ON THE COHEN-MACAULAY GRAPHS AND GIRTH

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Abstract. In this paper we investigate the Cohen-Macaulay property of graphs versus girth. In particular, we classify Cohen-Macaulay graphs of girth at least five and planar Gorenstein graphs of girth four.

Introduction

In this paper, all graphs will be assumed to be simple (i.e., a finite, undirected, loopless and without multiple edges). For a graph $G$ we denote $V(G)$ and $E(G)$ to be the vertex set and the edge set of $G$, respectively.

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring of variables $x_1, \ldots, x_n$ over the field $k$. Let $G$ be a graph with $V(G) \subseteq \{x_1, \ldots, x_n\}$. We associate to the graph $G$ a quadratic squarefree monomial ideal

$$I(G) = (x_ix_j \mid x_ix_j \in E(G)) \subseteq R,$$

which is called the edge ideal of $G$. $G$ is called a Cohen-Macaulay graph (resp. Gorenstein) if the edge ideal $I(G)$ is Cohen-Macaulay (resp. Gorenstein). Two vertices $u, v$ of $G$ are adjacent if $uv$ is an edge of $G$. An independent set in $G$ is a set of vertices no two of which are adjacent to each other. An independent set of maximum size will be referred to as a maximum independent set of $G$, and the independence number of $G$, denoted by $\alpha(G)$, is the cardinality of a maximum independent set in $G$. An independent set $S$ in $G$ is maximal (with respect to set inclusion) if the addition to $S$ of any other vertex in the graph destroys the independence. A graph is called well-covered if every maximal independent set has the same size (see [P1, P2]).

If $G$ is Cohen-Macaulay, then $G$ is well-covered by [Vi, Proposition 6.1.21]. Characterize combinatorial Cohen-Macaulay graphs is impossible because the Cohen-Macaulay property of graphs is dependent on the characteristic of the base field (see [Vi, Exercise 5.3.31]). Then, the work on Cohen-Macaulay graphs now has focused on certain class of graphs as: chordal graphs, bipartite graphs and so on (see [HH1, HHZ, MKY, W1, W2]). In here, we will give a combinatorial classification all Cohen-Macaulay graphs in some new classes.

The girth of a graph $G$, denoted by $\text{girth}(G)$, is the length of any shortest cycle in $G$ or in the case $G$ is a forest we consider the girth to be infinite. If $X \subseteq V(G)$,
then $G[X]$ is the subgraph of $G$ spanned by $X$. By $G \setminus W$, we mean the induced subgraph $G[V \setminus W]$ for some $W \subseteq V(G)$. The neighborhood of a vertex $v$ of $G$ is the set $N_G(v) = \{ w \mid w \in V(G) \text{ and } vw \in E(G) \}$, and let $N_G[v] = N_G(v) \cup \{ v \}$; if there is no ambiguity on $G$, we use $N(v)$ and $N[v]$, respectively.

A vertex $v$ of $G$ is said to be simplicial if the induced subgraph of $G$ on the set $N[v]$ is a complete graph and we say this complete graph to be a simplex of $G$. A 5-cycle $C_5$ of a graph $G$ is called basic, if $C_5$ does not contain two adjacent vertices of degree three or more in $G$ (see [FHN1]). We call a 4-cycle $C_4$ basic if it contains two adjacent vertices of degree two, and the remaining two vertices belong to a simplex or a basic 5-cycle of $G$. A graph $G$ is in the class $\mathcal{SQC}$, if $V(G)$ can be partitioned into three disjoint subsets $S, Q$, and $C$: The subset $S$ contains all vertices of the simplexes of $G$, and the simplexes of $G$ are vertex disjoint; the subset $C$ consists of the vertices of the basic 5-cycles and the basic 5-cycles form a partition of $C$; the remaining set $Q$ contains all vertices of degree two of the basic 4-cycles. The class $\mathcal{SQC}$ is a subclass of the class of well-covered graphs (see [RV]). The first main result of the paper, Theorem 2.3, proves that all graphs in the such class are vertex decomposable. Together this result and results of Finbow, Hartnell and Nowakowski about characterizing well-covered graphs with girths at least 5 ([FHN1]) and well-covered graphs in which triangles are allowed, but which have no 4-cycles nor 5-cycles ([FHN2]), we also can combinatorially characterize all Cohen-Macaulay graphs of girth at least five and Cohen-Macaulay graphs of girth three having no 4- nor 5-cycles (see Theorem 3.1 and 3.3).

A well-covered graph is called 1-well-covered if and only if the deletion of any point from the graph leaves a graph that is also well-covered. A well-covered graph is in the class $W_2$ if and only if any two disjoint independent sets in the graph can be extended to disjoint maximum independent sets. Staples [Sp] showed that a well-covered graph is 1-well-covered if and only if it is in the class $W_2$. Pinter [Pi1] characterized the planar graphs of girth 4 in the such class. Using this result, we also can classify all Gorenstein planar graphs in this class (Theorem 3.10).

The paper consists of three sections. In section 1, we set up some basic notations, terminologies for the simplicial complex and the graph. Section 2 is devoted to prove that all graphs in the class $\mathcal{SQC}$ are vertex decomposable. In the last section, we will prove the main results Theorem 3.1, 3.3 and 3.10.

1. Preliminaries

For the detailed information about combinatorial and algebraic background we get from [S], [HH] and [Vi]. We also will use some notation on graphs according to [D]. Let $\Delta$ be a simplicial complex on $\{ x_1, \ldots, x_n \}$. The Stanley-Reisner ideal of the simplicial complex $\Delta$ is a squarefree monomial ideal

$$I_\Delta = (x_{j_1} \cdots x_{j_i} \mid j_1 < \cdots < j_i \text{ and } \{ j_1, \ldots, j_i \} \notin \Delta) \subseteq R.$$

The $k$-algebra $k[\Delta] = R/I_\Delta$ is called the Stanley-Reisner ring of $\Delta$. We say that $\Delta$ is Cohen-Macaulay (resp. Gorenstein) (over $k$) if $k[\Delta]$ is Cohen-Macaulay (resp.
Gorenstein). The dimension of a face $F \in \Delta$ is $\dim F = |F| - 1$, where $|F|$ stands for the cardinality of $F$, and the dimension of $\Delta$ is $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$.

A simplicial complex $\Delta$ (not necessarily pure) is recursively defined to be vertex decomposable if it is either a simplex or else has some vertex $v$ so that:

1. both $\Delta \setminus v$ and $\text{lk}_\Delta v$ are vertex decomposable, and
2. no face of $\text{lk}_\Delta(v)$ is a facet of $\Delta \setminus v$.

Vertex decompositions were introduced in the pure case by Provan and Billera [PB] and extended to non-pure complexes by Björner and Wachs in [BW, Section 11].

Now, let $G$ be a simple graph and $\Delta(G)$ the set of all independent sets in $G$. Then $\Delta(G)$ is a simplicial complex and is the so-called the independence complex of $G$. Note that $I(G) = I_{\Delta(G)}$ and $\dim(\Delta(G)) = \alpha(G) - 1$. A graph is called a vertex decomposable graph if so is its independence complex. In order to study the vertex decomposability of a graph, it suffices to study the vertex decomposability of every its component.

**Lemma 1.1** (W1, Lemma 20). $G$ is vertex decomposable if and only if all its connected components are vertex decomposable.

For any vertex $v \in V(G)$, let $G_v = G \setminus N_G[v]$. We will use the following to check the vertex decomposable property of a graph.

**Lemma 1.2** (W1, Lemma 4). A graph $G$ is vertex decomposable if $G$ is a totally disconnected graph (with no edges) or if it has some vertex $v$ so that:

1. $G \setminus v$ and $G_v$ are both vertex decomposable, and
2. no independent set in $G_v$ is a maximal independent set in $G \setminus v$.

2. A class of vertex decomposable graphs

Recall that a 5-cycle $C_5$ of a graph $G$ is called basic, if $C_5$ does not contain two adjacent vertices of degree three or more in $G$; a 4-cycle $C_4$ is called basic, if it contains two adjacent vertices of degree two, and the remaining two vertices belong to a simplex or a basic 5-cycle of $G$. A graph $G$ is in the class $\mathcal{SQC}$ if there are simplicial vertices $x_1, \ldots, x_m$; basic 5-cycles $C_1^1, \ldots, C_s^s$; and basic 4-cycles $Q_1^1, \ldots, Q_t^t$ such that

$$V(G) = \bigcup_{j=1}^m N[x_j] \cup \bigcup_{j=1}^s V(C_j^i) \cup \bigcup_{j=1}^t B(Q_j^i)$$

and this forms a partition of $V(G)$, where $B(Q_j^i)$ is the set of two vertices of degree 2 of the basic 4-cycle $Q_j^i$. Such the graph $G$ is well-covered [RV, Theorem 3.1]. Moreover, since the proof of this result, we also have a formula to compute the cardinality of a maximum independent set of such the graph

$$\alpha(G) = m + 2s + t.$$  

The main result of this section says that all the graphs $G$ in the class $\mathcal{SQC}$ are vertex decomposable. The proof is divided some steps. First, one can see that this
assertion holds for well-covered simplicial graphs. A graph $G$ is said to be simplicial if every vertex of $G$ belongs to a simplex of $G$. Using a characterization due to Prisner, Topp and Vestergaard in [PTV, Lemma 2], one can see that all well-covered simplicial graph belong to the class $\mathcal{SQC}$.

**Lemma 2.1** (W2, Corollary 5.5). If $G$ is a (well-covered) simplicial graph, then $G$ is vertex decomposable.

Next, we will show that a graph is vertex decomposable if it belongs to the class $\mathcal{SC}$. A simple graph $G$ is called in the class $\mathcal{SC}$ if $V(G)$ can be partitioned into two disjoint subsets $S$ and $C$: the subset $S$ contains all vertices of the simplexes of $G$, and the simplexes of $G$ are vertex disjoint; the subset $C$ consists of the vertices of the basic 5-cycles and the basic 5-cycles form a partition of $C$. Obviously, the class $\mathcal{SC}$ is a subclass of the class $\mathcal{SQC}$.

**Lemma 2.2.** If $G$ is a graph in the class $\mathcal{SC}$, then $G$ is vertex decomposable.

**Proof.** We prove by induction on $|V(G)|$. If $|V(G)| < 5$, then $G$ is simplicial. Therefore, our assertion is followed from Lemma 2.1

Assume that $|V(G)| \geq 5$. If $G$ is disconnected, let $G_1, \ldots, G_m$ be components of $G$. Note that each $G_i$ is also in the class $\mathcal{SC}$. Since $|V(G_i)| < |V(G)|$, by the induction hypothesis, $G_i$ is vertex decomposable. Therefore, by [W1, Lemma 20], $G$ is also vertex decomposable.

If $G$ is connected. Let $C^1, \ldots, C^s$ be basic 5-cycles and $x_1, \ldots, x_t$ simplicial vertices of $G$ such that

$$V(C^1), \ldots, V(C^s), N[x_1], \ldots, N[x_t]$$

form a partition of $V(G)$. If $s = 0$, then the assertion is followed from Lemma 2.1

So, we may assume that $s \geq 1$. Write $C^1 = \{xy, yz, zu, uv, vx\}$ with $\deg_G(x) \geq 3$.

We first claim that $G \setminus x$ is vertex decomposable. Let $H = G \setminus x$. Since $C^1$ is a basic 5-cycle of $G$, we imply that $\deg_H(y) = \deg_H(v) = 1$. Therefore, $C^2, \ldots, C^s$ are also basic 5-cycles of $H$ and $x_1, \ldots, x_t, y, v$ are simplicial vertices of $H$. Clearly,

$$V(H) = V(C^2) \cup \cdots \cup V(C^s) \cup N_H[x_1] \cup \cdots \cup N_H[x_t] \cup N_H[y] \cup N_H[v]$$

and this is a partition of $V(H)$. In particular, $H$ belongs to $\mathcal{SC}$. Futhermore, $|V(H)| = |V(G)| - 1$, by induction, $H$ is vertex decomposable, as claimed.

We next claim that $G_x$ is vertex decomposable. Let $L = G_x$. Since $C^1$ is a basic 5-cycle, so either $z$ or $u$ has degree 2. Assume that $\deg_G(z) = 2$, so that $z$ is a simplicial vertex of $L$. Without loss of generality, we may assume that $C^2, \ldots, C^m$ are all basic 5-cycles which have vertices connect to $x$. Observe that $C^{m+1}, \ldots, C^s$ are basic 5-cycles of $L$ and $x_1, \ldots, x_t$ are simplicial vertices of $L$. For each $i = 2, \ldots, m$, let $c_i$ be a vertex of $C^i$ that is incident with $x$ in $G$. Let $u_i$ and $v_i$ be two adjacent vertices of $c_i$ in the cycle $C^i$. It implies that $u_i$ and $v_i$ are two simplicial vertices of $L$ (because their degree in $G$ equal 2) and

$$V(C^{m+1}), \ldots, V(C^s), N_L[z], N_L[u_2], N_L[v_2], \ldots, N_L[u_m], N_L[v_m], N_L[x_1], \ldots, N_L[x_t]$$
form a partition of \( V(L) \). In particular, \( L \) is in the class \( \mathcal{SC} \). Clearly, \( |V(L)| < |V(G)| \), so by induction, \( L \) is also vertex decomposable.

Since \( \alpha(H) = 2(s - 1) + (t + 2) = 2s + t \) and \( \alpha(L) = 2(s - m) + 1 + 2(m - 1) + t = 2s + t - 1 = \alpha(H) - 1 \), by Lemma 2.2 we have \( G \) is Cohen-Macaulay as required. □

We now in position to prove the main result of this section.

**Theorem 2.3.** If \( G \) is a graph in the class \( \mathcal{SQC} \), then \( G \) is vertex decomposable. In particular, the such graph is Cohen-Macaulay.

**Proof.** We prove by induction on \( |V(G)| \). If \( |V(G)| < 3 \), then \( G \) is a well-covered simplicial graph. It implies that \( G \) is vertex decomposable by Lemma 2.1.

Assume that \( |V(G)| \geq 3 \). Let \( C^1, \ldots, C^s \) be basic 5-cycles; \( x_1, \ldots, x_t \) simplicial vertices; and \( Q^1, \ldots, Q^m \) basic 4-cycles of \( G \) such that

\[
V(G) = \bigcup_{j=1}^{t} N[x_j] \cup \bigcup_{j=1}^{s} V(C^j) \cup \bigcup_{j=1}^{m} B(Q^j)
\]

and this is a partition of \( V(G) \), where \( B(Q^j) \) is the set of two vertices of degree 2 of the basic 4-cycle \( Q^j \). If \( m = 0 \), then \( G \) is in the class \( \mathcal{SC} \). Therefore, \( G \) is vertex decomposable by Lemma 2.2.

If \( m \geq 1 \), let \( Q^1 = \{a_1b_1, b_1c, cd_1, d_1a_1\} \) be a basic 4-cycle with

\[
\deg_G(a_1) = \deg_G(b_1) = 2; \deg_G(c) \geq 3; \deg_G(d_1) \geq 3.
\]

Without loss of generality, we may assume that \( c \in V(Q^i) \) for \( i = 1, \ldots, l \) and \( c \notin V(Q^i) \) for \( i = l + 1, \ldots, m \). Write \( Q^i = \{a_i b_i, b_i c, c d_i, d_i a_i\} \) with \( \deg_G(a_i) = \deg_G(b_i) = 2 \) for \( i = 2, \ldots, l \). Note that \( a_1, b_1, \ldots, a_l, b_l \) are distinct points, but \( d_1, \ldots, d_l \) may be not distinct points.

From definition, we distinguish two cases:

**Case 1:** \( c \) belongs to only one of the basic 5-cycles of \( G \), say

\[
C^1 = \{cu_1, u_1y_1, y_1z_1, z_1v_1, v_1c\}
\]

with \( \deg_G(u_1) = \deg_G(v_1) = 2 \) (because the degree of \( c \) is at least 3 in \( G \)). We now claim that \( H = G \setminus c \) is vertex decomposable. Since \( \deg_H(b_1) = \cdots = \deg_H(b_l) = 1 \), \( b_1, \ldots, b_l \) are simplicial vertices of \( H \). It is easy to check that \( u_1, v_1, b_1, \ldots, b_l, x_1, \ldots, x_t \) are all simplicial vertices; \( C^1, \ldots, C^{s-1} \) are basic 5-cycles; and \( Q^{l+1}, \ldots, Q^m \) are basic 4-cycles of \( H \). Moreover,

\[
V(H) = N_H[u_1] \cup N_H[v_1] \cup \bigcup_{j=1}^{l} N_H[b_j] \cup \bigcup_{j=1}^{t} N_H[x_j] \cup \bigcup_{j=2}^{s} V(C^j) \cup \bigcup_{j=l+1}^{m} B(Q^j)
\]

and this is a partition of \( V(H) \). This follows \( H \) is in the class \( \mathcal{SQC} \) and \( |V(H)| = |V(G)| - 1 \). Then, \( H \) is vertex decomposable by induction, as claimed. Moreover, we also obtain

\[
\alpha(H) = 1 + 1 + l + t + 2(s - 1) + (m - l) = t + 2s + m.
\]
We next claim that $L = G_c$ is vertex decomposable. It is clear that $a_1, \ldots, a_t$ are isolated vertices of $L$. It means that they are simplicial vertices of $L$. As $C^1$ is a basic 5-cycle, so either $y_1$ or $z_1$ has degree 2 in $G$. Assume $\deg_G(y_1) = 2$. Then, $\deg_G(y_1) \leq 1$ which implies that $y_1$ is a simplicial vertex of $L$.

Assume that each of $Q^{l+1}, \ldots, Q^{l+r}$ has at least one vertex being adjacent with $c$, and every $Q^{l+r+1}, \ldots, Q^m$ has no any vertices being adjacent with $c$. Write $Q^i = \{a_j, b_j, c_j, d_j, e_j\}$ with $\deg_G(a_j) = \deg_G(b_j) = 2$ and $c$ is adjacent with $c_j$ for all $j = l + 1, \ldots, l + r$. Hence, $b_{l+1}, \ldots, b_{l+r}$ are simplicial in $L$.

Assume that each of $C^2, \ldots, C^p$ has at least one vertex being adjacent with $c$, and every $C^{p+1}, \ldots, C^s$ has no any vertices being adjacent with $c$. For each $i = 2, \ldots, p$, let $C_i = \{u_i y_i, z_i, u_i, w_i, v_i\}$ with $c$ and $w_i$ are adjacent in $G$. So, $\deg_G(u_i) = \deg_G(v_i) = 2$. Hence, both of $u_i$ and $v_i$ are simplicial in $L$.

By definition of the class $\mathbb{SQC}$, $c \notin N_G[x_i]$ for all $i = 1, \ldots, t$. Then, $x_i$ is also simplicial in $L$ for all $i$.

In summary, $L$ has simplicial vertices

$$y_1, a_1, \ldots, a_t, b_{l+1}, \ldots, b_{l+r}, u_2, v_2, \ldots, u_p, v_p, x_1, \ldots, x_t;$$

basic 5-cycles $C^{p+1}, \ldots, C^{s-1}$; and basic 4-cycles $Q^{l+1}, \ldots, Q^m$. Moreover,

$$V(L) = N_L[y_1] \cup \bigcup_{j=1}^l N_L[a_j] \cup \bigcup_{j=1}^r N_L[b_{l+j}] \cup \bigcup_{j=2}^p (N_L[u_j] \cup N_L[v_j])$$

$$\cup \bigcup_{j=1}^t N_L[x_j] \cup \bigcup_{j=p+1}^s V(C^j) \cup \bigcup_{j=l+r+1}^m B(Q_m),$$

and this is a partition of $V(L)$. It implies that $L$ is also in the class $\mathbb{SQC}$. Hence, $\alpha(L) = 1 + l + r + 2(p - 1) + t + 2(s - p) + (m - l - r) = t + 2s + m - 1 = \alpha(H) - 1$.

Since $|V(L)| < |V(G)|$, also by induction, $L$ is vertex decomposable. Using Lemma \[2\] $G$ is vertex decomposable as required.

**Case 2:** $c$ belongs to only one of the simplices $N_G[x_1], \ldots, N_G[x_t]$. In the same way as the proof of case 1, $G$ is also vertex decomposable as required.

A vertex $v$ of a graph $G$ is called a cut vertex of $G$ if $G \setminus v$ has more components than $G$. A connected graph with no cut vertex is called a block. A block of a graph $G$ is a subgraph of $G$ which is itself a block and which is maximal with respect to that property. A graph $G$ is called a block-cactus graph if every block is complete or a cycle. As a consequence of our theorem, one has the following characterization of Cohen-Macaulay block-cactus graphs which is independent of the field.

**Corollary 2.4.** Let $G$ be a block-cactus graph. Then the following statements are equivalent:

1. $G$ is well covered and vertex decomposable.
2. $G$ is Cohen-Macaulay.
(3) $G$ is in the class $SQC$.

Proof. (1)$\implies$(2) has known in more general statement (see [S]).

(2)$\implies$(3): It suffices to prove for connected block-cactus graphs. Now, if $G$ is Cohen-Macaulay then $G$ is a well-covered block-cactus graph. By [RV, Theorem 3.2], $G$ belongs to the class $\{4$-cycle, $7$-cycle$\} \cup SQC$. But, both of $4$-cycle and $7$-cycle are not Cohen-Macaulay, so $G$ is in the class $SQC$.

(3)$\implies$(1): If $G$ is a block-cactus graph in the class $SQC$, then we are done by Theorem 2.3. $\square$

If every block of a connected block-cactus graph $G$ is a edge or a cycle, then $G$ is called a cactus graph. Equivalently, $G$ is a cactus graph if and only if it is connected and two cycles have at most one vertex in common. A $3$-cycle is called basic if it contains at least one vertex of degree two. Note that a pendant edge in a graph is any edge incident with a vertex of degree $1$. Now, Corollary 2.4 can restate more explicitly in a combinatorial way for Cohen-Macaulay cactus graphs as follows (see [MKY]).

Corollary 2.5. Let $G$ be a cactus graph. Then the following statements are equivalent:

1. $G$ is well covered and vertex decomposable.
2. $G$ is Cohen-Macaulay.
3. $G$ satisfies two following conditions:
   a. every vertex of degree $2$ is incident with only one pendant edge or one basic $3$-cycle or one basic $4$-cycle or one basic $5$-cycle;
   b. every vertex of degree at least $3$ is incident with only one pendant edge or one basic $3$-cycle or one basic $5$-cycle.

3. The Cohen-Macaulay Property versus girth

In this section, we characterize all Cohen-Macaulay graphs by their girths bases on the results due to Finbow, Hartnell and Nowakowski ([FHN1], [FHN2], and Pinter [Pi]). Recall that the girth of a graph $G$ is the length of any shortest cycle in $G$. In a given graph $G$, let $C(G)$ denote the set of all points which belong to basic $5$-cycles. Denote by $P(G)$ the set of vertices in $G$ which are incident with pendant edges in $G$. $G$ is said to belong to the class $PC$ if $V(G)$ can be expressed as $P(G) \cup C(G)$, where $P(G) \cap C(G) = \emptyset$ and the pendant edges form a perfect matching of $P(G)$. If $uv$ is a pendant edge in $G$ with $\text{deg}(u) = 1$, then $N[u] = \{u, v\}$. And then $u$ is a simplicial vertex in $G$. This implies that the class $PC$ is a subset of the class $SQC$.

The first main result of this section is the characterization of Cohen-Macaulay graphs of girth at least $5$.

Theorem 3.1. Let $G$ be a connected graph of girth at least $5$. Then, $G$ is Cohen-Macaulay if and only if $G$ is either a vertex or in the class $PC$.

Proof. If $G$ is a Cohen-Macaulay graph, then $G$ is well covered. By [FHN1], we have either $G$ is in the class $PC$ or $G$ is one of six exceptional graphs shown in Figure 3.
Among six exceptional graphs above, only $K_1$ is Cohen-Macaulay. Therefore, $G$ is either a vertex or in the class $PC$.

Conversely, if $G$ is one vertex, then $G$ is obviously Cohen-Macaulay. If $G$ is in $PC$, then $G$ is in $SQC$. By Theorem 2.3 $G$ must be Cohen-Macaulay. □

The following corollary is immediate.

**Corollary 3.2.** Let $G \neq K_1$ be a connected graph of girth at least 6. Then, $G$ is Cohen-Macaulay if and only if its pendant edges form a perfect matching of $G$.

Next, we will characterize all Cohen-Macaulay graphs in which triangles are allowed, but have no 4- nor 5-cycles. In particular, in such the graphs no cliques of size greater than 3 can exist.

**Theorem 3.3.** Let $G$ be a graph that contains neither 4-cycles nor 5-cycles. Then, the following conditions are equivalent:

1. $G$ is Cohen-Macaulay.
2. There are simplicial vertices $x_1, \ldots, x_m$ of $G$ such that $\deg_G(x_i) \leq 3$ for all $i$ and $N_G[x_1], \ldots, N_G[x_m]$ form a partition of $V(G)$.
3. $G$ is a well-covered simplicial graph such that every simplicial vertex has degree at most 3.

Proof. (1) $\implies$ (2) Since $G$ is a well-covered graph containing no 4-cycles nor 5-cycles, by [FHN2, Theorem 1.1], $G$ is either a well-covered simplicial graph such that every simplicial vertex has degree at most 3 or one of two exceptional graphs $C_7$ and $T_{10}$ shown in Figure 4. But both of $C_7$ and $T_{10}$ are not Cohen-Macaulay, so $G$ satisfies the condition as in the second statement. (2) $\iff$ (3) and (3) $\implies$ (1) hold true by [PTV, Theorem 1] and Lemma 2.1. □

Let $U$ be a subset of the vertex set of a simplicial complex $\Delta$. Let $\Delta \setminus U = \{F \in \Delta \mid F \cap U = \emptyset\}$ be a subcomplex of $\Delta$. If $U = \{x\}$, then we write $\Delta \setminus x$ stand for $\Delta \setminus \{x\}$. Define for $\Delta$ on the vertex set $V$, core$(V) = \{x \in V \mid \text{st}(x) = V\}$ where $\text{st}(x) = \{F \in \Delta \mid F \cup \{x\} \in \Delta\}$ and core$(\Delta) = \{F \subseteq \text{core}(V) \mid F \in \Delta\}$.

In [B], the doubly Cohen-Macaulay simplicial complexes are introduced. A simplicial complex $\Delta$ is called doubly Cohen-Macaulay (over $k$) if $\Delta$ is Cohen-Macaulay (over $k$) and for every vertex $x$ of $\Delta$ the subcomplex $\Delta \setminus x$ is also Cohen-Macaulay (over $k$) of the same dimension as $\Delta$. It is well known that if $\Delta$ is Gorenstein with core$(\Delta) = \Delta$, then $\Delta$ is doubly Cohen-Macaulay (see [S, Theorem II.5.1]).

Recall that $W_2$ is the class of well-covered graphs $G$ such that $G \setminus x$ are well-covered with $\alpha(G) = \alpha(G \setminus x)$ for all vertices $x$. Let $G$ be a Gorenstein graph without isolated vertices, so that core$(\Delta(G)) = \Delta(G)$. Hence, $\Delta(G)$ is doubly Cohen-Macaulay. It follows that for all vertices $x$ of $G$, $\Delta(G) \setminus x = \Delta(G \setminus x)$ is Cohen-Macaulay with $\alpha(G \setminus x) = \dim \Delta(G \setminus x) + 1 = \dim \Delta(G) + 1 = \alpha(G)$.

This implies that $G$ is in class $W_2$.  

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Pinter constructed a infinite family of graphs by a recursive procedure as follows:

(1) Begin with the graph $G_3$ shown in Figure 5;
(2) Given any graph $G$ in the construction, let $x$ and $y$ be two adjacent points of
degree 2 in $G$. Let $u$ be the neighbor of $x$ such that $u \neq y$. Then construct
a new graph $G'$ with precisely three more points than $G$ as follows. Let the
three new points be $a$, $b$ and $c$. Now join $a$ to $x$ and $b$, $b$ to $c$ and $c$ to $u$ and $y$
(see Figure 6).

Let us denote this infinite family of graphs by $\mathcal{G}$. Pinter proved the following result.

**Lemma 3.4.** A connected graph $G$ is a girth 4 planar member of class $W_2$ if and only
if $G$ is a member of the family $\mathcal{G}$.

In general, two graphs $G$ and $H$ are isomorphic, written $G \cong H$, if there are a
bijection map $\varphi : V(G) \to V(H)$ such that $uv \in E(G) \iff \varphi(u)\varphi(v) \in E(H)$ for all $u, v \in V(G)$. Although the graphs $G$ and $H$ are not identical, they have identical
structures if they are isomorphic.

**Definition 3.5.** For every integer $n \geq 1$, we define two graphs $G_n$ and $H_n$ as follows:

(1) $G_n$ is a graph with the vertex set $\{x_1, \ldots, x_{3n-1}\}$ and the edge set
\[
\{x_1x_2, \{x_{3k-1}x_{3k}, x_{3k}x_{3k+1}, x_{3k+1}x_{3k+2}, x_{3k+2}x_{3k-2}\}_{k=1,2,\ldots,n-1}, \{x_{3l-3}x_{3l}\}_{l=2,3,\ldots,n-1}\}
\]

(2) $H_n = G_n \setminus x_{3n-1}$. It means that $V(H_n) = \{x_1, \ldots, x_{3n-2}\}$ and
\[
E(H_n) = \begin{cases}
\{x_1\} & \text{if } n = 1, \\
\{x_1x_2, x_2x_3, x_3x_4\} & \text{if } n = 2, \\
E(G_{n-1}) \cup \{x_{3n-2}x_{3n-3}, x_{3n-3}x_{3n-4}, x_{3n-3}x_{3n-6}\} & \text{if } n \geq 3.
\end{cases}
\]

Roughly speaking, the family $\mathcal{G}$ is actually $\{G_n\}_{n \geq 3}$. In fact, from the construction
of $\mathcal{G}$ and Lemma 3.4, we will obtain the following.
Lemma 3.6. A connected planar graph $G$ of girth 4 is a member of $W_2$ if and only if $G \cong G_n$ for some $n \geq 3$.

Let $G$ be a connected graph in the class $W_2$. Pinter [Pi2] proved that if $G \neq K_2$ or $C_5$ then girth$(G) \leq 4$. Thus, connected Gorenstein graphs with girth at least 5 is one of three graphs $K_1, K_2$ and $C_5$. So the structure of connected Gorenstein graphs is non trivial only for the ones of girth $\leq 4$. In the last theorem, we will give a complete characterization of a Gorenstein connected planar graph of girth 4. Recall that the union of graphs $G$ and $H$ is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If $G$ and $H$ are disjoint, we refer to their union as a disjoint union, and generally denote it by $G \cup H$. Firstly, we have the following observation.

Remark 3.7. Let $G$ be a simple graph $G$ and a point $x \in V(G)$. If both of $G_x$ and $G \setminus x$ are well covered with $\alpha(G \setminus x) = \alpha(G_x) + 1$, then $G$ is also well covered with $\alpha(G) = \alpha(G \setminus x)$.

Secondly, we will prove the vertex decomposability of $G_n$ and $H_n$.

Lemma 3.8. For all integers $n \geq 1$, both of $G_n$ and $H_n$ are well covered and vertex decomposable with $\alpha(G_n) = \alpha(H_n) = n$. In particular, $G_n$ and $H_n$ are Cohen-Macaulay.

Proof. We will prove by induction on $n$. If $n = 1$ or $n = 2$, then our assertion holds true. If $n \geq 3$, since $H_n \setminus x_{3n-3} = G_{n-1} \cup \{x_{3n-2}\}$, so $H_n \setminus x_{3n-3}$ is well covered and vertex decomposable with $\alpha(H_n \setminus x_{3n-3}) = \alpha(G_{n-1}) + 1 = n$ by induction and Lemma 1.1. On the other hand, it is clear that $(H_n)_{x_{3n-3}} = G_{n-2} \cup \{x_{3n-5}\}$. Also by induction and Lemma 1.1 $(H_n)_{x_{3n-3}}$ is well covered and vertex decomposable with $\alpha((H_n)_{x_{3n-3}}) = \alpha(G_{n-2}) + 1 = n - 1 = \alpha(H_n \setminus x_{3n-3}) - 1$. Therefore, $H_n$ is well covered with $\alpha(H_n) = \alpha(H_n \setminus x_{3n-3}) = n$ (by Remark 3.7) and vertex decomposable (by Lemma 1.1).

Moreover, $G_n \setminus x_{3n-1} = H_n$ is well covered and vertex decomposable with $\alpha(G_n \setminus x_{3n-1}) = n$ has done. On the other hand, one can check that $(G_n)_{x_{3n-1}} \cong G_{n-1}$. By induction, $(G_n)_{x_{3n-1}}$ is vertex decomposable and $\alpha((G_n)_{x_{3n-1}}) = n - 1$. Thus, $G_n$ is vertex decomposable by Lemma 1.1. It is clearly that $G_n$ is well covered with $\alpha(G_n) = n$ which is complete the proof. \qed
Next, one can see that \( I(G_n)^2 \) is Cohen-Macaulay for all integers \( n \geq 1 \), as conjectured by G. Rinaldo, N. Terai and K. Yoshida \([RTY, \text{Conjecture 5.7}]\). The case \( n = 1 \) is known in \([MT, \text{Theorem 3.2}]\) and the case \( n = 2 \) is also mentioned in \([TY, \text{Theorem 3.8 (iv)}]\).

**Proposition 3.9.** For all integers \( n \geq 1 \), \( I(G_n)^2 \) are Cohen-Macaulay.

**Proof.** Note that \( G_n \) is a triangle-free graph, so \( I(G_n)^2 = I(G_n)^{(2)} \) (see \([SVV]\) or \([RTY, \text{Corollary 4.5}]\)). Therefore, it suffices to prove that \( I(G_n)^{(2)} \) is Cohen-Macaulay.

If \( n = 1 \) (resp. \( n = 2 \)) then \( G_n \) is an edge (resp. a pentagon). Thus \( I(G_n)^{(2)} \) is Cohen-Macaulay.

If \( n \geq 3 \), by Lemma \([3.8]\) and \([HMT, \text{Theorem 2.3}]\), it is enough to prove that \( (G_n)_{xy} \) is Cohen-Macaulay with \( \alpha((G_n)_{xy}) = n - 1 \) for every edge \( xy \in E(G_n) \); where \( (G_n)_{xy} \) stands for \( G_n \setminus (N_{G_n}(x) \cup N_{G_n}(y)) \). We distinguish six following cases:

**Case 1:** \( xy = x_1x_2 \). Clearly, \( (G_n)_{x_1x_2} \cong H_{n-1} \). Therefore, by Lemma \([3.8]\) \( (G_n)_{x_1x_2} \) is Cohen-Macaulay with \( \alpha((G_n)_{x_1x_2}) = n \).

**Case 2:** \( xy = x_{3k-1}x_{3k} \) for some \( k = 1, \ldots, n - 1 \). Observe that

\[
(G_n)_{x_{3k-1}x_{3k}} = \begin{cases} 
U_1 \cup \{x_5\} & \text{if } k = 1, \text{ where } U_1 \cong G_{n-2} \quad (1) \\
U_2 \cup \{x_{3n-1}\} & \text{if } k = n - 1, \text{ where } U_2 \cong G_{n-2} \quad (2) \\
U_3 \cup \{x_2\} \cup \{x_8\} & \text{if } k = 2, \text{ where } U_3 \cong G_{n-3} \quad (3) \\
M \cup N \cup \{x_{3k+2}\} & \text{if } 3 \leq k < n - 1, \quad (4) 
\end{cases}
\]

where \( M = G_n\{x_1, \ldots, x_{3k-6}, x_{3k-4}\} \) and \( N = G_n\{x_{3k+4}, \ldots, x_{3n-1}\} \).

In the three cases \((1) - (3)\), using Lemma \([3.8]\) and Lemma \([L.1]\) we have \( (G_n)_{x_{3k-1}x_{3k}} \) is always Cohen-Macaulay with \( \alpha((G_n)_{x_{3k-1}x_{3k}}) = n - 1 \). In the last case, we define the map \( \varphi : V(H_{k-1}) \to V(M) \) as follows:

If \( k = 3 \) then \( \varphi(i) = x_i \) for all \( i = 1, 2, 3 \) and \( \varphi(4) = x_5 \).

If \( k > 3 \) then \( \varphi(i) = x_i \) for all \( i = 1, \ldots, 3k - 9 \); \( \varphi(x_{3k-8}) = x_{3k-6}, \varphi(x_{3k-7}) = x_{3k-7}, \varphi(x_{3k-6}) = x_{3k-8} \) and \( \varphi(x_{3k-5}) = x_{3k-4} \).

Clearly, \( \varphi \) is an isomorphism of two graphs \( H_{k-1} \) and \( M \). Therefore, \( M \) must be Cohen-Macaulay with \( \alpha(M) = k - 1 \) by Lemma \([3.8]\). Similarly, we have a bijection map \( \psi : V(G_{n-k-1}) \to V(N) \) is defined by \( \psi(i) = x_{3k+3+i} \) for all \( i = 1, \ldots, 3n-3k-4 \). So \( G_{n-k-1} \cong N \). Using again Lemma \([3.8]\), \( N \) is Cohen-Macaulay with \( \alpha(N) = n - k - 1 \). Thus, \( (G_n)_{x_{3k-1}x_{3k}} \) is Cohen-Macaulay with \( \alpha((G_n)_{x_{3k-1}x_{3k}}) = (k - 1) + (n - k - 1) + 1 = n - 1 \).

By the same argument, we will obtain the following.

**Case 3:** \( xy = x_{3k}x_{3k+1} \) for some \( k = 1, \ldots, n - 1 \). Then,

\[
(G_n)_{x_{3k}x_{3k+1}} \cong \begin{cases} 
H_{n-2} \cup \{x_1\} & \text{if } k = 1 \\
G_{k-1} \cup G_{n-k-1} \cup \{x_{3k-2}\} & \text{if } k \geq 2.
\end{cases}
\]

**Case 4:** \( xy = x_{3k+1}x_{3k+2} \) for some \( k = 1, \ldots, n - 1 \). Then,

\[
(G_n)_{x_{3k+1}x_{3k+2}} \cong H_{k-1} \cup H_{n-k-1} \cup \{x_{3k-1}\}.
\]
Case 5: \( xy = x_{3k+2}x_{3k-2} \) for some \( k = 1, \ldots, n - 1 \). Then,
\[
(G_n)_x^{x_{3k+2}x_{3k-2}} \cong G_{k-1} \sqcup G_{n-k-1} \sqcup \{ x_{3k} \}.
\]

Case 6: \( xy = x_{3k}x_{3k-3} \) for some \( 2 \leq k \leq n - 1 \). Then,
\[
(G_n)_x^{x_{3k+2}x_{3k-2}} \cong G_{k-1} \sqcup G_{n-k-2} \sqcup \{ x_{3k-3} \} \sqcup \{ x_{3k+2} \}.
\]

From six cases above, for every edge \( xy \in E(G_n) \), we always obtain that \((G_n)_{xy}\) is Cohen-Macaulay with \( \alpha((G_n)_{xy}) = n - 1 \) which completes the proof.

Finally, one can see that the class of connected planar Gorenstein graphs with girth 4 is exactly \( \mathcal{G} \).

**Theorem 3.10.** Let \( G \) be a connected planar graph of girth 4. Then, \( G \) is Gorenstein if and only if \( G \) is in the family \( \mathcal{G} \).

**Proof.** If \( G \) is Gorenstein, then \( G \) is in the class \( W_2 \) (note that \( G \) has no isolated vertices). Using Lemma 3.4, \( G \) is a member of family \( \mathcal{G} \).

Conversely, if \( G \) is a member of \( \mathcal{G} \), we may assume that \( G = G_n \) for some \( n \geq 3 \). Because of Proposition 3.9, \( I(G_n)^2 \) is Cohen-Macaulay over any field \( k \). Hence, by [RTY] Theorem 2.1, one has \( G_n \) must be Gorenstein as required.

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**References**

[B] K. Baclawski, *Cohen-Macaulay connectivity and geometric lattices*, European J. Combinatorics 3 (1982), 293-305.

[BH] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics 39, Cambridge University Press, Cambridge 1993.

[BW] A. Björner and M. Wachs, *Shellable nonpure complexes and posets. II*, Trans. Amer. Math. Soc. 349 (1997), no. 10, 3945-3975.

[D] R. Diestel, *Graph theory*, 2nd. edition, Springer: Berlin/Heidelberg/New York/Tokyo, 2000.

[FHN1] A. Finbow, B. Hartnell and R. J. Nowakowski, *A characterization of well covered graphs of girth 5 or greater*, J. Combin. Theory Ser. B, 57 (1993), 44-68.

[FHN2] A. Finbow, B. Hartnell and R. J. Nowakowski, *A characterization of well covered graphs that contain neither 4- nor 5-cycles*, J. Graph Theory, 18 (1994), no. 7, 713-721.

[HH] J. Herzog and T. Hibi, *Monomial ideals*, Springer: London/Dordrecht/Heidelberg/New York, 2011.

[HH1] J. Herzog and T. Hibi, *Distributive lattices, bipartite graphs and Alexander duality*, J. Algebraic Combin. 22 (2006), no. 3, 289-302.

[HHZ] J. Herzogs, T. Hibi and X. Zheng, *Cohen-Macaulay chordal graphs*, J. Combin. Theory Ser. A, 113 (2006), no. 5, 911-916.

[HMT] D. T. Hoang, N. C. Minh and T. N. Trung, *Combinatorial characterizations of the Cohen-Macaulayness of the second power of edge ideals*, submitted 2011.
[MKY] F. Mohammadi, D. Kiani, and S. Yassemi, *Shellable cactus graphs*, Math. Scand. **106** (2010), no. 2, 161-167.

[MT] N. C. Minh and N. V. Trung, *Cohen-Macaulayness of powers of two-dimensional squarefree monomial ideals*, J. Algebra **322** (2009), 4219-4227.

[MV] S. Morey and R. Villarreal, *Edge ideals: Algebraic and combinatorial properties*, to appear in Progress in Commutative Algebra, arXiv:1012.5329.

[Pi1] M. R. Pinter, *A class of planar well-covered graphs with girth four*, J. Graph Theory, **19** (1995), no. 1, 69-81.

[Pi2] M. R. Pinter, *A class of well-covered graphs with girth four*, Ars Combin. **45** (1997), 241-255.

[P1] M. D. Plummer, *Some covering concepts in graphs*, Journal of Combinatorial Theory, **8** (1970), 91-98.

[P2] M. D. Plummer, *Well-covered graphs: a survey*, Quaestiones Math. **16** (1993), no. 3, 253-287.

[PB] J. Provan and L. Billera, *Decompositions of simplicial complexes related to diameters of convex polyhedra*, Math. Oper. Res. **5** (1980), no. 4, 576-594.

[PTV] E. Prisner, J. Topp and P. D. Vestergaard, *Well covered simplicial, chordal and circular arc graphs*, J. Graph Theory **21** (1996), no. 2, 113-119.

[RV] B. Randerath and L. Volkmann, *A characterization of well covered block-cactus graphs*, Australas. J. Combin. **9** (1994), 307-314.

[RTY] G. Rinaldo, N. Terai and K. Yoshida, *On the second powers of Stanley-Reisner ideals*, J. Commut. Algebra **3** (2011), no. 3, 405-430.

[SVV] A. Simis, W. Vasconcelos and R. H. Villarreal, *On the ideal theory of graphs*, J. Algebra **167** (1994), 389-416.

[S] R. Stanley, Combinatorics and Commutative Algebra, 2. Edition, Birkhäuser, 1996.

[Sp] J. W. Staples, *On some subclasses of well-covered graphs*, J. Graph Theory **3** (1979), 197-204.

[TyT] N. V. Trung and T. M. Tuan, *Equality of ordinary and symbolic powers of Stanley-Reisner ideals*, J. Algebra **328** (2011), 77-93.

[Vi] R. Villarreal, *Monomial Algebras*, Monographs and Textbooks in Pure and Applied Mathematics Vol. 238, Marcel Dekker, New York, 2001.

[W1] R. Woodroofe, *Vertex decomposable graphs and obstructions to shellability*, Proc. Amer. Math. Soc. **137** (2009), no. 10, 3235-3246, arXiv:0810.0311.

[W2] R. Woodroofe, *Chordal and sequentially Cohen-Macaulay clutters*, Electron. J. Combin. **18** (2011), no. 1, Paper 208, 20 pages, arXiv:0911.4697.
Figure 3.
Figure 4.

Figure 5.

Figure 6.
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