Logarithmic Chow theory

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Abstract

Log geometry was introduced by Kato in [Igu89] based on ideas of Illusie and Fontaine, who expanded on it in [Ill94] to control degenerations of varieties. This theory enhances a scheme with a sheaf of monoids over the structure sheaf. Many of the notions of algebraic geometry, such as smoothness and flatness, can be copied across to log geometry, as explained most vividly by Olsson in [Ols03]. This led us to ask whether other geometric notions can be transferred to this setting.

In this paper we describe a refined Chow theory for log varieties. This is motivated by the construction by Abramovich, Chen, Gross and Siebert of log Gromov-Witten invariants in [GS13]. This produces a dimension graded family of Abelian groups supporting a push-forward and pull-back along proper and log flat morphisms respectively.

Chow groups and rings have a central role in many areas of Algebraic Geometry. Classical enumerative problems find their most succinct statements in this language whilst problems in the Minimal Model Program require an intimate knowledge of the positivity of various intersections. The construction by Fulton and MacPherson of Chow cohomology in terms of bivariant theory explained many features of the theory which previously had been mysterious. This work was continued by Kresch to extend the theory to more and more unfamiliar worlds, culminating in a theory for Artin stacks.

Log geometry is an extension of algebraic geometry to include degeneration data. Most importantly the notion of smoothness generalises well to this setting, certain log schemes become smooth despite their underlying schemes not being classically smooth.

The key concept we exploit here is the categorical definition of a monomorphism. Monomorphisms of log schemes were first studied in work of Mochizuki [Moc15] although the relation to this work is unclear. The driving force behind this paper is that a cycle on a scheme should not be thought of as a closed immersion,
but rather as a proper monomorphism. That the two concepts are equivalent for schemes was proven by Grothendieck in [DG67]. In this paper we introduce the relevant concepts and construct the log Chow groups of a log scheme. In future work we will construct the appropriate bivariant theories and construct an intersection pairing for log smooth schemes.

An intuitive example is that of $\mathbb{P}^1$ with its toric log structure. The cycles classes here can be thought of as genuine rationally equivalent cycles where we avoid those parts of the rational equivalence where the log structure changes. To begin with the only possible zero cycles are points with trivial log structure. There is precisely one such class, any two trivial points are rationally equivalent. Then there are two possible types of one dimensional cycles, strict maps from the standard log point to the toric fixed points and the entirety of $\mathbb{P}^1$. The points are rigid since any nearby point has trivial log structure, the whole of $\mathbb{P}^1$ is rigid for dimension reasons. Therefore we find that the log Chow groups are given by $A^0_\dagger(\mathbb{P}^1) \cong \mathbb{Z}$ and $A^1_\dagger(\mathbb{P}^1) \cong \mathbb{Z}_3$.

For now let us introduce a motivating philosophy and explain the striking features it uncovers:

There are interesting proper log étale maps generically of degree one.

Examples of which include blowups along toric ideals of toric varieties. We call such morphisms log refinements. We will see that such blow ups are monomorphisms of log schemes, despite the fact that geometrically they potentially send whole divisors to a point. In fact we believe that these should produce isomorphisms on any sensible log geometric construction you make.

**Example 1.** We consider $\mathbb{A}^2$ with its toric log structure. Let $\pi : X \to \mathbb{A}^2$ be a weighted blow up of the origin and $i : E \to X$ be the strict inclusion of the exceptional curve. Then the composite $\pi i : E \to \mathbb{A}^2$ is a proper monomorphism, the logarithmic generalisation of a closed immersion, and so defines a cycle on $\mathbb{A}^2$.

Another miraculous part of this definition is that the category of log schemes has a form of image factorisation:

**Theorem 1.** Let $f : X \to Y$ be a morphism of log schemes. Then there exists refinements $\tilde{f} : \tilde{X} \to \tilde{Y}$ lifting $f$ and a factorisation $\tilde{p} : \tilde{X} \to \tilde{Z}$, $\tilde{i} : \tilde{Z} \to \tilde{Y}$ such that $\tilde{f} = \tilde{i} \tilde{p}$ and $\tilde{i}$ is a proper monomorphism. If $\tilde{q} : \tilde{X} \to \tilde{W}$ and $\tilde{j} : \tilde{W} \to \tilde{Y}$ is another factorisation of $\tilde{f}$ with $\tilde{j}$ a proper monomorphism then there is a proper monomorphism $\tilde{k} : \tilde{Z} \to \tilde{W}$ such that $\tilde{i} = \tilde{j} \tilde{k}$.

The choice of refinement dissappears as soon as one inverts these refinement morphisms.
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**Log Geometry**

**Basic definitions**

Let us begin by recalling the definitions of log geometry. Log varieties and log schemes were introduced in [Igu89] to encapsulate degeneration data of schemes in such a way as to make relatively nice singular schemes behave as if they are smooth. The data is encoded in a sheaf of monoids on the space as follows.

**Definition 1.** A pre-log scheme $X$ is a pair $(X, \alpha_X : \mathcal{M}_X \to \mathcal{O}_X)$ of a scheme $X$ (the underlying scheme), a sheaf of monoids on $X$ in the étale topology $\mathcal{M}_X$ and a homomorphism $\alpha_X$ of sheaves of monoids from $\mathcal{M}_X$ to the multiplicative monoid $\mathcal{O}_X$. Saying that this is a log scheme means that the restriction of $\alpha_X$ to the inverse image of $\mathcal{O}_X \times X$ is an isomorphism.

To any pre-log scheme there is an associated log structure given by taking the amalgamated sum

$$\mathcal{M}_X \oplus_{\alpha_X^{-1} \mathcal{O}_X^\times} \mathcal{O}_X^\times$$

A morphism of log schemes $\phi : X \to Y$ is a pair $(\phi, \phi^\#)$ with $\phi : X \to Y$ a morphism of schemes and $\phi^\#$ fitting into a commutative diagram:

$$\begin{array}{ccc}
\phi^{-1}\mathcal{M}_Y & \xrightarrow{\phi^\#} & \mathcal{M}_X \\
\downarrow & & \downarrow \\
\phi^{-1}\mathcal{O}_Y & \xrightarrow{\phi^\ast} & \mathcal{O}_X \\
\end{array}$$

Given a morphism of schemes $\phi : X \to Y$ and the structure of a log scheme on $Y$ the inverse image sheaf $\phi^{-1}(\mathcal{M}_Y)$ on $X$ is naturally a pre-log structure. We call the associated log structure the pull-back log structure, written $\phi^*\mathcal{M}_Y$. A morphism of log schemes $\phi : X \to Y$ is strict if the natural map $\phi^\# : \phi^*\mathcal{M}_Y \to \mathcal{M}_X$ is an isomorphism.

There is an interesting invariant of a log scheme which contains the combinatorial data not seen by classical geometry called the ghost or characteristic sheaf and is the quotient $\mathcal{M}_X/\mathcal{O}_X^\times$. 
Log structures are often used to study degenerations of pairs \((X, D)\), so we now explain how to construct a log structure from such a pair.

**Example 2.** Let \(X\) be a smooth scheme and \(D\) be a simple normal crossings divisor. Write \(U\) for the complement of \(D\) in \(X\) and let \(j\) be the inclusion of \(U\) into \(X\). Let \(\mathcal{M}_X\) be the sheaf \(j_!\mathcal{O}_U^* \cap \mathcal{O}_X\), and \(\alpha_X\) the inclusion. This defines a log scheme \(X\), the divisorial log structure on \(X\).

General log schemes exhibit pretty wild behaviour so we restrict those we consider. The objects we want to study are locally modelled on toric varieties together with their toric boundary, which has another intrinsic definition which we give below.

**Example 3.** Let \(P\) be a finitely generated integral monoid. Take \(\mathcal{P}\) to be the constant sheaf on \(\text{Spec} \ k[P]\), which maps to \(\mathcal{O}_{\text{Spec} \ k[P]}\) via \(p \mapsto z^p\). Let \(\text{Spec} \ ^{\dagger}k[P]\) denote the corresponding log scheme, the toric log scheme, whose log structure is referred to as the toric log structure. If \(T\) is any toric scheme then we can construct a log structure on \(T\) just by taking charts and glueing the toric log structures together. The log structure is non-trivial only over the big torus contained inside \(T\).

Let us describe the ghost sheaf on \(\mathbb{A}^2 \cong \text{Spec} \ k[\mathbb{N}^2]\) with its toric log structure. On the big torus the toric monomials are certainly invertible and so the ghost sheaf has trivial stalks at these points. Now along the two toric strata \(z^{(1,0)} = 0\) and \(z^{(0,1)} = 0\) those same monomials are not invertible. Since an element of \(\mathcal{M}_{\mathbb{A}^2}\) in a neighbourhood of one of these strata takes the form \(z^{(a,b)}f\) for \(f\) invertible and either \(b = 0\) or \(a = 0\) there is a contribution of an \(\mathbb{N}\) to the ghost sheaf. Drawing the stalks of the ghost sheaf we obtain the following picture:

```
 N

 N^2    N
```

Now the actual objects we want are locally modelled on this construction.

**Definition 2.** A log scheme \(X\) is called fine saturated (abbreviated as \(fs\)) if étale locally on \(X\) there are strict morphisms to toric log schemes.
A surprisingly important example is the standard log point, $\text{Spec } k^\dagger = (\text{Spec } k, \alpha : k^\times \oplus \mathbb{N} \to k)$ where $\alpha$ sends $(a, 0)$ to $a$ and $(a, 1)$ to 0. This is isomorphic to the origin inside $\mathbb{A}^1$ with the restriction of the toric log structure.

Now for all but the very final section on Kato-Nakayama spaces we will only work with fs log schemes and so it is worth a little revision of history: from now on whenever we refer to log schemes we mean fs log schemes. The category of fs log schemes has many fantastic properties.

**Theorem 2** (Log schemes have fibre products). The category of fs log schemes possesses fibre products. Given a diagram $X \to Z \leftarrow Y$ there is a canonical morphism $X \times_Z Y \to X \times_Z Y$ which is the composition of a finite surjective map followed by a closed immersion. If the map $X \to Z$ is strict then the above canonical morphism is an isomorphism and the induced map $X \times_Z Y \to Y$ is strict.

**Proof.** This is Corollary 2.1.6 of [Ogu].

Let us give an example where the above map is not an isomorphism.

**Example 4.** Let $\mathbb{A}^2_\dagger$ denote the plane with its toric log structure. There is a toric blowup of this, blowing up the origin, which we denote $\pi : \text{Bl} \to \mathbb{A}^2_\dagger$. Choose two different points of the exceptional curve $P \cong Q \cong (\text{Spec } k, k^\times \oplus \mathbb{N})$. We claim that the product $P \times_{\mathbb{A}^2_\dagger} Q$ is empty, but the product of the underlying schemes is a point.

To show that the product is empty note that it would be enough to prove that it doesn’t have any $k$-points. Now any fs $k$-valued point would have log structure $k^\times \oplus Q$, which by choosing an element we can reduce to having log structure $k^\times \oplus N$. This reduces the problem to one of algebra, what is the push-out of the diagram along the induced map:

\[
\begin{array}{ccc}
\mathbb{N}^2 & \longrightarrow & \mathbb{N} \\
\downarrow & & \\
\mathbb{N} & & 
\end{array}
\]

This push-out is zero, so the sections coming from $\mathbb{N}^2$ map to zero. But then the maps to $k$ do not commute and there is no such point.

The geometry of log schemes is highly related to the combinatorial data of the log structure and hence to the combinatorics of toric varieties. One way to organise this data is through fans and refinements of fans. Suppose we have lattices $M = \mathbb{Z}^n$ and $N = M^\vee$ contained inside vector spaces $M_\mathbb{R} = M \otimes \mathbb{R}$ and $N_\mathbb{R} = M_\mathbb{R}^\vee$. A cone $\sigma$ inside $N$ is a subset consisting of positive linear combinations of a finite collection
of rational vectors \( \phi_1, \ldots, \phi_n \). Such a cone is convex and rational polyhedral and has an associated dual cone given by

\[
\sigma^\vee = \{ v \in M_\mathbb{R} \mid \phi(v) \geq 0 \forall \phi \in N_\mathbb{R} \}.
\]

Associated to \( \sigma^\vee \) is the monoid \( \sigma^\vee \cap M \) and hence a monoid ring \( k[\sigma^\vee \cap M] \). After passing to \( \text{Spec} \ k[\sigma^\vee \cap M] \) we obtain a scheme and thus a covariant functor from cones and inclusions of cones to toric varieties and morphisms of toric varieties. This is not an equivalence as the image consists only of affine toric varieties. To include the glueing conditions we introduce the notion of a fan:

**Definition 3.** A fan \( \Sigma \) inside \( N \) is a simplicial complex of cones. It is complete if the collection covers all of \( N \). A morphism of fans \( f : \Sigma_1 \to \Sigma_2 \) is a linear map on underlying vector spaces mapping each cone of \( \Sigma_1 \) into a cone of \( \Sigma_2 \). There is a functor from the category of fans to toric varieties given by gluing together the dual fans along their natural inclusions.

According to [CLS11] this produces an equivalence of categories between the category of toric varieties and toric morphisms and the category of fans and their morphisms. Toric blowups are given by subdividing the corresponding fan. In particular there is a canonical subdivision, the barycentric subdivision of each cone in a fan.

To relate this to logarithmic geometry requires that we finesse some details. There is a direct way to move between a toric variety and its fan given by the Artin fan of [Ols03]. Let \( T \) be a toric variety with big torus \( G_m^n \). The action of the big torus allows us to take the stack quotient \( A(T) = [T/G_m^n] \). This is an irreducible Artin stack of dimension zero and there is a canonical morphism \( T \to A(T) \). The fan then is a topological realisation of the Artin fan. This is sufficiently functorial that given a log variety \( X \) one can take its Artin fan \( A(X) \) and again there is a canonical strict morphism \( X \to A(X) \). Refinements of the Artin fan correspond to log refinements of the log structure, blow ups induced by monoid ideals of the log structure.

**Lemma 1.** Let \( X \) be a log variety. Then there is a canonical log refinement given by barycentric subdivision of the Artin fan. This introduces a one dimensional cone inside each cone of the fan and we call the corresponding refinement the barycentric refinement.

**Proof.** This is Section 4.3 of [ACMW14].

One might think that strict closed immersions should form the basis of cycle groups in our theory, and whilst this does not capture all the possible data it is...
good intuition to bear in mind. Also of import is that given an étale morphism $U \to X$ there is an induced log structure on $U$ given by pull-back and the induced log morphism $U \to X$ is strict. We call such a morphism strict étale.

**Geometric properties of morphisms**

We begin by recalling the extension of étale and smooth morphisms to the world of log geometry. These two concepts formally satisfy some form of lifting criterion which we do not mention here.

**Definition 4.** A morphism $f : X \to Y$ is said to be log flat (resp. étale, smooth) if étale locally on $X$ and $Y$ there are charts for the log structure

$$
\begin{array}{ccc}
X & \longrightarrow & \text{Spec}_{k[P]} \\
\downarrow & & \downarrow \\
Y & \longrightarrow & \text{Spec}_{k[Q]}
\end{array}
$$

such that the map $\text{Spec}_{k[P]} \to \text{Spec}_{k[Q]}$ is induced by some morphism of monoids $m : Q \to P$ with the following properties:

1. $m$ is an inclusion.
2. The morphism $Q^{gp} \to P^{gp}$ has no special properties (resp. is an inclusion with finite cokernel of order coprime to the characteristic of $k$ for both).
3. The induced morphism $X \to Y \times_{\text{Spec} k[Q]} \text{Spec} k[P]$ is flat (resp. étale, smooth).

We say that a log étale morphism $\pi : X \to Y$ is a log refinement if there is an étale cover of $Y$ by $U_i \to Y$ and charts $U_i \to \text{Spec}_{k[Q_i]}$ with the following property. There are Zariski open subsets $U_{ij}$ of $X \times_Y U_i$ along with charts $U_{ij} \to \text{Spec}_{k[P_{ij}]}$ and monoid homomorphism $\iota : Q_i \to P_{ij}$ such that $\iota^{gp} : Q_i^{gp} \to P_{ij}^{gp}$ is an isomorphism and the canonical map $U_{ij} \to \text{Spec}_{k[P_{ij}]} \times_{\text{Spec}_{k[Q_i]}} U_i$ a strict Zariski open immersion. This includes strict Zariski open immersions, toric blowups and pullbacks of these.

There is a subtlety in the definition of log flatness which requires some explanation. The definition only states that for some choice of charts for the log structure is the induced morphism flat, not for every chart. Indeed one can easily construct charts where the induced morphism fails to be flat. The theory of neat charts of [Ogu] provides charts well suited to our purposes. Ogus defines a chart $\phi : X \to \text{Spec}_{k[P]}$ to be neat at a point $x$ if $x$ maps to the origin
inside $\text{Spec}_{\log} k[P]$. In Theorem III 1.2.7 he proves the existence of such charts compatible with morphisms. To be precise suppose that $f : X \to Y$ is a morphism and $x$ and $y$ points with $f(x) = y$. Then there are neat charts at $x$ and $y$ given by morphisms $\phi : X \to \text{Spec}_{\log} k[Q]$ and $\psi : Y \to \text{Spec}_{\log} k[P]$ and a morphism $a : P \to Q$ such that the following diagram commutes:

\[
\begin{array}{rcl}
X & \xrightarrow{\phi} & \text{Spec}_{\log} k[Q] \\
\downarrow{f} & & \downarrow{a} \\
Y & \xrightarrow{\psi} & \text{Spec}_{\log} k[P]
\end{array}
\]

In Theorem IV 4.1.7 Ogus proves that under these conditions, if $f$ is log flat, then the induced map $X \to Y \times_{\text{Spec}_{\log} k[Q]} \text{Spec}_{\log} k[P]$ is flat in a neighbourhood of $x$. We call such compatible charts, neat at $x$ and $y$ and flat near $x$ fabulous.

In the classical setting étale morphisms preserve dimension. We are inspired by this to make the following definition:

**Definition 5.** Let $X$ be a log scheme. The logarithmic dimension or simply dimension of $X$ is defined as follows. If $X$ has generically non-trivial log structure then the dimension is defined to be one more than the supremum of $\dim(U)$ for $U \to X$ a log étale morphism:

$$\logdim(X) = \sup_{\xrightarrow{Z \to X \text{ log étale}}} \dim Z + 1$$

If $X$ has generically trivial log structure we define the dimension to be the maximum of $\dim(X)$ and the previous dimension restricted to the locus where the log structure is non-trivial.

Of course to know that this definition is finite and useful we need to have a method to calculate it. This falls to the following lemma which explicitly describes the log dimension.

**Lemma 2** (An explicit formula for the log dimension). Let $X$ be a log scheme and $\zeta$ a scheme theoretic point. The potential log dimension of $\zeta$ is defined by

$$\dim \{ \zeta \} + \text{rk}(\mathcal{M}_{X,\zeta})$$

where the rank makes sense since this is a finite rank monoid. We define the potential log dimension of $X$ to be the supremum of the potential log dimensions of all scheme theoretic points of $X$. Then the log dimension of $X$ equals the potential log dimension of $X$. 
Proof. First we reduce to the case where \( X \) has globally non-trivial log structure. This result is trivial on the locus where \( X \) has trivial log structure and so we may pass to the closed locus on which it does not. Let \( \pi : Z \rightarrow X \) be a log étale morphism. We begin by proving that \( \dim(Z) \) is less than the potential log dimension.

Take \( U \subset X \) a connected component of a strata of the log structure on \( X \) such that the fibre product \( Z \times_X U \) is of dimension \( \dim Z \). Let \( \zeta \) be the generic point of the closure \( \overline{U} \) and \( \zeta \) the corresponding geometric point. The product \( \overline{\zeta} \times_X Z \) is a log étale cover of \( \zeta \), so is isomorphic to a disjoint union of toric varieties given by refining \( \overline{\mathcal{M}}_{X,\zeta} \) and possibly passing to a sub-lattice, thus is of dimension at most \( \text{rk}(\overline{\mathcal{M}}_{X,\zeta}) - 1 \). This shows that the relative dimension of \( Z \times_X U \) over \( U \) is at most \( \text{rk}(\overline{\mathcal{M}}_{X,\zeta}) - 1 \). By assumption this shows that \( \dim Z \) is at most \( \dim U + \text{rk}(\overline{\mathcal{M}}_{X,\zeta}) - 1 \).

Now we construct a log étale cover of \( X \) of dimension one less than the potential log dimension. To do this we choose a coherent refinement of the fans for \( \overline{\mathcal{M}}_{X,\zeta} \) over \( \zeta \) such that each refinement contains a one-cell, for example the barycentric subdivision of each fan. This gives a log refinement morphism and the generic fibre over \( \zeta \) is a toric variety of dimension \( \text{rk}(\overline{\mathcal{M}}_{X,\zeta}) - 1 \) with generic log structure \( \mathbb{N} \). Therefore the dimension of this cover is at least \( \dim \{ \zeta \} + \text{rk}(\overline{\mathcal{M}}_{X,\zeta}) \).

Neither of these definitions of log dimension is obviously invariant under log étale morphisms, but in fact they are. In this argument properness is not necessary, only log dominance. Once we have defined the latter notion the same proof applies.

**Lemma 3 (Dimension is log étale invariant).** Let \( f : X \rightarrow Y \) be a proper log étale morphism of log schemes. Then \( \logdim X = \logdim Y \).

**Proof.** From the definition it is clear that the log dimension of \( X \) is at most the log dimension of \( Y \), so suppose that we have \( U \rightarrow Y \) with \( \overline{U} \) of dimension \( n \). Then the morphism \( U \times_Y X \) is non-trivial, log étale and proper over \( U \) and hence the dimension of \( U \times_Y X \) is at least the dimension of \( U \). This is enough to prove that the dimension of \( X \) is equal to the dimension of \( Y \).

In the classical world flat morphisms between varieties are of constant relative dimension. Now that we have a notion of logarithmic dimension we can ask whether log flat morphisms have a well defined relative dimension. The following lemma shows that this is indeed a well formed analogy.

**Definition 6.** Let \( f : X \rightarrow Y \) be a morphism of log schemes. We say that \( f \) has constant relative log dimension \( d \) if for all maps \( i : Z \rightarrow Y \) with \( Z \) a point one has
Remark 1. This is analogous to the classical definition of having constant relative dimension in this context. By applying the formula for potential log dimension one can show that if \( f : X \to Y \) and \( g : Y \to Z \) have constant relative dimensions \( d \) and \( e \) then \( gf \) has constant relative log dimension \( d + e \).

Lemma 4 (Log flat morphisms have equidimensional fibres). Let \( f : X \to Y \) be a log flat morphism between log varieties with connected underlying varieties. Then \( f \) has constant relative log dimension \( d \). We say that \( d \) is the relative log dimension of \( f \).

Proof. We begin by proving the result for the special case of a strict point mapping to the origin of \( \text{Spec}_\log k[Q] \) and a morphism \( \text{Spec}_\log k[P] \to \text{Spec}_\log k[Q] \) induced by an inclusion of monoids \( Q \to P \). Let \( T \) be a torus invariant stratum of \( \text{Spec}_\log k[P] \) associated to a face \( \tau \) of \( P^\vee \). We claim that \( \log\dim T = \text{rk}(P_{gp}) \). This follows from the formula for the potential log dimension: \( T \) as a scheme is dimension \( \text{codim} \tau + \text{dim} \tau + \text{rk}(P_{gp}) = \text{rk}(P_{gp}) \). The fibre over the origin is the union of such strata and by the same techniques is of log dimension \( \text{rk}(Q_{gp}) \). Therefore the relative log dimension is \( \text{rk}(P_{gp}) - \text{rk}(Q_{gp}) \).

Now suppose that \( i : p \to \text{Spec}_\log k[Q] \) is not strict but still maps to the origin. Suppose that the log structure on \( p \) is \( k^\times \oplus M \) for some monoid \( M \). Then \( p \) is naturally isomorphic to the origin inside \( \text{Spec}_\log k[M] \) and there is an induced morphism \( \text{Spec}_\log k[M] \to \text{Spec}_\log k[Q] \) coming from the map of monoids. The fibre product of fs toric log varieties is given by taking the fs pushout of the associated monoids. We therefore have the following diagram:

\[
\begin{array}{ccc}
\text{Spec}_\log k[M \oplus_Q P] & \longrightarrow & \text{Spec}_\log k[P] \\
\downarrow & & \downarrow \\
\text{Spec}_\log k[M] & \longrightarrow & \text{Spec}_\log k[Q]
\end{array}
\]

where the fibre product we wish to calculate is the fibre over the origin inside \( \text{Spec}_\log k[M] \). Using the canonical isomorphism \( X \times_Y Z \times_Z W \cong X \times_Y W \) it is enough to take the fibre product over the morphism \( \text{Spec}_\log k[M \oplus_Q P] \to \text{Spec}_\log k[M] \). By the previous paragraph this is the union of toric strata all of relative log dimension \( \text{rk}((M \oplus_P Q)^{gp}) - \text{rk}(M^{gp}) \). Now by the construction of fs pushout one has \( \text{rk}((M \oplus_P Q)^{gp}) - \text{rk}(M^{gp}) = \text{rk}(Q^{gp}) - \text{rk}(P^{gp}) \). Therefore such morphisms are of constant relative log dimension \( \text{rk}(Q^{gp}) - \text{rk}(P^{gp}) \).
Now we suppose that we are in the case where étale locally on $X$ and $Y$ we have charts around $y$

\[
\begin{array}{c}
X \longrightarrow \text{Spec}_{\log} k[P] \\
\downarrow \\
Y \longrightarrow \text{Spec}_{\log} k[Q]
\end{array}
\]

with $y$ mapping to the origin and $X \cong Y \times_{\text{Spec}_{\log} k[Q]} \text{Spec}_{\log} k[P]$. Since étale morphisms preserve dimension we may replace $X$ and $Y$ by these étale open subsets. Then $y \times_Y X \cong y \times_{\text{Spec}_{\log} k[Q]} \text{Spec}_{\log} k[P]$ and we know already that $y \times_{\text{Spec}_{\log} k[Q]} \text{Spec}_{\log} k[P]$ is of relative dimension $rk(P^{gp}) - rk(Q^{gp})$ over $y$.

For the general case we may take charts for the log structure with $y$ mapping to the origin. By log flatness $X$ maps via a flat and strict morphism to $Y \times_{\text{Spec}_{\log} k[Q]} \text{Spec}_{\log} k[P]$. Since this map is strict all fibre products along it commute with passing to the underlying scheme. Such a map has a well defined relative dimension $e$ as a map of schemes and induces an isomorphism of log structures. Therefore $y \times_Y X$ is strict and flat of relative dimension $e$ over $y \times_{\text{Spec}_{\log} k[Q]} \text{Spec}_{\log} k[P]$. Therefore by the formula for potential log dimension we have $\logdim(y \times_Y X) - \logdim y = e + rkP^{gp} - rkQ^{gp}$. The integers $e$ and $rkP^{gp} - rkQ^{gp}$ are independent of the choice of charts and locally constant and so the result follows.

This hints at the existence of a log degree for finite proper morphisms, counting the length of a fibre. To make sense of this we must first explore what it means for a log scheme to be irreducible. This is a question about decompositions into subobjects.

**Proper monomorphisms and subobjects**

We recall that a monomorphism is a morphism $f : X \to Y$ such that for any two maps $a, b : Z \to X$ if $fa = fb$ then $a = b$. This is the correct generalisation of an injection to general category theory. The following result of [DG67] explains the geometrical significance:

**Theorem 3** (Proper monomorphisms of schemes are closed immersions). Let $f : X \to Y$ be a proper monomorphism of schemes, then $f$ is a closed immersion.

*Proof.* See Theorem 8.11.5 of [DG67].

Our philosophy is that the correct notion of subobject is not closed immersion but rather the equivalent proper monomorphism. This makes sense in that monomorphisms are the proper generalisation of injections of sets and properness
is a good substitute for being closed in a world without an underlying topology. Therefore let us make the following definition:

**Definition 7.** Let $i : X \to Y$ be a morphism of log schemes. We say that $X$ is a log subscheme of $Y$ if $i$ is proper as a morphism of schemes and $i$ is a monomorphism in the category of fs log schemes.

There is a natural family of examples of such morphisms, the proper log refinement morphisms. These should be treated as being isomorphisms of log varieties and the technically correct tool would be some form of localisation at a multiplicative system inside the category of log schemes. For the sake of geometric intuition we will not pursue this, though it will mean that many geometric results will carry a clause saying “up to log refinement”.

**Theorem 4 (Log refinement morphisms are monomorphisms).** Let $r : X \to Y$ be a log refinement morphism. Then it is a monomorphism in the category of log varieties.

**Proof.** We claim that $r$ is a monomorphism if the canonical diagonal map $X \to X \times_Y X$ is an isomorphism with inverse either projection map. Suppose that the diagonal map is an isomorphism and suppose that we have $f, g : Z \to X$ are morphisms with $rf = rg$. Then we have an induced morphism $f \times g : Z \to X \times_Y X$. Now $f$ and $g$ are the composites of this diagonal with left and right projection, but the two projection maps are equal, which is to say that $f = g$.

The diagonal being the identity is étale local for $Y$ and Zariski local for $X$, so we may reduce to taking the charts defined in the definition of a log refinement morphism and so a diagram

$$
\begin{array}{ccc}
X & \longrightarrow & \text{Spec}_{\log} k[P] \\
\downarrow & & \downarrow \\
Y & \longrightarrow & \text{Spec}_{\log} k[Q]
\end{array}
$$

with all the horizontal arrows strict and the right hand vertical morphism given by a refinement of fans. By definition the map $\bigoplus X \to \bigoplus Y \times_{\text{Spec } k[Q]} \text{Spec } k[P]$ is an open immersion and in particular the product $\bigoplus X \times_{\text{Spec } k[Q]} \text{Spec } k[P]$ is isomorphic to $X$ itself. We are now studying $(Y \times_{\text{Spec } k[Q]} \text{Spec } k[P]) \times_Y (Y \times_{\text{Spec } k[Q]} \text{Spec } k[P])$ which is canonically isomorphic to $Y \times_{\text{Spec } k[Q]} (\text{Spec } k[P] \times_{\text{Spec } k[Q]} \text{Spec } k[P])$. This then allows us to reduce to proving $\text{Spec } k[P] \cong \text{Spec } k[P] \times_{\text{Spec } k[Q]} \text{Spec } k[P]$. Working instead with the algebras this supposes that we have $i : Q \to P$ an injection of fs monoids such that $Q^{op} \cong P^{op}$ via $i^{op}$, and asks if $P \oplus_Q P \cong P$. This follows from the integral fibred sum of two monoids as follows. The sum is the quotient of $P \oplus P$ by the ideal generated by $(p_1 + q, p_2) (p_1, p_2 + q)$ for all
$p_1, p_2 \in P$ and $q \in Q^{sp}$. But by definition $Q^{sp} \cong P^{sp}$ and $P$ and $Q$ are both integral. This proves that étale locally the diagonal morphism $X \to X \times_Y X$ is an isomorphism, and therefore by descent it is globally. 

A scheme is irreducible if for any decomposition into subobjects the intersection between some pair of subobjects is in some sense thick. Let us introduce the notion of a covering family and then simply copy this definition.

**Definition 8.** Let $f_j : X_j \to X$ be a collection of log schemes and morphisms indexed by a finite set $J$, with all $X_j$ reduced. We say that such a collection is a log covering collection if for every log refinement morphism (equivalently every log étale morphism) $U \to X$ the morphism

$$\prod_i U \times_X X_i \to U$$

is surjective on points. Such a cover with $f_j$ strict Zariski open immersions will be called an open cover, whilst if $f_j$ are log étale (resp. log refinement) morphisms then the cover is called a log étale (resp. log refinement) cover.

We similarly say that a morphism is log dominant if it is dominant in the classical sense when pulled back along any log refinement morphism. Analogously a morphism is log surjective if it is classically surjective on scheme-theoretic points when pulled back along any log étale morphism.

This definition suggests a topos-theoretic approach to log geometry. Indeed such a theory exists and we will exploit it in the sequel to this paper. The following is an example of a family which is classically a covering collection but stops being one in the logarithmic setting.

**Example 5.** Let $P = (\text{Spec } k, k^\times \oplus \mathbb{N}^2)$ and $Q = (\text{Spec } k, k^\times \oplus \mathbb{N})$ and take the morphism $Q \to P$ induced by the identity on schemes and the map $(1, 1)$ on monoids. This then appears to be a covering collection but if one takes the étale cover of $P$ given by embedding $P$ into $k^2$ and blowing up the origin one obtains a cover of $P$ by $\mathbb{P}^1$. The fibre product of this over $Q$ however consists of a single point and does not cover.

By passing to all log refinement or log étale covers we gain the ability to ask topological questions which may not have a sensible answer otherwise. Let us give the definition of an irreducible log scheme.

**Definition 9.** Let $X$ be a log scheme whose underlying scheme is irreducible. We say that $X$ is physically reducible if there exist proper monomorphisms $i_1 : X_1 \to X$ and $i_2 : X_2 \to X$ such that they together form a covering collection for $X$, neither is a refinement of $X$ and $\logdim X > \logdim X_1 \times_X X_2$. We say that $X$ is physically irreducible if it is not physically reducible.
Our problem with this definition is that it is not invariant under log refinements. For instance take a double blowup of the origin inside $\mathbb{A}^2$, restricted to the fibre over the origin. Then the source of this morphism has underlying scheme the union of two copies of $\mathbb{P}^1$ meeting at a single point, so it is reducible, the target however is physically irreducible. We fix this with the following definition.

**Definition 10.** Let $X$ be a log scheme. We say that $X$ is logarithmically irreducible if it is a refinement of a physically irreducible scheme. If $X$ has generically trivial log structure then this is the same as physically irreducible.

Such schemes have nice properties, for instance the log degree of a log finite morphism, which we introduce later, is constant on the different components. Essentially these are the correct things to think of as being irreducible, but confusingly they still have top dimensional subspaces, and we will spend some time addressing this. Let us give some examples of these different concepts.

**Example 6.** Take $X$ to be a point with log structure $\mathbb{N}^2$. There is a log refinement $\tilde{X} \to X$ whose underlying scheme is the union of two copies of $\mathbb{P}^1$ meeting at a point given by taking two blow ups of the origin inside $\mathbb{A}^2$. If one takes $X_1, X_2$ to be the two irreducible components of $\tilde{X}$ with the induced log structures, then the log dimension of $X_1 \times_X X_2$ is 2, the same as the log dimension of $X$, so this does not provide a valid decomposition of $X$. In fact $X$ is physically irreducible, and this refinement is then also log irreducible, but not physically irreducible.

Instead take two copies of $\mathbb{P}^1$ meeting at a single marked point so that the stalk of the ghost sheaf at the common point is $\mathbb{N}$. This looks like the above example but is not log irreducible since it has no non-trivial log refinements. To understand why we want to avoid this situation suppose that instead we took a punctured elliptic curve meeting $\mathbb{A}^1$, and consider the chart to $\mathbb{A}^1$. The degree, if it has sensible properties, is not invariant on the two classical irreducible components of this.

One basic feature of irreducible spaces is that a surjection from an irreducible space demonstrates the irreducibility of the target. This continues to hold for physically irreducible spaces by the definition of log surjectivity. We would be interested to know if this continues to hold for logarithmic irreducibility.

The basic intuition is that irreducible log varieties do not have large jumps in the rank of their log structure. To explore this let us give an example of a reducible log scheme.

**Example 7.** Let $L$ be a line through the origin in $\mathbb{A}^2$ which is not one of the axes and take the induced log structure. We claim that this is reducible. There is a morphism $\mathbb{A}^1 \to L$ which over the origin is induced by the map of monoids $(a,b) \mapsto a + b$. This is certainly not a log refinement morphism. Now let $P \cong (\text{Spec } k, k^\times \oplus \mathbb{N}^2)$ be the origin with the induced log structure. The coproduct of
these maps is surjective on geometric points, and by construction every geometric point of $L$ has a geometric point mapping strictly to it which is enough to guarantee log surjectivity.

By construction these two form a covering collection for $L$ and neither is a log refinement of $L$.

We can provide a precise criterion for when a log scheme is irreducible.

**Lemma 5** (Jumping criterion for irreducibility). Let $X$ be a log scheme whose underlying scheme is irreducible. Let $J$ be the set of generic points of irreducible components of strata of the log structure on $X$. Then $X$ is log irreducible if and only if the potential log dimension is constant on $J$. Furthermore the potential log dimension is constant if $X$ is log irreducible, regardless of $X$ being irreducible or not.

**Proof.** We begin by proving that the potential log dimension is upper semi-continuous on $J$ under the inclusion topology. If $X = \text{Spec}_{\log} k[P]$ is a toric scheme then we have seen that the log dimension of each strata is equal to $\operatorname{rk}(P_{gp})$ and so the result holds. Now suppose that $\phi : X \to \text{Spec}_{\log} k[P]$ is a chart for the log structure. The dimension of the fibres of $\phi$ is upper semi-continuous. Now applying the formula for potential log dimension we see that the potential log dimension is upper semi-continuous on $X$ as follows. Let $X_1$ and $X_2$ be two irreducible strata with $X_1 \subset X_2$ and let $Y_1$, $Y_2$ be the closures images of $X_1$, $X_2$ in $\text{Spec}_{\log} k[P]$. These will be contained in closed strata $S_1$, $S_2$ of $\text{Spec}_{\log} k[P]$ where the ghost sheaf has generic stalks $P_1$ and $P_2$ respectively. The log dimension of $X_i$ is given by

$$\dim Y_i + \dim \phi|_{X_i} + \operatorname{rk}(P_i) = \dim \phi|_{X_i} - \operatorname{codim} Y_i/S_i + \operatorname{rk}(P),$$

where $\dim \phi|_{X_i}$ is the dimension of the generic fibre of $X_i \to Y_i$. By upper semi-continuity of the dimension of the fibres of $\phi$ we therefore have $\dim \phi|_{X_1} > \dim \phi|_{X_2}$. We also necessarily have $\operatorname{codim} Y_2/S_2 \geq \operatorname{codim} Y_1/S_1$. Combining these we obtain the desired result.

Let us assume that the potential log dimension jumps up along a strata with generic point $\zeta$. Take a neat chart $\phi : X \to \text{Spec} k[P]$ for the log structure around $\zeta$, so that $\zeta$ is mapped to the origin. The closure of the image of $\phi$ is a closed subvariety of $\text{Spec} k[P]$ passing through the origin defined by the ideal $\langle f_1, \ldots, f_n \rangle$. We will construct a toric blowup of $\text{Spec} k[P]$ such that the strict transform of $V(f_1)$ does not meet any of the zero strata of the blowup. Let $\{m_i\}$ be the set of monomials appearing in $f_1$ and $\text{Hull}(\{m_i\})$ the convex hull of these points inside $P$. We consider the monoid ideal defined by

$$\text{Span}(\{m_i\}) := \{p \in P \mid p = q + a, q \in \text{Hull}(\{m_i\}), a \in P\}$$
This defines a toric ideal inside $\text{Spec } k[P]$ and hence a log blow up $Bl_{k[P]}$ in the sense of [Kat99], Section 3. The zero strata of this blowup correspond to the zero strata of $\text{Span } \{m_i\}$ and by construction are contained inside the set $\{m_i\}$. Therefore the function $f_i$ does not vanish at any of these points since precisely one monomial in it is non-zero. In particular the strict transform of $V(f_1)$ does not contain any of these points. Take $X_1$ to be the strict transform of the image of $\phi$. By construction this does not meet any zero strata of the exceptional locus and maps via a monomorphism $i_1$ to $X$. Now take $X_2 = \{\zeta\}$. Together these two cover $X$ and neither is mapping via a proper refinement to $X$. We must now demonstrate that $\dim X_1 \times_X X_2 < \dim X$. The product $X_1 \times_X X_2$ consists of the fibre of $i_1$ over $\zeta$. We now apply the formula for potential log dimension. Let $\tilde{X}_2 = X_2 \times \text{Spec } k[P] \text{Bl}_{k[P]}$. Then $X_1 \times_X X_2$ is a strict proper closed subscheme of this and does not include any of the minimal strata of $\tilde{X}_2$. Choosing $\zeta$ to have maximal dimension amongst all the jump points we see that the potential log dimension must drop since the geometric dimension has decreased.

For the reverse implication let $i_1 : X_1 \to X$ and $i_2 : X_2 \to X$ be proper monomorphisms with $\dim X_1 \times_X X_2 < \dim X$ and assume that the potential log dimension is constant on $J$. To begin take the simultaneous integralisations of $i_1$ and $i_2$ following the construction of [Kat99] Theorem 3.16. These induce refinements of $X$, $X_1$, $X_2$ and $X_1 \times_X X_2$ preserving all this data. By the formula for potential log dimension we can work with this refinement. One of $X_1$ and $X_2$ maps to the open locus of $X$ with generic log structure, say $X_1$, and by properness of $i_1$ this map is surjective on points. We will show that $i_1$ is an isomorphism by showing that if $i_1$ is integral and monomorphic then it is strict. If $i_1$ were not strict then there would be at least a geometric point $x$ where it fails to be strict. By passing to the stalk of the ghost sheaf at $x$ we reduce to proving the following. Suppose that $(\text{Spec } k, k^\times \oplus N) \to (\text{Spec } k, k^\times \oplus M)$ is an integral monomorphism, then it is strict. From integrality there is a surjection of dual cones $N^\vee \to M^\vee$. If this surjection were not an isomorphism then there would be two points of $n^\vee$ mapping to the same point of $M^\vee$ and this cannot happen since $i_1$ is a monomorphism. This shows that $M \subset N$ with the saturation of $M$ equal to the whole of $N$, but since $M$ and $N$ are themselves fine saturated this is equivalent to saying that $N^\vee$ is saturated in $M^\vee$. Now suppose that there were $m \in M^\vee \setminus N^\vee$ with $km \in N^\vee$. Then take the two maps from $\text{Spec } k^\dagger$ defined on dual monoids by

$$(1, 1) \mapsto (1, n) \in (k^\times \oplus N^\vee)$$

and

$$(1, 1) \mapsto (\zeta, n) \in (k^\times \oplus N^\vee)$$

where $\zeta$ is a $k^{th}$ root of unity. These two maps are equalised by the map $(\text{Spec } k, k^\times \oplus N) \to (\text{Spec } k, k^\times \oplus M)$ but are distinct. This contradicts $i_1$.
being a monomorphism.

Since $i_1$ is strict and an isomorphism on underlying schemes it is an isomorphism, contradicting the assumption that $X_1, X_2$ provide a decomposition of $X$.

The second statement follows since the constancy of log dimension on these strata is preserved by log refinements. 

It is a triviality classically that any scheme is flat over a point. This stops being true in the world of log geometry: for example, take any log scheme with globally non-trivial log structure over the trivial log point. The failure to be log flat is deeply and intimately linked to questions of irreducibility, as the next theorem makes clear. Before we introduce this theorem we require three propositions about how flatness interacts with blow-ups.

There is a universe of generalisations of blow ups, moving from points, to subvarieties, to ideals, to relative blow ups over morphisms. Suppose that $I \subset \mathcal{O}_S$ is a sheaf of ideals on a scheme $S$. The sheaf of $\mathcal{O}_X$-algebras $\oplus I^*$ is a sheaf of graded ideals on $S$ and thus we may take relative Proj to produce a scheme proper over $S$. Taking the ideal sheaf $I$ to be the one described by a subvariety $Z \subset S$ we obtain the classical blow up.

Suppose that $\text{Bl}_Z(S)$ is the blow up of the variety $S$ along a closed subscheme $Z$. Let $D$ be a closed subvariety of $S$ not contained in $S$. The strict transform of $D$ is defined to be the closure of $D \setminus D \cap S$ inside $\text{Bl}_Z(S)$. In general suppose that $f : X \to S$ is a morphism of schemes, the strict transform of $X$ along $Z$ is defined to be the blow up of $X$ along $X \times_S Z$. This admits a morphism to $\text{Bl}_Z(S)$ and agrees with the above definition. The total transform of $X$ is defined to be the fibre product $X \times_S \text{Bl}_Z(S)$. General properties of blow ups are best explained in [Sta18], for instance one may blow up and take the strict transform of sheaves.

**Proposition 1.** Let $X$ be a separated scheme, $I$ a sheaf of ideals corresponding to a closed subscheme $Z$ and $M$ a sheaf of ideals corresponding to a closed point $x \in X$. Let $X_1$ be the scheme obtained by blowing up first $Z$ and $X_2$ the blow up of $X_1$ along the inverse image of $x$. Let $X_3$ be the blow up of $X$ at $x$. Then $X_1$ and $X_3$ have isomorphic open sets $U_1 \subset X_1$ with $X_3 \setminus U_3$ codimension two.

**Proof.** Firstly the inverse image of $x$ in both $X_1$ and $X_3$ are principal. Therefore by the universal property of blow ups there is a canonical morphism $\pi : X_1 \to X_3$. We will build an inverse to this morphism over the desired set. We may make several simplifications. $X$ is affine, equal to Spec $A$, the sheaves of ideals come from ideals $I$ and $m$ of $A$ and $m \supset I$. Let $f \in I \subset m$ be an element, then $X_3 \setminus D(F)$ has affine charts given by Spec $A[m/f]$. We construct affine charts on $X_1$ minus the strict transform of $f$ and $V(I)$ as follows. The scheme $X_2 \setminus V(f)$ has an affine chart given by Spec $A[I/f]$. In this scheme the strict transform of $I$ is principal,
equal to $\langle t \rangle$. The desired chart is then given by $\text{Spec } A[I/f][\text{im}(m)/t]$. But by construction $I \cdot 1/t \subset A$ and so this maps via multiplication to $\text{Spec } A[m/f]$. By varying $f$ we obtain a rational map $\sigma : X_3 \to X_1$ defined away from the strict transform of $V(I)$. The composites $\sigma \circ \pi$ and $\pi \circ \sigma$ are the identity away from $I$ and $x$ and so by separatedness of $X$ are the identity where they are defined.

Proposition 2. Let $f : X \to B$ be a morphism with $B$ a scheme which is not flat at a closed point $b \in B$. Let $f' : X' \to \text{Bl}_P(B)$ be the strict transform. Then the locus over which $f'$ is not flat intersects the exceptional divisor of $\text{Bl}_b(B)$ in at most codimension two.

Proof. We apply tag 080X of [Sta18]. This says that there is a sequence of blowups of $B$ along closed subvarieties such that the strict transform $\overline{X}$ of $X$ is flat over the blowup. Suppose that this sequence blows up $Z_1, \ldots, Z_n$, and let $B'$ be the scheme obtained by blowing up the inverse image of $P$ after blowing up all these. Let $B''$ be the scheme obtained by first blowing up $P$ then blowing up the strict transforms of all of the $Z_i$. Now let $X'$ and $X''$ be the corresponding strict transforms. Both the pair $B'$ and $B''$ and the pair $X'$ and $X''$ differ by a birational transform which is an isomorphism away from codimension two. In particular the fibre over $P$ is a divisor in both of these. By the result of tag 085S of [Sta18] $X' \to B'$ is flat and hence there is an open set in $B''$ over which $X''$ is flat containing all but a codimension two subset of the fibre over $P$. Now the blowdown map $B'' \to \text{Bl}_P(B)$ is an isomorphism on an open subset containing all but a codimension one subset of the fibre over $P$. In total this means that there is an open subset $U$ of $\text{Bl}_P(B)$ containing all but a codimension two subset of the exceptional fibre over $P$ such that the strict transform $\overline{X}$ is flat over $U$.

Proposition 3 (Total transforms preserve non-flatness). Let $f : X \to B$ be a morphism with $B$ a toric variety such that $f$ is not flat at a geometric point $b \in B$. 

\[
\begin{array}{ccc}
X' & \xleftarrow{\sim} & X'' \\
\downarrow & & \downarrow \\
\overline{X} & \xleftarrow{\sim} & \overline{X} \\
\downarrow & & \downarrow \\
B' & \xleftarrow{\sim} & B'' \\
\downarrow & & \downarrow \\
B & \xleftarrow{\sim} & \text{Bl}_P(B)
\end{array}
\]
Let $f' : X' \to \text{Bl}_b(B)$ be the total transform (so the fibre product) with respect to a weighted blow up of the origin. Then $X'$ is not flat over the exceptional divisor over $b$.

Proof. I must thank János Kollár for pointing me in the correct direction for a reference. We apply [DG67] Theorem 11.6.1, which when simplified to our setting states the following. Suppose that $f : A \to A'$ is an injective local morphism of local rings and of finite presentation, $A$ is geometrically unibranched and integral and $g : B \to A$ is a morphism of rings of finite presentation. Let $Y = \text{Spec } A$, $X = \text{Spec } B$, $Y' = \text{Spec } A'$, $X' = X \times_Y Y'$ fitting into a diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow{g'} & & \downarrow{g} \\
X & \xrightarrow{f} & Y
\end{array}
$$

Choose a point $x$ in $X$ mapping to the closed point of $Y$ and $x'$ a point in $X'$ mapping to $x$ in $X$ and the closed point of $Y'$ under the projections. Then $f$ is flat at $x$ if $f'$ is flat at $x'$.

To apply this to our situation we first note that toric varieties are geometrically unibranched. We are free to localise $B$ to $b$, $X$ to a closed point where $f$ is not flat and $\text{Bl}_b(B)$ to any point in the exceptional divisor. Then the non-flatness of $f$ implies the desired result.

Corollary 1. Let $f : X \to B$ be a morphism of schemes with $B$ a toric variety not flat at $0 \in B$. Let $\pi : \text{Bl}_0(B) \to B$ be a (possibly weighted) blow up of the origin. Then the strict transform of $X$ is a proper subscheme of the total transform of $X$.

Lemma 5 provides a strong restriction on the form of neat charts for log irreducible schemes. The most natural example of log reducible log schemes comes from taking a proper subvariety of a toric variety which passes through the origin. The potential log dimension is not constant and so the subvariety is reducible. In fact this is the source of all examples as the following lemma makes clear:

Lemma 6 (Neat charts have dense image). Let $X$ be a log irreducible log scheme and $\phi : X \to \text{Spec}_\log k[P]$ a chart for the log structure with $x_P \in X$ mapping to the origin and $x_0$ mapping to the big torus. Then the image of $X$ is dense.

Proof. Suppose this were not the case. Then the image of the points with trivial log structure is a non-empty proper subvariety of the big torus, of codimension $d$ say. Let $X_P$ be the log strata containing $x_P$ and $X_0$ the strata containing $x_0$. We may assume that the fibre of $\phi$ over $\phi(x_0)$ is of minimal dimension amongst fibres
of \( \phi \), equal to \( e \). Let us calculate the potential log dimension on these two strata. The dimension of \( X_P \) is at least \( e \) and the rank of the stalk of the log structure is \( \text{rk}(P^{gp}) \). The dimension of \( X_0 \) is at most \( \dim \text{Spec } k[P] - d + e = \text{rk}(P^{gp}) - d + e \) and carries the trivial log structure. This contradicts the assumption that \( X \) was irreducible.

Since the image of a morphism is a constructible set this in fact tells us that the image of such an \( X \) is dense inside the chart.

**Theorem 5** (Relating irreducibility and log flatness). Let \( X \) be a log scheme with generically trivial log structure and irreducible underlying scheme. Then it is log flat over the trivial log point if and only if it is log irreducible.

**Proof.** Suppose that \( s : X \to \text{Spec } k \) is log flat. Then we have charts

\[
\begin{array}{c}
X \\
\downarrow \\
\text{Spec } k \\
\downarrow \\
\text{Spec } k \\
\end{array}
\begin{array}{c}
\text{Spec } k[P] \\
\downarrow \\
\text{Spec } k \\
\downarrow \\
\text{Spec } k \\
\end{array}
\]

such that \( X \) is flat over \( \text{Spec } k[P] \). By the jumping criterion for irreducibility, Lemma 5, the potential log dimension of strata of \( \text{Spec}_{log} k[P] \) is constant, and by the equidimensionality of fibres of flat morphisms the same is true for the strata of \( X \). Thus we see that \( X \) is in fact log irreducible.

Conversely suppose that \( X \) is irreducible. Letting \( X^\circ \) be the generic log stratum of \( X \), this fits into a chart

\[
\begin{array}{c}
X^\circ \\
\downarrow \\
\text{Spec } k \\
\downarrow \\
\text{Spec } k \\
\end{array}
\begin{array}{c}
\text{Spec } k \\
\downarrow \\
\text{Spec } k \\
\downarrow \\
\text{Spec } k \\
\end{array}
\]

and hence is automatically flat. We now induct on the strata ordered by inclusion after taking closure. Let \( X_P \) denote the stratum on which the ghost sheaf has constant value \( P \). Suppose that \( s \) were not flat at some point on \( X_P \) and take a good chart for \( s \) in a neighbourhood of that point. Thus we have a diagram

\[
\begin{array}{c}
X \\
\downarrow \\
\text{Spec } k \\
\downarrow \\
\text{Spec } k \\
\end{array}
\begin{array}{c}
\text{Spec } k[P] \\
\downarrow \\
\text{Spec } k \\
\downarrow \\
\text{Spec } k \\
\end{array}
\]

Since \( X \) is log irreducible the scheme theoretic image of \( X \) inside \( \text{Spec}_{log} k[P] \) is dense, by Lemma 6. The morphism \( X \to \text{Spec } k[P] \) is generically flat, hence the
locus on which it is not flat must be a proper closed subset of \( \text{Spec } k[P] \) passing through the origin. The result then follows from Proposition 2, Proposition 3 and the construction of 5. Indeed, the weighted blowup of the origin of 5 induces a log refinement of \( X, \tilde{X} \). Let \( X_1 \) be the strict transform of \( X \) with respect to this refinement with the induced log structure and \( X_2 = X_P \). These two log schemes admit natural proper monomorphisms to \( X \) of which one is strict over each point of \( X \). Therefore they form a covering collection.

It remains to show that the log dimension of \( X_1 \times_X X_2 \) is less than the log dimension of \( X \). To see this note that the pullback of \( X_1 \times_X X_2 \) to the refinement cover \( \tilde{X} \) is a proper subvariety of the fibre not meeting any of the zero dimensional strata.

\[ \square \]

So this suggests that if we had a method to strip away the generic log structure we might be able to realise every log scheme with irreducible underlying scheme as being in some sense flat over a point. It is not necessary that the log structure be globally trivial: in fact we can get away with stripping it away locally. To this end a short exact sequence of monoids is a series of morphisms of monoids \( a : K \to M \) and \( b : M \to C \) such that \( a \) is a monomorphism of monoids and \( b : M \to C \) is the cokernel of \( a \).

Definition 11. Let \( f : X \to X' \) be a morphism of log schemes which is the identity on a common underlying scheme \( S \). We say that \( f \) is an extension morphism if there is a monoid \( P \) and an (étale) cover of \( S \) by \( U_i \) such that on each \( U_i \) there is an exact sequence \( M_{X'} \to M_X \to P \).

The canonical extension morphism is the inclusion of the subsheaf of \( M_X \) consisting of those elements whose support in the ghost sheaf is strictly smaller than \( X \). By taking charts for \( M_X \) one sees that this fits into a short exact sequence involving the log structure of \( X \) and the generic ghost stalk. We call this log structure the naked log structure on \( X \) and note that it certainly is fs. The induced short exact sequence need not split.

Combining the above definition and theorem we obtain the following.

Corollary 2 (General structure of irreducible log schemes). Let \( X' \) be a log irreducible log scheme. Then it admits a morphism \( X' \to X \) a log extension with \( X \) log flat over the trivial log point.

Proof. Our choice of \( X \) is the naked log structure on \( X' \). Since \( X' \) was log irreducible the potential log dimension is constant on the strata of \( X' \). Now since \( X' \to X \) is an extension morphism there is a constant \( d = \text{rk}(\overline{M}_{X', \text{gen}}) \) such that the potential log dimension of a strata of \( X' \) of the corresponding strata of \( X \) differ by \( d \). Therefore \( X \) is log irreducible and has trivial generic log structure.

\[ \square \]
These factorisations, of the structure morphism into a log extension followed by a log flat morphism, clearly are not symmetric since the extension need not globally split. But from this one might hope that a log flat morphism followed by an extension could be massaged into an extension morphism followed by a log flat morphism. Indeed this is possible in certain situations and we will need this later.

**Lemma 7.** Let \( f : X \to Y \) be a log morphism admitting a factorisation into

\[
X \xrightarrow{s} Y' \xrightarrow{e} Y
\]

with \( s \) log flat and \( e \) an extension morphism and suppose that \( Y \) is log irreducible and has generically trivial log structure. Then there is a commutative square

\[
\begin{array}{ccc}
X & \xrightarrow{s} & Y' \\
\downarrow{e'} & & \downarrow{e} \\
X' & \xrightarrow{s'} & Y
\end{array}
\]

where \( e' \) is an extension morphism and \( s' \) is log flat.

**Proof.** Our choice of \( X' \) will be the naked log structure on \( X \). We have already seen that this admits a morphism \( e' : X \to X' \) a log extension morphism. To construct the morphism \( s' \) we take the scheme theoretic morphism \( s \). This underlying morphism is not necessarily open but we will demonstrate that it at least has dense image. Since \( Y' \) is log irreducible the image of \( Y' \) inside any neat chart is dense, so suppose that we have a diagram of charts

\[
\begin{array}{ccc}
X & \xrightarrow{} & \text{Spec } k[P] \\
& & \downarrow{} \\
Y' & \xrightarrow{} & \text{Spec } k[Q]
\end{array}
\]

with \( X \) flat over \( Y' \times_{\text{Spec } k[Q]} \text{Spec } k[P] \) and the map \( Q \to P \) an inclusion. Since \( Q \to P \) is an inclusion the map on spectra \( \text{Spec } k[P] \to \text{Spec } k[Q] \) has dense image. Since the product \( Y' \times_{\text{Spec } k[Q]} \text{Spec } k[P] \) is non-empty it is a dense set inside a component of \( Y' \). By assumption \( Y' \) is log irreducible and so the image of \( Y' \times_{\text{Spec } k[Q]} \text{Spec } k[P] \) is dense inside \( Y' \). Hence the pullback of any section of the log structure on \( Y \) with support strictly smaller than \( Y \) will have support strictly smaller than \( X \). This shows that there is an induced morphism \( s' : X' \to Y \).

We now wish to show that \( s' \) is log flat. Let \( M \) be the generic log structure on \( X \) and \( N \) be the generic log structure on \( Y' \). By supposition \( s \) is flat, so locally on \( X \) and \( Y' \) there exist a fabulous chart for the morphism at \( x \) and \( y \)

\[
\begin{array}{ccc}
X & \xrightarrow{} & \text{Spec } k[P] \\
& & \downarrow{} \\
Y' & \xrightarrow{} & \text{Spec } k[Q]
\end{array}
\]
with \( X \) flat over \( Y' \times_{\text{Spec } k[Q]} \text{Spec } k[P] \). Let \( S \) be the monoid of sections of the ghost sheaf in the chosen neighbourhood of \( x \) whose support is of strictly smaller dimension, and \( T \) similarly for \( y \). The quotient maps \( k[P] \to k[S] \) induce closed embeddings \( \text{Spec } k[S] \to \text{Spec } k[P] \) of strata fitting into a diagram with strict horizontal arrows:

\[
\begin{array}{ccc}
X & \to & \text{Spec } k[S] \to \text{Spec } k[P] \\
\downarrow & & \downarrow \\
Y' & \to & \text{Spec } k[T] \to \text{Spec } k[Q]
\end{array}
\]

where we claim that \( \text{Spec } k[S] \times_{\text{Spec } k[T]} Y' \to \text{Spec } k[P] \times_{\text{Spec } k[Q]} Y' \) is a closed embedding of irreducible components. First we note that \( \text{Spec } k[S] \) is a closed subvariety of \( \text{Spec } k[P] \times_{\text{Spec } k[Q]} \text{Spec } k[T] \) by comparing both as closed subsets of \( \text{Spec } k[P] \). Therefore taking products \( \text{Spec } k[S] \times_{\text{Spec } k[T]} Y' \) is a closed subvariety of \( \text{Spec } k[P] \times_{\text{Spec } k[Q]} Y' \). Now \( X \) maps via a flat map to \( Y' \times_{\text{Spec } k[Q]} \text{Spec } k[P] \), so the image is open but this map factors through \( Y' \times_{\text{Spec } k[T]} \text{Spec } k[S] \). Therefore \( Y' \times_{\text{Spec } k[T]} \text{Spec } k[S] \) is the union of components of \( Y' \times_{\text{Spec } k[Q]} \text{Spec } k[P] \). Now consider the cartesian square

\[
\begin{array}{ccc}
X & \to & Y' \times_{\text{Spec } k[T]} \text{Spec } k[S] \\
\downarrow & & \downarrow \\
X & \to & Y' \times_{\text{Spec } k[Q]} \text{Spec } k[P]
\end{array}
\]

which shows that \( X \to Y' \times_{\text{Spec } k[T]} \text{Spec } k[S] \) is itself flat.

Now the induced morphisms to \( \text{Spec } k[S] \) and \( \text{Spec } k[T] \) form charts for the log structures on \( X' \) and \( Y \) respectively. But \( X \) and \( X' \) are isomorphic as schemes, as are \( Y \) and \( Y' \). Therefore \( X' \) is flat over \( Y \times_{\text{Spec } k[S]} \text{Spec } k[T] \). This completes the proof.

Now that we have worked through the consequences of irreducibility we can return to our original question of how to define a logarithmic degree.

**Definition 12.** We introduce some notation to generalise the function field of an irreducible variety. Let \( X \) be a physically irreducible scheme with irreducible underlying scheme. \( X \) then has generic log structure \( M \) and a generic point \( \zeta_X = \text{Spec } k(X) \). The log function field of \( X \) is defined to be \( k(X)(M^{\text{gp}}) \), the fraction field of \( k(X)[M^{\text{gp}}] \), and is written \( R(X) \). This is invariant under passing to open subsets.

Now let \( f : X \to Y \) be a dominant morphism between physically irreducible log varieties of the same log dimension and with irreducible underlying schemes.
The morphism $f$ induces a morphism $f^*: R^!(Y) \to R^!(X)$ between fields of the same transcendence degree. This therefore has a well defined degree which we write $\deg_{X/Y}$.

This definition should be invariant under refinement pullback. It indeed is, as the following lemma proves

**Lemma 8** (Degree is stable under log refinement pullback). Let $f: X \to Y$ be a dominant morphism between physically irreducible log varieties of the same dimension, $r: Y' \to Y$ a log refinement morphism with $Y'$ physically irreducible and $X' \cong X \times_Y Y'$. Then $\deg_{X/Y} = \deg_{X'/Y'}$.

**Proof.** We prove the following: if $g: Z' \to Z$ is a log refinement morphism with $Z'$ a log irreducible scheme then the morphism $g^*$ induces an isomorphism $R^!(Z) \to R^!(Z')$. Indeed, by passing to the generic point of $Z$ we may assume that $Z$ is a point. Let $M_Z$ denote the ghost sheaf on $Z$, and $M_{Z'}$ the ghost sheaf at the generic point of $Z'$. There is a morphism $g^\#_{gp}: M_{Z'} \to M_{Z'}$ with kernel $K$. By definition $R^!(Z) = k(Z)(M_{Z}^{gp})$ and $R^!(Z') = k(Z')(M_{Z'}^{gp})$. Now one can calculate $k(Z')$ by observing that there is an open dense subset of the generic fibre of $g$, isomorphic to Spec $k(Z)[K]$. Hence the field of fractions is isomorphic to $k(Z)(K)$. Hence $R^!(Z') = k(Z')(M_{Z'}^{gp}) = k(Z)(K)(M_{Z'}^{gp}) = k(Z)(M_{Z}^{gp}) = R^!(Z)$.

Now we claim that if $f: X \to Y$ is a morphism of log irreducible schemes and $\pi: Y' \to Y$ a log refinement, then the product $X \times_Y Y'$ is also log irreducible. The proof follows from the formula for potential log dimension.

The result now follows from the functoriality of $f^*$. Indeed the following square commutes:

$$
\begin{array}{ccc}
R^!(Y) & \cong & R^!(Y') \\
\downarrow f^* & & \downarrow f^* \\
R^!(X) & \cong & R^!(X')
\end{array}
$$

From this we also see that if $\tilde{Y} \to Y$ is a refinement with $Y$ physically irreducible and $Y_i$ a classically irreducible component of $\tilde{Y}$ with the induced log structure, then $R^!(Y_i) \cong R^!(Y)$, and the same is true if we pass to strata even. This ensures that log degree is preserved under passing to one of these components.

**Images of log varieties**

We have seen that there are more subobjects of a log scheme than expected. This changes the behaviour of image factorisations in the category and provides more
evidence that one should really invert the refinement morphisms. We suspect that these morphisms form the weak equivalences in some sort of model structure on the category of log varieties. Let us give an example of what can happen if we do not invert these morphisms.

**Example 8.** We suspect that the category of log varieties actually already carries an image factorisation, being a variant of “the scheme theoretic image inside a maximal log refinement of \( Y \)”. Let us give an example where one can construct an image but it certainly does not behave how one might want. Take a point with monoid \( \mathbb{N}^3 \), call this \( P \), a point with monoid \( \mathbb{N}^2 \), call this \( Q \), and a standard log point \( T \). We will construct a map \( f : Q \coprod Q \to P \) as the coproduct of two maps \( a \) and \( b \). The best way to describe these maps is via the dual monoids, so in this case two morphisms \( a^\lor, b^\lor : \mathbb{N}^2 \to \mathbb{N}^3 \). This process of dualising is often referred to as tropicalisation in the wider literature. Our choice of the maps will be \( a^\lor : (x, y) \mapsto (x, y, y) \) and \( b^\lor : (x, y) \mapsto (x, x, y) \). There is another morphism \( c : \mathbb{N} \to \mathbb{N}^2 \) equalising these two maps given by \( x \mapsto (x, x) \). In total this provides an equaliser diagram \( T \rightrightarrows Q \coprod Q \to P \).

We claim that the image of the map \( f \) is a point with log structure dual to the convex hull of the images of \( a^\lor \) and \( b^\lor \), call this \( \text{Im} \). Suppose that we have a factorisation \( p : Q \coprod Q \to I \) and \( i : I \to P \) with \( i \) a monomorphism. Replacing \( I \) by the image of \( p \) with the induced log structure from \( I \) we may assume that \( p \) is scheme theoretically surjective. Thus \( I \) consists of either one point or two points. If it consists of two points there is no way that the map \( i \) could be a monomorphism, since it would equalise the two morphisms \( T \rightrightarrows Q \coprod Q \). Therefore the underlying scheme of \( I \) is a single point. The image must be dual to a rational polyhedral complex inside \( \mathbb{N}^3 \) and contain the images of \( a^\lor \) and \( b^\lor \). In particular, it contains the convex hull of the images of \( a^\lor \) and \( b^\lor \), from which it follows that \( \text{Im} \) is indeed the image. This is now very unpleasant, as there are standard log points of \( \text{Im} \) not mapped to by any standard log point of \( Q \coprod Q \). By the formula for log dimension the image is even higher dimensional than the source! Of course once we allow ourselves to refine the source this issue disappears, and one can take an integralisation of this morphism instead. By passing to the integralisation we will see that the map on standard log points is now surjective.

**Theorem 6 (Images of log morphisms).** Let \( f : X \to Y \) be an integral morphism of log varieties. There exists a log refinement \( \psi : X' \to X \) and a log scheme \( \text{im}(X') \) fitting into a diagram \( X' \to \text{im}(X') \to Y \) which is an image factorisation for \( f \).

**Proof.** Let \( f : X \to Y \) be an integral morphism of log varieties. Let \( p : X \to \text{Im}(f) \) and \( i : \text{Im}(f) \to Y \) denote the scheme theoretic image of \( X \) inside \( Y \) and we give this the log structure \( J \) associated to the saturation of \( \text{Im}(i^{-1}M_Y \to p_*M_X) \). We claim that this is an fs log structure. It is by definition saturated. It is finitely
generated since $\mathcal{M}_Y$ is finitely generated and integral since $\mathcal{M}_X$ is integral. We denote this log scheme $\text{Im}(f)$. By construction there is a factorisation $p : X \to \text{Im}(f)$ and $i : \text{Im}(f) \to Y$ with $ip = f$ where to construct $p$ we use the universal property of saturations. We begin by proving that $i$ is a monomorphism. Let $a, b : Z \to \text{Im}(f)$ be two morphisms equalised by $i$. Since $i$ is a monomorphism of underlying schemes $a$ and $b$ can only differ in their effect on the log structure. But the induced map $i^#$ is surjective, hence $a$ and $b$ are in fact the same map.

We must now prove that this factorisations is universal amongst image factorisations via proper monomorphisms. Let $q : X \to J$ and $j : J \to Y$ be another factorisation of $f$ with $j$ a proper monomorphism. By replacing $J$ by the scheme theoretic image of $X$ inside $J$ with log structure pulled back from $J$ we may assume that $q$ is scheme theoretically surjective. We now claim that there is a morphism $\text{Im}(f) \to J$. To construct such a map it is enough to prove that the morphism $j$ is a scheme theoretic monomorphism. To this end let $a, b : U \to J$ be two maps equalised by $j$. We first claim that topologically $a$ and $b$ must be equal. Suppose that they differ on some point $U \ni \text{Spec } K \to J$. We extend this to a morphism of log schemes with source $(\text{Spec } K, K^\times \oplus \mathbb{N})$. First note that since $f$ induces a surjection on dual cones and $q$ is surjective we deduce that $j$ also gives a surjection on dual cones. Let $C$ be the dual cone of the stalk of the characteristic sheaf at the image of $\text{Spec } K$ inside $Y$, $C_a$ the dual cone of the stalk of the characteristic sheaf of $J$ at the image of $\text{Spec } K$ under $a$, and $C_b$ similarly for $b$. Choose an element of $C$, $c$, then by surjectivity we may find elements $c_a$ and $c_b$ of $C_a$ and $C_b$ mapping to $c$. These define morphisms of log schemes $(\text{Spec } K, K^\times \oplus \mathbb{N}) \to J$ equalised by $j$. Hence they define the same map.

Now we know that topologically these two morphisms agree we suppose that we have $a, b : U \to J$ distinct morphisms of classical schemes, equalised by $j$. We will give a log structure to $U$ such that these two morphisms extend to log morphisms, also equalised by $j$. The log structure we take is the log structure associated to the coproduct $a^{-1}\mathcal{M}_J \oplus b^{-1}\mathcal{M}_J$ using the fact that $a^{-1} = b^{-1}$ for sheaves. This clearly admits log morphisms extending $a$ and $b$, and these are by construction equalised by $j$. 
Since $j$ is a proper monomorphism of schemes now it is in fact a closed subspace of $Y$, and hence there is a canonical morphism $u : \text{Im}(f) \to J$. We now need to extend this to a log morphism. But this is easy since the log structure $u^{-1}M_J$ fits into a sequence

$$u^{-1}j^{-1}MY \to u^{-1}M_J \to f_*M_X$$

This induces a compatible morphism of log structures. This morphism is canonical and unique, hence defines the image.

Any morphism admits a non-canonical choice of integralisation. Choosing any integralisation then defines an image factorisation but these are non-canonical and have no universal property. However any two integralisations admit a roof given by log refinements and by the above construction these induce log refinements of the image factorisations. Inverting these refinement morphisms then produces a canonical functorial image factorisation. We promised again not to look too hard at this problem, so we will not spell out the details.

\begin{corollary}
The category of log varieties localised at the log refinement morphisms supports image factorisations.
\end{corollary}

Such a factorisation is functorial and has all the standard properties of an image.

The Geometrisation functor

We will want later a geometric object which somehow represents the log structure on a log scheme. We provide a construction here via embeddings into toric charts.

The idea is that the geometrisation should be a scheme over $X$ such that over a point with characteristic stalk $M$ the fibre is $\text{Spec } k[M]$. We can easily do this on a toric chart $\text{Spec } k[P]$ as it corresponds to the quotient by the ideal $\langle x^m(1 - y^m) \mid m \in P \rangle$ inside of $\text{Spec } k[P \oplus P]$ where the $x$ are coordinates on the first factors and $y$ coordinates on the second. The map to $\text{Spec } k[P]$ is then projection to the first factor. Let $\langle x^{m_1}, \ldots, x^{m_n} \rangle$ be a toric ideal corresponding to a face $\sigma$. The fibre over this is spanned by elements $y^m$ subject to $y^m - y^{m+n}$ if $x^n$ does not vanish on this strata, recovering the monoid ring of the quotient $P/\sigma$. For a general log scheme $X$ there are étale local charts for the log structure $X \to \text{Spec } k[P]$. Locally the geometrisation of $X$ is defined to be the product $X \times_{\text{Spec } k[P]} \text{Spec } k[P \oplus P]/\langle x^n(y^0 - y^n) \forall n \in P \rangle$.

Suppose that we have two different charts $X \to \text{Spec } k[P]$ and $X \to \text{Spec } k[Q]$. By taking products we can assume that they are defined over the same étale open of $X$. By restricting the étale further we may assume that $X$ is affine and the
natural map $\mathcal{O}_X(X) \to \mathcal{O}_{X,P}$ is an inclusion. Then both $X \times_{\text{Spec } k[P]} \text{Spec } k[P \oplus P]/\langle x^m(1 - y^m) \rangle$ and $X \times_{\text{Spec } k[Q]} \text{Spec } k[Q \oplus Q]/\langle x^m(1 - y^m) \rangle$ are the spectra of the ring $\mathcal{O}_X(X \otimes \mathbb{Z}[\mathcal{M}_{X,P}])/\langle \alpha(p)z^p \mid p \in \mathcal{M}_{X,P} \rangle$. Therefore the two constructions agree on the overlap. We now are glueing an étale equivalence relation on $X$ and hence produce an algebraic space. This is an algebraic space $^{1}$ $\text{Geom } (X)$ over $X$ admitting (via the morphism $\alpha$) a section called the geometrisation of $X$. This section embeds $X$ at each point as the unique zero strata inside the affine toric variety sitting over it.

This construction is functorial and by the formula for potential log dimension produces a scheme whose geometric dimension is the logarithmic dimension of $X$. We do not like working with algebraic spaces and there is an important classical invariant scheme, the generic geometrisation constructed as follows. Let $M$ be the stalk at the generic point of the characteristic sheaf. The generisation map gives a morphism from $\mathcal{O}_X[\mathcal{M}_X]$ to $\mathcal{M}$, the constant sheaf. Relative $\text{Spec}$ of this just reconstructs the scheme $X \times \text{Spec } k[M]$ and a canonical morphism $X \times \text{Spec } k[M] \to \text{Geom } (X)$. We write $\mathbb{T}_X$ for the scheme $X \times \text{Spec } k[M]$. If $f : X \to Y$ is a surjective morphism of irreducible log varieties then there is an induced morphism $\mathbb{T}_X \to \mathbb{T}_Y$.

The motivation for this definition is to study the behaviour of degree in families. For a physically irreducible scheme the function field of the generic geometrisation is isomorphic to the log function field of the original log scheme.

**Deformation equivalence**

**Cycles on a log scheme**

In the classical theory the Chow groups are built out of the free groups generated by cycles of a fixed dimension. This will remain true here and we will introduce in this section a pushforward and pullback for such cycles. Let us begin by introducing notation for these objects.

**Definition 13.** Let $X$ be a log scheme. The group of $k$-dimensional cycles is the free Abelian group generated by $i : Z \to X$ with $i$ a proper monomorphism and $Z$ a physically irreducible log scheme of log dimension $k$. It is denoted $Z^k_1(X)$.

We could take it to be generated by all the logarithmically irreducible classes, but this is the most natural statement. The desired group will be the quotient of this group by certain relations designed to mimic the classical notion of rational equivalence. For now we will show that there is a canonical way to associate

---

$^{1}$Indeed strictly an algebraic space and not a scheme since the log structure could carry some monodromy.
a cycle to any subscheme by first constructing a decomposition into irreducible components.

**Lemma 9** (Irreducible decompositions of log varieties). Let \( X \) be a log scheme. There is a finite covering collection \( X_1, \ldots, X_n \) with \( X_i \) a cycle on \( X \) and \( X_i \) physically irreducible. This covering collection is unique up to reordering and log refinement morphisms.

**Proof.** The uniqueness up to reordering is a standard result of irreducibility whilst the choice of log refinement comes from the fact that we are free to blow up the log structure. Therefore we must only prove existence. Such a covering collection is constructed as follows. Let \( J_n \) be the subset of jump points for the log strata \( J \) as defined in 5 where the component has dimension \( n \). There is some maximum \( n \) for which this is non-empty and equal to a finite set \( \{ \zeta_i \}_{i \in I} \). Perform the construction of the first part of 5 on each \( \zeta_i \) to obtain a log scheme \( \tilde{X} \) of strictly smaller dimension together with a collection \( X_i = \{ \zeta_i \} \). By induction this eventually terminates and produces the desired collection. \( \square \)

Now we simply follow the existing theory: given a closed subscheme of \( X \) the associated cycle is simply a weighted sum of the irreducible components of the subscheme. Let us define this now.

**Definition 14.** Let \( X \) be a log scheme and \( i : Z \to X \) a proper morphism. The associated cycle of \( Z \) is constructed as the weighted sum

\[
\sum_A (-1)^{|A|+1} \deg_{\cap A Z_i / \text{im}(\cap A Z_i)} [\cap A Z_i]
\]

where the sum is taken over all subsets of the set of irreducible components of \( Z \), \( A \), such that the product \( \cap A Z_i := Z_{a_1} \times_X Z_{a_2} \times_X \ldots \times_X Z_{a_n} \) over all elements of \( A \) is top dimensional.

**Lemma 10** (Existence of flat pull-back). Let \( f : X \to Y \) be a log flat morphism of relative dimension \( k \). There is a flat pullback of cycles \( f^* : Z^!(Y) \to Z^!_{s+k}(X) \) defined by

\[
f^*(i : Z \to Y) \mapsto [(i : Z \times_Y X \to X)]
\]

**Proof.** Note that proper monomorphisms are preserved under base-change. The remaining statement is vacuous. \( \square \)
Lemma 11 (Existence of extension pull-back). Let \( f : X' \to X \) be a log extension by \( P \) a finitely generated monoid of rank \( k \). There is an extension pullback of cycles \( f^* : Z^\dagger_k(X) \to Z^\dagger_{k+k'}(X') \) defined by

\[
    f^* : (i : Z \to X) \mapsto (i : Z \times_X X' \to X')
\]

Proof. We must show that the pullback is a cycle, so is an extension of a log scheme log flat over \( \text{Spec} \ k \), and that it is the correct dimension. That it is the correct dimension is clear since the underlying schemes of \( Z \) and \( Z \times_Y X' \) agree and we can apply our formula for the potential log dimension. It would be sufficient to prove that log extension morphisms are stable under composition. Let \( f : X'' \to X' \) and \( g : X' \to X \) be two log extension morphisms of a physically irreducible scheme \( X \) by \( P \) and by \( Q \) respectively. By definition that means that there is an étale cover of the underlying scheme \( X \) such that

\[
    \mathcal{M}_{X'} \to \mathcal{M}_{X''} \to P
\]

and

\[
    \mathcal{M}_X \to \mathcal{M}_{X'} \to Q
\]

are exact. We now consider the inclusion \( \mathcal{M}_X \to \mathcal{M}_{X'} \). The cokernel here contains \( Q \) with quotient \( P \). Since \( X \) was log irreducible this extension is constant and so defines a log extension.

Lemma 12 (Existence of proper pushforward). Let \( f : X \to Y \) be a proper morphism. There is a pushforward of cycles \( f_* : Z^\dagger_k(X) \to Z^\dagger_1(Y) \) defined by

\[
    f_* : (i : Z \to X) \mapsto \text{deg}_{Z/\text{im}(Z)}(i : \text{im}(Z) \to X)
\]

Proof. We need not worry ourselves with the degree here, as this correction factor only occurs to ensure that the map descends to the Chow group. We must show that the image of a cycle is a cycle. But \( Z \) surjects onto its image, so the physical irreducibility of \( Z \) ensures the irreducibility of the image.

Deformation Equivalence

We are finally in a position to define deformations of a subscheme. Classically two irreducible subvarieties of \( X \) are rationally equivalent if there is an irreducible family inside \( X \times \mathbb{P}^1 \) whose fibres over two points are the two subvarieties involved. Since \( \mathbb{P}^1 \) is a smooth curve this is equivalent to classifying subvarieties up to
flat deformation parameterized by a rational curve. Initially we expected flat deformation to be the correct concept, but our exploration of flatness, as detailed previously, in fact led us to conclude that it is irreducibility that is important.

We commented earlier that we should really invert the multiplicative system generated by refinement morphisms. This will be clear when we define refinement equivalence. It is however technically tedious to deal with such tools compared to dealing with the underlying geometric objects.

**Definition 15.** Let $X$ be a log scheme. A deformation family is a choice of $U \subset \mathbb{P}^1$ with trivial log structure and a commutative diagram

$$
\begin{array}{ccc}
W & \rightarrow & X \times U \\
\downarrow & & \downarrow \\
U & & U
\end{array}
$$

where we require that $i$ is a proper monomorphism and $f$ factors as the composition of a log refinement followed by a log flat morphism.

This definition enforces the equidimensionality of fibres and captures the idea that cycles are precisely those which are extensions of irreducible log varieties. This gives rise to the following definition:

**Definition 16.** Let $i_j : V_j \rightarrow X$ be proper monomorphisms for $j \in \{1, 2\}$. We say that these two are deformation equivalent if there exists a deformation family $W \rightarrow U$ and two points $P_j \in U$ such that the fibre over $P_j$ is isomorphic to $i_j : V_j \rightarrow X$.

We write $\text{Def}_k(X)$ for the subgroup of $\mathbb{Z}_k^\dagger(X)$ generated by $[i_1 : V_1 \rightarrow X] - [i_2 : V_2 \rightarrow X]$ wherever $i_j : V_j \rightarrow X$ are such that the associated cycles are pure $k$-dimensional.

Passing to a refinement of a cycle changes the cycle. We believe that this is simply an artifact of how we set up the theory. Let us quotient out the difference as follows

**Definition 17.** Let $i_j : V_j \rightarrow X$ for $j \in \{1, 2\}$ be two cycles. We say that $i_1 : V_1 \rightarrow X$ is a refinement of $i_2 : V_2 \rightarrow X$ if there is a refinement morphism $V_1 \rightarrow V_2$ over $X$. This extends to an equivalence relation on the set of cycles. We say that two cycles are refinement equivalent if they lie in the same equivalence class under this relation.

We write $\text{Ref}_k(X)$ for the subgroup of $\mathbb{Z}_k^\dagger(X)$ generated by $[i_1 : V_1 \rightarrow X] - [i_2 : V_2 \rightarrow X]$ for all pairs of refinement equivalent pairs $i_1 : V_1 \rightarrow X$ and $i_2 : V_2 \rightarrow X$. 
Definition 18. Let $X$ be a log scheme, the $k$-dimensional logarithmic Chow group, denoted $A^+_k(X)$ is defined to be the quotient

$$\frac{Z^+_k(X)}{\langle \text{Re}_k(X), \text{De}_k(X), \text{Br}_k(X) \rangle}$$

These groups contain an incredible amount of information whenever the log structure are even slightly non-trivial, e.g., Spec $k$ with monoid $\mathbb{N}^2$. Let us calculate this in two examples where we can get a handle on this information.

Example 9. Take $L = \mathbb{A}^1$ with the toric log structure. We claim that $A^+_0(L)$ is trivial whilst $A^+_1(L)$ is two dimensional. The only zero cycles are just points with trivial log structure and these can only embed into the part with trivial log structure. These are then equivalent to the trivial cycle as in the classical theory. There are two possible cycles of dimension one, the origin and the entire space. It is clear that neither of these can be deformed in any manner and so freely generate the Chow group.

Let $P$ be the origin in $\mathbb{A}^2$ with the restriction of the toric log structure. Since $P$ is log irreducible and two dimensional the only two dimensional cycles are given by choosing rational cones inside $\mathbb{R}^2_{\geq 0}$ subject to the natural glueing condition on such cones coming from choosing refinements. A one dimensional cycles is a choice of map from the standard log point to $P$, so a choice of weights $1 \mapsto (a, b)$. Any two points with the same weights are equivalent, this can be seen by taking the $(a, b)$-weighted blowup of $\mathbb{A}^2$ at which point these two points lift to points on the same component of the exceptional curve. Therefore $A^+_1(P) \cong \mathbb{Z}^\omega$. The group $A^+_0(P)$ vanishes for degree reasons.

We previously introduced push and pull morphisms for cycles. We need to prove that these descend to the Chow groups.

Lemma 13 (Push and pull morphisms). The following maps descend to the quotient Chow groups:

1. Pullback along a flat morphism $f : X \to Y$.
2. Pullback along an extension morphism $f : X \to Y$.
3. Pushforward along a proper morphism $f : X \to Y$.

Proof. The first two of these commute with taking products and we use this to show that they preserve deformation, and refinement equivalence. That they preserve refinement equivalence is clear since log refinement morphisms are stable under pullback. Let $F$ be a log deformation family on $Y$. We claim that the pullback
remains a log deformation family. This means that the pullback map $f^{-1}F \to X \times U$ is a proper monomorphism and the morphism $f^{-1}F \to U$ factors as an extension morphology followed by a log flat morphism. As we said both maps are given by pullback and pullback preserves both properness and monicity, hence the fact that the induced map $f^{-1}F \to X \times U$ is a proper monomorphism is automatic. This situation is described in Figure 1. We now break into separate cases.

Suppose that $f$ is a log extension morphism. We want to show that the induced morphism $f^{-1}F \to U$ factors as an extension morphism followed by a log flat morphism. But we already have a factorisation $f^{-1}F \to F \to U$ where the second map factors as an extension morphism followed by a log flat morphism. By 11 we see that the induced morphism indeed does have this property.

Instead we suppose that $f$ is a log flat morphism. Then we have a factorisation $f^{-1}F \xrightarrow{p} F \xrightarrow{e'} F' \xrightarrow{s} U$ where $e$ is an extension morphism and $p$ and $s$ are both log flat. By Lemma 7 we can refactorise this to $f^{-1}F \xrightarrow{e'} f^{-1}F' \xrightarrow{p'} F' \xrightarrow{s} U$ with $e'$ an extension morphism and $p'$ log flat. Since log flat morphisms are stable under composition we have the desired result.

That proper morphisms preserve refinement equivalence is clear. Choosing different refinements just give different refinements of the image factorisation.

A picture for deformation equivalence under proper morphisms is described in Figure 2. To prove that this preserves deformation equivalence it would be enough to prove the following statement. Suppose that $\pi : F \to U$ is a deformation family, $P \subset U$ a strictly embedded point, $D$ an irreducible component of $F \times_U P$ and $f : X \to Y$ a proper morphism, then $\deg_{D/f(D)} = \deg_{F/f(F)}$.

We make some reductions to this situation. Firstly we don’t need to mention $X$ or $Y$, we have $f : F \to G$ an surjective proper morphism between irreducible log varieties, $\pi : F \to U$ the composition of an extension and a log flat morphism to an irreducible $U \subset \mathbb{A}^1$ with trivial log structure and $\psi : G \to U$ similarly such that $\pi = \psi f$. Next the question is local on $U$, so we may localise it at the point.

\[
\begin{array}{ccc}
X \times U & \xrightarrow{f^{-1}F} & F & \xrightarrow{F} Y \times U \\
\downarrow & & \downarrow & & \downarrow \\
U & \cong & U & & U
\end{array}
\]

Figure 1
Figure 2

$P$ so that $U \cong \text{Spec } k[t]_{(0)}$ and take fibre products. We denote the central fibres of these families by $F_P$ and $G_P$.

We now turn back to our geometrisation functor. There is an induced morphism $T_F \to T_G$ between schemes of the same whose degree equals the degree of the generic points. Writing $T_G$ as $G \times \text{Spec } M$ and base changing to the generic point of $\text{Spec } M$ we obtain a proper flat morphism of relative dimension zero $T_F \to T_G \times_G \text{Spec } k(M)$ whose degree over the closed point of $T_G \times_G \text{Spec } k(M)$ is equal to $\deg_{\text{F/\text{im(F)}}}$ and whose degree over the generic point is $\deg_{\text{F/\text{im(F)}}}$. By the classical theory of degree these two are equal.

For classical étale morphisms there are push-pull formulae for how they interact with the Chow groups, these should hold also in the log setting.

**Lemma 14.** Let $f : X \to Y$ be a proper degree $d$ log étale morphism. Then the composite $f_\ast f^\ast$ is multiplication by $d$.

**Proof.** Degree of a map is invariant under pullback and so the result follows as in the classical case.

We have seen that the refinement morphisms are non-trivial proper degree one morphisms. In particular this shows that such maps induce isomorphisms between log Chow groups. Again this suggests that we could invert such morphisms.
Future work

Log trivial cycles

One problem with this construction is that there can be not enough cycles with trivial log structure. For example take $\mathbb{P}^1$ with the induced log structure and consider what happens to a trivial point as it approaches zero. The limit does not exist, there is no choice of log structure. Any log scheme admits a proper log morphism to the underlying scheme with the trivial log structure. It could well be the case that the correct definition is to take the direct sum of these log Chow groups with the quotient of the Chow groups of the underlying scheme by the quotient under pushforward.

Kato-Nakayama spaces

In the classical setting there is a forgetful map from the Chow homology of a scheme over $\mathbb{C}$ to the singular homology of the associated complex analytic space. In general this map need not be injective or surjective, but has certainly proved very useful for the study of these spaces. For example it leads to Hodge theory and the Lefschetz theorems. The paper [Ogu] introduces a construction mimicking the associated analytic space, the Kato-Nakayama space.

Construction 1. We write $\mathbb{D}$ for the log ringed space $(\text{Spec } \mathbb{C}, \mathbb{R}_{\geq 0} \times S^1)$ where the monoid factor is the multiplicative monoid and maps via the exponential map to $\mathbb{C}$. As a set the Kato-Nakayama space $X^{\log}$ is defined to be $\text{Hom}(\mathbb{D}, X)$, and if $X$ is the spectra of a monoid ring then this naturally carries a topology. Now if $X$ has toric charts then the topologies on each chart glue to give a topology on $X^{\log}$. The forgetful map gives a continuous proper morphism to $X^{an}$. There are two choices of sheaves of rings on this space, defined in [Ogu] and [KN99], but we do not need this structure for our conversation.

This construction is clearly functorial and continuous and preserves families. This is enough to show that our notion of deformation equivalence produces homotopic families. What it does not do is preserve the notion of dimension under refinement, for instance the Kato-Nakayama space of $(\text{Spec } k, k^* \oplus \mathbb{N}^2)$ is two real dimensional, but the Kato-Nakayama space of our favourite refinement by $\mathbb{P}^1$ has real dimension three. The correct way to handle this is to consider the fibre product $KN(X) \times_{X^{an}} KN(X)$, a real manifold of the correct dimension, though potentially one needs to swap the orientation on one of these. Passing to refinements does not produce isomorphic Kato-Nakayama spaces, but in examples one finds that they are at least homotopic. Similarly this construction preserves break equivalence too. This leads us to propose the following conjecture:
Conjecture 1. There is a dimension preserving forgetful functor from the category of log varieties localised at the refinement morphisms to the category of complex manifolds localised at the homotopy equivalences.

Our construction of this functor would induce a natural morphism from the logarithmic Chow groups to the singular homology of the associated Kato-Nakayama space. One could ask how much of the classical theory one could hope to recreate in this situation.

The bivariant theory

We have constructed a dimension graded series of Abelian groups. In the classical setting these groups are not quite the correct object, one often wants a cohomology theory rather than a homology theory. The correct notion in the algebraic world is to study the Chow cohomology groups, defined as follows

**Definition 19.** Let \( f : V \to W \) be a morphism of varieties. A bivariant class of dimension \( d \) relative to \( f \) is a collection of maps \( A_*(Y) \to A_{*-d}(Y \times_W V) \) compatible with pushforward, pullback and Gysin maps. These form a group \( A_d(f : V \to W) \) and have compatible composition maps given composable morphisms \( f : X \to Y \) and \( g : Y \to Z \). The Chow cohomology groups are the groups \( A_*(1 : V \to V) \) and the composition maps give rise to a ring structure on this group.

The canonical location to learn about these groups is [Ful98]. These behave as a cohomology theory in the sense it is a ring under composition, that for a smooth variety \( V \) there is a perfect pairing \( A_*(V) \times A^*(V) \to \mathbb{Z} \) and these are functorial in the same manner as topological cohomology theories.

To imitate this we want to first construct Gysin maps. To do so we need to create a log version of the normal cone. Such a cone should be functorial and in the case of a log regular embedding produce a vector bundle. Such a bundle will not naturally be a bundle in the Zariski or étale topologies, but rather in a logarithmic variant. Once we have this the construction of the Gysin maps, Chow cohomology and perfect pairings is standard. This would produce novel results such as a perfect intersection theory on logarithmically smooth scheme.

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