Darboux Coverings and Rational
Reductions of the KP Hierarchy

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Abstract

We use the method of Darboux coverings to discuss the invariant submanifolds
of the KP equations, presented as conservation laws in the space of monic Lau-
trent series in the spectral parameter (the space of the Hamiltonian densities).
We identify a special class of these submanifolds with the rational invariant
submanifolds entering matrix models of 2D–gravity, recently characterized by
Dickey and Krichever. Four examples of the general procedure are provided.

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1 Introduction

Constrained KP hierarchies were introduced as symmetry reductions of the KP hierarchy [17, 22, 10], the symmetry being generated by suitable combinations of a Baker–Akhiezer function and an adjoint one. These integrable systems have been recently widely studied [17, 14, 18, 4, 1] especially in connection with multi–matrix models of two–dimensional gravity and the theory of \( \mathcal{W} \)–algebras (see, e.g., [2, 5, 3, 6] and references quoted therein). For instance, their (bi–)Hamiltonian structures and free–field realizations [4, 1, 21], and their picture in the Segal–Wilson approach [23] are well under control. Their geometrical structure has been clarified in [14, 18] by proving that such hierarchies are the restriction of the KP flows to submanifolds \( \mathcal{K}_{m,n} \), in which the \( n \)–th power of the KP Lax operator \( L \) factors as the ratio of two purely differential operators, 

\[
L^n = L_{(m)}^{-1}(n+m)
\]

(whence the denomination of “rational reductions” of the KP theory).

In this Letter we want to present a different approach to constrained KP hierarchies, based on the geometrical concept of Darboux covering. Roughly speaking, a Darboux covering of a vector field \( X \) on a manifold \( M \) is a vector field \( Y \) on another manifold \( N \) which is doubly related to \( X \) by a pair of maps from \( N \) to \( M \). A Darboux covering of the KP equations has been constructed in [20], by introducing the Darboux–KP (DKP) equations. There it has been shown that these equations allow to relate in a natural way the KP and the modified KP (mKP) theories, and their discrete (Toda Lattice) counterparts. Here we want to show how to use such a covering in the study of the constrained KP theories, and in the clarification of the relations among the KP equations, the modified KP equations\(^1\), and the AKNS–type hierarchies [17, 18, 14].

A peculiarity the reader must be warned of is that our approach to the KP equations differs from the standard one based on the algebra of pseudo differential operators (see, e.g., [12, 13]). We pursue the bihamiltonian approach exposed in [7, 8, 9], in which the basic dynamical variable is the generating function \( h(z) \) of the Hamiltonian densities,

\[
h(z) = z + \sum_{i \geq 1} h_i(x)z^{-i},
\]

(1.1)

and the KP equations are the associated conservation laws

\[
\partial_{t_j} h = \partial_x H^{(j)}.
\]

(1.2)

In the same style, the DKP equations are written as a system

\[
\begin{cases}
\partial_{t_j} h = \partial_x H^{(j)} \\
\partial_{t_j} a = a(\tilde{H}^{(j)} - H^{(j)})
\end{cases}
\]

(1.3)

in a pair of Laurent series \((h, a)\). In our picture, the constrained KP hierarchies are the restriction of the DKP equations to suitable invariant submanifolds in the space of pairs \((h, a)\).

\(^1\)A possible connection between constrained KP and mKP hierarchies was noticed by L. A. Dickey in one of the preprints of [4].
The scheme of this Letter is as follows: in Sections 2 and 3 we recall from \[20\] the notion of Darboux covering, implement it for the KP theory, and find a family (parametrized by an integer number) of invariant submanifolds $S_l$. In Section 4 we present the Lax map connecting the standard picture of the KP hierarchy with the one summarized in equations (1.1) and (1.2), and perform the identification of $S_{n,n+m} = S_n \cap S_{n+m}$ with the Dickey–Krichever rational submanifolds $K_{m+1,n}$. Finally, in Section 5 we give the simplest examples of our construction. The last example, considering a reduction of the DKP system on the *triple* intersection $S_0 \cap S_1 \cap S_2$, gives rise to a non standard one–field reduction of the AKNS system.

### 2 Darboux coverings

Let us consider the dynamical systems described by the vector fields $X$, $Y$, and $Z$ on the manifolds $M$, $N$ and $P$, respectively. We shall say that $Y$ intertwines $X$ and $Z$ if there exists a pair of maps $\mu : N \rightarrow M$ and $\sigma : N \rightarrow P$ relating $Y$ to $X$ and $Z$, respectively. In the form of a diagram we have

$$
\begin{align*}
M & \xleftarrow{\mu} N \xrightarrow{\sigma} P \\
X & \xleftarrow{\mu^*} Y \xrightarrow{\sigma^*} Z
\end{align*}
$$

If $Z$ coincides with $X$ and $N$ is a fiber bundle over $M = P$ with canonical projection $\mu$, we shall say that $Y$ is a Darboux covering of $X$. So, a Darboux covering is a vector field $Y$ on a fiber bundle over $M$ that intertwines $X$ with itself by means of the canonical projection $\mu : N \rightarrow M$ and the Darboux map $\sigma : N \rightarrow M$.

The concept of Darboux covering described above may be used to construct solutions as well as invariant submanifolds of the vector field $X$. The construction of solutions has been detailed in \[20\]. In this Letter we will exemplify the use of Darboux coverings for the reduction of the dynamical system $X$. The reduction process rests on the elementary remark that any invariant submanifold $S \subset N$ of the vector field $Y$ projects into two submanifolds

$$
\begin{align*}
S' &= \mu(S) \\
S'' &= \sigma(S)
\end{align*}
$$

which are invariant by $X$ on the base space $M$. Thus, a possible strategy to discover invariant submanifolds of the vector field $X$ on $M$ is to look for the invariant submanifolds of its Darboux covering $Y$ on $N$.

### 3 Darboux–KP equations

Our main example of Darboux covering concerns the KP equations. Here the manifolds $M$ and $N$, the vector fields $X$ and $Y$, and the maps $\mu : N \rightarrow M$ and $\sigma : N \rightarrow P$ are defined as follows. The manifold $M$ is the infinite dimensional affine space of monic Laurent series

$$
h(z) = z + \sum_{j \geq 1} h_j z^{-j},
$$

(3.1)
whose coefficients are functions of a space variable $x$. This space is immersed into the commutative algebra $L$ of all formal Laurent series having the form

$$k(z) = \sum_{j \geq -n} k_j z^{-j}$$

for some integer $n$. The manifold $N$ is the product $N = M \times A$, of $M$ with the affine space $A$ of formal Laurent series of the form:

$$a(z) = z + \sum_{j \geq 0} a_j z^{-j}.$$  \hspace{1cm} (3.3)

The series in $A$ differ from the ones in $M$ since they contain the coefficient $a_0$ as well. This apparently trivial distinction is basic in the relation between KP and modified KP equations [19, 20]. The canonical projection

$$\mu(h, a) = h$$  \hspace{1cm} (3.4)

gives rise to the Miura map of the KP theory [20].

To define the KP equations on $M$ we introduce the Faà di Bruno iterates $h^{(j)}$ of $h(z)$ defined by

$$h^{(0)} = 1$$

$$h^{(j+1)} = (\partial_x + h)h^{(j)}, \; \forall j \geq 0.$$  \hspace{1cm} (3.5)

By linearly combining these iterates in the form

$$H^{(j)} = h^{(j)} + \sum_{l=0}^{j-2} p_l^{\text{i}}[h]h^{(l)}$$

where the coefficients $p_l^{\text{i}}[h]$ are allowed to be differential polynomials in the coefficients $h_j$ of $h$, we select those Laurent series $H^{(j)}$ having the asymptotic behaviour

$$H^{(j)} = z^j + O(z^{-1}) \; \text{ as } z \to \infty.$$ \hspace{1cm} (3.7)

For instance, if we consider

$$H^{(2)} = h^{(2)} + p_0^{\text{2}}[h]h^{(0)}$$

$$= h_x + h^2 + p_0^{\text{2}}[h] \cdot 1$$

$$= z^2 + (2h_1 + p_0^{\text{2}}[h]) + (h_{1x} + 2h_2)z^{-1} + O(z^{-2})$$

we immediately see that we have to choose $p_0^{\text{2}}[h] = -2h_1$ in order to get rid of the coefficient of $z^0$. Therefore,

$$H^{(2)} = h_x + h^2 - 2h_1 = z^2 + (h_{1x} + 2h_2)z^{-1} + O(z^{-2}).$$ \hspace{1cm} (3.9)

The Laurent series $H^{(j)}$ will be referred to as the currents of the KP theory. They define the local conservation laws

$$\partial_t^j h = \partial_x H^{(j)},$$ \hspace{1cm} (3.10)
which are a possible form of the KP equations. The well known commutability of the KP flows follows from the relations

$$\partial_t H^{(k)} = \partial_t H^{(j)},$$

proved in [9].

To define a Darboux covering of the KP equations we introduce the map $\sigma : N \to M$,

$$\sigma(h, a) = h + \frac{a_x}{a}$$

and we consider on $N$ the Darboux–KP equations (or briefly DKP equations),

$$\begin{cases}
\partial_t h = \partial_x H^{(j)} \\
\partial_t a = a(\tilde{H}^{(j)} - H^{(j)})
\end{cases}$$

where $\tilde{H}^{(j)}$ is the current $H^{(j)}$ evaluated at the point $\tilde{h} = \sigma(h, a)$. The proof that the DKP equations provide a Darboux covering of the KP hierarchy rests on the remark that if the pair $(h, a)$ is a solution of the DKP equations, then $\tilde{h} = \sigma(h, a)$ is a solution of the KP equations since

$$\partial_t \tilde{h} = \partial_t h + \partial_x (\tilde{H}^{(j)} - H^{(j)}) = \partial_x \tilde{H}^{(j)}.$$  

Furthermore, the relation

$$\left[ \frac{\partial}{\partial t_j}, \frac{\partial}{\partial t_k} \right] \cdot a = a \left( \frac{\partial}{\partial t_j} (\tilde{H}^{(k)} - H^{(k)}) - \frac{\partial}{\partial t_k} (\tilde{H}^{(j)} - H^{(j)}) \right) = 0$$

shows that the DKP equations define a commuting hierarchy of vector fields on $N$.

A less obvious property is that these equations admit a rich family of invariant submanifolds $S_l \subset N$, defined by the constraints

$$z^l a = H^{(l+1)} + \sum_{m=0}^{l} a_m H^{(l-m)}$$

for any integer $l \geq -1$.

In [20] the following two properties of the DKP equations have been proved.

**Proposition 3.1** The DKP vector fields are tangent to the submanifolds $S_l$.

**Proposition 3.2** The modified KP equations are the restrictions of the DKP equations to $S_0$.

To explain the last statement, we notice that the submanifold $S_0$ is defined by the equation

$$a(z) = h(z) + a_0.$$  

(3.17)
It shows that $S_0$ is a plane in $N$, which may be parameterized by $a(z)$. Thus the
DKP vector fields give rise to a system of equations on the space $A$. In [20] it has
been shown that these equations may be written in the form

$$\partial_t a = \partial_x A^{(j)}, \quad (3.18)$$

where the currents $A^{(j)}$ are defined similarly to the currents $H^{(j)}$: one first introduces
the Faà di Bruno iterates $a^{(j)}$ by

$$a^{(0)} = 1$$

$$a^{(j+1)} = (\partial_x + a) a^{(j)}, \quad \forall j \geq 0, \quad (3.19)$$

and then one considers the unique linear combinations

$$A^{(j)} = a^{(j)} + \sum_{l=1}^{j-1} q_l [a] a^{(l)} \quad (3.20)$$

with the asymptotic behavior

$$A^{(j)} = z^j + O(z^0) \quad \text{as } z \to \infty. \quad (3.21)$$

The resulting equations (3.18) are one of the possible forms of the mKP equations [16, 19].

Proposition 3.3 is our main source of invariant submanifolds of the KP equations. A
simple consequence is:

**Proposition 3.3** The submanifolds

$$S_{l,l+n} = \mu(S_l \cap S_{l+n}) \quad (3.22)$$

are invariant submanifolds for the KP equations.

Our task is to show that they indeed coincide with Kricever's rational invariant
submanifolds.

## 4 The Lax map

In this section we translate the previous results on the invariant submanifolds of the
DKP equations in the language of pseudodifferential operators, with the purpose of
comparing our results with those recently suggested by Dickey and Krichever. The
connection is established by means of a suitable map

$$\phi : L \to \Psi DO, \quad (4.1)$$

relating the commutative algebra $L$ of scalar–valued Laurent series $h(z)$ to the non–
commutative algebra $\Psi DO$ of pseudodifferential operators on the line. This map
is defined in three steps. First one introduces the negative iterates $h^{(-1)}; h^{(-2)} \ldots$. 
which are computed by solving backwards the Faà di Bruno recursion relations (3.5), starting from $h^{(0)} = 1$. One notices that the full set of the Faà di Bruno iterates $\{h^{(j)}\}_{j \in \mathbb{Z}}$ forms a basis in $L$ attached at the point $h(z)$. Then one defines the map $\phi$ on the Faà di Bruno iterates $h^{(j)}$, $j \in \mathbb{Z}$, by setting

$$\phi(h^{(j)}) = \partial_x^j.$$  \hfill (4.2)

Finally, one extends the map $\phi$ to the whole space $L$ by linearity, setting

$$\phi(fh^{(j)}) = f \cdot \phi(h^{(j)}) = f \cdot \partial_x^j,$$ \hfill (4.3)

where $f$ is an arbitrary scalar–valued function independent of $z$. Actually, $\phi$ is the inverse map to the one introduced in [11]. It will be referred to as the Lax map since it allows to give the KP theory the usual Lax formulation [12, 13]. The Lax operator $L$ of the KP hierarchy is defined as the image

$$L = \phi(z)$$ \hfill (4.4)

of the first element of the standard basis in $L$. In [11] the following property of $\phi$,

$$\phi(z^k \cdot h^{(j)}) = \partial^j \cdot L^k,$$ \hfill (4.5)

has been proved. Furthermore, it has been shown that

$$\phi(H^{(j)}) = (L^j)_+,$$ \hfill (4.6)

where $(L^j)_+$ denotes the differential part of the $j$–th power of $L$, and that the KP equations (3.10) admit the usual Lax representation

$$\frac{\partial L}{\partial t_j} = [L, (L^j)_+].$$ \hfill (4.7)

We presently use the map $\phi : L \rightarrow \Psi DO$ to write explicitly the equations of the submanifolds

$$S'_{l, l+n} = \mu(S_l \cap S_{l+n})$$ \hfill (4.8)

in the formalism of pseudodifferential operators. We notice that the submanifold $S_l \cap S_{l+n}$ in $N$ is defined by the pair of equations

$$z^l a = H^{(l+1)} + \sum_{k=0}^l a_k H^{(l-k)}$$ \hfill (4.9)

$$z^{l+n} a = H^{(l+n+1)} + \sum_{k=0}^{l+n} a_k H^{(l+n-k)}.$$ \hfill (4.10)

By eliminating $a$ we get

$$H^{(l+n+1)} + \sum_{k=0}^{l+n} a_k H^{(l+n-k)} = z^n (H^{(l+1)} + \sum_{k=0}^l a_k H^{(l-k)}).$$ \hfill (4.11)
Since the canonical projection \( \mu : N \to M \) is simply \( \mu(h, a) = h \), this equation may be seen as the equation defining \( S'_{l,l+n} \) in \( M \). To get the equation of this submanifold in the algebra \( \Psi DO \), let us use the map \( \phi \) to define the pair of differential operators:

\[
L_{(l+1)} : = \phi(H^{(l+1)} + \sum_{k=0}^{l} a_k H^{(l-k)})
\]

\[
L_{(l+n+1)} : = \phi(H^{(l+n+1)} + \sum_{k=0}^{l+n} a_k H^{(l+n-k)}).
\]

Then, by using the property (4.3) we obtain the operator equation

\[
L_{(l+n+1)} = L_{(l+1)} \cdot L^n.
\]

The argument above proves

**Proposition 4.1** The image of \( S'_{l,l+n} \) under \( \phi \) is the submanifold of the pseudodifferential operators \( \mathcal{L} \) verifying the equation

\[
\mathcal{L}^n = L_{(l+1)}^{-1} \cdot L_{(l+n+1)}.
\]

In this form, one easily recognizes that \( S'_{l,l+n} \) coincides with the rational invariant submanifold \( K_{l+1,l+n} \) introduced by Dickey and Krichever.

We end this section by two remarks:

1) As in the above-mentioned approach, we recover the \( n \)-GD manifolds as a special case of this construction. In fact, the equation for \( S_{-1} \) reduces to \( a \equiv z \), and hence the defining relations for \( S'_{-1,n} \) can be solved as \( z^n = H^{(n)} \), which, under the Lax map \( \phi \) translates into the well-known operator equation \( \mathcal{L}^n = (\mathcal{L}^n)^+ \).

2) The submanifolds \( S_l \cap S_{l+n} \) are merely a special class of invariant submanifolds of the DKP equations. Other invariant submanifolds of the KP equations are obtained by considering intersection of three or more invariant submanifolds, e.g.,

\[
S'_{l,p,q} = \mu(S_l \cap S_p \cap S_q).
\]

Obviously, these multiple intersections can be obtained as intersections of the double ones, i.e., as intersections of different Krichever’s invariant submanifolds. Nevertheless, in our formalism they can be more easily handled, as it will be shown by the example of Subsection 5.4.

### 5 Some Examples

In this section we will give four examples aiming to show how the formalism of Darboux coverings and Laurent series can be concretely used to find reductions of the KP equations. Preliminarily we write explicitly the KP currents \( H^{(j)} \) and the DKP equations (3.13) we shall need henceforth. These equations play a fundamental role in our construction of the constrained KP equations. We obtain the latter by reducing, first of all, the DKP equations on \( S_l \cap S_{l+n} \), and then by projecting the reduced equations on \( M \).
Since we are interested in the first few equations, we need only the first three positive Faà di Bruno iterates of \( h \):

\[
\begin{align*}
  h^{(1)} &= h \\
  h^{(2)} &= h_x + h^2 \\
  h^{(3)} &= h_{xx} + 3hh_x + h^3.
\end{align*}
\]  

Their linear combinations fulfilling condition (3.7) are the first three KP currents:

\[
\begin{align*}
  H^{(1)} &= h^{(1)} \\
  H^{(2)} &= h^{(2)} - 2h_1 \\
  H^{(3)} &= h^{(3)} - 3h_1h^{(1)} - 3(h_2 + h_{1,x}).
\end{align*}
\]  

Their expansions in powers of \( z \) are:

\[
\begin{align*}
  H^{(1)} &= z + h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3} + O(z^{-4}) \\
  H^{(2)} &= z^2 + (2h_2 + h_{1,x}) z^{-1} + (2h_3 + h_{2,x} + h_1^2) z^{-2} + O(z^{-3}) \\
  H^{(3)} &= z^3 + (3h_3 + 3h_{2,x} + h_{1,xx}) z^{-1} + O(z^{-2}).
\end{align*}
\]  

Then we expand in powers of \( z \) the Darboux map \( \sigma \). The first three components are:

\[
\begin{align*}
  \tilde{h}_1 &= h_1 + a_{0,x} \\
  \tilde{h}_2 &= h_2 + (a_1 - \frac{1}{2}a_0^2)_x \\
  \tilde{h}_3 &= h_3 + (a_2 - a_0a_1 + \frac{1}{3}a_0^3)_x.
\end{align*}
\]  

They allow to compute the currents \( \tilde{H}^{(1)}, \tilde{H}^{(2)}, \tilde{H}^{(3)} \) up to the order \(-1\) in \( z \). By inserting into the definition (3.13) of the DKP equations, we finally obtain for the first components:

**DKP2:**

\[
\begin{align*}
  \frac{\partial h_2}{\partial t_2} &= (2h_3 + h_1^2 + h_{2,x})_x \\
  \frac{\partial h_1}{\partial t_2} &= (2h_2 + h_{1,x})_x \\
  \frac{\partial a_0}{\partial t_2} &= (2a_1 + a_{0,x} - a_0^2)_x \\
  \frac{\partial a_1}{\partial t_2} &= (2a_2 + a_{1,x})_x + 2a_{0,x}(h_1 - a_1) \\
  \frac{\partial a_2}{\partial t_2} &= (2a_3 + a_{2,x} + a_1^2)_x + 2a_{0,x}(h_2 - a_2) + 2a_{1,x}(h_1 - a_1)
\end{align*}
\]  

**DKP3:**
\[
\begin{align*}
\frac{\partial h_1}{\partial t_3} &= (3h_3 + 3h_{2,x} + h_{1,xx})_x \\
\frac{\partial a_0}{\partial t_3} &= (a_{0,xx} - 3a_0a_{0,x} + a_0^3 - 3a_0a_1 + 3a_2 + 3a_{1,x})_x \\
\frac{\partial a_1}{\partial t_3} &= (3a_3 + 3a_{2,x} + a_{1,xx})_x + 3a_{0,x}(h_2 - a_2) \\
&\quad + 3(a_{1,x} - a_0a_{0,x})(h_1 - a_1) + 3(a_{0,x}(h_1 - a_1))_x
\end{align*}
\]  

(5.6)

5.1 The AKNS system

Let us consider the intersection

\[S_{0,1} = S_0 \cap S_1,\]

(5.7)

whose equations are

\[
\begin{align*}
a &= H^{(1)} + a_0 \\
za &= H^{(2)} + a_0H^{(1)} + a_1.
\end{align*}
\]

(5.8)

If we solve the first equation with respect to \( h \) and we substitute into the second one, we obtain the equation

\[za = a^{(2)} - a_0a^{(1)} - (a_1 + a_{0,x})\]

(5.9)

for \( a(z) \). It shows that \( (a_0, a_1) \) are free parameters on \( S_{0,1} \), while all the other coefficients can be written as differential polynomials in \( (a_0, a_1) \). In particular, we obtain

\[
\begin{align*}
h_1 &= a_1 \\
h_2 &= a_2 = -(a_{1,x} + a_0a_1) \\
h_3 &= a_3 = a_{1,xx} + a_{0,x}a_1 + 2a_0a_{1,x} - a_1^2 + a_0^2a_1.
\end{align*}
\]

(5.10)

Substituting these constraints into the first two DKP equations (5.5) and (5.6) we obtain

\[
\begin{align*}
\frac{\partial a_0}{\partial t_2} &= (2a_1 + a_{0,x} - a_0^2)_x \\
\frac{\partial a_1}{\partial t_2} &= -(a_{1,x} + 2a_0a_1)_x
\end{align*}
\]

(5.11)

and

\[
\begin{align*}
\frac{\partial a_0}{\partial t_3} &= (a_{0,xx} - 3a_0a_{0,x} + a_0^3 - 6a_0a_1)_x \\
\frac{\partial a_1}{\partial t_3} &= (a_{1,xx} + 3a_0a_{1,x} - 3a_1^2 + 3a_0^2a_1)_x.
\end{align*}
\]

(5.12)

They coincide with the \( t_2 \) and \( t_3 \) flows of the \((1|1)\)-KdV theory of \[5, 4\], which gives, after the identifications

\[a_0 = -\frac{r_x}{r}, \quad a_1 = -rq.\]

(5.13)
the classical AKNS hierarchy.

To see explicitly the connection with the rational reductions, we simply consider the constraint (5.8) as

$$z(h^{(1)} + a_0) = (h^{(2)} + a_0 h^{(1)} - a_1)$$

following from equation (5.9) of $S_{0,1}$, and we apply the map $\phi$ of Section 4 to get the equality

$$\mathcal{L} = (\partial_x + a_0)^{-1} \cdot (\partial_x^2 + a_0 \partial_x - a_1).$$

## 5.2 The Yajima–Oikawa system

The next example concerns the intersection

$$S_{0,2} = S_0 \cap S_2,$$

whose equations are

$$\begin{cases}
a = H^{(1)} + a_0 \\
z^2 a = H^{(3)} + a_0 H^{(2)} + a_1 H^{(1)} + a_2.
\end{cases}$$

(5.17)

They show that the submanifold $S_{0,2}$ is parameterized by three fields, say $(a_0, a_1, a_2)$, and provide the constraints

$$\begin{align*}
h_1 &= a_1 \\
h_2 &= a_2 \\
h_3 &= a_3 = -\frac{1}{2}(a_{1,xx} + a_{2,x} + 2a_0 a_2 + a_1^2).
\end{align*}$$

(5.18)

Therefore, the equations of motion of the second flow are

$$\begin{cases}
\frac{\partial a_0}{\partial t_2} = \partial_x(2a_1 - a_0^2 + a_{0,x}) \\
\frac{\partial a_1}{\partial t_2} = \partial_x(a_{1,x} + 2a_2) \\
\frac{\partial a_2}{\partial t_2} = -\partial_x(2a_{2,x} + a_{1,xx} + a_0(2a_2 + a_{1,x})),
\end{cases}$$

(5.19)

as it is easily checked by inserting the constraints (5.18) into the first DKP equations (5.3).

The explicit relation with the rational reduction to $K_{1,2}$ is easily seen taking into account the expressions (5.2) for the KP currents $H^{(j)}$. Indeed we can write the constraint equations (5.17) as

$$z^2(h^{(1)} + a_0) = h^{(3)} + a_0 h^{(2)} - 2a_1 h^{(1)} - (3a_{1,x} + 2a_2 + 2a_0 a_1),$$

(5.20)

so that the equations of $K_{1,2}$ are

$$\mathcal{L}^2 = (\partial_x + a_0)^{-1} \cdot (\partial_x^3 + a_0 \partial_x^2 - 2a_1 \partial_x - (3a_{1,x} + 2a_2 + 2a_0 a_1)).$$

(5.21)
Equations (5.19) coincide with the equations of the second flow of the Yajima–Oikawa hierarchy. The coordinate change relating our picture to the one of [17] can be obtained comparing (5.21) with

\[ L^2 = \partial_x^2 + u + \psi \partial^{-1} \psi^* \]  

(5.22)

and it reads:

\[ a_0 = -\frac{\psi_x}{\psi}, \quad a_1 = -\frac{u}{2}, \quad a_2 = -\frac{1}{4}u_x - \frac{1}{2}\psi\psi^*. \]  

(5.23)

We remark that thanks to Proposition 3.2, the AKNS and the Yajima–Oikawa hierarchies can be obtained as reductions of the modified KP equations, as well as of the KP ones.

### 5.3 The 2–AKNS system

We conclude our list of simple intersections of pairs of submanifolds considering \( S_{1,2} \), which turns out to be the 2–AKNS hierarchy, alias a special case of the three–boson hierarchy described, f.i., in [13]. The defining equations for \( S_{1,2} \) are

\[
\begin{align*}
za &= H^{(2)} + a_0 H^{(1)} + a_1 \\
z^2a &= H^{(3)} + a_0 H^{(2)} + a_1 H^{(1)} + a_2.
\end{align*}
\]

(5.24)

Hence, \( S_{1,2} \), is parameterized by four fields, which we will choose to be \((a_0, a_1, h_1, h_2)\).

Since

\[ h_3 = -h_{1,xx} - 2h_{2,x} + h_1^2 - a_1 h_1 - a_0(h_{1,x} + h_2), \]

(5.25)

the \( t_2 \) flow is given by

\[
\begin{align*}
\frac{\partial a_0}{\partial t_2} &= \partial_x(2a_1 - a_0^2 + a_{0,x}) \\
\frac{\partial a_1}{\partial t_2} &= 2a_{2,x} + a_{1,xx} + 2a_{0,x}(h_1 - a_1) \\
\frac{\partial h_1}{\partial t_2} &= \partial_x(2h_2 + h_{1,x}) \\
\frac{\partial h_2}{\partial t_2} &= -\partial_x(2h_{1,xx} + 3h_{2,x} - 3h_1^2 + 2a_0(h_{1,x} + h_2) + 2a_1 h_1).
\end{align*}
\]

(5.26)

The system (5.24) gives, in terms of the free parameters and of the Faà di Bruno polynomials of \( h \), the relation

\[
z \cdot (h^{(2)} + a_0 h^{(1)} + (a_1 - 2h_1)) = h^{(3)} + a_0 h^{(2)} + (a_1 - 3h_1) h^{(1)} + \\
+ (a_2 - 2a_0 h_1 - 3h_2 - 3h_{1,x}).
\]

(5.27)

Then the link with the rational reduction to \( K_{2,1} \) is obtained as in the previous cases.
5.4 A triple intersection

Finally, we consider the intersection of the three submanifolds we have so far examined, i.e.,

\[ S_{0,1,2} := S_0 \cap S_1 \cap S_2, \]

which is explicitly given, in terms of Faà di Bruno polynomials, by the system

\[
\begin{align*}
  a &= h + a_0 \\
  za &= h^{(2)} + a_0 h^{(1)} + (a_1 - 2h_1) \\
  z^2a &= h^{(3)} + a_0 h^{(2)} + (a_1 - 3h_1)h^{(1)} + (a_2 - 2a_0 h_1 - 3h_2 - 3h_{1,x}).
\end{align*}
\]

We already know (see Subsection 5.1) that \(S_0 \cap S_1\) is parameterized by \((a_0, a_1)\). In order to determine \(S_0 \cap S_1 \cap S_2\) we have to impose that \(z^2a\) belong to the linear span \(H_+\) of \(\{H^{(j)}\}_{j \geq 0}\) (i.e., to the linear span of \(\{h^{(j)}\}_{j \geq 0}\)). Using the equations of \(S_0 \cap S_1\) we obtain

\[
\begin{align*}
  z^2a &= z(h^{(2)} + a_0 h^{(1)} + (a_1 - 2h_1)) \\
  &= z[(\partial_x + h)(h) + a_0 h + (a_1 - 2h_1)] \\
  &= z[(\partial_x + h)(a - a_0) + a_0 h + (a_1 - 2a_1)] \\
  &= (\partial_x + h)(za) - z(a_{0,x} + a_1).
\end{align*}
\]

Since \(za \in H_+\) on \(S_1\), and \((\partial_x + h)(H_+) \subset H_+\) by the definition of the Faà di Bruno polynomials, equation (5.31) shows that \(z^2a \in H_+\) if and only if

\[ a_1 = -a_{0,x}. \]

Therefore the restriction of the DKP equations to the invariant submanifold \(S_{0,1,2}\) is a one–field system, which projects on a one–field reduction of KP. The first flows are immediately obtained from (5.11) and (5.12):

\[
\begin{align*}
  \frac{\partial a_0}{\partial t_2} &= -(a_{0,x} + a_0^2)_x \\
  \frac{\partial a_0}{\partial t_3} &= (a_{0,xx} + 3a_0 a_{0,x} + a_0^3)_x.
\end{align*}
\]

Since the reduction to \(S_{0,1}\) coincides with AKNS (see Subsection 5.1), we have that the reduction to \(S_{0,1,2}\) can be seen as a reduction of AKNS. In terms of the AKNS variables \((q, r)\), the constraint (5.31) takes the form

\[ r_x^2 - qr_{xx} + qr^3 = 0. \]

This is a non–standard constraint of the AKNS hierarchy, which could be hardly discovered by a direct inspection of the equations.

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