THE INTEGRABLE MULTIBOSON SYSTEMS
AND
ORTHOGONAL POLYNOMIALS

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Abstract
The strict relation between some class of multiboson hamiltonian systems and the corresponding class of orthogonal polynomials is established. The correspondence is used effectively to integrate the systems. As an explicit example we integrate the class of multiboson systems corresponding to q-Hahn polynomials.

Introduction
The results of this paper give an effective tools to integrate some class of quantum physical systems. Such systems play a prominent role in quantum many-body physics, nuclear physics as well as quantum optics. More information about these models and their physical content can be found in many papers and monographs on this subject and among others one may consult [A-I], [J1], [J2], [Kar1], [Kar2], [M-G], [Odz].

This paper is devoted to a detailed study of the mathematical structures underlying the multi-boson hamiltonian systems, whose dynamics is generated by the operators of the general form given in (1.1).

In the first section, it is shown, that the reduction procedure applied to the multiboson hamiltonian system leads to some interesting operator algebras $\mathcal{A}_\mathcal{R}$ (parametrized by a structural function $\mathcal{R}$) which generalize in a natural way Heisenberg algebra as well as $sl_q(2)$ and $SU_q(2)$ quantum algebras. It was demonstrated [Odz] that the algebras $\mathcal{A}_\mathcal{R}$ are on the one hand side strictly related to the integration of the quantum systems. Their description in terms of the coherent states leads on the other hand to the connection of these algebras with the theory of basic hypergeometric series and orthogonal polynomials.

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In Section 2 we establish the explicit relation between the spectral realization of the algebra \( A_R \) in the space \( L^2(\mathbb{R}, d\sigma) \) and the representation in the space of holomorphic function.

According to the spectral theorem, the problem of integration of the dynamical system is equivalent to the construction of the spectral measure \( d\sigma \) for the corresponding Hamiltonian. Under some assumptions on its form (see Section 3) the problem of this construction can be explicitly solved within the framework of q-Hahn’s polynomials theory. The measure is then uniquely determined by the solution of \( q \)-difference Pearson equation.

For the sake of completeness some elementary facts from \( q \)-analysis are presented in the Appendix A. The Appendix B contains the proofs of some fundamental properties of q-Hahn’s polynomials.

## 1 The multi-boson systems

This section is devoted to the detailed analysis of the symmetry properties of \((N + 1)\)-boson systems. The dynamics of these systems is assumed to be governed by the Hamiltonian operator of the form:

\[
H = h_0(a_0^a a_0, \ldots, a_N^a a_N) + g_0(a_0^a a_0, \ldots, a_N^a a_N) a_{k_0}^0 \ldots a_{k_N}^N + \sum_{k_0, \ldots, k_N, k_0, \ldots, k_N \in \mathbb{Z}} g_{k_0 \ldots k_N} (a_{k_0}^0 \ldots a_{k_N}^N) a_{k_0}^0 \ldots a_{k_N}^N, \tag{1.1}
\]

where \( a_0, \ldots, a_N \) and \( a_0^a, \ldots, a_N^a \) are bosonic annihilation and respectively creation operators with standard Heisenberg commutation relations:

\[
[a_i, a_j^*] = \delta_{ij}, \quad [a_i, a_j] = 0, \quad [a_i^*, a_j^*] = 0. \tag{1.2}
\]

The following notational convention is assumed in (1.1):

\[
a_i^{k_i} = \begin{cases} a_i^{k_i} & \text{for } k_i > 0 \\ 1 & \text{for } k_i = 0 \\ a_i^{k_i} = (a_i^{*})^{-k_i} & \text{for } k_i < 0. \end{cases} \tag{1.3}
\]

The monomial

\[
a_{k_0}^0 \ldots a_{k_N}^N, \quad k_0, \ldots, k_N \in \mathbb{Z} \tag{1.4}
\]

can be thought of as an operator which describes the subsequent creation and annihilation of the clusters of the bosonic modes.

The operator

\[
g_0 (a_0^a a_0, \ldots, a_N^a a_N) \tag{1.5}
\]

is a kind of generalization of the coupling constant. The coupling constant is replaced in our case by a function depending on the occupation number operators of the bosonic modes. The operator

\[
h_0 (a_0^a a_0, \ldots, a_N^a a_N) \tag{1.6}
\]

can always be chosen as a free Hamiltonian being a weighted sum of the occupation number operators of elementary modes \( a_0^a a_0, \ldots, a_N^a a_N \):

\[
h_0^{\text{free}} = \omega_0 a_0^a a_0 + \ldots + \omega_N a_N^a a_N. \tag{1.7}
\]

The considerations at this paper will be by no means restricted to this free case however. The Hamiltonian under consideration (1.1) is an elementary ingredient of the most general Hamiltonian operator

\[
H = \sum_{k_0, \ldots, k_N \in \mathbb{Z}} g_{k_0 \ldots k_N} (a_0^a a_0, \ldots, a_N^a a_N) a_{k_0}^0 \ldots a_{k_N}^N, \tag{1.8}
\]
with the functions \( g_{k_0...k_N} \) connected by the following conjugation rule.

\[
[g_{k_0...k_N} (a_0^*a_0, \ldots, a_N^*a_N)]^\ast = g_{-k_0...-k_N} (a_0^*a_0 - k_0, \ldots, a_N^*a_N - k_N).
\] (1.9)

The class of model hamiltonians \([1.1]\) corresponds to many important quantum physical systems. Their dynamics is generated by specific operators of the form \([1.1]\), \([1.2]\), \([Kar1]\), \([Kar2]\), \([A-I]\). For this reason the analysis of the system \([1.1]\) seems to be important and may shed new light on the unsolved problems of quantum physics.

In order to analyze the quantum system described by the Hamiltonian \([1.1]\), it is convenient to introduce the following operators:

\[
A := g_0 (a_0^*a_0, \ldots, a_N^*a_N) a_0^{k_0} \cdots a_N^{k_N}.
\] (1.10)

and

\[
A_i = A_i^\ast := \sum_{j=0}^N \alpha_{ij} a_j^* a_j,
\] (1.11)

where \(i = 0, 1, 2, \ldots, N\). One assumes that real \((N + 1) \times (N + 1)\)-matrix \(\alpha = (\alpha_{ij})\) satisfies the conditions

\[
det \alpha \neq 0
\] (1.12)

\[
\sum_{j=0}^N \alpha_{ij} k_j = \delta_{i0}.
\] (1.13)

The operators \(A_0, A_1, \ldots, A_N, A\) and \(A^\ast\) do satisfy the following commutation relations

\[
[A_0, A] = -A, \quad [A_0, A^\ast] = A^\ast
\] (1.14)

\[
[A, A_i] = 0 = [A^\ast, A_i]
\] (1.15)

for \(i = 1, \ldots N\), and

\[
[A_i, A_j] = 0
\] (1.16)

for \(i, j = 0, 1, \ldots, N\).

One has in addition

\[
A^\ast A = [g_0 (a_0^*a_0 - k_0, \ldots, a_N^*a_N - k_N)]^2 \mathcal{P}_{k_0} (a_0^*a_0 - k_0) \cdots \mathcal{P}_{k_N} (a_N^*a_N - k_N)
\] (1.17)

\[
AA^\ast = [g_0 (a_0^*a_0, \ldots, a_N^*a_N)]^2 \mathcal{P}_{k_0} (a_0^*a_0) \cdots \mathcal{P}_{k_N} (a_N^*a_N),
\] (1.18)

where \(\mathcal{P}_{k_0} (a_0^*a_0), \ldots, \mathcal{P}_{k_N} (a_N^*a_N)\) are polynomials:

\[
\mathcal{P}_k (a^\ast a) := a^k a^{-k} = \begin{cases} a^k (a^\ast)^k = (a^\ast a + 1) \cdots (a^\ast a + k) & \text{for } k > 0 \\ 1 & \text{for } k = 0 \\ (a^\ast)^{-k} a^{-k} = a^\ast a (a^\ast a - 1) \cdots (a^\ast a - k + 1) & \text{for } k < 0. \end{cases}
\] (1.19)

The operators \(A^\ast A\) and \(AA^\ast\) are diagonal in the standard Fock basis

\[
|n_0, n_1, \ldots, n_N\rangle = \frac{1}{\sqrt{n_0! \cdots n_N!}} (a_0^\ast)^{n_0} \cdots (a_N^\ast)^{n_N} |0\rangle,
\] (1.20)

where \((n_0, n_1, \ldots, n_N) \in \mathbb{Z}_+^{N+1} := (\mathbb{Z}_+ \cup \{0\}) \times \cdots \times (\mathbb{Z}_+ \cup \{0\}) (N + 1\) times.

Let us note that operators \(A_0, \ldots, A_N\) are unbounded. Whether \(A\) and \(A^\ast\) are bounded or not depends on the choice of the structural function \(g_0\). All of them are defined on the common domain \(D\) spanned by finite linear combinations

\[
|\psi\rangle = \sum_{(i_0, i_1, \ldots, i_N) \in F} c_{i_0,i_1,\ldots,i_N} |n_{i_0}, n_{i_1}, \ldots, n_{i_N}\rangle
\] (1.21)
of the Fock basis elements, where \( F \) is some finite set of multi indices.

Identifying the 1-dimensional spaces \( \mathbb{C} \langle n_0, n_1, \ldots, n_N \rangle \) with the elements of \( \mathbb{Z}_+^{N+1} \) we obtain the action of \( A \) and \( A^* \) (and their natural powers) on \( \mathbb{Z}_+^{N+1} \). It is easy to see that the orbits of these actions are located on one dimensional lines which are parallel to the vector \( (k_0, k_1, \ldots, k_N) \in \mathbb{Z}_+^{N+1} \).

If the function \( g_0 \) of (1.11) is regular and nonvanishing in all points of \( \mathbb{Z}_+^{N+1} \), we can make some simple but useful observations. The first one is that if \( \alpha_{ij} k_i < 0 \) for some \( i, j \in \{0, 1, \ldots, N\} \) then for any element \( \langle n_0, n_1, \ldots, n_N \rangle \) of the Fock basis there exists \( M \in \mathbb{N} \) such that

\[
A^M \langle n_0, n_1, \ldots, n_N \rangle = 0 \quad \text{and} \quad (A^*)^M \langle n_0, n_1, \ldots, n_N \rangle = 0. \tag{1.22}
\]

This means that orbits of \( A \) and \( A^* \) in \( \mathbb{Z}_+^{N+1} \) are finite. In the opposite case the orbits contain infinitely many points.

The second observation is that \( A \langle n_0, n_1, \ldots, n_N \rangle = 0 \) if and only if there exists \( i \in \{0, 1, \ldots, N\} \) such that \( k_i > 0 \) and \( n_i \in \{0, 1, \ldots, k_i - 1\} \). Hence each orbit of \( A \) and \( A^* \) in \( \mathbb{Z}_+^{N+1} \) has exactly one vacuum (the point annihilated by \( A \)). It is then natural to introduce the parametrization of the Fock basis which is in agreement with the above orbit decomposition of \( \mathbb{Z}_+^{N+1} \).

Replacing the occupation number operators \( a_0^* a_0, \ldots, a_N^* a_N \) by the operators \( A_0, A_1, \ldots, A_N \) in (1.17) and (1.18) one obtains:

\[
A^* A = \mathcal{G}(A_0 - 1, A_1, \ldots, A_N) \tag{1.23}
\]

\[
A A^* = \mathcal{G}(A_0, A_1, \ldots, A_N) \tag{1.24}
\]

with the function \( \mathcal{G} \) uniquely determined by \( g_0 \), polynomials \( P_{k_0}, \ldots, P_{k_N} \) and the linear map (1.11).

The Hamiltonian (1.1) can be reexpressed in terms of (1.10) and (1.11) in the following form:

\[
H = H_0(A_0, A_1, \ldots, A_N) + A + A^* \tag{1.25}
\]

It is clear that it admits \( N \) commuting integrals of motion \( A_1, \ldots, A_N \): \( [A_i, H] = 0 \) for \( i = 1, \ldots, N \). They commute with the operator \( A_0 \) too. This maximal system of commutative observables is diagonalized in the Fock basis and the eigenvalues of \( A_0, A_1, \ldots, A_N \) on \( \langle n_0, n_1, \ldots, n_N \rangle \) are given by

\[
\lambda_i = \sum_{j=0}^{N} \alpha_{ij} n_j \quad i = 0, 1, \ldots, N. \tag{1.26}
\]

The eigenvalues \( (\lambda_0, \lambda_1, \ldots, \lambda_N) \) form a discrete convex cone \( \Lambda_+ \subset \mathbb{R}^{N+1} \). It is spanned by the columns of the matrix \( (\alpha_{ij}) \) with entries from \( (\mathbb{Z}_+ \cup \{0\}) \): \( \Lambda_+ = \alpha(\mathbb{Z}_+^{N+1}) \). The sequences \( (\lambda_0, \lambda_1, \ldots, \lambda_N) \in \Lambda_+ \) will be used as a new parametrization \( \{\langle \lambda_0, \lambda_1, \ldots, \lambda_N \rangle \} \) of the Fock basis elements.

In order to integrate the system (1.1) one can reduce it to the eigen subspaces \( \mathcal{H}_{\lambda_1, \ldots, \lambda_N} \subset \mathcal{H} \) spanned by the eigenvectors \( \langle \lambda_0, \lambda_1, \ldots, \lambda_N \rangle \) with fixed \( \lambda_1, \ldots, \lambda_N \). This subspace is invariant with the respect to the algebra \( \mathcal{A}_{\text{red}} \), which is generated by the operators \( A, A^*, A_0 \). These operators do satisfy the following relations (1.23),(1.24),(1.14):

\[
[ A_0, A ] = - A, \quad [ A_0, A^* ] = A^*, \tag{1.27}
\]

\[
A^* A = \mathcal{G}(A_0 - 1, \lambda_1, \ldots, \lambda_N), \tag{1.28}
\]

\[
A A^* = \mathcal{G}(A_0, \lambda_1, \ldots, \lambda_N). \tag{1.29}
\]
Hence one can conclude that the problem of integration of the system (1.1) amounts to the integration of the system described by the reduced Hamiltonian:

\[ H_{\text{red}} = H_0 (A_0, \lambda_1, \ldots, \lambda_N) + A + A^* \]  
(1.30)

being an element of the algebra \( A_{\text{red}} \).

The orthonormal basis of the Hilbert subspace \( \mathcal{H}_{\lambda_1 \ldots \lambda_N} \) is formed by the vectors \( |\lambda_0, \lambda_1, \ldots, \lambda_N \rangle \) with \( \lambda_0 \) such that \((\lambda_0, \lambda_1, \ldots, \lambda_N) \in \Lambda_+ \). From (1.27-1.29) it follows that

\[ A_0 |\lambda_0, \lambda_1, \ldots, \lambda_N \rangle = \lambda_0 |\lambda_0, \lambda_1, \ldots, \lambda_N \rangle \] \hspace{1cm} (1.31)
\[ A |\lambda_0, \lambda_1, \ldots, \lambda_N \rangle = \sqrt{\mathcal{G} (\lambda_0 - 1, \ldots, \lambda_N)} |\lambda_0 - 1, \lambda_1, \ldots, \lambda_N \rangle \] \hspace{1cm} (1.32)
\[ A^* |\lambda_0, \lambda_1, \ldots, \lambda_N \rangle = \sqrt{\mathcal{G} (\lambda_0, \lambda_1, \ldots, \lambda_N)} |\lambda_0 + 1, \lambda_1, \ldots, \lambda_N \rangle . \] \hspace{1cm} (1.33)

Let us note here that if \((\lambda_0, \lambda_1, \ldots, \lambda_N) \in \Lambda_+\) then either \((\lambda_0 - 1, \ldots, \lambda_N) \in \Lambda_+ \) or

\[ A |\lambda_0, \lambda_1, \ldots, \lambda_N \rangle = 0 \quad (A^* |\lambda_0, \lambda_1, \ldots, \lambda_N \rangle = 0). \] \hspace{1cm} (1.34)

Because of (1.28-1.29) the conditions (1.34) are equivalent to

\[ \mathcal{G} (\lambda_0 - 1, \lambda_1, \ldots, \lambda_N) = 0 \quad (\mathcal{G} (\lambda_0, \lambda_1, \ldots, \lambda_N) = 0). \] \hspace{1cm} (1.35)

Since \( \Lambda_+ \) is a discrete convex cone we can easily see that the representation of \( A_{\text{red}} \) in \( \mathcal{H}_{\lambda_1 \ldots \lambda_N} \) splits into irreducible components. These components are generated out of vacuum (or antivacuum) states \(|\lambda_0, \lambda_1, \ldots, \lambda_N \rangle \). The vacuum states are parametrized by the solutions \( \lambda_0 \) of the equations (1.33). In all cases under consideration the operator \( A_0 \) is diagonal while the operator \( A \) is weighted unilateral shift operator. One does not exclude the case when the irreducible representations generated by \(|\lambda_0, \lambda_1, \ldots, \lambda_N \rangle \) are of finite dimension.

We can give now the following

**Proposition 1.1** If the structural function \( \mathcal{G} \) is regular then:

1) \( \dim \mathcal{H}_{\lambda_1 \ldots \lambda_N} < \infty \) if and only if there exists a pair \( i, j \in \{0, 1, \ldots, N\} \) such that \( \frac{k_i}{k_j} < 0. \)

2) if \( k_0 > 0 \) then the equation \( A |\lambda_0, \lambda_1, \ldots, \lambda_N \rangle = 0 \) is solved by

\[ \lambda_{0,l} := \frac{l}{k_0} - \frac{1}{k_0} \sum_{j=1}^{N} \beta_{0j} \lambda_j \] \hspace{1cm} (1.36)

where \( \beta_{ij} \) are matrix elements of \( \alpha^{-1} \), \( l \in L := \{0, \frac{1}{\kappa} k_0, \frac{2}{\kappa} k_0, \ldots, \frac{\kappa - 1}{\kappa} k_0\} \) and \( \kappa \) is the biggest common divisor of the numbers \( k_0, k_1, \ldots, k_N \). Moreover \( \mathcal{H}_{\lambda_1 \ldots \lambda_N} \) splits onto the irreducible components

\[ \mathcal{H}_{\lambda_1 \ldots \lambda_N} = \bigoplus_{l \in L} \mathcal{H}_{\lambda_1 \ldots \lambda_N}^l \] \hspace{1cm} (1.37)

where \( \mathcal{H}_{\lambda_1 \ldots \lambda_N}^l \) are generated by \( A_{\text{red}} \) out of the states \(|\lambda_{0,l}, \lambda_1, \ldots, \lambda_N \rangle \).

3) \( \dim \mathcal{H}_{\lambda_1 \ldots \lambda_N} = \infty \) if and only if \( \dim \mathcal{H}_{\lambda_1 \ldots \lambda_N}^l = \infty \) for all \( l \).

**Proof**: follows immediately from the observations above. Irreducibility of \( \mathcal{H}_{\lambda_1 \ldots \lambda_N}^l \) is a consequence of the fact that \(|\lambda_{0,l}, \lambda_1, \ldots, \lambda_N \rangle \) is the unique vacuum in this space. QED

The construction presented above generalizes the one of [Kar1] [Kar2], where the models with \( \mathcal{G} \) being a polynomial of special type are considered. In these papers the approximative methods of integration of the models were used.
As it was mentioned in the Introduction, our aim is to study some integrable family of Hamiltonians (1.30). Therefore, instead of the operator $A_0$, we will use the Hermitian operator $Q := q^{A_0 - \lambda_{0,t}}$, (1.38) where $0 < q < 1$.

The structural relations (1.27-1.29) acquire the following form in terms of the operators $A, A^*$ and $Q$

\[
\begin{align*}
QA^* &= qA^*Q, \quad qQA = AQ \\
A^*A &= \mathcal{R}(Q) \\
AA^* &= \mathcal{R}(qQ),
\end{align*}
\]

(1.39) (1.40) (1.41)

where structural function $\mathcal{R}$ is given by

\[
\mathcal{R}(Q) = G \left( \frac{\log Q}{\log q} + \lambda_{0,t} - 1, \lambda_1, \ldots, \lambda_N \right).
\]

(1.42)

The function $\mathcal{R}$ takes positive values in all points $\{q^n\}_{n=1}^\infty$ and moreover $\mathcal{R}(1) = 0$. The algebras of the type above were analyzed in [Odz]. The reduced Hamiltonian (1.30) can be rewritten as

\[
H_{\text{red}} = D(Q) + A + A^*,
\]

(1.43)

with the function $D$ given by

\[
D(Q) = H_0 \left( \frac{\log Q}{\log q} + \lambda_{0,t}, \lambda_1, \ldots, \lambda_N \right).
\]

(1.44)

by the use of (1.38).

The analysis below is restricted to infinite-dimensional case only. The discussion of finite dimensional case will be presented in the separate paper.

## 2 Spectral and holomorphic representations

Let $A_\mathcal{R}$ be the operator algebra generated by the operators $A, A^*$ and $Q$. In this section we describe two natural and, important from physical point of view, representations of the algebra $A_\mathcal{R}$.

The first representation, which we will call a holomorphic one, is related to the coherent states $|z\rangle$, $|z| < \mathcal{R}(0)$, of the anihilation operator $A \in A_\mathcal{R}$

\[
A |z\rangle = z |z\rangle.
\]

(2.1)

Let us consider the case when the orthonormal basis of the Hilbert space $H_{\text{red}} := \mathcal{H}_{l_{\lambda_1,\ldots,\lambda_N}}$

\[
|n\rangle := |\lambda_{0,t} + n, \lambda_1, \ldots, \lambda_N\rangle, \quad n \in \mathbb{N} \cup \{0\}
\]

(2.2)

is infinite. The basis vectors are generated by the operator $A^*$ out of the vacuum state $|0\rangle$. The action of the algebra generators on the vectors of this basis is:

\[
\begin{align*}
Q |n\rangle &= q^n |n\rangle \\
A |n\rangle &= \sqrt{\mathcal{R}(q^n)} |n - 1\rangle \\
A^* |n\rangle &= \sqrt{\mathcal{R}(q^{n+1})} |n + 1\rangle.
\end{align*}
\]

(2.3) (2.4) (2.5)
The coherent state $|z\rangle$ is thus given by

$$
|z\rangle := \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{R(q) \ldots R(q^n)}} |n\rangle,
$$

(2.6)

where $z \in \mathbb{D} = \{ z \in \mathbb{C} : |z| < R(0) \}$. The states $|z\rangle$, $z \in \mathbb{D}$ form a linearly dense subset in $\mathcal{H}_{red}$. Therefore, the map

$$
I(v) := \langle v | z \rangle
$$

(2.7)

where $v \in \mathcal{H}_{red}$ and $z \in \mathbb{D}$, is an antilinear and one-to-one map of the Hilbert space $\mathcal{H}_{red}$ into the vector space $\mathcal{O}(\mathbb{D})$ of holomorphic functions on the disc $\mathbb{D}$. It was shown in [Odz] that the image $I(\mathcal{H}_{red})$ is isomorphic to the Hilbert space $L^2(\mathcal{O}(\mathbb{D}), d\mu_R)$ containing holomorphic functions on $\mathbb{D}$ which are square integrable with respect to the measure

$$
d\mu_R(z, z) = \frac{1}{2\pi} \frac{1}{\text{Exp}_R(x) (1 - q; q)_\infty} \gamma \left( \frac{1}{x} \right) \lim_{a \to \infty} \frac{a \log q}{(ax; q)_\infty} d_q x d\varphi,
$$

(2.8)

where $z = \sqrt{x} e^{i\varphi}$, $d_q x$ is the Jackson measure (see Appendix A), $d\varphi$ is the Lebesque measure on the circle $S^1$. The $R$-exponential function $\text{Exp}_R$ (see [Odz]) and the function $\gamma$ are defined by

$$
\text{Exp}_R(x) := \langle z | z \rangle = \sum_{n=0}^{\infty} \frac{x^n}{\sqrt{R(q) \ldots R(q^n)}},
$$

(2.9)

$$
\gamma(z) := \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{R(q; q)_{n-k}}{(q; q)^k} (-1)^k q^{\frac{k}{2}} \right) z^k,
$$

(2.10)

with

$$
R(q; q)_k := R(q) R(q^2) \ldots R(q^k),
$$

(2.11)

$$
(q; q)_k := (1 - q)(1 - q^2) \ldots (1 - q^k),
$$

(2.12)

$$
(x; q)_\infty := \prod_{k=0}^{\infty} \left( 1 - q^k x \right).
$$

(2.13)

Let us remark here, that the existence of the morphism (2.7) is equivalent to the existence of the resolution of the unity of the type:

$$
\int_{\mathbb{D}} |z\rangle \langle z| d\mu_R(z, z) = 1.
$$

(2.14)

By holomorphic representation of the algebra $A_R$ we will understand the representation in the Hilbert space $L^2(\mathcal{O}(\mathbb{D}), d\mu_R)$. Straightforward calculation shows that:

$$
A \varphi(z) = \partial_R \varphi(z)
$$

(2.15)

$$
A^* \varphi(z) = z \varphi(z)
$$

(2.16)

$$
Q \varphi(z) = \varphi(qz)
$$

(2.17)

$$
H_{red} \varphi(z) = (D(Q) + z + \partial_R) \varphi(z),
$$

(2.18)

where $\varphi \in L^2(\mathcal{O}(\mathbb{D}), d\mu_R)$ and $\partial_R$ is a $R$-difference operator given by

$$
\partial_R \varphi(z) := R(qQ) \partial_0 \varphi(z),
$$

(2.19)

$$
\partial_0 \varphi(z) := \frac{\varphi(z) - \varphi(0)}{z},
$$

(2.20)
\[ \partial_R \text{ is a } q\text{-derivative } \partial_q, \text{ and the standard derivative } \frac{d}{dz} \text{ is obtained from } \partial_q \text{ in the limit } q \to 1. \]

The quantum algebra \( A_R \) of \( R \) given by (2.21) is a \( q \)-deformation of the Heisenberg algebra.

Hence the analytic realization of \( A_R \) introduced above is a natural generalization of the Bergman-Fock-Segal representation of the Heisenberg algebra.

The second representation of \( A_R \) is related to the spectral measure of a selfadjoint extension of the Hamiltonian (1.43). The action of \( H_{\text{red}} \) on the elements of the orthonormal basis \( \{ |n\rangle \}_{n=0}^{\infty} \) is given in terms of three-diagonal (Jacobi) matrix

\[
H_{\text{red}} |n\rangle = \sqrt{\mathcal{R}(q^n)} |n-1\rangle + D(q^n) |n\rangle + \sqrt{\mathcal{R}(q^{n+1})} |n+1\rangle.
\]  

(2.22)

We will call this matrix the Jacobi matrix of the operator \( H_{\text{red}} \).

The operator \( H_{\text{red}} \) is symmetric and its domain \( D_{H_{\text{red}}} \) contains all finite linear combinations of the basis elements \( \{ |n\rangle \}_{n=0}^{\infty} \). The theory of such type operators is strictly related with the theory of orthogonal polynomials [A-G], [A], [Ch], [Su].

Let \( K_\omega \) denote the deficiency subspace of \( H_{\text{red}} \) for \( \omega \in \mathbb{C} \) and \( \text{Im}\omega \neq 0 \)

\[
K_\omega := ((H_{\text{red}} - \omega 1) D_{H_{\text{red}}})^\perp.
\]  

(2.23)

The deficiency indices \((n_+, n_-)\):

\[
n_+ = \dim K_\omega, \text{ for } \text{Im}\omega > 0 \]  

(2.24)

\[
n_- = \dim K_\omega, \text{ for } \text{Im}\omega < 0 \]  

(2.25)

of the operator \( H_{\text{red}} \) are \((0,0)\) or \((1,1)\). In order to show this property, one should observe that \( |v\rangle \in K_\omega \) if and only if

\[
H_{\text{red}}^* |v\rangle = \omega |v\rangle.
\]  

(2.26)

where \( H_{\text{red}}^* \) is the Hermitian conjugate of \( H_{\text{red}} \). The vector

\[
|v\rangle = \sum_{n=0}^{\infty} P_n(\omega) |n\rangle \in \mathcal{H}_{\text{red}},
\]  

(2.27)

solves (2.26) if and only if coefficients \( P_n(\omega) \) do satisfy the three term recurrence equation

\[
\omega P_n(\omega) = \sqrt{\mathcal{R}(q^n)} P_{n-1}(\omega) + D(q^n) P_n(\omega) + \sqrt{\mathcal{R}(q^{n+1})} P_{n+1}(\omega)
\]  

(2.28)

\( n \in \mathbb{N}, \) with the initial conditions

\[
P_0(\omega) \equiv 1, \quad P_1(\omega) = \frac{\omega - D(1)}{\mathcal{R}(q)}
\]  

(2.29)

and

\[
\sum_{n=0}^{\infty} |P_n(\omega)|^2 < +\infty
\]  

(2.30)

Hence \( n_+ \) and \( n_- \) are equal to 0 or 1.

Because every \( P_n(\omega) \) is a real polynomial of degree \( n \) of the complex variable \( \omega \) one has \( n_+ = n_- \).

Following [A] we will call the Jacobi matrix of \( H_{\text{red}} \) to be of the type \( D \) or \( C \) if the deficiency indices of \( H_{\text{red}} \) are \((0,0)\) or \((1,1)\) respectively.
Proposition 2.1  

i) If
\[
\sum_{n=0}^{\infty} \frac{1}{\sqrt{R(q^n)}} = +\infty
\]  (2.31)

then the operator \( H_{\text{red}} \) has deficiency indices \((0, 0)\). This is equivalent to its essential selfadjointness.

ii) If the set of the coherent states \(|z\rangle\) of the annihilation operator \(A\) is parametrized by the disc \(\mathbb{D}\) of finite radius \(R(0) < +\infty\) then \(H_{\text{red}}\) is essentially selfadjoint.

iii) If the deficiency indices of \(H_{\text{red}}\) are \((1, 1)\) then coherent states \(|z\rangle\) of \(A\) exist for any \(z \in \mathbb{C}\).

Proof:

i) Let \(\{Q_n(\omega)\}_{n=0}^{\infty}\) be an another solution of the recurrence (2.28) with the initial conditions given by:
\[
Q_0(\omega) \equiv 0, \quad Q_1(\omega) \equiv \frac{1}{\sqrt{R(q)}}.
\]  (2.32)

One then has
\[
P_{k-1}(\omega) Q_k(\omega) - P_k(\omega) Q_{k-1}(\omega) = \frac{1}{\sqrt{R(q^k)}}
\]  (2.33)
where \(k \in \mathbb{N}\). Applying Schwartz inequality to (2.33), one finds
\[
\sum_{n=1}^{\infty} \frac{1}{\sqrt{R(q^n)}} \leq 2 \left( \sum_{n=0}^{\infty} |P_n(\omega)|^2 \right) \left( \sum_{n=0}^{\infty} |Q_n(\omega)|^2 \right).
\]  (2.34)

Moreover one can prove (see [A]) that if \(\{P_n(\omega)\}_{n=0}^{\infty}\) satisfies (2.30), then \(\{Q_n(\omega)\}_{n=0}^{\infty}\) satisfies it too. Therefore \(\sum_{n=0}^{\infty} \frac{1}{\sqrt{R(q^n)}} < +\infty\). This contradicts (2.31). Thus, one has
\[
\sum_{n=0}^{\infty} |P_n(\omega)|^2 = \infty,
\]  (2.35)
meaning that \(n_+ = n_- = 0\). We apply Th.VIII.3 from [R-S] vol.1.

ii) If \(R(0) < +\infty\) then the condition (2.35) follows immediately from (2.31). The statement ii) follows from the statement i).

iii) One proves it \textit{ad absurdum} using the previous statement. QED

Let \(\hat{H}_{\text{red}}\) be some selfadjoint extension of \(H_{\text{red}}\). In the case of type C such extensions are parametrized by points of \(S^1\) see [R-S] whereas in the case of type D the extension is unique. Let \(dE_{\mathbb{R}, \mathcal{D}}(\omega)\) be the spectral measure of \(\hat{H}_{\text{red}}\).

By the spectral representation of \(A_{\mathbb{R}}\) we will call the representation in the space \(L^2(\mathbb{R}, d\sigma_{\mathbb{R}, \mathcal{D}})\) where the measure \(d\sigma_{\mathbb{R}, \mathcal{D}}\) is given by
\[
d\sigma_{\mathbb{R}, \mathcal{D}}(\omega) := \langle 0|dE_{\mathbb{R}, \mathcal{D}}(\omega)|0\rangle
\]  (2.36)
and the operator \(\hat{H}_{\text{red}}\) acts by multiplication with the identity function on \(\mathbb{R}\). Let \(U_{PN} : H_{\text{red}} \rightarrow L^2(\mathbb{R}, d\sigma_{\mathbb{R}, \mathcal{D}})\) denote the intertwining operator for these two representations. Because the polynomials \(\{P_n(\omega)\}_{n=0}^{\infty}\) form an orthonormal basis in \(L^2(\mathbb{R}, d\sigma_{\mathbb{R}, \mathcal{D}})\), it is convenient to write the intertwining operator using the physical notation of Dirac
\[
U_{PN} = \sum_{n=0}^{\infty} P_n(\omega) \otimes \langle n|.
\]  (2.37)
The convergence is understood in the sense of weak topology. The relation of the holomorphic representation and the spectral representation is given by the isomorphism of Hilbert spaces

\[ U_{PZ} : L^2 \mathcal{O} (\mathbb{D}, d\mu_T) \rightarrow L^2 (\mathbb{R}, d\sigma_{\mathcal{R}, \mathcal{D}}) \] (2.38)

and \( U_{PZ} \) one can be written as

\[ U_{PZ} = \sum_{n=0}^{\infty} P_n (\omega) \otimes \frac{z^n}{\sqrt{R(q) \ldots R(q^n)}}, \] (2.39)

where \( \left\{ \frac{z^n}{\sqrt{R(q) \ldots R(q^n)}} \right\}_{n=0}^{\infty} \) is an orthonormal basis in \( L^2 \mathcal{O} (\mathbb{D}, d\mu_T) \). Similarly let

\[ U_{ZN} : \mathcal{H}_{red} \rightarrow L^2 (\mathbb{D}, d\mu) \] (2.40)

be given by

\[ U_{ZN} := \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{R(q) \ldots R(q^n)}} \otimes \langle n \rangle. \] (2.41)

We are thus getting the following commutative diagram

\[ \begin{array}{ccc}
\mathcal{H}_{red} & \xrightarrow{U_{PZ}} & L^2 (\mathbb{R}, d\sigma_{\mathcal{R}, \mathcal{D}}) \\
\uparrow & & \uparrow \\
L^2 (\mathbb{D}, d\mu) & \xrightarrow{U_{ZN}} & L^2 \mathcal{O} (\mathbb{D}, d\mu_T)
\end{array} \] (2.42)

of Hilbert space isomorphisms.

If the series

\[ V (\omega, z) := \sum_{n=0}^{\infty} P_n (\omega) \frac{z^n}{\sqrt{R(q) \ldots R(q^n)}} \] (2.43)

is pointwise convergent for all \( \omega \in [a, b] \) and \( z \in \mathbb{D} \), the isomorphism \( U_{PZ} \) can be represented as the integral transform

\[ (U_{PZ} \varphi) (\omega) = \int_{\mathbb{D}} V (\omega, z) \varphi (z) d\mu_T (z, \mathcal{D}). \] (2.44)

The kernel \( V (\omega, z) \) of this transform satisfies the \( R \)-difference equation

\[ (\omega - z) V (\omega, z) = D(Q) V (\omega, z) + \partial_R V (\omega, z). \] (2.45)

Of course, from the orthogonal polynomials theory point of view \( V (\omega, z) \) is nothing else than the generating function for the family of orthogonal polynomials under consideration.

The function \( V (\omega, z) \) is useful to calculate many important physical quantities of the system described by the Hamiltonian \( H_{red} \).

First of all, note that using (2.44) and (2.43) one obtains

\[ \langle v | H^n_{red} z \rangle = \int_{\mathbb{R}} \overline{V(\omega, v)} \omega^n V (\omega, z) d\sigma_{\mathcal{R}, \mathcal{D}} (\omega) = (z + D(Q) + \partial_R)^n \exp_R (\tau z). \] (2.46)

Putting in particular \( n = 0 \) we have

\[ \exp_R (\tau z) = \int_{\mathbb{R}} \overline{V(\omega, v)} V (\omega, z) d\sigma_{\mathcal{R}, \mathcal{D}} (\omega). \] (2.47)
which is the integral representation of the $\mathcal{R}$-exponential function $\text{Exp}_{\mathcal{R}} (\overline{v} z)$. This function satisfies the equation
\[
\partial_{\mathcal{R}} \text{Exp}_{\mathcal{R}} (\overline{v} z) = \overline{v} \text{Exp}_{\mathcal{R}} (\overline{v} z).
\] (2.48)
Let us recall that $\text{Exp}_{\mathcal{R}} (\overline{v}) \in L^2 \mathcal{O} (\mathbb{D}, d\mu_{\mathcal{R}})$ is the expression of the coherent state in holomorphic representation.

The evolution operator $U (t) := e^{i \hat{H}_{\text{red}} t}$ acts on the function from $L^2 (\mathbb{R}, d\sigma_{\mathcal{R}, \mathcal{D}})$ as multiplication by phase factors:
\[
U (t) \psi (\omega) = e^{i \omega t} \psi (\omega).
\] (2.49)
It enables us to calculate the transition amplitudes between coherent states:
\[
\langle v | U (t) | z \rangle = \int_{\mathbb{R}} V (\omega, v) e^{i \omega t} V (\omega, v) d\sigma_{\mathcal{R}, \mathcal{D}} (\omega).
\] (2.50)

The vacuum - vacuum transition amplitude is also important for physicists. It is given by:
\[
\langle 0 | U (t) | 0 \rangle = \int_{\mathbb{R}} e^{i \omega t} d\sigma_{\mathcal{R}, \mathcal{D}} (\omega) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \mu_n
\] (2.51)
where $\mu_n$ is the $n$-th moment of the measure $d\sigma_{\mathcal{R}, \mathcal{D}}$.

The above shows that the measure $d\sigma_{\mathcal{R}, \mathcal{D}}$ plays a significant role in the description of our physical system. The construction $d\sigma_{\mathcal{R}, \mathcal{D}}$ is one of the most important problems which has to be solved in order to recover the dynamics.

In order to give an example see [A] of the solution of this problem, let us recall the notion of a simple symmetric operator.

The Hilbert subspace $\mathcal{H}_1 \subset \mathcal{H}$ is a reducible subspace of the linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ if $\mathcal{H}_1$ and $\mathcal{H}_2 := \mathcal{H}_1^\perp$ are invariant subspaces for $T$ and for orthogonal projection $\Pi_1 : \mathcal{H} \rightarrow \mathcal{H}_1$ one has $\Pi_1 (D_T) \subset D_T$. The symmetric operator $T$ is simple if there does not exist the irreducible subspace of $T$ such that $T|_{\mathcal{H}_1}$ has selfadjoint extension in $\mathcal{H}_1$.

If the Jacobi matrix of the reduced Hamiltonian $H_{\text{red}}$ is of type $C$ then the series
\[
\sum_{n=0}^{\infty} |P_n (\omega)|^2 \quad \& \quad \sum_{n=0}^{\infty} |Q_n (\omega)|^2
\] (2.52)
are almost uniformly convergent on $\mathbb{C}$. Thus the functions $A (\omega), B (\omega), C (\omega)$ and $D (\omega)$ defined by
\[
A (\omega) = \omega \sum_{k=0}^{\infty} Q_k (0) Q_k (\omega),
\]
\[
B (\omega) = -1 + \omega \sum_{k=0}^{\infty} Q_k (0) P_k (\omega),
\]
\[
C (\omega) = 1 + \omega \sum_{k=0}^{\infty} P_k (0) Q_k (\omega),
\] (2.53)
\[
D (\omega) = \omega \sum_{k=0}^{\infty} P_k (0) P_k (\omega)
\]
are entire functions. If in addition $H_{\text{red}}$ is simple and closed operator then the spectral measure $dE_{\mathcal{R}, \mathcal{D}} (\omega)$ of its arbitrary selfadjoint extension $\hat{H}_{\text{red}}$ is localized at the nulls $\omega_i, \ i =$
1, 2, ... , of the function \( q(\omega) = B(\omega)t - D(\omega) \) for some \( t \in \mathbb{R} \). The steps \( \mu_i, i = 1, 2, \ldots \) of the measure \( d\sigma_{\mathcal{R}, \mathcal{D}}(\omega) := (0)dE_{\mathcal{R}, \mathcal{D}}(\omega)0 \) satisfy the following conditions

\[
\sum_{i=1}^{\infty} \frac{1}{\mu_i(1 + \omega_i^2)} |q'(\omega_i)|^2 < \infty \tag{2.54}
\]

\[
\sum_{i=1}^{\infty} \frac{1}{\mu_i |q'(\omega_i)|^2} = \infty. \tag{2.55}
\]

From the identity

\[
|n\rangle = P_n(H_{\text{red}})|0\rangle \tag{2.56}
\]

it follows that if the Jacobi matrix of \( H_{\text{red}} \) is of type \( D \) then \( H_{\text{red}} \) has simple spectrum. Conversely, one can prove (see [5]) that every selfadjoint operator \( H = H^* \) with simple spectrum may be represented in some orthonormal basis by the formula (2.22) where the Jacobi matrix is of the type \( D \).

Using (2.3-2.5) we can associate with it the algebra \( \mathcal{A}_\mathcal{R} \) with the proper structural function \( \mathcal{R} \). This fact indicates that the algebras of this kind are important tools to investigate the symmetry structures of the physical systems with dynamics generated by Hamiltonians with simple spectrum.

In the next section we will describe the situation when neither coherent states nor the kernel \( V(\omega, z) \) do exist.

### 3 The integrable systems related to q-Hahn’s polynomials

We will integrate quantum systems related to the algebras \( \mathcal{A}_\mathcal{R} \) and Hamiltonian \( H_{\text{red}} =: H_{AB} \) with the corresponding structural functions of the form

\[
\mathcal{R}_{AB}(x) = (1 - q)x[\partial_\eta \beta(x) - \beta(x) \partial_q \beta(x)] \tag{3.1}
\]

\[
\mathcal{D}_{AB}(x) = (1 - q)x\partial_q \beta(x). \tag{3.2}
\]

where

\[
\beta(x) = \frac{q(1 - x)[a_0(1 - q) - b_1]x + b_1q^2}{(1 - q)[a_1(1 - q) - b_2]x^2 + b_2q^2} \tag{3.3}
\]

\[
\eta(x) = \left\{ \begin{array}{ll}
\frac{[(a_0(1 - q) - b_1)x + b_1q^2][(a_0(1 - q) - b_1)x + b_1q]}{[(a_1(1 - q) - b_2)x^2 + b_2q^2][(a_1(1 - q) - b_2)x^2 + b_2q^2]} & \\
+ \frac{b_0(1 - q)}{(a_1(1 - q) - b_2)x^2 + b_2q^2} & q^3(1 - x)(1 - q^{-1}x) \\
\end{array} \right\} \frac{(1 - q)(1 - q^x)}{(1 - q)^2(1 + q)}. \tag{3.4}
\]

These functions depend on five real parameters \( a_0, a_1 \) and \( b_0, b_1, b_2 \). It will appear later on that it is natural to introduce the following two polynomials:

\[
A(\omega) = a_1\omega + a_0 \tag{3.5}
\]

\[
B(\omega) = b_2\omega^2 + b_1\omega + b_0. \tag{3.6}
\]

It is clear from (3.3-3.4) that the pairs of polynomials \( (A(\omega), B(\omega)) \) of degree one and two respectively, taken up to common overall real factor \( c \neq 0 \), parametrize the models under consideration. The only condition we will impose on the pair \( (A(\omega), B(\omega)) \) is the one given by \( \mathcal{R}_{AB}(q^n) > 0 \) for any \( n \in \mathbb{N} \).
Analogously to the theory of classical orthogonal polynomials there is a $q$-difference equation (an analog of Pearson equation (3.7)):

$$\partial_q (qB)(\omega) = (qA)(\omega)$$

(3.7)

associated with the pair $(A(\omega), B(\omega))$.

We will look for the solutions $\varphi(\omega)$ of (3.7) which do satisfy the boundary conditions

$$\varphi(a) B(a) = \varphi(b) B(b) = 0,$$

(3.8)

for some fixed $a, b$ such that $-\infty \leq a < b \leq \infty$. We thus have so called Pearson data $(A(\omega), B(\omega))$ on the interval $(a, b) \subset \mathbb{R}$.

**Proposition 3.1** Let $\varphi(\omega)$ be the solution of the $q$-Pearson equation (3.7) defined by $(A(\omega), B(\omega))$ and satisfying (3.8). Then $\varphi^{(k)}(\omega) := \varphi(q^k \omega) B(q^{\omega}) \ldots B(q^{\omega})$ is the solution of Pearson $q$-equation (3.3) associated with the pair

$$A^{(k)}(\omega) := q^k A(q^k \omega) + \frac{1 - q^k Q^k}{1 - q} \partial_q B(\omega)$$

(3.9)

$$B^{(k)}(\omega) := B(\omega).$$

(3.10)

where $k \in \mathbb{N}$. If $\varphi(\omega)$ satisfies the boundary conditions (3.3) then $\varphi^{(k)}(\omega)$ satisfies (3.8) too.

**Proof**: By straightforward calculation. QED

Let $L^2([a,b], d\sigma_{AB})$ be the Hilbert space of square-integrable functions with respect to the measure

$$d\sigma_{AB} (\omega) = \varphi(\omega) d\omega$$

(3.11)

where

$$d\omega = \sum_{k=0}^{\infty} (1 - q)^k \left[ b \delta(\omega - q^k b) - a \delta(\omega - q^k a) \right] d\omega$$

(3.12)

is the Jackson measure on the interval $[a,b]$. It will be assumed that the weight function $\varphi(\omega)$ does satisfy the Pearson $q$-equation (3.7) supplemented with boundary condition (3.8).

Applying the orthonormalization procedure to the monomials $\{\omega^n\}_{n=0}^{\infty} \subset L^2([a,b], d\sigma_{AB})$ we obtain the system of orthonormal polynomials (OPS) $\{P_n(\omega)\}_{n=0}^{\infty}$

$$\int_a^b P_n(\omega) P_m(\omega) d\sigma_{AB}(\omega) = \delta_{nm}$$

(3.13)

which is uniquely determined by the pair $(A(\omega), B(\omega))$ and interval. The polynomials $\{P_n(\omega)\}_{n=0}^{\infty}$ are called $q$-Hahn’s polynomials, see [G-R, H]. Let us denote by $\{\tilde{P}_n(\omega)\}_{n=0}^{\infty}$ the monic OPS associated with $\{P_n(\omega)\}_{n=0}^{\infty}$. (i.e. $\tilde{P}_n(\omega) = \alpha_n P_n(\omega)$ where $\alpha_n$ is the coefficient of the highest power in $P_n(\omega)$).

**Theorem 3.1** If $\{\tilde{P}_n(\omega)\}_{n=0}^{\infty}$ is the monic OPS corresponding to the Pearson data $(A(\omega), B(\omega))$ then the family of polynomials:

$$\left\{ \frac{1}{[n][n-1] \ldots [n-k]} \partial_q^k \tilde{P}_n(\omega) \right\}_{n=0}^{\infty}$$

(3.14)

where

$$[k] := \frac{1 - q^k}{1 - q},$$

forms the monic OPS corresponding to $(A^{(k)}(\omega), B^{(k)}(\omega))$, with the same boundary conditions (3.8).
Proof: For \( k \leq n - 2 \) we have

\[
\int_a^b \tilde{P}_n(\omega) \omega^k A(\omega) q(\omega) d_q \omega = 0. \tag{3.15}
\]

Using Leibnitz rule, (3.7) and (3.8) we obtain

\[
0 = \int_a^b \tilde{P}_n(\omega) \omega^k (\partial_q B)(\omega) d_q \omega = \\
\tilde{P}_n(\omega) \omega^k B(\omega) q(\omega) \bigg|_a^b - \int_a^b \partial_q \left( \tilde{P}_n(\omega) \omega^k \right) B(\omega) q(\omega) d_q \omega = \\
- \int_a^b \partial_q \left( \tilde{P}_n(\omega) \omega^k \right) (B(\omega) - (1 - q) \omega A(\omega)) q(\omega) d_q \omega = \\
- \int_a^b \partial_q \tilde{P}_n(\omega) (q \omega)^k (B(\omega) - (1 - q) \omega A(\omega)) q(\omega) d_q \omega - \\
\int_a^b \tilde{P}_n(\omega) [k] \omega^{k-1} (B(\omega) - (1 - q) \omega A(\omega)) q(\omega) d_q \omega.
\]

The degree of polynomial

\[
[k] \omega^{k-1} (B(\omega) - (1 - q) \omega A(\omega)) \tag{3.16}
\]

is \( k + 1 < n \), and from (3.8) one finds that:

\[
\int_a^b \partial_q \tilde{P}_n(\omega) \omega^k q^{(1)}(\omega) d_q \omega = 0 \tag{3.17}
\]

for \( k \leq n - 2 \). This shows that the polynomials \( \left\{ \frac{1}{[n]} \partial_q \tilde{P}_n(\omega) \right\}_{n=1}^\infty \) form a monic OPS for the Pearson data \((A^{(1)}(\omega), B^{(1)}(\omega)) \) given by (3.9-3.10). QED

Since, for the rational function \( \mathcal{R}_{AB}(x) \) of (3.1) one has \( \mathcal{R}_{AB}(0) = 0 \) the Proposition 2.1 and it’s consequences described in the previous section imply that in the case under consideration the following statements are true:

i) the operator \( \overline{H}_{AB} \) is selfadjoint and has simple spectrum;

ii) the coherent states do not exist for \( \mathcal{A}_{AB} \)

iii) the Hilbert space \( \mathcal{H}_{red} \) is unitary isomorphic to \( L^2 ([a,b], d\sigma_{AB}) \), with the isomorphism given by (2.37).

Hence, the Hahn’s polynomials \( \{P_n(\omega)\}_{n=0}^\infty \) form an orthonormal basis in \( L^2 ([a,b], d\sigma_{AB}) \). The measure \( d\sigma_{AB} \) is the expectation value of the spectral measure \( dE_{AB} \) in the vacuum state \( |0\rangle \).

It is then clear that the properties of Hahn’s polynomials are crucial for better understanding of the physical systems corresponding the algebra \( \mathcal{A}_{AB} \). The proposition below gives a description of some important properties of these polynomials, namely

**Theorem 3.2 (Hahn)** Fix a Pearson data \((A(\omega), B(\omega)) \) on the interval \([a,b] \subset \mathbb{R} \). Then the following statements are equivalent:

A. The family of polynomials \( \left\{ \tilde{P}_n(\omega) \right\}_{n=0}^\infty \) forms the monic OPS with respect to \((A(\omega), B(\omega)) \).
B. The polynomials are given by formula of Rodrigues:

\[ \tilde{P}_n (\omega) = c_n \frac{1}{q(\omega)} \partial_q^n \left[ q(\omega) B(\omega) B(q^{-1}\omega) \cdots B(q^{-(n-1)}\omega) \right] \]  

(3.18)

\( n \in \mathbb{N} \), where \( c_n \) is a normalization constant.

C. The polynomials \( \{ \tilde{P}_n (\omega) \}_{n=0}^\infty \) do satisfy the following \( q \)-difference equation (Hahn equation)

\[ (A(\omega) \partial_q + B(\omega) \partial_q Q^{-1} \partial_q) \tilde{P}_n (\omega) = \lambda_n \tilde{P}_n (\omega) \]  

(3.19)

where

\[
\lambda_n = a_1 [n] + b_2 [n] [n-1] q^{-(n-1)}; \quad n = 2, 3, \ldots \]  

(3.20)

D. Every polynomial of the system \( \{ \tilde{P}_n (\omega) \}_{n=0}^\infty \) is given by

\[ \tilde{P}_n (\omega) = \prod_{k=0}^{n-1} \frac{1}{a_1^{(k)} - b_2 [-n+1+k]} \left( A^{(0)}(\omega) + B(\omega) \partial_q Q^{-1} \right) \cdots \left( A^{(k-1)}(\omega) + B(\omega) \partial_q Q^{-1} \right) \]  

(3.21)

where the linear functions

\[ A^{(k)}(\omega) = a_1^{(k)} \omega + a_0^{(k)} \]  

(3.22)

are defined in (3.9).

E. The polynomials of the system \( \{ \tilde{P}_n (\omega) \}_{n=0}^\infty \) are related by the three term recurrence formula

\[ \tilde{P}_{n+1} (\omega) + R_{AB} (q^n) \tilde{P}_{n-1} (\omega) = (\omega - D_{AB} (q^n)) \tilde{P}_n (\omega) \]  

(3.23)

with the initial condition \( P_0 (\omega) \equiv 1 \).

The proofs of the equivalence of \( A, B, C \) may be found in the original paper of Hahn [1]. The recurrence formula (3.23) is considered there without specification of the form of the structural functions \( R_{AB}, D_{AB} \). The only assumption made is that they are rational functions of the parameter \( q \). A complete proof of this theorem is given in the Appendix B.

From the considerations of Section 2, it follows that the problem of integration of the multiboson system described in \( L^2 ([a, b], d\sigma_{AB}) \), is reduced to the construction of the measure \( d\sigma_{AB} \). According to (3.11) the measure \( d\sigma_{AB} \) is given by the density function \( \varrho(\omega) \) which is a solution of the \( q \)-difference Pearson equation (3.7). Let us therefore present all possible solutions of (3.7) from the class of the meromorphic functions. Using (3.5,3.6) we can rewrite (3.7) in the form

\[ \varrho(\omega) = \frac{B(q\omega)}{B(\omega) - (1-q)\omega A(\omega)} \quad \varrho(q\omega) = \frac{b_2 q^2 \omega^2 + b_1 q \omega + b_0}{(b_2 - (1-q) a_1) \omega^2 + (b_1 - (1-q) a_0) \omega + b_0} \]  

(3.24)

and after standard calculations we obtain the following classes of solutions depending on the values of the parameters \( b_2, b_1, b_0, a_1, a_0 \):

**Proposition 3.2** One has the following subcases of the solutions of the \( q \)-difference Pearson equation (3.7):
i) If \( b_0 \neq 0 \) and \( b_2 - (1-q)a_1 \neq 0 \), then

\[
g(\omega) = \left( \frac{q^w_a}{a}; q \right)_\infty \frac{(q^w_b; q)_\infty}{\left( \frac{x}{c}; q \right)_\infty}, \tag{3.25}
\]

where \( a \neq 0 \), \( b \neq 0 \) are roots of the polynomial \( B(\omega) \) and \( c \neq 0 \), \( d \neq 0 \) are roots of the polynomial \( B(\omega) - (1-q)\omega A(\omega) \).

ii) If \( b_0 \neq 0 \) and \( b_1 - (1-q)a_0 \neq 0 \) and \( b_2 - (1-q)a_1 = 0 \), then

\[
g(\omega) = \left( \frac{q^w_a}{a}; q \right)_\infty \frac{(q^w_b; q)_\infty}{\left( \frac{x}{c}; q \right)_\infty}, \tag{3.26}
\]

where \( a \neq 0 \), \( b \neq 0 \) are roots of the polynomial \( B(\omega) \) and \( c \neq 0 \) is the root of the polynomial \( B(\omega) - (1-q)\omega A(\omega) \).

iii) If \( b_0 \neq 0 \) and \( b_1 - (1-q)a_0 = 0 \) and \( b_2 - (1-q)a_1 = 0 \), then

\[
g(\omega) = \left( \frac{q^w_a}{a}; q \right)_\infty \frac{(q^w_b; q)_\infty}{\left( \frac{x}{c}; q \right)_\infty}, \tag{3.27}
\]

where \( a \neq 0 \), \( b \neq 0 \) are roots of the polynomial \( B(\omega) \).

iv) If \( b_0 = 0 \), \( b_1 \neq 0 \) and \( b_1 - (1-q)a_0 \neq 0 \) and \( b_2 - (1-q)a_1 \neq 0 \) and \( b_2 \neq 0 \), then

\[
g(\omega) = \omega^r \left( \frac{q^w_a}{a}; q \right)_\infty \left( \frac{x}{c}; q \right)_\infty, \tag{3.28}
\]

where \( a \neq 0 \) is the root of the polynomial \( B(\omega) \), \( c \neq 0 \) is the root of the polynomial \( B(\omega) - (1-q)\omega A(\omega) \) and \( q^{-r} = \left| \frac{q^b_{b_1 - (1-q)a_0}}{b_1 - (1-q)a_0} \right| \).

v) If \( b_0 = 0 \), \( b_1 \neq 0 \) and \( b_1 - (1-q)a_0 \neq 0 \) and \( b_2 - (1-q)a_1 = 0 \) and \( b_2 \neq 0 \), then

\[
g(\omega) = \omega^r \left( \frac{q^w_a}{a}; q \right)_\infty, \tag{3.29}
\]

where \( a \neq 0 \) is the root of the polynomial \( B(\omega) \) and \( q^{-r} = \left| \frac{q^b_{b_1 - (1-q)a_0}}{b_1 - (1-q)a_0} \right| \).

vi) If \( b_0 = b_1 - (1-q)a_0 = 0, b_1 \neq 0, b_2 \neq 0 \) and \( b_2 - (1-q)a_1 \neq 0 \), then

a) \[
g(\omega) = \omega^r \left( \frac{q^w_a}{a}; q \right)_\infty \left( -\omega; q \right)_\infty \left( -q\omega^{-1}; q \right)_\infty, \tag{3.30}
\]

for \( q^{-r} = \frac{q^b_{b_1 - (1-q)a_0}}{b_2 - (1-q)a_1} > 0 \);

b) \[
g(\omega) = \omega^r \frac{(q^w_a)_\infty}{(\omega; q)_\infty (q\omega^{-1}; q)_\infty}, \tag{3.31}
\]

for \( -q^{-r} = \frac{q^b_{b_1 - (1-q)a_0}}{b_2 - (1-q)a_1} < 0 \),

where \( a \neq 0 \) is the root of the polynomial \( B(\omega) \).
vii) If \( b_0 = b_1 = 0, b_1 - (1 - q)a_0 \neq 0 \) and \( b_2 \neq 0 \), then

\[
\varrho(\omega) = \omega^r \frac{(-\omega; q)_\infty (-q\omega^{-1}; q)_\infty}{(\frac{\omega}{c}; q)_\infty}
\]  

(3.32)

for \( q^r = \frac{q^2b_2}{b_1 - (1 - q)a_0} > 0 \);

b) \[
\varrho(\omega) = \omega^r \frac{(\omega; q)_\infty (q\omega^{-1}; q)_\infty}{(\frac{\omega}{c}; q)_\infty}
\]  

(3.33)

for \( -q^r = \frac{q^2b_2}{b_1 - (1 - q)a_0} < 0 \), where \( c \neq 0 \) is the root of the polynomial \( B(\omega) - (1 - q)\omega A(\omega) \).

viii) If \( b_0 = b_1 = b_1 - (1 - q)a_0 = 0, \) and \( b_2 - (1 - q)a_1 \neq 0 \) and \( b_2 \neq 0 \), then

\[
\varrho(\omega) = \omega^r
\]  

(3.34)

for \( q^r = \left| \frac{q^2b_2}{b_2 - (1 - q)a_1} \right| \).

Proof: The subcases i), ii) and iii) are easily obtained by iteration. The points iv)- viii) are proved by calculation of Laurent expansion coefficient and application of the Ramanujan’s identities (see [G-R]). QED

We can now determine the interval of integration in (3.13) and determine the conditions on polynomials \( A(\omega) \) and \( B(\omega) \) such that the measure \( d\sigma_{AB} \) is positive (ie \( R(q^n) > 0 \) for \( n \in \mathbb{N} \)). It will be convenient to express the conditions on \( A(\omega) \) and \( B(\omega) \) in terms of roots of the polynomials \( B(\omega) \) and \( B(\omega) - (1 - q)\omega A(\omega) \).

Proposition 3.3 The measure \( d\sigma_{AB} \) is positive and the condition (3.8) is fulfilled if and only if (in the notation and classification of Proposition 3.2)

i) The integration interval is \([a, b]\) with \( a < 0 < b \) and \( c, d \) satisfies one of the following conditions:

\[
\begin{align*}
\alpha) \ & c = \frac{a}{a}, \\
\beta) \ & c < a \text{ and } d > b, \\
\gamma) \ & c, d < a, \\
\delta) \ & \text{there exists } K \in \mathbb{N} \text{ such that } q^{K-1}a < c, d < q^Ka, \\
\epsilon) \ & c, d > b, \\
\zeta) \ & \text{there exists } K \in \mathbb{N} \text{ such that } q^Kb < c, d < q^{K-1}b.
\end{align*}
\]

ii) The integration interval is \([a, b]\) with \( a < 0 < b \) and \( c < a \) or \( c > b \).

iii) The integration interval is \([a, b]\) with \( a < 0 < b \).

17
iv) This case splits into two subcases:
   1) For \( a > 0 \) the integration interval is \([0, a]\) and \( c < 0 \) or \( c > a \).
   2) For \( a < 0 \) the integration interval is \([a, 0]\) and \( c < a \) or \( c > 0 \) and \( r \) has to be such that \( a^r > 0 \).

v) This case splits into two subcases:
   1) For \( a > 0 \) the integration interval is \([0, a]\).
   2) For \( a < 0 \) the integration interval is \([a, 0]\) and \( r \) has to be such that \( a^r > 0 \).

vi) This case splits into two subcases:
   1) For \( a > 0 \) the integration interval is \([0, a]\).
   2) For \( a < 0 \) the integration interval is \([a, 0]\) and \( r \) have to be such that \( a^r > 0 \).

vii) In this case \( R_{AB}(q^n) \) are not positive for \( n \) large anough.

viii) In this case \( R_{AB}(q^n) = 0 \) and \( D_{AB}(q^n) = 0 \) for \( n \in \mathbb{N} \).

Proof:

i) The equation \( B(\omega)d(\omega) = 0 \) is solved by \( aq^{-k+1} \) and by \( bq^{-k+1} \) for \( k \in \mathbb{N} \). For any function \( f(\omega) \) and any \( k, l \in \mathbb{N} \), using (3.11) and (3.12) one can obtain

\[
\int_{aq^{-k+1}}^{aq^{-l+1}} f(\omega)d\sigma_{AB}(\omega) = 0 = \int_{bq^{-k+1}}^{bq^{-l+1}} f(\omega)d\sigma_{AB}(\omega) \quad (3.35)
\]

and

\[
\int_{a}^{b} f(\omega)d\sigma_{AB}(\omega) = \int_{aq^{-k+1}}^{bq^{-l+1}} f(\omega)d\sigma_{AB}(\omega). \quad (3.36)
\]

Hence have the integration interval is \([a, b]\). The condition of positivity of \( d\sigma_{AB}(\omega) \)

\[
\int_{a}^{b} f(\omega)d\sigma_{AB}(\omega) > 0 \quad \text{for} \quad f > 0 \quad (3.37)
\]

is equivalent to

\[
a \varphi(q^{i}a) < 0 \quad \text{and} \quad b \varphi(q^{i}b) > 0 \quad \text{for} \quad i = 0, 1, \ldots. \quad (3.38)
\]

The continuity of \( \varphi \) at \( \omega = 0 \) gives \( a < 0 < b \) and the inequalities

\[
\varphi(q^{i}a) > 0, \quad \varphi(q^{i}b) > 0 \quad \text{for} \quad i = 0, 1, \ldots. \quad (3.39)
\]

which are solved by \( \alpha)-\zeta \).

The proofs of ii)-viii) are similar to the one above. QED

The above class of orthogonal polynomials, which we call the q-Hahn polynomials, contains, as a special cases the families of orthogonal polynomials well known from literature. Using a very good paper \([K-S]\) we obtain the following identification:

1) Putting in i)\( \beta \) \( d = 1 \) we have the Big q-Jacobi polynomials. If in addition we put \( b = q \) and \( c = \frac{2}{q} \) we obtain the Big q-Legendre polynomials.

2) Putting in ii) \( b < 1 \) and \( c = 1 \) we obtain the Big q-Laguerre polynomials.

3) Putting in iii) \( b = 1 \) we obtain the Al-Salam-Carlitz I polynomials. If in addition we assume \( a = -1 \) we obtain the Discrete q-Hermite I polynomials.
Putting in iv)1) $a = 1$ we obtain the Little $q$-Jacobi polynomials. If in addition we put $c = \frac{1}{q}$ and $r = 0$ we obtain the Little $q$-Legendre polynomials.

5) Putting in v)1) $a = 1$ we obtain the Little $q$-Laguerre/Wall polynomials.

6) Putting in vi)1) $r = 0$ we obtain the Alternative $q$-Charlier polynomials.

We will find now the equations for the moments

$$
\mu_n = \int_a^b \omega^n d\sigma_{AB}(\omega)
$$

(3.40)

of the measure $d\sigma_{AB}$.

From Section 2 it is clear that once the moments are known one may determine many important physical characteristic of the system under consideration.

Multiplying $q$-difference Pearson equation (3.7) by $\omega^n q^n$ and using Lebnitz rule for $q$-derivative we obtain the following three-term recurrence equation

$$
- [n] (b_2 \mu_{n+1} + b_1 \mu_n + b_0 \mu_{n-1}) = q^n (a_1 \mu_{n+1} + a_0 \mu_n)
$$

(3.41)

for $n \geq 1$, and

$$
a_1 \mu_1 + a_0 \mu_0 = 0.
$$

(3.42)

The initial rule $\mu_0 = \int_a^b d\sigma_{AB}(\omega)$ for this recurrence can be calculated in straightforward way. In terms of the notation and classification introduced in Proposiiton 3.2 we have

i)

$$
\mu_0 = (1 - q)(b - a) \frac{(q; q)_{\infty}(q^b; q)_{\infty}(q^a; q)_{\infty}(q^c; q)_{\infty}}{(q^a; q)_{\infty}(q^b; q)_{\infty}(q^c; q)_{\infty} (q^d; q)_{\infty}}
$$

(3.43)

ii)

$$
\mu_0 = (1 - q)(b - a) \frac{(q; q)_{\infty}(q^b; q)_{\infty}(q^a; q)_{\infty}}{(q^a; q)_{\infty}(q^b; q)_{\infty}}
$$

(3.44)

iii)

$$
\mu_0 = (1 - q)(b - a) (q; q)_{\infty}(q^b; q)_{\infty}(q^a; q)_{\infty}(q^c; q)_{\infty}
$$

(3.45)

iv)

$$
\mu_0 = (1 - q) a^{r+1} \frac{(q; q)_r}{(q^a; q)_{r+1}}
$$

(3.46)

v)

$$
\mu_0 = (1 - q) a^{r+1} (q; q)_r
$$

(3.47)
vi)  a) 
\[ \mu_0 = (1 - q) a^{r+1} \frac{(q; q)_\infty (-aq^{r+1}; q)_\infty}{(-a; q)_\infty (-\frac{q}{a}; q)_\infty} \]  
(3.48)

b) 
\[ \mu_0 = (1 - q) a^{r+1} \frac{q (aq^{r+1}; q)_\infty}{(a; q)_\infty (\frac{q}{a}; q)_\infty}. \]  
(3.49)

Let us note that replacing in iv)-vi) \( r \) by \( r + n, \ n \in \mathbb{N} \) we obtain the moments \( \mu_n \) for the corresponding cases.

In order to consider the cases i)-iii) let us introduce a real function \( \mu \) satisfying the equation 
\[ (1 - \omega) B(Q) \mu(\omega) + (1 - q) \omega Q A(Q) \mu(\omega) = 0. \]  
(3.50)

It is easy to check that \( \mu(q^{n+1}) \) satisfies the recurrence equation (3.41), hence \( \mu(q^{n+1}) = \mu_n \).

Reexpressing (3.50) in the form 
\[ \left( \frac{B(Q)}{B(q^{-1}Q) - (1 - q) q^{-1}Q A(q^{-1}Q)} - \omega \right) \mu(\omega) = 0 \]  
(3.51)

and the equation (3.7) in the form 
\[ \left( \frac{B(q \omega)}{B(\omega) - (1 - q) \omega A(\omega)} - Q \right) \varrho(\omega) = 0 \]  
(3.52)

one may observe some symmetry between the equation on \( \varrho \) and the equation for the moment function \( \mu \). After the substitution of the form \( Q \to q Q \) and \( \omega \to Q \) the operator from (3.51) transforms into the one of (3.52). The equation (3.51) as well as the equation (3.52) can be easily solved.

For example, if we assume that \( B(1) = 0 \) then (3.51) can be written in the form 
\[ \partial_R \mu(\omega) = \mu(\omega), \]  
(3.53)

where \( \partial_R \) is \( R \)-derivative. The function \( R \) is here given by 
\[ R(\omega) = \frac{B(\omega)}{B(q^{-1}\omega) - (1 - q) q^{-1} \omega A(q^{-1}\omega)}. \]  
(3.54)

Then one of the two linearly independent solutions of (3.51) is simply \( R \)-exponential \( Exp_R \). In this case it is given as the basic hypergeometric series 
\[ \mu_1(\omega) = Exp_R(\omega) = {}_3\Phi_2 \left( \begin{array}{c} \frac{1}{2}, 1, q ; q \end{array} \right) \]  
(3.55)

where \( a \neq 1 \) is the root of the polynomial \( B(\omega) \) and \( c, \ d \) are roots of the polynomial 
\[ B(q^{-1}\omega) - (1 - q) q^{-1} \omega A(q^{-1}\omega). \]  
The function \( {}_3\Phi_2 \) is defined in \( \text{G-R}, \text{K-S} \).

The second solution \( \mu_2(\omega) \) is related to \( \mu_1(\omega) \) by the following formula (\( q \)-version of Wronskian):
\[ \mu_2(\omega) \mu_1(q \omega) - \mu_2(q \omega) \mu_1(\omega) = x^\lambda \frac{(\alpha q; q)_\infty}{(\omega; q)_\infty} \]  
(3.56)

where \( q^\lambda = \frac{b_2}{b_0}, \ \alpha = \frac{(1-q)a_1-b_2}{b_0} \).
Any solution \( \mu (\omega) \) is a linear combination of \( \mu_1 (\omega) \) and \( \mu_2 (\omega) \). We are then getting the following formulae for moments

\[
\mu_n = \mu (q^{n+1}) = c_1 \mu_1 (q^{n+1}) + c_2 \mu_2 (q^{n+1})
\]

where the constants \( c_1, c_2 \) are determined by

\[
\mu_0 = \int_a^b d\sigma_{AB} = c_1 \mu_1 (q) + c_2 \mu_2 (q)
\]

\[
a_0 (c_1 \mu_1 (q) + c_2 \mu_2 (q)) + a_1 (c_1 \mu_1 (q^2) + c_2 \mu_2 (q^2)) = 0.
\]

### Appendix A

**The affine difference calculus and q-Hahn’s orthogonal polynomials**

In this section we present the preliminary considerations related to the calculus generated by the action of the affine group \( A_+ \) on the real line. Let us define the linear representation of \( A_+ \)

\[
(\mathfrak{L}_{q,h} \varphi) (x) := \varphi (qx + h), \quad (q,h) \in A_+ = \{(q,h) : q > 0, \quad h \in \mathbb{R}\}
\]

acting on the functions \( \varphi \) from the algebra \( \mathfrak{F} \). Since our consideration will be formal in its character we do not impose any additional conditions on \( \mathfrak{F} \).

According to \( \text{H} \) we introduce the derivative operator

\[
(\partial_{q,h} \varphi) (x) := \frac{\varphi (x) - \varphi (qx + h)}{x - (qx + h)}
\]

as a natural generalization of the q-derivative \( \partial_q := \partial_{q,0} \) and of the difference derivative \( \partial_h := \partial_{1,0} \).

The Leibnitz rule for the derivative \( \partial_{q,h} \) is

\[
(\partial_{q,h} \varphi \psi) (x) = (\partial_{q,h} \varphi) (x) \psi (x) + (\mathfrak{L}_{q,h} \varphi) (x) (\partial_{q,h} \psi) (x)
\]

There is also the following equivariance property:

\[
\mathfrak{L}_{c,t}^{-1} \circ \partial_{q,h} \circ \mathfrak{L}_{c,t} = c \partial_{q,ct+(1-q)t}
\]

and enables us to reduce \((q,h)\)–analysis to \(q\)–analysis. We have for example

\[
\partial_{q,h} = \mathfrak{L}_{1,\frac{h}{1-q}}^{-1} \circ \partial_q \circ \mathfrak{L}_{1,\frac{h}{1-q}}.
\]

Let us now solve the equation

\[
\partial_{q,h} \varphi = \varrho
\]

for the given function \( \varrho \in \mathfrak{F} \). In the order to do this, we apply the operator \( \mathfrak{L}_{q,h}^k \) to \((A.6)\), and we find that

\[
\mathfrak{L}_{q,h}^k \varphi (x) - \mathfrak{L}_{q,h}^{k+1} \varphi (x) = q^k [(1 - q) x - h] \mathfrak{L}_{q,h}^k \varrho (x).
\]

Summing up both sides of the identity \([A.7]\) with respect to \( k \) we get

\[
\varphi (x) - \varphi (x^\infty) = \sum_{k=0}^\infty [x - (qx + h)] q^k \varrho \left( q^k x + \frac{1 - q^k h}{1 - q} \right)
\]
where
\[ x^\infty = \lim_{k \to \infty} \left( q^k x + \frac{1 - q^k}{1 - q} h \right) = \frac{h}{1 - q}. \]  
(A.9)

The equation \((A.8)\), justifies the following definition of the \((q, h)\)–integral
\[
\int_{q, h} \varphi(x) = \int_{x^\infty}^x \varphi(t) \, dq, h t := \sum_{k=0}^{\infty} \left[ x - (qx + h) \right] q^k \varphi \left( q^k x + \frac{1 - q^k}{1 - q} h \right). 
\]  
(A.10)

The \((q, h)\)–integral operator is the right inverse of the \((q, h)\)–derivative operator
\[
\partial_{q, h} \circ \int_{q, h} = \text{id}, 
\]  
(A.11)
and, moreover,
\[
\int_{q, h} \circ \partial_{q, h} = \text{id} - \delta_{1, \infty}. 
\]  
(A.12)

The operator \(\delta_{1, \infty}\) is an idempotent operator defined by
\[
(\delta_{1, \infty} \varphi)(x) = \varphi(x^\infty), 
\]  
(A.13)
projecting the function on the constants.

Like in \((A.7)\) we have
\[
\int_{q, h} = \mathfrak{L}^{-1}_{1, \frac{h}{1-q}} \circ \int_q \mathfrak{L}_{1, \frac{h}{1-q}}, 
\]  
(A.14)
which reduces (by the translation automorphism \(\mathfrak{L}_{1, \frac{h}{1-q}}\)) the \((q, h)\)–integral to the Jackson integral \(\int_q = \int_{q, 0}\). The integration on the interval \([a, b]\) can be defined by
\[
\int_a^b \varphi(t) \, dq, h t = \int_{b^\infty}^b \varphi(t) \, dq, h t - \int_{a^\infty}^a \varphi(t) \, dq, h t. 
\]  
(A.15)

If \(q \to 1\), the calculus presented above corresponds to the difference calculus. For \(h \to 0\) one obtains \(q\)–difference calculus. The differential calculus will be obtained when \(q \to 1\), \(h \to 0\).

Let us mention finally that the identities \((A.3)\) and \((A.14)\) enable us to reduce \((q, h)\)-calculations to the \(q\)-calculations. This property motivates us to discuss the case of the \(q\)–analysis in this paper.

**Appendix B**

**Proof of the Theorem 3.2**

\(A \iff B\)

The monic OPS \(\{P_n(\omega)\}_{n=0}^{\infty}\) is uniquely defined by the weight function \(\varphi(\omega)\) and the interval \([a, b]\). In order to prove the equivalence of the properties \(A\) and \(B\) it is sufficient to show that the system of polynomials defined by \((3.18)\) is a monic OPS. In order to do that let us reexpress the function
\[
F_k(\omega; n) := \partial_q^k \left[ \varphi(\omega) B(\omega) B(q^{-1}\omega) \ldots B(q^{-(n-1)}\omega) \right] 
\]  
(B.1)
in the following way
\[
F_k(\omega; n) = \varphi(\omega) B(\omega) B(q^{-1}\omega) \ldots B(q^{-(n-1-k)}\omega) R_{k,n}(\omega) \quad k = 0, 1, \ldots, n - 1 
\]  
(B.2)
where $R_{k,n}(\omega)$ is a polynomial of degree not greater than $k \leq n$. These polynomials do and satisfy the recurrence formula

$$R_{k+1,n}(\omega) = A(\omega) R_{k,n}(q\omega) + B \left( q^{-(n-1-k)} - q^n - q^n B(\omega) \right) \partial_q R_{k,n}(\omega) + \frac{B (q^{-(n-1-k)} - B(\omega))}{(1-q)\omega} R_{k,n}(q\omega),$$

for $k = 0, 1, \ldots, n-1$ with the initial condition $R_{0,n}(\omega) \equiv 1$.

For $k < n$, applying (B.2) we have:

$$\int_a^b q^{\frac{k(k+1)}{2}} \omega^k F_n(\omega) \partial_q \omega \, dq\omega = c_n \int_a^b q^{\frac{k(k+1)}{2}} \omega^k \partial_q^n F_0(\omega; n) \, dq\omega =$$

$$= c_n q^{\frac{k(k+1)}{2}} \omega^k F_{k-1}(\omega; n) \bigg|_a^b - c_n [k] \int_a^b q^{\frac{k(k+1)}{2}} \omega^k \partial_q^{n-1} F_0(\omega; n) \, dq\omega =$$

$$= c_n q^{\frac{k(k+1)}{2}} \omega^k \partial_q^n F_0(\omega; n) \bigg|_a^b - \ldots =$$

$$= (-1)^k [k] \ldots [1] c_n \int_a^b \partial_q F_0(\omega; n) \, dq\omega =$$

$$= (-1)^k \ldots [1] \partial_q F_0(\omega; n) \bigg|_a^b =$$

$$= 0.$$

This shows, that polynomials $\tilde{P}_n(\omega)$, $n \in \mathbb{N} \cup \{0\}$, form an OPS. By the proper choice of the normalizing constants $c_n$ one can obtain the monic OPS.

$B \Rightarrow C$

We have proved the validity of the $q$– Rodrigues formula for any Pearson data. So by Theorem 3.1, we have

$$\frac{1}{n} \partial_q \tilde{P}_n(\omega) = c_n^{(1)} \frac{1}{\omega^{(1)}(\omega)} \partial_q^{n-1} \left[ \omega^{(1)}(\omega) B^{(1)}(\omega) \omega^{(n-1)} B^{(n-1)}(\omega) \right]$$

(B.6)

for the Pearson data $(A^{(1)}(\omega), B^{(1)}(\omega))$ given by (B.9). Using now the equality

$$Q q(\omega) B(\omega) \ldots B \left( q^{-(n-1)} \omega \right) = q^{(1)}(\omega) B^{(1)}(\omega) B^{(n-1)}(\omega) \ldots B^{(n-1)}(\omega)$$

(B.7)

and substituting (B.6) into (B.8) we find

$$\tilde{P}_n(\omega) = c_n \frac{1}{\omega^{(1)}(\omega)} \partial_q^n \left[ Q q^{(1)}(\omega) B^{(1)}(\omega) B^{(n-1)}(\omega) \ldots B^{(n-1)}(\omega) \right]$$

(B.8)

$$= c_n q^{-(n-1)} \frac{1}{\omega^{(1)}(\omega)} \partial_q Q^{-1} \partial_q^{n-1} \left[ q^{(1)}(\omega) B^{(1)}(\omega) B^{(n-1)}(\omega) \ldots B^{(n-1)}(\omega) \right] =$$

$$= c_n q^{-(n-1)} \frac{1}{\omega^{(1)}(\omega)} \partial_q Q^{-1} \left[ n \omega^{(1)}(\omega) \partial_q \tilde{P}_n(\omega) \right] =$$

$$= \frac{c_n}{c_n^{(1)}} q^{-(n-1)} \left[ A(\omega) \partial_q \tilde{P}_n(\omega) + B(\omega) \partial_q Q^{-1} \partial_q \tilde{P}_n(\omega) \right].$$
We have proved (3.19). In order to show (3.20) we compare the coefficients of $x^n$ on both sides of (3.19). Additionally we find the formula

$$c_n^{(1)} = [n] q^{-(n-1)} \alpha_n$$

(B.9)

for the normalizing coefficients. Later we will use (B.9) for the calculation of $c_n$.

$C \Rightarrow B$

The proof goes by the induction. It is easy to see that for $n = 1$ (3.18) follows from (3.19). Let us assume that it is true for $n - 1$. We prove it for $n$:

From our assumption and Theorem 3.1 we have

$$\frac{1}{[n]} \partial_q \tilde{P}_n (\omega) = c_n^{(1)} \frac{1}{[n]} \partial_q^{-1} (q^{(1)} (\omega) B (\omega) \ldots B (q^{-(n-2)} \omega)) .$$

(B.10)

Using (3.18) and the Proposition 3.1, we obtain thesis after a simple calculations.

$C \Leftrightarrow D$

From (3.20) we have

$$\tilde{P}_n (\omega) = \frac{[n]}{\lambda_n} (A (\omega) + B (\omega) \partial_q Q^{-1}) \frac{1}{[n]} \partial_q \tilde{P}_n (\omega) .$$

(B.11)

According to the Theorem 3.1, the polynomials \( \left\{ \frac{1}{[n]} \partial_q \tilde{P}_n (\omega) \right\}_0^\infty \) form the monic OPS with respect to the data \( (A(1) (\omega), B(1) (\omega)) \) given by (3.9-3.10). We can thus apply the formula (B.10) with \( (A(1) (\omega), B(1) (\omega)) \) to the polynomial \( \frac{1}{[n]} \partial_q \tilde{P}_n (\omega) \). Repeating this procedure \( n \)-times and using the formula

$$\lambda_n^{(k)} = a_1^{(k)} [n] + b_2 [n] [n-1] q^{-(n-1)}$$

(B.12)

where \( a_1^{(k)} \) is defined by (3.22), we obtain (3.21).

$B \Rightarrow E$

In order to prove that the recurrence formula (3.23) holds we use the identity

$$\partial_q^k \left[ q (\omega) B (\omega) B (q^{-1} \omega) \ldots B (q^{-(n-1)} \omega) \right] = q (\omega) B (\omega) B (q^{-1} \omega) \ldots B (q^{-(n-1-k)} \omega) R_{k,n} (\omega)$$

(B.13)

where the polynomial \( R_{k,n} (\omega) \) satisfies the recurrence equation (B.4). From the eq. (B.13) and Rodrigues formula one has

$$\tilde{P}_n (\omega) = c_n R_{n,n} (\omega) .$$

(B.14)

Let us denote by \( \alpha_k, \beta_k \) and \( \gamma_k \) the three highest coefficients of the polynomial

$$R_{k,n} (\omega) = \alpha_k \omega^k + \beta_k \omega^{k-1} + \gamma_k \omega^{k-2} + \ldots .$$

(B.15)

After substituting (B.15) into (B.14) and comparing the coefficients of the monomials \( \omega^{k+1}, \omega^k \) and \( \omega^{k-1} \) we obtain the following system of the recurrence equations

$$\begin{align*}
\alpha_{k+1} &= (a_1 - b_2 [-2n + 2 + k]) q^k \alpha_k \\
\beta_{k+1} &= (a_0 - b_1 [-n + 1]) q^k \alpha_k + (a_1 - b_2 [-2n + 3 + k]) q^{k-1} \beta_k \\
\gamma_{k+1} &= b_0 [k] \alpha_k + (a_0 - b_1 [-n + 2]) q^{k-1} \beta_k + (a_1 - b_2 [-2n + 4 + k]) q^{k-2} \gamma_k. 
\end{align*}$$

(B.16)
One can solve them by iteration:

\[ \alpha_k = q^{\frac{k(k-1)}{2}} \prod_{l=0}^{k} -1 \left( a_1 - b_2 [-2n + l] \right) \]  

(B.17)

\[ \beta_k = \frac{[n]}{q^{n-1}} \cdot \frac{a_0 - b_1 [-n + 1]}{a_1 - b_2 [-2n + 2]} \cdot \alpha_k \]  

(B.18)

\[ \gamma_k = \frac{(1 - q^n) (1 - q^{n-1})}{(1 - q)^2 (1 + q)} \cdot \frac{(a_0 - b_1 [-n + 2]) (a_0 - b_1 [-n + 1]) + b_0 (a_1 - b_2 [-2n + 2])}{(a_1 - b_2 [-2n + 2]) (a_1 - b_2 [-2n + 3])} \alpha_k \]  

(B.19)

Thus for the monic polynomial

\[ \bar{P}_n (\omega) = \omega^n + \beta (q^n) \omega^{n-1} + \gamma (q^n) \omega^{n-2} + ... \]  

(B.20)

we find that the coefficients

\[ \beta (q^n) := \frac{\beta_n}{\alpha_n} \]

\[ \gamma (q^n) := \frac{\gamma_n}{\alpha_n} \]

are given by the rational functions (3.3-3.4). Using the three terms recurrence relation (3.24) and (B.15) we obtain the formula (3.1-3.2) for the structural functions \( R \) and \( D \).

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