FURTHER EVALUATION OF WAHL VANISHING THEOREMS
FOR SURFACE SINGULARITIES IN CHARACTERISTIC \( p \)

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ABSTRACT. Let \((\text{Spec} \ R, \mathfrak{m})\) be a rational double point defined over an algebraically closed field \(k\) of characteristic \(p \geq 0\). We evaluate further the dimensions of the local cohomology groups which were treated by Wahl in 1975 as vanishing theorem C (resp. D) under the assumption that \(p\) is a very good prime (resp. good prime) with respect to \((\text{Spec} \ R, \mathfrak{m})\). We use Artin’s classification of rational double points and completely determine the dimensions \(\dim_k H^1_k(S_X)\), \(\dim_k H^1_k(S_X \otimes \mathcal{O}_X(E))\), supplementing Wahl’s theorems. In the proof we construct derivations concretely which do not lift to the minimal resolution \(X \to \text{Spec} \ R\), as well as non-trivial equisingular families which inject into a versal deformation of the rational double point \((\text{Spec} \ R, \mathfrak{m})\).

1. Introduction

In 1975 Jonathan Wahl proved the Grauert-Riemenschneider vanishing theorem along with three other types of vanishing theorems for a surface singularity \((\text{Spec} \ R, \mathfrak{m})\) defined over an algebraically closed field \(k\) of characteristic \(p \geq 0\) [13]. Last decades witnessed that these theorems on local cohomology groups have played influential roles in the theory of surface singularities. Among them are what he calls Theorems C, D which bear restrictions on the characteristic of the ground field \(k\).

**Theorem C** Let \(X \to \text{Spec} \ R\) be the minimal resolution of a rational double point. Then \(H^1_k(S_X) = 0\), and in particular the resolution is equivariant, except the following cases:

\[\begin{align*}
A_n & \quad p \mid n + 1, \\
D_n & \quad p = 2, \\
E_6 & \quad p = 2, 3, \\
E_7 & \quad p = 2, 3, \\
E_8 & \quad p = 2, 3, 5.
\end{align*}\]

**Theorem D** Let \(X \to \text{Spec} \ R\) be the minimal resolution of a rational double point. Then \(H^1_k(S_X(E)) = 0\), except in the following cases:

\[\begin{align*}
D_n & \quad p = 2, \\
E_6 & \quad p = 2, 3, \\
E_7 & \quad p = 2, 3, \\
E_8 & \quad p = 2, 3, 5.
\end{align*}\]

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In these theorems $E = \cup_i E_i$ denotes the exceptional divisor in $X$ and its irreducible decomposition, $S_X$ is the sheaf of logarithmic derivations $S_X = \Theta_X(-\log E)$ (cf. [14]) which fits in an exact sequence:

$$0 \to S_X \to \Theta_X \to \bigoplus_i N_{E_i/X} \to 0.$$
Wahl's vanishing theorems in characteristic \( p \)

ii) The dimension of \( H^1_E(S \otimes O_X(E)) \) is zero except:

\[
\begin{align*}
1 & \quad \text{for } E_6^0, E_7^1, E_8^3, \\
2 & \quad \text{for } E_7^2, E_8^2, \\
3 & \quad \text{for } E_7^0, E_8^1, \\
4 & \quad \text{for } E_8^0, \\
n - 1 - r & \quad \text{for } D_{2n}, D_{2n+1}.
\end{align*}
\]

iii) One has an isomorphism

\[
H^0(X \setminus E, \Theta_X)/H^0(X, \Theta_X) \cong H^1_E(S_X),
\]
whose dimension is zero with the following exceptions:

\[
\begin{align*}
1 & \quad \text{for } A_n \text{ with } 2 \mid (n + 1), \\
1 & \quad \text{for } E_6^0, E_7^2, E_8^3, \\
2 & \quad \text{for } E_7^2, E_8^2, \\
3 & \quad \text{for } E_7^1, E_8^1, \\
4 & \quad \text{for } E_8^0, \\
n + 1 - r & \quad \text{for } D_{2n}, \\
n - r & \quad \text{for } D_{2n+1}.
\end{align*}
\]

Corollary 1.1. For a rational double point (\( \text{Spec } R, m \)) of the following type, one has \( H^1_E(S_X) = 0 \), in particular, the minimal resolution \( X \to \text{Spec } R \) is equivariant.

\( E_8^3 \) in \( p = 5 \), \( E_7^1, E_8^2 \) in \( p = 3 \), \( E_6^1, E_8^1 \) in \( p = 2 \).

The assertion i) in both theorems gives an answer to the question asked in [13 (5.18.2)] for rational double points.

Wahl’s vanishing theorems have come into focus in the study of three dimensional canonical singularities in arbitrary characteristic and we needed further evaluation of dimensions of \( H^1_E(S_X), H^1_E(S_X \otimes O_X(E)) \). One has the smooth morphism from the simultaneous resolution functor of a versal deformation of a rational double point (\( \text{Spec } R, m \)) [1] to the deformation functor of its minimal resolution \( X \):

\[ \text{Res} \mathcal{X}/S \to \mathcal{D}_X. \]

Non-zero elements of \( H^1_E(S_X) \) correspond to families of resolutions which are non-trivial in \( \text{Res} \mathcal{X}/S \), but map to trivial deformations of \( X \). This is linked with phenomena of three dimensional canonical singularities peculiar to positive characteristic \( p \) which are recently observed in [4], [5], [6], [9].

2. Preliminaries

When we say (\( \text{Spec } R, m \)) is a surface singularity, it is understood that \( (R, m) \) is a two dimensional excellent normal local ring with the maximal ideal \( m \).

We use the term equisingular deformations of the resolution in the sense of Wahl [14].

The following proposition is in implicit form in Artin’s work [11 Corollary 4.6]. Here \( \mathfrak{R} \) denotes a locally quasi-separated algebraic space which represents the functor \( \text{Res} \mathcal{X}/S \) of simultaneous resolutions of families of a surface singularity (\( \text{Spec } R, m \)).
Proposition 2.1. Suppose $X/S$ is a versal deformation of a rational surface singularity $(\text{Spec } R, m)$ at $s_0 \in S$ and has minimum tangent space dimension there. Then the universal family $\mathcal{X}_{\mathcal{X}/S}$ is a versal deformation of $X$ with minimum tangent space dimension, if and only if the minimal resolution $X \to \text{Spec } R$ is equivariant.

Proof. $(\Rightarrow)$ Suppose that there exists a non-zero element $\theta \in H^0(X \setminus E, \Theta_X)/H^0(X, \Theta_X)$. Then one has a diagram

$$
\begin{array}{ccc}
X \times T & \xrightarrow{\varphi} & X \times T \\
\downarrow_{\pi \times \text{id}} & & \downarrow_{\pi \times \text{id}} \\
\text{Spec } R \times T & \to & \text{Spec } R \times T,
\end{array}
$$

where $T := \text{Spec } k[\varepsilon]/(\varepsilon^2)$ and $\varphi$ is an isomorphism given by sending $f \in R$ to $f + \theta(\varepsilon)\epsilon$. But no morphism $X \times T \to X \times T$ makes this diagram commutative. One may consider the map $\varphi \circ (\pi \times \text{id})$ as a resolution of $\text{Spec } R \times T$. This gives a nontrivial extension of $T \to S$ to $T \to \mathcal{R}$, although $X \times T$ is a trivial deformation of $X$. This contradicts the fact that $\mathcal{X}_{\mathcal{X}/S}$ has the minimum tangent space dimension as a versal deformation of $X$.

$(\Leftarrow)$ This is proved by Artin. \hfill $\Box$

The following is a refinement of the inequality which Shepherd-Barron used in [10] Proposition 3.1.

Proposition 2.2. For a rational double or triple point $(\text{Spec } R, m)$, one has the inequality

$$\dim_k H^1(\Theta_X) + \dim_k H^0(X \setminus E, \Theta_X)/H^0(X, \Theta_X) \leq \tau_0,$$

where $\tau_0$ is the Tjurina number of $(\text{Spec } R, m)$.

Proof. This is based on the fact that there is a smooth morphism of functors $\text{Res } \mathcal{X}/S \to D_X$ [11 Lemma 3.3], and the tangent space $\text{Res } \mathcal{X}/S(k[\varepsilon]/(\varepsilon^2))$ has its dimension $\tau_0$ [11 Theorem 3]. The inequality is the dimension formula for the surjective $k$-linear mapping $\text{Res } \mathcal{X}/S(k[\varepsilon]/(\varepsilon^2)) \to D_X(k[\varepsilon]/(\varepsilon^2))$. \hfill $\square$

Proposition 2.3. Let $(\text{Spec } R, m)$ be a rational surface singularity defined over an algebraically closed field $k$ of characteristic $p \geq 0$, and $\pi : X \to \text{Spec } R$ be its minimal resolution with the reduced exceptional divisor $E = \bigcup_i E_i$. Then we have the equalities

$$\dim_k H^1_E(S_X) = \dim_k H^1(S_X \otimes O_X(E + 2K_X)),$$

$$\dim_k H^2_E(S_X) = \dim_k H^1(S_X \otimes O_X(E + 2K_X)) = \dim_k H^1(S_X),$$

where $S_X$ is a locally free sheaf defined as the kernel of the surjection from the tangent to normal sheaves $\Theta_X \to \bigoplus_i \mathcal{N}_{E_i/X}$. In particular, the local cohomology groups $H^1_E(S_X)$, $H^2_E(S_X \otimes O_X(E + 2K_X))$ are finite dimensional $k$-vector spaces.

Proof. We use the exact sequence given by Wahl [15 (1.2)]

$$0 \to \Omega_X \to S_X^! \to \bigoplus_i O_{E_i} \to 0,$$

from which follows $\wedge^2(S_X^!) \cong O_X(E + K_X)$ hence $S_X^! \cong S_X \otimes O_X(E + K_X)$. We combine this with the Grothendieck local duality theorem, to have the equalities $\dim_k H^1_E(S_X) = \dim_k H^1(S_X^! \otimes K_X) = \dim_k H^1(S_X \otimes O_X(E + 2K_X))$ as well as
Theorem 2.4. For a rational double point (Spec R, m), we have the equality
\[ \dim_k H^1(\Theta_X) = \dim_k H^1_E(\Theta_X) = \#\{-2 \text{ curves in } E\} + h^1(S_X), \]
where \( \pi : X \to \text{Spec } R \) is the minimal resolution with the exceptional divisor E.

Proof. We repeat the argument in [13, Theorem 6.1]. One has
\[ 0 \to \Theta_X \to S_X(E) \to \Theta_E \otimes \mathcal{N}_{E/X} \to 0, \]
which gives
\[ 0 \to H^1_E(\Theta_E \otimes \mathcal{N}_{E/X}) \to H^1_E(\Theta_X) \to H^1_E(S_X(E)) \to 0. \]
The local duality theorem gives the second equality. For the first equality we use the local duality theorem and the standard isomorphism \( \wedge^2 \Omega_X \cong K_X \).

\[ \square \]

3. LOWER ESTIMATES OF \( \dim_k H^1(S_X) \), \( \dim_k H^1_E(S_X) \)

Observing Artin’s list of rational double points [1] enables one to get the lower bounds of dimensions of \( H^1(S_X) \) and \( H^1_E(S_X) \).

Proposition 3.1. Let \( X \to \text{Spec } R \) be the minimal resolution of a rational double point (Spec R, m). If (Spec R, m) is of the following type, then one has non-trivial equisingular deformations which provide the lower bound of \( \dim_k H^1(S_X) \) as
\[ \begin{cases} 
1 & \text{for } E^0_8 \text{ in } p = 5 \text{ and } E^1_8, E^0_6, E^0_7 \text{ in } p = 3, \\
2 & \text{for } E^0_8 \text{ in } p = 3.
\end{cases} \]
Moreover, the subspace these equisingular families generate does not collapse in the tangent map
\[ H^1(S_X) \to H^1(X \setminus E, S_X). \]

Proof. We give concrete one parameter deformations \( X \to \text{Spec } k[s] \) and construct simultaneous resolutions by blowing up the singular loci.

For \( E^0_8 \) in \( p = 5 \), we have a deformation \( z^2 + x^3 + y^5 + sxy^4 = 0 \). This has a simultaneous resolution with no base extension, and has \( E^0_8 \) singularity on the special fiber \( X_0 \), and \( E^1_8 \) singularity on a general fiber \( X_s \) (s \( \neq 0 \)).

Similarly we present non-trivial one-parameter deformations which admit simultaneous resolutions with no base extension. These result in equisingular deformations. For \( E^0_6 \) in \( p = 3 \), \( z^2 + x^3 + y^5 + sx^2y^2 = 0 \). For \( E^0_8 \) in \( p = 3 \), \( z^2 + x^3 + y^4 + sx^2y^2 = 0 \). For \( E^1_8 \) in \( p = 3 \), \( z^2 + x^3 + y^5 + sxy^4 = 0 \). For \( E^0_7 \) in \( p = 3 \), \( z^2 + x^3 + y^5 + s_1x^2y^3 = 0 \) and \( z^2 + x^3 + y^5 + s_2x^2y^3 = 0 \).

Each family is induced by an injection to a versal deformation of the rational double point (Spec R, m), and one gets the last assertion. \[ \square \]
Proposition 3.2. Let $X \to \text{Spec} R$ be the minimal resolution of a rational double point $(\text{Spec} R, m)$ of the following type. Then one calculates the dimension of $H^0(X \setminus E, S_X)/H^0(X, S_X)$ as

$$\begin{cases} 
1 & \text{for } A_n \text{ with } p \mid (n+1), \\
1 & \text{for } E^0_8 \text{ in } p = 5 \text{ and } E^0_7 \text{ in } p = 3, \\
2 & \text{for } E^0_6, E^0_9 \text{ in } p = 3.
\end{cases}$$

For $E^1_8, E^1_9$ in $p = 3$, one has

$$\dim_k H^0(X \setminus E, S_X)/H^0(X, S_X) \geq 1.$$  

Proof. A quadratic transformation $x' = x/y, y' = y, z' = z/y$ gives equalities of derivations $\partial/\partial x = (1/y')\partial/\partial x', \partial/\partial y = \partial/\partial y' - (x'/y')\partial/\partial x' - (z'/y')\partial/\partial z'$, $\partial/\partial z = (1/y')\partial/\partial z'$. As was pointed out by Burns and Wahl [3 Proposition 1.2], any derivation $D$ of $(R, m)$ which satisfies $D(m) \subset m$ can be extended to a regular derivation on $\text{Proj} \bigoplus_{i \geq 0} m^i$ (the point blow-up of $m$).

For $A_n : R \cong k[[x, y, z]]/(xy + z^{n+1})$ with $p \mid (n+1)$, we have the exact sequence (cf. [3 Theorem 25.2])

$$0 \to \left( \frac{\partial}{\partial z}, x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right) \to T_{k[x]} \otimes R \xrightarrow{(y, x)} \text{Hom}_R(I/I^2, R).$$

This is the Koszul complex associated with the regular sequence $x, y \in R$. The derivation $\frac{\partial}{\partial z} \in \text{Der}_k(R)$ does not lift to $\text{Proj} \bigoplus_{i \geq 0} m^i$ (the blow-up of $m$).

For $E^0_8 : R \cong k[[x, y, z]]/(z^2 + x^3 + y^5)$ in $p = 5$, one has the exact sequence

$$0 \to \left( \frac{\partial}{\partial y}, 2z \frac{\partial}{\partial x} - 3x^2 \frac{\partial}{\partial z} \right) \to T_{k[x]} \otimes R \xrightarrow{(3x^2, 0, 2z)} \text{Hom}_R(I/I^2, R).$$

The derivation $\frac{\partial}{\partial y} \in \text{Der}_k(R)$ does not lift to the point blow-up $\text{Proj} \bigoplus_{i \geq 0} m^i$.

For $E^0_9 : R \cong k[[x, y, z]]/(z^2 + x^3 + xy^3)$ in $p = 3$, one has the exact sequence

$$0 \to \left( \frac{\partial}{\partial y}, z \frac{\partial}{\partial x} + y^3 \frac{\partial}{\partial z} \right) \to T_{k[x]} \otimes R \xrightarrow{(y^3, 0, 2z)} \text{Hom}_R(I/I^2, R).$$

The derivation $\frac{\partial}{\partial y}$ does not lift to the point blow-up.

For $E^0_6 : R \cong k[[x, y, z]]/(z^2 + x^3 + y^4)$ in $p = 3$, one has

$$0 \to \left( \frac{\partial}{\partial y}, z \frac{\partial}{\partial x} + y^3 \frac{\partial}{\partial z} \right) \to T_{k[x]} \otimes R \xrightarrow{(0, y^2, 2z)} \text{Hom}_R(I/I^2, R).$$

Two derivations $\frac{\partial}{\partial x}, y \frac{\partial}{\partial x}$ do not lift to the minimal resolution $X$.

For $E^0_7 : R \cong k[[x, y, z]]/(z^2 + x^3 + y^5)$ in $p = 3$, one has

$$0 \to \left( \frac{\partial}{\partial x}, z \frac{\partial}{\partial y} - y^4 \frac{\partial}{\partial z} \right) \to T_{k[x]} \otimes R \xrightarrow{(0, y^4, z)} \text{Hom}_R(I/I^2, R).$$

Two derivations $\frac{\partial}{\partial x}, y \frac{\partial}{\partial x}$ do not lift to the minimal resolution $X$. 

For $E_8^1 : R \cong k[[x, y, z]]/(z^2 + x^3 + y^5 + x^2 y^3)$ in $p = 3$, we have the derivation $D := y \partial / \partial x - x \partial / \partial y \in \text{Der}_k(R)$. This satisfies $D(\mathfrak{m}) \subset \mathfrak{m}$, so this $D$ lifts to a point blow-up $\text{Proj} \bigoplus_{i=0}^{\infty} \mathfrak{m}^i$, on which lies a rational double point of type $E_8^0$. But this $D$ does not lift to the minimal resolution, because it is the very element considered in $E_8^0$ above.

For $E_6^1$ in $p = 3$, we have $R \cong k[[x, y, z]]/(z^2 + x^3 + y^4 + x^2 y^2)$. The derivation $D := (y - xy) \partial / \partial x + (x - y^2) \partial / \partial y + yz \partial / \partial z \in \text{Der}_k(R)$ does not lift to the minimal resolution. □

4. Proof of the main theorem

Theorem 4.1. Let $X \to \text{Spec} R$ be the minimal resolution of a rational double point defined over an algebraically closed field $k$ of characteristic $p \neq 2$. Then the following assertions hold.

i) The natural morphism $H^1(S_X) \to H^1(X \setminus E, S_X)$ is an inclusion.

ii) The dimension of $H^1_E(S_X \otimes \mathcal{O}_X(E))$ is zero except:

\[
\begin{aligned}
1 & \quad \text{for } E_8^0 \text{ in } p = 5 \text{ and } E_6^0, E_7^1, E_8^1 \text{ in } p = 3, \\
2 & \quad \text{for } E_8^0 \text{ in } p = 3.
\end{aligned}
\]

iii) One has an isomorphism $H^0(X \setminus E, \Theta_X)/H^0(X, \Theta_X) \cong H^1_E(S_X)$, whose dimension is zero with the following exceptions:

\[
\begin{aligned}
1 & \quad \text{for } A_n \text{ with } p | (n + 1), \\
1 & \quad \text{for } E_8^0 \text{ in } p = 5 \text{ and } E_6^0, E_7^1, E_8^1 \text{ in } p = 3, \\
2 & \quad \text{for } E_8^0, E_8^0 \text{ in } p = 3.
\end{aligned}
\]

Proof. If the characteristic $p$ is a good prime (resp. very good prime) for the type of the rational double point ($\text{Spec} R, \mathfrak{m}$), Wahl’s Theorem D (resp. Theorem C) provides the assertions i) and ii) (resp. iii)). If $p$ is not a very good prime, one needs to show that the lower estimates given in the previous section attain indeed the actual values. This is trivially verified, since one has the inequality coming from Proposition 2 and Theorem 1,

\[
\# \{-2 \text{ curves in } E\} + h^1(S_X) + \dim_k H^0(X \setminus E, \Theta_X)/H^0(X, \Theta_X) \leq \tau_0.
\]

The Tjurina numbers in Artin’s list [2] say this is indeed an equality. □

Remark 4.2. The pro-representable hull of equisingular deformations of $X$ injects into a versal deformation of the rational double point ($\text{Spec} R, \mathfrak{m}$). This forms a nonsingular subvariety whose dimension is as prescribed in Theorem 4.1 ii). One gets concrete one-parameter families given in the proof of Proposition 3.4. For $E_8^0$ in $p = 3$, one has a two-parameter family: $z^2 + x^3 + y^5 + s_1 x^2 y^3 + s_2 x^2 y^2 = 0$ over $\text{Spec} k[[s_1, s_2]]$. Two strata of dimension one and zero respectively can be observed in it.

5. Characteristic 2

As is often the case with characteristic 2, computation becomes more involved and demanding. However, we can complete our evaluation of dimensions essentially in the same way as before.
Theorem 5.1. Let $X \to \text{Spec } R$ be the minimal resolution of a rational double point defined over an algebraically closed field $k$ of characteristic 2. Then we have the following assertions.

i) The natural morphism $H^1(S_X) \to H^1(X \setminus E, S_X)$ is an inclusion.

ii) The dimension of $H^1_k(S_X \otimes \mathcal{O}_X(E))$ is zero except:

\[
\begin{align*}
1 & \quad \text{for } E_6^0, E_7^2, E_8^3, \\
2 & \quad \text{for } E_7^0, E_8^2, \\
3 & \quad \text{for } E_8^3, \\
4 & \quad \text{for } E_8^0, \\
\end{align*}
\]

\[\begin{align*}
n - 1 & \quad \text{for } D_{2n}^2, D_{2n+1}^2. \\
\end{align*}\]

iii) One has an isomorphism

\[H^0(X \setminus E, \Theta_X) / H^0(X, \Theta_X) \cong H^1_{E_k}(S_X),\]

whose dimension is zero with the following exceptions:

\[
\begin{align*}
1 & \quad \text{for } A_n \text{ with } 2 \mid (n + 1), \\
1 & \quad \text{for } E_6^0, E_7^3, E_8^3, \\
2 & \quad \text{for } E_7^0, E_8^2, \\
3 & \quad \text{for } E_8^3, \\
4 & \quad \text{for } E_8^0, \\
n + 1 & \quad \text{for } D_{2n}^2, \\
n & \quad \text{for } D_{2n+1}^2. \\
\end{align*}\]

First we take care of derivations of rational double points of type $D_n$.

Lemma 5.2. For a rational double point of type $D$ in characteristic 2, the following derivations do not lift to the minimal resolution.

$D_{2n}^0: R \cong k[[x, y, z]]/(z^2 + x^2y + xy^n)$.

\[
x \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, y \frac{\partial}{\partial z}, y^2 \frac{\partial}{\partial z}, \ldots, y^{n-1} \frac{\partial}{\partial z}.
\]

$D_{2n}^1: R \cong k[[x, y, z]]/(z^2 + x^2y + xy^n + xy^{n-r}z)$ with $r = 1, 2, \ldots, n - 1$.

\[
nxy^{n-r-1} \frac{\partial}{\partial x} + y^{n-r} \frac{\partial}{\partial y} + (x + ry^{n-r-1}z) \frac{\partial}{\partial z}, y2 \mathfrak{d}_1, y2 \mathfrak{d}_1, \ldots, y^{n-r-1} \mathfrak{d}_1
\]

with $\mathfrak{d}_1 := x \frac{\partial}{\partial x} + (y^r + z) \frac{\partial}{\partial z}$.

$D_{2n+1}^0: R \cong k[[x, y, z]]/(z^2 + x^2y + y^{n}z)$.

\[
\frac{\partial}{\partial x}, y \frac{\partial}{\partial x}, y^2 \frac{\partial}{\partial x}, \ldots, y^{n-1} \frac{\partial}{\partial x}.
\]

$D_{2n+1}^1: R \cong k[[x, y, z]]/(z^2 + x^2y + y^n z + xy^{n-r}z)$ with $r = 1, 2, \ldots, n - 1$.

\[
\mathfrak{d}_2, y \mathfrak{d}_2, y^2 \mathfrak{d}_2, \ldots, y^{n-r-1} \mathfrak{d}_2
\]

with $\mathfrak{d}_2 := (x + y^r) \frac{\partial}{\partial x} + z \frac{\partial}{\partial z}$.

Proof. Local calculation based on induction on $n \geq 2$. \qed
Proof. of Theorem 5.1 Each concrete one-parameter deformation \( X \to \text{Spec} \, k[s] \) presented below admits a simultaneous resolution without base extension, providing a non-trivial equisingular deformation of the minimal resolution \( X \).

For \( D^0_{2n} \), one chooses an integer \( k \in \{1, 2, \ldots, n-1\} \). The family is \( z^2 + x^2y + xy^n + sxy^{n-k}z = 0 \), which has the singularity \( D^0_{2n} \) on the special fiber \( X_0 \), and singularity \( D^k_2 \) on a general fiber \( X_s \) (\( s \neq 0 \)). For \( D^0_{2n+1} \) with \( 1 \leq r \leq n-2 \), one chooses an integer \( k \in \{r+1, r+2, \ldots, n-1\} \). The family is \( z^2 + x^2y + xy^n + xy^{n-r}z + sxy^{n-k}z = 0 \), which has \( D^0_{2n+1} \) on the special fiber \( X_0 \), and \( D^k_{2n+1} \) on a general fiber \( X_s \) (\( s \neq 0 \)).

For \( D^0_{2n+1} \) with \( 1 \leq r \leq n-2 \), one chooses an integer \( k \in \{r+1, r+2, \ldots, n-1\} \). The family is \( z^2 + x^2y + y^n z + sxy^{n-k}z = 0 \), which has \( D^0_{2n+1} \) on the special fiber \( X_0 \), and \( D^k_{2n+1} \) on a general fiber \( X_s \) (\( s \neq 0 \)).

For \( E^0_r \), one chooses \( z^2 + x^3 + y^2 z + sxyz = 0 \), which has the singularity \( E^0_r \) on \( X_0 \), and \( E^k_r \) on \( X_s \) (\( s \neq 0 \)).

For \( n \geq 0 \), one chooses an integer \( k \in \{r+1, r+2, \ldots, 4\} \). The deformation is \( z^2 + x^3 + y^3 + \theta r + s \theta r = 0 \), where \( \theta = 0, \theta_1 = x^3 y^2 z, \theta_2 = y^3 z, \theta_3 = x y z \). This has the singularity \( E^0_r \) on the special fiber \( X_0 \), and \( E^k_r \) on \( X_s \) (\( s \neq 0 \)).

Hereafter, we give derivations of \((R, m)\) which do not lift to the minimal resolution. For \( A_n \), the derivation is exactly of the same form as before, so we omit it.

For \( E^0_6 \), \( R \cong k[[x, y, z]]/(z^2 + x^3 + y^2 z) \), one has the exact sequence with \( I = (z^2 + x^3 + y^2 z) \),

\[
0 \rightarrow \left( y^2 \frac{\partial}{\partial x} + x^2 \frac{\partial}{\partial y} \right) \rightarrow T_{A^3} \otimes R \left( \frac{x^2 + y^2}{x^2 + y^2} \right) \rightarrow \text{Hom}_R(I/I^2, R).
\]

The derivation \( \frac{\partial}{\partial y} \in \text{Der}_k(R) \) does not lift to the point blow-up \( \text{Proj} \bigoplus_{i \geq 0} m^i \).

For \( E^0_7 \), \( R \cong k[[x, y, z]]/(z^2 + x^3 + xy^3) \), one has the exact sequence with \( I = (z^2 + x^3 + xy^3) \),

\[
0 \rightarrow \left( xy^2 \frac{\partial}{\partial x} + (x^2 + y^3) \frac{\partial}{\partial y} \right) \rightarrow T_{A^3} \otimes R \left( \frac{x^2 + y^3}{x^2 + y^3} \right) \rightarrow \text{Hom}_R(I/I^2, R).
\]

Four derivations \( \frac{\partial}{\partial z}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{y^2}{\partial z} \in \text{Der}_k(R) \) do not lift to the minimal resolution \( X \).

For \( E^0_8 \), \( R \cong k[[x, y, z]]/(z^2 + x^3 + y^5) \), one has the exact sequence

\[
0 \rightarrow \left( y^4 \frac{\partial}{\partial x} + x^2 \frac{\partial}{\partial y} \right) \rightarrow T_{A^3} \otimes R \left( \frac{x^2 + y^5}{x^2 + y^5} \right) \rightarrow \text{Hom}_R(I/I^2, R).
\]

Four derivations \( \frac{\partial}{\partial z}, \frac{y^2}{\partial z}, \frac{y^2}{\partial z}, x \frac{\partial}{\partial z} \) do not lift to the minimal resolution \( X \).
For $E_1^8 : R \cong k[[x, y, z]]/(z^2 + x^3 + y^5 + y^3z)$. The following three derivations do not lift to the minimal resolution.

\[
xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + (yz + x) \frac{\partial}{\partial z}, \vartheta_3, \ y\vartheta_4 \text{ with } \vartheta_3 = xz \frac{\partial}{\partial x} + (yz + x) \frac{\partial}{\partial y} + (z^2 + y) \frac{\partial}{\partial z}.
\]

For $E_2^8 : R \cong k[[x, y, z]]/(z^2 + x^3 + y^5 + y^3z)$. The following two derivations do not lift to the minimal resolution.

\[
y \frac{\partial}{\partial y} + (x^2 + z) \frac{\partial}{\partial z}, \ y \left( y \frac{\partial}{\partial y} + (x + z) \frac{\partial}{\partial z} \right)
\]

For $E_3^8 : R \cong k[[x, y, z]]/(z^2 + x^3 + y^5 + y^3z)$. The following derivation does not lift to the minimal resolution.

\[
y \frac{\partial}{\partial y} + (y^2 + z) \frac{\partial}{\partial z}
\]

We combine these with the previous lemma, inequalities in Preliminaries and the Tjurina numbers in Artin’s list [2] to get the assertions i), ii), iii). □

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REFERENCES

1. Artin, M.: Algebraic construction of Brieskorn’s resolutions, J. Algebra, 29, 330–348 (1974).
2. Artin, M.: Coverings of the rational double points in characteristic $p$, In: W. L. Baily Jr. and T. Shioda (eds.), Complex Analysis and Algebraic Geometry, 11–22, Cambridge Univ. Press, Cambridge (1977).
3. Burns, D. J., Wahl, J. M.: Local contributions to global deformations of surfaces, Invent. Math., 26, 67–88 (1974).
4. Hirokado, M.: Canonical singularities of dimension three in characteristic 2 which do not follow Reid’s rules, to appear in Kyoto J. Math.
5. Hirokado, M., Ito, H., Saito, N.: Three dimensional canonical singularities in codimension two in positive characteristic, J. Algebra 373 207–222 (2013).
6. Ishii, S., Reguera, A. J.: Singularities in arbitrary characteristic via jet schemes, Hodge Theory and $L^2$-analysis, ALM 39, 417–447 (2017).
7. Liedtke, C., Satriano, M.: On the birational nature of lifting, Adv. Math., 254, 118–137 (2014).
8. Matsumura, H.: Commutative Ring Theory, Cambridge Univ. Press, Cambridge (1986).
9. Sato, K., Takagi, S.: General hyperplane sections of threefolds in positive characteristic, preprint (2017).
10. Shepherd-Barron, N. I.: On simple groups and simple singularities, Israel J. Math., 123, 179–188 (2001).
11. Slodowy, P.: Simple singularities and simple algebraic groups, Lecture Notes in Math. 815, Springer, Berlin-Heidelberg-New York (1980).
12. Springer, T. A., Steinberg, R.: Conjugacy classes, In: Borel et alii (eds.), Seminar on algebraic groups and related finite groups, Lecture Notes in Math. 131, 167–266, Springer, Berlin-Heidelberg-New York (1970).
13. Wahl, J. M.: Vanishing theorems for resolutions of surface singularities, Invent. Math., 31, 17–41 (1975).
14. Wahl, J. M.: Equisingular deformations of normal surface singularities, I, Ann. Math., 104, 325–356 (1976).
15. Wahl, J. M.: A characterization of quasi-homogeneous Gorenstein surface singularities, Com- pos. Math., 55, 269–288 (1985).

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