GENERALIZED OPTIMAL LIQUIDATION PROBLEMS ACROSS MULTIPLE TRADING VENUES

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Abstract. In this paper, we generalize the Almgren-Chriss’s market impact model to a more realistic and flexible framework and employ it to derive and analyze some aspects of optimal liquidation problem in a security market. We illustrate how a trader’s liquidation strategy alters when multiple venues and extra information are brought into the security market and detected by the trader. This study gives some new insights into the relationship between liquidation strategy and market liquidity, and provides a multi-scale approach to the optimal liquidation problem with randomly varying volatility.

1. Introduction. The optimal liquidation problem of large trades has been studied extensively in the micro-structure literature. Two major sources of risk faced by large traders are: (i) inventory risk arising from uncertainty in asset value; and (ii) transaction costs arising from market friction. In a frictionless and competitive market, an asset can be traded with any amount at any rate without affecting the market price of the asset. The optimal liquidation problem then becomes an optimal stopping problem. In an incomplete market, the optimal liquidation problem involves delicate market-micro-structural issues. The impacts of transaction costs on optimal liquidation and optimal portfolio selection have been studied via various mechanisms [4, 1, 2, 3, 6, 9].

In this paper, we adopt a simple, but practical market impact model to study some aspects of the optimal liquidation problem. The model is phenomenological and not directly based on the fine details of micro-structure though it may primarily be related to some literature on market micro-structure. Following Almgren and Chriss [4], we decompose the price impact into temporary price impact and...
permanent price impact. Temporary impact refers to temporary imbalance in supply/demand caused by trading. It disappears immediately when trading activities cease. Permanent impact means changes in the “equilibrium” price due to trading, which lasts at least for the whole process of liquidation.

We generalize the work of Almgren and Chriss [4] to a more realistic and flexible framework in which multiple venues are available for the trader to submit his/her trades. We mainly consider short-term liquidation problem for a large trader who experiences temporary and permanent market impact. Two broad classes of problems are addressed in this paper which we believe are representative. The first one is the case with constant volatility. This assumption considerably simplifies the problem and allows us to exhibit the essential features of liquidation across multiple venues without losing ourselves in complexities. The second one is the case when volatility varies randomly throughout the trading horizon. In this case, we present a “pure” stochastic volatility approach, in which the volatility is modeled as an Itô process driven by a Brownian motion that has a component independent of the Brownian motion driving the asset price. In comparison with Almgren’s work in [3], we mainly focus on a special class of volatility model, the time-scale volatility model proposed in [7, 8]:

\[ d\nu_t = \epsilon (m - \nu_t) dt + \xi \sqrt{\epsilon} dW_t, \]

and work in the regime \( \epsilon \ll 1 \). The separation of time-scales provides an efficient way to identify and track the effect of stochastic volatility, which is desirable from the practical perspective.

The remainder of this paper is organized as follows. Section 2 devotes to model building for the execution problem in the presence of multiple trading venues. Section 3 presents a solution for the constant volatility approximation. Extension of the results to the stochastic volatility approximation are then discussed in Section 4. The last section summarizes the results.

2. Problem setup. In this section, we first present our liquidation problem in the presence of multiple trading venues based on the principle of no arbitrage. We then discuss the optimal liquidating strategies.

2.1. The trader’s liquidation problem. Consider an institutional trader who, starting at time \( t = 0 \), has a liquidation target of \( Q \) shares, which must be completed by time \( t = T \). Suppose there are \( N \) distinct venues for the trader to submit his/her orders. Let \( X_t \) denote the number of shares remaining to liquidate at time \( t \), \( \theta(n)_t \) the rate of liquidating in Venue \( n, n = 1, 2, \ldots, N \). Then, we have

\[ \theta^{(1)}_t + \cdots + \theta^{(N)}_t = -\frac{dX_t}{dt}. \]

\( \theta^{(n)}_t, n = 1, 2, \ldots, N, \) are the decision variables of the trader based on available information at time \( t \). For a liquidation program, we expect \( \theta^{(n)}_t \geq 0 \) (a buy program may be modeled similarly).

Consider a probability space \( (\Omega, \mathcal{F}, \mathcal{P}) \) endowed with a filtration \( \{\mathcal{F}_t\}_{t \geq 0} \), on which we introduce the notion of admissible control.

**Definition 2.1.** A \( N \)-dimensional stochastic process \( \theta(\cdot) = \{\theta^{(1)}(\cdot), \ldots, \theta^{(N)}(\cdot)\} \) is called an admissible control process if the following conditions hold:

(a) (Adaptivity) For each time \( t \in [0, T] \), \( \theta_t = (\theta^{(1)}_t, \ldots, \theta^{(N)}_t) \) is \( \mathcal{F}_t \)-adapted;
(ii): (Non-negativity) \( \theta_t \in \mathbb{R}_+^N \), where \( \mathbb{R}_+ \) is the set of all nonnegative real values;

(iii): (Consistency)

\[
\int_0^T \sum_{n=1}^N \theta_t^{(n)} dt \leq Q;
\]

(iv): (Square-integrability)

\[
E \left[ \int_0^T \left| \theta_t^{(1)} \right|^2 + \cdots + \left| \theta_t^{(N)} \right|^2 dt \right] < \infty;
\]

(v): (\( L_\infty \)-integrability)

\[
E \left[ \sup_{0 \leq t \leq T} \left| \theta_t^{(1)} + \cdots + \theta_t^{(N)} \right| \right] < \infty.
\]

Let \( \Theta_t \) denote the collection of admissible controls with respect to time \( t (< T) \), \( \hat{\Theta}_t \) the collection of controls satisfying conditions (i), (iv) and (v).

Suppose the stock price evolves over the time according to

\[
dS_t = \sigma_t dW_t
\]

where \( \{W_t\}_{t \geq 0} \) is a standard Brownian motion with \( \{\mathcal{F}_t\}_{t \geq 0} \) and \( \sigma_t (> 0) \) is the absolute volatility of the stock price, which can be (i) constant; (ii) time-varying; or (iii) stochastic. The drift term in Eq. (1) is set to be zero, which means that we expect no obvious trend in short-term future. This may make intuitive sense. Since the total time \( T \) is usually one day or less, the drift is generally not significant over such a short trading horizon.

2.2. The market impact model. Generally, risky assets, especially for those with low liquidity, exhibit price impacts due to the feedback effects of trader’s liquidating strategies. Small orders usually have insignificant impact, especially for liquid stocks. Large orders, however, may have a significant impact on the price. Investors, especially institutional investors, must keep the price impact in mind when making investment decisions.

Following Almgren and Chriss [4], we decompose the price impact into temporary price impact and permanent price impact:

\[
\left\{ \begin{array}{l}
  dS_t^I = dS_t - \eta^{per} f(\theta_t) dt, \\
  S_t^{I,(n)} = S_t - \eta^{tem,(n)} \theta_t^{(n)}, \quad n = 1, 2, \cdots, N.
\end{array} \right.
\]

Temporary impact refers to temporary imbalance in supply/demand caused by trading. It disappears immediately when trading activities cease. Permanent impact means changes in the “equilibrium” price due to trading, which lasts at least for the whole process of liquidation. \( S_t^I \) and \( S_t^{I,(n)} \), in Eq. (2), denote, respectively, the impact-adjusted “equilibrium” price and the actual price received on each trade at time \( t \) over Venue \( n \); \( f(\theta_t) \) describes the net order flow that has been absorbed by the system at time \( t \); \( \eta^{per} > 0 \) is the coefficient of permanent price impact over the “equilibrium” price; and \( \eta^{tem,(n)} \) is the coefficient of temporary price impact over Venue \( n \).

\[1\]When long-term strategies are considered, it is more reasonable to consider geometric rather than arithmetic Brownian motion. In this paper, we mainly focus on short-term liquidating strategies, the total fractional price changes over such a short time are relatively small, and hence the difference between arithmetic and geometric Brownian motions can be negligible.
Different from \[4\], where \(N\) is set to be 1 and hence \(f(\theta_t) = \theta_t\), each venue’s price under the multiple-venue setting depends not only on its internal transactions but also on its competitor’s deals. The following proposition provides a price impact model under the multiple-venue setting based on no-arbitrage argument.

**Proposition 1.** Assume that market liquidity remains unchanged over \([0, T]\), and that there exists no arbitrage opportunity in the financial market. Based on Almgren \([3]\)’s linear price impact model, the affected asset price in the presence of \(N\) trading venues is given by

\[
dS_t^I = dS_t - \eta \text{per} (\beta(1) \cdot \theta_t^{(1)} + \cdots + \beta(N) \cdot \theta_t^{(N)}) dt
\]

where

\[
\sum_{n=1}^{N} \beta(n) = 1, \quad \beta(n) \in (0, 1).
\]

**Proof.** Without loss of generality, we prove that Proposition 1 holds for \(N = 2\). Consider a small time interval \([t, t + \Delta t]\). Assume that trades occur immediately after time \(t\). With the assumption of linear price impact, stock prices in the two trading venues, immediately after the execution of the orders, are drawn down to \(p_t^{(1)} = S_t - \eta \text{per} \theta_t^{(1)} \Delta t\) and \(p_t^{(2)} = S_t - \eta \text{per} \theta_t^{(2)} \Delta t\), respectively. Assume that \(\theta_t^{(1)} < \theta_t^{(2)}\). Obviously, we have \(p_t^{(1)} > p_t^{(2)}\). It is absolutely a risk-less arbitrage opportunity. Other investors, being aware of this opportunity, will buy from Venue 2 and sell to Venue 1 to benefit from favorable prices across the two venues.

Suppose the financial market possess a tendency moving along with the “gravity” of the market of form

\[
E_t \left[ \frac{dp_t^{(i)}}{dt} \right] = G_i \times D_t, \quad i = 1, 2,
\]

where \(E_t [\cdot]\) denotes the conditional expectation with respect to the filtration \(F_t\), \(G_i\) is a constant and \(D_t\) stands for the difference between the two prices at time \(t\). The adjusted stock prices at time \(t + \Delta t\) shall be

\[
\begin{align*}
\begin{cases}
p_{t+\Delta t}^{(1)} = S_t - \eta \text{per} \theta_t^{(1)} \Delta t + \int_t^{t+\Delta t} \sigma_u dW_u - x(\eta \text{per} \theta_t^{(2)} - \eta \text{per} \theta_t^{(1)}) \Delta t, \\
p_{t+\Delta t}^{(2)} = S_t - \eta \text{per} \theta_t^{(2)} \Delta t + \int_t^{t+\Delta t} \sigma_u dW_u + y(\eta \text{per} \theta_t^{(2)} - \eta \text{per} \theta_t^{(1)}) \Delta t,
\end{cases}
\end{align*}
\]

with \(x, y > 0\). With no-arbitrage argument, we shall have \(p_{t+\Delta t}^{(1)} = p_{t+\Delta t}^{(2)}\), which yields

\[
\begin{cases}
x + y = 1 \\
S_t^{I+\Delta t} := p_{t+\Delta t}^{(1)} = p_{t+\Delta t}^{(2)} = S_t + \int_t^{t+\Delta t} \sigma_u dW_u - \eta \text{per} (y \theta_t^{(1)} + x \theta_t^{(2)}) \Delta t.
\end{cases}
\]

Proposition 1 holds when we set \(\Delta t \to 0\).

\[\square\]

**2.3. The gain/loss of trading.** Let \(C_t\) denote the cash flow accumulated by the trader by time \(t\), i.e.,

\[
C_t = \int_0^t (\tilde{S}_s^{I, (1)} \theta_s^{(1)} + \cdots + \tilde{S}_s^{I, (N)} \theta_s^{(N)}) ds, \quad t < T.
\]
Given the state variables \((S_t^i, C_t, X_t)\) the instant before the end of trading \(t = T^-\), we have one final liquidation (if necessary) so that the number of shares owned at \(t = T\) is \(X_T = 0\). The liquidation value \(C_T\) after this final trade is
\[
C_T = C_{T-} + X_{T-}(S_{T-}^i - C^o(X_{T-})),
\]
where \(C^o(q)\), a non-negative increasing function in \(q\), is the costs incurred when liquidating the outstanding position \(X_{T-}\) at time \(T\).

The gain/loss of trading \(\mathcal{R}_T\), after integrating by parts, is given by\(^2\)
\[
\mathcal{R}_T = C_{T-} - QS_0 = \int_0^{T-} \sigma_t X_t dW_t - X_{T-}C^o(X_{T-}) - \eta_{per} \int_0^{T-} X_t (\beta^{(1)} \theta^{(1)}_t + \cdots + \beta^{(N)} \theta^{(N)}_t) dt \\
- \int_0^{T-} \left[ \eta_{tem,(1)} (\theta^{(1)}_t)^2 + \cdots + \eta_{tem,(N)} (\theta^{(N)}_t)^2 \right] dt.
\]

\[ (3) \]

2.4. The objective. Generally speaking, investors are risk averse. The mean-variance criterion is useful when taking both return and risk into account. However, the mean-variance criterion may induce a potential problem of time inconsistency, i.e., planned and implemented policies are different, and hence complicate the problem. To avoid the time-inconsistent problem, we decide to use instead the quadratic variation term, \(\int_0^{T-} \sigma_u^2 X_u^2 du\). When properly normalized, the quadratic variation can also be interpreted as the average standard deviation per unit time, see, for instance, Brugiere [5].

Let \((S_t, \sigma_t, X_t) = (s, \sigma, q)\) be the initial state. At any time \(t \in [0, T)\), the optimal policy should solve the following optimization problem:
\[
J(t, s, \sigma, q) = \max_{(\theta^{(1)}, \cdots, \theta^{(N)}), \forall (\theta_{t}, s, \sigma, q) \in \Theta_t} \left\{ \mathbb{E}_t [\mathcal{R}_T - \mathcal{R}_t] - \lambda \cdot \mathbb{E}_t \left[ \int_t^{T-} \sigma_u^2 X_u^2 du \right] \right\}. \tag{4}
\]

Proposition 2 below discusses the time consistency of the optimal strategy. Its proof can be found in Appendix A.

**Proposition 2. (Time consistency).** Let \((t_1, s_1, \sigma_1, q_1)\) be some state at time \(t_1\), \(\theta_{t_1,s_1,\sigma_1,q_1}^* = (\theta^{(1)}_{t_1,s_1,\sigma_1,q_1}, \cdots, \theta^{(N)}_{t_1,s_1,\sigma_1,q_1})^T\) be the corresponding optimal strategy. Let \((t_2, s_2, \sigma_2, q_2)\) be some other state at time \(t_2 > t_1\) and \(\theta_{t_2,s_2,\sigma_2,q_2}^* = (\theta^{(1)}_{t_2,s_2,\sigma_2,q_2}, \cdots, \theta^{(N)}_{t_2,s_2,\sigma_2,q_2})^T\) be the corresponding optimal strategy. It follows that the optimal controls of Problem (4) are time-consistent in the sense that for the same state \((t', s', \sigma', q')\) at a later time \(t' > t_2\),
\[
\theta^*_{t_1,s_1,\sigma_1,q_1}(t', s', \sigma', q') = \theta^*_{t_2,s_2,\sigma_2,t_2}(t', s', \sigma', q'). \tag{5}
\]

We shall use Eq. (4) as our objective function from now on. We will employ standard techniques of optimal control to identify the PDE satisfied by the value function in the next section.

2.5. Hamilton-Jacobi-Bellman (HJB) equation. Since the optimal controls satisfy the Bellman’s principle of optimality as shown in Appendix A, dynamic programming (DP) approach can be directly applied to this problem. The optimal

\[\text{Relative to the arrival price benchmark.}\]
control $\theta^*$ can be obtained by solving the following Hamilton-Jacobi-Bellman (HJB) equation:

$$
\begin{align*}
\min_{\theta_t \in \Theta_t} \{ & \eta_{\text{per}}(\beta(1)\theta_t^{(1)} + \cdots + \beta(N)\theta_t^{(N)})q + \\
& \eta_{\text{tem}}(\theta_t^{(1)})^2 + \cdots + \eta_{\text{tem}}(\theta_t^{(N)})^2 + (\theta_t^{(1)} + \cdots + \theta_t^{(N)}) \partial_q J \} = 0
\end{align*}
$$

where $\mathcal{L}$ is the generator of the processes $(S_t, \sigma_t)_{t \geq 0}$.

Notice that the optimization problem in (HJB) is a constrained optimization problem. To solve, we might first relax the constraints in it (i.e., replacing $\Theta_t$ by $\hat{\Theta}_t$), and solve the associated unconstrained optimization problem instead. And then prove that the derived solution satisfies all the constrains listed in $\Theta_t$. As the first step, we derived the optimal control of the associated unconstrained optimization problem, which is given by

$$
\theta_t^{n,*} = -\frac{1}{2\eta_{\text{tem}}(n)}(\partial_t J + \eta_{\text{per}} \beta(n)q), \quad \text{for } n = 1, \cdots, N,
$$

where the associated value function $J$ solves the following partial differential equation (PDE):

$$
\begin{align*}
\min_{\theta_t \in \Theta_t} \{ & \eta_{\text{per}}(\beta(1)\theta_t^{(1)} + \cdots + \beta(N)\theta_t^{(N)})q + \\
& \eta_{\text{tem}}(\theta_t^{(1)})^2 + \cdots + \eta_{\text{tem}}(\theta_t^{(N)})^2 + (\theta_t^{(1)} + \cdots + \theta_t^{(N)}) \partial_q J \} = 0
\end{align*}
$$

The optimal control can then be rewritten in the form of

$$
\theta_t^{n,*} = -\frac{1}{2\eta_{\text{tem}}(n)}[(2h(t; \sigma) + \eta_{\text{per}} \beta(n)q + g(t; \sigma)].
$$

3. **Constant Volatility.** The most illuminating case is that $\sigma_t$ is constant, i.e., $\sigma_t \equiv \sigma > 0$. This considerably simplifies the problem and allows us to exhibit the essential features of liquidating across multiple venues without losing ourselves in complexities. With the constant volatility assumption, $J(t, s, \sigma, q)$ should be independent of $s$ since the terminal data does not depend on $s$ and (HJB) introduces no $s$-dependence.

Write $J(t, q; \sigma) := J(t, s, \sigma, q)$. We look for a solution of the form

$$
J(t, q; \sigma) = f(t; \sigma) + g(t; \sigma)q + h(t; \sigma)q^2.
$$

The first case has explicit liquidating formula. For the second case, there is a multi-scale argument that reduces the optimal liquidation PDE to a formal series expansion that can easily be solved explicitly. We shall present the argument and the accuracy of this approach in Section 4.3.
Following similar arguments in [11], we have

\[
\begin{aligned}
g(t; \sigma) &= 0, \\
h(t; \sigma) &= \begin{cases} \\
\sqrt{\Delta_N} \cdot \frac{ce^{-2a\Delta_N(T-t)}}{a} - \frac{b}{2a}, & \Delta_N \geq 0 \\
-\frac{\Delta_N}{a} \tan \left( \arctan \left( \frac{b - 2aK}{2\sqrt{-a\Delta_N}} \right) + \sqrt{-a\Delta_N} (T - t) \right) - \frac{b}{2a}, & \Delta_N < 0,
\end{cases}
\end{aligned}
\]

where

\[
a = \sum_{n=1}^{N} \eta_{\text{tem}(n)},
\]

\[
b = \sum_{n=1}^{N} \eta_{\text{per} \beta(n)},
\]

\[
c = \sum_{n=1}^{N} \frac{(\eta_{\text{per} \beta(n)})^2}{4\eta_{\text{tem}(n)}},
\]

\[
\Delta_N = \lambda \sigma^2 + \frac{b^2}{4a} - 4ac,
\]

\[
\varsigma = \left( 1 - \frac{2Ka - b}{2\sqrt{a\Delta_N}} \right) / \left( 1 + \frac{2Ka - b}{2\sqrt{a\Delta_N}} \right).
\]

It is worth noting that

\[
\begin{aligned}
\theta^{n,*}_t &= -\frac{1}{2\eta_{\text{tem}(n)}} \left( 2h(t; \sigma) + \eta_{\text{per} \beta(n)} \right) X_t, \\
\dot{X}_t &= -\sum_{n=1}^{N} \theta^{n,*}_t.
\end{aligned}
\]

And hence,

\[
X_t = Q \cdot \exp \left( \int_0^t \sum_{n=1}^{N} \frac{1}{2\eta_{\text{tem}(n)}} (2h(u; \sigma) + \eta_{\text{per} \beta(n)}) du \right).
\]

Meanwhile\(^3\), \(b^2 - 4ac \leq 0\), and the equality holds if and only if (i) \(N = 1\); or (ii) the trading venues have the same market efficiency, namely, \(\beta^{(1)} = \cdots = \beta^{(N)} = 1/N\). Proposition 3 below shows some properties of the solution. Its proof can be found in Appendix B.

**Proposition 3.** Assume that the model parameters satisfy the condition\(^4\):

\[
K > \frac{b}{2a} + \sqrt{\frac{|\Delta_N|}{a}}. \tag{10}
\]

If \(N = 1\) or the trading venues have the same market efficiency, then \(h(t; \sigma)\) is a decreasing function in \(t\) and \(2h(t; \sigma) + \eta_{\text{per} \beta(n)} \leq 0\) for \(n = 1, \cdots, N\), which implies that

(i): \(\theta^{n,*}_t \geq 0\), for any \(n = 1, \cdots, N\); and that

(ii): \(\int_0^T \sum_{n=1}^{N} \theta^{n,*}_t dt \leq Q\).

\(^3\)Hölder’s inequality.

\(^4\)That is, clearing fees associated with the outstanding position \(X_{T-}\) dominate the potential profit arising from arbitrage opportunities incurred by the permanent price impact and the potential position risk involved by price fluctuations.
That is, the control policy in Eq. (9) is the optimal control of (HJB)\(^5\).

3.1. Effect of available venues. Let \(\theta_t^{\text{single,\ast}}\) denote the optimal liquidating strategy under the single venue setting. Assume that there is no difference among the \(N\) available venues, namely, \(\beta^1 = \cdots = \beta^N = 1/N\) and \(\eta_{\text{tem}}(1) = \cdots = \eta_{\text{tem}}(N) = \eta_{\text{tem}}\). The following two conclusions can be drawn.

(i): (Effect on optimal liquidating speed). First, we have

\[
\theta_t^{\ast} = \theta_t^{\text{single,\ast}} := \theta_t^\ast(N)
\]

where

\[
\theta_t^\ast(N) = -\sqrt{\frac{\lambda \sigma^2}{N \eta_{\text{tem}}}} \times \frac{\varsigma(N)e^{-2\sqrt{\lambda \sigma^2 N/\eta_{\text{tem}}}(T-t) - 1}}{\varsigma(N)e^{-2\sqrt{\lambda \sigma^2 N/\eta_{\text{tem}}}(T-t) + 1}} X_t := \mathfrak{J}(t, N) X_t,
\]

with

\[
\varsigma(N) = \frac{2 \sqrt{\lambda \sigma^2 N/\eta_{\text{tem}}}}{(2K \frac{N}{\eta_{\text{tem}}} - b)N}.
\]

Notice that \(X_0 = Q\) and

\[
\lim_{N \rightarrow \infty} X_0 = - \lim_{N \rightarrow \infty} N \theta_0^\ast(N) = - \lim_{N \rightarrow \infty} N \mathfrak{J}(0, N) Q = \infty.
\]

For any fixed time \(t \in (0, T)\), one can verify that (the proof can be found in Appendix C)

\[
\lim_{N \rightarrow \infty} X_t = 0, \quad \text{and} \quad \lim_{N \rightarrow \infty} -\dot{X}_t = 0.
\]

That is to say, as \(N\) approaches infinity, the investor would immediately close his/her position at the beginning of the trading horizon.

(ii): (Effect on transaction cost). The two strategies: (i) liquidating in a single venue; and (ii) \(\{\theta_t^{(1)} = \theta_t^{(2)} = \cdots = \theta_t^{(n)} = \frac{1}{N} \theta_t^{\text{single,\ast}}\} \in [0, T]\), equally splitting the original target among \(N\) venues, transmit the same information to the market (i.e., have the same permanent impact), but involve different transaction costs,

\[
\int_0^T \sum_{n=1}^N (S_t^n - \overline{S}_t^{(n)}(\ast)) \theta_t^{(n)} dt = \int_0^T \sum_{n=1}^N \frac{\eta_{\text{tem}}}{N^2} (\theta_t^{\text{single,\ast}})^2 dt
\]

\[
\leq \int_0^T \frac{\eta_{\text{tem}}(\theta_t^{\text{single,\ast}})^2}{N} dt
\]

\[
= \int_0^T \frac{\eta_{\text{tem}}(\theta_t^{\text{single,\ast}})^2}{N} dt
\]

\[
= \int_0^T (S_t^n - \overline{S}_t^{(n)}) \theta_t^{\text{single,\ast}} dt.
\]

That is, liquidating schedule across multiple venues can indeed help to reduce transaction costs arising from market liquidity.

\(^5\)Satisfying all the constraints listed in \(\Theta_t\).
4. **Stochastic Volatility model.** The constant volatility assumption might be reasonable for large-cap US stocks. However, this assumption becomes defective when we move to the analysis of small and medium-capitalization stocks, whose volatilities vary randomly through the day. In this section, we relax the constant volatility assumption. Similarly, we relax the constraints in (HJB) and consider solving the associated unconstrained optimization problem. We then verify that the obtained optimal control does satisfy all the constraints in (HJB).

4.1. **Slow mean-reverting volatility model.** Consider the case in which stock prices are conditionally normal, and the volatility process is a positive and increasing function of a mean-reverting process:

\[
\begin{aligned}
    dS_t &= \phi(\nu_t) dW_t \\
    d\nu_t &= \epsilon(m - \nu_t) dt + \xi \sqrt{\epsilon} dB_t \\
    B_t &= \rho W_t + \sqrt{1 - \rho^2} Z_t.
\end{aligned}
\]

\{\nu_t\} in Eq. (11) is a simple building-block for a large class of stochastic volatility models described by choice of \(\phi(\cdot)\), \{W_t\} and \{Z_t\} are two independent one-dimensional Brownian motions, \(\rho \ (|\rho| < 1)\) describes the correlation between stock price and volatility shocks, \(m\) is the equilibrium or long-term mean of \{\nu_t\}, \(\epsilon \ (> 0)\) is the intrinsic time-scale of the process, and \(\xi\), combining with the scalar \(\sqrt{\epsilon}\), represents the degree of volatility around the equilibrium \(m\) caused by shocks.

According to Itô’s lemma,

\[
d\phi(\nu_t) = \epsilon \left[ \phi'(\nu_t)(m - \nu_t) + \xi^2 \phi''(\nu_t) \right] dt + \xi \phi'(\nu_t) \sqrt{\epsilon} dB_t.
\]

We call this model mean-reverting because the volatility is a monotonic function of \{\nu_t\} whose drift pulls it towards the mean \(m\). The volatility is correspondingly pulled towards approximately \(\phi(m)\).

Plenty of analysis on specific Itô models by numerical and analytical methods can be found in the literature [3, 7]. Our goal here is to identify and capture the relevant features of liquidating strategies for small- and medium-capitalization stocks. Our framework in Eq. (11) is adequate and efficient enough to capture this. Here we will work in the regime \(\epsilon \ll 1\) (slow mean-reverting). For simplicity, we consider the case of \(N = 1\). Let

\[
\mathcal{L} = \epsilon ((m - \nu) \partial_{\nu} + \frac{1}{2} \xi^2 \partial_{\nu\nu}).
\]

The value function \(J\) then solves the following PDE:

\[
\begin{aligned}
    (\partial_t + \mathcal{L}) J - \lambda \phi^2(\nu) q^2 + \frac{1}{4\eta^2 \text{erm}} (\partial_q J + \eta \phi'\nu q)^2 &= 0, \\
    J(T-, s, \nu, q) &= -K q^2.
\end{aligned}
\]

Similarly, we look for a solution in the form of \(^6\)

\[
J(t, s, \nu, q) = f(t, \nu) + g(t, \nu) q + h(t, \nu) q^2.
\]

The optimal liquidating schedule can then be calculated through Eq. (9). As before, we need to solve for \(g\) and \(h\) to get the optimal liquidating policy and they solve

\(^6\)Independent of \(s\).
the following PDEs:

\[
\begin{align*}
\left\{ \begin{array}{l}
(\partial_t + \mathcal{L})h - \lambda \phi^2(\nu) + \frac{1}{4\eta_{em}} (2h + \eta_{per})^2 = 0, \\
h(T-, \nu) = -K, \\
(\partial_t + \mathcal{L})g + \frac{1}{2\eta_{em}} (2h + \eta_{per})g = 0, \\
g(T-, \nu) = 0.
\end{array} \right.
\]

(14)

It is straightforward to verify that (Feynman-Kac formula) \( g(t, \nu) \equiv 0 \). Let

\[
Y_t = \frac{1}{2\eta_{em}} \int_0^t (2h(u, \nu_u) + \eta_{per})du,
\]

and

\[
\mathfrak{h}(t, \nu_t, Y_t) = e^{Y_t} (2h(t, \nu_t) + \eta_{per}).
\]

Via Itô’s lemma, we have

\[
d\mathfrak{h}(t, \nu_t, Y_t) = e^{Y_t} \left\{ 2(\partial_t + \mathcal{L})h + \frac{1}{2\eta_{em}} (2h + \eta_{per})^2 \right\} dt + 2 \xi \sqrt{\tau} \partial_\nu h dB_t.
\]

Therefore,

\[
\mathfrak{h}(t, \nu_t, Y_t) = e^{Y_T} - (2K + \eta_{per}) - 2\lambda \int_t^T e^{Y_\nu_0} \phi^2(\nu_u)du - 2 \int_t^T \xi \sqrt{\tau} \partial_\nu h(u, \nu_u)dB_u.
\]

Hence,

\[
\mathfrak{h}(t, \nu, y) = \mathbb{E}[\mathfrak{h}(t, \nu_t, Y_T)|\nu_t = \nu, Y_t = y] = e^{Y_T} - (2K + \eta_{per}) - 2\lambda \int_t^T e^{Y_\nu_0} \phi^2(\nu_u)du|\nu_t = \nu, Y_t = y].
\]

(15)

If condition (10) is satisfied, we then have \(-2K + \eta_{per} < 0, \mathfrak{h}(t, \nu, y) \leq 0, \forall (t, \nu, y) \in [0, T] \times \mathbb{R}^2\), and hence \(2h(t, \nu) + \eta_{per} \leq 0, \forall (t, \nu) \in [0, T] \times \mathbb{R}\). That is, the control policy derived in the unconstrained optimization problem does satisfy all the constraints in (HJB).

Eq. (15) only provides an implicit analytical solution for \( \mathfrak{h} \). For the general coefficients \((\phi, m, \epsilon, \xi)\), we do not have an explicit solution for it. Fortunately, the small-\(\epsilon\) regime gives rise to a regular perturbation expansion in the powers of \(\epsilon\). As did in [7], we shall seek an asymptotic approximation for \( \mathfrak{h} \):

\[
h(t, \nu) = h^{(0)}(t, \nu) + \epsilon h^{(1)}(t, \nu) + \epsilon^2 h^{(2)}(t, \nu) + \cdots
\]

(16)

and study the accuracy of it in the following subsections.

4.2. Asymptotics. We first construct a regular perturbation expansion in the powers of \(\epsilon\) by writing

\[
\mathcal{L} = \epsilon \mathcal{L}_0, \quad h(t, \nu) = \sum_{i \geq 0} \epsilon^i h^{(i)}(t, \nu)
\]

where

\[
\mathcal{L}_0 = (m - \nu)\partial_\nu + \frac{1}{2} \xi^2 \partial_{\nu\nu}.
\]
Substituting these expansions into Eq. (14) and grouping the terms of the powers of \( \epsilon \), we find that the lowest order equations of the regular perturbation expansion are
\[
O(1) : \begin{cases}
\partial_t h^{(0)} - \lambda \phi^2(\nu) + \frac{1}{4\eta_{tem}} (2h^{(0)} + \eta^{per})^2 = 0, \\
h^{(0)}(T-, \nu) = -K,
\end{cases}
\]
\[
O(\epsilon) : \begin{cases}
\partial_t h^{(1)} + L_0 h^{(0)} + \frac{1}{\eta_{tem}} (2h^{(0)} + \eta^{per}) h^{(1)} = 0, \\
h^{(1)}(T-, \nu) = 0.
\end{cases}
\]

The solutions of Eq. (17) can be obtained easily. They are given by
\[
h^{(0)}(t, \nu) = \sqrt{\eta_{tem}} \lambda \phi^2(\nu) \cdot \frac{\varsigma(\nu) e^{\frac{1}{2}\sqrt{\lambda \phi^2(\nu)/\eta_{tem}(T-t)}} - 1}{\varsigma(\nu) e^{\frac{1}{2}\sqrt{\lambda \phi^2(\nu)/\eta_{tem}(T-t)}} + 1} - \frac{1}{2} \eta^{per},
\]
and
\[
h^{(1)}(t, \nu) = \int_t^T D(r, t) L_0 h^{(0)}(r, \nu) dr,
\]
respectively, with \( \varsigma(\nu) \) and \( D(r, t) \) given by
\[
\varsigma(\nu) = \frac{\sqrt{\eta_{tem}} \lambda \phi^2(\nu) - (K - \eta^{per}/2)}{\sqrt{\eta_{tem}} \lambda \phi^2(\nu) + (K - \eta^{per}/2)},
\]
and
\[
D(r, t) = \exp \left( \frac{1}{\eta_{tem}} \int_t^T (2h^{(0)} + \eta^{per}) d\tau \right),
\]
respectively.

4.3. Optimal strategy. We now analyze and interpret how the principle expansion terms for the value function can be used in the expression for the optimal liquidation strategy \( \theta^* \), which leads to an approximate feedback policy of the form:
\[
\theta^*(t, \nu, q) = \tilde{\theta}^{(0),*}(t, \nu, q) + \epsilon \theta^{(1),*}(t, \nu, q) + \cdots + .
\]

4.3.1. Zeroth-order policy. First, we introduce the zeroth order terms \( h^{(0)} \) in the expansion for \( h \). This gives the zeroth order optimal liquidation strategy
\[
\tilde{\theta}^{(0),*}(t, \nu, q) := -\frac{1}{2\eta_{tem}} \left( 2h^{(0)}(t, \nu) + \eta^{per} \right) q,
\]
which corresponding to the case of constant volatility in Section 3.

4.3.2. First-order policy. The approximation to the optimal strategy can be more accurate by going into the higher order terms. Substituting the expansion of \( h \) up to terms in \( \epsilon \) gives
\[
\theta^*(t, \nu, q) = \tilde{\theta}^*(t, \nu, q) + \text{higher order terms}
\]
where
\[
\tilde{\theta}^*(t, \nu, q) = -\frac{1}{2\eta_{tem}} \left( 2h^{(0)}(t, \nu) + \eta^{per} \right) q - \frac{\epsilon}{\eta_{tem}} h^{(1)}(t, \nu) q.
\]

The term with \( \epsilon \) corresponds to the traders’ response to the risk arising from stochastic volatility, which can be regarded as the principle term hedging the risk factor.

**Theorem 4.1.** For the first-order policy, we have \( |\theta^*(t, \nu, q) - \tilde{\theta}^*(t, \nu, q)| = \mathcal{O}(\epsilon^2) \).
Proof. Notice that, at any time \( t < T \),

\[
|\theta^*(t, \nu, q) - \tilde{\theta}^*(t, \nu, q)| = \frac{|h - (h(0) + \epsilon h^{(1)})|}{\eta^\text{tem}} q.
\]

Let \( R^* = h - (h(0) + \epsilon h^{(1)}) \). It then solves the following PDE:

\[
\begin{cases}
(\partial_t + \mathcal{L}) R^* + \frac{1}{\eta^\text{tem}} R^*[h + h^{(0)} + \epsilon h^{(1)} + \eta^\text{per}] + \epsilon^2 \left\{ \mathcal{L}_0 h^{(1)} + \frac{1}{\eta^\text{tem}} (h^{(1)})^2 \right\} = 0 \\
R^*(T-, \nu) = 0.
\end{cases}
\]

Via Feynman-Kac formula, we have

\[
R^*(t, \nu) = \epsilon^2 \cdot \mathbb{E}_t \int_t^T D_2(r; t) \left\{ \mathcal{L}_0 h^{(1)} + \frac{1}{\eta^\text{tem}} (h^{(1)})^2 \right\} dr
\]

where

\[
D_2(r; t) = \exp \left( \frac{1}{\eta^\text{tem}} \int_t^r [h + h^{(0)} + \epsilon h^{(1)} + \eta^\text{per}] d\tau \right).
\]

Since the integrand in Eq. (19) is bounded in \([0, T]\), there exists a constant \( C \), such that \( |R^*(t, \nu)| \leq C\epsilon^2 \). This completes our proof. \( \square \)

4.4. Simulation results. As in Fouque et al. [7], we assume that \( \phi(\nu) = e^{\nu} \). We test the performance of (i) the zeroth-order policy; and (ii) the first-order policy. For the sake of simplicity, we refer to these strategies as “Constant-Vol” and “Vol-adjust”, respectively.

As far as our simulation is concerned, we use the following hypothetical values of the model parameters: \( T = 1 \) hour, \( \nu_0 = 0.5 \), \( \sigma_0 = \phi(\nu_0) = e^{0.5} \), \( m = 1 \), \( \epsilon = 0.01 \), \( \xi = 2 \), \( \rho = -0.4 \), \( \lambda = 0.1 \), \( \eta^\text{per} = 0.005 \), \( \eta^\text{tem} = 0.01 \). We run 1000 simulations to compare the performance of these strategies, primarily focusing on the shape of the profit and loss (P&L) profile. The table below shows the final results.

### Table 1. 1,000 simulations with \( Q = 100, s = 15 \), w.r.t. \( \mathcal{R}_T \).

| Statistics     | Constant-Vol | Vol-adjust |
|----------------|--------------|------------|
| Mean           | -300.70      | -288.46    |
| Std            | 54.76        | 27.45      |
| Skewness       | 1.03         | 0.24       |
| Kurtosis       | 5.26         | 3.05       |
| Objective function | -577.58    | -560.36    |

In this paper, we mainly focus on short-term liquidating strategies. The drift term of the stock price is set to be zero. Namely, we expect no obvious trend in short-term future and the only thing we care about is the transaction cost incurred by the strategy. As we can see from Table 1 and Figure 1, the differences between the two strategies are significant. Profiting from hedging the risk arising from stochastic volatility is possible. Compared with the “Constant-Vol” strategy, the “Vol-adjust” strategy obtains a lower cost and a smaller risk.
5. Conclusions. In this paper, we utilize a quantitative model to discuss the optimal liquidating problem in an illiquid market. We formulate the optimal liquidation problem with both temporary and permanent market impacts. Under the no-arbitrage assumption, we propose a linear model to determine the equilibrium price in a competitive market with multiple trading venues. We prove that when more trading venues spring up in the financial market, the liquidity of the asset enhances, which provides good supplies of liquidity to the investor. Multi-scale analysis method is applied in this paper to discuss the case of slow mean-reverting stochastic volatility. Different liquidation strategies are discussed and compared to shed light on the effect of stochastic volatility on investor’s profit and loss profile.

Appendix.

Appendix A.

Proof. By Eq. (3), the objective at any time $t$ becomes

$$
\mathbb{E}_t \left[ \int_t^{T-} \sigma_u X_u dW_u - \eta_{pecr} \int_t^{T-} X_u (\beta_u^{(1)} \theta_u^{(1)} + \cdots + \beta_u^{(N)} \theta_u^{(N)}) du 
- \int_t^{T-} \left( \eta_{tem,(1)} (\theta_u^{(1)})^2 + \cdots + \eta_{tem,(N)} (\theta_u^{(N)})^2 \right) du 
- \lambda \int_t^{T-} \sigma_u^2 X_u^2 du - X_T - C_o (X_T -) \right],
$$

Given initial value $X_t = q$,

$$
J(t, s, \sigma, q)
= \max_{\theta(.) \in \Theta_t} \mathbb{E}^{t, s, \sigma, q}_{\theta(.)} \left[ \int_t^{t+\Delta t} \sigma_u X_u dW_u - \eta_{pecr} \int_t^{t+\Delta t} X_u (\beta_u^{(1)} \theta_u^{(1)} + \cdots + \beta_u^{(N)} \theta_u^{(N)}) du 
- \int_t^{t+\Delta t} \left( \eta_{tem,(1)} (\theta_u^{(1)})^2 + \cdots + \eta_{tem,(N)} (\theta_u^{(N)})^2 \right) du 
+ \mathbb{E}^{t+\Delta t, s+\Delta s, \sigma+\Delta \sigma, q+\Delta q}_{\theta(.)} \left[ \int_{t+\Delta t}^{T-} \sigma_u X_u dW_u - \eta_{pecr} \int_{t+\Delta t}^{T-} X_u (\beta_u^{(1)} \theta_u^{(1)} + \cdots + \beta_u^{(N)} \theta_u^{(N)}) du 
- \int_{t+\Delta t}^{T-} \left( \eta_{tem,(1)} (\theta_u^{(1)})^2 + \cdots + \eta_{tem,(N)} (\theta_u^{(N)})^2 \right) du - X_T - C_o (X_T -) \right],
$$
where $\mathbb{E}_{\theta(\cdot)}^{t,s,q}[\cdot]$ is the conditional expectation conditioned on the control process $\theta(\cdot)$ and the initial state $(S_t, \sigma_t, X_t) = (s, \sigma, q)$.

Notice that for any control process $\theta(\cdot) \in \Theta_t$, we have

$$J(t+s,t+\Delta t, s+\Delta s, \sigma+\Delta \sigma, q+\Delta q) \leq \max_{\theta(\cdot) \in \Theta_t} J(t+s,t+\Delta t, s+\Delta s, \sigma+\Delta \sigma, q+\Delta q).$$

Thus, we have

$$J(t+s,t+\Delta t, s+\Delta s, \sigma+\Delta \sigma, q+\Delta q) \leq \max_{\theta(\cdot) \in \Theta_t} \mathbb{E}_{\theta(\cdot)}^{t,s,q}[J(t+s,t+\Delta t, s+\Delta s, \sigma+\Delta \sigma, q+\Delta q)]$$

Then, we obtain

$$J(t+s,t+\Delta t, s+\Delta s, \sigma+\Delta \sigma, q+\Delta q) \geq \mathbb{E}_{\theta(\cdot)}^{t,s,q}[J(t+s,t+\Delta t, s+\Delta s, \sigma+\Delta \sigma, q+\Delta q)]$$

Hence, we obtain

$$J(t+s,t+\Delta t, s+\Delta s, \sigma+\Delta \sigma, q+\Delta q) \geq \max_{\theta(\cdot) \in \Theta_t} \mathbb{E}_{\theta(\cdot)}^{t,s,q}[J(t+s,t+\Delta t, s+\Delta s, \sigma+\Delta \sigma, q+\Delta q)]$$

(21)
Proof. In the case of (i) Appendix B.

Programming Principle (DPP): Putting both inequalities, Eq.(20) and Eq.(21), together, we arrive at the Dynamic decreasing function in $t$ efficiency, $\Delta$

Recall that this principle.

The dynamic programming equation, HJB equation, is the infinitesimal version of this principle.

Appendix B.

\textbf{Proof.} In the case of (i) $N = 1$; or (ii) the trading venues have the same market efficiency, $\Delta_N = \lambda \sigma^2 > 0$. Hence,

$$h(t; \sigma) = \sqrt{\frac{\Delta_N}{a}} \cdot \varsigma e^{-2\sqrt{a\Delta_N}(T-t)} - 1 - \eta_{\text{per}} \beta(n).$$

Under the assumption that $K > \frac{b}{2a} + \sqrt{\frac{\Delta_N}{a}}$, $-1 < \varsigma < 0$. Therefore, $h$ is a decreasing function in $t$, and for any $n = 1, \ldots, N$

$$2h(t; \sigma) + \eta_{\text{per}} \beta(n) = \sqrt{\frac{\Delta_N}{a}} \cdot \varsigma e^{-2\sqrt{a\Delta_N}(T-t)} - 1 < 0. \tag{22}$$

Recall that

$$\begin{cases} X_t = -\sum_{i=1}^{N} \theta_t^{n, *} \\ \theta_t^{n, *} = -\frac{1}{2\eta_{\text{tem}}} \left[ 2h(t; \sigma) + \eta_{\text{per}} \beta(n) \right] X_t. \end{cases}$$

Hence, $X_t$ satisfies the following first-order ODE:

$$\begin{cases} \dot{X}_t = \left( \sum_{n=1}^{N} \frac{1}{2\eta_{\text{tem}}} \left[ 2h(t; \sigma) + \eta_{\text{per}} \beta(n) \right] \right) X_t \\ X_0 = Q, \end{cases}$$

which yields

$$X_t = Q \cdot \exp \left( \int_0^t \sum_{n=1}^{N} \frac{1}{2\eta_{\text{tem}}} \left[ 2h(u; \sigma) + \eta_{\text{per}} \beta(n) \right] du \right) \in (0, Q], \text{ for } t \in [0, T). \tag{23}$$

Combining the results in Eq. (22) and Eq. (23), we conclude that, for any $n = 1, \ldots, N$,

$$\theta_t^{n, *} = -\frac{1}{2\eta_{\text{tem}}} \left[ 2h(t; \sigma) + \eta_{\text{per}} \beta(n) \right] X_t \geq 0,$$
and
\[
\int_0^T \sum_{n=1}^N \theta_t^n \, dt = X_0 - X_T = Q \left[ 1 - \exp \left( \int_0^T \sum_{n=1}^N \frac{1}{2\eta_{tem}} \left[ 2h(u; \sigma) + \eta_{per}^u \beta^{(u)} \right] \, du \right) \right] \leq Q.
\]

**Appendix C.**

**Proof.**

\[
\lim_{N \to \infty} J(t, N) = 0 \quad \text{and} \quad \lim_{N \to \infty} \sqrt{N} J(t, N) = \sqrt{\frac{\lambda \sigma^2}{\eta_{tem}}},
\]

Hence
\[
\lim_{N \to \infty} X_t = \lim_{N \to \infty} Q e^{-\int_0^t N \bar{3}(u, N) \, du} = \lim_{N \to \infty} Q e^{-\sqrt{N} \times \int_0^t \sqrt{N} \bar{3}(u, N) \, du} = \lim_{N \to \infty} Q e^{-a_1 \sqrt{N}} = 0,
\]

where \(0 < a_1 = \lim_{N \to \infty} \int_0^t \sqrt{N} \bar{3}(u, N) \, du = \sqrt{\frac{\lambda \sigma^2}{\eta_{tem}}} t < \infty\). (24)

We also have
\[
\lim_{N \to \infty} -\dot{X}_t = \lim_{N \to \infty} N \theta_t^N(N) = \lim_{N \to \infty} Q N \bar{3}(t, N) \times e^{-\int_0^t N \bar{3}(u, N) \, du} = \lim_{N \to \infty} Q \sqrt{\frac{\lambda \sigma^2}{\eta_{tem}}} \times \frac{1}{a_1 e^{a_1 \sqrt{N}}} = 0.
\]

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\footnote{According to the structure of \(\bar{3}(t, N)\), there exists a finite number \(M > 0\) such that \(|\sqrt{N} \bar{3}(t, N)| \leq M\), for all \((t, N) \in [0, T) \times \mathbb{Z}_+\), where \(\mathbb{Z}_+\) is the set of all nonnegative integers. Directly applying the Dominated convergence theorem to \(\{\sqrt{N} \bar{3}(t, N)\}\), we obtain the result in Eq. (24).}
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