Vertex integrals in heavy-particle theories

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Abstract. We give the results of complete analytical computations of two- and three-point loop integrals occurring in heavy particle theories, with and without velocity change, for arbitrary values of external momenta and masses.

INTRODUCTION

In this talk we consider a class of one-loop integrals occurring in heavy-particle theories [1], with arbitrary real values for the external masses and residual momenta. We give the results of complete analytical computations of three-point loop integrals with and without velocity change, and two-point loop integrals. The details of the calculations are given in [2], and in a forthcoming paper.

LOOP INTEGRALS

The loop integrals we consider are of the form,

\[ J_{\alpha_1 \cdots \alpha_n}^2 = \frac{i\mu^{4-d}}{(2\pi)^d} \int d^d \ell \frac{\ell^{\alpha_1} \cdots \ell^{\alpha_n}}{(2v \cdot (\ell + k) - \delta M + i\varepsilon)(\ell^2 - m^2 + i\varepsilon)} \]

\[ J_{\alpha_1 \cdots \alpha_n}^3 = \frac{i\mu^{4-d}}{(2\pi)^d} \int d^d \ell \frac{\ell^{\alpha_1} \cdots \ell^{\alpha_n}}{(2v_1 \cdot (\ell + k_1) - \delta M_1 + i\varepsilon)(2v_2 \cdot (\ell + k_2) - \delta M_2 + i\varepsilon)} \times \frac{1}{(\ell^2 - m^2 + i\varepsilon)} \]

\[ H_{\alpha_1 \cdots \alpha_n}^3 = \frac{i\mu^{4-d}}{(2\pi)^d} \int d^d \ell \frac{\ell^{\alpha_1} \cdots \ell^{\alpha_n}}{(2v \cdot \ell - \delta M + i\varepsilon)((\ell - k_1)^2 - m^2 + i\varepsilon)} \times \frac{1}{((\ell - k_2)^2 - m^2 + i\varepsilon)} \]

Here \( v_i^\mu, i = 1, 2 \), are the velocities of the external heavy legs, \( k_i^\mu \) their residual momenta, and \( \delta M_i \) their mass splittings relative to the common heavy mass of the corresponding heavy quark symmetry multiplet. \( m \) and \( m' \) are the masses of the light particles within the loops, which in chiral theories are light pseudoscalar mesons. These integrals are defined in \( d = 4 - \varepsilon \) dimensions, \( \mu \) being the mass scale of dimensional regularization.
Their degrees of divergence are \( n + d - 3 \) for \( f_2^{\alpha_1 \cdots \alpha_n} \), \( n + d - 4 \) for \( f_3^{\alpha_1 \cdots \alpha_n} \) and \( n + d - 5 \) for \( f_3^{\alpha_1 \cdots \alpha_n} \). The factor of 2 in front of \( v_1^\mu \) corresponds to our normalization of the heavy-particle propagators.

Our method of calculation [2] is to obtain the integrals (1) as large-mass limits of ordinary loop integrals. We closely follow the approach of [3] for the computation of scalar integrals, which are greatly simplified in the large-mass limit, and the method of [4] to express tensor integrals in terms of scalar ones.

**TWO-POINT INTEGRALS**

Two-point integrals with one heavy propagator have been given in [2, 5, 6]. The scalar two-point integral \( f_2 \) is a function of \( m \) and \( \Delta = \delta M - 2v \cdot k \). We write it in terms of \( \xi = \Delta / (2m) \),

\[
f_2(\Delta, m) = \frac{\Delta}{32\pi^2} \left( \frac{2}{\epsilon} + \log \left( \frac{\mu^2}{m^2} \right) + 2 \right) + \frac{m}{16\pi^2} \mathcal{F}(\xi)
\]

with \( \mathcal{F}(x) = \sqrt{x^2 - 1 + i\epsilon} \left[ \log \left( x - \sqrt{x^2 - 1 + i\epsilon} \right) - \log \left( x + \sqrt{x^2 - 1 + i\epsilon} \right) \right] \). The coefficient of the dimensional regularization pole vanishes when \( \Delta = 0 \). This is due to the fact that the real part of the integrand in \( f_2 \) is parity-odd when \( \Delta = 0 \).

The vector two-point integral \( g_2^\mu(v^\alpha, \Delta, m) \) is given in terms of only one form factor, \( g_2^\mu(v^\alpha, \Delta, m) = F(\Delta, m) v^\mu \), with \( F(\Delta, m) = v_\mu g_2^\mu(v^\alpha, \Delta, m) = 1/2A_0(m) + \Delta/2f_2(\Delta, m) \), where \( A_0 \) is the standard one-point scalar integral (see the appendix of [2]). The second-rank tensor integral is computed analogously, explicit results being given in [2].

**THREE-POINT INTEGRALS WITH VELOCITY CHANGE**

The scalar three-point integral with velocity change, \( f_3 = f_3(v_1 \cdot v_2, \Delta_1, \Delta_2, m) \), where \( \Delta_j = \delta M_j - 2v_j \cdot k_j \), can be expressed in terms of four dilogarithms [2],

\[
f_3 = \frac{1}{64\pi^2} \frac{1 - \Omega^2}{\Omega} \left\{ \frac{2}{\epsilon} \log(\alpha) + \log^2(\alpha) + \sum_{k, \sigma} (-1)^{k+1} \left[ \frac{1}{2} \log^2 \left( \frac{\xi_{k\sigma}}{\mu} \right) + \text{Li}_2 \left( \frac{-y_0}{\xi_{k\sigma}} \right) \right] \right\}.
\]

The notation is as follows, \( \omega = v_1 \cdot v_2 \), \( \Omega = \sqrt{(\omega - 1)/(\omega + 1)} \), \( \alpha = \omega + \sqrt{\omega^2 - 1} = (1 + \Omega)/(1 - \Omega) \) and \( \mu \) is the mass unit in the MS scheme. The sum extends over \( k = 1, 2 \) and \( \sigma = \pm \), with

\[
y_0 = -\frac{1 + \Omega}{2\Omega} \left( \frac{1 + \Omega}{2} \Delta_1 - \frac{1 - \Omega}{2} \Delta_2 \right),
\]

\[
z_{1\pm} = \frac{1}{2} \left( \frac{1 + \Omega^2}{2\Omega} \Delta_1 - \frac{1 - \Omega^2}{2\Omega} \Delta_2 \pm \sqrt{\Delta_1^2 - 4m^2 + i\epsilon} \right),
\]

\[
z_{2\pm} = \frac{\alpha}{2} \left( \frac{1 - \Omega^2}{2\Omega} \Delta_1 - \frac{1 + \Omega^2}{2\Omega} \Delta_2 \pm \sqrt{\Delta_2^2 - 4m^2 + i\epsilon} \right).
\]
This expression for \( f_3 \) is equivalent to the result given in eq. (30) of [2], though it has been written in a more compact form by means of the identity

\[
\frac{1}{2} \log^2(z) - \log(z) \log(-z) = -\frac{\pi^2}{2} - \frac{1}{2} \log^2(-z),
\]

valid on the first Riemann sheet of the logarithm, and the identity (A.2) of [3] for the dilogarithm.

In order to compute the vector integral \( f_3^\mu \), we define two sets of form factors as \( f_3^\mu = I_1 v_1^\mu + I_2 v_2^\mu \) and \( F_{1,2} = v_1 \cdot f_3 \). These form factors are given by,

\[
I_1 = \frac{1 - \Omega^2}{4\Omega^2} \left[-(1 - \Omega^2)F_1 + (1 + \Omega^2)F_2 \right], \quad I_2 = \frac{1 - \Omega^2}{4\Omega^2} \left[(1 + \Omega^2)F_1 - (1 - \Omega^2)F_2 \right],
\]

with \( F_{1,2} = 1/2 f_2 (\Delta_{1,2}, m) + \Delta_{1,2}/2 f_3 (v_1, v_2, \Delta_{1,2}, m) \). These equations give an explicit expression for \( f_3^\mu \). For the sake of brevity, we omit here the results for the tensor integral \( f_3^{\mu\nu} \), which can be found in [2].

### THREE-POINT INTEGRALS WITH ONE HEAVY PROPAGATOR

The scalar three-point integral \( \mathcal{H}_3 = \mathcal{H}_3 (v \cdot q, q^2, \Delta, m, m') \), with \( q = (k_2 - k_1) / 2 \) and \( \Delta = \delta M - v \cdot (k_1 + k_2) \), can be expressed in terms of eight dilogarithms as,

\[
\mathcal{H}_3 = \frac{1}{(4\pi)^2} \frac{1}{4|q|} \sum_{j=1,2} (\mathcal{F}_1(y_j) + \mathcal{F}_2(x_j) - \mathcal{F}_3(z_j))
\]

\[
\mathcal{F}_1(x) = \text{Li}_2 \left( \frac{z_0 - 4|q|x}{z_0 - 4|q|x} \right) - \text{Li}_2 \left( \frac{z_0}{z_0 - 4|q|x} \right)
\]

\[
\mathcal{F}_2(x) = -\text{Li}_2 \left( \frac{z_0 - 4|q|x}{z_0 - 4|q|x} \right) - \frac{1}{2} \log^2 \left( \frac{z_0 - 4|q|x}{\mu^2} \right)
\]

\[
\mathcal{F}_3(x) = \mathcal{F}_1(x) + \mathcal{F}_2(x) = -\text{Li}_2 \left( \frac{z_0}{z_0 - 4|q|x} \right) - \frac{1}{2} \log^2 \left( \frac{z_0 - 4|q|x}{\mu^2} \right).
\]

The quantities entering these equations are \( |q| = \sqrt{(v \cdot q)^2 - q^2} \) (\( |q| \) is assumed to be real), \( \alpha = \alpha_+ \) with \( \alpha_+ = 2(v \cdot q \pm |q|) \), \( z_0 = -(m^2 - m^2) - \alpha_+ (\Delta - 2|q|) \), and,

\[
x_{1,2} = v \cdot k_2 + 2|q| - \frac{\delta M}{2} \pm \sqrt{\left( v \cdot k_1 - \frac{\delta M}{2} \right)^2 - m^2 + i\epsilon}
\]

\[
y_{1,2} = \frac{1}{2\alpha_+} \left( 4q^2 + m^2 - m^2 \pm \sqrt{(4q^2 - (m' + m)^2)(4q^2 - (m' - m)^2) + i\epsilon\sigma} \right)
\]

\[
z_{1,2} = v \cdot k_2 - \frac{\delta M}{2} \pm \sqrt{\left( v \cdot k_2 - \frac{\delta M}{2} \right)^2 - m^2 + i\epsilon},
\]
where in the expression for $y_j$ we denoted $\sigma \equiv \text{sgn}(q^2)$. $\mu$ is a positive constant with dimension of mass, analogous to the mass unit in dimensional regularization. It is not difficult to show that $\mathcal{H}_3$ does not depend on $\mu$, it appears there for purely dimensional reasons.

Tensor three-point integrals $\mathcal{H}_3^{\alpha_1\ldots\alpha_n} = \mathcal{H}_3^{\alpha_1\ldots\alpha_n}(v, k_1, k_2; \delta M, m, m')$ can be given in terms of integrals with smaller ranks and fewer points, by the well-known method of [4]. We will consider integrals of standard form $\mathcal{H}_3^{\alpha_1\ldots\alpha_n}(v, q; \Delta, m, m') = \mathcal{H}_3^{\alpha_1\ldots\alpha_n}(v, -q, q; \Delta + v \cdot (k_1 + k_2), m, m')$ in terms of which we can express $\mathcal{H}_3$ as,

$$\mathcal{H}_3^{\alpha_1\ldots\alpha_n} = \mathcal{H}_3^{\alpha_1\ldots\alpha_n}(v, q; \Delta, m, m') + \sum_{j=1}^{n} \mu^{\{\alpha_1 \ldots \alpha_j\}} \mathcal{H}_3^{\alpha_{j+1} \ldots \alpha_n}(v, q; \Delta, m, m'),$$

where $\mu \equiv 1/2(p' + p)^\mu$, and $A^{\{\alpha_1 \ldots \alpha_n\}} \equiv A^{\alpha_1 \alpha_2 \ldots \alpha_n} + A^{\alpha_2 \alpha_1 \alpha_3} + \cdots + A^{\alpha_3 \alpha_2 \alpha_1} + \cdots$.

Clearly, for the scalar integral we have $\mathcal{H}_3 = \mathcal{H}_3$.

For the vector integral we write $\mathcal{H}_3^{\alpha}(v, q; \Delta, m_1, m_2) = V^{\alpha} + Q_{q^\alpha}$, with $V = V(v \cdot q, q^2, \Delta, m_1, m_2)$ and similarly $Q$. If $(v \cdot q)^2 - q^2 = |q|^2 = 0$, then $q^\alpha \propto v^\alpha$ and we can set $Q = 0$. If $|q|^2 \neq 0$,

$$|q|^2 V = -q^2 v_\alpha H_3^\alpha - v \cdot q q_\alpha H_3^\alpha,$$

with,

$$v_\alpha H_3^\alpha = \frac{1}{2} B_0(4q^2, m_1, m_2) + \Delta \frac{1}{2} H_3,$$

$$q_\alpha H_3^\alpha = \frac{1}{4} j_2(\Delta - 2v \cdot q, m_2) - \frac{1}{4} j_2(\Delta + 2v \cdot q, m_1) + \frac{m_1^2 - m_2^2}{4} H_3.$$

We have omitted the arguments $(v, q; \Delta, m_1, m_2)$ of $\mathcal{H}_3^\alpha$ on both sides of these equations for brevity. $B_0$ is the standard scalar two-point integral, as given in the appendix of [2].

Higher-rank tensor integrals can be computed analogously.

The results presented in this section were obtained in collaboration with R. Flores Mendieta. A detailed derivation will be given elsewhere.

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