On the Field of a Binary Pulsar

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Abstract

In the present work, an exact solution of Einstein’s field equations representing the field of a binary system, is obtained. It is shown that this solution is a time-dependent one. This solution is a modified version of that obtained by Curzon, since it reduces to the later one in the static case. The equations of motion for a test particle in the field of a binary system are formulated and solved. Such equations can be used to study the motion of a third body, or a photon, in the field of a binary system.

Introduction

What is meant by the two body problem is the problem of two structurless non-spinning point-like particles, characterized by two mass parameters $m_1$ and $m_2$, moving under their mutual gravitational interaction. In order to formulate this problem, one aims to get an explicit expression of acceleration of the binaries in terms of their positions and velocities. To discuss this problem we can dissolve it into two aspects: i) obtaining the equations of motion of the two interacting bodies, ii) solving these equations.
In the context of Newtonian gravity the members of the system are considered as widely separated objects, such that the contribution of the non-linear effects can be neglected. Newtonian gravity has a linear structure that enables us to derive the equations of motion of the binary components, and also to get an exact solution for these equations. It gives a full treatment of the binary system. On the other hand, in General Relativity, The field equations have a non-linear hyperbolic structure, so that it is not easy even to derive the equations of motion for such systems. Since in General Relativity the equations of motion are embedded in the field equations, consequently it is very difficult to derive the field equations as a linear functional of the matter distribution independently of the equations of motion [1][5].

One may ask why we are obliged to deal with General Relativity. Actually the Newtonian theory is very limited, since it deals only with systems, have a widely separated and slowly moving objects. Therefore, this theory can not predict the behavior of systems with a small separation and fast moving objects. Moreover it can not account for the accumulating small feedback effects, e.g. the advance of perihelia of planetary orbits. Fortunately, in such cases we are saved by General Relativity, which gives a good agreement with the observational and experimental tests. In order to avoid the above mentioned difficulties, most of the relativistic treatments in the literature have used an approximation technique, i.e. the Parameterized Post Newtonian (PPN) method. This method expands the relativistic effects in power series of $v/c$, when only the relativistic expressions with order of $v^2/c^2$ added to the Newton’s law, so it is called first post-Newtonian (1 PN) approximation [10]. Recently, the equations of motion of binary systems have been derived at the third and half post-Newtonian (3.5 PN) order. However the beauty of this technique is its applicability for various classes of relativistic theories of gravity and also on its attractive mathematical formalism. The weak point of this technique is the breaking of the general covariance principle.

Exact solutions in General Relativity have played an important role to give a full description of some physical problems (e.g. Schwarzschild, kerr, Reissner-Nordström). The meaning of an “Exact Solution” is that the metric is represented by a coordinate system, in terms of the well-known analytical functions. Unfortunately, there are few exact solutions for real physical problems that have been established, as mentioned by Kinnersky c.f. [6] “Most of the known exact solutions describe situations which are frankly unphysical”.

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Although we have a lot of these non-physical solutions, one of the most important problems, which remained without an exact solution, is the physical field of two bodies. In general, for any physical problem we aim to construct a mathematical model; using some reasonable conditions; defined by a certain set of differential equations. Actually it is not easy to interpret the obtained solution in some physical theories such as General Relativity, because of its high non-linearity. But even if we could not understand the qualitative features of an exact solution, we can compare it with approximate results, which will be very useful, to check the validity of this solution.

The aim of the present work is to find an exact solution, of Einstein’s field equations, which can be used to represent the gravitational field of a binary systems. This certifies a difficult task, but we are going to simplify this problem in the following manner. In the literature there are many trials to solve two body problem in the frame work of General Relativity. The solution required is expected to represent the field of two revolving objects (singularities) in one system. In 1924 Curzon obtained a solution, to Einstein’s field equations, representing the field of two static singularities. Unfortunately, his solution has been classified among un-physical solutions, since it does not represent a real astrophysical system, we are going to adjust this solution in order to be suitable for describing a binary system. For this reason, we give a brief review of Curzon solution in §1 with its necessary background. In section §2 we are going to modify (adjust) this solution to represent the field of a real binary system. We test the solution by reducing to the flat space and schwarzschild solutions, under certain conditions, in §3. Motion in the modified Curzon field is treated in section §4. The work is discussed and concluded in §5.

We assumed that the two members of this system have the same masses and rotate in a circular orbit about the center of mass. We assumed further, that the line of sight is in the plane of the orbit. This represents a two body problem which represents many physical configurations especially PSR J0737-3039 [3, 7].
1 Curzon Solution

This solution is a Weyl class one, which was found soon after the birth of GR. It refers to two “monopoles” on the axis of symmetry. This is mentioned by Bonnor [2] as “Probably the most perspicacious of all exact solutions in GR”.

1.1 Standard Curzon Static Field

In this section, we aim to review the solution, obtained by Curzon (1924), of Einstein’s equations

\[ R_{\alpha\beta} = 0, \quad (1.1) \]

for an axial symmetric gravitational field produced by two singularities on the axis \( \bar{x}^1 \) separated by a distance \( 2a \). If the origin of a reference frame lies at the mid point between the two singularities on this axis, so for an arbitrary point \( P \) in the plane of \( \bar{x}^1 \) and \( \bar{x}^2 \) coordinates the direction of these singularities can be defined using bipolar coordinates \( \bar{r}_1 \) and \( \bar{r}_2 \), as shown in Figure 1:

![Figure 1](image)

**Figure 1:** The arbitrary point \( P \) on the plane of \( \bar{x}_1 \) and \( \bar{x}_2 \) coordinates is referred to the singularities \( m_1 \) and \( m_2 \) by bipolar coordinates \( \bar{r}_1 \) and \( \bar{r}_2 \), respectively.
where
\[
\bar{r}_1^2 = (\bar{x}^1 - a)^2 + (\bar{x}^2)^2
\]
\[
\bar{r}_2^2 = (\bar{x}^1 + a)^2 + (\bar{x}^2)^2
\]
(1.2)
The cylindrical coordinates used in this study are
\[
\bar{x}^1 = \bar{z}, \quad \bar{x}^2 = \bar{\rho}, \quad \bar{x}^3 = \bar{\phi}, \quad \bar{x}^4 = \bar{t}
\]
(1.3)
The metric characterizing the space with axial-symmetric static gravitational field [4],
\[
d\bar{s}^2 = -e^{-\bar{\mu}} (d\bar{z}^2 + d\bar{\rho}^2) - e^{-\bar{\nu}} \bar{\rho}^2 d\bar{\phi}^2 + e^{\bar{\nu}} d\bar{t}^2
\]
(1.4)
where \(\bar{\mu} \equiv \bar{\mu}(\bar{z}, \bar{\rho})\), \(\bar{\nu} \equiv \bar{\nu}(\bar{z}, \bar{\rho})\).
So that the metric coefficients \(\bar{g}_{\alpha\beta}\) are
\[
\bar{g}_{11} = -e^{\bar{\mu}}, \quad \bar{g}_{22} = -e^{\bar{\mu}}, \quad \bar{g}_{33} = -\bar{\rho}^2 e^{-\bar{\nu}}, \quad \bar{g}_{44} = e^{\bar{\nu}}
\]
\[
\bar{g}_{\alpha\beta} = 0, \quad \alpha \neq \beta
\]
The components of the contravariant tensor \(\bar{g}^{\alpha\beta}\) are given by
\[
\bar{g}^{\alpha\beta} = \begin{pmatrix}
-e^{-\bar{\mu}} & 0 & 0 & 0 \\
0 & -e^{-\bar{\mu}} & 0 & 0 \\
0 & 0 & -e^{\bar{\nu}/\bar{\rho}^2} & 0 \\
0 & 0 & 0 & e^{\bar{\nu}}
\end{pmatrix}
\]
Using the definition of Christoffel symbol,
\[
\{ \bar{\alpha} \bar{\beta} \bar{\gamma} \} \overset{def.}{=} \frac{1}{2} \bar{g}^{\alpha j} (\bar{g}_{\beta j,\gamma} + \bar{g}_{\gamma j,\beta} - \bar{g}_{\beta\gamma, j})
\]
(1.5)
we get for the space (1.4) the following non-vanishing components,
\[
\begin{align*}
\{ \bar{1} \bar{11} \} &= \frac{1}{2} \bar{\mu}_1 \\
\{ \bar{1} \bar{12} \} &= \frac{1}{2} \bar{\mu}_2 \\
\{ \bar{1} \bar{22} \} &= -\frac{1}{2} \bar{\mu}_1 \\
\{ \bar{1} \bar{33} \} &= \frac{1}{2} \bar{\rho}^2 e^{-(\bar{\nu} + \bar{\mu})} \bar{\nu}_1 \\
\{ \bar{1} \bar{44} \} &= \frac{1}{2} e^{(\bar{\nu} - \bar{\mu})} \bar{\nu}_1 \\
\{ \bar{2} \bar{11} \} &= -\frac{1}{2} \bar{\mu}_2 \\
\{ \bar{2} \bar{12} \} &= \frac{1}{2} \bar{\nu}_1 \\
\{ \bar{2} \bar{22} \} &= \frac{1}{2} \bar{\nu}_1 \\
\{ \bar{2} \bar{33} \} &= -\frac{1}{2} \bar{\nu}_2 \\
\{ \bar{2} \bar{44} \} &= -\frac{1}{2} (\bar{\nu}(-2 + \bar{\rho} \bar{\nu}_2))
\end{align*}
\]
where \( \bar{\mu}_1 \) denotes \( \frac{\partial \bar{\mu}}{\partial \bar{z}} \), \( \bar{\mu}_2 \) denotes \( \frac{\partial \bar{\mu}}{\partial \bar{\rho}} \), \( \tilde{\nu}_1 \) denotes \( \frac{\partial \tilde{\nu}}{\partial \bar{z}} \) and \( \tilde{\nu}_2 \) denotes \( \frac{\partial \tilde{\nu}}{\partial \bar{\rho}} \).

Now by using Einstein’s field equation in free space (1.1),

\[
\bar{R}_{\alpha\beta} \left( \text{def.} \right) \frac{\partial^2 \log \sqrt{-\bar{g}}}{\partial \bar{x}^\alpha \partial \bar{x}^\beta} - \left\{ \frac{\bar{\gamma}}{\alpha \beta} \right\} \frac{\partial \log \sqrt{-\bar{g}}}{\partial \bar{x}^\gamma} \left\{ \frac{\bar{\gamma}}{\alpha \beta} \right\} + \left\{ \frac{\bar{\gamma}}{\alpha \epsilon} \right\} \left\{ \frac{\bar{\epsilon}}{\gamma \beta} \right\} = 0,
\]

the following set of differential equations is obtained

\[
\begin{align*}
\bar{R}_{11} &= \frac{1}{2}(\bar{\mu}_{11} + \bar{\mu}_{22} + \bar{\nu}_1^2 + \bar{\mu}_2/\bar{\rho}) = 0, \\
\bar{R}_{12} &= \frac{1}{2}(\tilde{\nu}_1 \tilde{\nu}_2 - \tilde{\nu}_1/\bar{\rho} - \bar{\mu}_1/\bar{\rho}) = 0, \\
\bar{R}_{22} &= \frac{1}{2}(\bar{\mu}_{11} + \bar{\mu}_{22} + \bar{\nu}_2^2 - \bar{\mu}_2/\bar{\rho} - 2\bar{\nu}_2/\bar{\rho}) = 0, \\
\bar{R}_{33} &= -\frac{1}{2}e^{\bar{\mu}-\bar{\nu}^2}(\tilde{\nu}_{11} + \tilde{\nu}_{22} + \tilde{\nu}_2/\bar{\rho}) = 0, \\
\bar{R}_{44} &= -\frac{1}{2}e^{\bar{\mu}+\nu}(\tilde{\nu}_{11} + \tilde{\nu}_{22} + \tilde{\nu}_2/\bar{\rho}) = 0.
\end{align*}
\]

Equation (1.7.5) in this set is a mere repetition of equation (1.7.4). Thus we have only four equations for \( \bar{\mu} \) and \( \tilde{\nu} \). Since \( \tilde{\nu} \) is a function of \( \bar{z} \) and \( \bar{\rho} \), equation (1.7.5) can be written in the form,

\[
\nabla^2 \tilde{\nu} = 0,
\]

where \( \nabla^2 \text{ def.} = \frac{\partial^2}{\partial \bar{\rho}^2} + \frac{1}{\bar{\rho}} \frac{\partial}{\partial \bar{\rho}} + \frac{\partial^2}{\partial \bar{z}^2} \) in the cylindrical coordinates symmetrical about the \( \bar{z} \)-axis. It is clear that the above equation represents the classical (flat-space) Laplace’s equation, hence the solution of this equation is

\[
\tilde{\nu} = -2 \left[ \frac{m_1}{\bar{r}_1} + \frac{m_2}{\bar{r}_2} \right],
\]

where \( m_1 \) and \( m_2 \) are constants of the integration. One may realize how the contribution of the classical theory of gravity is represented by this linear equation. To obtain the relativistic effects of the curved space-time, let us introduce a new function \( \bar{\lambda} \), such that

\[
\bar{\lambda} = \bar{\mu} + \tilde{\nu},
\]

by using (1.7.2), we get

\[
\bar{\lambda}_1 = \bar{\mu}_1 + \tilde{\nu}_1 = \tilde{\nu}_1 \tilde{\nu}_2 \bar{\rho},
\]
by using (1.7.1) and (1.7.3), we get
\[ \bar{\lambda}_2 = \bar{\mu}_2 + \bar{\nu}_2 = \frac{\bar{\rho}}{2} (\bar{\nu}_2^2 - \bar{\nu}_1^2). \] (1.11)

Now,
\[ d\bar{\lambda} = \bar{\lambda}_1 d\bar{\varepsilon} + \bar{\lambda}_2 d\bar{\rho}, \] (1.12)
hence, substituting from (1.10) and (1.11) into (1.12), we get
\[ d\bar{\lambda} = \left(4\bar{\rho}^2 (\bar{\varepsilon} - a) d\bar{\varepsilon} + 2\bar{\rho} \left[\bar{\rho}^2 - (\bar{\varepsilon} - a)^2\right] d\bar{\rho}\right) \frac{m_1}{\bar{r}_1^6} \]
\[ + \left(4\bar{\rho}^2 (\bar{\varepsilon} + a) d\bar{\varepsilon} + 2\bar{\rho} \left[\bar{\rho}^2 - (\bar{\varepsilon} + a)^2\right] d\bar{\rho}\right) \frac{m_1}{\bar{r}_2^6} \]
\[ + 4 \left(\bar{\varepsilon} \bar{\rho} d\bar{\varepsilon} - \left[\bar{\varepsilon}^2 - a^2 - \bar{\rho}^2\right] d\bar{\rho}\right) \frac{m_1 m_2}{\bar{r}_1^3 \bar{r}_2^3}. \] (1.13)

Since the metric of the manifold given by equation (1.4), thus the integration of equation (1.13) should reduce to the flat space at infinity as \( \bar{r}_1 \) and \( \bar{r}_2 \to \infty \), so \( \bar{\lambda} \) tends to zero.
\[ \bar{\lambda} = -\frac{m_2^2}{\bar{r}_1^2} \bar{\rho}^2 - \frac{m_2^2}{\bar{r}_2^2} \bar{\rho}^2 + \frac{m_1 m_2}{a^2} \left[\left(\frac{\bar{\varepsilon}^2 + \bar{\rho}^2 - a^2}{\bar{r}_1 \bar{r}_2}\right) - 1\right]. \] (1.14)

One can realize that the function \( \bar{\lambda} \) is determined from \( \bar{\nu} \) by quadrature up to an additive constant. Also, the metric function \( \bar{\lambda} \) has the property that \( \lim_{\rho \to 0} \bar{\lambda} \neq 0 \), so there is some kind of a compression described by a canonical singularity between the particles holding particles apart [9]. From equations (1.8), (1.9) and (1.14), the function \( \bar{\mu} \) is given by
\[ \bar{\mu} = 2 \left[\frac{m_1}{\bar{r}_1} + \frac{m_2}{\bar{r}_2}\right] - \left[\frac{m_2^2}{\bar{r}_1^2} + \frac{m_2^2}{\bar{r}_2^2}\right] \bar{\rho}^2 + \frac{m_1 m_2}{a^2} \left[\left(\frac{\bar{\varepsilon}^2 + \bar{\rho}^2 - a^2}{\bar{r}_1 \bar{r}_2}\right) - 1\right]. \] (1.15)

Now we define two angles \( \bar{\alpha}_1 \) and \( \bar{\alpha}_2 \), as the angles between by \( AP, BP \) and the axis \( \bar{\varepsilon} \), respectively (see Figure 1). Finally the solution of (1.1), in the present case, can be written as
\[ \bar{\nu} = -2 \left[\frac{m_1}{\bar{r}_1} + \frac{m_2}{\bar{r}_2}\right], \] (1.16)
\[ \bar{\mu} = 2 \left[\frac{m_1}{\bar{r}_1} + \frac{m_2}{\bar{r}_2}\right] - \left[\frac{m_1^2}{\bar{r}_1^2} \sin^2 \bar{\alpha}_1 + \frac{m_2^2}{\bar{r}_2^2} \sin^2 \bar{\alpha}_2\right] \]
\[ - 2 \frac{m_1 m_2}{a^2} \sin^2 \left(\frac{\bar{\alpha}_1 - \bar{\alpha}_2}{2}\right). \] (1.17)
The solution given by (1.16) and (1.17) is the solution given by Curzon \[4\] in its standard form.

### 1.2 Stability of Curzon Solution

The relativistic solution, given above, can be rewritten in the following equivalent form \[9\].

\[
 ds^2 = e^\varphi dt^2 - e^{-\varphi} \left[ e^{\lambda} (dz^2 + d\rho^2) + \rho^2 d\phi^2 \right],
\]

where

\[
 \bar{\nu} = -2 \left[ \frac{m_1}{\bar{r}_1} + \frac{m_2}{\bar{r}_2} \right],
\]

\[
 \bar{\lambda} = -\frac{m_2}{\bar{r}_1^2} \rho^2 - \frac{m_2}{\bar{r}_2^2} \rho^2 + \frac{m_1 m_2}{a^2} \left[ \frac{z^2 + \rho^2 - a^2}{\bar{r}_1 \bar{r}_2} \right] - 1.
\]

To compute the force between the two singularities, one easily see that \(\bar{\lambda}(0) = 0\) along the \(z\)-axis between the two singularities, can be given as

\[
 \bar{\lambda} = \frac{m_1 m_2}{a^2} \left[ \frac{z^2 - a^2}{\bar{r}_1 \bar{r}_2} - 1 \right] = -2 \frac{m_1 m_2}{a^2} \tag{1.18}
\]

then the stress force between the two masses

\[
 F = -\frac{GM_1 M_2}{(2D)^2}, \tag{1.19}
\]

if we take the Newtonian limit \(D \gg m_1, m_2\), the medium between the two masses contains a compression merely the Newtonian force attraction \[9\].

### 2 Modification of Curzon Solution

Actually the static case, which had been studied by Curzon, does not represent any physical configuration (e.g. real binary systems in Nature) in view of the fact that the binary systems are always dynamical systems. The main aim of the present section is to show how to modify Curzon solution such that it can be used to represent the field of a real physical binary system.
In order to describe such systems, the effect of time should be introduced, by considering a co-rotating coordinate system, associated with the angular motion of the collinear singularities, with respect to a frame of reference fixed at the center. It is similar to the situation of using co-moving coordinate systems in cosmological applications.

2.1 General Outlines of the Modification

Curzon has chosen the point $P$ in the $(\bar{z}, \bar{\rho})$ plane. Since $A$ and $B$ are static they could not define a particular plane, he always can reorient the coordinates by choosing a particular value of an angle $\bar{\phi}$ to make $ABP$ plane always coincides with the $(\bar{z}, \bar{\rho})$ plane, without affecting generality. While the non-static case, such collinear singularities $A$ and $B$ are rotating around a common center $C$ by an angular velocity $\omega$. It is clear that the moving singularities are defining a particular plane (orbital plane), so the point $P$ should be an arbitrary point, not necessary in this plane. The separations between this point and the two singularities $(A, B)$ are respectively

$$\begin{align*}
r_1^2 &= (\bar{z} - a)^2 + \bar{\rho}^2 \sin^2 \bar{\phi}, \\
r_2^2 &= (\bar{z} + a)^2 + \bar{\rho}^2 \sin^2 \bar{\phi}.
\end{align*}$$

(2.1)

where the angle $\bar{\phi}$ represents the inclination of the orbital plane to the plane of the sky. As $\bar{\phi} \to \pi/2$ the binary system tends to be an eclipsing binary system with respect to an observer at the point $P$.

In general the singularities (binary components) are assumed to be separated from $C$ by distances $a$ and $b$ $(a \neq b)$, and the angular velocity $\omega$ in this case is a function of time. To simplify the problem we are going to assume that the motion is circular as following:

1. The masses of two singularities are comparable masses.
2. The distances, $a$ and $b$, are equal.
3. The angular velocity $\omega$ of the collinear singularities is constant.
4. The orbits of the two singularities will coincident on each other producing a circular orbit with a radius $a$. 

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Now before discussing the non-static case, we will return to standard form of Curzon metric (1.4), and show its form in other coordinate systems. This is done in order to facilitate comparison with some special cases.

### 2.2 A Particular Choice of Coordinate Systems

In order to refer the points of the manifold to the Cartesian coordinate system, as general covariance is still preserved since we use tensors in the formalism, we use the transformation

\[
TI: (\bar{z}, \bar{\rho}, \bar{\phi}, \bar{t}) \rightarrow (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})
\]

i.e.

\[
\begin{align*}
\bar{z} &= \tilde{z}, \\
\bar{\rho} &= \sqrt{\tilde{x}^2 + \tilde{y}^2}, \\
\bar{\phi} &= \tan^{-1}(\tilde{x}/\tilde{y}), \\
\bar{t} &= \tilde{t}.
\end{align*}
\]

(2.2)

Applying (2.2) to Curzon metric (1.4) and recalling that \(ds\) is a scalar, we get

\[
ds^2(\bar{x}^\beta) = ds^2(\tilde{x}^\alpha)
\]

So we can rewrite (1.4) in the form

\[
ds^2 = -\frac{\tilde{x}^2 e^{\mu} + \tilde{y}^2}{\tilde{x}^2 + \tilde{y}^2} d\tilde{x}^2 - \frac{2 \tilde{x} \tilde{y} (e^{\mu} - e^{\nu})}{\tilde{x}^2 + \tilde{y}^2} d\tilde{x} d\tilde{y} \\
- \frac{\tilde{y}^2 e^{\mu} + \tilde{x}^2 e^{-\nu}}{\tilde{x}^2 + \tilde{y}^2} d\tilde{y}^2 - e^{\tilde{\mu}} d\tilde{z}^2 + e^{\tilde{\nu}} d\tilde{t}^2
\]

(2.3)

where

\[
\tilde{\nu}(\tilde{x}, \tilde{y}, \tilde{z}) = -2 \left[ \frac{m_1}{\tilde{r}_1} + \frac{m_2}{\tilde{r}_2} \right]
\]

(2.4.1)

\[
\tilde{\mu}(\tilde{x}, \tilde{y}, \tilde{z}) = 2 \left[ \frac{m_1}{\tilde{r}_1} + \frac{m_2}{\tilde{r}_2} \right] - (\tilde{x}^2 + \tilde{y}^2) \left[ \frac{m_1}{\tilde{r}_1^3} + \frac{m_2}{\tilde{r}_2^3} \right] \\
+ \frac{m_1 m_2}{a^2} \left[ \frac{(\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 - a^2)}{\tilde{r}_1 \tilde{r}_2} - 1 \right]
\]

(2.4.2)
and

\[
\begin{align*}
\tilde{r}_1^2 &= \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 + a^2 - 2a\tilde{z} \\
\tilde{r}_2^2 &= \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 + a^2 + 2a\tilde{z}
\end{align*}
\]  
(2.5)

The metric (2.3) represents the gravitational field of two singularities, and has axial symmetry about the $\tilde{z}$-axis.

Again we are going to apply a second transformation to represent the metric (2.3) in spherical polar coordinate, as follows

\[
TII: (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}) \rightarrow (\hat{r}, \hat{\theta}, \hat{\phi}, \hat{t})
\]

i.e.

\[
\begin{align*}
\tilde{x} &= \hat{r} \cos \hat{\theta} \\
\tilde{y} &= \hat{r} \sin \hat{\theta} \cos \hat{\phi} \\
\tilde{z} &= \hat{r} \sin \hat{\theta} \sin \hat{\phi} \\
\tilde{t} &= \hat{t}
\end{align*}
\]  
(2.6)

where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. One can notice that the transformation $TII$ is not the conventional transformation. Consequently, the angle $\hat{\phi}$ is not the same angle $\bar{\phi}$, given in the standard Curzon solution.

Since,

\[
d\tilde{s}^2(\tilde{x}^\gamma) = d\bar{s}^2(\bar{x}^\alpha).
\]

Then, the metric coefficient will be given by

\[
g_{11} = -e^{\hat{\mu}}
\]  
(2.7.1)

\[
g_{22} = -\hat{r}^2 \frac{\cos^2 \hat{\theta} \sin^2 \hat{\phi} \cos \hat{\phi} e^{\hat{\mu}} + \cos^2 \hat{\phi} e^{-\hat{\nu}}}{\cos^2 \hat{\theta} + \sin^2 \hat{\theta} \cos^2 \hat{\phi}}
\]  
(2.7.2)

\[
g_{23} = -\hat{r}^2 \sin \hat{\theta} \sin \hat{\phi} \cos \hat{\theta} \cos \hat{\phi} \left( e^{\hat{\mu}} - e^{-\hat{\nu}} \right)
\]  
(2.7.3)

\[
g_{33} = -\hat{r}^2 \sin^2 \hat{\theta} \cos^2 \hat{\phi} \sin^2 \hat{\phi} e^{-\hat{\nu}} + \cos^2 \hat{\phi} e^{\hat{\mu}}
\]  
(2.7.4)

\[
g_{44} = e^{\hat{\nu}}
\]  
(2.7.5)
where

\[ \dot{\nu}(\hat{r}, \hat{\theta}, \hat{\phi}) = -2 \left[ \frac{m_1}{\dot{r}_1} + \frac{m_2}{\dot{r}_2} \right] \]  
(2.8.1)

\[ \dot{\mu}(\hat{r}, \hat{\theta}, \hat{\phi}) = 2 \left[ \frac{m_1}{\dot{r}_1} + \frac{m_2}{\dot{r}_2} \right] 
- \dot{r}^2 \left( \cos^2 \hat{\theta} + \sin^2 \hat{\theta} \cos^2 \hat{\phi} \right) \left[ \frac{m_1}{\dot{r}_1^4} + \frac{m_2}{\dot{r}_2^4} \right] 
+ \frac{m_1 m_2}{a^2} \left[ \left( \frac{\dot{r}^2 - a^2}{\dot{r}_1 \dot{r}_2} \right) - 1 \right] \]  
(2.8.2)

and

\[ \dot{r}_1^2 = (\dot{r} + a)^2 - 2a \dot{r} (1 + \sin \hat{\theta} \sin \hat{\phi}) \]  
\[ \dot{r}_2^2 = (\dot{r} + a)^2 - 2a \dot{r} (1 - \sin \hat{\theta} \sin \hat{\phi}) \]  
(2.9)

In view of the previous illustration, considering the non-static case assumptions, given in §2.1, we apply the third transformation

\[ TIII: (\hat{r}, \hat{\theta}, \hat{\phi}, \hat{t}) \longrightarrow (r, \theta, \phi, t) \]

i.e.

\[ \begin{align*}
\hat{r} &= r, \\
\hat{\theta} &= \theta, \\
\hat{\phi} &= \phi + \omega t, \\
\hat{t} &= t,
\end{align*} \]

(2.10)

which if combined with the transformation law of the scalar,

\[ ds^2(x^\sigma) = ds^2(x^\gamma), \]

would give the following non-vanishing components of the metric tensor in
terms of the new coordinate system \((r, \theta, \phi, t)\).

\[ g_{11} = -e^\mu, \quad (2.11.1) \]

\[ g_{22} = -r^2 \frac{\cos^2 \theta \sin^2(\phi + \omega t) e^\mu + \cos^2(\phi + \omega t) e^{-\nu}}{[\cos^2 \theta + \sin^2 \theta \cos^2(\phi + \omega t)]}, \quad (2.11.2) \]

\[ g_{23} = -r^2 \sin \theta \sin (\phi + \omega t) \cos \theta \cos (\phi + \omega t) \left( e^\mu - e^{-\nu} \right) \left( \cos^2 \theta + \sin^2 \theta \cos^2(\phi + \omega t) \right)^{1/2}, \quad (2.11.3) \]

\[ g_{24} = -\omega r^2 \sin \theta \sin (\phi + \omega t) \cos \theta \cos (\phi + \omega t) \left( e^\mu - e^{-\nu} \right) \left( \cos^2 \theta + \sin^2 \theta \cos^2(\phi + \omega t) \right)^{1/2}, \quad (2.11.4) \]

\[ g_{33} = -r^2 \sin^2 \theta \cos^2 \theta \left( e^\mu + \cos^2(\phi + \omega t) e^{-\nu} \right) \left[ \cos^2 \theta + \sin^2 \theta \cos^2(\phi + \omega t) \right], \quad (2.11.5) \]

\[ g_{34} = -\omega r^2 \sin^2 \theta \cos^2 \theta \left( e^\mu - e^{-\nu} \right) \left[ \cos^2 \theta + \sin^2 \theta \cos^2(\phi + \omega t) \right] / \left[ \cos^2 \theta + \sin^2 \theta \cos^2(\phi + \omega t) \right], \quad (2.11.6) \]

\[ g_{44} = -[\omega r^2 \sin^2 \theta \left( \cos^2 \theta \sin^2(\phi + \omega t) e^{-\nu} + \cos^2(\phi + \omega t) e^\mu \right) - \left( \cos^2 \theta + \sin^2 \theta \cos^2(\phi + \omega t) \right) e^\nu] / \left[ \cos^2 \theta + \sin^2 \theta \cos^2(\phi + \omega t) \right], \quad (2.11.7) \]

while (2.8.1), (2.8.2) and (2.9) now read:

\[ \nu (r, \theta, \phi, t) = -2 \left[ \frac{m_1}{r_1} + \frac{m_2}{r_2} \right], \quad (2.12.1) \]

\[ \mu (r, \theta, \phi, t) = 2 \left[ \frac{m_1}{r_1} + \frac{m_2}{r_2} \right] - r^2 \left( \cos^2 \theta + \sin^2 \theta \cos^2(\phi + \omega t) \right) \left[ \frac{m_1^2}{r_1^4} + \frac{m_2^2}{r_2^4} \right] + \frac{m_1 m_2}{a^2} \left[ \left( \frac{r^2 - a^2}{r_1 r_2} \right) - 1 \right], \quad (2.12.2) \]
and
\[
\begin{align*}
    r_1^2 &= (r + a)^2 - 2ar(1 + \sin \theta \sin (\phi + \omega t)), \\
    r_2^2 &= (r + a)^2 - 2ar(1 - \sin \theta \sin (\phi + \omega t)).
\end{align*}
\] (2.13)

Thus the set of quantities \([2.11.1]-[2.11.7]\) represents the field which is produced by a dynamical two-body system, at any chosen point outside the two singularities.

3 Check of the modified solution

The line element of the modified Curzon solution is given in terms of spherical polar coordinates \((r, \theta, \phi, t)\). In order to check this solution one should obtain the flat space metric and Schwarzschild metric as limiting cases under certain conditions.

3.1 First check: flat space metric

It can be easily shown that the \(\mu\) and \(\nu\) functions, given by \([2.12.1]\) and \([2.12.2]\), vanish as \(r_1, r_2 \to \infty\), i.e. at a large distance from the binary system. In this case the metric coefficients \([2.11.1]-[2.11.7]\) will become
\[
\begin{align*}
    g_{11} &= -1, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2 \theta, \quad g_{44} = 1
\end{align*}
\]
and the line element will reduce to
\[
    ds^2 = -dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + dt^2
\]
which describes the space in absence of the gravitational field.

3.2 Second check: Schwarzschild metric

Taking \(\omega = 0\), \(m_1 + m_2 = m\), and taking \(r\) large enough, \(r_1 \approx r_2 \approx r\), such that \(O\left(\frac{1}{r^2}\right) \to 0\). In this case the metric coefficients \([2.11.1]-[2.11.7]\) will become
\[
\begin{align*}
    g_{11} &= -e^\mu, \quad g_{22} = -r^2 e^\mu, \quad g_{33} = -r^2 \sin^2 \theta e^\mu, \quad g_{44} = e^\nu,
\end{align*}
\]
and the line element can be written as
\[
    ds^2 = -e^{2\mu}(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) + e^{2\nu} dt^2.
\]
Where
\[ \nu^* = -\left( \frac{m_1}{r_1} + \frac{m_2}{r_2} \right) = -\frac{m}{r} \]
\[ \mu^* = \left( \frac{m_1}{r_1} + \frac{m_2}{r_2} \right) + \mathcal{O} \left( \frac{1}{r^2} \right) = \frac{m}{r}, \]
then the line element can be rewritten as
\[
\begin{align*}
\text{ds}^2 &= -\left[ 1 + \frac{m}{r} + \mathcal{O} \left( \frac{1}{r^2} \right) \right]^2 \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi \right) \\
&\quad + \left[ 1 + \frac{m}{r} + \mathcal{O} \left( \frac{1}{r^2} \right) \right]^{-2} dt^2.
\end{align*}
\]
Apply the transformation \( R = r + m \), we get
\[
\text{ds}^2 = -\left( 1 - \frac{2m}{R} \right)^{-1} dR^2 - R^2 d\theta^2 - R^2 \sin^2 \theta d\phi^2 + \left( 1 - \frac{2m}{R} \right) dt^2,
\]
which represents the line element of Schwarzschild gravitational field in its standard form.

4 Motion in The Modified Curzon Field

“spacetime tells matter how to move, and matter tells spacetime how to curve” [8]. We have seen how matter tells spacetime how to curve, now we would like to search how spacetime tells matter how to move!! So we calculate the non-vanishing Christoffel symbol coefficients of the second kind (symmetric in first two indices) for the space represented by (2.11.1)-(2.11.7). By using the calculated values of the Christoffel symbols and apply the geodesic equation,
\[
\frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0,
\]
one can formulate the equations of motion of a test particle in the gravitational field of a binary system. In this way we can describe the motion of a test particle like a third body in the binary system, without adopting perturbation technique, or we can describe the motion of a massless particle (e.g. photon) in the field of the binary pulsar. To test planer motion, we put \( x_2 = \theta \) in the equations of motion and using the above values of the
Christoffel symbols, we can see that the differential equation for the angle $\theta$ can be written as,

$$\frac{d^2\theta}{ds^2} + \left[ A \frac{dr}{ds} + B \frac{d\theta}{ds} + C \frac{d\phi}{ds} + D \frac{dt}{ds} \right] \frac{d\theta}{ds} = 0,$$

where $A$, $B$, $C$, and $D$ are known functions. By taking initially $\theta = \theta_0 = \pi/2$, and $(\frac{d\theta}{ds})_0 = 0$, we get from the above equation

$$\frac{d^2\theta}{ds^2} = 0,$$

then

$$\dot{\theta} \equiv \frac{d\theta}{ds} = 0. \quad (4.1)$$

It is clear from this solution that the motion of a test particle is a planer motion. Now restricting ourselves by taking the plane $\theta = \pi/2$ for an eclipsing binary. In this case the line element will become

$$ds^2 = -e^\mu dr^2 - r^2 e^\mu d\phi^2 - 2\omega r^2 e^\mu d\phi dt + (e^\nu - \omega^2 r^2 e^\mu) dt^2. \quad (4.2)$$

For $x^3 = \phi$, the equation of motion becomes,

$$\frac{d}{ds} \left[ -2r^2 e^\mu \dot{\phi} - 2\omega r^2 e^\mu \dot{t} \right] = e^\nu (\nu_\phi - \mu_\phi) \dot{t}^2, \quad (4.3)$$

where $\mu_\phi = \frac{\partial \mu}{\partial \phi}$, and $\nu_\phi = \frac{\partial \nu}{\partial \phi}$. This leads to

$$r^2 e^\mu \left( \dot{\phi} + \omega \dot{t} \right) = - \frac{1}{2} \int e^\nu (\nu_\phi - \mu_\phi) \dot{t} \, dt + L, \quad (4.4)$$

where $L$ is an arbitrary constant.

Similarly, for $x^4 = t$, we get

$$\frac{d}{ds} \left[ -2\omega r^2 e^\mu \dot{\phi} + 2 (e^\nu - \omega^2 r^2 e^\mu) \dot{t} \right] = e^\nu (\nu_\phi - \mu_\phi) \dot{t}^2. \quad (4.5)$$

The above differential equation leads to

$$\omega r^2 e^\mu \left( \dot{\phi} + \omega \dot{t} \right) - e^\nu \dot{t} = - \frac{1}{2} \int e^\nu (\nu_\phi - \mu_\phi) \dot{t} \, dt - \mathcal{E}, \quad (4.6)$$
where $\mathcal{E}$ is an arbitrary constant.

Recalling equations (2.12.1) and (2.12.2), one can show that:

$$\omega \nu_{\phi} - \nu_t = \omega \mu_{\phi} - \mu_t = 0.$$  \hfill (4.7)

Multiplying (4.4) by the constant $\omega$, combining it with (4.6) and solving the equations for $\dot{t}$, we get

$$\dot{t} = \left( \mathcal{E} + \omega \mathcal{L} \right) e^{-\nu}. \hfill (4.8)$$

Similarly, solving for $\dot{\phi}$, we get

$$\dot{\phi} = \frac{\mathcal{L}}{2r^2} e^{-\mu} - \omega \left( \mathcal{E} + \omega \mathcal{L} \right) e^{-\nu} = \left( \mathcal{E} + \omega \mathcal{L} \right) F_{\phi}, \hfill (4.9)$$

where

$$F_{\phi} = \frac{e^{-\mu}}{2r^2} \int (\nu_{\phi} - \mu_{\phi}) \, dt.$$ 

It is easy to show that $F_{\phi} \approx \mathcal{O} \left( \frac{1}{r^3} \right)$, which can be ignored for the time being, therefore

$$\dot{\phi} \approx \frac{\mathcal{L}}{r^2} e^{-\mu} - \omega \left( \mathcal{E} + \omega \mathcal{L} \right) e^{-\nu} + \mathcal{O} \left( \frac{1}{r^3} \right). \hfill (4.10)$$

Recalling equation (4.2), and substituting from (4.8) and (4.10), we get for $\dot{r}$

$$\dot{r} = \sqrt{\left( \mathcal{E} + \omega \mathcal{L} \right)^2 e^{-\mu-\nu} - e^{-\mu} \left( B + \frac{\mathcal{L}^2}{r^2} e^{-\mu} \right)}, \hfill (4.11)$$

where the parameter $B$ is defined as

$$B = \begin{cases} 
0, & \text{for a photon;} \\
1, & \text{for a material particle.}
\end{cases}$$

In what follows, we are going to extract some physical features from the above results by considering some special cases.

### 4.1 Boundary conditions and physical meaning

- **The field far from the source:**

  Assuming that the source of field is too distant from the observer, then we can consider that $r \simeq r_1 \simeq r_2$. Also the total mass of the system is $m = m_1 + m_2$. 

By taking the approximation \( r \gg m \) so that \( \mathcal{O}\left(\frac{m^2}{r^2}\right) \to 0 \), therefore
\[
e^\mu = e^{-\nu} \approx \left(1 - \frac{2m}{r}\right)^{-1},
\]
and
\[
\dot{\theta} = 0, \quad \text{(4.12.1)}
\]
\[
i \approx \frac{\mathcal{E} + \omega L}{1 - \frac{2m}{r}}, \quad \text{(4.12.2)}
\]
\[
\dot{\phi} \approx \frac{\mathcal{L}}{r^2} - \omega \frac{\mathcal{E} + \omega L}{1 - \frac{2m}{r}}, \quad \text{(4.12.3)}
\]
\[
\dot{\tau} \approx \sqrt{\mathcal{E}^2 - \left(1 - \frac{2m}{r}\right) \left(\mathcal{B} + \frac{\mathcal{L}^2}{r^2}\right)}. \quad \text{(4.12.4)}
\]

It is clear that the above equations are similar to the Schwarzschild case except for some additional terms depending on the angular velocity \( \omega \). Also, from our experience in classical mechanics it is clear that the constants of integrations \( \mathcal{L} \) and \( \mathcal{E} \) are, respectively, representing the angular momentum and the energy of the moving test particle.

- **Static Field:**

Taking \( \omega = 0 \), we get the following set of differential equations:
\[
\dot{\theta} = 0, \quad \text{(4.13.1)}
\]
\[
i \approx \frac{\mathcal{E}}{1 - \frac{2m}{r}}, \quad \text{(4.13.2)}
\]
\[
\dot{\phi} \approx \frac{\mathcal{L}}{r^2}, \quad \text{(4.13.3)}
\]
\[
\dot{\tau} \approx \sqrt{\mathcal{E}^2 - \left(1 - \frac{2m}{r}\right) \left(\mathcal{B} + \frac{\mathcal{L}^2}{r^2}\right)}. \quad \text{(4.13.4)}
\]
which is now identical to the motion in Schwarzschild field.

After all, our attempt is to show how some measurable quantities (e.g. redshift and time delay in the field of binary systems) can be extracted by using of the solution of the equations of motion.

5 Prospective

The modified model enables us to go over to radiative spacetimes, considering some new developments, especially, impulsive gravitational waves generated by boosting various particles. Also, the more useful benefit of this model is its capability to calculate the redshift of the pulsations of highly relativistic systems (binary pulsars) like the recently discovered double-pulsar system PSR J0737-3039A & B \[3, 7\]. Furthermore, our model enables us to calculate the energy and the angular momentum of the binary pulsars by using one of the famous methods (e.g. Möller energy-momentum complex). In this way we can estimate the decay of the orbital period, coalescence rate and the gravitational radiation of the compact binary systems.

References

[1] Luc Blanchet, *On the two-body problem in general relativity*, C. R. Acad. Sci. Paris, t. 2, Série IV (2001), 1343–1352.

[2] W. B. Bonnor, *Physical interpretation of vacuum solutions of einstein's equations. Part I. time-independent solutions*, Gen. Rel. Grav. 24 (1992), 551.

[3] M. Burgay, N. D'Amico, A. Possenti, R.N. Manchester, A.G. Lyne, B.C. Joshi, M.A. McLaughlin, M. Kramer, J.M. Sarkisian, F. Camilo, V. Kalogera, C. Kim, and D.R. Lorimer, *An increased estimate of the merger rate of double neutron stars from observations of a highly relativistic system*, Nature 426 (2003), 531–533.

[4] H. E. J. Curzon, *Cylindrical solutions of Einstein's gravitation equations*, Proc. London Math. Soc. 23 (1924), 477–480.
[5] Thibaut Damour and Nathalie Deruelle, *Radiation reaction and angular momentum loss in small angle gravitational scattering*, Physics Letter **87A** (1981), no. 3, 81–84.

[6] D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact solutions of Einstein's field equations*, Cambridge University Press, Cambridge, 1980.

[7] A. G. Lyne, M. Burgay, M. Kramer, A. Possenti, R. N. Manchester, F. Camilo, M. A. McLaughlin, D. R. Lorimer, N. D’Amico, B. C. Joshi, J. Reynolds, and P. C. C. Freire, *A Double-Pulsar System: A Rare Laboratory for Relativistic Gravity and Plasma Physics*, Science **303** (2004), 1153–1157.

[8] C. W. Misner, K. S. Thorn, and J. A. Wheeler, *Gravitation*, Freeman, San Francisco, 1973.

[9] H. P. Robertson and Thomas W. Noonan, *Relativity and cosmology*, W. B. Saunders company, Philadelphia, 1968.

[10] Clifford M. Will, *Theory and experiments in gravitational physics*, revised ed., Cambridge University Press, Cambridge, 1993.