Quantum information is a common topic of research in many areas of quantum physics, such as quantum communication and quantum computation, as well as quantum thermodynamics. It can be encoded in discrete or continuous variable systems, with the appropriated formalism to treat it generally depending on the quantum system to be chosen. For continuous variable systems, it is convenient to employ a quasi-probability function to represent quantum states and non-classical signatures. The Wigner function is a special quasi-probability function because it allows to describe a quantum system in the phase space very similar to the classical one. This work aims to provide a self-contained phase-space treatment of quantum information, using the Wigner function as the quantum state and complex functions obtained from the Weyl transform to describe observables of a given quantum system. We present many ways to quantify quantum information, besides general examples of its dynamics for non-Gaussian and Gaussian states, and for unitary and dissipative dynamics, showing that a robust phase-space formalism may be useful for future developments concerning the manipulation and quantification of information in quantum devices. Our results show a conversion of the negativity of local Wigner functions in mutual information and that the coherence for Gaussian states may be a witness for non-Markovian dynamics, evidencing the usefulness in several protocols in quantum thermodynamics.

I. INTRODUCTION

The complete ability of quantification and manipulation of quantum information is a paramount challenge to design new micro and nano devices based on quantum properties of systems. It is expected that this fine control would have important impacts on quantum communication and quantum cryptography [1–4], quantum metrology [5, 6], quantum thermodynamics [7–9], and quantum computation [10–12]. These current and future implications emphasize the importance in understanding quantum information in as many points of view as possible, mainly because different experimental platforms require an appropriated set of tools to manipulate information. Encoding and processing information depend on the quantum system to be considered. There are basically two broad classes of quantum systems for these purposes, the two-level and continuous variable systems. The first of them has a large numbers of studies, ranging from the use of quantum information to reverse the direction of heat in quantum thermodynamics [13–15] to investigate backflow of information in non-Markovian dynamics [16–18]. The latter, despite of some relevant works as, for instance, in squeezed states [19–21] and in photon addition and subtraction [22], possesses relevant points to be addressed, as the quality of squeezing sources and detectors, and quantum computation architecture [23, 24].

In what concerns the case of quantum information in two-level systems, it is considered that the most adequate formalism is the wave function or, in general cases, the density operator approach [25, 26]. On the other hand, for continuous variable quantum systems, such as in quantum optics [27], trapped ions [28, 29], cold atoms [30] and cavity QED [31, 32], it is common to employ, besides the density operator, a quasi-probability function to represent the state of some system [33]. When dealing with the quantum to classical transition, the most suitable of these functions is the Wigner function [34, 35], which is represented in a phase space similarly to that of the classical physics. The Wigner function has given important contribution in quantum physics, for instance, in implementing teleportation of a quantum gate [36] and in general uncertainty relations in quantum systems [37–40]. However, in many cases it has been used more as a mean of visualization than to properly describe the state of the system. There exist, nonetheless, the so called phase-space formalism of quantum mechanics, in which the Wigner function is the state of some system and all the observables are complex functions represented in the phase space.

The core of the phase-space formalism of quantum mechanics is the Weyl transform [35, 42], which converts a given operator in a complex function. The Wigner function is then defined as the Weyl transform of the density operator, for pure or mixed states. The Moyal product [33] is also an important ingredient, once it provides the appropriated inner product, in order to guarantee the uncertainty relations of the quantum theory. Based on these main elements, a formalism to treat quantum information, non-classicality signatures, and coherence can be established. The goal of this work is to provide a self-contained formalism to treat quantum information in the phase space. This approach may be useful to provide new insights of manipulation as well as conversion of information, some of them with the help of classical counterparts. Moreover, our treatment can be fruitful in quantum thermodynamics, mainly in cases in which information play an important role, such as in information-driving engine models [43–45] and in Maxwell’s demon in a quantum system [46].

This manuscript is organized as follows. In section II we provide the basic properties of the Wigner function and the phase-space formalism of quantum mechanics. Section III is devoted to develop several ways of quantifying quantum information in phase space, for both non-Gaussian and Gaussian states as well as for unitary and open dynamics. In section IV we explore some general examples to illustrate how to use the phase-space formalism for studying quantum information. Finally, we draw our conclusions and final remarks in section V.
II. BASIC PROPERTIES OF THE WIGNER FUNCTION AND THE PHASE-SPACE FORMALISM

In this section we review some relevant properties of the Wigner function and the phase-space formalism of quantum mechanics. The standard formalism of quantum mechanics is based on operators that act on the Hilbert space, $\mathcal{H}$. A quantum system can be described by means of a wave function in the position representation or, through the Fourier transform, in the momentum representation. Moreover, there is the well-known commutation relation between position and momentum operators, $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$, where $i$ and $j$ run over all the Hilbert space.

The operators-based formalism is not the only way to describing quantum mechanics. The phase-space formalism of quantum mechanics (PSQM) is another interesting way to study quantum mechanical systems, being relevant in a large number of scenarios. In the PSQM, the Wigner function represents the state of a given system and has the important property of carrying out simultaneously information about position and momentum of the system. In order to introduce the Wigner function and the phase-space formalism, we consider the Weyl transform of an operator $\hat{O}(\hat{q}, \hat{p})$, defined as [33, 35],

$$A^W(q,p) = \int dq \, e^{-ipq/\hbar} \langle q + y/2 | \hat{O}(\hat{q}, \hat{p}) | q - y/2 \rangle,$$

(1)

where $W$ stands for Weyl transform. The Weyl transform converts an operator in a c-function and it is a natural connection between operators-based and phase-space formalisms.

Next, considering a system described by a pure state, $|\psi\rangle$, we can write the density operator, $\hat{\rho} = |\psi\rangle\langle\psi|$. The Wigner function can be defined just as the Weyl transform of the density operator,

$$W(q,p) = h^{-1} \int dq \, e^{-ipq/\hbar} \langle \psi(q + y/2) | \psi(q - y/2) \rangle.$$

(2)

Some important properties of the Wigner function are that it is normalized when integrated over all phase-space, additionally providing the marginal probability distribution for momentum $|\phi(p)|^2$, and position $|\psi(q)|^2$,

$$\int dq \, W(q,p) = |\phi(p)|^2, \quad \int dp \, W(q,p) = |\psi(q)|^2.$$

(3)

From the Wigner function, it is also possible to obtain expectation the value of an observable $\mathcal{O}$,

$$\langle \mathcal{O} \rangle = h^{-1} \int \int dq dp \, W(q,p) \mathcal{O}^W(q,p).$$

(4)

The generalization of the Wigner function for mixed states is straightforward. For a mixed state $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$, where $p_j$ is the probability of each state and it is always positive, with $\sum_j p_j = 1$, the Wigner function is simply given by,

$$W(q,p) = \sum_j p_j W_j(q,p),$$

(5)

with $W_j(q,p)$ the Wigner function associated to each part of the ensemble.

One of the basic ingredients in considering the PSQM formalism is the Moyal product, which is introduced once we are now dealing with c-functions and no longer with operators. The Moyal product, defined for position and momentum variables [33], reads $A^W(q,p) \star B^W(q,p)$, where,

$$\star = \exp \left[ \frac{i\hbar}{2} \left( \frac{\partial}{\partial q} \frac{\partial}{\partial \tilde{p}} - \frac{\partial}{\partial \tilde{p}} \frac{\partial}{\partial q} \right) \right].$$

(6)

In introducing the Moyal product into the PSQM formalism we obtain a set of important tools to treat quantum systems, most of them being very similar to the classical case. In particular, for a unitary dynamics, the time evolution of an observable $\mathcal{O}^W$ is dictated by the equation,

$$\dot{\mathcal{O}}^W(q,p; t) = \frac{i}{\hbar} [\mathcal{O}^W(q,p; 0), H^W(q,p; t)],$$

(7)

$$= \frac{i}{\hbar} \mathcal{O}^W(q,p; 0) \star H^W(q,p; t)$$

(8)

$$- \frac{i}{\hbar} H^W(q,p; t) \star \mathcal{O}^W(q,p; 0),$$

well known as Moyal equation. For a generalization of the Moyal equation for open quantum systems we refer to Ref. [47].

In addition, applying the Weyl tranfrom on the eigenvalue equation and using the Moyal product, we obtain the so called stargenvalue equation, given by [33, 42],

$$H^W(q,p) \star W(q,p) = EW(q,p),$$

(9)

where $H^W(q,p)$ is the Weyl transform of the Hamiltonian of a quantum system and $E$ sets for the eigenvalues of energy. In the case of a quantum system with $N$ dimensions, the calculation of a particular sub phase space $(q_k,p_k)$ is performed by integrating over the rest of the variables,

$$W(q_k,p_k) = \int \int dq_{N-k} dp_{N-k} W(q, p).$$

(10)

Additionally, we present how to obtain the time evolution of the Wigner function using the Moyal product in the case of a unitary operation. Once the initial Wigner function is obtained from Eq. (4), the time evolution is given by the unitary operator [33],

$$U_\ell(q, p; t) = e^{i\hbar \ell \hat{H}^W / \hbar} = 1 + (it/\hbar)H^W$$

(11)

$$+ \frac{(it/\hbar)^2}{2!} H_\ell^W \star H^W + ..., $$

(12)

with $\ell = 1, ..., N,$ and $N$ the dimension of the system, resulting in,

$$W(q, p; t) = U_\ell^{-1}(q, p; t) \star W(q, p; 0) \star U_\ell(q, p; t).$$

(13)

Finally, for the case in which the system is a two-mode Gaussian state, we can introduce a vector collecting all the
coordinates of the phase space, \( \vec{R}(q_1, p_1, q_2, p_2) \). The two-mode Gaussian state is then completely characterized in terms of its first moments and covariance matrix, defined as \( \bar{\delta} = ((q_1), \langle p_1 \rangle, \langle q_2 \rangle, \langle p_2 \rangle) \) and \( \sigma = \sigma_{11} \oplus \sigma_{22} \), respectively, with,

\[
\sigma_{ii} = \begin{pmatrix} \sigma_{Q_i, Q_i} & \sigma_{P_i, Q_i} \\ \sigma_{Q_i, P_i} & \sigma_{P_i, P_i} \end{pmatrix},
\]

and \( \sigma_{AB} = \langle AB + BA \rangle - 2\langle A \rangle \langle B \rangle \). This results in a convenient expression for obtaining the Wigner function [23, 24],

\[
W_G(\vec{R}) = \frac{\exp\left[ -\frac{1}{2}(\vec{R} - \bar{\delta})\sigma^{-1}(\vec{R} - \bar{\delta}) \right]}{(2\pi)^4 \sqrt{\text{Det}[\sigma]}},
\]

particularizing the expression for two modes.

### III. QUANTUM INFORMATION IN PHASE-SPACE WITH WIGNER FUNCTION

In this section, we present a detailed discussion of how to use the Wigner function to study quantum information aspects. For this purpose, our analysis is restricted for a bipartite system, described by a Hilbert space \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \). The first point to be addressed is the concept of entropy. Since a direct translation of the von Neumann entropy to the phase-space is not to be addressed is the concept of entropy. Since a direct translation of the von Neumann entropy to the phase-space is not to be addressed is the concept of entropy. Since a direct translation of the von Neumann entropy to the phase-space is not to be addressed is the concept of entropy. Since a direct translation of the von Neumann entropy to the phase-space is not.

**Mutual information.** The linear entropy is useful to define the mutual information between two parts of a bipartite quantum system. Let consider a quantum system described by a Wigner function \( W(q_i, p_i) \) where the parts may have some physical interaction. The mutual information shared between the two parts is defined as [37, 40, 48],

\[
S = 1 - (2\pi\hbar)^2 \int \int dq_1 dp_1 W^2(q_1, p_1),
\]

with \( i = 1, 2 \).

The Eq. (16) holds for pure and mixed quantum states and satisfies the relation \( 0 \leq S \leq 1 \), where \( S = 0 \) for a pure state.

**Fidelity.** The fidelity is a suitable way of comparing how similar (or distinct) two states are. Given two Gaussian states, characterized by their first and second moments (covariance matrix), \( \bar{\delta} \) and \( \delta \), respectively, such that we can write the states as \( W_1(\bar{d}_1, \sigma_1) \) and \( W_2(\bar{d}_2, \sigma_2) \), the fidelity is given by [60, 70],

\[
F(\sigma_1, \bar{d}_1; \sigma_2, \bar{d}_2) = \frac{2}{\sqrt{\Delta + \delta - \sqrt{\Delta}}} e^{-\frac{1}{2} \bar{\delta}^T \sigma_+^{-1} \bar{\delta}},
\]

where \( \Delta = \text{Det}[\sigma_1 + \sigma_2], \delta = (\text{Det}[\sigma_1] - 1)(\text{Det}[\sigma_2] - 1), \bar{d} \equiv \bar{d}_1 - \bar{d}_2, \) and \( \sigma_+ \equiv \sigma_1 + \sigma_2 \). The fidelity is bounded by \( 0 \leq F \leq 1 \), with \( F = 1 \) and \( F = 0 \) for identical and completely different states, respectively. Equation (18) has particular importance in quantum information processing with Gaussian states because it allows, for example, to quantify the quality of a given noisy communication channel, where we have Gaussian states as input and output [55]. Another important point is that the covariance matrix is experimentally accessible, in particular in quantum optical devices [57].

**Coherence.** An important quantum signature is the presence of quantum coherence in states. This quantity has been addressed in a series of recent articles in quantum information [58, 59], and also in quantum thermodynamics [7, 60, 61]. The quantification of coherence as a resource was firstly proposed in Ref. [59] and extended for Gaussian states in Ref. [62]. In order to quantify the coherence in a given one-mode Gaussian state \( \rho(\sigma, \bar{d}) \), where \( \sigma \) and \( \bar{d} \) are the covariance matrix and first moments, respectively, Ref. [62] introduces a quantifier defined as \( C[\rho(\sigma, \bar{d})] = \min \{ \text{S}[\rho(\sigma, \bar{d})]\|\|\delta] \} \), where \( \text{S}[\bullet \| \bullet] \) is the relative entropy and the minimum is evaluated over all thermal states \( \delta \). The coherence measure for one-mode Gaussian states assumes the following expression when minimized [62],

\[
C[\rho(\sigma, \bar{d})] = \nu - 1 \log_2 \left( \frac{\nu - 1}{2} \right) - \nu + 1 \log_2 \left( \frac{\nu + 1}{2} \right) + (\bar{n} + 1) \log_2 (\bar{n} + 1) - (\bar{n}) \log_2 (\bar{n}),
\]

where \( \nu = \sqrt{\sigma_{11} \sigma_{22} - \sigma_{12}^2}, \) and \( \bar{n} = (1/4)(\sigma_{11} + \sigma_{22} + d_1^2 + d_2^2 - 2) \). As we will observe in the following examples, this measure
is appropriated to capture the coherence of Gaussian states due to the displacement operator.

**Dissipative dynamics of Gaussian states.** When a system is in thermal contact with a Markovian environment, in general it is affected by decoherence effects [25]. In the particular case when the system is a one-mode Gaussian state \( \rho(\sigma, d) \) the complete characterization of dissipation effects is encoded into the dynamics of the first moments and covariance matrix [56, 63]. The time evolution of the first moments and covariance matrix during the thermalization process is obtained from the two uncoupled differential equations,

\[
\dot{\sigma} = \Gamma \sigma + \Gamma (2\bar{m} + 1) \mathbb{1}_{2 \times 2},
\]
\[
\dot{d} = -(\Gamma / 2)d,
\]

where \( \Gamma \) and \( \bar{m} \) are the decay rate and the mean number of photons of the thermal environment, respectively. The solutions for the above equations are straightforward and given by,

\[
\sigma(t) = e^{-\Gamma t} \sigma(0) + (1 - e^{-\Gamma t})(2\bar{m} + 1) \mathbb{1}_{2 \times 2},
\]
\[
\mathbb{1}_{\alpha}(t) = e^{-\Gamma t / 2} \mathbb{1}_{\alpha}(0),
\]

with \( \mathbb{1}_{\alpha}(0) \) and \( \sigma(0) \) the initial first moments and covariance matrix of the system. Naturally, when \( t \to \infty \), the first moments and the covariance matrix tend to the asymptotic thermal state.

**IV. EVOLUTION OF A BIPARTITE CONTINUOUS VARIABLE SYSTEM**

**Unitary evolution**

In order to investigate how to use the mutual information and the negativity of the Wigner function to understand correlations and non-classicality in bipartite systems, we consider the following Hamiltonian

\[
H(q_i, p_i) = \alpha^2 p_i + \beta^2 q_i^2 + \gamma (p_i q_2 - p_2 q_1),
\]

with \( \alpha^2 = 1 / (2m) \) and \( \beta^2 = m \omega^2 / 2 \), where \( m \) and \( \omega \) are the mass and frequency of the system, respectively, and \( \gamma \) represents the coupling constant between the two oscillators.

The Hamiltonian (22) illustrates many important scenarios in physics, ranging from fluctuation relations for a particle in a magnetic field [64] to general uncertainty relations [37, 38, 40, 42] and quantum heat engines [65, 66]. For example, Eq. (22) could describe a particle moving in a plane in the presence of an orthogonal uniform magnetic field or it can be useful in the study of noncommutativity of the phase-space [39]. In Appendix A we provide a detailed derivation of the Wigner function and associated eigenvalues for the Hamiltonian (22) using Eq. (9). Denoting by \( W_{n_1, n_2}(q_i, p_i) \) the Wigner function associated to the Hamiltonian (22) with eigenvalues \( E_{n_1, n_2} \) reads

\[
W_{n_1, n_2}(q_i, p_i) = \frac{(-1)^{n_1 + n_2}}{\pi^{2\hbar^2}} \exp \left[ -\frac{1}{\hbar} \left( \frac{\alpha^2 q_i^2}{\beta^2 p_i^2} + \frac{\beta^2 p_i^2}{\alpha^2 q_i^2} \right) \right]
\times L_{n_1}[\Omega_+ / \hbar] L_{n_2}[\Omega_- / \hbar],
\]

with \( \Omega_\pm = (\alpha / \beta) q_i^2 + (\beta / \alpha) p_i^2 + 2 \sum_{i,j=1}^2 (\epsilon_{ij} p_i q_j) \), \( n_{1,2} \) are integer and nonnegative numbers, \( L_{n_{1,2}} \) are the associated Laguerre polynomials, and the eigenvalues are,

\[
E_{n_1, n_2} = 2\hbar \alpha \beta (n_1 + n_2 + 1) + \hbar \gamma (n_1 - n_2).
\]

We are interested in studying the behavior of the mutual information and the negativity of the local Wigner functions for a bipartite system described by the states (23) in the case in which the dynamics of the whole system is unitary. From Eq. (7) we obtain a set of uncoupled four equations of motions for \( H(q_i, p_i) \),

\[
q_1(t) = x_0 \cos(\omega t) \cos(\gamma t) + y_0 \cos(\omega t) \sin(\gamma t)
\]
\[
\frac{\beta}{\alpha} [p_{20} \sin(\omega t) \sin(\gamma t) + p_{10} \sin(\omega t) \cos(\gamma t)],
\]
\[
q_2(t) = y_0 \cos(\omega t) \cos(\gamma t) - x_0 \cos(\omega t) \sin(\gamma t)
\]
\[
- \frac{\beta}{\alpha} [p_{20} \sin(\omega t) \sin(\gamma t) - p_{10} \sin(\omega t) \cos(\gamma t)],
\]
\[
p_1(t) = p_{20} \cos(\omega t) \cos(\gamma t) + p_{10} \cos(\omega t) \sin(\gamma t)
\]
\[
- \frac{\alpha}{\beta} [y \sin(\omega t) \sin(\gamma t) + x \sin(\omega t) \cos(\gamma t)],
\]
\[
p_2(t) = p_{20} \cos(\omega t) \cos(\gamma t) - p_{10} \cos(\omega t) \sin(\gamma t)
\]
\[
+ \frac{\alpha}{\beta} [x \sin(\omega t) \sin(\gamma t) - y \sin(\omega t) \cos(\gamma t)],
\]

where \( x_0, y_0, p_{10}, \) and \( p_{20} \) are arbitrary initial parameters.

The Wigner function in Eq. (23) is clearly stationary. From Eq. (7) we obtain a set of uncoupled four equations of motions for \( H(q_i, p_i) \),

\[
W_{k, l}(q_i, p_i) = \frac{(-1)^{k+l}}{\pi^{2\hbar^2}} \exp \left[ -\xi_{1(2)}^2 / \hbar \right] L_{k(l)}^{(0)} \left[ 2 \xi_{1(2)}^2 / \hbar \right].
\]

With

\[
\xi_1^2 = \left( \frac{\alpha^2 q_1^2 + \beta^2 p_1^2}{\beta^2 p_1^2} \right)^2 \cos(\gamma t)^2 + \left( \frac{\alpha^2 q_2^2 + \beta^2 p_2^2}{\alpha^2 q_2^2} \right)^2 \sin(\gamma t)^2
\]
\[
- \left( \frac{\alpha}{\beta} q_1 q_2 + \frac{\beta}{\alpha} p_1 p_2 \right)^2 \sin(2\gamma t),
\]
\[
\xi_2^2 = \left( \frac{\alpha^2 q_1^2 + \beta^2 p_1^2}{\alpha^2 q_1^2} \right)^2 \sin(\gamma t)^2 + \left( \frac{\alpha^2 q_2^2 + \beta^2 p_2^2}{\beta^2 p_2^2} \right)^2 \cos(\gamma t)^2
\]
\[
+ \left( \frac{\alpha}{\beta} q_1 q_2 + \frac{\beta}{\alpha} p_1 p_2 \right)^2 \sin(2\gamma t).
\]

Figure (1) shows the mutual information (black solid line) and the negativity of the local Wigner functions for two sets of quantum numbers, \( (k, l) = (1, 0) \) and \( (k, l) = (2, 1) \). In the first case, Fig. (1)-(a), we observe that the negativity of \( W_1(q_1, p_1) \) (blue dotted line) and \( W_2(q_2, p_2) \) (red dashed line) is initially non-zero and zeros, respectively, as expected, once these states are the first and ground (Gaussian) states of the harmonic oscillator. As the time evolves, the mutual information shared between the subsystems \( (q_1, p_1) \) and \( (q_2, p_2) \) increases up to the negativity of \( W_1(q_1, p_1) \) reaching the minimum value, resulting in a consumption of the negativity of \( W_1(q_1, p_1) \). Then, the mutual information starts to
Thermal environment

\[
\begin{array}{c}
\Gamma_i \quad \bar{m} \\
(q_1, p_1) \quad \bar{\gamma} \quad (q_2, p_2)
\end{array}
\]

Figure 2. Illustration of the system which is considered, \( W_{\text{sys}} = W_1(q_1, p_1) \), coupled to another quantum oscillator \((q_2,p_2)\) by the coupling \( \gamma \), and also coupled to a thermal environment with coupling \( \bar{\gamma} \). We are considering that the coupling between the thermal environment and the quantum oscillator \((q_2,p_2)\) is sufficiently weak such that any effect due to it is negligible for the evolution of the system \( W_{\text{sys}} \).

decrease while the negativity of \( W_2(q_2,p_2) \) increases up to the maximum value. In the second case, Fig. (1)-(b), we have two non-Gaussian initially states with non-zero initial values of negativity of the local Wigner functions. Similarly to the first case, the negativities decreases as the increase of the mutual information and, after a period of time, mutual information decreases while the negativies increases with inverse values. In both cases, we note the negativity-mutual information conversion, highlighting the role of the negativity in supplying correlations. Our results can be, in principle, experimentally implemented in continuous variable platforms, such as in specific ions trap in which there are two vibrational degrees of freedom accessible [41].

Dissipative dynamics - Gaussian states

For this purpose, we shall assume that the states of the subsystems \((q_1,p_1)\) and \((q_2,p_2)\) are Gaussian, i.e. \( W_1(q_1,p_1) \) and \( W_2(q_2,p_2) \) are completely characterized by their first moments and covariance matrix, and defining our system as being \( W_{\text{sys}} \equiv W_1(q_1,p_1) \). Besides, we consider that \( \Gamma \) and \( \gamma \) are constants mediating the coupling between the thermal reservoir with the subsystem \((q_1,p_1)\), and the subsystems \((q_1,p_1)\) and \((q_2,p_2)\), respectively. Figure (2) shows an illustration of the considered dissipative process. We restrict our attention to investigate the dynamics of \( W_{\text{sys}} \) and observe how the coupling \( \gamma \) impacts its dissipative dynamics when in thermal contact with a Markovian environment. From a physical point of view, we consider that the time evolution of \( W_{\text{sys}} \) is strictly Markovian when \( \gamma \to 0 \). Moreover, in tracing out the degree of freedom \((q_2,p_2)\) and restricting our attention only to \( W_{\text{sys}} \), we are assuming that the coupling between the thermal environment and the degree of freedom \((q_2,p_2)\) is sufficiently weak such that it does not cause any effect on the evolution of \( W_{\text{sys}} \).

Figure (3) shows the fidelity (black solid line) and normalized coherence (red dashed line) for a heating process, i.e. \( \bar{n} > \bar{m} \), with \( \bar{n} \) and \( \bar{m} \) the mean number of photons of the system and thermal environment, respectively. In order to guarantee that the environment itself generates a Markovian evolution on the system, we show the case in which \( \gamma = 0 \) Fig. (3)(inset), i.e. there is no coupling of the system with any other degree of freedom. In the case with \( \gamma = 0.1 \), we observe the non-monotonic behavior of the fidelity and the increase of the coherence in some time intervals.

Non-Markonian-like effect

In Fig. (3) we note that the fidelity for \( \gamma \neq 0 \) presents a non-monotonic behavior during the thermalization process with a thermal environment. This profile in the fidelity has been recently associated to a non-Markovian dynamics on the system characterizing an information backflow from the environment to the system [67, 68, 71, 72]. Here the considered thermal environment generates a Markovian dynamics on the system, as we note from the Fig. (3) (inset) for \( \gamma = 0 \). Thus, any non-Markovian-like behavior arises due to the coupling constant \( \gamma \neq 0 \). Physically, the fidelity is a witness of non-Markovianity and the condition is that \( F(\rho_1(\tau),\rho_2(\tau)) \leq F(\rho_1(t),\rho_2(t)) \), with \( \rho_{1(2)} \) two arbitrary states of the system and \( \tau \) and \( t \) arbitrary times such that \( \tau > t \) [71, 72]. Figure (3) presents basically two time intervals for \( \gamma = 0.1 \) with this behavior. Furthermore, the coherence increases during the same time intervals, evidencing a memory effect on the system. Again, it is worth to note that this behavior is exclusively due to the coupling \( \gamma \) and does not depend on the structure of the ther-
Figure 3. (Color online): Fidelity and normalized coherence for the system defined as $W_{sys} = W_I(q_1, p_1)$ as function of the dimensionless time of thermalization with the thermal environment for $\gamma = 0$. The inset shows the same quantities for $\gamma = 0.1$. The inset shows the same quantities with the thermal environment.

V. CONCLUSIONS

In this work we have considered the description of quantum information in the phase-space formalism of quantum mechanics. By defining the Weyl transform of a general operator as well as the Wigner function describing the state of a given system, it was possible to introduce several information quantifiers for different scenarios, in particular, we explored measures for Gaussian and non-Gaussian states, with unitary and dissipative dynamics.

We considered a Hamiltonian of a coupled bipartite system with high degree of flexibility to explore relevant examples. For unitary evolution, we showed that the behavior of the mutual information between two-coupled harmonic oscillator depends on the amount of negativity of the local Wigner functions associated to each subsystem, evidencing the negativity-mutual information conversion. On the other hand, for dissipative dynamics of Gaussian states, it was possible to note that for Gaussian states initially with coherence, the effect of the coupling between the two subsystems may generate some non-trivial behavior in the fidelity comparing the state of the system and the asymptotic one, similarly to a non-Markovian-like effect. Moreover, the same effect can be observed in the coherence during the time evolution, with coherence revivals in specific time intervals.

The description of quantum information in the phase-space formalism of quantum mechanics, besides to offer another possibility of studying quantum processes, is relevant for further theoretical and experimental investigations, specially due to the fact that most experimentally accessible devices, e.g., quantum optics, have high ability for generating Gaussian states. Moreover, our results may be useful in several protocols in quantum thermodynamics, in particular for quantum Otto heat machines. Thus, we hope that this work can contribute to unveil new features of quantum information in phase-space.

ACKNOWLEDGMENTS

Jonas F. G. Santos acknowledges CAPES (Brazil), Grant No. 88882.315250/2019-01 and São Paulo Research Foundation (FAPESP), Grant No. 2019/04184-5, for support. Carlos H. S. Vieira acknowledges CAPES (Brazil) for support. Pedro R. Dieguez acknowledges CAPES (Brazil), Grant No. 88882.315250/2019-01 and São Paulo Research Foundation (FAPESP), Grant No. 2019/04184-5, for support. Carlos H. S. Vieira acknowledges CAPES (Brazil) for support. Pedro R. Dieguez acknowledges CAPES (Brazil), Grant No. 88882.315250/2019-01 and São Paulo Research Foundation (FAPESP), Grant No. 2019/04184-5, for support.

Appendix A. APPENDIX: DERIVATION OF THE WIGNER FUNCTION FOR TWO-COUPLED HARMONIC OSCILLATORS IN PHASE-SPACE

In this appendix we derived with some details the Wigner function for two-coupled harmonic oscillators, described in the Hamiltonian (22). By introducing now the creation and annihilation operators,

$$a_i = \frac{\alpha}{\sqrt{2\hbar\alpha\beta}} q_i + i \frac{\beta}{\sqrt{2\hbar\alpha\beta}} p_i,$$

$$a_i^\dagger = \frac{\alpha}{\sqrt{2\hbar\alpha\beta}} q_i - i \frac{\beta}{\sqrt{2\hbar\alpha\beta}} p_i,$$

with $i = 1, 2$, these operators $a_i$ and $a_i^\dagger$ satisfy the following condition,

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = [a_j, a_i^\dagger] = 0.$$ 

Now we can write the equation (22) as,

$$H(q_i, p_i) = 2\hbar\alpha\beta \left( a_i^\dagger a_i + a_i a_i^\dagger \right) - i\hbar\gamma (a_1 a_2^\dagger - a_2 a_1^\dagger)$$

Note that still there exists a coupling between operators $a_1$ and $a_2$. Then, it is interesting to define the new set of creation and annihilation operators,

$$A_\pm = \frac{1}{\sqrt{2}} (a_1 \mp ia_2), \quad A_\pm^\dagger = \frac{1}{\sqrt{2}} (a_1^\dagger \pm ia_2^\dagger),$$

with these operators obeying the following commutation relations,

$$[A_\pm, A_\pm^\dagger] = [A_\pm, A_\mp^\dagger] = 0$$

$$[A_\pm^\dagger, A_\mp] = [A_\mp^\dagger, A_\pm] = 0,$$

$$[A_\pm, A_\pm] = [A_\mp, A_\mp] = 1$$

$$[A_\mp, A_\mp^\dagger] = [A_\pm, A_\pm^\dagger] = 0$$
Therefore, the Hamiltonian (34) is given by,

\[
H(q_i, p_i) = 2\hbar \alpha \beta \left( A_+^\dagger A_+ + A_-^\dagger A_- + 1 \right) - \hbar \gamma \left( A_+^\dagger A_+ - A_-^\dagger A_- \right).
\]

(36)

The next step is to rewrite the Moyal product (6) in terms of these new operators. Denoting by \( A_+^\dagger \) and \( A_\pm \) as the Weyl transform of the operators \( A_+^\dagger \) and \( A_\pm \), respectively, we can write the Moyal product as,

\[
\ast = \exp \left[ \frac{1}{2} \left( \frac{\partial}{\partial A_+} \frac{\partial}{\partial A_+^\dagger} - \frac{\partial}{\partial A_-} \frac{\partial}{\partial A_-^\dagger} \right) \right]
\times \exp \left[ \frac{1}{2} \left( \frac{\partial}{\partial A_-} \frac{\partial}{\partial A_-^\dagger} - \frac{\partial}{\partial A_+} \frac{\partial}{\partial A_+^\dagger} \right) \right],
\]

with \( A_+^\dagger A_+ = A_-^\dagger A_- - 1/2 \).

Defining new variables \( \eta_1 = A_+^\dagger A_+ \) and \( \eta_2 = A_-^\dagger A_- \), the Weyl transform of the Hamiltonian (36) reads,

\[
H^W(\eta_1, \eta_2) = (2\hbar \alpha \beta - \hbar \gamma)\eta_1 + (2\hbar \alpha \beta + \hbar \gamma)\eta_2.
\]

(37)

In addition, applying the so-called stargenvalue equation (9) for the system,

\[
H^W(\eta_1, \eta_2) \ast W(\eta_1, \eta_2) = EW(\eta_1, \eta_2),
\]

we have that,

\[
H^W(\eta_1, \eta_2) \ast W(\eta_1, \eta_2) = \left( (2\hbar \alpha \beta - \hbar \gamma)\eta_1 + (2\hbar \alpha \beta + \hbar \gamma)\eta_2 \right.
+ \frac{1}{2}(2\hbar \alpha \beta - \hbar \gamma) \left. + \frac{1}{2}(2\hbar \alpha \beta + \hbar \gamma) \right)
+ \frac{1}{2}(2\hbar \alpha \beta - \hbar \gamma) \left. + \frac{1}{2}(2\hbar \alpha \beta + \hbar \gamma) \right)
- \frac{1}{4}(2\hbar \alpha \beta - \hbar \gamma) \left. + \frac{1}{4}(2\hbar \alpha \beta + \hbar \gamma) \right)
= EW(\eta_1, \eta_2).
\]

(38)

After some mathematical manipulations we obtain two uncoupled differential equations given by,

\[
\left[ \eta_2 - \frac{1}{4} \left( \frac{\partial}{\partial \eta_2} \right) + \eta_2 \left( \frac{\partial^2}{\partial \eta_2^2} \right) - E_{1(2)} \right] \times \chi_{1(2)}(\eta_{1(2)}) = 0
\]

(39)

The explicit solutions of Eqs. (39) can be written in terms of the Laguerre polynomials,

\[
\chi_{1(2)}(\eta_{1(2)}) = (-1)^{n_{1(2)}} \exp \left[ -2\chi_{1(2)} \right] L_{n_{1(2)}} \left[ 4\chi_{1(2)} \right],
\]

(40)

with \( n_{1(2)} \) non-negative integers and, \( n_{1(2)} = (E_{1(2)} - 1/2) \), resulting in,

\[
E_{n_1, n_2} = 2\hbar \alpha \beta (n_1 + n_2 + 1) + \hbar \gamma (n_1 - n_2).
\]

(41)

Finally, writing as function of canonical phase-space variables, the Wigner function in (23) is obtained.

[1] A. Bermudez, T. Schaezt, and M. B. Plenio, Dissipation-Assisted Quantum Information Processing with Trapped Ions, Phys. Rev. Lett. 110, 110502 (2013).

[2] K. Marshall and C. Weedbrook, Device-independent quantum cryptography for continuous variables, Phys. Rev. A 90, 042311 (2014).

[3] P. Papanastasiou, C. Ottaviani, and S. Pirandola, Finite-size analysis of measurement-device-independent quantum cryptography with continuous variables, Phys. Rev. A 96, 042332 (2017).

[4] C. Ottaviani, S. Mancini, and S. Pirandola, Two-way Gaussian quantum cryptography against coherent attacks in direct reconciliation, Phys. Rev. A 92, 062323 (2015).

[5] M. Perarnau-Llobet, A. González-Tudela, J. I. Cirac, Multimode Fock states with large photon number: effective descriptions and applications in quantum metrology, arXiv:1910.03323.

[6] Pietro Liuzzo-Scorpo, Andrea Mari, Vittorio Giovannetti, and Gerardo Adesso, Optimal Continuous Variable Quantum Teleportation with Limited Resources, Phys. Rev. Lett. 119, 120503 (2017).

[7] P. A. Camati, J. F. G. Santos, and R. M. Serra, Coherence effects in the performance of the quantum Otto heat engine, Phys. Rev. A 99, 062103 (2019).

[8] R. Dann and R. Kosloff, Quantum Signatures in the Quantum Carnot Cycle, arXiv:1906.0694.
[9] S. Su, W. Shen, J. Du, and J. Chen, Coherence induced work in quantum heat engines with Larmor precession, arXiv:1908.06443.

[10] W. H. Zurek, Reversibility and Stability of Information Processing Systems, Phys. Rev. Lett. 53, 391 (1984).

[11] F. Arute, K. Arya, R. Babbush, et al. Quantum supremacy using a programmable superconducting processor. Nature 574, 505-510 (2019).

[12] Z. Gedik, I. A. Silva, B. Çakmak, G. Karpat, E. L. G. Vdoto, D. O. Soares-Pinto, E. R. deAzevedo and F. F. Fanchini, Computational speed-up with a single qubit. Sci. Rep. 5, 14671 (2015).

[13] M. H. Partovi, Entanglement versus stosszahlansatz: disappearance of the thermodynamic arrow in a high-correlation environment. Phys. Rev. E 77, 021110 (2008).

[14] S. Jevtic, D. Jennings, and T. Rudolph, Maximally and minimally correlated states attainable within a closed evolving system. Phys. Rev. Lett. 108, 110403 (2012).

[15] K. Micadei, J.P.S. Peterson, A.M. Souza et al. Reversing the direction of heat flow using quantum correlations. Nat Commun 10, 2456 (2019).

[16] S. Hamedani Raja, M. Borrelli, R. Schmidt, J. P. Pekola, S. Arvind, D. O. Soares-Pinto, H. Y. Su, W. Shen, J. Du, and J. Chen, Coherence induced of nonclassicality in cavity QED system, Physical Review A 98, 033828 (2018).

[17] M. A. Nielsen and I. L. Chuang, Quantum Computation and Information, Cambridge University Press, Cambridge, 2002.

[18] H. P. Breuer and F. Petruccione, The Theory of Open Quantum Systems, (Oxford University Press, Oxford, 2002).

[19] M. Walschaers, C. Fabre, V. Parigi, and N. Treps, Entanglement and entanglement validation, Phys. Rev. A 96, 022117 (2017).

[20] K.-P. Marzlin and S. Deering, The Moyal equation for open quantum evolution, Phys. Rev. A 91, 052324 (2014).

[21] M. Rosenbaum and J. D. Vergara, The quantum Otto refrigerator. In preparation.

[22] A. V. Dodonov, D. Valente, T. Werlang, Quantum power boost in a nonstationary cavity-QED quantum heat engine, J. Phys. A: Math. Theor. 51, 365302 (2018).

[23] C. K Zachos, D. B Fairlie, and T. L Curtright, Quantum Mechanics in Phase Space: An Overview with Selected Papers, (World Scientific Pub. Co. Inc., Singapure, 2005).

[24] E. Wigner, On the Quantum Correction For Thermodynamic Equilibrium, Phys. Rev. 40, 740 (1932).

[25] W. B. Case, Wigner functions and Weyl transforms for pedestrains, Am. J. Phys. 76, 937 (2008).

[26] K.S. Chou, J.Z. Blumoff, C.S. Wang et al. Deterministic teleportation of a quantum gate between two logical qubits. Nature 561, 368 (2018).

[27] J. F. G. Santos, A. E. Bernardini, and C. Bastos, Probing phase-space noncommutativity through quantum mechanics and thermodynamics of free particles and quantum rotors, Physica A 438, 340 (2015).

[28] J. F. G. Santos and A. E. Bernardini, Gaussian fidelity distorted by external fields, Physica A 445, 75 (2016).

[29] J. F. G. Santos, Noncommutative phase-space effects in thermal diffusion of Gaussian states, J. Phys. A: Math. Theor. 52, 405306 (2019).

[30] D. S. Su, C. R. Myers, and K. K. Sabapathy, Quantum power boost in a nonstationary cavity-QED quantum heat engine, J. Phys. A: Math. Theor. 51, 365302 (2018).

[31] A. E. Bernardini and O. Bertolami, Probing phase-space noncommutativity through quantum beating, missing information and the thermodynamic limit, Phys. Rev. A 88, 012101 (2013).

[32] J. Steinbach, J. Twamley, and P. L. Knight, Engineering two-mode interactions in ion traps, Phys. Rev. A 56, 4815 (1997).

[33] M. Rosenbaum and J. D. Vergara, The $\phi$-value equation and Wigner distributions in noncommutative Heisenberg algebras, Gen. Rel. Grav. 38, 607 (2006).

[34] S. Toyaba, T. Sagawa, M. Ueda et al. Experimental demonstration of information-to-energy conversion and validation of the generalized Jarzynski equality, Nature Phys 6, 988–992 (2010).

[35] A. Kenfack and K. Zyczkowski, Negativity of the Wigner function revisited, J. Phys. A: Math. Theor. 52, 204001 (2019).

[36] J. Steinbach, J. Twamley, and P. L. Knight, Engineering two-mode interactions in ion traps, Phys. Rev. A 56, 4815 (1997).

[37] G. Manfredi, and M. R. Feix, Entropy and Wigner functions, Phys. Rev. E 62, 4665 (2000).

[38] A. Kenfack and K. Zyczkowski, Negativity of the Wigner function as an indicator of non-classicality, J. Opt. B: Quantum Semiclass. Opt. 6 396 (2004).

[39] R. P. Rundle, P. W. Mills, Todd Tilma, J. H. Samson, and M. J. Everitt, Simple procedure for phase-space measurement and entanglement validation, Phys. Rev. A Phys. Rev. A 96, 022117 (2017).

[40] M. Walschaers, C. Fabre, V. Parigi, and N. Treps, Entanglement and Wigner Function Negativity of Multimode Non-Gaussian States, Phys. Rev. Lett. 119, 183601 (2017).

[41] I. I. Arkhipov, A.Barasiński, and J. Svozilík, Negativity volume for hybrid bipartite states, Sci. Rep. 8, 16955 (2018).

[42] G. Nogues, A. Rauschenbeutel, S. Osnaghi, P. Bertet, M. Brune, J. M. Raimond, S. Haroche, L. G. Lutterbach, and L. Davidovich, Measurement of a negative value for the Wigner function of radiation, Phys. Rev. A 62, 054101 (2000).

[43] G. Nogues, A. Rauschenbeutel, S. Osnaghi, P. Bertet, M. Brune, J. M. Raimond, S. Haroche, L. G. Lutterbach, and L. Davidovich, Measurement of a negative value for the Wigner function of radiation, Phys. Rev. A 62, 054101 (2000).

[44] R. Laflamme, D. E. Browne, N. Delfosse, C. Okag, and J. Bernelho-Vega, Contextuality and Wigner-function negativity in qubit quantum computation, Phys. Rev. A 95, 052334 (2014).

[45] X.-B. Wang, T. Hiroshima, A. Tomita, and M. Hayashi, Quantum information with Gaussian states, Physics Reports 448, (2007).
[56] A. Serafini, Quantum Continuous Variables. A primer of Theoretical Methods, (CRC Press, Boca Raton, 2017).
[57] J. DiGuglielmo, B. Hage, A. Franzen, J. Fiurášek, and Roman Schnabel, Experimental characterization of Gaussian quantum-communication channels, Phys. Rev. A 76, 012323 (2007).
[58] A. Winter and D. Yang, Operational Resource Theory of Coherence, Phys. Rev. Lett. 116, 120404 (2016).
[59] T. Baumgratz, M. Cramer, and M. B. Plenio, Quantifying coherence, Phys. Rev. Lett. 113, 140401 (2014).
[60] T. Feldmann and R. Kosloff, Quantum lubrication: Suppression of friction in a first-principles four-stroke heat engine, Phys. Rev. E 73, 025107(R) (2006).
[61] Y. Rezek and R. Kosloff, Irreversible performance of a quantum harmonic heat engine, New J. Phys. 8, 83 (2006).
[62] J. Xu, Quantifying coherence of Gaussian states, Phys. Rev. A 93, 032111 (2016).
[63] A. Carlini, A. Mari, and V. Giovannetti, Time-Optimal Thermalization of Single-Mode Gaussian States, Phys. Rev. A 90, 052324 (2014).
[64] A. M. Jayannavar and Mamata Sahoo, Charged particle in a magnetic field: Jarzynski equality, Phys. Rev. A 75, 032102 (2007).
[65] E. Muñoz and F. J. Peña, Magnetically driven quantum heat engine Phys. Rev. E 89, 052107 (2014).
[66] J. F. G. Santos, A. E. Bernardini, Quantum engines and the range of the second law of thermodynamics in the noncommutative phase-space, Eur. Phys. J. Plus 132, 260 (2017).
[67] Á. Rivas, S. F. Huelga, and M. B. Plenio, Entanglement and Non-Markovianity of Quantum Evolutions, Phys. Rev. Lett. 105, 050403 (2010).
[68] H.-P. Breuer, E.-M. Laine, and J. Piilo, Measure for the Degree of Non-Markovian Behavior of Quantum Processes in Open Systems, Phys. Rev. Lett. 103, 210401 (2009).
[69] A. S. Holevo, Some statistical problems for quantum Gaussian states, IEEE Trans. Inf. Theory 21, 533 (1975).
[70] H. Scutaru, Fidelity for displaced squeezed states and the oscillator semigroup, J. Phys. A 31, 3659 (1998).
[71] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, Colloquium: Non-Markovian dynamics in open quantum systems, Rev. Mod. Phys. 88, 021002 (2016).
[72] A. K. Rajagopal, A. R. Usha Devi, and R. W. Rendell, Kraus representation of quantum evolution and fidelity as manifestations of Markovian and non-Markovian forms A., Phys. Rev. A 82, 042107 (2010).
[73] Á. Cuevas, A. Geraldi, C. Liorni et al. All-optical implementation of collision-based evolutions of open quantum systems. Sci Rep 9, 3205 (2019).