GEOMETRIC COUNTING ON WAVEFRONT REAL SPHERICAL SPACES

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Abstract. We provide $L^p$-versus $L^\infty$-bounds for eigenfunctions on a real spherical space $Z$ of wavefront type. It is shown that these bounds imply a non-trivial error term estimate for lattice counting on $Z$. The paper also serves as an introduction to geometric counting on spaces of the mentioned type.
1. Introduction

Given a space $Z$ with a discrete subset $D$ and an increasing and exhausting family $(B_R)_{R>0}$ of compact subsets, it can be of considerable interest to know the expected number of points from $D$ inside $B_R$ for large $R$. In general terms this is called lattice counting on $Z$. This paper is about lattice counting on a homogeneous space $Z = G/H$, and methods from harmonic analysis to approach it. Here $G$ is a real reductive group and $H$ is a closed subgroup with finitely many components and real algebraic Lie algebra.

At this point we do not go into the specifics of the lattice count on $G/H$ and refer right away to Sections 2 and 4 where we give a self-contained exposition aimed at a wide audience. The terminology there is essentially taken from [6] but we emphasize more the underlying fiber bundle structure $H \to G \to G/H$ and relate counting on $Z$ to counting on the total space $G$ and fiber $H$.

The lattice counting problem including non-trivial error terms has a positive solution for all symmetric spaces $Z = G/H$. A central tool there is the so-called wavefront lemma, see [7]. The wavefront lemma however holds more generally for all real spherical spaces of wavefront type. In Section 3 we give an introduction to real spherical spaces of wavefront type and recall the proof of the wavefront lemma from [19].

The strive for solving the lattice counting problem triggered many interesting developments, perhaps more interesting than the problem itself. Specifically we mention here Selberg’s trace formula for the upper half plane. Here we wish to point out another connection to harmonic analysis. In [24] we have shown that non-trivial error terms for the lattice count are tied to $L^p$-versus $L^\infty$-bounds of eigenfunctions on the non-compact space $Z$.

We assume now that $Z = G/H$ is unimodular, i.e. carries a $G$-invariant positive Radon measure. Following [2] we measure volume growth on $Z$ via a volume-weight

$$v(z) = \text{vol}_Z(Bz) \quad (z \in Z)$$

where $B$ is some fixed neighborhood of 1 in $G$. Let $1 \leq p < \infty$. Then Bernstein’s invariant Sobolev lemma ([2] key lemma p. 686, or [27], Lemma 4.2) implies for all $f \in C^\infty(Z)$ that

$$|f(z)| \leq C v(z)^{-\frac{1}{p}} \|f\|_{p;k} \quad (z \in Z) \quad (1.1)$$

where $\| \cdot \|_{p;k}$ is a $k$-th Sobolev norm of the $L^p$-norm $\| \cdot \|_p$ and $k > \frac{\dim G}{p}$.

We recall from [25] that $v$ is uniformly bounded from below if and only if $H$ is reductive in $G$. Let us assume this in the sequel. Then we obtain in particular:

$$\|f\|_\infty \leq C \|f\|_{p;k} \quad (f \in C^\infty(Z)). \quad (1.2)$$

What is relevant for the lattice count are estimates in the other direction. We call $f \in C^\infty(Z)$ an eigenfunction if it is an eigenfunction for $Z(g)$, the center of the universal enveloping algebra $U(g)$ of $g = \text{Lie}(G)$. Now, given $1 \leq p' < p < \infty$ we ask whether there exist a number $l = l(Z) > 0$ and a constant $C > 0$ such that

$$\|f\|_p \leq C \|f\|_{\infty;l} \quad (1.3)$$
holds for all $L^p$-eigenfunctions $f$ on $Z$. It is of independent interest to classify all homogeneous spaces $Z = G/H$ which feature (1.3).

In Section 6 we establish (1.3) for all rank one real spherical spaces of wave-front type and eigenfunctions which are fixed under a maximal compact subgroup $K < G$. The approach relies on Harish-Chandra’s constant-term approximation of eigenfunctions which was recently obtained for real spherical spaces, see [5]. In Theorem 6.2 we cut down the techniques from [5] to the absolute necessary to give a proof of the constant term approximation in case of real rank one. Having obtained that we prove (1.3) for $K$-fixed functions and rank $R_Z = 1$. Combined with the harmonic analysis approach of [24] this then leads to a quantitative error bound for the lattice count on these spaces (see Theorem 5.5). Finally, in Section 7 we sketch a possible approach to (1.3) for all wavefront real spherical spaces.

2. Geometric counting

A setup for geometric counting needs:

- A locally compact space $X$.
- A notion of volume on $X$ given by a Radon measure $\mu$.
- A discrete set $D \subset X$.
- An increasing and exhausting family $B = (B_R)_{R > 0}$ of relatively compact sets $B_R \subset X$.

For $R > 0$ we then set

$$N_R(D, X) := \#\{d \in D \mid d \in B_R\}.$$ 

For a measurable subset $B \subset X$ we use the notation $\text{vol}(B) = |B| = \mu(B)$. We then ask to what extent $N_R(D, X)$ approximates $\text{vol}(B_R)$ for $R \to \infty$. We say that the quadruple $(X, \mu, D, B)$ satisfies main term counting (MTC) provided that

$$\lim_{R \to \infty} \frac{N_R(D, X)}{|B_R|} = 1.$$ 

The mother of all counting problems is the Gauß circle problem (GCP), that is $(\mathbb{R}^2, dx \wedge dy, \mathbb{Z}^2, B)$ with $B_R = B^{\text{Encl}}_R$ the round Euclidean ball of radius $R$. It is almost immediate that the GCP satisfies MTC.

In order to expect MTC in a general setup explained above one needs additional assumptions. In some sense the discrete set needs to be equidistributed at infinity. This might be satisfied if $D$ is freely homogeneous, that is:

- There is an infinite discrete group $\Gamma$ acting on $X$ properly, freely and volume-preserving.
- There is $x_0 \in X$ such that $\Gamma \simeq \Gamma \cdot x_0 = D$.
- There is a (locally closed) fundamental domain $F \subset X$ for the $\Gamma$-action with $\text{vol}(X/\Gamma) := \text{vol}(F) = 1$ and $x_0 \in F$.

At least for $F$ relatively compact and the family $(B_R)_{R > 0}$ exhausting the space in a homogeneous manner, we can imagine that $\text{vol}(B_R)$ is asymptotically approximated by

$$\#\{\gamma \in \Gamma \mid \gamma \cdot F \subset B_R\},$$

i.e. the number of tiles $\gamma \cdot F$ which lie in $B_R$. In fact, it is easy to construct a family $\mathcal{B}$ which satisfies MTC. For that let $(F_R)_{R > 0}$ be a relatively compact exhaustion of the fundamental domain $F$ and $(\Gamma_R)_{R > 0}$ an exhaustion of $\Gamma$ by finite subsets. Then $B_R := \Gamma_R \cdot F_R$ defines an
exhaustion of $X$ which satisfies MTC. In practice we certainly wish to take more geometric
exhaustions $\mathcal{B}$ than the one constructed above. Typically has a variety of interesting metrics
$d$ on $X$ and one would like to take for $\mathcal{B}$ the metric balls $B_R := \{x \in X \mid d(x, x_0) < R\}$. We
return to this issue later on.

Here is a large class of freely homogeneous examples where MTC holds. We let $X = G$ be
connected Lie group which admits a lattice $\Gamma < G$. We recall that a lattice in a Lie group
$G$ is a discrete subgroup $\Gamma < G$ with finite co-volume with respect to a Haar measure $\mu$ on
$G$. We take $D = \Gamma$ and normalize $\mu$ such that $\text{vol}(G/\Gamma) = 1$. For the exhausting family
$\mathcal{B}$ almost anything will do; a particular nice family would be balls of radius $R$ with respect
to a left invariant metric on $X$. Then there exists an exhausting compact family
estimate for this case, see Theorem 1.5 of [11]. Almost anything will do; a particular nice family would be balls of radius $R$ with respect to a left invariant metric on $X = G$. The quickest way to establish MTC for $G$ is via the
wavefront lemma applied to $G/H$ on $G$ viewed as a homogeneous space for the two-sided action of
$G \times G$, see [7] and Remark 3.7 after Lemma 3.5 below. A further study yields also an error
estimate for this case, see Theorem 1.5 of [11].

Starting with a freely homogeneous quadruple $(X, \mu, \Gamma, \mathcal{B})$ which satisfies MTC we let
$p : X \to Z := H\backslash X$ be a principal fiber bundle with fibre $H$. We assume that $H$ is a Lie
group. Let $z_0 = p(x_0)$ and identify $H$ with $H \cdot x_0 = p^{-1}(z_0)$. Let $\mu_H$ be a right Haar-measure
on $H$ and $\mu_X = \mu$. We request that there is a Radon measure $\mu_Z$ on $Z$ such that

\begin{equation}
(2.2) \quad \int_X f(x) \, d\mu_X(x) = \int_Z \int_H f(h\tilde{z}) \, d\mu_H(h) \, d\mu_Z(z) \quad (f \in C_c(X))
\end{equation}

where $Z \to X$, $z \mapsto \tilde{z}$ is some measurable cross section.

The following assumptions on the fibre are then natural:

- There is a discrete subgroup $\Gamma_H < \Gamma$ such that $H \cap (\Gamma \cdot x_0) = \Gamma_H \cdot x_0$.
- $\Gamma_H$ is a co-volume 1 lattice in $H$.
- $(H, \mu_H, \Gamma_H, \mathcal{B}^H)$ satisfies MTC where $\mathcal{B}^H := (B_R \cap H)_{R>0}$.

Let now $B^Z_R := p(B_R)$ and $\mathcal{B}^Z$ the corresponding family of balls. One might then ask
whether $p(\Gamma) \subset Z$ is discrete and $(Z, \mu_Z, p(\Gamma), \mathcal{B}^Z)$ satisfies MTC?

In case the principal bundle is homogeneous we have the following:

**Conjecture 2.1.** Let $G$ be a connected Lie group and $H < G$ a closed subgroup such that
$Z := G/H$ carries a $G$-invariant positive Radon measure $\mu_Z$ which satisfies (2.2) with respect
to some Haar measures $\mu_G$ and $\mu_H$ of $G$ and $H$. Assume that $H$ has finitely many connected
components. Further let $\Gamma < G$ be a lattice such that

- $\text{vol}(G/\Gamma) = 1$.
- $\Gamma_H := \Gamma \cap H$ is a lattice in $H$ such that $\text{vol}(H/\Gamma_H) = 1$.

Then there exists an exhausting compact family $\mathcal{B}^Z = (B^Z_R)_{R>0}$ of $Z$ such that the quadruple
$(Z, \mu_Z, \Gamma/\Gamma_H, \mathcal{B}^Z)$ satisfies MTC.

We point out here that the setup in the above open problem was taken from [6]. Notice
that the GCP falls in this setup as well as the Selberg circle problem (SCP) on the upper
half plane $Z = \text{SL}(2, \mathbb{R})/\text{SO}(2, \mathbb{R})$ on which we will comment in more detail later on.

We now give some evidence to the conjecture and show:

**Conjecture 2.1** holds true if $Y_H := H/\Gamma_H$ is compact and $G \to Z$ is trivial.
Proof. Recall that $G \to Z = G/H$ is trivial means $G \cong Z \times H$ as a right $H$-space. This is already an interesting class and typical examples arise in the following manner: We let $G$ be semisimple, $G = KAN$ be an Iwasawa decomposition and $N = N_1 \times N_2$. Then for $H = N_2$, the principal bundle $G \to Z$ is trivial, as $Z \cong K \times A \times N_1$ as manifolds.

To continue, we assume that $B^G_R = B^Z_R \times B^H_R$ are product balls. This yields in particular (2.3)
\[
\text{vol}(B^G_R) = \text{vol}(B^Z_R) \cdot \text{vol}(B^H_R).
\]

As we assume that $Y_H := H/\Gamma_H$ is surjective with fibers isomorphic to $\Gamma$, we deduce from (2.3) and (2.4) that MTC holds in case $G \cong Z \times H$ is trivial.

In fact, let $\tilde{G}$ avoids all lattice points. For the construction of such a $\tilde{G}$, one can employ the Mostow decomposition of $Z$ which is of particular importance for this article and which we now recall.

Let $K < G$ be a maximal compact subgroup of $G$ such that $K_H := K \cap H$ is a maximal compact subgroup of $H$. The Mostow decomposition (see [23]) of $Z = G/H$ then asserts the existence of a finite dimensional vector space and a $K_H$-module $V \subset \mathfrak{g}$ such that
\[
K \times_{K \cap H} V \to G/H, \ [k, X] \mapsto k \exp(X)H
\]
is a diffeomorphism. In particular, as $K$ is compact, the existence of such a trivializing set $Z'$ described above follows.

The coordinates given by (2.5) allow us to define a natural family of balls. For that let $N_K(H)$ the normalizer of $H$ in $K$ and observe that $V$ can be chosen to be $N_K(H)$-invariant. Let $\| \cdot \|$ be a $N_K(H)(H \cap K)$-invariant Euclidean norm on $V$ and define $K$-invariant balls as follows
\[
B_R := \{[k, X] \in G/H \mid \|X\| < R\}.
\]

Following [24] we call these balls intrinsic. Having this terminology we ask:

Problem 2.2. Does conjecture 2.1 hold true with $B^Z$ a family of intrinsic balls?
Here is a short history of MTC in the context of Conjecture 2.1. MTC was established via harmonic analysis in [6] for symmetric spaces G/H and certain families of balls, for lattices with \( Y_H \) compact. In subsequent work [7] the obstruction that \( Y_H \) is compact was removed and an ergodic approach was presented. The ergodic techniques were refined in [8] and it was discovered that MTC holds for a wider class of reductive spaces: For reductive algebraic groups G, H defined over \( \mathbb{Q} \) and arithmetic lattices \( \Gamma < G(\mathbb{Q}) \) it is sufficient to request that the identity component of H is not contained in a proper parabolic subgroup of G which is defined over \( \mathbb{Q} \) and that the balls \( B_R \) satisfy a certain condition of non-focusing.

In these works the balls \( B_R \) are constructed as follows. All spaces considered are affine in the sense that there exists a \( G \)-equivariant embedding of \( \mathbb{Z} \) into the representation module \( V \) of a rational representation of \( G \). For any such embedding and any norm on the vector space \( V \), one then obtains a family of balls \( B_R \) on \( \mathbb{Z} \) by intersection with the metric balls in \( V \). For symmetric spaces all families of balls produced this way are suitable for the lattice counting, but in general one needs to assume non-focusing in addition.

The core of the approach of [7] was a geometric lemma satisfied by symmetric spaces which the authors termed wavefront lemma. In their work on p-adic spherical spaces Sakellaridis and Venkatesh [33] coined the notion of a wavefront p-adic spherical space and showed that these spaces satisfy a p-adic version of the wavefront lemma. Wavefront real spherical spaces were introduced in [19] and it was shown in [19] Lemma 6.3 that they satisfy the wavefront lemma of [7]. In particular real spherical spaces of wavefront type satisfy MTC for all reasonable families of balls (see Theorem 3.6 below).

At this point we remark that all symmetric spaces are real spherical of wavefront type. The latter type of spaces is going to be the main player of this article. Before we continue with error term bounds for the lattice counting problem we insert a section on wavefront spaces and provide a proof of the wavefront lemma.

3. Real spherical spaces of wavefront type

The notational convention for this paper is that we denote Lie groups by upper case Latin letters, e.g. \( A, B, C \), and their corresponding Lie algebras with lower case German letters, e.g. \( a, b, c \).

We assume that \( Z = G/H \) is real spherical, that is, a minimal parabolic subgroup \( P \) of \( G \) admits an open orbit on it. By choosing \( P \) suitably we can then arrange that its orbit through the origin \( z_0 = H \in Z \) is open, or equivalently that \( \mathfrak{g} = \mathfrak{h} + \mathfrak{p} \). All symmetric spaces are known to be real spherical.

According to [21] there is a unique parabolic subgroup \( Q \supset P \) with the following two properties:

- \( QH = PH \).
- There is a Levi decomposition \( Q = L U \) with \( L_n \subset Q \cap H \subset L \).

Here \( L_n \) denotes the analytic subgroup of \( L \) for which the Lie algebra \( \mathfrak{l}_n \) is the sum of all non-compact simple ideals of \( \mathfrak{l} \).

We observe that \( L \cap P \) is a minimal parabolic subgroup in \( L \). It follows that we can choose an Iwasawa decomposition \( L = K_L A_L N_L \) such that \( A_L N_L \subset P \). Having fixed that we choose a compatible Iwasawa decomposition \( G = K A N \), i.e. \( K_L < K, A_L = A \) and \( N_L < N \). Then \( N \) is the unipotent radical of \( P \) and with \( M = Z_K(A) \) we have the Langlands decomposition \( P = MAN \) of \( P \).
Set \( A_H := A \cap H \) and put \( A_Z = A/A_H \). We recall that \( \dim A_Z \) is an invariant of the real spherical space, called the real rank (see [21]).

In [19], Section 6, we defined the notion of wavefront for real spherical spaces, which will now be recalled.

Attached to \( a \) and \( P \) are the root system \( \Sigma = \Sigma(a, g) \subset a^* \setminus \{0\} \) and its set \( \Sigma^+ \) of positive roots. For the associated root space decomposition \( g = \bigoplus_{\alpha \in \{0\} \cup \Sigma} g^\alpha \) we have \( g^0 = a \oplus m \) and \( n = \oplus_{\alpha \in \Sigma^+} g^\alpha \). We write \( \Sigma_u \subset \Sigma^+ \) for the subset of \( a \)-weights of \( u \), and let \( \overline{\pi} \) and \( \overline{\pi} \) denote the corresponding sum of negative root spaces. Then

\[
g = \overline{\pi} \oplus l \oplus u = \bigoplus_{\alpha \in \Sigma_u} g^{-\alpha} \oplus l \oplus \bigoplus_{\beta \in \Sigma_u} g^\beta
\]

Attached to \( Z \) is a geometric invariant, the so-called compression cone. It is a closed and convex subcone \( a^-Z \) of \( aZ \), defined as follows. According to [21] there exists a linear map

\[
(3.1) \quad T : \overline{\pi} \oplus l \oplus u \subset \bigoplus_{\beta \in \{0\} \cup \Sigma_u} g^\beta
\]

such that

\[
(3.2) \quad h = l \cap h \oplus \{X + T(X) \mid X \in \overline{\pi}\}.
\]

Here \( l_H \subset a \oplus m \) denotes the orthocomplement of \( l \cap h \) in \( l \). For \( \alpha \in \Sigma_u \) and \( \beta \in \{0\} \cup \Sigma_u \) we denote by \( T_{\alpha, \beta} : g^{-\alpha} \to g^\beta \) the map obtained from (3.1) by restriction of \( T \) to \( g^{-\alpha} \) and projection to \( g^\beta \). Then

\[
T = \sum_{\alpha, \beta} T_{\alpha, \beta}
\]

and by definition

\[
a^-_Z = \{Y \in a \mid (\alpha + \beta)(Y) \leq 0, \forall \alpha, \beta \text{ with } T_{\alpha, \beta} \neq 0\}.
\]

For each \( Y \in a_H \) and all \( \overline{X} \in \overline{\pi} \) we find

\[
h \ni [Y, \overline{X} + T_{\alpha, \beta}(\overline{X})] = -\alpha(Y)\overline{X} + \beta(Y)T_{\alpha, \beta}(\overline{X}).
\]

Hence if \( T_{\alpha, \beta} \neq 0 \) we see by comparing with (3.2) that \(-\alpha(Y) = \beta(Y)\). We conclude that \( a_H \subset a^-_Z \). Moreover, if we denote by \( a^- \subset a \) the closure of the negative Weyl chamber, then it is clear that \( a^- \subset a^-_Z \). Hence

\[
a^- + a_H \subset a^-_Z.
\]

Since \( T_{\alpha, \beta} \) can be zero for many pairs of roots \((\alpha, \beta)\), the above inclusion can be proper in general. By definition \( Z \) is called a wavefront space if in fact

\[
(3.3) \quad a^- + a_H = a^-_Z.
\]

At this point we note that if \( Z \) is symmetric, say with corresponding involution \( \sigma \) of \( g \), then the special Iwasawa decomposition which we requested above can be obtained by choosing a Cartan involution \( \theta \) that commutes with \( \sigma \). Then \( T(\overline{X}) = \sigma(\overline{X}) \) for all \( \overline{X} \in \overline{\pi} \), and \( T_{\alpha, \beta} \) is non-zero if and only if \( \beta = -\sigma \alpha \) in this case. From this it easily follows that we have the equality in (3.3), that is, all symmetric spaces are wavefront.
3.1. Examples of wavefront spaces. We first give a classification of all non-symmetric wavefront real spherical pairs \((g, h)\) with \(g\) simple and \(h\) reductive. These are the pairs of Table 1. The table is deduced from the classification in [17] together with the following result from [20] Thm. 6.3.

**Lemma 3.1.** Let \(Z = G/H\) be a wavefront real spherical space with \(g\) simple and \(h\) reductive. Let \(H < H^* < G\) be a closed subgroup such that \(Z^* := G/H^*\) is unimodular. Then \(H^*/H\) is compact.

**Remark 3.2.** We say that \(Z\) has real rank one if \(\dim a_Z = 1\). In Table 1 the following are of real rank one:

\[
\begin{align*}
(2), (6), (9) - (11), (12) \text{ and } (13) \text{ for } q = 1, (20) - (22) .
\end{align*}
\]

There are many more examples in case \(g\) is semi-simple and not simple (see the classification in [18]). However, with the exception of two cases, they are not interesting for the lattice count for the following reason: If \(G = G_1 \times \ldots \times G_n\) is a product of simple groups, then an irreducible lattice can only exist if the \(g_i \otimes \mathbb{R} \mathbb{C}\) are all isomorphic (see [15]). In view of the classification in [18] one is then left with the group case \((g, h) = (g_0 \oplus g_0, \text{diag}(g_0))\)

| \(g\) | \(h\) | Condition |
|------|------|-----------|
| \(su(p_1 + p_2, q_1 + q_2) \oplus su(p_1, q_1) + su(p_2, q_2)\) | \(su(p_1, q_1) \oplus su(p_2, q_2)\) | \((p_1, q_1) \neq (q_2, p_2)\) |
| \(su(n, 1)\) | \(su(n - 2q, 1) + sp(q) + \mathfrak{f} \subseteq u(1)\) | \(1 \leq q \leq \frac{n}{2}\) |
| \(sl(n, \mathbb{H})\) | \(sl(n - 1, \mathbb{H}) + \mathfrak{f}\) | \(\mathbb{R} \subseteq \mathfrak{f} \subseteq \mathbb{C}\) |
| \(sp(p, q)\) | \(sp(p - 1, q)\) | \(p \neq q\) |
| \(so(2p, 2q)\) | \(so(p, q)\) | \(p \neq q - 1, q\) |
| \(so(2p + 1, 2q)\) | \(so(p, q)\) | \(p \neq q - 1, q\) |
| \(so(n, 1)\) | \(so(n - 2q, 1) + su(q) + \mathfrak{f} \subseteq u(1)\) | \(2 \leq q \leq \frac{n}{2}\) |
| \(so(n, 1)\) | \(so(n - 4q, 1) + sp(q) + \mathfrak{f} \subseteq sp(1)\) | \(2 \leq q \leq \frac{n}{4}\) |
| \(so(n, 1)\) | \(so(n - 16, 1) + \text{spin}(9)\) | \(n \geq 16\) |
| \(so(n, q)\) | \(so(n - 7, q) + G_2\) | \(n \geq 7, q = 1, 2\) |
| \(so(n, q)\) | \(so(n - 8, q) + \text{spin}(7)\) | \(n \geq 8, q = 1, 2, 3\) |
| \(so(2n)\) | \(so(2n - 2)\) | \(n \geq 5\) |
| \(so(10)\) | \(so(6, 1) + sp(5, 2)\) | |
| \(so(10)\) | \(so(6, 1) + sp(5, 2)\) | |
| \(so(4, 3)\) | \(G_2\) | |
| \(so(7, \mathbb{C})\) | \(G_2^\mathbb{C}\) | |
| \(E_6^1\) | \(sl(3, \mathbb{H}) + \mathfrak{f}\) | \(\mathfrak{f} \subseteq u(1)\) |
| \(E_7^2\) | \(E_6^0\) or \(E_6^0\) | |
| \(F_4^2\) | \(sp(2, 1) + \mathfrak{f}\) | \(\mathfrak{f} \subseteq u(1)\) |
| \(G_2\) | \(sl(3, \mathbb{C})\) | |
| \(G_2\) | \(sl(3, \mathbb{R}), su(2, 1)\) | |

Table 1

Cases marked \(*\) result from a symmetric over-algebra \(h^* \supset h\) such that \(h + u(1) = h^*\).
and the triple spaces
\[ g = \mathfrak{so}(1, n) \oplus \mathfrak{so}(1, n) \oplus \mathfrak{so}(1, n) \quad \text{and} \quad h = \text{diag} \mathfrak{so}(1, n) \quad (n \geq 2). \]
The triple spaces feature a lot of interesting irreducible lattices. Here we review an example given in [24].

We let \( n = 2 \) and \( G_0 = \text{SO}_e(1, 2) \) and consider space \( Z = G/H \) where \( G = G_0 \times G_0 \times G_0 \) and \( H = \text{diag}(G_0) \).

We assume that \( G_0 = \text{SO}_e(1, 2) \) is defined by a quadratic form \( Q \) which has integer coefficients and is anisotropic over \( \mathbb{Q} \), for example
\[ Q(x_0, x_1, x_2) = 2x_0^2 - 3x_1^2 - x_2^2. \]
Then, according to Borel, \( \Gamma_0 = G_0(\mathbb{Z}) \) is a uniform lattice in \( G_0 \).

Next let \( k \) be a cubic Galois extension of \( \mathbb{Q} \). Note that \( k \) is totally real. An example of \( k \) is the splitting field of the polynomial \( f(x) = x^3 + x^2 - 2x - 1 \). Let \( \sigma \) be a generator of the Galois group of \( k|\mathbb{Q} \). Let \( \mathcal{O}_k \) be the ring of algebraic integers of \( k \). We define \( \Gamma < G = G_0^3 \) to be the image of \( G_0(\mathcal{O}_k) \) under the embedding
\[ G_0(\mathcal{O}_k) \ni \gamma \mapsto (\gamma, \gamma^\sigma, \gamma^{\sigma^2}) \in G. \]
Then \( \Gamma < G \) is a uniform irreducible lattice with \( H \cap \Gamma \simeq \Gamma_0 \) a uniform lattice in \( H \simeq G_0 \).

3.2. Property I. We briefly recall some results and notions from [20] and [24].

Let \( (\pi, \mathcal{H}_\pi) \) be a unitary irreducible representation of \( G \). We denote by \( \mathcal{H}_\pi^\infty \) the \( G \)-Fréchet module of smooth vectors and by \( \mathcal{H}_\pi^{-\infty} \) its dual. Elements in \( \mathcal{H}_\pi^{-\infty} \) are called distribution vectors. It is known (see [26]) that for a real spherical space the space \( (\mathcal{H}_\pi^{-\infty})^H \) of \( H \)-fixed distribution vectors is finite dimensional. The representation \( \pi \) is said to be \( H \)-distinguished if \( (\mathcal{H}_\pi^{-\infty})^H \neq \{0\} \).

Let \( \eta \in (\mathcal{H}_\pi^{-\infty})^H \) and \( H_\eta < G \) the stabilizer of \( \eta \). Note that \( H < H_\eta \) and set \( Z_\eta := G/H_\eta \). With regard to \( \eta \) and \( v \in \mathcal{H}_\pi^\infty \) we form the generalized matrix-coefficient
\[ m_{v,\eta}(gH) := \eta(\pi(g^{-1})v) \quad (g \in G) \]
which is a smooth function on \( Z_\eta \).

We recall the following facts from [20] Thm. 7.6 and Prop. 7.7:

Proposition 3.3. Let \( Z \) be a wavefront real spherical space with \( H < G \) reductive. Then the following assertions hold:

1. Every generalized matrix coefficient \( m_{v,\eta} \) as above is bounded.
2. Let \( (\pi, \mathcal{H}) \) be a unitary irreducible representation of \( G \) and let \( \eta \in (\mathcal{H}_\pi^{-\infty})^H \). Then \( Z_\eta \) is unimodular, i.e. carries a positive \( G \)-invariant Radon measure, and there exists \( 1 \leq p < \infty \) such that \( m_{v,\eta} \in L^p(Z_\eta) \) for all \( v \in \mathcal{H}_\pi^\infty \).

The property of \( Z = G/H \) that \( (2) \) is valid for all \( \pi \) and \( \eta \) as above is denoted Property (I) in [20]. Note that \( (1) \) and \( (2) \) together imply \( m_{v,\eta} \in L^q(Z_\eta) \) for \( q > p \). Assuming Property (I) we can then make the following notation.

Definition 3.4. Given \( \pi \) as above, we define \( p_H(\pi) \) as the smallest index \( \geq 1 \) such that all \( K \)-finite generalized matrix coefficients \( m_{v,\eta} \) with \( \eta \in (\mathcal{H}_\pi^{-\infty})^H \) belong to \( L^p(Z_\eta) \) for any \( p > p_H(\pi) \).
Notice that \( m_{\nu,\eta} \) belongs to \( L^p(Z_\eta) \) for all \( K \)-finite vectors \( \nu \) once that this is the case for some non-trivial such vector \( \nu \), see [20] Lemma 7.2. For example, this could be the trivial \( K \)-type, if it exists in \( \pi \).

It follows from Proposition 3.3 (2) and finite dimensionality of \((\mathcal{H}_{-\infty})^H\) that \( p_H(\pi) < \infty \). We say that \( \pi \) is \( H \)-tempered if \( p_H(\pi) = 2 \). Note that if \( \pi \) is not \( H \)-distinguished (that is, if \((\mathcal{H}_{-\infty})^H = 0) \) then \( p_H(\pi) = 1 \).

3.3. The polar decomposition and the wavefront lemma. Let us denote by \( z_0 = H \in Z \) the standard base point. It is convenient to assume that there are complex groups \( H_C \subset G_C \) such that \( G \subset G_C \) is a real form and \( H = G \cap H_C \). Set \( Z_C = G_C/H_C \) and observe that

\[
Z \hookrightarrow Z_C, \quad gH \mapsto gH_C
\]

costitutes a \( G \)-equivariant embedding.

We now recall from [19] (see also [16], Sect. 13) the polar decomposition for real spherical spaces

\[Z = \Omega A^- W \cdot z_0\]  (3.4)

where

- \( \Omega \) is a compact set of the type \( FK \) with \( F \subset G \) a finite set.
- \( W \subset G \) is a finite set with the property that \( W \cdot z_0 \subset T \cdot z_0 \cap Z \) where \( T = \exp(ia) \) and the intersection is taken in \( Z_C = G_C/H_C \). In particular, \( Pw \cdot z_0 \subset Z \) is open for all \( w \in W \).

Denote \( A^- = \exp(a^-) \) and notice that the wavefront property (3.3) implies that

\[A^- \cdot z_0 = A^- \cdot z_0.\]  (3.5)

With that we obtain a generalization of the “wavefront lemma” of Eskin-McMullen ([7] Theorem 3.1), see Lemma 6.3 in [19].

**Lemma 3.5.** (Wavefront Lemma) Suppose that \( Z = G/H \) is a wavefront real spherical space. Then there exists a closed subset \( E \subset G \) with the following properties.

1. \( E \rightarrow G/H \) is surjective.
2. The family of left translations of \( Z \) by elements \( g \in E \) is equicontinuous at \( z_0 \), that is, for every neighborhood \( V \) of \( 1 \) in \( G \), there exists a neighborhood \( U \) of \( 1 \) in \( G \) such that

\[z \in U \cdot z_0 \Rightarrow g \cdot z \in V g \cdot z_0\]

for all \( g \in E \).

**Proof.** Put

\[E = \Omega A^- W.\]

Then (1) follows from the wavefront assumption, by (3.4)-(3.5).

Let \( V \) be a neighborhood of \( 1 \) in \( G \). By compactness there exists a neighborhood \( V_1 \subset V \) such that \( \text{Ad}(x)V_1 \subset V \) for all \( x \in \Omega \). Then it suffices to establish the implication in (2) for \( g \in A^- W \) and with \( V_1 \) instead of \( V \).
Since conjugation by $A^-$ contracts $n$ there exists an open neighborhood $U_1 \subset V_1 \cap P$ of 1 in $P$ such that $\text{Ad}(a)U_1 \subset U_1$ for all $a \in A^-$. It follows from the openness of $Pw \cdot z_0$ for each $w$ that

$$\bigcap_{w \in W} w^{-1}U_1w \cdot z_0$$

is open. It hence contains $U \cdot z_0$ for some neighborhood $U$ of 1 in $G$. With that we obtain for $z \in U \cdot z_0$ that $w \cdot z \in U_1w \cdot z_0$ for all $w \in W$, and hence for $g = aw \in A^-W$ that

$$g \cdot z \in aU_1w \cdot z_0 \subset U_1aw \cdot z_0 \subset V_1g \cdot z_0.$$

as claimed. \hfill \Box

We refer to [6] or [7] for the notion of well-rounded balls.

**Theorem 3.6.** Let $Z = G/H$ be a wave-front real spherical space. Then MTC holds for any family $B$ of well-rounded balls. If in addition $H$ is reductive, then MTC holds for the intrinsic balls.

**Proof.** The first part follows from [7]. To be precise: Th. 1.2 (Equidistribution) in [7] only requires the wave-front lemma. MTC, that is [7] Th. 1.4, then follows from Th. 1.2 for any family of well-rounded balls. The last statement follows from [24], Section 2, where it was shown that the intrinsic balls are well rounded for $H$ reductive. \hfill \Box

**Remark 3.7.** The wavefront lemma holds for an arbitrary Lie group $G$ when considered as a homogeneous space $Z = G \times G/\Delta(G) \simeq G$ with isomorphism provided by the map

$$p : Z \to G, \quad (g_1, g_2)\Delta(G) \mapsto g_1 g_2^{-1}.$$  

We take $E = G \times \{1\} \subset G \times G$. Now for $(g, 1) \in E$ and $U = U_1 \times U_2$, a neighborhood of $(1, 1)$ in $G \times G$, we have

$$p((g, 1)U\Delta(G)) = gU_1U_2^{-1}.$$  

On the other hand for the given neighborhood $V = V_1 \times V_2$ we have similarly

$$p(V(g, 1)\Delta(G)) = V_1gV_2^{-1}$$

and this contains $gV_2^{-1}$. Thus we just have to require of $U$ that $U_1U_2^{-1} \subset V_2^{-1}$.

4. GENERALITIES ON COUNTING WITH ERROR TERMS

After the interlude on wavefront real spherical spaces, we pick up the discussion from Section 2 and continue with error terms for the main term count. Assume that we have a quadruple $(X, \mu, D, (B_R)_{R>0})$ which satisfies main term counting MTC. We then define the pointwise error term

$$\text{err}_{pt}(R, D) := |N_R(D, X) - |B_R|$$

and one might ask for the optimal $\alpha \leq 1$ such that

$$\text{err}_{pt}(R, D) \ll |B_R|^{1+\epsilon} \quad (\epsilon > 0).$$

For the GCP one knows that $\alpha \leq \frac{131}{416} = 0.3149\ldots$ and some believe that $\alpha = \frac{1}{4}$, the lower threshold of Hardy and Landau, is possible.

In this regard we mention that $\alpha = \frac{1}{2}$ can be achieved for every family of balls which are given by $B_R = R \cdot B$ for some absolutely convex bounded set $B \subset \mathbb{R}^2$ with $C^2$-boundary,
see [12], p.8. Also observe that the geometry of $B$ matters for the error count as $\alpha = \frac{1}{2}$ is obviously optimal for the square $B = [-1, 1]^2$.

Thus if it comes to error term bounds we should use balls which are as round as possible, for instance the intrinsic balls which we just introduced.

In obtaining the reasonably good bound of $\alpha = \frac{1}{3}$ for the GCP elementary techniques from Fourier analysis suffice. As an outsider one might ask at what threshold of $\alpha$ analysis converts to number theory. Before we come to the issue of $\alpha = \frac{1}{3}$ we pin down a more specific general setup.

For the remainder we only consider quadruples $(G/H, \mu, \Gamma/\Gamma_H, (B_R)_{R>0})$ which satisfy the MTC. Specifically $Z = G/H$ is a unimodular homogeneous space such that there is a lattice $\Gamma \subset G$ with $\Gamma H = \Gamma \cap H$ a lattice in $H$. We assume that the Haar measures on $G$ and $H$ are normalized such that $G/\Gamma$ and $H/\Gamma_H$ both have volume 1.

With this set-up there is then a double fibration

$$
\begin{array}{ccc}
G/\Gamma_H & \rightarrow & G/H \\
\downarrow & & \downarrow \\
Y = G/\Gamma & \rightarrow & Z = G/H
\end{array}
$$

By fibre-wise integration we obtain maps between functions on $Z$ and $Y$. From $L^\infty(Y)$ to $L^\infty(Z)$ we thus have $\phi \mapsto \phi^H$ defined by

$$
(4.1) \quad \phi^H(gH) := \int_{H/\Gamma_H} \phi(gh\Gamma) \, d(h\Gamma_H) \quad (\phi \in L^\infty(Y))
$$

with

$$
(4.2) \quad \|\phi^H\|_\infty \leq \|\phi\|_\infty.
$$

In the opposite direction, from $L^1(Z)$ to $L^1(Y)$ we have $f \mapsto f^\Gamma$ defined by

$$
(4.3) \quad f^\Gamma(g\Gamma) := \sum_{\gamma \in \Gamma/\Gamma_H} f(g\gamma H) \quad (f \in L^1(Z)),
$$

and likewise contractive

$$
(4.4) \quad \|f^\Gamma\|_1 \leq \|f\|_1.
$$

From Fubini’s theorem we obtain the following adjointness relation:

$$
(4.5) \quad \langle f^\Gamma, \phi \rangle_{L^2(Y)} = \langle f, \phi^H \rangle_{L^2(Z)}
$$

for all $\phi \in L^\infty(Y)$ and $f \in L^1(Z)$. In particular, applying (4.5) to $|f|$ and $\phi = 1_Y$ implies (4.4).

We write $1_R \in L^1(Z)$ for the characteristic function of $B_R$ and deduce

- $1_R^\Gamma(e\Gamma) = N_R(\Gamma, Z) := \# \{ \gamma \in \Gamma/\Gamma_H \mid \gamma \cdot z_0 \in B_R \}$,
- $\|1_R^\Gamma\|_{L^1(G/\Gamma)} = |B_R|$.

where the second equality follows from (4.5) with $\phi = 1_Y$.

**Example 4.1.** Once again we return to the GCP with $Z = G = \mathbb{R}^2$ and $\Gamma = \mathbb{Z}^2$. Then

$$
N_R(\mathbb{Z}^2, \mathbb{R}^2) = \sum_{\gamma \in \mathbb{Z}^2} 1_R(\gamma).
$$
Let us recall how to obtain the bound \( \alpha = \frac{1}{2} \) by means of harmonic analysis on \( Y = \mathbb{R}^2 / \mathbb{Z}^2 \), more precisely the Poisson summation formula. Informally we would like to apply this and deduce

\[
N_R(Z^2, \mathbb{R}^2) = \sum_{\gamma \in \Gamma^\wedge} \widehat{1}_R(\gamma)
\]

where \( \widehat{1}_R \) is the Fourier-transform of the the characteristic function \( 1_R \) and \( \Gamma^\wedge \) is the dual lattice. However, as the cutoff \( 1_R \) is not smooth, the sum in (4.6) is not absolutely convergent. The remedy is to smoothen \( 1_R \) by convolution with some radial \( \varphi \in C_c^\infty(G) \) of integral 1, i.e. set \( 1_{R,\varphi} := \varphi * 1_R \) and consider

\[
(4.7) \quad \sum_{\gamma \in \mathbb{Z}^2} 1_{R,\varphi}(\gamma) = \sum_{\gamma \in \Gamma^\wedge} \widehat{1}_{R,\varphi}(\gamma) = \widehat{1}_{R,\varphi}(0) + \sum_{\gamma \in \Gamma^\wedge, \gamma \neq 0} \widehat{1}_{R,\varphi}(\gamma).
\]

Now observe that

\[
\widehat{1}_{R,\varphi}(0) = \int_{\mathbb{R}^2} 1_{R,\varphi}(x) \, dx \wedge dy = |B_R|
\]

and \( \sum_{\gamma \in \mathbb{Z}^2, \gamma \neq 0} \widehat{1}_{R,\varphi}(\gamma) \) converges. By using for \( \varphi \) an approximation of the identity, one derives the estimate as follows.

Since the theory applies equally well in \( n \) dimensions we will work in this generality, and fix a non-negative function \( \varphi \in C_c^\infty(\mathbb{R}^n) \), supported in \( B_1 \) and with integral 1. Let \( \varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(\varepsilon^{-1} x) \), and observe that

\[
1_{R-\varepsilon} * \varphi_\varepsilon \leq 1_R \leq 1_{R+\varepsilon} * \varphi_\varepsilon.
\]

Hence if we denote \( N_{R,\varepsilon} = \sum_{\gamma \in \Gamma} 1_R * \varphi_\varepsilon(\gamma) \) we find

\[
(4.8) \quad N_{R-\varepsilon,\varepsilon} \leq N_R \leq N_{R+\varepsilon,\varepsilon}.
\]

By explicit calculation (see [34], Ch. IV.3) one finds

\[
\widehat{1}_R(\lambda) = \int_{||x|| \leq R} e^{-i\lambda \cdot x} \, dx = C_1 R^n \int_{-1}^{1} (1 - t^2)^{\frac{n-1}{2}} e^{-i|\lambda|Rt} \, dt = C_2 \left( \frac{R}{|\lambda|} \right)^{\frac{n}{2}} J_{\frac{n}{2}}(R|\lambda|)
\]

where \( J_{\frac{n}{2}} \) is a Bessel function and \( C_1, C_2 > 0 \) some constants, which depend only on \( n \). It is well-known ([34], Lemma IV.3.11) that the Bessel functions \( J_m(x) \) for all \( m \geq 0 \) behave like \( x^{-\frac{3}{2}} \) as \( x \to \infty \). Hence

\[
\widehat{1}_R(\lambda) = O(R^{\frac{n+1}{2}} |\lambda|^{\frac{n-1}{2}})
\]

as \( R|\lambda| \to \infty \).

Since \( 1_R * \varphi_\varepsilon \in C_c^\infty(\mathbb{R}^n) \) we can apply Poisson summation. By using \( \widehat{1}_{R,\varphi_\varepsilon} = \widehat{1}_R \widehat{\varphi_\varepsilon} \) we obtain from (4.7)

\[
N_{R,\varepsilon} = \sum_{\gamma \in \Gamma^\wedge} \widehat{1}_R(\gamma) \widehat{\varphi_\varepsilon}(\gamma) = |B_R| + \sum_{\gamma \in \Gamma^\wedge, \gamma \neq 0} \widehat{1}_R(\gamma) \widehat{\varphi_\varepsilon}(\gamma),
\]

where \( \Gamma^\wedge \) denotes the dual lattice. By the rapid decay of \( \widehat{\varphi_\varepsilon} \) there exists \( C > 0 \) such that

\[
\sum_{\gamma \in \Gamma^\wedge, \gamma \neq 0} |\gamma|^{-\frac{n+1}{2}} |\widehat{\varphi_\varepsilon}(\gamma)| \leq C \varepsilon^{\frac{1}{2}}
\]
for all $\epsilon > 0$, and hence also such that
\begin{equation}
|N_{R,\epsilon} - |B_R|| \leq CR^{\frac{n-1}{2}} \epsilon^{\frac{1}{2}}.
\end{equation}
Now $|B_{R,\epsilon}| - |B_R| \sim R^{n-1}\epsilon$, and by comparing with (4.8) we then conclude that
\begin{equation}
|N_R - |B_R|| \leq C((R + \epsilon)^{\frac{n-1}{2}} \epsilon^{\frac{1}{2}} + R^{n-1})
\end{equation}
for some $C > 0$ and all $\epsilon > 0$. The best possible value of $\epsilon$ is $\epsilon_0 = R^{\frac{1}{n+1}}$ and yields
\begin{equation}
|N_R - |B_R|| \leq CR^{\frac{n(n-1)}{n+1}}.
\end{equation}
For $n = 2$ this gives the mentioned bound by $O(R^2)$. For $n = 3$ it gives $O(R^3)$, compared to the best known upper bound $O(R^4 \log R)$. For $n \geq 4$ the error is known to behave essentially like $R^{n-2}$ (see [9]).

This method resembles to some extent the approach in [24] where error term bounds were obtained for wavefront spaces: Poisson summation was replaced by spectral analysis and Weyl’s law.

**Example 4.2.** As a second instance of a classical problem of counting lattice points we review the case of the hyperbolic space. This study was initiated by Delsarte, see [4]. Let the upper half plane $H = \{z = x + iy \in \mathbb{C} : y > 0\}$ be equipped with the invariant metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$ by which it has constant negative curvature. The Riemannian metric induces a distance function $d$ on $H$ which we use to define a family of balls
\begin{equation*}
B(R) = \{z \in H : d(z, i) < R\} \quad (R > 0).
\end{equation*}
Their volume is exponentially growing with the radius:
\begin{equation*}
\text{vol}(B(R)) := \int_{B(R)} \frac{dxdy}{y^2} = 4\pi \sinh^2(R/2) \sim \pi e^R.
\end{equation*}

To set up the counting problem, recall that the hyperbolic plane admits a group of symmetries $G := \text{Isom}(H) = \text{PSL}_2(\mathbb{R})$ and the role of the lattice is played by a discrete subgroup $\Gamma < \text{PSL}_2(\mathbb{R})$ of these symmetries. More precisely, we consider the set $D_\Gamma(z) = \Gamma \cdot z$, with $z \in H$. The hyperbolic lattice counting problem then consists of estimating
\begin{equation*}
N_R(z) := \#D_\Gamma(z) \cap B(R) = \#\{\gamma \in \Gamma : d(i, \gamma \cdot z) < R\},
\end{equation*}
as $R \to \infty$.

An interesting feature of this problem is that, unlike the Euclidean lattice point counting, a direct packing argument is not possible since most of the volume of a hyperbolic ball is located near its boundary. This so-called mass concentration phenomenon is a reflection of the exponential volume growth of the Haar measure on the semi-simple Lie group $G$. Thus, different techniques are required to approach this lattice counting problem and even obtaining the main term was a non-trivial achievement. In [4] the problem is studied for co-compact lattices and MTC is proved, that is, with the measure normalized as above,
\begin{equation*}
N_R(z) \sim \frac{1}{\text{vol}(\Gamma \backslash H)} \pi e^R
\end{equation*}
for all $z$, as $R \to \infty$. In [30] Selberg developed the trace formula as a tool of obtaining geometric information from spectral information and vice-versa.
In case \( \Gamma = \PSL(2, \mathbb{Z}) \), Selberg showed\cite{30} that

\[
N_R(i) \sim \frac{3}{\pi} \vol(B(R))
\]

and obtained an error term estimate,

\[
|N_R(i) - \frac{3}{\pi} \vol(B(R))| = O(\exp(2R/3)).
\]

Maybe the most significant result is the connection between the lattice counting problem for a general lattice of finite co-volume \( \Gamma \subset \PSL(2, \mathbb{Z}) \) and the so-called automorphic spectrum. More precisely, Selberg provided the following asymptotic formula expressing the number of lattice points as a sum over the eigenvalues of the Laplace operator on the Riemann surface \( \Gamma \backslash \mathbb{H} = \Gamma \backslash G/K \).

To formulate the exact formula we denote by \( \Delta \) the Laplace-Beltrami operator on the Riemannian manifold \( \Gamma \backslash \mathbb{H} \). This is sometimes called the hyperbolic Laplacian. In case \( \Gamma \backslash \mathbb{H} \) is compact the spectrum of \( \Delta \) is discrete. In the non-compact case, in addition to the discrete spectrum there is a contribution from the continuous spectrum. This spectrum consists of Eisenstein series which are parametrized by the interval \([\frac{1}{2}, \infty)\) and by cusps. In both cases, we denote by \( \{u_j\}_{j \geq 0} \) a complete system of orthonormal eigenfunctions for the discrete spectrum of the hyperbolic Laplacian corresponding to \( 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \). In particular \( u_0 = \vol(\Gamma \backslash \mathbb{H})^{-\frac{1}{2}} \).

Fix \( z, w \in \mathbb{H} \) and let

\[
N(R, z, w) = \#\{\gamma \in \Gamma : 2 \cosh(d(\gamma z, w)) < R\},
\]

thus \( N_R(z) = N(R', z, i) \) with \( R' = 2 \cosh(R) \sim e^R \). By using the spectral expansion of the automorphic kernel (see Theorem 7.1\cite{13}), Selberg showed in\cite{31} and\cite{32} (see also Theorem 2 of\cite{29} and Theorem 12.1 in\cite{13}) the following formula:

\[
N(R, z, w) = \sqrt{\pi} \sum_{\frac{1}{2} < s_j \leq 1} \frac{\Gamma(s_j - \frac{1}{2})}{\Gamma(s_j + 1)} u_j(z)u_j(w) R^{s_j} + O(R^{\frac{3}{4}})
\]

The relation between \( s_j \) and the eigenvalue \( \lambda_j \) is given by the usual translation \( \lambda_j := s_j(1 - s_j) \).

Notice that in the sum, the value \( s_0 = 1 \) corresponds to the constant function yielding the main term \( \frac{\pi}{\vol(\Gamma \backslash \mathbb{H})} R \). The rest of the sum is over exceptional eigenvalues, that is those eigenvalues of the Laplacian that satisfy \( \lambda_j := s_j(1 - s_j) < \frac{1}{4} \).

The moral of the last example is that even if one invokes the full force of the spectral decomposition of \( L^2(\Gamma \backslash \mathbb{H}) \) and the trace formula, understanding the error term in the lattice counting problem requires refined information about the automorphic spectrum. For example, when considering congruence lattices \( \Gamma(N) \), the Selberg-\( \frac{1}{4} \)-conjecture, that there are no exceptional eigenvalues, that is \( \lambda_1(\Gamma(N) \backslash \mathbb{H}) \geq \frac{1}{4} \), is relevant to determining the main term in the corresponding lattice point counting formula.

For the rest of this article we assume now that \( Z = G/H \) is a wavefront real spherical space, \( H \) is reductive and \( \mathcal{B} \) is constituted by the intrinsic balls. Further we restrict ourselves to the cases where the cycle \( H/\Gamma_H \) is compact.\footnote{After a theory for regularization of \( H \)-periods of Eisenstein series is developed, it is expected that one can drop this assumption.}
in addition that $\Gamma < G$ is irreducible, i.e. there do not exist non-trivial normal subgroups $G_1, G_2$ of $G$ and lattices $\Gamma_1, \Gamma_2$ such that $\Gamma_1 \Gamma_2$ has finite index in $\Gamma$.

The error we study is measure theoretic in nature, and will be denoted here as $\text{err}(R, \Gamma)$. Thus, $\text{err}(R, \Gamma)$ measures the deviation of two measures on $Y = \Gamma \backslash G$, the counting measure arising from lattice points in a ball of radius $R$, and the invariant measure $d\mu_Y$ on $Y$. More precisely, with $1_R$ denoting the characteristic function of $B_R$ we consider the densities

$$F^\Gamma_R(g\Gamma) := \frac{\sum_{\gamma \in \Gamma \cap T_R} 1_R(g\gamma H)}{|B_R|}.$$ 

Then, 

$$\text{err}(R, \Gamma) = \|F^\Gamma_R - d\mu_Y\|_1,$$

where $\|\cdot\|_1$ denotes the total variation of the signed measure. Notice that $|F^\Gamma_R(e\Gamma) - 1| = \frac{|N_\Gamma(\Gamma, Z) - |B_R||}{|B_R|}$ is essentially the error term for the point wise count.

Our results on the error term $\text{err}(R, \Gamma)$ allows us to deduce results toward the error term in the smooth counting problem, a classical problem that studies the quantity

$$\text{err}_{pt, \alpha}(R, \Gamma) = |B_R| |F^\Gamma_{\alpha, R}(e\Gamma) - 1|$$

where $\alpha \in C_c^\infty(G)$ is a positive smooth function of compact support (with integral one) and $F^\Gamma_{\alpha, R} = \alpha * F^\Gamma_R$. We refer to [24], Remark 5.2 for the comparison of $\text{err}(R, \Gamma)$ with $\text{err}_{pt, \alpha}(R, \Gamma)$.

Below we will introduce an exponent $p_H(\Gamma)$ (see [6.1]), which measures the worst $L^p$-behavior of any generalized matrix coefficient associated with a spherical unitary representation $\pi$, which is $H$-distinguished and occurs in the automorphic spectrum of $L^2(\Gamma \backslash G)$. In [24] we obtained the following error bound for triple spherical spaces.

**Theorem 4.3.** Let $Z = G_0^3 / \text{diag}(G_0)$ for $G_0 = \text{SO}_e(1, n)$ and assume that $H / \Gamma_H$ is compact. For all $p > p_H(\Gamma)$ there exists a $C = C(p) > 0$ such that

$$\text{err}(R, \Gamma) \leq C |B_R|^{-\frac{1}{(6n+3)p}}$$

for all $R \geq 1$.

5. COUNTING WITH ERROR TERMS ON A WAVEFRONT REAL SPHERICAL SPACE

We assume from now on that $G$ is semisimple with no compact factors.

First we need some notation. We denote by $\hat{G}$ the unitary dual of $G$, i.e. the set of equivalence classes of irreducible unitary representations of $G$. We recall the maximal compact subgroup $K < G$. By $\hat{G}_s \subset \hat{G}$ we understand the subset which corresponds to $K$-spherical representations, i.e. representations which have a non-zero $K$-fixed vector.

With that we define in continuation of Definition 3.3

$$p_H(\Gamma) := \sup \{p_H(\pi) \mid \pi \in \hat{G}_s \cap \text{supp} \ L^2(G/\Gamma) \}$$

and recall that $p_H(\Gamma) < \infty$ by [24], Lemma 6.3.

A subset $\Lambda \subset \hat{G}_s$ is called $L^p$-bounded provided that $p_H(\pi) < p$ for all $\pi \in \Lambda$. We set $\Lambda_p := \{\pi \in \hat{G}_s \mid p_H(\pi) < p\}$. We note that

$$\Lambda_p = \{\pi \in \hat{G}_s \mid (\forall \eta \in (H_\pi^{-\infty})^H)(\forall v \in H_\pi^{K-\text{fin}}) m_{v, \eta} \in L^p(Z_\eta)\}$$

and

$$\Lambda_p \subset \Lambda_{p'} \quad (p < p').$$
In [24] we formulated the following hypothesis.

**Hypothesis A:** For $1 \leq p < \infty$ there exists a compact subset $D \subset G$ and constants $c, C > 0$ such that the following assertions hold for all $\pi \in \Lambda_p$, $\eta \in (\mathcal{H}^{-\infty})^H$ and $v \in \mathcal{H}_\pi^K$:

\begin{enumerate}
  
  \item[(A1)] $\|m_{v,\eta}\|_{L^p(Z_\eta)} \leq C \|m_{v,\eta}\|_\infty$,
  
  \item[(A2)] $\|m_{v,\eta}\|_\infty \leq c \|m_{v,\eta}\|_{\infty,D}$.
\end{enumerate}

Here $\| \cdot \|_{\infty,D}$ denotes the supremum norm taken on the subset $D$.

Under this hypothesis we have then shown in [24] Thm. 7.3 that:

**Theorem 5.1.** Let $Z$ be wavefront real spherical space for which Hypothesis A is valid. Assume also

- $G$ is semisimple with no compact factors
- $H$ is reductive and the balls $B_R$ are intrinsic
- $\Gamma$ is an arithmetic and irreducible lattice
- $\Gamma_H = H \cap \Gamma$ is co-compact in $H$
- $p > p_H(\Gamma)$
- $s > \frac{1}{2} \text{rank} (G/K) + \dim (G/K) + 1$

Then there exists a constant $C = C(p, s) > 0$ such that

$$
\text{err}(R, \Gamma) \leq C |B_R|^{-\frac{1}{(2s+1)p}}
$$

for all $R \geq 1$. Moreover, if $Y = \Gamma \backslash G$ is compact one can replace the last condition by $s > \dim (G/K) + 1$.

In [24] the Hypothesis A was verified for the triple spaces, and thus we could derive Theorem 4.3 from Theorem 5.1.

When writing [24] the harmonic analysis on real spherical spaces was not sufficiently developed to obtain Hypothesis A (or suitable variants thereof) in general. In particular at that time we were not able to derive an error bound for all wavefront real spherical spaces. For symmetric spaces the existence of a non-quantitative error term was established in [1] and improved in [10].

In case of the hyperbolic plane our error term (5.2) is still far from the quality of the bound of A. Selberg. This is because the approach in [24] only uses a weak version of the trace formula, namely Weyl’s law, and uses simple soft Sobolev bounds between eigenfunctions on $Y$.

Here we shall prove a slightly weaker version of Hypothesis A for spaces $Z$ of real rank one. In order to prepare for it we draw from some notions which we used in the lecture notes [27]. Together with every $G$-continuous norm $p$ on a Harish-Chandra module $V$ there comes a family of Laplace-Sobolev norms $(p_k)_{k \in \mathbb{N}}$ on $V^\infty$. We briefly recall the definition of the $p_k$ for $k$ even (one can define Sobolev norms $p_k$ for any value of $k \in \mathbb{R}_{\geq 0}$). For that we let $\Delta \in \mathcal{U}(\mathfrak{g})$ be a Laplace element such that $\Delta = \mathcal{C} + 2\Delta_\mathfrak{t}$ where $\mathcal{C}$ is the Casimir element of $\mathfrak{g}$ and $\Delta_\mathfrak{t} \in \mathcal{U}(\mathfrak{t})$ is a Laplace element. Then

$$
p_{2k}(v) := \sum_{j=0}^{k} p(\Delta^j v) \quad (v \in V^\infty). \quad (5.3)
$$
Lemma 5.4. Let \( |\mathcal{A}_V| \in \mathbb{C} \) be the multiple by which the Casimir element \( \mathcal{C} \in \mathcal{Z}(\mathfrak{g}) \) acts on \( V \). In particular, if \( V \) is the Harish-Chandra module of a unitary irreducible representation \((\pi, \mathcal{H})\) we write \( |\pi| = |\mathcal{A}_V| \). The following lemma will be used frequently in the sequel to relate between \( p_k \) and \( p \).

Lemma 5.2. Let \( V \) be an irreducible Harish-Chandra module. Let \( k \in \mathbb{N}_0 \). Then there exists constants \( C_1, C_2 > 0 \), independent of \( V \), such that

\[
C_1(1 + |\mathcal{A}_V|)^{\frac{k}{2}} p(v) \leq p_k(v) \leq C_2(1 + |\mathcal{A}_V|)^{\frac{k}{2}} p(v) \quad (v \in V^K).
\]

Proof. (For \( k \) even) As \( v \in V^K \) is \( K \)-fixed we have \( \Delta_i v = 0 \) and thus for every \( j \in \mathbb{N} \)

\[
\Delta^j v = (C + \Delta_i)^j v = \chi^j v.
\]

Hence the assertion follows from (5.3). \( \square \)

In the weaker version of Hypothesis A we replace \([A1]\) by a Sobolev estimate, namely

Hypothesis B: Let \( 1 \leq p' < p < \infty \). Then there exist \( C > 0 \) and \( l \in \mathbb{N} \) such that

\[
\|m_{v,\eta}\|_{L^p(Z_\eta)} \leq C\|m_{v,\eta}\|_{\infty, l}.
\]

for all \( \pi \in \Lambda_{p'}, \eta \in (\mathcal{H}^{-\infty})^H \) and \( v \in \mathcal{H}^K_\pi \).

Remark 5.3. Assume \( \pi \) is non-trivial. Under the assumptions in Thm. 5.1 it follows from \([24]\) Lemma 6.2 that \( H_{\eta}/H \) is compact, and hence \( Z_\eta \) can be replaced by \( Z \) both in \([A1]\) and in \([B1]\).

In the hypothesis above it is unnecessary to include an analogue of \([A2]\), as such an analogue can be derived from \([B1]\). This is the content of the next lemma. For \( R > 0 \) we set

\[
A^-_R := \{ a \in A^\pm | \rho(\log a) \geq -R \}
\]

where \( \rho = \frac{1}{2} \text{tr} \text{ad}_a \in \mathfrak{a}^* \). Further we set

\[
A^-_{Z,R} := A^-_RA_H/A_H \subset A^-_Z
\]

and if \( D \subset G \) is a compact set

\[
D_R := DA^-_{Z,R} \mathcal{W} \cdot z_0 = DA^-_R \mathcal{W} \cdot z_0 \subset Z.
\]

Lemma 5.4. Let \( Z \) be a wavefront real spherical space for which Hypothesis B is valid. With \( p', p, \) and \( l \) as above let \( d = \frac{1}{4}(lp + \dim \mathfrak{a}_Z(l + \dim \mathfrak{g} + 1)) \). Then there exists a compact subset \( D \subset G \) such that

\[
\|m_{v,\eta}\|_\infty = \|m_{v,\eta}\|_{\infty, D_\log(1+|\pi|)}
\]

for all \( \pi \in \Lambda_{p'}, \eta \in (\mathcal{H}^{-\infty})^H \) and \( v \in \mathcal{H}^K_\pi \).

Proof. We recall the polar decomposition \( Z = \Omega A^\pm \mathcal{W} \cdot z_0 \) in (3.4). For \( f = m_{v,\eta} \) and \( g \in \Omega \)
we set \( f_g(z) = f(g\cdot z) \) for \( z \in Z \). We normalize \( \|f\|_\infty = 1 \). Let now \( g_0 \in \Omega, \mathcal{W} \in \mathcal{W} \) and \( X_0 \in \mathfrak{a}_Z \) be such that \( |f_{g_0}(\exp(X_0)\mathcal{W} \cdot z_0)| = \|f\|_\infty \). We recall the invariant Sobolev Lemma from (1.1):

\[
|\phi(z)| \leq C \mathcal{V}(z)^{-\frac{1}{p}} \|\phi\|_{p,s} \quad (z \in Z)
\]
for \( s > \frac{\dim G}{p} \) and all \( \phi \in L^p(Z)^\infty \). As \( Z \) is wavefront it follows from [20], Prop. 4.3, that \( v \) is bounded from below by a positive constant. Thus (5.5) applied specifically to \( \phi = L_Y f_g \) for \( g \in \Omega, Y \in \mathfrak{g} \) with \( \| Y \| = 1 \) and \( z = \exp(X)w \cdot z_0 \) with \( X \in a^-_Z \) yields

\[
(5.6) \quad |L_Y f_g(\exp(X)w \cdot z_0)| \leq C e^{\frac{1}{p} \rho(X)} \| f_g \|_{p, s+1} \leq C \| f \|_{p, s+1}
\]

for some \( C > 0 \).

Now define a function on \( K \times A^-_Z \) by

\[
F(k, X) = f_{g_0}(k \exp(X)w \cdot z_0)
\]

and observe that (5.6) implies

\[
\| dF(k, X) \| \leq C \| f \|_{p, s+1} \quad (k \in K, X \in a^-_Z).
\]

With (B1) and (5.4) we thus obtain

\[
\| dF(k, X) \| \leq C \| f \|_{\infty, l+s+1} \leq C (1 + |\pi|)^{\frac{l+s+1}{2}}.
\]

Set \( \delta := \frac{1}{4c} \left( 1 + |\pi| \right)^{-\frac{l+s+1}{2}} \). Then the mean value theorem implies that there exists a neighborhood \( U \) of \( 1 \) in \( K \) such that \( |F(k, X)| \geq \frac{1}{2} \) for \( k \in U \) and \( |X - X_0| < \delta \).

Next we note that

\[
\int_K \int_{A^-_Z} |\phi(kaw \cdot z_0)|^p J(a) \, da \, dk \leq \| \phi \|_p^p \quad (\phi \in L^p(Z))
\]

for a positive function \( J \) on \( A^-_Z \) which we may assume to satisfy the bound \( J(a) \geq Ca^{-2p} \) for all \( a \in A^-_Z \) (this has to do with a good choice of \( K \): We only need to request that \( \mathfrak{t} + a + \text{Ad}(a)\mathfrak{h}_I = \mathfrak{g} \) holds true for all \( a \in A^-_Z \) and all boundary degenerations \( \mathfrak{h}_I \) of \( \mathfrak{h} \), see (6.4) in [22]).

Hence we obtain for some \( C > 0 \) that

\[
\| f \|_p \geq C \left[ \text{vol}_K(U) \text{vol}_{a^{-}_Z}(\{ ||X - X_0|| \leq \delta \}) e^{-2\rho(X_0)} \right]^\frac{1}{p},
\]

or equivalently

\[
\| f \|_p \geq C (1 + |\pi|)^{-\frac{\dim a_-^Z}{2p}} e^{-\frac{2}{p} \rho(X_0)}.
\]

Together with (B1) this gives

\[
e^{-\frac{2}{p} \rho(X_0)} \leq C (1 + |\pi|)^{-\frac{l+s+1}{2} + \frac{\dim a^{-}_Z}{2p}} e^{-\frac{2}{p} \rho(X_0)}
\]

which shows

\[
|\rho(X_0)| \leq d \log(1 + |\pi|) + C
\]

for a constant \( C \) independent of \( \pi \) and

\[
d = \frac{1}{4} (lp + \dim a^-_Z (l + s + 1)).
\]

This shows that the maximum of \( |f| \) is attained in a region as asserted. \( \square \)
Theorem 5.5. Let $Z$ be a wavefront real spherical space for which Hypothesis B is valid, and assume all bulleted items in Thm. 5.1. Let $s$ be as in Theorem 5.1 and let $l, d$ be as above for $p' := p_H(\Gamma)$ and $p > p'$. Then there exists a constant $C = C(p, s) > 0$ such that

$$\text{err}(R, \Gamma) \leq C |B_R|^{-\frac{1}{2(2p'+1)p}}$$

for all $R \geq 1$, where $s' := s + l + 2d$.

The bound (5.2) is obtained in [24] Thm. 7.3 from the estimate of Prop. 6.5 via standard techniques quite in resemblance to the Euclidean case, see (4.9) - (4.11). With Hypothesis A replaced by B, the analogue of Prop. 6.5 becomes:

Proposition 5.6. Assume that $Z$ is wavefront real spherical and all the bulleted assumption s from above. Assume moreover, that Hypothesis B is valid. Let $p > p_H(\Gamma)$. Then the map

$$\text{Av}_H : C_\infty^b(Y)^K_{\text{van}} \to L^p(Z)^K, \ \phi \mapsto \phi^H; \ \phi^H(gH) = \int_{H/H_G} \phi(gh) \, d(hH_G)$$

is continuous. More precisely, for all

1. $k > \dim(G/K) + 1$ if $Y$ is compact.
2. $k > \frac{\text{rank}(G/K) + 1}{2} \dim(G/K) + 1$ if $Y$ is non-compact and $\Gamma$ is arithmetic

there exists a constant $C = C(p, k) > 0$ such that

$$\|\phi^H\|_{L^p(Z)} \leq C \|\phi\|_{\infty, k+l+2d} \quad (\phi \in C_\infty^b(Y)^K_{\text{van}})$$

Proof. For the sake of completeness we give the slightly modified proof. Let $s := \dim(G/K)$ and $r := \text{rank}(G/K)$. Let $\phi \in C_\infty^b(Y)^K_{\text{van}}$ and write $\phi = \phi_d + \phi_c$ for its decomposition in discrete and continuous Plancherel parts. We assume first that $\phi = \phi_d$.

If $Y$ is compact it follows from Weyl’s law that the multiplicities in $L^2(Y)$ satisfy

$$\sum_{|\pi| \leq R} m(\pi) \sim c_Y R^{s/2} \quad (R \to \infty),$$

for some constant $c_Y > 0$. It follows that

$$\sum_{\pi} m(\pi)(1 + |\pi|)^{-m} < \infty \quad (5.8)$$

for all $m > s/2 + 1$. In case $Y$ is non-compact, we let $\hat{G}_{\Gamma,d} \subset \hat{G}$ be the the discrete support of the Plancherel measure of $L^2(Y)$ and $m(\pi)$ the corresponding multiplicity of $\pi$. Assuming $\Gamma$ is arithmetic it is shown in [14] that

$$\sum_{\pi \in \hat{G}_{\Gamma,d} \atop |\pi| \leq R} m(\pi) \leq c_Y R^{r s/2} \quad (R > 0)$$

so that for $m > r s/2 + 1$ we again obtain (5.8).

Let $p > p_H(\Gamma)$. As $\phi$ is in the discrete spectrum we decompose it as $\phi = \sum_\pi \phi_\pi$ and obtain with [13] that

$$\|\phi^H\|_p \leq \sum_\pi \|\phi_\pi^H\|_p \leq C \sum_\pi \|\phi_\pi^H\|_{\infty, l}.$$
Note that $\pi$ is non-trivial in these sums since $\phi$ has vanishing integral. With (B2) we obtain further
$$\|\phi^H\|_p \leq C \sum_{\pi} \|\hat{\phi}_{\pi}^H\|_{\infty, \Omega \log(1+|\pi|), l}.$$ Let $B_{|\pi|} = \Omega A_{d \log(1+|\pi|)} W H_c \subset G$ where $H_c \subset H$ is a compact subset such that $H_c \Gamma_H = H$.

Note that $\|\hat{\phi}_{\pi}^H\|_{\infty, \Omega \log(1+|\pi|)} \leq \|\phi_{\pi}\|_{\infty, B_{|\pi|}}$ which together with (5.4) allows us to estimate the last sum as follows:
$$\sum_{\pi} \|\phi_{\pi}^H\|_{\infty, \Omega \log(1+|\pi|), l} \leq \sum_{\pi} \|\phi_{\pi}\|_{\infty, B_{|\pi|}, l} \leq C \sum_{\pi} (1 + |\pi|)^{-m/2} \|\phi_{\pi}\|_{\infty, B_{|\pi|}, m+l}$$
with $C > 0$ a constant depending only on $k$ (we allow the same symbol $C$ for universal positive constants, independently of their actual values). Applying the Cauchy-Schwartz inequality combined with (5.8) we obtain

$$\|\phi^H\|_p \leq C \left( \sum_{\pi} \|\phi_{\pi}\|_{\infty, B_{|\pi|}, m+l}^2 \right)^{1/2}$$

with $C > 0$.

In the sequel we view functions on $Y$ as right $\Gamma$-invariant functions on $G$. Recall from [27], Sect. 4, the notion of volume weights on a homogeneous space for $G$ and let $\nu_Y$ be a volume weight for $Y = G/\Gamma$. From Lemma 5.7 we now infer
$$\inf \nu_Y |_{B_{|\pi|}} \geq C (1 + |\pi|)^{-2d} \quad (\pi \in \hat{G}_s).$$
The invariant Sobolev lemma (see [27], Lemma 4.2) for $K$-invariant functions on $G/\Gamma$ then gives us with $n > \frac{d}{2}$ the bound
$$\|\phi_{\pi}\|_{B_{|\pi|}, \infty} \leq C (1 + |\pi|)^d \|\phi_{\pi}\|_{2, n} \leq C \|\phi_{\pi}\|_{2, n+2d},$$
again by use of (5.4). Thus we obtain from (5.9) that

$$\|\phi^H\|_p \leq C \left( \sum_{\pi} \|\phi_{\pi}\|_{2, m+n+l+2d}^2 \right)^{1/2} \leq C \|\phi\|_{2, m+n+l+2d} \leq C \|\phi\|_{\infty, m+n+l+2d}.$$

With $k = m + n$ this shows the asserted bound for the discrete spectrum. The treatment of the continuous spectrum is then analogous to the one of Prop. 6.5 in [24].

Let now $\Gamma < G$ be an arithmetic lattice. This means in particular that $G$ is set of real points of an algebraic group $G$ defined over $\mathbb{Q}$ and $\Gamma$ is commensurable to $G(\mathbb{Z})$. As $G$ is defined over $\mathbb{Q}$, the same can be assumed for $A$ and $Q$, see [10]. Let $y_0 = \Gamma \in Y$ be the standard base point and $\nu_Y$ be an associated volume weight.

**Lemma 5.7.** Let $\Omega \subset G$ be a compact subset. Then there exists a constant $C > 0$ such that
$$\inf_{y \in \Omega} \nu_Y (ag \cdot y_0) \geq C a^{2\rho} \quad (a \in A^\cdot).$$

**Proof.** By the Bruhat decomposition we may assume that $\Omega$ is contained in a set of the form $N_c A_c M N_c F$ with $N_c \subset N$ compact etc and $F \subset G(\mathbb{Q})$ a finite set. So we may as well assume that $\Omega = N_c A_c M N_c F$. Now $M, A_c$ all commute with $A_c$ and $N_c$ is compressed by $A^\cdot$. By the properties of volume weights this reduces the situation further to $\Omega = N_c F$. 

[24]
[27]
[10]
Let us first assume that $F = 1$ and $g = \pi \in N_c$. Let $B_A \subset A$ and $B_{N^*} \subset \overline{N}$ be fixed balls and note that
\[
v_Y(a\overline{\pi} \cdot y_0) \geq C \text{vol}_Y(KB_AB_{N^*} \cdot y_0) \quad (a \in A^-)
\]
We may assume that $a^{-1}B_{N^*} \subset B_{N^*}$ for all $a \in A^-$ and thus [3], Th. 15.4, from the theory of Siegel sets implies further that
\[
\text{vol}_Y(KB_AB_{N^*} \cdot y_0) \geq C \text{vol}_G(KB_AB_{N^*} \cdot a) \geq Ca^{2\rho} \quad (a \in A^-)
\]
where the last inequality follows from the standard integral formulas for the Iwasawa decomposition. This settles the case with $F = \{1\}$. The case where $F = \{f\}$ is obtained in a similar fashion when we work with $A_f := f^{-1}Af$ instead of $A$ (note that $A_f$ is defined over $\mathbb{Q}$ as well). Having said that the generalization to $F \subset G(\mathbb{Q})$ finite is then immediate.

\section{6. Proof of Hypothesis B in Case rank$_g Z = 1$}

In this section we assume that $Z$ is wavefront with rank$_g Z = 1$, i.e. dim $a_Z = 1$ (see Remark 3.2 for examples, and note in particular the cases (20) - (21), which are far from being symmetric).

\subsection*{6.1. Basic Geometry of Rank One Wavefront Spaces.}

We begin by showing that $a^-_Z$ is a half-line.

\textbf{Lemma 6.1.} Let $Z$ be a wavefront real spherical space of real rank one. Then $a^-_Z \neq a_Z$.

\textit{Proof.} We argue by contradiction and assume that $a^-_Z = a_Z$. This implies $a_Z$ equals the edge of the compression cone and thus normalizes $H$. The overgroup $\overline{H} := HA_Z$ is real spherical and the associated real spherical space $\overline{Z} = G/\overline{H}$ has real rank zero. But then [16], Cor. 8.5, implies that $\overline{H}$ contains a conjugate of $AN$ and this excludes $H$ from being wavefront. \qed

Set $a^+_Z := -a^-_Z$ and note that $a_Z = a^-_Z \cup a^+_Z$ with $a^-_Z \cap a^+_Z = \{0\}$.

The first important feature of rank one spaces is a simplified polar decomposition, namely: there exists a compact subset $\Omega_Z \subset Z$ such that

\begin{equation}
Z = KA_Z W \cdot z_0 \cup \Omega_Z.
\end{equation}

The reason for that is that the standard compactification of $Z$ has only one $G$-orbit in the boundary see [16], Th. 13.7.

In particular we obtain for any non-negative measurable function $f$ on $Z$ that
\[
\int_Z f(z) \, dz \leq \int_{\Omega_Z} f(z) \, dz + \int_{KA_Z W \cdot z_0} f(z) \, dz.
\]

Further we recall that from [22], proof of Th. 8.5,
\[
\int_{KA_Z W \cdot z_0} f(z) \, dz \simeq \sum_{w \in W} \int_K \int_{A_Z Z} f(kaw \cdot z_0) J(a) \, dkda
\]

with
\[
J(a) \simeq a^{-2\rho} \quad (a \in A^-_Z).
\]

Here we used that $\rho \in a^*$ factors to a functional on $a^-_Z$ (cf. Lemma 4.2 in [20]).

In particular, if in addition $f$ is $K$-invariant then we obtain that
As for proving Hypothesis B it is then not serious to assume that $Z = KA_Z W \cdot z_0$.

We now come to the main technical tool for the verification of Hypothesis B.

6.2. The constant term approximation. Our concern then is with matrix coefficients $f = m_{v,\eta}$ for a unitarizable irreducible Harish-Chandra module $V$ and $v \in V^K$ a $K$-fixed vector.

In general, if $V$ is Harish-Chandra module with smooth moderate growth completion $V^\infty$, then we refer to $(V, \eta)$ as a spherical pair provided that $\eta : V^\infty \to \mathbb{C}$ is a continuous $H$-invariant functional.

**Theorem 6.2** (Constant term approximation). Let $Z = G/H$ be a wave-front real spherical space of real rank one. There exist constants $C, c_0 > 0$ with the following properties.

Let $(V, \eta)$ be a spherical pair with $V$ irreducible, which satisfies an priori-bound

\[
|m_{v,\eta}(\omega w)| \leq a^{r\rho} p(v) \quad (v \in V^\infty; \omega \in \Omega, a \in A_Z^-, w \in W)
\]

for some $r > 0$ and a $G$-continuous norm $p$.

Then there exists a number $\mu \in \mathbb{C}$ such that the following holds. Let

\[
\{\lambda_1, \lambda_2\} = \{\rho \pm \mu \rho\} \subset a^*_Z \mathbb{C}.
\]

Then for all $v \in (V^\infty)^M$ and $w \in W$ there exist $c_1(v, w), c_2(v, w) \in \mathbb{C}$ such that

\[
|m_{v,\eta}(aw \cdot z_0) - \text{const}_w(v)(a)| \leq Ca^{(r+c_0)\rho} p_8(v) \quad (a \in A_Z^-, w \in W)
\]

where

\[
\text{const}_w(v)(a) := \begin{cases} c_1(v, w)a^{\lambda_1} + c_2(v, w)a^{\lambda_2} & \lambda_1, \lambda_2 \neq \rho \\ a^{\rho}(c_1(v, w) + c_2(v, w)\rho(\log a)) & \lambda_1 = \lambda_2 = \rho \end{cases} \quad (a \in A_Z^-).
\]

Moreover, let $I_0 := \{i = 1, 2 \mid c_i(v, w) \neq 0 \text{ for some } v, w\}$. Then

\[
\Re \lambda_i(X) \geq r\rho(X) \quad (X \in a^+_Z, i \in I_0),
\]

and

\[
|\text{const}_w(v)(a)| \leq Ca^{r'} p_8(v) \quad (v \in V^M; a \in A_Z^-, w \in W)
\]

for every $r \leq r' \leq r + c_0$ such that $\Re \lambda_i(X) \geq r' \rho(X)$ for all $X \in a^+_Z, i \in I_0$.

Finally, if $V$ is unitarizable, then $\mu \in \mathbb{R} \cup i\mathbb{R}$.

**Proof.** In what follows, the elements of $W$ can be dealt with on an equal footing and for our simplified discussion here we shall assume $W = \{1\}$. Furthermore, since all ideas are contained in the example of $g = \mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{h}$ one-dimensional, we restrict ourselves to that case. Let

\[
X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

We assume that $a_Z = \mathbb{R}X$ and $p = \text{span}\{X, E\}$. Then $X \in a_Z^+$. Let

\[
\phi(t) = m_{v,\eta}(\exp(-tX))
\]

for $t \in \mathbb{R}$.
The Casimir element of $g$ is given by
\[ C = \frac{1}{2}X^2 + EF + FE = \frac{1}{2}X^2 + X + 2FE. \]

In general for $Y \in g$ and $\psi \in C^\infty(Z)$ we set $(L_Y \psi)(z) = \frac{d}{ds}\bigg|_{s=0} \psi(\exp(-sY)z)$ for $z \in Z$.

Let now $c \in \mathbb{C}$ be such that $2C$ acts as $c \text{id}_V$ on $V$. Set
\[ \Phi(t) = \begin{pmatrix} \phi(t) \\ \phi'(t) \end{pmatrix}. \]

Now note with $f = m_{v,\eta}$ that
\[ \phi''(t) = (L_X f)(\exp(-tX)) = (L_{2c-2X-4FE} f)(\exp(-tX)) = c\phi(t) - 2\phi'(t) + r(t) \]
with
\[ r(t) = -4(L_{FE} f)(\exp(-tX)). \]

For any smooth function $\psi$ on $Z$, viewed as a right $H$-invariant function on $G$ we have for $a \in A^{-}_Z$ that
\[ (L_F \psi)(a) = -a^\alpha (R_F \psi)(a) \]
with $\alpha$ the positive root and $(R_F \psi)(a) = \frac{d}{dt} |_{t=0} \psi(a \exp(tf))$. Let now $\mathfrak{h} = \mathbb{R} Y$ and note that $g = \mathfrak{h} + \mathfrak{p}$ implies that $Y$ can be normalized such that $F = Y + d_1X + d_2E$ with $d_1, d_2 \in \mathbb{R}$. Thus we get
\[ (L_F \psi)(a) = d_1a^\alpha (L_X \psi)(a) + d_2a^{2\alpha}(L_E \psi)(a). \]

We let now $\psi = L_E f$ and thus obtain from (6.3) that
\[ |r(t)| \leq C e^{-t(r+c_0)} p_2(v), \quad t \geq 0, \]
for a universal constant $C \geq 0$ and $c_0 > 0$ (specifically for $g = \mathfrak{sl}(2, \mathbb{R})$ we have $c_0 = 2$ if $d_1 \neq 0$ and otherwise $c_0 = 4$, but in general it can be different).

Set
\[ R(t) := \begin{pmatrix} 0 \\ r(t) \end{pmatrix}. \]

Then our discussion shows that $\Phi$ satisfies the first order differential equation
\[ (6.8) \quad \Phi'(t) = A \Phi(t) + R(t), \]
where
\[ A = \begin{pmatrix} 0 & 1 \\ c & -2 \end{pmatrix}. \]

The characteristic polynomial of $A$ is given by $\lambda^2 + 2\lambda - c$ and thus has solutions
\[ (6.9) \quad \lambda_\pm = -1 \pm \mu \]
where
\[ (6.10) \quad \mu = \sqrt{1 + c}. \]

The general solution formula for (6.8) then is
\[ \Phi(t) = e^{tA} \Phi(0) + e^{tA} \int_0^t e^{-sA} R(s) \, ds. \]
Note that $A$ is semi-simple if and only if $\mu \neq 0$, i.e. $c \neq -1$. For $\lambda \in \text{spec } A = \{\lambda_-, \lambda_+\}$ we write $E(\lambda) \subset \mathbb{C}^2$ for the corresponding generalized eigenspace. Let $0 < \delta < c_0 + r$ and write

$$P : \mathbb{C}^2 \to \bigoplus_{\lambda \in \text{spec } A, -\Re \lambda \leq r + c_0 - \delta} E(\lambda)$$

for the projection along the complementary generalized eigenspaces. Next we note that

$$\int_0^t e^{-sA} R(s) \, ds = \int_0^t e^{-sA} P R(s) \, ds + \int_0^t e^{-sA} (1 - P) R(s) \, ds$$

$$= \int_0^\infty e^{-sA} P R(s) \, ds - \int_t^\infty e^{-sA} P R(s) \, ds + \int_0^t e^{-sA} (1 - P) R(s) \, ds$$

in which convergence is ensured by the definition of $P$, as $\delta > 0$.

Set

$$I_1(t) = e^{tA} \int_t^\infty e^{-sA} P R(s) \, ds = \int_0^\infty e^{-sA} P R(s + t) \, ds$$

$$I_2(t) = e^{tA} \int_0^t e^{-sA} (1 - P) R(s) \, ds = \int_0^t e^{sA} (1 - P) R(t - s) \, ds$$

This gives us

$$\Phi(t) = e^{tA} u - I_1(t) + I_2(t)$$

for $u \in \mathbb{C}^2$ given by

$$u = \Phi(0) + \int_0^\infty e^{-sA} P R(s) \, ds.$$ 

Next we estimate $I_1$ and $I_2$ for $t \geq 0$. For that we first recall the Gelfand-Shilov estimate for $\|e^{tA}\|$ for a complex $N \times N$-matrix with $\sigma := \max \{\Re \lambda \mid \lambda \in \text{spec } A\}$:

$$\|e^{tA}\| \leq e^{\sigma t} (1 + t \|A\| + \ldots + \frac{t^{N-1}}{(N-1)!} \|A\|^{N-1}) \quad (t \geq 0).$$

and in particular for $N = 2$:

$$\|e^{tA}\| \leq e^{\sigma t} (1 + t \|A\|) \quad (t \geq 0).$$

Then,

$$\|I_1(t)\| \leq \|P\| \int_0^\infty e^{s(r+c_0-\delta)} (1 + s \|A\|) \|R(s + t)\| \, ds$$

$$\leq C \|P\| p_2(v) \int_0^\infty e^{s(r+c_0-\delta)} (1 + s \|A\|) e^{-(r+c_0)(s+t)} \, ds$$

$$\leq \frac{C}{\delta^2} \|P\| (1 + \|A\|) p_2(v) e^{-(r+c_0)t}$$

$$\leq \frac{C}{\delta^2} \|P\| p_4(v) e^{-(r+c_0)t}. $$
For the last line we used the fact, obtained from (6.3), that
\[
(1 + \|A\|)p_2(v) \leq C(1 + |c|)p_2(v) \leq Cp_4(v) \quad (v \in V^\infty).
\]
Similarly
\[
\|I_2(t)\| \leq \|1 - P\| \int_0^t e^{s(-r - c_0 + \delta)}(1 + s\|A\|)\|R(t - s)\| \, ds
\leq C\|1 - P\|p_2(v)e^{-(r + c_0)t} \int_0^t e^{s\delta}(1 + s\|A\|) \, ds
\leq \frac{C}{\delta^2}\|P\|(1 + \|A\|)p_2(v)e^{-(r + c_0 - \delta)t}
\leq \frac{C}{\delta^2}\|P\|p_4(v)(1 + t)e^{-(r + c_0 - \delta)t}.
\]

With the Lemma (6.4) from below we can find a value \( \frac{\alpha}{4} \leq \delta \leq \frac{\alpha}{2} \) and a constant \( C > 0 \) such that \( \|P\| \leq C\|A\|^2 \). This yields then
\[
(6.13) \quad \|I_1(t)\| + \|I_2(t)\| \leq Ce^{-(r + \frac{\alpha}{2})t}p_8(v).
\]

We now define \( \text{const}_w(v)(\exp(-tX)) \) to be the first coordinate of \( e^{tA}u \). By expanding
\( u = c_+u_+ + c_-u_- \) into generalized eigenvectors of \( A \) to eigenvalues \( \lambda_\pm \) we obtain (6.3) for
\( \lambda_i = -\lambda_\pm \). Furthermore (6.4) follows from (6.11) and (6.13).

From (6.3) and (6.4) it follows that
\[
|\text{const}_w(v)(a)| \leq Ca^\nu p_8(v) \quad (v \in V^M; a \in A_\mathbb{Z}, w \in \mathbb{W}).
\]

Hence a coefficient \( c_\pm \) vanishes provided that \( \Re \lambda_\pm > -r \), as stated in (6.6).

From (6.12) we have \( u = \Phi(0) + I_1(0) \) and hence \( \|u\| \leq C p_8(v) \). If \( \Re \lambda_\pm \leq -r' \) this implies
\[
\|e^{tA}u\| \leq Ce^{-tr'}p_8(v)
\]
and hence (6.7).

Finally, we remark that if \( V \) is unitary then the Casimir eigenvalue \( c \) is real, and thus the final assertion is a consequence of (6.10).

\[\square\]

**Remark 6.3.** The alert reader might ask where the assumption \( v \in V^M \) entered the proof. For a general rank one space with parabolic \( q = I + u \) we recall the shape of the Casimir operator \( C \). For every root space \( g^a \subset u \) we let select a basis \( (Y_\alpha^j)_{1 \leq j \leq m_a} \) with \( m_a = \dim g^a \). Then we choose \( Y_{\alpha}^j \in g^{-a} \) such that \( \kappa(Y_\alpha^j, Y_{-\alpha}^k) = \delta_{jk} \) with \( \kappa \) the Cartan-Killing form. Set \( C_u := \sum_{\alpha \in \Sigma(a, u + \Pi)} Y_\alpha^j Y_{-\alpha}^j \) and note that
\[
C_u = 2 \sum_{\alpha \in \Sigma(g, u)} Y_\alpha^j Y_{-\alpha}^j + \sum_{\alpha \in \Sigma(g, u)} [Y_\alpha^j, Y_{-\alpha}^j] \quad \text{=:} C_u^\alpha \quad \text{=:} X_{2\rho_a, \cdot}
\]

We note that \( 2\rho_a = \kappa(X_{2\rho_a}, \cdot) \) as a functional on \( a \). The fact that \( Z \) is unimodular then implies that \( X_{2\rho_a} \in a_{ij} \) by [20], Lemma 4.2. Then
\[
C = C_1 + C_u
\]
with \( C \) a multiple of the Casimir of \( l \). Now \( l = l_n \oplus m_l \oplus (\mathfrak{g}(l) \cap a) \) with \( m_l < m \) and \( l_n < l \cap \mathfrak{h} \) by the local structure theorem. In particular if \( f \) is a left \( M \)-invariant function on \( Z = G/H \) and \( a \in A_Z \), then we have

\[
L_c f(a) = L_{c(\mathfrak{g}(l) + a)} + C_a f(a) = L_{\mathfrak{g}(l) + a} f(a) = L_{\mathfrak{g}(l) + \mathfrak{h}(a)} f(a)
\]

and the analogy to the \( sl(2) \)-case becomes apparent.

**Lemma 6.4.** Let \( 0 < \nu \leq 1 \), \( N \in \mathbb{N} \) and \( A \in \text{Mat}_N(\mathbb{C}) \) with \( \text{spec}(A) = \{ \lambda_1, \ldots, \lambda_r \} \) such that \( \text{Re} \lambda_1 \leq \ldots \leq \text{Re} \lambda_r \). For every \( 1 \leq j \leq r \) let \( V_j \subset \mathbb{C}^n \) be the generalized eigenspace of \( A \) associated to the eigenvalue \( \lambda_j \). For every \( 1 \leq k \leq r \) we let \( E_k = \bigoplus_{j=1}^k V_j \) and \( P_k : \mathbb{C}^N \to E_k \) be the projection along \( \bigoplus_{j=k+1}^r V_j \). Suppose for some \( 1 \leq k \leq r-1 \) that \( \text{Re} \lambda_{k+1} - \text{Re} \lambda_k \geq \nu \). Then there exists a constant \( C = C(\nu, N) > 0 \) such that

\[
\|P_k\| \leq C\|A\|^N.
\]

**Proof.** Let \( R \subset \mathbb{C} \) be the positively oriented and axes-parallel rectangle which intersects the imaginary axis in \( \pm i(\|A\| + 1) \), and the real axis in \( -(\|A\| + 1) \), respectively half way between \( \text{Re} \lambda_k \) and \( \text{Re} \lambda_{k+1} \).

Observe that:

- \( R \) surrounds \( \{ \lambda_1, \ldots, \lambda_k \} \) but not \( \{ \lambda_{k+1}, \ldots, \lambda_r \} \),
- \( \text{dist}(\text{spec} A, R) \geq \nu/2 \),
- \( |R| \leq 8\|A\| + 8 \).

Next we recall that

\[
P_k = \frac{1}{2\pi i} \oint_R (z - A)^{-1} \, dz.
\]

and thus

\[
\|P_k\| \leq \frac{1}{2\pi |R|} \max_{z \in R} \|(A - z)^{-1}\|.
\]

Cramer’s rule gives

\[
(A - z)^{-1} = \frac{1}{\det(A - z)}((-1)^{i+j} \det(A - z)_{ij})_{i,j}.
\]

Using the observations above we get

- \( |\det(A - z)| \geq C_1 \) for all \( z \in R \) with \( C_1 = C_1(N, \nu) \)
- \( |\det(A - z)_{ij}| \leq C_2\|A\|^{N-1} \) for all \( z \in R, 1 \leq i, j \leq N \), and a constant \( C_2 = C_2(N) \).

The assertion follows. \( \square \)

6.3. **Function spaces on** \( Z \). We recall the two standard weight functions \( v \), the volume weight, and \( r \), the radial weight, (see [27], Sect. 4 and 9) with the bounds:

\[
(6.14) \quad r(\omega a w \cdot z_0) \preceq (1 + \| \log a \|)
\]

\[
(6.15) \quad v(\omega a w \cdot z_0) \preceq a^{-2\rho}
\]

for all \( \omega \in \Omega, a \in A_Z, w \in W \).

Given \( 1 \leq p < \infty, m > 0 \) we consider the following norms on \( C_c^\infty(Z) \):

\[
\|f\|_{p,m} := \|f \, r^m\|_p = \left( \int_Z |f(z)|^p \, r(z)^m \, dz \right)^{\frac{1}{p}}
\]
and

\[ q_{p,m}(f) = \sup_{z \in \mathbb{Z}} |f(z)| v(z)^{\frac{1}{p}} r(z)^m. \]

If \( k \in \mathbb{N}_0 \) we denote by \( \| \cdot \|_{p,m,k} \), resp. \( q_{p,m,k} \), the \( k \)-the Sobolev norm of \( \| \cdot \|_{p,m} \), resp. \( q_{p,m} \) (see (5.3)). These norms are related as follows (see [27], Sect. 9):

\[
\|f\|_{p,m;k} \leq Cq_{p,m';k}(f) \quad (m' - m > 1).
\]

\[
q_{p,m;k}(f) \leq C\|f\|_{p,m;k+k'} \quad (k' > \frac{\dim G}{p}).
\]

In particular it follows from (6.15) and (6.17) for a matrix coefficient \( f = m_{v,\eta} \) which is \( L^p \)-integrable that

\[
\sup_{a \in \mathcal{A}^- \cap \mathcal{W}} |f(a w \cdot z_0)| a^{\frac{2p}{\rho}} \leq C\|f\|_{p,k} \quad (k > \frac{\dim G}{p})
\]

with \( \| \cdot \|_{p,k} := \| \cdot \|_{p,0,k} \).

Phrased differently all \( L^p \)-integrable \( f = m_{v,\eta} \) satisfy the a-priori bound

\[
|f(a w \cdot z_0)| \leq C a^{r/p} \|f\|_{p,k}
\]

for \( r = \frac{2}{p} > 0 \) and the \( G \)-continuous norm \( \| \cdot \|_{p,k} \) as required in (6.3).

6.4. Verification of Hypothesis B. We assume that \( \mathcal{W} = 1 \) for simplification of the exposition. Furthermore, under the specific assumptions in Theorem 5.1 we have for all non-trivial \( \pi \) that \( H_\eta / H \) is compact (see Remark 5.3) and thus we may as well assume that \( H = H_\eta \).

Fix now \( 1 \leq p' < \infty \) and \( p > p' \). We may assume that \( \frac{2}{p} \leq c_0 + \frac{2}{p} \) for the constant \( c_0 \) of Theorem 6.2.

Let \( f = m_{v,\eta} \) be associated to a representation \( \pi \in \Lambda_{p'} \), which we may assume is non-trivial. Moreover we assume that \( f \) is \( K \)-fixed. With the a priori bound (6.18) for \( r = \frac{2}{p} \), we can apply Theorem 6.2 and let \( c(f)(a) = \text{const}_v(a, 1) \) be a constant term for \( f \) (see (6.5)). Then the approximation bound (6.4) reads

\[
|f(a) - c(f)(a)| \leq C a^{(r+c_0)p} \|f\|_{p,k+8} \quad (a \in \mathcal{A}_Z^-).
\]

As \( f \) is \( K \)-fixed we may assume with (6.2) and the neglect of \( \Omega_Z \) that

\[
\|f\|_p = \left( \int_{\mathcal{A}_Z^-} |f(a)|^p a^{-2p} da \right)^{\frac{1}{p}}.
\]

We now identify \( \mathcal{A}_Z^- \) with \([0, \infty] \) via an element \( X \in \mathfrak{a}_Z^- \) with \( \rho(X) = -1 \), i.e.

\([0, \infty] \ni t \leftrightarrow a_t = \exp(tX) \in \mathcal{A}_Z^- \).

Then (6.20) translates into

\[
\|f\|_p = \left( \int_0^\infty |f(a_t)|^p e^{2t} dt \right)^{\frac{1}{p}}.
\]
We let $R > 0$, to be specified later, and begin with the estimate:

\[(6.22) \quad \|f\|_p = \|f\|_{p,[0,R]} + \|f\|_{p,[R,\infty]} \leq e^{\frac{2R}{p}} \|f\|_{\infty,[0,R]} + \|f - c(f)\|_{p,[R,\infty]} + \|c(f)\|_{p,[R,\infty]} .\]

Since $f$ is in fact $L^p'$-bounded (in particular satisfies the bound \((6.18)\) for $r = \frac{2}{p'}$), the leading coefficients $\lambda_i$ of $c(f)$ satisfy $\text{Re} \lambda_i(X) \leq -\frac{2}{p}$. Hence we obtain from $\frac{2}{p'} - \frac{2}{p} \leq c_0$ and \((6.7)\) that

\[(6.23) \quad \|c(f)\|_{p,[R,\infty]} \leq C \|f\|_{p,k+8} \left( \int_R^\infty e^{-\frac{2}{p'}t} e^{2t} dt \right)^{\frac{1}{p}} = C e^{-(\frac{2}{p'} - \frac{2}{p})R} \|f\|_{p,k+8} .\]

Note that $C$ does not depend on $\pi$.

Further, from \((6.19)\) we obtain

\[(6.24) \quad \|f - c(f)\|_{p,[R,\infty]} \leq C \left( \int_R^\infty e^{-(2+p_{c_0})t} e^{2t} dt \right)^{\frac{1}{2}} \|f\|_{p,k+8} \leq C e^{-Rc_0} \|f\|_{p,k+8} .\]

Set now $\delta := \frac{2}{p'} - \frac{2}{p}$ and note that $\delta \leq c_0$. Inserting the bounds from \((6.23)\) and \((6.24)\) into \((6.22)\) we thus obtain that

\[\|f\|_p \leq C (e^{\frac{2R}{p}} \|f\|_{\infty} + e^{-R\delta} \|f\|_{p,k+8})\]

which, in view of \((5.4)\), yields

\[(6.25) \quad \|f\|_p \leq C (e^{\frac{2R}{p}} \|f\|_{\infty} + e^{-R\delta} (1 + |\pi|) \|f\|_p) .\]

So far, $R > 0$ was arbitrary, but we now choose it such that

\[Ce^{-R\delta} (1 + |\pi|) \frac{1}{2^{\delta+4}} \leq \frac{1}{2}\]

holds for the constant $C$ of \((6.25)\), i.e.

\[(6.26) \quad R = R_{\pi} = \frac{k + 8}{2\delta} \log(1 + |\pi|) + \frac{\log(2C)}{\delta}\]

will do. Then we obtain from \((6.25)\)

\[\|f\|_p \leq 2Ce^{\frac{2R_{\pi}}{p}} \|f\|_{\infty},\]

and hence

\[\|f\|_p \leq C (1 + |\pi|) \frac{k+8}{2\delta} \|f\|_{\infty}\]

with $C > 0$ independent of $\pi$. It follows with $l := \frac{k+8}{2\delta p}$

\[(6.27) \quad \|f\|_p \leq C \|f\|_{\infty;l}\]

which is Hypothesis B.
7. Remarks for higher rank

The purpose of this section is to sketch a proof of (1.3) for general unimodular wavefront spaces. Observe that (1.3) is more general than Hypothesis B in two respects:

- The eigenfunction \( f \) is not \( K \)-fixed.
- The \( G \)-representation generated by \( f \) is not necessarily unitarizable.

However, our verification of Hypothesis B for the rank one cases does not really need the assumption that the eigenfunction is \( K \)-fixed (the argument given allows to sum up \( K \)-types). Further, the main ingredient is the constant term approximation, which does not require that the underlying Harish-Chandra module is unitarizable.

Regarding the lattice counting problem we note that (1.3) also allows counting for balls which are not \( K \)-invariant.

We now turn to the heart of the matter:

7.1. The constant term approximation in higher rank. Let \( t \subset \mathfrak{m} \) be a maximal torus and set \( \mathfrak{c} := \mathfrak{a} + \mathfrak{t} \). Then \( \mathfrak{c} < \mathfrak{g} \) is a Cartan algebra and we let \( \mathcal{W}_c \) be the Weyl group associated to the root system \( \Sigma(\mathfrak{c}, \mathfrak{g}) \). Further we choose a positive system \( \Sigma^+(\mathfrak{c}, \mathfrak{g}) \) of \( \Sigma(\mathfrak{c}, \mathfrak{g}) \) such that \( \Sigma^+(\mathfrak{c}, \mathfrak{g})|_{\mathfrak{a}} \setminus \{0\} = \Sigma^+(\mathfrak{a}, \mathfrak{g}) \).

For a subset of spherical roots \( I \subset S \) (see [27] for the definition of spherical roots) we set

\[
a_I := \{ X \in \mathfrak{a}_Z \mid (\forall \alpha \in I) \alpha(X) = 0 \},
\]

\[
a_I^- := \{ X \in a_I \mid (\forall \alpha \in S \setminus I) \alpha(X) < 0 \} \subset a_Z^-.
\]

Finally for an irreducible Harish-Chandra module \( V \) we let \( \Lambda_V \in \mathfrak{c}_C^* \) be the associated infinitesimal character. Then with the techniques provided in [5], but essentially the same proof, we obtain the following analogue of Theorem 6.2.

**Theorem 7.1** (Constant term approximation - higher rank). Let \( Z = G/H \) be a wave-front real spherical space. Let \( (V, \eta) \) be a spherical pair with \( V \) irreducible and which satisfies the a priori-bound

\[
|m_{v, \eta}(\omega w \cdot z_0)| \leq a^r p(v) \quad (v \in V^\infty; \omega \in \Omega, a \in A_Z^-, w \in W)
\]

for some \( r > 0 \) and a \( G \)-continuous norm \( p \). Let \( I \subset S \) and \( C_I \subset a_I^- \) be a compact subset. Then there exist constants \( C, c_0 > 0, k, d \in \mathbb{N} \) only depending on \( Z \) and \( C_I \) such that the following assertion holds true: There exist \( \mathcal{E}_V \subset \rho + \mathcal{W}_c \cdot \Lambda_V \) with

\[
(\forall \lambda \in \mathcal{E}_V, X \in a_Z^-) \quad \text{Re} \lambda(X) \leq r \rho(X)
\]

such that for all \( v \in V^\infty, w \in W \) and \( \lambda \in \mathcal{E}_V \) there exist polynomials \( c_{v, \lambda, w}^I \in \text{Pol}(a_I) \) of degree bounded by \( d \) such that the sum

\[
\text{const}_{w}^I(v)(a) := \sum_{\lambda \in \mathcal{E}_V} a^\lambda c_{v, \lambda, w}^I(\log a) \quad (a \in A_Z)
\]

satisfies for all \( X \in C_I, a \in A_Z^-, t \geq 0, v \in V^\infty \) and \( w \in W \) that

\[
|m_{v, \eta}(a \exp(tX)w \cdot z_0) - \text{const}_{w}^I(v)(a \exp(tX))| \leq C p_k(v) a^{r} e^{(r+c_0)\rho(X)}.
\]
If in addition there exists an \( r \leq r' \leq r + c_0 \) such that the improved bound

\[
(\forall \lambda \in \mathcal{E}_V, X \in a_Z^-) \quad \text{Re} \, \lambda(X) \leq r' \rho(X)
\]

holds, then

\[
|\text{const}_w^I(v)(a)| \leq Cp_k(v)a^{r'}(a \in A_Z^-; v \in V^\infty, w \in W).
\]

**Remark 7.2.** The constant term approximation in Theorem 7.1 deviates from the one in [5] in several aspects. The a priori-bound (7.1) used in [5] is

\[
|m_{v,\eta}(\omega a w \cdot z_0)| \leq a^{2p}(1 + \| \log a \|)^N \rho(v) \quad (v \in V^\infty; \omega \in \Omega, a \in A_Z, w \in W)
\]

for some \( G \)-continuous norm \( p \) and \( N \in \mathbb{N} \). If we ignore logarithmic terms, then this is in essence the case \( r = 2 \) above, and the presence of logarithmic terms does not pose additional technical difficulties.

The constant term in [5] is unique and obtained from the one above by replacing \( \mathcal{E}_V \) with \( \mathcal{E}'_V := \{ \lambda \in \mathcal{E}_V \mid \text{Re} \, \lambda = 0 \} \), i.e. by discarding all exponents \( \lambda \) for which \( \text{Re} \, \lambda \neq 0 \). As such the constant term approximation in [5] is only uniform if one fixes \( \text{Re} \, \Lambda_V \). The constant term approximation above however is uniform (i.e. holds for all \( \Lambda_V \)), but is non-unique and also allows exponents \( \lambda \) with \( \text{Re} \, \lambda \neq 0 \).

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