Local Exact Controllability of a Parabolic System of Chemotaxis*

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December 23, 2012

Abstract

This paper studies the controllability problem of a parabolic system of chemotaxis. The local exact controllability to trajectories of the system imposes one control force only is obtained by applying Kakutani’s fixed point theorem combined with the null controllability of the associated linearized parabolic system. The control function is shown to be in $L^\infty(Q)$, which is estimated by using the methods of maximal regularity and $L^p$-$L^q$ estimates of parabolic equations.

Keywords: local exact controllability, chemotaxis system, Carleman inequality, Kakutani’s fixed point theorem.

AMS subject classifications: 93B05, 93C20, 35B37.

1 Introduction and main results

Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a bounded domain with sufficient smooth boundary $\partial \Omega$. Let $\omega$ be a nonempty open subset of $\Omega$, and $T > 0$. We denote $Q = \Omega \times (0, T)$, $\Sigma = \partial \Omega \times (0, T)$ and $Q_\omega = \omega \times (0, T)$. Throughout this paper, we use $W^{s,q}(\Omega)$, $W^{2,1}_q(Q)$ and $C^\alpha(\overline{\Omega})$ ($s, \alpha \geq 0, 1 \leq q \leq \infty$) for the usual Sobolev spaces (e.g., [30]), and set $H^m(\Omega) = W^{m,2}(\Omega)$ for $m \in \mathbb{N}$. $L^p(\Omega)$ and $L^p(Q)$

\*This work was supported by Program for Innovative Research Team in UIBE, the National Natural Science Foundation of China, the National Basic Research Program of China (2011CB808002), the National Research Foundation of South Africa, the National Science Foundation of China (11201358), and the Fundamental Research Funds for the Central Universities.

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\((1 \leq p \leq \infty)\) are the usual Lebesgue function spaces with the norm \(| \cdot |_p\) and \(\| \cdot \|_p\), respectively. Moreover, let

\[
\begin{align*}
V^1(Q) &= \{y | y \in L^2(0, T; H^1(\Omega)), \partial_t y \in L^2(0, T; H^1(\Omega)^*)\}, \\
V^2(Q) &= \{y | y \in L^2(0, T; H^2(\Omega)), \partial_t y \in L^2(Q)\},
\end{align*}
\]

be equipped with their graph norms, where \(H^1(\Omega)^*\) denotes the dual space of \(H^1(\Omega)\). The duality between \(H^1(\Omega)^*\) and \(H^1(\Omega)\) is denoted by \(\langle \cdot, \cdot \rangle\).

In this paper, we are concerned with the following controlled parabolic system with state functions \(u \equiv u(x, t)\) and \(v \equiv v(x, t)\) :

\[
\begin{align*}
\partial_t u &= \nabla \cdot (\nabla u - \chi u \nabla v) + 1_\omega f \quad \text{in } Q, \\
\partial_t v &= \Delta v - \gamma v + \delta u \quad \text{in } Q, \\
\partial_\nu u &= 0, \partial_\nu v &= 0 \quad \text{on } \Sigma, \\
u(x, 0) &= u_0(x) \quad v(x, 0) = v_0(x) \quad x \in \Omega,
\end{align*}
\]

where \(\partial_t = \partial / \partial t\), and \(\partial_\nu = \partial / \partial \nu\) stands for the derivative with respect to the outer normal \(\nu\) of \(\partial \Omega\), \(1_\omega\) represents the characteristic function of \(\omega\), \(f \equiv f(x, t)\) is the control function so that \(1_\omega f\) is the control force acting from the outside on a portion of the domain \(\Omega\), \(u_0\) and \(v_0\) are the initial values, and \(\chi, \gamma\) and \(\delta\) are given positive constants.

A pair of functions \((u, v)\) with

\(u \in V^1(Q) \cap L^\infty(Q), v \in V^2(Q) \cap L^\infty(Q)\)

is called a weak solution of (1.1) if for all \(\varphi \in L^2(0, T; H^1(\Omega))\), the following identities hold:

\[
\begin{align*}
\int_0^T \langle \partial_t u, \varphi \rangle \, dt + \int_Q [(\nabla u - \chi u \nabla v) \cdot \nabla \varphi + 1_\omega f \varphi] \, dx \, dt &= 0, \\
\int_Q \varphi \partial_t v \, dx \, dt + \int_Q [(\nabla v \cdot \nabla \varphi + (-\gamma v + \delta u) \varphi] \, dx \, dt &= 0.
\end{align*}
\]

We write the free system of (1.1), that is, in the absence of \(f\), as follows:

\[
\begin{align*}
\partial_t \overline{u} &= \nabla \cdot (\nabla \overline{u} - \chi \overline{u} \nabla \overline{v}) \quad \text{in } Q, \\
\partial_t \overline{v} &= \Delta \overline{v} - \gamma \overline{v} + \delta \overline{u} \quad \text{in } Q, \\
\partial_\nu \overline{u} &= 0, \partial_\nu \overline{v} &= 0 \quad \text{on } \Sigma, \\
\overline{u}(x, 0) &= \overline{u}_0(x) \quad \overline{v}(x, 0) = \overline{v}_0(x) \quad x \in \Omega.
\end{align*}
\]

The system (1.2) is a prototype chemotaxis system so called Keller-Segel model which describes the aggregation process of slime mold resulting from chemotactic attraction. In (1.2), \(\overline{u}\) represents the density of the cellular slime mold, \(\overline{v}\) is the density of the chemical substance (see [29]). In the
last decade, there is a large number of works devoted to the mathematical analysis of the Keller-Segel system. Several topics on the Keller-Segel model for chemotaxis such as aggregation, blow-up of solutions, and chemotactic collapse, etc., have been concerned and some significant results have been achieved from different discipline perspectives. In Horstmann [27] and Hillen and Painter [26], it provides a detailed introduction into the mathematics of the Keller-Segel model for chemotaxis with abundant references therein. Here we would mention a few facts about the local and global existence of solutions for the Keller-Segel model. Generally speaking, the blow-up of solutions of Keller-Segel system in finite or infinite time depends strongly on the space dimension. In 1-d case, a finite time blow-up never occur, and the global solution exists and converges to the stationary solution as times goes to infinity (see [32]). But the blow-up may occur in finite or infinite time in n-dimensional case for \( n \geq 3 \) (see [11, 28]). For the 2-d case, several thresholds have been found. When the mass of the initial data is below some threshold value, the solution exists globally in time and its \( L^\infty \)-norm is uniformly bounded for all time. While the mass of the initial data is larger than some threshold value, the solution will blow up either in finite or in infinite time (see [9, 20, 35]).

Due to blow-up feature of solutions of the Keller-Segel model, it is interesting to consider some controllability problems. Let \((\overline{u}, \overline{v})\) be a trajectory, i.e., a solution of (1.2) corresponding to some initial value \((u_0, v_0)\). We say that the system (1.2) is \textit{locally exactly controllable to the trajectory} \((\overline{u}, \overline{v})\) \textit{at time} \(T\), if there exists a neighborhood \(O\) of \((u_0, v_0)\) such that for any initial data \((u_0, v_0)\) \(\in O\), the solution \((u, v)\) of (1.1) driven by some control function \(f\) satisfies

\[
    u(x, T) = \overline{u}(x, T), \quad v(x, T) = \overline{v}(x, T), \quad \text{for} \quad x \in \Omega \quad \text{a.e.,}
\]

where the neighborhood \(O\) and the control function space will be specified later.

In this paper, we suppose that \(\overline{u}, \overline{v}\) verify the following regularity properties:

\[
    \overline{u}, \overline{v} \in L^\infty(Q), \quad \nabla \overline{v} \in L^\infty(Q)^N. \tag{1.3}
\]

**Remark 1.1.** The solution \((\overline{u}, \overline{v})\) of the system (1.2) exists at least locally in time interval \([0, T_1]\) for \(T_1 < T_{\text{max}}\) with sufficiently small initial data \((\overline{u}_0, \overline{v}_0)\), where \(T_{\text{max}}\) is the maximal existence time (see [27] and reference therein). If the system is locally exactly controllable, then we can drive the state of the system by some control force to a given trajectory at time \(T \leq T_1\) before the time \(T_{\text{max}}\) to avoid blow-up. It is also worth indicating that the reason we consider the local exact controllability instead of exact controllability is that the solution may blow up when the mass of initial value is larger than some threshold value.

**Remark 1.2.** When \((\overline{u}, \overline{v}) = (0, 0)\), the local exact controllability is reduced to the local null controllability. If the system (1.1) is locally null controllable at time \(T\) with some control, then we can switch off the control after time \(T\) and the system will keep into zero afterwards.
This paper is devoted to the local exact controllability of the coupled parabolic system (1.1) via one control. The controllability of parabolic systems of coupled equations attracts intensive attention in the last few years. In Barbu [7], it studies the local exact controllability to steady states with controls acting on each equation of the system via the same interior domain. This could be done by taking it as a direct consequence of the controllability of the scalar parabolic equations. It is much more interesting and applicable to consider the controllability of a parabolic system with one control force imposed on one equation of the system. Ammar Kdjioda et al. [1] is the first work of this kind. They show that the phase-field system is locally exactly controllable to the trajectory by one control force. The series of works of Ammar Kdjodia et al. (2, 3, 4), and the works of González-Burgos et al. (22, 10, 14, 15), have extended such problem to more general cases. The survey paper [5] gives a comprehensive introduction to this topic. For more works of the controllability of parabolic equations, we also refer to [16, 17, 18, 19] and [12].

However, as to our best knowledge, very few results are available to the control problems of the system (1.1). In Ryu and Yagi [34], it considers an optimal control problem of the system (1.1) with the control to be distributed on the second equation of (1.1). The present paper can be considered as a first work on the controllability of the system (1.1). There are some other kinds of interesting control problems for the system (1.1). In the system (1.1), the chemotactic term $-\chi \nabla \cdot (u \nabla v)$ causes much more mathematical difficulties than the coupled parabolic systems aforementioned. The techniques presented in this paper would be useful for other forms of chemotaxis system such as the parabolic-elliptic chemotaxis system, and even for other coupled systems like drift-diffusion equations from the semiconductor device.

The idea of obtaining the controllability of (1.1) is somehow classical: We first establish the null controllability of the linearized system and then apply the fixed point theorem. Now we consider the null controllability of the linearized system of (1.1), which is written as follows:

$$
\begin{cases}
\partial_t y = \Delta y - \nabla \cdot (B y) - \nabla \cdot (a \nabla z) + 1_\omega f & \text{in } Q, \\
\partial_t z = \Delta z - \gamma z + \delta y & \text{in } Q, \\
\partial_\nu y = 0, \partial_\nu z = 0 & \text{on } \Sigma, \\
y(x, 0) = y_0(x) \quad z(x, 0) = z_0(x) & x \in \Omega,
\end{cases}
$$

(1.4)

where $a \in L^\infty(Q)$, $B \in L^\infty(Q)^N$ with $B \cdot \nu = 0$ on $\Sigma$, $f \in L^2(Q)$ is the control force, and $y_0, z_0 \in L^2(\Omega)$ are given initial data. To study the null controllability of (1.4), we are led to consider the observability of the adjoint system of (1.4):

$$
\begin{cases}
-\partial_t \phi = \Delta \phi + B \nabla \phi + \delta \theta & \text{in } Q, \\
-\partial_t \theta = \Delta \theta - \gamma \theta - \nabla \cdot (a \nabla \phi) & \text{in } Q, \\
\partial_\nu \phi = 0, \partial_\nu \theta = 0 & \text{on } \Sigma, \\
\phi(x, T) = \phi^T(x), \theta(x, T) = \theta^T(x) & x \in \Omega,
\end{cases}
$$

(1.5)
where \( \phi^T, \theta^T \in L^2(\Omega) \). It is well-known that the null controllability of (1.4) is equivalent to the observability inequality for system (1.5):

\[
|\phi(\cdot, 0)|^2_2 + |\theta(\cdot, 0)|^2_2 \leq C \int \int_{Q_\omega} |\phi|^2 \, dx \, dt
\]

for every solution \((\phi, \theta)\) of (1.5). However, in order to obtain the input space of \(L^\infty(Q)\), we need to establish instead an improved observability inequality of the following form:

\[
|\phi(\cdot, 0)|^2_2 + |\theta(\cdot, 0)|^2_2 \leq C \int \int_{Q_\omega} e^{2s\alpha} |\phi|^2 \, dx \, dt,
\]

which can be derived from a global Carleman inequality

\[
\int \int_Q e^{2s\alpha} (|\phi|^2 + |\theta|^2) \, dx \, dt \leq C \int \int_{Q_\omega} e^{\frac{3}{2}s\alpha} |\phi|^2 \, dx \, dt
\]

for every solution \((\phi, \theta)\) of (1.5). Here, in (1.6) and (1.7), \(C\) denotes some positive constant independent of \(\phi\) and \(\theta\), \(\alpha = \alpha(x, t)\) is a weight function which will be specified precisely in Section 2, and \(s\) is a real number considered as parameter. The basic idea for the inequality (1.7) comes originally from [14] and [22], where similar inequalities are obtained for some cascaded system and parabolic system of phase-field.

Now we state our first result.

**Theorem 1.1.** Let \(T > 0\). For any \((y_0, z_0) \in L^2(\Omega) \times L^2(\Omega)\), there exists a control \(f \in L^\infty(Q)\) such that the solution \((y, z)\) of system (1.4) corresponding to \(f\) satisfies \((y, z) \in V^1(Q) \times V^1(Q)\) and \(y(x, T) = 0, \ z(x, T) = 0\) for \(x \in \Omega\) almost everywhere. Moreover, the control \(f\) satisfies

\[
\|f\|_\infty \leq C^\kappa (|y_0|_2 + |z_0|_2)
\]

where \(C\) is a positive constant depending only on \(\Omega\) and \(\omega\), and

\[
\kappa = (1 + \|a\|_\infty^2 + \|B\|_\infty^2)T + \frac{1}{T} + 1 + \|a\|_\infty + \|B\|_\infty.
\]

The approach used here to obtain the \(L^\infty(Q)\) control is originally from [7] (see also [36]). We improve this approach to get the explicit representation of the bound with respect to \(T\) by adopting some techniques from semigroup theory such as \(L^p-L^q\) estimate and maximal \(L^p\)-regularity.

The main result of this paper is the following Theorem 1.2.

**Theorem 1.2.** Let \(p > N + 2\). Let \((\overline{u}, \overline{v})\) be a trajectory of (1.2) corresponding to \((\overline{u}_0, \overline{v}_0)\) and satisfy (1.3). Then, there exists a positive constant \(c_1\) independent of \(T\) such that for each \((u_0, v_0)\) that satisfies

\[
|u_0 - \overline{u}_0|_\infty + |v_0 - \overline{v}_0|_{W^{2(1 - \frac{1}{p}), p}(\Omega)} \leq e^{-c_1(T + \frac{1}{T})},
\]

there is a control \(f \in L^\infty(Q)\) such that system (1.4) admits a solution \((u, v)\) satisfying

\[
u \in V^1(Q) \cap L^\infty(Q), \ v \in V^2(Q) \cap L^\infty(Q),
\]

and \(u(x, T) = \overline{u}(x, T), \ v(x, T) = \overline{v}(x, T)\) for \(x \in \Omega\) almost everywhere.
We proceed as follows. In next section, Section 2, we give some preliminary results. Section 3 is devoted to the proof of the Theorem 1.1. The proof of Theorem 1.2 is presented in section 4.

It is pointed out that throughout the paper, we use $C$ to denote a positive constant that is independent of time $T$ in most cases but may be dependent of $\Omega, \omega$. In the later case we may write $C(\Omega, \omega)$ instead of a special specification.

2 Preliminaries

In this section, we collect some results that are needed in later sections. These results are particularly useful in the establishment of the regularity of linear parabolic system and the $L^\infty$-estimate of controls.

For $p \in (1, \infty)$, let $A := A_p$ denote the sectorial operator defined by
\[
A_p u := -\Delta u, \quad \forall \, u \in D(A_p) := \{ u \in W^{2,p}(\Omega) ; \partial_\nu u|_{\partial\Omega} = 0 \}.
\] (2.1)

Suppose that $\gamma$ is a positive constant.

(i) Let $\alpha \geq 0$ and $D((A + \gamma)^\alpha)$ be the function space endowed with the graph norm. Then $D((A + \gamma)^\alpha)$ is a Banach space with the following embedding properties ([24, p.39])
\[
D((A + \gamma)^\alpha) \hookrightarrow W^{1,p}(\Omega) \quad \text{if } \alpha > \frac{1}{2},
\] (2.2)
and
\[
D((A + \gamma)^\alpha) \hookrightarrow C^\gamma(\Omega) \quad \text{if } 0 \leq \gamma < 2\alpha - \frac{n}{p}.
\] (2.3)

(ii) Let $\{e^{-tA}\}_{t \geq 0}$ and $\{e^{-t(A+\gamma)}\}_{t \geq 0}$ be the analytic $C_0$-semigroups generated by $-A$ and $-(A + \gamma)$ on $L^p(\Omega) (1 < p < \infty)$, respectively. By standard $C_0$-semigroup theory, we have ([13, 33])
\[
\left| e^{-tA}u \right|_q \leq C m(t)^{-\frac{\alpha}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} |u|_p,
\] (2.4)
and
\[
\left| (A + \gamma)^\alpha e^{-t(A+\gamma)}u \right|_q \leq C t^{-\frac{\alpha}{2} \left( \frac{1}{p} - \frac{1}{q} \right) - \alpha} |u|_p.
\] (2.5)

for all $u \in L^p(\Omega), t > 0$ and $1 < p \leq q < \infty$, where $m(t) = \min\{1, t\}$.

(iii) Let $\alpha \geq 0$ and $1 < p < \infty$. Then for any $\varepsilon > 0$ there exists a constant $C_\varepsilon$ depending on $\Omega, \varepsilon$ and $p$ such that ([28, Lemma 2.1])
\[
\left| (A + \gamma)^\alpha e^{-tA}\nabla u \right|_p \leq C_\varepsilon t^{-\alpha - \frac{1}{2} - \varepsilon} |u|_p
\] (2.6)

for all $u \in L^p(\Omega), t > 0$.

As a consequence of (2.4) and (2.6), we have
(iv) For any $\varepsilon > 0$, there exists a constant $C_\varepsilon$ depending on $\Omega$, $\varepsilon$ and $p$, such that
\[
|e^{-tA}\nabla \cdot u|_q \leq C_\varepsilon m(t)^{-\frac{1}{2}-\varepsilon} \frac{1}{(p_\varepsilon - \frac{1}{2})} |u|_p
\]
for all $u \in L^p(\Omega)$, $t > 0$, $1 < p \leq q < \infty$.

(v) (Maximal regularity) Let $1 < p < \infty$. If $F \in L^p(\Omega)$ and $u_0 \in W^{2(1-\frac{1}{p})p}(\Omega)$ with $\partial_\nu u_0 = 0$ on $\partial \Omega$, then there exists a unique solution of
\[
\frac{du}{dt} = (A + \gamma)u + F \text{ for a.e. } t \in (0, T), \quad u(0) = u_0
\]
that satisfies
\[
\left\| \frac{du}{dt} \right\|_p^p + \|u\|_p^p \leq C \left( \|F\|_p^p + \|u_0\|_{W^{2(1-\frac{1}{p})p}(\Omega)}^p \right), \tag{2.8}
\]
where $C$ is a positive constant independent of $T$ and $F$.

Inequality (2.8) was first established as Theorem 9.1 of [30] in Chapter IV, but the independency of $C$ with respect to $T$ is given later as Theorem 1.1 of [31] (see also Theorem 2.3 of [21]).

Now we consider the well-posedness of the following linear parabolic system which contains (2.4) as its special case.

\[
\begin{aligned}
\partial_t y &= \Delta y - \nabla \cdot (By) - \nabla \cdot (a \nabla z) + F \quad \text{in } Q, \\
\partial_t z &= \Delta z - \gamma z + \delta y \quad \text{in } Q, \\
\partial_\nu y &= 0, \quad \partial_\nu z = 0 \quad \text{on } \Sigma, \\
y(x, 0) &= y_0(x) \quad \text{and } \quad z(x, 0) = z_0(x) \quad \text{for all } x \in \Omega,
\end{aligned}
\tag{2.9}
\]

**Proposition 2.1.** Let $a \in L^\infty(\Omega)$ and $B \in L^\infty(\Omega)^N$ with $B \cdot \nu = 0$ on $\Sigma$.

(i) If $y_0, z_0 \in L^2(\Omega)$ and $F \in L^2(\Omega)$, then system (2.9) admits a unique solution $(y, z) \in V^1(\Omega) \times V^1(\Omega)$ satisfying
\[
\|y\|_{V^1(\Omega)}^2 + \|z\|_{V^1(\Omega)}^2 \leq e^{C_K} \left( \|y_0\|_2^2 + \|z_0\|_2^2 + \|F\|_2^2 \right); \tag{2.10}
\]

(ii) Let $2 \leq p < \infty$. If $F \in L^p(\Omega)$, $y_0 \in L^p(\Omega)$ and $z_0 \in W^{2(1-\frac{1}{p})p}(\Omega)$ with $\partial_\nu z_0 = 0$ on $\partial \Omega$, then system (2.9) admits a unique solution $(y, z) \in L^p(\Omega) \times W^{2,1}_p(\Omega)$ satisfying
\[
\|y\|_p^p + \|z\|_p^{2,1} \leq e^{C_K} \left( \|y_0\|_p^p + \|z_0\|_{W^{2(1-\frac{1}{p})p}(\Omega)}^p + \|F\|_p^p \right); \tag{2.11}
\]

(iii) Let $p > N + 2$. If $F \in L^\infty(\Omega)$, $y_0 \in L^\infty(\Omega)$ and $z_0 \in W^{1,p}(\Omega)$ with $\partial_\nu z_0 = 0$ on $\partial \Omega$, then system (2.9) admits a solution $(y, z) \in L^\infty(\Omega) \times L^\infty(\Omega)$ satisfying
\[
\|y\|_\infty + \|z\|_\infty \leq e^{C_K} \left( \|y_0\|_\infty + \|z_0\|_{W^{1,p}(\Omega)} + \|F\|_\infty \right), \tag{2.12}
\]

7
where \( \kappa \) is given by (1.9) and \( C = C(\Omega) \).

**Proof.** The existence of solution with respect to \( y_0, z_0 \) and \( F \) in different function spaces can be deduced similarly as in (30) for which we omit here. We only show the required estimates with respect to time \( T \). Since the proof for (2.10) is similar to (2.11), we need only to show (2.11).

Multiply the first equation of (2.9) by \( |y|^{p-2}y \) and integrate over \( \Omega \), to get
\[
\frac{d}{dt} |y|_p^p + \int_\Omega |\nabla y|^2 |y|^{p-2} dx \leq C \left( 1 + \|a\|_\infty^2 + \|B\|_\infty^2 \right) |y|_p^p + C \|a\|_\infty^2 |\nabla z|_p^p + C |F|_p^p, \tag{2.13}
\]
and in the same way, to get from the second equation of (2.9) that
\[
\frac{d}{dt} |z|_p^p + \int_\Omega |\nabla z|^2 |z|^{p-2} dx + |z|_p^p \leq C |y|_p^p. \tag{2.14}
\]

Differentiate \( |\nabla z|_p^p \) with respect to \( t \) and take the second equation of (2.9) into account again to obtain
\[
\frac{d}{dt} |\nabla z|_p^p + \int_\Omega |\nabla z|^{p-2} |\Delta z|^2 dx \leq C |\nabla z|_p^p + C \left( |y|_p^p + |z|_p^p \right). \tag{2.15}
\]

The inequalities (2.13), (2.15) together with Gronwall’s inequality lead to
\[
|y(\cdot, t)|_p^p + |z(\cdot, t)|_p^p + |\nabla z(\cdot, t)|_p^p \leq e^{C_\kappa} \left( |y_0|_p^p + \|z_0\|_{W^{2, p}(\Omega)} \right) \tag{2.16}
\]
for all \( t \in [0, T] \). On the other hand, by the maximal regularity (2.8) for the second equation of (2.9), it follows that
\[
\| \partial_t z \|_p^p + \| \Delta z \|_p^p + \| z \|_p^p \leq C \left( \| z_0 \|_{W^{2(1-\frac{1}{p}), P}(\Omega)} + \| y \|_p^p + \| z \|_p^p \right),
\]
which together with (2.16) yields (2.11).

Now we turn to the \( L^\infty \)-estimate (2.12). We first assume that \( y_0 \in C (\overline{\Omega}) \) and \( F \in C (\overline{\Omega}) \). Let \( A \) be defined by (2.1), and let \( \{e^{-tA} \}_{t \geq 0} \) and \( \{e^{-t(A+\gamma)} \}_{t \geq 0} \) be the analytic \( C_0 \)-semigroups generated by \( -A \) and \( -(A+\gamma) \) in \( L^p(\Omega), 1 < p < \infty \), respectively. Then the solution \((y, z)\) of system (2.9) can be represented as follows
\[
y(\cdot, t) = e^{-tA} y_0 + \int_0^t e^{-(t-s)A} \left( -\nabla \cdot (By) - \nabla \cdot (a \nabla z) + F \right) (\cdot, s) ds, \tag{2.17}
\]
\[
z(\cdot, t) = e^{-t(A+\gamma)} z_0 + \delta \int_0^t e^{-(t-s)(A+\gamma)} y(\cdot, s) ds. \tag{2.18}
\]

Take the norm of \( C (\overline{\Omega}) \) on both sides of (2.17) to get
\[
\| y(\cdot, t) \|_{C(\overline{\Omega})} \leq \left\| e^{-tA} y_0 \right\|_{C(\overline{\Omega})} + \int_0^t \left\| e^{-(t-s)A} \nabla \cdot (By + a \nabla z) (\cdot, s) \right\|_{C(\overline{\Omega})} ds
\]
\[+ \int_0^t \left\| e^{-(t-s)A} F(\cdot, s) \right\|_{C(\overline{\Omega})} ds. \tag{2.19}
\]

To estimate (2.19), we first observe that the operator \(-A\) generates a bounded analytic semigroup on \( C (\overline{\Omega}) \) (16). It follows from the maximum principle that
\[
\left\| e^{-tA} y_0 \right\|_{C(\overline{\Omega})} \leq \| y_0 \|_{C(\overline{\Omega})}, \tag{2.20}
\]
and
\[
\left\| e^{-(t-s)A} F(\cdot, s) \right\|_{C(\overline{\Omega})} \leq \left\| F(\cdot, s) \right\|_{C(\overline{\Omega})} \tag{2.21}
\]
for any \(0 \leq s \leq t\). Since \(p > N + 2\), we can take \(\varepsilon\) and \(\alpha\) such that
\[
0 < \varepsilon < \frac{p - N - 2}{2p} \quad \text{and} \quad \frac{N}{2p} < \alpha < \frac{1}{2} - \frac{1}{p} - \varepsilon.
\]
Then, with the help of (2.3), (2.6) and the Hölder inequality, we have, for any \(t \in [0, T]\), that
\[
\int_0^t \left\| e^{-(t-s)A} \nabla \cdot (By + a \nabla z) \right\| ds \leq \left\| (A + \gamma)^{\alpha} e^{-(t-s)A} (By + a \nabla z) \right\|_{p} \tag{2.16}
\]
for any \(0 \leq t \leq T\). Since \(|y| \leq C\), we can take
\[
\int_0^T \left\| e^{-(t-s)A} \nabla \cdot (By + a \nabla z) \right\| ds \leq C (1 + \|a\|_{\infty} + \|B\|_{\infty}) \left(\|y\|_{p} + \|z\|_{p} \right) T^{\frac{1}{2} - \alpha - \frac{1}{p}}.
\]
This together with (2.16) gives
\[
\int_0^t \left\| e^{-(t-s)A} \nabla \cdot (By + a \nabla z) \right\| ds \leq C \kappa \left(\|y_0\|_{p} + \|z_0\|_{W^{1,p}(\Omega)} + \|F\|_{p} \right). \tag{2.22}
\]
By (2.19) - (2.22), we obtain
\[
\|y\|_{\infty} \leq C \kappa \left(\|y_0\|_{\infty} + \|z_0\|_{W^{1,p}(\Omega)} + \|F\|_{\infty} \right). \tag{2.23}
\]
Next, take the norm of \(W^{1,p}(\Omega)\) on both sides of (2.18) to get
\[
\|z(\cdot, t)\|_{W^{1,p}(\Omega)} \leq \left\| e^{-(t(A+\gamma))} z_0 \right\|_{W^{1,p}(\Omega)} + \delta \int_0^t \left\| e^{-(t-s)(A+\gamma)} y(\cdot, s) \right\|_{W^{1,p}(\Omega)} ds, \tag{2.24}
\]
for any \(t \in [0, T]\). To estimate (2.24), we first notice that
\[
\left\| e^{-(t(A+\gamma))} z_0 \right\|_{W^{1,p}(\Omega)} \leq C T \|z_0\|_{W^{1,p}(\Omega)} \tag{2.25}
\]
which can be obtained by the same energy method used in proving (2.16). Let \(\frac{1}{2} < \alpha < 1 - \frac{1}{p}\). By (2.7), (2.5) and the Hölder inequality, we have that for any \(t \in [0, T]\),
\[
\int_0^t \left\| e^{-(t-s)(A+\gamma)} y(\cdot, s) \right\|_{W^{1,p}(\Omega)} ds \leq C \int_0^t \left| (A + \gamma)^{\alpha} e^{-(t-s)(A+\gamma)} y(\cdot, s) \right|_{p} ds \tag{2.26}
\]
This together with (2.16) gives
\[
\int_0^t \left\| e^{-(t-s)(A+\gamma)} y(\cdot, s) \right\|_{W^{1,p}(\Omega)} ds \leq e^{C \kappa} \left(\|y_0\|_{p} + \|z_0\|_{W^{1,p}(\Omega)} + \|F\|_{p} \right). \tag{2.26}
\]
Finally, by (2.23)–(2.26) and the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ for $p > N$, we get

$$
\|z\|_\infty \leq e^{C\kappa} \left( \|y_0\|_\infty + \|z_0\|_{W^{1,p}(\Omega)} + \|F\|_\infty \right).
$$

To complete the proof, let us consider the general case that $y_0 \in L^\infty(\Omega)$ and $F \in L^\infty(Q)$. This can be done by smoothing the data and density argument. Precisely, let $\{y_{0n}\}_{n=1}^\infty \subset C(\bar{\Omega})$ and $\{F_n\}_{n=1}^\infty \subset C(\bar{\Omega})$ be such that $y_{0n} \to y_0$ in $L^2(\Omega)$, $F_n \to F$ in $L^2(Q)$ and $\|y_{0n}\|_\infty \leq \|y_0\|_\infty$, $\|F_n\|_\infty \leq \|F\|_\infty$. For each $n$, let $(y_n, z_n)$ be a solution of (2.9) corresponding to $y_{0n}$, $z_0$, $F_n$, which satisfies the inequalities (2.10) and (2.12) with $(y, z)$ replaced by $(y_n, z_n)$. Thus, by the uniformly boundedness, we can extract subsequences of $(y_n, z_n)$ such that it converges to $(y, z)$, which is a weak solution of (2.9) corresponding to $y_0$, $z_0$ and $F$. Moreover, $y, z$ satisfy the inequality (2.12).

\[\square\]

3 Proof of Theorem 1.1

To prove Theorem 1.1, we first establish a global Carleman inequality for the adjoint system (1.5).

Let $\omega' \subseteq \omega$, that is, $\overline{\omega'} \subseteq \omega$, be a nonempty open subset. Then, there is a function $\beta \in C^2(\bar{\Omega})$ such that $\beta(x) > 0$ for all $x \in \Omega$, and $\beta|_{\partial\Omega} = 0$, $|\nabla \beta(x)| > 0$ for all $x \in \bar{\Omega} \setminus \omega'$ (see [19, Lemma 1.1]). For $\lambda > 0$, set

$$
\varphi = \frac{e^{\lambda \beta}}{t(T-t)}, \quad \alpha = \frac{e^{\lambda \beta} - e^{2\lambda \|\beta\|_{C(\overline{\Omega})}}}{t(T-t)},
$$

and

$$
\gamma(\lambda) = e^{2\lambda \|\beta\|_{C(\overline{\Omega})}}.
$$

Lemma 3.1. Let $f_i \in L^2(Q)$, $i = 0, 1, \ldots, N$. Then there exists a constant $\lambda_0 = \lambda_0(\Omega, \omega') > 1$, such that for all $\lambda \geq \lambda_0$ and $s \geq \gamma(\lambda)(T + T^2)$,

$$
\begin{align*}
\iint_Q \left[ (s\varphi)^{1+d} |\nabla z|^2 + (s\varphi)^{3+d} |z|^2 \right] e^{2s\alpha} \, dx dt \\
\leq \quad C \left( \iint_Q (s\varphi)^d e^{2s\alpha} |f_0|^2 \, dx dt + \sum_{i=1}^N \iint_Q (s\varphi)^{2+d} e^{2s\alpha} |f_i|^2 \, dx dt \right. \\
+ \left. \iint_{Q_{\omega'}} (s\varphi)^{3+d} e^{2s\alpha} |z|^2 \, dx dt \right)
\end{align*}
$$

for all solutions $z$ to the equation

$$
\begin{cases}
\partial_t z - \Delta z = f_0 + \sum_{i=1}^N \frac{\partial f_i}{\partial x_i} \quad \text{in } Q, \\
\partial_\nu z = 0 \quad \text{on } \Sigma, \\
z(x, 0) = z_0(x) \quad x \in \Omega,
\end{cases}
$$

with $z_0 \in L^2(\Omega)$, where $C = C(\Omega, \omega')$, and $\gamma(\lambda)$ given by (3.2).
Essentially speaking, Lemma 3.1 has been proven in [23] (see also [22]) but the explicit independence of the constant \( C \) with respect to \( T \) is shown in a similar way as in [16] and [18]. For notational simplicity in the sequel, we introduce

\[
I_1(s, \lambda; \phi) = \int_Q \left[ (s \phi)^3 |\nabla \phi|^2 + (s \phi)^5 |\phi|^2 \right] e^{2s\alpha} dx dt, \tag{3.4}
\]

and

\[
I_2(s, \lambda; \theta) = \int_Q \left[ s \phi |\nabla \theta|^2 + (s \phi)^3 |\theta|^2 \right] e^{2s\alpha} dx dt. \tag{3.5}
\]

**Lemma 3.2.** There exists a positive constant \( \lambda_1 = C(\Omega, \omega')(1 + \|a\|_2^2 + \|B\|_{\infty}^2) \) satisfying \( \gamma(\lambda_1) \geq \lambda_1 > 1 \) such that for any \( \lambda \geq \lambda_1, s \geq \gamma(\lambda)(T + T^2) \) and \( \phi^T, \theta^T \in L^2(\Omega) \), the associated solution \((\phi, \theta)\) to (1.5) satisfies

\[
I_1(s, \lambda; \phi) + I_2(s, \lambda; \theta) \leq C_1 \int_{Q_\omega} \lambda^8(s \phi)^9 e^{2s\alpha} |\phi|^2 dx dt, \tag{3.6}
\]

where \( C_1 = C_1(\Omega, \omega', \omega) \).

**Proof.** Applying Lemma 3.1 to the first equation of (1.5) with \( d = 2 \) and the second one with \( d = 0 \), respectively, we obtain that there exist positive constants \( c_0(\Omega, \omega') \) and \( \lambda_1^0 \) satisfying

\[
\gamma(\lambda_1^0) \geq \lambda_1^0 = c_0(\Omega, \omega') \left( 1 + \|a\|_\infty^2 + \|B\|_{\infty}^2 \right) > 1 \tag{3.7}
\]

such that for all \( \lambda \geq \lambda_1^0 \) and \( s \geq \gamma(\lambda)(T + T^2) \),

\[
I_1(s, \lambda; \phi) + I_2(s, \lambda; \theta) \leq c_1 \int_{Q_{\omega'}} \left[ (s \phi)^5 |\phi|^2 + (s \phi)^3 |\theta|^2 \right] e^{2s\alpha} dx dt \tag{3.8}
\]

for all solutions \((\phi, \theta)\) to (1.5) with \( \phi^T, \theta^T \in L^2(\Omega) \), where and in what follows, the symbol \( c_i, i = 1, 2, \ldots, \) stand for some positive constants depending on \( \Omega, \omega' \) and \( \omega \).

Next, let \( \xi \in C_0^\infty(\Omega) \) be such that \( \xi = 1 \) in \( \omega' \), \( \xi = 0 \) in \( \Omega \setminus \bar{\omega} \), \( 0 \leq \xi \leq 1 \) in \( \omega \), and

\[
\Delta \xi \cdot \xi^{-1/2} \in L^\infty(\Omega), \nabla \xi \cdot \xi^{-1/2} \in L^\infty(\Omega)^N. \tag{3.9}
\]

The existence of such a function \( \xi \) is easy to obtain (see, for instance [14]). Set

\[
\eta = (s \phi)^3 e^{2s\alpha}.
\]

Multiply the first equation of (1.5) by \( \theta \eta \xi \) to get

\[
\delta \int_Q (s \phi)^3 e^{2s\alpha} |\theta|^2 \xi dx dt = \int_Q \eta \xi \theta [-\partial_t \phi - \Delta \phi - B \nabla \phi] dx dt
\]

\[
= \int_Q \{ \eta \xi [\Delta \theta + \gamma \theta + \nabla \cdot (a \nabla \phi)] + \phi \theta \xi (\partial_t \eta) + \eta \xi \theta (\Delta \phi - B \nabla \phi) \} dx dt.
\]

Integration by parts gives

\[
\delta \int_Q (s \phi)^3 e^{2s\alpha} |\theta|^2 \xi dx dt = \sum_{i=1}^7 J_i, \tag{3.10}
\]
where
\[
J_1 = \int_Q \phi \theta (\partial_t \eta + \gamma \eta) \xi dxdt, \quad J_2 = \int_Q \phi \nabla (\eta \xi) \cdot \nabla \theta dxdt,
\]
\[
J_3 = -\int_Q a \phi \nabla (\eta \xi) \cdot \nabla \phi dxdt, \quad J_4 = \int_Q \theta \nabla (\eta \xi) \cdot \nabla \phi dxdt,
\]
\[
J_5 = -\int_Q \theta \eta \xi B \nabla \phi dxdt, \quad J_6 = \int_Q \eta \xi \nabla \theta \cdot \nabla \phi dxdt,
\]
\[
J_7 = -\int_Q a |\nabla \phi|^2 \eta \xi dxdt.
\]

To estimate these integrals, we first observe by (3.1) and (3.9) that
\[
|\partial_t \eta| \leq (s \varphi)^5 e^{2s\alpha}; \quad |\nabla (\eta \xi)| \leq C (\xi^{1/2} s^{3 \varphi} + \xi \lambda s^{4 \varphi}) e^{2s\alpha}.
\]

This together with Cauchy’s inequality gives the estimation of \(J_i, i = 1, \ldots, 6\) as follows:

\[
J_1 \leq \varepsilon_1 I_2(s, \lambda; \theta) + \frac{C}{4 \varepsilon_1} \int_Q [(s \varphi)^3 + (s \varphi)^7] e^{2s\alpha} |\phi|^2 \xi dxdt; \quad (3.11)
\]

\[
J_2 \leq \varepsilon_1 I_2(s, \lambda; \theta) + \frac{C}{4 \varepsilon_1} \int_Q [(s \varphi)^5 + \lambda^2 (s \varphi)^7] e^{2s\alpha} |\phi|^2 \xi dxdt; \quad (3.12)
\]

\[
J_3 \leq \varepsilon_1 I_1(s, \lambda; \phi) + \frac{C ||a||^2_\infty}{4 \varepsilon_1} \int_Q [(s \varphi)^3 + \lambda^2 (s \varphi)^5] e^{2s\alpha} |\phi|^2 \xi dxdt; \quad (3.13)
\]

\[
J_4 \leq \varepsilon_1 I_2(s, \lambda; \theta) + \frac{c_2}{2 \varepsilon_1} \int_Q [(s \varphi)^3 + \lambda (s \varphi)^5] e^{2s\alpha} |\nabla \phi|^2 \xi dxdt; \quad (3.14)
\]

\[
J_5 \leq \varepsilon_1 I_2(s, \lambda; \theta) + \frac{c_3 \|B\|^2_\infty}{4 \varepsilon_1} \int_Q (s \varphi)^3 e^{2s\alpha} |\nabla \phi|^2 \xi dxdt; \quad (3.15)
\]

\[
J_6 \leq \varepsilon_1 I_2(s, \lambda; \theta) + \frac{c_4}{4 \varepsilon_1} \int_Q (s \varphi)^5 e^{2s\alpha} |\nabla \phi|^2 \xi dxdt; \quad (3.16)
\]

\[
J_7 \leq ||a||_\infty \int_Q (s \varphi)^3 e^{2s\alpha} |\nabla \phi|^2 \xi dxdt; \quad (3.17)
\]

where \(\varepsilon_1\) is an arbitrary positive constant which will be determined later.

The inequalities (3.14)-(3.17) lead to

\[
J_4 + J_5 + J_6 + J_7 \leq 3 \varepsilon_1 I_2(s, \lambda; \theta) + \frac{c_5}{\varepsilon_1} (1 + ||a||^2_\infty + \|B\|^2_\infty) \times \int_Q \lambda^2 (s \varphi)^5 e^{2s\alpha} |\nabla \phi|^2 \xi dxdt. \quad (3.18)
\]

Next, we estimate the integral on the right hand side of the inequality (3.18). Let
\[
\tilde{\eta} = \lambda^2 (s \varphi)^5 e^{2s\alpha}.
\]

Multiply the first equation of (1.5) by \(\tilde{\eta} \xi \phi\) and integrate over \(Q\) to obtain, by the integration by parts, that
\[
\int_Q \lambda^2 (s \varphi)^5 e^{2s\alpha} |\nabla \phi|^2 \xi dxdt = \sum_{i=8}^{11} J_i,
\]
where
\[ J_8 = -\frac{1}{2} \int_\Omega \phi^2 \partial_t \bar{\eta} dx dt, \quad J_9 = -\int_\Omega \phi \nabla (\bar{\eta} \xi) \cdot \nabla \phi dx dt, \]
\[ J_{10} = \int_\Omega \phi \bar{\eta} B \nabla \phi dx dt, \quad J_{11} = \delta \int_\Omega \theta \phi \bar{\eta} dx dt. \]

Since
\[ |\partial_t \bar{\eta}| \leq C\lambda^2 (s\varphi)^7 e^{2s\alpha}, \quad |\nabla (\bar{\eta} \xi)| \leq C (\xi^{1/2} \lambda^2 s^5 \varphi^5 + \lambda^3 s^6 \varphi^6 \xi) e^{2s\alpha}, \]
in the same way of estimating \( J_1 - J_7 \), we can get for any \( \varepsilon > 0 \) that
\[ J_8 \leq C \int_\Omega \lambda^2 (s\varphi)^7 e^{2s\alpha} |\phi|^2 \xi dx dt; \]  
(3.19)
\[ J_9 \leq \varepsilon_2 I_1(s, \lambda; \phi) + \frac{C}{2\varepsilon_2} \int_\Omega [\lambda^4 (s\varphi)^7 + \lambda^6 (s\varphi)^9] e^{2s\alpha} |\phi|^2 \xi dx dt; \]  
(3.20)
\[ J_{10} \leq \varepsilon_2 I_2(s, \lambda; \theta) + \frac{C}{4\varepsilon_2} \int_\Omega \lambda^4 (s\varphi)^7 e^{2s\alpha} |\phi|^2 \xi dx dt; \]  
(3.21)
\[ J_{11} \leq \varepsilon_2 I_1(s, \lambda; \phi) + \frac{C}{4\varepsilon_2} \int_\Omega \lambda^4 (s\varphi)^7 e^{2s\alpha} |\phi|^2 \xi dx dt. \]  
(3.22)

Finally, we take
\[ \varepsilon_1 = \frac{\delta}{10c_1}, \quad \text{and} \quad \varepsilon_2 = \frac{\delta}{10c_1 c_5 \left( 1 + \|a\|^2_\infty + \|B\|^2_\infty \right)} \times \frac{\delta}{20c_1} \]
to get, from (3.10)-(3.22), that
\[ I_1(s, \lambda; \phi) + I_2(s, \lambda; \theta) \leq c_0 \left( 1 + \|a\|^2_\infty + \|B\|^2_\infty \right)^2 \int_{Q_\omega} \lambda^6 (s\varphi)^9 e^{2s\alpha} |\phi|^2 \xi dx dt. \]

Thus there is a positive constant
\[ \gamma(\lambda_1) \geq \lambda_1 = c_0 \left( 1 + \|a\|^2_\infty + \|B\|^2_\infty \right) \geq \lambda_1^0 > 1 \]
such that for any \( \lambda \geq \lambda_1 \) and \( s \geq \gamma(\lambda)(T + T^2) \), the inequality (3.6) holds, where \( \lambda_1^0 \) is given by (3.7).

\[ \square \]

**Proposition 3.1.** There exist positive constants \( \lambda \) and \( s \) such that, for all \( T > 0, \phi^T, \theta^T \in L^2(\Omega) \), the solution \( (\phi, \theta) \) of the system (1.5) satisfies
\[ |\phi(\cdot, 0)|_2^2 + |\theta(\cdot, 0)|_2^2 \leq e^{C\kappa} \int_{Q_\omega} e^{\frac{4}{3} s\alpha} |\phi|^2 \xi dx dt, \]  
(3.23)

where \( \kappa \) is given by (1.9).

**Proof.** By integration by parts, we observe that
\[ -\frac{d}{dt} |\phi|^2_2 + |\nabla \phi|^2_2 \leq (1 + \|B\|^2_\infty) |\phi|^2_2 + \delta^2 |\theta|^2_2, \]  
(3.24)
and
\[ -\frac{d}{dt} |\theta|^2_2 + |\nabla \theta|^2_2 + 2\gamma |\theta|^2_2 \leq \|a\|^2_\infty |\nabla \phi|^2_2. \]  
(3.25)
Suppose first that \( \|a\|_\infty \geq 1 \). Multiply (3.24) by \( \|a\|^2_\infty \) to get by (3.25) that
\[
\frac{d}{dt} \left[ e^{C(1+\|a\|^2_\infty + \|B\|^2_\infty) t} \left( \|a\|^2_\infty + \|B\|^2_\infty \right) \right] \geq 0.
\]
Integrating above inequality over \([0, t]\) for any \( t \in (0, T) \) gives
\[
\|a\|^2_\infty \|\phi(\cdot, 0)\|^2_2 + \|\theta(\cdot, 0)\|^2_2 \leq e^{C(1+\|a\|^2_\infty + \|B\|^2_\infty) T} \left( \|a\|^2_\infty \|\phi(\cdot, t)\|^2_2 + \|\theta(\cdot, t)\|^2_2 \right),
\]
which implies that
\[
\|\phi(\cdot, 0)\|^2_2 + \|\theta(\cdot, 0)\|^2_2 \leq e^{C[1+\|a\|^2_\infty + \|B\|^2_\infty]T + \|a\|_\infty} \left( \|\phi(\cdot, t)\|^2_2 + \|\theta(\cdot, t)\|^2_2 \right)
\]
for any \( t \in (0, T) \). The integration of (3.20) on both sides over \([T/4, 3T/4] \) leads to
\[
\|\phi(\cdot, 0)\|^2_2 + \|\theta(\cdot, 0)\|^2_2 \leq \frac{2}{T} e^{C[1+\|a\|^2_\infty + \|B\|^2_\infty]T + \|a\|_\infty} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\Omega} |\phi|^2 + |\theta|^2 \, dxdt.
\]
Since
\[
(s\varphi)^{-5} e^{-2\alpha}, (s\varphi)^{-3} e^{-2\alpha} \leq \frac{C_\alpha}{T^2} \quad \text{in} \quad \Omega \times \left[ \frac{T}{4}, \frac{3T}{4} \right],
\]
it follows by (3.6) that
\[
\|\phi(\cdot, 0)\|^2_2 + \|\theta(\cdot, 0)\|^2_2 \leq \frac{2C_\alpha}{T} e^{C[1+\|a\|^2_\infty + \|B\|^2_\infty]T + \|a\|_\infty} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{Q^c_\omega} \lambda^8 (s\varphi)^9 e^{2\alpha} |\phi|^2 \, dxdt,
\]
where by taking \( \lambda = C \left( 1 + \|a\|^2_\infty + \|B\|^2_\infty \right) \) and \( s = C \left( 1 + \|a\|^2_\infty + \|B\|^2_\infty \right) (T + T^2) \),
we get (3.23).

Finally, if \( \|a\|_\infty < 1 \), then
\[
\frac{d}{dt} \left[ e^{C(1+\|B\|^2_\infty) t} \left( \|\phi\|^2_2 + \|\theta\|^2_2 \right) \right] \geq 0.
\]
is a direct consequence of (3.24) and (3.25). Thus, (3.26) verifies. In a similar argument as in the proof of \( \|a\|_\infty \geq 1 \), one can easily get (3.23). This completes the proof.

\[\square\]

**Proof of Theorem 1.1.** Let \( s \) and \( \lambda \) be such that the observability estimate (3.23) and
\[
\eta(\lambda) = e^{-\lambda \|\beta\|_C} \leq \frac{1}{2},
\]
hold. Let \( \varepsilon > 0 \) and consider the following optimal control problem
\[
\text{Minimize} \left\{ \int_{Q_\omega} |f|^2 e^{-\frac{3}{2} \alpha} \, dxdt + \frac{1}{\varepsilon} \left( |y(\cdot, T)|^2_2 + |z(\cdot, T)|^2_2 \right) \right\}
\]
subject to all \( f \in L^2(Q) \), where \((y, z)\) is the solution of (1.1) associated to \( f \). The existence of an optimal pair \((f_\varepsilon, y_\varepsilon, z_\varepsilon)\) to the above optimal control problem follows from the standard argument. By the Pontryagin maximum principle (S),
\[
f_\varepsilon = 1_{\omega} \phi e^{\frac{3}{2} \alpha}.
\]
Here, \((\phi_\varepsilon, \theta_\varepsilon)\) is the solution of the adjoint system following:

\[
\begin{align*}
-\partial_t \phi_\varepsilon &= \Delta \phi_\varepsilon + B \nabla \phi_\varepsilon + \delta \theta_\varepsilon & \text{in } Q, \\
-\partial_t \theta_\varepsilon &= \Delta \theta_\varepsilon - \gamma \theta_\varepsilon - \nabla \cdot (a \nabla \phi_\varepsilon) & \text{in } Q, \\
\partial_\nu \phi_\varepsilon &= 0, \partial_\nu \theta_\varepsilon &= 0 & \text{on } \Sigma, \\
(\phi_\varepsilon(x), \theta_\varepsilon(x, T)) &= -\frac{1}{\varepsilon}(y_\varepsilon, \varepsilon z_\varepsilon)(x, T) & x \in \Omega,
\end{align*}
\]

where \((y_\varepsilon, z_\varepsilon)\) is the solution of (1.4) with \(f = f_\varepsilon\). By (1.4), (3.28), (3.29), and Proposition 3.1 it follows that

\[
\int_Q \int_0^T |\phi_\varepsilon|^2 e^{\frac{4}{\varepsilon^2 \alpha_0 dx dt} + \frac{1}{\varepsilon}} \left(|y(\cdot, T)|^2 + |z(\cdot, T)|^2 \right) \leq e^{C_\varepsilon} \left(|y_0|^2 + |z_0|^2 \right). \tag{3.30}
\]

We can simply get from (3.28) and (3.30) that the control function \(f_\varepsilon\) satisfies

\[
\|f_\varepsilon\|_2^2 \leq e^{C_\varepsilon} \left(|y_0|^2 + |z_0|^2 \right). \tag{3.31}
\]

Next we show that \(f_\varepsilon\) can be taken in \(L^\infty(\Omega)\). To this end, let \(\tau\) be a sufficiently small positive constant and let \(\{\tau_j\}_{j=0}^{M+1}\) be a finite increasing sequence such that \(0 < \tau_j < \tau, j = 0, 1, \ldots, M, \tau_{M+1} = \tau\). Let \(\{p_i\}_{i=0}^1\) be another finite increasing sequence such that \(p_0 = 2, p_M > (N + 2)/2\) and,

\[
\left(\frac{N + 1}{2} \left(\frac{1}{p_i} - \frac{1}{p_{i+1}} \right) \leq \frac{1}{4}, i = 0, 1, \ldots, M - 1. \tag{3.32}
\]

Set

\[
\alpha_0 = \min_\Omega \frac{1 - e^{-2\|\alpha\|_\Omega \varepsilon_0}}{t(T - t)}.
\]

By (3.31),

\[
\alpha_0 \leq \alpha \leq \frac{\alpha_0}{1 + \eta(\lambda)} < 0,
\]

where \(\eta(\lambda)\) is defined by (3.27).

For each \(i, i = 0, 1, \ldots, M, M + 1\), define

\[
\zeta_i(x, t) = e^{(s + \tau_j)\alpha_0} \phi_\varepsilon(x, T - t), \\
\vartheta_i(x, t) = e^{(s + \tau_j)\alpha_0} \theta_\varepsilon(x, T - t), \\
G_i(x, t) = \left[\partial_t (e^{(s + \tau_j)\alpha_0}) \right] \phi_\varepsilon(x, T - t), \\
H_i(x, t) = \left[\partial_t (e^{(s + \tau_j)\alpha_0}) \right] \theta_\varepsilon(x, T - t),
\]

and

\[
\tilde{a}(x, t) = a(x, T - t), \tilde{B}(x, t) = B(x, T - t).
\]

Then for each \(i\), \((\zeta_i, \vartheta_i)\) solves the following system:

\[
\begin{align*}
\partial_t \zeta_i - \Delta \zeta_i &= \tilde{B} \nabla \zeta_i + \delta \vartheta_i + G_i & \text{in } Q, \\
\partial_t \vartheta_i - \Delta \vartheta_i &= -\gamma \vartheta_i + \nabla \cdot (\tilde{a} \nabla \zeta_i) + H_i & \text{in } Q, \\
\partial_\nu \zeta_i &= 0, \partial_\nu \vartheta_i &= 0 & \text{on } \Sigma, \\
\zeta_i(x, 0) &= 0, \vartheta_i(x, 0) &= 0 & x \in \Omega.
\end{align*}
\]
Now we apply the $L^p$-$L^q$ estimate to the above system. By the semigroup theory, the solution $(\zeta_i, \varrho_i)$, $i = 1, 2, \ldots, M + 1$, of (3.32) can be represented as

$$
\zeta_i(\cdot, t) = \int_0^t e^{-(t-s)A} \left[ \hat{B} \nabla \zeta_i + \delta \varrho_i + G_i \right] (\cdot, s) ds, \quad (3.33)
$$

$$
\varrho_i(\cdot, t) = \int_0^t e^{-(t-s)A} \left[ -\gamma \varrho_i - \nabla \cdot (\tilde{a} \nabla \zeta_i) + H_i \right] (\cdot, s) ds. \quad (3.34)
$$

Firstly, by (2.4) to (3.33), we have

$$
|\zeta_i(\cdot, t)|_{p_i} = C \left[ \int_0^t m(t-s)^{-\frac{N}{2} \left( \frac{1}{p_i-1} - \frac{1}{m} \right)} \left| \left( \hat{B} \nabla \zeta_i + \delta \varrho_i + G_i \right) (\cdot, s) \right|_{p_i} ds, \right.
$$

which can be estimated by Young’s convolution inequality (see, e.g. [13, p.3]) as

$$
\|\zeta_i\|_{p_i} \leq C \left[ \|B\|_{\infty} \left\| \nabla \zeta_i \right\|_{p_i-1} + \|\varrho_i\|_{p_i-1} + \|G_i\|_{p_i-1} \right] \times \left[ \int_0^T m(t)^{-\frac{N}{2} \left( \frac{1}{p_i-1} - \frac{1}{m} \right) \frac{1}{r_i}} dt \right]^\frac{1}{r_i}, \quad (3.35)
$$

where $r_i = 1/[1 - (1/p_i-1) + (1/p_i)]$. Similarly, applying (2.4) and (2.7) with $\varepsilon = \frac{1}{q}$ to (3.34), we have

$$
|\varrho_i(\cdot, t)|_{p_i} = C \left[ \int_0^t m(t-s)^{-\frac{N}{2} \left( \frac{1}{p_i-1} - \frac{1}{m} \right)} \left| \left(-\gamma \varrho_i + H_i \right) (\cdot, s) \right|_{p_i} ds \right.
$$

$$
+ C \|a\|_{\infty} \int_0^t m(t-s)^{-\frac{N}{2} \left( \frac{1}{p_i-1} - \frac{1}{m} \right) \frac{1}{r_i}} \left| \nabla \zeta_i (\cdot, s) \right|_{p_i} ds,
$$

which can also be estimated by Young’s convolution inequality as

$$
\|\varrho_i\|_{p_i} \leq C \left( \|\varrho_i\|_{p_i-1} + \|H_i\|_{p_i-1} \right) \left[ \int_0^T m(t)^{-\frac{N}{2} \left( \frac{1}{p_i-1} - \frac{1}{m} \right) \frac{1}{r_i}} dt \right]^\frac{1}{r_i} \times \left[ \int_0^T m(t)^{-\frac{N}{2} \left( \frac{1}{p_i-1} - \frac{1}{m} \right) \frac{1}{r_i}} dt \right]^\frac{1}{r_i}, \quad (3.36)
$$

Owing to (3.31), we also have

$$
\left[ \int_0^T m(t)^{-\frac{N}{2} \left( \frac{1}{p_i-1} - \frac{1}{m} \right) \frac{1}{r_i}} dt \right]^\frac{1}{r_i} \leq C \left[ (T + 1)^\frac{1}{r_i} + T^{-\left( \frac{N}{2} + 1 \right) \left( \frac{1}{m} - \frac{1}{p_i+1} \right) + 1} \right] \quad (3.37)
$$

and

$$
\left[ \int_0^T m(t)^{-\frac{N}{2} \left( \frac{1}{p_i-1} - \frac{1}{m} \right) \frac{1}{r_i}} dt \right]^\frac{1}{r_i} \leq C \left[ (T + 1)^\frac{1}{r_i} + T^{-\left( \frac{N}{2} + 1 \right) \left( \frac{1}{m} - \frac{1}{p_i+1} \right) + \frac{1}{2}} \right]. \quad (3.38)
$$

Secondly, we estimate the energy of solution $(\zeta_i, \varrho_i)$ to get the following $L^{p_i-1}$-estimate

$$
\|\zeta_i\|_{p_i-1} + \|\varrho_i\|_{p_i-1} + \|\nabla \zeta_i\|_{p_i-1} \leq e^{C\kappa} \left( \|G_i\|_{p_i-1} + \|H_i\|_{p_i-1} \right). \quad (3.39)
$$

This inequality together with (3.35), (3.36), (3.37), and (3.38) gives

$$
\|\zeta_i\|_{p_i} + \|\varrho_i\|_{p_i} \leq e^{C\kappa} \left( \|G_i\|_{p_i-1} + \|H_i\|_{p_i-1} \right). \quad (3.40)
$$
Since
\[ \|G_i\|_{p_{i-1}} \leq CT \|\zeta_{i-1}\|_{p_{i-1}} \quad \text{and} \quad \|H_i\|_{p_{i-1}} \leq CT \|\vartheta_{i-1}\|_{p_{i-1}}, \]
(3.41)
it follows from (3.40) that
\[ \|\zeta_i\|_{p_i} + \|\vartheta_i\|_{p_i} \leq e^{Ci} \left( \|\zeta_{i-1}\|_{p_{i-1}} + \|\vartheta_{i-1}\|_{p_{i-1}} \right), \]
(3.42)
where \( C_i = C_i(\Omega, \omega), i = 0, 1, \ldots, M, \) are positive constants. The iteration inequality (3.42) from 0 to \( M \) implies that
\[ \|\zeta_M\|_{p_M} + \|\vartheta_M\|_{p_M} \leq e^{CR} (\|\zeta_0\|_2 + \|\vartheta_0\|_2). \]
(3.43)
By the definition of \( \zeta_0 \) and \( \vartheta_0 \), we obtain from (3.30) and (3.43) that
\[ \|\zeta_M\|_{p_M} + \|\vartheta_M\|_{p_M} \leq e^{C\kappa} (\|y_0\|_2 + \|z_0\|_2). \]
(3.44)
Finally, we apply \( L^p \)-maximal regularity for the first equation of (3.32) for \( \zeta_{M+1} \) to get
\[ \|\partial_t \zeta_{M+1}\|_{p_M} + \|\Delta \zeta_{M+1}\|_{p_M} + \|\zeta_{M+1}\|_{p_M} \leq C \left( \|B\|_\infty \|\nabla \zeta_{M+1}\|_{p_M} + \|\vartheta_{M+1}\|_{p_M} + \|G_{M+1}\|_{p_M} \right). \]
This, by taking into account of (3.41) and (3.39), leads to
\[ \|\zeta_{M+1}\|_{W^{2,1}_{p_M}(Q)} \leq e^{C\kappa} \left( \|\zeta_M\|_{p_M} + \|\vartheta_M\|_{p_M} \right). \]
Hence, by the imbedding inequality ([30, Lemma 3.3,Ch.II])
\[ \|\zeta_{M+1}\|_{C(Q)} \leq e^{C(1+T+\frac{1}{2})} \|\zeta_{M+1}\|_{W^{2,1}_{p_M}(Q)}, \]
for \( p_M > (N+2)/2 \), and by (3.41), we get
\[ \|\zeta_{M+1}\|_\infty \leq e^{C\kappa} (\|y_0\|_2 + \|z_0\|_2). \]
That is
\[ \left\| \phi_e e^{(s+\tau)\alpha_0} \right\|_\infty \leq e^{C\kappa} (\|y_0\|_2 + \|z_0\|_2). \]
which together with (3.28) yields
\[ \left\| e^{-s(\frac{1}{2} - \eta(\lambda)) + \tau(1 + \eta(\lambda))} f_e \right\|_\infty \leq e^{C\kappa} (\|y_0\|_2 + \|z_0\|_2), \]
where \( \eta(\lambda) \) is given by (3.27). This gives, by choosing \( \tau \) small enough such that
\[ -s \left( \frac{1}{2} - \eta(\lambda) \right) + \tau(1 + \eta(\lambda)) < 0, \]
that
\[ \|f_e\|_\infty \leq e^{C\kappa} (\|y_0\|_2 + \|z_0\|_2). \]
The above inequality enables us to extract a subsequences of \( f_\varepsilon \), still denoted by itself, such that \( f_\varepsilon \to f \) weakly in \( L^2(\Omega) \), weakly* in \( L^\infty(\Omega) \) as \( \varepsilon \to 0 \). Let \((y_\varepsilon, z_\varepsilon)\) be the solution to the system associated to \( f_\varepsilon \). Then, by Proposition 2.1, we see that \( y_\varepsilon \) and \( z_\varepsilon \) are both bounded in \( V^1(\Omega) \). Thus, there exist subsequences \( y_\varepsilon \) and \( z_\varepsilon \), still denoted by themselves, such that

\[
y_\varepsilon \to y, \ z_\varepsilon \to z \quad \text{weakly in } V^1(\Omega) \; \text{strongly in } L^2(\Omega)
\]

for \((y, z) \in V^1(\Omega) \cap C([0,T];L^2(\Omega))\), which is the weak solution of the system corresponding to \( f \in L^\infty(\Omega) \), and \( y(x,T) = 0 \) and \( z(x,T) = 0 \) almost everywhere in \( \Omega \). This completes the proof. \( \square \)

4 Proof of Theorem 1.2

Let \((\mathbf{\pi}, \mathbf{\nu})\) be a trajectory of the system (1.2) with the initial value \((\mathbf{\pi}_0, \mathbf{\nu}_0)\), which satisfies (1.3). Set \( u = \mathbf{\pi} + y, \ v = \mathbf{\nu} + z, \ y_0 = u_0 - \mathbf{\pi}_0, \ z_0 = v_0 - \mathbf{\nu}_0 \). Then, \((y, z)\) solves the following parabolic system

\[
\begin{aligned}
\partial_t y &= \Delta y - \chi \nabla \cdot (y \nabla \mathbf{\nu}) - \chi \nabla \cdot (\mathbf{\pi} + y) \nabla z + 1_\omega f \quad \text{in } Q, \\
\partial_t z &= \Delta z - \gamma z + \delta y \quad \text{in } Q, \\
\partial_\nu y &= 0, \partial_\nu z = 0 \quad \text{on } \Sigma, \\
y(x,0) &= y_0(x) \quad z(x,0) = z_0(x) \quad x \in \Omega.
\end{aligned}
\]  

(4.1)

The local exact controllability of the system (1.1) is equivalent to the local null controllability of the system (4.1).

Let \( K = \{ \eta \in L^\infty(\Omega) \mid \| \eta \|_\infty \leq 1 \} \). For each \( \eta \in K \), we consider the following linearized system

\[
\begin{aligned}
\partial_t y &= \Delta y - \nabla \cdot (B y) - \nabla \cdot (a_\eta \nabla z) + 1_\omega f \quad \text{in } Q, \\
\partial_t z &= \Delta z - \gamma z + \delta y \quad \text{in } Q, \\
\partial_\nu y &= 0, \partial_\nu z = 0 \quad \text{on } \Sigma, \\
y(x,0) &= y_0(x) \quad z(x,0) = z_0(x) \quad x \in \Omega,
\end{aligned}
\]  

(4.2)

where \( a_\eta = \chi(\mathbf{\pi} + \eta) \) and \( B = \chi \nabla \mathbf{\nu} \). By (1.3), we see that

\[
a_\eta \in L^\infty(\Omega), \quad B \in L^\infty(\Omega)^N \quad \text{with } B \cdot \nu = 0 \quad \text{on } \Sigma.
\]

so system (4.2) is casted into the exact framework of system (4.1). Thus, we can apply Theorem 1.1 to obtain that for each \( \eta \in K \), there exists a pair \(((y, z), f)\) which solves system (4.2) with \( y(x,T) = 0, z(x,T) = 0 \) almost everywhere in \( \Omega \). Here and in what follows, we denote by \((y, z)\) the solution to system (4.1) corresponding to \( f \) and \( \eta \) if there is no ambiguity. By (1.8), we see that the control functions are bounded as follows:

\[
\| f \|_\infty \leq e^{C\kappa_0} (|y_0|_2 + |z_0|_2),
\]  

(4.3)
By (2.12) of Proposition 2.1 and (4.3), we have the following estimate

\[ \|y\|_{V^1(Q)} + \|z\|_{V^2(Q)} + \|y\|_\infty + \|z\|_\infty \leq e^{C\kappa_0} \left( |y_0|_\infty + \|z_0\|_{W^{1,q}(\Omega)} \right). \]  

(4.5)

For \( \eta \in K \), define a multi-valued mapping \( \Lambda : K \to 2^{L^2(Q)} \) by

\[
\Lambda(\eta) = \left\{ y \in L^2(Q) \mid \exists f \text{ satisfying (4.3) such that } (y, z) \text{ is the solution to (4.2) corresponding to } \eta \text{ and } f, \right. \\
\left. \quad \text{and } y(x, T) = z(x, T) = 0 \text{ a.e. in } \Omega \right\}.
\]

We apply Kakutani’s fixed-point theorem ([8, p.7]) to the map \( \Lambda \) to prove Theorem 1.2. First, it is clear that \( K \) is a convex subset of \( L^2(Q) \). By the argument above, we see that \( \Lambda(\eta) \) is nonempty and convex for each \( \eta \in K \). Moreover, by (4.5), \( \Lambda(\eta) \) is bounded in \( V^1(Q) \) for each \( \eta \in K \) and hence \( \Lambda(\eta) \) is a compact subset of \( L^2(Q) \) by the Aubin-Lions lemma ([8, p.17]).

Next, we show that \( \Lambda \) is upper semi-continuous. To this purpose, let \( \{\eta_n\}_{n=1}^\infty \) be a sequence of functions in \( K \) such that \( \eta_n \to \eta \) strongly in \( L^2(Q) \), and let \( y_n \in \Lambda(\eta_n) \) for each \( n \). Then, by the definition of \( \Lambda(\eta_n) \), there exists \( f_n \) for each \( n \) such that \( (y_n, z_n) \) solves the following system

\[
\begin{aligned}
\partial_t y_n &= \Delta y_n - \nabla \cdot (B y_n) - \nabla \cdot (a_{y_n} \nabla z_n) + 1_\omega f_n \quad \text{in } Q, \\
\partial_t z_n &= \Delta z_n - \gamma z_n + \delta y_n \quad \text{in } Q, \\
\partial_\nu y_n &= 0, \quad \partial_\nu z_n = 0 \quad \text{on } \Sigma, \\
y_n(x, 0) &= y_0(x), \quad z_n(x, 0) = z_0(x) \quad x \in \Omega,
\end{aligned}
\]

(4.6)

and \( y_n(x, T) = z_n(x, T) = 0 \) for \( x \in \Omega \) almost everywhere. Moreover, the control \( f_n \) satisfies

\[ \|f_n\|_\infty \leq e^{C\kappa_0} (|y_0|_2 + |z_0|_2). \]  

(4.7)

By (4.7) and Proposition 2.1, we obtain

\[ \|y_n\|_{V^1(Q)} + \|z_n\|_{V^2(Q)} \leq e^{C\kappa} \left( |y_0|_2 + \|z_0\|_{W^{1,2}(\Omega)} \right). \]  

(4.8)

By (4.7), (4.8) and applying the Aubin-Lions lemma again, we can get \( f \in L^\infty(Q) \), \( y \in V^1(Q) \), \( z \in V^2(Q) \) and the subsequences of \( f_n, y_n, z_n \), still denoted by themselves, such that

\[ f_n \to f \text{ weakly* in } L^\infty(Q), \text{ and weakly in } L^2(Q); \]
\[ y_n \to y \text{ weakly in } V^1(Q), \text{ and strongly in } L^2(Q); \]
\[ z_n \to z \text{ weakly in } V^2(Q), \text{ and strongly in } L^2(0, T; H^1(\Omega)). \]

Passing to the limit as \( n \to \infty \) in (4.6), we get that \( (y, z) \) is a weak solution of (4.6) corresponding to \( \eta \). We claim that that \( y \in \Lambda(\eta) \). Actually, let \( Y_n = y_n - y, Z_n = z_n - z \), and \( F_n = 1_\omega(f_n - f) \).
Then \((Y_n, Z_n)\) solves the following system

\[
\begin{aligned}
\partial_t Y_n &= \Delta Y_n - \nabla \cdot (BY_n) \\
&\quad - \nabla \cdot [a_m \nabla Z_n + (a_{n_n} - a_n) \nabla z] + F_n \quad \text{in } Q, \\
\partial_t Z_n &= \Delta Z_n - \gamma Z_n + \delta Y_n \quad \text{in } Q, \\
\partial_\nu Y_n &= 0, \quad \partial_\nu Z_n = 0 \quad \text{on } \Sigma, \\
Y_n(x, 0) &= 0, \quad Z_n(x, 0) = 0 \quad x \in \Omega.
\end{aligned}
\]

(4.9)

Multiply the first equation of (4.9) by \(Y_n\), and integrate over \(\Omega\), to give

\[
\frac{d}{dt} \|Y_n\|_2^2 + \|\nabla Y_n\|_2^2 \leq C \|B\|_\infty^2 \|Y_n\|_2^2 + C \|a_m\|_\infty^2 \|\nabla Z_n\|_2^2
\]

\[
+ C \int_\Omega \|\eta_n - \eta\|^2 \|\nabla z\|_2^2 \, dx + C \int_\Omega F_n Y_n \, dx.
\]

(4.10)

In the same way to the second equation of (4.9), we have

\[
\frac{d}{dt} \|Z_n\|_2^2 + \|\nabla Z_n\|_2^2 + \gamma \|Z_n\|_2^2 \leq C \|Y_n\|_2^2.
\]

(4.11)

Differentiate \(\|\nabla Z_n\|_2^2\) with respect to \(t\) to get, from the second equation of (4.9), that

\[
\frac{d}{dt} \|\nabla Z_n\|_2^2 + \|\Delta Z_n\|_2^2 + \gamma \|\nabla Z_n\|_2^2 \leq C \|Y_n\|_2^2.
\]

(4.12)

Since \(\|a_m\|_\infty \leq C\), it follows from (4.10)-(4.12) and Gronwall’s lemma that

\[
\|Y_n(\cdot, t)\|_2^2 + \|Z_n(\cdot, t)\|_2^2 + \|\nabla Z_n(\cdot, t)\|_2^2 \leq e^{C(1+\|B\|_\infty^2)T} \left( \int_\Omega \|\eta_n - \eta\|^2 \|\nabla z\|_2^2 \, dx + \int_\Omega F_n Y_n \, dx \right).
\]

(4.13)

On the other hand, since \((y, z)\) solves (4.2), by (ii) of Proposition 2.1, we get that

\[
\|z\|_{W^{2,1}_p(Q)} \leq C \left( \|y_0\|_p + \|z_0\|_{W^{2(1-\frac{1}{p})}_p(\Omega)} + \|\omega f\|_p \right),
\]

which together with (4.3) implies

\[
\|z\|_{W^{2,1}_p(Q)} \leq C \left( \|y_0\|_p + \|z_0\|_{W^{2(1-\frac{1}{p})}_p(\Omega)} \right).
\]

(4.14)

Since \(W^{2,1}_p(Q) \hookrightarrow C^1(\overline{Q})\) for \(p > N + 2\) ([30] Lemma 3.3, Ch II), it follows from (4.14) that

\[
\|\nabla z\|_{C(\overline{Q})^N} \leq C \left( \|y_0\|_p + \|z_0\|_{W^{2(1-\frac{1}{p})}_p(\Omega)} \right).
\]

(4.15)

Since \(\eta_n \rightharpoonup \eta\) strongly in \(L^2(\Omega)\), \(Y_n \rightarrow 0\) strongly in \(L^2(\Omega)\), and \(F_n \rightarrow 0\) weakly in \(L^2(\Omega)\), thus, by (4.15), we see that the right hand side of (4.13) tends to 0 as \(n \rightarrow \infty\). Hence, \(|Y_n(\cdot, t)|_2 \rightarrow 0\), \(|Z_n(\cdot, t)|_2 \rightarrow 0\) for all \(t \in [0, T]\). Since \(y_n(x, T) = z_n(x, T) = 0\) in \(\Omega\) almost everywhere, we get that
\( y(x, T) = z(x, T) = 0 \) in \( \Omega \) almost everywhere, which implies that \( y \in \Lambda(\eta) \). This shows that \( \Lambda \) is upper semi-continuous.

Now it remains to show that \( \Lambda(K) \subset K \). By Proposition 2.1 for any \( y \in \Lambda(K) \),

\[
\|y\|_{\infty} \leq e^{c_1\kappa_0} \left( |y_0|_{\infty} + \|z_0\|_{W^{2(1-\frac{1}{p})p}(\Omega)} \right),
\]

where \( c_1 \) is a positive constant. Take \( \delta = e^{-c_1\kappa_0} \) such that if \( |y_0|_{\infty} + \|z_0\|_{W^{2(1-\frac{1}{p})p}(\Omega)} \leq \delta \) which is exactly (1.10), then \( \|y\|_{\infty} \leq 1 \) and hence \( \Lambda(K) \subset K \). Therefore, the conditions of Kakutani’s fixed point are satisfied, that is, if the initial data \((u_0, v_0)\) satisfies (1.10), then there exists at least one fixed point \( y \), which together with \( z \), is the solution of (1.1) corresponding with some control \( f \) and satisfies \( y(x, T) = 0 \) and \( z(x, T) = 0 \) for \( x \in \Omega \) almost everywhere. This completes the proof. \( \square \)

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