Iterative Merging Algorithm for Cooperative Data Exchange

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Abstract—This paper considers the problem of finding the minimum sum-rate strategy in cooperative data exchange (CDE) systems. In a CDE system, there are a number of geographically close cooperative clients who send packets to help the others recover a packet set. A minimum sum-rate strategy is the strategy that achieves universal recovery (the situation when all the clients recover the whole packet set) with the the minimal sum-rate (the total number of transmissions). We propose an iterative merging (IM) algorithm that recursively merges client sets based on an estimate of the minimum sum-rate and achieves local recovery until the universal recovery is achieved. We prove that the minimum sum-rate and a corresponding strategy can be found by starting the IM algorithm with an initial lower estimate of the minimum sum-rate. We run an experiment to show that the complexity of the IM algorithm is lower than the complexity of existing deterministic algorithms.

I. INTRODUCTION

Due to the growing amount of data exchange over wireless networks and increasing number of mobile clients, the base-station-to-peer (B2P) links are severely overloaded. It is called the 'last mile' bottleneck problem in wireless transmissions. Cooperative peer-to-peer (P2P) communications is proposed for solving this problem. The idea is to allow mobile clients to exchange information with each other through P2P links instead of solely relying on the B2P transmissions. If the clients are geographically close to each other, the P2P transmissions could be more reliable and faster than B2P ones.

Consider the situation when a base station wants to deliver a set of packets to a group of clients. Due to the fading effects of wireless channels, after broadcast via B2P links, there may still exist some clients that do not obtain all the packets. However, the clients’ knowledge of the packet set may be complementary to each other. Therefore, instead of relying on retransmissions from the base station, the clients can broadcast linear packet combinations of the packets they know via P2P links so as to help the others recover the missing packets. We call this kind of transmission method cooperative data exchange (CDE) and the corresponding system CDE system.

Let the universal recovery be the situation that all clients obtain the entire packet set and the sum-rate be the total number of linear combinations sent by all clients. In CDE systems, the most commonly addressed problem is to find the minimum-sum rate strategy, the transmission scheme that achieves universal recovery and has the minimum sum-rate. This problem was introduced in [1]. Randomized and deterministic algorithms for solving this problem have been proposed in [2], [3] and [4], [5], respectively. The idea of the randomized algorithms in [2], [3] is to choose a client with the maximal or non-minimal rank of the received encoding vectors and let him/her transmit once by using random coefficients from a large Galois field. But, these randomized algorithms repetitively call the rank function, the complexity of which grows with both the number of clients and the number of packets. On the other hand, the authors in [4], [5] propose deterministic algorithms where the complexity only grows with the number of clients. However, there are two problems with the deterministic algorithms in [4], [5]. One is that they both rely on the submodular function minimization (SFM) algorithm, and the complexity of SFM algorithms is not low. The other is that the algorithms in [4], [5] cannot be implemented in a de-centralized manner.

In this paper, we propose an iterative merging (IM) algorithm, a deterministic algorithm, for finding the minimum sum-rate strategy in CDE systems. The IM algorithm starts with an initial estimate of the minimum sum-rate. It recursively merges the clients that require the least number of transmissions for both the local recovery and the recovery of the collectively missing packets. Here, local recovery means the merged clients exchange whatever missing in the packet set that they collectively know so that they share the same common knowledge and can be treated as a single entity. After the local recovery, the recovery of their collectively missing packets will be done by other clients. The IM algorithm iteratively achieves the local recovery in the merged clients until the universal recovery is finally reached. Also, the IM algorithm updates the estimate of minimum sum-rate whenever it finds that universal recovery is not achievable. We prove that the minimum sum-rate and a minimum sum-rate strategy can be found by starting the IM algorithm with a lower bound on the minimum sum-rate. By comparing the IM algorithm to the divide-and-conquer (DC) algorithm proposed in [4], we show that the IM algorithm is a bottom-up approach that can be implemented in a decentralized manner. We run an experiment to show that the complexity of the IM algorithm is $O(K^3 \cdot \gamma)$, where $K$ is the number of clients and $\gamma$ is the complexity of evaluating a submodular function.

There are many algorithms proposed for solving SFM problems. To our knowledge, the algorithm proposed in [2] has the lowest complexity $O(K^3 \cdot \gamma + K^6)$, where $K$ is the number of clients, and $\gamma$ is the complexity of evaluating a submodular function.
local recovery is how to let both client \( H \) clients in has-sets in order to help each other recover. First, we need to clarify some notations and definitions as follows.

\[
\frac{\alpha^*_S}{\alpha^*_S} \text{ for the local recovery in } \mathcal{S} \text{ is determined by } \mathcal{W}_S.
\]

\[
\alpha^*_S = \max\left\{ \sum_{X \in \mathcal{W}_S} \left| \frac{|H_S| - |H_X|}{|W_S| - 1} \right| : \mathcal{W}_S \text{ is a partition of } \mathcal{S} \right\} \quad \text{that satisfies } 2 \leq |W_S| \leq |S|.
\]  

Let \( \mathcal{W}_S^* \) be the minimizer of \( (1) \). We call \( \mathcal{W}_S^* \) the minimum sum-rate partition for local recovery in \( \mathcal{S} \), which imposes that a minimum sum-rate strategy must satisfy

\[
\sum_{j \in X} r_j = |H_S| - |H_{S \setminus X}|
\]

for all \( X \in \mathcal{W}_S^* \). It can be seen that the universal recovery is also the local recovery in \( \mathcal{S} = \mathcal{K} \). The minimum sum-rate and minimum sum-rate partition for universal recovery are \( \alpha^*_K \) and \( \mathcal{W}_K^* \), respectively. There exists a minimum sum-rate strategy satisfies \( \sum_{j \in X} r_j = L - |H_{S \setminus X}| \) for all \( X \in \mathcal{W}_K^* \). Also, for all \( \alpha \geq \alpha^*_K \), there exists a strategy that achieves the universal recovery and has the sum-rate equal to \( \alpha \).

**A. Iterative Merging Scheduling Method**

In this section, we propose a greedy scheduling method for the universal recovery in CDE systems, which is the basis for the IM algorithm we will describe in the next subsection. We assume that the clients in CDE system can form coalitions, or groups. A coalition can contain just one client, and each client must appear in no more than one coalition. Any form of coalition in \( \mathcal{K} \) can be represented by a partition \( \mathcal{W}_K \), and any \( k \)-subset \( Y \) of \( \mathcal{W}_K \) contains \( k \) coalitions in \( \mathcal{W}_K \).

The idea of this scheduling method is to iteratively merge coalitions and achieve local recovery until the universal recovery is achieved. At the beginning, we assume that each client forms one coalition, which can be denoted by a \( K \)-partition \( \mathcal{W}_K = \{ \{j\} : j \in \mathcal{K} \} \). Let \( \alpha \) be an estimate of \( \alpha^*_K \). For example, \( \alpha \) can be the lower bound on \( \alpha^*_K \) proposed in [2], [8] or the upper bound on \( \alpha^*_K \) proposed in [2]. We start an iterative procedure. In each iteration, we perform two steps:

1. Let \( k \in \{2, \ldots, |\mathcal{W}_K|\} \). We choose \( Y \) as a \( k \)-subset with the minimum value of \( k \) that satisfies the conditions

\[
\sum_{X \in Y} \frac{|H_{\hat{Y}} - H_X|}{|Y| - 1} + L - |H_{\hat{Y}}| < \alpha,
\]

\[
\sum_{X \in Y} \frac{|H_{\hat{Y}} - H_X|}{|Y| - 1} + L - |H_{\hat{Y}}| \leq \sum_{X \in Y'} \frac{|H_{\hat{Y'}} - H_X|}{|Y'| - 1} + L - |H_{\hat{Y'}}|,
\]

for all other subsets \( Y' \) such that \( |Y'| = |Y| \). Achieve local recovery in \( \hat{Y} \), and update \( \mathcal{W}_K \) by merging all coalitions \( X \) in \( \hat{Y} \) into one coalition.

\[\text{Eq. (1) was originally proposed in [8] for solving a secrecy generation problem. It is shown in [9] that Eq. (1) is also a method to determine the minimum sum-rate in CDE systems. Eq. (1) is based on the condition for the local recovery.}\]

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2. If \( \alpha < \left\lfloor \frac{\sum_{x \in W_K} L - |H_x|}{|W_K| - 1} \right\rfloor \), terminate iteration, increase \( \alpha \) by one and start the IM scheduling method (from \( K \)-partition) again; Otherwise, go to step 1 until the universal recovery is achieved.

In step 1, the interpretations of the conditions (3) and (4) are as follows. Based on Condition (4), \( Y \) must be the minimum sum-rate partition for the local recovery of the collectively known packets in \( \tilde{Y} \), i.e., \( Y = W^*_Y \). So, \( \sum_{x \in Y} |H_x - H_{\tilde{x}}| \) incurs the minimum sum-rate for the local recovery in \( Y \). \( L - |H_{\tilde{Y}}| \) is the number of collectively missing packets the recovery of which relies on the transmissions in client set \( K \setminus \tilde{Y} \). If condition (3) is breached, it means that universal recovery with sum-rate \( \alpha \) is not possible if the coalitions in \( Y \) are merged to form one coalition \( \tilde{Y} \). Therefore, it is better for them to work individually than together. Condition (4) means that \( Y \) require less number of transmissions for the recovery of both collectively known and collectively missing packets than any other \( Y' \) such that \( |Y'| = |Y| \).

As discussed in Section III if we find that condition
\[
\alpha \geq \left\lfloor \frac{\sum_{x \in W_K} L - |H_x|}{|W_K| - 1} \right\rfloor
\]  
(5)
does not hold for some partition \( W_K \) in step 2, it means \( \alpha < \alpha^*_K \) and universal recovery is not possible with the sum-rate \( \alpha \). Therefore, \( \alpha \) should be increased.

Example 3.1: Consider applying the IM scheduling method in the CDE system in Fig. 1 with \( \alpha = 6 \). The procedure is:
- Assume that each client works individually at the beginning and initiate \( W_K = \{\{1\}, \{2\}, \{3\}, \{4\}\} \). It can be shown that coalitions \{1\} and \{3\} require 3 transmissions for local recovery and recovery of the collectively missing packet set \( \{p_1, p_2\} \), which is less than any other two coalitions, i.e., 2-subset \( Y = \{\{1\}, \{3\}\} \). Hence condition (4) satisfies condition (3). Also, since \( \alpha < 6 \), \( Y = \{\{1\}, \{3\}\} \) satisfies the condition (3). Therefore, coalitions \{1\} and \{3\} should be merged to form one coalition \{1,3\} in the next iteration. The problem of the local recovery in coalition \{1,3\} is not difficult to solve. It is straightforward to see that \( \{\{1\}, \{3\}\} \) is the minimum sum-rate partition for local recovery in \{1,3\}, i.e., \( W^*_{\{1,3\}} = \{\{1\}, \{3\}\} \). According to (2), \( r_1 = |H_{\{1,3\}}| - |H_{\{3\}}| = 0 \) and \( r_4 = |H_{\{1,3\}}| - |H_{\{1\}}| = 1 \). After local recovery, client 1 and client 3 have the same has-sets. Therefore, the has-set of coalition \{1,3\} is \( H_{\{1,3\}} = \{p_3, \ldots, p_6\} \). At this moment, the transmission strategy is \( \gamma = (0.0, 1.0, 0) \). One can show that condition (5) holds for \( W_K \). So, we continue to the next iteration.
- For \( W_K = \{\{1,3\}, \{2\}, \{4\}\} \), one can show that \( Y = \{\{1,3\}, \{2\}\} \) is a 2-subset that satisfies the conditions (3) and (4). So, coalitions \{1,3\} and \{2\} are merged as \{1,2,3\}, and \( W_K \) is updated as \( W_K = \{\{1,2,3\}, \{4\}\} \). One can show that \( \{1,3\}, \{2\} \) is the minimum sum-rate partition for the local recovery in \{1,2,3\}, which imposes \( \sum_{j \in \{1,2,3\}} r_j = |H_{\{1,2,3\}}| - |H_{\{2\}}| = 3 \) and \( r_2 = |H_{\{1,2,3\}}| - |H_{\{1,3\}}| = 1 \). By comparing to the current transmission strategy \( \gamma = (0, 0, 1, 0) \), there should be 2 excessive transmissions from coalition \{1,3\} and 1 excessive transmissions from coalition \{2\}. We can directly update \( r_2 \) to 1. Since clients 1 and 3 have the same has-sets, the excessive 2 transmissions can be completed by either client. We assume the client 1 does the excessive transmissions, i.e., \( r_1 \) is updated to 2. We have transmission strategy \( \gamma = (2, 1, 1, 0) \). Since condition (5) holds for \( W_K \), we continue to the next iteration.
- For \( W_K = \{\{1,2,3\}, \{4\}\} \), we do not need to consider conditions (3) and (4) to merge the coalitions. The universal recovery can be achieved by the local recovery between coalitions \{1,2,3\} and \{4\}. One can show that \( W^*_K = \{\{1,2,3\}, \{4\}\} \), which imposes \( \sum_{j \in \{1,2,3\}} r_j = L - |H_{\{4\}}| = 5 \) and \( r_4 = L - |H_{\{1,2,3\}}| = 1 \). We let client 2 to complete the 1 excessive transmission in coalition \{1,2,3\} and set \( r_4 = 1 \). The transmission strategy is updated as \( \gamma = (2, 2, 1, 1) \). It can be shown (by other method) that the minimum sum-rate is \( \alpha^*_K = 6 \) and \( \{2,2,1,1\} \) is one of the minimum sum-rate strategies.

Example 3.2: Consider applying the IM scheduling method in the CDE system in Fig. 1 with \( \alpha = 5 \). It can be shown that \( W_K \) is updated as \( W_K = \{\{1,3\}, \{2\}, \{4\}\} \) in the first iteration. But, since \( \sum_{x \in W_K} \frac{L - |H_x|}{|W_K| - 1} = 6 > 5 \), condition (5) is breached. The iteration terminates, and \( \alpha \) is updated as \( \alpha = 6 \). We initiate \( W_K = \{\{1\}, \{3\}, \{2\}, \{4\}\} \) and start the IM scheduling method again, where the same procedure as in Example 3.1 is repeated.

Example 3.3: If the IM scheduling method is applied to the CDE system in Fig. 1 with \( \alpha = 7 \). One can show that the same procedure as in Example 3.1 is repeated since condition (5) satisfied at each iteration. At the end, we have the transmission strategy \( \gamma = (2, 2, 1, 1) \). But, \( \alpha = 7 \) imposes a sum-rate equals 7. Since the universal has been achieved already, the excessive 1 transmission can be added to any client \( j \in K \). For example, the strategy can be updated as \( \gamma = (2, 2, 2, 1) \).

B. Iterative Merging Algorithm

We describe the IM scheduling method as the IM algorithm in Algorithms 1, 2 and 3. In Algorithm 1, \( v_\alpha \) is defined as
\[
v_\alpha(x) = \alpha - L - |H_x|,
\]  
(6)
and \( \alpha > \sum_{x \in W_K} v_\alpha(x) \) is equivalent to the breach of condition (5). In Algorithm 2,
\[
\xi_\alpha(Y) = v_\alpha(Y) - \sum_{x \in Y} v_\alpha(x).
\]  
(7)
\( \xi_\alpha(Y) < 0 \) and \( \xi_\alpha(Y') \leq \xi_\alpha(Y) \) are equivalent to conditions (3) and (4), respectively. The set \( U \) returned by Algorithm 2 contains \( k \)-subsets such that the coalitions in each \( k \)-subset should be merged. For example, for the CDE system in Fig. 1 when \( W_K = \{\{1\}, \{2\}, \{3\}, \{4\}\} \) and \( \alpha = 6 \), FindMergeCond(\( W_K, \alpha \)) returns \( U = \{\{1\}, \{3\}\} \). So, \{1\}

\[\footnote{In this case, \( U \) contains one 2-subset of \( W_K \). It is possible that \( U \) contains several \( k \)-subsets. For example, if \( U = \{\{1\}, \{3\}\}, \{\{2\}, \{4\}\}\), coalitions \{1\} and \{3\} merges, and coalitions \{2\} and \{4\} merges.} \]
Algorithm 1: Iterative Merge (IM)  

\begin{algorithmic}[1]
\State \textbf{input} : sum-rate $\alpha$
\State \textbf{output} : sum-rate $\alpha$, transmission strategy $r$
\State Initialize a $K$-partition $W_K = \{\{j\} : j \in K\}$ and a $K$-dimension transmission strategy $r = (0, \ldots, 0)$;
\Repeat
\State $U = \text{FindMergeCand}(W_K, \alpha)$;
\State $r = \text{UpdateRates}(r, U)$;
\ForAll {the $Y \in U$ do}
\State update $W_K$ by merging all $X \in Y$;
\EndFor
\Until {$|W_K| = 2$, $U = \emptyset$ or $\alpha > \sum_{X \in W_K} \alpha(X)$;}
\If {$\alpha > \sum_{X \in W_K} \alpha(X)$}
\State $\alpha = \alpha + 1$;
\State go to step 1;
\Else
\State $r = \text{UpdateRates}(r; \{W_K\})$;
\State $r' = r' + \alpha - \sum_{j \in K} r_j$, where $j'$ is the client that is randomly chosen in set $K$;
\EndIf
\end{algorithmic}

Algorithm 2: FindMergeCand (find merging candidate)  

\begin{algorithmic}[1]
\State \textbf{input} : a partition of client set $W_K$, sum-rate $\alpha$
\State \textbf{output} : $U$, the set contains all candidates for merge
\State $k = 1$;
\Repeat
\State $k = k + 1$;
\ForAll {the $Y$ that is a $k$-subset of $W_K$ do}
\State $U = \emptyset$;
\If {$\xi_0(Y) < 0$ and $\xi_0(Y) \leq \xi_0(Y')$ for all other $k$-subsets $Y'$ such that $Y \cap Y' \neq \emptyset$, $Y \cap Z = \emptyset$ for all $Z$ such that $Z \in \mathcal{U}$ then}
\State $U = U \cup Y$;
\EndIf
\EndFor
\Until {$k = |W_K| - 1$ or $U \neq \emptyset$;}
\end{algorithmic}

and \{3\} merges. In Algorithm 3, $\Delta r$ calculates excessive number of transmissions (as described in Example 5.1).

**Theorem 3.4:** The IM algorithm returns $\alpha_K^*$ if $\alpha < \alpha_K^*$ and $\alpha$ if $\alpha \geq \alpha_K^*$.

**Proof:** Consider the Queyranne’s algorithm [10]  
\[ \mathcal{M} := \mathcal{M} \cup \{e\} \]
where $e = \arg \min \{v_a(\mathcal{M} \cup \{e\}) - v_a(\{e\}) : e \in K \setminus \mathcal{M}\}$. Let $W_K$ be a partition generated by the IM algorithm. For any $X \in W_K$ such that $X$ is not a singleton, if we start the Queyranne’s algorithm with $\mathcal{M}^{(0)} = S \subseteq X$, we will get $\mathcal{M}^{(|X| - |S|)} = X^\circ$. Due to the crossing submodularity of $v_a$, at any iteration $m \in \{2, \ldots, K - 1\}$ of Queyranne’s algorithm [10]  
\[ v_a(\mathcal{M}^{(m)}) + v_a(\{j\}) \leq v_a(\mathcal{M}^{(m)} \setminus S) + v_a(S \cup \{j\}), \quad (8) \]
for all $j \in K \setminus \mathcal{M}^{(m)}$ and $S$ such that $\emptyset \neq S \subseteq \mathcal{M}^{(m-1)}$.

Also, the clients in $Y$ merge only if $\xi_0(Y) < 0$. $W_K$ satisfies  
\[ \sum_{X \in W_K} v_a(X) \leq \sum_{X \in W_K} v_a(X) \]  
for all $\alpha_K^*$ such that $|W_K| = |W_K^*|$. Alternatively speaking, $W_K$ generated by the IM algorithm incurs the minimum values of $\sum_{X \in W_K} v_a(X)$. Recall that $\alpha \leq \sum_{X \in W_K} v_a(X)$ is equivalent to condition (5). If it satisfies for all $W_K$ in IM algorithm, $\alpha \geq \alpha^*$. Otherwise, $\alpha < \alpha^*$, and $\alpha$ is increased until $\alpha = \alpha^*$.

We then show the output $r$ achieves the universal recovery. According to (8), for all $Y \in U$, $Y = W^*$. So, for the local recovery in $Y$, there should be $|H_{Y'}| - |H_{Y'^{\prime}}|$ transmissions from client set $X$ for all $X \in Y$. It means that in addition to the current sum-rate $\sum_{j \in X} r_j$ in $X$ there should be  
\[ \Delta r = |H_{Y'}| - |H_{Y'^{\prime}}| - \sum_{j \in X} r_j, \]
more transmissions from $X$. $\Delta r$ can be added to any clients in $X$. Since $r$ is updated for the local recovery in each iteration, the final $r$ in step 13 achieves universal recovery and has a sum-rate equals $\alpha_K^*$. If $\alpha > \alpha_K^*$, the excessive transmission $\sum_{j \in K} r_j$ will be added to any client in $K$. Therefore, the final $r$ must achieves universal recovery. Theorem holds.

**Remark 3.5:** Based on Theorem 3.4, the minimum sum-rate and a minimum sum-rate transmission strategy can be found by starting the IM algorithm with input $\alpha$ being a lower bound on minimum sum-rate. It is interesting that the greedy IM scheduling method finally leads to the optimal solution in CDE systems.

IV. RELATIONSHIP WITH EXISTING WORK

The IM algorithm is closely related to the divide-and-conquer (DC) algorithm proposed in [4]. The DC algorithm starts with $1$-partition of the client set $K$. It iteratively breaks the non-singleton element in the current partition by calling 1-MAC algorithm. For all $S \subseteq K$, 1-MAC($S$) returns $\alpha_S^*$ and $W^*_S$ for the local recovery in $S$. We use the following example to show how the DC algorithm works.

**Example 4.1:** Consider the CDE system in Fig. 1. First call 1-MAC($\mathcal{K}$). It returns $\alpha^{(K)}_1 = 6$ and $W^{(K)}_1 = \{\{1, 2, 3\}, \{4\}\}$, which imposes transmission rates $\sum_{j \in \{1, 2, 3\}} r_j = 5$ and $r_4 = 1$. Then, the problem is to determine the exact rates of clients 1, 2 and 3. To do so, 1-MAC($\{1, 2, 3\}$) is called, which returns $\alpha^{(1, 2, 3)}_1 = 4$ and $W^{(1, 2, 3)}_1 = \{\{1, 3\}, \{\{2\}\}\}$. So, $\sum_{j \in \{1, 3\}} r_j = 3$ and $r_2 = 1$ are sufficient for the local recovery in $\{1, 2, 3\}$. In this case, there’s an excessive rate $\Delta r = 5 - 3 - 1 = 1$ in set $\{1, 2, 3\}$. $\Delta r$ will be added to any client in $\{1, 2, 3\}$, say, client 2, i.e., $r_2 = r_2 + 1 = 2$. Then, 1-MAC($\{1, 3\}$) is called. The results are $\alpha^{(1, 3)}_1 = 1$ and $W^{(1, 3)}_1 = \{\{1\}, \{3\}\}$, which means $r_1 = 0$ and $r_3 = 1$ are sufficient for the local recovery.
The complexity of the IM algorithm depends on two aspects. One is the value of \( \alpha - \alpha^*_K + 1 \) if \( \alpha < \alpha^*_K \), since the IM algorithm will be repeated for \( \alpha - \alpha^*_K + 1 \) times until it updates \( \alpha \) to \( \alpha^*_K \). The other is the complexity of Find\( \text{MergeCand} \) algorithm. The complexity of the IM algorithm is \( O(K \cdot \beta) \), where \( \beta \) is the complexity of Find\( \text{MergeCand} \) algorithm which may vary with different CDE systems. For example, if Find\( \text{MergeCand} \) returns \( U \) containing 2-subsets, \( \beta = O(K^2 \cdot \gamma) \); if Find\( \text{MergeCand} \) returns \( U \) containing 3-subsets, \( \beta = O(K^3 \cdot \gamma) \). Here, \( \gamma \) is the complexity of evaluating function \( v_\alpha \). To show the actual complexity of the IM algorithm, we do the following experiment.

We set the number of packets \( L = 50 \) and vary the number of clients \( K \) from 3 to 30. For each combination of \( K \) and \( L \), we repeat the procedure below for 1000 times.

- randomly generate the has-sets \( \mathcal{H}_j \) for all \( j \in K \) subject to the condition \( \bigcup_{j \in K} \mathcal{H}_j = \mathcal{P} \);
- set \( \alpha \) to be the lower bound on \( \alpha^*_K \) derived in [2], i.e., \( \alpha = \left\lceil \sum_{j \in K} \frac{|\mathcal{H}_j|}{K-1} \right\rceil \); run the IM algorithm to find the minimum sum-rate strategy; count the actual number of evaluations of function \( v_\alpha \) involved in the IM algorithm.

We plot the average number of evaluations of function \( v_\alpha \) over 1000 repetitions in Fig. 3. It shows that the average complexity is about \( O(K^3 \cdot \gamma) \).

The authors in [4], [5] also proposed the deterministic algorithms for searching the minimum sum-rate and minimum sum-rate strategy in CDE systems. The algorithm in [5] has the complexity \( O(K \cdot \text{SFM}(K)) \), and the algorithm in [4] has the complexity \( O(K^3 \cdot \text{SFM}(K)) \), where \( \text{SFM}(K) \) is the complexity of solving a submodular function minimization problem. To our knowledge, the algorithm proposed in [7] has the lowest complexity \( O(K^5 \cdot \gamma + K^6) \) for solving \( \text{SFM}(K) \). From the experiment results in Fig. 3, it can be seen that the complexity of the IM algorithm is much lower than the algorithms in [4], [5].

VI. CONCLUSION

This paper proposed an IM algorithm that found the minimum sum-rate and a minimum sum-rate strategy in CDE systems. The IM algorithm recursively formed client sets into coalitions and achieved local recovery until the universal recovery was reached. We showed that merging process of the IM was the reverse procedure of the DC algorithm. Based on experiment results, we showed that the complexity of the IM algorithm was lower than the complexity of existing algorithms.

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Additional Pages (Appendices)

APPENDIX A

THE PROOF AND EXAMPLES OF $M^{(|X|−|S|)} = X$ BY STARTING QUEYRANNE’S ALGORITHM WITH $\emptyset \neq M^{(0)} = S \subset X$ IN THE PROOF OF THEOREM 3.4

Let $S \subset K$ such that $|S| \leq K - 2$. We have

$$v_\alpha(S \cup \{u\}) - v_\alpha(\{u\}) = |\bigcup_{j \in S} H_j| - |H_u \cap (\bigcup_{j \in S} H_j)|. $$

So,$$ v_\alpha(S \cup \{u\}) - v_\alpha(\{u\}) \leq v_\alpha(S \cup \{u'\}) - v_\alpha(\{u'\})$$
equivalent to

$$|H_u \cap (\bigcup_{j \in S} H_j)| \geq |H_u' \cap (\bigcup_{j \in S} H_j)|. $$

Let $W_K$ be the partition of $K$ that is generated by IM algorithm (at any iteration). For any $X \in W_K$, since the clients $u' \in K \setminus X$ is not merged to $X$, $v_\alpha(S \cup \{u\}) - v_\alpha(\{u\}) \leq v_\alpha(S \cup \{u'\}) - v_\alpha(\{u'\})$ for all $\emptyset \neq S \subset X$, $u \in X \setminus S$ and $u' \in K \setminus X$. For example, in Example B.1 we have $W_K = \{\{1, 2, 3\}, \{4\}\}$ in the second iteration. Consider $X = \{1, 2, 3\}$. One can show that $|H_2 \cup H_4| \geq |H_2 \cup H_4|$ and $|H_1 \cup H_2 \cap H_3| \geq |H_1 \cup H_2 \cap H_4|$. They are equivalent to $v_\alpha(\{2\} \cup \{1\}) \leq v_\alpha(\{2\} \cup \{4\})$ and $v_\alpha(\{1, 2\} \cup \{3\}) \leq v_\alpha(\{1, 2\} \cup \{4\}$, respectively.

Therefore, if $M^{(0)} = S$, we will get $M^{(|X|−|S|)} = X$ at the $|X| − |S|$-th iteration. For example, for $W_K = \{\{1, 2, 3\}, \{4\}\}$ in Example 3.4 and $X = \{1, 2, 3\}$, it can be shown that: if we start the Queyranne’s algorithm with $M^{(0)} = \{1\}, \{2\}$ or $\{3\}$, we will get $M^{(2)} = \{1, 2, 3\};$ if we start the Queyranne’s algorithm with $M^{(0)} = \{1, 2\}, \{2, 3\}$ or $\{1, 3\}$, we will still get $M^{(1)} = \{1, 2, 3\}$.

APPENDIX B

EXAMPLES OF $W_K$ GENERATED BY THE IM ALGORITHM INCURRING THE MINIMUM VALUES OF $\sum_{X \in W_K} v_\alpha(X)$

Example B.1: Consider the CDE system in Fig. 1. We get $W_K = \{\{1, 2, 3\}, \{4\}\}$ in the second iteration of IM algorithm. By applying Queyranne’s algorithm with different $M^{(0)}$, we have the following results:

- If $M^{(0)} = \{1\}$ or $M^{(0)} = \{3\}$, then $M^{(1)} = \{1, 3\}$ and $M^{(2)} = \{1, 2, 3\}$. According to (8), we have

$$v_\alpha(\{1, 2, 3\}) + v_\alpha(\{4\}) \leq v_\alpha(\{2\}) + v_\alpha(\{1, 3, 4\})$$

$$+ v_\alpha(\{1, 2\}) + v_\alpha(\{3, 4\})$$

(9)

- If $M^{(0)} = \{2\}$, then $M^{(2)} = \{1, 2, 3\}$. According to (8), we have

$$v_\alpha(\{1, 2, 3\}) + v_\alpha(\{4\}) \leq v_\alpha(\{1, 3\}) + v_\alpha(\{2, 4\})$$

(10)

- If $M^{(0)} = \{1, 2\}$, then $M^{(1)} = \{1, 2, 3\}$. According to (8), we have

$$v_\alpha(\{1, 2, 3\}) + v_\alpha(\{4\}) \leq v_\alpha(\{3\}) + v_\alpha(\{1, 2, 4\})$$

(11)

#References

[10] M. Queyranne, “Minimizing symmetric submodular functions,” *Math. Programming*, vol. 82, no. 1-2, pp. 3–12, Jun. 1998.
If $M^{(0)} = \{2, 3\}$, then $M^{(1)} = \{1, 2, 3\}$. According to (8), we have

$$v_\alpha(\{2, 3\}) + v_\alpha(\{1\}) + v_\alpha(\{5\}) \leq v_\alpha(\{3, 4\}) + v_\alpha(\{1\}) + v_\alpha(\{5\}).$$

If $M^{(0)} = \{2\}$, then $M^{(1)} = \{2, 3\}$. According to (8), we have

$$v_\alpha(\{3\}) + v_\alpha(\{1\}) + v_\alpha(\{5\}) \leq v_\alpha(\{4\}) + v_\alpha(\{1\}) + v_\alpha(\{5\}).$$

If $M^{(0)} = \{2, 3\}$, then $M^{(1)} = \{2, 3, 4\}$. According to (8), we have

$$v_\alpha(\{2, 3\}) + v_\alpha(\{1\}) + v_\alpha(\{5\}) \leq v_\alpha(\{4\}) + v_\alpha(\{1\}) + v_\alpha(\{5\}) \leq v_\alpha(\{2, 3\}) + v_\alpha(\{4\}) + v_\alpha(\{1\}) + v_\alpha(\{5\}).$$

If $M^{(0)} = \{2, 4\}$, then $M^{(1)} = \{2, 3, 4\}$. According to (8), we have

$$v_\alpha(\{2, 3\}) + v_\alpha(\{1\}) + v_\alpha(\{5\}) \leq v_\alpha(\{4\}) + v_\alpha(\{1\}) + v_\alpha(\{5\}) \leq v_\alpha(\{3\}) + v_\alpha(\{1\}) + v_\alpha(\{2, 4\}).$$

APPENDIX C

IM ALGORITHM APPLIED TO CDE SYSTEM IN FIG. 1

Example C.1: Let $\alpha = 6$. We start IM algorithm with $\mathcal{W}_K = \{\{1\}, \{2\}, \{3\}, \{4\}\}$ and $r = (0, 0, 0, 0)$. The procedure is as follows.

- In the first iteration, we call FindNewPartition to determine $\mathcal{U}$. We first set $k = 2$ and consider all 2-subsets. $\{\{1\}, \{2\}\}, \{\{1\}, \{3\}\}$ and $\{\{2\}, \{3\}\}$ are the 2-subsets $\mathcal{Y}$ that satisfy $\xi_\alpha(\mathcal{Y}) < 0$. Since

$$\xi_\alpha(\mathcal{Y}) = \begin{cases} -1 & \mathcal{Y} = \{\{1\}, \{2\}\}, \\ -3 & \mathcal{Y} = \{\{1\}, \{3\}\}, \\ -1 & \mathcal{Y} = \{\{2\}, \{3\}\}. \end{cases}$$

$\mathcal{U} = \{\{1\}, \{3\}\} \neq \emptyset$. FindNewPartition returns $\mathcal{U} = \{\{1\}, \{3\}\}$ to IM algorithm. We call UpdateRates algorithm, where $r_1$ and $r_3$ in $r$ is updated as $r_1 = 0$ and $r_3 = 1$, respectively. So, $r = (0, 0, 1, 0)$. We then merge sets $\{1\}$ and $\{3\}$ and update $\mathcal{W}_K$ as $\mathcal{W}_K = \{\{1, 3\}, \{2\}, \{4\}\}$. Because $v_\alpha(\{1\}) + v_\alpha(\{2\}) + v_\alpha(\{4\}) + v_\alpha(\{1\}) = 7 > 6$, $\mathcal{U} \neq \emptyset$ and $|\mathcal{W}_K| \neq 2$, we continue the 'repeat' loop in IM algorithm.

- In the second iteration, with $\mathcal{W}_K = \{\{1, 3\}, \{2\}, \{4\}\}$, we have $\mathcal{U} = \{\{1\}, \{3\}\}$ and $\Delta r = 1$ for $\{2\}$. We choose $r_1$ to increase by two, and $r_2$ is directly updated as $r_2 = 1$. The transmission strategy is updated as $r = (2, 1, 0, 0)$, and $\mathcal{W}_K$ is updated as $\mathcal{W}_K = \{\{1, 2, 3\}, \{4\}\}$. Since $|\mathcal{W}_K| = 2$, the 'repeat' loop in IM algorithm terminates.

- Since $\alpha \leq \sum_{\mathcal{X} \in \mathcal{W}_K} v_\alpha(\mathcal{X})$ we call UpdateRates by the following $\mathcal{U}$ = $\{\mathcal{W}_K\} = \{\{1, 2, 3\}, \{4\}\}$. $\Delta r = 1$ for $\{1, 2, 3\}$ and $\Delta r = 1$ for $\{4\}$. We increase $r_2$ by one and set $r_4 = 1$. The transmission strategy is updated as $r = (2, 2, 1, 1)$. The IM algorithm finally returns $\alpha = 6$ and $r = (2, 2, 1, 1)$. It can be shown that 6 is the minimum sum-rate and $(2, 2, 1, 1)$ is one of the minimum sum-rate strategies.

Example C.2: Assume that we apply the IM algorithm to the CDE system in Fig. 1 with $\alpha = 5$. It can be shown that $\mathcal{W}_K = \{\{1, 3\}, \{2\}, \{4\}\}$ at the end of the first iteration and $v_\alpha(\{1\}) + v_\alpha(\{2\}) + v_\alpha(\{4\}) = 4 < 5$.
terminates, $\alpha$ is increased to 6 and the IM algorithm is started over again. With $\alpha = 6$, the same procedure as in Example C.1 is repeated.

**Example C.3:** Assume that we apply the IM algorithm to the CDE system in Fig. 1 with $\alpha = 7$. The same procedure as in Example C.1 is repeated. After step 13 in IM algorithm, we have $\alpha = 7$ and $r = (2, 2, 1, 1)$. Since $\alpha - \sum_{j \in K} r_j = 1$, there is an excessive transmission. We randomly choose any client in $K$, say, client 3, and add one to $r_3$. Therefore, the final transmission strategy is $r = (2, 2, 2, 1)$.

**APPENDIX D**

**EXAMPLE OF FINDMERCAND ALGORITHM**

In a four client CDE system, assume we call FindMergeCand algorithm with 4-partition $W_K = \{\{1\}, \ldots, \{4\}\}$. We first set $U = \emptyset$. We discuss how FindMergeCand algorithm works by showing three different cases as follows.

- Assume that we have $\xi_\alpha(\{\{2\}, \{3\}\}) = -1$, $\xi_\alpha(\{\{2\}, \{4\}\}) = -2$ and $\xi_\alpha(\{\{3\}, \{4\}\}) = -2$ and all other 2-subsets $Y$ do not satisfy the condition $\xi_\alpha(Y) < 0$. We then check if the condition $\xi_\alpha(Y) \leq \xi_\alpha(Y')$ for all other $k$-subsets $Y'$ such that $Y \cap Y' \neq \emptyset$. $\{\{2\}, \{3\}\}$ does not satisfy this condition since $\xi_\alpha(\{\{2\}, \{3\}\}) > \xi_\alpha(\{\{2\}, \{4\}\})$ and $\{\{2\}, \{3\}\} \cap \{\{2\}, \{4\}\} \neq \emptyset$. $\{\{2\}, \{4\}\}$ satisfies this condition and we update $U$ as $U = U \cup \{\{2\}, \{3\}\} = \{\{2\}, \{4\}\}$. $\{\{3\}, \{4\}\}$ also satisfies this condition, but $\{\{2\}, \{4\}\} \in U$ and $\{\{2\}, \{4\}\} \cap \{\{3\}, \{4\}\} \neq \emptyset$. So, $\{\{3\}, \{4\}\}$ will not be added to $U$. Since $U \neq \emptyset$, we finally terminate FindMergeCand algorithm and return $U = \{\{2\}, \{4\}\}$.

- Assume that we have $\xi_\alpha(\{\{1\}, \{3\}\}) = -1$, $\xi_\alpha(\{\{2\}, \{4\}\}) = -2$ and all other 2-subsets $Y$ do not satisfy the condition $\xi_\alpha(Y) < 0$. Then, FindMergeCand algorithm returns $U = \{\{1\}, \{3\}\}, \{\{2\}, \{4\}\}$.

- Assume that we have all 2-subsets $Y$ do not satisfy the condition $\xi_\alpha(Y) < 0$. We increase $k$ to 3 and consider all 3-subsets. The value of $k$ will be increased until $U \neq \emptyset$. 

