HOM–JORDAN–MALCEV–POISSON ALGEBRAS

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We provide and study a Hom-type generalization of Jordan–Malcev–Poisson algebras called Hom–Jordan–Malcev–Poisson algebras. We show that they are closed under twisting by suitable self-maps and give a characterization of admissible Hom–Jordan–Malcev–Poisson algebras. In addition, we introduce the notion of pseudo-Euclidian Hom–Jordan–Malcev–Poisson algebras and describe its $T^*$-extension. Finally, we generalize the notion of Lie–Jordan–Poisson triple system to the Hom setting and establish its relationships with the Hom–Jordan–Malcev–Poisson algebras.

Introduction

Nonassociative algebras become an important research field due to their importance in various problems related to physics and other branches of mathematics. The first occurrences of nonassociative Hom-algebras appeared in the study of quasideformations of the Lie algebras of vector fields. The Hom–Lie algebras were first introduced by Hartwig, Larsson, and Silvestrov in order to describe $q$-deformations of the Witt and Virasoro algebras with the help of $\sigma$-derivations (see [4]). The corresponding associative-type objects were called Hom-associative algebras. They were introduced by Makhlouf and Silvestrov in [7]. The Hom-alternative, Hom–Jordan, and Hom-flexible algebras were first introduced in [6] and then considered, as well as the Hom–Malcev algebras, in [11].

Poisson algebras form an important class of nonassociative algebras. They are used in numerous fields of mathematics and physics. Thus, they play a fundamental role in the Poisson geometry, quantum groups, deformation theory, Hamiltonian mechanics, and topological field theories. Poisson algebras were generalized in many ways. Thus, if we omit the property of commutativity of the associative structure, then we get the class of noncommutative Poisson algebras. Another way to generalize this class is to replace the associative structure by a Jordan product and the Lie bracket by a Malcev bracket. As a result, we get a new class of algebras called Jordan–Malcev–Poisson algebras (JMP algebras), which are defined by a triple $\mathcal{A} (\cdot, \cdot, \circ)$ consisting of a linear space equipped with a Malcev bracket and a Jordan structure satisfying the Leibniz rule:

$$[x, y \circ z] = [x, y] \circ z + y \circ [x, z].$$

They were introduced by Ait Ben Haddou, Benayadi, and Boumlane in [3]. These algebras can be described in terms of a single bilinear operation and are called admissible JMP algebras. This class contains alternative algebras. In a particular case, where $(\mathcal{A}, \circ)$ is associative and commutative, $(\mathcal{A}, [\cdot, \cdot], \circ)$ becomes a Malcev–Poisson algebra. This concept was first introduced by Shestakov in [9].

The aim of the present paper is to study a twisted generalization of the JMP algebras (they are called Hom–JMP algebras) and some other related algebraic structures (admissible Hom–JMP algebras). Moreover, we introduce...
and study pseudo-Euclidian Hom–JMP algebras, which are Hom–JMP algebras endowed with symmetric invariant nondegenerate bilinear forms. We provide a twist construction and extend the $T^*$-extension theory to this class of nonassociative Hom-algebras. We also construct generalized Hom-triple systems that are called Hom–Lie–Jordan–Poisson triple systems from admissible Hom–JMP algebras.

The paper is organized as follows: In Section 1, we summarize the definitions and some key constructions of the Hom–JMP algebras. In Section 2, we study and highlight the relationships between Hom–JMP algebras and admissible Hom–JMP algebras. In addition, it is shown that admissible Hom–JMP algebras are power Hom-associative. In Section 3, we introduce the notion of pseudo-Euclidian Hom–JMP algebras and describe its $T^*$-extension. Section 4 is devoted to the study of the Hom version of Lie–Jordan–Poisson triple system algebras. Moreover, its connection with admissible Hom–JMP algebras is provided.

1. Definitions and Preliminary Results

1.1. Basic Definitions. In this section, we introduce Hom–JMP algebras as a generalization of Hom–Poisson algebras, Malcev–Poisson algebras, and JMP algebras. We show that the Hom–JMP algebras are closed under suitable twisting by weak morphisms.

We first present basic definitions for the Hom-algebras. We work over a fixed commutative field $\mathbb{K}$ of characteristic 0.

**Definition 1.1.** Let $(A, \mu, \alpha)$ be a Hom-algebra.

1. The Hom-associator $as_A: A^{\otimes 3} \to A$ is defined as follows:

$$as_A(x, y, z) = \mu(\mu(x, y), \alpha(z)) - \mu(\alpha(x), \mu(y, z)).$$

2. The Hom-algebra $A$ is called a Hom–Jordan algebra if it is commutative and satisfies the Hom–Jordan identity

$$as_A(x^2, \alpha(y), \alpha(x)) = 0.$$

3. The Hom–Jacobiator $J_A: A^{\otimes 3} \to A$ is defined as

$$J_A(x, y, z) = \bigcirc_{x,y,z} \mu(x, y, \alpha(z)),$$

where $\bigcirc_{x,y,z}$ denotes the operation of cyclic summation over $x, y, z$.

4. A Hom–Malcev algebra is a Hom-algebra $(A, [,], \alpha)$ such that $[,]$ is skew-symmetric and the Hom–Malcev identity

$$J_A(\alpha(x), \alpha(y), [x, z]) = [J_A(x, y, z), \alpha^2(x)]$$

is satisfied for all $x, y, z \in A$.

5. A Hom–flexible algebra is a Hom-algebra $(A, \mu, \alpha)$ satisfying

$$as_A(x, y, z) + as_A(z, y, x) = 0$$

or, equivalently, $as_A(x, y, x) = 0$ for all $x, y, z \in A$. 


6. A Hom-alternative algebra is a Hom-algebra \((A, \mu, \alpha)\) satisfying

\[ as_A(x, y, z) + as_A(y, x, z) = as_A(x, y, z) + as_A(x, z, y) = 0 \]

or, equivalently, \( as_A(x, x, y) = as_A(x, y, y) = 0 \) for all \( x, y, z \in A \).

Note that any Hom-alternative algebra is Hom-flexible.

Let \((A, \mu, \alpha)\) be a Hom-algebra. We define a cyclic Hom-associator \( S_A \) by

\[ S_A(x, y, z) = \sum_{x, y, z} as_A(x, y, z). \]

We now recall the definition of a JMP algebra [5].

**Definition 1.2.** A JMP algebra \((A, \{\cdot,\}, \circ)\) consists of a Malcev algebra \((A, \{\cdot,\})\) and a Jordan algebra \((A, \circ)\) such that the Leibniz identity

\[ \{x, y \circ z\} = \{x, y\} \circ z + y \circ \{x, z\} \]

is satisfied for all \( x, y, z \in A \).

In a JMP algebra \((A, \{\cdot,\}, \circ)\), the bracket \( \{\cdot,\} \) is called the Poisson bracket and \( \circ \) is called the Jordan product. The Leibniz identity states that \( \{x, -\} \) is a derivation with respect to the Jordan product.

The Hom–Poisson algebras were first introduced in [8] by Makhlouf and Silvestrov. We now define the Hom-type generalization of a JMP algebra.

**Definition 1.3.** A Hom–JMP algebra \((A, \{\cdot,\}, \circ, \alpha)\) consists of a Hom–Malcev algebra \((A, \{\cdot,\}, \alpha)\) and a Hom–Jordan algebra \((A, \circ, \alpha)\) such that the Hom–Leibniz identity

\[ \{\alpha(x), y \circ z\} = \{x, y\} \circ \alpha(z) + \alpha(y) \circ \{x, z\} \]

is true for any \( x, y, z \in A \).

In the Hom–JMP algebra \((A, \{\cdot,\}, \circ, \alpha)\), the operations \( \{\cdot,\} \) and \( \circ \) are called the Hom–Poisson bracket and the Hom–Jordan product, respectively. In view of the skew-symmetry of the Hom–Poisson bracket \( \{\cdot,\} \), the Hom–Leibniz identity is equivalent to

\[ \{x \circ y, \alpha(z)\} = \{x, z\} \circ \alpha(y) + \alpha(x) \circ \{y, z\}. \]

A JMP algebra is exactly a Hom–JMP algebra with identity twisting map.

Let \((A, \{\cdot,\}, \circ_A, \alpha_A)\) and \((B, \{\cdot,\}, \circ_B, \alpha_B)\) be two Hom–JMP algebras. A weak morphism \( f : A \to B \) is a linear map such that

\[ f\{\cdot,\}_A = \{\cdot,\}_B f^\otimes and \quad f\circ_A = \circ_B f^\otimes. \]

A morphism \( f : A \to B \) is a weak morphism such that \( f\alpha_A = \alpha_B f \).

Note that a quadruple \((A, \{\cdot,\}, \circ, \alpha)\) is said to be multiplicative if and only if the twisting map \( \alpha : A \to A \) is a morphism.
The following result says that the Hom–JMP algebras are closed under twisting by weak self-morphisms.

**Theorem 1.1.** Let \((A, \{\cdot,\}, \circ, \alpha)\) be a Hom–JMP algebra and let \(\beta : A \to A\) be a weak morphism. Then

\[
A_\beta = \left( A, \{\cdot,\}_\beta = \beta\{\cdot,\}, \circ_\beta = \beta\circ, \beta\alpha \right)
\]

is also a Hom–JMP algebra. Moreover, if \(A\) is multiplicative and \(\beta\) is a morphism, then \(A_\beta\) is a multiplicative Hom–JMP algebra.

**Proof.** In [11] the author proved that \((A, \{\cdot,\}_\beta, \beta\alpha)\) is a Hom–Malcev algebra and \((A, \circ_\beta, \beta\alpha)\) is Hom–Jordan algebra.

It remains to establish the Hom–Leibniz identity. Let \(x, y, z \in A\). We know that

\[
\{\alpha(x), y \circ z\} = \{x, y\} \circ \alpha(z) + \alpha(y) \circ \{x, z\}.
\]

Thus, applying \(\beta^2\) to the previous identity, we obtain

\[
\{\beta^2\alpha(x), \beta^2(y) \circ \beta^2(z)\} = \{\beta^2(x), \beta^2(y)\} \circ \beta^2\alpha(z) + \beta^2\alpha(y) \circ \{\beta^2(x), \beta^2(z)\},
\]

i.e.,

\[
\{\beta\alpha(x), y \circ_\beta z\}_\beta = \{x, y\}_\beta \circ_\beta \beta\alpha(z) + \beta\alpha(y) \circ_\beta \{x, z\}_\beta.
\]

Therefore, \(A_\beta\) is a Hom–JMP algebra.

Theorem 1.1. is proved.

**Corollary 1.1.** Let \((A, \{\cdot,\}, \circ)\) be a JMP algebra and let \(\alpha : A \to A\) be a JMP morphism. Then

\[
(A, \{\cdot,\}_\alpha = \alpha\{\cdot,\}, \circ_\alpha = \alpha \circ, \alpha)\]

is a multiplicative Hom–JMP algebra.

**1.2. Admissible Hom–JMP Algebras.** Let \((A, \cdot, \alpha)\) be a Hom-algebra. We can define the following two new products:

\[
[x, y] = x \cdot y - y \cdot x \quad \text{and} \quad x \circ y = \frac{1}{2}(x \cdot y + y \cdot x) \quad \text{for all} \quad x, y, z \in A.
\]

We denote by \(A^-\) (resp., \(A^+\)) the algebra \(A\) with multiplication \([-,-]\) (resp., \(\circ\)).

**Lemma 1.1** [11]. Let \((A, \cdot, \alpha)\) be a Hom-flexible algebra. Then

\[
2S_A = J_{A^-}.
\]

**Lemma 1.2** [7]. A Hom-algebra \((A, \cdot, \alpha)\) is flexible if and only if

\[
[\alpha(x), y \circ z] = [x, y] \circ \alpha(z) + \alpha(y) \circ [x, z].
\]
Lemma 1.3. Let \((A, \cdot, \alpha)\) be a Hom-flexible algebra. Then
\[
J_{A^{-}}(x^2, \alpha(y), \alpha(x)) = 0 \quad \forall x, y \in A, \quad \text{where} \quad x^2 = x \cdot x = x \circ x.
\]

Proof. Let \(x, y \in A\). Thus,
\[
\begin{aligned}
[\alpha^2(x), \alpha(y)], \alpha^2(x)] + \alpha^2(x)] + \left[\alpha(x), x^2, \alpha^2(y)\right] = 0
\end{aligned}
\]
\[
= 2[\alpha^2(x), [y, x] \circ \alpha(x)] + 2[[y, x], \alpha(x)] \circ \alpha^2(x)
\]
\[
= 2\left\{\alpha^2(x)\cdot([y, x], \alpha(x)) + \alpha^2(x)\cdot(\alpha(x), [y, x]) - ([y, x], \alpha(x)) \cdot \alpha^2(x)
\right. 
\]
\[
\left. - (\alpha(x), [y, x]) \cdot \alpha^2(x) + ([y, x], \alpha(x)) \cdot \alpha^2(x) - (\alpha(x), [y, x]) \cdot \alpha^2(x)
\right.
\]
\[
+ \alpha^2(x)\cdot([y, x], \alpha(x)) - \alpha^2(x)\cdot(\alpha(x), [y, x]) \right\} = 0.
\]

Lemma 1.3 is proved.

The following result gives a characterization of Hom-flexible algebras.

Proposition 1.1. A Hom-algebra \((A, \cdot, \alpha)\) is flexible if and only if
\[
as_A(x, y, z) = \frac{1}{4}J_{A^{-}}(x, y, z) + \frac{1}{4}\left[\alpha(y), [z, x]\right] + as_{A^+}(x, y, z) \quad \text{for all} \quad x, y, z \in A.
\] (1.1)

Proof. If \((A, \cdot, \alpha)\) is Hom-flexible, then, by Lemma 1.1, we get
\[
J_{A^{-}}(x, y, z) + [\alpha(y), [z, x]] + 4as_{A^+}(x, y, z)
\]
\[
= 2S_A(x, y, z) + \alpha(y)(zx) - \alpha(y)(xz) - (zx)\alpha(y) + (xz)\alpha(y)
\]
\[
+ (xy)\alpha(z) + (yx)\alpha(z) + \alpha(z)(xy) + \alpha(z)(yx) - \alpha(x)(yz)
\]
\[
- \alpha(x)(yz) - (yz)\alpha(x) - (zy)\alpha(x)
\]
\[
= 2S_A(x, y, z) - as_A(y, z, x) + as_A(y, x, z) - as_A(z, x, y)
\]
\[
+ as_A(x, z, y) + as_A(x, y, z) - as_A(z, y, x)
\]
\[
= 4as_A(x, y, z).
\]

Conversely, suppose that Eq. (1.1) holds. Then
\[
as_A(x, y, x) = \frac{1}{4}J_{A^{-}}(x, y, x) + \frac{1}{4}\left[\alpha(y), [x, x]\right] + as_{A^+}(x, y, x)
\]
\[
\frac{1}{4} \left(\{x,y\}, \alpha(x) \right) + \{y,x\}, \alpha(x) \right) + \{x,x\}, \alpha(y) \right) \\
+ (x \circ y) \circ \alpha(x) - \alpha(x) \circ (y \circ x) \\
= 0 \quad \text{(since } \circ \text{ is commutative)}.
\]

Hence, \( A \) is Hom-flexible.

**Corollary 1.2.** Let \((A, \cdot, \alpha)\) be a Hom-flexible algebra. Then

\[
as_A(x^2, \alpha(y), \alpha(x)) = as_A(x^2, \alpha(y), \alpha(x)).
\]

**Proof.** Straightforward.

**Definition 1.4.** A Hom-algebra \((A, \cdot, \alpha)\) is said to be an admissible Hom–JMP algebra if \((A, \cdot, \circ, \alpha)\) is a Hom–JMP algebra.

**Remark 1.1.** For a given Hom–JMP algebra \((A, \{\}, \circ, \alpha)\), the vector space \( A \) endowed with the morphism \( \alpha \) and a product defined by \( x \cdot y := \frac{1}{2} \{x, y\} + x \circ y \), is an admissible Hom–JMP algebra.

**Proposition 1.2.** Let \((A, \cdot, \alpha)\) be an admissible Hom–JMP algebra and let \( \beta : A \to A \) be a weak morphism. Then

\[
A_\beta = (A, \cdot_\beta, \beta \alpha)
\]
is also an admissible Hom–JMP algebra, where \( x \cdot_\beta y = \beta(x) : \beta(y) \). Moreover, if \( A \) is multiplicative and \( \beta \) is a morphism, then \( A_\beta \) is a multiplicative admissible Hom–JMP algebra.

**Proof.** Straightforward.

**Example 1.1.** Every Hom-alternative algebra is an admissible Hom–JMP algebra.

**Remark 1.2.** Note that not all admissible Hom–JMP algebras are Hom-alternative algebras. Indeed, let \( A \) be the three-dimensional algebra defined with respect to the basis \( \{e_1, e_2, e_3\} \) by

|   | \( e_1 \) | \( e_2 \) | \( e_3 \) |
|---|---|---|---|
| \( e_1 \) | 0 | \( e_2 \) | \( -e_3 \) |
| \( e_2 \) | \( -e_2 \) | 0 | \( e_1 \) |
| \( e_3 \) | \( e_3 \) | \( -e_1 \) | 0 |

According to [3], \( A \) is an admissible JMP algebra. Consider a morphism \( \alpha : A \to A \) defined by

\[
\alpha(e_1) = e_1, \quad \alpha(e_2) = \lambda e_2, \quad \alpha(e_3) = \frac{1}{\lambda} e_3, \quad \lambda \in \mathbb{K}\backslash\{0\}.
\]
Then, in view of Proposition 1.2, \((A, \cdot, \alpha)\) is an admissible Hom–JMP algebra. On the other hand, 

\[ as_A(e_2, e_3, e_3) = (e_2 \cdot e_3) \cdot \alpha(e_3) - \alpha(e_2) \cdot (e_3 \cdot e_3) = -\frac{1}{\lambda^2} e_3 \neq 0. \]

Then \(A\) is not a Hom-alternative algebra.

**Remark 1.3.** Every admissible Hom–JMP algebra is Hom-flexible.

**Theorem 1.2.** Let \((A, \cdot, \alpha)\) be a Hom-flexible and Hom–Malcev admissible algebra. Then \(A\) is an admissible Hom–JMP algebra if and only if \((A, \cdot, \alpha)\) satisfies the identity

\[ R_{\alpha^2(x)} L_{x^2} \alpha = L_{\alpha(x^2)} R_{\alpha(x)} \alpha \quad \forall x \in A, \quad (1.2) \]

where \(L_x\) (resp., \(R_x\)) is the left multiplication (resp., right multiplication) by \(x\) in the algebra \((A, \cdot, \alpha)\).

**Proof.** Since \((A, \cdot, \alpha)\) is Hom-flexible, due to Corollary 1.2, we get

\[ as_A(x^2, \alpha(y), \alpha(x)) = as_A(x^2, \alpha(y), \alpha(x)). \]

This immediately yields the remaining part of the proof.

**Example 1.2.** Consider a five-dimensional Hom-algebra \((A, \cdot, \alpha)\) with respect to a basis \(\{e_1, \ldots, e_5\}\) and a multiplication table

|   | \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) |
|---|---|---|---|---|---|
| \(e_1\) | 0 | \(e_5 + \frac{1}{2} e_4\) | 0 | \(\frac{\nu}{2} e_1\) | 0 |
| \(e_2\) | \(e_5 - \frac{1}{2} e_4\) | 0 | 0 | \(-\frac{\nu^{-1}}{2} e_2\) | 0 |
| \(e_3\) | 0 | 0 | 0 | \(\frac{\lambda}{2} e_3\) | 0 |
| \(e_4\) | \(\frac{\nu}{2} e_1\) | \(\frac{\nu^{-1}}{2} e_2\) | \(\frac{\lambda}{2} e_3\) | \(-e_5\) | 0 |
| \(e_5\) | 0 | 0 | 0 | 0 | 0 |

where \(\alpha : A \to A\) is given by

\[ \alpha(e_1) = \nu e_1, \quad \alpha(e_2) = \nu^{-1} e_2, \quad \alpha(e_3) = \lambda e_3, \quad \alpha(e_4) = e_4, \quad \alpha(e_5) = e_5. \]

In [11], D. Yau proved that \((A, \cdot, \alpha)\) is a Hom-flexible and Hom–Malcev admissible algebra. In addition, by direct calculations, we can easily show that it satisfies condition (1.2). Hence, by using Theorem 1.2, we conclude that \((A, \cdot, \alpha)\) is an admissible Hom–JMP algebra.

We now prove that multiplicative admissible Hom–JMP algebras are power Hom-associative. The power associativity of admissible JMP algebras was shown in [3] (Proposition 2.2). We begin by recalling the definition of a power Hom-associative algebra.
Definition 1.5 [10]. Let \((A, \cdot, \alpha)\) be a Hom-algebra, let \(x \in A\), and let \(n\) be a positive integer.

1. The \(n\)th Hom-power \(x^n \in A\) is defined by
\[
x^1 = x, \quad x^n = x^{n-1} \cdot \alpha^{n-2}(x), \quad n \geq 2.
\]

2. \(A\) is called the \(n\)th power Hom-associative if
\[
x^n = \alpha^{n-i-1}(x^i) \cdot \alpha^{i-1}(x^{n-i})
\]
for all \(x \in A\) and \(i \in \{1, \ldots, n-1\}\).

3. \(A\) is called power Hom-associative if \(A\) is the \(n\)th power Hom-associative for all \(n \geq 2\). If the twisting map \(\alpha\) is the identity map, then the \(n\)th power Hom-associativity becomes
\[
x^n = x^i \cdot x^{n-i}.
\]

The class of power Hom-associative algebras contains multiplicative right Hom-alternative algebras and noncommutative Hom-Jordan algebras. Some other results for power Hom-associative algebras can be found in [10].

A well-known result of Albert [1] says that an algebra \((A, \cdot)\) is power associative if and only if it is third and fourth power associative, i.e., condition (1.3) holds for \(n = 3, 4\). Moreover, for (1.3) to be true for \(n = 3, 4\), it is necessary and sufficient that
\[
(x \cdot x) \cdot x = x \cdot (x \cdot x) \quad \text{and} \quad ((x \cdot x) \cdot x) \cdot x = (x \cdot x) \cdot (x \cdot x) \quad \text{for all} \quad x \in A.
\]

The Hom-versions of these statements were also proved in [10]. More precisely, a multiplicative Hom-algebra \((A, \cdot, \alpha)\) is power Hom-associative if and only if it is third and fourth power Hom-associative, which, in turn, is equivalent to
\[
x^3 = x^2 \cdot \alpha(x) = \alpha(x) \cdot x^2 \quad \text{and} \quad x^4 = x^3 \cdot \alpha^2(x) = \alpha(x^2) \cdot \alpha(x^2)
\]
for all \(x \in A\).

The following result is the Hom-version of Proposition 2.2 in [3].

Theorem 1.3. Every multiplicative admissible Hom–MJP algebra is power Hom-associative.

Proof. As indicated above, by virtue of a result from [10], it is sufficient to prove the two identities in (1.4). The Hom-flexibility implies that
\[
0 = a_{x^2} (x, x) = x^2 \cdot \alpha(x) - \alpha(x) \cdot x^2,
\]
which proves the first identity in (1.4). We prove the second identity in (1.4). Since \((A, \circ, \alpha)\) is Hom–Jordan, by virtue of Corollary 1.2, we get
\[
0 = a_{x^2} (\alpha(x), \alpha(x)) = (x^2 \cdot \alpha(x)) \cdot \alpha^2(x) - \alpha(x^2) \cdot (\alpha(x) \cdot \alpha(x)).
\]
\[= x^3 \cdot \alpha^2(x) - \alpha(x^2) \cdot \alpha(x^2).\]

The second identity in (1.4) is thus proved.

Theorem 1.3 is proved.

2. Pseudo-Euclidean Hom–JMP Algebras

In this section, we extend the notion of pseudo-Euclidean JMP algebra to the Hom–JMP algebras and provide some properties. Let \((A, \cdot, \alpha)\) be a Hom-algebra and let \(B: A \times A \to \mathbb{K}\) be a symmetric nondegenerate and invariant bilinear form on \(A\). We say that \((A, \cdot, \alpha, B)\) is a pseudo-Euclidean Hom–JMP algebra if, in addition,

\[B(\alpha(x), y) = B(x, \alpha(y)) \quad \forall x, y \in A.\]

**Definition 2.1.** Let \((A, \cdot, \circ, \alpha)\) be a Hom–JMP algebra and let \(B: A \times A \to \mathbb{K}\) be a symmetric nondegenerate and invariant bilinear form on \(A\). We say that \((A, \cdot, \circ, \alpha, B)\) is a pseudo-Euclidean Hom–JMP algebra if \((A, \cdot, \circ, \alpha, B)\) and \((A, \circ, \alpha, B)\) are pseudo-Euclidean Hom-algebras.

**Definition 2.2.** A Hom–JMP algebra \((A, \cdot, \circ, \alpha)\) is called Hom–pseudo-Euclidean if there exists \((B, \gamma)\), where \(B\) is a symmetric and nondegenerate bilinear form on \(A\) and \(\gamma: A \to A\) is a homomorphism such that

\[B(\alpha(x), y) = B(x, \alpha(y)), \quad B([x, y], \gamma(z)) = B(\gamma(x), [y, z]), \quad B(x \circ y, \gamma(z)) = B(\gamma(x), y \circ z)\]

for all \(x, y, z \in A\).

**Remark 2.1.** Note that we recover pseudo-Euclidean Hom–JMP algebras for \(\gamma = id\).

Let \((A, \cdot, \circ, B)\) be a pseudo-Euclidean JMP algebra. We denote by \(\text{Aut}_S(A, B)\) the set of symmetric automorphisms of \(A\) with respect to \(B\), i.e., automorphisms \(\beta: A \to A\) such that

\[B(\beta(x), y) = B(x, \beta(y)) \quad \forall x, y \in A.\]

**Proposition 2.1.** Let \((A, \cdot, \circ, B)\) be a pseudo-Euclidean JMP algebra and let \(\alpha \in \text{Aut}_S(A, B)\). Then \(A_{\alpha} = (A, \cdot, \circ, \alpha \circ \alpha, B_{\alpha})\) is a pseudo-Euclidean Hom–JMP algebra, where, for any \(x, y \in A\),

\[
\{x, y\}_{\alpha} = \{\alpha(x), \alpha(y)\}, \quad x \circ_{\alpha} y = \alpha(x) \circ \alpha(y), \quad B_{\alpha}(x, y) = B(\alpha(x), y).
\]

**Proof.** By Corollary 1.1, \((A, \cdot, \circ, \alpha, \alpha)\) is a Hom–JMP algebra. The bilinear form \(B_{\alpha}\) is nondegenerate because \(B\) is nondegenerate and \(\alpha\) is bijective. Now let \(x, y, z \in A\). Then

\[B_{\alpha}\{x, y\}_{\alpha}, z) = B(\alpha(\{x, y\}_\alpha), z) = B(\{\alpha(x), \alpha(y)\}, \alpha(z))\]
Proposition 2.2. Let \((A, \{\cdot\}, \circ, \alpha, B)\) be a pseudo-Euclidean Hom–JMP algebra. For any \(n \geq 0\), the quadruple \(A_n = (A, \{\cdot\}_n = \alpha^n \{\cdot\}, \circ_n = \alpha^n \circ, \alpha^{n+1}_n, B_n)\), where \(B_n\) is defined for \(x, y \in A\) by \(B_n(x, y) = B(\alpha^n(x), y)\), determines a pseudo-Euclidean Hom–JMP algebra.

Proof. Straightforward.

We now provide a construction of pseudo-Euclidean Hom–JMP algebra from an arbitrary Hom–JMP algebra (not necessarily pseudo-Euclidean).

Let \((A, \{\cdot\}, \circ)\) be a JMP-algebra and let \(A^*\) be the dual vector space of the underlying vector space of \(A\). In the vector space \(P = A \oplus A^*\), we define the following bracket: \(\{\cdot\}_P\) and the operation of multiplication \(\circ_P\) by

\[
\{x + f, y + g\}_P := \{x, y\} + f \ ad_y - g \ ad_x,
\]

\[
(x + f) \circ_P (y + g) := x \circ y + f \ L_y + g \ L_x \quad \forall (x, f), (y, g) \in P,
\]

where

\[
ad_x(y) = \{x, y\} \quad \text{and} \quad L_x(y) = x \circ y.
\]

Moreover, we consider a bilinear form \(B\) defined on \(P\) by

\[
B(x + f, y + g) = f(y) + g(x) \quad \forall (x, f), (y, g) \in P.
\]

Note that \((P, \{\cdot\}_P, \circ_P, B)\) is a pseudo-Euclidean JMP algebra called the \(T^*\)-extension of \(A\) by means of \(A^*\).
Proposition 2.3. Let \((A, \{\}, \circ)\) be a JMP algebra and let \(\alpha \in \text{Aut}(A)\). Then the endomorphism \(\beta = \alpha + t\alpha\) of \(P\) is an automorphism of \(P\) if and only if

\[
\text{Im}(\alpha^2 - Id) \subseteq Z_J(A) \cap Z_M(A),
\]

where \(Z_J(A)\) is the center of \((A, \circ)\) and \(Z_M(A)\) is the center of \((A, \{\})\). Further, if

\[
\text{Im}(\alpha^2 - Id) \subseteq Z_J(A) \cap Z_M(A)
\]

then \((P, \{\}, P, \beta, \circ_{P, \beta}, B_{\beta})\) is a regular pseudo-Euclidean Hom–JMP algebra.

**Proof.** Let \(x, y \in A\) and \(f, g \in A^*\). Then

\[
\beta(\{x + f, y + g\}_P) = \beta(\{x, y\} + fad_y - gad_x) = \alpha(\{x, y\}) + fad_y\alpha - gad_x\alpha,
\]

and

\[
\{\beta(x + f), \beta(y + g)\}_P = \{\alpha(x) + f\alpha, \alpha(y) + g\alpha\}_P = \{\alpha(x), \alpha(y)\} + foad_{\alpha(y)} - goad_{\alpha(x)}.
\]

Thus,

\[
\beta(\{x + f, y + g\}_P) = \{\beta(x + f), \beta(y + g)\}_P
\]

if and only if

\[
fad_y\alpha - gad_x\alpha = foad_{\alpha(y)} - goad_{\alpha(x)} \quad \forall x, y \in A,
\]

i.e., for all \(z \in A\),

\[
f(\{y, \alpha(z)\}) - g(\{x, \alpha(z)\}) = f(\alpha(\{\alpha(y), z\}) - g(\alpha(\{\alpha(x), z\}).
\]

Hence, \(\beta\) is an automorphism of \((P, \{\})_P\) if and only if

\[
f(\{x, \alpha(y)\}) = f(\alpha(\{\alpha(x), y\}) \quad \forall f \in A^* \quad \forall x, y \in A,
\]

which is equivalent to \(x, \alpha(y)\) = \(\alpha(x, y)\) \(\forall x, y \in A\).

As a consequence, \(\beta\) is an automorphism of \((P, \{\})_P\) if and only if \(\{\alpha^2(x) - x, \alpha(y)\} = 0 \forall x, y \in A\), i.e., \(\text{Im}(\alpha^2 - id) \subseteq Z_M(A)\) because \(\alpha \in \text{Aut}(A)\).

Similarly, \(\beta\) is an automorphism of \((P, \circ_P)\) if and only if \(\text{Im}(\alpha^2 - id) \subseteq Z_J(A)\). Then \(\beta \in \text{Aut}(P)\) if and only if \(\text{Im}(\alpha^2 - id) \subseteq Z_M(A) \cap Z_J(A)\).

In what follows, we show that \(\beta\) is symmetric with respect to \(B\). Indeed, let \(x, y \in A\) and \(f, g \in A^*\). Then

\[
B(\beta(x + f), y + g) = B(\alpha(x) + f\alpha, y + g) = f(\alpha(y)) + g(\alpha(x))
\]

\[
= B(x + f, \alpha(y) + g\alpha) = B(x + f, \beta(y + g)).
\]

The last assertion follows from the previous calculations and Proposition 2.1.

Proposition 2.3 is proved.

Corollary 2.1. Let \((A, \{\}, \circ)\) be a JMP algebra and let \(\theta \in \text{Aut}(A)\) be such that \(\theta^2 = id\) (\(\theta\) is an involution). Then \((P, \{\}, P, \beta, \circ_{P, \beta}, B_{\beta})\) is a regular pseudo-Euclidean Hom–JMP algebra, where \(\beta = \theta + t\theta\).
3. Hom–Lie–Jordan–Poisson Triple System

In this section, we generalize the notion of Lie–Jordan–Poisson triple system introduced in [3] to the Hom setting. We provide the relationships for this class of algebras with admissible Hom–JMP algebras. Finally, we endow it with a symmetric nondegenerate invariant bilinear form and present some key structures.

Definition 3.1. A Hom–Lie triple system is a triple \((L, [,], \alpha = (\alpha_1, \alpha_2))\) satisfying the following conditions:

(i) \([x, y, z] = -[y, x, z]\) (left skew-symmetry),

(ii) \([x, y, z] + [y, z, x] + [z, x, y] = 0\) (ternary Jacobi identity),

(iii) \([\alpha_1(x), \alpha_2(y), [u, v, w]] = [[x, y, u], \alpha_1(v), \alpha_2(w)] + [\alpha_1(u), [x, y, v], \alpha_2(w)] + [\alpha_1(u), \alpha_2(v), [x, y, w]]\)

for all \(u, v, w, x, y, z \in L\).

A particular situation (of interest for our setting) occurs when all twisting maps \(\alpha_i\) are equal, i.e., \(\alpha_1 = \alpha_2 = \alpha\) and \(\alpha([x, y, z]) = [\alpha(x), \alpha(y), \alpha(z)]\) for all \(x, y, z \in L\). The Hom–Lie triple system \((L, [,], \alpha)\) is said to be multiplicative.

Definition 3.2. A Hom–Lie–Jordan–Poisson triple system is a quadruple \((A, {,}, \circ, \alpha)\) such that

(i) \((A, \circ, \alpha)\) is a Hom–Jordan algebra,

(ii) \((A, {,}, \alpha)\) is a multiplicative Hom–Lie triple system,

(iii) \(\{\alpha(x), \alpha(y), z \circ t\} = \{x, y, z\} \circ \alpha(t) + \alpha(z) \circ \{x, y, t\} \ \forall x, y, z, t \in A\).

In the case where \((A, \circ, \alpha)\) is a commutative Hom-associative algebra, the quadruple \((A, {,}, \circ, \alpha)\) is called a Hom–Lie–Poisson triple system.

Lemma 3.1 [2]. Let \((A, {,}, \alpha)\) be a Hom–Malcev algebra. Then \((A, {,}, \alpha^2)\) is a multiplicative Hom–Lie triple system, where

\[
\{x, y, z\} = 2\{\{x, y\}, \alpha(z)\} - \{\{y, z\}, \alpha(x)\} - \{\{z, x\}, \alpha(y)\} \ \forall x, y, z \in A.
\]

Proposition 3.1. Let \((A, {,}, \circ, \alpha)\) be a Hom–JMP algebra. Then the quadruple \((A, {,}, \circ_\alpha, \alpha^2)\) is a Hom–Lie–Jordan–Poisson triple system, where

\[
\{x, y, z\} = 2\{\{x, y\}, \alpha(z)\} - \{\{y, z\}, \alpha(x)\} - \{\{z, x\}, \alpha(y)\} \ \forall x, y, z \in A,
\]

\[
x \circ_\alpha y = \alpha(x) \circ \alpha(y) \ \forall x, y \in A.
\]

Proof. Let \(x, y, z, t \in A\). We have

\[
\{\alpha^2(x), \alpha^2(y), z \circ_\alpha t\}
\]

\[
= 2\{\{\alpha^2(x), \alpha^2(y)\}, \alpha(z \circ_\alpha t)\} - \{\{\alpha^2(y), z \circ_\alpha t\}, \alpha^3(x)\} - \{\{z \circ_\alpha t, \alpha^2(x)\}, \alpha^3(y)\}
\]
Hom–Jordan algebra.

Let \( (A, \{\cdot, \cdot\}, \alpha) \) be a multiplicative Hom–Lie triple system and let \( B : A \times A \to \mathbb{K} \) be a symmetric nondegenerate bilinear form. We say that \( B \) is invariant if

\[
B(L(x,y)(z), t) = -B(z, L(x,y)(t)) \quad \forall x, y, z, t \in A,
\]

where \( L(x,y)(z) = \{x, y, z\} \). In this case, \( (A, \{\cdot, \cdot\}, \alpha, B) \) is called a pseudo-Euclidean Hom–Lie triple system if, in addition,

\[
B(\alpha(x), y) = B(x, \alpha(y)) \quad \forall x, y \in A.
\]

**Definition 3.3.** Let \( (A, \{\cdot, \cdot\}, \circ, \alpha) \) be a Hom–Lie–Jordan–Poisson triple system and let \( B : A \times A \to \mathbb{K} \) be a bilinear form on \( A \). We say that \( (A, \{\cdot, \cdot\}, \circ, \alpha, B) \) is a pseudo-Euclidean Hom–Lie–Jordan–Poisson triple system if \( (A, \{\cdot, \cdot\}, \alpha, B) \) is a pseudo-Euclidean Hom–Lie triple system and \( (A, \circ, \alpha, B) \) is a pseudo-Euclidean Hom–Jordan algebra.

**Definition 3.4.** A Hom–Lie–Jordan–Poisson triple system \( (A, \{\cdot, \cdot\}, \circ, \alpha) \) is called Hom–pseudo-Euclidean if there exists \( (B, \gamma) \), where \( B \) is a symmetric and nondegenerate bilinear form on \( A \) and \( \gamma : A \to A \) is a homomorphism such that

\[
B(\alpha(x), y) = B(x, \alpha(y)),
\]
\[ B(L(x, y)(z), \gamma(t)) = -B(\gamma(z), L(x, y)(t)), \]
\[ B(x \circ y, \gamma(z)) = B(\gamma(x), y \circ z) \]
for all \( x, y, z, t \in A. \)

**Corollary 3.1.** Let \((A, \{\}, \circ, \alpha, B)\) be a pseudo-Euclidean Hom–JMP algebra. Then the 6-uplet \((A, \{\}, \circ, \alpha, B, \alpha)\) is a Hom–pseudo-Euclidean Hom–Lie–Jordan–Poisson triple system, where \(\{\}, \circ\alpha\) are defined by (3.1) and (3.2).

**Proof.** Let \( x, y, z, t \in A. \) We get

\[
B(\{x, y, z\}, \alpha(t)) \\
= B(2\{x, y\}, \alpha(z)) - \{\{y, z\}, \alpha(x)\} - \{\{z, x\}, \alpha(y)\}, \alpha(t)) \\
= B(\{x, y\}, \alpha(z)), \alpha(t)) - B(\{y, z\}, \alpha(x), \alpha(t)) - B(\{z, x\}, \alpha(y), \alpha(t)) \\
= -B(\alpha(z), 2\{x, y\}, \alpha(t)) \circ - B(\{y, z\}, \alpha(x), \alpha(t)) - B(\{z, x\}, \alpha(y), \alpha(t)) \\
= -B(\alpha(z), 2\{x, y\}, \alpha(t)) \circ - B(\alpha(y), \alpha(z)), \{x, t\} \circ - B(\alpha(x), \alpha(z)), \{y, t\}) \\
= -B(\alpha(z), 2\{x, y\}, \alpha(t)) \circ - B(\alpha(z), \{x, t\}, \alpha(y)) \circ - B(\alpha(z), \{t, y\}, \alpha(x)) \\
= -B(\alpha(z), 2\{x, y\}, \alpha(t)) \circ - \{\{x, t\}, \alpha(y)\} \circ - \{\{t, y\}, \alpha(x)\} \\
= -B(\alpha(z), \{x, y, t\}).
\]

Corollary 3.1 is proved.

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