GLOBAL WELLPOSEDNESS OF HEDGEHOG SOLUTIONS FOR THE (3 + 1) SKYRME MODEL

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ABSTRACT. We consider the hedgehog solutions in the (3 + 1)-dimensional Skyrmee model which is an energy-supercritical problem. We introduce a new strategy to prove global wellposedness for arbitrarily large initial data.

1. Introduction

In this paper we consider the (3+1)-dimensional Skyrme model in quantum field theory. This nonlinear sigma model was first proposed by Skyrme [24, 25, 26] to incorporate baryons as stable field configurations in the description of low energy interaction of pions. Let \( U : \mathbb{R}^{3+1} \rightarrow SU(2) \) be a map into the isospin group with signature (+−−−). Define the \( su(2) \)-valued connection one-form \( A \) by (below \( U^\dagger \) denotes the Hermitian adjoint)

\[
A = U^\dagger dU = A_\mu dx^\mu,
\]

where \( x^0 = t, (x^j)_{1 \leq j \leq 3} = x \in \mathbb{R}^3 \). The Lagrangian density of the classical Skyrme model is given by

\[
L = -\frac{1}{4} f_2^2 \pi \text{Tr}(A_\mu A^\mu) + \frac{1}{4} \epsilon^2 \pi \text{Tr}\left([A_\mu, A_\nu][A^\mu, A^\nu]\right),
\]

where \( f_2^2 \) is the pion decay constant, and \( \epsilon > 0 \) is a coupling parameter. The actual value of \( f_2^2 \) does not play much role in our mathematical analysis and we will conveniently set it to be 2. Here \([·, ·]\) is the usual Lie bracket on \( su(2) \) and \( \text{Tr}(·) \) denotes the matrix trace.

The Euler-Lagrangian equation of (1.1) takes the form

\[
\partial_\mu \left( A^\mu - \epsilon^2 [A_\nu, [A^\mu, A^\nu]] \right) = 0.
\]

Let \( I_2 \) be the identity matrix and \( \sigma_j, 1 \leq j \leq 3 \) be the Pauli spin matrices. Introducing the angular variable \( \omega = \omega(t, x) \) and the spin vector \( n = (n_j) \in S^2 \), we write the group element \( U \in SU(2) \) as

\[
U(t, x) = \exp \left( \frac{\omega(t, x)}{2i} \sigma_j n_j(t, x) \right)
= I_2 \cos \left( \frac{\omega(t, x)}{2} \right) - i \left( \sigma_j n_j(t, x) \right) \sin \left( \frac{\omega(t, x)}{2} \right).
\]

We shall be mainly concerned with a special family of solutions known as hedgehog solutions. Under the hedgehog ansatz, we set \( r = |x|, n_j(x) = \frac{x_j}{r} \) and

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\( \omega(t, x) = 2f(r, t) \), where \( f \) is the unknown radial function. We then obtain from (1.2)–(1.3),

\[
\left( 1 + \epsilon^2 \frac{2 \sin^2 f}{r^2} \right) \left( \partial_{tt} - \partial_{rr} - \frac{2}{r} \partial_r \right) f = -\epsilon^2 \frac{4 \sin^2 f}{r^3} \partial_r f - \epsilon^2 \frac{\sin(2f)}{r^2} \left( (\partial_t f)^2 - (\partial_r f)^2 \right) - \frac{\sin(2f)}{r^2} - \epsilon^2 \frac{\sin^2 f \cdot \sin(2f)}{r^4}.
\]

(1.4)

Introduce the notations

\[
\Delta_d = \partial_{rr} + \frac{d-1}{r} \partial_r
\]

and

\[
\square_d = \partial_{tt} - \Delta_d = \partial_{tt} - \partial_{rr} - \frac{d-1}{r} \partial_r.
\]

For radial functions on \( \mathbb{R}^d \), \( \Delta_d \) and \( \square_d \) are simply the usual Laplacian and D'Alembertian in polar coordinates. In our work, it will be useful to lift the function \( f(r) \) to a radial function in \( \mathbb{R}^d \) for some convenient choices of the dimension \( d \).

Using the above notation, we write (1.4) compactly as

\[
\left( 1 + \epsilon^2 \frac{2 \sin^2 f}{r^2} \right) \square_d f = -\epsilon^2 \frac{4 \sin^2 f}{r^3} \partial_r f - \epsilon^2 \frac{\sin(2f)}{r^2} \left( (\partial_t f)^2 - (\partial_r f)^2 \right) - \frac{\sin(2f)}{r^2} - \epsilon^2 \frac{\sin^2 f \cdot \sin(2f)}{r^4}.
\]

(1.5)

The boundary conditions for \( f \) are

\[
\lim_{r \to 0} f(t, r) = N_1 \pi, \quad \lim_{r \to \infty} f(t, r) = 0,
\]

where \( N_1 \geq 0 \) is an integer.

The main result of this paper, roughly speaking, is that for smooth and arbitrarily large initial data the corresponding solution to (1.5)–(1.6) exists globally in time. The precise formulation of the results will be given in Section 2. The basic conservation law associated with (1.5) is given by the Skyrme energy

\[
E(t) = \frac{1}{2} \int_0^\infty \left( 1 + \epsilon^2 \frac{2 \sin^2 f}{r^2} \right) \left( (\partial_t f)^2 + (\partial_r f)^2 \right) r^2 dr + \int_0^\infty \frac{\sin^2 f}{r^2} (1 + \epsilon^2 \frac{\sin^2 f}{2r^2}) r^2 dr = E_0, \quad \forall t > 0.
\]

(1.7)

With respect to the Skyrme energy conservation, the main difficulty associated with the analysis of (1.5) is that it is energy-supercritical and no useful theory is readily available for such problems. We shall introduce a new (and special) strategy to overcome this difficulty and prove global wellposedness for arbitrarily large initial data. As far as we know, this is the first unconditional result on a physical energy-supercritical problem.

We summarize below the main points of the proof.

**Main steps of the proof**
In our analysis the value of $\epsilon$ does not play much role and we will henceforth set $\epsilon = 1$ in (1.5) for convenience.

Step 1. Local (in time) analysis and lifting to dimension 5.

The first step is to get a good local theory. Observe that the nonlinearity on the RHS of (1.5) has strong singularities near $r = 0$ which can only be balanced out by a good local asymptotics of $f$ as $r \to 0$. To kill this singularity we introduce $g = g(r,t)$ by the relation

$$f(r,t) = \phi(r,t) + rg(r,t), \quad (1.8)$$

where $\phi$ is a smooth cut-off function such that $\phi(r) \equiv N_1\pi$ for $r \leq 1$. We then regard $g$ as a radial function on $\mathbb{R}^5$ and obtain from (1.4), (1.8) an equation for $g$ of the form

$$\Box g = N(r,g,\partial_t g, \nabla g), \quad (1.10)$$

where $N$ is a smooth nonlinearity and no longer contains any singularities near $r = 0$. Local wellposedness in $H^k_{rad}(\mathbb{R}^5)$ then follows from energy estimates. From the local analysis, to continue the solution to all time, we only need to control the quantity

$$G(t) = \|\langle x \rangle g(t,x)\|_{L^\infty_x(\mathbb{R}^5)} + \|\langle x \rangle (|\partial_t g| + |\nabla g|)\|_{L^\infty_x(\mathbb{R}^5)}. \quad (1.9)$$

We shall achieve this in several steps.

Step 2. A nonlocal transformation and derivation of the $\Phi$-equation.

The blowup/continuation criteria (1.9) is supercritical with respect to the Skyrme energy (1.7). To nail down global wellposedness, we analyze in a deeper way the structure of (1.5). For this purpose, we introduce a nonlocal transformation of the form (see Section 3 for more details)

$$\Phi(r,t) = \int_0^{g(r,t)} \left( 1 + \frac{2 \sin^2(ry + \phi(r))}{r^2} \right) dy + \frac{1}{r^3} \phi_{\geq 1}(r), \quad (1.10)$$

where $\phi_{\geq 1}$ is a smooth cut-off function localized to the regime $r \gtrsim 1$. Regard $\Phi$ as a radial function on $\mathbb{R}^5$. For $\Phi$ we then obtain from (1.3), (1.10) a nonlocal equation of the form

$$\Box g = \frac{1}{r^3} \phi_{\geq 1} - \frac{3}{2} \Phi + \frac{1}{2} \int_0^{g(r,t)} \left( 3B^{\frac{3}{2}} + B^{-\frac{9}{2}} - B^{-\frac{7}{2}} \right) dy, \quad (1.11)$$

where

$$B = 1 + \frac{2 \sin^2(ry + \phi(r))}{r^2}. \quad \text{(1.7)}$$

The remarkable feature of this new system is that at the cost of nonlocality all derivative terms on the RHS of (1.4) have been eliminated.

Step 3. Control of $H^1$-norm of $\Phi$ and a non-blowup argument.

This includes the estimates of $\|\Phi\|_{L^2_t(\mathbb{R}^5)}$, $\|\partial_t \Phi\|_{L^2_t(\mathbb{R}^5)}$ and $\|\nabla \Phi\|_{L^2_t(\mathbb{R}^5)}$. This is an important first step to beat energy supercriticality. Due to the particular structure in (1.10), it is not difficult to check that the Skyrme energy (1.7) is insufficient to give any control of $\|\nabla \Phi\|_{L^2_t(\mathbb{R}^5)}$ which is a manifestation of energy supercriticality.
at the lowest level. A heuristic analysis (see the beginning of Section 4) shows that in the worst case scenario the linear part of (1.11) could take the form

\[ \Box_5 \Phi = -\frac{3}{2} \Phi + \frac{3}{r^2} \Phi \]

which is a wave operator with negative inverse square potential. Since \( d = 5 \) and \( 3 > \frac{(d-2)^2}{4} \), we cannot use Strichartz (cf. [5]). To solve this problem we resort to a nonlinear approach which exploits the fine structure of the equation. Let \( T \) be the first possible blowup time. By performing estimates directly on (1.10)–(1.11), we obtain

\[ \int_{\mathbb{R}^5} \left( \frac{1}{2} |\nabla \Phi(t)|^2 - \phi_{< r_0}(r) \cdot H(r, t) \right) \, dx \leq C(T), \quad \forall \, 0 \leq t < T, \quad (1.12) \]

where \( 0 < C(T) < \infty \) is a constant depending on \( T, r_0 < \frac{1}{2} \) is a small constant, \( \phi_{< r_0} \) is a smooth cut-off function localized to \( r \leq r_0 \), and

\[ H(r, t) = \frac{3}{2} \int_0^{g(r,t)} \left( 1 + \frac{2 \sin^2(rw)}{r^2} \right)^{\frac{1}{2}} \cdot \left( \int_0^w \left( 1 + \frac{2 \sin^2(ry)}{r^2} \right)^{\frac{1}{2}} \cdot \frac{2 \sin^2(ry)}{r^2} \, dy \right) \, dw. \]

By a detailed analysis on \( H \), we show that \( H \) admits the sharp bound

\[ H(r, t) \leq \frac{9}{4} \cdot \frac{1}{2} \frac{|\Phi(r, t)|^2}{r^2}. \]

From this and (1.12), we get

\[ 0 \leq \int_{\mathbb{R}^5} \left( |\nabla \Phi(t)|^2 - \frac{9}{4} \cdot \frac{|\Phi(t)|^2}{r^2} \right) \, dx \leq C(T), \quad \forall \, 0 \leq t < T, \quad (1.13) \]

where the positivity of the integral follows from Hardy’s inequality (see Lemma 4.3) on \( \mathbb{R}^5 \). The estimate (1.13) is the sharpest available and yet it is not coercive enough to give control of \( H^1 \) norm of \( \Phi \). The main reason is that there could exist a sequence

\[ \| \nabla \Phi(t_n) \|_{L^2_x(\mathbb{R}^5)} \to +\infty, \quad \left\| \frac{\Phi(t_n)}{r} \right\|_{L^2_x(\mathbb{R}^5)} \to +\infty, \]

but

\[ \int_{\mathbb{R}^5} \left( |\nabla \Phi(t_n)|^2 - \frac{9}{4} \cdot \frac{|\Phi(t_n)|^2}{r^2} \right) \, dx \to C_1, \quad \text{as} \ t_n \to T, \]

where \( C_1 \geq 0 \) is a finite constant. To rule out this blowup scenario, we shall analyze in detail the special structure of \( \Phi \) and perform a delicate limiting and contradiction argument (see in particular (4.21)–(4.27) in the proof of Proposition 4.4). The technical details are contained in the proof of Proposition 4.4 and as a result we can control the \( H^1 \)-norm of \( \Phi \).

Step 4. Nonlinear energy bootstrap and higher order estimates.

In this final step we upgrade the \( H^1 \) estimate of \( \Phi \) to \( H^4 \) estimates which are sufficient to give a priori bound of the quantity \( G(t) \) defined in (1.9) (and yielding global wellposedness). The main task is to interweave the Sobolev estimates of \( g \) and \( \Phi \) back and forth a number of times using in an essential way the structure of the nonlocal system (1.10)–(1.11). The estimates are organized in such a way that we first obtain temporal regularity and then use the structure of the equation to
HEDGEHOG SOLUTIONS

trade temporal regularity for spatial regularity. The technical details are given in Section 5.

The above four steps complete our proof of global wellposedness. To put things into perspective, we briefly review below some results connected with the Skyrme model.

Connection with other works.

(1) Prior to this work, progress has been slow on understanding the global dynamics of the Skyrme model. In [33] Wong analyzed in detail the dominant energy condition and the breakdown of hyperbolicity for the Skyrme model (see also Gibbons [16], Grutchfield and Bell [6]). In particular it follows that a small perturbation of a static Skyrmion configuration yields local wellposedness. After our work is completed, the author learned that Geba, Nakanishi and Rajeev [15] proved a small data global wellposedness and scattering result for the Skyrme wave map for initial data in critical Besov space.

(2) In [9, 10], Geba and Rajeev considered a semilinear Skyrme model introduced by Adkins and Nappi [1]. The equivariant solutions satisfy the following

\[ \partial_{tt} f - \partial_{rr} f - \frac{2}{r} \partial_r f + \frac{\sin(2f)}{r^2} + \frac{(f - \sin f \cos f)(1 - \cos 2f)}{r^4} = 0 \]

and has conserved energy

\[ E(f(t)) = \int_0^\infty \left( \frac{1}{2} (\partial_t f)^2 + (\partial_r f)^2 \right) + \frac{\sin^2 f}{r^2} + \frac{(f - \sin f \cos f)^2}{2r^4} r^2 dr. \]

They proved that near the first possible blowup time, the energy does not concentrate. But the issue of global wellposedness is still open.

(3) If \( \epsilon = 0 \) in (1.5), then we recover the equivariant wave map from \( \mathbb{R}^{3+1} \) to \( S^3 \) which is also an energy-supercritical problem. Generally smooth solutions will blow up in finite time. Indeed Shatah [23] constructed finite-time blowup solutions which is self-similar and has finite energy. This was extended to other target manifolds in [27] and higher dimensions \( d \geq 4 \) in [7]. In [3] Bizoń constructed a countable family of spherically symmetric self-similar wave maps from the 3+1 Minkowski spacetime into the 3-sphere. These constructions all rely on the existence of a nontrivial harmonic map.

(4) The (2 + 1)-dimensional analogue of the Skyrme model is known as baby Skyrme models. The technique developed in this paper can also be used to prove global wellposedness of corresponding hedgehog solutions. The details will be given in a future publication. In contrast, the \( \epsilon = 0 \) limit of the baby Skyrme model gives rise to the (2 + 1)-dimensional energy-critical equivariant wave map

\[ \partial_{tt} f - \partial_{rr} f - \frac{\partial_r f}{r} + k^2 \frac{\sin(2f)}{2r^2} = 0, \]

where \( k \geq 1 \) is an integer giving the homotopy index. It is known that (cf. [8, 27, 29]) for smooth initial data with energy \( E < E(Q) \), where \( Q(r) = 2 \arctan(r^k) \), the corresponding solution is global. Also by an argument of Struwe there is no blowup of self-similar type. The existence (and dynamics) of finite-time blowup solutions were obtained in [22] (\( k \geq 4 \))
and [20] \((k = 1)\) using different techniques and giving different blowup rates. For results and some recent developments on energy-critical wave maps from \((2 + 1)\) Minkowski space to general target manifolds we refer to [19, 18, 32, 4, 17, 30, 31, 21] and references therein.

(5) The technique introduced in this paper has been recently generalised and extended to many other important physical models. In [11], Geba and Grillakis have improved and streamlined the result of this paper to Sobolev \(H^s, s > 7/2\). In [12] and [14], Creek, and Geba-Grillakis have obtained large data global regularity for the 2 + 1-dimensional equivariant Faddeev model. We refer to the monograph [13] for an extensive overview of more recent developments.

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2. Reformulation and main results

As was already mentioned, the value of \(\epsilon\) will not play much role in our analysis as long as \(\epsilon > 0\). In the rest of this paper we shall set \(\epsilon = 1\) in (1.5).

Denote
\[
A_1 = 1 + \frac{2 \sin^2 f}{r^2}.
\]

Then
\[
\Box_3 f = -\frac{1}{A_1} \cdot \frac{4 \sin^2 f}{r^2} \cdot \partial_r f - \frac{1}{A_1} \cdot \frac{\sin(2f)}{r^2} \cdot \left((\partial_t f)^2 - (\partial_r f)^2\right)
- \frac{1}{A_1} \cdot \frac{\sin(2f)}{r^2} - \frac{1}{A_1} \cdot \frac{\sin^2 f \cdot \sin(2f)}{r^4}
=: N(r, f, f'),
\]

with boundary condition (1.6).

Let \(\phi\) be a smooth cut-off function such that \(\phi(r) = N_1 \pi\) for \(r \leq 1\) and \(\phi(r) = 0\) for \(r \geq 2\).

Define \(g(r, t)\) by
\[
f(r, t) = \phi(r) + r g(r, t).
\]

No boundary condition is needed for \(g\) at \(r = 0\).

Note that
\[
\Box_3 f = -\Delta_3 \phi + \Box_3 (rg)
= -\Delta_3 \phi + r \Box_5 g - \frac{2}{r} g.
\]

\(^1\) We shall regard \(g\) as a radial function on \(\mathbb{R}^5\) and construct a classical solution \(g \in H^k(\mathbb{R}^5)\). By radial Sobolev embedding, \(|g(r, t)| \lesssim r^{-2}\) as \(r \to \infty\). Hence the boundary condition \(f(\infty, t) = 0\) causes no trouble either.
By (2.1), (2.2), (2.3), the equation for \( g \) then takes the form

\[
\Box_5 g = \frac{2}{r^2} g + \frac{1}{r} \Delta_3 \phi + \frac{1}{r} \phi_{<1} \cdot N(r, rg, (rg)') \\
+ \frac{1}{r} \phi_{>1} \cdot N(r, \phi + rg, (\phi + rg)'),
\]

where \( \phi_{>1} = 1 - \phi_{<1} \), and \( \phi_{<1} \) is a smooth cut-off function such that \( \phi_{<1}(r) = 1 \) for \( r < \frac{1}{2} \); \( \phi_{<1}(r) = 0 \) for \( r \geq 1 \).

In more detail,

\[
\Box_5 g = \frac{\phi_{<1}}{1 + F_0(rg)^2} \left( \tilde{F}_1(rg) g^3 + \tilde{F}_2(rg) g^5 \\
- \tilde{F}_3(rg) \cdot g \cdot ((\partial_t g)^2 - (\partial_r g)^2) \\
+ \tilde{F}_4(rg) \cdot g^4 \cdot r \partial_r g \\
+ \phi_{>1} \cdot \frac{2}{r^2} g + \frac{1}{r} \Delta_3 \phi \\
+ \frac{1}{r} \phi_{>1} \cdot N(r, \phi + rg, (\phi + rg)'),
\]

where

\[
\tilde{F}_0(x) = 2 \left( \frac{\sin x}{x} \right)^2, \\
\tilde{F}_1(x) = \frac{2}{x^2} - \frac{\sin(2x)}{x^3}, \\
\tilde{F}_2(x) = \frac{\sin(2x)}{x^3} - \frac{\sin^2 x \sin(2x)}{x^5}, \\
\tilde{F}_3(x) = \frac{\sin(2x)}{x}, \\
\tilde{F}_4(x) = -\frac{4 \sin^2 x}{x^4} + \frac{2 \sin(2x)}{x^3}.
\]

It is not difficult to check that \( \tilde{F}_i(x) \), \( 0 \leq i \leq 4 \) are well-defined for all \( x \in \mathbb{R} \) with the help of power series expansion. Observe that the functions \( \tilde{F}_i \) can all be written as

\[
\tilde{F}_i(x) = F_i(x^2), \quad i = 0, \cdots, 4,
\]

where \( F_i \) are smooth functions satisfying

\[
\left\| \frac{d^k}{dx^k} F_i(x) \right\|_{L^\infty} \leq C_k, \quad \forall k \geq 0,
\]

where \( C_k \) are constants depending only on \( k \).

The reason that we write \( \tilde{F}_i(rg) = F_i(r^2 g^2) \) is that we shall regard \( F_i(r^2 g^2) = F_i(|x|^2 g^2) \) for \( x \in \mathbb{R}^5 \) which is smooth in \( x \). This will help local energy estimates in the local theory.

Now we lift \( g \) to be radial function on \( \mathbb{R}^5 \), clearly then

\[
r \partial_r g = \sum_{i=1}^5 x_i \cdot \partial_{x_i} g = x \cdot \nabla g.
\]
Thus we rewrite (2.5) as
\[ \Box g = \frac{\phi_{<1}}{1 + F_0(r^2 g^2)} \left( F_1(r^2 g^2) g^4 + F_2(r^2 g^2) g^5 \right. \\
- F_3(r^2 g^2) \cdot g \cdot ((\partial_t g)^2 - (\nabla g)^2) \\
+ F_4(r^2 g^2) \cdot g^4 \cdot (x \cdot \nabla g) \\
\left. + \phi_{>1} \cdot \frac{2}{r^2} g + \frac{1}{r} \Delta_3 \phi \\
+ \frac{1}{r} \phi_{>1} \cdot N(r, \phi + rg, (\phi + rg)') \right). \] (2.7)

For any integer \( k \), we shall denote by \( H^k_{\text{rad}}(\mathbb{R}^5) \) the usual \( H^k \) Sobolev space restricted to radial functions on \( \mathbb{R}^5 \).

**Proposition 2.1** (Local wellposedness and continuation criteria). Let \( k > \frac{5}{2} + 1 \) be an integer. Assume
\[ (g, \partial_t g) \bigg|_{t=0} = (g_0, g_1) \in H^k_{\text{rad}}(\mathbb{R}^5) \times H^{k-1}_{\text{rad}}(\mathbb{R}^5). \]
Then there exists \( T > 0 \) and a local solution \( g \in C([0, T), H^k_{\text{rad}}(\mathbb{R}^5)) \cap C^1([0, T), H^{k-1}_{\text{rad}}(\mathbb{R}^5)) \) to (2.7). Furthermore the solution can be continued past any \( T_1 \geq T \) as long as
\[ \sup_{0 \leq t < T_1} G(t) < \infty, \] (2.8)
where
\[ G(t) = \| \langle x \rangle g(t) \|_{L^\infty(\mathbb{R}^5)} + \| \langle x \rangle (|\partial_t g| + |\nabla g|) \|_{L^\infty(\mathbb{R}^5)}. \] (2.9)

The proof of Proposition 2.1 uses standard energy estimates and will be omitted here. Our main result is

**Theorem 2.2** (Global wellposedness for large data). Let \( k \geq 4 \) be an integer and assume
\[ (g, \partial_t g) \bigg|_{t=0} = (g_0, g_1) \in H^k_{\text{rad}}(\mathbb{R}^5) \times H^{k-1}_{\text{rad}}(\mathbb{R}^5). \]
Then the corresponding solution in Proposition 2.1 is global.

By Proposition 2.1, the proof of Theorem 2.2 reduces to showing that (2.8) holds for any \( T > 0 \). We shall achieve this by devising a new nonlinear energy bootstrap method.

3. **Nonlinear energy bootstrap: preliminary transformations**

Recall that (2.1) has the basic energy conservation
\[ E(t) = \frac{1}{2} \int_0^\infty \left( 1 + \frac{2 \sin^2 f}{r^2} \right) \left( (\partial_t f)^2 + (\partial_r f)^2 \right) r^2 dr \]
\[ + \int_0^\infty \frac{\sin^2 f}{r^2} \left( 1 + \frac{\sin^2 f}{2r^2} \right) r^2 dr \]
\[ = E_0, \quad \forall t > 0. \] (3.1)

The continuation criteria (2.8) is supercritical with respect to this basic energy conservation. To prove global wellposedness of (2.1) one certainly needs a new
strategy. In this section we explain the setup of our nonlinear energy bootstrap argument.

Define \( \hat{\Phi}_1 : (0, \infty) \times \mathbb{R} \to \mathbb{R} \) by

\[
\hat{\Phi}_1(\rho, z) = \int_{N_1 \pi}^{\rho} \left( 1 + \frac{2 \sin^2 y}{\rho^2} \right)^{\frac{1}{2}} dy.
\] (3.2)

The definition of \( \hat{\Phi}_1 \) takes into consideration of the boundary condition especially when \( N_1 \neq 0 \).

Define \( \Phi_1(r, t) = \hat{\Phi}_1(r, f(r, t)). \) (3.3)

Then \( \Box_3 \Phi_1 = (\partial_{zz} \hat{\Phi}_1)(r, f(r, t)) \left( (\partial_r f)^2 - (\partial_r f')^2 \right) + (\partial_{zz} \hat{\Phi}_1)(r, f(r, t)) \Box_3 f \)

\[ - \left( \Delta_{3, \rho} \hat{\Phi}_1 \right)(r, f(r, t)) - 2(\partial_{\rho} \partial_r \hat{\Phi}_1)(r, f(r, t)) \partial_r f. \] (3.4)

Here \( \Delta_{3, \rho} \) is the three-dimensional radial Laplacian in the \( \rho \) variable, i.e.

\[
(\Delta_{3, \rho} \hat{\Phi}_1)(\rho, z) = (\partial^2_{\rho} \hat{\Phi}_1)(\rho, z) + \frac{2}{\rho} (\partial_{\rho} \hat{\Phi}_1)(\rho, z).
\]

Recall \( A_1 = 1 + \frac{2 \sin^2 f}{r^2} \).

Easy to check that

\[
(\partial_{zz} \hat{\Phi}_1)(r, f(r, t)) + (\partial_{z} \hat{\Phi}_1)(r, f(r, t)) \cdot \left( -\frac{1}{A_1} \right) \cdot \frac{\sin(2f)}{r^2} = 0,
\]

\[
-2(\partial_{\rho} \partial_z \hat{\Phi}_1)(r, f(r, t)) - \frac{(\partial_{zz} \hat{\Phi}_1)(r, f(r, t))}{A_1} \cdot \frac{4 \sin^2 f}{r^3} = 0. \] (3.5)

Therefore by (3.4), (2.1) and (3.5), we get

\[
\Box_3 \Phi_1 = -A_1^{-\frac{1}{2}} \cdot \frac{\sin(2f)}{r^2} - A_1^{-\frac{1}{2}} \cdot \frac{\sin^2 f \cdot \sin(2f)}{r^4} - (\Delta_{3, \rho} \hat{\Phi}_1)(r, f(r, t)). \] (3.6)

Denote \( B_1 = 1 + \frac{2 \sin^2 y}{r^2} \). (3.7)

By a simple computation,

\[
\Delta_{3, \rho}(B_1^2) = \frac{1}{r^2} \left( B_1^{-\frac{1}{2}} - B_1^{-\frac{3}{2}} \right).
\]

Hence

\[
(\Delta_{3, \rho} \hat{\Phi}_1)(r, f(r, t)) = \frac{1}{r^2} \int_{N_1 \pi}^{f(r, t)} \left( B_1^{-\frac{1}{2}} - B_1^{-\frac{3}{2}} \right) dy. \] (3.8)

By a tedious calculation, we have

\[
A_1^{-\frac{1}{2}} \cdot \frac{\sin(2f)}{r^2} = \frac{1}{r^2} \int_{N_1 \pi}^{f(r, t)} \partial_y \left( B_1^{-\frac{1}{2}} \cdot \sin(2y) \right) dy
\]

\[ = \frac{1}{r^2} \int_{N_1 \pi}^{f(r, t)} B_1^{-\frac{1}{2}} \left( 2 - r^2(B_1^2 - 1) \right) dy. \] (3.9)
Similarly
\[
A_1^{\frac{1}{2}} \cdot \frac{\sin^2 f \cdot \sin(2f)}{r^4} = \frac{1}{r^2} \int_{N_{1\pi}}^{f(r,t)} \left( 2B_1^2 - B_1^{-\frac{3}{2}} - B_1^{-\frac{1}{2}} \right) dy + \frac{1}{2} \int_{N_{1\pi}}^{f(r,t)} B_1^{-\frac{7}{2}}(-3B_1^3 + 5B_1^2 - B_1 - 1) dy. \tag{3.10}
\]

Plugging (3.8), (3.9) and (3.10) into (3.6), we obtain
\[
\Box_3 \Phi_1 = -\frac{2}{r^2} \Phi_1 + \frac{1}{2} \int_{N_{1\pi}}^{f(r,t)} \left( 3B_1^2 - 3B_1^2 + B_1^{-\frac{1}{2}} - B_1^{-\frac{3}{2}} \right) dy. \tag{3.11}
\]

Equation (3.11) is still not very satisfactory since it contains terms of inverse square potential type. To remove such terms, one more transformation is needed.

Define \( \Phi_2(r,t) \) by
\[
\Phi_1(r,t) = r \Phi_2(r,t). \tag{3.12}
\]

Then
\[
\Box_5 \Phi_1 = \Box_5 (r \Phi_2) = r \Box_5 \Phi_2 - \frac{2}{r^2} \Phi_1. \tag{3.13}
\]

By (3.13), equation (3.11) expressed in the \( \Phi_2 \) variable now takes the form
\[
\Box_5 \Phi_2 = -\frac{3}{2} \Phi_2 + \frac{1}{2} \int_{N_{1\pi}}^{f(r,t)} \left( 3B_1^2 + B_1^{-\frac{1}{2}} - B_1^{-\frac{3}{2}} \right) dy. \tag{3.14}
\]

Although formally the RHS of (3.14) still contains \( 1/r \) terms which may be singular when \( r \to 0 \), it actually causes no trouble in our energy bootstrap estimates later. To see this, we bring back the \( g \)-function used in the local analysis.

Recall that
\[
f(r,t) = \phi(r) + rg(r,t), \tag{3.15}
\]
where \( \phi(r) \equiv N_1 \pi \) for \( r < 1 \) and \( \phi(r) = 0 \) for \( r \geq 2 \).

Define
\[
B_2 = 1 + \frac{2 \sin^2 (ry)}{r^2}. \tag{3.16}
\]

Observe that \( B_2 \) is a smooth function (see the discussion preceding the estimate (2.0)).

Let \( \phi_{<1} \) be a smooth cut-off function such that \( \phi_{<1}(r) = 1 \) for \( r \leq \frac{1}{2} \) and \( \phi_{<1}(r) = 0 \) for \( r > 1 \). By (3.15), (3.16), we have
\[
\frac{1}{2r} \phi_{<1}(r) \int_{N_{1\pi}}^{f(r,t)} \left( 3B_1^2 + B_1^{-\frac{1}{2}} - B_1^{-\frac{3}{2}} \right) dy = \frac{1}{2r} \phi_{<1}(r) \int_{N_{1\pi}}^{N_{1\pi} + rg(r,t)} \left( 3B_1^2 + B_1^{-\frac{1}{2}} - B_1^{-\frac{3}{2}} \right) dy = \frac{1}{2} \phi_{<1}(r) \int_0^{g(r,t)} \left( 3B_2^2 + B_2^{-\frac{1}{2}} - B_2^{-\frac{3}{2}} \right) dy. \tag{3.17}
\]

In the second equality above, we have performed a change of variable \( y \to N_{1\pi} + ry \). Clearly (3.17) is smooth as long as \( g \) is smooth since it has no singular terms in \( r \).
By using (3.17), we rewrite (3.14) as
\[ \square_3 \Phi_2 = -\frac{3}{2} \Phi_2 + \frac{1}{2} \phi_{<1} \int_{0}^{g(r,t)} \left( 3B_2^2 + B_2^2 - B_2^{-2} \right) dy \]
\[ + \frac{1}{2r} \phi_{>1} \int_{N_1 \pi}^{f(r,t)} \left( 3B_1^3 + B_1^{-\frac{3}{2}} - B_1^{-\frac{3}{2}} \right) dy, \] (3.18)
where \( \phi_{>1} = 1 - \phi_{<1} \) is localized to \( r \gtrsim 1 \).

Equation (3.18) is almost good for us since it no longer contains any derivative terms or singularities in \( r \). However there is one more problem.

By (3.2), (3.3), (3.12), we have
\[ \Phi_2(r,t) = \frac{1}{r} \int_{N_1 \pi}^{f(r,t)} \left( 1 + 2 \sin^2 \left( \frac{y}{2} \right) \right) dy. \] (3.19)

By (3.18) it is not difficult to check that \( \Phi_2 \) has no singularity near \( r \sim 0 \).

However for \( r \geq 2 \) by using energy conservation (3.1) and radial Sobolev embedding, we get \( |f(r,t)| \lesssim r^{-1} \). If \( N_1 > 0 \), then (3.19) asserts that
\[ \Phi_2(r,t) \sim \frac{\text{Const}}{r}, \] as \( r \to \infty \).

In particular \( \Phi_2 \notin L^2_2(\mathbb{R}^5) \) when we regard \( \Phi_2 \) as a radial function on \( \mathbb{R}^5 \). We therefore need to introduce one more transformation to kill this divergence.

To this end, we define
\[ \Phi(r,t) = \Phi_2(r,t) + \frac{1}{3} \phi_{>1} \cdot \frac{1}{r} \int_{N_1 \pi}^{f(r,t)} \left( 3B_1^3 + B_1^{-\frac{3}{2}} - B_1^{-\frac{3}{2}} \right) dy \] (3.20)
\[ = \frac{1}{r} \phi_{<1} \int_{N_1 \pi}^{f(r,t)} B_1^\frac{3}{2} dy \] (3.21)
\[ + \frac{1}{r} \phi_{>1} \int_{N_1 \pi}^{f(r,t)} B_1^\frac{3}{2} dy \] (3.22)
\[ + \frac{1}{r} \phi_{>1} \int_{N_1 \pi}^{f(r,t)} \left( B_1^3 - B_1^\frac{3}{2} + \frac{1}{3} B_1^{-\frac{3}{2}} - \frac{1}{3} B_1^{-\frac{3}{2}} \right) dy. \] (3.23)

Since \( \phi(r) \equiv N_1 \pi \) for \( r < 1 \), by (3.18), we have
\[ \text{(3.21)} = \phi_{<1} \int_{0}^{g(r,t)} \left( 1 + 2 \sin^2 \left( \frac{y}{2} \right) \right) dy. \] (3.24)

For (3.22), we have
\[ \text{(3.22)} = \phi_{>1} \int_{0}^{g(r,t)} \left( 1 + 2 \sin^2 \left( \frac{y + \phi(r)}{2} \right) \right) dy + \frac{1}{r} \phi_{>1} \int_{0}^{\phi(r)} B_1^\frac{3}{2} dy. \] (3.25)

Note that \( \phi(r) = 0 \) for \( r \geq 2 \), therefore we can write
\[ \frac{1}{r} \phi_{>1} \int_{0}^{\phi(r)} B_1^\frac{3}{2} dy = \phi_{<1}(r), \] (3.26)
where \( \phi_{<1} \) is a smooth cut-off function localized to \( r \sim 1 \).

For (3.24), observe that by (3.14)
\[ B_1^3 - B_1^\frac{3}{2} = O \left( \frac{1}{r^2} \right), \quad r \gtrsim 1, \]
and similarly
\[ \frac{1}{3}B_1^{-\frac{3}{2}} - \frac{1}{3}B_1^{-\frac{1}{2}} = O\left(\frac{1}{r^2}\right), \quad r \gtrsim 1. \]

Therefore we shall write
\[ (3.23) = \frac{1}{r^3}\phi_{\geq 1}(r), \quad (3.27) \]
where \( \phi_{\geq 1}(r) \) is a smooth cut-off function localized to \( r \gtrsim 1 \) and can vary from place to place.

By using (3.21)–(3.27), we obtain
\[
\Phi(r,t) = \int_0^{g(r,t)} \left(1 + \frac{2\sin^2(ry + \phi(r))}{r^2}\right)^{\frac{1}{2}} dy + \phi_{\sim 1} + \frac{1}{r^3}\phi_{\geq 1}(r).
\]

We can further include \( \phi_{\sim 1}(r) \) into \( \phi_{\geq 1}(r) \) and simply write
\[
\phi_{\sim 1}(r) + \frac{1}{r^3}\phi_{\geq 1}(r) = \frac{1}{r^3}\phi_{\geq 1}(r).
\]

Then
\[
\Phi(r,t) = \int_0^{g(r,t)} \left(1 + \frac{2\sin^2(ry + \phi(r))}{r^2}\right)^{\frac{1}{2}} dy + \frac{1}{r^3}\phi_{\geq 1}(r). \quad (3.28)
\]

On the other hand by (3.20) and a simple computation, we have
\[
\Phi(r,t) = \Phi_2(r,t) + \frac{1}{r^3}\cdot\phi_{\geq 1}(r). \quad (3.29)
\]

Plugging (3.29) into (3.18) and using (3.20), we get
\[
\Box_5 \Phi = \frac{1}{r^3} \cdot \phi_{\geq 1} - \frac{3}{2} \Phi + \frac{1}{2} \int_0^{g(r,t)} \left(3B_2^{\frac{3}{2}} + B_2^{-\frac{1}{2}} - B_2^{-\frac{3}{2}}\right) dy
\]
\[
+ \frac{1}{2} \cdot \phi_{\geq 1} \cdot \frac{1}{r} \int_0^{f(r,t)} \left(3B_1^{\frac{3}{2}} + B_1^{-\frac{1}{2}} - B_1^{-\frac{3}{2}}\right) dy. \quad (3.30)
\]

By using an argument similar to the derivation of (3.28), we further simplify (3.30) as
\[
\Box_5 \Phi = \frac{1}{r^3} \phi_{\geq 1} - \frac{3}{2} \Phi + \frac{1}{2} \int_0^{g(r,t)} \left(3B^\frac{3}{2} + B^{-\frac{1}{2}} - B^{-\frac{3}{2}}\right) dy, \quad (3.31)
\]
where
\[
B = 1 + \frac{2\sin^2(ry + \phi(r))}{r^2}. \quad (3.32)
\]

Formula (3.23) then takes the form
\[
\Phi(r,t) = \int_0^{g(r,t)} B^\frac{3}{2} dy + \frac{1}{r^3}\phi_{\geq 1}(r). \quad (3.33)
\]

We analyze (3.31), (3.33) in the next section.
4. Non-blowup of $H^1$-norm of $\Phi$

The first step in our analysis is to control the $H^1$-norm of $\Phi$. This includes $\|\nabla \Phi\|_{L^2(\mathbb{R}^5)}$, $\|\partial_t \Phi\|_{L^2(\mathbb{R}^5)}$ and $\|\nabla \Phi\|_{L^2(\mathbb{R}^5)}$. By (3.33), (2.2), we have

$$\partial_t \Phi = \frac{1}{r} \cdot \partial_t f \cdot \left(1 + \frac{2 \sin^2 f}{r^2}\right)^{\frac{1}{2}},$$

and therefore by (3.1), we get

$$\|\partial_t \Phi\|_{L^2(\mathbb{R}^5)} \lesssim 1.$$ (4.1)

By (3.32), (3.33), easy to see

$$|\Phi(r,t)| \lesssim |g(r,t)| + |g(r,t)|^2 + \frac{1}{r^3} |\phi_{\geq 1}(r)|.$$ (4.2)

By the assumption of Proposition 2.1 and Sobolev embedding, we have $\|g(0)\|_{L^4(\mathbb{R}^3)} \lesssim 1$. By (4.2), this gives $\|\Phi(0)\|_{L^2(\mathbb{R}^5)} \lesssim 1$. Using (4.1), we then have

$$\|\Phi(t)\|_{L^2(\mathbb{R}^5)} \leq \text{Const} \cdot t, \quad \forall t > 0.$$ (4.3)

By (3.1), we have

$$\|\partial_t f\|_{L^2(\mathbb{R}^3)} + \|\partial_r f\|_{L^2(\mathbb{R}^3)} \lesssim 1.$$ (4.4)

Since $\|f(0)\|_{L^2(\mathbb{R}^3)} \lesssim 1$, we get

$$\|f(t)\|_{H^1(\mathbb{R}^3)} \leq \text{Const} \cdot t, \quad \forall t > 0.$$ (4.5)

By (2.2) and Hardy’s inequality (see (4.13)), we get

$$\|g(t)\|_{H^1(\mathbb{R}^3)} \leq \text{Const} \cdot t, \quad \forall t > 0.$$ (4.6)

However it is not difficult to check that (3.1) and (4.5) are insufficient to bound $\|\nabla \Phi\|_{L^2(\mathbb{R}^5)}$.

One may try to do Strichartz. But there is one problem as we now explain.

Imagine that

$$g(r,t) \sim \frac{1}{r},$$

for a range of values of $r \ll 1$.

Then by (3.33),

$$\Phi(r,t) \sim \frac{\sqrt{2}}{r} g(r,t),$$

and

$$\frac{3}{2} \int_0^{g(r,t)} B \, dy \sim \frac{3\sqrt{2}}{r^3} g(r,t) \sim \frac{3}{r^2} \Phi(r,t).$$

Therefore for a range of values of $r \ll 1$, the linear part of (3.31) takes the form

$$\Box_5 \Phi = \frac{3}{2} \Phi + \frac{3}{r^2} \Phi.$$ (4.7)

Certainly (4.6) cannot hold for all $r \to 0$ since $g$ is assumed to be regular at $r = 0$. 

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2Certainly (4.6) cannot hold for all $r \to 0$ since $g$ is assumed to be regular at $r = 0$. 

Equation (4.7) is a wave operator with negative inverse square potential. Since $d = 5$ and
\[3 > \frac{(d-2)^2}{4},\]
o no Strichartz is available (cf. [5]). This destroys the hope of employing good linear estimates.

Therefore a new idea is required to establish $H^1$-norm bound of $\Phi$. In particular we shall use a nonlinear approach which exploits in an essential way the structure of the equation.

**Lemma 4.1.** There exists $r_0 > 0$ sufficiently small, such that for any $0 \leq r \leq r_0$, we have
\[F(\beta) = \int_0^\beta (r^2 + 2 \sin^2 y)^{\frac{1}{2}} \left(\frac{3}{4} - \sin^2 y\right) dy \geq 0, \quad \forall \beta \geq 0.\]
If $r > 0$, then the equality holds iff $\beta = 0$.

**Proof of Lemma 4.1.** By a simple calculation, we have
\[\int_0^\pi (\sin y) \cdot \left(\frac{3}{4} - \sin^2 y\right) dy = \frac{1}{6}.\]
Clearly there exists $r_1 > 0$ sufficiently small such that
\[\int_0^\pi (r^2 + 2 \sin^2 y)^{\frac{1}{2}} \left(\frac{3}{4} - \sin^2 y\right) dy \geq \frac{1}{12}, \quad \forall 0 \leq r < r_1. \tag{4.8}\]

Consider $m\pi \leq \beta < (m+1)\pi$ and $m$ is large. Then by (4.8), for $0 \leq r < r_1$, we have
\[\int_0^\beta (r^2 + 2 \sin^2 y)^{\frac{1}{2}} \left(\frac{3}{4} - \sin^2 y\right) dy \geq \frac{m}{12} + \int_{m\pi}^\beta (r^2 + 2 \sin^2 y)^{\frac{1}{2}} \left(\frac{3}{4} - \sin^2 y\right) dy \geq \frac{m}{12} - O(1) > \frac{1}{12},\]
if $m$ is taken to be sufficiently large.

Therefore we only need to consider $F(\beta)$ on a compact interval $[0, m\pi]$.

Observe that $F(0) = 0$, $F(m\pi) > \frac{1}{12}$. It suffices to consider critical points of $F$ in $(0, m\pi)$ and prove the positivity of $F$ at these points.

Solving $F'(\beta) = 0$ yields
\[\sin(\beta) = \pm \frac{\sqrt{3}}{2}.\]
Hence
\[\beta = j\pi + \frac{\pi}{3} \quad \text{or} \quad j\pi + \frac{2\pi}{3}, \quad j \geq 0, \; j \in \mathbb{Z}.\]
If $\beta = j\pi + \frac{\pi}{3}$, then for $0 \leq r < r_1$, by (4.8),

$$F(j\pi + \frac{\pi}{3}) = \int_{0}^{\frac{\pi}{3}} (r^2 + 2\sin^2 y)\left(\frac{3}{4} - \sin^2 y\right)dy \geq \frac{1}{12} + \int_{0}^{\frac{\pi}{3}} (r^2 + 2\sin^2 y)\left(\frac{3}{4} - \sin^2 y\right)dy > 0.$$ 

If $\beta = j\pi + \frac{2\pi}{3}$, then for $0 \leq r < r_1$,

$$F(j\pi + \frac{2\pi}{3}) \geq \frac{1}{12} + \int_{0}^{\frac{2\pi}{3}} (r^2 + 2\sin^2 y)\left(\frac{3}{4} - \sin^2 y\right)dy \geq \int_{0}^{\frac{2\pi}{3}} (r^2 + 2\sin^2 y)\left(\frac{3}{4} - \sin^2 y\right)dy.$$ 

Define

$$\tilde{F}(\rho) = \int_{0}^{\frac{2\pi}{3}} (\rho + \sin^2 y)\left(\frac{3}{4} - \sin^2 y\right)dy.$$ 

Easy to check $\tilde{F}(0) = 0$. 

On the other hand,

$$\tilde{F}(\rho) - \tilde{F}(0) = \int_{0}^{\frac{2\pi}{3}} \frac{1}{\sqrt{\rho + \sin^2 y + \sqrt{\sin^2 y}}} \cdot \left(\frac{3}{4} - \sin^2 y\right)dy > 0,$$ 

for $\rho$ sufficiently small.

Hence $F(j\pi + \frac{2\pi}{3}) > 0$ for $0 < r \leq r_0$, where $r_0$ is sufficiently small. \hfill \Box

Define $G_i : (0, \infty) \times \mathbb{R} \to \mathbb{R}$, $i = 0, 1, 2$ by

$$G_0(r, w) = \int_{0}^{w} \left(1 + \frac{2\sin^2 (ry)}{r^2}\right)\frac{1}{2} \cdot \frac{2\sin^2 (ry)}{r^2}dy,$$  

(4.9)

$$G_1(r, z) = \frac{3}{2} \int_{0}^{z} G_0(r, w)\left(1 + \frac{2\sin^2 (rw)}{r^2}\right)^\frac{1}{2}dw,$$  

(4.10)

$$G_2(r, w) = \int_{0}^{w} \left(1 + \frac{2\sin^2 (ry)}{r^2}\right)^\frac{1}{2}dy.$$  

(4.11)

**Corollary 4.2.** For any $0 < r \leq r_0$, $z \in \mathbb{R}$, we have

$$\left|G_1(r, z)\right| \leq \frac{9}{4} \cdot \frac{1}{2} \cdot \frac{(G_2(r, z))^2}{r^2}.$$  

(4.12)

**Proof of Corollary 4.2.** Since $G_1(r, z)$ is an even function of $z$, it suffices to consider the case $z > 0$. By Lemma 4.1 for $w \geq 0$,

$$0 \leq \frac{3}{2}G_0(r, w) \leq \frac{9}{4} \cdot \frac{1}{r^2} \cdot G_2(r, w).$$
Therefore
\[
0 \leq G_1(r,z) \leq \frac{9}{4} \cdot \frac{1}{r^2} \int_0^z G_2(r,w) \cdot \left(1 + \frac{2\sin^2(rw)}{r^2}\right)^{-\frac{1}{2}} dw
\]
\[
= \frac{9}{4} \cdot \frac{1}{r^2} \int_0^z G_2(r,w) \cdot (\partial_w G_2)(r,w) dw
\]
\[
= \frac{9}{4} \cdot \frac{1}{r^2} \cdot \left(\frac{G_2(r,z)}{r^2}\right)^2.
\]

\[\Box\]

**Lemma 4.3 (Hardy’s inequality).** Let \(d \geq 3\). Then
\[
\int_{\mathbb{R}^d} \frac{f^2}{|x|^2} dx \leq \frac{4}{(d-2)^2} \int_{\mathbb{R}^d} |\nabla f|^2 dx, \quad \forall f \in C_0^\infty(\mathbb{R}^d).
\]

The constant \(\frac{4}{(d-2)^2}\) is sharp.

The goal of this section is to prove the following

**Proposition 4.4 (Non-blowup of \(H^1\) norm of \(\Phi\)).** Let \(T > 0\) be the maximal lifespan of the local solution \(g\) constructed in Proposition 2.1. If \(T < \infty\), then
\[
\sup_{0 \leq t < T} \left(\|\Phi(t)\|_{H^1(\mathbb{R}^5)} + \|\partial_t \Phi(t)\|_{L^2_{x}(\mathbb{R}^5)}\right) < \infty.
\]

Before we begin the proof of Proposition 4.4, we set up some notations.

**Notation.** Throughout the rest of this paper, unless explicitly mentioned, we shall suppress the dependence of constants on the initial data or on the time \(T\). For example we shall write (4.14) simply as
\[
\|\Phi(t)\|_{H^1(\mathbb{R}^5)} + \|\partial_t \Phi(t)\|_{L^2_{x}(\mathbb{R}^5)} \lesssim 1, \quad \forall 0 \leq t < T.
\]

**Proof of Proposition 4.4.** By (4.1), (4.3), we only need to show
\[
\|\nabla \Phi(t)\|_{L^2_{x}(\mathbb{R}^5)} \lesssim 1, \quad \forall 0 \leq t < T.
\]

Let \(\psi \in C_c^\infty(\mathbb{R}^5), 0 \leq \psi \leq 1\) be a radial smooth cut-off function such that \(\psi(x) = 1\) for \(|x| \leq \frac{1}{2}\) and \(\psi(x) = 0\) for \(|x| \geq 1\). Choose \(r_0 \leq \frac{1}{2}\) as in Lemma 4.1 and define
\[
\phi_{<r_0}(x) = \psi\left(\frac{x}{r_0}\right),
\]
\[
\phi_{>r_0}(x) = 1 - \phi_{<r_0}(x).
\]

By (3.31)–(3.33), we have
\[
\Box_5 \Phi = \frac{1}{r^3} \phi_{\geq 1} + \frac{1}{2} \int_0^{g(r,t)} \left(B^{-\frac{7}{2}} - B^{-\frac{9}{2}}\right) dy + \frac{3}{2} \int_0^{g(r,t)} \left(B^{\frac{5}{2}} - B^{\frac{7}{2}}\right) dy
\]
\[
= \frac{1}{r^3} \phi_{\geq 1} + \frac{1}{2} \int_0^{g(r,t)} \left(B^{-\frac{7}{2}} - B^{-\frac{9}{2}}\right) dy + \frac{3}{2} \phi_{>r_0} \int_0^{g(r,t)} \left(B^{\frac{5}{2}} - B^{\frac{7}{2}}\right) dy
\]
\[
+ \frac{3}{2} \phi_{<r_0} \int_0^{g(r,t)} \left(1 + \frac{2\sin^2(ry)}{r^2}\right)^{-\frac{1}{2}} \cdot \frac{2\sin^2(ry)}{r^2} dy.
\]

(4.15)
Multiplying both sides of (4.15) by $\partial_t \Phi$ and integrating by parts, we obtain
\[\frac{d}{dt} \int_{\mathbb{R}^5} \left( \frac{1}{2} \partial_t \Phi^2 + \frac{1}{2} |\nabla \Phi|^2 - \phi_{<r_0}(x) \cdot G_1(r, g(r, t)) \right) dx \lesssim \|\partial_t \Phi\|_{L^2_x(\mathbb{R}^5)} \cdot \left( 1 + \|g(t)\|_{L^2_x(\mathbb{R}^5)} \right).\]  
(4.16)

Plugging (4.1), (4.5) into (4.16) and integrating in time, we get
\[\sup_{0 \leq t < T} \int_{\mathbb{R}^5} \left( \frac{1}{2} \partial_t \Phi^2 + \frac{1}{2} |\nabla \Phi|^2 - \phi_{<r_0}(x) \cdot G_1(r, g(r, t)) \right) dx \lesssim 1.\]  
(4.17)

In particular, this yields
\[\int_{\mathbb{R}^5} \left( |\nabla \Phi(t)|^2 - \frac{9}{4} \cdot \frac{\Phi(t)^2}{r^2} \right) dx \lesssim 1, \quad \forall 0 \leq t < T.\]  
(4.18)

Using Corollary 4.2 and (4.3), we get
\[\int_{\mathbb{R}^5} \left( |\nabla \Phi(t)|^2 - \frac{9}{4} \cdot \frac{\Phi(t)^2}{r^2} \right) dx \leq 1, \quad \forall 0 \leq t < T.\]  
(4.19)

By Hardy’s inequality (Lemma 4.3), we have
\[\int_{\mathbb{R}^5} \left( |\nabla \Phi(t)|^2 - \frac{9}{4} \cdot \frac{\Phi(t)^2}{r^2} \right) dx \geq 0.\]  
(4.20)

There is no hope to obtain (4.14) by using only (4.19), (4.20), since there could possibly exist a sequence $\Phi(t_n)$ with the property that
\[\|\nabla \Phi(t_n)\|_{L^2_x(\mathbb{R}^5)} \to \infty, \quad \left\| \frac{\Phi(t_n)}{r} \right\|_{L^2_x(\mathbb{R}^5)} \to \infty,\]
but
\[\int_{\mathbb{R}^5} \left( |\nabla \Phi(t_n)|^2 - \frac{9}{4} \cdot \frac{\Phi(t_n)^2}{r^2} \right) dx \to C_1, \quad as \ t_n \to T,\]
where $C_1 \geq 0$ is a finite constant.

Certainly a new argument is needed here.

To solve this problem, we shall proceed by exploiting in more detail the structure of $\Phi$.

Assume (4.14) does not hold. By (4.1) and (4.3), there exists $t_n \to T$ such that
\[\lim_{n \to \infty} \|\nabla \Phi(t_n)\|_{L^2_x(\mathbb{R}^5)} = +\infty.\]  
(4.21)

Define
\[\tilde{\Phi}(t_n) = \frac{\Phi(t_n)}{\|\nabla \Phi(t_n)\|_{L^2_x(\mathbb{R}^5)}}.\]  
(4.22)

Then
\[\left\| \nabla \tilde{\Phi}(t_n) \right\|_{L^2_x(\mathbb{R}^5)} = 1,\]  
(4.23)

and by (4.21), (4.19), (4.20),
\[\left\| \frac{\tilde{\Phi}(t_n)}{r} \right\|_{L^2_x(\mathbb{R}^5)} \to \frac{2}{3}, \quad as \ t_n \to T.\]  
(4.24)
Next consider (3.33). If \( r \gtrsim 1 \), then
\[
|\partial_r \Phi(r,t)| \lesssim |\partial_r g| + \frac{2}{r} + \frac{1}{r^3} \phi_1(r).
\]
Therefore by (4.5) and (4.3),
\[
\|\nabla \Phi(t)\|_{L^2(|x|>\frac{1}{2}, x \in \mathbb{R}^5)} + \|\frac{2}{r} \Phi(t)\|_{L^2(|x|>\frac{1}{2}, x \in \mathbb{R}^5)} \lesssim 1. \tag{4.25}
\]
For \( r \leq \frac{1}{2} \), by (3.33) and a short computation, we have
\[
\partial_r \Phi(r,t) + \frac{2}{r} \Phi(r,t) = \left(1 + \frac{2 \sin^2 f}{r^2}\right)^{\frac{1}{2}} \cdot \partial_r f - \frac{1}{r} \int_0^{g(r,t)} \left(1 + \frac{2 \sin^2 (ry)}{r^2}\right)^{-\frac{1}{2}} dy. \tag{4.26}
\]
By (3.1),
\[
\left\|\left(1 + \frac{2 \sin^2 f}{r^2}\right)^{\frac{1}{2}} \cdot \partial_r f\right\|_{L^2(\mathbb{R}^5)} \lesssim 1.
\]
Hence (4.25), (4.26) gives
\[
\left\|\partial_r \Phi(t) + \frac{2}{r} \Phi(t)\right\|_{L^2(\mathbb{R}^5)} \lesssim 1, \quad \forall \ 0 \leq t < T. \tag{4.27}
\]
By (4.21), (4.22), (4.27), we obtain
\[
\left\|\partial_r \tilde{\Phi}(t_n) + \frac{2}{r} \tilde{\Phi}(t_n)\right\|_{L^2(\mathbb{R}^5)} \to 0, \quad \text{as} \ t_n \to T.
\]
But this contradicts (4.23) and (4.24).
\( \square \)

Remark 4.5. In the above derivation, the contradiction (blow-up) argument is actually not needed. One can directly use (4.19) and (4.27) to get the desired uniform bound on \( \|\partial_r \Phi(t)\|_{L^2(\mathbb{R}^5)} \). We thank the anonymous referee for pointing this out.

5. Nonlinear energy bootstrap: more estimates

Let \( T > 0 \) be the same as in Proposition 4.4. Our goal in this section is to prove
\[
\sum_{|\alpha|+|\beta| \leq 4} \left\|\partial_x^\alpha \partial_t^\beta \Phi(t)\right\|_{L^2(\mathbb{R}^5)} \lesssim 1, \quad \forall \ 0 \leq t < T, \tag{5.1}
\]
and eventually
\[
\sup_{0 \leq t < T} G(t) < \infty, \tag{5.2}
\]
where \( G(t) \) is defined in (2.9). By Proposition 2.1, this implies global wellposedness.

We shall prove (5.1) in several steps.

First we get some decay estimates of \( \Phi \) and \( g \).

By Proposition 4.4 and radial Sobolev embedding, we have
\[
|\Phi(r,t)| \lesssim \min \left\{r^{-\frac{1}{2}}, r^{-2}\right\}, \quad \forall \ r > 0, \ 0 \leq t < T. \tag{5.3}
\]
We claim that
\[ |g(r,t)| \lesssim \min \left\{ r^{-\frac{3}{4}}, r^{-2} \right\}, \quad \forall r > 0, \ 0 \leq t < T. \quad (5.4) \]

By (3.28), it suffices to prove
\[ |g(r,t)| \lesssim r^{-\frac{3}{4}}, \quad \forall 0 < r \ll 1, \ 0 \leq t < T. \]

For \( r \ll 1 \), (3.33) gives
\[ \Phi(r,t) = \int_0^{g(r,t)} \left( 1 + \frac{2 \sin^2(ry)}{r^2} \right)^{\frac{1}{4}} dy. \quad (5.5) \]

Suppose for some \( 0 < r \ll 1 \), \( |g(r,t)| \gtrsim \frac{1}{r} \), then clearly
\[ \Phi(r,t) \sim \frac{g}{r}. \]

By (5.3), this would imply
\[ |g(r,t)| \lesssim r^{-\frac{3}{4}}, \]
which contradicts the assumption \( |g(r,t)| \gtrsim \frac{1}{r} \).

Therefore \( |g(r,t)| \lesssim \frac{1}{t} \) for all \( r \ll 1 \). By (5.3), we get
\[ |\Phi(r,t)| \sim \left| \int_0^{g(r,t)} \left( 1 + y^2 \right)^{\frac{1}{4}} dy \right| \gtrsim g(r,t)^2. \]

Hence by (5.3),
\[ g(r,t)^2 \lesssim r^{-\frac{3}{4}}, \quad \forall 0 < r \ll 1, \ 0 \leq t < T. \]

Therefore (5.4) is proved.

Before we continue, we need to introduce standard Strichartz for the wave operator.

**Definition 5.1.** Let \( d \geq 2 \). A pair \((q, r)\) is said to be wave admissible if
\[ 2 \leq q \leq \infty, \quad 2 \leq r < \infty, \]
and
\[ \frac{1}{q} + \frac{d - 1}{2r} = \frac{d - 1}{4}. \]

Note that the case \((q, r, d) = (2, \infty, 3)\) is not admissible.

**Lemma 5.2.** Let \( d \geq 2 \). Suppose \( u : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) solves
\[ \begin{cases} \partial_{tt} u - \Delta u = F, \\ (u, \partial_t u)_{t=0} = (u_0, u_1). \end{cases} \]

Let \((q, r), (\tilde{q}, \tilde{r})\) be wave admissible and satisfy the scaling condition
\[ \frac{1}{q} + \frac{d - 1}{2r} = \frac{1}{\tilde{q}} + \frac{d - 1}{\tilde{r}} - 2. \]

Then on the space-time slab \([0, T] \times \mathbb{R}^d\), we have
\[ \|u\|_{L^q_t L^r_x} + \|u\|_{C^1_t H^2_x} + \|\partial_t u\|_{C^0_t H^1_x} \leq \|u_0\|_{H^2} + \|u_1\|_{H^1} + \|F\|_{L^q_t L^r_x}. \]

Here \((\tilde{q}', \tilde{r}')\) are the conjugates of \((\tilde{q}, \tilde{r})\), i.e. \[ \frac{1}{	ilde{q}} + \frac{1}{\tilde{q}'} = \frac{1}{r} + \frac{1}{r'} = 1. \]
To simplify the presentation, we introduce more

**Notations.** For any \( z \in \mathbb{R}^d \), we use the Japanese bracket notation \( \langle z \rangle := (1+|z|^2)^{\frac{1}{2}} \).

For any space-time slab \([0, T_1] \times \mathbb{R}^5\), we shall use the notation

\[ \|u\|_{L^q_t L^r_x([0,T_1])} \]

to denote

\[ \|u\|_{L^q_t L^r_x([0,T_1] \times \mathbb{R}^5)}. \]

We will need to use the standard Littlewood-Paley projection operators. Let \( \tilde{\phi} \in C_c^\infty(\mathbb{R}^5) \) be a radial bump function supported in the ball \( \{ x \in \mathbb{R}^5 : |x| \leq \frac{25}{24} \} \) and equal to one on the ball \( \{ x \in \mathbb{R}^5 : |x| \leq 1 \} \). For any constant \( C > 0 \), denote \( \tilde{\phi}_{\leq C}(x) := \tilde{\phi}(\frac{x}{C}) \) and \( \tilde{\phi}_{> C} := 1 - \tilde{\phi}_{\leq C} \). For each dyadic \( N > 0 \), define the Littlewood-Paley projectors

\[ \hat{P}_{\leq N} f(\xi) := \tilde{\phi}_{\leq N}(\xi) \hat{f}(\xi), \]
\[ \hat{P}_{> N} f(\xi) := \phi_{> N}(\xi) \hat{f}(\xi), \]
\[ \hat{P}_{N} f(\xi) := (\tilde{\phi}_{\leq N} - \hat{\phi}_{\leq N}) \hat{f}(\xi), \]

and similarly \( P_{< N} \) and \( P_{\geq N} \).

Now we are ready to continue our estimates.

Taking the time derivative on both sides of (3.31), we get

\[ \Box_5 (\partial_t \Phi) = - \frac{3}{8} \partial_t \Phi + \frac{3}{4} A^\frac{1}{2} \partial_t g + \frac{1}{2} (A^{-\frac{1}{2}} - A^{-\frac{3}{2}}) \partial_t g, \]  

where

\[ A = 1 + \frac{2 \sin^2(r g(r, t) + \phi(r))}{r^2}. \]  

By (3.31), we have

\[ |A - 1| \lesssim \min \left\{ r^{-\frac{3}{2}}, r^{-4} \right\}. \]  

From (3.33), one has

\[ \partial_t \Phi = A^{\frac{1}{2}} \partial_t g. \]

Substituting (3.4) into (5.9), we get

\[ \Box_5 (\partial_t \Phi) = \left( \frac{3}{8} + \frac{1}{2} A^{-2} \right) (A - 1) \partial_t \Phi. \]  

By Strichartz (Lemma 5.2) and (5.8), we have for any \( 0 < T_1 < T \),

\[ \| P_{\geq 1} \partial_t \Phi \|_{L^q_t L^r_x([0,T_1])} \lesssim \| P_{\geq 1} \partial_t \Phi(0) \|_{H^\frac{2}{q}_x} + \| P_{\geq 1} \partial_t \Phi(0) \|_{H^\frac{2}{r}_x} \]
\[ + \| (A - 1) \partial_t \Phi \|_{L^q_x L^r_x([0,T_1])} \]
\[ \lesssim 1 + \| (A - 1) \|_{L^q_x L^r_x([0,T_1])} \cdot \| \partial_t \Phi \|_{L^q_x L^r_x([0,T_1])} \]
\[ \lesssim 1 + T_1^{\frac{1}{2}} \| \partial_t \Phi \|_{L^q_x L^r_x([0,T_1])}. \]  

Obviously

\[ \| P_{< 1} \partial_t \Phi \|_{L^q_x L^r_x([0,T_1])} \lesssim \| \partial_t \Phi \|_{L^q_x L^r_x} \lesssim 1. \]  

(5.11)
Using (5.11), (5.12), a continuity argument (see Appendix B) yields
\[ \| \partial_t \Phi \|_{L^3_t L^3_x([0,T])} \lesssim 1. \] (5.13)
Therefore
\[ \| \partial_t \Phi \|_{L^\infty_t H^{1/2}_x([0,T])} \lesssim 1. \] (5.14)
Using (5.10), we have
\[ \Box (\partial_{tt} \Phi) = \left( \frac{3}{2} + \frac{1}{2} A^{-2} \right) (A - 1) \partial_t \Phi \]
\[ + \left( -\frac{1}{2} A^{-2} + A^{-3} + \frac{3}{2} \right) \partial_t A \partial_t \Phi. \] (5.15)
By (5.7), observe that
\[ |\partial_t A| \lesssim |\partial_t \Phi|. \] (5.16)
Therefore by (5.13),
\[ \left\| \left( -\frac{1}{2} A^{-2} + A^{-3} + \frac{3}{2} \right) \partial_t A \partial_t \Phi \right\|_{L^3_t L^3_x([0,T])} \lesssim \| \partial_t \Phi \|_{L^3_t L^3_x([0,T])} \lesssim 1. \]
Denote
\[ G_3(r,t) = \frac{1}{2} \int_0^{g(r,t)} \left( 3B^{\frac{3}{4}} + B^{-\frac{1}{4}} - B^{-\frac{3}{4}} \right) dy. \] (5.17)
Then by (5.18), (5.14), we have
\[ |G_3(r,t)| \lesssim \begin{cases} |g(r,t)|, & \text{if } r \gtrsim 1, \\ |\Phi(r,t)|^2 + |\Phi(r,t)|, & \text{if } r \ll 1. \end{cases} \] (5.18)
Hence by (4.14),
\[ \| P_{\leq 1} G_3(t) \|_{L^2_t L^2_x(\mathbb{R}^3)} \lesssim 1, \quad \forall 0 \leq t < T. \] (5.19)
By essentially repeating the derivation of (5.13), (5.14) with \( \partial_t \Phi \) replaced by \( \partial_{tt} \Phi \), we get
\[ \| \partial_{tt} \Phi \|_{L^\infty_t H^{1/2}_x([0,T])} \lesssim 1. \] (5.20)
Note that the low frequency part of \( \partial_t \Phi \) causes no trouble since it can be controlled by \( \| P_{\leq 1} \Delta \Phi \|_{L^2_x} \lesssim \| \Phi \|_{L^2_x} \) using equation (3.31) together with (5.19).
Now by (3.31), we have
\[ -\Delta \Phi = -\partial_{tt} \Phi + \frac{1}{r^3} \phi_{\geq 1} - \frac{3}{2} \Phi + G_3, \] (5.21)
where \( G_3(r,t) \) was already defined in (5.17).
By (5.18) and (4.5),
\[ \| G_3(t) \|_{L^2_{t}(|x| > \frac{1}{2}, x \in \mathbb{R}^3)} \lesssim 1, \quad \forall 0 \leq t < T. \] (5.22)
By (5.18) and (4.14), we get
\[ \| \phi_{\leq \frac{1}{2}} G_3(t) \|_{L^2_t(\mathbb{R}^3)} \lesssim 1, \quad \forall 0 \leq t < T, \] (5.23)
where \( \phi_{\leq \frac{1}{2}} \) is a smooth cut-off function localized to \( r \leq \frac{1}{2} \).
Therefore (5.3), (5.4) and (5.8) can be refined to

By (5.15), (5.16) and Strichartz, we have for any $0 \leq t < T$,

$$
\left\| P_{>1} |\nabla|^{-\frac{1}{2}} \Delta \Phi(t) \right\|_{L^2_t(L^2_x)} \lesssim 1 + \left\| \nabla |^{-\frac{1}{2}} P_{>1} G_3 \right\|_{L^2_t(L^2_x)}
$$

Hence by (5.21) and (5.20), we obtain

$$
\left\| |\nabla|^{-\frac{1}{2}} \Phi(t) \right\|_{L^2_t(L^2_x)} \lesssim 1, \quad \forall 0 \leq t < T.
$$

By Sobolev embedding,

$$
\left\| \Phi(t) \right\|_{L^2_t(L^2_x)} \lesssim 1, \quad \forall 0 \leq t < T. \tag{5.24}
$$

By (5.18), (5.22), (5.24), we get

$$
\left\| G_3(t) \right\|_{L^2_t(L^2_x)} \lesssim 1, \quad \forall 0 \leq t < T.
$$

Hence by (5.21) and (5.20), we obtain

$$
\left\| \Phi(t) \right\|_{H^2_t(L^2_x)} \lesssim 1, \quad \forall 0 \leq t < T. \tag{5.25}
$$

By radial Sobolev embedding, we have

$$
\left\| |\nabla|^{\frac{1}{2}} \Phi(t) \right\|_{L^p_t(L^2_x)} \lesssim \left\| \Delta \Phi \right\|_{L^2_t(L^2_x)} \lesssim 1.
$$

Therefore (5.3), (5.4) and (5.8) can be refined to

$$
|\Phi(r,t)| \lesssim \min \left\{ r^{-\frac{1}{2}}, r^{-2} \right\}, \tag{5.26}
$$

$$
|g(r,t)| \lesssim \min \left\{ r^{-\frac{1}{2}}, r^{-2} \right\}, \tag{5.27}
$$

$$
|A - 1| \lesssim \min \left\{ r^{-\frac{1}{2}}, r^{-4} \right\}. \tag{5.28}
$$

By (5.19), (5.16) and Strichartz, we have for any $T_1 < T$,

$$
\left\| \partial_t \Phi \right\|_{L^\infty_t H^1_x([0,T_1])} + \left\| \partial_{tt} \Phi \right\|_{L^\infty_t L^2_x([0,T_1])} \lesssim \left\| \partial_t \Phi(0) \right\|_{H^1} + \left\| \partial_{tt} \Phi(0) \right\|_{L^2} + (A - 1) \left\| \partial_t \Phi \right\|_{L^2_t L^2_x([0,T_1])}
$$

$$
+ \left\| \partial_t A \cdot \partial_t \Phi \right\|_{L^2_t L^2_x([0,T_1])}
$$

$$
\lesssim 1 + T_1 \left\| (A - 1) \right\|_{L^\infty_x L^2_t([0,T_1])} \cdot \left\| \partial_t \Phi \right\|_{L^\infty_t H^1_x([0,T_1])}
$$

$$
+ \left\| \partial_t \Phi \right\|_{L^2_t L^2_x([0,T_1])}. \tag{5.29}
$$

By (5.28),

$$
\left\| (A - 1) \right\|_{L^\infty_t L^2_x} \lesssim 1. \tag{5.30}
$$

By (5.10), (5.28) and Strichartz, it is not difficult to check that

$$
\left\| \partial_t \Phi \right\|_{L^2_t L^2_x([0,T])} + \left\| \partial_{tt} \Phi \right\|_{L^\infty_t H^1_x([0,T])} \lesssim 1. \tag{5.31}
$$

Plugging (5.30), (5.31) into (5.29), a simple continuity argument then shows that

$$
\left\| \partial_t \Phi \right\|_{L^\infty_t H^1_x([0,T])} + \left\| \partial_{tt} \Phi \right\|_{L^2_t L^2_x([0,T])} \lesssim 1. \tag{5.32}
$$
By (5.10), (5.32), (5.28) and Hardy’s inequality, we then have
\[
\| \partial_t \Delta \Phi \|_{L^\infty_t L^2_x([0,T])} \lesssim \| \partial_{tt} \Phi \|_{L^\infty_t L^2_x([0,T])} + \| (A - 1) \partial_t \Phi \|_{L^\infty_t L^2_x([0,T])} \\
\lesssim 1 + \| \nabla \partial_t \Phi \|_{L^\infty_t L^2_x([0,T])} \\
\lesssim 1.
\] (5.33)

We can write (5.32), (5.33) collectively as
\[
\| \partial_{tt} \Phi \|_{L^\infty_t L^2_x([0,T])} + \| \partial_t \nabla \Phi \|_{L^\infty_t L^2_x([0,T])} + \| \partial_t \Delta \Phi \|_{L^\infty_t L^2_x([0,T])} \lesssim 1.
\] (5.34)

By (5.31) and (5.34), we have
\[
\| \nabla \Delta \Phi \|_{L^\infty_t L^2_x([0,T])} \\
\lesssim 1 + \left\| \partial_t \left( \int_0^t g(r,t) \left( 3B^\frac{3}{2} + B^{-\frac{3}{2}} - B^{-\frac{3}{2}} \right) dy \right) \right\|_{L^\infty_t L^2_x([0,T])} \\
\lesssim 1 + \left\| A^{\frac{3}{2}} \partial_r g \right\|_{L^\infty_t L^2_x([0,T])} \\
+ \left\| \int_0^t g(r,t) \left( \frac{9}{2} B^\frac{1}{2} - \frac{1}{2} B^{-\frac{3}{2}} + \frac{3}{2} B^{-\frac{3}{2}} \right) \partial_r B dy \right\|_{L^\infty_t L^2_x([0,T])}.
\] (5.35)

Observe that for \( r \leq \frac{1}{2} \),
\[
|\partial_r B(r,g)| \lesssim |g|^3.
\]

Therefore by (5.27),
\[
\left\| \int_0^t g(r,t) \left( \frac{9}{2} B^\frac{1}{2} - \frac{1}{2} B^{-\frac{3}{2}} + \frac{3}{2} B^{-\frac{3}{2}} \right) \partial_r B dy \right\|_{L^\infty_t L^2_x([0,T])} \\
\lesssim \| g \|_{L^\infty_t L^2_x([0,T])} \| g \|_{L^\infty_t L^2_x([0,T])} \\
\lesssim 1.
\] (5.36)

On the other hand, by (5.27), (5.7) and (4.3),
\[
\| A^{\frac{3}{2}} \partial_r g \|_{L^\infty_t L^2_x([0,T])} \lesssim \| \partial_r g \|_{L^\infty_t L^2_x([0,T])} + \| \phi < \frac{1}{2} \cdot r^{-\frac{3}{2}} \partial_r g \|_{L^\infty_t L^2_x([0,T])} \\
\lesssim 1 + \| \phi \cdot r^{-\frac{3}{2}} \partial_r g \|_{L^\infty_t L^2_x([0,T])}.
\] (5.37)

Plugging (5.30), (5.39) into (5.38), we get
\[
\| \nabla \Delta \Phi \|_{L^\infty_t L^2_x([0,T])} \lesssim 1 + \| \phi \cdot r^{-\frac{3}{2}} \partial_r g \|_{L^\infty_t L^2_x([0,T])}.
\] (5.38)

By (4.26), (5.26), (5.27) and Hardy’s inequality (see Appendix C), we have
\[
\| \phi \cdot r^{-\frac{3}{2}} \partial_r g \|_{L^\infty_t L^2_x([0,T])} \lesssim 1 + \| \phi \cdot r^{-\frac{3}{2}} \partial_r \Phi \|_{L^\infty_t L^2_x} \\
\lesssim 1 + \| \frac{1}{r} \nabla \Phi \|_{L^\infty_t L^2_x} \lesssim 1.
\]

Substituting it into (5.38), we get
\[
\| \nabla \Delta \Phi \|_{L^\infty_t L^2_x([0,T])} \lesssim 1.
\]
Hence together with (5.34), we have
\[
\|\partial_{ttt}\Phi\|_{L^\infty_t L^2_x([0,T])} + \|\partial_t \nabla \Phi\|_{L^\infty_t L^2_x([0,T])} \\
+ \|\partial_t \Delta \Phi\|_{L^\infty_t L^2_x([0,T])} + \|\nabla \Delta \Phi\|_{L^\infty_t L^2_x([0,T])} \lesssim 1. 
\] (5.39)
By Sobolev embedding we get
\[
\|\Phi\|_{L^\infty_t L^\infty_x([0,T])} \lesssim 1.
\]
Therefore we refine (5.26), (5.27), (5.28) to
\[
|\Phi(r,t)| \lesssim (r)^{-2},
\]
(5.40)
\[
|g(r,t)| \lesssim (r)^{-2},
\]
(5.41)
\[
|A - 1| \lesssim (r)^{-4}.
\]
(5.42)
By (4.26) (see Appendix C), we get
\[
\|\partial_r g\|_{L^\infty_t L^4_x([0,T])} \lesssim 1.
\]
(5.43)
By (5.9) and (5.39), we have
\[
\|\partial_t g\|_{L^\infty_t L^4_x([0,T])} + \|\partial_{tt} g\|_{L^\infty_t L^2_x([0,T])} \lesssim 1.
\]
(5.44)
Using (5.43), (5.44) and (2.7), we obtain
\[
\|\Delta g\|_{L^\infty_t L^2_x([0,T])} \lesssim 1.
\]
(5.45)
Also by Hardy’s inequality we get \(\|\frac{1}{r} \partial_r g\|_{L^\infty_t L^2_x([0,T])} \lesssim 1\) and hence
\[
\|\partial_{rr} g\|_{L^\infty_t L^2_x([0,T])} \lesssim 1.
\]
(5.46)
By (5.7), (5.41), (5.43), (5.45), (5.46), it follows that
\[
\|\nabla A\|_{L^\infty_t L^4_x([0,T])} + \|\Delta A\|_{L^\infty_t L^2_x([0,T])} \lesssim 1.
\]
(5.47)
Also it is not difficult to check that
\[
\left\| \Delta \left( \int_0^{g(r,t)} \left( 3B^{\frac{1}{2}} + B^{-\frac{1}{2}} - B^{-\frac{1}{2}} \right) dy \right) \right\|_{L^\infty_t L^2_x([0,T])} \lesssim 1.
\]
(5.48)
From (5.7), (5.9) and (5.39), we get
\[
\|\partial_{tt} A\|_{L^\infty_t L^2_x([0,T])} + \|\partial_{tt} A\|_{L^\infty_t L^2_x([0,T])} \lesssim 1.
\]
(5.49)
Differentiating (5.15) in time, we get
\[
\Box_5 (\partial_{tt} \Phi) = \left( \frac{3}{2} A - \frac{3}{2} + \frac{1}{2} A^{-1} - \frac{1}{2} A^{-2} \right) \partial_{tt} \Phi \\
+ (2A^{-3} - A^{-2} + 3) \partial_t A \cdot \partial_{tt} \Phi \\
+ (A^{-3} - \frac{1}{2} A^{-2} + \frac{3}{2}) \partial_t A \partial_{tt} \Phi \\
+ (-3A^{-4} + A^{-3}) (\partial_t A)^2 \partial_t \Phi.
\]
(5.50)
By Strichartz, (5.39), (5.49) and Sobolev, we get,
\[
\| \partial_{ttt} \Phi \|_{L^\infty_t H^1_x([0,T])} + \| \partial_{ttt} \Phi \|_{L^2_t L^2_x([0,T])} \\
\lesssim \| \partial_{tt} \Phi(0) \|_{H^1_x} + \| \partial_{tt} \Phi(0) \|_{L^2_x} \\
+ \| (A-1) \partial_t \Phi \|_{L^1_t L^2_x([0,T])} + \| \partial_t \Phi \cdot \partial_t \Phi \|_{L^1_t L^2_x([0,T])} \\
+ \| \partial_t A \cdot \partial_t \Phi \|_{L^1_t L^2_x([0,T])} + \| \partial_t \Phi \|_{L^3_t L^6_x([0,T])} \\
\lesssim 1.
\]
(5.51)

By (5.51), (3.31), we get
\[
\| \partial_{tt} \Delta \Phi \|_{L^\infty_t L^2_x([0,T])} \lesssim 1 + \| \partial_t \left( (3A^2 + A^{-1} - A^{-2}) \partial_t \Phi \right) \|_{L^\infty_t L^2_x([0,T])} \\
\lesssim 1.
\]
(5.52)

Using (5.31) again with the estimates (5.52) and (5.48), we finally obtain
\[
\| \Delta^2 \Phi \|_{L^\infty_t L^2_x([0,T])} \lesssim 1.
\]

In a similar way we have the estimate
\[
\| \partial_t \nabla \Delta \Phi \|_{L^\infty_t L^2_x([0,T])} \lesssim 1.
\]

Hence we have established
\[
\| \partial_{tt} \Phi \|_{L^\infty_t H^1_x([0,T])} + \| \partial_{ttt} \Phi \|_{L^2_t L^2_x([0,T])} \\
+ \| \partial_{tt} \Phi \|_{L^1_t H^2_x([0,T])} + \| \Phi \|_{L^\infty_t H^2_x([0,T])} \\
+ \| \partial_t \Phi \|_{L^\infty_t H^3_x([0,T])} \lesssim 1.
\]
(5.53)

This proved (5.1).

We are now ready to prove (5.2).

By (5.41)
\[
\| \langle x \rangle g(t) \|_{L^\infty_t L^\infty_x([0,T])} \lesssim 1.
\]
(5.54)

By (5.39), (5.56), Sobolev embedding and radial Sobolev embedding, we have
\[
\| \langle x \rangle \partial_t g \|_{L^\infty_t L^\infty_x([0,T])} \lesssim \| \langle x \rangle \partial_t \Phi \|_{L^\infty_t L^\infty_x([0,T])} \\
\lesssim \| \partial_t \Phi \|_{L^\infty_t H^2_x([0,T])} \lesssim 1.
\]
(5.55)

In a similar way, by using (4.26), we get
\[
\| \langle x \rangle \partial_r g \|_{L^\infty_t L^\infty_x([0,T])} \lesssim 1.
\]
(5.56)

Now (5.2) clearly follows from (5.54)–(5.56).

**Appendix A. Some technical estimates**

In this appendix we collect some useful technical estimates. Some of these estimates are rather pedestrian. Nevertheless, we include all the details here for the sake of completeness.

The following radial Sobolev embedding is well-known and dates back to Strauss [28]. We will often use it without explicit mentioning.
Lemma A.1 (Radial Sobolev embedding). Suppose \( d \geq 2 \) and \( h : \mathbb{R}^d \to \mathbb{R} \) is radial. If \( h \in C_0^\infty(\mathbb{R}^d) \), then for some constant \( C_d > 0 \) depending only on the dimension \( d \), we have
\[
r^\frac{d}{d+2} |h(r)| \leq C_d \|h\|_{H^1(\mathbb{R}^d)}, \quad \forall r > 0,
\]

Proof. Use the identity \( h(r)^2 = -2 \int_r^\infty h(\rho) \partial_\rho h(\rho) d\rho \) and observe \( r^{d-1} \leq r^{d-1} \).

In the rest of this section, we shall show that at \( t = 0 \), under the assumption that \((g_0, g_1) \in H^4_{\text{rad}}(\mathbb{R}^5) \times H^4_{\text{rad}}(\mathbb{R}^5)\), we have
\[
\sum_{j=0}^3 \|\partial_t^j \Phi(t = 0)\|_{H^1(\mathbb{R}^5)} + \|\partial_t^3 \Phi(t = 0)\|_{L^3(\mathbb{R}^5)} < \infty.
\]
These were used in Section 4 and Section 5.

We now give the details. We will proceed in 8 steps. Recall that
\[
\Phi(r, t) = \int_0^{g(r, t)} \left(1 + \frac{2 \sin^2 (ry + \phi(r))}{r^2}\right)^\frac{1}{2} dy + r^{-\frac{5}{2}} \phi \geq 1(r),
\]
where \( \phi(r) = N_1 \pi \) for \( r \leq 1 \) and \( \phi(r) = 0 \) for \( r \geq 2 \). Recall \( f(r, t) = \phi(r) + r g(r, t) \) and
\[
E(t) = \frac{1}{2} \int_0^\infty \left(1 + \frac{2 \sin^2 f}{r^2}\right) ((\partial_r f)^2 + (\partial_t f)^2) r^2 dr + \int_0^\infty \frac{\sin^2 f}{r^2} \left(1 + \frac{\sin^2 f}{2 r^2}\right) r^2 dr.
\]
Assume \((g, \partial_t g)_{t=0} = (g_0, g_1) \in H^4_{\text{rad}}(\mathbb{R}^5) \times H^4_{\text{rad}}(\mathbb{R}^5)\). By Hardy we have \( \|g_0\|_{L^2(\mathbb{R}^5)} < \infty \). By Sobolev embedding we have \( \|g_0\|_{\infty} + \|\nabla g_0\|_{\infty} + \|g_1\|_{\infty} < \infty \).

1. \( \|\partial_r f(t = 0)\|_{L^2(\mathbb{R}^5)} + \|\partial_t f(t = 0)\|_{L^2(\mathbb{R}^5)} < \infty \).
Since \( \partial_r f|_{t=0} = \phi'(r) + r \partial_r g_0 + g_0 \), we have
\[
\|\partial_r f(t = 0)\|_{L^2(\mathbb{R}^5)} \lesssim 1 + \|\partial_r g_0\|_{L^2(\mathbb{R}^5)} + \|g_0\|_{L^2(\mathbb{R}^5)} \lesssim 1 + \|g_1\|_{H^1(\mathbb{R}^5)} < \infty.
\]
The estimate for \( \|\partial_t f\|_{L^2(\mathbb{R}^5)} \) is similar and therefore omitted.

2. \( E(0) < \infty \).

We first consider the term \( \int_0^\infty \frac{\sin^2 f}{r^2} ((1 + \frac{\sin^2 f}{2 r^2}) r^2 dr \). Clearly the contribution of the part \( r \sim 1 \) is bounded. Therefore we only need to consider \( 0 < r \leq 1 \) and \( r \geq 2 \).

Then (below for simplicity of notation \( f, g \) and \( \partial_t f \) will all be evaluated at \( t = 0 \))
\[
\int_0^\infty \frac{\sin^2 f}{r^2} ((1 + \frac{\sin^2 f}{2 r^2}) r^2 dr \lesssim 1 + \int_0^\infty g^2 r^2 dr + \int_0^\infty g^4 r^2 dr
\]
\[
\lesssim 1 + \|g\|_{L^2(\mathbb{R}^5)}^2 + \|\partial_t g\|_{L^2(\mathbb{R}^5)}^2 \lesssim 1 + \|\nabla g\|_{L^2(\mathbb{R}^5)}^2 + \|\nabla (g^2)\|_{L^2(\mathbb{R}^5)}^2 \quad \text{(A.1)}
\]
Next for the first term in \( E \), we first deal with \( r \geq 1 \):
\[
\int_1^\infty \left(1 + \frac{2 \sin^2 f}{r^2}\right) ((\partial_r f)^2 + (\partial_t f)^2) r^2 dr \lesssim \|\partial_t f\|_{L^2(\mathbb{R}^5)}^2 + \|\partial_r f\|_{L^2(\mathbb{R}^5)}^2 < \infty.
\]
For the part $0 < r \leq 1$ we have
\[
\int_0^1 \left( 1 + \frac{2 \sin^2 f}{r^2} \right) \left( (\partial_r f)^2 + (\partial_r f)^2 \right) dr \\
\lesssim \int_0^1 \left( 1 + g^2 \right) (r^2 (\partial_r g)^2 + r^2 (\partial_r g)^2 + g^2) dr \\
\lesssim 1 + \int_0^1 g^2 (1 + g^2) r^2 dr + (1 + \|g\|_{L^2}^2) (\|\partial_r g\|_{L^2}^2 + \|\partial_r g\|_{L^2}^2) < \infty.
\]

3. $\|\nabla \Phi(t = 0)\|_{L^2(\mathbb{R}^5)} < \infty$.

First observe that $|\Phi(r, 0)| \lesssim |g(r, 0)| + |g(r, 0)|^2 + r^{-3} |\phi_{\geq 1}(r)|$, and
\[
\|\frac{\partial}{\partial r} \|_{L^2(\mathbb{R}^5)} \lesssim 1 + \|\frac{g}{r}\|_{L^2(\mathbb{R}^5)} + \|\frac{g^2}{r}\|_{L^2(\mathbb{R}^5)} < \infty,
\]
where we have used (A.1).

Next by a simple change of variable $ry \to y$, we rewrite $\Phi$ as
\[
\Phi(r, t) = r^{-2} \int_{rg(r,t)} (r^2 + 2 \sin^2(y + \phi(r))) \frac{1}{r} dy + r^{-3} \phi_{\geq 1}(r).
\]

Then
\[
\partial_r \Phi = -\frac{2}{r} \Phi + r^{-3} \phi_{\geq 1}(r) + r^{-2} \partial_r (rg)(r^2 + 2 \sin^2(rg + \phi(r))) \frac{1}{r} \\
\quad - r^{-2} \int_0^{rg} (r^2 + 2 \sin^2(y + \phi)) \frac{1}{r} (r + \sin(2y + 2\phi) \phi'(r)) dy.
\]

If $r \geq \frac{1}{2}$, then it follows that
\[
|\partial_r \Phi| \lesssim r^{-1} |\Phi| + r^{-3} |\phi_{\geq 1}(r)| + |\partial_r g| + r^{-1} |g|.
\]

If $0 < r < \frac{1}{2}$, then
\[
|\partial_r \Phi| \lesssim r^{-1} |\Phi| + r^{-2} |\partial_r (rg)| \cdot (r + r |g|) + r^{-1} |g| \\
\quad \lesssim r^{-1} |\Phi| + r^{-1} |g| + r^{-1} g^2 + |g| |\partial_r g| + |\partial_r g|.
\]

Thus we have $\|\partial_r \Phi\|_{L^2(\mathbb{R}^5)} < \infty$.

4. $\|\partial_t \Phi(t = 0)\|_{H^1(\mathbb{R}^5)} < \infty$.

First observe that
\[
\partial_t \Phi = \frac{1}{r} \partial_t f \left( 1 + \frac{2 \sin^2 f}{r^2} \right). \tag{A.2}
\]

Thus
\[
\partial_t \Phi \big|_{t = 0} = g_1 \cdot (1 + \frac{2 \sin^2 (rg_0 + \phi(r))}{r^2}) \frac{1}{r}. \tag{A.3}
\]

Since $g_1 \in H^3(\mathbb{R}^5)$ and $g_0 \in H^4(\mathbb{R}^5)$, we clearly have $\|\frac{2 \sin^2 (rg_0 + \phi(r))}{r^2}\|_{\infty} \lesssim 1$, and
\[
\|\partial_t \Phi(t = 0)\|_{L^2(\mathbb{R}^5)} < \infty.
\]

To bound the $\dot{H}^1$-norm, we consider first the regime $r \geq \frac{1}{2}$. Denote
\[
B_0 = 1 + \frac{2 \sin^2 (rg_0 + \phi(r))}{r^2}.
\]

\[\text{Recall that } \phi_{\geq 1} \text{ can vary from line to line.}\]
Clearly for \( r \geq \frac{1}{2} \), using \( \|g_0\|_\infty + \|\nabla g_0\|_\infty \lesssim 1 \), we have
\[
|\partial_r B_0| \lesssim 1 + r^{-2}(|\partial_r (rg_0)| + |\phi'(r)|) \lesssim 1. \tag{A.4}
\]
Next for \( r < \frac{1}{2} \), we have
\[
B_0 = 1 + \frac{2 \sin^2(rg_0)}{(rg_0)^2} g_0^2 = 1 + \tilde{G}(rg_0)g_0^2,
\]
where \( \tilde{G}(z) = 2z^{-2}\sin^2(z) \) has bounded derives of all orders. It follows that
\[
|\partial_r B_0| \lesssim |\partial_r (rg_0)| g_0^2 + |\partial_r g_0| \cdot |g_0| \lesssim r|\partial_r g_0| g_0^2 + |g_0|^3 + |\partial_r g_0| \cdot |g_0| \lesssim 1. \tag{A.5}
\]
where we used again the fact \( \|g_0\|_\infty + \|\nabla g_0\|_\infty \lesssim 1 \). It follows that
\[
\|\partial_r (\partial_t \Phi)\|_{L^2_t L^2_z(\mathbb{R}^3)} \lesssim \|\partial_t g_1\|_{L^2_t L^2_z(\mathbb{R}^3)} + \|g_1 B_0^{-\frac{1}{2}} \partial_r B_0\|_{L^2_t L^2_z(\mathbb{R}^3)} \lesssim 1.
\]
5. Denote \( A = 1 + \frac{2 \sin^2 f_0}{r^2} \) where \( f_0 = \phi(r) + rg_0 \). Then
\[
\|A\|_\infty + \|\partial_r A\|_\infty \lesssim 1, \quad \|\Delta_5 A\|_{L^2_t L^2_z(\mathbb{R}^3)} + \|\Delta_5 A\|_{L^\infty_t L^2_z(\mathbb{R}^3)} \lesssim 1. \tag{A.6}
\]
For \( r \geq \frac{1}{2} \), we use (A.4). For \( r < \frac{1}{2} \) we use (A.5). Thus the first inequality is obvious. The second inequality follows from a similar computation. One should note that by Sobolev embedding,
\[
\|\Delta_5 g_0\|_{L^\infty_t L^2_z(\mathbb{R}^3)} \lesssim \|g_0\|_{H^4(\mathbb{R}^3)} < \infty.
\]
6. \( \| (\partial_t \Phi)(t = 0) \|_{H^1(\mathbb{R}^3)} < \infty \).
First observe that
\[
\partial_t \Phi = \frac{1}{r} \partial_t f (1 + \frac{2 \sin^2 f}{r^2}) \phi + \frac{1}{r^3} (\partial_t f)^2 (1 + \frac{2 \sin^2 f}{r^2})^{-\frac{1}{2}} \sin(2f). \tag{A.7}
\]
We first consider the second term on the RHS. Since \( \partial_t f \big|_{t = 0} = rg_1 \), we have
\[
\frac{1}{r^3} (\partial_t f)^2 (1 + \frac{2 \sin^2 f}{r^2})^{-\frac{1}{2}} \sin(2f) \big|_{t = 0} = r^{-1} g_1^2 (1 + \frac{2 \sin^2 (\phi + rg_0)}{r^2})^{-\frac{1}{2}} \sin(2\phi + 2rg_0).
\]
For any \( r > 0 \), it is not difficult to check that (note below that for \( r \leq \frac{1}{2} \), one can write \( r^{-1} (1 + \frac{2 \sin^2 (\phi + rg_0)}{r^2})^{-\frac{1}{2}} \sin(2\phi + 2rg_0) = \tilde{F}(rg_0)g_0 \), where \( \tilde{F} \) has bounded derivatives)
\[
|\tilde{F}(rg_0)g_0| \lesssim 1 + |g_0(r)|,
\]
\[
|\partial_r \tilde{F}(rg_0)g_0| \lesssim 1 + |g_0(r)| + |g_0||\partial_r (rg_0(r))| + |\partial_r g_0(r)|.
\]
It follows easily that
\[
\|g_1^2 r^{-1} (1 + \frac{2 \sin^2 (\phi + rg_0)}{r^2})^{-\frac{1}{2}} \sin(2\phi + 2rg_0)\|_{H^1(\mathbb{R}^3)} \lesssim 1.
\]
It remains for us to check the first term on the RHS in \([A.7]\). Note that by \([A.6]\) the factor \((1 + \frac{2\sin^2 \frac{\pi}{2} f}{r})^\frac{1}{2}\) is harmless for us when estimating the \(H^1\)-norm. Therefore we only need to focus on the estimate of \(\|\frac{1}{r} \partial_t f\|_{H^1(\mathbb{R}^5)}\). Observe that

\[
\frac{1}{r} \partial_t f = \partial_t g = \Box_5 g + \Delta_5 g.
\]  \(\text{(A.8)}\)

Clearly \(\|\Delta_5 g_0\|_{H^1(\mathbb{R}^5)} \lesssim 1\). For \(\Box_5 g\) we use \(\Box 3\):

\[
\Box_5 g = \frac{\hat{F}_1(r)g^3 + \hat{F}_2(r)g^5}{1 + F_0(r)g^2} - \hat{F}_3(r) \cdot g \cdot (\partial_t g - (\partial_r g)^2) + \hat{F}_4(r) \cdot g^4 \cdot r \partial_r g + \phi_{>1} + \frac{2}{r^2} \Delta_3 \phi + 1 + \partial_t \phi \cdot N(r, \phi + rg, (\phi + rg)'), \quad (A.9)
\]

where \(\hat{F}_1(x) = F_1(x^2)\) and \(F_i\) has bounded derivatives of all orders. Clearly by using radial Sobolev embedding and \(\|g_0\|_{H^s} \lesssim 1\), we have

\[
\|\partial_r (F_1(r^2 g_0^2))\|_{\infty} \lesssim \|\partial_r (r^2 g_0^2)\|_{\infty} \lesssim 1.
\]

By a tedious calculation, it is not difficult to check then that the RHS of \([A.9]\) all have bounded \(H^1(\mathbb{R}^5)\)-norm. Thus \(\Box_5 g_0 \lesssim 1\) and consequently \(\|\partial_t (\Phi)(t = 0)\|_{H^1(\mathbb{R}^5)} < \infty\).

7. \(\|\partial_t (\Phi)(t = 0)\|_{H^1(\mathbb{R}^5)} < \infty\).

Here we use \(\Box 3\):

\[
\dot{\partial}_t \Phi|_{t=0} = \Delta_5 \dot{\partial}_t \Phi|_{t=0} - \frac{3}{2} \partial_t \Phi|_{t=0} + \frac{1}{2} \partial_t g(3B \frac{\partial^2 \Phi}{\partial t^2} + B^\frac{1}{2}, t = 0) - B^\frac{3}{2}, t = 0).
\]

where \(A\) is the same as in \([A.6]\). By \([A.6]\), the last term above clearly is bounded in \(H^1(\mathbb{R}^5)\). Also in Step 4 we have shown \(\|\partial_t \Phi|_{t=0}\|_{H^1(\mathbb{R}^5)} < \infty\). By \([A.3]\), we have

\[
\Delta_5 \dot{\partial}_t \Phi|_{t=0} = \Delta_5 (g_1 A^\frac{3}{2}) = \Delta_5 (g_1 A^\frac{3}{2}) + 2 g_1 A^\frac{3}{2} \cdot 2 A^\frac{1}{2} A^\frac{1}{2} + g_1 \Delta_5 (A^\frac{3}{2}).
\]

Clearly by \([A.6]\), it follows that \(\|\Delta_5 \dot{\partial}_t \Phi|_{t=0}\|_{L^2(\mathbb{R}^5)} \lesssim 1\).

8. \(\|\partial_t (\dot{\partial}_t \Phi)(t = 0)\|_{L^2(\mathbb{R}^5)} < \infty\). Here we use again \(\Box 3\):

\[
\dot{\partial}_t \Phi|_{t=0} = \Delta_5 \dot{\partial}_t \Phi|_{t=0} - \frac{3}{2} \partial_t \Phi|_{t=0} + \frac{1}{2} \partial_t g|_{t=0} (3A^\frac{3}{2} + A^\frac{1}{2} - A A^\frac{1}{2}) + \frac{1}{2} g_1 (A^\frac{3}{2} - A^\frac{1}{2} A^\frac{1}{2} + A^\frac{1}{2}) \cdot \sin(2rg_0 + 2\phi) \cdot (2rg_1).
\]

By the calculation in Step 6, we have \(\|\partial_t g|_{t=0}\|_{L^2(\mathbb{R}^5)} \lesssim 1\). The last three terms above clearly is \(L^2(\mathbb{R}^5)\)-bounded.

We now only to estimate \(\|\Delta_5 \partial_t f|_{t=0}\|_{L^2(\mathbb{R}^5)}\). By \([A.2]\), we have

\[
\dot{\partial}_t f|_{t=0} = \frac{1}{r} \dot{\partial}_t f|_{t=0} A^\frac{3}{2} + r^{-1} g_1^2 A^\frac{1}{2} \sin 2f_0.
\]
where \( \tilde{f}_0 = \phi + r g_0 \). Clearly by (A.6) and \( \| g_0 \|_{H^{\text{hi}}} + \| g_1 \|_{H^{\text{hi}}} \lesssim 1 \), we have
\[
\| \Delta g_0^2 A^{-2} \frac{\sin \frac{2 f_0}{r}}{r} \|_{L^2_x(\mathbb{R}^5)} \lesssim 1.
\]
By (A.8), we have
\[
\frac{1}{r} \partial_t f \big|_{t=0} A^{\frac{1}{2}} = A^{\frac{1}{2}} (\Box g + \Delta g) \big|_{t=0}.
\]
By (A.6), we have
\[
\| \Delta_5 (A^\frac{1}{2} \Delta g_0) \|_{L^2_x(\mathbb{R}^5)} \lesssim \| g_0 \|_{H^4} + \| \Delta_5 (A^\frac{1}{2}) \|_{L^2_x(\mathbb{R}^5)} \| \Delta_5 g_0 \|_{L^4_x(\mathbb{R}^5)} < \infty.
\]
Similarly we have (below we used the simple inequality \( \| h \|_{L^2_x(\mathbb{R}^5)} \lesssim \| h \|_{L^2_x(\mathbb{R}^5)} + \| \Delta_5 h \|_{L^2_x(\mathbb{R}^5)} \))
\[
\| \Delta_5 (A^\frac{1}{2} \Box g_0) \|_{L^2_x(\mathbb{R}^5)} \lesssim \| \Box g_0 \|_{L^2_x(\mathbb{R}^5)} + \| \nabla \Box g_0 \|_{L^2_x(\mathbb{R}^5)} + \| \Delta_5 \Box g_0 \|_{L^2_x(\mathbb{R}^5)} \lesssim \| \Box g_0 \|_{L^2_x(\mathbb{R}^5)} + \| \Delta_5 \Box g_0 \|_{L^2_x(\mathbb{R}^5)}.
\]
In Step 6 (see the estimates near (A.8), we have estimated \( \| \Box g_0 \|_{H^1(\mathbb{R}^5)} \). Thus we only need to deal with the term \( \| \Delta_5 \Box g_0 \|_{L^2_x(\mathbb{R}^5)} \). By using (A.3) and a tedious computation, it is not difficult to check that the RHS of (A.9) has finite \( H^2(\mathbb{R}^5) \)-norm. This then completes the estimate of \( \| \partial_t \Phi \|_{t=0} \|_{L^2_x(\mathbb{R}^5)} \).

**APPENDIX B. THE CONTINUITY ARGUMENT**

In this appendix we give more details of the continuity argument in the derivation of (5.13). Recall the main equation:
\[
\Box_5 (\partial_t \Phi) = \left( \frac{3}{2} + \frac{1}{2} A^{-2} \right) (A - 1) \partial_t \Phi,
\]
and
\[
|A - 1| \lesssim \min \{ r^{-\frac{2}{3}}, r^{-4} \}.
\]
Now denote \( u = \partial_t \Phi \). Our goal is to show that on the interval \( [0, T_1) \) (\( T_1 < T \) can be arbitrarily close to \( T \)), we have
\[
\| u \|_{L^1_t L^2_x([0,T_1])} + \| u \|_{C^0_t H^{\frac{1}{2}}_x([0,T_1])} \lesssim 1,
\]
where the implied constant is independent of \( T_1 \).

To this end we decompose \( [0, T_1] = \bigcup_{i=0}^{N_0} [t_i, t_{i+1}] \), where \( t_0 = 0, t_{N_0+1} = T_1 \), and \( t_{i+1} - t_i \) will be taken sufficiently small. The needed smallness will become clear in the argument below.

First observe that by using the estimates in Section 4, we have
\[
\| P_{< 1} u \|_{L^1_t L^2_x([0,T_1])} \lesssim \| u \|_{L^\infty_t L^2_x([0,T_1])} = \| \partial_t \Phi \|_{L^\infty_t L^2_x([0,T_1])} \lesssim 1.
\]
By Strichartz (Lemma (5.2) and (3.2)), we have on each \( [t_i, t_{i+1}] \),
\[
\| P_{\geq 1} u \|_{L^1_t L^2_x([t_i,t_{i+1}])} + \| P_{\geq 1} u \|_{C^0_t H^{\frac{1}{2}}_x([t_i,t_{i+1}])} \lesssim \| P_{\geq 1} u(t_i) \|_{H^{\frac{1}{2}}_x} + \| P_{\geq 1} \partial_t u(t_i) \|_{H^{\frac{1}{2}}_x} + \| (A - 1) u \|_{L^2_t L^2_x([t_i,t_{i+1}])} \lesssim \| P_{\geq 1} u(t_i) \|_{H^{\frac{1}{2}}_x} + \| P_{\geq 1} \partial_t u(t_i) \|_{H^{\frac{1}{2}}_x} + (t_{i+1} - t_i)^{\frac{1}{3}} \| u \|_{L^1_t L^2_x([t_i,t_{i+1}])}.
\]

Clearly if \((t_{i+1} - t_i)\) is sufficiently small, we have (using (B.4))
\[
\|P_{\geq 1} u\|_{L^3_t L^2_x([t_i, t_{i+1}])} + \|P_{\geq 1} \partial_t u\|_{C^0_t H^{\frac{1}{2}}_x([t_i, t_{i+1}])} \lesssim \|P_{\geq 1} u(t_i)\|_{H^{\frac{1}{2}}} + \|P_{\geq 1} \partial_t u(t_i)\|_{H^{-\frac{1}{2}}} + 1.
\]

Clearly by using the above estimate and iterating from \(i = 0\) to \(i = N_0\) (for the base step \(i = 0\), one can use the estimates in Appendix A to obtain
\[
\|P_{\geq 1} u(t = 0)\|_{\dot{H}^{\frac{1}{2}}} + \|P_{\geq 1} \partial_t u(t = 0)\|_{H^{-\frac{1}{2}}} \lesssim 1,
\]
we can obtain the estimate (B.3).

**Appendix C. Additional estimates**

This appendix is for the estimates in (5.38) and (5.43).

For \(r \leq \frac{1}{2}\), we have \(f = N_1 \pi + rg\). By using (4.26), we have
\[
\partial_r \Phi(r, t) + \frac{2}{r} \Phi(r, t) = \left(1 + \frac{2 \sin^2 \frac{f}{r^2}}{r^2}\right)^{\frac{1}{2}} \left(\partial_r g + \frac{g}{r}\right) + \frac{1}{r} \int_0^r \frac{g(r, t)}{r^2} \left(1 + \frac{2 \sin^2 \frac{rg}{r^2}}{r^2}\right)^{-\frac{1}{2}} dy.
\]
(C.1)

By (5.20) and (5.21), we have for \(r \leq \frac{1}{2}\),
\[
|\Phi(r)| \lesssim r^{-\frac{1}{2}}, \quad |g(r)| \lesssim r^{-\frac{1}{4}}.
\]
(C.2)

Thus for \(r \leq \frac{1}{2}\), plugging (C.2) into (C.1), we obtain
\[
|\partial_r g| \lesssim |\partial_r \Phi| + r^{-\frac{1}{2}}.
\]

This estimate is used in (5.38).

Next we turn to (5.43). By (5.40) and (5.41), we have
\[
|\Phi(r)| \lesssim \langle r \rangle^{-2}, \quad |g(r)| \lesssim \langle r \rangle^{-2}.
\]

By (C.2), we then have
\[
|\partial_r g| \lesssim |\partial_r \Phi| + r^{-3} \langle r \rangle^{-2}.
\]

Thus \(\|\partial_r g\|_{L^\infty_t L^1_x} \lesssim 1\).

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