Analytic Approach for Solving System of Fractional Differential Equations

Nabaa N. Hasan*, Zainab john

Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, IRAQ.

*Correspondent contact: alzaer1972@uomustansiriyah.edu.iq

ABSTRACT
In this paper, Sumudu transformation (ST) of Caputo fractional derivative formulae are derived for linear fractional differential systems. This formula is applied with Mittage-Leffler function for certain homogenous and nonhomogenous fractional differential systems with nonzero initial conditions. Stability is discussed by means of the system's distinctive equation.

KEYWORDS: Caputo derivatives; Sumudu transform; Mittage-Leffler function; asymptotically stable.

INTRODUCTION
Applications of fractional derivative in the present day includes fluid flow, dynamical process, electrical networks, probability and statistics, control theory and so on, [1]. ST first defined in 1993, which used to solve engineering control problems see [2]. However, ST solved fractional ordering differential equations and graph two-dimensional solutions. As shown in [3], the Taylor collection method was derived for solving fractional differential equations based on taking the truncated Taylor expansions of the vector-function solution. In [4] analytical solutions presented for systems of fractional differential equations using the differential transform method. As in [5], several sufficient criteria were established to ensure the Mittage-Leffler stability and asymptotic stability for the differential system of fractional order. In [6] study properties of stability, Mittage-Leffler stability, Liapchitz stability and comparison results of stability.

Preliminaries

Some important preliminaries of fractional calculus are given here.

Definition (2.1), [7]
Consider a set A defined as:

\[ A = \left\{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| \leq Me^{-\frac{|t|}{\tau_1}} if \ t \in (-1)X[0,\infty) \right\} \]

For all real t ≥ 0, and f(t) ∈ A, The Sumudu transform of f(t) is denoted by

\[ S[f(t)](u) = \int_0^\infty e^{-ut} f(ut)dt, \ u \in (\tau_1, \tau_2) \] (1)

Definition (2.2), [8]
The Caputo fractional differential operator \( D_t^\alpha \) of order \( \alpha \) is:

\[ D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau, \quad (2) \]

for \( n-1 < \alpha < n, \ n \in \mathbb{N}, \ t > 0. \] (2)
The Mittage Leffler function $E_v(Z)$ with $v > 0$, is define by the following series:

$$E_v(Z) = \sum_{n=0}^{\infty} \frac{Z^n}{(nv+1)}, \ v > 0, Z \in \mathbb{C}$$

**Definition (2.4), [1]**

Mittage-Leffler functions of one and two parameters are defined respectively:

$$E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)}, \alpha > 0, \ x \in \mathbb{C}$$

$$E_{\alpha\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)}, \alpha > 0, \beta > 0, \ x \in \mathbb{C}$$

**Definition (2.5), [1]**

The one parameters of Mittage-leffler function of the matrix $\mathcal{A} \in M_n$ ($M_n$ square matrix of order $nxn$) is defined for $\alpha > 0$,

$$E_{\alpha}(\mathcal{A}) = \sum_{k=0}^{\infty} \frac{\mathcal{A}^k}{\Gamma(\alpha k+1)},$$

$$E_{\alpha}(\mathcal{A}t^\alpha) = \sum_{k=0}^{\infty} \frac{\mathcal{A}^k t^{\alpha k}}{\Gamma(\alpha k+1)}, \quad (3)$$

**Remark (2.6)**

If $\mathcal{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a square Matrix of order $2 \times 2$, the Matrix Mittage-leffler function $E_{\alpha}(\mathcal{A}X^\alpha)$ is given by:

$$E_{\alpha}(\mathcal{A}X^\alpha) = \begin{bmatrix} E_{\alpha}(\lambda_1 X^\alpha) \\ E_{\alpha}(\lambda_2 X^\alpha) \end{bmatrix} = \begin{bmatrix} E_{\alpha}(\lambda_1 X^\alpha) + \frac{\lambda_2 - \lambda_1}{\lambda_1 - \lambda_2} E \\ \lambda_1^{1-\alpha} X^{1-\alpha} \end{bmatrix}$$

$$\begin{bmatrix} X^{1-\alpha} \frac{\lambda_2 - \lambda_1}{\lambda_1 - \lambda_2} E \\ \lambda_2^{1-\alpha} (\lambda_2 + \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2} E) \end{bmatrix}$$

where $E = E_{\alpha}(\lambda_1 X^\alpha) - E_{\alpha}(\lambda_2 X^\alpha)$ and $\lambda_1 \neq \lambda_2$ are the eigenvalues of $\mathcal{A}$.

**The Method:**

Method is derived by ST of Mittage-Leffler function for solving certain type of fractional differential equations.

**Lemma (3.1), [9]:**

Let $\alpha > 0$, $\beta > 0$, $\lambda \in \mathbb{R}$ and $u^{-\alpha} > |\lambda|$ then:

$$S[t^{\beta-1} E_{\alpha\beta}(\mathcal{A}t^\alpha) u] = \frac{u^{\beta-1}}{1 - \lambda u^\alpha}$$

where $E_{\alpha\beta}$ is Mittage-Leffler function in two parameters.

**Theorem (3.2), [9]:**

Let $n \in \mathbb{N}$ and $\alpha > 0$ be such that $n - 1 < \alpha < n$ and $F(u)$ be the ST of the function $f(t)$ then the ST Caputo $\alpha$ derivative of $f(t)$ is given by:

$$S[D^\alpha f(t)]u = u^{-\alpha} F(u) - \sum_{k=0}^{n-1} u^{\alpha-k} [f^{(k)}(0)]$$

**Example (3.3)**

Take into account the initial value problem (I.V. Problem) for a homogenous fractional differential equation

$$D^\alpha f(t) + af(t) = 0, \quad 0 < \alpha < 1,$$

$f(0) = c$

where $a$ and $c$ are constants, applying ST on both sides, hence

$$S(D^\alpha f(t))(u) + aS(f(t))(u) = 0$$

$$u^{-\alpha} F(u) - f(0) u^{-\alpha} + a F(u) = 0$$

$$(u^{-\alpha} + a) F(u) = f(0) u^{-\alpha}$$

since $f(0) = c$ then $F(u) = \frac{u^{-\alpha} c}{(u^{-\alpha} + a)} = \frac{c}{1 + au^{-\alpha}}$

by eq.(6) replacing $\beta = 1$

$$F(u) = S[E_{\alpha,1}(-at^\alpha)(u)]c$$

Taking inverse ST, we get

$$f(t) = c E_{\alpha}(-at^\alpha)$$

Now, we will generalize lemma(3.1) to solve a homogenous linear fractional system of order $0 < \alpha < 1$.

**Theorem (3.4)**

Let $\mathcal{A} \in M_n$ be a scalar matrix, $\eta \in M_{n,1}$ be a scalar vector and $y(t) \in M_{n,1}$ be unknown vector. The exact solution homogenous linear fractional system of order $0 < \alpha < 1$

$$D^\alpha y(t) = \mathcal{A} y(t), y(0) = \eta \quad (7)$$

is given by:

$$y(t) = E_{\alpha}(\mathcal{A}t^\alpha). \eta \quad (8)$$

Where $E_{\alpha}(\mathcal{A}t^\alpha)$ is the matrix Mittage-Leffler function.

**Proof**

Taking Sumudu transformation to both sides of eq. (7) and use the Sumudu transformation of the Caputo derivative to get

$$u^{-\alpha} Y(u) - u^{-\alpha} y(0) = \mathcal{A} Y(u)$$

$$(u^{-\alpha} I - \mathcal{A}) Y(u) = u^{-\alpha} \eta$$

$$Y(u) = \frac{u^{-\alpha} \eta}{u^{-\alpha} I - \mathcal{A}}$$

by lemma (3.1)

$$y(t) = S[E_{\alpha,1}(\mathcal{A}t^\alpha)(u)]. \eta$$

taking inverse of Sumudu transform we get eq. (8)

$$y(t) = E_{\alpha}(\mathcal{A}t^\alpha). \eta$$

**Example (3.5)**

Let the I.V. Problem for a fractional differential system of order $0 < \alpha < 1$

$$y(t) = \mathcal{A} y(t), \ y(0) = \eta$$

where $\mathcal{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\eta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
The eigenvalues of \( \mathcal{A} \) are \( \lambda_1 = 1 + i \), \( \lambda_2 = 1 - i \)

\[
E_\omega(\omega t^n) = \begin{cases}
E_\omega((1 + i)t^n) - E_\omega((1 - i)t^n) & t^n = t^{\alpha}(E_\omega((1 + i)t^n) - E_\omega((1 - i)t^n))
\end{cases}
\]

\[
y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \quad \text{and} \quad \eta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

Hence:
\[
y_1(t) = t^{1-\alpha} \left( E_\omega((1 + i)t^\alpha) - E_\omega((1 - i)t^\alpha) \right)
\]
\[
y_2(t) = t^{1-\alpha} \left( E_\omega((1 + i)t^\alpha) + E_\omega((1 - i)t^\alpha) \right) - \frac{1}{2} \left( E_\omega((1 + i)t^\alpha) - E_\omega((1 - i)t^\alpha) \right)
\]

To solve nonhomogeneous linear fractional system of order \( 0 < \alpha < 1 \), we first introduce the Sumudu transform convolution theorems.

**Theorem (3.5)**[10]

Let \( W_1(t) \) and \( W_2(t) \) functions in the set of functions \( \mathcal{A} \) having Sumudu transforms \( F(u) \) and \( G(u) \) respectively. Then the ST of the convolutions of \( W_1(t) \) and \( W_2(t) \), where

\[
(W_1 * W_2)(t) = \int_0^t W_1(\tau)W_2(t - \tau)d\tau
\]

is defined by:

\[
S((W_1 * W_2)(t)) = uF(u)G(u).
\]

Now, we will generalize lemma(3.1), theorem(3.4), and theorem(3.5) to solve nonhomogenous linear fractional systems of order \( 0 < \alpha < 1 \)

**Theorem (3.6)**

Let \( \mathcal{A} \in M_n \) be a scalar matrix, \( \eta \in M_{n,1} \) be a scaler vector \( W_1(t) \in M_{n,1} \) and \( y(t) \in M_{n,1} \) be unknown vector. The exact solution nonhomogenous linear fractional systems of order \( 0 < \alpha < 1 \),

\[
D^\alpha y(t) = \mathcal{A}y(t) + W_1(t) \quad y(0) = \eta
\]

is given by:

\[
y(t) = E_\alpha(\mathcal{A}t^\alpha)\eta + \int_0^t (t-s)^{\alpha-1}E_\alpha(\mathcal{A}(t-s)^{\alpha})W_1(s)ds
\]

where \( E_\alpha(\mathcal{A}t^\alpha) \) is the matrix Mittage-Leffler function.

**Proof**

Taking ST to both sides of eq.(10),

\[
u^{-\alpha}Y(u) - u^{-\alpha}y(0) = \mathcal{A}Y(u) + F(u)
\]

\[
Y(u) = \frac{u^{-\alpha}\eta}{u^{-\alpha}I - \mathcal{A}} + \frac{F(u)}{u^{-\alpha}I - \mathcal{A}}
\]

Applying the inverse ST to both sides of eq. (12), we have

Using eq. (8) and eq(4), we have

\[
S^{-1}(Y(u)) = S^{-1}\left\{ \frac{u^{-\alpha}\eta}{u^{-\alpha}I - \mathcal{A}} + S^{-1}(F(u)) \star S^{-1}(u^{-\alpha}I - \mathcal{A})^{-1} \right\}
\]

By substituting the Sumudu transform of the Mittage-Leffler function lemma (3.1) and theorem (3.6) we get the solution as in eq. (11).

**Example (3.6)**

Consider the initial value problem for a nonhomogeneous fractional differential system of order \( 0 < \alpha < 1 \)

\[
D^\alpha y(t) = \mathcal{A}y(t) + W_1(t) \quad y(0) = \eta
\]

where \( \mathcal{A} = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \eta = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, f(t) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} \)

hence by eq.(11):

\[
y(t) = \left( E_\alpha\left[ \begin{bmatrix} -2 \\ 0 \\ 0 \\ 3 \end{bmatrix}t^\alpha \right] \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right) + \int_0^t (t-s)^{\alpha-1}E_\alpha\left[ \begin{bmatrix} -2 \\ 0 \\ 0 \\ 3 \end{bmatrix} (t-s)^{\alpha} \right] \begin{bmatrix} \sin s \\ \cos s \end{bmatrix} ds
\]

Then

\[
y_1(t) = E_\alpha(-2t^\alpha) + \int_0^t (t-s)^{\alpha-1}E_\alpha(-2(t-s)^{\alpha}) \sin s ds
\]

\[
y_2(t) = -2E_\alpha(3t^\alpha) + \int_0^t (t-s)^{\alpha-1}E_\alpha(3(t-s)^{\alpha}) \cos s ds
\]

**STABILITY ANALYSIS**

Stability of the linear fractional differential system defined by the Caputo’s derivative \( 0 < \alpha < 1 \) is discussed here according to two theorems

**Theorem (4.1)**, [11]

The system eq.(7) is asymptotically stable if and only if the eigenvalues \( \lambda(A) \) of the matrix \( \mathcal{A} \) satisfy

\[
\frac{\cos(\lambda(A))}{\lim_{||A||}} < 1 - \alpha
\]

**Theorem (4.2)**, [12]

The system eq.(7) is asymptotically stable if and only if \( |\arg(spec(A))| > \alpha \frac{\pi}{2} \), where \( spec(A) \) is the spectrum of \( \mathcal{A} \).

Now discuss the stability of the linear system given in example (3.5) as follows:
Since $\mathcal{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, and $\lambda = 1 \mp i$

By theorem (4.1),
\[
\cos \lambda = \frac{1}{2} \quad \text{and} \quad ||\lambda|| = \sqrt{2}
\]
then
\[
\frac{\cos \lambda}{||\lambda||} = \frac{1}{2} < 1 - \alpha
\]
Hence $\alpha < 1/2$, there values of $\alpha$ get the system is asymptotically stable where $0 < \alpha < 1$.

By theorem (4.2):
\[
|\arg(\text{spec}(\mathcal{A}))| = |\theta| = 0.785
\]
0.785 $> \frac{\pi}{2}$

hence $\alpha < 0.5$, then the system (10) is asymptotically stable when $\alpha < 0.5$.

CONCLUSIONS
In this work, we studied and proved the ST operational transform method as shown in theorems, which are important in solving certain homogenous and non-homogenous fractional differential systems associating the Caputo fractional derivatives.

ACKNOWLEDGMENTS
I would like to thank Mustansiriyah University (www.uomustansiriyah.edu.iq) Baghdad-Iraq for its support in the present work.

REFERENCES
[1] Kilçman, A., & Altun, O. (2014). Some remarks on the fractional Sumudu transform and applications. Appl. Math, 8(6), pp. 1-8. http://dx.doi.org/10.12785/amis/080625
[2] Bulut, H., Baskonus, H. M., & Belgacem, F. B. M. (2013, January). The analytical solution of some fractional ordinary differential equations by the Sumudu transform method. In Abstract and Applied Analysis, Vol. (2013). https://doi.org/10.1155/2013/203875
[3] Sheikhani, A. H. R., & Mashoof, M. (2017). A Collocation Method for Solving Fractional Order Linear System. Journal of the Indonesian Mathematical Society, 23(1), pp. 27-42.

https://doi.org/10.22342/jims.23.1.257.27-42
[4] Ertürk, V. S., & Momani, S. (2008). Solving systems of fractional differential equations using differential transform method. Journal of Computational and Applied Mathematics, 215(1), pp. 142-151. https://doi.org/10.1016/j.cam.2007.03.029
[5] Li, X., Liu, S., & Jiang, W. (2018). q-Mittag-Leffler stability and Lyapunov direct method for differential systems with q-fractional order. Advances in Difference Equations, 2018(1), pp. 1-9. https://doi.org/10.1186/s13662-018-1502-5
[6] Skhail, E. S. E. A. (2018). Some Qualitative Properties of Fractional Order Differential Systems (Doctoral dissertation, Faculty of Science Department of Mathematics Some Qualitative Properties of Fractional Order Differential Systems Submitted by: Esmail Syaid Esmail Abu Skhail Supervisor Dr. Mohammed M. Matar Department of Mathematics, Faculty of Science, Al-Azhar University–Gaza).
[7] Takaci, D., Takaci, A., & Takaci, A. (2017). Solving fractional differential equations by using Sumudu transform and Mikusinski calculus. J. Inequal. Spec. Funct, 8(1), pp. 84-93.
[8] Al-Shammari, A. G. N., Abd AL-Hussein, W. R., & AL-Safi, M. G. (2018). A new approximate solution for the Telegraph equation of space-fractional order derivative by using Sumudu method. Iraqi Journal of Science, 59(3A), pp. 1301-1311. https://doi.org/10.24996/ijis.2018.59.3A.18
[9] Bodkhe, D. S., & Panchal, S. K. (2016). On Sumudu transform of fractional derivatives and its applications to fractional differential equations. Asian Journal of Mathematics and Computer Research, 11(1), pp. 69-77.
[10] Belgacem, F. B. M., Karaballi, A. A., & Kalla, S. L. (2003). Analytical investigations of the Sumudu transform and applications to integral production equations. Mathematical problems in Engineering, 2003. https://doi.org/10.1155/S1024123X03207018
[11] Li, H., Cheng, J., Li, H. B., & Zhong, S. M. (2019). Stability analysis of a fractional-order linear system described by the Caputo–Fabrizio derivative. Mathematics, 7(2), pp. 1-9. https://doi.org/10.3390/math7020200
[12] Chaid, A. R. K. K. M. (2016). Stability of Linear Multiple Different Order Caputo Fractional System. Control Theory and Informatics, 6(3), p.55-68.