Graph signal processing (GSP) is an important methodology for studying arbitrarily structured data. As acquired data is increasingly taking the form of multi-way tensors, new signal processing tools are needed to maximally utilize the multi-way structure within the data. We review modern signal processing frameworks generalizing GSP to multi-way data, starting from graph signals coupled to familiar regular axes such as time in sensor networks, and then extending to general graphs across all tensor modes. This widely applicable paradigm motivates reformulating and improving upon classical problems and approaches to creatively address the challenges in tensor-based data. We synthesize common themes arising from current efforts to combine GSP with tensor analysis and highlight future directions in extending GSP to the multi-way paradigm.

I. INTRODUCTION

Over the past decade, graph signal processing (GSP) [1, 2] has laid the foundation for generalizing classical Fourier theory for processing data measured on a regular grid, such as time, to handle irregularly structured data, such as network data. An existing gap in GSP is that graphs signals are processed independently of one another, i.e., single-way processing, while the topology underlying a collection of signals, e.g., the axis of time in a sensor network, is not taken into account. In the coming decade, generalizing GSP to handle multi-way data, represented by multidimensional arrays or tensors, will be essential. This survey discusses the burgeoning family of multi-way graph signal processing (MWGSP) methods for analyzing data tensors as a dependent collection of axes.

To introduce the concept of way, consider a network of $N$ sensors each measuring a signal sampled at $T$ time points. The classic signal processing perspective treats these signals as a collection of $N$ independent 1D time-series and the relation structure of the graph is ignored. On the other hand, the standard GSP perspective treats the data as a collection of $T$ independent 1D graph signals that describe the state of all sensors for a given time point $t_j \in T$. Both are single-way approaches that ignore the underlying geometry of the other way (also referred to as mode). As an example of a MWGSP framework, the time-vertex (T-V) framework [3] unifies these perspectives to form a 2-way framework that processes graph signals that are time-varying [4]. The T-V framework bridges the gap between classical signal processing and GSP, yet can be seen as a subset of a framework that considers the coupling of

\begin{enumerate}
  \item Note that the graph itself is static, the signals are time-varying
\end{enumerate}
multiple geometries, whether predefined temporal or spatial axes, or irregular graph-based axes. Such a general framework is by definition more versatile and is our main focus.

In a T-V signal one of the modes of the data is a regular grid – time. In general, however, a regular geometry may not underlie any of the modes of the data, e.g., genes and cells in sequencing data or users and items in recommendation systems [4–3]. Furthermore, in a wide range of applications, data consists of more than 2 ways, and is given as a multi-way data tensor. Mathematically, tensors generalize matrices to higher dimensions [9]. In this work the term tensors includes matrices (as they are 2-tensors). Examples of tensors includes video, hyperspectral imaging, MRI scans, multi-subject fMRI data, epigenetics, trial-based neural data, and higher-order sparse tensor data such as databases of crime incident reports, taxi rides or ad click data [10–15].

In this overview of multi-way data analysis, we present a broad viewpoint to simultaneously consider general graphs underlying all modes of a tensor. Thus, MWGSP is a non-trivial departure from classical signal processing, producing an opportunity to exploit joint structures and correlations across modes to more accurately model and process signals in real-world applications of current societal importance: climate, spread of epidemics and traffic, as well as complex systems in biology. Classical signal processing and GSP typically process 1D or 2D signals [1, 16–17, 2, 18, 3] and do not address higher-order tensor data. Here we lay the mathematical and theoretical foundations to develop a framework for higher-order signal processing of tensor data, and explore the challenges and algorithms that result when one imposes relational structure along all axes of data tensors.

While tensors are the primary structure for representing $D$-dimensional signals, research on tensors has primarily focused on methods for tensor factorization [9, 19, 20], while ignoring the underlying geometry on the tensor modes. We interpret multi-way analyses in light of graph-based signal processing to consider tensors as multi-way graph signals defined on multi-way graphs. GSP is a powerful framework in the multi-way setting because it leads to intuitive and uniform interpretations of operations on irregular geometry. At the heart of this framework is the graph Laplacian, which provides a basis for harmonic analysis of data in MWGSP and an important regularizer in modeling and recovery of multi-way graph signals. We illustrate the breadth of MWGSP by reinterpreting classic techniques, such as the 2-D discrete Fourier transform as a special case of MWGSP. Further, GSP can inspire novel multi-way regularizations that are not immediately obvious by viewing the data purely as a tensor. Thus, we band together a coherent family of recent and novel MWGSP methods across varied applications in inpainting, denoising, data completion, factor analysis, dictionary learning, and graph learning [12, 13, 5, 21–26].

The organization of this paper is as follows. Sec. [II] reviews standard GSP, which we refer to as single-way GSP. In Sec. [III] we introduce tensors and multilinear operators and construct multi-way graphs, transforms, and filters. Sec. [IV] briefly highlights two recent multiway frameworks: the time-vertex framework, a natural development of MWGSP that couples a known time axis to a graph topology, and the Generalized Graph Signal Processing framework which extends MWGSP by coupling non-discrete and arbitrary geometries into a single signal processing
framework. In Sec. V we move to multi-way signal modeling and recovery, where graph-based multi-way methods are used in a broad range of tasks. In Sec. VI we conclude the paper with open questions for future work.

II. SINGLE-WAY GSP

GSP generalizes classical signal processing from regular Euclidean geometries such as time and space, to arbitrary, irregular, and non-Euclidean geometries represented discretely by a graph. In this section, we review basic concepts.

a) Graphs: This tutorial considers undirected, connected, and weighted graphs \( G = \{V, E, W\} \) consisting of a finite vertex set \( V \), an edge set \( E \), and a weighted adjacency matrix \( W \). If two vertices \( v_i, v_j \) are connected, then \( (v_i, v_j) \in E \), and \( W_{i,j} = W_{j,i} > 0 \); otherwise \( W_{i,j} = W_{j,i} = 0 \). We employ a superscript parenthetical index to reference graphs and their accompanying characteristics from a set of graphs \( G^{(i)} \), i.e., \( G = \{G^{(i)} = (V^{(i)}, E^{(i)}, W^{(i)})\}_{i=1}^D \). Contextually we will refer to the cardinality of the vertex set of a graph \( G^{(i)} \) as \( |V^{(i)}| = n_i \). When parenthetical indexing is not used, we refer to a general graph \( G \) on \( n \) nodes.

In some applications, the graph is known a-priori: vertices represent physical locations or sensors, such as traffic intersection or routers in a wifi network, while connectivity encodes physically connected or proximal locations. In contrast, for other settings there is no a-priori graph - a vertex set and its connectivity must be learned from data [27]. See also the box “Graph construction.”

b) Graph Signals: A signal \( f : V \rightarrow \mathbb{R} \) on the vertices of a graph on \( n \) nodes may be represented as a vector \( f \in \mathbb{R}^n \), where \( f_i = f(v_i) \) is the signal value at vertex \( v_i \in V \). The graph Fourier transform decomposes a graph signal in terms of the eigenvectors of a graph shift operator. Many choices have been proposed for graph shifts, including the adjacency matrix \( W \) and various forms of the graph Laplacian \( L \), a second order difference operator over the edge set of the graph. In this paper we use the popular combinatorial graph Laplacian defined as \( L := D - W \), where the degree matrix \( D \) is diagonal with elements \( D_{ii} = \sum_j W_{ij} \). This matrix is real and symmetric. Its eigendecomposition is \( L = \Psi A_G \Psi^* \), where the columns of \( \Psi \) are a complete set of orthonormal eigenvectors \( \{\psi_\ell\}_{\ell=0}^{n-1} \), the diagonal of \( A_G \) are the real eigenvalues \( \{\lambda_\ell\}_{\ell=0}^{n-1} \) and * denotes the adjoint matrix. One may refer to the eigenvectors of \( L \) as graph harmonics and the eigenvalues as graph frequencies.

c) Graph Fourier Analysis: The Graph Fourier Transform (GFT) and its inverse are

\[
\hat{f}(\lambda_k) := \langle f, \psi_k \rangle = \sum_{k=1}^N f(k) \psi_k^*(k) \quad \text{and} \quad f(k) = \sum_{\ell=0}^{N-1} \hat{f}(\lambda_\ell) \psi_\ell(k),
\]

or in matrix form \( \text{GFT}\{f\} = \Psi^* f \). The GFT generalizes the classical Fourier transform since the former is the spectral expansion of a vector in the discrete graph Laplacian eigensystem while the latter is the spectral expansion of a function in the eigensystem of the continuous Laplacian operator. Indeed, the GFT is synonymous with the discrete Fourier transform (DFT) when the graph Laplacian is built on a path or ring graph.

\( ^2 \)A complete survey of GSP is provided in [2].
The quadratic form of the graph Laplacian quantifies the smoothness of a graph signal

\[ f^T L f = \sum_{(i,j) \in E} W_{i,j} (f_i - f_j)^2. \]  

Using the Courant-Fischer theorem, it is easy to show that the graph Laplacian eigenvectors are the smoothest choice of unit basis functions for graph signals. Consequently, it is typical to reinforce the classical Fourier analogy by indexing graph harmonics in ascending order according to their eigenvalues, such that the lowest indexed harmonics are the smoothest elements of the graph eigenbasis.

Despite these analogies, there are many difficulties in directly extending classical tools to arbitrary graphs. For example, there is no straightforward analogue of convolution in the time domain to convolution in the vertex domain of graph signals. Instead, filtering signals in the GFT domain is defined analogously to filtering in the frequency domain, with a filtering function \( \hat{h}(\cdot) \) applied to the eigenvalues \( \lambda_\ell \), which take the place of the frequencies:

\[ \tilde{f}(k) = \sum_{\ell=0}^{N-1} \hat{h}(\lambda_\ell) \hat{f}(\lambda_\ell) \psi_\ell(k), \]  

where \( \tilde{f} \) is the result of filtering \( f \) with the graph spectral filter \( h(\mathcal{L}) \). This spectral analogy is a common approach for generalizing classical notions that lack clear vertex interpretations.

III. EXTENDING GSP TO MULTI-WAY SPACES

Classical \( D \)-dimensional Fourier analysis provides a template for constructing unified geometries from various data sources. The \( D \)-dimensional Fourier transform applies a 1-dimensional Fourier transform to each axis of the data sequentially. For example, a 2D-DFT applied to an \( n_1 \times n_2 \) image \( X \) is

\[ \text{2D-DFT}\{X\} = \text{DFT}_c(\text{DFT}_r(X)) = \text{DFT}_r(\text{DFT}_c(X)) = U_{n_1} X U_{n_2}, \]  

where \( \text{DFT}_r \) (\( \text{DFT}_c \)) applies the DFT to the rows (columns) of \( X \) and \( U_n \) denotes a normalized \( n \)-point DFT matrix: \( U_n(t,k) = \frac{1}{\sqrt{n}} \exp\{-2\pi jtk/n\} \). This 2D transform decomposes the input into a set of plane waves.

The 2D-GFT is algebraically analogous to the 2D-DFT. For two graphs \( G^{(1)} \) and \( G^{(2)} \) on \( n_1 \) and \( n_2 \) vertices, a 2D graph Fourier transform of \( X \in \mathbb{R}^{n_1 \times n_2} \) is

\[ \text{2D-GFT}(X) = \text{GFT}_{n_1}(\text{GFT}_{n_2}(X))) = \text{GFT}_{n_2}(\text{GFT}_{n_1}(X)). \]  

[18] suggested the 2D-GFT as a method for efficiently processing big-data. We will elaborate its similarity to the DFT shortly but to provide brief intuition we note that when \( G^{(1)} = P^{n_1} \) and \( G^{(2)} = P^{n_2} \), i.e., they are path graphs on \( n_1 \) and \( n_2 \) vertices, this transform is equivalent to a 2D-DFT [18].

In the following, we describe MWGSP, a framework for \( D \)-dimensional signal processing on coupled and irregular domains, which enables holistic data analysis by considering relational structures on potentially all modes of a
multi-way signal. MWGSP encompasses standard GSP while extending fundamental GSP tools such as graph filters to $D$-dimensions. Furthermore, because graphs can be used to model discrete structures from classical signal processing, MWGSP forms an intuitive superset of discrete signal processing on domains such as images or video.

A. Tensors

Tensors are both a data structure representing $D$-dimensional signals, as well as a mathematical tool for analyzing multilinear spaces. We use both perspectives to formulate MWGSP. In this paper, we adopt the tensor terminology and notation used by [9].

1) Tensors as a $D$-dimensional array: The number of ways or modes of a tensor is its order. Vectors are tensors of order one and denoted by boldface lowercase letters, e.g., $a$. Matrices are tensors of order two and denoted by boldface capital letters, e.g., $A$. Tensors of higher-order, namely order three and greater, we denote by boldface Euler script letters, e.g., $\mathcal{A}$. If $\mathcal{A}$ is a $D$-way data array of size $n_1 \times \cdots \times n_D$, we say $\mathcal{A}$ is a tensor of order $D$.

**Order-3 tensors**

For simplicity, we briefly review some tensor terminology for the 3-way tensor $\mathbf{X} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$. The size of each mode is denoted by $n_i$, with $n_1$ being the number of columns, $n_2$ the number of rows, and $n_3$ the number of tubes [9]. Video and time-series recording of matrix valued signals are a common application for tensors of this form (Fig. 1(a)). In videos, the first and second modes of the tensor encode pixel values for each frame, while the third axis indexes the frames in time. We can slice a video tensor to produce different views of the data. A frontal slice is the matrix $\mathbf{X}_{i,:,k}$, formed by selecting the $k$-th frame of the video as in Fig. 1(b). Though each frame matrix is intuitively an image, we can also create matrices using horizontal slices, $\mathbf{X}_{i,:,}$, which for a video produces a matrix (viewable as an image) that shows the time evolution of the $i$-th row of pixels in the input as in Fig. 1(c); similarly, the lateral slice $\mathbf{X}_{:,j,:}$ contains the time evolution of one column of pixels of the input as in Fig. 1(b). Finally, one can consider 2D and 3D indexing of 3rd order tensors, with, for example, the fiber $\mathbf{X}_{i,j,:}$ returning a $n_3$ dimensional time-series vector of the value of the $i,j$th pixel of each frame shown in Fig. 1(e), while $\mathbf{X}_{i,j,k}$ is a single number corresponding to the value of pixel $i,j$ in frame $k$.

![Fig. 1.](image)

There are multiple operations to reshape tensors. **Vectorization** maps the elements of a matrix into a vector in column-major order. That is, for $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2}$,

$$\text{vec}(\mathbf{X}) = [\mathbf{X}_{1,1}, \ldots, \mathbf{X}_{n_1,1}, \mathbf{X}_{1,2}, \ldots, \mathbf{X}_{n_1,2}, \ldots, \mathbf{X}_{1,n_2}, \ldots, \mathbf{X}_{n_1,n_2}]^T.$$
A tensor mode-$d$ vectorization operator, \( \text{vec}_d(\mathbf{X}) \) is similarly defined by stacking the elements of \( \mathbf{X} \) in mode-$d$ major order. Let \( \text{ten}(\mathbf{x}, \ell, \{n_1, \ldots, n_D\}) = \mathbf{X} \) be the \( \ell \)-th tensorization of \( \mathbf{x} \), which is the inverse of the \( \ell \)-major vectorization of \( \mathbf{X} \). Denote by \( n \setminus \ell = \prod_{i=1}^{\ell-1} n_i \prod_{j=\ell+1}^{D} n_j \) the product of all factor sizes except for the \( \ell \)-th factor. Then, let \( \text{mat}(\mathbf{X}, \ell) = \mathbf{X}(\ell) \in \mathbb{R}^{n_\ell \times n \setminus \ell} \) be the mode-$\ell$ matricization of \( \mathbf{X} \) formed by setting the \( \ell \)-th mode of \( \mathbf{X} \) to the rows of \( \mathbf{X}(\ell) \), vectorizing the remaining modes to form the columns of \( \mathbf{X}(\ell) \) as in Fig. 1(f).

2) Tensor products: Up to this point we have avoided explicitly constructing \( D \)-dimensional transforms. In the 2D case, applying a two-dimensional transform is calculated via linear operators as in (4); generalizing to higher-order tensors requires multilinear operators. Therefore, we introduce the tensor product and its discrete form, the Kronecker product. These products are powerful tools for succinctly describing \( D \)-dimensional transforms.

The great utility of the tensor product is that it simultaneously transforms spaces alongside their linear operators. This is the so-called universal property of the tensor product. In brief, it states that the tensor product of two vector spaces \( V \) and \( W \) is the unique result of a bilinear map \( \varphi : V \times W \rightarrow V \otimes W \). The power in \( \varphi \) is that it uniquely factors any bilinear map on \( V \times W \) into a linear map on \( V \otimes W \). The universal property implies that the tensor product is symmetric: \( V \otimes W \) is a canonical isomorphism of \( W \otimes V \). Though the tensor product is defined in terms of two vector spaces, it can be applied repeatedly to combine many domains, so we generically refer to it as a product of many spaces.

In this paper, we are particularly concerned with the tensor product on Hilbert spaces \( \mathcal{H}^{(k)}, k = 1, \ldots, D \). These metric spaces include both continuous and discrete Euclidean domains from classical signal processing, as well as the non-Euclidean vertex domain. Since tensor products on Hilbert spaces produce Hilbert spaces, we can combine time, space, vertex, or other signal processing domains via the tensor product and remain in a Hilbert space. Under some constraints, an orthonormal basis for the product of \( D \) Hilbert spaces is admitted directly by the tensor product of the factor spaces.

To obtain a \( D \)-dimensional GFT, we note that 1) these discrete 1D transforms are linear, and 2) the domain of a \( D \)-dimensional GFT is the tensor product of \( D \) Hilbert spaces. Thus, by the universal property of tensor products, a multidimensional GFT can be obtained by a multilinear transformation of the 1D transforms. Remarkably, the new transform may be computed without computing a product space. However, as the GFT amounts to a change of basis, its multidimensional extensions immediately yield a basis that encodes the geometry of the product space.

Kronecker products: The Kronecker product produces the matrix of a tensor product with respect to a standard basis and generalizes the outer product of vectors \( \mathbf{x} \mathbf{y}^\ast \) for \( \mathbf{x} \in \mathbb{C}^m \) and \( \mathbf{y} \in \mathbb{C}^n \). For analogy, it is common to use the same notation to denote the Kronecker and tensor product. The Kronecker product is associative. Consequently, for a sequence of \( D \) matrices \( \mathbf{M}^{(k)} \in \mathbb{C}^{n_k \times n_k} \) for \( k = 1, \ldots, D \)

\[
    \mathbf{M} = \bigotimes_{k=1}^{D} \mathbf{M}^{(k)} = \mathbf{M}^{(1)} \bigotimes_{k=2}^{D} \mathbf{M}^{(k)} = \left( \bigotimes_{k=1}^{D-1} \mathbf{M}^{(k)} \right) \otimes \mathbf{M}^{(D)} = \mathbf{M}^{(1)} \otimes \cdots \otimes \mathbf{M}^{(D)}. \tag{6}
\]
It is important to note that the Kronecker product is in general non-commutative.

The Kronecker product has many convenient algebraic properties for computing multidimensional transforms. Vectorization enables one to bilinear matrix multiplication as a linear transformation

\[
\text{vec}(CXB) = (B^T \otimes C) \text{vec}(X),
\]

assuming that the dimensions of \(C, X, B\) are compatible such that \(CXB\) is a valid operation.

This identity is a discrete realization of the universal property of tensors, and shows that the Kronecker product corresponds to a bilinear operator. We will use this identity to 1) construct multi-dimensional discrete Fourier bases, and 2) decompose multi-way algorithms for computational efficiency.

3) Constructing multi-way Fourier bases: We now apply Equation (7) to construct a 2D-GFT. If \(\Psi^{(1)}\) and \(\Psi^{(2)}\) are Fourier bases for graph signals on any two graphs \(G_1\) and \(G_2\), a 2D-GFT basis is \(\Psi^{(2)} \otimes \Psi^{(1)}\). This is a single orthonormal basis of dimension \(|\mathcal{V}^{(1)}||\mathcal{V}^{(2)}|\times|\mathcal{V}^{(1)}||\mathcal{V}^{(2)}|\), which can be used to describe a 2D graph signal \(X \in \mathbb{R}^{n_1 \times n_2 \times n_3}\) in the geometry of a single graph by the GFT

\[
(\Psi^{(2)} \otimes \Psi^{(1)})^* \text{vec}(X) = \hat{x}.
\]

By constructing a 2D-GFT using Equation (7), the permutation equivalence of multidimensional Fourier transforms (see, e.g., Eqs. (5) and (5)) becomes apparent. This is because for any Kronecker product of two matrices \(B\) and \(C\) there always exists a pair of permutation matrices \(P_1\) and \(P_2\) such that \(P_1(B \otimes C)P_2 = (C \otimes B)\) [28].

Unlike the DFT, where it is clear that increasing dimension yields grids, cubes, and hypercubes, interpreting the geometry of \(\Psi^{(2)} \otimes \Psi^{(1)}\) is less intuitive. For this, we must turn to a graph product.

Product graphs: MWGSP relies on a graph underlying each mode of the given tensor data. The question is: What joint geometry arises from these graphs, and what multilinear operators exist on this joint graph structure? Our approach is to construct a multidimensional graph \(G\) over the entirety of a data \(X\) as the product graph of a set of factor graphs \(G = \{G^{(1)}, \ldots, G^{(D)}\}\).

For example, if \(X\) contains the results of a \(n_3\) sample longitudinal survey of \(n_2\) genes on a cohort of \(n_1\) patients, then the intramodal relationships of \(X \in \mathbb{R}^{n_1 \times n_2 \times n_3}\) are modeled by separate graphs \(G^{(1)}\), in which each patient is a vertex, \(G^{(2)}\), in which each gene is a vertex, and \(G^{(3)}\), which represents time as a path graph on \(n_3\) vertices. We will use this example throughout this section, though our derivation generalizes to any order tensor.

While one could treat matrix-valued slices of \(X\) as signals on each individual graph or perform a multilinear GFT over the data, we use the graph product to model \(X\) as a single graph signal. We begin by constructing the vertices of \(G\), which for all graph products is performed by assigning a single vertex to every element in the Cartesian product of the factor vertex sets i.e., \(\mathcal{V} = \mathcal{V}^{(1)} \times \ldots \times \mathcal{V}^{(D)}\). Thus, the vertex set of \(G\) is of cardinality \(n = \prod_{k=1}^{D} n_k\). For example, our longitudinal survey will be modeled by the product graph \(G\) on \(n = n_1n_2n_3\).
vertices. As a Cartesian product, the elements of \( \mathcal{V} \) can be expressed as \((\text{patient}, \text{gene}, \text{time})\). The experimental observation tensor can be modeled as a graph signal \( \mathbf{x} = \text{vec}(\mathbf{X}) \) in \( \mathbb{R}^n \).

Our next step is to learn the geometry of \( \mathcal{G} \). This geometry is given by a multilinear mapping of the edge sets (weights) of the factor graphs into a single set of product edges (weights). There are a variety of graph products, each of which differs from each other only in the construction of this map. We focus on the \textit{Cartesian graph product} as it is the most widely employed in multi-way algorithms. However, we note that other products such as the tensor and strong graph products each induce novel edge topologies that warrant further exploration for MWGSP \cite{29, 18}.

\textit{Cartesian graph products:} We denote the Cartesian product of \( D \) graphs as
\[
\mathcal{G} = \mathcal{G}(1) \square \cdots \square \mathcal{G}(D) .
\]

The Cartesian graph product is intuitively an XOR product since for any two vertices \( \{v = (v^{(1)}, \ldots, v^{(D)}), u = (u^{(1)}, \ldots, u^{(D)})\} \subset \mathcal{V} \), the edge \((v, u)\) exists if and only if there exists a single \( i \) such that \( (v_i, u_i) \in \mathcal{E}^{(i)} \) and \( v_\ell = u_\ell \) for all \( \ell \neq i \). In other words, the vertices of \( \mathcal{G} \) are connected if and only if exclusively one pair of factor vertices are adjacent and the remaining factor vertices are the same. Figure 2a illustrates the generation of an \( n_1 \times n_2 \) 2D grid graph via the product of two path graphs on \( n_1 \) and \( n_2 \) vertices.

The XOR product can induce topological properties such as regularity onto a graph. Since the path graph basis is well-characterized as a discrete Fourier basis, it is a convenient tool for including Euclidean domains in multi-way analysis. For example, we can model time series and longitudinal graph signals as a single vector using a path graph product. In the case of our gene expression data \( \mathbf{X} \), the product of the gene and patient mode graphs with a path on \( n_3 \) vertices, i.e., \( \mathcal{G}(1) \square \mathcal{G}(2) \square \mathcal{P}^{n_3} \), models the data by treating the temporal mode as a sequence. One can intu this operation as copying \( \mathcal{G}(1) \square \mathcal{G}(2) \) \( n_3 \) times and connecting neighborhoods between each copy.

\textit{Product graph matrices:} The Kronecker product links graph shift operators on Cartesian product graphs to the corresponding operators on the factors. The \textit{Kronecker sum} of \( D \) matrices \( \mathbf{A}^{(k)} \in \mathbb{C}^{n_k \times n_k} \) for \( k = 1, \ldots, D \) is
\[
\mathbf{A} = \bigoplus_{k=1}^D \mathbf{A}^{(k)} = \sum_{k=1}^D \mathbf{I}_{n_{>k}} \otimes \mathbf{A}^{(k)} \otimes \mathbf{I}_{n_{<k}}, \text{ where } n_{>k} = \prod_{\ell=k+1}^{D} n_\ell, \text{ and } n_{<k} = \prod_{\ell=1}^{k-1} n_\ell.
\]

The joint adjacency matrix \( \mathbf{A} \) and graph Laplacian \( \mathbf{L} \) are constructed by the Kronecker sum of their corresponding factor graph matrices. The eigensystem of a Kronecker sum is generated by the pairwise sum of the eigenvalues of the factors and the tensor product of the factor eigenbases \cite[Thm. 4.4.5]{28}. Thus, the Fourier basis for \( \mathcal{G} \) is immediate from the factors. For \( k = 1, \ldots, D \) let \( (\lambda_\ell, \mathbf{\Psi}_\ell) \) be the \( \ell \)th eigenpair of \( \mathbf{L}^{(k)} \) for \( 0 \leq \ell \leq n_k - 1 \)

\[3\text{We can do this because the vectorization vec}(\mathbf{X}) \text{ is isomorphic to } \mathbf{X}, \text{ which can be shown using Eq. (7).}\]
Then let $I_\ell = (\ell_1, \ldots, \ell_D) \in [n_1] \times \ldots \times [n_D]$ be a multi-index to the $\ell$th eigenpair of $\mathcal{L}$. The product graph Fourier basis is then

$$(\lambda_{I_\ell}, \Psi_{I_\ell}) = \left( \sum_{k=1}^{D} \lambda^{(k)}_{\ell_k} \bigotimes_{k=1}^{D} \Psi^{(k)}_{\ell_k} \right).$$

(10)

This spectral characterization of the Cartesian product graph implies that all graphs which admit a Cartesian factorization over the same set of graphs are isomorphic; consequently, Fourier analyses on these graphs produce the same results.

**Efficient MWGSP by graph factorization:** We define the Multi-way graph Fourier Transform (MWGFT) as

$$\mathbf{F} \mathbf{x} = \left( \bigotimes_{k=1}^{D} \Psi^{(\ell)} \right) \ast \text{vec}(\mathbf{x}).$$

(11)

On the surface, the computational cost of a MWGFT (and MWGSP in general) seems high as multi-way product graphs are often much larger than their individual factors; the cardinality of the product vertex set is the product of the number of vertices in each factor. However, the product graph structure actually yields efficient algorithms. With small adjustments to fundamental operations like matrix multiplication, in the best case one can effectively reduce the computational burden of an order $D$ tensor with $n = \prod_{\ell=1}^{D} n_\ell$ total elements to a sequence of problems on $n^{(1/D)}$ elements. The computational strategy is to apply Equation (7) and its order-$D$ generalization to avoid storing and computing large product graph operators.

We introduce the order-$D$ form of Equation (7) via an algorithm. Given a sequence of operators $\mathbf{M}^{(\ell)} \in \mathbb{R}^{n_\ell}$, $\ell = 1, \ldots, D$, an efficient algorithm for computing $\mathbf{y} = \bigotimes_{k=1}^{D} \mathbf{M}^{(\ell)} \ast \text{vec}(\mathbf{x})$ proceeds by applying each $\mathbf{M}^{(\ell)}$ to the corresponding mode-wise matricization of $\mathbf{x}$. An informal version of this algorithm is provided in Algorithm 1.

**Algorithm 1** $D$-tensor multilinear transformations

1. Initialize $\mathbf{Y} = \mathbf{X}$
2. for $\ell = 1, \ldots, D$ do
3. Matricize: $\mathbf{Y}^{(\ell)} = \text{mat}(\mathbf{Y}, \ell)$
4. Factor update: $\mathbf{Y}^{(\ell)} = \mathbf{M}^{(\ell)^T} \mathbf{Y}^{(\ell)}$
5. Reform tensor: $\mathbf{Y} = \text{ten}(\mathbf{Y}^{(\ell)}, \ell, \{n_1, \ldots, n_D\})$
6. end for
7. Vectorization: $\mathbf{y} = \text{vec}(\mathbf{Y})$.

As a sequential product of an $n_\ell \times n_\ell$ matrix with an $n_\ell \times n \setminus \ell$ matrix, this method can dramatically improve the cost of algorithms that depends on matrix multiplication. Further, the number of operations are only dependent on computations over smaller factor matrices, which enables one to perform costly computations on the product graph without computing and storing expensive operators.

As an example, consider the computational burden of applying an MWGFT for a product graph $\mathcal{G}$ on $n = \prod_{\ell=1}^{D} n_\ell$ nodes. In the worst case, Algorithm 1 is as fast as directly computing Equation (11). However, in the best-case scenario where $n_\ell = n_k$ for all $k = 1, \ldots, D$, one computes $D$ graph Fourier bases of size $\sqrt{n} \times \sqrt{n}$, which
requires $O\left(n^{3/D}\right)$ operations. Then, to compute an MWGFT using the factor bases, we use $\Psi^{(\ell)}$ as the sequence of operators in Algorithm 1 which costs $O\left(Dn^{1/D+1}\right)$ operations. This improves upon the standard GFT, which costs $O\left(n^3\right)$ to obtain an eigenbasis and $O\left(n^2\right)$ to apply. For example, when $D = 3$ and $n_1 = n_2 = n_3 = \sqrt[3]{n}$, we obtain an asymptotically linear factorization of a graph Fourier basis for $G$, and the corresponding MWGFT can be applied in $O\left(3n^{1/3+1}\right)$ operations.

**Edge density:** We briefly comment on how the graph edge density impacts the scalability of signal processing algorithms for multi-way data. Matrix equations can be efficiently solved by iteratively computing a sparse matrix-vector products. The computational complexity of such algorithms, which include fundamental techniques such as Krylov subspace methods and polynomial approximation, typically depend linearly on the number of edges in the graphs, e.g., [16, 30, 7, 3]. This dependency suggests using the sparsest possible graph that still captures the main similarity structure along each mode. Indeed, a common strategy is to construct sparse Laplacian matrices [30] or edge-incidence matrices [7], using $k$-nearest-neighbor graphs which produce edge sets whose cardinality is linear in the number of nodes. Additional strategies for sparse graph construction have also been proposed [27]. Yet, given sparse factors graph, there is no guarantee that the product will be sparse. Thus, major efficiency gains for multi-way algorithms can be made by replacing iterative matrix-vector multiplications (both sparse and dense) with a sequence of factor graph sparse matrix-vector multiplications using Algorithm 1.

Three immediate applications for such a factorization are multi-way filter approximations [see, e.g., 16], compressive spectral clustering [31], and fast graph Fourier transforms [32]. We detail the former, while briefly describing future directions for the latter. For filtering, one could spectrally define and exactly compute a multi-way product graph filter (see box on Multi-way Filters) using the MWGSP techniques described in the previous section. Yet, Chebyshev approximations [16] are an efficient, robust, and accurate technique for approximate filtering. Efficient multi-way Chebyshev approximation leverages the Kronecker sum definition for product graph Laplacians $L$. That is, by noting that $Lx = \sum_{k=1}^{D} \left( I_{n\geq k} \otimes L^{(k)} \otimes I_{n< k} \right)x$ is equivalent to computing

$$
\left( I_{n\geq 1} \otimes L^{(1)} \right)x + \left( I_{n\geq 2} \otimes L^{(2)} \otimes I_{n_1} \right)x + \ldots + \left( L^{(D)}I_{n<D} \right)x,
$$

it is clear that Chebyshev approximations of functions on $L$ (such as spectral graph wavelets) can be written as a sum of sparse matrix vector multiplications; the total operations are now dominated by the densest factor graph.

The efficiency of this approach cannot be understated, as it facilitates a number of algorithms, including the compressive spectral algorithm proposed in [31]. Indeed, it is increasingly common to estimate geometric and spectral qualities of the graph Laplacian by applying ideal filter approximations for eigencounting and coherence estimation. Finally, factor graph sparsity and Algorithm 1 could be combined with recently proposed approaches for approximate orthogonal decompositions [32, 33] to construct a fast product graph Fourier transform. This algorithm
would admit striking similarities to the classical fast Fourier transform.

Multi-way Filters

It is natural to define spectral filters for multi-way graph signals on the product $\mathcal{G}$ as a function over the product graph eigenvalues $h : \Lambda \rightarrow \mathbb{R}$ as if they are traditional spectrally defined GSP filters. For example, the filter in Fig. 2b illustrates the effect of the product adjacency matrix on an impulse signal. Since these functions operate on the product eigenvalues, they directly consider the edge topology induced by a particular choice of product. Yet, it is feasible to develop filters for multi-way graph signals on $\mathcal{G}$ that are defined by multivariate functions $h : \Lambda^{(1)} \times \cdots \times \Lambda^D \rightarrow \mathbb{R}$. These multivariate filters are split into two classes: separable and nonseparable.

Separable filters have multivariate response functions that can be written as the product of separate univariate functions. In the $D = 2$ case, a separable filter for the product graph could be written as $\left( \mathbf{T}^{(1)} \otimes \mathbf{T}^{(2)} \right) \mathbf{x}$ in which $\mathbf{T}^{(1)} = \Psi^{(1)}(\Lambda^{(1)}) \Psi^{(1)*}$ and $\mathbf{T}^{(2)} = \Psi^{(2)}(\Lambda^{(2)}) \Psi^{(2)*}$. Since this Kronecker product is permutation equivalent, we can treat its operation as an order-independent unimodal filtering of $\mathbf{x}$ (7). If $\mathbf{T}^{(1)}$ and $\mathbf{T}^{(2)}$ are both filters defined in a Laplacian eigenbasis of their respective graph, then the tensor product $\left( \mathbf{T}^{(1)} \otimes \mathbf{T}^{(2)} \right)$ is also diagonalized by the product eigenbasis. Thus, this filter is merely a reweighting of the product graph eigenbasis. Importantly, however, it adds a degree of freedom that cannot be obtained using the 1D eigenvalues of $\Lambda$. In Figure 2, we demonstrate the application of a product of mode-wise heat filters to a graph signal on a grid (Fig. 2i top) and to a time-vertex signal which is a dynamic mesh (Fig. 2i bottom).

While there is a choice of $\tau_1$ and $\tau_2$ such that certain regions of this filter can be computed from a heat kernel on the Cartesian product graph spectrum, such an approach abandons the flexibility of bilinear filtering. By separability, each mode can be analyzed independently of the other by setting the appropriate $\tau_k$ to 0. This enables analyzing a joint signal along each mode independently, for example by filtering out high frequency structure along one domain while preserving the frequency content of the other mode. A $D$-way separable filter applied to $\mathbf{x} = \text{vec}(\mathbf{X})$ is given by

$$\tilde{\mathbf{x}} = \Psi H \left( \prod_{k=1}^D \Lambda^{(k)} \right) \Psi^* \mathbf{x},$$

where $\Psi = (\otimes_{\ell=1}^D \Psi^{(\ell)})$ (see Equation (10) and surrounding discussion), and $H \left( \prod_{k=1}^D \Lambda^{(k)} \right)$ is a diagonal matrix whose elements are given by $\prod_{k=1}^D h^{(k)}(\lambda_{\ell_k})$, i.e., the product of separate spectral functions $h^{(k)}$ for each factor graph $\mathcal{G}^{(k)}$.

Nonseparable filters cannot be designed from separate univariate filters on each mode. This class of filters encompasses a broad class of functions that include many filters defined in terms of the product graph eigenvalues, as well as multivariate functions (Fig. 2). Indeed, [3] find that one cannot in general describe PDEs that describe diffusion, wave, or disease propagation with separable filters, as that the relation between frequencies is not independent.

IV. MWGSP FRAMEWORKS

Here we highlight two recent multi-way frameworks: time-vertex framework [3] and Generalized GSP [34].

A. Time-vertex framework

The joint time-vertex framework [3, 17] arose to address the limitations of GSP in analyzing dynamic data on graphs. This required generalizing harmonic analysis to a coupled time-graph setting by connecting a regular axis (time) to an arbitrary graph. The central application of these techniques are to analyze graph signals that
Fig. 2. Multi-way graphs, signals and spectral filters. (a) Product of two path graphs is a 2D grid. (b) Tensor product of two impulses is a 2D impulse (left). Filtering by the joint adjacency matrix propagates the impulse to neighboring vertices in each path (right). 2D graph spectral filtering. The input signal is (d, top) the tensor product of an impulse with a smooth function (c) or (d, bottom) a T-V signal: a dynamic mesh. (e) A separable diffusion filter is the product of domain-specific heat kernels. Separable filters can filter along both axes in unison (left), or each axis independently (middle/right). (f) Non-separable filter.

are time-series signals, for example, a time-series that reside on a sensor graph. Each time point of this series is itself a graph signal, while each vertex on the graph maps to a time-series of $T$ samples. This enables learning covariate structures from time-vertex (T-V) signals, which are bivariate functions on the vertex and time domain. Such sequence of graph signals are commonly collected longitudinally through sensor networks, video, health data, and social networks.

Signal processing in the T-V framework introduces the JFT $\mathcal{J}$ for a time-vertex signal $X \in \mathbb{R}^{V_T \times T}$ as

$$\mathcal{JFT}\{X\} = \Psi^* X U_T \quad \text{or} \quad \mathcal{JFT}\\{\text{vec}(X)\} = (U_T \otimes \Psi)^* \text{vec}(X).$$

The multi-way Fourier expansion of a T-V signal is a tensor product of the DFT basis with a GFT basis. Consequently, the JFT admits a fast transform in which one first performs an FFT along the time mode of the data before taking the GFT of the result, thus requiring only one Laplacian diagonalization.

Including the DFT basis in this framework immediately admits novel joint time-vertex structures that are based on classical tools. Grassi et al. $\mathcal{J}$ introduce variational norms that combine classical variation with graph variation. For efficient filter analysis, they also propose an FFT and Chebyshev based algorithm for computing fast T-V filters. This algorithm applies to both separable and non-separable filters, such as those demonstrated in Fig. 2 (d-f, bottom row) for a dynamic mesh in . Filtering along only one mode revealed either skeleton structure ($\tau_2 = 0$) or blurred dynamics of the figure ($\tau_1 = 0$), but joint separable or non-separable filtering reveals joint dependencies.
[3] also introduce overcomplete dictionary representations constructed as a tensor-like composition of graph spectral dictionaries with classical short-time Fourier transform (STFT) and wavelet frames. These joint dictionaries can be constructed to form frames, enabling the analysis and manipulation of data in terms of time-frequency-vertex-frequency localized atoms. [17] also introduce T-V spectral filtering, as well as define a T-V Kalman filter, with both batch and online function estimators. Further works have integrated ideas from classical signal processing such as stationarity [35–37]. Thus, recent developments in the T-V framework can serve as a road-map for the future development of general MWGSP methods.

B. Generalized Graph Signal Processing

Another recent development is that of the Generalized Graph Signal Processing [34] framework which extends the notions of MWGSP to arbitrary, non graphical geometries. Generalized GSP facilitates multivariate signal processing of interesting signals in which at least one domain lacks a discrete geometry. This framework recognizes that the key intuition of graph signal processing is the utility of irregular, non-Euclidean geometries for analyzing signals. However, where GSP techniques axiomatize a finite relational structure encoded by a graph shift operator, Generalized GSP extends classical Fourier analogies to arbitrary Hilbert spaces (i.e., complete inner product spaces) $\mathcal{H} \subseteq H$ equipped with a compact, self-adjoint operator $A$. This broad class of geometries contains GSP, as the standard space of square summable graph signals, i.e., $L^2(V) = \{f : V \mapsto \mathbb{C}, \|f\|_2 < \infty\}$ is itself a Hilbert space.

The geometries and corresponding signals that can be induced by Generalized GSP offer an intriguing juxtaposition of continuous and discrete topologies. As an example, consider the tensor product of a graph $G$ with the space of square integrable functions on an interval, e.g., $G \otimes L^2([-1, 1])$. Graph signals in this space map each vertex to a $L^2$ function. Conversely, $L^2$ functions can be mapped to specific vertices. To generate a Fourier basis for the product space, one simply takes the tensor product of the factor space eigenbases. This is a promising future direction for MWGSP, as it implies that one can, for instance, combine graph Fourier bases with generalized Fourier bases for innovative signal representations.

[13] proposed an early example of Generalized GSP, though under a different name. This work modeled videos and collections of related matrices as matrix-valued graph signals using matrix convolutional networks. The authors aimed to solve the challenging missing data problem of node undersampling: some matrix slices from the networks are completely unobserved. When matrices have a low-rank graph Fourier transform, the network’s graph structure enables recovery of missing slices. In light of the development of Generalized GSP, it is clear that [13] proposed an algorithm for denoising of multi-way signals on $G \otimes \mathbb{R}^{n_1 \times n_2}$.

V. SIGNAL PROCESSING ON MULTI-WAY GRAPHS

In the previous section, we focused on signal processing through the lens of harmonic analysis, using the graph Laplacian to analyze data in the spectral domain. In this section, we focus on signal modeling and recovery in the
multi-way setting through the lens of optimization, where the graph Laplacian serves the role of imposing signal smoothness. Including graph structures along the modes of multi-way matrices and higher-order tensors has led to more robust and efficient approaches for denoising, matrix completion and inpainting, collaborative filtering, recommendation systems, biclustering, factorization, and dictionary learning [5, 21, 23, 13, 12]. We begin with dual-graph modeling in the matrix setting and then extend to the higher-order tensor setting. In the tensor setting we review both using multi-way graph regularization in tensor factorization methods and in complementary fashion using tensor factorization in signal modeling and recovering to make graph regularization computationally tractable.

A. Signal processing on dual graphs

The typical model in this setting is to add dual row-column graph regularizers of the form $\gamma_r \text{Tr}(X^T L_r X) + \gamma_c \text{Tr}(X L_c X^T)$ to classical problem formulations; such regularization incentivizes the recovered signal to be smooth with respect to the underlying data graphs [2]. The matrices $L_r$ and $L_c$ denote the graph Laplacians on the rows and columns of $X$ respectively, and the nonnegative tuning parameters $\gamma_r$ and $\gamma_c$ typically trade off data fit with smoothness with respect to the row and column geometries encoded in $L_r$ and $L_c$ respectively. In [26], a multiscale Laplacian pyramid on coupled row and column graphs (analogous to the classical Laplacian pyramid on images), is introduced for data imputation.

Table I presents formulations of these different algorithms; multiple extensions and other methods exist in the literature. For the time-vertex framework [3], the graph on the columns is a temporal graph modeled explicitly with a ring graph Laplacian $L_T$. The mapping $P_\Theta$ is a projection operator on the set of observed entries $\Theta$ in missing data scenarios. Methods may differ in their fidelity term minimizing the Frobenius norm for denoising or 1-norm to impart robustness to outliers [30], and several methods assume a low-rank structure, either with a nuclear norm penalty [38] or with an explicit low-rank factorization of the data matrix $Y$ as $DX$, sometimes with additional constraints on the factor matrices (non-negativity [40], sparsity [21]). A few methods aim to solve a matrix completion problem (see Fig. 3(a)). Finally, while most instances of graph regularization rely on the quadratic penalty term $\text{Tr}(X^T L_r X) = \sum_{(i,j) \in E_r} w_{i,j} \|X_i - X_j\|_2^2$, the bi-clustering formulation in [7] [39] employs a penalty that is either

| Method            | Fidelity term | Graph regularizers | Constraints |
|-------------------|---------------|--------------------|-------------|
| MCG [38]          | $\|P_\theta(Y - X)\|_F^2$ | $\gamma_r \text{Tr}(X^T L_r X) + \gamma_c \text{Tr}(X L_c X^T)$ | $\gamma_0 \|X\|_F^2$ |
| CFG [5]           | $\|P_\theta(Y - DX)\|_F^2$ | $\gamma_r \text{Tr}(D^T L_r D) + \text{Tr}(X L_c X^T)$ | $\alpha \|D\|_F^2 + \beta \|X\|_F^2$ |
| DGRDL [21]        | $\|Y - DX\|_F^2$ | $\gamma_r \text{Tr}(X^T L_r X) + \gamma_c \text{Tr}(X L_c X^T)$ | $\|X\|_F^2$ |
| T-V Reg [3]       | $\|Y - X\|_F^2$ | $\gamma_r \text{Tr}(X^T L_r X) + \gamma_c \text{Tr}(X L_c X^T)$ | $\|X\|_F^2$ |
| T-V Inpaint [3]   | $\|P_\theta(Y - X)\|_F^2$ | $\gamma_r \text{Tr}(X^T L_r X) + \gamma_c \text{Tr}(X L_c X^T)$ | $\|X\|_F^2$ |
| Cvx Biclust [7]   | $\|Y - X\|_F^2$ | $\gamma_r \sum_{(i,j) \in E_r} w_{i,j} \|X_i - X_j\|_2 + \gamma_c \sum_{(i,j) \in E_c} \Omega(x_{i,j}) \|X_i - X_j\|_2$ | $\|X\|_F^2$ |
| Comani-missing [39]| $\|P_\Theta(Y - X)\|_F^2$ | $\gamma_r \text{Tr}(X^T L_r X) + \gamma_c \text{Tr}(X L_c X^T)$ | $\|X\|_F^2$ |
| FRPCAG [40]       | $\|Y - X\|_F^2$ | $\gamma_r \text{Tr}(X^T L_r X) + \gamma_c \text{Tr}(X L_c X^T)$ | $\|X\|_F^2$ |
| DNMF [40]         | $\|Y - DX\|_F^2$ | $\gamma_r \text{Tr}(D^T L_r D) + \gamma_c \text{Tr}(X L_c X^T)$ | $D \geq 0, X \geq 0$ |
linear in the $l_2$-norm or concave and continuously differentiable relying on the mapping $\Omega(\|X_i - X_j\|_2)$. The motivation there is that convex penalties, either when $\Omega$ is linear or quadratic, do not introduce enough smoothing for small differences and too much smoothing for large differences, resulting in poorer clustering results.

Typically an alternating optimization algorithm is used to solve the various problems in Table I. The T-V regularization problem is the only one with a closed form solution given by a joint non-separable low-pass filter (generalizing Tikhonov regularization to the T-V case). The graph Dual regularized Non-negative matrix factorization (DNMF) \cite{40} relies on an alternating optimization scheme for the non-negative factor matrices. Other solutions are computed with proximal methods such as Alternating-Direction Method of Multipliers (ADMM) to handle multiple regularization terms via variable splitting.

Fig. 3. (a) ORL data 50% missing (left) and after MCG (right). (b) Oracle (right) and noisy (left) adjacency matrices for row graph. (c) same for columns. Matrix completion relative error on (d) ORL, (e) Lung500 datasets.

Dual-graph regularized approaches have been shown to consistently out-perform their non-regularized or single-graph regularized counterparts across a wide range of applications and domains. In Fig. 3 (d)-(e) we compare several approaches for matrix completion \cite{38, 41, 30} with single way or multi-way graph regularization on the ORL and Lung500 datasets with 10% or 50% entries missing at random. The ORL \cite{42} dataset consists of 300 images of faces (30 people with 10 images per person), which are flattened into 2576 features. We used a row graph that connects similar images together and a column graph that ignores the natural 2D grid geometry and instead considers a wider geometry in the image plane. Lung500 is a gene expression dataset consisting of 56 samples (rows) and 500 genes (columns) \cite{7}. To set $\gamma_r, \gamma_c$, we ran each method for a range of values and selected the result with best performance. For comparison to single-way graph regularization, we also set $\gamma_c = 0$ in MCG \cite{38} and FRPCAG \cite{41} to ignore the graph on the feature (column) space. In general, $\gamma_r$ and $\gamma_c$ induce row and column smoothness at different levels and their choice should be driven by the trade-off in the smoothness of the data along the two modes and the aspect ratio of the matrix, or informed by cross-validation.

We report the relative reconstruction error on the missing values. The multi-way graph regularized approaches
out-performed their corresponding single-way versions \((\gamma_c = 0)\). In all cases MCG also out-performed RPCAG which is a single-way graph regularised problem, and FRPCAG out-performed RPCAG in most cases. The effect of using a graph constructed from the missing data (noisy graph) vs complete data (oracle graph) is discussed in the “Graph construction” box.

**Graph Construction**

A question that arises in graph-based methods is how to construct the graphs themselves. We describe common strategies for graph construction below.

**Graphs from side information:** Often supplementary information can be used to construct graphs. In some cases, there may be a natural geometry that easily translates into similarity graphs for rows and columns. For example, in [43] the authors estimate local regression models that exhibit smoothness with respect to regions in a brain. In other cases, other supplemental data sets may be leveraged to provide similarity structure among rows or columns. As an example in music recommender systems, in [6] the authors used a publicly available playlist categorization as well as sophisticated summary statistics extracted from the audio signal to construct a graph for estimating a latent association matrix between playlists and songs.

**Data-driven graph from noisy/missing data:** One difficulty that arises is that in the presence of noise, outliers and missing entries, constructing a graph from the data yields a corrupted graph. In our simulations, we construct a sparse \(k\)-nearest neighbor \((k = 7)\) graph. Fig. [3]b-c compares a ‘noisy graph’ construction to an ‘oracle graph’ for ORL data. The graph along the images (b) connects images of different people together while the feature graph (c) loses local pixel geometry. Results in Fig. [3]d-e demonstrate that the noisy graph clearly degrades the performance compared to the oracle graph, especially for higher missing percentages. A solution to such corrupted graphs is the metric repair method in [44].

**Graph learning:** In [21, 24] learning the graph on the feature space is incorporated into the optimization problem by minimizing over \(L\) in addition to the signal recovery. For a detailed review see [27].

**Dynamically varying graphs:** Graphs may not be static, presenting a current challenge in GSP, which is especially acute in time-vertex frameworks which admit time as one of components in the analysis [45]. Challenges include determining how to identify when a graph needs to be updated, i.e., when the underlying topology has changed. The challenge of accounting for dynamically varying graphs also poses computational questions, namely what are computationally efficient ways to update graphs within the processing framework that will minimally spawn artifacts at transitions?

**B. Tensor processing on graphs**

A challenge of many well-studied problems in signal processing and machine learning is that in general algorithm complexity increases exponentially when one considers tensors with three or more modes. Early multi-way data analysis approaches flattened data tensors to matrices and then applied classical two-way analysis techniques. Flattening, however, obscures higher-order patterns and interactions between the different modes of the data. Thus, multilinear tensor decompositions have been the main workhorse in tensor signal processing and data analysis, generalizing the notion of matrix factorizations to higher-order tensors, and have become common in applications such as hyperspectral and biomedical imaging.

While there is no single generalization of a spectral decomposition for tensors, the two most common tensor
decompositions are the CANDECOMP/PARAFAC (CP) decomposition (see Fig. 1(g)) and the Tucker decomposition [9]. Just as the singular value decomposition can be used to construct a lower-dimensional approximation to a data matrix, finding a coupled pair of lower dimensional subspaces for the rows and columns, these two decompositions can be used to construct lower dimensional approximations to a $D$-way tensor $X \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_D}$. Under mild conditions, the CP decomposition, which approximates $X$ by a sum of rank-one tensors, is unique up to scaling and permutations of the columns of its factor matrices [9], but the CP factor matrices typically cannot be guaranteed to have orthogonal columns. The Tucker decomposition permits orthonormal factor matrices but in general fail to have unique representations [9].

Much of the multi-way literature has focused on improving and developing new tensor factorizations. Graph-based regularizations along modes of the tensor are proving versatile for developing robust tensor and low-rank decompositions [11, 14, 15], as well as new approaches to problems in higher order data processing such as tensor completion, data imputation, recommendation system, feature selection, anomaly detection, and co-clustering [22, 13, 23, 25, 12]—a generalization of biclustering to tensors. Generalization of these problems to tensors incurs a higher computational cost than the equivalent matrix problems. Thus multi-way graph-regularized formulations typically combine a low-rank tensor factorization with graph-based regularization along the rows of the factor matrices; for example [25, 23] rely on a CP decomposition while [14] relies on a Tucker decomposition. In [46], a Tucker decomposition is used within MWGSP, to construct wavelets on multislice graphs in a two-stage approach.

An example of combining tensor decompositions with graph regularization is the following “low-rank + sparse” model for anomaly detection in internet traffic data [25]:

$$\min_{\mathcal{X}, \mathcal{E}(A^{(i)})} \| (Y - \mathcal{E}) - \mathcal{X} \|_F^2 + \sum_{i=1}^d \gamma_i \text{Tr}(A^{(i)}^T L_i A^{(i)}) \quad \text{s.t.} \quad \mathcal{X} = \sum_{i=1}^R a_i^{(1)} \circ a_i^{(2)} \circ a_i^{(3)}, \quad \| \mathcal{E} \|_0 \leq \epsilon, \quad (12)$$

where $Y$ is a data tensor and $\mathcal{E}$ is the tensor of sparse outliers. The equality constraint on $\mathcal{X}$ requires that $\mathcal{X}$ has a rank-$R$ CP decomposition where $a_i^{(d)}$ is the $i$th column of the $d$th factor matrix $A^{(d)} \in \mathbb{R}^{n_d \times R}$ and $\circ$ denotes an outer product. Note that the graph regularization terms in (12) are applied to the factor matrices $A^{(i)} \in \mathbb{R}^{n_i \times R}$, reducing the computational complexity of the estimation algorithm. Decomposing a data tensor into the sum of a low-rank and sparse components is also used in [11, 24, 14].

In [15], computational complexity is further reduced by pre-calculating $F_R^{(i)}$ mode-specific graph Laplacian eigenvectors of rank $R$ from the matricization of the tensor along each mode and using these in solving tensor-robust PCA. The solution relies on projecting the tensor onto a tensor product of the graph basis $\{ F_R^{(i)} \}$, resulting in a formulation to similar to the Tucker decomposition.

Co-clustering assumes the observed tensor is the sum of a “checkerbox” tensor (under suitable permutations along the modes) and additive noise. For example, Chi et al. [12] propose estimating a “checkerbox” tensor with
the minimizer to a convex criterion. In the case of 3-way tensor, the criterion is
\[
\frac{1}{2} \| \mathbf{Y} - \mathbf{X} \|_F^2 + \gamma \left[ \sum_{(i,j) \in \mathcal{E}(1)} w_{1,ij} \| \mathbf{X}_{i:} - \mathbf{X}_{j:} \|_F + \sum_{(i,j) \in \mathcal{E}(2)} w_{2,ij} \| \mathbf{X}_{i::} - \mathbf{X}_{j::} \|_F + \sum_{(i,j) \in \mathcal{E}(3)} w_{3,ij} \| \mathbf{X}_{i::} - \mathbf{X}_{j::} \|_F \right],
\]
where \( \mathcal{E}(d) \) is a set of edges for the mode-\( d \) graph, \( \gamma \) is a nonnegative tuning parameter, and \( w_{d,ij} \) is a weight encoding the similarity between the \( i \)th and \( j \)th mode-\( d \) slices. Minimizing the criterion in (13) can be interpreted as denoising all modes of the tensor simultaneously via vector-valued graph total-variation.

C. Manifold learning on multi-way data

Tensor factorization can fail to recover meaningful latent variables when nonlinear relationships exist among slices along each of modes [19]. Manifold learning overcomes such limitations by estimating nonlinear mappings from high-dimensional data to low-dimensional representations (embeddings). While GSP uses the eigenvectors of the graph Laplacian as a basis in which to linearly expand graph signals [1], manifold learning uses the eigenvectors \( \psi_\ell \) themselves as a new nonlinear low-dimensional representation \( \Psi \) for the datapoints \( \{ \mathbf{x}_i \} \) as
\[
\Psi : \mathbf{x}_i \rightarrow (\psi_1(i), \ldots, \psi_d(i)).
\]

A naïve strategy to apply manifold learning to the multi-way data is to take the \( D \) different matricizations of a \( D \)-way tensor and construct a graph Laplacian using a generic metric on each of the \( D \) modes independently, thereby ignoring the higher-order coupled structure in the tensor. Recent work [8, 10, 39], however, incorporate higher-order tensor structure in manifold learning by thoughtfully designing the similarity measures used to construct the graph weights \( \mathbf{W}^{(d)} \). The co-manifold learning framework, starting with the matrix organization approach, and most recently extended to tensors and the missing data setting, can be viewed as blending GSP and manifold learning together [47, 10, 39].

From a MWGSP perspective, the key contribution of this line of work is a new metric that is defined between tensor slices as the difference between a graph-based multiscale decomposition of each slice along its remaining modes; for example the distance between two horizontal slices in a 3-way tensor is
\[
d(\mathbf{X}_i, \mathbf{X}_j) = \| (\mathbf{M}^{(3)} \otimes \mathbf{M}^{(2)}) \text{vec} (\mathbf{X}_{i::} - \mathbf{X}_{j::}) \|_1 = \| \text{vec}(\mathbf{M}^{(2)}(\mathbf{X}_{i::} - \mathbf{X}_{j::})(\mathbf{M}^{(3)})^T) \|_1,
\]
where \( \mathbf{M}^{(k)} \) is a multiscale transform in the \( k \)th mode. This metric was shown to be a tree-based Earth-mover’s distance in the 2D setting [48]. The resulting similarity depends on a multi-way multiscale difference between slices, and has been successfully used in practice to construct weighted graphs in multiway data. The multiscale decompositions are constructed either from data-adaptive tree transforms [8] or through a series of multi-way graph-based co-clustering solutions [39].
VI. FUTURE OUTLOOK

As multi-way signal processing frameworks continue to mature, several challenges remain ahead. While novel techniques are continually introduced into single-way graph signal processing, one approach to developing multi-way techniques is to identify, extend, and adapt techniques which are particularly useful for multi-way signals. For example, the coupled geometric paradigm of multi-way data analysis could acutely benefit from the extension of techniques such as [49], which adapt the signal processing toolbox to be more effective for irregular data analysis. On the other hand, it is clear that the efficiency gains offered by single-way GSP’s march towards fast transforms [32, 33] are compounded in the multi-way setting.

From a theoretical perspective, additional open questions that remain include 1) Are there advantages to be gained by treating classical domains as lying on graphs themselves? 2) How to learn mode-specific or coupled graphs from data, in general and in dynamical settings? 3) Are such tensor datasets typically low-rank or high-rank? 4) How do we process data whose generative model is nonlinear across the different modes?

From a practical perspective, some real-world applications require online real-time processing. As such future directions include developing online and distributed versions of multi-way graph signal processing, especially in the presence of large-scale data, where streaming solutions are necessary (the data does not fit in memory). In addition, there is need for new optimization techniques to efficiently solve problems that combine tensors with graph-based penalties. Deep learning is also emerging as a framework to learn rather than design wavelet-type filterbanks in signal processing and these approaches can be extended to the graph and multi-way settings to learn joint multiscale decompositions. Finally, as the GSP community continues to address real-world data domains such as climate, traffic, and biomedical research, inter-disciplinary collaboration is essential to define relevant problems and demonstrate significant utility of these approaches within a domain.

REFERENCES

[1] D. I. Shuman, S. K. Narang, P. Frossard, A. Ortega, and P. Vandergheynst, “The emerging field of signal processing on graphs: Extending high-dimensional data analysis to networks and other irregular domains,” IEEE Signal Process. Mag., vol. 30, no. 3, pp. 83–98, 2013.
[2] A. Ortega, P. Frossard, J. Kovačević, J. M. Moura, and P. Vandergheynst, “Graph signal processing: Overview, challenges, and applications,” Proceedings of the IEEE, vol. 106, no. 5, pp. 808–828, 2018.
[3] F. Grassi, A. Loukas, N. Perraudin, and B. Ricaud, “A time-vertex signal processing framework: Scalable processing and meaningful representations for time-series on graphs,” IEEE Trans. Signal Process., vol. 66, no. 3, pp. 817–829, 2017.
[4] A. Bhattacharjee et al., “Classification of human lung carcinomas by mRNA expression profiling reveals distinct adenocarcinoma subclasses,” Proceedings of the National Academy of Sciences, vol. 98, no. 24, pp. 13 790–13 795, 2001.
[5] N. Rao, H.-F. Yu, P. K. Ravikumar, and I. S. Dhillon, “Collaborative filtering with graph information: Consistency and scalable methods,” in Adv. Neural Inf. Process. Syst., 2015, pp. 2107–2115.
[6] K. Benzi, V. Kalofolias, X. Bresson, and P. Vandergheynst, “Song recommendation with non-negative matrix factorization and graph total variation,” in ICASSP-2016, March 2016, pp. 2439–2443.
[7] E. C. Chi, G. I. Allen, and R. G. Baraniuk, “Convex Biclustering,” Biometrics, vol. 73, no. 1, pp. 10–19, 2017.
[8] G. Mishne, R. Talmon, I. Cohen, R. R. Coifman, and Y. Kluger, “Data-driven tree transforms and metrics,” IEEE Trans. Signal Inf. Process. Netw., vol. 4, pp. 451–466, 2018.
[9] T. G. Kolda and B. W. Bader, “Tensor decompositions and applications,” SIAM Review, vol. 51, no. 3, pp. 455–500, 2009.
[10] G. Mishne, R. Talmon, R. Meir, J. Schiller, M. Lavzin, U. Dubin, and R. R. Coifman, “Hierarchical coupled-geometry analysis for neuronal structure and activity pattern discovery,” IEEE J. Sel. Topics Signal Process., vol. 10, no. 7, pp. 1238–1253, 2016.
[11] Y. Nie, L. Chen, H. Zhu, S. Du, T. Yue, and X. Cao, “Graph-regularized tensor robust principal component analysis for hyperspectral image denoising,” Applied optics, vol. 56, no. 22, pp. 6094–6102, 2017.
[12] E. C. Chi, B. R. Gaines, W. W. Sun, H. Zhou, and J. Yang, “Provable convex co-clustering of tensors,” arXiv:1803.06518 [stat.ME], 2018.
[13] Q. Sun, M. Yan, D. Donoho, and S. Boyd, “Convolutional imputation of matrix networks,” in International Conf. on Machine Learning, 2018, pp. 4818–4827.
[14] K. Zhang, M. Wang, S. Yang, and L. Jiao, “Spatial–spectral-graph-regularized low-rank tensor decomposition for multispectral and hyperspectral image fusion,” IEEE J. Sel. Topics Appl. Earth Observ. Remote Sens., vol. 11, no. 4, pp. 1030–1040, 2018.

[15] N. Shahid, F. Grassi, and P. Vandergheynst, “Tensor robust PCA on graphs,” in IEEE Internat. Conf. Acoust. Speech and Signal Process. (ICASSP), 2019, pp. 5406–5410.

[16] D. K. Hammond, P. Vandergheynst, and R. Gribonval, “Wavelets on graphs via spectral graph theory,” Appl. Comput. Harmon. Anal., vol. 30, no. 2, pp. 129 – 150, 2011.

[17] D. Romero, V. N. Ioannidis, and G. B. Giannakis, “Kernel-based reconstruction of space-time functions on dynamic graphs,” IEEE Journal of Selected Topics in Signal Processing, vol. 11, no. 6, pp. 856–869, 2017.

[18] A. Sandryhaila and J. M. F. Moura, “Big data analysis with signal processing on graphs: Representation and processing of massive data sets with irregular structure,” IEEE Signal Processing Magazine, vol. 31, no. 5, pp. 80–90, 2014.

[19] E. Acar and B. Yener, “Unsupervised multiway data analysis: A literature survey,” IEEE Trans. Knowl. Data Eng., vol. 21, no. 1, pp. 6–20, 2009.

[20] A. Cichocki, D. Mandic, L. De Lathauwer, G. Zhou, Q. Zhao, C. Caiafa, and H. A. Phan, “Tensor decompositions for signal processing applications: From two-way to multiway component analysis,” IEEE Signal Process. Mag., vol. 32, no. 2, pp. 145–163, March 2015.

[21] Y. Yankelevsky and M. Elad, “Dual graph regularized dictionary learning,” IEEE Trans. Signal Inf. Process. Netw., vol. 2, no. 4, pp. 611–624, 2016.

[22] C. Li, Q. Zhao, J. Li, A. Cichocki, and L. Guo, “Multi-tensor completion with common structures,” in AAAI Conf. on Artificial Intelligence, 2015.

[23] V. N. Ioannidis, A. S. Zamzam, G. B. Giannakis, and N. D. Sidiropoulos, “Coupled graph and tensor factorization for recommender systems and community detection,” IEEE Trans. Knowl. Data Eng., pp. 1–1, 2019.

[24] Y. Su, X. Bai, W. Li, P. Jing, J. Zhang, and J. Liu, “Graph regularized low-rank tensor representation for feature selection,” J Vis Commun Image R, vol. 56, pp. 234–244, 2018.

[25] K. Xie, X. Li, X. Wang, G. Xie, J. Wen, and D. Zhang, “Graph based tensor recovery for accurate internet anomaly detection,” in IEEE INFOCOM, 2018, pp. 1502–1510.

[26] N. Rabin and D. Fishelov, “Two directional Laplacian pyramids with application to data imputation,” Advances in Computational Mathematics, pp. 1–24, 2019.

[27] G. Mateos, S. Segarra, A. G. Marques, and A. Ribeiro, “Connecting the dots: Identifying network structure via graph signal processing,” IEEE Signal Process. Mag., vol. 36, no. 3, pp. 16–43, 2019.

[28] R. A. Horn, R. A. Horn, and C. R. Johnson, Topics in matrix analysis. Cambridge university press, 1994.

[29] W. Imrich and S. Klavzar, Product graphs: structure and recognition. Wiley, 2000.

[30] N. Shahid, N. Perraudin, V. Kalofolias, G. Puy, and P. Vandergheynst, “Fast robust PCA on graphs,” IEEE J. Sel. Topics Signal Process., vol. 10, no. 4, pp. 740–756, 2016.

[31] N. Tremblay, G. Puy, R. Gribonval, and P. Vandergheynst, “Compressive spectral clustering,” in International Conference on Machine Learning, 2016, pp. 1002–1011.

[32] L. Le Magoarou, R. Gribonval, and N. Tremblay, “Approximate fast graph fourier transforms via multilayer sparse approximations,” IEEE Transactions on Signal and Information Processing over Networks, vol. 4, no. 2, pp. 407–420, 2017.

[33] T. Frerix and J. Bruna, “Approximating orthogonal matrices with effective givens factorization,” in International Conference on Machine Learning, 2019, pp. 1993–2001.

[34] F. Ji and W. P. Tay, “A Hilbert space theory of generalized graph signal processing,” IEEE Trans. Signal Process., vol. 67, no. 24, pp. 6188–6203, 2019.

[35] B. Girault, “Stationary graph signals using an isometric graph translation,” in 2015 23rd European Signal Processing Conference (EUSIPCO). IEEE, 2015, pp. 1516–1520.

[36] A. G. Marques, S. Segarra, G. Leus, and A. Ribeiro, “Stationary graph processes and spectral estimation,” IEEE Transactions on Signal Processing, vol. 65, no. 22, pp. 5911–5926, 2017.

[37] A. Loukas and N. Perraudin, “Stationary time-vertex signal processing,” EURASIP J. Adv. Signal Process., vol. 2019, no. 1, p. 36, 2019.

[38] V. Kalofolias, X. Bresson, M. Bronstein, and P. Vandergheynst, “Matrix completion on graphs,” arXiv preprint arXiv:1408.1717, 2014.

[39] G. Mishne, E. C. Chi, and R. R. Coifman, “Co-manifold learning with missing data,” in ICML, 2019.

[40] F. Shang, L. Jiao, and F. Wang, “Graph dual regularization non-negative matrix factorization for co-clustering,” Pattern Recognition, vol. 45, no. 6, pp. 2237–2250, 2012.

[41] N. Shahid, V. Kalofolias, X. Bresson, M. Bronstein, and P. Vandergheynst, “Robust principal component analysis on graphs,” in Proceedings of the IEEE International Conference on Computer Vision, 2015, pp. 2812–2820.

[42] F. S. Samaria and A. C. Harter, “Parameterisation of a stochastic model for human face identification,” in Proc. IEEE workshop on applications of computer vision. IEEE, 1994, pp. 138–142.

[43] Y. Hu and G. I. Allen, “Local-Aggregate Modeling for Big Data via Distributed Optimization: Applications to Neuroimaging,” Biometrics, vol. 71, no. 4, pp. 905–917, 2015.

[44] A. C. Gilbert and R. Sonthalia, “Unrolling swiss cheese: Metric repair on manifolds with holes,” in 56th Annual Allerton Conf. on Biometrics, 2018.

[45] K. Yamada, Y. Tanaka, and A. Ortega, “Time-varying graph learning with constraints on graph temporal variation,” arXiv preprint arXiv:2001.03346, 2020.

[46] N. Leonardi and D. Van De Ville, “Tight wavelet frames on multislice graphs,” IEEE Trans. Sig. Process., vol. 61, no. 13, pp. 3357–3367, July 2013.

[47] M. Gavish and R. R. Coifman, “Sampling, Denoising and Compression of Matrices by Coherent Matrix Organization,” Appl Comput Harmon Anal., vol. 33, pp. 354 – 369, 2012.

[48] R. R. Coifman and W. E. Leeb, “Earth mover’s distance and equivalent metrics for spaces with hierarchical partition trees,” Yale University, Tech. Rep., 2013, technical report YALEU/DCS/TR1482.

[49] B. Girault, A. Ortega, and S. S. Narayanan, “Irregularity-aware graph fourier transforms,” IEEE Transactions on Signal Processing, vol. 66, no. 21, pp. 5746–5761, 2018.