CLASSIFICATION OF SOLUTIONS TO GENERAL TODA SYSTEMS WITH SINGULAR SOURCES

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ABSTRACT. We classify all the solutions to the elliptic Toda system associated to a general simple Lie algebra with singular sources at the origin and with finite integrals. The solution space is shown to be parametrized by a subgroup of the corresponding complex Lie group. We also show the quantization result for the finite integrals. This work generalizes the previous works in [LWY12] and [Nie16] for Toda systems of types A and B, C. However, a more Lie-theoretic method is needed here for the general case, and the method relies heavily on the structure theories of the local solutions and of the W-invariants for the Toda system. This work will have applications to nonabelian Chern-Simons-Higgs gauge theory and to the mean field equations of Toda type.

1. INTRODUCTION

In this paper, we consider the following Toda systems on the plane. Let \( g \) be a complex simple Lie algebra of rank \( n \), and let \( (a_{ij}) \) be its Cartan matrix (see [Hel78, Kna02, FH91] and the Appendix for basic Lie theory). The Toda system associated to \( g \) with singular sources at the origin and with finite integrals is the following system of semilinear elliptic PDEs

\[
\begin{aligned}
\Delta u_i + 4 \sum_{j=1}^{n} a_{ij} e^{u_j} &= 4\pi \gamma_i \delta_0 \quad \text{on } \mathbb{R}^2, \quad \gamma_i > -1, \\
\int_{\mathbb{R}^2} e^{u_i} \, dx &< \infty, \quad 1 \leq i \leq n,
\end{aligned}
\]

where \( \delta_0 \) is the Dirac delta function at the origin. Here the solutions \( u_i \) are required to be real and well-defined on the whole \( \mathbb{R}^2 \) minus the origin.

When the Lie algebra \( g = A_1 = \mathfrak{sl}_2 \) whose Cartan matrix is (2), the Toda system becomes the Liouville equation

\[
\Delta u + 8e^u = 4\pi \gamma \delta_0 \quad \text{on } \mathbb{R}^2, \quad \gamma > -1, \quad \int_{\mathbb{R}^2} e^u \, dx < \infty.
\]

The Toda system (1.1) and the Liouville equation (1.2) arise in many physical and geometric problems. For example, in the Chern-Simons theory, the Liouville equation is related to the abelian gauge field theory, while the Toda system is related to nonabelian gauges (see [Yan01, Tar08]). On the geometric side, the Liouville equation is related to conformal metrics on \( S^2 \) with conical singularities whose Gaussian curvature is 1. The Toda systems are related to holomorphic curves in...
projective spaces [Dol97] and the Plücker formulas [GH78], and the periodic Toda systems are related to harmonic maps [Gue97].

From the analytic point of view, one would like to study the following mean field equation on a compact surface $M$ with a Riemannian metric $g$

$$(1.3) \quad \Delta_g u + \rho \left( \frac{h e^u}{\int_M h e^u} - \frac{1}{|M|} \right) = 4\pi \sum_{j=1}^N \gamma_j \left( \delta_{p_j} - \frac{1}{|M|} \right) \quad \text{on} \ M,$$

where $h$ is a positive smooth function on $M$ and $|M|$ is the volume of $M$ with respect to $g$. This equation again arises from both conformal change of metrics [KW74, Tro91] with prescribed Gaussian curvature and the Chern-Simons-Higgs theory on the compact surface $M$. There are intense interests and extensive literature on (1.3) concerning solvability, blow-up analysis and topological degrees [Lin14, Mal14, Tar10, CL03].

In general, we are interested in the following mean field equations of Toda type

$$\Delta_g u_i + \sum_{j=1}^n a_{ij} \rho_j \left( \frac{h_j e^{u_j}}{\int_M h_j e^{u_j}} - \frac{1}{|M|} \right) = 4\pi \sum_{j=1}^N \gamma_{ij} \left( \delta_{p_j} - \frac{1}{|M|} \right) \quad \text{on} \ M,$$

Such systems have been studied in [JW01, JLW06] for Lie algebras of type $A$. When carrying out the analysis of such systems, there often appears a sequence of bubbling solutions near blow-up points. For that purpose, the fundamental question is to completely classify all entire solutions of the Toda system with finite integrals and with singular sources at the origin as in (1.1).

The classification problem for the solutions to the Toda systems has a long history. For the Liouville equation (1.2), Chen and Li [CL91] classified their solutions without the singular source, and Prajapat and Tarantello [PT01] completed the classification with the singular source. For general $A_n = \mathfrak{sl}_{n+1}$ Toda systems, Jost and Wang [JW02] classified the solutions without singular sources, and Ye and two of the authors [LWY12] completed the classification with singular sources. This later work also invented the method of characterizing the solutions by a complex ODE involving the $W$-invariants of the Toda system. The work [LWY12] has also established the corresponding quantization result for the integrals and the non-degeneracy result for the corresponding linearized systems. The case of $G_2$ Toda system was treated in [ALW15]. In [Nie16], one of us generalized the classification to Toda systems of types $B$ and $C$ by treating them as reductions of type $A$ with symmetries and by applying the results from [Nie12].

In this paper, we complete the classification of solutions to Toda systems for all types of simple Lie algebras, and we also establish the quantization result for the corresponding integrals. We note that the remaining types of Toda systems can not be treated as reductions of type $A$, and a genuinely new method is needed for our purpose. We are able to achieve our goal by systematically applying and further developing the structure theories of local solutions to Toda systems (cf. [LS92, GL14]) and of the $W$-invariants (see [Nie14]). We furthermore note that it is the finite-integral conditions and the strength of the singularities that combine to greatly restrict the form of the solutions. The current work will lay the foundation for future applications to the Chern-Simons-Higgs theory and to the mean field equations.

Our approach of solving (1.1) will heavily use the complex coordinates and holomorphic functions. Let $x = (x_1, x_2)$ be the coordinates on $\mathbb{R}^2$, and we introduce the
complex coordinates $z = x_1 + i x_2$ and $\bar{z} = x_1 - i x_2$, thus identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. For simplicity, we write $\partial_z = \frac{\partial}{\partial z} = \frac{i}{2}(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2})$, and similarly $\partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2})$. The Laplace operator is then $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = 4 \partial_z \partial_{\bar{z}}$.

The coefficient 4 here is responsible for the slightly unconventional coefficient 4 on the left of (1.1), and this coefficient can be easily dealt with (see [Nie16, Remark 1.4]).

Furthermore, our results and proofs are more conveniently presented in a different set of dependent variables. Let $U_i = \sum_{j=1}^n a^{ij} u_j$ for $1 \leq i \leq n$, where $(a^{ij})$ is the inverse matrix of $(a_{ij})$. Then the $U_i$ satisfy

\[
\begin{align*}
U_{i,zz} + \exp\left(\sum_{j=1}^n a^{ij} U_j\right) = \pi \gamma_i \delta_0 & \quad \text{on } \mathbb{R}^2, \\
\int_{\mathbb{R}^2} e^{\sum_{j=1}^n a^{ij} U_j} dx < \infty,
\end{align*}
\]

where $\gamma_i = \sum_{j=1}^n a^{ij} \gamma_j$. The first equation can also be written as

\[
\Delta U_i + 4 e^{u_i} = 4 \pi \gamma_i \delta_0 \quad \text{on } \mathbb{R}^2.
\]

(The $\gamma_i$ were denoted by $\alpha_i$ in [LWY12, Nie16], but we will use $\alpha_i$ to denote the $i$th simple root of the Lie algebra $g$ in this paper.)

Throughout the paper, we use the Lie-theoretic setup detailed in the Appendix. Although for the Lie algebras of classical types $A, B, C$, and $D$, the setup can be made fairly concrete, we have chosen to present our result in a Lie-theoretic and intrinsic way, which automatically covers the Lie algebras of exceptional types $G_2, F_4, E_6, E_7$ and $E_8$. Here is our main theorem.

**Theorem 1.6.** Let $G$ be a connected complex Lie group whose Lie algebra is $g$ with the Cartan matrix $(a_{ij})$. Let $G = KAN$ be the Iwasawa decomposition of $G$ (see Eq. (A.7)) with $K$ maximally compact, $A$ abelian, and $N$ nilpotent. Let $N_r$ be the subgroup of $N$ (see Definition 7.2) determined by the $\gamma_i$.

Let $\Phi : \mathbb{C}\backslash \mathbb{R}_{\leq 0} \to N \subset G$ be the unique solution of

\[
\begin{align*}
\Phi^{-1} \Phi_z = \sum_{i=1}^n \bar{z}^i e_{-\alpha_i} & \quad \text{on } \mathbb{C}\backslash \mathbb{R}_{\leq 0}, \\
\lim_{z \to 0} \Phi(z) = Id,
\end{align*}
\]

where $Id \in G$ is the identity element, the limit exists because $\gamma_i > -1$, and the root vectors $e_{-\alpha_i} \in g_{-\alpha_i}$ are normalized as in (A.3) and (A.5).

Then the all solutions to (1.7) are

\[
U_i = -\log(|\Phi^* C^* \Lambda^2 C \Phi| i) + 2 \gamma_i \log |z|, \quad 1 \leq i \leq n,
\]

where $C \in N_r$ and $\Lambda \in A$. Here for $g \in G$, $g^* = (g^\theta)^{-1}$ and $\theta$ is the Cartan involution of $G$, and $|\langle i | \cdot | i \rangle|$ is the highest matrix coefficient for the $i$th fundamental representation (see the Appendix).

Consequently, all the solutions to (1.7) are

\[
u_i = -\sum_{j=1}^n a_{ij} \log|\langle j | \Phi^* C^* \Lambda^2 C \Phi| j \rangle| + 2 \gamma_i \log |z|, \quad 1 \leq i \leq n.
\]
Furthermore, they satisfy the following quantization result for the integrals (see (8.5))

\[ \sum_{j=1}^{n} a_{ij} \int_{\mathbb{R}^2} e^{u_j} dx = \pi (2 + \gamma_i - \kappa \gamma_i), \quad 1 \leq i \leq n, \]

where \( \kappa \) is the longest element in the Weyl group and if \( -\kappa \alpha_i = \alpha_k \), then \( -\kappa \gamma_i := \gamma_k \).

Sections 2 to 8 are devoted to the proof of this main theorem, and the approach can be summarized as follows. In Sections 2 and 3, we develop the structure theories of the local solutions and of the \( W \)-invariants for the Toda system, and we relate them. In Sections 4 and 5, we use the finite-integral conditions and the strength of the singularities to greatly restrict the forms of the \( W \)-invariants and hence of the solutions. In Section 6, we establish the close relationship of our current work with that of Kostant [Kos79] on Toda ODE systems. Then in Section 7, we take up the monodromy consideration for the solutions to be well-defined on the punctured plane. Finally in Section 8, we study the quantization result for the finite integrals.

In consistence with the results in [LWY12, Nie16] for the Lie algebras of types \( A, B \) and \( C \), the solution space to the Toda system is parametrized by the subgroup \( AN \) of a corresponding complex Lie group \( G \). When all the \( \gamma_i \) are integers, \( AN = AN \) (see Definition 7.2), and the solution space has the maximal dimension. Since \( N \) is a complex group, the real dimension of \( AN \) is the same as the real dimension of a real group corresponding to the real Lie algebra \( g_0 \) (A.3).

For an element \( g \) in a classical Lie group \( G \), we have that \( g^* = \bar{g}^t \) is the conjugate transpose. The abelian subgroup \( A \) can be chosen to consist of diagonal matrices with positive real entries, and the nilpotent subgroup \( N \) can be chosen to consist of unipotent lower-triangular matrices with complex entries. The fundamental representations are contained in the wedge products of the standard representations together with the spin representations for the \( B \) and \( D \) cases. In Section 7 of the paper, we will relate the above general theorem 1.6 to the previous results [LWY12, Nie16] in the \( A, B \) and \( C \) cases, and we will spell out more details for our general theorem in the \( D \) case.

The nondegeneracy of the linearized system in this general setting of the Toda system (1.1) for a simple Lie algebra will be pursued in a future work.

Acknowledgment. Z. Nie thanks the University of British Columbia and the National Taiwan University for hospitality during his visits in the summer of 2015, where part of this work was done. He also thanks Prof. L. Fehér for very useful correspondences on Section 3. Z. Nie acknowledges the Simons Foundation through Grant #430297. The research of J. Wei is partially supported by NSERC of Canada.

2. Local solutions using holomorphic functions

In this section, we show that the solutions to the Toda system (1.1) for a general simple Lie algebra locally all come from holomorphic data, which are generalizations of the developing map in the Liouville case. Our approach follows [LS79], [LS92 §3.1, §4.1], and [GL11 Appendix]. Since we only consider real-valued solutions to the Toda systems, we only need the holomorphic data of [LS79] and the compact real
form (unitary structure) comes into play. In [GL14], similar results were obtained for the periodic Toda systems and the loop groups.

Therefore this section generalizes the well-known fact that locally the solutions to the Liouville equation

\[ U_{z\bar{z}} = -e^{2U} \text{ in an open set } D \subset \mathbb{C} \]

are

\[ U(z) = \log \frac{|f'|}{1 + |f|^2}, \]

where \( f \) is a holomorphic function in \( D \) whose derivative is nowhere zero.

For simplicity, we introduce the notation \( \mathbb{C}^* = \mathbb{C}\{0\} \) and also recall that \( g^{-1} = \oplus_{i=1}^n g_{-\alpha_i} \) as in [A.14]. We use the terminology that a domain is a connected open set in \( \mathbb{C} \).

**Theorem 2.3.** Let \( \{U_i\} \) be a set of solutions to the Toda system (1.4). Then there exists a domain \( D \subset \mathbb{C}\{\mathbb{R}\leq 0\} \) containing 1 and a holomorphic map

\[ \eta : D \to g^{-1}; \quad \eta(z) = \sum_{i=1}^n f_i(z)e^{-\alpha_i}, \]

where the \( f_i \) are holomorphic and nowhere zero in \( D \), such that

\[ U_i = -\log \langle i|L^*L|i \rangle + \sum_{j=1}^n a_{ij} \log |f_j|^2, \]

where \( L : D \to N \subset G \) satisfies

\[ L^{-1}L_z = \eta, \quad L(1) = \text{Id}. \]

Here \( L^* = (L^0)^{-1} \), and \( \langle i| \cdot |i \rangle \) is the highest matrix coefficient between the \( i \)th fundamental representation (see the Appendix).

On the other hand, a holomorphic map \( \eta = \sum_{i=1}^n f_i e^{-\alpha_i} : D \to g_{-1} \) as in (2.4) in a simply-connected domain \( D \), where the \( f_i \) are nowhere zero, gives rise to a set of solutions \( \{U_i\} \) to the Toda system in \( D \).

**Proof.** Using the notation and the setup in the Appendix, Eq. (1.4) has the following zero-curvature equation on \( \mathbb{C}^* \)

\[ [\partial_z + A, \partial_{\bar{z}} + A^0] = 0, \quad \text{that is,} \]

\[ -A_{\bar{z}} + (A^0)_z + [A, A^0] = 0, \quad \text{where} \]

\[ A = -\sum_{i=1}^n \frac{1}{2} U_{i,z} h_{\alpha_i} + \sum_{i=1}^n \exp \left( \frac{i}{2} \sum_{j=1}^n a_{ij} U_j \right) e^{-\alpha_i}, \]

\[ A^0 = \sum_{i=1}^n \frac{i}{2} U_{i,\bar{z}} h_{\alpha_i} - \sum_{i=1}^n \exp \left( \frac{i}{2} \sum_{j=1}^n a_{ij} U_j \right) e_{\alpha_i}. \]

The zero-curvature equation can also be written as the Maurer-Cartan equation

\[ d\omega + \frac{1}{2} [\omega, \omega] = 0 \]

for the following Lie algebra valued differential form

\[ \omega = Adz + A^0 d\bar{z} \in \Omega^1(\mathbb{C}^*, g). \]
With $dz = dx_1 + i dx_2$ and $d\bar{z} = dx_1 - i dx_2$, it is also
\[ \omega = (A + A^\theta)dx_1 + i(A - A^\theta)dx_2. \]
Since the Cartan involution $\theta$ on $g$ is conjugate linear, we see that $\omega$ takes value in the fixed subalgebra $g^\theta = \mathfrak{t}$ (see Appendix Subsection A.3). Therefore by [Sha97, Theorems 6.1 and 7.14], there exists a map on the simply-connected domain
\[ F : \mathbb{C}\setminus \mathbb{R}_{\leq 0} \to K \subset G \]
to the compact subgroup $K = G^\theta$ such that
\[
\begin{cases}
F^{-1}dF = \omega \\
F(1) = Id.
\end{cases}
\]
Therefore,
\[ F^{-1}F_\bar{z} = A, \quad F^{-1}F_z = A^\theta. \]

The map $F$ has the following Gauss decomposition (A.2) in a domain $1 \in D \subset \mathbb{C}\setminus \mathbb{R}_{\leq 0}$
\[ F = LM \exp(H) \]
where $L : D \to N = N_-$ takes value in the negative nilpotent subgroup, and $M : D \to N_+$ takes value in the positive nilpotent subgroup. Furthermore $H = \sum_{i=1}^n b_i h_{\alpha_i} : D \to \mathfrak{h}$ takes value in the Cartan subalgebra, and exp : $\mathfrak{h} \to \mathcal{H}$ is the exponential map to the Cartan subgroup. From $F(1) = Id$ in (2.10), we see clearly that $L(1) = Id$.

Now we show that $L$ is holomorphic in $D$. By the second equation in (2.11), we have
\[
\exp(-H)M^{-1}(L^{-1}L_\bar{z})M \exp(H) + \exp(-H)M^{-1}M_\bar{z} \exp(H) + H_\bar{z} = A^\theta.
\]
In view of (2.8), the components in $n_- = \oplus_{\alpha \in \Delta^+} g_{-\alpha}$ (see (A.1)) of the above equation give
\[ L^{-1}L_\bar{z} = 0, \]
and the components in $\mathfrak{h}$ give
\[ b_{i,\bar{z}} = \frac{1}{2}U_{i,\bar{z}}, \quad 1 \leq i \leq n. \]
Thus we see that $b_{i,z\bar{z}} = \frac{1}{2}U_{i,z\bar{z}}$. Taking the conjugate, we also have $\bar{b}_{i,z\bar{z}} = \frac{1}{2}U_{i,z\bar{z}}$ since $U_i$ is real. Therefore,
\[ (b_{i} + \bar{b}_{i})z\bar{z} = U_{i,z\bar{z}}. \]
Hence we have, for $1 \leq i \leq n$,
\[ b_i + \bar{b}_i = U_i - p_i \]
for some real-valued harmonic function $p_i$ in $D$.

By the first equation in (2.11), we have
\[
\exp(-H)M^{-1}(L^{-1}L_z)M \exp(H) + \exp(-H)M^{-1}M_z \exp(H) + H_z = A.
\]
Since $A \in g_{-1} \oplus \mathfrak{h}$ by (2.8), we see that $L^{-1}L_z \in g_{-1}$. We denote it by $\eta$ and write it out in terms of the basis
\[ L^{-1}L_z = \eta = \sum_{i=1}^n f_i(z)e_{-\alpha_i}. \]
Then the \( f_i \) are holomorphic by (2.13). Furthermore, by (2.8) the component of \( A \) in \( g_{-1} \) is \( \sum_{i=1}^{n} \exp \left( \frac{1}{2} \sum_{j=1}^{n} a_{ij} U_{j} \right) e^{-\alpha_i} \). We also see that

the component of the LHS of (2.15) in \( g_{-1} = \exp(-H)(L^{-1}L_z) \exp(H) \)

\[= \exp(-H) \left( \sum_{i=1}^{n} f_i(z) e^{-\alpha_i} \right) \exp(H) = \sum_{i=1}^{n} f_i(z) e^{\alpha_i(H)} e^{-\alpha_i}.\]

Comparing the above, we see that the \( f_i \) in (2.16) are nowhere zero in \( D \). Thus we have shown (2.6) and (2.4).

Now following the physicists, we denote by \(|i\rangle\) a highest weight vector in the \( i \)th fundamental representation of \( G \), and \(|\rangle \) a lowest weight vector in its dual representation such that \(|\rangle Id|i\rangle = 1 \) (see Appendix Subsection A.3).

From (2.12), we have

\[ L = F \exp(-H)M^{-1}.\]

Therefore using the \(*\) operation from Appendix Subsection A.4 we have

\[(2.17) \langle i|L^*L|i\rangle = \langle i|(M^{-1})^* \exp(-\tilde{H})F^*F \exp(-H)M^{-1}|i\rangle \]

\[= \langle i| \exp \left( - \sum_{j=1}^{n} (b_j + \tilde{b}_j) h_{\alpha_j} \right) |i\rangle = e^{-(b_i + \tilde{b}_i)},\]

where we have used the following facts. First, by \( F \in K \) we have \( F^*F = Id \). Secondly, since \( M^{-1} \in N_+ \) and \(|i\rangle\) is a highest weight vector, we have \( M^{-1}|i\rangle = |i\rangle \). Similarly, \((M^{-1})^* \in N_- \) and \(|\rangle \) is a lowest weight vector, so \(|\rangle(M^{-1})^* = |\rangle \). Finally, we have \( h_{\alpha}|i\rangle = \delta_{ij}|i\rangle \) (see Eq. (A.17)). Eq. (2.17) actually shows that \(|i|L^*L|i\rangle\) is real, and this also follows from Appendix Subsection A.9. Therefore by (2.11),

\[(2.18) U_i = - \log\langle i|L^*L|i\rangle + p_i.\]

Now we show that for the above \( U_i \) to satisfy (1.4) with \( L \) from (2.16), we must have

\[p_i = \sum_{j=1}^{n} a_{ij} \log |f_j(z)|^2.\]

This follows from [LS92 §4.1.2] using the so-called Jacobi identity from [LS92 §1.6.4]. The identity says that for a general element \( g \in G^* \), the simply-connected Lie group with Lie algebra \( g \) (see Appendix Subsection A.8), we have

\[(2.19) \langle i|g|i\rangle \langle i|e_{\alpha_i} ge_{-\alpha_i}|i\rangle - \langle i|ge_{-\alpha_i}|i\rangle \langle i|e_{\alpha_i}g|i\rangle = \prod_{j \neq i} (j|g|j)^{-\alpha_{ij}}.\]

From (2.18) and that \( p_i \) is harmonic, we have

\[(2.20) U_{i,z\bar{z}} = - \frac{\langle i|L^*L|i\rangle \langle i|L^*L|i\rangle \bar{z}}{\langle i|L^*L|i\rangle^2} - \frac{\langle i|L^*L|i\rangle \bar{z}}{\langle i|L^*L|i\rangle}.\]

Now by (2.13) and (2.16), we have

\[\langle i|L^*L|i\rangle \bar{z} = \langle i|L^*L_0|i\rangle = f_i(z) \langle i|L^*Le_{-\alpha_i}|i\rangle,\]

where we have used that for the \( i \)th fundamental representation we have \( e_{-\alpha_i}|i\rangle = 0 \) for \( j \neq i \) (see Eq. (A.17)). Taking the \(*\) operation and noting (A.8), (2.16) also
gives

\begin{equation}
(L^*) \bar{z} (L^*)^{-1} = \eta^* = \sum_{i=1}^{n} f_i(z) e_{\alpha_i}.
\end{equation}

Therefore, similarly we have

\begin{equation}
\langle i | L^* L | i \rangle \bar{z} = \langle i | \eta^* L^* L | i \rangle = \bar{f}_i(z) \langle i | e_{\alpha_i} L^* L | i \rangle.
\end{equation}

Furthermore, we have

\begin{equation}
\langle i | L^* L | i \rangle \bar{z} = \langle i | \eta^* L^* L | i \rangle = f_i(z) \langle i | e_{\alpha_i} L^* L | i \rangle.
\end{equation}

Now applying the Jacobi identity \(^{(2.19)}\) to \(^{(2.20)}\) with \(g = L^* L\) gives

\begin{equation}
U_{i,z} = -|f_i|^2 \prod_{j=1}^{n} \langle j | L^* L | j \rangle^{-a_{ij}}
\end{equation}

by \(a_{ii} = 2\). By \(^{(2.18)}\), this is

\begin{equation}
U_{i,z} = -|f_i|^2 \exp \left( \sum_{j=1}^{n} a_{ij} U_j - \sum_{j=1}^{n} a_{ij} p_j \right) = - \exp \left( \log |f_i|^2 - \sum_{j=1}^{n} a_{ij} p_j \right) \exp \left( \sum_{j=1}^{n} a_{ij} U_j \right).
\end{equation}

Therefore for the \(U_i\) to satisfy \(^{(1.4)}\), we need \(\log |f_i|^2 - \sum_{j=1}^{n} a_{ij} p_j = 0\). This gives

\begin{equation}
p_i = \sum_{j=1}^{n} a_{ij} \log |f_j|^2
\end{equation}

and proves the formula \(^{(2.5)}\).

Now given a holomorphic map \(\eta = \sum_{i=1}^{n} f_i e_{-\alpha_i} : D \to g_{-1}\) as in \(^{(2.4)}\) in a simply-connected domain \(D\), where the \(f_i\) are nowhere zero, we can construct a \(L : D \to G\), which solves \(L^{-1} L_z = \eta\) as in \(^{(2.6)}\) but may not necessarily satisfy the condition \(L(1) = Id\). Construct the \(U_i\) as in \(^{(2.5)}\), and they are checked to satisfy the Toda system \(^{(1.4)}\) in the same way as above, where the important point is again the Jacobi identity \(^{(2.19)}\). \(\Box\)

3. \(W\)-Invariants of the Toda Systems

In this section, we first present the algebraic theories of \(W\)-invariants of Toda systems as developed in \cite{FF96, Nie14}. Then we present a result relating the \(W\)-invariants with the local solutions from the last section, following the approach of \cite{FOR+92}.

By definition, a \(W\)-invariant (also called a characteristic integral) for the Toda system \(^{(1.4)}\) is a polynomial in the \(U_i\) for \(k \geq 1\) and \(1 \leq i \leq n\) whose derivative with respect to \(\bar{z}\) is zero if the \(U_i\) are solutions.

For example, for the Liouville equation \(^{(2.1)}\),

\begin{equation}
W = U_{zz} - \frac{1}{2} |f|^2
\end{equation}

is a \(W\)-invariant since \(W_{\bar{z}} = 0\) for a solution \(U\). Furthermore, plugging in the local solution \(^{(2.2)}\), we have

\begin{equation}
W = \frac{1}{2} \left( \frac{f'}{f} - \frac{3}{2} \left( \frac{f''}{f} \right)^2 \right),
\end{equation}

where \(f = \tilde{f}_i(z) e_{\alpha_i}\).
that is, the $W$-invariant of the local solution becomes one half of the Schwarzian derivative of the holomorphic function $f$. We aim to generalize such results to general Toda systems in this section.

For a general Toda system associated to a simple Lie algebra of rank $n$, there are $n$ basic $W$-invariants (see [FF96]) so that the other $W$-invariants are differential polynomials in these. One of us in [Nie14] has given a concrete construction and a direct proof of the basic $W$-invariants $W_j$ for $1 \leq j \leq n$. They are obtained by conjugating one side of the following zero-curvature equation of the Toda system to its Drinfeld-Sokolov gauge [DSS], which is in turn related to a Kostant slice of the corresponding Lie algebra [Kos63].

For this purpose, we conjugate (2.7) by \(\exp\left(\frac{1}{2} \sum_{i=1}^{n} U_i h_{\alpha_i}\right)\) to arrive at another zero-curvature representation of the Toda system (1.4):

\[
\partial_z + \epsilon - \sum_{i=1}^{n} U_{i,z} h_{\alpha_i}, \partial_{\bar{z}} - \sum_{i=1}^{n} e^{u_i} e_{\alpha_i} = 0,
\]

where $\epsilon = \sum_{i=1}^{n} e^{-\alpha_i} \in \mathfrak{g}-1$ and $u_i = \sum_{j=1}^{n} a_{ij} U_j$. (The current zero-curvature equation has different signs from the version in [Nie14], but all results there continue to hold for the current version.)

Let $\mathfrak{s}$ be a Kostant slice of $\mathfrak{g}$, that is, a homogeneous subspace $\mathfrak{s}$ with respect to the principal grading (A.13) such that

\[
\mathfrak{g} = [\mathfrak{s}, \mathfrak{g}] \oplus \mathfrak{s}.
\]

Then it is known [Kos63] that $\mathfrak{s} \subset \mathfrak{n}_+$ in (A.1) and $\dim \mathfrak{s} = n$. Let $\{s_j\}_{j=1}^{n}$ be a homogeneous basis of $\mathfrak{s}$ ordered with nondecreasing gradings from (A.13).

We first state the following lemma about the existence and uniqueness of the so-called Drinfeld-Sokolov gauge.

**Lemma 3.4.** For $1 \leq i \leq n$, let $\phi_i$ be smooth functions on a domain $D \subset \mathbb{C}$.

(i) There exists a unique map $\tilde{M} : D \to N_+$ from $D$ to the positive nilpotent subgroup depending on the derivatives $\partial_z^k \phi_i$ for $1 \leq i \leq n$ and $k \geq 0$, such that

\[
\tilde{M}\left(\partial_z + \epsilon - \sum_{i=1}^{n} \phi_i h_{\alpha_i}\right) \tilde{M}^{-1} \in \partial_z + \epsilon + \mathfrak{s}.
\]

(ii) There exist differential polynomials (in the $\psi_i$ and their derivatives as indicated by the brackets)

\[
S_j([\psi_1], \cdots, [\psi_n]) = S_j(\psi_1, \cdots, \psi_n, \partial_z \psi_1, \cdots, \partial_z \psi_n, \cdots)
\]

for $1 \leq j \leq n$ that are independent of the $\phi_i$, such that

\[
\tilde{M}\left(\partial_z + \epsilon - \sum_{i=1}^{n} \phi_i h_{\alpha_i}\right) \tilde{M}^{-1} = \partial_z + \epsilon + \sum_{j=1}^{n} W_j s_j,
\]

where $W_j = S_j([\phi_1], \cdots, [\phi_n])$.

(iii) If the $\phi_i$ in (3.5) are holomorphic in $D$, then so is $\tilde{M}$.

We refer the reader to [FOR+92, p. 7] and [Nie17, two pages containing and after Eq. (11)] for the inductive proof of the existence and the uniqueness of $\tilde{M}$.
using (3.3) and that
\[(3.7) \quad \ker \text{ad}_r \cap \mathfrak{b}_+ = 0.\]
The holomorphy of \(\tilde{M}\) when the \(\phi_i\) are holomorphic is easily seen from the inductive proof.

**Theorem 3.8** ([Nie14]). Suppose that the \(U_i\) are a set of solutions of the following Toda field theory
\[U_{i,zz} + \exp \left( \sum_{j=1}^n a_{ij} U_j \right) = 0\]
in a domain \(D\) in \(\mathbb{C}\). Then there exists a unique map \(M_0 : D \to N_+\) depending on the derivatives \(\partial^k U_i\) for \(1 \leq i \leq n\) and \(k \geq 1\), such that
\[(3.9) \quad M_0 \left( \partial_z + \epsilon - \sum_{i=1}^n U_{i,z} \alpha_i \right) M_0^{-1} = \partial_z + \epsilon + \sum_{j=1}^n W_j s_j,\]
and the \(W_j = S_j(U_{1,z}, \ldots, U_{n,z})\) (with \(S_j\) from (3.6)) for \(1 \leq j \leq n\) are the basic \(W\)-invariants of the Toda system.

Here is the main result in this section which relates the \(W\)-invariants with the holomorphic functions \(f_i\) in \(\eta\) from (2.4) for local solutions.

**Theorem 3.10.** For the local solutions (2.5) in a simply-connected domain \(D\), there exists a unique holomorphic map \(M_1 : D \to N_+\) from \(D\) to the positive nilpotent subgroup whose coordinates depend on the derivatives \(\partial^k f_i\) for \(1 \leq i \leq n\) and \(k \geq 0\), such that
\[(3.11) \quad M_1 \left( \partial_z + \epsilon - \sum_{i=1}^n F_i \alpha_i \right) M_1^{-1} = \partial_z + \epsilon + \sum_{j=1}^n W_j s_j,\]
where
\[(3.12) \quad F_i = \sum_{j=1}^n a^{ij} \partial_z \log f_j = \sum_{j=1}^n a^{ij} \frac{f'_j}{f_j},\]
and the \(W_j\) are the \(W\)-invariants computed by (3.9) for the local solutions (2.5).

**Proof.** Our proof follows the general approach in [FWB+89, BFO+90, FOR+92] of treating Toda theories as conformally reduced WZNW theories. For that purpose, we choose
\[(3.13) \quad Q_1 = \exp \left( - \sum_{k=1}^n \left( \sum_{j=1}^n a^{kj} \log f_j \right) h_{\alpha_k} \right) : D \to \mathcal{H},\]
where \(\mathcal{H}\) is the Cartan subgroup of \(G\). Note that a single-valued branch of \(\log f_j\) can be chosen since \(D\) is simply-connected. Since \([h_{\alpha_k}, e_{-\alpha_i}] = -a_{ik} e_{-\alpha_i}\) by (A.10), we have
\[Q_1^{-1} e_{-\alpha_i} Q_1 = \exp \left( - \sum_{k=1}^n \left( \sum_{j=1}^n a^{kj} \log f_j \right) a_{ik} \right) e_{-\alpha_i} = \exp \left( - \log f_i \right) e_{-\alpha_i} = - \frac{1}{f_i} e_{-\alpha_i}.\]
It is also clear that
\[Q_1^{-1} \partial_z Q_1 = - \sum_{i=1}^n \left( \sum_{j=1}^n a^{ij} \partial_z \log f_j \right) h_{\alpha_i} = - \sum_{i=1}^n F_i h_{\alpha_i}.\]
Define
\[ \Psi = Q_1^*L^*LQ_1. \]
Then by (2.6), (2.4), and the above,
\[ \Psi^{-1}z = (LQ_1)^{-1}(LQ_1)z = Q_1^{-1}L^{-1}LzQ_1 + Q_1^{-1}\partial_zQ_1 = \epsilon - \sum_{i=1}^n F_i h_{\alpha_i}. \]
This quantity \( \Psi^{-1}z \) is called the current of \( \Psi \) and is denoted by \( J \) in the works [FWB+89, BFO+90, FOR+92], and the requirement that its component in \( n_- \) is \( \epsilon \) is the key idea in these works.

Furthermore, the local solutions in (2.5) has the following neat form
\[ U_i = -\log(\langle i|Q_1^*L^*LQ_1|i \rangle) = -\log(\langle i|\Psi|i \rangle). \]
Since \( \langle i|\Psi|i \rangle > 0 \) (see (A.20)), \( \Psi \) has Gauss decomposition (A.2) in the domain \( D \):
\[ \Psi = \Psi_+ \Psi_0 \Psi_-, \quad \Psi_0, \Psi_+ \in \mathcal{N}, \Psi_0 \in \mathcal{H}. \]
Then clearly \( U_i = -\log(\langle i|\Psi_0|i \rangle) \). Therefore, we see that
\[ \Psi_0 = \exp\left(-\sum_{i=1}^n U_i h_{\alpha_i}\right), \quad \Psi_0^{-1}z = -\sum_{i=1}^n U_i z h_{\alpha_i}. \]
Plugging (3.16) in (3.14), we get
\[ \Psi^{-1}z = \epsilon - \sum_{i=1}^n F_i h_{\alpha_i}. \]
Comparing the components of the above equality in \( n_- \), we see that
\[ \Psi_0^{-1}z = \epsilon. \]
Also in view of (3.17), Eq. (3.18) becomes
\[ \Psi_+^{-1}z + \Psi_0^{-1}z = \epsilon - \sum_{i=1}^n U_i z h_{\alpha_i}. \]
We owe this important equality to [FOR+92, Eq. (2.14a)]. Therefore,
\[ \Psi_+ \left( \partial + \epsilon - \sum_{i=1}^n F_i h_{\alpha_i} \right) \Psi_+^{-1}z = \partial + \epsilon - \sum_{i=1}^n U_i z h_{\alpha_i}. \]
By (3.9), we see that (3.11) holds with \( M_1 = M_0 \Psi_+ \). Since the \( F_i \) are holomorphic, we see that \( M_1 \) is holomorphic although neither \( \Psi_+ \) nor \( M_0 \) is (see Lemma 3.4 (iii)).

4. USE THE FINITE-INTEGRAL CONDITIONS

In this section, we adapt the analytical estimates from [BM91, LWW12] using the finite-integral conditions and the strength of the singularities to determine the simple forms of the \( W \)-invariants, which will be used to determine the solutions in the next section.

For a differential monomial in the \( U_i \), we call by its degree the sum of the orders of differentiation multiplied by the algebraic degrees of the corresponding factors. For example the above \( W = U_{zz} - U_z^2 \) in (3.1) for the Liouville equation has a homogeneous degree 2. It is known from [FF96] and also clear from (3.9) that the
$W$-invariants $W_j$ involve the $\partial^k U_i$ for $k \geq 1$ and that the homogeneous degree of $W_j$ is the same as the degree $d_j$ of the corresponding primitive adjoint-invariant function of the Lie algebra $\mathfrak{g}$ [Kos59]. We call such degrees the degrees of the simple Lie algebra and we have listed them in Appendix Subsection A.11.

**Proposition 4.1.** The $W$-invariants for the Toda system (1.4) are

\[ W_j = \frac{w_j}{z^{d_j}}, \quad z \in \mathbb{C}^*, \ 1 \leq j \leq n, \]

where the $d_j$ are the degrees of the Lie algebra $\mathfrak{g}$ and the $w_j$ are constants.

**Proof.** This proof is an adaption of the proof in [LWY12] of the corresponding assertion in their Eq. (5.9).

Following [LWY12, Eq. (5.10)], introduce

\[ V_i = U_i - 2\gamma_i \log |z|, \quad 1 \leq i \leq n. \]

Then system (1.4) becomes

\[
\begin{cases}
\Delta V_i = -4|z|^{2\gamma_i} \exp \left( \sum_{j=1}^{n} a_{ij} V_j \right), \\
\int_{\mathbb{R}^2} |z|^{2\gamma_i} \exp \left( \sum_{j=1}^{n} a_{ij} V_j \right) \, dx < \infty.
\end{cases}
\]

As $\gamma_i > -1$, applying Brezis-Merle’s argument in [BM91], we have that $V_i \in C^{0,\alpha}$ on $\mathbb{C}$ for some $\alpha \in (0,1)$ and that they are upper bounded over $\mathbb{C}$. Therefore we can express $V_i$ by the integral representation formula, and we have for $k \geq 1$

\[
\partial^k_z V_i(z) = O(1 + |z|^{2+2\gamma_i-k}) \quad \text{near 0},
\]

\[
\partial^k_z V_i(z) = O(|z|^{-k}) \quad \text{near } \infty.
\]

Therefore from (4.2), we have for $k \geq 1$

\[
\partial^k_z U_i(z) = O(|z|^{-k}) \quad \text{near 0},
\]

\[
\partial^k_z U_i(z) = O(|z|^{-k}) \quad \text{near } \infty.
\]

By $W_j, \bar{z} = 0$ and that $W_j$ has degree $d_j$, we see from the above estimates that $z^{d_j} W_j$ is holomorphic and bounded on $\mathbb{C}^*$. Therefore $z^{d_j} W_j = w_j$ is a constant by the Liouville theorem, and so (4.2) holds. \qed

**Theorem 4.6.** The $W$-invariants $W_j$ for the Toda system (1.4) are also computed by

\[ M_2 \left( \partial_z + \epsilon - \sum_{i=1}^{n} \frac{\zeta^i}{z} h_{\alpha_i} \right) M_2^{-1} = \partial_z + \epsilon + \sum_{j=1}^{n} W_j s_j, \]

where $M_2 : \mathbb{C}^* \to N_+$ is unique and holomorphic.

**Proof.** By (4.3),

\[ U_{i,z} = V_{i,z} + \partial_z (2\gamma_i \log |z|) = V_{i,z} + \frac{\zeta^i}{z}. \]

From (4.2), $W_j = \frac{w_j}{z^{d_j}}$. Since $\gamma_i > -1$, by (4.3) and (4.5) for the orders at 0, we see that all the terms involving $\partial^k_z V_i$ will not appear in the final form of $W_j$ since their
orders of pole are not high enough. Therefore in terms of (3.6), we have

\[(4.8) \quad W_j = S_j([U_1, z], \ldots, [U_n, z]) = S_j\left(\left[\frac{1}{z}\right], \ldots, \left[\frac{n}{z}\right]\right), \quad 1 \leq j \leq n,\]

where we recall that the brackets indicate that the solutions to the Toda systems of type (2.5) to be differential polynomials and depend on the derivatives of the arguments.

By Lemma 3.4 there exists a unique holomorphic \( M_2 : \mathbb{C}^* \rightarrow N_+ \) such that

\[M_2\left(-\partial_z + \epsilon + \sum_{i=1}^{n} \frac{\gamma_i}{z} h_{\alpha_i}\right) M_2^{-1} = -\partial_z + \epsilon + \sum_{j=1}^{n} W_j^f s_j,\]

where

\[W_j^f = S_j\left(\left[\frac{\gamma_1}{z}\right], \ldots, \left[\frac{n}{z}\right]\right) = W_j, \quad 1 \leq j \leq n,\]

by (4.8).

\[\square\]

5. The holomorphic functions in the local solutions

The \( W \)-invariants play essential roles in our approach of classifying the solutions. The work [LWY12] classified the solutions to the Toda systems of type A by relating them to an ODE whose coefficients are the \( W \)-invariants. In this section, we will use the \( W \)-invariants to largely restrict the holomorphic functions \( f_i(z) \) (2.4) in the local solutions (2.5) to be \( f_i(z) = z^{\gamma_i} \) as long as we allow some constant group element.

**Theorem 5.1.** The local solutions \( U_i \) (2.5) in a simply-connected domain \( D \subset \mathbb{C} \setminus \mathbb{R}_{\leq 0} \) that are solutions to (1.4) have the following form

\[(5.2) \quad U_i = -\log(|\Phi^* g^* g| i) + 2\gamma_i \log |z|, \quad 1 \leq i \leq n,\]

where \( \Phi \) satisfies (1.7) and \( g \in G \) is a constant group element.

**Proof.** As in (3.13), choose

\[(5.3) \quad Q_2 = \exp\left(-\sum_{k=1}^{n} \log z^{\gamma_k} h_{\alpha_k}\right) : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \rightarrow \mathcal{H}.\]

Then by (1.7),

\[(5.4) \quad (\Phi Q_2)^{-1}(\Phi Q_2)_z = Q_2^{-1}\left(\sum_{i=1}^{n} z^{\gamma_i} e_{-\alpha_i}\right) Q_2 + Q_2^{-1} Q_2 z = \epsilon - \sum_{i=1}^{n} \frac{\gamma_i}{z} h_{\alpha_i}.\]

Let \( M = M_2^{-1} M_1 : D \rightarrow N_+ \), where \( M_1 \) and \( M_2 \) are from Theorems 3.10 and 4.6. Then

\[M\left(\partial_z + \epsilon - \sum_{i=1}^{n} F_i h_{\alpha_i}\right) M^{-1} = \partial_z + \epsilon - \sum_{i=1}^{n} \frac{\gamma_i}{z} h_{\alpha_i},\]

which is

\[\epsilon - \sum_{i=1}^{n} F_i h_{\alpha_i} = M^{-1}\left(\epsilon - \sum_{i=1}^{n} \frac{\gamma_i}{z} h_{\alpha_i}\right) M + M^{-1} M_2.\]

This and (5.4) imply that

\[(\Phi Q_2 M)^{-1}(\Phi Q_2 M)_z = M^{-1}\left(\epsilon - \sum_{i=1}^{n} \frac{\gamma_i}{z} h_{\alpha_i}\right) M + M^{-1} M_2 = \epsilon - \sum_{i=1}^{n} F_i h_{\alpha_i}.\]
Comparison with \( 3.14 \) gives, on the basis that both \( LQ_1 \) and \( \Phi Q_2 M \) are holomorphic in \( D \), that
\[
(5.5) \quad LQ_1 = g\Phi Q_2 M,
\]
where \( g \in G \) is a constant element.

Therefore by \( 3.15 \), \( 5.5 \) and \( 5.3 \), \( U_i \) from \( 2.5 \) becomes
\[
U_i = -\log \langle i | Q_1^* L^* LQ_1 | i \rangle = -\log \langle i | M^* Q_2^* \Phi^* g^* g \Phi Q_2 M | i \rangle \\
= -\log \langle i | \Phi^* g^* g \Phi | i \rangle + 2\gamma^i \log |z|.
\]

□

**Proposition 5.6.** The local solutions in \( 5.2 \) are of the form \( 1.8 \) with \( \Lambda \in A \) and \( C \in N \) and are at least defined on \( \mathbb{C} \setminus \mathbb{R}_{\leq 0} \).

**Proof.** In the local solutions \( 5.2 \), write the Iwasawa decomposition \( \Lambda \) for \( g \in G \) as
\[
g = F\Lambda C
\]
with \( F \in K, \Lambda \in A \) and \( C \in N \). Then \( 5.2 \) becomes \( 1.8 \) by \( F^* F = \text{Id} \) and \( \Lambda^* = \Lambda \).

The solutions \( U_i \) in \( 1.8 \) with \( \Lambda \in A \) and \( C \in N \) are well-defined on \( \mathbb{C} \setminus \mathbb{R}_{\leq 0} \) after the branch cut for the functions \( z^{\gamma^i} \). Clearly the \( U_i \) satisfy the Toda system \( 1.4 \) by the converse direction in Theorem 2.3 and have the right strength of singularities at the origin. □

6. **Formula for \( \Phi \) using relation with Kostant’s work**

In this section, we obtain an explicit expression for the \( \Phi \) in \( 1.7 \) inspired by \[Kos79\] and thus establish a very concrete relationship of our work with that of Kostant for Toda ODEs. This expression is essential to the next two sections.

Following \[LWY12\], we denote
\[
\mu_i = \gamma_i + 1 > 0, \quad 1 \leq i \leq n.
\]

Inspired by \[Kos79\] and using \( A.11 \), we introduce the following notation
\[
w_0 = \sum_{i=1}^{n} \mu_i E_i \in \mathfrak{h}.
\]

We also introduce
\[
\zeta = \sum_{i=1}^{n} z^{\gamma_i} e_{-\alpha_i} \in \mathfrak{g}_{-1},
\]
\[
\xi = z\zeta = \sum_{i=1}^{n} z^{\mu_i} e_{-\alpha_i} \in \mathfrak{g}_{-1}.
\]

To find the concrete expression for \( \Phi \) in \( 1.7 \), we introduce the following setup after \[Kos79\].

Let \( S \) be the set of all finite sequences \( 6.4 \)
\[
s = (i_1, \ldots, i_k), \quad k \geq 0, \quad 1 \leq i_j \leq n.
\]
We write \(|s|\) for the length \(k\) of the element \(s \in \mathcal{S}\), and we also write

\[
\varphi(s) = \sum_{j=1}^{\lfloor |s| \rfloor} \alpha_{ij}, \quad \varphi(s, w_0) = \varphi(s)(w_0) = \langle \varphi(s), w_0 \rangle = \sum_{j=1}^{\lfloor |s| \rfloor} \mu_{ij},
\]

where \(\langle \cdot, \cdot \rangle\) is the natural pairing between \(\mathfrak{h}'_0\) and \(\mathfrak{h}_0\) (see (A.3)). Note that \(\varphi(s)\) is equal to the constant function 0 on \(\mathfrak{h}\) if \(|s| = 0\).

For \(0 \leq j \leq |s| - 1\), let \(s_j \in \mathcal{S}\) be the sequence obtained from \(s\) by “cutting off” the first \(j\) terms (different from \([\text{Kos79}]\))

\[
s_j = (i_{j+1}, \ldots, i_{|s|}),
\]

and define

\[
p(s, w_0) = \prod_{j=0}^{\lfloor |s| - 1 \rfloor} \langle \varphi(s_j), w_0 \rangle.
\]

Note that by convention, we have \(p(s, w_0) = 1\) when \(|s| = 0\). Clearly when \(|s| \geq 1\), we have

\[
p(s, w_0) = \varphi(s, w_0)p(s_1, w_0).
\]

Let \(U(n_1) = U(n_-)\) be the enveloping algebra of \(n_1\). For convenience write \(e_{-i} = e_{-\alpha_i}\) for \(i = 1, \ldots, n\). For \(s \in \mathcal{S}\) as in (6.4), put

\[
e_{-s} = e_{-i_k} \cdots e_{-i_2} e_{-i_1}.
\]

We note that the \(\xi\) in (6.3) is

\[
\xi = \sum_{i=1}^{n} z^{\varphi(t^i, w_0)} e_{-i},
\]

where the \(s\) are the simplest \((i)\) for \(1 \leq i \leq n\).

**Proposition 6.11.** In the space \(\hat{D}(N)\) (see Appendix Subsection A.8), we have

\[
\Phi = \sum_{s \in \mathcal{S}} \frac{z^{\varphi(s, w_0)} e_{-s}}{p(s, w_0)},
\]

where the right hand side converges in \(\hat{D}(N)\) by Proposition A.18.

**Proof.** We denote the right hand side of (6.12) by \(\Upsilon\). Now we show that \(\Upsilon\) satisfies (1.7), whose right hand side is \(\zeta\) from (6.3), and hence it is equal to \(\Phi\). Clearly \(\Upsilon(0) = 1\) corresponding to the empty \(s\). We also have

\[
\Upsilon_z = \frac{\sum_{s \in \mathcal{S}} \varphi(s, w_0) z^{\varphi(s, w_0) - 1} e_{-s}}{p(s, w_0)} = \frac{1}{z} \sum_{|s| \geq 1} \frac{z^{\varphi(s, w_0)} e_{-s}}{p(s_1, w_0)}
\]

by (6.8).

Let \(s \in \mathcal{S}\) and let \(\{t^1, \ldots, t^n\}\) be the set of all \(t \in \mathcal{S}\) such that \(t_1 = s\) according to (6.6). Then by (6.10) and (6.5), we have

\[
\frac{z^{\varphi(t^i, w_0)} e_{-t_i}}{p(s, w_0)} \xi = \sum_{i=1}^{n} \frac{z^{\varphi(t^i, w_0)} e_{-t_i}}{p(s, w_0)} = \sum_{i=1}^{n} \frac{z^{\varphi(t^i, w_0)} e_{-t_i}}{p(t^i, w_0)}.
\]
Therefore we have

\[ Υξ = \sum_{|s| \geq 1} \frac{e^{s|w_0|}e^{-s}}{p(s, w_0)}. \]

Combining (6.13) and (6.14), we see \( T_\gamma = \frac{1}{\xi} Υξ = Υξ \). Therefore \( Υ^{-1}T_\gamma = ξ \), and (6.12) is proved.

Remark 6.15. It can be shown that the \( Φ \) defined by a system of ODE from (1.7) is also uniquely characterized by the following algebraic condition inspired by [Kos79]:

\[
\begin{cases}
Φ^{-1}w_0Φ = w_0 - ξ, \\
Φ(0) = Id.
\end{cases}
\]

Now we use Proposition 6.11 to get information about the solutions (1.8) with \( λ \in A \) and \( C \in N \). Let \( V_i \) be the \( i \)th fundamental representation of \( g \) equipped with a Hermitian form \( \langle \cdot, \cdot \rangle \) which is invariant under the compact subgroup \( K^* \) of \( G^* \) (see Appendix Subsection A.9). Choose a highest weight vector \( v_\omega \in V_i \) with \( \{ v_\omega, v_\omega \} = 1 \). By (1.8) and (A.20), we have

\[ e^{-U_i} = |z|^{-2\gamma_i} \langle i|Φ^*C^*λ^2CΦ|i \rangle = |z|^{-2\gamma_i} \{|ACΦv_\omega, ACΦv_\omega\} \].

We know that

\[ \{ACΦv_\omega, ACΦv_\omega\} = \sum_{v \in V_i} |\{ACΦv_\omega, v\}|^2 = \sum_{v \in V_i} |\{Φv_\omega, C\lambda v\}|^2, \]

where the sum ranges over a unitary basis of \( V_i \) consisting of weight vectors and the second equality follows from (A.19).

In general, if \( v \) is a weight vector with weight \( \beta \) such that \( \omega_i - \beta = \sum_{i=1}^n m_i\alpha_i \), then by Proposition 6.11 and (A.15), we have

\[ \{Φv_\omega, v\} = d z \sum_{i=1}^n m_i\mu_i = d z \langle \omega_i - \beta, w_0 \rangle, \]

where \( d \) is a constant.

7. The monodromy consideration

We now want to show that for the \( U_i \) in (1.8) to be well-defined on \( C^* \), \( C \) needs to belong to a suitable subgroup \( N_\Gamma \) of \( N \). This subgroup was introduced in [Nie10] already in the classification result for Toda systems of types \( B \) and \( C \).

Using (6.2), consider the following element in the Cartan subgroup

\[ t_\Gamma := \exp(2π iw_0) ∈ H. \]

Clearly \( t_\Gamma^{-1} = t_\Gamma^* \).

Definition 7.2. The subgroup \( N_\Gamma ⊂ N \) is the centralizer of \( t_\Gamma \) in \( N \), that is,

\[ N_\Gamma = \{ C ∈ N | Ct_\Gamma = t_\Gamma C \}. \]

Remark 7.3. Here is a more concrete description of \( N_\Gamma \). For a positive root \( α ∈ Δ^+ \), write \( α = \sum_{i=1}^n m_i\alpha_i \) in terms of the positive simple roots \( \{ α_i \}_{i=1}^n \). Define the number \( α_\Gamma = \sum_{i=1}^n m_i\mu_i \), where we replace \( α_i \) by \( \mu_i \). Also define the subset \( Δ_\Gamma \) of \( Δ^+ \) as \( Δ_\Gamma = \{ α ∈ Δ^+ | α_\Gamma ∈ N \} \), and the Lie subalgebra \( n_\Gamma \) of \( n \) as \( n_\Gamma = \bigoplus_{α ∈ Δ_\Gamma} g_{-α} \). (Note that \( α_\Gamma = \sum_{i=1}^n m_i\mu_i ∈ N \) is equivalent to \( \sum_{i=1}^n m_i\gamma_i ∈ Z \).)
Then $N_\Gamma$ in the above definition is the subgroup of $N$ corresponding to $n_\Gamma$. The reason is that clearly $\alpha_\Gamma = \alpha(w_0)$ by (6.2) and (A.11), and so $\text{Ad}_{t_\Gamma} e_\alpha = \exp(2\pi i t_\Gamma) e_\alpha$. Hence we see that

\begin{equation}
(7.4) \quad n_\Gamma = n^\text{Ad}_{t_\Gamma} \quad \text{and} \quad N_\Gamma = N^\text{Ad}_{t_\Gamma}
\end{equation}

are the fixed point sets of the adjoint actions by $t_\Gamma$.

**Theorem 7.5.** The $U_j$ in (1.8) are well-defined on $\mathbb{C}^*$ if and only if

\[ C \in N_\Gamma. \]

**Proof.** For the $U_j$ in (1.8) to be well-defined on $\mathbb{C}^*$, we need that the $U_j$ are invariant under the change of $z \mapsto e^{-2\pi i z}$, that is, when one travels once (clockwise) around the origin.

By the definition (7.4), we have

\[ \sum_{j=1}^n (z^{2\pi i})^{j-i} e_{-\alpha_j} = \sum_{j=1}^n z^{\gamma_j} e^{-2\pi i \gamma_j} e_{-\alpha_j} = \sum_{j=1}^n z^{\gamma_j} e^{-2\pi i \mu_j} e_{-\alpha_j} = \text{Ad}_{t_\Gamma} \left( \sum_{j=1}^n z^{\gamma_j} e_{-\alpha_j} \right). \]

Hence the corresponding solution to

\[ \begin{cases} \frac{1}{i} \Psi^{-1} \Psi_z = -\sum_{j=1}^n (z e^{2\pi i})^{j-i} e_{-\alpha_j} & \text{on } \mathbb{C} \setminus \mathbb{R}_{\leq 0}, \\ \Phi(0) = \text{Id}, \end{cases} \]

is

\[ \Phi(z) = \text{Ad}_{t_\Gamma} \Psi(z) = t_\Gamma \Psi(z) t_\Gamma^{-1}. \]

Therefore the corresponding solution (1.8) to the Toda system is

\[ \dot{U}_i = -\log(i) \dot{\Phi}^* C^* \dot{C} \Phi(i) + 2\gamma^i \log |z| \]

\[ = -\log(i) \dot{\Phi}^* t_\Gamma^* C^* t_\Gamma \Phi(i) + 2\gamma^i \log |z|, \]

where we have used that $t_\Gamma^* = t_\Gamma^{-1}$. Note that $\Lambda t_\Gamma = t_\Gamma \Lambda$ since both belong to the Cartan subgroup $\mathcal{H}$.

Clearly if $C t_\Gamma = t_\Gamma C$, then $\dot{U}_i = U_i$ and the $U_j$ are well-defined on $\mathbb{C}^*$ for $1 \leq i \leq n$.

On the other hand, suppose $C \notin N_\Gamma$. We now show that the $U_j$ corresponding to the fundamental representation with the smallest dimension is not well-defined on $\mathbb{C}^*$. The fundamental representations of the smallest dimension are the first fundamental representations, except that for $E_6, E_7$ and $E_8$, they are the last fundamental representations, using our labeling of the roots in Appendix Subsection A.5.

Suppose $C = \exp(\sum_{\alpha \in \Delta^+} c_\alpha e_{-\alpha})$ such that $c_\alpha \neq 0$ for some $\alpha \notin \Delta_\Gamma$. Let $\alpha_0$ be such a root with the smallest principal grading according to (A.13). Therefore, if $\alpha_0 = \sum_{i=1}^m m_i \alpha_i$, then $\sum_{i=1}^m m_i \mu_i \notin \mathbb{N}$. An investigation of the weight diagrams of the fundamental representations of the smallest dimension of $\mathfrak{g}$ (see [Vav00]) confirms that to every root $\alpha \in \Delta^+$, there exists a weight $\beta$ for this fundamental representation such that $\alpha = \beta - \beta'$, where $\beta'$ is another weight for the fundamental representation. Furthermore, in view of $\mu_i > 0$ from (6.4), $\alpha$ is the only root with the given $\alpha_\Gamma$ among the roots as differences $\beta - \beta'$ for this weight $\beta$. Therefore, by
Proposition 6.11 and (6.18), the $U_j$ corresponding to this fundamental representation has a term in $\langle j | \Phi^* C^* \Lambda^j \Phi | j \rangle$ as

$$a |z|^{2(\omega_\beta - \beta, w_0)} \left[ 1 + c z^{-\sum_{i=1}^n m_i \mu_i} + \ldots \right]^2,$$

where $a > 0$, $c \neq 0$, $\sum_{i=1}^n m_i \mu_i \notin \mathbb{N}$, and the dots are other terms with different exponents. (See (5.3) for the example of $A_2$ Toda system.) This term is not well-defined on $\mathbb{C}^*$ because of the branch cut needed for $z^{-\sum_{i=1}^n m_i \mu_i}$.

\[\square\]

8. Quantization of the finite integrals

In this section, we use Section 6 to obtain the quantization result for the integrals of our solutions.

In general, let $\lambda$ be a dominant weight, and let $V^\lambda$ be the corresponding irreducible representation. Let $K$ be the longest Weyl group element which maps positive roots to negative roots. Then $\kappa \lambda$ is the lowest weight of $V^\lambda$ (see Appendix Subsection A.10). Throughout this paper, we denote the action of $K$ on $h^0_0$ without parentheses. In the following calculations, we use the Hermitian form $\{ \cdot, \cdot \}$ on $V^\lambda$ which is invariant under the compact subgroup $K^*$ of $G^*$ (see Appendix Subsection A.9). Choose vectors $v^\lambda \in V_\lambda$ and $v^{\kappa \lambda} \in V_{\kappa \lambda}$ in the one-dimensional highest and lowest weight spaces such that

$$\{ v^\lambda, v^\lambda \} = 1, \quad \{ v^{\kappa \lambda}, v^{\kappa \lambda} \} = 1.$$

(One can choose $v^{\kappa \lambda}$ to be $s_0(\kappa) v^\lambda$, where $s_0(\kappa) \in G^*$ induces the longest Weyl element $\kappa$. See [Kos79] Eq. (5.2.10)).

With the notation (6.3) and (6.9) and by (A.15), the vector $e^\varphi(s) v^\lambda$ is a weight vector with weight $-\varphi(s) + \lambda$. Since different weight spaces are orthogonal, by Proposition 6.11 we have

$$\{ \Phi v^\lambda, v^{\kappa \lambda} \} = \left( \sum_{s \in S^\lambda} \frac{c_{s,\lambda}}{p(s, w_0)} \right) e^{(\lambda - \kappa \lambda, w_0)},$$

where $S^\lambda = \{ s \in S | \varphi(s) = \lambda - \kappa \lambda \}$, and $c_{s,\lambda} = \{ e^\varphi(s) v^\lambda, v^{\kappa \lambda} \}$. It is known that coefficient in the parentheses is nonzero by [Kos79] Prop. 5.9.1.

The following theorem is the main result of this section. It provides the asymptotic behavior at $\infty$ of our solutions and the quantization result for our integrals.

**Theorem 8.2.** The solutions $U_i$ in (1.8) satisfy that

$$U_i = 2(\gamma_i - <\omega_i - \kappa \omega_i, w_0>) \log |z| + O(1), \quad as \ z \to \infty,$$

where $\omega_i$ is the $i$th fundamental weight. Consequently, the solutions $u_i$ in (1.9) satisfy that

$$u_i = -2(2 - \kappa \gamma_i) \log |z| + O(1), \quad as \ z \to \infty,$$

where $-\kappa \omega_i = \alpha_k$, then $-\kappa \gamma_i = \gamma_k$.

The finite integrals in (1.1) are quantized as

$$\sum_{j=1}^n a_{ij} \int_{\mathbb{R}^2} e^{u_j} dx = \pi (\mu_i - \kappa \mu_i),$$

where $-\kappa \mu_i = \langle -\kappa \omega_i, w_0 \rangle$. Therefore we have

$$\sum_{j=1}^n a_{ij} \int_{\mathbb{R}^2} e^{u_j} dx = 2\pi (1 + \gamma_i), \quad 1 \leq i \leq n,$$
except the following cases:

(i) When $g = A_n$, we recover the quantization in [LWY12]

\[
\sum_{j=1}^{n} a_{ij} \int_{\mathbb{R}^2} e^{\nu_i} \, dx = \pi(2 + \gamma_i + \gamma_{n+1-i}), \quad 1 \leq i \leq n.
\]

(ii) When $g = D_{2n+1}$ and $i = 2n$ or $2n + 1$, we have

\[
- \int_{\mathbb{R}^2} e^{\nu_{2n+1}} \, dx + 2 \int_{\mathbb{R}^2} e^{\nu_{2n}} \, dx = \pi(2 + \gamma_2n + \gamma_{2n+1}),
\]

\[
- \int_{\mathbb{R}^2} e^{\nu_{2n-1}} \, dx + 2 \int_{\mathbb{R}^2} e^{\nu_{2n}} \, dx = \pi(2 + \gamma_{2n} + \gamma_{2n+1}).
\]

(iii) When $g = E_6$ and using the labeling in Appendix Subsection A.5 we have

\[
\sum_{j=1}^{6} a_{ij} \int_{\mathbb{R}^2} e^{\nu_i} \, dx = \pi(2 + \gamma_1 + \gamma_6), \quad i = 1, 6,
\]

\[
\sum_{j=1}^{6} a_{ij} \int_{\mathbb{R}^2} e^{\nu_i} \, dx = \pi(2 + \gamma_3 + \gamma_5), \quad i = 3, 5.
\]

Proof. To show the asymptotic behavior (8.3), we prove

\[
e^{-U_i} = |z|^{-2\gamma_i} |z|^{2(\omega_i - \kappa\omega_i, w_0)} \langle c_i + o(1) \rangle \quad \text{as } z \to \infty.
\]

Recall our previous discussions in (6.16) and (6.17). By Proposition 6.11 and $\mu_i > 0$ from (6.4), we see that the highest power of $z$ appears in the norm squared of $\{\Lambda C\Phi v^{\omega_i}, v^{\omega_i}\}$, that is, the coordinate of $\Lambda C\Phi v^{\omega_i}$ in the lowest weight vector $v^{\omega_i}$. Furthermore, by (A.19) we have

\[
\{\Lambda C\Phi v^{\omega_i}, v^{\omega_i}\} = \{\Phi v^{\omega_i}, C^* \Lambda v^{\omega_i}\} = \Lambda^{\omega_i}(\{\Phi v^{\omega_i}, v^{\omega_i}\} + \text{lower order terms})
\]

since $C^* \in N_+$, where $\Lambda^{\omega_i} = \exp(\langle \kappa\omega_i, H \rangle)$ if $\Lambda = \exp(H)$ with $H \in \mathfrak{h}$. Using (6.1), we see that (8.10) holds with

\[
c_i = (\Lambda^{\omega_i})^2 \left| \sum_{s \in S_{w_0}} \frac{c_{s, \omega_i}}{p(s, w_0)} \right|^2,
\]

By $u_i = \sum_{j=1}^{n} a_{ij} U_j$, $\gamma_i = \sum_{j=1}^{n} a_{ij} \gamma_j$, $a_i = \sum_{j=1}^{n} a_{ij} \omega_j$, and $\langle a_i, w_0 \rangle = \mu_i$, and assuming that $-\kappa a_i = \alpha_k$ for some $1 \leq k \leq n$, we see from (8.3)

\[
u_i = \sum_{j=1}^{n} a_{ij} U_j = 2(\gamma_i - \langle a_i, \kappa a_i, w_0 \rangle) \log |z| + O(1)
\]

\[= 2(\gamma_i - \mu_i - \mu_k) \log |z| + O(1) = -2(2 + \gamma_k) \log |z| + O(1)
\]

\[= 2(2 - \kappa \gamma_i) \log |z| + O(1), \quad \text{as } z \to \infty,
\]

which is (8.4).

Integrating (1.5) and using (8.3), we have

\[
4 \int_{\mathbb{R}^2} e^{\nu_i} \, dx = 4\pi \gamma_i \lim_{R \to \infty} \int_{\partial B_R} \frac{\partial U_i}{\partial n} \, ds
\]

\[= 4\pi \langle \omega_i, \kappa\omega_i, w_0 \rangle.
\]
As in the proof of (8.4), the linear combinations of the above using the Cartan matrix clearly give (8.5), which implies the more concrete version (1.10) in Theorem 1.6.

Except three cases, \(-\kappa = \text{Id}\) and so (8.6) follows immediately. The three exceptions are for the Lie algebras of type \(A_n\), \(D_{2n+1}\) and \(E_6\), where \(-\kappa\) represents the symmetry of the Dynkin diagram of the Lie algebra. In the \(A_n\) case,

\[-\kappa \alpha_i = \alpha_{n+1-i}, \quad 1 \leq i \leq n,
\]

that is, \(-\kappa\) is the reflection of the Dynkin diagram about its center. Therefore we have (8.7), which is the quantization result in [LWY12, Theorem 1.3], up to the factor of 4.

In the \(D_{2n+1}\) case, we have

\[-\kappa \alpha_{2n} = \alpha_{2n+1}, \quad -\kappa \alpha_{2n+1} = \alpha_{2n}, \quad -\kappa \alpha_i = \alpha_i, \quad 1 \leq i \leq 2n-1,
\]

that is, \(-\kappa\) permutes the last two roots and preserves the others. This proves (8.8).

In the \(E_6\) case, using the labeling in Appendix Subsection A.5 we have

\[-\kappa \alpha_1 = \alpha_6, \quad -\kappa \alpha_6 = \alpha_1,
-\kappa \alpha_3 = \alpha_5, \quad -\kappa \alpha_5 = \alpha_3,
-\kappa \alpha_2 = \alpha_2, \quad -\kappa \alpha_4 = \alpha_4.
\]

This then proves (8.9). \(\Box\)

9. Examples and relations with previous results

For classical groups of types \(A\), \(B\), \(C\), and \(D\), we discuss the Lie-theoretic setups in more concrete terms in order to make our result more explicit especially to the analysts. We present the examples of \(A_2\) Toda systems and relate them with the previous results in [LWY12]. Then we will present the example of \(D_4\) to illustrate our new result in this paper.

It is known [FH91] that the Lie groups \(SL_{n+1}\mathbb{C}\) and \(Sp_{n}\mathbb{C}\) are simply-connected, but \(\pi_1(SO_m\mathbb{C}) \cong \mathbb{Z}/2\) for \(m \geq 3\) and its universal cover is \(Spin_m\mathbb{C}\). However, as discussed in Appendix Subsection A.8 all the calculations can be done in the simpler \(SO_m\mathbb{C}\).

For an element \(g\) in these classical groups, \(g^* = \bar{g}^t\) is the conjugate transpose. The compact subgroups \(K \subset G\) are characterized by \(g^{-1} = g^*\), and they are the intersections of \(G\) with the corresponding unitary groups.

Clearly for \(SL_{n+1}\mathbb{C}\), the nilpotent subgroup \(N\) consists of unipotent lower-triangular matrices with complex entries, and the abelian subgroup \(A\) consists of diagonal matrices with positive entries that multiply to 1.

We choose the symmetric and skew-symmetric bilinear forms \(\kappa_m\) and \(\Omega_{2n}\) for \(SO_m\mathbb{C}\) and \(Sp_{2n}\mathbb{C}\) as

\[
\kappa_m = \left( \begin{array}{ccc} 1 & & \\ & \ddots & \\ & & 1 \end{array} \right)_{m \times m}, \quad \Omega_{2n} = \left( \begin{array}{cc} -\kappa_n & \kappa_n \\ \kappa_n & -\kappa_n \end{array} \right).
\]

Then the nilpotent subgroups \(N \subset G\) in these cases consist of unipotent lower-triangular matrices with complex entries, which preserve the corresponding bilinear forms. The abelian subgroups \(A \subset G\) consist of diagonal matrices with positive
entries whose symmetric pairs with respect to the secondary diagonal multiply to 1.

The first fundamental representations of these classical groups are just their standard representations, that is, the standard actions of $SL_{n+1} \mathbb{C}$ on $\mathbb{C}^{n+1}$, $SO_n \mathbb{C}$ on $\mathbb{C}^n$, and $Sp_{2n} \mathbb{C}$ on $\mathbb{C}^{2n}$. The $i$th fundamental representations are the irreducible representations with the same highest weights as in the $i$th wedge products of the standard representations, except that at the end there is 1 spin representation for $B_n$ and two half-spin representations for $D_n$. Therefore the highest matrix coefficients $\langle i | \cdot | i \rangle$ are just the leading principal minors of rank $i$ of the matrices in the standard representations, except for the spin representations where one needs to do more work.

**Example 9.2 (A$_2$ Toda system).** The group is $G = SL_3 \mathbb{C}$, and the solution to

$$
\Phi^{-1} \Phi_z = \sum_{i=1}^{2} z^{\gamma_i} e_{-\alpha_i} = \begin{pmatrix}
0 & z^{\gamma_1} & 0 \\
z^{\gamma_2} & 0 & 0
\end{pmatrix}
$$

is $\Phi(0) = Id$

$$
\Phi(z) = \begin{pmatrix}
1 & \frac{z^{\mu_1}}{\mu_2 (\mu_1 + \mu_2)} & 1 \\
\frac{z^{\mu_1}}{\mu_1 (\mu_1 + \mu_2)} & 1 & \frac{z^{\nu_2}}{\mu_2} \\
\frac{z^{\nu_2}}{\mu_2} & \frac{z^{\nu_2}}{\mu_2} & 1
\end{pmatrix}.
$$

General elements $\Lambda \in A$ and $C \in N$ are of the form

$$
\Lambda = \begin{pmatrix}
\lambda_0 & \lambda_1 & \lambda_2 \\
\lambda_1 & \lambda_2 & 0
\end{pmatrix},
$$

$$
C = \begin{pmatrix}
c_0 & c_1 & 1 \\
c_1 & c_2 & 1
\end{pmatrix},
$$

where $\lambda_i > 0$ and $\lambda_0 \lambda_1 \lambda_2 = 1$. Then

$$
X := \Lambda C \Phi = \begin{pmatrix}
\lambda_0 \\
\lambda_1 (\frac{z^{\mu_1}}{\mu_1} + c_0) \\
\lambda_2 (\frac{z^{\mu_1}}{\mu_2 (\mu_1 + \mu_2)} + c_1 \frac{z^{\mu_1}}{\mu_1} + c_0)
\end{pmatrix} \begin{pmatrix}
\lambda_1 & \lambda_2 & 0 \\
\frac{z^{\nu_2}}{\mu_2} & \frac{z^{\nu_2}}{\mu_2} & 1
\end{pmatrix}.
$$

Therefore by Theorem 1.6 we see that

$$
e^{-U_1} = |z|^{-2 \gamma_1} \langle 1 | X^* X | 1 \rangle = |z|^{-2 \gamma_1} (X^* X)_{1,1}
$$

(9.3)

$$
= |z|^{-2 \gamma_1} \left( \lambda_0^2 + \lambda_1^2 \left( \frac{z^{\mu_1}}{\mu_1} + c_0 \right)^2 + \lambda_2^2 \left( \frac{z^{\mu_1}}{\mu_1 (\mu_1 + \mu_2)} + c_1 \frac{z^{\mu_1}}{\mu_1} + c_2 \right)^2 \right).
$$

This clearly matches with the main Theorem 1.1 in [LWY12], noting that our $\gamma^1$ is their $\alpha_1$, after some suitable correspondence.

The condition in [LWY12] Theorem 1.1 that if $\gamma_{j+1} + \cdots + \gamma_i \notin \mathbb{Z}$ for some $j < i$, then $c_{ij} = 0$ exactly matches our current condition that $C \in N_F$.

Similarly for Toda systems of type $C$ and $B$, our current results are related to those in [Nie16]. Now let us present the $D_4$ example in some detail, since this illustrates our new result.

**Example 9.4 (D$_4$ Toda system).** For simplicity, we present this example using concrete numbers. Let $\gamma_1 = -\frac{1}{4}, \gamma_2 = 1, \gamma_3 = 2, \gamma_4 = 3$, and then $\gamma^1 = 3, \gamma^2 = \frac{11}{2}, \gamma^3 = \frac{17}{4}, \gamma^4 = \frac{19}{4}$ by $\gamma^i = \sum_{j=1}^{n} a^{ij} \gamma_j$.
The solution to
\[
\Phi^{-1}\Phi z = \sum_{i=1}^{4} z^{\gamma_i} e^{-\alpha_i} = \begin{pmatrix} 2 \sqrt{2} \\ 0 \\ z \\ z^2 \\ z^3 \\ -z^3 - z^2 \\ -z \end{pmatrix}
\]
is
\[
\Phi(0) = \text{Id}
\]
\[
\Phi(z) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\sqrt{2} & 1 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{z} & \frac{1}{z^2} & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{z^4} & \frac{1}{z^5} & \frac{1}{z^6} & 1 & 0 & 0 & 0 \\
\frac{1}{z^8} & \frac{1}{z^9} & \frac{1}{z^{10}} & \frac{1}{z^{11}} & 1 & 0 & 0 \\
\frac{1}{z^{13}} & \frac{1}{z^{14}} & \frac{1}{z^{15}} & \frac{1}{z^{16}} & \frac{1}{z^{17}} & 1 & 0 \\
-\frac{64}{125} & -\frac{64}{125} & -\frac{64}{125} & -\frac{64}{125} & -\frac{64}{125} & -\frac{64}{125} & -2\sqrt{2}
\end{pmatrix}
\]

General elements \( A \in A \) and \( C \in N \) are of the form
\[
A = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_4, \lambda_3^{-1}, \lambda_2^{-1}, \lambda_1^{-1}), \quad \lambda_i > 0,
\]
\[
C = \begin{pmatrix}
c_{21} & 1 \\
c_{31} & c_{32} & 1 \\
c_{41} & c_{42} & c_{43} & 1 \\
c_{51} & c_{52} & c_{53} & 0 & 1 \\
c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & 1 \\
c_{71} & c_{72} & c_{73} & c_{74} & c_{75} & c_{76} & 1 \\
c_{81} & c_{82} & c_{83} & c_{84} & c_{85} & c_{86} & c_{87} & 1
\end{pmatrix}
\]

Solving \( C^t \kappa C = \kappa \) (see [14]), we see that the coordinates are the \( c \)'s above the secondary diagonal, that is, the \( c_{ij} \) for \( j < i \leq 8 - j, \ 1 \leq j \leq 3 \), and the other \( c \)'s are solved in these.

Furthermore, for the current \( \gamma \)'s, \( \alpha_1 \) in Remark [13] is not an integer if \( \alpha \) contains \( \alpha_1 \). Therefore the subgroup \( N \) in our case is obtained by letting \( c_{11} = 0 \) for \( 2 \leq i \leq 7 \), which are coordinates corresponding to the roots \( \alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 \).

The solution space in this case has dimension \( 16 = 4 + 2 \cdot 6 \) and is parametrized by the positive numbers \( \lambda_1, \ldots, \lambda_4 \) and the complex numbers \( c_{32}, c_{43}, c_{53}, c_{42}, c_{52}, c_{62} \), which are coordinates corresponding to the roots \( \alpha_2, \alpha_3, \alpha_4, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4 \).

Therefore with \( X = \Lambda C \Phi \), we see that
\[
e^{-U_1} = |z|^{-2\gamma_1}(X^*X)_{1,1},
\]
\[
e^{-U_2} = |z|^{-2\gamma_2}(X^*X)_{[1,2],[1,2]},
\]
where \( U_2 \) involves the leading principal \( 2 \times 2 \) minor.
Now to determine $U_3$ and $U_4$, we need the 3rd and the 4th fundamental representations of $D_4$, which are the two half-spin representations. (There is the triality, the symmetry for $\alpha_1$, $\alpha_3$ and $\alpha_4$ in the case of $D_4$, but we disregard that since that symmetry is not valid for higher $D_n$.) Then $U_3$ and $U_4$ are expressed in terms of the highest matrix coefficient of $X^*X$ for the spin representations. These can be concretely computed, with the help of a computer algebra system such as Maple, by the Lie algebra spin representations and the exponential map. (See [FH91] for more details on the spin representations.)

Appendix A. Background and setup from Lie theory

In this appendix, we provide the details for the Lie-theoretic background and setup needed in this paper. Our basic references are [Kna02, Hel78, FH91].

A.1. Cartan subalgebra and root space decomposition. Let $g$ be a complex simple Lie algebra. The Killing form $B(X,Y) = \text{Tr}(\text{ad}_X \text{ad}_Y)$ is a symmetric nondegenerate bilinear form on $g$, where $\text{ad}_X : g \to g; Z \mapsto [X,Z]$ is the adjoint action of $X$. Let $h$ be a fixed Cartan subalgebra, whose dimension $n$ is the rank of the Lie algebra.

Let $g = h \oplus \bigoplus_{\alpha \in \Delta} g_{\alpha}$ be the root space decomposition of $g$ with respect to $h$, where $\Delta$ denotes the set of roots. The roots are linear functions on the Cartan subalgebra $h$, and for $X_{\alpha} \in g_{\alpha}$ and $H \in h$, we have $[H, X_{\alpha}] = \alpha(H) X_{\alpha}$. It is known that $\text{dim}_C g_{\alpha} = 1$ for $\alpha \in \Delta$.

Let $\Delta = \Delta^+ \cup \Delta^-$ be a fixed decomposition of the set of roots into the sets of positive and negative roots, and let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be the set of positive simple roots.

We furthermore introduce the following standard subalgebras of $g$:

$$n = n_- = \bigoplus_{\alpha \in \Delta^+} g_{-\alpha}, \quad n_+ = \bigoplus_{\alpha \in \Delta^+} g_{\alpha}, \quad b_+ = h \oplus n_+.$$ (A.1)

A.2. Gauss decomposition. Let $G$ be a connected complex Lie group whose Lie algebra is $g$. Let $H$ be the Cartan subgroup of $G$ corresponding to $h$. Denote the subgroups of $G$ corresponding to $n = n_-$, $n_+$ and $b_+$ in (A.1) by $N = N_-, N_+$ and $B_+$. Here $B_+$ is called a Borel subgroup of $G$. Then by the Gauss decomposition (see [LS92, Eq. (1.5.6)] and [Kos79, Eq. (2.4.4)]), there exists an open and dense subset $G_r$ of $G$, called the regular part, such that

$$G_r = N_- N_+ H.$$ (A.2)

We note that $HN_+ = N_+ H$ by $hn = (hnh^{-1})h$, where $h \in H$ and $n \in N_+$. Clearly $G_r$ contains the identity element of $G$.

A.3. Split and compact real forms. By [Kna02, Theorem 6.6 and Corollary 6.10], the complex Lie algebra $g$ has a split real form

$$g_0 = h_0 \oplus \bigoplus_{\alpha \in \Delta} \mathbb{R} e_{\alpha},$$ (A.3)

where $h_0 = \{H \in h | \alpha(H) \in \mathbb{R}, \forall \alpha \in \Delta\}$, and the $e_{\alpha} \in g_{\alpha}$ form a Cartan-Weyl basis.

There exists a basis $\{H_{\alpha_i}\}_{i=1}^n$ of $h_0$ such that

$$B(H_{\alpha_i}, H) = \alpha_i(H), \quad \forall H \in h.$$ (A.4)
The positive definite Killing form $B$ on $\mathfrak{h}_0$ also introduces an inner product on the real dual space $\mathfrak{h}_0^* = \text{Hom}_\mathbb{R}(\mathfrak{h}_0, \mathbb{R})$ by $(\alpha_i, \alpha_j) = B(H_{\alpha_i}, H_{\alpha_j})$ for $1 \leq i, j \leq n$.

The split real form (A.3) defines a Cartan decomposition of the Lie algebra into vector subspaces
\begin{equation}
\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}, \quad \text{where} \quad \mathfrak{a} = \mathfrak{h}_0, \quad \mathfrak{n} = \mathfrak{n}^-.
\end{equation}
Then we have the corresponding Iwasawa decomposition on the group level [Kna02, Theorem 6.46]
\begin{equation}
G = KAN,
\end{equation}
where $K, A$ and $N$ are the subgroups in $G$ corresponding to $\mathfrak{t}, \mathfrak{a}$ and $\mathfrak{n}$. The subgroup $K$ is compact, the subgroup $A$ is abelian, and the subgroup $N$ is nilpotent. The groups $A$ and $N$ are simply-connected. (The usual version using $\mathfrak{n}_+ = \oplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ implies our current version using $\mathfrak{n}^-$ after one application of the Cartan involution $\theta$.)

The Cartan involution $\theta$ on $\mathfrak{g}$ also lifts to the group $G$ [Kna02, Theorem 6.31], and we continue to denote it by the same notation. Then $K = G^\theta$ is the subgroup fixed by $\theta$. For $g \in G$, define $g^* = (g^\theta)^{-1}$. Then $(gh)^* = h^*g^*$, and an element $F \in K$ if and only if $F^*F = I_d$. Furthermore, from (A.6) we see that $(N_+)^* = N_-.$

For the purpose of doing representation theory, we let $G^s$ be a connected and simply-connected Lie group with Lie algebra $\mathfrak{g}$. Then it is known that $G^s$ is the universal covering of a general $G$ whose Lie algebra is $\mathfrak{g}$. Let $G^s = K^sA^sN^s$ be its Iwasawa decomposition, while $G = KAN$ is the Iwasawa decomposition of $G$. Then since $A$ and $N$ are simply-connected, they are isomorphic to $A^s$ and $N^s$ respectively. Only $K^s$ is a covering of $K$.

A.5. Cartan matrices and classification of simple Lie algebras. We further specify the normalization of the $\varepsilon_{\alpha_i}$ and $\varepsilon_{-\alpha_i}$ for $1 \leq i \leq n$ in (A.3) by requiring that
\begin{equation}
\alpha_i(h_{\alpha_i}) = 2, \quad \text{where} \quad h_{\alpha_i} = [\varepsilon_{\alpha_i}, \varepsilon_{-\alpha_i}].
\end{equation}
By [Kna02, Eq. (2.93)], the relation of the $h_{\alpha_i}$ with the basis \([A.4]\) is that

\[
(A.9) \quad h_{\alpha_i} = \frac{2H_{\alpha_i}}{B(H_{\alpha_i}, H_{\alpha_i})} = \frac{2H_{\alpha_i}}{(\alpha_i, \alpha_i)}.
\]

The benefit of the choice of the $h_{\alpha_i}$ is that the Lie subalgebra generated by $e_{\alpha_i}, h_{\alpha_i}$, and $e_{-\alpha_i}$ is isomorphic to a copy of $\mathfrak{sl}_2$ with the standard basis

\[
e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

Then the Cartan matrix $(a_{ij})$ of $\mathfrak{g}$ is defined by

\[
(A.10) \quad a_{ij} = \alpha_i(h_{\alpha_j}) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}, \quad 1 \leq i, j \leq n.
\]

There are four infinite series of classical complex simple Lie algebras and five exceptional Lie algebras with the following Cartan matrices

\[
A_n = \mathfrak{sl}_{n+1} : \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \ddots \\ & & & -1 & 2 \end{pmatrix}, \quad B_n = \mathfrak{so}_{2n+1} : \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \\ & & & -1 & 2 \end{pmatrix},
\]

\[
C_n = \mathfrak{sp}_{2n} : \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \\ & & & -1 & 2 \end{pmatrix}, \quad D_n = \mathfrak{so}_{2n} : \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \\ & & & -1 & 2 \end{pmatrix},
\]

\[
G_2 : \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \quad F_4 : \begin{pmatrix} 2 & -1 & -2 & -1 \\ -1 & 2 & -1 & 2 \\ -1 & -2 & 1 & 2 \\ -2 & 1 & -2 & 1 \end{pmatrix},
\]

\[
E_6 : \begin{pmatrix} 2 & -1 & 1 & -1 & 1 & 1 \\ -1 & 2 & -1 & 2 & -1 & 1 \\ -1 & -2 & 1 & 2 & -1 & 1 \\ -1 & 2 & -1 & 2 & -1 & 1 \\ 1 & -2 & 1 & 2 & -1 & 1 \\ 1 & 2 & -1 & 2 & -1 & 1 \end{pmatrix}, \quad E_7 : \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ -1 & -2 & 1 & 2 & & & \\ -1 & 2 & -1 & 2 & -1 & & \\ -1 & -2 & 1 & 2 & -1 & 2 & \\ -2 & 1 & -2 & 1 & -2 & 1 & 1 \\ -2 & 1 & -2 & 1 & -2 & 1 & 1 \end{pmatrix},
\]

\[
E_8 : \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ -1 & -2 & 1 & 2 & & & & \\ -1 & 2 & -1 & 2 & -1 & & & \\ -1 & -2 & 1 & 2 & -1 & 2 & & \\ -1 & 2 & -1 & 2 & -1 & 2 & 1 & \\ -1 & -2 & 1 & 2 & -1 & 2 & 1 & 1 \\ 1 & -2 & 1 & 2 & -1 & 2 & 1 & 1 \end{pmatrix}.
\]

In the above, the labelling of the roots for the exceptional Lie algebras follows [Kna02, pp 180-1]. We require that $n \geq 2$ for $B_n$ and $C_n$, and $n \geq 3$ for $D_n$. Furthermore, we have the following isomorphisms

\[
B_2 \cong C_2, \quad D_3 \cong A_3.
\]

A.6. **Principal grading.** There exist $E_j \in \mathfrak{h}_0$ such that

\[
(A.11) \quad \alpha_i(E_j) = \delta_{ij}, \quad \text{for } 1 \leq i, j \leq n.
\]

By \([A.10]\), we have $E_j = \sum_{k=1}^{n} a^{kj} h_{\alpha_k}$, where $(a^{kj})$ is the inverse Cartan matrix. Define the so-called principal grading element

\[
(A.12) \quad E_0 = \sum_{j=1}^{n} E_j \in \mathfrak{h}_0, \quad \text{such that } \alpha_i(E_0) = 1, \quad \text{for } 1 \leq i \leq n.
\]
Define $g_j = \{x \in g \mid [E_0, x] = jx\}$. Then
\begin{equation}
(A.13) \quad g = \bigoplus_j g_j
\end{equation}
is the principal grading of $g$, and we have
\begin{equation}
(A.14) \quad g_{-1} = \bigoplus_{i=1}^n g_{-\alpha_i}.
\end{equation}

A.7. Representation spaces. The weight lattice of $g$ is $\Lambda_W = \{\beta \in h' \mid \beta(h_{\alpha_i}) \in \mathbb{Z}, \forall 1 \leq i \leq n\}$. A weight $\beta$ is called dominant if $(\beta, \alpha_i) \geq 0$ for all $1 \leq i \leq n$. The weight lattice is a lattice generated by the fundamental weights $\omega_i$ for $1 \leq i \leq n$ such that
\begin{equation}
\omega_i(h_{\alpha_j}) = \frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}.
\end{equation}

An irreducible representation $\rho$ of $g$ on a finite-dimensional complex vector space $V$ has the weight space decomposition $V = \bigoplus V_{\beta}$, where $\beta \in \Lambda_W$ and $\rho(H)(v) = \beta(H)v$ for $H \in h$ and $v \in V_{\beta}$. We have
\begin{equation}
(A.15) \quad \rho(g_{\alpha})V_{\beta} \subset V_{\alpha+\beta}.
\end{equation}
There exists a unique highest weight $\lambda$ with a one-dimensional highest weight space $V_{\lambda}$ such that $\rho(n_+)V_{\lambda} = 0$. All the weights of $V$ are of the form $\lambda - \sum_{i=1}^n m_i \alpha_i$ where the $m_i$ are nonnegative integers.

The Theorem of the Highest Weight [Kna02 Theorem 5.5] asserts that up to equivalence, the irreducible finite-dimensional complex representations of $g$ stand in one-one correspondence with the dominant weights which sends an irreducible representation to its highest weight. We denote the irreducible representation space corresponding to a dominant weight $\lambda$ by $V_{\lambda}$.

There is a canonical pairing between the dual space $V^* = \text{Hom}(V, \mathbb{C})$ and $V$ denoted by $\langle w, v \rangle \in \mathbb{C}$ with $v \in V$ and $w \in V^*$. $V^*$ has a right representation $\rho^*$ of $g$ defined by
\begin{equation}
(A.16) \quad \langle wp^*(X), v \rangle = \langle w, \rho(X)v \rangle, \quad X \in g.
\end{equation}
The representation corresponding to the $i$th fundamental weight $\omega_i$ is called the $i$th fundamental representation of $g$, which we denote by $V_i$. We choose a highest weight vector in $V_i$, and following the physicists [LS92] we called it by $|i\rangle$. We choose a vector $|i\rangle$ in the lowest weight space in $V_i^*$ and require that $\langle i|Id|i\rangle = 1$ for the identity element $Id \in G$. For simplicity, we will omit the notation $\rho$ for the representation.

We have (see [LS92 Eq. (1.4.19)])
\begin{equation}
(A.17) \quad X|i\rangle = 0, \forall X \in n_+; \quad h_{\alpha_j}|i\rangle = \delta_{ij}|i\rangle; \quad \text{and} \quad e_{-\alpha_j}|i\rangle = 0, \forall j \neq i.
\end{equation}
That is, in the $i$th fundamental representation, only $e_{-\alpha_i}$ may bring the highest weight vector down. Similarly we have $\langle i|Y = 0$ for $Y \in n = n_-$, and $\langle i|e_{\alpha_j} = 0$ if $j \neq i$.

A.8. Lift the representation to the group. Let $G^*$ be a connected and simply-connected Lie group whose Lie algebra is $g$. Then all the irreducible representations of $g$ lift to representations of $G^*$, and in particular the fundamental representations $V_i$ do. For $g \in G^*$, the pairing $\langle i|g|i\rangle$ is called the highest matrix coefficient [LS92 p. 45] because it is the matrix coefficient for the highest weight vector.
It is clear that \( g|i| = |i| \) for \( g \in N_+ \), \( \langle i | g = \langle i | \) for \( g \in N_- \), and \( \exp(H)|i| = e^{\omega(H)}|i| \) for \( H \in \mathfrak{h} \) from the corresponding Lie algebra facts in Subsection A.7.

The universal enveloping algebra of \( \mathfrak{g} \) is defined as
\[
U(\mathfrak{g}) = T(\mathfrak{g})/J,
\]
where \( T(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} T^k(\mathfrak{g}) \) is the tensor algebra, and \( J \) is the two-sided ideal generated by all \( X \otimes Y - Y \otimes X - [X,Y] \) with \( X \) and \( Y \) in \( T^1(\mathfrak{g}) \). A representation of \( \mathfrak{g} \) also leads to a representation for the universal enveloping algebra \( U(\mathfrak{g}) \) [Kna02].

For \( \mu, \nu \in U(\mathfrak{g}) \) and \( g \in G^s \), \( \langle i | \nu g u | i \rangle \) denotes the pairing of \( \langle i | \nu \) in \( V_i^* \) with \( g(\mu | i) \) in \( V_i \).

In our main Theorem 1.6, we can work with a general Lie group \( G \) whose Lie algebra is \( \mathfrak{g} \) instead of only the simply-connected \( G^s \). The reason is that the simply-connected compact subgroup \( K^s \) of \( G^s \) is used only in passing. Our results are expressed using \( N \) and \( A \), and they are simply-connected and the same for a general \( G \) and for the simply-connected \( G^s \) (see Subsection A.3).

In Sections 6 and 8 on asymptotic behaviors and quantization, we also need the following setup from [Kos79] §5.7. Let \( \mathbb{C}[G^s] \) be the group algebra of \( G^s \) where \( G^s \) is regarded as an abstract group. Let \( D(G^s) = \mathbb{C}[G^s] \otimes U(\mathfrak{g}) \).

Let \( D(G^s)' \) be the space of all representative functionals on \( D(G^s) \). We endow \( D(G^s) \) with the weak * topology so that \( D(G^s)' \) is its continuous dual. One notes then, for example, that the series \( \sum_{j=0}^{\infty} p_j \) converges to \( \exp^s x \) for any \( x \in \mathfrak{g} \). (For details, see [Kos79] p. 278.) Let \( \hat{D}(G^s) \) be its completion.

Also let \( D(N) = D(N_-) \) be the subalgebra of \( D(G^s) \) generated by \( N = N_- \) and \( \mathfrak{n} = \mathfrak{n}_- \), and let \( \hat{D}(N) = \hat{D}(N_-) \) be the closure of \( D(N) \) in \( \hat{D}(G^s) \). For its importance to us, we cite the following proposition from [Kos79].

Proposition A.18 ([Kos79] Prop. 5.7.1]). For every \( k \in \mathbb{Z}_+ \) choose an arbitrary element \( u_k \in U(\mathfrak{n})_k = \{ u \in U(\mathfrak{n}) | [u, E_0] = ku \} \) with \( E_0 \in \mathfrak{h} \) from (A.12). Then the infinite sum \( \sum_{k \in \mathbb{Z}_+} u_k \) converges in \( \hat{D}(N) \). Furthermore any \( u \in D(N) \) can be uniquely written as an infinite sum
\[
 u = \sum_{k \in \mathbb{Z}_+} u_k,
\]
where \( u_k \in U(\mathfrak{n})_k \).

Our Proposition 6.11 is one instance of this proposition, where an element in \( N \) is expressed in \( U(\mathfrak{n}) \).

A.9. Invariant Hermitian forms. There is a more concrete realization of the dual \( V_i^* \) in Subsection A.7. By the unitary trick, there exists a Hermitian form \( \langle \cdot, \cdot \rangle \) on \( V_i \) (conjugate linear in the second position) invariant under the compact group \( K^s \) of a simply-connected \( G^s \). The important property of this Hermitian form is that [Kos79] Eq. (5.11)]
\[
(A.19) \quad \{ g u, v \} = \{ u, g^* v \}, \quad g \in G^s, \ u, v \in V_i.
\]

Therefore we have an isomorphism \( V_i \to V_i^*; v \mapsto \{ , v \} \), where \( V_i \) is the vector space \( V_i \) with the conjugate scalar multiplication. Furthermore the right representation of \( G \) on \( V_i^* \) now corresponds to the right action \( \hat{V}_i \times G \to \hat{V}_i; (v, g) \to g^* v \) by comparing (A.16) and (A.19). Choose \( \omega_\psi \in V_i \) to be a highest weight vector for
the $i$th fundamental representation, and we require that $\{v^{\omega_i}, v^{\omega_i}\} = 1$. Then the term in (1.8) is, with $g = \Lambda C \Phi$,

\begin{equation}
\langle i | g^* g | i \rangle = \{g^* g v^{\omega_i}, v^{\omega_i}\} = \{g v^{\omega_i}, g v^{\omega_i}\} > 0.
\end{equation}

A.10. Weyl group. The Weyl group $W$ of a Lie algebra $\mathfrak{g}$ is the finite group generated by the reflections in the simple roots on $h_0^{\prime}$

\[ s_i(\beta) = \beta - \frac{2(\beta, \alpha_i)}{\langle \alpha_i, \alpha_i \rangle} \alpha_i, \quad 1 \leq i \leq n. \]

In the Weyl group, there is a unique element $\kappa \in W$ that is the longest element in the sense that when one writes it as a product of the simple reflections it has the maximal length. (Actually the maximal length is the number of positive roots.) This $\kappa$ on $h_0^{\prime}$ maps all the positives roots to the negative roots and vice versa.

The $\kappa$ is $-Id$ except in the following cases: $A_n$, $D_{2n+1}$ and $E_6$, where $-\kappa$ represents the outer-automorphism of the corresponding Lie algebra represented by the symmetry of its Dynkin diagram. (See [Dav08, Remark 13.1.8].)

Therefore $-\kappa \alpha_i$ is still a simple root and it is hence $\alpha_k$ for some $1 \leq k \leq n$. In fact $-\kappa \alpha_i = \alpha_i$ except the three cases mentioned above.

The weights of the irreducible representation $V^\lambda$ with a highest weight $\lambda$ are invariant under the Weyl group, and its lowest weight is $\kappa \lambda$.

A.11. Degrees of primitive adjoint-invariant functions on $\mathfrak{g}$. The degrees of the primitive homogeneous adjoint-invariant functions of the simple Lie algebras are listed as follows

| Lie algebras | degrees |
|-------------|---------|
| $A_n$       | $2, 3, \cdots, n + 1$ |
| $B_n$       | $2, 4, \cdots, 2n$ |
| $C_n$       | $2, 4, \cdots, 2n$ |
| $D_n$       | $2, 4, \cdots, 2n - 2, n$ |
| $G_2$       | $2, 6$ |
| $F_4$       | $2, 6, 8, 12$ |
| $E_6$       | $2, 5, 6, 8, 9, 12$ |
| $E_7$       | $2, 6, 8, 10, 12, 14, 18$ |
| $E_8$       | $2, 8, 12, 14, 18, 20, 24, 30$ |

References

[ALW15] Weiwei Ao, Chang-Shou Lin, and Juncheng Wei, *On Toda system with Cartan matrix $G_2$*, Proc. Amer. Math. Soc. **143** (2015), no. 8, 3525–3536.

[BFO+90] J. Balog, L. Fehér, L. O’Raifeartaigh, P. Forgács, and A. Wipf, *Toda theory and $W$-algebra from a gauged WZNW point of view*, Ann. Physics **203** (1990), no. 1, 76–136.

[BM91] H. Brezis and F. Merle, *Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions*, Comm. Partial Differential Equations **16** (1991), no. 8-9, 1223–1253.

[CL03] Chiun-Chuan Chen and Chang-Shou Lin, *Topological degree for a mean field equation on Riemann surfaces*, Comm. Pure Appl. Math. **56** (2003), no. 12, 1667–1727.

[CL91] Wen Xiong Chen and Congming Li, *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J. **63** (1991), no. 3, 615–622.

[Dav08] Michael W. Davis, *The geometry and topology of Coxeter groups*, London Mathematical Society Monographs Series, vol. 32, Princeton University Press, Princeton, NJ, 2008.
[Do97] Adam Doliwa, *Holomorphic curves and Toda systems*, Lett. Math. Phys. 39 (1997), no. 1, 21–32.

[DS84] V. G. Drinfel'd and V. V. Sokolov, *Lie algebras and equations of Korteweg-de Vries type*, Current problems in mathematics, Vol. 24, Itogi Nauki i Tekhniki, Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984, pp. 81–180.

[FOR+92] L. Fehér, L. O’Raifeartaigh, P. Ruelle, I. Tsutsui, and A. Wipf, *Generalized Toda theories and W-algebras associated with integral gradings*, Ann. Physics 213 (1992), no. 1, 1–20.

[FF96] Boris Feigin and Edward Frenkel, *Integrals of motion and quantum groups*, Integrable systems and quantum groups (Montecatini Terme, 1993), Lecture Notes in Math., vol. 1620, Springer, Berlin, 1996, pp. 349–418.

[FWB+89] P. Forgács, A. Wipf, J. Balog, L. Fehér, and L. O’Raifeartaigh, *Liouville and Toda theories as conformally reduced WZNW theories*, Phys. Lett. B 227 (1989), no. 2, 214–220.

[FH91] William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991. A first course; Readings in Mathematics.

[GH78] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

[Gue97] Martin A. Guest, *Harmonic maps, loop groups, and integrable systems* (Montecatini Terme, 1993), Lecture Notes in Math., vol. 1620, Springer, Berlin, 1996, pp. 349–418.

[GL14] Martin A. Guest and Chang-Shou Lin, *Analytic aspects of the tt* equations of Cecotti and Vafa*, J. Reine Angew. Math. 689 (2014), 1–32.

[Hel78] Sigurdur Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics, vol. 80, Academic Press, Inc. New York-London, 1978.

[JLW06] Jürgen Jost, Changshou Lin, and Guofang Wang, *Analytic aspects of the Toda system. II. Bubbling behavior and existence of solutions*, Comm. Pure Appl. Math. 59 (2006), no. 4, 526–558.

[JW01] Jürgen Jost and Guofang Wang, *Analytic aspects of the Toda system. I. A Moser-Trudinger inequality*, Comm. Pure Appl. Math. 54 (2001), no. 11, 1289–1319.

[JW02] Classification of solutions of a Toda system in $\mathbb{R}^2$, Int. Math. Res. Not. 6 (2002), 277–290.

[KW74] Jerry L. Kazdan and F. W. Warner, *Curvature functions for compact 2-manifolds*, Ann. of Math. (2) 99 (1974), 14–47.

[Kna02] Anthony W. Knapp, *Lie groups beyond an introduction*, 2nd ed., Progress in Mathematics, vol. 140, Birkhäuser Boston, Inc., Boston, MA, 2002.

[Kos59] Bertram Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math. 81 (1959), 797–1022.

[Kos63] Bertram Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. 85 (1963), 327–404.

[Kos79] Bertram Kostant, *The solution to a generalized Toda lattice and representation theory*, Adv. in Math. 34 (1979), no. 3, 195–338.

[Lez80] A. N. Leznov, *On complete integrability of a nonlinear system of partial differential equations in two-dimensional space*, Teoret. Mat. Fiz. 42 (1980), no. 3, 343–349.

[LS79] A. N. Leznov and M. V. Saveliev, *Representation of zero curvature for the system of nonlinear partial differential equations $x_{\alpha, z \bar{z}} = \exp(kx)_{\alpha}$ and its integrability*, Lett. Math. Phys. 3 (1979), no. 6, 489–494.

[LS92] Group-theoretical methods for integration of nonlinear dynamical systems, Progress in Physics, vol. 15, Birkhäuser Verlag, Basel, 1992. Translated and revised from the Russian; Translated by D. A. Leites.

[Lin14] Chang-Shou Lin, *Mean field equations, hyperelliptic curves and modular forms*, Proceedings of ICM Seoul III (2014), 331–344.

[LWY12] Chang-Shou Lin, JunCheng Wei, and Dong Ye, *Classification and nondegeneracy of $SU(n + 1)$ Toda system with singular sources*, Invent. Math. 190 (2012), no. 1, 169–207.

[Mal14] Andrea Malchiodi, *Liouville equations from a variational point of view*, Proceedings of ICM Seoul III (2014), 345–361.
Zhaohu Nie, Solving Toda field theories and related algebraic and differential properties, J. Geom. Phys. 62 (2012), no. 12, 2424–2442.

Zhaohu Nie, On characteristic integrals of Toda field theories, J. Nonlinear Math. Phys. 21 (2014), no. 1, 120–131.

Zhaohu Nie, Classification of solutions to Toda systems of types C and B with singular sources, Calc. Var. Partial Differential Equations 55 (2016), no. 3, 55:53.

Zhaohu Nie, Toda field theories and integral curves of standard differential systems, Journal of Lie Theory 27 (2017), no. 2, 377–395.

J. Prajapat and G. Tarantello, On a class of elliptic problems in $\mathbb{R}^2$: symmetry and uniqueness results, Proc. Roy. Soc. Edinburgh Sect. A 131 (2001), no. 4, 967–985.

R. W. Sharpe, Differential geometry, Graduate Texts in Mathematics, vol. 166, Springer-Verlag, New York, 1997. Cartan’s generalization of Klein’s Erlangen program; With a foreword by S. S. Chern.

Gabriella Tarantello, Selfdual gauge field vortices, Progress in Nonlinear Differential Equations and their Applications, 72, Birkhäuser Boston, Inc., Boston, MA, 2008. An analytical approach.

Gabriella Tarantello, Analytical, geometrical and topological aspects of a class of mean field equations on surfaces, Discrete Contin. Dyn. Syst. 28 (2010), no. 3, 931–973.

Marc Troyanov, Prescribing curvature on compact surfaces with conical singularities, Trans. Amer. Math. Soc. 324 (1991), no. 2, 793–821.

Nikolai Vavilov, A third look at weight diagrams, Rend. Sem. Mat. Univ. Padova 104 (2000), 201–250.

Yisong Yang, Solitons in field theory and nonlinear analysis, Springer Monographs in Mathematics, Springer-Verlag, New York, 2001.

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