GEOMETRY OF HESSENBERG VARIETIES WITH APPLICATIONS TO
NEWTON-OKOUNKOV BODIES

HIRAKU ABE, LAUREN DEDIEU, FEDERICO GALETTO, AND MEGUMI HARADA

ABSTRACT. In this paper, we study the geometry of various Hessenberg varieties in type A, as well as families thereof. Our main results are as follows. We find explicit and computationally convenient generators for the local defining ideals of indecomposable regular nilpotent Hessenberg varieties, allowing us to conclude that all regular nilpotent Hessenberg varieties are local complete intersections. We also show that certain flat families of Hessenberg varieties, whose generic fibers are regular semisimple Hessenberg varieties and whose special fiber is a regular nilpotent Hessenberg variety, have reduced fibers. In the second half of the paper we present several applications of these results. First, we construct certain flags of subvarieties of a regular nilpotent Hessenberg variety, obtained by intersecting with Schubert varieties, with well-behaved geometric properties. Second, we give a computationally effective formula for the degree of a regular nilpotent Hessenberg variety with respect to a Plücker embedding. Third, we explicitly compute some Newton-Okounkov bodies of the two-dimensional Peterson variety.

CONTENTS

1. Introduction 1
2. Background on Hessenberg varieties 2
3. The defining ideals of regular nilpotent Hessenberg varieties 4
4. One-parameter families of Hessenberg varieties 10
5. Flags of subvarieties in regular nilpotent Hessenberg varieties 11
6. An efficient formula for the degree of regular nilpotent Hessenberg varieties 13
7. Newton-Okounkov bodies of Peterson varieties 16

1. INTRODUCTION

In this paper we study Hessenberg varieties of various types and families thereof, with a view towards applications and connections to other areas. Throughout this paper, for simplicity we restrict to Lie type A although we suspect that our discussion generalizes to other Lie types.

Hessenberg varieties in type A are subvarieties of the full flag variety \( \text{Flags}(\mathbb{C}^n) \) of nested sequences of linear subspaces in \( \mathbb{C}^n \). Their geometry and (equivariant) topology have been studied extensively since the late 1980s \([12,13,14]\). This subject lies at the intersection of, and makes connections between, many research areas such as geometric representation theory (see for example \([45,21]\)), combinatorics (see e.g. \([17,35,14,25,11]\)), and algebraic geometry and topology (see e.g. \([32,8,40,28,40,41,2,10]\)). A special case of Hessenberg varieties called the Peterson variety \( \text{Pet}_n \) arises in the study of the quantum cohomology of the flag variety \([32,42]\), and more generally, geometric properties and invariants of many different types of Hessenberg varieties (including in Lie types other than A) have been widely studied.

We now describe the main results of this paper. (For definitions we refer to Section 2.)

1. We determine an explicit list of generators for the local defining ideals of indecomposable regular nilpotent Hessenberg varieties (Proposition 3.7).

Date: July 23, 2018.

2010 Mathematics Subject Classification. Primary: 14M17, 14M25; Secondary: 14M10.

Key words and phrases. Hessenberg varieties, Peterson varieties, flag varieties, local complete intersections, flat families, Schubert varieties, Newton-Okounkov bodies, degree.
(2) We prove that certain flat families of Hessenberg varieties over \( \mathbb{A}^1 \) (or \( \mathbb{P}^1 \)) have reduced fibers (Theorem 4.1). In the second part of the paper we give applications of the above. We were motivated from the theory of Newton-Okounkov bodies, but items (3) and (4) are also of independent interest.

(3) We construct families of flags \( Y_\bullet = \{Y_0 = \text{Hess}(N,h) \supseteq Y_1 \supseteq \cdots \supseteq Y_n\} \) of subvarieties in regular nilpotent Hessenberg varieties arising from intersections with (dual) Schubert varieties; the intersections are smooth at \( Y_n = \{pt\} \), where \( n = \dim \text{Hess}(N,h) \) (Theorem 5.4).

(4) We give a computationally efficient formula for the degree of an arbitrary indecomposable regular nilpotent Hessenberg variety with respect to a Plücker embedding associated to a weight \( \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \) as a polynomial in the \( \lambda_i \) (Theorem 6.2).

(5) We explicitly compute some Newton-Okounkov bodies associated to the Peterson variety in \( \text{Flags}(\mathbb{C}^3) \), a special case of regular nilpotent Hessenberg varieties (Theorem 7.4). Some remarks are in order. Firstly, our results in (1) generalize a result of Insko and Yong [28] for the case of Peterson varieties, and also a result of Insko [27] showing that regular nilpotent Hessenberg varieties are local complete intersections when the Hessenberg function is strictly increasing. Secondly, the family we consider in (2) is presumably the one hinted at in [3, Remark 7.3]. Thirdly, one reason for studying the flags of subvarieties in (4) is that well-behaved such flags are often a crucial ingredient in the construction of Newton-Okounkov bodies. Fourthly, the polynomial mentioned in (5) is called a volume polynomial in [4], where the authors also show that the natural Poincaré duality algebra associated to this polynomial is in fact isomorphic to the ordinary cohomology ring of the regular nilpotent Hessenberg variety. Finally, we view the results of (5) as a first case of a Newton-Okounkov-type computation for Hessenberg varieties.

Acknowledgements. We are grateful to Mikiya Masuda for his stimulating questions and his support and encouragement. We also thank Allen Knutson for pointing out to us the significance of the flat family of Hessenberg varieties over \( \mathbb{A}^1 \) (or \( \mathbb{P}^1 \)).

2. Background on Hessenberg varieties

In this section we recall some basic definitions used in the study of Hessenberg varieties. Since detailed exposition is available in the literature [46, 13] we keep the discussion brief.

By the flag variety we mean the homogeneous space \( \text{GL}_n(\mathbb{C})/B \), where \( B \) denotes the subgroup of upper-triangular matrices. This homogeneous space may also be identified with the space of nested sequences of linear subspaces of \( \mathbb{C}^n \), i.e.

\[
\text{Flags}(\mathbb{C}^n) := \{ V_\bullet = \{ \{0\} \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n \subseteq \mathbb{C}^n \mid \dim \mathbb{C}(V_i) = i \} \cong \text{GL}_n(\mathbb{C})/B;
\]

the identification with \( \text{GL}_n(\mathbb{C})/B \) takes a coset \( MB \), for \( M \in \text{GL}_n(\mathbb{C}) \), to the flag \( V_\bullet \) with \( V_i \) defined as the span of the leftmost \( i \) columns of \( M \).

We use the notation \([n] := \{1, 2, \ldots, n\}\) throughout. A Hessenberg function is a function \( h: [n] \to [n] \) satisfying \( h(i) \geq i \) for all \( 1 \leq i \leq n \) and \( h(i+1) \geq h(i) \) for all \( 1 \leq i < n \). We frequently denote a Hessenberg function by listing its values in sequence, \( h = (h(1), h(2), \ldots, h(n) = n) \). To a Hessenberg function \( h \) we associate a subspace of \( \mathfrak{gl}_n(\mathbb{C}) \) (the vector space of \( n \times n \) complex matrices) defined as

\[
H(h) := \{ (a_{i,j})_{i,j \in [n]} \in \mathfrak{gl}_n(\mathbb{C}) \mid a_{i,j} = 0 \text{ if } i > h(j) \},
\]
which we call the **Hessenberg subspace** $H(h)$. It is sometime useful to visualize this space as a configuration of boxes on a square grid of size $n \times n$ whose shaded boxes correspond to the $a_{i,j}$ which are allowed to be non-zero (see Figure 2.1).

![Figure 2.1. The picture of $H(h)$ for $h = (3,3,4,5,6,6)$.](image)

We can now define the central object of study.

**Definition 2.1.** Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be a linear operator and $h : [n] \to [n]$ a Hessenberg function. The **Hessenberg variety** associated to $A$ and $h$ is defined to be

$$\text{Hess}(A, h) := \{ V \in \text{Flags}(\mathbb{C}^n) \mid AV \subseteq V_{h(i)}, \forall i \}. $$

Equivalently, under the identification (2.1) and viewing $A$ as an element in $\text{gl}_n(\mathbb{C})$,

$$\text{Hess}(A, h) = \{ MB \in \text{GL}_n(\mathbb{C})/B \mid M^{-1}AM \in H(h) \}. $$

In particular, any Hessenberg variety $\text{Hess}(A, h)$ is, by definition, an algebraic subset of the flag variety $\text{Flags}(\mathbb{C}^n)$. It is straightforward to see that $\text{Hess}(A, h)$ and $\text{Hess}(gAg^{-1}, h)$ are isomorphic $\forall g \in \text{GL}_n(\mathbb{C})$, so we frequently assume without loss of generality that $A$ is in standard Jordan canonical form with respect to the standard basis on $\mathbb{C}^n$.

Since $H(h) \subseteq \text{gl}_n(\mathbb{C})$ is stable under the action of $B$, the quotient space $\text{gl}_n(\mathbb{C})/H(h)$ admits a $B$-action by $b \cdot X = \text{Ad}(b)X$ for $b \in B$ and $X \in \text{gl}_n(\mathbb{C})$, where $\overline{X}$ denotes the image of $X$ in $\text{gl}_n(\mathbb{C})/H(h)$. So we have the following vector bundle over $\text{GL}_n(\mathbb{C})/B$:

$$\text{GL}_n(\mathbb{C}) \times_B (\text{gl}_n(\mathbb{C})/H(h))$$

where $B$ acts on the product $\text{GL}_n(\mathbb{C}) \times (\text{gl}_n(\mathbb{C})/H(h))$ by $(M, X) \cdot b = (Mb, b^{-1}Xb)$ for $b \in B$ and $(M, X) \in \text{GL}_n(\mathbb{C}) \times (\text{gl}_n(\mathbb{C})/H(h))$. A matrix $A \in \text{gl}_n(\mathbb{C})$ defines a section of this vector bundle by

$$s_A : \text{GL}_n(\mathbb{C})/B \to \text{GL}_n(\mathbb{C}) \times_B (\text{gl}_n(\mathbb{C})/H(h)) ; \ MB \mapsto [M, M^{-1}AM]. $$

Now it is clear from (2.4) that

$$\text{Hess}(A, h) = \{ MB \in \text{GL}_n(\mathbb{C})/B \mid s_A(MB) = 0 \}. $$

Namely, $\text{Hess}(A, h)$ is the zero set $Z(s_A)$ of the section $s_A$.

In this manuscript we discuss two important special cases of Hessenberg varieties: the regular nilpotent Hessenberg varieties and the regular semisimple Hessenberg varieties.

**Definition 2.2.** A Hessenberg variety $\text{Hess}(A, h)$ is called **regular nilpotent** if $A$ is a principal nilpotent operator. Equivalently, the Jordan canonical form of $A$ has a single Jordan block with eigenvalue zero, i.e., up to a change of basis $A$ is of the form:

$$\begin{pmatrix}
0 & 1 & & \\
0 & 1 & & \\
& & \ddots & \\
0 & & & 1 \\
& & & 0
\end{pmatrix}$$

For the remainder of this paper we let $N$ denote the matrix (operator) above.
Regular nilpotent Hessenberg varieties are known to be irreducible \cite[Lemma 7.1]{5}, and they are the subject of Section 3 of this paper. When we study families of Hessenberg varieties in Section 4, the following type will also become relevant.

**Definition 2.3.** A Hessenberg variety Hess(A, h) is called **regular semisimple** if A is a semisimple operator with distinct eigenvalues. Equivalently, there is a basis of \( \mathbb{C}^n \) with respect to which A is diagonal with pairwise distinct entries along the diagonal.

We will need the following terminology from \cite[Definition 4.4]{15}.

**Definition 2.4.** Let \( h: [n] \to [n] \) be a Hessenberg function. If \( h(j) \geq j + 1 \) for \( j \in \{1, 2, \ldots, n - 1\} \), then we say that \( h \) is **indecomposable**.

Finally, we give the definition of a special case of a regular nilpotent Hessenberg variety which is studied in more detail in Section 7.

**Definition 2.5.** When \( h \) is of the form \( h(j) = j + 1 \) for \( j \in \{1, 2, \ldots, n - 1\} \), the corresponding regular nilpotent Hessenberg variety is called a **Peterson variety**.

### 3. The defining ideals of regular nilpotent Hessenberg varieties

In this section, we will show that the zero scheme \( Z(s_A) \) of the section \( s_A \) of the vector bundle \( GL_n(\mathbb{C}) \times_B (gl_n(\mathbb{C})/H(h)) \) described in \( \ref{2.4} \) is reduced as a scheme. As a corollary, we will provide explicit lists of (local) generators for the defining ideals of Hess\((N, h)\), considered as subvarieties of Flags\((\mathbb{C}^n)\). Since we already know that Hess\((N, h) = Z(s_A) \) in Flags\((\mathbb{C}^n) \) as in \( \ref{2.6} \), a local trivialization of the vector bundle above produces an explicit list of polynomials which cut out Hess\((N, h) \) set-theoretically; the issue which we must address is whether the ideal that these polynomials generate is radical, or, whether the relevant quotient ring is reduced. The main content of this section, recorded in Proposition 3.6 and Proposition 3.7 is to show that in fact the quotient rings associated to our lists of polynomials are reduced and thus we have found generators for the defining ideals of our varieties. We can then easily conclude that Hess\((N, h) \) is a local complete intersection (Corollary 3.17).

Recall from Definition 2.2 that \( N \) is the \( n \times n \) regular nilpotent matrix in Jordan canonical form. We define the following.

**Definition 3.1.** Let \( Z(N, h) \) denote the zero scheme in \( G/B \) of the section \( s_N \).

Locally, the section \( s_N \) is a collection of (local) regular functions, and the scheme \( Z(N, h) \) is locally the zero scheme of those functions (cf. also \cite[Appendix B.3.2]{20}). Note that, a priori, \( Z(N, h) \) may be nonreduced. We now describe an explicit local presentation of \( Z(N, h) \).

It is well-known that Flags\((\mathbb{C}^n)\) can be covered by affine coordinate patches, each isomorphic to \( \mathbb{A}^{n(n-1)/2} \). Let

\[
U^- := \left\{ M = \begin{pmatrix}
1 & * & 1 \\
* & \ddots & \ddots \\
* & \cdots & 1 \\
* & \cdots & * & 1
\end{pmatrix} \right\} \quad \text{with 1's along the diagonal}
\]

\[
\cong \mathbb{A}^{n(n-1)/2} \subseteq \text{Mat}(n \times n, \mathbb{C}).
\]

Then the map \( U^- \to \text{Flags}(\mathbb{C}^n) \cong GL_n(\mathbb{C})/B \) given by \( M \in U^- \mapsto MB \in GL_n(\mathbb{C})/B \), is an open embedding. By slight abuse of notation we denote also by \( U^- \) its image in Flags\((\mathbb{C}^n)\). The set of translates \( \{N_w := wU^- \} \) of \( U^- \) by the permutations \( w \in \Sigma_n \), along with the embeddings \( \Psi_w: U^- \cong \mathbb{A}^{n(n-1)/2} \cong N_w \subseteq \text{Flags}(\mathbb{C}^n) \) sending \( M \mapsto wMB \), form an open cover of Flags\((\mathbb{C}^n)\):

\[
\text{Flags}(\mathbb{C}^n) = \bigcup_{w \in \Sigma_n} N_w.
\]
Using the bijection $U^{-} \xrightarrow{\cong} \mathcal{N}_{w}$, a point in $\mathcal{N}_{w}$ is uniquely identified with the $w$-translate of a lower-triangular matrix with 1’s along the diagonal. Therefore a point in $\mathcal{N}_{w}$ is uniquely determined by a matrix $(x_{i,j})$ whose entries are subject to the following relations
\begin{equation}
\begin{aligned}
x_{w(j),j} &= 1, \quad \forall j \in [n], \\
x_{w(i),j} &= 0, \quad \forall i, j \in [n] : j > i.
\end{aligned}
\end{equation}
(3.1)
Thus the coordinate ring of $\mathcal{N}_{w}$, denoted by $\mathbb{C}[x_{w}]$, is isomorphic to the quotient of the polynomial ring $\mathbb{C}[x_{i,j}]$ by the relations (3.1). Observe that $\mathbb{C}[x_{w}]$ is isomorphic to a polynomial ring in the $n(n-1)/2$ variables $x_{i,j}$ not covered by the relations (3.1).

**Example 3.2.** Let $n = 4$ and $w = (2, 4, 1, 3) \in \mathfrak{S}_4$ in the standard one-line notation. An element $M$ of $\mathcal{N}_{w} = wU^{-}$ can be written as
\[
wM = \begin{pmatrix}
x_{1,1} & x_{1,2} & 1 & 0 \\
1 & 0 & 0 & 0 \\
x_{3,1} & x_{3,2} & x_{3,3} & 1 \\
x_{4,1} & 1 & 0 & 0
\end{pmatrix}.
\]
To describe the local presentation of the zero scheme $Z(N, h)$ in the neighborhood $\mathcal{N}_{w}$, we make the following definition.

**Definition 3.3.** Let $w \in \mathfrak{S}_n$ and let $i, j \in [n]$ with $i > h(j)$. We define the polynomial $f_{i,j}^{w} \in \mathbb{C}[x_{w}]$ by
\[
f_{i,j}^{w} := ((wM)^{-1}N(wM))_{i,j}
\]
where here the $(k, \ell)$-th matrix entries of $M$ for $k > \ell$ are viewed as variables. We also define the ideal $J_{w,h} := \langle f_{i,j}^{w} \mid i > h(j) \rangle \subseteq \mathbb{C}[x_{w}]$ to be the ideal in $\mathbb{C}[x_{w}]$ generated by the $f_{i,j}^{w}$.

From the explicit description of $s_{N}$ given in (2.4), the following Lemma is straightforward.

**Lemma 3.4.** Let $w \in \mathfrak{S}_n$ and $\mathcal{N}_{w} = \text{Spec} \mathbb{C}[x_{w}]$ as above. Then $Z(N, h) \cap \text{Spec} \mathbb{C}[x_{w}] \cong \text{Spec} \mathbb{C}[x_{w}]/J_{w,h}$. In particular, we obtain an open affine cover
\[
Z(N, h) = \bigcup_{w \in \mathfrak{S}_n} \text{Spec} \mathbb{C}[x_{w}]/J_{w,h}.
\]

**Example 3.5.** Let $n = 4$ and $w = (2, 4, 1, 3) \in \mathfrak{S}_4$, continued from Example 3.2. Then it is easy to check that
\[
(wM)^{-1} = \begin{pmatrix}
x_{1,1} & x_{1,2} & 1 & 0 \\
1 & 0 & 0 & 0 \\
x_{3,1} & x_{3,2} & x_{3,3} & 1 \\
x_{4,1} & 1 & 0 & 0
\end{pmatrix}^{-1} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & -x_{4,1} & 0 & 1 \\
1 & -x_{1,1} + x_{1,2}x_{4,1} & 0 & -x_{1,2} \\
x_{3,3} & y & 1 & -x_{3,2} + x_{1,2}x_{3,3}
\end{pmatrix}
\]
where $y = -x_{3,1} + x_{1,1}x_{3,3} + x_{4,1}(x_{3,2} - x_{1,2}x_{3,3})$. So, for example, we have
\[
f_{4,1}^{w} = ((wM)^{-1}N(wM))_{4,1}
\]
\[
= -x_{3,3} + x_{3,1}(-x_{3,1} + x_{1,1}x_{3,3} + x_{4,1}(x_{3,2} - x_{1,2}x_{3,3})) + x_{4,1},
\]
(3.2)
\[
f_{4,2}^{w} = ((wM)^{-1}N(wM))_{4,1}
\]
\[
= -x_{3,1} + x_{1,1}x_{3,3} + x_{4,1}(x_{3,2} - x_{1,2}x_{3,3}) + 1.
\]
So if $h = (3, 3, 4)$, then we have $J_{w,h} = \langle f_{4,1}^{w}, f_{4,2}^{w} \rangle$ with these polynomials.

We now state the main result of this section.
Proposition 3.6. Let $h: [n] \to [n]$ be an indecomposable Hessenberg function. Then the zero scheme $\mathcal{Z}(N, h)$ of the section $s_N$ described at (2.4) is reduced.

Combining this with the local description of $\mathcal{Z}(N, h)$ given at Lemma 3.4, we obtain the following.

Proposition 3.7. Let $h: [n] \to [n]$ be an indecomposable Hessenberg function. For every $w \in \mathfrak{S}_n$, the ring $\mathbb{C}\langle x_w \rangle / J_{w,h}$ is the coordinate ring of the subvariety $\mathcal{N}_{w,h} := \text{Hess}(N, h) \cap \mathcal{N}_w$ of $\mathcal{N}_w$. In particular, the ideal $J_{w,h}$ is radical and is the defining ideal of $\mathcal{N}_{w,h}$.

Remark 3.8. Using the language and techniques of degeneracy loci, it is shown in [5] that $\mathcal{Z}(N, h)$ is reduced, for the special case when the Hessenberg function is of the form $h = (k, n, \ldots, n)$ for some $2 \leq k \leq n$ [4, Theorem 7.6].

The necessity of the indecomposability hypothesis can be seen from a small example.

Example 3.9. Let $n = 2$ and $h = (1, 2)$. We have $J_{id,h} = \langle f_{1,1}^d \rangle \subseteq \mathbb{C}[x_{2,1}]$ where $f_{1,1}^d = x_{2,1}^2$. Clearly the ring $\mathbb{C}[x_{2,1}] / J_{id,h}$ is not reduced, so it is not the coordinate ring of $\mathcal{N}_{id,h}$.

We now prove Proposition 3.6. For this, we recall the following property of the regular nilpotent Hessenberg variety $\text{Hess}(N, h)$.

Proposition 3.10. ([5, Lemma 7.1]) Let $h: [n] \to [n]$ be a Hessenberg function. Then $\text{Hess}(N, h)$ is irreducible of dimension $\sum_{i=1}^{n} (h(i) - i)$.

This proposition implies that the zero scheme $\mathcal{Z}(N, h)$ is irreducible as well, and has the expected codimension. Hence the following is immediate from [16, §18.5 and Proposition 18.13] (cf. [18, Theorem 8.2]).

Lemma 3.11. The scheme $\mathcal{Z}(N, h)$ is a local complete intersection and hence Cohen-Macaulay.

Thus, to prove the reducedness of $\mathcal{Z}(N, h)$, it is enough to prove that $\mathcal{Z}(N, h)$ is generically reduced ([10, Exercise 18.9]). Since $\mathcal{Z}(N, h)$ is irreducible, we only need to find a single reduced point of the scheme $\mathcal{Z}(N, h)$. To do this, we focus our attention on the neighborhood $\mathcal{N}_{w_0} \cong \text{Spec} \mathbb{C}\langle x_{w_0} \rangle$ where $w_0$ is the longest element in $\mathfrak{S}_n$. The following is the most important computation in our argument.

Lemma 3.12. Let $h: [n] \to [n]$ be an indecomposable Hessenberg function. Then the ring $\mathbb{C}\langle x_{w_0} \rangle / J_{w_0,h}$ is isomorphic to a polynomial ring, hence it is reduced.

It is already known that the intersection $\text{Hess}(N, h) \cap \mathcal{N}_{w_0}$ of the variety $\text{Hess}(N, h)$ with the affine coordinate patch around $w_0$ is isomorphic to a variety to a complex affine space ([10] and [16]). The point of Lemma 3.12 is that $J_{w_0,h}$ is its defining ideal, and that its generators take a particular form. Before proving Lemma 3.12 we give some concrete examples.

Example 3.13. Let $n = 4$ and $h = (3, 3, 4, 4)$. The longest element of $\mathfrak{S}_4$ is the permutation $w_0 = (4, 3, 2, 1)$ in one-line notation. The coordinate ring of $\mathcal{N}_{w_0}$ is

$$\mathbb{C}\langle x_{w_0} \rangle \cong \mathbb{C}[x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{3,1}],$$

and a point in $\mathcal{N}_{w_0}$ is determined by a matrix

$$M = \begin{pmatrix}
  x_{1,1} & x_{1,2} & x_{1,3} & 1 \\
  x_{2,1} & x_{2,2} & 1 & 0 \\
  x_{3,1} & 1 & 0 & 0 \\
  1 & 0 & 0 & 0
\end{pmatrix}.$$

Given the form of $M$, it is easy to see that its inverse must have the form

$$M^{-1} = \begin{pmatrix}
  0 & 0 & 0 & 1 \\
  0 & 0 & 1 & y_{3,1} \\
  0 & 1 & y_{2,2} & y_{2,1} \\
  1 & y_{1,3} & y_{1,2} & y_{1,1}
\end{pmatrix}.$$

Starting from the matrix equality $M^{-1} M = (\delta_{i,j})$, and comparing entries we can obtain expressions for the $y_{i,j}$ in terms of the $x_{i,j}$. For example,

$$y_{1,3} = -x_{1,3},$$
$$y_{1,2} = -x_{1,2} - y_{1,3}x_{2,2} = -x_{1,2} + x_{1,3}x_{2,2}.$$
It is also straightforward to see that each $y_{i,j}$ depends only on the variables $x_{k,\ell}$ with $k \geq i$ and $\ell \geq j$. Graphically, this says that $y_{i,j}$ depends only on $x_{i,j}$ and variables located to the right or below $x_{i,j}$ in the matrix $M$; for example, $y_{1,2}$ depends only on the variables contained in the bounded region depicted in Figure 3.1.

![Figure 3.1. Variables appearing in the expression of $y_{1,2}$](image)

Now we describe the generators of $J_{w_0,h} = \langle f_{w_0}^{4,1}, f_{w_0}^{4,2} \rangle$. We have

$$M^{-1}NM = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & y_{3,1} & 1 \\
1 & y_{1,3} & y_{1,2} & y_{1,1}
\end{pmatrix} \begin{pmatrix}
x_{2,1} & x_{2,2} & 1 & 0 \\
x_{3,1} & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

and from this we get

$$f_{w_0}^{4,1} = (M^{-1}NM)_{4,1} = x_{2,1} + y_{1,3}x_{3,1} + y_{1,2} = x_{2,1} - x_{1,3}x_{3,1} - x_{1,2} + x_{1,3}x_{2,2},$$

$$f_{w_0}^{4,2} = (M^{-1}NM)_{4,2} = x_{2,2} + y_{1,3} = x_{2,2} - x_{1,3}.$$

We deduce that $x_{2,1}$ and $x_{2,2}$ are determined by the other variables and conclude that $\mathbb{C}[x_{w_0}]/J_{w_0,h} \cong \mathbb{C}[x_{1,1}, x_{1,2}, x_{1,3}, x_{1,4}]$ is a polynomial ring and in particular is reduced. It is possible to easily visualize, using the Hessenberg diagram, the variables which turn out to be dependent on other variables and hence “vanish” in the quotient $\mathbb{C}[x_{w_0}]/J_{w_0,h}$, as illustrated in Figure 3.2 for this example. Specifically, we can first cross out any box which is not contained in the Hessenberg diagram for $h = (3344)$; see the left diagram in Figure 3.2. We then flip the picture upside down (so that, in this case, the boxes in positions (1, 1) and (1, 2) are now crossed out), and finally shift the entire picture downwards by one row. In this case we end up with a picture, as in the right-hand diagram in Figure 3.2 with the boxes in positions (2, 1) and (2, 2) crossed out. Then the variables corresponding to the crossed-out boxes are the ones which vanish in the quotient, and in fact (by the computation above) they are dependent on the (non-crossed-out) variables appearing either below it within the same column, or in a column to its right in a row at most one above it.

![Figure 3.2. Variables killed in $\mathbb{C}[x_{w_0}]/J_{w_0,h}$](image)
Example 3.14. Let \( n = 5 \) and \( h = (3,4,4,5,5) \). The diagram in Figure 3.3 predicts that \( \mathbb{C}[x_{w_0}] / J_{w_0,h} \cong \mathbb{C}[x_{1,1}, x_{1,2}, x_{1,3}, x_{1,4}, x_{3,2}, x_{4,1}] \). Indeed the generators of \( J_{w_0,h} \) are

\[
f_{5,1}^0 = x_{2,1} - x_{1,2} - x_{1,3}x_{4,1} + x_{1,4}x_{3,2} - x_{1,4}x_{3,1} + x_{1,4}x_{2,3}x_{4,1} - x_{1,4}x_{2,3}x_{3,2} - 1,
\]

\[
f_{5,2}^0 = x_{2,2} - x_{1,3} - x_{1,4}x_{3,2} + x_{1,4}x_{2,3},
\]

\[
f_{5,3}^0 = x_{2,3} - x_{1,4},
\]

\[
f_{4,1}^0 = x_{3,1} - x_{2,2} - x_{2,3}x_{4,1} + x_{2,3}x_{3,2}.
\]

Again, we see that \( \mathbb{C}[x_{w_0}] / J_{w_0,h} \) is reduced. Following the method outlined in the previous example, we see that the variables which vanish in the quotient are \( x_{2,1}, x_{2,2}, x_{2,3} \) and \( x_{3,1} \). See Figure 3.3.

![Figure 3.3. Variables killed in \( \mathbb{C}[x_{w_0}] / J_{w_0,h} \)](image)

**Proof of Lemma 3.14.** Let \( M = (x_{i,j}) \) determine a point in \( \mathcal{N}_{w_0,h} \). Recall that, as elements of \( \mathbb{C}[x_{w_0}] \), the variables \( x_{i,j} \) are subject to the following relations:

- \( x_{i,n+1-i} = 1 \), \( \forall i \in [n] \);
- \( x_{i,j} = 0 \), \( \forall i, j \in [n] : i > n + 1 - j \).

For all \( i, j \in [n] \), we have \( (M^{-1}M)_{n+1-i,j} = \delta_{n+1-i,j} \). This equality can be written more explicitly as

\[
y_{i,j} + \sum_{k=1}^{n-j} y_{i,n+1-k,k,j} = \delta_{n+1-i,j},
\]

where \( y_{i,j} := (M^{-1})_{n+1-i,n+1-j} \) (see (3.3) or (3.5) below for visualizations of this indexing).

For all \( i, j \in [n] \), the polynomials \( y_{i,j} \) have the following properties:

- (i) \( y_{i,n+1-i} = 1 \);
- (ii) \( y_{i,j} = 0 \), whenever \( i > n + 1 - j \);
- (iii) \( y_{i,j} \) is a polynomial in the variables \( x_{k,l} \) with \( k \geq i \) and \( l \geq j \).

These properties follow from equation (3.4) using an elementary inductive argument. Using properties (i) and (ii) we deduce that

\[
M^{-1} = \\
\begin{pmatrix}
1 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & y_{1,2} & \ldots & y_{1,n-1} \\
y_{1,n-1} & \ldots & y_{1,2} & 1
\end{pmatrix}.
\]
Lemma 3.15. Let \( n \) be a free variable. The other variables are the \( y \). Let us denote by \( i,j \) with row index \( k \geq i \) and column index \( q \geq j + 1 \). This follows from property (iii) combined with the observation that \( k \leq n - j \) implies \( n + 1 - k \geq j + 1 \). We conclude that the summation appearing in equation (3.6) depends only on variables \( x_{p,q} \) with \( q \geq j + 1 \) and \( p \geq i \), or \( q = j \) and \( p \geq i + 2 \).

Finally, the above discussion and a simple inductive argument imply that setting \( f_{n+1-i,j}^{w_0} \) equal to 0 has the effect of eliminating the variables \( x_{i+1,j} \) from the quotient \( \mathbb{C}[x_{w_0}] / J_{w_0,h} \) and there are no further relations on the remaining variables. Namely, \( \mathbb{C}[x_{w_0}] / J_{w_0,h} \) is isomorphic to the polynomial ring

\[
\mathbb{C}[x_{i,j} \mid 1 \leq i, j \leq n - 1, i \notin \{2,3,\ldots, n + 1 - h(j)\}],
\]

which in particular is reduced, as was to be shown. It also follows that \( J_{w_0,h} \) is radical and is the defining ideal of \( N_{w_0,h} \) in \( N_{w_0} \).

Motivated by the proof of Lemma 3.12 we introduce the following terminology which will be useful in Section 4 the set \( \{x_{i,j} \mid 1 \leq i,j \leq n-1, i \in \{2,3,\ldots, n+1-j\}\} \) consists of the non-free variables and the indices \( (i,j) \) for \( 1 \leq i,j \leq n-1, i \in \{2,3,\ldots, n+1-j\} \) give the positions of the non-free variables. The other variables are the free variables. In particular, observe that \( x_{1,1} \) is always a free variable.

We also record the following fact which follows easily from the above analysis and which we use in Section 4.

**Lemma 3.15.** Let \( h : [n] \to [n] \) be an indecomposable Hessenberg function. Then, for each pair \((i,j)\) with \( n - i \geq j \), we have

\[
f_{n+1-i,j}^{w_0} = x_{i+1,j} - g,
\]

where \( g \) is a polynomial contained in the ideal of \( \mathbb{C}[x_{w_0}] \) generated by \( \{x_{i,\ell} \mid j + 1 \leq \ell \leq n - i\} \).

**Proof.** Let us denote by \( I_{i,j} \) the ideal mentioned in the claim. From the expression (3.6) of \( f_{n+1-i,j}^{w_0} \), it suffices to show that \( y_{i,\ell} \notin I_{i,\ell} \) for \( j + 1 \leq \ell \leq n - i \). We fix arbitrary \( 1 \leq i < n \) and \( j < n - i \), and prove
this by induction on ℓ with \( j + 1 \leq \ell \leq n - i \). Recall from (3.4) with the properties (i) and (ii) that we have

\[
y_{i,\ell} = -\sum_{k=i}^{n-\ell} y_{i,n+1-k} x_{k,\ell} = -x_{i,\ell} - \sum_{k=i+1}^{n-\ell} y_{i,n+1-k} x_{k,\ell},
\]

where the second equality follows from \( i \leq n - \ell \) and \( y_{i,n+1-i} = 1 \). So when \( \ell = n - i \), we have

\[
y_{i,n-i} = -x_{i,n-i} \in I_{i,n-i}.
\]

Now, by induction, we assume that \( y_{i,p} \in I_{i,p} (\ell + 1 \leq p \leq n - i) \), and we prove that \( y_{i,\ell} \in I_{i,\ell} \). Our polynomial \( y_{i,\ell} \) is described by the rightmost expression of (3.7). There we have \( x_{i,\ell} \in I_{i,\ell} \), and also \( y_{i,n+1-k} \in I_{i,n+1-k} \), by the inductive hypothesis, since \( \ell + 1 \leq n + 1 - k \leq n - i \). These inequalities also imply that we have \( I_{i,n+1-k} \subset I_{i,\ell} \), and hence we obtain \( y_{i,\ell} \in I_{i,\ell} \), as desired.

Having just proved directly that \( \mathbb{C}[x_{w_0}] / J_{w_0,h} \) is reduced, the reader may wonder why we do not do the same for all \( w \in S_n \). As the proof of Lemma 3.12 may suggest, the argument works out well for the particular form of the matrices \( w_0 M \) for \( M \in U \); for general \( w \in S_n \), it seems to be more complicated to analyze these ideals directly, as the following simple example illustrates.

**Example 3.16.** Let \( n = 4 \) and \( h = (3,3,4,4) \). Let \( w = (2,4,1,3) \in S_4 \) as in Example 3.2 and Example 3.5. The ideal \( J_{w,h} \) is generated by \( f_{3,1}^w \) and \( f_{2,2}^w \) described in (3.2). Although one can check computationally (using, say, Macaulay2 [23]) that this ideal is reduced, it does not seem so straightforward to prove it directly.

**Proof of Propositions 3.6 and 3.7.** As we discussed already, Lemma 3.11 and Proposition 3.12 show that the scheme \( Z(N, h) \) is reduced. This proves Proposition 3.6. Moreover, by Lemma 3.3 Spec \( \mathbb{C}[x_{w}] / J_{w,h} \) is reduced for each \( w \in S_n \). That is, we conclude that the ideal \( J_{w,h} \) is the defining ideal of Hess\((N, h) \cap N_w \). This proves Proposition 3.7.

As we saw in Lemma 3.11 the zero scheme \( Z(N, h) \) is a local complete intersection. Now we additionally know that it is reduced, i.e., \( Z(N, h) \) is the integral scheme associated to Hess\((N, h) \), and thus obtain the following corollary.

**Corollary 3.17.** Let \( h \colon [n] \to [n] \) be any Hessenberg function. Then the corresponding regular nilpotent Hessenberg variety Hess\((N, h) \) is a local complete intersection.

**Proof.** If the Hessenberg function \( h \) is indecomposable, the claim holds as we saw above. Now suppose that \( h \) is not indecomposable. Then by the definition of indecomposability we must have \( h(j) = j \) for some \( j \in \{2, 3, \ldots, n - 1\} \). In this case, Hess\((N, h) \) is isomorphic to a product of regular nilpotent Hessenberg varieties whose Hessenberg functions are indecomposable [15, Theorem 4.5]. Thus the claim holds in this case as well.

### 4. One-parameter families of Hessenberg varieties

Let \( h \colon [n] \to [n] \) be a Hessenberg function and let \( H(h) \subseteq \mathfrak{gl}_n(\mathbb{C}) \) be the corresponding Hessenberg space. The Hessenberg varieties (see Definition 2.1) with Hessenberg function \( h \) can be assembled into a family over \( \mathfrak{gl}_n(\mathbb{C}) \) defined as

\[
\{(MB,X) \in \text{GL}_n(\mathbb{C}) / B \times \mathfrak{gl}_n(\mathbb{C}) \mid M^{-1} XM \in H(h) \} \subseteq \text{Flags}(\mathbb{C}^n) \times \mathfrak{gl}_n(\mathbb{C}).
\]

We are interested in a smaller family which we define as follows. Throughout the discussion we fix pairwise distinct complex numbers \( \gamma_1, \gamma_2, \ldots, \gamma_n \). For \( t \in \mathbb{C} \), we define

\[
\Gamma_t := \begin{pmatrix}
t \gamma_1 & 1 \\
t \gamma_2 & 1 \\
0 & \ddots & \ddots \\
0 & 0 & \ddots & 1 \\
0 & 0 & \cdots & t \gamma_n
\end{pmatrix}.
\]

Viewing \( \mathbb{C} \) as the complex affine line \( \mathbb{A}^1 = \mathbb{A}^1_{\mathbb{C}} \), we define a family over \( \mathbb{A}^1 \) by setting

\[
\mathcal{X}_h := \{(MB, t) \in \text{Flags}(\mathbb{C}^n) \times \mathbb{A}^1 \mid M^{-1} \Gamma_t M \in H(h) \}
\]
which can be viewed as a subfamily of \([1,1]\) via the embedding \(\mathbb{A}^1 \hookrightarrow \mathfrak{gl}_n(\mathbb{C})\) by \(t \mapsto \Gamma_t\), and in particular there is a canonical projection

\[
\pi : \mathcal{X}_h \rightarrow \mathbb{A}^1, \quad (MB, t) \mapsto t.
\]

The fiber at \(t \neq 0\) is a regular semisimple Hessenberg variety, and the fiber at \(t = 0\) is a regular nilpotent Hessenberg variety. In particular, the fibers are irreducible (\([13\text{ Theorem 6 and Corollary 9]}\) and \([5\text{ Lemma 7.1]}\)). By construction, this morphism is projective, and hence proper. Thus, the irreducibility of the base space \(\mathbb{A}^1\) and the fibers imply that the total space \(\mathcal{X}_h\) is irreducible as well (cf. the proof of \([43\text{ 1. § 6.3, Theorem 1.26]}\)). In this section, we will prove the following geometric properties of \(\mathcal{X}_h\) where we implicitly think of \(\mathcal{X}_h\) and \(\mathbb{A}^1\) as their associated integral schemes.

**Theorem 4.1.** Suppose that \(h\) is indecomposable. The morphism \(p : \mathcal{X}_h \rightarrow \mathbb{A}^1\) is flat, and its scheme-theoretic fibers over the closed points of \(\mathbb{A}^1\) are reduced.

As in Section 2, our family \(\mathcal{X}_h\) coincides set-theoretically with the zero locus of a section of the vector bundle \((GL_n(\mathbb{C}) \times_B (\mathfrak{gl}_n(\mathbb{C})/H)) \times \mathbb{A}^1\) given by

\[
s_{\Gamma}(MB, t) = ([M, M^{-1}\Gamma_t M], t)
\]

for \((MB, t) \in GL_n(\mathbb{C})/B \times \mathbb{C}\). Let \(Z(h)\) be the zero scheme of the section \(s_{\Gamma}\) above. Then we have a morphism \(Z(h) \rightarrow \text{Spec} \mathbb{C}[t]\) of schemes corresponding to the projection \(p : \mathcal{X}_h \rightarrow \mathbb{A}^1\). Since \(\mathcal{X}_h\) is irreducible as we discussed above, the scheme \(Z(h)\) is irreducible as well. By \([13\text{ Theorem 6}]\) or \([5\text{ Lemma 7.1}]\), the zero locus of \(s_{\Gamma}\) in \(GL_n(\mathbb{C})/B \times \mathbb{A}^1\) has the expected codimension, namely \(\sum_{i=1}^n (n - h(i))\). Hence, the zero scheme \(Z(h)\) is Cohen-Macaulay. Thus the morphism \(Z(h) \rightarrow \text{Spec} \mathbb{C}[t]\) is flat \([34\text{ Section 23]}\) since the fibers of \(\mathcal{X}_h \rightarrow \mathbb{A}^1\) have the same dimension.

The product \(\text{Flags}(\mathbb{C}^n) \times \mathbb{A}^1\) is covered by the affine varieties \(N_w \times \mathbb{A}^1\), for \(w \in \mathfrak{S}_n\), with coordinate ring \(\mathbb{C}[x_{w,t}]\). The family \(\mathcal{X}_h\) is covered by \(\mathcal{X}_h \cap (N_w \times \mathbb{A}^1)\), for \(w \in \mathfrak{S}_n\), and if we define \(F_{i,j}^w := (M^{-1}\Gamma_t M)_{i,j} \in \mathbb{C}[x_{w,t}]\), then \(\mathcal{X}_h \cap (N_w \times \mathbb{A}^1)\) is set-theoretically cut out by the equations \(F_{i,j}^w = 0\), for all \(i, j \in [n]\) with \(i > h(j)\). Let \(\mathcal{J}_{w,h} \subseteq \mathbb{C}[x_{w,t}]\) denote the ideal generated by the \(F_{i,j}^w\), for all \(i, j \in [n]\) with \(i > h(j)\). One can easily prove that

\[
Z(h) \cap \text{Spec} \mathbb{C}[x_{w,t}] \cong \text{Spec} \mathbb{C}[x_{w,t}]/\mathcal{J}_{w,h}.
\]

This gives us an open cover of the scheme \(Z(h)\). In other words, we have

\[
Z(h) = \bigcup_{w \in \mathfrak{S}_n} \text{Spec} \mathbb{C}[x_{w,t}]/\mathcal{J}_{w,h}.
\]

We are ready to prove that the scheme-theoretic fibers of \(Z(h) \rightarrow \text{Spec} \mathbb{C}[t]\) over the closed points are reduced. For this purpose, let \(z \in \mathbb{C}\) be a closed point in \(\text{Spec} \mathbb{C}[t]\). The local ring of \(\text{Spec} \mathbb{C}[t]\) at \(z\) is the localization \(\mathbb{C}[t]_{(t-z)}\). Let \(k(z)\) denote its residue field. Recall that the scheme-theoretic fibre of \(p : Z(h) \rightarrow \text{Spec} \mathbb{C}[t]\) over \(z\) is

\[
Z(h)_z := Z(h) \times_{\text{Spec} \mathbb{C}[t]} \text{Spec} (k(z)).
\]

Since \(Z(h)\) is covered by the open affine schemes \(\text{Spec} \mathbb{C}[x_{w,t}]/\mathcal{J}_{w,h}\) for \(w \in \mathfrak{S}_n\), the fibre \(Z(h)_z\) has an open affine covering consisting of

\[
\text{Spec} \mathbb{C}[x_{w,t}]/\mathcal{J}_{w,h} \times_{\text{Spec} \mathbb{C}[t]} \text{Spec} (k(z)) \cong \text{Spec} ((\mathbb{C}[x_{w,t}]/\mathcal{J}_{w,h}) \otimes_{\mathbb{C}[t]} k(z)).
\]

Consider the ideal \(\mathcal{J}_{w,h}_{|t=z} := \{F_{i,j}^w_{|t=z} \mid i > h(j)\}\) of \(\mathbb{C}[x_{w}]\) whose generators are obtained from the generators of \(\mathcal{J}_{w,h}\) after setting the variable \(t\) equal to \(z\). Since the functor \(- \otimes_{\mathbb{C}[t]} k(z)\) has the effect of substituting \(t\) with \(z\), we have an isomorphism of rings \((\mathbb{C}[x_{w,t}]/\mathcal{J}_{w,h}) \otimes_{\mathbb{C}[t]} k(z) \cong \mathbb{C}[x_{w}]/(\mathcal{J}_{w,h}_{|t=z})\) and thus

\[
Z(h)_z = \bigcup_{w \in \mathfrak{S}_n} \text{Spec} (\mathbb{C}[x_{w}]/(\mathcal{J}_{w,h}_{|t=z})).
\]

In order to show that the fibres \(Z(h)_z\) are reduced, we will prove that the rings \(\mathbb{C}[x_{w}]/(\mathcal{J}_{w,h}_{|t=z})\) are reduced.

**Proof of Theorem 4.1.** We already saw in the above discussion that the morphism \(p : \mathcal{X}_h \rightarrow \mathbb{A}^1\) is flat. Now consider the scheme-theoretic fiber at \(z = 0\). For any \(w \in \mathfrak{S}_n\), we have

\[
F_{i,j}^w_{|t=0} = (M^{-1}\Gamma_0 M)_{i,j} = (M^{-1}NM)_{i,j} = F_{i,j}^w.
\]
where $F_{i,j}^{w_0}$ is a generator of the ideal $J_{w, h}$ as introduced in Section 3. Then we have an equality of ideals $J_{w, h}|_{z = 0} = J_{w, h}$ for all $w \in \mathfrak{S}_n$. It follows that the ring $\mathbb{C}[x_n]/(J_{w, h}|_{z = 0})$ is reduced by Proposition 3.7.

Next, consider the case $z \neq 0$. Focusing on the $u_0$ patch, the ideal $J_{w_0, h}|_{z = z}$ is generated by the polynomials $F_{i,j}^{w_0} = (M^{-1} \mathbb{G} M)_{i,j}$ with $i > h(j)$. Recall that we have $M^{-1} = (y_{i,j})$, with the $y_{i,j}$ satisfying equation (4.4) and enjoying the properties (i), (ii), and (iii) recorded in the proof of Proposition 3.12. For $i < n + 1 - j$, equation (4.4) together with properties (i) and (ii) and (iii) imply that

$$y_{i,j} = -\sum_{k=i+1}^{n-j} y_{i,n+1-k} x_{k,j} - x_{i,j}.$$ 

Hence, by property (iii) the polynomial

$$\tilde{y}_{i,j} := y_{i,j} + x_{i,j}$$

does not depend on the variable $x_{i,j}$.

From the definition of $F_{n+1-i,j}^{w_0}$ it follows that

$$F_{n+1-i,j}^{w_0} = (0 \ldots 0 \ 1 \ y_{i,n-i} \ldots y_{i,1}) \begin{pmatrix} z\gamma_1 x_{1,j} + x_{2,j} \\ \vdots \\ z\gamma_{n-j} x_{n-j,1} + x_{n,j} \\ z\gamma_{n-j} x_{n-j,j} + 1 \\ z\gamma_{n+1-j} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= (z\gamma_i x_{i,j} + x_{i+1,j}) + \sum_{k=i+1}^{n-j} (z\gamma_k x_{k,j} + x_{k+1,j}) y_{i,n+1-k} + z\gamma_{n+1-j} y_{i,j}.$$ 

Note that the first and last summand always appear because the indecomposability of $h$ implies that $i < n + 1 - h(j) \leq n + 1 - j$, hence $i < n + 1 - j$. The condition $i < n + 1 - j$ also guarantees that the variable $x_{i,j}$ appearing in the expression above is not 0 or 1 (cf. (3.1)). Using equation (4.4), we obtain

$$F_{n+1-i,j}^{w_0} = z(\gamma_i - \gamma_{n+1-j}) x_{i,j} + x_{i+1,j}$$

$$+ \sum_{k=i+1}^{n-j} (z\gamma_k x_{k,j} + x_{k+1,j}) y_{i,n+1-k} + z\gamma_{n+1-j} \tilde{y}_{i,j}.$$ 

The coefficient $z(\gamma_i - \gamma_{n+1-j})$ of $x_{i,j}$ in equation (4.5) is nonzero since $z \neq 0$ and we assume the $\gamma_k$ are pairwise distinct. With the exception of the first term, all the terms in equation (4.5) depend only on variables $x_{k,\ell}$ with $k > i$ and $\ell \geq j$, or $k \geq i$ and $\ell > j$.

Now a simple inductive argument based on the above observations shows that in $\mathbb{C}[x_{n}] / (J_{w_0, h}|_{z = z})$ the variables $x_{i,j}$ with $1 \leq j \leq n - 1$ and $1 \leq i \leq n - h(j)$ can be replaced by expressions involving the free variables $x_{i,j}$ with $1 \leq j \leq n - 1$ and $i > n - h(j)$. More formally, we have the following ring isomorphisms

$$\mathbb{C}[x_{n}] / (J_{w_0, h}|_{z = z}) \cong \mathbb{C}[x_{i,j} \mid 1 \leq j \leq n - 1, i > n - h(j)].$$

It follows that the ring $\mathbb{C}[x_{n}] / (J_{w_0, h}|_{z = z})$ is reduced.\qed

We end this section with an example showing that Theorem 4.1 does not hold when $h$ is decomposable.

**Example 4.2 (Non-reduced fiber when $h$ is decomposable).** Let $n = 2$ and $h = (1, 2)$. We consider the open subset $\mathfrak{X}_h \cap (N_{\text{id}} \times \mathbb{A}^1)$ of our family $\mathfrak{X}_h$ and its fiber at $t = 0$. We have $J_{\text{id}, h} = \langle F_{2,1}^{\text{id}} \rangle \subseteq \mathbb{C}[x_{1,1}, t]$, where

$$F_{2,1}^{\text{id}} = t(\gamma_2 - \gamma_1) x_{1,1} - x_{1,1}^2.$$ 

It is easy to see directly that the quotient ring $\mathbb{C}[x_{1,1}, t] / J_{\text{id}, h}$ is reduced. However, we have $J_{\text{id}, h}|_{t = 0} = \langle x_{1,1}^2 \rangle$. Thus the ring $\mathbb{C}[x_{1,1}] / (x_{1,1}^2)$ is not reduced. We conclude that scheme-theoretic fiber $(\mathfrak{X}_h)_0$ is not reduced.
Corollary 4.3. Suppose that $h$ is indecomposable. The regular nilpotent Hessenberg variety $\text{Hess}(N, h)$ and the regular semisimple Hessenberg variety $\text{Hess}(S, h)$ determine the same cycles in $H_*(\text{GL}_n(\mathbb{C})/\mathcal{B})$:

$$[\text{Hess}(N, h)] = [\text{Hess}(S, h)] \quad \text{in} \quad H_*(\text{GL}_n(\mathbb{C})/\mathcal{B}).$$

5. Flags of subvarieties in regular nilpotent Hessenberg varieties

The point of this section is to use results and techniques from Section 3 to show that, in the case of indecomposable regular nilpotent Hessenberg varieties, there is a choice of a sequence of (dual) Schubert varieties which is particularly well-behaved when intersected with $\text{Hess}(N, h)$. While the construction is interesting in its own right, we were motivated by the theory of Newton-Okounkov bodies. For a given algebraic variety $X$, the computation of Newton-Okounkov bodies associated to $X$ requires the choice of auxiliary data, one of which is a valuation on the rational functions on $X$. Natural candidates for such valuations are given by well-behaved flags of subvarieties of $X$. In general it is natural to choose such flags which are compatible with existing structures on $X$. For instance, for flag varieties $G/B$, Kaveh showed in [29] that flags of Schubert varieties give rise to Newton-Okounkov bodies with intimate connections to representation theory. Thus, for Hessenberg varieties, which are subvarieties of the flag variety $\text{Flags}(\mathbb{C}^n)$, it is natural to consider flags of subvarieties obtained by intersecting with Schubert varieties, as we discuss here.

Recall from [19, § 10.6, p.176] that the **dual Schubert variety** $\Omega_w \subseteq \text{Flags}(\mathbb{C}^n)$ for $w \in \mathfrak{S}_n$ is the set of $V \in \text{Flags}(\mathbb{C}^n)$ satisfying the condition

$$\dim(V_p \cap \tilde{F}_{n-q}) \geq |\{i \leq p \mid w(i) \geq q + 1\}|$$

for $q, p \in [n]$ where $\tilde{F}_j$ is the **anti-standard flag** given by $\tilde{F}_j := \mathbb{C}e_{n+1-j} \oplus \mathbb{C}e_{n+2-j} \oplus \cdots \oplus \mathbb{C}e_n$. Recall also that $\text{codim}(\Omega_w \subseteq \text{Flags}(\mathbb{C}^n)) = \ell(w)$ the length of $w \in \mathfrak{S}_n$ [19, § 10.2, p.159].

For a permutation $w \in \mathfrak{S}_n$, let us define the **rank matrix** $rk(w)$ by

$$rk(w)[q, p] := |\{i \leq p \mid w(i) \leq q\}|.$$ 

Evidently, $rk(w)[q, p]$ is the rank of the upper left $q \times p$ submatrix of the permutation matrix of $w$. Recall that the permutation matrix of $w \in \mathfrak{S}_n$ is the matrix which has $1$’s in the $(w(j), j)$-th entries for $1 \leq j \leq n$ and $0$’s elsewhere. For $V \in \text{Flags}(\mathbb{C}^n)$, let us consider the composition of the maps

$$V_p \hookrightarrow \mathbb{C}^n \twoheadrightarrow \mathbb{C}^n/\tilde{F}_{n-q}.$$ 

Then we have

$$\text{rank}(V_p \to \mathbb{C}^n/\tilde{F}_{n-q}) = \dim V_p - \dim \ker(V_p \to \mathbb{C}^n/\tilde{F}_{n-q}) = p - \dim(V_p \cap \tilde{F}_{n-q})$$

and

$$rk(w)[q, p] = |\{i \leq p \mid w(i) \leq q\}| = p - |\{i \leq p \mid w(i) \geq q + 1\}|.$$ 

Hence, we get

$$\Omega_w = \{V \in \text{Flags}(\mathbb{C}^n) \mid \text{rank}(V_p \to \mathbb{C}^n/\tilde{F}_{n-q}) \leq rk(w)[q, p] \text{ for } q, p \in [n]\}.$$ 

Now, let us write an element $V \in \text{Flags}(\mathbb{C}^n) = \text{GL}_n(\mathbb{C})/\mathcal{B}$ in the standard neighbourhood $\mathcal{N}_{w_0}(\subset \text{Flags}(\mathbb{C}^n))$ around $w_0 B$ by a matrix of the form

$$V = \begin{pmatrix}
    x_{1,1} & x_{1,2} & \cdots & x_{1,n-1} & 1 \\
    x_{2,1} & x_{2,2} & \cdots & 1 & \vdots \\
    \vdots & \vdots & \ddots & \vdots & \ddots \\
    x_{n-1,1} & 1 & \cdots & \cdots & \cdots \\
    1 & & & & \\
\end{pmatrix} B.$$ 

This is the notation from [11]. In contrast, [19] uses $r_w(q, p) = rk(w^{-1})[q, p]$.
Then [5.1] implies that the opposite Schubert variety $\Omega_w \cap N_{w_0}$ (in this neighbourhood) is described as the set of $V_\bullet \in \text{Flags}(\mathbb{C}^n)$ satisfying the condition:

$$\text{the upper-left } q \times p \text{ matrix in (5.2) has rank at most } rk(w)[q,p] \text{ for all } q,p \in [n].$$

The diagram $D(w)$ of a permutation $w \in S_n$ is obtained from the matrix of $w^{-1}$ by removing all cells in an $n \times n$ array which are weakly to the right and below a 1 in $w^{-1}$. The remaining cells form the diagram $D(w)$. Note that the cells of $D(w^{-1})$ are in bijection with the inversions in $w^{-1}$, and in particular, the Bruhat length $\ell(w) = \ell(w^{-1})$ of $w$ is equal to $|D(w^{-1})|$. For $w \in S_n$, we say that the diagram $D(w^{-1})$ forms a Young diagram if all of the boxes in the diagram are connected. From the definitions, the following lemma is immediate.

![Figure 5.1](image)

**Figure 5.1.** For $w = (4, 8, 6, 2, 7, 3, 1, 5)$ in one-line notation, $D(w)$ is the configuration of white boxes in the array above.

**Lemma 5.1.** Let $w \in S_n$ and suppose that $D(w^{-1})$ forms a Young diagram. Then we have

$$rk(w)[q,p] = 0 \text{ for } (q,p) \in D(w^{-1}).$$

**Lemma 5.2.** Suppose that $D(w^{-1})$ forms a Young diagram. Then the opposite Schubert variety $\Omega_w \cap N_{w_0}$ (in the affine chart $N_{w_0}$) is the set of $V_\bullet \in \text{Flags}(\mathbb{C}^n)$ satisfying the condition

$$x_{q,p} = 0 \text{ for } (q,p) \in D(w^{-1})$$

where $x_{i,j}$ are the coordinates for $N_{w_0}$ given in (5.2).

**Proof.** Let $Z \subseteq N_{w_0}$ be the (irreducible) Zariski closed subset of $V_\bullet \in N_{w_0} (\subseteq \text{Flags}(\mathbb{C}^n))$ satisfying

$$x_{q,p} = 0 \text{ for } (q,p) \in D(w^{-1}).$$

Then, it is clear from Lemma 5.1 that $\Omega_w \cap N_{w_0} \subseteq Z$. Also, we have

$$\text{codim } \Omega_w \cap N_{w_0} = \ell(w) = \ell(w^{-1}) = |D(w^{-1})| = \text{codim } Z$$

where the first equality uses the fact that $\Omega_w \cap N_{w_0} \neq \emptyset$. Hence $\dim \Omega_w \cap N_{w_0} = \dim Z$, and since $Z$ is irreducible, we obtain $\Omega_w \cap N_{w_0} = Z$. 

We now build a flag of subvarieties in indecomposable regular nilpotent Hessenberg varieties which looks like a flag of coordinate subspaces near the point $w_0$. The construction uses a particular sequence of dual Schubert varieties in Flags$(\mathbb{C}^n)$ which we now describe. First set

$$D := \dim \text{Flags}(\mathbb{C}^n) = \frac{1}{2} n(n-1)$$

and let $u_i \in S_n$ denote the permutation obtained by multiplying the right-most $i$ simple transpositions of the word

$$(s_1)(s_2s_1)(s_3s_2s_1)\cdots(s_{n-1}s_{n-2}\cdots s_2s_1),$$

where $s_i$ denotes the simple transposition exchanging $i$ and $i+1$, and we set $u_0 := \text{id}$. Note that $u_D (= w_0)$ is the longest element. It is not hard to check that the diagrams $D(u_i^{-1})$ form Young diagrams, and that the Young diagrams corresponding to the sequence $u_0^{-1}, u_1^{-1}, \ldots, u_D^{-1}, u_D^{-1} = u_D$ "grow" in sequence by adding boxes from left to right, starting at the top row. We illustrate with an example.
Example 5.3. Suppose \( n = 5 \). Then
\[
\begin{align*}
u_0 &= \text{id}, \\
u_1 &= s_1, \\
u_2 &= s_2s_1, \\
u_3 &= s_3s_2s_1, \\
u_4 &= s_4s_3s_2s_1, \\
u_5 &= s_1s_4s_3s_2s_1, \\
u_6 &= s_2s_1s_4s_3s_2s_1, \\
u_7 &= s_3s_2s_1s_4s_3s_2s_1, \\
u_8 &= s_1s_3s_2s_1s_4s_3s_2s_1, \\
u_9 &= s_2s_1s_3s_2s_1s_4s_3s_2s_1, \\
u_{10} &= s_1s_2s_1s_3s_2s_1s_4s_3s_2s_1.
\end{align*}
\]

The Young diagrams of \( u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10} \) are

```
\[
\begin{array}{c}
\emptyset \\
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\end{array}
\begin{array}{cccc}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\end{array}
\begin{array}{cccc}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\end{array}
\begin{array}{cccc}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\end{array}
\begin{array}{cccc}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\end{array}
\end{array}
\]
\end{align*}
```

We can now define a sequence of subvarieties of \( \text{Hess}(N, h) \) by intersecting with a sequence of dual Schubert varieties, as follows:
\[
\text{Hess}(N, h) = \Omega_{u_0} \cap \text{Hess}(N, h) \supseteq \Omega_{u_1} \cap \text{Hess}(N, h) \supseteq \cdots \supseteq \Omega_{u_{10}} \cap \text{Hess}(N, h) = \{ w_0B \}.
\]
(5.3)

This sequence is not proper in the sense that it may happen that \( \Omega_{u_i} \cap \text{Hess}(N, h) = \Omega_{u_{i+1}} \cap \text{Hess}(N, h) \) for some \( i \). Nevertheless, by omitting redundancies of the above form, we obtain a flag of subvarieties of \( \text{Hess}(N, h) \) with well-behaved geometric properties within the open dense subset \( N_{w_0} \). This is the content of the next theorem and is the main result of this section. Recall from Section 3 that the defining equations of \( N_{w_0, h} = \text{Hess}(N, h) \cap N_{w_0} \) in \( N_{w_0} \) have the property that some of the coordinates \( x_{i,j} \) are free and others are non-free variables (cf. remarks after proof of Lemma 3.12).

Theorem 5.4. Let \( h: \mathbb{N} \to \mathbb{N} \) be an indecomposable Hessenberg function. Let \( \{ u_\ell \}_{\ell=0}^D \) be the sequence in \( \mathcal{S}_n \) defined above, where \( D = n(n - 1)/2 \). Let \( N_{w_0, h} = \text{Hess}(N, h) \cap N_{w_0} \) be the open affine chart of \( \text{Hess}(N, h) \) around \( w_0B \). Then the subvarieties
\[
N_{w_0, h} = \Omega_{u_0} \cap N_{w_0, h} \supseteq \Omega_{u_1} \cap N_{w_0, h} \supseteq \cdots \supseteq \Omega_{u_D} \cap N_{w_0, h} = \{ w_0B \}
\]
(5.4)

satisfy the following:

1. if the lowest lower-right corner of the Young diagram formed by \( D(u_\ell^{-1}) \) is located at the position of a free variable, then \( \Omega_{u_{\ell-1}} \cap N_{w_0, h} \neq \Omega_{u_{\ell}} \cap N_{w_0, h} \) and
\[
\dim \Omega_{u_{\ell}} \cap N_{w_0, h} = \dim \Omega_{u_{\ell-1}} \cap N_{w_0, h} - 1;
\]

otherwise, \( \Omega_{u_{\ell-1}} \cap N_{w_0, h} = \Omega_{u_{\ell}} \cap N_{w_0, h} \);

2. each \( \Omega_{u_\ell} \cap N_{w_0, h} \) is isomorphic to an affine space, and in particular is non-singular and irreducible in \( N_{w_0, h} \).

Proof. Throughout this argument we use the explicit list of \( D = n(n - 1)/2 \) coordinates on \( N_{w_0} \cong \mathbb{A}^D = \mathbb{A}^{n(n-1)/2} \) given in (5.2), totally ordered by reading the variables from left to right and top to bottom, i.e.
\[
x_{1,1}, x_{1,2}, \ldots, x_{1,n-1}, x_{2,1}, x_{2,2}, \ldots, x_{2,n-2}, \ldots, x_{n-1,1}.
\]
(5.5)
Note also that there are exactly as many variables in the list above as there are elements in the sequence
\[ u_1, u_2, \ldots, u_D. \]
As already observed above, from the construction of the sequence \( u_\ell \) it follows that the associated diagrams \( D(u_\ell^{-1}) \) form Young diagrams, and for a given \( \ell \), \( 1 \leq \ell \leq D \), the Young diagram of \( D(u_\ell^{-1}) \) contains the boxes corresponding to the first \( \ell \) variables in the list (5.5). We already saw in Lemma 5.2 that \( \Omega_{u_\ell} \cap N_{\omega_0} \) is equal to the coordinate subspace given by \( \{ x_{q,p} = 0 \mid (q,p) \in D(u_\ell^{-1}) \} \), so it follows that the sequence of intersections \( \Omega_{u_\ell} \cap N_{\omega_0} \) can be described explicitly in coordinates by setting the first \( \ell \) variables in (5.5) equal to 0, i.e. we have
\[
N_{\omega_0} \supset \{ x_{1,1} = 0 \} \supset \{ x_{1,1} = x_{1,2} = 0 \} \supset \cdots \supset \{ x_{1,1} = x_{1,2} = \cdots = x_{n-1,1} = 0 \} = \{ \omega_0 B \}.
\]
In order to prove the statements in the theorem, we must now also analyze the intersection of these \( \Omega_{u_\ell} \cap N_{\omega_0} \) with \( \text{Hess}(N, h) \). We proceed by induction on \( \ell \).

For \( \ell = 1 \), we have \( \mathbb{C}[\Omega_{u_1} \cap N_{\omega_0}, h] \cong \mathbb{C}[N_{\omega_0}, h]/(x_{1,1}) \). As shown in Lemma 5.12 \( \mathbb{C}[N_{\omega_0}, h] \cong \mathbb{C}[x_{\omega_0}]/J_{\omega_0, h} \) is isomorphic to a polynomial ring. Moreover, \( D(u_1^{-1}) \) is a single box located at the position of \( x_{1,1} \), which is always a free variable. Therefore \( \mathbb{C}[\Omega_{1} \cap N_{\omega_0}, h] \) is isomorphic to a polynomial ring of dimension one less than \( \mathbb{C}[\Omega_{u_1} \cap N_{\omega_0}, h] \cong \mathbb{C}[N_{\omega_0}, h] \), and \( \Omega_{u_1} \cap N_{\omega_0} \) satisfies properties (1) and (2).

For \( \ell > 1 \), let \( x_{i,j} \) denote the \( \ell \)-th variable in the ordered list (5.5), so that \( \Omega_{u_{\ell-1}} \cap N_{\omega_0, h} \) is obtained from \( \Omega_{u_{\ell-1}} \cap N_{\omega_0, h} \) by setting \( x_{i,j} \) equal to 0. (Visually, the position \( (i,j) \) is the lowest lower-right corner of the Young diagram corresponding to \( D(u_{\ell-1}^{-1}) \).) First we consider the case when \( x_{i,j} \) is a free variable. Then it is clear that \( x_{i,j} = 0 \) places a new linear condition on \( \Omega_{u_{\ell-1}} \cap N_{\omega_0, h} \). Moreover, \( \mathbb{C}[\Omega_{u_{\ell-1}} \cap N_{\omega_0, h}] \) is irreducible by inductive hypothesis. Therefore the new condition \( x_{i,j} = 0 \) forces \( \Omega_{u_{\ell}} \cap N_{\omega_0, h} \neq \Omega_{u_{\ell-1}} \cap N_{\omega_0, h} \) and \( \dim \Omega_{u_{\ell}} \cap N_{\omega_0, h} = \dim \Omega_{u_{\ell-1}} \cap N_{\omega_0, h} - 1 \). Next suppose that \( x_{i,j} \) is a non-free variable. As we saw in Lemma 5.12 the defining equations of \( \text{Hess}(N, h) \) within the affine coordinate chart \( N_{\omega_0} \) take the form
\[
x_{i,j} = g
\]
where \( x_{i,j} \) is a non-free variable and where \( g \) is a polynomial in the free variables which is contained in the ideal generated by \( x_{i-1,t} \) for \( t > j \). Since the sequence (5.6) sets variables equal to 0 in order from left to right and top to bottom, we know that at this \( \ell \)-th step, all variables \( x_{i-1,t} \) for \( t > j \), which are contained in the row directly above that of \( x_{i,j} \), have already been set equal to 0, and hence \( x_{i,j} \) is already equal to 0 in \( \Omega_{u_{\ell-1}} \cap N_{\omega_0, h} \). Thus the placement of the additional condition \( x_{i,j} = 0 \) does not affect the intersection and we conclude that in this case \( \Omega_{u_{\ell}} \cap N_{\omega_0, h} = \Omega_{u_{\ell-1}} \cap N_{\omega_0, h} \), as was to be shown.

It follows from the above that each \( \Omega_{u_{\ell}} \cap N_{\omega_0, h} \) is isomorphic to an affine space with codimension equal to the number of free variables contained within the first \( \ell \) variables in the sequence (5.5). In particular, it is non-singular and irreducible. This completes the proof. \( \square \)

The practical consequence of the above discussion is the following. By omitting the redundancies in the sequence (5.4) caused by the non-free variables, we obtain a flag of subvarieties in \( \text{Hess}(N, h) \) (defined in a geometrically natural fashion by intersecting with dual Schubert varieties) such that, near \( \omega_0 B \), the flag is simply a sequence of affine coordinate subspaces. It would be interesting to compute Newton-Okounkov bodies of regular nilpotent Hessenberg varieties associated to this natural flag. Indeed, the computation of the special case of the Peterson variety in Section 7 uses the flag described above.

6. AN EFFICIENT FORMULA FOR THE DEGREE OF REGULAR NILPOTENT HESSENBerg VARIETIES

Let \( \text{Hess}(X, h) \) be a Hessenberg variety in \( \text{Flags}(\mathbb{C}^n) \) and consider a Plücker embedding of \( \text{Flags}(\mathbb{C}^n) \hookrightarrow \mathbb{P}(V_\lambda^*) \), where \( \lambda \) is a strict partition and \( V_\lambda \) is the irreducible representation of \( \text{GL}_n(\mathbb{C}) \) associated with \( \lambda \). It is then natural to consider the induced embedding \( \text{Hess}(X, h) \hookrightarrow \mathbb{P}(V_\lambda^*) \), and to ask for its degree. In this section, we give an efficient computation of the degree of \( \text{Hess}(N, h) \hookrightarrow \mathbb{P}(V_\lambda^*) \) for all indecomposable regular nilpotent Hessenberg varieties. Throughout this section, we let \( S \colon \mathbb{C}^n \rarrow \mathbb{C}^n \) be a semisimple operator with pairwise distinct eigenvalues, and we consider the associated regular semisimple Hessenberg variety \( \text{Hess}(S, h) \).

In Theorem 4.1 we showed that a certain family \( \mathbb{X}_h \to \mathbb{A}^1 \) of Hessenberg varieties is both flat and has reduced fibres. Since Hilbert polynomials are constant along fibres of a flat family [26, Theorem 9.9] and
because the special fibre is reduced, we can conclude the following. (We can also obtain this result from Corollary 4.3)

**Corollary 6.1.** Let $\lambda$ be a dominant weight and let $\text{Flags}(\mathbb{C}^n) \hookrightarrow \mathbb{P}(V_\lambda^*)$ be the corresponding Plücker embedding. By composing with the natural inclusion maps, we obtain embeddings $\text{Hess}(N, h) \hookrightarrow \mathbb{P}(V_\lambda^*)$ and $\text{Hess}(S, h) \hookrightarrow \mathbb{P}(V_\lambda^*)$. If $h$ is indecomposable, then the degrees of these two embeddings are equal, i.e.,

$$\deg(\text{Hess}(N, h) \hookrightarrow \mathbb{P}(V_\lambda^*)) = \deg(\text{Hess}(S, h) \hookrightarrow \mathbb{P}(V_\lambda^*)) .$$

It is known that regular semisimple Hessenberg varieties are smooth and are equipped with an action of the maximal torus $T$ of $\text{GL}_n(\mathbb{C})$ [13]. In what follows, we use the recent work of Abe, Horiguchi, Masuda, Murai, and Sato [4] as well as the classical Atiyah-Bott-Berline-Vergne formula to obtain a computationally efficient formula for the degree of the embedding $\text{Hess}(S, h) \hookrightarrow \mathbb{P}(V_\lambda^*)$, expressed as a polynomial in the components of $\lambda = (\lambda_1 > \lambda_2 > \ldots, \lambda_{n-1} > \lambda_n)$. By Corollary 6.1, the formula also computes the degree of $\text{Hess}(N, h)$.

We now turn to the details. Let $\lambda = (\lambda_1 > \lambda_2 > \ldots, \lambda_{n-1} > \lambda_n) \in \mathbb{Z}^n$ be a strict partition. It is well-known that there is a unique irreducible representation $\lambda$ of $\text{GL}_n(\mathbb{C})$ associated with $\lambda$, and a corresponding Plücker embedding

$$\text{Flags}(\mathbb{C}^n) \cong \text{GL}_n(\mathbb{C})/B \rightarrow \mathbb{P}(V_\lambda^*)$$

given by mapping $\text{Flags}(\mathbb{C}^n)$ to the $\text{GL}_n(\mathbb{C})$-orbit of the highest weight vector in $V_\lambda^*$. Composing with the canonical inclusion map $\text{Hess}(N, h) \hookrightarrow \text{Flags}(\mathbb{C}^n)$, this gives us a closed embedding of $\text{Hess}(N, h)$ into $\mathbb{P}(V_\lambda^*)$.

Define the **volume** of this embedding (or of the corresponding line bundle) by

$$\text{Vol}(\text{Hess}(N, h) \hookrightarrow \mathbb{P}(V_\lambda^*)) := \frac{1}{d!} \deg(\text{Hess}(N, h) \hookrightarrow \mathbb{P}(V_\lambda^*))$$

where $d := \dim \text{Hess}(N, h) = \sum_{j=1}^n (h(j) - j)$.

Using the result from [4] that the cohomology ring $H^*(\text{Hess}(N, h); \mathbb{Q})$ is a Poincaré duality algebra generated by degree 2 elements, the recent work of [4] relates the cohomology ring of $\text{Hess}(N, h)$ to other combinatorial and algebraic invariants; in particular, in [4, § 11] they define, purely algebraically, a certain polynomial (denoted $P_1$ in [4, § 11]) associated to $H^*(\text{Hess}(N, h); \mathbb{Q})$. The main result of this section is that this polynomial computes the volume $\text{Vol}(\text{Hess}(N, h) \hookrightarrow \mathbb{P}(V_\lambda^*))$. To state the result precisely, we first concretely define the polynomial (up to a scalar multiple) given in [4] for our special case of Lie type $A_{n-1}$. Let $\mathbb{Q}[x_1, \ldots, x_n]$ be a polynomial ring in $n$ variables and for any $i \in [n]$ let $\partial_{x_i}$ denote the usual derivative with respect to the variable $x_i$. Also for any $i, j \in [n]$ we define $\partial_{i,j} := \partial_{x_j} - \partial_{x_i}$. With this notation in place we may now define, following [4],

$$P_h(x_1, \ldots, x_n) := \left( \prod_{h(j) < i} \partial_{i,j} \right) \left( \prod_{1 \leq k < \ell \leq n} \frac{x_k - x_\ell}{\ell - k} \right) \in \mathbb{Q}[x_1, \ldots, x_n].$$

The theorem below is the main result of this section.

**Theorem 6.2.** Let $h : [n] \rightarrow [n]$ be an indecomposable Hessenberg function and let $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_n) \in \mathbb{Z}^n$ be a strict partition. Then

$$\text{Vol}(\text{Hess}(N, h) \hookrightarrow \mathbb{P}(V_\lambda^*)) = P_h(\lambda_1, \ldots, \lambda_n).$$

**Proof.** Consider the regular semisimple Hessenberg variety $\text{Hess}(S, h)$ corresponding to the same Hessenberg function $h$ and define the volume $\text{Vol}(\text{Hess}(S, h))$ by the same formula [6.1] (replacing $N$ by $S$). From the right-hand side of (6.1) and by Corollary 6.1 it follows that it suffices to prove that the volume of the regular semisimple Hessenberg variety is computed by $P_h$, i.e. it is enough to show

$$\text{Vol}(\text{Hess}(S, h) \hookrightarrow \mathbb{P}(V_\lambda^*)) = P_h(\lambda_1, \ldots, \lambda_n).$$

Since $\text{Hess}(S, h)$ is non-singular [13], the degree of a projective embedding is equal to its symplectic volume [24, § 1.3, pg. 171]:

$$\text{Vol}(\text{Hess}(S, h) \hookrightarrow \mathbb{P}(V_\lambda^*)) = \frac{1}{d!} \deg(\text{Hess}(S, h) \hookrightarrow \mathbb{P}(V_\lambda^*)) = \frac{1}{d!} \int_{\text{Hess}(S, h)} c_1(L_\lambda)^d$$

where $c_1(L_\lambda)$ is the first Chern class of the line bundle $L_\lambda$. It is known that $L_\lambda$ is a line bundle generated by homogeneous polynomials of degree 2, and writing $P_h$ in terms of $\partial_{i,j}$ suggests that it is the degree 2 component of $L_\lambda$.
where \(c_1(L_\lambda)\) is the first Chern class of the pullback line bundle \(L_\lambda\) on \(\text{Hess}(S, h)\) with respect to the Plücker embedding and \(d = \dim_{\mathbb{C}} \text{Hess}(S, h) = \sum_{j=1}^{n} (h(j) - j)\). Since the maximal torus \(T\) of \(GL_n(\mathbb{C})\) acts on \(\text{Hess}(S, h)\) [13], the Atiyah-Bott-Berline-Vergne localization formula [6, 8] computes this integral using the local data around the torus fixed points:

\[
\frac{1}{d!} \int_{\text{Hess}(S, h)} c_1(L_\lambda)^d = \frac{1}{d!} \sum_{\omega \in \mathfrak{g}_n} \lambda_{\omega} e_{\omega},
\]

where \(\lambda_{\omega}\) denotes the weight of the \(T\)-action on the fiber of \(L_\lambda\) at the fixed point \(wB\) and \(e_{\omega}\) denotes the \(T\)-equivariant Euler class of the normal bundle to the fixed point \(wB\), i.e., the product of the weights of the \(T\)-representation on the tangent space \(T_w \text{Hess}(S, h)\).

To proceed further we need a more explicit description of the line bundle \(L_\lambda\). Let \(L_i\) denote the \(i\)-th tautological line bundle over \(\text{Flags}(\mathbb{C}^n)\), i.e., the fiber of \(L_i\) at a flag \(V_\bullet \in \text{Flags}(\mathbb{C}^n)\) is \(V_i/V_{i-1}\). Then it is well-known [19, § 9.3] that

\[
L_\lambda \cong (L_1^*)^{\lambda_1} \otimes (L_2^*)^{\lambda_2} \otimes \cdots \otimes (L_n^*)^{\lambda_n}
\]

is the pullback to \(\text{Flags}(\mathbb{C}^n)\) of \(O(1) \to \mathbb{P}(V_\lambda^*)\). By slight abuse of notation we also denote by \(L_\lambda\) this line bundle restricted to \(\text{Hess}(S, h)\).

We can now compute the right-hand side of (6.4). Recall that the torus \(T\) in question is the diagonal torus \(T = \{ \text{diag}(t_1, t_2, \ldots, t_n) \mid t_i \in \mathbb{C}^\times \} \) of \(GL_n(\mathbb{C})\). In this context, \(T\)-weights are elements of \(\mathbb{Z}[t_1, \ldots, t_n]\) where each \(t_i\) denotes the weight \(T \to \mathbb{C}^\times\) defined by \(\text{diag}(t_1, t_2, \ldots, t_n) \mapsto t_i\). The weight of the \(i\)-th tautological line bundle \(L_i\) at the fixed point \(w \in \mathfrak{g}_n\) is given by \(t_{w(i)}\) since the fiber is spanned by \(e_{w(i)} \subset \mathbb{C}^n\) by definition of \(L_i\) where \(e_1, \ldots, e_n\) are the standard basis of \(\mathbb{C}^n\). Thus the weight \(\lambda_{w}\) is

\[
\lambda_{w} = - \sum_{i=1}^{n} \lambda_i t_{w(i)}.
\]

It is also known [13] that the weight \(e_{w}\) is given by

\[
e_{w} = \prod_{j < i \leq h(j)} (t_{w(i)} - t_{w(j)}) = (-1)^d \prod_{j < i \leq h(j)} (t_{w(i)} - t_{w(j)}).
\]

Putting together (6.3), (6.4), (6.6) and (6.7) we therefore obtain

\[
\text{Vol}(\text{Hess}(S, h) \to \mathbb{P}(V_\lambda^*)) = \frac{1}{d!} \sum_{\omega \in \mathfrak{g}_n} \prod_{j < i \leq h(j)} (t_{w(i)} - t_{w(j)})
\]

The essential idea of what follows, due to [1], is to now think of the right-hand side of (6.8) as a polynomial in the variables \(\lambda_i\). More precisely, let us define

\[
Q_{\text{Hess}(S, h)}(x_1, \ldots, x_n) := \frac{1}{d!} \prod_{\omega \in \mathfrak{g}_n} \prod_{j < i \leq h(j)} (x_{i} - x_{j})
\]

This is in fact a polynomial in \(\mathbb{R}[x_1, \ldots, x_n]\) since after taking the summation over \(\mathfrak{g}_n\) the right-hand side does not depend on \(t_1, \ldots, t_n\) [6, 8]. From the definition it follows that for any strict partition \(\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_n)\) we have

\[
\text{Vol}(\text{Hess}(S, h) \to \mathbb{P}(V_\lambda^*)) = Q_{\text{Hess}(S, h)}(\lambda_1, \ldots, \lambda_n).
\]

Now a straightforward computation shows that

\[
\partial_{i,j} \left( \sum_{i=1}^{n} x_i t_{w(i)} \right) = t_{w(j)} - t_{w(i)}.
\]

From this, it follows from an easy induction argument that

\[
Q_{\text{Hess}(S, h)}(x_1, \ldots, x_n) = \left( \prod_{h(j) < i} \partial_{i,j} \right) Q_{\text{Flags}(\mathbb{C}^n)}(x_1, \ldots, x_n)
\]
where we think of Flags($\mathbb{C}^n$) as the regular semisimple Hessenberg variety with $h = (n, \ldots, n)$. For a strict partition $\lambda$, the volume of Flags($\mathbb{C}^n$) with respect to the Plücker embedding into $\mathbb{P}(V_\lambda^*)$ is well-known to be the volume of the Gelfand-Cetlin polytope associated to $\lambda$, for which a formula is known (e.g. [36] and [39]), and we conclude

$$(6.11) \quad \text{Vol}(\text{Flags}($\mathbb{C}^n$)) = Q_{\text{Flags}($\mathbb{C}^n$)}(x_1, \ldots, x_n) = \prod_{1 \leq k < \ell \leq n} \frac{x_k - x_\ell}{\ell - k}.$$ 

From (6.10) and (6.11) we therefore deduce that

$$(6.12) \quad Q_{\text{Hess}(S,h)}(x_1, \ldots, x_n) = P_h(x_1, \ldots, x_n).$$

Thus, from (6.9) and (6.12), we conclude that for a strict partition $\lambda$

$$\text{Vol}(\text{Hess}(S,h)) = Q_{\text{Flags}($\mathbb{C}^n$)}(x_1, \ldots, x_n)$$

as was to be shown. \hfill \square

**Remark 6.3.** Since the line bundle $L$ is trivial, we have $L_n \cong L_1^* \otimes \cdots \otimes L_{n-1}^*$. So we can always assume that $\lambda_n = 0$.

We can use Theorem 6.2 to compute the volume of a special case of a regular nilpotent Hessenberg variety which is studied in Section 7.

**Example 6.4.** Let $n = 3$ and $h = (2,3,3)$, and consider the corresponding regular nilpotent Hessenberg variety $\text{Pet}_3 := \text{Hess}(N, h) \subset \text{Flags}(\mathbb{C}^3)$. Then

$$P_h(x_1, x_2, x_3) = (\partial_{x_1} - \partial_{x_2}) \left( \frac{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}{2} \right)$$

$$= \frac{1}{2} (x_1 - x_2)^2 + 2(x_1 - x_2)(x_2 - x_3) + \frac{1}{2} (x_2 - x_3)^2.$$

So we obtain

$$\text{Vol}(\text{Pet}_3 \hookrightarrow \mathbb{P}(V_\lambda^*)) = \frac{1}{2} (\lambda_1 - \lambda_2)^2 + 2(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3) + \frac{1}{2} (\lambda_2 - \lambda_3)^2$$

for any strict partition $\lambda = (\lambda_1 > \lambda_2 > \lambda_3)$. Let us introduce the notation $a_1 := \lambda_2 - \lambda_3$ and $a_2 := \lambda_1 - \lambda_2$ and set $\lambda_3 = 0$ following Remark 6.3. Then we have

$$\text{Vol}(\text{Pet}_3 \hookrightarrow \mathbb{P}(V_\lambda^*)) = \frac{1}{2} a_1^2 + 2a_1 a_2 + \frac{1}{2} a_2^2.$$ 

7. **Newton-Okounkov bodies of Peterson varieties**

The theory of Newton-Okounkov bodies gives a new method of associating combinatorial data to geometric objects. In the case of a toric variety $X$, the combinatorics of its moment map polytope $\Delta$ fully encodes the geometry of $X$, but this fails in the general case. Building on the work of Okounkov [37, 38], Kaveh-Khovanskii [30] and Lazarsfeld-Mustata [35] construct a convex body $\Delta$ in $\mathbb{R}^n$ associated to $X$ equipped with the auxiliary data of a divisor $D$ and a choice of valuation $\nu$ on the space of rational functions $\mathbb{C}(X)$. The theory of Newton-Okounkov bodies is powerful for several reasons. Firstly, it applies to an arbitrary projective algebraic variety, and secondly, under a mild hypothesis on the auxiliary data, the construction guarantees that the associated convex body $\Delta$ is maximal-dimensional, as in the classical setting of toric varieties. Hence one interpretation of the results of Lazarsfeld-Mustata and Kaveh-Khovanskii is that there is a combinatorial object of ‘maximal’ dimension associated to $X$, even when $X$ is not a toric variety. It is an interesting problem to compute new concrete examples of these bodies, and one of our motivations for this paper was to compute Newton-Okounkov bodies of Hessenberg varieties.

In this section we use results of Section 6,7 and 8 to give a concrete computation of the Newton-Okounkov bodies $\Delta(\text{Pet}_3, R(W_\lambda), \nu)$ of the Peterson variety $\text{Pet}_3$, where here $W_\lambda$ is the image of $H^0(\text{Flags}(\mathbb{C}^3), L_\lambda)$ in $H^0(\text{Pet}_3, L_\lambda|_{\text{Pet}_3})$ and $L_\lambda$ is the Plücker line bundle over $\text{Flags}(\mathbb{C}^3)$ corresponding to $\lambda$ (see [19] § 9.3 or [6.5]). For precise definitions of Newton-Okounkov bodies we refer the reader to e.g. [30]. We should note that since $\text{Pet}_3$ is a surface, it is already known [33] [22] that the Newton-Okounkov body is a polygon. The question is to determine precisely this polygon; in the present section, we describe it explicitly as a convex hull of a finite number of points. For the purpose of our argument below it is also useful to recall that the
volume of $\Delta(\text{Pet}_3, R(W_\lambda), \nu)$ is equal to the degree of $\text{Pet}_3$ (in the appropriate embedding to be recalled below).

We need some notation. Let $\lambda = (\lambda_1 > \lambda_2 > \lambda_3) \in \mathbb{Z}^3$ be a dominant weight where we may assume without loss of generality that $\lambda_3 = 0$. In fact it will be convenient to set the notation $a_1 := \lambda_2$ and $a_2 := \lambda_1 - \lambda_2$ so that $\lambda = (a_1 + a_2, a_1, 0)$. Let $L_\lambda$ denote the Plücker line bundle obtained from the Plücker embedding $\varphi_\lambda : \text{Flags}(\mathbb{C}^3) \to \mathbb{P}(V_\lambda^*)$ where $V_\lambda$ denotes the irreducible $\text{GL}_3(\mathbb{C})$-representation associated with $\lambda$. Let $W_\lambda$ denote the image of $H^0(\text{Flags}(\mathbb{C}^3), L_\lambda)$ in $H^0(\text{Pet}_3, L_\lambda|_{\text{Pet}_3})$ and let $R(W_\lambda)$ denote the corresponding graded ring. We use a geometric valuation on $\text{Pet}_3$ coming from the flag of subvarieties constructed in Section 5. More specifically, on the affine open chart $\mathcal{N}_{w_0}$ near the longest permutation $w_0 = (321) \in S_3$, it follows from the analysis in Section 3 of the defining equations of regular nilpotent Hessenberg varieties that $\text{Pet}^\lambda_3 := \mathcal{N}_{w_0} \cap \text{Pet}_3$ can be identified with matrices of the form

$$
\begin{pmatrix}
y & x & 1 \\
x & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
$$

for arbitrary $x, y \in \mathbb{C}$, and applying Theorem 5.4 in this case, we obtain the flag (restricted to $\text{Pet}^\lambda_3$)

$$
\text{Pet}^\lambda_3 \supset \{x = 0\} \supset \{x = y = 0\} = \{\text{pt}\}.
$$

Letting $\nu$ denote the valuation corresponding to the above flag, Theorem 7.6 of this section computes $\Delta(\text{Pet}_3, R(W_{(a_1+a_2, a_1, 0)}), \nu)$ for all values of $a_1, a_2 \in \mathbb{Z}_{\geq 0}$ (we argue separately the cases $a_2 \geq a_1$ and $a_1 \geq a_2$). It is not hard to see that for the usual lexicographic order on $\mathbb{Z}^2$ with $x > y$, the valuation $\nu$ is the lowest-term valuation.

We briefly recall a well-known basis for $H^0(\text{Flags}(\mathbb{C}^3), L_\lambda)$ and compute its restriction to $\text{Pet}^\lambda_3$ in terms of the variables $x$ and $y$ above. The following discussion is valid for more general flags and partitions but we restrict to our case for simplicity; see [19] for details. Let $\lambda = (a_1 + a_2, a_1, 0)$ as above. For each semistandard Young tableau $T$ with shape $\lambda$ there is an associated section $\sigma_T$ of $H^0(\text{Flags}(\mathbb{C}^3), L_\lambda)$ obtained by taking the product of the Plücker coordinates corresponding to each column of $T$. We illustrate with an example.

**Example 7.1.** Let $A$ be a matrix of the form (7.1) representing a flag and suppose $T = 321$. Then the left column corresponds to the determinant

$$
\det \begin{pmatrix} y & x \\ x & 1 \end{pmatrix}
$$

of the first and second rows of the left $3 \times 2$ submatrix of $A$, while the second column corresponds to the determinant $\det(1) = 1$ of the third row of the left $3 \times 1$ submatrix. Thus $\sigma_T = y - x^2$.

The following is well-known.

**Theorem 7.2.** ([19] § 8 and 9) The set $\{\sigma_T\}$ of all sections corresponding to semistandard Young tableaux of shape $\lambda$, as described above, is a basis for $H^0(\text{Flags}(\mathbb{C}^3), L_\lambda)$.

Motivated by the above theorem, we now analyze the set $\mathcal{S}_\lambda$ of all semistandard Young tableau of shape $\lambda = (a_1 + a_2, a_1, 0)$ with entries in $\{1, 2, 3\}$. First observe that, from the definition of $\lambda$, our Young tableau contains columns of length at most 2. Moreover, since columns must be strictly increasing, the only possible length-2 columns which can appear in $T \in \mathcal{S}_\lambda$ are $\begin{array}{c} 2 \\ 1 \end{array}$ and $\begin{array}{c} 3 \\ 1 \end{array}$. The only possible length-1 columns are $\begin{array}{c} 1 \\ 2 \end{array}$ and $\begin{array}{c} 1 \\ 3 \end{array}$. Moreover, because rows must be weakly increasing (from left to right), a column $\begin{array}{c} 3 \\ 2 \end{array}$ must appear to the left of a $\begin{array}{c} 3 \\ 1 \end{array}$ or a $\begin{array}{c} 2 \\ 1 \end{array}$, and a $\begin{array}{c} 2 \\ 3 \end{array}$ can only appear to the left of a $\begin{array}{c} 3 \\ 1 \end{array}$ and so on. Thus it is not hard to see that we can uniquely represent a semistandard Young tableau of shape $\lambda = (a_1 + a_2, a_1, 0)$ by recording the number of times each type of column appears. More formally, let

$$
k_{12}(T) := \text{ the number of times the column } \begin{array}{c} 1 \\ 2 \end{array} \text{ appears in } T
$$

and

$$
k_1(T) := \text{ the number of times the column } \begin{array}{c} 1 \end{array} \text{ appears in } T
$$

and similarly for $k_{13}(T), k_{32}(T), k_2(T)$ and $k_3(T)$. The following lemma is straightforward.
Lemma 7.3. Let $T \in S_\lambda$. Then:

1. $T$ is completely determined by the 6 integers $k_{12}(T), k_{13}(T), k_{23}(T), k_1(T), k_2(T)$ and $k_3(T)$;
2. we must have $k_{12}(T) + k_{13}(T) + k_{23}(T) = a_1$, \quad $k_1(T) + k_2(T) + k_3(T) = a_2$, and if $k_{23}(T) \neq 0$ then $k_1(T) = 0$.

Thus the set $S_\lambda$ is in bijective correspondence with the set

$$\{ (k_{12}, k_{13}, k_{23}, k_1, k_2, k_3) \in \mathbb{Z}_{\geq 0}^6 \mid k_{12} + k_{13} + k_{23} = a_1, \quad k_1 + k_2 + k_3 = a_2, \quad k_{23} \neq 0 \Rightarrow k_1 = 0 \}. \tag{7.2}$$

Proof. By definition, a Young tableau of shape $\lambda = (a_1 + a_2, a_1, 0)$, reading left to right, has $a_1$ columns of size 2 and $a_2$ columns of size 1. A semistandard Young tableau $T \in S_\lambda$ must have weakly increasing rows. Hence the only possible arrangement of the length-2 columns is to place (starting from the left) all the 3’s, then the 2 3’s, and then the 1 3 2’s. Since the diagram has $a_1$ many columns of length 2, it is immediate that $k_{12}(T) + k_{13}(T) + k_{23}(T) = a_1$. It also follows that the left $a_1$ columns are determined by these 3 integers. Next consider the length-1 columns. Again, since rows must be weakly increasing, all 1 2’s must be placed first, followed by 2 3’s, followed by the 1 3’s. Finally, if $k_{23}(T) \neq 0$, this means that there is already a 2 in the top row before reaching the length-1 columns, so there cannot be any 1’s among the length-1 columns, i.e. $k_1(T) = 0$ as claimed. Again it follows that these are completely determined by $k_1(T), k_2(T)$ and $k_3(T)$ and that $k_1(T) + k_2(T) + k_3(T) = a_2$. Moreover, it is clear that any 6 positive integers satisfying the conditions of (7.2) correspond to some $T \in S_\lambda$. \hfill \square

Based on the above lemma, henceforth we specify a semistandard Young tableau $T$ by a tuple of integers $(k_{12}, k_{13}, k_{23}, k_1, k_2, k_3)$ satisfying the conditions in (7.2), and we also use the notation

$$\lambda := (12)^{k_{12}}(13)^{k_{13}}(23)^{k_{23}}(1)^{k_1}(2)^{k_2}(3)^{k_3}. \tag{7.3}$$

Example 7.4. Suppose $\lambda = (5, 2, 0)$ so that $a_2 = 3$ and $a_1 = 2$. The tableau 1 1 2 2 3 corresponds to $(0, 2, 0, 0, 2, 1)$ and we also write it as $\lambda = (13)^2(2)^2(3)$.

We need the following computation.

Lemma 7.5. Let $T$ be a semistandard Young tableau

$$T := (12)^{k_{12}}(13)^{k_{13}}(23)^{k_{23}}(1)^{k_1}(2)^{k_2}(3)^{k_3}$$

as above. Then the section $\sigma_T$, restricted to $\text{Pet}_3^0$ and expressed in terms of the variables $x$ and $y$ in (7.2), takes the form

$$(y - x^2)^{k_{12}}(-x)^{k_{13}}(-1)^{k_{23}}y^{k_1}x^{k_2}1^{k_3}. \tag{7.4}$$

Proof. Let $A$ denote a $3 \times 3$ matrix as in (7.1). By its construction, the section $\sigma_T$ evaluated at $A$ takes the form

$$(P_{12})^{k_{12}}(P_{13})^{k_{13}}(P_{23})^{k_{23}}(P_1)^{k_1}(P_2)^{k_2}(P_3)^{k_3}$$

where

$$P_{12} = \begin{vmatrix} y & x \\ x & 1 \end{vmatrix} = y - x^2, \quad P_{13} = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix} = -x, \quad P_{23} = \begin{vmatrix} x & 1 \\ 1 & 0 \end{vmatrix} = -1,$$

$$P_1 = y, \quad P_2 = x, \quad P_3 = 1.$$

The result follows. \hfill \square

We now compute the Newton-Okounkov bodies $\Delta(\text{Pet}_3, R(W_{(a_1+a_2,a_1,0)}, \nu))$. Recall from the above discussion (cf. also [20] Corollary 3.2)) that if we can find vertices contained in $\Delta(\text{Pet}_3, R(W_{(a_1+a_2,a_1,0)}, \nu))$ whose convex hull $\Delta$ has volume equal to the degree of $\text{Pet}_3 \hookrightarrow \mathbb{P}(V_\lambda^*)$, then $\Delta = \Delta(\text{Pet}_3, R(W_{(a_1+a_2,a_1,0)}, \nu))$. Since we know the degree of $\text{Pet}_3 \hookrightarrow \mathbb{P}(V_\lambda^*)$ from Example 6.4, we take this approach in our arguments below.
Theorem 7.6. Let $\lambda = (a_1 + a_2, a_1, 0)$. If $a_2 \geq a_1$, then the corresponding Newton-Okounkov body $\Delta(\text{Pet}_3, R(W_{(a_1+a_2,a_1,0)}), \nu)$ is the convex hull of the vertices

$$\{(0,0), (2a_1 + a_2, 0), (0, a_1 + a_2), (3a_1, a_2 - a_1)\}.$$ 

If $a_1 \geq a_2$, then the corresponding Newton-Okounkov body $\Delta(\text{Pet}_3, R(W_{(a_1+a_2,a_1,0)}), \nu)$ is the convex hull of the vertices

$$(0,0), (0, a_1 + a_2), (2a_2 + a_1, 0), (3a_2, a_1 - a_2).$$

**Proof.** We begin with the case $a_2 \geq a_1$. First, notice that the area of the convex hull described in the statement of the theorem is

$$3a_1(a_2 - a_1) + \frac{1}{2}(3a_1)(2a_1) + \frac{1}{2}(a_2 - a_1)^2 = \frac{1}{2}a_1^2 + 2a_1a_2 + \frac{1}{2}a_2^2.$$ 

Therefore, as observed above, it suffices to show that the four stated vertices all lie in $\nu(W_{(a_1+a_2,a_1,0)})$. We deal with the four cases separately.

We begin with $(0,0)$. The semistandard Young tableau $(23)(3)^{a_2}$ corresponds to the polynomial 1 (by Lemma 7.5), and $\nu(1) = (0,0)$. Hence, $(0,0)$ is in the image $\nu(W_{(a_1+a_2,a_1,0)})$.

Next we consider $(0,a_1+a_2)$. The semistandard Young tableau $(12)(1)^{a_2}$ corresponds to the polynomial $(y-x^2)^{a_1}y^{a_2}$, and $\nu((y-x^2)^{a_1}y^{a_2}) = (0, a_1 + a_2)$.

Now we consider $(2a_1 + a_2,0)$, for which we look at the set of tableaux $(12)^k(13)^{a_1-k}(1)a_1-1-k(2)a_2-a_1+k$ for $0 \leq k \leq a_1$. Notice that these are valid tableaux because $a_2 \geq a_1$. By Lemma 7.5 these have corresponding polynomials (up to sign)

$$g_k := (y-x^2)^k x^{a_2} y^{a_1-k} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} y^{k-j} x^{2j} x^{a_2} y^{a_1-k} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} x^{a_2+2j} y^{a_1-j}.$$

Note that the set of $a_1 + 1$ monomials $x^ay^b$ that appear in the $a_1 + 1$ polynomials $\{g_0, \ldots, g_{a_1}\}$ is precisely:

$$\{x^{a_2} y^{a_1}, x^{a_2+2}, y^{a_1-1}, x^{a_2+4}, y^{a_1-2}, \ldots, x^{a_2+2a_1}\}$$

and also that, with respect to this ordered basis, the $(a_1 + 1) \times (a_1 + 1)$ matrix of coefficients of the $g_k$ is triangular and invertible. Thus $x^{a_2+2a_1}$ is equal to an appropriate linear combination of the $g_k$’s and in particular is in $W_{(a_1+a_2,a_1,0)}$. Since $\nu(x^{a_2+2a_1}) = (a_2 + 2a_1, 0)$ we see that this vertex lies in the image of $\nu$. 

**Figure 7.1.** $\Delta(\text{Pet}_3, R(W_{(a_1+a_2,a_1,0)}), \nu)$ for $a_2 \geq a_1$. 

22
Finally, for the case of the vertex $(3a_1, a_2 - a_1)$ we consider the tableaux $(12)^k(13)^{a_1-k}(1)^{a_2-k}(2)^k$ for $0 \leq k \leq a_1$. Notice that these are valid tableaux because $a_2 \geq a_1$. Again by Lemma 7.3 we can compute the corresponding polynomials $h_k$ to be

$$h_k := (y - x)^k x^{a_1} y^{a_2-k} = \left[ \sum_{j=0}^{k} (-1)^j \binom{k}{j} y^{k-j} x^{2j} \right] x^{a_1} y^{a_2-k} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} x^{a_1+2j} y^{a_2-j}.$$

By an argument similar to that above, we can see that there is an appropriate linear combination of the $h_k$ which equals $x^{3a_1} y^{a_2-a_1}$, and since $\nu(x^{3a_1} y^{a_2-a_1}) = (3a_1, a_2 - a_1)$ we conclude that it is in the image, as desired. This concludes the proof for $a_2 \geq a_1$.

Now suppose $a_1 \geq a_2$. We follow the same strategy so we will be brief. For $(0, 0)$ and $(0, a_1 + a_2)$ it suffices to consider the tableaux $(23)^a(3)^2$ and $(12)^a(1)^2$ respectively. For $(2a_2 + a_1, 0)$, the collection of tableaux of the form $(12)^k(13)^{a_1-k}(1)^{a_2-k}(2)^k$ for varying $k$ as in the proof of Theorem 7.6 does the job.

For the last case of $(3a_2, a_1 - a_2)$ we need the so-called truncated Pascal matrices. Recall that an upper-triangular Pascal matrix $T$ is an infinite matrix with $(i, j)$-th entry for $i, j \in \mathbb{Z}_{\geq 0}$ equal to the binomial coefficient $\binom{i}{j}$, where we take the convention that $\binom{i}{j} := 0$ if $i - j < -1$. A truncated Pascal matrix is a matrix obtained from an upper-triangular Pascal matrix $T$ by selecting some arbitrary finite subsets of the rows and columns of $T$ of equal size, i.e.

$$T(r, s) := \begin{pmatrix}
\begin{array}{ccc}
\{r_0, r_1, \ldots, r_i\} & \{s_0, s_1, \ldots, s_i\} \\
\vdots & \ddots & \ddots \\
\{r_0, r_1, \ldots, r_i\} & \{s_0, s_1, \ldots, s_i\}
\end{array}
\end{pmatrix},$$

for some sets $r = \{r_0 < r_1 < \cdots < r_d\}$ and $s = \{s_0 < s_1 < \cdots < s_d\}$, for $s_i, r_i \in \mathbb{N}$.

Now consider the tableaux $(12)^{a_1-a_2+k}(13)^{a_2-k}(1)^{a_2-k}(2)^k$ where $0 \leq k \leq a_2$. As before we can compute the corresponding polynomials $h_k$ to be

$$h_k = \sum_{j=0}^{a_1-a_2+k} (-1)^j \binom{a_1-a_2+k}{j} x^{a_2+2j} y^{a_1-j},$$

for $0 \leq k \leq a_2$.

There are $a_1 + 1$ many monomials $x^a y^b$ appearing in these $a_2 + 1$ polynomials; listed in increasing lex order, they are

$$(7.5) \quad \{x^{a_2} y^{a_1}, x^{a_2+2} y^{a_1-1}, x^{a_2+4} y^{a_1-2}, \ldots, x^{a_2+2a_1-2} y, x^{a_2+2a_1}\}.$$

The $(a_1 + 1) \times (a_2 + 1)$ matrix of coefficients of the $h_k$ with respect to the ordered basis (7.5) has $(j, k)$-th entry equal to $(-1)^j \binom{a_1-a_2+k}{j}$.

We wish to find a suitable linear combination of the $h_k$ so that its lowest term is a multiple of $x^{3a_2} y^{a_1-a_2}$. Some elementary linear algebra shows that it suffices to prove that the upper-left $(a_2+1) \times (a_2+1)$ submatrix $A$ of the matrix of coefficients above, with entries equal to $(-1)^j \binom{a_1-a_2+k}{j}$ for $0 \leq j, k \leq a_2$, is invertible. For this it suffices in turn to show that det $A \neq 0$. Let $A'$ denote the matrix obtained from $A$ by multiplying every other row by $(-1)$; then det $A' = \pm$ det $A$ so it suffices to show det $A' \neq 0$. Finally observe that $A'$ is (up to sign) a truncated Pascal matrix $T(r, s)$ for $r = \{0 < 1 < 2 < \cdots < a_2\}$ and $s = \{a_1 - a_2 < a_1 - a_2 + 1 < \cdots < a_1\}$. By our assumption that $a_1 \geq a_2$ we have that $r_i \leq s_i$ for all $i$. It is known 31 that a truncated Pascal matrix is invertible if and only if $r_i \leq s_i$ for all $i$, so we conclude that det $A' \neq 0$ as desired. This completes the proof. 

\section*{References}

[1] H. Abe and S. Billey, \textit{Consequences of the Lakshmibai-Sandhya Theorem: the ubiquity of permutation patterns in Schubert calculus and related geometry}, Adv. Stud. Pure Math., \textbf{71} (2016) 1-52.

[2] H. Abe and P. Crooks, \textit{Hessenberg varieties for the minimal nilpotent orbit}, arXiv:1510.02436. To appear in Pure and Applied Mathematics Quarterly.
[38] A. Okounkov, Why would multiplicities be log-concave? The orbit method in geometry and physics (Marseille, 2000), 329-347, Progr. Math., 213, Birkhäuser Boston, Boston, MA, 2003.

[39] A. Postnikov, Permutohedra, associahedra, and beyond. Int. Math. Res. Not. IMRN 2009, no. 6, 1026-1106.

[40] M. Precup, Affine pavings of Hessenberg varieties for semisimple groups. Sel. Math. New Series 19 (2013), 903-922.

[41] M. Precup, The connectedness of Hessenberg varieties. J. Algebra 437 (2015), 34-43.

[42] K. Rietsch, Totally positive Toeplitz matrices and quantum cohomology of partial flag varieties. J. Amer. Math. Soc. 16 (2003), no. 2, 363-392 (electronic).

[43] I. R. Shafarevich, Basic algebraic geometry. 1. Varieties in projective space. Second edition. Translated from the 1988 Russian edition and with notes by Miles Reid. Springer-Verlag, Berlin, 1994.

[44] J. Shareshian and M. L. Wachs, Chromatic quasisymmetric functions and Hessenberg varieties. Configuration spaces, 433-460, CRM Series, 14 Ed. Norm., Pisa, 2012.

[45] T. A. Springer, Trigonometric sums, Green functions of finite groups and representations of Weyl groups. Invent. Math. 36 (1976), 173-207.

[46] J. S. Tymoczko, Linear conditions imposed on flag varieties. Amer. J. Math. 128 (2006) no. 6, 1587-1604.

Department of Mathematics and Statistics, McMaster University, 1280 Main Street West, Hamilton, Ontario L8S4K1, Canada / Osaka City University Advanced Mathematical Institute, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558-8585, JAPAN
E-mail address: hirakuabe@globe.ocn.ne.jp

Department of Mathematics, Vincent Hall 9, University of Minnesota, 206 Church St. SE, Minneapolis, Minnesota 55455, U.S.A.
E-mail address: dedieu@umn.edu
URL: http://www.math.umn.edu/~dedieu/

Department of Mathematics and Statistics, McMaster University, 1280 Main Street West, Hamilton, Ontario L8S4K1, Canada
E-mail address: galettof@math.mcmaster.ca
URL: http://math.galetto.org

Department of Mathematics and Statistics, McMaster University, 1280 Main Street West, Hamilton, Ontario L8S4K1, Canada
E-mail address: Megumi.Harada@math.mcmaster.ca
URL: http://www.math.mcmaster.ca/Megumi.Harada/