\( \mathfrak{sl}(M+1) \) Construction of Quasi-solvable Quantum \( M \)-body Systems

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Abstract

We propose a systematic method to construct quasi-solvable quantum many-body systems having permutation symmetry. By the introduction of elementary symmetric polynomials and suitable choice of a solvable sector, the algebraic structure of \( \mathfrak{sl}(M+1) \) naturally emerges. The procedure to solve the canonical-form condition for the two-body problem is presented in detail. It is shown that the resulting two-body quasi-solvable model can be uniquely generalized to the \( M \)-body system for arbitrary \( M \) under the consideration of the \( GL(2, K) \) symmetry. An intimate relation between quantum solvability and supersymmetry is found. With the aid of the \( GL(2, K) \) symmetry, we classify the obtained quasi-solvable quantum many-body systems. It turns out that there are essentially five inequivalent models of Inozemtsev type. Furthermore, we discuss the possibility of including \( M \)-body (\( M \geq 3 \)) interaction terms without destroying the quasi-solvability.

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I. INTRODUCTION

Since the discovery of quasi-solvability in one-dimensional quantum mechanics [1], many new ideas and concepts have been discovered and developed. The progress until the early 1990s is summarized in Ref. [2]. So far, most of the known quasi-solvable quantum systems of one-degree of freedom are related to the ones constructed from the \( \mathfrak{sl}(2) \) generators [3]. This family of quasi-solvable models was completely classified under the consideration of the \( \text{GL}(2, \mathbb{R}) \) invariance of the models [4, 5]. Recently, a new type of orthogonal polynomial systems was discovered in connection with a quasi-solvable model [6]. Soon later, it was shown that similar type of polynomial systems, known as weakly orthogonal polynomial systems, can be constructed for all the \( \mathfrak{sl}(2) \) quasi-exactly solvable models [7, 8]. In the last couple of years, much attention has been paid to quasi-solvable models from the viewpoint of \( \mathcal{N} \)-fold supersymmetry [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. Furthermore, several realistic physical systems have been found, which can be reduced to one-dimensional quasi-solvable models [13, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32].

Unlike one-body systems, construction of quasi-solvable quantum many-body systems suffers from another difficult problem. The difficulty originates from the fact that second-order differential operators of several variables are, in general, no longer equivalent to the Schrödinger operators. Indeed, construction of quasi-solvable two-body problems by the naive extension of the \( \mathfrak{sl}(2) \) method to the rank two algebras, investigated extensively in the early 1990s [2, 5, 33, 34, 35, 36, 37, 38], led to the Schrödinger operators in curved space. In 1995, a breakthrough was achieved in Ref. [39], where the exact solvability of the rational and trigonometric \( A \) type Calogero–Sutherland (CS) models [40, 41, 42] for any finite number of particles were shown by a similar algebraic method. The key ingredient is to construct the generators of \( \mathfrak{sl}(M + 1) \) Lie algebra in terms of the elementary symmetric polynomials which reflect the permutation symmetry of the CS models. This idea was further employed to show the exact solvability of the rational and trigonometric \( A \) and \( BC \) type CS models and their supersymmetric generalizations [43], and to show the quasi-exact solvability of various deformed CS models [44, 45]. So, it is natural to ask what kinds of quasi-solvable quantum many-body systems can be constructed from the same algebraic procedure. Recently, this classification problem was partly accessed in Ref. [46], though the method depends on the specific ansatz employed. In our previous short letter [47], we have briefly reported a partial answer for this problem without recourse to any specific ansatz by restricting the consideration up to two-body interactions. In this paper, we will give the details and the further developments on the issues. Especially, we will see that the \( \text{GL}(2, K) \) symmetry plays a central role not only on the classification problem but also on the problem of what kind of many-body operators can exist in the quasi-solvable quantum Hamiltonian.

The paper is organized as follows. In the next section, we give the definition of quasi-solvability together with its important subclasses such as quasi-exact solvability, solvability and so on. Based on the definition, we present a procedure to construct a quasi-solvable model by introducing a set of differential operators. In Section III we show that, by the introduction of elementary symmetric polynomials and suitable choice of the set of differential operators which determines a solvable sector of the model, a family of quasi-solvable models can be constructed from a polynomial of the \( \mathfrak{sl}(M + 1) \) generators of second-degree. Section IV deals with the canonical-form condition, namely, under what conditions we can obtain the quasi-solvable quantum systems from the algebraic operators constructed in Section III. We explicitly show the procedure to solve the canonical-form condition for the
two-body problem. In Section V we explain the $GL(2, K)$ symmetry of the model. It turns out that this symmetry is so restrictive that the quasi-solvable quantum two-body model obtained in Section IV can be uniquely generalized to $M$-body ($M \geq 3$) systems. An interesting relation between (quasi-)solvability and supersymmetry is discussed. In Section VI, we show that the many-body quantum systems obtained solely by the consideration of the $GL(2, K)$ symmetry are actually quasi-solvable. This is achieved by expressing the models in terms of the elementary symmetric polynomials. In Section VII, we classify the models under the consideration of the $GL(2, K)$ symmetry, by essentially the same way as in Ref. [4]. The explicit forms of both the superpotentials and potentials are presented. In Section VIII, we investigate the possibility of including $M$-body ($M \geq 3$) interaction terms without destroying the quasi-solvability. Again, the $GL(2, K)$ symmetry plays an essential role in this problem. We give several concluding remarks in the final section. Some useful formulae are summarized in Appendices.

II. QUASI-SOLVABILITY IN MANY-BODY SYSTEMS

First of all, we will give the definition of quasi-solvability and some notions of its special cases based on Refs. [16, 38]. A linear differential operator $H$ of several variables $q = (q_1, \ldots, q_M)$ is said to be quasi-solvable if it preserves a finite dimensional functional space $V_N$ whose basis admits an explicit analytic form:

$$HV_N \subset V_N, \quad \dim V_N = n(N) < \infty, \quad V_N = \text{span} \{\phi_1(q), \ldots, \phi_{n(N)}(q)\}. \quad (2.1)$$

An immediate consequence of the above definition of quasi-solvability is that, since we can calculate finite dimensional matrix elements $S_{k,l}$ defined by,

$$H\phi_k = \sum_{l=1}^{n(N)} S_{k,l}\phi_l \quad (k = 1, \ldots, n(N)), \quad (2.2)$$

we can diagonalize the operator $H$ and obtain its spectra on the space $V_N$, at least, algebraically. Furthermore, if the space $V_N$ is a subspace of a Hilbert space $L^2(S)$ ($S \subset \mathbb{R}^M$) on which the operator $H$ is naturally defined, the solvable spectra and the corresponding vectors of $V_N$ give the exact eigenvalues and eigenfunctions of $H$, respectively. In this case, the operator $H$ is said to be quasi-exactly solvable (on $S$). Otherwise, the solvable spectra and the corresponding vectors of $V_N$ only give local solutions of the characteristic equation and have, at most, restrictive meanings in the perturbation theory defined on the physical space $L^2(S)$ [16].

A quasi-solvable operator $H$ of several variables is said to be solvable if it preserves an infinite flag of finite dimensional functional spaces $V_N$,

$$V_1 \subset V_2 \subset \cdots \subset V_N \subset \cdots, \quad (2.3)$$

whose bases admit explicit analytic forms, that is,

$$HV_N \subset V_N, \quad \dim V_N = n(N) < \infty, \quad V_N = \text{span} \{\phi_1(q), \ldots, \phi_{n(N)}(q)\}. \quad (2.4)$$

for $N = 1, 2, 3, \ldots$. Furthermore, if the sequence of the spaces (2.3) defined on $S \subset \mathbb{R}^M$ satisfies,

$$V_N(S) \rightarrow L^2(S) \quad (N \rightarrow \infty), \quad (2.5)$$

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the operator $H$ is said to be *exactly solvable* (on $S$).

In the case of a system of a single variable, it was proved that quasi-solvability is essentially equivalent to $\mathcal{N}$-fold supersymmetry [16]. In this context, it is convenient to formulate the quasi-solvability in terms of $\mathcal{N}$th-order linear differential operators. Let us introduce $\mathcal{N}$-fold supercharges by,

$$
Q_N = \sum_{\{i\}} P_N^{(i)\dagger} \psi_{\{i\}}, \quad Q_N^\dagger = \sum_{\{i\}} P_N^{(i)} \psi_{\{i\}}^\dagger,
$$

(2.6)

where $\psi_{\{i\}}$ and $\psi_{\{i\}}^\dagger$ are fermionic coordinates and $P_N^{(i)}$ are $\mathcal{N}$th-order linear differential operators of several variables $q_i$. Using these operators, we define the vector space $\mathcal{V}_N$ as,

$$
\mathcal{V}_N = \bigcap_{\{i\}} \ker P_N^{(i)}.
$$

(2.7)

Now, it is easy to see that an operator $H$ is quasi-solvable with the solvable sector (2.7) if the following quasi-solvability condition holds [16]:

$$
P_N^{(i)} H \mathcal{V}_N = 0 \quad \forall \{i\}.
$$

(2.8)

Therefore, the above formulation gives a concrete procedure to construct a quasi-solvable operator. In practice, however, the problem often becomes more tractable when we make a suitable similarity transformation on the vector space and on the operators and make a suitable change of variables. In the case of $M = 1$, the procedure was successfully employed to construct the type A $\mathcal{N}$-fold supersymmetric models [17, 18, 22]. In the following section, we will generalize the procedure to the case of several variables.

III. $\mathfrak{sl}(M + 1)$ ALGEBRAIZATION OF THE MODELS

Let us consider an $M$-body quantum Hamiltonian for a system of identical particles on a line,

$$
H_N = -\frac{1}{2} \sum_{i=1}^{M} \frac{\partial^2}{\partial q_i^2} + V(q_1, \ldots, q_M),
$$

(3.1)

which possesses permutation symmetry, that is,

$$
V(\ldots, q_i, \ldots, q_j, \ldots) = V(\ldots, q_j, \ldots, q_i, \ldots) \quad \forall i \neq j.
$$

(3.2)

To algebraize the Hamiltonian (3.1), we will proceed the following three steps. First, we make a *gauge* transformation on the Hamiltonian (3.1):

$$
\tilde{H}_N = e^{W(q)} H_N e^{-W(q)}.
$$

(3.3)

The function $W(q)$ is to be determined later and plays the role of the superpotential when the system Eq. (3.1) is supersymmetric. As in Eq. (3.3), we will hereafter attach tildes to both operators and vector spaces to indicate that they are quantities gauge-transformed from the original ones. In the next step, we change the variables $q_i$ to $h_i$ by a function
of a single variable; \( h_i = h(q_i) \). Note that the way of changing the variables preserves the permutation symmetry. The third step is the introduction of elementary symmetric polynomials of \( h_i \) defined by,

\[
\sigma_k(h) = \sum_{i_1 < \cdots < i_k} h_{i_1} \cdots h_{i_k} \quad (k = 1, \ldots, M),
\]

(3.4)

from which we further change the variables to \( \sigma_i \). Owing to the permutation symmetry, the gauged Hamiltonian (3.3) can be completely expressed in terms of these elementary symmetric polynomials (3.4). We choose a set of components of the \( N \)-fold supercharges in terms of the above variables \( \sigma_i \) as follows:

\[
\tilde{P}^{(i)} = \frac{\partial^N}{\partial \sigma_{i_1} \cdots \partial \sigma_{i_N}} \quad (1 \leq i_1, \ldots, i_N \leq M),
\]

(3.5)

where \( \{ i \} \) is an abbreviation of the set \( \{ i_1, \ldots, i_N \} \). Using these \( N \)-fold supercharges, we define the vector space by Eq. (2.7), in which,

\[
\tilde{V}_N = \text{span} \left\{ \sigma_{n_1} \cdots \sigma_{n_M} : n_i \in \mathbb{Z}_{\geq 0}, 0 \leq \sum_{i=1}^{M} n_i \leq N - 1 \right\}.
\]

(3.6)

For given \( M \) and \( N \), the dimension of the vector space (3.6) becomes,

\[
\dim \tilde{V}_N = \sum_{n=0}^{N-1} \frac{(n + M - 1)!}{n!(M-1)!} = \frac{(N + M - 1)!}{(N-1)!M!}.
\]

(3.7)

We will construct the system (3.3) to be quasi-solvable so that the solvable subspace is given by just Eq. (3.6). This can be achieved by imposing the quasi-solvability condition of the gauge-transformed form [16, 17],

\[
\tilde{P}^{(i)} \tilde{H}_N \tilde{V}_N = 0 \quad \forall \{ i \}.
\]

(3.8)

The above condition ensures \( \tilde{H}_N \tilde{V}_N \subset \tilde{V}_N \), that is, the subspace \( \tilde{V}_N \) is invariant under the action of \( \tilde{H}_N \). As a consequence, we can diagonalize the gauged Hamiltonian (3.3) in the subspace (3.6) with the finite dimension given by Eq. (3.7), and can obtain the corresponding spectra algebraically. Then, it is evident that they are also the spectra of the original Hamiltonian (3.1) in the subspace \( V_N \) given by,

\[
V_N = \text{span} \left\{ \sigma_1^{n_1} \cdots \sigma_M^{n_M} e^{-W(q)} : n_i \in \mathbb{Z}_{\geq 0}, 0 \leq \sum_{i=1}^{M} n_i \leq N - 1 \right\}.
\]

(3.9)

Thus, the system \( H_N \) turns to be quasi-solvable.

The general solution of Eq. (3.8) can be obtained in the same way as in Ref. [17]. As in the case of the one-body models, it is sufficient to find differential operators up to second-order as solutions for \( \tilde{H}_N \) since we are constructing a Schrödinger operator in the original
variables $q_j$. We can find that solutions expressed in terms of first-order derivatives are the following:

$$\frac{\partial}{\partial \sigma_i} \equiv E_{0i}, \quad (3.10a)$$

$$\sigma_j \frac{\partial}{\partial \sigma_j} \equiv E_{ij}, \quad (3.10b)$$

$$\sigma_i \left( N - 1 - \sum_{k=1}^{M} \sigma_k \frac{\partial}{\partial \sigma_k} \right) \equiv \sigma_i E_{00} \equiv E_{i0}. \quad (3.10c)$$

These $M^2 + 2M$ operators $E_{\kappa\lambda}(\kappa + \lambda > 0)$ together with $E_{00}$ constitute the $\mathfrak{gl}(M + 1)$ Lie algebra:

$$[E_{\kappa\lambda}, E_{\mu\nu}] = \delta_{\mu,\lambda} E_{\kappa\nu} - \delta_{\kappa,\nu} E_{\mu\lambda}. \quad (3.11)$$

Here we note that throughout this paper the Latin indices take values from 1 to $M$ while Greek indices from 0 to $M$, as in the standard convention in relativistic theories. Solutions involving second-order derivatives are of the following form:

$$\frac{\partial^2}{\partial \sigma_i \partial \sigma_j} = E_{0i} E_{0j}, \quad (3.12a)$$

$$\frac{\partial^2}{\partial \sigma_i \partial \sigma_k} = E_{ij} E_{0k}, \quad (3.12b)$$

$$\sigma_i \sigma_j \frac{\partial^2}{\partial \sigma_k \partial \sigma_l} = E_{ik} E_{jl} - \delta_{j,k} E_{il}, \quad (3.12c)$$

$$\sigma_i \sigma_j \left( N - 1 - \sum_{l=1}^{M} \sigma_l \frac{\partial}{\partial \sigma_l} \right) \sigma_j \frac{\partial}{\partial \sigma_i} = E_{i0} E_{jk}, \quad (3.12d)$$

$$\sigma_i \sigma_j \left( N - 2 - \sum_{k=1}^{M} \sigma_k \frac{\partial}{\partial \sigma_k} \right) \left( N - 1 - \sum_{l=1}^{M} \sigma_l \frac{\partial}{\partial \sigma_l} \right) = E_{i0} E_{j0}. \quad (3.12e)$$

Therefore, the general solution which contains up to the second-order differential operators reads,

$$\tilde{H}_N = - \sum_{\kappa,\lambda,\mu,\nu=0}^{M} A_{\kappa\lambda,\mu\nu} E_{\kappa\lambda} E_{\mu\nu} + \sum_{\kappa,\lambda=0}^{M} B_{\kappa\lambda} E_{\kappa\lambda} - C, \quad (3.13)$$

where $A_{\kappa\lambda,\mu\nu}$, $B_{\kappa\lambda}$, $C$ are arbitrary constants. Since $E_{00} + \sum_{i=1}^{M} E_{ii} = N - 1$, the algebra represented by Eqs. (3.10) is essentially $\mathfrak{sl}(M + 1)$ and we can set $A_{\kappa\lambda,00} = A_{00,\mu\nu} = 0 \forall \kappa,\lambda,\mu,\nu$ and $B_{00} = 0$. However, all the terms in Eq. (3.13) are not linearly independent and thus we can set some of the constants zero. From the Lie algebraic relations (3.11), we can put,

$$A_{i0,j0} = A_{i0,jl} = 0 \quad \forall i < j, \quad (3.14a)$$

$$A_{ik,jl} = 0 \quad \forall i < j \text{ or } \forall k < l, \quad (3.14b)$$

$$A_{ik,0l} = A_{0k,0l} = 0 \quad \forall k < l, \quad (3.14c)$$

$$A_{0k,jl} = A_{0k,j0} = A_{ik,j0} = 0 \quad \forall i, j, k, l. \quad (3.14d)$$
Furthermore, we can remove the remaining redundant terms as follows:

\[ E_{i0}E_{0l} = -\sum_{k=1}^{M} E_{ik}E_{kl} + (M - N + 1)E_{il} \Rightarrow A_{i0,0l} = 0 \quad \forall \ i, l. \quad (3.15) \]

After the substitutions (3.14) and (3.15), the remaining terms in Eq. (3.13) turn to be linearly independent. Substituting the generators (3.10) for Eq. (3.13) and taking Eqs. (3.14) and (3.15) into account, we obtain the general solution (3.13) in terms of \( \sigma_i \):

\[ \tilde{H}_N = -\sum_{k,l=1}^{M} \left[ A_0(\sigma)\sigma_k\sigma_l - A_k(\sigma)\sigma_l + A_{kl}(\sigma) \right] \frac{\partial^2}{\partial\sigma_k\partial\sigma_l} + \sum_{k=1}^{M} \left[ B_0(\sigma)\sigma_k - B_k(\sigma) \right] \frac{\partial}{\partial\sigma_k} - C(\sigma), \quad (3.16) \]

where \( A_{\kappa}, A_{kl}, B_{\kappa} \) and \( C \) are the following second-degree polynomials of several variables:

\[ A_{\kappa}(\sigma) = \sum_{i,j \geq j} A_{i0,j\kappa}\sigma_i\sigma_j, \quad (3.17a) \]

\[ A_{kl}(\sigma) = \sum_{i,j \geq j} A_{ik,jl}\sigma_i\sigma_j + \sum_{i=1}^{M} A_{i0k0l}\sigma_i + A_{0k0l} \quad (k \geq l), \quad (3.17b) \]

\[ B_0(\sigma) = 2(N - 2)A_0(\sigma) - \sum_{i=1}^{M} B_{i0}\sigma_i, \quad (3.17c) \]

\[ B_k(\sigma) = (N - 2)A_k(\sigma) + \sum_{i,j \geq j (\geq k)} A_{ij,jk}\sigma_i - \sum_{i=1}^{M} B_{ik}\sigma_i - B_{0k}, \quad (3.17d) \]

\[ C(\sigma) = (N - 1)(N - 2)A_0(\sigma) - (N - 1)\sum_{i=1}^{M} B_{i0}\sigma_i + C. \quad (3.17e) \]

Before closing the section, we will investigate the condition under which the quasi-solvable operator (3.16) becomes solvable. If we act the \( \mathfrak{sl}(M + 1) \) generators for a fixed \( N \) on the vector spaces \( \tilde{V}_k \) for any \( k = 1, 2, 3, \ldots \), we have,

\[ E_{i0}\tilde{V}_k \subset \tilde{V}_{k-1}, \quad E_{ij}\tilde{V}_k \subset \tilde{V}_k, \quad E_{i0}\tilde{V}_k \subset \tilde{V}_{k+1}. \quad (3.18) \]

Thus, the operator (3.13) preserves an infinite flag of the spaces \( \tilde{V}_k \) given by Eq. (3.6),

\[ \tilde{V}_1 \subset \tilde{V}_2 \subset \cdots \subset \tilde{V}_k \subset \cdots, \quad (3.19) \]

if it does not contain the operators such as \( E_{i0}E_{j0}, E_{i0}E_{jk} \) and \( E_{i0} \) at all. If we note from Eqs. (3.14d) and (3.15) that we can arrange the operator (3.13) so that it does not contain operators like \( E_{0k}E_{j0} \) and \( E_{i0}E_{0l} \), we conclude that the operator is solvable if it does not contain the raising operator \( E_{i0} \) at all. Thus, the solvability condition can be written as,

\[ A_{i0,j\kappa} = B_{i0} = 0 \quad \forall \ i, j, \kappa. \quad (3.20) \]
Finally, if we substitute the above condition for the set of polynomials (3.17), we get the solvability condition for the gauged Hamiltonian (3.16) as,

\[ A_\kappa(\sigma) = B_0(\sigma) = 0 \quad \forall \kappa. \]  

(3.21)

If this is the case, \( B_k(\sigma) \)'s become polynomials of at most first-degree while \( C(\sigma) \) becomes a constant:

\[ B_k(\sigma) = \sum_{i \geq j(\geq k)} A_{ij,jk} \sigma_i - \sum_{i=1}^M B_{ik} \sigma_i - B_{0k}, \]  

(3.22)

\[ C(\sigma) = C. \]  

(3.23)

**IV. CANONICAL-FORM CONDITION**

It is evident from the construction that any operators obtained from Eq. (3.16) by gauge transformations are quasi-solvable. However, this does not necessarily fulfill our purpose at the quantum level. One of the most difficult problems one would encounter in the algebraic approach to the quasi-solvable quantum many-body systems is to solve the canonical-form condition:

\[ H_N = e^{-W(q)} \tilde{H}_N e^{W(q)} = -\frac{1}{2} \sum_{i=1}^M \frac{\partial^2}{\partial q_i^2} + V(q). \]  

(4.1)

If the Hamiltonian (3.16) is gauge-transformed back to the original one, it in general does not take the canonical form of the Schrödinger operator like Eq. (3.1) and one can hardly solve, for arbitrary \( M \), the conditions under which a gauge-transform of Eq. (3.16) could be cast in the Schrödinger form. This difficulty can, however, be overcome in our case by considering the symmetries.

Suppose we can solve the canonical-form condition for an \( M \). Then we would obtain a quasi-solvable Hamiltonian which might contain up to \( M \)-body interactions:

\[ H_N = -\frac{1}{2} \sum_{i=1}^M \frac{\partial^2}{\partial q_i^2} + g_1(M) \sum_{i=1}^M V_1(q_i) + g_2(M) \sum_{i<j}^M V_2(q_i, q_j) + \cdots + g_m(M) \sum_{i_1<\cdots<i_m}^M V_m(q_{i_1}, \ldots, q_m) + \cdots + g_M(M) V_M(q_1, \ldots, q_M), \]  

(4.2)

where each \( g_m(M)(1 \leq m \leq M) \) denotes the coupling constant of the corresponding \( m \)-body interaction and may depend on the number of the particle \( M \). Then, we can get a quasi-solvable \( M \)-body model with up to \( M' \)-body interactions \( (M' < M) \) if we turn off all the coupling constants \( g_m(M) \) for \( m > M' \):

\[ H_N = -\frac{1}{2} \sum_{i=1}^M \frac{\partial^2}{\partial q_i^2} + g_1(M) \sum_{i=1}^M V_1(q_i) + g_2(M) \sum_{i<j}^M V_2(q_i, q_j) + \cdots + g_{M'}(M) \sum_{i_1<\cdots<i_{M'}}^M V_{M'}(q_{i_1}, \ldots, q_{i_{M'}}), \]  

(4.3)
The resultant model should be, when we put $M = M'$, one of the models constructed from the $\mathfrak{sl}(M' + 1)$. This is a consequence of the permutationally symmetric construction. Actually, the gauged Hamiltonian (3.13) composed of the $\mathfrak{sl}(M' + 1)$ generators is reduced to the one composed of the generators of the subalgebra $\mathfrak{sl}(M' + 1) \subset \mathfrak{sl}(M + 1)$ if we set $h_i = 0$ for $i > M'$ involved in both the generators (3.10) and the representation space (3.6).

The above observation means that, we can know the functional form of the $M$-body quasi-solvable potential which involves up to $M'$-body interactions ($M' < M$) if we can solve the canonical-form condition for the $M'$-body case. Furthermore, as we will later see in section V, the $M$-dependence of each coupling constant $g_m(M)$ can be uniquely determined by considering another symmetry, namely, $GL(2, K)$ symmetry. Therefore, it is necessary and sufficient to solve the $M'$-body canonical-form condition as far as we are concerned with the quasi-solvable models containing up to $M'$-body interactions. The simplest but non-trivial case is the two-body case. In the followings, we will see that we can actually solve the canonical-form condition for the two-body problem.

A. The Conditions for the Two-body Case

To solve the condition, we must first transform the variables from $\sigma_i$ back to $h_i$. For this purpose it is convenient to note that $h_1$ and $h_2$ are the solutions of the algebraic equation:

$$h_i^2 - \sigma_1 h_i + \sigma_2 = 0 \quad (i = 1, 2). \quad (4.4)$$

From the differential of the above relation:

$$\text{d}h_i = \frac{h_i}{2h_i - \sigma_1} \text{d}\sigma_1 - \frac{1}{2h_i - \sigma_1} \text{d}\sigma_2 \quad (i = 1, 2), \quad (4.5)$$

we can easily obtain:

$$\frac{\partial}{\partial \sigma_1} = \sum_{i \neq j}^2 \frac{h_i}{h_i - h_j} \frac{\partial}{\partial h_i}, \quad \frac{\partial}{\partial \sigma_2} = - \sum_{i \neq j}^2 \frac{1}{h_i - h_j} \frac{\partial}{\partial h_i}. \quad (4.6)$$

Substituting Eqs. (3.4) and (4.6) for the gauged Hamiltonian (3.13), we can express it in terms of $h_i$ as follows:

$$\hat{H}_N = - \frac{1}{(h_1 - h_2)^2} \sum_{i=1}^2 \left[ A_0 h_i^4 - A_1 h_1^3 + (A_2 + A_{11})h_i^2 - A_{21} h_i + A_{22} \right] \frac{\partial^2}{\partial h_i^2}$$

$$+ \frac{1}{(h_1 - h_2)^2} \left[ 2A_0 \sigma_1^2 - A_1 \sigma_2 \sigma_1 + A_2 \sigma_1^2 - 2(A_2 - A_{11}) \sigma_2 - A_{21} \sigma_1 + 2A_{22} \right] \left( \frac{\partial^2}{\partial h_1 \partial h_2} + \sum_{i \neq j} \frac{1}{h_i - h_j} \frac{\partial}{\partial h_i} \right)$$

$$+ \sum_{i \neq j}^2 \frac{B_0 h_i^2 - B_1 h_i + B_2}{h_i - h_j} \frac{\partial}{\partial h_i} - C. \quad (4.7)$$
Let us first consider the canonical-form condition with respect to the second-order differential operators. Recalling the relations:

\[ e^{-W} \frac{\partial}{\partial q_i} e^W = \frac{1}{h_i'} \left( \frac{\partial^2}{\partial q_i^2} + \sum_{i,j} \frac{\partial W}{\partial q_i} \frac{\partial W}{\partial q_j} + \frac{\partial^2 W}{\partial q_i^2} \right), \quad (4.8a) \]

\[ e^{-W} \frac{\partial^2}{\partial h_1 \partial h_2} e^W = \frac{1}{(h_1')^2} \left( \frac{\partial^2}{\partial q_1^2} + \left( \frac{2}{h_1'} \frac{\partial W}{\partial q_1} - \frac{h_2''}{h_1'} \right) \frac{\partial}{\partial q_1} + \frac{\partial^2 W}{\partial q_1^2} + \left( \frac{\partial W}{\partial q_1} \right)^2 - \frac{h_2''}{h_1'} \frac{\partial W}{\partial q_1} \right), \quad (4.8b) \]

\[ e^{-W} \frac{\partial^2}{\partial h_i^2} e^W = \frac{1}{(h_i')^2} \left( \frac{\partial^2}{\partial q_i^2} + \left( \frac{2}{h_i'} \frac{\partial W}{\partial q_i} - \frac{h_2''}{h_i'} \right) \frac{\partial}{\partial q_i} + \frac{\partial^2 W}{\partial q_i^2} + \left( \frac{\partial W}{\partial q_i} \right)^2 - \frac{h_2''}{h_i'} \frac{\partial W}{\partial q_i} \right), \quad (4.8c) \]

and requiring the second-order differential operator to be gauge-transformed back to the Laplacian in the flat space, we obtain the following conditions:

\[ 2A_0 \sigma_2^2 - A_1 \sigma_2 \sigma_1 + A_2 \sigma_1^2 - 2(A_2 - A_{11}) \sigma_2 - A_{21} \sigma_1 + 2A_{22} = 0, \quad (4.9) \]
\[ A_0 h_i^2 - A_1 h_i^3 + (A_2 + A_{11}) h_i^2 - A_{21} h_i + A_{22} = \frac{1}{2}(h_1 - h_2)^2 (h_i')^2. \quad (4.10) \]

Once the above conditions satisfied, the gauged Hamiltonian is transformed to,

\[ e^{-W} \tilde{H}_N e^W = -\frac{1}{2} \sum_{i=1}^2 \left[ \frac{\partial^2}{\partial q_i^2} + \left( \frac{2}{h_i'} \frac{\partial W}{\partial q_i} - \frac{h_2''}{h_i'} \right) \frac{\partial}{\partial q_i} + \frac{\partial^2 W}{\partial q_i^2} + \left( \frac{\partial W}{\partial q_i} \right)^2 - \frac{h_2''}{h_i'} \frac{\partial W}{\partial q_i} \right] + \sum_{i \neq j} \frac{B_0 h_i^2 - B_1 h_i + B_2}{(h_i - h_j) h_i'} \left( \frac{\partial}{\partial q_i} + \frac{\partial W}{\partial q_i} \right) - C. \quad (4.11) \]

The condition that the above operator does not contain the first-order differential operator reads,

\[ \frac{\partial W}{\partial q_i} = \frac{h_i''}{2h_i'} + \frac{B_0 h_i^2 - B_1 h_i + B_2}{(h_i - h_j) h_i'} \quad (i \neq j). \quad (4.12) \]

Finally, with the condition (4.9), (4.10) and (4.12) satisfied, the original Hamiltonian takes the desirable Schrödinger form as,

\[ H_N = e^{-W} \tilde{H}_N e^W = -\frac{1}{2} \sum_{i=1}^2 \frac{\partial^2}{\partial q_i^2} + \frac{1}{2} \sum_{i=1}^2 \left[ \left( \frac{\partial W}{\partial q_i} \right)^2 - \frac{\partial^2 W}{\partial q_i^2} \right] - C(\sigma(h_i)). \quad (4.13) \]

In the following, we shall solve the obtained canonical-form conditions (4.9), (4.10) and (4.12) in order.

**B. The first condition**

The first condition (4.9) is an algebraic identity and thus is satisfied if and only if all the coefficients of the polynomial vanish, i.e.,

\[ A_{20,20} = 0, \quad 2A_{20,10} = A_{20,21}, \quad 2A_{10,10} = A_{20,11} - A_{20,22}, \]
\[ A_{10,11} = A_{20,12}, \quad A_{10,12} = 0, \quad A_{20,22} = A_{21,21}, \]
\[ 2A_{20,12} = 2A_{21,11} - A_{22,21}, \quad 2A_{10,12} = 2A_{11,11} - A_{22,12}, \quad A_{12,11} = 0, \quad (4.14) \]
\[ A_{21,01} = -A_{22,22}, \quad A_{22,01} = 2A_{11,01} + 2A_{22,12}, \quad A_{12,01} = 2A_{12,12}, \]
\[ A_{01,01} = -A_{22,02}, \quad A_{02,01} = 2A_{12,02}, \quad A_{02,02} = 0. \]
There are 27 parameters \( A_{\kappa\lambda,\mu
u} \) in Eq. (3.10) for \( M = 2 \). The above conditions reduce the number of these free parameters to 12.

C. The second condition

With the first condition (4.14) satisfied, the l.h.s. of the second condition (4.10) becomes,
\[
A_0 h_i^4 - A_1 h_i^3 + (A_2 + A_{11}) h_i^2 - A_{21} h_i + A_{22}
\]
\[
= (A_{20,10} \sigma_2 h_i^4 + A_{10,10} \sigma_1 h_i^4 - A_{20,22} \sigma_2 h_i^3 + A_{21,11} \sigma_2 h_i^2 + A_{11,11} \sigma_1 h_i^2
\]
\[- A_{22,22} \sigma_1 h_i^2 - A_{22,12} \sigma_2 - A_{22,02} h_i - A_{12,12} \sigma_1 - A_{12,02}) (h_i - h_j)
\]
\[- A_{20,12} h_i^3 (h_i + 2h_j) (h_i - h_j) \quad (i \neq j).
\]
(4.15)

The differential equation (4.10) has a solution if and only if Eq. (4.15) is divided by the factor \((h_1 - h_2)^2\). This factorization condition reads,
\[
A_{20,10} = 0, \quad A_{20,22} = 2A_{10,10}, \quad A_{21,11} = 3A_{20,12},
\]
\[
A_{22,22} = 2A_{11,11}, \quad A_{22,12} = A_{11,01}, \quad A_{22,02} = -2A_{12,12}, \quad A_{12,02} = 0,
\]
(4.16)

from which the number of the free parameters \( A_{\kappa\lambda,\mu
u} \) is further reduced to 5. Then, the second condition turns to be,
\[
\frac{1}{2} (h'_i)^2 = A_{10,10} h_i^4 - A_{20,12} h_i^3 + A_{11,11} h_i^2 + A_{11,01} h_i + A_{12,12}
\]
\[\equiv a_4 h_i^4 + a_3 h_i^3 + a_2 h_i^2 + a_1 h_i + a_0 \equiv P(h_i).
\]
(4.17)

Under the conditions (4.14) and (4.16), the polynomials \( A_\kappa \) and \( A_{kl} \) are expressed in terms of the five parameters \( a_p \) \( (p = 0, \ldots, 4) \) as follows:
\[
A_0(\sigma) = a_4 \sigma_1^2,
\]
(4.18a)
\[
A_1(\sigma) = 4a_4 \sigma_2 \sigma_1 - a_3 \sigma_1^2,
\]
(4.18b)
\[
A_2(\sigma) = 2a_4 \sigma_2^2 - a_3 \sigma_2 \sigma_1,
\]
(4.18c)
\[
A_{11}(\sigma) = 2a_4 \sigma_2^2 - 3a_3 \sigma_2 \sigma_1 + a_2 \sigma_1^2 - 2a_2 \sigma_2 + a_1 \sigma_1 + 2a_0,
\]
(4.18d)
\[
A_{21}(\sigma) = -4a_3 \sigma_2^2 + 2a_2 \sigma_2 \sigma_1 + 4a_1 \sigma_2 + 2a_0 \sigma_1,
\]
(4.18e)
\[
A_{22}(\sigma) = 2a_2 \sigma_2^2 + a_1 \sigma_2 \sigma_1 + a_0 \sigma_1^2 - 2a_0 \sigma_2.
\]
(4.18f)

D. The third condition

The third condition (4.12) is a set of partial differential equations with two variables. In order that it has a solution, we must impose the integrability condition:
\[
\frac{\partial \partial W}{\partial q_2 \partial q_1} = \frac{\partial \partial W}{\partial q_1 \partial q_2}.
\]
(4.19)

Using Eq. (4.12), we get,
\[
\frac{\partial \partial W}{\partial q_j \partial q_i} = \frac{\partial}{\partial q_j} \left[ h''_i \left\{ h'_i + \frac{B_0 h_i^2 - B_1 h_i + B_2}{(h_i - h_j)h'_i} \right\} \right]
\]
\[= \frac{h'_i \sum_{p=0}^4 a_p h_p^p}{h'_i (h_i - h_j)^2} \quad (i \neq j),
\]
(4.20)
where \( a_p \ (p = 0, \ldots, 4) \) are given by,
\[
\begin{align*}
\bar{a}_0 &= -B_{02}, & \bar{a}_1 &= B_{01} - 2B_{12}, \\
\bar{a}_2 &= 2B_{11} - B_{22}, & \bar{a}_3 &= (2\mathcal{N} + 3)a_3 - 2B_{10} + B_{21}, \\
\bar{a}_4 &= 2(\mathcal{N} - 2)a_4 - B_{20}.
\end{align*}
\]
(4.21)

Therefore the integrability condition (4.19) is equivalent to,
\[
\sum_{p=0}^{4} \bar{a}_p h_1^p = \sum_{p=0}^{4} \bar{a}_p h_2^p
\]
(4.22)

It is evident that the above relation holds if and only if both of the quantities are a constant, 
\(-2c\). Finally, the integrability condition reads,
\[
\bar{a}_p = -2ca_p \quad (p = 0, \ldots, 4).
\]
(4.23)

Combining Eqs. (4.21) and (4.23), we obtain,
\[
\begin{align*}
B_{02} &= 2ca_0, & B_{01} - 2B_{12} &= -2ca_1, \\
2B_{11} - B_{22} &= -2ca_2, & 2B_{10} - B_{21} &= (2\mathcal{N} + 3 + 2c)a_3, \\
B_{20} &= 2(\mathcal{N} - 2 + c)a_4,
\end{align*}
\]
(4.24)

from which the number of the free parameters \( B_{\kappa\lambda} \) reduces to 3. When the integrability condition is fulfilled, the third condition (4.12) turns to be,
\[
\frac{\partial \mathcal{W}}{\partial q_i} = \frac{h_i'}{2h_i} + \frac{1}{h_i} \left[ \frac{\mathcal{N} - 2 + c}{2} P'(h_i) - Q(h_i) \right] - \frac{ch_i'}{h_i - h_j} \quad (i \neq j),
\]
(4.25)

where \( Q(h_i) \) is defined by,
\[
\begin{align*}
Q(h_i) &\equiv \left( \frac{\mathcal{N} - 2 - c}{2} a_3 + B_{10} \right) h_i^2 + \left( (\mathcal{N} - 1 - c)a_2 - B_{11} \right) h_i + \frac{\mathcal{N} - 2 + c}{2} a_1 - B_{12} \\
&\equiv b_2 h_i^2 + b_1 h_i + b_0.
\end{align*}
\]
(4.26)

The parameters \( B_{\kappa\lambda} \) are expressed in terms of the new three parameters \( b_p \) and the five \( a_p \) as,
\[
\begin{align*}
B_{10} &= b_2 - \frac{\mathcal{N} - 2 - c}{2} a_3, & B_{20} &= 2(\mathcal{N} - 2 + c)a_4, \\
B_{11} &= -b_1 + (\mathcal{N} - 1 - c)a_2, & B_{21} &= 2b_2 - (3\mathcal{N} + 1 + c)a_3, \\
B_{12} &= -b_0 + \frac{\mathcal{N} - 2 + c}{2} a_1, & B_{22} &= -2b_1 + 2(\mathcal{N} - 1)a_2, \\
B_{01} &= -2b_0 + (\mathcal{N} - 2 - c)a_1, & B_{02} &= 2ca_0.
\end{align*}
\]
(4.27a-d)

We now can easily integrate the differential equation (4.23) to obtain,
\[
\mathcal{W}(q_1, q_2) = -\sum_{i=1}^{2} \int dh_i \frac{Q(h_i)}{2P(h_i)} + \frac{\mathcal{N} - 1 + c}{2} \sum_{i=1}^{2} \ln |h_i'| - c \ln |h_1 - h_2|,
\]
(4.28)

where the integral constant is omitted.
E. Summary

In summary, we have derived the canonical-form conditions for two-body case, Eqs. (4.9), (4.10) and (4.12), and have solved them to obtain Eqs. (4.17), (4.18) and (4.26)–(4.28). Substituting Eqs. (4.9), (4.10), (4.12) and (4.25) for Eq. (4.7), we see that the gauged Hamiltonian \(\tilde{H}_N\) which satisfies the canonical-form condition must have the following form:

\[
\tilde{H}_N(h) = -\sum_{i=1}^2 P(h_i) \frac{\partial^2}{\partial h_i^2} + \sum_{i=1}^2 \left[ \frac{N - 2 + c}{2} P'(h_i) - Q(h_i) \right] \frac{\partial}{\partial h_i} - 2c \sum_{i \neq j}^2 \frac{P(h_i)}{h_i - h_j} \frac{\partial}{\partial h_i} - C(\sigma(h)).
\]

In the above, \(C\) is calculated by substituting Eqs. (4.18a) and (4.27a) for Eq. (3.17e) and reads,

\[
C(\sigma(h)) = \frac{N - 1}{12} \left( N - 2 + 2c \right) \sum_{i=1}^2 P''(h_i) - \frac{N - 1}{2} \sum_{i=1}^2 Q'(h_i) - \frac{N - 1}{2} c \sum_{i \neq j}^2 \frac{P'(h_i)}{h_i - h_j} + R,
\]

where the constant \(R\) is given by,

\[
R = \frac{(N - 1)(N - 2 - c)}{3} a_2 + (N - 1)b_1 + C.
\]

V. \(GL(2, K)\) SHAPE INVARIANCE

It was shown that the one-body \(\mathfrak{su}(2)\) quasi-solvable models can be classified using the shape invariance of the Hamiltonian under the action of \(GL(2, K)\) \((K = \mathbb{R} \text{ or } \mathbb{C})\) of linear fractional transformations \([4, 5]\). We can see that the two-body Hamiltonian (4.29) also has the same property of shape invariance. The linear fractional transformation of \(h_i\) is introduced by,

\[
h_i \mapsto \hat{h}_i = \frac{\alpha h_i + \beta}{\gamma h_i + \delta} \quad (\alpha, \beta, \gamma, \delta \in K; \; \Delta \equiv \alpha \delta - \beta \gamma \neq 0).
\]

Then, it turns out that the Hamiltonian (4.29) is shape invariant under the following transformation (with \(M = 2\)) induced by Eq. (5.1),

\[
\tilde{H}_N(h) \mapsto \tilde{\tilde{H}}_N(h) = \prod_{i=1}^M (\gamma h_i + \delta)^{N-1} \tilde{H}_N(\hat{h}) \prod_{i=1}^M (\gamma h_i + \delta)^{-(N-1)},
\]

where the polynomials \(P(h_i)\) and \(Q(h_i)\) in the \(\tilde{H}_N(h)\) are transformed according to,

\[
P(h_i) \mapsto \hat{P}(h_i) = \Delta^{-2} (\gamma h_i + \delta)^4 P(\hat{h}_i),
\]
\[
Q(h_i) \mapsto \hat{Q}(h_i) = \Delta^{-1} (\gamma h_i + \delta)^2 Q(\hat{h}_i).
\]
This shape invariance originally comes from the $GL(2, K)$ invariance of the solvable subspace (3.6):

$$\tilde{\mathcal{V}}_N(h) \mapsto \tilde{\mathcal{V}}_N'(h) \equiv \text{span} \left\{ \prod_{i=1}^{M} (\gamma h_i + \delta)^{N-1} \phi : \phi \in \tilde{\mathcal{V}}_N(h) \right\} = \tilde{\mathcal{V}}_N(h),$$

(5.4)

where $\tilde{\mathcal{V}}_N(h)$ is given by,

$$\tilde{\mathcal{V}}_N(h) = \text{span} \left\{ \sigma_1(h)^n \cdots \sigma_M(h)^{n_M} : n_i \in \mathbb{Z}_{\geq 0}, 0 \leq \sum_{i=1}^{M} n_i \leq N - 1 \right\}.$$  (5.5)

From the invariance (5.4), the gauged Hamiltonian (3.16) must be invariant under the $GL(2, K)$ transformation (5.2) because it is constructed so that $\tilde{H}_N \tilde{\mathcal{V}}_N \subset \tilde{\mathcal{V}}_N$. Therefore, the gauged Hamiltonian (4.29) constitutes an invariant subspace of the $GL(2, K)$ invariant operators for $M = 2$ which satisfy the canonical-form condition.

From the above consideration, the $M$-body extension ($M > 2$) of the two-body gauged Hamiltonian (4.29) should have the same shape invariance property under the $GL(2, K)$ transformation (5.2). From the functional form of the two-body $\mathfrak{sl}(3)$ quasi-solvable system (4.29) and also from the form of the one-body $\mathfrak{sl}(2)$ quasi-solvable system (3.16), we can cast the $M$-body quasi-solvable gauged Hamiltonian which contains up to two-body interactions as,

$$\tilde{H}_N(h) = -\sum_{i=1}^{M} P(h_i) \frac{\partial^2}{\partial h_i^2} + \sum_{i=1}^{M} \left[ C_1(M) P'(h_i) - Q(h_i) \right] \frac{\partial}{\partial h_i} - 2c \sum_{i \neq j}^{M} \frac{P(h_i)}{h_i - h_j} \frac{\partial}{\partial h_i} - \left[ \frac{N - 1}{12} C_2(M) \sum_{i=1}^{M} P''(h_i) - \frac{N - 1}{2} C_3(M) \sum_{i \neq j}^{M} \frac{P'(h_i)}{h_i - h_j} + R \right],$$

(5.6)

where $C_1, C_2$ and $C_3$ satisfy the following initial conditions:

$$C_1(1) = \frac{N - 2}{2}, \quad C_2(1) = \frac{N - 2 + c}{2},$$

(5.7a)

$$C_2(1) = N - 2, \quad C_2(2) = N - 2 + 2c,$$

(5.7b)

$$C_3(2) = c.$$  (5.7c)

If we make the $GL(2, K)$ transformation on the above $\tilde{H}_N$ according to Eq. (5.2), we obtain,

$$\tilde{H}_N'(h) = -\sum_{i=1}^{M} \hat{P}(h_i) \frac{\partial^2}{\partial h_i^2} + \sum_{i=1}^{M} \left[ C_1(M) \hat{P}_1(h_i) - \hat{Q}(h_i) \right] \frac{\partial}{\partial h_i} - 2c \sum_{i \neq j}^{M} \frac{\hat{P}(h_i)}{h_i - h_j} \frac{\partial}{\partial h_i} - \left[ \frac{N - 1}{12} C_2(M) \sum_{i=1}^{M} \hat{P}_2(h_i) - \frac{N - 1}{2} C_3(M) \sum_{i \neq j}^{M} \frac{\hat{P}_3(h_i)}{h_i - h_j} + R \right],$$

(5.8)
where $P_1$, $P_2$ and $P_3$ are given by,

\[
P_1(h_i) = \Delta^{-1}(\gamma h_i + \delta)^2 P'(\hat{h}_i) + 2 \frac{\mathcal{N} - 2 + (M - 1)c}{C_1(M)} \Delta^{-2} \gamma (\gamma h_i + \delta)^3 P(\hat{h}_i),
\]

\[
P_2(h_i) = P''(\hat{h}_i) + 6 \frac{2C_1(M) + (M - 1)C_3(M)}{C_2(M)} \Delta^{-1} \gamma (\gamma h_i + \delta) P'(\hat{h}_i)
\]
\[+ 12 \frac{\mathcal{N} - 2 + 2(M - 1)c}{C_2(M)} \Delta^{-2} \gamma (\gamma h_i + \delta)^2 P(\hat{h}_i),
\]

\[
P_3(h_i) = \Delta^{-1}(\gamma h_i + \delta)^2 P'(\hat{h}_i) + \frac{4c}{C_3(M)} \Delta^{-2} \gamma (\gamma h_i + \delta)^3 P(\hat{h}_i).
\]

(5.9a)

(5.9b)

(5.9c)

Useful formulae needed in the above calculation are summarized in Appendix A. Hence, the requirement that the $M$-body gauged Hamiltonian is shape invariant under Eqs. (5.2) and (5.3) reads,

\[
P_1(h_i) = P_3(h_i) = \hat{P}'(h_i), \quad P_2(h_i) = \hat{P}''(h_i).
\]

(5.10)

Comparing Eqs. (5.9) with (A5), we obtain conditions for the invariance:

\[
C_1(M) = \frac{\mathcal{N} - 2 + (M - 1)c}{2}, \quad C_2(M) = \mathcal{N} - 2 + 2(M - 1)c, \quad C_3(M) = c,
\]

(5.11)

which in turn satisfy the required initial conditions (5.7). Thus, the requirement of the $GL(2,\mathcal{K})$ invariance uniquely determines the $M$-dependence of the coupling constants as was mentioned before. Finally, we obtain,

\[
\hat{H}_{\mathcal{N}}(h) = - \sum_{i=1}^{M} P'(h_i) \frac{\partial^2}{\partial h_i^2} + \sum_{i=1}^{M} \left[ \mathcal{N} - 2 + (M - 1)c \frac{P'(h_i) - Q(h_i)}{2} \right] \frac{\partial}{\partial h_i}
\]
\[- 2c \sum_{i \neq j}^{M} \frac{P(h_i)}{h_i - h_j} \frac{\partial}{\partial h_i} - C(\sigma(h)),
\]

(5.12)

where $C$ is given by,

\[
C(\sigma(h)) = \frac{\mathcal{N} - 1}{12} [\mathcal{N} - 2 + 2(M - 1)c] \sum_{i=1}^{M} P''(h_i)
\]
\[- \frac{\mathcal{N} - 1}{2} \sum_{i=1}^{M} Q'(h_i) - \frac{\mathcal{N} - 1}{2} c \sum_{i \neq j}^{M} \frac{P'(h_i)}{h_i - h_j} + R.
\]

(5.13)

The $P$ and $Q$ in Eqs. (5.12) and (5.13) are again a fourth- and a second-degree polynomial, respectively:

\[
P(h_i) = a_4 h_i^4 + a_3 h_i^3 + a_2 h_i^2 + a_1 h_i + a_0,
\]

(5.14a)

\[
Q(h_i) = b_2 h_i^2 + b_1 h_i + b_0.
\]

(5.14b)

There are 10 parameters, namely, $a_p$ ($p = 0, \ldots, 4$), $b_p$ ($p = 0, 1, 2$), $c$, $R$, which characterize the quasi-solvable Hamiltonian (5.12). The function $h(q)$, which determines the change of variables, is given by a solution of the differential equation,

\[
h'(q)^2 = 2P(h(q)).
\]

(5.15)
If we transform back the gauged Hamiltonian (5.12) with \( h(q) \) satisfying Eq. (5.15), the original Hamiltonian becomes the Schrödinger type,

\[
H_N = -\frac{1}{2} \sum_{i=1}^{M} \frac{\partial^2}{\partial q_i^2} + \frac{1}{2} \sum_{i=1}^{M} \left[ \left( \frac{\partial W(q)}{\partial q_i} \right)^2 - \frac{\partial^2 W(q)}{\partial q_i^2} \right] - C(\sigma(h)),
\]

and the superpotential \( W(q) \) is given by,

\[
W(q) = -\sum_{i=1}^{M} \int d h_i Q(h_i) + \frac{N - 1 + (M - 1) c}{2} \sum_{i=1}^{M} \ln | h_i' | - c \sum_{i<j}^{M} \ln | h_i - h_j |. 
\]

By construction, the elements of the vector space (3.9) give the solvable wave functions \( \psi(q) \) of the Hamiltonian (5.16):

\[
\psi(q) \in \text{span} \left\{ \sigma_1^{n_1} \cdots \sigma_M^{n_M} e^{-W(q)} : n_i \in \mathbb{Z}_{\geq 0}, 0 \leq \sum_{i=1}^{M} n_i \leq N - 1 \right\}. 
\]

From the form of the Hamiltonian (5.16), we can observe an interesting feature of the (quasi-)solvability. As we have examined in the end of Section III, the system is solvable if Eq. (3.21) is fulfilled. The resulting equation (3.23) together with the form of the Hamiltonian (5.16) indicate that the system becomes of supersymmetric form (except for the irrelevant constant term \(-C\)) \[48, 49\] if it is not only quasi-solvable but also solvable. A system is always quasi-solvable if it is supersymmetric, since the ground state is always solvable. From the above result, we can conclude that a system is always supersymmetric if it is solvable and all its states have the form (5.18). We note that this interesting feature does not depend on the normalizability of the solvable sector, that is, it holds whether or not supersymmetry is dynamically broken.

VI. INVERSE PROBLEM

In the preceding section, we have derived the \( M \)-body Hamiltonian containing up to the two-body interactions (5.12), without directly solving the canonical-form condition for \( M > 2 \). To confirm that the Hamiltonian (5.12) is indeed quasi-solvable, i.e., it preserves the vector space (3.6) for any integer \( M \), we shall see in this section that it can be surely written in terms of the elementary symmetric polynomials \( \sigma_i \) as the form of Eq. (3.16) with the set of the specific form of the polynomials (3.17). In order to express the Hamiltonian
In terms of $\sigma_i$, we need to know the following transformation laws from $h_i$ to $\sigma_i$: 

\[
\sum_{i=1}^{M} h_i^p \frac{\partial^2}{\partial h_i^2} = \sum_{k,l=1}^{M} A_{k,l}^{(p)}(\sigma) \frac{\partial^2}{\partial \sigma_k \partial \sigma_l} \quad (p = 0, \ldots, 4), \tag{6.1a}
\]

\[
\sum_{i=1}^{M} h_i^p \frac{\partial}{\partial h_i} = \sum_{k=1}^{M} B_k^{(p)}(\sigma) \frac{\partial}{\partial \sigma_k} \quad (p = 0, \ldots, 3), \tag{6.1b}
\]

\[
\sum_{i=1}^{M} h_i^p = B^{(p)}(\sigma) \quad (p = 0, 1, 2), \tag{6.1c}
\]

\[
\sum_{i \neq j}^{M} \frac{h_i^p}{h_i - h_j} \frac{\partial}{\partial h_i} = \sum_{k=1}^{M} C_k^{(p)}(\sigma) \frac{\partial}{\partial \sigma_k} \quad (p = 0, \ldots, 4), \tag{6.1d}
\]

\[
\sum_{i \neq j}^{M} \frac{h_i^p}{h_i - h_j} = C^{(p)}(\sigma) \quad (p = 0, \ldots, 3). \tag{6.1e}
\]

Substituting the above for Eqs. (5.12) and (5.13), we rewrite the gauged Hamiltonian as,

\[
\tilde{H}_N = -\sum_{k,l=1}^{M} \left( \sum_{p=0}^{4} a_p A_{k,l}^{(p)} \right) \frac{\partial^2}{\partial \sigma_k \partial \sigma_l} - \sum_{k=1}^{M} \left[ \sum_{p=0}^{2} b_p B_k^{(p)} \right] \frac{\partial}{\partial \sigma_k} - C(\sigma), \tag{6.2}
\]

with,

\[
C(\sigma) = \frac{N - 1}{12} \left[ N - 2 + 2(M - 1)c \right] \sum_{p=0}^{2} (p + 2)(p + 1) a_{p+1} B^{(p)}_{-} + 2c \sum_{p=0}^{4} a_p C_k^{(p)} \frac{\partial}{\partial \sigma_k} - C(\sigma). \tag{6.3}
\]

In Appendix B, we give the explicit procedure to obtain the transformation formulae (6.1).

Let us begin with investigating $C(\sigma)$. Applying the formulae (B9) and (B18) to Eq. (6.3), we obtain,

\[
C(\sigma) = (N - 1)(N - 2)a_4 \sigma_1^2 - 2(N - 1)(N - 2 + c)a_4 \sigma_2
\]

\[
- (N - 1) \left( b_2 - \frac{N - 2 - (M - 1)c}{2} a_3 \right) \sigma_1
\]

\[
+ (N - 1) \frac{N - 2 - (M - 1)c}{6} Ma_2 - \frac{N - 1}{2} Mb_1 + R. \tag{6.4}
\]

Comparing the result (6.4) with Eq. (3.17e), we can easily see that the $C(\sigma)$ in the Hamil-
tonian (5.12) is indeed of the form (3.17e) with,

\[ A_0(\sigma) = a_4\sigma^2, \]

\[ \sum_{i=1}^{M} B_{i0}\sigma_i = 2(N - 2 + c)a_4\sigma_2 + \left( b_2 - \frac{N - 2 - (M - 1)c}{2}a_3 \right)\sigma_1. \]  

(6.6)

\[ C = (N - 1)\frac{N - 2 - (M - 1)c}{6}Ma_2 - \frac{N - 1}{2}Mb_1 + R. \]  

(6.7)

These results are in agreement, in the case of \( M = 2 \), with those obtained in Section IV, namely, Eqs. (4.18a), (4.27a) and (4.31).

In the next, we will examine the structure of the second-order derivative operator. Applying the formulae (B37) to the first term of Eq. (6.2), we obtain,

\[ \sum_{p=0}^{4} a_p A_{k,l}^{(p)} = a_4\sigma^2_1\sigma_{k}\sigma_l - \left[ (2a_4\sigma_2 - a_3\sigma_1)\sigma_k + a_4\sigma_1\sigma_{k+1} \right] \sigma_l \]

\[ - a_4\sigma_1\sigma_k\sigma_{l+1} + A_{kl}(\sigma), \]  

(6.8)

where \( A_{kl}(\sigma) \) represents the part involving up to the quadratic terms in \( \sigma_i \) and is given by,

\[ A_{kl}(\sigma) = a_4 \sum_{m=0}^{2} (k - m + 2)\sigma_{k-m+2}\sigma_{l+m} + \sum_{m=3}^{k+2} (k - l - 2m + 2)\sigma_{k-m+2}\sigma_{l+m} \]

\[ - a_3 \sum_{m=0}^{1} (k - m + 1)\sigma_{k-m+1}\sigma_{l+m} + \sum_{m=3}^{k+1} (k - l - 2m + 1)\sigma_{k-m+1}\sigma_{l+m} \]

\[ + a_2 \left[ k\sigma_k\sigma_l + \sum_{m=1}^{k} (k - l - 2m)\sigma_{k-m}\sigma_{l+m} \right] - a_1 \sum_{m=0}^{k-1} (k - l - 2m - 1)\sigma_{k-m-1}\sigma_{l+m} \]

\[ + a_0 \left[ (M - l + 1)\sigma_{k-1}\sigma_{l-1} + \sum_{m=0}^{k-2} (k - l - 2m - 2)\sigma_{k-m-2}\sigma_{l+m} \right]. \]  

(6.9)

In Eq. (6.9), we note that \( \sigma_0 = 1 \) and \( \sigma_{-k} = 0 \) for \( k > 0 \). Comparing Eq. (6.8) with Eq. (3.16) and (3.17a)–(3.17c), we can observe that the second-order derivative operator in Eq. (5.12) are expressed in the form of the first line in Eq. (3.16) with the substitution (6.5) and

\[ A_k(\sigma) = (2a_4\sigma_2 - a_3\sigma_1)\sigma_k + 2a_4\sigma_1\sigma_{k+1}, \]  

(6.10)

\[ A_{kk}(\sigma) = A_{kk}(\sigma), \]  

(6.11)

\[ A_{kl}(\sigma) = A_{kl}(\sigma) + A_{lk}(\sigma) \quad (k > l). \]  

(6.12)

In the case of \( M = 2 \), these results are also in agreement with Eqs. (4.18) obtained in Section IV.

Finally, if we apply the formulae (B15) and (B28) to the second term of Eq. (6.2), we can find that the first-order operator in the gauged Hamiltonian (5.12) has the form of the
first term in the second line of Eq. (3.16) with the following relation:

\[ \sum_{i \geq j \geq k} M A_{ij,jk} \sigma_i - \sum_{i=1}^{M} B_{ik} \sigma_i - B_{0k} = -(k + 2) \left[ 2(N - 2) + (k + 1)c \right] a_4 \sigma_{k+2} \]

\[- (k + 1) \left( b_2 - \frac{3(N - 2) + (2k - M + 1)c}{2} a_3 \right) \sigma_{k+1} \]

\[+ k \left( b_1 - \left[ N - 2 + (k - M)c \right] a_2 \right) \sigma_k \]

\[- (k - M - 1) \left( b_0 - \frac{N - 2 + (2k - M - 1)c}{2} a_1 \right) \sigma_{k-1} \]

\[- (k - M - 1) (k - M - 2) c a_0 \sigma_{k-2}. \] (6.13)

From this relation, we can determine \( B_{ik} \) and \( B_{0k} \) uniquely since the coefficients \( A_{ij,jk} \) in the l.h.s. of Eq. (6.13) are fixed by Eqs. (3.17b) and (6.11)–(6.12). In the case of \( M = 2 \), we can again check that \( B_{ik} \) and \( B_{0k} \) calculated by Eq. (6.13) are in complete agreement with Eqs. (4.27b)–(4.27d) derived in Section IV.

Summarizing the analyses, we have shown that the \( M \)-body gauged Hamiltonian (5.12) can be written in terms of \( \sigma \) as the form of Eq. (3.16), which ensures the quasi-solvability with the solvable sector (3.6). In addition, we can find out a reason why the form of the quasi-solvable gauged Hamiltonian should be restricted to Eq. (5.12). For example, the formulae (B9) and (B11) indicate,

\[ \sum_{i=1}^{M} h_i p_i \frac{\partial}{\partial h_i} \sim \sum_{k=1}^{M} \sigma_i^{p-1} \sigma_k \frac{\partial}{\partial \sigma_k} + \cdots. \] (6.14)

From the fact that \( B_0(\sigma) \) in Eq. (3.16) is a polynomial of at most second-degree, the highest degree term in the r.h.s. of Eq. (6.14) can be fit in the form of Eq. (3.16) only if \( p \leq 3 \). This explains why the coefficient of the first-order derivative in Eq. (5.12) must be a polynomial of at most third-degree in \( h_i \). Similar observation on the other parts of the operators can also explain the structure of Eq. (5.12).

Finally, we will return to the condition for the solvability. As we have examined in Section III, the operator (3.16) becomes solvable when the condition (3.21) is satisfied. Comparing Eq. (4.21) with Eqs. (6.5), (6.6) and (6.10), we obtain the solvability condition in terms of the free parameters:

\[ a_3 = a_4 = b_2 = 0. \] (6.15)

Since the system has the \( GL(2, K) \) shape invariance (5.2), it is solvable if the polynomials \( P \) and \( Q \) can be transformed according to Eqs. (5.3) so that the conditions (6.15) are satisfied. In the next section, we will see what kind of solvable models as well as quasi-solvable models can be constructed from the general form of the Hamiltonian (5.16).

VII. CLASSIFICATION OF THE MODELS

For a given \( P(h) \), the function \( h(q) \) is determined by Eq. (5.15) and a particular model is obtained by substituting this \( h(q) \) for Eqs. (5.13), (5.16) and (5.17). The solution of
Eq. (5.15) is easily obtained as an elliptic integral:

\[ |q| = \int \frac{dh}{\sqrt{2P(h)}}. \]  

(7.1)

We note that the above solution has the reflection and translational symmetry; if \( h(q) \) is a solution of Eq. (5.15), \( h(-q) \) and \( h(q + q_0) \) also satisfy Eq. (5.15). Furthermore, this elliptic integral also has \( GL(2, K) \) invariance discussed in Section V:

\[
\int \frac{dh}{\sqrt{2P(h)}} \mapsto \int \frac{d\hat{h}}{\sqrt{2P(\hat{h})}} = \int \frac{d\hat{h}}{\sqrt{2P(\hat{h})}} \Delta \left( \frac{1}{2} \right) \frac{1}{\sqrt{2P(\hat{h})}}
\]  

(7.2)

The elliptic integral (7.1) is classified according to the distribution of the zeros of \( P(h) \), e.g., multiplicity of the zeros. Since the transformation (5.3a) induced by Eq. (5.1) does not alter the distribution, the shape invariance (7.2) together with (5.2) enable us to classify the resultant quasi-solvable models (5.16) according to the distribution of the roots of \( P(h) \).

This idea was first introduced in Ref. [4] to classify the one-body \( sl(2) \) quasi-solvable models.

Under the transformation (5.3a) of \( GL(2, \mathbb{R}) \) or \( GL(2, \mathbb{C}) \), every quartic polynomial \( P(h) \) with real or complex coefficients is equivalent to one of the eight or five forms, respectively, shown in Table I. In Table I \( \nu, g_2, g_3 \in K \) according to the transformation group \( GL(2, K) \),

| Case | \( GL(2, \mathbb{R}) \) | \( GL(2, \mathbb{C}) \) |
|------|-----------------|-----------------|
| I    | 1/2             | 1/2             |
| II   | 2h              | 2h              |
| III  | \( 2\nu h^2 \)  | \( 2\nu h^2 \)  |
| III' | \( \nu(h^2 + 1)^2/2 \) | \( 2\nu(h^2 - 1) \) |
| IV   | \( 2\nu(h^2 - 1) \) | \( 2\nu(h^2 + 1) \) |
| IV'  | \( h^3 - g_2 h/2 - g_3/2 \) | \( 2h^3 - g_2 h/2 - g_3/2 \) |
| V    | \( \nu(h^2 + 1) [(1 - k^2) h^2 + 1]/2 \) | \( \nu(h^2 + 1) [(1 - k^2) h^2 + 1]/2 \) |
| V'   | \( \nu(h^2 + 1) [(1 - k^2) h^2 + 1]/2 \) | \( \nu(h^2 + 1) [(1 - k^2) h^2 + 1]/2 \) |

TABLE I: The representatives of \( P(h) \) under the \( GL(2, \mathbb{R}) \) and \( GL(2, \mathbb{C}) \) transformations.

and \( \nu \neq 0, 0 < k < 1, g_2^3 - 27g_3^2 \neq 0 \). Since case III' under the \( GL(2, \mathbb{R}) \) can be obtained from case III by a \( GL(2, \mathbb{C}) \) transformation, the potential in case III' is regarded as another real function representation of the complex potential real-valued on \( \mathbb{R}^M \) in case III. The same relation holds between case IV and IV', and between case V and V', respectively. The functional form of the potential is obtained by substituting the superpotential (5.17) for
Eq. (5.16), which reads (up to a constant term),

\[
V(q) = \frac{1}{2} \sum_{i=1}^{M} \left( \frac{\partial W}{\partial q_i} \right)^2 - \left( \frac{\partial^2 W}{\partial q_i^2} \right) - C(\sigma(h))
\]

\[
= \sum_{i=1}^{M} \frac{1}{4P(h_i)} \left[ \frac{N - 1 + (M - 1)c}{2} P'(h_i) - Q(h_i) \right] \times
\]

\[
\times \left[ \frac{N + 1 + (M - 1)c}{2} P'(h_i) - Q(h_i) \right] - a_4(M, \mathcal{N}) \sum_{i=1}^{M} h_i^2 - a_3(M, \mathcal{N}) \sum_{i=1}^{M} h_i
\]

\[
+ c(c - 1) \sum_{i<j}^{M} \left[ \frac{P(h_i) + P(h_j)}{(h_i - h_j)^2} - 2a_4h_ih_j \right],
\]

(7.3)

where the coupling constants \(a_4(M, \mathcal{N})\) and \(a_3(M, \mathcal{N})\) are given by,

\[
a_4(M, \mathcal{N}) = \left[ \mathcal{N}^2 + 2\mathcal{N}(M - 1)c + M(M - 1)c^2 - (M - 1)c - 1 \right] a_4, \quad (7.4a)
\]

\[
a_3(M, \mathcal{N}) = \left[ \mathcal{N}^2 + 2\mathcal{N}(M - 1)c + (M^2 - 1)c^2 - 2(M - 1)c - 1 \right] \frac{a_3}{2}
\]

\[- \left[ \mathcal{N} + (M - 1)c \right] b_2. \quad (7.4b)
\]

For each case in Table I, we have specific values of the parameters \(\nu\) and the transformation function \(h(q)\) obtained by Eq. (7.1). Substituting all of them for Eq. (7.3), we obtain a quasi-solvable potential in terms of \(q_i\) in each the case. Due to the double poles at \(h_i = h_j (i \neq j)\) in the last term of Eq. (7.3), the two-body potentials in all the cases have singularities at \(q_i = q_j (i \neq j)\). Hence, each of the models is naturally defined on a Weyl chamber,

\[
(0 <) q_M < \cdots < q_1 < \infty,
\]

(7.5)

if the potential is non-periodic on \(\mathbb{R}^M\), or on a Weyl alcove,

\[
0 < q_M < \cdots < q_1 < \Omega,
\]

(7.6)

if the potential is periodic on \(\mathbb{R}^M\) with a real period \(\Omega\). As we will see below, case I, II, III and IV(\(\nu\)) with \(\nu > 0\), and III′ with \(\nu < 0\) correspond to the former while the other cases, namely, case III and IV(\(\nu\)) with \(\nu < 0\), III′ with \(\nu > 0\), and V(\(\nu\)) correspond to the latter. In the latter case, a system can be quasi-exactly solvable unless a pole of a one-body potential in the system exists and is in the Weyl alcove. On the other hand, quasi-exact solvability in the former case depends mainly on the asymptotic behavior of Eq. (5.18) at \(|q_i| \to \infty\). In both the cases, finiteness of the contribution from \(q_i \sim q_j (i \neq j)\) to the norm of the wave functions (5.18) requires \(c > -1/2\).

In the following, we will show the functional forms of both the \(W(q)\) and \(V(q)\) without irrelevant constant terms.

### A. Case I

In this case, \(a_0 = 1/2\) and \(a_1 = a_2 = a_3 = a_4 = 0\). From Eq. (7.1), we have,

\[
h(q) = q.
\]

(7.7)
The $W(q)$ and $V(q)$ are calculated as follows:

$$W(q) = \sum_{i=1}^{M} \left( -\frac{b_2}{3} q_i^3 - \frac{b_1}{2} q_i^2 - b_0 \sum_{i=1}^{M} q_i - c \sum_{i<j}^{M} \ln |q_i - q_j| \right), \quad (7.8)$$

$$V(q) = \sum_{i=1}^{M} \left( \frac{1}{2} \left( b_2 q_i^2 + b_1 q_i + b_0 \right)^2 + \left[ N + (M - 1)c \right] b_2 \sum_{i=1}^{M} q_i ight) + c(c - 1) \sum_{i<j}^{M} \frac{1}{(q_i - q_j)^2}. \quad (7.9)$$

This is the rational $A$ type Inozemtsev model \cite{51, 52, 53}. Inozemtsev models are known as a family of deformed CS models which preserve the classical integrability. The main difference between quantum and classical case is that the quantum quasi-solvability holds only for quantized values of the parameters, namely, for integer values of $M$ and $N$, while the classical integrability holds for any continuous values. This is one of the common features quantum quasi-solvable models share. The quantum solvability is realized when $b_2 = 0$. In this case, the model reduces to the rational $A$ type CS model with the harmonic frequency $b_1$. The parameter $b_0$ corresponds to the freedom of the translational invariance and is irrelevant.

**B. Case II**

In this case, $a_1 = 2$ and $a_0 = a_2 = a_3 = a_4 = 0$. From Eq. (7.1), we have,

$$h(q) = q^2. \quad (7.10)$$

The $W(q)$ and $V(q)$ are calculated as follows:

$$W(q) = \sum_{i=1}^{M} \left( \frac{b_2}{8} q_i^4 - \frac{b_1}{4} q_i^2 - c_1 \sum_{i=1}^{M} \ln |q_i| - c \sum_{i<j}^{M} \ln |q_i^2 - q_j^2| \right), \quad (7.11)$$

$$V(q) = \frac{1}{8} \sum_{i=1}^{M} q_i^2 \left( b_2 q_i^2 + b_1 q_i \right) + \frac{1}{4} \left[ 4N + 4(M - 1)c + 2c_1 - 1 \right] b_2 \sum_{i=1}^{M} q_i^2 + c_1(c_1 - 1) \sum_{i<j}^{M} \left[ \frac{1}{(q_i - q_j)^2} + \frac{1}{(q_i + q_j)^2} \right]. \quad (7.12)$$

The parameter $c_1$ is introduced according to,

$$2c_1 = b_0 - N + 1 - (M - 1)c. \quad (7.13)$$

This is the rational $BC$ type Inozemtsev model. The quasi-exactly solvable model reported in Ref. \cite{45} is just this case. The solvability is realized when $b_2 = 0$. In this case, the model reduces to the rational $BC$ type CS model with the harmonic frequency $b_1/2$. 

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C. Case III

In this case, \( a_2 = 2\nu \) and \( a_0 = a_1 = a_3 = a_4 = 0 \). From Eq. (7.1), we have,

\[
h(q) = e^{2\sqrt{\nu}q}. \tag{7.14}
\]

The \( \mathcal{W}(q) \) and \( V(q) \) are calculated as follows:

\[
\mathcal{W}(q) = -\frac{b_2}{4\nu} \sum_{i=1}^{M} e^{2\sqrt{\nu}q_i} + \frac{b_0}{4\nu} \sum_{i=1}^{M} e^{-2\sqrt{\nu}q_i} - \sqrt{\nu}(\bar{b}_1 - \mathcal{N} + 1) \sum_{i=1}^{M} q_i
\]
\[
- c \sum_{i<j} M \ln |\sinh \sqrt{\nu}(q_i - q_j)|,\tag{7.15}
\]

\[
V(q) = \frac{b_2^2}{8\nu} \sum_{i=1}^{M} e^{4\sqrt{\nu}q_i} + \frac{b_2}{2}[\bar{b}_1 + \mathcal{N} + (M - 1)c] \sum_{i=1}^{M} e^{2\sqrt{\nu}q_i}
\]
\[
+ \frac{b_0^2}{8\nu} \sum_{i=1}^{M} e^{-4\sqrt{\nu}q_i} + \frac{b_0}{2}[\bar{b}_1 + \mathcal{N} - (M - 1)c] \sum_{i=1}^{M} e^{-2\sqrt{\nu}q_i}
\]
\[
+ \nu c(c - 1) \sum_{i<j} M \frac{1}{\sinh^2 \sqrt{\nu}(q_i - q_j)}. \tag{7.16}
\]

This is the hyperbolic (\( \nu > 0 \)) and trigonometric (\( \nu < 0 \)) A type Inozemtsev model. The solvability is realized when \( b_2 = 0 \). In this case, the model reduces to the hyperbolic (trigonometric) A type CS model in the external Morse potential. The parameter \( \bar{b}_1 = b_1 / 2\nu \) is related to the translational invariance of the model and is irrelevant.

D. Case III'

In this case, \( a_0 = \nu / 2, a_2 = \nu, a_4 = \nu / 2 \) and \( a_1 = a_3 = 0 \). From Eq. (7.1), we have,

\[
h(q) = \tan \sqrt{\nu}q. \tag{7.17}
\]

The \( \mathcal{W}(q) \) and \( V(q) \) are calculated as follows:

\[
\mathcal{W}(q) = -2\sqrt{\nu}c_3 \sum_{i=1}^{M} q_i - c_2 \sum_{i=1}^{M} \sin 2\sqrt{\nu}q_i - c_1 \sum_{i=1}^{M} \cos 2\sqrt{\nu}q_i
\]
\[
- (\mathcal{N} - 1) \sum_{i=1}^{M} \ln |\cos \sqrt{\nu}q_i| - c \sum_{i<j} \ln |\sin \sqrt{\nu}(q_i - q_j)|, \tag{7.18}
\]
\[ V(q) = (c_2^2 - c_1^2) \nu \sum_{i=1}^{M} \cos 4\sqrt{\nu}q_i + 2\nu \left\{ 2c_2c_3 - [N + (M - 1)c] c_1 \right\} \sum_{i=1}^{M} \cos 2\sqrt{\nu}q_i \]
\[ - 2c_1c_2\nu \sum_{i=1}^{M} \sin 4\sqrt{\nu}q_i - 2\nu \left\{ 2c_1c_3 + [N + (M - 1)c] c_2 \right\} \sum_{i=1}^{M} \sin 2\sqrt{\nu}q_i \]
\[ + \nu c(c - 1) \sum_{i<j}^{M} \frac{1}{\sin^2 \sqrt{\nu}(q_i - q_j)}. \]

(7.19)

The parameters are introduced according to,
\[ c_1 = -\frac{b_1}{4\nu}, \quad c_2 = -\frac{b_2 - b_0}{4\nu}, \quad c_3 = \frac{b_2 + b_0}{4\nu}. \]

(7.20)

The solvability is realized when \( c_2 = c_3 \). The parameter \( c_3 \) is related to the translational invariance of the model and is irrelevant. As was mentioned before, this case gives another real function representation of case III.

E. Case IV

In this case, \( a_0 = -2\nu, a_2 = 2\nu \) and \( a_1 = a_3 = a_4 = 0 \). From Eq. (7.1), we have,
\[ h(q) = \cosh 2\sqrt{\nu}q. \]

(7.21)

The \( W(q) \) and \( V(q) \) are calculated as follows:
\[ W(q) = -c_3 \sum_{i=1}^{M} \cosh 2\sqrt{\nu}q_i - c_2 \sum_{i=1}^{M} \ln \left| \sinh 2\sqrt{\nu}q_i \right| - c_1 \sum_{i=1}^{M} \ln \left| \sinh \sqrt{\nu}q_i \right| \]
\[ - c \sum_{i<j}^{M} \ln \left| \sinh \sqrt{\nu}(q_i - q_j) \sinh \sqrt{\nu}(q_i + q_j) \right|, \]

(7.22)

\[ V(q) = 2\nu c_3 \sum_{i=1}^{M} \sinh^2 2\sqrt{\nu}q_i + 4\nu c_3 \left[ c_1 + 2c_2 + 2(M - 1)c \right. \]
\[ + 2N - 1 \right] \sum_{i=1}^{M} \sinh^2 \sqrt{\nu}q_i + 2\nu c_2(c_2 - 1) \sum_{i=1}^{M} \frac{1}{\sinh^2 \sqrt{\nu}q_i} \]
\[ + \frac{\nu}{2} c_1(c_1 + 2c_2 - 1) \sum_{i=1}^{M} \frac{1}{\sinh^2 \sqrt{\nu}q_i} \]
\[ + \nu c(c - 1) \sum_{i<j}^{M} \left[ \frac{1}{\sinh^2 \sqrt{\nu}(q_i - q_j)} + \frac{1}{\sinh^2 \sqrt{\nu}(q_i + q_j)} \right]. \]

(7.23)

The parameters \( c_i \) are introduced according to,
\[ c_1 = \frac{b_0 + b_2}{2\nu}, \quad c_2 = \frac{b_1 - b_0 - b_2}{4\nu} - \frac{1}{2} [N - 1 + (M - 1)c], \quad c_3 = \frac{b_2}{4\nu}. \]

(7.24)
This is the hyperbolic \((\nu > 0)\) and trigonometric \((\nu < 0)\) BC type Inozemtsev model. The solvability is realized when \(b_2 = 0\), that is, \(c_3 = 0\). In this case, the model reduces to the hyperbolic (trigonometric) BC type CS model. The quasi-solvable models constructed by an ansatz method in Ref. [54] are all included in case I, II, III, IV and IV'.

**F. Case IV'**

In this case, \(a_0 = 2\nu\), \(a_2 = 2\nu\) and \(a_1 = a_3 = a_4 = 0\). From Eq. (7.4), we have,

\[
h(q) = \sinh 2\sqrt{\nu}q. \tag{7.25}
\]

The \(W(q)\) and \(V(q)\) are calculated as follows:

\[
W(q) = -c_3 \sum_{i=1}^{M} \sinh 2\sqrt{\nu}q_i - c_2 \sum_{i=1}^{M} \text{gd} 2\sqrt{\nu}q_i - c_1 \sum_{i=1}^{M} \ln |\cosh 2\sqrt{\nu}q_i|
- c \sum_{i<j}^{M} \ln |\sinh \sqrt{\nu}(q_i - q_j) \cosh \sqrt{\nu}(q_i + q_j)|, \tag{7.26}
\]

\[
V(q) = 2\nu c_3^2 \sum_{i=1}^{M} \cosh^2 2\sqrt{\nu}q_i + 2\nu c_3 [2c_1 + 2(M-1)c + 2N - 1] \sum_{i=1}^{M} \sinh 2\sqrt{\nu}q_i
+ 2\nu c_2 (2c_1 - 1) \sum_{i=1}^{M} \sinh 2\sqrt{\nu}q_i \cosh^2 2\sqrt{\nu}q_i - 2\nu [c_2 - c_1 (c_1 - 1)] \sum_{i=1}^{M} \frac{1}{\cosh^2 2\sqrt{\nu}q_i}
+ \nu c (c - 1) \sum_{i<j}^{M} \left[ \frac{1}{\sinh^2 \sqrt{\nu}(q_i - q_j)} - \frac{1}{\cosh^2 \sqrt{\nu}(q_i + q_j)} \right], \tag{7.27}
\]

where \(\text{gd} q = \arctan(\sinh q)\) is the Gudermann function. The parameters \(c_i\) are introduced according to,

\[
c_1 = \frac{b_1}{4\nu} - \frac{1}{2} [N - 1 + (M - 1)c], \quad c_2 = \frac{b_0 - b_2}{4\nu}, \quad c_3 = \frac{b_2}{4\nu}. \tag{7.28}
\]

This form is neither the Inozemtsev nor the Olshanetsky–Perelomov type [50] even in the solvable case, \(b_2 = c_3 = 0\). However, it can be expressed as the one in case IV if the parameters in the potential are extended to be complex. The quasi-solvable models constructed in Ref. [46] are all included in case I, II, III, IV and IV'.

**G. Case V**

In this case, \(a_0 = -g_3/2\), \(a_1 = -g_2/2\), \(a_3 = 2\) and \(a_2 = a_4 = 0\). From Eq. (7.1), we have,

\[
h(q) = \wp(q; g_2, g_3), \tag{7.29}
\]
where \( \varphi(q; g_2, g_3) \) is the Weierstrass elliptic function with the invariants \( g_2 \) and \( g_3 \), and will be abbreviated to \( \varphi(q) \) hereafter. The \( W(q) \) and \( V(q) \) are calculated as follows:

\[
W(q) = -b_1 \sum_{i=1}^{M} \int dq_i \frac{\varphi(q_i)}{\varphi'(q_i)} - \bar{b}_0 \sum_{i=1}^{M} \int dq_i \frac{1}{\varphi'(q_i)} - c_1 \sum_{i=1}^{M} \ln |\varphi'(q_i)| - c \sum_{i<j}^{M} \ln |\varphi(q_i) - \varphi(q_j)|.
\]

\[
V(q) = \alpha_1 \sum_{i=1}^{M} \frac{\varphi''(q_i)}{\varphi'(q_i)^2} + \alpha_2 \sum_{i=1}^{M} \frac{\varphi(q_i)}{\varphi'(q_i)^2} + \alpha_3 \sum_{i=1}^{M} \frac{1}{\varphi'(q_i)^2} + c_2(M, N) \sum_{i=1}^{M} \varphi(q_i) + 2c_1(c - 1) \sum_{i=1}^{M} \varphi(2q_i) + c(c - 1) \sum_{i<j}^{M} [\varphi(q_i - q_j) + \varphi(q_i + q_j)].
\]

The parameters \( \bar{b}_0, c_1 \) are given by,

\[
\bar{b}_0 = b_0 + \frac{b_2 g_2}{12}, \quad c_1 = \frac{b_2}{6} - \frac{1}{2} [N - 1 + (M - 1)c],
\]

and the coupling constants \( \alpha_i \) and \( c_2(M, N) \) are defined by,

\[
\alpha_1 = \frac{1}{12} \left[ 6\bar{b}_0 (2c_1 - 1) + b_1^2 \right],
\]

\[
\alpha_2 = \frac{b_1}{2} [ (2c_1 - 1) g_2 + 2\bar{b}_0 ],
\]

\[
\alpha_3 = \frac{1}{24} \left[ 18b_1 (2c_1 - 1) g_3 + 12\bar{b}_0^2 + b_1^2 g_2 \right],
\]

\[
c_2(M, N) = [N - 1 + (M - 1)c + 2c_1] [2N - 1 + 2(M - 1)c + 2c_1].
\]

The above expressions \((7.30) - (7.31)\) can be arranged into a more familiar form if we introduce the values of \( \varphi(q) \) at the half of the fundamental periods \( 2\omega_l \):

\[
e_l = \varphi(\omega_l) \quad (l = 1, 2, 3).
\]

Utilizing the well-known formulae of the Weierstrass function:

\[
2P(h) = \varphi'(q) = 4(\varphi(q) - e_1)(\varphi(q) - e_2)(\varphi(q) - e_3),
\]

\[
2P'(h) = 2\varphi''(q) = 12\varphi(q)^2 - g_2,
\]

and the addition theorem,

\[
\varphi(q_i + \omega_l) = \frac{H_l^2}{\varphi(q_i) - e_l} + e_l,
\]
we have the following forms of $W(q)$ and $V(q)$:

\[
W(q) = \sum_{l=1}^{3} \frac{b_1 e_l - \bar{b}_0}{4 H_l^2} \sum_{i=1}^{M} \ln |\varphi(q_i) - e_l| - c \sum_{i=1}^{M} \ln |\varphi'(q_i)| \\
- c \sum_{i<j}^{M} \ln |\varphi(q_i) - \varphi(q_j)| .
\] (7.38)

\[
V(q) = \sum_{l=1}^{3} \frac{2 \alpha_1 H_l^2 - \alpha_2 e_l + \alpha_3}{4 H_l^2} \sum_{i=1}^{M} \varphi(q_i + \omega_l) + c_2 (M, N) \sum_{i=1}^{M} \varphi(q_i) \\
+ 2 c_1 (c_1 - 1) \sum_{i=1}^{M} \varphi(2 q_i) + c (c - 1) \sum_{i<j}^{M} [\varphi(q_i - q_j) + \varphi(q_i + q_j)].
\] (7.39)

In the above, $H_l$ are defined by,

\[
H_l^2 = (e_l - e_m)(e_l - e_n) = 3 e_l^2 - \frac{g_2}{4} \quad (l = 1, 2, 3; \ l \neq m \neq n \neq l).
\] (7.40)

The potential form (7.39) shows that this case is the elliptic $BC$ type Inozemtsev model. Contrary to all the previous cases I–IV', the solvability is not realized for any values of the free parameters because $a_3 \neq 0$. On the other hand, the model becomes the elliptic $BC$ type CS model when $\alpha_1 = \alpha_2 = \alpha_3 = 0$, or equivalently, $\bar{b}_0 = b_1 = 0$. Therefore, in contrast to the fact that the rational and hyperbolic (trigonometric) $A$ and $BC$ type CS models are solvable as quantum systems, the elliptic $BC$ type CS model can be only quasi-solvable but is not solvable as a quantum system. The quantum elliptic model investigated in Ref. 58, 59 is just the CS model with the translation $q_i \rightarrow q_i + i \beta$. We note that from the formula,

\[
\varphi(q_i + w_l) = \varphi_{l^{(1/2)}}(q_i) - \varphi(q_i) + e_l,
\] (7.41)

where $\varphi_{l^{(1/2)}}(q)$ denotes the Weierstrass function with one of the fundamental periods $2 \omega_l$ replaced by $\omega_l$, the above model is related to the twisted elliptic CS models 55, 56, 57.

**H. Case V'**

In this case, $a_0 = \nu/2$, $a_2 = \nu(2 - k^2)/2$, $a_4 = \nu(1 - k^2)/2$ and $a_1 = a_3 = 0$. From Eq. (7.41), we have,

\[
h(q) = \frac{\text{sn} (\sqrt{\nu} q | k)}{\text{cn} (\sqrt{\nu} q | k)},
\] (7.42)
where \( \text{sn}(\sqrt{\nu}q|k) \) etc. are the Jacobian elliptic functions with the modulus \( k \), and will be abbreviated to \( \text{sn}(\sqrt{\nu}q) \) etc. hereafter. The \( \mathcal{W}(q) \) and \( V(q) \) are calculated as follows:

\[
\mathcal{W}(q) = -\frac{b_2}{\sqrt{\nu}} \sum_{i=1}^{M} \sum_{l=1}^{3} \int dq_i \frac{\text{sn}^2 \sqrt{\nu}q_i}{\text{dn} \sqrt{\nu}q_i} - \frac{b_0}{\sqrt{\nu}} \sum_{i=1}^{M} \int dq_i \frac{\text{cn}^2 \sqrt{\nu}q_i}{\text{dn} \sqrt{\nu}q_i} \\
+ \left( \frac{b_1}{k^2 \nu} + \frac{N-1+c_M}{2} \right) \sum_{i=1}^{M} \ln \left| \text{dn} \sqrt{\nu}q_i \right| - (N-1) \sum_{i=1}^{M} \ln \left| \text{cn} \sqrt{\nu}q_i \right|
\]

\[-c \sum_{i<j} \ln \left| \text{sn} \sqrt{\nu}q_i \text{ cn} \sqrt{\nu}q_j - \text{cn} \sqrt{\nu}q_i \text{ sn} \sqrt{\nu}q_j \right|, \quad (7.43)\]

\[
V(q) = \alpha_1 \sum_{i=1}^{M} \text{sn} \sqrt{\nu}q_i \text{ cn} \sqrt{\nu}q_i + \alpha_2 \sum_{i=1}^{M} \text{sn}^2 \sqrt{\nu}q_i + \sum_{i=1}^{M} \frac{\alpha_3 \text{sn} \sqrt{\nu}q_i \text{ cn} \sqrt{\nu}q_i + \alpha_4}{\text{dn}^2 \sqrt{\nu}q_i} \\
+ \frac{\nu c (c-1)}{2} \sum_{i<j} \left( \text{sn} \sqrt{\nu}q_i \text{ cn} \sqrt{\nu}q_j - \text{cn} \sqrt{\nu}q_i \text{ sn} \sqrt{\nu}q_j \right)^2, \quad (7.44)\]

where \( c_M = (M-1)c \) and the coupling constants \( \alpha_i \) are given by,

\[
\alpha_1 = \frac{b_0 - b_2}{2k^2 \nu} \left[ 2b_1 - \left( N + c_M \right) k^2 \nu \right], \quad (7.45)\]

\[
\alpha_2 = \frac{1}{2k^2 \nu} \left[ \left( b_1 - \frac{N-1+c_M k^2 \nu}{2} \right) \left( b_1 - \frac{N+1+c_M k^2 \nu}{2} \right) - (b_0 - b_2)^2 \right], \quad (7.46)\]

\[
\alpha_3 = -\frac{k^2 b_0 - b_2}{2k^2 \nu} \left[ 2b_1 + \left( N + c_M \right) k^2 \nu \right], \quad (7.47)\]

\[
\alpha_4 = -\frac{1}{2k^4 \nu} \left[ k^2 \left( b_1 + \frac{N-1+c_M k^2 \nu}{2} \right) \left( b_1 + \frac{N+1+c_M k^2 \nu}{2} \right) - \left( k^2 b_0 - b_2 \right)^2 \right], \quad (7.48)\]

where \( k^2 = 1 - \lambda^2 \). The solvability is not realized for any values of the free parameters because \( a_4 \neq 0 \). As was mentioned before, this case gives another real function representation of case V.

**VIII. POSSIBILITY BEYOND TWO-BODY INTERACTIONS**

So far, we have only concerned with up to two-body interactions. In this section, we will investigate what kind of many-body interactions can be constructed from our procedure. It can be achieved, in principle, by solving the canonical-form condition \((4.1)\) for \( M = 3, 4, 5, \ldots \). However, we will see that the \( GL(2, K) \) symmetry provide a powerful tool for this problem.

Let us first consider the \( M = 3 \) case. From the same procedure as in Eqs. \((4.1) \text{–} (4.6)\), we obtain the transforms of the differential operators:

\[
\frac{\partial}{\partial \sigma_l} = \sum_{i=1}^{3} \frac{(-h_i)^{3-l}}{(h_i - h_j)(h_i - h_k)} \frac{\partial}{\partial h_i} \quad (l = 1, 2, 3; \ i \neq j \neq k \neq i). \quad (8.1)\]

28
As we have seen in Section [IV], the second-order differential operators in the gauged Hamiltonian must be one-body in order to satisfy the canonical-form condition. So, three-body operators can only exist as first- and zeroth-order differential operators. From the form of the gauged Hamiltonian Eq. (3.16) for \( M = 3 \) with Eq. (8.1), we can conclude that only the following form of the three-body operator may be added as first-order differential operators to the \( M \)-body gauged Hamiltonian (5.12):

\[
\sum_{i \neq j \neq k \neq i}^{\text{M}} \frac{O(h_i)}{(h_i - h_j)(h_i - h_k)} \frac{\partial}{\partial h_i}.
\]  

(8.2)

where \( O(h_i) \) denotes a polynomial. On the other hand, \( \tilde{H}_N \) as a whole should be shape invariant under the \( GL(2, K) \) transformation (5.2). From the formulae (A2) and (A3), the operator (8.2) is transformed as,

\[
\text{Eq. (8.2)} \mapsto \sum_{i \neq j \neq k \neq i}^{\text{M}} \frac{\Delta^{-3}(\gamma h_i + \delta)^6 O(\hat{h}_i)}{(h_i - h_j)(h_i - h_k)} \frac{\partial}{\partial h_i} - 2(M - 2) \sum_{i \neq j}^{\text{M}} \frac{\Delta^{-3}(\gamma h_i + \delta)^5 O(\hat{h}_i)}{h_i - h_j} \frac{\partial}{\partial h_i} \\
+ (M - 1)(M - 2) \sum_{i=1}^{M} \Delta^{-3}(\gamma h_i + \delta)^4 O(\hat{h}_i) \frac{\partial}{\partial h_i} + (\text{0th-order ops.}), \tag{8.3}
\]

where (0th-order ops.) denotes the terms which do not involve any differential operators. From the first term of the r.h.s. in Eq. (8.3), the requirement of the \( GL(2, K) \) invariance determines the transformation of the \( O(h_i) \):

\[
O(h_i) \mapsto \hat{O}(h_i) = \Delta^{-3}(\gamma h_i + \delta)^6 O(\hat{h}_i). \tag{8.4}
\]

This means that \( O(h) \) should be a polynomial of sixth-degree. To complete the invariance, the second and third terms of the r.h.s. in Eq. (8.3) should be absorbed into the other two-body and one-body operators, respectively. We find that these operators should be of the following forms:

\[
\sum_{i \neq j}^{\text{M}} \frac{O'(h_i)}{h_i - h_j} \frac{\partial}{\partial h_i}, \quad \sum_{i=1}^{\text{M}} O''(h_i) \frac{\partial}{\partial h_i}, \tag{8.5}
\]

since the derivatives of \( O(h_i) \) transform according to Eq. (A6). Similar observation tells us what kind of zeroth-order operators should simultaneously exist. Finally, we find the \( GL(2, K) \) invariant combination of the three-body operators:

\[
\tilde{H}_N^{(3)} = \sum_{i \neq j \neq k \neq i}^{\text{M}} \frac{O(h_i)}{(h_i - h_j)(h_i - h_k)} \frac{\partial}{\partial h_i} - \frac{M - 2}{3} \sum_{i \neq j}^{\text{M}} \frac{O'(h_i)}{h_i - h_j} \frac{\partial}{\partial h_i} \\
+ \frac{(M - 1)(M - 2)}{30} \sum_{i=1}^{M} O''(h_i) \frac{\partial}{\partial h_i} - (N - 1) \left[ \frac{1}{6} \sum_{i \neq j \neq k \neq i}^{\text{M}} \frac{O'(h_i)}{(h_i - h_j)(h_i - h_k)} \\
- \frac{M - 2}{15} \sum_{i \neq j}^{\text{M}} \frac{O''(h_i)}{h_i - h_j} + \frac{(M - 1)(M - 2)}{120} \sum_{i=1}^{M} O''(h_i) \right]. \tag{8.6}
\]
Therefore, the three-body terms should be added to the gauged Hamiltonian \( \tilde{H}_N \) as the constant multiple of the combination \( g_3 \). The form of the total gauged Hamiltonian reads,

\[
\tilde{H}_N + g_3 \tilde{H}_N^{(3)} = -\sum_{i=1}^{M} P(h_i) \frac{\partial^2}{\partial h_i^2} + \sum_{i=1}^{M} S_i(h) \frac{\partial}{\partial h_i} - T(h),
\]

(8.7)

where \( S_i(h) \) is given by,

\[
S_i(h) = \frac{N - 2 + (M - 1)c}{2} P'(h_i) - Q(h_i) - 2c \sum_{j(\neq i)}^{M} \frac{P(h_i)}{h_i - h_j} + g_3 \left[ \sum_{(i \neq j, k(\neq i))}^{M} \frac{O(h_i)}{(h_i - h_j)(h_i - h_k)} - \frac{M - 2}{3} \sum_{j(\neq i)}^{M} \frac{O'(h_i)}{h_i - h_j} + \frac{(M - 1)(M - 2)}{30} O''(h_i) \right].
\]

(8.8)

In order that the total gauged Hamiltonian (8.7) can be cast in the Schrödinger form by a gauge transformation, we must have,

\[
\frac{\partial W}{\partial q_i} = \frac{h''_i}{2h'_i} + \frac{S_i(h)}{h'_i} \quad (i = 1, \ldots, M),
\]

(8.9)

in addition to Eq. (5.15). Thus, the integrability condition for \( W(q) \) becomes,

\[
\frac{\partial}{\partial q_l} \frac{\partial W}{\partial q_i} = \frac{\partial}{\partial q_l} \frac{\partial W}{\partial q_i} \quad \Leftrightarrow \quad \frac{h'_i}{h'_i} \frac{\partial S_i(h)}{\partial h_l} = \frac{h'_l}{h'_l} \frac{\partial S_i(h)}{\partial h_i} \quad (\forall i \neq l).
\]

(8.10)

However, this condition cannot be satisfied unless \( g_3 = 0 \) since we obtain from Eq. (8.8),

\[
\frac{h'_i}{h'_i} \frac{\partial S_i(h)}{\partial h_l} = -c \frac{h'_i h'_i}{(h_i - h_l)^2} + \frac{g_3}{(h_i - h_l)^2} h'_i \left[ \sum_{j(\neq i)}^{M} \frac{2O(h_i)}{h_i - h_j} - \frac{M - 2}{3} O'(h_i) \right].
\]

(8.11)

In other words, the \( \mathfrak{sl}(M+1) \) quasi-solvable operator (3.16) which satisfies the canonical-form condition cannot contain the three-body operators.

The generalization of the above argument to the \( M \geq 3 \) cases is straightforward and results in the same conclusion that the existence of \( M \)-body operators for \( M \geq 3 \) is prohibited. Therefore, the quasi-solvable quantum many-body systems constructed from the procedure in Section III are exhausted by the ones classified in the preceding section.

**IX. CONCLUDING REMARKS**

In conclusion, we have proposed a systematic method to construct quasi-solvable quantum many-body systems. In Table II, we summarize the complete list of the quasi-solvable models as well as their special subclass of the solvable models constructed in this paper.

We would like to close this paper by giving several remarks on the future problems.
TABLE II: Classification of the (quasi-)solvable quantum many-body systems.

| Case | Quasi-solvable | Solvable |
|------|----------------|----------|
| I    | rational $A$ Inozemtsev | rational $A$ Calogero–Sutherland |
| II   | rational $BC$ Inozemtsev | rational $BC$ Calogero–Sutherland |
| III  | hyp. (trig.) $A$ Inozemtsev | hyp. (trig.) $A$ Calogero–Sutherland + external Morse potential |
| IV   | hyp. (trig.) $BC$ Inozemtsev | hyp. (trig.) $BC$ Calogero–Sutherland |
| V    | elliptic $BC$ Inozemtsev (elliptic $BC$ Calogero–Sutherland) | |

1. We should stress that quasi-solvable quantum many-body models which can be constructed from $\mathfrak{sl}(M+1)$ generators are not limited to the ones presented here. This is because it depends on the way of changing of the variables and on the choice of the solvable subspace $V_N$. In Ref. [44], several $M$-body quasi-solvable quantum models with up to $M$- and up to 6-body interactions were constructed from $\mathfrak{sl}(M+1)$ generators. The Calogero–Marchioro–Wolfes model [60, 61] or the rational $G_2$ type CS model and the trigonometric $G_2$ type CS model without the two-body interaction, all of which have three-body interactions, were shown to have $\mathfrak{sl}(3)$ algebraizations and thus to be solvable in Refs. [62, 63, 64]. Therefore, there may be another family of quasi-solvable quantum many-body systems which can be constructed from a different $\mathfrak{sl}(M+1)$ algebraization.

2. From Table II, we notice that, among the classical type CS models, only the elliptic $A$ type CS model cannot be obtained from the present procedure. As far as we know, the quantum (quasi-)solvability of this model has not been confirmed yet. We note that this model is also a special in the sense that any deformation of the model preserving the integrability have not been discovered yet in contrast to the other classical type CS models.

3. Table II indicates that classical integrability may have more intimate relation with quantum quasi-solvability rather than with quantum solvability and quasi-exact solvability. The latter is understood because classical integrability does not depend on whether the particles are moving inside a bounded region or not. So, it will be an interesting problem to study the relation between classical integrability and quantum quasi-solvability.

4. Although the paper only deals with scalar problems without any internal degree of freedom, there have been much progress in constructing quasi-solvable spin systems recently. Among them, the generalized Dunkl operator approaches have provided one of the most powerful tools for the issues [65, 66, 67]. Then, it is quite interesting to note that the spin CS models constructed from the $A_M$-type Dunkl operators [65] coincide, in the scalar limit, with the (scalar) models constructed from the $\mathfrak{sl}(M+1)$ generators in this paper. Thus, there should be some intimate relations underlying between the two approaches although the algebraic structure of the $A_M$-type Dunkl operators and that of the $\mathfrak{sl}(M+1)$ are quite different from each other. Another
interesting problem is the possibility to generalize the Dunkl operators to include $M$-body ($M \geq 3$) operators. The conclusion in Section VIII however indicates little possibility for such a generalization, at least, of the $A_M$-type.

5. In this paper, we imposed the canonical-form condition (4.1) in order to obtain the ordinary Schrödinger operators. However, it may be possible to obtain another class of quasi-solvable second-order operators by imposing another type of condition instead of Eq. (4.1). If we can solve the canonical-form condition on a specific non-trivial metric or on a particular manifold, we will obtain a gravitational deformation of the quasi-solvable quantum models.

6. A generalization of the Bender–Dunne polynomials to the models obtained here will be an interesting problem. Although it was argued in Ref. [7] that they exist for any quasi-solvable quantum systems regardless of the number of particles, we have not appreciated whether their arguments can be naively applicable to our case.

7. As we have discussed in the end of Section IV all the solvable models constructed in this paper are supersymmetric. It was shown in Refs. [43, 68, 69] that the rational and hyperbolic (trigonometric) $A$ and $BC$ type CS models are of the supersymmetric form. Table I shows that the hyperbolic (trigonometric) $A$ type CS model can be deformed without destroying both the solvability and supersymmetry by adding the external Morse potential. Then, a natural question arises whether there is a solvable quantum model which is not of the supersymmetric form.

8. In the case of $M = 1$, it was shown that quasi-solvability and $\mathcal{N}$-fold supersymmetry is essentially equivalent [16]. However, the relation between them has not been understood in many-body quantum systems although $\mathcal{N}$-fold supersymmetry in many-body case was briefly formulated in Ref. [16]. We expect that they also have the similar intimate relation to each other in the many-body case.

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APPENDIX A: FORMULAE FOR $GL(2, K)$ TRANSFORMATION

Here we will list the formulae useful for the calculation of $GL(2, K)$ transformation in Section VII. Under the linear fractional transformation (5.1), we have,

$$h_i - h_j \mapsto \hat{h}_i - \hat{h}_j = \frac{\Delta(h_i - h_j)}{(\gamma h_i + \delta)(\gamma h_j + \delta)},$$  \quad (A1)

$$\frac{1}{h_i - h_j} \mapsto \frac{1}{\hat{h}_i - \hat{h}_j} = \Delta^{-1}(\gamma h_i + \delta)^2 \left[ \frac{1}{h_i - h_j} - \gamma(\gamma h_i + \delta)^{-1} \right]; \quad (A2)$$
The differential operators in the gauged Hamiltonian are transformed according to,

\[
\frac{\partial}{\partial h_i} \mapsto \prod_{j=1}^{M} (\gamma h_j + \delta)^{N-1} \frac{\partial}{\partial h_i} \prod_{j=1}^{M} (\gamma h_j + \delta)^{-(N-1)}
\]

\[
= \Delta^{-1}(\gamma h_i + \delta)^2 \left[ \frac{\partial}{\partial h_i} - (N-1)\gamma(\gamma h_i + \delta)^{-1} \right], \quad (A3)
\]

and,

\[
\frac{\partial^2}{\partial h_i^2} \mapsto \prod_{j=1}^{M} (\gamma h_j + \delta)^{N-1} \frac{\partial^2}{\partial h_i^2} \prod_{j=1}^{M} (\gamma h_j + \delta)^{-(N-1)}
\]

\[
= \Delta^{-2}(\gamma h_i + \delta)^4 \left[ \frac{\partial^2}{\partial h_i^2} - 2(N-2)\gamma(\gamma h_i + \delta)^{-1} \frac{\partial}{\partial h_i} 
\right. 
\]

\[+ (N-1)(N-2)\gamma^2(\gamma h_i + \delta)^{-2} \right]. \quad (A4)
\]

From the transformation rule \((A3)\) for the polynomials \(P(h)\) and \(Q(h)\), the derivatives of the transformed polynomials are calculated as,

\[
\hat{P}'(\hat{h}_i) = \Delta^{-1}(\gamma h_i + \delta)^2 P'(\hat{h}_i) + 4\Delta^{-2}\gamma(\gamma h_i + \delta)^3 P(\hat{h}_i), \quad (A5a)
\]

\[
\hat{P}''(\hat{h}_i) = P''(\hat{h}_i) + 6\Delta^{-1}\gamma(\gamma h_i + \delta)P'(\hat{h}_i) + 12\Delta^{-2}\gamma^2(\gamma h_i + \delta)^2 P(\hat{h}_i), \quad (A5b)
\]

\[
\hat{Q}'(\hat{h}_i) = Q'(\hat{h}_i) + 2\Delta^{-1}\gamma(\gamma h_i + \delta)Q(\hat{h}_i). \quad (A5c)
\]

Similarly, the derivatives of the transformed sextic polynomial \(\hat{O}(h)\) given by Eq. \((8.4)\) are calculated as,

\[
\hat{O}'(\hat{h}_i) = \Delta^{-2}(\gamma h_i + \delta)^4 O'(\hat{h}_i) + 6\Delta^{-3}\gamma(\gamma h_i + \delta)^5 O(\hat{h}_i), \quad (A6a)
\]

\[
\hat{O}''(\hat{h}_i) = \Delta^{-1}(\gamma h_i + \delta)^2 O''(\hat{h}_i) + 10\Delta^{-2}\gamma(\gamma h_i + \delta)^3 O'(\hat{h}_i) 
\]

\[+ 30\Delta^{-3}\gamma^2(\gamma h_i + \delta)^4 O(\hat{h}_i), \quad (A6b)
\]

\[
\hat{O}'''(\hat{h}_i) = O'''(\hat{h}_i) + 12\Delta^{-1}\gamma(\gamma h_i + \delta)O''(\hat{h}_i) 
\]

\[+ 60\Delta^{-2}\gamma^2(\gamma h_i + \delta)^3 O'(\hat{h}_i) + 120\Delta^{-3}\gamma^3(\gamma h_i + \delta)^3 O(\hat{h}_i). \quad (A6c)
\]

**APPENDIX B: TRANSFORMATION FORMULAE BETWEEN \(h_i\) AND \(\sigma_i\)**

In this appendix, we will show a systematic method to calculate the transformation formulae between the variables \(h_i\) and \(\sigma_i\) needed in Section \[VI\] and \[VIII\]. The method presented here is essentially based on the one in Ref. \[44\]. The starting point is that, for the elementary symmetric polynomials \(\sigma_k\) and the Newton polynomials \(s_n\) defined by,

\[
\sigma_k(h) = \sum_{i_1 < \cdots < i_k} h_{i_1} \cdots h_{i_k}, \quad \sigma_0 = 1; \quad s_n(h) = \sum_{i=1}^{M} h_i^n, \quad (B1)
\]
the following relation holds:

\[ G(t) = \sum_{k=0}^{M} \sigma_k t^k = \exp \left( \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n} s_n t^n \right). \]  \hspace{1cm} (B2)

Thus, \( G(t) \) gives the generating function for both the elementary symmetric polynomials and the Newton ones. The derivatives of \( G(t) \) with respect to \( h_i \) and \( t \) are,

\[ \frac{\partial}{\partial h_i} G(t) = G(t) \sum_{n=1}^{\infty} (-)^{n+1} h_i^{n-1} t^n, \]  \hspace{1cm} (B3a)

\[ \frac{\partial}{\partial t} G(t) = G(t) \sum_{n=1}^{\infty} (-)^{n+1} t^{n-1} s_n = \sum_{k=0}^{M} k \sigma_k t^{k-1}, \]  \hspace{1cm} (B3b)

\[ \frac{\partial^2}{\partial t^2} G(t) = G(t) \left( \sum_{m,m'=1}^{\infty} (-)^{m+m'+2} t^{m+m'-2} s_m s_{m'} - \sum_{m=1}^{\infty} (-)^{m+1} t^{m-1} m s_{m+1} \right). \]  \hspace{1cm} (B3c)

We will see that most of the formulae involving differential operators can be derived with the aid of the generating function \( G(t) \). Here we derive a useful formula for the later purposes. Let us define a generating function \( B(t) \) as,

\[ B(t) = \sum_{k=0}^{M} \frac{\partial \sigma_k}{\partial h_i} t^k. \]  \hspace{1cm} (B4)

From Eq. (B3a), we have another expression for \( B(t) \):

\[ B(t) = \frac{\partial}{\partial h_i} G(t) = t G(t) - \sum_{n=2}^{\infty} (-)^{n} h_i^{n-1} t^n G(t) \]

\[ = \left( 1 - h_i \frac{\partial}{\partial h_i} \right) t G(t) = \left( 1 - h_i \frac{\partial}{\partial h_i} \right) \sum_{k=0}^{M} \sigma_{k-1} t^k. \]  \hspace{1cm} (B5)

Comparing Eq. (B4) and the last expression in Eq. (B5), we have the following formula,

\[ \frac{\partial \sigma_k}{\partial h_i} = \sigma_{k-1} - h_i \frac{\partial \sigma_{k-1}}{\partial h_i}, \]  \hspace{1cm} (B6)

which facilitates the derivation of the several formulae.

1. **Calculation of** \( B_{-}^{(p)}(\sigma) \)

The definition of the quantity \( B_{-}^{(p)} \) is given by,

\[ \sum_{i=1}^{M} h_i^p = s_p(h) = B_{-}^{(p)}(\sigma). \]  \hspace{1cm} (B7)
From the formula \(B_{3b}\), we can immediately obtain,

\[
B^{(p)}_\pm = \frac{(-)^{p-1}}{(p-1)!} \frac{\partial^{p-1}}{\partial t^{p-1}} \left[ G(t) \left( \frac{\partial}{\partial t} G(t) \right) \right]_{t=0}.
\]  

(B8)

The first few formulae read,

\[
\begin{align*}
B^{(0)}_\pm &= s_0 = M, \\
B^{(1)}_\pm &= s_1 = \sigma_1, \\
B^{(2)}_\pm &= s_2 = \sigma_1^2 - 2\sigma_2, \\
B^{(3)}_\pm &= s_3 = \sigma_1^3 - 3\sigma_2\sigma_1 + 3\sigma_3.
\end{align*}
\]

(B9a) \hspace{1cm} (B9b) \hspace{1cm} (B9c) \hspace{1cm} (B9d)

2. Calculation of \(B^{(p)}_k(\sigma)\)

The definition of the quantity \(B^{(p)}_k\) is given by,

\[
\sum_{i=1}^{M} h^p_i \frac{\partial}{\partial h^p_i} \sum_{k=1}^{M} B^{(p)}_k \frac{\partial}{\partial \sigma_k} = \sum_{j=1}^{M} h^p_j \frac{\partial \sigma_j}{\partial h^p_j}.
\]

(B10)

The usefulness of the formula \(B_{3b}\) is understood if we note that the following recursion relation can be derived from it:

\[
B^{(p+1)}_k = \sum_{i=1}^{M} h^p_i \left( \sigma_k - \frac{\partial \sigma_{k+1}}{\partial h^p_i} \right)
= B^{(p)}_\pm \sigma_k - B^{(p)}_{k+1}.
\]

(B11)

The above relation enable us to calculate all the \(B^{(p)}_k\) needed, once one of them is known. The calculation of \(B^{(1)}_k\) will be the easiest. Let us define a generating function \(B^{(1)}(t)\) as,

\[
B^{(1)}(t) = \sum_{k=0}^{M} B^{(1)}_k t^k.
\]

(B12)

From the formulae \(B_{3b}\), we have another expression for \(B^{(1)}(t)\):

\[
B^{(1)}(t) = \sum_{i=1}^{M} h_i \frac{\partial}{\partial h_i} G(t) = \sum_{n=1}^{\infty} (-)^{n+1} t^n s_n G(t)
= t \frac{\partial}{\partial t} G(t) = \sum_{k=0}^{M} k\sigma_k t^k.
\]

(B13)

By comparing Eq. \(B_{12}\) with the last expression in Eq. \(B_{13}\), we obtain,

\[
B^{(1)}_k = k\sigma_k.
\]

(B14)
Applying the recursion relation (B11) to Eq. (B14) successively, we finally get the formulae for $B_k^{(p)}$:

\[ B_k^{(0)} = (M - k + 1)\sigma_{k-1}, \tag{B15a} \]
\[ B_k^{(1)} = k\sigma_k, \tag{B15b} \]
\[ B_k^{(2)} = \sigma_1\sigma_k - (k + 1)\sigma_{k+1}, \tag{B15c} \]
\[ B_k^{(3)} = (\sigma_1^2 - 2\sigma_2)\sigma_k - \sigma_1\sigma_{k+1} + (k + 2)\sigma_{k+2}. \tag{B15d} \]

3. Calculation of $C_{-}^{(p)}(\sigma)$

The quantity $C_{-}^{(p)}$ is defined by,

\[ \sum_{i \neq j}^{M} \frac{h_i^p}{h_i - h_j} = C_{-}^{(p)}(\sigma). \tag{B16} \]

A modification of Eq. (B16) reads,

\[ C_{-}^{(p)} = \frac{1}{2} \sum_{i \neq j}^{M} \frac{h_i^p - h_j^p}{h_i - h_j} = \frac{1}{2} \sum_{i \neq j}^{M} \sum_{l=0}^{p-1} h_i^l h_j^{p-1-l} \]
\[ = \frac{1}{2} \left( \sum_{l=0}^{p-1} s_l s_{p-l} - p s_{p-1} \right). \tag{B17} \]

With the aid of the formulae (B9) we easily obtain, for instance,

\[ C_{-}^{(0)} = 0, \tag{B18a} \]
\[ C_{-}^{(1)} = \frac{M}{2} (M - 1), \tag{B18b} \]
\[ C_{-}^{(2)} = (M - 1)\sigma_1, \tag{B18c} \]
\[ C_{-}^{(3)} = (M - 1)\sigma_1^2 - (2M - 3)\sigma_2. \tag{B18d} \]

4. Calculation of $C_{k}^{(p)}(\sigma)$

The quantity $C_{k}^{(p)}$ is defined by,

\[ \sum_{i \neq j}^{M} \frac{h_i^p}{h_i - h_j} \frac{\partial}{\partial h_i} = \sum_{k=1}^{M} C_{k}^{(p)} \frac{\partial}{\partial \sigma_k}; \quad C_{k}^{(p)} = \sum_{i \neq j}^{M} \frac{h_i^p}{h_i - h_j} \frac{\partial \sigma_k}{\partial h_i}. \tag{B19} \]

In order to calculate $C_{k}^{(p)}$, we define a generating function $C^{(p)}(t)$ as,

\[ C^{(p)}(t) = \sum_{k=0}^{M} C_{k}^{(p)} t^k. \tag{B20} \]
From the formula \([B3a]\) we have,

\[
2C^{(p)}(t) = 2 \sum_{k=0}^{M} C^{(p)}_k t^k = \sum_{i \neq j}^{M} \frac{1}{h_i - h_j} \left( h_i^p \frac{\partial}{\partial h_i} - h_j^p \frac{\partial}{\partial h_j} \right) G(t)
\]

\[
= \sum_{i \neq j}^{M} \sum_{m=1}^\infty (-)^{m+1} t^m \sum_{l=0}^{p+m-2} h_i^l h_j^{p+m-2-l} G(t).
\]  

(B21)

On the other hand, we obtain for \(q \geq 0\) from the formula \([B3c]\),

\[
t^q \frac{\partial^2}{\partial t^2} G(t) = G(t) \sum_{m=q}^{\infty} \sum_{l=0}^{m-q+2} (-)^{m-q+2l} s_l s_{m-q+2-l} - G(t) \sum_{m=1}^{\infty} (-)^{m+1} t^{m+q-1} m s_{m+1}
\]

\[
= G(t) \sum_{m=q}^{\infty} (-)^{m-q+2l} t^m \sum_{l=0}^{m-q+2} s_l s_{m-q+2-l}
\]

\[
- G(t) \sum_{m=1}^{\infty} (-)^{m+1} t^{m+q-1} (2M + m) s_{m+1}.
\]

(B22)

The coefficient of \(G(t)\) in the second line of Eq. \([B22]\) is further deformed as,

\[
\sum_{m=q}^{\infty} (-)^{m-q+2l} t^m \sum_{l=0}^{m-q+2} s_l s_{m-q+2-l} = \sum_{i \neq j}^{M} \sum_{m=1}^\infty (-)^{m-q+2l} t^m \sum_{l=0}^{m-q+2} h_i^l h_j^{m-q+2-l}
\]

\[
+ \sum_{m=1}^{\infty} (-)^{m+1} t^{m+q-1} (m + 2) s_{m+1} + f_q^C(t),
\]

(B23)

where \(f_q^C(t)\) is given by,

\[
f_q^C(t) = \begin{cases} 
(2M - 3)s_2 + s_1^2, & \text{if } q = 0, \\
\sum_{m=1}^{q-1} (-)^{m-q+2l} t^m \sum_{l=0}^{m-q+2} (s_{m-q+2} - s_l s_{m-q+2-l}), & \text{if } q \geq 1.
\end{cases}
\]

(B24)

Substituting Eq. \([B23]\) for Eq. \([B22]\) and applying the formula \([B3b]\), we obtain,

\[
t^q \frac{\partial^2}{\partial t^2} G(t) = \sum_{i \neq j}^{M} \sum_{m=1}^{\infty} (-)^{m-q+2l} t^m \sum_{l=0}^{m-q+2} h_i^l h_j^{m-q+2-l} G(t)
\]

\[
- 2(M - 1)t^{q-1} \left( -\frac{\partial}{\partial t} + s_1 \right) G(t) + f_q^C(t) G(t).
\]

(B25)

Combining Eqs. \([B20]\) and \([B25]\), we get another expression for \(C^{(p)}(t)\) \((0 \leq p \leq 4)\):

\[
2C^{(p)}(t) = (-)^{p+1} \left[ t^{q-p} \frac{\partial^2}{\partial t^2} - 2(M - 1)t^{3-p} \left( \frac{\partial}{\partial t} - \sigma_1 \right) - f_{4-p}^C(t) \right] G(t)
\]

\[
= (-)^{p+1} \sum_{k=0}^{M} \left[ k(k - 2M + 1)t^{2-p} + 2(M - 1)\sigma_1 t^{3-p} - f_{4-p}^C(t) \right] \sigma_k t^k.
\]

(B26)
Comparison of Eq. (B20) with Eq. (B26) enable us to calculate $C_k^{(p)}$ for $0 \leq p \leq 4$. The recursion relation derived from Eq. (B6) in this case is,

$$C_k^{(p+1)} = \sigma_k C_k^{(p)} - C_{k+1}^{(p)}.$$  \hfill (B27)

The calculation of $C_k^{(2)}$ may be the easiest. From Eqs. (B26) and (B27), we obtain the formulae for $C_k^{(p)}$:

$$2C_k^{(0)} = -(k - M - 1)(k - M - 2)\sigma_{k-2},$$  \hfill (B28a)

$$2C_k^{(1)} = (k - M)(k - M - 1)\sigma_{k-1},$$  \hfill (B28b)

$$2C_k^{(2)} = -k(k - 2M + 1)\sigma_k,$$  \hfill (B28c)

$$2C_k^{(3)} = 2(M - 1)\sigma_1\sigma_k + (k + 1)(k - 2M + 2)\sigma_{k+1},$$  \hfill (B28d)

$$2C_k^{(4)} = 2(M - 1)\sigma_2\sigma_k - 2(2M - 3)\sigma_2\sigma_k - 2(M - 1)\sigma_1\sigma_{k+1} - (k + 2)(k - 2M + 3)\sigma_{k+2}.$$  \hfill (B28e)

5. Calculation of $A_{k,l}^{(p)}(\sigma)$

The definition of the quantity $A_{k,l}^{(p)}$ is given by,

$$\sum_{i=1}^{M} h_i^p \frac{\partial^2}{\partial h_i^2} = \sum_{k,l=1}^{M} A_{k,l}^{(p)} \frac{\partial^2}{\partial \sigma_k \partial \sigma_l}; \quad A_{k,l}^{(p)} = \sum_{i=1}^{M} h_i^p \frac{\partial \sigma_k}{\partial h_i} \frac{\partial \sigma_l}{\partial h_i}.$$  \hfill (B29)

In order to calculate $A_{k,l}^{(p)}$, we introduce a generating function $A^{(p)}(t, u)$ defined by,

$$A^{(p)}(t, u) = \sum_{k,l=0}^{M} A_{k,l}^{(p)} t^k u^l.$$  \hfill (B30)

Then, it immediately reads,

$$A^{(p)}(t, u) = \sum_{i=1}^{M} h_i^p \frac{\partial G(t)}{\partial h_i} \frac{\partial G(u)}{\partial h_i}$$

$$= \sum_{m', n'=1}^{\infty} (-)^{m'+n'+2} s_{m'+n'+p-2} t^{m'} u^{n'} G(t) G(u).$$  \hfill (B31)

If we make the substitution $m = m' + n'$ and $n = m' - n'$, the expression of $A^{(p)}(t, u)$ is rewritten as,

$$A^{(p)}(t, u) = \sum_{m=2}^{\infty} (-)^{m} s_{m+p-2} (tu)^{m/2} \sum_{n=2-m}^{m-2} \left( \frac{t}{u} \right) n/2 G(t) G(u)$$

$$= \frac{tu}{t-u} \sum_{m=1}^{\infty} (-)^{m+1} s_{m+p-1} (t^m - u^m) G(t) G(u).$$  \hfill (B32)
On the other hand, utilizing the formula (B31) we have for \( p \geq 0 \),

\[
t^{2-p} \frac{\partial}{\partial t} G(t) = \sum_{m=1}^{\infty} (-1)^{m+1} t^{m-p+1} s_m G(t)
\]

\[
= (-1)^{p+1} \sum_{m=1}^{\infty} (-1)^{m+1} t^{m} s_{m+p-1} G(t) + f_p^A(t) G(t),
\]

where \( f_p^A(t) \) is given by,

\[
f_p^A(t) = \begin{cases} 
Mt, & p = 0, \\
\sum_{m=1}^{p-1} (-1)^{m+1} t^{m-p+1} s_m, & p \geq 1.
\end{cases}
\]

From Eqs. (B32) and (B33), we obtain another expression for \( A^{(p)}(t, u) \):

\[
A^{(p)}(t, u) = (-1)^{p+1} \frac{tu}{t-u} \left[ t^{2-p} \frac{\partial}{\partial t} - u^{2-p} \frac{\partial}{\partial u} - f_p^A(t) + f_p^A(u) \right] G(t)G(u)
\]

\[
= (-1)^{p+1} \frac{M}{t-u} \sum_{k,l=0}^{M} \left[ k t^{1-p} - l u^{1-p} - f_p^A(t) + f_p^A(u) \right] \sigma_k \sigma_l t^{k+1} u^{l+1}. \quad (B35)
\]

Comparison of Eq. (B30) with Eq. (B35) enable us to calculate \( A_{k,l}^{(p)} \). The recursion relation derived from the formula (B36) in this case is,

\[
A_{k,l}^{(p+1)} = B_k^{(p)} \sigma_l - A_{k,l+1}^{(p)} = \sigma_k B_{l+1}^{(p)} - A_{k+1,l}^{(p)}. \quad (B36)
\]

The calculation of \( A_{1,l}^{(1)} \) may be the easiest. From Eqs. (B35) and (B36) we obtain,

\[
A_{1,l}^{(0)} = (M - l + 1) \sigma_{k-1} \sigma_{l-1} + \sum_{m=0}^{k-2} (k - l - 2m - 2) \sigma_{k-m-2} \sigma_{l+m}, \quad (B37a)
\]

\[
A_{1,l}^{(1)} = - \sum_{m=0}^{k-1} (k - l - 2m - 1) \sigma_{k-m-1} \sigma_{l+m}, \quad (B37b)
\]

\[
A_{1,l}^{(2)} = k \sigma_k \sigma_l + \sum_{m=1}^{k} (k - l - 2m) \sigma_{k-m} \sigma_{l+m}, \quad (B37c)
\]

\[
A_{1,l}^{(3)} = \sigma_1 \sigma_k \sigma_l - \sum_{m=0}^{k-1} (k - m + 1) \sigma_{k-m+1} \sigma_{l+m} - \sum_{m=2}^{k+1} (k - l - 2m + 1) \sigma_{k-m+1} \sigma_{l+m}, \quad (B37d)
\]

\[
A_{1,l}^{(4)} = (\sigma_1^2 - 2\sigma_2) \sigma_k \sigma_l - \sigma_1 (\sigma_{k+1} \sigma_l + \sigma_k \sigma_{l+1})
+ \sum_{m=0}^{2} (k - m + 2) \sigma_{k-m+2} \sigma_{l+m} + \sum_{m=3}^{k+2} (k - l - 2m + 2) \sigma_{k-m+2} \sigma_{l+m}. \quad (B37e)
\]

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