Moving vortex in relativistic irrotational perfect fluid or superfluid

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Abstract

Irrotational relativistic vortex configurations in uniform subsonic motion with respect to a surrounding perfect fluid are analysed for the purpose of application to superfluid layers in neutron stars. Asymptotic solutions are found by asymptotically expanding the flow equation at large distances from the vortex core and then by solving it order by order. The asymptotic effective tension and energy density that are needed for an averaged macroscopic description are thus obtained as functions of the vortex velocity, the vortex circulation, the asymptotic chemical potential and of parameters depending on the equation of state.

I. INTRODUCTION

The aim of this work is to obtain large scale average values of physically important quantities such as energy and tension for stationary longitudinally invariant “vortex” solutions for a relativistic irrotational perfect fluid in an asymptotically uniform background in a flat spacetime. The present analysis is restricted to the case in which only a single vortex is present. This analysis constitutes an essential first step towards the large scale treatment of cases in which, as discussed in the concluding section, there is an extended array of vortices.

The main application we have in mind is the interior layers of a rotating neutron star, in which the bulk of the matter can be represented to a good approximation as a relativistic superfluid at zero temperature, whose local behaviour on a microscopic scale is that of an irrotational perfect fluid. This means that the vorticity resulting from the stellar rotation will be concentrated in discrete vortex cores which may be subject to external drag forces that cause them to move relative to the ambient superfluid. In order to evaluate the effect of such relative movement, the present study generalises results of an earlier investigation that was restricted to the axisymmetric case in which the vortex is at rest with respect to the uniform background, so that an exact analytic description was possible even for the inner regions of the vortex. In that previous work, the fluid under investigation was a
superfluid at non zero temperature, which can be described by the relativistic extension of Landau’s two fluid model. Thanks to the numerous symmetries of the configuration, namely invariance under time translation, invariance under longitudinal translation (along the axis of the vortex) and axisymmetry, it was possible to obtain exact analytical results for a “cool” superfluid (Bose condensate plus a perfect gas of phonons). Here, we still assume invariance under time translation and longitudinal translation but axisymmetry is lost because of the motion of the vortex with respect to the surrounding fluid.

This loss of axisymmetry makes it difficult to obtain exact analytic results except in the extreme limit of a “stiff” fluid – in which the speed of sound is equal to that of light – which is relevant for various cosmological contexts, notably the case in which the superfluid is formed as an axionic field condensate whose vortices constitute what are known as “global” cosmic strings. The present work is intended for the rather different astrophysical context of neutron star interiors – in which the relevant sound speeds are expected to be considerably (albeit not incomparably) smaller than the speed of light. In these circumstances it is not easy to provide an exact analytic treatment of the inner regions of a moving vortex. However the present work shows that it is possible to provide a very satisfactory description of the outer regions by an asymptotic expansion. This description is quite sufficient for the purpose of calculating the large scale averages (of energy, tension et cetera) that are needed in practice for the usual astrophysical applications.

It will be convenient to work with cylindrical coordinates with respect to which the spacetime metric $g_{\mu\nu} (\rho, \sigma = 0, 1, 2, 3)$ is given by

$$g_{\rho\sigma}d\rho^\rho d\rho^\sigma = -c^2 dt^2 + d\ell^2 + dr^2 + r^2 d\theta^2 ,$$

where $c$ is the speed of light, so that the postulated stationarity and longitudinal invariance will be expressible as the condition that the relevant physical quantities should be independent of $t$ and $\ell$. It is to be remarked that the neglect of general relativistic curvature effects in this work is justified in so far as most neutron star (and even cosmological) applications are concerned by the consideration that the gravitational curvature radii involved will be extremely large compared with the lengthscales – of which the most important are the inter-vortex separation distances – that are relevant for the local effects considered here. It will be supposed that the latter distances are themselves very large compared with the lengthscales characterising the central vortex core, whose details may consequently be neglected without significant loss of accuracy. (Neglect of the core would not of course be justifiable if we were concerned with cosmic strings of “local” as opposed to “global” type, nor of the analogous phenomena of the quantised magnetic flux tubes in neutron star matter.)

The present analysis will not allow for thermal effects, whose treatment would require the use of the generalised Landau type two constituent superfluid model, but whose consequences are not very important in typical neutron star applications for which the zero temperature limit approximation is sufficient. Our attention will be restricted here to superfluid models of the simplest “barotropic” kind for which a complete set of independent physical variables is constituted just by the components of the 4-momentum covector $\mu_\sigma$ whose dynamics is governed by the irrotationality condition

$$\mu_\sigma = \nabla_\sigma \varphi ,$$

where $\varphi$ is a gauge dependent scalar – interpretable as a phase angle in the underlying bosonic condensate – with period $2\pi$ in units such that the Dirac Plank constant $h$ is set
to unity. The complete system of the equations of motion depends on the specification of
the appropriate equation of state, giving the fluid pressure $P$ as a function of the square
of the effective mass or “relativistic chemical potential” $\mu$ – defined as the magnitude of
the momentum covector – which determines the corresponding dilatonic amplitude scalar $\Phi$
according to the prescription
\[
\Phi^2 = \frac{2}{c^2} \frac{dP}{d\mu^2}, \quad c^2 \mu^2 = -\mu^\sigma \mu_\sigma = -\varphi_\sigma \nabla^\sigma \varphi. \tag{1.3}
\]
This in turn determines the corresponding particle current vector $n^\sigma$, which is given by
\[
n^\rho = \Phi^2 \mu^\rho, \tag{1.4}
\]
while the associated stress energy momentum density tensor will be given by
\[
T^{\rho\sigma} = \Phi^2 \mu^\rho \mu^\sigma + P g^{\rho\sigma}. \tag{1.5}
\]
In conjunction with (1.2), all that is needed to complete the system of dynamical equations
of the superfluid (thereby automatically ensuring the conservation of the stress momentum
energy tensor) is the equation expressing the conservation just of the current (1.4). This
will evidently be presentable as a non-linear wave equation in the form
\[
\nabla_\sigma (\Phi^2 \nabla^\sigma \varphi) = 0. \tag{1.6}
\]
The special “stiff” (Zel’dovich type) case is characterised by the special condition that $\Phi^2$
simply be constant (which arises for an equation of state of the form $P \propto \mu^2 - m^2$, where $m$
is a fixed mass per particle), so that (1.6) reduces to a linear equation of the familiar Dalem-
bertian form, for which the bicharacteristic speed $c_s$ of propagation of small perturbations
is evidently equal to the speed of light $c$. For the more general class of equations of state
covered by the present treatment, the corresponding bicharacteristic sound speed $c_s$ will be
given by the formula
\[
(c/c_s)^2 = 2 \frac{\mu^2}{\Phi^2} \frac{d\Phi^2}{d\mu^2} + 1. \tag{1.7}
\]
For an equation of state of “relativistic polytropic” type $P \propto \mu^{N+1}$ where $N$ is a fixed
“polytropic index” number, one obtains the fixed value $(c_s/c)^2 = 1/N$.

In order to determine the asymptotically contributions to the modifications of energy and
tension due to the presence of a vortex, the only information about the uniform background
state that was found to be needed in the axisymmetric non-moving case \[\text{[2]}\] consisted of
the asymptotic limit values, $\mu^2_\infty$ and $\Phi^2_\infty$, say, of the squared chemical potential $\mu^2$ and
the squared amplitude $\Phi^2$, together with the corresponding limit value $c^2_\infty$ of the squared sound
speed $c_s^2$, which it will be convenient to express in terms of a dimensionless non-negative
parameter $\delta$ defined as a measure of deviation from “stiffness” by the formula
\[
\delta = \frac{1}{2} ((c/c_\infty)^2 - 1) = \left( \frac{\mu^2}{\Phi^2} \frac{d\Phi^2}{d\mu^2} \right)_\infty. \tag{1.8}
\]
In the case of a simple relativistic polytrope it will be given by $2\delta = N - 1$, which means
that it will be equal to unity, $\delta = 1$, in the case $N = 3$ for which the equation of state is that
of the standard model for a gas of ultrarelativistic particles, which has been commonly used as a crude first approximation for the treatment of neutron star matter. In the case of a relatively moving vortex, it will be shown here that this information is not quite sufficient, but that it is also necessary to know one more parameter, characterising the next order of differentiation of the equation of state. This extra parameter, $\delta^\natural$ say, is conveniently specifiable in the form

$$\delta^\natural = \frac{1}{2} \left( \mu^2 \left( \frac{d(c/c_s)^2}{d\mu^2} \right) \right)_{\infty} = - \left( \mu^2 \frac{d}{d\mu^2} \left( \frac{\mu^2}{\Phi^2} \right) \right)_{\infty},$$

(1.9)

so as to be interpretable as a dimensionless measure of deviation from the relativistic polytropic case. Although it is conceivable that it might be negative, one would expect in practice that for realistic equations of state this parameter $\delta^\natural$ would usually be positive.

In addition to this minimal information about the uniform background state, the only other parameter needed to characterise the vortex in the non-moving axisymmetric case is the corresponding circulation integral $\kappa$ which (since we have set $\hbar = 1$) must be a multiple of $2\pi$, or equivalently the corresponding angular momentum $A$ say per unit mass, taking the latter to be the asymptotic value $\mu_\infty$ of the relativistic chemical potential $\mu$, provided it is taken for granted that the value of the net current input $I$ per unit length is zero. These conserved quantities will be expressible as averages over a circle of radius $r$ (whose value can be chosen arbitrarily without affecting the result) by the formulae

$$A = \frac{\kappa}{2\pi \mu_\infty}, \quad \kappa = \oint d\varphi = 2\pi \varphi'$$

and

$$I = 2\pi r n^\sigma \nu^\sigma,$$

(1.10)

(1.11)

where $\left\langle \right\rangle$ indicates angular averaging over the chosen circle of radius $r$, and a prime indicates differentiation with respect to the angle $\theta$ round the circle, while $\nu^\sigma$, with components $(0,0,1,0)$ in the system (1.1), is the outgoing unit normal to the circle. Finally, in the case of a moving vortex, the specification of $A$, and if necessary of $I$, must evidently be supplemented by the specification of the velocity, $\beta$ say, of the relative motion in order for the characterisation of the vortex to be complete. It is conceivable in principle that $I$ might be given a non-zero value by an artificially contrived injection process in a laboratory experiment, but in the natural context of neutron star vortices it is reasonable to assume that – as will be postulated in the last sections of the present work – the central core input should vanish: $I = 0$.

The main results of the present work are the calculation of the asymptotic form of the tension, i.e. the integrated longitudinal stress, $T$ say, and the corresponding energy per unit length $U$ say, over a circular cross section of radius $r$ through the vortex, as functions of the background parameters $\mu_\infty^2$, $\Phi_\infty^2$, $\delta$ and $\delta^\natural$ and of the parameters $A$ and $\beta$ that specify the amplitude and velocity of the vortex itself, subject to the assumption that the injection rate $I$ is zero. As in the familiar non-moving case these quantities are found to have a logarithmic radial dependence.
II. THE ASYMPTOTIC EXPANSION AND ITS LOWEST ORDER

Our objective is to evaluate large scale averages of energy and tension for stationary, longitudinally invariant, asymptotically uniform solutions for an irrotational barotropic relativistic perfect fluid in flat space. It is not possible to solve explicitly the superfluid dynamical equation (1.6) except in the special “stiff” case where this equation becomes linear. Since we are ultimately interested by the asymptotic effect of the vortex and not the details of the flow near the core of the vortex, we shall try to solve this equation by using an asymptotic expansion at large $r$. Prior to this, it is convenient to reexpress (1.6) in the form

\[ 2\nabla_\sigma \nabla^\sigma \varphi + ((c/c_s)^2 - 1)\mu^{-2}(\nabla_\sigma \mu^2)\nabla^\sigma \varphi = 0. \tag{2.1} \]

At this stage, one can invoke the time translation symmetry and the longitudinal symmetry, to which one can associate two Killing vectors $k^\sigma$ and $l^\sigma$ generating respectively time and longitudinal space translations and corresponding to the ignorable coordinates $t$ and $\ell$. These two Killing vectors provide us with two constants of motion

\[ k^\sigma \mu_\sigma = -E, \quad l^\sigma \mu_\sigma = \mathcal{L}, \tag{2.2} \]

where $E$ is interpretable as the effective energy per particle, and $\mathcal{L}$ is interpretable as an effective longitudinal momentum per particle. Consequently, the potential $\varphi$ has a linear dependence on time $t$ and longitudinal distance $\ell$ in cylindrical coordinates. Since there is no loss of generality in eliminating the $\ell$ dependence altogether, i.e. setting $\mathcal{L} = 0$, by a longitudinal boost, the potential will be conveniently expressible in the form

\[ \varphi = E\left(\frac{u}{c^2} - t\right) \tag{2.3} \]

where $u$ is a function only of $r$ and $\theta$.

In order to analyse that asymptotic form of the flow at large $r$, we now postulate that $u$ has an expansion of the form

\[ u = \nu r + w \ln r + x + y \frac{\ln r}{r} + z \frac{1}{r} + o\left(\frac{1}{r}\right), \tag{2.4} \]

where $\nu, w, x, y, \text{ and } z$ are functions of $\theta$ only. In order to proceed, we need the first and second partial derivatives of $u$ with respect to $r$ and with respect to $\theta$. To the relevant asymptotic order, the former will be expressible as

\[ u_r = \nu + \frac{w}{r} - y \frac{\ln r}{r^2} + \frac{y - z}{r^2} + o\left(\frac{1}{r^2}\right), \tag{2.5} \]

\[ u_{rr} = -\frac{w}{r^2} + 2y \frac{\ln r}{r^3} + \frac{2z - 3y}{r^3} + o\left(\frac{1}{r^2}\right), \tag{2.6} \]

while the latter will be expressible as

\[ u_\theta = \nu' r + w' \ln r + x' + y' \frac{\ln r}{r} + z' \frac{1}{r} + o\left(\frac{1}{r}\right), \tag{2.7} \]

\[ u_{\theta\theta} = \nu'' r + w'' \ln r + x'' + y'' \frac{\ln r}{r} + z'' \frac{1}{r} + o\left(\frac{1}{r}\right), \tag{2.8} \]
using a prime to denote straight differentiation with respect to $\theta$.

According to (1.3), the effective mass $\mu$ will be given by

$$c^2 \mu^2 = \frac{E^2}{c^2} (c^2 - u_r^2 - \frac{1}{r^2} u_\theta^2). \quad (2.9)$$

This can be evaluated with the help of the asymptotic forms of the first partial derivatives (2.5) and (2.7) and we obtain

$$\frac{\partial \mu^2}{\partial r} = c^2 - (v^2 + v'^2) - 2v' w' \ln \frac{r}{r} - 2\left(\frac{vw + v'x'}{r} - \frac{w' \ln r}{r^2}\right) - \frac{2(vz - v'z' - v' \ln r)}{r^2} - \frac{2(vw + v'x')}{r^2} + o\left(\frac{1}{r^2}\right). \quad (2.10)$$

As well as the expression for $\mu^2$ itself, we shall use its partial derivatives with respect to $r$ and $\theta$ for the evaluation to the required order of both the quadratic first order differential contribution

$$(\mu^2)_{,r} \nabla^r \varphi = \frac{E}{c^2} \left(\mu^2_r \, u_r + \frac{1}{r^2} \mu^2_\theta \, u_\theta\right), \quad (2.11)$$

and the second order Laplacian contribution

$$\nabla_{,r} \nabla^r \varphi = \frac{E}{c^2} \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{r\theta}\right), \quad (2.12)$$

in the dynamical equation (2.1).

Independently of the particular form of the equation of state, it can be seen that the vanishing of the coefficient of the leading term on the left of (2.1), namely the term of order $1/r$, will give the requirement

$$v'' + v = 0, \quad (2.13)$$

whose first integral gives

$$v^2 + v'^2 = V^2, \quad (2.14)$$

where $V$ is a constant of integration that will be interpretable as the flow velocity of the uniform background. There will be no loss of generality in arranging for the direction of this flow to be aligned with the axis $\theta = 0$, which means that the local solution for $v$ will be given explicitly by

$$v = V \cos \theta. \quad (2.15)$$

Still independently of the particular form of the equation of state function given by (1.7), it can also be seen that the vanishing of the coefficient of the next leading term on the left of (2.1), namely the term of order $\ln r/r^2$, will give the further requirement

$$w' = 0 \quad (2.16)$$

as the only possibility compatible with global regularity, which simply means that (like $\beta$) $w$ must be a constant. We shall restrict our attention in subsequent sections to the cases in which $w$ is restricted to be zero, which is the condition for there to be no source or sink in the vortex, i.e. for which there is no net inflow or outflow, but for the purposes of the next section we shall provisionally continue to consider configurations of the most general kind for which the constant $w$ is unrestricted.
III. CONDITIONS FOR EQUILIBRIUM AT INTERMEDIATE ORDER.

In order to proceed beyond the lowest order, we must take account of the particular form of the function given by (1.7), which will have an asymptotic expansion that can be seen from (2.10) to take the form

\[ \frac{1}{2} \left( (c/c_s)^2 - 1 \right) = \delta + 2\delta^2 \gamma^2 \left( \frac{vw + v'x'}{r} \right) + o\left( \frac{1}{r^2} \right), \]  

(3.1)

using the usual abbreviation \( \gamma^2 = 1/(1-\beta^2) \) with \( \beta = V/c \), where \( \delta \) and \( \delta^2 \) are dimensionless constants defined in (1.8) and (1.9) in terms of asymptotic values of respectively second and third derivatives of the equation of state function.

On substitution of the leading order dynamical conditions (2.14) and (2.16), the expansion formula (2.10) simplifies to

\[ \frac{c^6 \mu^2}{E^2} = \frac{c^2}{\gamma^2} - 2 \frac{wv + x'v'}{r} - 2(\nu'v')' \frac{\ln r}{r^2} - \frac{w^2 + x'^2 + 2y\nu + 2(zv')'}{r^2} + o\left( \frac{1}{r^2} \right). \]  

(3.2)

It is now apparent from (3.1) that the vanishing of the coefficient of the leading surviving term the left of (2.1), namely the term of order \( 1/r^2 \), will give the requirement

\[ \left( \frac{c^2}{\gamma^2} - 2\delta v'^2 \right) x''' + 4\delta vv'x' + 2w\delta(v^2 - v'^2) = 0. \]  

(3.3)

This provides a decoupled second order differential equation for \( x \), for which one immediately obtains a first integral expressible as

\[ \left( \frac{c^2}{\gamma^2} - 2\delta v'^2 \right) x' - 2w\delta v v' = A \frac{c^2}{\gamma^3} \sqrt{1 - \alpha^2}, \]  

(3.4)

where \( A \) is an arbitrary constant of integration which is interpretable as proportional to the corresponding value of the total circulation \( \kappa \) round the vortex, in terms of which it is given by the relation

\[ A = \frac{\kappa \gamma c^2}{2\pi E}, \quad \kappa = \oint d\varphi, \]  

(3.5)

in which \( \alpha \) is a predetermined constant given by

\[ \alpha^2 = 2\beta^2 \gamma^2 \delta = \frac{(c/c_s)^2 - 1}{(c/V)^2 - 1}, \]  

(3.6)

which, assuming the asymptotic flow velocity \( V = c\beta \) to be subsonic, will necessarily be less than unity:

\[ V^2 < c_s^2 \quad \Rightarrow \quad \alpha^2 < 1. \]  

(3.7)

Subject to this subsonicity condition, which will be taken for granted in all that follows, the first order differential equation (3.4) can in its turn be integrated straightforwardly to
give the explicit solution for $x$ which is found to be simply expressible in terms of a modified angle variable $\psi$ by

$$x = A \gamma^{-1} \psi + w \ln \sqrt{1 - \alpha^2 \sin^2 \theta}, \quad \psi = \arctan \{ \sqrt{1 - \alpha^2 \tan \theta} \} \quad (3.8)$$

The absence of a second arbitrary constant of integration from this expression does not imply any loss of generality, because the absence of radial dependence in the corresponding term in (2.4) means that it is only the derivative of $x$ but not its absolute value that is physically relevant. For the same reason (unlike the other angular dependence functions $v, w, y, z$ which must all be strictly continuous) the function $x$ can be discontinuous—and indeed it necessarily will be so somewhere unless the circulation $\kappa$ vanishes. It can be observed that while the physically meaningful derivative $x'$ is a continuous function of $\theta$, the expression (3.8) gives a value of the potential $x$ itself that has jump discontinuities at $\theta = \pm \pi/2$. (Insertion and adjustment of an additive constant of integration on the right of (3.8) could be used to make gauge changes of $\varphi$ by which the discontinuities could be displaced elsewhere, or by which one—or both—of them could be removed altogether.)

Again using (3.1), we can go on to evaluate the coefficient of the next leading term in (2.1), namely the one of order $\ln r/r^3$, whose vanishing can be seen to be expressible as the condition

$$\left( \frac{c^2}{\gamma^2} - 2\delta v'^2 \right) y'' + 8\delta v v' y' + \left( \frac{c^2}{\gamma^2} + 2\delta (v'^2 - v^2) \right) v y = 0 \quad (3.9)$$

We thus again obtain a decoupled second order equation, this time for $y$, for which it can be seen that there is a first integral given by

$$\left( 1 - 2\delta \frac{\gamma^2}{c^2} v'^2 \right) v' y' + \left( 1 + 2\delta \frac{\gamma^2}{c^2} v'^2 \right) v y = V B, \quad (3.10)$$

where $B$ is a constant of integration. The complete solution is hence finally obtainable in the explicit form

$$y = \frac{B \cos \theta + C \sin \theta}{1 - \alpha^2 \sin^2 \theta}, \quad (3.11)$$

where $C$ is another constant of integration. The regularity of this solution is guaranteed by the subsonicity condition (3.7) which ensures that the denominator does not vanish anywhere. It is to be noted that unlike the solution (3.8) for $x$, the solution (3.11) for $y$ is not affected by the presence of a non vanishing source coefficient $w$.

Although the values of the integration constants $C$ and $B$ appear at this stage to be arbitrary, it transpires that they are in fact predetermined by the condition of existence of a globally regular solution for the still unknown function $z$. The latter is governed by a higher order equilibrium equation which we must now work out, even though it turns out that the integrals we shall need for evaluating the asymptotically dominant contribution to the energy and tension can be evaluated without any specific knowledge of the actual functional form of the higher order coefficient $z$ itself.
IV. CONDITIONS FOR EQUILIBRIUM AT THE RELEVANT HIGHER ORDER.

In view of the automatic cancellation expressed by (4.1), the dominant surviving contribution to the integrals with which we are concerned will come from the next term in (3.2), which is only of order $1/r^2$. In order to obtain the information required for evaluating this contribution, we now consider the next leading term in (2.1) which (since we have already dealt with all the terms up to $\ln r/r^3$) is the one of order $1/r^3$. The vanishing of the corresponding coefficient, as evaluated using (3.1), is found to be expressible as the condition

\[
\left(\frac{c^2}{\gamma^2} - 2\delta v'^2\right)z'' + 8\delta v v'z' + \left(\frac{c^2}{\gamma^2} + 2\delta(v'^2 - 2v^2)\right)z - 2\left(\frac{c^2}{\gamma^2} + \delta(v'^2 - 3v^2)\right)y
\]

\[-4\delta v v'y' - 2\left(1 + 2\delta + 2\delta^2\gamma^2 c^{-2} v'^2\right)v'x'x'' + 4\left(\delta + 2\delta^2\gamma^2 c^{-2} v'^2\right)v x'^2
\]

\[= 2v x''w + 4\delta^2\gamma^2 c^{-2}(vv'x'' + (v'^2 - 3v^2)x')v'w - 4\left(\delta + \delta^2\gamma^2 c^{-2}(v^2 - v')\right)vw^2.\]  

(4.1)

in which the right hand side will vanish when the source coefficient $w$ is set to zero.

For the sake of simplicity, let us from now on restrict our attention to the physically important source free case, meaning that we shall impose the postulate

\[w = 0.\]  

(4.2)

In these circumstances the right hand side of (4.1) will drop out, leaving an equation for $z$ that can be rewritten, with the unknown terms grouped on the left, in the form

\[
\left(\left(\frac{c^2}{\gamma^2} - 2\delta v'^2\right)vz' - \frac{c^2}{\gamma^2} - 2\delta(V^2 + v^2)\right)z' = \left((1 + 2\delta + 2\delta^2\gamma^2 c^{-2} v'^2)vv'x'^2 + 4\delta v^2v'y\right)'
\]

\[+ \left(v^2 - v'^2 - 2V^2(2\delta + \delta^2\gamma^2 c^{-2} v'^2)\right)x'^2 + 2\left(\frac{c^2}{\gamma^2} - \delta(V^2 + 2v'^2)\right)y .\]  

(4.3)

The $y$ dependent term and the term proportional to $x'^2$ at the end of the preceding expression can be evaluated in the form of a derivative. We can thus obtain a first integrated version of the equation (4.3) for $z$ in the form

\[
(1 - \alpha^2 \sin^2 \theta) \cos \theta z' + (1 - \alpha^2(1 + \cos^2 \theta)) \sin \theta z = \mathcal{F} ,
\]

(4.4)

in terms of a known source function $\mathcal{F}$, from which the fully integrated solution will be obtainable the form

\[
z = \frac{f \cos \theta}{1 - \alpha^2 \sin^2 \theta},
\]

(4.5)

where the function $f$ is calculable by direct quadrature as a solution of the differential relation

\[f' = \frac{\mathcal{F}}{\cos^2 \theta}.\]  

(4.6)

It can be seen that the inhomogeneous source term on the right of (4.4) will be given by an expression of the form
\[ F = P + Q + R , \]

in which the first term is an automatically periodic symmetric contribution expressible by

\[ P = \kappa + C \left( \ln \sqrt{1 - \alpha^2 \sin^2 \theta} - \frac{\cos^2 \theta (1 + \alpha^2 \sin^2 \theta)}{1 - \alpha^2 \sin^2 \theta} \right) \]

(4.8)

where \( \kappa \) is a new constant of integration, while the second term is an antisymmetric contribution, with a manifestly periodic form expressible by

\[ Q = \frac{V A^2 \sin 2\theta}{c^2} \left( \frac{1 + \beta^2 \gamma^2 (2\delta^2 - \delta + \delta^2)}{2(1 - \alpha^2 \sin^2 \theta)} - \frac{(1 - \alpha^2)(1 + 2\delta + 2\delta \gamma \beta^2 \sin^2 \theta)}{2(1 - \alpha^2 \sin^2 \theta)^2} \right) + B \sin 2\theta \frac{1 - \alpha^2(1 + \cos^2 \theta)}{2(1 - \alpha^2 \sin^2 \theta)} . \]

(4.9)

What matters most for our present purpose is the third term in (4.7), which is an antisymmetric remainder with a not automatically periodic form given by

\[ R = -\frac{A^2 V (2\delta - \beta^2 \gamma^2 (2\delta^2 - \delta - \delta^2))}{c^2(1 - \alpha^2)^{1/2}} \psi + B \sqrt{1 - \alpha^2} \psi . \]

(4.10)

Unlike the other terms, the second contribution \( R \) will be subject, for generic parameter values, to a mismatch \( [R]_{0}^{2\pi} \) between its value at departure from the axis of motion, when \( \psi = 0 \), and its value at return to this axis, when \( \psi = 2\pi \), since it is evident from (4.11) that we shall have

\[ [R]_{0}^{2\pi} = 2\pi \left( -\frac{A^2 V (2\delta - \beta^2 \gamma^2 (2\delta^2 - \delta - \delta^2))}{c^2(1 - \alpha^2)^{1/2}} + B \sqrt{1 - \alpha^2} \right) . \]

(4.11)

The only physically admissible solutions are those for which \( z \) and hence also \( F \) satisfies the condition of being a regular periodic function of the angle \( \theta \). Since this condition is automatically satisfied by the other contributions \( P \) and \( Q \) it must therefore also be satisfied by the third term \( R \) in (4.7), which means that the mismatch (4.11) must be made to vanish. It can thus be seen that the integration constant \( B \) cannot be chosen independently but that – as a necessary condition for global regularity – it must be specified in terms of the constant \( A \) (which according to (3.5) is proportional to the circulation \( \kappa \)) by the relation

\[ B = V \left( \frac{A}{c} \right)^2 \frac{(2\delta - \beta^2 \gamma^2 (2\delta^2 - \delta - \delta^2))}{(1 - \alpha^2)} . \]

(4.12)

This not only eliminates the mismatch but actually gets rid of the remainder term altogether, so that we have

\[ R = 0 \, , \quad F = P + Q \, , \]

(4.13)

and it can also be seen to have the important secondary effect of removing the singularity that would otherwise have arisen from the the contribution to \( z \) via (4.10) and (4.11) from the antisymmetric term \( Q \) which, by substitution of (4.12), is rather miraculously simplified so as to be given by
\[
\frac{Q}{\cos^2 \theta} = \frac{A^2V}{c^2} \sin 2\theta \left( \frac{\beta^2\gamma^2(2\delta^2 + \delta + \delta^2)}{(1 - \alpha^2\sin^2 \theta)^2} - \frac{\alpha^2(2\delta - \beta^2\gamma^2(2\delta^2 - \delta - \delta^2))}{2(1 - \alpha^2)(1 - \alpha^2\sin^2 \theta)} \right). \tag{4.14}
\]

It is evident that the corresponding contribution to the quadrature for \(f\) as given by (4.6) will thus be satisfactorily regular and periodic. It is to be noted however that in order for \(z\) as given by (4.5) to be appropriately regular and periodic, while it is of course necessary that \(f\) should also be periodic, it is nevertheless not necessary that \(f\) should be regular in the strictest sense, since it is permissible for it to have a simple pole where \(\cos \theta\) vanishes. Such a pole will in general arise from the contribution to (4.5) from \(P\), which can be seen to be expressible in the form

\[
\frac{P}{\cos^2 \theta} = \left( \tan \theta (K + C \ln \sqrt{1 - \alpha^2\sin^2 \theta}) - C \frac{\psi}{\sqrt{1 - \alpha^2}} \right)'. \tag{4.15}
\]

In addition to the admissible pole term proportional to \(\tan \theta\), the differentand on the right of (4.15) also contains an inadmissibly non-periodic term proportional to the modified angle coordinate \(\psi\) whenever the coefficient \(C\) is non zero. It can thus be seen that as well as fixing the constant of integration \(B\) by the condition (4.12) the requirement of global regularity also fixes the other constant of integration introduced in (3.11) by the simpler condition

\[C = 0. \tag{4.16}\]

The solution for \(z\) that is thus finally obtained from (4.5) and (4.6) will be given explicitly by

\[
z = V A^2 \cos \theta \left( \frac{\beta^2\gamma^2(2\delta^2 + \delta + \delta^2)}{\alpha^2(1 - \alpha^2\sin^2 \theta)^2} + \frac{(2\delta - \beta^2\gamma^2(2\delta^2 - \delta - \delta^2)) \ln \sqrt{1 - \alpha^2\sin^2 \theta}}{(1 - \alpha^2)(1 - \alpha^2\sin^2 \theta)} \right)
\]
\[+ \frac{\kappa \sin \theta + L \cos \theta}{1 - \alpha^2\sin^2 \theta}, \tag{4.17}\]

where \(L\) is yet another constant of integration, which, like \(\kappa\), presumably depends on the nature of the inner core. However what matters for the purpose of the following sections is not the particular functional form of the higher order angular coefficient \(z\), but just the conditions (4.12) and (4.16) that are necessary and sufficient for its global regularity, and that specify the specific form of the intermediate order angular coefficient \(y\) which is what we really need to know.

V. ASYMPTOTIC AVERAGES.

Our purpose is to obtain asymptotic averages of quantities \(Q\) that will of the kind discussed in the appendix of the preceding work [4] in the sense that – as in the prototype case of the pressure \(P\) – they can be postulated to have a smooth functional dependence on the chemical potential, and will thus have a deviation from the corresponding uniform asymptotic limit value \(Q_\infty\) that will have a form given by

\[
Q - Q_\infty = \left. \frac{dQ}{d\mu^2} \right|_\infty (\mu^2 - \mu^2_\infty) + \frac{1}{2} \left. \frac{d^2Q}{(d\mu^2)^2} \right|_\infty (\mu^2 - \mu^2_\infty)^2 + o((\mu^2 - \mu^2_\infty)^2). \tag{5.1}
\]
The relevant deviation of the square of the chemical potential (or effective mass) variable $\mu$ will itself be given, according to (3.2), by

$$\frac{c^2(\mu^2 - \mu_{\infty}^2)}{\gamma^2 \mu_{\infty}^2} = -2 \frac{v'y'}{r} - 2(v'y')^2 \ln \frac{r}{r^2} - \frac{x'^2 + 2vy' + 2(v'z')}{r^2} + o\left(\frac{1}{r^2}\right), \quad (5.2)$$

This differs substantially from the corresponding expression in our previous analysis [2] in which both $v$ and $y$ vanished (thereby also incidentally eliminating the term involving $z$) so that the leading terms of order $1/r$ and of order $\ln r/r^2$ were absent, leaving just the term of order $1/r^2$ with coefficient simply given by $x'^2$. Its effect is to introduce new leading terms of order $1/r$ and of order $\ln r/r^2$ in the resulting expression for the asymptotic deviation of the quantity $Q$, which will be given by

$$Q - Q_{\infty} = -\frac{\gamma^2}{c^2} \frac{dQ}{d\mu^2} \cdot \ln \frac{r}{r^2} \quad + \quad \frac{2\gamma^4}{c^4} \frac{d^2Q}{(d\mu^2)^2} - \frac{2}{r^2} \frac{(v'y')^2}{\mu_{\infty}^4} + o\left(\frac{1}{r^2}\right), \quad (5.3)$$

The long range leading order terms in (5.3) will however cancel out, leaving a non zero contribution only from the short range term of order $1/r^2$ when we take the relevant average $\overline{Q}$, as evaluated over a circular section of radius $r$ through the vortex. Such an average will be given by

$$\overline{Q} = \frac{2}{r^2} \int_{r_{\circ}}^{r} \widehat{Q} \, r dr = \frac{r_{\circ}^2}{r^2} \overline{Q}_{\circ} + \frac{2}{r^2} \int_{r_{\circ}}^{r} \widehat{Q} \, r dr \quad (5.4)$$

where $\overline{Q}_{\circ}$ is the average over an inner core of radius $r_{\circ}$ and where, for any given radius $r$, the quantity $\widehat{Q}$ is the corresponding angular average, as given by

$$\widehat{Q} = \frac{1}{2\pi} \int Q \, d\theta. \quad (5.5)$$

Since the coefficient of term of order $1/r$ is antisymmetric with respect to $\theta$ its angular average will evidently vanish. The angular averages of the term of order $\ln r/r^2$ and of the term involving $z$ will also vanish because they have the form of derivatives of periodic functions. The angular average of the deviation in which we are interested will thus be given just by

$$\overline{Q} - Q_{\infty} = -\left(\mu_{\infty}^2 \frac{dQ}{d\mu^2}\right) \frac{\gamma^2}{c^2} \frac{x'^2 + 2vy'}{\mu_{\infty}^4} - \frac{2\gamma^4}{c^4} \frac{d^2Q}{(d\mu^2)^2} + \frac{\gamma^4}{c^4} \frac{v'^2x'^2}{\mu_{\infty}^4} \frac{1}{r^2} + o\left(\frac{1}{r^2}\right). \quad (5.6)$$

Since the preceeding outcome is only of order $1/r^2$, it can be seen that the required average will have an asymptotic behaviour of exactly the same kind as was obtained in the previous analysis [2] of the non-moving case, i.e. it will be described by an expression of the form

$$\overline{Q} - Q_{\infty} = \frac{Q A^2}{c^2} \frac{\ln r}{r^2} + o\left(\frac{\ln r}{r^2}\right), \quad (5.7)$$
with a constant coefficient $\tilde{Q}$ which will have the same dimensionality as the quantity $Q$ itself, where $A$ is a constant normalisation factor, interpretable as the *average angular momentum per unit mass*, that is specified independently of $Q$ by the formula

$$A = \frac{\kappa}{2\pi \mu_\infty},$$

(5.8)

which means that it is identifiable with the integration constant introduced in (3.3). The explicit formula for the asymptotic deviation coefficient $\tilde{Q}$ that is defined in this way will however be significantly more complicated in general than in the non-moving case, for which it was found to be given simply by $\tilde{Q} = -2(\mu^2 dQ/d\mu^2)|_\infty$. The generalisation that is needed for the case of a moving vortex, i.e. one with a non-zero value of $\beta$, can be read out from (62) as

$$\tilde{Q} = -2\mathcal{Y}\mu^2 \frac{dQ}{d\mu^2}|_\infty + 4\mathcal{Z}\mu^4 \frac{d^2Q}{(d\mu^2)^2}|_\infty,$$

(5.9)

with dimensionless coefficients $\mathcal{Y}$ and $\mathcal{Z}$ given by

$$\mathcal{Y} = \frac{\gamma^2}{A^2} (\overrightarrow{x'^2} + 2\overrightarrow{vy}) \quad \quad \mathcal{Z} = \frac{\gamma^4}{c^2 A^2} \overrightarrow{y'^2 x'^2}.$$

(5.10)

To evaluate these coefficients, we use the explicit expressions obtained from (3.4) and from (3.11) and (4.12). We get

$$\mathcal{Z} = \frac{\beta^2 \gamma^2}{2(1 - \alpha^2)^{1/2}},$$

(5.11)

and

$$\mathcal{Y} = \frac{2 - \alpha^2}{2(1 - \alpha^2)^{1/2}} + \frac{(1 - \sqrt{1 - \alpha^2})(\alpha^2 - \alpha^4 + \gamma^2 \alpha^2 + 2\beta^4 \gamma^4 \delta^5)}{\alpha^2(1 - \alpha^2)}. $$

(5.12)

The latter expression is rather unwieldy even in the “polytropic” case characterised by $\delta^5 = 0$, but in the stiff limit characterised by $\delta = \delta^5 = 0$ it simply gives $\mathcal{Y} = 1$, independently of the velocity $\beta$.

The applications with which we are principally concerned are the pressure $P$ for which (5.9) gives

$$\tilde{P} = -c^2 \mu^2 \Phi^2 \mathcal{Y} - 2\mathcal{Z}\delta,$$

(5.13)

and the squared amplitude $\Phi^2$ itself for which (5.3) gives

$$\tilde{\Phi}^2 = -2\Phi^2 \left(\mathcal{Y}\delta - 2\mathcal{Z}(\delta^2 - \delta^5)\right).$$

(5.14)

These quantities are all that is needed to evaluate the average deviations of the longitudinal components

$$n_i = \Phi^2 \mu_i$$

(5.15)
of the conserved current and

\[ T_{ij} = \Phi^2 \mu_i \mu_j + P g_{ij} \tag{5.16} \]

of the stress energy momentum tensor, where the Latin indices run over the values 0,1 in the coordinate system \([1,1]\), with respect to which the gradient components

\[ \mu_i = \varphi, i \tag{5.17} \]

take the constant values

\[ \mu_0 = -E, \quad \mu_1 = 0, \tag{5.18} \]

so that we simply obtain

\[ \tilde{n}_i = \tilde{\Phi}^2 \mu_i \tag{5.19} \]

and

\[ \tilde{T}_{ij} = \tilde{\Phi}^2 \mu_i \mu_j + \tilde{P} g_{ij}. \tag{5.20} \]

The general expression \((5.7)\) is not limited to scalar functions of \(\mu^2\) but can be also applied to other quantities. This will however require more work because the intermediate formulas such as \((5.6)\) are no longer applicable. Interesting examples of this kind are the transverse current components and the transverse-transverse components of the stress-momentum tensor. Although the resulting asymptotic coefficients are mathematically well defined, the physical interpretation of these quantities is less obvious than the components treated above so that we have preferred to relegate these results in an appendix.

VI. CONCLUSIONS

The present work has given asymptotic solutions for a relativistic vortex moving with respect to a surrounding irrotational perfect fluid (or, equivalently, a zero temperature superfluid). We have obtained the effective energy density, which can be expressed in the form

\[ T_{\text{eff}}(r) = \frac{\kappa^2}{4\pi^2} \Phi^2_\infty (\mathcal{Y} - 2\delta \mathcal{Z}) \frac{\ln r}{r^2} + o\left(\frac{\ln r}{r^2}\right) \tag{6.1} \]

and the effective tension, which can be given by

\[ U_{\text{eff}}(r) = \frac{\kappa^2}{4\pi^2} \Phi^2_\infty \left[ (1 - 2\gamma^2 \delta) \mathcal{Y} + (4\gamma^2 (\delta^2 - \delta - \delta^*) - 2\delta) \mathcal{Z} \right] \frac{\ln r}{r^2} + o\left(\frac{\ln r}{r^2}\right), \tag{6.2} \]

where \(\mathcal{Y}\) and \(\mathcal{Z}\) are given by \((5.11)\) and \((5.12)\).

The present work constitutes an intermediate step that is needed for the construction of a realistic modelization of the relativistic dynamics of a neutron star interior. Like a superfluid in a rotating bucket, the neutron superfluid in a rotating neutron star minimizes its free energy by forming an array of quantized vortices that, on a scale larger than the
intervortex separation length, simulates a rigid body rotation. Because of the huge number of vortices in a neutron star, it is not practicable to keep track of all individual vortices, so a macroscopic description requires the use of a model involving a vorticity 2-form

\[ w_{\rho\sigma} = 2\nabla_{[\rho\mu\sigma]}, \quad (6.3) \]

that is not supposed to be the small scale local vorticity field, (which vanishes outside the microscopic vortex cores) but that is to be interpreted as the large scale average over a neighbourhood extending across a many vortices.

The previous analysis [2] of a relativistic vortex configuration in a background without relative motion provided the basis for the specification of a particularly simple model [8] within the general category that is needed for such a description. Such models are derivable from a Lagrangian type master function \( \Lambda\{n^\rho, w_{\rho\sigma}\} \) that depends on the three scalar quantities that can be built from the (macroscopically averaged) current density \( n^\rho \) and the (macroscopically averaged) vorticity \( w_{\rho\sigma} \), namely the magnitude \( n \) of the particle current vector \( n^\rho \) itself, the scalar magnitude \( w \) of the vorticity covector \( w_{\rho\sigma} \) and the magnitude \( \zeta \) of the associated Joukowsky lift force density vector \( \zeta_\rho \) as defined by

\[ n^2 = -n^\rho n_\rho, \quad w^2 = \frac{1}{2} w_{\rho\sigma} w^{\rho\sigma}, \quad \zeta^2 = \zeta_\rho \zeta^\rho, \quad (6.4) \]

where the Joukowsky vector is defined as

\[ \zeta_\rho = w_{\rho\sigma} n^\sigma. \quad (6.5) \]

This vector is interpretable as representing the volume density of force that would be exerted on the vortices as an expression of the Magnus effect, by the relative flux of the fluid according to the simple formula originally derived by Joukowsky for flow past a long aerofoil. The earlier analysis [2] was sufficient to unambiguously determine the appropriate form for the equation of state for the function \( \Lambda\{n, w, \zeta\} \) only in the limit for which \( \zeta \) vanishes.

As convenient ansatz for use as a provisional approximation, it has been suggested [8] that this special limit form should be extrapolated by assuming that there is no dependence on \( \zeta \) at all, even when \( \zeta \) is non-zero, a supposition that is mathematically justifiable for a fluid obeying the “stiff” Zel’dovich type equation of state, but not in general. To obtain a more accurate treatment for an arbitrarily compressible fluid it will ultimately be necessary to use a more sophisticated ansatz taking account of the effect of the relative flow that will be present when \( \zeta \) is non-zero. The present analysis of the effect of relative flow on an individual vortex is an indispensible first step towards the achievement of what is required. What still remains to be done is to extrapolate the present analysis to the case in which there is not just one vortex but an extended array of parallelly aligned vortices. In the absence of relative flow at large distance, i.e. in the case corresponding to vanishing \( \zeta \), such an extrapolation was almost trivial, at least in the large separation limit that is relevant, because of the axial symmetry of the individual vortex solution. However for the non axisymmetric configurations considered in the present work the required extrapolation will not be so straightforward.

It is to be remarked that the effect analysed in the present work is not the only kind that needs to be examined in greater detail in order to improve the accuracy of the recently
proposed model [8] for the treatment of relative flow past the vortices in neutron stars. Whereas the modification of the stress - momentum - energy tensor considered here is likely to be very small, a potentially more important correction arises from the friction between the vortices and the “normal” matter that will be present, not just due to thermal effects (which in typical neutron stars will be very small) but also, even in the zero temperature limit, due to the fact that the matter will not be entirely constituted just by neutrons, but will also include a small but significant fraction in the form of protons, together with a gas of charge neutralising electrons whose scattering from the vortices can provide a dynamically important friction drag [1].
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VII. APPENDIX: ASYMPTOTIC AVERAGES OF TRANSVERSE CURRENT COMPONENTS.

In this appendix, we compute the asymptotic formula (5.7) for quantities which are not scalar functions of $\mu^2$ and therefore for which the method used in the main body does not apply. The first important example is that of the tranverse components with respect to Minkowskian coordinates characterised by

$$x^0 = t , \quad x^1 = \ell , \quad x^2 = r \cos \theta , \quad x^3 = r \sin \theta$$

(7.1)

of the conserved current $n^\sigma$. Since these components have no dependence on the longitudinal coordinates $x^i$ ($i = 0, 1$) the current conservation law automatically reduces from four to two dimensional form,

$$\nabla_\sigma n^\sigma = 0 \Rightarrow \nabla_a n^a = 0 ,$$

(7.2)

so that we obtain

$$\nabla_b (x^a n^b) = n^a ,$$

(7.3)

using early Latin letters for the transverse indices, $a, b = 2, 3$. For any 2-dimensional transverse section $\Sigma$ bounded by a curve $s$ with outgoing unit normal having components $\nu_a$, Green’s theorem thus provides us with the relation

$$\int_\Sigma n^a d\Sigma = \oint_s x^a n^b \nu_b ds .$$

(7.4)

In the case of a circular section with boundary characterised by

$$x^a = r \nu^a , \quad \nu^2 = \cos \theta , \quad \nu^3 = \sin \theta$$

(7.5)

so that $ds = rd\theta$, the preceding formula can be used to see that the corresponding average – defined as in (5.4) – will be expressible just in terms of an angular average – as defined in (5.5) – over the section boundary in the form
\[ \overline{n^a} = 2 \nu^a \nu_b n^b . \] (7.6)

In the same way, for the stress energy momentum tensor we obtain
\[ \nabla_a T^\sigma_\rho = 0 \quad \Rightarrow \quad \nabla_a T^a_\rho = 0 , \] (7.7)

and hence
\[ \overline{T^a_\rho} = 2 \nu^a \nu_b T^b_\rho . \] (7.8)

Using
\[ n_a = \Phi^2 \mu_a , \quad \mu_a \nu^a = \frac{E}{c^2} u_r \] (7.9)

we obtain
\[ n^a \nu_a = \gamma_\infty \Phi^2 \left( \nu - 2 \delta^2 \frac{x' \nu' \nu'}{r} - (y + 2 \delta^2 \frac{y \nu'}{c^2} \ln \frac{r}{r^2}) \right) + O \left( \frac{1}{r^2} \right) . \] (7.10)

Since it is evident from symmetry considerations that the angular average \( x' \nu' \nu' \nu^a \) must vanish, we are left with a relation of the standard form (5.7), namely
\[ \overline{n^a - n^a_\infty} = \tilde{n}^a \frac{A^2 \ln r}{c^2 r^2} + o \left( \frac{\ln r}{r^2} \right) , \] (7.11)

with
\[ \tilde{n}^a = -n_\infty \frac{\gamma c^2}{A^2} 2 (y + 2 \delta^2 \frac{y \nu'}{c^2} \nu) \nu^a , \] (7.12)

in which the components of the relevant angular average are calculable from (2.15) and (3.11). Using the formula (4.12) for \( B \), and noting that \( C \) vanishes by (4.16), one finally obtains
\[ \tilde{n}^x = \frac{2 (2 \delta - \beta^2 \gamma^2 (2 \delta^2 - \delta - \delta^2))}{(1 - \alpha^2)} \left( 1 - \sqrt{1 - \alpha^2} \frac{\alpha^2}{\gamma} \right) n^x_\infty , \quad \tilde{n}^y = 0 . \] (7.13)

Having arrived at this result, it behoves us to remark however that the physical significance of the transverse current component average \( \overline{n^a} \) thus obtained is rather less obvious than that of the corresponding longitudinal average \( n^x \) obtained in the previous section, which is directly interpretable as being proportional to the current flux through the circular section across the cylindrical vortex cell under consideration. The corresponding transverse parts of the current flux associated with cylindrical vortex cells in a honeycomb lattice are also of obvious physical interest, but they are not directly definable as averages of the kind considered above, so their evaluation will require a rather different kind of calculation that will be left for a future article.

The same question of physical significance arises for the averages of the transverse components of the stress energy momentum tensor, for which an analysis of the same kind as
that given in the preceding paragraph also leads to an equation of the standard form, namely

\[
\tilde{T}_\rho^a - T_{\infty \rho}^a = \tilde{T}_\rho^a \frac{A^2}{c^2} \frac{\ln r}{r^2} + o\left(\frac{\ln r}{r^2}\right).
\]  

(7.14)

In the particular case of the the cross components \(T^{ai}\) this is obvious because they are expressible in the form

\[
T^{ai} = \mu^a n^i = n^a \mu^i
\]

(7.15)
in which the longitudinal momentum components \(\mu^i\) are constant. It is thus evident that the corresponding averages – which have a direct physical interpretation in terms of longitudinal fluxes of transverse momentum – will simply be proportional to those of the current, being given by

\[
\tilde{T}^{ai} = -\frac{2(2\delta - \beta^2 \gamma^2 (2\delta^2 - \delta - \delta^2))}{(1 - \alpha^2)} \left(1 - \frac{\sqrt{1 - \alpha^2}}{\alpha^2}\right) n^a \mu^i.
\]  

(7.16)

For the derivation of the corresponding formulae for the purely transverse components \(T^{a_b}\) – which do not have such a straightforward physical interpretation – a little more work is required since (7.4) no longer applies. One must then consider the product of the asymptotic expansion of \(\Phi^2\), given by (5.4), with that of the \(\mu_a\), given by (2.5) and (2.7), in order to obtain an asymptotic expansion at the required order for the part \(\Phi^2 \mu_a \mu_b\) of the components \(T_{ab}\), the pressure part being already known via (5.13). One obtains finally

\[
\tilde{T}_{xx} = c^2 \mu^2 \Phi^2 \left[\frac{\alpha^4 - 2\alpha^2 - 2\beta^4 \gamma^4 (\delta + \delta^2)}{\sqrt{1 - \alpha^2}} + (1 - \sqrt{1 - \alpha^2}) (2 + \beta^2 \gamma^2 - \alpha^2 - 2(\beta^4 \gamma^4 \delta^2/\alpha^2))\right],
\]  

(7.17)

\[
\tilde{T}_{xy} = 0,
\]  

(7.18)

\[
\tilde{T}_{yy} = -c^2 \mu^2 \Phi^2 \frac{1 - \sqrt{1 - \alpha^2}}{\alpha^2 (1 - \alpha^2)} (\alpha^2 - \alpha^4 + \gamma^2 \alpha^2 + 2\beta^4 \gamma^4 \delta^2).
\]  

(7.19)

In the no-motion limit \(\beta = 0\) or in the stiff fluid limit \((\delta = \delta^\ast = 0)\), the coefficients \(\tilde{T}_{xx}\) and \(\tilde{T}_{yy}\) vanish as one could have expected from the previous work [2].