A hybrid conjugate gradient method for optimization problems

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Received 14 October 2010; revised 18 November 2010; accepted 22 November 2010.

ABSTRACT

A hybrid method of the Polak-Ribièere-Polyak (PRP) method and the Wei-Yao-Liu (WYL) method is proposed for unconstrained optimization problems, which possesses the following properties: i) This method inherits an important property of the well known PRP method: the tendency to turn towards the steepest descent direction if a small step is generated away from the solution, preventing a sequence of tiny steps from happening; ii) The scalar \(0 \leq \beta_k \) holds automatically; iii) The global convergence with some line search rule is established for nonconvex functions. Numerical results show that the method is effective for the test problems.

Keywords: Line Search; Unconstrained Optimization; Conjugate Gradient Method; Global Convergence

1. INTRODUCTION

We are interested to consider the unconstrained optimization problem

\[
\min_{x \in \mathbb{R}^n} f(x),
\]

(1.1)

where \( f: \mathbb{R}^n \to \mathbb{R} \) is continuously differentiable. It is well known that there are many methods for solving optimization problems (see [24,26,28-32,34] etc.), where the conjugate gradient (CG) method is a powerful line search method because of its simplicity and its very low memory requirement, especially for the large scale optimization problems [22,23,27], which can avoid, like steepest descent method, the computation and storage of some matrices associated with the Hessian of objective functions. The following iterative formula is often used by the nonlinear CG method

\[
x_{k+1} = x_k + \alpha_k d_k, k = 0,1,2,\ldots
\]

(1.2)

for (1.1), where \( x_k \) is the current iterate point, \( \alpha_k > 0 \) is a steplength, and \( d_k \) is the search direction designed by

\[
d_k = \begin{cases} -g_k + \beta_k d_{k-1}, & \text{if } k \geq 1, \\ -g_k, & \text{if } k = 0, \end{cases}
\]

(1.3)

where \( \beta_k \in \mathbb{R} \) is a scalar which determines the different conjugate gradient methods \([4,5,8,9,12,13,15,16,18,20,21,25,33]\) etc., and \( g_k \) is the gradient of \( f(x) \) at the point \( x_k \). The well-known formula for \( \beta_k \) from the computation point of view is the following PRP method

\[
\beta_k^{PRP} = \frac{\langle g_k \rangle}{\|g_k\|^2} \left( g_{k+1} - g_k \right),
\]

(1.4)

where \( g_k \) and \( g_{k+1} \) are the gradients \( \nabla f(x_k) \) and \( \nabla f(x_{k+1}) \) of \( f(x) \) at the point \( x_k \) and \( x_{k+1} \), respectively, and \( \| \cdot \| \) denotes the Euclidean norm of vectors. Throughout this paper, we also denote \( f(x_k) \) by \( f_k \). Polak and Ribière [18] proved that this method with the exact line search is globally convergent when the objective function is convex. Powell [19] gave a counter example to show that there exist nonconvex functions on which the PRP method does not converge globally even the exact line search is used. He suggested that \( \beta_k \) should not be less than zero. Considering this suggestion, Gilbert and Nocedal [10] proved that the modified PRP method \( \beta_k^* = \max \{0, \beta_k^{PRP} \} \) is globally convergent with the weak Wolfe-Powell (WPP) line search technique and the assumption of sufficient descent condition. However, the global convergence of the PRP method is still open under the WPP line search rule.

Recently, Wei, Yao, and Liu (WYL) [21] propose a new conjugate gradient formula

\[
\beta_k^{WYL} = \frac{\langle g_{k+1} \rangle}{\|g_k\|^2} \left( g_{k+1} - \frac{\|g_k\|^2}{\|g_k\|^2} g_k \right)
\]

(1.5)

It is not difficult to deduce that

\[ x_{k+1} = x_k + \alpha_k d_k, k = 0,1,2,\ldots \]
\[ \beta_{k}^{WYL} = \frac{\|g_{k+1} - \|g_{k+1}^T g_{k}\| g_{k}\|}{\|g_{k}\|} \]

where \( \beta_{k}^{W-P} = \max\{\beta_{k}^{PRP}, \beta_{k}^{WYL}\} \).

Step 5: Set \( k := k + 1 \), and go to Step 2.

Remark i) If \( x_{k+1} \approx x_{k} \), we have \( g_{k+1} \approx g_{k} \) and \( \|g_{k+1}\| \approx \|g_{k}\| \), which imply that \( \beta_{k}^{PRP} \to 0 \), and \( \beta_{k}^{WYL} \to 0 \), which means that \( \beta_{k}^{W-P} \to 0 \) if a small step is generated for all \( k \geq 0 \). Thus the given method inherits the better property of the PRP method: the directions will turn out to be the steepest descent directions if the tiny steps from happening.

ii) By the definition of the new formula \( \beta_{k}^{W-P} \), we have

\[ \beta_{k}^{W-P} = \max\{\beta_{k}^{PRP}, \beta_{k}^{WYL}\} \geq \beta_{k}^{WYL} \]

Thus the given method is competitive to the PRP method and the WYL method.

3. THE GLOBAL CONVERGENCE

The following assumptions are often needed to prove the convergence of the nonlinear conjugate gradient methods (see [5,9,10,20,21] etc.).

Assumption 3.1 i) The function \( f(x) \) has a lower bound on the level set \( \Omega = \{x \in \mathbb{R}^n \mid f(x) \leq f(x_0)\} \), where \( x_0 \) is a given point and \( \Omega \) is bounded.

ii) In an open convex set \( \Omega_0 \) that contains \( \Omega, J \) is differentiable and its gradient \( g \) is Lipschitz continuous, namely, there exists a constants \( L > 0 \) such that

\[ \|g(x) - g(y)\| \leq L\|x - y\|, \quad \forall x, y \in \Omega_0. \]  

3.1. The global Convergence with the Weak Wolfe-Powell Line Search

The weak Wolfe-Powell (WWP) search rule is to find a step length \( \alpha_k \) such that

\[ f(x_k + \alpha_k d_k) \leq f_k + \delta \alpha_k g^T_k d_k \]  

and

\[ g(x_k + \alpha_k d_k)^T d_k \geq \sigma g^T_k, \]  

where \( \delta \in (0,1/2), \sigma \in (\delta,1) \). This line search technique is often used to study the convergence of conjugate gradient algorithms [6,27,34]. At present, the global convergence of the PRP method with the WWP line search is still open.

Lemma 3.1 Suppose that Assumption 3.1 holds. Let the sequence \( \{g_k\} \) and \( \{d_k\} \) be generated by Algorithm 1. \( g^T_k d_k \leq 0 \), and the stepsize \( \alpha_k \) be determined
by the WWP line search (3.2) and (3.3) Then the zoutendijk condition [34]

\[ \sum_{k=0}^{\infty} \left( \frac{g_k^T d_k}{\|d_k\|^2} \right)^2 < +\infty \]  

(3.4)

holds.

**Proof.** By (3.3) and Assumption 3.1 ii), we have

\[ -(1 - \sigma) g_k^T d_k \leq (g_{k+1} - g_k)^T d_k \leq \alpha_k L \|d_k\|^2, \]

this means that \( \alpha_k \geq -(1 - \sigma) g_k^T d_k / L \|d_k\|^2 \), which together with \( g_k^T d_k \leq 0 \), and (3.2) implies that

\[ \frac{(1 - \sigma) g_k^T d_k}{L \|d_k\|^2} \leq f_k - f_{k+1}, \]

summing up this inequality from \( k = 0 \) to \( \infty \), and using Assumption 3.1 i), we can obtain this lemma. This completes the proof.

We will prove the global convergence of Algorithm 1 by contradiction. Then we assume that there exists a positive constant \( \gamma > 0 \) such that

\[ \|g_k\| \geq \gamma, \quad \forall k \geq 0. \]  

(3.5)

Using (3.5) deduces a contradiction to obtain our conclusion.

Similar to Lemma 3.3.1 in [6], based on Assumption 3.1, Lemma 3.1, the fact \( \beta_k^{p,w} \geq 0 \), and (3.5), we can get the following lemma.

**Lemma 3.2** Let Assumption 3.1 hold and the sequences \( \{g_k\} \) and \( \{d_k\} \) be generated by Algorithm 1. The sufficient descent condition (1.6) holds, and the stepsize \( \alpha_k \) is determined by (3.2) and (3.3). Suppose that the inequalities (3.5) is true. Then we have \( d_k \neq 0 \) and

\[ \sum_{k=0}^{\infty} \left\| u_{k+1} - u_k \right\|^2 < \infty, \]

where \( u_k = \frac{d_k}{\|d_k\|} \).

**Proof.** These two inequalities (1.6) and (3.5) imply that \( d_k \neq 0 \) is true, for otherwise \( g_k = 0 \), then \( u_k = d_k / \|d_k\| \) is reasonable. Denote

\[ r_{k+1} = \frac{g_{k+1}}{d_{k+1}}, \quad \delta_k = \beta_k^{p,w} \frac{\|d_k\|}{\|d_{k+1}\|}. \]

By (2.1), for \( k \geq 0 \), we have

\[ u_{k+1} = r_{k+1} + \delta_k u_k, \]

this combining with \( \|u_{k+1}\| = \|u_k\| = 1 \) shows that

\[ \|u_{k+1} - u_k\| = \|r_{k+1}u_k - u_k\| = \delta_k u_{k+1} - u_k \]  

(3.6)

The inequality \( \beta_k^{p,w} \geq 0 \) implies that \( \delta_k \geq 0 \) is true, then it follows that from (3.6) and triangular inequality

\[ \|u_{k+1} - u_k\| \leq \|u_{k+1} - u_{k+1} \| + \|u_{k+1} - u_k\| \]  

(3.7)

By (1.6) and (3.4), we get

\[ \sum_{k=0}^{\infty} \left\| u_{k+1} \right\|^2 = \sum_{k=0}^{\infty} \left\| u_{k+1} \right\|^2 < \infty \]

Which together with (3.5), we obtain

\[ \sum_{k=0}^{\infty} \left\| r_{k+1} \right\|^2 < \infty \]

The above inequality and (3.7), we get this lemma. The proof is complete.

The following property (*) was introduced by Gilbert and Nocedal [10], which pertains to the WYL formula also has this property. Now we show that this property (*) pertains to our method.

**Property** (*). Suppose that

\[ 0 < \tau_1 \leq \|g_k\| \leq \tau_2. \]  

(3.8)

We say that the method has Property (*), if for all \( k \), there exists constants \( b > 1 \) and \( \lambda > 0 \) such that

\[ \|\beta_k\| \leq b \]  

and

\[ \|s_k\| \leq \lambda \Rightarrow \|\beta_k\| \leq \frac{1}{2b}. \]

**Lemma 3.3** Let Assumption 3.1 hold and the sequences \( \{g_k\} \) and \( \{d_k\} \) be generated by Algorithm 1. Then the new formula \( \beta_k^{p,w} \) possesses property (*).

**Proof.** The result of this lemma is proved by the following two cases.

**Case i:** we consider \( \beta_k^{p,w} \). By (3.1), we have

\[ \|\beta_k^{p,w}\| = \left\| g_{k+1}^T (g_{k+1} - g_k) / L \|g_{k+1}\|^2 \right\| < \infty. \]

(3.9)

From Assumption 3.1 i), then there exists a constant \( M_1 > 0 \) such that

\[ \|s_k\| \leq M_1. \]  

(3.10)

Let \( b = \max \left\{ 2 \left( L_2 / s_k^2 \right) M_1 \right\} > 1 \) and \( \lambda = \gamma_2 / 2b (L_2 / s_k) \), it follows that

\[ \|\beta_k^{p,w}\| \leq b \]  

and
Then the PRP formula $\beta_k^{PRP}$ has this property (*).

Case ii: let us consider $\beta_k^{WYL}$. Denote $Y_k = g_{k+1} - \frac{\|g_k\|}{\|g_{k+1}\|}g_k$, by (3.1), we get

$$\|Y_k\| \leq \frac{\|g_k\|}{\|g_{k+1}\|} \leq \frac{2L\gamma_2}{\gamma_1^2} \leq \frac{2L\gamma_2}{\gamma_1} \lambda = \frac{1}{2b}.
$$

Thus, the formula $\beta_k^{WYL}$ also has the property (*).

Let $b = \max \left\{ \frac{1}{2 \lambda}, \frac{1}{2 b L \gamma_2} \right\}$ and $\lambda = \frac{\gamma_2}{2 b L \gamma_2}$, it follows that (3.12) and the definition of $b$ and $\lambda$ that $b > 1$

$$\|\beta_k^{WYL}\| \leq b, \quad \text{and} \quad \|\beta_k^{WYL}\| \leq \frac{2L\gamma_2}{\gamma_1} \leq \frac{1}{2b}.
$$

Thus the formula $\beta_k^{WYL}$ also has the property (*).

Using the definition of the $\beta_k^{P-W} = \max \{\beta_k^{WYL}, \beta_k^{PRP}\}$, we conclude that the formula $\beta_k^{P-W}$ possesses the property (*). The proof is complete.

By Lemma 3.3, similar to Lemma 3.3.2 in [6], it is not difficult to prove the following result. Here we only state it as follows, but omit the proof.

**Lemma 3.4 (Lemma 3.3.2 in [6])** Let the sequences $\{g_k\}$ and $\{d_k\}$ be generated by Algorithm 1 and the conditions in Lemma 3.3 hold. If $\beta_k^{P-W} > 0$ and has property (*), then there exists a constant $\xi_0 > 0$ such that, for any $N \in \mathbb{N}$ and any index $k_0$ there is an index $k > k_0$ satisfying

$$k_{\xi_0, \Delta}^k > \frac{\lambda}{2},$$

where $k_{\xi_0, \Delta}^k = \{i \in \mathbb{N} : k \leq i \leq k + \Delta - 1, \beta_i^{P-W} > \lambda\}$. $N$ denotes the set of positive integers, and $k_{\xi_0, \Delta}^k$ denotes the numbers of elements in $k_{\xi_0, \Delta}^k$.

Finally, by Lemma 3.2 and Lemma 3.4, we present the global convergence theorem of Algorithm 1 with the WWP line search. Similar to Theorem 3.3.3 in [6], it is not difficult to prove the result, here we also give the process of the proof.

**Theorem 3.1** Let the sequence $\{g_k, d_k\}$ be generated by Algorithm 1 with the weak Wolfe-Powell line search and the conditions in Lemma 3.3 hold. Then $\lim_{i \to \infty} \inf \|g_i\| = 0$.

**Proof.** We will get this theorem by contradiction. Suppose that (3.5) is true, then the conditions in Lemma 3.2 and 3.3 hold. By Assumption 3.1 i), then there exists a constant $\xi_0 > 0$ such that

$$\|x\| \leq \xi_0, \forall x \in \Omega \quad (3.13)$$

We also denote $u_i = d_i / \|d_i\|$, then for all integers $l, k (i \geq k)$, we have

$$x_i - x_{k+1} = \sum_{j=k+1}^{i} \|\xi_j\| u_i = \sum_{j=k+1}^{i} \|\xi_j\| (u_{j-i} - u_{k+1}).
$$

Taking the norm in both sides of the above equality, and using (3.13) we get

$$\sum_{j=k+1}^{i} \|\xi_j\| \leq 2\xi_0 + \sum_{j=k+1}^{i} \|\xi_j\| \|u_{j-i} - u_{k+1}\| \leq 2\xi_0 + \sum_{j=k+1}^{i} \|\xi_j\|.$$ 

Let $\Delta = \left[ \frac{8\xi_0}{\lambda} \right]$ be the smallest integer where $\Delta$ does not less than $8\xi_0 / \lambda$. By Lemma 3.2, there exists an index $k_0$ such that

$$\sum_{j=k_0+1}^{i} \|\xi_j\| \leq \frac{1}{4\Delta} \quad (3.14)$$

On the other hand, by Lemma 3.3, there exists $k \geq k_0$ satisfying

$$\|u_{k+1} - u_{k+1}\| \leq \frac{1}{4\Delta} \quad (3.15)$$

For all $k \in [k, k + \Delta - 1]$, by Cauchy-Schwarz inequality and (3.14), we obtain

$$\|u_{i+1} - u_{k+1}\| \leq \left( \sum_{j=k+1}^{i} \|u_{j+1} - u_{j+1}\| \right)^{1/2} \leq \left( i-k \right)^{1/2} \left( \sum_{j=k+1}^{i} \|u_{j+1} - u_{j+1}\| \right)^{1/2} \leq \Delta \left( \frac{1}{4\Delta} \right)^{1/2} = \frac{1}{2}.$$

By the above inequality, (3.15) and (3.13), we have

$$2\xi_0 \geq \sum_{j=k_0+1}^{i} \|\xi_j\| \geq \frac{\lambda}{2} \sum_{j=k_0+1}^{i} \|\xi_j\| \geq \frac{\lambda\Delta}{4}.$$

Thus $\Delta < \frac{8\xi_0}{\lambda}$, this contradicts with the definition of $\Delta$. Therefore, the conclusion of this theorem is right. This completes the proof.

**4. NUMERICAL RESULTS**

In this section, we report some numerical experiments.
The unconstrained optimization problems with the given initial points can be found at:
www.ici.ro/camo/neculai/SCALCG/testuo.pdf,
which were collected by Neculai Andrei. Since this new method is the hybrid method of the PRP method and the WYL method, we test Algorithm 1 with the WWP line search and compare its performance with those of the WYL [21] and the PRP [18] methods. The stop criterions are given below: we stop the program if the inequality
\[ \|g(x_i)\| \leq \epsilon \] is satisfied or the inequality
\[ \|g(x_i)\| \leq \epsilon (1 + |f(x_i)|) \]
is satisfied, where \( \epsilon = 1.0 \cdot D - 5 \). All the codes were written in Fortran and run on PC with 2.60 GHz CPU processor and 256 MB memory and Windows XP operation system. In the experiments, the parameters were chosen as \( \delta = 1.0 \cdot D - 2 \), \( \sigma = 1.0 \cdot D - 1 \). The dimension of the test problems is from 500 to 5000. The detailed numerical results are listed on the web site:
http://210.36.18.9:8018/publication.asp?id=35392.

In Figure 1, “WYL”, “PRP”, and “MPRP-WYL” stand for the WYL method, the PRP method, and the new method, respectively.

Figure 1 shows the performance of these methods relative to the iterative number of the function and gradient(NFN), which were evaluated using the profiles of Dolan and Moré [7]. It is easy to see that the MPRP-WYL is predominant among these three methods and the new method can solve about 99% of the test problems successfully. The PRP method is better than the WYL method for \( 1 \leq t \leq 1.2 \) and the WYL method is better than the PRP method for \( 1.2 \leq t \leq 6 \). Moreover, the PRP method solves about 98% of the test problems and the WYL method solve about 99% of the test problems successfully, respectively. In a word, the given method is competitive to the other two methods and the hybrid formula is notable.

5. CONCLUSION

This paper gives a hybrid conjugate gradient method for solving unconstrained optimization. The global convergence for nonconvex functions with the WWP line search is established. The numerical results show that the given method is competitive to the other standard conjugate gradient methods for the test problems.

For further research, we should study the convergence of the new algorithm under other line search rules. Moreover, more numerical experiments and testing environments (such that [3]) for large practical problems should be done in the future.

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