Input-output Analysis of Quantum Finite-level Systems in Response to Single Photon States

Yu Pan\textsuperscript{a}, Guofeng Zhang\textsuperscript{b}, Matthew R. James\textsuperscript{c}

\textsuperscript{a}Research School of Engineering, Australian National University, Canberra, ACT 0200, Australia
\textsuperscript{b}Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong
\textsuperscript{c}ARC Centre for Quantum Computation and Communication Technology, Research School of Engineering, Australian National University, Canberra, ACT 0200, Australia

Abstract

Single photon states, which carry quantum information and coherently interact with quantum systems, are vital to the realization of all-optical quantum networks and quantum memory. In this paper we derive the conditions that enable an exact analysis of the response of passive quantum finite-level systems under the weak driving of single photon input. We show that when a class of finite level systems is driven by single photon inputs, expressions for the output states may be derived exactly using linear systems transfer functions. This removes the need for physical approximations such as weak excitation limit in the analysis of quantum nonlinear systems under single photon driving. We apply this theory to the analysis of a single photon switch. The input-output relations are consistent with the existing results in the study of few photon transport through finite-level systems.

Key words: Quantum finite-level systems; System response; Single photon state

\textsuperscript{*} This paper was not presented at any IFAC meeting. Corresponding author Y. Pan. Tel. +02-612-58601.

Email addresses: yu.pan.83.yp@gmail.com (Yu Pan), Guofeng.Zhang@polyu.edu.hk (Guofeng Zhang), matthew.james@anu.edu.au (Matthew R. James).
1 Introduction

The coherent interaction between single photons and quantum systems as the photons move within a coherent network, is of fundamental importance to quantum communication and on-chip quantum electronics. Critical photonic components and circuits can be invented by exploiting the behaviour of single photons in interaction with finite-level control systems [6,22,23]. For example, it is possible to build a quantum switch [27,4,24,21] which either controls the transmission of photons, or uses the single photon itself as the control to switch on and off a physical process. In particular, the ability to control the flow of single photons using on-chip finite-level systems could give birth to a new generation of light transistors [3,16]. Additionally, the light-matter interaction is also widely studied for its use in the storage of quantum information [14,29,17], where the media used to implement the memory are often two-level atoms. For these reasons, an exact analysis of the interaction between single photon and finite-level systems is necessary. A typical finite-level system is a quantum two-level system (qubit), whose dynamics would be governed by a complicated and nonlinear stochastic differential equation if the input to the system is an arbitrary state.

Nonlinear dynamics is difficult to analyze and normally there exists no analytical solution. As a result, linearization procedure is often needed in simplifying the model. Taking the simplest atom-photon interaction for example, a lot of efforts have been devoted to simplifying the model into a solvable one. The linearization could be done by assuming the atom remains at the ground state at all times during the interaction, which is called weak excitation limit. This assumption fixes the mean value of the system operators and thus reduces the bilinear terms in the dynamical equations into linear ones [26,8]. However in some cases, the excitation of the atom can be significant due to the single photon input, which makes the weak excitation assumption inappropriate. In recent years, several approaches were proposed to derive analytical solutions to single photon response [23,8] beyond the weak excitation limit. These approaches were developed for specific physical models, and their derivations rely on the existence of scattering matrix or scattering modes.

In this paper we are concerned with the exact solutions to the time-domain and frequency-domain system response of general finite-level systems. We are motivated by the observation that the expressions for the output states of a two-level system can be derived exactly using linear systems transfer functions under single photon driving. This linear systems transfer function approach is very favorable, as we can then exploit the abundant tools from quantum linear system theory for solving the response. For example, we could make use of the transfer function approach which was proposed to analyze linear quantum networks [31]. Moreover, the response of linear system to single photon
input has been solved in [35,32], where the solutions are obtained using a sta-
ble inversion technique [35,25]. Our calculations of the response of finite-level
systems to single photon input are thus developed upon [35]. The purpose of
this paper is to simplify the modelling and controlling of linear and nonlinear
quantum systems [9,7,19] using a unified linear approach in certain circum-
stances. This simplification could be especially useful for the engineering of
quantum coherent networks that involve quantum systems interconnected via
single photon signals. Therefore, this work is not an analysis of nonlinear
input-output relation, as did in [37,36]. Also, our work differs from [30,29,17].
In those works, the systems of concern are just assumed to be linear.

In Section 3 we find the exact solution of a two-level system in response to
a single photon input using linear systems transfer functions. That is, we are
able to transform the bilinear stochastic differential equation to a linear one
under single photon driving. The obtained input-output relation in frequency
domain is justified to be equivalent to the input-output relation for linear
systems. In particular, the transfer function as we calculated is consistent with
the existing physical findings [22,38,8]. The single photon inverting pulse [6,21]
which was invented to excite an atom is also rediscovered from our analysis,
using the zero-frequency principle [30]. Next, we consider an important class
of on-chip interaction which is between an artificial two-level atom and two
travelling modes within a waveguide. We can obtain the analytic form of the
steady-state output for these systems. Details are given in Section 4. There is
also a discussion on the implications of our results on a single photon switch.
In the last, we generalize the results to finite-level systems in Section 6 so that
the exact input-output analysis could apply to a coherent network containing
finite-level components.

2 Notations and preliminaries

We use $X^\dagger$ to denote the adjoint of an operator $X$ which is defined on a
Hilbert space $\mathcal{H}$. The notation $*$ is used for indicating the complex conjugate
of a complex number. $X^\dagger = X^*$ if $X$ is a one-dimensional scalar. We also define
the doubled-up column vector of operators as $\vec{X} = [X, X^\dagger]^T$. The doubled-up
matrix $\Delta(U, V)$ is given by

$$\Delta(U, V) = \begin{bmatrix} U & V \\ V^\dagger & U^\dagger \end{bmatrix},$$

where $U$ and $V$ are operators on $\mathcal{H}$. The commutator of two operators is given
by $[A, B] = AB - BA$. 

3
An open quantum system often involves a plant interacting with external fields, which can be properly modelled using a triplet \((S, L, H_0)\). \(S\) is a scattering matrix. The plant is coupled to the fields through an operator \(L\). In general \(L\) could be a column vector of operators when the system has \(K\) multiple inputs, e.g., \(L = [L_1 \ L_2 \ \cdots \ L_k \ \cdots \ L_K]^T\). \(H_0\) is the Hamiltonian of the plant. The total system is governed by the overall Hamiltonian \(H\), which includes \(H_0\), the internal Hamiltonian of the fields, and the interaction Hamiltonian between the plant and fields. The states of the total system undergo a unitary evolution generated by a unitary operator \(U(t, t_0) = \exp(-iH(t-t_0))\). \(t_0\) is the initial time of the evolution. In Heisenberg picture, the evolution of an operator \(X\) is defined as \(X(t) = U^\dagger(t, t_0)XU(t, t_0)\). Driven by bosonic fields, the dynamics of \(X(t)\) is described by quantum stochastic differential equations of the form

\[
\dot{X}(t) = \mathcal{G}_t(X) + b(t)^\dagger S(t)[X(t), L(t)] \\
+ [L^\dagger(t), X(t)]S(t)b(t), \\
b_{\text{out}}(t) = L(t) + S(t)b(t),
\]

where the generator \(\mathcal{G}_t(X)\) is defined as

\[
\mathcal{G}_t(X) = -i[X, H_0] \\
+ \sum_k (L_k^\dagger X L_k - \frac{1}{2} L_k^\dagger L_k X - \frac{1}{2} X L_k^\dagger L_k).
\]

Here \(b_{\text{out}}(t) = U^\dagger(t, t_0)b(t)U(t, t_0)\) defines the annihilation operators of the output fields. \(b(t)\) could be a single annihilation operator for a single channel input, or a column vector of annihilation operators for multiple input fields. For multiple input fields, we can write \(b(t)\) as \(b(t) = [b_1(t) \ b_2(t) \ \cdots \ b_K(t)]^T\). The noise term \(b(t)^\dagger S(t)[X(t), L(t)] + [L^\dagger(t), X(t)]S(t)b(t)\) in (2) is generally nonlinear in \(b(t)\), which would make the integration of (2) nontrivial.

A single photon input is defined by

\[
|\xi\rangle = B^\dagger(\xi)|0\rangle = \int_{-\infty}^{\infty} \xi(r)b^\dagger(r)|0\rangle dr,
\]

where \(\xi(r)\) represents the pulse shape of the single photon in the time domain. \(|\xi(r)|^2 dr\) is the probability of finding the photon during \(t = [r, r + dr]\), and we have a normalization condition \(\int_{-\infty}^{\infty} |\xi(r)|^2 dr = 1\).

The Fourier transform is defined by \(\mathcal{F}(f(t)) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt\), and the Laplace transform is defined by \(\mathcal{L}(f(t)) = \int_{-\infty}^{\infty} e^{-\omega t} f(t) dt\). We use \(\xi'(t)\) to denote the pulse shape of the output state. The transfer function \(G(s)\) of the system is
defined as the input-output transfer of the pulse shapes:

\[ \xi'(s) = G(s)\xi(s). \]  

\(|0_s\rangle\) is the ground state of an \(N\)-level system, and \(|j_s\rangle, j = 1, 2, ..., N\} are the excited states. \(|0_s\rangle\) contains no photon while \(|j_s\rangle, j = 1, 2, ..., N\} are single photon and multi-photon states. If there are more than one qubits, we use \(|j_s\rangle_i\) to denote the state of the \(i\)th qubit. Similarly, we use \(|0\rangle\) to denote a single-channel vacuum input. For multiple channel input, we use \(|0\rangle_i\) to denote a vacuum input from the \(i\)th channel. The state of the field is obtained by taking the partial trace over the finite-level system, that is, \(\rho_{field} = \text{Tr}_s(\rho) = \sum_{j=0}^{N} \langle j_s | \rho | j_s \rangle\). Here \(\rho\) is the state of plant-field combined system and \(\rho_{field}\) is called the reduced state.

The Pauli operators for a qubit are defined as \(\sigma_z = |1_s\rangle\langle 1_s| - |0_s\rangle\langle 0_s|, \sigma_x = |0_s\rangle\langle 1_s| + |1_s\rangle\langle 0_s|, \sigma_y = -i|1_s\rangle\langle 0_s| + i|0_s\rangle\langle 1_s|\). The raising and lowering operators for the qubit are given by \(\sigma_+ = |1_s\rangle\langle 0_s|, \sigma_- = |0_s\rangle\langle 1_s|\) respectively.

3 The response of quantum two-level system to single photon input: one-channel

3.1 Time-domain Analysis

We consider a qubit interacting with a single input field. Here by one-channel we mean the incoming photon can only approach the two-level system from one direction, and the output photon travels in one direction only. The internal Hamiltonian of the two-level system is defined as \(H_0 = \omega_c^2 \sigma_z\). \(\omega_c\) is the transition frequency between the ground and excited states. The qubit is interacting with the photon via a coupling operator \(L = \sqrt{\kappa} \sigma_-, \kappa \geq 0\). \(\kappa\) is a parameter defined by \(\kappa = 2\pi g^2\), where \(g\) is the coupling strength between the system and field (See Appendix [A]). The total system is described by the triplet

\[ (S, L, H_0) = \left( I, \sqrt{\kappa} \sigma_-, \frac{\omega_c}{2} \sigma_z \right), \]  

and hence the combined system is governed by the quantum stochastic differential equations \[28, 34\]

\[ \dot{\sigma}_-(t) = -\left(\frac{\kappa}{2} + i\omega_c\right)\sigma_-(t) + \sqrt{\kappa} \sigma_z(t)b(t), \]  

\[ b_{out}(t) = \sqrt{\kappa} \sigma_-(t) + b(t), \quad t \geq t_0. \]
Fig. 1. A schematic representation of the one-channel interaction. The input and output fields travel in the same direction.

\( b(t) \) is the annihilation operator defined on the input channel satisfying \([b(t), b(t')] = \delta(t - t')\), and (8) gives the input-output relation between the incoming and outgoing fields.

The directions of the input and output fields for the stochastic equations (7)-(8) are different from the directions commonly seen in a quantum optics setting. In quantum optics, a single-channel dissipation is often modelled by assuming one mirror of a cavity is leaking while the other mirror is perfectly reflecting. This is referred to as one-sided cavity [28]. A two-sided cavity often have both mirrors leaking and so there are more than one dissipation channels. Therefore, one-channel input usually corresponds to one-sided cavity in quantum optics. For one-sided cavity, the input signal is reflected and so the output field travels in an opposite direction compared to the input. However, as shown in Figure 1, the input and output fields of (8) propagate in the same direction, which models a transmission process. More details about the directions can be found in Appendix A. As we will show in this paper, this difference in direction only brings an extra negative sign to the output states.

If the initial state is \( \rho_0 = \vert 0 \rangle \langle 0 \rangle \otimes \langle 0_s \vert \langle 0 \vert \), then we have

\[
\rho_{\infty} = \lim_{t \to \infty, t_0 \to -\infty} U(t, t_0) \rho_0 U(t, t_0)^\dagger = \vert 0 \rangle \langle 0 \rangle \otimes \langle 0_s \vert \langle 0 \vert. \tag{9}
\]

This is easy to see by noting that the two-level system is passive. That is, there is no active element within the combined system and so the total energy is preserved during the evolution. \( U(t, t_0) \) is the unitary evolution operator acting on the combined system. Since the input is a vacuum state and the qubit is in its ground state, there will be no photon in the qubit and field at all times. For this reason, not only the steady state \( \rho_{\infty} \) is exactly the initial state when letting \( t \to \infty \), but also the combined system will remain at the state \( \vert 0 \rangle \langle 0 \rangle \otimes \langle 0_s \vert \langle 0 \vert \) for any \( t \). As a result, we can further write this evolution as

\[
\rho_t = U(t, t_0) \rho_0 U(t, t_0)^\dagger = \vert 0 \rangle \langle 0 \rangle \otimes \langle 0_s \vert \langle 0 \vert , \quad t \geq t_0. \tag{10}
\]

This property will be used later to convert (7) to a linear dynamical equation.
following lemma:

**Lemma 1** The steady-state output state of a two-level system in response to a single photon input $|\xi\rangle$ in (4) can be expressed as

$$\rho_{\text{out}} := \sum_{j=0,1} \langle j_s | \int_{-\infty}^{\infty} dr \xi(r) f^\dagger(r, -\infty) | 0_s \rangle | j_s \rangle$$

$$= \int_{-\infty}^{\infty} dr \xi(r) f^\dagger(r, -\infty) | 0_s \rangle | j_s \rangle,$$

where we have used the notation

$$f(t, t_0) = U(t, t_0) b(t) U^\dagger(t, t_0).$$

**Proof.** Replacing the vacuum input with the single photon input defined in (4), the final state of the combined system is calculated by

$$\rho_{\infty} = \lim_{t \to \infty, t_0 \to -\infty} U(t, t_0) B^\dagger(\xi) \rho_0 B(\xi) U^\dagger(t, t_0)$$

$$= \lim_{t \to \infty, t_0 \to -\infty} U(t, t_0) B^\dagger(\xi) U^\dagger(t, t_0) \rho_{\infty g}$$

$$= \lim_{t \to \infty, t_0 \to -\infty} U(t, t_0) B(\xi) U^\dagger(t, t_0).$$

The operator in (12) can be simplified as

$$\lim_{t \to \infty, t_0 \to -\infty} U(t, t_0) B^\dagger(\xi) U^\dagger(t, t_0)$$

$$= \lim_{t \to \infty, t_0 \to -\infty} U(t, t_0) \int_{-\infty}^{\infty} dr \xi(r) b^\dagger(r) U^\dagger(t, t_0)$$

$$= \lim_{t \to \infty, t_0 \to -\infty} U(t, t_0) \int_{t_0}^{t} dr \xi(r) b^\dagger(r) U^\dagger(t, t_0)$$

$$= \lim_{t \to \infty, t_0 \to -\infty} \int_{t_0}^{t} dr \xi(r) \int_{-\infty}^{\infty} dr \xi(r) U(r, t_0) b^\dagger(r) U^\dagger(r, t_0)$$

$$= \int_{-\infty}^{\infty} dr \xi(r) U(r, -\infty) b^\dagger(r) U^\dagger(r, -\infty)$$

$$= \int_{-\infty}^{\infty} dr \xi(r) f^\dagger(r, -\infty).$$

The output state is obtained by taking the partial trace in the qubit subspace...
\[ \rho_{\text{out}} = \text{Tr}_s(\rho_\infty) = \sum_{j=0,1} \langle j_s | \int_{-\infty}^{\infty} dr \xi(r) f^\dagger(r, -\infty) |0\rangle_0 \langle 0_s | \int_{-\infty}^{\infty} dr \xi^*(r) f(r, -\infty) |j_s\rangle. \] (14)

The next thing to do is to solve for \( \langle 0_s | 0 f(t, -\infty) |0_s \rangle \) and \( \langle 0_s | 0 f(t, -\infty) |1_s \rangle \). The key is to solve for the \( \langle 0_s | 0 f(t, -\infty) \) which occurs in both terms. We need to refer to the following lemma, which is a generalization of (10):

**Lemma 2** The unitary operator \( U(t, t_0) \) which generates (7)-(8) satisfies

\[ U(t, t_0) |0\rangle_0 |0_s\rangle = \exp(i \alpha_t) |0\rangle_0 |0_s\rangle, \] (15)

and

\[ U^\dagger(t, t_0) |0\rangle_0 |0_s\rangle = \exp(-i \alpha_t) |0\rangle_0 |0_s\rangle, \] (16)

where \( \exp(i \alpha_t) \) denotes a phase shift.

**Proof.** The system-field couplings are given by \( L^\dagger(t) b(t) = [\sigma^+ b](t) \) and \( L(t) b^\dagger(t) = [\sigma^- b^\dagger](t) \). Since \( \sigma^+ b |0\rangle_0 |0_s\rangle = \sigma^- b^\dagger |0\rangle_0 |0_s\rangle = 0 \), there is no interaction between the field and the system at all times, which proves the Lemma.

To be more precise, we can take a closer look at the overall Hamiltonian generating (7)-(8) (See more details in Appendix A)

\[ H = \frac{1}{2} \omega_c \sigma_z + \int_{-\infty}^{\infty} d\omega \omega b^\dagger(\omega) b(\omega) + i g \int_{-\infty}^{\infty} d\omega (\sigma_+ b(\omega) - \sigma_- b^\dagger(\omega)). \] (17)

The field is composed of oscillators defined on continuous frequency domain, where \( b(\omega) \) is the annihilation operator for mode \( \omega \). \( b(\omega) \) can be regarded as the Fourier transformation of \( b(t) \). \( g \) is the coupling strength. Recall that \( U(t, t_0) = \exp(-iH(t - t_0)) \). By \( \sigma_+ b(\omega) |0\rangle_0 |0_s\rangle = 0 \) and \( \sigma_- b^\dagger(\omega) |0\rangle_0 |0_s\rangle = 0 \), there will be no system-field interaction and hence no photon in the combined system at all times under the action of \( U(t, t_0) \). In other words, the two-level system remains in its ground state \( |0\rangle_0 |0_s\rangle \) always, irrespective of a global phase shift generated by the non-interacting Hamiltonian \( \frac{1}{2} \omega_c \sigma_z + \int_{-\infty}^{\infty} d\omega \omega b^\dagger(\omega) b(\omega) \).

We can easily obtain (16) using (15):

\[ U^\dagger(t, t_0) U(t, t_0) |0\rangle_0 |0_s\rangle = U^\dagger(t, t_0) \exp(i \alpha_t) |0\rangle_0 |0_s\rangle \Rightarrow U^\dagger(t, t_0) |0\rangle_0 |0_s\rangle = \exp(-i \alpha_t) |0\rangle_0 |0_s\rangle. \] (18)
Based on Lemma 2, if $\langle 0_s \rvert 0 \rangle A = 0$ holds for an operator $A$, then this equality would not be violated under a rotation which is defined as

$$\langle 0_s \rvert 0 \rangle U(t, t_0)A U^\dagger(t, t_0) = 0.$$  \hfill (19)

Now we are in position to solve for $\langle 0_s \rvert 0 \rangle f(t, -\infty) \rvert 0_s \rangle$ and $\langle 0_s \rvert 0 \rangle f(t, -\infty) \rvert 1_s \rangle$.

Lemma 3 Define

$$g_{G^-}(t) = \begin{cases} -\kappa e^{-\left(\frac{\tau}{2} + i \omega_c\right)t} + \delta(t), & t \geq 0, \\ 0, & t < 0 \end{cases}$$  \hfill (20)

as an impulse response function, we have

$$\langle 0 \rvert \langle 0_s \rvert f(r, -\infty) \rvert j_s \rangle = \int_{-\infty}^{\infty} g_{G^-}(t - r) \langle 0_s \rvert j_s \rangle \langle 0 \rvert b(t) \rangle dt, \quad j = 0, 1.$$  \hfill (21)

Proof. First we show that the integration of (7) transforms to the integration of a linear equation under single-photon driving. Applying $\langle 0_s \rvert 0 \rangle$ to the left of (7) yields

$$\langle 0 \rvert \langle 0_s \rvert \sigma_-(t) \rangle = \langle 0 \rvert \langle 0_s \rvert - (\frac{\kappa}{2} + i \omega_c) \sigma_-(t) + \sqrt{\kappa} \sigma_z(t) b(t) \rangle + \langle 0 \rvert \langle 0_s \rvert \sqrt{\kappa} \sigma_z(t) b(t) \rangle = \langle 0 \rvert \langle 0_s \rvert - (\frac{\kappa}{2} + i \omega_c) \sigma_-(t) \rangle + \langle 0 \rvert \langle 0_s \rvert \sqrt{\kappa} \sigma_z(t) b(t) \rangle = \langle 0 \rvert \langle 0_s \rvert - (\frac{\kappa}{2} + i \omega_c) \sigma_-(t) \rangle - \langle 0 \rvert \langle 0_s \rvert \sqrt{\kappa} \sigma_z(t) b(t) \rangle.$$  \hfill (22)

The last line of (22) is obtained due to the relation

$$\sigma_z U(t, t_0) \rvert 0_s \rangle \rvert 0 \rangle = -U(t, t_0) \rvert 0_s \rangle \rvert 0 \rangle,$$  \hfill (23)

which is implied by Lemma 2.

Based on (22), we can take (7) as a linear stochastic differential equation under the action of $\langle 0 \rvert \langle 0_s \rvert$, and integrate it to be

$$\langle 0 \rvert \langle 0_s \rvert \sigma_-(t) \rangle = \langle 0 \rvert \langle 0_s \rvert [e^{-\left(\frac{\tau}{2} + i \omega_c\right)(t-t_0)} \sigma_-(t_0) - \sqrt{\kappa} \int_{t_0}^{t} e^{-\left(\frac{\tau}{2} + i \omega_c\right)(t-r)} b(r) dr]. \hfill (24)$$
Due to the relation $b_{out}(t) = U^\dagger(t, t_0) b(t) U(t, t_0)$ and (8), we have

$$
\langle 0 | \langle 0_s | U^\dagger(t, t_0) b(t) U(t, t_0)
= \langle 0 | \langle 0_s | \sqrt{\kappa} e^{-\frac{\kappa}{2} + i \omega_c (t-t_0)} \sigma_-(t_0)
- \kappa \int_{t_0}^t e^{-\left(\frac{\kappa}{2} + i \omega_c\right)(t-r)} b(r) dr + b(t).
$$

(25)

Using (19) we have

$$
\langle 0 | \langle 0_s | b(t)
= \langle 0 | \langle 0_s | \sqrt{\kappa} e^{-\frac{\kappa}{2} + i \omega_c (t-t_0)} U(t, t_0) \sigma_-(t_0) U^\dagger(t, t_0)
- \langle 0 | \langle 0_s | \kappa \int_{t_0}^t e^{-\left(\frac{\kappa}{2} + i \omega_c\right)(t-r)} f(r, t_0) dr + \langle 0 | \langle 0_s | f(t, t_0).
$$

Letting $t_0 \to -\infty$, we arrive at

$$
\langle 0 | \langle 0_s | b(t) = \int_{-\infty}^t -\kappa e^{-\left(\frac{\kappa}{2} + i \omega_c\right)(t-r)} \langle 0 | \langle 0_s | f(r, -\infty) dr + \langle 0 | \langle 0_s | f(t, -\infty).
$$

(26)

By (26), we have

$$
\langle 0 | \langle 0_s | b(t) | j_s \rangle = \int_{-\infty}^\infty g_G^{-1}(t-r) \langle 0 | \langle 0_s | f(r, -\infty) | j_s \rangle dr.
$$

(27)

Then

$$
\langle 0 | \langle 0_s | \bar{b}(t) | j_s \rangle = \int_{-\infty}^\infty g_G(t-r) \langle 0 | \langle 0_s | \bar{f}(r, -\infty) | j_s \rangle dr,
$$

where we have used the notation

$$
g_G(t) = \Delta(g_G^{-1}(t), 0).
$$

Consequently,

$$
\langle 0 | \langle 0_s | f(r, -\infty) | j_s \rangle
= \left[ 1 \quad 0 \right] \int_{-\infty}^\infty g_G^{-1}(r-t) \langle 0 | \langle 0_s | \bar{b}(t) | j_s \rangle dt,
$$

(28)

where $g_G^{-1}(t)$ is the stable inverse function of $g_G(t)$. By Lemma 1 in [35], we have
\[
\begin{bmatrix}
1 & 0
\end{bmatrix}
\int_{-\infty}^{\infty} g_{G}(r-t)\hat{b}(t)dt
= \begin{bmatrix}
1 & 0
\end{bmatrix}
\int_{-\infty}^{\infty} \begin{bmatrix}
g_{G}(t-r)^\dagger & 0 \\
0 & g_{G}(t-r)
\end{bmatrix}
\begin{bmatrix}
b(t) \\
\hat{b}(t)
\end{bmatrix}
dt
= \int_{-\infty}^{\infty} g_{G}(t-r)^\dagger \hat{b}(t)dt.
\]

Eq. (28) becomes
\[
\langle 0 \mid \langle 0_s \mid f(r, -\infty) \mid j_s \rangle \\
= \int_{-\infty}^{\infty} g_{G}(t-r)^\dagger \langle 0 \mid \langle 0_s \mid \hat{b}(t) \mid j_s \rangle dt, \\
= \int_{-\infty}^{\infty} g_{G}(t-r)^\dagger \langle 0 \mid \hat{b}(t) \langle 0_s \mid j_s \rangle dt.
\]

We have successfully applied the stable inversion technique, which is commonly used for linear systems, to a two-level quantum system. \(g_{G}(t)\) is a function of \(t\) and so \(g_{G}(t-r)^\dagger = g_{G}(t-r)^*\). Now we go to the main result of this section:

**Theorem 4** The one-channel output state of two-level system in response to a single photon input \(\xi(t)\) can be expressed as

\[
\rho_{out} = B^\dagger(\xi^\prime) |0\rangle \langle 0| B(\xi^\prime). 
\]  (29)

The output pulse shape is given by

\[
\xi^\prime(t) := \int_{-\infty}^{\infty} g_{G}(t-r)\xi(r)dr. 
\]  (30)

**Proof.** We make use of Lemma 3 to calculate the terms in Eq. (11) as

\[
\langle 0 \mid \langle 0_s \mid \int_{-\infty}^{\infty} \xi^*(r)f(r, -\infty)dr \mid 0_s \rangle \\
= \langle 0 \mid \int_{-\infty}^{\infty} \xi^*(r)dr \int_{-\infty}^{\infty} g_{G}(t-r)^\dagger \hat{b}(t)dt \langle 0_s \mid 0_s \rangle \\
= \langle 0 \mid \int_{-\infty}^{\infty} \hat{b}(t)dt \int_{-\infty}^{\infty} g_{G}(t-r)^\dagger \xi^*(r)dr \\
= \langle 0 \mid \int_{-\infty}^{\infty} \xi^*(t)\hat{b}(t)dt \\
= \langle 0 \mid B(\xi^\prime). 
\]  (31)

Similarly, by Lemma 3 we have
\[\langle 0| \xi^*(r) f(r, -\infty) dr |1_s\rangle dr = \langle 0| \xi^*(r) dr \int_{-\infty}^{\infty} g_{G^-}(t - r) b(t) dt \langle 0_s|1_s\rangle = 0.\]

Eq. (14) becomes (29).

The output of the system is still a pure state which is expressed as
\[|\Psi\rangle_{out} = \int_{-\infty}^{\infty} \xi'(t) b^\dagger(t) dt |0\rangle.\] (32)

In particular, the output pulse shape is the input pulse convolved with the
impulse response function \(g_{G^-}(t)\).

The single photon response to two-level systems has been extensively studied
in physics, see e. g. [22,38]. We have exactly solved this problem using linear
systems transfer functions, and have obtained the analytic form of the output
state without making any physical approximations such as weak excitation
limit or scattering modes. Compared to the results obtained in [35], the out-
put state is analogous to the output of a single-mode linear system in response
to a single photon input, under the condition that the linear system has fre-
quency \(\omega_c\) and decay rate \(k\). This observation is consistent with the existing
results from [22,23,38], where the authors have found that the transmission
and reflection spectrums for the single-photon transport through a two-level
system are analogous to the scattering spectrums for linear cavities.

3.2 Frequency-domain Analysis

Since the output pulse shape (30) is a convolution of \(\xi(t)\) and \(g_{G^-}(t)\), we can
easily obtain the Fourier transform of (30) using the convolution theorem:

\[\xi'(\omega) = \mathcal{F}(\xi(t)) \mathcal{F}(g_{G^-}(t)) = \xi(\omega)\frac{\kappa}{2} + i(\omega + \omega_c)\]
\[= \xi(\omega) \frac{\kappa}{2} + i(\omega + \omega_c)\]
\[\xi'(\omega) = \xi(\omega) G(i\omega),\] (33)

where we have used
\[\mathcal{F}(-\kappa e^{-(\frac{\kappa}{2} + i\omega) t} u(t)) = -\frac{\kappa}{2} + i(\omega + \omega_c).\] (34)

\(u(t)\) is the Heaviside step function. The frequency-domain input-output trans-
fer function \(G(i\omega)\) matches the calculations in [22,38]. (33) resembles the
linear transfer function of a linear cavity. For a one-sided cavity, the transfer function should have a negative sign in front of $\omega_c$ as shown in [28]. However in our case, the output field is defined in an oppositive direction and so it is a positive sign before $\omega_c$.

Now we turn to zero-dynamics principle [30] for studying the full inversion of the atomic states. The transfer function in $s$-domain is

$$G(s) = \frac{-\kappa + s + i\omega_c}{\frac{\kappa}{2} + s + i\omega_c}. \quad (35)$$

By the zero-dynamics principle, if the input state can remove the zeros from the transfer function, then no output will be observed when the input photon is interacting with the system. As a result, the single photon input will be fully absorbed into the system at the end of the input pulse. To remove the zero from (35), we choose the input as $\xi(s) = \sqrt{\kappa}/(-\frac{\kappa}{2} + s + i\omega_c)$. The output pulse shape is then given by

$$\xi'(s) = \frac{\sqrt{\kappa}}{\frac{\kappa}{2} + s + i\omega_c}. \quad (36)$$

The inverse Laplace transform of (36) yields

$$\xi'(t) = \sqrt{\kappa}e^{(-\frac{\kappa}{2} - i\omega_c)t}u(t). \quad (37)$$

The pulse shape $\xi'(t)$ vanishes for $t < 0$ and begins at $t = 0$. In contrast, the input pulse is given by

$$\xi(t) = \mathcal{L}^{-1}(\xi(s)) = \begin{cases} -\sqrt{\kappa}e^{(\frac{\kappa}{2} - i\omega_c)t}, & t \leq 0, \\ 0, & t > 0, \end{cases} \quad (38)$$

which is exponentially rising but with a resonant phase component till $t = 0$. The input photon is fully absorbed into the two-level system, and the system state is inverted to $|1_s\rangle$ at $t = 0$. After $t = 0$, the photon begins to leak out due to spontaneous emission. The inverting single-photon pulse (38) matches the results from [6][21].

4 The response of quantum two-level system to single photon input: two-channel

4.1 Time-domain Analysis

In this section, we consider the model which allows the photons to be either transmitted or reflected at the same time, as shown in Figure 2. This
Fig. 2. The input could come from two different directions, namely, left-going and right-going. The field \( l_{\text{in}} \) comes from the right and propagates to the left. \( r_{\text{in}} \) comes in from an opposite direction.

A model could be used to characterize the on-chip interaction between a two-level system and waveguide, since the photons can travel within the photonic waveguide in both directions \[22,3,8,16\]. The total system is described by the triplet

\[
(S, L, H_0) = \left( I_2, \begin{bmatrix} \sqrt{\kappa_1} \\ \sqrt{\kappa_2} \end{bmatrix} \sigma_-, \frac{\omega_c}{2} \sigma_z \right).
\] (39)

Based on (39), we can write down the dynamical equations as

\[
\dot{\sigma}_-(t) = -\left( i\omega_c + \frac{\kappa_1 + \kappa_2}{2} \right) \sigma_-(t) + \sigma_z(t) \left( \sqrt{\kappa_1} l_{\text{in}}(t) + \sqrt{\kappa_2} r_{\text{in}}(t) \right),
\] (40)

\[
l_{\text{out}}(t) = U^\dagger(t, t_0) l_{\text{in}}(t) U(t, t_0) = \sqrt{\kappa_1} \sigma_- + l_{\text{in}}(t),
\] (41)

\[
r_{\text{out}}(t) = U^\dagger(t, t_0) r_{\text{in}}(t) U(t, t_0) = \sqrt{\kappa_2} \sigma_- + r_{\text{in}}(t).
\] (42)

Here we use slightly different notations for the input fields as

\[
b(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix} = \begin{bmatrix} l_{\text{in}}(t) \\ r_{\text{in}}(t) \end{bmatrix}.
\] (43)

We use the notation \( l_{\text{in}}(t) \) to emphasize our assumption that the input to the first channel comes from the right and goes to the left. Similarly, the photon in the second channel is travelling to the right. Let the input be

\[
|1_\xi \rangle_1 \otimes |0\rangle_2 = B^\dagger_1(\xi)|0\rangle = \int_{-\infty}^{\infty} \xi(r) l_{\text{in}}^\dagger(r)|0\rangle dr,
\] (44)

where \( |0\rangle = |0\rangle_1 \otimes |0\rangle_2 \). That is, initially there is one photon as input in the left-going channel and no photon in the right-going channel.
Similar to Section 3, a strict linear systems transfer function approach is also applicable here.

**Theorem 5** The steady-state two-channel output state of the two-level system (39) in response to a single photon input (44) is given by

\[
|\Psi_{\text{out}}\rangle = \left(\int_{-\infty}^{\infty} (\xi(t) - \kappa_1 \eta(t)) l_{\text{in}}^\dagger(t) dt |0\rangle_1 \right) \otimes |0\rangle_2 \\
+ |0\rangle_1 \otimes \left(\int_{-\infty}^{\infty} \sqrt{\kappa_1 \kappa_2} \eta(t) r_{\text{in}}^\dagger(t) dt |0\rangle_2 \right),
\]

(45)

where \(\eta(t)\) is expressed as

\[
\eta(t) := \int_{-\infty}^{t} e^{-(i\omega_c + \frac{\kappa_1 + \kappa_2}{2})(t-r)} \xi(r) dr.
\]

(46)

**Proof.** The steady-state output field state is of the following form

\[
\rho_{\text{out}} := \text{Tr}_s[\rho_\infty] \\
:= \sum_{j=0,1} \langle j_s | \int_{-\infty}^{\infty} \xi(r) f_1^\dagger(r, -\infty) dr |0_s\rangle |0\rangle \\
\langle 0_s | (0_s | \int_{-\infty}^{\infty} \xi^*(r) f_1(r, -\infty) dr |j_s\rangle
\]

(47)

where we define

\[
f(t, t_0) = \begin{bmatrix} f_1(t, t_0) \\ f_2(t, t_0) \end{bmatrix} = \begin{bmatrix} U(t, t_0)l_{\text{in}}(t)U^\dagger(t, t_0) \\ U(t, t_0)r_{\text{in}}(t)U^\dagger(t, t_0) \end{bmatrix}.
\]

(48)

By Eq. (40), we have

\[
\langle 0_s | (0_s | \hat{\sigma}_-(t) = -\left(i\omega_c + \frac{\kappa_1 + \kappa_2}{2}\right) \langle 0_s | (0_s | \hat{\sigma}_-(t) \\
+ \langle 0_s | (0_s | \hat{\sigma}_z(t) (\sqrt{\kappa_1} l_{\text{in}}(t) + \sqrt{\kappa_2} r_{\text{in}}(t)) \\
= -\left(i\omega_c + \frac{\kappa_1 + \kappa_2}{2}\right) \langle 0_s | (0_s | \hat{\sigma}_-(t) \\
- \langle 0_s | (0_s | \hat{\sigma}_z(t) (\sqrt{\kappa_1} l_{\text{in}}(t) + \sqrt{\kappa_2} r_{\text{in}}(t)).
\]

(49)

Integrating Eq. (49) from \(t_0\) to \(t\) yields
\[
\langle 0_s|0\rangle \sigma_-(t) = \langle 0_s|0\rangle \left[ e^{-i(\omega_c + \frac{\kappa_1 + \kappa_2}{2})(t-t_0)} - \int_{t_0}^{t} e^{-i(\omega_c + \frac{\kappa_1 + \kappa_2}{2})(t-r)} \left( \sqrt{\kappa_1} l_{in}(r) + \sqrt{\kappa_2} r_{in}(r) \right) dr \right].
\]

(50)

This, together with (41)-(42), gives

\[
\langle 0_s|0\rangle b(t) = \langle 0_s|0\rangle \left\{ -\int_{-\infty}^{t} \left[ \sqrt{\kappa_1} \sqrt{\kappa_2} \right] e^{-i(\omega_c + \frac{\kappa_1 + \kappa_2}{2})(t-r)} \left[ \sqrt{\kappa_1} \sqrt{\kappa_2} \right] f(r, -\infty)dr + f(t, -\infty) \right\},
\]

(51)

by sending \( t_0 \to -\infty \). The output state can be solved using the stable inversion of (51). Define the impulse response function as

\[
g_G(t) = \begin{cases} 
\delta(t) - \left[ \sqrt{\kappa_1} \right] e^{-i(\omega_c + \frac{\kappa_1 + \kappa_2}{2})t} \left[ \sqrt{\kappa_1} \sqrt{\kappa_2} \right], & t \geq 0, \\
0, & t < 0,
\end{cases}
\]

then Eq. (51) can be re-written as

\[
\langle 0_s|0\rangle b(t) = \langle 0_s|0\rangle \int_{-\infty}^{t} g_G(t-r) f(r, -\infty)dr.
\]

(52)

So we have

\[
\langle 0|b(t) = \langle 0|b(t)\langle 0_s|0_s\rangle = \int_{-\infty}^{t} g_G(t-r) \langle 0|f(r, -\infty)\rangle \langle 0_s|dr.
\]

(53)

By the stable inversion technique we can express \( f_1(r, -\infty) \) in terms of \( b(t) \):

\[
\langle 0_s|\int_{-\infty}^{t} \xi^*(r)f_1(r, -\infty)dr \rangle = \langle 0\int_{-\infty}^{t} dt \int_{-\infty}^{t} dr \left( g_G(t-r) \begin{bmatrix} \xi(r) \\ 0 \end{bmatrix} \right)^{\dagger} b(t)dt.
\]

An explicit calculation shows

(54)
\[ \int_{-\infty}^{\infty} g_G(t-r) \begin{bmatrix} \xi(r) \\ 0 \end{bmatrix} dr \]

\[ = \begin{bmatrix} \xi(t) \\ 0 \end{bmatrix} - \int_{-\infty}^{t} e^{-(i\omega_c + \frac{\kappa_1 + \kappa_2}{2})(t-r)} \begin{bmatrix} \kappa_1 \xi(r) \\ \sqrt{\kappa_1 \kappa_2} \xi(r) \end{bmatrix} dr \]

\[ = \begin{bmatrix} \xi(t) - \kappa_1 \eta(t) \\ -\sqrt{\kappa_1 \kappa_2} \eta(t) \end{bmatrix}. \quad (55) \]

Using (55), Eq. (54) can be re-written as

\[ \langle 0 | \langle 0_s | \int_{-\infty}^{\infty} \xi^*(r) f_1(r, -\infty) dr | 0_s \rangle \]

\[ = \langle 0 | \int_{-\infty}^{\infty} dt \left[ \xi^*(t) - \kappa_1 \eta^*(t) - \sqrt{\kappa_1 \kappa_2} \eta^*(t) \right] b(t) \]

Similarly, it is easy to prove

\[ \langle 0 | \langle 0_s | \int_{-\infty}^{\infty} \xi^*(r) f_1(r, -\infty) dr | 1_s \rangle = 0. \quad (57) \]

Thus the steady-state output state is a pure state of the form

\[ |\Psi_{out}\rangle = \langle 0_s | \int_{-\infty}^{\infty} \xi(r) f_1^\dagger(r, -\infty) dr | 0_s \rangle | 0 \rangle \]

\[ = \int_{-\infty}^{\infty} dt \left[ \xi(t) - \kappa_1 \eta(t) - \sqrt{\kappa_1 \kappa_2} \eta(t) \right] b^\dagger(t) | 0 \rangle \]

\[ = \left( \int_{-\infty}^{\infty} (\xi(t) - \kappa_1 \eta(t)) r_{in}^\dagger(t) dt | 0 \rangle_1 \right) \otimes | 0 \rangle_2 \]

\[ - | 0 \rangle_1 \otimes \left( \int_{-\infty}^{\infty} \sqrt{\kappa_1 \kappa_2} \eta(t) r_{in}^\dagger(t) dt | 0 \rangle_2 \right) . \]

By Theorem 5, the probability of finding the photon in the left-going output channel is \( \int_{-\infty}^{\infty} |\xi(t) - \kappa_1 \eta(t)|^2 dt \) while the probability of finding the photon in the right-going output channel is \( \int_{-\infty}^{\infty} \left| \sqrt{\kappa_1 \kappa_2} \eta(t) \right|^2 dt \).

4.2 Frequency-domain Analysis

The shapes of the output pulses in these two channels are given by

\[ \xi_1'(t) = \xi(t) - \kappa_1 \eta(t), \quad \xi_2'(t) = \sqrt{\kappa_1 \kappa_2} \eta(t), \quad (59) \]
respectively. $\xi'_1(t)$ is left-going, while $\xi'_2(t)$ is the shape of the output pulse that travels to the right. Since $\eta(t)$ can be written in the form of a convolution, again we can easily obtain $\eta(\omega)$ using convolution theorem. Define two transfer functions as

$$
\xi'_1(\omega) = G_1(i\omega)\xi(\omega), \quad \xi'_2(\omega) = G_2(i\omega)\xi(\omega).
$$

(60)

Simple calculation yields

$$
G_1(i\omega) = \frac{-\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} + i(\omega + \omega_c),
$$

(61)

$$
G_2(i\omega) = \frac{\sqrt{\kappa_1\kappa_2}}{\kappa_1 + \kappa_2} + i(\omega + \omega_c).
$$

(62)

Obviously, $G_1(i\omega)$ corresponds to the transmission process and $G_2(i\omega)$ is related to reflection. The two-level system is usually coupled to the left-going and right-going fields of the waveguide with the same strength, which implies $\kappa_1 = \kappa_2$. In that case, we have $G_1(-i\omega_c) = 0$ and $|G_2(-i\omega_c)| = 1$. The component of the input pulse with frequency $\omega_c$ is all reflected. However, since a single photon pulse with a single frequency is infinite in temporal extent, a single photon pulse of finite width inevitably contains frequency components other than $\omega_c$. As a result, a single photon input as defined in (4) can never be completely reflected. The scattering process is shown in Figure 3 where we have $G_{11} = G_1(i\omega)$ and $G_{12} = G_2(i\omega)$. If the single photon input is from the right-going channel, it is easy to conclude $G_{22} = G_1(i\omega)$ and $G_{21} = G_2(i\omega)$ due to the symmetry between the inputs and outputs. Note that these four transfer functions are valid only for single photon input. Two-photon scattering may induce significant nonlinear dynamics [18], and hence these input-output relations will break down if there are more than one photons in the input channels.

The transfer functions in $s$-domain are

$$
G_1(s) = \frac{-\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} + is(\omega + \omega_c),
$$

$$
G_2(s) = \frac{\sqrt{\kappa_1\kappa_2}}{\kappa_1 + \kappa_2} + is(\omega + \omega_c).
$$

(63)

(64)
Fig. 4. The upper figure depicts a system interacting with a waveguide having two open ends. The lower figure depicts a waveguide with a closed end, and the system is placed at the end of the waveguide.

\[ G_1(s) = \frac{-\frac{\kappa_1 - \kappa_2}{2} + s + i\omega_c}{\frac{\kappa_1 + \kappa_2}{2} + s + i\omega_c}, \]  

(63)

\[ G_2(s) = \frac{\sqrt{\kappa_1 \kappa_2}}{\frac{\kappa_1 + \kappa_2}{2} + s + i\omega_c}. \]  

(64)

The transfer function (63) has a zero. We could still try to make use of the zero-dynamics principle to remove transmission signals till \( t = 0 \) by choosing \( \xi(s) = \sqrt{\frac{\kappa_1 - \kappa_2}{2}} / (-\frac{\kappa_1 - \kappa_2}{2} + s + i\omega_c). \) If \( \kappa_1 = \kappa_2 \), we have \( \xi(t) = 0 \) which is not a single photon pulse. So the zero-dynamics principle becomes trivial when \( \kappa_1 = \kappa_2 \). In other words, the zero-dynamics principle may not hold for purely imaginary zeros. Nevertheless, if we can make \( \kappa_1 > \kappa_2 \) in the physical system, it is easy to see that the transmission signals will be suppressed until \( t = 0 \) with the input \( \xi(s) = \sqrt{\frac{\kappa_1 - \kappa_2}{2}} / (-\frac{\kappa_1 - \kappa_2}{2} + s + i\omega_c). \)

\( G_2(s) \) has no zeros, so the reflection signal always exists as long as the input photon is interacting with the two-level system. Due to the constant loss in the reflection channel, it is impossible to fully excite the two-level system from \( |0_s\) to \( |1_s\) using a single photon. One way to circumvent this constraint is to split the single photon into two parts and symmetrically couple these two parts to the qubit, as shown in [3,21]. The other way is to place the qubit at the end of the waveguide, and so the qubit will be interacting with only one channel (See Figure 4). As we studied in Section 3, a full inversion is possible using a one-channel single photon. Particularly, the authors in [16] have mentioned that this type of design may enhance the absorption of photon.
5 The switching behaviour of quantum two-level system to single photon input

The switching behaviour of single photon controlled by a two-level system can be easily seen by looking at the transfer functions $G_1(i\omega)$ and $G_2(i\omega)$. As we have shown in the last section, the all-pass frequency components are determined by

$$|G_1(i\omega)|^2 = 1.$$  \hspace{1cm} \text{(65)}

Further calculation shows

$$|G_1(i\omega)|^2 = 1 - \frac{4\kappa_1\kappa_2}{(\kappa_1 + \kappa_2)^2 + 4(\omega + \omega_c)^2}.$$  \hspace{1cm} \text{(66)}

By (66), if $\omega$ is largely detuned from the transition frequency $\omega_c$ of the qubit, we have $|G_1(i\omega)|^2 \approx 1$ and the photon will transmit with high probability.

On the other hand, a perfect reflection happens if

$$|G_2(i\omega)|^2 = 1$$  \hspace{1cm} \text{(67)}

holds. More explicitly, the condition (67) can be calculated to be

$$|G_2(i\omega)|^2 = \frac{4\kappa_1\kappa_2}{(\kappa_1 + \kappa_2)^2 + 4(\omega + \omega_c)^2} = 1.$$  \hspace{1cm} \text{(68)}

Since $(\kappa_1 + \kappa_2)^2 \geq 4\kappa_1\kappa_2$ is always true, (67) admits a solution only if $\kappa_1 = \kappa_2$ and $\omega = -\omega_c$. In other words, only the frequency component in resonance with the qubit can be reflected if the two channels are coupled to the qubit with the same strength. If $\kappa_1 \neq \kappa_2$, there is no frequency component that can be perfectly reflected.

In the next we analyze a switching system which could switch on and off the transmission of a single photon along a waveguide. To enable a switching behaviour, the system should possess two states, and the photon may be transmitted or reflected depending on different states of the system. This function could be realized by using an ancillary two-level system. The total switching system is modelled as \[16\]

$$(S, L, H_0) = \left( I_2, \begin{bmatrix} \sqrt{\kappa_1} \\ \sqrt{\kappa_2} \end{bmatrix} \sigma_z^{1}, \frac{\omega_c}{2} \sigma_z^{1} + \frac{\omega_c}{2} \sigma_z^{2} + J \sigma_z^{1} \sigma_z^{2} \right).$$  \hspace{1cm} \text{(69)}

$J$ is the coupling strength between the two-level systems. Here for simplicity we assume the second two-level system is well decoupled from noise and its state can only be adjusted by external control.

Letting $\kappa_1 = \kappa_2 = \kappa$, the input-output relations for (69) can be derived as
\[\dot{\sigma}_{-1}(t) = -(\kappa + i\omega_{c} + 2iJ\sigma_{z}^{2}(t))\sigma_{-}^{1}(t) + \sqrt{\kappa}\sigma_{z}^{1}(t)(l_{in}(t) + r_{in}(t)),\]

\[r_{out}(t) = \sqrt{\kappa}\sigma_{z}^{1}(t) + r_{in}(t),\]

\[l_{out}(t) = \sqrt{\kappa}\sigma_{z}^{1}(t) + l_{in}(t),\]  

(70)

which are analogous to (40)-(42), except that there is an additional term 

\[-2J_{i}\sigma_{z}^{2}(t)\sigma_{z}^{1}(t)\]  

that results from the interaction between the qubits. Here \(\sigma_{z}^{i}, i = 1, 2\) denotes the Pauli operator on the \(i\)th qubit. The interaction Hamiltonian \(J\sigma_{z}^{1}\sigma_{z}^{2}\) preserves the energy of the second qubit, so \(|0_{s}\rangle_{2}\) and \(|1_{s}\rangle_{2}\) are steady states of the second qubit. Generally speaking, if \(\sigma_{z}^{2}(t)\) is time dependent, there is no closed-form solution to (70). However, since we assume the second system is either in the ground state or the excited state, we could use similar approach as in Sec. 3 and Sec. 4 to reduce (70) to linear stochastic differential equation. For example, for the second system being at \(|1_{s}\rangle_{2}\), we have \(\sigma_{z}^{2}(t) = \sigma_{z}^{2}\) and so the following relation

\[\langle 0\rangle\langle 0\rangle|\langle 0\rangle\langle 1_{s}\rangle_{2}\dot{\sigma}_{z}^{1}(t)\]

\[= \langle 0\rangle\langle 0\rangle|\langle 0\rangle\langle 1_{s}\rangle_{2}[-(\kappa + i\omega_{c} + 2iJ)\sigma_{z}^{1}(t) - \sqrt{\kappa}(l_{in}(t) + r_{in}(t))]\]

(71)

holds. The solution to (70) is analogous to the one derive in Sec. 4 with the output state being

\[|\Psi_{out}\rangle = \int_{-\infty}^{\infty} (\xi(t) - \kappa\eta_{1}(t))l_{in}^{\dagger}(t)dt |0\rangle - \int_{-\infty}^{\infty} \kappa\eta_{1}(t)r_{in}^{\dagger}(t)dt |0\rangle ,\]

where \(\eta_{1}(t)\) is given by

\[\eta_{1}(t) = \int_{-\infty}^{t} e^{-i(\omega_{c} + \kappa)J(t-r)}\xi(r)dr.\]  

(72)

The corresponding transmission probability is

\[P_{T_{1}} = \int_{-\infty}^{\infty} |\xi(t) - \kappa\eta_{1}(t)|^{2} dt.\]  

(73)

Conversely, if the ancillary system is in the ground state \(|0_{s}\rangle_{2}\), the output state is given by

\[|\Psi_{out}\rangle = \int_{-\infty}^{\infty} (\xi(t) - \kappa\eta_{0}(t))l_{in}^{\dagger}(t)dt |0\rangle - \int_{-\infty}^{\infty} \kappa\eta_{0}(t)r_{in}^{\dagger}(t)dt |0\rangle .\]
In this case, \( \eta_0(t) \) is defined as
\[
\eta_0(t) = \int_{-\infty}^{t} e^{-(i\omega_c + \kappa - 2iJ)(t-r)} \xi(r) dr. \tag{74}
\]
The corresponding transmission probability is
\[
P_{T_0} = \int_{-\infty}^{\infty} \left| \xi(t) - \kappa \eta_0(t) \right|^2 dt. \tag{75}
\]
The efficiency of the switch can be simply characterized by
\[
P_e = P_{T_1} - P_{T_0}. \tag{76}
\]
If \( |P_e| = 1 \), then a perfect switching is achieved. However, as we discussed before, the perfect switching is not possible, and what we can do is to make \( |P_e| \) as close to 1 as possible by optimizing the input pulse shape.

**Example 6** Suppose we want to block the transmission of the photon when the second qubit is in \( |1_s\rangle_2 \), and let the photon pass through if the second qubit is at the ground state \( |0_s\rangle_2 \). For the second qubit being at \( |1_s\rangle_2 \), the transfer function characterizing the transmission process is given by
\[
G_1(i\omega) = \frac{i(\omega + \omega_c + 2J)}{\kappa + i(\omega + \omega_c + 2J)}. \tag{77}
\]
\( G_1(i\omega) \) has one zero, however as we have shown in Subsection 4.2, the zero-removing input in the form of \( \xi(i\omega) = 1/[i(\omega + \omega_c + 2J)] \) does not lead to feasible solution. So instead we use the input
\[
\xi(i\omega) = \frac{\sqrt{2\kappa}}{-\kappa + i(\omega + \omega_c + 2J)}, \tag{78}
\]
which corresponds to single photon pulse of finite width ending at \( t = 0 \). Since the input pulse is peaked at \( \omega_c + 2J \) which is the transition frequency of the two-qubit system, we expect a large portion of its frequency components will be reflected.

Similarly, when the second qubit is in \( |0_s\rangle_2 \), the transfer function for the transmission process is changed to
\[
G_1(i\omega) = \frac{i(\omega + \omega_c - 2J)}{\kappa + i(\omega + \omega_c - 2J)}. \tag{79}
\]
If the coupling strength \( J \) is large, a large portion of the frequency components of the input \( \xi(i\omega) \) will be largely detuned from \( \omega_c - 2J \) and so they can pass through the system with high probability.

We numerically calculated the transmitted pulses for different switching states \( |1_s\rangle_2 \) and \( |0_s\rangle_2 \) using Theorem 5. The simulation results are shown in Figure 22.
We let $\omega_c = 1$ and then choose $\kappa = 0.0025$, which is consistent with the life time of an on-chip atom. A strong coupling $J = 0.01$ can also be achieved with the state-of-the-art on-chip design. Since $\kappa = 0.0025$ is relatively small, the input pulse lasts for a long time. During that time, as we can see in Figure 5, the transmission is suppressed if the second qubit is in $|1_s\rangle_2$. Some portion of the photon is absorbed into the qubit during the interaction, which explains the nonzero output after $t = 0$. Therefore, the output photon has a large occurring probability in the reflection channel while it may be difficult to detect it in the other output direction. When the second qubit is in $|0_s\rangle_2$, the single photon passes through the qubit nearly without any distortion.

6 The response of quantum finite-level system to single photon input

In this section, we derive the conditions that enable us to solve the response of a general finite-level system using linear systems transfer functions. For simplicity, a single channel input is considered. Still, we define the initial state as $\rho_0 = |0\rangle\langle 0_s| \otimes |0_s\rangle\langle 0|$. The excited basis states $|j_s\rangle, j \geq 1$ are one-photon or multi-photon states and so we have $\langle 0_s|j_s\rangle = 0, j \geq 1$. The following theorem generalizes the conditions regarding the applicability of the linear systems transfer functions.

**Theorem 7** Assume the interaction between a passive finite-level system and the input-output field is given by the triplet $(I, L, H_0)$. The input-output relation of the finite-level system in response to a single photon input $\xi(t)$ defined in (4) is equivalent to that of a linear system determined by
\[ \dot{L}(t) = aL(t) + cb(t), \]
\[ b_{\text{out}}(t) = L(t) + b(t), \]  
(80)

if
\[ G_t(L) = aL(t), \quad a < 0, \]  
(81)

and
\[ [L^\dagger, L]|0_s\rangle = c|0_s\rangle \]  
(82)

hold. c is an eigenvalue of \([L^\dagger, L]\).

**Proof.** The final state of the combined system can still be written in the following form

\[ \rho_{\infty} = \lim_{t \to \infty, t_0 \to -\infty} U(t, t_0)B^\dagger(\xi)U^\dagger(t_0)\rho_{\infty g} \]
\[ \lim_{t \to \infty, t_0 \to -\infty} U(t, t_0)B(\xi)U^\dagger(t_0), \]  
(83)

with

\[ \lim_{t \to \infty, t_0 \to -\infty} U(t, t_0)B^\dagger(\xi)U^\dagger(t_0) \]
\[ = \int_{-\infty}^{\infty} dr \xi(r)f^\dagger(r, -\infty). \]  
(84)

The output state is obtained by taking the partial trace as

\[ \rho_{\text{out}} = \text{Tr}_s(\rho_{\infty}) \]
\[ = \sum_j \langle j_s | \int_{-\infty}^{\infty} dr \xi(r)f^\dagger(r, -\infty)|0_s\rangle \langle 0 | \]
\[ = \langle 0 | \int_{-\infty}^{\infty} dr \xi^\ast(r)f(r, -\infty) | j_s \rangle. \]  
(85)

Therefore, we need to solve for \( \langle 0 | \langle 0_s | f(t, -\infty) | j_s \rangle \). Employing the \((I, L, H_0)\) representation for the finite-level system, the stochastic differential equations of the total system can be written as

\[ \dot{X}(t) = G_t(X) + b^\dagger(t)[X(t), L(t)] \]
\[ + [L^\dagger(t), X(t)]b(t), \]  
(86)

\[ b_{\text{out}}(t) = L(t) + b(t). \]  
(87)

Particularly by (86) we have

\[ \dot{L}(t) = G_t(L) + [L^\dagger(t), L(t)]b(t). \]  
(88)
According to (81), $G_t(L)$ is linear in $L(t)$, and hence the nonlinearity can only lie in the term $[L^\dagger(t), L(t)]b(t)$. Similar to Sec. 3, whether or not the system can be solved using linear approach depends on the following expression

$$
\langle 0 | \langle 0_s | \dot{L}(t) = \langle 0 | \langle 0_s | G_t(L) + \langle 0 | \langle 0_s | [L^\dagger(t), L(t)]b(t) \\
= \langle 0 | \langle 0_s | G_t(L) + \langle 0 | \langle 0_s | U^\dagger(t, t_0)[L^\dagger, L]U(t, t_0)b(t). \tag{89}
$$

The system-field interaction is described by the couplings $L^\dagger(t)b(t)$ and $L(t)b^\dagger(t)$. Recall that we have concluded in Lemma 2 that

$$
U(t, t_0)|0⟩|0_s⟩ = \exp(i\alpha_t)|0⟩|0_s⟩, \tag{90}
$$

if

$$
L^\dagger b|0⟩|0_s⟩ = 0, \quad L^\dagger b|0⟩|0_s⟩ = 0. \tag{91}
$$

In particular, (90) is always true if the finite-level system is not coupled to other active sources. Since we have assumed the finite-level system is passive, the system will remain at the ground state due to the conservation of energy. As a result, the Hermitian operator $[L^\dagger, L]$ acts directly on the ground state $|0_s⟩$ in Eq. (89). Making use of (82) we arrive at

$$
\langle 0 | \langle 0_s | U^\dagger(t, t_0)[L^\dagger, L]U(t, t_0)b(t) = \langle 0 | \langle 0_s | cb(t). \tag{92}
$$

The dynamical equations of the system are simplified as

$$
\langle 0 | \langle 0_s | \dot{L}(t) = \langle 0 | \langle 0_s | [aL(t) + cb(t)], \\
b_{out}(t) = L(t) + b(t). \tag{93}
$$

Since $a$ is strictly negative, the stable inversion technique is applicable after the integration of (93). Therefore similar to the proof of Lemma 3 we can prove $\langle 0 | \langle 0_s | f(t, -\infty)|j⟩_s = 0, j \geq 1$. The remaining term $\langle 0 | \langle 0_s | f(t, -\infty)|0⟩_s$ determines the output state, which is the same as the output state of the linear system (80) in response to $ξ(t)$.

It is worth mentioning that by following the procedures from Section 4, Theorem 7 can be easily extended for multi-channel inputs and outputs.

Theorem 7 could be used to simplify the input-output analysis of finite-level coherent networks that are driven by single photons. Below is an application of Theorem 7 to the gradient echo quantum memory which is often made of atoms.

**Example 8** Consider the finite input-output model for gradient echo memories as

$$
G = G_1 \triangleright G_2 \triangleright G_3 \triangleright \cdots \triangleright G_N, \tag{94}
$$
where each $G_n, n \in \{1, 2, 3, ..., N\}$ is a two-level system represented by

$$G_n = (I, \sqrt{\kappa} \sigma_n^-, H_n).$$  \hspace{1cm} (95)

The notation $\triangleright$ is for the interconnection of two systems by series product \cite{11,33}. \cite{94} models an atomic memory by assuming the atoms are put in series and connected using a single channel. The $(S, L, H_0)$ representation of $G$ is thus given by

$$G = (I, \sum_n \sqrt{\kappa} \sigma_n^-, H_0).$$  \hspace{1cm} (96)

The input-output relation for this system can be calculated using \cite{86}-\cite{87}. It is easy to see that \cite{82} and \cite{81} hold for this system. For example, we can verify \cite{82} as

\begin{align*}
[L^\dagger, L]|0_s\rangle &= \sum_n \sqrt{\kappa} \sigma_n^+, \sum_n \sqrt{\kappa} \sigma_n^-|0_s\rangle \\
&= \kappa \sum_n [\sigma_n^+, \sigma_n^-]|0_s\rangle = \kappa \sum_n \sigma_n^z|0_s\rangle = \kappa(-1)^N|0_s\rangle.
\end{align*}

(97)

As a result, we can treat $G$ as a series interconnection of linear cavities, simply by replacing each $\sigma_n^-$ with $a_n^-$. $a_n^-$ is the annihilation operator for the $n$th cavity. Particularly, we have $[\sigma_n^+, \sigma_n^-]|0_s\rangle = -|0_s\rangle = [a_n^+, a_n^-]|0_s\rangle$ due to $[a_n^+, a_n^-] = -1$.

A so-called weak atomic excitation limit is introduced in \cite{20,14} to approximate the atoms by linear cavities in this memory model. Here we have proven that this approximation is not necessary under single-photon driving.

7 Conclusion

In this paper we have investigated the response of a class of quantum finite-level systems which in general should be governed by nonlinear stochastic differential equations. We found the analytical solutions to these dynamics when the input state is single photon state. The future research would include the application of this work to the hybrid coherent quantum networks driven by single photons.

Acknowledgements

This research is supported by Australian Research Council and AFOSR Grant FA2386-12-1-4075. This research is also supported in part by National Natural
Science Foundation of China grant (No. 61374057, 11404113) and Hong Kong RGC grant (No. 531213).

References

[1] C. Altafini and F. Ticozzi. Modeling and control of quantum systems: An introduction. *Automatic Control, IEEE Transactions on*, 57(8):1898–1917, 2012.

[2] S. Baur, D. Tiarks, G. Rempe, and S. Dürr. Single-photon switch based on rydberg blockade. *Phys. Rev. Lett.*, 112:073901, 2014.

[3] D. E. Chang, A. S. Sørensen, E. A. Demler, and M. D. Lukin. A single-photon transistor using nanoscale surface plasmons. *Nature Physics*, 3:807–812, 2007.

[4] W. Chen, K. M. Beck, R. Bcker, M. Gullans, M. D. Lukin, H. Tanji-Suzuki, and V. Vuleti?. All-optical switch and transistor gated by one stored photon. *Science*, 341(6147):768–770, 2013.

[5] Y. Chen, M. Wubs, J. M?rk, and A. F. Koenderink. Coherent single-photon absorption by single emitters coupled to one-dimensional nanophotonic waveguides. *New Journal of Physics*, 13(10):103010, 2011.

[6] J. I. Cirac, P. Zoller, H. J. Kimble, and H. Mabuchi. Quantum state transfer and entanglement distribution among distant nodes in a quantum network. *Phys. Rev. Lett.*, 78:3221–3224, 1997.

[7] D. Dong and I. R. Petersen. Quantum control theory and applications: A survey. *IET Control Theory Appl.*, 4:2651–2671, 2010.

[8] S. Fan, Ş. E. Kocabas, and J. T. Shen. Input-output formalism for few-photon transport in one-dimensional nanophotonic waveguides coupled to a qubit. *Phys. Rev. A*, 82:063821, 2010.

[9] C. Gardiner and P. Zoller. *Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics*. Springer Series in Synergetics. Springer, 2004.

[10] J. Gough. Quantum stratonovich calculus and the quantum wong-zakai theorem. *Journal of Mathematical Physics*, 47(11), 2006.

[11] J. Gough and M. R. James. The series product and its application to quantum feedforward and feedback networks. *Automatic Control, IEEE Transactions on*, 54(11):2530–2544, 2009.

[12] J. E. Gough, M. R. James, H. I. Nurdin, and J. Combes. Quantum filtering for systems driven by fields in single-photon states or superposition of coherent states. *Phys. Rev. A*, 86:043819, 2012.
[13] R. L. Hudson and K. R. Parthasarathy. Quantum ito’s formula and stochastic evolutions. *Communications in Mathematical Physics*, 93(3):301–323, 1984.

[14] M. R. Hush, A. R. R. Carvalho, M. Hedges, and M. R. James. Analysis of the operation of gradient echo memories using a quantum input-output model. *New Journal of Physics*, 15(8):085020, 2013.

[15] G. J. Milburn. Coherent control of single photon states. *The European Physical Journal Special Topics*, 159(1):113–117, 2008.

[16] L. Neumeier, M. Leib, and M. J. Hartmann. Single-photon transistor in circuit quantum electrodynamics. *Phys. Rev. Lett.*, 111:063601, 2013.

[17] H. I. Nurdin and J. E. Gough. Modular Quantum Memories Using Passive Linear Optics and Coherent Feedback. *arXiv:1409.7473 e-prints*, 2014.

[18] A. Nysteen, P. Trost Kristensen, D. P. S. McCutcheon, P. Kaer, and J. Mørk. Scattering of two photons on a quantum emitter in a one-dimensional waveguide: Exact dynamics and induced correlations. *ArXiv e-prints:1409.1256*, 2014.

[19] I. R. Petersen. Analysis of Linear Quantum Optical Networks. *arXiv:1403.6214 e-prints*, 2014.

[20] I. Protsenko, P. Domokos, V. Lefèvre-Seguin, J. Hare, J. M. Raimond, and L. Davidovich. Quantum theory of a thresholdless laser. *Phys. Rev. A*, 59:1667–1682, 1999.

[21] E. Rephaeli, J. T. Shen, and S. Fan. Full inversion of a two-level atom with a single-photon pulse in one-dimensional geometries. *Phys. Rev. A*, 82:033804, 2010.

[22] J. T. Shen and S. Fan. Coherent photon transport from spontaneous emission in one-dimensional waveguides. *Opt. Lett.*, 30(15):2001–2003, 2005.

[23] J. T. Shen and S. Fan. Coherent single photon transport in a one-dimensional waveguide coupled with superconducting quantum bits. *Phys. Rev. Lett.*, 95:213001, 2005.

[24] I. Shomroni, S. Rosenblum, Y. Lovsky, O. Bechler, G. Guendelman, and B. Dayan. All-optical routing of single photons by a one-atom switch controlled by a single photon. *Science*, 345(6199):903–906, 2014.

[25] T. Sogo. On the equivalence between stable inversion for nonminimum phase systems and reciprocal transfer functions defined by the two-sided laplace transform. *Automatica*, 46(1):122 – 126, 2010.

[26] R. J. Thompson, G. Rempe, and H. J. Kimble. Observation of normal-mode splitting for an atom in an optical cavity. *Phys. Rev. Lett.*, 68:1132–1135, Feb 1992.

[27] T. Volz, A. Reinhard, M. Winger, A. Badolato, K. J. Hennessy, E. L. Hu, and A. Imamoglu. Ultrafast all-optical switching by single photons. *Nature Photonics*, 6(9):605–609, 2012.
Here we present a more physical approach to solve the two-channel single photon response. We start with the Hamiltonian of the total system which is written as

$$H = \frac{1}{2}\omega_c \sigma_z + \int_{-\infty}^{\infty} d\omega (\omega l_{in}^\dagger(\omega)l_{in}(\omega) - \omega r_{in}^\dagger(\omega)r_{in}(\omega))$$

$$+ ig \int_{-\infty}^{\infty} d\omega [\sigma_+ (l_{in}(\omega) + r_{in}(\omega))$$

$$- \sigma_- (l_{in}^\dagger(\omega) + r_{in}^\dagger(\omega))].$$

(A.1)
For simplicity, we use the same coupling strength $g$ for both channels. $r_{in}$ is the field annihilation operator for the signals travelling to the right, and $l_{in}$ is the field annihilation operator for the left-going signals. The negative sign in front of $\omega r_{in}(\omega)r_{in}(\omega)$ indicates an opposite group velocity of the right-going waves.

The total Hamiltonian (A.1) can be divided into interacting and non-interacting parts as [8]

$$H = \frac{1}{2}\omega_c\sigma_z + \int_{-\infty}^{\infty} d\omega (\omega \tilde{b}^\dagger(\omega)\tilde{b}(\omega) + \omega \hat{b}^\dagger(\omega)\hat{b}(\omega))$$

$$+ i\sqrt{2}g \int_{-\infty}^{\infty} d\omega (\sigma_+ \tilde{b}(\omega) - \sigma_- \hat{b}^\dagger(\omega)),$$

(A.2)

where the field operators are redefined as

$$\tilde{b}(\omega) = \frac{l_{in}(\omega) + r_{in}(-\omega)}{\sqrt{2}}, \quad \hat{b}(\omega) = \frac{l_{in}(\omega) - r_{in}(-\omega)}{\sqrt{2}}.$$ (A.3)

It is obvious from (A.2) that $\hat{b}(\omega)$ is interaction free, and thus we can consider the field operator $\tilde{b}(\omega)$ only.

The Heisenberg equations of motion under (A.1) are

$$\dot{l}_{in}(\omega, t) = -i\omega l_{in}(\omega, t) - g\sigma_- (t),$$ (A.4)

$$\dot{r}_{in}(\omega, t) = i\omega r_{in}(\omega, t) - g\sigma_- (t),$$ (A.5)

$$\dot{\sigma}_-(t) = -i\omega \sigma_-(t)$$

$$- g \int_{-\infty}^{\infty} d\omega \sigma_z(t)(l_{in}(\omega, t) + r_{in}(\omega, t)).$$ (A.6)

Note that (A.4) and (A.5) have opposite signs for the unitary parts, which suggests the following definitions for the time-domain input operators

$$l_{in}(t) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega l_{in}(\omega, t_0)e^{-i\omega(t-t_0)},$$

$$r_{in}(t) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega r_{in}(\omega, t_0)e^{i\omega(t-t_0)}.$$ (A.7)

The time-domain output operators need to be defined as

$$l_{out}(t) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega l_{in}(\omega, t)e^{-i\omega(t-t_1)},$$

$$r_{out}(t) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega r_{in}(\omega, t)e^{i\omega(t-t_1)}.$$ (A.8)
in order to match (40)-(42). \(t_1\) denotes the final time of the evolution. An extra negative sign is used in the definition of the output field operators, as compared to the conventional definitions in quantum optics [28].

The interaction and interaction-free field operators \(\tilde{b}(t)\) and \(\hat{b}(t)\) are related to \(l_{in}(t)\) and \(r_{in}(t)\) by

\[
\tilde{b}(t) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \tilde{b}(\omega, t_0) e^{-i\omega(t-t_0)} = -\frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} d\omega \left( \frac{l_{in}(\omega, t_0) + r_{in}(-\omega, t_0)}{\sqrt{2}} \right) e^{-i\omega(t-t_0)} = l_{in}(t) + r_{in}(t),
\]

\[
\hat{b}(t) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \tilde{b}(\omega, t_0) e^{-i\omega(t-t_0)} = -\frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} d\omega \left( \frac{l_{in}(\omega, t_0) - r_{in}(-\omega, t_0)}{\sqrt{2}} \right) e^{-i\omega(t-t_0)} = l_{in}(t) - r_{in}(t),
\]

This reduces (A.2) to single-channel interaction, with an additional scaling factor \(\sqrt{2}\) associated with the coupling strength \(g\). More explicitly, the dynamics of \(\sigma_-\) follows

\[
\dot{\sigma}_-(t) = -(\kappa + i\omega_c)\sigma_- + \sqrt{2\kappa \sigma_z(t)}\tilde{b}(t),
\]

\[
\tilde{b}_{out}(t) = \sqrt{2\kappa}\sigma_- + \tilde{b}(t),
\]

where we used the notation \(\kappa = 2\pi g^2\).

\(\hat{b}(t)\) satisfies the same commutation relations as the white-noise input operators \(l_{in}(t)\) and \(r_{in}(t)\). Taking \(\tilde{b}(t)\) as the input, the input-output relation can be written as

\[
\hat{b}_{out}(t) = \hat{b}(t),
\]

according to (A.4)-(A.5). This implies the relation

\[
U^\dagger(t, t_0)\hat{b}(t)U(t, t_0) = \hat{b}(t),
\]

which provides another evidence that \(\hat{b}(t)\) is interaction-free.

Now we consider there is a left-going single-photon input, and the output state can be computed by

\[
\rho_{out} = \sum_{j=0,1} \langle j_s | \int_{-\infty}^{\infty} dr \xi(r) f_1^\dagger(r, -\infty) |0_s\rangle |0\rangle \langle 0| \int_{-\infty}^{\infty} dr \xi^*(r) f_1(r, -\infty) |j_s\rangle.
\]

(A.14)
We denote $h(t, t_0) = U(t, t_0)\tilde{b}(t)U^\dagger(t, t_0)$. The impulse response function is redefined as

$$g_{G^-}(t) = \begin{cases} -2\kappa e^{-(\kappa + i\omega_c) t} + \delta(t), & t \geq 0, \\ 0, & t < 0. \end{cases} \tag{A.15}$$

The stable inverse of (A.15) can be calculated using Lemma 3. We can then solve (A.11) to get

$$\langle 0|\langle 0_s|h(t, -\infty)|0_s\rangle = \int_{-\infty}^{\infty} g_{G^-}(r - t)^\dagger \langle 0|\tilde{b}(r)dr. \tag{A.16}$$

Next, we will make use of the following relation

$$\langle 0|\langle 0_s|f_1(t, -\infty)|0_s\rangle = \int_{-\infty}^{\infty} g_{G^-}(r - t)^\dagger \langle 0|\tilde{b}(r)dr + \langle 0|\tilde{b}(t)\rangle \tag{A.17}$$

to write $\langle 0|\langle 0_s|f_1(t, -\infty)|0_s\rangle$ as

$$\langle 0|\langle 0_s|f_1(t, -\infty)|0_s\rangle = \frac{\sqrt{2}}{2} \int_{-\infty}^{\infty} g_{G^-}(r - t)^\dagger \langle 0|\tilde{b}(r)dr + \langle 0|\tilde{b}(t)\rangle \tag{A.18}$$

Similarly, we can prove

$$\langle 0|\langle 0_s|f_1(t, -\infty)|1_s\rangle = 0. \tag{A.19}$$

Using (A.18) and (A.19), the output state can be expressed as a pure state
\[ |\Psi\rangle_{out} = \frac{1}{2} \int_{-\infty}^{\infty} \xi(t) [ \int_{-\infty}^{\infty} g_{G^-}(r-t) l^\dagger_{in}(r) dr + l^\dagger_{in}(t) dt ] |0\rangle \\
+ \frac{1}{2} \int_{-\infty}^{\infty} \xi(t) [ \int_{-\infty}^{\infty} g_{G^-}(r-t) r^\dagger_{in}(r) dr - r^\dagger_{in}(t) dt ] |0\rangle \\
= \left[ \int_{-\infty}^{\infty} \xi(t) l^\dagger_{in}(t) dt \\
- \kappa \int_{-\infty}^{r} \xi(t) e^{-(\kappa+i\omega_c)(r-t)} dl^\dagger_{in}(r) dr |0\rangle \\
- \kappa \int_{-\infty}^{r} \xi(t) e^{-(\kappa+i\omega_c)(r-t)} dr r^\dagger_{in}(r) dr |0\rangle \right], \quad (A.20) \]

which is consistent with Theorem 5.