A Hardy-Littlewood-Sobolev type inequality for variable exponents and applications to quasilinear Choquard equations involving variable exponent

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Abstract

In this work, we have proved a version of the Hardy-Littlewood-Sobolev inequality for variable exponents. After we use the variational method to establish the existence of solution for a class of Choquard equations involving the $p(x)$-Laplacian operator.

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1 Introduction

The stationary Choquard equation

$$-\Delta u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x-y|^\lambda} |u|^{p-2}u \right) \text{ in } \mathbb{R}^N \tag{1.1}$$

where $N \geq 3$, $0 < \lambda < N$, arises in many interesting physical situations in quantum theory and plays particularly an important role in the theory of Bose-Einstein condensation where it accounts for the finite-range many-body interactions. For $N = 3$, $p = 2$ and $\lambda = 1$, it was investigated by Pekar in [45] to study the quantum theory of a polaron at rest. In [35], Choquard applied it as approximation to Hartree-Fock theory of one-component plasma. This equation was also proposed by Penrose in [41] as a model of selfgravitating matter and is known in that context as the Schrödinger-Newton equation.

Motivated by these facts, at the last years a lot of articles have studied the existence and multiplicity of solutions for some equations which are in some sense related to the problem (1.1), we would like to cite the articles due to Ackermann [3], Alves & Yang [12, 13], Cingolani, Secchi & Squassina [17], Gao & Yang [33], Lions [37], Ma & Zhao [38], Moroz & Van Schaftingen [42, 43, 44] and their references.

In all the above mentioned papers the authors have used variational methods to show the existence of solution. This method works well thanks to a Hardy-Littlewood-Sobolev type inequality [36] which has the following statement

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**Proposition 1.1** (Hardy-Littlewood-Sobolev inequality). Let \( t, r > 1 \) and \( 0 < \lambda < N \) with \( 1/t + \lambda/N + 1/r = 2 \), \( f \in L^t(\mathbb{R}^N) \) and \( h \in L^r(\mathbb{R}^N) \). There exists a sharp constant \( C(t, N, \mu, r) \), independent of \( f \), \( h \), such that

\[
\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^{\lambda}} \, dx \, dy \right| \leq C(t, N, \mu, r) \| f \|_{L^t(\mathbb{R}^N)} \| h \|_{L^r(\mathbb{R}^N)}. \tag{1.2}
\]

Motivated by the above papers, we intend to study the existence of solution for the following class of quasilinear problem

\[
\begin{cases}
-\Delta_{p(x)} u + V(x)|u|^{p(x)-2}u = \left( \int_{\mathbb{R}^N} \frac{F(x,u(x))}{|x-y|^{\lambda(x,y)}} \right) f(y,u(y)) \quad \text{in } \mathbb{R}^N, \\
u \in W^{1,p(x)}(\mathbb{R}^N),
\end{cases}
\tag{1.3}
\]

where \( V, p : \mathbb{R}^N \to \mathbb{R}, \lambda : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \) and \( f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \) are continuous functions, \( F(x,t) \) is the primitive of \( f(x,t) \), that is,

\[
F(x,t) = \int_0^t f(x,s) \, ds
\]

and \( \Delta_{p(x)} \) denotes the \( p(x) \)-Laplacian given by

\[
\Delta_{p(x)} u = \text{div}(|\nabla u|^{p(x)-2} \nabla u).
\]

Our intention, as in the papers mentioned above, is to use variational methods. Having this in mind, we must prove a version of the Hardy-Littlewood-Sobolev type inequality which works well for variable exponents. One of the main difficulty is to show that the energy functional associated with (1.3) given by

\[
J(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \left( |\nabla u|^{p(x)} + V(x)|u|^{p(x)} \right) - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x,u(x))F(y,u(y))}{|x-y|^{\lambda(x,y)}} \, dx \, dy
\]

is well defined and belongs to \( C^1(W^{1,p(x)}(\mathbb{R}^N), \mathbb{R}) \). In fact the main difficulty is to prove that the functional \( \Psi : W^{1,p(x)}(\mathbb{R}^N) \to \mathbb{R} \) given by

\[
\Psi(u) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x,u(x))F(y,u(y))}{|x-y|^{\lambda(x,y)}} \, dx \, dy
\]

belongs to \( C^1(W^{1,p(x)}(\mathbb{R}^N), \mathbb{R}) \) with

\[
\Psi'(u)v = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x,u(x)f(y,u(y))v(y))}{|x-y|^{\lambda(x,y)}} \, dx \, dy, \quad \forall u, v \in W^{1,p(x)}(\mathbb{R}^N).
\]

The \( p(x) \)-Laplacian operator possesses more complicated nonlinearity than the \( p \)-Laplacian. For instance, it is inhomogeneous and in general, it has no first eigenvalue, that is, the infimum of the eigenvalues of \( p(x) \)-Laplacian equals 0 (see [34]). Thus, transposing the results obtained with the \( p \)-Laplacian to problems with the \( p(x) \)-Laplacian operator is not
an easy task. The study of these problems are often very complicated and require relevant topics of nonlinear functional analysis, especially the theory of variable exponent Lebesgue and Sobolev spaces (see, e.g., [18] and its abundant reference).

Partial differential equations involving the \( p(x) \)-Laplacian arise, for instance, as a mathematical model for problems involving electrorheological fluids and image restorations, see [1, 2, 14, 15, 16, 46]. This explains the intense research on this subject in the last decades, see for example the papers [4, 5, 6, 7, 8, 10, 19, 20, 21, 23, 26, 27, 30, 31, 32, 40] and their references.

The plan of the paper is as follows: In Section 2 we recall some facts involving the variable exponent Sobolev space and prove our version of the Hardy-Littlewood-Sobolev inequality for variable exponents. In Section 3 we show that \( \Psi \) is \( C^1 \) and in Section 4 we study the existence of solution of (1.3) by assuming some conditions on \( V(x) \) and \( f(x,t) \).

2 Variable exponent Sobolev space

In this section we recall some results on variable exponent Sobolev spaces. The reader is referred to [27, 22] and the references therein for more details.

In the sequel, we set

\[
C^+(\mathbb{R}^N) := \{ h \in C(\mathbb{R}^N) : 1 < h^- \leq h^+ < +\infty \}
\]

where

\[
h^+ := \sup_{x \in \mathbb{R}^N} h(x) \quad \text{and} \quad h^- := \inf_{x \in \mathbb{R}^N} h(x).
\]

For \( p \in C^+(\mathbb{R}^N) \), we consider the Lebesgue space

\[
L^{p(x)}(\mathbb{R}^N) = \left\{ u; \text{ } u \text{ is a measurable real-valued function, } \int_{\mathbb{R}^N} |u(x)|^{p(x)} \, dx < +\infty \right\}
\]

which becomes a Banach space when endowed with the Luxemburg norm

\[
\|u\|_{L^{p(x)}(\mathbb{R}^N)} = \inf \left\{ \alpha > 0; \int_{\mathbb{R}^N} \left( \frac{u(x)}{\alpha} \right)^{p(x)} \, dx \leq 1 \right\}.
\]

Proposition 2.1. The functional \( \rho_p : L^{p(x)}(\mathbb{R}^N) \to \mathbb{R} \) defined by

\[
\rho_p(u) = \int_{\mathbb{R}^N} |u(x)|^{p(x)} \, dx
\]

has the following properties:

(i) \( \|u\|_{L^{p(x)}(\mathbb{R}^N)} < 1(=1;>1) \iff \rho_p(u) < 1(=1;>1) \)

(ii) \( \|u\|_{L^{p(x)}(\mathbb{R}^N)} > 1 \implies \|u\|_{L^{p(x)}(\mathbb{R}^N)}^+ \leq \rho_p(u) \leq \|u\|_{L^{p(x)}(\mathbb{R}^N)}^+ \)

\( \|u\|_{L^{p(x)}(\mathbb{R}^N)} < 1 \implies \|u\|_{L^{p(x)}(\mathbb{R}^N)}^- \leq \rho_p(u) \leq \|u\|_{L^{p(x)}(\mathbb{R}^N)}^- \)
Proposition 2.2. We have the following generalized Hölder inequality.

\[ g \] are continuous functions. The following embeddings will be used in this work.

\[(27)\] equipped with the norm \( p \) \( r > 0 \) can be found in \([29]\). For \( y \) \( r > 0 \) works well for some problems like \((1.3)\).

Let \( u \), \( p \)

\[ \frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \text{ a.e } x \in \mathbb{R}^N. \]

We have the following generalized Hölder inequality.

**Proposition 2.2 ([39]).** For any \( u \in L^{p(x)}(\mathbb{R}^N) \) and \( v \in L^{p'(x)}(\mathbb{R}^N) \),

\[ \left| \int_{\mathbb{R}^N} u(x)v(x)dx \right| \leq 2\|u\|_{L^{p(x)}(\mathbb{R}^N)}\|v\|_{L^{p'(x)}(\mathbb{R}^N)}. \]

The Banach space \( W^{1,p(x)}(\mathbb{R}^N) \) is defined as

\[ W^{1,p(x)}(\mathbb{R}^N) := \{ u \in L^{p(x)}(\mathbb{R}^N); |\nabla u| \in L^{p(x)}(\mathbb{R}^N) \} \]

equipped with the norm

\[ \|u\|_{W^{1,p(x)}(\mathbb{R}^N)} := \|u\|_{L^{p(x)}(\mathbb{R}^N)} + \|\nabla u\|_{L^{p(x)}(\mathbb{R}^N)}. \]

In what follows, we denote by \( h \ll g \) provided \( \inf \{h(x) - g(x); x \in \mathbb{R}^N\} > 0 \) where \( h \) and \( g \) are continuous functions. The following embeddings will be used in this work.

**Proposition 2.3 ([27]).** Let \( p : \mathbb{R}^N \to \mathbb{R} \) be a Lipschitz continuous function with \( 1 < p^- \leq p^+ < N \) and \( s \in C^+(\mathbb{R}^N) \).

\( (i) \) If \( p \leq s \leq p^* \), then there is a continuous embedding \( W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{s(x)}(\mathbb{R}^N) \).

\( (ii) \) If \( p \leq s \ll p^* \) then there is a compact embedding \( W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{s(x)}_{\text{loc}}(\mathbb{R}^N) \),

where \( p^*(x) := Np(x)/(N - p(x)) \) for all \( x \in \mathbb{R}^N \).

We also need the following variable exponent generalization of the Lions’ Lemma that can be found in \([29]\). For \( r > 0 \) and \( y \in \mathbb{R}^N \) we denote by \( B_r(y) \) the open ball in \( \mathbb{R}^N \) with center \( y \) and radius \( r \).

**Lemma 2.1.** Let \( p : \mathbb{R}^N \to \mathbb{R} \) be a Lipschitz continuous function with \( 1 < p^- \leq p^+ < N \). If \( (u_n) \) is a bounded sequence in \( W^{1,p(x)}(\mathbb{R}^N) \) such that

\[ \lim_{n \to +\infty} \sup_{y \in \mathbb{R}^N} \int_{B_r(x)} |u_n(x)|^{p(x)}dx = 0 \]

for some \( r > 0 \), then \( u_n \to 0 \) in \( L^{q(x)}(\mathbb{R}^N) \) for any \( q \in C^+(\mathbb{R}^N) \) satisfying \( p \ll q \ll p^* \).

Next, we prove a Hardy-Littlewood-Sobolev type inequality for variable exponents, which works well for some problems like \([13]\).
Proposition 2.4 (Hardy-Littlewood-Sobolev type inequality for variable exponents). Let $p, q \in C^+(\mathbb{R}^N)$ and consider functions $h \in L^{p^+}(\mathbb{R}^N) \cap L^{p^-}(\mathbb{R}^N)$ and $g \in L^{q^+}(\mathbb{R}^N) \cap L^{q^-}(\mathbb{R}^N)$. Let $\lambda : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function such that $0 < \lambda^- \leq \lambda^+ < N$ and

$$\frac{1}{p(x)} + \frac{\lambda(x, y)}{N} + \frac{1}{q(y)} = 2,$$  
for all $x, y \in \mathbb{R}^N$.

Then,

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{h(x)g(y)}{|x - y|^{\lambda(x, y)}} \, dx \, dy \right| \leq C \left( \|h\|_{L^{p^+}(\mathbb{R}^N)} \|g\|_{L^{p^+}(\mathbb{R}^N)} + \|h\|_{L^{p^-}(\mathbb{R}^N)} \|g\|_{L^{q^-}(\mathbb{R}^N)} \right),$$

where $C > 0$ is a constant that does not depend on $h$ and $g$.

Proof. First of all, note that

$$\lambda(x, y) = 2N \left( 1 - \frac{1}{2p(x)} - \frac{1}{2q(y)} \right) \leq 2N \left( 1 - \frac{1}{2p^+} - \frac{1}{2q^+} \right) \quad \text{for all } x, y \in \mathbb{R}^N.$$

Therefore,

$$\lambda^+ = \sup_{x, y \in \mathbb{R}^N} \lambda(x, y) \leq 2N \left( 1 - \frac{1}{2p^+} - \frac{1}{2q^+} \right).$$

Now, if $(x_n), (y_n) \subset \mathbb{R}^N$ are sequences satisfying

$$p(x_n) \to p^+ \quad \text{and} \quad q(y_n) \to q^+$$

we see that

$$\lambda(x_n, y_n) \to 2N \left( 1 - \frac{1}{2p^+} - \frac{1}{2q^+} \right)$$

from where it follows

$$\lambda^+ = 2N \left( 1 - \frac{1}{2p^+} - \frac{1}{2q^+} \right)$$

or equivalently

$$\frac{1}{p^+} + \frac{\lambda^+}{N} + \frac{1}{q^+} = 2. \quad (2.1)$$

A similar reasoning provides

$$\frac{1}{p^-} + \frac{\lambda^-}{N} + \frac{1}{q^-} = 2. \quad (2.2)$$

Once

$$\frac{1}{|x - y|^{\lambda(x, y)}} \leq \frac{1}{|x - y|^{\lambda^+}} + \frac{1}{|x - y|^{\lambda^-}} \quad \forall x, y \in \mathbb{R}^N,$$

we derive that

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{h(x)g(y)}{|x - y|^{\lambda(x, y)}} \, dx \, dy \right| \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|h(x)||g(y)|}{|x - y|^{\lambda^+}} \, dx \, dy + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|h(x)||g(y)|}{|x - y|^{\lambda^-}} \, dx \, dy$$

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Gathering (2.1), (2.2) and Proposition 1.2 we get
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|h(x)||g(y)|}{|x-y|^\lambda} \, dx \, dy \leq C \|h\|_{L^p^+ (\mathbb{R}^N)} \|g\|_{L^{q^+} (\mathbb{R}^N)}
\]
and
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|h(x)||g(y)|}{|x-y|^\lambda} \, dx \, dy \leq C \|h\|_{L^p^- (\mathbb{R}^N)} \|g\|_{L^{q^-} (\mathbb{R}^N)}.
\]
From the last two inequalities,
\[
\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{h(x)g(y)}{|x-y|^\lambda(x,y)} \, dx \, dy \right| \leq C(\|h\|_{L^p^+ (\mathbb{R}^N)} \|g\|_{L^{q^+} (\mathbb{R}^N)} + \|h\|_{L^p^- (\mathbb{R}^N)} \|g\|_{L^{q^-} (\mathbb{R}^N)})
\]
and the result is proved.

The next corollary is a key point in our arguments.

**Corollary 2.1.** Let \( q \in C^+(\mathbb{R}^N) \) and \( \lambda : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \) be a function satisfying
\[
\frac{1}{q(x)} + \frac{\lambda(x,y)}{N} + \frac{1}{q(y)} = 2, \; \forall x, y \in \mathbb{R}^N. \tag{2.3}
\]
If \( u \in W^{1,p(x)}(\mathbb{R}^N) \) and \( r \in \mathcal{M} \) where
\[
\mathcal{M} = \left\{ r \in C^+(\mathbb{R}^N) : p(x) \leq r(x)q^- \leq r(x)q^+ \leq p^*(x), \; \forall x \in \mathbb{R}^N \right\}, \tag{2.4}
\]
then \( U(x) = |u(x)|^r(x) \in L^{q^-}(\mathbb{R}^N) \cap L^{q^+}(\mathbb{R}^N) \). Moreover,
\[
\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^r(x)|u(y)|^r(y)}{|x-y|^\lambda(x,y)} \, dx \, dy \right| \leq C(\|u\|_{L^{2r^+}(\mathbb{R}^N)}^{2r^+} + \|u\|_{L^{2r^-}(\mathbb{R}^N)}^{2r^-}) + \|u\|_{L^{2r^+}(\mathbb{R}^N)}^{2r^+} + \|u\|_{L^{2r^-}(\mathbb{R}^N)}^{2r^-}).
\]

**Proof.** Using the Sobolev embedding, \( u \in L^{s(x)}(\mathbb{R}^N) \) for all \( s \in C^+(\mathbb{R}^N) \) with
\[
p(x) \leq s(x) \leq p^*(x), \; \forall x \in \mathbb{R}^N.
\]
Thereby, \( U(x) = |u(x)|^r(x) \in L^{q^+} \cap L^{q^-}(\mathbb{R}^N) \), because \( r \in \mathcal{M} \). Now, we use the Proposition 2.4 with \( p(x) = q(x) \) and \( h(x) = g(x) = U(x) \) to obtain the desired result.

Before continuing our study, we would like point out some important properties of the function \( \lambda(x,y) \) given in (2.3):

**Remark 2.1.**
1) The function \( \lambda \) is symmetric, that is,
\[
\lambda(x,y) = \lambda(y,x) \; \forall x, y \in \mathbb{R}^N.
\]
ii) If \( q \) is \( \mathbb{Z}^N \)-periodic, that is,
\[
q(x + y) = q(x) \quad \forall x \in \mathbb{R}^N \quad \text{and} \quad \forall y \in \mathbb{Z}^N,
\]
then \( \lambda \) is \( \mathbb{Z}^N \times \mathbb{Z}^N \)-periodic, that is,
\[
\lambda(x + z, y + w) = \lambda(x, y) \quad \forall x, y \in \mathbb{R}^N \quad \text{and} \quad \forall z, w \in \mathbb{Z}^N.
\]

iii) If \( q \) is radial, that is,
\[
q(x) = q(|x|), \quad \forall x \in \mathbb{R}^N
\]
then
\[
\lambda(x, y) = \lambda(|x|, |y|) \quad \forall x, y \in \mathbb{R}^N.
\]

The item i) in Remark 2.1 will be crucial in the proof of the differentiability of the functional \( \Psi \).

### 3 Differentiability of the functional \( \Psi \).

In this section, we will study the differentiability of the functional \( \Psi \). To this end, we must assume some conditions on \( f \). First of all, we fix \( q \in C^+(\mathbb{R}^N) \) and \( \lambda : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \) satisfying (2.3), that is,
\[
\frac{1}{q(x)} + \frac{\lambda(x, y)}{N} + \frac{1}{q(y)} = 2, \quad \forall x, y \in \mathbb{R}^N.
\]

The function \( f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \) is a continuous function verifying the following growth condition
\[
|f(x, t)| \leq C_1(|t|^{r(x)-1} + |t|^{s(x)-1}), \quad \forall (x, t) \in \mathbb{R}^N \times \mathbb{R} \quad (f_1)
\]
where \( C_1 > 0 \) and \( r, s \in \mathcal{M} \).

Note that the function \( F(x, t) := \int_0^t f(x, s)ds \) is continuous and
\[
|F(x, t)| \leq C_2(|t|^{r(x)} + |t|^{s(x)}), \quad \forall (x, t) \in \mathbb{R}^N \times \mathbb{R} \quad (F)
\]
for some positive constant \( C_2 \).

In the proof of differentiability of \( \Psi \) we will use the lemma below whose the proof we omit because it is very simple.

**Lemma 3.1.** Let \( E \) be a normed vectorial space and \( J : E \to \mathbb{R} \) be a functional verifying the following properties:

i) The directional derivative \( \frac{\partial J(u)}{\partial v} \) exists for all \( u, v \in E \).

ii) For each \( u \in E \), \( \frac{\partial J(u)}{\partial (\cdot)} \in E' \).

iii) \[
\begin{align*}
\text{if} \; & u_n \to u \; \text{in} \; E \implies \frac{\partial J(u_n)}{\partial (\cdot)} \to \frac{\partial J(u)}{\partial (\cdot)} \; \text{in} \; E'.
\end{align*}
\]
that is,
\[ u_n \to u \quad \text{in} \quad E \implies \sup_{\|v\| \leq 1} \left| \frac{\partial J(u_n)}{\partial v} - \frac{\partial J(u)}{\partial v} \right| \to 0. \]

Then, \( J \in C^1(E, \mathbb{R}) \) and
\[ J'(u)v = \frac{\partial J(u)}{\partial v}, \quad \forall u, v \in E. \]

After the above commentaries we are ready to prove the differentiability of \( \Psi \).

**Lemma 3.2.** The functional
\[ \Psi(u) := \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, u(x))F(y, u(y))}{|x - y|^\lambda(x,y)} dxdy, \]
for all \( u \in W^{1,p(x)}(\mathbb{R}^N) \) is well defined, for all \( u, v \in W^{1,p(x)}(\mathbb{R}^N) \) with
\[ \Psi'(u)v = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, u(x)f(y, u(y))v(y))}{|x - y|^\lambda(x,y)} dxdy, \]
for all \( u, v \in W^{1,p(x)}(\mathbb{R}^N) \).

**Proof.** Note that by Proposition 2.4 the functional \( \Psi \) is well defined. In the sequel, we will show that \( \Psi \) satisfies the assumptions of Lemma 3.1. To this end, we will divide the proof into three steps:

**Step 1: Existence of the directional derivative:**

Let \( u, v \in W^{1,p(x)}(\mathbb{R}^N) \) and \( t \in \mathbb{R} \) small enough. Note that
\[
\frac{\Psi(u + tv) - \Psi(u)}{t} = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, u(x) + tv(x))F(y, u(y) + tv(y)) - F(x, u(x))F(y, u(y))}{t|x - y|^\lambda(x,y)} dxdy.
\]

Denoting by \( I \) the integrand in (3.1), we have
\[
I = \frac{F(x, u(x) + tv(x))(F(y, u(y) + tv(y)) - F(y, u(y))}{t} + \frac{F(y, u(y))(F(x, u(x) + tv(x)) - F(x, u(x))}{t}.
\]

By the Mean Value Theorem there exists \( \theta(x, t), \eta(y, t) \in [0, 1] \), such that
\[
F(y, u(y) + tv(y)) - F(y, u(y)) = f(y, u(y) + \eta(y, t)tv(y))v(y)t
\]
and
\[
F(x, u(x) + tv(x)) - F(x, u(x)) = f(x, u(x) + \theta(x, t)tv(x))v(x)t.
\]

The relation (3.1) allows us to estimate
\[
\left| \frac{\Psi(u + tv) - \Psi(u)}{t} - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, u(x)f(y, u(y))v(y))}{|x - y|^\lambda(x,y)} dxdy \right| \leq |B_1| + |B_2|
\]
Such property combined with Fubini’s Theorem implies that
\[ B_1^t := \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F(x, u(x) + tv(x)) f(y, u(y) + \eta(y, t)tv(y))v(y) - F(x, u(x))f(y, u(y))v(y)\frac{dxdy}{|x-y|^\lambda(x,y)} \]
and
\[ B_2^t := \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F(y, u(y)) f(x, u(x) + \theta(x, t)tv(x))v(x)\frac{dxdy}{|x-y|^\lambda(x,y)} - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F(x, u(x))f(y, u(y))v(y)\frac{dxdy}{|x-y|^\lambda(x,y)} \]
By Remark 2.1, the function \( \lambda(x, y) \) is symmetric, that is,
\[ \lambda(x, y) = \lambda(y, x) \quad \forall x, y \in \mathbb{R}^N. \]
Such property combined with Fubini’s Theorem implies that
\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F(x, u(x))f(y, u(y))v(y)\frac{dxdy}{|x-y|^\lambda(x,y)} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F(y, u(y)) f(x, u(x))v(x)\frac{dydx}{|x-y|^\lambda(y,x)} \]
\[ = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F(y, u(y)) f(x, u(x))v(x)\frac{dxdy}{|x-y|^\lambda(y,x)} \]
\[ = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F(y, u(y)) f(x, u(x))v(x)\frac{dxdy}{|x-y|^\lambda(x,y)}. \]
Therefore
\[ B_2^t := \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F(y, u(y)) f(x, u(x) + \theta(x, t)tv(x))v(x) - F(y, u(y))f(x, u(x))v(x)\frac{dxdy}{|x-y|^\lambda(x,y)}. \]
Since \( \theta(x, t) \in [0, 1] \) and \( t \) is small,
\[ |f(x, u(x) + \theta(t, x)tv(x))v(x) - f(x, u(x))v(x)|^q \leq C(|u(x)|^{q^+(r(x)-1)}|v(x)|^{q^+} + |v(x)|^{q^+r(x)}) \]
\[ + C(|u(x)|^{q^+(s(x)-1)}|v(x)|^{q^+} + |v(x)|^{q^+s(x)} + |v(x)|^{q^+(r(x)-1)}|v(x)|^{q^+} + |u(x)|^{q^+(s(x)-1)}|v(x)|^{q^+}) \]
\[ \tag{3.2} \]
The growth conditions (2.4) ensure that the right side of the inequality (3.2) is an integrable function. Thus, the Lebesgue’s Theorem gives
\[ \|f(., u + \theta(., t)v)v - f(., u)v\|_{L^{q^+}(\mathbb{R}^N)} \rightarrow 0 \text{ as } t \rightarrow 0. \]
A similar argument provides
\[ \|f(., u + \theta(., t)v)v - f(., u)v\|_{L^{q^-}(\mathbb{R}^N)} \rightarrow 0 \text{ as } t \rightarrow 0. \]
Thereby, by Proposition 2.4
\[ |B_2^t| \leq C\|F(., u)\|_{L^{q^+}(\mathbb{R}^N)}\|f(., u + \theta(., t)v)v - f(., u)v\|_{L^{q^+}(\mathbb{R}^N)} \]
\[ + C\|F(., u)\|_{L^{q^-}(\mathbb{R}^N)}\|f(., u + \theta(., t)v)v - f(., u)v\|_{L^{q^-}(\mathbb{R}^N)} \]
which implies that $B'_2 \to 0$ as $t \to 0$. Related to $B'_1$, we have
\[
|B'_1| \leq \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |F(x, u(x))| \left| f(y, u(y) + \eta(y, t)tv(y))v(y) - f(y, v(y))v(y) \right| |x - y|^{\lambda(x,y)} dxdy \\
+ \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left| f(y, u(y) + \eta(y, t)tv(y))v(y) \right| \left| F(x, u(x) + tv(x)) - F(x, u(x)) \right| |x - y|^{\lambda(x,y)} dxdy.
\]
Arguing as above,
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|F(x, u(x))| \left| f(y, u(y) + \eta(y, t)tv(y))v(y) - f(y, v(y))v(y) \right| |x - y|^{\lambda(x,y)}}{dxdy} \to 0
\]
as $t \to 0$. On the other hand, the Lebesgue’s Theorem also implies that
\[
\|F(\cdot, u + tv) - F(\cdot, u)\|_{L^q(\mathbb{R}^N)} \to 0 \quad \text{as} \quad t \to 0 \tag{3.3}
\]
and
\[
\|F(\cdot, u + tv) - F(\cdot, u)\|_{L^q(\mathbb{R}^N)} \to 0 \quad \text{as} \quad t \to 0. \tag{3.4}
\]
As in (3.2), the quantities $\|f(\cdot, u + \eta(\cdot,t)tv)\|_{L^q(\mathbb{R}^N)}$ and $\|f(\cdot, u + \eta(\cdot,t)tv)\|_{L^q(\mathbb{R}^N)}$ are uniformly bounded by a constant which does not depend on $t$ small. Thus, the Proposition \ref{pro:2.4} combined with (3.3) and (3.4) give
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(y, u(y) + \eta(y, t)tv(y))v(y) - f(y, v(y))v(y)|}{|x - y|^{\lambda(x,y)}} dxdy \to 0
\]
as $t \to 0$, and so, $B'_1 \to 0$ as $t \to 0$. From the above analysis,
\[
\lim_{t \to 0} \frac{\Psi(u + tv) - \Psi(u)}{t} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, u(x)f(y, u(y))v(y))}{|x - y|^{\lambda(x,y)}} dxdy,
\]
showing the existence the existence of the directional derivative $\frac{\partial \Psi(u)}{\partial v}$.

**Step 2:** $\frac{\partial \Psi(u)}{\partial (\cdot)} \in (W^{1,p(x)}(\mathbb{R}^N))'$ for all $u \in W^{1,p(x)}(\mathbb{R}^N)$.

The linearity is simple to verify. We must show that
\[
\left| \frac{\partial \Psi(u)}{\partial v} \right| \leq C_u \|v\|, \quad \forall v \in W^{1,p(x)}(\mathbb{R}^N),
\]
for some positive constant $C_u$. By Proposition \ref{pro:2.4},
\[
\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, u(x))f(y, u(y))v(y)}{|x - y|^{\lambda(x,y)}} dxdy \right| \leq C \|F(\cdot, u)\|_{L^{q^+}(\mathbb{R}^N)} \|f(\cdot, u)v\|_{L^q(\mathbb{R}^N)}
\]
\[
+ C \|F(\cdot, u)\|_{L^{q^+}(\mathbb{R}^N)} \|f(\cdot, u)v\|_{L^q(\mathbb{R}^N)} \tag{3.5}
\]
Suppose that $\|v\|_{W^{1,q(\cdot)}(\mathbb{R}^N)} \leq 1$. The continuous embeddings $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{p+r(x)}(\mathbb{R}^N)$ and $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{p+s(x)}(\mathbb{R}^n)$ combined with the Hölder inequality gives
\[
\int_{\mathbb{R}^N} |f(y, u(y))v(y)|^{q^+} dy \leq C \|u\|^{q^+(r(x) - 1)} \|f(y)\|_{L^{q(\cdot)}(\mathbb{R}^N)} \|v\|^{q^+} \|L^{q^+}(\mathbb{R}^N)}
\]
\[
+ C \|u\|^{q^+(s(x) - 1)} \|f(y)\|_{L^{q(\cdot)}(\mathbb{R}^N)} \|v\|^{q^+} \|L^{s(x)}(\mathbb{R}^N)}
\]
\[ \leq C_u \left( \max \left( \|v\|_{L^{q^+}(\mathbb{R}^N)}, \|v\|_{L^{q^+}(\mathbb{R}^N)} \right) + \max \left( \|v\|_{L^{q^+}(\mathbb{R}^N)}, \|v\|_{L^{q^+}(\mathbb{R}^N)} \right) \right) \]

\[ \leq C_u \left( \max \left( \|v\|_{L^{q^+}(\mathbb{R}^N)}, \|v\|_{L^{q^+}(\mathbb{R}^N)} \right) + \max \left( \|v\|_{L^{q^+}(\mathbb{R}^N)}, \|v\|_{L^{q^+}(\mathbb{R}^N)} \right) \right) \]

\[ \leq C_u \]

where

\[ C_u := K_1 \left( \max \left( \left( \int_{\mathbb{R}^N} |u(y)|^{q^+ r(y)} dy \right)^{\left( \frac{r-1}{r} \right)^+}, \left( \int_{\mathbb{R}^N} |u(y)|^{q^+ s(y)} dy \right)^{\left( \frac{r-1}{r} \right)^-} \right) \right) \]

\[ + K_1 \left( \max \left( \left( \int_{\mathbb{R}^N} |u(y)|^{q^+ s(y)} dy \right)^{\left( \frac{r-1}{r} \right)^+}, \left( \int_{\mathbb{R}^N} |u(y)|^{q^+ s(y)} dy \right)^{\left( \frac{r-1}{r} \right)^-} \right) \right) \]

and \( K_1 \) is a constant which does not depend on \( u \) and \( v \). Using the same type ideas,

\[ \|f(\cdot, u)v\|_{L^{q^-}(\mathbb{R}^N)} \leq C_{u_2}, \text{ for all } v \in W^{1,p(x)}(\mathbb{R}^N) \text{ with } \|v\|_{W^{1,p(x)}(\mathbb{R}^N)} \leq 1 \]

(3.7)

where

\[ C_{u_2} := K_2 \left( \max \left( \left( \int_{\mathbb{R}^N} |u(y)|^{q^- r(y)} dy \right)^{\left( \frac{r-1}{r} \right)^+}, \left( \int_{\mathbb{R}^N} |u(y)|^{q^- s(y)} dy \right)^{\left( \frac{r-1}{r} \right)^-} \right) \right) \]

\[ + K_2 \left( \max \left( \left( \int_{\mathbb{R}^N} |u(y)|^{q^- s(y)} dy \right)^{\left( \frac{r-1}{r} \right)^+}, \left( \int_{\mathbb{R}^N} |u(y)|^{q^- s(y)} dy \right)^{\left( \frac{r-1}{r} \right)^-} \right) \right) \]

with \( K_2 \) being a constant which does not depend on \( u \) and \( v \). The inequalities (3.5), (3.6) and (3.7) justify the Step 2.

**Step 3:**

\[ \sup_{\|v\|_{W^{1,p(x)}(\mathbb{R}^N)} \leq 1} \left| \frac{\partial \Psi(u_n)}{\partial v} - \frac{\partial \Psi(u)}{\partial v} \right| \to 0 \text{ if } u_n \to u. \]

(3.8)

Consider \( v \in W^{1,p(x)}(\mathbb{R}^N) \) with \( \|v\|_{W^{1,p(x)}(\mathbb{R}^N)} \leq 1 \) and note that

\[ \left| \frac{\partial J(u_n)}{\partial v} - \frac{\partial J(u)}{\partial v} \right| \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|F(x, u_n(x)) - F(x, u(x))||f(y, u_n(y))v(y)|}{|x - y|^\lambda(x,y)} \, dx \, dy \]

\[ + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|F(x, u(x))||f(y, u_n(y))v(y) - f(y, u(y))v(y)|}{|x - y|^\lambda(x,y)} \, dx \, dy \]

\[ = B_f^n + B_F^n \]

The sequences \( \|f(\cdot, u_n)v\|_{L^{q^+}(\mathbb{R}^N)} \) and \( \|f(\cdot, u_n)v\|_{L^{q^-}(\mathbb{R}^N)} \) are uniformly bounded (see (3.6) and (3.7)). By Proposition 2.4

\[ B_f^n \leq C \|F(\cdot, u_n) - F(\cdot, u)\|_{L^{q^+}(\mathbb{R}^N)} \|f(\cdot, u_n)v\|_{L^{q^+}(\mathbb{R}^N)} \]

\[ + C \|F(\cdot, u_n) - F(\cdot, u)\|_{L^{q^-}(\mathbb{R}^N)} \|f(\cdot, u_n)v\|_{L^{q^-}(\mathbb{R}^N)} \]
and since
\[ \|F(\cdot, u_n) - F(\cdot, u)\|_{L^{q^+}(\mathbb{R}^N)}, \|F(\cdot, u_n) - F(\cdot, u)\|_{L^{q^+}(\mathbb{R}^N)} \to 0, \]  
we have
\[ B^n_F \to 0 \quad \text{as} \quad n \to +\infty. \] (3.10)

Now we will estimate \( B^n_F \). A similar argument works to prove that
\[ \|f(\cdot, u_n) - f(\cdot, u)v\|_{L^{q^+}(\mathbb{R}^N)}, \|f(\cdot, u_n) - f(\cdot, u)v\|_{L^{q^+}(\mathbb{R}^N)} \to 0 \]
Then, by Proposition 2.4
\[ B^n_F \leq C \|F(\cdot, u)\|_{L^{q^+}(\mathbb{R}^N)} \|f(\cdot, u_n) - f(\cdot, u)v\|_{L^{q^+}(\mathbb{R}^N)} \]
from where it follows that
\[ B^n_F \to 0 \quad \text{as} \quad n \to +\infty. \] (3.11)

Now, (3.8) follows from (3.10)-(3.11). This completes the proof. \( \square \)

4 An application

In this section we will illustrate how we can use the Proposition 2.4 to prove the existence of a solution for problem like (1.3). In what follows, we will consider in the condition (f1) that \( r, s \in \mathcal{M} \) with
\[ r^-, s^- > p^+/2. \] (4.1)

Finally, we consider the Ambrosetti-Rabinowitz type condition:
\[ 0 < \theta F(x, t) \leq 2f(x, t)t, \quad \forall t > 0 \quad (f_2) \]
for some \( \theta > p^+ \) with \( F(x, l) \geq c_l \) for all \( x \in \mathbb{R}^N \) for some \( l > 0 \) and for some constant \( c_l = c(l) \).

Related to the potential \( V : \mathbb{R}^N \to \mathbb{R} \), we assume that
\[ \inf_{x \in \mathbb{R}^N} V(x) := V_0 > 0. \] \( (V_0) \)

The main result of this section is the following:

**Theorem 4.1.** Assume \((f_1) - (f_2), (V_0), (2.3)\) and one of the conditions below:
\( V_1) \) \( V, p, q \) and \( f(\cdot, t) \) are \( \mathbb{Z}^N \)-periodic functions for all \( t \in \mathbb{R} \).
\( V_2) \) The embeddings
\[ E \hookrightarrow L^{s}(\mathbb{R}^N) \]
are compact for \( s \in C^+(\mathbb{R}^N) \) with
\[ p \ll s \ll p^*. \]
where \( E \) is Banach, reflexive space defined by
\[ E = \left\{ u \in W^{1,p}(\mathbb{R}^N) : \eta_p(u) := \int_{\mathbb{R}^N} \left( |\nabla u|^{p(x)} + V(x)|u(x)|^{p(x)} \right) dx < +\infty \right\} \]
with the norm
\[ \|u\| = \inf \left\{ \lambda > 0; \int_{\mathbb{R}^N} \left( |\nabla u(x)|^{p(x)} + V(x) \left| \frac{u(x)}{\lambda} \right|^{p(x)} \right) dx \leq 1 \right\}. \]

Then, problem (1.3) has a nontrivial solution.

We would like point out that the condition \( V_2 \) holds if potential \( V \) verifies one of the conditions below:

a) \( V \) is coercive, that is,
\[ V(x) \to +\infty \quad \text{as} \quad |x| \to +\infty. \]

b) For all \( M > 0, \)
\[ \text{med}(\{x \in \mathbb{R}^N : V(x) \leq M\}) < +\infty, \]
where \( \text{med} \) denotes the Lebesgue measure in \( \mathbb{R}^N \). For more details regarding to the above commentaries see [11].

In the proof of Theorem 4.1 we will use variational methods. A direct adaptation of the arguments of the previous section allow us to prove that \( \Psi \) is of class \( C^1 \) under the conditions \( V_1 \) and \( V_2 \).

The energy functional \( J : E \to \mathbb{R} \) associated with (1.3) is given by,
\[ J(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + V(x)|u(x)|^{p(x)}) dx - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F(x, u(x))F(y, u(y)) \frac{1}{|x-y|^\lambda(x,y)} dxdy, \]
that is,
\[ J(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + V(x)|u(x)|^{p(x)}) dx - \Psi(u). \]

By study made in the previous section, we know that \( J \in C^1(E, \mathbb{R}) \) with
\[ J'(u)v = \int_{\mathbb{R}^N} |\nabla u|^{p(x)-2}\nabla u \nabla v dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F(x, u(x)f(y, u(y))v(y)) \frac{1}{|x-y|^\lambda(x,y)} dxdy, \quad \forall u, v \in E. \]

Our first lemma establishes the mountain pass geometry.

**Lemma 4.1.** The functional \( J \) verifies the following properties:

(i) There exists \( \rho > 0 \) small such that \( J(u) \geq \eta \) for \( u \in E \) with \( \|u\| = \rho \) for some \( \eta > 0 \).

(ii) There exists \( e \in E \) such that \( \|e\| \rho \) and \( J(e) < 0 \).

**Proof.** i) By Proposition 2.4 and \( (F) \),
\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, u(x))F(y, u(y))}{|x-y|^\lambda(x,y)} dxdy \leq C(\|F(., u)\|_{L^{q^+}(\mathbb{R}^N)}^2 + \|F(., u)\|_{L^{q^-}(\mathbb{R}^N)}^2) \]
for all \( u \in E \). Note that
\[ \|F(., u)\|_{L^{q^+}(\mathbb{R}^N)} \leq C \left( \int_{\mathbb{R}^N} |u(x)|^{q^+r(x)} + |u(x)|^{q^+s(x)} dx \right)^{\frac{1}{q^+}} \]
\[ \leq C \left( \int_{\mathbb{R}^N} |u(x)|^{q^+r(x)} dx \right)^{\frac{1}{q^+}} + C \left( \int_{\mathbb{R}^N} |u|^{q^+s(x)} dx \right)^{\frac{1}{q^+}} \]
\[
\leq C \left( \max \left( \|u\|_{L^{q,+}(\mathbb{R}^N)}^{r+}, \|u\|_{L^{q,-}(\mathbb{R}^N)}^{r-} \right) + \max \left( \|u\|_{L^{q,+}(\mathbb{R}^N)}^{s+}, \|u\|_{L^{q,-}(\mathbb{R}^N)}^{s-} \right) \right)
\]

and
\[
\|F(\cdot, u)\|_{L^{q,-}(\mathbb{R}^N)} \leq C \max \left( \|u\|_{L^{q,-}(\mathbb{R}^N)}^{r-}, \|u\|_{L^{q,-}(\mathbb{R}^N)}^{s-} \right) + C \max \left( \|u\|_{L^{q,+}(\mathbb{R}^N)}^{s+}, \|u\|_{L^{q,-}(\mathbb{R}^N)}^{r-} \right).
\]

Considering \(\|u\|\) small and using the fact that
\[
\|u\|_{L^p(\mathbb{R}^N)} + r(x) \leq C \max \left( \|u\|_{L^{q,+}(\mathbb{R}^N)}^{r-}, \|u\|_{L^{q,-}(\mathbb{R}^N)}^{r-} \right)
\]
\[
\|u\|_{L^p(\mathbb{R}^N)} - r(x) \leq C \max \left( \|u\|_{L^{q,+}(\mathbb{R}^N)}^{s+}, \|u\|_{L^{q,-}(\mathbb{R}^N)}^{s-} \right).
\]

we get
\[
J(u) \geq \int_{\mathbb{R}^N} \frac{1}{q^+} \left( |\nabla u(x)|^{p(x)} + V_0 |u(x)|^{p(x)} \right) \, dx - C \max(\|u\|^{2r-}, \|u\|^{2r+})
- C \max(\|u\|^{2s-}, \|u\|^{2s+})
\geq C(\|\nabla u\|^{p^+}_{L^{p^+}(\mathbb{R}^N)} + \|u\|^p_{L^p(\mathbb{R}^N)}) - \overline{C}(\|u\|^{2r-} + \|u\|^{2s-}).
\]

Since \(2r-, 2s- > p^+\) the result is proved.

ii) The condition \((f_2)\) implies that
\[
F(x, t) \geq Ct^q \quad \forall (x, t) \in \mathbb{R}^N \times \mathbb{R} \quad \text{and} \quad t \geq l,
\]
where \(C\) depends only on \(l\). Now, considering a nonnegative function \(\varphi \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}\) the last inequality permits to conclude that \(J(t\varphi) < 0\) for \(t\) large enough. This finishes the proof.

Using the Mountain Pass Theorem without the Palais-Smale condition, we have that there is a sequence \((u_n) \subset E\) such that
\[
J(u_n) \to d \quad \text{and} \quad J'(u_n) \to 0,
\]
where \(d > 0\) is the mountain pass level.

Regarding with such sequence we have the next result.

**Lemma 4.2.** The sequence \((u_n)\) is bounded.

**Proof.** If \(n\) is large, we have
\[
J(u_n) - \frac{J'(u_n).u_n}{\theta} \leq d + 1 + \|u_n\|.
\]
On the other hand, supposing by contradiction that \((u_n)\) is unbounded, we must have for some subsequence

\[
J(u_n) - \frac{J'(u_n)u_n}{\theta} = \int_{\mathbb{R}^N} \left( \frac{1}{p(x)} - \frac{1}{\theta} \right) (|\nabla u_n(x)|^{p(x)} + V(x)|u_n(x)|^{p(x)})\,dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, u_n(x))}{|x-y|^{\lambda(x,y)}} \left( \frac{f(y, u_n(y))u_n(y)}{\theta} - \frac{F(y, u_n(y))}{2} \right)\,dxdy
\]

\[
\geq C \int_{\mathbb{R}^N} |\nabla u_n(x)|^{p(x)} + V(x)|u_n(x)|^{p(x)}\,dx
\]

\[
\geq \|u_n\|^{p^-}
\]

which is absurd since \(p^- > 1\).

\[\square\]

Since \((u_n)\) is bounded, up to a subsequence, \(u_n \rightharpoonup u\) in \(E\) for some \(u \in E\). The following lemma will be needed to prove that \(u\) is a critical point of \(J\).

**Lemma 4.3.**

(i) \[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, u_n(x))f(y, u(y))v(y)}{|x-y|^{\lambda(x,y)}}\,dxdy \rightarrow \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, u(x))f(y, u(y))v(y)}{|x-y|^{\lambda(x,y)}}\,dxdy
\]

for all \(v \in C_0^\infty(\mathbb{R}^N)\).

(ii) \[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, u_n(x))(f(y, u_n(y))v(y) - f(y, u(y))v(y))}{|x-y|^{\lambda(x,y)}}\,dxdy \rightarrow 0
\]

for all \(v \in C_0^\infty(\mathbb{R}^N)\).

(iii) \[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, u_n(x))f(y, u_n(y))v(y)}{|x-y|^{\lambda(x,y)}}\,dxdy \rightarrow \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, u(x))f(y, u(y))v(y)}{|x-y|^{\lambda(x,y)}}\,dxdy
\]

for all \(v \in C_0^\infty(\mathbb{R}^N)\).

**Proof.** i) Once \(L^{q_+}(\mathbb{R}^N)\) and \(L^{q_-}(\mathbb{R}^N)\) are uniformly convex, the Banach space \((L^{q_+}(\mathbb{R}^N) \cap L^{q_-}(\mathbb{R}^N), \|\cdot\|_{L^{q_+}(\mathbb{R}^N)}, \|\cdot\|_{L^{q_-}(\mathbb{R}^N)})\) is uniformly convex (therefore reflexive). The growth of \(F\) and the fact that \((u_n)\) is bounded in \(E\) implies that the sequence \((F(\cdot, u_n(\cdot)))\) is bounded in \(L^{q_+}(\mathbb{R}^N) \cap L^{q_-}(\mathbb{R}^N)\).

We claim that \(F(\cdot, u_n) \rightharpoonup F(\cdot, u)\) in \(L^{q_+}(\mathbb{R}^N) \cap L^{q_-}(\mathbb{R}^N)\). Since \((F(\cdot, u_n))\) is bounded in \(L^{q_+}(\mathbb{R}^N) \cap L^{q_-}(\mathbb{R}^N)\), there exists \(L \in L^{q_+}(\mathbb{R}^N) \cap L^{q_-}(\mathbb{R}^N)\) such that \(F(\cdot, u_n) \rightharpoonup L\) in \(L^{q_+}(\mathbb{R}^N) \cap L^{q_-}(\mathbb{R}^N)\). Fix \(\varphi \in C_0^\infty(\mathbb{R}^N)\) and consider the continuous linear functional \(I_\varphi(w) := \int_{\mathbb{R}^N} w\varphi \,dx, w \in L^{p^+}(\mathbb{R}^N) \cap L^{p^-}(\mathbb{R}^N)\). Then,

\[
I_\varphi(F(\cdot, u_n)) \rightarrow \int_{\mathbb{R}^N} L\varphi \,dx.
\]
Using \[7, \text{Proposition 2.6}\], we get \( F(\cdot, u_n) \to F(\cdot, u) \) in \( L^{q_+}(\mathbb{R}^N) \), and so,

\[
\int_{\mathbb{R}^N} F(x, u_n(x)) \varphi dx \to \int_{\mathbb{R}^N} F(x, u(x)) \varphi dx.
\]

Thereby

\[
\int_{\mathbb{R}^N} F(x, u(x)) \varphi(x) dx = \int_{\mathbb{R}^N} L \varphi(x) dx \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N),
\]

showing that

\[
L(x) = F(x, u(x)) \quad \text{a.e in } \mathbb{R}^N.
\]

By Proposition \[2.4\], the application

\[
H(w) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{w(x) f(y, u(y)) v(y)}{|x - y|^{\lambda(x,y)}} \, dx \, dy, \quad w \in L^{q_+}(\mathbb{R}^N) \cap L^{q_-}(\mathbb{R}^N)
\]

is a continuous linear functional. Since \( F(\cdot, u_n) \to F(\cdot, u) \) in \( L^{q_+}(\mathbb{R}^N) \cap L^{q_-}(\mathbb{R}^N) \), we find

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, u_n(x)) f(y, u_n(y)) v(y)}{|x - y|^{\lambda(x,y)}} \, dx \, dy \to \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, u(x)) f(y, u(y)) v(y)}{|x - y|^{\lambda(x,y)}} \, dx \, dy
\]

which proves the result.

ii) Denote by \( I \) the integral described in ii). Then,

\[
|I| \leq C \|F(x, u_n(x))\|_{L^{p_+}(\mathbb{R}^N)} \|f(y, u_n) v - f(\cdot, u) v\|_{L^{p_+}(\mathbb{R}^N)}
\]

\[
\leq C \|F(\cdot, u_n)\|_{L^{p_-}(\mathbb{R}^N)} \|f(\cdot, u) v - f(\cdot, u) v\|_{L^{p_-}(\mathbb{R}^N)}
\]

Since \((u_n)\) is bounded in \( E \), \((F(\cdot, u_n))\) is bounded sequence in \( L^{q_+}(\mathbb{R}^N) \cap L^{q_-}(\mathbb{R}^N) \). Let \( v \in C_0^\infty(\mathbb{R}^N) \) and consider an open set \( \Omega \) containing the support of \( v \). The boundedness of \((u_n)\) \( E \) combined with the Proposition \[2.3\] implies that, up to a subsequence,

- \( u_n(x) \to u(x) \) a.e in \( \mathbb{R}^N \),
- \( |u_n(x)| \leq h_1(x) \) a.e in \( \Omega \) for some \( h_1 \in L^{q_+ r(x)}(\Omega) \),
- \( |u_n(x)| \leq h_2(x) \) a.e in \( \Omega \) for some \( h_2 \in L^{q_- s(x)}(\Omega) \).

These informations combined with Lebesgue Dominated Convergence Theorem give

\[
\|f(\cdot, u_n) v - f(\cdot, u) v\|_{L^{q_+}(\mathbb{R}^N)} \to 0.
\]

A similar reasoning provides

\[
\|f(\cdot, u_n) v - f(\cdot, u) v\|_{L^{q_-}(\mathbb{R}^N)} \to 0
\]

and the proof is over. iii) is a direct consequence of i) and ii).

Regarding to the pointwise convergence of \((\nabla u_n)\) we have the following Lemma.

**Lemma 4.4.** For a subsequence the two properties below holds
(i) $\nabla u_n(x) \to \nabla u(x)$ a.e in $\mathbb{R}^N$.

(ii) $|\nabla u_n|^{p(x)-2}\nabla u_n \to |\nabla u_n|^{p(x)-2}\nabla u_n$ in $(L^{\frac{p(x)}{p(x)-1}}(\mathbb{R}^N))^N$.

Proof. Fix $R > 0$ and $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\varphi(x) = 1$ for $x \in B_R(0)$. Since $I'(u_n) \to 0$ in $E'$ and $(u_n)$ is bounded in $E$, we get $I'(u_n)(u_n\varphi) = o_n(1) = I'(u_n)(u\varphi)$. Setting

$$P_n(x) := (|\nabla u_n(x)|^{q(x)-2}\nabla u_n(x) - |\nabla u(x)|^{q(x)-2}\nabla u(x)).(\nabla(u_n - u)(x)), x \in \mathbb{R}^N.$$ 

standard arguments combined with Lemma 4.3 ensure that

$$\int_{\mathbb{R}^N} P_n(x)\varphi(x) \, dx \to 0$$

which proves the claim. Therefore $\nabla u_n \to \nabla u$ in $L^{p(x)}(B_R(0))$ for all $R > 0$ large which implies $\nabla u_n(x) \to \nabla u(x)$ a.e in $\mathbb{R}^N$ for some subsequence.

(ii) Using the fact that $|\nabla u_n|^{p(x)-2}\nabla u_n$ is bounded in $L^{\frac{p(x)}{p(x)-1}}(\mathbb{R}^N)$ and the pointwise convergence $|\nabla u_n|^{p(x)-2}\nabla u_n(x) \to |\nabla u|^{p(x)-2}\nabla u(x)$ a.e in $\mathbb{R}^N$ we have by [7, Proposition 2.6] that

$$|\nabla u_n|^{q(x)-2}\nabla u_n \to |\nabla u_n|^{q(x)-2}\nabla u_n \text{ in } (L^{\frac{q(x)}{q(x)-1}}(\mathbb{R}^N))^N.$$ 

Now, we are ready to prove that $u$ is a critical point of $J$.

**Lemma 4.5.** The function $u$ is a critical point of $J$, that is, $J'(u) = 0$.

Proof. We claim that

$$J'(u_n)v \to J'(u)v, \quad \forall v \in C_0^\infty(\mathbb{R}^N).$$

In order to verify such limit, note that

$$J'(u_n)v = \int_{\mathbb{R}^N} |\nabla u_n(x)|^{q(x)-2}\nabla u_n(x)\nabla v(x) + V(x)u_n(x)v(x) \, dx$$

$$- \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, u_n(x))f(y, u_n(y))v(y)}{|x-y|^\lambda(x,y)} \, dxdy$$

By Lemma 4.3,

$$\int_{\mathbb{R}^N} |\nabla u_n(x)|^{q(x)-2}\nabla u_n(x)\nabla v(x) \, dx \to \int_{\mathbb{R}^N} |\nabla u(x)|^{q(x)-2}\nabla u(x) . \nabla v(x) \, dx \quad (4.2)$$

and

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, u_n(x))f(y, u_n(y))v(y)}{|x-y|^\lambda(x,y)} \, dxdy \to \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, u(x))f(y, u(y))v(y)}{|x-y|^\lambda(x,y)} \, dxdy. \quad (4.3)$$

Moreover,

$$\int_{\mathbb{R}^N} V(x)|u_n(x)|^{p(x)-2}u_n(x)v(x) \, dx \to \int_{\mathbb{R}^N} V(x)|u(x)|^{p(x)-2}u(x)v(x) \, dx$$

which combined with the relations (4.2) (4.3) justifies the claim. From the claim we have $J'(u)v = 0$ for all $v \in C_0^\infty(\mathbb{R}^N)$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $E$ we conclude the proof. $\square$
4.1 Proof of Theorem 4.1

In the sequel, we will divide the proof into two cases, which are related to the conditions \( V_1 \) and \( V_2 \).

**Case 1:** \( V_1 \) holds:

If \( u \neq 0 \), then \( u \) is a nontrivial solution which finishes the proof for this case. If \( u = 0 \), we must find another solution \( v \in W^{1,p(x)}(\mathbb{R}^N) \setminus \{0\} \) for the equation (1.3). For such purpose, the claim below is crucial in our argument.

**Claim 4.1.** There exist \( r > 0, \beta > 0 \) and a sequence \((y_n)\) in \( \mathbb{R}^N \) such that

\[
\liminf_{n \to +\infty} \int_{B_r(y_n)} |u_n(x)|^{p(x)} \, dx \geq \beta > 0.
\]

**Proof.** Suppose that the claim is false. Then, by Lemma 2.1

\[ u_n \to 0 \quad \text{in} \quad L^{t(x)}(\mathbb{R}^n) \quad (4.4) \]

for all \( t \in C^+(\mathbb{R}^N) \) with \( p \ll t \ll p^* \). Applying Proposition 2.4,

\[
\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, u_n(x)) f(y, u_n(y)) u_n(y)}{|x - y|^{\lambda(x,y)}} \, dx \, dy \right| \leq C \left( \|F(x, u_n(x))\|_{L^{t^+}(\mathbb{R}^N)} \|f(y, u_n(y)) u_n(y)\|_{L^{t^+}(\mathbb{R}^N)} + C \|F(x, u_n(x))\|_{L^{t^-}(\mathbb{R}^N)} \|f(y, u_n(y)) u_n(y)\|_{L^{t^-}(\mathbb{R}^N)} \right).
\]

By \( (f_1), (F) \) and (4.4),

\[
\int_{\mathbb{R}^N} |F(x, u_n(x))|^{q^+} \, dx \to 0
\]

\[
\int_{\mathbb{R}^N} |F(x, u_n(x))|^{q^-} \, dx \to 0
\]

\[
\int_{\mathbb{R}^N} |f(y, u_n(y)) u_n(y)|^{q^+} \, dy \to 0
\]

and

\[
\int_{\mathbb{R}^N} |f(y, u_n(y)) u_n(y)|^{q^-} \, dy \to 0.
\]

Therefore

\[
\int_{\mathbb{R}^N} \frac{F(x, u_n(x)) f(y, u_n(y)) u_n(y)}{|x - y|^{\lambda(x,y)}} \, dx \, dy \to 0.
\]

The above limit together with the fact that \( J'(u_n) u_n = o_n(1) \) give

\[
\int_{\mathbb{R}^N} (|\nabla u_n(x)|^{p(x)} + V(x)|u_n(x)|^{p(x)}) \, dx \to 0.
\]

Thus,

\[ u_n \to 0 \quad \text{in} \quad E \]

which leads to \( J(u_n) \to 0 \). We have a contradiction because \( J(u_n) \to d \) and \( d > 0 \).
By using standard arguments, we can assume in Claim 4.1 that \((y_n) \subset \mathbb{Z}^N\). Once \(q\) is \(\mathbb{Z}^N\)-periodic, the Remark 2.4 yields \(\lambda\) is \(\mathbb{Z}^N \times \mathbb{Z}^N\)-periodic. This fact combined with the periodicity of \(p, V, f(t)\) and \(F(\cdot, t)\) guarantees that the function \(v_n(x) = u_n(x + y_n)\) satisfies

\[ J(v_n) = J(u_n), \quad \|J'(v_n)\| = \|J'(u_n)\| \quad \text{and} \quad \|u_n\| = \|v_n\| \quad \forall n \in \mathbb{N}. \]

From the above information, \((v_n)\) is a \((PS)_d\) sequence for \(J\). Since \((v_n)\) is bounded in \(E\), we have up to a subsequence, \(v_n \to v\) in \(L^{p(x)}(B_r(0))\) for some \(v \in E\). In order to verify that \(v \neq 0\), note that by Claim 4.1 we have

\[ 0 < \beta \leq \lim_{n \to +\infty} \int_{B_r(y_n)} |u_n(x)|^{p(x)} \, dx = \lim_{n \to +\infty} \int_{B_r(0)} |v_n(x)|^{p(x)} \, dx = \int_{B_r(0)} |v(x)|^{p(x)} \, dx. \]

Reasoning as in the proof of Lemma 4.3 we prove the theorem for the case where \(V_2\) holds.

**Case 2:** \(V_2\) holds:

Repeating the same arguments explored until moment, there is a sequence \((u_n) \subset E\) such that

\[ J(u_n) \to d \quad \text{and} \quad J'(u_n) \to 0, \]

where \(d > 0\) is the mountain pass level. Moreover, we can assume that for some subsequence of \((u_n)\), still denote by itself, there is \(u \in E\) such that

\[ u_n \rightharpoonup u \quad \text{in} \quad E, \]

\[ u_n(x) \to u(x) \quad \text{a.e. in} \quad \mathbb{R}^N. \]

By \(V_2\),

\[ u_n \to u \quad \text{in} \quad L^{s(x)}(\mathbb{R}^N) \quad \forall s \in \mathcal{M}. \]

The above limit combined with \((f_1) - (f_2)\) and \((F)\) give

\[ F(\cdot, u_n) \to F(\cdot, u) \quad \text{in} \quad L^{q^+}(\mathbb{R}^N), \]

\[ F(\cdot, u_n) \to F(\cdot, u) \quad \text{in} \quad L^{q^-}(\mathbb{R}^N), \]

\[ f(\cdot, u_n)u_n \to f(\cdot, u)u \quad \text{in} \quad L^{q^+}(\mathbb{R}^N) \]

and

\[ f(\cdot, u_n)u_n \to f(\cdot, u)u \quad \text{in} \quad L^{q^-}(\mathbb{R}^N). \]

The above limits combine with Proposition 2.4 imply that

\[ \int_{\mathbb{R}^N} \frac{F(x, u_n(x))f(y, u_n(y))u_n(y)}{|x - y|^\lambda(x,y)} \, dxdy \to \int_{\mathbb{R}^N} \frac{F(x, u(x))f(y, u(y))u(y)}{|x - y|^\lambda(x,y)} \, dxdy. \]

Now, gathering the last limit with the limits \(J'(u_n)u_n = o_n(1), J'(u_n)u = o_n(1)\), we can ensure that

\[ \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)} + V(x)|u_n(x)|^{p(x)} \, dx \to \int_{\mathbb{R}^N} |\nabla u|^{p(x)} + V(x)|u(x)|^{p(x)} \, dx, \]
from where it follows that
\[ u_n \to u \quad \text{in} \quad E, \]
showing that \( u \) is a critical point of \( J \) with \( J(u) = d > 0 \). This completes the proof.

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