PERFECT SNAKE-IN-THE-BOX CODES FOR RANK MODULATION

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Abstract. For odd $n$, the alternating group on $n$ elements is generated by the permutations that jump an element from any odd position to position 1. We prove Hamiltonicity of the associated directed Cayley graph for all odd $n \neq 5$. (A result of Rankin implies that the graph is not Hamiltonian for $n = 5$.) This solves a problem arising in rank modulation schemes for flash memory. Our result disproves a conjecture of Horovitz and Etzion, and proves another conjecture of Yehezkeally and Schwartz.

1. Introduction

The following questions are motivated by applications involving flash memory. Let $S_n$ be the symmetric group of permutations $\pi = [\pi(1), \ldots, \pi(n)]$ of $[n] := \{1, \ldots, n\}$, with composition defined by $(\pi \rho)(i) = \pi(\rho(i))$. For $2 \leq k \leq n$ let

$$\tau_k := [k, 1, 2, \ldots, k-1, k+1, \ldots, n] \in S_n$$

be the permutation that jumps element $k$ to the start. Let $S_n$ be the directed Cayley graph of $S_n$ with generators $\tau_2, \ldots, \tau_n$, i.e. the directed graph with vertex set $S_n$ and a directed edge, labelled $\tau_i$, from $\pi$ to $\pi \tau_i$ for each $\pi \in S_n$ and each $i = 2, \ldots, n$.

We are concerned with self-avoiding directed cycles (henceforth referred to simply as cycles) in $S_n$. (A cycle is self-avoiding if it visits each vertex at most once). In applications to flash memory, a permutation represents the relative ranking of charges stored in $n$ cells. Applying $\tau_i$ corresponds to the operation of increasing the $i$th charge to make it the largest, and a cycle is a schedule for cycling through a set of distinct charge arrangements via such operations. Schemes of this kind were originally proposed in [6].

One is interested in maximizing the length of such a cycle, since this maximizes the information that can be stored. It is not difficult to show

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that $S_n$ has a directed **Hamiltonian** cycle, i.e. one that includes *every* permutation exactly once; see e.g. [4, 6, 7]. However, for the application it is desirable that the cycle should not contain any two permutations that are within a certain fixed distance $r$ of each other, with respect to some metric $d$ on $S_n$. The motivation is to avoid errors arising from one permutation being mistaken for another [6, 9]. The problem of maximizing cycle length for given $r, d$ combines notions of Gray codes [12] and error-detecting/correcting codes [1], and is sometimes known as a snake-in-the-box problem.

The main result of this article is that, in the case that has received most attention, there is a ‘perfect’ cycle that has the maximum size even among arbitrary sets of permutations satisfying the distance constraint.

Our focus is following case considered in [3, 16, 17]. Let $r = 1$ and let $d$ be the **Kendall tau** metric [8], which is defined by setting $d(\pi, \sigma)$ to be the inversion number of $\pi^{-1}\sigma$, i.e. the minimum number of adjacent (or elementary) transpositions needed to get from $\pi$ to $\sigma$. (The $i$th adjacent transposition swaps the permutation elements in positions $i, i+1$, where $1 \leq i \leq n-1$.) Thus, the cycle is not allowed to contain any two permutations that are related by a single elementary transposition. The primary object of interest is the maximum possible length $M_n$ of such a directed cycle in $S_n$.

It is easy to see that $M_n \leq n!/2$. Indeed, any set of permutations satisfying the above distance constraint includes at most one from the pair $\{\pi, \pi\tau_2\}$ for every $\pi$, but these pairs partition $S_n$. To get a large cycle, an obvious approach is to restrict to the **alternating group** $A_n$ of all even permutations. Since an elementary transposition changes the parity, this guarantees that the distance condition is satisfied. The generator $\tau_k$ lies in $A_n$ if and only if $k$ is odd. Therefore, if $n$ is odd, this approach reduces to the problem of finding a maximum directed cycle in the directed Cayley graph $A_n$ of $A_n$ with generators $\tau_3, \tau_5, \ldots, \tau_n$. Yehezkeally and Schwartz [16] conjectured that for odd $n$ the maximum cycle length $M_n$ is attained by a cycle of this type; our result will imply this. (For even $n$ this approach is less useful, since without using $\tau_n$ we can access only permutations that fix $n$.) As in [3, 16, 17], we focus mainly on odd $n$.

For small odd $n$, it is not too difficult to find cycles in $A_n$ with length reasonably close to the upper bound $n!/2$, by ad-hoc methods. Finding systematic approaches that work for all $n$ is more challenging. Getting all the way to $n!/2$ apparently involves a fundamental obstacle, but we will show how it can be overcome.
Specifically, it is obvious that $M_3 = 3!/2 = 3$. For general odd $n \geq 5$, Yehezkeally and Schwartz [16] proved the inductive bound $M_n \geq n(n-2)M_{n-2}$, leading to $M_n \geq \Omega(n!/\sqrt{n})$ asymptotically. They also showed by computer search that $M_5 = 5!/2 - 3 = 57$. Horowitz and Etzion [5] improved the inductive bound to $M_n \geq (n^2 - n - 1)M_{n-2}$, giving $M_n = \Omega(n!)$. They also proposed an approach for constructing a longer cycle of length $n!/2 - n + 2(= (1 - o(1))n!/2)$, and showed by computer search that it works for $n = 7, 9$. They conjectured that this bound is optimal for all odd $n$. Zhang and Ge [17] proved that the scheme of [5] works for all odd $n$, establishing $M_n \geq n!/2 - n + 2$, and proposed another scheme aimed at improving the bound by 2 to $n!/2 - n + 4$. Zhang and Ge proved that their scheme works for $n = 7$, disproving the conjecture of [5] in this case, but were unable to prove it for general odd $n$.

The obvious central question here is whether there exists a ‘perfect’ cycle of length $n!/2$ for any odd $n \geq 3$. As mentioned above, Horovitz and Etzion [5] conjectured a negative answer for all such $n$, while the authors of [16, 17] also speculate that the answer is negative. We prove a positive answer for all $n \neq 5$.

**Theorem 1.** For all odd $n \geq 7$, there exists a directed Hamiltonian cycle of the directed Cayley graph $A_n$ of the alternating group $A_n$ with generators $\tau_3, \tau_5, \ldots, \tau_n$. Thus, $M_n = n!/2$.

Besides being the first of optimal length, our cycle has a simpler construction than those in [3, 17]. It may be described via a fairly straightforward explicit rule that specifies which generator should immediately follow each permutation $\pi$, as a function of $\pi$. (See [4, 15] for other Hamiltonian cycles with this property). While the improvement from $n!/2 - n + 2$ to $n!/2$ is in itself unlikely to be important for applications, our methods are quite general, and it is hoped that they will prove useful for related problems.

We briefly discuss even $n$. Clearly, one can simply leave the last element $\pi(n)$ of the permutation fixed, and use the cycle in $A_{n-1}$, which gives $M_n \geq M_{n-1}$ for even $n$. Horovitz and Etzion [3] asked for a proof or disproof that this is optimal. In fact, one can do much better. We expect that $M_n \geq (1 - o(1))n!/2$ asymptotically as $n \to \infty$ (an $n$-fold improvement over $(n - 1)!/2$), and perhaps even $M_n \geq n!/2 - O(n^2)$. We will outline an approach to showing a bound of this sort, although we expect that a full proof for general even $n$ would be rather messy. When $n = 6$ we use this approach to show $M_6 \geq 315 = 6!/2 - 45$, improving the bound $M_6 \geq 57$ of [3] by more than a factor of 5.
Hamiltonian cycles of Cayley graphs have been extensively studied, although general results are relatively few. See e.g. [2, 10] for surveys. In particular, it is unknown whether every undirected Cayley graph is Hamiltonian. Our key construction (described in the next section) appears to be novel in the context of this literature.

Central to our proof are techniques having their origins in change ringing (English-style church bell ringing). Change ringing is also concerned with self-avoiding cycles in Cayley graphs, and change ringers discovered key aspects of group theory considerably before mathematicians did – see [14]. As we shall see, the fact that $A_5$ has no Hamiltonian cycle (so that we have the strict inequality $M_5 < 5!/2$) follows from a theorem of Rankin [11, 13] that was originally motivated by change ringing.

2. Breaking the parity barrier

Here we explain the key obstruction that frustrated the previous attempts at a Hamiltonian cycle of $A_n$ in [5, 16, 17]. We then explain how it can be overcome.

By a cycle cover of a directed Cayley graph we mean a set of self-avoiding directed cycles whose vertex sets partition the vertex set of the graph. A cycle or a cycle cover can be specified in several equivalent ways: we can list the vertices or edges encountered by a cycle in order, or we can specify a starting vertex of a cycle and list the generators it uses in order, or we can specify which generator immediately follows each vertex – i.e. the label of the unique outgoing edge that belongs to the cycle or cycle cover. It will be useful to switch between these alternative viewpoints.

A standard approach to constructing a Hamiltonian cycle is to start with a cycle cover, and then make local modifications that unite several cycles into one, until we have a single cycle. However, in $A_n$ and many other natural cases, there is a serious obstacle involving parity. The order $\text{order}(g)$ of a group element $g$ is the smallest $t \geq 1$ such that $g^t = \text{id}$, where id is the identity. In our case, let $\tau_k, \tau_\ell$ be two distinct generators of $A_n$, and observe that their ratio $\rho := \tau_k / \tau_\ell$ has order $q := |k - \ell| + 1$, which is odd; see Figure 1 (right) for an example. More precisely, $\rho$ is simply the permutation that jumps element $k$ to position $\ell$. Let $\pi$ be a permutation and consider the right coset $Q = \{\pi, \pi \rho, \ldots, \pi \rho^{q-1}\}$ of the cyclic group generated by $\rho$. Suppose that we have a cycle cover in which every element of $Q$ is followed immediately by $\tau_k$, and moreover suppose that the $q$ elements of $Q$ lie in distinct cycles. (An example is the cycle cover that uses $\tau_k$ from every vertex).
We can then remove the outgoing $\tau_k$-edge from each element of $Q$, and instead use $\tau_\ell$ from each element of $Q$. It is easy to see that this results in a new cycle cover in which $q$ cycles have been united into one – see Figure 1. Since $q$ is odd, this does not change the parity of the total number of cycles. It turns out that this parity is preserved even if the elements of $Q$ were not all in distinct cycles. This is a problem, because many cycle covers that one might start with have an even number of cycles – this holds for the cycle cover that uses $\tau_k$ everywhere (for $n \geq 3$), and for the one arising from the obvious inductive approach to proving Theorem 1. Thus we can (apparently) never get to a single cycle.

The above ideas in fact lead to the following rigorous condition for non-existence of Hamiltonian cycles. The result was proved by Rankin [11], based on an 1886 proof by W. H. Thompson of a special case arising in change ringing; Swan [13] later gave a simpler version of the proof.

**Theorem 2.** Consider the directed Cayley graph $G$ of a finite group with two generators $a, b$. If $\text{order}(ab^{-1})$ is odd and $|G|/\text{order}(a)$ is even then $G$ has no directed Hamiltonian cycle.

An immediate consequence is that $A_5$ has no directed Hamiltonian cycle (so $M_n < n!/2 = 60$), and indeed $A_n$ has no directed Hamiltonian cycle using only two generators for odd $n \geq 5$.

To break the parity barrier, we must use at least three generators in a fundamental way. The problem was that $\text{order}(\tau_k \tau_\ell^{-1})$ is odd: we
Figure 2. The key construction. Left: replacing a suitable combination of generators $\tau_{n-2}$ and $\tau_{n-4}$ with $\tau_n$ links 6 cycles into one, breaking the parity barrier. Right: an example of the permutations appearing around the inner cycle when $n = 9$, with the identity being the circled vertex.

We will prove Theorem 1 by induction, using the above linkage in the inductive step, and using it in the reverse direction (replacing 6 $\tau_n$-edges with $\tau_{n-2}$, $\tau_{n-4}$, ...) for the base case.
3. Hypergraph spanning

The other main ingredient for our proof is a systematic way of organizing the various linkages. For this the language of hypergraphs will be convenient. Similar hypergraph constructions were used in [3, 17]. A hypergraph \((V, H)\) consists of a vertex set \(V\) and a set \(H\) of nonempty subsets of \(V\), which are called hyperedges. A hyperedge of size \(r\) is called an \(r\)-hyperedge.

The incidence graph of a hypergraph \((V, H)\) is the bipartite graph with vertex set \(V \cup H\) and edge between \(v \in V\) and \(h \in H\) if \(v \in h\). A component of a hypergraph is a component of its incidence graph, and a hypergraph is connected if it has one component. We say that a hypergraph is acyclic if its incidence graph is acyclic. Note for example that if two hyperedges \(h\) and \(h'\) share two distinct vertices \(v\) and \(v'\) then the hypergraph is not acyclic. (Several non-equivalent notions of acyclicity for hypergraphs have been considered—the notion we use here is sometimes called Berge-acyclicity—see e.g. [3]).

We are interested in hypergraphs of a particular kind that are related to the linkages considered in the previous section. Let \([n]^{(k)}\) be the set of all \(n!/k!\) ordered \(k\)-tuples of distinct elements of \([n]\). If \(t = (a, b, c) \in [n]^{(3)}\) is a triple, define the triangle \(T(t) = T(a, b, c) = \{(a, b), (b, c), (c, a)\} \subset [n]^{(2)}\) of pairs respecting the cyclic order. (Note that \(T(a, b, c) = T(c, a, b) \neq T(c, b, a)\)).

**Proposition 3.** Let \(n \geq 3\). There exists an acyclic hypergraph with vertex set \([n]^{(2)}\), with all hyperedges being triangles \(T(t)\) for \(t \in [n]^{(3)}\), and with exactly two components: one containing precisely the 3 vertices of \(T(3, 2, 1)\), and the other containing all other vertices.

**Proof.** We give an explicit inductive construction. We \(n = 3\) we simply take as hyperedges the two triangles \(T(3, 2, 1)\) and \(T(1, 2, 3)\). Now let \(n \geq 4\). Start with the hyperedges of the hypergraph already constructed for \(n - 1\): thus, we have the component \(T(3, 2, 1)\), a component \(K\) containing all other vertices in \([n - 1]^{(2)}\), and the 2\(n\) isolated vertices \((i, n), (n, i)\) for \(i < n\). Now add the \(n - 1\) further hyperedges

\[
T(1, 2, n), T(2, 3, n), T(3, 4, n), \ldots, T(n - 2, n - 1, n), T(n - 1, 1, n).
\]

It is easy to check that each of these contains one vertex of \(K\) and two of the isolated vertices, with each isolated vertex appearing exactly once. Thus, the resulting hypergraph has the required properties. \(\square\)
Corollary 4. Let $n \geq 5$ and let $a, b, c, d, e \in [n]$ be distinct. There exists a connected acyclic hypergraph with vertex set $[n]^{(2)}$ such that one hyperedge is the 6-hyperedge $T(a, b, c) \cup T(c, d, e)$, and all others are triangles $T(t)$ for $t \in [n]^{(3)}$.

Proof. By symmetry, it is enough to prove this for any one choice of $(a, b, c, d, e)$; we choose $(2, 1, 3, 4, 5)$. The result follows from Proposition 3, on noting that $T(3, 4, 5)$ is a hyperedge of the hypergraph constructed there: we simply unite it with $T(3, 2, 1) = T(2, 1, 3)$ to form the 6-hyperedge. □

4. The Hamiltonian cycle

We now prove Theorem 1 by induction on (odd) $n$. We give the inductive step first, followed by the base case $n = 7$. The following simple observation will be used in the inductive step.

Lemma 5. Let $n$ be odd and consider any Hamiltonian cycle of $A_n$. For every $i \in [n]$ there exists a permutation $\pi \in A_n$ with $\pi(n) = i$ that is immediately followed by $\tau_n$ in the cycle.

Proof. The last element $\pi(n)$ of the permutation (and indeed the final pair $(\pi(n - 1), \pi(n))$) can change only at a $\tau_n$-edge. Since the cycle visits all permutations, the result follows. □

Proof of Theorem 1, inductive step. We will prove by induction on odd $n \geq 9$ that there exists a Hamiltonian cycle of $A_n$ with the additional property that it uses at least one $\tau_{n-2}$-edge. As mentioned above, we postpone the proof of the base case $n = 7$. For distinct $a, b \in [n]$ define the set of permutations ending $[\ldots, a, b]$:

$$A_n(a, b) := \{ \pi \in A_n : (\pi(n - 1), \pi(n)) = (a, b) \}.$$

Let $n \geq 9$, and let $L = (\tau_{s(1)}, \tau_{s(2)}, \ldots, \tau_{s(m)})$ be the sequence of generators used by the Hamiltonian cycle of $A_{n-2}$ guaranteed by the inductive hypothesis, in the order that they are encountered starting from $\text{id} \in A_n$ (where $m = (n - 2)!/2$ and each $s(i) \in \{3, 5, \ldots, n - 2\}$).

If we start from any permutation of $A_n(a, b)$ and apply the sequence of generators $L$, we obtain a cycle comprising precisely all the permutations of $A_n(a, b)$. The idea is to start with a cycle cover comprising one such cycle for each pair $(a, b) \in [n]^{(2)}$, and link them together by substituting the generator $\tau_n$ (which is not present in $L$) at appropriate points, as discussed in Section 2. This will be done using the hypergraph in Corollary 4—each 3-hyperedge will indicate a substitution of $\tau_n$ for $\tau_{n-2}$ in 3 cycles, linking them together, while the 6-hyperedge...
will correspond to the parity-breaking linkage in which \( \tau_n \) is substituted for occurrences of both \( \tau_{n-2} \) and \( \tau_{n-4} \).

We will describe our cycle via the sequence of generators. For a generator \( \tau_k \) that occurs in \( L \), fix any location \( j \) where it occurs, so \( s(j) = k \), and let \( L[\tau_k] \) be the sequence obtained by starting at that location and omitting it from the cycle:

\[
L[\tau_k] := (\tau_{s(j+1)}, \tau_{s(j+2)} \ldots, \tau_{s(m)}, \tau_{s(1)} \ldots, \tau_{s(j-1)}).
\]

Note that composition of the elements of \( L[\tau_k] \) in order is \( \tau_k^{-1} \).

We start with the linkage corresponding to the 6-hyperedge. Starting from \( \text{id} \in A_n \), we apply the sequence of generators

\[
\tau_n, L[\tau_{n-2}], \tau_n, L[\tau_{n-4}], \tau_n, L[\tau_{n-4}],
\]

(\( \text{where commas denote concatenation} \)). Recall that \( L[\tau_{n-2}] \) and \( L[\tau_{n-4}] \) indeed exist by Lemma 5 and the inductive hypothesis respectively. By (1) this gives a cycle. Each \( L[\cdot] \) sequence covers precisely the permutations of one set \( A_n(a,b) \), and it is easy to check that the pairs 6 pairs \((a,b)\) that arise are precisely \((a,b) \in T(n-2, n-1, n) \cup T(n-4, n-3, n) \).

Figure 2 (right) gives the sequence of permutations before and after the \( \tau_n \)’s for \( n = 9 \) – here the pairs \((a,b)\) are 89, 78, 97, 69, 56, 95.

Starting from the cycle constructed above, we next link in the other sets \( A_n(a,b) \), two at a time. Choose an acyclic connected hypergraph according to Corollary 4, with the 6-hyperedge \( h_0 := T(n-2, n-1, n) \cup T(n-4, n-3, n) \). By connectedness, we can enumerate the hyperedges as \( h_0, h_1, \ldots, h_N \) in such a way that any initial segment \( h_0, h_1, \ldots, h_i \) induces a connected sub-hypergraph. By acyclicity, each \( h_i \) shares exactly one vertex with \( \{h_0, \ldots, h_{i-1}\} \).

We will process the hyperedges \( h_1, h_2, \ldots \) in turn, augmenting the cycle as we go. The hyperedge \( h_i = T(a, b, c) \) will link \( A_n(a,b), A_n(b,c) \) and \( A_n(c,a) \). After processing \( h_1, \ldots, h_i \), the current cycle will contain precisely the permutations of the \( A_n(u,v) \) for all vertices \((u,v) \in [n]^{(2)} \) belonging to the hyperedges \( h_0, \ldots, h_i \).

Here is the procedure. Suppose that the next hyperedge to be considered is \( h_i = T(a, b, c) = \{(a,b), (b,c), (c,a)\} \). For exactly one of these 3 pairs, say \((b,c)\), the permutations of \( A_n(b,c) \) are already present in the current cycle. We choose an occurrence of \( \tau_{n-2} \) in the current cycle with the property that the permutation immediately preceding it is of the form \([\ldots, a, b, c]\); we explain below why this is always possible. Then we delete this \( \tau_{n-2} \) from the current cycle and replace it with the sequence

\[
\tau_n, L[\tau_{n-2}], \tau_n, L[\tau_{n-2}], \tau_n.
\]
The effect of this is to add in $A_n(a, b)$ and $A_n(c, a)$ – see Figure 1 (right).

To check that a suitable $\tau_{n-2}$ exists as claimed above, note that any vertex $(b, c)$ of the hypergraph is contained in at most one hyperedge (or triangle comprising half a 6-edge) of the form $T(u, b, c)$ for each $u \in [n] \setminus \{b, c\}$ (so at most $n - 2$ hyperedges in total). Consider our augmenting procedure from the point of view of the edge sets of the cycles. Each time we add a new set $A_n(b, c)$ (including at the initial step), we can imagine that we first introduce a full cycle $C(b, c)$ starting from some permutation of $A_n(b, c)$ and using the sequence $L$, but then we immediately delete one edge from it, because actually we inserted only the sequence $L[\cdot]$. At subsequent steps, further $\tau_{n-2}$-edges of $C(b, c)$ may be deleted. By Lemma 5 applied to the inductive hypothesis, $C(b, c)$ contains a permutation $[\ldots, u, b, c]$ immediately followed by a $\tau_{n-2}$ for each such $u \notin \{b, c\}$. By the above observation about the hypergraph, we are never required to delete a $\tau_{n-2}$ that has already been deleted. Moreover, $\tau_{n-4}$-edges are deleted only in the initial step (one each from 4 different cycles), so they cannot interfere with later deletions of $\tau_{n-2}$-edges.

After processing all hyperedges we obtain a Hamiltonian cycle of $A_n$ as required. Finally, we must check that it contains a $\tau_{n-2}$ as required for the inductive hypothesis. In fact it contains many. For example, in the last step we inserted $L[\tau_{n-2}]$, and never deleted anything further from it; but by Lemma 5, $L$ contains at least $n - 2 (> 1)$ occurrences of $\tau_{n-2}$.

**Proof of Theorem 1, base case.** For the base case of the induction, we give an explicit directed Hamiltonian cycle of $A_7$ that includes $\tau_5$ at least once. (In fact the latter condition must be satisfied, since, as remarked earlier, Theorem 2 implies that there is no Hamiltonian cycle using only $\tau_3$ and $\tau_7$.)

Table 1 specifies which generator the cycle uses immediately after each permutation of $A_7$, in terms of the permutation itself. The skeptical reader may simply check by computer that these rules generate the required cycle. But the rules were constructed by hand; below we briefly explain how.

First suppose that from every permutation of $A_7$ we apply the $\tau_7$ generator, as specified in row 7 of the table. This gives a cycle cover comprising $|A_7|/7 = 360$ cycles of size 7. Now consider the effect of replacing some of these $\tau_7$’s according to rows 1–6 in succession. Row 1 links the cycles in sets of 3 to produce 120 cycles of length 21, each containing exactly one permutation of the form 67***** or 76****.
Table 1. Rules for generating a directed Hamiltonian cycle of \( A_7 \). Permutations of the given forms should be followed by the generator in the same row of the table. The symbol \( * \) denotes an arbitrary element of \([7]\), and \( \pi \) denotes any element other than \( a \).

| row | permutations | generator |
|-----|--------------|-----------|
| 1   | 6777****, 7776*** | \( \tau_5 \) |
| 2   | 676666, 766666 | \( \tau_3 \) |
| 3   | 567777, 576777 | \( \tau_5 \) |
| 4   | 256777, 457677 | \( \tau_5 \) |
| 5   | 5671234, 5612347, 5623714, 5637142 | \( \tau_3 \) |
| 6   | 5623471, 5671423 | \( \tau_5 \) |
| 7   | otherwise | \( \tau_7 \) |

Row 2 then links these cycles in sets of 5 into 24 cycles of length 105, each containing exactly one permutation of the form 675**** or 765****. Rows 3 and 4 link various sets of three cycles, permuting elements 1234, to produce 6 cycles. Finally, rows 5 and 6 break the parity barrier as discussed earlier, uniting these 6 cycles into one. \( \square \)

5. Even size

We briefly discuss a possible approach for even \( n \). Recall that \( M_n \) is the maximum length of a cycle \( S_n \) in which no two permutations are related by an adjacent transposition.

To get a cycle longer than \( M_{n-1} \) we must use \( \tau_n \). But this is an odd permutation, so we cannot remain in the alternating group \( A_n \). We suggest following \( \tau_n \) immediately by another odd generator, say \( \tau_{n-2} \), in order to return to \( A_n \) (note that \( \tau_2 \) is forbidden). To get every element into the last position of the permutation, we need to perform such a transition (at least) \( n \) times in total in our cycle. In the \( i \)th transition we visit one odd permutation, \( \alpha_i \) say, between the generators \( \tau_n \) and \( \tau_{n-2} \). For the remainder of the cycle we propose using only generators \( \tau_k \) for odd \( k \), so that we remain in \( A_n \). Thus the problem is essentially reduced to the odd case \( n-1 \), except that we must avoid all odd permutations that differ from the permutations \( \alpha_i \) by an adjacent transposition \( (O(n^2) \) permutations), and we need not a cycle but a path, with its endpoints chosen to link the \( n \) transitions.
described above. Given the flexibility available in $\mathcal{A}_{n-1}$ (especially for $n-1 \geq 7$), these requirements do not appear particularly problematic.

We used a computer search based on the above approach to obtain a cycle of length 315 for $n = 6$, answering a question of [5]. The search space was reduced by quotienting the graph $S_6$ by a group of order 3 to obtain a Schreier graph, giving a cycle in which the sequence of generators is repeated 3 times. The cycle uses the sequence of generators $(\tau_{k(i)})$ where $(k(i))^{315}_{i=1}$ is the sequence

$$
\left(6455353355535553555355355553555535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355535553555355
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