TRUNCATIONS OF THE RING OF NUMBER-THEORETIC FUNCTIONS

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Abstract. We study the ring $\Gamma$ of all functions $\mathbb{N}^+ \to K$, endowed with the usual convolution product. $\Gamma$, which we call the ring of number-theoretic functions, is an inverse limit of the “truncations”

$$\Gamma_n = \{ f \in \Gamma | \forall m > n : f(m) = 0 \} .$$

Each $\Gamma_n$ is a zero-dimensional, finitely generated $K$-algebra, which may be expressed as the quotient of a finitely generated polynomial ring with a stable (after reversing the order of the variables) monomial ideal. Using the description of the free minimal resolution of stable ideals given by Eliahou-Kervaire, and some additional arguments by Aramova-Herzog and Peeva, we give the Poincaré-Betti series for $\Gamma_n$.

1. Introduction

Cashwell and Everett [2] studied “the ring of number-theoretic functions”

$$\Gamma = \{ f | \mathbb{N}^+ \to K \} \quad (1)$$

where $\mathbb{N}^+$ is the set of positive natural numbers (we denote by $\mathbb{N}$ the set of all natural numbers) and $K$ is a field containing the rational numbers. $\Gamma$ is endowed with component-wise addition and multiplication with scalars, and with the convolution (or Cauchy) product

$$fg(n) = \sum_{(a,b) \in (\mathbb{N}^+)^2} f(a)g(b) \quad (a,b) \in (\mathbb{N}^+)^2 \quad (a+b=n) \quad (2)$$

With these operations, $\Gamma$ becomes a commutative $K$-algebra. It is immediate that it is a local domain; less obvious is the fact that it is a unique factorisation domain. Cashwell and Everett proved this in [2] using the isomorphism

$$\Phi : \Gamma \to K[[X]]$$

$$f \mapsto \sum f(n)x_1^{a_1}x_2^{a_2}\cdots \quad (3)$$

where $X = \{x_1, x_2, x_3, \ldots \}$, $K[[X]]$ is the “large” power series ring of all functions from the free abelian monoid $\mathcal{M} = [X]$ (the free abelian monoid generated by $X$) to $K$, and where the summation extends over all $n = p_1^{a_1}p_2^{a_2}\cdots \in \mathbb{N}^+$. Here, and henceforth, we denote by $p_i$ the $i$’th prime number, with $p_1 = 2$, and by $\mathcal{P}$ the set of all prime numbers. That (3) is an isomorphism is immediate from the following isomorphism of commutative monoids, implied by the fundamental theorem of arithmetics:

$$(\mathbb{N}^+, \cdot) \simeq \prod_{p \in \mathcal{P}} (\mathbb{N}, +) \quad (4)$$

The following number-theoretic functions are of particular interest (whenever possible, we use the same notation as in [2]):

1. The multiplicative unit $\epsilon$ given by $\epsilon(1) = 1$, $\epsilon(n) = 0$ for $n > 1$,
2. \( \lambda : \mathbb{N}^+ \to \mathbb{N} \) given by \( \lambda(1) = 0 \), \( \lambda(q_1 \cdots q_l) = l \) if \( q_1, \ldots, q_l \) are any (not necessarily distinct) prime numbers.

3. \( \tilde{\lambda} : \mathbb{N}^+ \to \mathbb{N} \) given \( \tilde{\lambda}(1) = 0 \), \( \tilde{\lambda}(p_1^{a_1} \cdots p_r^{a_r}) = \sum a_r p_r \).

4. The Möbius function \( \mu(1) = 1 \), \( \mu(n) = (-1)^v \) if \( n \) is the product of \( v \) distinct prime factors, and 0 otherwise.

5. For any \( i \in \mathbb{N}^+ \), \( \chi_i(p_i) = 1 \), and \( \chi_i(m) = 0 \) for \( m \neq p_i \). Note that under the isomorphism \( \tilde{\Phi}_i, \Phi(\chi_i) = x_i \).

The topic of this article is the study of the “truncations” \( \Gamma_n \), where for each \( n \in \mathbb{N}^+ \),

\[
\Gamma_n = \{ f \in \Gamma \mid m > n \implies f(m) = 0 \} \tag{5}
\]

With the modified multiplication given by

\[
fg(n) = \sum_{(a,b) \in \{1, \ldots, n\} \times \{1, \ldots, n\}} f(a)g(b) \tag{6}
\]

\( \Gamma_n \) becomes a \( K \)-algebra, isomorphic to \( \Gamma / J_n \), where \( J_n \) is the ideal

\[
J_n = \{ f \in \Gamma \mid \forall m \leq n : f(m) = 0 \}.
\]

If we define

\[
\pi_n : \Gamma \to \Gamma_n \tag{7}
\]

\( \pi_n(f)(m) = \begin{cases} f(m) & m \leq n \\ 0 & m > n \end{cases} \tag{8} \]

then \( \pi_n \) is a \( K \)-algebra epimorphism, and \( J_n \) is the kernel of \( \pi_n \). We note furthermore that \( J_n \) is generated by monomials in the elements \( x_i \).

To describe the main idea of this paper, we need a few additional definitions. First, for any \( n \in \mathbb{N}^+ \) we denote by \( r(n) \in \mathbb{N} \) the largest integer such that \( p_r(n) \leq n \). In other words, \( r(n) \) is the number of prime numbers \( \leq n \) (this number is often denoted \( \pi(n) \)). Secondly, for a monomial \( m = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \), we define the support \( \text{Supp}(m) \) as the set of positive integers \( i \) such that \( \alpha_i > 0 \). We define \( \max(m) \) and \( \min(m) \) as the maximal and minimal elements in the support of \( m \).

**Definition 1.1.** A monomial ideal \( I \subset K[x_1, \ldots, x_r] \) is said to be strongly stable if whenever \( m \) is a monomial such that \( x_j m \in I \), then \( x_i m \in I \) for all \( i \leq j \). If this condition holds at least for all \( i \leq j = \max(m) \) then \( I \) is said to be stable.

We can now state our main theorem:

**Theorem 1.2.** Let \( n \in \mathbb{N}^+ \) and \( r = r(n) \). Then the following holds:

1. \( \Gamma_n \simeq \frac{K[x_1, \ldots, x_r]}{I_n} \), where \( I_n \) is a strongly stable monomial ideal, with respect to the reverse order of the variables.

2. \( \Gamma_n \) is artinian, with \( \dim K(\Gamma_n) = n \). Furthermore, if it is given the natural grading with \( |\chi_i| = 1 \), then its Hilbert series is \( \sum d_i t^i \) where \( d_i \) is the number of \( w \leq n \) with \( \lambda(w) = i \).

3. There is a 1-1 bijection between the minimal monomial generators of \( I_n \) of minimal support \( v \), and the solutions in non-negative integers to the equation

\[
\log n - \log p_v < \sum_{i=v}^r b_i \log p_i \leq \log n \tag{9}
\]

4. If we denote by \( C_{n,v} \), the number of such solutions, then the Poincaré-Betti series of the free minimal resolution of \( K \) as a cyclic module over \( \Gamma_n \) is the following rational function:

\[
P(\text{Tor}_*^\Gamma(n, K), t) = \frac{(1+t)^r}{1 - t^2(\sum_{i=1}^r (1+t)(i-1)C_{n,v-i+1})} \tag{10}
\]

We will show this result, and also give the graded Poincaré-Betti series. For this, we define the number \( C_{n,v,d} \) which counts the number of minimal generators of \( I_n \) of minimal support \( v \) and total degree \( d \). We determine some elementary properties of the numbers \( C_{n,v,d} \) and \( C_{n,v} \).
2. The ring of number-theoretic functions and its truncations

2.1. Norms, degrees, and multiplicativity. For a monomial \( M \ni m = x_1^{a_1} \cdots x_n^{a_n} \), we define the weight of \( m \) as \( w(m) = p_1^{a_1} \cdots p_n^{a_n} \) (we put \( w(1) = 1 \)). Hence \( w \) gives a bijection between \( M \) and \( \mathbb{N}^+ \). Furthermore, we can define a term order on \( M \) by \( m > m' \) iff \( w(m) > w(m') \). If we define the initial monomial \( \text{in}(f) \) of \( f \in K[[X]] \) as the monomial in \( \text{Supp}(f) \) minimal with respect to \( > \), then \( \text{in}(f) \) is easily seen to correspond to the norm \( N(\alpha) \) of a number-theoretic function \( \alpha \), defined as the smallest \( n \) such that \( \alpha(n) \neq 0 \). Here, we must use \( w \) and \( \Phi \) to identify \( M \) and \( \mathbb{N}^+ \) and \( K[[X]] \) and \( \Gamma \). As observed in \([2]\), the norm is multiplicative: \( N(\alpha \beta) = N(\alpha)N(\beta) \).

Cashwell and Everett also define the degree \( D(\alpha) \) to mean the smallest \( d \) such that there exists an \( n \) with \( \lambda(n) = d \) and \( \alpha(n) \neq 0 \). This corresponds the smallest total degree of a monomial in \( \text{Supp}(f) \). Furthermore, the norm \( M(\alpha) \), defined as the smallest integer \( n \) with \( \lambda(n) = D(\alpha) \), \( \alpha(n) \neq 0 \), corresponds to the initial monomial of \( f \) under the term order obtained by refining the total degree partial order with the term order \( > \).

A multiplicative function is an element \( \alpha \in \Gamma \) such that \( \alpha(1) = 1 \) and \( \alpha(ab) = \alpha(a)\alpha(b) \) whenever \( a \) and \( b \) are relatively prime. Cashwell and Everett observes that a multiplicative function is necessarily a unit in \( \Gamma \). One can further observe that if \( \alpha \) is multiplicative, then \( f = \Phi(\alpha) \) can be written

\[
f(x_1, x_2, x_3, \ldots) = f_1(x_1)f_2(x_2)f_3(x_3) \cdots
\]

where each \( f_i(x_i) \in K[[x_i]] \) is invertible. In particular, the constant function \( \Gamma \ni \nu_0 \) with \( \nu_0(n) = 1 \) for all \( n \), corresponds to

\[
\sum_{m \in M} m = \frac{1}{1 - x_1} \frac{1}{1 - x_2} \frac{1}{1 - x_3} \cdots
\]

Since the Möbius function is defined to be the inverse of this function, we get that it corresponds to

\[
(1 - x_1)(1 - x_2)(1 - x_3) \cdots = 1 - \left( \sum_{i=1}^{\infty} x_i \right) + \left( \sum_{i<j} x_i x_j \right) - \left( \sum_{i<j<k} x_i x_j x_k \right) + \cdots
\]

2.2. Truncations of the ring of number-theoretic functions. Let \( n, n' \in \mathbb{N}^+ \), \( n' > n \). Then there is a \( K \)-algebra epimorphism

\[
\varphi_n' : \Gamma_{n'} \to \Gamma_n
\]

\[
\varphi_n'(f)(m) = \begin{cases} f(m) & m \leq n \\ 0 & m > n \end{cases}
\]

Hence, the \( \Gamma_n \)'s form an inverse system.

**Lemma 2.1.** \( \lim \Gamma_n \cong \Gamma \).

**Proof.** Given any \( f \in \Gamma \), the sequence \( (\pi_1(f), \pi_2(f), \pi_3(f), \ldots) \) is coherent. Conversely, given any coherent sequence \( (g_1, g_2, g_3, \ldots) \), we can define \( g : \mathbb{N} \to K \) by \( g(m) = g_i(m) \) where \( i \geq m \). \( \square \)

As a side remark, we note that

**Lemma 2.2.** The decreasing filtration

\[
J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots
\]

is separated, that is, \( \cap_n J_n = (0) \).

**Definition 2.3.** We define

\[
I_n = K[[X]] \{ m \in M \, | \, w(m) > n \},
\]

that is, as the monomial ideal in \( K[[X]] \) generated by all monomials of weight strictly higher than \( n \). We put \( A_n = \frac{K[[X]]}{I_n} \).
Theorem 2.7. The bi-graded Hilbert series of $K$ is bihomogeneous, this grading is inherited by $\Gamma$. Proof. As a vector space, $K[[X]] \cong U \oplus I_n$, where $U$ consists of all functions supported on monomials of weight $\leq n$. It follows that $A_n \cong U$ as $K$ vector spaces. Of course, there are exactly $n$ monomials of weight $\leq n$. Finally, if $s > r$ then $w(x_s) = p_s > n$, hence $x_s \in I_n$. This establishes part II of the main theorem. Part I of the main theorem is now proved.

Proposition 2.6. A $K$-basis of $A_n$ is given by all monomials of weight $\leq n$. Hence $A_n$ is an artinian algebra, with $\dim_K(A_n) = n$. Putting $r = r(n)$, we have that

$$A_n = \frac{K[[X]]}{I_n} \cong \frac{K[x_1, \ldots, x_r]}{I_n \cap K[x_1, \ldots, x_r]}$$

Proof. We will abuse notations and identify $I_n$ and its contraction $I_n \cap K[x_1, \ldots, x_r]$.

Lemma 2.5. $\Gamma_n \cong A_n$.

Proof. Since $A_n$ has a $K$-basis is given by all monomials of weight $\leq n$, the two $K$-algebras are isomorphic as $K$-vector spaces. The multiplication in $A_n$ is induced from the multiplication in $K[[X]]$, with the extra condition that monomials of weight $> n$ are truncated. This is the same multiplication as in $\Gamma_n$.

Proposition 2.6. $I_n$ is a strongly stable ideal, with respect to the reverse order of the variables.

Proof. We must show that if $m \in I_n$, and $x_i | m$, then $mx_i/x_i \in I$ for $i \leq j \leq r$. We have that $w(mx_i/x_i) = w(m)p_j/p_i > w(m) > n$. Part II of the main theorem is now proved.

We give $K[x_1, \ldots, x_r]$ an $\mathbb{N}^2$-grading by giving the variable $x_i$ bi-degree $(1, p_i)$. Since each $I_n$ is bihomogeneous, this grading is inherited by $A_n$.

Theorem 2.7. The bi-graded Hilbert series of $A_n$ is given by

$$A_n(t, u) = \sum_{i,j} c_{ij} t^i u^j,$$

where $c_{ij}$ is the number of $p_1^{a_1} \cdots p_r^{a_r} \leq n$ with $\sum a_r = i$ and $\sum a_r p_r = j$. Furthermore,

$$A_n(t, 1) = \sum_i d_i t^i$$

$$A_n(1, u) = \sum_j c_j u^j$$

where $d_i$ is the number of $w \leq n$ with $\lambda(w) = i$, and $c_i$ is the number of $w \leq n$ with $\lambda(w) = i$. In particular, the $t^i$-coefficient of $A_n(t, 1)$ is the number of prime numbers $\leq n$.

Proof. The monomial $x_1^{a_1} \cdots x_r^{a_r}$ has bi-degree $(\sum_{i=1}^n a_i, \sum a_i p_i)$.

This establishes part II of the main theorem.

3. Minimal generators for $I_n$

Let $n \in \mathbb{N}^+$, and let $r = r(n)$. We have that

$$x_1^{a_1} \cdots x_r^{a_r} = m \in I_n \iff w(m) > n \iff \prod_{i=1}^r p_i^{a_i} > n. \quad (14)$$

We denote by $G(I_n)$ the set of minimal monomial generators of $I_n$. For $m = x_1^{a_1} \cdots x_r^{a_r}$ to be an element of $G(I_n)$ it is necessary and sufficient that $m \in I_n$ and that for $1 \leq v \leq r$, $x_v | m \implies m/x_v \notin I_n$. In other words,

$$1 \leq j \leq n, a_j > 0 \implies n < \prod_{i=1}^r p_i^{a_i} \leq p_j n. \quad (15)$$
\textbf{Definition 3.1.} For \( n, v, d \) positive integers, we define:
\begin{align*}
C_n &= \# G(I_n) \quad (16) \\
C_{n,v} &= \# \{ m \in G(I_n) \mid \min(m) = v \} \quad (17) \\
C_{n,v,d} &= \# \{ m \in G(I_n) \mid \min(m) = v, |m| = d \} \quad (18)
\end{align*}

\textbf{Theorem 3.2.} \( C_{n,v} \) is the number of solutions \( (b_1, \ldots, b_r) \in \mathbb{N}^r \) to the equation
\begin{equation}
\log n - \log p_v < \sum_{i=v}^{r} b_i \log p_i \leq \log n. 
\end{equation}
Equivalently, \( C_{n,v} \) is the number of integers \( x \) such that \( n/p_v < x \leq n \) and such that no prime factors of \( x \) are smaller than \( p_v \).

Similarly, \( C_{n,v,d} \) is the number of solutions \( (b_1, \ldots, b_r) \in \mathbb{N}^r \) to the system of equations
\begin{equation}
\log n - \log p_v < \sum_{i=v}^{r} b_i \log p_i \leq \log n \\
\sum_{i=1}^{r} b_i = d - 1.
\end{equation}

or equivalently, \( C_{n,v,d} \) is the number of integers \( x \) such that \( n/p_v < x \leq n \) and such that no prime factors of \( x \) are smaller than \( p_v \), and with the additional constraint that \( \lambda(x) = d \).

\textbf{Proof.} We have that \( a_v > 0, a_w = 0 \) for \( w < v \). Hence equation (15) implies that
\begin{equation}
n < \prod_{j=v}^{r} p_i^{a_j} \leq p_v n.
\end{equation}
Putting \( b_v = a_v - 1, b_j = a_j \) for \( j > v \) we can write this as
\begin{equation}
n < p_v \prod_{j=v}^{r} p_i^{b_i} \leq p_v n \iff n/p_v < \prod_{j=v}^{r} p_i^{b_i} \leq n
\end{equation}
from which (19) follows by taking logarithms. This implies (20) as well. \( \blacksquare \)

We have now proved part 1 of the main theorem.

\textbf{Example 3.3.} The first few \( I_n \)'s are as follows: \( I_2 = (x_1^2) \), \( I_3 = (x_1^2, x_2^2, x_1 x_2) \), \( I_4 = (x_1^3, x_2^2, x_1 x_2) \), \( I_5 = (x_1^3, x_2^2, x_1 x_2, x_3^2, x_1 x_3, x_2 x_3) \).

We tabulate \( C_{n,i} \) and \( C_{n,i,j} \), the latter in form of the polynomial \( u^{-2} \sum_j \lambda_{m, j} u^j \) in the tables 3 and 4.

\textbf{Theorem 3.4.} (1) \( C_{n,v} = 0 \) for \( v > r(n) \)
(2) \( \forall n \in \mathbb{N} : \forall v \leq r(n) : C_{n,1+r(n)-v} \geq v \)
(3) \( \forall n \in \mathbb{N} : C_n \geq \binom{r(n)+1}{2} \)
(4) \( \forall v \in \mathbb{N} : \exists N : \forall n \geq N : C_{n,1+r(n)-v} = v \)
(5) If \( n \) is even, then \( C_{n,v} = C_{n-1,v} \) for all \( v \)
(6) \( C_{n,1} = \lfloor n/2 \rfloor \).

\textbf{Proof.} (1) Obvious.
(2) and (3) It suffices to show that for any subset \( S \subset \{1, \ldots, r\} \) of cardinality 1 or 2, there is an \( m \in G(I_n) \) with \( \text{Supp}(m) = S \). If \( S = \{ i \} \) then there is an unique positive integer \( a \) such that \( p_i^{a-1} \leq n < p_i^a \), and \( m = x_i^a \) is the desired generator. If \( S = \{ i, j \} \) with \( i < j \) then we claim that there is a positive integer \( a \) such that \( x_i^a x_j \in G(I_n) \). Namely, choose \( b \) such that \( p_j^{b-1} \leq n < p_j^b \), then since \( p_i < p_j \) one has \( n < p_i^{a-1} p_j \). Hence \( x_i^{p_j^{b-1}} x_j \in I_n \), so it is a multiple of some minimal generator. By the definition of \( b \), this minimal generator must be of the form \( x_i^a x_j \) for some \( a \), which establishes the claim.
We must show that the number of solutions in $\mathbb{N}^r$ to
\[
\frac{n}{2} < \prod_{i=1}^{r} p_i^{b_i} \leq n
\]
is precisely $\lceil \frac{n}{2} \rceil$. Obviously, any integer $x \in (\frac{n}{2}, n]$ fits the bill; there are $\lceil \frac{n}{2} \rceil$ of those.

The case $v = 1$ follows from (3). Hence, it suffices to show that if $v > 1$, $x \in (\frac{n}{p_v}, n] \cap \mathbb{N}$, and if $x$ has no prime factor $< p_v$, then $x \in (\frac{n}{p_v}, n-1] \cap \mathbb{N}$. The only way this can fail to happen is if $x = n$, but then $x$ is even, and has the prime factor $2 = p_1 < p_v$, a contradiction.

For large enough $n$, the only integers $x \leq n$ with all prime factors $\geq 1 + r(n) - v$ are $p_1 + r(n)-v, \ldots, p_r(n)$. There is $v$ of these, and they are all $> \frac{n}{p_v}$.

\begin{theorem}
1. $C_{n,v,d} = 0$ for $v > r(n)$, and for $d < 2$,
2. $\forall v \in \mathbb{N} : \exists N : \forall n \geq N : C_{n,1+r(n)-v,2} = v$, $C_{n,1+r(n)-v,d} = 0$ for $d \not= 2$,
3. $\binom{r(n)}{2} = \# \{ m \in \mathbb{N}^+ | m \leq n, \lambda(m) = 2 \}$.
\end{theorem}

\begin{proof}
The first and the last assertions are obvious. The second one follows from the proof of (4) in the previous lemma.
\end{proof}

4. Poincaré series

In (3), a minimal free multi-graded resolution of an ideal $I$ over $S$ is given, where $S = K[x_1, \ldots, x_r]$ is a polynomial ring, and $I \subset (x_1, \ldots, x_r)^2$ is a stable ideal. As a consequence, the following formula for the Poincaré-Betti series is derived:
\[
P(\text{Tor}_{s}^{S}(I, K), t) = \sum_{a \in G(I)} (1 + t)^{\max(a)-1}
\]
where $G(I)$ is the minimal generating set of $I$. Since the resolution is multi-graded, (21) can be modified to yield a formula for the graded Poincaré-Betti series (we here consider $S$ as $\mathbb{N}$-graded, with each variable given weight 1):
\[
P(\text{Tor}_{s}^{S}(I, K), t, u) = \sum_{a \in G(I)} u^{\lceil a \rceil}(1 + t)^{\max(a)-1}
\]
We will use the following variant of this result:

\begin{theorem}[Eliahou-Kervaire]
Let $I \subset (x_1, \ldots, x_r)^2 \subset K[x_1, \ldots, x_r] = S$ be a stable monomial ideal. Put
\[
b_{i,d} = \# \{ m \in G(I) | \max(m) = i, |m| = d \}
\]
\[
b_i = \# \{ m \in G(I) | \max(m) = i \}
\]

Then
\[
P(\text{Tor}_{s}^{S}(I, K), t) = \sum_{i=1}^{r} b_i (1 + t)^{(i-1)}
\]
\[
P(\text{Tor}_{s}^{S}(I, K), t, u) = \sum_{i=1}^{r} \left( 1 + tu \right)^{(i-1)} \sum_{j} b_{i,j} u^j.
\]

For the Betti-numbers we have that
\[
\beta_i = \dim_K (\text{Tor}_{q}^{S}(I, K)) = \sum_{i=1}^{r} b_{i} \left( i - 1 \right) / q.
\]

From Proposition (2.6) we have that the ideals $I_n$ are stable after reversing the order of the variables. Hence, replacing max by min, and hence $b_i$ with $C_{n,1+r-i}$, we get:
Corollary 4.2. Let \( n \in \mathbb{N}^+ \), \( r = r(n) \), \( S = K[x_1, \ldots, x_r] \). Then
\[
P(\text{Tor}^S(I_n, K), t) = \sum_{i=1}^{r} C_{n,1+r-i}(1 + t)^{(i-1)}
\]
(28)
\[
P(\text{Tor}^S(I_n, K), t, u) = \sum_{i=1}^{r}(1 + tu)^{(i-1)} \sum_{j} C_{n,1+r-i,j} u^j.
\]
(29)

For the Betti-numbers we have that
\[
\beta_q = \sum_{i=1}^{r} C_{n,1+r-i}(i - 1) q^i.
\]
(30)

In [4, 5] it is shown that if \( S = K[x_1, \ldots, x_r] \) and \( I \) is a stable monomial ideal in \( S \), then \( S/I \) is a Golod ring. Hence, from a result of Golod [4] (see also [5]), it follows that
\[
P(\text{Tor}^S(I_n, K), t) = \frac{(1 + t)^r}{1 - t^2 P(\text{Tor}^S(I_n, K), t)}
\]
(31)

Regarding \( S \) as an \( \mathbb{N} \)-graded ring, one can show that in fact
\[
P(\text{Tor}^S(I_n, K), t, u) = \frac{(1 + tu)^r}{1 - t^2 P(\text{Tor}^S(I_n, K), t, u)}
\]
(32)

The following theorem is an immediate consequence:

Theorem 4.3 (Herzog-Aramova, Peeva). Let \( S = K[x_1, \ldots, x_r] \), and suppose that \( I \) is a stable monomial ideal in \( S \). Put
\[
b_{i,d} = \# \{ x \in G(I) \mid \text{max}(x) = i, |x| = d \}
\]
\[
b_i = \# \{ x \in G(I) \mid \text{max}(x) = i \}
\]

Then, for \( R = S/I \), we have that
\[
P(\text{Tor}^R(I_n, K), t) = \frac{(1 + t)^r}{1 - t^2 \sum_{i=1}^{r}(1 + t)^{(i-1)} \sum_{j} b_i}
\]
(33)
\[
P(\text{Tor}^R(I_n, K), t, u) = \frac{(1 + tu)^r}{1 - t^2 \sum_{i=1}^{r}(1 + tu)^{(i-1)} \sum_{j} b_{i,j} u^j}
\]
(34)

Specialising to the case of \( A_n \), we obtain:

Corollary 4.4. Let \( n \in \mathbb{N}^+ \), and let \( r = r(n) \). Regard \( A_n \) as a naturally graded \( K \)-algebra, with each \( x_i \) given weight 1, and regard \( K \) as a cyclic \( A \)-module. Then
\[
P(\text{Tor}^A(I_n, K), t) = \frac{(1 + t)^r}{1 - t^2 \sum_{i=1}^{r}(1 + t)^{(i-1)} C_{n,r-i+1}}
\]
(35)
\[
P(\text{Tor}^A(I_n, K), t, u) = \frac{(1 + tu)^r}{1 - t^2 \left( \sum_{i=1}^{r}(1 + tu)^{(i-1)} \sum_{j} C_{n,r-i+j} u^j \right)}
\]
(36)

Part IV of the main theorem is now proved.

Example 4.5. We consider the case \( n = 5 \), then \( r = r(n) = 3 \), so \( S = K[x_1, x_2, x_3] \) and \( I = I_5 = \langle x_1^3, x_1 x_2, x_1 x_3, x_2^3, x_2 x_3, x_3^3 \rangle \). We get that \( C_{5,1} = 3, C_{5,2} = 2, C_{5,3} = 1 \). According to our formula\(^{1}\) we have
\[
P^S_I(t) = 1 + 2(1 + t) + 3(1 + t)^2 = 6 + 8t + 3t^2
\]
\[
P^{S/I}_K(t) = \frac{(1 + t)^r}{1 - t^2 P^S_I(t)} = \frac{1}{1 - 3t}
\]

\(^{1}\)Here, we have used the abbreviation \( P^S_I(t) = P(\text{Tor}^S(I_n, K), t) \), we will also write \( P^{S/I}_K(t) = P(\text{Tor}^{S/I}_n(K, K), t) \) et cetera.
When we consider the grading by total degree, we have that $C_{5,1,2} = 2$, $C_{5,1,3} = 1$, $C_{5,2,2} = 2$, $C_{5,3,2} = 1$. Hence, our formulas yield

$$P^S_I(t, u) = u^2 + 2u^2(1 + t) + (2u^2 + u^3)(1 + t)^2$$

$$= 5u^2 + u^3 + (6u^2 + 2u^3)t + (2u^2 + u^3)t^2$$

$$P^{S/I}_K(t, u) = -\frac{1 + tu}{u^3t^2 + 2t^2u^2 + 2tu - 1}$$

We list the first few Poincaré-Betti series $P(\text{Tor}^A_n(K, K), t, u)$ in table 3.

**Conjecture 4.6.** $P(\text{Tor}^A_n(K, K), t) = \frac{(1+t)^{\ell_1(n)}}{q_n(t)}$, $q_n(t) = \sum_{i=0}^{\ell_2(n)} h_i(n)t^i$, with

1. $q_n(-1) \neq 0$,
2. $\ell_1(n)$ is the number of odd primes $p$ such that $p^2 \leq n$,
3. $\ell_2(n) = \ell_1(n) + 1$,
4. $h_0(n) = -1$,
5. $h_1(n) = r(n) - \ell_1(n)$,
6. $h_{\ell_2(n)}(n) = C_{n,1} = [n/2]$.

5. **Acknowledgements**

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Figure 1. The numbers $C_n$ and $C_{n,i}$.

| $n$ | $i = 1$ | $i = 2$ | $i = 3$ | $i = 4$ | $i = 5$ | $i = 6$ | $i = 7$ | $i = 8$ | $i = 9$ | $i = 10$ |
|-----|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 2   | 1       | 1       |         |         |         |         |         |         |         |         |
| 3   | 1       | 2       | 1       |         |         |         |         |         |         |         |
| 4   | 3       | 2       | 1       | 1       |         |         |         |         |         |         |
| 5   | 6       | 3       | 2       | 1       | 1       |         |         |         |         |         |
| 6   | 6       | 3       | 2       | 1       | 1       | 1       |         |         |         |         |
| 7   | 10      | 4       | 3       | 2       | 1       | 1       | 1       |         |         |         |
| 8   | 10      | 4       | 3       | 2       | 1       | 1       | 1       | 1       |         |         |
| 9   | 11      | 5       | 3       | 2       | 1       | 1       | 1       | 1       | 1       |         |
| 10  | 11      | 5       | 3       | 2       | 1       | 1       | 1       | 1       | 1       | 1       |
| 11  | 16      | 6       | 4       | 3       | 2       | 1       | 1       | 1       | 1       | 1       |
| 12  | 16      | 6       | 4       | 3       | 2       | 1       | 1       | 1       | 1       | 1       |
| 13  | 22      | 7       | 5       | 4       | 3       | 2       | 1       | 1       | 1       | 1       |
| 14  | 22      | 7       | 5       | 4       | 3       | 2       | 1       | 1       | 1       | 1       |
| 15  | 23      | 8       | 5       | 4       | 3       | 2       | 1       | 1       | 1       | 1       |
| 16  | 23      | 8       | 5       | 4       | 3       | 2       | 1       | 1       | 1       | 1       |
| 17  | 30      | 9       | 6       | 5       | 4       | 3       | 2       | 1       | 1       | 1       |
| 18  | 30      | 9       | 6       | 5       | 4       | 3       | 2       | 1       | 1       | 1       |
| 19  | 38      | 10      | 7       | 6       | 5       | 4       | 3       | 2       | 1       | 1       |
| 20  | 38      | 10      | 7       | 6       | 5       | 4       | 3       | 2       | 1       | 1       |
| 21  | 39      | 11      | 7       | 6       | 5       | 4       | 3       | 2       | 1       | 1       |
| 22  | 39      | 11      | 7       | 6       | 5       | 4       | 3       | 2       | 1       | 1       |
| 23  | 48      | 12      | 8       | 7       | 6       | 5       | 4       | 3       | 2       | 1       |
| 24  | 48      | 12      | 8       | 7       | 6       | 5       | 4       | 3       | 2       | 1       |
| 25  | 50      | 13      | 9       | 7       | 6       | 5       | 4       | 3       | 2       | 1       |
| 26  | 50      | 13      | 9       | 7       | 6       | 5       | 4       | 3       | 2       | 1       |
| 27  | 51      | 14      | 9       | 7       | 6       | 5       | 4       | 3       | 2       | 1       |
| 28  | 51      | 14      | 9       | 7       | 6       | 5       | 4       | 3       | 2       | 1       |
| 29  | 61      | 15      | 10      | 8       | 7       | 6       | 5       | 4       | 3       | 2       |
| 30  | 61      | 15      | 10      | 8       | 7       | 6       | 5       | 4       | 3       | 2       |

Figure 2. The numbers $C_{n,i,g}$.

| $n$ | $i = 1$ | $i = 2$ | $i = 3$ | $i = 4$ | $i = 5$ | $i = 6$ | $i = 7$ | $i = 8$ | $i = 9$ | $i = 10$ |
|-----|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 2   | 1       | 1       |         |         |         |         |         |         |         |         |
| 3   | 2       | 1       |         |         |         |         |         |         |         |         |
| 4   | u+1     | 1       |         |         |         |         |         |         |         |         |
| 5   | u+2     | 2       | 1       |         |         |         |         |         |         |         |
| 6   | 2u+1    | 2       | 1       |         |         |         |         |         |         |         |
| 7   | 2u+2    | 3       | 2       | 1       |         |         |         |         |         |         |
| 8   | u^2+u+2 | 3       | 2       | 1       |         |         |         |         |         |         |
| 9   | u^2+2u+2| u+2     | 2       | 1       |         |         |         |         |         |         |
| 10  | u^2+3u+1| u+2     | 2       | 1       |         |         |         |         |         |         |
| 11  | u^2+3u+2| u+3     | 3       | 2       | 1       |         |         |         |         |         |
| 12  | 2u^2+2u+2| u+3   | 3       | 2       | 1       |         |         |         |         |         |
| 13  | 2u^2+2u+3| u+4   | 4       | 3       | 2       | 1       |         |         |         |         |
| 14  | 2u^2+3u+2| u+4   | 4       | 3       | 2       | 1       |         |         |         |         |
| 15  | 2u^2+4u+2| 2u+3  | 4       | 3       | 2       | 1       |         |         |         |         |
| 16  | u^3+u^2+4u+2| 2u+3  | 4       | 3       | 2       | 1       |         |         |         |         |
| 17  | u^3+u^2+4u+3| 2u+4  | 5       | 4       | 3       | 2       | 1       |         |         |         |
| 18  | u^3+u^2+3u+3| 2u+4  | 5       | 4       | 3       | 2       | 1       |         |         |         |
| 19  | u^3+2u^2+3u+4| 2u+5  | 6       | 5       | 4       | 3       | 2       | 1       |         |         |
| 20  | u^3+3u^2+4u+4| 2u+5  | 6       | 5       | 4       | 3       | 2       | 1       |         |         |
| 21  | u^3+3u^2+4u+4| 3u+4  | 6       | 5       | 4       | 3       | 2       | 1       |         |         |
| 22  | u^3+3u^2+4u+4| 3u+4  | 6       | 5       | 4       | 3       | 2       | 1       |         |         |
| 23  | u^3+3u^2+4u+4| 3u+5  | 7       | 6       | 5       | 4       | 3       | 2       | 1       |         |
| 24  | u^3+2u^2+4u+4| 3u+5  | 7       | 6       | 5       | 4       | 3       | 2       | 1       |         |
| 25  | u^3+2u^2+5u+4| 4u+5  | 8       | 7       | 6       | 5       | 4       | 3       | 2       | 1       |
| 26  | u^3+2u^2+6u+3| 4u+5  | 8       | 7       | 6       | 5       | 4       | 3       | 2       | 1       |
| 27  | u^3+3u^2+6u+3| u+6   | 6       | 5       | 4       | 3       | 2       | 1       |         |         |
| 28  | u^3+4u^2+5u+4| u+6   | 6       | 5       | 4       | 3       | 2       | 1       |         |         |
| 29  | u^3+4u^2+5u+4| u+7   | 7       | 6       | 5       | 4       | 3       | 2       | 1       |         |
| 30  | u^3+5u^2+6u+4| u+7   | 7       | 6       | 5       | 4       | 3       | 2       | 1       |         |
\begin{tabular}{|c|c|c|}
\hline
\(n\) & \textit{Graded} & \textit{Non-graded} \\
\hline
2 & \(- (tu - 1)^{-1}\) & \(- (t - 1)^{-1}\) \\
3 & \(- (2tu - 1)^{-1}\) & \(- (2t - 1)^{-1}\) \\
4 & \(- \frac{u+tu}{1+tu}tu^{-1}\) & \(- \frac{2}{1+tu}\) \\
5 & \(- \frac{u^2+2u}{1+tu}tu^{-1}\) & \(- \frac{5}{4}+\frac{1}{t+1}\) \\
6 & \(- \frac{(u^2+u)tu}{1+tu}\) & \(- \frac{5}{1+4}\) \\
7 & \(- \frac{2u+u}{1+tu}tu^{-1}\) & \(- \frac{5}{4}+\frac{1}{t+1}\) \\
8 & \(- \frac{u^2+2u}{1+tu}tu^{-1}\) & \(- \frac{5}{4}+\frac{1}{t+1}\) \\
9 & \(- \frac{u^2+2u+2u^3}{1+2tu+2u^2}tu^{-1}\) & \(- \frac{5}{4}+\frac{1}{t+1}\) \\
10 & \(- \frac{(u^2+4u+3u^3)tu}{1+2tu+2u^2}tu^{-1}\) & \(- \frac{5}{4}+\frac{1}{t+1}\) \\
11 & \(- \frac{(u^2+4u+3u^3)tu}{1+2tu+2u^2}tu^{-1}\) & \(- \frac{5}{4}+\frac{1}{t+1}\) \\
12 & \(- \frac{(u^2+2u+2u^3)tu}{1+2tu+2u^2}tu^{-1}\) & \(- \frac{5}{4}+\frac{1}{t+1}\) \\
13 & \(- \frac{(u^2+2u+3u^3)tu}{1+2tu+2u^2}tu^{-1}\) & \(- \frac{5}{4}+\frac{1}{t+1}\) \\
14 & \(- \frac{(u^2+3u+2u^3)tu}{1+2tu+2u^2}tu^{-1}\) & \(- \frac{5}{4}+\frac{1}{t+1}\) \\
15 & \(- \frac{(u^2+4u+2u^3)tu}{1+2tu+2u^2}tu^{-1}\) & \(- \frac{5}{4}+\frac{1}{t+1}\) \\
16 & \(- \frac{(u^2+4u+6u^3)tu}{1+2tu+2u^2}tu^{-1}\) & \(- \frac{5}{4}+\frac{1}{t+1}\) \\
17 & \(- \frac{(u^2+4u+6u^3)tu}{1+2tu+2u^2}tu^{-1}\) & \(- \frac{5}{4}+\frac{1}{t+1}\) \\
18 & \(- \frac{(u^2+2u+3u^3)tu}{1+2tu+2u^2}tu^{-1}\) & \(- \frac{5}{4}+\frac{1}{t+1}\) \\
19 & \(- \frac{(u^2+3u+4u^3)tu}{1+2tu+2u^2}tu^{-1}\) & \(- \frac{5}{4}+\frac{1}{t+1}\) \\
20 & \(- \frac{(u^2+3u+4u^3)tu}{1+2tu+2u^2}tu^{-1}\) & \(- \frac{5}{4}+\frac{1}{t+1}\) \\
21 & \(- \frac{(u^2+3u+4u^3)tu}{1+2tu+2u^2}tu^{-1}\) & \(- \frac{5}{4}+\frac{1}{t+1}\) \\
22 & \(- \frac{(u^2+3u+4u^3)tu}{1+2tu+2u^2}tu^{-1}\) & \(- \frac{5}{4}+\frac{1}{t+1}\) \\
23 & \(- \frac{(u^2+3u+4u^3)tu}{1+2tu+2u^2}tu^{-1}\) & \(- \frac{5}{4}+\frac{1}{t+1}\) \\
24 & \(- \frac{(u^2+3u+4u^3)tu}{1+2tu+2u^2}tu^{-1}\) & \(- \frac{5}{4}+\frac{1}{t+1}\) \\
25 & \(- \frac{(u^2+3u+4u^3)tu}{1+2tu+2u^2}tu^{-1}\) & \(- \frac{5}{4}+\frac{1}{t+1}\) \\
\hline
\end{tabular}

**Figure 3.** Graded and non-graded Poincaré-Betti series of the minimal free resolution of \(K\) over \(A_n\).