Exact results for the behavior of the thermodynamic Casimir force in a model with a strong adsorption

Daniel M Dantchev, Vassil M Vassilev and Peter A Djondjorov

Institute of Mechanics, Bulgarian Academy of Sciences,
Acad. G. Bonchev St., Building 4, 1113 Sofia, Bulgaria
E-mail: daniel@imbm.bas.bg

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Abstract. When massless excitations are limited or modified by the presence of material bodies one observes a force acting between them generally called Casimir force. Such excitations are present in any fluid system close to its true bulk critical point. We derive exact analytical results for both the temperature and external ordering field behavior of the thermodynamic Casimir force within the mean-field Ginzburg–Landau Ising type model of a simple fluid or binary liquid mixture. We investigate the case when under a film geometry the boundaries of the system exhibit strong adsorption onto one of the phases (components) of the system. We present analytical and numerical results for the (temperature-field) relief map of the force in both the critical region of the film close to its finite-size or bulk critical points as well as in the capillary condensation regime below but close to the finite-size critical point.

Keywords: Casimir effect, classical phase transitions, exact results, finite-size scaling
In a recent article [1], we have derived exact results for both the temperature $T$ and external ordering field $h$ behavior of the order parameter profile and the corresponding response functions—local and total susceptibilities—within the three-dimensional continuum mean-field Ginzburg–Landau Ising type model of a simple fluid or binary liquid mixture for a system with a film geometry $\infty^2 \times L$. In the current article we extend them to derive exact results for the thermodynamic Casimir force within the same model. We concentrate in the region of the parametric space in $(T, h)$ plane close to the critical point of the fluid or close to the demixing point of the binary liquid mixture. We recall that for classical fluids in the case of a simple fluid or for binary liquid mixtures the wall generically prefers one of the fluid phases or one of the components. Because of that in the current article we study the case when the bounding surfaces of the system strongly prefer one of the phases of the system. Since in such systems one observes also the phenomena of the capillary condensation close below the critical point for small negative values of the ordering field $(\beta h)(L/a) = O(1), h < 0$, we also study the behavior of the force between the confining surfaces of the system in that parametric region. Here $\beta = 1/(k_BT)$, the field $h$ is measured in units of Bohr magneton $\mu_B$, and $a$ is some characteristic microscopic length, say, the average distance between the constituents of the fluid. Let us recall that the model we are going to consider is a standard model within which one studies phenomena like critical adsorption [2–15], wetting or drying [12, 13, 16–19], surface phenomena [20, 21], capillary condensation [3, 7, 8, 10, 17, 22, 23], localization-delocalization phase transition [22, 24, 25], finite-size behavior of thin films [7, 22, 24, 26–33], the thermodynamic Casimir effect [10, 34–39], etc. The results of the model have been also used to calculate the Casimir forces in systems with chemically or topographically patterned substrates, as well as, coupled with the Derjaguin approximation, for studies on interactions of colloids—see, e.g. the review [40] and the literature cited therein. Until very recently, i.e. before [1], the results for the case $h = 0$ were derived analytically [34, 35, 37, 41] while the $h$-dependence was
studied numerically either at the bulk critical point of the system $T = T_c$, or along some specific isotherms—see, e.g. [10, 15, 25, 36, 41–43]. In the current article we are going to improve this situation with respect to the Casimir force by deriving exact analytical results for it in the $(T, h)$ plane.

In 1948 [44], after a discussion with Niels Bohr [45], the Dutch physicist Casimir realized that the zero-point fluctuations of the electromagnetic field in vacuum lead to a force of attraction between two perfectly conducting parallel plates and calculated this force. In 1978 Fisher and De Gennes [46] pointed out that a very similar effect exists in fluids with the fluctuating field being the field of its order parameter, in which the interactions in the system are mediated not by photons but by different type of massless excitations such as critical fluctuations or Goldstone bosons (spin waves). Nowadays one usually terms the corresponding Casimir effect the critical or the thermodynamic Casimir effect [33].

Currently the Casimir, and Casimir-like, effects are object of studies in quantum electrodynamics, quantum chromodynamics, cosmology, condensed matter physics, biology and, some elements of it, in nano-technology. The interested reader can consult the existing impressive number of reviews on the subject [14, 33, 40, 47–81]. So far the critical Casimir effect has enjoyed only two general reviews [33, 77] and few concerning specific aspects of it [14, 40, 78–81].

The critical Casimir effect has been already directly observed, utilizing light scattering measurements, in the interaction of a colloid spherical particle with a plate [82] both of which are immersed in a critical binary liquid mixture. Very recently the nonadditivity of critical Casimir forces has been experimentally demonstrated in [83]. Indirectly, as a balancing force that determines the thickness of a wetting film in the vicinity of its bulk critical point the Casimir force has been also studied in $^4$He [84, 85], as well as in $^4$He–$^3$He mixtures [86]. In [87] and [88] measurements of the Casimir force in thin wetting films of binary liquid mixture are also performed. The studies in this field have also enjoined a considerable theoretical attention. Reviews on the corresponding results can be found in [14, 40, 78–81].

Before turning exclusively to the behavior of the Casimir force, let us briefly remind some basic facts of the theory of critical phenomena. In the vicinity of the bulk critical point $(T_c, h = 0)$ the bulk correlation length of the order parameter $\xi$ becomes large, and theoretically diverges: $\xi^+ = \xi(T \to T_c^+, h = 0) \simeq \xi_0^+ t^{-\nu}$, $t = (T - T_c)/T_c$, and $\xi_h \equiv \xi(T = T_c, h \to 0) \simeq \xi_{0,h} |h/k_BT_c|^{-\nu/\Delta}$, where $\nu$ and $\Delta$ are the usual critical exponents and $\xi_0^+$ and $\xi_{0,h}$ are the corresponding nonuniversal amplitudes of the correlation length along the $t$ and $h$ axes. If in a finite system $\xi$ becomes comparable to $L$, the thermodynamic functions describing its behavior depend on the ratio $L/\xi$ and take scaling forms given by the finite-size scaling theory. For such a system the finite-size scaling theory [30–33, 77, 89] predicts:

- For the Casimir force
  
  \[ F_{\text{Cas}}(t, h, L) = L^{-d} X_{\text{Cas}}(x_t, x_h); \]  

- For the order parameter profile
  
  \[ \phi(z, T, h, L) = a_t L^{-\beta/\nu} X_{\phi}(z/L, x_t, x_h), \]  

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where \( x_t = a_tL^{1/\nu} \), \( x_h = a_hL^{\Delta/\nu} \). In equations (1) and (2), \( \beta \) is the critical exponent for the order parameter, \( d \) is the dimension of the system, \( a_t \) and \( a_h \) are nonuniversal metric factors that can be fixed, for a given system, by taking them to be, e.g. \( a_t = 1/[(\xi_0^{+})^{1/\nu}] \), and \( a_h = 1/[(\xi_{0,h})^{\Delta/\nu}] \).

2. The Ginzburg–Landau mean-field model and the Casimir force

2.1. Definition of the model

Here, as in [1], we consider a critical system of Ising type in a film geometry \( \infty^2 \times L \), where \( L \) is supposed to be along \( z \) axis, described by the minimizers of the standard \( \phi^4 \) Ginzburg–Landau functional

\[
\mathcal{F}[\phi; \tau, h, L] = \int_0^L \mathcal{L}(\phi, \phi')dz, \tag{3}
\]

where

\[
\mathcal{L} \equiv \mathcal{L}(\phi, \phi') = \frac{1}{2}\phi'^2 + \frac{1}{2}\tau\phi^2 + \frac{1}{4}g\phi^4 - h\phi. \tag{4}
\]

Here \( L \) is the film thickness, \( \phi(z; \tau, h, L) \) is the order parameter assumed to depend on the perpendicular position \( z \in (0, L) \) only, \( \tau = (T - T_c)/T_c(\xi_0^+)^{-2} \) is the bare reduced temperature, \( h \) is the external ordering field, \( g \) is the bare coupling constant and the primes indicate differentiation with respect to the variable \( z \).

2.2. Basic expression for the Casimir force

The thermodynamic Casimir force is the excess pressure over the bulk one acting on the boundaries of the system which is due to the finite size of the system. To derive this excess pressure there are several ways but probably the most straightforward one is to apply the corresponding mathematical results of the variational calculus. For example, following Gelfand and Fomin [90, pp 54–56] it is easy to show that the functional derivative of \( \mathcal{F} \) with respect to the independent variable \( z \) at \( z = L \) is

\[
-\left( \frac{\delta \mathcal{F}}{\delta z} \right)_{z=L} = -\left( \phi' \frac{\partial \mathcal{L}}{\partial \phi'} - \mathcal{L} \right)_{z=L}. \tag{5}
\]

Having in mind equation (4), one derives explicitly

\[
\left( \frac{\delta \mathcal{F}}{\delta z} \right)_{z=L} = \left( \frac{1}{2}\phi'^2 - \frac{1}{4}g\phi^4 - \frac{1}{2}\tau\phi^2 + h\phi \right)_{z=L} \equiv p_L(\tau, h). \tag{6}
\]

This derivative has the meaning of a force acting on the surface of the system at \( z = L \) and, since \( \mathcal{F} \) is normalized per unit area, it has a meaning of a pressure acting on that surface. That is why, the notation \( p_L(\tau, h) \) is used. The above is actually the procedure

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used in [34] where the authors perform the corresponding variational calculations on
their own. Another common way to proceed is to use the apparatus based on the stress
tensor operator (see, e.g. [36], [91] and [92])

\[ T_{kl} = \frac{\partial L}{\partial (\partial_i \Phi)} (\partial_k \Phi) - \delta_{kl} L. \]  

(7)

It is elementary to check that the expression in the parentheses in the right-hand-side of equation (6) coincides with \( T_{zz} \) component of the stress tensor. In [1], we have shown that this expression is a first integral of the considered system, and therefore \( T_{zz} \) and \( p_L(\tau, h) \) do not depend on the coordinate \( z \) at which they are calculated.

In the bulk system, within the mean-field theory the gradient term in \( L \) is absent and instead of \( p_L \) one obtains

\[ p_b(\tau, h) = -\frac{1}{4} g \phi_b^4 - \frac{1}{2} \tau \phi_b^2 + h \phi_b, \]  

(8)

where \( \phi_b \) is the order parameter of the bulk system. Clearly, \( \phi_b \) is determined by the cubic equation \(-\phi_b [\tau + g \phi_b^2] + h = 0 \), \( \phi_b \) being such that \( L_b = \frac{1}{2} \tau \phi_b^2 + \frac{1}{4} g \phi_b^4 - h \phi_b \) attains its minimum. Now, one can immediately determine the Casimir force as

\[ F_{Cas}(\tau, h, L) = p_L(\tau, h) - p_b(\tau, h). \]  

(9)

When \( F_{Cas}(\tau, h, L) < 0 \) the excess pressure will be inward of the system that corresponds to an attraction of the surfaces of the system towards each other and to a repulsion if \( F_{Cas}(\tau, h, L) > 0 \).

In the light of the above it is evident that once the order parameter profile \( \phi \) is known in analytic form for given values of the parameters \( \tau \) and \( h \), then the corresponding Casimir force is determined exactly. It is noteworthy that the above expressions do not depend on the specific choice of the boundary conditions. In the current article we specialize to the so-called \((+,+)\) boundary conditions under which one requires that \( \lim_{z \to 0} \phi(z) = \lim_{z \to L} \phi(z) = +\infty \). The exact solution for \( \phi(z, \tau, h, L) \) for this case has been determined in [1]. In what follows we will study the properties of the force \( F_{Cas}(\tau, h, L) \) using this exact solution.

3. Exact results for the Casimir force

Since the thermodynamic Casimir force is normally presented in terms of the scaling variables

\[ l_t \equiv \text{sign}(\tau) \frac{L}{\xi^+_t} = \text{sign}(\tau) \frac{L}{\sqrt{\left| \tau \right|}}, \]  

(10)

\[ l_h \equiv \text{sign}(h) \frac{L}{\xi_h} = L \sqrt{\frac{3}{\sqrt{\nu}} \frac{h}{\Delta}}, \]  

(11)

in the remainder we are going to use such variables as the basic parameters determining the behavior of the force. In the above we have taken into account that for the model considered here \( \xi_0, l_h / \xi_0^+ \) = 1/\( \sqrt{3} \) [36], \( \nu = 1/2 \) and \( \Delta = 3/2 \). The phase diagram of
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Figure 1. Phase diagram. The critical point of the finite system is \( T_{cL} = (\ell_t^c, \ell_h^c) = (-5.06935, -9.53633) \). We recall that at this point the susceptibility of the finite system diverges. The inset on the right shows the pre-capillary-condensation curve, determined in [1], where above \( T_{cap} = (-5.13834, -9.55252) \) and below \( T_{cL} \) the jump of the order parameter at the middle of the system is from a less dense gas to a more dense one. For \( T \leq T_{cap} \) the system jumps from a ‘gas’ to a ‘liquid’ state.

This model has been studied in details in [1]—see there figure 3 and the text around it. Here, for the convenience of the reader, it is depicted in figure 1 in terms of the scaling variables \( l_t \) and \( l_h \).

3.1. Exact analytical results for the Casimir force

In terms of the scaling variables given in equations (10) and (11), the value \( p_L(\tau, h) \) of the first integral, see equation (6), becomes

\[
p_L(\tau, h) = \frac{1}{gL^4} p(l_t, l_h),
\]

where the constant \( p(l_t, l_h) \) is

\[
p(l_t, l_h) = X^2 - X^4 - \text{sign}(l_t) \ell_t^2 X^2 + \frac{2}{3\sqrt{6}} \ell_h^3 X.
\]

Here

\[
X(\zeta | l_t, l_h) = \sqrt{\frac{g}{2}} L^{3/2} \phi(z)
\]

is the scaling function of the order parameter \( \phi \), \( \beta = 1/2 \) and hereafter the prime means differentiation with respect to the variable \( \zeta = z/L, \zeta \in [0, 1] \). Similarly, for the bulk system, see equation (8), one has

\[
p_b(\tau, h) = \frac{1}{gL^4} p_b(l_t, l_h),
\]
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where

\[ p_b(l_t, l_h) = -X_b^4 - \text{sign}(l_t) l_t^2 X_b^2 + \frac{2}{3\sqrt{6}} l_h^3 X_b. \]  

(16)

From equations (12) and (15) for the Casimir force (9) one obtains

\[ F_{\text{Cas}}(\tau, h, L) = \frac{1}{g L^4} X_{\text{Cas}}(l_t, l_h), \]  

(17)

where its scaling function \( X_{\text{Cas}} \) is

\[ X_{\text{Cas}}(l_t, l_h) = p(l_t, l_h) - p_b(l_t, l_h). \]  

(18)

Given \( l_t \) and \( l_h \), the determination of \( p_b(l_t, l_h) \) is evident, while \( p(l_t, l_h) \) is given by the expression

\[ p(l_t, l_h) = x_m \left( \frac{2}{3\sqrt{6}} l_h^3 - x_m^3 - \text{sign}(l_t) l_t^2 x_m \right), \]  

(19)

see equation (3.15) in [1]. As shown in [1], \( x_m \) is to be determined from

\[ 12\wp \left( \frac{1}{2}; g_2, g_3 \right) - \text{sign}(l_t) l_t^2 - 6x_m^2 = 0 \]  

(20)

so that it gives rise to a continuous order parameter profile in the interval (0,1), and satisfies the condition

\[ 6\sqrt{3} x_m (\text{sign}(l_t) l_t^2 + 2x_m^2) - \sqrt{2} l_h^3 > 0. \]  

(21)

In equation (20) \( \wp(\xi; g_2, g_3) \) is the Weierstrass elliptic function whose invariants \( g_2 \) and \( g_3 \) are given by the expressions

\[ g_2 = \frac{1}{12} l_t^4 + p(l_t, l_h), \]  

(22)

\[ g_3 = -\frac{1}{216} [l_h^6 + l_t^6 - 36 p(l_t, l_h) l_t^2 l_h^4]. \]  

(23)

Thus, in order to determine \( p(l_t, l_h) \) for the regarded (+, +) boundary conditions at given values of the parameters \( l_t \) and \( l_h \), one should find all the solutions \( x_m \) of the transcendental equation (20) which meet the above requirements. If there is more than one such solution \( x_m \), as explained in detail in [1], we take that one which leads to an order parameter profile that corresponds to the minimum of the energy functional (3)

\[ \mathcal{E} = \frac{1}{g L^3} \int_0^1 f(X, X') d\zeta, \]  

(24)

where

\[ f(X, X') = X'^2 + X^4 + \text{sign}(l_t) l_t^2 X^2 - \frac{2}{3\sqrt{6}} l_h^3 X. \]  

(25)

The precise mathematical procedure how this can be achieved, despite the divergence of the energy (see equation (3.27) in [1] and the text around it), is also explained in

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details in [1]. Let us note that $x_m$ has a clear physical meaning—it is the value of the scaling function of the order parameter profile at the middle of the system, i.e. $X(1/2|l_t, b_h) = x_m(l_t, b_h)$.

From equations (16) and (19), once $X_b$ and $x_m$ are determined, the scaling function of the Casimir force takes the form

$$X_{\text{Cas}}(l_t, b_h) = X_b^4 - x_m^4 + \text{sign}(l_t) b_0^2 (X_b^2 - x_m^2) - \frac{2}{3 \sqrt{6}} b_M^3 (X_0 - x_m).$$

When $h = 0$, i.e. $b_h = 0$, the behavior of the Casimir force under $(+, +)$ boundary conditions has been analytically studied in [35] and [34]. In [34] the value of the so-called Casimir amplitude, i.e. the result for $l_t = b_h = 0$ is obtained, while in [35] the behavior of the force as a function of $l_t$ has been studied.

When $h \neq 0$ the behavior of the Casimir force has been studied only numerically. In [36] and [92] it has been obtained only for $T = T_c$ for some chosen values of $l_t$. Below we present its behavior as a function of both $l_t$ and $b_h$ in $(l_t, b_h)$ plane by evaluating numerically the analytical expressions given above.

### 3.2. Numerical evaluation of the analytical expressions

Using the derived exact analytical expressions described above in the current section we determine the Casimir force in the critical and in the capillary condensation regimes at temperatures below but relatively close to $T_{c,L}$. It should be pointed out that the solutions $x_m$ of the transcendental equation (20) that correspond to certain values of the parameters $l_t$ and $b_h$ are to be obtained numerically identifying by inspection those of them that obey the conditions formulated above.

The behavior of the normalized finite-size scaling function $X_{\text{Cas}}(l_t, b_h) = X_{\text{Cas}}(l_t, b_h) / |X_{\text{Cas}}(0, 0)|$ of the Casimir force is shown in figures 2 and 3.

The relief map of the Casimir force, as a function of both $l_t$ and $b_h$, is shown in figure 2 where the upper part presents the force in a larger scale, while the lower one is a blow up of the region close to the bulk critical point. The only other model we are aware of where such a relief map as a function of both relevant scaling variables is available is that one of the three-dimensional spherical model under periodic boundary conditions [93, 94]. One observes a valley in this map with its deepest point at the dashed line shown in figure 4.

Figures 3(a) and (b) present cross-sections of the foregoing 3d figures for given fixed values of $l_t$, or $b_h$, as a function of $b_h$, or $l_t$, respectively. Please note that they cover region of parameters that goes beyond the one covered in figure 2. Figure 3 shows the behavior of $X_{\text{Cas}}$ as a function of $l_t$ for $b_h = 0, \pm 4.19$. Note that $X_{\text{Cas}}$ is negative and for $l_t = 3.749$ above $T_c$, as in the case of the 2d Ising model [95]. The value of the minimum is 1.411 times deeper than the corresponding value of the force at $T = T_c$, which agrees with the results of [35, 96]. The overall behavior of the curves is similar to that one obtained for the $d = 2$ Ising model [97] via density-matrix renormalization group (DMRG) calculations. Let us, nevertheless, note that for bulk $d = 2$ Ising type systems there is no true phase transition for finite $L$. It has been shown, however, that something similar does exist—a line of very weakly rounded first-order transitions ending in a pseudo-critical point [98]. Let us note that for $T_{c,L} < T < T_c$,
For such values of $l h$ one does not cross the capillary condensation line when lowering the temperature—see figure 1. Let us also note that, of course, one formally can consider temperatures well below $T_{c,L}$ but then the model will no longer deliver physically sensible information. We recall that in the current study when taking $T<T_{c,L}$ we do consider temperatures below but quite close to $T_{c,L}$.

Figure 3(b) depicts the behavior of $X_{Cas}$ as a function of $l h$ for $l t = 4.47, 0, -7.75, -10.95, -14.14$. Note that the minimum of the function $X_{Cas}(l t, l h)$ is again negative, it is attained at $l h = -8.405$, and is 10.052 times deeper than the corresponding value at the bulk critical point. The markers on the curves, including the inset curve representing the blow-up in the case $l t = 0$, show an excellent agreement of the numerical results obtained in [36] (filled markers), and in [92] (empty squares) with the analytic results (solid lines) presented here. We observe that the Casimir force exhibits a discontinuous jump on crossing the coexistence line as predicted in [7, 99] and as also shown in [10, 36, 100]. It is easy to estimate the magnitude of this jump. One way of arguing is through direct formal use of the Kelvin equation [101]—as it is done in [100], which leads to the conclusion that the jump $\Delta F_{Cas} \simeq 2\Phi_b(T) h_{cap}(T) \simeq -2\sigma(T)/L$ where $\Phi_b(T)$ is the bulk spontaneous order parameter, $h_{cap}(T)$ is the field on the capillary coexistence line and $\sigma(T)$ is the interfacial tension between the coexisting ‘+’ and ‘−’ phases of the fluid. Here we have taken into account that according to the Kelvin equation for a fixed $T<T_{c,L}$ and large $L$ coexistence happen at $h_{cap} \sim \sigma(T)/[L\Phi_b(T)]$ [100, 101]. The above expression for $\Delta F_{Cas}$ implies that the magnitude of the jump decreases
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Since the jumps are to an exponentially small force upon crossing the capillary condensation line the above estimate for the jump of the force can be also considered as a very rough estimation of the maximal value of the force for a given $T$ near the line of coexistence of the two phases. Let us also note that for $T$ below but close to $T_{c,L}$ the above expressions for $\Delta F_{\text{Cas}}$ take a scaling form: $2\Phi_h(T)h_{\text{cap}}(T) \sim |t|^\beta h \sim x^\beta t x_h L^{-d}$ and $\sigma(T)L \sim |t|^{(d-1)\nu} L \sim x_1^{(d-1)} L^{-d}$, as it is actually also clear from the scaling relation we have derived for that region of thermodynamic parameters. We recall that within the mean-field approach $\beta = \nu = 1/2$ and, formally, $d = 4$ in the scaling relations. We see

Figure 3. Plots of cross-sections of the Casimir force for given fixed values of $l_t$, or $l_h$, as a function of $l_h$, or $l_t$, respectively. (a) The dependence of the normalized finite-size scaling function $\bar{X}_{\text{Cas}}(l_t, l_h)$ of the Casimir force on the scaling variable $l_t$ for three values of the scaling variable $l_h$: $l_h = 0, l_h = \pm 4.19$. (b) The dependence of the normalized thermodynamic Casimir force $\bar{X}_{\text{Cas}}(l_t, l_h)$ on the field scaling variable $l_h$ for several values of the temperature scaling variable $l_t$. 

\[ l_t = \text{sign}(\tau) \frac{L}{\xi_t^+} \]

\[ l_h = \text{sign}(h) \frac{L}{\xi_h} \]
that, as argued in [100], in a fluid confined by identical, strongly adsorbing walls the Casimir force is strongly influenced by capillary condensation. The DMRG results presented there (see figure 1 in [100]) and in [102] for $d = 2$ Ising strip subject to identical surface fields also support this. Let us note that for relatively small negative values of $l_h$ the jump in the force when one crosses the phase coexistence curve and the minimal Casimir force actually occur at different values of the field $l_h$. The last implies that by increasing the temperature one passes through the minimum of the force which then continuously increases till crossing the coexistence curve at which the force jumps to negative values much closer to 0. For $l_t \leq -10$ the minimal value of the force is practically achieved at the phase coexisting line.

We conclude that our results given in figures 2, 3(a) and (b) are in full conformity with the statement made in [100] that in a fluid confined by identical, strongly adsorbing walls the Casimir force is much stronger for states which lie slightly off bulk coexistence, with $h < 0$.

**4. Summary and concluding remarks**

We have derived exact analytical results for the thermodynamic Casimir force, see equation (26), in a widely used model in the theory of phase transitions. In this model, the value $x_m$ of the order parameter in the middle of the system is a solution of equation (20) obeying the condition (21). If there is more than one solution $x_m$ satisfying the above requirements, we take the one that leads to an order parameter profile corresponding to the minimum of the energy functional (24), as explained in details in [1]. The obtained results allow us to plot the relief map of the force, see figure 2, as a function of the both relevant scaling variables—the temperature and field. In addition, figures 3(a) and (b) present cross-sections of the behavior of the force for given fixed
values of $l_{\mu}$ or $l_{\nu}$ as a function of $l_{\delta}$, or $l_{\gamma}$ respectively. Finally, figure 4 presents the loci of the minima of the force in the $(l_{\mu}, l_{\nu})$ plane. The inspection of the results convincingly demonstrates that for $h < 0$ the Casimir force in the capillary condensation regime is much more attractive than that found at the critical point. The comparison there of the numerical evaluation of our analytical expressions with the available numerical results reported in [36, 92] shows an excellent agreement between each other. Furthermore, the analysis of the available DMRG numerical results [97, 100, 102] for the behavior of the Casimir force for the 2d Ising model, as presented in section 3.2, led us to conclusion that our mean-field results, which formally correspond to systems with spacial dimension $d \geq 4$, are quite similar in the capillary condensation regime to those of the two-dimensional Ising model. Thus, the results for the capillary condensation regime are quite robust with respect to the influence of the dimensionality. Of course, the situation is different in the parametric space close to the bulk critical point, as well as very close to the critical point of the finite system. We recall that this later critical point shall show the singularities of the corresponding $(d - 1)$-dimensional system. Within the mean-field approach, as it is well known, the role of the fluctuations is not properly taken into account. If one wants to go beyond the mean-field approximation for, say, $d = 3$ dimensional systems one shall either relay on numerical methods like in [92, 103], or use methods based on renormalization group approach. However, let us recall that the mean-field results serve as a starting point there for renormalization group calculations [21, 35, 89]. Thus, our results shall be helpful for such future analytical studies on the thermodynamic Casimir force. Finally, let us also remind that in physical chemistry and, more precisely, in colloid sciences fluid mediated interactions between two surfaces or large particles are usually referred to as solvation forces or disjoining pressure [3, 95, 99]. Thus, our results can be also considered as pertaining to a particular case of such forces when the fluid is near its critical point.

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