Unextendible product bases and locally unconvertible bound entangled states

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(Dated: October 31, 2018)

PACS numbers: 03.65.Bz, 03.67.-a, 89.70.+c

I. INTRODUCTION AND SUMMARY OF RESULTS

One of the most challenging problems in the field of quantum information is to understand what transformations of multipartite pure states has been extensively studied. In the case of pure states LOCC transformation reduces to local filtering operations [12]. The question of convertibility by local filtering operations (LFO) can be investigated using the theory of normal forms and entanglement monotones which are invariant under LFO, see Ref. [4].

Conditions for stochastic convertibility of mixed states are less understood. It was conjectured in Ref. [5] that a two-qubit mixed state $\rho_S$ can be stochastically converted into a state $\rho_T$ iff the Bell-diagonal normal form of $\rho_T$ equals to convex sum of some separable state and the Bell-diagonal normal form of $\rho_S$.

In this paper we consider only mixed states of a very special form, namely bound entangled states associated with unextendible product bases in a system of three qubits. The notion of unextendible product basis (UPB) was originally introduced in Ref. [6].

In our setting UPB is a family of vectors $|S_1\rangle, \ldots, |S_n\rangle \in (\mathbb{C}^2)^\otimes 3$ such that

- Each vector $|S_j\rangle$ has a product form $|S_j\rangle = |A_j\rangle \otimes |B_j\rangle \otimes |C_j\rangle$ for some one-qubit states $|A_j\rangle$, $|B_j\rangle$, and $|C_j\rangle$.
- $\langle S_i | S_j \rangle = \delta_{ij}$ for all $i$ and $j$,
- The orthogonal complement to the space spanned by the vectors $|S_1\rangle, \ldots, |S_n\rangle$ does not contain product vectors.

The vectors $|S_j\rangle$ are refered to as members of the UPB. If $S = \{|S_j\rangle\}$ is a UPB, the linear space spaned by its members will be denoted $\mathcal{H}_S$. A mixed state $\rho_S$ associated with a UPB $S$ is defined as a properly normalized projector onto an orthogonal complement of $\mathcal{H}_S$. In our setting it is

$$\rho_S = \frac{1}{2^n} - \frac{1}{n} \left( I - \sum_{j=1}^n |S_j\rangle \langle S_j| \right).$$

(1)

It can not be a separable state, since its range does not contain product vectors. However, as was pointed out in...
Ref. [8], in the case of three qubits $\rho_S$ is separable with respect to any bipartite cut, for instance $ABC = A \cup BC$ (we will use letters A, B, and C to label the qubits). In particular it is not possible to distill some pure entanglement between any two qubits starting from many copies of $\rho_S$ and applying only LOCC transformations. For that reason $\rho_S$ is referred to as a bound entangled state.

In section [II] we build a complete classification of three-qubit UPBs. It appears that each UPB has exactly four members. By local unitaries and permutations of the members we can bring any UPB into the following form:

$$
\begin{align*}
|S_1\rangle &= |0\rangle \otimes |0\rangle \otimes |0\rangle, \\
|S_2\rangle &= |1\rangle \otimes |B\rangle \otimes |C\rangle, \\
|S_3\rangle &= |A\rangle \otimes |1\rangle \otimes |C^\perp\rangle, \\
|S_4\rangle &= |A^\perp\rangle \otimes |B^\perp\rangle \otimes |1\rangle,
\end{align*}
$$

(2)

where

$$
\begin{align*}
|A\rangle\langle A| &= (1/2) \left[ I + \cos (\theta_A) \sigma^z + \sin (\theta_A) \sigma^x \right], \\
|B\rangle\langle B| &= (1/2) \left[ I + \cos (\theta_B) \sigma^z + \sin (\theta_B) \sigma^x \right], \\
|C\rangle\langle C| &= (1/2) \left[ I + \cos (\theta_C) \sigma^z + \sin (\theta_C) \sigma^x \right].
\end{align*}
$$

(3)

(Here and throught the paper we use a designation $|A\rangle$ for a state orthogonal to the state $|A\rangle$.) The family given by Eq. (2) specifies a UPB for an arbitrary triple $\theta_A, \theta_B, \theta_C \neq 0 \mod \pi$. However some triples must be identified since the for each UPB can be matched by local unitaries and permutation of the UPB’s members (in this case the associated states are related by local unitaries). We show that the “fundamental” region of parameters corresponds to $\theta_A, \theta_B, \theta_C \in (0, \pi)$. Restricting ourselves to this region we count each UPB exactly one time.

Our main conclusion is that stochastic approximate LOCC conversion (in either direction) of states $\rho_S$ and $\rho_T$ associated with UPBs $S$ and $T$ is impossible, unless $\rho_S$ and $\rho_T$ are related by local unitaries. Following Ref. [7] we obtain the necessary conditions for LOCC convertibility finding the necessary conditions for convertibility by separable superoperators. So the statement which we have actually proved is following:

**Theorem 1.** Suppose $S$ and $T$ are three-qubit UPBs which are not related by local unitary operators and permutation of the members. Let $\rho_S$ and $\rho_T$ be the bound entangled states associated with $S$ and $T$. There exists a finite precision $\epsilon(S, T) > 0$ such that for any separable completely positive superoperator $E$ satisfying $E(\rho_S) \neq 0$ one has

$$
F\left(\rho_T, \frac{E(\rho_S)}{\text{Tr}[E(\rho_S)]}\right) \leq 1 - \epsilon(S, T).
$$

The paper is organized in the following way. In section [II] we build a classification of three-qubit UPBs and reveal some useful facts about them. The most important fact is Lemma [5] which says that members of UPB are the only product vectors in the spanning space of a UPB.

In section [IV] we address stochastic (exact) convertibility of mixed states associated with UPBs. In section [V] we examine a simplified version of the problem, namely approximate convertibility by local filtering operations. Section [VI] contains a proof of Theorem [IV]. In Conclusion we summarize the results obtained in the paper and discuss possible application of our method to UPBs in a system $\mathbb{C}^3 \otimes \mathbb{C}^3$.

**II. UNEXTENDIBLE PRODUCT BASES FOR THREE QUBITS**

In this section we put forward a complete classification of UPBs for a system of three qubits and prove some useful facts about them. The qubits will be referred to as A, B, C. Let $|S_j\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ be members of UPB:

$$
|S_j\rangle = |A_j\rangle \otimes |B_j\rangle \otimes |C_j\rangle, \quad j = 1, \ldots, n.
$$

(4)

By definition $\langle S_i|S_j\rangle = \delta_{ij}$ for all $i$ and $j$. Let $H_S$ be a linear $n$-dimensional space spanned by the basis vectors $S_j$ and $\rho_S$ be a density operator proportional to a projector onto orthogonal complement of $H_S$, see Eq. (2). As an example consider a UPB ”Shifts” suggested in Ref. [6]. Its members are

$$
\begin{align*}
|S_1\rangle &= |0\rangle \otimes |0\rangle \otimes |0\rangle, \\
|S_2\rangle &= |1\rangle \otimes |0\rangle \otimes |0\rangle, \\
|S_3\rangle &= |0\rangle \otimes |1\rangle \otimes |0\rangle, \\
|S_4\rangle &= |0\rangle \otimes |0\rangle \otimes |1\rangle.
\end{align*}
$$

(Here we use a standard notations $|\pm\rangle = 2^{-1/2}(|0\rangle \pm |1\rangle)$.) If some product vector $|S\rangle = |A\rangle \otimes |B\rangle \otimes |C\rangle$ extends this basis, then one of vectors $|A\rangle, |B\rangle, |C\rangle$ must be orthogonal to at least two vectors from the family $\{|0\rangle, |1\rangle, |+\rangle, |\rangle\rangle\}$, which is impossible.

An important characteristic of an UPB is its orthogonality graph, see Ref. [8]. For our purposes it will be convenient to introduce more general definition of an orthogonality graph.

**Definition 1.** Let $M = \{\Psi_1, \ldots, \Psi_N\} \in \mathcal{H}$ be a family of vectors in a finite-dimensional Hilbert space $H$. An orthogonality graph $G = (V, E)$ of the family $M$ has a set of vertices $V = \{1, \ldots, N\}$ with one vertex assigned to each state $|\Psi_j\rangle$. A pair of vertices $(i, j)$ is an edge, $(i, j) \in E$, iff $\langle \Psi_i | \Psi_j \rangle = 0$.

For example, an orthogonality graph of any family of vectors in $\mathbb{C}^2$ is a collection of disjoint edges and isolated vertices. This definition will help us to prove the following statement.

**Lemma 1.** Any three-qubit UPB has four members.

**Proof.** First of all we note that the number of members $n$ cannot be greater than 5. Indeed, if $n \geq 6$, we have $Rk(\rho_S) \leq 2$. From Ref. [8] we know that a PPT three-qubit state with a rank two or one must be separable.
But $\rho_S$ is not separable by definition. Let us prove that an existence of a UPB with five members also leads to a contradiction. Let $S$ be a such UPB with members given by Eq. (4). Consider orthogonality graphs $G_A = (V, E_A)$, $G_B = (V, E_B)$, and $G_C = (V, E_C)$ for the families $\{|A_j\}$, $\{|B_j\}$, and $\{|C_j\}$ respectively. Here $V = \{1, \ldots, 5\}$ and

\[
(i, j) \in E_A \text{ iff } \langle A_i | A_j \rangle = 0, \\
(i, j) \in E_B \text{ iff } \langle B_i | B_j \rangle = 0, \\
(i, j) \in E_C \text{ iff } \langle C_i | C_j \rangle = 0,
\]

Orthogonality of the basis vectors $\langle S_i | S_j \rangle = \delta_{ij}$ implies that any pair of vertices $(i, j)$ belongs to at least one of the sets $E_A, E_B, E_C$. Thus

\[
|E_A| + |E_B| + |E_C| \geq \frac{n(n-1)}{2} = 10
\]

and at least one of the sets $E_A, E_B, E_C$ contains 4 or more edges. Assume that $|E_A| \geq 4$ and focus on the graph $G_A$. Observe that $G_A$ can not have a vertex with a valence 3 or greater. Indeed, if, say, $(1,2) \in E_A$, $(1,3) \in E_A$, and $(1,4) \in E_A$ then a state $|S_6\rangle = |A_1\rangle \otimes |B_4\rangle \otimes |C_3\rangle$ extends a basis which is impossible. Besides, $G_A$ can not contain cycles with odd number of edges (this constraint comes from two-dimensional geometry). Summarizing, $G_A$ must match the following restrictions:

- there are at least 4 edges
- a valence of any vertex may be 0,1,2 only
- there are no odd cycles

Up to permutations of the vertices there are only two graphs satisfying all the restrictions. They are shown on FIG. 1. For the graph on the left the basis can be extended by a state $|S_6\rangle = |A_2\rangle \otimes |B_4\rangle \otimes |C_3\rangle$. For the graph on the right the basis can be extended by a state $|S_6\rangle = |A_2\rangle \otimes |B_3\rangle \otimes |C_4\rangle$. Thus an assumption that UPB contains 5 members leads to a contradiction.

For our purposes the order of UPB members will not be important. Besides we would like to identify UPBs which can be matched by local unitary operators. Let us introduce the following equivalence relation:

**Definition 2.** UPBs $S$ and $S'$ are equivalent, $S \sim S'$, iff $|S_j\rangle = U_A \otimes U_B \otimes U_C |S'_j\rangle$, $j = 1, 2, 3, 4$, for some unitary operators $U_A, U_B, U_C$ and some permutation $\sigma \in S_4$.

We will see later (corollary to Lemma 3) that UPBs $S$ and $S'$ are equivalent iff the corresponding states $\rho_S$ and $\rho_{S'}$ are related by local unitaries. Classification of UPB with 4 members is given by

**Lemma 2.** Each equivalence class of UPBs has a representative of the form

\[
|S_1\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle, \\
|S_2\rangle = |1\rangle \otimes |B\rangle \otimes |C\rangle, \\
|S_3\rangle = |A\rangle \otimes |1\rangle \otimes |C\rangle, \\
|S_4\rangle = |A\rangle \otimes |B\rangle \otimes |1\rangle. 
\] (5)

For arbitrary $|A\rangle, |B\rangle, |C\rangle \notin \{0,1\}$ the set of states given by Eq. (6) is a UPB.

**Proof.** Suppose $S = \{|S_j\rangle\}$ is a UPB with four members given by Eq. (4). The unextendability implies that neither of the sets $\{|A_j\}$, $\{|B_j\}$, $\{|C_j\}$ can contain two equal states. Indeed, if, say, $|A_1\rangle = |A_2\rangle$, then a state $|S_6\rangle = |A_1\rangle \otimes |B_4\rangle \otimes |C_3\rangle$ extends the basis. One remains to take into account mutual orthogonality of the basis members. Applying local unitary operators we can make $|S_1\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle$. It follows that each of $|S_2\rangle$, $|S_3\rangle$, $|S_4\rangle$ contains at least one factor $|1\rangle$. Since neither of the sets $\{|A_j\}$, $\{|B_j\}$, $\{|C_j\}$ can contain two copy of $|1\rangle$, by appropriate permutation of the members $|S_2\rangle$, $|S_3\rangle$, $|S_4\rangle$ we can always make

\[
|S_1\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle, \\
|S_2\rangle = |1\rangle \otimes |B\rangle \otimes |C\rangle, \\
|S_3\rangle = |A\rangle \otimes |1\rangle \otimes |C\rangle, \\
|S_4\rangle = |A\rangle \otimes |B\rangle \otimes |1\rangle. 
\] (6)

Orthogonality $\langle S_2 | S_3 \rangle = 0$ translates into $\langle C_2 | C_3 \rangle = 0$. Analogously, $\langle S_2 | S_4 \rangle = 0$ implies that $\langle B_2 | B_4 \rangle = 0$ and $\langle S_3 | S_4 \rangle = 0$ gives us $\langle A_3 | A_4 \rangle = 0$. Introducing $|A\rangle = |A_3\rangle$, $|B\rangle = |B_2\rangle$, and $|C\rangle = |C_2\rangle$ we arrive to a representation declared in Eq. (6).

Consider an orthogonal basis $\{|S_j\}\rangle$ as in Eq. (5) and suppose that $|A\rangle, |B\rangle, |C\rangle \notin \{0,1\}$. Assume that a state $|a\rangle \otimes |b\rangle \otimes |c\rangle$ extends the basis. It is orthogonal to each of $|S_j\rangle$, $j = 1, 2, 3, 4$ either on the qubit $A$, or on the qubit $B$, or on the qubit $C$. But each of the states $|a\rangle$, $|b\rangle$, $|c\rangle$ may provide orthogonality to at most one member of the family $\{|S_j\}\rangle$. Thus the basis is unextendible.

Let us count, how many real parameters we need to parameterize equivalence classes of UPBs. Applying local phase shifts to the qubits $A$, $B$, $C$, we can always make

\[
|A\rangle \langle A| = (1/2)(I + \cos (\theta_A) \sigma^z + \sin (\theta_A) \sigma^\tau), \\
|B\rangle \langle B| = (1/2)(I + \cos (\theta_B) \sigma^z + \sin (\theta_B) \sigma^\tau), \\
|C\rangle \langle C| = (1/2)(I + \cos (\theta_C) \sigma^z + \sin (\theta_C) \sigma^\tau). 
\] (7)
Different triples \((\theta_A, \theta_B, \theta_C)\) may still represent the same equivalence class. For example, applying \(\sigma^z\) to the qubit \(A\), we should identify \(\theta_A\) and \(-\theta_A\) (analogously, \(\theta_B \equiv -\theta_B\) and \(\theta_C \equiv -\theta_C\)). Thus all equivalence classes of UPBs are contained in the region \(\theta_A, \theta_B, \theta_C \in (0, \pi)\). Although it is not necessary for the following discussion, in Appendix we prove that all triples \((\theta_A, \theta_B, \theta_C)\) inside this region represent different equivalence classes.

By definition, an orthogonal complement of a spanning space \(H_S\) of any UPB \(S\) does not contain product vectors. The set of product vectors in the space \(H_S\) itself however is not empty. It always contains at least \(n\) vectors — the members of UPB. Generally, the space \(H_S\) contains more than \(n\) product vectors (some examples are given in Conclusion section). Amazingly, in the system of three qubits the members of UPB are the only product vectors in \(H_S\), which significantly simplify analysis of convertibility issues.

**Lemma 3.** Let \(S\) be an arbitrary UPB for three qubits. The basis members \(|S_j\rangle, j = 1, 2, 3, 4\) are the only product vectors in the spanning space \(H_S\).

**Proof.** Let \(S = \{|S_j\rangle\}\) be a UPB with four members given by Eq. (5). We will prove more strong statement, namely, that the basis members \(|S_j\rangle, j = 1, 2, 3, 4\) are the only product vectors in \(H_S\) with respect to any bipartite cut of three qubits. Consider for example a cut \(ABC = A \cup BC\). Suppose that for some \(|\Psi_A\rangle \in \mathbb{C}^2\) and some \(|\Psi_{BC}\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2\) there exists a decomposition

\[
|\Psi_A\rangle \otimes |\Psi_{BC}\rangle = \sum_{j=1}^4 \alpha_j |S_j\rangle, \quad \sum_{j=1}^4 |\alpha_j|^2 = 1.
\]

(8)

Tracing out the qubits B and C we get:

\[
|\Psi_A\rangle \langle \Psi_A| = \rho_{12} + \rho_{34},
\]

(9)

where

\[
\rho_{12} = \begin{pmatrix}
|\alpha_1|^2 & \alpha_1 \alpha_2 \langle B|0\rangle \langle C|0\rangle \\
\alpha_1 \alpha_2 \langle 0|B\rangle \langle C|0\rangle & |\alpha_2|^2
\end{pmatrix},
\]

and

\[
\rho_{34} = \begin{pmatrix}
|\alpha_3|^2 & \alpha_3 \alpha_4 \langle B^\perp|1\rangle \langle C^\perp|1\rangle \\
\alpha_3 \alpha_4 \langle 1|B^\perp\rangle \langle C^\perp|1\rangle & |\alpha_4|^2
\end{pmatrix}.
\]

Here \(\rho_{12}\) is represented by its matrix elements in \(|0\rangle\langle 0|\) basis, while \(\rho_{34}\) is represented by its matrix elements in \(|A\rangle\langle A|\) basis. Since \(\rho_{12}\) and \(\rho_{34}\) are non-negative Hermitian operators, the equality in Eq. (9) is possible only if \(\rho_{12}\) and \(\rho_{34}\) are both proportional to the projector \(|\Psi_A\rangle \langle \Psi_A|\). In particular it implies that \(\det \rho_{12} = 0\) and \(\det \rho_{34} = 0\). According to Lemma 2, the bases \(|\langle B|B^\perp\rangle\rangle\) and \(|\langle C|C^\perp\rangle\rangle\) do not coincide with the basis \(|0\rangle\langle 0|\). It means that the determinant

\[
\det \rho_{12} = |\alpha_1 \alpha_2|^2 (1 - |\langle B|0\rangle|^2 |\langle C|0\rangle|^2)
\]

equals zero only if \(\alpha_1 \alpha_2 = 0\). Analogously, \(\det \rho_{34} = 0\) only if \(\alpha_3 \alpha_4 = 0\). Since the basis \(|\langle A|A^\perp\rangle\rangle\) does not coincide with \(|\langle 0|\langle 1|\rangle\rangle\), the equality in Eq. (9) is possible only if at most one of the coefficients \(\alpha_1, \alpha_2, \alpha_3, \alpha_4\) is nonzero. Thus the state \(|\Psi_A\rangle \otimes |\Psi_{BC}\rangle\) coincides with one of the basis members.

**Corollary 1.** UPBs \(S\) and \(T\) are equivalent iff \(\rho_S\) and \(\rho_T\) are related by local unitaries.

Indeed, suppose that \(\rho_T = U \rho_S U^\dagger\), for some unitary product operator \(U = U_A \otimes U_B \otimes U_C\). Then \(U \mathcal{H}_S = \mathcal{H}_T\) and thus vectors \(U|S_j\rangle, j = 1, 2, 3, 4\) are some product vectors in \(\mathcal{H}_T\). Lemma 3 says that these vectors are \(|T_j\rangle\), up to some permutation. Thus \(S \sim T\). The reverse statement is trivial.

**Corollary 2.** If UPBs \(S\) and \(T\) are not equivalent, they can not be converted to each other by local filtering operations.

To prove this corollary we will need two facts pointed out in Ref. 3, namely: 1) An orbit of any mixed state under LFO (or a closure of this orbit) contains a normal form, i.e. a state which has all one-particle marginals proportional to the identity. 2) Inside each orbit there is a unique (up to local unitaries) normal form. From the representation Eq. (5) it is clear that a state \(\rho_S\) associated with an arbitrary UPB \(S\) automatically comes in a normal form. Now the statement we need follows from the previous corollary and the two facts mentioned above. To investigate convertibility under general LOCC and approximate convertibility this results however is not strong enough. We will need one more lemma.

**Lemma 4.** Let \(X = X_A \otimes X_B \otimes X_C\) be a non-degenerated (full rank) factorized 3-qubit operator such that

\[
X \cdot H_S = H_T
\]

for some UPBs \(S\) and \(T\). Then \(S \sim T\) and \(X\) is proportional to a unitary operator: \(X = rU, r \in \mathbb{R},UU^\dagger = I\).

**Proof.** Obviously, multiplying \(X\) on factorized unitary operators on the left and on the right is equivalent to a choice of representatives in equivalence classes of \(S\) and \(T\). So we can assume that

\[\bullet X_A, X_B, \text{ and } X_C\] are Hermitian operators

\[\bullet \text{ Members of the UPB } S \text{ are given by Eq. (5)}\]

Members of the UPB \(T\) will be denoted as

\[
|T_1\rangle = |A_1\rangle \otimes |B_1\rangle \otimes |C_1\rangle,
|T_2\rangle = |A_2\rangle \otimes |B_2\rangle \otimes |C_2\rangle,
|T_3\rangle = |A_3\rangle \otimes |B_3\rangle \otimes |C_3\rangle,
|T_4\rangle = |A_4\rangle \otimes |B_4\rangle \otimes |C_4\rangle.
\]

(10)

If \(|\psi\rangle \in \mathcal{H}_S\) is a product vector, then its image \(X|\psi\rangle \in \mathcal{H}_T\) is also a product vector. Combining Lemma 3 and
the fact that $X$ is non-degenerate operator, we conclude that
\begin{align}
X|S_j⟩ &= x_j|T_{σ(j)}⟩, \quad j = 1, 2, 3, 4,
\end{align}
for some coefficients $x_j ≠ 0$ and some permutation $σ ∈ S_4$. Let us concentrate on the qubit A. From Eq. (11) we infer
\begin{align}
X_A|0⟩ &= a_1|A_{σ(1)}⟩, \\
X_A|1⟩ &= a_2|A_{σ(2)}⟩, \\
X_A|A⟩ &= a_3|A_{σ(3)}⟩, \\
X_A|A⟩ &= a_4|A_{σ(4)}⟩,
\end{align}
where $a_j$ are some non-zero coefficients. For each $j = 1, 2, 3, 4$ choose some unital vector $|A_j⟩$ orthogonal to $|A_j⟩$. Taking into account the fact that $X_A$ is Hermitian and non-degenerated, we can rewrite Eq. (12) as
\begin{align}
X_A|A_{σ(1)}⟩ &= b_1|1⟩, \\
X_A|A_{σ(2)}⟩ &= b_2|0⟩, \\
X_A|A_{σ(3)}⟩ &= b_3|A⟩, \\
X_A|A_{σ(4)}⟩ &= b_4|A⟩,
\end{align}
for some non-zero coefficients $b_j$. Applying the classification Lemma 2 to the UPB $T$ we conclude that the family $\{ |A⟩ \}$ and the family $\{ |A⟩ \}$ coincide up to permutation of the elements and some phase adjustment for each element. Moreover, from this Lemma we can learn that an orthogonality graph for the family $\{ |A⟩ \}$ must be one of three graphs shown on FIG. 2.

![FIG. 2: Possible orthogonality graphs for the family $\{ |A⟩ \}$.](image)

(We choose an ordering of vertices which depends upon $σ$, since this ordering appears in Eq. 12.) Whatever graph is chosen, the equations Eq. (12) guarantee that an operator $(X_A)^4$ is diagonal both in the basis $\{ |0⟩, |1⟩ \}$ and the basis $\{ |A⟩, |A⟩ \}$. By definition, these bases do not coincide. Thus $(X_A)^4$ is proportional to the identity, $(X_A)^4 = x_A I$, for some coefficient $x_A$. Since $X_A$ is Hermitian, this is possible only if $X_A$ is proportional to a unitary operator. Applying the same arguments to the qubits B and C, we conclude that $X$ is proportional to a unitary operator. Finally, Lemma 3 implies that $S ∼ T$.

### III. EXACT CONVERTIBILITY OF BOUND ENTANGLED STATES

Consider a pair of 3-qubit mixed states $ρ_S$ and $ρ_T$ associated with a pair of non-equivalent UPBs $S$ and $T$ respectively. Non-equivalence assumption implies that we can not convert $ρ_S$ into $ρ_T$ exactly by LFO, see the corollary to Lemma 3. Now we are in position to prove more strong statement, namely that $ρ_S$ can not be converted into $ρ_T$ exactly by a separable superoperators. To simplify notations, let us denote $Ω_f$ a set of 3-qubit product operators, i.e.
\begin{align}
Ω_f = \{ X = X_A ⊗ X_B ⊗ X_C, \quad X_A, X_B, X_C ∈ ℂ^2 \).
\end{align}

**Lemma 5.** Let $\{ X_i ∈ Ω_f \}_{i=1,...,D}$ be a family of product operators such that $∑_{i=1}^D X_i ⊗ X_i ≤ I$. Suppose that for some UPBs $S$, $T$ and real number $p > 0$ one has the equality
\begin{align}
∑_{i=1}^D X_i ⊗ X_i = p ρ_T.
\end{align}

Then the UPBs $S$ and $T$ are equivalent.

**Remarks:** 1) The factor $p$ can be regarded as a probability for conversion to succeed, since
\begin{align}
p = Tr(∑_{i=1}^D X_i X_i) ∈ [0, 1].
\end{align}

2) Without loss of generality we can choose $D$ as a real dimension of a space of all superoperators on three qubits, which is $D = 213$. Indeed, superoperators $E(ρ) = ∑_{i=1}^D X_i X_i$ discussed in the lemma constitute a compact convex subset in the linear space of all superoperators on three qubits.

**Proof.** Denote
\begin{align}
ρ_i = X_i ⊗ ρ_S ⊗ X_i⊥,
\end{align}

Since $ρ_i$ is a positive semidefinite operator, the equality Eq. (14) tells us that
\begin{align}
Rg(ρ_i) ⊆ Rg(ρ_T), \quad i = 1, . . . , D,
\end{align}
or, equivalently,
\begin{align}
X_i ⊗ Rg(ρ_S) ⊆ Rg(ρ_T),
\end{align}
where $Rg(ρ)$ is a range of the operator $ρ$. Suppose that for some $i$ we have $ρ_i ≠ 0$ and the operator $X_i$ is degenerated. Let us show that this assumption leads to a contradiction. By definition, $X_i$ has a product form, $X_i = X_A ⊗ X_B ⊗ X_C$, so at least one of the factors $X_A$, $X_B$, $X_C$ has a rank one. Consider the case $Rk(X_A) = 1$, i.e. $X_A = |ψ⟩⟨φ|$ for some vectors $|ψ⟩, |φ⟩ ∈ ℂ^2$. Then
\begin{align}
ρ_i = |ψ⟩⟨ψ| ⊗ ρ_B C, \\
ρ_B C = X_B ⊗ X_C ⟨ψ|ρ_S ⟨φ| X_B ⊗ X_C.
\end{align}
Peres criteria \[10, 11\] it implies that \(\rho_{BC}\) is separable. Let us choose some product vector \(|\psi_B\rangle \otimes |\psi_C\rangle\) from its range. Then a product vector \(|\psi\rangle \otimes |\psi_B\rangle \otimes |\psi_C\rangle\) belongs to the range of \(\rho_l\) and thus to the range of \(\rho_T\), see Eq. (19). Since \(\rho_T\) has no product vectors in its range, we conclude that for each \(l\) either \(X_l\) is non-degenerated or \(\rho_l = 0\).

Let us focus on some \(l\) with \(\rho_l \neq 0\). According to classification Lemma 4 the spaces \(\text{Rg}(\rho_S)\) and \(\text{Rg}(\rho_T)\) have the same dimension. Since \(X_l\) is non-degenerated, Eq. (10) actually means that \(X_l \cdot \text{Rg}(\rho_S) = \text{Rg}(\rho_T)\), or equivalently, that

\[ X_l^\dagger \cdot \mathcal{H}_T = \mathcal{H}_S. \]  

(18)

According to Lemma 4 this is possible only if \(S \sim T\). \(\square\)

Since stochastic LOCC transformations are described by separable superoperators, we have proved the following theorem:

**Theorem 2.** Let \(S\) and \(T\) be unextendible product bases for three qubits. Consider bound entangled states \(\rho_S\) and \(\rho_T\) associated with \(S\) and \(T\). If some LOCC transformation maps \(\rho_S\) into \(\rho_T\) with a non-zero probability, then the states \(\rho_S\) and \(\rho_T\) coincide up to local unitary transformation.

**IV. APPROXIMATE CONVERTIBILITY: SIMPLIFIED SCENARIO**

Consider a three-qubit UPB \(S = \{|S_j\rangle\}\) with the members given by Eq. (2). Let \(T\) be another UPB which is not equivalent to \(S\). In this section we will study a simplified version of the stochastic approximate convertibility problem. Namely, we will consider approximate convertibility by local filtering operations, i.e. transformations like

\[ \rho_S \to \rho = \frac{X \rho_S X^\dagger}{\rho_S[X]}, \quad \rho_S[X] = \text{Tr}(X \rho_S X^\dagger). \]  

(19)

where \(X \in \Omega_f\) is a product operator such that \(\rho_S[X] > 0\). Our goal is to prove that there exists a finite precision \(\epsilon > 0\) such that all states \(\rho\) which can appear in Eq. (19) lie outside \(\epsilon\)-neighborhood of \(\rho_T\) (we use fidelity to quantify the distance). Denote a set of achievable states \(\rho\) as \(M_S:\)

\[ M_S = \left\{ \rho = \frac{X \rho_S X^\dagger}{\rho_S[X]} : X \in \Omega_f, \quad \rho_S[X] > 0 \right\}. \]  

(20)

The results of the previous section imply that \(\rho_T \notin M_S\). Note however that \(M_S\) is not a compact set since \(\rho_S[X] = 0\) for some product operators \(X\) (for instance, \(\rho_S[|S_j\rangle\langle S_j|] = 0\) for any member of the UPB). Thus there might exist a sequence of product operators \(\{X_n\}_{n \geq 0}\), \(\rho_S[X_n] > 0\), such that the corresponding sequence \(\rho(X_n)\) converges to some operator \(\rho\) which does not belong to \(M_S\). If \(\rho = \rho_T\), it would imply that stochastic convertibility with an arbitrary small precision \(\epsilon\) is possible (although the success probability may turn to zero as \(\epsilon \to 0\)). Amazingly, this is not the case.

**Lemma 6.** Let \(S\) be an arbitrary UPB. Consider a set of achievable states \(M_S\) as in Eq. (20) and an arbitrary convergent operator sequence \(\{Y_n \in M_S\}_{n \geq 0}\). Denote

\[ \hat{Y} = \lim_{n \to \infty} Y_n. \]

Then either \(\hat{Y} \in M_S\) or \(\hat{Y}\) is separable.

**Proof.** Suppose that \(\hat{Y} \notin M_S\). Obviously, in the definition of \(M_S\) one suffices to consider normalized operators \(X\). Let us agree that

\[ X = X_A \otimes X_B \otimes X_C, \quad ||X_A|| = ||X_B|| = ||X_C|| = 1. \]

Let us choose an arbitrary sequence of normalized operators \(\{X_n \in \Omega_f\}_{n \geq 0}\), \(\rho_S[X_n] > 0\), such that \(Y_n = X_n \rho_S X_n^\dagger / \rho_S[X_n]\). Without lose of generality we can assume that the sequence \(\{X_n\}\) is also convergent (since \(X_n\) are taken from a bounded manifold, we can always extract a convergent subsequence). Denote

\[ \hat{X} = \lim_{n \to \infty} X_n = \hat{X}_A \otimes \hat{X}_B \otimes \hat{X}_C. \]

(21)

Note that \(\rho_S[\hat{X}] = 0\), since otherwise \(\hat{Y} \in M_S\). It means that \(\hat{X}_S \hat{X}^\dagger = 0\), or, equivalently, \(\rho_S \hat{X}^\dagger = 0\) which implies \(\text{Im}(\hat{X}^\dagger) \subseteq \text{Ker}(\rho_S)\). Therefore

\[ \text{Im}(\hat{X}_A^\dagger) \otimes \text{Im}(\hat{X}_B^\dagger) \otimes \text{Im}(\hat{X}_C^\dagger) \subseteq \mathcal{H}_S. \]

But according to Lemma 3 the only product vectors in \(\mathcal{H}_S\) are the members of \(S\). Thus \(\hat{X} = |a, b, c\rangle\langle S_j|\) for some \(j \in [1, 4]\) and some one-qubit normalized states \(|a\rangle, |b\rangle, |c\rangle\) Without loss of generality, we can assume that \(j = 1\), i.e.

\[ \hat{X}_A = |a\rangle\langle 0|, \quad \hat{X}_B = |b\rangle\langle 0|, \quad \hat{X}_C = |c\rangle\langle 0|, \]

(22)

(see Eq. (3)). The elements of the sequence \(\{X_n\}_{n \geq 0}\) can always be written as

\[ X_n = X_{A,n} \otimes X_{B,n} \otimes X_{C,n}, \]

\[ X_{A,n} = |a_n\rangle\langle 0| + |\alpha_n\rangle\langle 1|, \]

\[ X_{B,n} = |b_n\rangle\langle 0| + |\beta_n\rangle\langle 1|, \]

\[ X_{C,n} = |c_n\rangle\langle 0| + |\gamma_n\rangle\langle 1|, \]

(23)

where \(|a_n\rangle, |b_n\rangle, |c_n\rangle\) and \(|\alpha_n\rangle, |\beta_n\rangle, |\gamma_n\rangle\) are some one-qubit states. The requirements Eq. (21) translate into

\[ \lim_{n \to \infty} |a_n\rangle = |a\rangle, \quad \lim_{n \to \infty} |b_n\rangle = |b\rangle, \quad \lim_{n \to \infty} |c_n\rangle = |c\rangle, \]

and

\[ \lim_{n \to \infty} \langle \alpha_n |a_n\rangle = \lim_{n \to \infty} \langle \beta_n |b_n\rangle = \lim_{n \to \infty} \langle \gamma_n |c_n\rangle = 0. \]
Denote
\[ Y_n = \frac{X_n \rho_S X_n^\dagger}{\rho_S[X_n]} . \] (24)

To compute a limit of \( Y_n \) for \( n \to \infty \) we can keep only the leading terms in the expression for \( X_n \) (see a comment below), namely
\[
\begin{align*}
X_n & \approx X_n^{(0)} + X_n^{(1)}, \\
X_n^{(0)} & = |\alpha_n, \beta_n, \gamma_n \rangle \langle 0, 0, 0|, \\
X_n^{(1)} & = |\alpha_n, \beta_n, \gamma_n \rangle \langle 0, 0, 0| + |\alpha_n, \beta_n, \gamma_n \rangle \langle 1, 0, 0|. 
\end{align*}
\] (25)

Obviously, \( X_n^0 \) does not yield any contribution to \( Y_n \), so that
\[ X_n \rho_S X_n^\dagger \approx X_n^{(1)} \rho_S X_n^{(1)} \dagger. \]

Now we can substitute \( \alpha_n, \beta_n, \gamma_n \) in the expression for \( X_n^{(1)} \) by \( a, b, c \) respectively, since we would like to keep only leading order terms. Using the list of the UPB’s members Eq. (5) one can easily check that matrix elements \( \langle 1, 0, 0 | \rho_S | 1, 0, 0 \rangle \) (and their cyclic permutations) vanish. Therefore we arrive to
\[
\begin{align*}
X_n^{(1)} & \rho_S X_n^{(1)} \dagger \approx \\
& \approx \langle 0, 1, 0 | \rho_S | 0, 0, 1 \rangle \cdot |\alpha_n, b, c \rangle \langle \alpha_n, b, c | \\
& + \langle 0, 1, 0 | \rho_S | 0, 0, 1 \rangle \cdot |\alpha_n, b, c \rangle \langle \alpha_n, b, c | \\
& + \langle 0, 1, 0 | \rho_S | 0, 0, 1 \rangle \cdot |\alpha_n, b, c \rangle \langle \alpha_n, b, c |. 
\end{align*}
\] (26)

Note that this is a separable state and that the matrix elements
\[
\begin{align*}
& \langle 1, 0, 0 | \rho_S | 1, 0, 0 \rangle, \\
& \langle 0, 1, 0 | \rho_S | 0, 1, 0 \rangle, \\
& \langle 0, 0, 1 | \rho_S | 0, 0, 1 \rangle,
\end{align*}
\]
are strictly positive, since Lemma 3 tells us that the states \( |1, 0, 0 \rangle, |0, 1, 0 \rangle, \) and \( |0, 0, 1 \rangle \) do not belong to \( \mathcal{H}_S \).

It justifies that the leading terms in the expansion for \( X_n \rho_S X_n^\dagger \) do not vanish and all terms we have disregarded are indeed small compared with the terms we keep. In particular,
\[
\begin{align*}
\text{Tr}(X_n^{(1)} \rho_S X_n^{(1)} \dagger) & \approx \\
& \approx \langle 1, 0, 0 | \rho_S | 1, 0, 0 \rangle \langle \alpha_n | \alpha_n \rangle \\
& + \langle 0, 1, 0 | \rho_S | 0, 1, 0 \rangle \langle \beta_n | \beta_n \rangle \\
& + \langle 0, 0, 1 | \rho_S | 0, 0, 1 \rangle \langle \gamma_n | \gamma_n \rangle. 
\end{align*}
\] (27)

We conclude that
\[ \hat{Y} = \lim_{n \to \infty} Y_n = \lim_{n \to \infty} \frac{X_n^{(1)} \rho_S X_n^{(1)} \dagger}{\text{Tr}(X_n^{(1)} \rho_S X_n^{(1)} \dagger)}, \] (28)
where approximations Eq. (26,27) should be substituted into nominator and denominator. But from Eq. (26) we infer that \( \hat{Y} \) is separable.

Denote \( \overline{M} \) a closure of the set \( M \). It is a compact set. A fidelity \( F(\rho, \rho_T) \) is a continuous function of a state \( \rho \).

According to Lemma 3 \( F(\rho, \rho_T) < 1 \) for all \( \rho \in M_S \). Lemma 4 implies that \( F(\rho, \rho_T) < 1 \) even for all \( \rho \in \overline{M}_S \).

Thus there exists a finite precision \( \epsilon > 0 \) such that
\[ F(\rho, \rho_T) \leq 1 - \epsilon, \quad \text{for any } \rho \in M_S. \]

Of course the precision \( \epsilon \) may depend upon \( S \) and \( T \), in particular \( \epsilon \to 0 \) as \( S \) turns to \( T \).

V. APPROXIMATE CONVERTIBILITY IN A GENERAL CASE

We now are prepared to analyse approximate convertibility by arbitrary completely positive separable super-operators. Such operators correspond to probabilistic mixtures of LFOs, so we can exploit the results of section XV. A proof of Theorem 11 is contained in the following lemma.

Lemma 7. Let \( S \) and \( T \) be non-equivalent UPBs. Suppose \( \{X_l \in \Omega_f\}_{l=1,...,D} \) is a family of product operators such that \( \rho_S[X_l] \equiv \text{Tr}(X_l \rho_S X_l^\dagger) > 0 \) for all \( l \). Denote \( \rho_S = \sum_{l=1}^D \rho_S[X_l] \). There exist a finite precision \( \epsilon = \epsilon(S,T) > 0 \) depending upon \( S \) and \( T \) only, such that
\[ F\left(\rho_T, \frac{1}{\rho_S} \sum_{l=1}^D X_l \rho_S X_l^\dagger \right) \leq 1 - \epsilon(S,T). \] (29)

Proof. We start from introducing an auxiliary proximity measure between \( \rho_T \) and the final state. Let \( |T_j \rangle \), \( j = 1, 2, 3, 4 \) be the members of the UPB \( T \). Consider a functional
\[ f_T(\rho) = \sum_{j=1}^4 \langle T_j | \rho | T_j \rangle. \] (30)

It is clear that
\[ f_T(\rho) = 0 \quad \text{iff} \quad \text{Rg}(\rho) \subseteq \text{Rg}(\rho_T). \] (31)

Thus the equality \( f_T(\rho) = 0 \) is necessary for \( \rho = \rho_T \).

Define \( \rho \) as
\[ \rho = \frac{1}{\rho_S} \sum_{l=1}^D X_l \rho_S X_l^\dagger. \]

Let us first prove that there exist a finite precision \( \delta > 0 \) depending upon \( S \) and \( T \) only, such that
\[ f_T(\rho) \geq \delta. \] (32)

Using linearity of \( f_T \) we can write
\[ f_T(\rho) = \sum_{l=1}^D \frac{\rho_S[X_l]}{\rho_S[X_l]} f_T(Y_l), \]
\[ Y_l = \frac{X_l \rho_S X_l^\dagger}{\rho_S[X_l]} \in M_S, \] (33)
Eq. (32) into an upper bound on fidelity $F$. From Eq. (31) we infer that $f_T(Y) > 0$ for any $Y \in M_S$. Moreover, $f_T(Y) > 0$ for any $Y$ belonging to a closure of the set $M_S$. Indeed, Lemma 5 tells us that such $Y$ either belongs to $M_S$ or is separable. In the latter case the inclusion $\text{Rg}(Y) \subseteq \text{Rg}(\rho_T)$ is impossible, since $\text{Rg}(\rho_T)$ is free of product vectors. Summarizing, there exists a finite precision $\delta$, such that

$$f_T(Y) \geq \delta, \text{ for any } Y \in M_S.$$  

(34)

Returning to Eq. (33) we immediately get the estimate Eq. (32).

To conclude the proof we need to turn inequality Eq. (32) into an upper bound on fidelity $F(\rho_T, \rho)$. It can be done as follows. We start from the standard definition of fidelity:

$$F(\rho_T, \rho) = \text{Tr} \left( \sqrt{\rho_T \rho} \sqrt{\rho_T \rho} \right).$$  

(35)

Introduce orthogonal projectors $P_T$ and $P_T^\perp$ onto the spanning space $\mathcal{H}_T$ and its orthogonal complement respectively. By definition, $P_T + P_T^\perp = I$. Then $\rho_T = (1/4)P_T$ and thus

$$F(\rho_T, \rho) = \frac{1}{2} \text{Tr} \sqrt{P_T^\perp \rho P_T^\perp}.$$  

(36)

Note that $f_T(\rho) = \text{Tr}(\rho P_T)$. From an identity

$$\text{Tr} \left( P_T^\perp \rho P_T^\perp \right) + \text{Tr} (P_T \rho P_T) = 1$$

and from the estimate Eq. (32) we get

$$\text{Tr} \left( P_T^\perp \rho P_T^\perp \right) \leq 1 - \delta.$$  

(37)

Taking into account that the operator $P_T^\perp \rho P_T^\perp$ has a rank at most four and applying Cauchy-Schwarz inequality we arrive to

$$\text{Tr} \sqrt{P_T^\perp \rho P_T^\perp} \leq \sqrt{4 \text{Tr} \left( P_T^\perp \rho P_T^\perp \right)} \leq 2\sqrt{1 - \delta}.$$  

(38)

Substituting it to Eq. (36) gives us

$$F(\rho_T, \rho) \leq \sqrt{1 - \delta} \leq 1 - \frac{\delta}{2}.$$  

(39)

We have proved the lemma.

\section{VI. CONCLUSION}

We have studied local convertibility of three-qubit mixed states associated with unextendible product bases. A complete classification of three-qubit UPBs is suggested. This family of UPBs is shown to have some nice mathematical properties which allow to investigate convertibility question completely. We proved that for any non-equivalent UPBs $S$ and $T$ the stochastic approximate conversion of the associated states $\rho_S$ and $\rho_T$ is impossible.

It would be interesting to apply our method to UPBs in some other low-dimension systems. The system of two qutrits $\mathbb{C}^3 \otimes \mathbb{C}^3$ is of particular interest, since for this system a complete UPBs classification has been already found in Ref. 5. It is known that all two-qutrit UPBs are characterized by the same orthogonality graphs and consist of five members. Unfortunately a set of product vectors in a spanning space of a UPB is generally larger than a set of the UPB’s members. We have checked it for two particular two-qutrit UPBs, called in Ref. 6 as "Tiles" and "Pyramid". Using the designations of this reference, the extra product vectors in the spanning spaces of "Pyramid" and "Tiles" respectively can be written as $|0 \rangle \otimes |0 \rangle$ and $\frac{1}{9}(2|0 \rangle - |1 \rangle + 2|2 \rangle) \otimes (2|0 \rangle - |1 \rangle + 2|2 \rangle)$.

Exploiting the symmetry of these particular UPBs one can show that there are exactly six product vectors in their spanning spaces. However the presence of extra product vectors as well as three-dimensional geometry makes the convertibility analysis very complicated.

\section*{Acknowledgments}

We would like to thank Guifre Vidal for supporting ideas which significantly simplify the proof in Section III. We also acknowledge useful conversations with Patrick Hayden, Alexei Kitaev, and Federico Spedalieri.

\section*{APPENDIX}

In this section we prove that all UPBs given by Eq. (5) with parameters $\theta_A, \theta_B, \theta_C \in (0, \pi)$ are not equivalent to each other in the sense of Definition 2. Consider UPBs $S = \{|S_j\rangle\}$ and $S' = \{|S'_j\rangle\}$ with the members

$$|S_1\rangle = |0 \rangle \otimes |0 \rangle \otimes |0 \rangle,$$

$$|S_2\rangle = |1 \rangle \otimes |B \rangle \otimes |C \rangle,$$

$$|S_3\rangle = |A \rangle \otimes |1 \rangle \otimes |C' \rangle,$$

$$|S_4\rangle = |A^\perp \rangle \otimes |B^\perp \rangle \otimes |1 \rangle,$$  

(A.1)

and

$$|S'_1\rangle = |0 \rangle \otimes |0 \rangle \otimes |0 \rangle,$$

$$|S'_2\rangle = |1 \rangle \otimes |B' \rangle \otimes |C' \rangle,$$

$$|S'_3\rangle = |A' \rangle \otimes |1 \rangle \otimes |C'' \rangle,$$

$$|S'_4\rangle = |A'^\perp \rangle \otimes |B'^\perp \rangle \otimes |1 \rangle.$$  

(A.2)
Since the overall phase of the vectors is not important, the parameterization Eq. (3) is equivalent to

\[ |A⟩ = \cos (θ_A/2) |0⟩ + \sin (θ_A/2) |1⟩, \]
\[ |B⟩ = \cos (θ_B/2) |0⟩ + \sin (θ_B/2) |1⟩, \]
\[ |C⟩ = \cos (θ_C/2) |0⟩ + \sin (θ_C/2) |1⟩, \quad (A.3) \]

and

\[ |A′⟩ = \cos (θ_A′/2) |0⟩ + \sin (θ_A′/2) |1⟩, \]
\[ |B′⟩ = \cos (θ_B′/2) |0⟩ + \sin (θ_B′/2) |1⟩, \]
\[ |C′⟩ = \cos (θ_C′/2) |0⟩ + \sin (θ_C′/2) |1⟩. \quad (A.4) \]

Suppose that

\[ U_A \otimes U_B \otimes U_C |S_j⟩ = |S_{σ(j)}⟩ \]

for some one-qubit unitary operators \( U_A, U_B, U_C \) and permutation \( σ \in S_4 \). If \( σ(1) = 1 \) then orthogonality implies \( σ(j) = j \) for all \( j \). It means that \( |⟨0|A⟩| = |⟨0|A′⟩|, |⟨0|B⟩| = |⟨0|B′⟩|, \) and \( |⟨0|C⟩| = |⟨0|C′⟩| \). From Eq. (A.3) and Eq. (A.4) we have \( |⟨1|A⟩| = |⟨1|A′⟩|, |⟨0|B⟩| = |⟨0|B′⟩|, \) and \( |⟨0|C⟩| = |⟨0|C′⟩| \). Thus \( \langle θ_A, θ_B, θ_C⟩ \) and \( \langle θ_A′, θ_B′, θ_C′⟩ \) coincide.

If \( σ(2) = 1 \) then orthogonality implies \( σ(2) = 1, σ(3) = 4, \) and \( σ(4) = 3 \). It means that \( |⟨0|A⟩| = |⟨0|A′⟩|, |⟨0|B⟩| = |⟨0|B′⟩|, \) and \( |⟨0|C⟩| = |⟨0|C′⟩| \). Thus \( \langle θ_A, θ_B, θ_C⟩ \) and \( \langle θ_A′, θ_B′, θ_C′⟩ \) coincide.

If \( σ(3) = 1 \) then orthogonality implies \( σ(1) = 3, σ(2) = 4, \) and \( σ(4) = 2 \). It means that \( |⟨0|A⟩| = |⟨0|A′⟩|, |⟨0|B⟩| = |⟨1|B⟩⟩, \) and \( |⟨0|C⟩| = |⟨0|C′⟩| \). Thus \( \langle θ_A, θ_B, θ_C⟩ \) and \( \langle θ_A′, θ_B′, θ_C′⟩ \) coincide.

Finally, if \( σ(4) = 1 \) then orthogonality implies \( σ(1) = 4, σ(2) = 3, \) and \( σ(3) = 2 \). It means that \( |⟨0|A⟩| = |⟨0|A′⟩|, |⟨0|B⟩| = |⟨0|B′⟩|, \) and \( |⟨0|C⟩| = |⟨1|C′⟩| \). Thus \( \langle θ_A, θ_B, θ_C⟩ \) and \( \langle θ_A′, θ_B′, θ_C′⟩ \) coincide.

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[12] By local filtering operations we mean a transformation \( ρ \to X_ρ X^† \), where \( X = X_A \otimes X_B \otimes \cdots \) is an arbitrary product operator.