MODERN SINGULAR INTEGRAL THEORY WITH MILD KERNEL REGULARITY

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ABSTRACT. We present a framework based on modified dyadic shifts to prove multiple results of modern singular integral theory under mild kernel regularity. Using new optimized representation theorems we first revisit a result of Figiel concerning the UMD-extensions of linear Calderón–Zygmund operators with mild kernel regularity and extend our new proof to the multilinear setting improving recent UMD-valued estimates of multilinear singular integrals. Our other results include weighted bi-parameter estimates and two-weight bi-commutator estimates with modified Dini-type assumptions.

1. INTRODUCTION

The usual definition of a singular integral operator (SIO)

\[ Tf(x) = \int_{\mathbb{R}^d} K(x, y) f(y) \, dy \]

involves a Hölder-continuous kernel \( K \) with a power-type continuity-modulus \( t \mapsto t^\gamma \). However, many results continue to hold with significantly more general assumptions. Such kernel regularity considerations become non-trivial especially in connection with results that go beyond the classical Calderón–Zygmund theory – an example is the \( A_2 \) theorem of Hytönen [33] with Dini-continuous kernels by Lacey [41].

The fundamental question concerning the \( L^2 \) (or \( L^p \)) boundedness of an SIO \( T \) is usually best answered by so-called \( T1 \) theorems, where the action of the operator \( T \) on the constant function 1 is key. We study kernel regularity questions specifically in situations that are very tied to the \( T1 \) type arguments and the corresponding structural theory – a prominent example is the theory of SIOs acting on functions taking values in infinite-dimensional Banach spaces. Another major line of investigation concerns the theory of bi-parameter SIOs based on the representation theorem [50]. The distinction between one-parameter and multi-parameter SIOs has to do with the classification of SIOs according to the size of the singularity of the kernel \( K \). We will return to the specifics later.

A concrete definition of kernel regularity is as follows. It concerns the required regularity of the continuity-moduli \( \omega \) appearing in the various kernel estimates, such as,

\[ |K(x, y) - K(x', y)| \leq \omega \left( \frac{|x - x'|}{|x - y|}, \frac{1}{|x - y|^{d+1}}, |x - x'| \leq |x - y|/2. \]

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Recently, Grau de la Herrán and Hytönen [25] proved that the modified Dini condition
\[ \|\omega\|_{\text{Dini,} \alpha} := \int_0^1 \omega(t) \left(1 + \log \frac{1}{t}\right)^\alpha \frac{dt}{t} \]
with \( \alpha = \frac{1}{2} \) is sufficient to prove a \( T_1 \) theorem even with an underlying measure \( \mu \) that can be non-doubling. This matches the best known sufficient condition for the classical homogeneous \( T_1 \) theorem [15] – such results are implicit in Figiel [30] and explicit in Deng, Yan and Yang [16]. The exponent \( \alpha = 1/2 \) has a fundamental feeling in all of the existing arguments – it seems very difficult to achieve a \( T_1 \) theorem with a weaker assumption.

The proofs of \( T_1 \) theorems display a fundamental structural decomposition of SIOs into their cancellative parts and so-called paraproducts. It is this structure that is extremely important for obtaining further estimates beyond the initial scalar-valued \( L^p \) boundedness. The original dyadic representation theorem of Hytönen [32, 33] (extending an earlier special case of Petermichl [56]) provides a decomposition of the cancellative part of an SIO into so-called dyadic shifts. These are suitable generalisations of dyadic martingale transforms, also known as Haar multipliers

\[ f = \sum_{Q \in \mathcal{D}} \langle f, h_Q \rangle h_Q \mapsto \sum_{Q \in \mathcal{D}} \lambda_Q \langle f, h_Q \rangle h_Q, \quad |\lambda_Q| \leq 1, \]  

where \( h_Q \) is a cancellative Haar function on a cube \( Q \) and \( \mathcal{D} \) is a dyadic grid.

In [25] a new type of representation theorem appears, where the key difference to the original representation theorems [32, 33] is that the decomposition of the cancellative part is in terms of different operators that package multiple dyadic shifts into one and offer more efficient bounds when it comes to kernel regularity. Some of the ideas of the decomposition in [25] are rooted in the work of Figiel [29, 30].

Our work begins by giving a streamlined version of the new type of one-parameter representation theorem [25]. Importantly, we extend it to the multilinear setting and are able to identify an explicit and clear exposition of the appearing dyadic model operators that we call modified dyadic shifts. For example, in the linear case the usual generalization of (1.1) takes the form of a dyadic shift

\[ S_{i,j}f = \sum_{K \in \mathcal{D}} \sum_{I(\text{k}) = J(\text{j}) = K} a_{IJK} \langle f, h_I \rangle h_J, \]

while we replace these by the modified dyadic shifts

\[ Q_k f = \sum_{K \in \mathcal{D}} \sum_{I(\text{k}) = J(\text{K}) = K} a_{IJK} \langle f, h_I \rangle H_{I,J}. \]

Here \( I(\text{k}) \in \mathcal{D} \) is the \( k \)th parent of \( I \). The difference is that some more complicated functions \( H_{I,J} \) appear – on the other hand, we have \( i = j = k \).

The UMD – unconditional martingale differences – property of a Banach space \( X \) is a well-known necessary and sufficient condition for the boundedness of various singular integrals on \( L^p(\mathbb{R}^d; X) = L^p(X) \). A Banach space \( X \) has the UMD property if \( X \)-valued martingale difference sequences converge unconditionally in \( L^p \) for some (equivalently, all) \( p \in (1, \infty) \). By Burkholder [6] and Bourgain [5] we have that \( X \) is a UMD space if and only if a particular SIO – the Hilbert transform – admits an \( L^p(X) \)-bounded extension.
This theory soon advanced up to the vector-valued $T1$ theorem of Figiel [30]. It is important to understand that the fundamental result that all $L^2$ bounded SIOs (at least with the power-type continuity-modulus $t \mapsto t^\gamma$) can be extended to act boundedly on $L^p(X)$, $p \in (1, \infty)$, goes through this $T1$ theorem. The deep work of Figiel also contains estimates for the required kernel regularity $\alpha$ in terms of some characteristics of the UMD space $X$ – in general UMD spaces the threshold $\alpha = 1/2$ has to be replaced by a more complicated expression. We revisit these results using the modified dyadic shifts and obtain a modern proof of the following theorem. In our terminology, a Calderón–Zygmund operator (CZO) is an SIO that satisfies the $T1$ assumptions (equivalently, $T : L^2 \to L^2$ boundedly).

1.2. Theorem. Let $T$ be a linear $\omega$-CZO and $X$ be a UMD space with type $r \in (1, 2]$ and cotype $q \in [2, \infty)$. If $\omega \in \text{Dini}_{\omega(r,q)}$, we have

$$\|Tf\|_{L^p(X)} \lesssim \|f\|_{L^p(X)}, \quad p \in (1, \infty).$$

See the main text for the exact definitions. If $X$ is a Hilbert space, then $r = q = 2$ and we get the usual $\alpha = 1/2$. This theory is mostly relevant in UMD spaces that go beyond the function lattices, such as, non-commutative $L^p$ spaces.

A major part of our arguments has to do with the extension of these results to the multilinear setting. A basic model of an $n$-linear SIO $T$ in $\mathbb{R}^d$ is obtained by setting

$$T(f_1, \ldots, f_n)(x) = U(f_1 \otimes \cdots \otimes f_n)(x, \ldots, x), \quad x \in \mathbb{R}^d, \quad f_i : \mathbb{R}^d \to \mathbb{C},$$

where $U$ is a linear singular integral operator in $\mathbb{R}^{nd}$. See e.g. Grafakos–Torres [24] for the basic theory. Multilinear SIOs appear in applications ranging from partial differential equations to complex function theory and ergodic theory. For example, $L^p$ estimates for the homogeneous fractional derivative $D^\alpha f = \mathcal{F}^{-1}(|\xi|^\alpha \hat{f}(\xi))$ of a product of two or more functions – the fractional Leibniz rules – are used in the area of dispersive equations. Such estimates descend from the multilinear Hörmander-Mihlin multiplier theorem of Coifman-Meyer [10] – See e.g. Kato–Ponce [40] and Grafakos–Oh [23].

The original bilinear representation theorem with the usual power-type continuity-modulus is by some of us together with K. Li and Y. Ou [48]. An $n$-linear extension of [48] – even to the operator-valued setting – is by some of us together with K. Li and F. Di Plinio [18]. The structural theory in the $n$-linear setting is quite delicate – however, the finer details of the proofs of the representation theorems appear to be now converging to their final and most elegant form and we are able to present a short argument. This also cleans up some technicalities of [48].

After the structural theorems we prove the multilinear analogue of Theorem 1.2, see Theorem 4.43. Until recently, vector-valued extensions of multilinear SIOs had mostly been studied in the framework of $\ell^p$ spaces and function lattices, rather than general UMD spaces – see e.g. [7, 22, 44, 45, 53]. Taking the work [20] much further, the paper [17] finally established $L^p$ bounds for the extensions of $n$-linear SIOs to tuples of UMD spaces tied by a natural product structure, such as, the composition of operators in the Schatten-von Neumann subclass of the algebra of bounded operators on a Hilbert space. In [19] the bilinear case of [17] was applied to prove UMD-extensions for modulation invariant singular integrals, such as, the bilinear Hilbert transform. With new refined methods, in this paper we are able to prove Figiel type results in the multilinear setting.

The one-parameter kernels that we have seen thus far are singular when $x = y$ (in the linear case). Bi-parameter SIOs have kernels with singularities on $x_1 = y_1$ or $x_2 = y_2$, ...
where \( x, y \in \mathbb{R}^d \) are written as \( x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \) for a fixed partition \( d = d_1 + d_2 \). For \( x, y \in \mathbb{C} = \mathbb{R} \times \mathbb{R} \), compare e.g. the one-parameter Beurling kernel \( 1/((x-y)^2) \) with the bi-parameter kernel \( 1/[(x_1-y_1)(x_2-y_2)] \) – the product of Hilbert kernels in both coordinate directions. A bi-parameter \( T \) theorem was first achieved by Journé [39], and recovered by one of us [50] through a bi-parameter dyadic representation theorem. The multi-parameter extension of this is by Y. Ou [54].

In part due to the failure of bi-parameter sparse domination methods, see [3] (see also [4] however), representation theorems are even more important in bi-parameter than in one-parameter. For example, the dyadic methods have proved very fruitful in connection with bi-parameter commutators and weighted analysis, see Holmes–Petermichl–Wick [38], Ou–Petermichl–Strouse [55] and [46]. See also [1, 2]. In particular, the original bi-parameter weighted estimates of Fefferman–Stein [28] and Fefferman [26, 27] were quite difficult in the sense that reaching the natural \( A_p \) class instead of \( A_{p/2} \) required an involved bootstrapping argument.

We also prove a new version of the bi-parameter representation theorem [50] following the new modified shift idea. An inherent complication of bi-parameter analysis is the appearance of certain hybrid combinations of shifts and paraproducts that are new compared to the one-parameter case. They are even more complicated now that we are using modified shifts instead of the regular shifts. Some corollaries include the weighted boundedness of bi-parameter CZOs with a \( \text{Dini} \)–type continuity, and new estimates for various commutators \( [b, T]: f \mapsto bTf - T(bf) \) with similarly mild kernel regularity.

1.3. Theorem. Suppose that \( \mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \) is the underlying bi-parameter space, \( p \in (1, \infty) \), \( \mu, \lambda \in A_p(\mathbb{R}^d) \) are bi-parameter weights and \( \nu = \mu^{1/p}\lambda^{-1/p} \in A_2(\mathbb{R}^d) \).

1. If \( T_i, i = 1, 2, \) is an \( \omega_i \)-CZO on \( \mathbb{R}^{d_i} \), where \( \omega_i \in \text{Dini}_{1/2} \), then
   \[
   \|[T_1, [T_2, b]]\|_{L^p(\mu) \to L^p(\lambda)} \lesssim \|b\|_{\text{BMO}_{\text{prod}}(\nu)}.
   \]

2. Suppose that \( T \) is a bi-parameter \( (\omega_1, \omega_2) \)-CZO, where \( \omega_i \in \text{Dini}_{1/2} \). Then we have
   \[
   \|Tf\|_{L^p(\omega)} \lesssim \|f\|_{L^p(\omega)}
   \]

   and
   \[
   \|[bm, \cdots [b_2, [b_1, T]] \cdots]\|_{L^p(\mu) \to L^p(\mu)} \lesssim \prod_{j=1}^m \|b_j\|_{\text{bmo}}.
   \]

Under slightly higher kernel regularity we have the Bloom version
   \[
   \|[bm, \cdots [b_2, [b_1, T]] \cdots]\|_{L^p(\mu) \to L^p(\lambda)} \lesssim \prod_{j=1}^m \|b_j\|_{\text{bmo}(\mu^{1/m})}.
   \]

See again the main text for all of the definitions. These Bloom-style two-weight estimates have recently been one of the main lines of development concerning commutators, see e.g. [1, 2, 37, 38, 42, 43, 46, 47] for a non-exhaustive list.

Lastly, we mention that in all of our settings it is possible to recover from our results – and this is most difficult in the multilinear setting – an efficient representation of SIOs with the usual dyadic shifts. This is because we ensure that we always use such modified operators that can be split into a sum of standard dyadic model operators.

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2. Basic Notation and Fundamental Estimates

Throughout this paper $A \lesssim B$ means that $A \leq CB$ with some constant $C$ that we deem unimportant to track at that point. We write $A \sim B$ if $A \lesssim B \lesssim A$.

Dyadic notation. Given a dyadic grid $\mathcal{D}$, $I \in \mathcal{D}$ and $k \in \mathbb{Z}$, $k \geq 0$, we use the following notation:

1. $\ell(I)$ is the side length of $I$.
2. $I^{(k)}$ is the $k$th parent of $I$, i.e., $I \subset I^{(k)}$ and $\ell(I^{(k)}) = 2^k \ell(I)$.
3. $\text{ch}(I)$ is the collection of the children of $I$, i.e., $\text{ch}(I) = \{ J \in \mathcal{D} : J^{(1)} = I \}$.
4. $E_I f = \langle f \rangle_I f$ is the averaging operator, where $\langle f \rangle_I f = \frac{1}{|I|} \int_I f$.
5. $E_{I,k} f$ is defined via $E_{I,k} f = \sum_{J \in \mathcal{D}} J^{(k)} = I E_J f$.
6. $\Delta_I f$ is the martingale difference $\Delta_I f = \sum_{J \in \text{ch}(I)} E_J f - E_I f$.
7. $\Delta_{I,k} f$ is the martingale difference block $\Delta_{I,k} f = \sum_{J \in \mathcal{D}} J^{(k)} = I \Delta_J f$.
8. $P_{I,k} f$ is the following sum of martingale difference blocks $P_{I,k} f = \sum_{j=0}^k \Delta_{I,j} f$.

A fundamental fact is that we have the square function estimate

\[ \| S_D f \|_{L^p} \sim \| f \|_{L^p}, \quad p \in (1, \infty), \quad S_D f := \left( \sum_{I \in \mathcal{D}} |\Delta_I f|^2 \right)^{1/2}. \]

See e.g. [11, 13] for even weighted $\| S_D f \|_{L^p(w)} \sim \| f \|_{L^p(w)}$, $w \in A_p$, square function estimates and their history. A weight $w$ (i.e. a locally integrable a.e. positive function) belongs to the weight class $A_p(\mathbb{R}^d)$, $1 < p < \infty$, if

\[ [w]_{A_p(\mathbb{R}^d)} := \sup_Q \frac{1}{|Q|} \int_Q w \left( \frac{1}{|Q|} \int_Q w^{1-p'} \right)^{p-1} < \infty, \]

where the supremum is taken over all cubes $Q \subset \mathbb{R}^d$.

We will also have use for the Fefferman–Stein inequality

\[ \left\| \left( \sum_k |M f_k|^2 \right)^{1/2} \right\|_{L^p} \lesssim \left\| \left( \sum_k |f_k|^2 \right)^{1/2} \right\|_{L^p}, \quad p \in (1, \infty), \]

where $M$ is the Hardy–Littlewood maximal function. However, most of the time we can make do with the lighter Stein’s inequality

\[ \left\| \left( \sum_{I \in \mathcal{D}} |E_I f_I|^2 \right)^{1/2} \right\|_{L^p} \lesssim \left\| \left( \sum_{I \in \mathcal{D}} |f_I|^2 \right)^{1/2} \right\|_{L^p}, \quad p \in (1, \infty). \]
The distinction is relevant, for example, in UMD-valued analysis. We will introduce the required vector-valued machinery later.

For an interval $J \subset \mathbb{R}$ we denote by $J_1$ and $J_2$ the left and right halves of $J$, respectively. We define $h^0_J = |J|^{-1/2}1_J$ and $h^1_J = |J|^{-1/2}(|J_1| - 1_{J_2})$. Let now $I = I_1 \times \cdots \times I_d \subset \mathbb{R}^d$ be a cube, and define the Haar function $h^0_I, \eta = (\eta_1, \ldots, \eta_d) \in \{0, 1\}^d$, by setting

$$h^0_I = h^0_{I_1} \otimes \cdots \otimes h^0_{I_d}.$$ 

If $\eta \neq 0$ the Haar function is cancellative: $\int h^0_I = 0$. We exploit notation by suppressing the presence of $\eta$, and write $h_I$ for some $h^0_I, \eta \neq 0$. Notice that for $I \in \mathcal{D}$ we have $\Delta_I f = \langle f, h_I \rangle h_I$ (where the finite $\eta$ summation is suppressed), $\langle f, h_I \rangle := \int f h_I$.

### 3. One-parameter singular integrals

Let $\omega$ be a modulus of continuity: an increasing and subadditive function with $\omega(0) = 0$. A relevant quantity is the modified Dini condition

$$\|\omega\|_{\text{Dini}^\alpha} := \int_0^1 \omega(t) \left(1 + \log \frac{1}{t}\right)^\alpha \frac{dt}{t}, \quad \alpha \geq 0. \tag{3.1}$$

In practice, the quantity (3.1) arises as follows:

$$\sum_{k=1}^\infty \omega(2^{-k})k^\alpha = \sum_{k=1}^\infty \frac{1}{\log 2} \int_{2^{-k}}^{2^{-k+1}} \omega(2^{-k})k^\alpha \frac{dt}{t} \leq \int_0^1 \omega(t) \left(1 + \log \frac{1}{t}\right)^\alpha \frac{dt}{t}. \tag{3.2}$$

For many standard arguments $\alpha = 0$ is enough. For the $T1$ type arguments we will – at the minimum – always need $\alpha = 1/2$. When we do UMD-extensions beyond function lattices, we will need a bit higher $\alpha$ depending on the so-called type and cotype constants of the underlying UMD space $X$.

**Multilinear singular integrals.** A function

$$K: \mathbb{R}^{d(n+1)} \setminus \Delta \to \mathbb{C}, \quad \Delta = \{x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{d(n+1)} : x_1 = \cdots = x_{n+1}\},$$

is called an $n$-linear $\omega$-Calderón–Zygmund kernel if it holds that

$$|K(x)| \leq \frac{C_K}{\left(\sum_{m=1}^n |x_{n+1} - x_m|\right)^d},$$

and for all $j \in \{1, \ldots, n+1\}$ it holds that

$$|K(x) - K(x')| \leq \omega\left(\sum_{m=1}^n |x_j - x'_j|\right) \frac{1}{\left(\sum_{m=1}^n |x_{n+1} - x_m|\right)^d}$$

whenever $x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{d(n+1)} \setminus \Delta$ and $x' = (x_1, \ldots, x_{j-1}, x'_j, x_{j+1}, \ldots, x_{n+1}) \in \mathbb{R}^{d(n+1)}$ satisfy

$$|x_j - x'_j| \leq 2^{-1} \max_{1 \leq m \leq n} |x_{n+1} - x_m|.$$
3.3. Definition. An \( n \)-linear operator \( T \) defined on a suitable class of functions – e.g. on the linear combinations of cubes – is an \( n \)-linear \( \omega \)-SIO with an associated kernel \( K \), if we have

\[
\langle T(f_1, \ldots, f_n), f_{n+1} \rangle = \int_{\mathbb{R}^d(n+1)} K(x_{n+1}, x_1, \ldots, x_n) \prod_{j=1}^{n+1} f_j(x_j) \, dx
\]

whenever \( \text{spt} f_i \cap \text{spt} f_j = \emptyset \) for some \( i \neq j \).

3.4. Definition. We say that \( T \) is an \( n \)-linear \( \omega \)-CZO if the following conditions hold:

- \( T \) is an \( n \)-linear \( \omega \)-SIO.
- We have that
  \[
  \|T^{m_1}(1, \ldots, 1)\|_{\text{BMO}} := \sup_{D} \sup_{I \in D} \left( \frac{1}{|I|} \sum_{J \in I} |\langle T^{m_1}(1, \ldots, 1), h_J \rangle| \right)^{1/2} < \infty
  \]
  for all \( m \in \{0, \ldots, n\} \). Here \( T^{0_1} := T \), \( T^{m_1} \) denotes the \( m \)th, \( m \in \{1, \ldots, n\} \), adjoint
  \[
  \langle T(f_1, \ldots, f_n), f_{n+1} \rangle = \langle T^{m_1}(f_1, \ldots, f_{m-1}, f_{n+1}, f_{m+1}, \ldots, f_n), f_m \rangle
  \]
  of \( T \), and the pairings \( \langle T^{m_1}(1, \ldots, 1), h_J \rangle \) have a standard \( T1 \) type definition with the aid of the kernel \( K \).
- We have that
  \[
  \|T\|_{\text{WBP}} := \sup_{D} \sup_{I \in D} |I|^{-1/2} |\langle T(1_1, \ldots, 1_I), 1_I \rangle| < \infty.
  \]

Model operators. Let \( i = (i_1, \ldots, i_{n+1}) \), \( i_j \in \{0,1,\ldots\} \), and let \( D \) be a dyadic lattice in \( \mathbb{R}^d \). An operator \( S_i \) is called an \( n \)-linear dyadic shift if it has the form

\[
\langle S_i(f_1, \ldots, f_n), f_{n+1} \rangle = \sum_{K \in D} \langle A_K(f_1, \ldots, f_n), f_{n+1} \rangle,
\]

where

\[
\langle A_K(f_1, \ldots, f_n), f_{n+1} \rangle = \sum_{l_1, \ldots, l_{n+1} \in D} \alpha_{K,(l_j)} \prod_{j=1}^{n+1} \langle f_j, \tilde{h}_{l_j} \rangle.
\]

Here \( \alpha_{K,(l_j)} = \alpha_{K,l_1,\ldots,l_{n+1}} \) is a scalar satisfying the normalization

\[
|\alpha_{K,(l_j)}| \leq \frac{\prod_{j=1}^{n+1} |I_j|^{1/2}}{|K|^n},
\]

and there exist two indices \( j_0, j_1 \in \{1, \ldots, n+1\}, j_0 \neq j_1 \), so that \( \tilde{h}_{l_{j_0}} = h_{l_{j_0}}, \tilde{h}_{l_{j_1}} = h_{l_{j_1}} \) and for the remaining indices \( j \notin \{j_0, j_1\} \) we have \( \tilde{h}_{l_j} \in \{h_{l_j}^0, h_{l_j}^1\} \).

A modified \( n \)-linear shift \( Q_k, k \in \{1,2,\ldots\} \), has the form

\[
\langle Q_k(f_1, \ldots, f_n), f_{n+1} \rangle = \sum_{K \in D} \langle B_K(f_1, \ldots, f_n), f_{n+1} \rangle,
\]
where

\[
\langle B_K(f_1, \ldots, f_n), f_{n+1} \rangle = \sum_{I_1^{(k)} = \ldots = I_{n+1}^{(k)} = K} a_{K,I_1^{(j)}} \left[ \prod_{j=1}^{n} \langle f_j, h_{I_j}^j \rangle - \prod_{j=1}^{n} \langle f_j, h_{I_{n+1}}^{n+1} \rangle \right] \langle f_{n+1}, h_{I_{n+1}} \rangle,
\]

or $B_K$ has one of the other symmetric forms, where the role of $f_{n+1}$ is replaced by some other $f_j$. The coefficients satisfy the same (but now $|I_0| = \ldots = |I_{n+1}|$) normalization

\[
|a_{K,I_1^{(j)}}| \leq \frac{|I_1|^{(n+1)/2}}{|K|^n}.
\]

An $n$-linear dyadic paraproduct $\pi = \pi_D$ also has $n + 1$ possible forms, but there is no complexity associated to them. One of the forms is

\[
\langle \pi(f_1, \ldots, f_n), f_{n+1} \rangle = \sum_{I \in D} a_I \prod_{j=1}^{n} \langle f_j \rangle_I \langle f_{n+1}, h_I \rangle,
\]

where the coefficients satisfy the usual BMO condition

\[
\sup_{I_0 \in D} \left( \frac{1}{|I_0|^n} \sum_{I \subset I_0} |a_I|^2 \right)^{1/2} \leq 1.
\]

In the remaining $n$ alternative forms the cancellative Haar function $h_I$ is in a different position.

When we represent a CZO we will have modified dyadic shifts $Q_k$, standard dyadic shifts of the very special form $S_k, \ldots, k$ and paraproducts $\pi$. Dyadic shifts $S_k, \ldots, k$ are simply easier versions of the operators $Q_k$. Paraproducts do not involve a complexity parameter and are thus inherently not even relevant for the kernel regularity considerations (we just need their boundedness).

At least in the linear situation, we can easily unify the study of shifts $S_k, \ldots, k$ and modified shifts $Q_k$. This viewpoint could work in the multilinear generality also (with some tensor product formalism), but we did not pursue it. We can understand a modified linear shift to have the more general form $Q_k, k = 0, 1, \ldots$, where

\[
\langle Q_k f, g \rangle = \sum_{K \in D} \sum_{I^{(k)} = J^{(k)} = K} a_{IJK} \langle f, h_I \rangle \langle g, H_{I,J} \rangle
\]

or

\[
\langle Q_k f, g \rangle = \sum_{K \in D} \sum_{I^{(k)} = J^{(k)} = K} a_{IJK} \langle f, H_{I,J} \rangle \langle g, h_J \rangle,
\]

the constants $a_{IJK}$ satisfy the usual normalization and and the functions $H_{I,J}$ satisfy

1. $H_{I,J}$ is supported on $I \cup J$ and constant on the children of $I$ and $J$, i.e., we have

\[
H_{I,J} = \sum_{L \in \text{ch}(I) \cup \text{ch}(J)} b_L 1_L, \ b_L \in \mathbb{R},
\]

2. $|H_{I,J}| \leq |I|^{-1/2}$ and

3. $\int H_{I,J} = 0$.  

In practice we have \( H_{I,J} \in \{ h^0, h_1, h_1 - h^0, h_I, h_J \} \), but this abstract form contains enough information to bound the operators.

An important property of the functions \( H_{I,J} \) is the following. Let \( I^{(k)} = J^{(k)} = K \). Since \( H_{I,J} \) is constant on the children of \( I \) and \( J \) there holds that \( \langle g, H_{I,J} \rangle = \langle E_{K,k} g, H_{I,J} \rangle \).

We have the expansion
\[
E_{K,k+1} g = E_K g + P_{K,k} g
\]
and the zero average of \( H_{I,J} \) over \( K \) implies that \( \langle E_K g, H_{I,J} \rangle = 0 \). Thus, we have the key property
\[
\langle g, H_{I,J} \rangle = \langle P_{K,k} g, H_{I,J} \rangle.
\]
Also, there clearly holds that \( \langle f, h_I \rangle = \langle \Delta_{K,k} f, h_I \rangle \). Analogous steps will be taken in the general multilinear situation as well, even though the functions \( H \) do not explicitly appear.

### 3.1. The threshold \( \alpha = 1/2 \)

We quickly explain the role of the regularity threshold \( \alpha = 1/2 \), which appears naturally in the fundamental scalar-valued theory. The estimate of the next lemma and identities like (3.10) are at the heart of the matter.

#### 3.11. Lemma

Let \( p \in (1, \infty) \). There holds that
\[
\left\| \left( \sum_{K \in D} |P_{K,k} f|^2 \right)^{1/2} \right\|_{L^p} \sim \sqrt{k + 1} \| f \|_{L^p}, \quad k \in \{0, 1, 2, \ldots \}.
\]

**Proof.** If \( f_i \in L^p \) then
\[
\left\| \left( \sum_{i=0}^\infty \sum_{I \in D} |\Delta_I f_i|^2 \right)^{1/2} \right\|_{L^p} \sim \left\| \left( \sum_{i=0}^\infty |f_i|^2 \right)^{1/2} \right\|_{L^p}.
\]
This can be proved by using random signs and the Kahane-Khinchine inequality (scalar- and \( \ell^2 \)-valued) or by extrapolating the corresponding weighted \( L^2 \) version of (3.12), which just follows from \( \| S_D f \|_{L^2(w)} \sim \| f \|_{L^2(w)} \) \( w \in A_2 \). Recall that the classical extrapolation theorem of Rubio de Francia says that if \( \| h \|_{L^{p_0}(w)} \lesssim \| g \|_{L^{p_0}(w)} \) for some \( p_0 \in (1, \infty) \) and all \( w \in A_{p_0} \), then \( \| h \|_{L^p(w)} \lesssim \| g \|_{L^p(w)} \) for all \( p \in (1, \infty) \) and all \( w \in A_p \).

Let \( K \in D \). We have that
\[
\sum_{I \in D} |\Delta_I P_{K,k} f|^2 = \sum_{j=0}^k |\Delta_{K,j} f|^2.
\]
Thus, (3.12) gives that
\[
\left\| \left( \sum_{K \in D} |P_{K,k} f|^2 \right)^{1/2} \right\|_{L^p} \sim \left\| \left( \sum_{K \in D} \sum_{j=0}^k |\Delta_{K,j} f|^2 \right)^{1/2} \right\|_{L^p}
\]
\[
= \left\| \left( \sum_{j=0}^k \sum_{I \in D} |\Delta_I f|^2 \right)^{1/2} \right\|_{L^p} \sim \sqrt{k + 1} \| f \|_{L^p}.
\]
\[\square\]
It follows quite easily that we now e.g. have the linear estimate

\[(3.13) \quad \|Q_kf\|_{L^p} \lesssim \sqrt{k+1}\|f\|_{L^p}, \quad k \in \{0, 1, 2, \ldots\}, \quad p \in (1, \infty).\]

As our main focus in the one-parameter situation is to prove much more general UMD-valued versions, see Section 4, we omit the details. To understand the elementary proof of this basic estimate, see Section 5, where we explicitly prove analogous estimates for the more complicated multi-parameter versions of the modified dyadic model operators. The estimate \(\|Q_kf\|_{L^p} \lesssim \sqrt{k+1}\|f\|_{L^p}\) together with (3.2) yield why we always at least need that \(\omega \in \text{Dini}_{1/2}^{1/2}\).

**Modified shifts are sums of standard shifts.** The standard linear shifts satisfy the complexity free bound

\[\|S_{i_1,i_2}f\|_{L^p} \lesssim \|f\|_{L^p}, \quad p \in (1, \infty).\]

Similar estimates hold in the multilinear generality – for example, the following complexity free bilinear estimate is true

\[
\|S_{i_1,i_2,i_3}(f_1, f_2)\|_{L^{p_1}} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}},
\]

\[\forall 1 < p_1, p_2 \leq \infty, \quad \frac{1}{q_3} < q_3 < \infty, \quad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_3}.\]

Such estimates cannot be proved via weak \((1, 1)\) type estimates (or sparse domination), as those estimates are not complexity free. They can, however, be obtained with direct \(L^p\) estimates or by extrapolation [22]. In any case, we need roughly \(nk\) shifts to represent an \(n\)-linear modified shift \(Q_k\) as a sum of standard shifts (see Lemma 3.14 below), and therefore, these estimates lose to estimates like (3.13).

On the other hand, more involved estimates, such as some commutator estimates, can be difficult to carry out directly with the operators \(Q_k\). Bounds via the route of representing \(Q_k\) using ordinary shifts still lead to the modified Dini condition (3.1) with some \(\alpha\), and this is still quite efficient. Therefore, Lemma 3.14 is of practical and philosophical use, but should not be resorted to when more efficient estimates can be obtained by the direct study of the operators \(Q_k\).

**Lemma.** Let \(Q_k, k \in \{1, 2, \ldots\},\) be a modified \(n\)-linear shift of the form (3.6). Then for some \(C \lesssim 1\) we have

\[Q_k = \sum_{m=1}^{n} \sum_{j=0}^{k-1} S_{0,0,j,k,\ldots,k} - C \sum_{m=1}^{n} \sum_{j=0}^{k-1} S_{0,,0,1,j} - C\]

where in the first sum there are \(m-1\) zeroes and in the second sum \(m\) zeroes in the complexity of the shift.

**Proof.** Write \(b_{K,(I_j)} = |I_j|^{n/2}a_{K,(I_j)}\) so that

\[a_{K,(I_j)} \left[ \prod_{j=1}^{n} (f_j, h_{I_j}^0) - \prod_{j=1}^{n} (f_j, h_{I_{n+1}}^0) \right] = b_{K,(I_j)} \left[ \prod_{j=1}^{n} (f_j)_{I_j} - \prod_{j=1}^{n} (f_j)_{I_{n+1}} \right].\]

We then write

\[\prod_{j=1}^{n} (f_j)_{I_j} - \prod_{j=1}^{n} (f_j)_{I_{n+1}} = \left[ \prod_{j=1}^{n} (f_j)_{I_j} - \prod_{j=1}^{n} (f_j)_{K} \right] + \left[ \prod_{j=1}^{n} (f_j)_{K} - \prod_{j=1}^{n} (f_j)_{I_{n+1}} \right].\]
We start working with the first term. Notice that

\[ \langle f_j \rangle_{I_j} = \langle E_{K,k} f_j \rangle_{I_j} = \langle P_{K,k-1} f_j \rangle_{I_j} + \langle f_j \rangle_K. \]

Using this we can write

\[
\prod_{j=1}^{n} \langle f_j \rangle_{I_j} - \prod_{j=1}^{n} \langle f_j \rangle_K = (P_{K,k-1} f_1)_{I_1} \prod_{j=2}^{n} \langle f_j \rangle_{I_j} + (P_{K,k-1} f_2)_{I_2} \prod_{j=3}^{n} \langle f_j \rangle_{I_j} + \cdots + \prod_{j=1}^{n-1} \langle f_j \rangle_K (P_{K,k-1} f_n)_{I_n}.
\]

Consider now, for \( m \in \{1, \ldots, n\} \), the following part of the modified shift

\[
\langle A_m(f_1, \ldots, f_n), f_{n+1} \rangle := \sum_{K} \sum_{L^{(j)}=K} b_{K,(I_j)}
\]

\[
\times \prod_{j=1}^{m-1} \langle f_j \rangle_{K} \cdot \langle P_{K,k-1} f_m \rangle_{I_m} \cdot \prod_{j=m+1}^{n} \langle f_j \rangle_{I_j} \cdot \langle f_{n+1}, h_{I_{n+1}} \rangle.
\]

Next, write

\[
\langle P_{K,k-1} f_m \rangle_{I_m} = \sum_{j=0}^{k-1} \sum_{L^{(j)}=K} \langle \Delta L f_m \rangle_{I_m} = \sum_{j=0}^{k-1} \sum_{L^{(j)}=K} \langle f_m, h_L \rangle \langle h_L \rangle_{I_m}.
\]

We can write \( \langle A_m(f_1, \ldots, f_n), f_{n+1} \rangle \) in the form

\[
\sum_{j=0}^{k-1} \sum_{L^{(j)}=K} \sum_{L^{(m+1)}= \cdots = L^{(k)}=K} \left( \sum_{L^{(1)}= \cdots = L^{(k)}=K} b_{K,(I_j)} \right) \langle K \rangle^{-(m-1)/2} \langle h_L \rangle_{I_m} \langle I_{n+1} \rangle^{-(n-m)/2}
\]

\[
\times \prod_{j=1}^{m-1} \langle f_j, h_K \rangle^0 \cdot \langle f_m, h_L \rangle \cdot \prod_{j=m+1}^{n} \langle f_j, h_K \rangle^0 \cdot \langle f_{n+1}, h_{I_{n+1}} \rangle.
\]

Notice the normalization estimate

\[
\sum_{L^{(1)}= \cdots = L^{(k)}=K} b_{K,(I_j)} \langle K \rangle^{-(m-1)/2} \langle L \rangle^{1/2} \langle I_{n+1} \rangle^{-(n-m)/2}
\]

\[
\leq \prod_{j=1}^{m-1} \langle K \rangle^1 |K|^{1/2} \cdot \langle L \rangle^{1/2} \cdot \prod_{j=m+1}^{n} |I_j|^{1/2}.
\]

We also have two cancellative Haar functions, so for every \( j \in \{0, \ldots, k-1\} \), the inner sum in \( A_m \) is a standard \( n \)-linear shift of complexity \((0, \ldots, 0, j, k, \ldots, k)\), where the \( j \) is in the \( m \)th slot:

\[
\langle A_m(f_1, \ldots, f_n), f_{n+1} \rangle = \sum_{j=0}^{k-1} \langle S_{0, \ldots, 0, j, k, \ldots, k}, f_1, \ldots, f_m, f_{n+1} \rangle.
\]
We now turn to the part of the modified shift associated with
\[
\prod_{j=1}^{n} \langle f_j \rangle_{I_{n+1}} - \prod_{j=1}^{n} \langle f_j \rangle_{K} = \sum_{i=0}^{k-1} \left( \prod_{j=1}^{n} \langle f_j \rangle_{I_{n+1}^{(i)}} - \prod_{j=1}^{n} \langle f_j \rangle_{I_{n+1}^{(i+1)}} \right).
\]

Further, we write
\[
\prod_{j=1}^{n} \langle f_j \rangle_{I_{n+1}^{(i)}} - \prod_{j=1}^{n} \langle f_j \rangle_{I_{n+1}^{(i+1)}} = \langle \Delta_{n+1}^{(i+1)} f_1 \rangle_{I_{n+1}^{(i)}} \prod_{j=2}^{n} \langle f_j \rangle_{I_{n+1}^{(i)}} + \langle f_1 \rangle_{I_{n+1}^{(i+1)}} \langle \Delta_{n+1}^{(i+1)} f_2 \rangle_{I_{n+1}^{(i)}} \prod_{j=3}^{n} \langle f_j \rangle_{I_{n+1}^{(i)}} + \ldots + \prod_{j=1}^{n} \langle f_j \rangle_{I_{n+1}^{(i+1)}} \langle \Delta_{n+1}^{(i+1)} f_n \rangle_{I_{n+1}^{(i)}}.
\]

Consider now, for \(m \in \{1, \ldots, n\}\), the following part of the modified shift
\[
\langle U_{m}(f_1, \ldots, f_n), f_{n+1} \rangle = \sum_{i=0}^{k-1} \sum_{K} \sum_{L^{(k)}=L} \left( \sum_{I_{n+1}^{(k)}=K} b_{K,(I_j)} |L^{(1)}|^{-{(m-1)/2}} (\hat{h}_{L^{(1)}}) L |L|^{-{(n-m)/2}} \right)
\times \prod_{j=1}^{m-1} \langle f_j, \hat{h}_{L^{(1)}} \rangle \cdot \langle f_m, \hat{h}_{L^{(1)}} \rangle \cdot \prod_{j=m+1}^{n} \langle f_j, \hat{h}_{L^{(1)}} \rangle \cdot \langle f_{n+1}, \hat{h}_{I_{n+1}} \rangle.
\]

We can write \(\langle U_{m}(f_1, \ldots, f_n), f_{n+1} \rangle\) in the form
\[
\sum_{i=1}^{k-1} \sum_{K} \sum_{L^{(k)}=L} \left( \sum_{I_{n+1}^{(k)}=K} b_{K,(I_j)} |L^{(1)}|^{-{(m-1)/2}} (\hat{h}_{L^{(1)}}) L |L|^{-{(n-m)/2}} \right)
\times \prod_{j=1}^{m-1} \langle f_j, \hat{h}_{L^{(1)}} \rangle \cdot \langle f_m, \hat{h}_{L^{(1)}} \rangle \cdot \prod_{j=m+1}^{n} \langle f_j, \hat{h}_{L^{(1)}} \rangle \cdot \langle f_{n+1}, \hat{h}_{I_{n+1}} \rangle.
\]

Notice the normalization estimate
\[
\sum_{I_{n+1}^{(k)}=K} |b_{K,(I_j)}||L^{(1)}|^{-{(m-1)/2}} |L^{(1)}|^{-1/2} |L|^{-{(n-m)/2}} \lesssim \frac{|L^{(1)}| m/2 |L|^{(n-m)/2} |I_{n+1}|^{1/2}}{|L^{(1)}|^n}.
\]

Therefore, for some constant \(C \lesssim 1\) we get that
\[
\langle U_{m}(f_1, \ldots, f_n), f_{n+1} \rangle = C \sum_{i=0}^{k-1} \langle S_{0,\ldots,0,1,\ldots,1} f_1, \ldots, f_m, f_{n+1} \rangle,
\]
where there are \(m\) zeroes in \(S_{0,\ldots,0,1,\ldots,1}\).
The representation theorem. Let \( \sigma = (\sigma^i)_{i \in \mathbb{Z}} \), where \( \sigma^i \in \{0, 1\}^d \). Let \( \mathcal{D}_0 \) be the standard dyadic grid on \( \mathbb{R}^d \),
\[
\mathcal{D}_0 := \{ 2^{-k}([0,1]^d + m) : k \in \mathbb{Z}, m \in \mathbb{Z}^d \}.
\]
We define the new dyadic grid
\[
\mathcal{D}_\sigma = \{ I + \sum_{i : 2^{-i} < \ell(I)} 2^{-i} \sigma^i : I \in \mathcal{D}_0 \} = \{ I + \sigma : I \in \mathcal{D}_0 \},
\]
where we simply have defined
\[
I + \sigma := I + \sum_{i : 2^{-i} < \ell(I)} 2^{-i} \sigma^i.
\]
It is straightforward that \( \mathcal{D}_\sigma \) inherits the key nestedness property of \( \mathcal{D}_0 \); if \( I, J \in \mathcal{D}_\sigma \), then \( I \cap J \in \{ I, J, \emptyset \} \). Moreover, there is a natural product probability measure \( \mathbb{P}_\sigma = \mathbb{P} \)
on \( \{0,1\}^d \mathbb{Z} \) – this gives us the notion of random dyadic grids \( \sigma \mapsto \mathcal{D}_\sigma \) over which we take the expectation \( \mathbb{E}_\sigma \) below.

3.17. Remark. The assumption \( \omega \in \text{Dini}_{1/2} \) in the theorem below is only needed to have a converging series. The regularity is not explicitly used in the proof of the representation. It is required due to the estimates of the model operators briefly discussed above. We will state the T1 type corollaries, including the UMD-extensions, carefully later.

3.18. Theorem. Suppose that \( T \) is an \( n \)-linear \( \omega \)-CZO, where \( \omega \in \text{Dini}_{1/2} \). Then we have
\[
\langle T(f_1, \ldots, f_n), f_{n+1} \rangle = C_T \mathbb{E}_{\sigma} \sum_{k=0}^{\infty} \sum_{u=0}^{c_d} \omega(2^{-k}) \langle V_{k,u,\sigma}(f_1, \ldots, f_n), f_{n+1} \rangle,
\]
where \( V_{k,u,\sigma} \) is always either a standard \( n \)-linear shift \( S_{k,\ldots,k} \), a modified \( n \)-linear shift \( Q_k \) or an \( n \)-linear paraproduct (this requires \( k = 0 \)) in the grid \( \mathcal{D}_\sigma \). Moreover, we have
\[
|C_T| \lesssim \sum_{m=0}^{n} \| T^{m*}(1, \ldots, 1) \|_{\text{BMO}} + \| T \|_{\text{WBP}} + C_K + 1.
\]

Proof. We begin with the decomposition
\[
\langle T(f_1, \ldots, f_n), f_{n+1} \rangle = \mathbb{E}_{\sigma} \sum_{I_1,\ldots,I_{n+1}} \langle T(\Delta_{I_1} f_1, \ldots, \Delta_{I_n} f_n), \Delta_{I_{n+1}} f_{n+1} \rangle
\]
\[
= \sum_{j=1}^{n+1} \mathbb{E}_{\sigma} \sum_{I_1,\ldots,I_{n+1}} \langle T(\Delta_{I_1} f_1, \ldots, \Delta_{I_n} f_n), \Delta_{I_{n+1}} f_{n+1} \rangle + \mathbb{E}_{\sigma} R_{\sigma},
\]
where \( I_1, \ldots, I_{n+1} \in \mathcal{D}_\sigma \) for some \( \sigma \in \{0,1\}^d \mathbb{Z} \). We deal with the remainder term \( R_{\sigma} \) later, and now focus on dealing with one of the main terms
\[
\Sigma_{j,\sigma} = \sum_{I_1,\ldots,I_{n+1}} \langle T(\Delta_{I_1} f_1, \ldots, \Delta_{I_n} f_n), \Delta_{I_{n+1}} f_{n+1} \rangle,
\]
where \( j \in \{1, \ldots, n+1\} \).
The main terms are symmetric, and we choose to handle $\Sigma_{\sigma} := \Sigma_{n+1,\sigma}$. After collapsing the sums

$$\sum_{I_i: \ell(I_i) > \ell(I_{n+1})} \Delta_I f_I = \sum_{I_i: \ell(I_i) = \ell(I_{n+1})} E_I f_I,$$

we have

$$\Sigma_{\sigma} = \sum_{\ell(I_i) = \cdots = \ell(I_{n+1})} T(E_{I_1} f_{I_1}, \ldots, E_{I_n} f_n, \Delta_{I_{n+1}} f_{I_{n+1}}).$$

Further, we write

$$\langle T(E_{I_1} f_{I_1}, \ldots, E_{I_n} f_n, \Delta_{I_{n+1}} f_{I_{n+1}}) \rangle_{\ell(I_{n+1})}$$

$$= \langle T(h_{I_1}^0, \ldots, h_{I_n}^0), h_{I_{n+1}} \rangle \prod_{j=1}^n (f_j, h_{I_j}^0) \langle f_{n+1}, h_{I_{n+1}} \rangle$$

$$= \langle T(h_{I_1}^0, \ldots, h_{I_n}^0), h_{I_{n+1}} \rangle \left[ \prod_{j=1}^n (f_j, h_{I_j}^0) - \prod_{j=1}^n (f_j, h_{I_{n+1}}^0) \right] \langle f_{n+1}, h_{I_{n+1}} \rangle$$

$$+ \langle T(1_{I_1}, \ldots, 1_{I_n}), h_{I_{n+1}} \rangle \prod_{j=1}^n (f_j, h_{I_{n+1}}) \langle f_{n+1}, h_{I_{n+1}} \rangle.$$

We define the abbreviation

$$\varphi_{I_1, \ldots, I_{n+1}} := \langle T(h_{I_1}^0, \ldots, h_{I_n}^0), h_{I_{n+1}} \rangle \prod_{j=1}^n (f_j, h_{I_j}^0) - \prod_{j=1}^n (f_j, h_{I_{n+1}}^0) \rangle \langle f_{n+1}, h_{I_{n+1}} \rangle.$$

If we now sum over $I_1, \ldots, I_{n+1}$ we may express $\Sigma_{\sigma}$ in the form

$$\Sigma_{\sigma} = \sum_{\ell(I_1) = \cdots = \ell(I_{n+1})} \varphi_{I_1, \ldots, I_{n+1}} + \sum_I \langle T(1, \ldots, 1), h_I \rangle \prod_{j=1}^n (f_j) \langle f_{n+1}, h_I \rangle = \Sigma_{\sigma}^1 + \Sigma_{\sigma}^2,$$

where we recognize that the second term $\Sigma_{\sigma}^2$ is a paraproduct. Thus, we only need to continue working with $\Sigma_{\sigma}^1$.

Since $\varphi_{I_1, \ldots, I} = 0$, we have that

$$\Sigma_{\sigma}^1 = \sum_{m_1, \ldots, m_n \in \mathbb{Z}^d} \sum_{\ell(I) \neq (0, \ldots, 0)} \varphi_{I+m_1 \ell(I), \ldots, I+m_n \ell(I)} I.$$

As in [25] we say that $I$ is $k$-good for $k \geq 2$ and denote this by $I \in \mathcal{D}_{\sigma, \text{good}}(k)$ if $I \in \mathcal{D}_{\sigma}$ satisfies

$$d(I, \partial I^k) \geq \ell(I^k) \frac{k}{4} = 2^{k-2} \ell(I).$$

Notice that for all $I \in \mathcal{D}_0$ we have

$$\mathbb{P}(\{|\sigma: I + \sigma \in \mathcal{D}_{\sigma, \text{good}}(k)\}) = 2^{-d}.$$
Thus, by the independence of the position of $I$ and the $k$-goodness of $I$ we have
\[
\mathbb{E}_\sigma \Sigma_\sigma^1 = 2^d \mathbb{E}_\sigma \sum_{k=2}^{\infty} \sum_{\max|m_i| \in \mathcal{D}_{\sigma, \text{good}}(k)} \sum_{I \in (2^{k-1}, 2^{k-2}]^d} \varphi_{I+m_1 \ell(I), \ldots, I+m_n \ell(I), I}
\]
\[
= 2^d \mathbb{E}_\sigma \sum_{k=2}^{\infty} \omega(2^{-k}) \langle \tilde{Q}_k(f_1, \ldots, f_n), f_{n+1} \rangle,
\]
where
\[
\langle \tilde{Q}_k(f_1, \ldots, f_n), f_{n+1} \rangle := \frac{1}{\omega(2^{-k})} \max|m_i| \sum_{I \in (2^{k-1}, 2^{k-2}]^d} \varphi_{I+m_1 \ell(I), \ldots, I+m_n \ell(I), I}.
\]

Next, the key implication of the $k$-goodness is that
\[(I + m \ell(I))^{(k)} = I^{(k)} = K\]
if $|m| \leq 2^{k-2}$ and $I \in \mathcal{D}_{\sigma, \text{good}}(k)$. Indeed, notice that e.g. $c_1 + m \ell(I) \in [I + m \ell(I)] \cap K$ (so that $[I + m \ell(I)] \cap K \neq \emptyset$) which is enough as
\[
d(c_1 + m \ell(I), K^c) \geq d(c_1, K^c) - |m|\ell(I) > d(I, \partial K) - |m|\ell(I) \geq 2^{k-2}\ell(I) - 2^{k-2}\ell(I) = 0.
\]

Therefore, to conclude that $C^{-1} \tilde{Q}_k$ is a modified $n$-linear shift it only remains to prove the normalization
\[
(3.19) \quad \frac{|\langle T(h^0_{I+m_1 \ell(I)} \cdots, h^0_{I+m_n \ell(I)}), h_I \rangle|}{\omega(2^{-k})} \lesssim |I|^{(n+1)/2}/|K|^n.
\]

Suppose first that $k \sim 1$. Recall that $(m_1, \ldots, m_n) \neq (0, 0)$ and assume for example that $m_1 \neq 0$. We have using the size estimate of the kernel that
\[
|\langle T(h^0_{I+m_1 \ell(I)} \cdots, h^0_{I+m_n \ell(I)}), h_I \rangle| \lesssim \int_{\mathbb{R}^{(n+1)d}} h^0_{I+m_1 \ell(I)}(x_1) \cdots h^0_{I+m_n \ell(I)}(x_n) |h_I(x_{n+1})| \\ dx
\]
\[
\lesssim \frac{1}{|I|^{(n+1)/2}} \int_{C \setminus I} \int_I \frac{dx_1 \ dx_{n+1}}{|x_{n+1} - x_1|^d} \lesssim \frac{1}{|I|^{(n-1)/2}},
\]
where in the second step we repeatedly used estimates of the form
\[
\int_{\mathbb{R}^d} \frac{dz}{(r + |z_0 - z|)^{d+\alpha}} \lesssim \frac{1}{r^\alpha}.
\]

Notice that this is the right upper bound (3.19) in the case $k \sim 1$.

Suppose then that $k$ is large enough so that we can use the continuity assumption of the kernel. In this case we have that if $x_{n+1} \in I$ and $x_1 \in I + m_1 \ell(I), \ldots, x_n \in I + m_n \ell(I)$,
then $\sum_{m=1}^{n} |x_{n+1} - x_m| \sim 2^k \ell(I) = \ell(K)$. Thus, there holds that

\[
\|T(h_0^{I+\ell(I)} \cdots, h_0^{I+m\ell(I)}, h_I)\| = \left| \int_{\mathbb{R}^{n+1}} (K(x_{n+1}, x_1, \ldots, x_n) - K(c_1, x_1, \ldots, x_n)) \prod_{j=1}^{n} h_0^{I+m_j \ell(I)}(x_j)h_I(x_{n+1}) \, dx \right|
\]

\[
\lesssim \int_{\mathbb{R}^{n+1}} \omega(2^{-k}) \frac{1}{|K|^n} \prod_{j=1}^{n} h_0^{I+m_j \ell(I)}(x_j) |h_I(x_{n+1})| \, dx
\]

\[
= \omega(2^{-k}) \frac{1}{|K|^n} |I|^{n+1} |J|^{-(n+1)/2} = \omega(2^{-k}) \left| \frac{|I|^{(n+1)/2}}{|K|^n} \right|
\]

We have proved (3.19). This ends our treatment of $E_\sigma \Sigma_\sigma$.

We now only need to deal with the remainder term $E_\sigma R_\sigma$. Write

\[ R_\sigma = \sum_{(I_1, \ldots, I_{n+1}) \in I_\sigma} \langle T(\Delta I_1 f_1, \ldots, \Delta I_n f_n), \Delta I_{n+1} f_{n+1} \rangle, \]

where each $(I_1, \ldots, I_{n+1}) \in I_\sigma$ satisfies that if $j \in \{1, \ldots, n + 1\}$ is such that $\ell(I_j) \leq \ell(I_i)$ for all $i \in \{1, \ldots, n + 1\}$, then $\ell(I_j) = \ell(I_{i_0})$ for at least one $i_0 \in \{1, \ldots, n + 1\} \setminus \{j\}$. The point why the remainder is simpler than the main terms is that we can split this summation so that there are always at least two sums which we do not need to collapse – that means we will readily have two cancellative Haar functions. To give the idea, it makes sense to explain the bilinear case $n = 2$. In this case we can, in a natural way, decompose

\[
\sum_{(I_1, I_2, I_3) \in I_\sigma} = \sum_{\ell(I_2) = \ell(I_3) < \ell(I_1)} + \sum_{\ell(I_1) = \ell(I_3) < \ell(I_2)} + \sum_{\ell(I_1) = \ell(I_2) < \ell(I_3)} + \sum_{\ell(I_1) = \ell(I_2) = \ell(I_3)}
\]

which – after collapsing the relevant sums – gives that

\[
R_\sigma = \sum_{\ell(I_1) = \ell(I_2) = \ell(I_3)} \langle T(E_{I_1} f_1, \Delta I_2 f_2, \Delta I_3 f_3), \Delta I_3 f_3 \rangle + \langle T(\Delta I_1 f_1, E_{I_2} f_2, \Delta I_3 f_3) \rangle
\]

\[
+ \langle T(\Delta I_1 f_1, \Delta I_2 f_2, E_{I_3} f_3) \rangle + \langle T(\Delta I_1 f_1, \Delta I_2 f_2, \Delta I_3 f_3) \rangle = \sum_{i=1}^{4} R_{\sigma}^i.
\]

These are all handled similarly (the point is that there are at least two martingale differences remaining in all of them) so we look for example at

\[
R_{\sigma}^2 = \sum_{\ell(I_1) = \ell(I_2) = \ell(I_3)} \langle T(\Delta I_1 f_1, E_{I_2} f_2, \Delta I_3 f_3) \rangle
\]

\[
= \sum_{\ell(I_1) = \ell(I_2) = \ell(I_3)} \langle T(\Delta I_1 f_1, E_{I_2} f_2, \Delta I_3 f_3) \rangle + \sum_{I_1 \neq I_3 \text{ or } I_2 \neq I_3} \langle T(\Delta I_1 f_1, E_{I_2} f_2, \Delta I_3 f_3) \rangle.
\]

We can represent these terms as sums of standard bilinear shifts of the form $S_{k,k,k}$. The first term is handled exactly like $\Sigma_k^1$ above. The second term is readily a zero complexity shift – the estimate for the shift coefficient follows from the size estimate of the kernel of $T$ and the weak boundedness property

\[ \langle T(1_I, 1_I) \rangle \lesssim |I|. \]
The general $n$-linear remainder term $R_\sigma$ is analogous and only yields standard $n$-linear shifts $S_{i_1,\ldots,i_n}$. We are done. \hfill\Box

4. UMD-Valued Extensions of Multilinear CZOs

Preliminaries of Banach space theory. An extensive treatment of Banach space theory is given in the books [34, 35] by Hytönen, van Neerven, Veraar and Weis.

We say that $\{\varepsilon_i\}_i$ is a collection of independent random signs, where $i$ runs over some index set, if there exists a probability space $(\mathcal{M}, \mu)$ so that $\varepsilon_i: \mathcal{M} \to \{-1, 1\}$, $\{\varepsilon_i\}_i$ is independent and $\mu(\{\varepsilon_i = 1\}) = \mu(\{\varepsilon_i = -1\}) = 1/2$. Below, $\{\varepsilon_i\}_i$ will always denote a collection of independent random signs.

Suppose $X$ is a Banach space. We denote the underlying norm by $\| \cdot \|_X$. The Kahane-Khintchine inequality says that for all $x_1, \ldots, x_N \in X$ and $p, q \in (0, \infty)$ there holds that

$$\left( \mathbb{E} \left| \sum_{i=1}^{N} \varepsilon_i x_i \right|^p_X \right)^{1/p} \sim \left( \mathbb{E} \left| \sum_{i=1}^{N} \varepsilon_i x_i \right|^q_X \right)^{1/q}. \quad (4.1)$$

Definitions related to Banach spaces often involve such random sums and the definition may involve some fixed choice of the exponent – but the choice is irrelevant by the Kahane-Khintchine inequality.

The Kahane contraction principle says that if $(a_i)_{i=1}^{N}$ is a sequence of scalars, $x_1, \ldots, x_N \in X$ and $p \in (0, \infty]$, then

$$\left( \mathbb{E} \left| \sum_{i=1}^{N} \varepsilon_i a_i x_i \right|^p_X \right)^{1/p} \lesssim \max \left\{ |a_i| \left( \mathbb{E} \left| \sum_{i=1}^{N} \varepsilon_i x_i \right|^p_X \right)^{1/p} \right\}. \quad (4.2)$$

Actually, if $p \in [1, \infty]$ and $a_i \in \mathbb{R}$, then (4.2) holds with “$\leq$” in place of “$\lesssim$”, see [34] for more details.

4.3. Definition. Let $X$ be a Banach space, let $r \in [1, 2]$ and $q \in [2, \infty]$.

(1) The space $X$ has type $r$ if there exists a finite constant $\tau \geq 0$ such that for all finite sequences $x_1, \ldots, x_N \in X$ we have

$$\left( \mathbb{E} \left| \sum_{i=1}^{N} \varepsilon_i x_i \right|^r_X \right)^{1/r} \leq \tau \left( \sum_{i=1}^{N} |x_i|^r_X \right)^{1/r}. \quad (4.3)$$

(2) The space $X$ has cotype $q$ if there exists a finite constant $c \geq 0$ such that for all finite sequences $x_1, \ldots, x_N \in X$ we have

$$\left( \sum_{i=1}^{N} |x_i|^q_X \right)^{1/q} \leq c \left( \mathbb{E} \sum_{i=1}^{N} \varepsilon_i x_i \right)^{1/q}. \quad (4.4)$$

For $q = \infty$ the usual modification is used.

The least admissible constants are denoted by $\tau_{r,X}$ and $c_{q,X}$ – they are the type $r$ constant and cotype $q$ constant of $X$.

In [35, Section 7] the reader can find the basic theory of types and cotypes. We only need a few basic facts, however.

If $X$ has type $r$ (cotype $q$), then it also has type $u$ for all $u \in [1, r]$ (cotype $v$ for all $v \in [q, \infty]$), and we have $\tau_{u,X} \leq \tau_{r,X}$ ($c_{v,X} \leq c_{q,X}$). It is also trivial that always
We say that $X$ has non-trivial type if $X$ has type $r$ for some $r \in (1,2]$ and finite cotype if it has cotype $q$ for some $q \in [2,\infty)$.

For the types and cotypes of $L^p$ spaces we have the following: if $X$ has type $r$, then $L^p(X)$ has type $\min(r,p)$, and if $X$ has cotype $q$, then $L^p(X)$ has cotype $\max(q,p)$.

The UMD property is a necessary and sufficient condition for the boundedness of various singular integral operators on $L^p(\mathbb{R}^d; X) = L^p(X)$, see [34, Sec. 5.2.c and the Notes to Sec. 5.2].

4.4. Definition. A Banach space $X$ is said to be a UMD space, where UMD stands for unconditional martingale differences, if for all $p \in (1,\infty)$, all $X$-valued $L^p$-martingale difference sequences $(d_i)_{i=1}^N$ and all choices of fixed signs $\epsilon_i \in \{-1,1\}$ we have

$$\left\| \sum_{i=1}^N \epsilon_i d_i \right\|_{L^p(X)} \leq \left\| \sum_{i=1}^N d_i \right\|_{L^p(X)}.$$ \hfill (4.5)

The $L^p(X)$-norm is with respect to the measure space where the martingale differences are defined.

A standard property of UMD spaces is that if (4.5) holds for one $p_0 \in (1,\infty)$ it holds for all $p \in (1,\infty)$ [34, Theorem 4.2.7]. Moreover, if $X$ is UMD then so is the dual space $X^*$ [34, Prop. 4.2.17]. Importantly, UMD spaces have non-trivial type and a finite cotype.

Stein’s inequality says that for a UMD space $X$ we have

$$\mathbb{E} \left[ \left\| \sum_{I \in \mathcal{D}} \epsilon_I f_I 1_I \right\|_{L^p(X)} \right] \leq \mathbb{E} \left[ \left\| \sum_{I \in \mathcal{D}} \epsilon_I f_I \right\|_{L^p(X)} \right], \quad p \in (1,\infty).$$

This UMD-valued version of Stein’s inequality is by Bourgain, for a proof see e.g. Theorem 4.2.23 in the book [34].

We now introduce some definitions related to the so called decoupling estimate. For $K \in \mathcal{D}$ denote by $Y_K$ the measure space $(K, \text{Leb}(K), \nu_K)$. Here $\text{Leb}(K)$ is the collection of Lebesgue measurable subsets of $K$ and $\nu_K = dx |K|/|K|$, where $dx |K|$ is the $d$-dimensional Lebesgue measure restricted to $K$. We then define the product probability space

$$(Y, \mathcal{A}, \nu) := \prod_{K \in \mathcal{D}} Y_K.$$ 

If $y \in Y$ and $K \in \mathcal{D}$, we denote by $y_K$ the coordinate related to $Y_K$.

In our upcoming estimates, it will be important to separate scales using the following subgrids. For $k \in \{0,1,\ldots\}$ and $l \in \{0,\ldots,k\}$ define

$$\mathcal{D}_{k,l} := \{K \in \mathcal{D} : \ell(K) = 2^{m(k+1)+l} \text{ for some } m \in \mathbb{Z}\}.$$ \hfill (4.6)

The following proposition concerning decoupling is a special case of Theorem 3.1 in [36]. It is a result that can be stated in the generality of suitable filtrations, but we prefer to only state the following dyadic version.

4.7. Proposition. Let $X$ be a UMD space, $p \in (1,\infty)$, $k \in \{0,1,\ldots\}$ and $l \in \{0,\ldots,k\}$. Suppose $f_K, K \in \mathcal{D}_{k,l}$, are functions such that

(1) $f_K = 1_K f_K$,
(2) $\int f_K = 0$ and
(3) $f_K$ is constant on those $K' \in \mathcal{D}_{k,l}$ for which $K' \subseteq K$. 

Suppose $f_K, K \in \mathcal{D}_{k,l}$, are functions such that
Then we have
\begin{equation}
\int_{\mathbb{R}^d} \left| \sum_{K \in D_{k,l}} f_K(x) \right|_X^p \, dx \sim \mathbb{E} \int_{\mathbb{R}^d} \left| \sum_{K \in D_{k,l}} \varepsilon_K K(x) f_K(y_K) \right|_X^p \, dx \, d\nu(y),
\end{equation}
where the implicit constant is independent of $k, l$.

Hilbert spaces are the only Banach spaces with both type $2$ and cotype $2$. Below we will prove estimates for modified shifts like
\[ \|Q_k f\|_{L^2(X)} \lesssim (k + 1)^{1/\min(r,q')} \|f\|_{L^2(X)}, \]
where the UMD space $X$ has type $r$ and cotype $q$. Therefore, in the Hilbert space case – and thus in the scalar-valued case – these estimates recover the best possible regularity $\alpha = 1/2$. The presented estimates are efficient in completely general UMD spaces – however, in UMD function lattices it should be more efficient to mimic the scalar-valued theory and use square function estimates instead. In this paper we are only interested on these deeper general UMD-valued estimates. For the same reason we will not pursue UMD theory in the bi-parameter setting, as it is known that then the additional “property (\alpha)” is required, see [35, Sec. 8.3.e]. The function lattice assumption is formally stronger, but for main concrete examples practically the same as the “property (\alpha)” assumption.

**The linear case.** We feel that it is too difficult to jump directly into the multilinear estimates, as they are quite involved. Thus, we first study the linear case. We show that the framework of modified dyadic shifts gives a modern and convenient proof of the results of Figiel [29, 30] concerning UMD-extensions of CZOs with mild kernel regularity.

Before moving to the main $X$-valued estimate for $Q_k$, we state the following result for paraproducts. We have
\begin{equation}
\|\pi f\|_{L^p(X)} \lesssim \|f\|_{L^p(X)}
\end{equation}
whenever $p \in (1, \infty)$ and $X$ is UMD. We understand that this is usually attributed to Bourgain – in any case, a simple proof can now be found in [36].

**4.10. Remark.** The estimate in Proposition 4.11 below is best used for $p = 2$, since then e.g. $\min(r,p) = r$, if $r \in (1,2]$ is an exponent such that $X$ has type $r$. Indeed, it is efficient to only move the $p = 2$ estimate for the CZO $T$, and then interpolate to get the $L^p$ boundedness under a modified Dini type assumption that is independent of $p$. On the $Q_k$ level improving an $L^2$ estimate into an $L^p$ estimate with good dependency on the complexity does not seem so simple. Interpolation would introduce some additional complexity dependency, since the weak $(1,1)$ inequality of $Q_k$ is not complexity free.

**4.11. Proposition.** Let $p \in (1, \infty)$ and $X$ be a UMD space. If $Q_k$ is a modified shift of the form (3.9), then
\[ \|Q_k f\|_{L^p(X)} \lesssim (k + 1)^{1/\min(r,p)} \|f\|_{L^p(X)}, \]
where $r \in (1,2]$ is an exponent such that $X$ has type $r$. If $Q_k$ is a modified shift of the form (3.8), we have
\[ \|Q_k f\|_{L^p(X)} \lesssim (k + 1)^{1/\min(q',q')} \|f\|_{L^p(X)}, \]
where $q \in [2, \infty)$ is an exponent such that $X$ has cotype $q$. 

Proof. We assume that \( Q_k \) has the form (3.9) – the other result follows by duality. This uses that if the UMD space \( X \) has cotype \( q \), then the dual space \( X^* \) has type \( q' \) – see [35, Proposition 7.4.10].

If \( K \in \mathcal{D} \) we define

\[
B_K f := \sum_{I^{(k)} = J^{(k)} = K} a_{IJK} \langle f, H_I, J \rangle h_I.
\]

Recall the lattices \( \mathcal{D}_{k,l} \) from (4.6) and write \( Q_k f = \sum_{l=0}^k \sum_{K \in \mathcal{D}_{k,l}} B_K f. \) By using the UMD property of \( X \) and the Kahane–Khintchine inequality we have for all \( s \in (0, \infty) \) that

\[
\| Q_k f \|_{L^p(X)} \sim \left( \mathbb{E} \left\| \sum_{l=0}^k \sum_{K \in \mathcal{D}_{k,l}} B_K f \|_{L^p(X)}^s \right\|^{1/s} \right)^{1/s}.
\]

We use this with the choice \( s := \min(r, p) \), since \( L^p(X) \) has type \( s \). Using this we have

\[
\left( \mathbb{E} \left\| \sum_{l=0}^k \sum_{K \in \mathcal{D}_{k,l}} B_K f \|_{L^p(X)}^s \right\|^{1/s} \right)^{1/s} \leq \left( \mathbb{E} \left\| \sum_{l=0}^k \sum_{K \in \mathcal{D}_{k,l}} B_K f \|_{L^p(X)}^s \right\|^{1/s} \right)^{1/s}.
\]

To end the proof, it remains to show that

\[
(4.12) \quad \left\| \sum_{K \in \mathcal{D}_{k,l}} B_K f \right\|_{L^p(X)} \lesssim \| f \|_{L^p(X)}
\]

uniformly on \( l \).

Recall that \( \langle f, H_I, J \rangle = \langle P_{K,k} f, H_I, J \rangle \). We then see that

\[
B_K f = \sum_{I^{(k)} = J^{(k)} = K} a_{IJK} \langle P_{K,k} f, H_I, J \rangle h_I = \sum_{I^{(k)} = J^{(k)} = K} a_{IJK} \langle P_{K,k} f, 1_J H_{I,J} \rangle h_I + \sum_{I^{(k)} = J^{(k)} = K, I \neq J} a_{IJK} \langle P_{K,k} f, 1_I H_{I,J} \rangle h_I.
\]

Accordingly, this splits the estimate of (4.12) into two parts.

We consider first the part related to \( \langle P_{K,k} f, 1_I H_{I,J} \rangle \). By the UMD property and the Kahane–Khintchine inequality we have

\[
\left\| \sum_{K \in \mathcal{D}_{k,l}} \sum_{I^{(k)} = J^{(k)} = K, I \neq J} a_{IJK} \langle P_{K,k} f, 1_I H_{I,J} \rangle h_I \right\|_{L^p(X)} \sim \left( \mathbb{E} \left\| \sum_{K \in \mathcal{D}_{k,l}} \epsilon_K \sum_{I^{(k)} = J^{(k)} = K, I \neq J} a_{IJK} \|_{L^p(X)}^p \right\|^{1/p} \right)^{1/p}.
\]

Notice then that for

\[
a_K(x, y) := |K| \sum_{I^{(k)} = J^{(k)} = K, I \neq J} a_{IJK} 1_I(y) H_{I,J}(y) h_J(x)
\]
we have
\[ \sum_{I^{(k)}=J^{(k)}=K} a_{IJK} \langle P_{K,k} f, 1_1 H_{I,J} \rangle h_J(x) = \frac{1}{|K|} \int_K a_K(x,y) P_{K,k} f(y) \, dy \]
\[ = \int_{Y_K} a_K(x,y_K) P_{K,k} f(y_K) \, d\nu_K(y_K) \]
\[ = \int_{Y} a_K(x,y_K) P_{K,k} f(y_K) \, d\nu(y). \]

Using this we have by Hölder’s inequality (recalling that \( \nu \) is a probability measure) that
\[ \mathbb{E} \left\| \sum_{K \in D_{k,t}} \epsilon_K \sum_{I^{(k)}=J^{(k)}=K} a_{IJK} \langle P_{K,k} f, 1_1 H_{I,J} \rangle h_J \right\|_{L^p(X)}^p \]
\[ \leq \mathbb{E} \int_{\mathbb{R}^d} \int_Y \left| \sum_{K \in D_{k,t}} \epsilon_K a_K(x,y_K) P_{K,k} f(y_K) \right|_X^p \, d\nu(y) \, dx. \]

Notice now that \( |a_K(x,y)| \leq 1_K(x) \). Thus, the Kahane contraction principle implies that for fixed \( x \) and \( y \) there holds that
\[ \mathbb{E} \left| \sum_{K \in D_{k,t}} \epsilon_K a_K(x,y_K) P_{K,k} f(y_K) \right|_X^p \leq \mathbb{E} \left| \sum_{K \in D_{k,t}} \epsilon_K 1_K(x) P_{K,k} f(y_K) \right|_X^p. \]

Using this we are left with
\[ \mathbb{E} \int_{\mathbb{R}^d} \int_Y \left| \sum_{K \in D_{k,t}} \epsilon_K a_K(x,y_K) P_{K,k} f(y_K) \right|_X^p \, d\nu(y) \, dx \sim \mathbb{E} \int_{\mathbb{R}^d} \left| \sum_{K \in D_{k,t}} P_{K,k} f(x) \right|_X^p \, dx \]
\[ = \|f\|_{L^p(X)}^p, \]

where we used the decoupling estimate (4.8) and noticed that \( \sum_{K \in D_{k,t}} P_{K,k} f = f \).

We turn to the part related to the terms \( \langle P_{K,k} f, 1_1 H_{I,J} \rangle \). We begin with
\[ \left\| \sum_{K \in D_{k,t}} \sum_{I^{(k)}=J^{(k)}=K} a_{IJK} \langle P_{K,k} f, 1_1 H_{I,J} \rangle h_J \right\|_{L^p(X)} \]
\[ \sim \mathbb{E} \left\| \sum_{K \in D_{k,t}} \sum_{I^{(k)}=J^{(k)}=K} \epsilon_J a_{IJK} \langle P_{K,k} f, 1_1 H_{I,J} \rangle \frac{1_J}{|J|^{1/2}} \right\|_{L^p(X)}^p, \]

where we used the UMD property to introduce the random signs and then the fact that we can clearly replace \( h_J \) by \( |h_J| = 1_J/|J|^{1/2} \) due to the random signs (the relevant random variables are identically distributed). We write the inner sum as
\[ \sum_{I^{(k)}=J^{(k)}=K} \epsilon_J a_{IJK} \langle P_{K,k} f, 1_1 H_{I,J} \rangle \frac{1_J}{|J|^{1/2}} = \sum_{J^{(k)}=K} \epsilon_J \langle P_{K,k} f \sum_{I^{(k)}=K} a_{IJK} |J|^{1/2} H_{I,J} \rangle 1_J. \]

Notice also that
\[ \left| \sum_{I^{(k)}=K} a_{IJK} |J|^{1/2} H_{I,J} \right| \leq \frac{1}{|K|} \sum_{I^{(k)}=K} |I| = 1. \]
We continue from (4.14). Applying Stein’s inequality we get that the last term in (4.14) is dominated by
\[
E \left\| \sum_{K \in D_{k,l}} \sum_{J^{(k)} = K} \varepsilon J P_{K,k} f \sum_{J^{(k)} = K} a_{I,J^{K}} |J|^{1/2} H_{I,J} f_J \right\|_{L^p(X)}^p 
\lesssim E \left\| \sum_{K \in D_{k,l}} \sum_{J^{(k)} = K} \varepsilon J P_{K,k} f_J \right\|_{L^p(X)}^p.
\]
where we used the Kahane contraction principle. From here the estimate is easily concluded by
\[
E \left\| \sum_{K \in D_{k,l}} \sum_{J^{(k)} = K} \varepsilon J P_{K,k} f_J \right\|_{L^p(X)}^p = E \left\| \sum_{K \in D_{k,l}} \varepsilon K \sum_{J^{(k)} = K} P_{K,k} f_J \right\|_{L^p(X)}^p 
= E \left\| \sum_{K \in D_{k,l}} \varepsilon K P_{K,k} f \right\|_{L^p(X)}^p \sim \|f\|_{L^p(X)}^p.
\]
Here we first changed the indexing of the random signs (using that for a fixed \(x\) for every \(K\) there is at most one \(J\) as in the sum for which \(1_J(x) \neq 0\)) and then applied the UMD property. This finishes the proof. \(\square\)

4.15. Theorem. Let \(T\) be a linear \(\omega\)-CZO and \(X\) be a UMD space with type \(r \in (1, 2]\) and cotype \(q \in [2, \infty)\). If \(\omega \in \text{Dini}_{1/\min(r,q')},\) we have
\[
\|Tf\|_{L^p(X)} \lesssim \|f\|_{L^p(X)}
\]
for all \(p \in (1, \infty)\).

Proof. Apply Theorem 3.18 to simple vector-valued functions. By Proposition 4.11 and the \(X\)-valued boundedness of the paraproducts (4.9) we conclude that
\[
\|Tf\|_{L^2(X)} \lesssim \|f\|_{L^2(X)}.
\]
As the weak type \((1, 1)\) follows from this even with just the assumption \(\omega \in \text{Dini}_0\), we can conclude the proof by the standard interpolation and duality method. \(\square\)

The multilinear case. Let \((X_1, \ldots, X_{n+1})\) be UMD spaces and \(Y_{n+1}^* = X_{n+1}\). Assume that there is an \(n\)-linear mapping \(X_1 \times \cdots \times X_n \to Y_{n+1}\), which we denote with the product notation \((x_1, \ldots, x_n) \mapsto \prod_{j=1}^n x_j\), so that
\[
\prod_{j=1}^n x_j \in Y_{n+1} \leq \prod_{j=1}^n |x_j| x_j.
\]
With just this setup it makes sense to extend an \(n\)-linear SIO (or some other suitable \(n\)-linear operator) \(T\) using the formula
\[
T(f_1, \ldots, f_n)(x) = \sum_{j_1, \ldots, j_n} T(f_{1,j_1}, \ldots, f_{n,j_n})(x) \prod_{k=1}^n e_{k,j_k}, \quad x \in \mathbb{R}^d,
\]
(4.16)
\[
f_k = \sum_{j_k} e_{k,j_k} f_{k,j_k}, \quad f_{k,j_k} \in L^\infty, \quad e_{k,j_k} \in X_k.
\]
In the bilinear case \(n = 2\) the existence of such a product is the only assumption that we will need. The bilinear case is somewhat harder than the linear case, but the \(n \geq 3\) case is
by far the most subtle. Indeed, for \( n \geq 3 \) we will need a more complicated setting for the tuple of spaces \((X_1, \ldots, X_{n+1})\) – the idea is to model the Hölder type structure typical of concrete examples of Banach \( n \)-tuples, such as that of non-commutative \( L^p \) spaces with the exponents \( p \) satisfying the natural Hölder relation. We will borrow this setting from [17]. In [17] it is shown in detail how natural tuples of non-commutative \( L^p \) spaces fit to this abstract framework. While we borrow the setting, the proof is significantly different. First, we have to deal with the more complicated modified shifts. Second, even in the standard shift case the proof in [17] is – by its very design – extremely costly on its complexity dependency. To circumvent this we need a new strategy.

For \( m \in \mathbb{N} \) we write \( \mathcal{J}_m := \{1, \ldots, m\} \) and denote the set of permutations of \( \mathcal{J} \subset \mathcal{J}_m \) by \( \Sigma(\mathcal{J}) \). We write \( \Sigma(m) = \Sigma(\mathcal{J}_m) \).

Next, we fix an associative algebra \( A \) over \( \mathbb{C} \), and denote the associative operation \( A \times A \to A \) by \((e, f) \mapsto ef\). We assume that there exists a subspace \( L^1 \) of \( A \) and a linear functional \( \tau : L^1 \to \mathbb{C} \), which we refer to as trace. Given an \( m \)-tuple \((X_1, \ldots, X_m)\) of Banach subspaces \( X_j \subset A \), we construct the seminorm

\[
(4.17) \quad |e|_{Y(X_1, \ldots, X_m)} = \sup \left\{ \left| \tau \left( e \prod_{\ell=1}^m e_{\sigma(\ell)} \right) \right| : \sigma \in \Sigma(m), |e_j|_{X_j} = 1, j = 1, \ldots, m \right\}
\]

on the subspace

\[
(4.18) \quad Y(X_1, \ldots, X_m) = \left\{ e \in A : e \prod_{\ell=1}^m e_{\sigma(\ell)} \in L^1 \quad \forall \sigma \in \Sigma(m), e_j \in X_j, j = 1, \ldots, m \right\}.
\]

For a Banach subspace \( X \subset A \) and \( y \in Y(X) \) we can define the mapping \( \Lambda_y \in X^* \) by the formula \( \Lambda_y(x) := \tau(yx) \), since by the definition \( yx \in L^1 \) and \( |\tau(yx)| \leq |y|_{Y(X)} |x|_X \). We say that a Banach subspace \( X \) of \( A \) is admissible if the following holds.

1. \( Y(X) \) is a Banach space with respect to \(| \cdot |_{Y(X)} \).
2. The mapping \( y \mapsto \Lambda_y \) from \( Y(X) \) to \( X^* \) is surjective.
3. For each \( x \in X, \ y \in Y(X) \), we also have \( xy \in L^1 \) and

\[
(4.19) \quad \tau(xy) = \tau(yx).
\]

If \( X \) is admissible, then the map \( y \mapsto \Lambda_y \) is an isometric bijection from \( Y(X) \) onto \( X^* \), and we identify \( Y(X) \) with \( X^* \). The following is [17, Lemma 3.10].

4.20. Lemma. Let \( X \) be admissible and reflexive (for instance, \( X \) is admissible and UMD). If \( Y(X) \) is also admissible, then \( Y(Y(X)) = X \) as sets and \( |x|_{Y(Y(X))} = |x|_X \) for all \( x \in X \).

If \( X, X_1, \ldots, X_m \) are Banach spaces we write \( X = Y(X_1, \ldots, X_m) \) to mean that \( X \) and \( Y(X_1, \ldots, X_m) \) coincide as sets, \( Y(X_1, \ldots, X_m) \) is a Banach space with the norm \( | \cdot |_{Y(X_1, \ldots, X_m)} \) and that the norms are equivalent, that is, \( |x|_X \sim |x|_{Y(X_1, \ldots, X_m)} \) for all \( x \in X \).

4.21. Definition (UMD Hölder pair). Let \( X_1, X_2 \) be admissible spaces. We say that \( \{X_1, X_2\} \) is a UMD Hölder pair if \( X_1 \) is a UMD space and \( X_2 = Y(X_1) \).

4.22. Definition (UMD Hölder \( m \)-tuple, \( m \geq 3 \)). Let \( X_1, \ldots, X_m \) be admissible spaces. We say that \( \{X_1, \ldots, X_m\} \) is a UMD Hölder \( m \)-tuple if the following properties hold.

P1. For all \( j_0 \in \mathcal{J}_m \) there holds

\[
X_{j_0} = Y(\{X_j : j \in \mathcal{J}_m \setminus \{j_0\}\}).
\]
P2. If $1 \leq k \leq m - 2$ and $J = \{j_1 < j_2 < \cdots < j_k\} \subset J_m$, then $Y(X_{j_1},\ldots,X_{j_k})$ is an admissible Banach space with the norm (4.17) and

(4.23) \[ \{X_{j_1},\ldots,X_{j_k}, Y(X_{j_1},\ldots,X_{j_k})\} \]

is a UMD Hölder $(k + 1)$-tuple.

The following is a key consequence of the definition. Let $m \geq 3$ and $\{X_1,\ldots,X_m\}$ be a UMD Hölder $m$-tuple. Then according to P2 the pair $\{X_{j_0}, Y(X_{j_0})\}$ is a UMD Hölder pair, which by Definition 4.21 implies that $X_{j_0}$ and $Y(X_{j_0})$ are UMD spaces. The inductive nature of the definition then ensures that each $Y(X_{j_1},\ldots,X_{j_k})$ appearing in (4.23) is a UMD space.

Notice also the following. Let $m \geq 2$ and $\{X_1,\ldots,X_m\}$ be a UMD $m$-Hölder tuple. Let $e_j \in X_j$ for $j \in J_m$. For each $\sigma \in \Sigma(m)$, as $X_{\sigma(1)} = Y(X_{\sigma(2)},\ldots,X_{\sigma(m)})$, we necessarily have $\prod_{j=1}^m e_{\sigma(j)} \in L^1$ and

\[
|\tau(e_{\sigma(1)} \cdots e_{\sigma(m)})| \leq |e_{\sigma(1)}|Y(X_{\sigma(2)},\ldots,X_{\sigma(m)}) \prod_{j=2}^m |e_{\sigma(j)}|X_{\sigma(j)} \sim \prod_{j=1}^m |e_j|X_j.
\]

Moreover, by (4.19) we have

(4.24) \[ \tau(e_1 \cdots e_m) = \tau(e_me_1 \cdots e_{m-1}) = \tau(e_{m-1}e_me_1 \cdots e_{m-2}) = \cdots . \]

We have the following Hölder type inequality.

4.25. Lemma. Let $\{X_1,\ldots,X_m\}$ be a UMD Hölder tuple. Then we have

\[ |ev|Y(X_m) \lesssim |e|Y(X_1,\ldots,X_k)|v|Y(X_{k+1},\ldots,X_{m-1}) . \]

Proof. To estimate $|ev|Y(X_m)$ we need to estimate

\[ |\tau(eve_m)| \]

with an arbitrary $e_m$ with $|e_m|X_m = 1$. First, we bound this with

\[ |e|Y(X_1,\ldots,X_k)|v|e_m|Y(X_1,\ldots,X_k) . \]

To estimate $|ve_m|Y(X_1,\ldots,X_k)$ we need to estimate

\[ |\tau(ve_m u_{\sigma_1(1)} \cdots u_{\sigma_1(k)})| \]

with arbitrary $u_j|X_j = 1$ and $\sigma_1 \in \Sigma(k)$. We estimate this with

\[ |v|Y(X_{k+1},\ldots,X_{m-1})|e_m u_{\sigma_1(1)} \cdots u_{\sigma_1(k)}|Y(X_{k+1},\ldots,X_{m-1}) . \]

To estimate $|e_m u_{\sigma_1(1)} \cdots u_{\sigma_1(k)}|Y(X_{k+1},\ldots,X_{m-1})$ we need to estimate

\[ |\tau(e_m u_{\sigma_1(1)} \cdots u_{\sigma_1(k)} u_{\sigma_2(k+1)} \cdots u_{\sigma_2(m-1)})| \]

with an arbitrary permutation $\sigma_2$ of $\{k+1,\ldots,m-1\}$ and $u_j|X_j = 1$. Finally, we estimate this with

\[ |e_m|Y(X_1,\ldots,X_{m-1}) \sim |e_m|X_m = 1 . \]

In all of the statements below an arbitrary UMD Hölder tuple $\{X_1,\ldots,X_{n+1}\}$ is given.
4.26. **Proposition.** Suppose that $Q_k$ is an $n$-linear modified shift and $f_j : \mathbb{R}^d \to X_j$. Let $1 < p_j < \infty$ with $\sum_{j=1}^{n+1} 1/p_j = 1$. Then we have

$$|\langle Q_k(f_1, \ldots, f_n), f_{n+1} \rangle| \lesssim (k+1)^\alpha \prod_{j=1}^{n+1} \|f_j\|_{L^{p_j}(X_j)},$$

where

$$\alpha = \frac{1}{\min(p'_1, \ldots, p'_{n+1}, s_1', \ldots, s_{n+1}')},$$

and $X_j$ has cotype $s_j$.

**Proof.** We will assume that $Q_k$ is of the form (3.6) – the other cases follow by duality using the property (4.24). We follow the ideas of the decomposition from the proof of Lemma 3.14: we will estimate the terms $A_m$ and $U_m$, $m \in \{1, \ldots, n\}$, from there separately.

First, we estimate the part $A_m$ defined in (3.15). We have that

$$A_m(f_1, \ldots, f_n) = \sum_K A_m,K(f_1, \ldots, f_n),$$

where

$$A_m,K(f_1, \ldots, f_n) := \sum_{I_m^{(k)} = \ldots = I_{n+1}^{(k)} = K} b_{m,K,(I_j)} \prod_{j=1}^{m-1} \langle f_j \rangle_{K} \cdot \langle P_{K,k-1} f_m \rangle_{I_m} \cdot \prod_{j=m+1}^{n} \langle f_j \rangle_{I_j} \cdot h_{I_{n+1}},$$

and

$$b_{m,K,(I_j)} = b_{m,K,I_m,\ldots,I_{n+1}} := \sum_{I_m^{(k)} = \ldots = I_{n+1}^{(k)} = K} b_{K,(I_j)}.$$

Here we have the normalization

$$|b_{m,K,(I_j)}| \leq \frac{|I_m|^{n+1/2}}{|K|^{n}} \sum_{I_m^{(k)} = \ldots = I_{n+1}^{(k)} = K} 1 = \frac{|I_m|^{n-m+3/2}}{|K|^{n-m+1}}.$$

Recall the grids $D_{k,l}$, $l \in \{0, 1, \ldots, k\}$, from (4.6). Let

$$Z_{1,\ldots,n} := Y(X_{n+1}) = Y(Y(X_1, \ldots, X_n)).$$

Recalling that $X_{n+1}$ is identified with $Y(X_{n+1})$, we have that $L_{p_{n+1}}(Z_{1,\ldots,n})$ has type $s := \min(p'_{n+1}, s_{n+1}')$. Thus, there holds that

$$\|A_m(f_1, \ldots, f_n)\|_{L_{p_{n+1}}(Z_{1,\ldots,n})} \sim \left(\mathbb{E} \left[ \sum_{l=0}^{k} \varepsilon_l \sum_{K \in D_{k,l}} A_m,K(f_1, \ldots, f_n) \right]^{s} \right)^{1/s} \lesssim \left( \sum_{l=0}^{k} \sum_{K \in D_{k,l}} A_m,K(f_1, \ldots, f_n) \right)^{s}.$$
In the first step we used the UMD property of \( Z_{1,\ldots,n} \) and the Kahane–Khintchine inequality. We see that to prove the claim it suffices to show the uniform bound

\[
(4.29) \quad \left\| \sum_{K \in \mathcal{D}_{k,l}} A_{m,K}(f_1, \ldots, f_n) \right\|_{L^{p_{n+1}}(Z_{1,\ldots,n})} \lesssim \prod_{j=1}^n \|f_j\|_{L^{p_j}(X_j)}.
\]

We turn to prove (4.29). To avoid confusion with the various \( Y \) spaces, we denote the decoupling space by \((W, \nu)\) in this proof. The decoupling estimate (4.8) gives that the left hand side of (4.29) is comparable to

\[
\left( \mathbb{E} \int_W \int_{\mathbb{R}^d} \left| \sum_{K \in \mathcal{D}_{k,l}} \varepsilon_K 1_K(x) \sum_{I_m^{(1)} = \ldots = I_{m+1}^{(1)}} b_{m,K,(I_j)} \right| \prod_{j=1}^{m-1} \langle f_j \rangle_K \cdot \langle P_{K,k-1} f_m \rangle_I_m \cdot \prod_{j=m+1}^n \langle f_j \rangle_{I_j} \cdot h_{I_{n+1}}(w_K) \right|_{Z_{1,\ldots,n}}^{p_{n+1}} \, d\nu(w) \right)^{1/p_{n+1}}.
\]

Suppose that \( m > 1 \); if \( m = 1 \) one can start directly from (4.32) below. Now, with a fixed \( w \in W \) can use Stein’s inequality with respect to the function \( f_1 \) to have that the previous term is dominated by

\[
(4.30) \quad \left( \mathbb{E} \int_W \int_{\mathbb{R}^d} \left| f_1(x) \sum_{K \in \mathcal{D}_{k,l}} \varepsilon_K 1_K(x) \sum_{I_m^{(1)} = \ldots = I_{m+1}^{(1)}} b_{m,K,(I_j)} \right| \prod_{j=2}^{m-1} \langle f_j \rangle_K \cdot \langle P_{K,k-1} f_m \rangle_I_m \cdot \prod_{j=m+1}^n \langle f_j \rangle_{I_j} \cdot h_{I_{n+1}}(w_K) \right|_{Z_{1,\ldots,n}}^{p_{n+1}} \, d\nu(w) \right)^{1/p_{n+1}}.
\]

To move forward, notice that by Lemma 4.25 we have

\[
(4.31) \quad \left| e_1 \sum_{k} \prod_{j=2}^n e_{j,k}\right|_{Z_{1,\ldots,n}} = \left| e_1 \sum_{k} \prod_{j=2}^n e_{j,k}\right|_{Y(X_{n+1})} \lesssim \left| e_1 \right|_{X_1} \sum_{k} \prod_{j=2}^n e_{j,k}\right|_{Z_{2,\ldots,n}}.
\]

where \( Z_{2,\ldots,n} = Y(Y(X_2, \ldots, X_n)) \). Having established (4.31) we can now dominate (4.30) with

\[
\|f_1\|_{L^{p_1}(X_1)} \left( \mathbb{E} \int_W \int_{\mathbb{R}^d} \left| \sum_{K \in \mathcal{D}_{k,l}} \varepsilon_K 1_K(x) \sum_{I_m^{(1)} = \ldots = I_{m+1}^{(1)}} b_{m,K,(I_j)} \right| \prod_{j=2}^{m-1} \langle f_j \rangle_K \cdot \langle P_{K,k-1} f_m \rangle_I_m \cdot \prod_{j=m+1}^n \langle f_j \rangle_{I_j} \cdot h_{I_{n+1}}(w_K) \right|_{Z_{2,\ldots,n}}^{q_{2,\ldots,n}} \, dx \, d\nu(w) \right)^{1/q_{2,\ldots,n}},
\]

where \( q_{2,\ldots,n} \) is defined by \( 1/q_{2,\ldots,n} = \sum_{j=2}^n 1/p_j \). We can continue this process. In the next step we argue as above but using the Hölder tuple \((X_2, \ldots, X_n, Y(X_2, \ldots, X_n))\). Below we write \( Z_{k_1,\ldots,k_2} = Y(Y(X_{k_1}, \ldots, X_{k_2})) \) and \( 1/q_{k_1,\ldots,k_2} = \sum_{j=k_1}^{k_2} 1/p_j \). Iterating this we
arrive at $\prod_{j=1}^{m-1} \|f_j\|_{L^p(X_j)}$ multiplied by

$$
\left( \mathbb{E} \int_W \int_{\mathbb{R}^d} \left| \sum_{K \in \mathcal{D}_{k,l}} \varepsilon_K 1_K(x) \sum_{I_m^{(k)} = \cdots = I_{n+1}^{(k)} = K} b_{m,K,(I_j)} (P_{K,k-1} f_m) I_m \prod_{j=m+1}^n \langle f_j I_j \cdot h_{I_{n+1}} (w_K) \rangle^{q_m,\ldots,n} \right| Z_{m,\ldots,n} \right)^{1/q_m,\ldots,n}.
$$

(4.32)

We assume that $m < n$; if $m = n$ one is already at (4.36) below. Let $K \in \mathcal{D}_{k,l}$. We define the kernel

$$
b_{m,K}(x,y_m,\ldots,y_n) := \sum_{I_m^{(k)} = \cdots = I_{n+1}^{(k)} = K} |K|^{n-m+1} b_{m,K,(I_j)} \prod_{j=m}^n \frac{I_j(y_j)}{|I_j|} h_{I_{n+1}}(x)
$$

so that

$$
\sum_{I_m^{(k)} = \cdots = I_{n+1}^{(k)} = K} b_{m,K,(I_j)} (P_{K,k-1} f_m) I_m \prod_{j=m+1}^n \langle f_j I_j \cdot h_{I_{n+1}} (w_K) \rangle = \frac{1}{|K|^{n-m+1}} \int_{K^{n-m+1}} b_{m,K}(w_K,y_m,\ldots,y_n) P_{K,k-1} f_m(y_m) \prod_{j=m+1}^n f_j(y_j) \, dy.
$$

Using this representation in (4.32) we can use Stein’s inequality with respect to the function $f_n$ to have that the term in (4.32) is dominated by

$$
\left( \mathbb{E} \int_W \int_{\mathbb{R}^d} \left| \sum_{K \in \mathcal{D}_{k,l}} \varepsilon_K \frac{1}{|K|^{n-m}} \int_{K^{n-m}} b_{m,K}(w_K,y_m,\ldots,y_n-1,x) P_{K,k-1} f_m(y_m) \prod_{j=m+1}^{n-1} f_j(y_j) \, dy \right| f_n(x) \left| Z_{m,\ldots,n} \right| \, dx \, dv(w) \right)^{1/q_m,\ldots,n},
$$

which is (again by Lemma 4.25) dominated by $\|f_n\|_{L^p(X_n)}$ multiplied with

$$
\left( \mathbb{E} \int_W \int_{\mathbb{R}^d} \left| \sum_{K \in \mathcal{D}_{k,l}} \varepsilon_K \frac{1}{|K|^{n-m}} \int_{K^{n-m}} b_{m,K}(w_K,y_m,\ldots,y_n-1,x) P_{K,k-1} f_m(y_m) \prod_{j=m+1}^{n-1} f_j(y_j) \, dy \right| \left| Z_{m,\ldots,n} \right| \, dx \, dv(w) \right)^{1/q_m,\ldots,n-1}.
$$

Next, we fix the point $w \in W$ and consider the term

$$
\mathbb{E} \int_{\mathbb{R}^d} \left| \sum_{K \in \mathcal{D}_{k,l}} \varepsilon_K \frac{1}{|K|^{n-m}} \int_{K^{n-m}} b_{m,K}(w_K,y_m,\ldots,y_n-1,x) P_{K,k-1} f_m(y_m) \prod_{j=m+1}^{n-1} f_j(y_j) \, dy \right| \left| Z_{m,\ldots,n-1} \right| \, dx.
$$

(4.34)
To finish the estimate of (4.32) it is enough to dominate (4.34) by \( \prod_{j=m}^{n-1} \| f_j \|_{L^{p_j}(X_j)}^{q_{m,...,n-1}} \) uniformly in \( w \in W \). Recalling (4.33) we can write (4.34) as

\[
\mathbb{E} \int_{\mathbb{R}^d} \sum_{K \in \mathcal{D}_{k,l}} \sum_{I_m^{(k)} = \ldots = I_{n+1}^{(k)} = K} |K| b_{m,K,(I_j)} \langle P_{K,k-1} f_m \rangle I_m \prod_{j=m+1}^{n-1} \langle f_j \rangle I_j h_{I_{n+1}}(w_K) \bigg|_{Z_{m,...,n-1}}^{q_{m,...,n-1}} \, dx,
\]

which is further comparable to

\[(4.35) \int_{\mathbb{R}^d} \sum_{K \in \mathcal{D}_{k,l}} \sum_{I_m^{(k)} = \ldots = I_{n+1}^{(k)} = K} |K| \sum_{I_{n+1}^{(k)} = K} \frac{|K|}{|I_n|^{1/2}} b_{m,K,(I_j)} h_{I_{n+1}}(w_K) \langle P_{K,k-1} f_m \rangle I_m \prod_{j=m+1}^{n-1} \langle f_j \rangle I_j h_{I_n}(x) \bigg|_{Z_{m,...,n-1}}^{q_{m,...,n-1}} \, dx.
\]

In the last step we were able to replace \( 1_{I_n}/|I_n|^{1/2} \) with \( h_{I_n} \) because of the random signs, after which we removed the signs using UMD. Recalling the size of the coefficients \( b_{m,K,(I_j)} \) from (4.27) we see that

\[
\left| \sum_{I_{n+1}^{(k)} = K} \frac{|K|}{|I_n|^{1/2}} b_{m,K,(I_j)} h_{I_{n+1}}(w_K) \right| \leq \frac{|I_n|^{n-m+1/2}}{|K|^{n-m}},
\]

since there is only one \( I_{n+1} \) such that \( h_{I_{n+1}}(w_K) \neq 0 \). It is seen that after applying decoupling (4.35) is like (4.32) but the degree of linearity is one less. Therefore, iterating this we see that (4.34) satisfies the desired bound if we can estimate

\[(4.36) \mathbb{E} \int_{\mathbb{R}^d} \sum_{K \in \mathcal{D}_{k,l}} \sum_{I_m^{(k)} = \ldots = I_{n+1}^{(k)} = K} b_{K,I_m,I_{m+1}} \langle P_{K,k-1} f_m \rangle I_m h_{I_{m+1}}(x) \bigg|_{X_m}^{p_m} \, dx,
\]

where

\[
|b_{K,I_m,I_{m+1}}| \leq \frac{|I_m|^{3/2}}{|K|},
\]

by \( \| f_m \|_{L^{p_m}(X_m)}^{p_m} \). This is a linear estimate and bounded exactly like the right hand side of (4.13). Therefore, we get the desired bound \( \| f_m \|_{L^{p_m}(X_m)}^{p_m} \). This finally finishes our estimate for the term \( A_m \).

We turn to estimate the parts \( U_m \) from (3.16). Recall that \( U_m(f_1, \ldots, f_n) \) is by definition

\[
\sum_{K} \sum_{I_m^{(k)} = \ldots = I_{n+1}^{(k)} = K} b_{K,(I_j)} \sum_{i=0}^{k-1} \left( \prod_{j=m+1}^{n-1} \langle f_j \rangle_{I_{n+1}^{(i)}} \cdot \langle \Delta_{I_{n+1}^{(i)}} f_m \rangle_{I_{n+1}^{(i)}} \cdot \prod_{j=m+1}^{n} \langle f_j \rangle_{I_{n+1}^{(i)}} \right) h_{I_{n+1}^{(i)}},
\]
Similarly as with the operators $A_m$, we use the fact that $L^{p_{n+1}}(Z_{1,...,n})$ has type $s = \min(p_{n+1}, \epsilon_{n+1})$ to reduce to controlling the term

$$\sum_{K \in D, i} \sum_{I^{(k)} = \cdots = I^{(k)} = K} b_{K, I^{(k)}} \left( \prod_{j=0}^{k-1} \langle f_j \rangle_{I^{(j+1)}} \cdot \langle \Delta f_{I^{(j+1)}} f_m \rangle_{I^{(j)}} \cdot \prod_{j=m+1}^{n} \langle f_j \rangle_{I^{(j)}} \right) h_{I^{(k)+1}}$$

(4.37)

uniformly on $l$. For every $I_{n+1}$ such that $I_{n+1} = K$ there holds that

$$\left| \sum_{I^{(k)} = \cdots = I^{(k)} = K} b_{K, I^{(k)}} \right| \leq |I_{n+1}|^{1/2}.$$

Therefore, using the UMD property and the Kahane contraction principle the $L^{p_{n+1}}(Z_{1,...,n})$-norm of (4.37) is dominated by

$$E \left\| \sum_{K \in D, i} \varepsilon_K \sum_{I^{(k)} = K} \sum_{i=0}^{k-1} \prod_{j=1}^{m-1} \langle f_j \rangle_{I^{(j+1)}} \cdot \langle \Delta f_{I^{(j+1)}} f_m \rangle_{I^{(j)}} \cdot \prod_{j=m+1}^{n} \langle f_j \rangle_{I^{(j)}} \right\|_{L^{p_{n+1}}(Z_{1,...,n})}.$$

If all the averages were on the “level $i+1$” we could estimate this directly. Since there are the averages on the “level $i$” we need to further split this. There holds that

$$\prod_{j=m+1}^{n} \langle f_j \rangle_{I^{(j)}} = \langle f_{m+1} \rangle_{I^{(m+1)}} \prod_{j=m+2}^{n} \langle f_j \rangle_{I^{(j)}} + \langle \Delta f_{I^{(m+1)}} f_m \rangle_{I^{(m+1)}} \prod_{j=m+2}^{n} \langle f_j \rangle_{I^{(j)}}.$$

Then, both of these are expanded in the same way related to $f_{m+2}$ and so on. This gives that

$$\prod_{j=m+1}^{n} \langle f_j \rangle_{I^{(j)}} 1_I = \sum_{\varphi} \prod_{j=m+1}^{n} D^{\varphi(j)} f_j 1_I.$$

Here the summation is over functions $\varphi: \{m+1, \ldots, n\} \rightarrow \{0, 1\}$ and for a cube $I \in D$ we defined $D^{\varphi(j)} = E_I$ and $D_I = \Delta_I$. We also used the fact that $\langle \Delta f_{I^{(m+1)}} f_m \rangle_{I^{(m+1)}} = \Delta f_{I^{(m+1)}} f_I$.

Finally, we take one $\varphi$ and estimate the related term. It can be written as

$$E \left\| \sum_{K \in D, i} \varepsilon_K \sum_{\ell \in L, \ell(K) \geq 2^{-k} \ell(K)} \prod_{j=1}^{m-1} \langle f_j \rangle_{L} \cdot \Delta_L f_m \cdot \prod_{j=m+1}^{n} D^{\varphi(j)} f_j \right\|_{L^{p_{n+1}}(Z_{1,...,n})}.$$

If $\varphi(j) = 0$ for all $j = m+1, \ldots, n$, then this can be estimated by a repeated use of Stein’s inequality (similarly as above) related to the functions $f_j, j \neq m$.

Suppose that $\varphi(j) \neq 0$ for some $j$. Notice that

$$\sum_{K \in D, i} \varepsilon_K \sum_{\ell \in L, \ell(K) > 2^{-k} \ell(K)} \prod_{j=1}^{m-1} \langle f_j \rangle_{L} \cdot \Delta_L f_m \cdot \prod_{j=m+1}^{n} D^{\varphi(j)} f_j$$

$$= E' \left( \sum_{K \in D, i} \varepsilon_K \sum_{\ell \in L, \ell(K) > 2^{-k} \ell(K)} \varepsilon_L \prod_{j=1}^{m-1} \langle f_j \rangle_{L} \cdot \Delta_L f_m \left( \sum_{L} \varepsilon_L' \prod_{j=m+1}^{n} D^{\varphi(j)} f_j \right) \right).$$
Therefore, by Lemma 4.25 the term in (4.38) is dominated by

\[(4.39) \quad \mathbb{E}\left(\left(\sum_{K \in D_{k,l}} \varepsilon_K \sum_{L \subseteq K, \ell(L) > 2^{-k} \ell(K)} \varepsilon'_L \prod_{j=1}^{m-1} \langle f_j \rangle_L \cdot \Delta_L f_m \right)^{q_1,\ldots,q_m} \right)^{1/q_1,\ldots,q_m},\]

multiplied by

\[(4.40) \quad \left(\mathbb{E}\left(\sum_L \varepsilon'_L \prod_{j=m+1}^{n} D^{(j)}_L f_j \|_{L^q_{m+1,\ldots,n}(Z_{m+1,\ldots,n})} \right)^{1/q_{m+1,\ldots,n}} \right)^{1/q_1,\ldots,q_m},\]

The term in (4.39) can be estimated by a repeated use of Stein’s inequality. The term in (4.40) is like the term in (4.38). If there is only one martingale difference, then we can again estimate directly with Stein’s inequality. If there is at least two martingale differences, then one can split into two as we did when we arrived at (4.39) and (4.40). This process is continued until one ends up with terms that contain only one martingale difference, and such terms we can estimate.

The proof of Proposition 4.26 is finished. □

With the essentially same proof as for the terms $A_m$ above we also have the following result.

4.41. **Proposition.** Suppose that $S_{k_1,\ldots,k}$ is an $n$-linear shift of complexity $(k, \ldots, k)$ and $f_j : \mathbb{R}^d \to X_j$. Let $1 < p_j < \infty$ with $\sum_{j=1}^{n+1} 1/p_j = 1$. Then we have

\[|\langle S_{k_1,\ldots,k}(f_1, \ldots, f_n), f_{n+1} \rangle| \lesssim (k + 1)^\alpha \prod_{j=1}^{n+1} \|f_j\|_{L^{p_j}(X_j)},\]

where

\[\alpha = \frac{1}{\min(p_1', \ldots, p_{n+1}', s_1', \ldots, s_{n+1}')}\]

and $X_j$ has cotype $s_j$.

The following is [17, Theorem 5.3]. This is a significantly simpler argument than the shift proof and consists of repeated use of Stein’s inequality until one is reduced to the linear case (4.9).

4.42. **Proposition.** Suppose that $\pi$ is an $n$-linear paraproduct and $f_j : \mathbb{R}^d \to X_j$. Let $1 < p_j < \infty$ with $\sum_{j=1}^{n+1} 1/p_j = 1$. Then we have

\[|\langle \pi(f_1, \ldots, f_n), f_{n+1} \rangle| \lesssim \prod_{j=1}^{n+1} \|f_j\|_{L^{p_j}(X_j)}.\]

Finally, we are ready to state our main result concerning the UMD extensions of $\omega$-CZOs.

4.43. **Theorem.** Suppose that $T$ is an $n$-linear $\omega$-CZO. Suppose $\omega \in \text{Dini}_\alpha$, where

\[\alpha = \frac{1}{\min((n+1)/n, s_1', \ldots, s_{n+1}')}\]
and $X_j$ has cotype $s_j$. Then for all exponents $1 < p_1, \ldots, p_n \leq \infty$ and $1/q_{n+1} = \sum_{j=1}^{n} 1/p_j > 0$ we have

$$
\|T(f_1, \ldots, f_n)\|_{L^{n+1}(X_{n+1}^*)} \lesssim \prod_{j=1}^{n} \|f_j\|_{L^{p_j}(X_j)}.
$$

Proof. The important part is to establish the boundedness with a single tuple of exponents. We may e.g. conclude from the boundedness of the model operators and Theorem 3.18 that

$$
|\langle T(f_1, \ldots, f_n), f_{n+1} \rangle| \lesssim \prod_{j=1}^{n+1} \|f_j\|_{L^{n+1}(X_j)}
$$

if we choose $\alpha$ as in the statement of the theorem. It is completely standard how to improve this to cover the full range: we can e.g. prove the end point estimate $T: L^1(X_1) \times \cdots \times L^1(X_n) \rightarrow L^{1/n, \infty}(X_{n+1}^*)$, see [49], and then use interpolation or good-$\lambda$ methods. See e.g. [24, 51]. For such arguments the spaces $X_j$ no longer play any role (the scalar-valued proofs can readily be mimicked). \hfill \Box

4.44. Remark. The exponent $(n+1)/n$ in the definition of $\alpha$ is slightly annoying, since now the exponent $\alpha = 1/2$ valid in the scalar-valued case $X_1 = \cdots = X_{n+1} = \mathbb{C}$ does not follow from this result. Of course, it is way more simple to prove scalar-valued (or suitable lattice-valued) estimates directly with other methods anyway (see Section 3.1).

On the other hand, in some non-trivial situations the presence of $(n+1)/n$ is not a restriction. Suppose each space $X_j$ is some $L^{p_j}$ space (a non-commutative $L^p$ space $L^{p_j}(M)$, say) and $\sum_{j=1}^{n+1} 1/p_j = 1$, $1 < p_j < \infty$. Then the cotype of $X_j$ is $s_j = \max(2, p_j) \geq p_j$ so that

$$
1 = \sum_{j=1}^{n+1} \frac{1}{p_j} \geq \sum_{j=1}^{n+1} \frac{1}{s_j} = n + 1 - \sum_{j=1}^{n+1} \frac{1}{s_j},
$$

and so there has to be an index $j$ so that $s_j \leq (n+1)/n$ anyway.

5. Bi-parameter SIOs. Let $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and consider a linear operator $T$ on $\mathbb{R}^d$. We define what it means for $T$ to be a bi-parameter SIO. Let $\omega_i$ be a modulus of continuity on $\mathbb{R}^{d_i}$. Let $f_j = f_1^j \otimes f_2^j$, $j = 1, 2$.

Full kernel representation. Here we assume that $\text{spt} f_1^i \cap \text{spt} f_2^i = \emptyset$ for both $i = 1$ and $i = 2$. In this case we demand that

$$
\langle T f_1, f_2 \rangle = \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} K(x, y) f_1(y) f_2(x) \, dx \, dy,
$$

where $K: \mathbb{R}^{2d} \setminus \{(x, y) \in \mathbb{R}^{2d}: x_1 = y_1 \text{ or } x_2 = y_2 \} \rightarrow \mathbb{C}$ is a kernel satisfying a set of estimates which we specify next. Note that this will imply kernel representations also for $T_1^*, T_2^*$ and $T^*$, where the partial adjoint $T_i^*$ is defined via $\langle T_i^* (f_3^1 \otimes f_3^2), f_1^1 \otimes f_2^1 \rangle = \langle T(f_2^1 \otimes f_3^2), f_1^1 \otimes f_2^1 \rangle$ and $T_2^* = (T_1^*)^*$. We denote their natural full kernels by $K_1^*, K_2^*$ and $K^*$.

The kernel $K$ is assumed to satisfy the size estimate

$$
|K(x, y)| \lesssim \frac{1}{|x_1 - y_1|^{d_1}} \frac{1}{|x_2 - y_2|^{d_2}},
$$
the Hölder estimate
\[
|K(x, y) - K((x_1, x_2'), y) - K((x_1, x_2), y) + K(x', y)|
\leq \omega_1 \left( \frac{|x_1 - x_1'|}{|x_1 - y_1|} \right) \frac{1}{|x_1 - y_1|^{d_1}} \frac{1}{|x_2 - y_2|^{d_2}}
\]
whenever $|x_1 - x_1'| \leq |x_1 - y_1|/2$ and $|x_2 - x_2'| \leq |x_2 - y_2|/2$, and the mixed Hölder and size estimates
\[
|K(x, y) - K((x_1', x_2), y)| \leq \omega_1 \left( \frac{|x_1 - x_1'|}{|x_1 - y_1|} \right) \frac{1}{|x_1 - y_1|^{d_1}} \frac{1}{|x_2 - y_2|^{d_2}}
\]
whenever $|x_1 - x_1'| \leq |x_1 - y_1|/2$ and
\[
|K(x, y) - K((x_1, x_2'), y)| \leq \frac{1}{|x_1 - y_1|^{d_1}} \frac{1}{|x_2 - y_2|^{d_2}}
\]
whenever $|x_2 - x_2'| \leq |x_2 - y_2|/2$. These estimates are also assumed from the kernels $K_1^*$, $K_2^*$ and $K^*$ (with some of these dual assumptions simply repeating the above estimates).

**Partial kernel representations.** Suppose now only that $\text{spt } f_1^1 \cap \text{spt } f_2^2 = \emptyset$. Then we assume that
\[
\langle Tf_1, f_2 \rangle = \int_{\mathbb{R}^d_1} \int_{\mathbb{R}^d_1} K_{f_1^2, f_2^2}(x_1, y_1) f_1^1(y_1) f_2^2(x_1) dx_1 dy_1,
\]
where $K_{f_1^2, f_2^2}$ is an $\omega_1$-Calderón–Zygmund kernel with a constant depending on the fixed functions $f_1^2, f_2^2$. This means that we have the size condition
\[
|K_{f_1^2, f_2^2}(x_1, y_1)| \leq C(f_1^2, f_2^2) \frac{1}{|x_1 - y_1|^{d_1}}
\]
and the Hölder estimate
\[
|K_{f_1^2, f_2^2}(x_1, y_1) - K_{f_1^2, f_2^2}(x_1', y_1)| \leq C(f_1^2, f_2^2) \omega_1 \left( \frac{|x_1 - x_1'|}{|x_1 - y_1|} \right) \frac{1}{|x_1 - y_1|^{d_1}}
\]
whenever $|x_1 - x_1'| \leq |x_1 - y_1|/2$. The analogous Hölder estimate in the $y_1$ slot is also assumed. We assume the following $T1$ type control on the constant $C(f_1^2, f_2^2)$. We have
\[
C(1_{I_2}, 1_{I_2}) + C(a_{I_2}, 1_{I_2}) + C(1_{I_2}, a_{I_2}) \lesssim |I_2|
\]
for all cubes $I_2 \subset \mathbb{R}^{d_2}$ and all functions $a_{I_2}$ satisfying $a_{I_2} = 1_{I_2}, |a_{I_2}| \leq 1$ and $\int a_{I_2} = 0$.

Analogous partial kernel representation is assumed when $\text{spt } f_1^2 \cap \text{spt } f_2^2 = \emptyset$.

**5.1. Definition.** If $T$ is a linear operator with full and partial kernel representations as defined above, we call $T$ a bi-parameter $(\omega_1, \omega_2)$-SIO.

**Bi-parameter CZOs.** We say that $T$ satisfies the weak boundedness property if
\[
|\langle T(1_{I_1} \otimes 1_{I_2}), 1_{I_1} \otimes 1_{I_2} \rangle| \lesssim |I_1||I_2|
\]
for all cubes $I_i \subset \mathbb{R}^{d_i}$.

An SIO $T$ satisfies the diagonal BMO assumption if the following holds. For all cubes $I_i \subset \mathbb{R}^{d_i}$ and functions $a_{I_i}$ with $a_{I_i} = 1_{I_i}, |a_{I_i}| \leq 1$ and $\int a_{I_i} = 0$ we have
\[
|\langle T(a_{I_1} \otimes 1_{I_2}), 1_{I_1} \otimes 1_{I_2} \rangle | + |\langle T(1_{I_1} \otimes 1_{I_2}), a_{I_1} \otimes 1_{I_2} \rangle |
+ |\langle T(1_{I_1} \otimes a_{I_2}), 1_{I_1} \otimes 1_{I_2} \rangle | + |\langle T(1_{I_1} \otimes 1_{I_2}), 1_{I_1} \otimes a_{I_2} \rangle | \lesssim |I_1||I_2|.
\]
The product BMO space is originally by Chang and Fefferman \cite{8, 9}, and it is the right bi-parameter BMO space for many considerations. An SIO $T$ satisfies the product BMO assumption if it holds $S(1, 1) \in \text{BMO}_{\text{prod}}$ for all the choices

$$S \in \{T, T_1^*, T_2^*, T^*\}.$$ 

This can be interpreted in the sense that

$$\|S1\|_{\text{BMO}_{\text{prod}}} = \sup_{D_1, D_2} \Omega \sup_{I_1 \times I_2 \subset \Omega} \left( \frac{1}{|\Omega|} \sum_{I_i \in D_i'} \|\langle S1, h_{I_1} \otimes h_{I_2}\rangle\|^2 \right)^{1/2} < \infty,$$

where the supremum is over all dyadic grids $D_i'$ on $\mathbb{R}^d$ and open sets $\Omega \subset \mathbb{R}^d = \mathbb{R}^d_1 \times \mathbb{R}^d_2$ with $0 < |\Omega| < \infty$, and the pairings $\langle S1, h_{I_1} \otimes h_{I_2}\rangle$ can be defined, in a standard way, using the kernel representations.

### 5.2. Definition

A bi-parameter $(\omega_1, \omega_2)$-SIO $T$ satisfying the weak boundedness property, the diagonal BMO assumption and the product BMO assumption is called a bi-parameter $(\omega_1, \omega_2)$-Calderón–Zygmund operator ($(\omega_1, \omega_2)$-CZO).

**General bi-parameter notation and basic operators.** A weight $w(x_1, x_2)$ (i.e. a locally integrable a.e. positive function) belongs to the bi-parameter weight class $A_p(\mathbb{R}^d_1 \times \mathbb{R}^d_2)$, $1 < p < \infty$, if

$$[w]_{A_p(\mathbb{R}^d_1 \times \mathbb{R}^d_2)} := \sup_R \frac{1}{|R|} \int_{R} w^{p-1} \left( \frac{1}{|R|} \int_{R} w^{1-p'} \right)^{p-1} < \infty,$$

where the supremum is taken over $R = I_1 \times I_2$ and each $I_i \subset \mathbb{R}^d_i$ is a cube with sides parallel to the axes. We simply call such $R$ rectangles. Thus, this is the one-parameter definition but cubes are replaced by rectangles.

We have

$$[w]_{A_p(\mathbb{R}^d_1 \times \mathbb{R}^d_2)} < \infty \iff \max \left( \sup_{x_1 \in \mathbb{R}^d_1} [w(x_1, \cdot)]_{A_p(\mathbb{R}^d_2)}, \sup_{x_2 \in \mathbb{R}^d_2} [w(\cdot, x_2)]_{A_p(\mathbb{R}^d_1)} \right) < \infty,$$

and that

$$\max \left( \sup_{x_1 \in \mathbb{R}^d_1} [w(x_1, \cdot)]_{A_p(\mathbb{R}^d_2)}, \sup_{x_2 \in \mathbb{R}^d_2} [w(\cdot, x_2)]_{A_p(\mathbb{R}^d_1)} \right) \leq [w]_{A_p(\mathbb{R}^d_1 \times \mathbb{R}^d_2)},$$

while the constant $[w]_{A_p}$ is dominated by the maximum to some power. For basic bi-parameter weighted theory see e.g. \cite{38}.

We denote a general dyadic grid in $\mathbb{R}^d_1$ by $D^1$. We denote cubes in $D^1$ by $I_1, J_1, K_i, \text{ etc.}$

If $A$ is an operator acting on $\mathbb{R}^d$, we can always let it act on the product space $\mathbb{R}^d = \mathbb{R}^d_1 \times \mathbb{R}^d_2$ by setting $A^1 f(x) = A(f(\cdot, x_2))(x_1)$. Similarly, we use the notation $A^2 f(x) = A(f(x_1, \cdot))(x_2)$ if $A$ is originally an operator acting on $\mathbb{R}^d$. Our basic bi-parameter dyadic operators – martingale differences and averaging operators – are obtained by simply chaining together relevant one-parameter operators. For instance, a bi-parameter martingale difference is $\Delta_R f = \Delta^1_{I_1} \Delta^2_{I_2} f$, $R = I_1 \times I_2$. Bi-parameter estimates, such as the square function bound

$$\left\| \left( \sum_{I_i \in D^1} \|\Delta^1_{I_1} \Delta^2_{I_2} f\|^2 \right)^{1/2} \right\|_{L^p(w)} \sim \|f\|_{L^p(w)},$$
where \( w \) is a bi-parameter \( A_p \) weight, are easily obtained using vector-valued versions of the corresponding one-parameter estimates. The required vector-valued estimates, on the other hand, follow simply by extrapolating the obvious weighted \( L^2(w) \) estimates.

When we integrate with respect to only one of the parameters we may e.g. write

\[
\langle f, h_{I_1} \rangle_{1}(x_2) := \int_{\mathbb{R}^{d_1}} f(x_1, x_2) h_{I_1}(x_1) \, dx_1.
\]

**Bi-parameter model operators.** As the bi-parameter CZOs are modelled after tensor products, we expect that their representation by model operators should involve generalisations of \( Q_{k_1} \otimes Q_{k_2} \) (a modified bi-parameter shift), of \( Q_{k_1} \otimes \pi \) and \( \pi \otimes Q_{k_2} \) (a modified partial paraproduct) and \( \pi \otimes \pi \) (a bi-parameter full paraproduct).

A modified bi-parameter shift \( Q_{k_1, k_2} \) (with respect to a grid \( D = D^1 \times D^2 \)) takes the form

\[
\langle Q_{k_1, k_2} f, g \rangle = \sum_{K_1, K_2} a_{I_1, I_2, K_1, K_2} \langle f, h_{I_1} \otimes h_{I_2} \rangle \langle g, H_{I_1, K_1} \otimes H_{I_2, K_2} \rangle
\]

or one of the three other possible forms, where \( h_{I_i} \) and \( H_{I_i, K_i} \) can be interchanged. Here the coefficients are assumed to satisfy

\[
|a_{I_1, I_2, K_1, K_2}| \leq \frac{|I_1|}{|K_1|} \frac{|I_2|}{|K_2|}
\]

and the functions \( H_{I_i, K_i} \) are like in the linear one-parameter situation.

**5.4. Proposition.** For every \( p \in (1, \infty) \) and bi-parameter \( A_p \) weight \( w \) we have

\[
\|Q_{k_1, k_2} f\|_{L^p(w)} \lesssim \sqrt{k_1 + 1} \sqrt{k_2 + 1} \|f\|_{L^p(w)}, \quad k_i \in \{0, 1, 2, \ldots \}.
\]

**Proof.** Assume that the shift is e.g. of the form

\[
\langle Q_{k_1, k_2} f, g \rangle = \sum_{K_1, K_2} a_{I_1, I_2, K_1, K_2} \langle f, h_{I_1} \otimes H_{I_2, K_2} \rangle \langle g, H_{I_1, K_1} \otimes h_{I_2} \rangle.
\]

We write for the moment that \( \varphi_{K_1, K_2} f := |\Delta_{K_1, k_1, 1}^1 P_{K_2, k_2}^2 f| \) and \( \phi_{K_1, K_2} g := |P_{K_1, k_1}^1 \Delta_{K_2, k_2}^2 g| \).

First, recalling (3.10) we estimate

\[
|\langle Q_{k_1, k_2} f, g \rangle| \leq \sum_{K_1, K_2} \sum_{I_1, I_2} \frac{|I_1|}{|K_1|} \frac{|I_2|}{|K_2|} \langle \varphi_{K_1, K_2} f, h_{I_1}^0 \otimes (h_{I_2}^0 + h_{I_2}^0) \rangle \langle \phi_{K_1, K_2} g, (h_{I_1}^0 + h_{I_1}^0) \otimes h_{I_2}^0 \rangle.
\]

This is split into four terms according to the sums inside the pairings. For brevity we explicitly demonstrate the estimate only with the term

\[
\sum_{K_1, K_2} \sum_{I_1, I_2} \frac{|I_1|}{|K_1|} \frac{|I_2|}{|K_2|} \langle \varphi_{K_1, K_2} f, h_{I_1}^0 \otimes h_{I_2}^0 \rangle \langle \phi_{K_1, K_2} g, h_{I_1}^0 \otimes h_{I_2}^0 \rangle.
\]
which can be written as
\[
\sum_{K_1, K_2} \langle E_{K_1, K_2}^1 E_{K_2, K_2}^2 \varphi_{K_1, K_2} f , \phi_{K_1, K_2} g \rangle.
\]
This is dominated by
\[
\left\| \left( \sum_{K_1, K_2} \langle E_{K_1, K_2}^1 E_{K_2, K_2}^2 \varphi_{K_1, K_2} f \rangle^2 \right)^{1/2} \right\|_{L^p(w)} \left\| \left( \sum_{K_1, K_2} \varphi_{K_1, K_2}^2 \right)^{1/2} \right\|_{L^{p'}(w^{1-p'})}.
\]
From here the proof is finished by using weighted Stein’s inequality and weighted square function estimates.

\[\Box\]

A modified partial paraproduct \(U_{k_1}\) with the paraproduct component on \(\mathbb{R}^{d_2}\) takes the form
\[
\langle U_{k_1} f, g \rangle = \sum_{I_1, I_2, J_1, J_2} a_{I_1, I_2, J_1, J_2} \langle f, h_{I_1} \otimes h_{I_2} \rangle \langle g, H_{I_1, J_1} \otimes \frac{1}{K_2} \rangle
\]
or one of the other three possible forms, where \(h_{I_1}\) and \(H_{I_1, J_1}\) can be interchanged and \(h_{I_2}\) and \(1_{K_2}/|K_2|\) can be interchanged. Here the coefficients are assumed to satisfy
\[
\| a_{I_1, J_1, K_1, K_2} \|_{BMO} \leq \frac{|I_1|}{|K_1|}
\]
for every fixed \(I_1, J_1, K_1\) satisfying \(I_1^{(k_1)} = J_1^{(k_1)} = K_1\) (see (3.7) for the definition of the BMO norm). A partial paraproduct with the paraproduct component on \(\mathbb{R}^{d_1}\) takes the symmetric form.

5.6. Proposition. For every \(p \in (1, \infty)\) and bi-parameter \(A_p\) weight \(w\) we have
\[
\| U_{k_1} f \|_{L^p(w)} \lesssim \sqrt{k_1 + 1} \| f \|_{L^p(w)}, \quad k_1 \in \{0, 1, 2, \ldots \}.
\]
Proof. We assume that \(U_{k_1}\) is of the form (5.5). The other forms are handled in the same way. From the one-parameter \(H^1\)-BMO duality
\[
\sum_I |a_I| b_I \lesssim \| (a_I) \|_{BMO} \left\| \left( \sum_I |b_I| \frac{1}{|I|} \right)^{1/2} \right\|_{L^1}
\]
we have that
\[
|\langle U_{k_1} f, g \rangle| \lesssim \sum_{K_1} \sum_{I_1^{(k_1)} = J_1^{(k_1)} = K_1} |I_1| \left\| f, h_{I_1} \otimes h_{K_2} \right\|_{L^p(w)} \left( \sum_{K_2} \langle g, H_{I_1, J_1} \otimes \frac{1}{K_2} \rangle \right)^{1/2} \frac{1}{|K_2|} \left( \sum_{K_2} \frac{1}{|K_2|} \right)^{1/2}
\]
\[
\lesssim \sum_{K_1} \sum_{I_1^{(k_1)} = J_1^{(k_1)} = K_1} |I_1| \left( \int_{\mathbb{R}^{d_2}} S_{D^2} f, h_{I_1} \right) M_{D^{2}}(g, H_{I_1, J_1}).
\]
Here \(M_{D^2}\) is the dyadic maximal function.

Notice that we have
\[
S_{D^2}(f, h_{I_1})_1 = S_{D^2}(\Delta_{K_1, k_1}^1 f, h_{I_1})_1 \leq \langle S_{D^2}^2 \Delta_{K_1, k_1}^1 f, h_{I_1}^0 \rangle_1.
\]
By (3.10) we have \((g, H_{I_1, J_1})_1 = (P_{K_1, k_1}^1 g, H_{I_1, J_1})_1\) and thus
\[
M_{D^2}(g, H_{I_1, J_1})_1 \leq \langle M_{D^2}^2 P_{K_1, k_1}^1 g, h_{I_1}^0 + h_{J_1}^0 \rangle_1 = \langle M_{D^2}^2 P_{K_1, k_1}^1 g, h_{I_1}^0 \rangle_1 + \langle M_{D^2}^2 P_{K_1, k_1}^1 g, h_{J_1}^0 \rangle_1.
\]
We split the main estimate into two according to this last sum. First, we consider the part

\[ \sum_{K_1} \int_{\mathbb{R}^d} \sum_{I_1(k_1) = J_1(k_1) = K_1} \left| I_1 \right| \langle S_{D_2}^2 \Delta_{K_1,k_1}^1 f, h_{I_1}^0 \rangle \langle M_{D_2}^2 P_{K_1,k_1}^1 g, h_{I_1}^0 \rangle. \tag{5.8} \]

Fix \( K_1 \) and notice that

\[ \sum_{I_1(k_1) = J_1(k_1) = K_1} \left| I_1 \right| \langle S_{D_2}^2 \Delta_{K_1,k_1}^1 f, h_{I_1}^0 \rangle \langle M_{D_2}^2 P_{K_1,k_1}^1 g, h_{I_1}^0 \rangle = \sum_{I_1(k_1) = K_1} \langle S_{D_2}^2 \Delta_{K_1,k_1}^1 f \rangle_{1,1} \langle M_{D_2}^2 P_{K_1,k_1}^1 g, h_{I_1}^0 \rangle = \langle E_{K_1,k_1}^1 S_{D_2}^2 \Delta_{K_1,k_1}^1 f, M_{D_2}^2 P_{K_1,k_1}^1 g \rangle. \]

Using the last identity gives that

\[ \sum_{K_1} \int_{\mathbb{R}^d} \left| I_1 \right| \langle S_{D_2}^2 \Delta_{K_1,k_1}^1 f \rangle \langle M_{D_2}^2 P_{K_1,k_1}^1 g \rangle \leq \left\| \left( \sum_{K_1} \left| I_1 \right| \langle S_{D_2}^2 \Delta_{K_1,k_1}^1 f \rangle \right)^2 \right\|^{1/2} \left\| \left( \sum_{K_1} \langle M_{D_2}^2 P_{K_1,k_1}^1 g \rangle \right)^2 \right\|^{1/2} \left\| L^p(w) \right\| \left\| L^{p'}(w^1 - p') \right\|. \tag{5.8} \]

From here the estimate can be concluded by using weighted versions of Stein’s inequality, square function estimates and Fefferman–Stein inequality. Notice that the estimate related to \( g \) produces the factor \( \sqrt{k_1 + 1} \).

Next, we consider the remaining part

\[ \sum_{K_1} \int_{\mathbb{R}^d} \sum_{I_1(k_1) = J_1(k_1) = K_1} \left| I_1 \right| \langle S_{D_2}^2 \Delta_{K_1,k_1}^1 f, h_{I_1}^0 \rangle \langle M_{D_2}^2 P_{K_1,k_1}^1 g, h_{I_1}^0 \rangle \] of the main estimate. This time there holds that

\[ \sum_{I_1(k_1) = J_1(k_1) = K_1} \left| I_1 \right| \langle S_{D_2}^2 \Delta_{K_1,k_1}^1 f, h_{I_1}^0 \rangle \langle M_{D_2}^2 P_{K_1,k_1}^1 g, h_{I_1}^0 \rangle = \langle S_{D_2}^2 \Delta_{K_1,k_1}^1 f \rangle_{1,1} \langle M_{D_2}^2 P_{K_1,k_1}^1 g, 1 \rangle. \]

This gives that

\[ \sum_{K_1} \int_{\mathbb{R}^d} \langle S_{D_2}^2 \Delta_{K_1,k_1}^1 f \rangle_{K_1,1} \langle M_{D_2}^2 P_{K_1,k_1}^1 g \rangle \leq \left\| \left( \sum_{K_1} 1_{K_1} \otimes \langle S_{D_2}^2 \Delta_{K_1,k_1}^1 f \rangle_{K_1,1}^2 \right)^{1/2} \right\| \left\| \left( \sum_{K_1} \langle M_{D_2}^2 P_{K_1,k_1}^1 g \rangle^2 \right)^{1/2} \right\| \left\| L^p(w) \right\| \left\| L^{p'}(w^1 - p') \right\|. \tag{5.9} \]

From here the estimate can be finished as above.

\[ \square \]

A full paraproduct \( \Pi \) takes the form

\[ \langle \Pi f, g \rangle = \sum_{K_1,K_2} a_{K_1,K_2} \langle f, h_{K_1} \otimes \frac{1_{K_1}}{|K_1|} \rangle \langle g, \frac{1_{K_1}}{|K_1|} \otimes h_{K_2} \rangle, \]
or one of the three other possible forms, where \( h_{K_i} \) and \( 1_{K_i}/|K_i| \) can be interchanged. The coefficients are assumed to satisfy

\[
\| (a_{K_1K_2}) \|_{\text{BMO}_{\prod}} = \sup_{\Omega} \left( \frac{1}{|\Omega|} \sum_{K_1 \times K_2 \subset \Omega} |a_{K_1K_2}|^2 \right)^{1/2},
\]

where the supremum is over open sets \( \Omega \subset \mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \) with \( 0 < |\Omega| < \infty \). It is entirely standard, as we will next explain, that \( \| f \|_{L^p(w)} \lesssim \| f \|_{L^p(w)} \) whenever \( p \in (1, \infty) \) and \( w \) is a bi-parameter \( A_p \) weight. While the product BMO is a very complicated space, it is not so hard to deal with in practice. This is due to the fact that the following bi-parameter \( H^1\)-BMO duality (a weighted version appears in Proposition 4.1 of [38])

\[
\sum_{K_1, K_2} |a_{K_1K_2}| |b_{K_1K_2}| \lesssim \| (a_{K_1K_2}) \|_{\text{BMO}_{\prod}} \left( \sum_{K_1, K_2} |b_{K_1K_2}|^2 \frac{1_{K_1 \times K_2}}{|K_1 \times K_2|} \right)^{1/2}
\]

holds – using this it is e.g. easy to see the weighted boundedness of the full paraproducts. See for instance [38].

**Comparison to the usual model operators.** A bi-parameter shift \( S_{i_1,j_1}^{i_2,j_2} \) takes the form

\[
\langle S_{i_1,j_1}^{i_2,j_2} f, g \rangle = \sum_{K_1, K_2} \sum_{I_{1i_1}^{j_1} = j_1^{i_1} = K_1} a_{I_1, I_2, J_2} \langle f, h_{I_1} \otimes h_{I_2} \rangle \langle g, h_{J_1} \otimes h_{J_2} \rangle,
\]

where

\[
|a_{I_1, I_2, J_2}| = \frac{|I_1|^{1/2} |J_1|^{1/2} |J_2|^{1/2} |J_2|^{1/2}}{|K_1|}.
\]

A partial paraproduct \( W_{i_1,j_1} \) with the paraproduct component on \( \mathbb{R}^{d_2} \) takes the form

\[
\langle W_{i_1,j_1} f, g \rangle = \sum_{K_1, K_2} \sum_{I_{1i_1}^{j_1} = j_1^{i_1} = K_1} a_{I_1, I_2, K_2} \langle f, h_{I_1} \otimes h_{J_2} \rangle \langle g, h_{J_1} \otimes \frac{1_{K_2}}{|K_2|} \rangle,
\]

or the symmetric form, where \( h_{K_2} \) and \( 1_{K_2}/|K_2| \) are interchanged. Here the coefficients are assumed to satisfy

\[
\| (a_{I_1, I_2, K_2}) \|_{\text{BMO}} \leq \frac{|I_1|^{1/2} |J_1|^{1/2}}{|K_1|}.
\]

Similarly as in the one-parameter case (Lemma 3.14), it is possible to represent the modified model operators as sums of the original operators. We omit the proof as the idea is the same as in the one-parameter case.

**5.11. Lemma.** Let the modified shift \( Q_{k_1,k_2} \) e.g. have the form (5.3). Then we have that

\[
\langle Q_{k_1,k_2} f, g \rangle = \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} \left( \langle S_{k_1-j_1,0,j_2}^{k_2-j_2,0} f, g \rangle + \langle S_{k_1-j_1,0}^{k_2-j_2} f, g \rangle + \langle S_{k_1-j_1,0}^{k_2-j_2,0} f, g \rangle + \langle S_{k_1-j_1}^{k_2-j_2} f, g \rangle \right).
\]

Let the modified partial paraproduct \( U_{k_1} \) e.g. have the form (5.5). Then we have that

\[
\langle U_{k_1} f, g \rangle = \sum_{j_1=0}^{k_1} \left( \langle W_{k_1-j_1,0} f, g \rangle + \langle W_{k_1,j_1} f, g \rangle \right).
\]
**Bi-parameter representation theorem.** We set
\[
\sigma = (\sigma_1, \sigma_2) \in (\{0, 1\}^{d_1})^Z \times (\{0, 1\}^{d_2})^Z, \quad \sigma_i = (\sigma_i^k)_{k \in \mathbb{Z}},
\]
and denote the expectation over the product probability space by
\[
\mathbb{E}_\sigma = \mathbb{E}_{\sigma_1} \mathbb{E}_{\sigma_2} = \mathbb{E}_{\sigma_2} \mathbb{E}_{\sigma_1} = \iint d\mathbb{P}_{\sigma_1} d\mathbb{P}_{\sigma_2}.
\]
We also set \( D_0 = D_0^1 \times D_0^2 \), where \( D_0^1 \) is the standard dyadic grid of \( \mathbb{R}^{d_1} \). As in the one-parameter case we use the notation
\[
I_i = \{ I_i^k \}_{k \in \mathbb{Z}} \in D_0^1 \quad \text{or} \quad I_i^d = \{ I_i^{d, k}_m \}_{k \in \mathbb{Z}}, \quad I_i^d \in D_0^2.
\]
Given \( \sigma = (\sigma_1, \sigma_2) \) and \( R = I_1 \times I_2 \in D_0 \) we set
\[
R + \sigma = (I_1 + \sigma_1) \times (I_2 + \sigma_2) \quad \text{and} \quad D_\sigma = \{ R + \sigma : R \in D_0 \} = D_{\sigma_1} \times D_{\sigma_2}.
\]

**5.12. Theorem.** Suppose that \( T \) is a bi-parameter \((\omega_1, \omega_2)\)-CZO, where \( \omega_i \in \text{Dini}_{1/2} \). Then we have
\[
\langle T f, g \rangle = C \mathbb{E}_\sigma \sum_{k = (k_1, k_2) \in \mathbb{N}^2} \sum_{u = 0}^{c_d} \omega_1(2^{-k_1}) \omega_2(2^{-k_2}) \langle V_{k, u, \sigma} f, g \rangle,
\]
where \( V_{k, u, \sigma} \) is always either a modified bi-parameter shift \( Q_{k_1, k_2} \), a modified partial paraproduct \( U_{k_1} \) or \( U_{k_2} \) (this requires \( k_1 = 0 \) or \( k_2 = 0 \)) or a full paraproduct (this requires \( k = 0 \)) in the grid \( D_\sigma \).

**5.13. Remark.** We do not write the dependence of the constant \( C \) on the various kernel and \( T_1 \) assumptions as explicitly as in the one-parameter case, but the dependence is analogous.

**Proof of Theorem 5.12.** We expand \( \langle T f, g \rangle \) in the form
\[
\langle T f, g \rangle = \mathbb{E}_\sigma \left( \sum_{\ell(I_1) < \ell(J_1)} + \sum_{\ell(I_1) > \ell(J_1)} + \sum_{\ell(I_1) = \ell(J_1)} \right) \left( \sum_{\ell(I_2) < \ell(J_2)} + \sum_{\ell(I_2) > \ell(J_2)} + \sum_{\ell(I_2) = \ell(J_2)} \right) \langle T \Delta_{I_1} \Delta_{I_2} f, \Delta_{I_1} \Delta_{I_2} g \rangle,
\]
where \( I_i, J_i \in D_{\sigma_i} \). We start dealing with one of the appearing terms
\[
\Sigma_{<, >, \sigma} := \sum_{\ell(I_1) < \ell(J_1)} \sum_{\ell(I_2) > \ell(J_2)} \langle T \Delta_{I_1} \Delta_{I_2} f, \Delta_{I_1} \Delta_{I_2} g \rangle = \sum_{\ell(I_1) = \ell(J_1)} \sum_{\ell(I_2) > \ell(J_2)} \langle T \Delta_{I_1} E_{I_2}^2 f, E_{I_1}^1 \Delta_{I_2} g \rangle = \sum_{m_1 \in \mathbb{Z}^{d_1}, I_1, I_2} \sum_{m_2 \in \mathbb{Z}^{d_2}} \langle T \Delta_{I_1} E_{I_2}^2 f, E_{I_1 + m_1} \Delta_{I_2} g \rangle.
\]
We further write \( \Sigma_{<,\sigma} = \Sigma'_{<,\sigma} + \Sigma''_{<,\sigma} + \Sigma'''_{<,\sigma} + \Sigma''''_{<,\sigma} \), where
\[
\Sigma'_{<,\sigma} := \sum_{m_1 \in \mathbb{Z}^d \setminus \{0\}} \sum_{l_1, l_2} \langle T(\langle \Delta_1, f \rangle l_2 - \langle \Delta_1, f \rangle l_2, 2) \otimes 1_{l_2 + m_2 \ell(l_2)},
\]
\[
1_{l_1 + m_1 \ell(l_1)} \otimes [\langle \Delta_2, g \rangle l_1 + m_1 \ell(l_1), 1 - \langle \Delta_2, g \rangle l_1, 1]\rangle,
\]
\[
\Sigma''_{<,\sigma} := \sum_{m_1 \in \mathbb{Z}^d \setminus \{0\}} \sum_{l_1, l_2} \langle T(\langle \Delta_1, f \rangle l_2, 2 \otimes 1_{\mathbb{R}^d_2}),
\]
\[
1_{l_1 + m_1 \ell(l_1)} \otimes [\langle \Delta_2, g \rangle l_1 + m_1 \ell(l_1), 1 - \langle \Delta_2, g \rangle l_1, 1]\rangle,
\]
\[
\Sigma'''_{<,\sigma} := \sum_{m_2 \in \mathbb{Z}^d \setminus \{0\}} \sum_{l_1, l_2} \langle T(\langle \Delta_1, f \rangle l_2 + m_2 \ell(l_2), 2 - \langle \Delta_1, f \rangle l_2, 2) \otimes 1_{l_2 + m_2 \ell(l_2)},
\]
\[
1_{\mathbb{R}^d_1} \otimes [\langle \Delta_2, g \rangle l_1, 1]\rangle,
\]
and
\[
\Sigma''''_{<,\sigma} := \sum_{l_1, l_2} \langle T(\langle \Delta_1, f \rangle l_2, 2 \otimes 1_{\mathbb{R}^d_2}), 1_{\mathbb{R}^d_1} \otimes [\langle \Delta_2, g \rangle l_1, 1]\rangle.
\]

Notice that
\[
\Sigma''''_{<,\sigma} = \sum_{l_1, l_2} \langle T^* 1_{h}, h, h_2 \rangle \langle f, h_1 \otimes \frac{1}{|l_2|} \rangle \langle g, \frac{1}{|l_1|} \otimes h_2 \rangle
\]
is a full paraproduct, so that we only have to deal with \( \Sigma'_{<,\sigma}, \Sigma''_{<,\sigma} \) and \( \Sigma'''_{<,\sigma} \).

We write \( E_{\sigma} \Sigma''_{<,\sigma} \) similarly as in the proof of Theorem 3.18 in the form
\[
2^{d_1} E_{\sigma} \sum_{k_1 = 2}^{\infty} \sum_{0 < |m_2| < 2^{k_1 - 3}} \sum_{l_2 \in D_{\mathbb{R}^d_2}} \frac{\langle T(h_1 \otimes 1_{\mathbb{R}^d_2}), h_1 + m_1 \ell(l_1) \otimes h_2 \rangle}{\omega_1(2^{-k_1}) |l_1|^{1/2}} \langle f, h_1 \otimes \frac{1}{|l_2|} \rangle \langle g, |h_1 + m_1 \ell(l_1) - h_1 | \otimes h_2 \rangle.
\]

As we know from the proof of Theorem 3.18 that here \( (I_1 + m_1 \ell(l_1))^{(k_1)} = I_1^{(k_1)} =: K_1 \), to realize that this is a modified full paraproduct we only need to show that
\[
\left| \langle T(h_1 \otimes 1_{\mathbb{R}^d_2}), h_1 + m_1 \ell(l_1) \otimes h_2 \rangle \right| \lesssim \omega_1(2^{-k_1}) |I_1|^{1/2} |I_2| / |K_1|
\]
with fixed \( I_1 \) and \( m_1 \) as above. For this, it is enough to fix a cube \( I_2 \subset \mathbb{R}^d_2 \) and show that
\[
|\langle T(h_1 \otimes 1_{\mathbb{R}^d_2}), h_1 + m_1 \ell(l_1) \otimes a_{l_2} \rangle| \lesssim \omega_1(2^{-k_1}) |I_1|^{1/2} |I_2| / |K_1|
\]
whenever \( a_{l_2} = 1_{I_2} a_{l_2} |a_{l_2}| < 1 \) and \( \int a_{l_2} = 0 \).

To prove (5.15) we first estimate
\[
|\langle T(h_1 \otimes 1_{\mathbb{R}^d_2}), h_1 + m_1 \ell(l_1) \otimes a_{l_2} \rangle| \leq |\langle T(h_1 \otimes 1_{C_{l_2}}), h_1 + m_1 \ell(l_1) \otimes a_{l_2} \rangle| + |\langle T(h_1 \otimes 1_{C_{l_2}}), h_1 + m_1 \ell(l_1) \otimes a_{l_2} \rangle| := B_1 + B_2.
\]
Suppose first that $k_1$ is large enough so that $|x_1 - c_{I_1}| \gtrsim |m_1\ell(I_1)|$ for the center $c_{I_1}$ of $I_1$ and $x_1 \in I_1 + m_1\ell(I_1)$. Using that $\int h_{I_1} = 0$ and $\int a_{I_2} = 0$ we write

$$B_2 = \left| \iint_{(I_1 + m_1\ell(I_1)) \times I_2} \iint_{I_1 \times (C I_2)^c} W(x, y) h_{I_1}(y_1) a_{I_2}(x_2) \, dy \, dx \right|,$$

where

$$W(x, y) = K(x, y) - K((x, c_{I_2}), y) - K((x, (c_{I_1}, y_2)) + K((x, c_{I_2}), (c_{I_1}, y_2)).$$

We have by the Hölder estimate of the full kernel that here

$$|W(x, y)| \lesssim \omega_1(2^{-k_1}) \frac{1}{|K_1|} \omega_2 \left( \frac{|x_2 - c_{I_2}|}{|y_2 - c_{I_2}|} \right) \frac{1}{|y_2 - c_{I_2}|^2}.$$

This gives that

$$|B_2| \lesssim \omega_1(2^{-k_1}) |I_1|^{1/2} \frac{|I_1|}{|K_1|} \int_{I_1} \int_{(C I_2)^c} \omega_2 \left( \frac{|x_2 - c_{I_2}|}{|y_2 - c_{I_2}|} \right) \frac{1}{|y_2 - c_{I_2}|^2} \, dy_2 \, dx_2$$

$$\lesssim \omega_1(2^{-k_1}) |I_1|^{1/2} \frac{|I_1|}{|K_1|} \int_{I_1} \sum_{k_2=1}^\infty \omega_2(2^{-k_2}) \, dx_2 \lesssim \omega_1(2^{-k_1}) |I_1|^{1/2} \frac{|I_1|}{|K_1|}|I_2|.$$

We now prove the same bound for $|B_1|$ in the case that $k_1$ is large enough:

$$|B_1| = \left| \int_{I_1 + m_1\ell(I_1)} \int_{I_1} [K_{C I_2} a_{I_2}(x_1, y_1) - K_{C I_2} a_{I_2}(x_1, c_{I_1})] h_{I_1}(y_1) \, dy_1 \, dx_1 \right|$$

$$\lesssim \omega_1(2^{-k_1}) |I_1|^{1/2} \frac{|I_1|}{|K_1|} C(1_{C I_2}, a_{I_2}) \lesssim \omega_1(2^{-k_1}) |I_1|^{1/2} \frac{|I_1|}{|K_1|}|I_2|.$$

Next, we assume that $k_1 \sim 1$ so that $|m_1| \sim 1$. This time we have by the mixed Hölder and size estimate of the full kernel $K$ that

$$B_2 = \left| \iint_{(I_1 + m_1\ell(I_1)) \times I_2} \iint_{I_1 \times (C I_2)^c} [K(x, y) - K((x, c_{I_2}), y)] h_{I_1}(y_1) a_{I_2}(x_2) \, dy \, dx \right|$$

$$\lesssim \left( |I_1|^{-1/2} \int_{C I_1 \setminus I_1} \int_{I_1} \frac{dy_1 \, dx_1}{|x_1 - y_1|^d} \right) \left( \int_{I_1} \int_{(C I_2)^c} \omega_2 \left( \frac{|x_2 - c_{I_2}|}{|y_2 - c_{I_2}|} \right) \frac{1}{|y_2 - c_{I_2}|^2} \, dy_2 \, dx_2 \right)$$

$$\lesssim |I_1|^{1/2} |I_2|,$$

which is the desired bound in this situation. For $B_1$ we have

$$|B_1| = \left| \int_{I_1 + m_1\ell(I_1)} \int_{I_1} K_{C I_2} a_{I_2}(x_1, y_1) h_{I_1}(y_1) \, dy_1 \, dx_1 \right|$$

$$\lesssim \left( |I_1|^{-1/2} \int_{C I_1 \setminus I_1} \int_{I_1} \frac{dy_1 \, dx_1}{|x_1 - y_1|^d} \right) C(1_{C I_2}, a_{I_2}) \lesssim |I_1|^{1/2} |I_2|.$$

We are done with the proof of (5.15), and thus done showing that

$$E_\sigma \Sigma''_{<,>,\sigma} = C E_\sigma \sum_{k_1=2}^\infty \omega_1(2^{-k_1}) \langle Q_{k_1}, \sigma f, g \rangle,$$

where $Q_{k_1,\sigma}$ is a modified partial paraproduct with the paraproduct component on $\mathbb{R}^{d_2}$. The term $\Sigma''_{<,>,\sigma}$ is symmetric. Moreover, with similar but slightly simpler arguments (as
there are no BMO considerations) we can show that
\[ E_p \sum_{\gamma} = C E_p \sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \omega_1(2^{-k_1})\omega_2(2^{-k_2}) \langle Q_{k_1,k_2}, f, g \rangle, \]
where \( Q_{k_1,k_2,\sigma} \) is a modified bi-parameter shift.

Thus, we have controlled one of the nine terms appearing in the decomposition (5.14). However, clearly the terms \( \Sigma_{<,\sigma}, \Sigma_{>,\sigma} \) and \( \Sigma_{>,<,\sigma} \) (with the obvious definitions) can be handled completely analogously. If there is an equality in one (or both) of the parameters, we resort to arguments as in the end of the proof of Theorem 3.18. We are done. \( \square \)

5.16. **Corollary.** Suppose that \( T \) is a bi-parameter \((\omega_1, \omega_2)\)-CZO, where \( \omega_i \in D_{n_i}/2 \). Then
\[ \|T f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)} \]
whenever \( p \in (1, \infty) \) and \( w \in A_p \) is a bi-parameter weight.

6. **Commutator estimates**

The basic form of a commutator is \([b, T] : f \mapsto bTf - T(bf)\). We are interested in various iterated versions in the multi-parameter setting and with mild kernel regularity.

For a bi-parameter weight \( w \in A_2(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) \) and a locally integrable function \( b \) we define the weighted product BMO norm
\[ \|b\|_{\text{BMO}_{\text{prod}}(w)} = \sup_{D} \sup_{\Omega} \left( \frac{1}{w(\Omega)} \sum_{R \subset \Omega} \frac{|\langle b, h_R \rangle|^2}{\langle w \rangle_R} \right)^{\frac{1}{2}}, \]
where the supremum is over all dyadic grids \( D \) on \( \mathbb{R}^{d_1} \) and \( D = D^1 \times D^2 \), and over all open sets \( \Omega \subset \mathbb{R}^{d} := \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \) for which \( 0 < w(\Omega) < \infty \). The following theorem, which is the two-weight Bloom version of [14], was proved in [47] with \( \omega_i(t) = t^{\alpha} \).

6.2. **Theorem.** Suppose that \( T_i \) is an \( \omega_i \)-CZO, where \( \omega_i \in D_{n_i}/2 \). Let \( b : \mathbb{R}^d \to \mathbb{C}, p \in (1, \infty), \mu, \lambda \in A_p(\mathbb{R}^d) \) be bi-parameter weights and \( \nu = \mu^{1/p}\lambda^{-1/p} \in A_2(\mathbb{R}^d) \) be the associated bi-parameter Bloom weight. Then we have
\[ \| [T_1, T_2, b] \|_{L^p(\mu) \to L^p(\lambda)} \lesssim \|b\|_{\text{BMO}_{\text{prod}}(\nu)}. \]

**Proof.** Let \( \|b\|_{\text{BMO}_{\text{prod}}(\nu)} = 1 \). By Theorem 3.18 we need to e.g. bound \( \|Q_{k_1}, Q_{k_2}, b\|_{L^p(\lambda)} \). It seems non-trivial to fully exploit the operators \( Q_k \) here and we content on using Lemma 3.14 to reduce to bounding
\[ \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} \| [S_{k_1,j_1}, S_{k_2,j_2}, b] f \|_{L^p(\lambda)} \]
and other similar terms. Reaching \( D_{n_i} \) would require replacing this step with a sharper estimate.

On page 11 of [47] it is recorded that
\[ \| [S_{u_1,v_1}, S_{u_2,v_2}, b] f \|_{L^p(\lambda)} \lesssim (1 + \max(u_1, v_1))(1 + \max(u_2, v_2)) \|f\|_{L^p(\mu)}. \]
Controlling commutators like \([b, S_{i,j}]\) is crucial. To get started, we define the one-parameter paraproducts (with some implicit hand side in this one-parameter situation. The equivalence follows from the weighted parameter situation

\[\|S_{u_1,v_1}, S_{u_2,v_2}, b\|_{L^p(\lambda)} \lesssim (1 + \max(u_1, v_1))^{1/2} (1 + \max(u_2, v_2))^{1/2}\|f\|_{L^p(\mu)}.\]

We will get back to this after completing the proof. Therefore, we have

\[\sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} \|S_{k_1,j_1}, S_{k_2,j_2}, b\|_{L^p(\lambda)} \lesssim (1 + k_1)^{3/2} (1 + k_2)^{3/2}\|f\|_{L^p(\mu)}.\]

Handling the other terms of the shift expansion of \([Q_{k_1}, [Q_{k_2}, b]]\) similarly, we get

\[\|Q_{k_1}, [Q_{k_2}, b]\|_{L^p(\lambda)} \lesssim (1 + k_1)^{3/2} (1 + k_2)^{3/2}\|f\|_{L^p(\mu)}.\]

Controlling commutators like \([Q_{k_1}, \pi, b]\) similarly we get the claim.

We return to (6.3) now. Decompositions are very involved in the bi-commutator case, and we prefer to give the idea of the improvement (6.3) by studying the simpler one-parameter situation \([b, S_{i,j}]\), where \(S_{i,j}\) is a one-parameter shift on \(\mathbb{R}^d\) and \(b \in \text{BMO}(\nu)\);

\[\|b\|_{\text{BMO}(\nu)} := \sup_{I \subset \mathbb{R}^d, \text{cube}} \frac{1}{\nu(I)} \int_I |b - \langle b \rangle_I| \sim \sup_{D} \sup_{I_0 \in D} \left( \frac{1}{\nu(I_0)} \sum_{I_0 \in D} \frac{|\langle b, h_I \rangle|^2}{\langle h_I \rangle^2} \right)^{1/2} < \infty.\]

Interestingly, this part of the argument can be improved: there actually holds that

\[\|b\|_{\text{BMO}(\nu)} \lesssim (1 + \max(I_0))^{1/2} (1 + \max(I))^{1/2}\|f\|_{L^p(\mu)}.\]

Here we only have use for the expression on the right-hand side, which is the analogue of the bi-parameter definition (6.1). However, it is customary to define things as on the left-hand side in this one-parameter situation. The equivalence follows from the weighted John-Nirenberg [52]

\[\sup_{I \subset \mathbb{R}^d, \text{cube}} \frac{1}{\nu(I)} \int_I |b - \langle b \rangle_I| \sim \sup_{I \subset \mathbb{R}^d, \text{cube}} \left( \frac{1}{\nu(I)} \int_I |b - \langle b \rangle_I|^2 \nu^{-1} \right)^{1/2}, \quad \nu \in A_2.\]

Of course, one-parameter commutators \([b,T]\) can be handled even with Dini, but e.g. sparse domination proofs [42, 43] are restricted to one-parameter, unlike these decompositions. To get started, we define the one-parameter paraproducts (with some implicit dyadic grid)

\[A_1(b,f) = \sum_I \Delta_I b \Delta_I f, \quad A_2(b,f) = \sum_I \Delta_I b E_I f, \quad A_3(b,f) = \sum_I E_I b \Delta_I f.\]

By writing \(b = \sum_I \Delta_I b\) and \(f = \sum_J \Delta_J f\), and collapsing sums such as \(1_I \sum_J I \subseteq J \Delta_J f = E_I f\), we formally have

\[bf = \sum_I \Delta_I b \Delta_I f + \sum_{I \subseteq J} \Delta_I b \Delta_J f + \sum_{J \not\subseteq I} \Delta_I b \Delta_J f = \sum_{k=1}^{3} A_k(b,f).\]

We now decompose the commutator as follows

\[[b, S_{i,j}]f = bS_{i,j}f - S_{i,j}(bf) = \sum_{k=1}^{2} A_k(b, S_{i,j}f) - \sum_{k=1}^{2} S_{i,j}(A_k(b,f)) + [A_3(b, S_{i,j}f) - S_{i,j}(A_3(b,f))].\]
We have the well-known fact that \( \|A_k(b, f)\|_{L^p(\lambda)} \lesssim \|b\|_{\text{BMO}(\nu)} \|f\|_{L^p(\mu)} \) for \( k = 1, 2 \) – this can be seen by using the weighted \( H^1\)-BMO duality \([57]\) (with \( a_1 = \langle b, h_1 \rangle \))

\[
(6.4) \quad \sum_I |a_I| |b_I| \lesssim \|(a_I)\|_{\text{BMO}(\nu)} \left( \sum_I |b_I|^2 \frac{1}{|I|} \right)^{1/2} \|f\|_{L^1(\nu)},
\]

where

\[
\|(a_I)\|_{\text{BMO}(\nu)} = \sup_{I_0 \in \mathcal{D}} \left( \frac{1}{\nu(I_0)} \sum_{I \subset I_0} |a_I|^2 \right)^{1/2}.
\]

Combining this with the well-known estimate \( \|S_{i,j} f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)} \) for all \( w \in A_p \) it follows that

\[
\left\| \sum_{k=1}^2 A_k(b, S_{i,j} f) - \sum_{k=1}^2 S_{i,j}(A_k(b, f)) \right\|_{L^p(\lambda)} \lesssim \|b\|_{\text{BMO}(\nu)} \|f\|_{L^p(\mu)}.
\]

The complexity dependence is coming from the remaining term

\[
A_3(b, S_{i,j} f) - S_{i,j}(A_3(b, f)) = \sum_K \sum_{f^{(i)} = f^{(j)} = K} \|b\|_{L^p} \langle b, h \rangle \langle f, h \rangle h_j.
\]

There are many ways to bound this, but the following way based on the \( H^1\)-BMO duality – and executed in the particular way that we do below – gives the best dependence that we are aware of:

\[
\|A_3(b, S_{i,j} f) - S_{i,j}(A_3(b, f))\|_{L^p(\lambda)} \lesssim (1 + \max(i, j))^{1/2} \|b\|_{\text{BMO}(\nu)} \|f\|_{L^p(\mu)}.
\]

We write

\[
\langle b\rangle_j - \langle b\rangle_I = [\langle b\rangle_j - \langle b\rangle_K] - [\langle b\rangle_I - \langle b\rangle_K],
\]

where we further write

\[
\langle b\rangle_j - \langle b\rangle_K = \sum_{J \subseteq L \subset K} \langle \Delta_L b \rangle_J = \sum_{J \subseteq L \subset K} \langle b, h_L \rangle \langle h_L \rangle_J,
\]

and similarly for \( \langle b\rangle_I - \langle b\rangle_K \). We dualize and e.g. look at

\[
\sum_K \sum_{f^{(i)} = f^{(j)} = K} \sum_{J \subseteq L \subset K} \|b, h_L\| \langle |h_L| \rangle_J |a_{iJK}| \|f, h_I\| \langle g, h_J \rangle |
\]

\[
= \sum_K \sum_{L \subset K} \sum_{\ell(L) > 2^{-j} \ell(K)} \|b, h_L\| |L|^{-1/2} \sum_{J \subseteq L \subset K} |a_{iJK}| |f, h_I\| |g, h_J\| |
\]

\[
\lesssim \|b\|_{\text{BMO}(\nu)} \sum_K \int \left( \sum_{L \subset K} \int_{\ell(L) > 2^{-j} \ell(K)} \frac{1}{|L|^2} \left[ \sum_{J \subseteq L \subset K} |a_{iJK}| |f, h_I\| |g, h_J\| \right]^2 \right)^{1/2} \nu,
\]

where we used the weighted \( H^1\)-BMO duality. Here

\[
\sum_{J \subseteq L \subset K} |a_{iJK}| |f, h_I\| |g, h_J\| \leq \frac{1}{|K|} \int_K |\Delta_{K,j} f| \int_L |\Delta_{K,j} g|,
\]
and we can bound
\[ \sum_K \int \left( \sum_{L \subseteq K} 1_L \langle |\Delta_{K,i}f| \rangle_{L^2}^2 \langle |\Delta_{K,j}g| \rangle_{L^2}^2 \right)^{1/2} \nu \]
\[ \leq j^{1/2} \sum_K \int (M \Delta_{K,i}f)(M \Delta_{K,j}g) \nu \]
\[ \leq j^{1/2} \left\| \left( \sum_K |M \Delta_{K,i}f|^2 \right)^{1/2} \right\|_{L^p(\mu)} \left\| \left( \sum_K |M \Delta_{K,j}g|^2 \right)^{1/2} \right\|_{L^{p'}(\lambda^{1-p'})} \]
\[ \lesssim j^{1/2} \|f\|_{L^p(\mu)} \|g\|_{L^{p'}(\lambda^{1-p'})}. \]

We are done with the one-parameter case – the desired bi-parameter case can now be done completely similarly by tweaking the proof in [47] using the above idea.

6.5. Remark. The previous way to use the $H^1$-BMO duality was to look at
\[ \sum_K \sum_{L \subseteq K} \| (b, h_L) \|_{L^1}^{-1/2} \sum_{J \subseteq L} \| a_{IJK} \| \langle f, h_I \rangle \| \langle g, h_J \rangle \|, \]
where $l = 0, \ldots, j - 1$ is fixed, and to apply the $H^1$-BMO duality to the whole $K, L$ summation. With $l$ fixed this yields a uniform estimate, and there is also a curious ‘extra’ cancellation present – we can even bound
\[ \sum_{I^{(i)}=J^{(j)}=K} \| a_{IJK} \| \langle f, h_I \rangle \| g, h_J \| \leq \frac{1}{|K|} \int_K |\Delta_{K,j}f| \int_L |g|, \]
that is, forget the $\Delta_{K,j}$ from $g$. Then it remains to sum over $l$ which yields the dependence $j$ instead of $j^{1/2}$. The way in our proof above is more efficient and we see that we utilize all of the cancellation as well.

6.6. Remark. An interesting question is can we have $\alpha = 1$ instead of $\alpha = 3/2$ by somehow more carefully exploiting the operators $Q_k$ – this would appear to be the optimal result theoretically obtainable by the current methods.

We also note that it is certainly possible to handle higher order commutators, such as, $[T_1, [T_2, [b, T_3]]]$.

We will continue with more multi-parameter commutator estimates – the difference to the above is that now even the singular integrals are allowed to be multi-parameter.

For a weight $w$ on $\mathbb{R}^d := \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ we say that a locally integrable function $b: \mathbb{R}^d \to \mathbb{C}$ belongs to the weighted little BMO space $\text{bmo}(w)$ if
\[ \|b\|_{\text{bmo}(w)} := \sup_{R} \frac{1}{w(R)} \int_{R} |b - \langle b \rangle_R| < \infty, \]
where the supremum is over rectangles $R = I_1 \times I_2 \subset \mathbb{R}^d$. If $w = 1$ we denote the unweighted little BMO space by $\text{bmo}$. There holds that
\[ \|b\|_{\text{bmo}(w)} \sim \max \left( \text{ess sup}_{x_1 \in \mathbb{R}^{d_1}} \|b(x_1, \cdot)\|_{\text{BMO}(w(x_1), \cdot)}, \text{ess sup}_{x_2 \in \mathbb{R}^{d_2}} \|b(\cdot, x_2)\|_{\text{BMO}(w(\cdot, x_2))} \right), \]
see [38]. Here BMO(w(x1, ·)) and BMO(w(·, x2)) are the one-parameter weighted BMO spaces. For example,
\[ \|b(x_1, \cdot)\|_{\text{BMO}(w(x_1, \cdot))} := \sup_{I_2} \frac{1}{w(x_1, \cdot)(I_2)} \int_{I_2} |b(x_1, y_2) - \langle b(x_1, \cdot) \rangle_{I_2}| \, dy_2, \]
where the supremum is over cubes \( I_2 \subset \mathbb{R}^d \).

We first record the following one-weight estimate, which works with the Dini_{1/2} assumption.

**6.8. Theorem.** Suppose that \( T \) is a bi-parameter \((\omega_1, \omega_2)\)-CZO, where \( \omega_i \in \text{Dini}_{1/2} \). Then for all \( p \in (1, \infty) \) and all bi-parameter \( A_p \) weights \( w \) we have
\[ \| [b_m, \cdots [b_2, [b_1, T]] \cdots] \|_{L^p(w) \to L^p(w)} \lesssim \prod_{j=1}^m \|b_j\|_{\text{BMO}}. \]

**Proof.** The claim follows from Corollary 5.16 by the well-known Cauchy trick [12] for commutators. Here it is key that Corollary 5.16 is a weighted estimate, even if we would just be interested in the unweighted estimate for the commutator. Equally important is that the BMO space \( \text{bmo} \) is a simple one here – it behaves like the one-parameter BMO.

We give the details for the reader’s convenience. Some tricks regarding the fact that here we have different functions \( b_1, \ldots, b_m \) (instead of \( b = b_1 = \cdots = b_m \)) are taken from [21]. For \( z = (z_1, \ldots, z_m) \in \mathbb{C}^m \) define the operator
\[ F(z)f = \exp \left( \sum_{j=1}^m b_j z_j \right) T \left( \exp \left( - \sum_{j=1}^m b_j z_j \right) f \right). \]

Next, we write
\[ [b_m, \cdots [b_2, [b_1, T]] \cdots] = \partial_{z_1} \cdots \partial_{z_m} F(0) = C \oint \cdots \oint F(z) \frac{dz_1 \cdots dz_m}{z_1^2 \cdots z_m^2}, \]
where each integral is over some closed path around the origin in the corresponding variable. Of course, here we used the Cauchy integral formula. It follows that, for any \( \delta_1, \ldots, \delta_m \), we have
\[
\| [b_m, \cdots [b_2, [b_1, T]] \cdots] \|_{L^p(w) \to L^p(w)} \\
\lesssim \oint_{|z_1| = \delta_1} \cdots \oint_{|z_m| = \delta_m} \| F(z) \|_{L^p(w) \to L^p(w)} \frac{|dz_1| \cdots |dz_m|}{|z_1|^2 \cdots |z_m|^2} \\
\lesssim \oint_{|z_1| = \delta_1} \cdots \oint_{|z_m| = \delta_m} \| T \|_{L^p(\exp(p \text{Re}(\sum_{j=1}^m b_j z_j))) w \to L^p(\exp(p \text{Re}(\sum_{j=1}^m b_j z_j)) w)} \frac{|dz_1| \cdots |dz_m|}{\delta_1^2 \cdots \delta_m^2} \\
\lesssim \oint_{|z_1| = \delta_1} \cdots \oint_{|z_m| = \delta_m} C \left( \exp \left( p \text{Re} \left( \sum_{j=1}^m b_j z_j \right) \right) w \right)_{A_p} \frac{|dz_1| \cdots |dz_m|}{\delta_1^2 \cdots \delta_m^2},
\]
where \( C \) is increasing and we used Corollary 5.16.

Now, it remains to choose the radii \( \delta_1, \ldots, \delta_m \) intelligently. This is based on the following standard fact. There is an \( \epsilon \) (depending only on \( p \) and \( d \)) so that
\[ (6.9) \quad [e^{\text{Re}(bz)} w]_{A_p} \leq C([w]_{A_p}) \]
whenever \( z \in \mathbb{C} \) satisfies\[ |z| \leq \frac{\epsilon}{(w)_{A_p} \|b\|_{\text{BMO}}}, \quad (w)_{A_p} := \max([w]_{A_p}, [w^{1-p'}]_{A_p}) = [w]_{A_p}^{\max(1,p'-1)}.
\]
This follows from Lemma 2.1 of [31], which is the above statement with the usual \( \text{BMO} \) and one-parameter weights \( w \), and the right-hand side of (6.9) can even be replaced with \( C_{p,d}[w]_{A_p} \). Indeed, simply notice that \[ [e^{\text{Re}(b z) w}]_{A_p} \leq \max \left( \text{ess sup}_{x_1 \in \mathbb{R}^d_1} [e^{\text{Re}(b(x_1, \cdot) z) w(x_1, \cdot)}]_{A_p(\mathbb{R}^d_2)}, \text{ess sup}_{x_2 \in \mathbb{R}^d_2} [e^{\text{Re}(b(\cdot, x_2) z) w(\cdot, x_2)}]_{A_p(\mathbb{R}^d_1)} \right)^\gamma, \]
where e.g. by Lemma 2.1 of [31] we have \[ [e^{\text{Re}(b(x_1, \cdot) z) w(x_1, \cdot)}]_{A_p(\mathbb{R}^d_2)} \leq [w(x_1, \cdot)]_{A_p(\mathbb{R}^d_2)} \leq [w]_{A_p} \]
whenever \[ |z| \leq \frac{\epsilon}{(w(x_1, \cdot))_{A_p(\mathbb{R}^d_2)} \|b(x_1, \cdot)\|_{\text{BMO}(\mathbb{R}^d_2)}}. \]
It remains to notice that\[ (w)_{A_p} \|b\|_{\text{BMO}} \geq (w(x_1, \cdot))_{A_p(\mathbb{R}^d_2)} \|b(x_1, \cdot)\|_{\text{BMO}(\mathbb{R}^d_2)}. \]
For a suitable \( \epsilon \) we then set\[ \delta_1 = \frac{\epsilon}{(w)_{A_p} \|b_1\|_{\text{BMO}}}. \]
For \( j \geq 2 \) we define recursively \[ \delta_j = \frac{\epsilon}{\sup_{|z_{j-1}| = \delta_{j-1}} \left( \left| \exp \left( p \text{Re} \left( \sum_{k=1}^{j-1} b_k z_k \right) \right) w \right|_{A_p} \right) \|b_j\|_{\text{BMO}}}. \]
By iterating (6.9) we see that \[ \left| \exp \left( p \text{Re} \left( \sum_{j=1}^{m} b_j z_j \right) \right) w \right|_{A_p} \lesssim C([w]_{A_p}) \]
if \( |z_j| = \delta_j \) for all \( j = 1, \ldots, m \). We now get that \[ \|\|b_m, \ldots [b_2, [b_1, T]] \cdots \|\|_{L^p(w) \to L^p(w)} \lesssim C([w]_{A_p}) \frac{1}{\delta_1 \cdots \delta_m} \lesssim C([w]_{A_p}) \prod_{j=1}^{m} \|b_j\|_{\text{BMO}}. \]

The following theorem was proved in [46] with \( \omega_j(t) = t^n \). The first order case \([b, T]\) appeared before in [38]. This two-weight Bloom case requires a proof based on the analysis of the commutators of model operators, and this requires a higher \( \alpha \) in the Dini, than what is required in Theorem 6.8, which is based on the Cauchy trick and Corollary 5.16. See also [47] for the optimality of the space \( \text{bmo}(\nu^{1/m}) \) in the case \( b_1 = \cdots = b_m = b \).

6.10. Theorem. Let \( p \in (1, \infty), \mu, \lambda \in A_p \) be bi-parameter weights and \( \nu := \mu^{1/p} \lambda^{-1/p} \). Suppose that \( T \) is a bi-parameter \((\omega_1, \omega_2)\)-\( \text{CZO} \) and \( m \in \mathbb{N} \). Then we have\[ \|\|b_m, \ldots [b_2, [b_1, T]] \cdots \|\|_{L^p(\mu) \to L^p(\lambda)} \lesssim \prod_{i=1}^{m} \|b_i\|_{\text{bmo}(\nu^{1/m})}. \]

if one of the following conditions holds:

1. $T$ is paraproduct free and $\omega_i \in \text{Dini}_{m/2+1}$;
2. $m = 1$ and $\omega_i \in \text{Dini}_{3/2}$;
3. $\omega_i \in \text{Dini}_{m+1}$.

Proof. The proof is similar in spirit to that of Theorem 6.2. We use Lemma 5.11 and estimates for the commutators of the usual bi-parameter model operators. If we use the bounds from [46] directly, we e.g. immediately get

$$
\| [b_m, \cdots [b_2,[b_1, Q_{k_1,k_2}]] \cdots ] \|_{L^p(\mu) \rightarrow L^p(\lambda)} 
\lesssim (1 + k_1)(1 + k_2)(1 + \max(k_1, k_2)) \prod_{i=1}^m \| b_i \|_{bmo(\nu^{1/m})}.
$$

Similarly, we can read an estimate for all the other model operators from [46]. This gives us the result under the higher regularity assumption (3). Indeed, when using the estimate (6.11) in connection with the representation theorem one ends up with the series

$$
\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \omega_1(2^{-k_1}) \omega_2(2^{-k_2})(1 + k_1)(1 + k_2)(1 + \max(k_1, k_2))^m.
$$

We split this into two according to whether $k_1 \leq k_2$ or $k_1 > k_2$ and, for example, there holds that

$$
\sum_{k_1=0}^{\infty} \omega_1(2^{-k_1})(1 + k_1) \sum_{k_2=k_1}^{\infty} \omega_2(2^{-k_2})(1 + k_2)^m \lesssim \sum_{k_1=0}^{\infty} \omega_1(2^{-k_1})(1 + k_1) \| \omega_2 \|_{\text{Dini}_{m+1}}
\lesssim \| \omega_1 \|_{\text{Dini}_{1}} \| \omega_2 \|_{\text{Dini}_{m+1}}.
$$

The first order case $m = 1$ with the desired regularity (assumption (2)) follows as the papers [1, 2, 38] dealing with commutators of the form $[T_1, [T_2, \ldots [b, T_k]]\ldots]$, where each $T_k$ can be multi-parameter, include the proof of the first order case with the $H^1$-BMO duality strategy. And this strategy can be improved to give the additional square root save as in Theorem 6.2.

For $m \geq 2$ the new square root save becomes tricky. The paper [46] is not at all based on the $H^1$-BMO duality strategy on which this save is based on (see the proof of Theorem 6.2). We can improve the strategy of [46] for shifts. Thus, we are able to make the square root save for paraproduct free $T$ (assumption (1)). By this we mean that (both partial and full) paraproducts in the dyadic representation of $T$ vanish, which could also be stated in terms of (both partial and full) “$T1 = 0$” type conditions. The reader can think of convolution form SIOs.

We start considering $[b_2, [b_1, S^{j_2,j_1}]]$. The reductions in pages 23 and 24 of [46] (Section 5.1) give that we only need to bound the key term

$$
\langle U^{b_1,b_2} f, g \rangle := \sum_{K_1,K_2} \sum_{I_{1}^{(1)} = J_{1}^{(1)} = K_1} a_{J_1,K_1,I_1,J_2,K_2} \langle (b_1)_1 \times (b_1)_2 - \langle b_1 \rangle_{I_1} \times (b_1)_2 \rangle \langle (b_2)_1 \times (b_2)_2 - \langle b_2 \rangle_{I_1} \times (b_2)_2 \rangle \langle f, h_{J_1} \otimes h_{J_2} \rangle \langle g, h_{J_1} \otimes h_{J_2} \rangle.
$$
We write
\[
b_1K_1 \times L_2 - \langle b_1 \rangle K_1 \times K_2 = \sum_{I_2 \subseteq L_2 \subseteq K_2} \langle \Delta L_2 \langle b_1 \rangle K_1, I_2 \rangle \langle hL_2, I_2 \rangle.
\]
(6.12)
and
\[
b_2K_1 \times L_2 - \langle b_2 \rangle K_1 \times K_2 = \sum_{I_1 \subseteq L_1 \subseteq K_1} \langle \Delta L_1 \langle b_2 \rangle L_2, I_1 \rangle = \sum_{I_1 \subseteq L_1 \subseteq K_1} \langle b_2, hL_1 \otimes \frac{1}{I_2} \rangle \langle hL_1, I_1 \rangle.
\]

Writing \( \langle b_1, \frac{1}{K_1} \otimes hL_2 \rangle = \int_{\mathbb{R}^d} \langle b_1, hL_2 \rangle \frac{1}{|K_1|} \) and similarly for \( \langle b_2, hL_1 \otimes \frac{1}{|I_2|} \rangle \) we arrive at
\[
\int_{\mathbb{R}^d} \sum_{K_1, K_2} \sum_{L_1 \subseteq K_1} \sum_{L_2 \subseteq K_2} |\langle b_1, hL_2 \rangle| \langle L_2 \rangle^{-1/2} |\langle b_2, hL_1 \rangle| |L_1|^{-1/2}
\]
\[
\sum_{I_1 \subseteq L_1 \subseteq K_1} \sum_{I_2 \subseteq L_2 \subseteq K_2} a_{I_1, I_2} \langle b_1, hL_1 \otimes hL_2 \rangle \langle f, hL_2 \rangle^-\langle g, hL_1 \otimes hL_2 \rangle |\frac{1}{K_1}| \frac{1}{I_2} \langle hL_1 \rangle \langle hL_2 \rangle.
\]
The last line can be dominated by
\[
|L_1| \langle M^2 \Delta^1_{K_1,j_1} \Delta^2_{K_2,j_2} f \rangle_{L_1,1} \langle \Delta^1_{K_1,j_1} \Delta^2_{K_2,j_2} g \rangle_{K_1 \times K_2} \frac{1}{|K_1|} \frac{1}{I_2} \langle hL_2 \rangle.
\]
We have now reached the term
\[
\int_{\mathbb{R}^d} \sum_{K_1, K_2} \langle \Delta^1_{K_1,j_1} \Delta^2_{K_2,j_2} g \rangle_{K_1 \times K_2} \frac{1}{|K_1|} \sum_{L_2 \subseteq K_2} \langle b_1, hL_2 \rangle \langle L_2 \rangle^{-1/2} \langle hL_1 \rangle \langle hL_2 \rangle
\]
\[
\sum_{L_1 \subseteq K_1} \langle b_2, hL_1 \rangle |\langle L_1 \rangle|^{1/2} \langle M^2 \Delta^1_{K_1,j_1} \Delta^2_{K_2,j_2} f \rangle_{L_1,1}.
\]
Recall that with fixed $x_2$ we have $b(\cdot, x_2) \in \text{BMO}(\nu^{1/2}(\cdot, x_2))$, see (6.7). By weighted $H^1$-BMO duality we now have that

$$
\sum_{L_1 \subset K_1 \atop \ell(L_1) > 2^{-1} \ell(K_1)} |\langle b_2, h_{L_1}(x_2) \rangle| L_1^{1/2} \langle M^2 \Delta_{K_{1,i_1}}^1 \Delta_{K_{2,i_2}}^2 f \rangle_{L_1,1}(x_2)
$$

\[ \lesssim \|b_2\|_{\text{bmo}(\nu^{1/2})} \int_{\mathbb{R}^d} \left( \sum_{L_1 \subset K_1 \atop \ell(L_1) > 2^{-1} \ell(K_1)} 1_{L_1}(y) \langle \langle M^2 \Delta_{K_{1,i_1}}^1 \Delta_{K_{2,i_2}}^2 f \rangle_{L_1,1}(x_2) \rangle^2 \right)^{1/2} \nu^{1/2}(y, x_2) \, dy_1
\]

\[ \leq i_1^{1/2} \|b_2\|_{\text{bmo}(\nu^{1/2})} |K_1| \langle M^1 M^2 \Delta_{K_{1,i_1}}^1 \Delta_{K_{2,i_2}}^2 f \cdot \nu^{1/2} \rangle_{K_1,1}(x_2).
\]

The term $i_1^{1/2} \|b_2\|_{\text{bmo}(\nu^{1/2})}$ is fine and we do not drag it along in the following estimates. We are left with the task of bounding

\[ \int_{\mathbb{R}^d} \sum_{K_{1,i_1}, K_{2,i_2}} \langle \Delta_{K_{1,i_1}}^1 \Delta_{K_{2,i_2}}^2 g \rangle_{K_{1,i_1} \times K_{2,i_2}} \sum_{L_2 \subset K_2 \atop \ell(L_2) > 2^{-1} \ell(K_2)} \langle \langle b_1, h_{L_2} \rangle \rangle_{L_2} \langle M^1 M^2 \Delta_{K_{1,i_1}}^1 \Delta_{K_{2,i_2}}^2 f \cdot \nu^{1/2} \rangle_{L_2,2}.
\]

We now put the $\int_{\mathbb{R}^d}$ inside and get the term

\[ \int_{\mathbb{R}^d} 1_{L_2} M^1 \langle M^1 M^2 \Delta_{K_{1,i_1}}^1 \Delta_{K_{2,i_2}}^2 f \cdot \nu^{1/2} \rangle = |L_2| \langle M^1 \langle M^1 M^2 \Delta_{K_{1,i_1}}^1 \Delta_{K_{2,i_2}}^2 f \cdot \nu^{1/2} \rangle \rangle_{L_2,2}.
\]

Then, we are left with

\[ \int_{\mathbb{R}^d} \sum_{K_{1,i_1}, K_{2,i_2}} \langle \Delta_{K_{1,i_1}}^1 \Delta_{K_{2,i_2}}^2 g \rangle_{K_{1,i_1} \times K_{2,i_2}} \sum_{L_2 \subset K_2 \atop \ell(L_2) > 2^{-1} \ell(K_2)} \langle \langle b_1, h_{L_2} \rangle \rangle_{L_2} \langle M^1 \langle M^1 M^2 \Delta_{K_{1,i_1}}^1 \Delta_{K_{2,i_2}}^2 f \cdot \nu^{1/2} \rangle \rangle_{L_2,2}.
\]

By weighted $H^1$-BMO duality we have analogously as above that

\[ \sum_{L_2 \subset K_2 \atop \ell(L_2) > 2^{-1} \ell(K_2)} \langle \langle b_1, h_{L_2} \rangle \rangle_{L_2} \langle M^1 M^2 \Delta_{K_{1,i_1}}^1 \Delta_{K_{2,i_2}}^2 f \cdot \nu^{1/2} \rangle_{L_2,2} \]

\[ \lesssim i_2^{1/2} \|b_1\|_{\text{bmo}(\nu^{1/2})} \int_{\mathbb{R}^d} 1_{K_2} M^2 M^1 \langle M^1 M^2 \Delta_{K_{1,i_1}}^1 \Delta_{K_{2,i_2}}^2 f \cdot \nu^{1/2} \rangle_{L_2,2} \nu^{1/2}.
\]

Forgetting the factor $i_2^{1/2} \|b_1\|_{\text{bmo}(\nu^{1/2})}$, which is as desired, we are then left with

\[ \int_{\mathbb{R}^d} \sum_{K_{1,i_1}, K_{2,i_2}} \langle \Delta_{K_{1,i_1}}^1 \Delta_{K_{2,i_2}}^2 g \rangle_{K_{1,i_1} \times K_{2,i_2}} 1_{K_1} 1_{K_2} M^2 M^1 \langle M^1 M^2 \Delta_{K_{1,i_1}}^1 \Delta_{K_{2,i_2}}^2 f \cdot \nu^{1/2} \rangle_{L_2,2} \nu^{1/2}
\]

\[ \leq \int_{\mathbb{R}^d} \sum_{K_{1,i_1}, K_{2,i_2}} M^2 M^1 \langle M^1 M^2 \Delta_{K_{1,i_1}}^1 \Delta_{K_{2,i_2}}^2 f \cdot \nu^{1/2} \rangle \cdot M^1 M^2 \Delta_{K_{1,i_1}}^1 \Delta_{K_{2,i_2}}^2 g \cdot \nu^{1/2}.
\]

Writing $\nu^2 = \frac{1}{\beta} \lambda^\frac{1}{2} \cdot \lambda^\frac{1}{2}$ we bound this with

\[ \left\| \left( \sum_{K_{1,i_1}, K_{2,i_2}} [M^2 M^1 \langle M^1 M^2 \Delta_{K_{1,i_1}}^1 \Delta_{K_{2,i_2}}^2 f \cdot \nu^{1/2} \rangle]^2 \right)^{1/2} \right\|_{L^p(\mu^{1/2}, \lambda^{1/2})}.
\]
multiplied by

\[
\left\| \left( \sum_{K_1, K_2} |M^1 M^2 \Delta_{K_1, j_1}^1 \Delta_{K_2, j_2}^2 g| \right)^2 \right\|_{L^{p'}(\lambda^{1 - p'})}^{1/2}.
\]

It remains to use square function bounds together with the Fefferman–Stein inequality. For the more complicated term with the function \(f\) the key thing to notice is that first \(\mu^{1/2} \lambda^{1/2} \in A_p\) and then that \(\nu^{p/2} \mu^{1/2} \lambda^{1/2} = \mu\). We have controlled \(\langle U_{3,4}^{b_1, b_2} f, g \rangle\).

The bound for \(\langle U_{1,1}^{b_1, b_2} f, g \rangle\) follows by handling the other similar terms \(U_{m_1, m_2}^{b_1, b_2}\). There is a slight variation in the argument needed, for example, in the following term

\[
\langle U_{1,1}^{b_1, b_2} f, g \rangle := \sum_{K_1, K_2} \sum_{I_1^{(1)} = I_1^{(1)} = K_1} a_{I_1, I_2, J_1, J_2} \left[ \langle b_1 \rangle_{J_1, J_2} - \langle b_1 \rangle_{K_1} \right]
\]

\[
\left[ \langle b_2 \rangle_{J_1, J_2} - \langle b_2 \rangle_{K_1} \right] \langle f, h_{I_1} \otimes h_{J_2} \rangle \langle g, h_{J_1} \otimes h_{J_2} \rangle.
\]

We expand the differences of averages as

\[
\left[ \langle b_1 \rangle_{J_1, J_2} - \langle b_1 \rangle_{K_1} \right] \left[ \langle b_2 \rangle_{J_1, J_2} - \langle b_2 \rangle_{K_1} \right] = \sum_{J_1 \subseteq U_1 \subseteq K_1} \sum_{J_2 \subseteq V_1} \left\langle b_1, h_{U_1} \otimes \frac{1}{J_2} \right\rangle \left\langle b_2, h_{V_1} \otimes \frac{1}{J_2} \right\rangle \left\langle h_{V_1}, J_1 \right\rangle.
\]

The key difference to the above term \(U_{3,4}^{b_1, b_2}\) is that we need to further split this into two by comparing whether we have \(V_1 \subset U_1\) or \(U_1 \subset V_1\). The related two terms are handled symmetrically. The absolute value of the one coming from “\(V_1 \subset U_1\)” can be written as

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sum_{K_1, K_2} \sum_{U_1 \subseteq K_1} \sum_{V_1 \subseteq U_1} \sum_{J_1 \subseteq V_1} \sum_{I_1^{(1)} \in (2^{-2} K_1) \cap (I_1 \cap V_1)} \sum_{J_2 \subseteq V_1} \sum_{I_2^{(2)} = I_2 \subseteq V_1} \left| a_{I_1, I_2, J_1, J_2} \right| \left\langle f, h_{I_1} \otimes h_{J_2} \right\rangle \left\langle g, h_{J_1} \otimes h_{J_2} \right\rangle \frac{1}{J_2} \frac{1}{J_2}.
\]

The last line can be dominated by

\[
\left\langle |\Delta_{K_1, j_1}^1 \Delta_{K_2, j_2}^2 g| \right\rangle_{V_1 \times J_2} \sum_{J_2} \left\langle |\Delta_{K_1, j_1}^1 \Delta_{K_2, j_2}^2 g| \right\rangle_{V_1 \times J_2} \frac{1}{J_2} \frac{1}{J_2}.
\]

Using the weighted \(H^1\)-BMO duality as above we have

\[
\int_{\mathbb{R}^d} \left| \langle b_2, h_{V_1} \rangle_{j_2} \right| \left| V_1 \right|^{1/2} \left| \Delta_{K_1, j_1}^1 \Delta_{K_2, j_2}^2 g \right|_{V_1 \times J_2} \frac{1}{J_2} \frac{1}{J_2} dy_2
\]

\[
\leq J_1^{1/2} \left\| b \right\|_{\text{bmo}(\nu^{1/2})} \left| V_1 \right| \frac{1}{J_2} \frac{1}{J_2} \left( M^1 M^2 \Delta_{K_1, j_1}^1 \Delta_{K_2, j_2}^2 g \cdot \nu^{1/2} \right)_{V_1 \times J_2}.
\]
Forgetting the factor \( j_1^{1/2} \| b_2 \|_{bmo(\nu^{1/2})} \) we have reached the term
\[
\int_{\mathbb{R}^d} \sum_{K_1,K_2} \langle \Delta^1_{K_1,j_1} \Delta^2_{K_2,j_2} f \rangle |K_1 \times K_2 | \sum_{J \subseteq K_1} \sum_{U \subseteq K_2} |\langle b_1, h_{U_1} \rangle_1| |U_1|^{1/2} \langle M^1 M^2 \Delta^1_{K_1,j_1} \Delta^2_{K_2,j_2} g \cdot \nu^{1/2} \rangle_{U_1 \times J_2},
\]
which – after using the \( H^1 \)-BMO duality – produces \( j_1^{1/2} \| b_1 \|_{bmo(\nu^{1/2})} \) multiplied by
\[
\int_{\mathbb{R}^d} \sum_{K_1,K_2} \langle \Delta^1_{K_1,j_1} \Delta^2_{K_2,j_2} f \rangle |K_1 \times K_2 | M^1 M^2 (M^1 M^2 \Delta^1_{K_1,j_1} \Delta^2_{K_2,j_2} g \cdot \nu^{1/2})_{U_1 \times J_2}.
\]

Similarly as with \( T_{3,4}^{b_1,b_2} \), this term is under control. The term with \( U_1 \subseteq V_1 \) is symmetric, and so we are also done with \( U_{1,1}^{b_1,b_2} \).

This ends our treatment of \( U_{1,1}^{b_1,b_2} \), since the above arguments showcased the only major difference between the various terms \( U_{m_1,m_2}^{b_1,b_2} \). Thus, we are done with \( [b_2, [b_1, S_{1,1,j_2}^{b_1}]] \). By Lemma 5.11 we conclude that
\[
\| [b_2, [b_1, Q_{k_1,k_2}]] \|_{L^p(\mu) \to L^p(\lambda)} \lesssim (1 + k_1)(1 + k_2)(1 + \max(k_1, k_2)) \prod_{i=1}^2 \| b_i \|_{bmo(\nu^{1/2})}.
\]
By handling the higher order commutators similarly, we get the claim related to assumption (1). We omit these details.

\[ \square \]

6.13. Remark. The new square root save from the \( H^1 \)-BMO arguments reduces the required regularity from \( m + 1 \) to \( m/2 + 1 \). In these higher order commutators this is more significant than the save that could theoretically be obtained by not using Lemma 5.11. This could change the \(+1\) to \(+1/2\).

Theorem 6.2 involves only one-parameter CZOs in its estimate
\[
\| [T_1, [T_2, b]] \|_{L^p(\mu) \to L^p(\lambda)} \lesssim \| b \|_{BMO_{prod}(\nu)},
\]
while the basic estimate
\[
\| [b, T] \|_{L^p(\mu) \to L^p(\lambda)} \lesssim \| b \|_{bmo(\nu)}
\]
of Theorem 6.10 involves a bi-parameter CZO \( T \). A joint generalization – considered in the unweighted case in [55] – is an estimate for
\[
\| [T_1, [T_2, \ldots, [b, T_k]]] \|_{L^p(\mu) \to L^p(\lambda)},
\]
where each \( T_i \) can be a completely general \( m \)-parameter CZO. Then the appearing BMO norm is some suitable combination of little BMO and product BMO. See [1, 2] for a fully satisfactory Bloom type upper estimate in this generality – however, only for CZOs with the standard kernel regularity. The general case of [1, 2] is hard to digest, but let us formulate a model theorem of this type with mild kernel regularity.

6.14. Theorem. Let \( \mathbb{R}^d = \prod_{i=1}^4 \mathbb{R}^{d_i} \) be a product space of four parameters and let \( \mathcal{I} = \{I_1, I_2\} \), where \( I_1 = \{1, 2\} \) and \( I_2 = \{3, 4\} \), be a partition of the parameter space \( \{1, 2, 3, 4\} \). Suppose that \( T_i \) is a bi-parameter \((\omega_{1,i}, \omega_{2,i})\)-CZO on \( \prod_{j \in I_i} \mathbb{R}^{d_j} \), where \( \omega_{j,i} \in \text{Dini}_{3/2} \). Let \( b : \mathbb{R}^d \to \mathbb{C}, \)
Let $p \in (1, \infty)$, $\mu, \lambda \in A_p(\mathbb{R}^d)$ be $4$-parameter weights and $\nu = \mu^{1/p} \lambda^{-1/p}$ be the associated Bloom weight. Then we have

$$\| [T_1, [T_2, b]] \|_{L^p(\mu) \to L^p(\lambda)} \lesssim \| b \|_{\text{bmo}^T(\nu)}.$$ 

Here $\text{bmo}^T(\nu)$ is the following weighted little product BMO space:

$$\| b \|_{\text{bmo}^T(\nu)} = \sup_{\bar{u}} \| b \|_{\text{BMO}^\prod(\nu)},$$

where $\bar{u} = (u_i)_{i=1}^4$ is such that $u_i \in I_i$ and $\text{BMO}^\prod(\nu)$ is the natural weighted bi-parameter BMO space on the parameters $\bar{u}$. For example,

$$\| b \|_{\text{BMO}^{(1,3)}(\nu)} := \sup_{x_2 \in \mathbb{R}^d, x_4 \in \mathbb{R}^d} \| b(\cdot, x_2, \cdot, x_4) \|_{\text{BMO}^\prod(\nu)}.$$

where the last weighted product BMO norm is defined in (6.1).

The proof is again a combination of Lemma 5.11 with the known estimates for the commutators of standard model operators [1, 2]. However, there is again the additional square root save. There are no new significant challenges with this, which was not the case with Theorem 6.10 above, since these references are completely based on the $H^1$-BMO strategy. In this regard the situation is closer to that of Theorem 6.2.

REFERENCES

[1] E. Airta, Two-weight commutator estimates: general multi-parameter framework, Publ. Mat. 64 (2020) 681–729.
[2] E. Airta, K. Li, H. Martikainen, E. Vuorinen, Some new weighted estimates on product spaces, Indiana Univ. Math. J., to appear, arXiv:1910.12546, 2019.
[3] A. Barron, J. M. Conde-Alonso, G. Rey, Y. Ou, Sparse domination and the strong maximal function, Adv. Math. 345 (2019) 1–26.
[4] A. Barron, J. Pipher, Sparse domination for bi-parameter operators using square functions, preprint, arXiv:1709.05009, 2017.
[5] J. Bourgain, Some remarks on Banach spaces in which martingale difference sequences are unconditional, Ark. Mat. 21 (1983) 163–168.
[6] D. L. Burkholder, A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions. In Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981), Wadsworth Math. Ser., pages 270–286. Wadsworth, Belmont, CA, 1983.
[7] A. Culiuc, F. Di Plinio, Y. Ou, Domination of multilinear singular integrals by positive sparse forms, J. Lond. Math. Soc. 98 (2) (2018) 369–392.
[8] S.-Y. A. Chang, R. Fefferman, A continuous version of duality of $H^1$ with BMO on the Bidisc, Ann. of Math. 112 (1980) 179–201.
[9] S.-Y. A. Chang, R. Fefferman, Some recent developments in Fourier analysis and $H^p$ theory on product domains, Bull. Amer. Math. Soc. 12 (1985) 1–43.
[10] R. Coifman, Y. Meyer, Au delà des opérateurs pseudo-différentiels, Astérisque 57 (1978) 1–185.
[11] D. Cruz-Uribe, J. M Martell, C. Pérez, Sharp weighted estimates for classical operators, Adv. Math. 229 (2012) 408–441.
[12] R. Coifman, R. Rochberg, G. Weiss, Factorization theorems for Hardy spaces in several variables, Ann. of Math. (2) 103 (1976) 611–635.
[13] S.-Y. A. Chang, J. M. Wilson, T. H. Wolff, Some weighted norm inequalities concerning the Schrödinger operators, Comment. Math. Helv. 60 (1985) 217–246.
[14] L. Dalenc, Y. Ou, Upper bound for multi-parameter iterated commutators Publ. Mat. 60 (2016) 191–220.
[15] G. David, J.-L. Journé, A boundedness criterion for generalized Calderón-Zygmund operators, Ann. of Math. 120 (1984) 371–397.
[16] D. Deng, L. Yan, Q. Yang, Blocking analysis and $T(1)$ theorem, Sci. China Ser. A 41 (1998) 801–808.
[17] F. Di Plinio, K. Li, H. Martikainen, E. Vuorinen, Multilinear singular integrals on non-commutative $L^p$ spaces, preprint, arXiv:1905.02139, 2019.

[18] F. Di Plinio, K. Li, H. Martikainen, E. Vuorinen, Multilinear operator-valued Calderón-Zygmund theory, J. Funct. Anal. 279 (8) 108666 (2020).

[19] F. Di Plinio, K. Li, H. Martikainen, E. Vuorinen, Banach-valued multilinear singular integrals with modulation invariance, preprint, arXiv:1909.07236, 2019.

[20] F. Di Plinio, Y. Ou, Banach-valued multilinear singular integrals, Indiana Univ. Math. J. 67 (2018) 1711–1763.

[21] X.T. Duong, J. Li, Y. Ou, J. Pipher, B. Wick, Commutators of multi-parameter flag singular integrals and applications, Anal. PDE 12 (2019) 1325–1355.

[22] L. Grafakos, J.M. Martell, Extrapolation of weighted norm inequalities for multivariable operators and applications, J. Geom. Anal. 14 (2004) 19–46.

[23] L. Grafakos, S. Oh, The Kato-Ponce inequality, Comm. Partial Differential Equations 39 (2014) 1128–1157.

[24] L. Grafakos, R. Torres, Multilinear Calderón–Zygmund theory, Adv. Math. 165 (2002) 124–164.

[25] A. Grau de la Herrán, T. Hytönen, Dyadic representation and boundedness of non-homogeneous Calderón–Zygmund operators with mild kernel regularity, Michigan Math. J. 67 (2018) 757–786.

[26] R. Fefferman, Harmonic analysis on product spaces, Ann. of Math. 126 (1987) 109–130.

[27] R. Fefferman, $A^p$ weights and singular integrals, Amer. J. Math. 110 (1988) 975–987.

[28] R. Fefferman, E. Stein, Singular integrals on product spaces, Adv. Math. 45 (1982) 117–143.

[29] T. Figiel, On equivalence of some bases to the Haar system in spaces of vector-valued functions, Bull. Polish Acad. Sci. Math. 36 (1988) 119–131.

[30] T. Hytönen, The Holmes–Wick theorem on two-weight bounds for higher order commutators revisited, Arch. Math. (Basel), 107 (2016) 389–395.

[31] T. Hytönen, Representation of singular integrals by dyadic operators, and the $A_2$ theorem, Expo. Math. 35 (2017)166–205.

[32] T. Hytönen, Sharp weighted bound for general Calderón-Zygmund operators, Ann. of Math. 175 (2012) 1473–1506.

[33] T. Hytönen, J. van Neerven, M. Veraar, L. Weis, Analysis in Banach Spaces, Volume I: Martingales and Littlewood-Paley Theory, Springer–Verlag, 2016.

[34] T. Hytönen, J. van Neerven, M. Veraar, L. Weis, Analysis in Banach Spaces, Volume II: Probabilistic Methods and Operator Theory, Springer-Verlag, 2017.

[35] T. Hänninen, T. Hytönen, Operator-valued dyadic shifts and the $T(1)$ theorem, Monatsh. Math. 180 (2016) 212–253.

[36] T. Hänninen, T. Hytönen, The Holmes–Wick theorem on two-weight bounds for higher order commutators revisited, Arch. Math. (Basel), 107 (2016) 389–395.

[37] I. Holmes, M. Lacey, B. Wick, Commutators in the two-weight setting, Math. Ann. 367 (2017) 51–80.

[38] I. Holmes, S. Petermichl, B. Wick, Weighted little bmo and two-weight inequalities for Journé commutators, Anal. PDE 11 (2018) 1693–1740.

[39] J.-L. Journé, Calderón-Zygmund operators on product spaces, Rev. Mat. Iberoam. 1 (1985) 55–91.

[40] T. Kato, G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, Comm. Pure Appl. Math. 41 (1988) 891–907.

[41] M. Lacey, An elementary proof of the $A_2$ bound, Israel J. Math. 217 (2017) 181–195.

[42] A. Lerner, S. Ombrosi, I. Rivera-Ríos, On pointwise and weighted estimates for commutators of Calderón-Zygmund operators, Adv. Math. 319 (2017) 153–181.

[43] A. Lerner, S. Ombrosi, I. Rivera-Ríos, Commutators of singular integrals revisited, Bull. London Math. Soc. 51 (2019) 107–119.

[44] K. Li, J. Martell, S. Ombrosi, Extrapolation for multilinear Muckenhoupt classes and applications to the bilinear Hilbert transform, preprint, arXiv:1802.03338, 2018.

[45] K. Li, J.M. Martell, H. Martikainen, S. Ombrosi, E. Vuorinen, End-point estimates, extrapolation for multilinear Muckenhoupt classes, and applications, Trans. Amer. Math. Soc., to appear, arxiv:1902.04951, 2019.
[46] K. Li, H. Martikainen, E. Vuorinen, Bloom type inequality for bi-parameter singular integrals: efficient proof and iterated commutators, Int. Math. Res. Not. IMRN (2019), rnz072, https://doi.org/10.1093/imrn/rnz072.

[47] K. Li, H. Martikainen, E. Vuorinen, Bloom type upper bounds in the product BMO setting, J. Geom. Anal. 30 (2020) 3181–3203.

[48] K. Li, H. Martikainen, Y. Ou, E. Vuorinen, Bilinear representation theorem, Trans. Amer. Math. Soc. 371 (6) (2019) 4193–4214.

[49] G. Lu, P. Zhang, Multilinear Calderón-Zygmund operators with kernels of Dini’s type and applications, Nonlinear Anal. 107 (2014) 92–117.

[50] H. Martikainen, Representation of bi-parameter singular integrals by dyadic operators, Adv. Math. 229 (2012) 1734–1761.

[51] H. Martikainen, E. Vuorinen, Dyadic-probabilistic methods in bilinear analysis, Mem. Amer. Math. Soc., to appear, arXiv:1609.01706, 2016.

[52] B. Muckenhoupt and R. L. Wheeden, Weighted bounded mean oscillation and the Hilbert transform, Studia Math. 54 (1975/76) 221–237.

[53] B. Nieraeth, Quantitative estimates and extrapolation for multilinear weight classes, Math. Ann. 375 (2019) 453–507.

[54] Y. Ou, Multi-parameter singular integral operators and representation theorem, Rev. Mat. Iberoam. 33 (2017) 325–350.

[55] Y. Ou, S. Petermichl, E. Strouse, Higher order Journé commutators and characterizations of multi-parameter BMO, Adv. Math. 291 (2016) 24–58.

[56] S. Petermichl, The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical $A_p$ characteristic, Amer. J. Math. 129 (5) (2007) 1355–1375.

[57] S. Wu, A wavelet characterization for weighted Hardy spaces, Rev. Mat. Iberoam. 8 (1992) 329–349.