On projective representations of finitely generated groups

Sumana Hatui\(^a\), E. K. Narayanan\(^b\), and Pooja Singla\(^c\)

\(^a\)School of Mathematical Sciences, National Institute of Science Education and Research, An OCC of HBNI, Bhubaneswar, Odisha, India; \(^b\)Department of Mathematics, Indian Institute of Science, Bangalore, India; \(^c\)Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Kanpur, India

**ABSTRACT**

We prove a characterization of monomial projective representations of finitely generated nilpotent groups. We also characterize polycyclic groups whose projective representations are finite dimensional.

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1. Introduction

The study of projective representations has a long history starting with the pioneering work of Schur for finite groups [12–14]. It involves understanding homomorphisms from a group into the projective general linear groups. Let \( G \) be a group and \( V \) be a complex vector space. A projective representation of a group \( G \) is a homomorphism \( \rho \) from \( G \) to the projective general linear group \( \text{PGL}(V) \). More precisely, \( \rho \) is a map from \( G \) to the general linear group \( \text{GL}(V) \) with \( \rho(1) = \text{Id}_V \) and such that

\[
\rho(xy) = \alpha(x, y)\rho(x)\rho(y) \quad \forall \ x, y \in G
\]

where \( \alpha : G \times G \to \mathbb{C}^\times \) is a 2-cocycle. Such a \( \rho \) is called an \( \alpha \)-representation. If \( \alpha(x, y) = 1 \) for all \( x, y \in G \) then \( \rho \) is an ordinary representation. A key concept in the study of projective representations is the representation group to which these representations lift as ordinary representations. Schur [12] proved that for every finite group \( G \) there exists a group \( \tilde{G} \), nowadays called Schur cover or representation group of \( G \), such that the \( \alpha \)-representations of \( G \) are obtained from the ordinary representations of \( \tilde{G} \). The representation groups for the symmetric group \( S_n \) for \( n \geq 4 \), were classified by Schur [14]. See [8, Section 3.3] for more examples of Schur covers for several finite groups. For a finitely generated group \( G \), under the assumption that \( H_2(G, \mathbb{Z}) \) is finitely generated, there is a finitely generated representation group \( \tilde{G} \) (see [2, Chapter II, Proposition 3.2]).

We next define induction for the projective representations of a group. A definition of this appeared in [3, Section 2.2]. Below, we give a slight variant of this one that fits better in our discussion.

**Definition 1.1** (Induced projective representation). Let \( \alpha \) be a 2-cocycle of \( G \), \( H \) be a subgroup of \( G \) and \( (\phi, W) \) be an \( \alpha \)-representation of \( H \). The induced projective representation \( (\tilde{\phi}, V) \) of \( \phi \), denoted by \( \text{Ind}_H^G(\phi) \), is defined as follows: The space \( V \) consists of the functions \( f : G \to W \) such that
(i) \( f(hg) = (h, g)^{-1} \phi(h)f(g) \) for all \( h \in H, g \in G \).

(ii) The support of \( f \) is contained in a union of finitely many right cosets of \( H \) in \( G \).

The projective representation \( \tilde{\phi} : G \to GL(V) \) is defined by \( \tilde{\phi}(g)f(x) = \alpha(x, g)f(xg) \) for all \( x, g \in G \).

It is easy to see that \( \tilde{\phi}(g)f \in V \) and \( \tilde{\phi} \) is an \( \alpha \)-representation of \( G \). For \( \alpha = 1 \), the above definition coincides with usual induction for discrete groups, see [9, Definition 2.1].

Our first result is a characterization of the irreducible monomial projective representations of finitely generated nilpotent groups. Let \( \alpha \in Z^2(G, \mathbb{C}^\times) \) and let \( \rho \) be an \( \alpha \)-representation of \( G \). Then \( \rho \) is said to be monomial if there exists a subgroup \( H \subset G \) and an \( \alpha \)-representation \( \psi : H \to \mathbb{C}^\times \) such that \( \rho \) is equivalent to \( \text{Ind}_{H}^{G}(\psi) \). In the context of ordinary representations of finitely generated nilpotent groups, monomial representations are characterized by the finite weight property (see [1, 9]). Motivated by this result, we define the following:

**Definition 1.2.** An \( \alpha \)-representation \( (\rho, V) \) of a group \( G \) is said to have finite weight if there exists a subgroup \( H \subset G \) and an \( \alpha \)-representation \( \psi : H \to \mathbb{C}^\times \) such that the space

\[
V_{H}(\psi) = \{ v \in V : \rho(h)v = \psi(h)v \forall h \in H \}\]

is a nontrivial finite dimensional space.

The following result gives a complete characterization of monomial irreducible projective representations of a finitely generated nilpotent group.

**Theorem 1.3.** An irreducible \( \alpha \)-representation \( \rho \) of a finitely generated nilpotent group is monomial if and only if it has finite weight.

For a proof of this result, see Section 3. To prove this result, we show that the characterization of monomial irreducible projective representations of \( G \) can be obtained from the corresponding characterization for the ordinary representations of its representation group.

Our next result is a generalization of the well-known result of Hall [4, Theorem 3.2, Theorem 3.3] which states that a polycyclic group \( G \) has all irreducible representations finite dimensional if and only if \( G \) is abelian by finite. We extend this result to projective representations. For a 2-cocycle \( \alpha \) of \( G \), we say \( G \) is \( \alpha \)-finite if every irreducible \( \alpha \)-representation of \( G \) is finite dimensional. We obtain the following characterization of \( \alpha \)-finite polycyclic groups.

**Theorem 1.4.** Let \( G \) be a polycyclic group and \( \alpha \) be a 2-cocycle of \( G \). Then \( G \) is \( \alpha \)-finite if and only if there is a normal abelian subgroup \( N \) of \( G \) such that \( [\alpha]_{N \times N} \) is of finite order and \( G/N \) is a finite group.

The proof of this result is included in Section 4 and is built on generalizing the ideas of Hall for \( \alpha = 1 \) case. We conclude by characterizing the finite dimensional irreducible \( \alpha \)-representations of discrete Heisenberg groups of rank one.

### 2. Preliminaries

In this section we recall some standard definitions and results regarding projective representations of discrete groups. We refer the reader to [8] for related results in the case of finite groups. Recall that the second cohomology group \( H^2(G, \mathbb{C}^\times) \) is defined to be the abelian group \( Z^2(G, \mathbb{C}^\times)/B^2(G, \mathbb{C}^\times) \), where \( Z^2(G, \mathbb{C}^\times) \) is the set of all 2-cocycles of \( G \) which form an abelian group under the pointwise multiplication and \( B^2(G, \mathbb{C}^\times) \) is the collection of all 2-coboundaries on \( G \). For any \( \alpha \in Z^2(G, \mathbb{C}^\times) \), its image in \( H^2(G, \mathbb{C}^\times) \) is denoted by \( [\alpha] \). Two 2-cocycles \( \alpha_1, \alpha_2 \in Z^2(G, \mathbb{C}^\times) \) are called cohomologous if
Let $V$ be a complex vector space. Recall, a projective representation of a group $G$ is a map $\rho : G \to \text{GL}(V)$ such that
\[ \rho(x)\rho(y) = \alpha(x,y)\rho(xy) , \text{ for all } x, y \in G, \]
for suitable scalars $\alpha(x, y) \in \mathbb{C}^\times$. By the associativity of $\text{GL}(V)$, the map $(x, y) \mapsto \alpha(x, y)$ gives a 2-cocycle of $G$, that is, $\alpha$ satisfies the following:
\[ \alpha(x, y)\alpha(xy, z) = \alpha(x, yz)\alpha(y, z) , \text{ for all } x, y, z \in G. \]
In this case, we say $\rho$ is an $\alpha$-representation. Two $\alpha$-representations $\rho_1 : G \to \text{GL}(V)$ and $\rho_2 : G \to \text{GL}(W)$ are called linearly equivalent if there is an invertible $T \in \text{Hom}(V, W)$ such that
\[ T\rho_1(g)T^{-1} = \rho_2(g) , \text{ for all } g \in G. \]
Recall that an $\alpha$-representation of $G$ for $\alpha(x, y) = 1$ for all $x, y \in G$ is called an ordinary representation of $G$. At times we shall call this just as a representation of $G$, omitting the word ordinary, whenever our meaning is clear from the context.

Let $\text{Irr}(G)$ denote the set of all linearly inequivalent ordinary irreducible representations of $G$ over $\mathbb{C}$ and $\text{Irr}^\alpha(G)$ denote the set of all linearly inequivalent irreducible $\alpha$-representations of $G$ over $\mathbb{C}$. We remark that for $\alpha, \alpha' \in Z^2(G, \mathbb{C}^\times)$ such that $[\alpha] = [\alpha']$, the sets $\text{Irr}^\alpha(G)$ and $\text{Irr}^{\alpha'}(G)$ are in bijective correspondence and can be easily obtained from each other. Therefore to study irreducible projective representations of $G$, we will pick a representative $\alpha$ for each element of $H^2(G, \mathbb{C}^\times)$ and study the corresponding $\alpha$-representations.

For a group $G$ and a 2-cocycle $\alpha$ of $G$, the set $\mathbb{C}^\alpha G$, called the twisted group algebra of $G$ with 2-cocycle $\alpha$, is a $\mathbb{C}$-algebra with its vector space basis given by the set $\{e_g \mid g \in G\}$. The multiplication of basis elements of $\mathbb{C}^\alpha G$ is given by the following
\[ e_g e_h = \alpha(g, h)e_{gh}, \text{ for all } g, h \in G \]
and is extended linearly to the whole set. Parallel to the ordinary representations of $G$, notions of twisted group algebra appears in [8] for finite groups and in [10, 11] for infinite groups.

We next recall the definition of transgression and inflation homomorphisms. For a central extension,
\[ 1 \to A \to \tilde{G} \to G/A \to 1, \]
the Hochschild-Serre spectral sequence [6] for cohomology of groups yields the following exact sequence
\[ \text{Hom}(\tilde{G}, \mathbb{C}^\times) \xrightarrow{\text{res}} \text{Hom}(A, \mathbb{C}^\times) \xrightarrow{\text{tra}} H^2(\tilde{G}/A, \mathbb{C}^\times) \xrightarrow{\text{inf}} H^2(\tilde{G}, \mathbb{C}^\times), \tag{2.0.1} \]
where $\text{tra} : \text{Hom}(A, \mathbb{C}^\times) \to H^2(\tilde{G}/A, \mathbb{C}^\times)$ given by $f \mapsto \text{tra}(f) = [\alpha]$, with
\[ \alpha(x, y) = \frac{f(\mu(x)\mu(y)\mu(x\bar{y})^{-1})}{\mu(\bar{x}y)}, \text{ for all } x, y \in \tilde{G}/A, \]
for a section $\mu : \tilde{G}/A \to \tilde{G}$, denotes the transgression homomorphism. The inflation homomorphism, $\text{inf} : H^2(\tilde{G}/A, \mathbb{C}^\times) \to H^2(\tilde{G}, \mathbb{C}^\times)$ is given by $[\alpha] \mapsto \text{inf}(\mu) = [\beta]$, where $\beta(x, y) = \alpha(xA, yA)$, for all $x, y \in \tilde{G}$.
Throughout this article, while making a choice of a section map we will always choose one that maps identity to identity. We end this section with some required results regarding representation group of $G$.

**Definition 2.1** (Representation group of $G$). A group $\tilde{G}$ is called a representation group of $G$, if there is a central extension
\[ 1 \to A \to \tilde{G} \to G \to 1 \]
such that corresponding transgression map
\[ \text{tra} : \text{Hom}(A, \mathbb{C}^\times) \to H^2(G, \mathbb{C}^\times) \]
is an isomorphism.
Lemma 2.2. Let $G$ be a polycyclic group. Then $G$ has a representation group over $\mathbb{C}$ which is finitely generated. Furthermore if $G$ is finitely generated nilpotent, then $G$ has a representation group which is finitely generated nilpotent.

Proof. For any group $G$, there exists a central extension $1 \to H_2(G, \mathbb{Z}) \to \tilde{G} \to G \to 1$ such that $\tilde{G}$ is a representation group of $G$ (see [2, Chapter II, Proposition 3.2]). In particular, if both $G$ and $H_2(G, \mathbb{Z})$ are finitely generated then $\tilde{G}$ is finitely generated. It is well-known that polycyclic groups are finitely presented. Let $G = F/R$ be a free presentation of polycyclic group $G$. By Hopf formula [7], $H_2(G, \mathbb{Z}) = \frac{[F,F]}{[F,R]}$ is a subgroup of $R/[F,R]$ and $R/[F,R]$ is a finitely generated abelian group. Hence $H_2(G, \mathbb{Z})$ is a finitely generated group. This proves the existence of a finitely generated representation group of a polycyclic group $G$. The result for nilpotent groups follows because every nilpotent group is a polycyclic group.

In [5], the authors described a representation group for finitely generated abelian groups as well as for discrete Heisenberg groups and their $t$-variants. Then, by [5, Corollary 3.3], it follows that the sets $\text{Irr}(\tilde{G})$ and $\bigcup_{\alpha \in H_2(G, \mathbb{C} \times)} \text{Irr}_\alpha(G)$ are in bijective correspondence.

Lemma 2.3. Let $1 \to A \to \tilde{G} \to G \to 1$ be a central extension such that $\tilde{G}$ is a representation group of $G$. Then $A \subseteq [\tilde{G}, \tilde{G}]$, where $[\tilde{G}, \tilde{G}]$ denotes the commutator subgroup of $\tilde{G}$.

Proof. By the definition of the representation group and the exactness of (2.0.1), we have $\text{res}: \text{Hom}(\tilde{G}, \mathbb{C} \times) \to \text{Hom}(A, \mathbb{C} \times)$ is trivial. Hence, $A \subseteq [\tilde{G}, \tilde{G}]$.

3. Monomial and finite weight projective representations

In this section we explore the monomial projective representations of a finitely generated group $G$ and its relations to the monomial representations of a representation group $\tilde{G}$. As a consequence we show that, for a finitely generated nilpotent group $G$, an irreducible projective representation $\rho$ is monomial if and only if $\rho$ is of finite weight (see Definition 1.2). We need the following results.

Lemma 3.1. Let $H$ and $K$ be two subgroups of $G$ and $\chi : H \to \mathbb{C}^\times$, $\delta : K \to \mathbb{C}^\times$ be characters of $H$ and $K$ respectively such that,

1. $kHk^{-1} \subseteq H$ for all $k \in K$, i.e. $K$ normalizes $H$.
2. $\chi(khk^{-1}) = \chi(h)$ for all $h \in H$ and $k \in K$.
3. $\chi|_{H \cap K} = \delta|_{H \cap K}$

Then $\chi \delta : HK \to \mathbb{C}^\times$ defined by $\chi \delta(hk) = \chi(h)\delta(k)$ for all $h \in H$ and $k \in K$ is a character of $HK$ such that $\chi \delta|_H = \chi$.

Proof. See Lemma 2.9 in [9].

Theorem 3.2. [Frobenius reciprocity] The induction functor is left adjoint to the restriction functor, i.e.,

$$\text{Hom}_G(\text{Ind}_G^G(\rho), \delta) = \text{Hom}_K(\rho, \text{Res}_K^G(\delta)),$$

where $\delta$ is any representation of $G$, $\rho$ is a representation of a subgroup $H$ of $G$ and $\text{Res}_K^G(\delta)$ is the restriction of $\delta$ to $K$.

Proof. See [15, Chapter 1]
Theorem 3.3. Let $G$ be a finitely generated nilpotent group. An irreducible representation $\rho$ of $G$ is monomial if and only if $\rho$ is of finite weight.

Proof. See the main results in [1] and [9].

Let $G$ be a group and $\tilde{G}$ be its representation group. Thus by Lemma 2.3, we have a subgroup $A$ of $\tilde{G}$, $A \subset [\tilde{G}, \tilde{G}] \cap Z(\tilde{G})$ and a central extension

$$1 \rightarrow A \rightarrow \tilde{G} \rightarrow G \rightarrow 1$$

such that the map $\tau$ is an isomorphism. Fix a section $s : G \rightarrow \tilde{G}$. Let $\alpha \in H^2(G, \mathbb{C}^\times)$ and $\chi : A \rightarrow \mathbb{C}^\times$ be such that

$$\alpha(x, y) = \chi(s(x)s(y)s(xy)^{-1}), \quad \text{for all } x, y \in G.$$  

For any $\alpha$-representation $(\rho, V)$ of $G$, define $\tilde{\rho} : \tilde{G} \rightarrow \text{GL}(V)$ by $\tilde{\rho}(g)s(g)) = \chi(g)\rho(g)$. Then $(\tilde{\rho}, V)$ is an ordinary representation of $\tilde{G}$. It is well-known that the map $\rho \mapsto \tilde{\rho}$ preserves irreducibility and gives an equivalence between categories of all $\alpha$-representations of $G$ and the category of all representations of $\tilde{G}$ lying above $\chi$.

Lemma 3.4. Let $\tilde{H}$ be a subgroup of $\tilde{G}$ and $\tilde{\psi}$ be a one dimensional ordinary representation of $\tilde{H}$. If $\text{Ind}_{\tilde{H}}^G(\tilde{\psi})$ is an ordinary irreducible representation of $\tilde{G}$, then $A \subseteq \tilde{H}$.

Proof. Suppose $A \not\subseteq \tilde{H}$. The character $\tilde{\psi}$ restricted to $\tilde{H} \cap A$ can be extended to a character $\delta$ of $A$ (since $A$ is abelian). By Lemma 3.1 we obtain an extension of the character $\tilde{\psi}$ to $\tilde{\psi} \delta : HA \rightarrow \mathbb{C}^\times$. It follows from Frobenius reciprocity (Theorem 3.2) that

$$\text{Hom}_{HA}^{\tilde{H}}(\text{Ind}_{\tilde{H}}^A(\tilde{\psi}), \tilde{\psi} \delta) = \text{Hom}_{\tilde{H}}^{\tilde{H}}(\tilde{\psi}, \tilde{\psi} \delta|_{\tilde{H}}) \neq 0.$$  

Hence $\text{Ind}_{\tilde{H}}^A(\tilde{\psi})$ is not irreducible. On the other hand $\text{Ind}_{\tilde{H}}^G(\tilde{\psi}) \cong \text{Ind}_{HA}^{\tilde{H}}(\text{Ind}_{\tilde{H}}^A(\tilde{\psi}))$ implies that $\text{Ind}_{\tilde{H}}^A(\tilde{\psi})$ is irreducible. This is a contradiction. Hence $A \subseteq \tilde{H}$.

Proof of Theorem 1.3. By Lemma 2.2, a finitely generated nilpotent group $G$ has a representation group $\tilde{G}$ which is also finitely generated nilpotent. Let $\rho : G \rightarrow \text{GL}(V)$ be an irreducible monomial $\alpha$-representation. Then there exists a subgroup $H \subset G$ and an $\alpha$-representation $\psi : H \rightarrow \mathbb{C}^\times$ such that $\rho = \text{Ind}_H^G(\psi)$. Let $\tilde{H} = \pi^{-1}(H)$, where $\pi$ is the surjective homomorphism from $\tilde{G}$ to $G = \tilde{G}/A$. Then $\tilde{H}$ is a subgroup of $\tilde{G}$ and every element $\tilde{h} \in \tilde{H}$ can be uniquely written as $\tilde{h} = a_h s(h)$ for some $h \in H$ and $a_h \in A$. Define $\tilde{\psi} : \tilde{H} \rightarrow \mathbb{C}^\times$ by

$$\tilde{\psi}(\tilde{h}) = \tilde{\psi}(a_h s(h)) = \chi(a_h)\psi(h).$$  

The map $\tilde{\psi}$ is a one dimensional representation of $\tilde{H}$. By definition of $\tilde{\rho}$ and induced representation, we obtain that $(\tilde{\rho}, V)$ and $\text{Ind}_{\tilde{H}}^G(\tilde{\psi})$ are isomorphic as $\tilde{G}$ representations. By Frobenius reciprocity,

$$\text{Hom}_{\tilde{H}}(\tilde{\psi}|_{\tilde{H}}, \tilde{\rho}|_{\tilde{H}}) = \text{Hom}_{\tilde{G}}(\tilde{\rho}, \tilde{\psi}) = 0.$$  

Since $\tilde{\rho}$ is irreducible, $V_{\tilde{H}}(\tilde{\psi})$ is one dimensional. By definition of $\tilde{\rho}$, $V_{\tilde{H}}(\tilde{\psi}) = V_{\tilde{H}}(\psi)$, hence $V_{\tilde{H}}(\psi)$ is finite dimensional. This implies that $\rho$ is a finite weight representation of $G$. Conversely, suppose $(\rho, V)$ is a finite weight irreducible $\alpha$-representation of $G$. Then there exists a subgroup $H$ and an $\alpha$-representation $\psi : H \rightarrow \mathbb{C}^\times$ such that $V_{\tilde{H}}(\psi)$ is finite dimensional. Let $\tilde{H} = \pi^{-1}(H)$ and $\tilde{\psi} : \tilde{H} \rightarrow \mathbb{C}^\times$ given by

$$\tilde{\psi}(\tilde{h}) = \tilde{\psi}(a_h s(h)) = \chi(a_h)\psi(h)$$  

is a one dimensional character of $\tilde{H}$. We note that $(\tilde{\rho}, V)$ is an irreducible representation of $\tilde{H}$ such that $V_{\tilde{H}}(\tilde{\psi}) = V_{\tilde{H}}(\psi)$. Therefore $\tilde{\rho}$ is a finite weight representation. By Theorem 3.3, $\tilde{\rho}$ is a monomial
representation. Let \( \tilde{\rho} \cong \text{Ind}_{\tilde{H}}^{G}(\tilde{\psi}) \). By Lemma 3.4 \( \alpha \subset \tilde{H} \). Therefore \( \rho \cong \text{Ind}_{H}^{G}(\psi) \) and hence a monomial representation.

4. Criterion for finite dimensional projective representations

Let \( G \) be a polycyclic group. It is a well-known result due to Hall that every irreducible representation of \( G \) is finite dimensional if and only if \( G \) is abelian by finite (see [4, Theorems 3.2 and 3.3]). That is, such groups \( G \) are characterized by the condition that there exists an abelian normal subgroup \( N \) of \( G \) such that \( G/N \) is finite. By closely following some of the arguments in [4], we extend this result to projective representations. For \( \alpha \in Z^{2}(G, \mathbb{C}^{\times}) \), we show that every irreducible \( \alpha \)-representation of \( G \) is finite dimensional if and only if there exists an abelian normal subgroup \( N \) of \( G \) such that \( G/N \) is finite and \( [\alpha_{N \times N}] \) is of finite order. Notice that the finite order condition is automatically satisfied if \( [\alpha] = [1] \).

As a consequence of this main result we show that every irreducible projective representation of a finitely generated polycyclic group \( G \) is finite dimensional if and only if there is a normal abelian subgroup \( N \) of \( G \) such that \( G/N \) is finite and \( [\alpha]_{N \times N} \) is of finite order for all \( \alpha \in Z^{2}(G, \mathbb{C}^{\times}) \).

Recall that for \( \alpha \in Z^{2}(G, \mathbb{C}^{\times}) \), the group \( G \) is \( \alpha \)-finite if every irreducible \( \alpha \)-representation of \( G \) is finite dimensional. We need the following results:

Theorem 4.1. Let \( N \) be a finitely generated abelian group. Then there exists a central extension

\[
1 \to \mathbb{Z} \to N^{*} \to N \to 1,
\]

such that \( \mathbb{Z} = Z(N^{*}) = [N^{*}, N^{*}] \) and \( N^{*} \) is a representation group of \( N \). Moreover, \( N^{*} \) is a finitely generated two step nilpotent group.

Proof. See [5, Theorem 1.4] and its proof. The proof of [5, Theorem 1.4] is done by showing that the corresponding transgression map is an isomorphism.

Theorem 4.2. Let \( G \) be a finitely generated two step nilpotent group and \( \pi \) an irreducible representation of \( G \). Then, \( \pi \) is finite dimensional if and only if the character obtained by restricting \( \pi \) to \( [G, G] \) is of finite order.

Proof. See [9, Theorem 1.3].

We start with the following lemma:

Lemma 4.3. Let \( N \) be a finitely generated abelian group and \( \alpha \in Z^{2}(N, \mathbb{C}^{\times}) \). Then \( N \) is \( \alpha \)-finite if and only if \( [\alpha] \) is of finite order.

Proof. Let \( \rho \) be an irreducible \( \alpha \)-representation of \( N \) and \( \tilde{\rho} \) be its lift to \( N^{*} \) (see Theorem 4.1). Let \( s : N \to N^{*} \) be a section such that \( s(1) = 1 \). By Theorem 4.1, we have \( \chi : \mathbb{Z} \to \mathbb{C}^{\times} \) such that \( \text{tra}(\chi) = [\alpha] \).

For \( n_{1}, n_{2} \in N \),

\[
\alpha(n_{1}, n_{2}) = \rho(n_{1})\rho(n_{2})\rho(n_{1}n_{2})^{-1} = \tilde{\rho}(s(n_{1})s(n_{2})s(n_{1}n_{2})^{-1}).
\]

It follows from the injectivity of the map \( \text{tra} \) that the character obtained by restricting the irreducible representation \( \tilde{\rho} \) to the center \( Z = [N^{*}, N^{*}] \) equals \( \chi \). Since orders of \( \chi \) and \( [\alpha] \) are equal and \( \rho \) is finite dimensional if and only if \( \tilde{\rho} \) is finite dimensional, the proof now follows from Theorem 4.2.

Let \( N \) be a subgroup of \( G \) and \( \alpha \) be a 2-cocycle of \( G \). The restriction of cocycle \( \alpha \) to the set \( N \times N \) gives a 2-cocycle of \( N \) and this is denoted by either \( \alpha|_{N \times N} \) or by \( \alpha \) itself.
**Lemma 4.4.** Let $G$ be $\alpha$-finite and $N$ be a normal subgroup of $G$. Then $N$ is $\alpha|_{N \times N}$-finite.

**Proof.** Let $Q$ be a maximal ideal of the twisted group algebra $\mathbb{C}^\alpha N$. Then there exists a maximal ideal $P$ of $\mathbb{C}^\alpha G$ containing $Q$. Consider a natural non-zero morphism $\lambda : \mathbb{C}^\alpha N/Q \to \mathbb{C}^\alpha G/P$ of $\mathbb{C}^\alpha N$-modules that sends the class of 1 to the class of 1. By condition of the lemma, the vector space $\mathbb{C}^\alpha G/P$ is finite-dimensional. Next, maximality of $Q$ implies that $\lambda$ is injective. This shows finite-dimensionality of $\mathbb{C}^\alpha N/Q$.

The following lemma will be crucially used in the proof of Theorem 1.4.

**Lemma 4.5.** Let $G$ be a finitely generated polycyclic group with cyclic series

$$1 = G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n = G.$$ 

Assume that $G_{n-1}$ has an abelian subgroup of finite index and $G_n$ does not have an abelian subgroup of finite index. Then there exists a free abelian subgroup $A$ of finite rank, normal in $G$, and of finite index in $G_{n-1}$, satisfying the following:

1. $\text{rank}(A) \geq 2$.
2. There exists $z \in G_n$ such that $L = \langle A, z \rangle$ has finite index in $G$ and $L/A$ is infinite.
3. There exists a subgroup $B$ of $A$ such that $A = B \oplus \langle t \rangle$ for some $t \in A$ and $N_L(B) > A$, where $N_L(B)$ is the normalizer of $B$ in $L$.

**Proof.** A proof of the above result is included in the proof of [4, Theorem 3.3 (p. 615)].

Now we are in a position to prove Theorem 1.4. We provide necessary and sufficient conditions for a polycyclic group to be $\alpha$-finite.

**Proof of Theorem 1.4.** We first suppose that $N$ is a normal abelian subgroup of $G$ of finite index such that every irreducible $\alpha$-representation of $N$ is finite dimensional. Let $V$ be an irreducible $\mathbb{C}^\alpha G$-module. We will prove that $V$ is finite dimensional. Suppose not. Since $V$ is also $\mathbb{C}^\alpha N$-module and $G$ is finitely generated, there exists a maximal $\mathbb{C}^\alpha N$-submodule of $V$, say $W$. Then $V/W$ is an irreducible $\mathbb{C}^\alpha N$-module and hence finite dimensional.

Let $\{x_1 = 1, x_2, \ldots, x_i\}$ be a set of left coset representatives of $N$ in $G$. Now $N(x_iW) \subseteq x_i(NW) \subseteq x_iW$ and $\dim(V/x_iW) = \dim(V/W)$. Consider $W_0 = \cap_{i=1}^{n} x_iW$. Then $V/W_0$ is a finite dimensional $\mathbb{C}^\alpha N$-module. If $x_iW = x_iNW$ and $w \in W_0$, $x_i(wx_iW) = x_iNW \in W_0$. Hence $W_0$ is a $\mathbb{C}^\alpha G$-submodule of $V$. Since $V$ is irreducible, either $W_0 = V$ or $W_0 = 0$. Either case leads to finite dimensionality of $V$.

Conversely, Suppose $G$ is $\alpha$-finite. We prove that there is an abelian normal subgroup $N$ of $G$ such that $G/N$ is finite. Since $G$ is a finitely generated polycyclic group, $G$ has a series

$$1 = G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n = G$$

such that $G_i/G_{i-1}$ is cyclic. We use induction on $n$. The group $G$ is cyclic for $n = 1$. So the result is true in this case. Now assume $n > 1$. We consider the following cases separately:

(a) $G_{n-1}$ does not have an abelian subgroup of finite index.
(b) $G_{n-1}$ has an abelian subgroup of finite index.

**Case (a):** First suppose that $G_{n-1}$ has no abelian subgroup of finite index. Hence by induction hypothesis $G_{n-1}$ has an infinite dimensional irreducible $\alpha$-representation. So by Lemma 4.4, $G$ has an infinite dimensional irreducible $\alpha$-representation and we are done.

**Case (b):** In this case, we prove the result by contradiction. Assume that $G$ does not have an abelian subgroup of finite index. Let $A$ be the normal, abelian subgroup of $G$ obtained by Lemma 4.5. If $[\alpha|_{A \times A}]$ is
not of finite order, then by Lemma 4.3, A is not $\alpha$-finite. By Lemma 4.4, G is not $\alpha$-finite, a contradiction. Hence $[\alpha|_{A \times A}]$ is of finite order. By Lemma 4.3, A is $\alpha$-finite. By Theorem 4.1, we have a central extension

$$1 \to Z \to A^* \to A \to 1$$

such that $A^*$ is a representation group of A, which is a two-step nilpotent group such that $Z = [A^*, A^*]$. There is a character $\chi$ of Z such that $\text{tra}(\chi) = [\alpha_{A \times A}]$. Since A is $\alpha$-finite, irreducible ordinary representations of $A^*$ lying above $\chi$ are finite dimensional and by [9, Theorem 1.1], they are monomial. Thus irreducible $\alpha_{A \times A}$-representations of A are finite dimensional and monomial. So for $\rho \in \text{Irr}^A(A)$, there exists a finite index subgroup $H$ of A and a character $\psi : H \to \mathbb{C}^\times$ such that $\alpha(h_1, h_2) = \psi(h_1)\psi(h_2)\psi(h_1h_2)^{-1}$ and $\rho = \text{Ind}_H^A(\psi)$. Define a map $\mu : G \to \mathbb{C}^\times$ by

$$\mu(g) = \begin{cases} \psi^{-1}(g) & \text{for } g \in H, \\ 1 & \text{for } g \notin H. \end{cases}$$

Then take $\alpha'(g_1, g_2) = \alpha(g_1, g_2)\mu(g_1)\mu(g_2)\mu(g_1g_2)^{-1}$ for all $g_1, g_2 \in G$. The cocycles $\alpha'$ and $\alpha$ are cohomologous and $\alpha'|_{\text{H} \times H} = 1$. Let $[A : H] = \ell$. Consider the subgroup C of A generated by $\ell$-th power of elements of A. Clearly $C \subset H$. By definition, C is a characteristic subgroup of A of finite index such that $\alpha'|_{C \times C} = 1$.

Let $M = \langle C, z \rangle$. We show that M has an irreducible $\alpha'$-representation of infinite dimension. Since M is a finitely generated polycyclic group, we have a central extension

$$1 \to J \to M^* \to M \to 1$$

such that $M^*$ is a representation group of M, existence follows from Theorem 2.2. Then there is a character $\chi : J \to \mathbb{C}^\times$ such that $\text{tra}(\chi) = [\alpha'|_{M \times M}]$. Let $\tilde{C} = \pi^{-1}(C)$. Consider the central extension $1 \to J \to \tilde{C} \to C \to 1$. Every element $\tilde{c} \in \tilde{C}$ can be written as $\tilde{c} = (c, j), c \in C, j \in J$, where s is a section from M to $M^*$.

Recall that $A = B \oplus \langle t \rangle$ such that $A/B$ is infinite cyclic. Since $A/C$ is of finite index, there is a smallest positive integer k such that $t^k \in C$. Now let $\lambda, \beta \in \mathbb{C}^\times$ which is not a root of unity. For each integer $h = 0, \pm 1, \pm 2, \ldots$, we define a function $\rho_h$ on $\tilde{C}$ by the rule that, for any $\tilde{c} \in \tilde{C}$, $\rho_h(\tilde{c}) = \rho_h(c)j = \chi(j)\lambda^\beta$, where the integer $\beta = \beta_h(c)$ is defined by the condition that $z^h \tilde{e}z^{-h} = (t^k)^\beta (\text{mod } B)$. Let $\tilde{c}_i = j_i s(c_i), i = 1, 2$. Now we have $\tilde{c}_1 \tilde{c}_2 = j_1 j_2 s(c_1) s(c_2) s(c_1 c_2)^{-1} s(c_1 c_2).$ so,

$$\rho_h(\tilde{c}_1 \tilde{c}_2) = \chi(j_1)\chi(j_2)\chi(s(c_1)s(c_2))s(c_1 c_2)^{-1}\lambda^{\beta_1 + \beta_2}$$

$$= \chi(j_1)\chi(j_2)\alpha'(c_1 c_2)\lambda^{\beta_1 + \beta_2}$$

$$= \chi(j_1)\chi(j_2)\lambda^{\beta_1 + \beta_2}$$

$$= \rho_h(\tilde{c}_1)\rho_h(\tilde{c}_2).$$

Hence $\rho_h$ are one dimensional ordinary representations of $\tilde{C}$ such that $\rho_h|_J = \chi$. Observe that $\{\rho_h|h = 0, \pm 1, \pm 2, \ldots\}$ are not equivalent.

Since $M/C \cong \langle z \rangle$, we have $M^*/C \cong \langle s(z) \rangle$. Now we define $V = \oplus_{\infty} \mathbb{C}[v_m]$, where $v_m$ is a generator of space $\rho_m$ for $m = 0, \pm 1, \pm 2, \ldots$. Define

$$s(z)v_m = v_{m+1}, \quad \tilde{c}v_m = \rho_m(\tilde{c})v_m.$$

Then V is an infinite dimensional ordinary representation of $M^*$ lying above $\chi$. The representation V is easily seen to be irreducible. Thus V is an infinite dimensional irreducible $\alpha'$-representation of M. So by Lemma 4.4, G will have an infinite dimensional irreducible $\alpha'$-representation. The cocycles $\alpha'$ and $\alpha$ are cohomologous, so G will have an infinite dimensional irreducible $\alpha$-representation, which is a contradiction.

Corollary 4.6. Every irreducible projective representation of a polycyclic group G is finite dimensional if and only if there is a normal abelian subgroup N of G such that for any $\alpha \in Z^2(G, \mathbb{C}^\times), [\alpha|_{N \times N}]$ is of finite order and $G/N$ is a finite group.
**Proof.** Let $G$ be a polycyclic group. If there is an abelian normal subgroup $N$ such that for any $\alpha \in \mathbb{Z}^2(G, \mathbb{C}^\times)$, $[\alpha]_{N\times N}$ is of finite order and $G/N$ is finite, then by Theorem 1.4 every irreducible projective representation of $G$ is finite dimensional.

Conversely, suppose every irreducible projective representation of $G$ is finite dimensional. By Theorem 1.4, it follows that, there is an abelian normal subgroup $N$ such that $G/N$ is finite. If for some $\alpha \in \mathbb{Z}^2(G, \mathbb{C}^\times)$, $[\alpha]_{N\times N}$ is not of finite order, then by Lemmas 4.3 and 4.4, there exists an infinite dimensional irreducible $\alpha$-representation of $G$, which is a contradiction. □

### 4.1. Examples

In this section, we discuss some examples of $\alpha$-finite groups.

**Example 1.** Consider the group $G = (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}) \rtimes \mathbb{Z}$, where the multiplication is defined by

$$(m_1, n_1, p_1)(m_2, n_2, p_2) = (m_1 + m_2 + p_1 n_2 (\text{mod } n), n_1 + n_2, p_1 + p_2).$$

By [5, Lemma 2.2(ii)], it follows that every 2-cocycle $\alpha \in \mathbb{Z}^2(G, \mathbb{C}^\times)$, up to cohomologous, is of the following form:

$$\sigma((m_1, n_1, p_1), (m_2, n_2, p_2)) = \lambda^{(m_2 p_1 + m_1 p_2 (p_1 - 1)/2)} \mu^{(n_1 m_2 + p_1 m_2 (n_2 - 1)/2 + p_1 n_1 n_2)},$$

where $\lambda, \mu \in \mathbb{C}^\times$ such that $\lambda^2 = \mu^2 = 1$. Hence every 2-cocycle is of finite order and there is a normal subgroup $(\mathbb{Z}/n\mathbb{Z} \times n\mathbb{Z}) \times \mathbb{Z}$ of $G$ such that quotient group is finite. So by Corollary 4.6, every projective representation of $G$ is finite dimensional.

**Example 2.** Our next example is of generalized discrete Heisenberg groups. These are finitely generated two-step nilpotent groups of rank $2n + 1$ with rank 1 center. Given an $n$-tuple $(d_1, d_2, \ldots, d_n)$ of positive integers with $d_1 | d_2 \cdots | d_n$ we write

$$G = \mathbb{H}_{2n+1}(d_1, d_2, \ldots, d_n) = \{(a, b, c) | a \in \mathbb{Z}, b, c \in \mathbb{Z}^n\},$$

where the group operation is defined by

$$(a, b_1, b_2, \ldots, b_n, c_1, \ldots, c_n)(a', b_1', b_2', \ldots, b_n', c_1', \ldots, c_n') = (a + a' + \sum_{i=1}^n d_i b_i' c_i, b_1 + b_1', b_2 + b_2', \ldots, b_n + b_n', c_1 + c_1', \ldots, c_n + c_n').$$

Consider $H = \mathbb{H}_3(d_1)$ and $K = \mathbb{H}_{2n-1}(d_2, \ldots, d_n)$. Then $G$ is a central product of normal subgroups $H$ and $K$ with $Z = [H, H] \cap [K, K] = d_2 \mathbb{Z}$. Consider the set

$$H_3^{d_1}(d_2) = \{(m, n, p) | m \in \mathbb{Z}/d_2 \mathbb{Z}, n, p \in \mathbb{Z}\},$$

with the group operation defined by

$$(m_1, n_1, p_1)(m_2, n_2, p_2) = (m_1 + m_2 + d_1 p_1 n_2, n_1 + n_2, p_1 + p_2).$$

Then

$$G/Z \cong H_3^{d_1}(d_2) \times \mathbb{Z}^{2n-2}.$$
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