Third-order chromatic dispersion stabilizes Kerr frequency combs

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Using numerical simulations of an extended Lugiato-Lefever equation, we analyze the stability and nonlinear dynamics of Kerr frequency combs generated in microresonators and fiber resonators taking into account third-order dispersion effects. We show that cavity solitons underlying Kerr frequency combs, normally sensitive to oscillatory and chaotic instabilities, are stabilized in a wide range of parameter space by third-order dispersion. Moreover, we demonstrate how the snaking structure organizing compound states of multiple cavity solitons is qualitatively changed by third-order dispersion, promoting an increased stability of Kerr combs underlined by a single cavity soliton.

Optical frequency combs permit to measure light frequencies and time intervals with exquisite accuracy, leading to numerous key applications. An octave of bandwidth is however typically required for self-referencing \cite{1,2}. Kerr microresonators support on-chip generation of such broadband frequency combs, with the potential to lead to very small footprints \cite{3}. In fact, octave spanning "Kerr combs" have been demonstrated in both silica microtoroids \cite{4} and silicon nitride microresonators \cite{5}. With such large bandwidths, it is important to take third-order chromatic dispersion (TOD) into account, which leads to asymmetric frequency combs. Comb generation in these conditions can be modeled using a simple generalized mean-field Lugiato-Lefever equation (LLE), as recently shown in \cite{6}. That study and others \cite{7} have highlighted a link between Kerr combs and temporal cavity solitons (CSs) \cite{3}. But CSs are known to exhibit many instabilities \cite{9}. Although the instabilities of Kerr frequency combs have been recently intensely studied \cite{10–13} and some effects of fourth-order dispersion have been uncovered in the LLE \cite{14,15}, the influence of TOD on the dynamics of CSs, and by association of Kerr combs, has not been investigated in detail. Addressing this issue is the goal of the present Letter.

Here we show that TOD in microresonators can lead to suppression of dynamical regimes such as oscillations and chaos, effectively stabilizing Kerr combs (a fact already hinted in \cite{16}). Furthermore, we discuss the underlying dynamical mechanism behind this stabilization. We relate the dynamics of Kerr combs in the presence of TOD to the snaking structure organizing single- and multi-peak CSs. Not only are single CSs stabilized when introducing TOD, but multi-peak solutions are generally unstable and are only stable for large amounts of TOD. Together this leads to an increased stability of Kerr combs with TOD, based on a stable underlying single CS.

The mean-field LLE that describes comb generation in microresonators has been described by many \cite{6,7,11,12,20,21}. Using the normalization of \cite{8}, and including dispersion up to third-order, that equation reads,

\[
\partial_t u = -(1 + i\theta)u + i|u|^2u + u_0 + i\beta_2 u + d_3 \partial^3_t u. \tag{1}
\]

Here \(t\) is the slow-time describing the evolution of the intra-cavity field \(u(t, \tau)\) at the scale of the cavity photon lifetime, while \(\tau\) is a fast-time that describes the temporal structure of that field along the resonator roundtrip. The first term on the right-hand side describes cavity losses (the system is dissipative by nature); \(\theta\) measures the cavity detuning between the frequency of the input pump and the nearest cavity resonance; the cubic term represents the Kerr nonlinearity, with the sign set so that it corresponds to the self-focusing case; \(u_0\) is the amplitude of the homogeneous (continuous-wave) driving field or pump; and the fast-time derivatives model chromatic dispersion (dispersion is assumed anomalous at the pump frequency), with \(d_3\) the relative strength of the TOD. \(d_3\) can be calculated from the physical parameters of the system, \(d_3 = (1/3)(2\alpha/L)^{1/2}\beta_3/|\beta_2|^{3/2}\) \cite{22}, where \(\alpha\) is half the percentage of power lost per round-trip (the cavity finesse \(\mathcal{F} = \pi/\alpha\)), \(L\) is the cavity length, and \(\beta_2\) (\(\beta_3\)) is the second (third) order dispersion coefficient. Table \ref{tab:1} shows typical values of these parameters and corresponding relative TOD strength \(d_3\) for three different physical systems, namely crystalline magnesium fluoride (MgF\(_2\)) \cite{17} and silicon nitride (Si\(_3\)N\(_4\)) \cite{5,6} (see also \cite{23}) microresonators, as well as cavities made of a combination of standard and dispersion-shifted optical fibers \cite{18,19} and in which temporal CSs in the presence of TOD have recently been observed \cite{19}. The values

\begin{table*}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\rowcolor{Gray} Parameter & MgF\(_2\) & Si\(_3\)N\(_4\) & Fiber \cite{18,19} \\
\hline
\(\alpha\) & \(4.31 \times 10^{-3}\) & \(8.89 \times 10^{-3}\) & 0.16 \\
\hline
\(L\) & 6.2 mm & 628 \(\mu\)m & 105 m \\
\hline
\(|\beta_2|\) (ps\(^2\)/km) & 5.9 & 48.7 & 0.1–0.54 \\
\hline
\(|\beta_3|\) (ps\(^3\)/km) & –0.35 & –0.14 & 0.12 \\
\hline
\(d_3\) & –0.045 & –0.034 & 0.18–2.2 \\
\hline
\end{tabular}
\caption{Physical parameters and normalized TOD coefficient \(d_3\) for three different optical systems. \(\beta_2 < 0\) in all cases.}
\end{table*}
the evolution of the temporal intensity profile of the intracav-
ized in the captions. The pseudocolor plots at the top show
is sufficiently large the dynamical instabilities are completely
the comb spectrum, which is perfectly steady. The TOD breaks
from the pump due to spectral recoil from the emission of
dispersive waves \[6, 16, 19\], which are clearly seen both in
Fig. 1. Evolution of (a) the temporal intensity profile of an
oscillating CS over successive roundtrips (top) and its asso-
ciated comb spectrum in dB (bottom) in the absence of TOD
\( (d_3 = 0) \). (b) With \( d_3 = 0.15 \), the system is stable. The profiles
at time \( t = 5 \) are shown on top of each graph. \( \theta = 6.1, u_0 = 4 \).

An example of oscillatory and chaotic behaviors of an iso-
culated CS in the absence of TOD (second-order dispersion
only, \( d_3 = 0 \)) is shown in Figs. 1(a) and 1(a), respectively. Only
the pump amplitude differs in these two simulations as indi-
cated in the captions. The pseudocolor plots at the top show
the evolution of the temporal intensity profile of the intracav-
ity field while the bottom ones are the corresponding spectra.
Figures 1(b) and 2(b) reveal that when the magnitude of TOD
is sufficiently large the dynamical instabilities are completely
suppressed: the CS is stable, albeit in a moving reference
frame. Note that this motion has no practical effect on the
combspectrum, which is perfectly steady. The TOD breaks
the reflection reversibility \( \tau \rightarrow -\tau \), which leads to asymme-
tries in the temporal and spectral profiles \[16, 18, 19, 24\]. This
asymmetry is also responsible from the observed constant-
velocity spatial drift: the CS carrier frequency is shifted from
the pump due to spectral recoil from the emission of
dispersive waves \[6, 16, 19\], which are clearly seen both in
Figs. 1(b) and 2(b). Accordingly, the group-velocity of the CS
differs slightly from that of the pump. In fiber cavities, a sim-
ilar change in group-velocity occurs through acoustic effects
and leads to long-range CS interactions \[25\].

In order to verify whether the stabilization of the CS and
the corresponding comb is a general feature in the presence of
TOD, we analyzed the stability of CSs in the whole param-
eter space \( (\theta, u_0) \) for various values of the TOD. The result
of this analysis is shown in Fig. 3. To interpret this figure,
let us first recall that the homogeneous steady state (HSS) \( u_s \)
of Eq. (1) is given by \( I_e[1 + (\theta - \theta_c)^2] = I_0 \), where \( I_e = |u_s|^2 \)
and \( I_0 = |u_0|^2 \). For \( \theta < \sqrt{3} \), only one HSS exists, hence the
system is monostable. For \( \theta > \sqrt{3} \) three HSS states appear,
one of which is a saddle point (unstable), hence this regime
is referred as bistable. These homogeneous solutions are con-
ected through saddle-node (SN) bifurcations shown as green
dotted lines in Fig. 3. On this figure we have also indicated
the cusp point \( C \) at \( \theta = \sqrt{3} \). On top of HSSs, it is well known
that the LLE admits extended periodic pattern solutions that
appear through modulational instability (MI) \[20, 26\]. When
the detuning is sufficiently large \( \theta > 41/30 \) in the absence
of TOD, these patterns emerge subcritically from the HSSs,
hence the patterned solution and a stable HSS coexist. It is
this generalized bistability that makes possible the existence
of multiple stationary temporal CSs in the LLE. They coexist
with the patterned solutions within a so-called snaking or pin-
ing region delimited by two SN bifurcation points \[27, 28\].

These points, referred to as \( SN_1 \) and \( SN_2 \), are plotted using
thick blue solid lines in Figs. 3(a)–(c) for increasing values of
TOD. In the absence of TOD, a detailed overview of the
regions of multistability between the HSSs, CSs, compound
CS states, and extended patterns can be found in \[9, 11, 12\]. In
these works, the organization of various dynamical regimes,
including oscillations and chaos, was also discussed.

From Fig. 3 it is clear that while the region of existence of
the HSSs is independent of TOD, the snaking region in which
CSs can be found (between the blue lines) shrinks with in-
creasing values of the TOD. To highlight this point, we plot in
Fig. 3(d) the width of the snaking region \( |u_0(SN_2) - u_0(SN_1)| \)
versus the TOD strength $d_3$ for a fixed detuning $\theta = 6.1$. Here it can be seen that the shrinkage, while initially rapid, somewhat saturates at higher $d_3$ such that a region admitting CS solutions can be found independent of the TOD strength $d_3$.

Figures 3(a)–(c) also illustrate the dependence of various regions of instabilities of a single CS as a function of the TOD strength $d_3$. CSs are stable in region I while they are found to exhibit a time-oscillatory behavior [as was illustrated in Fig. 2(a)] in region II (light-gray colored). These oscillatory solutions emerge through a Hopf bifurcation H (thin red line). In the absence of TOD, this Hopf bifurcation has theoretically been demonstrated to originate in a Gavrilov-Guckenheimer codimension-2 point [12] and has been experimentally observed using fiber resonators [9]. Above region II, for increasing values of pump power and detuning, we find that the temporal evolution of the CSs lead to spatio-temporal chaos [as was illustrated in Fig. 2(a)]. The part of parameter space where this chaotic behavior is located is referred to as region III. Figs 3(a)–(c) demonstrate that both the oscillatory (II) and chaotic (III) regions of instabilities shrink and shift to higher values of the detuning $\theta$, confirming that the stabilization of CSs and Kerr combs in the presence of TOD, which was exemplified in Figs 2 and 2 is a general feature.

The dynamical regimes discussed above only concern Kerr combs underlined by a single CS. However, in the absence of TOD, multistability between many different stationary solutions is known to exist [12]. These solutions consist of multiple CSs and can be understood as bound states of single CSs. We therefore proceed to studying the effect of TOD on the stability and bifurcation structure of multi-peak solutions. Fig. 4 shows the typical bifurcation structure of CSs plotted in terms of their energy, for $d_3 = 0$ (solid lines) and $d_3 > 0$ (dashed lines). Energy is calculated with the homogeneous background $u_0$, removed, i.e., using the norm $|u - u_0|^2 = \int |u(\tau) - u_0|^2 d\tau$. Blue (red) lines are stable (unstable) solutions, respectively. Detuning is fixed at $\theta = 1.5$. Without TOD, $d_3 = 0$, two branches of CSs bifurcate from the HSS together with the extended pattern at the MI point (zero norm point at the bottom right of the Figure). These two branches are related to solutions with, respectively, an odd and even number of peaks. Initially these solutions correspond to small amplitude unstable states. Following these branches “snaking” upwards (increasing the norm), these states grow in amplitude and energy, and successively gain and lose stability through SN bifurcations. At the SN points, extra peaks are also added symmetrically on both sides of the temporal structures resulting in new bound states of CSs of higher norm (see insets in Fig. 4). The even and odd state branches are connected through a branch corresponding to different kind of structures called rung states [29]. The back and forward oscillation of the branches is referred to as homoclinic snaking or a snakes-and-ladders structure [50]. This snaking or pinning region is defined by the asymptotic location of the SNs higher up the snaking structure (SN$_{1,2}$). The dependence of the pinning region on the various system parameters was shown in Fig. 5. However, as revealed by Fig. 4, TOD not only changes the size of the pinning region, but also alters the whole snaking structure of CSs.

As the $\tau$-reversibility symmetry is broken by TOD, the bifurcations responsible for the rung states become imperfect and the snaking structure breaks up. This occurs in two different fashions depending on the detuning $\theta$. For low values of the detuning, as exemplified in Fig. 4, the break-up leads to a stack of isolas. Examples of temporal intensity profiles associated to such isolas for $d_3 = 0.05$ are shown in inset. The creation of isolas is well known in conservative systems harboring localized structures when breaking reversibility [29]. For a fixed value of the detuning, these isolas shrink with increasing values of $d_3$, until they eventually disappear when the cusp of CSs (the point C where SN$_1$ and SN$_2$ meet in Fig. 3) moves beyond the actual value of the detuning. At this point the system no longer admits CS solutions.

For higher values of the detuning $\theta$, which are more relevant to practical Kerr comb generation, the situation is different. This case is illustrated in Fig. 6 that shows the snaking structure for a detuning $\theta = 6.1$ and increasing strength of the TOD. In this regime all the multi-peak CSs organized in the snaking structure are unstable in the absence of TOD and for small values of TOD. CS branches corresponding to even and odd numbers of peaks merge in a type of “mixed snaking.” A similar transition between isolas and mixed snaking has recently been studied in detail in the context of the Swift-Hohenberg equation [31]. Figs 5(b)–(d) show that TOD increasingly stabilizes the multiple peak solutions, starting with the one peak branch (b), and then gradually stabilizing the two-peak one (c), three-peak one (d), etc. This stabilization process seems to involve multiple Hopf bifurcations as most clearly seen in Fig. 5(c). Examples of typical solutions are shown in inset of Fig. 5(d), showing an increased amplitude of the oscillatory tails. For TOD values corresponding to microresonators (see Table 1), this stabilization process may however not be sufficiently strong. The detuning is typically ramped up in experiments (i.e., going from Fig. 4 to Fig. 5), and in that process the single-peak CS solution may typically become the preferred remaining stable solution. The results presented here therefore could help explain the emergence of
a single CS observed both experimentally and theoretically in recent works exploring the route to stable Kerr frequency combs \cite{17,32}. We finally remark that other solutions (not shown here) such as multiple displaced single CSs connected via their oscillatory tails can also exist and we aim to investigate their bifurcation structure and stability in future work.

In summary we have shown that the stability, dynamics, and bifurcation structure of CSs and Kerr combs is largely modified in the presence of TOD. TOD tends to suppress dynamical instabilities of the single CS such as oscillations and chaos. Moreover the so-called snaking structure, organizing the single and multiple CS solutions, is altered by TOD. Our analysis has revealed that despite multi-peak solutions can be stabilized by TOD, such stabilization requires an increasing amount of TOD as the number of peaks increases. Altogether, by promoting their increased stability, these TOD effects thus especially lead to Kerr combs underlined by a single CS.

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