SCHEMES AS FUNCTORS ON TOPOLOGICAL RINGS

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ABSTRACT. In this text, we extend the known definitions of a topology on the set $X(R)$ of $R$-rational points from topological fields, local rings and adele rings to any ring $R$ with a topology. This definition is functorial in both $X$ and $R$, and it does not rely on any restriction on $X$ like separability or finiteness conditions. Besides establishing basic facts and showing the equivalence with classical approaches, we characterize properties of $R$, such as being a topological Hausdorff ring, a local ring or with a totally disconnected spectrum, in terms of functorial properties of the topology of $X(R)$.

INTRODUCTION

The aim of this text is to generalize the definition of a topology for the set $X(R)$ of $R$-rational points of a scheme $X$ from known cases of schemes $X$ and topological rings $R$ to all schemes and all topological rings. We will give a general definition and investigate its functorial properties. We show that it recovers the known topologies, and we characterize certain properties of $R$, such as being a topological Hausdorff ring, a local ring or with a totally disconnected spectrum, in terms of functorial properties of the topologies on $X(R)$.

We begin with recalling the known constructions of topologies on sets of rational points.

The strong topology. Let $k$ be a topological field with closed points. It is a classical fact that one can endow the sets $X(k)$ of $k$-rational points with a topology in unique way for all varieties $X$ over $k$ such that the following properties hold for all varieties $U_i$, $X$, $Y$ and $Z$ over $k$ (cf. [6, Ch. I.10]).

(S0) Every morphism $X \to Y$ yields a continuous map $X(k) \to Y(k)$.
(S1) The canonical bijection $X \times_k Y(k) \to X(k) \times_k Y(k)$ is a homeomorphism.
(S2) The canonical bijection $k^1(k) \to k$ is a homeomorphism.
(S3) A closed immersion $Y \to X$ yields a closed embedding $Y(k) \to X(k)$ of topological spaces.
(S4) An open immersion $Y \to X$ yields an open embedding $Y(k) \to X(k)$ of topological spaces.
(S5) Given an affine open covering $\{U_i\}$ of $X$, a subset $W$ of $X(k)$ is open if and only if $W \cap U_i(k)$ is open in $U_i(k)$ for every $i$, and $X(k) = \bigcup U_i(k)$.

Since (S3) implies that this topology is stronger than the Zariski topology on $X(k)$, it is often called the strong topology for $X(k)$.

Knowing that such a unique topology for the sets $X(k)$ exists, it is clear how to find an explicit description: if $X$ is a closed subscheme of $\mathbb{A}^n$, the set $X(k)$ inherits the subspace topology from $\mathbb{A}^n(k) = k^n$; if $X$ is a general scheme, then we can cover it with affine opens $U_i$ and endow $X(k)$ with the finest topology such that all the inclusions $U_i(k) \to X(k)$ are continuous.

The basic properties of $k$ that are used to show the independence of this construction from the ambient affine spaces $\mathbb{A}^n$ and from the covering $U_i$ of $X$ are the following:

(i) $k$ is a Hausdorff ring, i.e. $k$ is a topological ring that is Hausdorff;
(ii) $k$ is with open unit group, i.e. the set $k^*$ of units are open in $k$ and a topological group with the subspace topology;
(iii) $k$ is a local ring.
A proof of this can be found in [11 Prop. 3.1]. The conclusion is that the construction of the strong topology extends to all local Hausdorff rings \( R \) with open unit group. The necessity of the properties (S1)-(S3) is easily seen. That \( k \) has to be a Hausdorff ring can be easily derived from (S1)-(S3). That \( k \) has to be with open unit group is dictated by (S4), which yields an open topological embedding \( k^\times = \mathbb{G}_m(k) \to \mathbb{A}^1(k) = k \). That \( k \) is a local ring follows from (S5); namely, every point of \( X(k) \) has to be contained in \( U(k) \) for an affine open subscheme \( U \) of \( X \).

**Weil’s construction for adèle rings.** One obstacle to extend the definition of the strong topology in terms of open coverings to other classes of topological rings is that the axioms (S4) and (S5) cannot be true anymore. If \( R \) is not with open unit group, then the open immersion \( \mathbb{G}_m \to \mathbb{A}^1 \) yields an injection \( R^\times = \mathbb{G}_m(R) \to \mathbb{A}^1(R) = R \), which is not open. If \( R \) is not local, then there are varieties \( X \) that have \( R \)-rational points that are not contained in any affine open subscheme. This makes clear that we have to discard axioms (S4) and (S5) if we want to find a definition of a topology for \( X(R) \) for more general topological rings \( R \) than local Hausdorff rings with open unit group.

An example of a ring whose units are not open is the adèlic ring \( A \) of a global field \( k \). For \( k \)-varieties \( X \), André Weil constructs in [11] a topology for \( X(A) \) by a different method. We recall Weil’s construction in the following.

We denote places of \( k \) by \( v \), the completion w.r.t. \( v \) by \( k_v \) and, in case of a non-archimedean place \( v \), the ring of integers in \( k_v \) by \( O_v \). Note that all local fields \( k_v \) are local Hausdorff rings with with open unit group, and so are the rings \( O_v \). Therefore the sets \( X(k_v) \) come with the strong topology for every \( k \)-variety \( X \), or, more generally, for every \( k \)-scheme \( X \) of finite type.

Let \( S \) be a finite set of places containing all the archimedean ones, and let \( O_S \) be the \( S \)-integers in \( k \). If \( X_S \) is an \( O_S \)-model of \( X \) and \( v \notin S \), then also \( X_S(O_v) = \text{Hom}_{O_v} (\text{Spec} O_v, X) \) is equipped with the strong topology. Note that for every finite type \( k \)-scheme \( X \), there exist a finite set \( S \) of places and a finite type \( O_S \)-model \( X_S \) of \( X \).

Let \( A_S = \prod_{v \in S} k_v \times \prod_{v \notin S} O_v \) be the \( S \)-adèles. We equip the set

\[
X_S(A_S) = \prod_{v \in S} X_S(k_v) \times \prod_{v \notin S} X_S(O_v)
\]

with the product topology, which we call the \( S \)-adèlic topology. For a finite set \( S' \) of places that contains \( S \), we denote by \( X_{S'} \) the base extension of \( X_S \) to the \( S' \)-integers \( O_{S'} \). By [3] 8.14.2, we have

\[
X(A) = \colim_{S \subseteq S'} X_{S'}(A_{S'}),
\]

which is equipped with the colimit topology, i.e. the finest topology such that all the maps \( X_{S'}(A_{S'}) \to X(A) \) are continuous. We call this topology in the following the adelic topology.

It is well-known that the adelic topology satisfies the analogues of properties (S0)-(S3), but fails to satisfy (S4) since the idèle group \( A^\times \) is not open in \( A \). This means that for a closed subscheme \( U \) of \( \mathbb{A}^n \), the adelic topology of \( U(A) \) is the subspace topology of \( \mathbb{A}^n(A) = A^n \), but that the adelic topology on \( X(A) \) is not induced by an affine open covering \( \{ U_i \} \) of \( X \) in case of a variety \( X \) that is not affine.

**Grothendieck’s construction for affine schemes.** In the case of affine schemes \( X = \text{Spec} A \) over a base ring \( k \), Grothendieck constructs a topology on \( X(R) = \text{Hom}_k(A,R) \) for any topological \( k \)-algebra \( R \). Namely, we endow \( X(R) \) with the coarsest topology such that for all elements \( a \in A \), the evaluation map

\[
ev_a : \text{Hom}_k(A,R) \to R \quad (f : A \to R) \mapsto f(a)
\]

is continuous. We call this topology on \( X(R) \) the affine topology.
Note that this definition does neither require any compatibility of the topology of $R$ with the ring operation, nor a Hausdorff property nor open unit group nor a unique maximal ideal.

The affine topology coincides with the strong topology on $X(k)$ in case of a local Hausdorff ring $k$ with open unit group and a finite type $k$-scheme $X$. The affine topology also coincides with the adelic topology on $X(A)$ when $X$ is a variety over a global field $k$, cf. Corollary 2.2.

**The fine topology.** Let $k$ be a ring, $X$ a $k$-scheme and $R$ a $k$-algebra with a topology, by which we mean a $k$-algebra equipped with a topology that we do not assume to be compatible with the ring operations in any way.

Thefine topology on $X(k)$ is the finest topology such that any $k$-morphism $\varphi : U \to X$ from an affine $k$-scheme to $X$ induces a continuous map $\varphi_R : U(R) \to X(R)$ where $U(R)$ is considered with the affine topology.

Besides being a generalization from certain classical cases of topologies on $X(R)$ to all $k$-schemes $X$ and all $k$-algebras $R$ with a topology, the definition of the fine topology comes in a language that transfers easily to other scheme-like theories, which includes algebraic spaces and stacks, analytic spaces and tropical schemes. In particular, we make use in a subsequent work of the very same concept to define a topology on the set of tropical points of a tropical scheme, which coincides as a point set with a tropical variety in the classical sense; cf. [2].

**Content of this paper.** In the subsequent sections of this text, we will prove the following statements. We show that the fine topology generalizes all the other concepts of topologies as explained above; namely, the fine topology is equal to the affine topology (Lemma 1.2), the strong topology (Corollary 5.3) and the (S-)adelic topology (Theorem 7.3) whenever the latter topologies are defined for $X(R)$.

We will see that the fine topology of $X(R)$ is functorial in $X$ and $R$ (Proposition 1.3), which can be seen as the generalization of property (S0) of the strong topology. We will investigate a series of further axioms for the fine topology, as explained in the following.

Let $C$ be a class of $k$-schemes. We say that $R$ satisfies (F1)–(F6) for all schemes in $C$ if the following hypotheses are satisfied for all $X$, $Y$ and $Z$ in $C$ w.r.t. the fine topology:

(F1) The canonical bijection $X \times_Z Y(R) \to X(R) \times_{Z(R)} Y(R)$ is a homeomorphism.

(F2) The canonical bijection $A^1_k(R) \to R$ is a homeomorphism.

(F3) A closed immersion $Y \to X$ yields a closed embedding $Y(R) \to X(R)$ of topological spaces.

(F4) An open immersion $Y \to X$ of $k$-schemes yields an open embedding $Y(R) \to X(R)$ of topological spaces.

(F5) Let $\{U_i\}_{i \in I}$ be an affine open covering of $X$. Then $X(R) = \bigcup_{i \in I} U_i(R)$, and a subset $W$ of $X(R)$ is open if and only if $W \cap U_i(R)$ is open in $U_i(R)$ for all $i \in I$.

(F6) Let $\{U_i\}_{i \in I}$ be a finite affine open covering of $X$ and $U = \coprod_{i \in I} U_i$. Let $\Psi : U \to X$ be the associated morphism. Then the map $\Psi_R : U(R) \to X(R)$ is surjective and open.

While (F1)–(F5) are direct analogues of the properties (S1)–(S5) of the strong topology, (F6) is a variant of (F5) that remedies the fact that the adelic topology cannot be determined by affine open coverings. Namely, the adèle ring $R = A$ of a global field $k$ satisfies (F6) for all $k$-schemes of finite type (Lemma 7.2). The same holds true for the $S$-adèles $A_X$ where $S$ is a finite set of places of $k$ (Lemma 7.1).

It is clear from the previous discussions that the axioms (F1)–(F6) do not hold for all $k$-algebras $R$ with topology, but we will find necessary and sufficient conditions for various combinations of these axioms:

(i) $R$ is a topological ring if and only if (F1) and (F2) are satisfied for all $k$-schemes (of finite type) (Proposition 2.1).
(ii) A topological ring $R$ is Hausdorff if and only if (F3) is satisfied for all $k$-schemes (of finite type) (Proposition 5.1).

(iii) A topological ring $R$ is with open unit group if and only if (F4) is satisfied for all $k$-schemes of finite type (Proposition 5.2).

(iv) A ring is a local ring if and only if $X(R) = \bigcup_{i \in I} U_i(R)$ for all $k$-schemes $X$ and $U_i$ as in (F5) (Lemma 5.1).

(v) A local Hausdorff ring with open unit group satisfies axioms (F1)–(F5) for all $k$-schemes of finite type (Corollary 5.3).

(vi) The spectrum of maximal ideals of $R$ is totally disconnected if and only if $\Psi_R : U(R) \to X(R)$ is surjective for all $k$-schemes $X$ and $U$ as in (F6) (Theorem 6.3).

The last section of this text is concerned with locally compact topologies. Namely, if $R$ is a locally compact Hausdorff ring that satisfies (F6) for all $k$-schemes $X$ of finite type, then $X(R)$ is locally compact in the fine topology (Lemma 8.1). We conclude with a question about the connection between complete varieties and compact sets of $R$-rational points.

**Acknowledgements.** We are indebted to Brian Conrad who showed us a proof of the equivalence of the fine topology and the adelic topology.

### 1. Functoriality

Throughout the paper, let $k$ be a ring and $R$ a $k$-algebra with a topology, which we do not assume to be compatible with the ring operations unless stated explicitly. In this section, we will show that the fine topology for $X(R)$ is functorial in $X$ and $R$. We begin with recalling the corresponding fact for the affine topology.

**Lemma 1.1.** The affine topology of $X(R)$ is functorial in both $X$ and $R$ where $X$ is an affine $k$-scheme.

**Proof.** Let $X = \text{Spec} A$. By its definition, the affine topology of $X(R)$ is generated by the open subsets

$$U_{V,a} = \{ f : A \to R \mid f(a) \in V \}$$

where $a \in A$ and $V$ is an open subset of $R$. It hence suffices to show that the inverse images of subsets of the form $U_{V,a}$ are open to verify the continuity of the maps in question.

Let $Y = \text{Spec} B$ and $\varphi : B \to A$ a homomorphism of $k$-algebras that corresponds to a morphism $\varphi : X \to Y$ of affine $k$-schemes. It is easily verified that $\varphi^{-1}(U_{V,b}) = U_{\varphi^{-1}(V),\varphi(b)}$, which shows that $\varphi : X(R) \to Y(R)$ is continuous.

Let $f : R \to S$ be a continuous homomorphism of $k$-algebras with topologies. It is easily verified that $f^{-1}(U_{V,a}) = U_{f^{-1}(V),a}$, which shows that $f_X : X(R) \to X(S)$ is continuous. $\square$

**Lemma 1.2.** Let $X$ be an affine $k$-scheme. Then the affine and the fine topology for $X(R)$ coincide.

**Proof.** Since $X$ is affine, the identity morphism $\text{id} : X \to X$ yields a continuous map $\text{id}_R : X(R) \to X(R)$ w.r.t. the affine topology for the domain and the fine topology for the image. This shows that the affine topology is finer than the fine topology.

Every morphism $U \to X$ from an affine $k$-scheme $U$ to $X$ factors through the identity $\text{id} : X \to X$, and $U(R) \to X(R)$ is continuous w.r.t. the affine topology for domain and image by Lemma 1.1. Therefore the fine topology is at least as fine as the affine topology, which completes the proof of the lemma. $\square$

By virtue of this lemma, we do not have to specify the topology anymore when we are considering affine schemes. We verify the functoriality of the fine topology for $X(R)$ where $X$ can be an arbitrary $k$-scheme.

**Proposition 1.3.** The fine topology on $X(R)$ is functorial in both $X$ and $R$. 

Proof. Let \( \varphi : X \rightarrow Y \) be a morphism of \( k \)-schemes and \( \varphi_R : X(R) \rightarrow Y(R) \) the induced map. Let \( W \subset Y(R) \) be open. We have to show that \( Z = \varphi^{-1}_R(W) \) is open in \( X(R) \), which is the case if \( \alpha_R^{-1}(Z) \) is open in \( U(R) \) for every morphism \( \alpha : U \rightarrow X \) from an affine \( k \)-scheme \( U \) to \( X \).

Since \( \varphi \circ \alpha : U \rightarrow Y \) is a \( k \)-morphism from the affine scheme \( U \) to \( Y \), the inverse image 
\[
\alpha_R^{-1}(W) = (\varphi \circ \alpha)_R^{-1}(W)
\]

of \( W \) in \( U(R) \) is indeed open. This shows that the fine topology of \( X(R) \) is functorial in \( X \).

Let \( f : R \rightarrow S \) be a continuous homomorphism of \( k \)-algebras with topologies and \( f_S : X(R) \rightarrow X(S) \) the induced map. Let \( W \subset X(S) \) be open. We have to show that \( Z = f_S^{-1}(W) \) is open in \( X(R) \), which is the case if \( \alpha_R^{-1}(Z) \) is open in \( U(R) \) for every morphism \( \alpha : U \rightarrow X \) from an affine \( k \)-scheme \( U \) to \( X \).

By Lemma \( \text{[1]} \) the homomorphism \( f : R \rightarrow S \) induces a continuous map \( f_U : U(R) \rightarrow U(S) \). Since \( \alpha^{-1}_S(W) \) is open in \( U(S) \), the inverse image \( \alpha^{-1}_R(Z) = f^{-1}_U(\alpha^{-1}_S(W)) \) is open in \( U(R) \). This shows that the fine topology of \( X(R) \) is functorial in \( R \). \( \square \)

The only other property from the axioms (S0)–(S4) that survives in the generality of \( k \)-algebras with any topology and all \( k \)-schemes is the following.

Lemma 1.4. Let \( R \) be a \( k \)-algebra with topology and \( \varphi : Y \rightarrow X \) a closed immersion of affine \( k \)-schemes. Then \( \varphi_R : Y(R) \rightarrow X(R) \) is a topological embedding.

Proof. Let \( X = \text{Spec} A \) and \( Y = \text{Spec}(A/I) \) for some ideal \( I \) of \( A \). The basic closed subsets of \( Y(R) \) are of the form
\[
U_{V, \bar{a}} = \{ f : A/I \rightarrow R \mid f(\bar{a}) \in V \}
\]
where \( V \) is a closed subset of \( R \) and \( \bar{a} = a + I \) with \( a \in A \). It is clear that \( U_{V, \bar{a}} = \varphi^{-1}_R(U_{V, \bar{a}}) \). This proves the lemma. \( \square \)

Remark 1.5. The closure of \( U_{V, \bar{a}} \) in \( X(R) \) is \( U_{V, \bar{a}} \cap \bigcap_{b \in I} \{ f : A \rightarrow R \mid f(b) \in \{0\} \} \), where \( \{0\} \) is the closure of \( \{0\} \) in \( R \).

2. Topological rings

In this section, we will show that \( R \) is a topological ring if and only if it satisfies the following axioms for the class of all \( k \)-schemes.

(F1) The canonical bijection \( X \times_Z Y(R) \rightarrow X(R) \times_{Z(R)} Y(R) \) is a homeomorphism.

(F2) The canonical bijection \( \mathbb{A}_k^1(R) \rightarrow R \) is a homeomorphism.

Proposition 2.1. Given a ring \( k \) and a \( k \)-algebra \( R \) equipped with a topology. Then the following are equivalent.

(i) \( R \) is a topological ring.

(ii) \( R \) satisfies (F1) and (F2) for all schemes in \( \mathcal{C} \) where \( \mathcal{C} \) can be the class of all \( k \)-schemes, the class of all \( k \)-schemes of finite type or the class of all affine schemes of finite type over \( k \).

Proof. If \( R \) satisfies (F1) and (F2) for all \( k \)-schemes, then this is in particular true for all \( k \)-schemes of finite type. If \( R \) satisfies (F1) and (F2) for all \( k \)-schemes of finite type, then also for all affine schemes of finite type over \( k \).

Assume that \( R \) satisfies (F1) and (F2) for all affine \( k \)-schemes of finite type. Then \( \mathbb{A}_k^1(R) = R \) by (F2) and \( \mathbb{A}_k^2(R) = R \times R \) by (F1). Therefore addition and multiplication, which define morphisms \( \mathbb{A}_k^2 \rightarrow \mathbb{A}_k^1 \) of \( k \)-schemes, yield continuous maps \( R \times R = \mathbb{A}_k^2(R) \rightarrow \mathbb{A}_k^1(R) = R \) by the functoriality of the fine topology, cf. Proposition \( \text{[1]} \). This shows that \( R \) is a topological ring.

The theorem is proven once we have shown that a topological ring \( R \) satisfies (F1) and (F2) for all \( k \)-schemes. The proof of (F2) can be found in \( \text{[1]} \) Prop. 2.1. The same holds for
WY affine proof). We show that (F3) for affine
true for all affine
specializes from a more general class of schemes to a more restrictive class. If (F3) holds
(F1) and (F2) hold true for any of the considered classes. It is clear that property (F3)
topological embedding
all morphisms
of closed subschemes of an affine space coincides with the other concepts of topologies
This follows from the openness of
Proposition 3.1.
Recall that we call a
ing Hausdorff ring if and only if it satisfies the following axiom for the class of all
k-schemes. Then the fine topology coincides with the strong topology for
X
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(F1) A closed immersion
Y → X yields a closed embedding
Y(R) → X(R) of topological
 spaces.
Proposition 3.1. Given a ring k and a k-algebra R equipped with a topology. Then the
following are equivalent.
(i) R is a Hausdorff ring.
(ii) R satisfies (F1)–(F3) for all schemes in C where C can be the class of all k-
schemes, the class of all k-schemes of finite type or the class of all affine schemes
of finite type over k.
Proof. We know already from Proposition 2.1 that R is a topological ring if and only if
(F1) and (F2) hold true for any of the considered classes. It is clear that property (F3)
specializes from a more general class of schemes to a more restrictive class. If (F3) holds
true for all affine k-schemes of finite type, the diagonal \( \Delta : A^1_k \to A^2_k \) induces a closed
topological embedding \( R = A^1_k(R) \to A^2_k(R) = R^2 \), which shows that R is Hausdorff.
It is proven in [1] Prop. 2.1] that a Hausdorff ring R satisfies (F3) for the class of all
affine k-schemes. (Note that the finiteness assumption in [1] is not used in this part of
the proof). We show that (F3) for affine k-schemes implies (F3) for all k-schemes.
Let \( \varphi : Y \to X \) be a closed immersion of \( k \)-schemes and \( Z \subset Y(R) \) a closed subset. Let \( W = \varphi_*\langle Z \rangle \) be the image in \( X(R) \). We have to show that for every morphism \( \alpha : U \to X \) from an affine \( k \)-scheme \( U \) to \( X \), the inverse image \( \alpha^{-1}_R(W) \) is closed in \( U(R) \). The pullback \( \varphi^*U = U \times_X Y \) of \( U \) along \( \varphi \) is an affine \( k \)-scheme that comes with a morphism \( \alpha' : \varphi^*U \to Y \) and a closed immersion \( \varphi' : \varphi^*U \to U \). Since \( Z \) is closed in \( Y(R) \), the inverse image \( (\alpha'_R)^{-1}(Z) \) is closed in \( \varphi^*U \). By the result for closed immersions of affine \( k \)-schemes, \( \alpha_R^{-1}(W) = \varphi'_R((\alpha'_R)^{-1}(Z)) \) is closed in \( U(R) \). This shows that \( W \) is closed in \( X(R) \) and finishes the proof of property (F3).

4. RINGS WITH OPEN UNIT GROUP

Recall that we say that a \( k \)-algebra \( R \) with topology is with open unit group if the subset \( R^\times \) of units is open in \( R \) and a topological group w.r.t. the subspace topology. Note that in the case of a topological ring \( R \), the units \( R^\times \) form a topological group w.r.t. the subspace topology if and only if the inversion \( (-)^{-1} : R^\times \to R^\times \) is continuous.

In this section, we will show that a topological ring \( R \) is with open unit group if and only if it satisfies the following axiom for the class of all \( k \)-schemes.

\[ \text{(F4)} \quad \text{An open immersion } Y \to X \text{ of } k \text{-schemes yields an open embedding } Y(R) \to X(R) \text{ of topological spaces.} \]

We will need the following reduction to \( k \)-schemes of finite type.

**Lemma 4.1.** Every morphism \( \varphi : U \to X \) from an affine \( k \)-scheme \( U \) factors through an affine \( k \)-scheme \( U' \) of finite type.

**Proof.** If \( X \) is affine, then \( \varphi : U \to X \) corresponds to a morphism of \( k \)-algebras \( f : A \to B \). Since \( X \) is of finite type over \( k \), \( A \) is finitely generated over \( k \), and so is its image \( f(A) \). Thus \( U' = \text{Spec } f(A) \) is an affine \( k \)-scheme of finite type, as desired. In the general case, note that \( \varphi \) can be presented as the colimit of finitely many homomorphisms \( f : A \to B \) of \( k \)-algebras, so we obtain \( U' \) by gluing finitely many affine pieces of the form \( \text{Spec } f(A) \).

**Proposition 4.2.** Given a ring \( k \) and a \( k \)-algebra \( R \) with a topology. Then the following are equivalent.

(i) \( R \) is a topological ring with open unit group.

(ii) \( R \) satisfies (F1), (F2) and (F4) for all \( k \)-schemes of finite type.

**Proof.** We know already from Theorem 2.1 that \( R \) is a topological ring if and only if (F1) and (F2) hold true for all \( k \)-schemes of finite type. If (ii) holds true, then the open immersion \( \mathbb{G}_m \to \mathbb{A}^1_k \) yields an open map \( R^\times = \mathbb{G}_m(R) \to \mathbb{A}^1_k(R) = R \), and \( R^\times = \mathbb{G}_m(R) \) is a topological group since \( \mathbb{G}_m \) is a group scheme. This shows that \( R \) is with open unit group.

It is shown in [1, Prop. 2.1] that a topological ring with open unit group satisfies (F4) for the class of all affine \( k \)-schemes of finite type. We verify (F4) for arbitrary \( k \)-schemes of finite type in the following.

For an open immersion \( \iota : Y \to X \) of finite type \( k \)-schemes \( X \) and \( Y \), consider an open subset \( Z \) of \( Y(R) \) and its image \( W = \iota_*\langle Z \rangle \) in \( X(R) \). We have to show that \( \alpha^{-1}_R(W) \) is open in \( U(R) \) for every morphism \( \alpha : U \to X \) from an affine \( k \)-scheme \( U \) to \( X \). By Lemma 4.1, we can assume that \( U \) is of finite type over \( k \). The pullback \( \iota^*U = U \times_X Y \) of \( U \) along \( \iota \) is an affine scheme that comes with an open immersion \( \iota' : \iota^*U \to U \) and a morphism \( \alpha' : \iota^*U \to Y \). Since \( Z \) is open in \( Y(R) \), the inverse image \( (\alpha'_R)^{-1}(Z) \) is open in \( \iota^*U(R) \).

By the result for open immersions of affine \( k \)-schemes, \( \alpha_R^{-1}(W) = \iota'_R((\alpha'_R)^{-1}(Z)) \) is open in \( U(R) \). This finishes the proof of the proposition.
5. Local rings

In order to compare the fine topology with the strong topology, we are interested in cases where the fine topology on $X(R)$ can be defined in terms of an affine open covering. More precisely, we are interested in $k$-algebras $R$ with topology that satisfy the following axiom for all $k$-schemes $X$ in a class $\mathcal{C}$.

(F5) Let $\{U_i\}_{i \in I}$ be an affine open covering of $X$. Then $X(R) = \bigcup_{i \in I} U_i(R)$, and a subset $W$ of $X(R)$ is open if and only if $W \cap U_i(R)$ is open in $U_i(R)$ for all $i \in I$.

The latter property $X(R) = \bigcup_{i \in I} U_i(R)$ of (F5) depends only on the algebraic structure of $R$. Namely, we have the following characterization.

**Lemma 5.1.** Given a ring $k$ and a $k$-algebra $R$ equipped with a topology. Then the following are equivalent.

(i) $R$ is a local ring.

(ii) For every $k$-scheme $X$ and every open covering $\{U_i\}_{i \in I}$ of $X$, we have $X(R) = \bigcup_{i \in I} U_i(R)$.

*Proof.* If $R$ is a local ring, $X$ a $k$-scheme and $\{U_i\}_{i \in I}$ an open covering. Every morphism $\text{Spec} R \to X$ maps the unique closed point to a scheme-theoretic point of $X$, which is contained in one of the open subschemes $U_i$. Since generalizations are mapped to generalizations and open subsets are closed under generalizations, the image of $\text{Spec} R \to X$ must be contained in $U_i$. This shows that (i) implies (ii).

Assume that $R$ is not local. Then $R$ has two distinct maximal ideals $m_1$ and $m_2$. Let $U_1$ and $U_2$ be affine opens neighborhoods of $m_1$ and $m_2$ that are contained in the complements of the closed points $m_1$ and $m_2$ in $X = \text{Spec} R$, respectively. Then the $R$-rational point $\text{id} : \text{Spec} R \to X$ is contained in neither $U_1(R)$ nor $U_2(R)$. This shows that (ii) implies (i). □

**Proposition 5.2.** A local $k$-algebra $R$ with topology that satisfies (F4) for all affine $k$-schemes (of finite type) satisfies (F5) for all $k$-schemes (of finite type).

*Proof.* Let $X$ be a $k$-scheme and $\{i_i : U_i \to X\}_{i \in I}$ an affine open covering of $X$. Consider a subset $W$ of $X(R)$. If $W$ is open in $X(R)$, then $i_i^{-1}(W)$ is open in $U_i(R)$ for all $i \in I$ by the definition of the fine topology. Thus (F5) follows.

Assume, conversely, that $i_i^{-1}(W)$ is open in $U_i(R)$ for all $i \in I$. Consider a morphism $\alpha : U \to X$ from an arbitrary affine $k$-scheme $U$ to $X$. We have to show that $\alpha^{-1}(W)$ is open in $U(R)$. Since every morphism of $k$-schemes is locally affine, we find an affine open covering $\{V_i\}_{i \in I}$ of $U$ such that $\alpha : V_i \to X$ restricts to morphisms $\alpha_i : V_i \to U_i$ of affine $k$-schemes. Thus $i_i^{-1}(V_i^{-1}(W))$ is open in $U_i(R)$ for all $i \in I$. By Lemma 5.1. every point of $W$ is contained in some $U_i(R)$, and therefore $\alpha^{-1}(W) = \bigcup_{i \in I} \alpha_i^{-1}(i_i^{-1}(W))$, which is an open subset of $U(R)$ since $R$ satisfies (F4) for affine $k$-schemes.

In case, $R$ satisfies (F4) only for $k$-schemes of finite type, we use Lemma 4.1 to reduce the above arguments to finite type schemes $U$. Then the same proof shows that $R$ satisfies (F5) for all $k$-schemes of finite type. □

Since the axioms (F1)–(F5) determine the fine topology for $k$-schemes of finite type, this shows at once the equivalence of the fine topology with the strong topology in case of a local Hausdorff ring $R = k$ with open unit group.

**Corollary 5.3.** Let $k$ be a local Hausdorff ring with open unit group and $X$ a $k$-scheme of finite type. Then the fine topology of $X(k)$ coincides with strong topology, and $k$ satisfies (F1)–(F5) for all $k$-schemes of finite type. □

**Example 5.4** (Zariski topology). Let $k$ be a field with the topology of finite closed subsets. Then the fine topology for $X(k)$ for a $k$-scheme $X$ of finite type is equal to the Zariski topology. This can be seen as follows.
In the affine case $X = \text{Spec}(k[T_1, \ldots, T_n]/I)$, a basic closed subset is of the form

$$U_{\mathcal{T}_I} = \{ f : k[T_1, \ldots, T_n]/I \to k \mid f(\mathcal{T}) \in \{ c \} \}$$

where $\mathcal{T} = P + I$ is the class of a polynomial $P \in k[T_1, \ldots, T_n]$ and $c \in k$. If $a_i = f(T_i)$, then $f(\mathcal{T}) = P(a_1, \ldots, a_n)$. Thus $f(\mathcal{T}) \in \{ c \}$ means that $P(a_1, \ldots, a_n) - c = 0$, and $U_{\mathcal{T}_I}$ corresponds with the set $V(P - c)(k)$ of $k$-rational points of the vanishing set of $P - c$. If, conversely, $V(J)$ is a closed subscheme of $X$, defined by an ideal $J$, then

$$V(J)(k) = \bigcap_{\mathcal{T} \in J} U_{\mathcal{T}_0}.$$ 

This shows that the fine topology is equal to the Zariski topology for affine $k$-schemes of finite type.

Since a morphism $U \to X$ of $k$-schemes of finite type induces a continuous map $U(k) \to X(k)$ w.r.t. Zariski topology, the fine topology is finer than the Zariski topology. Since the Zariski topology is the finest topology such that the inclusions $U_i \to X$ for a fixed affine open covering $\{ U_i \}$ of $X$ yield continuous maps $U_i(k) \to X(k)$, the fine topology for $X(k)$ is indeed equal to the Zariski topology.

Note that if $k$ is infinite, it is neither a topological ring nor Hausdorff nor with an open unit group. The only property that remains valid from the ones that we are investigating in this text is that $k^\times$ is open in $k$. However, $k$ satisfies axioms (S2)–(S5) for all $k$-schemes of finite type; only (S1) does not hold.

**Example 5.5** (The patchwork topology). Let $X$ be a $k$-scheme of finite type. If $R$ is a Hausdorff ring with open unit group, i.e. $R$ satisfies (F1)–(F4) for all $k$-schemes, then we could attempt to define a *patchwork topology* for $X(R)$: we say that $W \subset X(R)$ is patchwork open if and only if $W \cap U(R)$ is affine open in $U(R)$ for all affine open subschemes $U$ of $X$.

Without requiring $R$ to be local, this definition entails the following problems.

First of all, the patchwork topology cannot be defined by a single fixed covering $\{ U_i \}$ of $X$ since there might be $R$-rational points that are not contained in any of the $U_i(R)$, but in $U(R)$ for some affine open $U$ of $X$ that is not among the $U_i$. This ambiguity can be resolved by considering the maximal atlas $\{ U_i \}$ of all affine open subschemes of $X$.

The more serious problem is the following. Let $k$ be a field. Then there are varieties $X$ over $k$ with pairs of $k$-rational points $\alpha$ and $\beta$ that are not contained commonly in any affine subset of $X$. Arguably the most prominent example of such a variety is Hironaka’s threefold, see [4].

Let $R = k \times k$ together with a topology. If, for instance, $k$ is a topological field that is Hausdorff and $R$ has the product topology, then $R$ is a Hausdorff ring with open unit group. The $R$-rational points of $X$ are pairs of $k$-rational points. By the mentioned property of $X$ w.r.t. the $k$-rational points $\alpha$ and $\beta$, the $R$-rational point $(\alpha, \beta)$ is not contained in $U(R)$ for any affine open subscheme $U$ of $X$. By the definition of the patchwork topology, the point $(\alpha, \beta)$ is thus an isolated point in $X(R)$.

By Chow’s lemma, there exists a birational map $\mathbb{P}^n \to X$ for some $n \geq 0$. Unless $R$ is discrete, $\mathbb{P}^n(R)$ does not have isolated components. Therefore the induced map $\mathbb{P}^n(R) \to X(R)$ cannot be continuous in the patchwork topology.

6. Rings with totally disconnected spectrum

In this section, we consider a weaker version of axiom (F5) that still allows us to deduce the topology of $X(R)$ from an affine open covering of $X$, but that applies to a wider class of rings than (F5). In particular, we will show in section [7] that the adèle ring of a global field satisfies the following axiom.

Namely, we are interested in $k$-algebras $R$ with topology that satisfy the following axiom for all $k$-schemes $X$ and $U_i$ in a class $\mathcal{C}$. 

(F6) Let \( \{ U_i \}_{i \in I} \) be a finite affine open covering of \( X \) and \( U = \bigsqcup_{i \in I} U_i \). Let \( \Psi : U \to X \) be the associated morphism. Then the map \( \Psi_R : U(R) \to X(R) \) is surjective and open.

Note that (F6) implies the following description of open subsets of \( X(R) \) w.r.t. to an arbitrary affine open covering of \( X \).

**Lemma 6.1.** Suppose that \( R \) satisfies (F6) for the all \( k \)-schemes. Let \( \{ \iota_i : U_i \to X \}_{i \in I} \) be an affine open covering of \( X \) and denote by \( \Psi_J : U_J \to X \) the induced morphism from the disjoint union \( U_J = \bigsqcup_{i \in J} U_i \) to \( X \) where \( J \) is a finite subset of \( I \). Then a subset \( W \) of \( X(R) \) is open if and only if \( \Psi_{J,R}'(W) \) is open in \( U_J(R) \) for every finite subset \( J \) of \( I \).

**Proof.** Clearly the inverse image of an open affine \( W \) of \( X(R) \) is open in any \( U_J(R) \). Assume conversely that the inverse image of a subset \( W \) of \( X(R) \) is open in \( U_J(R) \) for any finite subset \( J \) of \( I \). We will show that \( W \) is open. For this purpose, let \( \alpha : V \to X \) be a morphism from an affine \( k \)-scheme \( V \) to \( X \). Since \( V \) is compact, the image of \( \alpha \) is contained in the union \( X_I = \bigsqcup_{i \in I} U_i \) of finitely many \( U_i \). Therefore \( \alpha^{-1}(W) = \alpha^{-1}(W') \) where \( W' = W \cap X_I(R) \). For \( W' \) and the finite covering \( \{ U_i \}_{i \in I} \) of \( X_I \), we can apply axiom (F6) to conclude that \( W' \) is fine open in \( X_I(R) \) and thus \( \alpha^{-1}(W) \) open in \( V(R) \). \( \square \)

The surjectivity of \( \Psi_R : U(R) \to X(R) \) in axiom (F6) depends on a purely algebraic property of \( R \). We will investigate this in the following.

We say that a ring \( R \) is **with totally disconnected spectrum** if every two closed points of \( \text{Spec} R \) have respective open neighborhoods \( U_1 \) and \( U_2 \) such that \( \text{Spec} R = U_1 \sqcup U_2 \) as a topological space. This is equivalent to the condition that the subspace \( \text{Spec} \max R \) of closed points of \( \text{Spec} R \) is totally disconnected.

**Remark 6.2.** Since a specialization \( y \) of a point \( x \) in \( \text{Spec} R \) cannot be separated from \( x \) by an open neighborhood, \( \text{Spec} R \) is typically not totally disconnected as a topological space. Therefore, the above definition makes only sense if we consider closed points of \( \text{Spec} R \).

Examples of rings with totally disconnected spectrum are local rings or products of local rings, as can be deduced from condition (iii) in the following theorem. For the same reason, adele rings are with totally disconnected spectrum.

**Theorem 6.3.** The following conditions on \( R \) are equivalent.

(i) \( R \) is with totally disconnected spectrum.

(ii) Let \( X \) be a \( k \)-scheme with a finite affine open covering \( \{ U_i \} \). Let \( U = \bigsqcup U_i \) and \( \Psi : U \to X \) the induced morphism. Then \( \Psi_R : U(R) \to X(R) \) is surjective.

(iii) For every equality in \( R \) of the form \( 1 = h_1 + \cdots + h_n \), there are idempotent elements \( e_i \in (h_i) \) such that \( 1 = e_1 + \cdots + e_n \).

**Proof.** We show that (ii) implies (iii). Let \( X \) be a \( k \)-scheme and \( \{ U_i \} \) a finite affine open covering. Let \( U = \bigsqcup U_i \) and \( \Psi : U \to X \) the induced morphism. A point of \( X(R) \) is a morphism \( \alpha : \text{Spec} R \to X \). We want to show that \( \alpha \) factors through \( U \).

There exists a finite affine open covering \( \{ V_j \} \) of \( \text{Spec} R \) and a map \( j \mapsto i(j) \) between the indices of the \( V_j \) and the \( U_i \), respectively, such that \( \alpha \) restricts to morphisms \( \alpha : V_j \to U_{i(j)} \) for all \( j \). After relabeling indices and counting the \( U_i \) multiple times if necessary, we might assume that \( i = j \). We claim that we can refine the covering \( \{ V_j \} \) of \( \text{Spec} R \) to an affine open covering \( \{ V_j' \} \) of \( \text{Spec} R \) such that \( \text{Spec} R = \bigsqcup V_j' \). Once we know this, we can conclude that \( \alpha : \text{Spec} R \to X \) factors through \( \bigsqcup V_j' \to \bigsqcup U_i = U \).

We prove the claim by induction on the number \( n \) of open subsets \( V_i \) of \( \text{Spec} R \) where we allow \( X \) to vary. For \( n = 1 \), there is nothing to prove. We proceed with the case \( n = 2 \), which is the critical step in our induction.

In this case, we have affine open coverings \( \text{Spec} R = V_1 \cup V_2 \) and \( X = U_1 \cup U_2 \) with restrictions \( \alpha : V_i \to U_i \) for \( i = 1, 2 \). Let \( Z_1 \) and \( Z_2 \) be the respective complements of \( V_1 \) and \( V_2 \) in \( Z = \text{Spec} \max R \), which are Zariski closed subsets. We have to show that there exist...
neighborhoods $Z'_1$ of $Z_1$ in $Z$ such that $Z = Z'_1 \amalg Z'_2$. If both $Z_1$ and $Z_2$ consist of a single closed point of $\text{Spec} R$, then this follows from (iii) by the definition of a totally disconnected topological space.

Assume that $Z_1$ consists of a single point $x$, and $Z_2$ is an arbitrary closed subset of $Z$. Then there exist neighborhoods $Z'_{1,x,y}$ of $x$ and $Z'_{2,x,y}$ of $y$ with $Z = Z'_{1,x,y} \amalg Z'_{2,x,y}$ for every $y \in Z_2$. Since the closed subset $Z_2$ of the compact space $Z$ is compact, $Z_2$ is covered by finitely many of the $Z'_{2,x,y}$, say, by $Z'_{2,x,y_k}$ for $k = 1, \ldots, r$. Then $Z'_{1,x} = \bigcap Z'_{1,x,y_k}$ is a neighborhood of $x$ and $Z'_{2,x} = \bigcup Z'_{2,x,y_k}$ is a neighborhood of $Z_2$ and $Z = Z'_{1,x} \amalg Z'_{2,x}$, as desired.

We consider the general case of two closed subsets $Z_1$ and $Z_2$ of $Z$. Then there are neighborhoods $Z'_{1,x}$ of $x$ and $Z'_{2,x}$ of $Z_2$ with $Z = Z'_{1,x} \amalg Z'_{2,x}$ for every $x \in Z_1$. Since $Z_1$ is compact, we can cover it with finitely many of $Z'_{1,x}$, say, with $Z'_{1,l}$ for $l = 1, \ldots, s$. Then $Z_1 = \bigcup Z'_{1,l}$ is a neighborhood of $Z_1$ and $Z_2 = \bigcap Z'_{2,l}$ is a neighborhood of $Z_2$ and $Z = Z_1 \amalg Z_2$ as desired. This completes the case $n = 2$.

Let $n > 2$. Assume that the indices of the $V_i$ are $i = 1, \ldots, n$. We have that the covering $V_1 \cup V_2$ with $V_{i+2} = V_2 \cup \cdots \cup V_n$ has a refinement $V'_i \subset V_i$ and $V'_{i+2} \subset V_{i+2}$ such that $\text{Spec} R = V'_1 \amalg V'_{n+2}$. By the induction hypothesis, there is a refinement $\{V''_i\}$ for the covering $\{V_2, \ldots, V_n\}$ of $V_2$, such that $V_{i+2} = V''_i \amalg \cdots \amalg V''_n$. If we define $V'_i = V''_i \cap V'_{i+2}$, then $X = V'_1 \amalg \cdots \amalg V'_n$ as desired. This shows that (iii) follows from (i).

We continue with the implication (ii) to (iii). An equality $1 = h_1 + \cdots + h_n$ corresponds to the covering of $X = \text{Spec} R$ by principal opens $U_i = \text{Spec} R[h_i^{-1}]$. Let $U = \bigcup U_i$ and $\Psi : U \to X$ the induced morphism. The identity map $\text{id} : \text{Spec} R \to \text{Spec} R$ is a point of $X(R)$. By (iii), it factors through a morphism $\text{Spec} R \to U$, which is only possible if there exists a refinement $\{V_i\}$ of $\{U_i\}$ such that $\text{Spec} R = \bigcup V_i$. If $R_i$ is the coordinate ring of $V_i$, then we have that $R = \bigcap R_i$. If we denote the image of $1$ under $R_i \to R$ by $e_i$, which is an idempotent element of $R$, then we have $V_i = \text{Spec} R_i = \text{Spec} R[e_i^{-1}]$. Since $X = \bigcup V_i$, we have $1 = e_1 + \cdots + e_n$ as desired. This shows (iii).

We conclude with the implication (iii) to (ii). Consider two closed points $x$ and $y$ in $Z = \text{Spec} \text{max} R$, which are maximal ideals of $R$. We have to find respective neighborhoods $Z_1$ and $Z_2$ with $Z = Z_1 \amalg Z_2$. To do so, we consider two elements $h_1 \in x$ and $h_2 \in y$ with $1 = h_1 + h_2$. By (iii), there are idempotent elements $e_1 \in (h_1)$ and $e_2 \in (h_2)$ with $1 = e_1 + e_2$. Since $e_1 \in x$ and $e_2 \in y$, the principal open subsets $Z'_1 = \text{Spec} \text{max} R[e_i^{-1}]$ $(i = 1, 2)$ are neighborhoods of $x$ and $y$, respectively. Since $e_1 + e_2 = 1$, we have $Z = Z'_1 \amalg Z'_2$. This shows that $\text{Spec} R$ is totally disconnected and finishes the proof of the theorem.

7. The adelic topology

In this section, we shall show that the fine topology coincides with the adelic and the $S$-adic topology when $R$ is the adèle ring $A$ of a global field $k$ or the $S$-adèle ring $A_S$, respectively.

Let $S$ be a finite set of places of $k$ containing all the archimedean ones and let $\mathcal{O}_S$ be the $S$-integers of $k$. We consider a finite type $\mathcal{O}_S$-scheme $X_S$ and a finite affine open covering $\{U_{S,i}\}$. Let $U_S = \bigcup U_{S,i}$ and $\Psi_S : U_S \to X_S$ the induced morphism.

**Lemma 7.1.** The induced map $\Psi_{S,A_S} : U_S(A_S) \to X_S(A_S)$ is continuous, surjective and open w.r.t. the $S$-adic topology.

**Proof.** By the functoriality of the $S$-adic topology, $\Psi_A$ is continuous. By Theorem 6.3, $\Psi_{A_S}$ is surjective.

For a place $v$ of $k$, we define $R_v$ as the completion $k_v$ of $k$ at $v$ if $v \in S$ and as the integers $\mathcal{O}_v$ of $k_v$ if $v \notin S$. Since $R_v$ is a local Hausdorff ring with open unit group, we know by Corollary 5.3 that the fine topology coincides with the strong topology for $X_S(R_v)$.

**By Theorem 6.3,** $\Psi_{S,R_S} : U_S(R_S) \to X_S(R_S)$ is continuous, surjective and open in the fine topology, which coincides with the strong topology in this case. Therefore the product
the induced map $Ψ_A : U(A) → X(A)$ is open w.r.t. the adelic topology.

Proof. By the functoriality of adelic topologies, $Ψ_A$ is continuous. By Theorem 6.3, $Ψ_A$ is surjective.

Let $Z$ be an open subset of $U(A)$. We want to show that $W = Ψ_A(Z)$ is open in $X(A)$. Choose a finite subset $S$ of places containing all the archimedean ones and an $𝒪_S$-model $Ψ_S : U_S → X_S$ of $Ψ : U → X$ where $U_S$ and $X_S$ are finite type $𝒪_S$-models of $U$ and $X$, respectively. For finite sets $S'$ of places containing $S$, we denote by $ι_{U,S'} : U_S → U(A)$ and $ι_{X,S'} : X_S → X(A)$ the canonical maps induced by $𝒪_{S'} → A$.

By the definition of the adelic topology, $W$ is adelic open in $U(A)$ if and only if $W_{S'} = ι_{U,S'}^{-1}(W)$ is $S'$-adelic open in $U_S(A_{S'})$ for all finite $S'$ containing $S$. By Lemma 7.1, the image $Z_{S'} = Ψ_A(ι_{U,S'}^{-1}(W))$ is $S'$-adelic open in $X_S(A_{S'})$ for all $S'$. Since $Z_{S'} = ι_{X,S'}^{-1}(W)$, we conclude that $W$ is adelic open in $X(A)$. This completes the proof of the lemma. □

Theorem 7.3. Let $k$ be a global field, $S$ a finite set of places containing all the archimedean ones and $𝒪_S$ the $S$-integers.

(i) Let $A_S$ be the $S$-adèles of $k$ and $X_S$ a $𝒪_S$-scheme of finite type. Then the fine topology and the $S$-adic topology for $X_S(A_S)$ coincide.

(ii) Let $A$ be the adèles of $k$ and $X$ a $k$-scheme of finite type. Then the fine topology and the adelic topology for $X(A)$ coincide.

Proof. Since the proofs of (i) and (ii) are completely analogous, we present only the proof of (i). We consider an adelic open subset $W$ of $X(A)$ and want to show that for every morphism $α : U → X$ from an affine $k$-scheme $U$ to $X$, the inverse image $Z = α_A^{-1}(W)$ is fine open in $U(A)$. By Lemma 4.1, we can restrict ourselves to affines $U$ of finite type over $k$. Since the adelic topology is functorial in finite type $k$-schemes, $Z$ is adelic open in $U(A)$, and by Corollary 7.4, it is affine open. This shows that $W$ is fine open in $X(A)$.

Assume conversely that $W$ is fine open in $X(A)$. Choose a finite affine open covering $\{U_i\}$ of $X$, let $U = \coprod U_i$ be the disjoint union and $Ψ : U → X$ the induced morphism. Then $Z = Ψ_A^{-1}(W)$ is fine open in $U(A)$, and therefore adelic open by Corollary 7.4. By Lemma 7.2, $W = Ψ_A(Z)$ is adelic open in $X(A)$. This finishes the proof of (i). □

Corollary 7.4. Both $A_S$ and $A$ satisfy axiom (F6) for the class of all $k$-schemes of finite type.

Proof. Axiom (F6) follows from Lemma 7.1 and part (i) of Theorem 7.3 for $A_S$, and from Lemma 7.2 and part (ii) of Theorem 7.3 for $A$. □

8. Locally compact rings

In this concluding section, we point out that axiom (F6) is enough to ensure that a locally compact Hausdorff ring defines a functor from the category of $k$-schemes of finite type to the category of locally compact topological spaces.

Lemma 8.1. If $R$ is a locally compact Hausdorff ring over $k$ with (F6), then $X(R)$ is locally compact for every $k$-scheme $X$ of finite type.

Proof. It is shown in [1] Prop. 2.1] that $X(R)$ is locally compact if $X$ is an affine $k$-scheme of finite type. For a general $k$-scheme $X$ of finite type, we choose a finite affine open covering $\{U_i\}$ and let $Ψ : U → X$ be the associated morphism. We want to find a compact
neighborhood of a point $\alpha \in X(R)$. By (F6), there is a point $\alpha' \in U(R)$ such that $\Psi_R(\alpha') = \alpha$. By the result for affine $k$-schemes, $\alpha'$ has a compact neighborhood $Z$, and by (F6) and the continuity of $\Psi_R$, the image $W = \Psi_R(Z)$ is a compact neighborhood of $\alpha$ in $X(R)$. □

A natural task is to study the relation between completions of varieties and compactifications of the sets $X(R)$. It is a classical theorem that a complex variety $X$ is complete if and only if $X(\mathbb{C})$ is compact in the strong topology (cf. [5]). More generally for a local field $k$, a $k$-variety $X$ is complete if and only if for all finite field extensions $K$ of $k$, the set $X(K)$ is compact in its strong topology (cf. [5]). The proof in [5] can easily adopted to a proof for adelic points: a variety $X$ over a global field $k$ is complete if and only if the set $X(\mathbb{A}_K)$ is compact for the adèles $\mathbb{A}_K$ of every finite field extension $K$ of $k$.

We pose the following

**Question.** Let $R$ be a locally compact Hausdorff ring over $k$ with (F6). Assume that $\mathbb{P}^1(R)$ is compact, but that $R$ is not. Is it true that a morphism $X \to \text{Spec} \, k$ is proper if and only if $X(S)$ is compact for all continuous homomorphisms $R \to S$ into a locally compact Hausdorff ring $S$ with (F6)?

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