The space of volume forms

Xiuxiong Chen; Weiyong He

Abstract

Donaldson introduced an interesting geometric structure (the Donaldson metric) on the space of volume forms for any compact Riemannian manifold, which has nonpositive sectional curvature formally. The geodesic equation and its perturbed equations are fully nonlinear elliptic equations. These equations are also equivalent to two free boundary problems of the Laplacian equation and it also has relationship with many interesting problems, such as Nahm’s equation. In this paper we solve these equations and demonstrate the geometric structure of the space of volume forms; in particular, we show that the space of volume forms with the Donaldson metric is a metric space with non-positive curvature in Alexanderov sense.

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1 Introduction

On any compact Kähler manifold, Mabuchi [20], Semmes [21], and Donaldson [8] introduced a Weil-Peterson type metric in the space of Kähler metrics and proved that it is a formally non-positively curved symmetric
space of “non-compact” type. According to [21], the geodesic equation can be transformed into a homogenous complex Monge-Ampere equation. In [8], Donaldson proposed an ambitious program relating the geometry of this infinite dimensional space to the core problems in Kähler geometry, such as the uniqueness and the existence problems for constant scalar curvature Kähler metrics and its relation to the stability of the underlying polarization. In [5], the first author solved the geodesic equation in $C^{1,1}$ sense with intriguing applications in Kähler geometry. This provides somewhat technical foundation to this ambitious program of Donaldson. Exciting progress is achieved in the last few years in that subject and the readers are encouraged to read [9, 1, 10, 6]... for further references in that subject.

Is there a corresponding story on Riemannian side? After all, in complex dimension 1, complex and real geometry coincide. Perhaps with a bit more imagination, this “dual” or “companion version” of Donaldson’s original program might also have important applications in Riemannian geometry. In a recent paper [11], Donaldson tells a potentially exciting story in the space of volume forms, more or less, mirror to his program in Kähler geometry. It of courses comes with twists of new ideas. He pointed out that the existence problem for geodesic segment is related to a few renowned problems in PDE such as regularity for some free boundary problems, Nahm’s equation etc. From PDE point of view, the geodesic equation is similar to its analogues equation in Kähler setting, but has difference in a significant way.

Let us be more specific. In [11], Donaldson introduced a Weil-Peterson type metric on the space of volume forms (normalized) on any Riemannian manifold $(X, g)$ with fixed total volume. This infinite dimension space can be parameterized by smooth functions such that

$$\mathcal{H} = \{ \phi \in C^\infty(X) : 1 + \triangle_g \phi > 0 \}.$$ 

This is a locally Euclidean space. The tangent space is exactly $C^\infty(X)$, up to addition of a constant. For any $\delta \phi \in T_{\phi} \mathcal{H}$, the metric is given by

$$\|\delta \phi\|^2 = \int_X (\delta \phi)^2 (1 + \triangle_g \phi) dg.$$ 

The energy function on a path $\Phi(t) : [0, 1] \to \mathcal{H}$ is defined as

$$E(\Phi(t)) = \int_0^1 \int_X |\dot{\Phi}|^2 (1 + \triangle \Phi) dg.$$ 

Then the geodesic equation is

$$\Phi_{tt}(1 + \triangle \Phi) - |\nabla \Phi_t|^2_g = 0.$$ 

(1.1)

This is a fully nonlinear degenerated elliptic equation. To approach this equation, Donaldson introduced a perturbed equation of the geodesic equation

$$\Phi_{tt}(1 + \triangle \Phi) - |\nabla \Phi_t|^2_g = \epsilon.$$ 

(1.2)
for any $\epsilon > 0$. The equation (1.2) can be also formulated as the other two equivalent free boundary problems according to [11].

As usual, the geodesic equation (1.1) tells exactly how to define the Levi-Civita connection in $\mathcal{H}$. Donaldson showed that $\mathcal{H}$ is formally a non-positively curved space. Donaldson asked if there exists a smooth geodesic segment between any two points in $\mathcal{H}$. In this paper, we give a partial answer to this question.

**Theorem 1.1.** For any two points $\phi_0, \phi_1 \in \mathcal{H}$ and any $\epsilon > 0$, there exists a smooth solution of (1.2), $\Phi(t) : [0, 1] \rightarrow \mathcal{H}$ which connects $\phi_0, \phi_1$.

In particular, we can prove that the (weak) $C^2$ bound of the solution is independent of $\inf \epsilon$. The weak $C^2$ means $\Delta \Phi, \Phi_{tt}, \nabla \Phi_t$ are bounded, while $\nabla^2 \Phi$ might not. Hence,

**Theorem 1.2.** For any two points $\phi_0, \phi_1 \in \mathcal{H}$, there exists a weakly $C^2$ geodesic segment $\Phi : [0, 1] \rightarrow \overline{\mathcal{H}}$ which connects $\phi_0, \phi_1 \in \mathcal{H}$, where $\overline{\mathcal{H}}$ is the closure of $\mathcal{H}$ under the weak $C^2$ topology.

Following [4], we can also prove that

**Theorem 1.3.** The infinite dimensional space $\mathcal{H}$ is a non-positively curved metric space in the sense of Alexandrov.

According to Donaldson, the geodesic equation (1.1) and the equation (1.2) are relevant to many other interesting problems, especially the Nahm’s equation. Also the space of volume forms is of fundamental interest in many subjects, such as optimal transportation theory. Theorems 1.1, 1.2 might have some applications in these related subjects.

We derive the a priori estimates and use the method of continuity to solve (1.1) and (1.2). For any function $\Phi \in C^2(M \times [0, 1])$, define

$$Q(D^2\Phi) = \Phi_{tt}(1 + \Delta \Phi) - |\nabla \Phi_t|^2.$$  

One can derive the a priori estimates for

$$Q(D^2\Phi) = f > 0.$$  

The estimates can actually be done; for example, $|\Phi|_{C^1}$ will depend on $|\nabla f^{1/2}|$ (c.f [17]). One can also choose paths as follows. Set

$$P(s, D^2 \Phi) = s Q(D^2 \Phi) + (1 - s)(\Phi_{tt} + \Delta \Phi).$$ (1.3)  

We want to solve the following equation for any $s \in [0, 1]$ and $\epsilon > 0$

$$P(D^2 \Phi(\cdot, t, s, \epsilon)) = \epsilon,$$ (1.4)  

with the boundary condition

$$\Phi(\cdot, 0, s, \epsilon) = \phi_0, \Phi(\cdot, 1, s, \epsilon) = \phi_1.$$  

These kind of paths are also used often to deal with fully nonlinear equations; for example, see [13] and [3]. In our case the right hand side of (1.3) becomes a constant and the dependence on $\nabla f^{1/2}$ is automatically
gone. Otherwise two cases are similar (P is just a slight modification of Q as an operator and has the similar properties, see (3.19) for example); for simplicity we shall still use (1.4).

When \( s = 0 \), (1.4) is a standard Laplacian equation and the Dirichlet problem is always solvable with respect to any \( \epsilon > 0 \) and any smooth \( \phi_0, \phi_1 \in H \). For simplicity of notations, we always assume that \( \epsilon \leq 1 \). Otherwise, the estimates depend on \( \epsilon \) when \( \epsilon \) is large. The main result in this paper is the following a priori estimates

**Theorem 1.4.** For any smooth solution to the equation (1.4), there is a uniform bound on \( \|\Phi\|_{C^0}, \|\Phi\|_{C^1}, \Delta \Phi, \Phi_{tt} \) and \( |\nabla \Phi_t| \), independent of both \( \epsilon \) and \( s \).

**Organization:** In Section 2 we summarize Donaldson’s theory on the space of volume forms. In Section 3 we derive the a priori estimates to solve the equations. We assume first that the Ricci curvature of the background metric is non-negative. With this assumption the computation is straightforward. In Section 4 we derive the a priori \( C^2 \) estimates without Ricci assumption. In Section 5 we discuss the geometric structure of \( H \). In particular we prove that \( H \) is a non-positively curved metric space in the sense of Alexandrov.

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## 2 Geometric structure of \( H \)

In this section we briefly summarize Donaldson’s theory \([11]\) on the space of volume forms. The readers are encouraged to refer \([8, 5]\) for more details. Given a compact Riemannian manifold \((X, g)\), let \( H \) be the set of smooth functions \( \phi \) on \( X \) such that \( 1 + \Delta \phi > 0 \). (We use the sign convention that \( \Delta \) is a negative operator, which is opposite to the sign convention in \([11]\).) We now introduce a \( L^2 \) metric in this space. Clearly, the tangent space \( TH \) is \( C^\infty(X) \). Define the norm of tangent vector \( \delta \phi \) at a point \( \phi \) by

\[
\| \delta \phi \|_\phi^2 = \int_X (\delta \phi)^2 (1 + \Delta \phi) dg.
\]

Thus a path \( \phi(t) \) in \( H \), parameterised by \( t \in [0, 1] \) say, is simply a function on \( X \times [0, 1] \) and the “energy” of the path is

\[
E(\phi(t)) = \frac{1}{2} \int_0^1 \int_X \left( \frac{\partial \phi}{\partial t} \right)^2 (1 + \Delta \phi) dg dt. \tag{2.1}
\]
It is straightforward to write down the Euler-Lagrange equations associated to the energy (2.1). These are

\[ \ddot{\phi} = \frac{\|\nabla \dot{\phi}\|^2}{1 + \Delta \phi}. \]

These equations define the geodesics in \( H \). We can read off the Levi-Civita connection of the metric from this geodesic equation, as follows. Let \( \phi(t) \) be any path in \( H \) and let \( \psi(t) \) be another function \( X \times [0,1] \), which we regard as a vector field along the path \( \phi(t) \). Then the covariant derivative of \( \psi \) along the path is given by

\[ D_t \psi = \frac{d\psi}{dt} + (W_t, \nabla X \psi), \]

where

\[ W_t = -\frac{1}{1 + \Delta \phi} \nabla X \dot{\phi}. \]

This has an important consequence for the holonomy group of the manifold \( H \). It is shown [11] that the holonomy group of \( H \) is contained in the group of volume-preserving diffeomorphisms of \( (X, d\mu_0) \), where \( d\mu_0 = (1 + \Delta \phi_0) dg \). Also Donaldson proved that the sectional curvature of the manifold \( H \) is formally non-positive. Let \( \phi \) be a point of \( H \) and let \( \alpha, \beta \) be tangent vectors to \( H \) at \( \phi \), so \( \alpha, \beta \) are just functions on \( X \). The curvature \( R_{\alpha,\beta} \) is a linear map from tangent vectors to tangent vectors. Donaldson showed the following

**Theorem A.** The curvature of \( H \) is given by

\[ R_{\alpha,\beta}(\psi) = (\nu_{\alpha,\beta}, \nabla \psi), \]

where the vector field

\[ \nu_{\alpha,\beta} = \frac{1}{1 + \Delta \phi} \text{curl} \left( \frac{1}{1 + \Delta \phi} \nabla \alpha \times \nabla \beta \right). \]

The sectional curvature is defined by

\[ K_{\alpha,\beta} = (R_{\alpha,\beta}(\alpha), \beta). \]

In particular

\[ K_{\alpha,\beta} = -\int_X \frac{1}{1 + \Delta \phi} |d\alpha \wedge d\beta|^2 dg \leq 0. \]

We define a functional on \( H \) for paths by

\[ V(\phi) = \int_X \phi dg. \]

This function is convex along geodesics in \( H \), since the geodesic equation implies that \( \dot{\phi} \geq 0 \). Now introduce a real parameter \( \epsilon \geq 0 \) and consider the functional on paths in \( H \):

\[ E_\epsilon(\phi(t)) = E(\phi(t)) + \epsilon \int_0^1 V(\phi) dt, \]
corresponding to the motion of a particle in the potential $-\epsilon V$. The Euler-Lagrange equations are
\[ \ddot{\phi} = \frac{|\nabla X \dot{\phi}|^2 + \epsilon}{1 + \Delta \phi}, \]
This equation is equivalent to the other two free boundary problems. For the detailed discussion of this equation and its relation to free boundary problems and many other interesting problems, we refer the readers to Donaldson [11].

3 Existence of the solution

In this section we derive the a priori estimates to solve equation (1.1). For the easiness of presentation and convenience of the first time readers, we assume the background metric has non-negative Ricci curvature. In this case, the calculation is more streamliner which explains the main idea. The general case will be deferred to Section 4.

Theorem 3.1. Assume that $(X, g)$ has non-negative Ricci curvature. For any $\phi_0, \phi_1 \in \mathcal{H}$, the equation
\[ P(D^2 \Phi(\cdot, t, s, \epsilon)) = \epsilon, \] (3.1)
with the boundary condition
\[ \Phi(\cdot, 0, s, \epsilon) = \phi_0, \Phi(\cdot, 1, s, \epsilon) = \phi_1, \] (3.2)
has a unique smooth solution with $\Phi \in \mathcal{H}$ at $s = 1$. Moreover,
\[ |\Phi|_{C^0}, |\Phi|_{C^1}, |\triangle \Phi|, |\Phi_{tt}|, |\nabla \Phi| \]
are uniformly bounded, independent of $s$ and $\epsilon$.

3.1 Concavity and the continuity method

The foundation relies on a convexity of the nonlinear operator $Q$ which is showed by Donaldson in [11]. It is also used to show the uniqueness of the equation (1.2) in [11]. Let $A$ be a symmetric $(n+1) \times (n+1)$ matrix with entries $A_{ij}, 0 \leq i, j \leq n$. Define
\[ Q(A) = A_{00} \sum_{i=1}^{n} A_{ii} - \sum_{i=1}^{n} A_{i0}^2. \]
Thus $Q$ is a quadratic function on the vector space of symmetric $(n+1) \times (n+1)$ matrices.

Lemma 3.2. (Donaldson [11]) 1. If $A > 0$, then $Q(A) > 0$ and if $A \geq 0$, $Q(A) \geq 0$.

2. If $A, B$ are two matrices with $Q(A) = Q(B) > 0$, and if the entries $A_{00}, B_{00}$ are positive then for any $s \in [0, 1],
\[ Q(sA + (1-s)B) \geq Q(A), Q(A - B) \leq 0. \]
Moreover, the equality holds if and only if $A_{ij} = B_{ij}, A_{i0} = B_{i0}$. 6
Lemma 3.2 is shown by Donaldson [11] using some Lorentz geometry. One can also prove this through elementary calculus. This lemma is equivalent to the following concavity of $\log(Q(D^2\Phi))$.

**Lemma 3.3.** Consider the function

$$f(x, y, z_1, \cdots, z_n) = \log \left( xy - \sum z_i^2 \right).$$

Then $f$ is concave when $x > 0, y > 0$ and $xy - \sum z_i^2 > 0$.

**Proof.** We can restate Lemma 3.2 as follows (taking $s = 1/2$). Let $x > 0, y > 0$ and $a, b > 0$, if

$$xy - \sum z_i^2 = ab - \sum c_i^2 > 0,$$

then

$$\frac{1}{4} \left( (x + a)(y + b) - \sum (z_i + c_i)^2 \right) \geq xy - \sum z_i^2. \quad (3.3)$$

Obviously $f$ is smooth when $x > 0, y > 0, xy - \sum z_i^2 > 0$. It suffices to prove that

$$\frac{1}{2} \left( f(x, y, z_i) + f(\tilde{x}, \tilde{y}, \tilde{z}_i) \right) \leq f \left( \frac{x + \tilde{x}}{2}, \frac{y + \tilde{y}}{2}, \frac{z_i + \tilde{z}_i}{2} \right).$$

Namely, we need to show that

$$\log \left( xy - \sum z_i^2 \right) + \log \left( \tilde{x}\tilde{y} - \sum \tilde{z}_i^2 \right) \leq 2 \log \left( \left( \frac{x + \tilde{x}}{2} \right) \left( \frac{y + \tilde{y}}{2} \right) - \sum \left( \frac{z_i + \tilde{z}_i}{2} \right)^2 \right).$$

We can rewrite the above as

$$\sqrt{(xy - \sum z_i^2)(\tilde{x}\tilde{y} - \sum \tilde{z}_i^2)} \leq \left( \left( \frac{x + \tilde{x}}{2} \right) \left( \frac{y + \tilde{y}}{2} \right) - \sum \left( \frac{z_i + \tilde{z}_i}{2} \right)^2 \right).$$

Suppose that for some $\lambda > 0$,

$$\tilde{x}\tilde{y} - \sum \tilde{z}_i^2 = \lambda^2 (xy - \sum z_i^2).$$

Let

$$\tilde{x} = \lambda a, \quad \tilde{y} = \lambda b, \quad \tilde{z}_i = \lambda c_i,$$

we get that

$$ab - \sum c_i^2 = xy - \sum z_i^2.$$

The above reads

$$\lambda(xy - \sum z_i^2) \leq \frac{1}{4} \left( (x + \lambda a)(y + \lambda b) - \sum (z_i + \lambda c_i)^2 \right). \quad (3.4)$$

The right hand side of (3.4) is

$$\frac{1}{4} \left( 1 + \lambda^2 \right) (xy - \sum z_i^2) + \frac{\lambda}{4} (xb + ya - 2z_i c_i).$$

Note when $\lambda = 1$, (3.4) is exactly (3.3). We can get that from (3.3)

$$2(xy - \sum z_i^2) \leq (xb + ya - 2z_i c_i).$$

Hence (3.4) follows from (3.3). Note the equality holds if and only if $\lambda = 1, x = a, y = b, z_i = c_i$, namely $x = \tilde{x}, y = \tilde{y}, z_i = \tilde{z}_i$. □
Replacing $y$ by $\sum_i y_i$, the above argument shows that

$$h(x, y_1, \cdots, y_n, z_1, \cdots, z_n) = \log \left( x(\sum_i y_i) - \sum_i z_i^2 \right)$$

is also a concave function for $x > 0, \sum_i y_i > 0$ and $x(\sum_i y_i) - \sum_i z_i^2 > 0$. It follows that $\log Q(D^2\Phi)$ is a concave functional on $D^2\Phi$.

When $s = 0$, the equation (3.1) reads

$$\Phi_{tt} + \Delta \Phi = \epsilon.$$ 

There is a unique smooth solution to this Laplace equation with boundary condition (3.2). It is a uniformly elliptic linear equation and its linearization has zero kernel with zero boundary data. Hence the equation (3.1) can be solved uniquely for $s$ sufficiently close to 0. We can define

$$s_0 = \sup \{ \tilde{s} : P(s, \Phi) = \epsilon \text{ has a unique smooth solution for } s \in [0, \tilde{s}) \}.$$ 

Note that $s_0$ is uniformly bounded away from zero. Actually we have

**Proposition 3.4.** There exists a positive constant $\delta = \delta(X, \phi_0, \phi_1)$ such that

$$P(D^2\Phi) = \epsilon$$

has a smooth unique solution $\Phi$ for any $s \in [0, \delta]$. Moreover, $|\Phi|_{C^k}$ is uniformly bounded when $s \in [0, \delta]$.

**Proof.** The proof is an inverse theorem for the operator $P$. Consider

$$P : [0, 1] \times C^{k,\alpha} \to C^{k-2,\alpha}$$

for any $k \geq 2, \alpha \in [0, 1)$. We know there exists a unique smooth function $\Phi$ such that $P(0, \Phi) = \epsilon$. Since $dP_0$ is invertible at $(0, \Phi)$, the proposition follows from the inverse function theorem in the Banach space.

Without loss of generality, we will assume that $s \geq \delta > 0$ for the continuity family (3.1). We observe that the linearity $dP$ is elliptic for any $s \in [0, s_0)$, where

$$dP(h) = (s\Phi_{tt} + (1-s))\Delta h + (s(1+\Delta \Phi) + (1-s))h_{tt} - 2s\Phi_{tk}h_{tk}. \quad (3.5)$$

**Proposition 3.5.** For any $s \in [0, s_0)$, if $\Phi$ is the unique solution of (3.1), then

$$\Phi_{tt} + 1 + \Delta \Phi > 0.$$ 

It follows that $s\Phi_{tt} + (1-s), s(1+\Delta \Phi) + (1-s)$ are both positive. Moreover, $dP$ is an elliptic operator.

**Proof.** Let $s$ be the first value such that at some point $p = (x_0, t_0) \in X \times [0, 1]$

$$\Phi_{tt} + 1 + \Delta \Phi = 0,$$

where $\Phi$ is the solution of

$$P(s, D^2\Phi) = \epsilon.$$
It follows that
\[ Q(D^2\phi)(x_0, t_0) = \phi_{tt}(1 + \Delta \phi) - |\nabla \phi|^2 \leq 0. \]

It follows that
\[ P(s, D^2\phi)(x_0, t_0) = sQ(D^2\phi) + (1 - s)(\phi_{tt} + \Delta \phi) \leq 0. \]

Contradiction. Since \( \phi_{tt} + 1 + \Delta \phi > 0 \), it follows that both \( s\phi_{tt} + (1 - s), s(1 + \Delta \phi) + (1 - s) \) are positive. To show \( dP \) is elliptic, we see that
\[ (s\phi_{tt} + (1 - s))(s(1 + \Delta \phi) + (1 - s)) - s^2\phi_{tt} \leq s(1 - s)^2. \]

It follows that the quadratic form
\[ (s\phi_{tt} + (1 - s))\sum_k \xi_k^2 + (s(1 + \Delta \phi) + (1 - s))\xi_0^2 - 2s\phi_{tk}\xi_k > 0 \]
for any nonzero vector \( \xi = (\xi_0, \ldots, \xi_{n+1}) \). In particular, \( dP \) is elliptic. \( \blacksquare \)

Remark 3.6. Note that we do not have \( Q(D^2\phi) > 0 \) along the continuous family. At \( s = 0 \), the solution \( \phi(x, t) \) is not necessarily in \( H\). But if we have a smooth solution at \( s = 1 \), then \( \phi(x, t) \in H\).

We shall then derive the \textit{a priori} estimates and prove Theorem 3.1 at the end of this section. We assume \( 0 < \epsilon \leq 1 \) throughout the paper.

3.2 \( C^0 \) estimate

The \( C^0 \) estimate follows from Lemma 3.2 and the maximum principle.

Proposition 3.7. Let \( \phi(x, t) \) be a solution of the equation (3.1) for \( s \in [0, s_0) \). Denote \( \psi_a(x, t) = a(1 - t)t + (1 - t)\phi_0 + t\phi_1 \) where \( a \) is a fixed constant. Then \( \phi \) satisfies the following \textit{a priori} \( C^0 \) estimate
\[ \psi_{-a} \leq \phi \leq \psi_a, \]
when \( a \) is sufficiently big.

Proof. Assume contrary; then there is some point \( p = (x_0, t_0) \) such that \( \phi(x_0, t_0) > \psi_a(x_0, t_0) \) and so \( \phi - \psi_a \) obtains its maximum interior at \( q \). At the point \( q \), we have
\[ \phi_{tt} \leq \psi_{a,tt} = -2a, \Delta \phi \leq \Delta \psi_a = (1 - t)\Delta \phi_0 + t\Delta \phi_1. \]

However by Proposition 3.5
\[ \phi_{tt} > -\frac{(1 - s)}{s}. \]

Note we assume that \( s \geq \delta > 0 \) by Proposition 3.4. Contradiction if \( a \) is sufficiently big. It follows that for some big \( a \)
\[ \phi \leq \psi_a. \]

If there is some point \( p = (x_0, t_0) \) such that \( \phi(x_0, t_0) < \psi_{-a}(x_0, t_0) \), then \( \phi - \psi_{-a} \) obtains its minimum interior at \( q \). At the point \( q \), \( D^2(\phi - \psi_{-a}) \)
\( \Psi_{-a} \) is nonnegative. In particular at the point \( q \) \( \Phi_{tt} \geq \Psi_{-a,tt} = 2a, \triangle \Phi(q) \geq \triangle \Psi_{-a} = t \triangle \phi_1 + (1-t) \triangle \phi_0 \). Suppose at point \( q \)

\[
Q(D^2 \Phi) = \Phi_{tt}(1 + \triangle \Phi) - \Phi_{tk}^2 \geq Q(D^2 \Psi_{-a}),
\]

where

\[
Q(D^2 \Psi_{-a}) = 2a(1 + t \triangle \phi_1 + (1-t) \triangle \phi_0) - |\nabla \phi_0 - \nabla \phi_1|^2 > 0
\]

when \( a \) is big enough. Then

\[
P(s, D^2 \Phi)(q) = sQ(D^2 \Phi) + (1-s)(\Phi_{tt} + 1 + \triangle \Phi) \geq sQ(D^2 \Psi_{-a}) + (1-s)(2a + t \triangle \phi_1 + (1-t) \triangle \phi_0) > 1
\]

when \( a \) is big enough. Contradiction. It follows that at \( q \),

\[
Q(D^2 \Phi) < Q(D^2 \Psi_{-a}).
\]

By (3.1) we know that \( \lambda > 1 \). It follows from Lemma 3.2 that \( Q(B - A) \leq 0 \). But \( B - A \) is semi-positive definite, \( Q(B - A) \geq 0 \). Contradiction.

3.3 \( C^1 \) estimates

At any point \( p \in X \times [0,1] \), take local coordinates \((x_1, \cdots, x_n, t)\). We can always choose a coordinates such that the metric tensor \( g \) satisfies \( g_{ij} = \delta_{ij}, \partial_k g_{ij} = 0 \) at one point. We will also use, for any smooth function \( f \) on \( X \times [0,1] \), the following notations

\[
\triangle f_i = \triangle (f_i), \quad \triangle f_{ij} = \triangle (f_{ij}), \quad \triangle f,_{i} = (\triangle f),_{i} \text{ and } \triangle f,_{ij} = (\triangle f),_{ij}.
\]

By Weitzenbock’s formula, we have

\[
\triangle f_i = \triangle (f_i) = R_{ij} f_j + \triangle f,_{i} + R_{ij} f_j,
\]

where \( R_{ij} \) is the Ricci tensor of the metric \( g \). The derivatives \( f_i, f_{ij} \) etc are all covariant derivatives.

**Lemma 3.8.** Assume \((X, g)\) has nonnegative Ricci curvature. If \( \Phi \) is a solution of (3.1), then \( \Phi \) satisfies the following a priori estimates

\[
|\nabla \Phi| \leq C, \quad |\Phi_t| \leq C,
\]

where \( C \) is a universal constant, independent of \( s \) and \( \epsilon \).
Proof. Note that
\[ \Phi_{tt} + \frac{(1 - s)}{s} > 0. \]
By Proposition 3.4 we can assume that \( s \geq \delta \). It follows that
\[ \Phi_t + \frac{(1 - s)}{s} \]
is a \( t \)-increasing function. It implies that
\[ \Phi_t(x, 0) - \frac{(1 - s)}{s} < \Phi_t(x, 1) + \frac{(1 - s)}{s} - \frac{(1 - s)}{s}. \]
Namely
\[ |\Phi_t| \leq \max_{\partial (X \times [0,1])} |\Phi_t| + C. \]
On the boundary, by the \( C^0 \) estimates in Proposition 3.7, we have
\[ |\Phi_t(x, 0)| = \lim_{t \to 0} \frac{\Phi(x, t) - \Phi(x, 0)}{t} \leq \lim_{t \to 0} \frac{at(1 - t)}{t} + |\phi_1 - \phi_0| = a + |\phi_1 - \phi_0|. \]
Similarly, one can bound \( \Phi_t(x, 1) \) by \( a + |\phi_1 - \phi_0| \), where \( a \) is the same constant as in Proposition 3.7.
To bound \( |\nabla \Phi| \), take
\[ h = \frac{1}{2} |\nabla \Phi|^2. \]
Taking derivative, we get
\[
\begin{align*}
h_t &= \Phi_{tk} \Phi_k, \quad h_k = \Phi_{tk} \Phi_t \\
h_{tt} &= \Phi_{ttk} \Phi_k + \Phi^2_{tk}, \quad h_{tk} = \Phi_{ttk} \Phi_t + \Phi_{tk} \Phi_t \\
\triangle h &= \Phi_{ttk} \Phi_i + \Phi^2_{tk} = \triangle \Phi_t \Phi_i + \Phi^2_{tk} \\
&= \triangle \Phi_{ij} \Phi_i + \Phi^2_{ik} + R_{ij} \Phi_i \Phi_j, \quad (3.8)
\end{align*}
\]
where \( R_{ij} \) is the Ricci curvature of \( (X, g) \). If \( \Phi \) solves the equation \( (5.1) \), by taking derivative, we can get that
\[
\begin{align*}
(s \Phi_{tt} + (1 - s)) \triangle \Phi_t + (s(1 + \triangle \Phi) + (1 - s)) \Phi_{ttt} - 2s \Phi_{tk} \Phi_{ttk} &= 0, \quad (3.9) \\
(s \Phi_{tt} + (1 - s)) \triangle \Phi_{tk} + (s(1 + \triangle \Phi) + (1 - s)) \Phi_{ttk} - 2s \Phi_{ti} \Phi_{tik} &= 0 \quad (3.10)
\end{align*}
\]
It follows that
\[
\begin{align*}
dP(h) &= (s \Phi_{tt} + (1 - s)) \triangle h + (s(1 + \triangle \Phi) + (1 - s)) h_{tt} - 2s \Phi_{tk} h_{tk} \\
&= (s \Phi_{tt} + (1 - s)) (\triangle \Phi_t \Phi_i + \Phi^2_{tk}) \\
&\quad + (s(1 + \triangle \Phi) + (1 - s)) (\Phi_{ttk} \Phi_k + \Phi^2_{tk}) \\
&\quad - 2s \Phi_{tk} (\Phi_{tk} \Phi_i + \Phi_{tk} \Phi_t) + (s \Phi_{tt} + (1 - s)) R_{ij} \Phi_i \Phi_j. \quad (3.11)
\end{align*}
\]
\[ 11 \]
By (3.9), (3.10) and (3.11), we have
\[ dP(h) = (s\Phi_{tt} + (1 - s))\Phi_{ik}^2 + (s(1 + \triangle \Phi) + (1 - s))\Phi_{ik}^2 
- 2s\Phi_{ik}\Phi_{i} \Phi_{k} + (s\Phi_{tt} + (1 - s))R_{ij}\Phi_{i}\Phi_{j}. \] (3.12)

If $|\nabla \Phi|^2$ achieves its maximum value in the interior, we can assume that
\[ \max |\nabla \Phi|^2 = \max_{\partial X \times [0,1]} |\nabla \Phi|^2 + \eta, \]
where $\eta > 0$. Then $h + \lambda t^2$ takes its maximum in the interior for any positive constant $\lambda \ll \eta$. Assume the point is $p$. At point $p$, $D^2(h + \lambda t^2) \leq 0$. It follows that $dP(h + \lambda t^2) \leq 0$. On the other hand, by (3.12), we have
\[ dP(h + \lambda t^2) = (s\Phi_{tt} + (1 - s))\Phi_{ik}^2 + (s(1 + \triangle \Phi) + (1 - s))\Phi_{ik}^2 
- 2s\Phi_{ik}\Phi_{i} \Phi_{k} + (s\Phi_{tt} + (1 - s))R_{ij}\Phi_{i}\Phi_{j} + 2\lambda(s(1 + \triangle \Phi) + (1 - s)) > 0. \]
Contradiction. It implies that $|\nabla \Phi|^2$ obtains its maximum on the boundary. By the boundary condition (3.2), we know that $|\nabla \Phi|$ is uniformly bounded.

### 3.4 $C^2$ estimates

First we have the following interior $C^2$ estimates.

**Lemma 3.9.** Assume $(X, g)$ has nonnegative Ricci curvature. If $\Phi$ is a solution of (3.1), then $\Phi$ satisfies the following a priori estimate
\[ -C \leq 1 + \triangle \Phi \leq C, \quad -C \leq \Phi_{tt} \leq \max_{\partial(X \times [0,1])} |\Phi_{tt}| + C, \]
where $C$ is a universal constant, independent of $s$ and $\epsilon$.

**Proof.** By Proposition 3.5, we just need to show the upper bound. If $1 + \triangle \Phi$ achieves its maximum in the interior, at the point $p$, we can assume that
\[ 1 + \triangle \Phi(p) = 1 + \max_{\partial(X \times [0,1])} \triangle \Phi + \eta, \]
where $\eta > 0$. For any positive constant $\lambda \ll \eta$, $1 + \triangle \Phi + \lambda t^2$ achieves its maximum value in the interior, at the point $\tilde{p}$. Then $D^2(1 + \triangle \Phi + \lambda t^2)(\tilde{p}) \leq 0$. It follows that $dP(1 + \triangle \Phi + \lambda t^2)(\tilde{p}) \leq 0$. Let $h = 1 + \triangle \Phi + \lambda t^2$. Taking derivative, we have
\[ h_t = \Delta \Phi_t + 2\lambda, \quad h_k = \Delta \Phi_{k}, \quad h_{tt} = \Delta \Phi_{tt} + 2\lambda, \quad h_{kk} = \Delta \Phi_{kk}, \quad \triangle h = \Delta^2 \Phi. \] (3.13)

Note at the point $\tilde{p}$, $h_t = \Delta \Phi_t + 2\lambda = 0, h_k = \Delta \Phi_{k} = 0$. Taking derivative of (3.10), we have, at the point $\tilde{p}$,
\[ (s\Phi_{tt} + (1 - s))\Delta^2 \Phi + (s(1 + \triangle \Phi) + (1 - s))\Delta \Phi_{tt} - 2s\Phi_{tt} \Delta(\Phi_{tt}) - 2s\Phi_{tt}^2 = 0. \]
It follows from above and \((3.13)\) that
\[
dP(h)(\bar{p}) = (s \Phi_{tt} + (1 - s)) \Delta h + (s(1 + \Delta \Phi) + (1 - s)) h_{tt} - 2s \Phi_{tk} h_{tk} \\
= (s \Phi_{tt} + (1 - s)) \Delta^2 \Phi \\
+ (s(1 + \Delta \Phi) + (1 - s)) (\Delta \Phi_{tt} + 2\lambda) - 2s \Phi_{tk} \Delta \Phi_{tk} \\
= 2\lambda(s(1 + \Delta \Phi) + (1 - s)) + 2s \Phi_{tik}^2 + 2s R_{ij} \Phi_{tt} \Phi_{ij} > 0.
\]
Contradiction. To bound \(\Phi_{tt}\), consider
\[
h = \Phi_{tt} + 1 + \Delta \Phi.
\]
If \(h\) obtains its maximum on the boundary, we are done. If \(h\) obtains its maximum in the interior, at the point \(p\), we can assume that
\[
h(p) = \max_{\partial X \times [0,1]} h + \eta,
\]
where \(\eta\) is positive. Take \(\bar{h} = h + \lambda \xi^2\) for some positive constant \(\lambda \ll \eta\), then \(\bar{h}\) achieves its maximum in the interior, say, at the point \(\bar{p}\). It follows that \(D^2 \bar{h}(\bar{p}) \leq 0, dP(\bar{h})(\bar{p}) \leq 0\). Taking derivative, we have
\[
\bar{h}_t = \Phi_{ttt} + \Delta \Phi_t + 2\lambda, \quad \bar{h}_{tt} = \Phi_{tttt} + \Delta \Phi_{tt} + 2\lambda \\
\bar{h}_{tk} = \Phi_{ttk} + \Delta \Phi_{tk}, \quad \Delta \bar{h} = \Delta \Phi_{tt} + \Delta^2 \Phi, \quad \bar{h}_{tk} = \Phi_{tttk} + \Delta \Phi_{tk}. \tag{14}
\]
We calculate
\[
dP(\bar{h}) = (s \Phi_{tt} + (1 - s)) \Delta \bar{h} + (s(1 + \Delta \Phi) + (1 - s)) \bar{h}_{tt} - 2s \Phi_{tk} \bar{h}_{tk} \\
= (s(1 + \Delta \Phi) + (1 - s))(\Phi_{tttt} + \Delta \Phi_{tt} + 2\lambda) \\
+ (s \Phi_{tt} + (1 - s))(\Delta \Phi_{tt} + \Delta^2 \Phi) - 2s \Phi_{tk}(\Phi_{tttk} + \Delta \Phi_{tk}). \tag{15}
\]
Taking derivative of \((3.9)\) and \((3.10)\), we have
\[
(s \Phi_{tt} + (1 - s)) \Delta \Phi_{tt} + (s(1 + \Delta \Phi) + (1 - s)) \Phi_{tttt} - 2s \Phi_{tk} \Phi_{ttk} \\
+ 2s \Phi_{ttt} \Delta \Phi_t - 2s \Phi_{tk}^2 = 0 \tag{16}
\]
\[
(s \Phi_{tt} + (1 - s)) \Delta^2 \Phi + (s(1 + \Delta \Phi) + (1 - s)) \Delta \Phi_{tt} - 2s \Phi_{tt} \Phi_{ttk} \\
+ 2s \Phi_{ttk} \Delta \Phi_{tk} - 2s \Phi_{tk}^2 = 0. \tag{17}
\]
It follows from \((3.15), (3.16)\) and \((3.17)\) that
\[
dP(\bar{h}) = 2\lambda(s(1 + \Delta \Phi) + (1 - s)) + 2s \Phi_{tik}^2 + 2s \Phi_{tk}^2 \\
- 2s \Phi_{ttt} \Delta \Phi_t - 2s \Phi_{ttk} \Delta \Phi_{tk} - 2s R_{ij} \Phi_{tt} \Phi_{ij} \geq 0.
\]
Denote
\[
L = \Phi_{tik}^2 - \Phi_{ttt} \Delta \Phi_t, \quad M = \Phi_{tij}^2 - \Phi_{ttk} \Delta \Phi_{tk}. \tag{18}
\]
Using \((3.9)\) and \((3.10)\), we can get that
\[
(s \Phi_{tt} + (1 - s)) L = (s \Phi_{tt} + (1 - s)) \Phi_{tik}^2 + (s(1 + \Delta \Phi) + (1 - s)) \Phi_{ttk}^2 - 2s \Phi_{tk} \Phi_{ttk} \Phi_{ttt}.
\]
\[
(s \Phi_{tt} + (1 - s)) M = (s \Phi_{tt} + (1 - s)) \Phi_{tij}^2 + (s(1 + \Delta \Phi) + (1 - s)) \Phi_{tjk}^2 - 2s \Phi_{jk} \Phi_{ttk} \Phi_{ttt}.
\]
It follows that \(L, M \geq 0\). Hence,
\[
dP(\bar{h}) \geq 2\lambda(s(1 + \Delta \Phi) + (1 - s)) > 0.
\]
Contradiction.
Then we derive the boundary estimates of $\triangle \Phi, \Phi_{tt}$ and $\nabla \Phi_t$ by careful construction of some barrier functions. This type of construction of barrier functions follows from [15] and [14].

**Lemma 3.10.** Assume $(X, g)$ has nonnegative Ricci curvature. If $\Phi$ is a solution of (3.1), then $\Phi$ satisfies the following a priori estimate

$$|\triangle \Phi| \leq C, \quad |\Phi_{tk}| \leq C, \quad |\Phi_{tt}| \leq C,$$

where $C$ is a universal constant, independent of $s$ and $\epsilon$. Note that the proof of boundary estimates do not depend on the Ricci curvature assumption if we get the interior $C^2$ estimates.

**Proof.** In light of Proposition [3.5] we assume $s \geq \delta$. By Lemma 3.9, $\triangle \Phi$ is uniformly bounded. If $\Phi$ solves equation (3.1), we have

$$|\Phi_{tk}|^2 = \Phi_{tt}(1 + \triangle \Phi) + \frac{(1 - s)}{s}(\Phi_{tt} + \triangle \Phi) - \frac{\epsilon}{s} \leq C|\Phi_{tt}| + 1, \quad |\Phi_{tt}| \leq C\max_{\partial(X \times [0,1])} |\Phi_{tt}| + 1.$$

To bound $\max_{\partial(X \times [0,1])} |\Phi_{tt}|$, observe that

$$(s(1 + \triangle \Phi) + (1 - s))\Phi_{tt} = s|\Phi_{tk}|^2 - (1 - s)\triangle \Phi + \epsilon.$$  

Since $\Phi(x, 0) = \phi_0, \Phi(x, 1) = \phi_1, 1 + \triangle \Phi$ is positive and uniformly bounded away from zero on the boundary. Thus $s(1 + \triangle \Phi) + (1 - s)$ is uniformly bounded away from zero on the boundary. It follows that

$$\max_{\partial(X \times [0,1])} |\Phi_{tt}| \leq C\left(\max_{\partial(X \times [1,1])} |\Phi_{tt}|^2 + 1\right).$$

We finish the proof by showing

$$\max_{\partial(X \times [0,1])} |\Phi_{tk}|$$

is bounded. We just consider $|\Phi_{tk}|$ on $X \times \{0\}$. The case on $X \times \{1\}$ follows similarly. For any point $p$ on $X \times \{0\}$, we can choose a local coordinates around $p$ such that $p = (0, \cdots, 0, 0)$, and

$$g_{ij}(0) = \delta_{ij}, \quad \partial g_{ij}(0) = 0.$$  

Let $B_0(\rho) \subset X$ be a small ball with radius $\rho$. For any $x \in B_0(\rho)$, we have

$$g_{ij}(x) = (1 + o(|x|))\delta_{ij}, \quad \partial g_{ij}(x) = o(|x|).$$

Take $\Omega = B_0(\rho) \times [0, \kappa]$, where $\kappa$ is a small positive number. Let $h$ be a function on $\Omega$ defined by

$$h = (\Phi - \phi_0), k$$

for any $k = 1, 2, \cdots, n$. Note $h$ is only a local defined function. Take

$$h = A(\Phi - \phi_0 + At - At^2) + B(\sum x_i^2) + h,$$

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where $A \gg B$ are two big fixed positive constants. Since $h$ is bounded, it is easy to see that $\tilde{h} \geq 0$ on $\partial \Omega$. Taking derivative, we have

$$
\begin{align*}
 h_t &= \Phi_{tk}, \quad h_i = \Phi_{ki} - (\phi_0)_{ki}, \quad h_{tt} = \Phi_{ttk} \\
 h_{ti} &= \Phi_{tki}, \quad \Delta h = \Delta \Phi_k - \Delta \phi_0 k.
\end{align*}
$$

It follows from above and (3.10) that

$$
dP(h) = (s\Phi_{tt} + (1 - s))\Delta h + (s(1 + \Delta \Phi) + (1 - s))h_{tt} - 2s\Phi_{ti}h_{ti},
$$

and

$$
dP(-At^2) = -2A(s(1 + \Delta \Phi) + (1 - s)).
$$

Since on $X \times \{0\}$, $s(1 + \Delta \Phi) + (1 - s)$ is uniformly bounded away from zero, we can pick up $\kappa$ small enough such that on $X \times [0, \kappa]$,

$$
A(s(1 + \Delta \Phi) + (1 - s)) > 2\epsilon.
$$

We should emphasize that $A, B$ are uniformly bounded even though $\kappa$ might depend on $s$. It follows that

$$
dP(\Phi - \phi_0) = -(1 + \Delta \phi_0)(s\Phi_{tt} + (1 - s)) - (1 - s)(\Phi_{tt} + 1 + \Delta \Phi) + 2\epsilon,
$$

and

$$
dP(-At^2) = -2A(s(1 + \Delta \Phi) + (1 - s)).
$$

We can calculate

$$
dP(\sum x_i^2) < (2n + 1)(s\Phi_{tt} + (1 - s)).
$$

And

$$
\Delta(\sum x_i^2) = 2n + o(\rho) < 2n + 1,
$$

we can calculate

$$
dP(\sum x_i^2) < (2n + 1)(s\Phi_{tt} + (1 - s)).
$$

It follows from above that in $\Omega$,

$$
dP(\tilde{h}) < -(A(1 + \Delta \phi_0) + (2n + 1)B + C)(s\Phi_{tt} + (1 - s)) < 0.
$$

By the maximum principle, we know that $\tilde{h} \geq 0$ in $\Omega$. Since $\tilde{h}(0) = 0$,

$$
\frac{\partial \tilde{h}}{\partial t}|_{t=0} \geq 0.
$$

It follows that on $X \times \{0\}$,

$$
\Phi_{tk} \geq -C.
$$
It follows similarly that
\[ \Phi_{tk} \leq C, \]
by taking \( h = (\phi_0 - \Phi)_k \), and
\[ \tilde{h} = A(\Phi - \phi_0 + At - At^2) + B(\sum x_i^2) + h. \]

Hence we have proved the weakly \( C^2 \) bound of \( \Phi \) for any smooth solution of (3.1). When \( \epsilon > 0 \), for any \( s \in [0, 1] \), the equation (3.1) is uniformly elliptic. We can rewrite the equation for \( s > 0 \)
\[ \log \left( \left( \Phi_{tt} + \frac{(1 - s)}{s} \left( \Delta \Phi + 1 + \frac{(1 - s)}{s} \right) - |\nabla \Phi|^2 \right) \right) = \log \frac{s\epsilon + (1 - s)^2}{s^2}. \]
(3.19)
It is clear that the equation is concave for any \( s \in [0, 1] \) and \( \epsilon > 0 \) by Lemma 3.3. It then follows from the Evans-Krylov theory to get that the Hölder estimates of \( D^2 \Phi \). In fact, the assumption of \( \Phi \in C^{1,\beta} \) for some \( \beta \in (0, 1) \) is sufficient to get global \( C^{2,\alpha} \) regularity for a uniformly elliptic and concave fully nonlinear equation provided sufficient smooth boundary data and the right hand side (see Theorem 7.3 in [7] for example).

Remark 3.11. In a previous version, first we prove that \( |D^2 \Phi| \) is bounded by using the weak maximum principle (c.f. Gilbarg-Trudinger [13], Section 9.7, Theorem 9.20 and Section 9.9 Theorem 9.26) and then use Evans-Krylov theory to obtain Hölder estimate of \( |D^2 \Phi| \). We would like to thank the referee for pointing out that this not necessary for using Evans-Krylov theory.

Once we have the Hölder estimates of \( D^2 \Phi \), the Schauder theory gives all the higher derivatives bound. To see this, if \( \Phi \) solves (3.1), then we have
\[ dP(\Phi_t) = 0. \]
Since the coefficients are all Hölder continuous, the Schauder theory applies and we get that \( \Phi_t \) is \( C^{2,\alpha} \). The similar discussion holds for \( \Phi_k \), namely,
\[ dP(\Phi_k) = R_k \Phi_t. \]
Then the boot-strapping argument allows us to conclude that all higher derivatives are bounded and \( \Phi \in C^\infty \) follows. We should emphasize that the above discussion holds only for \( \epsilon > 0 \). Summarize above, we have

Lemma 3.12. If \( \Phi \) solves the equation (3.11), then the following estimates hold independent of \( s \),
\[ |D^l \Phi| \leq C = C(\epsilon, l) \]
for any \( l \in \mathbb{N} \).

We are in the position to prove Theorem 3.1.

Proof. First we show that \( s_0 = 1 \). Otherwise, assume \( s_0 < 1 \). Then the continuous family (3.1) has a unique solution for \( 0 \leq s < s_0 \). Consider a sequence of \( s_i \rightarrow s_0 \) and the solutions \( \Phi^i \). In light of Proposition 3.5 we
can assume $s_i \geq \delta$. By the a priori estimates derived above, for any $s_i$, the solutions $\Phi^i$ of (3.1) satisfy
\[ |\Phi^i|, |\nabla \Phi^i|, |\Phi^i_t|, |\Phi^i_{tt}|, |\Delta \Phi^i| \leq C. \]

In particular, $dP$ is uniformly elliptic for any $s_i$. When $\epsilon > 0$, we still have the global Hölder estimates for second derivatives. Then by the standard elliptic argument, we obtain the higher derivatives estimates for $\Phi^i$. Then the solutions $\Phi^i$ converge to a smooth solution for $s = s_0$. Again $dP$ is a uniform elliptic operator at $s = s_0$ and $dP$ has zero kernel with zero boundary data, one can solve the equation (3.1) for $s$ is sufficiently close to $s_0$, but this contradicts with the definition of $s_0$. So $s_0 = 1$. Moreover, the solutions $\Phi^i$ satisfy the estimates
\[ |\Phi^i|, |\nabla \Phi^i|, |\Phi^i_t|, |\Phi^i_{tt}|, |\Delta \Phi^i| \leq C. \]

The standard elliptic theory gives the higher derivatives estimates. So $\Phi^i$ sub-converge to a smooth function $\Phi(x, t)$ solving the equation
\[ Q(D^2 \Phi) = \Phi_{tt}(1 + \Delta \Phi) - |\nabla \dot{\Phi}|^2 = \epsilon. \]

In particular, we know that $\Phi_{tt} + 1 + \Delta \Phi$ is bounded away from zero. When $s = 1$, it follows that $\Phi_{tt}, 1 + \Delta \Phi > 0$ and $\Phi \in \mathcal{H}$. \qed

4 Without assumption on Ricci

In this section we drop the non-negative assumption of Ricci to prove Theorem 3.1. Note that $C^0$ estimates, the boundary $C^1, C^2$ estimates, and higher derivative estimates do not depend on the Ricci curvature assumption. We just need to establish interior $C^1$ and $C^2$ estimates.

4.1 $C^1$ estimate

Lemma 4.1. If $\Phi$ is a solution of (3.1), then $\Phi$ satisfies the following a priori estimates
\[ |\nabla \Phi| \leq C, \quad |\Phi_t| \leq C, \]
where $C$ is a universal constant, independent of $s$ and $\epsilon$.

Proof. Note that $|\Phi_t|$ is uniformly bounded without Ricci assumption. If $\Phi$ solves (3.1) with boundary condition (3.8), then $\tilde{\Phi} = \Phi + at$ solves (3.1) with boundary condition
\[ \tilde{\Phi}(x, 0) = \phi_0, \quad \tilde{\Phi}(x, 1) = \phi_1 + A, \]
where $A$ is any constant. Thus we can assume that
\[ \Phi_t^2 \gg \Phi > 0 \]
by choosing the normalizing condition on $\phi_0, \phi_1$. Take
\[ h = \frac{1}{2} \left( |\nabla \Phi|^2 + b\Phi^2 \right). \]
One can calculate that
\[
\frac{1}{2} dP(\|\nabla \Phi\|^2) = (s\Phi_{tt} + (1-s))\Phi_{ik}^2 + (s(1 + \Delta \Phi) + (1-s))\Phi_{ik}^2 - 2s\Phi_{ik}\Phi_{ti}\Phi_{ik}
\]
\[\quad + (s\Phi_{tt} + (1-s))R_{ij}\Phi_i\Phi_j.\]

And
\[
dP(\frac{\Phi^2}{2}) = (s\Phi_{tt} + (1-s))\Delta \left( \frac{\Phi^2}{2} \right) + (s(1 + \Delta \Phi) + (1-s))(\Phi_{tt}^2 + \Phi_i^2)
\]
\[\quad - 2s\Phi_{ik}(\Phi_{ik} + \Phi_i\Phi_k)
\]
\[
= (s\Phi_{tt} + (1-s))\Phi_{ik} + (s(1 + \Delta \Phi) + (1-s))(\Phi_{tt}^2 - 2s\Phi_{ik}\Phi_{ti}\Phi_{ik}
\]
\[\quad + (s\Phi_{tt} + (1-s))R_{ij}\Phi_i\Phi_j + b(s\Phi_{tt} + (1-s))|\nabla \Phi|^2
\]
\[\quad - b\Phi((s\Phi_{tt} + (1-s)) + (1-s)\Phi_{tt} + \Delta \Phi))
\]
\[\quad + b(s(1 + \Delta \Phi) + (1-s))\Phi_i^2 - 2bs\Phi_{ik}\Phi_i\Phi_k. \quad (4.1)\]

It follows that
\[
dP(h) = (s\Phi_{tt} + (1-s))\Phi_{ik} + (s(1 + \Delta \Phi) + (1-s))\Phi_{ik}^2 - 2s\Phi_{ik}\Phi_{ti}\Phi_{ik}
\]
\[\quad + (s\Phi_{tt} + (1-s))R_{ij}\Phi_i\Phi_j + b(s\Phi_{tt} + (1-s))|\nabla \Phi|^2
\]
\[\quad - b\Phi((s\Phi_{tt} + (1-s)) + (1-s)\Phi_{tt} + \Delta \Phi))
\]
\[\quad + b(s(1 + \Delta \Phi) + (1-s))\Phi_i^2 - 2bs\Phi_{ik}\Phi_i\Phi_k. \quad (4.2)\]

We want to show that
\[
h \leq \max_{\partial H \times [0,1]} h + C,
\]
where $C$ is a universal constant. If not, $h$ obtains its maximum in the interior, at the point $p$. Note at the point $p$,
\[
h_t = \Phi_{ik}\Phi_k + b\Phi_t\Phi = 0.
\]

By (4.3), we get at the point $p$,
\[
dP(h) = (s\Phi_{tt} + (1-s))\Phi_{ik} + (s(1 + \Delta \Phi) + (1-s))\Phi_{ik}^2 - 2s\Phi_{ik}\Phi_{ti}\Phi_{ik}
\]
\[\quad + (s\Phi_{tt} + (1-s))R_{ij}\Phi_i\Phi_j + b(s\Phi_{tt} + (1-s))|\nabla \Phi|^2
\]
\[\quad - b\Phi((s\Phi_{tt} + (1-s)) + (1-s)\Phi_{tt} + \Delta \Phi))
\]
\[\quad + b(s(1 + \Delta \Phi) + (1-s))\Phi_i^2 + 2bs\Phi_{ik}\Phi_i\Phi_k. \quad (4.2)\]

Since at the point $p$, $D^2h \leq 0$, then $dP(h)(p) \leq 0$. By (1.12), we have
\[
dP(h) > ((b - C)|\nabla \Phi|^2 - b\Phi)(s\Phi_{tt} + (1-s))
\]
\[\quad + b\Phi_i^2(s(1 + \Delta \Phi) + (1-s)) - b\Phi((s\Phi_{tt} + \Delta \Phi).\]

We assume $s \geq \delta$ and $\Phi_i^2 \gg \Phi > 0$. Pick constant $b$ big enough, and if $|\nabla \Phi|^2$ is too big, we have
\[
dP(h) > 0.
\]

Contradiction.

\[\square\]
4.2 \( C^2 \) estimate

We have the following interior estimates

**Lemma 4.2.** If \( \Phi \) is a solution of (3.1), then \( \Phi \) satisfies the following a priori estimate

\[
0 < \triangle \Phi + 1 + \Phi_{tt} \leq C \left( \max_{\partial X \times [0,1]} |\Phi_{tt}|^2 + 1 \right),
\]

where \( C \) is a universal constant, independent of \( s \) and \( \epsilon \).

**Proof.** Denote

\[
f = \Phi_{tt} + 1 + \triangle \Phi, \quad h = \frac{1}{2}b \Phi^2 - b\Phi + A,
\]

where \( b, A \) are positive constants such that \( h > 0 \). Take

\[
\tilde{h} = f \exp(h).
\]

To get the interior estimates we want to show that

\[
\tilde{h} \leq \max_{\partial X \times [0,1]} \tilde{h} + C \tag{4.3}
\]

for some uniformly bounded constant \( C \). Note that if (4.3) holds then one can proceed to prove the boundary estimate saying that \( f \) is actually uniformly bounded. If (4.3) does not hold, \( \tilde{h} \) obtains its maximum in the interior, at the point \( p \). It follows that \( D^2 \tilde{h}(p) \leq 0 \) and \( dP(\tilde{h})(p) \leq 0 \).

Taking derivative,

\[
\tilde{h}_t = \exp(h)(fh_t + f_t), \quad \tilde{h}_k = \exp(h)(fk + fh_k),
\]

and

\[
\tilde{h}_{tt} = \exp(h)(fh_t^2 + 2f_t h_t + f h_{tt} + f_t), \quad \tilde{h}_{kk} = \exp(h)(fh_k^2 + 2fk h_k + fh_{kk} + fh_k).
\]

Also we have

\[
\tilde{h}_{tk} = \exp(h)(fk h_t + fh_k + fh_k h_t + fh_{tk}).
\]

Note also at the point \( p \), we have

\[
\tilde{h}_t = 0, \quad \tilde{h}_k = 0.
\]

It follows that

\[
f h_t + f_t = 0, \quad fh_k + f_k = 0.
\]

One can calculate that at the point,

\[
dP(\tilde{h}) = (s\Phi_{tt} + (1-s))\Delta \tilde{h} + (s(1 + \Delta \Phi) + (1-s))\tilde{h}_{tt} - 2s\Phi_{tk}\tilde{h}_{tk}
\]

\begin{align*}
&= (s\Phi_{tt} + (1-s))\exp(h)(fh_t^2 + 2f_t h_t + fh_{tt} + f_t) \\
&\quad + (s(1 + \Delta \Phi) + (1-s))\exp(h)(fh_k^2 + 2fk h_k + fh_{kk} + fh_k) \\
&\quad - 2s\Phi_{tk}\exp(h)(fk h_t + fh_k + fh_k h_t + fh_{tk}) \\
&\quad + \exp(h)((s\Phi_{tt} + (1-s))h_{kk} + (s(1 + \Delta \Phi) + (1-s))h_{tt} - 2s\Phi_{tk} h_{tk}) \\
&\quad + \exp(h)((s\Phi_{tt} + (1-s))f_k + (s(1 + \Delta \Phi) + (1-s))f_t - 2s\Phi_{tk} f_{tk}) \\
&\quad + \exp(h)f((s\Phi_{tt} + (1-s))h_{kk} + (s(1 + \Delta \Phi) + (1-s))h_{tt} - 2s\Phi_{tk} h_{tk}) \\
&\quad + (s\Phi_{tt} + (1-s))\exp(h)(fh_k^2 + 2fk h_k) + (s(1 + \Delta \Phi)) \\
&\quad + (1-s)\exp(h)(fh_k^2 + 2fk h_k + fh_k) - 2s\Phi_{tk}\exp(h)(fh_t + f_k h_k + fh_{k} h_t) \\
&\quad = \exp(h)(fdP(h) + dP(f) - Q(h, f)), \tag{4.4}
\end{align*}

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where
\[ Q(h, f) = f((s\Phi_{tt} + (1 - s))h_k^2 + (s(1 + \Delta\Phi) + (1 - s))h_i^2 - 2s\Phi_{ik}h_i h_k). \]

(4.5)

Now we carry out \(dP(f), dP(h)\). We know that (c. f. (3.18))
\[ dP(f) = s(L + M) + 2sR_{ij}\Phi_{ti}\Phi_{tj} \geq 2sR_{ij}\Phi_{ti}\Phi_{tj}, \]

It is easy to get that at the point \(p\),
\[ dP(h) = b(\Phi_{tt} + 1 + \Delta\Phi) - 2b\epsilon \geq b(\Phi_{tt} + 1 + \Delta\Phi) - 2b. \]

(4.6)

By (4.4) and (4.6) we have
\[ dP(h) \geq b\exp(h)(\Phi_{tt} + 1 + \Delta\Phi)\]

(4.7)

It is easy to see that
\[ R_{ij}\Phi_{ti}\Phi_{tj} > -C(\Phi_{tt} + 1 + \Delta\Phi)^2. \]

By (4.5) it is clear that
\[ Q(h, f) \leq C(\Phi_{tt} + 1 + \Delta\Phi)^2. \]

It follows that
\[ dP(h) \geq \exp(h) \left\{ (b - C)(\Phi_{tt} + 1 + \Delta\Phi)^2 - 2b(\Phi_{tt} + 1 + \Delta\Phi) \right\}. \]

We can assume that
\[ \Phi_{tt} + 1 + \Delta\Phi > \frac{2b}{b - C} \]

at the point \(p\). Contradiction with \(dP(h) \leq 0\) at the point \(p\). \(\square\)

5 The space \(H\)

In this section, we discuss the property of the space \(H\) by using the weak solution of the geodesic equation. The discussion below follows closely Chen [5], Calabi-Chen [4] in the case of the space of Kähler metrics.

5.1 Uniqueness of the weak solution

For the weak solution obtained for the geodesic equation, it fits the standard notion of viscosity solution developed in fully nonlinear equations. But since we obtain a global approximation of the weak solution, we will use the following approximation instead of viscosity solution.

**Definition 5.1.** A continuous function \(\Phi\) in \(X \times [0, 1]\) is a weak \(C^0\) solution to the geodesic equation \(f\) with prescribed boundary data if for any \(\epsilon > 0\), there exists a smooth function \(\tilde{\Phi} \in \mathcal{H}\) such that
\[ |\Phi - \tilde{\Phi}| < \delta = \delta(\epsilon) \]
and \(\tilde{\Phi}\) solves
\[ Q(D^2\tilde{\Phi}) = \epsilon \]
with the same boundary data, where \(\delta = \delta(\epsilon)\) is a constant depending on \(\epsilon\) and \(\delta \to 0\) when \(\epsilon \to 0\).
Obviously, the weak solution we obtain is a weak $C^0$ solution of the geodesic equation (1.1).

**Theorem 5.2.** Suppose $\Phi, \Psi$ are two $C^0$ weak solutions to the geodesic equation (1.1) with prescribed boundary data $(\phi_0, \phi_1)$ and $(\psi_0, \psi_1)$. Then

$$\max_{X \times [0,1]} |\Phi - \Psi| \leq \max_{\partial (X \times [0,1])} (|\phi_0 - \psi_0|, |\phi_1 - \psi_1|).$$

**Proof.** For any $\epsilon > 0$, we have $\tilde{\Phi}, \tilde{\Psi}$ solving $Q(D^2 \tilde{\Phi}) = Q(D^2 \tilde{\Psi}) = \epsilon$ with respect to the boundary condition. Then we know that $|\Phi - \tilde{\Phi}| < \delta, |\Psi - \tilde{\Psi}| < \delta$.

We want to show that

$$\max_{X \times [0,1]} |\tilde{\Phi} - \tilde{\Psi}| \leq \max_{\partial (X \times [0,1])} (|\phi_0 - \psi_0|, |\phi_1 - \psi_1|).$$

If the boundary conditions are the same, namely, $\phi_0 = \psi_0, \phi_1 = \psi_1$, we have $\tilde{\Phi} = \tilde{\Psi}$. In general, we can use the same trick as in Proposition 3.7 to get the inequality above. If $\max(\Phi - \tilde{\Phi}) > \max_{\partial (X \times [0,1])} (|\phi_0 - \psi_0|, |\phi_1 - \psi_1|)$, then $\Phi - \tilde{\Psi} - \lambda t (1 - t)$ obtain its maximum in interior, where $\lambda > 0$ is small enough. It follows that $D^2 \tilde{\Psi} > D^2 \Phi$. But $Q(D^2 \Phi) = Q(D^2 \Psi) = \epsilon$ implies that $Q(D^2 \Psi - D^2 \Phi) < 0$, contradiction. Switch $\tilde{\Phi}, \tilde{\Psi}$, we can get the inequality as promised.

It follows that

$$|\Phi - \Psi| < \max_{\partial (X \times [0,1])} (|\phi_0 - \psi_0|, |\phi_1 - \psi_1|) + 2\delta.$$

Let $\epsilon \to 0$, we get that

$$|\Phi - \Psi| < \max_{\partial (X \times [0,1])} (|\phi_0 - \psi_0|, |\phi_1 - \psi_1|).$$

□

As a direct consequence we have

**Corollary 5.3.** The weak solution of the geodesic equation (1.1) is unique with the fixed boundary data.

### 5.2 $H$ is a metric space

In this section we want to prove that $H$ is a metric space and the weak $C^2$ geodesic realizes the global minimum of the length over all paths. For simplicity, for any $\Phi \in H$, we fix the normalization

$$\int_M \Phi dg = 0.$$

Since the solution we obtain does not have enough derivatives, we use the solution $Q(D^2 \tilde{\Phi}) = \epsilon$ to approximate the weak solution.
Definition 5.4. Let $\phi_1, \phi_2$ be two points in $\mathcal{H}$. Then we know there exists a unique weak geodesic connecting these two points. Define the geodesic distance

$$d(\phi_1, \phi_2) = \int_0^1 dt \sqrt{\int_M \dot{\Phi}^2 (1 + \Delta \Phi) \, dg}$$

as the length of this geodesic, where $\Phi$ solves the geodesic equation weakly with boundary condition $\phi_1, \phi_2$.

Then we have the following

Lemma 5.5. Suppose $\Phi(t)$ is a weak $C^2$ geodesic in $\mathcal{H}$ from 0 to $\phi$ and we normalize $\Phi$ such that

$$\int_M \Phi \, dg = 0.$$ 

Also we can assume that $Vol(M, g) = 1$. Then we have

$$d(0, \phi) \geq \max \left( \sqrt{\int_{\phi > 0} \phi^2 (1 + \Delta \phi) \, dg}, \sqrt{\int_{\phi < 0} \phi^2 \, dg} \right).$$

In other words, the length of any weak $C^2$ geodesic is strictly positive.

Proof. Suppose $\tilde{\Phi}$ is the solution of

$$Q(D^2 \tilde{\Phi}) = \epsilon$$

with the same boundary condition. It is easy to see that

$$d(0, \phi) = \lim_{\epsilon \to 0} d_{\epsilon}(0, \phi),$$

where

$$d_{\epsilon} = \int_0^1 dt \sqrt{\int_M \dot{\Phi}^2 (1 + \Delta \Phi) \, dg}.$$ 

Denote the energy element as

$$E_{\epsilon}(t) = \int_M \dot{\Phi}^2 (1 + \Delta \Phi) \, dg.$$ 

It is easy to see that

$$\left| \frac{d}{dt} E_{\epsilon}(t) \right| \leq C \epsilon,$$

where $C$ is a universal constant. It follows that

$$|E_{\epsilon}(t_1) - E_{\epsilon}(t_2)| \leq C \epsilon |t_1 - t_2| \leq C \epsilon.$$

In particular, we have $E(t)$ is a constant for any $t$ even $\Phi$ has no enough derivative. Note that if $\Phi$ is smooth this is a direct consequence of the geodesic equation. Since $\ddot{\Phi} > 0$, it follows that

$$\dot{\Phi}(0) < \phi < \dot{\Phi}(1).$$ 

It implies

$$E_{\epsilon}(0) = \int_M \dot{\Phi}^2(0) \, dg > \int_{\phi < 0} \phi^2 \, dg.$$ 

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and

\[ E_\epsilon(1) = \int_M \dot{\Phi}^2(1 + \Delta \phi) dg > \int_{\phi > 0} \phi^2(1 + \Delta \phi) dg. \]

Thus,

\[ d_\epsilon(0, \phi) > \max \left( \sqrt{\int_{\phi < 0} \phi^2 \, dg} - C_\epsilon, \sqrt{\int_{\phi > 0} \phi^2(1 + \Delta \phi) \, dg} - C_\epsilon \right). \]

So we can get

\[ d(0, \phi) \geq \max \left( \sqrt{\int_{\phi > 0} \phi^2(1 + \Delta \phi) \, dg}, \sqrt{\int_{\phi < 0} \phi^2 \, dg} \right). \]

We need the next geodesic approximation lemma.

**Lemma 5.6.** Suppose \( C_i : \phi_i(s) : [0, 1] \to H(i = 0, 1) \) are two smooth curves in \( H \). For any \( \epsilon > 0 \), there exist two parameter families of smooth curves \( \Phi(t, s, \epsilon) \) solving

\[ Q(D^2 \Phi) = \epsilon \]

with boundary condition

\[ \Phi(0, s, \epsilon) = \phi_0(s), \quad \Phi(1, s, \epsilon) = \phi_1(s) \]

satisfying the following:

1. There exists a uniformly bounded constant \( C = C(M, \phi_0, \phi_1) \) such that

   \[
   |\Phi| + \left| \frac{\partial \Phi}{\partial t} \right| + \left| \frac{\partial \Phi}{\partial s} \right| \leq C; \quad 0 < \frac{\partial^2 \Phi}{\partial t^2} \leq C; \quad \frac{\partial^2 \Phi}{\partial s^2} \leq C.
   \]

2. The convex curve \( C(s, \epsilon) \) converges to the unique geodesic between \( \phi_0(s) \) and \( \phi_1(s) \) in the weak \( C^2 \) topology.

3. Define the energy element along \( C(s, \epsilon) \) as

   \[ E(t, s, \epsilon) = \int_M \left| \frac{\partial \Phi}{\partial t} \right|^2 (1 + \Delta \Phi) \, dg. \]

   There exists a uniform constant \( C \)

   \[
   \left| \frac{\partial E}{\partial t} \right| \leq C_\epsilon.
   \]

**Proof.** The lemma is clear except

\[
\left| \frac{\partial \Phi}{\partial s} \right| \leq C, \quad \frac{\partial^2 \Phi}{\partial s^2} \leq C.
\]

The inequalities above follow from the maximum principle directly since

\[
dQ \left( \frac{\partial \Phi}{\partial s} \right) = 0, \quad dQ \left( \frac{\partial^2 \Phi}{\partial s^2} \right) \geq 0,
\]

where the last inequality is a consequence of concavity of the operator \( Q \).
Next we show that $d$ is a continuous function in $\mathcal{H}$. First we have

**Lemma 5.7.** Suppose $\phi_0 \in \mathcal{H}$, then for any $\phi \in \mathcal{H}$, $d(\phi_0, \phi) \to 0$ when $\phi \to \phi_0$ in $C^k$ topology for $k \geq 4$.

*Proof.* This is really a consequence of Theorem 5.2. $\Phi_0(t) \equiv \phi_0$ is a geodesic with length zero. Let $\Phi(t)$ be the weak $C^2$ geodesic which connects $\phi_0$ and $\phi$, then by Theorem 5.2 $|\Phi(t) - \Phi_0(t)| \leq |\phi_0 - \phi|$. For any $x$ fixed, apply the interpolation inequality for $\Phi(t, x) - \Phi_0(x)$ in $[0, 1]$ such that, for any $\epsilon_1 > 0$,

$$
\left| \frac{\partial}{\partial t} (\Phi(t, x) - \phi_0(x)) \right| \leq C(\epsilon_1) \max \frac{\partial^2}{\partial t^2} (\Phi(t) - \phi_0).
$$

In the case of $t \in [0, 1]$ the proof of the above interpolation inequality is straightforward. We can assume that $|\phi - \phi_0|_{C^4} \leq 1$, hence $|\Phi_t| \leq C_1$ for $C_1$ depending only on $M, g, \phi_0$. It then follows that

$$
|\Phi_t| \leq C(\epsilon_1)|\phi(x) - \phi_0(x)| + C_1 \epsilon_1,
$$

It then implies that when $|\phi - \phi_0| \to 0$, $|\Phi_t| \to 0$. Hence $d(\phi_0, \phi) \to 0$. \(\square\)

We shall then prove the triangle inequality.

**Theorem 5.8.** Suppose $C$ is a smooth simple curve defined by $\phi(s) : s \in [0, 1] \to \mathcal{H}$. Let $\psi \in \mathcal{H}$ be a point which is not on $C$. For any $s$,

$$
d(\psi, \phi(s)) \leq d(\psi, \phi(0)) + d_C(\phi(0), \phi(s)),
$$

where $d_C$ denotes the length along the curve $C$. In particular, we have the following triangle inequality

$$
d(\psi, \phi(1)) \leq d(\psi, \phi(0)) + d(\phi(0), \phi(1)).
$$

*Proof.* For any $\epsilon > 0$, we can get a two parameter families of smooth curve $C(t, s, \epsilon) : \Phi(t, s, \epsilon) \in \mathcal{H}$ solving

$$
Q(D^2 \Phi) = \epsilon
$$

with the boundary conditions corresponding $(\psi, \phi(s))$. Denote

$$
L(s, \epsilon) = d_C(\psi, \phi(s)) = \int_0^1 dt \sqrt{\int_M \left| \frac{\partial \Phi}{\partial t} \right|^2 (1 + \triangle \Phi) dg}
$$

and

$$
l(s) = d_C(\phi(0), \phi(s)) = \int_0^s d\tau \sqrt{\int_M \left| \frac{\partial \Phi}{\partial \tau} \right|^2 (1 + \triangle \Phi) dg}.
$$

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Let $F(s, \epsilon) = L(s, \epsilon) + l(s)$. Taking derivatives,

$$
\frac{dL(s, \epsilon)}{ds} = \int_0^1 \frac{dt}{\sqrt{E(t, s, \epsilon)}} \int_M \left( \frac{\partial \Phi}{\partial t} \frac{\partial^2 \Phi}{\partial t \partial s} + \frac{1}{2} \Phi_s^2 \Delta \Phi_s \right) dg
$$

$$
= \int_0^1 \frac{dt}{\sqrt{E(t, s, \epsilon)}} \left\{ \frac{\partial}{\partial t} \int_M \Phi_t \Phi_s (1 + \Delta \Phi) dg - \epsilon \int_M \Phi_s dg \right\}
$$

$$
= \left[ \frac{1}{\sqrt{E}} \int_M \Phi_t \Phi_s (1 + \Delta \Phi) dg \right]_0^1 - \epsilon \int_0^1 \frac{dt}{\sqrt{E(t, s, \epsilon)}} \int_M \Phi_s dg
$$

$$
+ \int_0^1 dt \left( E(t, s, \epsilon) - \frac{1}{2} \int_M \Phi_t \Phi_s (1 + \Delta \Phi) dg \right) \int_M \Phi_t \Phi_s dg
$$

$$
\geq \left[ \frac{1}{\sqrt{E}} \int_M \Phi_t \Phi_s (1 + \Delta \Phi) dg \right]_0^1 - C\epsilon
$$

$$
= \frac{1}{\sqrt{E(1, s, \epsilon)}} \int_M \Phi_t (1, s, \epsilon) \Phi_s (1, s, \epsilon) (1 + \Delta \phi(s)) dg - C\epsilon,
$$

where we use the fact $\Phi_s(0, s, \epsilon) = 0$. Also we have

$$
\frac{dl(s)}{ds} = \sqrt{\int_M \Phi_s^2 (1 + \Delta \phi(s)) dg}.
$$

By the Schwartz inequality, we have

$$
\sqrt{E(1, s, \epsilon)} \sqrt{\int_M \Phi_s^2 (1 + \Delta \phi(s)) dg} \geq \left| \int_M \Phi_t (1, s, \epsilon) \Phi_s (1, s, \epsilon) (1 + \Delta \phi(s)) dg \right|,
$$

it follows that

$$
\frac{dF(s, \epsilon)}{ds} \geq -C\epsilon.
$$

So $F(s, \epsilon) - F(0, \epsilon) \geq -C\epsilon$. Let $\epsilon \to 0$, we get that

$$
d(\psi, \phi(1)) \leq d(\psi, \phi(0)) + d_c(\phi(0), \phi(1)).
$$

To get the triangle inequality, let $C$ be the curve $\Phi(s)$ solving

$$
Q(D^2 \Phi) = \epsilon.
$$

And we can get that

$$
d(\psi, \phi(1)) \leq d(\psi, \phi(0)) + d_t(\phi(0), \phi(1)).
$$

Let $\epsilon \to 0$ again, we get

$$
d(\psi, \phi(1)) \leq d(\psi, \phi(0)) + d(\phi(0), \phi(1)).
$$

\[\square\]

**Remark 5.9.** Theorem 5.8 is used to show that the length of any simple curve is always longer than the length of the geodesic between two end points (see Corollary 5.10). This will follow directly from Theorem 5.8 with the assumption that $\psi$ is not on the curve $C$ and Lemma 5.7. This lapse of the argument was kindly pointed out by the anonymous referee.
Indeed, the previous version of the paper did not assume that $\psi$ is not on the curve. We thank the referee for pointing out this. However, this does not affect all the main results in this section in view of the result in Lemma 5.7.

**Corollary 5.10.** The geodesic distance between any two points in $\mathcal{H}$ realizes the minimum of the lengths over all possible paths.

**Proof.** We just need to prove that the result holds for any smooth simple curve $C : \phi(s) : [0, 1] \to \mathcal{H}$. Take $\psi \to \phi(0)$ in $C^4$ such that $\psi$ is not on the curve $C$. By Theorem 5.8, we get that

$$d(\psi, \phi(1)) \leq d(\psi, \phi(0)) + d_C(\phi(0), \phi(1)).$$

Now let $\psi \to \phi(0)$ and by Lemma 5.7, we get

$$d(\phi(0), \phi(1)) \leq d_C(\phi(0), \phi(1)).$$

$\square$

**Theorem 5.11.** The space $(\mathcal{H}, d)$ is a metric space. Moreover, the distance function is at least $C^1$.

**Proof.** The only thing we need to show is the differentiability of the distance function. Follow from the proof of Theorem 5.8 for any $\epsilon > 0$ we have

$$\left| \frac{dL(s, \epsilon)}{ds} - \frac{1}{\sqrt{E(1, s, \epsilon)}} \int_M \Phi_t \Phi_s (1 + \Delta \Phi) dg \right| \leq C\epsilon.$$

It easily follows from above that

$$\lim_{s \to s_0} \frac{d(\psi, \phi(s)) - d(\psi, \phi(s_0))}{s - s_0} = \frac{1}{\sqrt{E(1, s_0, \epsilon)}} \int_M \Phi_t (1, s_0) \Phi_s (1, s_0) (1 + \Delta \phi(s_0)) dg.$$

$\square$

### 5.3 The curvature of $\mathcal{H}$

Donaldson [11] has shown that the space $\mathcal{H}$ has non-positive sectional curvature formally with the natural metric. However, we can only demonstrate that the geodesic equation has a weak $C^2$ solution. To overcome this difficulty, we show that the space $\mathcal{H}$ has non-positive curvature in the Alexandrov’s sense by following Calabi-Chen [4].

**Theorem 5.12.** The space $(\mathcal{H}, d)$ is a non-positive curved space on any Riemannian manifold $(X, g)$ in the following sense. Let $A, B, C$ be three points in $\mathcal{H}$. For any $\lambda \in [0, 1]$, let $P$ be the point on the geodesic path connecting $B$ and $C$ such that $d(B, P) = \lambda d(B, C)$ and $d(P, C) = (1 - \lambda)d(B, C)$. Then the following inequality holds:

$$d^2(A, P) \leq (1 - \lambda)d^2(A, B) + \lambda d^2(A, C) - \lambda(1 - \lambda)d^2(B, C).$$

To prove this theorem, we need the following lemma, which in essence says that the Jacobi vector field along any geodesic grows super-linearly.
Lemma 5.13. Let $\Phi(t, s, \epsilon)$ be the two parameter families of approximation of geodesics as in Lemma 5.6. Let $Y = \Phi_s$ be the deformation vector fields and $X = \Phi_t$ be the tangent vector fields along the approximating geodesic. Then we have

$$\langle DX DX Y, Y \rangle \geq 0.$$ 

Note $D$ is the covariant derivative defined in Section 2. Moreover, if we assume that $Y(0, s, \epsilon) = 0$, we have at $t = 1$

$$\langle Y, DX Y \rangle \geq \langle Y, Y \rangle$$

Proof. By definition, the length of $Y$ is given by

$$|Y|^2 = \langle Y, Y \rangle = \int_M \Phi_s^2(1 + \Delta \Phi) dg.$$ 

It follows that

$$\frac{1}{2} \frac{\partial}{\partial t} |Y|^2 = \langle DX Y, Y \rangle = \langle DX X, Y \rangle.$$ 

Let $K(X, Y)$ be the sectional curvature of the space $H$ at point $\Phi(t, s, \epsilon)$. By the calculation of Donaldson [11], we know that $K(X, Y) \leq 0$.

We should emphasize that the calculation in [11] is algebraic and since $\Phi(t, s, \epsilon) \in H$, so we can use in our setting. Therefore, we have

$$\frac{1}{2} \frac{\partial^2}{\partial t^2} |Y|^2 = \langle DX Y, DX Y \rangle + \langle DX DX X, Y \rangle \geq 0.$$ 

By definition, it is easy to see that

$$DX X = \Phi_{tt} - \frac{1}{1 + \Delta \Phi} \langle \nabla \Phi_t, \nabla \Phi_t \rangle = \frac{\epsilon}{1 + \Delta \Phi}. $$

Also we can get

$$DX X = \frac{\epsilon}{1 + \Delta \Phi} \frac{\partial}{\partial s} \left( \frac{\epsilon}{1 + \Delta \Phi} \right) = \frac{\epsilon \Delta \Phi_s}{(1 + \Delta \Phi)^2} - \frac{1}{1 + \Delta \Phi} \nabla \Phi_s \cdot \nabla \left( \frac{\epsilon}{1 + \Delta \Phi} \right).$$

It follows that

$$\langle DX DX X, Y \rangle = \int_M \Phi_s \left\{ - \frac{\epsilon \Delta \Phi_s}{(1 + \Delta \Phi)^2} - \frac{1}{1 + \Delta \Phi} \nabla \Phi_s \cdot \nabla \left( \frac{\epsilon}{1 + \Delta \Phi} \right) \right\} (1 + \Delta \Phi) dg$$

$$= \epsilon \int_M \left\{ - \frac{\epsilon \Delta \Phi_s}{1 + \Delta \Phi} - \Phi_s \nabla \Phi_s \nabla \left( \frac{1}{1 + \Delta \Phi} \right) \right\} dg$$

$$= \epsilon \int_M \frac{\nabla \Phi_s^2}{1 + \Delta \Phi} dg = \epsilon \left\langle \frac{\nabla \Phi_s}{1 + \Delta \Phi}, \frac{\nabla \Phi_s}{1 + \Delta \Phi} \right\rangle \geq 0.$$
In particular, we have
\[ \langle D_X D_X Y, Y \rangle = \langle D_Y D_X X, Y \rangle - K(X, Y) \geq 0. \]
It follows that
\[ \frac{1}{2} \frac{\partial^2}{\partial t^2} |Y|^2 \geq |D_X Y|^2. \]
Note that
\[ \frac{1}{2} \frac{\partial^2}{\partial t^2} |Y|^2 = |Y| \frac{\partial^2}{\partial t^2} |Y| + \left( \frac{\partial}{\partial t} |Y| \right)^2, \]
and it is easy to see that
\[ |D_X Y|^2 \geq \left( \frac{\partial}{\partial t} |Y| \right)^2, \]
we get that
\[ \frac{\partial^2}{\partial t^2} |Y| \geq 0. \]
Namely, \(|Y|\) is a convex function of \(t\). If \(Y(0) = 0\), it follows that
\[ \frac{\partial}{\partial t} |Y(t)|_{t=1} \geq |Y(1)|, \]
namely at \(t = 1\),
\[ \langle D_X Y, Y \rangle \geq \langle Y, Y \rangle. \]

Now we are in the position to prove Theorem 5.12.

Proof. For any \(A, B, C \in H\), take \(\phi_0(s) \equiv A, \phi_1(s) \) to be an approximation of geodesic connecting \(B, C\), then apply Lemma 5.6, we can get a two parameter families \(\Phi(t, s, \epsilon) \in H\). Denote \(P(s)\) to be the point \(\Phi(1, s, \epsilon)\). Let \(E(s)\) be the energy of curve \(\Phi(t, s, \epsilon)\) connection \(A\) and \(P(s)\). When \(\epsilon \to 0\), it is easy to see that \(E(s)\) is the square of the geodesic distance from \(A\) to \(P(s)\). We have
\[ E(s) = \int_0^1 \int_M \Phi_t^2 (1 + \Delta \Phi) dt. \]
As in Theorem 5.8, it is easy to get that
\[ \frac{1}{2} \frac{d}{ds} E \left( \frac{dE}{ds} \right) = \int_M \Phi_t \Phi_s (1 + \Delta \Phi) d\mu - \epsilon \int_0^1 \int_M \Phi_s d\mu. \]
Use the notation in Lemma 5.13, we get
\[ \frac{1}{2} \frac{d}{ds} E \left( \frac{dE}{ds} \right) = \langle X, Y \rangle_{t=1} - \epsilon \int_0^1 \int_M \Phi_s d\mu. \]
Then it follows that
\[ \frac{1}{2} \frac{d^2 E}{ds^2} = \frac{d}{ds} \langle X, Y \rangle_{t=1} - \epsilon \int_0^1 \int_M \Phi_s d\mu dt \]
\[ = \langle D_Y X, Y \rangle_{t=1} + \langle X, D_Y Y \rangle_{t=1} - \epsilon \int_0^1 \int_M \Phi_s d\mu dt \]
\[ \geq \langle Y, Y \rangle_{t=1} + \langle X, D_Y Y \rangle_{t=1} - C\epsilon, \]
\[ \geq \langle Y, Y \rangle_{t=1} - C\epsilon, \]
and
\[ \frac{1}{2} \frac{d^2 E}{ds^2} = \frac{d}{ds} \langle X, Y \rangle_{t=1} - \epsilon \int_0^1 \int_M \Phi_s d\mu dt \]
\[ = \langle D_Y X, Y \rangle_{t=1} + \langle X, D_Y Y \rangle_{t=1} - \epsilon \int_0^1 \int_M \Phi_s d\mu dt \]
\[ \geq \langle Y, Y \rangle_{t=1} + \langle X, D_Y Y \rangle_{t=1} - C\epsilon, \]
\[ \geq \langle Y, Y \rangle_{t=1} - C\epsilon, \]
and
\[ \frac{1}{2} \frac{d^2 E}{ds^2} = \frac{d}{ds} \langle X, Y \rangle_{t=1} - \epsilon \int_0^1 \int_M \Phi_s d\mu dt \]
\[ = \langle D_Y X, Y \rangle_{t=1} + \langle X, D_Y Y \rangle_{t=1} - \epsilon \int_0^1 \int_M \Phi_s d\mu dt \]
\[ \geq \langle Y, Y \rangle_{t=1} + \langle X, D_Y Y \rangle_{t=1} - C\epsilon, \]
where we use Lemma 5.13. Note at $t = 1$,

$$D_Y Y = \frac{\epsilon}{1 + \Delta \Phi}$$

We have

$$\langle X, D_Y Y \rangle_{t=1} = \epsilon \int_M \Phi_t dg.$$  

Also we have

$$\langle Y, Y \rangle = \int_M \Phi^2_s (1 + \Delta \Phi) dg \geq E(\Phi(1, \cdot, \epsilon)) - C\epsilon,$$

where $E(\Phi(1, \cdot, \epsilon))$ is the energy of the path $\Phi(1, s, \epsilon)$. We get that

$$\frac{1}{2} \frac{d^2 E}{ds^2} \geq E(\Phi(1, \cdot, \epsilon)) - C\epsilon.$$  

It follows that

$$E(s) \leq (1 - s) E(0) + s E(1) - s(1 - s) E(\Phi(1, \cdot, \epsilon)) - C\epsilon.$$  

Now fix $s$, let $\epsilon \to 0$. Each energy element approaches the square of the length of the path. Thus we get

$$\left| AP(s) \right|^2 \leq (1 - s)|AB|^2 + s|AC|^2 - s(1 - s)|BC|^2.$$

References

[1] C. Arezzo, G. Tian, *Infinite geodesic rays in the space of Kähler potentials*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 2 (2003), no. 4, 617–630.

[2] Z. Blocki, *On geodesics in the space of Kähler metrics*, preprint.

[3] L. A. Caffarelli, X. Cabrè, *Fully nonlinear elliptic equations*, American Mathematical Society Colloquium Publications, 43. American Mathematical Society, Providence, RI, 1995.

[4] E. Calabi, X. X. Chen, *The space of Kähler metrics II*, J. Differential. Geom. 61 (2002), no.2, 173-193.

[5] X. X. Chen, *The space of Kähler metrics*, J. Differential. Geom. 56 (2000), no.2, 189-234.

[6] X. X. Chen, *Space of Kähler metrics III–On the lower bound of the Calabi energy and geodesic distance*, arXiv:math/0606228.

[7] Y. Chen, L. Wu; *Second order elliptic equations and systems of elliptic equations*, Translations of Mathematical Monographs, Amer. Math. Soc., 1998.

[8] S. Donaldson, *Symmetric spaces, Kähler geometry and Hamiltonian dynamics*, Northern California Symplectic Geometry Seminar, 13–33, Amer. Math. Soc. Transl. Ser. 2, 196, Amer. Math. Soc., Providence, RI, 1999.
[9] S. K. Donaldson, *Scalar curvature and stability of toric varieties*, J. Differential Geom. 62 (2002), no. 2, 289–349.

[10] S. K. Donaldson, *Lower bound of the Calabi functional*, J. Differential Geom. 70 (2005), no. 3, 453–472.

[11] S. Donaldson, *Nahm’s equations and free-boundary problems*, arXiv:0709.0184.

[12] L. Evans, *Classical solutions of fully nonlinear, convex, second order elliptic equations*, Acta Math. 148 (1982), 47–157.

[13] D. Gilbarg, N. Trudinger, *Elliptic partial differential equations of second order*.

[14] B. Guan, *The Dirichlet problem for complex Monge-Ampère equations and regularity of the pluri-complex Green function*. Comm. Anal. Geom. 6 (1998), no. 4, 687–703.

[15] B. Guan, J. Spruck, *Boundary-value problems on $S^n$ for surfaces of constant Gauss curvature*. Ann. of Math. (2) 138 (1993), no. 3, 601–624.

[16] P. F. Guan, *$C^2$ a priori estimates for degenerate Monge-Ampère equations*. Duke Math. J. 86 (1997), no. 2, 323–346.

[17] W.Y. He, *The Donaldson equation*, http://arxiv.org/abs/0810.4123.

[18] N. Krylov, *Boundedly inhomogeneous elliptic and parabolic equations*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 46 (1982), no. 3, 487–523. English Translation in Math. USSR Izv. 20 (1983).

[19] N. Krylov, *Boundedly inhomogeneous elliptic and parabolic equations in a domain*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 47 (1983), no. 1, 75–108. English Translation in Math. USSR Izv. 22, 67-97 (1984).

[20] T. Mabuchi, *Some symplectic geometry on compact Kähler manifolds*, I. Osaka J. Math. 24 (1987), no. 2, 227–252.

[21] S. Semmes, *Complex Monge-Ampère and symplectic manifolds*, Amer. J. Math. 114 (1992), no. 3, 495–550.

[22] Y. Yuan, *A priori estimates for solutions of fully nonlinear special Lagrangian equations*, Ann. Inst. H. Poincare Anal. Non Lineaire 18 (2001), 261-270.

[23] Y. Yuan, *A Bernstein problem for special Lagrangian equations*. Invent. Math. 150 (2002), no. 1, 117–125.

Xiuxiong Chen
xxchen@math.wisc.edu
Department of Mathematics
University of Wisconsin-Madison

Weiyong He
whe@uoregon.edu
Department of Mathematics
University of Oregon