Path Integral Quantization of the Poisson-Sigma Model

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Abstract

We apply the antifield quantization method of Batalin and Vilkovisky to the calculation of the path integral for the Poisson-Sigma model in a general gauge. For a linear Poisson structure the model reduces to a nonabelian gauge theory, and we obtain the formula for the partition function of two-dimensional Yang-Mills theory for closed oriented two-dimensional manifolds.

Keywords: Path integral quantization, Poisson-Sigma model

1 Introduction

The Poisson-Sigma model associates to any Poisson structure on a finite-dimensional manifold a two-dimensional field theory [1]. Choosing different Poisson structures leads to specific models which include most of the topological and semi-topological field theories which have been of interest in recent years. These include gravity models, non-abelian gauge theories and the Wess-Zumino-Witten model. Under some natural restrictions these models are completely integrable.

When the two-dimensional spacetime is a cylinder the Poisson-Sigma model can be quantized with the canonical quantization procedure [2]. In this paper we use the path integral method in order to perform the quantization for arbitrary world sheet topologies.

In the language of gauge theories the models considered here involve an open gauge algebra, i.e. the algebra closes only on-shell. In such cases the Faddeev-Popov method of path integral quantization fails. Quantization procedures which rely on the BRST symmetry of the extended action are in principle more powerful [3]. We find that for our purposes the antifield formalism of Batalin and Vilkovisky [4] is the most effective quantization method.

The path integral approach for the Poisson-Sigma model was first discussed in a preliminary way by Schaller and Strobl [2]. In a recent paper Cattaneo and Felder [5] use the
perturbation expansion of the path integral in the covariant gauge to elucidate Kontsevich’s formula for the deformation quantization of the algebra of functions on a Poisson manifold [4]. Kummer, Liebl and Vassilevich have investigated the special case of 2d dilaton gravity in the temporal gauge, and they have calculated the generating functional using BRST methods [4]. In further work they have studied the coupling to matter fields [8]. We present here a complete and general derivation of the partition function for the Poisson-Sigma model for an arbitrary gauge. In the study of the Chern-Simons topological gauge theory it is known that different choices of the gauge-fixing function lead to different integral representations of the associated knot invariants: the light-cone gauge leads to the universal Kontsevich integral [9], the Landau gauge to the covariant integrals of Bott and Taubes [10], [11] and the temporal gauge to the singular integrals studied by Labastida and Pérez [12]. The relation between these various representations is at present not well-understood, in particular no one has been able to reproduce the necessary Kontsevich normalization factor from a path integral approach. We hope that the techniques developed here may be helpful in that context.

Our paper is structured as follows. In Section 2 we give a concise but self-contained review of the Batalin-Vilkovisky quantization procedure for gauge theories. In Section 3 we apply the method to the Poisson-Sigma model. In Section 4 we show how the general model reduces under certain circumstances to the more familiar Yang-Mills case, and that we reproduce in this case the formula for the partition function [13]. Section 5 contains our conclusions and an outlook for further research.

2 The Antifield Formalism for the Quantization of Gauge Theories

2.1 The Structure Equations of Gauge Theories

The Batalin-Vilkovisky formalism has a beautiful geometric interpretation, first discovered by Witten [14], and recently described in the paper of Alexandrov et. al. [15]. Here we are not concerned with these aspects; we just want to show how the formalism can be applied. We restrict our account to irreducible dynamical systems. For further details we refer to the recent review by Gomis, Paris and Stuart [16].

We consider a system whose dynamics is governed by a classical action $S_0[\phi^i]$ which depends on the fields $\phi^i(x), \ i = 1, \ldots, n$. In the following we shall use a compact notation in which the multi-index $i$ may denote the various fields involved, the discrete indices on which they may depend, and the dependence on the spacetime variables as well. The generalized summation convention then means that a repeated index may denote not only a sum over discrete variables, but also integration over the spacetime variables. $\epsilon_i = \epsilon(\phi^i)$ will denote the Grassman parity of the fields $\phi^i$. Fields with $\epsilon_i = 0$ are called bosonic, fields with $\epsilon_i = 1$ fermionic. The graded commutation rule is

$$\phi^i(x)\phi^j(y) = (-1)^{\epsilon_i\epsilon_j}\phi^j(y)\phi^i(x). \quad (2.1)$$

For a gauge theory the action is invariant under a set of $m$ gauge transformations with infinitesimal form

$$\delta\phi^i = R^i_\alpha \epsilon^\alpha, \quad \alpha = 1 \text{ or } 2 \text{ or } \ldots m. \quad (2.2)$$
This is compact notation for
\[
\delta \phi^i(x) = (R^i_\alpha(\phi)\varepsilon^\alpha)(x)
= \sum_\alpha \int dy R^i_\alpha(x, y) \varepsilon^\alpha(y).
\] (2.3)

The \(\varepsilon^\alpha(x)\) are the infinitesimal gauge parameters and the \(R^i_\alpha(\phi)\) the generators of the gauge transformations. When \(\varepsilon_\alpha = \epsilon(\varepsilon^\alpha) = 0\) we have an ordinary symmetry, when \(\varepsilon_\alpha = 1\) a supersymmetry. The Grassman parity of \(R^i_\alpha\) is \(\epsilon(R^i_\alpha) = \epsilon_i + \epsilon_\alpha \mod 2\). When the gauge generators are independent the theory is said to be irreducible, otherwise it is reducible. For our purposes it will be sufficient to consider the irreducible case.

A subscript index after a comma denotes the right derivative with respect to the corresponding field, and in general when a derivative is indicated it is to be understood as a right derivative unless specifically noted to be otherwise. The field equations may then be written as
\[
S_{0,i} = 0.
\] (2.4)

The classical solutions \(\phi_0\) are determined by \(S_{0,i}|_{\phi_0} = 0\). The Noether identities are
\[
S_{0,i} R^i_\alpha = 0.
\] (2.5)

The general solution to the Noether identity \(S_{0,i} \lambda^i = 0\) is
\[
\lambda^i = R^i_\alpha T^\alpha + S_{0,j} E^{ji}.
\] (2.6)

The commutator of two gauge transformations is
\[
[\delta_1, \delta_2] \phi^i = (R^i_{\alpha,j} R^j_\beta - (-1)^{\epsilon_\alpha \epsilon_\beta} R^i_{\beta,j} R^j_\alpha) \varepsilon^\beta \varepsilon^\alpha.
\] (2.7)

Since this commutator is a symmetry of the action it satisfies the Noether identity
\[
S_{0,i}(R^i_{\alpha,j} R^j_\beta - (-1)^{\epsilon_\alpha \epsilon_\beta} R^i_{\beta,j} R^j_\alpha) = 0,
\] (2.8)

which by Eq. (2.4) implies that
\[
R^i_{\alpha,j} R^j_\beta - (-1)^{\epsilon_\alpha \epsilon_\beta} R^i_{\beta,j} R^j_\alpha = R^i_\gamma T^\gamma_{\alpha \beta} - S_{0,j} E^{ji}_{\alpha \beta}.
\] (2.9)

Eqs. (2.7) and (2.9) lead to the following condition:
\[
[\delta_1, \delta_2] \phi^i = (R^i_\gamma T^\gamma_{\alpha \beta} - S_{0,j} E^{ji}_{\alpha \beta}) \varepsilon^\beta \varepsilon^\alpha.
\] (2.10)

The tensors \(T^\gamma_{\alpha \beta}\) are called the structure constants of the gauge algebra, although they depend in general on the fields of the theory. When \(E^{ij}_{\alpha \beta} = 0\) the gauge algebra is said to be closed, otherwise it is open. Eq. (2.10) defines a Lie algebra if the algebra is closed and the \(T^\gamma_{\alpha \beta}\) are independent of the fields.

The gauge tensors have the following graded symmetry properties:
\[
E^{ij}_{\alpha \beta} = -(-1)^{\epsilon_\alpha \epsilon_\beta} E^{ji}_{\alpha \beta} = -(-1)^{\epsilon_\alpha \epsilon_\beta} E^{ij}_{\beta \alpha},
\] (2.11)
\[
T^\gamma_{\alpha \beta} = -(-1)^{\epsilon_\alpha \epsilon_\beta} T^\gamma_{\beta \alpha}.
\] (2.12)
The Grassman parities are

\[ \epsilon(T_{\alpha \beta}^\gamma) = \epsilon_\alpha + \epsilon_\beta + \epsilon_\gamma \pmod{2} \]  

(2.13)

and

\[ \epsilon(E_{\alpha \beta}^{ij}) = \epsilon_i + \epsilon_j + \epsilon_\alpha + \epsilon_\beta \pmod{2}. \]  

(2.14)

Various restrictions are imposed by the Jacobi identity

\[ \sum_{\text{cyclic}(123)} [\delta_1, [\delta_2, \delta_3]] = 0. \]  

(2.15)

These restrictions are

\[ \sum_{\text{cyclic}(123)} (R_\delta^i A^\delta_{\alpha \beta \gamma} - S_{0,j} B^{ji}_{\alpha \beta \gamma}) \epsilon_1^\gamma \epsilon_2^\beta \epsilon_3^\alpha = 0, \]  

(2.16)

where

\[ 3A^\delta_{\alpha \beta \gamma} \equiv (T^\delta_{\alpha \beta,k} R^k_{\gamma} - T^\delta_{\alpha \eta,k} T^\eta_{\gamma}) \]

\[ + (-1)^{\epsilon_\alpha (\epsilon_\beta + \epsilon_\gamma)} (T^\delta_{\beta \gamma,k} R^k_{\alpha} - T^\delta_{\beta \eta,k} T^\eta_{\alpha}) + (-1)^{\epsilon_\gamma (\epsilon_\alpha + \epsilon_\beta)} (T^\delta_{\gamma \alpha,k} R^k_{\beta} - T^\delta_{\gamma \eta,k} T^\eta_{\beta}) \]

(2.17)

and

\[ 3B^{ji}_{\alpha \beta \gamma} \equiv (E^{ji}_{\alpha \beta,k} R^k_{\gamma} - E^{ji}_{\alpha \eta,k} T^\eta_{\gamma}) - (-1)^{\epsilon_\iota (\epsilon_\alpha + \epsilon_\beta)} R^i_{\alpha,k} E^{kj}_{\beta \gamma} + (-1)^{\epsilon_\iota (\epsilon_\iota + \epsilon_\alpha)} R^i_{\iota,k} E^{kj}_{\gamma \beta} \]

\[ + (-1)^{\epsilon_\alpha (\epsilon_\beta + \epsilon_\gamma)} (\alpha \rightarrow \beta, \ \beta \rightarrow \gamma, \ \gamma \rightarrow \alpha) \]

\[ + (-1)^{\epsilon_\gamma (\epsilon_\alpha + \epsilon_\beta)} (\alpha \rightarrow \gamma, \ \beta \rightarrow \alpha, \ \gamma \rightarrow \beta). \]  

(2.18)

As in the familiar Faddeev-Popov procedure it is useful to introduce ghost fields \(C^\alpha\) with opposite Grassman parities to the gauge parameters \(\epsilon_\alpha\);  

\[ \epsilon(C^\alpha) = \epsilon_\alpha + 1, \]  

(2.19)

and to replace the gauge parameters by ghost fields. One must then modify the graded symmetry properties of the gauge structure tensors according to

\[ T_{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \ldots} \rightarrow (-1)^{\epsilon_\alpha_2 + \epsilon_\alpha_4 + \ldots} T_{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \ldots}. \]  

(2.20)

The Noether identities then take the form

\[ S_{0,i} R^i_{\alpha} C^\alpha = 0, \]  

(2.21)

and the structure relations (2.9) become

\[ (2R^i_{\alpha,\beta} R^j_{\gamma} - R^i_{\gamma} T^\gamma_{\alpha \beta} + S_{0,j} E^{ji}_{\alpha \beta}) C^\beta C^\alpha = 0. \]  

(2.22)
2.2 Introducing the Antifields

We incorporate the ghost fields into the field set \( \Phi^A = \{ \phi^i, C^\alpha \} \), where \( i = 1, \ldots, n \) and \( \alpha = 1, \ldots, m \). Clearly \( A = 1, \ldots, N \), where \( N = n + m \). One then further increases the set by introducing an antifield \( \Phi^*_A \) for each field \( \Phi^A \). The Grassman parity of the antifields is

\[
\epsilon(\Phi^*_A) = \epsilon(\Phi^A) + 1 \pmod{2}. \tag{2.23}
\]

We also assign to each field a **ghost number**, with

\[
\text{gh}[\phi^i] = 0, \tag{2.24}
\]
\[
\text{gh}[C^\alpha] = 1, \tag{2.25}
\]
\[
\text{gh}[\Phi^*_A] = -\text{gh}[\Phi^A] - 1. \tag{2.26}
\]

In the space of fields and antifields the **antibracket** is defined by

\[
(X, Y) = \frac{\partial_r X}{\partial \Phi^A} \frac{\partial Y}{\partial \Phi^*_A} - \frac{\partial_r X}{\partial \Phi^*_A} \frac{\partial Y}{\partial \Phi^A}, \tag{2.27}
\]

where \( \partial_r \) denotes the right, \( \partial_l \) the left derivative. The antibracket is graded antisymmetric;

\[
(X, Y) = -(-1)^{\epsilon X + 1}(\epsilon Y + 1)(Y, X). \tag{2.28}
\]

It satisfies a graded Jacobi identity

\[
((X, Y), Z) + (-1)^{\epsilon X + 1}(\epsilon Y + \epsilon Z)((Y, Z), X) + (-1)^{\epsilon Z + 1}(\epsilon X + \epsilon Y)((Z, X), Y) = 0. \tag{2.29}
\]

It is a graded derivation

\[
(X, Y Z) = (X, Y)Z + (-1)^{\epsilon Y \epsilon X}(X, Z)Y, \tag{2.30}
\]
\[
(X Y, Z) = X(Y, Z) + (-1)^{\epsilon X \epsilon Y}(X, Z)Y.
\]

It has ghost number

\[
\text{gh}[(X, Y)] = \text{gh}[X] + \text{gh}[Y] + 1 \tag{2.31}
\]

and Grassman parity

\[
\epsilon((X, Y)) = \epsilon(X) + \epsilon(Y) + 1 \pmod{2}. \tag{2.32}
\]

For bosonic fields

\[
(B, B) = 2 \frac{\partial B}{\partial \Phi^A} \frac{\partial B}{\partial \Phi^*_A}, \tag{2.33}
\]

for fermionic fields

\[
(F, F) = 0, \tag{2.34}
\]

and for any \( X \)

\[
((X, X), X) = 0. \tag{2.35}
\]
If we group the fields and antifields together into the set
\[ z^a = \{ \Phi^A, \Phi^*_A \}, \quad a = 1, \ldots, 2N, \] (2.36)
then the antibracket is seen to define a symplectic structure on the space of fields and antifields;
\[ (X, Y) = \frac{\partial X}{\partial z^a} \zeta^{ab} \frac{\partial Y}{\partial z^b} \] (2.37)
with
\[ \zeta^{ab} = \begin{pmatrix} 0 & \delta^A_B \\ -\delta^A_B & 0 \end{pmatrix}. \] (2.38)
The antifields can be thought of as conjugate variables to the fields, since
\[ (\Phi^A, \Phi^*_B) = \delta^A_B. \] (2.39)

### 2.3 The Classical Master Equation

Let \( S[\Phi^A, \Phi^*_A] \) be a functional of the fields and antifields with the dimensions of an action, vanishing ghost number and even Grassman parity. The equation
\[ (S, S) = 2 \frac{\partial S}{\partial \Phi^A} \frac{\partial S}{\partial \Phi^*_A} = 0 \] (2.40)
is the \textit{classical master equation}. Solutions of the classical master equation with suitable boundary conditions turn out to be generating functionals for the gauge structures of the gauge theory. \( S \) is also the starting point for the quantization of the theory.

One denotes by \( \Sigma \) the subspace of stationary points of the action in the space of fields and antifields:
\[ \Sigma = \left\{ z^a \left| \frac{\partial S}{\partial z^a} = 0 \right. \right\}. \] (2.41)
Given a classical solution \( \phi_0 \) of \( S_0 \) one possible stationary point is
\[ \phi^i = \phi^i_0, \quad C^a = 0, \quad \Phi^*_A = 0. \] (2.42)

An action \( S \) which satisfies the classical master equation has its own set of invariances:
\[ \frac{\partial S}{\partial z^a} R^a_b = 0, \] (2.43)
with
\[ R^a_b = \zeta^{ac} \frac{\partial \eta \partial_c S}{\partial z^c \partial z^b}. \] (2.44)
This equation implies
\[ R^c_a R^a_b \big|_{\Sigma} = 0. \] (2.45)
We see that $R^a_0$ is nilpotent on-shell. A nilpotent $2N \times 2N$ matrix has rank less than or equal to $N$. Let $r$ be the rank of the hessian of $S$ at the stationary point:

$$r = \text{rank } \frac{\partial l}{\partial r} \frac{\partial S}{\partial z^a} \frac{\partial z^b}{\Sigma}.$$  \hspace{1cm} (2.46)

We then have $r \leq N$. The relevant solutions of the classical master equation are those for which $r = N$. In this case the number of independent gauge invariances of the type in Eq. (2.43) equals the number of antifields. When at a later stage the gauge is fixed the non-physical antifields are eliminated.

To ensure the correct classical limit the proper solution must contain the classical action $S_0$ in the sense that

$$S[\Phi^A, \Phi^*_A] |_{\Phi^*_A=0} = S_0[\phi^i].$$  \hspace{1cm} (2.47)

The action $S[\Phi^A, \Phi^*_A]$ can be expanded in a series in the antifields, while maintaining vanishing ghost number and even Grassman parity:

$$S[\Phi, \Phi^*] = S_0 + \phi^*_i R^i_{\alpha \gamma} C^\alpha + C^*_{\alpha \beta} \frac{1}{2} T^\alpha_{\beta \gamma} (-1)^{\epsilon_{\beta}} C^\gamma C^\beta + \phi^*_i \phi^*_{j} (-1)^{\epsilon_{i}} \frac{1}{4} E^j_{\alpha \beta} (-1)^{\epsilon_{\alpha}} C^\beta C^\alpha + \ldots .$$  \hspace{1cm} (2.48)

When this is inserted into the classical master equation one finds that this equation implies the gauge structure of the classical theory (see e.g. Eq. (3.21) below).

2.4 Gauge-fixing and Quantization

Eq. (2.43) shows that the action $S$ still possesses gauge invariances, and hence is not yet suitable for quantization via the path integral approach: a gauge-fixing procedure is necessary. In the Batalin-Vilkovisky approach the gauge is fixed, and the antifields eliminated, by use of a gauge-fixing fermion $\Psi$ which has Grassman parity $\epsilon(\Psi) = 1$ and $\text{gh}[\Psi] = -1$. It is a functional of the fields $\Phi^A$ only; its relation to the antifields is

$$\Phi^*_A = \frac{\partial \Psi}{\partial \Phi^A}.$$  \hspace{1cm} (2.49)

We define a surface in functional space

$$\Sigma_{\Psi} = \left\{ (\Phi^A, \Phi^*_A) \left| \Phi^*_A = \frac{\partial \Psi}{\partial \Phi^A} \right. \right\},$$  \hspace{1cm} (2.50)

so that for any functional $X[\Phi, \Phi^*]$

$$X |_{\Sigma_{\Psi}} = X \left[ \Phi, \frac{\partial \Psi}{\partial \Phi} \right].$$  \hspace{1cm} (2.51)

To construct a gauge-fixing fermion $\Psi$ of ghost number $-1$ one must again introduce additional fields. The simplest choice utilizes a trivial pair $\tilde{C}_\alpha, \tilde{\pi}_\alpha$ with

$$\epsilon(\tilde{C}_\alpha) = \epsilon_\alpha + 1, \epsilon(\tilde{\pi}_\alpha) = \epsilon_\alpha, \hspace{1cm} \text{gh}[\tilde{C}_\alpha] = -1, \hspace{1cm} \text{gh}[\tilde{\pi}_\alpha] = 0.$$  \hspace{1cm} (2.52)
The fields $\bar{C}_\alpha$ are the Faddeev-Popov antighosts. Along with these fields we include the corresponding antifields $\bar{C}^{*\alpha}$, $\bar{\pi}^{*\alpha}$. Adding the term $\bar{\pi}_\alpha \bar{C}^{*\alpha}$ to the action $S$ does not spoil its properties as a proper solution to the classical master equation, and we get the non-minimal action

$$S^{\text{non}} = S + \bar{\pi}_\alpha \bar{C}^{*\alpha}. \quad (2.54)$$

The simplest possibility for $\Psi$ is

$$\Psi = \bar{C}_\alpha \chi^\alpha(\phi), \quad (2.55)$$

where $\chi^\alpha$ are the gauge-fixing conditions for the fields $\phi$. We denote the gauge-fixed action by

$$S_\Psi = S^{\text{non}} \mid_{\Sigma_\Psi}. \quad (2.56)$$

Quantization is performed using the path integral to calculate a correlation function $X$, with the constraint (2.49) implemented by a $\delta$-function:

$$I_{\Psi}(X) = \int \mathcal{D}\Phi \mathcal{D}\Phi^* \delta \left( \Phi^*_A - \frac{\partial \Psi}{\partial \Phi_A^*} \right) \exp \left( \frac{i}{\hbar} W[\Phi, \Phi^*] \right) X[\Phi, \Phi^*]. \quad (2.57)$$

Here $W$ is the quantum action, which reduces to $S$ in the limit $\hbar \to 0$. An admissible $\Psi$ leads to well-defined propagators when the path integral is expressed as a perturbation series expansion.

The results of a calculation should be independent of the gauge-fixing. Consider the integrand in Eq. (2.57),

$$\mathcal{I}[\Phi, \Phi^*] = \exp \left( \frac{i}{\hbar} W[\Phi, \Phi^*] \right) X[\Phi, \Phi^*]. \quad (2.58)$$

Under an infinitesimal change in $\Psi$

$$I_{\Psi + \delta \Psi}(X) - I_{\Psi}(X) \approx \int \mathcal{D}\Phi \Delta \mathcal{I} \delta \Psi, \quad (2.59)$$

where the Laplacian $\Delta$ is

$$\Delta = (-1)^{\epsilon_A+1} \frac{\partial}{\partial \Phi^*_A} \frac{\partial}{\partial \Phi_A}. \quad (2.60)$$

Obviously the integral $I_{\Psi}(X)$ is independent of $\Psi$ if $\Delta \mathcal{I} = 0$. For $X = 1$ one gets the requirement

$$\Delta \exp \left( \frac{i}{\hbar} W \right) = \exp \left( \frac{i}{\hbar} W \right) \left( \frac{i}{\hbar} \Delta W - \frac{1}{2\hbar^2} (W, W) \right) = 0. \quad (2.61)$$

The formula

$$\frac{1}{2} (W, W) = i\hbar \Delta W \quad (2.62)$$
is the quantum master equation. A gauge-invariant correlation function satisfies

\begin{equation}
(X, W) = i\hbar \triangle X.
\end{equation}

The terms of higher order in \(\hbar\) by which the quantum action \(W\) may differ from the solution of the classical master equation \(S\) correspond to the Counterterms of the renormalizable gauge theory if

\begin{equation}
\triangle S = 0.
\end{equation}

One must of course use a regularization scheme which respects the symmetries of the theory. For \(W = S + O(\hbar)\) the quantum master equation (2.62) reduces in this case to the classical master equation \((S, S) = 0\). Hence, up to possible counter terms, one may simply choose \(W = S\). This is the case for the systems we are considering in this paper.

To implement the gauge-fixing we use for the action \(W = S^{\text{non}}\). For the path integral \(Z = I_{\Psi}(X = 1)\) we perform the integration over the antifields in Eq. (2.57) by using the \(\delta\)-function. The result is

\begin{equation}
Z = \int D\Phi \exp \left(\frac{i}{\hbar} S_{\Psi}\right).
\end{equation}

3 The Path Integral for the Poisson-Sigma Model

3.1 The Classical Theory

A Poisson manifold \(N\) is a smooth manifold equipped with a Poisson structure \(P \in \Lambda^2 TN\). In local coordinates \(X^i\) on \(N\)

\begin{equation}
P = \frac{1}{2} P^{ij}(X) \partial_i \wedge \partial_j,
\end{equation}

and \(P^{ij}\) satisfies the condition

\begin{equation}
P^{ij} P^{lk} \epsilon_{ijkl} = 0,
\end{equation}

which reflects the vanishing of the corresponding Schouten-Nijenhuis bracket for \(P\) with itself. Here the bracketed indices denote an antisymmetric sum. In the notation of Poisson brackets

\begin{equation}
\{f(X), g(X)\} = P^{ij} f_i g_j
\end{equation}

and the Jacobi identity follows from Eq. (3.2):

\begin{equation}
\{f, \{g, h\}\} + \text{cyclic} = 0.
\end{equation}

The Poisson bracket satisfies the Leibniz derivation rule:

\begin{equation}
\{h, fg\} = \{h, f\}g + f\{h, g\}.
\end{equation}

\(P\) is in general degenerate, in which case it does not induce a symplectic structure on \(N\), and the map \(T^*N \to TN\) induced by \(P\), which maps a one-form \(\alpha_i dX^i\) on \(N\) to the vector field
\[ \alpha_i P^{ij} \partial_j \], is not surjective. However, as a consequence of the Jacobi identity, the image of this map forms an involutive system of vector fields. It then turns out that the characteristic distribution \( S(N) \) of the Poisson manifold \( N \) is completely integrable and the Poisson structure \( P \) induces symplectic structures on the leaves \( S \), i.e. a nondegenerate symplectic structure \( \Omega_S \) on \( S \).

Indeed, the splitting theorem of Weinstein \[18\] states that for a regular Poisson manifold, i.e. the Poisson tensor has constant rank, there exist so-called Casimir-Darboux coordinates on the Poisson manifold \( N \). For \( P \) degenerate there are functions \( f \) on \( N \) whose Hamiltonian vector fields \( X_f = f \rho^{ij} \partial_j \) vanish. These functions are called Casimir functions. Let \( \{ C^I \} \) be a maximal set of independent Casimir functions. Then \( C^I(X) = \text{const.} = C^I(X_0) \) defines a level surface through \( X_0 \) whose connected components may be identified with the symplectic leaves. According to Darboux’s theorem there are local coordinates \( X^\alpha \) on \( S \) such that the symplectic form \( \Omega_S \) is given by

\[ \Omega_S = dX^1 \wedge dX^2 + dX^3 \wedge dX^4 + \ldots . \]  

(3.6)

Together with the Casimir functions we then have a coordinate system \( \{ X^I, X^\alpha \} \) on \( N \) with \( P^{IJ} = P^{I\alpha} = 0 \) and \( P^{\alpha\beta} = \text{constant} \).

We now consider a field theory on a two dimensional world sheet \( M \) without boundary \[1\]. The theory involves a set \( X^i \) of bosonic scalar fields, which can be interpreted as a set of maps \( X^i : M \to N \). In addition one has a 1-form \( A \) on the world sheet \( M \) which takes values in \( T^*N \), i.e. a 1-form on \( M \) which is simultaneously the pullback of a section of \( T^*N \) by \( X(x) \), where \( \{ x \} \) are coordinates on \( M \). This field \( A = A_{\mu i} dx^\mu \wedge dX^i \) reduces in the case of a linear Poisson structure, which leads to the Yang-Mills theory, to an ordinary gauge field. In these coordinates the action of the semi-topological Poisson-Sigma model is

\[ S_0[X, A] = \int_M \mu \left[ \epsilon^{\mu\nu}(A_{\mu i} \partial_\nu X^i + \frac{1}{2} P^{ij}(X) A_{\mu i} A_{\nu j}) + C(X) \right] , \]  

(3.7)

where \( \mu \) is the volume form on \( M \) and \( C(X) \) is a Casimir function.

The gauge transformations are

\[ \delta X^i = P^{ij}(X) \delta_j , \quad \delta A_{\mu i} = D_{\mu i} \delta_j , \]  

(3.8)

where \( D_{\mu i} = \partial_\mu \delta_i + P^{kj} A_{jk} \). The equations of motion are

\[ \epsilon^{\mu\nu} D_{\mu i} A_{\nu j} + \frac{\partial C(X)}{\partial X^i} = 0 \]  

(3.9)

and

\[ \epsilon^{\mu\nu} (\partial_\nu X^i + P^{ij} A_{ij}) = \epsilon^{\mu\nu} D_\nu X^i = 0 \]  

(3.10)

The gauge algebra is given by

\[ [\delta(\epsilon_1), \delta(\epsilon_2)] X^i = P^{ij}(P^{mn} \epsilon^j_{1n} \epsilon^2_{2m}) , \]  

(3.11)

\[ [\delta(\epsilon_1), \delta(\epsilon_2)] A_{\mu i} = D_{\mu i}(P^{mn} \epsilon^j_{1n} \epsilon^2_{2m}) - \epsilon^{\mu\nu} D_\mu X^j \epsilon_{\nu \mu} P^{mn} \epsilon^j_{1n} \epsilon^2_{2m} \]  

(3.12)
In the notation of Section 2 the generators of the gauge transformations $R$ are here $P^{ij}$ and $D^{j}_{\mu i}$. The gauge tensors $T$ and $E$ are $P^{ij, k}$ and $\epsilon_{\nu \rho} P^{mn, j}_{, i}$. The higher order gauge tensors $A$ and $B$ vanish.

We denote the ghost fields again by $C^{i}$. The Noether identities are then

$$\int_{M} \mu \left( (\epsilon^{\mu \nu} D^{\mu}_{\nu} A_{\nu j} + \frac{\partial C(X)}{X^{i}}) P^{k i} + (e^{\mu \nu} D_{\nu} X^{i}) D^{k}_{\mu i} \right) C_{k} = 0. \quad (3.13)$$

Considering the commutator of two gauge transformations leads to (see Eqs. (2.7-2.9))

$$\int_{M} \mu \left( 2 P^{mi, j}_{, P^{nj}} - P^{ij} P^{mn, j}_{, } \right) C_{m} C_{n} = 0 \quad (3.14)$$

$$\int_{M} \mu \left( 2 (P^{jk, l}_{, D^{l}_{\mu j} + P^{mk, i}_{, ij} A_{\mu m} P^{j i}} - D^{m}_{\mu i} P^{k l}_{, m} + (e^{\mu \nu} D_{\nu} X^{i}) \epsilon_{\nu \mu} P^{k l}_{, j i}) C_{l} C_{k} = 0. \right. \quad (3.15)$$

The Jacobi identity is

$$P^{ij, m} P^{mk} C_{i} C_{j} C_{k} = 0. \quad (3.16)$$

We shall later need the first derivative of the Jacobi identity:

$$(P^{ij, mn} P^{mk} + P^{ij, m} P^{mk, n}) C_{i} C_{j} C_{k} = 0, \quad (3.17)$$

as well as the second derivative

$$(P^{ij, mnp} P^{mk} + P^{ij, mn} P^{mk, p} + P^{ij, mp} P^{mk, n} + P^{ij, m} P^{mk, np}) C_{i} C_{j} C_{k} = 0. \quad (3.18)$$

**The Antifields of the Poisson-Sigma Model**

The fields and antifields of the model are

$$\Phi^{A} = \{A^{ij}, X^{i}, C_{i} \} \text{ and } \Phi_{A}^{*} = \{A^{i*}, X_{i}^{*}, C^{i*} \}. \quad (3.19)$$

The extended action is

$$S = \int_{M} \mu \left[ \epsilon^{\mu \nu} (A_{\mu i} \partial_{\nu} X^{i} + P^{ij} (X) A_{\mu i} A_{\nu j}) + C(X) + A^{i*} D^{j}_{\mu i} X_{j} + X_{i}^{*} P^{ij} (X) C_{j} \right. \quad (3.20)$$

$$+ \frac{1}{2} C^{i*} P^{jk, i}(X) C_{j} C_{k} + \frac{1}{4} A^{i*} A^{j*} \epsilon_{\mu \nu} P^{kl}_{, ij} (X) C_{k} C_{l} \right].$$
The classical master equation is

\[
(S, S) = \int_M \mu \left[ (\epsilon^\nu (D_\nu X^m) D^j m + (\epsilon^\nu \partial_\nu A_\nu) + \partial C(X) P^{jm} \right) C_j - (X^*_j P_{ij, m} P^{km} - X^*_j P^{jm} \frac{1}{2} P_{jk, m}) C_j C_k + \epsilon_{\mu\nu}(D_\mu X^m)A^{\nu j \ast} \frac{1}{2} P^{kl, mj} C_k C_l - A^{\mu i \ast} P^{j k, im} A^{\nu j} \frac{1}{2} P^{ln, mj} C_k C_l + \frac{1}{2} C^{\mu i \ast} P^{j k, im} C_{j k} C_k + C^{\mu i \ast} P^{j k, im} C_{j k} C_k C_k \right. \\
\left. \left. + A^{\mu i \ast} A^{\nu j \ast} \epsilon_{\mu\nu}(\frac{1}{4} P^{kl, ijm} C_k C_l P^{mn} C_n + \frac{1}{4} P^{mi, ij} C_i P^{kl, mn} C_k C_l - \frac{1}{2} P^{mn, ijm} C_k C_l P^{kl, mn} C_k C_l) \right] = 0. \right]
\]

(3.21)

Eqs. (3.13)-(3.18) ensure that the extended action (3.20) is a solution of the classical master equation (3.21).

Gauge-fixing

We shall use gauge-fixing conditions of the form \( \chi_i(A, X) \), so that the gauge fermion (2.55) becomes \( \Psi = \bar{C}_i \chi_i(A, X) \). The antifields are then fixed to be

\[
A^{\mu i \ast} = \bar{C}_j \frac{\partial \chi_j(A, X)}{\partial A_{\mu i}}, \\
X^*_i = \bar{C}_j \frac{\partial \chi_j(A, X)}{\partial X^i}, \\
C^*_i = 0, \\
C^{\ast i} = \chi_i(A, X).
\]

(3.22)

The gauge-fixed action is

\[
S\Psi = \int_M \mu \left[ \epsilon^{\mu\nu}(A_{\mu i} \partial_\nu X^i + \rho_{ij}(X)A_{\mu i} A_{\nu j}) + C(X) \right. \\
\left. + \bar{C}_k \frac{\partial \chi_k(A, X)}{\partial A_{\mu i}} D^j m \frac{\partial \chi_k(A, X)}{\partial X^i} P^{jm} C_j + \bar{C}_k \frac{\partial \chi_k(A, X)}{\partial X^i} \epsilon_{\mu\nu} P^{kl, ijm} C_k C_l + \pi^i \chi_i(A, X) \right].
\]

(3.23)

We now consider different gauge conditions:

(i) First, the Landau gauge for the gauge potential \( \chi_i = \partial_\mu A_{\mu i} \), so that the gauge fermion
becomes \( \Psi = \bar{C}^i \partial^\mu A_{\mu i} \). The antifields are fixed to be:

\[
A^{\ast\mu i} = \partial^\mu \bar{C}^i,
X^*_i = C^{\ast i} = 0,
\bar{C}^*_i = \partial^\mu A_{\mu i}.
\]

(3.24)

For this gauge choice the gauge-fixed action is:

\[
S_\Psi = \int_M \mu \left[ \epsilon^{\mu\nu} (A_{\mu i} \partial_\nu X^i + P^{ij} (X) A_{\mu i} A_{\nu j}) + C(X) + \bar{C}^i \partial^\mu D^{ij}_{\mu i} C_j \\
+ \frac{1}{4} (\partial^\mu \bar{C}^i) (\partial^\nu \bar{C}^j) \epsilon_{\mu\nu} P^{kl,ij} (X) C_k C_l - \bar{\pi}^i (\partial^\mu A_{\mu i}) \right].
\]

(3.25)

Translating this action into the notation of Cattaneo and Felder \cite{Cattaneo90} one sees that it is exactly the expression they use to derive the perturbation series.

(ii) Now we consider the temporal gauge \( \chi_i = A_{0i} \). In this case the gauge fermion is given by \( \Psi = \bar{C}^i A_{0i} \). The antifields are fixed to:

\[
A^{\ast 0 i} = \bar{C}^i,
A^{\ast 1 i} = 0,
X^*_i = C^{\ast i} = 0,
\bar{C}^*_i = A_{0 i}.
\]

(3.26)

The gauged-fixed action is:

\[
S_\Psi = \int_M \mu \left[ \epsilon^{\mu\nu} (A_{\mu i} \partial_\nu X^i + P^{ij} (X) A_{\mu i} A_{\nu j}) + C(X) + \bar{C}^i \partial^\mu D^{ij}_{0 i} C_j - \bar{\pi}^i (A_{0i}) \right].
\]

(3.27)

(iii) Finally we consider the Schwinger-Fock gauge \( \chi_i = x^\mu A_{\mu i} \). Then the antifields are fixed to be:

\[
A^{\ast \mu i} = x^\mu \bar{C}^i,
X^*_i = C^{\ast i} = 0,
\bar{C}^*_i = x^\mu A_{\mu i}.
\]

(3.28)

For this gauge choice the gauge-fixed action is:

\[
S_\Psi = \int_M \mu \left[ \epsilon^{\mu\nu} (A_{\mu i} \partial_\nu X^i + P^{ij} (X) A_{\mu i} A_{\nu j}) + C(X) + \bar{C}^i x^\mu D^{ij}_{\mu i} C_j - \bar{\pi}^i (\partial^\mu A_{\mu i}) \right].
\]

(3.29)

Notice that in the non-covariant gauges (ii) and (iii) the action simplifies, in that the term which arose because of the non-closed gauge algebra vanishes.
Gauge fixing in Casimir-Darboux coordinates

Important simplifications occur when we write the action in Casimir-Darboux coordinates $X^i \rightarrow \{X^I, X^\alpha\}$, so we go through the gauge-fixing procedure again for these coordinates. The extended action is

$$ S = \int_M \mu \left[ \epsilon^{\mu\nu} (A_{\mu I} \partial_\nu X^I + A_{\mu\alpha} \partial_\nu X^\alpha + P^{\alpha\beta}(X^I) A_{\mu\alpha} A_{\nu\beta}) + C(X) + A^{\mu I*} \partial_\mu C_I 
+ A^{\mu\alpha*} \partial_\mu C_\alpha + X^*_\alpha P^{\beta\alpha}(X^I) C_\beta \right]. \quad (3.30) $$

This extended action still possesses gauge invariances, so one has to introduce a nonminimal sector. The non-minimal action is

$$ S^{\text{non}} = \int_M \mu \left[ \epsilon^{\mu\nu} (A_{\mu I} \partial_\nu X^I + A_{\mu\alpha} \partial_\nu X^\alpha + P^{\alpha\beta}(X^I) A_{\mu\alpha} A_{\nu\beta}) + C(X) + A^{\mu I*} \partial_\mu C_I 
+ A^{\mu\alpha*} \partial_\mu C_\alpha + X^*_\alpha P^{\beta\alpha}(X^I) C_\beta - \bar{\pi}^I \bar{C}_I - \bar{\pi}^\alpha \bar{C}_\alpha \right]. \quad (3.31) $$

In these coordinates the gauge freedom of the maps $X^i : M \rightarrow N$ is reduced to the freedom of the maps $X^\alpha : M \rightarrow S$, where $S$ is a symplectic leaf of the Poisson manifold $N$. The gauge transformations $\delta_\varepsilon X^i = P^{ij} \varepsilon_j$ reduce to

$$ \delta_\varepsilon X^\alpha = P^{\alpha\beta} \varepsilon_\beta, \quad \delta_\varepsilon X^I = 0. \quad (3.32) $$

After gauge fixing we need to consider only the homotopy classes $[X^\alpha]$.

It is now possible to decompose the gauge condition into a part depending only on $A_{\mu I}$ and another part depending only on $X^\alpha$, so that the gauge-fixing of the gauge fields is implemented by gauge conditions of the form $\chi_I(A_I)$ and $\chi_\alpha(X^\alpha)$. The gauge fermion may be written as

$$ \Psi = \int_M \mu \left[ \bar{C}_I \chi_I(A_I) + \bar{C}_\alpha \chi_\alpha(X^\alpha) \right]. \quad (3.33) $$

The gauge conditions as expressed through the gauge fermion are

$$ A^{\mu I*} = \bar{C}_J \frac{\partial \chi_J(A_I)}{\partial A_{\mu I}}, $$

$$ A^{\mu\alpha*} = 0, $$

$$ X^*_\alpha = \bar{C}_\beta \frac{\partial \chi_\beta(X^\alpha)}{\partial X^\alpha}, $$

$$ C^*_I = 0, $$

$$ \bar{C}_I = \chi_I(A_I), $$

$$ \bar{C}_\alpha = \chi_\alpha(X^\alpha). \quad (3.34) $$
The gauge-fixed action in Casimir-Darboux coordinates takes the form
\[
S_\psi = \int_M \mu \left[ \epsilon^{\mu\nu}(A_{\mu I}\partial_{\nu}X^I + A_{\mu \alpha}\partial_{\nu}X^{\alpha} + P^{\alpha\beta}A_{\mu \alpha}A_{\nu \beta}) + C(X^I) \right. \\
+ \tilde{C}_I \left. \partial_{\lambda I}(A_J) \partial_{\mu}C_I + \tilde{C}_I^{\alpha} \partial_{\lambda I}(X^{\alpha}) \partial\chi^{I} - \tilde{C}_I^{\alpha} \partial_{\lambda I}(X^{\alpha}) \right].
\]
(3.35)

### 3.2 The Path Integral for the Poisson-Sigma Model

Using Eq. (2.65), the path integral for the Poisson-Sigma model in Casimir-Darboux coordinates is
\[
Z = \int_{\Sigma^{(3)}} D\pi D\bar{\pi} D\chi D\bar{\chi} \det \left( \frac{\partial}{\partial A_{\mu I}} \chi^{I} \right)_{\Omega_0(M)} \det \left( \frac{\partial}{\partial X^{\alpha}} \chi^{I} \right)_{\Omega_0(M)}
\]
\[
\exp \left( -\frac{1}{\hbar} \int_M \left[ \epsilon^{\mu\nu}(A_{\mu I}\partial_{\nu}X^I + A_{\mu \alpha}\partial_{\nu}X^{\alpha} + P^{\alpha\beta}A_{\mu \alpha}A_{\nu \beta}) + C(X^I) \right] \right),
\]
(3.36)

where we have performed the usual Wick rotation $t = i\tau$, so that the exponent of the path integral is now real. When the model is integrable we expect to be able to carry out the functional integrations successively, in order to obtain a closed expression for the path integral. We shall indeed be able to achieve this goal for the special case described in Section (4).

Integrating over the ghost and antighost fields yields the Faddeev-Popov determinants:
\[
Z = \int_{\Sigma^{(3)}} D\pi D\bar{\pi} D\chi D\bar{\chi} \det \left( \frac{\partial}{\partial A_{\mu I}} \chi^{I} \right)_{\Omega_0(M)} \det \left( \frac{\partial}{\partial X^{\alpha}} \chi^{I} \right)_{\Omega_0(M)}
\]
\[
\exp \left( -\frac{1}{\hbar} \int_M \left[ \epsilon^{\mu\nu}(A_{\mu I}\partial_{\nu}X^I + A_{\mu \alpha}\partial_{\nu}X^{\alpha} + P^{\alpha\beta}A_{\mu \alpha}A_{\nu \beta}) + C(X^I) - \pi^{I} \chi^{I}(A) \right] \right),
\]
(3.37)

where the subscripts $\Omega^k(M)$ indicate that the determinant results from an integration over k-forms on $M$. The integrations over $\pi_I$ and $\pi_\alpha$ yield $\delta$-functions which implement the gauge conditions.

\[
Z = \int_{\Sigma^{(3)}} D\pi D\bar{\pi} D\chi D\bar{\chi} \det \left( \frac{\partial}{\partial A_{\mu I}} \chi^{I} \right)_{\Omega_0(M)} \det \left( \frac{\partial}{\partial X^{\alpha}} \chi^{I} \right)_{\Omega_0(M)}
\]
\[
\exp \left( -\frac{1}{\hbar} \int_M \left[ \epsilon^{\mu\nu}(A_{\mu I}\partial_{\nu}X^I + A_{\mu \alpha}\partial_{\nu}X^{\alpha} + P^{\alpha\beta}A_{\mu \alpha}A_{\nu \beta}) + C(X^I) \right] \right),
\]
(3.38)

where from now on the integrations extend only over the degrees of freedom which respect the gauge-fixing conditions. The integration over $A_{\mu \alpha}$ is gaussian, it yields
\[
Z = \int_{\Sigma^{(3)}} D\pi D\bar{\pi} D\chi D\bar{\chi} \det \left( \frac{\partial}{\partial A_{\mu I}} \chi^{I} \right)_{\Omega_0(M)} \det \left( \frac{\partial}{\partial X^{\alpha}} \chi^{I} \right)_{\Omega_0(M)}
\]
\[
\det^{-1/2} \left( P^{\alpha\beta}(X^I) \right)_{\Omega_0(M)} \exp \left( -\frac{1}{\hbar} \int_M \left[ \epsilon^{\mu\nu}(A_{\mu I}\partial_{\nu}X^I + \Omega_{\alpha\beta}\partial_{\mu}X^{\alpha}\partial_{\nu}X^{\beta}) + C(X^I) \right] \right).
\]
(3.39)
Besides the term in the exponent the only dependence on \( A_{\mu I} \) is in the relevant Faddeev-Popov determinant. If we choose a gauge condition linear in \( A_{\mu I} \) this determinant becomes independent of the fields, and can be absorbed into a normalization factor. The integration over \( A_{\mu I} \) then yields a \( \delta \)-function for \( \partial_\nu X^I \). When this \( \delta \)-function is implemented the fields \( X^I \) become independent of the coordinates \( \{ x^\mu \} \) on \( M \). Hence the Casimir functions are constants. The constant modes of the Casimir coordinates \( X^I_0 \) count the symplectic leaves.

The path integral is now

\[
Z = \int_{\Sigma_\Psi} dX^I_0 \mathcal{D}X^\alpha \det \left( \frac{\partial \chi^\alpha(X^\alpha)}{\partial X^\gamma} P^{\gamma\beta}(X^I_0) \right)_{\Omega^\alpha(M)} \det^{-1/2} \left( P^{\alpha\beta}(X^I_0) \right)_{\Omega^1(M)} \exp \left( -\frac{1}{\hbar} \int_M \mu \Omega_{\alpha\beta} dX^\alpha dX^\beta \right) \exp \left( -\frac{1}{\hbar} \int_M \mu C(X^I_0) \right). \tag{3.40}
\]

The gauge-fixing of the fields \( X^\alpha \) reduces the integral \( \mathcal{D}X^\alpha \) to a sum over the homotopy classes:

\[
Z = \int_{\Sigma_\Psi} dX^I_0 \sum_{[M \to S(X^I_0)]} \det \left( \frac{\partial \chi^\alpha(X^\alpha)}{\partial X^\gamma} P^{\gamma\beta}(X^I_0) \right)_{\Omega^\alpha(M)} \det^{-1/2} \left( P^{\alpha\beta}(X^I_0) \right)_{\Omega^1(M)} \exp \left( -\frac{1}{\hbar} \int_M \mu \Omega_{\alpha\beta} dX^\alpha dX^\beta \right) \exp \left( -\frac{1}{\hbar} \int_M \mu C(X^I_0) \right). \tag{3.41}
\]

Since the \( C(X^I_0) \) are independent of the coordinates on \( M \) the last exponent simplifies to

\[
\exp \left( -\frac{1}{\hbar} \int_M \mu C(X^I_0) \right) = \exp \left( -\frac{1}{\hbar} A_M C(X^I_0) \right), \tag{3.42}
\]

where \( A_M \) is the surface area of \( M \). The form of the path integral then becomes

\[
Z = \int_{\Sigma_\Psi} dX^I_0 \sum_{[M \to S(X^I_0)]} \det \left( \frac{\partial \chi^\alpha(X^\alpha)}{\partial X^\gamma} P^{\gamma\beta}(X^I_0) \right)_{\Omega^\alpha(M)} \det^{-1/2} \left( P^{\alpha\beta}(X^I_0) \right)_{\Omega^1(M)} \exp \left( -\frac{1}{\hbar} \int_M \mu \Omega_{\alpha\beta} dX^\alpha dX^\beta \right) \exp \left( -\frac{1}{\hbar} A_M C(X^I_0) \right). \tag{3.43}
\]

Note that we have now arrived at an almost closed expression for the partition function for the Poisson-Sigma model, i.e. all the functional integrations have been performed.

### 4 SU(2) Yang-Mills Theory

To make further progress we consider the special case where the Poisson manifold \( N = \mathbb{R}^3 \), and the Poisson structure is linear: \( P^{ij} = \epsilon^{ij}_k X^k \). The quadratic Casimir operator is \( C(X) = \sum_i X^i X^i \). If we use this Casimir operator in the classical action (3.7) we may integrate out the \( X^i \) fields to obtain the action for the two-dimensional Yang-Mills theory. If we omit the Casimir term in the action the same procedure yields the topological BF-theory.
Because of the Jacobi identity the structure constants $c^{ij}_k$ define a Lie Algebra structure on the dual space $G$ of $N$. For this reason the linear Poisson structure is also called a Lie-Poisson structure on $N$. We are here interested in the case where the Lie algebra is three-dimensional, and the structure constants are those of the group $SU(2)$. The Poisson structure is degenerate and the symplectic leaves are two-dimensional spheres characterized, in the Casimir-Darboux coordinates, by their radius $X^I_0$. Weinstein [18] has shown that the symplectic leaves of a linear Poisson structure are the co-adjoint orbits of the corresponding compact, connected Lie group $G$ of $G$. By a theorem of Kirillov these orbits can in turn be identified with the irreducible unitary representations of $G$ [19].

These considerations can be used to further reduce the expression for the path integral. Consider the homotopy classes of the maps $X^\alpha : M \rightarrow S(X^I_0)$. The Hopf theorem tells us that the mappings $f, g : M \rightarrow S(X^I_0)$ are homotopic if and only if the degree of the mapping $f$ is the same as the degree of $g$. This means that the sum over the homotopy classes of the maps $[X^\alpha]$ can be expressed as a sum over the degrees $n = \text{deg}[X^\alpha]$:  

$$
\sum_{[X^\alpha]} \rightarrow \sum_{n \in \mathbb{Z}}
$$

For a map $f : X \rightarrow Y$, where $X$ and $Y$ are $k$-dimensional oriented manifolds and $\omega$ a $k$-form on $Y$, the degree of the mapping is given by  

$$
\int_X f^* \omega = \text{deg}[f] \int_Y \omega.
$$

Using this formula yields:  

$$
\int_M \mu \Omega_{\alpha\beta} dX^\alpha dX^\beta = n \int_S \Omega_S(X^I_0),
$$

where $\Omega_S(X^I_0)$ is the symplectic form on the corresponding leaf $S$. This gives for the partition function of Eq. (3.43)  

$$
Z = \int_{\Sigma_\Psi} dX^I_0 \sum_{n \in \mathbb{Z}} \det \left( \frac{\partial X^\alpha(X)}{\partial X^\gamma} P_{\gamma\beta}(X^I_0) \right)_{\Omega^0(M)} \det^{-1/2} \left( P_{\alpha\beta}(X^I_0) \right)_{\Omega^1(M)}
\times \exp\{ - n \int_S \Omega_S(X^I_0) \} \exp\{ - \frac{1}{\hbar} A_M C(X^I_0) \}.
$$

The sum over $n$ yields a periodic $\delta$-function:  

$$
Z = \int_{\Sigma_\Psi} dX^I_0 \sum_{n \in \mathbb{Z}} \det \left( \frac{\partial X^\alpha(X)}{\partial X^\gamma} P_{\gamma\beta}(X^I_0) \right)_{\Omega^0(M)} \det^{-1/2} \left( P_{\alpha\beta}(X^I_0) \right)_{\Omega^1(M)}
\times \delta \left( \int_S \Omega_S(X^I_0) - n \right) \exp\{ - \frac{1}{\hbar} A_M C(X^I_0) \}.
$$

The $\delta$-function says that the symplectic leaves must be integral. By the identification of the leaves with the co-adjoint orbits, the orbits must also be integral. The fact that the orbits are integral reduces the number of the co-adjoint orbits to a countable set, which we label by $O(\Omega)$. 


We now consider the two determinants in the path integral. We choose the “unitary gauge” $\chi_\alpha(X^\alpha) = X^\alpha$, so that $\partial_\chi_\alpha(X)/\partial X^\gamma = \delta^\alpha_\gamma$, and the two determinants have the same form. The restriction of the scalar fields to the Casimir-Darboux coordinates $X^I$ corresponds to the restriction of the scalar fields to the invariant Cartan subalgebra considered by Blau and Thompson in [20], so we may adopt their argumentation concerning the powers to which the determinants occur for a manifold with Euler characteristic $\chi(M)$. The result is a factor

$$\det(P^{\alpha\beta}(X^I_0))^{\chi(M)}.$$  

The determinant of a mapping equals the volume of the image of that mapping, hence the determinant $\det(P^{\alpha\beta}(X^I_0))$ corresponds to the symplectic volume of the leaf, which we denote by $\text{Vol}(\Omega_S(X^I_0))$. The path integral then takes the form:

$$Z = \int_{\Sigma} \sum_{n \in \mathbb{Z}} \text{Vol}(\Omega_S(X^I_0))^{\chi(M)} \delta \left( \int_S \Omega_S(X^I_0) - n \right) \exp\left(-\frac{1}{\hbar} A_M C(X^I_0)\right).$$  

Implementing the $\delta$-function by integrating over $X^I_0$ the sum over the mapping degrees becomes a sum over the set $\mathcal{O}(\Omega)$ of the integral orbits:

$$Z = \sum_{\mathcal{O}(\Omega)} \text{Vol}(\Omega_S(X^I_0))^{\chi(M)} \exp\left(-\frac{1}{\hbar} A_M C(X^I_0)\right).$$  

Because of the identification of the integral orbits with the irreducible unitary representations this leads to a sum over the representations. A special form of the character formula of Kirillov [21] says that the symplectic volume of the co-adjoint orbit equals the dimension of the corresponding irreducible unitary representation. So the final form of the partition function is

$$Z = \sum_{\lambda} d(\lambda)^{\chi(M)} \exp\left(-\frac{1}{\hbar} A_M C(\lambda)\right),$$  

where $\lambda$ denotes the irreducible unitary representation corresponding to the co-adjoint orbit, and $d(\lambda)$ is the dimension of this representation. This is exactly the partition function for the two-dimensional Yang-Mills theory [13]. When we omit the Casimir term in the action we get just a sum over the dimensions of the representations, which is the correct result for the BF-theory, see e.g. [20].

5 Conclusions and Outlook

The Poisson-Sigma model is more than a unified framework for different topological and semi-topological field theories. Due to its reformulation of the degrees of freedom of the theories in terms of the coordinates of a Poisson manifold it achieves a description in terms of the natural variables of general dynamical systems. Gauge theories, which are characterized by singular Lagrangians, cannot in general be described in terms of symplectic manifolds; the foliation which is characteristic for Poisson manifolds is necessary.

The advantages of such a description of these field theories is at least twofold. First, it allows one to discuss the quantization of the classical field theory by a direct application
of the techniques of deformation quantization. Second, for integrable systems the general dynamical concepts of integrability may be utilized in order to reduce the partition function of the theory.

To some extent the above remarks are illustrated in the present work. The use of Casimir-Darboux coordinates allowed essential simplifications. In Section (4) we achieved a full reduction of the path integral in a special case through the use of concepts and theorems involving the symmetries of general dynamical systems.

We believe that further research will uncover ways of utilizing these structures even more thoroughly. The techniques used here should in principle be applicable in more general situations than the particular case we considered in Section (4). We also hope to be able to treat more general manifolds. This would allow in particular a more direct comparison with canonical quantization procedures. Finally, as already discussed in the Introduction, an understanding of the mechanisms active in the general case should allow the resolution of problems encountered in particular field theories.

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