A NOTE ON ADDITIVITY OF POLYGAMMA FUNCTIONS

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Abstract. In the note, the functions $\psi^{(i)}(e^x)$ for $i \in \mathbb{N}$ are proved to be sub-additive on $(\ln \theta_i, \infty)$ and super-additive on $(-\infty, \ln \theta_i)$, where $\theta_i \in (0, 1)$ is the unique root of equation $2|\psi^{(i)}(\theta)| = |\psi^{(i)}(\theta^2)|$.

1. Introduction

Recall [3, 5, 7] that a function $f$ is said to be sub-additive on $I$ if

$$f(x + y) \leq f(x) + f(y)$$

holds for all $x, y \in I$ such that $x + y \in I$. If the inequality (1) is reversed, then $f$ is called super-additive on $I$.

The sub-additive and super-additive functions play an important role in the theory of differential equations, in the study of semi-groups, in number theory, and also in the theory of convex bodies. A lot of literature for the sub-additive and super-additive functions can be found in [3, 5] and related references therein.

It is well-known that the classical Euler gamma function $\Gamma(x)$ may be defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt.$$ (2)

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi or digamma function, and $\psi^{(k)}(x)$ for $k \in \mathbb{N}$ are called the polygamma functions. It is common knowledge that these functions are fundamental and important and that they have much extensive applications in mathematical sciences.

In [4], the function $\psi(a + x)$ is proved to be sub-multiplicative with respect to $x \in [0, \infty)$ if and only if $a \geq a_0$, where $a_0$ denotes the only positive real number which satisfies $\psi(a_0) = 1$.

In [5], the function $|\Gamma(x)|^\alpha$ was proved to be sub-additive on $(0, \infty)$ if and only if $\frac{\ln 2}{\ln \Delta} \leq \alpha \leq 0$, where $\Delta = \min_{x \geq 0} \frac{\Gamma(2x)}{\Gamma(x)}$.

In [2, Lemma 2.4], the function $\psi(e^x)$ was proved to be strictly concave on $\mathbb{R}$.

In [7, Theorem 3.1], the function $\psi(a + e^x)$ is proved to be sub-additive on $(-\infty, \infty)$ if and only if $a \geq c_0$, where $c_0$ is the only positive zero of $\psi(x)$.

In [6, Theorem 1], among other things, it was presented that the function $\psi^{(k)}(e^x)$ for $k \in \mathbb{N}$ is concave (or convex, respectively) on $\mathbb{R}$ if $k = 2n - 2$ (or $k = 2n - 1$, respectively) for $n \in \mathbb{N}$.

In this short note, we discuss sub-additive and super-additive properties of polygamma functions $\psi^{(i)}(x)$ for $i \in \mathbb{N}$.

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Our main result is the following Theorem 1.

Theorem 1. The functions \(|\psi^{(i)}(x^2)|\) for \(i \in \mathbb{N}\) are super-additive on \((-\infty, \ln \theta_i)\) or sub-additive on \((\ln \theta_i, \infty)\), where \(\theta_i \in (0, 1)\) is the unique root of equation
\[2|\psi^{(i)}(\theta)| = |\psi^{(i)}(\theta^2)|.\]  (3)

2. Proof of Theorem 1

Let
\[f(x, y) = |\psi^{(i)}(x)| + |\psi^{(i)}(y)| - |\psi^{(i)}(xy)|\]  (4)
for \(x > 0\) and \(y > 0\), where \(i \in \mathbb{N}\). In order to show Theorem 1, it is sufficient to prove the positivity or negativity of the function \(f(x, y)\). Direct differentiation yields
\[\frac{\partial f(x, y)}{\partial x} = y|\psi^{(i+1)}(xy)| - |\psi^{(i+1)}(x)| = \frac{1}{x}[xy|\psi^{(i+1)}(xy)| - x|\psi^{(i+1)}(x)|].\]  (5)

In [1, Lemma 1] and [8, 9], among other things, the functions \(x^\alpha|\psi^{(i)}(x)|\) are proved to be strictly increasing on \((0, \infty)\) if and only if \(\alpha \geq i + 1\) and strictly decreasing if and only if \(\alpha \leq i\). From this monotonicity, it follows easily that \(\frac{\partial f(x, y)}{\partial x} \geq 0\) if and only if \(y \leq 1\), which means that the function \(f(x, y)\) is strictly increasing for \(y < 1\) and strictly decreasing for \(y > 1\) in \((0, \infty)\). Since
\[\lim_{x \to \infty} f(x, y) = |\psi^{(i)}(y)| > 0,
\] then the function \(f(x, y)\) is positive in \(x \in (0, \infty)\) for \(y > 1\).

For \(y < 1\), by virtue of the increasing monotonicity of \(f(x, y)\), it is deduced that
\[
\begin{align*}
(1) & \text{ if } x > 1, \text{ then } f(1, y) = |\psi^{(i)}(1)| < f(x, y) < |\psi^{(i)}(y)|; \\
(2) & \text{ if } x < 1, \text{ then } f(x, y) < f(1, y) = |\psi^{(i)}(1)|; \\
(3) & \text{ if } y < x < 1, \text{ then } f(y, y) < f(x, y); \\
(4) & \text{ if } x < y < 1, \text{ then } f(x, x) < f(x, y).
\end{align*}
\]
This implies that
\[f(\theta, \theta) = 2|\psi^{(i)}(\theta)| - |\psi^{(i)}(\theta^2)| < f(x, y)\]  (6)
for \(y < 1\), where \(\theta < 1\) with \(\theta < x\) and \(\theta < y\). Since \(f(\theta, \theta)\) is strictly increasing on \((0, 1)\) such that \(f(1, 1) = |\psi^{(i)}(1)| > 0\) and \(\lim_{\theta \to 0^+} f(\theta, \theta) = -\infty\), then the function \(f(\theta, \theta)\) has a unique zero \(\theta_i \in (0, 1)\) such that \(f(\theta, \theta) > 0\) for \(1 > \theta > \theta_i\).

In conclusion, the function \(f(x, y)\) is positive for \(x, y > \theta_i\) or negative for \(0 < x, y < \theta_i\). The proof of Theorem 1 is complete.

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