma^\mathbb{Z}$,

$$S_\Sigma((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}}, \quad (x_i)_{i \in \mathbb{Z}} \in \Sigma^\mathbb{Z}.$$ 

$S_\Sigma$-invariant closed subsets $X$ of $\Sigma^\mathbb{Z}$ (more precisely, with $S_X$ denoting the restriction of $S_\Sigma$ to $X$, the dynamical systems $(X, S_X)$ are called subshifts. These are the subject of symbolic dynamics. For an introduction to symbolic dynamics see [Ki] or [LM].

A word is called admissible for a subshift $X \subset \Sigma^\mathbb{Z}$ if it appears in a point of $X$. We denote the set of admissible words of a subshift $X \subset \Sigma^\mathbb{Z}$ by $L(X)$. The language $L(X)$ is factorial and bi-extensible, and every factorial and bi-extensible language is the set of admissible words of a unique subshift.

Let $\bullet$ be a symbol that is not in $\Sigma$, and consider a subshift $X \subset \Sigma^\mathbb{Z}$. Denote by $\varphi(\sigma)$ the mapping that assigns to a word $a \in L(X)$ the word that is obtained from $a$ by carrying out the substitution that replaces the symbol $\sigma$ by the word $\sigma \bullet$. The set of subwords of the words in $\varphi(\sigma)(L(X))$ is a factorial and bi-extensible language, and we denote the subshift that it determines by $X(\sigma)$. One says that the subshift $X(\sigma)$ arises from the subshift $X$ by symbol expansion. In Section 2 we describe some effects of symbol expansion.

Subshifts $X \subset \Sigma^\mathbb{Z}$ and $\tilde{X} \subset \tilde{\Sigma}^\mathbb{Z}$ are called flow equivalent if there exists a sequence $Z_k, 1 \leq k \leq K, K \in \mathbb{N}$, of subshifts, such that $X = Z_1$ and $\tilde{X} = Z_K$, and such that $Z_k$ is topologically conjugate to $Z_{k-1}$, or $Z_k$ is obtained from $Z_{k-1}$ by symbol expansion, or $Z_{k-1}$ is obtained from $Z_k$ by symbol expansion, $1 < k \leq K$. Flow equivalence was introduced by Parry and Sullivan in 1975 [PS]. Next to topological conjugacy it is one of the fundamental equivalence relations for subshifts.

The notions of $\mathcal{R}$-graph, $\mathcal{R}$-graph semigroup, and $\mathcal{R}$-graph shift were introduced in [Kr2]. The class of $\mathcal{R}$-graph shifts contains the class of Markov-Dyck shifts [M3]. In Section 5 we show for certain $\mathcal{R}$-graphs, that the flow equivalence of their $\mathcal{R}$-graph shifts implies their isomorphism. This extends a result of Costa and Steinberg [CS] for Markov-Dyck shifts. The proof uses Property (A) and the semigroup that is associated to subshifts with Property (A) [Kr1]. In Section 3 we prove invariance under flow equivalence of Property (A) and in Section 4 we prove invariance under flow equivalence of the associated semigroup. For an extension of the theory beyond subshifts with Property (A) see Costa and Steinberg [CS].

In Section 5 we consider $\mathcal{R}$-graph shifts. In [HK] there was given a necessary and sufficient condition for an $\mathcal{R}$-graph to have an $\mathcal{R}$-graph shift with Property (A), whose associated semigroup is the $\mathcal{R}$-graph semigroup of the $\mathcal{R}$-graph. Under
We denote the bijection of \( \Omega(X) \). For precision we note, that one has, with \( \ell(a) \) also, we denote by \( \phi(x) \), and we use similar notation in the case that indices range in semi-infinite intervals. From the context it becomes clear, if such an identification is made.) We set
\[ \Gamma(a) = \{ x^+ \in X \mid a x^+ \in X \}, \quad a \in x_{[i,j]}, \quad i, j \in \mathbb{Z}, i \leq j. \]
The notation \( \Gamma(a) \) has the symmetric meaning. We also set
\[ \omega_X(a) = \bigcap_{x^- \in \Gamma(a)} \{ x^+ \in \Gamma(a) : x^- a x^+ \in X \}, \quad a \in x_{[i,j]}, \quad i, j \in \mathbb{Z}, i \leq j. \]
The notation \( \omega_X(a) \) has the symmetric meaning. And we set
\[ \Gamma_X(a) = \{ (x^-, x^+) \in \Gamma(a) \times \Gamma(a) : x^- a x^+ \in X \}, \quad a \in x_{[i,j]}, \quad i, j \in \mathbb{Z}, i \leq j. \]
Let \( \sigma \in \Sigma \), let \( \bullet \) be a symbol that is not in \( \Sigma \), and consider for a subshifts \( X \subset \Sigma^\mathbb{Z} \) the subshift \( X^{(}\sigma\rangle \subset (\Sigma \cup \{\bullet\})^\mathbb{Z} \). We denote by \( \varphi^{(}\sigma\rangle (\varphi^{(}\sigma\rangle ) \) the mapping that assigns to \( x^+ \in \chi_{\langle \infty,0 \rangle} (x^+ \in \chi_{[0,\infty)} \) the point \( x^{(}\sigma\rangle \varphi^{(}\sigma\rangle \chi_{\langle -\infty,0 \rangle} (x^{(}\sigma\rangle \chi_{[0,\infty)} \) that is obtained from \( x^- \) \( (x^+ \) ) by carrying out the substitution that replaces the symbol \( \sigma \) by the word \( \sigma \bullet \). Also we denote by \( \varphi^{(}\sigma\rangle \) the mapping that assigns to a point \( x \in X \) the point in \( X^{(}\sigma\rangle \), that is given by
\[ \varphi^{(}\sigma\rangle (x_{\langle -\infty,0 \rangle}) = \varphi^{(}\sigma\rangle (x_{\langle -\infty,0 \rangle}), \quad \varphi^{(}\sigma\rangle (x_{[0,\infty)}) = \varphi^{(}\sigma\rangle (x_{[0,\infty)}). \]
One observes that
\[ \varphi^{(}\sigma\rangle (O_X(x)) \subset O_X(\varphi^{(}\sigma\rangle (x)), \quad x \in X. \]
For precision we note, that one has, with \( \ell^- (x,n)(\ell^+ (x,n)) \) denoting the length of \( \varphi^{(}\sigma\rangle (x_{\langle -n,0 \rangle})(\varphi^{(}\sigma\rangle (x_{[0,n]})), \) that
\[ \varphi^{(}\sigma\rangle (S_{\langle -n \rangle} \chi_{\langle -\infty,0 \rangle} (\varphi^{(}\sigma\rangle (x)), \quad \varphi^{(}\sigma\rangle (S_{\langle -n \rangle} \chi_{\langle -\infty,0 \rangle} (\varphi^{(}\sigma\rangle (x)), \quad n \in \mathbb{N}. \]
Also,
\[ \varphi^{(}\sigma\rangle (X) \cup S_{\langle -n \rangle} \chi_{\langle -\infty,0 \rangle} (\varphi^{(}\sigma\rangle (X)). \]
We denote the bijection of \( \Omega(X) \) onto \( \Omega(X^{(}\sigma\rangle \) that assigns to the \( S_X \)-orbit of \( x \in X \) the \( S_X^{(}\sigma\rangle \)-orbit of \( \varphi^{(}\sigma\rangle (x) \) by \( \xi_n \).

**Lemma 2.1.** For a subshift \( X \subset \Sigma^\mathbb{Z} \) and for \( \sigma \in \Sigma, a \in L(X) \), one has
\[ \varphi^{(}\sigma\rangle (\omega_X^+ (a)) = \varphi^{(}\sigma\rangle (\varphi^{(}\sigma\rangle (a)). \]
Lemma 3.1. For a subshift $X \subset \Sigma^\mathbb{Z}$ and for $a \in \Sigma$, one has
$$\Gamma_X(a) = \Gamma_X(a),$$
if and only if
$$\Gamma_X(a) = \Gamma_X(a).$$
Proof. The lemma follows from
$$\Gamma_X(a) = \Gamma_X(a), \quad \Gamma_X(a) = \Gamma_X(a), \quad a \in \mathcal{L}(X).$$

3. Property (A)

Given a subshift $X \subset \Sigma^\mathbb{Z}$ we define a subshift of finite type $A_n(X)$ by
$$A_n(X) = \bigcap_{i \in \mathbb{Z}} \{ x \in X : x_{i+n} \in \omega^+_{X}(x_{i+n}) \} \cap \{ x \in X : x_{i+n} \in \omega^-_{X}(x_{i+n}) \}, \quad n \in \mathbb{N},$$
and we set
$$A(X) = \bigcup_{n \in \mathbb{N}} A_n(X).$$

Lemma 3.1. For a subshift $X \subset \Sigma^\mathbb{Z}$, and for $\sigma \in \Sigma$, one has

(1) \quad $\xi_\sigma(\Omega(A_n(X))) \subseteq \Omega(A_{2n}(X^{\sigma})), \quad n \in \mathbb{N},$

and

(2) \quad $\xi_\sigma^{-1}(\Omega(A_n(X^{\sigma}))) \subseteq \Omega(A_n(X)), \quad n \in \mathbb{N}.$

Proof. We show (1). Let $n \in \mathbb{N}$, let $x \in A_n(X)$, and let $i \in \mathbb{Z}$. Let $\mu$ be the number of times that the symbol $\bullet$ appears in $\varphi_\sigma(x)_{[i,i+2n]}$. Assume that neither $x_i^{(\sigma)} = \bullet$, nor $x_{i+2n-1}^{(\sigma)} = \bullet$. Then
$$\varphi_\sigma(x_{i,[i+2n-\mu]}) = \varphi_\sigma(x_{[i,i+2n]}).$$
From
$$x_{i,[i+2n-\mu,\infty)} \in \omega^+_{X}(x_{i,[i+2n-\mu]}),$$
it follows then by Lemma 2.1, that
$$\varphi_\sigma(x)_{[i,i+2n,\infty)} \in \omega^+_{X}(\varphi_\sigma(x)_{[i,i+2n]}).$$
In the case that $x_i^{(\sigma)} = \bullet$, necessarily $x_{i-1}^{(\sigma)} = \sigma$, and in the case that $x_{i+2n-1}^{(\sigma)} = \sigma$, necessarily $x_{i+2n}^{(\sigma)} = \bullet$, and in both cases it is seen that (3) also holds.

For (2) one has a similar argument. □
We recall from [KT1] the definition of Property (A). For $n \in \mathbb{N}$ a subshift $X \subset \Sigma^\mathbb{Z}$, has property $(a, n, H)$, $H \in \mathbb{N}$, if for $h, h' \geq 3H$ and for $I_-, I_+, I_-^*, I_+^* \in \mathbb{Z}$, such that
\[ I_+ - I_- + I_-^* - I_+^* \geq 3H, \]
and for \( a \in A_n(X)_{(I_-, I_+)}, \tilde{a} \in A_n(X)_{(I_-, I_+)} \),

such that
\[ a(I_-, I_+ + H) = \tilde{a}(I_-, I_+ + H), \]
\[ a(I_+, I_-) = \tilde{a}(I_+, I_-), \]
one has that
\[ \Gamma_X(a) = \Gamma_X(\tilde{a}). \]

It is assumed, that $A(X) \neq \emptyset$. The subshift $X \subset \Sigma^\mathbb{Z}$ has property (A) if there are $H_n, n \in \mathbb{N}$, such that $X$ has the properties $(a, n, H_n), n \in \mathbb{N}$.

**Theorem 3.2.** For a subshift $X \subset \Sigma^\mathbb{Z}$ and for $\sigma \in \Sigma$, one has that $X$ has Property (A) if and only if $X^{(\sigma)}$ has Property (A).

**Proof.** The theorem follows from Lemma 2.2 and Lemma 3.1. \qed

4. THE ASSOCIATED SEMIGROUP

Consider a subshift $X \subset \Sigma^\mathbb{Z}$ with Property (A). We denote the set of periodic points in $A(X)$ by $P(A(X))$. We introduce a preorder relation $\geq(X)$ into the set $P(A(X))$ where for $q, r \in P(A(X)), q \geq(X) r$, means that there exists a point in $A(X)$ that is left asymptotic to the orbit of $q$ and right asymptotic to the orbit of $r$.

The equivalence relation on $P(A(X))$ that results from the preorder relation $\geq(X)$ we denote by $\approx(X)$. We denote the set of $\approx(X)$-equivalence classes by $\mathfrak{P}(X)$.

**Lemma 4.1.** For a subshift $X \subset \Sigma^\mathbb{Z}$, for $\sigma \in \Sigma$, $q, r \in P(A(X))$, and for $\sigma \in \Sigma$, one has
\[ q \geq(X) r, \]
if and only if
\[ \varphi(\sigma)(q) \geq(X^{(\sigma)}) \varphi(\sigma)(r). \]

**Proof.** This follows from Lemma 3.1. \qed

We recall the construction of the associated semigroup. For a property (A) subshift $X \subset \Sigma^\mathbb{Z}$ we denote by $Y(X)$ the set of points in $X$ that are left asymptotic to a point in $P(A(X))$ and also right asymptotic to a point in $P(A(X))$. Let $y, \tilde{y} \in Y(X)$, let $y$ be left asymptotic to $q \in P(A(X))$ and right asymptotic to $r \in P(A(X))$, and let $\tilde{y}$ be left asymptotic to $\tilde{q} \in P(A(X))$ and right asymptotic to $\tilde{r} \in P(A(X))$. Given that $X$ has the properties $(a, n, H_n), n \in \mathbb{N}$, we say that $y$ and $\tilde{y}$ are equivalent, $y \approx(X) \tilde{y}$, if $q \approx(X) \tilde{q}$ and $r \approx(X) \tilde{r}$, and if for $n \in \mathbb{N}$ such that $q, r, \tilde{q}, \tilde{r} \in P(A_n(X))$ and for $I, J, \tilde{I}, \tilde{J} \in \mathbb{Z}, I < J, \tilde{I} < \tilde{J}$, such that
\[ y(-\infty, I) = q(-\infty, I), \quad y(J, \infty) = r(J, \infty), \]
\[ \tilde{y}(-\infty, \tilde{I}) = \tilde{q}(-\infty, \tilde{I}), \quad \tilde{y}(\tilde{J}, \infty) = \tilde{r}(\tilde{J}, \infty), \]

one has for $h \geq 3H_n$ and for
\[ a \in X_{(I-h, J+h)}, \tilde{a} \in X_{(\tilde{I}-h, \tilde{J}+h)}, \]

such that
\[ a(I-h, J+h) = \tilde{a}(I-h, J+h), \]
\[ a(I-h, J+H_n) = \tilde{a}(I-h, J+H_n), \]
\[ a(J+h-H_n, J+h) = \tilde{a}(J+h-H_n, J+h), \]

\[ a(I-h, J+H_n) = \tilde{a}(I-h, J+H_n), \]
\[ a(J+h-H_n, J+h) = \tilde{a}(J+h-H_n, J+h), \]

\[ a(I-h, J+H_n) = \tilde{a}(I-h, J+H_n), \]
\[ a(J+h-H_n, J+h) = \tilde{a}(J+h-H_n, J+h), \]

\[ a(I-h, J+H_n) = \tilde{a}(I-h, J+H_n), \]
\[ a(J+h-H_n, J+h) = \tilde{a}(J+h-H_n, J+h), \]
and such that

\[ a_{(i-h,i]} \in A_n(X)_{(i-h,i]}, \quad \tilde{a}_{(j-h,j]} \in A_n(X)_{(j-h,j]} \]

and

\[ a_{(j,j+h]} \in A_n(X)_{(j,j+h]}, \quad \tilde{a}_{(j,j+h]} \in A_n(X)_{(j,j+h]} \]

that

\[ \Gamma_X(a) = \Gamma_X(\tilde{a}). \]

To give \([Y(X)]_{\approx(X)}\) the structure of a semigroup, let \(u, v \in Y(X)\), let \(u\) be right asymptotic to \(q \in P(A(X))\) and let \(v\) be left asymptotic to \(r \in P(A(X))\). If here \(q \gtrsim (X)r\), then \([u]_{\approx(X)} [v]_{\approx(X)}\) is set equal to \([y]_{\approx(X)}\), where \(y\) is any point in \(Y\) such that there are \(n \in \mathbb{N}, I, J, \bar{I}, \bar{J} \in \mathbb{Z}, I < J, \bar{I} < \bar{J}\), such that \(q, r \in A_n(X)\), and such that

\[ u_{(-\infty, i]} = q_{(-\infty, i]}, \quad v_{(-\infty, j]} = r_{(-\infty, j]}, \]

\[ y_{(-\infty, i+H_n]} = u_{(-\infty, j+H_n]}, \quad y_{(j-H_n, \infty)} = v_{(j-H_n, \infty)}, \]

and

\[ y_{(i, j]} \in A_n(X)_{(i,j]}, \]

provided that such a point \(y\) exists. If such a point \(y\) does not exist, \([u]_{\approx(X)} [v]_{\approx(X)}\) is equal to zero. Also, in the case that one does not have \(q \gtrsim (X)r\), \([u]_{\approx(X)} [v]_{\approx(X)}\) is equal to zero.

Consider a subshift \(X \subset \Sigma^\mathbb{Z}\) with Property (A). For \(p \in \mathcal{P}(X)\), we choose a \(d(p) \in p\), and we set

\[ D = \{ d(p) : p \in \mathcal{P}(X) \}. \]

In order to facilitate the proof of its invariance under flow equivalence we give an alternate description of the semigroup that is associated to \(X\) in terms of the system \(D \subset Y_X\) of representatives of the equivalence relation \(\approx(X)\). For \(y \in O_X(d(p)), p \in \mathcal{P}(X)\), we define a \(J(y, d(p)) \in \mathbb{Z}\) by

\[ S_X^{-J(y, d(p))}(y) = d(p), \quad 0 \leq \pi(d(p)) < \pi(d(p^2)). \]

For \(p \in \mathcal{P}(X)\) we set

\[ H(d(p)) = \min \{ H \in \mathbb{N} : \Gamma_X(p_{[0, H \pi(p)]}) = \Gamma_X(p_{[0, (H+1) \pi(d(p))]} \} \].

We denote by \(Y_X^{-}(D)\), the set of points in \(Y_X\), that are left asymptotic to the orbit of a point in \(D\), and also right asymptotic to the orbit of a point in \(D\). More precisely, we denote by \(Y_X^{-}(d(p))(Y_X^{+}(d(p)))\), the set of points in \(Y_X\), that are left (right) asymptotic to the orbit of \(d(p), p \in \mathcal{P}(X)\). For

\[ y \in Y_X^{-}(d(q)) \cap Y_X^{+}(d(r)), \quad q, r \in \mathcal{P}(X), \]

we set

\[ I^-(y) = \begin{cases} J(y, d(q)), & \text{if } y \in O_X(d(q)), \\ \max \{ I \in \mathbb{Z} : y_{(-\infty, l]} = d(q)_{(-\infty, 0]} \}, & \text{if } y \notin O_X(d(q)), \end{cases} \]

and

\[ I^+(y) = \begin{cases} J(y, d(r)), & \text{if } y \in O_X(d(r)), \\ \min \{ I \in \mathbb{Z} : y_{l, \infty} = d(r)_{[0, \infty]} \}, & \text{if } y \notin O_X(d(r)). \end{cases} \]

We say that \(O, O' \in \Omega(Y_X^{-}(D))\) are \(\approx(D)\)-equivalent, if \(O\) and \(O'\) are left asymptotic to the same periodic orbit, and also right asymptotic to the same periodic orbit, and if, with \(q \in \mathcal{P}\) such that \(y\) and \(y'\) are right asymptotic to the orbit of \(d(q)\) and with
\( r \in \mathfrak{B} \) such that \( y \) and \( y' \) are left asymptotic to the orbit of \( d^{(\tau)} \), there exist \( y \in O \) and \( y' \in O' \) such that

\[
\Gamma_X (d^{(q)}_{[0,H(d^{(p)}) \pi (d^{(\tau)})]} y_{1-I^- (y), I^+ (y)}) d^{(\tau)}_{[0,H(d^{(\tau)}) \pi (d^{(\tau)})]}) = \\
\Gamma_X (d^{(q)}_{[0,H(d^{(p)}) \pi (d^{(\tau)})]} y_{1-I^- (y'), I^+ (y')}) d^{(\tau)}_{[0,H(d^{(\tau)}) \pi (d^{(\tau)})]})
\]

To give \( \Omega(Y^{(D)}_X) \) the structure of a semigroup, let \( q, p, r \in \mathfrak{B}, \) and let for points

\[
u = Y^{+}_X (p) \cap Y^{+}_X (r), \quad \nu = Y^{+}_X (p) \cap Y^{+}_X (r),
\]

in case, that the word (3) is not admissible for \( X \), let a point \( y[u, v] \in Y^{+}_X (q) \cap Y^{+}_X (r) \) be given by

\[
y[u, v]_{(-\infty, 0)} = d^{(q)}_{[0,\infty, 0]},
\]

and

\[
y[u, v]_{[0,\infty)} = d^{(q)}_{[0,H(d^{(p)}) \pi (d^{(\tau)})]} y_{1-I^- (u), I^+ (u)} d^{(p)}_{[0,H(d^{(p)}) \pi (d^{(\tau)})]} y_{1-I^- (v), I^+ (v)} d^{(\tau)}_{[0,\infty, 0]},
\]

and set

\[
[O(u)]_{\equiv (D)} [O(v)]_{\equiv (D)} = [O(y[u, v])]_{\equiv (D)}.
\]

In case, that the word (3) is not admissible for \( X \), set

\[
[O(u)]_{\equiv (D)} [O(v)]_{\equiv (D)} = 0.
\]

Also, for \( q, r \in \mathfrak{B} \), if

\[
Y^{+}_X (q) \cap A(X) \cap Y^{+}_X (r) \neq \emptyset,
\]

define a \( \approx (D) \)-equivalence class \( \gamma(q, r) \) by

\[
\gamma(q, r) = [O(y)]_{\equiv (D)}, \quad y \in Y^{+}_X (q) \cap A(X) \cap Y^{+}_X (r).
\]

As a consequence of Property (A) of \( X \) the \( \approx (D) \)-equivalence class \( \gamma(q, r) \) is well defined. If

\[
Y^{+}_X (q) \cap A(X) \cap Y^{+}_X (r) = \emptyset,
\]

set

\[
\gamma(q, r) = 0.
\]

Identify \( p \in \mathfrak{B} \) with \( \gamma(p, p) \). Finally for \( q, r \in \mathfrak{B} \), and for \( u \in Y^{+}_X (q), v \in Y^{+}_X (r) \), set

\[
[O(u)]_{\equiv (D)} [O(v)]_{\equiv (D)} = [u]_{\equiv (D)} \gamma(q, r)[v]_{\equiv (D)}.
\]

An isomorphism \( \eta_{\sigma, D} \) of \( [Y_X]_{\equiv (X)} \) onto \( [\Omega(Y^{(D)}_X)]_{\equiv (D)} \) is obtained by choosing out of every \( \approx (X) \)-equivalence class \( \alpha \) a point \( \eta(\alpha) \in Y^{(D)}_X \), and by setting

\[
\eta^{(D)}_X (\alpha) = [\eta(\alpha)]_{\equiv (D)}.
\]

**Theorem 4.2.** For a subshift \( X \subset \Sigma^2 \) with Property (A) and for \( \sigma \in \Sigma \), the semigroups, that are associated to \( X \) and \( X^{(\sigma)} \), are isomorphic.

**Proof.** Set

\[
d^{(p^{(\sigma)})} = \varphi^{(\sigma)} (d^{(p)}), \quad p \in \mathfrak{B}.
\]

One has

\[
\pi (d^{(p^{(\sigma)})}) = \pi (d^{(p)}), \quad p \in \mathfrak{B},
\]

and one has by Lemma 2.2, that

\[
H(d^{(p^{(\sigma)})}) = H(d^{(p)}), \quad p \in \mathfrak{B}.
\]
Setting
\[ D^{(\sigma)} = \{ d^{(\sigma)}(p) : p \in \mathcal{P} \}. \]
yields a system of representatives of the \( (X^{(\sigma)}) \)-equivalence classes in \( \mathcal{P}(X^{(\sigma)}) \).
By construction
\[ \varphi^{(\sigma)}([y[u, v])] = y[\varphi^{(\sigma)}(u), \varphi^{(\sigma)}(v)]], \quad u, v \in Y_X. \]
Also, by Lemma 3.1, for \( q, r \in \mathcal{P} \),
\[ Y_X^{-}(q) \cap A(X) \cap Y_X^{-}(r) \neq \emptyset, \]
if and only if
\[ Y_X^{-}(q^{(\sigma)}) \cap A(X^{(\sigma)}) \cap Y_X^{-}(r^{(\sigma)}) \neq \emptyset. \]
It follows that an isomorphism \( \psi_{\sigma,D} \) of \( [Y_X^{(D)}]_{\approx(D)} \) onto \( [Y_X^{(D^{(\sigma)})}]_{\approx(D^{(\sigma)})} \) is given by setting
\[ \psi_{\sigma,D}([y]) = [\varphi^{(\sigma)}(y)], \quad y \in Y_X^{-} \]
and one obtains an isomorphism \( \Xi^{(\sigma)} \) of \( [Y_X]_{\approx(X)} \) onto \( [Y_X^{(\sigma)}]_{\approx(X^{(\sigma)})} \) by setting
\[ \Xi^{(\sigma)} = \eta_{\sigma,D}^{-1} \psi_{\sigma,D} \eta_{\sigma,D}. \quad \square \]

For the invariance of the associated semigroup under flow equivalence, under the assumption that \( A(X) \) is dense in \( X \), or in the sofic case, see also [CS, Theorem 9.20]).

The semigroup \( [Y_X^{(D)}]_{\approx(D)} \) is a set of equivalence classes of orbits. As originally done in [Kr1], we have introduced the associated semigroup of a subshift with Property (A) in terms of equivalence classes of points, rather than equivalence classes of orbits. However, since points in \( Y_X \), that are in the same orbit, are \( \approx (X) \)-equivalent, one can define the associated semigroup in the first place as a set of equivalence classes of orbits. The same remark applies to the set of idempotents \( \mathcal{P} \). When the associated semigroup is introduced as a set of equivalence classes of orbits, then the mapping \( \xi_{\sigma} \) is seen to induce the isomorphism of the associated semigroup of \( X \) onto the associated semigroup of \( X^{(\sigma)} \).

5. \( \mathcal{R} \)-graph shifts

Given finite sets \( \mathcal{E}^{-} \) and \( \mathcal{E}^{+} \) and a relation \( \mathcal{R} \subset \mathcal{E}^{-} \times \mathcal{E}^{+} \), we set
\[ \mathcal{E}^{-}(\mathcal{R}) = \{ e^{-} \in \mathcal{E}^{-} : \mathcal{E}^{-} \times \mathcal{E}^{+} \subset \mathcal{R} \}, \quad \mathcal{E}^{+}(\mathcal{R}) = \{ e^{+} \in \mathcal{E}^{+} : \mathcal{E}^{-} \times \mathcal{E}^{+} \subset \mathcal{R} \}. \]
and
\[ \Omega_{\mathcal{R}}^{+}(e^{-}) = \{ e^{+} \in \mathcal{E}^{+} : \mathcal{E}^{-} \times \mathcal{E}^{+} \subset \mathcal{R} \}, \quad e^{-} \in \mathcal{E}^{-}, \]
\[ \Omega_{\mathcal{R}}^{-}(e^{+}) = \{ e^{-} \in \mathcal{E}^{-} : \mathcal{E}^{-} \times \mathcal{E}^{+} \subset \mathcal{R} \}, \quad e^{+} \in \mathcal{E}^{+}. \]

We recall from [Kr2] the notion of an \( \mathcal{R} \)-graph. Let there be given a finite directed graph with vertex set \( \mathcal{P} \) and edge set \( \mathcal{E} \). Assume also given a partition
\[ \mathcal{E} = \mathcal{E}^{-} \cup \mathcal{E}^{+}. \]
With \( s \) and \( t \) denoting the source and the target vertex of a directed edge we set
\[ \mathcal{E}^{-}(q, r) = \{ e^{-} \in \mathcal{E}^{-} : s(e^{-}) = q, \ t(e^{-}) = r \}, \]
\[ \mathcal{E}^{+}(q, r) = \{ e^{-} \in \mathcal{E}^{+} : s(e^{-}) = r, \ t(e^{-}) = q \}, \quad q, r \in \mathcal{P}. \]
We assume that \( \mathcal{E}^{-}(q, r) \neq \emptyset \) if and only if \( \mathcal{E}^{+}(q, r) \neq \emptyset \), \( q, r \in \mathcal{P} \), and we assume that the directed graph \( (\mathcal{P}, \mathcal{E}^{-}) \) is strongly connected, or, equivalently, that the directed graph \( (\mathcal{P}, \mathcal{E}^{+}) \) is strongly connected. Let there further be given relations
\[ \mathcal{R}(q, r) \subset \mathcal{E}^{-}(q, r) \times \mathcal{E}^{+}(q, r), \quad q, r \in \mathcal{P}, \]
and set
\[ \mathcal{R} = \bigcup_{q, r \in \mathcal{P}} \mathcal{R}(q, r). \]

The resulting structure, for which we use the notation \( G_{\mathcal{R}}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+) \), is called an \( \mathcal{R} \)-graph.

We also recall the construction of a semigroup (with zero) \( S_{\mathcal{R}}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+) \) from an \( \mathcal{R} \)-graph \( G_{\mathcal{R}}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+) \) as described in [Kr2]. The semigroup \( S_{\mathcal{R}}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+) \) contains idempotents \( 1_p, p \in \mathcal{P} \), and has \( \mathcal{E} \) as a generating set. Besides \( 1_p = 1_p, p \in \mathcal{P} \), the defining relations are:

\[ f^- g^+ = 1_q, \quad f^- \in \mathcal{E}^-(q, r), \quad g^+ \in \mathcal{E}^+(q, r), \quad (f^-, g^+) \in \mathcal{R}(q, r), \quad q, r \in \mathcal{P}, \]

and

\[ 1_q e^- = e^- 1_r = e^-, \quad e^- \in \mathcal{E}^-(q, r), \]
\[ 1_r e^+ = e^+ 1_q = e^+, \quad e^+ \in \mathcal{E}^+(q, r), \quad q, r \in \mathcal{P}, \]

\[ f^- g^+ = \begin{cases} 1_q, & \text{if } (f^-, g^+) \in \mathcal{R}(q, r), \\ 0, & \text{if } (f^-, g^+) \notin \mathcal{R}(q, r), \end{cases} \]

\( f^- \in \mathcal{E}^-(q, r), \quad g^+ \in \mathcal{E}^+(q, r), \quad q, r \in \mathcal{P}, \)

and

\[ 1_q 1_r = 0, \quad q, r \in \mathcal{P}, q \neq r. \]

The semigroup \( S_{\mathcal{R}}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+) \) is called an \( \mathcal{R} \)-graph semigroup.

The \( \mathcal{R} \)-graph shift \( M_{\mathcal{R}}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+) \) of the \( \mathcal{R} \)-graph \( G_{\mathcal{R}}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+) \) is the subshift

\[ M_{\mathcal{R}}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+) \subset \left( \{ \mathcal{E}^- \cup \mathcal{E}^+ \} \right)^\mathbb{Z} \]

with the admissible words \( (\sigma_i)_{1 \leq i \leq I} \), \( I \in \mathbb{N} \), of \( M_{\mathcal{R}}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+) \) given by the condition

\[ \prod_{1 \leq i \leq I} \sigma_i \neq 0. \]

For an \( \mathcal{R} \)-graph \( G_{\mathcal{R}}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+) \) we denote by \( \mathcal{V}(1) \) the set of vertices in \( \mathcal{P} \) that have a single predecessor vertex in \( \mathcal{E}^- \), or, equivalently, that have a single successor vertex in \( \mathcal{E}^+ \). For \( p \in \mathcal{V}(1) \) the predecessor vertex of \( p \) in \( \mathcal{E}^- \), which is identical to the successor vertex of \( p \) in \( \mathcal{E}^+ \), is denoted by \( \kappa(p) \). We set

\[ \mathcal{E}^-_{\mathcal{R}} = \bigcup_{p \in \mathcal{V}(1)} \mathcal{E}^-(\mathcal{R}(\kappa(p), p)), \quad \mathcal{E}^+_{\mathcal{R}} = \bigcup_{p \in \mathcal{V}(1)} \mathcal{E}^+(\mathcal{R}(\kappa(p), p)), \]

and

\[ \mathcal{V}(1) = \{ p \in \mathcal{V}(1) : \mathcal{R}(\kappa(p), p) = \mathcal{E}^-_{\mathcal{R}}(\kappa(p), p) \times \mathcal{E}^+_{\mathcal{R}}(\kappa(p), p) \}. \]

We formulate conditions (a), (b), (c) and (d) on an \( \mathcal{R} \)-graph \( G_{\mathcal{R}}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+) \) as follows:

(a–) \( \Omega^+_{\mathcal{R}(q, r)}(e^-) \neq \Omega^+_{\mathcal{R}(q, r)}(\bar{e}^-), \quad e^-, \bar{e}^- \in \mathcal{E}^-(q, r), e^- \neq \bar{e}^- \), \( q, r \in \mathcal{P} \),

(a+) \( \Omega^-_{\mathcal{R}(q, r)}(e^+) \neq \Omega^-_{\mathcal{R}(q, r)}(\bar{e}^+), \quad e^+, \bar{e}^+ \in \mathcal{E}^+(r, q), e^+ \neq \bar{e}^+ \), \( q, r \in \mathcal{P} \).

(b–) There is no non-empty cycle in \( \mathcal{E}^-_{\mathcal{R}} \),

(b+) There is no non-empty cycle in \( \mathcal{E}^+_{\mathcal{R}} \).

(c) For \( p \in \mathcal{V}(1) \) such that \( \kappa(p) \neq p \), \( \mathcal{E}^-_{\mathcal{R}}(p) = \emptyset \), or \( \mathcal{E}^+_{\mathcal{R}}(p) = \emptyset \),

(d) For \( q, r \in \mathcal{V}(1) \), \( q \neq r \), there do not simultaneously exist a path in \( \mathcal{E}^-_{\mathcal{R}} \) from \( q \) to \( r \) and a path in \( \mathcal{E}^+_{\mathcal{R}} \) from \( q \) to \( r \),
Theorem 5.1. For $\mathcal{R}$-graphs $\mathcal{G}_\mathcal{R}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+)$ that satisfy the Conditions (a), (b), (c) and (d) the flow equivalence of the $\mathcal{R}$-graph shifts $D_\mathcal{R}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+)$ implies the isomorphism of the $\mathcal{R}$-graphs $\mathcal{G}_\mathcal{R}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+)$.

Proof. By Theorem 2.3 of [HK] and Theorem 6.1 of [HK] the conditions imply that the $\mathcal{R}$-graph shift $D_\mathcal{R}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+)$ has property (A), and that the semigroup, that is associated to it, is $S_\mathcal{R}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+)$. By Theorem 4.2 the flow equivalence of the shifts $D_\mathcal{R}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+)$ implies the isomorphism of the $\mathcal{R}$-graph semigroups $S_\mathcal{R}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+)$, which in turn, by Theorem 2.1 of [Kri2], implies the isomorphism of the $\mathcal{R}$-graphs $\mathcal{G}_\mathcal{R}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+)$. □

Theorem 5.1 extends the result of Costa and Steinberg, that the flow equivalence of Markov-Dyck shifts of finite irreducible directed graphs, in which every vertex has at least two incoming edges, implies the isomorphism of the graphs (see [CS] and Theorem 8.6).

Let for $K > 1$, $B_K$ denote the full shift on $K$ symbols, and let $D_2$ denote the Dyck shift on four symbols. The shifts $D_2 \times B_K$, $K > 1$, belong to the class of $\mathcal{R}$-graph shifts. They arise from the one-vertex $\mathcal{R}$-graphs $\mathcal{G}_\mathcal{R}(\{p\}, \mathcal{E}^-, \mathcal{E}^+)$, where

$$\mathcal{E}^- = \{e^- m, \beta) : 1 \leq m \leq K, \beta = 0,1\}, \mathcal{E}^+ = \{e^- (l, \beta) : 1 \leq l \leq K, \beta = 0,1\},$$

and where

$$(e^- m, (\beta^-), e^- (l, \beta^+)) \in \mathcal{R}(p, p),$$

if and only if

$$\beta^- = \beta^+, \ 1 \leq m \leq K, 1 \leq l \leq K.$$

These $\mathcal{R}$-graphs do not satisfy the conditions of Theorem 5.1, but the $\mathcal{R}$-graph shifts $D_2 \times B_K$, $K > 1$, have Property (A), and the flow equivalence of their $\mathcal{R}$-graph shifts $D_2 \times B_K$, $K > 1$, still implies the isomorphism of these $\mathcal{R}$-graphs. This can be seen from the invariance under flow equivalence of the $\mathcal{K}$-groups of subshifts as shown by Matsumoto in [M1], and from

$$K_0(D_2 \times B_K) = \mathbb{Z}[\frac{1}{\mathcal{K}}] \infty, \ K > 1,$$

as also shown by Matsumoto [M2] Section 8]. Note that the associated semigroup of $D_2 \times B_K$, $K > 1$, is the Dyck inverse monoid $D_2$.

References

[CS] A. Costa and B. Steinberg, A categorical invariant of flow equivalence of shifts, arXiv: 1304.3487 [math.DS]

[HK] T. Hamachi and W. Krieger, A construction of subshifts and a class of semigroups, arXiv: 1303.4158 [math.DS]

[Ki] B. P. Kitchens, Symbolic dynamics, Springer, Berlin, Heidelberg, New York (1998)

[Kri1] W. Krieger, On a syntactically defined invariant of symbolic dynamics, Ergod. Th. & Dynam. Sys. 20 (2000), 501 – 516

[Kri2] W. Krieger, On subshift presentations, arXiv: 1209.2578 [math.DS]

[LM] D. Lind and B. Marcus, An introduction to symbolic dynamics and coding, Cambridge University Press, Cambridge (1995)

[M1] K. Matsumoto, Bowen-Franks groups as an invariant for flow equivalence of subshifts, Ergod. Th. & Dynam. Sys. 21 (2001), 1831 – 1842

[M2] K. Matsumoto, K-theoretic invariants and conformal measures of the Dyck shifts, International J. of Mathematics 16 (2005), 213 – 248

[M3] K. Matsumoto, C*-algebras arising from Dyck systems of topological Markov chains, Math. Scand. 109 (2011), 31 – 54

[PS] B. Parry and D. Sullivan, A topological invariant for flows on one-dimensional spaces, Topology 14 (1975), 297 – 299

Wolfgang Krieger
Institute for Applied Mathematics,
University of Heidelberg,
Im Neuenheimer Feld 294,
69120 Heidelberg,
Germany
krieger@math.uni-heidelberg.de