Original Article

Interior continuity, continuity up to the boundary, and Harnack’s inequality for double-phase elliptic equations with nonlogarithmic conditions

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Abstract
We prove continuity and Harnack’s inequality for bounded solutions to elliptic equations of the type
\[
\text{div} (|\nabla u|^{p-2} \nabla u + a(x)|\nabla u|^{q-2} \nabla u) = 0, \quad a(x) \geq 0,
\]
\[
|a(x) - a(y)| \leq A|x - y|^\delta \mu(|x - y|), \quad x \neq y,
\]
\[
\text{div} ([|\nabla u|^{p-2} \nabla u(1 + \ln(1 + b(x)|\nabla u|))] = 0, \quad b(x) \geq 0,
\]
\[
|b(x) - b(y)| \leq B|x - y| \mu(|x - y|), \quad x \neq y,
\]
\[
\text{div} ([|\nabla u|^{p-2} \nabla u + c(x)|\nabla u|^{q-2} \nabla u[1 + \ln(1 + |\nabla u|)]^\beta] = 0,
\]
\[
c(x) \geq 0, \quad \beta \geq 0, |c(x) - c(y)| \leq C|x - y|^{q-p} \mu(|x - y|), \quad x \neq y,
\]
under the precise choice of \( \mu \).

KEYWORDS
continuity of solutions, double-phase elliptic equations, Harnack’s inequality, nonlogarithmic conditions, regularity of a boundary point

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1 INTRODUCTION AND MAIN RESULTS

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n, n \geq 2 \). In this paper, we are concerned with elliptic equations of the type
\[
\text{div} A(x, \nabla u) = 0, \quad x \in \Omega. \tag{1.1}
\]
We suppose that the functions \( A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n \) are such that \( A(\cdot, \xi) \) are Lebesgue measurable for all \( \xi \in \mathbb{R}^n \), and \( A(x, \cdot) \) are continuous for almost all \( x \in \Omega \). We assume also that the following structure conditions are satisfied:
\[
A(x, \xi) \xi \geq K_1 g(x, |\xi|) |\xi|, \tag{1.2}
\]
\[
|A(x, \xi)| \leq K_2 g(x, |\xi|),
\]
where \( K_1, K_2 \) are positive constants.
This type of equations belongs to a wide class of elliptic equations with generalized Orlicz growth. In terms of the function $g$, this class can be characterized as follows. Let $g(x, v): \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ be a nonnegative function satisfying the following conditions:

1. $g_0$: For any $x \in \Omega$ the function $v \to g(x, v)$ is increasing, $\lim_{v \to 0^+} g(x, v) = 0$, $\lim_{v \to +\infty} g(x, v) = +\infty$, and $c_0^{-1} \leq g(x, 1) \leq c_0$ with some positive constant $c_0$.

2. $g_1$: There exist $1 < p < q$ such that for $x \in \Omega$, and for $w \geq v > 0$, there holds

$$\left(\frac{w}{v}\right)^{p-1} \leq \frac{g(x, w)}{g(x, v)} \leq \left(\frac{w}{v}\right)^{q-1}.$$

3. $g_2$: Fix $R > 0$ such that $B_R(x_0) \subset \Omega$. There exist constants $c, c_1 > 0$ and a positive, continuous, and nondecreasing function $\lambda(r)$ on the interval $(0, R)$, $\lambda(r) \leq 1$, $\lim_{r \to 0^+} r^{1-\delta_0}/\lambda(r) = 0$, and $\lambda(r) \leq (3/2)^1-\delta_0 \lambda(r/2)$ with some $\delta_0 \in (0, 1)$, such that for any $K > 0$, there holds

$$g(x, v/r) \leq c K c_1 g(y, v/r),$$

for any $x, y \in B_r(x_0) \subset B_R(x_0)$ and for all $r \leq v \leq K\lambda(r)$.

We note that condition $(g_1)$ and conditions $(Inc)_{p-1}$, $(Inc)_{q-1}$ from [22] coincide. Moreover, in the case $\lambda(r) = 1$, condition $(g_2)$ and condition $(A1-n)$ from [22] are equivalent.

Sometimes, we will assume that condition $(g_2)$ holds for $x_0 \in \partial \Omega$, in this case, we will assume that there exists $R > 0$, such that for every $v > 0$, the function $g(\cdot, v)$ is defined in $B_R(x_0)$ and for any $x, y \in B_r(x_0) \subset B_R(x_0)$, condition $(g_2)$ is valid.

**Remark 1.1.** The function $g_a(x)(v):= v^{p-1} + a(x)v^{q-1}$, $v > 0$, where $a(x) \geq 0$ in $\Omega$,

$$|a(x) - a(y)| \leq A|x - y|^{q-p} \mu(|x - y|), \ x, y \in \Omega, \ x \neq y,$$

$$A > 0, \ 1 < p < q, \ \lim_{r \to 0} \mu(r) = +\infty, \ \lim_{r \to 0} r^{\beta-p} \mu(r) = 0,$$

satisfies condition $(g_2)$ with $\lambda(r) = \mu^{-\frac{1}{q-p}}(r)$. Indeed,

$$g_a(x)(v/r) - g_a(y)(v/r) \leq A \mu(r) v^{q-p}(v/r)^{p-1} \leq AK^{q-p}(v/r)^{p-1} \leq AK^{q-p} g_a(y)(v/r) \text{ if } r \leq v \leq K\lambda(r).$$

We still need the following clarification. In the case $a(x_0) > 0$, we can take $\lambda(r) \equiv 1$ in condition $(g_2)$. Indeed, if $a(x_0) > 0$, then we choose $R$ from the condition $AR^{q-p} \mu(R) = \frac{1}{2}a(x_0)$. This choice guarantees that

$$\frac{1}{2} a(x_0) \leq a(x) \leq \frac{3}{2} a(x_0) \text{ for } x \in B_R(x_0).$$

Then, for all $x, y \in B_R(x_0)$ and $r \leq v \leq K$, we have

$$g_a(x)(v/r) \leq g_{3a(x_0)/2}(v/r) \leq 3g_{a(x_0)/2}(v/r) \leq 3g_a(y)(v/r).$$

From which the required $(g_2)$ follows with $\lambda(r) \equiv 1$.

The function $g_b(x)(v):= v^{p-1} [1 + \ln (1 + b(x)v)], v > 0$, where $b(x) \geq 0$ in $\Omega$,

$$|b(x) - b(y)| \leq B|x - y| \mu(|x - y|), \ x, y \in \Omega, \ x \neq y, \ B > 0, \ \lim_{r \to 0} \mu(r) = +\infty, \ \lim_{r \to 0} r \mu(r) = 0,$$
satisfies condition \((g_2)\) with \(\lambda(r) = 1/\mu(r)\). Indeed,
\[
\frac{\mathcal{g}_b(x)}{r} \left( \frac{\lambda(r)v}{r} \right) - \frac{\mathcal{g}_b(y)}{r} \left( \frac{\lambda(r)v}{r} \right) \\
\leq \left( \frac{\lambda(r)v}{r} \right)^{p-1} \ln \left( \frac{1 + b(x)\lambda(r)v}{1 + b(y)\lambda(r)v} \right) \\
\leq \left( \frac{\lambda(r)v}{r} \right)^{p-1} \ln \left( \frac{1 + |b(x) - b(y)|\lambda(r)v}{r} \right) \\
\leq \left( \frac{\lambda(r)v}{r} \right)^{p-1} \ln (1 + B\lambda(r)\mu(r)) \\
\leq \left( \frac{\lambda(r)v}{r} \right)^{p-1} \ln (2 + BK) \text{ if } r\mu(r) \leq v \leq K.
\]

If \(b(x_0) > 0\), then choosing \(R\) from the condition \(BR\mu(R) = \frac{1}{2}b(x_0)\), as in the previous case, we can take \(\lambda(r) \equiv 1\) in condition \((g_2)\).

Similarly, the function \(\mathcal{g}_c(x)(v) := v^{p-1} + c(x)v^{q-1}[1 + \ln(1+v)]^{\beta}\), \(v > 0\), \(\beta \geq 0\), \(c(x) \geq 0\),
\[
|c(x) - c(y)| \leq C|x - y|^{q-p}\mu(|x - y|), \quad x, y \in \Omega, \quad x \neq y, \quad C > 0, \quad \lim_{r \to 0} \mu(r) = +\infty, \quad \lim_{r \to 0} r^{q-p}\mu(r) = 0,
\]
satisfies condition \((g_2)\) with \(\lambda(r) = \left( \frac{\mu(r)\ln r^{-1}}{r} \right)^{\frac{1}{q-p}}\).

**Remark 1.2.** Let us consider the functions \(g_1(x, v) = v^{p-1}(1 + b(x)\ln(1 + v)), v > 0, b(x) \geq 0\) in \(\Omega\),
\[
|p(x) - p(y)| + |b(x) - b(y)| \leq \frac{\mu(r)}{\ln r^{-1}}, \quad x, y \in B_r(x_0), \quad \lim_{r \to 0} \mu(r) = +\infty, \quad \lim_{r \to 0} \frac{\mu(r)}{\ln r^{-1}} = 0.
\]
It is obvious that the functions \(g_1, g_2\) satisfy condition \((g_2)\) with
\[
\lambda(r) = r \exp \left( \frac{\ln r^{-1}}{\mu(r)} \right) \leq r \exp \left( \frac{1}{2} \ln r^{-1} \right) = r^{1/2}
\]
for sufficiently small \(r\), and hence our main condition (see equality \(1.6\) below) is violated. In what follows, in the case of functions \(g_1, g_2\), we will assume that \(\mu(r) \equiv \text{const}\). So, in this case condition \((g_2)\) is equivalent to the logarithmic Zhikov’s condition. The qualitative properties of solutions to elliptic equations with logarithmic growth are well known (see, e.g., [1–3, 5, 6, 18, 19, 21–24]).

The aim of this paper is to establish basic qualitative properties, such as interior continuity of bounded solutions, their continuity up to the boundary, and Harnack’s inequality for nonnegative bounded solutions to Equation \((1.1)\).

Before formulating the main results, let us recall the definition of a bounded weak solution to Equation \((1.1)\). We set \(G(x,v) := g(x,v)v\) for \(x \in \Omega\) and \(v \geq 0\) and write \(W(\Omega)\) for the class of functions \(u \in W^{1,1}(\Omega)\) with \(\int_{\Omega} G(x,|\nabla u|) dx < +\infty\). We also need a class of functions \(W_0(\Omega)\), which consists of functions \(u \in W^{1,1}_0(\Omega)\), such that \(\int_{\Omega} G(x,|\nabla u|) dx < +\infty\).

**Definition 1.3.** We say that a function \(u \in W(\Omega) \cap L^\infty(\Omega)\) is a bounded weak sub(super)-solution to Equation \((1.1)\), if
\[
\int_{\Omega} A(x,\nabla u) \nabla \varphi dx \leq (\geq) 0,
\]
holds for all nonnegative test functions \(\varphi \in W_0^1(\Omega)\).

**Remark 1.4.** Note that we are dealing only with bounded solutions. To prove the boundedness, we need additional assumptions either on the function \(g\) (see, e.g., [11, 14, 15, 22, 23]), or on the solution itself (see [12, 32]).

Our first main result of this paper reads as follows:
Theorem 1.5. Fix $x_0 \in \Omega$, let the assumptions $(g_0)$, $(g_1)$, and $(g_2)$ be fulfilled and let $u \in W(\Omega) \cap L^\infty(\Omega)$ be a bounded weak solution to Equation (1.1). Then, there exists positive number $C_1$ depending only on $K_1$, $K_2$, $n$, $p$, $q$, $c$, $c_0$, $c_1$, $\delta_0$ and $M := \text{ess sup}_\Omega |u|$ such that

$$\text{osc } u \leq 2M \exp \left( -C_1 \int_2^\rho \frac{\lambda(t)}{t} \frac{dt}{t} \right) + C_1 \frac{\rho}{\lambda(\rho)}$$

(1.5)

for any $0 < 2r < \rho < R$. If additionally

$$\int_0^\rho \frac{\lambda(t)}{t} \frac{dt}{t} = +\infty,$$

(1.6)

then the solution $u$ is continuous at $x_0$.

Remark 1.6. For the function $g_{a(x)}(\cdot)$ (see Remark 1.1) condition (1.6) can be rewritten as

$$\int_0^\rho -\frac{1}{\mu_p(r)} \frac{dr}{r} = +\infty.$$

This condition holds, for example, if $\mu(r) = \ln^\beta r^{-1}$ and $0 \leq \beta \leq q - p$.

For the function $g_{b(x)}(\cdot)$ in Remark 1.1, condition (1.6) can be rewritten as

$$\int_0^\rho \frac{1}{\mu(r)} \frac{dr}{r} = +\infty.$$

The function $\mu(r) = \ln r^{-1}$ satisfies the above condition.

Finally, for the function $g_{c(x)}(\cdot)$ in Remark 1.1, condition (1.6) can be rewritten as

$$\int_0^\rho \left( \mu(r) \ln^\beta r^{-1} \right)^{-\frac{1}{q-p}} \frac{dr}{r} = +\infty.$$

The function $\mu(r) = \ln^\beta_1 r^{-1}$, $\beta_1 \geq 0$, $\beta_1 + \beta \leq q - p$ satisfies the above condition.

To formulate our next result, we need the following definition:

**Definition 1.7** (capacity). Fix $x_0 \in \mathbb{R}^n$ and let $E \subset B_r(x_0) \subset B_{\rho}(x_0)$. For any $m > 0$, we set

$$C(E, B_{8\rho}(x_0); m) := \inf_{\varphi \in \mathfrak{R}(E)} \int_{B_{8\rho}(x_0)} g(x, m|\nabla \varphi|) |\nabla \varphi| \, dx,$$

where the infimum is taken over the set $\mathfrak{R}(E)$ of all functions $\varphi \in W_0(B_{8\rho}(x_0))$ with $\varphi \geq 1$ on $E$.

If $m = 1$, this definition leads to the standard definition of $C_C(E, B_{8\rho}(x_0))$ capacity (see, e.g., [21]).

Note that if $E = B_\rho(x_0)$ and $\rho \leq m \leq \lambda(8\rho)$, then $C(B_\rho(x_0), B_{8\rho}(x_0); m) \asymp \rho^{n-1} g(x_0, m/\rho)$, where the notation $f \asymp g$ means that there exists a constant $\gamma > 0$ such that $\gamma^{-1} f \leq g \leq \gamma f$. Indeed, let $\varphi \in W_0(B_{8\rho}(x_0))$ be such that $0 \leq \varphi \leq 1$, $\varphi = 1$ in $B_\rho(x_0)$ and $|\nabla \varphi| \leq 8/\rho$. Then, by $(g_1)$ and $(g_2)$ with $K = 1$, we obtain

$$C(B_\rho(x_0), B_{8\rho}(x_0); m) \leq \int_{B_{8\rho}(x_0)} g(x, m|\nabla \varphi|) |\nabla \varphi| \, dx \leq \frac{8^q}{\rho} \int_{B_{8\rho}(x_0)} g(x, m/\rho) \, dx \leq \gamma_{c,q} g(x_0, m/\rho) \rho^{n-1}.$$

For the opposite inequality, we need the following analog of the Young inequality:

$$g(x, a)b \leq \varepsilon g(x, a) + g(x, b/\varepsilon)b, \quad \varepsilon, a, b > 0, \ x \in \Omega.$$  

(1.7)
By \((g_1), (g_2)\) with \(K = 1\), the definition of 1-capacity and our choice of \(m\), we have
\[
C_1(B_{\rho}(x_0), B_{8\rho}(x_0)) g(x_0, m/\rho)
\leq g(x_0, m/\rho) \int_{B_{8\rho}(x_0)} |\nabla \varphi| \, dx
\leq \int_{B_{8\rho}(x_0)} g(x, m/\rho) |\nabla \varphi| \, dx
\leq \frac{\varepsilon m}{\rho} \int_{B_{8\rho}(x_0)} g(x, m/\rho) \, dx
+ \int_{B_{8\rho}(x_0)} g(x, |\nabla \varphi|/\varepsilon) |\nabla \varphi| \, dx
\leq \varepsilon \gamma \gamma_{c, q} mg(x_0, m/\rho)\rho^{n-1}
+ \int_{B_{8\rho}(x_0)} g(x, |\nabla \varphi|/\varepsilon) |\nabla \varphi| \, dx.
\]
Choosing \(\varepsilon = \varepsilon_1/m, \varepsilon_1 \in (0, 1)\), from the last inequality, we obtain
\[
C_1(B_{\rho}(x_0), B_{8\rho}(x_0)) g(x_0, m/\rho)
\leq \varepsilon_1 \gamma \gamma_{c, q} g(x_0, m/\rho)\rho^{n-1}
+ \frac{\varepsilon_1^{1-q}}{1-q} \int_{B_{8\rho}(x_0)} g(x, |\nabla \varphi|) |\nabla \varphi| \, dx.
\]
Since \(C_1(B_{\rho}(x_0), B_{8\rho}(x_0)) \asymp \rho^{n-1}\), choosing \(\varepsilon_1\) sufficiently small, from this, we arrive at the required inequality.

Further, we also need a following definition.

**Definition 1.8** (regular boundary points). We say that \(x_0 \in \partial \Omega\) is a regular boundary point of the domain \(\Omega\) for Equation (1.1) if for every bounded weak solution \(u \in W(\Omega) \cap L^\infty(\Omega)\) of Equation (1.1) satisfying the condition \(u - f \in W_0(\Omega)\), \(f \in \mathcal{C}(\Omega) \cap W(\Omega)\), the following equality holds:
\[
\lim_{\Omega \ni x \to x_0} u(x) = f(x_0).
\]

Our next main result of this paper reads as follows:

**Theorem 1.9.** Let conditions \((g_0), (g_1), \) and \((g_2)\) be fulfilled in \(\overline{\Omega}\). Assume also the following strict monotonicity condition:
\[
(A(x, \xi) - A(x, \eta))(\xi - \eta) > 0 \text{ if } x \in \Omega, \xi, \eta \in \mathbb{R}^n, \xi \neq \eta.
\]
Then, \(x_0 \in \partial \Omega\) is a regular boundary point of the domain \(\Omega\) for Equation (1.1), provided that
\[
\int_0^{g_{x_0}^{-1} \left( C(B_{r}(x_0) \setminus \Omega, B_{8r}(x_0); \lambda(r)) \right)} r^{n-1} \, dr = +\infty.
\]
Here, we use the notation \(g_{x_0}^{-1}(\cdot)\) for the inverse function to the function \(g(x_0, \cdot)\).

It seems that Theorem 1.9 is new even in the logarithmic case (i.e., if \(\lambda(r) \equiv 1\) in \((g_2)\)). In the case of the \(p(x)\)-Laplace equation and \(\lambda(r) \equiv 1\) (see Remark 1.2), Theorem 1.9 was proved in [7], see also [3]. In the case where \(\lambda(r) \equiv 1\) and
\[
C_G(B_{r}(x_0) \setminus \Omega, B_{8r}(x_0)) \geq \gamma^{-1} C_G(B_{8r}(x_0)) \asymp g(x_0, 1/r)r^{n-1},
\]
Theorem 1.9 was proved in [21]. Note also that the inequality
\[
C(B_{r}(x_0) \setminus \Omega, B_{8r}(x_0); \lambda(r)) \geq \gamma^{-1} C(B_{r}(x_0), B_{8r}(x_0); \lambda(r)) \asymp g(x_0, \lambda(r)/r)r^{n-1}
\]
implies that
\[
g_{x_0}^{-1} \left( C(B_{r}(x_0) \setminus \Omega, B_{8r}(x_0); \lambda(r)) \right) \geq \gamma^{-1} \frac{\lambda(r)}{r}.
\]
Therefore, in this case, condition \(\int_0^{\lambda(r)} r^{-1} \, dr = +\infty\) implies (1.9).

The following remark shows how the condition (1.9) can be rewritten for the double-phase elliptic equation.
Remark 1.10. Let $g(x, \cdot) = g_a(x)(\cdot)$ (see Remark 1.1). Let us consider separately two cases: $a(x_0) = 0$ and $a(x_0) > 0$. First, if $a(x_0) = 0$, then evidently we have

$$
\frac{C(B_r(x_0) \setminus \Omega, B_{sr}(x_0); \lambda(r))}{r^{n-1}} \geq \lambda(r) \left( \frac{C_p(B_r(x_0) \setminus \Omega, B_{sr}(x_0))}{r^{n-1}} \right)^{-\frac{1}{p-1}}.
$$

So, in the case $a(x_0) = 0$, condition

$$
\int_0 \lambda(r) \left( \frac{C_p(B_r(x_0) \setminus \Omega, B_{sr}(x_0))}{r^{n-p}} \right)^{-\frac{1}{p-1}} \, dr = +\infty
$$

implies (1.9).

Now, let $a(x_0) > 0$ and choose $\rho$ from the condition $A_\rho = \mu(\rho) = \frac{1}{2} a(x_0)$ (cf. Remark 1.1). This choice of $\rho$ guarantees that $\frac{1}{2} a(x_0) \leq a(x) \leq \frac{3}{2} a(x_0)$ for $x \in B_\rho(x_0)$. In this case, as it was noted in Remark 1.1, we can choose $\lambda(r) \equiv 1$. So, we have

$$
\frac{C(B_r(x_0) \setminus \Omega, B_{sr}(x_0); 1)}{r^{n-1}} \geq \frac{1}{2} a(x_0) \frac{C_q(B_r(x_0) \setminus \Omega, B_{sr}(x_0))}{r^{n-1}}.
$$

Using the definition of the function $g_a(x_0)(\cdot)$ (see Remark 1.1) and the Young inequality, we obtain

$$
g_a(x_0) \left( \frac{C_q(B_r(x_0) \setminus \Omega, B_{sr}(x_0))}{r^{n-1}} \right)^{-\frac{1}{q-1}} \leq 1 + a(x_0) \frac{C_q(B_r(x_0) \setminus \Omega, B_{sr}(x_0))}{r^{n-1}},
$$

and hence

$$
\left( \frac{C_q(B_r(x_0) \setminus \Omega, B_{sr}(x_0))}{r^{n-1}} \right)^{-\frac{1}{q-1}} \leq \gamma + \gamma g_a^{-1} \left( 1 + a(x_0) \frac{C_q(B_r(x_0) \setminus \Omega, B_{sr}(x_0))}{r^{n-1}} \right)
$$

$$
\leq \gamma + \gamma a(x_0) g_a^{-1} \left( \frac{C(B_r(x_0) \setminus \Omega, B_{sr}(x_0); 1)}{r^{n-1}} \right).
$$

Therefore, in the case $a(x_0) > 0$, condition

$$
\int_0 \left( \frac{C_q(B_r(x_0) \setminus \Omega, B_{sr}(x_0))}{r^{n-q}} \right)^{-\frac{1}{q-1}} \, dr = +\infty
$$

implies (1.9).

Our next result is the Harnack inequality for positive bounded solutions to Equation (1.1). To formulate this result, we need another property that characterizes equations with nonlogarithmic conditions.

(g3) Fix $R > 0$ such that $B_R(x_0) \subset \Omega$. We assume that for any $K > 0$, there exists $C(K) > 0$ and positive, continuous, and nonincreasing function $\delta(r)$ on the interval $(0, R)$, $\delta(r) \geq 1$, $\lim_{r \to 0} r^{1-\delta_0} \delta(r) = 0$, and $\delta(r/2) \leq (3/2)^{1-\delta_0} \delta(r)$ with some $\delta_0 \in (0, 1)$, such that

$$
g(x, v/r) \leq C(K) \delta(r) g(y, v/r)
$$

for any $x, y \in B_r(x_0) \subset B_R(x_0)$ and for all $r \leq v \leq K$.

Remark 1.11. The function $g_a(x)(\cdot)$ defined in Remark 1.1 satisfies condition (g3) with $\delta(r) = \mu(r)$. Indeed,

$$
g_a(x)(v/r) - g_a(y)(v/r) \leq A \mu(r) v^{q-p} (v/r)^{p-1} \leq AK^{q-p} \delta(r) g_a(y)(v/r) \quad \text{if} \ r \leq v \leq K.
$$
Similarly, the function $g_{b(x)}(\cdot)$ in Remark 1.1 satisfies condition (g3) with $\vartheta(r) = \ln \mu(r)$. Indeed, the following inequality holds:

$$g_{b(x)}(v/r) - g_{b(y)}(v/r) \leq (v/r)^{p-1} \ln (2 + |b(x) - b(y)| v/r)$$

$$\leq (v/r)^{p-1} \ln (2 + BK \mu(r))$$

$$\leq \gamma(B, K)(v/r)^{p-1} \ln \mu(r) \leq \gamma(B, K) \ln \mu(r) g_{b(y)}(v/r)$$

if $x, y \in B_r(x_0) \subset B_R(x_0)$, $r \leq v \leq K$, and $R > 0$ is small enough.

Note also that due to Remark 1.1 we can take $\vartheta(r) \equiv 1$ in condition (g3) for the functions $g_{a(x)}(\cdot)$ with $a(x_0) > 0$ and $g_{b(x)}(\cdot)$ with $b(x_0) > 0$. Obviously, the function $g_{c(x)}(\cdot)$ in Remark 1.1 satisfies condition (g3) with $\vartheta(r) = \mu(r) \ln \beta \nu^{-1}$.

**Theorem 1.12.** Let conditions (g0), (g1) be satisfied, and let conditions (g2) and (g3) be fulfilled in some ball $B_R(x_0) \subset \Omega$, and

$$\int_0^\infty \vartheta^{- \frac{2n}{p-1}}(r) \lambda(r) \frac{dr}{r} = +\infty.$$

Assume also that conditions (1.8), (1.9) hold, and let $u \in W(\Omega) \cap L^\infty(\Omega)$ be a bounded nonnegative weak solution to Equation (1.1). Then, there exists positive number $C_2$ depending only on $M, n, p, q, K_1, K_2, c_0, c, c_1, \delta_0,$ and $C(M)$ such that either

$$u(x_0) \leq C_2 \rho \vartheta^{- \frac{2n}{p-1}}(\rho) / \lambda(\rho) \quad (1.10)$$

or

$$u(x_0) \leq C_2 \vartheta^{- \frac{2n}{p-1}}(\rho) \frac{\inf_{B_\rho(x_0)} u}{\lambda(\rho)} \quad (1.11)$$

for all $0 < \rho < R/8$.

**Remark 1.13.** We note that in the case, where $g(x, \cdot) = g_{a(x)}(\cdot)$ (see Remark 1.1) and

$$\mu(\rho) = (\ln \rho^{-1})^\beta, \quad \lambda(\rho) = (\ln \rho^{-1})^{- \frac{\beta}{q-\beta}}, \quad \vartheta(\rho) = (\ln \rho^{-1})^\beta, \quad 0 \leq \beta \leq \frac{q-p}{1 + \frac{2n}{p-1}(q-p)},$$

inequalities (1.10), (1.11) translate into

$$u(x_0) \leq C_2 \ln \frac{1}{\rho} \left( \inf_{B_\rho(x_0)} u + \rho \right).$$

In addition, note that in the case $g(x, \cdot) = g_{b(x)}(\cdot)$ and $\mu(\rho) = (\ln \rho^{-1})^\beta, \lambda(\rho) = (\ln \rho^{-1})^{- \beta}, \vartheta(\rho) = \beta \ln \ln \rho^{-1}, \beta \in (0, 1)$, inequalities (1.10) and (1.11) can be rewritten as

$$u(x_0) \leq C_2 \ln \frac{1}{\rho} \left( \inf_{B_\rho(x_0)} u + \rho \right).$$

We also observe that a similar inequality holds in the case $g(x, \cdot) = g_{c(x)}(\cdot)$ and $\mu(\rho) = (\ln \rho^{-1})^\beta, \lambda(\rho) = (\ln \rho^{-1})^{- \beta}, \vartheta(\rho) = (\ln \rho^{-1})^{\beta + \beta_1}, 0 \leq (\beta + \beta_1) \left( 1 + \frac{2n}{p-1}(q-p) \right) \leq q - p.$

Finally, due to Remarks 1.1 and 1.11, we note that in the cases $g(x, \cdot) = g_{a(x)}(\cdot)$ and $a(x_0) > 0$ or $g(x, \cdot) = g_{b(x)}(\cdot)$ and $b(x_0) > 0$ inequalities (1.10), (1.11) can be rewritten as

$$u(x_0) \leq C_2 \left( \inf_{B_\rho(x_0)} u + \rho \right).$$
These conditions on $\mu(\rho)$ are worse than in Theorems 1.5 and 1.9, but they are much better than condition which was known earlier (see (1.12) below).

Before describing the method of proof, we say a few words concerning the history of the problem. The study of regularity of minima of functionals with nonstandard growth has been initiated by Zhikov [43–46, 48], Marcellini [30, 31], and Lieberman [29], and in the last 30 years, the qualitative theory of elliptic and parabolic equations with so-called log-condition (i.e., if $\lambda(r) = 1$ and $0 < \theta(r) \leq L < +\infty$) is being actively developed (see, e.g., [1, 3, 12, 13, 17, 20–23, 38, 39, 41] for references). Equations of this type and systems of such equations arise in various problems of mathematical physics (see, e.g., the monographs [8, 20, 33, 42] and references therein).

Double-phase elliptic equations under the logarithmic condition were studied by Colombo, Mingione [14–16], and by Baroni, Colombo, and Mingione [9–11]. Particularly, $C_{0,\beta}^{1}(\Omega)$, $C_{1,\beta}^{0}(\Omega)$ regularity and Harnack’s inequality were obtained in the case $\lambda(r) = 1$ and $\theta(r) = 1$, under the precise conditions on the parameters $\alpha, p, q$.

The case when conditions (g2) or (g3) hold differs substantially from the logarithmic case. To our knowledge, there are few results in this direction. Zhikov [47] obtained a generalization of the logarithmic condition, which guarantees the density of smooth functions in the Sobolev’s space $W^{1,p}(\Omega)$. Particularly, this result holds if $1 < p \leq p(x)$ and

$$|p(x) - p(y)| \leq L \frac{\ln|\ln|x - y||}{\ln|x - y||}, \quad x, y \in \Omega, \quad x \neq y, \quad 0 < L < p/n.$$  

Later, Zhikov and Pastukhova [49] under the same condition proved higher integrability of the gradient of solutions to the $p(x)$-Laplace equation. Interior continuity, continuity up to the boundary, and Harnack’s inequality to the $p(x)$-Laplace equation were proved by Alkhutov, Krasheninnikova [4], Alkhutov, Surnachev [7], and Surnachev [40] under the condition (g3), where $\theta(r)$ satisfies

$$\int_0^\infty \exp(-\gamma \theta(r)) \frac{dr}{r} = +\infty,$$  

with some positive constants $\gamma, c > 1$. Particularly, the function $\theta(r) = (\ln \ln r)^\gamma$, $L < 1/c$ satisfies the above condition. These results were generalized in [34, 39] for a wide class of elliptic and parabolic equations with nonlogarithmic Orlicz growth. Particularly, it was proved in [39] that under assumptions (g1), (g3), and (1.12) functions from the corresponding De Giorgi’s $B_1(\Omega)$ classes are continuous and moreover, it was shown that the solutions of the correspondent elliptic and parabolic equations with nonstandard growth belong to these classes.

In this paper, we substantially refine the results of [34, 39]. We would like to mention the approach taken in this paper. To prove interior continuity, we use De Giorgi’s approach. We consider De Giorgi’s $B_1(\Omega)$ classes and we prove the continuity of the functions belonging to these classes. The main difficulty arising in the proof of the main results is related to the so-called theorem on the expansion of positivity. Roughly speaking, having information on the measure of the “positivity set” of $u$ over the ball $B_r(\bar{x})$:

$$\left| \{x \in B_r(\bar{x}) : u(x) \leq N \} \right| \leq (1 - \beta(r)) |B_r(\bar{x})|$$

with some $r > 0$, $N > 0$ and $\beta := \beta(r) \in (0, 1)$, and using the standard De Giorgi’s or Moser’s arguments, we inevitably come to the estimate

$$u(x) \geq N r^{-1} e^{-\gamma \beta^{-1}(r)}, \quad x \in B_{2r}(\bar{x}),$$

with some $\gamma, \bar{c} > 1$. This estimate leads us to condition (1.12) (see, e.g., [34, 39]). To avoid this, we use a workaround that goes back to Landis’s works [27, 28] and his so-called “growth” lemma. So, we use the following auxiliary solutions. Fix $x_0 \in \mathbb{R}^n$ and let $E \subset B_{r}(x_0) \subset B_{2r}(x_0)$. We consider the solution $v := v(x, m)$ of the following problem:

$$\text{div}A(x, \nabla v) = 0, \quad x \in D := B_{8\rho}(x_0) \setminus E,$$  

$$v - m\psi \in W_0^1(D), \quad \rho \leq m \leq 2M\lambda(8\rho),$$

where $m$ is some fixed number, and $\psi \in W_0^1(B_{8\rho}(x_0))$, $\psi = 1$ on $E$.

The main step in the proof of Theorems 1.9 and 1.12 is the following theorem:
Theorem 1.14. Let conditions \((g_0), (g_1),\) and \((g_2)\) be fulfilled and let \(v\) be a solution to \((1.13), (1.14)\). Then, there exists a positive constant \(C_3\) depending only on \(n, p, q, K_1, K_2, c_0, c_1, \delta_0,\) and \(M\) such that either
\[
C(E, B_{8\rho}(x_0); m) \leq C_3 \rho^{n-1}
\]
(1.15)
or
\[
C_3^{-1} \frac{g_{x_0}^{-1}}{\rho^{n-1}} \left( \frac{C(E, B_{8\rho}(x_0); m)}{\rho^{n-1}} \right) \rho \leq v(x) \leq C_3 \frac{g_{x_0}^{-1}}{\rho^{n-1}} \left( \frac{C(E, B_{8\rho}(x_0); m)}{\rho^{n-1}} \right) \rho
\]
(1.16)
for a.a. \(x \in B_{2\rho}(x_0) \setminus B_{\rho}(x_0)\).

The rest of the paper contains the proof of the above theorems.

In Section 2, we consider elliptic \(B_1(\Omega)\) classes and prove interior continuity of functions from these classes. Section 3 contains the upper and lower bounds for auxiliary solutions of \((1.13), (1.14)\). In Section 4, there is continuity up to the boundary and Harnack’s inequality, proofs of Theorems 1.9, 1.12. We also note that our proofs do not require studying the special properties of Orlicz spaces. Finally, we note that in the proof of the main results, we do not distinguish between the cases of so-called \(p\)-growth (i.e., if \(a(x) = 0\) or \(b(x) = 0\)) and \((p, q)\)-growth (i.e., if \(a(x) > 0\) or \(b(x) > 0\)). Moreover, it seems that even in the case when \(\mu(r) = 1\), Equation \((1.1)\) with \(g(x, \cdot) = g_b(\cdot)(\cdot)\) and \(g(x, \cdot) = g_c(x)(\cdot)\) is considered here for the first time.

## 2 ‖ ELLIPTIC \(B_1\) CLASSES AND LOCAL CONTINUITY: PROOF OF THEOREM 1.5

In this section, we prove the continuity of functions from the corresponding \(B_1\) classes. As it was already mentioned, these classes were practically defined in [39].

Definition 2.1. We say that a measurable function \(u : \Omega \to \mathbb{R}\) belongs to the elliptic class \(B_{1, g}(\Omega)\) if \(u \in W^{1,1}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)\), \(\text{ess sup}_{\Omega} |u| \leq M\), and there exist numbers \(1 < p < q, c_2 > 0\) such that for any ball \(B_{2\rho}(x_0) \subset \Omega\), any \(k, l \in \mathbb{R}, k < l, |k|, |l| < M\), any \(\varepsilon \in (0, 1]\), any \(\sigma \in (0, 1)\), and for any \(\zeta \in C_0^\infty(\mathbb{B}_{\rho}(x_0))\), \(0 \leq \zeta \leq 1\), \(\zeta = 1\) in \(\mathbb{B}_{\rho}(1-\sigma)(x_0)\), the following inequalities hold:
\[
\int_{A_{k, \rho}^+} g\left(x, \frac{M_+(k, \varphi)}{\varphi}\right) |
\n(2.1)
\]
\[
\int_{A_{k, \rho}^-} g\left(x, \frac{M_-(l, \varphi)}{\varphi}\right) |
\n(2.2)
\]
\[
\int_{A_{k, \rho}^+} g\left(x, \frac{M_+(k, \varphi)}{\varphi}\right) |
\n\]
\[
\int_{A_{k, \rho}^-} g\left(x, \frac{M_-(l, \varphi)}{\varphi}\right) |
\n\]
\[
\text{here } (u - k)_\pm := \max\{\pm(u - k), 0\}, A_{k, \rho}^\pm := B_{\rho}(x_0) \cap \{u - k\}_\pm > 0\}, M_\pm(k, \varphi) := \text{ess sup}_{B_{\rho}(x_0)}[u - k]_\pm.
\]

Our main result of this section reads as follows:

Theorem 2.2. Fix \(x_0 \in \Omega\), let conditions \((g_0), (g_1), (g_2)\), and \((1.6)\) be fulfilled. Let \(u \in B_{1, g}(\Omega)\), then \(u\) is continuous at \(x_0\).

We note that the solutions of Equation (1.1) belong to the corresponding \(B_{1, g}(\Omega)\) classes. The proofs of inequalities (2.1) and (2.2) are completely similar to the proof of [39, section 2, inequalities (2.1), (2.2)]. Therefore, Theorem 1.5 follows from Theorem 2.2.

Theorem 2.2 is an immediate consequence of the following two lemmas.
**Lemma 2.3** (De Giorgi-type lemma). Let conditions \((g_0), (g_1), \) and \((g_2)\) be fulfilled, \(u \in B_{1,q}(\Omega), B_{g_0}(x_0) \subset B_{g}(x_0) \subset \Omega\), and let \(\mu_+^r\) and \(\mu_-^r\) be numbers such that \(\mu_+^r \geq \text{ess sup}_{B_r(x_0)} u, \mu_-^r \leq \text{ess inf}_{B_r(x_0)} u\). We set \(v_+ = \mu_+^r - u, v_- = u - \mu_-^r, \omega_r = \mu_+^r - \mu_-^r, \) and fix \(\xi \in (0, 1)\). Then, there exists \(v_1 \in (0, 1)\) depending only on \(n, p, q, c_0, c_1, c_2, \delta_0, \) and \(M\) such that if

\[
\left| \{ x \in B_r(x_0) : v_\pm(x) \leq \xi \lambda(r) \omega_r \} \right| \leq v_1 |B_r(x_0)|, \tag{2.3}
\]

then either

\[
\frac{\xi \omega_r}{r} \leq \frac{4}{\lambda(r)} \tag{2.4}
\]

or

\[
v_\pm(x) \geq \frac{\xi}{4} \lambda(r) \omega_r \quad \text{for a.a. } x \in B_{r/2}(x_0). \tag{2.5}
\]

**Proof.** We provide the proof of (2.5) for \(v_+\), while the proof for \(v_-\) is completely similar. For \(j = 0, 1, 2, \ldots\) we set

\[
r_j := \frac{r}{2} \left(1 + 2^{-j}\right), \quad k_j := \mu_+^r - \frac{\xi \lambda(r) \omega_r}{2} - \frac{\xi \lambda(r) \omega_r}{2^{j+1}}.
\]

We assume that \(M_+(k_\infty, r/2) \geq \xi \lambda(r) \omega_r/4\), because in the opposite case, the required (2.5) is evident. If (2.4) is violated, then \(M_+(k_\infty, r/2) \geq r\). In addition, since

\[
M_+(k_j, r) \leq 2^{-j} \lambda(r) \omega_r \leq 2 M \lambda(r),
\]

then condition \((g_2)\) with \(K = 2 M \) is applicable, and we obtain that

\[
g \left( \frac{x, M_+(k_j, r_j)}{r_j} \right) \simeq g \left( \frac{x_0, M_+(k_j, r)}{r} \right), \quad x \in B_r(x_0).
\]

Therefore, setting in inequality (2.1), \(k = k_j, l = k_{j+1}, \varphi = r_j, \psi(1 - \sigma) = r_{j+1}, \) and \(\epsilon = 1\), we can rewrite it as follows:

\[
\int_{A_{k_j, r_{j+1}} \setminus A_{k_{j+1}, r_{j+1}}} |
\nabla u |
\ dx \leq \gamma \frac{2^{j+1} \lambda(r) \omega_r}{r} |A_{k_j, r_j}^+|.
\]

From this, using the Sobolev embedding theorem completely similar to that of [26, chapter 2, Lemma 6.2], we arrive at the required (2.5), which completes the proof of the lemma. \(\square\)

**Lemma 2.4** (expansion of positivity). Let conditions \((g_0), (g_1), (g_2)\) be fulfilled, \(u \in B_{1,q}(\Omega), B_{g_0}(x_0) \subset B_{g}(x_0) \subset \Omega, \) and \(\xi \in (0, 1)\). Assume that with some \(\alpha \in (0, 1)\) there holds

\[
\left| \{ x \in B_{3r/4}(x_0) : v_\pm(x) \leq \xi \lambda(r) \omega_r \} \right| \leq (1 - \alpha) |B_{3r/4}(x_0)|. \tag{2.6}
\]

Then, there exists number \(C_\alpha\) depending only on \(n, p, q, c_0, c_1, c_2, M, \alpha, \) and \(\xi\) such that either

\[
\omega_r \leq C_\alpha r/\lambda(r) \tag{2.7}
\]

or

\[
v_\pm(x) \geq C_\alpha^{-1} \lambda(r) \omega_r \quad \text{for a.a. } x \in B_{r/2}(x_0). \tag{2.8}
\]

**Proof.** We provide the proof of (2.8) for \(v_+\), while the proof for \(v_-\) is completely similar. We set \(k_s := \mu_+^r - 2^{-s} \lambda(r) \omega_r, \) \(s = 1, 2, \ldots, s\), where \(s\) is large enough to be chosen later. For \(1 \leq s \leq s\), we assume that \(M_+(k_s, r/2) \geq \lambda(r) \omega_r / 2^{s+1},\) since otherwise inequality (2.8) is evident. If (2.7) is violated, then \(M_+(k_s, r/2) \geq C_s r/2^s \geq r\) if \(C_s \geq 2^s\). In addition, since \(M_+(k_s, r) \leq 2^{-s} \lambda(r) \omega_r \leq 2 M \lambda(r), s = 1, \ldots, s\), then condition \((g_2)\) with \(v = M_+(k_s, r), K = 2 M \) is applicable in \(B_r(x_0)\), and
we obtain that
\[ g \left( x, \frac{M_+(k_\sigma, r)}{r} \right) \leq \gamma(M) g \left( x_0, \frac{M_+(k_\sigma, r)}{r} \right), \quad x \in B_r(x_0). \]

Therefore, setting in inequality (2.1) \( k = k_\sigma, l = k_{\sigma+1}, \varphi = r, \) and \( \varepsilon = \left( \frac{|A_{k_\sigma, r}^+ \setminus A_{k_{\sigma+1}, r}^+|}{|B_r(x_0)|} \right)^{1/p} \), we can rewrite it as follows:
\[
\int_{A_{k_\sigma, r}^+ \setminus A_{k_{\sigma+1}, r}^+} |\nabla u| \xi^q \, dx \leq \gamma^2 \lambda(r) \omega_r \left( \frac{|A_{k_\sigma, r}^+ \setminus A_{k_{\sigma+1}, r}^+|}{|B_r(x_0)|} \right)^{\frac{p-1}{p}} |B_r(x_0)|,
\]
where \( \xi \in C_0^\infty(B_r(x_0)), 0 \leq \xi \leq 1, \xi = 1 \) in \( B_{3r/4}(x_0), |\nabla \xi| \leq 4/r \). From this, using (2.6), De Giorgi–Poincaré inequality [26, chapter 2, Lemma 3.9] and Lemma 2.3, we arrive at the required (2.8), which completes the proof of the lemma.

To complete the proof of Theorem 2.2, we note that the following two alternative cases are possible:
\[
\left\{ x \in B_{3\rho/4}(x_0) : u(x) \geq \mu_\rho^+ - \frac{\omega_\rho}{2} \right\} \leq \frac{1}{2} |B_{3\rho/4}(x_0)|
\]
or
\[
\left\{ x \in B_{3\rho/4}(x_0) : u(x) \leq \mu_\rho^- + \frac{\omega_\rho}{2} \right\} \leq \frac{1}{2} |B_{3\rho/4}(x_0)|
\]
for \( 0 < \rho < R \). Assume, for example, the first one. Then, by Lemma 2.4, using the fact that
\[
\left\{ x \in B_{3\rho/4}(x_0) : u(x) \geq \mu_\rho^+ - \frac{\lambda(\rho) \omega_\rho}{2} \right\} \subset \left\{ x \in B_{3\rho/4}(x_0) : u(x) \geq \mu_\rho^+ - \frac{\omega_\rho}{2} \right\},
\]
we obtain
\[
\omega_{\rho/2} \leq (1 - C_{-1} \lambda(\rho)) \omega_\rho + \frac{C_+ \rho}{\lambda(\rho)}.
\]
Iterating this inequality, we have for any \( j \geq 1 \)
\[
\omega_{\rho_j} \leq \omega_{\rho} \prod_{i=0}^{j-1} \left( 1 - C_{-1} \lambda(\rho_i) \right) + C_+ \sum_{i=0}^{j-1} \frac{\rho_i}{\lambda(\rho_i)} \leq \omega_{\rho} \exp \left( -\gamma \sum_{i=0}^{j-1} \lambda(\rho_i) \right) + \frac{\gamma \rho}{\lambda(\rho)},
\]
where \( \rho_j = 2^{-j} \rho, j = 0, 1, \ldots \). This completes the proof of Theorem 2.2.

3 UPPER AND LOWER ESTIMATES OF AUXILIARY SOLUTIONS: PROOF OF THEOREM 1.14

We refer to the parameters \( M, K_1, K_2, n, p, q, \delta_0, c, c_0, c_1, \) and \( C(M) \) as our structural data, and we write \( \gamma \) if it can be quantitatively determined a priori in terms of the above quantities. The generic constant \( \gamma \) may change from line to line.

In this section, we prove upper and lower bounds for auxiliary solutions \( v = v(x, m) \) to problem (1.13), (1.14). The existence of the solutions \( v \) follows from the general theory of monotone operators. We will assume that the following integral identity holds:
\[
\int_D A(x, \nabla v) \nabla \varphi \, dx = 0 \quad \text{for any } \varphi \in W_0(D).
\]
Testing (3.1) by $\varphi = (v - m)_+$ and by $\varphi = v_-$ and using condition (1.8), we obtain that

$$0 \leq v \leq m \quad \text{in } D. \quad (3.2)$$

Further, we will assume that inequality (1.15) is violated, that is,

$$C(E,B_{8\rho}(x_0);m) \geq \bar{c}\rho^{n-1}, \quad (3.3)$$

where $\bar{c} \geq 1$ is to be chosen later depending only on the data.

### 3.1 Upper bound for the function $v$

We note that in the standard case (i.e., if $p = q$) the upper bound for the function $v$ was proved in [35] (see also [36, chapter 8, section 3], [37]).

For $i, j = 0, 1, 2, \ldots$ we set $k_j := k(1 - 2^{-j})$, where $k > 0$ to be chosen later, $\rho_{i,j} := 2^{-i-j-3}\rho$,

$$M_i := \text{ess sup}_F v, \quad F_i := \left\{ x \in D : \frac{\rho}{4} (1 + 2^{-i}) \leq |x - x_0| \leq \frac{\rho}{2} (3 - 2^{-i}) \right\}.$$

Fix $i, j = 0, 1, \ldots, x \in F_i$ and suppose that $(v(x) - k)_+ \geq \rho$, then

$$M_{i,j}(k_{j+1}) := \text{ess sup}_{B_{\rho_{i,j}}(x)} (v - k_{j+1}) \geq (v(x) - k)_+ \geq \rho \geq \rho_{i,j}. \quad (3.4)$$

Note that our assumptions on $\lambda(r)$ (see condition (g$_2$)) guarantees the inequality

$$\lambda(R_1) \leq 2 \left( \frac{R_1}{R_2} \right)^{1-\delta_0} \lambda(R_2) \quad \text{if } 0 < R_2 < R_1 < R,$$

which together with (1.14) and (3.2), implies that

$$M_{i,j}(k_{j+1}) \leq m \leq 2M\lambda(8\rho) \leq \gamma 2^{(i+j)\gamma} \lambda(\rho_{i,j}).$$

In turn, it follows from this inequality and (3.4) that $\rho_{i,j} \leq M_{i,j}(k_{j+1}) \leq \gamma 2^{(i+j)\gamma} \lambda(\rho_{i,j})$. Therefore, condition (g$_2$) with $v = M_{i,j}(k_{j+1})$, $K = \gamma 2^{(i+j)\gamma}$ is applicable in $B_{\rho_{i,j}}(x)$, and we have that

$$g(x, \frac{M_{i,j}(k_{j+1})}{\rho_{i,j}}) \leq 2^{(i+j)\gamma} g \left( \frac{x}{\rho_{i,j}}, \frac{M_{i,j}(k_{j+1})}{\rho_{i,j}} \right), \quad x \in B_{\rho_{i,j}}(x). \quad (3.5)$$

Now, let $\xi_{i,j} \in C_0^\infty(B_{\rho_{i,j}}(x))$, $0 \leq \xi_{i,j} \leq 1$, $\xi_{i,j} = 1$ in $B_{\rho_{i,j}+\epsilon}(x)$, $|\nabla \xi_{i,j}| \leq 2^{i+j+4}/\rho$. Testing (3.1) by $\varphi = (v - k_{j+1})_+$ $\xi_{i,j}^q$ and using conditions (1.2), the Young inequality (1.7) and relation (3.5), we obtain

$$\int_{B_{\rho_{i,j}}(x)} G(x, |\nabla (v - k_{j+1})_+|) \xi_{i,j}^q \, dx \leq \frac{\gamma 2^{(i+j)\gamma}}{\rho} g \left( \frac{x}{\rho_{i,j}}, \frac{M_{i,j}(k_{j+1})}{\rho_{i,j}} \right) \int_{B_{\rho_{i,j}}(x)} (v - k_{j+1})_+ \, dx.$$
From this, using again (3.5) and (1.7), we get
\[
g\left(\bar{x}, \frac{M_{i,j}(k_{j+1})}{\rho_{i,j}}\right) \int_{B_{\rho_{i,j}}(\bar{x})} |\nabla (u - k_{j+1})| \zeta_{i,j}^q \, dx
\leq \gamma 2^{(j+1)\gamma} \int_{B_{\rho_{i,j}}(\bar{x})} g\left(x, \frac{M_{i,j}(k_{j+1})}{\rho_{i,j}}\right) \frac{\rho_{i,j}}{\rho} \int_{A_{\rho_{i,j}}(k_{j+1})} g\left(x, \frac{M_{i,j}(k_{j+1})}{\rho_{i,j}}\right) \left(\frac{M_{i,j}(k_{j+1})}{k} + 1\right) \int_{B_{\rho_{i,j}}(\bar{x})} (u - k_{j+1})_+ \, dx,
\]
here \(A_{\rho_{i,j},k_{j+1}} := B_{\rho_{i,j}}(\bar{x}) \cap \{u > k_{j+1}\}\). The last inequality can be rewritten as
\[
\int_{B_{\rho_{i,j}}(\bar{x})} |\nabla (u - k_{j+1})_+| \zeta_{i,j}^q \, dx \leq \gamma 2^{(j+1)\gamma} \int_{B_{\rho_{i,j}}(\bar{x})} \frac{\rho_{i,j}}{\rho} \int_{A_{\rho_{i,j}}(k_{j+1})} \left(\frac{M_{i,j}(k_{j+1})}{k} + 1\right) \int_{B_{\rho_{i,j}}(\bar{x})} (u - k_{j+1})_+ \, dx.
\]
From this, by standard arguments (see, e.g., [26, chapter 2]), choosing \(k\) from the condition
\[
k = \gamma 2^{j\gamma} \rho^{-n} \int_{B_{\rho/2^{j+3}}(\bar{x})} v \, dx + \gamma 2^{j\gamma} M_{i+1}^{n+1} \left(\rho^{-n} \int_{B_{\rho/2^{j+3}}(\bar{x})} v \, dx\right)^{\frac{1}{n+1}},
\]
and keeping in mind our assumption that \(v(\bar{x}) \geq k + \rho\), we arrive at
\[
v(\bar{x}) \leq \gamma 2^{j\gamma} \rho^{-n} \int_{B_{\rho/2^{j+3}}(\bar{x})} v \, dx + \gamma 2^{j\gamma} M_{i+1}^{n+1} \left(\rho^{-n} \int_{B_{\rho/2^{j+3}}(\bar{x})} v \, dx\right)^{\frac{1}{n+1}} + \gamma \rho.
\]
Since \(\bar{x} \in F_i\) is an arbitrary point, using the Young inequality, from the previous we obtain for any \(\varepsilon \in (0, 1)\)
\[
M_i \leq \varepsilon M_{i+1} + \frac{\varepsilon^2}{\varepsilon_1^2 \rho^n} \int_{F_{i+1}} v \, dx + \gamma \rho, \quad i = 0, 1, 2, \ldots \tag{3.6}
\]
Let us estimate the second term on the right-hand side of (3.6). We set \(v_{M_{i+1}} := \min\{v, M_{i+1}\}\) and apply Poincaré inequality, (1.7) and (1.3) to obtain
\[
\int_{F_{i+1}} v \, dx = \int_{F_{i+1}} v_{M_{i+1}} \, dx \leq \gamma \rho \int_D |\nabla v_{M_{i+1}}| \, dx
\]
\[
= \gamma \rho \int_D |\nabla v_{M_{i+1}}| \frac{g(x, M_{i}/\rho)}{g(x, M_{i}/\rho)} \, dx \leq \varepsilon_1 M_{i} \rho^n + \gamma_2 \rho \int_D G(x, |\nabla v_{M_{i+1}}|) \frac{g(x, M_{i}/\rho)}{g(x, M_{i}/\rho)} \, dx, \tag{3.7}
\]
with arbitrary \(\varepsilon_1 \in (0, 1)\). In addition, we can assume that \(M_0 \geq \rho\), because otherwise, by (3.3) the upper estimate (1.16) is evident. In view of (3.2) and (1.14), this assumption gives the inequalities \(\rho \leq M_0 \leq M_1 \leq m \leq 2M\lambda(8\rho), \quad i = 0, 1, 2, \ldots,\)
which allow us to apply condition (g2) with \(v = M_i\) and \(K = 2M\) to obtain the estimate:
\[
\int_D G(x, |\nabla v_{M_{i+1}}|) \frac{g(x, M_{i}/\rho)}{g(x_0, M_{i}/\rho)} \, dx \leq \gamma \rho \int_D G(x, |\nabla v_{M_{i+1}}|) \frac{g(x, M_{i}/\rho)}{g(x_0, M_{i}/\rho)} \, dx.
\]
Collecting this inequality, (3.6) and (3.7), and choosing \( \epsilon_1 \) from the condition \( \gamma 2^{i} \epsilon^{-\gamma} \epsilon_1 = \epsilon \), we obtain

\[
M_i \leq 2\epsilon M_{i+1} + \frac{\gamma 2^{i} \epsilon^{-\gamma} \rho^{1-n}}{g(x_0, M_i/\rho)} \int_D G(x, |\nabla v_{M_{i+1}}|) \, dx + \gamma \rho,
\]

which by (1.7) implies

\[
M_i g(x_0, M_i/\rho) \leq 3\epsilon M_{i+1} g(x_0, M_{i+1}/\rho) + \gamma 2^{i} \epsilon^{-\gamma} \rho^{1-n} \int_D G(x, |\nabla v_{M_{i+1}}|) \, dx + \gamma \rho, \quad i = 0, 1, 2, \ldots. \quad (3.8)
\]

Let \( \psi \in \mathcal{D}(E) \) be such that

\[
\int_{B_\rho(x_0)} g(x, m|\nabla \psi|) |\nabla \psi| \, dx \leq C(E, B_\rho(x_0); m) + \rho^n.
\]

Testing identity (3.1) by \( \varphi = v - m \psi \), and using (1.2), (1.7), and the previous inequality, we obtain

\[
\int_D G(x, |\nabla v|) \, dx \leq \gamma m \int_{B_\rho(x_0)} g(x, m|\nabla \psi|) |\nabla \psi| \, dx \leq \gamma m (C(E, B_\rho(x_0); m) + \rho^n). \quad (3.9)
\]

Testing (3.1) by \( \varphi = v - m \psi \), and using again (1.2), (1.7), and (3.9), we have

\[
\int_D G(x, |\nabla v_{M_{i+1}}|) \, dx \leq \gamma M_{i+1} \int_D G(x, |\nabla v|) \, dx \leq \gamma M_{i+1} (C(E, B_\rho(x_0); m) + \rho^n).
\]

This inequality and (3.8) imply that

\[
M_i g(x_0, M_i/\rho) \leq 3\epsilon M_{i+1} g(x_0, M_{i+1}/\rho) + \gamma 2^{i} \epsilon^{-\gamma} M_{i+1} \rho + \gamma 2^{i} \epsilon^{-\gamma} M_{i+1} \rho^{n-1}, \quad i = 0, 1, 2, \ldots,
\]

which yields for any \( \epsilon_2 \in (0, 1) \)

\[
g(x_0, M_i/\rho) \leq \frac{1}{\epsilon^2_2} \frac{M_i}{M_{i+1}} g(x_0, M_{i+1}/\rho) + \epsilon_2^{p-1} g(x_0, M_{i+1}/\rho)
\]

\[
\leq \left( \frac{3\epsilon_2 + \epsilon_2^{p-1}}{\epsilon_2^{p-1}} \right) g(x_0, M_{i+1}/\rho) + \frac{\gamma 2^{i} \epsilon^{-\gamma} M_{i+1}}{(\epsilon_2^{p-1})} \left( \frac{C(E, B_\rho(x_0); m)}{\rho^{n-1}} + \rho \right)
\]

\[
\leq \left( \frac{3\epsilon_2 + \epsilon_2^{p-1}}{\epsilon_2^{p-1}} \right) g(x_0, M_{i+1}/\rho) + \frac{\gamma 2^{i} \epsilon^{-\gamma} M_{i+1}}{(\epsilon_2^{p-1})} \left( \frac{C(E, B_\rho(x_0); m)}{\rho^{n-1}} \right), \quad i = 0, 1, 2, \ldots,
\]

here we also used inequality (3.3). Iterating the last inequality and choosing \( \epsilon_2 \) and \( \epsilon \) small enough, we arrive at

\[
g(x_0, M_0/\rho) \leq \gamma \frac{C(E, B_\rho(x_0); m)}{\rho^{n-1}},
\]

which proves the upper bound of the function \( v \).

### 3.2 Lower bound for the function \( v \)

The main step in the proof of the lower bound in (1.16) is the following lemma:

**Lemma 3.1.** There exist numbers \( \epsilon, \theta \in (0, 1) \) depending only on the data such that

\[
\left| \left\{ x \in K_{\rho/4, 2\rho} : v(x) \leq \epsilon \rho g_{x_0} \left( \frac{C(E, B_\rho(x_0); m)}{\rho^{n-1}} \right) \right\} \right| \leq (1 - \theta) |K_{\rho/4, 2\rho}|, \quad (3.10)
\]

where \( K_{\rho_1, \rho_2} := B_{\rho_2}(x_0) \setminus B_{\rho_1}(x_0) \).
Proof. Let $\zeta_1 \in C_0^\infty(B_\rho(x_0))$, $0 \leq \zeta_1 \leq 1$, $\zeta_1 = 1$ in $B_{\rho/2}(x_0)$, $|\nabla \zeta_1| \leq 2/\rho$. Testing (3.1) by $\varphi = v - m \zeta_1^q$ and using condition (g$_2$) with $K = 2M$ and the Young inequality (1.7), we obtain for any $\rho < \varepsilon_1 \leq 2ML(8\rho)$,

$$
\int_D G(x, |\nabla v|) \, dx \leq \frac{ym}{\rho} \int_{K_{\rho/2,\rho}} G(x, |\nabla v|) \zeta_1^{q-1} \, dx
$$

$$
\leq \frac{ym}{\varepsilon_1} \int_{K_{\rho/2,\rho}} G(x, |\nabla v|) \, dx + \frac{ym}{\rho} \int_{K_{\rho/2,\rho}} g(x, \varepsilon_1/\rho) \, dx
$$

$$
\leq \frac{ym}{\varepsilon_1} \int_{K_{\rho/2,\rho}} G(x, |\nabla v|) \, dx + \gamma m g(x_0, \varepsilon_1/\rho) \rho^{n-1}.
$$

Let $\zeta_2 \in C_0^\infty(K_{\rho/4,2\rho})$, $0 \leq \zeta_2 \leq 1$, $\zeta_2 = 1$ in $K_{\rho/2,\rho}$, $|\nabla \zeta_2| \leq 4/\rho$. Testing (3.1) by $\varphi = v \zeta_2^q$ and using the Young inequality (1.7), we estimate the first term on the right-hand side of the previous inequality as follows:

$$
\int_{K_{\rho/2,\rho}} G(x, |\nabla v|) \, dx \leq \int_{K_{\rho/4,2\rho}} G(x, |\nabla v|) \zeta_2^q \, dx \leq \gamma \int_{K_{\rho/4,2\rho}} G(x, v/\rho) \, dx \leq \gamma \int_{K_{\rho/4,2\rho}} G_{K_{\rho/4,2\rho}}^+(v/\rho) \, dx,
$$

where we used the notation $G_{K_{\rho/4,2\rho}}^+(v/\rho) = \sup_{x \in K_{\rho/4,2\rho}} G(x, v/\rho)$, $v \geq 0$. Combining the last two inequalities and using the definition of capacity, we obtain

$$
C(E, B_{8\rho}(x_0); m) \leq \frac{1}{m} \int_D G(x, |\nabla v|) \, dx \leq \frac{\gamma}{\varepsilon_1} \int_{K_{\rho/4,2\rho}} G_{K_{\rho/4,2\rho}}^+(v/\rho) \, dx + \gamma g(x_0, \varepsilon_1/\rho) \rho^{n-1}.
$$

Choosing $\varepsilon_1$ from the condition

$$
g(x_0, \varepsilon_1/\rho) = \bar{c}_1 \frac{C(E, B_{8\rho}(x_0); m)}{\rho^{n-1}}, \quad \bar{c}_1 \in (0, 1),
$$

by (3.3), we obtain $\varepsilon_1 \geq 8\rho$ if $\bar{c} = c(\bar{c}_1)$ is large enough. To apply condition (g$_2$), we still need to check the inequality $\varepsilon_1 \leq 2ML(8\rho)$. Indeed, let $\psi \in C_0^\infty(B_{2\rho}(x_0))$, $0 \leq \psi \leq 1$, $\psi = 1$ in $B_{\rho}(x_0)$ and $|\nabla \psi| \leq 2/\rho$, then taking into account the inequality in (1.14), and using (g$_1$) and (g$_2$) with $K = 2M$, we have

$$
C(E, B_{8\rho}(x_0); m) \leq C(B_{\rho}(x_0), B_{8\rho}(x_0); m)
$$

$$
\leq \int_{B_{8\rho}(x_0)} g(x, m|\nabla \psi|) |\nabla \psi| \, dx \leq \frac{2}{\rho} \int_{B_{2\rho}(x_0)} g(x, 2m/\rho) \, dx \leq \gamma g(x_0, m/\rho) \rho^{n-1},
$$

and therefore, if $\bar{c}_1 = 1/2\gamma$, then $\varepsilon_1 \leq \rho g_0^{-1}(\bar{c}_1 \gamma g(x_0, m/\rho)) \leq m \leq 2ML(8\rho)$. So, by our choice (3.12), from (3.11), we obtain

$$
C(E, B_{8\rho}(x_0); m) \leq \frac{\gamma}{\varepsilon_1} \int_{K_{\rho/4,2\rho}} G_{K_{\rho/4,2\rho}}^+(v/\rho) \, dx.
$$

(3.13)

Let us estimate the term on the right-hand side of (3.13). For this, we decompose $K_{\rho/4,2\rho}$ as $K_{\rho/4,2\rho} = K_{\rho/4,2\rho}' \cup K_{\rho/4,2\rho}''$, where

$$
K_{\rho/4,2\rho}' := K_{\rho/4,2\rho} \cap \left\{ \frac{C(E, B_{8\rho}(x_0); m)}{\rho^{n-1}} \right\}, \quad K_{\rho/4,2\rho}'' := K_{\rho/4,2\rho} \setminus K_{\rho/4,2\rho}',
$$

and $\varepsilon \in (0, 1)$ is small enough to be determined later. By our choice of $\varepsilon_1$ in (3.12)

$$
\varepsilon_1 \geq \bar{c}_1^{1/(p-1)} \rho g_0^{-1} \left( \frac{C(E, B_{8\rho}(x_0); m)}{\rho^{n-1}} \right) = \gamma \rho g_0^{-1} \left( \frac{C(E, B_{8\rho}(x_0); m)}{\rho^{n-1}} \right),
$$

$$
C(E, B_{8\rho}(x_0); m) \leq \frac{\gamma}{\varepsilon_1} \int_{K_{\rho/4,2\rho}'} G_{K_{\rho/4,2\rho}'}(v/\rho) \, dx
$$

(3.14)
so, if \( \varepsilon \geq c_0 \varepsilon^{1-q} \), then by \((g_0)\), \((g_1)\), and (3.3),
\[
\rho \leq \varepsilon \rho \langle g_{x_0}^{-1} \left( \frac{C(E, B_{8\rho}(x_0); m)}{\rho^{n-1}} \right) \rangle \leq 2M\lambda(8\rho),
\]
and by \((g_2)\) with \( K = 2M \), we have
\[
\frac{\gamma}{\varepsilon_1} \int_{K_{\rho/4,2\rho}} G^+_{K_{\rho/4,2\rho}} (v/\rho) \, dx \leq \frac{\gamma}{\varepsilon_1} G^+_{K_{\rho/4,2\rho}} \left( \langle g_{x_0}^{-1} \left( \frac{C(E, B_{8\rho}(x_0); m)}{\rho^{n-1}} \right) \rangle \right) \left| K'_{\rho/4,2\rho} \right| \leq \gamma\varepsilon_1 C(E, B_{8\rho}(x_0); m) \rho^n |K'_{\rho/4,2\rho}|.
\]
Similarly, using the upper bound (1.16) for the function \( v \), (3.3) and our choice of \( \varepsilon_1 \) by (3.12), we obtain
\[
\frac{\gamma}{\varepsilon_1} \int_{K''_{\rho/4,2\rho}} G^+_{K''_{\rho/4,2\rho}} (v/\rho) \, dx \leq \gamma(\varepsilon) \frac{C(E, B_{8\rho}(x_0); m)}{\rho^n} |K''_{\rho/4,2\rho}|.
\]
Collecting estimates (3.13)–(3.15), we obtain
\[
C(E, B_{8\rho}(x_0); m) \leq \varepsilon\gamma C(E, B_{8\rho}(x_0); m) + \gamma(\varepsilon) \frac{C(E, B_{8\rho}(x_0); m)}{\rho^n} |K''_{\rho/4,2\rho}|,\]
choosing \( \varepsilon \) from the condition \( \varepsilon \gamma = 1/2 \), we arrive at \( |K''_{\rho/4,2\rho}| \geq \gamma^{-1} \rho^n \), which completes the proof of the lemma. \( \square \)

Repeating the similar arguments as in Section 2, by Lemma 2.4, we arrive at (1.16), which completes the proof of Theorem 1.14.

4 | BOUNDARY CONTINUITY AND HARNACK’S INEQUALITY, PROOF OF THEOREMS 1.9 AND 1.12

In this section, we will prove Theorems 1.9 and 1.12.

4.1 | Boundary continuity, proof of Theorem 1.9

4.1.1 | Auxiliary propositions

Further, we will need the following two simple lemmas.

**Lemma 4.1** (weak comparison principle). Suppose that \( u \) is a bounded weak supersolution and \( v \) is a bounded weak subsolution to Equation (1.1) in \( \Omega \). Assume also that (1.8) holds and \( v \leq u \) on \( \partial \Omega \) in the sense of \( W(\Omega) \), that is, \((v - u)_+ \in W_0(\Omega)\), then \( v \leq u \) a.e. in \( \Omega \).

**Proof.** If not, test the correspondent identities for \( v \) and \( u \) by \((v - u)_+\), subtracting the resulting inequalities and using condition (1.8), we arrive to a contradiction. This proves the lemma. \( \square \)

**Lemma 4.2.** Let \( u \in W(\Omega) \cap L^\infty(\Omega) \) be a bounded weak solution to Equation (1.1) satisfying the condition \( u - f \in W_0(\Omega) \), \( f \in C(\Omega) \cap W(\Omega) \), and assume that (1.8) holds. Fix \( x_0 \in \partial \Omega \), \( \rho > 0 \) and take any number \( k \geq \sup_{B_{\rho}(x_0) \cap \partial \Omega} f \) and define
\[
\langle u^+ - k \rangle := \begin{cases} (u - k)_+ & \text{if } x \in B_{\rho}(x_0) \cap \Omega, \\ 0 & \text{if } x \in B_{\rho}(x_0) \setminus \Omega. \end{cases}
\]
Then, \( u^+_k \) is a bounded weak subsolution to Equation (1.1) in the ball \( B_\rho(x_0) \). The same conclusion holds for the zero extension of \( u^-_k = (u-k)^- \) for the levels \( k \leq \inf_{B_\rho(x_0) \cap \partial \Omega} f \).

Proof. Let \( \varphi \in W_0^0(B_\rho(x_0)) \), \( \varphi \geq 0 \) be arbitrary, and test (1.4) by \( \frac{u^+_k}{u^+_k + \varepsilon} \varphi \), \( \varepsilon > 0 \), so we obtain

\[
\int_{B_\rho(x_0)} A(x, \nabla u^+_k) \frac{u^+_k}{u^+_k + \varepsilon} \nabla \varphi \, dx = -\varepsilon \int_{B_\rho(x_0)} A(x, \nabla u^+_k) \nabla u^+_k (u^+_k + \varepsilon)^2 \varphi \, dx \leq 0,
\]

letting \( \varepsilon \to 0 \), we arrive at the required statement. \( \Box \)

4.1.2 Proof of Theorem 1.9

Let \( x_0 \in \partial \Omega \) be an arbitrary boundary point, and all assumptions and notation of Theorem 1.9 are satisfied. In particular, there exists \( R > 0 \) so that condition (g2) is fulfilled in \( B_R(x_0) \). Let \( \rho \in (0, R) \) be arbitrary. Further, we will assume that

\[
\sum_{i \in \mathbb{N}} \tau(r_i, \lambda(r_i)) = +\infty, \quad r_i := \rho/2^i,
\]

where

\[
\tau(r, \lambda(r)) := r g^{-1}_{x_0} \left( \frac{C(B_{r/4}(x_0) \setminus \Omega, B_r(x_0); \lambda(r))}{r^{n-1}} \right) \text{ for } r \in (0, R).
\]

Under the condition (4.1), we need to prove that

\[
\lim_{\Omega \ni x \to x_0} u(x) = f(x_0).
\]

This equality will be established if we show that \( f(x_0) \leq \liminf_{\Omega \ni x \to x_0} u(x) \) and \( \limsup_{\Omega \ni x \to x_0} u(x) \leq f(x_0) \). The proof of the both inequalities is completely similar and we will prove only the second one.

To prove the inequality \( \limsup_{\Omega \ni x \to x_0} u(x) \leq f(x_0) \), we argue by contradiction and assume that \( \limsup_{\Omega \ni x \to x_0} u(x) > f(x_0) \). Let \( k \) be an arbitrary number such that \( f(x_0) < k < \limsup_{\Omega \ni x \to x_0} u(x) \), choose a number \( R_0 = R_0(k) \) so that \( k \geq \sup_{\partial \Omega \cap B_{R_0}(x_0)} f \) and assume without loss that \( \rho \leq R_0 \). We consider the function \( u^+_k \), which was defined in Lemma 4.2. We set \( M_k(r) = \sup_{\partial B_r(x_0)} u^+_k \) for \( r \in (0, \rho] \). By our choices,

\[
\lim_{r \to 0} M_k(r) = a > 0.
\]

We fix a sufficiently large positive number \( C_4 \), which will be specified later and choose \( \bar{r}_0 \in (0, \rho) \) from the condition

\[
\tau(\bar{r}_0, \lambda(\bar{r}_0)) \geq C_4 \bar{r}_0.
\]

Condition (4.3) can always be realized, since otherwise, we would have for all \( r \in (0, \rho) \) that \( \tau(r, \lambda(r)) \leq C_4 r \), and consequently \( \sum_{i \in \mathbb{N}} \tau(r_i, \lambda(r_i)) \leq \gamma \), reaching a contradiction to (4.1). By (g1) and (4.2), inequality (4.3) can be rewritten as

\[
\frac{C(B_{\bar{r}_0/4}(x_0) \setminus \Omega, B_{\bar{r}_0}(x_0); M_k(\bar{r}_0)\lambda(\bar{r}_0))}{\bar{r}_0^{n-1}} \geq \gamma(a) g(x_0, C_4) \geq \gamma(a) c_0^{-1} C_4^{p-1}.
\]

Let us construct an auxiliary solution \( v = v(x, M_k(\bar{r}_0)\lambda(\bar{r}_0)) \) of the problem (1.13), (1.14) in \( D_0 = B_{\bar{r}_0}(x_0) \setminus E_0 \), \( E_0 = B_{\bar{r}_0/4}(x_0) \setminus \Omega \). If \( \gamma(a) c_0^{-1} C_4^{p-1} \geq C_3 \), then condition (1.15) is violated and by Theorem 1.14, we have with some \( \eta \in (0, 1) \),

\[
v(x) \geq \eta \tau(\bar{r}_0, M_k(\bar{r}_0)\lambda(\bar{r}_0)) \quad \text{for a.a. } x \in B_{\bar{r}_0/2}(x_0) \setminus B_{3\bar{r}_0/8}(x_0).
\]
Consider the function $\tilde{u}_{k,0} = M_k(\tilde{r}_0) - u_k^+$, evidently $\tilde{u}_{k,0}$ is a weak supersolution to Equation (1.1) in $B_{\tilde{r}_0}(x_0)$. Moreover, $\tilde{u}_{k,0} \geq v$ on $\delta D_0$, so by Lemma 4.1, $\tilde{u}_{k,0} \geq v$ in $D_0$, and from the previous, we obtain

$$M_k(\tilde{r}_0) - M_k(\tilde{r}_0/2) \geq \eta \tau(\tilde{r}_0, M_k(\tilde{r}_0)\lambda(\tilde{r}_0)).$$  (4.5)

If $\tau(\tilde{r}_0/2, \lambda(\tilde{r}_0/2)) \geq C_4 r_0/2$, we set $\tilde{r}_1 = \tilde{r}_0/2$. If $\tau(\tilde{r}_0/2, \lambda(\tilde{r}_0/2)) \leq C_4 r_0/2$, similarly to (4.3), we find a number $i \geq 2$ such that $\tau(\tilde{r}_0/2^i, \lambda(\tilde{r}_0/2^i)) \geq C_4 r_0/2^i$. Let $i_1 \geq 2$ be the maximal number satisfying the above condition, in this case, we set $\tilde{r}_1 = \tilde{r}_0/2^{i_1}$, then inequality (4.5) can be rewritten in the form

$$M_k(\tilde{r}_0) - M_k(\tilde{r}_1) \geq \eta \tau(\tilde{r}_0, M_k(\tilde{r}_0)\lambda(\tilde{r}_0)).$$

Further, we define the sequence $\{\tilde{r}_j\}$ by induction. Suppose we have chosen $\tilde{r}_0, \tilde{r}_1, \ldots, \tilde{r}_j$ such that

$$\tau(\tilde{r}_i, \lambda(\tilde{r}_i)) \geq C_4 \tilde{r}_i, \quad i = 0, 1, \ldots, j,$$

(4.6)

$$M_k(\tilde{r}_{i+1}) \leq M_k(\tilde{r}_i) - \eta \tau(\tilde{r}_i, M_k(\tilde{r}_i)\lambda(\tilde{r}_i)), \quad i = 0, 1, \ldots, j - 1.$$  (4.7)

Let us show how to choose $\tilde{r}_{j+1}$. By (4.6) similarly to (4.4) we obtain

$$\tau(\tilde{r}_j, M_k(\tilde{r}_j)\lambda(\tilde{r}_j)) \geq \gamma(a) g(x_0, C_4) \geq \gamma(a) c^{-1}_0 C_{p-1}^p.$$  (4.8)

Let us construct an auxiliary solution $v = v(x, M_k(\tilde{r}_j)\lambda(\tilde{r}_j))$ of the problem (1.13), (1.14) in $D_j = B_{\tilde{r}_j}(x_0) \setminus E_j$, $E_j = B_{\tilde{r}_j/4}(x_0) \setminus \Omega$ and consider the function $\tilde{u}_{k,j} = M_k(\tilde{r}_j) - u_k^+$. Since $\tilde{u}_{k,j} \geq v$ on $\delta D_j$, by Lemma 4.1, we obtain $\tilde{u}_{k,j} \geq v$ in $D_j$. If $\gamma(a) c^{-1}_0 C_{p-1}^p \geq C_3$, then condition (1.15) is violated and by Theorem 1.14, we have

$$M_k(\tilde{r}_j) - M_k(\tilde{r}_j/2) \geq \eta \tau(\tilde{r}_j, M_k(\tilde{r}_j)\lambda(\tilde{r}_j)).$$  (4.9)

If $\tau(\tilde{r}_j/2, \lambda(\tilde{r}_j/2)) \geq C_4 \tilde{r}_j/2$, we set $\tilde{r}_{j+1} = \tilde{r}_j/2$. And if $\tau(\tilde{r}_j/2, \lambda(\tilde{r}_j/2)) \leq C_4 \tilde{r}_j/2$, similarly to (4.3) we find a number $i \geq 2$ such that $\tau(\tilde{r}_j/2^i, \lambda(\tilde{r}_j/2^i)) \geq C_4 \tilde{r}_j/2^i$. Let $i_{j+1} \geq 2$ be the maximal number satisfying the above condition, in this case we set $\tilde{r}_{j+1} = \tilde{r}_j/2^{i_{j+1}}$. Then, inequality (4.9) can be rewritten as

$$M_k(\tilde{r}_j) - M_k(\tilde{r}_{j+1}) \geq \eta \tau(\tilde{r}_j, M_k(\tilde{r}_j)\lambda(\tilde{r}_j)).$$  (4.10)

Now, we have

$$\sum_{i \in \mathbb{N}} \tau(\tilde{r}_i, \lambda(\tilde{r}_i)) = +\infty.$$  (4.11)

Indeed, by our choices

$$\sum_{i \in \mathbb{N}} \tau(\tilde{r}_i, \lambda(\tilde{r}_i)) = \sum_{i \in \mathbb{N}} \tau(\tilde{r}_i, \lambda(\tilde{r}_i)) + \sum_{i \in \mathbb{N}} \tau(\tilde{r}_i, \lambda(\tilde{r}_i))$$

$$\leq \gamma \sum_{i \in \mathbb{N}} \frac{\rho}{2^i} + \sum_{i \in \mathbb{N}} \tau(\tilde{r}_i, \lambda(\tilde{r}_i)) \leq \gamma \rho + \sum_{i \in \mathbb{N}} \tau(\tilde{r}_i, \lambda(\tilde{r}_i)),$$

from which the required (4.11) follows.

To complete the proof of Theorem 1.9, we sum up inequality (4.10) for $j = 0, 1, 2, \ldots, l$. By (4.2) and (g1), we obtain that

$$\eta \gamma(a) \sum_{j=0}^{l} \tau(\tilde{r}_j, \lambda(\tilde{r}_j)) \leq \sum_{j=0}^{l} (M_k(\tilde{r}_j) - M_k(\tilde{r}_{j+1})) \leq M_k(r_0) \leq 2M.$$  (4.12)

This implies that $\sum_{i \in \mathbb{N}} \tau(\tilde{r}_i, \lambda(\tilde{r}_i)) \leq \gamma(a, M) < +\infty$, which contradicts (4.11). This completes the proof of Theorem 1.9.
4.2 Harnack’s inequality for double-phase elliptic equations, proof of Theorem 1.12

4.2.1 De Giorgi-type lemma under assumptions \((g_0), (g_1), (g_3)\)

In order to prove Theorem 1.12, we need the following version of De Giorgi-type lemma, which was proved in [39, Lemma 2.3].

**Lemma 4.3.** Let \( u \in W(\Omega) \cap L^\infty(\Omega) \) be a bounded weak solution to Equation (1.1) under assumptions \((g_0), (g_1), \) and \((g_3)\), and let \( \bar{x} \in \Omega \) be an arbitrary point. Consider a ball \( B_{\rho}(\bar{x}) \subset \Omega \) and denote by \( \mu_\pm \) and \( \omega \) nonnegative numbers such that

\[
\mu_+ \geq \operatorname{ess sup}_{B_{\rho}(\bar{x})} u, \quad \mu_- \leq \operatorname{ess inf}_{B_{\rho}(\bar{x})} u, \quad \omega = \mu_+ - \mu_-.
\]

Fix \( \xi, a \in (0, 1) \), then there exists number \( \nu \in (0, 1) \) depending only on the data and \( a \) such that if

\[
\left| \left\{ x \in B_{\rho}(\bar{x}) : u(x) \geq \mu_+ - \xi \omega \right\} \right| \leq \nu \theta^{-2n}(r) |B_{\rho}(\bar{x})|,
\]

then either

\[
\xi \omega \leq r
\]

or

\[
u u(x) \leq \mu_+ - a \xi \omega \quad \text{for a.a. } x \in B_{\rho/2}(\bar{x}).
\]

Likewise, if

\[
\left| \left\{ x \in B_{\rho}(\bar{x}) : u(x) \leq \mu_- + \xi \omega \right\} \right| \leq \nu \theta^{-2n}(r) |B_{\rho}(\bar{x})|,
\]

then either (4.13) holds, or

\[
u u(x) \geq \mu_- + a \xi \omega \quad \text{for a.a. } x \in B_{\rho/2}(\bar{x}).
\]

4.2.2 Proof of Theorem 1.12

We assume that all the hypotheses and notation of Theorem 1.12 are in force. This means that conditions \((g_0), (g_1), \) are in force, assumptions \((g_2), (g_3)\) are fulfilled in some ball \( B_{\rho}(x_0) \subset \Omega \), and \( u \in W(\Omega) \cap L^\infty(\Omega) \) is a bounded nonnegative weak solution to Equation (1.1).

We set \( u_0 := u(x_0) \) and construct the ball \( B_{\rho}(x_0) \) for \( 0 < \rho < R \) and \( \tau \in (0, 1) \). Following Krylov and Safonov [25], we consider the equation

\[
\max_{B_{\rho \tau}(x_0)} u = \frac{u_0}{2} (1 - \tau)^{-\frac{n}{p-1}} \left( \frac{\theta(1 - \tau \rho)}{\theta(\rho)} \right)^{\frac{2n}{p-1}}.
\]

Further, we will assume that

\[
u u_0 \geq C_3 \rho \frac{\theta^{2n}(\rho)}{\lambda(\rho)}.
\]

Let \( \tau_0 \in (0, 1) \) be the maximal root of the above equation. Fix \( \bar{x} \in B_{\rho \tau_0}(x_0) \) by the condition

\[
u u(\bar{x}) = \max_{B_{\rho \tau_0}(x_0)} u = \frac{u_0}{2} (1 - \tau_0)^{-\frac{n}{p-1}} \left( \frac{\theta((1 - \tau_0) \rho)}{\theta(\rho)} \right)^{\frac{2n}{p-1}}.
\]
Since $B_{\rho(1-\tau_0)/2}(x)$, by our choice of $\tau_0, x$, and by assumptions on $\theta(r)$ in (g$_3$), we have

$$\max_{B_{\rho(1-\tau_0)/2}(x)} u \leq \frac{2^{n-1} u_0}{(1-\tau_0)^{p-1}} \left( \frac{\theta \left( \frac{1-\tau_0}{2} \rho \right)}{\theta(\rho)} \right)^{\frac{2n}{p-1}} = 2^{n-1} u(x) \left( \frac{\theta \left( \frac{1-\tau_0}{2} \rho \right)}{\theta((1-\tau_0)\rho)} \right)^{\frac{2n}{p-1}} \leq \frac{3n}{4} u(x).$$

**Claim.** There exists a positive number $\nu \in (0, 1)$ depending only on the known data such that

$$\left| \left\{ x \in B_{\rho(1-\tau_0)/2}(x) : u(x) \geq \frac{u(x)}{2} \right\} \right| \geq \nu \theta^{-2n} \left( \frac{1-\tau_0}{2} \rho \right) \left| B_{\rho(1-\tau_0)/2}(x) \right|.$$  \hfill (4.19)

Indeed, in the opposite case, we apply (4.12)–(4.14) with the choices

$$\mu_+ = 2^{n-1} u(x), \quad \xi \omega = \left( 2^{n-1} - \frac{1}{2} \right) u(x), \quad a = \frac{2^{n-1} - \frac{3}{4}}{2^{n-1} - \frac{1}{2}} \in (0, 1).$$

The condition (4.17) obviously implies $\xi \omega \geq (1-\tau_0)\rho/2$. Therefore, we can conclude that

$$u(x) \leq \max_{B_{\rho(1-\tau_0)/2}(x)} u \leq \frac{3}{4} u(x)$$

reaches a contradiction, which proves the claim.

We set

$$r = \frac{1}{2} (1-\tau_0)\rho, \quad E = B_r(x) \cup \left\{ u \geq \frac{1}{2} u(x)\lambda(\rho) \right\},$$

then inequality (4.19) translates into

$$|E| \geq \nu \theta^{-2n}(r) |B_r(x)|.$$  \hfill (4.21)

Next, we will apply Theorem 1.14. For this, we consider an auxiliary solution

$$v = v(x, m), \quad m = \frac{1}{2} u(x)\lambda(\rho), \quad x \in D = B_{4\rho}(x) \setminus E,$$

of problem (1.13), (1.14) in $D = B_{4\rho}(x) \setminus E$. By our choices and by (4.17), we have $\rho \leq m \leq \frac{1}{2} M\lambda(\rho)$. We need to check the inequality

$$C(E, B_{4\rho}(x); m) \geq C_3 \rho^{n-1}.$$  \hfill (4.23)

For every $v > 0$, we set

$$g_{B_{4\rho}(x)}(v) = \inf_{x \in B_{4\rho}(x)} g(x, v), \quad \Phi(v) := \int_0^v g_{B_{4\rho}(x)}(s) \, ds.$$
Let \( \varphi \in W_0(B_{4\rho}(\bar{x})) \) and \( \varphi \geq 1 \) on \( E \). Using the Poincaré inequality, condition \((g_1)\), and the Young inequality \((1.7)\), we obtain

\[
\frac{1}{\rho} \int_{B_{4\rho}(\bar{x})} g_{B_{4\rho}(\bar{x})}(m\varphi/\rho) \varphi \, dx \\
\leq \frac{\gamma}{\rho} \int_{B_{4\rho}(\bar{x})} \Phi(m\varphi/\rho) \, dx \leq \gamma \int_{B_{4\rho}(\bar{x})} g_{B_{4\rho}(\bar{x})}(m\varphi/\rho) |\nabla \varphi| \, dx \\
\leq \frac{1}{2\rho} \int_{B_{4\rho}(\bar{x})} g_{B_{4\rho}(\bar{x})}(m\varphi/\rho) \varphi \, dx + \gamma \int_{B_{4\rho}(\bar{x})} g_{B_{4\rho}(\bar{x})}(m|\nabla \varphi|) |\nabla \varphi| \, dx \\
\leq \frac{1}{2\rho} \int_{B_{4\rho}(\bar{x})} g_{B_{4\rho}(\bar{x})}(m\varphi/\rho) \varphi \, dx + \gamma \int_{B_{4\rho}(\bar{x})} g(x, m|\nabla \varphi|) |\nabla \varphi| \, dx,
\]

which implies that

\[
\int_{B_{4\rho}(\bar{x})} g(x, m|\nabla \varphi|) |\nabla \varphi| \, dx \geq \frac{\gamma^{-1}}{\rho} \int_{B_{4\rho}(\bar{x})} g_{B_{4\rho}(\bar{x})}(m\varphi/\rho) \varphi \, dx
\]

\[
\geq \gamma^{-1} g_{B_{4\rho}(\bar{x})}(m/\rho) \left| E \right| \geq \gamma^{-1} \nu \theta^{-2n(r)} g_{B_{4\rho}(\bar{x})}(m/\rho) \left( \frac{r}{\rho} \right)^n \rho^{n-1}.
\]

And hence,

\[
C(E, B_{4\rho}(\bar{x}); m) \geq \gamma^{-1} \nu \left( \frac{r}{\rho} \right)^n \rho^{n-1} \theta^{-2n(r)} g_{B_{4\rho}(\bar{x})}(m/\rho).
\]

(4.24)

By our choice of \( m, u(\bar{x}), r \) (see \((4.22), (4.18), \) and \((4.20)\), respectively), and by \((4.17)\), we have

\[
g_{B_{4\rho}(\bar{x})}(m/\rho) = g_{B_{4\rho}(\bar{x})}( \frac{\theta(2r)}{\theta(\rho)} )^{\frac{2n}{p-1}} \left( \frac{\rho}{2r} \right)^{\frac{n}{p-1}} u_0 \lambda(\rho) \left( \frac{C_2}{4} \left( \frac{\rho}{2r} \right)^{\frac{n}{p-1}} \theta^{-2n(2r)} \right),
\]

and continuing to estimate from below using the inequality \( \theta(2r) \geq \frac{2}{3} \theta(r) \) from \((g_3)\) and conditions \((g_1)\) and \((g_0)\), we obtain

\[
g_{B_{4\rho}(\bar{x})}(m/\rho) \geq \frac{C_2}{4} \left( \frac{2}{9} \right)^{\frac{n}{p-1}} \left( \frac{r}{\rho} \right)^{\frac{n}{p-1}} \theta^{-2n(r)} g_{B_{4\rho}(\bar{x})}(m/\rho)\left( \frac{r}{\rho} \right)^n \rho^{n-1}.
\]

Therefore, choosing \( C_2 \) so that \( \gamma^{-1} \nu (C_2/4)^{p-1}(2/9)^n \rho^{n-1} \geq C_3 \), by \((4.24)\), we arrive at the required inequality \((4.23)\).

Now, using \((4.23)\) and Theorem 1.14, we conclude that

\[
u(x) \geq \gamma^{-1} \rho g_{x}^{-1} \left( \frac{C(E, B_{4\rho}(\bar{x}); m)}{\rho^{n-1}} \right) \text{ for a.a. } x \in B_{2\rho}(\bar{x}) \setminus B_{\rho}(\bar{x}).
\]

Inequality \((4.24)\) and condition \((g_1)\) imply that

\[
g_{x}^{-1} \left( \frac{C(E, B_{4\rho}(\bar{x}); m)}{\rho^{n-1}} \right) \geq \gamma^{-1} \nu \left( \frac{1}{\rho} \right)^{\frac{1}{p-1}} \theta^{-2n(r)} \left( \frac{r}{\rho} \right)^{\frac{n}{p-1}} \rho^{n-1}.
\]
And using our choices of \( m \) and \( u(\bar{x}) \), see (4.22) and (4.18), respectively, from the previous, we obtain

\[
\nu(x) \geq \gamma^{-1} \Theta^{\frac{m}{p-1}}(\rho) \lambda(\rho) u_0, \quad x \in B_{2\rho}(\bar{x}) \setminus B_{\rho}(\bar{x}).
\]

(4.25)

By our construction \( u \geq v \) on \( \partial D \) (see (4.20), (4.22)), and therefore, by Lemma 4.1 and (4.25), we have

\[
\inf_{B_{\rho}(x_0)} u \geq \inf_{B_{2\rho}(\bar{x})} u \geq \inf_{\partial B_{2\rho}(\bar{x})} u \geq \inf_{\partial B_{\rho}(\bar{x})} v \geq \gamma^{-1} \Theta^{\frac{m}{p-1}}(\rho) \lambda(\rho) u_0,
\]

which completes the proof of Theorem 1.12.

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CONFLICT OF INTEREST STATEMENT

The authors declare no potential conflicts of interests.

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