A Riemann–Kempf singularity theorem for higher rank Brill–Noether loci

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Abstract

Given a vector bundle \( V \) of rank \( r \) over a curve \( X \), we define and study a surjective rational map \( \text{Hilb}^k(\mathbb{P}V) \to \text{Quot}^{0,d}(V) \) generalising the natural map \( \text{Sym}^d X \to \text{Quot}^{0,d}(\mathcal{O}_X) \). We then give a generalisation of the geometric Riemann–Roch theorem to vector bundles of higher rank over \( X \). We use this to give a geometric description of the tangent cone to the higher rank Brill–Noether locus \( B^{k}_{r,d} \) at a suitable bundle \( E \). This gives a generalisation of the Riemann–Kempf singularity theorem. As a corollary, if \( k = r \) and \( h^0(X,E) = r + n \), we show that the \( n \)th secant variety of the rank one locus of \( \mathbb{P} \text{End} E \) is contained in the tangent cone to \( B^{k}_{r,d} \).

1. Introduction

Let \( X \) be a projective smooth curve of genus \( g \geq 2 \) and \( D \) an effective divisor of degree \( d \) on \( X \). By the geometric Riemann–Roch theorem, \( \dim |D| \) is exactly the defect of \( D \) on the canonical model of \( X \) in \( |K_X|^* \). If \( X \) is general and \( d \leq g - 1 \), the line bundle \( \mathcal{O}_X(D) \) defines a point of multiplicity \( h^i(X,\mathcal{O}_X(D)) \) of the Brill–Noether locus \( W_d(X) \). The Riemann–Kempf singularity theorem (see [1, p. 241]) states that the projectivised tangent cone to \( W_d(X) \) at \( \mathcal{O}_X(D) \) is precisely the union of the secants Span(\( \phi_{K_X}(D') \)) \( \subset |K_X|^* \) for \( D' \in |D| \). Using this picture, in [8, 16, 18] new proofs of Torelli’s theorem were given using the infinitesimal geometry of Brill–Noether loci.

In recent years, higher rank analogues of \( W_d(X) \) have been the subject of much attention. We denote by \( U_X(r,d) \) the moduli space of stable vector bundles of rank \( r \) and degree \( d \) over \( X \), and consider the higher rank Brill–Noether locus

\[
B^{k}_{r,d} = \{ E \in U_X(r,d) : h^0(X,E) \geq k \}.
\]

See [9] for a summary of relevant results. Our interest is primarily in the infinitesimal geometry of \( B^{k}_{r,d} \) at singular points. As motivation, we note that in [14, 21], the infinitesimal geometry of generalised theta divisors associated to higher rank vector bundles (examples of twisted Brill–Noether loci) was used to prove ‘Torelli-type’ theorems (recovering the curve and the bundle, respectively). It seems therefore natural to investigate what can be recovered from the tangent cones \( T_E B^{k}_{r,d} \) at singular points \( E \). Note that [6] gives a comprehensive introduction and many interesting results on the singular loci of higher rank Brill–Noether loci and twisted Brill–Noether loci.

The projectivised tangent space of \( U_X(r,d) \) at \( E \) is \( \mathbb{P}H^1(X,\text{End} E) \). It follows from [17] that there is a natural map \( \psi : \mathbb{P} \text{End} E \to \mathbb{P}H^1(X,\text{End} E) \), generalising the canonical curve, which is an embedding for general \( X \) and \( E \). In light of this, as a first step towards finding analogues of the above results on line bundles, one can seek generalisations of the geometric Riemann–Roch and Riemann–Kempf theorems for bundles of higher rank, given in terms of

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the geometry of the scroll \( \psi(\mathbb{P}End E) \). One such generalisation was given in [12] for bundles of Euler characteristic zero, where the tangent cones of

\[ B_{r, r(g-1)}^1 = \{ E \in U_X(r, r(g-1)) : h^0(X, E) \geq 1 \} \subset U_X(r, r(g-1)), \]

another ‘generalised theta divisor’, were described geometrically.

In the present work, we generalise the picture in another way. Returning to the opening example, we note that the sequence \( 0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_X(D)_D \to 0 \) realises \( \mathcal{O}_X(D) \) as an elementary transformation of the trivial bundle. Let \( V \) be a vector bundle of rank \( r \) and \( \pi : \mathbb{P}V \to X \) the associated projective bundle. We consider elementary transformations \( 0 \to V \to \tilde{V} \to \tau \to 0 \); that is, bundles \( \tilde{V} \) of rank \( r \) containing \( V \) as a locally free subsheaf of full rank. If the support of \( \tau \) is reduced and of degree \( d \), then the choice of \( \tilde{V} \) is canonically equivalent to a choice of \( d \) points \( \nu_1, \ldots, \nu_d \) of \( \mathbb{P}V \) belonging, respectively, to fibres over distinct points \( x_1, \ldots, x_d \) of \( X \). Generalising the definition of \( \mathcal{O}_X(D) \), one can realise the sheaf \( \tilde{V} \) as the sheaf of rational sections of \( V \) with poles bounded by the \( x_i \) and in the direction corresponding to the \( \nu_i \). Then \( \tilde{V}^* \) is the subsheaf of \( V^* \) of sections whose values at \( x_i \) belong to the hyperplane determined by \( \nu_i \).

Our first goal is to systematise and extend this construction to the case where \( \text{Supp}(\tau) \) may be nonreduced. Let \( Z \) be a subscheme of \( \mathbb{P}V \) of dimension zero and length \( d \), with ideal sheaf \( \mathcal{I}_Z \). Generalising the operation \( D \mapsto \mathcal{O}_X(D) \), we define

\[ V_Z := (\pi_*(\mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}V}(1)))^* \]

The association \( Z \mapsto V_Z \) gives a map \( \alpha : \text{Hilb}^d(\mathbb{P}V) \to \text{Quot}^0, d(V^*) \), generalising the natural map \( \text{Sym}^d X \to \text{Quot}^0, d(\mathcal{O}_X) \) given by \( D \mapsto \mathcal{O}_X(-D) \). In Theorem 2.2 and Theorem 2.6 we show that the restriction of \( \alpha \) to the component of \( \text{Hilb}^d(\mathbb{P}V) \) containing reduced subschemes is surjective and generically injective.

In contrast to the line bundle case, if \( r \geq 2 \), then \( \deg(V_Z) \) may be strictly less than \( \deg V + d \) in special cases. This leads us to the notion of \( (\mathcal{O}_{\mathbb{P}V}(1), \pi) \)-nondefectivity (Definition 3.4). Much can be said about \( (\mathcal{O}_{\mathbb{P}V}(1), \pi) \)-nondefective subschemes, but we limit ourselves here to what is necessary for the present applications.

In §4, we link the elementary transformations \( V_Z \) with the extrinsic geometry of the image of \( \mathbb{P}V \) by the natural map \( \psi : \mathbb{P}V \to |\mathcal{O}_{\mathbb{P}V}(1) \otimes \pi^* K_X|^* \). The main results are as follows.

**Proposition A** (Proposition 4.2). Let \( Z \) be an element of \( \text{Hilb}^d(\mathbb{P}V) \). Then \( \text{Span}(\psi(Z)) \) can be identified with the projectivisation of the image of the coboundary map in

\[ 0 \to H^0(X, V) \to H^0(X, V_Z) \to H^0(X, \tau_Z) \to H^1(X, V) \to H^1(X, V_Z) \to 0. \]

We then give the following generalisation of the geometric Riemann–Roch theorem for bundles of higher rank, relating \( h^0(X, V_Z) \) with the secant defect of \( \psi(Z) \) (Definition 3.1).

**Theorem B** (Theorem 4.3). Let \( 0 \to V \to \tilde{V} \to \tau \to 0 \) be an elementary transformation. Then for any \( (\mathcal{O}_{\mathbb{P}V}(1), \pi) \)-nondefective \( Z \) such that \( V_Z \cong V \) as vector bundles, we have

\[ h^0(X, \tilde{V}) - h^0(X, V) = \text{def}(\psi(Z)). \]

For applications to Brill–Noether loci, it is necessary (and straightforward) to have also ‘relative’ versions of these results, given in terms of the geometry of the rank one locus of \( \psi(\mathbb{P}(V \otimes F)) \) in \( \mathbb{P}H^1(X, V \otimes F) \) for any bundle \( F \). These are proven in §4.2. It should be noted that a similar situation is studied in [4]; see Remark 4.12 for discussion.

In §5, we apply these results to the study of the generalised Brill–Noether loci \( B_{k,r}^r \). Suppose \( E \) is a stable, generically generated bundle. This means that for generic \( \Pi \in \text{Gr}(r, H^0(X, E)) \), the bundle \( E \) is naturally an elementary transformation of \( \mathcal{O}_X \otimes \Pi \). By Theorem 2.2, for each
such Π there exists $Z_Π \in \text{Hilb}^d(\mathbb{P}E^*)$ such that $\mathcal{O}_X \otimes Π = (E^*)_{Z_Π}$. Combining this observation with results from [6], we obtain the following generalisation of the Riemann–Kempf singularity theorem. In what follows, for any $ℓ \geq r$ we denote by $U^ℓ$ the open subset of $\text{Gr}(ℓ, H^0(X, E))$ of subspaces which generically generate $E$.

**Theorem C** (Theorem 5.6). Suppose $k \geq r$. Let $E$ be a stable, generically generated bundle with $h^0(X, E) \geq k$ which is Petri $k$-injective (Definition 5.1).

(a) For any $Π \in U^r$, we have $\text{def}(\psi(Z_Π \times_X \mathbb{P}E)) = r \cdot h^0(X, E) - 1$.

(b) The projectivised tangent cone $\mathbb{P}T_E B_{r,d}^k$ is the Zariski closure of

$$\bigcup_{Λ \in U^k} \left( \bigcap_{Π \in U^r \cap \text{Gr}(r, Λ)} \text{Span}(\psi(Z_Π \times_X \mathbb{P}E)) \right).$$

If $E$ is globally generated, then $Z_Π$ is even reduced for general $Π$; see Theorem 5.6(c). An interesting corollary in the case $k = r$ is that if $h^0(X, E) = r + n$, then $\mathbb{P}T_E B_{r,d}^r$ contains the $n$th secant variety of the image of the rank one locus of $\mathbb{P}\text{End} E$ (Theorem 5.9). This generalises the fact that the tangent cone to the Riemann theta divisor at a point of multiplicity $n + 1$ contains the $n$th secant variety of the canonical curve (see [1, p. 232]).

Finally, as it is not always clear even that $B_{r,d}^r$ is nonempty, in §5.4 we show that bundles $E$ satisfying the hypotheses of Theorem C exist for several values of $r$, $d$, and $k$.

We hope that these results may be useful in proving new ‘Torelli-type’ statements.

We work over the complex field $\mathbb{C}$, although many arguments are valid more generally.

2. **Elementary transformations and finite length subschemes**

Let $V$ be a vector bundle of rank $r$ over a projective smooth curve $X$, and denote by $π : \mathbb{P}V \to X$ the corresponding projective bundle of lines in $V$. The latter is naturally isomorphic to the Quot scheme $\text{Quot}^{0,1}(V^*)$ parametrising elementary transformations of the form $0 \to V \to V^* \to τ \to 0$, where $τ$ is a skyscraper sheaf of length 1; equivalently, elementary transformations of the form $0 \to V \to V \to τ \to 0$. More generally, in [22] a tower of projective bundles was constructed parametrising such $V$ for $τ$ of degree $d \geq 1$ (see also [7, Lemma 4.2]). Here, we study an alternative way of parametrising these elementary transformations.

Let $Z \subset \mathbb{P}V$ be a subscheme of dimension zero and length $d$, corresponding to a point of the Hilbert scheme $\text{Hilb}^d(\mathbb{P}V)$. Tensoring the sequence $0 \to \mathcal{I}_Z \to \mathcal{O}_{\mathbb{P}V} \to \mathcal{O}_Z \to 0$ by $\mathcal{O}_{\mathbb{P}V}(1)$ and taking direct images, we obtain an exact sequence

$$0 \to π_*(\mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}V}(1)) \to V^* \to π_*(\mathcal{O}_Z \otimes \mathcal{O}_{\mathbb{P}V}(1)) \to \cdots$$

(2.1)

Set $V_Z := (π_*(\mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}V}(1)))^*$. Then $0 \to V \to V_Z \to V_Z/V \to 0$ is an elementary transformation. We write $τ_Z := V_Z/V$.

**Remark 2.1.** We will switch freely between elementary transformations of the form $V \subset \tilde{V}$ and $\tilde{V}^* \subset V^*$, depending on what is more convenient at a given time. It will be necessary to distinguish between the statement that $V_Z \cong \tilde{V}$ as vector bundles and the stronger statement that $V_Z \subset V^*$ and $\tilde{V}^* \subset V^*$ define the same point of the Quot scheme $\text{Quot}^{0,d}(V^*)$. To this end, we will occasionally abuse language and speak of an elementary transformation of the form $0 \to V \to \tilde{V} \to τ \to 0$ as ‘an element of $\text{Quot}^{0,d}(V^*)$’.

If $r = 1$, then $Z$ is a divisor of degree $d$ on $\mathbb{P}V = X$, and then $V_Z^* = V^* \otimes \mathcal{O}_X(−Z)$ as points of $\text{Quot}^{0,d}(V^*)$. For $r \geq 2$ we only have the inequality $\text{deg}(V_Z) \leq \text{deg}(V) + d$, which may be
strict. This will be discussed further in § 3.1. For now, let us give the main result of the present section.

**Theorem 2.2.** Let $0 \to V \to \tilde{V} \to \tau \to 0$ be an elementary transformation, where $\tau$ has degree $d \geq 1$.

(a) There exists $Z \in \text{Hilb}^d(\mathbb{P}V)$ such that $\tilde{V}^* = V^*_Z$ as elements of $\text{Quot}^{0,d}(V^*)$.

(b) If $\tau$ has reduced support on $X$, then $Z$ is reduced and uniquely determined.

Let us give an overview of the proof. First, note that (b) is almost obvious: In this case, $\tilde{V}$ is determined by specifying a line $\ker(V|_x \to \tilde{V}|_x)$ for each $x \in \text{Supp}(\tau)$, so we obtain naturally $d$ points of $\mathbb{P}V$.

If $\tau$ has nonreduced support, then it emerges that a certain choice of local frames for $V^*$ and $\tilde{V}^*$ determines a zero-dimensional scheme $Z \subset \mathbb{P}V$ of length $d$ such that $\tilde{V}^* = V^*_Z$ as elements of $\text{Quot}^{0,d}(V^*)$. As in the proof of [12, Theorem 3.1], we will view sections of $V^* \to X$ as sections of $\mathcal{O}_{\mathbb{P}V}(1) \to \mathbb{P}V$, and show how the linear conditions satisfied by the chosen frame for $\tilde{V}^* \subset V^*$ determine a suitable $Z$. Note, however, that in general $Z$ is not unique if $\text{Supp}(\tau)$ is nonreduced.

We begin with a general statement on direct image sheaves.

**Proposition 2.3.** Let $\pi: Y \to S$ be a projective morphism. Let $L$ be a sheaf of $\mathcal{O}_Y$-modules such that $\pi^*L$ is locally free, and let $W$ be a locally free subsheaf of $\pi^*L$. Write $\varphi$ for the composed map $\pi^*W \to \pi^*\pi_*L \to L$, and set $F := \text{Im}(\varphi)$.

(a) The sheaves $\pi_*F$ and $W$ are isomorphic.

(b) If moreover $L$ is invertible, then there is a closed subscheme $Z \subseteq Y$ such that $F = \mathcal{I}_Z \otimes L$.

**Proof.** (a) It follows from the definitions that $\pi_*F = \pi_*\text{Im}(\varphi) = \text{Im}(\pi_*\varphi)$. Thus, let us show that $W \cong \text{Im}(\pi_*\varphi)$.

Since $\pi$ is projective, $\pi_*\mathcal{O}_Y = \mathcal{O}_S$. Thus, for any locally free sheaf $G$ on $S$, we have $\pi_*\pi^*G \cong G$ by the projection formula. Therefore, there is a commutative diagram

\[
\begin{array}{ccc}
W & \rightarrow & \pi_*L \\
\downarrow & & \downarrow \\
\pi_*\pi^*W & \rightarrow & \pi_*\pi^*(\pi_*L) \cong \pi_*L
\end{array}
\]

where the composition of the maps in the lower row is $\pi_*\varphi$. Hence, $W \cong \text{Im}(\pi_*\varphi)$.

(b) If $L$ is invertible, then $F \otimes L^{-1}$ is contained in $\mathcal{O}_Y$, so is of the form $\mathcal{I}_Z$ for some closed subscheme $Z \subseteq Y$. Part (b) follows. \qed

The proof of the following is left to the reader.

**Lemma 2.4.** Let $\pi: Y \to S$ be any morphism. Let $L$ be a sheaf on $Y$. For any finite collection $F_1, \ldots, F_\ell$ of subsheaves of $L$, we have

\[
\pi_*\left(\bigcap_{i=1}^\ell F_i\right) = \bigcap_{i=1}^\ell \pi_*F_i.
\]

**Proof of Theorem 2.2.** (a) Let $0 \to V \to \tilde{V} \to \tau \to 0$ be an elementary transformation, where $\deg(\tau) = d$. First, suppose that $\tau$ is supported at a single point $x \in X$. We consider the dual
elementary transformation $\tilde{V}^* \to V^*$. Taking stalks at $x$, we obtain an inclusion $\tilde{V}_x^* \to V_x^*$ of free modules of rank $r$ over the principal ideal domain $O_{X,x}$. By the theory of modules over a PID (see, for example, [19, p. 153]) there exists a basis $f_1, \ldots, f_r$ of $V_x^*$ together with elements $a_1, \ldots, a_r \in O_{X,x}$ such that

$$\tilde{V}_x^* = \bigoplus_{j=1}^r (a_j) \cdot f_j,$$

and satisfying $a_j | a_{j-1}$ for $2 \leq j \leq r$. Let $z$ be a local parameter at $x$. As $O_{X,x}$ is a local ring with maximal ideal generated by $z$, we may assume that $a_j = z^{k_j}$, where $k_1 \geq \cdots \geq k_r \geq 0$. Let $s \leq r$ be the largest index satisfying $k_s \geq 1$. Then

$$\frac{V_x^*}{V_x^*} \cong \bigoplus_{j=1}^s O_{X,x}(z^{k_j}).$$

It follows that $k_1 + \cdots + k_s = \deg(\tilde{V}/V) = d$.

Next, for $1 \leq j \leq s$, let $W_j \subset V^*$ be the elementary transformation generated at $x$ by

$$f_1, \ldots, f_{j-1}, z^{k_j} \cdot f_j, f_{j+1}, \ldots, f_r,$$

and equal to $V^*$ away from $x$. Clearly, $\tilde{V}^* = \bigcap_{j=1}^s W_j$.

For $1 \leq j \leq s$, let $F_j$ be the image of the composed map

$$\varphi_j: \pi^* W_j \to \pi^* V^* \to O_{PV}(1).$$

By Proposition 2.3, we have $\pi_* F_j = W_j$, and moreover there exists a closed subscheme $Z_j$ such that $F_j = I_{Z_j} \otimes O_{PV}(1)$. Let us describe $Z_j$.

For $1 \leq j \leq s$, let $\nu_j \in PV|_x$ be the point defined by the vanishing of $f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_r$. As $F_j$ is generated by the images of the pullbacks of (2.2) in $O_{PV}(1)$, for any $\nu \in PV \setminus \{\nu_j\}$ the map $\varphi_j|_{\nu}$ is surjective. Hence, $Z_j$ is supported exactly at $\nu_j$. Furthermore, the images of the pullbacks of $f_j, z f_j, \ldots, z^{k_j-1} f_j$ form a $\mathbb{C}$-basis for $H^0(Z_j, O_{Z_j} \otimes O_{PV}(1)) \cong H^0(Z_j, O_{Z_j})$, so $Z_j$ has length $k_j$.

Now, set $Z := Z_1 \cup \cdots \cup Z_s$, a subscheme of dimension zero and length $\sum_{j=1}^s k_j = d$. As the $\nu_j$ clearly form a basis of $PV|_x$, in particular they are distinct. It follows that $I_Z = I_{Z_1} \cap \cdots \cap I_{Z_s}$. Then by definition of $V_Z$, we have

$$V_Z^* = \pi_*(I_Z \otimes O_{PV}(1)) = \pi_* \left( \bigcap_{j=1}^s I_{Z_j} \otimes O_{PV}(1) \right) \otimes O_{PV}(1) \tag{2.3}$$

$$= \pi_* \left( \bigcap_{j=1}^s (I_{Z_j} \otimes O_{PV}(1)) \right) \text{ since } O_{PV}(1) \text{ is locally free}$$

$$= \bigcap_{j=1}^s \pi_*(I_{Z_j} \otimes O_{PV}(1)) \text{ by Lemma 2.4}$$

$$= \bigcap_{j=1}^s W_j = \tilde{V}^*.$$

This proves part (a) in the case where $\tau$ is supported at a single point of $X$.

For the general case: Note that $\tau$ is the direct sum $\bigoplus_{x \in \text{Supp}(\tau)} \tau_x$. For each $x \in \text{Supp}(\tau)$, set $W(x) := \text{Ker}(V^* \to \tau_x)$. Then clearly $\tilde{V}^* = \bigcap_{x \in \text{Supp}(\tau)} W(x)$. For each $x$, the above
construction yields a zero-dimensional subscheme \(Z(x) \subset \mathbb{P}V_x\) of length \(\deg(\tau_x)\) and an ideal sheaf \(I_{Z(x)}\) satisfying \(\pi_*((I_{Z(x)} \otimes \mathcal{O}_V(1))) = W(x)\). Let \(Z\) be the union
\[
\bigcup_{x \in \text{Supp}(\tau)} Z(x),
\]
and then \(I_Z = \bigcap_{x \in \text{Supp}(\tau)} I(x)\). By a similar argument to (2.3), we obtain \(V'_Z = V^*\). This completes the proof of (a) in general.

(b) If \(\tau\) has reduced support, then length \((\tau_x) = 1\) for each \(x \in \text{Supp}(\tau)\). At each such \(x\), in this case, \(k_1 = 1\) is the only \(k_j\) different from zero, and \(Z \cap \mathbb{P}V_x\) is supported at \(\nu_1\) and has length 1. Hence, \(Z\) is reduced of length \(d\). For uniqueness, we note that in this case \(\nu_1 \in \mathbb{P}V\) corresponds to the hyperplane \(\text{Im}(V^*_x) = V^*_x)\).

\[\square\]

**Remark 2.5.** If length \((\tau_x) \geq 2\) for some \(x\), then the choice of \(f_1, \ldots, f_r\) and the points \(\nu_j\) may not be unique. For example, consider the elementary transformation \(0 \to V \to V \otimes \mathcal{O}_X(x) \to \tau \to 0\). Distinct frames for \(V^* \otimes \mathcal{O}_X(-x) \subset V^*\) near \(x\) determine in general different bases \(\nu_1, \ldots, \nu_r\). Note also that the number \(s\) of generators of \(\tau_x\) over \(\mathcal{O}_X\) (as distinct from over \(\mathbb{C}\)) is not determined by length \((\tau_x)\). For example, if \(\tau_x \cong \mathcal{O}_{V_x}^{\oplus 2}\), then two generators are required over \(\mathcal{O}_X\), whereas if \(\tau_x \cong \mathcal{O}_{2x}\) then one suffices.

**Hilbert schemes of points**

Let us now give a refinement of Theorem 2.2. The scheme \(\text{Hilb}^d(\mathbb{P}V)\) is connected since \(\mathbb{P}V\) is, but by [5] it is not irreducible for \(r \geq 4\) and \(d \geq 8\). We denote by \(\text{Hilb}^d(\mathbb{P}V)_0\) the irreducible component containing reduced subschemes.

**Theorem 2.6.** (a) The association \(Z \mapsto \pi_*((I_Z \otimes \mathcal{O}_V(1)))\) defines a rational map
\[
\alpha: \text{Hilb}^d(\mathbb{P}V) \longrightarrow \text{Quot}^{0, d}(V^*)
\]
whose restriction to \(\text{Hilb}^d(\mathbb{P}V)_0\) is surjective.

(b) The restriction of \(\alpha\) to the subset
\[
\{ Z \in \text{Hilb}^d(\mathbb{P}V)_0 : \pi(Z) \text{ is reduced} \} \subset \text{Sym}^d \mathbb{P}V
\]
is bijective. In particular, the restriction of \(\alpha\) to \(\text{Hilb}^d(\mathbb{P}V)_0\) is a birational equivalence.

(c) No other component of \(\text{Hilb}^d(\mathbb{P}V)\) dominates \(\text{Quot}^{0, d}(V^*)\).

**Proof.** (a) It is straightforward to globalise the construction \(V_Z = \pi_*((I_Z \otimes \mathcal{O}_V(1)))^*\) and obtain a family of elementary transformations of \(V\) parametrised by the locus
\[
\{Z \in \text{Hilb}^d(\mathbb{P}V) : \deg(V_Z) = \deg V + d\} \subset \text{Hilb}^d(\mathbb{P}V).
\]
As the rank and degree are constant and \(X\) has dimension 1, this family is flat. The existence of \(\alpha\) then follows from the universal property of Quot schemes.

To conclude, by Theorem 2.2 it will suffice to show that if \(Z = Z_1 \cup \cdots \cup Z_s\) is a nonreduced scheme of length \(d \geq 2\) arising from the construction in the proof of Theorem 2.2, then \(Z\) is smoothable. Clearly, \(Z\) is smoothable if each \(Z_j\) is smoothable. Thus, it will suffice to prove the smoothability of a \(Z\) supported at a single point \(\nu_1 \in \mathbb{P}V_x\) with length \(k \geq 2\). In this case, \(I_Z \otimes \mathcal{O}_V(1)\) is generated near \(\nu_1\) by
\[
z^k \cdot f_1, f_2, \ldots, f_r,
\]
with notation as in the proof of Theorem 2.2. Hence, \(\mathcal{O}_Z\) is generated by the images of \(1, \pi^*z, \ldots, \pi^*z^{k-1}\). Therefore, \(\pi_*\mathcal{O}_Z\) is supported at \(x\) and generated by the images of
1, z, ..., z^{k-1}. Thus, the map \( O_X \rightarrow \pi_*O_Z \) is surjective, whence \( \pi: \mathbb{P}V \rightarrow X \) restricts to a closed embedding \( Z \hookrightarrow X \). Hence, \( Z \) is curvilinear and hence smoothable.

Part (b) follows from Theorem 2.2 (b). For the rest, clearly \( V_Z/V \) has reduced support only if \( Z \in \text{Hilb}^d(\mathbb{P}V)_0 \). As quotients with reduced support are dense in Quot^{0,d}(V^*), we obtain (c). \( \square \)

In the next section, we will describe the indeterminacy locus of \( \alpha \) in more detail.

3. Defective secants

In this section, we study the defect of a subvariety with respect to a line bundle in two distinct situations. At the end, we will give a more general definition of defectivity covering both cases.

Let \( Y \) be a variety, proper over \( \mathbb{C} \), equipped with a line bundle \( \mathcal{L} \) and associated map \( \psi: Y \rightarrow |\mathcal{L}|^* \) (not necessarily base point free). If \( Z \subseteq Y \) is a subscheme, then \( \text{Span}(\psi(Z)) \) is the projective linear subspace

\[
\mathbb{P}\text{Ker}(H^0(Y, \mathcal{L})^* \rightarrow H^0(Y, I_Z \otimes \mathcal{L})^*). \tag{3.1}
\]

If \( Z \) is of dimension zero and reduced, then \( \text{Span}(\psi(Z)) \) is the secant spanned by the images of the points of \( Z \). If \( Z \) is not reduced, \( \text{Span}(\psi(Z)) \) is a subspace of the span of the union of certain osculating spaces to \( \psi(Y) \) at points of \( \text{Supp}(Z) \).

**Definition 3.1.** Suppose \( Z \subset Y \) has dimension zero. We recall that the defect of \( \psi(Z) \) is

\[
\text{length } Z - 1 - \dim \text{Span}(\psi(Z)) =: \text{def}(\psi(Z)).
\]

We say that \( Z \) is \( \mathcal{L} \)-nondefective if \( \text{def}(\psi(Z)) = 0 \), and \( \mathcal{L} \)-defective otherwise. (If \( \text{Span}(\psi(Z)) \) is empty, we follow the convention that \( \dim \text{Span}(\psi(Z)) = -1 \).)

**Remark 3.2.** The map \( \psi: Y \rightarrow |\mathcal{L}|^* \) is base point free if and only if all \( y \in Y \) are \( \mathcal{L} \)-nondefective, and an embedding if and only if all \( Z \in \text{Hilb}^d(Y) \) are \( \mathcal{L} \)-nondefective.

**Remark 3.3.** Recall that the \( n \)th secant variety \( \text{Sec}^n(Y) \) of a nondegenerate variety \( Y \subset \mathbb{P}^N \) is the Zariski closure of the union of the linear spans of all subsets of \( n \) points of \( Y \). In general, \( Y \) is said to be ‘secant defective’ if some \( \text{Sec}^nY \) has less than the expected dimension. The above definition of an \( \mathcal{L} \)-defective scheme of dimension zero is a special case of this.

In § 4, we will use \( \mathcal{L} \)-defectivity to formulate a generalisation of the geometric Riemann–Roch theorem. Before doing so, we digress for a moment to show how the phenomenon of secant defect is also related to the indeterminacy locus of the map \( \alpha: \text{Hilb}^d(\mathbb{P}V) \rightarrow \text{Quot}^{0,d}(V^*) \) defined in the previous section.

3.1. Relatively nondefective subschemes

We return to the situation of Theorem 2.6. The map \( \alpha: \text{Hilb}^d(\mathbb{P}V) \rightarrow \text{Quot}^{0,d}(V^*) \) is defined at \( Z \) if and only if \( \deg V_Z = \deg V + d \). This is clearly the case for generic \( Z \in \text{Hilb}^d(\mathbb{P}V)_0 \), in particular if \( \text{Supp}(\pi(Z)) \) consists of \( d \) distinct points of \( X \). However, if, for example, \( Z \) is a union of \( r + 1 \) points \( \nu_1, \ldots, \nu_{r+1} \) in general position in a fibre \( \mathbb{P}V|_x \), the evaluation map

\[
(\pi_*O_{\mathbb{P}V}(1))|_x = V^*|_x \rightarrow \pi_*([O_Z \otimes O_{\mathbb{P}V}(1)]_x) = \bigoplus_{i=1}^{r+1} \nu_i^*.
\]
is not surjective, and \( \deg V_Z < \deg V + \text{length} Z \). Indeed, in this case \( V_Z \cong V \otimes \mathcal{O}_{X}(x) \). This motivates a definition.

**Definition 3.4.** Let \( Z \subset \mathbb{P}V \) be a subscheme of dimension zero. Recall that we have defined \( \tau_Z := V_Z/V \). The subscheme \( Z \) will be called \((\mathcal{O}_{\mathbb{P}V}(1), \pi)\)-nondefective if the following equivalent conditions obtain:

- the map \( V^* \rightarrow \pi_*(\mathcal{O}_Z \otimes \mathcal{O}_{\mathbb{P}V}(1)) \) in (2.1) is surjective;
- the map \( V^*/V^*_Z \rightarrow \pi_*(\mathcal{O}_Z \otimes \mathcal{O}_{\mathbb{P}V}(1)) \) is an isomorphism;
- \( \deg V_Z = \deg V + \text{length} (Z) \); equivalently \( \deg(\tau_Z) = \text{length} (Z) \).

Otherwise, \( Z \) will be called \((\mathcal{O}_{\mathbb{P}V}(1), \pi)\)-defective.

Let us discuss the relevance of \((\mathcal{O}_{\mathbb{P}V}(1), \pi)\)-nondefectivity for Theorem 2.6.

**Remark 3.5.** (a) The locus of \((\mathcal{O}_{\mathbb{P}V}(1), \pi)\)-defective subschemes in \( \text{Hilb}^d(\mathbb{P}V) \) is exactly the indeterminacy locus of \( \alpha : \text{Hilb}^d(\mathbb{P}V)/(\text{axisshort/axisshort/arrowaxisright Quot}_0, d)(V^*) \).

(b) The scheme \( Z \) constructed in the proof of Theorem 2.2 is clearly \((\mathcal{O}_{\mathbb{P}V}(1), \pi)\)-nondefective.

(c) As \( R^1 \pi_* \mathcal{O}_{\mathbb{P}V}(1) = 0 \), a subscheme \( Z \) is \((\mathcal{O}_{\mathbb{P}V}(1), \pi)\)-nondefective if and only if

\[ R^1 \pi_*(I_Z \otimes \mathcal{O}_{\mathbb{P}V}(1)) = 0. \]

(d) By either (a) or (c) and the universal property of the Hilbert scheme, it follows that \((\mathcal{O}_{\mathbb{P}V}(1), \pi)\)-nondefectivity is an open property in (flat) families of zero-dimensional subschemes of length \( d \) of \( \mathbb{P}V \).

**Remark 3.6.** Geometrically speaking, \( Z \) is \((\mathcal{O}_{\mathbb{P}V}(1), \pi)\)-defective if and only if for at least one point \( x \in X \), the image of \( Z|_x \rightarrow \mathbb{P}V|_x \) is defective. Now, for any \( L \in \text{Pic}(X) \), the map \( Z|_x \rightarrow |\mathcal{O}_{\mathbb{P}V}(1) \otimes \pi^*L|_x^* \) factorises

\[ Z|_x \rightarrow \mathbb{P}V|_x = \mathbb{P}(V \otimes L^{-1})|_x \rightarrow |\mathcal{O}_{\mathbb{P}V}(1) \otimes \pi^*L|_x^* \]

when the last space is nonempty. This means that an \((\mathcal{O}_{\mathbb{P}V}(1), \pi)\)-defective \( Z \) is secant defective in \( |\mathcal{O}_{\mathbb{P}V}(1) \otimes \pi^*L|_x^* \) for any \( L \in \text{Pic}(X) \).

Finally, to unify the two types of defect discussed in this section, we make a **relative** definition of defect with respect to a line bundle.

**Definition 3.7.** Let \( f : Y \rightarrow S \) be a proper map of varieties, and let \( \mathcal{L} \) be a line bundle over \( Y \). We define a closed subscheme \( Z \subset Y \) to be \((\mathcal{L}, f)\)-nondefective if the natural map

\[ f_* \mathcal{L} \rightarrow f_*(\mathcal{O}_Z \otimes \mathcal{L}) \]  

(3.2)
is surjective, and \((\mathcal{L}, f)\)-defective otherwise.

**Remark 3.8.** Clearly, this generalises Definition 3.4. Moreover, if \( f : Y \rightarrow \text{Spec} (\mathbb{C}) \) is the structure map, then (3.2) reduces to the restriction map \( H^0(Y, \mathcal{L}) \rightarrow H^0(Z, \mathcal{O}_Z \otimes \mathcal{L}) \). If \( Z \) has dimension zero, then one checks using (3.1) that \( \text{def}(\psi(Z)) \) is the dimension of the cokernel of (3.2).
4. Geometric Riemann–Roch for vector bundles

Let \( V \to X \) be a vector bundle of rank \( r \) with \( h^1(X, V) \geq 1 \), and \( \pi : \mathbb{P}V \to X \) the corresponding \( \mathbb{P}^{r-1} \)-bundle. We begin by describing a map \( \mathbb{P}V \to \mathbb{P}H^1(X, V) \). This is a slight generalisation of a construction in [17, §3], also used in various guises in [4, 7, 12] and elsewhere.

By Serre duality and the projection formula, there are identifications

\[
H^1(X, V) \cong H^0(X, K_X \otimes V^*) = H^0(\mathbb{P}V, \pi^* K_X \otimes \mathcal{O}_V(1))^*.
\]

(4.1)

By standard algebraic geometry, we obtain a map \( \psi : \mathbb{P}V \to \mathbb{P}H^1(X, V) \). We now generalise [17, Theorem 3.1], which itself is a generalisation of a standard fact on line bundles over curves.

**Proposition 4.1.** The map \( \psi \) is an embedding if and only if for all effective degree two divisors \( x + y \) on \( X \) we have

\[
h^0(X, K_X(-x - y) \otimes V^*) = h^0(X, K_X \otimes V^*) - 2r;
\]
equivalently, \( h^0(X, V(x + y)) = h^0(X, V) \).

**Proof.** Suppose \( \nu \in \mathbb{P}V \), and consider the elementary transformation \( V_Z \) with \( Z \) the single point \( \nu \). By [13, Corollary 3.4], (with \( V = K_X M^{-1} \otimes E \)), the point \( \psi(\nu) \) is the projectivised image of the coboundary map in

\[
H^0(X, V) \to H^0(X, V_\nu) \to H^0(X, V_\nu/V) \to H^1(X, V)
\]

and, similarly, the embedded tangent space to \( \psi(\mathbb{P}V) \) at \( \psi(\nu) \) is the projectivised image of

\[
H^0\left(X, \frac{V_\nu \otimes \mathcal{O}_X(x)}{V}\right) \to H^1(X, V).
\]

(4.2)

Hence, \( \psi \) separates points if and only if for any distinct \( \nu, \nu' \in \mathbb{P}V \) the map

\[
H^0(X, V_\nu/V) \oplus H^0(X, V_{\nu'}/V) \to H^1(X, V)
\]

is injective. It is easy to see that this is equivalent to the injectivity of the coboundary map

\[
H^0(X, V(x + y)/V) \to H^1(X, V)
\]

for all pairs of distinct points \( x, y \in X \).

Moreover, the differential of \( \psi \) is everywhere injective if and only if (4.2) is injective for all \( \nu \). As above, it is easy to see that this is the case if and only if the coboundary map

\[
H^0(X, V(2x)/V) \to H^1(X, V)
\]

is injective for all \( x \in X \). The statement now follows from Serre duality together with exactness of

\[
0 \to H^0(X, V) \to H^0(X, V(x + y)) \to H^0(X, V(x + y)/V) \to H^1(X, V) \to H^1(X, V(x + y)/V) \to 0.
\]

\( \square \)

4.1. Geometric Riemann–Roch for elementary transformations

The situation discussed in this subsection is in fact a special case of that in §4.2, but the details are more transparent in this simpler setting, and the generalisation is only technically more involved.
**Proposition 4.2.** Let $Z$ be an element of $\text{Hilb}^d(\mathbb{P}V)$, not necessarily $(\mathcal{O}_{\mathbb{P}V}(1), \pi)$-nondefective. Then $\text{Span}(\psi(Z))$ is the projectivisation of the image of the coboundary map $\partial_Z$ of the sequence

$$0 \to H^0(X, V) \to H^0(X, V_Z) \to H^0(X, \tau_Z) \xrightarrow{\partial_Z} H^1(X, V) \to H^1(X, V_Z) \to 0. \tag{4.3}$$

**Proof.** By Serre duality, the map $H^1(X, V) \to H^1(X, V_Z)$ is identified with

$$H^0(X, K_X \otimes V^*)^* \to H^0(X, K_X \otimes V_Z^*)^*.$$ 

By the projection formula and since $V_Z^* = \pi_*(\mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}V}(1))$, this becomes in turn

$$H^0(\mathbb{P}V, \mathcal{O}_{\mathbb{P}V}(1) \otimes \pi^*K_X)^* \to H^0(\mathbb{P}V, \mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}V}(1) \otimes \pi^*K_X)^*.$$ 

The statement now follows from (3.1) by exactness of (4.3). \hfill \Box

Now, we can generalise the geometric Riemann–Roch theorem.

**Theorem 4.3** (Geometric Riemann–Roch for elementary transformations). Let $0 \to V \to \tilde{V} \to \tau \to 0$ be an elementary transformation. Then for any $(\mathcal{O}_{\mathbb{P}V}(1), \pi)$-nondefective $Z$ such that $V_Z \cong \tilde{V}$ as vector bundles, we have $h^0(X, V) - h^0(X, V) = \text{def}(\psi(Z))$.

Note that by Theorem 2.2, such a $Z$ always exists.

**Proof.** Let $Z$ be a zero-dimensional $(\mathcal{O}_{\mathbb{P}V}(1), \pi)$-nondefective subscheme of $\mathbb{P}V$ such that the vector bundles $\tilde{V}$ and $V_Z$ are isomorphic. We have

$$\dim(\text{Span}(\psi(Z))) = \dim(\text{Im}(\partial_Z)) - 1 \text{ by Proposition 4.2}$$

$$= h^0(X, \tau_Z) - (h^0(X, \tilde{V}) - h^0(X, V)) - 1 \text{ by exactness and since } V_Z \cong \tilde{V}$$

$$= \text{length}(Z) - (h^0(X, \tilde{V}) - h^0(X, V)) - 1 \text{ by (4.3) -nondefectivity.}$$

Therefore, $\text{length}(Z) - \dim(\text{Span}(\psi(Z))) = h^0(X, \tilde{V}) - h^0(X, V)$. As the expression on the left is exactly $\text{def}(\psi(Z))$, the statement follows. \hfill \Box

Note that the vector bundle $\tilde{V}$ alone does not always determine $Z$; for example, distinct linearly equivalent effective divisors define isomorphic line bundles. If $V$ is a line bundle, the condition that $V^*_Z = \tilde{V}^*$ in $\text{Quot}^{0,d}(V^*)$ uniquely determines $Z$. This is no longer true for $r \geq 2$ (Remark 2.5). However, we can give a necessary geometric condition for the equality $V^*_Z = V^*_Z$ in $\text{Quot}^{0,d}(V^*)$.

**Proposition 4.4.** Suppose $Z$ and $Z'$ are such that $V^*_Z$ and $V^*_Z$, define the same point of $\text{Quot}^{0,d}(V^*)$. Then $\text{Span}(\psi(Z)) = \text{Span}(\psi(Z'))$.

**Proof.** By hypothesis and by definition of the Quot scheme, $V^*_Z = V^*_Z$; as subsheaves of $V^*$. A diagram chasing argument shows that

$$\text{Ker}(H^1(X, V) \to H^1(X, V_Z)) = \text{Ker}(H^1(X, V) \to H^1(X, V_{Z'})). \tag{4.4}$$

But by Proposition 4.2 and exactness, for any zero-dimensional $Z \subset \mathbb{P}V$ we have

$$\text{Span}(\psi(Z)) = \mathbb{P} \text{Im} \left( \Gamma(\tau_Z) \to H^1(X, V) \right) = \mathbb{P} \text{Ker}(H^1(X, V) \to H^1(X, V_Z)).$$

Putting this together with (4.4), we see that $\text{Span}(\psi(Z)) = \text{Span}(\psi(Z'))$. \hfill \Box
Remark 4.5. (a) Proposition 4.4 does not require that $Z$ and $Z'$ be $(\mathcal{O}_\Pi V(1), \pi)$-nondefective, or of the same length. For example, suppose $V$ has rank 2 and $Z$ is any finite reduced subscheme of length $d \geq 2$ of a fibre $\mathbb{P}V|_x$. Then $\text{Span}(Z)$ is the fibre $\mathbb{P}V|_x$ and $V_Z$ is isomorphic to $V \otimes \mathcal{O}_X(x)$, independently of length $(Z)$.

(b) The converse of the proposition does not hold. For example, if $h^1(X, V) = 1$, then $\psi$ is constant.

Remark 4.6. Suppose $V = \mathcal{O}_X$. Then any elementary transformation $0 \to V \to \tilde{V} \to \tau \to 0$ is necessarily of the form $0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D(D) \to 0$ for some $D \in \text{Sym}^2 X = \text{Hilb}^d(\mathbb{P}V)$. Here, $\pi : \mathbb{P}V \to X$ is the identity map and, trivially, all $D \in \text{Sym}^2 X$ are $(\mathcal{O}_\Pi V(1), \pi)$-nondefective. Moreover, $\psi : \mathbb{P}V \to \mathbb{P}H^1(X, \mathcal{O}_X)$ coincides with the canonical map $\phi_{K_X} : X \to [K_X]^*$. Hence, by Theorem 4.3, for each $D \in \text{Sym}^2 X$ we have

$$h^0(X, \mathcal{O}_X(D)) - h^0(X, \mathcal{O}_X) = \text{def}(\phi_{K_X}(D)),$$

that is,

$$h^0(X, \mathcal{O}_X(D)) - 1 = (d - 1) - \dim \text{Span}(\phi_{K_X}(D)),$$

so we recover the standard geometric Riemann–Roch theorem.

Before proceeding, we recall that a vector bundle $V \to X$ is stable if for each proper subbundle $W \subset V$ we have $\frac{\text{deg}(W)}{\text{rk}(W)} < \frac{\text{deg}(V)}{\text{rk}(V)}$. As is well known, stability is an open condition on families of vector bundles of fixed rank and degree. We denote by $U_X(r, e)$ the moduli space of stable bundles of rank $r$ and degree $e$ over $X$.

Remark 4.7 (A generalised Abel–Jacobi map). For $d \geq 1$, let $\alpha_d : \text{Sym}^d X \to \text{Pic}^d(X)$ be the Abel–Jacobi map $D \to \mathcal{O}_X(D)$. As was classically known, $\alpha_d^{-1}(L)$ is exactly the linear series $|L|$. More generally; fix a bundle $V$ of rank $r$ and degree $e - d$. Sending a length $d$ subscheme $Z \subset \mathbb{P}V$ to the moduli point of $V_Z$ in $U_X(r, e)$ defines a rational map

$$\alpha_{V,d} : \text{Hilb}^d(\mathbb{P}V) \dashrightarrow U_X(r, e),$$

generalising the Abel–Jacobi map. In particular, if $V = \mathcal{O}_X^{\oplus r}$, then as in the rank one case, the image of $\alpha_{V,d}$ is exactly the locus of stable, generically generated bundles. Moreover, by Theorem 4.3, the preimage of

$$\{ E \in U_X(r, d) : E \text{ is generically generated and } h^0(X, E) \geq r + 1 \} \subseteq B_{r,d}^r$$

is exactly the locus

$$\left\{ Z \in \text{Hilb}^d(\mathbb{P}(\mathcal{O}_X^{\oplus r})) : \psi(Z) \text{ is defective in } \mathbb{P}H^1(X, \mathcal{O}_X^{\oplus r}) \text{ and } (\mathcal{O}_X^{\oplus r})_Z \text{ is stable} \right\}.$$

For $r \geq 2$, the situation is complicated by the requirement of stability (but see [3]) and the presence of nontrivial automorphisms of $X \times \mathbb{P}^r$. Nonetheless, viewing sums of points on $X$ as zero-dimensional subschemes (as opposed to codimension one subschemes), Theorems 2.6 and 4.3 give a natural generalisation of the picture for line bundles and linear series on $X$ to bundles of higher rank.

4.2. A relative version

For applications to Brill–Noether loci, we will need a more general version of Theorem 4.3. Let $V$ and $F$ be vector bundles over $X$. For any $Z \in \text{Hilb}^d(\mathbb{P}V)$, we have an exact sequence

$$0 \to V \otimes F \to V_Z \otimes F \to \tau_Z \otimes F \to 0.$$
Inside the projective bundle $\mathbb{P}(V \otimes F)$ we have the rank one locus $\Delta := \mathbb{P}V \times_X \mathbb{P}F$, also called the decomposable locus. There is a commutative diagram

$$
\begin{array}{ccc}
\Delta & \xrightarrow{\bar{\omega}} & \mathbb{P}V \\
\downarrow{\pi} & & \downarrow{\pi} \\
\mathbb{P}F & \xrightarrow{\omega} & X.
\end{array}
$$

(4.5)

Since $\mathbb{P}F \to X$ is flat, for any closed subscheme $Z \subset \mathbb{P}V$ we have $I_{Z \times_X \mathbb{P}F} = \bar{\omega}^* I_Z$. Thus, if $\mathcal{L} \to \Delta$ is a line bundle and $\psi: \Delta \to |\mathcal{L}|^*$ the associated map, in view of (3.1) we obtain

$$
\text{Span}(\psi(Z \times_X \mathbb{P}F)) = \mathbb{P}\ker(H^0(\Delta, \mathcal{L})^* \to H^0(\Delta, \bar{\omega}^* I_Z \otimes \mathcal{L})^*). \quad (4.6)
$$

We observe that $\Delta$ is just the projective bundle $\mathbb{P}(\pi^* F) \to \mathbb{P}V$. If $Z \in \text{Hilb}^d(\mathbb{P}V)$ is reduced, $Z \times_X \mathbb{P}F$ is a union of $d$ fibres of $\mathbb{P}F$.

Now, we can generalise the results of §4.1. We write $\mathcal{L}$ for the line bundle

$$(\bar{\pi}^* \omega^* K_X) \otimes (\bar{\pi}^* O_{\mathbb{P}F}(1)) \otimes (\bar{\omega}^* O_{\mathbb{P}V}(1)) \to \Delta.$$

**Proposition 4.8.** (a) There is a natural identification $H^1(X, V \otimes F) \xrightarrow{\sim} H^0(\Delta, \mathcal{L})^*$. In particular, there is a natural map $\Delta \to \mathbb{P}^1(X, V \otimes F)$, which we again denote $\psi$.

(b) Via the above identification, $\text{Span}(\psi(Z \times_X \mathbb{P}F))$ coincides with the projectivised image of the coboundary map in the sequence

$$
\begin{align*}
0 & \to H^0(X, V \otimes F) \to H^0(X, V_Z \otimes F) \to H^0(X, \tau_Z \otimes F) \\
& \to H^1(X, V \otimes F) \to H^1(X, V_Z \otimes F) \to 0. \quad (4.7)
\end{align*}
$$

(c) The dimension of $\text{Span}(\psi(Z \times_X \mathbb{P}F))$ is at most $\text{rk}(F) \cdot \text{length}(Z) - 1$.

**Proof.** (a) This is a technical but straightforward computation. By Serre duality and repeated use of the projection formula, we get an identification

$$
H^1(X, V \otimes F) \xrightarrow{\sim} H^0(\mathbb{P}V, \pi^* \{\omega_* (\omega^* K_X \otimes O_{\mathbb{P}F}(1)) \otimes O_{\mathbb{P}V}(1)\}^*). \quad (4.8)
$$

As $\pi: \mathbb{P}V \to X$ is flat, by [10, Proposition III.9.3] there is a canonical isomorphism

$$
\pi^* (\omega_* \mathcal{S}) \xrightarrow{\sim} \bar{\omega}_* (\bar{\pi}^* \mathcal{S})
$$

for any coherent sheaf $\mathcal{S}$ on $\mathbb{P}F$. Setting $\mathcal{S} = \omega_* K_X \otimes O_{\mathbb{P}F}(1)$ in (4.8), we obtain

$$
H^0(\mathbb{P}V, \bar{\omega}_* (\bar{\pi}^* (\omega^* K_X \otimes O_{\mathbb{P}F}(1)) \otimes O_{\mathbb{P}V}(1))^*).
$$

Using the projection formula, we see easily that this is identified with

$$
H^0(\Delta, (\bar{\pi}^* \omega^* K_X) \otimes (\bar{\pi}^* O_{\mathbb{P}F}(1)) \otimes (\bar{\omega}^* O_{\mathbb{P}V}(1)))^*,
$$

which is exactly $H^0(\Delta, \mathcal{L})^*$.

(b) A calculation similar to that in (a) shows that via the identification in (a), the subspace

$$
H^1(X, V_Z \otimes F)^* \cong H^0(X, K_X \otimes F^* \otimes V_Z^*) \subset H^0(X, K_X \otimes F^* \otimes V^*)
$$

is identified with $H^0(\Delta, \bar{\omega}^* I_Z \otimes \mathcal{L}) \subset H^0(\Delta, \mathcal{L})$. Then the statement follows from (4.6).

(c) We have $h^0(X, \tau_Z \otimes F) \leq \text{length}(Z) \cdot \text{rk}(F)$ (with equality if and only if $Z$ is $(O_{\mathbb{P}V}(1), \pi)$-nondefective). Now, the statement follows from part (b). \[\square\]

In view of Proposition 4.8(c), the following definition makes sense.
**Definition 4.9.** For $V$, $F$, $Z$, and $\psi$ as above, the defect of $\psi(Z \times_X \mathbb{P}F)$ is
\[
def(\psi(Z \times_X \mathbb{P}F)) := (\text{rk}(F) \cdot \text{length}(Z) - 1) - \dim \text{Span}(\psi(Z \times_X \mathbb{P}F)).
\]

Now, we can give the desired relative version of Theorem 4.3.

**Theorem 4.10** (Relative generalised geometric Riemann–Roch). For any $(\mathcal{O}_V(1), \pi)$-nondefective $Z$ such that $V_Z \cong \hat{V}$ as vector bundles, we have
\[
h^0(X, \hat{V} \otimes F) - h^0(X, V \otimes F) = \text{def}(\psi(Z \times_X \mathbb{P}F)).
\]

**Proof.** Using Proposition 4.8, this can be proven exactly as Theorem 4.3. \qed

We also have a generalisation of Proposition 4.4, with a virtually identical proof:

**Proposition 4.11.** Suppose $Z$ and $Z'$ are such that $V_Z$ and $V_{Z'}$ define the same point of $\text{Quot}^{0,d}(V^*)$. Then $\text{Span}(Z \times_X \mathbb{P}F) = \text{Span}(Z' \times_X \mathbb{P}F)$.

**Remark 4.12.** The idea behind Theorem 4.10 is present in a recent work of Brivio [4]. Let $E$ be a general bundle in $U_X(r, r(2g-1))$ and $D = x_1 + \cdots + x_g$ an effective divisor of degree $g$. The condition of interest in [4] is that $h^0(X, E(-D)) = 1$. As $\chi(X, E(-D)) = 0$, this is equivalent to $h^0(X, K_X \otimes E^*(D)) = 1$. These spaces appear in the cohomology sequence
\[
0 \to H^0(X, K_X \otimes E^*) \to H^0(X, K_X \otimes E^*(D)) \to H^0(X, K_X \otimes E^*(D)|_D)
\]
\[
\to H^0(X, E^*) \to H^0(X, E(-D))^* \to 0.
\]

This is exactly (4.7) with $V = K_X(-D)$ and $Z = D$, and $F = E^*$. Then by Proposition 4.8(b), we can interpret the coboundary map as the restriction of the tautological model of [4] to the fibres of $\mathbb{P}(K_X \otimes E^*) \cong \mathbb{P}E^*$ along $D$. (Note that $\mathbb{P}E^*$ is denoted by $\mathbb{P}E$ in [4].) Then Theorem 4.10 gives another proof that $h^0(X, E(-D)) = h^0(X, K_X \otimes E^*(D)) = 1$ if and only if the fibres $\mathbb{P}E^*|_{x_1}, \ldots, \mathbb{P}E^*|_{x_g}$ span a space of dimension one less than expected; equivalently, that there exist
\[
(\eta_1, \ldots, \eta_g) \in \mathbb{P}E^*|_{x_1} \times \cdots \times \mathbb{P}E^*|_{x_g},
\]
which are linearly dependent in $\mathbb{P}H^0(X, E)^*$. (Compare with [4, Lemma 5.1, Proposition 7.2].)

5. Tangent cones of higher rank Brill–Noether loci

Suppose $L \to X$ is a line bundle of degree $d \leq g$ with $h^0(X, L) \geq k$, so defining a point of the Brill–Noether locus
\[
W^{k-1}_d(X) = \{ L \in \text{Pic}^d(X) : h^0(X, L) \geq k \}.
\]
The projectivised tangent cone $\mathbb{P}T_L W^{k-1}_d(X)$ at $L$ belongs to
\[
\mathbb{P}T_L \text{Pic}^d(X) = \mathbb{P}H^1(X, \mathcal{O}_X) = |K_X|^*.
\]
Then the Riemann–Kempf singularity theorem (see [1, Chapter VI.2]) states that
\[
\mathbb{P}T_L W^{k-1}_d(X) = \bigcup_{g \leq 1} \left( \bigcap_{D \in g^{-1} \text{Pic}(K_X)} \text{Span}(\phi_K(D)) \right),
\]
where as before $\phi_K$ is the canonical map. We will use the results in §4 to generalise this picture to bundles of higher rank.
5.1. Higher rank Brill–Noether loci

Here, we recall briefly some essential facts, referring the reader to [6, 9] for details. The moduli space $U_X(r, d)$ of stable bundles of rank $r$ and degree $d$ over $X$ is an irreducible quasi-projective variety of dimension $r^2(g - 1) + 1$. The Brill–Noether locus $B^k_{r, d}$ is defined set-theoretically by

$$B^k_{r, d} = \{ E \in U_X(r, d) : h^0(X, E) \geq k \}.$$

This is a determinantal subvariety of $U_X(r, d)$, with expected codimension $k(k - d + r(g - 1))$, when this is nonnegative. Suppose $E \in U_X(r, d)$ satisfies $h^0(X, E) = k$. Then the Zariski tangent space $T_E B^k_{r, d}$ is exactly $\text{Im} (\mu) \perp$, where $\mu$ is the Petri map

$$H^0(X, E) \otimes H^0(X, K_X \otimes E^\ast) \rightarrow H^0(X, K_X \otimes \text{End} E).$$

Equivalently, via Serre duality we have

$$T_E B^k_{r, d} = \text{Ker}(\cup : H^1(X, \text{End} E) \rightarrow \text{Hom}(H^0(X, E), H^1(X, E))).$$

It follows that $B^k_{r, d}$ is smooth and of the expected dimension at $E$ if and only if $\mu$ is injective; equivalently, if $\cup$ is surjective.

More generally, suppose $h^0(X, E) \geq k$. To ease notation, for any $\ell$, we denote the Grassmannian $\text{Gr}(\ell, H^0(X, E))$ simply by $Gr^k$. Write $S^k$ for the universal subbundle of $Gr^k \times H^0(X, E)$, and consider the diagram of bundles over $Gr^k$ as follows:

\[
\begin{array}{ccc}
\text{Gr}^k \times H^1(X, \text{End} E) & \xrightarrow{\text{Id} \times \cup} & \text{Gr}^k \times \text{Hom}(H^0(X, E), H^1(X, E)) \\
\cup \downarrow & & \downarrow \\
\text{Hom}(S^k, Gr^k \times H^1(X, E)) & & \\
\end{array}
\]

**Definition 5.1** (cf. [6, §2]). Let $E$ be as above. If the restricted Petri map

$$\mu_\Lambda := \mu|_{\Lambda \otimes H^0(X, K_X \otimes E^\ast)}$$

is injective for all subspaces $\Lambda \in \text{Gr}^k$, then $E$ is said to be Petri $k$-injective. Equivalently, via Serre duality, $E$ is Petri $k$-injective if and only if

$$\cup_\Lambda : H^1(X, \text{End} E) \rightarrow \text{Hom}(\Lambda, H^1(X, E))$$

is surjective for all $\Lambda \in \text{Gr}^k$.

If $E$ is Petri $k$-injective, then $\text{Ker}(\cup)$ is a vector subbundle of $\text{Gr}^k \times H^1(X, \text{End} E)$. By [6, Theorem 2.4 (4)], the tangent cone $T_E B^k_{r, d}$ is, set-theoretically,

$$\bigcup_{\Lambda \in \text{Gr}^k} \text{Im}(\mu_\Lambda) \perp = \bigcup_{\Lambda \in \text{Gr}^k} \text{Ker}(\cup_\Lambda),$$

the second expression following from Serre duality as above. Projectivising, we obtain the following.

**Lemma 5.2.** Suppose $E$ is a stable bundle with $h^0(X, E) \geq k$ which is Petri $k$-injective. Then the projectivised tangent cone $\mathbb{P}T_E B^k_{r, d}$ is the image of the projective subbundle

$$\mathbb{P}\text{Ker}(\cup) \subseteq \text{Gr}^k \times \mathbb{P}H^1(X, \text{End} E)$$

by the projection to $\mathbb{P}H^1(X, \text{End} E)$. In particular, $\mathbb{P}T_E B^k_{r, d}$ is irreducible.
Note that as all fibres of (5.1) contain the subspace
\[ \mathbb{P} \text{Ker}(\cup): H^1(X, \text{End} E) \to \text{Hom}(H^0(X, E), H^1(X, E)) = \mathbb{P} T_E B^0_{r,d}(X,E), \]
if \( h^0(X, E) > k \), then the map \( \mathbb{P} \text{Ker}(\cup) \to \mathbb{P} H^1(X, \text{End} E) \) is an embedding only if \( \cup \) is injective; equivalently, if \( \mu \) is surjective.

5.2. Riemann–Kempf for generically generated bundles

Throughout this section, \( E \) will be a stable bundle with \( h^0(X, E) \geq k \geq r \), and which is generically generated and Petri \( k \)-injective. Setting \( V = E^* \) and \( F = E \), we consider the map \( \psi: \mathbb{P} \text{End} E \to \mathbb{P} H^1(X, \text{End} E) \) defined in §4.

Remark 5.3. Although we will not require this fact, we mention that if \( g \geq 5 \), then \( \psi \) is an embedding for general nonhyperelliptic \( X \) and general \( E \in U_X^{r,d} \). For \( d = r(g - 1) \), this is [12, Theorem 4.3] (which relies in an essential way on [17, Proposition 3.2]), and the proof is valid for any value of \( d \).

Let \( \Pi \subseteq H^0(X, E) \) be a subspace of dimension \( r \) which generically generates \( E \). Transposing the evaluation map \( \mathcal{O}_X \otimes \Pi \to E \), we obtain an elementary transformation
\[ 0 \to E^* \to \mathcal{O}_X \otimes \Pi^* \to \tau_\Pi \to 0, \]
where \( \tau_\Pi \) is a torsion sheaf of degree \( d \), supported along a divisor belonging to the linear series \([\text{det}(E)]\). Hence, by Theorem 2.2, there exists \( Z_\Pi \in \text{Hilb}^d(\mathcal{P} E^*) \) such that the elementary transformation \( \mathcal{O}_X \otimes \Pi^* \) coincides with \( (E^*)_{Z_\Pi} \) as points of \( \text{Quot}^{0,d}(E) \) (cf. Remark 2.1). Moreover, there is a commutative diagram
\[
\begin{array}{ccc}
H^1(X, \text{End} E) & \overset{\cup_\Pi}{\longrightarrow} & \text{Hom}(\Pi, H^1(X, E)), \\
H^1(X, \Pi^* \otimes E) & \underset{\sim}{\longrightarrow} & \\
\end{array}
\]
whence \( \text{Ker}(\cup_\Pi) = \text{Ker}(H^1(X, \text{End} E) \to H^1(X, \Pi^* \otimes E)) \). Therefore, by exactness and by Proposition 4.8(b), with \( V = E^* \) and \( V_Z = (E^*)_{Z_\Pi} = \mathcal{O}_X \otimes \Pi^* \) and \( F = E \), we have
\[ \mathbb{P} \text{Ker}(\cup_\Pi) = \text{Span}(Z_\Pi \times_X \mathcal{P} E). \tag{5.2} \]
Note that although \( Z_\Pi \) may not be unique, \( \text{Span}(Z_\Pi \times_X \mathcal{P} E) \) is independent of the choice of \( Z_\Pi \) by Proposition 4.11.

Let us introduce some more notation. For any \( \ell \geq r \), set
\[ U_\ell := \{ \Lambda \in \text{Gr}^\ell : \Lambda \text{ generically generates } E \}. \]
This is an open subset of \( \text{Gr}^\ell \), which by hypothesis is nonempty and therefore dense. Furthermore, for \( \ell = r \), consider the set
\[ U^r_\text{red} := \{ \Pi \in U^r : \text{Supp}(\tau_\Pi) \text{ is reduced} \}. \]
This is an open subset of \( U^r \), being the inverse image of the complement of the big diagonal in \( \text{Sym}^d X \) by the map \( \Pi \mapsto \text{Supp}(\tau_\Pi) \). Thus, \( U^r_\text{red} \) is also open in \( \text{Gr}^r \).
If \( \Pi \in U^r_{\text{red}} \), then by Theorem 2.2(b) the scheme \( Z_\Pi \) is reduced and uniquely determined. Explicitly, \( Z_\Pi = \{ \nu_1, \ldots, \nu_d \} \), where \( \nu_i = \text{Ker}(E^*|_{x_i} \to \Pi) \) for distinct \( x_1, \ldots, x_d \in X \). Equivalently, \( \nu_i \in E^*|_{x_i} \) defines the hyperplane \( \text{Im}(\Pi \to E|_{x_i}) \). In this case,

\[
Z_\Pi \times_X \mathbb{P}E = \bigcup_{i=1}^d \mathbb{P}(\nu_i \otimes E|_{x_i}).
\]

More generally, suppose \( k \geq r \). For \( \Lambda \in \text{Gr}^k \), note that \( \text{Gr}(r, \Lambda) \) is naturally contained in \( \text{Gr}^r \).

Define a linear subseries \( U^k_{\text{red}} := \{ \Lambda \in U^k : \text{Gr}(r, \Lambda) \cap U^r_{\text{red}} \text{ is dense in } \text{Gr}(r, \Lambda) \} \).

**Lemma 5.4.** (a) If \( E \) is globally generated, then \( U^r_{\text{red}} \) is nonempty.

(b) Suppose \( U^r_{\text{red}} \) is nonempty. Then for \( k \geq r \), the locus \( U^k_{\text{red}} \) is constructible and dense in \( \text{Gr}^k \).

**Proof.** (a) If \( E \) is generated and stable, then we must have \( d \geq 1 \) and \( h^0(X, E) \geq r + 1 \). Thus, we may choose a subspace \( \Omega \subseteq H^0(X, E) \) of dimension \( r + 1 \) which generates \( E \). Note that \( \text{Gr}(r, \Omega) = \mathbb{P}^* \). The divisors

\[
\{ \text{Supp}(\tau_\Pi) \in \text{Sym}^dX : \Pi \in U^r \cap \mathbb{P}^* \}
\]

define a linear subseries \( g_\delta \) of \( |\det(E)| \). As \( E \) is generated, for any \( x \in X \) one can find \( \Pi \in \mathbb{P}^* \) which generates \( E|_x \), and so \( x \notin \text{Supp}(\frac{E}{\mathbb{P}^*}) = \text{Supp}(\tau_\Pi) \). Hence, this \( g_\delta \) is base point free. Therefore, the set \{ \( D \in g_\delta : D \) nonreduced \} has codimension at least one in \( g_\delta \). Thus, the support of \( \tau_\Pi \) is reduced for generic \( \Pi \in \mathbb{P}^* \), so such a \( \Pi \) belongs to \( U^r_{\text{red}} \).

(b) If \( k = r \) and \( U^r_{\text{red}} \) is nonempty, then it is dense in \( \text{Gr}^r \) since it is open. Suppose \( k \geq r \). Let \( S^k \subseteq \text{Gr}^k \times H^0(X, E) \) be the universal subbundle as before. We have also the relative Grassmannian bundle \( \text{Gr}(r, S^k) \subseteq \text{Gr}^k \times \text{Gr}^r \). The subset

\[
\{ (\Lambda, \Pi) \in \text{Gr}(r, S^k) : \Pi \in U^r_{\text{red}} \} \subseteq \text{Gr}(r, S^k).
\]

As \( U^r_{\text{red}} \) is open and dense in \( \text{Gr}^r \), it is easy to see that (5.4) is open and dense in \( \text{Gr}(r, S^k) \). As \( U^k_{\text{red}} \) is the image of (5.4) via the projection \( \text{Gr}(r, S^k) \to \text{Gr}^k \), we see that \( U^k_{\text{red}} \) is constructible and dense in \( \text{Gr}^k \). \qed

We require one more lemma, whose proof is left to the reader.

**Lemma 5.5.** Suppose \( \Lambda \in \text{Gr}^k \), and let \( U' \) be an open dense subset of \( \text{Gr}(r, \Lambda) \). Then

\[
\text{Ker}(\bigcup_{\Pi \in U'} \text{Ker}(\bigcup_{\Pi : H^1(X, \text{End} E) \to \text{Hom}(\Pi, H^1(X, E)))} \big)
\]

Now, we can give the generalised Riemann–Kempf singularity theorem for \( B^k_{r,d} \).

**Theorem 5.6.** Suppose \( E \) is a stable bundle of degree \( d \leq r \) with \( h^0(X, E) \geq k \geq r \) which is Petri \( k \)-injective and generically generated.

(a) For any \( \Pi \in U^r \), we have \( \text{def}(\psi(Z_\Pi \times_X \mathbb{P}E)) = r \cdot h^0(X, E) - 1 \).

(b) The projectivised tangent cone \( \mathbb{PT}_E B^k_{r,d} \) is the Zariski closure of

\[
\bigcup_{\Lambda \in U^k} \left( \bigcap_{\Pi \in U^r \cap \text{Gr}(r, \Lambda)} \text{Span}(\psi(Z_\Pi \times_X \mathbb{P}E)) \right).
\]
(c) Suppose $U^r_{\text{red}}$ is nonempty (for example, if $E$ is globally generated). Write $\pi : \mathbb{P}E^* \to X$ for the projection. Then $\mathbb{P}T_{E}B^k_{r,d}$ is the Zariski closure of

$$
\bigcup_{\Lambda \in U^k_{\text{red}}} \left( \bigcap_{\Pi \in U^r_{\text{red}} \cap \text{Gr}(r,\Lambda)} \text{Span} \left( \bigcup_{\nu \in Z_{\Pi}} \psi(\mathbb{P}(\nu \otimes E|_{\pi(\nu)}) \right) \right).
$$

(5.6)

Remark 5.7. Note that our hypotheses imply that $B^k_{r,d}$ is a proper sublocus of $U_X(r,d)$. For, since $d < r g$, we have $\chi(X, E) < r$. As $k \geq r$, any element of $B^k_{r,d}$ is a special bundle. Deforming a suitable direct sum of nonspecial line bundles, we see that a general bundle in $U_X(r,d)$ is nonspecial, so $B^k_{r,d}$ cannot be dense in $U_X(r,d)$.

Proof of Theorem 5.6. (a) As $E$ is stable, $h^0(X, \text{End } E) = 1$. By Theorem 4.10, we have

$$
def(Z_{\Pi} \times_X \mathbb{P}E) = h^0(X, \Pi^* \otimes E) - h^0(X, E^* \otimes E) = r \cdot h^0(X, E) - 1.
$$

(b) Since $E$ is generically generated and $k \geq r$, the locus $U^k$ is dense in $\text{Gr}^k$. Thus, by Lemma 5.2 and a topological argument, $T_{E}B^k_{r,d}$ is the Zariski closure

$$
\bigcup_{\Lambda \in U^k} \text{Ker}(\tilde{\cup}_{\Lambda} : H^1(X, \text{End } E) \to \text{Hom}(\Lambda, H^1(X, E))).
$$

(5.7)

Now, it is easy to see that $\Lambda \in U^k$ if and only if $U^r \cap \text{Gr}(r, \Lambda)$ is open and dense in $\text{Gr}(r, \Lambda)$. Therefore, by Lemma 5.5, the locus (5.7) coincides with

$$
\bigcup_{\Lambda \in U^k} \left( \bigcap_{\Pi \in U^r \cap \text{Gr}(r,\Lambda)} \text{Ker}(\tilde{\cup}_{\Pi} : H^1(X, \text{End } E) \to \text{Hom}(\Pi, H^1(X, E))) \right).
$$

Now, every $\Pi$ appearing in this expression belongs to $U^r$. Therefore, projectivising and using (5.2) we see, as desired, that $\mathbb{P}T_{E}B^k_{r,d}$ coincides with

$$
\bigcup_{\Lambda \in U^k} \left( \bigcap_{\Pi \in U^r \cap \text{Gr}(r,\Lambda)} \text{Span}(\psi(\mathbb{P}_{\Pi \times X} \mathbb{P}E)) \right).
$$

(c) By hypothesis and by Lemma 5.4(b), the locus $U^k_{\text{red}}$ is dense in $\text{Gr}^k$. As in part (a) above, $T_{E}B^k_{r,d}$ is the Zariski closure

$$
\bigcup_{\Lambda \in U^k_{\text{red}}} \text{Ker}(\tilde{\cup}_{\Lambda} : H^1(X, \text{End } E) \to \text{Hom}(\Lambda, H^1(X, E))).
$$

(5.8)

By definition of $U^k_{\text{red}}$, for each $\Lambda \in U^k_{\text{red}}$ the locus $U^r_{\text{red}} \cap \text{Gr}(r, \Lambda)$ is open and dense in $\text{Gr}(r, \Lambda)$. Hence, by Lemma 5.5, the locus (5.8) coincides with

$$
\bigcup_{\Lambda \in U^k_{\text{red}}} \left( \bigcap_{\Pi \in U^r_{\text{red}} \cap \text{Gr}(r,\Lambda)} \text{Ker}(\tilde{\cup}_{\Pi} : H^1(X, \text{End } E) \to \text{Hom}(\Pi, H^1(X, E))) \right).
$$
Now, every $\Pi$ appearing in this expression belongs to $U^r_{\text{red}}$. Projectivising and using (5.3), we have as desired

$$\mathbb{P}T_E B^r_{e,d} = \bigcup_{\Delta \in U^r_{\text{red}}} \left( \bigcap_{\Pi \in U^r_{\text{red}} \cap \text{Gr}(r,\Lambda)} \text{Span} \left( \bigcup_{\nu \in \mathbb{Z}_n} \psi(\mathbb{P}(\nu \otimes E_{|\pi(\nu)})) \right) \right).$$

$$\square$$

**Remark 5.8.** The hypothesis of Petri $k$-injectivity is only required in the proofs of (b) and (c). In general, the loci (5.5) and (5.6) are contained in $\mathbb{P}T_E B^r_{e,d}$.

### 5.3. Secant varieties

In [6, §5], it is shown that the tangent cones to certain generalised theta divisors contain secant varieties of the curve $X$. Here, we set $k = r$ and deduce a similar statement for $\mathbb{P}T_E B^r_{e,d}$ using Theorem 5.6. The following is a direct generalisation of the fact (see [1, Theorem VI.1.6.1 (i)]) that if $h^0(X, L) = n + 1 \geq 2$ for some $L \in \text{Pic}^{r-1}(X)$, then $\mathbb{P}T_L B^1_{1,\beta-1}$ contains $\text{Sec}^n(\phi_{KX}(X))$. Note that here we do not require $E$ to be Petri $k$-injective.

**Theorem 5.9.** Suppose $E$ is a stable, generically generated bundle of degree $d < rg$ with $h^0(X, E) = n + r$ for some $n \geq 1$. Then $\mathbb{P}T_E B^r_{e,d}$ contains $\text{Sec}^n(\psi(\Delta))$.

**Proof.** We begin by showing that the condition that $s \in H^0(X, E)$ annihilate $n$ sufficiently general points of $\mathbb{P}E^*$ defines a subspace of dimension $r$ which generically generates $E$. Let $\beta: \text{Sym}^n \mathbb{P}E^* \to \text{Gr}^r$ be defined by

$$\{\nu_1, \ldots, \nu_n\} \mapsto \{s \in H^0(X, E) : s|_{\nu_j} = 0 \text{ for } 1 \leq j \leq n\}.$$

Let us show that $\beta$ is defined on an open subset of $\text{Sym}^n \mathbb{P}E^*$, and that a general point of the image belongs to $U^r$.

First, suppose $E$ is generated by global sections, so that $U^r_{\text{red}}$ is dense in $\text{Gr}^r$ by Lemma 5.4. Let $\Pi$ be any element of $U^r_{\text{red}}$. Then $Z_\Pi$ is a collection of points $\nu_1, \ldots, \nu_d$ of $\mathbb{P}E^*$ lying over distinct points $x_1, \ldots, x_d$ of $X$, and moreover there is an elementary transformation $0 \to \mathcal{O}_X \otimes \Pi \to E \xrightarrow{(\nu_1, \ldots, \nu_d)} \bigoplus_{i=1}^d \mathcal{O}_{x_i} \to 0$. Taking global sections, we see that

$$\Pi = \{s \in H^0(X, E) : s|_{\nu_i} = 0 \text{ for } 1 \leq i \leq d\}.$$

Now, it is easy to see that $d \geq n$. Thus, by linear algebra, we can find $\nu_{i_1}, \ldots, \nu_{i_n}$ among the above $\nu_i$ such that

$$\Pi = \{s \in H^0(X, E) : s|_{\nu_{j}} = 0 \text{ for } 1 \leq j \leq n\} = \beta(\{\nu_{i_1}, \ldots, \nu_{i_n}\}).$$

It follows that $\beta$ is defined at a general point of $\text{Sym}^n \mathbb{P}E^*$, and that the inverse image of $U^r$ is dense since it is nonempty. (In fact we have shown that $\beta$ is surjective to $U^r_{\text{red}}$.)

If $E$ is only generically generated, then let $E_1$ be the rank $r$ subsheaf generated by global sections. Then $H^0(X, E) = H^0(X, E_1)$. Define

$$\beta_1: \text{Sym}^n \mathbb{P}E_1^* \to \text{Gr}(r, H^0(X, E_1)) = \text{Gr}^r$$

analogously to $\beta$ above. By the above argument for the globally generated case, a general point of the image of $\beta_1$ is a subspace generically generating $E_1$ and $E$. As the restrictions of $\beta$ and $\beta_1$ to

$$\text{Sym}^n \left( \mathbb{P}E^*|_{X\backslash \text{Supp}(E/E_1)} \right) = \text{Sym}^n \left( \mathbb{P}E_1^*|_{X\backslash \text{Supp}(E/E_1)} \right)$$

are canonically identified, if $\nu_1, \ldots, \nu_n$ are sufficiently general points of $\mathbb{P}E^*|_{X\backslash \text{Supp}(E/E_1)}$, then $\beta(\{\nu_1, \ldots, \nu_n\})$ belongs to $U^r$. 


Now, we can prove the theorem. Let $\nu_1 \otimes e_1, \ldots, \nu_n \otimes e_n$ be general points of $\Delta \subseteq \mathbb{P}(E^* \otimes E)$ supported over general $x_1, \ldots, x_n \in X$. By the preceding argument, we may assume that

$$\beta(\{\nu_1, \ldots, \nu_n\}) = \{s \in H^0(X, E) : s|_{x_j} = 0 \text{ for } 1 \leq j \leq n\} =: \Pi$$

is a subspace of dimension $r$ generically generating $E$. As before, there is an exact sequence

$$0 \to E^* \to O_X \otimes \Pi^* \to \tau_\Pi \to 0$$

giving an identification $O_X \otimes \Pi^* = (E^*)_\mathcal{Z}_\Pi$. Moreover, in this case $\mathcal{Z}_\Pi$ contains the points $\nu_1, \ldots, \nu_n$. By Theorem 5.6 (b), the tangent cone $\mathbb{P}T_E B_{r,d}^e$ contains

$$\text{Span}(Z_\Pi \times_X \mathbb{P}E) = \text{Span}(\bigcup_{i=1}^n \mathbb{P}(\nu_i \otimes E|_{x_i})),$$

hence in particular the secant spanned by $\nu_1 \otimes e_1, \ldots, \nu_n \otimes e_n$. Since the $\nu_i$ were chosen generally, $\mathbb{P}T_E B_{r,d}^e$ contains a dense subset of $\text{Sec}^n(\psi(\Delta))$, and hence all of $\text{Sec}^n(\psi(\Delta))$ since $\mathbb{P}T_E B_{r,d}^e$ is closed. \hfill $\square$

5.4. Existence of good singular points

It is nontrivial to establish that there exist bundles $E$ satisfying the hypotheses of Theorem 5.6. For $k = r + 1$, sufficient conditions on $d$ and $g$ are given in [2, Proposition 6.6]. For other values of $r$, $d$ and $k$, stable and generically generated $E$ with can be constructed using a method in [20], and shown to be Petri $k$-injective using a method in [15] adapted from [11]. We recall the construction and sketch the proof for the reader’s convenience, while noting that a more general statement for twisted Brill–Noether loci is proven in [15, Theorem 1.2].

For practicality in referring to [15], in what follows, we change the notation $B_{r,d}^e$ to $B_{n,e}^k$.

**Proposition 5.10.** Let $X$ be a Petri curve of genus $g \geq 2$. Let $n, e_0, e_1$ and $k_0$ be integers satisfying

$$1 \leq e_0 \leq g - 1, \quad 1 \leq e_1 \leq n, \quad \text{and} \quad g - k_0(k_0 - e_0 + g - 1) \geq 1.$$

Then for $e = ne_0 + e_1$ and for $n \leq k \leq nk_0$, the locus $B_{n,e}^k$ has a component $Y^k$ (which is generically smooth and of the expected dimension) of which a general element $E$ is a generically generated, Petri $k$-injective bundle with $h^0(X, E) = k$.

**Proof.** As $g - k_0(k_0 - e_0 + g - 1) \geq 1$, the rank one Brill–Noether locus $B_{nk_0}^{k_0} \subset \text{Pic}^{e_0}(X)$ is of positive dimension. Thus, we may choose mutually nonisomorphic line bundles $L_1, \ldots, L_n$ of degree $e_0$ with $h^0(X, L_i) = k_0$. Let

$$0 \to L_1 \oplus \cdots \oplus L_n \to E_0 \to \tau \to 0 \tag{5.9}$$

be an elementary transformation, where $\tau$ is a torsion sheaf of degree $e_1$. If $E_0$ defines a general point of $\text{Quot}^{e_1}(\bigoplus L_i^{-1})$, then $E_0$ is stable by [20, Théorème A-5]. Moreover, using the fact that $1 \leq e_0 \leq g - 1$, it is not hard to check that for a general $p \in X$ we have $h^0(X, L_i(p)) = h^0(X, L_i)$ for each $i$. Then by the proof of [15, Proposition 7.1], we may assume $h^0(X, E_0) = nk_0$.

As $X$ is Petri, we can apply [15, Lemma 7.2] (compare also [11, Théorème 1.2]) with $V = O_X$ and $A_i = H^0(X, L_i)$. Therefore, $E_0$ is Petri $nk_0$-injective. Thus, $B_{nk_0}^{nk_0}$ has a component $Y^{nk_0}$ which is generically smooth and of the expected dimension. Hence, by [15, Proposition 3.12], for $\chi(X, E) \leq k \leq nk_0$ the locus $B_{n,e}^k$ has a component $Y^k$ which is generically smooth and of the expected dimension; equivalently, a component whose general element $E$ has $h^0(X, E) = k$ and is Petri $k$-injective.

It remains to show that a general $E \in Y^k$ is generically generated for $k \geq r$. By the proof of [15, Proposition 3.12], we may assume that $Y^{k+1} \subset Y^k$. As $k_0 \geq 1$, clearly $E_0$ is generically
generated. As the property is open in families, for \( k \geq r \) a general \( E \in Y^k \) is generically generated. This completes the proof.

\[ \square \]

**Remark 5.11.** Suppose \( g \geq 4 \) and \( n \geq 2 \). Then one computes that

\[
\min\{e_0 : \dim B^2_{1,e_0} \geq 1\} = \left\lfloor \frac{g+3}{2} \right\rfloor = e'_0. \]

Then for \( e_0 \geq e'_0 \) we have \( nk_0 \geq 2n \); and so, for \( ne'_0 + 1 \leq e \leq ng - 1 \), Proposition 5.10 gives bundles defining singular points of \( B^k_{n,e} \) satisfying the hypotheses of Theorem 5.6 (b) for \( n \leq k \leq nk_0 - 1 \).

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