DISCRETE LAGRANGIAN SYSTEMS ON THE VIRASORO GROUP AND CAMASSA-HOLM FAMILY

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Abstract. We show that the continuous limit of a wide natural class of the right-invariant discrete Lagrangian systems on the Virasoro group gives the family of integrable PDE’s containing Camassa-Holm, Hunter-Saxton and Korteweg-de Vries equations. This family has been recently derived by Khesin and Misiołek as Euler equations on the Virasoro algebra for $H^1_{\alpha,\beta}$ metrics. Our result demonstrates a universal nature of these equations.

1. Introduction

Let $M$ be a manifold and $L$ be a function on $M \times M$. The discrete Lagrangian system with the Lagrangian $L$ is the system of difference equations

$$\delta S = 0,$$

which describes the stationary points of the functional $S = S(X)$ defined on the space of sequences $X = (x_k), x_k \in M, k \in \mathbb{Z}$ by a formal sum

$$S = \sum_{k \in \mathbb{Z}} L(x_k, x_{k+1}).$$

Sometimes this system has a continuous limit; in that case it is called a discrete version of the corresponding system of differential equations.

For the integrable equations it is natural to ask for the discretizations, which are also integrable. The theory of integrable Lagrangian discretizations of the classical integrable systems was initiated by J. Moser and one of the authors in [1, 2].

In particular, it was shown that the discrete Lagrangian system on the orthogonal group $O(n)$ with the Lagrangian $L(X, Y) = \text{tr}(X J Y^T), X, Y \in O(n), J = J^T$ can be considered as an integrable discrete version of the Euler-Arnold top [3].

The first attempt to generalize this approach to the infinite-dimensional situation was done in [4], where the case of the group of area-preserving plane diffeomorphisms $\text{SDiff}(\mathbb{R}^2)$ was considered. This was followed by the papers [5], [6], [7] where the discrete Lagrangian systems on the Virasoro group have been discussed. The interest to the Virasoro group was motivated by the important observation due to Khesin and Ovsienko [8] that the Korteweg-de Vries equation can be interpreted as an Euler equation on the Virasoro algebra. The goal was to find an integrable Lagrangian discretization of the KdV equation. The question is still open but some good candidates for the answer have been found (see [6]).

In this note inspired by [7] and the recent, very interesting Khesin-Misiołek paper [9] we are looking at the same problem but from a different angle. We consider a
wide natural class of the discrete Lagrangian systems on the Virasoro group and look
what we can get in the continuous limit.

The result turns out to be surprising: in spite of the fact that the class of discrete
systems we consider is quite general in the continuous limit we have the following
family of integrable (!) equations:

\[(1) \quad \alpha(v_t + 3vv_x) - \beta(v_{xxt} + 2v_x v_{xx} + vv_{xxx}) - bv_{xxx} = 0,\]

\[\alpha, \beta \text{ and } b \text{ are arbitrary constant parameters.}\]

This family appeared in Khesin and Misio\'lski paper [9] as the Euler equations
on the Virasoro algebra for a natural two-parameter family of metrics. We will call
it Camassa-Holm family because for the non-zero \(\alpha\) and \(\beta\) (i.e. in generic case)
(1) is equivalent to the Camassa-Holm equation [10]. In the degenerate cases we
have the Hunter-Saxton equation [11] corresponding to \(\alpha = 0, \beta \neq 0\) and the KdV
equation (when \(\beta = 0, \alpha \neq 0, b \neq 0\)). As it was shown in [9] these three cases
correspond precisely to the three different types of generic Virasoro orbits.

2. Discrete Lagrangian systems on the Virasoro group and their
continuous limit

Let \(\text{Diff}_+(S^1)\) be the group of orientation preserving diffeomorphisms of \(S^1\). We
will represent an element of \(\text{Diff}_+(S^1)\) as a function \(f : \mathbb{R} \to \mathbb{R}\) such that

\[(1) \quad f \in C^\infty(\mathbb{R}),\]
\[(2) \quad f'(x) > 0,\]
\[(3) \quad f(x + 2\pi) = f(x) + 2\pi.\]

Of course, such representation is not unique: the functions \(f\) and \(f + 2\pi\) represent
the same element of \(\text{Diff}_+(S^1)\).

This group has nontrivial central extension defined by the so-called Bott cocycle,
which is unique up to an isomorphism. This extension is called the Virasoro group
(also known as Bott-Virasoro group) and is denoted as Vir. Elements of Vir are
pairs \((f, F)\), where \(f \in \text{Diff}_+(S^1), F \in \mathbb{R}\). The product of two elements in Vir is
defined as

\[(f, F) \circ (g, G) = (f \circ g, F + G + \int_0^{2\pi} \log(f \circ g)' \, d\log g').\]

The unit element \(e\) of Vir is \((id, 0)\). The inverse element of \((f, F)\) is \((f^{-1}, -F)\).
Let us describe the class of discrete Lagrangian systems on Vir we are going to
consider.

The main property we impose is the right-invariance of the Lagrangian:

\[(2) \quad L(X, Y) = L(Xg, Yg), \quad X, Y, g \in \text{Vir}.\]

Because of this property one can rewrite the Lagrangian in the form

\[(3) \quad L(X, Y) = L(XY^{-1}, e) = H(XY^{-1}),\]

where \(H(X) = L(X, e)\). Thus any right-invariant discrete Lagrangian \(L\) is determined
by the corresponding function \(H\) on this group.

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\[\footnote{We should warn the reader that there is another family containing Camassa-Holm equation
discussed recently by Degasperis, Holm and Hone in [12]. Their family besides Camassa-Holm
equation contains only one more integrable case, so in general is non-integrable.}\]
We will consider the discrete Lagrangian systems on Vir corresponding to the functions \( H \) of the following form

\[
H((f, F)) = F^2 + \int_{0}^{2\pi} V(f(x) - x, f'(x)) \, dx,
\]

where \( f \) is a diffeomorphism, \( F \in \mathbb{R} \), and \( V(x_1, x_2) \) is an arbitrary, 2\( \pi \)-periodic in \( x_1 \) function of two variables, which satisfies the condition:

\[
V_1(0, 1) = 0.
\]

Here and below the function with the indices 1, 2 or 3 stands for the corresponding partial derivatives: \( V_1 = \frac{\partial V}{\partial x_1}, V_{11} = \frac{\partial^2 V}{\partial x_1^2} \) etc.

Periodicity of \( V \) is related to the fact that \( f(x) \) and \( f(x) + 2\pi \) represent the same diffeomorphism of \( S^1 \). The difference \( f(x) - x \), which is a 2\( \pi \)-periodic function, is as natural as \( f(x) \) itself, so the only property which might look artificial is the last condition on the partial derivative of \( V \). In order to explain its role and the choice of the Lagrangians let us consider more general functionals:

\[
H((f, F)) = F^2 + \int_{0}^{2\pi} U(f(x), f'(x), x) \, dx,
\]

where \( U(x_1, x_2, x_3) \) is an arbitrary function 2\( \pi \)-periodic with respect to the first argument. Thus, we consider the functional

\[
S = \sum_{k \in \mathbb{Z}} L((f_k, F_k), (f_{k+1}, F_{k+1})),
\]

where \( \{(f_k, F_k)\} \) is a sequence of points on Vir and \( L \) is defined using the function \( H \) as described above:

\[
L((f_k, F_k), (f_{k+1}, F_{k+1})) = H((f_k, F_k) \circ (f_{k+1}, F_{k+1})^{-1}).
\]

The discrete Euler-Lagrange equations for this functional have the form

\[
-\Omega_k + \Omega_{k+1} = 0,
\]

and

\[
-2\Omega_k [\log(\omega_k')]' - U_1(\omega_k, \omega_k', x)\omega_k' + U_{12}(\omega_k, \omega_k', x)(\omega_k')^2 + U_{22}(\omega_k, \omega_k', x)\omega_k''\omega_k' + U_{23}(\omega_k, \omega_k', x)\omega_k' + 2\Omega_{k+1} [\log((\omega_{k+1}')^{-1})]' + U_1(x, \frac{1}{(\omega_{k+1}')}, \omega_{k+1}^{-1})(\omega_{k+1}^{-1})' = -U_{12}(x, \frac{1}{(\omega_{k+1}')}, \omega_{k+1}^{-1}) + U_{22}(x, \frac{1}{(\omega_{k+1}')}, \omega_{k+1}^{-1})(\omega_{k+1}^{-1})'^2 - U_{23}(x, \frac{1}{(\omega_{k+1}')}, \omega_{k+1}^{-1})(\omega_{k+1}^{-1})' = 0,
\]

where \( (\omega_k, \Omega_k) \) are discrete analogues of angular velocities:

\[
(\omega_k, \Omega_k) = (f_{k-1}, F_{k-1}) \circ (f_k, F_k)^{-1}, \quad k \in \mathbb{Z}.
\]

The first equation simply says that \( \Omega_{k+1} = \Omega_k \), so \( \Omega_k \) is an integral of our discrete Lagrangian system.
Let us now find a continuous limit of our Euler-Lagrange equations. To do this we suppose that the angular velocity is of the form

\[ (\omega_1, \Omega_l) = (i \varepsilon + \varepsilon v_l(x), \varepsilon A_l), \]

i. e. the angular velocity is the identity element of Vir up to \( O(\varepsilon) \). Also we assume that

\[ \begin{align*}
  v_k(x) &= v(x,t), \\
  A_k &= A(t), \\
  v_{k+1}(x) &= v(x,t + \varepsilon), \\
  A_{k+1} &= A(t + \varepsilon).
\end{align*} \tag{8} \]

We should substitute these formulae in the discrete Euler-Lagrange equations and take the term of lowest order with respect to \( \varepsilon \) in the corresponding Taylor series. This is what we mean by the continuous limit of our discrete system.

It is easy to see that the continuous limit of the first Euler-Lagrange equation \( \bullet \) is \( A_1 = 0 \), i.e. \( A(t) = A \) is a constant. But when we consider the continuous limit of the second Euler-Lagrange equation \( \exists \) we have the following problem: the leading \( \varepsilon^1 \)-term gives us an ordinary (but not a partial) differential equation. Indeed a straightforward calculation shows that this term has a form

\[ \begin{align*}
  U_{233}(x,1,x) &- 2U_1(x,1,x)v_x + 2U_{23}(x,1,x)v_x + 2U_{123}(x,1,x)v + \\
  + U_{112}(x,1,x)v - U_{11}(x,1,x)v + 2U_{12}(x,1,x)v_x &- U_{13}(x,1,x)v.
\end{align*} \tag{9} \]

Since we want to have a more interesting continuous limit in \( x \), we have to eliminate this term and look at the next \( \varepsilon^2 \)-term. The condition that \( \varepsilon^1 \)-term is equal to zero is equivalent to some relations for partial derivatives of \( U \) in the points \((x,1,x)\). It is not clear how to resolve these equations in a general case, but one can easily check that there is one important particular class of solutions:

\[ U(x_1, x_2, x_3) = V(x_1 - x_3, x_2), \]

where \( V(x_1, x_2) \) is an arbitrary \( 2\pi \)-periodic in \( x_1 \) function with the only condition:

\[ V_1(0,1) = 0. \]

Thus we arrive at the Lagrangians of the form \( \bullet \). For them \( \exists \) is identically equal to zero and after a very long but straightforward calculation we end up with the second order term, which gives

\[ V_{11}(0,1)(v_t - 3v_x) - V_{22}(0,1)(v_{xt} - 2v_{xx}v_x - vv_{xxx}) - 4Av_{xxx} = 0. \]

If we change \( t \mapsto -t \) and introduce

\[ \alpha = V_{11}(0,1), \beta = V_{22}(0,1), b = -4A, \]

we come to the Camassa-Holm family as it appears in Khesin-Misiołek paper (eq. 3.7 in \( \bullet \)):

\[ \alpha(v_t + 3v_x) - \beta(v_{xt} + 2v_xv_{xx} + vv_{xxx}) - bv_{xxx} = 0. \tag{10} \]

This is the continuous limit of the discrete systems with the Lagrangians \( \bullet \).\( \bullet \)

Formally this family has three parameters, but we have a freedom of multiplication of the equation by a non-zero constant, the two-dimensional scaling symmetry group \( v \mapsto \lambda v, t \mapsto \mu t, x \mapsto \lambda \mu x \) and the Galilean group \( v \mapsto v + c, x \mapsto x + dt, t \mapsto t \). Modulo these symmetries we have just one generic orbit, containing the equation with \( \alpha = 1, \beta = 1, b = 0 \):

\[ v_t - v_{xxx} + 3v_xv_x - 2v_xv_{xx} - vv_{xxx} = 0, \]
which is one of the canonical forms of the *Camassa-Holm shallow-water equation* \[10\]

\[ v_t + 2\kappa v_x + \gamma v_{xxx} - v_{xxt} + 3v_xv_x - 2v_x^2v_x - vv_{xxx} = 0. \]

We have also four degenerate orbits. When \( \alpha \neq 0, \beta = 0, b \neq 0 \) the equation \[10\] is equivalent to the KdV equation:

\[ v_t + 3vv_x + v_{xxx} = 0. \]

Further degeneration \( \alpha \neq 0, \beta = 0, b = 0 \) leads to the dispersionless KdV equation (sometimes called also Hopf equation):

\[ v_t + 3v_x = 0. \]

When \( \alpha = 0, \beta \neq 0 \) we have the *Hunter-Saxton equation* \[11\]

\[ v_{xxx} + 2v_xv_x + vv_{xxx} = 0. \]

Finally if both \( \alpha \) and \( \beta \) are zero (but \( b \) is not) we simply have

\[ v_{xxx} = 0. \]

3. Discussion

The fact that the continuous limit of a wide natural class of the discrete right-invariant Lagrangian systems on the Virasoro group is integrable seems to be remarkable. This universality of the Camassa-Holm family is closely related with the results \[9, 13\] by Khesin and Misiolek who interpreted this family as the Euler equations on the Virasoro algebra corresponding to the special family of the right-invariant metrics on the Virasoro group (\( H_{1,\alpha,\beta} \) metrics). The heuristic explanation of our result is that these equations can be considered as nonlinear analogues of the harmonic oscillators on the Virasoro group: in the first approximation all Hamiltonian systems near equilibriums behave like harmonic oscillators.

The integrability of the Camassa-Holm family can be shown in different ways (see e.g. \[9,14\]) but the underlying reasons for that seem to be deep and deserve better understanding. In particular, it would be interesting to consider the discrete right-invariant Lagrangians depending on the derivatives of a diffeomorphism \( f \) up to the second order to see if similar phenomenon holds in that case or not. Another interesting question is what happens for other infinite-dimensional groups (e.g for SDiff(\( \mathbb{R}^2 \))).

It is instructive to compare the situation with the finite-dimensional case. For example, for the orthogonal group \( O(n) \) we may have as a continuous limit of a right-invariant discrete Lagrangian system any right-invariant geodesic flow, which is known for \( n > 3 \) to be in general non-integrable (see e.g. \[15\]). There are integrable cases of the Euler equations on \( O(n) \) (for example the Manakov metrics \[16\]) but no analogues of our result are known for them.

It is interesting to mention that from Khesin-Misiolek results (see section 5 in \[9\]) it follows that \( H_{1,\alpha,\beta} \)-metrics can be considered as the analogues of the very special cases of the Manakov metrics. The question what are the analogues for the general Manakov metrics (if there are any) seems to be open.
ACKNOWLEDGMENTS

We are grateful to Boris Khesin and Andy Hone for useful and stimulating discussions. We would like also to thank a referee for useful comments.

One of the authors (A. P.) is very grateful to the Centre de Recherches Mathématiques (CRM) for its hospitality.

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