ON IN Variant 1-DIMENSIONAL REPRESENTATIONS OF A FINITE W-ALGEBRA

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ABSTRACT. Let $g$ be a simple Lie algebra over $\mathbb{C}$ and $G$ be the corresponding simply connected algebraic group. Consider a nilpotent element $e \in g$, the corresponding element $\chi = (e, \bullet)$ in $g^*$, and the coadjoint orbit $O = G\chi$. We are interested in the set $\mathfrak{J}^l(W)$ of codimension 1 ideals $J \subset W$ in a finite $W$-algebra $W = U(g, e)$. We have a natural action of the component group $\Gamma = Z_G(\chi) / Z_G^\circ(\chi)$ on $\mathfrak{J}^l(W)$. Denote the set of $\Gamma$-stable points of $\mathfrak{J}^l(W)$ by $\mathfrak{J}^l(W)^\Gamma$. For a classical $g$ Premet and Topley in [PT14] proved that $\mathfrak{J}^l(W)^\Gamma$ is isomorphic to an affine space. In this paper we will give an easier and shorter proof of this fact.

1. Introduction

1.1. Let $g$ be a simple Lie algebra over $\mathbb{C}$ and $G$ be the corresponding simply connected algebraic group. Let $e \in g$ be a nilpotent element and $\chi = (e, \bullet)$ be the dual element in $g^*$, where $\langle \bullet, \bullet \rangle$ stands for the Killing form. We fix the coadjoint orbit $O = G\chi$. In [Pre02] Premet defined the finite $W$-algebra $W = U(g, e)$ associated to the nilpotent element $e$. In [Los10b] Losev gave an alternative definition that will be recalled in Section 3.4.

Let $\mathfrak{J}(W)$ and $\mathfrak{J}^l(W)$ be the sets of ideals and codimension 1 ideals in $W$ respectively. Note that every codimension 1 ideal in $W$ contains the two-sided ideal $[W, W]$ generated by the commutator $[W, W]$. Therefore the set $\mathfrak{J}^l(W)$ is isomorphic to the set of codimension 1 ideals in the commutative algebra $W_{ab} := W/([W, W])$. By the Hilbert Nullstellensatz such ideals are classified by the points of the maximal spectrum $\text{Spec}(W_{ab})$.

We set $\Gamma := Z_G(e) / Z_G^\circ(e)$ to be the component group of the centralizer $Z_G(e)$ of $e$ and $Q$ to be the reductive part of $Z_G(e)$. The action of $Q$ on $W$ preserves the ideal $([W, W])$ and induces an action on $\text{Spec}(W_{ab})$. Recall that $\Gamma$ is also equal to $Q/\mathfrak{q}^\circ$, and the action of $\mathfrak{q}^\circ$ on any $J \in \mathfrak{J}^l(W)$ is trivial because of the embedding $g \to W$, see [Pre07, Section 2] for more details. Therefore we have an action of $\Gamma$ on $\text{Spec}(W_{ab})$. Let $\mathfrak{J}^l(W)^\Gamma$ be the set of $\Gamma$-invariant codimension 1 ideals in $W$. The main goal of this paper is to give a geometric proof of the following theorem.

**Theorem 1.1.** [PT14, Theorem 2] Let $g$ be classical. Then $\mathfrak{J}^l(W)^\Gamma$ is in bijection with the set of points of an affine space.

1.2. The plan of the paper is as follows. It is known [Los10a] that there is a natural bijection between the set of $\Gamma$-invariant ideals $J \in \mathfrak{J}^l(W)^\Gamma$ and the space $Q(\mathfrak{O})$ of formal graded $G$-equivariant quantizations of $\mathfrak{O}$. We want to relate the sets of the formal graded quantizations of the coadjoint orbit $\mathfrak{O}$ and of its affinization $\text{Spec}(\mathfrak{C}(\mathfrak{O}))$. The set of formal graded quantizations of the affinization was computed by Losev in [Los16]. In Section 2 we consider the isomorphism classes of formal graded quantizations of $\text{Spec}(\mathfrak{C}(\mathfrak{O}))$. By [Los16] these are classified by the points of $\mathfrak{P}/W$ for some affine space $\mathfrak{P}$ and a finite group $W$. In Section 3 following [Los10a] we construct a natural bijection between the set of $\Gamma$-invariant ideals $J \in \mathfrak{J}^l(W)^\Gamma$ and the space $Q(\mathfrak{O})$ of formal $G$-equivariant quantizations of $\mathfrak{O}$ mentioned above. In Sections 4 and 5 we show that $Q(\mathfrak{O})$ and $\mathfrak{P}/W$ are in a natural bijection. This is the main part of this paper. The construction of Section 3 gives us an algebraic quantization of an affine subscheme of $g^*$ with the reduced scheme $\mathfrak{O}$. We show that it can be extended to a holomorphic one (i.e. a quantization in the complex-analytic
topology). After that, we lift that holomorphic quantization to a quantization of Spec(\(C[\mathcal{O}]\)). That gives us a bijection between \(\mathcal{Q}(\mathcal{O})\) and \(\mathfrak{P}/W\) as sets. We deduce the main theorem from that.

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2. Quantizations of Spec(\(C[\mathcal{O}]\))

2.1. Graded formal quantizations. Let \(A\) be a finitely generated Poisson algebra equipped with a grading \(A = \bigoplus_{i=0}^{\infty} A_i\) such that \(A_0 = C\), and the Poisson bracket has degree \(-d\), where \(d \in \mathbb{Z}_{>0}\).

Let \(A_h\) be a \(C[[h]]\)-algebra, flat over \(C[[h]]\) and complete and separated in the \(h\)-adic topology. Assume that \(A_h\) is equipped with a \(C^\times\)-action by \(C\)-algebra automorphisms that is rational on all quotients \(A_h/(h^k)\) and satisfies \(t.h = th\). Moreover, suppose that \([A_h, A_h] \subset h^d A_h\). All \(C[[h]]\)-algebras in this paper are assumed to satisfy all of these conditions. We have a skew-symmetric bracket on \(A_h\) given by \(\{a, b\} = \frac{[a, b]}{h^d}\) that induces a Poisson bracket on \(A_h/(h)\). The action of the torus on \(A_h\) induces a grading on the quotient \(A_h/(h)\). Suppose that there is an isomorphism \(\theta : A_h/(h) \simeq A\) of graded Poisson algebras. We say that \(A_h\) together with an isomorphism \(\theta\) is a graded formal quantization of \(A\).

Example 2.1. The completed homogeneous universal enveloping algebra \(U_h(\mathfrak{g}) = T(\mathfrak{g})[[h]]/(\xi \otimes \eta - \eta \otimes \xi - h^2[\xi, \eta] \mid \xi, \eta \in \mathfrak{g})\) is a formal quantization of the Poisson algebra \(S(\mathfrak{g})\).

By a Poisson scheme we mean a scheme \(X\) over Spec(\(C\)) whose structure sheaf \(\mathcal{O}_X\) is equipped with a Poisson bracket. For example, for any finitely generated commutative Poisson algebra \(A\) the affine scheme \(X = \text{Spec}(A)\) is a Poisson scheme. By a formal quantization of \(X\) we understand a sheaf \(D\) of \(C[[h]]\)-algebras such that \(D\) is flat over \(C[[h]]\) and complete and separated in the \(h\)-adic topology together with an isomorphism of sheaves of Poisson algebras \(\theta : D/hD \simeq \mathcal{O}_X\). In what follows we assume that \(X\) is conical. We introduce the conical topology on \(X\), where, by definition, a subset \(U\) is open if and only if \(U\) is Zariski open and \(C^\times\)-stable. Note that if \(X\) is normal then every point admits a \(C^\times\)-stable affine open neighborhood. From now on we consider quantizations in the conical topology and demand the isomorphism \(\theta : D/hD \simeq \mathcal{O}_X\) to be a \(C^\times\)-invariant isomorphism of sheaves of Poisson algebras.

A morphism of two formal quantizations \((D_1, \theta_1)\) and \((D_2, \theta_2)\) is a \(C^\times\)-equivariant morphism \(\phi : D_1 \to D_2\) of \(C[[h]]\)-algebras such that \(\theta_1 = \theta_2 \circ \phi\). Any such morphism is automatically an isomorphism of formal quantizations.

We will need the following standard lemma.

Lemma 2.2. Let \(A\) be a Poisson algebra and \(X = \text{Spec}(A)\). The map \(D \to \Gamma(X, D)\) gives a bijection between the set of formal quantizations of \(X\) and the set of formal quantizations of \(A\).

Let \(X\) be a complex analytic space such that the sheaf \(\mathcal{H}_X\) of holomorphic functions is equipped with a Poisson bracket. We will call such \(X\) a Poisson analytic space. Suppose that we have an action of \(C^\times\) on \(X\). Similarly to the notion of a formal quantization of \(X\) we define a formal holomorphic quantization of \(X\) to be a sheaf \(D\) of \(C[[h]]\)-algebras such that \(D\) is flat over \(C[[h]]\) and complete and separated in the \(h\)-adic topology together with a \(C^\times\)-invariant isomorphism of sheaves of Poisson algebras \(\theta : D/hD \simeq \mathcal{H}_X\).

Example 2.3. Let \(G\) be a complex Lie group. Let \(D_G\) be a sheaf of differential operators on \(G\). For any open \(U\) we set \(D_{G,h}(U) = R^h_0(D_G(U))\). For every inclusion of open sets \(V \subset U\) we have a restriction map \(D_G(U) \to D_G(V)\) that extends to a map \(R^h_0(D_G(U)) \to R^h_0(D_G(V))\) on Rees completions. The resulting sheaf \(D_{G,h}\) can be microlocalized to a formal quantization \(\mathfrak{D}_{G,h}\) on the cotangent bundle \(T^*G\).
Recall from [KR08] the notion of a $W$-algebra on a symplectic manifold $X$. In Section 2.2.3 loc.cit. the canonical $W$-algebra $W_{T^*G}$ coming from the sheaf of holomorphic differential operators on $G$ is constructed. The sheaf $\mathcal{D}_{G,hol,h} = W_{T^*G}(0)$ is a formal holomorphic quantization of $T^*G$.

**Example 2.4.** Set $X = \mathfrak{g}^*$ with the canonical structure of a Poisson variety. The left-invariant vector fields on $G$ give a trivialization of the cotangent bundle $T^*G \simeq G \times \mathfrak{g}^*$. Therefore we have a map $\pi : T^*G \to \mathfrak{g}^*$. We set $\mathcal{D} := (\pi_*\mathcal{D}_{G,h})^G$ and $\mathcal{D}_{hol} := (\pi_*\mathcal{D}_{G,hol,h})^G$. These are formal and formal holomorphic quantizations of $\mathfrak{g}^*$ correspondingly. The formal quantization of $S(\mathfrak{g})$ corresponding to $\mathcal{D}$ is $\Gamma(\mathfrak{g}^*, \mathcal{D}) = U_h(\mathfrak{g})$.

2.2. **Quantizations of Spec$(\mathbb{C}[\mathfrak{g}])$.** Let $X$ be a normal Poisson algebraic variety such that the regular locus $X^{reg}$ is a symplectic variety. Let $\omega^{reg}$ be the symplectic form on $X^{reg}$. Suppose that there exists a projective resolution of singularities $\rho : \hat{X} \to X$ such that $\rho^*(\omega^{reg})$ extends to a regular (not necessarily symplectic) form on $\hat{X}$. Following Beauville [Bea00], we say that $X$ has symplectic singularities. We will be mostly interested in the following example.

**Example 2.5.** [Pan91] Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{g}^* \subset \mathfrak{g}^*$ be a nilpotent orbit. Then $X = \text{Spec}(\mathbb{C}[\mathfrak{g}])$ has symplectic singularities.

We say that an affine Poisson variety $X$ is conical if its algebra of functions is endowed with a grading $\mathbb{C}[X] = \bigoplus_{i=0}^{\infty} \mathbb{C}[X]_i$ and a positive integer $d$ such that $\mathbb{C}[X]_0 = \mathbb{C}$, and for any $i, j$ and $f, g \in \mathbb{C}[X]_i, g \in \mathbb{C}[X]_j$ we have $\{f, g\} \in \mathbb{C}[X]_{i+j-d}$. Note that Spec$(\mathbb{C}[\mathfrak{g}])$ is conical with $d = 1$.

Let $X$ be an affine conical Poisson variety that has symplectic singularities. Following Namikawa, we can construct a certain affine space $\mathfrak{P}$ (denoted by $\mathfrak{b}$ in [Nam11]) and a finite group $W$ acting on $\mathfrak{P}$ by reflections. In [Los16] Losev showed that formal graded quantizations of $X$ are classified by $\mathfrak{P}/W$. Together with the Chevalley-Shepherd-Todd’s theorem this imply the following description of the isomorphism classes of formal graded quantizations of $X$.

**Proposition 2.6.** Isomorphism classes of formal graded quantizations of Spec$(\mathbb{C}[\mathfrak{g}])$ are classified by the points of an affine space.

3. **Quantizations of coadjoint orbits.**

3.1. **Hamiltonian $G$-equivariant quantizations.** In this section we recall constructions and results from [Los10a]. All quantizations are assumed to be formal. Let $X$ be a smooth symplectic variety over $\mathbb{C}$. Then both the structure sheaf $\mathcal{O}_X$ and the sheaf of holomorphic functions $\mathcal{H}_X$ admit Poisson brackets $\{\cdot, \cdot\}$ induced from the symplectic form. Therefore we may consider (holomorphic) quantizations of the variety $X$. In this section the main example is $X = \mathcal{O}$, the nilpotent coadjoint orbit with the Kostant-Kirillov symplectic form $\omega_X(\mu, \eta) = \chi([\mu, \eta])$. The induced Poisson bracket has degree $-1$.

Suppose that we have an action of a connected algebraic group $G$ on $X$. We say that a (holomorphic) quantization $\mathcal{D}$ is $G$-equivariant if the action of $G$ on $\mathcal{O}_X$ (resp. $\mathcal{H}_X$) lifts to the action on $\mathcal{D}$ by algebra automorphisms such that $h$ is $G$-invariant. For $\xi \in \mathfrak{g}$ let $\xi_X$ be the derivation of $\mathcal{O}_X$ (resp. $\mathcal{H}_X$) induced by the action of $G$.

Recall that a map $\mu : X \to \mathfrak{g}^*$ is called the moment map if $\{\mu^*(\xi), \cdot\} = \xi_X$. From now on suppose that $X$ is equipped with a moment map $\mu$. For $X = \mathcal{O}$ the natural embedding $\mathcal{O} \hookrightarrow \mathfrak{g}^*$ is a moment map. Let $\mathcal{D}$ be a $G$-equivariant quantization of $X$. We say that a map $\phi_h : \mathfrak{g} \to \Gamma(X, \mathcal{D})$ is a quantum comoment map if $\frac{1}{h} [\phi_h(\xi), \cdot] = \xi_D$ for all $\xi \in \mathfrak{g}$ and $\phi_h$ coincides with $\mu^*$ modulo $h$. We say that $\mathcal{D}$ is Hamiltonian if it is equipped with a quantum comoment map $\phi_h$.

In this section we will study $G$-equivariant Hamiltonian (holomorphic) quantizations of $\mathcal{O}$. Since the action of $G$ is transitive, the idea is to reduce a quantization to its fiber at a point. To do this we will need to use quantum jet bundles.
3.2. Quantum jet bundles. Let us first recall the notion of the jet bundle $J^\infty O_X$. Let $I_\Delta \subset O_{X \times X}$ be the ideal of the diagonal $X \subset X \times X$. We set $O_{X \times X}^\infty = \varprojlim_{k \to \infty} O_{X \times X}/I_\Delta^k$. Let $\pi_1, \pi_2$ be projection maps from $X \times X$ to its factors. We set $J^\infty O_X = \pi_1^*(O_{X \times X}^\infty)$. Note that the fiber of $J^\infty O_X$ at any point is non-canonically isomorphic to the algebra of formal series in $\dim X$ variables. Let $\nabla$ be a connection on $\pi_1^*(O_{X \times X})$ given by $\nabla_\xi (f \otimes g) = \xi(f) \otimes g$. We can uniquely extend it to a continuous in $\Delta$-adic topology flat connection on $J^\infty O_X$. To simplify notations we will also denote it by $\nabla$.

Recall that $X$ is a symplectic variety. The sheaf $O_{X \times X}$ has a natural $O_X$-linear (corresponding to the first factor) Poisson bracket that extends to one on $J^\infty O_X$. The bracket on the sheaf $O_X$ of flat sections coincides with the initial one. The fiber of $J^\infty O_X$ at a point $x$ is the algebra $A = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, where $n = \frac{1}{2} \dim(X)$ with the Poisson bracket given by $\{x_i, x_j\} = 0$, $\{y_i, y_j\} = 0$ and $\{x_i, y_j\} = \delta_{ij}$. Note that $A$ is the completion of the stalk $O_x$ with respect to the maximal ideal.

Analogously we can define the holomorphic jet bundle $J^\infty H_X$. The fiber of $J^\infty H_X$ will be a completion of the stalk $H_x$ with respect to the maximal ideal. It is a standard fact that this completion is the same algebra $A$.

Now let $\mathcal{D}$ be a quantization of $X$. We can define $J^\infty \mathcal{D}$ in the following way. We have the quotient map $O_X \otimes \mathcal{D} \to O_X \otimes O_X$. Set $I_\Delta$ to be the inverse image of $I_\Delta$. We define $(O_X \otimes \mathcal{D})^\wedge = \varprojlim_{k \to \infty} (O_X \otimes \mathcal{D})/I_\Delta^k$ and $J^\infty \mathcal{D} = \pi_1^*((O_X \otimes \mathcal{D})^\wedge)$.

**Definition 3.1.** A quantum jet bundle is a triple $(\mathcal{D}, \nabla, \Theta)$, where:

- $\mathcal{D}$ is a pro-coherent sheaf (i.e. limit of coherent sheaves) of $O_X[[h]]$-algebras such that $[\mathcal{D}, \mathcal{D}] \subset h^2 \mathcal{D}$;
- $\nabla$ is a flat $\mathbb{C}[[h]]$-linear connection on $\mathcal{D}$ such that $\nabla_\xi$ is a derivation of $\mathcal{D}$ for all $\xi \in \text{Vect}(X)$;
- $\Theta$ is an isomorphism $\mathcal{D}/h \mathcal{D} \to J^\infty O_X$ of sheaves of Poisson algebras that intertwines the connections.

A morphism of two quantum jet bundles $(\mathcal{D}_1, \nabla_1, \Theta_1)$ and $(\mathcal{D}_2, \nabla, \Theta)$ is a morphism $\Phi : \mathcal{D}_1 \to \mathcal{D}_2$ of sheaves of $O_X[[h]]$-algebras intertwining the connections, and such that $\Theta_1 = \Theta_2 \circ \Phi$. By Nakayama’s lemma any morphism of two quantum jet bundles is an isomorphism.

One can check that for any quantization $\mathcal{D}$ the corresponding jet bundle $J^\infty \mathcal{D}$ is a quantum jet bundle. Moreover, there is the following proposition.

**Proposition 3.2.** [BK04, Lemma 3.4] The assignments

- $\mathcal{D} \to J^\infty \mathcal{D}$,
- $\mathcal{D} \to \mathcal{D}^{\nabla}$

define mutually quasi-inverse equivalences between the category of quantizations of $X$ and the category of quantum jet bundles on $X$.

3.3. Hamiltonian quantum jet bundles. We are interested in the quantum jet bundles corresponding to Hamiltonian quantizations. We say that a quantum jet bundle $(\mathcal{D}, \nabla, \Theta)$ is $G$-equivariant if $\mathcal{D}$ is equipped with an action of $G$ such that $\nabla$ and $h$ are $G$-invariant and $\Theta : \mathcal{D}/h \mathcal{D} \to J^\infty O_X$ is $G$-equivariant. One can check that assignments from Proposition 3.2 define quasi-inverse equivalences between the category of $G$-equivariant quantizations of $X$ and the category of $G$-equivariant quantum jet bundles on $X$.

Now suppose that $X$ is endowed with a moment map $\mu : X \to g^*$. For $\xi \in g$ we set $\Phi(\xi) = \pi_2^*(\mu(\xi)) \in \Gamma(X, J^\infty O_X)^\nabla$. We have the left, right and diagonal actions of $G$ on $X \times X$ which give the maps $\xi \to \xi_x x, \xi_x x : x \mapsto \xi_x x + \xi_x \cdot \xi_x' \in g \to \text{Der}(O_{X \times X})$. The left action on the connection on $O_{X \times X}$, the right one comes from the moment map. Therefore we have $\xi_{J^\infty O_X} = \nabla_{\xi x} + \{\mu^*(\xi), \cdot\}$. We say that a $G$-equivariant quantum jet bundle $(\mathcal{D}, \nabla, \Theta)$ is Hamiltonian if it is
equipped with a map \( \Phi_h : g \to \Gamma(X, \mathcal{D}) \) such that \( \xi_\mathcal{D} = \bar{\nabla}_\xi + \frac{1}{\hbar}[\Phi_h(\xi), \bullet] \) and \( \Theta(\Phi_h(\xi)) = \Phi(\xi) \). By Proposition 3.2 they correspond to Hamiltonian quantizations.

**Lemma 3.3.** [Los10b, Proposition 3.4] Suppose that \( H^1_{DR}(X) = 0 \). Then every \( G \)-equivariant quantization \( \mathcal{D} \) of \( X \) admits a unique quantum comoment map \( \phi_h : g \to \Gamma(X, \mathcal{D}) \).

Note that for the homogeneous space \( X = \mathcal{O} \) of the group \( G \) we have \( H^1_{DR}(X) = 0 \). Therefore any \( G \)-equivariant quantization jet bundle \( (\mathcal{D}, \bar{\nabla}, \Theta) \) on \( \mathcal{O} \) admits a unique quantum comoment map \( \Phi_h : g \to \Gamma(X, \mathcal{D}) \).

The map \( \Phi_h \) extends to an algebra homomorphism \( U_h(g) \to \Gamma(X, \mathcal{D}) \). By \( \mathcal{O}_X \)-linearity we extend it to a homomorphism of sheaves \( \mathcal{O}_X \otimes U_h(g) \to \mathcal{D} \). We have a map \( (\mathcal{O}_X \otimes U_h(g))/h(\mathcal{O}_X \otimes U_h(g)) \simeq \mathcal{O}_X \otimes S(g) \to \mathcal{O}_X \otimes \mathbb{C}[X] \), where the last map is induced by \( \mu^* : g \to \mathbb{C}[X] \). Let \( \bar{I}_{\mu, \Delta} \subset \mathcal{O}_X \otimes U_h(g) \) be the inverse image of \( I_{\Delta} \subset \mathcal{O}_X \otimes \mathbb{C}[X] \). We set \( J^\infty U_h = \lim_{\Delta \to \infty} \mathcal{O}_X \otimes U_h(g)/\bar{I}_{\mu, \Delta} \).

It is endowed with a natural flat connection \( \bar{\nabla}_U \). The quantum comoment map \( \Phi_h \) extends to a continuous homomorphism \( \Phi_h : J^\infty U_h \to \mathcal{D} \) intertwining the connections. This map plays a crucial role in the classification of \( G \)-equivariant quantizations of \( \mathcal{O} \).

### 3.4. Finite W-algebra

In this section we will give a definition of the \( W \)-algebra \( W_h \) following [Los10b].

Recall from Example 2.1 that \( U_h(g) \) is a formal quantization of \( S(g) \), so \( U_h(g)/(h) \simeq S(g) \). Let \( m_\chi \subset S(g) \simeq \mathbb{C}[g^*] \) be the maximal ideal corresponding to the point \( \chi \) and \( \bar{m}_\chi \) be its preimage in \( U_h(g) \). We have the completion of \( U_h(g) \) in the point \( \chi \) defined by \( U_h^\wedge = \lim_{k \to \infty} U_h(g)/\bar{m}_\chi^k \). Recall that for a finite-dimensional vector space \( V \) with a symplectic form \( \omega \) we have the homogeneous \( \text{Weyl algebra} A_h(V) := T(V)[[h]]/(h^2(0) \subset \mathbb{C}[V]) \).

We denote the completion of \( A_h(V) \) with respect to the ideal \( A_h^+ \) by \( A_h^+(V) \).

We have a natural action of the centralizer \( Z_G(\text{e}) \subset G \) on \( U_h(g) \) that fixes the ideal \( \bar{m}_\chi \) and therefore extends to \( U_h^\wedge \). Set \( Q \) to be the reductive part of \( Z_G(\text{e}) \). We have the following decomposition due to Losev [Los10b, Theorem 3.3.1].

**Proposition 3.4.** There exists a \( \mathbb{C}[[h]] \)-algebra \( W_h \) with an action of \( Q \times \mathbb{C}^X \) and a \( Q \times \mathbb{C}^X \)-invariant isomorphism \( U_h^\wedge \simeq A_h^+(V)[[h]]W_h \) for the tangent space \( V = T_{\chi}\mathcal{O} \) with a symplectic action of \( Q \).

We denote the action of \( \mathbb{C}^X \) on \( W_h \) by \( \gamma \). Let \( W_{h,\text{fin}} \) be the subalgebra of \( W_h \) spanned by the elements \( w \in W_h \) such that \( \gamma(t)w = t^i w \) for some \( i \). We set \( W = W_{h,\text{fin}}/(h-1) \).

For an ideal \( J \subset \mathfrak{d}(W) \) let \( J_h = \bigoplus_{k \geq 0} J \cap W_{\leq k}h^k \) be the corresponding ideal in the Rees algebra \( W_{h,\text{fin}} \). We set \( J_h^\wedge \subset W_h \) to be the completion of \( J_h \).

We have the following important criteria of an ideal in \( W \) to be of codimension 1.

**Proposition 3.5.** [Los10a, Proposition 3.4] For an ideal \( J \subset \mathfrak{d}(W) \) we set \( I_h^\wedge = A_h^+(V)[[h]]J_h^\wedge \). Then \( J \) is of codimension 1 if and only if \( (U_h^\wedge/I_h^\wedge)/h(U_h^\wedge/I_h^\wedge) \) coincides with the completion \( \mathbb{C}[[\mathcal{O}]^\wedge \chi \simeq \mathbb{C}[[\mathcal{O}] \chi \) at \( \chi \).

### 3.5. Ideals in W-algebra vs quantization of orbits

Let us apply the techniques discussed above to study the set of quantizations of \( \mathcal{O} = G\chi \). Let \( H = G\chi \) be the stabilizer of \( \chi \) and \( \Gamma = H/H^0 \) its component group.

Recall from Section 3.3 that for any graded \( G \)-equivariant quantization \( \mathcal{D} \) of \( \mathcal{O} \) we have a quantum jet bundle \( \mathcal{D} := J^\infty \mathcal{D} \) and the quantum comoment map of quantum jet bundles \( \Phi_h : J^\infty U_h \to \mathcal{D} \).

**Lemma 3.6.** [Los10a, Lemma 5.2] The morphism \( \Phi_h \) is surjective.

We will need the classification of ideals of \( J^\infty U_h \). We say that an ideal \( I \) of a flat \( \mathbb{C}[[h]] \)-algebra \( A \) is \( h \)-saturated if \( ha \in I \) implies \( a \in I \). Equivalently, \( I \) is \( h \)-saturated if \( A/I \) is a flat \( \mathbb{C}[[h]] \)-algebra. We have the following description of the set of homogeneous \( G \)-stable \( h \)-saturated ideals in \( J^\infty U_h \).
Lemma 3.7. [Los10a, Lemma 5.1] (1) The fiber of $J^\infty U_h$ at $\chi$ is naturally identified with $U_h^\wedge$.
(2) Let $\mathfrak{J}(J^\infty U_h)$ be the set of homogeneous $G$-stable $h$-saturated ideals in $J^\infty U_h$. Taking the fiber of an ideal at $\chi$ defines a bijection between $\mathfrak{J}(J^\infty U_h)$ and $\mathfrak{J}(U_h^\wedge)^G$. The inverse map is given by $I_h^\wedge \rightarrow \pi^*(\mathcal{O}_G \otimes I_h^\wedge)^H$, where $\pi$ stands for the projection $G \rightarrow G/H = X$.
(3) Any element of $\mathfrak{J}(J^\infty U_h)$ is stable with respect to the connection $\nabla_U$.

Following [Los10a], we can prove the following theorem.

Theorem 3.8. [Los10a, Theorem 1.1] There is a natural bijection between the set of $G$-equivariant quantizations of $\mathcal{O}$ and the set $\mathfrak{J}^1(W)^G$.

Proof. Let $\mathcal{D}$ be a $G$-equivariant quantization of $\mathcal{O}$, $\mathcal{D}$ be the corresponding quantum jet bundle, and $\mathfrak{J}$ be the kernel of the map $\Phi_h$. Since $\mathcal{D}$ is $G$-equivariant, flat, and with a $\mathbb{C}^\times$-action, $\mathfrak{J}$ is a homogeneous $G$-stable $h$-saturated ideal of $J^\infty U_h$. Let $I_h^\wedge \in \mathfrak{J}(U_h^\wedge)^G$ be the fiber of $\mathfrak{J}$ at $\chi$.

Note that $\Theta$ induces an isomorphism $\Theta : \mathfrak{J} \otimes h \mathcal{D}_Y \simeq J^\infty \mathcal{O}_{X,\chi} \simeq \mathbb{C}[X]_\chi$ on stalks of the point $\chi$. But $\mathcal{D}_Y \simeq U_h^\wedge / I_h^\wedge$, so we have $(U_h^\wedge / I_h^\wedge)/h(U_h^\wedge / I_h^\wedge) \simeq \mathbb{C}[X]_\chi$. The ideal $\mathfrak{J} = (I_h^\wedge \cap \mathcal{W})_{fin}/(h - 1)$ lies in $\mathfrak{J}^1(W)$ by Proposition 3.5 and is $\Gamma$-invariant by construction.

In the opposite direction, for an ideal $\mathcal{J} \in \mathfrak{J}(W)$ let $\mathcal{J}_h^\wedge$ be the corresponding ideal in the Rees completion $\mathcal{W}_h$. We denote the ideal $\mathfrak{J}_h^\wedge \otimes \mathcal{J}_h^\wedge \subset U_h^\wedge$ by $I_h^\wedge$. Set $\mathfrak{J} = \pi^*(\mathcal{O}_G \otimes I_h^\wedge)^H$ and $\mathcal{D}_h = J^\infty U_h / \mathfrak{J}$. Since $\mathcal{J}$ is stable with respect to the connection $\nabla_U$, there is an induced connection $\nabla$ on $\mathcal{D}_h$. The map $\theta : \mathcal{D}_h / h \mathcal{D}_h \simeq (U_h^\wedge / I_h^\wedge)/h(U_h^\wedge / I_h^\wedge) \simeq \mathbb{C}[X]_\chi$ gives rise to the isomorphism $\Theta : \mathcal{D}_h / h \mathcal{D}_h \rightarrow J^\infty \mathcal{O}_X$.

One can easily check that the maps $\mathcal{D} \rightarrow \mathcal{D}_h$ and $\mathcal{J} \rightarrow \mathcal{D}_h$ are mutually inverse. Combining this result with Proposition 3.2 we get the theorem.

Let $\mathcal{D}$ be a $G$-equivariant quantization of $\mathcal{O}$ and $I_h^\wedge$, $I_h$ and $I$ be the corresponding ideals of $U_h^\wedge$, $U_h$ and $U(\mathfrak{g})$ respectively. The algebra $U_h/I_h$ is a formal quantization of $\text{gr}[U(\mathfrak{g})]/I$. The affine Poisson scheme $Y = \text{Spec}(\text{gr}[U(\mathfrak{g})]/I)$ is generically reduced, and its reduced scheme is $\mathfrak{Y}$. We can microlocalize $U_h/I_h$ to a formal quantization of $Y$. Restricting to the open subset $\mathcal{O} \subset Y$ we get a quantization $\mathcal{D}'$ of $\mathcal{O}$. Note that the stalks of $J^\infty \mathcal{D}$ and $J^\infty \mathcal{D}'$ at the point $\chi \in \mathcal{O}$ are both equal to $(U_h^\wedge / I_h^\wedge)$. Therefore we have proved the following lemma.

Lemma 3.9. The quantizations $\mathcal{D}'$ and $\mathcal{D}$ coincide. Therefore $\mathcal{D}$ extends to a quantization of $Y$.

Let us explain the structure of the proof of [Theorem 1.1]. The affine Poisson scheme $Y$ may not be normal or even reduced. Therefore we want to lift quantizations to a normal scheme $\text{Spec}(\mathbb{C}[\mathcal{O}])$. By Proposition 5.1 it is enough to lift quantizations to symplectic leaves of codimension less then 4. The smooth parts of both varieties coincide, so we need to look locally at points on codimension 2 symplectic leaves. To lift a quantization it is more convenient to work in the complex-analytic topology. In Theorem 4.10 we microlocalize a quantization of $Y$ to a holomorphic quantization of $Y$. In Section 5 we construct a holomorphic quantization of $\text{Spec}(\mathbb{C}[\mathcal{O}])$ corresponding to one of $Y$ and consider the corresponding algebraic quantization. In Section 5.3 we show that the main theorem follows from the constructions discussed above.

4. Analytification of Quantization

4.1. GAGA. Let $I$ be a two-sided ideal of $U(\mathfrak{g})$. Then $U(\mathfrak{g})/I$ is a filtered quantization of $\text{gr}[U(\mathfrak{g})]/I$. Consider the corresponding formal quantization of $\text{gr}[U(\mathfrak{g})]/I$ and microlocalize it to a formal quantization $\mathcal{D}_I$ of $\text{Spec}(\text{gr}[U(\mathfrak{g})]/I)$. The resulting sheaf $\mathcal{D}_I$ is a quotient of the quantization $\mathcal{D}$ of $\mathfrak{g}^*$ from Example 2.4. We want to establish a connection between modules over $\mathcal{D}$ and $\mathcal{D}_{hol}$ similar to the connection between coherent sheaves and coherent analytic sheaves. First, we need to recall this connection from [Ser56].
Let $X$ be an algebraic scheme over $\mathbb{C}$. The set of closed points of $X$ can be endowed with the structure of an analytic space $X_{hol}$. The natural embedding $i : X_{hol} \to X$ is continuous, where we consider Zariski topology on $X$ and analytic topology on $X_{hol}$. For a closed point $x \in X$ we have a local ring $\mathcal{O}_x$ with maximal ideal $m_x$. We define the completion $\hat{\mathcal{O}}_x$ as a limit $\lim_{k \to \infty} \mathcal{O}_x/m^k_x$. Analogously, for a point $x \in X_{hol}$ we consider the analogous completion $\hat{\mathcal{H}}_x = \lim_{k \to \infty} \mathcal{H}_x/m^k_{x,hol}$. Since every regular function in the neighborhood of $x$ gives a germ of holomorphic function at $x$ we have the embedding $\theta_x : \mathcal{O}_x \to \mathcal{H}_x$ on stalks that extends to a map $\hat{\theta}_x : \hat{\mathcal{O}}_x \to \hat{\mathcal{H}}_x$.

**Proposition 4.1.** ([Ser56, Proposition 3]) The homomorphism $\hat{\theta}_x \colon \hat{\mathcal{O}}_x \to \hat{\mathcal{H}}_x$ is an isomorphism.

**Corollary 4.2.** The algebra $\mathcal{H}_x$ is faithfully flat over $\mathcal{O}_x$.

For a coherent algebraic sheaf $\mathcal{F}$ on $X$ we define a holomorphic sheaf $\mathcal{F}_{hol}$ on $X_{hol}$ by $\mathcal{F}_{hol} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}_x$.

**Remark 4.3.** In fact, $\mathcal{F}$ and $\mathcal{O}_X$ are sheaves on $X$, not $X_{hol}$, so we have to replace them by their inverse images $i^{-1}\mathcal{F}$ and $i^{-1}\mathcal{O}_X$. In what follows we will omit such details.

**Corollary 4.4.** (i) The functor $\bullet_{hol}$ is faithfully flat;

(ii) The functor $\bullet_{hol}$ sends coherent algebraic sheaves to coherent holomorphic sheaves.

Now let $Y \subset X$ be a subscheme. Then we have the corresponding ideal sheaf $I_Y \subset \mathcal{O}_X$. The following is a well-known fact.

**Proposition 4.5.** The sheaf $I_{Y,hol} \subset \mathcal{H}_X$ is the ideal sheaf corresponding to the analytic subset $Y$.

### 4.2. Analytification of modules over quantization.

We want to establish the same connection for modules over quantizations. Recall the sheaves $\mathcal{D}$ and $\mathcal{D}_{hol}$ from [Example 2.4]. Since every algebraic differential operator is a holomorphic one, we have a natural map $\mathcal{D}_{G,h} \to \mathcal{D}_{G,hol,h}$ that induces a map $i : \mathcal{D} \to \mathcal{D}_{hol}$. Let $x \in \mathfrak{g}^*$ be any point and $\mathcal{D}_{hol,x}, D_x$ be the corresponding stalks. We have the isomorphisms $\mathcal{D}_{hol,x}/h\mathcal{D}_{hol,x} \simeq \mathcal{H}_x$ and $\mathcal{D}_x/h\mathcal{D}_x \simeq \mathcal{O}_x$. Let $m_{hol,x}$ and $m_x$ be the inverse images of the maximal ideals, and $D_{hol,x}$ and $D_x$ be the corresponding completions. The embedding $i$ extends to a map $\hat{i} : \hat{\mathcal{D}}_x \to \hat{\mathcal{D}}_{hol,x}$.

**Lemma 4.6.** The map $\hat{i}$ is an isomorphism.

**Proof.** First, note that the induced by $\hat{i}$ map $\hat{\mathcal{D}}_x/h\hat{\mathcal{D}}_x \to \hat{\mathcal{D}}_{hol,x}/h\hat{\mathcal{D}}_{hol,x}$ coincide with the map $\hat{\theta}$ from [Corollary 4.2]. The algebra $\mathcal{D}_{hol,x}$ is $\mathbb{C}[h]$-flat by construction. [ES10, Lemma A.2] implies that $\mathcal{D}_{hol,x}$ is $\mathbb{C}[[h]]$-flat. So $\hat{i}$ is a map from the complete and separated $\mathbb{C}[[h]]$-module $\hat{\mathcal{D}}_x$ to the flat $\mathbb{C}[[h]]$-module $\hat{\mathcal{D}}_{hol,x}$, and $\hat{i}$ induces an isomorphism on quotients by ($h$). Standard argument implies that $\hat{i}$ is an isomorphism.

A module $M$ over a sheaf of algebras $\mathcal{F}$ on $X$ is called of finite type if for every point $x \in X$ there is an open neighborhood $U \ni x$ such that $M_U$ admits a surjective morphism from a free finitely generated $\mathcal{F}_U$-module $\mathcal{F}_U^{\oplus n}$. A module $M$ is called coherent if it is of finite type and for any open subset $U$ and any surjective map $\phi : \mathcal{F}_U^{\oplus n} \to M_U$ the kernel of $\phi$ is of finite type.

We define the functor $\bullet_{hol}$ from the category of right coherent $\mathcal{D}$-modules to the category of right $\mathcal{D}_{hol}$-modules by $\mathcal{F}_{hol} = \mathcal{F} \otimes \mathcal{D}_{hol}$ (in the sense of [Remark 4.3]). Note that this functor sends $\mathcal{D}$ to $\mathcal{D}_{hol}$.

**Proposition 4.7.** The functor $\bullet_{hol}$ is faithfully flat and sends coherent $\mathcal{D}$-modules to coherent $\mathcal{D}_{hol}$-modules.
Proof. Analogously to Corollary 4.4, it is enough to show that $\mathcal{D}_{\text{hol},x}$ is faithfully flat over $\mathcal{D}_x$. By Lemma 4.6, for any $\mathcal{D}_x$-module $\mathcal{M}_x$ we have an isomorphism $\mathcal{M}_x \otimes_{\mathcal{D}_x} \mathcal{D}_x \cong \mathcal{M}_x \otimes_{\mathcal{D}_x} \mathcal{D}_{\text{hol},x} \otimes_{\mathcal{D}_{\text{hol},x}} \mathcal{D}_{\text{hol},x}$. Since the completion functor is faithfully flat, both $\bullet \otimes_{\mathcal{D}_x} \mathcal{D}_x$ and $\bullet \otimes_{\mathcal{D}_{\text{hol},x}} \mathcal{D}_{\text{hol},x}$ are faithfully flat. The proposition follows.

Let $\mathcal{D}$-mod and $\mathcal{D}_{\text{hol}}$-mod be the categories of coherent $\mathcal{D}$ and $\mathcal{D}_{\text{hol}}$-modules correspondingly. Taking the quotient by the ideal $(h)$ gives functors $\mathcal{D}$-mod $\rightarrow$ Coh$(X)$ and $\mathcal{D}_{\text{hol}}$-mod $\rightarrow$ Coh$_{\text{hol}}$(X) to the categories of coherent and coherent holomorphic sheaves correspondingly.

Proposition 4.8. The following diagram is commutative.

\[
\begin{array}{ccc}
\mathcal{D}$-mod & \overset{\bullet}{\rightarrow} & \mathcal{D}_{\text{hol}}$-mod \\
\downarrow / (h) & & \downarrow / (h) \\
\text{Coh}(X) & \overset{\bullet}{\rightarrow} & \text{Coh}_{\text{hol}}(X)
\end{array}
\]

Proof. Let $\mathcal{M}$ be a coherent $\mathcal{D}$-module.

Since $\mathcal{H}_X \cong \mathcal{D}_{\text{hol}} / h \mathcal{D}_{\text{hol}}$, we have $\mathcal{M}_{\text{hol}} / h \mathcal{M}_{\text{hol}} = \mathcal{M}_{\text{hol}} \otimes_{\mathcal{D}_{\text{hol}}} \mathcal{H}_X = (\mathcal{M} \otimes_{\mathcal{D}} \mathcal{D}_{\text{hol}}) \otimes_{\mathcal{D}_{\text{hol}}} \mathcal{H}_X = \mathcal{M} \otimes_{\mathcal{D}} \mathcal{H}_X$. Analogously, $(\mathcal{M} / h \mathcal{M})_{\text{hol}} = (\mathcal{M} \otimes_{\mathcal{D}} \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{H}_X = \mathcal{M} \otimes_{\mathcal{D}} \mathcal{H}_X$. 

4.3. **Analytification of $\mathcal{J}$**. Let $\mathcal{D}'_\mathcal{O}$ be a $G$-equivariant Hamiltonian quantization of $\mathcal{O}$ and $I$ be the corresponding ideal of $U(\mathfrak{g})$. Let $I_h \subset U_h$ be the corresponding ideal. The algebra $U_h / I_h$ is a quantization of $\text{gr}(U(\mathfrak{g}) / I)$, and let $\mathcal{D}'$ be the corresponding quantization of an affine scheme $X = \text{Spec} \mathcal{O}(\mathfrak{g}^* / I')$. From Section 3 we have a surjective map $J^\infty U_h \rightarrow J^\infty \mathcal{D}'_\mathcal{O}$ with the kernel $\mathcal{J}$. Let $I_h^\chi$ be the kernel of a corresponding map $U_h^\chi \cong J^\infty U_{h,\chi} \rightarrow J^\infty \mathcal{D}'_\chi$ on stalks of the point $\chi$ and $I_h = I_{h,\text{fin}}$ be the corresponding ideal in $U_h = \Gamma(\mathfrak{g}^*, \mathcal{D})$. We set $\mathcal{J} \subset \mathcal{D}$ to be the two-sided ideal generated by $I_h$. Since $I_h$ is a finitely presented $U_h$-module, we have $\mathcal{J} \subset \mathcal{D}$-mod.

Note that $\Gamma(\mathcal{D} / \mathcal{J}) = U_h / I_h$, so restricting to the affine scheme $X$ the quantization $\mathcal{D} / \mathcal{J}$ is a microlocalization of $U_h / I_h$, i.e. $\mathcal{D}'$. The discussion above implies the following proposition.

Proposition 4.9. Let $i$ be the natural embedding of $X$ in $\mathfrak{g}^*$. Then we have the following isomorphism $\mathcal{D} / \mathcal{J} \cong i_*(\mathcal{D}')$.

The main theorem of this section is as follows.

Theorem 4.10. The sheaf $\mathcal{J}_{\text{hol}}$ is an $h$-saturated two-sided ideal in $\mathcal{D}_{\text{hol}}$. Moreover, the quotient $\mathcal{D}_{\text{hol}} / \mathcal{J}_{\text{hol}}$ is a holomorphic quantization of $X$ as an analytic subspace.

Let us check the first part of the theorem. We have the following more specific statement.

Proposition 4.11. The sheaf $\mathcal{J}_{\text{hol}}$ is a two-sided ideal in $\mathcal{D}_{\text{hol}}$ generated by $\mathcal{J}_x$.

Proof. From the construction of the functor $\bullet_{\text{hol}}$ the sheaf $\mathcal{J}_{\text{hol}}$ is a right ideal in $\mathcal{D}_{\text{hol}}$ generated by $\mathcal{J}_x$. We have to check that for every point $x \in \mathfrak{g}^*$ the stalk $\mathcal{J}_{\text{hol},x}$ is a two-sided ideal.

By Lemma 4.6, $\mathcal{J}_x \cong \mathcal{J}_{\text{hol},x}$. We set $\mathcal{I}_x = \mathcal{J}_x \cap \mathcal{D}_{\text{hol},x}$. Note that $\mathcal{J}_{\text{hol},x} \subset \mathcal{I}_x$ and $\mathcal{I}_x$ is a two-sided ideal. From the universal property of completions the embedding map $i : \mathcal{I}_x \rightarrow \hat{\mathcal{I}}_x$ factors through $f : \hat{\mathcal{I}}_x \rightarrow \hat{\mathcal{J}}_x$. Note that $f$ is injective. Therefore $\hat{\mathcal{J}}_{\text{hol},x} \subset \hat{\mathcal{I}}_x \subset \hat{\mathcal{J}}_x$, so $\hat{\mathcal{I}}_x = \hat{\mathcal{J}}_x = \hat{\mathcal{J}}_{\text{hol},x}$.

Let $g : \hat{\mathcal{J}}_{\text{hol},x} \rightarrow \hat{\mathcal{J}}_x$ be the natural embedding and set $C = \text{Coker}(g)$. The completion functor $\hat{\bullet} = \hat{\bullet} \otimes_{\mathcal{D}_{\text{hol},x}} \mathcal{D}_{\text{hol},x}$ is exact on finitely generated $\mathcal{D}_{\text{hol},x}$-modules, so $\hat{C} = \text{Coker}(\hat{g}) = 0$. From the faithfulness of the completion functor $C = 0$, so $\hat{\mathcal{I}}_x = \mathcal{J}_{\text{hol},x}$, and the latter one is 2-sided.

Proof of Theorem 4.10. We have to check that $\mathcal{J}_{\text{hol},x}$ is $h$-saturated for every $x \in \mathfrak{g}^*$. Recall that $\mathcal{J}$ and therefore $\mathcal{J}_x$ are $h$-saturated. Hence the intersection $\mathcal{I}_x = \mathcal{J}_x \cap \mathcal{D}_{\text{hol},x}$ is $h$-saturated. Indeed, if $hf \in \mathcal{J}_x \cap \mathcal{D}_{\text{hol},x}$ for some $f \in \hat{\mathcal{D}}_x$ then $f \in \mathcal{J}_x$. By [Mat80, p. 2.4.C] $h\hat{\mathcal{D}}_x \cap \mathcal{D}_{\text{hol},x} = h\mathcal{D}_{\text{hol},x}$, so
Let $f \in \mathcal{D}_{\text{hol},x}$. But we proved that $\mathcal{J}_{\text{hol},x} = \mathcal{I}_x$, so $\mathcal{J}_{\text{hol},x}$ is $h$-saturated. Let $I_X \subset \mathcal{O}_g^*$ be the ideal sheaf corresponding to the closed subscheme $X \subset g^*$. Since $\mathcal{D}'_{\text{hol}} = \mathcal{D}_{\text{hol}}/\mathcal{J}_{\text{hol}}$ is $\mathbb{C}[[h]]$-flat, the following diagram is commutative.

\[
\begin{array}{c}
\text{0} \\
\downarrow \\
\text{0} \\
\downarrow \\
0 \rightarrow h\mathcal{J}_{\text{hol}} \rightarrow h\mathcal{D}_{\text{hol}} \rightarrow h\mathcal{D}'_{\text{hol}} \rightarrow \text{0} \\
\downarrow \\
\downarrow \\
\downarrow \\
0 \rightarrow \mathcal{J}_{\text{hol}} \rightarrow \mathcal{D}_{\text{hol}} \rightarrow \mathcal{D}'_{\text{hol}} \rightarrow \text{0} \\
\downarrow \\
\downarrow \\
\downarrow \\
0 \rightarrow I_{\text{hol},X} \rightarrow \mathcal{H} \rightarrow i_*(\mathcal{H}_X) \rightarrow \text{0} \\
\downarrow \\
\text{0} \\
\downarrow \\
\text{0} \\
\end{array}
\]

Therefore $\mathcal{D}'_{\text{hol}}$ restricted to $X$ is a holomorphic quantization of $X$. Theorem 4.10 is proved.

\[\square\]

5. Lift to the normalization

5.1. Singularities in codimension 2. Let $X$ be the same as in Section 4. In this section we show that any holomorphic quantization $\mathcal{D}_{\text{hol}}$ of $X$ can be lifted to a holomorphic quantization $\mathcal{D}'_{\text{hol}}$ of the normalization $\text{Spec}(\mathbb{C}[\mathcal{O}])$. We set $\pi : \text{Spec}(\mathbb{C}[\mathcal{O}]) \rightarrow X$ to be the normalization map. We will denote $\text{Spec}(\mathbb{C}[\mathcal{O}])$ by $X$ and omit indexes $\text{hol}$. To avoid confusion we will denote sheaves of holomorphic functions on $X$ and $\tilde{X}$ by $\mathcal{H}$ and $\mathcal{H}$ correspondingly. Recall the following well-known fact that we will prove in the Appendix.

Proposition 5.1. Let $X$ be a normal Cohen-Macaulay affine Poisson variety with finitely many symplectic leaves. Let $X_2$ be the union of the smooth part $X^{\text{reg}}$ and all symplectic leaves of codimension 2. Then any (holomorphic) quantization of $X_2$ uniquely extends to a (holomorphic) quantization of $X$.

Therefore it is enough to lift a holomorphic quantization of $X$ to one of $\tilde{X}_2$. The smooth locus of $X$ is the orbit $\mathcal{O}$, so it is enough to construct a sheaf of $\mathbb{C}[[h]]$-algebras $\mathcal{D}$ on $\tilde{X}_2$ that extends the restriction $\mathcal{D}_\mathcal{O}$. Let $i : \mathcal{O} \rightarrow \tilde{X}$ be the natural embedding. We set $\tilde{\mathcal{D}} = i_*\mathcal{D}_\mathcal{O}$.

Proposition 5.2. [KP82, Theorem 2] Let $\mathcal{O}$ be a nilpotent coadjoint orbit in a simple classical Lie algebra and $\mathcal{O}'$ an open orbit in the boundary $\delta \mathcal{O} = \overline{\mathcal{O}} - \mathcal{O}$. If $\mathcal{O}'$ is of codimension 2 then the singularity of $\overline{\mathcal{O}}$ in $\mathcal{O}'$ is smoothly equivalent to an isolated surface singularity of type $A_k$, $D_k$ or $A_k \cup A_k$, where the last one is the non-normal union of two singularities of type $A_k$ meeting transversally in the singular point.

By two singularities meeting transversally in the point we mean the following. The Kleinian singularity $A_k$ is an affine scheme $\text{Spec}(A)$, where $A = \mathbb{C}[x,y,z]/(xy - z^{k+1})$. We have a maximal ideal $(x,y,z)$, corresponding to the singular point 0 \in Spec(A), and a quotient map $A \rightarrow \mathbb{C}$. Consider a map $A \oplus A \rightarrow \mathbb{C}$ and set $B$ to be its kernel. We define the singularity $A_k \cup A_k$ as $\text{Spec}(B)$.

Let $\mathcal{O}'$ be a codimension 2 orbit in $\overline{\mathcal{O}}$, and $S$ be the corresponding singularity. Consider a point $x \in \mathcal{O}' \subset X$. We say that the open neighborhood $V \subset X$ of $x$ is standard if $V^{\text{reg}} = U \times S/\{0\}$.
for some open $U \subset \mathcal{O}'$. Note that standard neighborhoods of $x$ form a base of open subsets of $X$ containing $x$.

If $S$ is of type $A_k$ or $D_k$, the corresponding singularity in the normalization $\tilde{X}$ is equivalent to the one in $\mathcal{O}$. Consider a point $x \in \mathcal{O}' \subset X$, and set $y = \pi^{-1}(x)$. We have a standard open neighborhood $V \subset X$ of $x$ and set $Y = \pi^{-1}(V)$ to be an open neighborhood of $y$. Then $V = U \times S \subset \tilde{X}$, and $\mathcal{D}(V) = \mathcal{D}(V^{reg})$ by construction. We need to show that $\mathcal{D}_y/h\mathcal{D}_y \simeq \mathcal{H}_y$.

Assume that the singularity $S$ of the codimension $2$ orbit $\mathcal{O}' \subset \mathcal{O}$ is of type $A_k \cup A_k$. Let $\mathcal{O}'$ be the corresponding orbit of codimension $2$. For a point $x \in \mathcal{O}' \subset X$ we have a standard open neighborhood $V \subset X$. Then $V^{reg} = V_1 \cup V_2$, where $V_i = U \times (S_i \setminus \{0\}) \subset V^{reg}$, where $U \subset \mathcal{O}'$ is open, and $S_i$ is the singularity of type $A_k$. Let $x_1, x_2 \in \tilde{X}$ be the two preimages of $x$. We have $\pi^{-1}(V) = \tilde{V}_1 \cup \tilde{V}_2$, where $\tilde{V}_i = U \times S_i$ is an open neighborhood of $x_i$. By construction, $\mathcal{D}(\tilde{V}_i) = \mathcal{D}(V_i)$. We will show that $\mathcal{D}_{x_i}/h\mathcal{D}_{x_i} \simeq \mathcal{H}_{x_i}$. The proof for the singularity $S$ of type $A_k$ or $D_k$ can be obtained in the same way.

We have a natural map $\mathcal{D}(\tilde{V}_1 \cup \tilde{V}_2) \simeq \mathcal{D}(V_1 \cup V_2) \to \mathcal{H}(V_1 \cup V_2) \simeq \mathcal{H}(\pi^{-1}(V_1 \cup V_2))$ with the kernel $h\mathcal{D}(\tilde{V}_1 \cup \tilde{V}_2)$. Note that by the Hartogs extension theorem $\mathcal{H}(\pi^{-1}(V_1 \cup V_2)) \simeq \mathcal{H}(\tilde{V}_1 \cup \tilde{V}_2)$. Standard open neighborhoods $V^i$ of $x$ form a base of open subsets containing the point $x$. Therefore we can define stalks of sheaves at points $x, x_1, x_2$ as limits of open neighborhoods $V^i, \tilde{V}^i_1, \tilde{V}^i_2$ correspondingly. We have an induced map on stalks $\mathcal{D}_{x_1} \oplus \mathcal{D}_{x_2} \to \mathcal{H}_{x_1} \oplus \mathcal{H}_{x_2}$. It is enough to show that this map is surjective.

For every $V^i$ we have a natural map $\mathcal{D}(V^i) \to \mathcal{D}(V^i_1 \cup V^i_2) \simeq \mathcal{D}(V^{reg})$. Taking the limit as $i$ goes to $\infty$ we get a map $\mathcal{D}_x \to \mathcal{D}_{x_1} \oplus \mathcal{D}_{x_2}$. Analogously, we have maps $\mathcal{H}(V^i) \to \mathcal{H}(V^i_1 \cup V^i_2)$ and an isomorphism $\mathcal{H}(V^i_1 \cup V^i_2) \simeq \mathcal{H}(V^i_1 \cup V^i_2)$ from the Hartogs extension theorem. Taking the limit we get a map $\mathcal{H}_x \to \mathcal{H}_{x_1} \oplus \mathcal{H}_{x_2}$. We have the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{D}_x & \longrightarrow & \mathcal{D}_{x_1} \oplus \mathcal{D}_{x_2} \\
\downarrow & & \downarrow \\
\mathcal{H}_x & \longrightarrow & \mathcal{H}_{x_1} \oplus \mathcal{H}_{x_2} \\
\end{array}
$$

Here $\mathcal{H}_{x,\mathcal{O}'}$ stands for the ring of germs of analytic functions on $\mathcal{O}'$ at the point $x$. Note that the lower sequence is exact at the middle. Since $\mathcal{D}$ is a holomorphic quantization of $X$ the left vertical map is surjective. Therefore it is enough to show that the composition $\phi : \mathcal{D}_{x_1} \oplus \mathcal{D}_{x_2} \to \mathcal{H}_{x_1} \oplus \mathcal{H}_{x_2} \to \mathcal{H}_{x,\mathcal{O}'}$ is surjective. Let us choose an element $f \in \mathcal{H}_{x,\mathcal{O}'}$. We have a surjective map $\mathcal{H}_{x,X} \to \mathcal{H}_{x,\mathcal{O}'}$, and set $g$ to be a preimage of $f$. Consider a standard open neighborhood $V \subset X$ of $x$, such that there is a function $g \in \mathcal{D}(V)/h\mathcal{D}(V) \subset \mathcal{H}(V)$ giving the germ $g \in \mathcal{H}_{x,X}$. We can choose a lift $\bar{g} \in \mathcal{D}(V)$. Since $V_1 \subset V$ is an open subset, we have the restriction map $r : \mathcal{D}(V) \to \mathcal{D}(V_1) \simeq \mathcal{D}(\tilde{V}_1)$, and set $\bar{h} = r(\bar{g})$. Consider the germ of $\mathcal{D}_{x_1}$ that is represented by $\tilde{f}$. Then $\phi(\tilde{f}) = f$. So the right vertical arrow in Diagram 1 is surjective and $\mathcal{D}$ is a quantization of $\tilde{X}$.

5.2. From an analytic quantization to an algebraic one. Let $\mathcal{D}$ be an analytic quantization of $\text{Spec}(\mathbb{C}[\mathcal{O}])$ and $\mathcal{A} = \Gamma(\text{Spec}(\mathbb{C}[\mathcal{O}]), \mathcal{D})$ be the algebra of global sections. We have an action of $\mathbb{C}^\times$ on $\mathcal{A}$ and set $\mathcal{A}_{alg}$ to be the $h$-adic completion of $\mathcal{A}_{fin}$.

**Proposition 5.3.** $\mathcal{A}_{alg}$ is an algebraic formal quantization of $\mathbb{C}[\mathcal{O}]$.

**Proof.** $\mathcal{A}_{alg}$ is a complete and separated $\mathbb{C}[[h]]$-algebra with a $\mathbb{C}^\times$-action by algebra automorphisms induced from the $\mathbb{C}^\times$-action on $\mathcal{A}$. This action is rational on all quotient $\mathcal{A}_{alg}/(h^k)$, and $[\mathcal{A}_{alg}, \mathcal{A}_{alg}] \subset h\mathcal{A}_{alg}$. Therefore it is enough to show that $\mathcal{A}_{alg}/h\mathcal{A}_{alg} \simeq \mathbb{C}[\mathcal{O}]$.

$\text{Spec}(\mathbb{C}[\mathcal{O}])$ is an affine scheme and therefore a Stein analytic space. By [Car57], $H^1(\text{Spec}(\mathbb{C}[\mathcal{O}]), \mathcal{H}) = 0$. Therefore $\mathcal{A} = \Gamma(\text{Spec}(\mathbb{C}[\mathcal{O}]), \mathcal{D})$ is a formal quantization of $\Gamma(\text{Spec}(\mathbb{C}[\mathcal{O}]), \mathcal{H})$. Note that
Theorem 5.5. [PT14, Theorem 2] Let

Proof of Theorem 1.1. 5.3. Spec( )

Every \( f \) is a polynomial function, so \( f \) is a regular function. Therefore \( \Gamma(\text{Spec}(\mathbb{C}[O]), \mathcal{H})_{\text{fin}} = \mathbb{C}[O] \) and \( \mathcal{A}_{\text{alg}}/h\mathcal{A}_{\text{alg}} \simeq \mathcal{A}_{\text{fin}}/h\mathcal{A}_{\text{fin}} \). Therefore \( \mathcal{A}_{\text{fin}}/h\mathcal{A}_{\text{fin}} \simeq \mathcal{A}_{\text{fin}} \cap h\mathcal{A} \). Therefore \( \mathcal{A}_{\text{fin}}/h\mathcal{A}_{\text{fin}} \simeq (\mathcal{A}/h\mathcal{A})_{\text{fin}} = \Gamma(\text{Spec}(\mathbb{C}[O]), \mathcal{H})_{\text{fin}} \). For any regular function \( f \in \mathbb{C}[O] \) we have \( f \in \Gamma(\text{Spec}(\mathbb{C}[O]), \mathcal{H})_{\text{fin}} \). In the opposite direction, for any \( f \in \Gamma(\text{Spec}(\mathbb{C}[O]), \mathcal{H})_{\text{fin}} \) consider its germ at 0. It can be \( \mathbb{C}^\times \)-equivariantly lifted to a germ \( \tilde{f} \) of a function on \( \mathbb{C}^n \) at 0 for some \( n \). If \( f \) is \( \mathbb{C}^\times \)-finite then \( \tilde{f} \) is expressed by a polynomial function, so \( f \) is a regular function. Therefore \( \Gamma(\text{Spec}(\mathbb{C}[O]), \mathcal{H})_{\text{fin}} = \mathbb{C}[O] \) and \( \mathcal{A}_{\text{alg}}/h\mathcal{A}_{\text{alg}} \simeq \mathbb{C}[O] \).

We can microlocalize \( \mathcal{A}_{\text{alg}} \) to an algebraic formal quantization \( \mathcal{D}_{\text{alg}} \) of \( \text{Spec}(\mathbb{C}[O]) \). Theorem 4.10, Section 5.1 and Proposition 5.3 imply the following theorem.

Theorem 5.4. Every \( G \)-equivariant quantization of \( O \) can be uniquely extended to a \( G \)-equivariant quantization of \( \text{Spec}(\mathbb{C}[O]) \).

5.3. Proof of Theorem 1.1. Now we can proof the theorem stated in the introduction.

Theorem 5.5. [PT14, Theorem 2] Let \( \mathfrak{g} \) be classical. Then \( \mathfrak{z} \mathfrak{d}^1(W)^\Gamma \) is in bijection with the set of points of an affine space.

Proof. Suppose that \( \mathcal{D} \) is a graded formal quantization of \( \text{Spec}(\mathbb{C}[O]) \). By [Los16, Section 3.6] the action of \( G \) on the sheaf \( O \) uniquely extends to the action of \( G \) on \( \mathcal{D} \) such that \( \mathcal{D} \) becomes a \( G \)-equivariant quantization. Restricting to the open subset \( O \) we get a \( G \)-equivariant quantization \( \mathcal{D} \) of \( O \). By Theorem 5.3 it has a unique \( G \)-equivariant extension to \( \text{Spec}(\mathbb{C}[O]) \) that is \( \mathcal{D} \). Therefore the constructed maps \( \mathcal{D} \to \mathcal{D}, \mathcal{D} \to \mathcal{D} \) between the sets of \( G \)-equivariant quantizations of \( O \) and quantizations of \( \text{Spec}(\mathbb{C}[O]) \) are inverse to each other. By Proposition 2.6 the set of \( G \)-equivariant quantizations of \( \text{Spec}(\mathbb{C}[O]) \) is in bijection with the set of points on an affine space. From Theorem 3.8 the set of quantizations of \( O \) is in bijection with \( \mathfrak{z} \mathfrak{d}^1(W)^\Gamma \). The present theorem follows.

Remark 5.6. In fact, Premet and Topley considered an algebra \( U(\mathfrak{g}, e)^{ab} \) that is the quotient of \( \mathcal{W}_{ab} \) by the ideal \( I_\Gamma \) generated by all \( \phi - \phi^\gamma \) with \( \phi \in \mathcal{W}_{ab} \) and proved that \( U(\mathfrak{g}, e)^{ab} \) is a polynomial algebra. Then \( \mathfrak{z} \mathfrak{d}^1(W)^\Gamma \) is identified with the set of points of \( \text{Spec}(U(\mathfrak{g}, e)^{ab}) \) that is an affine space.

6. Appendix

In this section we give a proof of Proposition 5.1

Proposition 6.1. Let \( X_2 \) be a normal Cohen-Macaulay affine Poisson variety with finitely many symplectic leaves. Let \( X_2 \) be the union of the smooth part \( X_2^{\text{reg}} \) and all symplectic leaves of codimension 2. Then any (holomorphic) quantization of \( X_2 \) uniquely extends to a (holomorphic) quantization of \( X \).

Proof. That is a well-known fact for the algebraic case, we will prove the proposition for holomorphic quantizations. Let \( i : X_2 \to X \) be the natural embedding. Let \( \mathcal{D} \) be a holomorphic quantization of \( X_2 \). We will show that \( i_* \mathcal{D} \) is a holomorphic quantization of \( X \). Consider an open subset \( Y \subset X \). We set \( Y_2 = Y \cap X_2 \) and define \( A = \Gamma(Y, \mathcal{H}_X) \). We want to show that \( \Gamma(Y_2, \mathcal{D}) = \Gamma(Y, i_* \mathcal{D}) \) is a quantization of \( A \) for every Stein \( Y \). Then we have the short exact sequence \( 0 \to \Gamma(Y, i_* \mathcal{D}) \to \Gamma(Y, \mathcal{D}) \to \Gamma(Y, \mathcal{H}_X) \to 0 \) for all Stein open subsets \( Y \). Such subsets form a base of topology for \( X \), so it implies the short exact sequence of sheaves \( 0 \to h_i \mathcal{D} \to i_* \mathcal{D} \to \mathcal{H}_X \to 0 \), and therefore \( i_* \mathcal{D} \) is a quantization of \( X \). So it is enough to show that \( \Gamma(Y_2, \mathcal{D}) \) is a quantization of \( A \).

We have a short exact sequence \( 0 \to h \mathcal{D} \to \mathcal{D} \to \mathcal{H}_{Y_2} \to 0 \) of sheaves on \( Y_2 \). Applying the left exact functor \( \Gamma \), we get a long exact sequence of cohomology \( 0 \to \Gamma(Y_2, h \mathcal{D}) \to \Gamma(Y_2, \mathcal{D}) \to \Gamma(Y_2, \mathcal{H}_{Y_2}) \to H^1(Y_2, h \mathcal{D}) \to H^1(Y_2, \mathcal{D}) \to \ldots \). From the Hartogs extension theorem \( \Gamma(Y_2, \mathcal{H}_{Y_2}) \simeq A \). We will show that \( H^1(Y_2, \mathcal{H}_{Y_2}) = 0 \) and deduce that the map \( H^1(Y_2, h \mathcal{D}) \to H^1(Y_2, \mathcal{D}) \) is an isomorphism. First, we need the following lemma.
Lemma 6.2. Let $X$ be a Cohen-Macaulay algebraic scheme. Then $X_{hol}$ is a Cohen-Macaulay analytic space.

Proof. Consider a point $x \in X_{hol}$. We need to show that the local ring $\mathcal{H}_{X,x}$ is Cohen-Macaulay. Let $\hat{\mathcal{H}}_{X,x}$ be the completion of $\mathcal{H}_{X,x}$ with respect to the maximal ideal. By [Mat87, Theorem 17.5], $\mathcal{H}_{X,x}$ is Cohen-Macaulay if and only if $\hat{\mathcal{H}}_{X,x}$ is Cohen-Macaulay. The same argument shows that $\hat{\mathcal{O}}_{X,x}$ is Cohen-Macaulay. By Proposition 4.1 $\hat{\mathcal{H}}_{X,x} \simeq \hat{\mathcal{O}}_{X,x}$. Lemma follows. $\square$

Lemma 6.3. $H^1(Y_2, \mathcal{H}_{Y_2}) = 0$.

Proof. We set $Z = Y - Y_2$ to be the intersection of the union of all symplectic leaves of codimension greater or equal 4 with $Y$. We have a long exact sequence of sheaf cohomology

\[ \ldots \rightarrow H^1(Y, \mathcal{H}_Y) \rightarrow H^1(Y_2, \mathcal{H}_{Y_2}) \rightarrow H^2_Z(Y, \mathcal{H}_Y) \rightarrow \ldots. \]

$Y$ is a Stein analytic scheme, so $H^1(Y, \mathcal{H}_Y) = 0$. Analogously to [Har67, Theorem 3.8], $H^i_Z(Y, \mathcal{H}_Y) = 0$ for all $i < \text{depth}_Z(\mathcal{H}_Y)$. By Lemma 6.2, $X_{hol}$ is Cohen-Macaulay, so $\text{depth}_Z(\mathcal{H}_Y) = \text{codim } Z$ and $H^2_Z(Y, \mathcal{H}_Y) = 0$. The first and third terms in (1) are zero, so $H^1(Y_2, \mathcal{H}_{Y_2}) = 0$. $\square$

We have a short exact sequence $0 \rightarrow hD \rightarrow D \rightarrow \mathcal{H}_{Y_2} \rightarrow 0$ on $Y_2$ that induces the long exact sequence of cohomology

\[ 0 \rightarrow \Gamma(Y_2, hD) \rightarrow \Gamma(Y_2, D) \rightarrow A \rightarrow H^1(Y_2, hD) \rightarrow H^1(Y_2, D) \rightarrow H^1(Y_2, \mathcal{H}_{Y_2}) \]

The last term in (2) is trivial, so we have a surjective map $H^1(Y_2, hD) \rightarrow H^1(Y_2, D)$.

Lemma 6.4. Let $D$ be a sheaf of complete and separated $\mathbb{C}[[\hbar]]$-algebras on a compact analytic space $X$. Suppose that the natural map $f : H^d(X, hD) \rightarrow H^d(X, D)$ is surjective for some $d$. Then $f$ is an isomorphism.

Proof. We use an argument from [GL14, Lemma 5.6.3]. Since $X$ is compact, we can compute sheaf cohomology by Cech complex. Consider a Leray cover $\{U_i\}$ of $X$. Let $C^i$, $Z^i$ and $B^i$ be the cochains, cocycles and coboundaries for the Cech cohomology of $D$ with respect to $\{U_i\}$. The natural map $f : hZ^d/hB^d = \check{H}^d(X, hD) \rightarrow \check{H}^d(X, D) = Z^d/B^d$ is surjective, so $hZ^d + B^d = Z^d$. We want to show that $hZ^d \cap B^d = hB^d$.

Consider an element $x = hz_0 \in hZ^d \cap B^d$. We can decompose $z_0$ into a sum $hz_1 + b_1$ for some $z_1 \in Z^d$ and $b_1 \in B^d$. Then $x = h^2z_1 + h^2b_1$, and we set $x_1 = hb_1$. Analogously, we have a decomposition $z_1 = hz_2 + b_2$, and then $x = h^3z_2 + h^2b_2 + hb_1$. We set $x_2 = hb_2 + hb_1$. Applying the same procedure we get a sequence $x_k$, such that $x = x_k + h^kz_k$ and $x_k \in hB^d$. Since $Z^d$ is complete and separated, this sequence converges to $x$. Therefore $x \in hB^d$, and $hZ^d \cap B^d = hB^d$.

Then $h(Z^d/B^d) = hZ^d/(B^d \cap hZ^d) = hZ^d/hB^d$, and the natural embedding $h(Z^d/B^d) \rightarrow Z^d/B^d$ is surjective. Therefore $f$ is an isomorphism. $\square$

Applying the lemma above to (2) we get that the map $H^1(Y_2, hD) \rightarrow H^2(Y_2, D)$ is an isomorphism, and $\Gamma(Y_2, D)$ is a quantization of $A$. $\square$

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