Reduction of Abelian Varieties and Grothendieck’s Pairing

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Abstract

We prove that abelian varieties of small dimension over discrete valuated, strictly henselian ground fields with perfect residue class field obtain semistable reduction after a tamely ramified extension of the ground field. Using this result we obtain perfectness results for Grothendieck’s pairing.

1 Introduction

Let $R$ be a discrete valuation ring with field of fractions $K := \text{quot} R$ and with residue class field $k := R/m$. Furthermore let $A_K$ be an abelian variety over $K$ with its dual $A'_K$ and Néron models $A$ and $A'$, respectively. Their component groups are denoted by $\phi$ and $\phi'$. The duality between $A_K$ and $A'_K$ is reflected by the Poincaré bundle $\mathcal{P}$ on $A_K \times_K A'_K$. It has the property that the induced maps

$$A'_K \longrightarrow \text{Pic}^0 A_K, \quad a \mapsto \mathcal{P}|_{A_K \times \{a\}}$$

and vice versa are isomorphisms. We wish to extend $\mathcal{P}$ to the level of the associated Néron models, in order to study the relationship between $A$ and $A'$. An appropriate setting for this is the notion of biextensions. We briefly review the needed theory (cf. [SGA7], VII).

The canonical sequence

$$0 \longrightarrow \mathbb{G}_{m,R} \longrightarrow \mathcal{G} \longrightarrow i_*\mathbb{Z} \longrightarrow 0,$$

where $\mathcal{G}$ denotes the Néron model of $\mathbb{G}_m$, gives rise to an exact sequence

$$\text{Biext}^1(A_R, A'_R, \mathcal{G}) \longrightarrow \text{Biext}^1(A_R, A'_R, i_*\mathbb{Z}) \longrightarrow \text{Biext}^1(A_R, A'_R, i_*\mathbb{Z}).$$
After canonical identifications ([Bo97], section 4), this sequence can be written as

\[
\text{Biext}^1(A_R, A'_R, G_{m,R}) \rightarrow \text{Biext}^1(A_K, A'_K, G_{m,K}) \rightarrow \text{Hom}(\phi \otimes \phi', \mathbb{Q}/\mathbb{Z}).
\]

We regard the bundle \( \mathcal{P} \) as an element of the group \( \text{Biext}^1(A_K, A'_K, G_{m,K}) \).

Now, Grothendieck’s Pairing, \( \text{GP} \) for short, is defined to be the image of \( \mathcal{P} \) in the group \( \text{Hom}(\phi \otimes \phi', \mathbb{Q}/\mathbb{Z}) \). It represents the obstruction to extend \( \mathcal{P} \) to a biextension of \( G_{m,R} \) with \( A, A' \). Grothendieck conjectured this pairing to be perfect. Indeed, for perfect residue class field it can be shown that it is perfect\(^1\) in all cases except of the case of a discrete valuation ring of equal characteristic \( p \neq 0 \) with infinite residue class field \( k \).

1.1 Proposition Let \( A_K \) be an abelian variety. Then the prime-to-\( p \)-part of Grothendieck’s pairing is perfect. If \( A_K \) has semistable reduction, the whole pairing is perfect.

Proof. The first part is [Be01], Theorem 3.7, the second [We97]. \( \square \)

Grothendieck’s pairing induces a morphism \( \phi' \rightarrow \text{Hom}(\phi, \mathbb{Q}/\mathbb{Z}) =: \phi^* \).

It fits into the following diagram:

1.2 Proposition There is the following commutative diagram of sheaves with respect to the smooth topology

\[
\begin{array}{cccccc}
0 & \rightarrow & A^0 & \rightarrow & A' & \rightarrow & i_* \phi' & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{E}xt^1(A, G_m) & \rightarrow & \mathcal{E}xt(A, \mathcal{I}) & \rightarrow & \mathcal{E}xt^1(A, i_* \mathcal{Z}) = i_* \phi^*, & \text{GP}
\end{array}
\]

in which the sheaves in the second line can be represented uniquely by smooth schemes. In particular, the sheaf \( \mathcal{E}xt^1(A, G_m) \) can be represented by an open and closed subscheme of \( A' \), whose components correspond to the elements of \( \text{ker} \text{GP} \). This is a \( p \)-group.

Proof. For existence of the diagram see [Bo97], section 5, the last assertion is the snake lemma and [Be01], Theorem 3.7. \( \square \)

\(^1\)For mixed characteristic and perfect residue class field cf. [Beg], for finite \( k \) cf. [McC86]. If \( k \) is not perfect, counterexamples can be found, [BB02], corollary 2.5.
Let us denote by GP both the pairing $\phi \times \phi' \to \mathbb{Q}/\mathbb{Z}$ itself and the induced morphism $\phi' \to \phi^*$ (or $\phi \to (\phi')^*$, respectively). Since $\phi$ and $\phi'$ are finite groups, the induced morphisms are bijective, i.e. Grothendieck’s pairing is perfect, if and only if the induced morphisms are injective. Consequently, GP is perfect, if and only if the schemes representing $\mathcal{E}xt^1(A, \mathbb{G}_m)$ and $\mathcal{E}xt^1(A', \mathbb{G}_m)$ are connected.

2 Weil Restriction

As before, let $R$ be a discrete valuation ring, $K := \text{quot } R$ its field of fractions and let $A_K$ be an abelian variety over $K$ with dual $A'_K$, and Néron models $A_R$ and $A'_R$ over $R$. It is known that there exists a Galois extension $L/K$ such that $A_L := A_K \otimes_K L$ has semistable reduction ([SGA7, IX, 3.6]). In this case, Grothendieck’s pairing for $A_L$ and $A'_L$ is perfect ([We97]). We want to examine the relationship between Grothendieck’s pairing for $A_K$ and $A_L$: Let $S$ be the integral closure of $R$ in $L$. It induces a residue class field extension $\ell/k$. The Néron models of $A_L$ and $A'_L$ will be denoted by $A_S$ and $A'_S$. The pairings of groups of components of the Néron models over $R$ and $S$ can be summarised by the following commutative diagram of $\text{Gal}(k_s/\ell)$-modules.

\[
\begin{array}{ccc}
\phi_{AR} \times \phi_{A'R} & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\
\downarrow & & \downarrow e \\
\phi_{AS} \times \phi_{A'S} & \longrightarrow & \mathbb{Q}/\mathbb{Z},
\end{array}
\]

where $e$ is the ramification index of $L/K$. ([SGA7, XVII, 7.3.5]). Unfortunately, we cannot conclude that the first pairing is perfect if the second one is. However, we will use the technique of \textit{Weil restriction} to infer a partial result. All Weil restrictions we will encounter are representable by smooth group schemes, [BLR], 7.6, theorem 4. Let

$$X_K := \mathcal{R}_{L/K} A_L \quad \text{and} \quad X_R := \mathcal{R}_{S/R} A_S$$

be the Weil restriction of the abelian variety $A_L$ and its Néron model $A_S$.

2.1 Proposition \textit{In this situation, $X_R$ is the Néron model of $X_K$.}

Proof. Let $T$ be a smooth $R$-scheme. The calculation

$$\text{Hom}_R(T, X_R) = \text{Hom}_S(T \otimes_R S, A_S)$$

$$= \text{Hom}_L(T \otimes_R K \otimes_K L, A_L)$$

$$= \text{Hom}_K(T \otimes_R K, X_K)$$
shows that $X_R$ has the universal property of the Néron model of $X_K$. □

By $\phi_{X_R}$, $\phi_{A_S}$ etc., we denote the corresponding groups of components. In this situation, we have the following proposition:

**2.2 Proposition** *The canonical morphism $X_R \otimes_R S \to A_S$ induces a morphism*

$$
\phi_{X_R} \otimes_k \ell \longrightarrow \phi_{A_S}.
$$

*If $\ell/k$ is purely inseparable, this is an isomorphism.*

**Proof.** [BB02], proposition 1.1. □

From now on, we assume $R$ to be a strictly henselian discrete valuation ring with perfect, i.e., algebraically closed residue class field of characteristic $p \neq 0$. In this case, every finite extension of $k$ is trivial. In particular, it is purely inseparable and we can identify $\phi_{A_S}$ and $\phi_{X_R}$ by means of this proposition.

In this situation, we can allow a tamely ramified extension of $K$ to test whether or not Grothendieck’s pairing is perfect:

**2.3 Proposition** *Let $R$ be as stated above and let $L/K$ be a tamely ramified Galois extension. Consider the following assertions:

(i) Grothendieck’s pairing for $A_S$ and $A'_S$ is perfect.

(ii) Grothendieck’s pairing for $X_R$ and $X'_R$ is perfect.

(iii) Grothendieck’s pairing for $A_R$ and $A'_R$ is perfect.

Then we have $(i) \iff (ii) \implies (iii)$. 

We sketch the proof as given in [BB02], lemma 2.2 and corollary 3.1:

**Proof.** Let $n := [L : K]$. Since $k$ is algebraically closed, we have $n = e_{L/K}$. Since $L/K$ is tamely ramified, $p$ does not divide $n$. As shown in [BB02], Lemma 2.2 we have the following equality:

$$
[e_{L/K}] \circ \text{GP}_{X_R} = [n] \circ \text{GP}_{A_S},
$$

where $[n]$ denotes the $n$-multiplication on the appropriate group of components. It follows that the kernels of both compositions coincide (after identification as in proposition [2.2]). Since GP is an isomorphism on the prime-to-$p$-part and since $n$ is prime to $p$, the kernels of $\text{GP}_{X_R}$ and $\text{GP}_{A_S}$ coincide. This shows the equivalence of $(i)$ and $(ii)$. 

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Consider the canonical morphisms $A_K \hookrightarrow X_K$ and the norm map $X_K \rightarrow A_K$. Their composition is multiplication with $n$, cf. the proof of [BB02], corollary 3.1. These morphisms give rise to the following morphisms of Néron models:

$$A_R \longrightarrow X_R \longrightarrow A_R,$$

such that their compositions is the multiplication with $n$. Hence, there are morphisms of the smooth schemes which represent $\text{Ext}^1(-, \mathbb{G}_m)$ and of their component groups. Since $\ker GP$ is isomorphic to the group of components of the smooth scheme which represents $\text{Ext}^1(A_R, \mathbb{G}_m)$, proposition 1.2 we have two morphisms

$$\ker GP_{A_R} \longrightarrow \ker GP_{X_R} \longrightarrow \ker GP_{A_R},$$

such that the composition is multiplication by $n$. As these groups are $p$-groups by theorem 1.2 and $n$ is prime to $p$, the first morphism is injective; hence, (ii) $\Rightarrow$ (iii).

\[\square\]

3 Abelian Varieties of Small Dimension

We have seen that perfectness of Grothendieck’s pairing can be tested after a tamely ramified extension of the ground field. We would like to derive a property of abelian varieties which acquire semistable reduction after a tamely ramified extension $L/K$. As Grothendieck’s pairing of $A_L$ is perfect, we can conclude that Grothendieck’s pairing of $A_K$ is perfect, too. We will prove that abelian varieties of small dimension, depending on the residue class field characteristic $p$, achieve semistable reduction after a tamely ramified extension. This provides another new clue that Grothendieck’s conjecture is true if the residue class field $k$ is perfect. As before, let $R$ be a strictly henselian, discrete valuation ring with algebraically closed residue class field. In this case, the integral closure $S$ of $R$ in $L$ is a strictly henselian discrete valuation ring with algebraically closed residue class field. In particular, the inertia subgroups of $\text{Gal}(K_s/K)$ and $\text{Gal}(K_s/L)$ for a fixed separable closure $K_s$ of $K$ coincide with the absolute Galois groups of $K$ and $L$.

3.1 Definition (Tate module) Let $A_K$ be an abelian variety and let $\ell \neq p$ be a prime. The Tate module of $A_K$ is the $\text{Gal}(K_s/K)$-module

$$T_\ell(A_K) := \lim_{\longrightarrow} A_{K_s^\ell^n}(K_s).$$
As an abelian group, the Tate module is isomorphic to \( \mathbb{Z}_\ell^{2g} \), where \( g \) is the dimension of \( A_K \).

Let \( G := \text{Gal}(K_s/K) \) denote the absolute Galois group of \( K \) and let \( I \subseteq G \) denote the inertia subgroup. As we have seen, they coincide. Nonetheless, we use this notation for a coherent statement of the following theorems.

Our point of departure is the Galois criterion for semistable reduction (cf. [SGA7], IX, 3.5):

**3.2 Proposition (Galois Criterion for Semistable Reduction)** Let \( A_K \) be an abelian variety over \( K \) and let \( \ell \neq p \) be a prime. Then the following statements are equivalent:

(i) \( A_K \) has semistable reduction.

(ii) There exists an \( I \)-submodule \( T' \subseteq T := T_\ell(A_K) \), such that \( I \) operates trivially on \( T' \) and \( T/T' \).

\[ \square \]

As the inertia subgroup \( I \subseteq G \) coincides with the whole Galois group \( G \), we will not distinguish between them. Using the Galois criterion, we can formulate the semistable reduction theorem in terms of Galois theory:

**3.3 Proposition** Let \( A_K \) be an abelian variety. There exists a normal subgroup \( G' \subseteq G \) of finite index with the property (ii), i.e. there exists a subgroup \( T' \subseteq T \), stable under the action of \( G' \), such that \( G' \) operates trivially on \( T' \) and \( T/T' \).

**Proof.** It is known that \( A_K \) acquires semistable reduction after a finite Galois extension \( L/K \), corresponding to a normal subgroup \( G' := \text{Gal}(K_s/L) \subseteq G \) of finite index, cf. [SGA7], IX, 3.6. Since \( R \) is strictly henselian with algebraically closed residue class field, the integral closure \( S \) of \( R \) in \( L \) is strictly henselian, again. Therefore, the inertia subgroup of \( G' \) coincides with \( G' \) and we can apply the Galois criterion.

Our strategy is to enlarge \( G' \) by an appropriate pro-\( p \)-group, such that the resulting field extension is tamely ramified. Since \( R \) is strictly henselian, the theory of tamely ramified extension of \( K \) reduces to the following:

**3.4 Proposition** Let \( R \) be a strictly henselian discrete valuation ring with field of fractions \( K \) and residue class field \( k \) of characteristic \( p \neq 0 \).

(i) A finite extension \( L/K \) is tamely ramified if and only if \( p \nmid [L : K] \). Each such extension is a cyclic Galois extension.

(ii) Let \( L/K \) be any finite Galois extension with corresponding Galois group \( G = \text{Gal}(L/K) \). Then there exists a unique \( p \)-Sylow-subgroup \( G_p \subseteq G \).
Consequently, any Galois extension $L/K$ with Galois group $G$ can be splitted up into a tamely ramified Galois extensions $L^{\text{tr}}/K$ with cyclic Galois group and a wildly ramified Galois extension $L/L^{\text{tr}}$ with Galois group $G_p$, where $G_p \subseteq G$ is the unique $p$-Sylow-subgroup of $G$.

Proof. Since $k$ is separably closed, every finite extension $\ell/k$ is purely inseparable and, hence, its degree is a power of $p$. Due to the fundamental equation $[L : K] = e_{L/K} \cdot [\ell : k]$ the first part of (i) is obvious.

Since $R$ is henselian, we can lift the roots of the polynomial $X^n - 1$ from $k$ to $R$. Thus, we can conclude that the $n$-th roots of unity are contained in $R$ and hence in $K$. Any extension $L/K$ of degree $n$ prime to $p$ can easily be shown to be a Kummer extension, isomorphic to

$$L \cong K[X]/(X^n - \pi)$$

for a suitable uniformising element $\pi$ of $K$. Now, $\sigma \mapsto \sigma(X)$ constitutes an isomorphism $\text{Gal}(L/K) \cong \mu_n(K)$. Since the $n$-th roots of unity are contained in $R$ and thus in $K$, the group $\mu_n(K)$ is (non canonically) isomorphic to $\mathbb{Z}/n$. This settles assertion (i). To show (ii), let $G_p$ be any $p$-Sylow subgroup of $G$. The degree of the corresponding field extension $L^{G_p}/K$ is prime to $p$. Hence it is a Galois extension by (i), which implies that $G_p$ is normal. □

We set $K_s^{\text{tr}}$ for the union of all tamely ramified extensions of $K$ in $K_s$. This field is tamely ramified over $K$ and the above proposition can be generalized to the situation of profinite Galois groups as follows:

3.5 Corollary Let $R$ be as above. Then there exists an exact sequence

$$0 \rightarrow P \rightarrow \text{Gal}(K_s/K) \rightarrow \prod_{\ell \neq p} \hat{\mathbb{Z}}_{\ell} \rightarrow 0$$

with a pro-$p$-group $P = \text{Gal}(K_s/K_s^{\text{tr}})$. The last term is isomorphic to the Galois group $\text{Gal}(K_s^{\text{tr}}/K)$. □

Let $L/K$ be a finite Galois extension with Galois group $G'$. We have seen that it gives rise to normal field extensions $K \subseteq L^{\text{tr}} \subseteq L$. We now want to describe the Galois extension $K_s/L^{\text{tr}}$ and its Galois group:

3.6 Proposition In this situation, the field $L^{\text{tr}}$ can be written as $L \cap K_s^{\text{tr}}$. Thus we have $\text{Gal}(K_s/L^{\text{tr}}) = G' \cdot P$ for the pro-$p$-group $P = \text{Gal}(K_s/K_s^{\text{tr}})$.  

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Proof. Since $K_s^{\text{tr}}$ is the union of all tamely ramified extensions of $K$ in $K_s$, it is a tamely ramified extension of $K$. Thus $K_s^{\text{tr}} \cap L$ is the maximal tamely ramified extension of $K$ in $L$. □

In the light of this proposition, we are to study the action of the group $P$ on the Tate module. Since the Galois group acts on each of the groups $A_{K_s}(K_s) = (\mathbb{Z}/\ell^n)^{2g}$, it suffices to study the action of $G$ and $P$ on these groups to understand the action on the entire Tate module.

An action of $G$ on these groups can be regarded as a homomorphism

$$G \to \text{Aut}((\mathbb{Z}/\ell^n)^{2g}).$$

In a first step, we investigate the order of this automorphism group.

3.7 Lemma Let $\ell$ be a prime and let $Z = \mathbb{Z}/\ell \times \ldots \times \mathbb{Z}/\ell^n$.

(i) $\text{ord Aut } Z = \ell^s \cdot \prod_{i=1}^r (\ell^{d_i} - \ell^{i-1})$ for some $d_i \in \{i, \ldots, r\}$ and $\eta \in \mathbb{N}$.

(ii) $\text{ord Aut}((\mathbb{Z}/\ell^n)^r) = \ell^n \cdot \prod_{i=1}^r (\ell^i - 1)$ for some integer $\eta \in \mathbb{N}$.

(iii) If $H \subseteq (\mathbb{Z}/\ell^n)^r$ is a subgroup, then each prime divisor of $\text{ord Aut } H$ is a prime divisor of $\text{ord Aut}((\mathbb{Z}/\ell^n)^r)$.

Proof. The order of the automorphism group of a finite abelian $\ell$-group is computed in [Ra], Theorem 15. For a modern account, see [HR], Theorem 4.1.

If $H$ is a subgroup as in (iii), then it is isomorphic to

$$H \cong \prod_{i=1}^s \mathbb{Z}/\ell^{n_i} \quad \text{for some } s \leq r \text{ and } n_i \leq n,$$

and we can write each factor $\ell^{d_i} - \ell^{i-1}$ of $\text{ord Aut } H$ as $\ell^{i-1}(\ell^{d_i-i+1} - 1)$ for some $d_i \in \{i, \ldots, s\}$. Now assertion (iii) follows. □

The existence of elements of order $p$ in a finite group $G$ is equivalent to $p \mid \text{ord } G$. Therefore, it is sufficient to show that the product of factors $\ell^i - 1$ is not divisible by $p$ in order to prove that the automorphism group in mind has no elements of order $p$. Furthermore, if $\text{ord Aut } G$ is not divisible by $p$, the same is true for the order of the automorphism group of every subgroup of $G$.

Let $\mathbb{P}$ denote the set of primes, $\mathbb{P} = \{2, 3, 5, \ldots\}$. Then we can formulate Dirichlet’s prime number theorem as follows:

3.8 Lemma Let $p$ be a prime. Then the canonical map $\mathbb{P} \setminus \{p\} \to (\mathbb{Z}/p)^*$ is surjective. □
We are now ready to prove the following reduction theorem:

3.9 Theorem Let $R$ be a strictly henselian discrete valuation ring with residue class field of characteristic $p \neq 0$. Let $K$ be the field of fractions of $R$, and let $A_K$ be an abelian variety over $K$ of dimension $g$. If $2g + 3 \leq p$, then $A_K$ obtains semistable reduction over a tamely ramified extension of $K$.

Proof. We are going to show that with these assumptions there is no non-trivial $P$-action on $A_{K,\ell^n}(K_s) = (\mathbb{Z}/\ell^n)^{2g}$ for a suitable prime $\ell$ and every pro-$p$-group $P$. We do this by investigating the order of the corresponding automorphism groups. Following lemma 3.7, neither $\text{Aut}((\mathbb{Z}/\ell^n)^{2g})$ nor $\text{Aut} H$ for any subgroup $H \subseteq (\mathbb{Z}/\ell^n)^{2g}$ do have elements of order $p$, if $p \nmid \ell^i - 1$ for all $i \in \{1, \ldots, 2g\}$. Without loss of generality, we can restrict ourselves to the group $(\mathbb{Z}/\ell^n)^{2g}$. The last condition can be formulated as

$$p \nmid \text{ord} \text{Aut}((\mathbb{Z}/\ell^n)^{2g}) \iff \ell^i \equiv 1 \pmod{p} \text{ for all } i \in \{1, \ldots, 2g\}.$$

The group $(\mathbb{Z}/p)^*$ is cyclic and of order $p-1$. With lemma 3.8 we can choose a prime $\ell$ which generates $(\mathbb{Z}/p)^*$. Then $p-1$ is the minimal exponent with the property that

$$\ell^{p-1} \equiv 1 \pmod{p}.$$

Therefore, it is minimal with $p \nmid \ell^{p-1} - 1$. Consequently, if

$$2g < p - 1,$$

then $\text{Aut}((\mathbb{Z}/\ell^n)^{2g})$ does not have any elements of order $p$. As this inequality cannot be true for $p = 2$, the above inequality leads to $2g + 3 \leq p$.

With the semistable reduction theorem, 3.3 and the Galois criterion, 3.2 we choose a finite Galois extension $L/K$ with Galois group $G' := \text{Gal}(K_s/L) \subseteq G := \text{Gal}(K_s/K)$ and some $G'$-submodule $T' \subseteq T := T_\ell(A_K)$ such that $G'$ acts trivially on both $T'$ and $T/T'$.

Now, we consider the tamely ramified extension $L^\text{tr}/K$. We have seen in proposition 3.6 that the corresponding Galois group $\text{Gal}(K_s/L^\text{tr})$ is of the form $G' \cdot P$ for some pro-$p$-group $P \subseteq G$. Now the computation (the limit varies over all open subgroups $P' \subseteq P$)

$$H^0(P, (\mathbb{Z}/\ell^n)^{2g}) = \lim_{P'} H^0(P/P', ((\mathbb{Z}/\ell^n)^{2g})^{P'}) = \lim_{P'} \lim_{((\mathbb{Z}/\ell^n)^{2g})^{P'}} = (\mathbb{Z}/\ell^n)^{2g}$$

shows that every pro-$p$-group $P$ acts trivially on $T_\ell(A_K)$ (and, by lemma 3.7 on all its subgroups). Thus the product $G' \cdot P$ acts trivially on $T'$ and $T/T'$. Therefore, $A_{L^\text{tr}}$ has semistable reduction. □
We can infer numerous corollaries from this theorem. Amazingly, the threshold \(2g + 3 \leq p\) for the dimension \(g = \dim A_K\) has many consequences for the groups of components and for the canonical morphism \(\phi_{A_K} \rightarrow \phi_{A_L}\). With the methods of this proof we can show:

**3.10 Proposition** Let \(A_K\) be a variety of dimension \(g\) with \(2g + 3 \leq p\). Let \(L/K\) be a minimal Galois extension with the property that \(A_L\) reaches semistable reduction, then every prime divisor \(q\) of \([L : K]\) is smaller than \(2g + 3\).

*Proof.* We know that there exists a finite, tamely ramified extension \(L/K\) such that \(A_L\) reaches semistable reduction. The extension \(L/K\) induces an exact sequence

\[
0 \longrightarrow \text{Gal}(K_s/L) \longrightarrow \text{Gal}(K_s/K) \overset{f}{\longrightarrow} \mathbb{Z}/n \longrightarrow 0,
\]

where \(n := [L : K]\) and where \(\mathbb{Z}/n\) is isomorphic to the Galois group of \(L/K\). Now, let \(q \geq 2g + 3\) be a prime. As in the proof of the theorem, we can choose a prime \(\ell\) such that there is no non-trivial action of a pro-\(q\)-group on \(T_\ell(A_K)\) – and on all of its subgroups. Let \((\mathbb{Z}/n)_q\) be the \(q\)-part of \(\mathbb{Z}/n\). Since \(\mathbb{Z}/n\) is abelian, this corresponds to Galois extensions \(K \subseteq L' \subseteq L\). Then the preimage \(f^{-1}((\mathbb{Z}/n)_q) = \text{Gal}(K_s/L) \cdot Q\) for some pro-\(q\)-group \(Q\) is the Galois group \(\text{Gal}(K_s/L')\). Now, we can conclude as in the proof of the theorem. \(\square\)

Our main application of the above reduction theorem is the following result regarding Grothendieck’s pairing:

**3.11 Corollary** Let \(R\) be a discrete valuation ring with perfect residue class field of characteristic \(p\). If \(A_K\) is an abelian variety over \(K := \text{quot } R\) of dimension \(g\) with \(2g + 3 \leq p\), then Grothendieck’s pairing for \(A_K\) is perfect.

*Proof.* Without loss of generality we can assume \(R\) to be strictly henselian ([SGA7], IX, 1.3.1). Since \(k\) is assumed to be perfect, it is algebraically closed. According to the above theorem, \(A_K\) acquires semistable reduction after a tamely ramified Galois extension of \(L/K\). As Grothendieck’s pairing is perfect for \(A_L\), cf. [We97], we can conclude by means of proposition 2.3 that Grothendieck’s pairing of \(A_K\) is perfect. \(\square\)

Following Poincaré’s reducibility theorem ([Mum], IV, 18, theorem 1), every abelian variety is isogenous to a product of simple abelian varieties and we can prove:
3.12 Corollary Let $R$ be as in the above corollary and let $A_K$ be isogenous to the product $B_{K,1} \times \ldots \times B_{K,n}$ with $2 \cdot \dim B_{K,i} + 3 \leq p$ for every $i \in \{1, \ldots, n\}$, then Grothendieck’s pairing of $A_K$ is perfect.

Proof. Each factor $B_{K,i}$ reaches semistable reduction after a tamely ramified extension $K_i$ of $K$. If we take $K'$ to be the composite field of all $K_i$, then $K'/K$ is tamely ramified and the product of the $B_{K,i}$ reaches semistable reduction over $K'$. Since $A_K$ and $\prod B_{K,i}$ are isogenous, $A_{K'}$ has semistable reduction if and only if each $B_{K',i}$ has semistable reduction by [BLR], 7.3, corollary 7. Now, proposition 2.3 completes the proof.

□

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