

REPRESENTATIONS OF TWO-PARAMETER QUANTUM GROUPS AND SCHUR-WEYL DUALITY

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Abstract. We determine the finite-dimensional simple modules for two-parameter quantum groups corresponding to the general linear and special linear Lie algebras \( \mathfrak{gl}_n \) and \( \mathfrak{sl}_n \), and give a complete reducibility result. These quantum groups have a natural \( n \)-dimensional module \( V \). We prove an analogue of Schur-Weyl duality in this setting: the centralizer algebra of the quantum group action on the \( k \)-fold tensor power of \( V \) is a quotient of a Hecke algebra for all \( n \) and is isomorphic to the Hecke algebra in case \( n \geq k \).

Introduction

In this work we study the representations of two two-parameter quantum groups \( \tilde{U} = U_{r,s}(\mathfrak{gl}_n) \) and \( U = U_{r,s}(\mathfrak{sl}_n) \). Our Hopf algebra \( \tilde{U} \) is isomorphic as an algebra to Takeuchi’s \( U_{r,s}^{-1} \) (see [T]), but as a Hopf algebra, it has the opposite coproduct. As an algebra, \( \tilde{U} \) has generators \( e_j, f_j, (1 \leq j < n) \), and \( a_i^{\pm 1}, b_i^{\pm 1} \) \((1 \leq i \leq n)\), and defining relations given in (R1)-(R7) below. The elements \( a_i^{\pm 1}, b_i^{\pm 1} \) generate a commutative subalgebra \( \tilde{U}^0 \), and the elements \( e_j, f_j, \omega_j^{\pm 1}, (\omega_j')^{\pm 1} \) \((1 \leq j < n)\), where \( \omega_j = a_j b_{j+1} \) and \( \omega_j' = a_{j+1} b_j \), generate the subalgebra \( U = U_{r,s}(\mathfrak{sl}_n) \).

The structure of these quantum groups was investigated in [BW], where we realized both \( \tilde{U} \) and \( U \) as Drinfel’d doubles of certain Hopf subalgebras and constructed an R-matrix for \( \tilde{U} \) and \( U \). In particular, for any two \( \tilde{U} \)-modules in category \( \mathcal{O} \) (defined in Section 3), there is an isomorphism \( R_{M',M} : M' \otimes M \to M \otimes M' \). The construction of \( R_{M',M} \) is summarized in Section 4 of this note. In Sections 2 and 3, we classify the finite-dimensional simple \( \tilde{U} \)-modules when \( rs^{-1} \) is not a root of unity and prove that all finite-dimensional \( \tilde{U} \)-modules on which \( \tilde{U}^0 \) acts semisimply are completely reducible. These results hold equally well for \( U \). The hypothesis on \( \tilde{U}^0 \) is necessary: we provide examples of finite-dimensional modules that are not completely reducible. Our complete reducibility proof uses a quantum Casimir operator defined in [BW] and parallels the argument in [L].

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These elements satisfy (R5)-(R7) along with the following relations:

\[(1.1) \quad \omega_j = a_j b_{j+1} \quad \text{and} \quad \omega_j' = a_{j+1} b_j.\]

These elements satisfy (R5)-(R7) along with the following relations:
(R1') The $\omega_i^{\pm 1}, \omega_j^{\pm 1}$ all commute with one another and $\omega_i \omega_i^{-1} = \omega_j^{\prime} (\omega_j^{\prime})^{-1} = 1$.

(R2') $\omega_i e_j = r^{(\epsilon_i, \alpha_j)} s^{(\epsilon_j, \alpha_i)} e_j \omega_i$ and $\omega_i f_j = r^{- (\epsilon_i, \alpha_j)} s^{- (\epsilon_j, \alpha_i)} f_j \omega_i$.

(R3') $\omega'_i e_j = r^{(\epsilon_i+1, \alpha_j)} s^{(\epsilon_j, \alpha_i)} e_j \omega'_i$ and $\omega'_i f_j = r^{- (\epsilon_i+1, \alpha_j)} s^{- (\epsilon_j, \alpha_i)} f_j \omega'_i$.

(R4') $[e_i, f_j] = \delta_{i,j} \frac{q - 1}{q} (\omega_i - \omega_i^{\prime})$.

When $r = q$ and $s = q^{-1}$, the algebra $U_{r,s}(\mathfrak{gl}_n)$ modulo the ideal generated by the elements $b_i - a_i^{-1}, 1 \leq i \leq n$, is just the quantum general linear group $U_q(\mathfrak{gl}_n)$, and $U_{r,s}(\mathfrak{sl}_n)$ modulo the ideal generated by the elements $\omega_j^{\prime} - \omega_j^{-1}, 1 \leq j < n$, is $U_q(\mathfrak{sl}_n)$.

The algebras $\tilde{U}$ and $U$ are Hopf algebras, where the $a_i^{\pm 1}, b_i^{\pm 1}$ are group-like elements, and the remaining Hopf structure is given by

$$
\Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i, \quad \Delta(f_i) = 1 \otimes f_i + f_i \otimes \omega_i,
$$

$$
\epsilon(e_i) = \epsilon(f_i) = 0, \quad S(e_i) = -\omega_i^{-1} e_i, \quad S(f_i) = -f_i (\omega_i^{\prime})^{-1}.
$$

Let $\Lambda = \mathbb{Z} \epsilon_1 + \cdots + \mathbb{Z} \epsilon_n$, the weight lattice of $\mathfrak{gl}_n$, and $Q = \mathbb{Z} \Phi$ the root lattice. Corresponding to any $\lambda \in \Lambda$ is an algebra homomorphism $\hat{\lambda}$ from the subalgebra $U^0$ of $U$ generated by the elements $a_i^{\pm 1}, b_i^{\pm 1}$ ($1 \leq i \leq n$) to $\mathbb{K}$ given by

$$
\hat{\lambda}(a_i) = r^{(\epsilon_i, \lambda)} \quad \text{and} \quad \hat{\lambda}(b_i) = s^{(\epsilon_i, \lambda)}.
$$

The restriction $\hat{\lambda}: U^0 \to \mathbb{K}$ of $\hat{\lambda}$ to the subalgebra $U_0$ of $U$ generated by $\omega_j^{\pm 1}, (\omega'_j)^{\pm 1}$ ($1 \leq j < n$) satisfies

$$
\hat{\lambda}(\omega_j) = r^{(\epsilon_j, \lambda)} s^{(\epsilon_{j+1}, \lambda)} \quad \text{and} \quad \hat{\lambda}(\omega'_j) = r^{(\epsilon_{j+1}, \lambda)} s^{(\epsilon_j, \lambda)}.
$$

Let $M$ be a module for $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$ of dimension $d < \infty$. As $\mathbb{K}$ is algebraically closed, we have

$$
M = \bigoplus_{\chi} M_{\chi},
$$

where each $\chi: \tilde{U}^0 \to \mathbb{K}$ is an algebra homomorphism, and $M_{\chi}$ is the generalized eigenspace given by

$$
M_{\chi} = \{ m \in M \mid (a_i - \chi(a_i) 1)^d m = 0 = (b_i - \chi(b_i) 1)^d m, \quad \text{for all } i \}.
$$

When $M_{\chi} \neq 0$ we say that $\chi$ is a weight and $M_{\chi}$ is the corresponding weight space. (If $M$ decomposes into genuine eigenspaces relative to $\tilde{U}^0$ (resp. $U^0$), then we say that $\tilde{U}^0$ (resp. $U^0$) acts semisimply on $M$.)

From relations (R2) and (R3) we deduce that
where $\tilde{\alpha}_j$ is as in (1.3), and $\chi \cdot \psi$ is the homomorphism with values $(\chi \cdot \psi)(a_i) = \chi(a_i)\psi(a_i)$ and $(\chi \cdot \psi)(b_i) = \chi(b_i)\psi(b_i)$. In fact, if $(a_i - \chi(a_i))1^k m = 0$, then applying relation (R2) yields $(a_i - \chi(a_i))1^{(e_i, \tilde{\alpha}_j)}1^k e_j m = 0$, and similarly for $f_j$. Therefore, the sum of eigenspaces is a submodule of $M$, and if $M$ is simple this sum must be $M$ itself. Thus, in (1.5), we may replace the power $d$ by 1 whenever $M$ is simple, and $\tilde{U}^0$ must act semisimply in this case. We also can see from (1.6) that for each simple $M$ there is a homomorphism $\chi$ so that all the weights of $M$ are of the form $\chi \cdot \zeta$, where $\zeta \in Q$.

It is shown in [BW, Prop. 3.5] that if $\hat{\zeta} = \hat{\eta}$, then $\zeta = \eta$ ($\zeta, \eta \in Q$) provided $rs^{-1}$ is not a root of unity. As a result, we have the following proposition.

**Proposition 1.7.** [BW, Cor. 3.14] Let $M$ be a finite-dimensional module for $U_{r,s}(\mathfrak{gl}_n)$ or for $U_{r,s}(\mathfrak{sl}_n)$. If $rs^{-1}$ is not a root of unity, then the elements $e_i, f_i$ ($1 \leq i < n$) act nilpotently on $M$.

When $rs^{-1}$ is not a root of unity, a finite-dimensional simple module $M$ is a highest weight module by Proposition 1.7 and (1.6). Thus there is some weight $\psi$ and a nonzero vector $v_0 \in M_\psi$ such that $e_j v_0 = 0$ for all $j = 1, \ldots, n - 1$, and $M = \tilde{U}. v_0$. It follows from the defining relations that $\tilde{U}$ has a triangular decomposition: $\tilde{U} = U^- U^0 U^+$, where $U^+$ (resp., $U^-$) is the subalgebra generated by the elements $e_i$ (resp., $f_i$). Applying this decomposition to $v_0$, we see that $M = \oplus_{\zeta \in Q^+} M_{\psi, (\zeta, \hat{\zeta})}$, where $Q^+ = \sum_{i=1}^{n-1} \mathbb{Z}_{\geq 0} \alpha_i$.

When all the weights of a module $M$ are of the form $\lambda$, where $\lambda \in \Lambda$, then for brevity we say that $M$ has weights in $\Lambda$. Rather than writing $M_\lambda$ for the weight space, we simplify the notation by writing $M_\lambda$. Note then (1.6) can be rewritten as $e_j M_\lambda \subseteq M_{\lambda + \alpha_j}$ and $f_j M_\lambda \subseteq M_{\lambda - \alpha_j}$. Any simple $\tilde{U}$-module having one weight in $\Lambda$ has all its weights in $\Lambda$.

Next we give an example of a simple $\tilde{U}$-module with weights in $\Lambda$, which is the analogue of the natural representation for $\mathfrak{gl}_n$.

**The natural representation for $U_{r,s}(\mathfrak{gl}_n)$ and $U_{r,s}(\mathfrak{sl}_n)$**.

Consider an $n$-dimensional vector space $V$ over $\mathbb{K}$ with basis $\{v_j \mid 1 \leq j \leq n\}$. We define an action of the generators of $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$ by specifying their matrices relative to this basis:

\begin{align*}
e_j &= E_{j, j+1} , & f_j &= E_{j+1, j} & (1 \leq j < n) \\
a_i &= r E_{i, i} + \sum_{k \neq i} E_{k, k} , & (1 \leq i \leq n) \\
b_i &= s E_{i, i} + \sum_{k \neq i} E_{k, k} & (1 \leq i \leq n).
\end{align*}
It follows that \( \omega_j = a_j b_{j+1} = r E_{j,j} + s E_{j+1,j+1} + \sum_{k \neq j,j+1} E_{k,k} \) and \( \omega'_j = a_{j+1} b_j = s E_{j,j} + r E_{j+1,j+1} + \sum_{k \neq j,j+1} E_{k,k} \). Now to verify that this extends to an action of \( \widetilde{U} \), (hence of \( U = U_{r,s}(\mathfrak{g}l_n) \)), we need to check that the relations hold. We present an illustrative example and leave the remainder to the reader:

\[
a_i e_j = (r E_{i,i} + \sum_{k \neq i} E_{k,k}) E_{j,j+1}
\]

\[
= \begin{cases} 
  r E_{j,j+1} & \text{if } j = i \\
  E_{j,j+1} & \text{if } j \neq i.
\end{cases}
\]

This can be seen to equal \( r^{(\varepsilon_i, \alpha_j)} E_{j,j+1} (r E_{i,i} + \sum_{k \neq i} E_{k,k}) \), which confirms that \( a_i e_j = r^{(\varepsilon_i, \alpha_j)} e_j a_i \) holds.

It follows from the fact that \( a_i v_j = r^{(\varepsilon_i, \varepsilon_j)} v_j \) and \( b_i v_j = s^{(\varepsilon_i, \varepsilon_j)} v_j \) for all \( i, j \) that \( v_j \) corresponds to the weight \( \varepsilon_j = \varepsilon_1 - (\alpha_1 + \cdots + \alpha_{j-1}) \). Thus, \( V = \bigoplus_{j=1}^n V_{\varepsilon_j} \) is the natural analogue of the \( n \)-dimensional representation of \( \mathfrak{gl}_n \) and \( \mathfrak{sl}_n \), and it is a simple module for both \( \widetilde{U} \) and \( U \). When \( r = q \) and \( s = q^{-1} \), \( b_i \) acts as \( a_i^{-1} \) on \( V \), and so \( V \) is a module for the quotient \( U_q(\mathfrak{g}l_n) \) of \( U_{q,q^{-1}}(\mathfrak{g}l_n) \) by the ideal generated by \( b_i - a_i^{-1} \) (\( 1 \leq i \leq n \)). This is the natural module for the one-parameter quantum group \( U_q(\mathfrak{g}l_n) \). A similar statement is true for \( U_q(\mathfrak{sl}_n) \).

\section{Classification of finite-dimensional simple modules}

Results will be stated for \( \widetilde{U} \)-modules, but everything holds as well for \( U \)-modules.

Let \( \widetilde{U}^0 \) denote the subalgebra of \( \widetilde{U} \) generated by \( a_i, b_i \) (\( 1 \leq i \leq n \)) and \( \varepsilon_i \) (\( 1 \leq i < n \)). Let \( \psi \) be any algebra homomorphism from \( \widetilde{U}^0 \) to \( \mathbb{K} \), and \( V^\psi \) be the one-dimensional \( \widetilde{U}^0 \)-module on which \( e_i \) acts as multiplication by 0 (\( 1 \leq i < n \)), and \( \widetilde{U}^0 \) acts via \( \psi \). We define the Verma module \( M(\psi) \) with highest weight \( \psi \) to be the \( \widetilde{U} \)-module induced from \( V^\psi \), that is

\[
M(\psi) = \widetilde{U} \otimes_{\widetilde{U}^0} V^\psi.
\]

Let \( v_\psi = 1 \otimes v \in M(\psi) \), where \( v \) is any nonzero vector of \( V^\psi \). Then \( e_i v_\psi = 0 \) (\( 1 \leq i < n \)) and \( a_i v_\psi = \chi(a) v_\psi \) for any \( a \in \widetilde{U}^0 \) by construction.

Notice that \( \widetilde{U}^0 \) acts semisimply on \( M(\psi) \) by relations (R2) and (R3). If \( N \) is a \( \widetilde{U} \)-submodule of \( M(\psi) \), then \( N \) is also a \( \widetilde{U}^0 \)-submodule of the \( \widetilde{U}^0 \)-module \( M(\psi) \), and so \( \widetilde{U}^0 \) acts semisimply on \( N \) as well. If \( N \) is a proper submodule, it must be that \( N \subseteq \sum_{\mu \in Q^+ \setminus \{0\}} M(\psi)_{\psi,-(\mu)} \) by (1.6), as \( M(\psi)_{\psi} = \mathbb{K} v_\psi \) generates \( M(\psi) \). Therefore \( M(\psi) \) has a unique maximal submodule, namely the sum of all proper submodules, and a unique simple quotient, \( L(\psi) \). In fact, all finite-dimensional simple \( \widetilde{U} \)-modules are of this form, as the following theorem demonstrates.
Theorem 2.1. Let \( \psi : \widetilde{U}^0 \rightarrow \mathbb{K} \) be an algebra homomorphism. Let \( M \) be a \( \widetilde{U} \)-module, on which \( \widetilde{U}^0 \) acts semisimply and which contains an element \( m \in M_\psi \) such that \( e_im = 0 \) for all \( i \) \((1 \leq i < n)\). Then there is a unique homomorphism of \( \widetilde{U} \)-modules \( F : M(\psi) \rightarrow M \) with \( F(v_\psi) = m \). In particular, if \( rs^{-1} \) is not a root of unity and \( M \) is a finite-dimensional simple \( \widetilde{U} \)-module, then \( M \cong L(\psi) \) for some weight \( \psi \).

Proof. By the hypothesis on \( m \), \( \mathbb{K}m \) is a one-dimensional \( \widetilde{U}^{-0} \)-submodule of \( M \), considered as a \( U^{-0} \)-module by restriction. In fact, mapping \( v_\psi \) to \( m \) yields a \( \widetilde{U}^{-0} \)-homomorphism from \( V_\psi \) to \( \mathbb{K}m \). By the definition of \( M(\psi) \), we have \( \text{Hom}_{\widetilde{U}}(M(\psi), M) \cong \text{Hom}_{\widetilde{U}^{-0}}(V_\psi, M) \), so there is a unique \( \widetilde{U} \)-module homomorphism \( F : M(\psi) \rightarrow M \) with \( F(v_\psi) = m \), namely \( F(u \otimes v) = u.m \) for all \( u \in \widetilde{U} \).

For the final assertion, note that \( \widetilde{U}^0 \) acts semisimply on any finite-dimensional simple module \( M \), and by (1.6) and Proposition 1.7, there is some nonzero vector \( m \in M_\psi \) such that \( e_im = 0 \) \((1 \leq i < n)\). By the first part, \( M \) is a quotient of \( M(\psi) \), and so \( M \cong L(\psi) \), as \( L(\psi) \) is the unique simple quotient of \( M(\psi) \). \( \square \)

As a special case, we will consider the modules \( L(\lambda) = L(\lambda) \) where \( \lambda \in \Lambda \). Let \( \Lambda^+ \subset \Lambda \) be the subset of dominant weights, that is

\[ \Lambda^+ = \{ \lambda \in \Lambda \mid \langle \alpha_i, \lambda \rangle \geq 0 \text{ for } 1 \leq i < n \} \]

We will show that if \( L(\lambda) \) is finite-dimensional, then \( \lambda \in \Lambda^+ \). This requires an identity for commuting \( e_i \) past powers of \( f_i \). For \( k \geq 1 \), let

\[ [k] = \frac{s^k - r^k}{r - s}. \]  

Lemma 2.3. If \( k \geq 1 \), then

\[
e_i f_i^k = f_i^k e_i + [k] f_i^{k-1} \frac{r^{1-k} \omega_i - s^{1-k} \omega_i'}{r - s} 
\]

\[
e_i^k f_i = f_i e_i^k + [k] e_i^{k-1} \frac{s^{1-k} \omega_i - r^{1-k} \omega_i'}{r - s}. 
\]

Proof. For \( k = 1 \), the above equations are just one of the defining relations of \( U \). Assume that \( k > 1 \) and

\[
e_i f_i^{k-1} = f_i^{k-1} e_i + [k - 1] f_i^{k-2} \frac{r^{2-k} \omega_i - s^{2-k} \omega_i'}{r - s}. 
\]

Then

\[
e_i f_i^k = \left( f_i^{k-1} e_i + [k - 1] f_i^{k-2} \frac{r^{2-k} \omega_i - s^{2-k} \omega_i'}{r - s} \right) f_i 
\]

\[
= f_i^{k-1} \left( f_i e_i + \frac{\omega_i - \omega_i'}{r - s} \right) + [k - 1] f_i^{k-1} \left( \frac{r^{1-k} s \omega_i - r s^{1-k} \omega_i'}{r - s} \right) 
\]

\[
= f_i^k e_i + \frac{f_i^{k-1}}{r - s} (1 + [k - 1] r^{1-k} s \omega_i - (1 + [k - 1] r s^{1-k}) \omega_i') 
\]

\[
= f_i^k e_i + \frac{f_i^{k-1}}{r - s} ([k] r^{1-k} \omega_i - [k] s^{1-k} \omega_i'). 
\]
The argument for the second equation can be done similarly. □

**Lemma 2.4.** Assume $rs^{-1}$ is not a root of unity. Let $M$ be a nonzero finite-dimensional $\tilde{U}$-module on which $\tilde{U}^0$ acts semisimply, and $\lambda \in \Lambda$. Suppose there is some nonzero vector $v \in M$ with $e_i.v = 0$ for all $i$ ($1 \leq i < n$). Then $\lambda \in \Lambda^+$.

**Proof.** Proposition 1.7 implies that for any given value of $i$ there is some $k \geq 0$ such that $f_i^{k+1}.v = 0$ and $f_i^kv \neq 0$. Applying $e_i$ to $f_i^{k+1}.v = 0$ and using Lemma 2.3 and the fact that $e_i.v = 0$, we have

$$0 = [k + 1]f_i^k r^{-k}\omega_i - s^{-k}\omega'_i \rightarrow 0,$$

Now $[k + 1]/(r - s) \neq 0$ as $rs^{-1}$ is not a root of unity. Therefore, since $f^k.v \neq 0$,

$$r^{-k}\hat{\lambda}(\omega_i) = s^{-k}\hat{\lambda}(\omega'_i).$$

Equivalently,

$$r^{-k}\rho(\epsilon_i, \lambda)s(\epsilon_{i+1}, \lambda) = s^{-k}\rho(\epsilon_{i+1}, \lambda)s(\epsilon_i, \lambda),$$

or

$$r^{-k}(\alpha_i, \lambda) = s^{-k}(\alpha_i, \lambda).$$

Again, because $rs^{-1}$ is not a root of unity, this forces $\langle \alpha_i, \lambda \rangle = k \geq 0$. Therefore $\lambda \in \Lambda^+$. □

**Corollary 2.5.** When $rs^{-1}$ is not a root of unity, any finite-dimensional simple $\tilde{U}$-module with weights in $\Lambda$ is isomorphic to $L(\lambda)$ for some $\lambda \in \Lambda^+$.

We will show next that all modules $L(\lambda)$ with $\lambda \in \Lambda^+$ are indeed finite-dimensional, and that all other finite-dimensional simple $\tilde{U}$-modules are shifts of these by one-dimensional modules. In doing this, it helps to consider first the special case of simple $U_{r,s}(\mathfrak{sl}_2)$-modules.

**Highest weight modules for** $U = U_{r,s}(\mathfrak{sl}_2)$. For simplicity we drop the subscripts and just write $e, f, \omega, \omega'$ for the generators of $U = U_{r,s}(\mathfrak{sl}_2)$. Any homomorphism $\phi : \tilde{U}^0 \rightarrow \mathbb{K}$ is determined by its values on $\omega$ and $\omega'$. By abuse of notation, we adopt the shorthand $\phi = \phi(\omega)$ and $\phi' = \phi(\omega')$.

Corresponding to each such $\phi$, there is a Verma module $M(\phi) = U \otimes_{U^{\geq 0}} \mathbb{K}v$ with basis $v_j = f^j \otimes v$ ($0 \leq j < \infty$) such that the $U$-action is given by:

$$v_j = f^j \otimes v \quad (0 \leq j < \infty)$$

$$e.v_j = [j] \phi r^{-j+1} - \phi' s^{-j+1} \quad (v_{-1} := 0)$$

$$\omega.v_j = \phi r^{-j}(\epsilon_1, \alpha_1) s^{-j}(\epsilon_2, \alpha_1) v_j = \phi r^{-j} s^j v_j$$

$$\omega'.v_j = \phi' r^{-j}(\epsilon_2, \alpha_1) s^{-j}(\epsilon_1, \alpha_1) v_j = \phi' r^j s^{-j} v_j.$$
Note that \(M(\phi)\) is a simple \(U\)-module if and only if \([j]\frac{\phi r^{-j+1} - \phi' s^{-j+1}}{r - s} \neq 0\) for any \(j \geq 1\).

Suppose \([\ell + 1]\frac{\phi r^{-\ell} - \phi' s^{-\ell}}{r - s} = 0\) for some \(\ell \geq 0\). Then either \(r^{\ell+1} = s^{\ell+1}\), which implies \(rs^{-1}\) is a root of unity, or \(\phi' = \phi r^{-\ell} s^{\ell}\). Assuming that \(rs^{-1}\) is not a root of unity and \(\phi' = \phi r^{-\ell} s^{\ell}\), we see that the elements \(v_i, i \geq \ell + 1\), span a maximal submodule. The quotient is the \((\ell + 1)\)-dimensional simple module \(L(\phi)\), which we can suppose is spanned by \(v_0, v_1, \ldots, v_\ell\) and has \(U\)-action given by

\[
\begin{align*}
f.v_j &= v_{j+1}, & (v_{\ell+1} = 0) \\
e.v_j &= \phi r^{-\ell}[j][\ell + 1 - j]v_{j-1} & (v_{-1} = 0) \\
\omega.v_j &= \phi r^{-j} s^j v_j \\
\omega'.v_j &= \phi r^{-\ell+j} s^{\ell-j} v_j.
\end{align*}
\]  

(2.7)

When \(M(\phi)\) is not simple and \(rs^{-1}\) is not a root of unity, \(j = \ell + 1\) is the unique value such that \([j]\frac{\phi r^{-j+1} - \phi' s^{-j+1}}{r - s} = 0\). In this case, \(M(\phi)\) has a unique proper submodule, namely the maximal submodule generated by \(v_{\ell+1}\) as above.

We now have the following classification of simple modules for \(U_{r,s}(\mathfrak{sl}_2)\).

**Proposition 2.8.**

(i) Assume \(U = U_{r,s}(\mathfrak{sl}_2)\), where \(rs^{-1}\) is not a root of unity. Let \(\phi : U^0 \to K\) be an algebra homomorphism such that \(\phi(\omega') = \phi(\omega)r^{-\ell} s^{\ell}\) for some \(\ell \geq 0\). Then there is an \((\ell + 1)\)-dimensional simple \(U\)-module \(L(\phi)\) spanned by vectors \(v_0, v_1, \ldots, v_\ell\) and having \(U\)-action given by (2.7). Any \((\ell + 1)\)-dimensional simple \(U\)-module is isomorphic to some such \(L(\phi)\).

(ii) If \(\nu = \nu_1 e_1 + \nu_2 e_2 \in \Lambda^+\), then \(\nu_1 - \nu_2 = \ell\) for some \(\ell \in \mathbb{Z}_{\geq 0}\), and \(\nu(\omega') = r^{\nu_2} s^{\nu_1} = r^{\nu_1 - \ell} s^{\nu_2 + \ell} = \nu(\omega)r^{-\ell} s^{\ell}\) in this case. Thus, the module \(L(\nu)\) is \((\ell + 1)\)-dimensional and has \(U\)-action given by (2.7) with \(\phi = r^{\nu_1} s^{\nu_2} = r^{\nu_1 - \ell} s^{\nu_2 + \ell}\).

**Finite-dimensionality of \(L(\lambda)\) for \(\lambda \in \Lambda^+\).**

We show below that the simple \(U\)-modules \(L(\lambda)\) with \(\lambda \in \Lambda^+\) are finite-dimensional. For this it suffices to prove that \(M(\lambda)\) has a \(U\)-submodule of finite codimension, as \(L(\lambda)\) is the quotient of \(M(\lambda)\) by its unique maximal submodule.

As \(\lambda\) is dominant, \(k_i = (\alpha_i, \lambda)\) for \(i = 1, \ldots, n - 1\), are nonnegative integers. Define a \(U\)-submodule \(M'(\lambda)\) of \(M(\lambda)\) by

\[
M'(\lambda) = \sum_{i=1}^{n-1} \tilde{U} f_i^{k_i+1} v_{\lambda}.
\]  

(2.9)

Our goal is to prove that the module \(L'(\lambda) = M(\lambda)/M'(\lambda)\) is nonzero and finite-dimensional.
By Lemma 2.3 we have
\[
e_i f_i^{k+1} v_\lambda = [k_i + 1] f_i^k \frac{r^{-k_i} \omega_i - s^{-k_i} \omega'_i}{r - s} v_\lambda
\]
\[
= [k_i + 1] f_i^k \frac{r^{-(\alpha_i, \lambda)} p(\epsilon_{i+1}, \lambda) s(\epsilon_{i+1}, \lambda) - s^{-\alpha_i} p(\epsilon_{i+1}, \lambda) s(\epsilon_{i+1}, \lambda)}{r - s} v_\lambda
\]
\[
= [k_i + 1] f_i^k \frac{r^{(\epsilon_{i+1}, \lambda)} s(\epsilon_{i+1}, \lambda) - s^{-\epsilon_{i+1}} p(\epsilon_{i+1}, \lambda)}{r - s} v_\lambda = 0.
\]

If \( j \neq i \), \( e_j f_i^{k+1} v_\lambda = f_i^{k+1} e_j v_\lambda = 0 \) by the defining relations. Consequently, by Theorem 2.1, \( U f_i^{k+1} v_\lambda \) is a homomorphic image of \( M(\lambda - (k_i + 1)\alpha_i) \), and so all its weights are less than or equal to \( \lambda - (k_i + 1)\alpha_i \). This implies that \( v_\lambda \notin M'(\lambda) \), hence \( L'(\lambda) \neq 0 \).

**Lemma 2.10.** The elements \( e_j, f_j \) (1 \( \leq j < n \)) act locally nilpotently on \( L'(\lambda) \).

**Proof.** As the Verma module \( M(\lambda) \) is spanned over \( \mathbb{K} \) by all elements \( x_1 \cdots x_t v_\lambda \) where \( x_1, \ldots, x_t \in \{ f_1, \ldots, f_n \} \), \( t \in \mathbb{Z}_{\geq 0} \), it is enough to argue by induction on \( t \) that a sufficiently high power of \( e_j \) (resp., \( f_j \)) takes such an element to \( M'(\lambda) \).

If \( t = 0 \), then \( e_j v_\lambda = 0 \in M'(\lambda) \), and \( f_j^{k_j+1} v_\lambda \in M'(\lambda) \) by construction. Now assume that there are positive integers \( N_j \) such that
\[
e_j^{N_j} x_2 \cdots x_t v_\lambda \in M'(\lambda) \quad \text{and} \quad f_j^{N_j} x_2 \cdots x_t v_\lambda \in M'(\lambda).
\]

Suppose that \( x_1 = f_t \). If \( j \neq i \), then
\[
e_j^{N_j} x_1 \cdots x_t v_\lambda = e_i e_j^{N_j} x_2 \cdots x_t v_\lambda \in M'(\lambda).
\]

Otherwise by Lemma 2.3,
\[
e_i^{N_i+1} x_1 \cdots x_t v_\lambda = f_j e_i^{N_i+1} x_2 \cdots x_t v_\lambda + [N_i + 1] e_i^{N_i} s^{-N_i} \omega_i - r^{-N_i} \omega'_i x_2 \cdots x_t v_\lambda.
\]

Applying relation (R2') and the induction hypothesis, we see that these terms are both in \( M'(\lambda) \).

Now \( f_j^{N_j} x_1 \cdots x_t v_\lambda = f_i^{N_i} x_2 \cdots x_t v_\lambda \in M'(\lambda) \), and if \( |i - j| > 1 \), we also have \( f_j^{N_j} x_1 \cdots x_t v_\lambda = f_i f_j^{N_j} x_2 \cdots x_t v_\lambda \in M'(\lambda) \). Finally, we need to show that if \( |i - j| = 1 \), then \( f_j^{N_j+1} x_1 \cdots x_t v_\lambda \in M'(\lambda) \). This will follow from the induction hypothesis once we know that \( f_j^{N_j+1} f_i \in \mathbb{K} f_j f_i f_j^{N_j} + \mathbb{K} f_i f_j^{N_j+1} \).

We argue by induction on \( m \geq 1 \) that
\[
f_j^{m+1} f_i \in \mathbb{K} f_j f_i f_j^{m} + \mathbb{K} f_i f_j^{m+1}.
\]

Indeed if \( m = 1 \), this follows from relation (R7), but if \( m > 1 \), then by induction and (R7),
\[
f_j^{m+1} f_i \in f_j(\mathbb{K} f_j f_i f_j^{m-1} + \mathbb{K} f_i f_j^{m}) \subseteq \mathbb{K} f_j f_i f_j^{m} + \mathbb{K} f_i f_j^{m+1}. \quad \square
\]
Lemma 2.11. Assume $rs^{-1}$ is not a root of unity, and let $V$ be a module for
$U = U_{r,s}(sl_2)$ on which $U^0$ acts semisimply. Assume $V = \oplus_{j \in \mathbb{Z}_{\geq 0}} V_{\lambda - j\alpha}$ for some
weight $\lambda \in \Lambda$, each weight space of $V$ is finite-dimensional, and $e$ and $f$ act locally
nilpotently on $V$. Then $V$ is finite-dimensional, and the weights of $V$ are preserved
under the simple reflection taking $\alpha$ to $-\alpha$.

Proof. Let $\mu = \mu_1 e_1 + \mu_2 e_2$ be a weight of $V$, and $v \in V_{\mu} \setminus \{0\}$. As $e$ acts locally
nilpotently on $V$, there is a nonnegative integer $k$ such that $e^{k+1}.v = 0$ and $e^k.v \neq 0$.
By Theorem 2.1, $Ue^k.v$ is a homomorphic image of $M(\mu + k\alpha)$. But since $f$ acts
locally nilpotently on $Ue^k.v$, this image cannot be isomorphic to $M(\mu + k\alpha)$. Thus
because $M(\mu + k\alpha)$ has a unique proper submodule, $Ue^k.v \cong L(\mu + k\alpha)$, and so it is
finite-dimensional. Corollary 2.5 implies that $\mu + k\alpha$ is dominant. As there are only
finitely many dominant weights less than or equal to the given weight $\lambda$, and each
weight space is finite-dimensional, it must be that $V$ itself is finite-dimensional.

In particular, $V$ has a composition series with factors isomorphic to $L(\nu)$ for some
$\nu \in \Lambda^+$. Any weight $\mu$ of $V$ is a weight of some such $L(\nu)$ with $\nu = \nu_1 e_1 + \nu_2 e_2 \in \Lambda^+$.
By (ii) of Proposition 2.8, $L(\nu)$ has weights $\nu, \nu - \alpha, \ldots, \nu - \ell \alpha$ where $\ell = \nu_1 - \nu_2$.
Thus, $\mu = \nu - j\alpha$ for some $j \in \{0, 1, \ldots, \ell\}$. But then $\mu - \langle \mu, \alpha \rangle \alpha = \nu - (\ell - j)\alpha$
is a weight of $L(\nu)$ since $\ell - j \in \{0, 1, \ldots, \ell\}$, hence it is a weight of $V$. Thus, the
weights of $V$ are preserved under the simple reflection taking $\alpha$ to $-\alpha$. $\square$

Lemma 2.12. Assume that $rs^{-1}$ is not a root of unity, and let $\lambda \in \Lambda^+$. Then
$L(\lambda)$ is finite-dimensional.

Proof. This follows once we show that $L'(\lambda) = M(\lambda)/M'(\lambda)$, where $M'(\lambda)$ is as
in (2.9), is finite-dimensional. We will prove that the set of weights of $L'(\lambda)$ is
preserved under the action of the symmetric group $S_n$ (the Weyl group of $gl_n$) on $\Lambda$
which is generated by the simple reflections $s_i : \mu \rightarrow \mu - \langle \mu, \alpha_i \rangle \alpha_i$ ($1 \leq i < n$). Each
$S_n$-orbit contains a dominant weight, and there are only finitely many dominant
weights less than or equal to $\lambda$. As the weights in $M(\lambda)$ are all less than or equal to
$\lambda$, and the weight spaces are finite-dimensional, the same is true of $L'(\lambda)$. Therefore
$L'(\lambda)$ is finite-dimensional.

To see that $s_i$ preserves the set of weights of $L'(\lambda)$, let $\mu = \mu_1 e_1 + \cdots + \mu_n e_n$ be a
weight of $L'(\lambda)$. Consider $L'(\lambda)$ as a module for the copy $U_i$ of $U_{r,s}(sl_2)$ generated
by $e_i, f_i, \omega_i, \omega_i^+$, and let $L'_i(\mu)$ be the $U_i$-submodule of $L'(\lambda)$ generated by $L'(\lambda)_\mu$.
As all weights of $L'(\lambda)_{\mu}$ are less than or equal to $\lambda$, we have

$$L'_i(\mu) = \bigoplus_{j \in \mathbb{Z}_{\geq 0}} L'_i(\mu)_{\lambda' - j\alpha_i}$$

for some weight $\lambda' \leq \lambda$. By Lemmas 2.10 and 2.11, the simple reflection $s_i$ preserves
the weights of $L'_i(\mu)$, so in particular, $s_i(\mu)$ is also a weight of $L'(\lambda)$. $\square$

Remark 2.13. It will follow from Lemma 3.7 in the next section that $L(\lambda) \cong L'(\lambda)$,
since $L(\lambda)$ is the unique simple quotient of $M(\lambda)$, $L'(\lambda)$ is a finite-dimensional
quotient of $M(\lambda)$, and by that lemma, every finite-dimensional quotient is simple.
Corollary 2.14. Assume that $rs^{-1} \neq 1$ is not a root of unity. The finite-dimensional simple $\widetilde{U}$-modules having weights in $\Lambda$ are precisely the modules $L(\lambda)$ where $\lambda \in \Lambda^+$. Moreover, $L(\lambda) \cong L(\mu)$ if and only if $\lambda = \mu$.

Proof. The first statement is a consequence of Corollary 2.5 and Lemma 2.12. Assume there is an isomorphism of $\widetilde{U}$-modules from $L(\lambda)$ to $L(\mu)$. The highest weight vector of $L(\lambda)$ must be sent to a weight vector of $L(\mu)$, so $\lambda \leq \mu$. As a similar argument shows that $\mu \leq \lambda$, we have $\lambda = \mu$. □

Shifts by one-dimensional modules.

Suppose now that we have a one-dimensional module $L$ for $\widetilde{U} = U_{r,s}(\mathfrak{gl}_n)$. Then by Theorem 2.1, $L = L(\chi)$ for some algebra homomorphism $\chi : \widetilde{U}^0 \to \mathbb{K}$, with the elements $e_i, f_i$ ($1 \leq i < n$) acting as multiplication by 0. Relation (R4) yields

\begin{equation}
(2.15) \quad \chi(\omega_i) = \chi(a_i b_{i+1}) = \chi(a_{i+1} b_i) = \chi(\omega'_i) \quad (1 \leq i < n).
\end{equation}

Conversely, if an algebra homomorphism $\chi$ satisfies this equation, then $L(\chi)$ is one-dimensional by relation (R4). We will write $L_\chi = L(\chi)$ to emphasize that the module is one-dimensional.

Proposition 2.16. Assume $rs^{-1}$ is not a root of unity and $L(\psi)$ is the finite-dimensional simple module for $\widetilde{U} = U_{r,s}(\mathfrak{gl}_n)$ with highest weight $\psi$. Then there exists a homomorphism $\chi : \widetilde{U}^0 \to \mathbb{K}$ such that (2.15) holds and an element $\lambda \in \Lambda^+$ so that $\lambda = \chi \cdot \hat{\lambda}$. Thus, the weights of $L(\psi)$ belong to $\chi \cdot \hat{\Lambda}$.

Proof. When $L(\psi)$ is viewed as a module for the copy $U_i$ of $U_{r,s}(\mathfrak{sl}_2)$ generated by $e_i, f_i, \omega_i, \omega'_i$, it has a composition series whose factors are simple $U_i$-modules as described by Proposition 2.8. As the highest weight vector of $L(\psi)$ gives a highest weight vector of some composition factor, there is a weight $\phi_i$ of $U_i$ and a nonnegative integer $\ell_i$ so that $\psi(\omega_i) = \phi_i(\omega_i)$ and $\psi(\omega'_i) = \phi_i(\omega'_i) = \phi_i(\omega_i)r^{-\ell_i} s^{\ell_i} = \tilde{\chi}(\omega_i)r^{-\ell_i} s^{\ell_i}$.

Set $\ell_n = 0$ and define $\lambda_i = \ell_i + \cdots + \ell_n$ for $i = 1, \ldots, n$. Let $\lambda = \sum_{i=1}^n \lambda_i e_i$, which belongs to $\Lambda^+$. Now we define $\chi : \widetilde{U}^0 \to \mathbb{K}$ by the formulas

\[
\chi(a_i) = \psi(a_i) r^{-(\epsilon_i, \lambda)} = \psi(a_i) r^{-(\ell_i, + \cdots + \ell_n)}, \quad \chi(b_i) = \psi(b_i) s^{-(\epsilon_i, \lambda)} = \psi(b_i) s^{-(\ell_i, + \cdots + \ell_n)}.
\]

Then it follows that

\[
\chi(\omega'_i) = \chi(a_{i+1} b_i) = \psi(\omega'_i) r^{-(\ell_{i+1}, + \cdots + \ell_n)} s^{-(\ell_i, + \cdots + \ell_n)} = \psi(\omega_i) r^{-\ell_i} s^\ell_s r^{-(\ell_{i+1}, + \cdots + \ell_n)} s^{-(\ell_i, + \cdots + \ell_n)} = \chi(a_i b_{i+1}) = \chi(\omega_i)
\]

for $i = 1, \ldots, n - 1$, and $\psi = \chi \cdot \hat{\lambda}$ as desired. □

Remark 2.17. If $M$ is any finite-dimensional module, then $M = \bigoplus_{i=1}^m \bigoplus_{\lambda \in \hat{\Lambda}} M_{\psi_i, \hat{\lambda}}$ for some weights $\psi_i$ such that $\psi_i : \hat{\Lambda}$ (1 \leq i \leq m) are distinct cosets in $\text{Hom}(\widetilde{U}^0, \mathbb{K})/\hat{\Lambda}$.
(viewed as a $\mathbb{Z}$-module under the action $k \cdot \psi = \psi^k$). Then $M_i := \bigoplus_{\lambda \in \Lambda} M_{\psi_i, \lambda}$ is a submodule, and $M = \bigoplus_{i=1}^m M_i$. Therefore, if $M$ is an indecomposable $\widetilde{U}$-module, $M = \bigoplus_{\lambda \in \Lambda} M_{\psi, \lambda}$ for some $\psi \in \text{Hom}(\widetilde{U}_0, \mathbb{K})$. A simple submodule $S$ of $M$ has weights in $\psi \cdot \hat{\Lambda}$. By replacing $\psi$ with the homomorphism $\chi$ for $S$ given by Proposition 2.16, we may assume that for any indecomposable module $M$, there is a $\chi$ satisfying (2.15) so that $M = \bigoplus_{\lambda \in \Lambda} M_{\chi, \lambda}$.

**Lemma 2.18.** Let $\chi : \widetilde{U}_0 \to \mathbb{K}$ be an algebra homomorphism with $\chi(\omega_i) = \chi(\omega'_i)$ ($1 \leq i < n$). Let $M$ be a finite-dimensional $\widetilde{U}$-module whose weights are all in $\chi \cdot \hat{\Lambda}$. If $\widetilde{U}_0$ acts semisimply on $M$, then

$$M \cong L_\chi \otimes N$$

for some $\widetilde{U}$-module $N$ whose weights are all in $\Lambda$.

**Proof.** Let $\chi^{-1} : \widetilde{U}_0 \to \mathbb{K}$ be the algebra homomorphism defined by $\chi^{-1}(a_i) = \chi(a_i^{-1}) = (\chi(a_i))^{-1}$ and $\chi^{-1}(b_i) = \chi(b_i^{-1}) = (\chi(b_i))^{-1}$ for $1 \leq i \leq n$. Note that $L_\chi \otimes L_{\chi^{-1}}$ is isomorphic to the trivial module $L_\varepsilon$ corresponding to the counit. Let

$$N = L_{\chi^{-1}} \otimes M.$$ 

Then $M \cong L_\chi \otimes N$ as $L_\varepsilon$ is a multiplicative identity (up to isomorphism) for $\widetilde{U}$-modules. The weights of $N$ are all in $\chi^{-1} \cdot \chi \cdot \hat{\Lambda} = \hat{\Lambda}$. □

We now have a classification of finite-dimensional simple $\widetilde{U}$-modules.

**Theorem 2.19.** Assume $rs^{-1}$ is not a root of unity. The finite-dimensional simple $\widetilde{U}$-modules are precisely the modules

$$L_\chi \otimes L(\lambda),$$

where $\chi : \widetilde{U}_0 \to \mathbb{K}$ is an algebra homomorphism with $\chi(\omega_i) = \chi(\omega'_i)$ ($1 \leq i < n$), and $\lambda \in \Lambda^+.$

**Proof.** Let $M$ be a finite-dimensional simple $\widetilde{U}$-module. By Theorem 2.1, Proposition 2.16, and Lemma 2.18, $M \cong L_\chi \otimes N$ for some $\chi$ satisfying (2.15) and some simple module $N$ with weights in $\Lambda$. By Corollary 2.5, $N \cong L(\lambda)$ for some $\lambda \in \Lambda^+$. Conversely, any $\widetilde{U}$-module of this form is finite-dimensional by Lemma 2.12 and simple by its construction. □

**Remark 2.20.** If $r = q$ and $s = q^{-1}$ for some $q \in \mathbb{K}$, the classification of finite-dimensional simple $U_q(\mathfrak{sl}_n)$-modules is a consequence of Theorem 2.19 applied to $U_{q, q^{-1}}(\mathfrak{sl}_n)$: The simple $U_q(\mathfrak{sl}_n)$-modules are precisely those simple $U_{q, q^{-1}}(\mathfrak{sl}_n)$-modules on which $\omega'_i$ acts as $\omega^{-1}_i$, so that

$$\chi(\omega_i) = \chi(\omega'_i) = \chi(\omega^{-1}_i).$$
This implies $\chi(\omega_i) = \pm 1$ ($1 \leq i < n$). Each choice of algebra homomorphism
$\chi : U^0 \to K$ with $\chi(\omega_i) = \chi(\omega'_i) = \pm 1$ yields a one-dimensional $U_{q,q^{-1}}(\mathfrak{sl}_n)$-module $L_\chi$, and so the simple $U_q(\mathfrak{sl}_n)$-modules are the $L_\chi \otimes L(\lambda)$ with $\lambda \in \Lambda^+$ and $\chi$ as above. (Compare with [Ja, §5.2, Convention 5.4, and Thm. 5.10].)

**Remark 2.21.** We can interpret Proposition 2.8 in light of Theorem 2.19: Let $L(\phi)$ be the simple $U_{r,s}(\mathfrak{sl}_2)$-module described in the proposition. Let $\lambda = \ell \epsilon_1 \in \Lambda^+$ and define $\chi : U^0 \to K$ by $\chi(\omega) = \phi(\omega)r^{-\ell}$, $\chi(\omega') = \phi(\omega')s^{-\ell} = \phi(\omega)r^{-\ell}s^\ell s^{-\ell} = \chi(\omega)$.

Then $\phi = \chi \cdot \lambda$ and $L(\phi) \cong L_\chi \otimes L(\lambda)$.

§3. Complete reducibility

In this section we will establish complete reducibility of all finite-dimensional
$\tilde{U}$-modules on which $\tilde{U}^0$ acts semisimply. However, it is helpful to work in a more
general context.

Let $O$ denote the category of modules $M$ for $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$ which satisfy the
conditions:

1. $\tilde{U}^0$ acts semisimply on $M$, and the set $\text{wt}(M)$ of weights of $M$ belongs to $\Lambda$:

   $M = \bigoplus_{\lambda \in \text{wt}(M)} M_\lambda$, where $M_\lambda = \{m \in M \mid a_i.m = r^{(\epsilon_i,\lambda)}, b_i.m = s^{(\epsilon_i,\lambda)} \}

   $\text{for all $i$}$;

2. $\dim_K M_\lambda < \infty$ for all $\lambda \in \text{wt}(M)$;

3. $\text{wt}(M) \subseteq \bigcup_{\mu \in F} (\mu - Q^+)$ for some finite set $F \subset \Lambda$.

The morphisms in $O$ are $\tilde{U}$-module homomorphisms.

All finite-dimensional $\tilde{U}$-modules which satisfy (1) belong to category $O$, as do
all highest weight modules with weights in $\Lambda$ such as the Verma modules $M(\lambda)$.

We recall the definition of the quantum Casimir operator [BW, Sec. 4]. It is a
consequence of (R2) and (R3) that the subalgebra $U^+$ of $\tilde{U}$ (or of $U = U_{r,s}(\mathfrak{sl}_n)$)
generated by $e_i$ ($1 \leq i < n$) has the decomposition $U^+ = \oplus_{\zeta \in Q} U^+_\zeta$ where

$$U^+_\zeta = \{z \in U^+ \mid a_{i}z = r^{(\epsilon_i,\zeta)}a_{i}, b_{i}z = s^{(\epsilon_i,\zeta)}b_{i} (1 \leq i < n)\}.$$

The weight space $U^+_\zeta$ is spanned by all the monomials $e_{i_1} \cdots e_{i_r}$ such that $\alpha_{i_1} + \cdots + \alpha_{i_r} = \zeta$. Similarly, the subalgebra $U^-$ generated by $1$ and the $f_i$ has the
decomposition $U^- = \oplus_{\zeta \in Q} U^-_{-\zeta}$. The spaces $U^+_\zeta$ and $U^-_{-\zeta}$ are nondegenerately
paired by the Hopf pairing defined by

$$\langle f_i, e_j \rangle = \frac{\delta_{i,j}}{s-r}$$

$$\langle \omega'_i, \omega_j \rangle = r^{(\epsilon_i,\alpha_i)}s^{(\epsilon_j,\alpha_i)}$$

$\langle b_n, a_n \rangle = 1$, $\langle b_n, \omega_j \rangle = s^{-(\epsilon_n,\alpha_j)}$, $\langle \omega'_i, a_n \rangle = r^{(\epsilon_n,\alpha_i)}$.

(See [BW, Sec. 2].) The Hopf algebras $\tilde{U}$ and $U$ are Drinfel’d doubles of certain
Hopf subalgebras with respect to this pairing [BW, Thm. 2.7]. Let $d_\zeta = \dim_K U^+_\zeta$. 
Assume \( \{ u_\zeta^d \}_{\zeta, k=1} \) is a basis for \( U_+^\zeta \), and \( \{ v_\zeta^d \}_{\zeta, k=1} \) is the dual basis for \( U_-\zeta \) with respect to the pairing.

Now let

\[
\Omega = \sum_{\zeta \in Q^+} \sum_{k=1}^{d_\zeta} S(v_\zeta^k) u_\zeta^k,
\]

where \( S \) denotes the antipode. All but finitely many terms in this sum will act as multiplication by 0 on any weight space \( M_\lambda \) of \( M \in \mathcal{O} \). Therefore \( \Omega \) is a well-defined operator on such \( M \).

The second part of the Casimir operator involves a function \( g : \Lambda \to \mathbb{K}^\# \) defined as follows. If \( \rho \) denotes the half sum of the positive roots, then \( 2\rho = \sum_{j=1}^n (n+1-2j)\epsilon_j \in \Lambda \). For \( \lambda \in \Lambda \), set

\[
g(\lambda) = (rs^{-1})^{\frac{1}{2}(\lambda+2\rho, \lambda)}.
\]

When \( M \) is a \( \tilde{U} \)-module in \( \mathcal{O} \), we define the linear operator \( \Xi : M \to M \) by

\[
\Xi(m) = g(\lambda)m
\]

for all \( m \in M_\lambda, \lambda \in \Lambda \). Then we have the following result from [BW].

**Proposition 3.4.** [BW, Thm. 4.20] The operator \( \Omega \Xi : M \to M \) commutes with the action of \( \tilde{U} \) on any \( \tilde{U} \)-module \( M \in \mathcal{O} \).

We require the next lemma in order to prove complete reducibility.

**Lemma 3.5.** Assume \( rs^{-1} \) is not a root of unity, and let \( \lambda, \mu \in \Lambda^+ \). If \( \lambda \geq \mu \) and \( g(\lambda) = g(\mu) \), then \( \lambda = \mu \).

**Proof.** Because \( \lambda \geq \mu \), we may suppose \( \lambda = \mu + \beta \) where \( \beta = \sum_{i=1}^{n-1} k_i\alpha_i \) and \( k_i \in \mathbb{Z}_{\geq 0} \). By assumption we have

\[
(rs^{-1})^{\frac{1}{2}(\lambda+2\rho, \lambda)} = g(\lambda) = g(\mu) = (rs^{-1})^{\frac{1}{2}(\mu+2\rho, \mu)},
\]

and as \( rs^{-1} \) is not a root of unity, it must be that \( \langle \lambda + 2\rho, \lambda \rangle = \langle \mu + 2\rho, \mu \rangle \), or equivalently, \( 2\langle \mu + \rho, \beta \rangle + \langle \beta, \beta \rangle = 0 \). Since \( \mu \in \Lambda^+ \), \( \mu = \mu_1\epsilon_1 + \mu_2\epsilon_2 + \cdots + \mu_n\epsilon_n \) where \( \mu_i \in \mathbb{Z} \) for all \( i \) and \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \). Then

\[
0 = 2\mu_2 + 2\rho, \beta \rangle + \langle \beta, \beta \rangle \]

\[
= \sum_{i=1}^{n-1} k_i \left( 2\mu_i + (n+1-2i) - 2\mu_{i+1} - (n+1-2(i+1)) \right) + \sum_{i=1}^{n} (k_i - k_{i-1})^2 \quad (k_0 = 0 = k_n)
\]

\[
= \sum_{i=1}^{n-1} 2k_i (\mu_i - \mu_{i+1} + 1) + \sum_{i=1}^{n} (k_i - k_{i-1})^2.
\]

The only way this can happen is if \( k_i = 0 \) for all \( i \) and \( \lambda = \mu \). \( \square \)
Lemma 3.6. Assume that $rs^{-1}$ is not a root of unity.

(i) $\Omega \Xi$ acts as multiplication by the scalar $g(\lambda) = (rs^{-1})^{\frac{1}{2}(\lambda + 2\rho, \lambda)}$ on the Verma module $M(\lambda)$ with $\lambda \in \Lambda$, hence on any submodule or quotient of $M(\lambda)$.

(ii) The eigenvalues of the operator $\Omega \Xi : M \to M$ are integral powers of $(rs^{-1})^{\frac{1}{2}}$ on any finite-dimensional $M \in \mathcal{O}$.

Proof. By its construction, $\Omega \Xi$ acts by multiplication by $g(\lambda) = (rs^{-1})^{\frac{1}{2}(\lambda + 2\rho, \lambda)}$ on the maximal vector $v_\lambda$ of $M(\lambda)$. But since $M(\lambda) = \tilde{U}.v_\lambda$ and $\Omega \Xi$ commutes with $\tilde{U}$ on modules in $\mathcal{O}$, $\Omega \Xi$ acts as multiplication by $(rs^{-1})^{\frac{1}{2}(\lambda + 2\rho, \lambda)}$ on all of $M(\lambda)$.

If $M \in \mathcal{O}$ is finite-dimensional, it has a composition series. Each factor is a finite-dimensional simple $\tilde{U}$-module with weights in $\Lambda$, and in particular, is a quotient of $M(\lambda)$ for some $\lambda \in \Lambda$. On such a factor, $\Omega \Xi$ acts as multiplication by $g(\lambda)$. Therefore the action of $\Omega \Xi$ on $M$ may be expressed by an upper triangular matrix with each diagonal entry equal to $g(\lambda)$ for some $\lambda \in \Lambda$. \qed

Lemma 3.7. Assume $rs^{-1}$ is not a root of unity. Let $\lambda \in \Lambda$ and $M$ be a nonzero finite-dimensional quotient of the Verma module $M(\lambda)$. Then $M$ is simple.

Proof. First observe that by Lemma 2.4, $\lambda \in \Lambda^+$. Assume $M'$ is a proper submodule of $M$. As $M$ is generated by its one-dimensional subspace $M_\lambda$, we must have $M'_\lambda = 0$. Let $\mu \in \Lambda$ be maximal such that $M'_\mu \neq 0$, and note that $\mu < \lambda$. Let $m'$ be a nonzero vector of $M'_\mu$. By maximality of $\mu$, we have $e_i.m' = 0$ for all $i$ ($1 \leq i < n$). Letting $M'' = \tilde{U}.m'$, a nonzero finite-dimensional quotient of $M(\mu)$, we see that $\mu \in \Lambda^+$ as well. By Lemma 3.6 (i), $\Omega \Xi$ acts as multiplication by $g(\lambda)$ on $M$, and by $g(\mu)$ on $M''$. This forces $g(\lambda) = g(\mu)$, which contradicts Lemma 3.5 as $\mu < \lambda$. \qed

Theorem 3.8. Assume $rs^{-1}$ is not a root of unity. Let $M$ be a nonzero finite-dimensional $\tilde{U}$-module on which $\tilde{U}^0$ acts semisimply. Then $M$ is completely reducible.

Proof. We will establish the result first in the case $M$ has weights in $\Lambda$. Write $M$ as a direct sum of generalized eigenspaces for $\Omega \Xi$. Note that by Proposition 3.4, this is a direct sum decomposition of $M$ as a $\tilde{U}$-module. Therefore we may assume $M$ is itself a generalized eigenspace of $\Omega \Xi$, so that $(\Omega \Xi - (rs^{-1})^c)^d(M) = 0$ for some $c \in \frac{1}{2}\mathbb{Z}$, $d = \dim_{K^\mu} M$, by Lemma 3.6 (ii).

Let $P = \{m \in M \mid e_i.m = 0 \ (1 \leq i < n)\}$, and note that $P = \oplus_{\lambda \in \Lambda} P_\lambda$, $P_\lambda = P \cap M_\lambda$. If $m \in P_\lambda - \{0\}$, the $\tilde{U}$-submodule $\tilde{U}.m$ of $M$ is a nonzero quotient of $M(\lambda)$ by Theorem 2.1. By Lemma 3.7, each such $\tilde{U}.m$ is a simple $\tilde{U}$-module, and so the $\tilde{U}$-submodule $M'$ of $M$ generated by $P$ is a sum of simple $\tilde{U}$-modules. That is, $M'$ is completely reducible. Let $M'' = M/M'$.

Assuming $M'' \neq 0$, there is a weight $\mu \in \Lambda$ maximal such that $M'_\mu \neq 0$. Let $m'' \in M'_\mu - \{0\}$. By maximality of $\mu$, we have $e_i.m'' = 0$ for all $i$ ($1 \leq i < n$). By Lemma 2.4, we have $\mu \in \Lambda^+$, and by Theorem 2.1 and Lemma 3.6, $\Omega \Xi$ acts
as multiplication by \( g(\mu) \) on the \( U \)-module \( U.m'' \) generated by \( m'' \). This implies \( g(\mu) = (rs^{-1})^c \).

Let \( m \in M_\mu \) be a representative for \( m'' \in (M/M')_\mu \), and \( M_1 = \tilde{U}.m \). Then the module \( M_1 \) is a direct sum of its intersections with the weight spaces of \( M \), so there is a weight \( \eta \in \Lambda \) maximal such that \( M_1 \cap M_\eta \neq 0 \). Let \( m_1 \in M_1 \cap M_\eta - \{0\} \), so that \( e_i.m_1 = 0 \) for all \( i \) \( (1 \leq i < n) \). Again applying Theorem 2.1 and Lemmas 2.4 and 3.6, we have \( \eta \in \Lambda^+ \) and \( \Omega\Xi(m_1) = g(\eta)m_1 \). Therefore \( g(\eta) = (rs^{-1})^c \).

We now have \( g(\mu) = g(\eta) \), where \( \eta, \mu \in \Lambda^+ \), and \( \eta \geq \mu \) by construction. By Lemma 3.5, \( \eta = \mu \), so \( M_1 \) is the one-dimensional space spanned by \( m \), and \( e_i.m = 0 \) \( (1 \leq i < n) \), that is \( m \in P \). This implies \( m'' = 0 \), a contradiction to the assumption that \( M'' \neq 0 \). Therefore \( M'' = 0 \), and \( M = M' \) is completely reducible.

Finally, we consider the case that \( M \) does not have weights in \( \Lambda \). We may assume that \( M \) is indecomposable. By Remark 2.17, \( M \) has all its weights in \( \chi \cdot \hat{\Lambda} \) for some \( \chi \) satisfying (2.15). By Lemma 2.18, \( M \cong L_\chi \otimes N \) for some \( \tilde{U} \)-module \( N \) whose weights are all in \( \Lambda \). Note that \( \tilde{U}^{0} \) acts semisimply on \( N \) as well \( (N = L_{\chi^{-1}} \otimes M) \), and so \( N \) is completely reducible by the above argument. As the tensor product of modules distributes over direct sums, \( M \) is itself completely reducible. □

**Remark 3.9.** It is necessary to include the hypothesis that \( \tilde{U}^{0} \) acts semisimply in Theorem 3.8, as the next examples illustrate. (Recall that \( \tilde{U}^{0} \) does indeed act semisimply on any simple \( \tilde{U} \)-module, as remarked in the text following (1.6).) Let \( V = \mathbb{K}^m \) for \( m \geq 2 \) and \( \xi, \xi' \in \mathbb{K} \setminus \{0\} \). We define a \( \tilde{U} \)-module structure on \( V \) by requiring that \( e_i, f_i \) act as multiplication by \( 0 \) and \( a_i, b_i \) act via the same matrix. The remaining relations hold as \( e_i, f_i \) act as multiplication by \( 0 \). The scalars \( \xi, \xi' \) may be chosen so that \( V \) has weights in \( \Lambda \), for example choose an integer \( c \), let \( \lambda = c(\epsilon_1 + \cdots + \epsilon_n) \), and set \( \xi = r^c = \lambda(a_i) \), \( \xi' = s^c = \lambda(b_i) \). Clearly \( V \) is not completely reducible as the Jordan blocks are not diagonalizable. A related example for \( U_{r,s}(\mathfrak{sl}_n) \) is given by sending \( \omega_i, \omega'_i \) to the same Jordan block with diagonal entries \( \xi_i \in \mathbb{K} - \{0\} \) \( (1 \leq i < n) \).

§4. The \( R \)-matrix

In this section we recall the definition of the \( R \)-matrix from [BW, Sec. 4] and use it to prove a more general result on commutativity of the tensor product of finite-dimensional modules than was given there (compare [BW, Thm. 4.11] with Theorem 4.2 below). Let \( M, M' \) be \( \tilde{U} \)-modules in category \( \mathcal{O} \). We define an isomorphism of \( \tilde{U} \)-modules \( R_{M', M} : M' \otimes M \to M \otimes M' \) as follows. If \( \lambda = \sum_{i=1}^{n} \lambda_i \alpha_i \in \Lambda \), where \( \alpha_n = \epsilon_n \), set

\[
\omega_\lambda = \omega_1^{\lambda_1} \cdots \omega_{n-1}^{\lambda_{n-1}} a_n^{\lambda_n}
\]

\[
\omega'_\lambda = (\omega'_1)^{\lambda_1} \cdots (\omega'_{n-1})^{\lambda_{n-1}} b_n^{\lambda_n}.
\]

Also let

\[
\Theta = \sum_{\zeta \in \mathcal{Q}^+} \sum_{k=1}^{d_\zeta} u_k^\zeta \otimes u_k^\zeta,
\]
where the notation is as in the paragraph following (3.1). Define
\[ R_{M',M} = \Theta \circ \tilde{f} \circ P \]
where \( P(m' \otimes m) = m \otimes m' \), \( \tilde{f}(m \otimes m') = (\omega'_\mu, \omega_\lambda)^{-1}(m \otimes m') \) when \( m \in M_\lambda \) and \( m' \in M'_\mu \), and the Hopf pairing \( (\ , \ ) \) is defined in (3.1). Then \( R_{M',M} \) is an isomorphism of \( \tilde{U} \)-modules that satisfies the quantum Yang-Baxter equation and the hexagon identities [BW, Thms. 4.11, 5.4, and 5.7].

We will show that the tensor product of any two finite-dimensional \( \tilde{U} \)-modules in \( \mathcal{O} \) is commutative (up to module isomorphism). We first prove this in the special case that one of the modules is a one-dimensional module \( L_\chi = L(\chi) \), as defined in Section 2.

**Lemma 4.1.** Let \( M \) be a \( \tilde{U} \)-module in category \( \mathcal{O} \), and let \( L_\chi \) be a one-dimensional \( \tilde{U} \)-module. Then
\[ L_\chi \otimes M \cong M \otimes L_\chi. \]

*Proof.* Fix a basis element \( v \) of \( L_\chi \). Define a linear function \( F : L_\chi \otimes M \to M \otimes L_\chi \) as follows. If \( m \in M_\lambda \), where \( \lambda = -\sum_{i=1}^n c_i \alpha_i \), then
\[ F(v \otimes m) = \chi_1^{c_1} \cdots \chi_n^{c_n} m \otimes v, \]
where \( \chi_i = \chi(\omega_i) = \chi(\omega'_i) \) (1 \( \leq i \leq n \)) and \( \chi_n = \chi(a_n) \). Clearly \( F \) is bijective, and we check that \( F \) is a \( \tilde{U} \)-homomorphism:
\[ e_i F(v \otimes m) = \chi_1^{c_1} \cdots \chi_n^{c_n} (e_i \otimes 1 + \omega_i \otimes e_i)(m \otimes v) \]
\[ = \chi_1^{c_1} \cdots \chi_n^{c_n} e_i m \otimes v. \]

On the other hand, as \( e_i m \in M_{\lambda + \alpha_i} \), we have
\[ F(e_i.(v \otimes m)) = F((e_i \otimes 1 + \omega_i \otimes e_i)(v \otimes m)) \]
\[ = \chi_i F(v \otimes e_i m) \]
\[ = \chi_i (\chi_1^{c_1} \cdots \chi_i^{c_i-1} \cdots \chi_n^{c_n}) e_i m \otimes v \]
\[ = e_i F(v \otimes m). \]
Similarly, \( F \) commutes with \( f_i \). As the action by \( a_i, b_i \) preserves the weight spaces, \( F \) commutes with \( a_i, b_i \) (1 \( \leq i \leq n \)) as well. Therefore \( F \) is an isomorphism of \( \tilde{U} \)-modules. \( \square \)

**Theorem 4.2.** Let \( M \) and \( M' \) be finite-dimensional \( \tilde{U} \)-modules with \( \tilde{U}^0 \) acting semisimply. Then
\[ M \otimes M' \cong M' \otimes M. \]

*Proof.* As the tensor product distributes over direct sums, we may assume that \( M \) and \( M' \) are indecomposable. Therefore the weights of \( M \) are all in \( \chi \cdot \hat{\Lambda} \) for some algebra homomorphism \( \chi : \tilde{U}^0 \to \mathbb{K} \) with \( \chi(\omega_i) = \chi(\omega'_i) \). (See Remark 2.17.) By Lemma 2.18, \( M \cong L_\chi \otimes N \) for some module \( N \) with weights in \( \Lambda \). Similarly, \( M' \cong L'_{\chi'} \otimes N' \) for some \( \chi' \). By Lemma 4.1 and [BW, Thm. 4.11],
\[ M \otimes M' \cong (L_\chi \otimes N) \otimes (L_{\chi'} \otimes N') \cong (L_\chi \otimes L_{\chi'}) \otimes (N \otimes N') \]
\[ \cong (L_{\chi'} \otimes L_\chi) \otimes (N' \otimes N) \]
\[ \cong (L_{\chi'} \otimes N') \otimes (L_\chi \otimes N) \cong M' \otimes M. \quad \square \]
§5. Tensor powers of the natural module

In this section we consider tensor powers $V^\otimes k = V \otimes V \otimes \cdots \otimes V$ ($k$ factors) of the
natural module $V$ for $\bar{U}$ (defined in Section 1). Set $R = R_{V,V}$, and for $1 \leq i < k,$
let $R_i$ be the $\bar{U}$-module isomorphism on $V^\otimes k$ defined by

$$R_i(z_1 \otimes z_2 \otimes \cdots \otimes z_k) = z_1 \otimes \cdots \otimes z_{i-1} \otimes R(z_i \otimes z_{i+1}) \otimes z_{i+2} \otimes \cdots \otimes z_k.$$ 

Then it is a consequence of the quantum Yang-Baxter equation that the braid relations

$$R_i \circ R_{i+1} \circ R_i = R_{i+1} \circ R_i \circ R_{i+1} \quad \text{for} \quad 1 \leq i < k$$

$$R_i \circ R_j = R_j \circ R_i \quad \text{for} \quad |i-j| \geq 2$$

hold.

We would like to argue that

$$R_i^2 = (1 - rs^{-1}) R_i + rs^{-1} \text{Id}$$

for all $i = 1, \ldots, k - 1$. For this it suffices to work with the 2-fold tensor product $V \otimes V$.

Proposition 5.3. Whenever $s \neq -r$, the module $V \otimes V$ decomposes into two
simple submodules, $S^2_{r,s}(V)$ (the $(r,s)$-symmetric tensors) and $\Lambda^2_{r,s}(V)$ (the $(r,s)$-
antisymmetric tensors). These modules are defined as follows:

(i) $S^2_{r,s}(V)$ is the span of $\{v_i \otimes v_j | 1 \leq i \leq n\} \cup \{v_i \otimes v_j + sv_j \otimes v_i | 1 \leq i < j \leq n\}$.

(ii) $\Lambda^2_{r,s}(V)$ is the span of $\{v_i \otimes v_j - rv_j \otimes v_i | 1 \leq i < j \leq n\}$.

Proof. The following computations can be used to see that $S^2_{r,s}(V)$ and $\Lambda^2_{r,s}(V)$ are
submodules:

$$e_k.(v_i \otimes v_i) = \delta_{i,k+1}(v_k \otimes v_{k+1} + sv_{k+1} \otimes v_k)$$

$$f_k.(v_i \otimes v_i) = \delta_{i,k}(v_k \otimes v_{k+1} + sv_{k+1} \otimes v_k)$$

$$e_k.(v_i \otimes v_j + sv_j \otimes v_i) = \begin{cases} 
0 & \text{if } k+1 \neq i,j \\
(v_k \otimes v_j + sv_j \otimes v_k) & \text{if } k+1 = i \\
v_i \otimes v_k + sv_k \otimes v_i & \text{if } k+1 = j, k \neq i \\
(r+s)v_k \otimes v_k & \text{if } k+1 = j, k = i
\end{cases}$$

$$f_k.(v_i \otimes v_j + sv_j \otimes v_i) = \begin{cases} 
0 & \text{if } k \neq i,j \\
v_i \otimes v_{k+1} + sv_{k+1} \otimes v_i & \text{if } k = j \\
v_{k+1} \otimes v_j + sv_j \otimes v_{k+1} & \text{if } k = i, k+1 \neq j \\
(r+s)v_{k+1} \otimes v_{k+1} & \text{if } k = i, k+1 = j
\end{cases}$$
Proposition 5.5. The minimum polynomial of $v$ spans a one-dimensional module. Modulo the resulting two-dimensional module, $V$ has one vector that is not complemented in $V$. Proof. $v$ is a highest weight vector, and it is easy to see that given any other vector in (i), there is an element of $U$ taking it to $v_1 \otimes v_1$. Therefore $S^2_{r,s}(V)$ is simple. A similar argument proves that $\Lambda^2_{r,s}(V)$ is simple, with highest weight vector $v_1 \otimes v_2 - rv_2 \otimes v_1$. □

Remark 5.4. The $s = -r$ case is “nongeneric,” and in this exceptional case, $V \otimes V$ need not be completely reducible. For example, when $n = 2$ what happens is that $v_1 \otimes v_2 - rv_2 \otimes v_1$ spans a one-dimensional module (as it does for $n = 2$ generic) that is not complemented in $V \otimes V$. Modulo that submodule, $v_1 \otimes v_1$ spans a one-dimensional module. Modulo the resulting two-dimensional module, $v_1 \otimes v_2 + rv_2 \otimes v_1$ and $v_2 \otimes v_2$ span a two-dimensional module.

Proposition 5.5. The minimum polynomial of $R = R_{V,V}$ on $V \otimes V$ is $(t - 1)(t + rs^{-1})$ if $s \neq -r$.

Proof. It follows from the definition of $R$ that $R(v_1 \otimes v_1) = v_1 \otimes v_1$ and $R(v_1 \otimes v_2 - rv_2 \otimes v_1) = -rs^{-1}(v_1 \otimes v_2 - rv_2 \otimes v_1)$. By Proposition 5.3, $S^2_{r,s}(V)$ and $\Lambda^2_{r,s}(V)$ are simple, and in fact, $v_1 \otimes v_1$ and $v_1 \otimes v_2 - rv_2 \otimes v_1$ are the highest weight vectors. In particular, each is a cyclic module generated by its highest weight vector. As $Ra(v_1 \otimes v_1) = aR(v_1 \otimes v_1) = a(v_1 \otimes v_1)$ for all $a \in U$, this implies that $S^2_{r,s}(V)$ is in the eigenspace of $R$ corresponding to eigenvalue 1. Analogously, $\Lambda^2_{r,s}(V)$ corresponds to the eigenvalue $-rs^{-1}$, and since $V \otimes V$ is the direct sum of those submodules, we have the desired result. □

From Proposition 5.5 it follows that $R$ acts as

$$(5.6) \quad r \sum_{i<j} E_{j,i} \otimes E_{i,j} + s^{-1} \sum_{i<j} E_{i,j} \otimes E_{j,i} + (1 - rs^{-1}) \sum_{i<j} E_{j,i} \otimes E_{i,j} + \sum_i E_{i,i} \otimes E_{i,i}$$

on $V \otimes V$. Indeed, (5.6) is a linear operator that acts on $S^2_{r,s}(V)$ as multiplication by 1, and on $\Lambda^2_{r,s}(V)$ as multiplication by $-rs^{-1}$. By Proposition 5.5, $R$ has the same properties, and so $R$ is equal to this sum on $V \otimes V$. 

§6. QUANTUM SCHUR-WEYL DUALITY

Assume \(r, s \in \mathbb{K}\). Let \(H_k(r, s)\) be the unital associative algebra over \(\mathbb{K}\) with generators \(T_i\), \(1 \leq i < k\), subject to the relations:

(H1) \(T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}\), \(1 \leq i < k\)
(H2) \(T_iT_j = T_jT_i\), \(\mid i - j \mid \geq 2\)
(H3) \(T_i^2 = (s-r)T_i + rs1\).

When \(r \neq 0\), the elements \(t_i = r^{-1}T_i\) satisfy the braid relations (H1), (H2), along with the relation

(H3') \(t_i^2 = (q-1)t_i + q1\),

where \(q = r^{-1}s\). The Hecke algebra \(H_k(q)\) (of type \(A_{k-1}\)) is generated by elements \(t_i\), \(1 \leq i < k\), which satisfy (H1), (H2), (H3'). It has dimension \(k\) and is semisimple whenever \(q\) is not a root of unity. At \(q = 1\), the Hecke algebra \(H_k(q)\) is isomorphic to \(\mathbb{K}S_k\), the group algebra of the symmetric group \(S_k\), where we may identify \(t_i\) with the transposition \((i \ i + 1)\).

The two-parameter Hecke algebra \(H_k(r, s)\) defined above is isomorphic to \(H_k(r^{-1}s)\) whenever \(r \neq 0\). Thus, it is semisimple whenever \(r^{-1}s\) is not a root of unity. For any \(\sigma \in S_k\), we may define \(T_\sigma = T_{i_1} \cdots T_{i_{\ell}}\) where \(\sigma = t_{i_1} \cdots t_{i_{\ell}}\) is a reduced expression for \(\sigma\) in terms of transpositions. It follows from (H1) and (H2) that \(T_\sigma\) is independent of the reduced expression and these elements give a basis.

The results of Section 5 show that the \(\tilde{U}\)-module \(V^{\otimes k}\) affords a representation of the Hecke algebra \(H_k(r, s)\):

\[
H_k(r, s) \to \text{End}_{\tilde{U}}(V^{\otimes k})
\]

\[
T_i \mapsto sR_i \quad (1 \leq i < k).
\]

When \(k = 2\) and \(s \neq -r\), \(V^{\otimes 2} = S^2_{r,s}(V) \oplus \wedge^2_{r,s}(V)\) is a decomposition of \(V^{\otimes 2}\) into simple \(\tilde{U}\)-modules by Proposition 5.3. The maps \(p_1 = (sR_1 + r)/(s + r)\) and \(p_2 = (s - sR_1)/(s + r)\), \((R_1 = R_{V\wedge V})\), are the corresponding projections onto the simple summands. Thus, the map in (6.1) is an isomorphism for \(k = 2\). More generally, we will show next that it is surjective whenever \(rs^{-1}\) is not a root of unity, and it is an isomorphism when \(n \geq k\). This is the two-parameter version of the well-known result of Jimbo [Ji] that \(H_k(q) \cong \text{End}_{U_{gl_n}}(V^{\otimes k})\) and is the analogue of classical Schur-Weyl duality, \(\mathbb{K}S_k \cong \text{End}_{gl_n}(V^{\otimes k})\) for \(n \geq k\). It requires the following lemma. The case \(n < k\) is dealt with separately, and uses the isomorphism \(H_k(r, s) \cong \text{End}_{\tilde{U}}(V^{\otimes k})\) of the case \(n = k\).

Lemma 6.2. If \(n \geq k\) and \(V\) is the natural representation of \(\tilde{U}\), then \(V^{\otimes k}\) is a cyclic \(\tilde{U}\)-module generated by \(v_1 \otimes \cdots \otimes v_k\).

Proof. Let \(\bar{v} = v_1 \otimes \cdots \otimes v_k\). We begin by showing that if \(\sigma \in S_k\), then \(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \in \tilde{U} \cdot \bar{v}\).
Suppose we have an arbitrary permutation $x_1 \otimes \cdots \otimes x_k$ ($x_i \in \{v_1, \ldots, v_k\}$ for all $i$) of the factors of $\omega_i$. For some $\ell < m$, assume that $x_\ell = v_j$ and $x_m = v_{j+1}$. Then because of the formulas

$$
\Delta^{k-1}(e_j) = \sum_{i=1}^{k} \omega_j \otimes \cdots \otimes \omega_{j-1} e_j \otimes 1 \otimes \cdots \otimes 1
$$

(6.3)

$$
\Delta^{k-1}(f_j) = \sum_{i=1}^{k} 1 \otimes \cdots \otimes 1 \otimes f_j \otimes \omega_{j-1} \otimes \cdots \otimes \omega_i
$$

there are nonzero scalars $c$ and $c'$ such that

$$(ce_j f_j + c').(x_1 \otimes \cdots \otimes x_k) = x_1 \otimes \cdots \otimes x_m \otimes \cdots \otimes x_\ell \otimes \cdots \otimes x_k.
$$

Similarly, there are nonzero scalars $d$ and $d'$ such that

$$(de_j f_j + d').(x_1 \otimes \cdots \otimes x_m \otimes \cdots \otimes x_\ell \otimes \cdots \otimes x_k) = x_1 \otimes \cdots \otimes x_k.
$$

As the transpositions $(j, j+1)$ generate $S_k$, $v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \in \widetilde{U}_\omega$ for all $\sigma \in S_k$.

Next we will use induction on $k$ to establish the following. For any $k$ elements $i_1, \ldots, i_k \in \{1, \ldots, n\}$ satisfying $i_1 \leq i_2 \leq \cdots \leq i_k$, there is a $u \in \widetilde{U}$ such that $u_\omega = v_{i_1} \otimes \cdots \otimes v_{i_k}$ and $u$ does not contain any terms with factors of $e_{m'}, e_{m'+1}, \ldots, e_{n-1}, f_{m'+1}, f_{m'+2}, \ldots, f_{n-1}$, or $f_{m'}$ where $m = \max\{i_k, k\}$. If $k = 1$, we may apply $f_{m-1} \cdots f_1 \otimes v = v_1$ to obtain $v_m$ for any $m \in \{1, \ldots, n\}$. If $k > 1$, let $\ell$ be such that $i_\ell < i_k$, $i_{\ell+1} = i_{\ell+2} = \cdots = i_k$. (If no such $\ell$ exists, that is if $i_1 = \cdots = i_k$, then set $\ell = 0$ and apply $u'$ from (6.5) below to $v_1 \otimes \cdots \otimes v_k$ to obtain a nonzero scalar multiple of $v_{i_1} \otimes \cdots \otimes v_{i_k}$.) By induction, there is an element $u \in \widetilde{U}$ such that

$$
u, (v_1 \otimes \cdots \otimes v_\ell) = v_{i_1} \otimes \cdots \otimes v_{i_\ell},
$$

where $u$ has no terms with factors of $e_{m'}, e_{m'+1}, \ldots, e_{n-1}, f_{m'+1}, \ldots, f_{n-1}$ ($m' = \max\{i_\ell, \ell\}$).

Suppose initially that $i_\ell \leq \ell$. Then $m' = \ell$, and so $u, (v_1 \otimes \cdots \otimes v_k)$ is a nonzero scalar multiple of $(v_{i_1} \otimes \cdots \otimes v_\ell) \otimes (v_{\ell+1} \otimes \cdots \otimes v_k)$. Now apply

$$
u' = \begin{cases}
(f_{i_{\ell-1}} f_{i_{\ell-2}} \cdots f_1) \cdots (f_{i_{k-1}} f_{i_{k-2}}) (e_{i_k} e_{i_{k+1}} \cdots e_{k-1}) (e_{i_{k-1}} \cdots e_{i_{k+1}}) (e_{i_k}) & \text{if } i_k < k \\
(f_{i_{\ell-1}} f_{i_{\ell-2}} \cdots f_1) \cdots (f_{i_{k-1}} f_{i_{k-2}} f_{k-1}) (f_{i_k} f_{i_{k-2}} \cdots f_{k}) & \text{if } i_k \geq k
\end{cases}
$$

(6.5)

to obtain a nonzero scalar multiple of $v_{i_1} \otimes \cdots \otimes v_{i_k}$, as desired. (Note that we did not use any factors of $e_m, e_{m+1}, \ldots, e_{n-1}, f_{m+1}, \ldots, f_{n-1}$ for $m = \max\{i_k, k\}$.)

If on the other hand, $i_\ell > \ell$ (so that $m' = i_\ell$ and $i_k > i_{\ell+1}$), first apply $u'$ from (6.5) to $v_1 \otimes \cdots \otimes v_k$ to obtain a nonzero scalar multiple of

$$(v_1 \otimes \cdots \otimes v_\ell) \otimes (v_{i_k} \otimes \cdots \otimes v_{i_k}),$$
and then apply $u$ from (6.4) to obtain a nonzero scalar multiple of $v_{i_1} \otimes \cdots \otimes v_{i_k}$, as desired.

Finally, if $i_1, \ldots, i_k \in \{1, \ldots, n\}$ are any $k$ elements (not necessarily in nondecreasing numerical order), let $\sigma \in S_k$ be a permutation such that

\[ i_{\sigma(1)} \leq i_{\sigma(2)} \leq \cdots \leq i_{\sigma(k)}. \]

By the first paragraph of the proof, there is an element of $\tilde{U}$ taking $v$ to $v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$. Now we may apply $u'$ from (6.5) in the appropriate order (as above) to $v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$ to obtain a nonzero scalar multiple of $v_{i_1} \otimes \cdots \otimes v_{i_k}$. □

This leads to the two-parameter analogue of Schur-Weyl duality.

**Theorem 6.6.** Assume $rs^{-1}$ is not a root of unity. Then:

(i) $H_k(r, s)$ maps surjectively onto $\text{End}_{\tilde{U}}(V^\otimes k)$.

(ii) If $n \geq k$, then $H_k(r, s)$ is isomorphic to $\text{End}_{\tilde{U}}(V^\otimes k)$.

**Proof.** We establish part (ii) first. Assume $F \in \text{End}_{\tilde{U}}(V^\otimes k)$ and $v = v_1 \otimes \cdots \otimes v_k$. As $F$ commutes with the action of $\tilde{U}$, $F(v)$ must have the same weight as $v$, that is, $\epsilon_1 + \cdots + \epsilon_k$. The only vectors of $V^\otimes k$ with this weight are the linear combinations of the permutations of $v_1 \otimes \cdots \otimes v_k$, so that

\[ F(v) = \sum_{\sigma \in S_k} c_{\sigma} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}, \]

for some scalars $c_{\sigma} \in \mathbb{K}$. We will show that there is an element $R^\sigma$ in the image of $H_k(r, s)$ in $\text{End}_{\tilde{U}}(V^\otimes k)$ such that $R^\sigma(v) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$. (Previously we constructed an element $u \in \tilde{U}$ with this property.)

We begin with the transposition $\tau = t_j = (j, j + 1)$. For any tensor product $v_{i_1} \otimes \cdots \otimes v_{i_k}$ of distinct basis vectors, we have by (5.6) that

\[ v_{i_{\tau(1)}} \otimes \cdots \otimes v_{i_{\tau(k)}} = \begin{cases} r^{-1} R_j (v_{i_1} \otimes \cdots \otimes v_{i_k}) & \text{if } i_j < i_{j+1} \\ (sR_j + (r-s)\text{Id})(v_{i_1} \otimes \cdots \otimes v_{i_k}) & \text{if } i_j > i_{j+1}. \end{cases} \]

Therefore, if $\sigma = t_{j_1} \cdots t_{j_m}$, a product of such transpositions, we can set $R^{j_1} \cdots R^{j_m} := r^{-m} R_{j_m} \cdots R_{j_1}$, depending on the numerical order of the appropriate indices $i_{j_t}$ and $i_{j_{t+1}}$ in $r^{m-1} \cdots r^{j_1} \otimes v$. Then defining $R^\sigma = R^{j_1} \cdots R^{j_m} \in \text{End}_{\tilde{U}}(V^\otimes k)$, we have the desired map such that $R^\sigma(v) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$.

Now let $F_0 = F - \sum_{\sigma \in S_k} c_{\sigma} R^\sigma \in \text{End}_{\tilde{U}}(V^\otimes k)$ (with the $c_{\sigma}$ coming from (6.7)), and note that $F_0(\tilde{u}) = 0$. As $F_0$ commutes with the action of $\tilde{U}$, we have $F_0(\tilde{U} \tilde{v}) = \tilde{U} F_0(\tilde{v}) = 0$. By Lemma 6.2, $\tilde{U} \tilde{v} = V^\otimes k$. Therefore $F_0$ is the 0-map, which implies $F = \sum_{\sigma \in S_k} c_{\sigma} R^\sigma$ is in the image of $H_k(r, s)$. Consequently, the map $H_k(r, s) \to \text{End}_{\tilde{U}}(V^\otimes k)$ in (6.1) is a surjection, and $\text{End}_{\tilde{U}}(V^\otimes k)$ is the $\mathbb{K}$-linear
span of \( \{ R^\sigma \mid \sigma \in S_k \} \). Now suppose that \( \sum_{\sigma \in S_k} c_\sigma R^\sigma = 0 \) for some scalars \( c_\sigma \in \mathbb{K} \). Then in particular,
\[
0 = \sum_{\sigma \in S_k} c_\sigma R^\sigma = \sum_{\sigma \in S_k} c_\sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.
\]
The vectors \( \{ v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \mid \sigma \in S_k \} \) are linearly independent, so \( c_\sigma = 0 \) for all \( \sigma \in S_k \). This implies that \( \{ R^\sigma \mid \sigma \in S_k \} \) is a basis for the vector space \( \text{End}_{\mathbb{K}}(V^{\otimes k}) \) and \( \text{dim}_{\mathbb{K}} \text{End}_{\mathbb{K}}(V^{\otimes k}) = k! \). Because \( H_k(r,s) \) is isomorphic to \( H_k(r^{-1}s) \), it has dimension \( k! \) also. Therefore, \( H_k(r,s) \) is isomorphic to \( \text{End}_{\mathbb{K}}(V^{\otimes k}) \) for \( n \geq k \), as asserted.

Next we turn to the proof of (i) and assume here that \( n < k \). For \( i = n, k \), let \( \tilde{U}_i = U_{r,s}(gl_i) \), let \( \Lambda_i \) be the weight lattice of \( gl_i \), and let \( V_i \) be the natural \( \tilde{U}_i \)-module. By (ii), we may identify \( H_k(r,s) \) with \( \text{End}_{\tilde{U}_k}(V_k^{\otimes k}) \). We will show that \( H_k(r,s) \) maps surjectively onto \( \text{End}_{\tilde{U}_n}(V_n^{\otimes k}) \).

Consider \( V_n^{\otimes k} \) as a \( \tilde{U}_n \)-module via the inclusion of \( \tilde{U}_n \) into \( \tilde{U}_k \), and regard \( V_n^{\otimes k} \) as a \( \tilde{U}_n \)-submodule of \( V_k^{\otimes k} \) in the obvious way. Now \( V_n^{\otimes k} \) is a finite-dimensional \( \tilde{U}_n \)-module on which \( \tilde{U}_n^0 \) acts semisimply, so by Theorem 3.8, it is completely reducible. Therefore,
\[
(6.8) \quad V_n^{\otimes k} = L_1 \oplus \cdots \oplus L_t
\]
for simple \( \tilde{U}_n \)-modules \( L_i \). It suffices to show that the projections onto the simple summands \( L_i \) can be obtained from \( H_k(r,s) \).

Consider
\[
(6.9) \quad \tilde{U}_k.V_n^{\otimes k} = \tilde{U}_k.L_1 + \cdots + \tilde{U}_k.L_t,
\]
the \( \tilde{U}_k \)-submodule of \( V_n^{\otimes k} \) generated by \( V_n^{\otimes k} \). By Corollary 2.5, each \( L_i \) is isomorphic to some \( L(\lambda_j) \), \( \lambda_j \in \Lambda_n^+ \), and in particular is generated by a highest weight vector \( m_i \) with \( e_j.m_i = 0 \) for all \( j \), \( 1 \leq j < n \). We claim that \( e_j.m_i = 0 \) as well when \( n \leq j < k \). This follows from the expression for \( \Delta^{k-1}(e_j) \) in (6.3) and the action of \( e_j \) on the natural module \( V_k \) for \( \tilde{U}_k \) given by \( e_j.v_i = \delta_{i,j+1} v_j \), because \( m_i \) must be some linear combination of vectors \( v_{i_1} \otimes \cdots \otimes v_{i_k} \) with \( i_1, \ldots, i_k \in \{1, \ldots, n\} \). Therefore \( m_i \) is also a highest weight vector for the finite-dimensional \( \tilde{U}_k \)-module \( \tilde{U}_k.L_i \). By Theorem 2.1 and Lemma 3.7, \( \tilde{U}_k.L_i = \tilde{U}_k.m_i \) is a simple \( \tilde{U}_k \)-module. Therefore (6.9) must be a direct sum:
\[
\tilde{U}_k.V_n^{\otimes k} = \tilde{U}_k.L_1 \oplus \cdots \oplus \tilde{U}_k.L_t.
\]

Because \( V_k^{\otimes k} \) is a completely reducible \( \tilde{U}_k \)-module, there is some complementary \( \tilde{U}_k \)-submodule \( W \) such that
\[
(6.10) \quad V_k^{\otimes k} = \tilde{U}_k.L_1 \oplus \cdots \oplus \tilde{U}_k.L_t \oplus W.
\]
Let \( \pi_i \in H_k(r,s) \) be the projection of \( V_k^{\otimes k} \) onto \( \tilde{U}_k.L_i \). Then, \( \pi_i \) commutes with the \( \tilde{U}_k \)-action, and acts as the identity map on \( \tilde{U}_k.L_i \) and as 0 on the other summands in (6.10). Since \( L_j \subseteq \tilde{U}_k.L_j \) for all \( j \), the map \( \pi_i \) restricted to \( V_n^{\otimes k} \) commutes with the \( \tilde{U}_n \)-action and is the projection onto \( L_i \). Thus, \( H_k(r,s) \to \text{End}_{\mathbb{K}}(V_n^{\otimes k}) \) is onto. \( \square \)
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