Singular integrals in the rational Dunkl setting

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Abstract
On $\mathbb{R}^N$ equipped with a normalized root system $R$ and a multiplicity function $k \geq 0$ let us consider a (not necessarily radial) kernel $K(x)$ satisfying $|\partial^\beta K(x)| \lesssim \|x\|^{-N-|\beta|}$ for $|\beta| \leq s$, where $N$ is the homogeneous dimension of the system $(\mathbb{R}^N, R, k)$. We additionally assume that

$$\sup_{0 < a < b < \infty} \left| \int_a^{\min(b, \|x\|)} K(x) \, dw(x) \right| < \infty,$$

where $dw$ is the associated measure. We prove that if $s$ large enough then a singular integral Dunkl convolution operator associated with the kernel $K(x)$ is bounded on $L^p(dw)$ for $1 < p < \infty$ and of weak-type $(1,1)$. Furthermore, we study a maximal function related to the Dunkl convolutions with truncation of $K$.

Keywords Dunkl convolutions · Dunkl transforms · Singular integrals · Maximal functions

Mathematics Subject Classification 42B20 · 42B25 · 47B38 · 47G10

1 Introduction

The aim of this note is to study singular integral convolution operators in the Dunkl setting. We fix a normalized root system $R$ in $\mathbb{R}^N$ and a multiplicity function $k \geq 0$. Let $dw(x)$ denote the associated measure and $N$ the homogeneous dimension (see

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Sect. 2). For a positive integer $s$ we consider a kernel $K \in C^s(\mathbb{R}^N \setminus \{0\})$ such that

$$\sup_{0<a<b<\infty} \left| \int_{a<\|x\|<b} K(x) \, dw(x) \right| < \infty, \quad (A)$$

and

$$\left| \frac{\partial^\beta}{\partial x^\beta} K(x) \right| \leq C \|x\|^{-N-|\beta|} \quad \text{for all } |\beta| \leq s. \quad (D)$$

Set

$$K^{[t]}(x) = K(x)(1 - \phi(t^{-1}x)),$$

where $\phi$ is a fixed radial $C^\infty$-function supported by the unit ball $B(0, 1)$ such that $\phi(x) = 1$ for $\|x\| < 1/2$. We prove that if $s$ is sufficiently large, then there are constants $C_\rho > 0$ independent of $t > 0$ such that

$$\|f * K^{[t]}\|_{L^p(dw)} \leq C_\rho \|f\|_{L^p(dw)} \quad \text{for } 1 < p < \infty$$

and

$$w(\{x \in \mathbb{R}^N : |f * K^{[t]}(x)| > \lambda \}) \leq C_1 \lambda^{-1} \|f\|_{L^1(dw)}$$

(Theorems 4.1 and 4.2), where the symbol $*$ denotes the Dunkl convolution. We also prove that under the additional assumption

$$\lim_{\varepsilon \to 0} \int_{\varepsilon<|x|<1} K(x) \, dw(x) = L \in \mathbb{C}, \quad (L)$$

the limit $\lim_{t \to 0} f * K^{[t]}(x)$ exists and defines a bounded operator on $L^p(dw)$ for $1 < p < \infty$, which is of weak-type $(1,1)$ as well (Theorem 4.3, see also Theorem 3.7). Moreover, in this case, the maximal operator

$$\mathcal{K}^* f(x) = \sup_{t>0} |f * K^{[t]}(x)|$$

is bounded on $L^p(dw)$ for $1 < p < \infty$ and of weak-type $(1,1)$ (Theorem 5.1).

If $k \equiv 0$, then $dw$ is the Lebesgue measure in $\mathbb{R}^N$ and the Dunkl convolution reduces to the classical one. So the the above results are well known and $s = 1$ suffices in this case (see i.e. [10, Chapter 5], [20], [21]). However, in the general case of $R$ and $k$ the main difficulty which one faces trying to study singular integral operators in the Dunkl setting lies in the lack of knowledge about boundedness of the so called Dunkl translations $\tau_x$ on $L^p(dw)$-spaces for $p \neq 2$. Consequently, it is not known if for a fixed non-radial $L^1$-function $f$ the Dunkl convolution operator $g \mapsto f * g$ is bounded on $L^p(dw)$. Recent observations made in [12] allow us to obtain some knowledge for the functions $\tau_y f(x)$, provided $f$ satisfies certain regularity conditions in smoothness and decay. In the present paper we explore and extend these ideas of [12] for proving boundedness of singular integral convolution operators provided $s = s_0$ in (D), where $s_0$ is the smallest even integer bigger than $N/2$. 

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2 Preliminaries and notation

The Dunkl theory is a generalization of the Euclidean Fourier analysis. It started with the seminal article [7] and developed extensively afterwards (see e.g. [5], [6], [8], [9], [13], [15], [16], [17], [22], [24]). In this section we present basic facts concerning the theory of the Dunkl operators. For details we refer the reader to [7], [18], and [19].

We consider the Euclidean space $\mathbb{R}^N$ with the scalar product $\langle x, y \rangle = \sum_{j=1}^{N} x_j y_j$, where $x = (x_1, ..., x_N)$, $y = (y_1, ..., y_N)$, and the norm $\|x\|^2 = \langle x, x \rangle$. For a nonzero vector $\alpha \in \mathbb{R}^N$, the reflection $\sigma_\alpha$ with respect to the hyperplane $\alpha \perp$ orthogonal to $\alpha$ is given by

$$\sigma_\alpha(x) = x - 2\frac{\langle x, \alpha \rangle}{\|\alpha\|^2} \alpha.$$

In this paper we fix a normalized root system in $\mathbb{R}^N$, that is, a finite set $R \subset \mathbb{R}^N \setminus \{0\}$ such that $\sigma_\alpha(R) = R$ and $\|\alpha\| = \sqrt{2}$ for all $\alpha \in R$. The finite group $G$ generated by the reflections $\sigma_\alpha \in R$ is called the Weyl group (reflection group) of the root system. A multiplicity function is a $G$-invariant function $k : R \to \mathbb{C}$ which will be fixed and $\geq 0$ throughout this paper. Let

$$dw(x) = \prod_{\alpha \in R} |\langle x, \alpha \rangle|^{k(\alpha)} \, dx$$

be the associated measure in $\mathbb{R}^N$, where, here and subsequently, $dx$ stands for the Lebesgue measure in $\mathbb{R}^N$. We denote by $N = N + \sum_{\alpha \in R} k(\alpha)$ the homogeneous dimension of the system. Clearly,

$$w(B(tx, tr)) = t^N w(B(x, r)) \quad \text{for all } x \in \mathbb{R}^N, \ t, r > 0,$$

where $B(x, r) = \{y \in \mathbb{R}^N : \|y - x\| < r\}$. Moreover,

$$\int_{\mathbb{R}^N} f(x) \, dw(x) = \int_{\mathbb{R}^N} t^{-N} f(x/t) \, dw(x) \quad \text{for } f \in L^1(dw) \text{ and } t > 0.$$

Observe that there is a constant $C > 0$ such that

$$C^{-1} w(B(x, r)) \leq r^N \prod_{\alpha \in R} (|\langle x, \alpha \rangle| + r)^{k(\alpha)} \leq C w(B(x, r)), \quad (2.1)$$

so $dw(x)$ is doubling, that is, there is a constant $C > 0$ such that

$$w(B(x, 2r)) \leq C w(B(x, r)) \quad \text{for all } x \in \mathbb{R}^N, \ r > 0. \quad (2.2)$$
For $\eta \in \mathbb{R}^N$, the Dunkl operators $T_\eta$ are the following $k$-deformations of the directional derivatives $\partial_\eta$ by a difference operator:

$$T_\eta f(x) = \partial_\eta f(x) + \sum_{\alpha \in \mathbb{R}} \frac{k(\alpha)}{2} \langle \alpha, \eta \rangle \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}.$$ 

The Dunkl operators $T_\eta$, which were introduced in [7], commute and are skew-symmetric with respect to the $G$-invariant measure $dw$. Suppose that $\eta \neq 0$, $f, g \in C^1(\mathbb{R}^N)$ and $g$ is radial. The following Leibniz rule can be confirmed by a direct calculation:

$$T_\eta(fg) = f(T_\eta g) + g(T_\eta f).$$

Let $\{e_j\}_{1 \leq j \leq N}$ denote the canonical orthonormal basis in $\mathbb{R}^N$ and let $T_j = T_{e_j}$. For a multi-index $\beta = (\beta_1, \beta_2, \ldots, \beta_N) \in \mathbb{N}_0^N$, we set

$$|\beta| = \beta_1 + \beta_2 + \ldots + \beta_N,$$

$$\partial^\beta = \partial_1^{\beta_1} \circ \partial_2^{\beta_2} \circ \ldots \circ \partial_N^{\beta_N},$$

$$T^\beta = T_1^{\beta_1} \circ T_2^{\beta_2} \circ \ldots \circ T_N^{\beta_N}.$$ 

If $f(x, y)$ is a suitable function of two variables $x, y \in \mathbb{R}^N$, then by $\partial^\beta, x f(x, y)$, $T_j^x f(x, y)$, $T^\beta, x f(x, y)$ we denote the actions of the operators $\partial^\beta$, $T_j$, $T^\beta$ respectively on the function $x \mapsto f(x, y)$ with fixed $y$.

For a fixed $y \in \mathbb{R}^N$ the Dunkl kernel $E(x, y)$ is a unique analytic solution to the system

$$T_\eta f = \langle \eta, y \rangle f, \quad f(0) = 1. \quad (2.3)$$

The function $E(x, y)$, which generalizes the exponential function $e^{(x,y)}$, has the unique extension to a holomorphic function on $\mathbb{C}^N \times \mathbb{C}^N$. Moreover, it satisfies $E(x, y) = E(y, x)$ for all $x, y \in \mathbb{C}^N$. In particular, for $x, y, \xi = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{R}^N$, we have

$$T_j^x E(x, y) = y_j E(x, y), \quad T_j^x E(i\xi, x) = i\xi_j E(i\xi, x). \quad (2.4)$$

In our further consideration we shall need the following lemma.

**Lemma 2.1** For $x, \xi \in \mathbb{R}^N$ and $\beta \in \mathbb{N}_0^N$ we have

$$|\partial^\beta, \xi E(i\xi, x)| \leq \|x\|^{|eta|}.$$ 

In particular,

$$|E(i\xi, x)| \leq 1 \quad \text{for all } \xi, x \in \mathbb{R}^N. \quad (2.5)$$
Corollary 2.2 There is a constant \( C > 0 \) such that for all \( x, \xi \in \mathbb{R}^N \) we have

\[ |E(i\xi, x) - 1| \leq C \|x\| \|\xi\|. \] (2.6)

The Dunkl transform

\[ \mathcal{F} f(\xi) = c_k^{-1} \int_{\mathbb{R}^N} E(-i\xi, x) f(x) \, dw(x), \]

where

\[ c_k = \int_{\mathbb{R}^N} e^{-\frac{\|x\|^2}{2}} \, dw(x) > 0, \]

originally defined for \( f \in L^1(dw) \), is an isometry on \( L^2(dw) \), i.e.,

\[ \|f\|_{L^2(dw)} = \|\mathcal{F} f\|_{L^2(dw)} \text{ for all } f \in L^2(dw), \] (2.7)

and preserves the Schwartz class of functions \( S(\mathbb{R}^N) \) (see [4]). Its inverse \( \mathcal{F}^{-1} \) has the form

\[ \mathcal{F}^{-1} g(x) = c_k^{-1} \int_{\mathbb{R}^N} E(i\xi, x) g(\xi) \, dw(\xi). \]

The Dunkl translation \( \tau_x f \) of a function \( f \in S(\mathbb{R}^N) \) by \( x \in \mathbb{R}^N \) is defined by

\[ \tau_x f(y) = c_k^{-1} \int_{\mathbb{R}^N} E(i\xi, x) E(i\xi, y) \mathcal{F} f(\xi) \, dw(\xi). \]

In particular, for \( f \in S(\mathbb{R}^N) \), we have

\[ \int_{\mathbb{R}^N} \tau_x f(y) \, dw(y) = \mathcal{F}(\tau_x f)(0) = \mathcal{F} f(0) E(0, x) = \int_{\mathbb{R}^N} f(y) \, dw(y). \] (2.8)

The Dunkl translations are contraction on \( L^2(dw) \), however it is an open problem if they are bounded operators on \( L^p(dw) \) for \( p \neq 2 \).

The following specific formula was obtained by Rösler [17] for the Dunkl translations of (reasonable) radial functions \( f(x) = \tilde{f}(\|x\|) \):

\[ \tau_x f(-y) = \int_{\mathbb{R}^N} (\tilde{f} \circ A)(x, y, \eta) \, d\mu_x(\eta) \text{ for all } x, y \in \mathbb{R}^N. \] (2.9)

Here

\[ A(x, y, \eta) = \sqrt{\|x\|^2 + \|y\|^2 - 2\langle y, \eta \rangle} = \sqrt{\|x\|^2 - \|\eta\|^2 + \|y - \eta\|^2} \]
and $\mu_x$ is a probability measure, which is supported in the set $\text{conv} \ O(x)$, where $O(x) = \{ \sigma(x) : \sigma \in G \}$ is the orbit of $x$. Formula (2.9) implies that for all radial $f \in L^1(dw)$ and $x \in \mathbb{R}^N$ we have

$$\| \tau_x f(y) \|_{L^1(dw(y))} \leq \| f(y) \|_{L^1(dw(y))}.$$  

The Dunkl convolution $f \ast g$ of two reasonable functions (for instance Schwartz functions) is defined by

$$(f \ast g)(x) = c_k \mathcal{F}^{-1}[(\mathcal{F} f)(\mathcal{F} g)](x)$$

$$= \int_{\mathbb{R}^N} (\mathcal{F} f)(\xi) (\mathcal{F} g)(\xi) E(x, i\xi) dw(\xi) \text{ for } x \in \mathbb{R}^N,$$

or, equivalently, by

$$(f \ast g)(x) = \int_{\mathbb{R}^N} f(y) \tau_x g(-y) dw(y) = \int_{\mathbb{R}^N} f(y) g(x, y) dw(y) \text{ for all } x \in \mathbb{R}^N,$$

where, here and subsequently, $g(x, y) = \tau_x g(-y)$. Clearly, $f \ast g = g \ast f$.

By an interpolation argument, if $1 \leq p \leq 2$ and $q = 2p/(2-p)$, then

$$\| f \ast g \|_{L^q(dw)} \leq \| f \|_{L^2(dw)} \| g \|_{L^p(dw)}.$$  

The Dunkl Laplacian associated with $R$ and $k$ is the differential-difference operator

$$\Delta = \sum_{j=1}^N T_j^2,$$

which acts on $C^2(\mathbb{R}^N)$-functions by

$$\Delta f(x) = \Delta_{\text{eucl}} f(x) + \sum_{\alpha \in R} k(\alpha) \delta_\alpha f(x),$$

$$\delta_\alpha f(x) = \frac{\partial^2 f(x)}{\langle \alpha, x \rangle^2} - \frac{\| \alpha \|^2}{2} f(x) - f(\sigma_\alpha(x)).$$

Obviously, $\mathcal{F}(\Delta f)(\xi) = -\| \xi \|^2 \mathcal{F} f(\xi)$. The operator $\Delta$ is essentially self-adjoint on $L^2(dw)$ (see for instance [2, Theorem 3.1]) and generates the semigroup $e^{t\Delta}$ of linear self-adjoint contractions on $L^2(dw)$. The semigroup has the form

$$e^{t\Delta} f(x) = \mathcal{F}^{-1}(e^{-t\|\xi\|^2} \mathcal{F} f(\xi))(x) = \int_{\mathbb{R}^N} h_t(x, y) f(y) dw(y),$$

where the heat kernel

$$h_t(x, y) = \tau_x h_t(-y), \quad h_t(x) = \mathcal{F}^{-1}(e^{-t\|\xi\|^2})(x) = c_k^{-1}(2t)^{-N/2} e^{-\|x\|^2/(4t)} \text{ (2.10)}$$

is a $C^\infty$-function of all variables $x, y \in \mathbb{R}^N$, $t > 0$, and satisfies

$$0 < h_t(x, y) = h_t(y, x),$$

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\[
\int_{\mathbb{R}^N} h_t(x, y) \, dw(y) = 1.
\]

Set
\[
V(x, y, t) = \max(w(B(x, t)), w(B(y, t))).
\]

Let
\[
d(x, y) = \min_{\sigma \in G} \| \sigma(x) - y \|
\]
be the distance of the orbit of \( x \) to the orbit of \( y \). We shall need the following estimates for \( h_t(x, y) \). The estimate (2.11) was announced by W. Hebisch (with an outline of a proof which used a Poincaré inequality). An elementary and complete two-step proof of (2.11) and (2.12), which is based on (2.9), can be found in [1, Theorem 4.1] and [11, Theorem 3.1].

**Theorem 2.3** There are constants \( C, c > 0 \) such that for all \( x, y \in \mathbb{R}^N \) and \( t > 0 \) we have
\[
h_t(x, y) \leq C \left( 1 + \frac{\|x - y\|}{\sqrt{t}} \right)^{-2} V(x, y, \sqrt{t})^{-1} e^{-cd(x,y)^2/t}. \tag{2.11}
\]

Moreover, if \( \|y - y'\| \leq \sqrt{t} \), then
\[
|h_t(x, y) - h_t(x, y')| \leq C \frac{\|y - y'\|}{\sqrt{t}} \left( 1 + \frac{\|x - y\|}{\sqrt{t}} \right)^{-2} V(x, y, \sqrt{t})^{-1} e^{-cd(x,y)^2/t}. \tag{2.12}
\]

Theorem 2.3 and (2.9) imply the following lemma (see [11, Corollary 3.5]).

**Lemma 2.4** Suppose that \( \varphi \in C_c^\infty(\mathbb{R}^N) \) is radial and supported by the unit ball. Then there is \( C > 0 \) such that for all \( x, y \in \mathbb{R}^N \) and \( t > 0 \) we have
\[
|\varphi_t(x, y)| \leq C \left( 1 + \frac{\|x - y\|}{t} \right)^{-2} V(x, y, t)^{-1} \chi_{[0, 1]}(d(x, y)/t). \tag{2.13}
\]

Let us emphasise that the presence of the Euclidean distance \( \|x - y\| \) in the estimate (2.13) will play an important role in the proof of a Cotlar type inequality in Sect. 5.

### 3 \( L^2(dw) \) estimates

In the present section we assume that \( K(x) \) satisfies (A) and (D) with \( s = 1 \), that is,
\[
|\partial^\beta K(x)| \leq C_\beta \|x\|^{-N-|\beta|} \text{ for } |\beta| \leq 1. \tag{D(1)}
\]
Our aim is to prove that the convolution operators with the truncated kernels $K^{[r]}$ are uniformly bounded on $L^2(dw)$. Then we add the assumption (L) and show the $L^2$-bound of the limiting operator.

We start by the easy observation that (D(1)) implies

$$|T^β K(x)| ≤ C_β \|x\|^{-N-|β|} \text{ for } |β| ≤ 1.$$  \hspace{1cm} (D'1)

Recall that $φ$ denotes a fixed $C^∞(ℝ^N)$ radial function supported in the unit ball such that $φ(x) = 1$ for all $x \in ℝ^N$ such that $\|x\| ≤ 1/2$.

For $0 < a ≤ b < ∞$, let

$$K_{a,∞}(x) = K(x)χ_{\{y:a≤\|y\|\}}(x) \quad \text{and} \quad K_{a,b}(x) = K(x)χ_{\{y:a≤\|y\|≤b\}}(x),$$

$$K^{[a]}(x) = K(x)(1 - φ(a^{-1}x)) \quad \text{and} \quad K^{[a,b]}(x) = K^{[a]}(x) - K^{[b]}(x). \hspace{1cm} (3.1)$$

Let us list the following easily proved properties of the truncated kernels which follow from (D'1) and (A):

$$K_{a,b}, K^{[a,b]} \in L^1(dw) \cap L^2(dw), \quad K_{a,∞}, K^{[a]} \in L^2(dw),$$

$$\text{supp } K^{[a,b]} ⊆ \{x ∈ ℝ^N : a/2 ≤ \|x\| ≤ b\},$$

$$\sup_{0 < a < b < ∞} \|K_{a,b} - K^{[a,b]}\|_{L^1(dw)} = C_0 < ∞,$$

$$\lim_{b → ∞} \|K_{a,b} - K_{a,∞}\|_{L^2(dw)} = 0, \quad \lim_{b → ∞} \|K^{[a,b]} - K^{[a]}\|_{L^2(dw)} = 0. \hspace{1cm} (3.2)$$

Consequently, by the Plancherel’s identity (see (2.7)),

$$\lim_{b → ∞} \|\mathcal{F}K_{a,b} - \mathcal{F}K_{a,∞}\|_{L^2(dw)} = 0, \quad \lim_{b → ∞} \|\mathcal{F}K^{[a,b]} - \mathcal{F}K^{[a]}\|_{L^2(dw)} = 0. \hspace{1cm} (3.3)$$

Moreover, there are constants $A', C > 0$ such that for all $0 < a < b < ∞$ one has

$$|K^{[a,b]}(x)| ≤ C \|x\|^{-N}, \quad |K^{[a]}(x)| ≤ C \|x\|^{-N},$$

$$|T_j K^{[a,b]}(x)| ≤ C \|x\|^{-N-1}, \quad |T_j K^{[a]}(x)| ≤ C \|x\|^{-N-1},$$

$$\left| \int_{ℝ^N} K^{[a,b]}(x) dw(x) \right| ≤ A'.$$  \hspace{1cm} (3.4)

**Lemma 3.1** (D’1) and (A) imply that there is a constant $C > 0$ such that for all $0 < a < b < ∞$ and all $ξ ∈ ℝ^N$ one has $|\mathcal{F}K_{a,b}(ξ)| ≤ C$ and $|\mathcal{F}K^{[a,b]}(ξ)| ≤ C$.

**Proof** Thanks to (3.2) it suffices to prove the second inequality. Assume first that $ξ ∈ ℝ^N$ is such that $a ≤ \|ξ\|^{-1} ≤ b$. Put $t = \|ξ\|^{-1}$. We have $K^{[a,b]} = K^{[a,t]} + K^{[t,b]}$ and, consequently,
In order to estimate $I_1$, we write
\[
I_1 = c_k^{-1} \int K^{[a,t]}(x)(E(-i\xi, x) - 1) \, dw(x) + c_k^{-1} \int K^{[a,t]}(x) \, dw(x) =: I_{1,1} + I_{1,2}.
\]
Clearly, by (3.4) we get $|I_{1,2}| \leq C$. For $I_{1,1}$, by Corollary 2.2 and (D'(1)), we obtain
\[
|I_{1,1}| \leq C \|\xi\| \int_{\|x\| \leq t} \|x\|^{-N+1} \, dw(x) \leq C.
\]

We now turn to estimate $I_2$. Choose $j \in \{1, 2, \ldots, N\}$ such that $|\xi_j| \geq N^{-1/2} \|\xi\|$. Then, thanks to (2.4), (2.5), and (D'(1)), we have
\[
|I_2| \leq C \sqrt{N} \|\xi\|^{-1} \int_{\mathbb{R}^N} K^{[t,b]}(x) T_j E(-i\xi, x) \, dw(x) \leq C.
\]

The cases $\|\xi\|^{-1} < a$ or $\|\xi\|^{-1} > b$ can be treated similarly (we have to deal with just one integral in (3.5)).

From (3.3) and Lemma 3.1 we easily deduce the following corollary.

**Corollary 3.2** (D'(1)) and (A) imply that there is a constant $C > 0$ such that for every $0 < a < b < \infty$ we have
\[
\|\mathcal{F}K^{[a]}\|_{L^\infty} \leq C, \quad \text{and} \quad \|\mathcal{F}K_{a,\infty}\|_{L^\infty} \leq C.
\]

Set
\[
K^{[a,b]} f = f * K^{[a,b]}, \quad K_{a,b} f = f * K^{[a,b]}, \quad K^{[a]} f = f * K^{[a]}, \quad K_{a,\infty} f = f * K_{a,\infty}.
\]

Let us recall that for $f \in L^2(dw)$ the convolution $*$ of $f$ with any of the kernels is commutative.

**Corollary 3.3** There is a constant $C > 0$ such that for all $0 < a < b < \infty$ we have
\[
\|K^{[a,b]} f\|_{L^2(dw)} + \|K_{a,b} f\|_{L^2(dw)} + \|K^{[a]} f\|_{L^2(dw)} + \|K_{a,\infty} f\|_{L^2(dw)} \leq C \|f\|_{L^2(dw)}.
\]
Moreover, for every $a > 0$ and $f \in L^2(dw)$ we have
\[
\lim_{b \to \infty} \|K^{[a,b]} f - K^{[a]} f\|_{L^2(dw)} = 0. \tag{3.6}
\]

**Proof** The corollary follows directly from (3.3), Lemma 3.1, Corollary 3.2, the Plancherel’s identity (2.7), and the definition of the Dunkl convolution. \qed

From now on to the end of the paper we assume additionally that (L) is satisfied by $K$ as well.

**Lemma 3.4** Property (L) implies
\[
\lim_{a \to 0} \int_{\|x\| < 1} K^{[a]}(x) \, dw(x) = L.
\]

**Proof** Let us define $\tilde{\phi}(|x|) = \phi(x)$. For $0 < r \leq 1$ we set
\[
F(r) = \int_{r<\|x\|<1} K(x) \, dw(x) = \int_r^1 \int_{\mathbb{S}^{N-1}} r_1^{N-1} K(r_1 \omega) \prod_{\alpha \in R} |\langle r_1 \omega, \alpha \rangle|^{k(\alpha)} d\sigma(\omega) \, dr_1,
\]
where $\sigma$ is the spherical measure. Note that $F$ is differentiable and satisfies
\[
F'(r) = -\int_{\mathbb{S}^{N-1}} r^{N-1} K(r \omega) \prod_{\alpha \in R} |\langle r \omega, \alpha \rangle|^{k(\alpha)} d\sigma(\omega).
\]

Consequently, by the definition of $K^{[a]}$ and integration by parts, we get
\[
\int_{\|x\| < 1} K^{[a]}(x) \, dw(x) = \int_0^1 (1 - \tilde{\phi}(a^{-1} r)) \int_{\mathbb{S}^{N-1}} r^{N-1} K(r \omega) \prod_{\alpha \in R} |\langle r \omega, \alpha \rangle|^{k(\alpha)} d\sigma(\omega) \, dr
= \int_0^1 (1 - \tilde{\phi}(a^{-1} r))( - F'(r)) \, dr = -\int_0^1 (\tilde{\phi}(a^{-1} r))' F(r) \, dr.
\]

We write
\[
-\int_0^1 (\tilde{\phi}(a^{-1} r))' F(r) \, dr = -\int_0^1 (\tilde{\phi}(a^{-1} r))' (F(r) - L) \, dr - L \int_0^1 (\tilde{\phi}(a^{-1} r))' \, dr =: S_1(a) + S_2(a).
\]
\[
\tag{3.8}
\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Graphical representation of Lemma 3.4}
\end{figure}
Clearly, for all $0 < a < 1/4$, we have

$$S_2(a) = L. \quad (3.9)$$

Note that $\text{supp} \left( \tilde{\phi}(a^{-1} \cdot) \right)' \subseteq [a/2, a]$, hence

$$|S_1(a)| \leq \int_0^1 |(\tilde{\phi}(a^{-1} r))'||F(r) - L| \, dr = \int_{a/2}^a |(\tilde{\phi}(a^{-1} r))'||F(r) - L| \, dr \leq \max_{r \in [a/2, a]} |F(r) - L| \int_{a/2}^a |a^{-1}\tilde{\phi}'(a^{-1} r)| \, dr \leq C \|\tilde{\phi}'\|_{L^\infty} \max_{r \in [0, a]} |F(r) - L|.$$

Consequently, thanks to (L) and (3.7) we obtain $\lim_{a \to 0} S_1(a) = 0$. Combining this fact with (3.9) and (3.8) we get the claim. □

**Lemma 3.5** Under (D’(1)), (A), and (L) for almost every $\xi \in \mathbb{R}^N$, the limit

$$\lim_{a \to 0} \mathcal{F}K_{a, \infty}(\xi)$$

exists and defines a bounded function denoted by $\mathcal{F}K(\xi)$.

**Proof** According to Corollary 3.2, it suffices to show that $\mathcal{F}K_{a, \infty}(\xi)$ is a Cauchy sequence as $a \to 0$. To this end we write

$$|\mathcal{F}K_{a_1, \infty}(\xi) - \mathcal{F}K_{a_2, \infty}(\xi)| \leq c_k^{-1} \left| \int_{a_1 \leq \|x\| \leq a_2} K(x) \left( E(-i\xi, x) - 1 \right) \, dw(x) \right| + c_k^{-1} \left| \int_{a_1 \leq \|x\| \leq a_2} K(x) \, dw(x) \right| =: I_1 + I_2.$$

Thanks to Corollary 2.2,

$$I_1 \leq C \|\xi\| \int_{a_1 \leq \|x\| \leq a_2} \|x\|^{-N+1} \, dw(x) \to 0$$
as $a_1$ and $a_2$ tend to $0^+$. The convergence of $I_2$ to 0 is a consequence of (L). □

From of Corollary 3.2 and Lemma 3.5 we obtain the following theorem.

**Theorem 3.6** Under (D’(1)), (A), and (L), for every $f \in L^2(dw)$ the limit

$$\lim_{a \to 0} K_{a, \infty} f$$

exists in $L^2(dw)$ and defines an operator, which is bounded on $L^2(dw)$ and which will be denoted by $K f$. Moreover,

$$K f = \mathcal{F}^{-1}(\mathcal{F}K(\cdot)\mathcal{F}f(\cdot)).$$
Theorem 3.7 Under \((D'(1)), (A), \) and \((L)\) for almost every \(\xi \in \mathbb{R}^N\),

\[
\lim_{a \to 0} \mathcal{F} K^{[a]}(\xi) = \mathcal{F} K(\xi),
\]

where \(\mathcal{F} K(\xi)\) is defined in Lemma 3.5. Moreover, for every \(f \in L^2(dw)\) the limit

\[
\lim_{a \to 0} K^{[a]} f
\]

exists in \(L^2(dw)\) and it is equal to \(Kf\) (see Theorem 3.6).

Proof Let \(J(a, \xi) = \mathcal{F}(K^{[a]})(\xi) - \mathcal{F}(K_{a, \infty})(\xi)\). Thanks to Corollary 3.2, Lemma 3.5, and Theorem 3.6 it suffices to show that \(\lim_{a \to 0^+} J(a, \xi) = 0\) for all \(\xi \in \mathbb{R}^N\). Similarly to the proof of Lemma 3.5, we write

\[
J(a, \xi) = c_k^{-1} \int_{\mathbb{R}^N} \left( K^{[a]}(x) - K_{a, \infty}(x) \right) \left( E(-i\xi, x) - 1 \right) dw(x)
\]

\[
+ c_k^{-1} \int_{\mathbb{R}^N} \left( K^{[a]}(x) - K_{a, \infty}(x) \right) dw(x) =: J_1(a, \xi) + J_2(a, \xi).
\]

By Lemma 3.4 we have \(\lim_{a \to 0^+} J_2(a, \xi) = 0\). To deal with \(J_1(a, \xi)\) we note that

\[
\text{supp} \left( K^{[a]} - K_{a, \infty} \right) \subseteq \{ x \in \mathbb{R}^N : a/2 \leq \|x\| \leq a \}
\]

and \(\|K^{[a]}(x) - K_{a, \infty}(x)\| \leq \frac{C}{\|x\|^N}\).

Hence, using Corollary 2.2, we get

\[
|J_1| \leq C \int_{a/2 \leq \|x\| \leq a} \|\xi\| \|x\|^{-N+1} dw(x) \to 0 \quad \text{as} \quad a \to 0.
\]

\[\square\]

4 \(L^p(dw)\) estimates

The purpose of this section is to study singular integrals operators on \(L^p(dw)\). For this purpose we need to make the following stronger assumption on the kernel \(K\), namely that \((D)\) holds for \(|\beta| \leq s_0\), where \(s_0\) is the smallest even positive integer bigger than \(N/2\), that is,

\[
|\partial^\beta K(x)| \leq C_\beta \|x\|^{-N-|\beta|} \text{ for all } |\beta| \leq s_0. \quad (D(s_0))
\]

Clearly, \((D(s_0))\) implies

\[
|T^\beta K(x)| \leq C_\beta \|x\|^{-N-|\beta|} \text{ for all } |\beta| \leq s_0. \quad (4.1)
\]
The goal of this section is to prove the following theorems.

**Theorem 4.1** Under \((D(s_0))\) and \((A)\), there is a constant \(C > 0\) such that for all \(0 < a < b < \infty, \lambda > 0, \text{ and } f \in L^1(dw) \cap L^2(dw)\) we have

\[
\begin{align*}
    w\bigl(\{x \in \mathbb{R}^N : |K^{[a,b]} f(x)| > \lambda\}\bigr) & \leq C \lambda^{-1} \|f\|_{L^1(dw)}, \quad (4.2) \\
    w\bigl(\{x \in \mathbb{R}^N : |K^{[a]} f(x)| > \lambda\}\bigr) & \leq C \lambda^{-1} \|f\|_{L^1(dw)}. \quad (4.3)
\end{align*}
\]

**Theorem 4.2** Let \(1 < p < \infty\). Under \((D(s_0))\) and \((A)\), there is a constant \(C_p = C_p > 0\) such that for all \(0 < a < b < \infty, \text{ and } f \in L^p(dw) \cap L^2(dw)\) we have

\[
\begin{align*}
    \|K^{[a,b]} f\|_{L^p(dw)} & \leq C \|f\|_{L^p(dw)}, \quad (4.4) \\
    \|K^{[a]} f\|_{L^p(dw)} & \leq C \|f\|_{L^p(dw)}. \quad (4.5)
\end{align*}
\]

Moreover, the operators \(K^{[a,b]}\) converge strongly to \(K^{[a]}\) in \(L^p(dw)\) as \(b \to \infty\).

As the consequence of Theorems 4.1 and 4.2 we obtain the following theorem for the operator \(K\) defined in Theorem 3.6.

**Theorem 4.3** Let \(1 \leq p < \infty\). Under \((D(s_0)), (A), \text{ and } (L)\) there is a constant \(C = C_p > 0\) such that for all \(\lambda > 0, \text{ and } f \in L^p(dw) \cap L^2(dw)\) we have

\[
\begin{align*}
    w\bigl(\{x \in \mathbb{R}^N : |K^{[a]} f(x)| > \lambda\}\bigr) & \leq C \lambda^{-1} \|f\|_{L^1(dw)} \text{ if } p = 1, \quad (4.6) \\
    \|K^{[a]} f\|_{L^p(dw)} & \leq C \|f\|_{L^p(dw)} \text{ if } 1 < p < \infty. \quad (4.7)
\end{align*}
\]

Moreover, if \(1 < p < \infty\), the operators \(K^{[a]}\) converge strongly to \(K\) in \(L^p(dw)\) as \(a \to 0\).

### 4.1 Bessel potential

Let \(s > 0\). The Bessel potential \((I - \Delta)^{-s/2}\) can be defined by means of the heat semigroup \(e^{t \Delta} f = f * h_t\) by

\[
(I - \Delta)^{-s/2} f = \Gamma\left(\frac{s}{2}\right)^{-1} \int_0^\infty e^{-t \frac{s}{2}} f * h_t \frac{dt}{t} = f * J^{[s]}, \quad (4.8)
\]

where

\[
J^{[s]}(x) = \Gamma\left(\frac{s}{2}\right)^{-1} \int_0^\infty e^{-t \frac{s}{2}} h_t(x) \frac{dt}{t} \quad (4.9)
\]

(see [23, Sect. 4]). Recall that \(h_t(x)\) denotes the heat kernel (see (2.10)). The function \(J^{[s]}\) is radial, positive and belongs \(L^1(dw)\). Moreover,

\[
\mathcal{F} J^{[s]}(\xi) = (1 + \|\xi\|^2)^{-s/2}.
\]
As a consequence of (4.9) and (2.10) (see e.g [14, Chapter 1]), we get the following proposition.

**Proposition 4.4** Let $M > 0$. There is a constant $C = C_{s,M} > 0$ such that

$$0 < J^{[s]}(x) \leq C \begin{cases} \|x\|^{s-N} & \text{if } \|x\| \leq 1/2, \ 0 < s < N, \\ -\ln \|x\| & \text{if } \|x\| \leq 1/2, \ s = N, \\ 1 & \text{if } \|x\| \leq 1/2, \ s > N, \\ (1 + \|x\|^2)^{-M} & \text{if } \|x\| > 1/2, \ 0 < s. \end{cases}$$

Let $J^{[s]}(x, y) = \tau_x(J^{[s]})(-y)$. Clearly, by (2.9), $0 < J^{[s]}(x, y)$ for all $x, y \in \mathbb{R}^N$ and

$$\int_{\mathbb{R}^N} J^{[s]}(x, y) \, dw(x) = \int_{\mathbb{R}^N} J^{[s]}(x, y) \, dw(y) = \int_{\mathbb{R}^N} J^{[s]}(x) \, dw(x).$$

**Lemma 4.5** Let $0 < \delta < s$ and $0 < \delta \leq 1$. Then there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^N} |J^{[s]}(x, y) - J^{[s]}(x, y')| \, dw(x) \leq C \min(1, \|y - y'\|^{\delta}) \text{ for all } y, y' \in \mathbb{R}^N.$$

**Proof** By (4.9) we have

$$J^{[s]}(x, y) - J^{[s]}(x, y') = \Gamma\left(\frac{s}{2}\right)^{-1} \int_0^\infty e^{-t} (h_t(x, y) - h_t(x, y')) t^{s/2} \frac{dt}{t}. \quad (4.10)$$

Moreover, by (2.12) we get

$$\int_{\mathbb{R}^N} |h_t(x, y) - h_t(x, y')| \, dw(x) \leq C \min\left(1, \|y - y'\|/\sqrt{t}\right)^\delta. \quad (4.11)$$

The lemma is a direct consequence of (4.10) and (4.11). \qed

### 4.2 Auxiliary estimates on $K^{[a,b]}$

The following theorem and proposition were proved in [12, Theorem 1.7] and [12, Proposition 4.4] respectively.

**Theorem 4.6** Let $f \in L^2(dw)$, $\text{supp } f \subseteq B(0, r)$, and $x \in \mathbb{R}^N$. Then

$$\text{supp } \tau_x f(-\cdot) \subseteq \mathcal{O}(B(x, r)), \quad (4.12)$$

where $\mathcal{O}(B(x, r)) = \bigcup_{\sigma \in \mathcal{G}} B(\sigma(x), r)$ is the orbit of the Euclidean closed ball $B(x, r) = \{y \in \mathbb{R}^N : \|x - y\| \leq r\}$. \hfill \textcircled{3}
Proposition 4.7 There is a constant $C > 0$ such that for any $r_1, r_2 > 0$, any $f \in L^1(dw)$ such that $\text{supp } f \subseteq B(0, r_2)$, any continuous radial function $\varphi$ such that $\text{supp } \varphi \subseteq B(0, r_1)$, and for all $y \in \mathbb{R}^N$ we have

$$\| \tau_y (f * \varphi) \|_{L^1(dw)} \leq C (r_1 + r_2)^N \| \varphi \|_{L^\infty} \| f \|_{L^1(dw)}.$$

Proposition 4.8 Let $0 \leq \delta \leq 1$ be such that $N/2 + \delta < s_0$. Suppose that a function $K$ satisfies (D(s_0)). There is a constant $C > 0$ such that for all $j \in \mathbb{Z}$ we have

$$\int_{\mathbb{R}^N} \sup_{2j-1 \leq a \leq b < 2j+1} |K_{[a,b]}(x, y)| d(x, y)^\delta dw(y) \leq C 2^j \delta$$

(4.13)

\[ \text{for all } x \in \mathbb{R}^N, \]

$$\int_{\mathbb{R}^N} \sup_{2j-1 \leq a \leq b < 2j+1} |K_{[a,b]}(x, y) - K_{[a,b]}(x, y')| d(x, y) \leq C 2^{-j} \delta \|y - y'\| \delta$$

(4.14)

\[ \text{for all } y, y' \in \mathbb{R}^N, \]

$$\int_{\mathbb{R}^N} \sup_{2j-1 \leq a \leq b < 2j+1} |K_{[a,b]}(y, x) - K_{[a,b]}(y', x)| d(x) \leq C 2^{-j} \delta \|y - y'\| \delta$$

(4.15)

\[ \text{for all } y, y' \in \mathbb{R}^N, \]

$$\int_{\mathbb{R}^N} \sup_{2j-1 \leq a \leq b < 2j+1} |K_{[a]}(y, x) - K_{[a]}(y', x)| d(x) \leq C 2^{-j} \delta \|y - y'\| \delta$$

(4.16)

\[ \text{for all } y, y' \in \mathbb{R}^N. \]

Proof We first prove (4.13)–(4.15) for $j = 0$. Recall that $\phi$ is the fixed $C^\infty$ radial function supported in the unit ball such that $\phi(x) = 1$ for $\|x\| \leq 1/2$ which defines the truncated kernels $K_{[a]}$ and $K_{[a,b]}$ (see (3.1)). Let $\bar{\phi} : [0, \infty) \to \mathbb{R}$ be defined by the relation $\bar{\phi}(\|y\|) = \phi(y)$. Then $\bar{\phi} \in C^\infty_c[0, \infty)$, $\bar{\phi}(x) = 0$ for $x \geq 1$, and $\bar{\phi}(x) = 1$ for $0 \leq x \leq 1/2$. Set $\Phi(t, y) = -K(y)t^{-2}\|y\|\bar{\phi}'(t^{-1}\|y\|)$. Assume that $2^{-1} \leq a \leq b \leq 2$, then

$$K_{[a,b]}(y) = K(y)\bar{\phi}(b^{-1}\|y\| - \bar{\phi}(a^{-1}\|y\|))$$

$$= \int_a^b K(y) \frac{d}{dt}(\bar{\phi}(t^{-1}\|y\|)) \ dt = \int_a^b \Phi(t, y) \ dt,$$
where the integral converges in $L^2(dw)$, because $t \mapsto \Phi(t, \cdot)$ is a continuous function from $(0, \infty)$ to $L^2(dw)$. Let us denote $\Phi(t, x, y) = \tau_x(\Phi(t, \cdot))(y)$. Since the Dunkl translation $\tau_x$ is continuous on $L^2(dw)$, for fixed $x \in \mathbb{R}^N$ we have

$$K^{[a,b]}(x, y) = \int_a^b \Phi(t, x, y) \, dt,$$

where the integral converges in $L^2(dw(y))$.

Note that $\text{supp} \Phi(t, \cdot) \subseteq \{y : t/2 \leq \|y\| \leq t\} \subseteq B(0, 2)$, because $1/2 \leq a \leq t \leq b \leq 2$. Let $N/2 < s_1 \leq s_0$. Set

$$F(t, y) = \widetilde{\Phi}(t, \cdot) * (J^{[s_1]})(y), \quad \text{ where } \widetilde{\Phi}(t, y) = (I - \Delta)^{s_0/2} \Phi(t, y).$$

By the assumption $(D(s_0))$ (see also (4.1)), supp $\widetilde{\Phi}(t, \cdot) \subseteq B(0, 2)$ and $|\widetilde{\Phi}(t, x)| \leq C'$ for $1/2 \leq t \leq 2$, where the constant $C'$ depends only on the constants $C_\beta$ in $(D(s_0))$ and the (fixed) function $\phi$. Consequently, $\widetilde{\Phi}(t, \cdot) \in L^p(dw)$ for $1 \leq p \leq \infty$. In particular there is $C'' > 0$ independent of $j$ such that

$$\sup_{1/2 \leq t \leq 2} \|\widetilde{\Phi}(t, \cdot)\|_{L^1(dw)} \leq C''.$$

(4.17)

Let $\psi(y) = \phi(y) - \phi(2y)$. We write

$$J^{[s_1]}(y) = \sum_{\ell \in \mathbb{Z}} \psi(2^{-\ell} y)(J^{[s_1]})(y) =: \sum_{\ell \in \mathbb{Z}} \psi^{[\ell]}(y),$$

where the convergence is pointwise and in $L^1(dw)$. Recall that $\psi^{[\ell]}$ are radial functions. Hence,

$$F(t, \cdot) = \sum_{\ell \in \mathbb{Z}} \widetilde{\Phi}(t, \cdot) * \psi^{[\ell]}$$

with convergence in $L^2(dw)$. Therefore for all $x \in \mathbb{R}^N$,

$$F(t, x, y) := \tau_x F(t, -y) = \sum_{\ell \in \mathbb{Z}} \tau_x(\widetilde{\Phi}(t, \cdot) * \psi^{[\ell]})(-y),$$

where the series converges in $L^2(dw)$. We shall show that the convergence is in $L^1(dw)$ as well and the $L^1$-norm of the sum is uniformly bounded for $t \in [1/2, 2]$ and $x \in \mathbb{R}^N$. To this end we apply (4.17) together with Proposition 4.7 and Proposition 4.4 (with $M > N$) and obtain

$$\sum_{\ell \in \mathbb{Z}} \|\tau_x(\widetilde{\Phi}(t, \cdot) * \psi^{[\ell]})(\cdot)\|_{L^1(dw)} \leq C \sum_{\ell \in \mathbb{Z}} 2^{\ell N/2}(2 + 2^{\ell})^{N/2} \|\widetilde{\Phi}(t, \cdot)\|_{L^1(dw)} \|\psi^{[\ell]}\|_{L^\infty}$$

$$\leq C' \sum_{\ell \leq 0} 2^{\ell N/2} 2^{\ell(s_1-N)} + C' \sum_{\ell > 0} 2^{\ell N - \ell M} < \infty.$$
Thus,
\[
\sup_{1/2 \leq t \leq 2} \int_{\mathbb{R}^N} |F(t, x, y)| \, dw(y) \leq C'' < \infty,
\]
where \(C''\) depends on the constants in \((D(s_0))\). By the same arguments,
\[
\sup_{1/2 \leq t \leq 2} \int_{\mathbb{R}^N} |F(t, x, y)| \, dw(x) \leq C'' < \infty.
\]
Note that if \(s_1 = s_0\) then \(F(t, \cdot) = \Phi(t, \cdot)\) and
\[
\sup_{1/2 \leq a \leq b \leq 2} |K^{[a,b]}(x, y)| = \sup_{1/2 \leq a \leq b \leq 2} \left| \int_{a}^{b} \Phi(t, x, y) \, dt \right| \leq \int_{1/2}^{2} |\Phi(t, x, y)| \, dt.
\]
Consequently, applying (4.18) with \(F(t, \cdot) = \Phi(t, \cdot)\), we obtain
\[
\int \sup_{1/2 \leq a \leq b \leq 2} |K^{[a,b]}(x, y)| \, dw(y) \leq \int_{\mathbb{R}^N} \int_{1/2}^{2} |\Phi(t, x, y)| \, dt \, dw(y) \leq C'''.
\]
Now (4.13) for \(j = 0\) follows from (4.20), since, thanks to Theorem 4.6, \(K^{[a,b]}(x, y) = 0\) if \(d(x, y) > 2\).

In order to prove (4.14) we take \(s_1, s > 0\) such that \(N/2 < s_1\) and \(s_0 = s_1 + s\). Then
\[
\Phi(t, \cdot) = \tilde{\Phi}(t, \cdot) * J^{[s_1]} * J^{[s]} = F(t, \cdot) * J^{[s]}.
\]
Thus,
\[
\begin{align*}
\sup_{1/2 \leq a \leq b \leq 2} |K^{[a,b]}(x, y) - K^{[a,b]}(x, y')| \\
= \sup_{1/2 \leq a \leq b \leq 2} \left| \int_{a}^{b} \left( \Phi(t, x, y) - \Phi(t, x, y') \right) \, dt \right| \\
\leq \sup_{1/2 \leq a \leq b \leq 2} \int_{a}^{b} \left| F(t, x, z) \right| J^{[s]}(z, y) - J^{[s]}(z, y') \, dw(z) \, dt \\
\leq \int_{1/2}^{2} \int \left| F(t, x, z) \right| J^{[s]}(z, y) - J^{[s]}(z, y') \, dw(z) \, dt.
\end{align*}
\]
Integrating (4.21) with respect to \(dw(x)\) and using (4.19) together with Lemma 4.5, we obtain (4.14) for \(j = 0\). The proof of (4.15) for \(j = 0\) is identical.

In order to prove (4.13), (4.14), and (4.15) for arbitrary \(j \in \mathbb{Z}\) we use scaling. To this end we fix \(j \in \mathbb{Z}\) and write \(G_j(x) = 2^{jN} K(2^j x)\). Then \(G_j\) satisfies \((D(s_0))\) with
the same constants $C_\beta$. Moreover, one can easily check that

$$K^{[a,b]}(x) = 2^{-jN}G_j^{[2^{-j}a,2^{-j}b]}(2^{-j}x),$$

and, consequently,

$$K^{[a,b]}(x, y) = 2^{-jN}G_j^{[2^{-j}a,2^{-j}b]}(2^{-j}x, 2^{-j}y).$$

Now if $2^{j-1} \leq a \leq b \leq 2^{j+1}$, then $1/2 \leq 2^{-j}a \leq 2^{-j}b \leq 2$ and we apply the already proved results to $G_j$ and obtain the desired inequalities.

We now turn to prove (4.16). If $2^{j-1} \leq a \leq 2^{j+1}$, then we write

$$K^{[a]} = K^{[a,2^{j+1}]} + \sum_{\ell=j+1}^{\infty} K^{[2^{j},2^{j+1}]}(y, x),$$

where the convergence is in $L^2(dw)$. Since the translations $\tau_y$ are contractions on $L^2(dw)$, for fixed $y, y' \in \mathbb{R}^N$ we have

$$K^{[a]}(y, x) - K^{[a]}(y', x) = K^{[a,2^{j+1}]}(y, x) - K^{[a,2^{j+1}]}(y', x) + \sum_{\ell=j+1}^{\infty} \left( K^{[2^{j},2^{j+1}]}(y, x) - K^{[2^{j},2^{j+1}]}(y', x) \right),$$

where the convergence is in $L^2(dw(x))$. Consequently,

$$\left| K^{[a]}(y, x) - K^{[a]}(y', x) \right| \leq \left| K^{[a,2^{j+1}]}(y, x) - K^{[a,2^{j+1}]}(y', x) \right| + \sum_{\ell=j+1}^{\infty} \left| K^{[2^{j},2^{j+1}]}(y, x) - K^{[2^{j},2^{j+1}]}(y', x) \right| (4.22)$$

$dw(x)$-almost everywhere. Integrating the inequality (4.22) and using (4.15) we obtain

$$\int \left| K^{[a]}(y, x) - K^{[a]}(y', x) \right| dw(x) \leq C2^{-j\delta} \|y - y'\|\delta + \sum_{\ell=j+1}^{\infty} C2^{-\delta}\ell \|y - y'\|\delta.$$

which gives (4.16).

For a cube $Q \subset \mathbb{R}^N$, let $c_Q$ be its center and $\text{diam}(Q)$ be the length of its diameter. Let $Q^*$ denote the cube with the same center $c_Q$ such that $\text{diam}(Q^*) = 2\text{diam}(Q)$. Recall

$$O(Q^*) = \{\sigma(x) : x \in Q^*, \sigma \in G\}.$$

The following corollary is a direct consequence of Proposition 4.8.
Corollary 4.9 There are constants $C, \delta > 0$ such that for any $j \in \mathbb{Z}$ and any cube $Q \subset \mathbb{R}^N$, and $y, y' \in Q$ we have

$$
\int_{\mathbb{R}^N \setminus \mathcal{O}(Q^*)} \sup_{2^{-1} \leq 2^j \leq 2^{j+1}} |K^{[a,b]}(x, y) - K^{[a,b]}(x, y')| \, dw(x)
\leq C \min(2^{-j}\delta \text{diam}(Q)^{\delta}, 2^{\delta j}(\text{diam}(Q))^{-\delta}).
$$

4.3 Proofs of Theorems 4.1, 4.2, and 4.3

Proof (Proof of Theorem 4.1) Having the estimates on the kernels $K^{[a,b]}$ and $\mathcal{F}K^{[a,b]}$ already established, the proof of (4.2) goes by a Calderón-Zygmund technique (cf. [3]). To this end take any $0 < a \leq b < \infty$. There is a constant $C_1 > 1$, which depends on the doubling constant in (2.2) and $N$, such that $w(Q) \leq C_1 w(Q')$, where $Q'$ is any sub-cube of $Q$ such that $\text{diam}(Q') = \text{diam}(Q)/2$.

Let $f \in L^1(dw) \cap L^2(dw)$. Fix $\lambda > 0$. We denote by $Q_\lambda$ the collection of all maximal (disjoint) dyadic cubes $Q_\ell$ in $\mathbb{R}^N$ satisfying

$$\lambda < \frac{1}{w(Q_\ell)} \int_{Q_\ell} |f(x)| \, dw(x).$$

Then

$$\frac{1}{w(Q_\ell)} \int_{Q_\ell} |f(x)| \, dw(x) \leq C_1 \lambda.$$

Set $\Omega = \bigcup_{Q_\ell \in Q_\lambda} Q_\ell$. Then $w(\Omega) \leq \lambda^{-1} \|f\|_{L^1(dw)}$. Form the corresponding Calderón–Zygmund decomposition of $f$, namely, $f = g + b$, where

$$
g(x) = f \chi_{\Omega^c}(x) + \sum_\ell w(Q_\ell)^{-1} \left( \int_{Q_\ell} f(y) \, dw(y) \right) \chi_{Q_\ell}(x),
$$

$$
b(x) = \sum_\ell b_\ell(x), \text{ where } b_\ell(x) = \left( f(x) - w(Q_\ell)^{-1} \int_{Q_\ell} f(y) \, dw(y) \right) \chi_{Q_\ell}(x).
$$

Clearly, $g(x), b(x) \in L^1(dw(x)) \cap L^2(dw(x))$, $|g(x)| \leq C_1 \lambda$, $\|g\|^2_{L^2(dw)} \leq C_1 \lambda \|f\|_{L^1(dw)}$, and $\sum_\ell \|b_\ell\|_{L^1(dw)} \leq C \|f\|_{L^1(dw)}$. Furthermore,

$$w(\{x \in \mathbb{R}^N : |K^{[a,b]} f(x)| > \lambda\}) \leq w(\{x \in \mathbb{R}^N : |K^{[a,b]} g(x)| > \lambda/2\})
+ w(\{x \in \mathbb{R}^N : |K^{[a,b]} b(x)| > \lambda/2\}).$$

Since $\|K^{[a,b]}\|_{L^2(dw) \to L^2(dw)} \leq C_2$, where $C_2 > 0$ is independent of $0 < a, b < \infty$ (see Corollary 3.3), we obtain

$$w(\{x \in \mathbb{R}^N : |K^{[a,b]} g(x)| > \lambda/2\}) \leq \frac{4}{\lambda^2} C_2^2 \|g\|^2_{L^2(dw)} \leq \frac{C_3}{\lambda} \|f\|_{L^1(dw)}.$$
Define $\Omega^* = O\left(\bigcup_{Q_\ell \in \mathcal{Q}_n} Q_\ell^*\right)$. There is a constant $C'_2 > 1$, which depends on the Weyl group, doubling constant, and $N$, such that

$$w(\Omega^*) \leq C_2 w(\Omega) \leq C'_2 \lambda^{-1} \|f\|_{L^1(dw)}.$$ 

Thus it suffices to estimate $K^{[a,b]} b(\mathbf{x})$ on $\mathbb{R}^N \setminus \Omega^*$. Let $n_0, n_1 \in \mathbb{Z}$ be such that $2^{n_0} \leq a < 2^{n_1}$, $2^{n_1} \leq b < 2^{n_1+1}$. If $n_0 < n_1$, we write

$$K^{[a,b]} = K^{[a,2^{n_0}]} + \left(\sum_{j=n_0}^{n_1-1} K^{[2^j,2^{j+1}]}\right) + K^{[2^{n_1},b]},$$

(4.23)

otherwise $2^{n_0-1} \leq a \leq b \leq 2^{n_0+1}$ and we consider just the single kernel $K^{[a,b]}$.

Since $\sum_\ell b_\ell$ converges to $b$ in $L^2(dw)$, $K^{[a,b]} b = \sum_\ell K^{[a,b]} b_\ell$ with convergence in $L^2(dw)$. So we have

$$|K^{[a,b]} b(\mathbf{x})| \leq \sum_\ell \left\{ |K^{[a,2^{n_0}]} b_\ell(\mathbf{x})| + \left(\sum_{j=n_0}^{n_1-1} |K^{[2^j,2^{j+1}]} b_\ell(\mathbf{x})|\right) + |K^{[2^{n_1},b]} b_\ell(\mathbf{x})| \right\}.$$

By the fact that $\text{supp } b_\ell \subseteq Q_\ell$ and $\int_{\mathbb{R}^N} b_\ell(\mathbf{y}) \, dw(\mathbf{y}) = 0$, we get

$$\int_{\mathbb{R}^N \setminus \Omega^*} |K^{[2^j,2^{j+1}]} b_\ell(\mathbf{x})| \, dw(\mathbf{x})$$

$$= \int_{\mathbb{R}^N \setminus \Omega^*} \left| \int_{Q_\ell} K^{[2^j,2^{j+1}]}(\mathbf{x}, \mathbf{y}) b_\ell(\mathbf{y}) \, dw(\mathbf{y}) \right| \, dw(\mathbf{x})$$

$$\leq \int_{\mathbb{R}^N \setminus \mathcal{O}(Q_\ell^\delta)} \left| \int_{Q_\ell} \left( K^{[2^j,2^{j+1}]}(\mathbf{x}, \mathbf{y}) - K^{[2^j,2^{j+1}]}(\mathbf{y}, c_{Q_\ell^\delta}) \right) b_\ell(\mathbf{y}) \, dw(\mathbf{y}) \right| \, dw(\mathbf{x})$$

$$\leq C \min \left(2^{-j \delta} \text{diam}(Q_\ell^\delta), 2^{j \delta} \text{diam}(Q_\ell)^{-\delta}\right) \|b_\ell\|_{L^1(dw)},$$

(4.24)

where in the last inequality we have used Corollary 4.9 with $\delta > 0$ small enough. Similarly,

$$\int_{\mathbb{R}^N \setminus \Omega^*} |K^{[a,2^{n_0}]} b_\ell(\mathbf{x})| \, dw(\mathbf{x}) \leq C \min \left(2^{-n_0 \delta} \text{diam}(Q_\ell^\delta), 2^{n_0 \delta} \text{diam}(Q_\ell)^{-\delta}\right) \|b_\ell\|_{L^1(dw)},$$

(4.25)

$$\int_{\mathbb{R}^N \setminus \Omega^*} |K^{[2^{n_1},b]} b_\ell(\mathbf{x})| \, dw(\mathbf{x}) \leq C \min \left(2^{-n_1 \delta} \text{diam}(Q_\ell^\delta), 2^{n_1 \delta} \text{diam}(Q_\ell)^{-\delta}\right) \|b_\ell\|_{L^1(dw)}.$$
Obviously,
\[
\sum_{j \in \mathbb{Z}} \min \left( 2^{-\delta j} \operatorname{diam}(Q_{\ell})^\delta, 2^{\delta j} \operatorname{diam}(Q_{\ell})^{-\delta} \right) = \sum_{j \geq \log_2(\operatorname{diam}(Q_{\ell}))} 2^{-\delta j} \operatorname{diam}(Q_{\ell})^\delta + \sum_{j < \log_2(\operatorname{diam}(Q_{\ell}))} 2^{\delta j} \operatorname{diam}(Q_{\ell})^{-\delta} \leq C < \infty
\]  
(4.27)

with $C$ independent of $\ell$. Hence, summing up the inequalities (4.24)–(4.26) over $\ell \in \mathbb{Z}$ we end up with
\[
\int_{\mathbb{R}^N \setminus \Omega^*} |\mathcal{K}^{[a,b]} b(x)| \, dw(x) \leq C' \sum_{\ell \in \mathbb{Z}} \|b_\ell\|_{L^1(dw)} \leq C \|f\|_{L^1(dw)}
\]
with a constant $C$ independent of $0 < a < b < \infty$. Consequently, by the Chebyshev’s inequality, this completes the proof of the weak type $(1, 1)$ bound of the operator $\mathcal{K}^{[a,b]}$.

In order to prove (4.3), let us note that for any $f \in L^2(dw) \cap L^1(dw)$ and any $a > 0$ there is an increasing sequence $m_j \to \infty$, $m_j > a$, such that $\mathcal{K}^{[a]} f = \lim_{j \to \infty} \mathcal{K}^{[a,m_j]} f$ with convergence in $L^2(dw)$ and almost everywhere. Therefore, up to a set of $dw$-measure zero, we have
\[
\{x \in \mathbb{R}^N : |\mathcal{K}^{[a]} f(x)| > \lambda\} = \bigcup_{n=1}^{\infty} \bigcap_{j \geq n} \{x \in \mathbb{R}^N : |\mathcal{K}^{[a,m_j]} f(x)| > \lambda\}. 
\]
(4.28)

So, thanks to (4.28) and (4.2), we have
\[
w(\{x \in \mathbb{R}^N : |\mathcal{K}^{[a]} f(x)| > \lambda\}) \leq \sup_{a \leq b} w(\{x \in \mathbb{R}^N : |\mathcal{K}^{[a,b]} f(x)| > \lambda\}) \leq C \frac{1}{\lambda} \|f\|_{L^1(dw)}.
\]

Proof (Proof of Theorem 4.2) We first note that by (4.23) and (4.13) with $\delta = 0$ we have
\[
\int |\mathcal{K}^{[a,b]}(x, y)| \, dw(x) \leq C_{a,b}, \quad \int |\mathcal{K}^{[a,b]}(x, y)| \, dw(y) \leq C_{a,b}.
\]
Thus the operators $\mathcal{K}^{[a,b]}$ are bounded on $L^p(dw)$. Moreover, $(\mathcal{K}^{[a,b]})^* = (\mathcal{K}^*)^{[a,b]}$, where $\mathcal{K}^*(x) = \mathcal{K}(-x)$. Now for fixed $1 < p < \infty$ the uniform bound of the operators $\mathcal{K}^{[a,b]}$ on $L^p(dw)$ follows from interpolation, Corollary 3.3, Theorem 4.1, and duality.

Furthermore, (4.5) and the strong convergence of $\mathcal{K}^{[a,b]}$ to $\mathcal{K}^{[a]}$ on the space $L^p(dw)$ for $1 < p < 2$ as $b \to \infty$ also follows from the Marcinkiewicz interpolation theorem and (3.6) of Corollary 3.3. In order to prove (4.5) for $p > 2$, let us show first that for $f \in S(\mathbb{R}^N)$ the function $(a, \infty) \ni b \mapsto \mathcal{K}^{[a,b]} f \in L^p(dw)$ satisfies the Cauchy condition as $b \to \infty$. Indeed,
\[ \| K^{[a,b_1]} f - K^{[a,b_2]} f \|_{L^p(dw)} \leq \| K^{[a,b_1]} f - K^{[a,b_2]} f \|_{L^2(dw)}^{1/(p-1)} \| K^{[a,b_1]} f - K^{[a,b_2]} f \|_{L^2(dw)}^{1-1/(p-1)}. \]

Thanks to (3.6) we have

\[ \lim_{b_1,b_2 \to \infty} \| K^{[a,b_1]} f - K^{[a,b_2]} f \|_{L^2(dw)}^{1/(p-1)} = 0. \]

Moreover, by (4.4), we get

\[ \| K^{[a,b_1]} f - K^{[a,b_2]} f \|_{L^2(dw)}^{1-1/(p-1)} \leq C \| f \|_{L^2(dw)}^{1-1/(p-1)} < \infty. \]

Consequently,

\[ \lim_{b_1,b_2 \to \infty} \| K^{[a,b_1]} f - K^{[a,b_2]} f \|_{L^p(dw)} = 0 \] (4.29)

for \( f \in \mathcal{S}(\mathbb{R}^N) \). In order to prove (4.29) for \( f \in L^p(dw) \), it is enough to take a sequence \( \mathcal{S}(\mathbb{R}^N) \ni f_\ell \to f \) in \( L^p(dw) \) and write

\[ \| K^{[a,b_1]} f - K^{[a,b_2]} f \|_{L^p(dw)} \leq \| K^{[a,b_1]} (f - f_\ell) \|_{L^p(dw)} + \| K^{[a,b_1]} f_\ell - K^{[a,b_2]} f_\ell \|_{L^p(dw)} + \| K^{[a,b_2]} (f_\ell - f) \|_{L^p(dw)}, \]

then use (4.4) for the first and the third summand and (4.29) for the second one. \( \square \)

**Proof (Proof of Theorem 4.3)** In order to prove (4.6), we use the same arguments as in the second part of the proof of Theorem 4.1 (see (4.28)). For the proof of (4.7), see the proof of Theorem 4.2. \( \square \)

### 5 Maximal function associated with singular integral

Recall that \( \mathcal{K}^* f(x) = \sup_{a > 0} |K^{[a]} f(x)| \). The goal of this section is to prove the following theorem.

**Theorem 5.1** Under \( (D(s_0)) \), \( (A) \), and \( (L) \), the operator \( \mathcal{K}^* \) is of weak type \((1,1)\) and it is bounded on \( L^p(dw) \) for \( 1 < p < \infty \).

#### 5.1 Cotlar type inequality

Let

\[ \mathcal{M}_{HL} f(x) = \sup_{x \in B} \frac{1}{w(B)} \int_B |f(y)| dw(y), \]
where the supremum is taken over all Euclidean balls \( B \) which contain \( x \), be the non-centered Hardy-Littlewood maximal function defined on the space of homogeneous type \( (\mathbb{R}^N, \|x - y\|, dw) \). The following lemma, which is in the spirit of Cotlar’s inequality (cf. [10, Lemma 5.15]), plays a crucial role in the proof of Theorem 5.1.

**Lemma 5.2** Let \( p \in [1, \infty) \). There is a constant \( C > 1 \) such that for all \( f \in L^p(dw) \cap L^\infty \) and \( x \in \mathbb{R}^N \) we have

\[
\mathcal{K}^* f(x) \leq C \left( \sum_{\sigma \in \Gamma} (\mathcal{M}_{HL} \mathcal{K} f)(\sigma(x)) + \|f\|_{L^\infty} \right).
\]

**Proof** We assume additionally that \( f \in L^2(dw) \). Then this assumption can be easily relaxed by a density argument. Let \( \varphi \in C_c^\infty(\mathbb{R}^N) \) be a radial function such that \( \text{supp} \varphi \subseteq B(0, 1) \) and \( \int_{\mathbb{R}^N} \varphi \, dw = 1 \). Fix \( a > 0 \) and define the operator \( \check{\mathcal{K}}[a] := \mathcal{K} - \mathcal{K}^a \). Recall \( \varphi_a(x) = a^{-N} \varphi(x/a) \). Then

\[
\mathcal{K}^a f = (K^a - \varphi_a \ast \varphi_a \ast K^a) \ast f + \varphi_a \ast \varphi_a \ast (Kf) - ((\varphi_a \ast (\check{\mathcal{K}}[a] \varphi_a)) \ast f
\]

\[=: J_1 + J_2 - J_3.\]

Clearly, by Lemma 2.4, we have

\[
|(\varphi_a \ast \varphi_a)(x, y)| = |(\varphi \ast \varphi)_a(x, y)|
\]

\[
\leq C \left( 1 + \frac{\|x - y\|}{a} \right)^{-2} V(x, y, a)^{-1} \chi_{[0, 2]}(d(x, y)/a).
\]

Hence, \( |J_2| \leq C \sum_{\sigma \in \Gamma} (\mathcal{M}_{HL} \mathcal{K} f)(\sigma \cdot) \). In order to estimate \( J_3 \) we note that \( \text{supp} \check{\mathcal{K}}[a] \varphi_a \subseteq B(0, 2a) \), so by the Cauchy–Schwarz inequality, Theorem 4.2 together with Lemma 2.4 and (2.1) we have

\[
\|\check{\mathcal{K}}[a] \varphi_a\|_{L^1(dw)} \leq C w(B(0, 2a))^{1/2} \|\varphi_a\|_{L^2(dw)} \leq C.
\]

Applying Proposition 4.7 we get

\[
\int_{\mathbb{R}^N} |(\varphi_a \ast (\check{\mathcal{K}}[a] \varphi_a))(x, y)| \, dw(y) \leq C a^N \|\varphi_a\|_{L^\infty} \cdot \|\check{\mathcal{K}}[a] \varphi_a\|_{L^1(dw)} \leq C,
\]

which, together with (5.3), implies \( |J_3| \leq C \|f\|_{L^\infty} \). Finally, in order to evaluate \( J_1 \), we consider the integral kernel \( K^a(x, y) - (\varphi_a \ast \varphi_a \ast K^a)(x, y) \) of the operator \( f \mapsto (K^a \ast f - \varphi_a \ast \varphi_a \ast K^a \ast f) \). Note that

\[
\int_{\mathbb{R}^N} (\varphi_a \ast \varphi_a)(x, z) \, dw(z) = \int_{\mathbb{R}^N} \varphi_a \ast \varphi_a(z) \, dw(z) = \mathcal{F}(\varphi_a \ast \varphi_a)(0) = \mathcal{F} \varphi_a(0)^2 = 1
\]
Thus, using (4.16) and (5.2), we obtain

\[
\int_{\mathbb{R}^N} |K^a(x, y) - (\varphi_a \ast \varphi_a \ast K^a)(x, y)| \, dw(y)
\]

\[
= \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} (\varphi_a \ast \varphi_a)(x, z) (K^a(x, y) - K^a(z, y)) \, dw(z) \right| \, dw(y)
\]

\[
\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |(\varphi_a \ast \varphi_a)(x, z)| |K^a(x, y) - K^a(z, y)| \, dw(y) \, dw(z)
\]

\[
\leq \int_{\mathbb{R}^N} \left| (\varphi_a \ast \varphi_a)(x, z) \right| a^{-\delta} \|x - z\| \, dw(z)
\]

\[
\leq \int_{O(B(x, 2a))} w(B(x, 4a))^{-1} \left( 1 + \frac{|x - z|}{a} \right)^{-2} a^{-\delta} \|x - z\| \, dw(z) \leq C,
\]

which gives \(|J_1| \leq C\|f\|_{L^\infty}.
\]

\[\square\]

5.2 Proof of Theorem 5.1

Proof It is enough to prove that \(\mathcal{K}^\ast\) is of weak type \((p, p)\) for \(1 \leq p < \infty\). Let \(f \in L^p(dw) \cap L^2(dw)\). Fix \(\lambda > 0\). We denote by \(Q_\lambda\) the collection of all maximal (disjoint) dyadic cubes \(Q_\ell\) in \(\mathbb{R}^N\) satisfying

\[
\lambda^p < \frac{1}{w(Q_\ell)} \int_{Q_\ell} |f(x)|^p \, dw(x).
\]

Then

\[
\frac{1}{w(Q_\ell)} \int_{Q_\ell} |f(x)|^p \, dw(x) \leq C_1 \lambda^p.
\]

Set \(\Omega = \bigcup_{Q_\ell \in Q_\lambda} Q_\ell\). Thanks to (5.4) we have

\[
\sum_{\ell} w(Q_\ell) = w(\Omega) \leq \lambda^{-p} \|f\|_{L^p(dw)}^p.
\]

Form the corresponding Calderón–Zygmund decomposition of \(f\), namely, \(f = g + b\), where

\[
g(x) = f \chi_{\Omega^c}(x) + \sum_{\ell} \left( w(Q_\ell)^{-1} \int_{Q_\ell} f(y) \, dw(y) \right) \chi_{Q_\ell}(x),
\]

\[
b(x) = \sum_{\ell} b_\ell(x), \text{ where } b_\ell(x) = \left( f(x) - w(Q_\ell)^{-1} \int_{Q_\ell} f(y) \, dw(y) \right) \chi_{Q_\ell}(x).
\]

Clearly, \(\|g\|_{L^p(dw)} + \|b\|_{L^p(dw)} \leq C \|f\|_{L^p(dw)}, \|g\|_{L^2(dw)} + \|b\|_{L^2(dw)} \leq C \|f\|_{L^2(dw)}\), and \(|g(x)| \leq C_1^{1/p} \lambda\). Further,
Thus it suffices to estimate $\mathcal{X}^* b(x)$ on $\mathbb{R}^N \setminus \Omega^*$. Note that $\sum_{\ell} b_{\ell}$ converges to $b$ in $L^2(dw)$. Recall $c_{Q_{\ell}}$ is the center of $Q_{\ell}$. We write

\[
\|\mathcal{X}^* b\|_{L^1(\mathbb{R}^N \setminus \Omega^*, dw)} = \int_{\mathbb{R}^N \setminus \Omega^*} \sup_{a > 0} \left| \sum_{\ell} \int_{Q_{\ell}} K^{[a]}(x, y) b_{\ell}(y) \, dw(y) \right| \, dw(x)
\]

\[
= \int_{\mathbb{R}^N \setminus \Omega^*} \sup_{a > 0} \left| \sum_{\ell} \int_{Q_{\ell}} (K^{[a]}(x, y) - K^{[a]}(x, c_{Q_{\ell}})) b_{\ell}(y) \, dw(y) \right| \, dw(x)
\]

\[
\leq \sum_{\ell} \int_{Q_{\ell}} |b_{\ell}(y)| \int_{\mathbb{R}^N \setminus \Omega(Q_{\ell})} \sup_{a > 0} |K^{[a]}(x, y) - K^{[a]}(x, c_{Q_{\ell}})| \, dw(x) \, dw(y),
\]

where for the second equality we have used $\int_{\mathbb{R}^N} b_{\ell}(y) \, dw(y) = 0$. Using (4.22) we get

\[
\int_{\mathbb{R}^N \setminus \Omega(Q_{\ell})} \sup_{a > 0} |K^{[a]}(x, y) - K^{[a]}(x, c_{Q_{\ell}})| \, dw(x)
\]

\[
\leq \int_{\mathbb{R}^N \setminus \Omega(Q_{\ell})} \sup_{b > a > 0} |K^{[a,b]}(x, y) - K^{[a,b]}(x, c_{Q_{\ell}})| \, dw(x)
\]

\[
\leq \sum_{j \in \mathbb{Z}} \frac{2^{j-1} \leq a' < b' \leq 2^{j+1}} \sup_{b > a > 0} |K^{[a',b']}(x, y) - K^{[a',b']}(x, c_{Q_{\ell}})| \, dw(x)
\]

\[
\leq C \sum_{j \in \mathbb{Z}} \min \left( 2^{-\delta j} \text{diam}(Q_{\ell})^{\delta}, 2^{\delta j} \text{diam}(Q_{\ell})^{-\delta} \right) \leq C,
\]

where in the third inequality we have used Corollary 4.9 with $\delta > 0$ small enough and in the last inequality we have applied (4.27). So, taking together (5.8) and (5.9) we obtain

\[
\|\mathcal{X}^* b\|_{L^1(\mathbb{R}^N \setminus \Omega^*, dw)} \leq C \sum_{\ell} \|b_{\ell}\|_{L^1(dw)}
\]
Furthermore, by Hölder’s inequality and (5.5) we have
\[ \sum_{\ell} \| b_\ell \|_{L^1(dw)} \leq \sum_{\ell} w(Q_\ell) \left( w(Q_\ell)^{-1} \int_{Q_\ell} |b_\ell(x)|^p \, dw(x) \right)^{1/p} \leq C_\lambda \sum_{\ell} w(Q_\ell). \]  
(5.11)

Combining (5.11) and (5.6) we get
\[ \| K^* b \|_{L^1(R^N \setminus \Omega^2, dw)} \leq C_\lambda^{-p+1} \| f \|_{L^p}^p. \]  
(5.12)

Chebyshev’s inequality applied to (5.12) together with (5.7) imply
\[ S_1 \leq C_\lambda^{-p} \| f \|_{L^p}^p. \]

In order to estimate \( S_2 \), thanks to the fact that \( \| g \|_{L^\infty} \leq C_4^{1/p} \lambda \) and the Cotlar type inequality (5.1), we have
\[ w \left( \{ x \in R^N : |K^* g(x)| > 2C_4^{1/p} C_3 \lambda \} \right) \leq w \left( \{ x \in R^N : \sum_{\sigma \in G} M_{HL} K g(\sigma(x)) > C_4^{1/p} \lambda \} \right). \]

If \( p > 1 \), then recall \( M_{HL} \) and \( K \) are bounded operators on \( L^p(dw) \) (see Theorem 4.3), so \( S_2 \leq C_\lambda^{-p} \| f \|_{L^p}^p \).

If \( p = 1 \), then \( \| g \|_{L^2(dw)}^2 \leq C_\lambda \| f \|_{L^1(dw)} \) and, consequently,
\[ w \left( \{ x \in R^N : \sum_{\sigma \in G} M_{HL} K g(\sigma(x)) > C_1 \lambda \} \right) \leq C_\lambda^{-2} \| g \|_{L^2(dw)}^2 \leq C_4' \lambda^{-1} \| f \|_{L^1(dw)}. \]

\[ \square \]

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