Crossover from critical orthogonal to critical unitary statistics at the Anderson transition

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We report a novel scale-independent, Aharonov-Bohm flux controlled crossover from critical orthogonal to critical unitary statistics at the disorder induced metal-insulator transition. Our numerical investigations show that at the critical point the level statistics are definitely distinct and determined by fundamental symmetries. The latter is similar to the behavior of the metallic phase known from random matrix theory. The Aharonov-Bohm flux dependent crossover is characteristic of the critical ensemble.

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In analogy with classical phase transitions it is commonly believed that the critical behavior at the disorder induced metal-insulator transition (MIT) is determined by fundamental symmetries. This is not only the case for the critical exponent of the localization length and the dc-conductivity, but also for the statistical properties of energy eigenvalues and the wave functions. It was in particular the discovery of a new critical universality class of spectral correlations [1,2] that challenged our current understanding of the localization-delocalization transition.

Random matrix theory (RMT) [3] provides a general statistical description of the eigenvalue fluctuations and correlations in the metallic phase [1]. The corresponding universal energy level statistics only depend on fundamental symmetries of the system under consideration. For time reversal symmetry orthogonal (GOE) and symplectic (GSE) Gaussian ensembles are appropriate, whereas in case of broken time reversal symmetry the unitary (GUE) Gaussian ensemble describes the level statistics. In contrast to the metallic phase, the uncorrelated energy levels of the localized states in the insulating regime are characterized by the Poisson statistics, independent of the symmetry.

A critical statistics was not only found at the MIT in 3d in the presence of time reversal symmetry [1,2,3], but also for 2d symplectic systems [4,5], where spin rotational invariance is broken but time reversal symmetry is conserved. In both cases the critical level statistics for small energy level separations closely resemble the distributions corresponding to the metallic phases. The probability density distribution $P_c(s)$ of the normalized energy separation, $s = |E_n - E_{n+1}|/\Delta$, is still $P_c(s) \sim s^\beta$ for small $s$, where $\beta = 1$ or 4 in the orthogonal and symplectic case, respectively. Here, $E_n$ and $E_{n+1}$ are two successive eigenenergies and $\Delta$ is the mean level spacing. This reflects the energy level repulsion due to the strong overlap of the corresponding eigenstates. From these results at the critical point and the universal behavior in the metallic phase for all possible symmetries, one expects that $P_c(s) \sim s^\beta$ with $\beta = 2$ for the critical unitary case, when time reversal symmetry is broken.

Recently, however, it has been claimed [6] that the critical level statistics are independent of the presence or absence of time reversal symmetry. The authors based their assertion on numerical calculations of the nearest-neighbor energy spacing distribution $P_c(s)$ and the $\Delta_3$-statistics (see below). This unexpected result seems to support an observation [7] that the critical exponent $\nu$ of the localization length is not changed by a magnetic field. Because of the far reaching consequences for general theories concerning the MIT, a more careful and comprehensive study is indispensable.

In this letter we present results of numerical investigations which demonstrate that there is a system size independent crossover between the critical orthogonal and the critical unitary statistics which is controlled by the magnitude of an applied Aharonov-Bohm (AB) flux. Besides the AB-flux model, we investigate two further mechanisms for breaking time reversal symmetry: a homogeneous magnetic field and a spatially randomly fluctuating magnetic flux with zero mean (random flux model). The same limiting critical unitary level spacing statistics $P_u(s)$ is found in all three cases. Our results also show unambiguously that there exists a critical statistics for 3d unitary disordered systems that is clearly different from the critical orthogonal one.

The dynamics of non-interacting electrons in a 3d disordered system in the presence of processes that break the time reversal symmetry can be investigated by using the Anderson Hamiltonian

$$H = \sum_r \epsilon_r |r\rangle\langle r| + \sum_{r \neq r'} V_{r,r'} |r\rangle\langle r'|.$$  (1)

The vectors $r$ denote the sites of a simple cubic lattice with lattice constant $a$, and periodic boundary conditions are applied in all directions. The uncorrelated random energies $\epsilon_r$ are distributed with constant probability within the interval $-W/2 \leq \epsilon_r \leq W/2$, where $W$ denotes the magnitude of the disorder.
The non–diagonal transfer matrix elements between nearest neighbors, \( V_{r,r'} \), contain the symmetry breaking term: (i) for a constant magnetic field \( B \), \( V_{r,r'} = V \exp(i2\pi n) \), if \( r - r' = \pm e_y \), with \( r \cdot e_y = n \), and \( V_{r,r'} = V \) else. The magnetic field is chosen to be commensurate with the lattice. (ii) for a random magnetic flux \( V_{r,r'} = V \exp(i2\pi \theta_{r,r'}) \), where the random phases \( \theta_{r,r'} \) are drawn from an interval \([-\theta_0/2, \theta_0/2] \) with uniform probability \( p(\theta_{r,r'}) = 1/\theta_0 \). (iii) for the Aharonov-Bohm flux model \( V_{r,r'} = V \exp(\pm i2\pi \phi_0/L) \), \( r - r' = \pm e_j \), and \( j = x,y,z \), where the AB-flux \( \phi_{AB} = \phi_0 \) is given in units of the flux quantum \( \phi_0 = h/e \), and \( L \) is the linear size of the system.

In the presence of time reversal symmetry, the critical disorder at \( (E/V = 0) \) corresponds to \( W_c/V \approx 16.4 \). It separates localized \((W > W_c)\) from metallic states \((W < W_c)\). We find that this critical disorder does not change when an AB-flux is applied. In contrast, \( W_c \) increases with increasing strength of a magnetic field.

We start to present our results with case (i), a 3d-Anderson model with applied strong constant magnetic field corresponding to \( \alpha = eB \pi^2 \hbar \approx 1/5 \) flux quanta per plaquette. The critical disorder is \( W_c/V = 18.1 \). No dependence of the level spacing distribution on the system size could be detected within the error bars for \( L/a = 5,10,20 \) as expected for the critical ensemble. Similar results have also been obtained for magnetic field strengths corresponding to \( \alpha = 1/4,1/10 \) and 1/20.

Fig. 1 displays the critical level spacing distribution \( P^{(c)}(s) \) in the presence of a constant magnetic field in comparison with the critical data without magnetic field. The difference in the heights of the maxima and in the small-\( s \) behavior is clearly visible. This difference is much larger than the small fluctuations in the numerically obtained distributions. The small-\( s \) behavior is quadratic as expected for the critical unitary case, in contrast to the linear increase in the critical orthogonal ensemble. We tried to fit a suggested \( \text{GUE} \) critical level spacing distribution \( P_{\text{KL}}(s) = A s^\beta \exp(-Bs^\gamma) \) to our data with \( \gamma = 1 + 1/(\nu d) \), where \( \nu \) and \( d \) are the critical exponents of the localization length and the Euclidean dimension, respectively. While it is possible to fit the bulk of our data with \( \gamma \approx 1.2 \) which leads to \( \nu \approx 1.66 \), the deviation for larger \( s \) is obvious. The large-\( s \) behavior, which is shown on a logarithmic scale in the inset of Fig. 1, is best fitted by an exponential decay \( \sim \exp(-\kappa s) \), with \( \kappa \approx 1.9 \) having approximately the same value as in the critical orthogonal case \( \text{GUE} \). A similar exponential decay with \( \kappa \approx 4 \) has been reported for the 2d symplectic system \( \text{GUE} \).

The \( \Delta_3 \)-statistics is defined as the least square deviation of the integrated density of states, \( N(E) \), from a linear behavior, averaged over an energy range \( k \) times the mean level spacing \( \Delta \) centered at \( E \)

\[
\Delta_3(k) = \left\langle \min_{A,B} \frac{1}{R} \int_{-k/2}^{k/2} (N(E + \varepsilon) - A - Be)^2 \, d\varepsilon \right\rangle, \tag{2}
\]

where \( \left\langle \right\rangle \) denotes an ensemble average. The result shown in Fig. 2 for an energy interval around the critical point is again independent of the system sizes studied. The rigidity of the spectrum is enhanced in the critical unitary as compared to the critical orthogonal ensemble. The difference between the two critical curves is significantly larger than the error bars, but smaller than the difference between GOE and GUE. The influence of the magnetic field to make the spectrum more rigid in the

![FIG. 1. The critical unitary energy spacing distribution \( P^{(u)}(s) \) in the presence of a magnetic field \( (\alpha = 1/5) \) for systems of size \( L/a = 5 \) (o), \( L/a = 10 \) (s), and \( L/a = 20 \) (o). For comparison, the critical orthogonal \( P^{(c)}(s) \) and the GUE result are also shown. The inset shows the large \( s \) behavior which can be fitted by an exponential decay \( \exp(-\kappa s) \) with \( \kappa = 1.87 \) (dotted line).](image)
metallic regime, is weakened at the critical point due to stronger disorder. Our results for $P(s)$ and the $\Delta_3$-statistics clearly indicate the existence of a critical unitary statistics in the case of an applied strong magnetic field.

We now turn to the Aharonov-Bohm flux model (iii) for which the absence of a distinct critical unitary statistics has been asserted [9]. Here, we have to distinguish whether a magnetic flux is applied along one, two or three perpendicular directions. Depending on the magnitude of the flux and the number of flux directions we observe different scale independent level statistics for each set of parameters. To illustrate this unexpected behavior, the change of the maximum $P_m(\phi)$ of the critical level spacing distributions is shown in Fig. 3, where the normalized maximum $\Gamma_{\text{max}}(\phi) = (P_m(\phi) - P^0_m)/(P_m^0 - P^0_m)$ is plotted versus the Aharonov-Bohm flux $\phi$. We find a smooth crossover starting from the maximum of the critical orthogonal $P_m^0$ to the maximum of the critical unitary ensemble $P_m^u$ when the flux is increased along two or three directions. Hence, in the limit of large flux there exists the same critical unitary level statistics $P_c^u(s)$ known from the random flux model and from the strong magnetic field case also for the 2- and 3-component AB-flux model. For one flux line in a single direction, however, this critical unitary distribution is not reached. Increasing the magnetic flux beyond $\phi = 0.25$ decreases $P_m(\phi)$ because of a $\phi_0/2$-periodicity in the flux: at $\phi = 0.5$ all phases in $V_{r,r'}$ are equal to unity and the corresponding Hamiltonian is real. This behavior is known as “false T-breaking” [19]. In all cases, within the error bars, the results are independent of the system sizes investigated. We find a similar crossover also for the $\Delta_3$-statistics.

\[ l_b^2 = \frac{\hbar}{(eB)} = \frac{\alpha^2}{2\pi(1/L^3 \sum_r (\Phi_r^p/\phi_0)^2)^{1/2}}, \]  

where the magnetic flux $\Phi_r^p$ through a plaquette at site $r$, e.g. within the $xy$-plane, is calculated from the corresponding random phases $\Phi_r^p = \phi_0(\theta_{r+e_x} + \theta_{r+e_y} + \theta_{r+e_x+e_y} + \theta_{r+e_y}).$ The total flux per cross-section perpendicular to each of the three lattice axis is $\Phi_T = \sum_r \Phi_r^p = 0.$

![ Diagram showing $\Delta_3$-statistics for different models. ]

**FIG. 2.** The $\Delta_3$-statistics of the critical unitary ensemble ($W_c/V = 18.1, \alpha = 0.2$) for system sizes $L/a = 5$ (○), $L/a = 10$ (⋆), and $L/a = 20$ (○) in comparison with the critical orthogonal case, $L/a = 20$ (●). The known behavior the system size has to be taken to infinity as $B \to 0$.

In order to investigate whether or not a weak magnetic field changes this behavior one must consider a different model, because due the requirement of commensurability the system size has to be taken to infinity as $B \to 0$. Therefore, we used the random flux model (ii) which allows to study the effect of weak local magnetic fields with global magnetic field being zero. The strength of these local magnetic fields can be tuned by varying $\theta_0$, the width of the interval from which the random phases are chosen. We find the same critical unitary distribution $P_c^u(s)$ as in the strong magnetic field case, provided the condition $L/l_b \gg 1$ is fulfilled. The magnetic length $l_b$ which serves as a measure for the strength of the random local magnetic fields can be defined in analogy with the usual magnetic length $l_B^2 = \hbar/(eB) = a^2/(2\pi\alpha)$. It is related to the standard deviation, $b$, of the local magnetic fields,

\[ l_b^2 = \frac{\hbar}{(eB)} = \frac{\alpha^2}{2\pi(1/L^3 \sum_r (\Phi_r^p/\phi_0)^2)^{1/2}}, \]  

where the magnetic flux $\Phi_r^p$ through a plaquette at site $r$, e.g. within the $xy$-plane, is calculated from the corresponding random phases $\Phi_r^p = \phi_0(\theta_{r+e_x} + \theta_{r+e_y} + \theta_{r+e_x+e_y} + \theta_{r+e_y}).$ The total flux per cross-section perpendicular to each of the three lattice axis is $\Phi_T = \sum_r \Phi_r^p = 0.$
The small-s behavior of the critical $P(s, \phi)$ for various values of the AB-flux applied in all three directions is shown in Fig. 4. While for small $\phi$ the overall deviation of the level spacing distribution from the critical orthogonal curve is hardly visible, the small-s dependence is clearly quadratic. This holds for all fluxes considered, although the range of $s$ values showing the $s^2$-behavior decreases with decreasing flux. Only for $\phi = 0$ the linear relation, $P(s) \sim s$, characteristic of the orthogonal ensemble, is obtained.

In the metallic phase, only the basic symmetry determines the characteristics of the spectral fluctuations even though the mechanisms to break the time reversal symmetry differ. In a magnetic field each closed electron-orbit not lying in a plane parallel to the magnetic field direction picks up a magnetic flux, which is in contrast to the smaller set of those closed trajectories with winding number larger than zero, e.g. trajectories that circle round the single flux line in the AB-flux model. The latter situation was shown to be the reason for a flux controlled GOE $\rightarrow$ GUE crossover in Aharonov-Bohm chaotic billiards [21]. Nevertheless, this crossover becomes abrupt in the semiclassical limit $L \rightarrow \infty$. A similar discontinuous transition is known from RMT [3] for infinite order matrices and also from numerical studies on metallic samples [21] where the conductance diverges with $L$.

Our results for the MIT show that the critical unitary statistics depend on how the time reversal symmetry is broken. This is due to the finite critical conductance in combination with the system size independent Aharonov-Bohm flux $\phi_{AB} = \phi h/e$. Hence, the crossover becomes scale invariant at the metal-insulator transition point. This is different in the magnetic field case, because here the magnetic flux $\phi_a = BL^2$ increases with system size.

In conclusion, we have investigated the energy level statistics at the critical point of a 3d Anderson model when time reversal symmetry is broken by either a constant magnetic field, a spatially fluctuating magnetic flux, or an Aharonov-Bohm flux. A critical unitary level spacing distribution has been found that is distinct from the critical orthogonal one. We further showed that there exists a scale invariant AB-flux controlled crossover regime so that the critical level statistics depend on how the time reversal symmetry is broken.

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