Shannon theory for quantum systems and beyond: information compression for fermions

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We address the task of compression of fermionic quantum information. Due to the parity superselection rule, differently from the case of encoding of quantum information in qubit states, part of the information carried by fermionic systems is encoded in their delocalised correlations. As a consequence, reliability of a compression protocol must be assessed in a way that necessarily accounts also for the preservation of correlations. This implies that input/output fidelity is not a satisfactory figure of merit for fermionic compression schemes. We then discuss various aspects regarding the assessment of reliability of an encoding scheme, and show that entanglement fidelity in the fermionic case is capable of evaluating the preservation of correlations, thus revealing itself strictly stronger than input/output fidelity, unlike the qubit case. We then introduce a fermionic version of the source coding theorem showing that, as in the quantum case, the von Neumann entropy is the minimal rate for which a fermionic compression scheme exists, that is reliable according to the entanglement fidelity criterion.

I. INTRODUCTION

The task of data compression addresses the primary question in information theory as to how redundant is the information contained in a message and to what extent the message can then be compressed.

In classical information theory this question is answered by the source coding theorem [1], which establishes the fundamental role of Shannon entropy in information theory and its operational interpretation. The coding theorem recognizes the Shannon entropy as the fundamental limit for the compression rate in the i.i.d. setting. This means that if one compresses at a rate above the Shannon entropy, then the compressed data can be recovered perfectly in the asymptotic limit of infinitely long messages, while this is not possible for compression rate below the Shannon entropy. As a result the Shannon entropy, which can be intuitively thought of as the uncertainty about the outcome of an experiment that we are going to perform on a classical system, quantifies the information contained in a message and to what extent the message can then be compressed.

In quantum information theory the Shannon entropy is replaced by the von Neumann entropy. In particular, the quantum source coding theorem [2] identifies von Neumann entropy as the rate at which quantum compression can be reliably achieved. Consider a quantum information source described by a system A and density operator \( \rho \in \text{St}(A) \), with \( \text{St}(A) \) the set of states of system A. The density operator describes the preparation of a state \( \sigma_i \) from any ensemble \( \{p_i, \sigma_i\} \), with probabilities \( p_i \), such that \( \sum_i p_i \sigma_i = \rho \). A quantum message of \( N \) letters can now be understood in terms of \( N \) uses of the quantum source, that output a sequence of \( N \) states \( p_i, \sigma_j \), with \( 1 \leq j \leq N \), drawn independently. One instance of this preparation protocol thus produces \( \sigma_i := \bigotimes_{j=1}^{N} \sigma_{i_j} \), with probability \( p_{\text{i}} := \prod_{j} p_{i_j} \). Each of the \( N \) systems has density operator \( \rho \), and the density operator of the entire message is then given by \( \rho^{\otimes N} \). A compression scheme for messages from the above described source consists of two steps. Encoding: Alice encodes the system \( A^{\otimes N} \) according to a compression map given by a channel \( \mathcal{E} : \text{St}(A^{\otimes N}) \to \text{St}(B) \), where B is generally a system with dimension \( d_B(N) \) smaller than \( A^{\otimes N} \). The compression rate is defined as the asymptotic quantity \( R = \lim_{N\to\infty} \log_2 d_B(N)/N \). Typically, one estimates the “size” of the compressed message through the capacity of system B given in terms of \( \log_2 d_B(N) \), namely the number of qubits that are needed to simulate B. Alice then sends the system B to Bob using \( NR \) noiseless qubit channels. Decoding: Finally, Bob applies a decompression map \( \mathcal{D} : \text{St}(B) \to \text{St}(A^{\otimes N}) \) to the message encoded in system B, with the purpose of recovering the original message as reliably as possible.

As one might expect, the above scheme generally introduces an error in the decoding: we now discuss the figure of merit by which we estimate the error introduced by the compression scheme. In order to understand the operational meaning of the figure of merit, think of a referee (Charlie) who prepares the states \( \sigma_1 \) with probability \( p_1 \), and receives the final states \( \mathcal{D}\mathcal{E}(\sigma_1) \). The figure of merit that we use corresponds to the probability that, after receiving Bob’s final state, Charlie is able to distinguish it from the input one. For a single instance, this probability is a linear function of the trace-norm distance \( ||\sigma_1 - \mathcal{D}\mathcal{E}(\sigma_1)||_1 \). The probability of successful discrimination is thus evaluated to \( \sum_i p_i ||\sigma_i - \mathcal{D}\mathcal{E}(\sigma_i)||_1 = ||\rho^{\otimes N} - \mathcal{D}\mathcal{E}(\rho^{\otimes N})||_1 \). The protocol has then error \( \epsilon \) if the compressed and decompressed states \( \mathcal{D}\mathcal{E}(\sigma_1) \) are \( \epsilon \)-close to the original states \( \sigma_1 \) in the trace-norm distance. In
the case of qubits the above quantity is equivalent to fidelity, thanks to the Fuchs-van der Graaf inequalities.  

The optimal quantum encoding will then make the error arbitrarily small for $N$ large enough, with rate $R$ as small as possible. Schumacher’s quantum source coding theorem shows that the optimal rate is equal to the von Neumann entropy $S(\rho)$ of the state $\rho$.

Another way to evaluate the error for a compression scheme is the following: Charlie prepares a purification of the density operator $\rho^\otimes N$ and sends the $N$ copies of system $A$ to Alice. Alice then sends her share of the pure state to Bob, sending as few qubits to Bob as possible. After decompressing the received qubits, Bob shares an entangled state with Charlie. The quality of the compression scheme can then be evaluated considering how well Charlie can distinguish the initial state from the final one, after receiving Bob’s $N$ systems. The probability that Charlie detects a compression error can be evaluated through the input/output fidelity. Again, Schumacher’s theorem states that Alice can transfer her share of the pure state to Bob by sending $NS(\rho)$ qubits and achieving arbitrarily good fidelity, increasing the length $N$ of the message. This second perspective answers the question whether the compression protocol preserves the correlations that system $A^\otimes N$ has with a remote system $C$.

Equivalence of the two approaches in assessing the quality of a compression scheme shows that the ability to send quantum superpositions is equivalent to the ability to send entanglement. In other terms, the amount of quantum information preserved by a compression scheme represents the dimension of the largest Hilbert space whose superpositions can be reliably compressed, or equivalently the amount of entanglement that can be reliably compressed.

According to the above discussion a crucial point in the compression protocol is to quantify the reliability of the compression map $\mathcal{C} := \mathcal{D} \mathcal{E}$, which in the asymptotic limit of $N \to \infty$ must coincide with the identity map. In quantum theory checking the reliability of the compression map looking only at its local action, namely via the fidelity between states $\mathcal{C}(\rho^\otimes N)$ and $\rho^\otimes N$, or at the effects on correlations, namely via entanglement fidelity, is equivalent. This follows from the local process tomography of quantum theory where, given a map $\mathcal{C}$ on system $A$ one has

\[
(\mathcal{C} \otimes \mathcal{I}_C)(\Psi) = \Psi \quad \forall \Psi \in \text{St} (\text{AC})
\]

\[
\iff \mathcal{C}(\rho) = \rho \quad \forall \rho \in \text{St} (A). \tag{1}
\]

This equivalence is due to local discriminability of quantum theory, where the discrimination of bipartite quantum states can always be performed using local measurements only (this property is equivalent to the one known in the literature as local tomography, or tomographic locality). However, in the absence of local discriminability, a map preserving local states still can affect correlations with remote systems. This raises a crucial issue if one aims at studying the compression task beyond quantum theory, where the reliability of a protocol generally needs to be verified on extended systems. Indeed, in general, testing a compression scheme using ancillary systems is strictly stronger than testing them with local schemes.

As a first step in the direction of generalizing the compression protocol to an arbitrary information theory, in this paper we consider the case of fermionic systems as carriers of information. Fermionic computation has been proposed in Ref. 12 and later studied in several works. Differently from quantum systems, fermions obey the parity superselection rule. As a consequence, fermionic information theory does not satisfy local discriminability, thus providing a physically relevant example of a theory where the task of compression is not straightforward. Indeed, in the case of study, a map $\mathcal{C}$ that acts as the identity on local states $\rho^\otimes N$ could still destroy the correlations with remote systems, and then be mistakenly considered as a reliable compression map.

After reviewing the structure of fermionic quantum information, we prove that the entanglement fidelity is a valid criterion for the reliability of a fermionic compression map. We then show an analogous of the quantum source coding theorem in the fermionic scenario, showing that the minimal compression rate for which a reliable compression scheme exists is the von Neumann entropy of the fermionic state. We conclude therefore that the von Neumann entropy provides the informational content of the state also in the case of fermionic theory, namely in the presence of parity superselection. The above result, however, is not a straightforward consequence of Schumacher’s source coding theorem.

\section{Fermionic Information Theory}

We now briefly review fermionic information theory. The systems of the theory are made by local fermionic modes (LFMs). A LFM is the counterpart of the qubit in quantum theory, and can be thought of as a cell that can be either empty or occupied by a fermionic excitation. An $L$-LFMs system, denoted $L$, is described by $L$ fermionic fields $\phi_i$, satisfying the canonical anticommutation rule (CAR) $\{\phi_i, \phi_j^\dagger\} = \delta_{ij} I$, $\{\phi_i, \phi_j\} = 0$ where $i, j = 1, \ldots, L$. With these fields one constructs the occupation number operators $\phi_i^\dagger \phi_i$, which can be easily proved to have only eigenvalues 0 and 1. The common eigenvector $|\Omega\rangle$ of the operators $\phi_i^\dagger \phi_i$, $i = 1, \ldots, L$ with eigenvalue 0 defines the vacuum state $|\Omega\rangle |\Omega\rangle$ of $L$, representing the state in which all the modes are not excited. The fermionic vacuum state in terms of the field operators is given by $|\Omega\rangle |\Omega\rangle = \prod_{i=1}^{L-1} \phi_i^\dagger \phi_i^\dagger$. By applying the operators $\phi_i^\dagger$ to $|\Omega\rangle$ the corresponding $i$-th mode is excited and, by raising $|\Omega\rangle$ in all possible ways, we get the $2^L$ orthonormal vectors forming the Fock basis in the occupation number representation: a generic element of
this basis is
\[ |n_1, \ldots, n_L\rangle := (\varphi_1^1)^{n_1} \cdots (\varphi_L^1)^{n_L} |\Omega\rangle, \]
with \(n_i = \{0, 1\}\) corresponding to the occupation number at the \(i\)-th site. The linear span of these vectors corresponds to the antisymmetric Fock space \(\mathcal{F}_L\) of dimension \(2^L\). Notice that the Fock space \(\mathcal{F}_L\) is isomorphic to the Hilbert space \(\mathcal{H}_L\) of \(L\) qubits, by the trivial identification of the occupation number basis with the qubit computational basis. This correspondence lies at the basis of the Jordan-Wigner isomorphism typically used in the literature to map fermionic systems to qubits systems and vice-versa. We recall here the definition of the Jordan-Wigner map
\[ J_L(\varphi_i) = \left(\bigotimes_{i=1}^{l-1} \sigma_i^z\right) \otimes \sigma_l^z \otimes \left(\bigotimes_{k=l+1}^{L} I_k\right), \]
\[ J_L(\varphi_i^1) = J_L(\varphi_i)^1, \]
\[ J_L(XY) = J_L(X)J_L(Y), \]
\[ J_L(aX + bY) = aJ_L(X) + bJ_L(Y), \]
with \(X, Y\) linear combinations of products of field operators on the L-LFM, and where we used the standard notation for Pauli sigma operators. In the following we will drop the dependence on the number of LFM in the Jordan-Wigner map, namely we will write \(J_L(\varphi_i)\) in place of \(J_L(\varphi_i)(X)\), when it will be clear from the context. Notice that the Jordan-Wigner isomorphism is implicitly defined in Eq. \(\Box\), and, as such, it depends on the arbitrary ordering of the modes. All such representations are unitarily equivalent.

Differently from standard qubits, fermionic systems satisfy the parity superselection rule. One can decompose the Fock space \(\mathcal{F}_L\) of system \(L\) in the direct sum \(\mathcal{F}_L = \mathcal{F}_L^R \oplus \mathcal{F}_L^L\), with \(\mathcal{F}_L^R\) and \(\mathcal{F}_L^L\) the spaces generated by vectors with even and odd total occupation number, respectively. The convex set of states \(\text{St}(\mathcal{F}_L)\) corresponds, in the Jordan-Wigner representation, to the set of density matrices on \(\mathcal{F}_L\) of the form \(\rho = \rho_e + \rho_o\), with \(\rho_e, \rho_o \succeq 0\), \(\text{Tr}[\rho_e] + \text{Tr}[\rho_o] \leq 1\) and with \(\rho_e\) and \(\rho_o\) having support on \(\mathcal{F}_L^L\) and \(\mathcal{F}_L^R\), respectively, and pure states are represented by rank one density operators. Moreover, the density matrices representing the states represent linear combinations of products of an even number of field operators (see appendix \(#A#\) and \(#B#\) for further details). Viceversa, every linear combination of products of an even number of field operators that is represented by a density matrix is an admissible state. Analogously, effects in the set \(\text{Eff}(\mathcal{F}_L)\) are represented by positive operators on \(\mathcal{F}_L\) of the form \(a = a_e + a_o\), with \(a_e\) and \(a_o\) having support on \(\mathcal{F}_L^L\) and \(\mathcal{F}_L^R\), respectively. Notice that set of states and effects of system \(L\) have dimension \(d_{\mathcal{F}_L^R}^2 = 2^{2L-1}\), compared to the quantum case where the set of states and effects associated to the Hilbert space \(\mathcal{H}_L\) of \(L\) qubits has dimension \(d_{\mathcal{H}_L}^2 = 2^{2L}\).

Given a state \(\rho \in \text{St}(\mathcal{F}_L)\) we define the refinement set of \(\rho\) as \(\text{Ref}(\rho) := \{\sigma \in \text{St}(\mathcal{F}_L) | \exists \tau \in \text{St}(\mathcal{F}_L) : \rho = \sigma + \tau\}\), and a state is pure when all the elements in the refinement are proportional to the state itself. In the following we will denote by \(\text{PurSt}(\mathcal{F}_L)\) and \(\text{St}_1(\mathcal{F}_L)\) the set of pure states and the set of normalized states (of trace one) of system \(L\), respectively.

Given two fermionic systems \(L_F\) and \(M_F\), we introduce the composition of the two as the system made of \(K = L + M\) LFs, denoted with the symbol \(K_F := L_F \boxtimes M_F\) or simply \(K_F := L_F \boxtimes M_F\). We use the symbol \(\boxtimes\) to distinguish the fermionic parallel composition rule from the quantum one, corresponding to the tensor product \(\otimes\).

Given a state \(\Psi \in \text{St}(L_F \boxtimes M_F)\), one can discard the subsystem \(M_F\) and consider the marginal state, which we denote by \(\sigma := \text{Tr}_{M_F}(\Psi)\). We use the symbol \(\text{Tr}_{M_F}\) to denote the fermionic partial trace on the subsystem \(M_F\).

This is computed by performing the following steps (see ref. \(#24#\) for further details): (i) drop all those terms in \(\Psi\) containing an odd number of field operators in any of the LFM in \(M_F\); (ii) remove all the field operators corresponding to the LFM in \(M_F\) from the remaining terms. The fermionic trace \(\text{Tr}_F(\rho)\) of a state \(\rho \in \text{St}(M_F)\) is then defined as a special case of the partial one, corresponding to the case in which \(L = 0\).

Finally, the set of transformations from \(L_F\) to \(M_F\), denoted by \(\text{Tr}(L_F \rightarrow M_F)\), is given by completely positive maps from \(\text{St}(L_F)\) to \(\text{St}(M_F)\) in the Jordan-Wigner representation. Moreover, we denote by \(\text{Tr}_I(\text{L}_F \rightarrow \text{M}_F)\) the set of deterministic transformations, also called channels, from \(L_F\) to \(M_F\), corresponding to trace-preserving completely positive maps. Like in quantum theory, any fermionic transformation \(\mathcal{C} \in \text{Tr}(L_F \rightarrow M_F)\) can be expressed in Kraus form \(\mathcal{C}(\rho) = \sum_i C_i \rho C_i^\dagger\), with deterministic transformations having Kraus operators \(\{C_i\}\) such that \(J(\sum_i C_i^\dagger C_i) = I_{\mathcal{H}_L}\), \(I_{\mathcal{H}_L}\) denoting the identity operator on \(\mathcal{H}_L\). For a map \(\mathcal{C} \in \text{Tr}(L_F \rightarrow M_F)\) with Kraus operators \(\{C_i\}\), we define its Jordan-Wigner representative \(J(\mathcal{C})\) as the quantum map with Kraus operators \(\{J(C_i)\}\).

Now, given two transformations \(\mathcal{C} \in \text{Tr}(L_F \rightarrow M_F)\) and \(\mathcal{D} \in \text{Tr}(K_F \rightarrow N_F)\), we denote by \(\mathcal{C} \boxtimes \mathcal{D} \in \text{Tr}(L_F \boxtimes K_F \rightarrow M_F \boxtimes N_F)\) the parallel composition of \(\mathcal{C}\) and \(\mathcal{D}\), with Kraus operators \(\{C_i D_j\}\), where \(\{C_i\}\) are Kraus operators for \(\mathcal{C}\) and \(\{D_j\}\) for \(\mathcal{D}\). We observe that in the Jordan-Wigner representation one generally has \(J_{L+N}(C_i D_j) \neq J_L(C_i) \otimes J_K(D_j)\), and \(J_{L+N}(C_i \boxtimes D_j) \neq J_L(C_i) \otimes J_K(D_j)\). If \(\mathcal{C}\) is a transformation in \(\text{Tr}(L_F \rightarrow M_F)\), its extension to a composite system \(L_FN_F\), is given by \(\mathcal{C} \boxtimes I\), with \(I\) the identity map of system \(N_F\)—whose Jordan-Wigner representative is the quantum identity map—and its Kraus operators involve field operators on the \(L_F\) modes only. It is worth noticing that, despite the Jordan-Wigner representative of this map is not necessarily of the form \(J_L(\mathcal{C}) \boxtimes I\), upon suitable choice of the ordering of the LFM that defines the representation, one can always reduce to the case where, actually, \(J_{L+N}(\mathcal{C} \boxtimes I) = J_L(\mathcal{C}) \boxtimes I\).

As a special case of the above composition rule, one can define \(\rho \boxtimes \sigma := \rho \sigma \in \text{St}(L_F \boxtimes M_F)\) for the parallel composition of states \(\rho \in \text{St}(L_F)\) and \(\sigma \in \text{St}(M_F)\), and similarly.
\[ a \boxtimes b := ab \in \text{Eff}(L_F M_F) \] for the parallel composition of effects \( a \in \text{Eff}(L_F) \) and \( b \in \text{Eff}(M_F) \).

A useful characterization of fermionic maps in \( \text{Tr}(L_F \rightarrow L_F) \), proved in Ref. [25], is the following:

**Proposition II.1** (Fermionic transformations). All the transformations in \( \text{Tr}(L_F \rightarrow L_F) \) with Kraus operators being linear combinations of products of either an even number or an odd number of field operators are admissible fermionic transformations. Vice versa, every admissible fermionic transformation in \( \text{Tr}(L_F \rightarrow L_F) \) has Kraus operators being superpositions of products of either an even number or an odd number of field operators.

**Corollary II.1** (Fermionic effects). Fermionic effects are positive operators bounded by the identity operator that are linear combinations of products of an even number of field operators. Vice versa, every linear combination of products of an even number of field operators that is represented by a positive operator bounded by the identity is a fermionic effect.

The corollary follows immediately from Proposition II.1 since an effect \( A \) is obtained as a fermionic transformation \( \mathcal{A} \) followed by the discard map, i.e. the trace. Thus

\[ \text{Tr}[\rho A] = \text{Tr}[\mathcal{A}(\rho)] = \sum_i \text{Tr}[K_i \rho K_i^\dagger] = \text{Tr}[\rho \sum_i K_i^\dagger K_i], \]

namely \( A = \sum_i K_i^\dagger K_i \). Having the polynomial \( K_i \) a definite parity (though not necessarily the same for every \( i \)), \( A \) is an even polynomial.

In the following we denote by \( \mathcal{L}(\mathcal{H}_L) \) the set of linear operators on the Hilbert space \( \mathcal{H}_L \) of \( L \)-qubits and by \( \mathcal{L}(\mathcal{H}_L, \mathcal{H}_M) \) the set of linear operators from \( \mathcal{H}_L \) to \( \mathcal{H}_M \). It is useful to introduce the isomorphism between operators \( X \) in \( \mathcal{L}(\mathcal{H}_L, \mathcal{H}_M) \) and vectors \( |X\rangle \) in \( \mathcal{H}_M \otimes \mathcal{H}_L \) given by

\[ |X\rangle = (X \otimes I_{\mathcal{H}_L})|I_{\mathcal{H}_L}\rangle = (I_{\mathcal{H}_M} \otimes X^T)|I_{\mathcal{H}_L}\rangle, \]

where \( I_{\mathcal{H}_L} \) is the identity operator in \( \mathcal{H}_L \), \( |I_{\mathcal{H}_L}\rangle \in \mathcal{H}_L^{\otimes 2} \) is the maximally entangled vector \( |I_{\mathcal{H}_L}\rangle = \sum_i |i\rangle|i\rangle \) (with \( \{|i\rangle\} \) a fixed orthonormal basis for \( \mathcal{H}_L \)), and \( X^T \in \mathcal{L}(\mathcal{H}_M, \mathcal{H}_L) \) is the transpose of \( X \) with respect to the two fixed bases chosen in \( \mathcal{H}_L \) and \( \mathcal{H}_M \). Notice also the useful identity

\[ Y \otimes Z |X\rangle = |YXZ^T\rangle, \]

where \( X \in \mathcal{L}(\mathcal{H}_L, \mathcal{H}_M), Y \in \mathcal{L}(\mathcal{H}_M, \mathcal{H}_N) \) and \( Z \in \mathcal{L}(\mathcal{H}_N, \mathcal{H}_L) \). Moreover, for \( X, Y \in \mathcal{L}(\mathcal{H}_L, \mathcal{H}_M) \), one has \( \text{Tr}_{\mathcal{H}_L}||X\rangle\langle Y|| = XY^\dagger \), and \( \text{Tr}_{\mathcal{H}_L}||X\rangle\langle Y|| = X^TY^* \). We remark that, in the above paragraph, we are dealing with abstract linear operators on a Hilbert space, disregarding their possible interpretation as Jordan-Wigner representatives of some fermionic operator.

A notion that will be used in the following is that of states dilation.

**Definition II.1** (Dilation set of a state \( \rho \)). For any \( \rho \in \text{St}(L_F) \), we say that \( \Psi_\rho \in \text{PurSt}(L_F M_F) \) for some system \( M_F \) is a dilation of \( \rho \) if \( \rho = \text{Tr}_F(\Psi_\rho) \). We denote by \( D_\rho \) the set of all possible dilations of \( \rho \). A pure dilation \( \Psi_\rho \in \text{PurSt}(L_F M_F) \) of \( \rho \) is called a purification.

Naturally, any purification of \( \rho \) belongs to \( D_\rho \), more precisely the set of purifications of \( \rho \) is the subset of \( D_\rho \) containing pure states. A main feature of quantum theory that is valid also for fermionic systems is the existence of a purification of any state, that is unique modulo channels on the purifying system.

**Proposition II.2** (Purification of states). For every \( \rho \in \text{St}(L_F) \), there exists a purification \( \Psi_\rho \in \text{PurSt}(L_F M_F) \) of \( \rho \) for some system \( M_F \). Moreover, the purification is unique up to channels on the purifying system: if \( \Psi_\rho \in \text{PurSt}(L_F M_F) \) and \( \Phi_\rho \in \text{PurSt}(L_F K_F) \) are two purifications of \( \rho \) then there exists a channel \( \mathcal{V} \in \text{Tr}_F(M_F \rightarrow K_F) \) such that \( (\mathcal{A}_F \boxtimes \mathcal{V})(\Psi_\rho) = \Phi_\rho \).

**Proof.** It can be easily verified that every purification of \( \rho \in \text{St}(L_F) \), having even part \( \rho_e \) and odd part \( \rho_o \), can be obtained in terms of the minimal one \( J^{-1}(|F\rangle\langle F|) \in \text{PurSt}(L_F M_F) \), with \( F = J(\rho) \frac{1}{2}, M = \lfloor \log_2 2r \rfloor \) and \( r = \max(\text{rank}(\rho_e),\text{rank}(\rho_o)) \). Now, let \( \Psi_\rho \in \text{PurSt}(L_F M_F) \) and \( \Phi_\rho \in \text{PurSt}(L_F K_F) \) be two purifications of \( \rho \). If \( M = K \), let us choose the ordering defining the Jordan-Wigner isomorphism of Eq. (2) in such a way that the modes in the purifying systems \( M_F \) precede the modes of \( L_F \). Then, using the quantum purification theorem, we know that there exists a reversible map \( \mathcal{W} \) with unitary Kraus \( U \) such that \( |F\rho\rangle\langle F\rho| = (U \otimes I)|P\rho\rangle\langle P\rho| = J(\Psi_\rho) \).

The unitary \( U \) can be chosen in such a way that \( J^{-1}(\mathcal{W}) \) is an admissible fermionic map, namely in such a way that it respects the parity superselection rule (see Lemma [13] in Appendix [14]). Moreover, due to Lemma [13] in Appendix [10], \( J^{-1}(U \otimes I) \) cannot contain field operators on the modes in \( L_F \), and is then local on the purifying system \( K_F \). Now, let \( K > M \). Then, we can consider a pure state \( \omega \) on the \( K - M \) modes and take the parallel composition \( \Psi_\rho \boxtimes \omega \). This is still a purification of \( \rho \), and by the previous argument, there exists a reversible channel \( \mathcal{W} \in \text{Tr}_F(K_F \rightarrow K_F) \) such that \( \Phi_\rho = (\mathcal{A}_F \boxtimes \mathcal{W})(\Psi_\rho \boxtimes \omega) = (\mathcal{A}_F \boxtimes \mathcal{V})(\Psi_\rho) \) where \( \mathcal{V} \) is the channel defined by \( \mathcal{V} := \mathcal{W}(\mathcal{I}_F \boxtimes \omega) \). If \( K < M \), we consider \( \Phi_\rho \boxtimes \omega \), where \( \omega \) is any pure state on \( N = M - K \) modes system, and we have \( \Phi_\rho \boxtimes \omega = (\mathcal{A}_K \boxtimes \mathcal{W})(\Psi_\rho) \). Now we discard the additional modes, and the channel connecting the purifications is the sequential composition of \( \mathcal{W} \) and the discarding map: \( \mathcal{V} := (\mathcal{A}_K \boxtimes \text{Tr}_{N_F})\mathcal{W} \).

The main difference between fermionic and quantum information lies in the notion of what Kraus operators correspond to local maps. While in the case of qubit systems local maps acting on the \( i \)-th qubit of a composite system have Kraus operators that can be factorized.
as a non trivial operator on the i-th tensor factor $\mathbb{C}^2$ of the total Hilbert space, in the case of the fermionic Fock space $\mathcal{F}_L$, a local transformation on the i-th mode can be represented in the Jordan-Wigner isomorphism by operators that act non trivially on factors $\mathbb{C}^2$ different from the i-th one. This fact is the source of all the differences between the theory of qubits and fermionic theory, including supersel]ection and features that it affects, such as the notion of entanglement \cite{[20,21]. Due to parity supersel]ection, fermionic theory does not satisfy local process tomography, namely the property stating that two transformations $\mathcal{E}_1, \mathcal{E}_2 \in \text{Tr}(\mathcal{L}_F \rightarrow \mathcal{M}_F)$ are equal iff they act in the same way on the local states in $\text{St}(\mathcal{L}_F)$, namely $\mathcal{E}_1(\rho) = \mathcal{E}_2(\rho)$ for every $\rho \in \text{St}(\mathcal{L}_F)$ (see for example Eq. (11) in the introduction on the equality between the compression map $\mathcal{C}$ and the identity map). As a consequence, fermionic theory also violates local tomography. A typical example of a transformation that is locally equivalent to the identity but differs from it when extended to multipartite systems is the parity transformation, as shown in the following. Let us consider a single fermionic mode system 1F, whose possible states are constrained to be of the form $J(\rho) = \gamma_0 |0\rangle \langle 0| + \gamma_1 |1\rangle \langle 1|$ by the parity supersel]ection rule. Let $P_0$ and $P_1$ be the projectors on $|0\rangle$ and $|1\rangle$ respectively, on the even and odd sector of the Fock space. The parity transformation $\mathcal{P}$, that in the Jordan-Wigner representation $J(\mathcal{P})$ has Kraus operators $P_0$ and $P_1$, acts as the identity $\mathcal{I}_F$ when applied to states in $\text{St}(1F)$. However, taking the system 2F and considering the extended transformation $\mathcal{P} \otimes \mathcal{I}_{1F}$ on $\text{St}(2F)$ one notices that $\mathcal{P}$ differs from the identity map $\mathcal{I}_{2F}$. Indeed, the state $J^{-1}(|\Psi\rangle \langle \Psi|)$, with $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is a legitimate fermionic state in $\text{St}(2F)$, and one can straightforwardly verify that

\begin{equation}
(\mathcal{P} \otimes \mathcal{I}_{1F})[J^{-1}(|\Psi\rangle \langle \Psi|)] = \frac{1}{2} J^{-1}(|00\rangle \langle 00| + |11\rangle \langle 11|) \neq J^{-1}(|\Psi\rangle \langle \Psi|).
\end{equation}

A. Identical channels upon-input of $\rho$

In the following we will be interested in quantitatively assessing how closely a channel (the compression map) resembles another one (the identity map), provided that we know that the input state corresponds to a given $\rho$. To this end we introduce the notion of identical fermionic channels upon-input of $\rho$.

Given two fermionic channels $\mathcal{E}_1, \mathcal{E}_2 \in \text{Tr}_1(\mathcal{L}_F \rightarrow \mathcal{M}_F)$ and a state $\rho \in \text{St}(\mathcal{L}_F)$, we say that $\mathcal{E}_1$ and $\mathcal{E}_2$ are equal upon-input of $\rho$ if

\begin{equation}
(\mathcal{E}_1 \otimes \mathcal{I})(\Sigma) = (\mathcal{E}_2 \otimes \mathcal{I})(\Sigma) \quad \forall \Sigma \in \text{Ref}(D)\rho.
\end{equation}

Operationally, this means that one cannot discriminate between $\mathcal{E}_1$ and $\mathcal{E}_2$ when applied to any dilation $\Psi_\rho$ of the state $\rho$, independently of how $\Psi_\rho$ has been prepared.

Suppose that $\Psi_\rho \in D_\rho$ was prepared as $\Psi_\rho = \sum_i \Sigma_i$, for some refinement of $\Psi_\rho$. Even using the knowledge of the preparation, one cannot distinguish between $\mathcal{E}_1$ and $\mathcal{E}_2$. Notice that, differently from the quantum case here it is necessary to check the identity between channels on bipartite systems.

Following the above definition, one can quantify how close two channels are. One has that $\mathcal{E}_1$ and $\mathcal{E}_2$ are close upon-input of $\rho$ if

\begin{equation}
\sum_i \|((\mathcal{E}_1 - \mathcal{E}_2) \otimes \mathcal{I})(\Sigma_i))\|_1 \leq \varepsilon \quad \forall \{\Sigma_i\} \in D_\rho.
\end{equation}

where $\|X\|_1$ is the 1-norm of $J(X)$. One can straightforwardly prove that the trace distance $d(\rho, \sigma) := \frac{1}{2}\|\rho - \sigma\|_1$ has a clear operational interpretation in terms of the maximum success probability of discrimination between the two states $\rho$ and $\sigma$. Eq. \cite{[11] provides then an upper bound for the probability of discriminating between $\mathcal{E}_1$ and $\mathcal{E}_2$ when applied to the dilations of $\rho$, including their refinements: $\mathcal{E}_1$ and $\mathcal{E}_2$ cannot be distinguished with a success probability bigger than $\frac{1}{2} + \varepsilon$. Accordingly, a sequence of channels $\mathcal{E}_N \in \text{Tr}_1(\mathcal{L}_F \rightarrow \mathcal{M}_F)$ converges to the channel $\mathcal{C} \in \text{Tr}_1(\mathcal{L}_F \rightarrow \mathcal{M}_F)$ upon-input of $\rho$ if

\begin{equation}
\lim_{N \rightarrow \infty} \|((\mathcal{E}_N - \mathcal{C}) \otimes \mathcal{I})(\Sigma)\|_1 = 0 \quad \forall \Sigma \in \text{Ref}(D_\rho).
\end{equation}

III. FERMIONIC COMPRESSION

Consider now a system $L_F$ and let $\rho \in \text{St}(L_F)$ be the generic state of the system. As usual the source of fermionic information is supposed to emit $N$ independent copies of the state $\rho$. A fermionic compression scheme $(\mathcal{E}_N, \mathcal{D}_N)$ consists of the following two steps:

1. Encoding: Alice encodes the system $L_F^{\otimes N}$ via a channel $\mathcal{E}_N : \text{St}(L_F^{\otimes N}) \rightarrow \text{St}(M_F)$, where the target system is generally a system of $M$-LMFs. The map $\mathcal{E}_N$ produces a fermionic state $\mathcal{E}(\rho^{\otimes N})$ with support $\text{Supp}(\mathcal{E}(\rho^{\otimes N}))$ on a Fock space $\mathcal{F}_M$ of dimension $d_{\mathcal{F}_M}(N)$ smaller than the one of the original state $\rho^{\otimes N}$. The compression rate is defined as the quantity

\begin{equation}
R = \log_2 d_{\mathcal{F}_M}(N)/N.
\end{equation}

Alice sends the system $M_F$ to Bob using $N[R] \text{ noiseless fermionic channels}.

2. Decoding: Finally Bob sends the system $M_F$ through a decompression channel $\mathcal{D}_N : \text{St}(M_F) \rightarrow \text{St}(L_F^{\otimes N})$.

The scheme $(\mathcal{E}_N, \mathcal{D}_N)$ overall transforms the $L_F^{\otimes N}$ LMFs, with a compression map $\mathcal{E}_N := \mathcal{D}_N \mathcal{E}_N$. The latter can be more or less “good” (in a sense that will be precisely defined) in preserving the information which is contained in the system, depending on $\rho$ itself. The goal now is to define the notion of reliable compression scheme once we are provided with an information source $\rho$. 

A. Reliable compression scheme

The aim of a compression scheme, besides reducing the amount of information carriers used, is to preserve all the information that is possibly encoded in a given state $\rho$. What we actually mean is not only to preserve the input state and keep track of the correlations of our system with an arbitrary ancilla, but also to preserve these informations for any procedure by which the input system and its correlations have been prepared. In other words, even the agent that prepared the system along with possible ancillary systems, must have a small success probability in detecting the effects of compression on the original preparation. This amounts to require that the compression channel $\mathcal{E}_N$ must be approximately equal to the identity channel upon-input of $\rho$, and more precisely that in the limit of $N \to \infty$ the two channels must coincide upon-input of $\rho$.

In Section II.A we introduced the notion of $\varepsilon$-close channels upon-input of $\rho$. This notion can now be used to quantify the error, say $\varepsilon$, introduced by the map $\mathcal{E}_N$ in a compression protocol given the source $\rho$. According to Eq. (7) we have indeed the following definition of a reliable compression scheme

**Definition III.1** (Reliable compression scheme). Given a state $\rho \in \text{St}(\mathcal{L}_F)$, a compression scheme $(\mathcal{E}_N, J_N)$ is $\varepsilon$-reliable if $\sum_i \left\| \mathcal{E}_N \otimes \mathcal{F}(\Sigma_i) - \mathcal{E}_N \right\| < \varepsilon$ for every $\{\Sigma_i\}$ such that $\sum_i \Sigma_i \in D_{\rho, 2N}$, where $\mathcal{E}_N := J_N \mathcal{E}_N J_N$.

It is clear from the definition that in order to check the reliability of a fermionic compression map one should test it on states of an arbitrary large system, since the dilation set $D_{\rho, 2N}$ includes dilations on any possible ancillary system. It is then necessary to find a simpler criterion to characterize the reliability of a compression scheme. Let us start with a preliminary definition.

**Definition III.2.** Let $\rho \in \text{St}(\mathcal{L}_F)$. We define its square root $\rho^{1/2}$ as follows

$$\rho^{1/2} := J^{-1}[J(\rho)^{1/2}].$$

One can easily prove that the square root of a fermionic state is well defined, i.e. it does not depend on the particular Jordan-Wigner representation $J$ chosen (see Appendix C). In the following we show that a useful criterion for reliability can be expressed via entanglement fidelity:

**Definition III.3** (Entanglement fidelity). Let $\rho \in \text{St}_1(\mathcal{L}_F)$, $\mathcal{E} \in \text{Tr}_1(\mathcal{L}_F \to \mathcal{M}_F)$ and $\Phi_\rho \in \text{PurSt}(\mathcal{L}_F \mathcal{M}_F)$ be any purification of $\rho$. The entanglement fidelity is defined as and $F(\rho, \mathcal{E}) = F(\Phi_\rho, (\mathcal{E} \otimes \mathcal{F})(\Phi_\rho))$, where $F(\rho, \sigma) := \text{Tr}[J(\rho^{1/2}\sigma\rho^{1/2})^{1/2}]$ denotes the Uhlmann's fidelity between states $\rho, \sigma \in \text{St}_1(\mathcal{L}_F)$.

We notice that the Uhlmann fidelity of fermionic states is well defined, namely it is independent of the ordering of the fermionic modes (see Appendix C). As a consequence also the Entanglement fidelity, given in terms of the Uhlmann one, must be well defined.

By definition the Uhlmann fidelity of fermionic states coincides with the one of their Jordan-Wigner representatives and the same for their trace-norm distance, given $\rho, \sigma \in \text{St}(\mathcal{L}_F)$, the Fuchs-van der Graaf inequalities hold as a trivial consequence of their quantum counterparts

$$1 - F(\rho, \sigma) \leq \frac{1}{2}\|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho, \sigma)^2}. \quad (9)$$

The following proposition summarizes the main properties of fermionic entanglement fidelity that will be used in the remainder.

**Proposition III.1.** Let $\rho \in \text{St}_1(\mathcal{L}_F)$, $\mathcal{E} \in \text{Tr}_1(\mathcal{L}_F \to \mathcal{L}_F)$ and $\Phi_\rho \in \text{PurSt}(\mathcal{L}_F \mathcal{M}_F)$ be any purification of $\rho$. Entanglement fidelity has the following properties.

1. $F(\rho, \mathcal{E})$ is independent of the particular choice for the purification $\Phi_\rho$.

2. If the ordering is chosen in such a way that the $L$ modes are all before the purifying ones, the following identity holds:

$$F(\rho, \mathcal{E}) = \sum_i \text{Tr}[J(\rho)C_i]^2 \quad (10)$$

for arbitrary Kraus decomposition $J(\mathcal{E}) = \sum_i C_i \cdot C_i^\dagger$ of the Jordan-Wigner representative $J(\mathcal{E})$. From the second inequality in (9) it follows that, if $F(\rho, \mathcal{E}) \geq 1 - \delta$, one has

$$\|((\mathcal{E} \otimes \mathcal{F})(\Phi_\rho)) - \Phi_\rho\|_1 \leq 2\sqrt{\delta} \quad (11)$$

for every purification $\Phi_\rho$ of $\rho$.

**Proof.** Let $\Phi_\rho \in \text{PurSt}(\mathcal{L}_F \mathcal{M}_F)$ be a purification of $\rho$. If we choose the trivial ordering for the LFs, the Kraus of $J(\mathcal{E} \otimes \mathcal{F})$ are of the form $C_i \otimes I$. Moreover, since the minimal purification $|F\rangle \langle F|$ (introduced in the proof of proposition II.2) and $J(\Phi_\rho)$ both purify the same quantum state, they are connected through an isometry $V$. Recalling that for quantum states $|\psi\rangle \langle \psi|$ and $\sigma$ the quantum Uhlmann fidelity is given by $F(|\psi\rangle \langle \psi|, \sigma) = \langle \psi| \sigma |\psi\rangle^{1/2}$, we find

$$F(\rho, \mathcal{E}) = \sum_i \text{Tr}[|FV^T\rangle \langle FV^T|C_iFV^T]\langle C_iFV^T|]$$

$$= \sum_i \text{Tr}(|C_iFV^T\rangle \langle FV^T|)^2 = \sum_i \text{Tr}[J(\rho)C_i]^2,$$

namely, the claimed formula in (11). Since $\Phi_\rho$ is arbitrary, this also implies independence from the choice of the purification. \qed


In quantum theory a compression scheme \((\mathcal{E}_N, \mathcal{F}_N)\) is reliable when the entanglement fidelity \(F(\rho^\otimes N, \mathcal{E}_N)\), with \(\mathcal{E}_N := \mathcal{D}_N \mathcal{E}_N\), approaches 1 as \(N \to \infty\). Here we prove an analogous reliability criterion for the fermionic case.

We can now prove the following proposition and the subsequent corollary providing a simple reliability criterion for fermionic compression.

**Proposition III.2.** Given a state \(\rho \in \text{St}_1(\mathcal{L}_F)\) and a channel \(\mathcal{E} \in \text{Tr}_1(\mathcal{L}_F \to \mathcal{L}_F)\), \(\forall \varepsilon > 0\) there exists \(\delta > 0\) such that if \(F(\rho, \mathcal{E}) > 1 - \delta\) then
\[
\sum_i \|((\mathcal{E} - \mathcal{J}) \otimes \mathcal{F}_N)(\rho)\|_1 \leq \varepsilon
\]
for every \(\{\Sigma_i\}\) such that \(\sum_i \Sigma_i \in D_\rho\).

**Proof.** Firstly we observe that, given a set of states \(\{\Sigma_i\}\) such that \(\sum_i \Sigma_i \in D_\rho\), considering any purification \(\Psi_\rho \in \text{PurSt}(\mathcal{L}_F \mathcal{L}_K \mathcal{F}_N)\) of \(\Sigma := \sum_i \Sigma_i\), one can find a POVM \(\{b_i\} \in \text{Eff}(\mathcal{N}_F)\) such that \(\Sigma_i = \text{Tr}_{N_F}[(\mathcal{I}_F \mathcal{K}_F \mathcal{F}_N \otimes b_i)\Psi_\rho]\). As a consequence, we have
\[
\sum_i \|((\mathcal{E} - \mathcal{J}) \Sigma_i)\|_1 = \sum_i \|\text{Tr}_{N_F}[(\mathcal{E} - \mathcal{J}) \otimes \mathcal{F}_N](\Psi_\rho) (\mathcal{I}_F \mathcal{K}_F \otimes b_i)\|_1 \leq \sum_i \|((\mathcal{E} - \mathcal{J}) \otimes \mathcal{F}_N)(\Psi_\rho)\|_1,
\]
where the last inequality follows from the equivalent definition of the 1-norm for \(X \in \text{St}_2(\mathcal{L}_F)\)
\[
\|X\|_1 = \max_{b \in \text{Eff}(\mathcal{L}_F)} \text{Tr}[Xb],
\]
and from the fact that, for \(\{a_i\} \subseteq \text{Eff}(\mathcal{L}_F)\) such that
\[
\|\text{Tr}_{N_F}[(\mathcal{E} - \mathcal{J}) \otimes \mathcal{F}_N](\Psi_\rho) (\mathcal{I}_F \mathcal{K}_F \otimes a_i)\|_1 = \text{Tr}_{N_F}[(\mathcal{E} - \mathcal{J}) \otimes \mathcal{F}_N](\Psi_\rho)(a_i \otimes b_i),
\]
one can write
\[
\sum_i \|((\mathcal{E} - \mathcal{J}) \Sigma_i)\|_1 = \sum_i \text{Tr}_{N_F}[(\mathcal{E} - \mathcal{J}) \otimes \mathcal{F}_N](\Psi_\rho)(a_i \otimes b_i) = \text{Tr}_{N_F}[(\mathcal{E} - \mathcal{J}) \otimes \mathcal{F}_N](\Psi_\rho)(A),
\]
where \(A := \sum_i (a_i \otimes b_i)\). Now, by the Fuchs-van der Graaf inequalities, if \(F(\rho, \mathcal{E}) \geq 1 - \delta\), then
\[
\|((\mathcal{E} - \mathcal{J}) \otimes \mathcal{F}_N)(\Psi_\rho)\|_1 \leq 2\sqrt{1 - F(\rho, \mathcal{E})} \leq 2\sqrt{\delta}.
\]
The thesis is then obtained just taking \(\delta \leq \varepsilon^2/4\). \(\square\)

**Corollary III.1 (Reliable compression scheme).** Given a state \(\rho \in \text{St}(\mathcal{L}_F)\), a compression scheme \((\mathcal{E}_N, \mathcal{F}_N)\) is \(\varepsilon\)-reliable if one has \(F(\rho^\otimes N, \mathcal{E}_N) > 1 - \delta\), where \(\delta = \varepsilon^2/4\), and \(\mathcal{E}_N := \mathcal{D}_N \mathcal{E}_N\).

**B. Fermionic typical subspace**

At the basis of the quantum source coding theorem lies the notion of typical subspace, that in turn generalizes to the quantum case that of typical sequences and typical sets of classical information. We now introduce the notion of typical subspace also for fermionic systems and use it to show that, like in quantum theory, the von Neumann entropy of a fermionic state is the rate that separates the region of rates for which a reliable compression scheme exists from that of unachievable rates. In order to do this we have to verify that the compression map given in terms of the projection on the typical subspace represents an admissible fermionic map.

We start by defining the notion of logarithm of a fermionic state

**Definition III.4.** Let \(\rho\) be a fermionic state. We define its logarithm as
\[
\log_2 \rho = J^{-1}[\log_2 J(\rho)].
\]

Then we define the von Neumann entropy of a fermionic state via its Jordan-Wigner representative.

**Definition III.5.** Given a fermionic state \(\rho\), its von-Neumann entropy is defined as
\[
S_f(\rho) := S(J(\rho)) = -\text{Tr}(J(\rho) \log_2 J(\rho)).
\]

These definitions are independent of the particular Jordan-Wigner transform corresponding to a given ordering of the modes (see Appendix C).

When we use the orthonormal decomposition for \(J(\rho) = \sum x_i |x_i\rangle \langle x_i|\), this reduces to the Shannon entropy of the classical random variable \(X\) that takes values in \(\text{Rng}(X) = \{x_1, x_2, \ldots, x_n\}\), called range of \(X\), with probability distribution \((p_1, p_2, \ldots, p_n)\): \(S_f(\rho) = H(X) = -\sum p_i \log_2 p_i\). We remind that \(N\) i.i.d. copies of the state \(\rho\) are represented as \(J(\rho^\otimes N) = J(\rho)^\otimes N = \sum_{x_i \in \text{Rng}(X)^N} p_i|\langle x_i|\rangle|_{N}\). With \(T_{N, \varepsilon}(\rho)\) we will denote the typical set of the random variable \(X\).

**Definition III.6 (Typical subspace).** Let \(\rho \in \text{St}(\mathcal{L}_F)\) with orthonormal decomposition \(J(\rho) = \sum_{x_i \in \text{Rng}(X)} p_i |x_i\rangle \langle x_i|\). The \(\varepsilon\)-typical subspace \(F_{N, \varepsilon}(\rho)\) of \(\mathcal{H}_L^\otimes N\) is defined as
\[
F_{N, \varepsilon}(\rho) := \text{Span}\{|x_i| \mid x_i \in T_{N, \varepsilon}(X)\},
\]
where \(|x_i| := |x_{i_1}| |x_{i_2}| \cdots |x_{i_N}|\), and \(X\) is the random variable with \(\text{Rng}(X) = \{x_i\}\) and \(P_X(x_i) := p_i\).

It is an immediate consequence of the definition of typical subspace that
\[
F_{N, \varepsilon}(\rho) := \text{Span}\left\{|x_i| \mid |\frac{1}{N} \log_2 \frac{1}{P_X(x_i)} - S_f(\rho)| \leq \varepsilon\right\}.
\]
We will denote the projector on the typical subspace as

\[ P_{N,\epsilon}(\rho) := \sum_{x_i \in T_{N,\epsilon}(X)} |x_i\rangle\langle x_i| \]

\[ = \sum_{x_i \in T_{N,\epsilon}(X)} |x_{i_1}\rangle\langle x_{i_1}| \otimes \cdots \otimes |x_{i_N}\rangle\langle x_{i_N}|, \tag{15} \]

and we have that \( \dim(F_{N,\epsilon}(\rho)) = \text{Tr}[P_{N,\epsilon}(\rho)] = |T_{N,\epsilon}(X)|. \)

Notice that some of the superpositions of vectors in the typical subspace might not be legitimate fermionic pure states, as their parity might be different. However, up to now, we only defined the typical subspace as a mathematical tool, and it does not need a consistent physical interpretation. We will come back to this point later (see Lemma III.1), when we will discuss the physical meaning of the projection \( P_{N,\epsilon}(\rho) \). Now, it is immediate to see that

\[ \text{Tr}[P_{N,\epsilon}(\rho)J(\rho)^{\otimes N}] = \sum_{x_i \in T_{N,\epsilon}(X)} P_{XN}(x_i) = P_{XN}[x_i \in T_{N,\epsilon}(X)]. \tag{16} \]

As in quantum theory, also the fermionic typical subspace has the following features:

**Proposition III.3 (Typical subspace).** Let \( \rho \in \text{St}(L_F) \).
The following statements hold:

1. For every \( \epsilon > 0 \) and \( \delta > 0 \) there exists \( N_0 \) such that for every \( N \geq N_0 \)
   \[ \text{Tr}[P_{N,\epsilon}(\rho)J(\rho)^{\otimes N}] \geq 1 - \delta. \tag{17} \]

2. For every \( \epsilon > 0 \) and \( \delta > 0 \) there exists \( N_0 \) such that for every \( N \geq N_0 \)
   the dimension of the typical subspace \( F_{N,\epsilon}(\rho) \) is bounded as
   \[ (1 - \delta)2^{N(S_j(\rho) - \epsilon)} \leq \dim(F_{N,\epsilon}(\rho)) \leq 2^{N(S_j(\rho) + \epsilon)}. \tag{18} \]

3. For given \( N \), let \( S_N \) denote an arbitrary orthogonal projection on a subspace of \( F_L^{\otimes N} \) with dimension
   \( \text{Tr}(S_N) < 2^{NR}, \) with \( R < S_j(\rho) \) fixed. Then for every \( \delta > 0 \) there exists \( N_0 \) such that for every \( N \geq N_0 \)
   and every choice of \( S_N \)
   \[ \text{Tr}[S_NJ(\rho)^{\otimes N}] \leq \delta. \tag{19} \]

The proof of the above properties is exactly the same as the one of quantum theory (see for instance [28]). However, order to exploit the same scheme proposed by Schumacher for the quantum case, one has to check that the encoding and decoding channels given in the constructive part of the proof are admissible fermionic maps. In particular, the encoding channel makes use of the projector \( P_{N,\epsilon}(\rho) \) as a Kraus operator, therefore, we have to show that it is a legitimate Kraus for a fermionic map. This is proved in the following lemma based on characterization of fermionic transformations of Proposition II.1.

**Lemma III.1.** Let \( \rho \) be a fermionic state. The projector \( P_{N,\epsilon}(\rho) \) of eq (17) is the Kraus operator of an admissible fermionic transformation.

**Proof.** By proposition II.1 the projector on the typical subspace \( P_{N,\epsilon}(\rho) \) is a legitimate fermionic Kraus if it is the sum of products of either an even or an odd number of fermionic fields. Let us consider the single projection \( |x_i\rangle\langle x_i| \). This is given by the tensor product \( |x_{i_1}\rangle\langle x_{i_1}| \otimes \cdots \otimes |x_{i_N}\rangle\langle x_{i_N}| \), where each \( |x_{i_k}\rangle \) is an eigenvector of the density matrix \( J(\rho) \) representing the fermionic state \( \rho \), and, as such, it has a definite parity. Thus, each factor in the above expression of \( |x_i\rangle\langle x_i| \) is the Jordan-Wigner representative of an even polynomial, and also the projection \( |x_i\rangle\langle x_i| \) is thus the representative of an even polynomial for every \( i \), which is given, in detail, by the product \( J^{-1}(|x_i\rangle\langle x_i|) = \prod_{j=1}^N J^{-1}(|x_{i_j}\rangle\langle x_{i_j}|) \). Now, by Proposition II.1 \( P_{N,\epsilon}(\rho) \) is the Jordan-Wigner representative of a legitimate fermionic Kraus operator. \( \square \)

**C. Fermionic source coding theorem**

We can now prove the source coding theorem for fermionic information theory.

**Theorem III.1 (Fermionic source coding).** Let \( \rho \in \text{St}(L_F) \) be a state of system \( L_F \). Then for every \( \delta > 0 \) and \( R < S_j(\rho) \) there exists \( N_0 \) such that for every \( N \geq N_0 \) one has a compression scheme \( \{\delta_N, D_N\} \) with rate \( R \), and \( F(\rho^{\otimes N}, D_N, \delta_N) \geq 1 - \delta \). Conversely, for every \( R < S_j(\rho) \) there is \( \delta \geq 0 \) such that for every compression scheme \( \{\delta_N, D_N\} \) with rate \( R \) one has \( F(\rho^{\otimes N}, D_N, \delta_N) \leq \delta \).

The proof follows exactly the lines of the original proof for standard quantum compression, that can be found e.g. in Ref. [28]. As the direct proof is constructive, we only need to take care of the legitimacy of the compression protocol as a fermionic map. To this end, we recapitulate the construction here.

1. **Encoding:** Perform the measurement \( \{P_{N,\epsilon}(\rho), I - P_{N,\epsilon}(\rho)\} \). If the outcome corresponding to \( P_{N,\epsilon}(\rho) \) occurs, then leave the state unchanged. Otherwise, if the outcome corresponding to \( I - P_{N,\epsilon}(\rho) \) occurs, replace the state by a standard state \( |S\rangle\langle S| \), with \( |S\rangle \in F_{N,\epsilon}(\rho) \). Such an map is described by the channel \( \mathcal{M}_N : L_F^{\otimes N} \rightarrow L_F^{\otimes N} \) given by

   \[ J(\mathcal{M}_N)(\sigma) := P_{N,\epsilon}(\rho)\sigma P_{N,\epsilon}(\rho) + \text{Tr}[I - P_{N,\epsilon}(\rho)]\sigma |S\rangle\langle S| \]

Notice that this is a well defined transformation since by Lemma III.1 the projector on the typical subspace is a legitimate fermionic Kraus operator. The second term is a measure and prepare channel, which is also a legitimate fermionic transformation. Then consider a system \( M_F \) made of \( M := N[R] \) LFs and the
2. Decoding: For the decoding channel, we simply choose the fermionic case. Thus, the quantum proof applies to fermionic systems, showing the fermionic counterpart of the quantum source coding theorem. In spite of parity superselection rule and the non locality of the Jordan-Wigner representation of fermionic operators, the von Neumann entropy of fermionic states can still be interpreted as their information content, providing the minimal rate for which a reliable compression is achievable.

The novelty in this paper is the analysis of compression in the absence of local tomography. Here, the properties of a map, and in the specific case of study of the compression map, cannot be accessed locally. This poses stronger constraints on the set of reliable compression maps.

IV. DISCUSSION

We have studied information compression for fermionic systems, showing the fermionic counterpart of the quantum source coding theorem. As we learn from classical, quantum, and now also fermionic information theory, the task of information compression is intimately related to the notion of entropy. However, it is known that information theories beyond quantum exhibit inequivalent notions of entropy. This is the main issue one has to face in order to introduce the notion of information content in the general case. On one side one has to provide a definition of information content including a broad class of probabilistic theories. On the other side one can compare such a notion with the different notions of entropy, identifying the one that plays the same role of Shannon entropy in the compression task.

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in its even and odd part as follows
\[ \rho = \sum_{e} E_e \ket{\Omega} \bra{\Omega} E_e^\dagger + \sum_{o} O_o \ket{\Omega} \bra{\Omega} O_o^\dagger, \]
where $E_e$ and $O_o$ are linear combinations of products of even and odd number of field operators respectively. By recalling that $\ket{\Omega} \bra{\Omega} = \prod_{i=1}^{L} \phi_i \phi_i^\dagger$, one can easily realize that $\rho$ can be written as combination of products of even number of field operators. Moreover, by using the CAR, the generic state can be written as follows
\[ \rho = \sum_{s,t} \rho_{st} \prod_{i=1}^{L} \phi_i^{s_i} \phi_i \phi_i^\dagger, \]
where $s,t \in \{0,1\}^L$ and $\rho_{st} \in \mathbb{C}$.

Appendix B: Technical Lemmas

Here we show two lemmas that are used in the proof of Proposition 112 in the main text.

As a preliminary notion we define quantum states with definite parity. Let $\mathcal{H}_L$ be a Hilbert space of $L$-qubits and let $\text{St}(\mathcal{H}_L)$ be the corresponding set of states. The vectors of the computational basis
\[ |s_1, s_2, \ldots, s_L\rangle, \quad s_i = \{0,1\}, \quad i = 1, \ldots, L, \] (B1)
can be divided into even $p = 0$ and odd $p = 1$ vectors according to their parity $p := \oplus_{i=1}^{L} s_i$. Denoting by $\mathcal{H}_L^0$ and $\mathcal{H}_L^1$, with $\mathcal{H}_L = \mathcal{H}_L^0 \oplus \mathcal{H}_L^1$, the spaces generated by even and odd vectors respectively, one says that a state $\rho \in \text{St}(\mathcal{H}_L)$ has definite parity if it is of the form $\rho = \rho_0 + \rho_1$, with $\rho_0$ and $\rho_1$ having support on $\mathcal{H}_L^0$ and $\mathcal{H}_L^1$ respectively. As a special case, a pure state of definite parity $\rho$ must have support only on $\mathcal{H}_L^0$. We can now prove the following lemma.

Lemma B.1. Consider a quantum state $\rho \in \text{St}(\mathcal{H}_L)$ and two purifications $\Psi, \Phi \in \text{St}(\mathcal{H}_L, \mathcal{H}_M)$ with definite parity. Then it is always possible to find a unitary channel $\mathcal{U}$ that maps states of definite parity into states of definite parity and such that $(\mathcal{I} \otimes \mathcal{U}) (\Psi) = \Phi$.

Proof. Let $|\Psi\rangle \in \mathcal{H}_L^0 \otimes \mathcal{H}_M$ and $|\Phi\rangle \in \mathcal{H}_L^q \otimes \mathcal{H}_M$, for $p, q \in \{0,1\}$. Since the two states are purification of the same state $\rho \in \text{St}(\mathcal{H}_L)$ their Schmidt decomposition can always be taken as follows
\[ |\Psi\rangle = \sum_i \lambda_i \ket{i} |\Psi_i\rangle, \quad |\Phi\rangle = \sum_i \lambda_i \ket{i} |\Phi_i\rangle, \]
where $\{|i\rangle\} \in \mathcal{H}_L$ is the same orthonormal set for the two states, while $\{|\Psi_i\rangle\}, \{|\Phi_i\rangle\} \in \mathcal{H}_M$ are two generally different orthonormal sets. Notice that, since $\Psi$ and $\Phi$ are pure states of definite parity, any element in the above orthonormal sets must be a vector of definite parity. Within the set $\{|i\rangle\}$ one can separate even $\{|i_0\rangle\}$ and odd $\{|i_0\rangle\}$ parity vectors, and then

Appendix A: Fermionic States

In a L-LFM system, fermionic states in $\text{St}(\mathcal{F}_L)$ are represented by density matrices on the antisymmetric Fock space $\mathcal{F}_L$ satisfying the parity superselection rule. As such they can be written as combinations of products of field operators. Indeed, a fermionic state $\rho$ can be split
\[ \rho = \sum_{e} E_e \ket{\Omega} \bra{\Omega} E_e^\dagger + \sum_{o} O_o \ket{\Omega} \bra{\Omega} O_o^\dagger, \]
where $E_e$ and $O_o$ are linear combinations of products of even and odd number of field operators respectively.
write \( \Psi \) and \( \Phi \) (respectively of parity \( p \) and \( q \)) as

\[
|\Psi \rangle = \sum_{i_0} \lambda_{i_0} |i_0 \rangle |\Psi^p_{i_0} \rangle + \sum_{i_1} \lambda_{i_1} |i_1 \rangle |\Psi^p_{i_1} \rangle , \\
|\Phi \rangle = \sum_{i_0} \lambda_{i_0} |i_0 \rangle |\Phi^q_{i_0} \rangle + \sum_{i_1} \lambda_{i_1} |i_1 \rangle |\Phi^q_{i_1} \rangle ,
\]

where \( \bar{r} = r \oplus 1 \), and in the orthonormal sets \( \{ |\Psi^p_{i_0} \rangle , |\Psi^p_{i_1} \rangle \} \) and \( \{ |\Phi^q_{i_0} \rangle , |\Phi^q_{i_1} \rangle \} \) we separated vectors according to their parity. We can now complete the above two sets to orthonormal bases in such a way that all vectors in both bases have definite parity. Let us take for example the basis \( \{ |\Psi^p_{i_0} \rangle , |\Psi^p_{i_1} \rangle \} \) and \( \{ |\Phi^q_{i_0} \rangle , |\Phi^q_{i_1} \rangle \} \) with \( r(k), t(k) \in \{ 0, 1 \} \). It is now straightforward to see that the unitary map \( \mathcal{U} \) having Kraus operator

\[
U = \sum_{i_0} |\Psi^p_{i_0} \rangle \langle \Phi^q_{i_0} | + \sum_{i_1} |\Psi^p_{i_1} \rangle \langle \Phi^q_{i_1} | + \sum_k |\Psi^p_{r(k)} \rangle \langle \Phi^q_{t(k)} |
\]

is such that \( (I \otimes U) |\Psi \rangle = |\Phi \rangle \). Moreover \( \mathcal{U} \) maps states of definite parity into states of definite parity. \( \square \)

**Lemma B.2.** Let \( N_F := L_F K_F \) and \( \mathcal{C} \in \text{Tr}(N_F \to N_F) \) be a single Kraus transformation with Kraus \( C \) having Jordan-Wigner representative \( J(C) = U \otimes K_F, U \) acting on the first \( L \) qubits. Then \( \mathcal{C} \) is local on the first \( L \) modes.

**Proof.** Due to Proposition B.1 the Kraus operator of \( \mathcal{C} \) can be written as \( C = \sum_i C_i \), where either each \( C_i \) is a product of an even number of field operators, or each \( C_i \) is a product of an odd one. The set \( \{ C_i \} \) can be taken to be linearly independent without loss of generality. Let us assume by contradiction that \( \mathcal{C} \) is not local on the first \( L \) modes. Therefore, since a set of independent operators generating the algebra of the \( j \)-th mode is \( \{ \varphi_j, \varphi_j^+, \varphi_j^\dagger \varphi_j, \varphi_j^\dagger \varphi_j^\dagger \varphi_j \} \), there exists at least one product \( C_i \) that contains one of the factors \( \varphi_j, \varphi_j^+, \varphi_j^\dagger \), or \( \varphi_j^\dagger \varphi_j \), for some mode \( j \) of the system \( K_F \). Let \( j(i) \) be the mode with largest label in the chosen ordering of the \( N = L + K \) modes, such that the corresponding factor in the product \( C_i \) is not the identity (i.e. \( \varphi_j^\dagger \varphi_j + \varphi_j^\dagger \varphi_j \)). Accordingly, one has that the Jordan-Wigner representation of \( C_i \) is of the form

\[
J(C_i) = K \otimes O_{j(i)} \otimes \left( \bigotimes_{l=j(i)+1}^{N} I_l \right),
\]

where \( K \) is an operator on the first \( 1, \ldots, j(i) - 1 \) qubits, and \( O_{j(i)} \) is one of the factors \( \sigma_{j(i)}^+, \sigma_{j(i)}^-, \sigma_{j(i)}^\dagger \sigma_{j(i)} \) on the \( j \)-th qubit. This contradicts the hypothesis on the form of \( J(C) \). \( \square \)

**Appendix C: Jordan-Wigner independence**

In this appendix we show the consistency of definitions III.2, III.3, III.4 and III.5 given in text. In particular, we check that they are independent of the particular choice of the order of the fermionic modes, which defines the Jordan-Wigner transform. We remember that all Jordan-Wigner representations are unitarily equivalent.

**Lemma C.1.** Let \( \rho \) be a fermionic state. The square root and the logarithm of \( \rho \) are well defined.

**Proof.** Once we have fixed the ordering of the modes, the square root of a fermionic state \( \rho \) is defined via its Jordan-Wigner representative as follows

\[
\rho^\frac{1}{2} := J^{-1}[J(\rho)^\frac{1}{2}]
\]

If \( J \) is the Jordan-Wigner isomorphism associated to a different ordering, then consider \( X := J^{-1}[J(\rho)^\frac{1}{2}] \). We can now prove that \( X = \rho^\frac{1}{2} \) and then independence of the square root from the ordering. Indeed, one has

\[
J(X)^2 = J(\rho) = U J(\rho) U^\dagger,
\]

with \( U \) unitary. It follows that

\[
J(\rho) = U^\dagger J(X) U U^\dagger J(X) U = J(X)^2 \implies J(X) = J(\rho)^\frac{1}{2}.
\]

Since \( J \) is an isomorphism, by taking \( J^{-1} \) we finally get

\[
X = J^{-1}[J(\rho)^\frac{1}{2}] = \rho^\frac{1}{2}.
\]

Analogously, the logarithm of a fermionic state is defined though its Jordan-Wigner representative

\[
\log_2(\rho) := J^{-1}[\log_2(J(\rho))]
\]

Again, let \( J \) be the Jordan-Wigner isomorphism corresponding to a different ordering, and let \( X = J^{-1}[\log_2(J(\rho))] \). First we notice that

\[
\log_2[J(\rho)] = \log_2[U J(\rho) U^\dagger] = U \log_2[J(\rho)] U^\dagger
\]

since \( U \) is unitary (we remind that the logarithm of a positive operator is defined via its spectral decomposition, and a unitary map preserves the spectrum). Therefore, we find

\[
J(X) = \log_2[J(\rho)] \implies X = J^{-1}[\log_2(J(\rho))] = \log_2(\rho),
\]

that concludes the proof. \( \square \)

Based on the above lemma we have the following proposition.

**Proposition C.1.** Let \( \rho \) and \( \sigma \) be two fermionic states. The Uhlmann fidelity \( F(\rho, \sigma) \) and the von Neumann entropy \( S_f(\rho) \) of definitions III.3 and III.4 are well defined.

**Proof.** These two quantities are given by a trace of two well defined operators, as proved in the previous lemma. Moreover, since a reordering of the modes corresponds to a unitarily change of basis, the trace is Jordan-Wigner independent, and so are \( F(\rho, \sigma) \) and \( S_f(\rho) \). \( \square \)