CONGRUENCE FOR RATIONAL POINTS OVER FINITE FIELDS AND CONIVEAU OVER LOCAL FIELDS

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Abstract. If the $\ell$-adic cohomology of a projective smooth variety, defined over a local field $K$ with finite residue field $k$, is supported in codimension $\geq 1$, then every model over the ring of integers of $K$ has a $k$-rational point. For $K$ a $p$-adic field, this is [3, Theorem 1.1]. If the model $X$ is regular, one has a congruence $|X(k)| \equiv 1$ modulo $|k|$ for the number of $k$-rational points ([7, Theorem 1.1]). The congruence is violated if one drops the regularity assumption.

1. Introduction

Let $X$ be a projective variety defined over a local field $K$ with finite residue field $k = \mathbb{F}_q$. Let $R$ be the ring of integers of $K$. A model of $X/K$ is a flat projective morphism $X \to \text{Spec}(R)$, with $X$ an integral scheme, such that tensored with $K$ over $R$, it is $X \to \text{Spec}(K)$. As in [7] and [3], we consider $\ell$-adic cohomology $H^i(\bar{X})$ with $\mathbb{Q}_\ell$-coefficients. Recall briefly that one defines the first coniveau level

$$N^1H^i(X) = \{\alpha \in H^i(X), \exists \text{ divisor } D \subset X \text{ s.t. } 0 = \alpha|_{X \setminus D} \in H^i(X \setminus D)\}.$$

As $H^i(\bar{X})$ is a finite dimensional $\mathbb{Q}_\ell$-vector space, one has by localization

$$\exists D \subset X \text{ s.t. } N^1H^i(\bar{X}) = \text{Im}(H^i_D(\bar{X}) \to H^i(\bar{X})),
$$

where $D \subset X$ is a divisor. One says that $H^i(\bar{X})$ is supported in codimension 1 if $N^1H^i(\bar{X}) = H^i(\bar{X})$. The purpose of this note is twofold. We show the following theorem.

Theorem 1.1. Let $X$ be a smooth, projective, absolutely irreducible variety defined over a local field $K$ with finite residue field $k$. Assume that $\ell$-adic cohomology $H^i(\bar{X})$ is supported in codimension $\geq 1$ for all $i \geq 1$. Let $X$ be a model of $X$ over the ring of integers $R$ of $K$. Then there is a projective surjective morphism $\sigma : Y \to X$ of $R$-schemes such that

$$|Y(k)| \equiv 1 \text{ mod } |k|.$$

In particular, any model $X/R$ of $X/K$ has a $k$-rational point.

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This generalizes [8, Theorem 1.1] where the theorem is proven under the assumption that $K$ has characteristic 0. On the other hand, assuming that $X$ is regular, we showed in [7, Theorem 1.1] that the number of $k$-rational points $|X(k)|$ is congruent to 1 modulo $|k|$. It was in fact the way to show that $k$-rational points exist on $X$, as surely $|k|$, being a $p$-power, where $p$ is the characteristic of $k$, is $>1$. We show that if we drop the regularity assumption, there are models which, according to Theorem 1.1, have a rational point, but do not satisfy the congruence.

**Theorem 1.2.** Let $X_0 = \mathbb{P}^2$ over $K_0 := \mathbb{Q}_p$ or $\mathbb{F}_p((t))$. Then there is a finite field extension $K \supset K_0$, which can be chosen to be unramified, and there is a normal model $\mathcal{X}/R$ of $X := X_0 \otimes_{K_0} K$, such that $|\mathcal{X}(k)|$ is not congruent to 1 modulo $|k|$.

The proof of Theorem 1.1 follows closely the one in unequal characteristic in [8, Theorem 1.1], and, aside of Deligne’s integrality theorem [5, Corollaire 5.5.3] and [7, Appendix] and purity [9], relies strongly on de Jong’s alteration theorem as expressed in [4]. However, we have to replace the trace argument we used there by a more careful analysis of the Leray spectral sequence stemming from de Jong’s construction. The construction of the examples in Theorem 1.2 uses Artin’s contraction theorem as expressed in [1] and is somewhat inspired by Kollár’s construction exposed in [2, Section 3.3].

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2. **Proof of Theorem 1.1**

This section is devoted to the proof of Theorem 1.1.

Let $K$ be a local field with finite residue field $k$. Let $R \subset K$ be its valuation ring. Let $\mathcal{X} \to \text{Spec } R$ be a model of a projective variety $X \to \text{Spec } K$. We do not assume here that $X$ is absolutely irreducible, nor do we assume that $X/K$ is smooth. Then by [4 Corollary 5.15], there is a diagram

$$
\begin{array}{ccc}
\mathcal{Z} & \xrightarrow{\pi} & \mathcal{Y} \\
\downarrow & & \uparrow
\end{array}
\begin{array}{c}
\mathcal{X} \\
\sigma \\
\downarrow
\end{array}
\begin{array}{c}
\text{Spec } R
\end{array}
$$

and a finite group $G$ acting on $\mathcal{Z}$ over $\mathcal{Y}$ with the properties

1. $\mathcal{Z} \to \text{Spec } R$ and $\mathcal{Y} \to \text{Spec } R$ are flat,
2. $\sigma$ is projective, surjective, $K(\mathcal{X}) \subset K(\mathcal{Y})$ is a purely inseparable field extension,
3. $\mathcal{Y}$ is the quotient of $\mathcal{Z}$ by $G$,
4. $\mathcal{Z}$ is regular.
We want to show that this $Y$ does it. Let us set

$$Y = \mathcal{Y} \otimes K, \ Z = \mathcal{Z} \otimes K.$$  

The only difference with [8, (2.1)] is that $K(X) \subset K(Y)$ may be a purely inseparable extension rather than an isomorphism. Thus, the argument there breaks down as one does not have traces as in [8, (2.3), (2.4)]. We do not have [8, (2.5)] a priori, and we can’t conclude [8, Claim 2.1].

Let us overtake the notations of loc. cit.: we endow all schemes considered (which are $R$-schemes) with the upper subscript $u$ to indicate the base change $\otimes_R R^u$ or $\otimes_K K^u$, where $K^u \supset K$ is the maximal unramified extension, and $R^u \supset R$ is the normalization of $R$ in $K^u$. Likewise, we write $\bar{\ }$ to indicate the base change $\otimes_R \bar{R}, \otimes_K \bar{K}, \otimes_{k^u} k^u$, where $\bar{K} \supset K$, $\bar{k} \supset k$ are the algebraic closures and $\bar{R} \supset R$ is the normalization of $R$ in $\bar{K}$. We consider as in [7, (2.1)] the $F$-equivariant exact sequence ([6, 3.6(6)])

$$\ldots \rightarrow H^i_B(\mathcal{Y}^u) \xrightarrow{i} H^i(\bar{B}) = H^i(\mathcal{Y}^u) \xrightarrow{sp^u} H^i(Y^u) \rightarrow \ldots ,$$

where $F \in \text{Gal}(\bar{k}/k)$ is the geometric Frobenius, and $B = Y \otimes k$. We have [8, Claim 2.2] unchanged:

**Claim 2.1.** The eigenvalues of the geometric Frobenius $F \in \text{Gal}(\bar{k}/k)$ acting on $i(H^i_B(\mathcal{Y}^u)) \subset H^i(\bar{B})$ lie in $q \cdot \bar{\mathbb{Z}}$ for all $i \geq 1$.

So the problem is to show that the eigenvalues of $F$ acting on $\text{Im}(sp^u) \subset H^i(Y^u)$ lie in $q \cdot \bar{\mathbb{Z}}$ as well. Let us decompose the morphism $\sigma$ as

$$\sigma : Y \xrightarrow{\tau} X_1 \xrightarrow{\epsilon} X$$

where $X_1$ is the normalization of $X$ in $K(Y)$. Thus in particular, $\tau$ is birational, $\epsilon$ is finite and purely inseparable. Let us denote by $U \subset X$ a non-empty open such that $\tau|_{\epsilon^{-1}(U)}$ is an isomorphism, and let us set $D := X \setminus U$. We define

$$\mathcal{C} := \text{cone}(\mathbb{Q}_\ell \rightarrow R\tau_*\mathbb{Q}_\ell)[-1]$$

as an object in the bounded derived category of $\mathbb{Q}_\ell$-constructible sheaves on $X_1$. Since $\tau_*\mathbb{Q}_\ell = \mathbb{Q}_\ell$, the cohomology sheaves of $\mathcal{C}$ are in degree $\geq 1$, and have support in $D_1 := D \times_X X_1$. We conclude

$$H^i_{D^u_1}(X^u_1, \mathcal{C}) = H^i(X^u_1, \mathcal{C}) \forall i \geq 0.$$
One has the commutative diagram of exact sequences

\[
\begin{array}{c}
H^{i+1}_{D^u}(X^u) \\
\downarrow \\
H^i_{D^u}(X_1^u, C) \\
\downarrow \\
H^i(Y^u) \\
\downarrow \\
H^i(X^u)
\end{array}
\]

where \( E = \sigma^{-1}(D) \). By [7, Theorem 1.5 and Appendix] the eigenvalues of \( F \) on \( H^i(X^u) = H^i(X_1^u) \) and on \( H^{i+1}_{D^u}(X_1^u) = H_{D^u}^{i+1}(X^u) \) lie in \( q \cdot \bar{\mathbb{Z}} \). For the latter cohomology, one has to argue again by purity on \( X^u \) before applying loc. cit.: by purity one is reduced to considering cohomology of the type \( H^a(\Sigma^u)(-1) \) for a regular scheme \( \Sigma \) over \( K \) and \( a \geq 0 \). It remains to consider the eigenvalues of \( F \) on \( H^i_{E^u}(Y^u) = H^i_{L^u}(Z^u)^G \) where \( L = D \times X \). This is again the argument by purity and then loc. cit. So we conclude

**Claim 2.2.** The eigenvalues of the geometric Frobenius \( F \in \text{Gal}(\bar{k}/k) \) acting on \( H^i(Y^u) \), and therefore on \( \text{Im}(sp^u) \subset H^i(Y^u) \), lie in \( q \cdot \bar{\mathbb{Z}} \) for all \( i \geq 1 \).

So we conclude now as usual that all the eigenvalues of \( F \) acting on \( H^i(\bar{B}) \) lie in \( q \cdot \bar{\mathbb{Z}} \) for \( i \geq 1 \), thus the Grothendieck-Lefschetz trace formula applied to \( H^*(\bar{B}) \), together with the absolute irreducibility of \( B \), imply the congruence. This finishes the proof of Theorem 1.1.

### 3. Construction of examples

This section is devoted to the proof of Theorem 1.2.

Let us first recall that if \( E \) is a smooth genus 1 curve over a finite field \( \mathbb{F}_q \), it is always an elliptic curve, which means that it always carries a \( \mathbb{F}_q \)-rational point. Furthermore one has

**Claim 3.1.** Given an elliptic curve \( E/\mathbb{F}_q \), there is a finite field extension \( \mathbb{F}_{q^n} \supset \mathbb{F}_q \) such that \( |E(\mathbb{F}_{q^n})| \) is not congruent to 1 modulo \( q^n \).

**Proof.** By the trace formula, \( |E(\mathbb{F}_{q^n})| \) being congruent to 1 modulo \( q^n \) for all \( n \geq 1 \) is equivalent to saying that the eigenvalues of \( F^n \) acting on \( H^i(\bar{E}) \) lie in \( q^n \cdot \bar{\mathbb{Z}} \) for all \( n \geq 1 \) and \( i \geq 1 \). By purity (which in dimension 1 is Weil’s theorem), this is equivalent to saying that the eigenvalues of \( F^n \) acting on \( H^1(\bar{E}) \) lie in \( q^n \cdot \bar{\mathbb{Z}} \) for all \( n \geq 1 \). On the other hand, by duality, if \( \lambda \) is an eigenvalue, then \( \frac{q^n}{\lambda} \) is
also an eigenvalue. This is then impossible that both \( \lambda \) and \( \frac{\varepsilon}{\lambda} \) be \( q^n \)-divisible as algebraic integers.

□

We now construct the following scheme. Let us set \( \mathcal{P}_0 := \mathbb{P}^2 \) over \( R_0 := \mathbb{Z}_p \) or over \( \mathbb{F}_p[[t]] \). Choose an elliptic curve \( E_0 \subset \mathcal{P} \otimes \mathbb{F}_p = \mathbb{P}_p^2 \) defined over \( \mathbb{F}_p \). Let \( k \supset \mathbb{F}_p \) be a finite field extension such that \( |E_0(k)| \) is not \( k \)-divisible (Claim 3.1). Set \( E := E_0 \otimes_{\mathbb{F}_p} k \), \( \mathcal{P} := \mathcal{P}_0 \otimes_{R_0} R \), with \( R = W(k) \) or \( \mathbb{F}_q[[t]] \), and \( K = \text{Frac}(R) \). Choose a smooth projective curve \( C \subset \mathcal{P} \) of strictly increasing degree as \( a_n \) belongs to \( \mathcal{P} \) be the blow up of \( \Sigma \), \( n \) is degree \( \geq n \) for \( C \) trivial.\( \mathcal{P} \).\( H^2 \), \( N \) a locally free filtered sheaf, with associated graded a sum of ample line bundles \( \mathcal{E} \), \( I \subset \mathcal{O}_Y \) be the normalization of \( \Sigma \) in \( Y \) by the condition on the degree of \( \Sigma \). Then the conormal bundle \( N_{E/Y} \) of \( E \) in \( Y \) is an extension of the conormal bundle \( N_{E/Y} \) of \( E \) in \( Y \) by the restriction to \( E \) of the conormal bundle \( N_{Y/Y} \) of \( Y \) in \( Y \), both ample line bundles on \( E \) by the condition on the degree of \( \Sigma \).

Let \( I \subset \mathcal{O}_Y \) be the ideal sheaf of \( E \). For a coherent sheaf \( \mathcal{F} \) on \( Y \), we denote by \( I^n/I^{n+1} \cdot \mathcal{F} \) the image of \( I^n/I^{n+1} \otimes_{\mathcal{O}_Y} \mathcal{F} \) in \( \mathcal{F} \), where \( n \in \mathbb{N} \).

**Claim 3.2.** For every coherent sheaf \( \mathcal{F} \) on \( Y \), one has \( H^1(E, I^n/I^{n+1} \cdot \mathcal{F}) = 0 \) for all \( n \in \mathbb{N} \) large enough.

**Proof.** As by definition one has a surjection \( I^n/I^{n+1} \otimes_{\mathcal{O}_Y} \mathcal{F} \rightarrow I^n/I^{n+1} \cdot \mathcal{F} \), it is enough to show \( H^1(E, I^n/I^{n+1} \otimes_{\mathcal{O}_Y} \mathcal{F}) = 0 \) for \( n \) large enough. As \( I^n/I^{n+1} \) is locally free, \( I^n/I^{n+1} \otimes_{\mathcal{O}_Y} \mathcal{F} \) is an extension of \( I^n/I^{n+1} \otimes_{\mathcal{O}_Y} \mathcal{F}_0 \) by \( I^n/I^{n+1} \otimes_{\mathcal{O}_Y} \mathcal{T} \), where \( \mathcal{T} \subset \mathcal{F} \) is the maximal torsion subsheaf and \( \mathcal{F}_0 = \mathcal{F}/\mathcal{T} \) is locally free. As \( H^1(E, I^n/I^{n+1} \otimes_{\mathcal{O}_Y} \mathcal{T}) = 0 \), we may assume that \( \mathcal{F} \) is locally free. As \( I^n/I^{n+1} \) is a locally free filtered sheaf, with associated graded a sum of ample line bundles of strictly increasing degree as \( n \) grows, we have \( H^1(E, \text{gr}(I^n/I^{n+1}) \otimes_{\mathcal{O}_Y} \mathcal{F}) = 0 \) for \( n \) large enough, and thus \( H^1(E, I^n/I^{n+1} \otimes_{\mathcal{O}_Y} \mathcal{F}) = 0 \) as well.

□

Artin’s contraction criterion [11, Theorem 6.2] applied to \( E \rightarrow \text{Spec}(k) \), together with Artin’s existence theorem [11, Theorem 3.1] show the existence of a contraction

\[
(3.1) \quad a_1 : \mathcal{Y} \rightarrow \mathcal{X}_1
\]

where \( \mathcal{X}_1 \) is an algebraic space over \( R \), \( a_1|_{\mathcal{Y}\backslash E} \) is an isomorphism and \( a_1(E) = \text{Spec}(k) \). Let \( \mathcal{X} \rightarrow \mathcal{X}_1 \) be the normalization of \( \mathcal{X}_1 \) in \( K(\mathcal{Y}) = K(\mathcal{P}) \). This is a
normal algebraic space over $R$. One has a diagram

$$\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{a_1} & \mathcal{X} \\
\downarrow b & & \downarrow \nu \\
\mathcal{P} & \quad & \mathcal{X}_1
\end{array}$$

**Claim 3.3.** $|\mathcal{X}(k)|$ is not congruent to 1 modulo $|k|$.  

**Proof.** By [1, Theorem 1.1] (or by a simple computation in this case), $|\mathcal{Y}(k)|$ is congruent to 1 modulo $|k|$. By Claim 3.1 and the choice of $E$, $|\mathcal{X}_1(k)|$ is not congruent to 1 modulo $|k|$. On the other hand, as the fibers of $a_1$ are absolutely irreducible, $\nu$ has to be a homeomorphism. Thus $|\mathcal{X}(k)| = |\mathcal{X}_1(k)|$. This finishes the proof. 

In order to finish the proof of Theorem 1.2, it remains to show

**Claim 3.4.** $\mathcal{X} \to \text{Spec}(R)$ is a model of $X = \mathbb{P}^2/K$.

**Proof.** We have to show that $\mathcal{X} \to \text{Spec}(R)$ is a flat projective morphism. Since $\mathcal{X}$ is reduced, $\text{Spec}(R)$ is regular of dimension 1, then [10, IV Proposition 14.3.8] allows to conclude that $\mathcal{X}/R$ is flat. Thus we just have to show that $\mathcal{X}/R$ is projective. To this aim, we want to descend a line bundle from $\mathcal{Y}$ to $\mathcal{X}$. Let us define the line bundle $\mathcal{M} := b^* \mathcal{O}_\mathcal{P}(C)(-P_\Sigma)$ on $\mathcal{Y}$. By definition, one has

$$\mathcal{M}|_E \cong \mathcal{O}_E.$$  

**Claim 3.5.** The line bundle $\mathcal{M}$ descends to $\mathcal{X}$, that is there is a line bundle $\mathcal{L}$ on $\mathcal{X}$ with $a^* \mathcal{L} = \mathcal{M}$.

**Proof of Claim 3.5.** The proper morphism of algebraic spaces $a : \mathcal{Y} \to \mathcal{X}$, with $a_* \mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$, has the property that $a^{-1}(E) = E$ set-theoretically, that $a|_{\mathcal{Y}\setminus E} : \mathcal{Y}\setminus E \to \mathcal{X}\setminus a(E)$ is an isomorphism, and that $H^1(E, I^n/I^{n+1}) = 0$ for $n \geq 1$. So Keel’s theorem [11, Lemma 1.10] asserts that some positive power $\mathcal{M}^{\otimes r}$ descends to $\mathcal{X}$ if the following condition is fulfilled

$$\forall m > 0, \exists r(m) > 0 \text{ s.t. } \mathcal{M}^{\otimes r(m)}|_{E_m} \text{ descends to } a(E_m)$$  

where $E_m := \text{Spec}(\mathcal{O}_\mathcal{Y}/I^{m+1})$.

So we just have to check that (3.4) is fulfilled with $r = 1$ in our situation. The scheme $a(E_m)$ has Krull dimension 0. Thus by Hilbert 90’s theorem (see e.g. [12, Corollary 11.6]) one has

$$\text{Pic}(a(E_m)) = 0.$$  

We conclude that to check (3.4) is equivalent to checking that $\mathcal{M}^{\otimes r(m)}|_{E_m} \cong \mathcal{O}_{E_m}$ for some positive power $r(m)$. In fact one has

$$\mathcal{M}|_{E_m} \cong \mathcal{O}_{E_m} \forall m \geq 1.$$
For $m = 1$, this is (3.3). We argue by induction and assume that for $m > 1$, we have a trivializing section $s_m : \mathcal{O}_{E_m} \cong \mathcal{M}|_{E_m}$. We want to show that it lifts to a trivializing section $s_{m+1} : \mathcal{O}_{E_{m+1}} \cong \mathcal{M}|_{E_{m+1}}$. One has an exact sequence

\[(3.7) \quad 0 \to I^{m+1}/I^{m+2} \to \mathcal{M}|_{E_{m+1}} \to \mathcal{M}|_{E_m} \to 0.\]

Since $H^1(E, I^{m+1}/I^{m+2}) = 0$, as $m \geq 0$, the trivializing section $s_m : \mathcal{O}_{E_m} \cong \mathcal{M}|_{E_m}$ lifts to a section $s_{m+1} : \mathcal{O}_{E_{m+1}} \to \mathcal{M}|_{E_{m+1}}$, and likewise, its inverse $t_m : \mathcal{M}|_{E_m} \cong \mathcal{O}_{E_m}$ lifts to $t_{m+1} : \mathcal{M}|_{E_{m+1}} \to \mathcal{O}_{E_{m+1}}$. The composite $t_{m+1} \circ s_{m+1} : \mathcal{O}_{E_{m+1}} \to \mathcal{O}_{E_{m+1}}$ lifts the identity of $\mathcal{O}_{E_m}$. Therefore it is invertible. This shows that $s_{m+1}$ trivializes. The proof of Keel’s theorem (see (2) after [11, (1.10.1)]) shows then that one can take $r = 1$.

In order to finish the proof of Claim 3.4, it remains to see that $\mathcal{L}$ on $\mathcal{X}$ is ample. First, $\mathcal{L}|_{X \otimes k}$ is ample because by [11 Corollary 0.3], this is enough to see that the linear system associated to $\mathcal{L}|_{X \otimes k}$ does not contract any curve, which is true by construction. So by Serre vanishing theorem, for sufficiently large $m$, $H^1(X \otimes k, \mathcal{L}|_{X \otimes k}) = 0$. Base change implies $H^1(X, \mathcal{L}^\otimes m) \otimes k = 0$ ([10 III Theorem 7.7.5]), thus by Nakayama’s lemma, one has

\[(3.8) \quad H^1(X, \mathcal{L}^\otimes m) = 0 \text{ for } m \text{ large enough.}\]

As $\mathcal{L}$ is invertible, multiplication $\mathcal{L}^\otimes m \xrightarrow{\pi} \mathcal{L}^\otimes m$ by the uniformizer $\pi$ is injective, with quotient $\mathcal{L}|_{X \otimes k}$. Thus (3.8) implies surjectivity $H^0(X, \mathcal{L}^\otimes m) \to H^0(X \otimes k, \mathcal{L}|_{X \otimes k})$ for $m$ large enough. Thus $H^0(X, \mathcal{L}^\otimes m)$ is a free $R$-module, and the linear system $H^0(X, \mathcal{L}^\otimes m)$ maps base point freely $X$ to $\mathbb{P}^N_R$, with $N + 1 = \text{rank}_R H^0(X, \mathcal{L}^\otimes m)$. As it embeds $X \otimes k$, it embeds $X$ as well. This finishes the proof.

\[\square\]

4. Dimension 1

Remark 4.1. In Theorem 1.1 if $X/K$ has dimension 1, which means concretely if $X/K = \mathbb{P}^1/K$, then any normal model $\mathcal{X}/R$ satisfies the congruence $|\mathcal{X}(k)| \equiv 1$ modulo $|k|$. Thus the examples of Theorem 1.2 have the smallest possible dimension.

Proof. Indeed, using (2.1), the only thing to check is that $H^1(A)$, which is equal to $H^1(\mathcal{X}^u)$, injects via $\sigma^*$ into $H^1(B) = H^1(\mathcal{Y}^u)$. Here $A := \mathcal{X} \otimes_R k$. Let us denote by $\mathcal{X}'$ the normalization of $\mathcal{X}$ in $K(\mathcal{Y})$, with factorization

\[
\begin{aligned}
\mathcal{Y} & \xrightarrow{\sigma} \mathcal{X}' \xrightarrow{\nu} \mathcal{X} \\
\end{aligned}
\]
and set $A' := A \times_{X} X'$. Then $\sigma'$ induces an isomorphism $K(X') \xrightarrow{\sim} K(Y)$. Furthermore, $X' \xrightarrow{\nu'} X$ and $A' \xrightarrow{\nu A} A$ are homeomorphisms. Thus $H^1(X^u) = H^1(\bar{A}) \xrightarrow{\nu^u} H^1((X')^u) = H^1(\bar{A}')$ is an isomorphism. On the other hand, since $\sigma^*_\nu Q_\ell = Q_\ell$, the Leray spectral sequence for $\sigma'$ applied to $H^1(Y^u)$ yields an inclusion $H^1((X')^u) = H^1(\bar{A}') \hookrightarrow H^1(Y^u) = H^1(\bar{B})$. This finishes the proof. □

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