Canonical, Lie-algebraic and quadratic twist deformations of Galilei group

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Abstract

New Galilei quantum groups dual to the Hopf algebras proposed in [1] are obtained by the nonrelativistic contraction procedures. The corresponding Lie-algebraic and quadratic quantum space-times are identified with the translation sectors of considered algebras.
1 Introduction.

Recently, there were found arguments based on quantum gravity [2], [3] and string theory [4], [5] indicating that space-time at Planck scale should be noncommutative, i.e. it should have a quantum nature. On the other side, there appeared a lot of papers dealing with classical ([6]-[10]) and quantum ([11]-[15]) mechanics, Doubly Special Relativity frameworks ([16]-[19]), and field theoretical models ([20]-[30]), in which noncommutative space-time plays a prominent role.

In accordance with general classification of all possible deformations of relativistic and nonrelativistic symmetries ([31], [32]) one can distinguish three kinds of quantum spaces:

1) Canonical \((\theta^{\mu\nu}\)-deformed) space-time
\[
[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} \quad ; \quad \theta_{\mu\nu} = \text{const},
\]  
(1)

considered in [33]-[36]. The corresponding twist deformation of Poincaré Hopf algebra \(U_\theta(P)\) has been proposed in [34], while its dual quantum group \(P_\theta\) in [33] and [36]. There were also provided two \(\theta^{\mu\nu}\)-deformed Galilei Hopf algebras [1] as the contraction limits of twisted Poincaré group \(U_\theta(P)\).

2) Lie-algebraic modification of classical space
\[
[\hat{x}_\mu, \hat{x}_\nu] = i\theta^{\rho}_{\mu\nu} \hat{x}_\rho ,
\]  
(2)

with particularly chosen coefficients \(\theta^{\rho}_{\mu\nu}\) being constants. There exist two explicit realizations of such a noncommutativity - \(\kappa\)-Poincaré Hopf algebra \(U_\kappa(P)\) [37], [38] and twisted Poincaré group \(U_\kappa(P)\) [39] (see also [40]). Their dual partners \(P_\kappa\) and \(P_\zeta\) have been recovered in [41] and [39], respectively. Besides, the so-called \(\kappa\)-Galilei group has been provided by nonrelativistic contraction of \(\kappa\)-Poincaré Hopf algebra in [42], and its dual quantum partner has been described in [43]. The remaining Galilei algebras were recovered in [1] by various contractions of twisted Poincaré group \(U_\kappa(P)\).

3) Quadratic deformation of Minkowski space
\[
[\hat{x}_\mu, \hat{x}_\nu] = i\theta^{\rho\tau}_{\mu\nu} \hat{x}_\rho \hat{x}_\tau ,
\]  
(3)

with coefficients \(\theta^{\rho\tau}_{\mu\nu}\) being constants. This type of noncommutativity has been proposed as the translation sector of Poisson-Lie structure \(P_\zeta\). The explicit form of its nonrelativistic counterpart remains unknown.

In this article we perform three nonrelativistic contractions (see Section 2) of twisted Poincaré groups \(P_\theta\), \(P_\zeta\) and \(P_\kappa\), respectively. In such a way we recover six Galilei Hopf algebras dual to the quantum (Galilei) groups proposed in [1]. Two of them correspond

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1They were obtained by contractions of \(U_\theta(P)\), \(U_\zeta(P)\) and \(U_\kappa(P)\) twisted Poincaré Hopf algebras.
to canonical (1)), three - to Lie-algebraic (2)), and one - to quadratic (3)) type of space-time noncommutativity. We show that in the Lie-algebraic case their translation sectors can be identified with the nonrelativistic space-times, introduced in [1] as a quantum representation space (a Hopf module) of twisted (Galilei) algebras. Consequently, we reproduce three Lie-algebraically deformed space-times: two with quantum space and classical time, and one with classical space and quantum time. In such a way we also recover a new quadratic Galilei space-time (a translation sector) with quantum space and classical time.

It should be mentioned that presented groups can be recovered with use of two other Hopf-algebraic methods [44], [45]. First of them, so-called FRT procedure [44], uses quantum R-matrix associated with the considered algebra, while the second one, leads to quantum group by canonical quantization of a suitable Poisson-Lie structure [45], [32]. It should be noted, however, that contraction scheme used in this article has one advantage - it gives an additional information about relativistic counterparts of recovered algebras, i.e. we get our Galilei groups as a contraction limit of existing Poincaré Hopf structures.

The knowledge of explicit form of Galilei Hopf algebra \((U(G))\) as well as its dual quantum group \((G(U))\) permits us to analyze the basic nonrelativistic dynamical systems. Using so-called Heisenberg double procedure [46] one can provide a proper phase-space associated with the considered Hopf algebras. Such a construction in the case of relativistic symmetries has been presented in [47], [48] for \(\kappa\)-Poincaré algebra, and in [49] for Lie-algebraically twisted Poincaré group. Moreover, the Heisenberg uncertainty principle corresponding to the above (quantum) phase-spaces, has been provided in [47] and [49] for \(\kappa\)- and twist-deformed symmetries respectively. The analogous investigations at nonrelativistic level already has been undertaken.

The paper is organized as follows. In second Section we describe three contraction procedures used in this article - one \(c\)-independent and two with \(c\)-dependent parameter of deformation. In Sections 3, 4 and 5 we find canonical, Lie-algebraic and quadratic twist deformations of quantum Galilei group, respectively. The results are summarized and discussed in the last Section.

### 2 Contraction procedures.

Let us consider the following redefinition of rotation and translation generators of quantum Poincaré group [50] (see also [43]).

\[
\Lambda^0_0 = \left( 1 + \frac{v^2}{c^2} \right)^{\frac{1}{2}}, \tag{4}
\]

\[
\Lambda^i_0 = \frac{v^i}{c}, \tag{5}
\]

\(^2\)The light-velocity \(c\) plays a role of contraction parameter.
\[ \Lambda^0_i = \frac{v^k R^k_i}{c}, \]  
(6)  
\[ \Lambda^k_i = \left( \delta^k_l + \left( 1 + \frac{\tau^2}{c^2} \right)^{\frac{1}{2}} - 1 \right) \frac{v^k v^l}{v^2} R^l_i, \]  
(7)  
\[ a^i = b^i, \quad a^0 = c \tau, \]  
(8)

where \( R^i_j, v^i, \tau, b^i \) denote the generators of Galilei quantum group - rotations, boosts and translations, respectively. In this article we consider three nonrelativistic contractions:

i) Standard (Inönü-Wigner) contraction with \( c \)-independent parameter of deformation \([50]\), i.e. we use the redefinition (4)-(8) and take the \( c \to \infty \) limit.

ii) Contraction with \( c \)-dependent parameter \( \kappa \) \([\kappa] = (\text{length})^{-1}\) such that \( \kappa = \hat{\kappa}/c \) \([\hat{\kappa}] = (\text{time})^{-1}\) \([51], [52]\). Then, in the \( c \to \infty \) limit we get the quantum group with deformation parameter \( \hat{\kappa} \).

iii) The parameter \( \kappa \) is replaced by \( \kappa = \bar{\kappa} c \) \([\bar{\kappa}] = (\text{time}) \times (\text{length})^{-2}\) (see e.g. [51]), and the contraction limit \( c \to \infty \) leads to the quantum group with parameter \( \bar{\kappa} \).

As we mentioned in Introduction, we shall perform three contractions in the case of canonical \( \mathcal{P}_\theta \), Lie-algebraic \( \mathcal{P}_\zeta \) and quadratic \( \mathcal{P}_\xi \) Poincaré groups. They were provided in [34], [39] together with their dual partners \( \mathcal{U}(\mathcal{P}_\theta), \mathcal{U}(\mathcal{P}_\zeta) \) and \( \mathcal{U}(\mathcal{P}_\xi) \), respectively.

### 3 Canonical deformation.

Let us start with canonical deformation of relativistic symmetries. The \( \theta^{\mu\nu} \)-deformed Poincaré group \( \mathcal{P}_\theta \) has been proposed some years ago in [33] and rediscovered recently in [36]. Its algebraic sector looks as follows:\footnote{\( \eta_{\mu\nu} = (-,+,+,+). \)}

\[
[a^\mu, a^\nu] = i \theta^{\rho\sigma} (\Lambda^\mu_{\rho}, \Lambda^\nu_{\sigma} - \delta^\mu_{\rho} \delta^\nu_{\sigma}),
\]  
(9)  
\[
[\Lambda^\mu_{\tau}, \Lambda^\nu_{\rho}] = [a^\mu, a^\nu] = 0,
\]  
(10)

while coproducts remain undeformed

\[
\Delta(a^\mu) = \Lambda^\mu_{\nu} \otimes a^\nu + a^\mu \otimes 1, \quad \Delta(\Lambda^\mu_{\nu}) = \Lambda^\mu_{\rho} \otimes \Lambda^\rho_{\nu}.
\]  
(11)

The corresponding classical r-matrix has the form \([31]\)

\[
r_\theta = \frac{1}{2} \theta^{\mu\nu} P_\mu \wedge P_\nu,
\]  
(12)
with fourmomentum generators $P_\mu$ dual to the translations $a^\mu$. Obviously, the r-matrix (12) satisfies the classical Yang-Baxter (CYBE) equation

$$[[r_\theta, r_\theta]] = [r_{\theta 12}, r_{\theta 13} + r_{\theta 23}] + [r_{\theta 13}, r_{\theta 23}] = 0,$$

(13)

where symbol $[[\cdot, \cdot]]$ denotes the Schouten bracket and $r_{\theta 12} = \frac{1}{2} \theta^{\mu\nu} P_\mu \wedge P_\nu \wedge 1$, $r_{\theta 13} = \frac{1}{2} \theta^{\mu\nu} 1 \wedge P_\mu \wedge P_\nu$.

In the case of simplest contraction of $\mathcal{P}_\theta$ group (see (i)), for parameter $c$ running to infinity, we get the following algebraic sector

$$[b^i, b^j] = i \theta^{kl} (R^k_{\ i} R^l_{\ j} - \delta^i_k \delta^j_l) ,$$

(14)

$$[\tau, b^i] = [\tau, v^i] = [b^i, v^j] = [v^i, v^j] = 0 ,$$

(15)

$$[R^i_{\ j}, R^k_{\ l}] = [v^i, R^k_{\ l}] = [\tau, R^i_{\ j}] = [b^i, R^k_{\ l}] = 0 ,$$

(16)

supplemented by the classical ( undeformed) coproducts

$$\Delta(R^i_{\ j}) = R^i_{\ k} \otimes R^k_{\ j} ,$$

(17)

$$\Delta(v^i) = R^i_{\ j} \otimes v^j + v^i \otimes 1 ,$$

(18)

$$\Delta(\tau) = \tau \otimes 1 + 1 \otimes \tau ,$$

(19)

$$\Delta(b^i) = R^i_{\ j} \otimes b^j + v^i \otimes \tau + b^i \otimes 1 .$$

(20)

The relations (14)-(20) define softly deformed Galilei group $G_\theta$ dual to the Hopf algebra $U_\theta(G)$ associated with the following classical $r_\theta$-matrix (see (1))

$$r_\theta = \frac{1}{2} \theta^{kl} \Pi_k \wedge \Pi_l , \quad \Pi_i - dual to the translations b^i .$$

(21)

Let us turn to the contraction (ii) of $\mathcal{P}_\theta$ group ($\theta^{\mu\nu} = \hat{\theta}^{\mu\nu}/\kappa$) with $c$-dependent deformation parameter $\kappa = \hat{\kappa}/c$. In such a case, for $\hat{\theta}^{ij} = 0$ and $\hat{\theta}^{0i} = \hat{\theta}^{0i}/\kappa$, in the contraction limit $c \to \infty$ we get

$$[b^i, b^j] = i \frac{\hat{\theta}^{0k}}{\kappa} (v^i R^k_{\ j} - R^i_{\ k} v^j) ,$$

(22)

$$[\tau, b^i] = i \frac{\hat{\theta}^{0k}}{\kappa} (R^i_{\ k} + \delta^i_k) ,$$

(23)

4 All mentioned in this article classical r-matrices satisfy the classical Yang-Baxter equation (13), i.e. they correspond to twist-deformed Galilei Hopf algebras.
\[
\begin{align*}
\tau, v^i &= [b^i, v^j] = [v^i, v^j] = 0, \\
R^i_j, R^k_l &= [v^i, R^k_l] = [\tau, R^i_j] = [b^i, R^k_l] = 0, 
\end{align*}
\] (24)

with the classical coproduct \((17)-(20)\) and corresponding classical \(r_\hat{\kappa}\)-matrix

\[
r_\hat{\kappa} = \frac{\hat{\theta}_{\kappa}}{\kappa} \Pi_0 \wedge \Pi_\kappa; \quad \Pi_0 - \text{dual to the generator } \tau.
\] (26)

The relations \((22)-(25)\) and \((17)-(20)\) define softly deformed Galilei group \(G_{\hat{\kappa}}\) dual to the algebra \(U_{\hat{\kappa}}(G)\) (see \([1]\)). One can also check that for \(\theta^{ij} \neq 0\) the contraction \(ii\) becomes divergent, while in the case of contraction \(iii\) \((\kappa = \kappa c)\), for arbitrary value of parameter \(\theta^{\mu\nu}\), we get the undeformed Galilei quantum group \(G_0\).

4 Lie-algebraic deformation.

The Lie-algebraic twist deformation of Poincaré group \(P_\zeta\) has been studied in \([39]\). Its algebraic sector looks as follows

\[
\begin{align*}
[a^\mu, a^\nu] &= i \zeta^\nu (\delta^{\alpha}_\beta a^\beta - \delta^{\beta}_\alpha a^\alpha) + i \zeta^\mu (\delta^{\nu}_\beta a^\beta - \delta^{\beta}_\nu a^\nu), \\
[a^\mu, \Lambda^\nu_\rho] &= i \zeta^\lambda \Lambda^\nu_\lambda (\eta_{\beta\rho} \Lambda^\nu_\alpha - \eta_{\alpha\rho} \Lambda^\nu_\beta) + i \zeta^\mu (\delta^{\nu}_\beta \Lambda^\alpha_\rho - \delta^{\nu}_\alpha \Lambda^\beta_\rho), \\
[\Lambda^\mu_\nu, \Lambda^\rho_\tau] &= 0,
\end{align*}
\] (27)

while coproducts remain classical (see \((11)\)). The corresponding \(r_\zeta\)-matrix has the form

\[
r_\zeta = \frac{1}{2} \zeta^\lambda P_\lambda \wedge M_{\alpha\beta}; \quad \lambda \neq \alpha, \beta - \text{fixed},
\] (30)

where generators \(M_{\mu\nu}\) are dual to the rotations \(\Lambda^\mu_\nu\).

In the case of ”space-like” carrier \(\{M_{kl}, P_\gamma; \; \gamma \neq k, l, 0\}\) the \(c\)-independent contraction \(i)\) leads to the following algebraic sector

\[
\begin{align*}
b^i, b^j &= i \zeta \delta^{i j} (\delta^i_k b_l - \delta^j_k b_l) + i \zeta \delta^{i j} (\delta^j_k b_k - \delta^i_k b_k), \\
b^i, v^j &= i \zeta \delta^{i j} (\delta^j_k v_k - \delta^i_k v_k), \\
b^i, R^\rho_\tau &= i \zeta R^i_\gamma (\delta^i_k R^\rho_\kappa - \delta^\rho_k R^i_\kappa) + i \zeta \delta^i_\gamma (\delta^\rho_k R^\kappa_\tau - \delta^\kappa_k R^i_\tau), \\
\tau, b^i &= [\tau, v^i] = [v^i, v^j] = 0, \\
R^i_j, R^k_l &= [v^i, R^k_l] = [\tau, R^i_j] = 0,
\end{align*}
\] (31) (32) (33) (34) (35)
and undeformed coproducts (17)-(20). The above relations define twisted Galilei group $G_\zeta$ dual to the Hopf algebra $U_\zeta(G)$ (see [1]). In the case of carrier $\{M_{kl}, P_0\}$ we get undeformed Galilei quantum group $G_0$, while for "boost-like" carrier $\{M_{k0}, P_l ; k \neq l\}$ the commutators of boosts with translations become divergent.

It should be noted that Hopf algebra $U_\zeta(G)$ has been provided with use of the following twist factor

$$K_\zeta = \exp \frac{i}{2}(\zeta \Pi, \wedge K_{kl}) ; \quad K_{kl} - \text{dual to the generator} \quad R^k_l .$$

(36)

In such a case one can define the corresponding nonrelativistic space-time as its quantum representation space - a Hopf module [53]. It looks as follows [1]

$$\begin{align*}
[x_i, x_j]_{\zeta} &= i\zeta \delta_{ij} (\delta_{kl}x_l - \delta_{li}x_k), \\
[t, x_i]_{\zeta} &= 0,
\end{align*}$$

(37)

(38)

where the $\star_{\zeta}$-multiplication of two functions is given by

$$f(t, x) \star_{\zeta} g(t, x) := \omega \circ (K^{-1}_\zeta \triangleright f(t, x) \otimes g(t, x)) ,$$

(39)

with $K_\zeta = \exp \left(-\frac{i}{2} \zeta \partial_\gamma \wedge (x_k \partial_l - x_l \partial_k)\right)$ and $\omega \circ (a \otimes b) = a \cdot b$. Hence, we see, that after the substitution

$$\tau \leftrightarrow t , \quad b^i \leftrightarrow x^i ,$$

(40)

the translation sector (31), (34) can be identified with the nonrelativistic quantum space-time (37), (38).

Let us now turn to the contraction $\ii$ of the Poincare group $P_\zeta$ with $\zeta = \frac{1}{\kappa}$. Then, for $\{M_{kl}, P_0\}$ the corresponding Galilei quantum group $G_\kappa$ ($\kappa = \kappa/c$) looks as follows

$$\begin{align*}
[\tau, b^i] &= \frac{i}{\kappa}(\delta^i_k b_l - \delta^i_l b_k), \\
[\tau, v^i] &= \frac{i}{\kappa}(\delta^i_k v_l - \delta^i_l v_k), \\
[\tau, R^p_\tau] &= \frac{i}{\kappa}(\delta^p_k R^o_l - \delta^p_l R^o_k) + \frac{i}{\kappa}(\delta^o_k R^p_l - \delta^o_l R^p_k), \\
[b^i, R^p_\tau] &= \frac{i}{\kappa}v^i(\delta^p_l R^o_k - \delta^p_k R^o_l), \\
[b^i, b^j] &= [b^i, v^j] = [v^i, v^j] = 0, \\
[R^i_j, R^k_l] &= [v^i, R^k_l] = 0,
\end{align*}$$

(41)

(42)

(43)

(44)

(45)

(46)

$^5[a, b]_{\zeta} = a_{\zeta} b - b_{\zeta} a.$
while coproducts remain classical. For carriers \( \{ M_{kl}, P_\gamma \,; \, \gamma \neq k, l, 0 \} \) and \( \{ M_{k0}, P_l \,; \, k \neq l \} \) the contraction \( ii \) becomes divergent. Besides, it should be noted that a proper (dual) Hopf algebra \( \mathcal{U}_\kappa(\mathcal{G}) \) and the corresponding \( \hat{\kappa} \)-deformed space-time\(^6\) have been provided in \([1]\). We see, that after substitution \((40)\) the relations \((47)\) and \((41), (45)\) become identical.

Let us now consider the contraction \( iii \). Then, for carrier \( \{ M_{kl}, P_\gamma \,; \, \gamma \neq k, l, 0 \} \) we obtain the classical (undeformed) Galilei Hopf algebra \( \mathcal{G}_0 \).

In the case \( \{ M_{k0}, P_l \,; \, k \neq l \} \) the situation is more complicated, i.e. one can check that after contraction we get the following Galilei quantum group \( \mathcal{G}_\kappa (\kappa = \kappa_c) \)

\[
[b^i, b^j] = \frac{i}{\kappa} \tau (\delta^i_k \delta^j_l - \delta^i_l \delta^j_k),
\]

\[
[v^i, b^j] = \frac{i}{\kappa} (R^j_i R^i_k + \delta^j_i \delta^i_k),
\]

\[
[\tau, b^j] = [\tau, v^i] = [\tau, R^i_j] = [v^i, v^j] = 0,
\]

\[
[b^i, R^\alpha_j] = [R^j_i, R^\alpha_j] = [v^i, R^\alpha_j] = 0,
\]

with trivial coproduct \((17)-(20)\).

The corresponding (dual) Hopf algebra \( \mathcal{U}_\kappa(\mathcal{G}) \) has been recovered with use of the following twist factor

\[
\mathcal{K}_\kappa = \exp \frac{i}{2\kappa} (\Pi_l \land V_k); \quad V_k - \text{dual to the boost generator } v_k. \tag{52}
\]

One can observe that its nonrelativistic space-time is exactly the same as the translation sector \((48), (50)\) (see \([1]\))

\[
[x_i, x_j]_{\kappa} = \frac{i}{\kappa} t (\delta_{li} \delta_{kj} - \delta_{lj} \delta_{ki}),
\]

\[
[t, x_i]_{\kappa} = 0, \tag{53}
\]

where \( \ast_\kappa \)-multiplication is given by

\[
f(t, \tau) \ast_\kappa g(t, \tau) := \omega \circ (\mathcal{K}_\kappa^{-1} \triangleright f(t, \tau) \otimes g(t, \tau)) \quad \mathcal{K}_\kappa = \exp \left( \frac{i}{2\kappa} \partial_l \land t\partial_k \right),
\]

i.e. we can identify the translation sector \((48), (50)\) with nonrelativistic space-time \((53)\).

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\(^6\)The above space-time is equipped with quantum time and classical space. For its \( N = 1 \) supersymmetric counterpart see \([54]\).

\(^7\)The corresponding twist factor looks as follows \( \mathcal{K}_\hat{\kappa} = \exp \left( \frac{i}{\hat{\kappa}} \Pi_0 \land \hat{K}_{kl} \right) \).
5 Quadratic deformation.

The quadratic deformation of Poincaré group $\mathcal{P}_\xi$ has been investigated in [39]. It is given by the following algebraic sector

$$
[\Lambda_\mu, \Lambda_\nu] = (1 - \cosh \xi) \sum_{k=\alpha, \beta} \langle \delta^\mu \langle \delta^\nu \rangle \rangle \Lambda^{(k \Lambda^l)}_\mu \Lambda_\rho - \delta^\nu \langle \delta^\mu \rangle \Lambda^{(k \Lambda^l)}_\mu \Lambda_\rho + \right)
$$

$$
+ \ i \sinh \xi [\eta_{\beta \rho} \Lambda_\alpha - \eta_{\alpha \rho} \Lambda_\beta] (\eta_{\delta \tau} \Lambda_\gamma - \eta_{\gamma \tau} \Lambda_\delta) +
$$

$$
+ (\eta_{\beta \rho} \Lambda_\gamma - \eta_{\gamma \rho} \Lambda_\beta) (\eta_{\alpha \tau} \Lambda_\mu - \eta_{\mu \tau} \Lambda_\alpha) +
$$

$$
+ (\eta_{\gamma \rho} \Lambda_\delta - \eta_{\delta \rho} \Lambda_\gamma) (\eta_{\alpha \tau} \Lambda_\mu - \eta_{\mu \tau} \Lambda_\alpha) +
$$

$$
[ a^\mu, a^\nu ] = \left[ p \right] \sum_{k=\alpha, \beta} \langle \delta^\mu \rangle \langle \delta^\nu \rangle \left[ a_k, a_l \right]
$$

$$
[ a^\mu, \Lambda_\rho ] = (1 - \cosh \xi) \sum_{k=\alpha, \beta} \delta^\mu \langle \delta^\nu \rangle \langle \delta^\mu \rangle \Lambda^{(k \Lambda^l)}_\mu \Lambda_\rho + \right)
$$

$$
+ \ i \sinh \xi [\delta^\mu \delta^\nu \Lambda_\alpha - \delta^\mu \delta^\nu \Lambda_\beta] (\delta^\rho \Lambda_\gamma - \delta^\rho \Lambda_\delta) +
$$

$$
+ (\delta^\mu \delta^\nu \Lambda_\gamma - \delta^\mu \delta^\nu \Lambda_\delta) (\delta^\rho \Lambda_\alpha - \delta^\rho \Lambda_\beta) +
$$

$$
[ a^\mu, \Lambda_\rho ] = (1 - \cosh \xi) \sum_{k=\alpha, \beta} \delta^\mu \langle \delta^\nu \rangle \langle \delta^\mu \rangle \Lambda^{(k \Lambda^l)}_\mu \Lambda_\rho + \right)
$$

$$
+ \ i \sinh \xi [\delta^\mu \delta^\nu \Lambda_\alpha - \delta^\mu \delta^\nu \Lambda_\beta] (\delta^\rho \Lambda_\gamma - \delta^\rho \Lambda_\delta) +
$$

$$
+ (\delta^\mu \delta^\nu \Lambda_\gamma - \delta^\mu \delta^\nu \Lambda_\delta) (\delta^\rho \Lambda_\alpha - \delta^\rho \Lambda_\beta) +
$$

$$
with \delta^\mu \langle \delta^\nu \rangle \langle \delta^\rho \rangle = \delta^\mu \langle \delta^\nu \rangle \langle \delta^\rho \rangle = \delta^\mu \delta^\nu \delta^\rho, and the classical coproduct [11]. The corresponding $r_\xi$-matrix looks as follows

$$
r_\xi = \frac{1}{2} \xi M_{\alpha \beta} \Lambda M_{\gamma \delta},
$$

where indices $\alpha, \beta, \gamma, \delta$ are all different and fixed.

One can see that for $\alpha = i, \beta = 0, \gamma = k$ and $\delta = l$ the contraction $ii$) becomes divergent in the $c \to \infty$ limit. Similarly, for $\mathcal{P}_\xi$ with $\xi = \hat{\xi}/\kappa$ the contraction $ii$) $($$\kappa = \hat{\kappa}/c$$)$
does not exist. In the case \( iii \) (\( \kappa = \kappa c \)) situation appears less trivial, i.e. in the \( c \to \infty \) limit we get the following algebraic sector

\[
[v^\rho, v^\tau] = -\frac{i\xi}{2\kappa}\delta^{\rho\tau}(\delta^{i\tau}v^l - \delta^{i\tau}v^k) + \frac{i\xi}{2\kappa}\delta^{\tau\rho}(\delta^{i\rho}v^l - \delta^{i\rho}v^k),
\]

\[
[R^\rho_\tau, v^j] = -\frac{i\xi}{2\kappa}R^i_j(\delta_{i\tau}R^\rho_k - \delta_{k\tau}R^\rho_i) + \frac{i\xi}{2\kappa}\delta^{ij}(\delta^{k\rho}R^l_\tau - \delta^{l\rho}R^k_\tau),
\]

\[
[b^\rho, b^\tau] = \frac{i\xi}{2\kappa}(\delta^{\rho\tau} - i\xi\delta^{i\tau} \delta_{i\rho}) \{\tau, b_k\} - \frac{i\xi}{2\kappa}(\delta^{\rho\tau} - i\xi\delta^{i\tau} \delta_{i\rho}) \{\tau, b_k\},
\]

\[
[b^j, R^\rho_\tau] = -\frac{i\xi}{2\kappa}(\delta^{j\rho} - i\xi\delta^{j\tau} \delta_{j\rho}) \{\tau, b_l\} - \frac{i\xi}{2\kappa}(\delta^{j\rho} - i\xi\delta^{j\tau} \delta_{j\rho}) \{\tau, b_l\}.
\]

(58) (59) (60) (61) (62) (63)

and classical coproducts. The above relations define quadratic Galilei group \( G_{\kappa} \) equipped with the following classical \( r_{\kappa} \)-matrix

\[
r_{\kappa} = \frac{\xi}{2\kappa}(V_l \wedge K_{kl}).
\]

(64)

Let us now turn to nonrelativistic space-time corresponding to the quantum group \( (58)-(63) \). As it was mentioned, it is defined as a quantum representation space of Galilei group \( G_{\kappa} \). The action of dual generators \( V_l \) and \( K_{kl} \) on such a space is given by

\[
K_{kl} \triangleright f(t, \vec{x}) = i(x_k \partial_l - x_l \partial_k)f(t, \vec{x}), \quad V_l \triangleright f(t, \vec{x}) = it \partial_i f(t, \vec{x}),
\]

while the \( \star_{\kappa} \)-multiplication looks as follows

\[
f(t, \vec{x}) \star_{\kappa} g(t, \vec{x}) := \omega \circ (K_{\kappa}^{-1} \triangleright f(t, \vec{x}) \otimes g(t, \vec{x})), \quad K_{\kappa} = \exp\left(\frac{i\xi}{2\kappa}t \partial_i \wedge (x_k \partial_l - x_l \partial_k)\right).
\]

Hence, we have

\[
[x^\rho, x^\tau]_{\star_{\kappa}} = \frac{i\xi}{2\kappa}(\delta^{\rho\tau} - i\xi\delta^{i\tau} \delta_{i\rho}) \{t, x_l\}_{\star_{\kappa}} - \frac{i\xi}{2\kappa}(\delta^{\rho\tau} - i\xi\delta^{i\tau} \delta_{i\rho}) \{t, x_k\}_{\star_{\kappa}}, \quad [t, x^j]_{\star_{\kappa}} = 0.
\]

(66)

We see that the noncommutative space-time \( (66) \) is exactly the same as the translation sector \( (60) \) attached to quantum space and classical time.
6 Final remarks.

In this article we propose canonical and Lie-algebraic twist deformations of quantum groups dual to the recovered in [1] Galilei Hopf algebras $U_G$. Besides, we also obtained a new quadratic deformation of Galilei Hopf structure as well as the corresponding non-commutative (quantum) space-time. In such a way we show that the translation sectors of dual groups are identical with Hopf-modules (space-times) of corresponding Galilei Hopf algebras [1].

It should be mentioned that presented results complete our studies in [1] on the contractions of twisted Poincaré groups. Nevertheless, they can be extended in various ways. First of all, one can consider basic dynamical models corresponding to considered Galilei algebras [55]. Besides, it seems interesting to find other $D = 3 + 1$ dimensional nonrelativistic space-times, corresponding Hopf algebras describing symmetry, and dual groups, predicted by the general classification of all Galilean Poisson-Lie structures [32] (see also for two-dimensional case [56] and [57]). One can also ask about a "superposition" of discussed quantum deformations as well as their supersymmetric $\mathcal{N} = 1$ extensions (see e.g. [58] and [57]). Finally, as it was mentioned in Introduction, the corresponding quantum phase-spaces can be generated with use of the Heisenberg double procedure [46]. The investigation of these problems are under study.

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