Reaching consensus via coordinated groups

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ABSTRACT
Models of opinion dynamics have been used to depict how the opinions of agents within a social network evolve based on their interconnections and interactions. The collective behaviour of agents can also exhibit certain social phenomena such as convergence to a consensus or disagreement. In this study, we propose a gossip-based model for the opinion dynamics of interconnected groups and describe how they can reach consensus. Through analysis of the convergence properties of its expected dynamics, we show that the number of groups can affect the rate by which the network reaches consensus. We also include numerical examples to demonstrate our proposed model and its properties.

1. Introduction
A growing area in the field of social network analysis is the study of how opinions propagate under varying conditions, such as the underlying relationships among the social actors and the type of interactions involved [1,2]. By developing models of opinion dynamics, the effects of attitudes on collective behaviours can be examined while having the flexibility to make adjustments that are not possible when investigating actual social networks. In such models, different types of social phenomena can be observed depending on the design considerations that are implemented. In the field of control, multi-agent systems are utilized to achieve this purpose.

A common theme among several studies in this area is finding out if a group of agents can attain consensus and the conditions that make it possible. In fact, one of the earliest works on opinion dynamics can be traced back to DeGroot’s model for reaching consensus [3]. The relevance of consensus goes beyond opinion dynamics and is also studied in other applications, such as decentralized computers, sensor networks and distributed systems [4].

In the DeGroot model, consensus is attained by an iterative process of synchronously updating opinions using their weighted combinations. Other popular opinion dynamics models employ a similar deterministic system [5]. More recent models incorporate some form of time-varying dynamics or randomization [2]. A particular randomization method used in distributed networks that is also applied in modelling opinion formation is gossiping [6,7]. In this approach, interactions are represented by random pairwise exchange of opinions, which is meant to emulate how communications take place in real-world settings.

Research on opinion dynamics are often concerned with identifying the behaviour of an entire network based on the characteristics of individual agents. However, social networks may include smaller groups that also exhibit important patterns. Exploring group dynamics is especially relevant during this age when people are becoming more and more connected via technology, leading to the rise of communities with shared ideologies which help shape different facets of our societies.

One way to explore group behaviours is through a bottom up approach where there are no predefined groupings, but they may later surface as a consequence of how agents interact. Examples that follow this principle are several bounded confidence models, which describe the opinion dynamics of social networks where only like-minded individuals can influence one another [8,9]. The dynamics of these models may lead to clustering depending on the range of opinions that the agents are willing to accept. Another approach is to design a model with an a priori assumption that there are already explicit groups in place before observing the resulting dynamics [10,11]. This makes it possible to analyse how intergroup dynamics affect an entire social network.

In this research, we propose a novel model for the opinion dynamics of social networks with intragroup and intergroup interactions. The model, which employs a gossiping scheme, enables multiple groups to reach consensus via the presence of agents that act as links between groups. By analysing its expected dynamics,
we show how the number of groups may affect the rate by which the entire network achieves consensus. Additionally, we provide simulations to demonstrate the behaviour of the model.

This paper is organized as follows. Our notation and preliminaries are specified in Section 2. We describe our proposed model for reaching consensus via coordinated groups in Section 3. In Section 4, we analyse the expected dynamics of our proposed model. Numerical examples are provided in Section 5. We state our conclusion in Section 6.

2. Notation and preliminaries

We denote a directed graph as $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$ is the set of nodes and $E \subseteq V \times V$ is the set of edges. The neighbours of agent $i$ is given by $N_i$. A path is a sequence of edges that connects a distinct set of nodes. A directed graph is strongly connected if there is a path between every pair of nodes. We use $e_i \in \mathbb{R}^n$ to denote a standard basis vector, where the $i$th element is 1 while the rest are zeroes. The vector of ones is given by 1. The $i$th eigenvalue of matrix $A$ is denoted by $\lambda_i(A)$, and the eigenvalues of $A$ are ordered as $\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)$.

3. Gossip-based model

In this section, we describe a gossip-based model for the opinion dynamics of interconnected groups with coordinators and followers. In communication networks, gossiping refers to asynchronous pairwise interactions between agents. More recent opinion dynamics models utilize gossiping to approximate how communications take place in the real world. Our model implements a scheme similar to the symmetric gossip model in [4].

We represent a social network as a directed graph $G = (V, E)$ consisting of $n$ agents, given by $V = \{1, 2, \ldots, n\}$, whose communication lines are denoted by $E \subseteq V \times V$. We are concerned with changes in opinions brought about by exchanges of views, hence the edges are bidirectional which means that $(i,j) \in E$ if only if $(j,i) \in E$. Each agent belongs to a group $G_i = (V_i, E_i)$, where $V_i \subseteq V$ and $E_i \subseteq E$. We consider that each $G_i$ is a strongly connected subgraph, otherwise it is decomposed into multiple groups. However, groupings may be selected arbitrarily. The groups are labelled as $s = 1, 2, \ldots, m$, where $m$ is the number of groups in $G$. To avoid trivialities, we assume that $m \geq 2$. While in reality, individuals may belong in multiple groups, this study is restricted to cases when agents belong in exactly one group only. Figure 1 shows an example of a social network composed of interconnected groups.

In our model, we classify the members of a group as either a coordinator or a follower. A coordinator is an agent $i \in V^C \subseteq V$ that is connected by an edge to a member of another group. All the remaining agents are considered as followers. We assume that each group contains at least one coordinator and there is a path between any pair of coordinators, thus the graph induced by the set of coordinators, $G^C = (V^C, E^C)$, is also strongly connected. Additionally, we assume that every coordinator has at least one follower neighbour, that is $|N_i^C \setminus V^C| > 0$. The total number of coordinators is given by $c = |V^C|$. The sets $N_i^C \subseteq N_i$ (Figure 2a) and $N_i^C \subseteq N_i$ (Figure 2b) respectively denote the neighbours of $i$ that belong on the same group and the neighbours that are coordinators.

The opinions of all the agents at time $k \in \mathbb{Z}_{\geq 0}$ are contained in the vector $x(k) \in \mathbb{R}^n$. Starting with the initial opinions $x(0)$, opinions are updated based on the following algorithm.

Let $q \in (0, 1)$ be the weight for opinion exchange. Let $\rho \in (0, 1)$ be the frequency which agents interact with members from other groups. At each time $k \geq 0$:

1. Agent $i$ is chosen with uniform probability from $V$.

Let $G_i$ be the group of $i$, i.e. $i \in V_i$. 

Figure 1. A social network with three groups containing five, four and three members, respectively. Dashed edges connect the coordinators.

Figure 2. Neighbours of agent $i$. (a) Neighbours of $i$ within the same group. (b) Neighbours of $i$ that are coordinators. (c) All the neighbours of $i$ in the entire network. (a)$N_i^s$, (b)$N_i^C$ and (c)$N_i$. 

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(2) If \( i \notin V^C \), agent \( j \in N_i^\circ \) is chosen with probability \( 1/|N_i^\circ| \). If \( i \in V^C \), agent \( j \in N_i^\circ \) is chosen with probability \( (1-\rho)/|N_i^\circ| \), while agent \( j \in N_i^c \) is chosen with probability \( \rho/|N_i^c| \).

(3) The opinion of \( i \) and \( j \) are updated as
\[
x_i(k+1) = (1-q)x_i(k) + qx_j(k),
\]
\[
x_j(k+1) = (1-q)x_j(k) + qx_i(k),
\]
while others remain the same.

The algorithm above represents a random sequence of exchanges between a pair of coordinators, a pair of followers or a coordinator–follower pair. Coordinators make it possible for opinions to spread across the entire network. Manipulating the value of \( \rho \) can increase or decrease the frequency of interactions between different groups.

The algorithm can be compactly expressed as linear discrete-time system by introducing a time-varying matrix. Let \( M^\circ \in \mathbb{R}^{n \times n} \) be the random matrix based on the selected pair of agents \( i \) and \( j \) at time \( k \), and defined as
\[
M^\circ = I - q(e_i-e_j)(e_i-e_j)^T.
\]
The proposed model can then be written as
\[
x(k+1) = M(k)x(k)
\]
where \( M(k) = M^\circ \). Since the communication lines between agents are activated independently based on a fixed probability distribution, then the sequence \( \{M(k)\}_{k \geq 0} \) is independent and identically distributed (i.i.d.). Additionally, observe that \( M^\circ 1 = 1 \) and \( 1^T M^\circ = 1^T \) for any \( i, j \in V \), hence \( M^\circ \) is always doubly stochastic. We now state our first main result.

**Theorem 3.1:** The model (1) converges almost surely to the limit
\[
\lim_{k \to \infty} x(k) = x^*_a,
\]
where
\[
x^*_a = \frac{1}{n} 1 1^T x(0).
\]

**Proof:** The sequence \( \{M(k)\}_{k \geq 0} \) is i.i.d, where each \( M(k) \) is a doubly stochastic matrix with a positive diagonal. Since \( 0 < \rho < 1 \), for any pair \( i,j \in V \), \( \mathbb{P}[M(k) = M^\circ] > 0 \), which also implies \( \mathbb{E}[M(k)] > 0 \). Note that \( G \) is strongly connected, thus \( \mathbb{E}[M(k)] \) is irreducible. Additionally, \( M(k) \) is doubly stochastic for any \( k \), therefore \( \mathbb{E}[M(k)] \) is also doubly stochastic. By the Perron–Frobenius theorem, \( \lambda_{n-1}(\mathbb{E}[M(k)]) < \lambda_n(\mathbb{E}[M(k)]) = 1 \). Given the previous statements, the limit (2) can be reached based on the proofs in [12].

Theorem 3.1 shows that our model belongs to a class of averaging dynamics over multi-agent networks and can be interpreted as multiple groups coming to an agreement through the presence of coordinators.

### 4. Expected dynamics

We now analyse the expected dynamics of the model (1) in order to further understand the effects of the frequency of interactions among groups. We start with a key technical lemma.

**Lemma 4.1:** Let \( \bar{x}(k) = \mathbb{E}[x(k)] \) and \( \bar{M} = \mathbb{E}[M(k)] \) which is given by
\[
\bar{M} = (1-\alpha)W + \alpha H,
\]
where
\[
W = \frac{1}{n-\rho c} \left( \sum_{i \in V^c} \frac{1}{|N_i^c|} \sum_{j \in N_i^c} M_{ij} - \rho \sum_{i \in V} \frac{1}{|N_i^c|} \sum_{j \in N_i^c} M_{ij} \right)
\]
\[
H = \frac{1}{c} \sum_{i \in V^c} \frac{1}{|N_i^c|} \sum_{j \in N_i^c} M_{ij}
\]
\[
\alpha = \frac{\rho c}{n},
\]
and both \( W \) and \( H \) are doubly stochastic matrices. Then, the expected dynamics of the model (1) is
\[
\bar{x}(k+1) = \bar{M} \bar{x}(k).
\]

**Proof:** The proof of this lemma can be obtained by direct computation
\[
\mathbb{E}[M(k)] = \frac{1}{n} \sum_{i \in V} \sum_{j \in N_i^c} M_{ij} + \frac{1}{n} \sum_{i \in V^c} \sum_{j \in N_i^c} M_{ij} - \rho \sum_{i \in V} \sum_{j \in N_i^c} \frac{1}{|N_i^c|} \sum_{j \in N_i^c} M_{ij}.
\]

In fact, if we define \( \alpha \) as in the lemma, the third term becomes \( \alpha H \), where we see the non-negative matrix \( H \) satisfies \( H1 = 1 \) and \( 1^T H = 1^T \), which means \( H \) is
doubly stochastic. The first term
\[ \frac{1}{n} \sum_{i \in V} \sum_{j \in N_i^c} M^i_j = W_s \]
is clearly doubly stochastic, and the second term can be rewritten as
\[ \frac{\rho}{n} \sum_{i \in V \cup C} \sum_{j \in N_i^c} M^i_j \]
\[ = \alpha \left( \frac{1}{c} \sum_{i \in V \cup C} \sum_{j \in N_i^c} M^i_j \right) = \alpha W_C, \]
where \( W_C \) is also doubly stochastic. This means that \((W_s - \alpha W_C)1 = (1 - \alpha)1\) and \(1^T(W_s - \alpha W_C) = (1 - \alpha)1^T\) hold true. We therefore see that \(W_s - \alpha W_C\) can be represented as \((1 - \alpha)W\), where \(W\) should be a doubly stochastic matrix. This completes the proof. \(\blacksquare\)

The dynamics (3) can be alternatively viewed as a deterministic version of the model (1) which represents the evolution of the opinion profile of a social network based on an iterative process of intragroup and intergroup interactions. The matrix \(W\) provides the weight of the opinions of members within the same group, while the matrix \(H\) specifies how much the coordinators influence each other. The latter matrix enables opinions to spread across groups. The parameter \(\alpha\) can be seen as a weight that determines how much outside influence groups will accept per iteration. It can also be viewed as the frequency of interactions among group representatives, while \(1 - \alpha\) is the frequency of group discussions. Since \(\alpha\) is proportional to \(c\), the number of coordinators significantly contribute to the impact of groups on one another during each round of interactions. Next, we describe the limit of (3).

**Corollary 4.1:** The expected dynamics (3) converges to the limit
\[ \lim_{k \to \infty} \tilde{x}(k) = \tilde{x}_s \]
where
\[ \tilde{x}_s = \frac{1}{n} 11^T x(0). \] (4)

Corollary 4.1 is an intuitive extension of Theorem 3.1 on (3), which always converges to the same limit. While (3) have the same limit as (1), it enables us to analyse the eigenvalues of \(\tilde{M}\), which can give us a better understanding of how the convergence rate can be affected by the various groups in \(G\) and the frequency by which coordinators interact with one another. This leads us to the main result of this study.

**Theorem 4.1:** Suppose that the number of groups and the number of coordinators satisfy the inequality \(m \leq n - c + 1\). Then
\[ \max(1 - \alpha, \alpha) \leq \lambda_i(\tilde{M}) \leq 1 \quad i = n - m + 1, \ldots, n. \]

**Proof:** Since \(W, H\) and \(\tilde{M}\) are stochastic, their dominant eigenvalues are equal to 1. Note that \(W\) can be rearranged as
\[ W = \begin{bmatrix} W^1 & 0 & \cdots & 0 \\ 0 & W^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W^m \end{bmatrix} \]
where each \(W^i\) corresponds to agents of the same group, and \(H\) can be rearranged as
\[ H = \begin{bmatrix} \tilde{H}^C & 0 \\ 0 & 1 \end{bmatrix} \]
where \(\tilde{H}^C\) contains only the agents in \(V^C\). It can be directly inferred that \(W\) and \(H\) have \(m\) and \(n - c + 1\) dominant eigenvalues, respectively. Based on the eigenvalue inequalities of the sum of two Hermitian matrices stated in Theorem 4.3.1 of [13],
\[ \lambda_{i-j+1}((1 - \alpha)W) + \lambda_j(\alpha H) \leq \lambda_i(\tilde{M}) \leq 1 \quad j = 1 \ldots i \]
for \(i = 1 \ldots n\). From here, Theorem 4.1 can be obtained. \(\blacksquare\)

The theorem above implies that the dynamics (3) has \(m\) eigenvalues that are close or equal to 1 if \(\alpha\) is sufficiently small, which means that the number of groups

**Figure 3. Social network with \(m = 3\).**

**Figure 4. Social network with \(m = 5\).**
Figure 5. Result of model (1). (a) $m = 3, \rho = 0.3$ and (b) $m = 5, \rho = 0.3$.

Figure 6. Expected dynamics ($m = 3$). (a) $\rho = 0.15$ ($\alpha = 0.03$), (b) $\rho = 0.3$ ($\alpha = 0.06$) and (c) $\rho = 0.45$ ($\alpha = 0.09$).

Figure 7. Expected dynamics ($m = 5$). (a) $\rho = 0.15$ ($\alpha = 0.05$), (b) $\rho = 0.3$ ($\alpha = 0.1$) and (c) $\rho = 0.45$ ($\alpha = 0.15$).
Figure 8. Eigenvalues of (a) the social network with $m = 3$ and (b) the social network with $m = 5$.

is an important index for how interconnected groups reach consensus.

5. Numerical examples

In this section, we use numerical examples to demonstrate our proposed model and its expected dynamics. We consider a collection of 15 agents, whose initial opinions are given by

$$x(0) = \begin{bmatrix} 0.8675 & 0.2112 & 0.0941 & 0.5497 & 0.9685 \\ 0.7306 & 0.4245 & 0.1417 & 0.0792 & 0.3052 \\ 0.2776 & 0.0328 & 0.9055 & 0.8321 & 0.1522 \end{bmatrix}.$$  

The agents are organized into two different social networks, one with three groups (Figure 3) and the other with five groups (Figure 4). In both networks, groups contain one coordinator each. For the opinion updates, we use $\eta = 0.6$.

First, our proposed model (1) is applied on the given social networks. Figure 5 shows the resulting dynamics for both networks when $\rho = 0.3$. Despite the randomness of interactions, the two networks converge to the same consensus, which is the average of the initial opinions. The vertical lines denote the exchange of views between two agents.

For the expected behaviour of the model (3), we compare the results on the two social networks when $\rho$ is set to 0.15, 0.3, and 0.45. This corresponds to $\alpha$ values of 0.03, 0.06 and 0.09 for the first social network, and 0.05, 0.1 and 0.15 for the second social network. The results are shown in Figures 6 and 7. Similar to the gossip-based examples, the expected dynamics with varying configurations converged to the same limit. In these set of examples, the evolution of opinions are more noticeable, especially how opinions from the same group become close to one another even before they reach the consensus. Here, we can see the implication of Theorem 4.1, where the $m$ largest eigenvalues can signify the convergence of each group when $\alpha$ is significantly small. The corresponding eigenvalues of the given social networks are shown in Figure 8, which emphasizes the characterization of the $m$ largest eigenvalues as described in Theorem 4.1.

6. Conclusion

In this paper, we proposed a gossip-based model for achieving consensus in a social network composed of multiple groups. The model identifies the members of the network as either followers or coordinators. The presence of coordinators enables opinions to be transmitted across different groups, making it possible for the entire network to reach a consensus. As such, our proposed model can also be interpreted as a combination of intergroup and intragroup opinion dynamics.

Our research includes the analysis of the convergence properties of both the gossip-based model and its expected dynamics. We also demonstrated the model’s behaviour via simulations. Future works on this model can be done by extending it or performing other types of analyses.

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No potential conflict of interest was reported by the author(s).

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