An $O(n\sqrt{m})$ algorithm for the weighted stable set problem in \{claw, net\}-free graphs with $\alpha(G) \geq 4$

Paolo Nobili · Antonio Sassano

Abstract In this paper we show that a connected \{claw, net\}-free graph $G(V, E)$ with $\alpha(G) \geq 4$ is the union of a strongly bisimplicial clique $Q$ and at most two clique-strips. A clique is strongly bisimplicial if its neighborhood is partitioned into two cliques which are mutually non-adjacent and a clique-strip is a sequence of cliques $\{H_0, \ldots, H_p\}$ with the property that $H_i$ is adjacent only to $H_{i-1}$ and $H_{i+1}$. By exploiting such a structure we show how to solve the Maximum Weight Stable Set Problem in such a graph in time $O(|V|\sqrt{|E|})$.

Keywords claw-free graphs · net-free graphs · stable set · matching

1 Introduction

The Maximum Weight Stable Set Problem (MWSSP) in a graph $G(V, E)$ with node-weight function $w : V \rightarrow \mathbb{R}$ asks for a maximum weight subset of pairwise non-adjacent nodes.

For each graph $G(V, E)$ we denote by $V(F)$ the set of end-nodes of the edges in $F \subseteq E$, by $E(W)$ the set of edges with end-nodes in $W \subseteq V$ and by $N(W)$ (neighborhood of $W$) the set of nodes in $V \setminus W$ adjacent to some node in $W$. If $W = \{w\}$ we simply write $N(w)$. We denote by $N[W]$ and $N[w]$ the sets $N(W) \cup W$ and $N(w) \cup \{w\}$ and by $\delta(W)$ the set of edges having exactly one end-node in $W$; if $\delta(W) = \emptyset$ and $W$ is minimal with this property we say that $W$ is (or induces) a connected component of $G$. We denote by $G - F$ the subgraph of $G$ obtained by removing from $G$ the edges in $F \subseteq E$. A clique is a complete subgraph of $G$ induced by some set of nodes $K \subseteq V$. With a little abuse of notation we also regard the set $K$ as a clique and, for any edge $uv \in E$, both $uv$ and $\{u, v\}$ are
said to be a clique. A node \( w \) such that \( N(w) \) is a clique is said to be simplicial. By extension, a clique \( K \) such that \( N(K) \) is a clique is also said to be simplicial. A claw is a graph with four nodes \( w, x, y, z \) with \( w \) adjacent to \( x, y, z \) and \( x, y, z \) mutually non-adjacent. To highlight its structure, it is denoted as \((w : x, y, z)\). A \( P_k \) is a (chordless) path induced by \( k \) nodes and will be denoted as \((u_1, \ldots, u_k)\). A subset \( T \subseteq V \) is null (universal) to a subset \( W \subseteq V \setminus T \) if and only if \( N(T) \cap W = \emptyset \) (\( N(T) \cap W = W \)). We denote by \( U[w] \) the set of nodes universal to \( N(w) \) (note that \( w \in U[w] \)). Two nodes \( u, v \in V \) are said to be twins if \( N(u) \setminus \{v\} = N(v) \setminus \{u\} \).

We can always remove a twin from \( V \) without affecting the value of the optimal solution of MWSSP. In fact, if \( uv \in E \) we can remove the twin with minimum weight, while if \( uv \notin E \) we can replace \( w(v) \) by \( w(u) + w(v) \). The complexity of finding all the twins is \( \mathcal{O}(|V| + |E|) \) ([8], [2]) and hence we assume throughout the paper that our graphs have no twins. A net is a graph with four nodes \( w, x, y, z \) with \( x, y, z \) mutually non-adjacent nodes in the set of nodes universal to \( N(w) \) (note that \( w \in U[w] \)). Two nodes \( u, v \in V \) are said to be twins if \( N(u) \setminus \{v\} = N(v) \setminus \{u\} \). A maximal clique \( Q \) with \( N(w) \) is called diagonals.

A maximal clique \( Q \) is reducible if \( \alpha(N(Q)) \leq 2 \). If \( Q \) is a maximal clique, two non-adjacent nodes \( u, v \in N(Q) \) are said to be \( Q \)-distant if \( N(u) \cap N(v) \cap Q = \emptyset \) and \( Q \)-close otherwise \( (N(u) \cap N(v) \cap Q \neq \emptyset) \). A maximal clique \( Q \) is normal if it has three independent neighbors that are mutually \( Q \)-distant. In [6] Lovász and Plummer proved the following useful properties of a maximal clique in a claw-free graph.

**Proposition 11** Let \( G(V, E) \) be a claw-free graph. If \( Q \) is a maximal clique in \( G \) then:

(i) If \( u \) and \( v \) are \( Q \)-close then \( Q \subseteq N(u) \cup N(v) \);
(ii) If \( u, v, w \) are mutually non-adjacent nodes in \( N(Q) \) and two of them are \( Q \)-distant then any two of them are \( Q \)-distant and hence \( Q \) is normal.

\( \square \)
Theorem 11 Let \( G(V, E) \) be a claw-free graph and \( u \) a regular node in \( V \) whose closed neighborhood is covered by two maximal cliques \( Q \) and \( \overline{Q} \). If \( Q \) \((\overline{Q})\) is not reducible then it is normal.

Proof. Suppose, by contradiction, that \( Q \) is not normal and \( \alpha(N(Q)) \geq 3 \). Let \( v_1, v_2, v_3 \) be three mutually non-adjacent nodes in \( N(Q) \). If \( u \) is not adjacent to \( v_1, v_2, v_3 \), then by (i) of Proposition 11 we have that \( v_1, v_2, v_3 \) are mutually distant with respect to \( Q \) and hence \( Q \) is normal, a contradiction. Consequently, without loss of generality, we can assume that \( u \) is adjacent to \( v_1 \) and so \( v_1 \) belongs to \( \overline{Q} \). The nodes \( v_2 \) and \( v_3 \) do not belong to \( Q \cup \overline{Q} \) and hence are not adjacent to \( u \). It follows, again by (i) of Proposition 11, that \( v_2 \) and \( v_3 \) are distant with respect to \( Q \). But then, by (ii) of Proposition 11, \( Q \) is a normal clique, a contradiction. \( \square \)

2 Wings and similarity classes

Let \( S \) be a stable set of \( G(V, E) \). Any node \( s \in S \) is said to be stable; any node \( v \in V \setminus S \) satisfies \( |N(v) \cap S| \leq 2 \) and is called superfree if \( |N(v) \cap S| = 0 \), free if \( |N(v) \cap S| = 1 \) and bound if \( |N(v) \cap S| = 2 \). For each free node \( u \) we denote by \( S(u) \) the unique node in \( S \) adjacent to \( u \). Observe that, by claw-freeness, a bound node \( b \) cannot be adjacent to a node \( u \in V \setminus S \) unless \( b \) and \( u \) have a common neighbor in \( S \).

We denote by \( F(T) \) the set of free nodes with respect to \( S \); adjacent to some node in \( T \subseteq S \), to simplify our notation we will always write \( F(s) \) instead of \( F(\{s\}) \). A node \( z \in F(T) \) is said to be an outer node for \( T \) if there exists a free node \( x \in V \setminus F \) adjacent to \( z \) and to a node \( q \in S \setminus T \); the set of outer nodes for \( T \) is denoted by \( O(T) \). Every node of \( F(T) \) which is not an outer node for \( T \) is said to be an inner node for \( T \); the set of inner nodes for \( T \) is denoted by \( I(T) \).

We denote by \( I = \bigcup_{s \in S} I(s) \) and \( O = \bigcup_{s \in S} O(s) \) the set of all nodes that are inner or outer nodes for some node \( s \in S \). In the rest of the paper a node \( z \in I \) or \( z \in O \) will be simply called inner or outer. Evidently, \( F(T) = I(T) \cup O(T) \) and \( F = I \cup O \). Moreover, we write \( O(s), I(s), O(s, t) \) and \( I(s, t) \) instead of \( O(\{s\}), I(\{s\}), O(\{s, t\}), I(\{s, t\}) \).

A bound-wing defined by \( \{s, t\} \subseteq S \) is the set \( W^B(s, t) = \{u \in V \setminus S : N(u) \cap S = \{s, t\}\} \). A free-wing defined by the ordered pair \( (s, t) \) \((s, t) \in S \) is the set \( W^F(s, t) = \{u \in F(s) : N(u) \cap F(t) \neq \emptyset\} \). Observe that, by claw-freeness, any bound node is contained in a single bound-wing. On the contrary, a free node can belong to several free-wings. Moreover, while \( W^B(s, t) \equiv W^B(t, s) \), we have \( W^F(s, t) \neq W^F(t, s) \).

By slightly generalizing the definition due to Minty \([9]\), we call wing defined by \( (s, t) \) \((s, t) \in S \) the set \( W(s, t) = W^B(s, t) \cup W^F(s, t) \). Observe that \( W(s, t) = W(t, s) \). The nodes \( s \) and \( t \) are said to be the extrema of the wing \( W(s, t) \). We say that \( W(s, t) \) is a clique-wing if both \( W(s, t) \cap N(s) \) and \( W(s, t) \cap N(t) \) are cliques.

A node \( s \in S \) is said to be \( k \)-winged if \( s \) defines wings with \( k \) distinct nodes \( s_1, s_2, \ldots, s_k \in S \); if such wings are all clique-wings the node \( s \) is said to be \( k \)-clique-winged. The number of wings defined by \( s \) is also said to be the wing number of \( s \) and denoted by \( k_s \). For each regular node \( s \in S \), the set \( N[s] \) can be covered by two maximal cliques, say \( C_s \) and \( \overline{C}_s \). For each wing \( W(s, t) \) defined
by a regular node $s \in S$, let $C_s(t) = C_s \cap W(s,t)$ and $\overline{C}_s(t) = \overline{C}_s \cap W(s,t)$. We call $C_s(t)$ and $\overline{C}_s(t)$ the partial wings defined by $t$ in $C_s$ and $\overline{C}_s$. The number of partial wings defined by a regular node $s \in S$ is said to be the partial wing number of $s$. For each node $s \in S$ we regard the sets $C_s(s) = \{s\} \cup (I(s) \cap C_s)$ and $\overline{C}_s(s) = \{s\} \cup (I(s) \cap \overline{C}_s)$ as degenerate partial wings. Observe that both $C_s$ and $\overline{C}_s$ are partitioned into partial wings. Moreover, each bound node $u$ is contained in at most four, possibly coincident, partial wings (namely $C_s(t), C_t(s), \overline{C}_s(t), \overline{C}_t(s)$ where $s$ and $t$ are stable nodes adjacent to $u$ and each free node $u$ is contained in at most two partial wings (possibly coincident and/or degenerate). This implies that the total number of partial wings is $O(|V|)$.

We say that two nodes $u$ and $v$ in $V \setminus S$ are similar ($u \sim v$) if $N(u) \cap S = N(v) \cap S$ and dissimilar ($u \not\sim v$) otherwise. Clearly, similarity induces an equivalence relation on $V \setminus S$ and a partition in similarity classes. Similarity classes can be bound or free in that they are entirely composed by nodes that are bound or free with respect to $S$. Bound similarity classes are precisely the bound-wings defined by pairs of nodes of $S$, while each free similarity class contains the (free) nodes adjacent to the same node of $S$. Let $V_F$ be the set of nodes that are free with respect to $S$ and let $G_F(V_F, E_F)$ be the graph with edge-set $E_F = \{uv \in E : u, v \in V_F, u \not\sim v\}$ (free dissimilarity graph).

**Definition 21** Let $G(V,E)$ be a claw-free graph, $S$ a maximal stable set in $G$ and $G_F$ the free dissimilarity graph of $G$ with respect to $S$. A connected component of $G_F$ inducing a maximal clique in $G$ is said to be a free component of $G$ with respect to $S$.

**Theorem 21** Let $G(V,E)$ be a claw-free graph and $S$ a maximal stable set of $G$. Then a connected component of $G_F$ intersecting three or more free similarity classes induces a maximal clique in $G$ and hence is a free component.

**Proof.** We first claim that the nodes of any chordless path $P$ in $G_F$, the free dissimilarity graph of $G$ with respect to $S$, connecting two dissimilar nodes $u, v$ belong only to the similarity classes of $u$ and $v$. In fact, two consecutive nodes of $P$ necessarily belong to different classes. If a node in a third class existed in $P$ we would necessarily have three consecutive nodes $x, y, z$ of $P$ in three different classes. But then $(y : S(y), x, z)$ would be a claw in $G$, a contradiction. Suppose now that a connected component $X$ of $G_F$, intersecting three or more similarity classes, is not a clique in $G$ and let $u, z \in X$ be two nonadjacent nodes in $G$. Suppose first that $S(u)$ and $S(z)$ are two distinct nodes of $S$ and, consequently, that $u \not\sim z$. Let $v \in X$ be a node with $S(v) \notin \{S(u), S(z)\}$. It exists since we assumed that $X$ intersects more than two similarity classes. Let $P_{ux}$ and $P_{uz}$ be chordless paths connecting $u$ to $v$ and, respectively, $v$ to $z$ in $G_F$. By the above claim, $P_{ux}$ contains only nodes in the similarity classes of $u$ and $v$, while $P_{uz}$ contains only nodes in the similarity classes of $v$ and $z$. Let $W_{ux}$ be the path connecting $u$ to $z$ obtained by chaining $P_{ux}$ and $P_{uz}$ and let $P_{uz}$ be any chordless path connecting $u$ to $z$ whose nodes belong to $W_{ux}$. Since $uz \not\in E$, $P_{ux}$ contains at least one node in the similarity class of $v$ and hence contains nodes in three different similarity classes, contradicting the hypothesis that $P_{ux}$ is chordless. It follows that $u$ and $z$ belong to the same similarity class. More generally, any two dissimilar nodes in $X$ are adjacent. Let $v \in X$ be a node with $S(v) \not\equiv S(u) \equiv S(z)$. It follows that $uv, vz \in E$ and hence $(v : S(v), u, z)$ is a claw, a contradiction. It follows that $X$ is a clique in $G$. To
prove that it is also maximal, assume by contradiction that there exists some node $u \in N(X)$ universal to $X$. The node $u$ is not free for, otherwise, it would belong to $X$ and is not stable since $X$ intersects more than one similarity class. It follows that $u$ is bound and adjacent to two nodes $s, t \in S$. Moreover, there exists some node $z \in N(X) \cap S$ with $z \neq s, t$. But then, for each node $x \in X \cap N(z)$, $(u : s, t, x)$ is a claw in $G$, a contradiction. □

3 Canonical Stable Sets

Here we introduce a special class of maximal stable sets in $G$ which will be instrumental in our paper. In what follows a free node $x$ with $N(x) \supset N(S(x)) \setminus \{x\}$ will be called a dominating free node, while the node $S(x)$ will be said to be dominated by $x$.

Definition 31 Let $G(V,E)$ be a connected claw-free graph. A maximal stable set $S$ of $G$ is said to be canonical if and only if $G$ does not contain:

(i) an augmenting $P_3$ with respect to $S$;
(ii) a dominating free node.

A stable set satisfying conditions (i) and (ii) of Definition 31 can be easily obtained from a maximal stable set $S_0$ of a connected claw-free graph $G$ by repeatedly applying the following two operations:

(a) augmentation along a $P_3(x, s, y)$, where $x$ and $y$ are free and $s$ is stable with respect to $S$;
(b) alternation along a $P_2(x, S(x))$ (an edge), where $x$ is a dominating free node of $S(x)$ with $|N[x]| \geq |N[x']|$ for each free node $x'$ dominating $S(x)$.

Theorem 31 Let $G(V,E)$ be a claw-free graph and $S_0$ a maximal stable set of $G$. Then a canonical stable set can be obtained from $S_0$ by repeatedly applying operations (a) and (b) in time $O(|E|)$.

Proof. Let $S$ be any stable set of $G$. We first prove the following claims.

Claim (i). Let $T$ be a stable set obtained from $S$ by applying operation (a) along a $P_3(\bar{x}, s, \bar{y})$; then the set of free nodes with respect to $T$ is a proper subset of the set of free nodes with respect to $S$ and every $P_3$ augmenting with respect to $T$ is also augmenting with respect to $S$, in particular it does not contain $\bar{x}$ or $\bar{y}$.

Proof. Suppose first that there exists some node $x$ which is free with respect to $T$ but is not free with respect to $S$. Since $s$ is bound with respect to $T$ and $S$ is maximal, we have that $x \neq s$ is bound with respect to $S$. The node $x$ is adjacent to $s$ (otherwise it would also be bound also with respect to $T$) and to some other stable node $\tilde{s} \in S$. Moreover, since $x$ is free with respect to $T$ and is adjacent to $\tilde{s} \in T$, it is non-adjacent to $\bar{x}$ and to $\bar{y}$. But then $(s : \bar{x}, \bar{y}, x)$ is a claw in $G$, a contradiction.

Suppose now that there exists an augmenting $P_3(x, t, y)$ with respect to $T$ that is not augmenting with respect to $S$. Since $x$ and $y$ are free nodes also with respect to $S$, we have that $t$ is in $T \setminus S$, $x$ and $y$ are non-adjacent to any node in $S \cap T$ and hence are both adjacent to $s$. Hence, without loss of generality, we can assume
free nodes, since every node adjacent to \( S \) is also a free node. Observe that the alternation along the path \((x, S(x))\) does not create new free nodes, since every node adjacent to \( S(x) \) is also adjacent to \( x \). Assume that an augmenting \( P_3 \) \((y, t, z)\) exists with respect to \( T \), while no augmenting \( P_3 \) exists with respect to \( S \). We have that \( y \) and \( z \) are also free with respect to \( S \). If \( t \) is in \( S \), then \((y, t, z)\) is also augmenting with respect to \( S \). Otherwise, \( t \) is also free with respect to \( S \). In both cases we contradict the assumption that no augmenting \( P_3 \) exists with respect to \( S \). Let now \( t \) be a dominating free node with respect to \( T \). Since \( t \) is also a free node with respect to \( S \), if \( T(t) \neq x \) then \( t \) is dominating free also with respect to \( S \). Assume, conversely, that \( T(t) = x \). Since \( t \) dominates \( x \) with respect to \( T \), it satisfies \( N(t) \supseteq N(x) \setminus \{i\} \). In particular, \( t \) is adjacent to \( S(x) \) and all of its neighbors and so it dominates \( S(x) \) with respect to \( S \). But this violates the assumption that \( x \) has maximum degree among the free nodes dominating \( S(x) \).

The claim follows.

End of Claim (ii).

Proof. Observe that the alternation along the path \((x, S(x))\) does not create new free nodes, since every node adjacent to \( S(x) \) is also adjacent to \( x \). Assume that an augmenting \( P_3 \) \((y, t, z)\) exists with respect to \( T \), while no augmenting \( P_3 \) exists with respect to \( S \). We have that \( y \) and \( z \) are also free with respect to \( S \). If \( t \) is in \( S \), then \((y, t, z)\) is also augmenting with respect to \( S \). Otherwise, \( t \) is also free with respect to \( S \). In both cases we contradict the assumption that no augmenting \( P_3 \) exists with respect to \( S \). Let now \( t \) be a dominating free node with respect to \( T \). Since \( t \) is also a free node with respect to \( S \), if \( T(t) \neq x \) then \( t \) is dominating free also with respect to \( S \). Assume, conversely, that \( T(t) = x \). Since \( t \) dominates \( x \) with respect to \( T \), it satisfies \( N(t) \supseteq N(x) \setminus \{i\} \). In particular, \( t \) is adjacent to \( S(x) \) and all of its neighbors and so it dominates \( S(x) \) with respect to \( S \). But this violates the assumption that \( x \) has maximum degree among the free nodes dominating \( S(x) \).

The claim follows.

End of Claim (ii).

Let \( S_0 = \{s_1, s_2, \ldots, s_q\} \) be a maximal stable set of \( G(V, E) \) and let \( F_0 \) be the set of free nodes with respect to \( S_0 \). We now prove that a stable set \( Z_0 \) such that no augmenting \( P_3 \) exists in \( G \) with respect to it can be obtained from \( S_0 \) by repeatedly applying operation \((a)\) to a current stable set \( S \) (initialized as \( S := S_0 \)) in overall time \( O(|E|) \). Let \( F \) be the set of free nodes with respect to \( S \). At any stage of the procedure we examine a node \( s_i \in S_0 \). Let \( G_i(V_i, E_i) \) be the subgraph of \( G \) induced by \( N[s_i] \). Observe that, by \([5]\), \( |V_i| = O(\sqrt{|E_i|}) \). We scan the set \( V_i \cap F \) looking for a pair of non-adjacent nodes. This can be done in time \( O(|E_i|) \).

If we find an augmenting \( P_3 \) \((x_i, s_i, y_i)\), we update the stable set \( S \) by performing operation \((a)\) \((S := S \setminus \{s_i\} \cup \{x_i, y_i\})\). Moreover, the set \( F \) is updated by removing \( x_i, y_i \), any node adjacent to \( x_i \) or \( y_i \) and not to \( s_i \) (every such node is necessarily free with respect to the former stable set and becomes bound after operation \((a)\) is performed) and any node adjacent to both \( x_i \) and \( y_i \) (every such node must be free with respect to the former stable set, is adjacent to \( s_i \) and becomes bound after operation \((a)\) is performed). It follows that \( F \) can be updated in time \( O(|\delta(x_i) \cup \delta(y_i)|) \). Observe that, for each node \( u \in F \setminus V_i \), \( S(u) \) is not changed by operation \((a)\). In addition, Claim (i) ensures that no new augmenting \( P_3 \) is produced by the operation. In particular, no \( P_3 \) augmenting with respect to \( S \) exists with \( x_i (y_i) \) as stable node. This implies that we have only to check the nodes in \( S_0 \) as stable nodes in augmenting \( P_3 \) and we can avoid to update \( S(u) \) for any free node \( u \in N(s_i) \). It follows that the overall complexity of the procedure is \( O(\sum_{i=1}^q (|E_i| + |\delta(x_i) \cup \delta(y_i)|)) = O(|E|) \).

Let \( \{s_1, s_2, \ldots, s_q\} \) be the nodes in \( Z_0 \) and let \( F_0 \) be the set of free nodes with respect to \( Z_0 \). We now prove that a canonical stable set can be obtained from
Definition 32 A maximal clique \( Q \) in a graph \( G(V, E) \) is said to be bisimplicial if \( N(Q) \) is partitioned into two cliques \( K_1, K_2 \). The clique \( Q \) is said to be strongly bisimplicial if \( K_1 \) is null to \( K_2 \) and dominating if there exists an edge \( uv \in E \) with the property that \( u \in K_1, v \in K_2 \) and \( N(\{u, v\}) \supseteq V \setminus Q \). \qed

Observe that, in particular, a maximal simplicial clique is strongly bisimplicial.

4 The structure of \{claw, net\}-free graphs

Theorem 41 Let \( G(V, E) \) be a \{claw, net\}-free graph, \( S \) a canonical stable set in \( G \). Then each node \( s \in S \) has the property that \( k_s \leq 2 \).

Proof. Assume, by contradiction, that there exists a node \( s \in S \) with \( k_s \geq 3 \) and let \( C_s, \overline{C}_s \) be any pair of maximal cliques covering \( N[s] \). Let \( W(s, t_i) \) (\( i = 1, 2, 3 \)) be three wings intersecting \( N(s) \). Without loss of generality, we can assume that two of them, say \( W(s, t_1) \) and \( W(s, t_2) \), intersect \( C_s \).

Claim (i). For each pair \( W(s, t_i), W(s, t_j) \) (\( i \neq j \)) intersecting \( C_s \) and nodes \( x, y \in N(C_s) \setminus \overline{C}_s \) with \( x \in \{t_1\} \cup W(s, t_1) \) and \( y \in \{t_2\} \cup W(s, t_2) \) we have that \( x \) and \( y \) are \( C_s \)-distant.

Proof. Since \( x, y \in N(C_s) \setminus \overline{C}_s \) we have that \( x \) and \( y \) are not bound. If either \( x = t_i \) or \( y = t_j \) then \( x \) and \( y \) are not adjacent and have no common neighbor in \( C_s \), so the claim easily follows. If, conversely, both \( x \) and \( y \) are free, let \( \bar{x} \) and \( \bar{y} \) (possibly coincident), be free nodes in \( C_s \) adjacent to \( x \) and \( y \), respectively. If \( x \) and \( y \) are adjacent we have that \( x, y, \bar{x}, \bar{y} \) belong to the similarity classes of nodes \( s, t_i \) and \( t_j \). But then, by Theorem 21, they belong to a free component \( Q \) containing the net \( (\bar{x}, x, y, s, t_i, t_j) \), a contradiction. Hence \( x \) and \( y \) are not adjacent and do not have a common neighbor \( z \in C_s \) (otherwise \( z : x, y, s \) is a claw). The claim follows.

End of Claim (i).

Let \( x, y \in N(C_s) \setminus \overline{C}_s \) with \( x \in \{t_1\} \cup W(s, t_1) \) and \( y \in \{t_2\} \cup W(s, t_2) \). By Claim (i), \( x \) and \( y \) are \( C_s \)-distant. If there exists a node \( z \in \overline{C}_s \setminus C_s \) which is
non-adjacent to both \( x \) and \( y \) then, by (ii) of Proposition 11, \( C_s \) is normal, a contradiction. If, conversely, every node in \( C_a \setminus C_s \) is either adjacent to \( x \) or to \( y \) then \( C_a \setminus C_s \subseteq W(s,t_1) \cup W(s,t_2) \). In fact, if a node \( v \in C_a \setminus C_s \) is adjacent to \( x \) \((y)\) then it belongs to \( W(s,t_1) \) \((W(s,t_2))\). But then \( W(s,t_3) \) intersects \( C_s \) and there exists a node \( z \in N(C_s) \setminus C_a \) with \( z \in \{t_3\} \cup W(s,t_3) \). By Claim (i), \( x \), \( y \) and \( z \) are mutually \( C_s \)-distant and hence \( C_s \) is normal, again a contradiction. The theorem follows. \( \Box \)

**Lemma 41** Let \( G(V,E) \) be a connected \( \{\text{claw, net}\} \)-free graph with \( \alpha(G) \geq 4 \). Then \( G \) is quasi-line.

**Proof.** Assume, by contradiction, that \( G \) contains an irregular node \( a \) and let \( H = \{v_1, \ldots, v_{2k+1}\} \) be an odd anti-hole in \( N(a) \). Observe that \( \{v_i, v_{i+1}, \ldots, v_{i+k-1}\} \) is a clique for each \( i \in \{1, \ldots, 2k+1\} \) (sums taken modulo \( 2k+1 \)). We claim that \( N^2(H) = N(N(H)) \setminus H \) is empty. Otherwise, let \( x \) be a node in \( N^2(H) \) and let \( y \in N(H) \) be a node adjacent to \( x \). By claw-freeness, \( N(y) \cap H \) is a clique \( Q \). Without loss of generality, we can assume \( Q \subseteq \{v_1, \ldots, v_k\} \). Let \( v_i, v_j \) be the nodes in \( Q \) with smallest and largest index, respectively. If \( i = j \) \((|Q| = 1)\) then \( (y : v_{i-k+1}, v_{i+k-1}) \) (sums taken modulo \( 2k+1 \)) is a claw in \( G \), a contradiction. Hence we have \( i + 1 \leq j \leq i + k - 1 \). But then \( (y, v_i, v_j : x, v_{j-k}, v_{i+k}) \) is a net in \( G \), again a contradiction. It follows that \( V = H \cup N(H) \), contradicting the fact (proved in [6] by Lovász and Plummer) that \( \alpha(\{a\} \cup N(a) \cup N^2(a)) \leq 3 \). \( \Box \)

**Lemma 42** Let \( G(V,E) \) be a connected \( \{\text{claw, net}\} \)-free graph, \( Q \) a dominating bisimplicial clique in \( G \) whose neighborhood is partitioned into the cliques \( K_1 \) and \( K_2 \). Then the set \( P = V \setminus N[Q] \) is a clique.

**Proof.** Let \( uv \) be an edge in \( E \) with \( u \in K_1 \) and \( v \in K_2 \). Assume by contradiction that there exist two non-adjacent nodes \( x, y \in P \). Since \( Q \) is dominating, the set \( P \) is contained in \( N(\{u,v\}) \). If both \( x \) and \( y \) were adjacent to \( u \) we would have the claw \((u : x, y, z)\) where \( z \in Q \) is a node adjacent to \( u \). It follows that, without loss of generality, we can assume that \( x \) is adjacent to \( u \) and non-adjacent to \( v \) while, symmetrically, \( y \) is adjacent to \( v \) and non-adjacent to \( u \). Let \( z \) be any node in \( Q \) adjacent to \( u \) and assume that \( z \) is non-adjacent to \( v \). But then \((u : z, x, v)\) is a claw in \( G \), a contradiction. It follows that each node in \( Q \cap N(\{u,v\}) \) is adjacent to both \( u \) and \( v \). But now, if \( N(\{u,v\}) \supseteq Q \) then \( Q \cup \{u,v\} \) is a clique, contradicting the maximality of \( Q \). On the other hand, if \( N(\{u,v\}) \supseteq Q \), let \( t \) be a node in \( Q \setminus N(\{u,v\}) \) and \( z \) a node in \( N(\{u,v\}) \cap Q \). Then \((z, u, v : t, x, y)\) is a net in \( G \), again a contradiction. Hence, we have that \( P \) is a clique. \( \Box \)

**Definition 41** A connected graph \( G(V,E) \) is a clique-strip if there exists a partition \( \{H_0, \ldots, H_p\} \) of \( V \) in cliques (not necessarily maximal) with the property that \( H_i \) is adjacent to \( H_j \) only if \(|i − j| \leq 1 \) \(i, j \in \{0, 1, \ldots, p\} \). The clique-strip is said to be defined by \( \{H_0, \ldots, H_p\} \).

**Lemma 43** Let \( G(V,E) \) be a connected \( \{\text{claw, net}\} \)-free graph and assume that \( G \) contains a bisimplicial clique \( Q \) whose neighborhood is partitioned into the cliques \( X \) and \( Y \) and which is either dominating or strongly bisimplicial. Then \( G − X \) \((G − Y)\) is the union of at most two clique-strips defined by clique families which can be constructed in \( O(|E|) \) time.
Proof. Consider first the case in which $X$ is not null to $Y$ and let $x \in X$ and $y \in Y$ be adjacent nodes. Since $Q$ is dominating, $N[\{x, y\}] = V \setminus Q$ and, by Lemma 42, $P = V \setminus N[Q]$ is a clique. Since $P$ is null to $Q$ we have that $G - X$ is a clique-strip defined by $\{Q, Y, P\}$ ($G - Y$ is a clique-strip defined by $\{Q, X, P\}$) and the lemma follows.

Assume now that $X$ is null to $Y$ ($Q$ is strongly bisimplicial) and let $G_X$ ($G_Y$) be the connected component containing $X$ ($Y$) of the subgraph obtained by removing the nodes in $Q$ from $G$. Let $X_t$ be the set of nodes at distance $i$ from $X$ in $G_X$ (with $X_0 \equiv X$) and let $p$ be the largest integer such that $X_p \neq \emptyset$. Observe that, since $X$ is null to $Y$ and $X_1 \subset N(X_0)$, $X_1$ does not intersect $Y$ and hence is null to $Q$.

We first claim that, for each $i \in \{0, \ldots, p\}$, $X_i$ is a clique. The claim is true for $i = 0$. Assume that there exists some integer $t \geq 0$ such that the claim is true for $i \leq t$ but false for $i = t + 1$ and let $x_1$, $x_2$ be two non-adjacent nodes in $X_{t+1}$.

We first observe that $x_1$ and $x_2$ have no common neighbor in $X_t$. Suppose the contrary, let $z$ be such a node and $\varpi$ a node adjacent to $z$ in $X_{t-1}$ if $t \geq 1$ or in $Q$ if $t = 0$. Observe that in both cases $\varpi$ is not adjacent to $x_1$ or $x_2$. In fact, if $t \geq 1$, $\varpi$ belongs to $X_{t-1}$ which is null to $X_{t+1}$; if $t = 0$, $\varpi$ belongs to $Q$ which is null to $X_{t+1} \equiv X_1$. But then $(z : x_1, x_2, \varpi)$ is a claw in $G$, a contradiction. It follows that $x_1$ and $x_2$ (adjacent to $X_t$) have no common neighbor in $X_t$ and, hence, are not universal to $X_t$. We have two cases: either $x_1$ and $x_2$ have no common neighbor universal to $X_t$ or they do. In the first case, let $\hat{X}$ be any maximal clique in $G$ containing $X_t$ and observe that $x_1$ and $x_2$ belong to $N(\hat{X})$ and have no common neighbor in $\hat{X}$. In the second case, by claw-freeness, we have $X_t \subseteq N(x_1) \cup N(x_2)$.

Let $q$ be a node in $X_{t-1}$ (or in $Q$ if $t = 0$) adjacent to some node $\tilde{x} \in X_t$ and observe that $q$ is non-adjacent to both $x_1$ and $x_2$. Without loss of generality we can assume that $\tilde{x}$ is adjacent to $x_1$. Let $\tilde{y}$ be any node adjacent to $x_2$ in $X_t$ and observe that $\tilde{x}$ is non-adjacent to $x_2$ and $\tilde{y}$ is non-adjacent to $x_1$. If $q\tilde{y} \notin E$ we have that $(\tilde{x} : q, x_1, \tilde{y})$ is a claw in $G$, a contradiction. It follows that $q$ is adjacent to all the nodes in $N(x_2) \cap X_t$. A symmetric argument shows that $q$ is also universal $N(x_1) \cap X_t$ and hence it is universal to $X_t$. Moreover, any node $\hat{q}$ adjacent to both $x_1$ and $x_2$ is non-adjacent to $q$, otherwise $(\hat{q} : q, x_1, x_2)$ would be a claw in $G$. Let $\hat{X}$ be a maximal clique containing $X_t \cup \{\hat{q}\}$. Since $x_1$ and $x_2$ are not adjacent to $q$, they belong to $N(\hat{X})$ and have no common neighbor in $\hat{X}$.

In both cases, $x_1$ and $x_2$ belong to $N(\hat{X})$ and have no common neighbor in $\hat{X}$. Now, we claim that there exists a node $y$ in $N(\hat{X})$ which is non-adjacent to both $x_1$ and $x_2$. In fact, if $t \geq 1$ and $X_{t-1} \not\subseteq \hat{X}$ such a node clearly exists in $X_{t-1}$. If $X_{t-1} \subseteq \hat{X}$ and $t \geq 2$ such a node clearly exists in $X_{t-2}$. If $t = 0$, since $Q \cup X_0$ is not a clique and $Q$ is null to $X_1$, there exists at least one node $y \in Q$ which is non-adjacent to both $x_1$ and $x_2$ and not universal to $X_0$ and hence belongs to $N(X)$. Hence, we are left with the case $t = 1$ and $X_0 \equiv X \subseteq X$. Suppose that every node in $\hat{Q} = Q \cap N(\hat{X})$ is adjacent to either $x_1$ or $x_2$. If both $x_1$ and $x_2$ belong to $N(\hat{Q})$ then $x_1$ and $x_2$ both belong to $Y$, contradicting the assumption that $Y$ is a clique. It follows that either $x_1$ or $x_2$ is universal to $\hat{Q}$. Without loss of generality, assume that $x_1$ is universal to $\hat{Q}$. Since $Q$ is a maximal clique, we have that there exists a node $u \in Q \setminus \hat{Q}$. Let $v$ be any node in $Q$ and $h \in \hat{X}$ be a node adjacent to $v$. But then $(v : u, h, x_1)$ is a claw, a contradiction. It follows that also in this case there exists a node $y$ in $N(\hat{X})$ which is non-adjacent to both
\(x_1\) and \(x_2\). Hence \(x_1, x_2, y\) are three mutually non-adjacent nodes in \(N(\hat{X})\) with \(x_1, x_2\) distant. But then, by \((ii)\) of Proposition 11 \(\hat{X}\) is normal, contradicting the hypothesis that \(G\) (and hence \(G_X\)) is net-free.

Hence, we have that for each \(i \in \{0, \ldots, p\}\), \(X_i\) is a clique, each node in \(X_i\) is adjacent to some node in \(X_{i-1}\) for \(i \geq 1\) and each node in \(X_0\) is adjacent to some node in \(Q\). It is easy to check that \(G_X\) is a clique strip defined by a family \(\mathcal{X} = \{X_0 \equiv X, X_1, \ldots, X_p\}\). Analogously, also \(G_Y\) is a clique-strip, possibly coincident with \(\mathcal{Y} = \{Y_0 \equiv Y, Y_1, \ldots, Y_t\}\). Moreover, \(\mathcal{X}\) and \(\mathcal{Y}\) can be constructed in \(O(|E|)\) time. If \(G_X \neq G_Y\), then \(\mathcal{X} \setminus \{X\}\) and \(\mathcal{Y} \cup \{Q\}\) are clique families defining two clique-strips partitioning \(G - X\). Analogously, \(\mathcal{Y} \setminus \{Y\}\) and \(\mathcal{X} \cup \{Q\}\) are clique families defining two clique-strips partitioning \(G - Y\). If, conversely, \(G_X \equiv G_Y\), then \(G - X\) is a clique-strip defined by the family \(\mathcal{X} \cup \{Q\}\). In both cases the lemma follows.

**Theorem 42** Let \(G(V, E)\) be a connected \{(claw, net)\}-free graph and let \(S = \{s_1, s_2, \ldots, s_t\}\) be a canonical stable set of \(G\) \((t \geq 4)\). Then \(G\) contains a clique \(X\) such that \(G - X\) is the union of at most two clique-strips. The clique \(X\) and the clique families defining the clique-strips can be found in \(O(|E|)\) time.

**Proof.** Since \(\alpha(G) \geq 4\), by Lemma 41 \(G\) is quasi-line and hence each node \(s_i \in S\) is regular. It follows that \(N[s_i]\) is covered by two cliques, say \(C_{s_i}\) and \(\overline{C}_{s_i}\). In addition, by Theorem 41, we can assume without loss of generality that each node \(s_i \in S\) defines wings only with \(s_{i-1}\) and \(s_{i+1}\) (sums taken modulo \(t\)) with the wing \(W(s_i, s_1)\) possibly empty. We first prove a claim on the adjacency structure of \(G\).

**Claim (i).** A node in \(N[s_i]\) is only adjacent to nodes in \(N_i = N[s_{i-1}] \cup N[s_i] \cup N[s_{i+1}]\) \((i \in \{2, \ldots, t - 1\})\).

**Proof.** Let \(u\) be any node in \(N[s_i]\). Since \(s_i\) defines wings only with \(s_{i-1}\) and \(s_{i+1}\), we have that \(N(u) \cap S \subseteq \{s_{i-1}, s_i, s_{i+1}\}\). Assume now, by contradiction, that there exists a node \(z \notin N_i\) adjacent to \(u\) (note that this implies that \(u\) is not \(s_i\) and \(z\) is not a stable node). If \(z\) is a bound node adjacent to \(s_j, s_k \in S \setminus \{s_{i-1}, s_i, s_{i+1}\}\) we have that \((z : s_j, s_k, u)\) is a claw in \(G\), a contradiction. If \(u\) is bound suppose, without loss of generality, that it is adjacent to \(s_{i-1}\). But then \((u : s_{i-1}, s_i, z)\) is a claw in \(G\), again a contradiction. It follows that both \(u\) and \(z\) are free. But then the wing \(W(s_i, S(z))\) is non-empty, contradicting the assumption that \(s_i\) is 2-winged.

**End of Claim (i).**

**Claim (ii).** Let \(Q\) be a maximal clique in \(N[s_2]\) containing \(s_2\) and \(W(s_2, s_3) \cap N[s_2]\). Then \(Q\) is a bisimplicial clique which is either dominating or strongly bisimplicial.

**Proof.** By Claim (i), \(N(Q)\) is contained in \(N[s_1] \cup N[s_2] \cup N[s_3]\). Let \(X = N(Q) \cap (N[s_1] \cup N[s_2])\) and \(Y = N(Q) \cap N[s_3]\). Since \(N[s_1] \cap N[s_2] \subseteq W(s_1, s_2)\) is empty and \(N[s_2] \cap N[s_3]\) is contained in \(Q\) (by assumption) we have that \((X, Y)\) is a partition of \(N(Q)\). Observe that both \(X\) and \(Y\) are non-empty and that any node in \(X\) belongs to \(N[s_1]\) or to \(N[s_2]\) but not to \(N[s_3]\); on the other hand, any node in \(Y\) belongs to \(N[s_3]\) but not to \(N[s_1]\) or \(N[s_2]\). Assume that \(X\) is not a clique and let \(x_1, x_2\) be two non-adjacent nodes in \(X\). Since both \(x_1\) and \(x_2\) do not belong to \(N[s_2]\), we have that there exists a node \(y \in Y\) such that \(x_1, x_2\) and \(y\) are three mutually non-adjacent nodes in \(N(Q)\). If \(x_1, y\) or \(x_2, y\) are \(Q\)-distant then, by \((ii)\)
of Proposition 11, \( Q \) is a normal clique, a contradiction. It follows that there exist nodes \( t_1, t_2 \in Q \) with \( t_1 \in N(x_1) \cap N(y) \). By claw-freeness, \( t_1 \) is not adjacent to \( x_2 \) and \( t_2 \) is not adjacent to \( x_1 \).

Suppose first that \( s_4 \) does not belong to \( Y \). It follows that \( Q \cap N(Y) \) contains only free nodes in \( F(s_2) \) and hence \( y \) is a free node in \( F(s_3) \). But then, \((y, t_1, t_2 : s_3, x_1, x_2)\) is a net in \( G \), a contradiction. It follows that \( s_3 \) belongs to \( Y \) and we can choose \( y \equiv s_3 \). Let \( q \) be a node in \( W(s_3, s_4) \cap N(s_3) \). The node \( q \) is non-adjacent to both \( x_1 \) and \( x_2 \). If \( q \) is also non-adjacent to both \( t_1 \) and \( t_2 \) then we have the net \((t_1, t_2 : s_3, x_1, x_2, q)\), a contradiction. Assume, without loss of generality, that \( q \) is adjacent to \( t_1 \). It follows that \( q \) is also adjacent to \( t_2 \) (otherwise \((t_1 : x_1, t_2, q)\) would be a claw in \( G \)). Since \( q \) belongs to \( W(s_3, s_4) \), there exists a node \( \bar{q} \) adjacent to \( q \) which is either \( s_4 \) (if \( q \) is bound) or a free node in \( F(s_4) \) (if \( q \) is free). In both cases, \( \bar{q} \) is non-adjacent to \( x_1, x_2 \in N[s_1] \cup N(s_2) \) and to the bound nodes \( t_1, t_2 \in W(s_2, s_3) \). But then \((q, t_1, t_2 : \bar{q}, x_1, x_2)\) is a net in \( G \), a contradiction. Hence \( X \) is a clique and, since it contains \( N(s_2) \setminus Q \), we also have that \( Q \) is one of two maximal cliques covering \( N[s_2] \). If \( Y \) contains two non-adjacent nodes \( y_1 \) and \( y_2 \) we have that \( y_1, y_2 \) and \( s_1 \) are three mutually non-adjacent nodes in \( N(Q) \), and hence, by Theorem 11, \( Q \) is a normal clique, contradicting the hypothesis that \( G \) is net-free. It follows that \( X \) and \( Y \) are both cliques. If \( X \) is null to \( Y \) then the maximal clique \( Q \) is strongly bisimplicial as claimed.

Assume conversely that \( X \) is not null to \( Y \) and hence there exist adjacent nodes \( x \in X \) and \( y \in Y \). In this case we will show that \( Q \) is a dominating bisimplicial clique. We have that \( x \) is not stable (otherwise we would have \( x \equiv s_1 \), contradicting \( y \not\in N[s_1] \)) and \( y \) is not stable (otherwise we would have \( y \equiv s_3 \), contradicting \( x \not\in N[s_3] \)). Moreover, \( x \) and \( y \) are not both free (otherwise \( x \) would either belong to \( W(s_1, s_3) \), which is empty, or to \((W(s_2, s_3) \cap N(s_2)) \setminus Q \) which is also empty). If \( x \) is free and \( y \) is bound, then we have \( x \in F(s_1) \cup F(s_2) \) and \( y \in W(s_3, s_4) \); but then \((y : x, s_3, s_4)\) is a claw in \( G \), a contradiction. If \( x \) is bound and \( y \) is free, then we have \( x \in W(s_1, s_4) \) and \( y \in F(s_3) \); but then \((x : y, s_1, s_4)\) is a claw in \( G \), again a contradiction. It follows that both \( x \) and \( y \) are bound and we have \( x \in W(s_1, s_4) \) and \( y \in W(s_3, s_4) \).

Now assume that \( x \) and \( y \) have a common neighbor \( z \in Q \). But then \((x, y, z : s_1, s_3, s_2)\) is a net in \( G \), a contradiction. It follows that there exist nodes \( z', z'' \in Q \) such that \( x \) is adjacent to \( z' \) and non-adjacent to \( z'' \) while \( y \) is adjacent to \( z'' \) and non-adjacent to \( z' \). Moreover, \( z' \) is adjacent to \( s_1 \) (otherwise \((x : s_1, z', s_4)\) would be a claw in \( G \)) and \( z'' \) is adjacent to \( s_3 \) (otherwise \((y : s_3, z'', s_4)\) would be a claw in \( G \)). It follows that \( H = \{x, y, z', z''\} \) induces a \( C_4 \) in \( G \), with \( x \in W(s_1, s_4) \), \( y \in W(s_3, s_4) \), \( z' \in W(s_2, s_3) \) and \( z'' \in W(s_1, s_2) \).

We claim that \( N[H] = V \) (\( H \) dominates \( V \)). Suppose conversely that there exists a node \( t_0 \not\in N[H] \) and let \( P = (t_0, t_1, \ldots, t_p) \) be a shortest path connecting \( t_0 \) to \( t_p \in N(H) \) with \( p \geq 1 \). By claw-freeness, the node \( t_p \) is adjacent to at least two consecutive nodes of \( H \). On the other hand, since \( t_{p-1} \not\in N[H] \), the node \( t_p \) is adjacent to at most two consecutive nodes of \( H \). It follows that \( t_p \) is adjacent to exactly two consecutive nodes of \( H \), without loss of generality assume that they are \( z' \) and \( z'' \). We have that \( t_{p-1} \) is adjacent to \( s_1 \) (otherwise \((t_p, z', z'' : t_{p-1}, s_1, y)\) would be a net in \( G \)). Analogously, \( t_{p-1} \) is adjacent to \( s_3 \) (otherwise \((t_p, z', z'' : t_{p-1}, x, s_3)\) would be a net in \( G \)). If \( t_{p-1} \) is adjacent to \( s_2 \) we have the claw \((t_{p-1} : s_1, s_2, s_3)\), a contradiction. It follows that \( t_{p-1} s_2 \not\in E \) and hence
$s_2 \neq t_p$. We also have $t_ps_2 \in E$ (otherwise $(z': s_2, t_p, x)$ would be a claw in $G$). But then $(s_3, t_{p-1}, t_p : y, s_1, s_2)$ is a net in $G$, a contradiction. It follows that $N[H] = V$ as claimed.

To complete the proof of Claim (ii), we have to show that if a node $t$ is non-adjacent to both $x$ and $y$ then it belongs to $Q$. To this purpose observe that $t$ is adjacent to both $z'$ and $z''$. Suppose, by contradiction, that $t$ does not belong to $Q$ and hence is not adjacent to some node $\bar{t} \in Q$. It follows that $\bar{t}$ is adjacent to $x$ (otherwise $(z': t, \bar{t}, x)$ would be a claw in $G$) and to $y$ (otherwise $(z': t, \bar{t}, y)$ would be a claw in $G$). Consequently, $\bar{t}$ is universal to $H$ and adjacent to $s_2$. Since $W(s_2, s_4)$ is empty, $\bar{t}$ is not adjacent to $s_4$ and hence is adjacent to $s_1$ (otherwise $(x : s_1, s_4, \bar{t})$ would be a claw). But then $t_{s_3} \notin E$ (otherwise $(t : s_1, s_2, s_3)$ would be a claw) and hence $(y : s_3, s_4, \bar{t})$ is a claw in $G$, a contradiction.

*End of Claim (ii).*

Let $A = C_{s_2} \cap W(s_1, s_2)$, $\overline{A} = C_{s_2} \cap W(s_1, s_2)$, $B = C_{s_3} \cap W(s_3, s_4)$ and $\overline{B} = C_{s_3} \cap W(s_3, s_4)$. The sets $C_{s_2}, \overline{C}_{s_2}, C_{s_3}, \overline{C}_{s_3}, A, \overline{A}, B, \overline{B}$ can be constructed in $O(|N(s_2) \cup N(s_3)|^2) = O(|E|)$. If $A$ is empty then $C_{s_2} \setminus C_{s_2}$ intersects a single wing (namely $W(s_1, s_2)$) and hence $C_{s_3}$ is a maximal clique containing $s_2$ and $W(s_2, s_3) \cap N(s_2)$. Then, by Claim (ii), $\overline{C}_{s_3}$ is a strongly bisimplicial clique. In this case, by Lemma 43, we are done. Hence, we can assume that $A$ is non-empty and, analogously, also $A, B, \overline{B}$ are non-empty.

**Claim (iii).** If $A, \overline{A}, B, \overline{B}$ are all non-empty then $W(s_2, s_3) \cap N(s_2)$ is a clique.

**Proof.** Suppose that $W(s_2, s_3) \cap N(s_2)$ is not a clique. Let $x \in C_{s_2}$ and $y \in C_{s_3}$ be two non-adjacent nodes in $W(s_2, s_3) \cap N(s_2)$. Observe that either both $x$ and $y$ are bound nodes or one is bound and the other is free. Assume, without loss of generality, that $x$ is bound and $x \in C_{s_2}$. Since $G$ is connected, we have that $W(s_1, s_2)$ and $W(s_3, s_4)$ are non-empty. Let $t_1$ be a node in $A$ and $t_2$ a node in $B$. Observe that both $t_1$ and $t_2$ are adjacent to $x$ and that $t_1$ belongs to $W(s_1, s_2)$. If $t_1$ is bound let $\overline{t}_1 \equiv s_1$ while if $t_1$ is free let $\overline{t}_1$ be any free node in $F(s_1) \cap N(t_1)$. In both cases, if $\overline{t}_1$ is adjacent to $y$ we have that either $(y : \overline{t}_1, s_2, s_3)$ (if $y$ is bound) or $(y : \overline{t}_1, s_2, u)$ (if $y$ is free and $u$ is any node in $F(s_3) \cap N(y)$) is a claw in $G$, a contradiction. It follows that $\overline{t}_1$ is not adjacent to $y$. Moreover, $\overline{t}_1$ is not adjacent to $t_2$, for otherwise $s_3$ would be 3-winged, contradicting the hypothesis. Analogously, if $t_2$ is bound let $\overline{t}_2 \equiv s_4$ and if $t_2$ is free let $\overline{t}_2$ be any free node in $F(s_4) \cap N(t_2)$. In both cases, if $\overline{t}_2$ is adjacent to $y$ we have that $(t_2 : \overline{t}_2, x, y)$ is a claw in $G$, a contradiction. It follows that $\overline{t}_2$ is not adjacent to $y$. But then $\overline{t}_1$, $y$ and $\overline{t}_2$ are three mutually non-adjacent nodes in $N(C_{s_2})$ implying that $C_{s_2}$ is not reducible and hence, by Theorem 11, normal, contradicting the hypothesis.

*End of Claim (iii).*

Now, observe that the set $W = (W(s_2, s_3) \cap N(s_2)) \cup \{s_2\}$ can be constructed in $O(|N(s_2) \cup N(s_3)|^2) = O(|E|)$ time. Moreover, a maximal clique $Q$ containing $W$ can also be constructed in $O(|E|)$ time (recall that the size of $N(v)$ is $O(\sqrt{|E|})$ for any node $v \in V$). Hence, the theorem follows by Claim (ii) and by Lemma 43. □
5 The Maximum Weight Stable Set Problem on \{claw, net\}-free graphs

In this section we reduce the maximum weight stable set problem in a node-weighted \{claw, net\}-free graph \(G(V, E)\) to \(\sqrt{|E|} + 1\) maximum weight stable set problems in suitably defined interval graphs. In [11] Olariu proved that a graph is an interval graph if and only if its nodes admit a consistent ordering, defined as follows.

**Definition 51** Let \(G(V, E)\) be a connected graph. An ordering \(\{v_1, v_2, \ldots, v_n\}\) of \(V\) is said to be consistent if each triple \(\{i, j, k\}\) with \(1 \leq i < j < k \leq n\) and \(v_i v_k \in E\) satisfies \(v_j v_k \in E\).

We first prove that a claw-free \(C_4\)-free clique-strip is an interval graph. In particular, the next lemma will show that a \(C_4\)-free clique-strip \(G(V, E)\) admits a consistent ordering which can be found in \(O(|E| + |V| \log |V|)\) time.

**Lemma 51** Let \(G(V, E)\) be a connected claw-free \(C_4\)-free clique-strip defined by the family of cliques \(\{K_1, K_2, \ldots, K_p\}\). Then for any pair of nodes \(v_h, v_k \in K_t\) \((t \in \{1, \ldots, p - 1\})\) either \(N(v_h) \cap K_{t+1} \subseteq N(v_k) \cap K_{t+1}\) or \(N(v_k) \cap K_{t+1} \subseteq N(v_h) \cap K_{t+1}\). Moreover any ordering \(\{v_1, v_2, \ldots, v_n\}\) of \(V\) with \(v_h < v_k\) if:

(i) \(v_h \in K_t\) and \(v_k \in K_l\) for some \(1 \leq t < l \leq p\); or
(ii) \(v_h, v_k \in K_t\) for some \(t \in \{1, \ldots, p - 1\}\) and \(N(v_h) \cap K_{t+1} \subseteq N(v_k) \cap K_{t+1}\); is a consistent ordering.

**Proof.** Assume that there exists a pair of nodes \(v_h, v_k\) in some clique \(K_t\) \((t < p)\) such that neither \(N(v_h) \cap K_{t+1} \subseteq N(v_k) \cap K_{t+1}\) nor \(N(v_k) \cap K_{t+1} \subseteq N(v_h) \cap K_{t+1}\). Let \(v_h\) be a node in \(K_{t+1}\) which is non-adjacent to \(v_h\) and adjacent to \(v_k\). Analogously, let \(v_k\) be a node in \(K_{t+1}\) which is non-adjacent to \(v_k\) and adjacent to \(v_h\). Then, the four nodes \(\{v_h, v_k, v_h, v_k\}\) induce a \(C_4\) in \(G[K_t \cup K_{t+1}]\), a contradiction. Now, let \(\{v_1, v_2, \ldots, v_n\}\) be an ordering of \(V\) with \(v_h < v_k\) if condition (i) or (ii) is satisfied. Suppose that the ordering is not consistent and let \(v_i, v_j, v_k\) be three nodes such that \(1 \leq i < j < k \leq n\), \(v_i v_k \in E\) and \(v_j v_k \notin E\). If \(v_i, v_k\) belong to the same clique \(K_t\) then, by property (i), also \(v_j\) belongs to \(K_t\), contradicting the assumption that \(v_j v_k \notin E\). Hence, without loss of generality, we can assume \(v_i \in K_t\), \(v_k \in K_{t+1}\) and \(v_j \in K_t \cup K_{t+1}\) for some \(t \in \{1, \ldots, p - 1\}\). Since \(v_j v_k \notin E\), we have \(v_j \in K_t\). Moreover, since \(v_j \neq v_i\), by property (ii) we have \(N(v_j) \cap K_{t+1} \not\subseteq N(v_i) \cap K_{t+1}\) and hence \(N(v_i) \cap K_{t+1} \subseteq N(v_j) \cap K_{t+1}\). But this contradicts the assumption that \(v_i v_k \in E\) and \(v_j v_k \notin E\) and the lemma follows.

We now show how to remove all the \(C_4\) in a claw-free clique-strip while preserving at least one maximum weight stable set of the original graph. The core operation of this algorithm is the addition of edges having both end-points in a semi-homogeneous pair of cliques defined by Oriolo et al. as follows.

**Definition 52** [12] Let \(G(V, E)\) be a graph. A pair of cliques \((X, Y)\) in \(G\) is semi-homogeneous if for all \(u \in V\) not in \(X \cup Y\), \(u\) is either universal to \(X\) or universal to \(Y\) or null to \(X \cup Y\).
Our algorithm will exploit the properties of two special kind of semi-homogeneous pairs of cliques, described in the following theorem. We recall that a pair of cliques \((H, K)\) is said to be proper if each node in \(H\) \((K)\) is neither null nor universal to \(K\) \((H)\).

**Theorem 51** Let \(G(V, E)\) be a connected claw-free graph and let \(H\) and \(K\) be a proper pair of cliques in \(G\) with the property that each node in \(V \setminus (H \cup K)\) is either null to \(H\) or to \(K\). Then the following properties hold:

(i) the pair of cliques \((X, Y)\) with \(X = \{x_1, x_2\} \subseteq H\) and \(Y = \{y_1, y_2\} \subseteq K\) such that the set \(\{x_1, x_2, y_1, y_2\}\) induces a \(C_4\) in \(G\) is a semi-homogeneous pair;

(ii) the pair of cliques \((X, Y)\) with \(X = \{x_1\} \subseteq H\) and \(Y = K \setminus N(x_1)\) where \(x_1\) is some node in \(H\) having the maximum number of adjacent nodes in \(K\) is a semi-homogeneous pair;

(iii) if \((X, Y)\) is a semi-homogeneous pair as above any pair of non-adjacent nodes \(x \in X\) and \(y \in Y\) is a diagonal of some \(C_4\) in \(G\) and the graph \(G'\) obtained by adding any set of edges between \(X\) and \(Y\) is claw-free.

**Proof.** Let \((X, Y)\) be a pair of cliques of type (i) and assume, by contradiction, that there exists some node \(z \in V \setminus (X \cup Y)\) which is (without loss of generality) adjacent to \(x_1\) and non-adjacent to \(x_2\). Observe that by hypothesis \(z\) must be null to \(K\) and hence is non-adjacent also to \(y_1\) and \(y_2\). But then \((x_1 : x_2, y_1, z)\) is a claw in \(G\), a contradiction. It follows that \((X, Y)\) is semi-homogeneous.

Consider now a pair of cliques \((X, Y)\) of type (ii). If \(|Y| = 1\) then trivially \((X, Y)\) is semi-homogeneous, hence assume \(|Y| \geq 2\) and let \(y_1, \bar{y}\) be any pair of nodes in \(Y\). Assume, by contradiction, that there exists some node \(z \in V \setminus (X \cup Y)\) which is (without loss of generality) adjacent to \(\bar{y}\) and non-adjacent to \(x_1\) and \(y_1\). Observe that \(z\) does not belong to \(H \cup K\) and is not null to \(K\) so, by hypothesis, \(z\) must be null to \(H\). Let \(\bar{h} \in H\) be a node adjacent to \(\bar{y}\) (it exists, since \(\bar{y}\) is not null to \(H\)) and \(k' \in K\) a node adjacent to \(x_1\) and non-adjacent to \(\bar{h}\) (it exists, since \(x_1 \in H\) has the maximum number of adjacent nodes in \(K\) and \(\bar{y} \in K\) is adjacent to \(\bar{h}\) and non-adjacent to \(x_1\)). Observe that \(z\) is non-adjacent to \(\bar{h}\) and that \(y_1, \bar{y}\) and \(k'\) are mutually adjacent. We have \(k' \in E\) (otherwise \((\bar{y}, k', \bar{h}, z)\) would be a claw in \(G\)). But then \((k' : x_1, y_1, z)\) is a claw in \(G\), a contradiction. It follows that also in this case \((X, Y)\) is semi-homogeneous.

We observe that also in case (iii), letting \(y_1\) be any node in \(Y\), there exist nodes \(x_2 \in H\) and \(y_2 \in K\) such that the set \(\{x_1, x_2, y_1, y_2\}\) induces a \(C_4\) in \(G\). In fact, let \(x_2 \in H\) be a node adjacent to \(y_1\) (it exists, since \(y_1\) is not null to \(H\)). The node \(x_2\) has at most as many adjacent nodes in \(Y\) as the node \(x_1\); since \(y_1\) is adjacent to \(x_2\) and non-adjacent to \(x_1\), we have that there exists some node \(y_2 \in K\) which is adjacent to \(x_1\) and non-adjacent to \(x_2\). It follows that \(\{x_1, x_2, y_1, y_2\}\) induces a \(C_4\) in \(G\) as claimed.

Finally, assume that \((X, Y)\) is a pair of cliques of type (i) or (ii) and that the addition of some edge \(x_1, y_1\) with \(x_1 \in X\) and \(y_1 \in Y\) produces (without loss of generality) the claw \((x_1 : y_1, z_1, z_2)\). Since \(z_1\) and \(z_2\) are non-adjacent to \(y_1\) they belong to \(V \setminus K\), so the edges \(x_1 z_1\) and \(x_2 z_2\) belong to the original graph. \(z_1\) and \(z_2\) cannot both belong to the clique \(H\). If both belong to \(V \setminus (H \cup K)\) then they are (in \(G\)) null to \(K\). But since \(x_1\) is adjacent in \(G\) to some node \(k \in K\), we have that \((x_1 : z_1, z_2, k)\) is a claw in \(G\), a contradiction. It follows that one of the nodes \(z_1, z_2\)
belongs to $H$, and the other to $V \setminus (H \cup K)$, say (without loss of generality) $z_1 \in H$ and $z_2 \in V \setminus (H \cup K)$. Let $x_2 \in H$ and $y_2 \in K$ be nodes such that $\{x_1, x_2, y_1, y_2\}$ induce a $C_4$ in $G$ (they exist as observed above). Observe that $z_2$ is null to $K$, by assumption. Hence we have $z_1x_2 \in E$; moreover, we also have $z_2x_2 \in E$ (otherwise $(x_1 : x_2, y_2, z_2)$ would be a claw in $G$). But then $(x_2 : y_1, z_1, z_2)$ is a claw in $G$, a contradiction. Hence $G'$ is claw-free and the theorem follows. \hfill \square

We are now ready to describe the procedure which removes all the $C_4$ in a claw-free clique-strip $G(V, E)$. The procedure takes a claw-free clique-strip $G(V, E)$ defined by a clique family $\{K_1, K_2, \ldots, K_p\}$ and a node-weighting vector $w$.

Since any $C_4$ in $G$ is contained in two consecutive cliques of the family, the procedure iterates through such pairs. In particular, for each pair of consecutive cliques $K_i$ and $K_{i+1}$, the procedure does the following (inner loop): it initializes two sets \(X \equiv K_i, Y \equiv K_{i+1}\) and adds, for each $C_4$ induced in $X \cup Y$, at least one edge having as end-points the nodes of a diagonal (diagonal edge). By Property (iii) of Theorem 51 the addition of diagonal edges does not add new claws and hence the resulting graph remains a claw-free clique-strip. Two nodes $x \in X$ and $y \in Y$ are said to be paired if they are non-adjacent, $x$ is universal to $Y \setminus \{y\}$ and $y$ is universal to $X \setminus \{x\}$. Evidently, every $C_4$ containing $x$ must also contain $y$ (and viceversa); moreover, two couples of paired nodes induce a $C_4$. The inner loop makes use of two data structures that are maintained throughout: a set $P$ of paired nodes (initially empty) and an integer vector $d[\cdot]$ whose entries are associated with the nodes in $X \cup Y$ and such that for each $x \in X$ $d[x]$ is equal to the number of nodes in $Y$ adjacent to $x$ (for each $y \in Y$ $d[y]$ is equal to the number of nodes in $X$ adjacent to $y$). The initialization of the data structure $d[\cdot]$ can be done in $O(|K_i \cup K_{i+1}|^2)$ time.

The inner loop iterates through stages. At each stage, either (i) a node which cannot possibly belong to an induced $C_4$ in $X \cup Y$ is removed from $X \cup Y$; or (ii) one or two new couples of paired nodes are singled out; or (iii) a semi-homogeneous pair $X' \subset X$ and $Y' \subset Y$ is found and all the edges between $X'$ and $Y'$ are added, with the exception of the heaviest one.

At each stage let $x_{\min}, x_{\max}, y_{\min}, y_{\max}$ be the nodes having minimum and maximum degree in the sets $X$ and $Y$ (excluding the paired nodes in $P$). By using the data structure $d[\cdot]$ the four nodes can be found in time $O(|K_i \cup K_{i+1}|)$.

Case (i) applies when there exists some node $x \in X$ that is either null or universal to $Y$ ($x \equiv x_{\min}$, $d[x_{\min}] = 0$ or $x \equiv x_{\max}$, $d[x_{\max}] = |Y|$) or there exists some node $y \in Y$ that is either null or universal to $X$ ($y \equiv y_{\min}$, $d[y_{\min}] = 0$ or $y \equiv y_{\max}$, $d[y_{\max}] = |X|$). In such cases $x$ or $y$ is removed from $X \cup Y$ by calling Procedure Remove, which updates accordingly the data structure $d[\cdot]$. This task can be accomplished in time $O(|K_i \cup K_{i+1}|)$.

Case (ii) applies when there exists a node $x \in X$ which is adjacent to all the nodes in $Y$ but one ($x \equiv x_{\max}$, $d[x_{\max}] = |Y| - 1$) and a node $y$ which is adjacent to all the nodes in $X$ but one ($y \equiv y_{\min}$, $d[y_{\min}] = |X| - 1$). In this case $x$ is paired to some node $\bar{y} \in Y$ and $y$ is paired to some node $\bar{x} \in X$. The paired nodes are found by a call to Procedure FindMate which accomplishes the task in time $O(|K_i \cup K_{i+1}|)$. Observe that possibly $\bar{y} \equiv y$ (in this case we also have $\bar{x} \equiv x$) so
in each iteration of case (ii) we either find one or two couples of paired nodes. The set \( P \) is updated accordingly.

Case (iii) applies when one of two subcases arises. In subcase (iii-a) \( P \) contains four nodes (observe that \( P \) always contains an even number of nodes); in subcase (iii-b) \( X \) \( (Y) \) does not contain nodes which are null or universal to \( Y \) \( (X) \) (otherwise case (i) applies) and either \( d[x_{max}] < |Y| - 1 \) or \( d[y_{max}] < |X| - 1 \). If (subcase (iii-a)) \( P \) contains two couples of paired nodes (say \((x_1, y_1)\) and \((x_2, y_2)\)) then the \( C_4 \) induced by \((x_1, x_2, y_1, y_2)\) is removed by a call to Procedure \( KillC4 \) which adds the diagonal edge with smallest weight and updates accordingly the data structure \( d[\cdot] \) and the set \( P \). The procedure runs in time \( \mathcal{O}(|K_i \cup K_{i+1}|) \). If (subcase (iii-b)) \( x_{max} \) \( (y_{max}) \) has at least two non-adjacent nodes in \( Y \) \( (X) \), we call Procedure \( KillDiags \) which adds all the missing edges between \( x_{max} \) \( (y_{max}) \) and the nodes in \( Y \) \( (X) \) excluding only the heaviest one. The procedure also updates the data structure \( d[\cdot] \) and runs in \( \mathcal{O}(|K_i \cup K_{i+1}|) \) time. Observe that in subcases (iii-a) and (iii-b) we have the semi-homogeneous pairs of cliques described, respectively, in (i) and (ii) of Theorem 51 and hence the addition of the new edges preserves claw-freeness.

In each iteration of the inner loop, Case (i) can occur at most \(|K_i \cup K_{i+1}| \) times. Moreover, the occurrence of subcase (iii-a) produces a node in \( X \) universal to \( Y \) and a node in \( Y \) universal to \( X \) which are subsequently removed. It follows that also subcase (iii-a) can occur at most \( \mathcal{O}(|K_i \cup K_{i+1}|) \) times. The occurrence of case (ii) adds to \( P \) at least one couple of paired nodes. As soon as \( P \) contains four nodes, subcase (iii-a) applies, so case (ii) can occur at most twice as often as subcase (iii-a) and hence \( \mathcal{O}(|K_i \cup K_{i+1}|) \) times. The occurrence of subcase (iii-b) produces at least one couple of paired nodes, so an occurrence of case (ii) will follow. As a consequence, also subcase (iii-b) can occur at most \( \mathcal{O}(|K_i \cup K_{i+1}|) \) times.

In conclusion, the number of stages of the inner loop is \( \mathcal{O}(|K_i \cup K_{i+1}|) \). Moreover, by the above discussion we have that each stage is performed in \( \mathcal{O}(|K_i \cup K_{i+1}|) \) time. It follows that the inner loop has a complexity of \( \mathcal{O}((|K_i \cup K_{i+1}|)^2) \) time while the overall complexity of Procedure \( \text{Interval}(G) \) is \( \mathcal{O}(\sum_{i=1}^{p-1} (|K_i \cup K_{i+1}|)^2) = \mathcal{O}(|E|) \) time.

In [12] Oriolo et al. prove a useful lemma concerning semi-homogeneous pairs of cliques. The following is a rephrasing of such lemma.

**Lemma 52** Let \((X, Y)\) be a semi-homogeneous pair of cliques in a graph \( G(V, E) \) and let \( v \) be a node in \( V \setminus (X \cup Y) \). Then \( v \) is either adjacent to every stable set of \( G[X \cup Y] \) of cardinality 2 or to none of them.

**Proof.** If \( v \) is universal to \( X \) or to \( Y \) then it is trivially adjacent to every stable set of \( G[X \cup Y] \) of cardinality 2 and we are done. Hence assume that \( v \) is neither universal to \( X \) nor to \( Y \). Consequently, since \((X, Y)\) is a semi-homogeneous pair of cliques, \( v \) is null to \( X \cup Y \). The lemma follows. \( \square \)

**Theorem 52** Let \( G(V, E) \) be a node-weighted \{claw, net\}-free graph and \( X \) a clique in \( G \) such that \( G^X \equiv G - X \) is partitioned in two (possibly coincident)
clique-strips. For each node \( v \in X \), let \( G^v = G - N[v] \). Let \( \overrightarrow{G^X} \) be the interval graph obtained from \( G^X \) by applying Procedure Interval and \( \overrightarrow{G^v} = \overrightarrow{G^X} - N[v] \) (\( v \in X \)). Then \( \alpha_w(G^X) = \alpha_w(\overrightarrow{G^X}) \) and \( \alpha_w(G^v) = \alpha_w(\overrightarrow{G^v}) \), for each \( v \in X \).

Proof. The procedure Interval produces a sequence of graphs \( (G_0, \ldots, G_q) \) with \( G_0 \equiv G^X \) and \( G_q \equiv \overrightarrow{G^X} \). Each graph \( G_i \) (\( i = 1, \ldots, q \)) is obtained from \( G_{i-1} \) by turning into edges all the pair of non-adjacent nodes in \( G_{i-1}[H_i \cup K_i] \) with the exception of the heaviest one, where \( (H_i, K_i) \) is a semi-homogeneous pair of cliques. Evidently, \( \alpha_w(G_{i-1}) \geq \alpha_w(G_i) \). Moreover, for each \( v \in X \), let \( G^v_i = G_i \setminus N[v] \). We have \( G^0 \equiv G^X \) and \( G_q^v \equiv \overrightarrow{G^X} \).

We first prove that \( \alpha_w(G^X) = \alpha_w(\overrightarrow{G^X}) \). Conversely, suppose \( \alpha_w(G^X) > \alpha_w(\overrightarrow{G^X}) \). Let \( j \geq 1 \) be the smallest index with the property that \( \alpha_w(G_j) < \alpha_w(G^X) \) and let \( S \) be a maximum weight stable set of \( G_j \). Since \( S \) is not a stable set of \( G_j \), there exist nodes \( h \in H_j \cap S \) and \( k \in K_j \cap S \) with the property that \( hk \) is a new edge in
By construction there exist non-adjacent nodes in $G_j$ with $h' \in H_j, k' \in K_j$ and $w(h') + w(k') \geq w(h) + w(k)$. But then by Lemma 52 the set $S \setminus \{h, k\} \cup \{h', k'\}$ is a stable set of $G_j$ having weight not smaller than $\alpha_w(G_{j-1})$, a contradiction.

We now prove that $\alpha_w(G^o) = \alpha_w(G^c)$. Conversely, suppose $\alpha_w(G^o) > \alpha_w(G^c)$.

Let $j \geq 1$ be the smallest index with the property that $\alpha_w(G^o_j) < \alpha_w(G^c)$ and let $S$ be a maximum weight stable set of $G_{j-1}^c$. Since $S$ is not a stable set of $G_{j-1}^c$, there exist nodes $h \in H_j \cap S \setminus N[v]$ and $k \in K_j \cap S \setminus N[v]$ with the property that $hk$ is a new edge in $G_{j-1}^o$. By construction there exist non-adjacent nodes in $G_j$ with $h' \in H_j, k' \in K_j$ and $w(h') + w(k') \geq w(h) + w(k)$. Since $v$ is non-adjacent to $h$ and $k$, by Lemma 52 $v$ is also non-adjacent to $h'$ and $k'$. It follows that $h'$ and $k'$ belong to $G_j^o$. But then, again by Lemma 52, the set $S \setminus \{h,k\} \cup \{h',k'\}$ is a stable set of $G_j^o$ having weight not smaller than $\alpha_w(G_{j-1})$, a contradiction. The theorem follows.

We conclude by giving a streamlined description of the procedure for solving the Maximum Weight Stable Set Problem in a {claw, net}-free graph $G(V,E)$ with $\alpha(G) \geq 4$ in time $O(|V| \sqrt{|E|})$. First, using the results of [10] we either verify that $\alpha(G) \leq 3$ or construct a stable set $S_0$ of cardinality at least 4 in time $O(|E|)$. Next, By Theorem 31, we construct, in $O(|E|)$ time, a canonical stable set of cardinality at least 4 from $S_0$. Subsequently, in $O(|E|)$ time, by Theorem 42 we find a bisimplicial clique $Q$ which is either dominating or strongly bisimplicial and whose neighborhood is partitioned into two cliques $X$ and $Y$. Then, by Lemma 43, we can construct two (possibly coincident) clique families $H = \{H_1, \ldots, H_p\}$ and $K = \{K_1, \ldots, K_q\}$ defining clique-strips partitioning $G^X = G - X$. Finally, we apply Procedure Interval to the clique-strips in $G^X$ and turn $G^X$ into an interval graph $G^X$ in $O(|E|)$ time. By Lemma 51, a consistent ordering of $G^X$ can be found in time $O(|E| + |V| \log |V|)$. Now, for each node $v \in X$, let $G^v = G - N[v]$ and $\overline{G}^v = \overline{G^X} - N[v]$. Evidently, $\alpha_w(G) = \max \{\alpha_w(G^X), \max_{v \in X} \{\alpha_w(G^v) + w_v\}\}$. By Theorem 52, we have $\alpha_w(G^X) = \alpha_w(G^c)$ and $\alpha_w(G^v) = \alpha_w(\overline{G}^v)$, for each $v \in X$. Since $\overline{G}^v (v \in X)$ is an induced subgraph of $\overline{G^X}$, it is also an interval graph and inherits from $G^X$ the consistent ordering of its nodes. Hence, the maximum weight stable set problem on $G$ can be reduced to solving $|X| + 1$ maximum weight stable set problems on interval graphs.

In [7] Mannino, Oriolo, Ricci and Chandran proved that, given a consistent ordering, the maximum weight stable set problem on an interval graph can be solved in linear time. It follows that the maximum weight stable set problem in $G$ can be solved in time $O(|E| + |V| \log |V| + (|X| + 1)|V|) = O(|V| \sqrt{|E|})$.

References

1. Brandsttadt, A., Dragan, F.F.: On linear and circular structure of (claw, net)-free graphs. Discrete Applied Mathematics 129(23), 285 – 303 (2003)
2. Courner, A., Habib, M.: A new linear algorithm for modular decomposition. In: S. Tison (ed.) CAAP, Lecture Notes in Computer Science, vol. 787, pp. 68–84. Springer (1994)
3. Faenza, Y., Oriolo, G., Pietropaoli, U., Stauffer, G.: An algorithmic decomposition theorem for claw-free graphs. Tech. rep., University of Rome "Tor Vergata" (2010)
4. Faenza, Y., Oriolo, G., Stauffer, G.: Solving the weighted stable set problem in claw-free graphs via decomposition. J. ACM 61(4), 20 (2014)
5. Kloks, T., Kratsch, D., Müller, H.: Finding and counting small induced subgraphs efficiently. Inf. Process. Lett. 74(3-4), 115–121 (2000)
6. Lovász, L., Plummer, M.: Matching theory. Annals of Discrete Mathematics, 29. North-Holland Mathematics Studies, 121. Amsterdam etc.: North-Holland. XXXIII, 544 p. (1986)
7. Mannino, C., Oriolo, G., Ricci, F., Chandran, S.: The stable set problem and the thinness of a graph. Oper. Res. Lett. 35(1), 1–9 (2007)
8. McConnell, R.M., Spinrad, J.: Linear-time modular decomposition and efficient transitive orientation of comparability graphs. In: D.D. Sleator (ed.) SODA, pp. 536–545. ACM/SIAM (1994)
9. Minty, G.J.: On maximal independent sets of vertices in claw-free graphs. J. Comb. Theory, Ser. B 28, 284–304 (1980)
10. Nobili, P., Sassano, A.: An $O(m \log n)$ algorithm for the weighted stable set problem in claw-free graphs with $\alpha(G) \leq 3$. Submitted (2014)
11. Olariu, S.: An optimal greedy heuristic to color interval graphs. Inf. Process. Lett. 37(1), 21–25 (1991)
12. Oriolo, G., Pietropaoli, U., Stauffer, G.: A new algorithm for the maximum weighted stable set problem in claw-free graphs. In: IPCO, pp. 77–96 (2008)
13. Pulleyblank, W.R., Shepherd, F.B.: Formulations for the stable set polytope of a claw-free graph. In: IPCO, pp. 267–279 (1993)