First-order transition in small-world networks

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The small-world transition is a first-order transition at zero density $p$ of shortcuts, whereby the normalized shortest-path distance $\ell/L$ undergoes a discontinuity in the thermodynamic limit. On finite systems the apparent transition is shifted by $\Delta p \sim L^{-d}$. Equivalently a “persistence size” $L^* \sim p^{-1/d}$ can be defined in connection with finite-size effects. Assuming $L^* \sim p^{-\tau}$, simple rescaling arguments imply that $\tau = 1/d$. We confirm this result by extensive numerical simulation in one to four dimensions, and argue that $\tau \geq 1/d$ implies that this transition is first-order.

A physically meaningful characterisation of disordered connection networks is given by their average chemical distance or shortest-path distance $\bar{\ell} = 1/N \sum_{i=1}^{N} \ell_{ij}$, where $\ell_{ij}$ is the minimum number of links that must be traversed to join sites $i$ and $j$, and $< >$ means average over disorder realizations. If the network connectivity is topologically $d$-dimensional, $\bar{\ell}$ is proportional to $L$, its linear dimension. But on randomly bonded networks containing $L^d$ sites one has $\bar{\ell} \sim \log(L)$ much shorter than $L$. Small-world networks $\tilde{L}$ interpolate between these two cases. They consist of a regular $d$-dimensional lattice with a small fraction $p$ of long-range bonds, or shortcuts, per site (i.e. there is a total of $B = pL^d$ long-range bonds on top of a regular lattice). These systems have received attention recently [1], following the observation [1] that for system size $L$ one has $\bar{\ell} \sim \log(L)$ and therefore $\tau < 1$ provides evidence for the importance of “nonlinear effects”.

Some of their conclusions have been subject to criticism recently. Firstly Barrat [2] stressed the obvious fact that $\tau$ cannot be smaller than 1 in one dimension, otherwise the total number of long-range bonds at the “crossover point” would go to zero with system size. Barrat confirmed by numerical simulation on large systems that $\tau_d = 1$. Similar arguments can be used to conclude that $\tau \geq 1/d$ in $d$ dimensions.

A second controversial point concerns the nature of the small-world transition, which is the subject of this letter. Newman and Watts (thereafter NW) [3] recently suggested that the transition from logarithmic to linear scaling of $\bar{\ell}$ is a normal second-order phase transition at $p = 0$, i.e. one with a diverging characteristic length $\xi(p) \sim p^{-\nu}$, and derive $\nu = 1/d$ by renormalisation-group (RG) analysis.

In this Letter we show that the small-world transition is a first-order transition at $p = 0$, where the “order parameter” $L(p) = \bar{\ell}/L$ undergoes a discontinuity in the thermodynamic limit. Within this picture, $L^* \sim p^{-\tau}$ in [4] is a persistence size associated with finite-size corrections. We derive $\tau = 1/d$ by rescaling arguments similar to those of NW and confirm this result by extensive numerical simulation in one to four dimensions. The fact that $L^* \sim p^{-1/d}$ implies that, on finite systems, there is a shift or broadening of order $\Delta p \sim L^{-d}$ in the apparent transition point. In support of our claim for a first-order phase transition we recall earlier work of Nienhuis and Nauenberg [4] and of Fisher and Berker [5], who first discussed the relationship between first-order transitions and finite-size corrections of order $L^{-d}$.

Let us in the first place show that, if Eq. (1) holds with $L^* \sim p^{-\tau}$, then $\tau = 1/d$. The first steps of our derivation are similar to those in NW [4]. Consider a decimation transformation with rescaling parameter $b \ll L$, i.e. define “blocks” of linear size $b$, each containing $b^d$ sites of the original lattice. After rescaling we are left with a lattice of linear dimension $L = L/b$. Since all linear dimensions are rescaled by $b$, one also has that $\bar{\ell}^* = L^*/b$. Two blocks $I$ and $J$ in the decimated lattice are connected by a long-range bond if and only if any pair of original sites $i \in I$ and $j \in J$ are connected by a long-range bond. Since the total number $B$ of long-range bonds is invariant under rescaling (because their typical length is $\sim L$, much larger than $b$), one has $pL^d = \tilde{p}L^d$, and therefore $\tilde{p} = pb^d$. After rescaling one must have $L^* \sim \tilde{p}^{-\tau}$ and therefore $b = L^*/L^* = (p/\tilde{p})^{-\tau} = b^{d\tau}$, which implies $\tau = 1/d$.

Our extensive numerical investigations confirm this theoretical prediction. We used hypercubic lattices with helicoidal boundary conditions, of sizes up to 250 000 lattice sites in one to four dimensions, to which random bonds were added with probability $p$ per site. Since in this problem all bond costs are unity, shortest-path distances $\ell_{ij}$ can be determined efficiently by means of Breadth-First-Search (BFS) [6], whose time-complexity
is $O(N)$, and is simple to program. We calculate $\bar{\ell}(L,p)$ by averaging over 1000 – 2000 disorder realizations for several lattice sizes and $p$ values. From Eq. (1), the asymptotic derivative $\partial \bar{\ell}/\partial \log L$ becomes proportional to $L^\tau$ as $L \to \infty$. We thus fit a straight line to the $\bar{\ell}$ vs. $\log L$ curve, using the four largest sizes simulated for each value of $p$, and obtain the values of $L^\tau(p)$ displayed in Fig. 1. Fitting these data we obtain $\tau = 1/d$ within 5% error in all cases.

![Persistence size $L^*$ vs $p$ in one to four dimensions. Our fits for $\tau$ such that $L^* \sim p^{-\tau}$ agree, within 5% error in all cases, with $\tau = 1/d$, the value predicted by rescaling arguments.](image)

Now consider the order parameter $\mathcal{L}(p) = \lim_{L \to \infty} \bar{\ell}/L$, which is zero for all $p > 0$, but a (lattice-dependent) constant for $p = 0$. This suggests that it is reasonable to identify the small-world transition as a discontinuous, or first-order phase-transition at $p = 0$. Our characterisation of the transition as first-order is however not based on the discontinuity of $\mathcal{L}$, but a consequence of having $\tau = 1/d$ as we now discuss.

As first shown by Nienhuis and Nauenberg [3] in the context of the Renormalisation Group, and later more generally by Fisher and Berker [10], at a first-order fixed point one eigen-exponent must take the value $y = d$ (this is necessary in order to have phase-coexistence) and therefore finite-size corrections of order $L^{-d}$ can be expected. Conversely, if $y = d$ then some first-derivative of the free-energy density undergoes a discontinuity at the transition, i.e. $y = d$ implies that the transition is first-order. The existence of an eigen-exponent $y = d$ gives rise to finite-size corrections (e.g. the shift in the apparent transition point) which are of order $L^{-d}$ [10]. We see then that finite-size corrections of order $L^{-d}$ are a signature of first-order transitions.

For pedagogical reasons let us first review how finite-size corrections of order $L^{-d}$ arise at a typical first-order transition, and later make the analogy with small-world transitions explicit. Consider an Ising-type $d$-dimensional ferromagnet of linear size $L$ at very low temperature, subject to an external magnetic field $h$. Assume that a single spin (e.g. the central one) is “pinned down”. This system undergoes a first-order phase transition at $h = 0$. In the thermodynamic limit, the magnetisation is $m(h) \approx +1$ for $h > 0$, and $m(h) \approx -1$ for $h \leq 0$. For $L$ finite and $h > 0$, most of the spins will point up if the bulk energy $hL^d$ associated with the field is larger than the energy $J_\gamma$ associated with breaking the $\gamma$ bonds around the pinned spin. Therefore on finite systems of size $L$, the transition no longer happens at $h = 0$ but at $hL^d - J_\gamma = 0$, i.e. it suffers a shift $\Delta h$ of order $L^{-d}$.

Equivalently, for each finite $h \to 0^+$ a persistence size $L^* \sim h^{-1/d}$ can be defined such that $m < 0$ for $L < L^*$, while $m > 0$ for $L > L^*$.

In the case of small-world systems, the order parameter $\mathcal{L}(p)$ is a step function at $p = 0$ when $L \to \infty$. But for finite sizes $L$, $\mathcal{L}(p,L) = \bar{\ell}(p,L)/L$ is nonzero only not at $p = 0$ but for all $p < \Delta p \sim L^{-d}$, or equivalently for $L < L^* = p^{-1/d}$ when $p$ is fixed. The situation is clearly the same as in our previous example, i.e. the transition suffers a shift of order $L^{-d}$ on finite systems.

Thus $L^*$ is not a “crossover length” [3] but reflects the existence of boundary-dependent finite-size corrections around a first-order transition. We thus prefer to call $L^*$ “persistence size” [11].

As discussed in the previous paragraph, by showing that the finite-size corrections exponent $\tau$ equals $1/d$ we have also demonstrated that the small-world transition is first-order. But contrary to what is sometimes assumed, the first-order character of a transition does not preclude the existence of critical behaviour. Notice that $\mathcal{L}$ behaves in the same way as the magnetisation density $m(T)$ of a one-dimensional Ising model, or the spanning-cluster density $P_\infty(p)$ in a one-dimensional percolation model. These two cases do display critical behaviour, and constitute simple examples of first-order critical points, respectively at $T = 0$ and $p = 1$. First-order critical points have been briefly discussed by Fisher and Berker (FB) [10].

It is usual (though not entirely correct) to refer to 1d percolation or 1d Ising as displaying “second-order” transitions. The correct designation is “first-order critical”. Around a FOCP finite size corrections $\Delta p$ are of order $L^{-d}$ because this is true for any discontinuous transition. Since at a critical point one also has $\Delta p \sim L^{-1/\nu}$ this in turn means that $\nu = 1/d$ for a FOCP [10]. Thus the fact that the global observable $\bar{\ell}$ for finite small-world networks is a function of a single scaling variable $x = Lp^{1/d}$ is compatible both with a non-critical first-order transition (in which case $x = L/L^*$ with $L^*$ a persistence size) and with a first-order critical point (in this case $x = L/\xi$ with $\xi \sim p^{-1/d}$ an internal characteristic length).

Newman and Watts [18] characterise the small-world
transition as a “normal second-order” phase transition, i.e. they assume critical behaviour (“first-order critical” would be the appropriate term if there were critical behaviour). As said, the finite size behaviour of $\overline{r}(L,p)$ cannot be used to distinguish between a normal first-order point and a FOCP. A possible way to prove or disprove the existence of critical behaviour in small-world networks is to look at bilocal observables, and see whether an internal correlation length can be identified that governs the physics of the transition and diverges at $p_c$. We study in this work the behaviour of a particular bilocal observable: $\ell(r)$, the average shortest-path distance between two points separated by an Euclidean distance $r$. The basic question we wish to answer is whether the behaviour of $\ell(r)$ in the thermodynamic limit is dictated by a finite characteristic length $\xi(p)$ that diverges at $p = 0$.

Newman and Watts [6] recently argued that the characteristic length $\xi$ for the small-world transition is the mean separation $s(p) \sim p^{-1/d}$ between shortcut-ends. Their interpretation is consistent but they apparently base this claim on the sole fact that the global observable $\overline{r}$ is a function of $L/s(p)$ only. No analysis is done of any bilocal observable, which we think is the only way to assess the existence of critical behaviour.

We next show that $s(p)$ is not a lengthscale dictating the behaviour of shortest-path lengths $\ell(r)$. We will find that a characteristic length $r_c$ does in fact exist for $\ell(r)$, but that this characteristic length is $r_c \sim s(p) \log L$, i.e. it diverges with system size, for all $p > 0$.

Consider the shortest-path distance $\ell(r)$ between two points $a$ and $b$ separated by an Euclidean distance $r$. If $r << s(p) = p^{-1/d}$ then clearly $\ell(r) = r$. Let us then discuss the case $s(p) << r << L$. Imagine dividing the lattice into blocks of linear size $r$ as shown in Fig. 2. The resulting rescaled system can be viewed as a random graph made up of $\overline{N} = (L/r)^d$ nodes, randomly connected by a total of $B = pL^d$ long-range bonds. The average coordination number of a node (the number of shortcuts connected to it) is $k \sim pr^d$.

Consider now a path from $a$ to $b$ through one or more shortcuts (Fig. 2). In order to determine the “cost” of such path, we will associate a cost zero with each shortcut, and only count the total distance travelled on the Euclidean lattice within each visited node (dashed lines in Fig. 2). There are typically no shortcuts with both ends in the same block because $r << L$. Thus a path $(a,b)$ through shortcuts must contain two or more outgoing shortcuts and visit one or more extra nodes. Thus we are looking for closed “loops” in the random graph, that start and end in the original node (the one that contains $a$ and $b$). Now we know that, starting from any given node, after $n$ shortcut-steps a total of $\sim k^n$ nodes can be visited in average. A closed loop will exist if one of these visited nodes is the original one. The probability for this to happen is simply $k^n/\overline{N}$, which is negligible unless $n \sim \log \overline{N}/\log pr^d$. Thus the shortest closed loop in the random graph of blocks will typically contain $O(\log \overline{N}/\log pr^d)$ or more nodes. The cost associated with visiting a node can be bounded to be at most $O(r)$ and at least $O(s(p))$. Taking a conservative estimate let us say it is $s(p)$. Then the lowest cost of going from $a$ to $b$ through shortcuts is typically $\sim s(p) \log \overline{N}/\log pr^d$. Thus it will be “cheaper” to travel from $a$ to $b$ directly on the Euclidean lattice than through shortcuts (and consequently $\ell(r) = r$) if $r < s(p) \log \overline{N}/\log pr^d$.

This approximate scaling argument is confirmed in its main aspects by recent analytical [8] and numerical [7] studies (see Fig. 3), which show that $\ell(r)$ behaves as

$$\ell(r) \sim \begin{cases} r & \text{for } r < r_c = p^{-1/d} \log(p^{1/d}L) \\ r_c & \text{for } r > r_c \end{cases}$$

In other words, the characteristic length for shortest-path distances $\ell(r)$ is not $s(p)$ but $r_c$ above, which diverges with $L$ for all $p$. 

FIG. 2. We divide the system into blocks (shaded) of size $r$ (the Euclidean distance between $a$ and $b$) in order to estimate the average shortest-path distance $\ell(r)$ when $r$ is much larger than the mean separation $p^{-1/d}$ between shortcut-ends (white dots). By considering the distance travelled on the lattice (dashed lines) along any path from $a$ to $b$ through shortcuts (black lines), one finds that $\ell(r) \sim r$ unless $r > p^{-1/d} \log L$ (see text). 

FIG. 3.
In summary, we showed that the small-world transition is neither a "crossover phenomenon" nor a second-order phase transition, but a first-order phase transition at $p = 0$. Finite size corrections are dictated by a non-critical $\xi(p) \sim p^{1/d}\log L$ diverges with system size.

A relatively simple scaling argument, confirmed by more rigorous studies, shows that the characteristic length $r_c \sim p^{-1/d}\log L$ diverges with system size. For a non-critical first order transition despite the fact that $s(p)$ diverges at $p = 0$. We feel that a similar phenomenon happens in the small-world transition since the average distance between shortcut ends $s(p)$ diverges at $p = 0$ but plays no role in determining shortest paths.

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