Isotone maps on lattices

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Abstract. Let \( \mathcal{L} = (L_i \mid i \in I) \) be a family of lattices in a nontrivial lattice variety \( \mathbf{V} \), and let \( \varphi_i : L_i \rightarrow M \), for \( i \in I \), be isotone maps (not assumed to be lattice homomorphisms) to a common lattice \( M \) (not assumed to lie in \( \mathbf{V} \)). We show that the maps \( \varphi_i \) can be extended to an isotone map \( \varphi : L \rightarrow M \), where \( L = \text{Free}_\mathbf{V} \mathcal{L} \) is the free product of the \( L_i \) in \( \mathbf{V} \). This was known for \( \mathbf{V} = \mathbf{L} \), the variety of all lattices.

The above free product \( L \) can be viewed as the free lattice in \( \mathbf{V} \) on the partial lattice \( P \) formed by the disjoint union of the \( L_i \). The analog of the above result does not, however, hold for the free lattice \( L \) on an arbitrary partial lattice \( P \). We show that the only codomain lattices \( M \) for which that more general statement holds are the complete lattices. On the other hand, we prove the analog of our main result for a class of partial lattices \( P \) that are not-quite-disjoint unions of lattices.

We also obtain some results similar to our main one, but with the relationship lattices : orders replaced either by semilattices : orders or by lattices : semilattices. Some open questions are noted.

1. Introduction

By Yu. I. Sorkin [15, Theorem 3], if \( \mathcal{L} = (L_i \mid i \in I) \) is a family of lattices and \( \varphi_i : L_i \rightarrow M \) are isotone maps of the lattices \( L_i \) into a lattice \( M \), then there exists an isotone map \( \varphi \) from the free product \( \text{Free} \mathcal{L} \) of the \( L_i \) to \( M \) that extends all the \( \varphi_i \). (There are some difficulties with Sorkin’s original proof; but a simple, correct proof is given by G. Grätzer, H. Lakser and C. R. Platt in [11, §4].)

Our main result, proved in Section 2, is a generalization of this fact, with \( \text{Free} \mathcal{L} \) replaced by \( \text{Free}_\mathbf{V} \mathcal{L} \), the free product of the \( L_i \) in any nontrivial variety \( \mathbf{V} \) of lattices containing them—though not necessarily containing \( M \). (In Section 3, we explore some variants of our proof of this result.)

We may regard \( \text{Free}_\mathbf{V} \mathcal{L} \) as the free lattice in \( \mathbf{V} \) on the partial lattice \( P \) given by the disjoint union of the \( L_i \). Does the analog of the above result hold for more general partial lattices \( P \) and their free lattices \( L \)? In Section 4, we find that the lattices \( M \) such that this statement holds for all partial lattices \( P \) are the complete lattices. On the other hand, we describe in Section 5 a class of partial lattices \( P \), related to but distinct from the class considered in Section 2, for which the full analog of the result of that section holds.
Since semilattices lie between orders and lattices, it is plausible that statements similar to our main result should hold, either with “lattice” weakened to “semilattice”, or with “lattice” unchanged but “isotone map” strengthened to “semilattice homomorphism”. In Section 6, we shall find that the former statement is easy to prove. In that section and Section 7, we obtain several approximations to the latter statement; we do not know whether the full statement holds.

The reader familiar with the concepts of quasi-variety and pre-variety will find that the proofs given in this note for varieties of lattices in fact work for those more general classes. However, varieties are not sufficient for the result of Section 7, so we develop that in terms of prevarieties.

In Section 8, we note some open questions.

For general definitions and results in lattice theory, see [8] or [9].

For another context in which isotone maps among lattices have appeared in the literature—in this case, isotone sections of lattice homomorphisms—see R. Freese, J. Ježek, and J. B. Nation [5, pp. 99–107, 133] and R. Freese [6].

We are indebted to R. Freese for pointing us to those works, and for the related observation following Corollary 2.4 below; and to A. V. Kravchenko for an extensive correspondence regarding Sorkin’s original argument.

2. Extending isotone maps to free product lattices

Let $V$ be a nontrivial lattice variety, that is, a variety $V$ of lattices having a member with more than one element. Let $L = (L_i \mid i \in I)$ be a family of lattices in $V$, and $L = \text{Free}_V L$ their free product in $V$. Finally, let $(\varphi_i : L_i \to M \mid i \in I)$ be a family of isotone maps into a lattice $M$, not assumed to lie in $V$.

To show that the $\varphi_i$ have a common extension to $L$, it suffices, by the universal property of $L$, to find some $L' \in V$ such that each map $\varphi_i$ factors $L_i \to L' \to M$, where the first map is a lattice homomorphism, and the second an isotone map not depending on $i$. So let us, for now, forget free products, and obtain such a lattice $L'$.

We now map $L'$ to $M$ using the isotone map $\psi$ given by

$$\psi(x) = \bigvee (\bar{\varphi}_i(x_i) \mid i \in I) \quad \text{for} \quad x = (x_i \mid i \in I) \in L'. \quad (2.1)$$