Finite groups acting on coherent sheaves and Galois covers

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Abstract

Let $G$ be a finite group and $\rho : G \rightarrow \text{End}(E)$ be a group representation of $G$ on a coherent sheaf over an integral scheme. The purpose of this paper shall give a decomposition theorem of such representations in non-splitting components and apply this results to the study of Galois covers of a variety.

1 Introduction

In this paper by a variety we mean an integral separated scheme of finite type over an algebraically closed field $k$. All the varieties are projective. All the groups that we will consider are finite and $\text{char}(k) \nmid |G|$.

Let $X$ be a variety and $G$ be a finite group, let $\pi : E \rightarrow X$ be a vector bundle with $G$ as subgroup of $\text{Aut}_X(E)$. Then we have a representation $p^{-1}(x)$ of $G$ over $k$ for any closed point $x \in X$. This is a natural generalization of the representation theory of $G$ over $k$; in fact we recover this last when $X = \text{Spec}(k)$.

Now, to have a representation $G$ on the vector bundle is equivalent to have a structure of $\mathcal{O}_X[G]$-module in the sheaf of sections of $E$. Then in this paper we focus our attention to the abelian category of free torsion coherent sheaves with a group action. The main purpose of the first three section shall give a classification theorem of such objects.

The first motivation of that study is the relationship between Galois covers of varieties and representation theory over a field, for see this let $Y$ be a projective variety and $G$ be a finite group acting on $Y$, let $\pi : Y \rightarrow Y/G = X$ the quotient map. Then $\pi_*\mathcal{O}_Y$ is a coherent free torsion $\mathcal{O}_X$-module, such that, for any open set $U \subset X$, $\pi_*\mathcal{O}_Y(U)$ is an $\mathcal{O}_X(U)[G]$-module, furthermore in the generic point $\pi_*\mathcal{O}_{Y,\overline{\eta}}$ is isomorphic to $K_X[G]$ as representation of $G$ over the rational function field $K_X$ of $X$. In section 5, structure theorems for $\pi_*\mathcal{O}_X$ are given, many of them are generalizations from the cyclic case.

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2 General Theorems.

Let $A$ be a semisimple $k$-algebra (not necessary commutative) of finite dimension over $k$. It is very known that $A = e_1 A \oplus \ldots \oplus e_r A$ with $e_i A$ a simple algebra and $e_i$ idempotent, let $n_i = \text{dim}_k e_i A$.

**Definition 1** Let $(X, \mathcal{O}_X)$ be a ringed space. Define $\mathcal{O}_X(A) := \mathcal{O}_X \otimes_k A$, this is a sheaf of $\mathcal{O}_Y$-algebras, not necessary commutative, over $X$ but as an $\mathcal{O}_X$-module is free of rank $n = \text{dim}_k A$. An $\mathcal{O}_X(A)$-module is a pair consisting of an $\mathcal{O}_X$-module $E$ together with a $k$-morphism of rings $\rho : A \to \text{End}(E)$. A morphism of $\mathcal{O}_X(A)$-modules is a morphism $\phi : E \to F$ of $\mathcal{O}_X$-modules such that the next diagram commute for any $a \in A$

$$
\begin{array}{ccc}
E & \xrightarrow{\phi} & F \\
\rho(a) \downarrow & & \downarrow \rho(a) \\
E & \xrightarrow{\phi} & F
\end{array}
$$

In this case we say that $\phi$ is $A$-invariant.

**Remark and notation 1** Clearly from this definition, the kernel, cokernel, and image of a morphism of $\mathcal{O}_X(A)$-modules is again an $\mathcal{O}_X(A)$-module. Any direct sum, direct product, direct limit or inverse limit of $\mathcal{O}_X(A)$-modules is an $\mathcal{O}_X(A)$-module. If $x \in X$, $E_x$ have a natural structure of $\mathcal{O}_{X,x}(A)$-module. Observe, that this is an abelian category.

If $E$ and $F$ are $\mathcal{O}_X(A)$-modules, we denote by $\text{Hom}_A(E,F)$ the group formed by the morphisms of $\mathcal{O}_X(A)$-modules. Let $U \subset X$ be an open set of $X$ and $E,F$ two $\mathcal{O}_X(A)$-modules then we define by $\text{Hom}_A(E,F)$ the sheaf

$$
U \mapsto \text{Hom}_A(E|_U,F|_U)
$$

although, it is a sheaf of $\mathcal{O}_X$-modules, it is not necessarily of $\mathcal{O}_X(A)$-modules.

Recall that $A$ is a semisimple algebra, then we have only a finite number of simple $A$-modules, let $V_1, \ldots, V_r$ be this modules. Now is very known that for any finitely generated $A$-module $M$ it decompose in a direct sum $M_1 \oplus \ldots \oplus M_r$ where each $M_i$ is isomorphic to $V_i^{n_i}$, and there exist $\{e_1, \ldots, e_r\} \subset A$ such that $e_ie_j = e_\delta_{i,j}e_1 + \ldots + e_r = 1$ and $e_i M = M_i$.

**Proposition 1** If $M$ is an $\mathcal{O}_X(A)$-module, not necessarily finite generated, then $M = M_1 \oplus \ldots \oplus M_r$ where $M_i = e_i M$, furthermore if $M$ is locally free then each $M_i$ is locally free.

**Definition 2** Let $E$ an $\mathcal{O}_X(A)$-module, we define the isotypical decomposition of $E$, by the decomposition

$$
E = E_1 \oplus \ldots \oplus E_r
$$

obtained in the above proposition.
**Proposition 2** Let $E$ and $F$ be two $O_X(A)$-modules and $\phi : E \to F$ be an $O_X(A)$-morphism. If $e_1E \oplus ... \oplus e_rE$ and $e_1F \oplus ... \oplus e_rF$ are the isotypical decomposition of $E$ and $F$ respectively, then $\phi$ decompose in morphisms

$$\phi_i : e_iE \to e_iF$$


**Corollary 1** Let $E$ and $F$ be two $O_X(A)$-modules and $\phi : E \to F$ be an $O_X(A)$-morphism. Then $\text{Hom}_A(E, F) = \bigoplus \text{Hom}_A(e_iE, e_iF)$. ♦

**Corollary 2** Any exact sequence of $O_X(A)$-modules

$$... \to E_{i-1} \to E_i \to E_{i+1} \to ...,$$

decompose in exact sequences

$$... \to e_jE_{i-1} \to e_jE_i \to e_jE_{i+1} \to ...$$

with $j \in \{1, ..., r\}$. ♦

3  $O_X[G]$-Modules.

Since we are interested on sheaves with group action, from now on we suppose that $A$ is the group algebra $k[G]$, where $\text{char}(k) \not| |G|$. We denote by $O_X[G] := O_X(k[G])$ and by $V_0, ..., V_r$ the irreducibles representations of $G$ over $k$, these are the simples $k[G]$-modules, and by $O_X \otimes_k V$ the $O_X[G]$-module definite by $O_X \otimes_k V$, where $V$ is a representation of $G$ over $k$. We will use $V_0$ to mean the trivial representation of $G$.

Under the hypothesis $A = k[G]$, if $E$ and $F$ are $O_X[G]$-modules, then the $O_X$-modules $E \otimes F$ and $\text{Hom}(E, F)$ have a natural structure of $O_X[G]$-modules, given by

$$g(x \otimes y) = (gx) \otimes (gy), \ (\forall \ g \in G, x \in E, y \in F)$$

and

$$(g\sigma)(e) = g(\sigma)(g^{-1}e)$$

respectively

**Notation 2:** By convention, a locally free $O_X[G]$-module $E$ means that it is locally free as an $O_X$-module.

**Lemma 1** Let $E$ be a locally free $O_X[G]$-module of finite rank. Give us to $E^\vee$ the natural structure of $O_X[G]$-module. Then we have the next natural isomorphisms between $O_X[G]$-modules:

a) $(E^\vee)^\vee \cong E$.

b) For any $O_X[G]$-module $F$, $\text{Hom}(E, F) \cong E^\vee \otimes F$

c) For any $O_X[G]$-modules $F, G$, $\text{Hom}_{O_X}(E \otimes F, G) \cong \text{Hom}_{O_X}(F, \text{Hom}(E, G))$. ♦
Now we proceed to prove the main theorem of this section,

**Theorem 1** Let $X$ be an integral scheme over an algebraically closed field $k$, let $K$ be the rational function field of $X$ and $e$ the generic point. Let $E$ be a free torsion coherent $O_X[G]$-module, If $E_e \simeq (V_0^n \oplus \ldots \oplus V_r^n) \otimes_k K$ is the representation of $G$ in the generic point of $X$, then, the natural decomposition

$$E = e_0 E \oplus \ldots \oplus e_r E$$

satisfies

i) $\text{rank}(e_i E) = n_i \times \text{dim}(V_i)$

ii) The natural inclusion $(e_i E) \hookrightarrow E$ induce an isomorphism of representations $(e_i E)_e \rightarrow V_i^n \otimes_k K$

**Proof:** As the question is local, let be $U = \text{Spec}(A)$ an affin open set of $X$ where $E$ is trivial i.e $E|_U \cong A^n$, and consider the isotypical decomposition

$$E = E_1 \oplus \ldots \oplus E_r$$

where $E_i|_U \cong e_i E|_U$, on the other hand $X$ is an integral scheme, and so $A$ is an integral domain and $K(X)$ is the quotient field of $A$, taken localization in the generic point we have that $E_X = A^n \otimes_A K(X) = K(X)^n$. Now, the next commutative diagram

$$\begin{array}{ccc}
G & \longrightarrow & \text{Aut}(K(X)^n) \\
\downarrow & & \downarrow \\
\text{Aut}(A^\otimes n) & & \\
\end{array}$$

say us that the decomposition (1) is determinate in the generic point by the representation of $G$ in $K(X)^n$, from this, the decomposition of $E$ is determinate by the given in the generic point as representation of $G$. Further, the action of $G$ on $B^n$ is given by the restriction of the action of $G$ on $K(X)^n$. Now, the theorem follows from the next

**Lemma 2** Let $k$ be an algebraically closed field and $G$ be a finite group with $\text{char}(k)| |G|$. Then, for any field extension $K$ of $k$, a representation of $G$ over $K$ is of the form $V \otimes_k K$ with $V$ a representation of $G$ over $k$.

**Proof:** It is sufficient to prove the lemma for irreducible representations. By the corollary 3.61 in [1] page 68, any representation of the form $V \otimes_k K$ is irreducible over $K$ if $V$ is irreducible over $k$, and by theorem 30.15 in [2] page 214 all this are different irreducible representations, then this are all the irreducible representations. ◊

**Definition 3** Let $X$ be an integral scheme over an algebraically closed field $k$, $K$ be the function field of $X$ and $e$ be the generic point. Let $E$ be a free torsion coherent $O_X[G]$-module, and $E_e \simeq V \otimes_k K$ be the representation of $G$ in the generic point, we define the type of the representation of $G$ on $E$ as the representation $V$
Corollary 3 Let $X$ and $E$ be as in the above theorem and $V$ be the type of the representation in $E$. Then the representation of $G$ in the fiber of a closed point of $X$ is generically $V$.

Corollary 4 Let $X$ be an integral scheme over an algebraically closed field $k$. If $E$ and $F$ are $\mathcal{O}_X[G]$-modules with type $V$ and $W$ respectively, Then

i) $E \otimes \mathcal{O}_X F$ have the type $V \otimes W$.

ii) $\Lambda^r E$ have the type $\Lambda^r V$, and

iii) $S^n E$ have the type $S^n V$.

Proof: Just we need to observe that if $\epsilon$ is the generic point of $X$ then $(E \otimes \mathcal{O}_X F)_\epsilon = E_\epsilon \otimes_{K(\mathcal{O}_X)} F_\epsilon$, $(\Lambda^r E)_\epsilon = \Lambda^r (E_\epsilon)$ and $(S^n E)_\epsilon = S^n (E_\epsilon)$, the conclusion is immediately from the general theory of representation over a field.

Corollary 5 Let $E_i, E_j$ be two locally free $\mathcal{O}_X[G]$-modules of type $V_i, V_j$, respectively, both different irreducible representations of $G$. Then $\text{Hom}_G(E_i, E_j) = 0$

Proof: Let $\epsilon$ be the generic point of $X$. From the theorem 1, the rank of $\text{Hom}_G(E_i, E_j)$ is the dimension of the part of type $V_0$ of $(\mathcal{E}_i \otimes \mathcal{O}_X \mathcal{E}_j)_\epsilon$, but it is zero by the general theory of representation over a field. Now $\text{Hom}_G(E_i, E_j)$ is a locally free sheaf, then we conclude.

Corollary 6 Let $k = \mathbb{C}$ and $E_1, E_2$ be two locally free $\mathcal{O}_X[G]$-modules with type $V_i$. Then

$$\text{rank} \text{Hom}_G(E_1, E_2) = \text{rank} E_1 \times \text{rank} E_2 / \text{dim} V_i^2$$

4 Irreducibles $\mathcal{O}_X[G]$-Modules

Let $\mathcal{F}$ be a free torsion sheaf over an integral scheme. A natural question is: when $\mathcal{F}$ admit an $\mathcal{O}_X[G]$-module structure? As we see posterior, this will be possible only if the decomposition of $\mathcal{F}$ have certain structure.

Lemma 3 Let $V$ be an irreducible representation of $G$ over $k$, and suppose that $\text{dim}_k V \geq 2$. Then, there are a subgroup $H \leq G$ such that $V \downarrow^G_H$ have at least two isotypical components

Proof: Suppose that it is false. Then for any $g \in G$, $V \downarrow^{<g>}$ have only one isotypical component, that means, any element $g$ acts on $V$ by multiplication of a constant, in this case, any subspace $V$ of dimension 1 is $G$ invariant, but this is a contradiction with our hypothesis because $V$ is irreducible of dimension great than 1.

Lemma 4 Let $\mathcal{F}$ be a non-splitting coherent $\mathcal{O}_X$-module, if $\mathcal{F}$ is an $\mathcal{O}_X[G]$-module for some finite group $G$, then the representation type is $V = W^n$ for some irreducible $W$ of dimension 1. Furthermore, any element $g \in G$ acts by constant multiplication.
Proof: As $\mathcal{F}$ is irreducible, the type of representation must be $V = W^n$ for some irreducible $W$. Furthermore, when we restrict the action to any subgroup of $G$, the structure must be preserved. But, by the above lemma $W$ must have dimension 1, and in particular, if $g \in G$, this acts in the generic point by a constant multiplication, then acts globally in the same way. $\diamond$

Remark 3 Let $V$ be an irreducible representation of $G$ over $k$. Then by general theory, the dimension of the type $V_0$ part of $V^* \otimes V^*$ is equal to $s$, where $V^*$ is the dual representation of $V$. Thus by lemma 2 the same is true for representation over extensions of $k$.

Remark 4 Using the lemma 2, we obtain that the number of irreducible representations of $G$ over $K$ is the same than over $k$.

Now we are in position of classify the non-splitting $\mathcal{O}_X[G]$-modules

Theorem 2 Let $X$ be an integral scheme, let $\mathcal{E}$ be a non-splitting free torsion coherent $\mathcal{O}_X[G]$-module of $W$ type. Then $\mathcal{E} \simeq \mathcal{O}(V) \otimes \mathcal{F}$ with $\mathcal{F}$ a non-splitting $\mathcal{O}_X$-module and $W \simeq V^{\text{rank } \mathcal{F}}$, with $V$ an irreducible representation.

Proof: Let $\mathcal{E} = \mathcal{F}_1^{n_1} \oplus \ldots \oplus \mathcal{F}_r^{n_r}$ be the decomposition of $\mathcal{E}$ on non-splitting $\mathcal{O}_X$-modules with $\mathcal{F}_i \neq \mathcal{F}_j$, if $i \neq j$. This decomposition is unique up to permutations. From this, if $g \in G$, $g\mathcal{E} = g\mathcal{F}_1^{n_1} \oplus \ldots \oplus g\mathcal{F}_r^{n_r}$, then $g\mathcal{F}_i = \mathcal{F}_j$ for some $j$, but this imply $\mathcal{F}_i \simeq \mathcal{F}_j$ and $i = j$. Then, any $\mathcal{F}_i^{n_i}$ is $G$-invariant but $\mathcal{E}$ is a non-splitting $\mathcal{O}_X[G]$-module, and so $r = 1$, i.e. $\mathcal{E} = \mathcal{F}^n$ with $\mathcal{F}$ a non-splitting $\mathcal{O}_X$-module. Therefore the type of the representation is $V^s$ with $V$ irreducible.

The next step is to show that $s = \text{rank } \mathcal{F}$ and $\dim V = n$.

Now, we consider the part of type $V_0$ of $\mathcal{O}(V^*) \otimes \mathcal{E}$, where $V^*$ is the dual representation of $V$. This is a direct component of $\mathcal{O}(V^*) \otimes \mathcal{E} \simeq \mathcal{F}^{\dim V}$, so this component must be $\mathcal{F}^{i}$ for some $i$. In other hand, let $\epsilon$ be the generic point of $X$, so $\mathcal{E}_\epsilon = V^* \otimes K(X)$, then by remark 3, $s = \text{irank } \mathcal{F}$. Now, let $g \in G$, and consider the representation of $<g>$ obtained by restriction, then $\mathcal{E}$ have an isotypical decomposition $\mathcal{E} = \mathcal{F}_{\chi_1}^{n_1} \oplus \ldots \oplus \mathcal{F}_{\chi_r}^{n_r}$ with $\chi_i$ the irreducible representations of $<g>$. Then, by the above lemma, in each component $<g>$ acts by constant multiplication, then $\mathcal{E} = [\mathcal{O}(V_{\chi_1})^{n_1} \oplus \ldots \oplus \mathcal{O}(V_{\chi_r})^{n_r}] \otimes \mathcal{F}$, and in the generic point we must have $\mathcal{E}_\epsilon = [V_{\chi_1}^{n_1} \oplus \ldots \oplus V_{\chi_r}^{n_r}] \otimes \mathcal{F}_\epsilon$, in other hand, the restricted representation is given by $\mathcal{E}_\epsilon = (V \downarrow_{<g>})^i \otimes K(X)^{\dim \mathcal{F}}$, now using the uniqueness of the decomposition in irreducible representations over a field, we obtain $(V \downarrow_{<g>})^i \otimes K(X)^{\dim \mathcal{F}} = [V_{\chi_1}^{n_1} \oplus \ldots \oplus V_{\chi_r}^{n_r}] \otimes \mathcal{F}_\epsilon$ and using the fact that the characters are a basis for $K[G]$ when $K$ is an extension of an algebraically closed field (see remark 4 and [2] page 213 Theorem 30.12), then $[V_{\chi_1}^{n_1} \oplus \ldots \oplus V_{\chi_r}^{n_r}] \simeq (V \downarrow_{<g>})^i \simeq (V^* \downarrow_{<g>})^i \simeq (V^*)^i \downarrow_{<g>}$. Then $\mathcal{E} = \mathcal{O}(V^*) \otimes \mathcal{F}$, but $\mathcal{E}$ is a non-splitting $\mathcal{O}_X(G)$-module, so $i = 1$ and in consequence, $s = \text{rank } \mathcal{F}$, $\dim V = n$ and $\mathcal{E} \simeq \mathcal{F}^{\dim V}$.

Now, in the last step we want to describe the representation of $G$ on $\mathcal{E}$. For this case, consider again the part of type $V_0$ of $\mathcal{O}(V^*) \otimes \mathcal{E}$, where $V^*$ is the dual representation of $V$. This is a direct component of $\mathcal{O}(V^*) \otimes \mathcal{E} \simeq \mathcal{F}^{n_2}$ and by the above considerations, his rank is $\text{rank } \mathcal{F}$, and so this component must be $\mathcal{F}$.
Consider the natural inclusion
\[ \mathcal{O}(V) \otimes \mathcal{F} \longrightarrow \mathcal{O}(V) \otimes \mathcal{O}(V^\vee) \otimes \mathcal{E} \]
and the morphism
\[ \mathcal{O}(V) \otimes \mathcal{O}(V^\vee) \otimes \mathcal{E} \longrightarrow \mathcal{E} \]
given by
\[ a \otimes \delta \otimes e \mapsto \delta(a)e \]
then we have the natural \( G \)-morphisms
\[ \mathcal{O}(V) \otimes \mathcal{F} \longrightarrow \mathcal{E} \cong \mathcal{F}^{\text{dim}V} \]
that is a \( G \)-isomorphisms in the generic point, and using the integral hypothesis we obtain that the kernel is a torsion subsheaf of \( \mathcal{E} \), but it is a free torsion coherent sheaf by hypothesis, so we have that the morphism is injective. Now using the hypothesis of projective, we can consider the Hilbert polynomial of the cokernel (with respect to some ample sheaf), then it must be zero, so the morphism is surjective and then it is a \( G \)-isomorphism. ♦

Now we have the classification theorem of \( \mathcal{O}_X[G] \)-module.

**Theorem 3** Let \( \mathcal{E} \) be a free torsion coherent \( \mathcal{O}_X[G] \)-module of type \( V_0^{n_0} \oplus ... \oplus V_r^{n_r} \). Then the isotypical decomposition of \( \mathcal{E} \) is given by
\[ \mathcal{E} \cong \mathcal{O}(V_0) \otimes \mathcal{F}_0 \oplus ... \oplus \mathcal{O}(V_r) \otimes \mathcal{F}_r \]
when \( \mathcal{F}_i \) is an \( \mathcal{O}_X \)-module of rank \( n_i \) ♦

**Proposition 3** Let
\[ \mathcal{E}_i = \mathcal{O}(V_0) \otimes \mathcal{F}_{0,i} \oplus ... \oplus \mathcal{O}(V_r) \otimes \mathcal{F}_{r,i}, \quad i = 1, 2. \]
Then
\[ \text{Hom}_G(\mathcal{E}_1, \mathcal{E}_2) \cong \bigoplus_{j=0}^r \text{Hom}_{\mathcal{O}_X}(\mathcal{F}_{j,1}, \mathcal{F}_{j,2}) \]
**Proof:** By corollary 1
\[ \text{Hom}_G(\mathcal{E}_1, \mathcal{E}_2) = \bigoplus_{j=0}^r \text{Hom}_G(\mathcal{O}(V_j) \otimes \mathcal{F}_{j,1}, \mathcal{O}(V_j) \otimes \mathcal{F}_{j,2}) \]
then we can suppose, without lost of generality, that \( \mathcal{E}_i = \mathcal{O}(V) \otimes \mathcal{F}_i \) with \( V \) an irreducible representation of \( G \).

Let define the natural map
\[ \sigma : \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \mapsto \text{Id} \otimes \sigma : \mathcal{O}(V) \otimes \mathcal{F}_1 \longrightarrow \mathcal{O}(V) \otimes \mathcal{F}_2 \]
and define the inverse map in the next way: Let $V^\vee$ be the dual representation of $V$, then $V \otimes V^\vee \sim V_0 \oplus W$, then $\mathcal{O}(V) \otimes \mathcal{O}(V^\vee) \sim \mathcal{O}_{X} \oplus \mathcal{O}(W)$. Thus if $\rho : \mathcal{O}(V) \otimes F_1 \rightarrow \mathcal{O}(V) \otimes F_2$ is a $G$-morphisms we have a natural $G$-morphisms

$$Id \otimes \rho : \mathcal{O}(V^\vee) \otimes \mathcal{O}(V) \otimes F_1 \rightarrow \mathcal{O}(V^\vee) \otimes \mathcal{O}(V) \otimes F_2$$

let define $\psi$ by taken the restriction to the component corresponding to the trivial type. It is clear that $\sigma$ and $\psi$ are inverse maps.

**Proposition 4** Let $\mathcal{E}, F_0, ..., F_r$ be free torsion coherent sheaves over $X$, $W = V_0 \oplus ... \oplus V_r$ be a $G$ representation over $k$ and suppose that

$$\phi : \mathcal{O}(W) \otimes \mathcal{E} \rightarrow (\mathcal{O}(V_0) \otimes F_0) \oplus ... \oplus (\mathcal{O}(V_r) \otimes F_r)$$

is a $G$-invariant morphism. Then there exist a natural morphisms

$$\bar{\phi} : \mathcal{E} \rightarrow F_0 \oplus ... \oplus F_r$$

Moreover $\phi$ is injective if and only if $\bar{\phi}$ is.

**Proof:** By the above proposition $\text{Hom}_G(\mathcal{O}(W) \otimes \mathcal{E}, (\mathcal{O}(V_0) \otimes F_0) \oplus ... \oplus (\mathcal{O}(V_r) \otimes F_r))$ is natural isomorphic to $\bigoplus_{i=0}^{r} \text{Hom}(\mathcal{E}, F_i)$, let $\bar{\phi}$ be the image of $\phi$ by this map.

Now $\phi$ is injective iff

$$\phi_i : \mathcal{O}(V_i) \otimes \mathcal{E} \rightarrow \mathcal{O}(V_i) \otimes F_i$$

it is for $i = 0, ..., r$, iff

$$Id \otimes \phi_i : \mathcal{O}(V_i^\vee) \otimes \mathcal{O}(V_i) \otimes \mathcal{E} \rightarrow \mathcal{O}(V_i^\vee) \otimes \mathcal{O}(V_i) \otimes F_i$$

it is for $i = 0, ..., r$, iff

$$\bar{\phi}_i : \mathcal{E} \rightarrow F_i$$

for $i = 0, ..., r$.


5 Structure theorems for Galois covers.

Let $X \rightarrow Y$ be a Galois cover with $\mathcal{O}_{X/Y} = G$. Then $\pi_* \mathcal{O}_X$ have a natural structure of $\mathcal{O}_Y[G]$-module, the objective of this section shall give structure theorems for this object.

**Theorem 4** Let $X$ be a variety over an algebraically closed field $k$ and suppose $X$ with an action of a finite group $G$ (char($k$) $\nmid |G|$) Let $Y$ be the quotient variety. Then

$$\pi_* \mathcal{O}_X = \mathcal{O}_Y \oplus (\mathcal{O}(V_1) \otimes \mathcal{E}_{V_1}) \oplus ... \oplus (\mathcal{O}(V_r) \otimes \mathcal{E}_{V_r})$$

when $V_i$ are the irreducibles representation of $G$ over $k$ and rank $\mathcal{E}_{V_i} = \dim V_i$
Proof: Let $\pi : X \to X/G = Y$ be the natural quotient map, this is a Galois cover of $Y$. Then the field extension $K(X) : K(Y)$ is Galois with Galois group $G$. Now, by the normal basis theorem, there exist an element $\zeta \in K(X)$ such that $\{g\zeta\}_{g \in G}$ is a basis of $K(X)$ as a vector space over $K(Y)$. Thus $K(X)$ is the regular representation of $G$ over $K(Y)$, in other hand this is the action of $G$ on $(\pi_*O_X)_{\text{c}(Y)}$. Then the representation of $G$ in the generic point of $Y$ is the regular one, and using the theorem 3 we have the desired conclusion. ♦

**Corollary 7** Let $X$ and $Y$ be as above, let $H \leq G$ and $\phi : X/H \to Y$ be the natural morphisms. Then

$$\phi_*O_{X/H} = (\pi_*O_X)^H = O_Y \oplus (O(V_1^H) \otimes \mathcal{E}_{V_1}) \oplus \ldots \oplus (O(V_r^H) \otimes \mathcal{E}_{V_r}).$$

**Proof:** See [4] pages 65-69

Let $\pi : X \to Y$ be a Galois cover with $\text{Gal}(X/Y) = G$ and let $\mathcal{L}$ be an $O_X$-module. Let $\phi : \pi_*\mathcal{L} \otimes \pi_*O_X \to \bigoplus_{g \in G} \pi_*g^*\mathcal{L}$ be the map given by

$$m \otimes b \mapsto (m \cdot g^{-1}b)_g.$$

Then we have

**Lemma 5** Let $X \to Y$ be a Galois cover. Then

$$0 \to \pi_*\pi^*\pi_*O_X \xrightarrow{\phi} \bigoplus_{g \in G} \pi_*\pi^*g^*\mathcal{O}_X \to D \to 0$$

is an exact sequence of $O_Y(G)$-modules, where $\text{supp } D \subset \text{supp } \pi_*\pi^*\pi_*\Omega_{X/Y}$.

**Proof:** Just we need to proof the injective part of the sequence.

Let $y \in Y - \text{supp}(\pi_*\Omega_{X/Y})$ a closed point of $Y$. Then the localized sequence in $y$ is exact, so the sequence is exact and $\text{supp } D \subset \text{supp}(\pi_*\Omega_{X/Y})$. ♦

**Corollary 8** Let $\mathcal{F}$ be a locally free sheaf on $X$. Then

$$0 \to \pi_*\pi^*\pi_*\mathcal{F} \to \bigoplus_{g \in G} \pi_*\pi^*g^*\mathcal{F} \to D \to 0$$

is an exact sequence of $O_Y(G)$-modules, where $\text{supp } D \subset \text{supp}(\pi_*\Omega_{X/Y})$. ♦

Now we observe that $\pi_*\mathcal{F} \otimes \pi_*O_X$ and $\bigoplus_{g \in G} \pi_*g^*\mathcal{F}$ have a natural structure of $O_Y(G)$-module given by

$$m \otimes a \xrightarrow{h} m \otimes h(a)$$

and

$$(m_g)_g \xrightarrow{h} (m_g)_{h_g}$$

respectively, and the morphism $\phi$ is $G$-invariant.
Proposition 5 Let \( X \rightarrow Y \) be a Galois cover with Galois group \( G \), and \( F \) be a locally free sheaf on \( X \). Then we have natural injective morphism

\[
0 \rightarrow \pi_* F \otimes \mathcal{O}_Y \otimes_{\pi_* \mathcal{O}_Y} \mathcal{E}_{V_i} \rightarrow \pi_* \mathcal{F}^{\dim V_i}.
\]

Proof: Now by theorem 4 \( \pi_* \mathcal{O}_X = \mathcal{O}_Y \oplus (\mathcal{O}(V_1) \otimes \mathcal{E}_{V_1}) \oplus \ldots \oplus (\mathcal{O}(V_r) \otimes \mathcal{E}_{V_r}) \) is the isotypical decomposition of \( \pi_* \mathcal{O}_X \). Then we have that

\[
\pi_* \mathcal{F} \otimes \pi_* \mathcal{O}_X = \pi_* \mathcal{F} \oplus \bigoplus_{i=1}^r \otimes \mathcal{E}_{V_i} \otimes \mathcal{F}
\]

is the isotypical decomposition of it.

On the other hand, the action of \( G \), in the generic point \( \epsilon \) of \( Y \), in \( \bigoplus_{g \in G} \pi_* g^* \mathcal{F} \) is given by \( k[G] \otimes_k (\bigoplus_{g \in G} \pi_* g^* \mathcal{F})_\epsilon \). Then the isotypical decomposition of \( \bigoplus_{g \in G} \pi_* g^* \mathcal{F} \) is given by

\[
\bigoplus_{g \in G} \pi_* g^* \mathcal{F} = \pi_* \mathcal{F} \oplus \mathcal{O}(V_1)^{\dim V_1} \otimes \pi_* \mathcal{F} \oplus \ldots \oplus \mathcal{O}(V_r)^{\dim V_r} \otimes \pi_* \mathcal{F}
\]

Now, recall that \( \phi \) is \( G \)-invariant, then \( \phi \) decompose on morphism

\[
\phi : \mathcal{O}(V_i) \otimes [\mathcal{E} \otimes \pi_* \mathcal{F}] \rightarrow \mathcal{O}(V_i) \otimes (\pi_* \mathcal{F})^{\dim V_i}
\]

and applying the proposition 4 we have the injective morphisms

\[
\phi_i : \mathcal{E} \otimes \pi_* \mathcal{F} \rightarrow (\pi_* \mathcal{F})^{\dim V_i}
\]

and the proposition is proved. ♦

Now we are interested in the case when \( F \) is \( G \)-stable (see [4] pages 65-69). In this case the direct image of \( F \) have a natural structure of \( \mathcal{O}_X[G] \)-module. So we have in \( \pi_* \mathcal{F} \otimes \pi_* \mathcal{O}_X \) and \( \bigoplus_{g \in G} \pi_* g^* \mathcal{F} \) two others actions of \( G \) given by

\[
m \otimes a \mapsto \hat{h}(m) \otimes a
\]

and

\[
\bigoplus_{g \in G} (a_g)_g \mapsto \hat{h} \bigoplus_{g \in G} (ha_g)_{gh^{-1}},
\]

respectively, and \( h \circ \hat{h} = \hat{h} \circ h \) then we have that \( G \times G \) acts on both sheaves and again \( \phi \) is \( G \times G \)-invariant.

On the other hand, let consider in \( \mathcal{O}_Y[G] \) the actions given by

\[
(a_g)_g \mapsto (a_g)_{hg}
\]

and

\[
(a_g)_g \mapsto (a_g)_{gh^{-1}}
\]
then $h \circ \hat{h} = \hat{h} \circ h$ and again we have a $G \times G$ action, and the isotypical decomposition of $\mathcal{O}_Y[G]$ is given by

$$\mathcal{O}_Y[G] = \bigoplus_{V \in \mathcal{I}} \mathcal{O}(V) \otimes \mathcal{O}(V^\vee)$$

where $\mathcal{I}$ is the set of irreducible representations of $G$ over $k$.

**Lemma 6** The natural isomorphism

$$j : \mathcal{O}_Y[G] \otimes \pi_* \mathcal{F} \longrightarrow \bigoplus_{g \in G} \pi_* \mathcal{F}$$

is $G \times G$ invariant

**proof:** Let $\delta_h(g_0)$ be 1 if $h = g_0$ and zero in other case. Then is sufficient to proof that

$$j[h \times \hat{h}((\delta_g(g_0))g \otimes m)] = h \times \hat{h}[j((\delta_g(g_0))g \otimes m)]$$

but this is immediately by direct calculation. ♦

Let $\phi_i$ be the map defined in \ref{5} Then

**Proposition 6** $\phi_i$ is $G$-invariant with the right action

**Proof:** Recall that the morphism

$$\phi : \pi_* \mathcal{F} \otimes \pi_* \mathcal{O}_X \longrightarrow \mathcal{O}_Y[G] \otimes \pi_* \mathcal{F}$$

is $G \times G$ invariant. In particular we have

$$\phi : \pi_* \mathcal{F} \otimes \mathcal{O}(V) \otimes \mathcal{E}_V \longrightarrow \mathcal{O}(V) \otimes \mathcal{O}(V^\vee) \otimes \pi_* \mathcal{F}$$

is $G \times G$ invariant. Then

$$\phi : \pi_* \mathcal{F} \otimes \mathcal{E}_V \longrightarrow \mathcal{O}(V^\vee) \otimes \pi_* \mathcal{F}$$

is $G$ invariant under the right action. ♦

**Proposition 7** Let $X \longrightarrow Y$ be a Galois cover with Galois group $G$, and $\mathcal{F}$ be a locally free sheaf on $X$, $G$-stable, let

$$\pi_* \mathcal{F} = \mathcal{F}_0 \oplus \mathcal{O}(V_1) \otimes \mathcal{F}_1 \oplus ... \oplus \mathcal{O}(V_r) \otimes \mathcal{F}_r$$

be the isotypical decomposition of $\pi_* \mathcal{F}$. Then there exist natural injective morphisms

$$\mathcal{F}_i \otimes \mathcal{E}_V \longrightarrow \bigoplus_{l=0}^r \mathcal{F}_i^{n_l}$$

where

$$V_i \otimes V_j = \bigoplus_{l=0}^r V_i^{n_l}$$

11
Proof: From the above proposition we have the next $G$ invariant morphism

$$0 \to \pi_\ast \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E}_{V_j} \to \mathcal{O}(V_j^\vee) \otimes \pi_\ast \mathcal{F}$$

And this decompose in injective morphism

$$(\mathcal{O}(V_i) \otimes \mathcal{F}_{V_i}) \otimes \mathcal{E}_{V_j} \to (\mathcal{O}(V_j^\vee) \otimes \pi_\ast \mathcal{F})$$

$$\simeq \mathcal{O}(V_j^\vee) \otimes [\mathcal{F}_0 \oplus \mathcal{O}(V_1) \otimes \mathcal{F}_1 \oplus \ldots \mathcal{O}(V_r) \otimes \mathcal{F}_r]$$

$$\simeq [\mathcal{O}(V_j^\vee) \otimes \mathcal{F}_0 \oplus \mathcal{O}(V_j^\vee) \otimes \mathcal{O}(V_1) \otimes \mathcal{F}_1 \oplus \ldots \mathcal{O}(V_j^\vee) \otimes \mathcal{O}(V_r) \otimes \mathcal{F}_r]$$

each one factorizing by the $V_i$ part. Now we introduce some notation Let $W$ be a representation of $G$ over $k$ and $V$ an irreducible representation; then we define $<W,V>$:=dimension that $V$ appear on $W$.

Let be $s_i^l = <V_j^\vee \otimes V_i, V_l>$ then the $V_i$ part is determinated by

$$\mathcal{O}(V_i) \otimes \bigoplus_{l=1}^r \mathcal{F}_i^{s_i^l}$$

from this we have the injective morphism

$$\mathcal{O}(V_i) \otimes (\mathcal{F}_i \otimes \mathcal{E}_{V_j}) \to \mathcal{O}(V_i) \otimes \bigoplus_{l=0}^r \mathcal{F}_i^{s_i^l}$$

and using the proposition 4 we have the desired injective morphism

$$\mathcal{F}_i \otimes \mathcal{E}_{V_j} \to \bigoplus_{l=0}^r \mathcal{F}_i^{s_i^l}$$

on the other hand, by the theory of representation over a field we have $<V_j^\vee \otimes V_i, V_i>=<V_j \otimes V_i, V_i>=n_i$ where

$$V_j \otimes V_i = \bigoplus_{l=1}^r V_i^{n_i}$$

and then we conclude. ♦

Theorem 5 Let $X \to Y$ be a Galois cover with Galois group $G$, let

$$\pi_\ast \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{O}(V_1) \otimes \mathcal{E}_{V_1} \oplus \ldots \oplus \mathcal{O}(V_r) \otimes \mathcal{E}_{V_r}$$

be the isotypical decomposition of $\pi_\ast \mathcal{O}_X$. Then the algebra structure determine the natural injective morphisms

$$\mathcal{E}_{V_i} \otimes \mathcal{E}_{V_j} \to \bigoplus_{l=0}^r \mathcal{E}_i^{n_i}_{V_l}$$
where
\[ V_i \otimes V_j = \bigoplus_{l=0}^{r} V_i^l. \]

Further, this are isomorphisms in the unramified case.

For the end, we study the stability of the sheaf \( \pi_* \mathcal{O}_X \) in the case that \( \pi : X \to Y \) is unramified and \( X \) is a smooth variety. Then in this case we have that \( Y \) and \( \pi \) are smooth. Moreover \( \pi^* \pi_* \mathcal{O}_X \simeq \mathcal{O}_Y^{[G]} \) and \( \pi^* \Omega_{Y/k} \simeq \Omega_{X/k} \) so
\[ \omega_X = \Lambda^\dim X \Omega_{X/k} \simeq \Lambda^\dim X \pi^* \Omega_{Y/k} \simeq \pi^* \Lambda^\dim Y \Omega_{Y/k} \simeq \pi^* \omega_Y. \]

Let \( \mathcal{O}_Y(1) \) be an ample line bundle on \( Y \) then \( \pi^* \mathcal{O}_Y(1) \) is an ample line bundle on \( X \), then we have the next

**Lemma 7** Let \( F \) be an \( n \)-dimensional coherent sheaf on \( X \). Then \( F \) is \( \mu \)-polystable if and only if \( \pi^* F \) is \( \mu \)-polystable.

**Proof:** See [3] pages 62-63. \( \diamond \)

**Theorem 6** Let \( \pi : X \to Y \) as above. Then in the isotypical decomposition
\[ \pi_* \mathcal{O}_X = \bigoplus_{V \in \mathcal{I}} \mathcal{O}(V) \otimes \mathcal{E}_V \]
each \( \mathcal{E}_V \) is \( \mu \)-stable, \( \mathcal{E}_{V_i} \not\sim \mathcal{E}_{V_j} \) if \( i \neq j \) and \( \mathcal{E}_{V^*} \simeq \mathcal{E}^*_V \).

**Proof:** By the above lemma, we have that \( \pi_* \mathcal{O}_X \) is \( \mu \)-polystable. Then, just we need to prove that each \( \mathcal{E}_V \) is simple i.e. \( \text{End}(\mathcal{E}_V) = k \), and then \( \mu \)-stable. For that,
\[ \text{Hom}(\pi_* \mathcal{O}_X, \pi_* \mathcal{O}_X) \supseteq \bigoplus_{i=0}^{r} \text{Hom}(\mathcal{O}(V_i) \otimes \mathcal{E}_{V_i}, \mathcal{O}(V_i) \otimes \mathcal{E}_{V_i}) \]
\[ = \bigoplus_{i=0}^{r} \text{Hom}(\mathcal{E}_{V_i}, \mathcal{E}_{V_i})^{(\dim V_i)^2} \]
now \( h^0(\text{Hom}(\mathcal{E}_{V_i}, \mathcal{E}_{V_i})) \geq 1 \) so we have that
\[ h^0(\text{Hom}(\pi_* \mathcal{O}_X, \pi_* \mathcal{O}_X)) \geq \sum_{i=0}^{r} (\dim V_i)^2 = |G| \]
on the other hand
\[ h^0(\pi_* \mathcal{O}_X \otimes \pi_* \mathcal{O}_X) = h^0(\pi^* \pi_* \mathcal{O}_X) = h^0(\mathcal{O}_X^{[G]}) = |G| \]
and \( (\pi_* \mathcal{O}_X)^\vee \sim \pi_* \mathcal{O}_X \) then
\[ \dim \text{Hom}(\pi_* \mathcal{O}_X, \pi_* \mathcal{O}_X) = |G|, \]
but this equality imply \( h^0(\text{Hom}(\mathcal{E}_{V_i}, \mathcal{E}_{V_i})) = 1 \) so each \( \mathcal{E}_{V_i} \) is simple and from here is \( \mu \)-stable. Now observe that \( h^0(\text{Hom}(\mathcal{E}_{V_i}, \mathcal{E}_{V_j})) = 0 \) for \( i \neq j \), this imply that \( \mathcal{E}_{V_i} \not\sim \mathcal{E}_{V_j} \) if \( i \neq j \). Now just we need to see that \( (\mathcal{E}_{V_i})^\vee \sim \mathcal{E}_{V_i}^\vee \), but this is consequence from the fact that the trace map \( \pi_* \mathcal{O}_X \overset{Tr}{\to} (\pi_* \mathcal{O}_X)^\vee \) is a \( G \)-invariant isomorphism. \( \diamond \)
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