GENERALIZATIONS OF A RESULT OF JARNIK ON SIMULTANEOUS APPROXIMATION

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Abstract. Consider a non-increasing function $\Psi$ from the positive reals to the positive reals with decay $o(1/x)$ as $x$ tends to infinity. Jarnik proved in 1930 that there exist real numbers $\zeta_1, \ldots, \zeta_k$ together with 1 linearly independent over $\mathbb{Q}$ with the property that all $q\zeta_j$ have distance to the nearest integer smaller than $\Psi(q)$ for infinitely many positive integers $q$, but not much smaller in a very strict sense. We give an effective generalization of this result to the case of successive powers of real $\zeta$. The method also allows to generalize corresponding results for $\zeta$ contained in special fractal sets such as the Cantor set.

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1. Best constants for approximation of real numbers

1.1. Introduction. We study the simultaneous approximation properties of vectors of the form $(\zeta, \zeta^2, \ldots, \zeta^k)$ for real $\zeta$, by rational numbers with coinciding denominator. Dirichlet’s Theorem on simultaneous Diophantine approximation asserts, that for any given $\zeta = (\zeta_1, \ldots, \zeta_k) \in \mathbb{R}^k$ the inequality

$$\max_{1 \leq j \leq k} |q\zeta_j - p_j| \leq q^{-\frac{1}{k}}$$

has a solution $(q, p_1, \ldots, p_k) \in \mathbb{N} \times \mathbb{Z}^k$ with arbitrarily large $q$, where $\mathbb{N} = \{1, 2, \ldots\}$ throughout. Clearly, assuming $q > 0$ is no restriction, since we may multiply any vector $(q, p_1, \ldots, p_k)$ by $-1$ without affecting the absolute values in (1). Furthermore, if $q$ tends to infinity so do the $p_j$ and vice versa. We will not explicitly mention these facts in the sequel in similar settings. Property (1) is in particular true for vectors of successive powers, i.e. $\zeta_j = \zeta^j$.

A question studied by Jarnik, in the general setting of vectors $\zeta \in \mathbb{R}^k$, can be roughly explained as follows. Consider a function $\Psi : \mathbb{R}_{>0} \mapsto \mathbb{R}_{>0}$ that decreases sufficiently fast that (1) can be satisfied. Is it possible to find vectors $\zeta \in \mathbb{R}^k$ for which

$$\max_{1 \leq j \leq k} |q\zeta_j - p_j| \leq \Psi(q)$$

has arbitrarily large solutions $(q, p_1, \ldots, p_k) \in \mathbb{N} \times \mathbb{Z}^k$, but this is no longer true if we replace $\Psi$ by some certain slightly smaller function $\psi$, i.e. for which $\psi(x) < \Psi(x)$ for
all \(x\). In fact, he made the usual additional assumption that \(\zeta\) is \(\mathbb{Q}\)-linearly independent together with 1. In particular the case \(\psi(x) = c\Psi(x)\) for some constant \(c \in (0,1)\) is of interest. For precise definitions see Section 1.2. Jarnik established results on this question in [5]. Satz 5 in [5] establishes results in the somehow most general case \(\Psi(x) = o(x^{-1/k})\), Satz 6 provides stronger results considering only functions \(\Psi(x) = o(x^{-1})\), where \(x \to \infty\) is meant in both cases. We will formulate Satz 6 in Section 1.2. The present paper aims to generalize Satz 6 to the case of successive powers in several ways. Moreover, Section 4 deals with simultaneous approximation of numbers in fractal sets, and generalizes a result by Bugeaud [1].

1.2. Formulation of the main results. We use a similar notation to the one in [1] in the sequel. We should mention we prefer to use the notion of linear forms instead of rational approximations, which essentially implies a change in the \(x\)-exponent of the function \(\Psi(x)\) we will consider to the one in [1] by 1.

Throughout, let \(\Psi : \mathbb{N} \mapsto \mathbb{R}_{>0}\), i.e. \((\Psi(q))_{q \geq 1}\) induces a sequence of positive reals numbers. In the sequel one may always consider a continuation to a function \(\Psi : \mathbb{R}_{>0} \mapsto \mathbb{R}_{>0}\) with similar properties, however only the values for integers will be of interest and for technical reasons, in particular (B2), we restrict to \(\mathbb{N}\). For our purposes, we will consider functions \(\Psi\) which satisfy some condition concerning both decay and local monotonicity. More precisely, we will consider \(\Psi\) to have some (A)-property and some (B)-property, the most frequently used defined as follows.

\[
\begin{align*}
(A1) & \quad \Psi(x) = o(x^{-1}), \quad x \to \infty, \\
(A2) & \quad \Psi(x) < dx^{-1}, \quad \text{for some fixed small } d > 0, \text{ and all } x \geq \hat{x}, \\
(A3) & \quad \Psi(x) < \frac{1}{2}x^{-1}, \quad \text{for all } x \geq \hat{x}, \\
\end{align*}
\]

and

\[
\begin{align*}
(B1) & \quad x \leq y \implies \Psi(x) \geq \Psi(y), \\
(B2) & \quad \Psi(lx) \leq l\Psi(x), \quad l, x \in \mathbb{N}.
\end{align*}
\]

Condition (A2) depends on \(d\) and is a priori not an exact definition. An effective constant \(d\) will appear in the context of the results, however. Assuming \(d < 1/2\), it is evident that (A1) \(\implies\) (A2) \(\implies\) (A3), and clearly (B1) \(\implies\) (B2). Define

\[
\mathcal{K}_1(\Psi) = \{\zeta \in \mathbb{R} \setminus \mathbb{Q} : |\zeta q - p| \leq \Psi(q) \text{ for infinitely many } (q,p) \in \mathbb{N} \times \mathbb{Z}\},
\]

and let \(\mathcal{K}_1^*(\Psi)\) be such as \(\mathcal{K}_1(\Psi)\) but with the restriction of relatively prime vectors \((q,p)\). Observe that assuming \(\Psi\) tends to 0 and satisfies (B2), we have \(\mathcal{K}_1(\Psi) = \mathcal{K}_1^*(\Psi)\). Indeed, if \((q,p)\) satisfies the corresponding inequality so does \((q',p') = (q/d,p/d)\), where we have put \(d = \gcd(p,q)\), since

\[
|\zeta q - p| = |\zeta dq' - dp'| = d|q'\zeta - p'|.
\]

As \(\zeta \not\in \mathbb{Q}\), infinitely many distinct pairs \((p',q')\) are obtained from infinitely many pairs \((p,q)\) this way if \(\Psi\) tends to 0, which shows \(\mathcal{K}_1(\Psi) \subset \mathcal{K}_1^*(\Psi)\). The other inclusion is obvious.
The special case \( s = 1 \) of Satz 6 in Jarnik [5], translated into the present notation, asserts the following.

**Theorem 1.1 (Jarnik).** Let \( \Psi \) have the properties (A1), (B1). Then

\[
\mathcal{H}_1(\Psi) \setminus \cup_{c<1} \mathcal{H}_1(c\Psi) \neq \emptyset.
\]

In other words, for any suitable function \( \Psi \), there are elements in \( \mathcal{H}_1(\Psi) \) that do not belong to the set \( \mathcal{H}_1(c\Psi) \) for any \( c < 1 \). Note that \( \mathcal{H}_1(c\Psi) \) get larger as \( c \) increases.

Condition (A1) on \( \Psi \) is very natural in this context due to (1) for \( k = 1 \), and it seems it cannot be weakened in a reasonable way, as we briefly carry out. Basic facts on one-dimensional Diophantine approximation show

\[
|\zeta q - p| \leq \frac{1}{\sqrt{5}} q^{-1}
\]

has infinitely many integer solutions \( p, q \), see [8]. Thus (4) cannot hold if \( \Psi(x) > dx^{-2} \) holds for some \( d > 1/\sqrt{5} \) and arbitrarily large \( x \), for example for \( \Psi(x) = (1/2)x^{-1} \).

Moreover, it cannot hold for \( \Psi(x) = dx^{-2} \) for \( d \in (1/\sqrt{8}, 1/\sqrt{5}) \) for instance, due to facts connected to the Lagrange spectrum, see [4].

Condition (B1) avoids some problems and most likely cannot be dropped completely.

We point out that we will mostly only need the weaker assumption (B2), though.

The construction in the proof of Theorem 1.1 in fact shows more. Jarnik's proof shows that \( \zeta = [b_0; b_1, b_2, \ldots] \) with convergents \( p_n/q_n = [b_0; b_1, \cdots, b_n] \) belongs to (1), provided that

\[
b_{n+1} \Psi(q_n) q_n > 1, \quad n \geq 1,
\]

\[
\lim_{n \to \infty} b_{n+1} \Psi(q_n) q_n = 1.
\]

These assumptions can be satisfied since \( \Psi(x) = o(x^{-1}) \), moreover \( \lim_{n \to \infty} b_n = \infty \).

However, to obtain a better result in Theorem 1.4 later, we apply subtle modifications. It is easy to see that it is sufficient to have (6) for \( n \geq n_0 \). Hence, one may choose the initial partial quotients of \( \zeta = [b_0; b_1, b_2, \cdots] \) up to any \( b_m \) arbitrarily. Furthermore, it becomes evident that the partial quotients of a suitable \( b_n \) can be individually altered to \( b_n + 1 \) without affecting the result. Combining these facts, the suitable set of \( \zeta \) in the difference set in (4) is uncountable in any real interval. Moreover, we show that we can weaken (B1) to (B2) in the conditions of Theorem 1.1. So assume (B2), (A1) for \( \Psi \) and repeat the above construction starting from (6), (7). It is evident by the construction that for given \( c < 1 \), all convergents in lowest terms \( p_n/q_n \) of \( \zeta \) of sufficiently large index \( n \geq \hat{n}(c) \) have the property

\[
c \Psi(q_n) < |q_n \zeta - p_n| < \Psi(q_n).
\]

Moreover, it is well-known that for \( (p, q) \) linearly independent to all \( (p_n, q_n) \) we have

\[
|q \zeta - p| > (1/2)q^{-1},
\]

see Satz 11 in [8], so for large \( q \) condition (A1) yields

\[
|q \zeta - p| > \frac{1}{2q} > \Psi(q) > c \Psi(q).
\]
Hence we may restrict to the case \((p, q)\) are integral multiples of \((p_n, q_n)\) with \(p_n/q_n\) a convergent in lowest terms. However, for those pairs \((B2), (3)\) and \((8)\) imply \(c\Psi(q) < |\zeta q - p|\) as well. Combination of these facts indeed show the assertion and this argument also shows that when we restrict to coprime pairs \((p, q)\), condition \((A1)\) alone is sufficient.

Summing up all the above extensions of Theorem 1.1, we formulate the result as a theorem.

**Theorem 1.2.** Let \(\Psi\) satisfy \((A1), (B2)\). Let \(I\) be any non-empty open real interval. Then the set

\[
(\mathcal{K}_1(\Psi) \setminus \bigcup_{c<1} \mathcal{K}_1(c\Psi)) \cap I
\]

is uncountable. The same is true for \(\mathcal{K}_1^*(\Psi)\), where we may drop the condition \((B2)\).

We want to generalize this concept. Let \(k \geq 1\) be an integer, \(\Psi\) a function and \(\zeta = (\zeta_1, \ldots, \zeta_k) \in \mathcal{C}_k \subset \mathbb{R}^k\) where \(\mathcal{C}_k\) denotes the set of vectors in \(\mathbb{R}^k\) that are linearly independent over \(\mathbb{Q}\) together with 1. Define the set

\[
\mathcal{K}_k(\Psi) = \{\zeta \in \mathcal{C}_k : \max_{1 \leq j \leq k} |\zeta_j q - p_j| \leq \Psi(q)\text{ for infinitely many } (q, p_1, \ldots, p_k) \in \mathbb{N} \times \mathbb{Z}^k\}.
\]

This definition implies that any element in \(\mathcal{K}_k(\Psi) \subset \mathbb{R}^k\) gives rise to several elements in \(\mathcal{K}_l(\Psi) \subset \mathbb{R}^l\) for any \(1 \leq l < k\) by taking subsets. Apriori an analogue result is not clear for any \(l > k\). The general assertion of Satz 6 in [5] says the following.

**Theorem 1.3** (Jarnik). For any positive integer \(k\) and any \(\Psi\) satisfying \((A1), (B1)\), we have

\[
\mathcal{K}_k(\Psi) \setminus \bigcup_{c<1} \mathcal{K}_k(c\Psi) \neq \emptyset.
\]

In other words, there exists a real vector \(\zeta\) such that the system is again approximable to degree \(\Psi\) and no better. Jarnik uses induction on \(k\) to infer \((10)\) from \((4)\), which in fact shows that an extension of an element in \(\mathcal{K}_k(\Psi) \subset \mathbb{R}^k\) to elements in \(\mathcal{K}_l(\Psi) \subset \mathbb{R}^l\) for \(l > k\) indeed exists. Note also, that the restriction to \(\mathcal{C}_k\) is reasonable, since else any element of \(\mathcal{K}_1(\Psi)\) gives rise to an element in \(\mathcal{K}_k(\Psi)\) in a trivial way by \(\zeta = (\zeta, \zeta, \ldots, \zeta) \in \mathbb{R}^k\), and Theorem 1.2 is trivially implied by Theorem 1.1.

Based on \((9)\), we want to generalize \((10)\) to simultaneous approximation of vectors of the form \(\zeta = (\zeta, \zeta^2, \ldots, \zeta^k)\) with a completely different approach. For this reason, define

\[
\mathcal{K}_k(\Psi) = \{\zeta \in \mathcal{C}_k : \max_{1 \leq j \leq k} |\zeta_j q - p_j| \leq \Psi(q)\text{ for infinitely many } (q, p_1, \ldots, p_k) \in \mathbb{N} \times \mathbb{Z}^k\},
\]

where \(\mathcal{C}_k \subset \mathbb{R}\) is defined as the real numbers not algebraic of degree \(\leq k\). Clearly, for any function \(\Psi\) (no assumptions are required) the inclusions

\[
\cdots \subset \mathcal{K}_3(\Psi) \subset \mathcal{K}_2(\Psi) \subset \mathcal{K}_1(\Psi) = \mathcal{K}_1(\Psi)
\]

hold, and any element in \(\mathcal{K}_k(\Psi) \subset \mathbb{R}\) gives rise to some element in \(\mathcal{K}_k(\Psi) \subset \mathbb{R}^k\), but the reverse is (in general) false. Define \(\mathcal{K}_k^*(\Psi)\) similarly to \(k = 1\). The above clearly holds for \(\mathcal{K}_k^*(\Psi)\) too, and assuming \((B2)\) we again have \(\mathcal{K}_k(\Psi) = \mathcal{K}_k^*(\Psi)\). As indicated, we
want to extend both Theorems 1.2, 1.3. It turns out that we can even weaken \((A1)\) for \(k \geq 2\). We prove the following.

**Theorem 1.4.** Let \(k \geq 2\) be an integer, the function \(\Psi\) satisfy \((B2)\) and \(J \subset (-1/2, 1/2)\), \(J \subset \mathbb{R}\) be non-empty open intervals.

- If \(\Psi\) additionally satisfies \((A3)\), then the set 
  \[
  (\mathcal{K}_k(\Psi) \setminus \cup_{c<1} \mathcal{K}_k(c\Psi)) \cap I 
  \]
  is uncountable.
- If \(\Psi\) additionally satisfies \((A2)\), for any fixed \(c_0 < 1\) the set 
  \[
  (\mathcal{K}_k(\Psi) \setminus \mathcal{K}_k(c_0\Psi)) \cap J 
  \]
  is uncountable. An effective constant \(d\) in \((A2)\) depending only on \(J\) can be given.

In both assertions, the same is true for \(\mathcal{K}^*_k\), where we may drop the condition \((B2)\).

Elements in the sets can effectively be determined.

**Remark 1.5.** As indicated before, at least the first assertion is wrong for \(k = 1\), since (5) has arbitrarily large solutions. Recall also the counterexamples and reference subsequent to (5).

We will utilize results from Section 2 to infer this result from Theorem 1.2 in Section 3. The effectiveness of the proof of the first part admits an interesting result concerning functions \(\Psi\) which satisfy slightly more rigid restrictions. For technical reasons, we now assume \(\Psi\) to be defined on \(\mathbb{R}_{>0}\).

**Definition 1.6.** For a positive integer \(k\), say a function \(\Psi : \mathbb{R}_{>0} \mapsto \mathbb{R}_{>0}\) is **admissible of degree \(k\)** if it satisfies 

- \((A0.k)\) \[\Psi(x) = o(x^{-2k+1}), \quad x \to \infty.\]
- \((B0.k)\) \[
  \frac{\Psi(x)}{\Psi(y)} \geq \left(\frac{y}{x}\right)^{k-1}, \quad \text{for all large } x \leq y.\]

We call \(\Psi\) strictly admissible if it is admissible for any \(k\).

Observe that the admissibility condition becomes stronger as \(k\) increases, and for \(k = 1\) it is equivalent to \((A1), (B1)\). More precisely, 

\[
\cdots \implies (A0.2) \implies (A0.1) = (A1) \implies (A2) \implies (A3), \quad \cdots \implies (B0.2) \implies (B0.1) = (B1) \implies (B2).\]

We will discuss how strong the condition of (strict) admissibility is in Section 3.

**Theorem 1.7.** Let \(\Psi\) be admissible of degree \(k\) and \(I \subset (-1/2, 1/2)\) a non-empty open interval. Then there exist functions \(\Psi_1 = \Psi, \Psi_2, \Psi_3, \ldots, \Psi_k\) that satisfy \((A1), (B1)\) and can effectively be determined, such that

\[
(12) \quad (\mathcal{K}_1(\Psi) \setminus \cup_{c<1} \mathcal{K}_1(c\Psi)) \cap I = \bigcap_{1 \leq j \leq k} (\mathcal{K}_j(\Psi_j) \setminus \cup_{c<1} \mathcal{K}_j(c\Psi_j)) \cap I.
\]
More precisely, the difference sets coincide within $I$ for $1 \leq j \leq k$. Furthermore, the set in (12) is uncountable. If $\Psi$ is strictly admissible, then all $\Psi_j$ can be chosen strictly admissible as well and (12) holds for the infinite intersection, with coinciding sets.

All of this is true for $\mathcal{K}_j^*$ too, where we may drop the condition (B0,k) on $\Psi$.

**Remark 1.8.** It might be possible to relax the conditions on $\Psi$ to obtain functions $\Psi_j$ with a weaker conditions as well, similarly to Theorem 1.2, that satisfy the assertions of the theorem. We will not deal with this question.

Roughly speaking, the theorem tells us that under moderate assumptions on $\Psi$, if $\zeta \in \mathbb{R}$ can be approximated to some degree $\Psi$ and no better, then its powers $\zeta, \zeta^2, \ldots, \zeta^k$ can be simultaneously approximated to some modified degree $\Psi_k$, that can effectively be determined, and no better.

Note that it is of no importance that (B0,k) holds only for large $x \leq y$, since the set $\mathcal{K}_j^*$ is not influenced by the behavior of $\Psi$ in bounded intervals $(0, T)$. Also, restricting $\Psi$ to integers, one can restrict to assume (B0,k) for $y = x + 1$.

**1.3. Consequences and the relation to known results.** Before we turn to the proof, we want to discuss the assertion of Theorem 1.4. Estimate (1) suggests that Theorem 1.4 might hold under the weaker condition $\Psi(x) = o(x^{-1/k})$. However, for functions whose decay does not satisfy an (A)-condition, even much weaker claims are unknown. Bugeaud and Laurent [3] introduced the exponent $\lambda_k(\zeta)$ as the supremum of real $\eta$ such that

$$\max_{1 \leq j \leq k} |\zeta^j q - p_j| \leq q^{-\eta}$$

has infinitely many solutions $(q, p_1, \ldots, p_k) \in \mathbb{N} \times \mathbb{Z}^k$. Denote $\text{Spec}(\lambda_k)$ the spectrum of $\lambda_k(\zeta)$ as $\zeta$ runs through all real numbers not algebraic of degree $\leq k$. By virtue of (1) we have $\text{Spec}(\lambda_k) \subset [1/k, \infty]$. It is still unknown if actually $\text{Spec}(\lambda_k) = [1/k, \infty]$, which is Question 1 in [2]. Obviously, if for $\Psi(x) := x^{-\nu}$ and some constant $c \in (0, 1)$, or more general some function $c = c(x)$ with $c(x) < 1$ for all $x$ and $c(x) \neq o(x^{-\epsilon})$ as $x \to \infty$ for some $\epsilon > 0$, we would have

$$\mathcal{K}_k(\Psi) \setminus \mathcal{K}_k(c \Psi) \neq \emptyset,$$

then $\nu \in \text{Spec}(\lambda_k)$. This shows that a result in the spirit of Theorem 1.4 under the weaker condition $\Psi(x) = o(x^{-1/k})$ would imply a positive answer on the spectrum problem, but not vice versa. More general, it is not hard to see that if we let

$$\underline{\omega}(\Psi) := \liminf_{x \to \infty} - \frac{\log \Psi(x)}{\log x}, \quad \overline{\omega}(\Psi) := \limsup_{x \to \infty} - \frac{\log \Psi(x)}{\log x},$$

then for $c$ as above

$$\mathcal{K}_k(\Psi) \setminus \mathcal{K}_k(c \Psi) \subset \{\zeta \in \mathbb{R} : \underline{\omega}(\Psi) \leq \lambda_k(\zeta) \leq \overline{\omega}(\Psi)\}.$$

For a result concerning functions $\Psi$ with slower decay than $o(x^{-1})$, in the case of $\zeta \in \mathbb{R}^k$ linearly independent together with 1 (but not successive powers!), related to Theorem 1.3 Jarnik’s Satz 5 and its consequences discussed preceding Satz 6 (which is basically Theorem 1.3) in [5] give some information. However, the results are considerably weaker
than those of Theorem 1.1 or Theorem 1.4. Due to Roy’s recent progress \cite{9} in dynamic simultaneous approximation in the \( \mathbb{Q} \)-linearly independent case, it should be not too hard to improve the mentioned results in \cite{3} to obtain (13) for some effective constant \( c = c(k) \) that depends only on \( k \), or maybe something stronger.

Let us return to the case of \( \Psi \) that satisfies an (A)-condition. It was known that \( \text{Spec}(\lambda_k) \supset [1, \infty] \) by an explicit construction, see Theorem 2 in \cite{2}. An explicit construction of \( \zeta \) with prescribed exponent \( \lambda_k(\zeta) \geq 1 \) for \( k > 1 \), where \( \zeta \) can be chosen in the Cantor set or similar fractal sets, was established in \cite{10}, improving a slightly weaker result from \cite{1}. The inclusion \( \text{Spec}(\lambda_k) \supset [1, \infty] \) is equivalent to the fact that for fixed \( \nu \in [1, \infty) \) there exists \( \zeta \in \mathbb{R} \) such that

\[
\max_{1 \leq j \leq k} |\zeta^j q - p_j| \leq q^{-\nu + \delta}
\]

has arbitrarily large integer solutions \( q, p_1, \ldots, p_k \) for any \( \delta > 0 \) but for no \( \delta < 0 \), and similarly for \( \nu = \infty \). However, for \( k \geq 2 \), no versions with an explicit constant have been known. By this we mean an explicit \( c > 0 \) for which (13) holds with \( \Psi(x) := x - \nu \).

Thus Theorem 1.4 is a lot stronger than \( \text{Spec}(\lambda_k) \supset [1, \infty] \) and a satisfactory result for \( \Psi(x) = o(x^{-1}) \).

\section{Preparatory results}

The proofs in Section 3 will rely heavily on the following easy observation.

\begin{lemma}
Let \( k \geq 2 \) be an integer and \( \zeta \) a real number. Suppose \( |\zeta - p/q| = dq^{-k} \) with \( d < 1/2 \) and integers \( p, q \) with \( q \) sufficiently large holds (then \( p \) is large too). In case of \( \zeta \in (0, 1/2) \), we have

\[
\max_{1 \leq j \leq k} \|q^k \zeta^j\| = q^k \left| \zeta - \frac{p}{q} \right| = q^{k-1} |q \zeta - p| = d.
\]

In any case, we have

\[
\max_{1 \leq j \leq k} \|q^k \zeta^j\| = |L_k + o(1)| \cdot q^k \left| \zeta - \frac{p}{q} \right| = |L_k + o(1)| \cdot q^{k-1} |q \zeta - p|,
\]

where \( L_k = L_k(\zeta) := \max_{1 \leq j \leq k} (j \zeta^{j-1}) \), as \( q \to \infty \).
\end{lemma}

\begin{proof}
The condition \( |\zeta - p/q| = dq^{-k} \) with \( d < 1/2 \) implies

\[
|\zeta^k - pq^{-k}| = d < 1/2.
\]

In particular \( pq^{-k} \) is the closest integer to \( \zeta^k \). More general, by the assumption \( \zeta < 1/2 \) and since \( |\zeta - p/q| \) is very small by assumption, for large \( q \) we also have \( 0 < p/q < 1/2 \). It is easy to check that

\[
j(1/2)^{j-1} \leq 1, \quad j \in \{1, 2, 3, \ldots\}.
\]

This implies

\[
|\zeta^{j-1} + \cdots + \frac{p^{j-1}}{q^{j-1}}| < j(1/2)^{j-1} \leq 1, \quad 2 \leq j \leq k
\]
for large $q$. Hence the calculation
\[ |q^j \zeta^j - p/q^j| = q^j |\zeta^j - p/q| = q^k |\zeta - p/q| \zeta^{j-1} + \cdots + \frac{p^{j-1}}{q^j}, \quad 2 \leq j \leq k \]
and (16) shows that $p/q^j$ is the closest integer to $q^j \zeta_j$ for $1 \leq j \leq k$ and furthermore the minimum of $\|\zeta/q\|$ among $j \in \{1, 2, \ldots, k\}$ is obtained for $j = 1$. Thus (10) indeed proves (14). For (15) one can proceed very similarly, using that the left hand side of (18) tends to $L_k$ as $q \to \infty$ since $p/q$ tends to $\zeta$.

\( \square \)

**Remark 2.2.** Due to Satz 11 in [8] mentioned already preceding Theorem 1.2, the assumption of Lemma 2.1 implies $p/q$ must be a convergent in the continued fraction expansion of $\zeta$.

**Remark 2.3.** The remainder term in (15) can be estimated in dependence of $q$.

We look at the values $L_k(\zeta)$ more closely. The following proposition gives the most important properties of this quantity and will be helpful not only in the proofs in this section but also will be directly applied in the proof of the second assertion of Theorem 1.3.

**Proposition 2.4.** Let $k \geq 2$ be an integer. Consider $L_k = L_k(\zeta)$ from Lemma 2.1 as a function of $\zeta \in \mathbb{R}_{>0}$. Then $L_k$ is continuous, has image $[1, \infty)$, is constant $L_k(\zeta) = 1$ in $(0, 1/2]$ and strictly increasing in $(1/2, \infty)$.

**Proof.** Apart from $j = 1$, any expression $j \zeta^{j-1}$ involved in the maximum is continuous, strictly increasing and tends to infinity as a function of $\zeta$. It follows that the maximum $L_k = L_k(\zeta)$ is continuous, non-decreasing and strictly increases unless it is obtained for $j = 1$, which in view of (17) is easily seen to be equivalent to $L_k = 1$ and $\zeta \in (0, 1/2]$. \( \square \)

We quote Lemma 2.4 and Corollary 3.1 in [10] in slightly modified versions, such as the following additional results. To Lemma 2.4 in [10] we add the result (20) which was inferred within its proof in [10] but not explicitly mentioned in the Theorem. Furthermore, in view of Proposition 2.4, we can improve the original bound $C_0$ from Lemma 2.4 similarly to the proof of Lemma 2.1 to any constant smaller than $(1/2)L_k(\zeta)^{-1}$, when restricting to sufficiently large integers only. In case of $\zeta \in (0, 1/2)$ the constant can be put $1/2$. See also Remark 2.5 in [10], where a weaker bound is given however.

**Lemma 2.5** ([S., 2014]). Let $k$ be a positive integer and $\zeta$ be a positive real number. For an integer $z$ and $1 \leq j \leq k$, denote by $y_j$ the closest integer to $\zeta^j z$. There exists a constant $C = C(k, \zeta) > 0$ such that for any large integer $z \geq \hat{z} > 0$ the estimate

\[ \max_{1 \leq j \leq k} \|\zeta^j z\| < C \cdot z^{-1}, \]

implies $y_1/z = y_0/z_0$ for integers $(z_0, y_0) = 1$ and $z_0^k$ divides $z$. A suitable choice for $C$ is given by $C = 1/2$ if $\zeta \in (0, 1/2)$ and $C = C_0 := (1/2)L_k(\zeta)^{-1} - \epsilon$ with $L_k$ from Lemma 2.1 and arbitrary $\epsilon > 0$ (where $\hat{z}$ above depends on $\epsilon$).

Moreover, $y_0/z_0^j$ is a convergent of the continued fraction expansion of $\zeta^j$ for $1 \leq j \leq k$. Furthermore, provided (19) holds for some pair $(z, C)$, then it holds for any pair $(z', C)$.
with $z'$ a positive integral multiple of $z_0^k$ not larger than $z$, and the best possible value $C$ in (19) is obtained for $z' = z_0^k$. More precisely, we have

$$(20) \quad (z, y_1, \ldots, y_k) = M \cdot (z_0^k, z_0^{k-1} y_0, \ldots, y_0^k)$$

for some positive integer $M$ for any solution of (19).

**Corollary 2.6** (S., 2014). Let $k \geq 2$ be an integer, $\zeta$ be a real number. For any fixed $T > 1$, there exists $\hat{z} = \hat{z}(T, \zeta)$, such that the estimate

$$\max_{1 \leq j \leq k} \| \zeta^j z \| \leq z^{-T}$$

for an integer $z \geq \hat{z}$ implies that for $z_0, y_0$ as in Lemma 2.5 we have

$$(21) \quad |\zeta z_0 - y_0| \leq z_0^{-kT-k+1}.$$  

Similarly, if for $C_0 = C_0(k, \zeta)$ from Lemma 2.5 the inequality

$$\max_{1 \leq j \leq k} \| \zeta^j z \| < C_0 \cdot z^{-1}$$

has an integer solution $z > 0$, then (21) holds with $T = 1$.

We will only need special aspects of Lemma 2.5 and Corollary 2.6 for the proofs in Section 3 which will be summarized in the following Corollary 2.7 in a way that allows convenient quotation. Concretely, Lemma 2.5 in combination with the $T = 1$ case of Corollary 2.6 yield Corollary 2.7.

**Corollary 2.7.** Let $k \geq 2$ be an integer and $\zeta$ be real. Define $L_k$ as in Lemma 2.1. Further let the function $\Psi$ satisfy (B2) and additionally either condition (A3) for $\zeta \in (0, 1/2)$ or (A2) with $d = (1/2) L_k(\zeta)^{-1} - \epsilon$ for arbitrary $\epsilon > 0$ in case of arbitrary $\zeta$. All solutions $(z, y_1, \ldots, y_k) \in \mathbb{N} \times \mathbb{Z}^k$ of

$$(22) \quad \max_{1 \leq j \leq k} |\zeta^j z - y_j| \leq \Psi(z)$$

with large $z$, are integral multiples of solutions of the form $(z_0^k, z_0^{k-1} y_0, \ldots, y_0^k)$ with $z_0, y_0$ in Lemma 2.5. In particular, if (22) has arbitrarily large solutions, then it has solutions with the additional property $z = q^k$ for $q \in \mathbb{Z}$ with the property $\| \zeta q \| \leq q^{-2k+1}$.

**Proof.** The conditions (A2) respectively (A3) are chosen such that the assumptions of Lemma 2.5 and Corollary 2.6 for $T = 1$ are satisfied for large $z$ for which (22) holds. Due to (20), similarly to (3), we have

$$\max_{1 \leq j \leq k} |\zeta^j z - y_j| = M \max_{1 \leq j \leq k} |\zeta^j z_0^k - z_0^{k-j} y_0^j|,$$

so in view of (B2) indeed a solution of (22) leads to a primitive solution of the claimed form. Corollary 2.6 yields $|\zeta z_0 - y_0| \leq z_0^{-2k+1}$. It remains to put $q = z_0$ in view of (20).  \[\square\]
3. Proof of Theorems 1.4, 1.7

The considered sets $\mathcal{K}(.)$ are symmetric with respect to 0, for $(q, p_1, p_2, \ldots, p_k)$ satisfies the inequality within the definition of $\mathcal{K}$ or $\mathcal{K}^*$ for $\zeta$ if and only if $(q, -p_1, p_2, \ldots, (-1)^k p_k)$ satisfies the inequality for $-\zeta$. Thus we may restrict to $I \subset (0, 1/2), J \subset (0, \infty)$ in the proofs. This enables us to apply Proposition 2.4, which makes thing a little less technical. We use the abbreviation of an *everywhere uncountable set* for a set that has uncountable intersection with any non-empty open interval of $\mathbb{R}$ in the proof. We point out that the uncountable cardinality of real numbers that we will construct within the proof of Theorem 1.4 ensures that the restriction of $\mathcal{K}$ to numbers in $\mathcal{C}$, i.e. not algebraic of small degree, will not be relevant.

The proof of the first assertion of Theorem 1.4, where the result is stronger anyway, will be not too complicated to derive from Theorem 1.2 with aid of the results from Section 2. The proof of the second assertion will be more technical, since we have to apply (15) instead of (14) when applying Lemma 2.1. As $L_k$ depends on $\zeta$, we get a weaker result in this case. The remainder term in (15) makes the proof of this assertion slightly more technical as well.

**Proof of Theorem 1.4.** We start with the first assertion. Let $\Psi$ be arbitrary with (A3), (B2). Write $\Psi(x) = \Delta(x)x^{-1}$ with a function $\Delta(x)$ which obviously has the property $\Delta(x) < 1/2$ for all large $x$ and moreover

$$\Delta(lx)(lx)^{-1} < l \cdot \Delta(x)x^{-1}, \quad l, x \in \mathbb{N}. \quad (23)$$

We want that (9) is uncountable for the function $\tilde{\Psi}(x) = \Delta(x)x^{-2k+1}$. We have to check that (B2), (A1) are satisfied for $\tilde{\Psi}$ in order to apply Theorem 1.2. Applying (23) to $x, l$ leads after simplification to

$$\Delta(l^k x^k) \leq l^2 k \Delta(x^k), \quad l, x \in \mathbb{N}. \quad (9)$$

This indeed yields

$$\tilde{\Psi}(lx) = \Delta(l^k x^k)(lx)^{-2k+1} \leq l \cdot \Delta(x^k)x^{-2k+1} = l \cdot \tilde{\Psi}(x).$$

On the other hand, since $k \geq 2$ we have $x^{-2k+1} = o(x^{-1})$, and since $\Delta(x) < 1/2$ this implies $\tilde{\Psi}(x) = o(x^{-1})$ as $x \to \infty$, which we identify as (A1).

Thus (9) is indeed uncountable. In other words, there exists an uncountable set of $\zeta \in I$, such that for any fixed $c < 1$, we have

$$c\tilde{\Psi}(q) = c\Delta(q^k)q^{-2k+1} < \|\zeta q\| < \Delta(q^k)q^{-2k+1} = \tilde{\Psi}(q) \quad (24)$$

for arbitrarily large integers $q$, and

$$\|\zeta q\| > c\Delta(q^k)q^{-2k+1} = c\tilde{\Psi}(q) \quad (25)$$

for all sufficiently large integers $q \geq \hat{q}(\tilde{\Psi}, c)$. Using that $\zeta \in (0, 1/2)$, the same choice of $\zeta$ will be suitable for the function $\Psi(x)$, as we shall show. This also implies the effectiveness, since the proof of Theorem 1.2 is constructive.
Restricting to \( \zeta \in I \subset (0, 1/2) \), it follows from \( \Delta(x^k) < 1/2 \) for large \( x \) and \( 2k - 1 > k \) that we may apply (14) from Lemma 2.1 to the \( q \) that satisfy (24). It yields for those \( q \) the relation \( \max_{1 \leq j \leq k} \| q^j \zeta^j \| = q^{-k} \| \zeta \| \). Thus (24) further implies

\[
c \Psi(q^k) = c \Delta(q^k) q^{-k} < \max_{1 \leq j \leq k} \| q^j \zeta^j \| < \Delta(q^k) q^{-k} = \Psi(q^k).
\]

Hence, if we let \( z := q^k \), for any fixed \( c < 1 \) and \( z \) large enough indeed we have

\[
(26) \quad c \Psi(z) = c \Delta(z) z^{-1} < \max_{1 \leq j \leq k} \| z^j \zeta^j \| < \Delta(z) z^{-1} = \Psi(z).
\]

For the first assertion concerning \( \mathcal{X}_k \), it remains to show

\[
(27) \quad \max_{1 \leq j \leq k} \| z^j \zeta^j \| > c \Delta(z) z^{-1} = c \Psi(z)
\]

for all large integers \( z \) (that are not necessarily \( k \)-th powers of an integer). Assume the opposite, i.e. there exist arbitrarily large integers \( z \) that violate (27). Recall \( L_k(\zeta) = 1 \) for \( \zeta \in I \) by Proposition 2.4. Hence \( c \Psi(x) < \Psi(x) < (1/2)x^{-1} \) for large \( x \geq \delta(\epsilon) \), and application of Corollary 2.7 to the function \( c \Psi(x) \) implies that (27) is violated also for arbitrarily large \( z \) of the form \( z = q^k \) and additionally \( \| q \zeta \| \leq q^{-2k+1} \). For the assertion on \( \mathcal{X}_{k}^* \), in view of (20) it is sufficient to consider such \( z \) too. To sum up, we obtain a sequence of values \( q \) with the properties

\[
\max_{1 \leq j \leq k} \| q^j \zeta^j \| \leq c \Delta(q^k) q^{-k} = c \Psi(q^k)
\]

\[
\| q \zeta \| \leq q^{-2k+1}.
\]

Since \( q^{-2k+1} \leq (1/2)q^{-k+1} \) for \( q > 1 \), we may apply Lemma 2.1 more precisely (14) since \( \zeta \in (0, 1/2) \). It yields

\[
\| q \zeta \| = q^{-k} \max_{1 \leq j \leq k} \| q^j \zeta^j \| \leq c \Delta(q^k) q^{-2k+1} = c \tilde{\Psi}(q)
\]

contradicting (25). Hence (27) holds and the first assertion is proved.

We show the second claim. Consider \( c_0 < 1 \), an open interval \( J \) which we can assume to be bounded, and a function \( \Psi \) satisfying (12) and (12) for some \( d \) to be determined later, fixed. Write \( \Psi(x) = \Delta(x^k)x^{-1} \) with a function \( \Delta(x) \leq dx^{-1} \) for large \( x \). Let \( c \in (c_0, 1) \) be arbitrary. Pick any \( \zeta_0 \in J \) and define \( L^0 := L_k(\zeta_0) \) with \( L_k \) from Lemma 2.1. We may assume \( L^0 > 1 \), otherwise by Proposition 2.4 we have \( \zeta_0 \in J \cap (0, 1/2) \), in particular \( J \) contains a subinterval of \((0, 1/2)\) and the claim follows from the first part of the theorem.

Define \( \tilde{\Psi}(x) := L^0 \tilde{\Psi}(x) \) with \( \tilde{\Psi}(x) = \Delta(x^k)x^{-2k+1} \) as above. Obviously \( \tilde{\Psi} \) satisfies the conditions of Theorem 1.2 for the same reasons as \( \Psi \). Similar to (24), (25), we obtain

\[
(28) \quad c \tilde{\Psi}(q) = c L^0 \Delta(q^k) q^{-2k+1} < \|
\zeta q \| < L^0 \Delta(q^k) q^{-2k+1} = \tilde{\Psi}(q)
\]

for arbitrarily large integers \( q \), and

\[
(29) \quad \| \zeta q \| > c L^0 \Delta(q^k) q^{-2k+1} = c \tilde{\Psi}(q)
\]

for all \( q \geq \hat{q}(\tilde{\Psi}, c) \), for an uncountable set of values \( \zeta \in J \).

Note that non-empty pre-images of real open intervals under monotonic continuous maps are open intervals again and thus have uncountable intersection with any everywhere
Corollary 2.7 applied to the function $\zeta$ we may apply this to the map locally a well-defined strictly increasing continuous function. In view of Proposition 2.4, uncountable set. Moreover, if the function is strictly increasing, then the pre-image is for arbitrarily small fixed $\delta > 0$, for arbitrarily small $\nu > 0$ we can choose uncountably many $\zeta_1 \in (\zeta_0, \zeta_0 + \nu)$ that satisfy the above conditions (28), (29), and have the additional property $L^0 < L^1 < L^0 + \delta$ with $L^1 := L_k(\zeta_1)$. Making $\nu$ smaller if necessary, we may assume $\zeta_1 \in J$. We will treat $\zeta_1$ as fixed in the sequel, so $L^1$ is fixed too.

Clearly we may apply (15) to the integers $q$ that satisfy (28). Denoting the involved remainder terms by $\epsilon(q)$, we infer
\[ L^0 c \Delta(q^k) q^{-k} = (L^1 + \epsilon(q)) \max_{1 \leq j \leq k} \|q^k \zeta^j_1\| < L^0 \Delta(q^k) q^{-k} \]
which we can rewrite as
\[ \frac{L^0}{L^1 + \epsilon(q)} c \Psi(q^k) < \max_{1 \leq j \leq k} \|q^k \zeta^j_1\| < \frac{L^0}{L^1 + \epsilon(q)} \Psi(q^k). \]
Note that since $L^0 < L^1$ and $\epsilon(q) = o(1)$ as $q \to \infty$, we have $L^0/(L^1 + \epsilon(q)) < 1$ for large $q$ and the quotients tend to $L^0/L^1$ as $q \to \infty$. We can still choose the parameter $\delta$, and the quotient $L^0/L^1$ tends to 1 as $\delta \to 0$. Since we have strict inequality $c > c_0$ and $\epsilon(q) = o(1)$, choosing $\delta$ in dependence of $c, c_0$ sufficiently small, putting $z := q^k$ from (30) we indeed infer
\[ c_0 \Psi(z) < \max_{1 \leq j \leq k} \|z \zeta^j_1\| < \Psi(z) \]
for arbitrarily large integers $z$. It remains to prove
\[ \max_{1 \leq j \leq k} \|z \zeta^j_1\| > c_0 \Psi(z) \]
for all sufficiently large integers $z$ (not necessarily $k$-th powers). As in the first assertion, assume the opposite. Let $d$ in (A2) be arbitrary in the interval
\[ 0 < d < \frac{1}{2 \cdot \sup_{t \in J} L_k(t)}, \]
which is equivalent to $0 < d < (1/2)L_k(\gamma)^{-1}$ for $\gamma = \sup J$ by Proposition 2.4. Again Corollary 2.7 applied to the function $c_0 \Psi(x)$, which is smaller than $(1/2)L_k(\gamma)^{-1} x^{-1}$ for any $\zeta \in J$ and large $x$, yields that (31) is violated for arbitrarily large $z = q^k$ and additionally $\|q \zeta_1\| \leq q^{-2k+1}$. We further obtain
\[ \max_{1 \leq j \leq k} \|q^k \zeta^j_1\| \leq c_0 \Delta(q^k) q^{-k} = c_0 \Psi(q^k) \]
\[ \|q \zeta_1\| \leq q^{-2k+1} \]
for arbitrarily large integers $q$. Again we may apply (15) to get
\[ \|q \zeta_1\| = \frac{q^{-1-k}}{L^1 + \epsilon(q)} \max_{1 \leq j \leq k} \|q^k \zeta^j_1\| \leq \frac{c_0}{L^1 + \epsilon(q)} \Delta(q^k) q^{-2k+1} = \frac{c_0}{L^1 + \epsilon(q)} \tilde{\Psi}(q). \]
Recall (29) holds for $\zeta = \zeta_1$, so combination with (32) yields
\[ cL^0 < \frac{c_0}{L^1 + \epsilon(q)}. \]
Since we have \( c > c_0, L^1 > L^0 > 1 \) and \( \epsilon(q) \) tends to 0, this cannot hold for large \( q \), though. Again we conclude \((31)\). As \( c_0, J, \Psi \) were arbitrary under the given restrictions and we have shown the above can be done for uncountably many \( \zeta_1 \in J \), the second assertion is proved. \( \square \)

As indicated in Section 1, Theorem \( \textbf{1.7} \) is established very similar. Recall that in Theorem \( \textbf{1.7} \) the function \( \Psi \) is defined on \( \mathbb{R}_{>0} \), which guarantees that all quantities that will appear are well-defined.

**Proof of Theorem \( \textbf{1.7} \)** For given \( \Psi \) as in the theorem, for any positive integer \( j \) define
\[
\Delta_j(x) = \Psi(x^{1/j})x^{(2j-1)/j},
\]
such that \( \Psi(x) = \Delta_j(x^j)x^{-2j+1} \). Define \( \Psi_j(x) = \Delta_j(x)x^{-1} \). First we show the properties of \( \Psi_j \). By \((A0.k)\) we infer \( \Delta_j(x) = o(1) \) and hence indeed \( \Psi_j(x) = o(x^{-1}) \) as \( x \to \infty \) for \( 1 \leq j \leq k \). Similarly, it is easy to check that strictly admissible \( \Psi \) gives rise to \( \Psi_j \) satisfying \((A0.k)\) for any \( k \). We show \( \Psi_j \) is non-increasing. Since any map \( t \mapsto t^{1/j} \) increases on the positive reals, for arbitrary \( 0 < x \leq y \) the estimate \((B0.k)\) implies
\[
\frac{\Psi(x^{1/j})}{\Psi(y^{1/j})} \geq \left( \frac{y}{x} \right)^{(j-1)/j}, \quad 1 \leq j \leq k.
\]
This indeed leads to
\[
\Psi_j(x) = \Psi_j(x^{1/j})x^{(j-1)/j} \geq \Psi_j(y^{1/j})y^{(j-1)/j} = \Psi_j(y).
\]
It remains to prove that \( \Psi_j \) satisfy a \((B0.k)\) type relation for any exponent \( \mu > 0 \), provided that \( \Psi \) has this property. Let \( \mu > 0 \) arbitrary and put \( \eta = k\mu + k - 1 \). By strict admissibility of \( \Psi \), we have
\[
\frac{\Psi_j(x^{1/k})}{\Psi_j(y^{1/k})} \geq \left( \frac{y}{x} \right)^{\eta/k}
\]
for all large \( x_0(\eta) < x \leq y \), and further
\[
\frac{\Psi_j(x)}{\Psi_j(y)} = \frac{\Psi_j(x^{1/k})x^{(k-1)/k}}{\Psi_j(y^{1/k})y^{(k-1)/k}} \geq \left( \frac{y}{x} \right)^{(\eta+1-k)/k} = \left( \frac{y}{x} \right)^{\mu}.
\]
Recall that being an admissible function, \( \Psi \) satisfies \((A1)\), \((B1)\) and since \((B1)\) implies \((B2)\), Theorem \( \textbf{1.2} \) holds for \( \Psi \). Moreover, we just proved that the functions \( \Psi_j \) satisfy the properties of the function \( \Psi \) in Theorem \( \textbf{1.4} \) since \((A3)\), \((A2)\) both imply \((A1)\), and \((B1)\) implies \((B2)\). Hence, we can now proceed as in the proof of the first assertion of Theorem \( \textbf{1.4} \), where the \( \Psi_j \) play the role of \( \Psi \) from Theorem \( \textbf{1.4} \) and the present \( \Psi \) the role of \( \tilde{\Psi} \), to infer that the difference sets contain the set for \( j = 1 \) and thus \((12)\). Reversing the proof of Theorem \( \textbf{1.4} \) with Lemma \( \textbf{2.5} \) shows that there is actually equality, we omit the details. The cardinality result follows again from Theorem \( \textbf{1.2} \). \( \square \)

We give some examples of admissible functions. Let \( c > 0, \epsilon > 0, \sigma > 0 \) arbitrary generically. Any function \( \Psi(x) = cx^{-2k+1-\epsilon} \) is admissible of degree \( k \). More general, any map \( \Psi(x) = x^{-2k+1}\varphi(x) \) is admissible of degree \( k \) for any function \( \varphi : \mathbb{R}_{>0} \to \mathbb{R}_{>0} \) which tends to 0 monotonically for large \( x \). Any map \( \Psi(x) = c \cdot \exp(-cx) \) is strictly admissible,
which is equivalent to the fact that $x \mapsto c \cdot \exp(\epsilon x) x^{-k}$ increases for $x \geq x_0$. More general, any map $\Psi(x) = c \cdot \exp(-\epsilon x^2)$ is strictly admissible.

4. Best constants for fractal sets

We turn towards approximation of Cantor set type numbers by rationals. Recall the Cantor set can be defined as the numbers in $[0, 1]$ that allow a representation

$$c_1 3^{-1} + c_2 3^{-2} + c_3 3^{-3} + \cdots, \quad c_i \in \{0, 2\}.$$ 

So apart from special rational numbers with denominator a power of 3, whose ternary representation is not unique, it coincides with the numbers who have no 1 in the unique ternary representation. For sets with similar missing digit properties, Bugeaud’s Theorem 1 in [1], whose proof originates in a special form of the Folding Lemma [11], says the following.

**Theorem 4.1** (Bugeaud). For an integer $b \geq 2$, let $J(b) \subset \{0, 1, \ldots, b - 1\}$ with at least two elements. Denote by $K_J(b)$ the numbers in $[0, 1]$ whose base $b$ expansion contains only digits in $J(b)$. Let $\Psi$ satisfy (A1), (B1). Then for any $c < 1/b$, the set

$$(\mathcal{X}_1(\Psi) \setminus \mathcal{X}_1(c \Psi)) \cap K_{J(b)}$$

is uncountable.

**Remark 4.2.** Choosing $v$ in the proof this theorem large, it is possible to show that the set is still uncountable in any neighborhood of $\{0\}$. Compare this to the following Theorem 4.6.

It is not obvious that the proof of Theorem 4.1 can be modified in the way Theorem 1.1 was modified to obtain Theorem 1.2 to deduce (B1) can be weakened to (B2). We will return to $k = 1$ later.

We turn to the case $k \geq 2$ now. Thanks to Corollary 2.7 this case is indeed easier and proceeding very similar to the proof of (the first assertion of) Theorem 1.4 yields the following.

**Theorem 4.3.** Let $k \geq 2$, $b \geq 2$ be integers and $K_J(b)$ be as in Theorem 4.1. Assume the function $\Psi$ satisfies (A3), (B2). For any $c < 1/b$, the set

$$(\mathcal{X}_k(\Psi) \setminus \mathcal{X}_k(c \Psi)) \cap K_J(b)$$

is uncountable. The same holds for $\mathcal{X}_k^*$, where we may drop the condition (B2).

However, we can do slightly better. Using an approach similar to Theorem 1.26 in [10] with slight refinements, we can improve the bound $1/b$ essentially to $1/(b - 1)$. We will apply Lemma 2.7 in [10], which can be (almost) equivalently stated in the following way.

**Lemma 4.4** (S., 2014). Let $k \geq 2$, $b \geq 2$ be integers, and $\zeta = \sum_{n \geq 1} a_n b^{-n}$ for some positive integer sequence $(a_n)_{n \geq 1}$ that satisfies $\liminf_{n \to \infty} a_{n+1}/a_n > k$. Then for $(x, y) \in \mathbb{Z}^2$ with sufficiently large $x$, the estimate

$$|\zeta x - y| \leq x^{-\frac{k}{k-1}}$$
implies \((x, y)\) is linearly dependent to some
\[x_n := (x_n, y_n) := (b_{a_n}, \sum_{j \leq n} b^{a_j}).\]

We actually replaced the limes in the original condition by the limes inferior, which is easily seen to be permitted analyzing the proof. The proof of Lemma 4.4 with Minkowski’s Theorem, see [6], in [10] allows another extension. Assume the digits 1 in base \(b\) of \(\zeta\) are no longer isolated, but there are integer blocks \(I_1, I_2, \ldots\) where one has free choice of digits within \(I = \bigcup I_n\) and everywhere else we put the digit 0. Provided the block lengths are sufficiently small compared to the gaps between the blocks, Lemma 4.4 still holds.

We will utilize this generalization without rigorous proof in the proof of Theorem 4.6 and omit precise calculations on what sufficiently small in the above sense means for simplicity. This block construction will allow more flexibility in the construction and will lead to the improvement from \(c < 1/b\) to \(c < 1/(b - 1)\). We incorporate the following technical proposition, which gives an estimate how well real numbers in given intervals can be approximated by numbers in \(K_{J(b)}\) that satisfy the block property described above.

We restrict to the interesting case \(b \geq 3\), for \(b = 2\) similar bounds can be given, see Remark 4.8.

**Proposition 4.5.** Let \(b \geq 3\) and \(R \geq 1\) be integers. Let \(e, f\) be integers with \(R < e < f\). Let \(a \in \mathbb{R}\) with the property
\[(33) \quad \frac{a}{b} \leq b^{R-e} < a.\]

Define \(\mathcal{H}\) the set of all numbers of the form \(\chi_{e+1}b^{e-1} + \cdots + \chi_f b^{-f}\), where \(\chi_l \in \{0, 1\}\) for \(e + 1 \leq l \leq f\). Then, there exists \(\kappa \in \mathcal{H}\) such that
\[(34) \quad \frac{1}{b-1} \cdot (1 - b^{-e-f-1}) \leq b^R \cdot \frac{\kappa + b^{-e}}{a} < 1.\]

**Proof.** Define \(\kappa\) as the largest element in \(\mathcal{H}\) for which the right hand side of (33) holds. This is well-defined since \(\kappa = 0\) is a suitable choice and in this case (33) holds by assumption (33). Put \(v := b^{-e} + b^{-e-1} + \cdots + b^{-f}\). We separate two cases.

Case 1: \(a > b^R v\). Then \(\kappa = v - b^{-e}\) and the right inequality in (34) holds by construction. Moreover
\[b^R \frac{\kappa + b^{-e}}{a} = b^R \frac{\kappa + b^{-e} b}{a} \geq b^{R-1} (\kappa + b^{-e}) b^{e-R} = b^{e-1} (\kappa + b^{-e}) = b^{e-1} \sum_{j=e}^f b^{-j},\]
and the right hand side equals the left hand side in (34).

Case 2: \(a \leq b^R v\). Then, by (33), there exists a largest index \(t \in [e, f - 1]\) such that the inequality \(b^R u_t < a\) holds, where \(u_t := b^{-e} + b^{-e-1} + \cdots + b^{-t}\). By definition \(\kappa \geq u_t - b^{-e}\) and the right inequality in (34) holds. Moreover, by maximality of \(t\) we infer
\[b^R \cdot \frac{\kappa + b^{-e}}{a} \geq b^R u_t \geq \frac{b^{-e} + b^{-e-1} + \cdots + b^{-t}}{b^{-e} + b^{-e-1} + \cdots + b^{-t} + b^{-e-1}}.\]
The right hand side is easily seen to increase as \( t \) does. Consequently for \( b \geq 3 \) indeed
\[
b^R \cdot \frac{\kappa + b^{-e}}{a} \geq \frac{b^{-e}}{b^{-e} + b^{-e-1}} = \frac{b}{b+1} > \frac{1}{b+1} > \frac{1}{b-1} \cdot (1 - b^{-f-1}).
\]

For the convenience of the reader, we first give a detailed proof of the weaker assertion with bound \( c < \frac{1}{b} \), where the outline of the proof is easier to detect. Subsequently, we will describe how to generalize the result, where we will only sketch some parts that can be deduced very similarly to the weaker assertion. This avoids an overly technical proof.

**Theorem 4.6.** Let \( k \geq 2, b \geq 3 \) be integers and \( K_{J(b)} \) be as in Theorem 4.1. Assume the function \( \Psi \) satisfies (A3), (B2). Then the set
\[
(35) \quad \left( K_k(\Psi) \setminus \bigcup_{c < 1} K_k(c/\Psi) \right) \cap K_{J(b)}
\]
is uncountable in any neighborhood of \( \{0\} \). The same holds for \( K_k^* \), where we may drop the condition (B2). Elements in (35) can be effectively constructed.

**Proof.** Without loss of generality assume \( J(b) = \{0,1\} \). We first prove the result for \( c < 1/b \) in the union.

Consider an at the moment arbitrary function \( \delta : \mathbb{N} \mapsto \mathbb{R}_{>0} \) which tends to 0, to be determined later. Write also \( \delta_n := \delta(n) \). Define the disturbed function
\[
(36) \quad \Psi_n(x) = (1 + \delta(x))\Psi(x).
\]
Put
\[
(37) \quad \zeta = \sum_{n \geq 1} b^{-a_n}
\]
with an increasing sequence of positive integers \( a_n \) depending on \( \Psi \), defined recursively. First let \( a_1 \geq 3 \) be arbitrary. Since \( b \geq 2 \), construction (37) implies \( \zeta \in (0,1/2) \). Now determine \( a_{n+1} \) by
\[
(38) \quad \frac{\tilde{\Psi}(b^{ka_n})}{b} \leq b^{ka_n-a_n+1} < \tilde{\Psi}(b^{ka_n}).
\]
Note that \( a_{n+1} \) is almost independent from the exact choice of the function \( \delta \) for large \( n \), since a closer look at (38) shows a small perturbation of \( \tilde{\Psi} \) can effect a change of \( a_{n+1} \) by at most 1. By (A3) we have
\[
(39) \quad a_{n+1} \geq 2ka_n, \quad n \geq n_0.
\]
Next we prove
\[
(40) \quad \max_{1 \leq j \leq k} \| \zeta_j b^{ka_n} \| = b^{ka_n-a_n+1}(1 + O(b^{-2a_n+1}))
\]
as \( n \to \infty \). Write \( \zeta = S_n + \epsilon_n \) with
\[
S_n = \sum_{i=1}^n b^{-a_i}, \quad \epsilon_n = \sum_{i=n+1}^{\infty} b^{-a_i}.
\]
Since $S_n < 1$, there exists a positive real greater than 0 and the binomial coefficients are bounded above by $k!$, we have that
\[
\zeta^j = \sum_{i=0}^{j} \binom{j}{i} S_{n}^{i} e_{n}^{-i} = S_{n}^{j} + j S_{n}^{j-1} e_{n} + O(e_{n}^{2}), \quad 1 \leq j \leq k,
\]
as $n \to \infty$. Note now that $b^{k_{n}} S_{n}^{j}$ is an integer for $1 \leq j \leq k$ by construction. Moreover, the remaining terms converge to 0 and since $0 < S_{n} < \zeta < 1/2$ are maximized for $j = 1$ by Proposition 2.4 at least for large $n$, such that $j S_{n}^{j-1} = 1$. Thus in view of (39) we have
\[
\max_{1 \leq j \leq k} \| b^{k_{n}} \zeta^j \| = \| b^{k_{n}} \zeta \| = b^{k_{n} - a_{n} + 1}(1 + O(b^{-2a_{n}+1}))
\]
so (40) is proved.

Denote the remainder terms as a sequence $(\epsilon_{n})_{n \geq 1}$ of positive reals that tends to 0 and observe it is by the above remarks almost independent of the exact choice of the function $\delta$. Combination of (38), (40) shows that for $z_{n} := b^{k_{n}}$, we have
\[
\frac{1 + \epsilon_{n}}{b} < \max_{1 \leq j \leq k} \| \zeta^j z_{n} \| < \tilde{\Psi}(z_{n})(1 + \epsilon_{n}).
\]
Let $\delta$ tend to 0 sufficiently slowly such that $\epsilon_{n} < \delta_{n}$. For any $\sigma > 0$ and large enough $n \geq \tilde{n}(\sigma)$ we have $b^{-1}(1 + \delta_{n})^{-1}(1 + \epsilon_{n}) > (b + \sigma)^{-1}$. Inserting this in (41) in view of $\delta(x) \to 0$ in (36) yields
\[
\frac{1}{b + \sigma} \Psi(z_{n}) \leq \max_{1 \leq j \leq k} \| \zeta^j z_{n} \| < \Psi(z_{n}).
\]
It remains to show
\[
\frac{1}{b + \sigma} \Psi(z) \leq \max_{1 \leq j \leq k} \| \zeta^j z \|
\]
for all sufficiently large integers $z \geq \tilde{z}(\sigma)$. By the assumption (32), due to Corollary 2.7 we may restrict to $z = q^k$ where $q$ is a denominator of the continued fraction expansion of $\zeta$ and $\| q \zeta \| \leq q^{-2k+1} \leq q^{-3}$ (here we need $k \geq 2$). It is not hard to check only values $q$ of the form $q = b^{m_{n}}$ have this property. Concretely it follows from (39) and Lemma 1.4 since $k/(k-1) \leq k < 2k$ such as $k/(k-1) < 3$, for $k \geq 2$. The implied numbers $z = q^k$ of those $q$ are just $z_{n}$ as above, for which we have shown (42), though.

Since $a_{1}$ can be chosen arbitrarily large, the set is condensed at 0, and in view of (39) all constructed numbers $\zeta$ are very well approximable and thus transcendental (note that $\mathcal{X}_{k}$ is defined for $\mathcal{C}_{k}$ only). Finally, we modify the construction to show it is indeed uncountable. Define $a_{2n+1}$ given $a_{2n}$ as in (35), but $a_{2n} \in [(k+1)a_{2n-1}, 2ka_{2n-1} - 1]$ arbitrary, and define $\zeta$ by (37). Indeed, any $z_{2n}$ satisfies (42), and by virtue of Corollary 2.7, Lemma 2.7 in [10] and (A.3), we infer (12) very similarly. Clearly this method yields uncountably many such numbers. This finishes the proof of the weaker assertion where the union is taken over $c < 1/b$.

For the stronger result, consider $\zeta \in (0, 1/2)$ with base $b$ digits in $\{0, 1\}$ whose 1 digits are not isolated as in the proof above, but are restricted by the following. For any $n \in \mathbb{N}$ we will define an integer block $I_{n} = \{e_{n}, e_{n}+1, \ldots, f_{n}\}$, and the $e_{n}$-th base $b$ digit of $\zeta$ equals 1 and we have free base $b$ digit choice 0 or 1 within $I_{n} \setminus \{e_{n}+1\}$. Put $I = \bigcup I_{n}$ and put 0 in the base $b$ decimal places within $\mathbb{N} \setminus I$, which means at decimal places of the
form $f_n + 1, f_n + 2, \ldots, e_{n+1} - 1$. Suppose that the lengths of $I_n$ are given as $f_n - e_n = n$, such that they tend to infinity but rather slowly. Let $e_{n+1}$ be defined recursively from $f_n$ via

$$\frac{\widetilde{\Psi}(b^{kf_n})}{b} \leq b^{kf_n - e_{n+1}} < \widetilde{\Psi}(b^{kf_n}).$$

Given any $I_1 = \{e_1, e_1 + 1\}$ with large $e_1$, the sets $I_n$ are now well-defined and disjoint, so the class of numbers $\zeta$ constructed as above is well-defined too. By (A3) again we have

$$e_{n+1} \geq 2kf_n, \quad n \geq n_0.$$

For the moment, let $\kappa_n$ be any rational number in $[0, 1)$ of the form

$$\kappa_n := \chi \epsilon_{n+1} b^{-e_{n+1} - 1} + \ldots + \chi_{f_n+1} b^{-f_n+1},$$

where $\chi_l \in \{0, 1\}$ for $e_{n+1} + 1 \leq l \leq f_n$, which we will specify soon. Recall we put 1 for the $e_{n+1}$-th base $b$ digit. Hence any such choice of $\kappa_n$ determines the choice of 0s and 1s in $I_{n+1}$. For any $\zeta$ constructed this way, similarly to (40) we infer

$$\max_{1 \leq j \leq k} \| \zeta^j b^{kf_n} \| = b^{kf_n} (b^{-e_{n+1}} + \kappa_n + O(b^{-2f_{n+1}}))$$

as $n \to \infty$ with positive remainder term. Choose $\kappa_n$ largest possible such that

$$\frac{\widetilde{\Psi}(b^{kf_n})}{b} \leq b^{kf_n} (b^{-e_{n+1}} + \kappa_n) < \widetilde{\Psi}(b^{kf_n}).$$

Such a choice is clearly possible, for if we let all digits within $I_{n+1} \backslash \{e_{n+1}\}$ vanish and thus $\kappa_n = 0$, (47) follows from (44). We need a better lower bound for the quotient $b^{kf_n} (b^{-e_{n+1}} + \kappa_n) / \widetilde{\Psi}(b^{kf_n})$. We apply Proposition 4.5 with

$$a := \widetilde{\Psi}(b^{kf_n}), \quad R := kf_n, \quad e := e_{n+1}, \quad f := f_{n+1}.$$ 

It yields

$$b^{kf_n} (b^{-e_{n+1}} + \kappa_n) \geq \frac{1}{b - 1} \cdot (1 - b^{e_{n+1} - f_{n+1} - 1}),$$

where the worst case scenario is that $b^{kf_n - e_{n+1}}$ is very close to the lower bound $\widetilde{\Psi}(b^{kf_n})/b$ in (41) or equivalently $\widetilde{\Psi}(b^{kf_n})$ is close to $b^{kf_n - e_{n+1} + 1}$, and in this case the optimal choice is $\kappa_n = b^{-e_{n+1} - 1} + b^{-e_{n+1} - 2} + \ldots + b^{-f_{n+1}}$.

Since $f_n - e_n$ tends to infinity, the right hand side expression in (48) tends to $1/(b - 1)$. Similarly to the special case $c < 1/b$, it follows from (46) and (36) if $\delta(x) \to 0$ sufficiently slow, that for any $\sigma > 0$ there are arbitrarily large integers $z_n = b^{kf_n}$ for which

$$\frac{1}{b - 1 + \sigma} \Psi(z_n) \leq \max_{1 \leq j \leq k} \| \zeta^j z_n \| < \Psi(z_n).$$

Finally, the relation

$$\frac{1}{b + \sigma} \Psi(z) \leq \max_{1 \leq j \leq k} \| \zeta^j z \|$$
for all sufficiently large integers $z \geq \hat{z}(\sigma)$ can be inferred very similarly to the special case $c < 1/b$ with the slight generalization of Lemma 4.4 mentioned preceding the proof, since the growth of $f_n - \gamma_n = n$ is sufficiently slow compared with the large gaps in (15). □

Remark 4.7. Observe that a density result in the spirit of Theorem 1.4 cannot hold for the set in (35) for $b \geq 3$ by definition of $K_{j(b)}$.

Remark 4.8. Let $b = 2$. A very similar proof works and yields another proof the first assertion of Theorem 1.4 where the binary digit expansion of the implied $\zeta$ instead of the continued fraction expansion is determined. Concretely, if we let $b = 2$ within the assumptions of Proposition 4.10, the proof of its case 1 works precisely as for $b \geq 3$ and is applicable to Theorem 4.6 in the same way. For case 2, we obtain a bound as follows. By the assumption of case 2, the binary expansion of $a$ is given as $a = \tau_0 2^{R-\epsilon} + \tau_1 2^{R-\epsilon-1} + \cdots$ with $\tau_i \in \{0, 1\}$ for all $i$, and $\tau_0 = 1$ by (33). Since $\chi_l$ are arbitrary in $\{0, 1\}$, we may put $\chi_{-1} = \tau_1, \chi_{-2} = \tau_2, \ldots, \chi_f = \tau_{f-\epsilon}$. Then

$$2^R \cdot \frac{\kappa + 2^{-\epsilon}}{a} = \frac{\tau_0 2^{R-\epsilon} + \tau_1 2^{R-\epsilon-1} + \cdots + \tau_{f-\epsilon} 2^{-f}}{\tau_0 2^{R-\epsilon} + \tau_1 2^{R-\epsilon-1} + \cdots} \geq \frac{2^{-\epsilon} + \tau_1 2^{R-\epsilon-1} + \cdots + \tau_{f-\epsilon} 2^{-f}}{2^{-\epsilon} + \tau_1 2^{R-\epsilon-1} + \cdots + \tau_{f-\epsilon} 2^{-f} + 2^{-f}}.$$

The right hand side is smallest if all $\tau_j$ of positive index vanish, and thus

$$b^R \cdot \frac{\kappa + 2^{-\epsilon}}{a} = 2^R \cdot \frac{\kappa + 2^{-\epsilon}}{a} \geq \frac{2^{-\epsilon}}{2^{-\epsilon} + 2^{-f}} = \frac{2^{-\epsilon}}{2^{-\epsilon} + 1}.$$  

(51)

The most right expression tends to 1 as $f - \epsilon$ tends to infinity, such that we can apply (51) similarly to (48) in the proof of Theorem 4.6.

Remark 4.9. Functions $\Psi$ that lead to what was called the worst case scenario in the proof asymptotically for all large $q \in \mathbb{N}$ can readily be constructed, for example

$$\Psi(x) = x^{-N} + \exp(-x), \quad N \in \mathbb{N}.$$  

The bound $1/(b-1)$ seems to be close to the optimal value that can be obtained with the current methods, in particular restricting to approximation by rationals that are in the missing digit set $K_{j(b)}$ as well.

We return to $k = 1$. Let $\gamma = (1 + \sqrt{5})/2$ the golden ratio. If we restrict to functions $\Psi$ with the stronger decay condition

$$(A') \quad \Psi(x) = o(x^{-\gamma}), \quad \text{for some (arbitrarily small)} \quad \epsilon > 0, \quad \text{for all} \quad x \geq \hat{x},$$  

the result of Theorem 4.6 can be extended to $k = 1$. For this we need a slight generalization of Lemma 4.4 to non-integral bounds for the lim sup inferior.

Lemma 4.10. Let $b \geq 2$ be an integer, and $\zeta = \sum_{n \geq 1} b^{-a_n}$ for some positive integer sequence $(a_n)_{n \geq 1}$ that satisfies $\liminf_{n \to \infty} a_{n+1}/a_n =: \omega > 2$. Define

$$x_n := (x_n, y_n) := (b^{a_n}, \sum_{j \leq n} b^{a_j - a_n}).$$  

Then for all sufficiently large $n$ we have

$$|\zeta x_n - y_n| < x_n^{-\omega+1}.$$
Moreover, for \((x, y) \in \mathbb{Z}^2\) with sufficiently large \(x\), the estimate
\[
|\zeta x - y| \leq x^{-\frac{\omega}{\omega - 1}}
\]
implies \((x, y)\) is linearly dependent to some \(x_n\).

We omit the proof, as the first assertion can be derived straightforward and the second very similarly to Lemma 4.4 with Minkowski’s second Theorem. Lemma 4.10 can be extended to the case of blocks of non-vanishing base \(b\) digits analogous to Lemma 4.4.

**Theorem 4.11.** Let \(b \geq 3\) be an integer and \(K_{J(b)}\) be as in Theorem 4.1. Assume the function \(\Psi\) satisfies \((A')\), \((B2)\). Then the set
\[
\left(\mathcal{K}_1(\Psi) \setminus \cup_{c < b-1} \mathcal{K}_1(c\Psi)\right) \cap K_{J(b)}
\]
is uncountable in any neighborhood of \(\{0\}\). The same holds for \(\mathcal{K}_1^*\), where we may drop the condition \((B2)\). Elements in \((52)\) can be effectively constructed.

**Proof.** Proceed as in the proof of Theorem 4.6 with \(k = 1\) and observe that the stronger condition \((A')\) implies the stronger estimate
\[
a_{n+1} > (\gamma + 1 + \epsilon)a_n
\]
instead of \((39)\) for large \(n\), with a suitable \(\epsilon\) from \((A')\). We can infer \((12)\) for \(k = 1\) precisely as in the case \(k \geq 2\). Concerning \((43)\) for \(k = 1\), note that for \(\omega > \gamma + 1\) we have \(\omega/(\omega - 1) < \omega - 1\). By virtue of \((53)\), Lemma 4.10 shows approximations of order \(\|q\zeta\| < q^{-\gamma}\) are induced precisely by integers of the form \(q = b^{ka_n} = b^n\). However, due to \((A')\), the estimate \(\|q\zeta\| < q^{-\gamma}\) is necessary for a (large) counterexample to \((43)\), and we conclude as in the case \(k \geq 2\) (note that the generalization of Lemma 4.10 to blocks holds again, as mentioned subsequent to the lemma).

We close with some remarks.

**Remark 4.12.** Theorem 4.11 extends to \(b = 2\) similarly to Remark 4.8 and leads to explicit binary expansions of numbers \(\zeta\) that satisfy Theorem 1.2 provided \(\Psi\) satisfies \((A')\).

**Remark 4.13.** With a concise combination of the block method of the proof of Theorem 4.11 with the Folding Lemma instead of the unsubtle Lemma 4.10, condition \((A')\) can certainly be weakened in Theorem 4.11. It seems reasonable that \((A1)\) is sufficient, which would unconditionally improve Theorem 4.1.

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