Noise-sustained and controlled synchronization of stirred excitable media by external forcing

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Abstract. Most of the previous studies on constructive effects of noise in spatially extended systems have focused on static media, e.g., of the reaction–diffusion type. Because many active chemical or biological processes occur in a fluid environment with mixing, we investigate here the interplay among noise, excitability, mixing and external forcing in excitable media advected by a chaotic flow, in a two-dimensional FitzHugh–Nagumo model described by a set of reaction–advection–diffusion equations. In the absence of external forcing, noise may generate sustained coherent oscillations of the media in a range of noise intensities and stirring rates. We find that these noise-sustained oscillations can be synchronized by external periodic signals much smaller than the threshold. Analysis of the locking regions in the parameter space of the signal period, stirring rate and noise intensity reveals that the mechanism underlying the synchronization behaviour is a matching between the time scales of the forcing signal and the noise-sustained oscillations. The results demonstrate that, in the presence of a suitable level of noise, the stirred excitable media act as self-sustained oscillatory systems and become much easier to be entrained by weak external forcing. Our results may be verified in experiments and are useful to understand the synchronization of population dynamics of oceanic ecological systems by annual cycles.
1. Introduction

In this work, we study synchronization phenomena in complex nonlinear systems subject to an external forcing signal, a topic of fundamental importance in nonlinear sciences, with applications to various fields [1, 2] ranging from communication to ecology and neurobiology.

Most of the previous studies on synchronization behaviour have focused on self-sustained oscillatory systems. However, many biological and chemical systems are rather characterized by excitability. There is a threshold below which an external signal cannot generate a significant response in the system, and the system remains at a quiescent state. A signal that exceeds the threshold, on the other hand, can evoke a strong, spike-like response with a refractory period \(T_r\) before the system returns to its quiescent state. In recent years, a great deal of attention has been paid to study the response of excitable systems in the presence of noise [3]. In isolated excitable cells, noise alone can generate a sequence of spikes, separated by fluctuating interspike intervals, \(T\). Importantly, there is an optimal intensity of noise where the fluctuation of \(T\) is reduced and the spiking sequences become quite regular (periodic) with \(T \sim T_r\), a phenomenon known as coherence resonance [4, 5]. In excitable neurons, the internal noise resulting from random-gating dynamics of ion channels in membrane patches of finite size, thus, can lead to coherence resonance with respect to patch size [6].

Recently, the phenomena of frequency and phase synchronizations have also been investigated for noise-sustained oscillations in threshold systems. Stochastic resonance [7] occurs when the noise-controlled mean switching interval \(\langle T \rangle\) is close to the period \(T_s\) of the signal and the response becomes optimal [8]–[11]. An effective stochastic frequency and phase locking has been demonstrated with a suitable definition of phases of stochastic signals [11]–[13]. Similarly, channel noise has been shown to have important effects on the synchronization of spiking activity of excitable membrane by external signals [14]. Such an analysis has been also applied to oscillatory systems in the presence of noise [12, 13]. We have shown recently that noise can enhance the phase synchronization of chaotic oscillators [15].
Recently, research interest has shifted to the effects of noise in spatially extended systems [16]. In one-dimensional (1D) or 2D arrays of coupled excitable cells, a cooperative interplay between noise-induced nucleation and propagation of excitation can induce spatiotemporal stochastic resonance [17], or can enhance spatial synchronization [18] and temporal coherence [19, 20] of noise-sustained oscillations. Such an array-enhanced coherence can generate a characteristic time scale in the system, similar to the natural period of self-sustained oscillators. As a result, noise can enhance synchronization of the system to weak external forcing [21] and can generate resonant standing-wave patterns [22]. Noise can also support wave propagations [23]–[25] in sub-excitable media due to a noise-induced transition [16, 26].

In these studies, the media are static, and transport is governed by diffusion. In many systems, the media are not static, but subject to a motion, for example, when stirred by a flow. This occurs especially in chemical reactions taking place in a fluid environment, conversion of pollutants in atmospheric flows or bloom of plankton in oceanic currents. In this case, usually diffusive transport dominates only at small spatial scales while mixing due to the flow is much faster at large scales. We have shown recently that in inhomogeneous self-sustained oscillatory media, an increasing rate of mixing can lead to a transition to global synchronization of the whole media [27].

Excitability and mixing are two features relevant to many physical, chemical and biological systems [28, 29]. For example, the bloom of plankton in ocean fertilization experiments [30, 31] can be described by an initial value problem of excitable media subjected to turbulent ocean currents [32]. In such systems, noise from some natural variability is inevitable. The birth and death processes of individuals are intrinsically stochastic, which becomes especially pronounced when the number of individuals is small. The interaction of oceanic zooplankton with fish, which are far from being uniformly distributed, also introduces randomness [33]. In addition, several parameters can fluctuate irregularly in space and time. For example, the depth of the mixed layer, within which there is a strong vertical mixing, controls the average amount of light received and, thus, the growth rate of the phytoplankton. Another parameter crucial for the phytoplankton productivity is iron concentration, which can be elevated in surface water after rain [34].

Recently, we have shown [35] that the interplay among excitability, noise, diffusion and mixing can generate various pattern formation regimes in a 2D FitzHugh–Nagumo (FHN) model subject to advection by a chaotic flow. Especially, there is a regime in the parameter space of the stirring rate and noise intensity where noise can generate coherent global excitations and sustained oscillations of the media. Outside this regime, noise-induced excitations are either homogenized by strong mixing or develop into non-coherent excitations of the media in the presence of weak mixing.

In addition to unavoidable stochasticity, there are also important regular changes of the parameters in many systems due to external influences. For example, the oceanic plankton bloom is strongly influenced by seasonal changes of the sea surface temperature, sun light intensity, etc. In self-sustained (periodic or chaotic) oscillators with a characteristic mean natural period $T_0$, the system adjusts its time scale to achieve phase locking to the external forcing signal. This phenomenon of phase locking due to a time scale matching mechanism is characterized by Arnold tongue locking regions in the parameter space of the amplitude $A$ and period $T_e$ of the signal; locking can be achieved with almost vanishing $A$ values when the time scales match, i.e., $mT_e \approx nT_0$ for some integers $m$ and $n$. How do noise-sustained oscillations in stirred excitable media respond to weak external forcing? The purpose of this paper is to investigate
synchronization of noise-sustained oscillations due to weak periodic forcing. Such a study is important e.g. for understanding the role of stochasticity in synchronization of plankton bloom by seasonal changes. The paper is organized as follows: in section 2, we describe the model used in our numerical simulations. In section 3, we briefly discuss different regimes of noise-induced pattern formation in the absence of external forcing. In section 4, we study in detail synchronization of the system to weak periodic forcing. Finally, we conclude with a discussion in section 5.

2. Model

The FHN model employed in our numerical simulations is described by the following reaction–advection–diffusion equations:

\begin{align}
\frac{\partial C_1}{\partial t} + \mathbf{v}(\mathbf{r},t) \cdot \nabla C_1 &= C_1(a - C_1)(C_1 - 1) - C_2 + A \sin \frac{2\pi}{T_e} t + D \nabla^2 C_1, \\
\frac{\partial C_2}{\partial t} + \mathbf{v}(\mathbf{r},t) \cdot \nabla C_2 &= \epsilon(C_1 - 3C_2) + \xi(\mathbf{r},t) + D \nabla^2 C_2,
\end{align}

where $a$ is the excitation threshold, $D$ is the diffusion constant and $\epsilon \ll 1$ controls the time scale of the slow variable $C_2$. $A$ is the amplitude and $T_e$ the period of the external forcing. In the absence of forcing, $A = 0$, the spatially homogeneous system has a stable fixed point at $(C_1, C_2) = (0, 0)$, while it generates a large excursion to a maximum $C_1 \approx 1$ when perturbed over the threshold $a$. The noise $\xi(\mathbf{r},t)$ is Gaussian white in space and time, satisfying $\langle \xi(\mathbf{r},t)\xi(\mathbf{r}_1,t_1) \rangle = 2\Gamma \delta(\mathbf{r} - \mathbf{r}_1)\delta(t - t_1)$, where $\Gamma$ is its intensity.

Mixing by the flow $\mathbf{v}(\mathbf{r},t)$ is described by a well-known standard model of chaotic advection [36], which is assumed to be independent of the reaction:

\begin{align}
v_x(x,y,t) &= L \sin \left( \frac{2\pi y}{L} + \phi_i \right) , & v_y(x,y,t) &= 0, & nT_f < t \leq nT_f + T_f/2, \\
v_x(x,y,t) &= 0, & v_y(x,y,t) &= L \sin \left( \frac{2\pi x}{L} + \phi_{i+1} \right) , & nT_f + T_f/2 < t \leq (n + 1)T_f,
\end{align}

describing a velocity field formed by a sinusoidal shear flow alternating along the $x$- and $y$-directions for the first and second half of the period $T_f$, respectively. In such a flow model, if we fix the phases $\phi_i$ and $\phi_{i+1}$ in equations (3) and (4), there are two vertical and two horizontal lines in the medium which are motionless ($\mathbf{v} = 0$) and the neighbourhoods of these lines are subjected to small flow velocities. These lines then will form transport barriers, so that the whole medium does not have a uniform mixing property. To avoid such transport barriers typically present in time-periodic flows [36], the phase $\phi_i$ in each half-period is assumed to be a random variable uniformly distributed in $[0, 2\pi]$. Under the advection of such a flow, the fluid elements separate exponentially at a rate given by the largest Lyapunov exponent [36], having a value proportional to the stirring rate $\nu = 1/T_f$ [28]. The results below should be generic to this class of unsteady laminar flows.
Due to the stirring of the flow, a particle in the medium can move from one spatial position to other remote areas and start to interact with other particles there due to diffusion. In this sense, mixing introduces a long-range interaction among particles in the medium, in sharp contrast to motionless media where the interaction is mediated locally by diffusion. If there is no excitable dynamics and noise, i.e.,

$$\frac{\partial C_i}{\partial t} + \mathbf{v}(\mathbf{r}, t) \cdot \nabla C_i = D \nabla^2 C_i, \quad i = 1, 2,$$

the initial non-uniform concentration is homogenized by mixing, and the spatial fluctuations decay exponentially [27],

$$\langle (C - \langle C \rangle)^2 \rangle \sim \exp(-2\alpha t),$$

where the decaying rate $\alpha$ increases with the stirring rate $v$. This is equivalent to the case that the whole medium is coupled globally by the mean field with a coupling strength $\alpha$:

$$\frac{\partial C_i}{\partial t} = \alpha (\langle C_i \rangle - C_i), \quad i = 1, 2.$$

So it is often assumed that strong stirring leads to spatially uniform concentrations and the dynamical oscillators are supposed to be globally coupled by the mean field when the decaying rate $\alpha$ is sufficiently larger than typical time scales of the local dynamics, such as in experiments of synchronization of biological cells [37]. However, it has been shown that stirring is more efficient for synchronization when compared to global coupling with the same $\alpha$ values [27].

The interplay between mixing and excitability has recently been investigated [28, 29]. It has been shown that there is a range of stirring rates $v$, where a localized initial perturbation can develop into filaments with a characteristic width $\sim \sqrt{(D/v)}$. When the stirring rate exceeds some critical value, the initial perturbation decays. In the presence of noise, different dynamical pattern formations can be observed [35]; especially, noise can generate coherent global excitations of the media and the focus of the present work is on the synchronization of noise-induced oscillations by weak external forcing.

As we will see, the period $T_f$ of the flow is much smaller than the typical refractory period $T_r$ of the excitable dynamics and the period $T_e$ of the external signal. Since the phases $\phi_i$ and $\phi_{i+1}$ in equations (3) and (4) are independent random variables, the velocity at each spatial position fluctuates independently in each half-period of the flow. In a more general context, the stirring term of equations (1) and (2) can be regarded as a high-frequency and noisy temporal forcing on the systems, which induces random motion of the particles when viewed in longer time scales. Such a random forcing is similar to a type of multiplicative noise. It might be possible to interpret the transition among different dynamical regimes from the viewpoint of noise-induced transition [16, 26], if one can map the random motion of the particles to an equivalent multiplicative noise term acting on static media. Such an analysis is an interesting topic for future research.

In this work, we rely on numerical simulations of the model system of equations (1) and (2). We consider this system on the unit square ($L = 1$) with doubly periodic boundaries in the weak diffusion case $L^2/(DT_f) \gg 1$. The parameters used in our simulations are $a = 0.25$, $D = 10^{-5}$ and $\epsilon = 10^{-3}$. The system is integrated initially from the homogeneous steady state (HSS) $(C_1, C_2) = (0, 0)$, using a semi-Lagrangian scheme for the deterministic part; then an
Figure 1. Noise-induced behaviour in static media ($v \equiv 0$) at the noise level $\Gamma = 2^{6} \times 10^{-10}$. (a) A typical snapshot of the random excitation pattern. The grey scale (from black to white) is in $[-0.5, 1.0]$. (b) Time series of the mean concentration $\langle C_1 \rangle$.

Noise-induced pattern formation in the absence of external forcing, $A = 0$, has been discussed recently in [35]. For the convenience of discussion about synchronization of the noise-sustained oscillations by external forcing, here we present a brief description of the noise-sustained dynamics, both in the static and stirred media.

3. Dynamics in the absence of forcing: $A = 0$

Noise-sustained pattern formation in the absence of external forcing, $A = 0$, has been discussed recently in [35]. For the convenience of discussion about synchronization of the noise-sustained oscillations by external forcing, here we present a brief description of the noise-sustained dynamics, both in the static and stirred media.

3.1. Static media

In general, for the weak diffusion $D$ considered here, noise alone cannot generate a large-scale coherent behaviour in the static media ($v \equiv 0$). When noise is rather weak, the nucleation of excitation is very infrequent and thus is rare in the domain; the excitations spread in the form of target waves. With increasing noise intensity, many simultaneous noise-induced excitation centres diffuse to form a large-scale excitation of the domain from the initial HSS. After this transient, the domain relaxes to random patterns of small excited patches, as seen by a typical snapshot at a noise level with $\alpha = 6.0$ (figure 1(a)). The number of excitation centres increases at larger noise intensities; however the patches do not grow to a scale comparable to the domain size $L^2$ because noise also breaks up the wave fronts. In figure 1(b), we show the time evolution of the spatial mean concentration $\langle C_1 \rangle$. After the transient, the mean concentration relaxes to fluctuate slightly around a constant value, indicating that repeated large scale coherent excitations of the media are not possible. Such rather non-coherent patterns of excitation induced by noise at small $D$ is quite different from coherent, almost global oscillations in some regime of larger $D$ [18]–[20], [24].
3.2. Stirred media

Now we discuss the dynamics of the media under different stirring rates $\nu$, with a fixed noise intensity $\Gamma = 2^6 \times 10^{-10}$ (i.e., $\alpha = 6.0$). The competition between stirring and diffusion results in characteristic spatial and time scales. There exists a minimal characteristic length scale $l_0 \sim \sqrt{D/\nu}$, over which there is fast homogenization by mixing. A region of size $l_0$ is homogenized by diffusion on a time scale $\tau \sim l_0^2/D \sim 1/\nu$. The noise level controls the density of super-threshold perturbations created within a patch of size $l_0$ during the time $\tau$. When $\nu$ is small, the mixing is not strong enough, and both $l_0$ and $\tau$ are relatively large. In this case, there are possibly many locations with super-threshold perturbations within a patch of size $l_0$, and the patch may stay super-threshold even after the homogenization by diffusion, forming a survived excitation centre. If the mixing is sufficiently strong, then the amount of perturbations within the smaller patch of size $l_0$ during the shorter time $\tau$ may become sub-threshold by the homogenization and it decays. Due to this competition, we have observed different regimes of pattern formation of the media depending on the stirring rate $\nu$, as discussed in the following subsections.

3.2.1. Non-coherent excitation. When the stirring rate $\nu$ is small, e.g. $\nu = 0.02$, mixing by the flow is not strong compared to the diffusion. Many noise-induced excitations at different spatial positions survive the mixing and spread similar to those in the static media. The time series of the mean concentration $\langle C_1 \rangle$ is quite similar to the static media, except for some small oscillations, as seen in figure 2(a). With an increasing stirring rate, e.g. $\nu = 0.06$, the mixing becomes stronger, so that some noise-induced excitations decay to the background state instead of growing through diffusion. However, a few excitations can develop at different locations and at different times. The mean concentration $\langle C_1 \rangle$ displays repeated oscillations with fluctuating amplitudes, as shown in figure 2(b).
3.2.2. Coherent global excitation. With further increase in stirring rate, $\nu \gtrsim 0.09$, most of the local noise-induced super-threshold perturbations decay when mixed quickly with the neighbouring sub-threshold areas. However, at some moment, one or a few large-enough excitation centres survive the mixing almost simultaneously (figure 3, $t = 20$). They are elongated along the trajectory of the flow and develop into filaments with a characteristic width $l_0$ (figure 3, $t = 40$). The filaments become denser and denser to fill the whole domain (figure 3, $t = 40$–140), resulting in a coherent global excitation. Later on, the whole domain relaxes synchronously (figure 3, $t = 260$). Unlike in the regime of non-coherent excitation, now mixing is strong enough to keep the domain almost fully synchronized till it comes back to the close vicinity of HSS, where another round of simultaneous local excitations occurs to evoke another coherent global excitation. The process repeats to generate a noise-sustained coherent oscillation of the mean field $\langle C_1 \rangle$ (figure 2(c)); $\langle C_1 \rangle$ is now composed of a train of large spikes with almost periodic intervals $T$ and constant amplitudes $\langle C_1 \rangle \approx 1$. 

Figure 3. Typical process of development of filaments and formation of a coherent global excitation. The lower panel shows the time series of the mean concentration $\langle C_1 \rangle$ (——) corresponding to the snapshots of the patterns. The grey scale (from black to white) is in $[-0.5, 1.0]$ and is the same for all the snapshots. The stirring rate $\nu = 0.10$ and the noise level $\alpha = 6.0$. 

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In this regime of coherent global excitation, noise is strong enough to induce sufficiently large excitation centres, but importantly, it does not affect much the growth of the filaments to form a globally synchronized excitation of the whole domain. Even more importantly, mixing is strong enough to maintain the synchronized relaxation back to HSS, which is essential for sustained coherent oscillations of the media. So the dynamics of coherent global excitation can be divided into a stochastic and a deterministic component. The development of an excitation centre is a stochastic process with a characteristic waiting time $T_w$. Afterwards, the dynamics becomes essentially deterministic with an almost fixed refractory time $T_r$, as it is dominated by the excitable reaction as in the noise-free media. With increasing $v$, on average it takes a longer waiting time till a global excitation can develop in the stronger mixing case, so that the interval between repeated excitations $T = T_w + T_r$ increases, as seen in figure 2(d) for $v = 0.16$, compared to that in figure 2(c) for $v = 0.10$. Close to a threshold value $v \approx 0.22$, $T$ increases very rapidly as a function of $v$ and becomes more erratic; afterwards, the system moves into an almost homogeneous regime.

3.2.3. Homogenization. For $v \gtrsim 0.22$, the mixing becomes so strong that noise-induced excitation cannot survive. $\langle C_1 \rangle$ shows only small, unstructured fluctuations around the fixed point $\langle C_1 \rangle = 0$ (figure 2(e)). The domain is fairly homogeneous, displaying some small fluctuations around the HSS. Occasionally, local perturbations can develop into very narrow excited filaments, which decay after some time without leading to a large scale excitation.

These three regimes of dynamical pattern formation, homogenization, coherent global excitation and non-coherent excitation can also be observed when the noise intensity $\Gamma$ is increased at a fixed stirring rate $v$, as has been demonstrated in [35]. A phase diagram of these three regimes in the parameter space of the noise intensity and the stirring rate has been presented in [35]. In the coherent global excitation regime, the period of the noise-sustained oscillation increases both in its mean value and in its variability when approaching the boundary between the regimes of homogenization and coherent global excitation, either when the stirring rate is increased or when the noise intensity is decreased [35].

4. Periodic forcing

In the previous section, we have shown that there is a regime where noise can generate a coherent oscillation of the media. The period $T$ of the oscillations, composed of a fluctuating stochastic waiting time and a deterministic refractory period, is controlled by the stirring rate or the noise intensity. An interesting question is whether such noise-sustained oscillations can be entrained by a weak external forcing. This is investigated here. We will study synchronization behaviour as a function of the signal period $T$, signal amplitude $A$, stirring rate $v$ and noise intensity $\alpha$.

4.1. The synchronization region versus signal period $T$ and amplitude $A$

Without noise, the spatially homogeneous media do not respond to periodic signals when the amplitude $A$ is below a threshold $A_{th}$ whose value depends on the signal period $T_e$. Above the threshold, the system may produce one spike within each of the $m$ ($m = 1, 2, 3, \ldots$) periods of the signal, called $m : 1$ resonant response. We have shown in figure 4(a) the $1 : 1$, $2 : 1$ and $3 : 1$ super-threshold response regions of the noise-free, spatially homogeneous media.
Now let us fix the parameters in the regime of coherent global excitation: $\nu = 0.10$ and $\alpha = 6.0$. Without forcing, $A = 0$, the noise-sustained oscillations (figure 2(c)) have a fluctuating period $T = 668 \pm 30$, representing a 4–5% degree of variability.

We first apply weak, sub-threshold signals with an amplitude $A = 0.005$. As depicted in figure 5, the media can generate $1 : 1$ (a), $2 : 1$ (b) and $3 : 1$ (c) locked oscillations, depending on the signal period $T_e$. The spiking sequences are not fully periodic due to the stochastic nature of the system; however, in all these examples, the fluctuation of the inter-spike interval $T$ is less than 1% of the mean values.

In figure 6, we have shown a typical pattern formation process in the $2:1$ locking regime ($A = 0.005$ and $T_e = 320$). At the moment (a), the medium is close to the HSS; however, some small excitations already develop. As in the unforced media (figure 3), these local excitations grow quickly in the form of filaments (b) and lead to a coherent global excitation of the whole media (c). At the moment (d), we can observe filaments again when the medium relaxes quickly, as is also the case in the absence of forcing. Afterwards, during the second period of the signal, the medium is almost fully synchronized and relaxes slowly back to the vicinity of the HSS at moment (k); note that when the forcing signal is maximal around (g) and (h), we observe a temporal development of filaments. The external signal at (k) and (l) repeats that at (a) and (b), respectively; however, the mean concentration $\langle C_1 \rangle$ does not exactly repeat due to a small fluctuation of the phase difference. The patterns will not repeat even in the noise-free case, because of the chaotic flow (the random phase $\phi_i$ in equations (3) and (4)). This property is significantly different from the static media [22], where the noise-sustained $2:1$ resonant pattern...

**Figure 4.** (a) Small symbols: the $m:1$ ($m = 1, 2, 3$) super-threshold locking regions of the noise-free spatially homogeneous media. Large symbols and solid lines: the locking regions of the media at the noise level $\alpha = 6.0$ and the stirring rate $\nu = 0.1$. (b, c, d) The ratio $R = T/T_e$ ($\circ$) between the inter-spike interval $T$ and the signal period $T_e$ is shown as a function of $T_e$ for different signal amplitudes $A = 0.005$ (b), 0.010 (c) and 0.025 (d). The vertical broken lines in (a)–(c) denote $1/m$ ($m = 1, 2, 3$) of the average period $T_0 = \langle T \rangle \approx 668$ of the noise-sustained oscillations at $A = 0$. The thick solid line in (d) shows the super-threshold locking regions of the noise-free media at $A = 0.025$ as in (a). Note the log scale for $T_e$. 

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is composed of standing waves with almost stationary front and repeats effectively for every two periods of the forcing signal.

In stochastic systems, the phase cannot be locked perfectly. There should exist a fluctuation of the phase difference and occasionally a phase slip may occur [11]–[15]. To assess the synchronization properties in a statistical manner, usually we would check the phase difference over a long time interval consisting of a large number of signal periods, over a broad range of the parameter space. However, the numerical simulations of the present system are very time-consuming. To get a more comprehensive picture of the locking behaviour, we record the inter-spike interval $T$ (detected by threshold-crossing at $\langle C_1 \rangle = 0.5$) for a long time interval $t_0 = 5000$ ($t_0 = 10 000$ if $T_e > 1000$). This enables us to get 5–15 records of $T$ values for each value of $T_e$ in spite of the large amount of computation time. We then plot the ratio $R = T/T_e$ for all the obtained intervals. If the noise-sustained oscillations are effectively locked by the external signal, this ratio $R$ should take some values close to an integer $m$. In this way, we can detect the synchronization regions of $T_e$ where the ratio is almost constant. The results of $R$ as a function of the signal period $T_e$ are shown in figures 4(b), (c) and (d) for different signal amplitudes, $A = 0.005$, $0.01$ and $0.025$, respectively. We observe regions of $T_e$ where the ratio $R$ is around $m = 1, 2, 3$, exhibiting the $m : 1$ locking behaviour. These locking regions become wider for larger signal amplitudes. In the centre of the locking region, the symbols coincide almost perfectly and they exhibit a stronger variability when moving to the boundaries of the locking region, suggesting that there is an optimal response to the signal. Between the locking regions of two different ratios, the spiking sequences have strongly scattered intervals, corresponding to an unlocked, chaotic-like spiking behaviour.

To understand the mechanism underlying the synchronization behaviour, in figures 4(b) and (c) we compare the locking regions to the characteristic period $T_0/m$ (vertical broken lines) of the unforced, noise-sustained oscillations. Interestingly, for weak signal amplitudes, the $m : 1$

**Figure 5.** Various resonant spiking sequences of the mean concentration $\langle C_1 \rangle$ of the noisy media (- - -, periodic signals).
locking regions are centred around this characteristic period, i.e., $T_e \approx T_0 / m$. Compared to the unforced oscillations in figure 2(c), in the presence of a suitable weak external forcing, the time taken to get survived initial local excitations is advanced if $m T_e < T_0$ and is delayed otherwise, so that the oscillation period is adjusted effectively to match that of the signal, and synchronization is achieved. The dynamics during the refractory period is mainly unaffected by the weak signal. This observation suggests that synchronization for weak signals is due to a matching between the time scales of the forcing signal and the noise-sustained oscillations. The noise-sustained oscillations thus perform similarly to self-sustained oscillations when responding to weak external forcing [38]. Compared to noise-free media, noise significantly enhances the response sensitivity of the system to otherwise sub-threshold signals.

Figure 6. Typical pattern formation of the forced noisy media in the 2 : 1 locking region at $A = 0.005$ and $T_e = 320$. The lower panel shows the time series of the mean concentration $\langle C_1 \rangle$ (—) and external forcing (----) corresponding to the snapshots of the patterns. The grey scale (from black to white) is in $[-0.5, 1.0]$ for all the snapshots.
Note that the signal amplitude $A = 0.025$ is super-threshold for $120 \lesssim T_e \lesssim 1010$, as can be seen in figures 4(a) and (d). Here, we have observed broader $1:1$ and $2:1$ locking regions. Compared to the locking regions of the noise-free media (figure 4(d), solid line), we find that the locking behaviour in the presence of noise is quite different. For example, over the range $250 < T_e < 1000$, the noise-free media have a $2:1$ locking, while in the presence of noise, the locking ratio becomes $1:1$, again around the noise-sustained natural period $T_0$. More interestingly, for $T_e \gtrsim 1300$, we have observed a $1:2$ locking region. An example is shown in figure 5(d), where $\langle C_1 \rangle$ displays two spikes within each period $T_e$ of the signal, with alternating small and large intervals $T_1 + T_2 = T_e$. Note that in this region, the signal is sub-threshold. In fact, the $1:2$ locking regime is not present in the noise-free media at all. In self-sustained oscillators, the $1:2$ locking regime is usually observed due to time-scale matching. This comparison confirms again that, in the presence of noise, the excitable media is shifted effectively to an oscillatory regime as in the self-sustained oscillators.

The effective locking regions in figures 4(b)–(d) are also depicted in figure 4(a) for a comparison with those of the noise-free media. The tongue-like structures, extending to almost vanishing signal amplitudes, demonstrate clearly that noise-sustained oscillatory behaviour in the stirred excitable media acts as self-sustained oscillatory behaviour when subjected to weak external forcing. Noise thus can sustain synchronization to sub-threshold signals and can also significantly control the response to super-threshold signals.

4.2. Synchronization regions versus stirring rate $\nu$ and noise intensity $\alpha$

As mentioned above and also shown in figures 2(c) and (d) and in [35], in the coherent global excitation regime, the noise-sustained characteristic oscillation period $T_0$ is controlled by the stirring rate $\nu$ and the noise intensity $\alpha$. According to the time scale matching property, we expect that for a fixed signal amplitude $A$ and period $T_e$, there should exist a region of $\nu$ and $\alpha$ where the noise-sustained oscillations are locked by the external signal. This is actually observed in simulations with $A = 0.01$ and $T_e = 700$. The effective locking region with respect to $\nu$ for $\alpha = 6.0$ is shown in figure 7(a). Outside the synchronization region, if $\nu \lesssim 0.09$, the noise-sustained oscillations are non-coherent and are not locked by the signal (figure 7(b), $\nu = 0.04$). For a stronger stirring rate $\nu \gtrsim 0.22$, the unforced media would be in the homogenization region without displaying a collective oscillation. In the presence of the signal, they become oscillatory; however, the oscillations are not locked by the signal: skipping of spikes occurs occasionally for some signal periods as shown typically in figure 7(c) for $\nu = 0.26$. When the stirring rate becomes sufficiently large, we retain the homogenization regime. A similar behaviour is observed for variation of the noise intensity $\alpha$ at a fixed stirring rate $\nu = 0.10$. As shown in figure 8(a), the oscillations are locked effectively in the range $4.5 \lesssim \alpha \lesssim 7.0$. The variability of the ratio $R$ decreases when approaching the centre of the locking region at $\alpha \approx 6.0$, indicating an optimal response, similar to the well-known stochastic resonance [7]. For $\alpha \gtrsim 7.0$, the noise-sustained oscillations are non-coherent and not synchronized by the weak signal (figure 8(c), $\alpha = 7.5$). For weaker noise levels, addition of the signal can make the otherwise homogenized media oscillatory, but the oscillations are not locked, again due to occasional skipping of spikes (figure 8(b), $\alpha = 3.5$); for even smaller noise levels, the system again moves into the homogenization regime. These results demonstrate that, in the presence of weak external forcing, we still have the three regimes of pattern formation: homogenization, coherent global excitation and non-coherent excitation. The coherent global excitation regime now is extended to larger $\nu$ or smaller $\alpha$. 

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regions where the media are homogenized in the absence of the forcing signal. And this regime is composed of synchronized and non-synchronized regions.

It is important to compare the noise-sustained and controlled synchronizations in the stirred excitable media to noise-enhanced response and synchronization in conventional stochastic resonance of over-damped bistable systems [7, 8]. Stochastic resonance occurs when the noise-controlled mean switching interval $T_0 = \langle T \rangle$ is close to the period $T_s$ of the signal and the response becomes optimal [8]–[11]. Unlike the excitable media in the present paper, noise-sustained oscillations of over-damped bistable systems do not establish a natural period. Nevertheless, an effective locking of phase and mean switching period is achieved for sufficiently slow signals, in a tongue-like parameter region of the noise intensity $\Gamma$ and the signal amplitude $A$, for $A$ rather close to the threshold [11]. Such effective locking, however, is significantly different from the time scale matching condition. In fact, the optimal noise intensity of the signal-to-noise ratio is independent of the signal period $T_s$ for a slow-enough signal, and stochastic resonance can also occur for aperiodic signals [39]–[41]. While the coherence and synchronization measures

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure7}
\caption{(a) The ratio $R = T/T_s$ versus the stirring rate $\nu$. (b) Non-synchronized oscillation behaviour for $\nu = 0.04$. (c) Non-synchronized oscillation behaviour for $\nu = 0.26$. Other parameters are $\alpha = 6.0$, $A = 0.01$ and $T_s = 700$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure8}
\caption{(a) The ratio $R = T/T_s$ versus the noise intensity $\alpha$. (b) Non-synchronized oscillation behaviour for $\alpha = 3.5$. (c) Non-synchronized oscillation behaviour for $\alpha = 7.5$. Other parameters are $\nu = 0.10$, $A = 0.01$ and $T_s = 700$.}
\end{figure}

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exhibit a resonance behaviour with change in the noise intensity $\Gamma$, they do not display a locking and a resonance behaviour with respect to the signal period $T_e$ [11, 41]. Consequently, a higher order $m:n$ locking does not occur when the signal period moves to higher order matching points, $T_e \approx nT_0/m$ [11].

5. Discussion and outlook

We have shown that the interplay among mixing, diffusion, excitability and noise can generate three types of dynamical patterns: (i) homogenization, when strong enough stirring dilutes quickly those noise-induced local excitations, which in the motionless media will result in rather irregular patterns; (ii) coherent global excitation, when the flow enhances the coherence by propagating the excited phase in the form of filaments, leading to a synchronized excitation of the whole domain; and (iii) non-coherent excitation, if the stirring is not strong enough to prevent non-synchronized excitations. These results provide an explanation for the experimentally observed stirring effects in the excitable Belousov–Zhabotinsky (BZ) reaction [42, 43]. Fluctuations of the local reaction rates or heat release can induce oscillations of larger and more erratic periods with increasing stirring rates, which become quenched at strong stirring rates [43].

These regimes are also observed when the media are subjected to a weak external signal. More interestingly, since the noise-sustained coherent oscillations establish a characteristic period, which is controlled by the stirring rate and noise intensity, the media become entrained by an external forcing much weaker than the threshold. The mechanism of this enhanced synchronization is the time scale matching between the forcing signal and the noise-sustained oscillations, as in self-sustained oscillatory systems. Due to this mechanism, tongue-like synchronization regions for various locking ratios are observed when the time scale of the forcing signal is varied, or when the time scale of the noise-sustained oscillations is changed by the stirring rate or the noise intensity. Compared to noise-free media, the presence of noise, when interplaying constructively with mixing and excitability, has generated significantly different and much more sensitive resonant responses of the media to the signal. Noise-sustained pattern formation and synchronization demonstrated here in the FHN model should be generic in excitable media subjected to mixing, irrespective of details of the excitable and mixing dynamics. We thus expect this behaviour to be observed in experiments, for example, in stirred light-sensitive BZ reactions under periodic illumination.

It is interesting to apply our findings of noise-sustained oscillation and synchronization to the problem of oceanic plankton bloom. In this system, noise is significantly present. To understand the effects of noise on plankton bloom, we would like to employ a more realistic excitable model for the plankton dynamics [44]. We also should consider a more sophisticated model for the oceanic turbulent flows. Hopefully, in such an investigation, we can identify the effects of noise on the synchronization of plankton bloom to external forcing of seasonal cycles. Research on this topic is currently underway.

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