Induced interaction in a spin-polarized Fermi gas

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We study the effect of the induced interaction on the superfluid transition temperature of a spin-polarized Fermi gas. In the BCS limit, the polarization is very small in the superfluid state, and the effect of the induced interaction is almost the same as in the spin-balanced case. The temperature $T_i$ and the polarization $P_i$ of the tricritical point are both reduced from mean-field results by a factor about 2.22. This reduction is also significant beyond the BCS limit. In the unitary limit, we find $(P_i, T_i/T_F) = (0.42, 0.16)$, in comparison with mean-field and experimental results.

I. INTRODUCTION

Ultra-cold spin-polarized Fermi gases have attracted a lot of attentions from both experimental and theoretical sides in the past few years. In a spin-balanced Fermi gas, BEC-BCS crossover can be achieved at low temperatures when the scattering between atoms is tuned through a Feshbach resonance \cite{1}, i.e. the system can evolve smoothly from a BCS superfluid phase to a molecular-BEC phase \cite{2,3}. However, this crossover picture is no longer accurate with spin polarization, and phase separation between normal and superfluid phases can take place at very low temperatures \cite{4,5}. In addition, other exotic superfluid phases, such as the FFLO phase \cite{6,7}, may appear \cite{8}.

The phase separation takes place when the chemical potential difference between two spin species $h = (\mu_\uparrow - \mu_\downarrow)/2$, reaches a critical value $h_c$. This first-order transition was first investigated by Clogston and Chandrasekhar in the context of BCS superconductors \cite{9,10}. In the BCS limit, i.e. with a weakly-attractive interaction, they found that at zero temperature $T = 0K$, $h_c$ is given by $h_c = \Delta/\sqrt{2}$, where $\Delta$ is the energy gap for $h = 0$. Since the superfluid transition in the spin-balanced case is a second-order transition, a tricritical point is expected in the $T - h$ phase diagram. In the unitarity limit at the resonance, the tricritical point was observed experimentally \cite{11}, with both the first-order and second-order transition lines located. In recent years, the phase diagram of the spin-imbalanced Fermi gas was theoretically studied by mean-field method \cite{12,13}, variational method \cite{14}, pairing fluctuation theory \cite{15,16}, Quantum Monte-Carlo simulations \cite{17,18}, and renormalization-group approach \cite{19}.

In a Fermi gas, the fluctuation in the particle-hole channel generates the induced interaction which was first pointed out by Gorkov and Melik-Barkhudarov (GMB) \cite{20}. In a spin-balanced Fermi gas with a BEC-BCS crossover, it suppresses pairing considerably. In the BCS limit, the superfluid transition temperature $T_c$ is reduced from the mean-field value by a factor about 2.22. The induced interaction is important in both the BEC limit and unitary region \cite{21}. In the BEC limit, the effect of the induced interaction is negligible due to disappearance of Fermi surface. The induced interaction has also been studied for a spin-balanced Fermi gas in optical lattice \cite{22,23} and a homogeneous three-components Fermi gas \cite{24}. In this paper, we study the induced interaction in a spin-imbalanced Fermi gas and investigate its impact on the superfluid transition temperature from the BCS limit to the unitary region.

II. INDUCED INTERACTION A SPIN POLARIZED FERMI GAS

A spin-polarized Fermi gas with a wide Feshbach resonance can be described by a single-channel model,

$$
\mathcal{H} = \sum_\sigma \frac{\hbar^2}{2m} |\nabla \psi_\sigma|^2 + g_\sigma \psi_\sigma^\dagger \psi_\uparrow \psi_\downarrow - \mu (\psi_\sigma^\dagger \psi_\uparrow + \psi_\sigma^\dagger \psi_\downarrow) - h (\psi_\uparrow^\dagger \psi_\uparrow - \psi_\downarrow^\dagger \psi_\downarrow),
$$

where the coupling constant is given by $g = 4\pi \hbar^2 a_s/m$, $a_s$ is the scattering length, $\mu = (\mu_\uparrow + \mu_\downarrow)/2$, $\mu_\sigma$ is the chemical potential for spin component $\sigma$, and $h = (\mu_\uparrow - \mu_\downarrow)/2$ is an effective Zeeman field. In this following, we consider the homogeneous case with spin-up atoms as the majority component.

In the original work by Gorkov and Melik-Barkhudarov \cite{20}, the induced interaction was obtained in the BCS limit by the second-order perturbation \cite{21}. GMB’s treatment can be extended to the region with a strong interaction

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When interacting Fermi gas is given by
\[ U_{\text{ind}}(p_1, p_2; p_3, p_4) = -\frac{g^2 \chi(p_1 - p_4)}{1 + g\chi(p_1 - p_4)}, \]
where \( p_i = (k_i, \omega_i) \) is a vector in the space of wave-vector \( k \) and fermion Matsubara frequency \( \omega_i = (2l + 1)\pi/(\hbar\beta) \), \( \beta = 1/(k_B T) \). The polarization function \( \chi \) is given by
\[ \chi(p') = \frac{1}{\hbar^2 \beta V} \sum_p G_{0\uparrow}(p)G_{0\downarrow}(p + p') \int \frac{d^3k}{(2\pi)^3} \frac{f_{k\uparrow} - f_{k + k'\downarrow}}{\hbar \Omega_i + \xi_{k\uparrow} - \xi_{k + k'\downarrow}}, \]
where \( p' = (k', \Omega_i) \), \( \Omega_i = 2l\pi/(\hbar\beta) \) is the Matsubara frequency of a boson, \( V \) is the volume, \( f_{k\sigma} = 1/[1 + \exp(\beta \xi_{k\sigma})] \) is the Fermi distribution function, \( \xi_{k\sigma} = \epsilon_k - \mu_\sigma \), and \( \epsilon_k = \hbar^2 k^2/2m \). The Matsubara Green’s function of a non-interacting Fermi gas is given by \( G_{0\sigma}(p) = \hbar/(i\hbar \omega - \xi_{k\sigma}) \).

Including the induced interaction, the effective pairing interaction between atoms with different spins is given by
\[ U_{\text{tot}}(p_1, p_2; p_3, p_4) = g + U_{\text{ind}}(p_1, p_2; p_3, p_4) = \frac{g}{1 + g\chi(p_1 - p_4)}. \]

Although the effective interaction is a function of both momentum and frequency, only its s-wave part plays an important role on pairing at low temperatures. As in GMB’s work, we approximate this s-wave component \( g' \) by averaging the polarization function \( \chi_s = \langle \chi \rangle \),
\[ g' = \frac{g}{1 + g\chi_s}. \]

In this work, the possibility of FFLO state is ignored and we only consider pairing between atoms with opposite momentum. The average of the polarization function \( \chi_s \) is obtained by setting the frequencies to zero, setting wavevectors to \( k_1 = -k_2, k_3 = -k_4, k_1 = k_2 = k_3 = k_4 = k_F \), and integrating over the angle \( \theta \) between \( k_1 \) and \( k_4 \),
\[ \chi_s = \frac{m}{8\pi^2\hbar^2} \int_{-1}^{1} d\cos \theta \int_{0}^{\infty} dk \int \frac{k}{k_F} \left[ k_F^2 \ln \left| \frac{k^2 - 2kk' + 4mh}{k^2 + 2kk' + 4mh} \right| + f_{k\uparrow} \ln \left| \frac{k^2 - 2kk' + 4mh}{k^2 + 2kk' + 4mh} \right| \right], \]
where \( k = (3\pi^2 n)^{1/3} \), \( n \) is the total density, and the variable \( k' \) is a function of \( \theta \), \( k' = |k_1 - k_4| = k_F \sqrt{2(1 + \cos \theta)} \). When \( h = 0 \), Eq. (6) recovers the result in spin-balanced case [23]. As indicated by Eq. (4), the effective s-wave interaction is determined by the average of the polarization function \( \chi_s \).

In the BCS limit, \( k_F a_s \to 0^- \), the superfluid transition takes place at temperatures much lower than the Fermi temperature \( T_F \). In this zero-temperature limit, the average of the polarization function \( \chi_s \) is given by
\[ \chi_s = -\frac{m}{8\pi^2\hbar^2} \frac{1}{k_F^2} \int_{0}^{2k_F} dk' \left[ \left( \frac{k_F^2}{2} - \frac{(k^2 + 4mh)^2}{8k'^2} \right) \ln \left| \frac{k^2 + 2k_F k' + 4mh}{k^2 - 2k_F k' + 4mh} \right| + \frac{k^2 + 4mh}{2k'} \right] _{k_F^+} ^{k_F^+} + \left( \frac{k_F^2}{2} - \frac{(k^2 - 4mh)^2}{8k'^2} \right) \ln \left| \frac{k^2 + 2k_F k' - 4mh}{k^2 - 2k_F k' - 4mh} \right| + \frac{k^2 - 4mh}{2k'} \right] _{k_F^+} ^{k_F^+}, \]
where \( k_F^+ = \sqrt{2m\mu_\uparrow}/\hbar \) and \( k_F^- = \sqrt{2m\mu_\downarrow}/\hbar \). For given total density \( n = n_\uparrow + n_\downarrow \), the average \( \chi_s \) is shown as a function of spin polarization \( P = (n_\uparrow - n_\downarrow)/(n_\uparrow + n_\downarrow) \) in Fig. 1. When \( P = 0 \), GMB’s result for the spin-balanced case,
\[ \chi_{s0} = -\frac{1}{3} \ln(4e)N(E_F), \]
is recovered, where \( N(E_F) = mk_F/(2\pi^2\hbar^2) \) is the density of states at Fermi energy. As the polarization \( P \) increases, the absolute value of \( \chi_s \) decreases, indicating that the induced interaction becomes weaker. The average \( \chi_s \) varies very slowly for small polarization, for example \( \chi_s = 0.96\chi_{s0} \) at \( P = 0.5 \). For a nearly full-polarized system, \( P \to 1 \), the particle-hole fluctuation still exists with a non-zero \( \chi_s \). In this limit, the system is described by the picture of Fermi polarons [24], rather than the superfluid transition.

It is worth to note that besides the method described above, we can use other methods to compute the average of the polarization function \( \chi_s \). Since the dominant contribution to pairing comes from atoms near the two Fermi
The first-order and second-order phase transition lines meet at a tricritical point \( \chi^* \) which yields
\[
E_F = g^{-1} + \chi_s.
\]

Even for \( P \ll 1 \), there is very little difference among these three results. The effective s-wave interaction is given by
\[
g' \approx g - \chi_s.
\]

III. SUPERFLUID TRANSITION

We replace the coupling constant \( g \) by the effective s-wave interaction \( g' \) from Eq. (4). When pairing fluctuations are ignored, the thermodynamic potential is given by
\[
\Omega = - \frac{1}{\beta V} \sum_k \left[ \ln(1 + e^{-\beta E_k}) + \ln(1 + e^{-\beta (E_k - h)}) \right] + \frac{1}{V} \sum_k (\xi_k - E_k) - \frac{|\Delta|^2}{g'},
\]
where \( E_k = \sqrt{\xi_k^2 + |\Delta|^2} \) is the energy of the quasi-particle, \( \Delta = g' \langle \psi_\uparrow \psi_\downarrow \rangle \) is the order parameter, \( \xi_k = \epsilon_k - \mu \). The difference between Eq. (6) and the mean-field expression of the thermodynamic potential is that the particle-hole fluctuation has been taken into account through the effective interaction \( g' \).

The second-order superfluid transition is determined by the condition
\[
\lim_{\Delta \to 0} \frac{1}{\Delta} \frac{\partial \Omega}{\partial \Delta^2} = 0,
\]
which yields
\[
\frac{m}{4\pi \hbar^2 a_s} + \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1 - f_{k\uparrow} - f_{k\downarrow}}{2\xi_k} - \frac{1}{2\epsilon_k} \right] + \chi_s = 0,
\]
where the last term in the integrand on l.-h.-s. of this equation is a counter term due to vacuum renormalization. If \( \mu_\uparrow = \mu_\downarrow \), Eq. (6) is just the \( T_c \)-equation for the unpolarized Fermi gas, which in the BCS limit produces the superfluid transition temperature in GMB theory \( T_c = \frac{\gamma}{\pi} \left( \frac{2}{\pi e} \right)^{7/3} T_F \exp \left( \frac{\pi}{2 k_F a_s} \right) \).

In a spin-polarized Fermi gas, the superfluid transition is a first-order phase transition at very low temperatures. The first-order and second-order phase transition lines meet at a tricritical point \( (h, T) = (h_t, T_t) \). At the tricritical
point, in addition to Eq. (7), we have
\[
\lim_{\Delta \to 0} \frac{1}{\Delta^2} \frac{\partial^2 \Omega}{\partial \Delta^2} = 0. \tag{10}
\]
In Landau’s theory about phase transition, these two equations determine zero points of the first two coefficients in expansion of thermodynamic potential in terms of the order parameter. Eq. (10) can be explicitly written as
\[
\int \frac{d^3k}{(2\pi)^3} \left[ 1 - f_{k\uparrow} - f_{k\downarrow} + \beta \frac{\text{sech}^2 \frac{\beta \xi_k^\uparrow}{2} + \text{sech}^2 \frac{\beta \xi_k^\downarrow}{2}}{4\xi_k^2} \right] = 0. \tag{11}
\]
For a given averaged-chemical-potential $\mu$, the tricritical point $(h_t, T_t)$ can be determined from coupled equations (8) and (11).

In the BCS limit, both $T_t$ and $h_t$ are proportional to the superfluid transition temperature $T_c$ given by Eq. (3) for the spin-balanced Fermi gas with the same total density and scattering length. From Eq. (11), we obtain
\[
h_t = 1.911 k_B T_t, \tag{12}
\]
which leads to
\[
P_t = \frac{3h_t}{2E_F} = 2.867 \frac{T_t}{T_F}. \tag{13}
\]
Putting Eq. (12) into Eq. (8), we have
\[
T_t = 0.561 T_c. \tag{14}
\]
Since the induced interaction reduces $T_t$ from the mean-field result by a factor about 2.22, both temperature and polarization at the tricritical point are also reduced by the same factor,
\[
T_t = 0.561 T_c = 0.156 T_F e^{\pi/2 k_F a_s}, \tag{15}
\]
\[
P_t = 1.608 T_c/T_F = 0.446 e^{\pi/2 k_F a_s}. \tag{16}
\]
This result is the same if we choose to compute the average of the polarization function $\chi_s$ at $k_F \uparrow$ or $k_F \downarrow$, because the polarization $P_t$ approaches zero in the BCS limit.

Numeric solutions of tricritical polarization $P_t$ and tricritical temperature $T_t$ beyond the BCS limit are showed in Fig. 2 as a function of $1/k_F a_s$. Both $P_t$ and $T_t$ are reduced significantly from the mean-field results due to the induced interaction. At unitary, we obtain $(P_t, T_t/T_F) = (0.42, 0.16)$, considerably smaller than mean-field results $(P_t, T_t/T_F) = (0.70, 0.30)$, but still larger than experimental result $(P_t, T_t/T_F) \approx (0.20, 0.08)$ [11]. The discrepancy between experimental and our results are probably due to pairing fluctuations ignored in our approach which are important in the unitary region and on the BEC side. In the renormalization-group approach [23] with both particle-hole and particle-particle scattering considered, the tricritical point was obtained close to the experimental result.

IV. DISCUSSION AND CONCLUSION

In a spin-balanced fermi gas, pairing fluctuations can be considered in the approach pioneered by Nozières and Schmitt-Rink (NSR) [30]. However, in a spin imbalanced Fermi gas, simple applications of NSR theory failed in the unitary region [19, 20]. Hence, a more sophisticated consideration of this problem is needed in the future. Another interesting issue is how to generalize the induced interaction to the broken symmetry state where the average polarization function $\chi_s$ can be quite different from the present form due to the large pairing gap $\Delta$ and the effect of the induced interaction may be more complicated.

In conclusion, we study the effect of the induced interaction on the superfluid transition temperature in a spin-polarized Fermi gas with a wide Feshbach resonance. In the BCS limit, the absolute value of the induced interaction decreases as the polarization $P$ increases, but this change is very small for $P < 0.5$. Both temperature and polarization at the tricritical point are reduced from mean-field results by a factor 2.22. Beyond the BCS limit, reductions of the tricritical polarization $P_t$ and tricritical temperature $T_t$ are also significant. In the unitary limit, the tricritical point is found at $(P_t, T_t/T_F) = (0.42, 0.16)$, and the discrepancy with the experimental result indicates the importance of pairing fluctuations at unitarity.
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