DONALDSON INVARIANTS OF SYMPLECTIC MANIFOLDS

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ABSTRACT. We prove that symplectic 4-manifolds with $b_1 = 0$ and $b^+ > 1$ have nonvanishing Donaldson invariants, and that the canonical class is always a basic class. We also characterize in many situations the basic classes of a Lefschetz fibration over the sphere which evaluate maximally on a generic fiber.

1. Introduction

The purpose of this paper is to prove a nonvanishing result for Donaldson invariants of symplectic 4-manifolds. Donaldson [2] proved that his polynomial invariants are nonzero for large powers of a hyperplane class on a simply connected complex projective surface, and so it is natural to ask if this result generalizes to symplectic manifolds. For $X$ with symplectic form $\omega$ representing an integral homology class, Donaldson [4] showed that some large multiple of $[\omega]$ is Poincaré dual to the fibers of a Lefschetz pencil on $X$, and so $[\omega]$ is analogous to a hyperplane class on a projective surface. We therefore prove the following theorem:

Theorem 1.1. Let $X$ be a closed 4-manifold of Donaldson simple type with $b_1(X) = 0$ and $b^+(X) > 1$, and let $\omega$ be an integral symplectic form on $X$ with Poincaré dual $h \in H_2(X; \mathbb{Z})$. Then the Donaldson series $D_w^X(h)$ is nonzero for any $w \in H^2(X; \mathbb{Z})$. In fact, the canonical class $K_X$ is a basic class of $X$, and all basic classes $K$ satisfy

$$|K \cdot [\omega]| \leq K_X \cdot [\omega]$$

with equality if and only if $K = \pm K_X$.

The above inequality was observed by Donaldson [3], but here we prove that it is sharp. The analogous inequality and conditions for equality were proved for Seiberg–Witten basic classes by Taubes [32]. Note that by Witten’s conjecture [33] and work of Taubes [31, 30] we expect that symplectic manifolds with $b_1 = 0$ and $b^+ > 1$ automatically have Donaldson simple type and their Donaldson and Seiberg–Witten basic classes coincide; in Section 8 we will prove analogous results in case $X$ does not have simple type.

Theorem 1.1 is a straightforward consequence of the following theorem concerning Donaldson invariants of Lefschetz fibrations over the 2-sphere. It was announced in [19, Section 7.9] by Kronheimer and Mrowka, whose proof was explained in slides from Kronheimer’s talk at the 2009 Georgia International Topology Conference [18]. Here we give a new proof using an entirely different strategy, following the ideas of [17] instead.

Theorem 1.2. Let $X$ be a closed 4-manifold of Donaldson simple type with $b_1(X) = 0$ and $b^+(X) > 1$, and suppose that $X$ admits a relatively minimal Lefschetz fibration over $S^2$ with
generic fiber $\Sigma$ of genus $g \geq 2$. Let $w \in H^2(X; \mathbb{Z})$ be any class whose pairing with $h = [\Sigma]$ is odd. Then the Donaldson series $D^w_X(h)$ is nonzero and has leading term of order $e^{2g-2}$.

We will first establish Theorem 1.2 for all $g \geq 8$, deduce Theorem 1.1 as a corollary, and then use Theorem 1.1 to prove the remaining cases of Theorem 1.2. If $X$ does not have simple type, we will prove analogously (Theorem 8.4) that there is some $c \neq 0$ for which $D^w_X(h^n)$ is asymptotic to $c(2g-2)^n$ for all large $n \equiv -w^2 - \frac{3}{2}(b^+(X) + 1) \pmod{4}$.

In Section 3 we will prove the following theorem, which may be of independent interest; the analogous result for the Heegaard Floer 4-manifold invariants was proved by Ozsváth and Szabó [27], but we could not find a complete proof of the Seiberg–Witten version in the literature.

**Theorem 1.3.** Let $X \to S^2$ be a relatively minimal Lefschetz fibration with generic fiber $\Sigma$ of genus $g \geq 2$, and suppose that $b^+(X) > 1$. Then the canonical class $K_X$ is the unique Seiberg–Witten basic class of $X$ satisfying $K \cdot \Sigma = 2g - 2$.

We cannot prove the analogous uniqueness result for Donaldson basic classes, assuming that $b_1(X) = 0$ as well. However, we will show that if a basic class $K$ satisfies $K \cdot \Sigma = 2g - 2$ then either $K = K_X$ modulo torsion or $K^2 < K^2_X$, which we expect to be impossible; and we will also show (Proposition 3.7) that if every component of every fiber intersects some section of square $-1$, then $K \cdot \Sigma = 2g - 2$ implies that $K = K_X$ up to torsion. This last statement is in fact strong enough to complete the proof of Theorem 1.1.

As mentioned above, our approach to Theorems 1.1 and 1.2 is based on the strategy used by Kronheimer and Mrowka [17] in their celebrated proof of the Property P conjecture. They used some specific cases of Witten’s conjecture [33] relating the Donaldson and Seiberg–Witten invariants, together with known facts about Seiberg–Witten theory, to prove that certain symplectic manifolds had nonzero Donaldson invariants; this immediately implied nonvanishing results for the instanton Floer homology of some 3-manifolds which separate them. In this paper we use the same known cases of Witten’s conjecture (applying it to a slightly larger class of 4-manifolds) together with a gluing theorem of Muñoz [20] which determines the Donaldson invariants of a fiber sum in terms of the invariants of each summand. In particular, all of our techniques were available at the time [17] was published. Kronheimer and Mrowka’s proof [19, 18] of Theorem 1.2 proceeds instead via excision for instanton Floer homology, following the strategy used by Ozsváth and Szabó in [27].

The organization of this paper is as follows. In Section 2 we review the necessary background on Donaldson invariants, simple type, and the relation to Seiberg–Witten invariants. Section 3 is devoted to studying the basic classes of Lefschetz fibrations. In Sections 4 and 5 we prove Theorem 1.2 for $g \geq 8$ by applying Muñoz’s gluing theorem [20] and known cases of Witten’s conjecture to fiber sums of a given Lefschetz fibration with other suitable fibrations. In Section 6 we use the existence of Lefschetz pencils on symplectic manifolds [4] to prove Theorem 1.1 and in Section 7 we complete the proof of Theorem 1.2. Finally, in Section 8 we discuss the analogous results for symplectic manifolds which do not have simple type.

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2. Background on Donaldson invariants

2.1. Basic classes and simple type. Suppose that $X$ is a closed 4-manifold with $b_1(X) = 0$ and $b^+(X) > 1$ odd and a fixed homology orientation, and fix a class $w \in H^2(X;\mathbb{Z})$. Let $\mathbb{A}(X)$ be the graded symmetric algebra on $H_2(X;\mathbb{R}) \oplus H_0(X;\mathbb{R})$, where the positive generator $x \in H_0(X;\mathbb{Z})$ has degree 4 and $H_2(X;\mathbb{R})$ lies in degree 2. Then the Donaldson invariants \cite{16} corresponding to $U(2)$ bundles $E \to X$ with $c_1(E) = w$ form a linear map $D^w_X : \mathbb{A}(X) \to \mathbb{R}$ as in \cite{16}, and this map is zero on any homogeneous element of degree $d$ unless $d = -2w^2 - 3(b^+(X) + 1) \pmod{8}$.

Definition 2.1 (\cite{16}). The manifold $X$ has simple type if $D^w_X(x^2z) = 4D^w_X(z)$ for all $z \in \mathbb{A}(X)$.

Many complex surfaces with $b_1 = 0$ and $b^+ > 1$ are known to have simple type, including elliptic surfaces and complete intersections. In fact, we expect all symplectic manifolds with $b_1 = 0$ and $b^+ > 1$ to have simple type, by Witten’s conjecture (\cite{33}, see also Section 2.2) and work of Taubes \cite{30} in Seiberg-Witten theory.

For manifolds of simple type, given $h \in H_2(X;\mathbb{Z})$ we can form the power series

$$D^w_X(h) = D^w_X \left( \left( 1 + \frac{x}{2} \right) e^h \right) = \sum_{i=0}^{\infty} \frac{D^w_X(h^i)}{i!} + \frac{1}{2} \sum_{i=0}^{\infty} \frac{D^w_X(xh^i)}{i!}.$$  

Kronheimer and Mrowka proved the following structure theorem:

Theorem 2.2 (\cite{16} Theorem 1.7). If $X$ has simple type and Betti numbers $b_1(X) = 0$ and odd $b^+(X) > 1$, then there are finitely many classes $K_1, \ldots, K_s \in H^2(X;\mathbb{Z})$ such that

$$D^w_X(h) = \exp \left( \frac{Q(h)}{2} \right) \sum_{r=1}^{s} (-1)^{(w^2 + K_r \cdot w)/2} \beta_r e^{K_r \cdot h}$$

as a function on $H^2(X;\mathbb{R})$. Here $Q$ is the intersection form of $X$, viewed as a quadratic function, and the $\beta_r$ are nonzero rational numbers.

The $K_r$ are called the basic classes of $X$, or possibly the Donaldson basic classes to emphasize that these are not necessarily the same as the Seiberg-Witten basic classes. We remark that the Donaldson basic classes are only well-defined up to torsion, so throughout this paper we will actually think of them as elements of $H^2(X;\mathbb{Z})/\text{torsion}$.

Proposition 2.3 (\cite{16} Theorem 8.1). If $b_1(X) = 0$ and $b^+(X) > 1$ is odd, and $X$ contains an embedded surface $S$ with $|S|^2 = 2g(S) - 2 > 0$, then $X$ has simple type.

Such a surface is sometimes called a tight surface. For example, if $X$ as above is symplectic and contains a closed Lagrangian surface $L$ of genus at least 2, then $L$ is necessarily tight and so $X$ has simple type. Similarly, if $X$ has a Lagrangian torus in an odd homology class, then it must have simple type: this follows from a result of Muñoz \cite{21} Proposition 9.3, which says
in part that if \( b_1(X) = 0 \) and \( b^+(X) > 1 \), and \( X \) contains a homologically odd surface \( \Sigma \) of self-intersection 0 and genus at most 2, then \( X \) has simple type. The cited result also implies that \( X \) has simple type if it admits a genus 2 Lefschetz fibration over \( S^2 \).

Finally, we remark that Muñoz [21] has shown that all 4-manifolds with \( b_1 = 0 \) and \( b^+ > 1 \) have finite type of some order \( n \geq 0 \), meaning that \( D^u_X((x^2 - 4)^n z) = 0 \) for all \( z \). In this case there is still a notion of basic class and related structure theorem [23, 25]. These basic classes satisfy an adjunction inequality [24, Theorem 1.7] of the form

\[
|K \cdot \Sigma| + \Sigma^2 + 2d(K) \leq 2g - 2
\]

whenever \( \Sigma \) is an embedded surface of genus \( g \geq 1 \) and either \( \Sigma^2 > 0 \) or \( \Sigma^2 = 0 \) and \( \Sigma \) represents an odd homology class; here \( d(K) \geq 0 \) is the “order of finite type” of \( K \). (When \( X \) has simple type, all basic classes have \( d(K) = 0 \) and so this reduces to the usual adjunction inequality \( |K \cdot \Sigma| + \Sigma^2 \leq 2g - 2 \) as in [16].)

2.2. Witten’s conjecture. Witten [33] conjectured the following relationship between Donaldson and Seiberg–Witten invariants.

**Conjecture 2.4.** Suppose that \( X \) is a closed, oriented 4-manifold with \( b_1(X) = 0 \) and \( b^+(X) > 1 \) odd. If \( X \) has Seiberg–Witten simple type, then it has Donaldson simple type as well, the Seiberg–Witten and Donaldson basic classes coincide, and

\[
D^u_X(h) = c(X) \exp \left( \frac{Q(h)}{2} \right) \sum_{r=1}^{s} (-1)^{(w^2 + K \cdot w)/2} SW(K_r)e^{K_r \cdot h},
\]

where \( K_1, \ldots, K_s \) are the basic classes of \( X \) and \( c(X) \) is a nonzero rational number.

The full conjecture also predicts that \( c(X) = 2^{2 + (7e + 11\sigma)/4} \) where \( e \) and \( \sigma \) are the Euler characteristic and signature of \( X \). In any case, we note that this is exactly the formula of Theorem 2.2 with \( \beta_r = c(X) \cdot SW(K_r) \).

Using work of Feehan and Leness [7], Kronheimer and Mrowka established the following special case of Witten’s conjecture:

**Theorem 2.5** ([17, Corollary 7]). Suppose that \( X \) as above contains a tight surface and a sphere of self-intersection \(-1\), and that \( X \) has the same Euler characteristic and signature as a smooth hypersurface in \( \mathbb{CP}^3 \) of even degree at least 6. Then \( X \) satisfies Conjecture 2.4.

**Remark 2.6.** The proof of [17, Corollary 7] used the following facts about such hypersurfaces \( X_* \subset \mathbb{CP}^3 \), in addition to the requirements that \( b_1(X_*) = 0, b^+(X_*) > 1 \) is odd, and \( H^2(X_*; \mathbb{Z}) \) has no 2-torsion:

1. The only Seiberg–Witten basic classes are \( \pm K_{X_*} \), and \( SW(\pm K_{X_*}) = 1 \).
2. The Donaldson invariants of \( X_* \) are not identically zero.
3. \( X_* \) is spin.
4. \( X_* \) contains a tight surface and a symplectic surface \( S \) with \( S \cdot S \geq 0 \) and \( K_{X_*} \cdot S \neq 0 \).

We will make use of this observation in Section 7 by finding other 4-manifolds which have the same properties and thus also satisfy Conjecture 2.4, in order to complete the proof of Theorem 1.2.
Finally, in some cases we can prove that Witten’s conjecture still holds after blowing down a 4-manifold which satisfies the conjecture. In order to do so, we recall how the Donaldson and Seiberg–Witten invariants behave under blowups.

**Theorem 2.7 ([10][16]).** If $X$ has Donaldson simple type, then $\tilde{X} = X \# \mathbb{CP}^2$ does as well and their Donaldson series satisfy

$$D^w_X(h) = D^w_{\tilde{X}}(h) \cdot \exp \left(-\frac{(E \cdot h)^2}{2}\right) \cosh(E \cdot h)$$

$$D^{w+E}_X(h) = -D^w_{\tilde{X}}(h) \cdot \exp \left(-\frac{(E \cdot h)^2}{2}\right) \sinh(E \cdot h)$$

where $E$ is dual to the exceptional divisor. If $X$ has basic classes $\{K_r\}$, then it follows that the basic classes of $\tilde{X}$ are $\{K_r \pm E\}$.

**Theorem 2.8 ([9]).** If $X$ has Seiberg–Witten simple type with basic classes $\{K_r\}$, then $\tilde{X} = X \# \mathbb{CP}^2$ has basic classes $\{K_r \pm E\}$ where $E$ is dual to the exceptional divisor, and $SW_X(K_r) = SW_{\tilde{X}}(K_r \pm E)$.

The following is then a straightforward consequence of Theorems 2.7 and 2.8, as well as the immediate corollary of Fintushel and Stern’s blowup formula [10] that if $X \# \mathbb{CP}^2$ has Donaldson simple type then so does $X$.

**Proposition 2.9.** Let $X$ be a 4-manifold with $b_1 = 0$ and $b^+ > 1$ odd, and suppose that $X$ has Seiberg–Witten simple type. If $\tilde{X} = X \# \mathbb{CP}^2$ satisfies Conjecture 2.4, then so does $X$, and furthermore $c(X) = 2c(\tilde{X})$.

### 3. Lefschetz fibrations and basic classes

Let $X \to S^2$ be a Lefschetz fibration with generic fiber $\Sigma$ of genus $g \geq 2$. Then $X$ admits a symplectic form $\omega$ for which each fiber is symplectic [15], and so it has a canonical class $K_X \in H^2(X; \mathbb{Z})$. Throughout this section we will assume that $b^+(X) > 1$, and when we discuss Donaldson invariants we will also assume that $b_1(X) = 0$.

Our goal in this section is to determine some strong restrictions on the set of basic classes satisfying $K \cdot \Sigma = 2g - 2$, which is maximal by the adjunction inequality $|K \cdot \Sigma| + \Sigma \cdot \Sigma \leq 2g - 2$. The main result for Seiberg–Witten basic classes, Theorem 3.5, was claimed by Finashin [8, Section 3], but the argument given there is incomplete (see Remark 3.3) so we include a full proof here following the same ideas. An analogous result was proved by Ozsváth and Szabó [27, Theorem 5.1] for their 4-manifold invariants by different means.

Throughout this section we will abuse notation by identifying surfaces inside $X$ with their homology classes, but this should not cause any confusion.

**Lemma 3.1.** Suppose that $X$ has reducible singular fibers $F_1 \cup G_1, \ldots, F_k \cup G_k$ where $F_i$ and $G_i$ are the components of the $i$th reducible singular fiber. Then $\Sigma$, $F_i$, and $G_i$ all are primitive nonzero elements of $H_2(X; \mathbb{Z})$, both $F_i$ and $G_i$ have self-intersection $-1$, and $\Sigma, F_1, \ldots, F_k$ are linearly independent in $H_2(X; \mathbb{Z})$. 
Proof. The fact that $F_i$ and $G_i$ are primitive follows immediately from $F_i \cdot G_i = 1$, and Stipsicz [29] Theorem 1.4 showed that $\Sigma$ is primitive as well. Since $\Sigma$ is homologous to $F_i \cup G_i$ and disjoint from both $F_i$ and $G_i$, we have $F_i \cdot F_i = F_i \cdot (\Sigma - G_i) = -1$ and likewise for $G_i$. Finally, if there is a linear relation of the form $$n\Sigma + c_1 F_1 + \cdots + c_k F_k = 0,$$ then pairing both sides with $F_i$ gives $-c_i = 0$, so we are left with $n\Sigma = 0$ and this implies $n = 0$ as well. \hfill \Box

Proposition 3.2. Let $K$ be a basic class of $X$, either for Seiberg–Witten invariants or for Donaldson invariants if $b_1(X) = 0$. If $K \cdot \Sigma = 2g - 2$, then the Poincaré dual of $K_X - K$ can be expressed up to torsion as a sum $c_0 \Sigma + c_1 F_1 + \cdots + c_k F_k$ with $c_i \in \mathbb{Z}$ and $c_0 \geq 0$.

Proof. Since $\Sigma$ and the $F_i$ are primitive and linearly independent, we can extend them to a basis $F_0 = \Sigma, F_1, \ldots, F_k, F_{k+1}, \ldots, F_n$ of $H_2(X; \mathbb{Q})$. Let $\alpha_0, \ldots, \alpha_n$ be the dual basis of $(H_2(X; \mathbb{Q}))^* \cong H^2(X; \mathbb{Q})$.

Recall (e.g. from [15] Section 10.2) that we can give the Lefschetz fibration $f : X \to S^2$ a symplectic form $\omega = f^*(\omega_{S^2}) + t\eta$, where $\omega_{S^2}$ is an area form on the base and $f^*(\omega_{S^2})$ is Poincaré dual to $\Sigma$; $[\eta]$ is any integral class on $X$ which evaluates positively on every component of every fiber, including the generic fiber $\Sigma$; and $t > 0$ is sufficiently small. More specifically, the condition on $[\eta]$ means that it must be positive on $\Sigma$ and on each $F_i$ and $G_i$ for $1 \leq i \leq k$.

By the inequality $K \cdot [\omega] \leq K_X \cdot [\omega]$, which holds for all basic classes in both Donaldson theory [3] and Seiberg–Witten theory [32], and the fact that $K \cdot \Sigma = 2g - 2 = K_X \cdot \Sigma$ (the latter equality follows from the adjunction formula), we have the necessary condition

$$(K_X - K) \cdot [\eta] \geq 0.$$ 

Now suppose that $PD(K_X - K) = \sum_{i=0}^n c_i F_i$ for some $c_i \in \mathbb{Q}$. Then given any symplectic form on $\omega$ as above, we can replace $[\eta]$ with $[\eta'] = [\eta] - d(c_{k+1} \alpha_{k+1} + \cdots + c_n \alpha_n)$ without changing its evaluation on $\Sigma$ or any of the $F_i$ or $G_i = \Sigma - F_i$ for $i \leq k$. In particular, $[\eta']$ is still positive on these classes, so it yields a new symplectic form and we have

$$0 \leq (K_X - K) \cdot [\eta'] = (K_X - K) \cdot [\eta] - d \left( \sum_{i=k+1}^n c_i \alpha_i \right) \left( \sum_{j=0}^n c_j F_j \right)$$

$$= (K_X - K) \cdot [\eta] - d \sum_{i=k+1}^n c_i^2.$$ 

Taking $d$ to be arbitrarily large forces the right hand side to be negative unless $\sum_{i=k+1}^n c_i^2 = 0$, so we must have $c_i = 0$ for all $i > k$, as desired.

Finally, we note that $[\eta] = d\alpha_0 + \alpha_1 + \cdots + \alpha_k$ gives rise to a symplectic form for all $d \geq 2$, and $(K_X - K)[\eta] = c_0 d + c_1 + \cdots + c_k$ must remain nonnegative for all large $d$, so $c_0 \geq 0$. Since $\Sigma$ and the $F_i$ are primitive, linearly independent (over $\mathbb{Q}$) integral classes and $K_X - K$ is integral, it also follows that the $c_i$ must all actually be integral as well. \hfill \Box
Remark 3.3. The proof of Proposition 3.2 was suggested by Finashin in [8, A3(2)]. However, the proof there claimed that $\eta$ could be any class for which $\eta \cdot \Sigma > 0$, and this is only true if there are no reducible singular fibers. Otherwise we cannot a priori eliminate the possibility that $PD(K_X - K)$ has nontrivial $F_i$-components for some of $F_1, \ldots, F_k$.

Proposition 3.4. Suppose that $X \to S^2$ is relatively minimal and $K$ is a Seiberg–Witten or Donaldson basic class for which $K \cdot \Sigma = 2g - 2$. Then $K^2 \leq K_X^2$, with equality if and only if $K = K_X$ modulo torsion.

Proof. Let $K$ be such a basic class, and write $PD(K_X - K) = n\Sigma + \sum_{i=1}^k c_i F_i$ up to torsion. If $c_i < 0$ for any $i$, then we observe that $c_i F_i = c_i \Sigma + (-c_i) G_i$, so by replacing $F_i$ in our basis with $G_i$ we may assume without loss of generality that $c_i \geq 0$ for all $i$, and we will still have $n \geq 0$ as in the previous proposition. We then compute

$$K^2 = \left( PD(K_X) - n\Sigma - \sum_{i=1}^k c_i F_i \right)^2 = K_X^2 - 2n(K_X \cdot \Sigma) - 2 \sum_{i=1}^k c_i (K_X \cdot F_i) - \sum_{i=1}^k c_i^2.$$

Now the adjunction formula says that $K_X \cdot S + S \cdot S = 2g(S) - 2$ whenever $S$ is a symplectic surface, and by construction $\Sigma$ and each of the $F_i$ are symplectic, so

$$K_X \cdot \Sigma = 2g - 2, \quad K_X \cdot F_i = 2g(F_i) - 1$$

since $\Sigma$ and $F_i$ have self-intersection 0 and $-1$ respectively. In particular, we assumed that $g \geq 2$ and $X \to S^2$ is relatively minimal, which implies $g(F_i) \geq 1$ for each $i$, so both $K_X \cdot \Sigma$ and $K_X \cdot F_i$ are strictly positive. But this implies that

$$K_X^2 - K^2 = 2n(K_X \cdot \Sigma) + 2 \sum_{i=1}^k c_i (K_X \cdot F_i) + \sum_{i=1}^k c_i^2$$

is a sum of nonnegative terms which vanish only if $n$ and the $c_i$ are all zero, as desired. □

We remark that nothing we have proved so far in this section requires $X$ to have Donaldson simple type. Indeed, the only facts we have used about basic classes are the adjunction inequality, which still holds when $X$ has finite type, and the inequality $K \cdot [\omega] \leq K_X \cdot [\omega]$, which follows from the adjunction inequality and the existence of symplectic surfaces dual to $k[\omega]$ for large $k$.

At this point we specialize to Seiberg–Witten theory, where we can apply several theorems of Taubes [31, 32] to completely determine which basic classes evaluate maximally on the fiber $\Sigma$.

Theorem 3.5. Let $X \to S^2$ be a relatively minimal Lefschetz fibration of genus $g \geq 2$ satisfying $b^+(X) > 1$. Then the canonical class $K_X$ is the unique Seiberg–Witten basic class $K$ of $X$ for which $K \cdot \Sigma = 2g - 2$. 
Proof. Let $K$ be a Seiberg–Witten basic class with $K \cdot \Sigma = 2g - 2$, and note that $K_X$ is one such class \cite{31}. We cannot have $K^2 < K_X^2 = 3\tau(X) + 2\chi(X)$, because then the associated moduli space of monopoles has negative expected dimension, which makes it generically empty and so $K$ would not be a basic class. (In fact, since $X$ has Seiberg–Witten simple type \cite{30}, every basic class satisfies $K^2 = K_X^2$.) Therefore $K_X - K$ is torsion, which implies that $K_X \cdot [\omega] = K \cdot [\omega]$, hence $K = \pm K_X$ by \cite{32}. Clearly we cannot have $K = -K_X$, because the fact that $K_X \cdot \Sigma = 2g - 2 > 0$ would contradict the claim that $K_X - K = 2K_X$ is torsion, so it follows that $K = K_X$. \hfill $\square$

The same argument does not show that a Donaldson basic class $K$ of this form must be the canonical class, because we do not know (although we expect it to be true) that all Donaldson basic classes satisfy $K^2 \geq K_X^2$. However, in certain situations we can still reach the same conclusion.

**Lemma 3.6.** Let $X \to S^2$ be a relatively minimal genus $g \geq 2$ Lefschetz fibration with $b_1 = 0$, $b^+ > 1$, and generic fiber $\Sigma$, not necessarily of Donaldson simple type, and let $K$ be a basic class of $X$ such that $K \cdot \Sigma = 2g - 2$. Then $K \cdot E = -1$ for any section $E$ of square $-1$.

**Proof.** The class $\Sigma + E$ is represented by a genus $g$ surface of square $\Sigma^2 + 1 = 1$ obtained by smoothing the point of intersection of $\Sigma \cup E$, and so $K \cdot (\Sigma + E) + 1 \leq 2g - 2$ by the adjunction inequality. Since $K \cdot \Sigma = 2g - 2$ by assumption, we see that $K \cdot E \leq -1$.

We now use a fact proved by Muñoz \cite{25} Remark 3 about blow-ups: if $X_0$ is the manifold obtained by blowing down $X$ along $E$, then $K$ has the form $K = L \pm (2n + 1)E$ where $L$ is a basic class of $X_0$ and $n \geq 0$, and furthermore $d(K) = d(L) - n(n + 1)$. Since $K \cdot E \leq -1$, we actually have $K = L + (2n + 1)E$. In fact, since $K \cdot \Sigma + \Sigma \cdot \Sigma = 2g - 2$ the adjunction inequality \cite{24} Theorem 1.7] says that $d(K) = 0$, hence $d(L) = n(n + 1)$. If $\Sigma_0 \subset X_0$ is the surface with proper transform $\Sigma \subset X$, so that $\Sigma_0^2 = 1$ and $\Sigma = \Sigma_0 - E$, then

$$2g - 2 = K \cdot \Sigma = (L + (2n + 1)E) \cdot (\Sigma_0 - E) = L \cdot \Sigma_0 + (2n + 1)$$

and so the adjunction inequality applied to $L$ and $\Sigma_0$ gives

$$2g - 2 \geq L \cdot \Sigma_0 + \Sigma_0 \cdot \Sigma_0 + 2d(L) = (2g - 2 - (2n + 1)) + 1 + 2n(n + 1)$$

or equivalently $2n^2 \leq 0$. We conclude that $n = 0$, and so $K = L + E$ and $K \cdot E = -1$. \hfill $\square$

**Proposition 3.7.** Let $X \to S^2$ be a relatively minimal genus $g \geq 2$ Lefschetz fibration for which every component of every fiber intersects at least one section of square $-1$, and suppose that $b_1(X) = 0$ and $b^+(X) > 1$. Then any Donaldson basic class $K$ of $X$ such that $K \cdot \Sigma = 2g - 2$ must equal $K_X$ up to torsion.

**Proof.** Let $F_i \cup G_i$ be a reducible singular fiber, and let $E$ be a $(-1)$–section intersecting $F_i$. Make the section $E$ symplectic by taking the parameter $t$ in the proof of Proposition 3.2 sufficiently small. Then the class $F_i + E$ is represented by a genus $g(F_i)$ surface of square $F_i^2 + 1 = 0$ obtained by smoothing $F_i \cup E$, and so $K \cdot (F_i + E) \leq 2g(F_i) - 2$ by the adjunction inequality. Since both $F_i$ and $E$ are symplectic, the adjunction formula says that $K_X \cdot F_i = \Sigma = \Sigma X.$
2g(F_1) - 1 and K_X \cdot E = -1, and so K \cdot (F_1 + E) \leq K_X \cdot (F_1 + E). Therefore K \cdot F_i \leq K_X \cdot F_i, and likewise we can show that K \cdot G_i \leq K_X \cdot G_i. Combining the two inequalities, we get

\[ K \cdot \Sigma = K \cdot F_i + K \cdot G_i \leq K_X \cdot F_i + K_X \cdot G_i = K_X \cdot \Sigma, \]

and by assumption the left and right sides are equal, so in fact \( K \cdot F_i = K_X \cdot F_i \). Pairing both sides of PD(\( K_X - K \)) = n\Sigma + \sum c_j F_j \) with \(-F_i\), we see that \( c_i = 0 \).

Finally, we pair both sides of PD(\( K_X - K \)) = n\Sigma with a symplectic \((-1)\)-section \( E \) to get \( n = n\Sigma \cdot E = (K_X - K) \cdot E = 0 \), and so we conclude that \( K_X = K \) as desired. \( \square \)

**Remark 3.8.** We have not actually shown in the proof of Proposition 3.7 that \( K_X \) is a Donaldson basic class, only that \( K \cdot \Sigma = 2g - 2 \) implies \( K = K_X \) under the given hypotheses. However, we will establish that \( K_X \) is a basic class for \( X \) of simple type during the proof of Theorem 4.1 and for general \( X \) in Corollary 8.6.

### 4. A fiber sum which satisfies Witten’s conjecture

Let \( X \to S^2 \) be a symplectic Lefschetz fibration with fiber \( \Sigma \) of genus \( g \geq 8 \), and let \( X^\circ = X \setminus N(\Sigma) \) be the complement of a neighborhood of a regular fiber. Our goal is to prove the following, cf. [17, Proposition 15]:

**Theorem 4.1.** We can find a symplectic, relatively minimal Lefschetz fibration \( Z \to S^2 \) with fiber genus \( g \) and \( b^+(Z) > 1 \) so that the fiber sum \( W = X \#_\Sigma Z \) satisfies \( H_1(W; \mathbb{Z}) = 0 \), the restriction map \( H^2(W; \mathbb{Z}) \to H^2(X^\circ; \mathbb{Z}) \) is surjective, and:

1. There is a smooth hypersurface \( W_* \subset \mathbb{CP}^3 \) of even degree at least 6 such that \( b^- (W) < b^-(W_*) \) and \( b^+(W) = b^+(W_*) \).
2. \( Z \) contains a tight genus 2 surface disjoint from a generic fiber, hence so does \( W \).

Much of the proof follows the same lines as in [17]. We start by constructing a relatively minimal Lefschetz fibration \( Z \to S^2 \) with fiber \( \Sigma \) whose vanishing cycles include a generating set for \( H_1(\Sigma) \), so that \( H_1(Z) = 0 \). Then the union

\[ W = X^\circ \cup_{\Sigma} Z^\circ = X \#_\Sigma Z, \]

where \( \#_\Sigma \) denotes the Gompf fiber sum [14] along regular fibers \( \Sigma \), is also a symplectic manifold admitting a genus \( g \) Lefschetz fibration over \( S^2 \). We note that \( \Sigma \) is homologically essential in both \( X \) and \( Z \), since it is a symplectic surface of genus at least 2, and so the homology class \([\Sigma]\) is primitive in both \( X \) and \( Z \) by [29, Theorem 1.4].

We will prove Theorem 4.1 by modifying \( Z \), and hence \( W \), via a sequence of fiber sums.

**Lemma 4.2** (cf. [17, Lemmas 11–12]). If \( H_1(Z) = 0 \), then \( H_1(W) = 0 \) and the restriction map \( H^2(W) \to H^2(X^\circ) \) is surjective.

**Proof.** For the first claim, we note that \( H_1(Z) = 0 \) if and only if the vanishing cycles of \( Z \to S^2 \) generate \( H_1(\Sigma) \), and hence the vanishing cycles of \( W \to S^2 \) do as well. For the second claim, we have \( H^3(Z^\circ, \partial Z^\circ) = 0 \) by Poincaré duality, and thus \( H^3(W, X^\circ) = 0 \) by excision, so the restriction \( H^2(W) \to H^2(X^\circ) \) is surjective by the long exact sequence of the pair \((W, X^\circ)\). \( \square \)
**Lemma 4.3.** Let $V \to S^2$ be a genus $g$ Lefschetz fibration with generic fiber $\Sigma$. If we replace $Z$ with the fiber sum $Z\#_\Sigma V$, then this preserves the property $H_1(Z) = 0$ and hence the conclusion of Lemma 4.2. Furthermore, if we let

$$n^\pm(V) = b^\pm(V) - b_1(V) + 2g - 1$$

then this fiber sum operation increases $b^+(W)$ and $b^-(W)$ by $n^+(V)$ and $n^-(V)$, respectively.

**Proof.** The claim $H_1(Z) = 0$ is immediate, because the vanishing cycles of $Z$ generate $H_1(\Sigma)$ and hence the same is true of $Z\#_\Sigma V$. To compute $b^\pm(\tilde{W})$, where $\tilde{W} = W\#_\Sigma V = X\#_\Sigma (Z\#_\Sigma V)$ is the result of modifying $X$ by this fiber sum, we recall from [14] the formulas

$$\sigma(\tilde{W}) = \sigma(W) + \sigma(V),$$
$$\chi(\tilde{W}) = \chi(W) + \chi(V) - 2\chi(\Sigma).$$

From the second equation and the fact that $b_1(\tilde{W}) = b_1(W) = 0$, we have

$$2 + b^+(\tilde{W}) + b^-(\tilde{W}) = (2 + b^+(W) + b^-(W)) + (2 - 2b_1(V) + b^+(V) + b^-(V)) - 2(2 - 2g)$$

and by adding the first equation this simplifies to

$$b^+(\tilde{W}) = b^+(W) + b^+(V) - b_1(V) + 2g - 1.$$

Subtracting the first equation from this gives the analogous formula for $b^-$. \qed

**Lemma 4.4.** If $g \geq 8$ then there are relatively minimal genus $g$ Lefschetz fibrations $V_1 \to S^2$ and $V_2 \to S^2$ such that $\frac{n^-(V_1)}{n^+(V_1)} < 2$ and $\gcd(n^+(V_1), n^+(V_2)) = 2$.

**Proof.** Suppose that $g = 2k$ is even. Stipsicz [29] Section 5] has constructed a genus $g$ Lefschetz fibration on $V_1 = (\Sigma_k \times S^2) \# 4\mathbb{CP}^2$ by realizing it as the resolution of the double cover of $\Sigma_k \times S^2$ branched over the singular curve

$$(\{p_1, p_2\} \times S^2) \cup (\Sigma_k \times \{q_1, q_2\})$$

and composing the cover $V_1 \to \Sigma_k \times S^2$ with the projection to $S^2$; the generic fiber is the double cover of $\Sigma_k$ branched at two points, which is indeed $\Sigma_g$. Similarly, if we take the branched double cover of $\Sigma_{k-1} \times S^2$ with branch locus $(\{p_1, \ldots, p_6\} \times S^2) \cup (\Sigma_{k-1} \times \{q_1, q_2\})$ and resolve singularities, the resulting $V_2 \cong (\Sigma_{k-1} \times S^2) \# 12\mathbb{CP}^2$ admits a genus $g$ Lefschetz fibration. The resulting triples of Betti numbers $(b_1, b^+, b^-)$ are $(g, 1, 5)$ and $(g - 2, 1, 13)$ for $V_1$ and $V_2$ respectively, so we have $\gcd(n^+(V_1), n^+(V_2)) = \gcd(g + 2, 2)$ and $\frac{n^-(V_1)}{n^+(V_1)} = \frac{g + 4}{g} < 2$ for all even $g > 4$.

Now suppose instead that $g = 2k + 1$ is odd. Stipsicz [29] constructs a genus $g$ Lefschetz fibration on $V_1 = (\Sigma_k \times S^2) \# 8\mathbb{CP}^2$ by the same method as in the even case, using the double cover $\Sigma_g \to \Sigma_k$ with 4 branch points, and if we use the double cover $\Sigma_g \to \Sigma_{k-1}$ with 8 branch points we get another such fibration on $V_2 = (\Sigma_{k-1} \times S^2) \# 16\mathbb{CP}^2$. Then $V_1$ and $V_2$ have Betti numbers $(b_1, b^+, b^-)$ equal to $(g - 1, 1, 9)$ and $(g - 3, 1, 17)$ respectively, so $\gcd(n^+(V_1), n^+(V_2)) = \gcd(g + 1, g + 3) = 2$ and $\frac{n^-(V_1)}{n^+(V_1)} = \frac{g + 9}{g + 1} < 2$ for all odd $g > 7$. \qed
It may be possible to construct appropriate $V_1$ and $V_2$ by other means for most values of $g < 8$; in fact, the above construction already works for $g = 6$. However, it turns out that we cannot do this when $g = 2$:

**Proposition 4.5.** There is no genus 2 Lefschetz fibration $V_1 \to S^2$ with $\frac{n_+(V_1)}{n_-(V_1)} \leq 2$.

**Proof.** Since $n^\pm = \frac{e \pm \sigma}{2} + 2g - 2$, where $e$ and $\sigma$ are the Euler number and signature of $V_1$, the condition $\frac{n_-}{n^+} \leq 2$ is equivalent to $3\sigma + e \geq 4 - 4g$. In the case $g = 2$, Özbäğcn [26, Corollary 10] has shown that $V_1$ must satisfy $c_1^2 \leq 6\chi_h - 3$, where $c_1^2 = 3\sigma + 2e$ and $\chi_h = \frac{\sigma + e}{4}$; but this is equivalent to $3\sigma + e \leq -6$, which makes the desired inequality $3\sigma + e \geq 4$ impossible. □

**Lemma 4.6.** Let $V \to S^2$ be any nontrivial Lefschetz fibration with fiber $\Sigma$ of genus at least 1. Then the fiber sum $V \#_\Sigma V$ contains a tight surface of genus 2 which is disjoint from a fiber of the induced fibration.

**Proof.** Pick two critical values $x_1, x_2$ of the fibration $f : V \#_\Sigma V \to S^2$ which correspond to the same critical value of $V \to S^2$, so that the fibers over them have the same vanishing cycle $c \subset \Sigma$, and let $\alpha \subset S^2$ be a matching path [28] with endpoints at $x_1$ and $x_2$. Then there is a Lagrangian sphere $L$ lying above $\alpha$, and in particular $L \cdot L = -2$.

Now take a closed curve $\gamma \subset S^2$ which intersects $\alpha$ once and separates the sphere into two disks corresponding to the bases of each copy of $V$. The monodromy along $\gamma$ is trivial, so $f^{-1}(\gamma) \cong S^1 \times \Sigma$. The fiber $\Sigma$ above $\gamma \cap \alpha$ intersects $L$ in $c$, and if $c' \subset \Sigma$ is a curve which intersects $c$ once then $T = S^1 \times c' \subset f^{-1}(\gamma)$ is a torus of self-intersection zero which intersects $L$ in a point. As remarked in [16, Corollary 8.5], we can take $L$ and two parallel copies of $T$ and smooth their intersections to get a tight surface of genus 2 in $V \#_\Sigma V$ which is disjoint from every fiber over a point outside a small neighborhood of $\alpha \cup \gamma$. □

We are now ready to finish proving the main theorem of this section.

**Proof of Theorem 4.1.** We proceed exactly as in [17, Lemma 13], starting with $W = X \#_\Sigma Z$ for an appropriate $Z$ as above, but without having to increase the genus of the given fibration $X \to S^2$. Note that any two symplectic manifolds with $b_1 = 0$ have values of $b^+$ differing by an even number, since $b^+$ is necessarily odd for each of them.

Take two genus $g$ Lefschetz fibrations $V_1$ and $V_2$ over $S^2$ such that $\gcd(n^+(V_1), n^+(V_2)) = 2$ and $\frac{n_-(V_1)}{n^+(V_1)} < 2$, as in Lemma 4.3. Then any sufficiently large even number can be written as a nonnegative linear combination $m = k_1 \cdot n^+(V_1) + k_2 \cdot n^+(V_2)$ with $k_2 < n_+(V_1)$. If we replace $Z$ with its fiber sum with $k_1$ copies of $V_1$ and $k_2$ copies of $V_2$, the result will be to increase $b^+(W)$ by $m$ while keeping $H_1(W) = 0$, as in Lemma 4.3 and as $m$ gets large we will have $\frac{b^-(W)}{b^+(W)} \to \frac{n_-(V_1)}{n^+(V_1)} < 2$ since $k_2$ is bounded independently of $m$. If we also insist that $k_1 \geq 2$, then it follows that $b^+(Z) > 1$ since $n_+(V_1)$ is positive, and furthermore $Z$ (and hence $W$) contains a tight genus 2 surface by Lemma 4.6. In addition, $Z$ is relatively minimal, since the same is true of $V_1, V_2$ and the initial choice of $Z$ by construction.
In particular, since smooth hypersurfaces $W_\ast \subset \mathbb{C}P^3$ of degree $d$ satisfy $b^-(W_\ast) \to 2$ as $\deg(W_\ast) \to \infty$, we can take $d$ to be even and sufficiently large and then this construction will provide $W$ with $b^-(W) < b^-(W_\ast)$ and $b^+(W) = b^+(W_\ast)$, as desired. \hfill \Box

5. Donaldson invariants of fiber sums

Let $X \to S^2$ be a symplectic, relatively minimal Lefschetz fibration with fibers $\Sigma$ of genus $g \geq 8$. Take a symplectic Lefschetz fibration $Z \to S^2$ as constructed in Theorem 4.1 and let $W = X \#_\Sigma Z$ be their fiber sum; note that $W$ is also relatively minimal. Then $W$ has Donaldson simple type because it contains a tight surface and Seiberg–Witten simple type hence by Proposition 2.9 so does $W$. Therefore the Donaldson series of $W$ satisfies

$$D_W^w(h) = c(W) \exp \left( \frac{Q(h)}{2} \right) \sum_{r=1}^s (-1)^{w^2 + K_r \cdot w} \exp \left( SW(K_r) e^{K_r \cdot h} \right)$$

where $c(W)$ is a nonzero rational number.

**Proposition 5.1.** If $h \in H^2(W; \mathbb{Z})$ is the class of the fiber $\Sigma$, then $D_W^w(h)$ is nonzero with leading term of order $e^{2g-2}$ for all $w \in H^2(W; \mathbb{Z})$.

**Proof.** Let $K_W$ be the canonical class of $W$. Taubes [31] proved that $K_W$ is a Seiberg–Witten basic class and $SW(K_W) = \pm 1$. Furthermore, we know from Theorem 3.5 that $K_W$ is the only Seiberg–Witten basic class $K$ for which $K \cdot \Sigma = 2g - 2$, and so the coefficient of $e^{2g-2}$ in the series $D_W^w(h)$ is equal to $\pm c(W)$, which is nonzero. Therefore $D_W^w(h) \neq 0$, and there are no terms of higher order by the adjunction inequality. \hfill \Box

We now wish to apply a theorem of Muñoz concerning Donaldson invariants of fiber sums of manifolds with simple type; this includes $Z$ and $W$ by construction. We say that classes $w_W \in H^2(W; \mathbb{Z})$, $w_X \in H^2(X; \mathbb{Z})$, and $w_Z \in H^2(Z; \mathbb{Z})$ are compatible if they are all odd when evaluated on $\Sigma$; $w_W$ agrees with $w_X$ and $w_Z$ on $X^0 = X \setminus N(\Sigma)$ and $Z^0 = Z \setminus N(\Sigma)$; and $w_W^2 \equiv w_X^2 + w_Z^2$ (mod 4). This last congruence always holds mod 2 and can be achieved mod 4 by replacing $w_W$ with $w_W + PD(\Sigma)$ if necessary.

Next, we define $\mathcal{H} \subset H_2(W)$ to be the subspace of all classes $D$ such that $D|_{\partial X^0}$ is a multiple of the class $[S^1 \times \{\ast\}] \in H_1(\partial X^0) = H_1(S^1 \times \Sigma)$. Choose a linear map $\mathcal{H} \mapsto H_2(X) \oplus H_2(Z)$ so that if $D \mapsto (D_X, D_Z)$, then $D^2 = D_X^2 + D_Z^2$ and $D|_{X^0} = D_X|_{X^0}$ and likewise for $D$. Finally, give $X$ and $Z$ homology orientations, and let $W$ have the induced homology orientation as in [20, Remark 8]. Muñoz’s theorem now says the following, assuming $X$ also has simple type:

**Theorem 5.2 (Theorem 9).** Let $w_W \in H^2(W; \mathbb{Z})$ and pick compatible classes $w_X \in H^2(X; \mathbb{Z})$ and $w_Z \in H^2(Z; \mathbb{Z})$. Write the Donaldson series for $X$ and $Z$ as

$$D_X^{w_X}(\alpha) = \exp \left( \frac{Q_X(\alpha)}{2} \right) \sum_j a_{j,w_X} e^{K_j \cdot \alpha}$$

$$D_Z^{w_Z}(\beta) = \exp \left( \frac{Q_Z(\beta)}{2} \right) \sum_k b_{k,w_Z} e^{L_k \cdot \beta}.$$
Then

\[ D_W^{uw}(tD) = \exp \left( \frac{Q_W(tD)}{2} \right) \left( \sum_{K_j \cdot \Sigma = L_k \cdot \Sigma = 2g-2} -2^{7g-9} a_{j,w_X} b_{k,w_Z} e^{(K_j \cdot D_X + L_k \cdot D_Z + 2\Sigma \cdot D)t} \right) \]

\[ + \sum_{K_j \cdot \Sigma = L_k \cdot \Sigma = -(2g-2)} (-1)^{g} 2^{7g-9} a_{j,w_X} b_{k,w_Z} e^{(K_j \cdot D_X + L_k \cdot D_Z - 2\Sigma \cdot D)t} \].

Proof of Theorem 1.1 for \( g \geq 8 \). Let \( w_X = w \) be a cohomology class with \( w \cdot \Sigma \) odd. Recall from Theorem 1.1 that the restriction map \( H^2(W; \mathbb{Z}) \to H^2(X^\circ; \mathbb{Z}) \) is surjective, so we can lift \( w_X \mid \Sigma^0 \) to a class \( w_W \in H^2(W; \mathbb{Z}) \). This in turn provides us with a class in \( H^2(Z^\circ; \mathbb{Z}) \) by restriction, and this class can be extended across \( N(\Sigma) \subset Z \) to an element \( w_Z \in H^2(Z; \mathbb{Z}) \) because its restriction to \( \partial N(\Sigma) \), which can itself be extended across \( N(\Sigma) \). Clearly \( w_W, w_X, w_Z \) are compatible, except that the congruence \( w_W^2 \equiv w_X^2 + w_Z^2 \) may only be satisfied mod 2 rather than mod 4. According to [20, Remark 10], this only changes the formula of Theorem 5.2 by a sign, namely \( \epsilon = (-1)^{(g-1)(w_W^2 - w_X^2 - w_Z^2)/2} \), and since we only wish to prove a nonvanishing result we can ignore this.

Let \( D_X \) and \( D_Z \) be represented by a generic fiber \( \Sigma \subset X^0 \) and a tight genus 2 surface \( T \subset Z^0 \), respectively, and \( D = [\Sigma] + [T] \). Then \( \Sigma \cdot \Sigma = 0 \) and \( D \cdot D = T \cdot T = 2 \), and \( K \cdot T = 0 \) for every basic class \( K \) of \( W \) by the adjunction inequality, so if \( h \in H_2(W; \mathbb{Z}) \) is the class of a generic fiber then \( D_W^{uw}(D) = e \cdot D_W^{uw}(h) \) by Theorem 2.2. By Theorem 5.2 we have

\[ \frac{D_W^{uw}(D)}{e \cdot 2^{7g-9}} = \sum_{K_j \cdot \Sigma = L_k \cdot \Sigma = 2g-2} -a_{j,w_X} b_{k,w_Z} e^{2g-2} + \sum_{K_j \cdot \Sigma = L_k \cdot \Sigma = -(2g-2)} (-1)^g a_{j,w_X} b_{k,w_Z} e^{-(2g-2)} \].

By Proposition 5.1 however, we know that \( D_W^{uw}(h) \) is an element of \( \mathbb{Q}[\epsilon^{\pm 1}] \) whose coefficient of \( e^{2g-2} \) is \( \pm c(W) e^{2g-2} \), and so by equating coefficients we get (up to a sign)

\[ \frac{\pm c(W)}{2^{7g-9}} = \sum_{K_j \cdot \Sigma = L_k \cdot \Sigma = 2g-2} a_{j,w_X} b_{k,w_Z} = \left( \sum_{K_j \cdot \Sigma = 2g-2} a_{j,w_X} \right) \left( \sum_{L_k \cdot \Sigma = 2g-2} b_{k,w_Z} \right) \]

which is nonzero. The two factors in parentheses are exactly the \( e^{2g-2} \)-coefficients of \( D_X^{uw}(h) \) and \( D_Z^{uw}(h) \), so both of these coefficients are nonzero as well. Therefore the series \( D_X^{uw}(h) \) is nonempty, with leading term of order \( e^{2g-2} \), and the subset \( \{ K \mid K \cdot \Sigma = 2g-2 \} \) of the basic classes of \( X \) is nonempty, as desired.

\[ \square \]

6. Proof of Theorem 1.1

Let \( (X, \omega) \) and \( w \in H^2(X; \mathbb{Z}) \) satisfy the hypotheses of Theorem 1.1, namely that \( X \) has simple type, \( b_1(X) = 0 \), \( b^+(X) > 1 \), and the class \( [\omega] \) is integral. Donaldson [4] proved that for any sufficiently large integer \( k \), there is a Lefschetz pencil on \( X \) whose generic fibers \( \Sigma \) are symplectic submanifolds Poincaré dual to \( k[\omega] \). If this pencil has reducible singular fibers, then for any such fiber with components \( F_i \cup G_i \) we know that \( F_i \) is symplectic and so \( \Sigma \cdot F_i = k([\omega] \cdot F_i) > 0 \), hence \( F_i \) contains some base points of the pencil and likewise for \( G_i \).
The adjunction formula applied to the fiber $\Sigma$ tells us that if $g$ is the fiber genus, then $2g - 2 = k^2[\omega]^2 + k(K_X \cdot [\omega])$, and so by taking $k$ large we can insist that $g \geq 8$. If $\Sigma^2 = n$, then we can blow up $X$ at the $n > 0$ base points of the pencil, and the manifold 

$$\tilde{X} = X \# n\mathbb{CP}^2$$

will admit a genus $g$ Lefschetz fibration with $n$ sections $E_i$ of self-intersection $-1$, some of which intersect any component of any fiber, and with generic fiber $\tilde{\Sigma} = \Sigma - (E_1 + \cdots + E_n)$ the proper transform of $\Sigma$. The $g \geq 8$ case of Theorem 1.2 says that $D_X^\omega([\Sigma])$ is nonzero for some $\omega$, hence $D_X^\omega$ is not identically zero by the blowup formula.

Now we know that $\tilde{X}$ has at least one basic class $\tilde{K}$ for which $\tilde{K} \cdot \tilde{\Sigma} = 2g - 2$, hence Proposition 5.7 ensures that $K_{\tilde{X}}$ is the only such class. Furthermore, the class $K_{\tilde{X}}$ can be uniquely written as $K + \sum \sigma_i PD(E_i)$, where $K$ is a basic class on $X$ and $\sigma_i = \pm 1$ for each $i$. Since $K_{\tilde{X}} = K_X + \sum PD(E_i)$, we conclude that $K_X$ is a basic class of $X$.

Finally, suppose that $K$ is some basic class on $X$ for which $K \cdot [\omega] = K_X \cdot [\omega]$, or equivalently $K \cdot [\Sigma] = K_X \cdot [\Sigma]$. Then $\tilde{K} = K + \sum PD(E_i)$ is a basic class of $\tilde{X}$, and

$$\tilde{K} \cdot \tilde{\Sigma} = K \cdot \Sigma + n = K_X \cdot \Sigma + n = 2g - 2$$

by the adjunction formula, so it follows that $\tilde{K} = K_{\tilde{X}}$ and then $K = K_X$. Likewise, if $K \cdot [\omega] = -K_X \cdot [\omega]$ then the basic class $-K$ must be $K_{\tilde{X}}$, and so $|K \cdot [\omega]| = K_X \cdot [\omega]$ if and only if $K = \pm K_X$.

We conclude that if $h$ is Poincaré dual to $[\omega]$ and $K_X \cdot [\omega] > 0$, then the Donaldson series

$$D_X^\omega(h) = \exp \left( \frac{Q(h)}{2} \right) \sum_{r=1}^{s} (-1)^{(w^2+K_r \cdot w)/2} \beta_r e^{K_r \cdot h}$$

has exactly one nonzero highest-order term, namely the one where $K_r = K_X$. It follows that $D_X^\omega(h)$ is nonzero.

If instead $K_X \cdot [\omega] = 0$, then we know that $\pm K_X$ are the only basic classes. If $K_X$ is nonzero then according to Taubes [30] it is Poincaré dual to a nonempty, embedded symplectic curve $S$ (note that this requires $b^+(X) > 1$), hence $K_X \cdot [\omega] = \int_S \omega > 0$ which is a contradiction. Therefore $K_X = 0$ and we have $D_X^\omega = c \cdot (-1)^{w^2/2} e^{Q/2}$ for some rational $c$. Since $D_X^\omega$ is not identically zero, we must have $c \neq 0$ and therefore $D_X^\omega(h) \neq 0$ as desired.

**Remark 6.1.** The condition $K_X \cdot [\omega] = 0$ with $[\omega]$ integral actually forces $X$ to have simple type: if $\Sigma$ is an embedded symplectic surface which is Poincaré dual to $k[\omega]$ for $k$ large, then $g(\Sigma) \geq 2$ and the adjunction formula says that $\Sigma \cdot \Sigma = 2g(\Sigma) - 2$, so $\Sigma$ is a tight surface.

7. **Proof of Theorem 1.2**

We begin with some observations about the geography of symplectic 4-manifolds.

**Lemma 7.1.** For $n \geq 2$, the elliptic surface $E(n)$ contains a tight surface of genus 2 which is disjoint from both a generic elliptic fiber and a section of self-intersection $-n$. 
Proof. It suffices to prove this for \( n = 2 \), since for larger \( n \) we know that \( E(n) \) is a fiber sum of \( E(2) \) and \( E(n - 2) \) and the \((-n)\)-section is obtained by stitching together a \((-2)\)-section and a \((-n-2)\)-section of the respective fibrations. Since \( E(2) \) is also a fiber sum \( E(1) \#_2 E(1) \), with \((-2)\)-section obtained by gluing together \((-1)\)-sections of each \( E(1) \cong \mathbb{C}P^2 \# 9\mathbb{C}P^2 \), we can construct the desired tight surface just as in Lemma \[4.6\].

**Proposition 7.2.** For any fixed \( r < \frac{11}{3} \), there are simply connected, spin, symplectic manifolds \( X \) with \( \frac{b^+(X)}{\omega(X)} > r \) and \( b^+(X) \) arbitrarily large such that:

1. The only Seiberg–Witten basic classes of \( X \) are \( \pm K_X \), and \( SW(\pm K_X) = 1 \).
2. The Donaldson invariants of \( X \) are not identically zero.
3. \( X \) contains a tight surface and a symplectic surface \( S \) with \( S \cdot S = 0 \) and \( K_X \cdot S \neq 0 \).

Proof. Using the fact that \( b^+(X) = \frac{\sigma(X)}{2} - 1 \) whenever \( b_1(X) = 0 \), the condition \( \frac{b^+(X)}{\omega(X)} > r \) is equivalent to \( (r - 1)e + (r + 1)\sigma < 2(r - 1) \). In terms of the invariants \( c_1^2 = 3\sigma + 2e \) and \( \chi_h = \frac{\sigma + \sigma}{4} \), then, this is also equivalent to

\[
c_1^2 + 2(r - 10)\chi_h < r - 1.
\]

Let \( K \subset S^3 \) be a knot with Seifert genus \( g \), and suppose that the Alexander polynomial \( \Delta_K(t) \) has degree \( g \). Fintushel and Stern \[12\] construct a simply connected, spin 4-manifold \( Z_K \) with Seiberg–Witten simple type and exactly one basic class \( \kappa \) up to sign.

The first step in the construction of \( Z_K \) is to perform knot surgery \[11\] along a generic fiber \( T \) of the elliptic fibration on \( E(2n) \) to get a manifold \( E(2n)_K \). Inside \( E(2n)_K \), there is a genus \( g \) surface \( S' \) of self-intersection \(-2n\), such that \( S' \cap (E(2n)_K \setminus N(T)) \) is contained in a \((-2n)\)-section of the fibration on \( E(2n) \). Therefore we have a tight genus 2 surface \( F \subset E(2n)_K \) which is disjoint from both \( S' \) and another generic fiber \( T' \subset E(2n)_K \setminus N(T) \subset E(2n)_K \), hence from the surface \( \Sigma' \) of genus \( g + n \) and self-intersection 0 formed by smoothing out \( S' + nT' \). Then \( Z_K \) is the fiber sum \( E(2n)_K \#_{\Sigma'}= Y \) for an appropriate symplectic \( Y \) with embedded symplectic surface \( C \), and \( F \) remains a tight genus 2 surface inside \( Z_K \).

Now suppose that \( K \) is fibered and has genus \( g \). Then the surgered manifold \( E(2n)_K \) is symplectic as well, and so \( X = Z_K \) admits a symplectic structure. It follows from Taubes \[31\] that \( \kappa \) must be the canonical class, i.e. that \( SW(\pm K_X) = \pm 1 \) and that there are no other basic classes; if we can ensure that \( b^+(X) \equiv 3 \pmod{4} \), or equivalently that \( \chi_h(X) = \chi \) is even, then we will have \( SW(\pm K_X) = 1 \) for the canonical homology orientation on \( X \). The Donaldson invariants of \( X \) are nonzero by Theorem \[1.1\] and \( X \) contains the tight surface \( F \) and the symplectic surface \( S \) obtained as a parallel copy of \( C \subset Y \); then \( S^2 = 0 \) and \( K_X \cdot S = 2g(S) - 2 \) is nonzero because \( g(S) = g + n \geq 2 \).

It only remains to determine the invariants \( c_1^2(X) \) and \( \chi_h(X) \). According to \[12\], this construction yields \( c_1^2 = 8(g + n - 1) \) and \( \chi_h = 3n + g - 1 \), and so

\[
c_1^2 + (2r - 10)\chi_h = (6r - 22)n + (2r - 2)(g - 1).
\]
For \( r < \frac{1}{r} \), the coefficient of \( n \) is negative, and so for fixed \( g \) and any large enough \( n \) we will have \( c_2^2 + (2r - 10)x_h \) \( r - 1 \). If we also insist that \( n + g \) be odd then \( \chi_h = \frac{b^+(X) + 1}{2} \) will be even, and as \( n \) goes to infinity so does \( b^+(X) \), as desired. \( \square \)

We now complete the proof of Theorem \( 1.2 \). Let \( X \) be a manifold of Donaldson simple type with \( b_1(X) = 0 \) and \( b^+(X) > 1 \) all odd, and suppose that we have a relatively minimal Lefschetz fibration \( X \to S^2 \) of genus \( g \geq 2 \). By repeating the arguments of Section 4, we can find a relatively minimal genus \( g \) Lefschetz fibration \( Z \to S^2 \) so that the fiber sum \( W = X \#_\Sigma Z \) has \( H_1(W; \mathbb{Z}) = 0 \), the map \( H^2(W; \mathbb{Z}) \to H^2(X^c; \mathbb{Z}) \) is surjective, \( W \) and \( Z \) both contain a tight genus 2 surface, and we have a manifold \( w \) as in Proposition 7.2 such that \( b^-(W) < b^-(W) \) and \( b^+(W) = b^+(W) \).

To check that we can find such a \( w \), note that Lemma 4.4 still provides Lefschetz fibrations \( V_1 \) and \( V_2 \) with \( \frac{n^-(V_1)}{n^+(V_1)} \leq 3 \) and \( \gcd(n^+(V_1), n^+(V_2)) = 2 \) even when \( 2 \leq g < 8 \), and so there is some \( N_0 \) such that for any odd \( N \geq N_0 \) we can achieve \( b^+(W) = N \) and \( b^-(W) < \frac{N}{8} \). Now applying Proposition 7.2 with \( r = \frac{7}{2} \) guarantees that we can take \( w \) to have odd \( b^+(W) > N_0 \) and \( b^-(W) > \frac{N}{8} \) and so obtain the desired \( W \).

It now follows from Theorem 2.5 and Remark 2.6, together with Proposition 2.9, that the fiber sum \( W = X \#_\Sigma Z \) satisfies Conjecture 2.4. We proceed exactly as in Section 5 to conclude that if \( w \cdot \Sigma \) is odd, then \( D^w_X((\Sigma)) \) is nonzero with a leading term of order \( e^{2g-2} \).

Remark 7.3. By Theorem 2.2 we have proved that for any Lefschetz fibration \( X \to S^2 \) with fiber class \( h = [\Sigma] \) and \( w \in H^2(X; \mathbb{Z}) \) satisfying the conditions of Theorem 2.2 there is a constant \( c \neq 0 \) such that \( D_X^w(h^d) \) is asymptotic to \( c \cdot (2g-2)^d \) for large \( d \equiv -w^2 - \frac{3}{2}(b^+(X) + 1) \) (mod 4). In Donaldson’s original notation, this says that \( q_{k,X}(h, \ldots, h) \sim c' \cdot (2g-2)^k \) for some nonzero \( c' \) and all sufficiently large \( k \), cf. [2, Theorem C].

8. Lefschetz fibrations which do not have simple type

In this section we use our previous results to study Lefschetz fibrations of genus at least 2 which do not have simple type, and thus prove a nonvanishing theorem for Donaldson invariants of symplectic manifolds in general. We recall that according to Witten’s conjecture, all symplectic 4-manifolds should have simple type, which would render this section unnecessary.

To 4-manifolds \( X \) with boundary \( Y \) and classes \( w \in H^2(X; \mathbb{Z}) \) for which the instanton Floer homology group \( I_*(Y)_w \) is well-defined (as in [5]), though we follow Kronheimer and Mrowka’s notation from [19] and also confuse \( w\big|_Y \) with the Hermitian line bundle over \( Y \) having first Chern class \( w \), one can often assign relative Donaldson invariants which satisfy nice gluing theorems [3, 11]. In general these associate to elements of \( H_2(X, Y) \) with image \( \gamma \) in \( H_1(Y) \) an element of the Fukaya–Floer homology group \( HFF(Y, \gamma) \) [13], but in the cases where \( \gamma = 0 \) the relative invariants form a map \( \phi^\gamma_X : \mathbb{A}(X) \to I_*(Y)_w \) with values in the ordinary instanton Floer homology of \( Y \).

Lemma 8.1. Let \( X \to D^2 \) be a Lefschetz fibration of genus at least 2 over the disk. Then we can extend \( X \to D^2 \) to a Lefschetz fibration \( W \to S^2 \) such that \( H_1(W; \mathbb{Z}) = 0 \), \( b^+(W) > 1 \), the
map $H^2(W) \to H^2(X)$ is surjective, and $W$ has Donaldson simple type, and if $X$ is relatively minimal then so is $W$.

Proof. Let $\Sigma$ denote the generic fiber of $X$ and let $Y = \partial X$, and construct a Lefschetz fibration $Z_0 \to D^2$ with fiber $\Sigma$ and boundary $-Y$ such that the vanishing cycles of $Z_0$ are all nonseparating and generate $H_1(\Sigma; \mathbb{Z})$. Then we take $W_0 = X \cup_Y Z_0$ and $W = W_0 \#_\Sigma W_0 = X \cup_Y (Z_0 \#_\Sigma W_0)$.

Since $H_1(W_0) = 0$, Lemmas 4.3 and 4.6 imply that $b_1(W) = 0$, $b^+(W) > 1$, and $W$ has simple type. Finally, the surjectivity of $H^2(W) \to H^2(X)$ is implied by $H^3(W,X) = 0$, which by excision and Poincaré duality is equivalent to $H_1(W\setminus X) = 0$ and this is immediate from $\overline{W\setminus X} = Z_0 \#_\Sigma W_0$.

Given $(Y, w)$ for which $I_s(Y)_w$ is defined and a closed surface $R \subset Y$, there is a natural operator $\mu(R)$ of degree $-2$ on $I_s(Y)_w$, and we can decompose $I_s(Y)_w$ into the generalized eigenspaces $V_\lambda$ of $\mu(R)$; Kronheimer and Mrowka [19] define the group $I_s(Y|R)_w$ to be $V_{2g-2}$. In this notation, the goal of the following proposition is to show that the relative invariant $\phi_X^w(1)$ of a sufficiently nice Lefschetz fibration $X \to D^2$ projects to a nonzero element of $I_s(\partial X|\Sigma)_w$, where $\Sigma$ is a generic fiber.

Proposition 8.2. Let $X \to D^2$ be a relatively minimal Lefschetz fibration with generic fiber $\Sigma$ of genus $g \geq 2$ and boundary $Y$. If $w \cdot \Sigma$ is odd, then the relative invariant $\phi_X^w(1) \in I_s(Y)_w$ has nonzero $V_{2g-2}$-component.

Proof. We extend $X \to D^2$ to a closed, relatively minimal Lefschetz fibration $W \to S^2$ as in Lemma 6.1 and let $w \in H^2(X; \mathbb{Z})$ to a class in $H^2(W; \mathbb{Z})$ which we also denote by $w$. Let $Z = \overline{W\setminus X}$, and let $h \in H_2(W; \mathbb{Z})$ denote the class $[\Sigma]$. Then we have an equation

$$D_W^W(h^n) = \langle \phi_X^w(h^n), \phi_Z^w(h^{n-k}) \rangle$$

for the Poincaré duality pairing $\langle \cdot, \cdot \rangle : I_s(Y)_w \otimes I_s(-Y)_w \to \mathbb{C}$ (see [5, Theorem 6.7]).

Define a polynomial $f_0(t)$ by the formula

$$f_0(t) = (t + (2g-2))(t^2 + (2g-2)^2) \prod_{k=0}^{g-2} (t^4 - (2k)^4)$$

so that the roots of $f_0(t)$ are precisely all numbers of the form $i^r(2k)$ where $0 \leq r \leq 3$ and $0 \leq k \leq g - 1$, except for $2g - 2$. We also let $f_1(t)$ be the characteristic polynomial of the action of $\mu(\Sigma)$ on $I_s(Y)_w$, divided by $(t - (2g-2))^{\dim(V_{2g-2})}$. Finally, we let $f(t) = f_0(t)f_1(t)$, and we observe that $f(\mu(\Sigma))$ annihilates all generalized eigenspaces $V_\lambda$ except possibly when $\lambda = 2g-2$, since the same is true of its factor $f_1(\mu(\Sigma))$. Thus we can write $f(\mu(\Sigma)) = \psi \circ \pi_{2g-2}$, where $\pi_{2g-2} : I_s(Y)_w \to V_{2g-2}$ is the projection operator and $\psi$ is some endomorphism of $V_{2g-2}$. Since eigenspaces of $\mu(\Sigma)$ with different eigenvalues are orthogonal with respect to the pairing $\langle \cdot, \cdot \rangle$, it follows that

$$D_W^W(f(h)) = \langle \phi_X^w(1), f(\mu(\Sigma)) \cdot \phi_Z^w(1) \rangle = \langle \pi_{2g-2}(\phi_X^w(1)), \psi \circ \pi_{2g-2}(\phi_Z^w(1)) \rangle$$
and so $\pi_{2g-2}(\phi_X^w(1))$ (and the pairing $\langle \cdot, \cdot \rangle$ on $V_{2g-2}$) must be nonzero if $D_W^w(f(h)) \neq 0$.

Let $d_0 = -w^2 - \frac{3}{2}(b^+(W) + 1)$. Since $W$ has simple type, we can write

$$2D_W^w(e^{\theta h}) = \sum_j a_{j,w} e^{(K_j \cdot h)} + (-i)^{d_0} \sum_j a_{j,w} e^{(iK_j \cdot h)}$$

as a special case of [16 Equation 1.10] (with $\lambda = 0$ and $Q(h) = 0$, and writing $a_{j,w} = (-1)^{(w^2 + K_j \cdot w)/2}\beta_j$ for convenience). Comparing $i^k$–coefficients gives us

$$2D_W^w(h^k) = \sum_j a_{j,w}(K_j \cdot h)^k + (-i)^{d_0} \sum_j a_{j,w}(iK_j \cdot h)^k$$

and so it is easy to show that

$$2D_W^w(f(h)) = \sum_j a_{j,w} f(K_j \cdot h) + (-i)^{d_0} \sum_j a_{j,w} f(iK_j \cdot h).$$

Now by construction we have $f(K_j \cdot h) = 0$ unless $K_j \cdot h = 2g - 2$, and $f(iK_j \cdot h) = 0$ for all $j$; this is because $K_j \cdot h \equiv h^2 \pmod{2}$ is even, and the factor $f_0$ of $f$ vanishes on all numbers of the form $i^r \cdot (2k)$ with $0 \leq r \leq 3$ and $0 \leq k \leq g - 1$ except for $2g - 2$. Therefore

$$2D_W^w(f(h)) = f(2g - 2) \sum_{K_j \cdot h = 2g - 2} a_{j,w},$$

and the right hand side is nonzero by Theorem 1.2 since the sum is the coefficient of $e^{2g-2}$ in $D_W^w(h)$, so $\pi_{2g-2}(\phi_X^w(1)) \neq 0$ as desired. \hfill $\Box$

**Remark 8.3.** The function $f(t)$ is often more complicated than necessary: in the case where $Y \cong S^1 \times \Sigma$ and $w|_Y$ is Poincaré dual to the $S^1$ factor, Muñoz [22] determined the structure of the closely related variant $I_s(S^1 \times \Sigma)_w$, and in particular the spectrum of the operator $\mu(\Sigma)$ on it. Kronheimer and Mrowka [19 Section 7] observed that as a consequence of these results, the spectrum of $\mu(\Sigma)$ on $I_s(S^1 \times \Sigma)_w$ is in this case exactly the set $\{i^r \cdot (2k) \mid 0 \leq r \leq 3, 0 \leq k \leq g - 1\}$, so it would have sufficed to take $f = f_1$, and that the generalized eigenspace $V_{2g-2}$ is 1–dimensional. Furthermore, since $\mu(\Sigma)$ acts with degree $-2$ on a $(\mathbb{Z}/8\mathbb{Z})$–graded vector space, they observed that $V_\lambda \cong V_\lambda$ for each eigenvalue $\lambda$ and so $\text{dim}(V_{i^r(2g-2)}) = 1$ for each $r$ as well. Our argument above does not make any use of this information.

Finally, in the following theorem we can lift the restriction that $H_1(X) = 0$. In this case, the Donaldson invariants can be defined on an algebra $A(X)$ containing the symmetric algebra on $H_0(X) \oplus H_2(X)$. Furthermore, given a homology class $h \in H_2(X; \mathbb{Z})$, the Donaldson invariants $D_X^w(h^n)$ can be nonzero only if $n \equiv -w^2 - \frac{3}{2}(b^+(X) - b_1(X) + 1) \pmod{4}$.

**Theorem 8.4.** Let $X \to S^2$ be a relatively minimal Lefschetz fibration with generic fiber $\Sigma$ of genus $g \geq 2$ such that $b^+(X) > 1$. Let $\Delta \cong D^2 \times \Sigma$ be a small neighborhood of a regular fiber, with boundary $Y \cong S^1 \times \Sigma$, and let $w \in H^2(X; \mathbb{Z})$ be a class for which $w|_Y = PD(S^1)$. Then there is a nonzero $c$ such that $D_X^w(h^n) \sim c \cdot (2g - 2)^n$ for all large $n \equiv d_0 \pmod{4}$, where $d_0 = -w^2 - \frac{3}{2}(b^+(X) - b_1(X) + 1)$ and $h$ is the homology class of $\Sigma$. 

Theorem 8.3). We conclude that $c(1)$ is nonzero on $V_{2g-2}$, which is 1–dimensional (see Remark 8.3), we conclude that $c(1) \neq 0$.

Let $p(t)$ denote the product $p(2g-2)$, which is 1–dimensional, and as multiplication by $p(2g-2) = 1$ on each 1–dimensional eigenspace $V_{2g-2}$. It follows that

\[
D^w_W(p(h)^n) = \langle p(\mu(\Sigma))\mu(\Sigma)^n \cdot \phi^w_{X_0}(1), \phi^w_{\Delta}(1) \rangle = \sum_{r=0}^{3} \langle \pi_{r}^{(2g-2)}(\mu(\Sigma)^n \cdot \phi^w_{X_0}(1)) \rangle_{\pi_{r}^{(2g-2)}}(\phi^w_{\Delta}(1))
\]

It is clear from this that we have $D^w_W(p(h)^{n+4}) = (2g-2)^4 D^w_W(p(h)^n)$ for all integers $n$, and furthermore we can express the relations for $0 \leq n \leq 3$ by the matrix equation

\[
\begin{pmatrix}
D^w_W(p(h)^0)/(2g-2)^0 \\
D^w_W(p(h)^1)/(2g-2)^1 \\
D^w_W(p(h)^2)/(2g-2)^2 \\
D^w_W(p(h)^3)/(2g-2)^3
\end{pmatrix} =
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & i & i^2 & i^3 \\
1 & i^2 & i^4 & i^6 \\
1 & i^3 & i^6 & i^9
\end{pmatrix}
\begin{pmatrix}
c_0 \\
c_1 \\
c_2 \\
c_3
\end{pmatrix}.
\]

The $4 \times 4$ matrix is invertible, and since the $c_i$ are not all zero it follows that for some $n$ we must have $D^w_W(p(h)^n) \neq 0$. In fact, we must have $n \equiv d_0$ (mod 4), since otherwise every monomial $h^e$ of $p(h)^n$ has degree $e \equiv n \equiv d_0$ (mod 4) and so $D^w_W(h^e) = 0$. Therefore

\[
D^w_X(p(h)^n) = C(2g-2)^n \quad \text{for all } n \equiv d_0 \pmod{4}
\]

where $C$ is a nonzero constant, and $D^w_X(p(h)^n) = 0$ for all other $n$.

Consider the generating function $F(t) = \sum_{j=0}^{\infty} D^w_X(h^j)t^j$. If we write $p(t) = \sum_{j=0}^{d} a_j t^j$, where $d = \deg(p)$, then for any $j \geq 0$ the $t^{d+j}$–coefficient of

\[
(a_d + a_{d-1}t + \cdots + a_0 t^d) \cdot F(t) = t^{d} p\left(\frac{1}{t}\right) \cdot F(t)
\]

is equal to $D^w_X(a_d h^{d+j} + a_{d-1}h^{d-1+j} + \cdots + a_0 h^j)$, or $D^w_X(p(h)^j)$. Thus there is a polynomial $q$ of degree less than $d + 4$ such that

\[
(1 - (2g-2)^4 t^d) \cdot t^{d} p\left(\frac{1}{t}\right) \cdot F(t) = q(t),
\]
and since \( t^d p(1/t) = p_0 \cdot \prod_{k=0}^{g-2}(1 - (2k)^4 t^4)^{\dim(V_{2k})} \) we can solve for \( F \) and expand into partial fractions of the form

\[
F(t) = \frac{a_0 + a_1 t + a_2 t^2 + a_3 t^3}{1 - (2g-2)^4 t^4} + \left( r(t) + \sum_{k=0}^{g-2} \frac{q_k(t)}{(1 - (2k)^4 t^4)^{\dim(V_{2k})}} \right)
\]

where the \( a_i \) are constants and \( r, q_k \) are polynomials. If \( a_{d_0} = 0 \) (interpreting the subscript modulo 4) then for all large \( n \equiv d_0 \pmod 4 \) we must have \( |D_X^n(h^n)| \ll (2g-2)^n \), and so

\[
D_X^w(p(h)h^n) = C(2g-2)^n \text{ cannot hold for nonzero } C. \text{ This is a contradiction, so } a_{d_0} \neq 0 \text{ and } D_X^w(h^n) \sim a_{d_0}(2g-2)^n \text{ for all large } n \equiv d_0 \pmod 4. \]

\[ \square \]

**Remark 8.5.** Kronheimer and Mrowka \cite{19} also showed that for any \( \Sigma \)-bundle \( Y \to S^1 \) with fiber genus \( g \geq 2 \) and class \( w \in H^2(Y; \mathbb{Z}) \) such that \( w \cdot \Sigma \) is odd, the \( (2g-2) \)-eigenspace of \( \mu(\Sigma) \) on \( I_w(Y) \) is 1–dimensional. Thus we may repeat the proof of Theorem 8.4 verbatim to show that its conclusion still holds for any class \( w \in H^2(X; \mathbb{Z}) \) with \( w \cdot \Sigma \) odd.

**Corollary 8.6.** Let \((X,\omega)\) be a symplectic manifold with \( b_1 = 0 \) and \( b^+ > 1 \). If \( h \in H_2(X; \mathbb{Z}) \) is Poincaré dual to the class of an integral symplectic form, then the Donaldson invariants \( D_X^w(h^n) \) of \( X \) are nonzero for large \( n \) congruent to \( d_0 = -w^2 - \frac{3}{2}(b^+(X) + 1) \pmod 4 \), and \( K_X \) is a basic class.

**Proof.** We mostly repeat the proof in the case where \( X \) has simple type: Take a Lefschetz pencil \( X \to S^2 \) with fibers \( \Sigma \) in the class \( kh \) for \( k \) large and having genus \( g \geq 2 \). Let \( \tilde{X} \to S^2 \) be the Lefschetz fibration obtained by blowing up \( X \) at the \((kh)^2\) base points of the pencil and having generic fiber \( \tilde{\Sigma} \). Then for an appropriate choice of \( \tilde{w} \) we have \( D_X^w(\Sigma^n) \sim c_\tilde{w} \cdot (2g-2)^n \) for nonzero \( c_\tilde{w} \) and all \( n \equiv d_0 \pmod 4 \). Hence by \cite{23}, Theorem 6 there is at least one basic class \( \tilde{K} \) on \( \tilde{X} \) for which \( \tilde{K} \cdot \tilde{\Sigma} = 2g-2 \). Since at least one \((-1)\)–section intersects every component of every fiber of \( \tilde{X} \to S^2 \), Proposition \ref{p6.4} says that \( \tilde{K} = K_{\tilde{X}} \), and since \( \tilde{K} \) is unique the claim that \( D_{\tilde{X}}^w(\Sigma^n) \sim c_{\tilde{w}} \cdot (2g-2)^n \) for some nonzero \( c_{\tilde{w}} \) actually holds for all \( \tilde{w} \), regardless of the parity of \( \tilde{w} \cdot \tilde{\Sigma} \) or its restriction to any embedded \( S^1 \times \tilde{\Sigma} \) (see \cite{25}, Theorem 1).

Finally, since \( K_{\tilde{X}} = K_X + \sum E_i \) where the \( E_i \) are the exceptional divisors, we know from the blow-up formula \cite{10} and the description of the basic classes of a blow-up \cite{25}, Remark 3 that \( D_X^w \) is nonzero for any \( w \) and that \( K_X \) must be a basic class of \( X \). \[ \square \]

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