Uncertainty quantification for robust variable selection and multiple testing

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Abstract: We study the problem of identifying the set of active variables, termed in the literature as variable selection or multiple hypothesis testing, depending on the pursued criteria. For a general robust setting of non-normal, possibly dependent observations and a generalized notion of active set, we propose a procedure that is used simultaneously for both tasks, variable selection and multiple testing. The procedure is based on the risk hull minimization method, but can also be obtained as a result of an empirical Bayes approach or a penalization strategy. We address its quality via various criteria: the Hamming risk, FDR, FPR, FWER, NDR, FNR, and various multiple testing risks, e.g., MTR=FDR+NDR; and discuss a weak optimality of our results. Finally, we introduce and study, for the first time, the uncertainty quantification problem in the variable selection and multiple testing context in our robust setting.

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1. Introduction

We are concerned with the problem of identifying the set of active (or significant) variables. This task appears in a wide variety of applied fields as genomics, functional MRI, neuroimaging, astrophysics, among others. Such data is typically available on a large number of observation units, which may or may not contain a signal; the signal, when present, may be relatively faint and is dispersed across different observation units in an unknown fashion (i.e., the sparsity pattern is unknown to the observer).

A prototypical application is GWAS (genome-wide association studies), where millions of genetic factors are examined for their potential influence on phenotypic traits. Although the number of tested genomic locations sometimes exceeds $10^5$ or even $10^6$, it is often believed that only a small set of genetic locations have tangible influences on the outcome of the disease or the trait of interest. This is well modeled by the stylized assumption of signal sparsity.

Depending on pursued criteria the problem is termed in the literature as either variables selection, (also termed as recovery of the sparsity pattern), or multiple testing problem. Commonly, the problems of variable selection and multiple testing are studied separately in the literature, although there are conceptual similarities and connections between them. In fact, a variable selection method determines the corresponding multiple testing procedure and vice versa, the difference lies merely in different criteria for inference procedures.
1.1. The observations model and the context

Suppose we observe a high-dimensional ($\mathbb{R}^n$-valued) vector $X = (X_1, \ldots, X_n) \sim P_\theta$ such that $(X - \theta)/\sigma$ satisfies Condition (A1) (for some $\sigma > 0$), where $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ is an unknown high-dimensional signal. Actually, $P_\theta = P_{\theta,\sigma}$, but we will omit the dependence on $n$ and $\sigma$ in the sequel. In other words, we observe

$$X_i = \theta_i + \sigma \xi_i, \quad i \in [n] \triangleq \{1, \ldots, n\},$$

(1)

where $\xi = (\xi_1, \ldots, \xi_n) \triangleq (X - \theta)/\sigma$ is the “noise” vector and $\sigma > 0$ is the known “noise intensity”. We emphasize that we pursue a general distribution-free setting; the distribution of $\xi$ is arbitrary, satisfying only Condition (A1) below. The purpose of introducing $\sigma$ is that certain extra information can be converted into a smaller noise intensity $\sigma$, a “more informative” model. For example, suppose we originally observed $X_{ij}$ with $E_\theta X_{ij} = \theta_i$ and $\text{Var}_\theta(X_{ij}) = 1$, such that $(X_{ij}, j \in [m])$ are independent for each $i \in [n]$, for some $m = m_n \to \infty$ as $n \to \infty$. By taking $X_i = \frac{1}{m_n} \sum_{j=1}^{m_n} X_{ij}$, we obtain the model (1) with $\sigma^2 = \frac{1}{m_n} \to 0$ as $n \to \infty$. We are interested in non-asymptotic results, which imply asymptotic ones if needed. Possible asymptotic regime is high-dimensional setup $n \to \infty$, the leading case in the literature for high dimensional models. Another possible asymptotics is $\sigma \to 0$, accompanying $n \to \infty$ or on its own.

The general goal is (for now, loosely formulated) to select the active (or significant) coordinates $I_*(\theta) \subseteq [n]$ of the signal $\theta$, based on the data $X$. In the sequel, we will need to properly formalize the notion of active set $I_*(\theta)$. In particular, in this paper we let $I_*(\theta)$ be not necessarily the support of $\theta$, $S(\theta) = \{i \in [n] : \theta_i \neq 0\}$. The main motivation for this is that we may want to qualify some relatively small (but non-zero) coordinates of $\theta$ as “inactive”, with the threshold depending on the number of such coordinates. On the other hand, if the non-zero coordinates $\theta_i$ are allowed to be arbitrarily close (relative to the noise intensity $\sigma$) to zero, then it is clearly impossible to recover the signal support. So, even when relaxing the notion of active set, there are still principal limitations as no method should be able to distinguish between $|\theta_i| \asymp \sigma$ and $\theta_i = 0$. These limitations will be quantified by establishing an appropriate lower bound. To make the problem feasible, one needs either to impose some kind of strong signal condition on $\theta$ (typically done in the literature on variable selection), or somehow adjust (relax) the criterion that measures the procedure quality (typically done in the literature on the multiple testing). For example, certain procedures can control more tolerant criteria like FDR or NDR without any condition, but, as we show below, their sum can be controlled again only under some strong signal condition.

1.2. Variable selection and multiple testing in the literature

The standard, most studied situation in the literature is the particular case of (1):

$$X_i \sim N(\theta_i, \sigma^2), \quad i \in [n]; \quad I_*(\theta) = S(\theta) \triangleq \{i \in [n] : \theta_i \neq 0\},$$

(2)
where the support \( S(\theta) \) plays the role of active coordinates of \( \theta \), and \( \theta \) is assumed to be sparse in the sense that \( \theta \in \ell_0[s] = \{ \theta \in \mathbb{R}^n : |S(\theta)| \leq s \} \) with \( s = s_n = o(n) \) as \( n \to \infty \).

Considering the situation (2) for now, there is a huge literature on the active set recovery problem studied from various perspectives. Let \( \hat{I} = \tilde{I}(X) \subseteq [n] \) be a data dependent selector of the active set, \( \eta_I = (\eta_i(I), i \in [n]) = \{1\{i \in I\}, i \in [n]\} \) be the binary representation of \( I \subseteq [n] \). The historically first approach is via the probability of wrong recovery \( \mathbb{P}_\theta(\hat{I} \neq S(\theta)) \). The Hamming distance between \( I_1 \) and \( I_2 \) is \( |\eta_{I_1} - \eta_{I_2}| \triangleq \sum_{i \in [n]} |\eta_i(I_1) - \eta_i(I_2)| = |I_1 \setminus I_2| + |I_2 \setminus I_1| \). One common measure of the quality of \( \hat{I} \) is the expected Hamming loss (which we will call Hamming risk)

\[
R_H(\hat{I}, I_*) = \mathbb{E}_\theta |\eta_{\hat{I}} - \eta_{I_*}| = \mathbb{E}_\theta |\hat{I} \setminus I_*| + \mathbb{E}_\theta |I_* \setminus \hat{I}| = R_{FP}(\hat{I}, I_*) + R_{FN}(\hat{I}, I_*),
\]

where \( R_{FP} \) and \( R_{FN} \) are the false positives and false negatives terms (in a way, Type I and Type II errors), respectively. Note that \( \mathbb{P}_\theta(\hat{I} \neq I_*) = \mathbb{P}_\theta(|\eta_{\hat{I}} - \eta_{I_*}| \geq 1) \leq \mathbb{E}_\theta |\eta_{\hat{I}} - \eta_{I_*}| = R_H(\hat{I}, I_*) \), which means that the approach based on the Hamming risk provides stronger results.

As is already well understood in many papers in related situations, in order to be able to recover \( S(\theta) \), the non-zero signal \( \theta_{S(\theta)} = (\theta_i, i \in S(\theta)) \) has to satisfy some sort of strong signal condition. If \( \theta \in \ell_0[s] \) with polynomial sparsity parametrization \( s = n^\beta, \beta \in (0,1) \) in the normal model (2), [12] and [2] (see further references therein) express this condition in the form of the right scaling for the active coordinates \( \theta_i^2 \geq \sigma^2 \log(n), i \in S(\theta) \), for the signal to be detectable. Actually, [12] studied an idealized chi-squared model \( Y_i \overset{\text{iid}}{\sim} \chi^2_\nu(\lambda_i), i \in [n] \), where \( \chi^2_\nu(\lambda_i) \) is a chi-square distributed random variable with \( \nu \) degrees of freedom and non-centrality parameter \( \lambda_i \). Squaring both the sides of (2), we arrive at the above chi-squared model with non-centrality parameters \( \lambda_i = \theta_i^2 \) and degree-of-freedom parameter \( \nu = 1 \).

Polynomial sparsity \( s = n^\beta \) has been well investigated, especially in the normal model (2), with active coordinates \( \theta_i^2 \geq \sigma^2 \log(n), i \in S(\theta) \). The situation with arbitrary signal sparsity \( s \) is more complex: the right scaling for the active coordinates of \( \theta \in \ell_0[s] \) becomes essentially (assuming \( s = s_n \to \infty \) as \( n \to \infty \)) \( \theta_i^2 \geq \sigma^2 \log(n/s), i \in S(\theta) \). A recent important reference on this topic is [8], see also [18], [9]. More on this is in Section 4.

Inference on the active set can also be looked at from the multiple testing perspective. In classical multiple testing problem for the situation (2), one considers the following sequence of tests:

\[
H_{0,i} : \theta_i = 0 \ 	ext{versus} \ H_{1,i} : \theta_i \neq 0, \ i \in [n].
\]

To connect to variable selection, notice that in multiple testing language, a variable selector \( \hat{I} \) gives the multiple testing procedure which rejects the corresponding null hypothesis \( H_{0,i}, i \in \hat{I} \), whereas \( I_*(\theta) \) encodes which null hypothesis do not hold. Typically, in multiple testing framework, one is interested in controlling (up to some prescribed level) of a type I error. The most popular one is the so called False Discovery Rate (FDR): \( \text{FDR}(\hat{I}, I_*) = \)
Eθ | ˆI|/|I*| (with the convention 0/0 = 0), the averaged proportion of errors among the selected variables; in multiple testing terminology, the expected ratio of incorrect rejections to total rejections. This criterion, introduced in [6], has become very popular because it is “tolerant” and “scalable” in the sense that the more rejections are possible, the more false positives are allowed. It also delivers an adaptive signal estimator, see [1].

Besides FDR, we will study other known multiple testing criteria: Non-Discovery Rate (NDR), k-Family-wise Error Rate (k-FWER) introduced by [19], False Non-Discovery Rate (FNR) introduced by [15] and False Positive Rate (FPR). The FDR, k-FWER and FPR have the flavor of type I error as they deal with the control of false positives, NDP and FNR have the flavor of type II error as they deal with the control of false negatives. A weak optimality of our results in relation to [8] is discussed in Section 4.

In the multiple testing setting, most of theoretical studies rely on the fact that the null distribution is exactly known. In practice, it is often unreasonable to assume this, instead the null distribution is commonly (e.g., in genomics) implicitly defined as the ”background noise” of the measurements, and is adjusted via some pre-processing steps. The issue of finding an appropriate null distribution has been popularized by a series of papers by Efron, (see [13]-[14] and further references therein), where the concept of empirical null distribution was introduced. A recent reference on this topic is [21], see also further references therein. Our robust setting actually aligns well with the fact that the null hypothesis distribution is unknown, in fact, we avoid the problem of estimating the null distribution and obtain results that are robust over a certain (rather general) family of null distributions.

1.3. Multiple testing risk, strong signal condition again

In multiple testing setting, controlling type I error only is clearly not enough to characterize the quality of the procedures. For example, taking ˆI = ∅ gives the perfect FDR control: FDR(∅, I*) = 0, but this is of course an unreasonable procedure. One needs to control also some type II error, for example, the so called Non-Discovery Rate (NDR) NDR( ˆI, I*) = Eθ | ˆI|/|I*| (or, the false Non-Discovery Rate (FNR)). Again, if considered as the only criterion, the NDR can be easily controlled simply by taking another unreasonable procedure ˆI = [n], yielding NDR([n], I*) = 0. Thus, it is relevant to control these errors together, see [2], [20], [22], [10] and further references in these papers. For example, a criterion to control is the multiple testing risk MTR( ˆI, I*) = FDR( ˆI, I*) + NDR( ˆI, I*). Other choices of MTR are possible, as long as it is a combination of type I and type II errors. Apart from FDR+NDR, these are FDR+FNR, FPR+NDR and FPR+FNR.

Relating the MTR to the Hamming risk RH, notice that, the both MTR and RH, although being different, always combine some sort of Types I and II errors, in essence controlling the false positives and false negatives simultaneously. Clearly, the MTR is a more mild criterion as it is always a proportion: MTR ≪ RH.

It is desirable that MTR is as small as possible, e.g., converging to zero as n → ∞. However, as is shown in related settings in [18], [2], [20], [22], [8], [10], this is in general
impossible, which is not surprising because the same kind of principal limitation occurs for the Hamming risk in the case of variable selection. Again some sort of strong signal condition is unavoidable. The Hamming risk $R_H$ is a more severe quality measure than MTR, so its convergence to zero should occur under a more severe strong/sparse signal condition. In some papers in related settings this difference is referred to as exact and approximate recovery of the active set. Yet another type of recovery, the so called almost full recovery is studied in [8], this is the convergence of $R_H/|I_\ast|$ to zero. Below we introduce all these notions more precisely.

1.4. The scope

In this paper, we generalize the standard normal setting (2) to a more general setting (1) in that we pursue the robust inference in the sense that the distribution of the error vector $\xi$ is unknown, but assumed to satisfy only certain condition, Condition (A1) below. In particular, the $\xi_i$'s can be non-normal, not identically distributed, of non-zero mean, and even dependent; their distribution may depend on $\theta$.

Next, we generalize the notion of active set $I_\ast(\theta)$ that is now not necessarily the sparsity class $\ell_0(s)$ and not necessarily with all large non-zero coordinates. We propose to express strong signal conditions as a scale of classes $\{\Theta(K), K > 0\}$ for the signal $\theta$, for the both problems simultaneously, variable selection and multiple testing problems. For the problem of determining the active set to be well defined, the parameter $\theta$ has to possess a distinct set $I_\ast(\theta)$ of active coordinates, which is ensured by the condition $\theta \in \Theta(K)$ for some (sufficiently large) $K > 0$. The sparsity is expressed by $|I_\ast(\theta)|$ and the strong signal condition by the fact that $\theta \in \Theta(K)$ for sufficiently large $K$ (depending on the constants from Condition (A1)). Varying the “goodness” of $\theta$ (combination sparsity/strong signal) exposes the so called phase transition effect, separating the impossibility and possibility to recover the active set, shortly discussed in Section 4.2.

Finally, in this paper we address the new problem of uncertainty quantification (UQ) for the active set $\eta_\ast = \eta_\ast(\theta) = \eta(I_\ast(\theta))$, this is to be distinguished from the uncertainty quantification for the parameter $\theta$. For the Hamming loss $|\cdot|$ on $\{0, 1\}^n$, a confidence ball is $B(\hat{\eta}, \hat{r}) = \{\eta \in \{0, 1\}^n : |\hat{\eta} - \eta| \leq \hat{r}\}$, where the center $\hat{\eta} = \hat{\eta}(X) : \mathbb{R}^n \mapsto \{0, 1\}^n$ and radius $\hat{r} = \hat{r}(X) : \mathbb{R}^n \mapsto \mathbb{R}_+ = [0, +\infty]$ are measurable functions of the data $X$. The goal is to construct such a confidence ball $B(\hat{\eta}, C\hat{r})$ that for any $\alpha_1, \alpha_2 \in (0, 1]$ and some function $r(\eta_\ast(\theta))$, $r : \mathbb{R}^n \rightarrow \mathbb{R}_+$, there exist $C, c > 0$ such that

$$\sup_{\theta \in \Theta_{\text{cov}}} \mathbb{P}_{\theta}(\eta_\ast(\theta) \notin B(\hat{\eta}, C\hat{r})) \leq \alpha_1, \quad \sup_{\theta \in \Theta_{\text{size}}} \mathbb{P}_{\theta}(\hat{r} \geq cr(\eta_\ast(\theta))) \leq \alpha_2,$$  \tag{3}

for some $\Theta_{\text{cov}}, \Theta_{\text{size}} \subseteq \mathbb{R}^n$. The function $r(\eta_\ast(\theta))$, called the radial rate, is a benchmark for the effective radius of the confidence ball $B(\hat{\eta}, C\hat{r})$. The first expression in (3) is called coverage relation and the second size relation. To the best of our knowledge, there are no results on uncertainty quantification with the Hamming loss (3) for the active set $\eta_\ast(\theta)$. It is desirable to find the smallest $r(\eta_\ast(\theta))$, the biggest $\Theta_{\text{cov}}$ and $\Theta_{\text{size}}$ such that (3) holds and
r(\eta_*(\theta)) \asymp R(\Theta_{\text{size}}), where \( R(\Theta_{\text{size}}) \) is the optimal rate in estimation problem for \( \eta_*(\theta) \).

We derive some UQ results for the proposed selector \( \hat{I} \).

Typically, the so called deceptiveness issue is pertinent to the UQ problem, meaning that the confidence set of the optimal size and high coverage can only be constructed for non-deceptive parameters (in particular, \( \Theta_{\text{cov}} \) cannot be the whole set \( \mathbb{R}^n \)). Being non-deceptive is expressed by imposing some condition on the parameter; for example, the EBR (excessive bias restriction) condition \( \Theta_{\text{cov}} = \Theta_{\text{ebr}} \subset \mathbb{R}^n \), see [3–5]. Interestingly, there is no deceptiveness issue as such for our UQ problem. An intuition behind this is as follows: the problem of active set recovery is more difficult than the UQ-problem in a sense that solving the former problem implies solving the latter. Then the condition \( \theta \in \Theta(K) \) for the parameter to have distinct active coordinates implies also that the parameter is non-deceptive. In our case, we will have \( \Theta_{\text{cov}} = \Theta_{\text{size}} = \Theta(K) \) for some \( K > 0 \).

1.5. Organization of the rest of the paper

In Section 2 we introduce some notation, the generalized notion of active coordinates, and describe the criteria and procedures for variable selection and multiple testing. In Section 3 we present the main results of the paper. In Section 4 we shortly discuss a weak optimality of our results and a phase transition effect. The proofs of the theorems are collected in Section 5, which, despite generality of the setting and results, we could keep completely self-contained and relatively compact.

2. Preliminaries

2.1. Notation

Denote the probability measure of \( X \) from the model (1) by \( \mathbb{P}_\theta \), and by \( E_\theta \) the corresponding expectation. For the notational simplicity, we skip the dependence on \( \sigma \) and \( n \) of these quantities and many others. Denote by \( 1\{s \in S\} = 1_S(s) \) the indicator function of the set \( S \), by \( |S| \) the cardinality of the set \( S \), the difference of sets \( S \setminus S_0 = \{s \in S : s \notin S_0\} \). Let \( [k] = \{1, \ldots, k\} \) and \( [k]_0 = \{0\} \cup [k] \) for \( k \in \mathbb{N} = \{1, 2, \ldots\} \). For \( I \subseteq [n] \) define \( I^c = [n] \setminus I \).

If random quantities appear in a relation, this relation should be understood in a sure sense. For two nonnegative sequences \( (a_l) \) and \( (b_l) \), \( a_l \lesssim b_l \) means \( a_l \leq c b_l \) for all \( l \) (its range should be clear from the context) with some absolute \( c > 0 \), and \( a_l \asymp b_l \) means that \( a_l \lesssim b_l \) and \( b_l \lesssim a_l \). The symbol \( \triangleq \) will refer to equality by definition, \( \Phi(x) = \mathbb{P}(Z \leq x) \) for \( Z \sim \mathcal{N}(0,1) \). Throughout we assume the conventions: \( |\emptyset| = 0 \), \( \sum_{I \in \emptyset} a_I = 0 \) for any \( a_I \in \mathbb{R} \) and \( 0 \log(a/0) = 0 \) (hence \( (a/0)^0 = 1 \) for any \( a > 0 \)), in all the definitions whenever \( 0/0 \) occurs we set by default \( 0/0 = 0 \). Introduce the function \( \ell(x) = \ell_q(x) \triangleq x \log(qn/x) \), \( x, q > 0 \), increasing in \( x \in [0, n] \) for all \( q \geq 1 \). Finally, introduce the notation of ordered \( \theta_2^2, \ldots, \theta_q^2, \theta_{[1]}^2 \geq \theta_{[2]}^2 \geq \ldots \geq \theta_{[n]}^2 \); define additionally \( \theta_{[0]}^2 = \infty \) and \( \theta_{[n+1]}^2 = 0 \).

Recall the binary representation of an \( I \subseteq [n] \):

\[
\eta_I = (\eta_I(i), i \in [n]) = (1\{i \in I\}, i \in [n]) \in \{0,1\}^n.
\]
Let $\hat{\eta} = \eta(I)$ be some data dependent selector (for some measurable $I = I(X)$) which is supposed to estimate
\[ \eta_* = \eta_*(\theta) = \eta_{I_*} = (1\{i \in I_*\}, i \in [n]), \]
for some “true” active set $I_* = I_*(\theta)$. The Hamming distance between $\hat{\eta}$ and $\eta_*$ is determined by the number of positions at which $\hat{\eta}$ and $\eta_*$ differ:
\[ |\hat{\eta} - \eta_*| = \sum_{i=1}^n |\hat{\eta}_i - \eta_*(i)| = \sum_{i=1}^n 1\{\hat{\eta}_i \neq \eta_*(i)\} = |\tilde{I} \setminus I_*| + |I_* \setminus \tilde{I}|. \tag{4} \]

Now we give some definitions for the multiple testing framework. For a variable selector $\hat{I} = \hat{I}(X) \subseteq [n]$ (which is seen now as multiple testing procedure) and the active set $I_*$ (which is seen now as the set of the true null hypothesis), introduce the quantities characterizing the quality of the multiple testing procedure $\hat{I}$. The convention $0/0 = 0$ is used in the following definitions. The false discovery proportion (FDP) and false discovery rate (FDR) are
\[ \text{FDP}(\hat{I}) = \text{FDP}(\hat{I}, I_*) = \frac{|\tilde{I} \setminus I_*|}{|\tilde{I}|}, \quad \text{FDR}(\hat{I}) = \text{FDR}(\hat{I}, I_*) = E_\theta \text{FDP}(\hat{I}). \]
The false positive proportion and false positive rate are
\[ \text{FPP}(\hat{I}) = \text{FPP}(\hat{I}, I_*) = \frac{|\tilde{I} \setminus I_*|}{n - |I_*|}, \quad \text{FPR}(\hat{I}) = \text{FPR}(\hat{I}, I_*) = E_\theta \text{FPP}(\hat{I}). \]
The non-discovery proportion and non-discovery rate are
\[ \text{NDP}(\hat{I}) = \text{NDP}(\hat{I}, I_*) = \frac{|I_* \setminus \tilde{I}|}{|I_*|}, \quad \text{NDR}(\hat{I}) = \text{NDR}(\hat{I}, I_*) = E_\theta \text{NDP}(\hat{I}). \]
The false non-discovery proportion and false non-discovery rate are
\[ \text{FNP}(\hat{I}) = \text{FNP}(\hat{I}, I_*) = \frac{|I_* \setminus \tilde{I}|}{n - |\tilde{I}|}, \quad \text{FNR}(\hat{I}) = \text{FNR}(\hat{I}, I_*) = E_\theta \text{FNP}(\hat{I}). \]

Introduce the multiple testing risks (MTR) as all possible sums of the probabilities of type I and type II errors. The first MTR is the sum of the FDR and the NDR:
\[ \text{MTR}_1(\hat{I}) = \text{MTR}_1(\hat{I}, I_*) = \text{FDR}(\hat{I}, I_*) + \text{NDR}(\hat{I}, I_*). \]
The other MTR’s are defined similarly: they are always the sums of two rates (out of 4) whose numerators must be the quantities $|\tilde{I} \setminus I_*|$ and $|I_* \setminus \tilde{I}|$. Apart from MTR$_1$, these are MTR$_2 = \text{FPR} + \text{FNR}$, MTR$_3 = \text{FPR} + \text{NDR}$ and MTR$_4 = \text{FPR} + \text{FNR}$.

Finally introduce the $k$-family-wise error (k-FWER) and $k$-family-wise non-discovery (k-FWNR) rates:
\[ \text{k-FWER}(\hat{I}, I_*) = P_\theta(|\tilde{I} \setminus I_*| \geq k), \quad \text{k-FWNR}(\hat{I}, I_*) = P_\theta(|I_* \setminus \tilde{I}| \geq k). \]
In multiple testing settings, the k-FWER is the probability of rejecting at least $k$ true null hypotheses. The case $k = 1$ reduces to the control of the usual FWER.
2.2. **Criterion for selecting active variables**

Consider for the moment the estimation problem of $\theta$ when we use the projection estimators $\hat{\theta}(I) = (X_i1\{i \in I\}, i \in [n])$, $I \subseteq [n]$. The quadratic loss of $\hat{\theta}(I)$ gives a theoretical criterion

$$C_1^{th}(I) = ||\hat{\theta}(I) - \theta||^2 = \sum_{i \in I^c} \theta_i^2 + \sigma^2 \sum_{i \in I} \xi_i^2.$$ 

The best choice of $I$ would be the one minimizing $C_1^{th}(I)$. However, neither $\theta$ nor $\xi$ are observed. Substituting unbiased estimator $X_i^2 - \sigma^2$ instead of $\theta_i^2$, $i \in I^c$, leads to the quantity

$$C_2^{th}(I) = \sum_{i \in I^c} X_i^2 + \sigma^2 |I| + \sigma^2 \sum_{i \in I} \xi_i^2 = ||X - \hat{\theta}(I)||^2 + \sigma^2 |I| + \sigma^2 \sum_{i \in I} \xi_i^2$$

to minimize with respect to $I \subseteq [n]$, however still not usable in view of the term $\sigma^2 \sum_{i \in I} \xi_i^2$. If instead of $\sigma^2 \sum_{i \in I} \xi_i^2$, we use its expectation $\sigma^2 |I|$, we arrive essentially at Mallows’s $C_p$-criterion (and AIC in the normal case)

$$C_{\text{Mallows}}(I) = ||X - \hat{\theta}(I)||^2 + 2\sigma^2 |I|.$$ 

However, it is well known that the $C_p$-criterion leads to overfitting. An intuitive explanation is that using the expectation $\sigma^2 |I|$ as penalty in $C_{\text{Mallows}}(I)$ is too optimistic to control oscillations of its stochastic counterpart $\sigma^2 \sum_{i \in I} \xi_i^2$.

The next idea is to use some quantity $p(I)$ (instead of $|I|$) that majorizes $\sum_{i \in I} \xi_i^2$ in the more strict sense that for some $K, H_0, \alpha > 0$ and all $M \geq 0$

$$\mathbb{P}_\theta \left( \sup_{I \subseteq [n]} \left( \sum_{i \in I} \xi_i^2 - Kp(I) \right) \geq M \right) \leq H_0e^{-\alpha M}. \tag{5}$$

**Remark 1.** This idea is borrowed from the **risk hull minimization method** developed by Golubev in several papers; see [11] and references therein.

Thus, in view (5), using $K\sigma^2 p(I)$ instead of $\sigma^2 \sum_{i \in I} \xi_i^2$, we obtain a more adequate criterion $C_3(I) = ||X - \hat{\theta}(I)||^2 + \sigma^2 |I| + K\sigma^2 p(I)$. Since typically $|I| \lesssim p(I)$, the second term can be absorbed into the third, and we finally derive the criterion

$$C(I) = ||X - \hat{\theta}(I)||^2 + K\sigma^2 p(I), \tag{6}$$

for sufficiently large constant $K > 0$. It remains to determine $p(I)$, preferably smallest possible, for which (5) holds. First we state an assumption.

**Assumption (A1).** For some $p_0(I)$ such that $p_0(I) \leq C_\xi |I|$, for some $C_\xi > 0, H_\xi, \alpha_\xi > 0$ and any $M \geq 0$,

$$\sup_{\theta \in \mathbb{R}^\alpha} \mathbb{P}_\theta \left( \sum_{i \in I} \xi_i^2 \geq p_0(I) + M \right) \leq H_\xi e^{-\alpha_\xi M}, \quad I \subseteq [n]. \tag{A1}$$
If the distribution of $\xi$ does not depend on $\theta$ (in some important specific cases), there is no supremum over $\theta \in \mathbb{R}^n$. For independent $\xi_i$’s, (A1) holds with $p_0(I) \propto |I|$, so that indeed $p_0(I) \leq C_\xi |I|, I \subseteq [n]$, for some $C_\xi > 0$.

**Remark 2.** For fixed constants $C_\xi, H_\xi, \alpha_\xi$, one can think of (A1) with $p_0(I) = C_\xi |I|$ as description of a class of possible measures, in the sequel denoted by $\mathcal{P}_A$, see Remark 12 below.

**Remark 3.** Condition (A1) is of course satisfied for independent normals $\xi_i \overset{\text{ind}}{\sim} N(0,1)$ and for bounded (arbitrarily dependent) $\xi_i$’s. Recall that the $\xi_i$’s are not necessarily of zero mean, but for normals Condition (A1) is the weakest if $E_0 \xi_i = 0$. In a way, Condition (A1) prevents too much dependence, but it still allows some interesting cases of dependent $\xi_i$’s. For example, one can show (in the same way as in [5]) that this conditions is fulfilled for $\xi_i$’s that follow an autoregressive model AR(1) with normal white noise.

Let $\eta = \sup_{I \subseteq [n]} \left( \sum_{i \in I} \xi_i^2 - C_\xi |I| - \alpha_\xi^{-1} |I| + \log \binom{n}{|I|} \right)$. By (A1) and the union bound, it is not difficult to derive

$$P_\theta(\eta \geq M) \leq \sum_{I \subseteq [n]} P_\theta \left( \sum_{i \in I} \xi_i^2 \geq C_\xi |I| + \alpha_\xi^{-1} |I| + \log \binom{n}{|I|} + M \right)$$

$$\leq H_\xi e^{-\alpha_\xi M} \sum_{I \subseteq [n]} e^{-|I| \binom{n}{|I|}^{-1}} = H_\xi e^{-\alpha_\xi M} \sum_{k=0}^n e^{-k} \leq H_0 e^{-\alpha_\xi M}. \quad (7)$$

where $H_0 = H_\xi/(1 - e^{-1})$. As $\binom{n}{k} \leq \left( \frac{en}{K} \right)^k, k \in [n]$, we have

$$C_\xi |I| + \alpha_\xi^{-1} |I| + \log \binom{n}{|I|} \leq (C_\xi + \alpha_\xi^{-1}) |I| + \alpha_\xi^{-1} |I| \log \frac{q_\xi}{|I|} = C_\xi |I| \log \frac{q_\xi}{|I|},$$

where $q_\xi = e^{C_\xi \alpha_\xi + 2}$. The last two displays imply the following relation (that we will need later): for appropriate $M_\xi > 0$ and $q = e^2$

$$\sup_{\theta \in \mathbb{R}^n} \sum_{I \subseteq [n]} P_\theta \left( \sum_{i \in I} \xi_i^2 \geq M_\xi |I| \log \left( \frac{q_\xi}{|I|} \right) + M \right) \leq H_0 e^{-\alpha_\xi M}. \quad (8)$$

Although we will use the relation (8) only for $q = e^2$, it is also implied by (A1) for any $q > 1$ with appropriately chosen $M_\xi = M_\xi(q)$, certainly if $M_\xi(q) \geq \alpha_\xi^{-1} \left( \frac{\log q_\xi}{\log q} + 1 \right)$.

In view of (7) with (8), we see that under (A1), the criterion (5) is satisfied with $p(I) = \ell_q(|I|) = |I| \log \left( \frac{q_\xi}{|I|} \right)$. According to (6), this motivates the definition of the so called *preselector*

$$\tilde{I} = \tilde{I}(K) = \arg \min_{I \subseteq [n]} \left\{ \sum_{i \in I^c} X_i^2 + K \sigma^2 p(I) \right\}, \quad (9)$$

where $p(I) = p_q(I) \triangleq \ell_q(|I|) = |I| \log \left( \frac{q_\xi}{|I|} \right)$, for some $K > 0$ and $q = e^2$. If $\tilde{I}$ is not unique, take, say, the one with the biggest sum $\sum_{i \in I} (n - i)$.
Remark 4. Actually, an interesting interplay between constants $K$ and $q$ is possible, making certain constant in the proofs sharper. But we fix the second constant $q = e^2$ in (9) for the sake of mathematical succinctness.

Notice that if $X_i^2 > K\sigma^2 \log \left( \frac{qn}{|\tilde{I}|+1} \right)$ (with $q = e^2$), then $i \in \tilde{I}$. On the other hand, if $i \in \tilde{I}$, then

$$X_i^2 \geq K\sigma^2 \left[ \log \left( \frac{qn}{|\tilde{I}|} \right) - (|\tilde{I}| - 1) \log \left( 1 + \frac{1}{|\tilde{I}| - 1} \right) \right]$$

$$\geq K\sigma^2 \left[ \log \left( \frac{qn}{|\tilde{I}|} \right) - \frac{|\tilde{I}|-1}{|\tilde{I}|} \right] \geq K\sigma^2 \left[ \log \left( \frac{qn}{|\tilde{I}|} \right) - 1 \right] = K\sigma^2 \log \left( \frac{en}{|\tilde{I}|} \right).$$

Next, define the selector $\hat{\eta} = (\hat{\eta}_i, i \in [n])$ (and respectively $\hat{I}$) of significant coordinates as

$$\hat{\eta}_i = \hat{\eta}_i(K) = 1 \left\{ X_i^2 \geq K\sigma^2 \log \left( \frac{qn}{|\tilde{I}|} \right) \right\}, \quad \hat{I}(K) = \{ i \in [n] : \hat{\eta}_i = 1 \}. \quad (10)$$

Notice that the selector is always a subset of the preselector: $\hat{I}(K) \subseteq \tilde{I}(K)$.

Remark 5. We have already mentioned that this procedure can be related to the risk hull minimization (RHM) method developed by Golubev. In principle, (almost) the same procedures can be derived as a result of the empirical Bayes approach with appropriately chosen prior, or as a result of the penalization strategy with appropriately chosen penalty, see [5]. It is interesting that several methodologies deliver akin procedures.

2.3. The notion of active set

Suppose we consider an arbitrary $\theta$ and would still like somehow divide all the entries of $\theta$ into the groups of active and inactive coordinates. Clearly, as active group, the traditional support set $S(\theta) = \{ i \in [n] : \theta_i \neq 0 \}$ is not sensible for arbitrary $\theta$, because nonzero but smallish coordinates $\theta_i$ should possibly be assigned to the inactive group.

For an arbitrary $\theta \in \mathbb{R}^n$, define the active set $I_s(A, \theta) = I_s(A, \theta, \sigma^2)$ as follows: with $q = e^2$,

$$I_s(A, \theta) = \arg \min_{I \subseteq [n]} r^2(I, \theta), \quad \text{where} \quad r^2(I, \theta) \triangleq \sum_{i \in I^c} \theta_i^2 + A\sigma^2 |I| \log \left( \frac{qn}{|I|} \right) \quad (11)$$

and $I_s(A, \theta)$ is with the smallest sum $\sum_{i \in I_s(A, \theta)} i$ if the minimum is not unique. By the definition (11),

$$r^2(\theta) \triangleq r^2(I_s(A, \theta), \theta) \leq r^2(I, \theta) \quad \text{for any} \quad I \subseteq [n].$$

This implies that

if $\theta_i^2 \geq A\sigma^2 \log \left( \frac{qn}{|I|+1} \right)$, then $i \in I_s \subseteq [n], \quad (12)$
where we denoted for brevity $I_\ast = I_\ast(A, \theta)$. Conversely, if $i \in I_\ast$, then
\[
\theta_i^2 \geq A\sigma^2 \left[ \log \left( \frac{q_n}{|I_\ast|} \right) - \left( |I_\ast| - 1 \right) \log \left( 1 + \frac{1}{|I_\ast| - 1} \right) \right] \\
\geq A\sigma^2 \left[ \log \left( \frac{q_n}{|I_\ast|} \right) - \frac{|I_\ast| - 1}{|I_\ast|} \right] \geq A\sigma^2 \left[ \log \left( \frac{q_n}{|I_\ast|} \right) - 1 \right] = A\sigma^2 \log \left( \frac{q_n}{|I_\ast|} \right).
\]

We will use the last property later on. Also notice that \( A\sigma^2 \) depends on the product \( A^2 \) rather than just on \( A \). If we consider the asymptotic regime \( A \to \infty \), it is instructive to fix the product \( A\sigma^2 \) so that \( A \to \infty \), which can be interpreted as if \( \theta \) satisfies more and more stringent strong signal condition.

**Remark 6.** To get an idea what \( I_\ast(A, \theta) \) means, suppose in (11) we had \( A\sigma^2 |I| \log(qn) \) instead of \( A\sigma^2 |I| \log \left( \frac{q_n}{|I_\ast|} \right) \). Then active coordinates would have been all \( i \in [n] \) corresponding to “large” \( \theta_i^2 \geq A\sigma^2 \log(qn) \). The definition (11) does kind of the same, but the requirement for being active becomes slightly more lenient if there are more “large” coordinates. The function \( x \log(qn/x) \) is increasing (in fact, for all \( q \geq e \)) for \( x \in [1, n] \) slightly slower than \( x \log(qn) \), creating the effect of “borrowing strength” via the number of active coordinates: the more such coordinates, the less stringent the property of being active becomes.

The family \( \mathcal{I} = \mathcal{I}(\theta) = \{I_\ast^{\text{vsp}}(\theta), k \in [n]_0\} \), with \( I_\ast^{\text{vsp}}(\theta) = \{i \in [n]: \theta_i^2 \geq \theta_{[k]}^2\} \), is called **variable selection path** (VSP). It consists of at most \( n + 1 \) embedded sets:
\[
\emptyset = I_\ast^{\text{vsp}}(\theta) \subseteq I_1^{\text{vsp}}(\theta) \subseteq \ldots \subseteq I_n^{\text{vsp}}(\theta) = [n].
\]
Clearly, \( I_\ast(A, \theta) = I_i^{\text{vsp}}(\theta), \theta \in \mathbb{R}^n \). The function \( g(A) = |I_\ast(A, \theta)| : \mathbb{R}_+ \to \{0\} \cup \mathbb{N} \) is a non-increasing right continuous step function taking values \( |S(\theta)|, \ldots , 0 \), as \( A \) increases from 0 to infinity. If some of \( \theta_{[k]} \) coincide, the corresponding sets \( I_i^{\text{vsp}}(\theta) \) in the variable selection path \( \mathcal{I}(\theta) \) merge. Accounting for these merges, notice that the true support \( S(\theta) \) is the last set in the variable selection path \( \mathcal{I}(\theta) \).

**Remark 7.** We state some further properties of the active set \( I_\ast \) and the variable selection path \( \mathcal{I} \).

(a) The family \( \{I_\ast(A, \theta), A \geq 0\} \) reproduces the variable selection path \( \mathcal{I} \)
\[
\{I_\ast(A, \theta), A \geq 0\} = \mathcal{I}(\theta).
\]
(b) For any \( 0 \leq A_1 \leq A_2 \) and any \( \theta \in \mathbb{R}^n \), we have
\[
\emptyset \subseteq I_\ast(A_2, \theta) \subseteq I_\ast(A_1, \theta) \subseteq S(\theta) \subseteq [n].
\]
(c) If for some \( I \subseteq [n], \theta_i^2 \geq A\sigma^2 \log(qn/|I|) \) for all \( i \in I \) and \( \theta_i^2 \leq A\sigma^2 \log(q) \) for all \( i \in I^c \), then \( I_\ast(A, \theta) = I \). In particular, if \( \theta_i^2 \geq A\sigma^2 \log(qn/|S(\theta)|) \) for all \( i \in S(\theta) \) and some \( A > 0 \), then \( I_\ast(A', \theta) = S(\theta) \) for all \( A' \leq A \).

3. Main results

In this section we present the main results.
3.1. Control of the preselector \( \tilde{I} \)

First, we establish the results on over-size and under-size control of the preselector \( \tilde{I} = \tilde{I}(K) \) defined by (9). Recall that \( \ell(x) = \ell_q(x) = x \log(\frac{q}{x}) \), \( x \geq 0, \ q = e^2. \)

**Theorem 1.** Let \( \tilde{I} = \tilde{I}(K) \) be defined by (9), \( I_s(A, \theta) \) be defined by (11), let \( H_0 \) be from (8). Then for any \( A_0 \) there exist (sufficiently large) \( K_0 \) and constants \( M_0, \alpha_0 > 0 \) (depending on \( A_0 \)) such that for any \( M \geq 0, \)

\[
\sup_{\theta \in \mathbb{R}^n} \mathbb{P}_\theta(\ell(|\tilde{I}(K_0)|)) \geq M_0 \ell(|I_s(A_0, \theta)|) + M \leq H_0 e^{-\alpha_0 M}.
\]

In particular, this implies that there exists \( M_1 > 0 \) such that for any \( M \geq 0 \)

\[
\sup_{\theta \in \mathbb{R}^n} e^{\alpha_0 \ell(|I_s(A_0, \theta)|)} \mathbb{P}_\theta(|\tilde{I}(K_0)|) \geq M_1 |I_s(A_0, \theta)| + M \leq H_0 e^{-\alpha_0 M/2}.
\]

For any \( K_1 > 0, \delta \in [0, 1) \), there exist \( A_1, \alpha_1, \alpha'_1 > 0 \) (depending on \( \delta, K_1 \)) such that for any \( M \geq 0 \)

\[
\sup_{\theta \in \mathbb{R}^n} e^{\alpha'_1 \ell(|I_s(A_1, \theta)|)} \mathbb{P}_\theta(\ell(|\tilde{I}(K_1)|)) \leq \delta \ell(|I_s(A_1, \theta)|) - M \leq H_0 e^{-\alpha_1 M}.
\]

In particular, this implies that there exist \( A_1, \alpha'_1 > 0 \) such that

\[
\sup_{\theta \in \mathbb{R}^n} e^{\alpha'_1 \ell(|I_s(A_1, \theta)|)} \mathbb{P}_\theta(|\tilde{I}(K_1)|) \leq \delta |I_s(A_1, \theta)| \leq H_0.
\]

**Remark 8.** A couple of remarks are in order.

(a) We can obtain another formulation of the property (i) (and respectively (i')): for any sufficiently large \( K_0 \) (e.g., \( K_0 > 2M_\xi \)) there exist \( A_0 > 0 \) and constants \( M_0, \alpha_0 > 0 \) such that for any \( M \geq 0, \) (i) holds.

(b) We can also derive another formulation of the property (ii) (and respectively (ii')):

for any \( A_1 > 0 \) there exist (sufficiently small) \( K_1 > 0 \) and \( \delta \in [0, 1] \) such that for any \( M \geq 0, \) (ii) holds.

(c) Similarly to (i'), we could establish (ii') in the following form:

\[
\sup_{\theta \in \mathbb{R}^n} e^{\alpha'_1 \ell(|I_s(A_1, \theta)|)} \mathbb{P}_\theta(\ell(|\tilde{I}(K_1)|) \leq \delta |I_s(A_1)| - M) \leq H_0 e^{-\alpha_1 M}.
\]

From now on we fix some sufficiently large \( K > 0 \) (such that, according to Remark 8, there exists \( A_0 \) for which (i) and (i') are fulfilled) and compute the corresponding preliminary selector \( \tilde{I}(K) \). For this \( K \), the properties (i) and (ii) (or, (i') and (ii')) of Theorem 1 provide separately over-size and under-size control of \( \tilde{I} \), with some \( A_0(K) \) and \( A_1(K) \), respectively. By analyzing the proof, we see that always \( A_0(K) \leq A_1(K) \). Hence \( I_s(A_1, \theta) \subseteq I_s(A_0, \theta) \) (as it should be), forming a shell \( I_s(A_0, \theta) \setminus I_s(A_1, \theta) \) in the VSP.

**Remark 9.** If we fix some distribution of \( \xi \) satisfying Condition (A1), another way of defining \( K \) is to think of it as the smallest constant \( K_0 \) such that there exists \( A_0(K_0) \) (by (a) of Remark 8) for which (i) is fulfilled for some \( H_0 \leq H_0' < \infty \) and \( \alpha_0 \geq \alpha'_0 > 0 \).
3.2. Set of signals with distinct active coordinates

In the light of Theorem 1, we can say informally that ˜I “lives in an inflated shell” between $I_\ast(A_1(K), \theta)$ and $I_\ast(A_0(K), \theta)$, which can be thought of as indifference zone for the selector $\tilde{I} = \tilde{I}(K)$. In general, the sets $I_\ast(A_1(K), \theta) \subseteq I_\ast(A_0(K), \theta)$ can be far apart, and $\tilde{I}$ may have too much room to vary. This means that the corresponding signal $\theta$ does not have distinct active and inactive coordinates, active coordinates as such are not identifiable.

The values of the constants $A_0(K)$, $A_1(K)$ evaluated in the proof of the above theorem are of course far from being optimal as we use rather rough bounds in the course of our argument. However, the main message of Theorem 1 is that constants exist such that (i’) and (ii’) are fulfilled. This motivates the following definition of the set of signals with distinctive active and inactive coordinates.  

**Definition.** Fix some $K, M_1, \delta > 0$, and define $A_0(K)$ to be the biggest constant for which (i’) holds with some $M_1' \leq M_1$ and $A_1(K)$ be the smallest constant for which (ii’) is fulfilled with some $\delta' \geq \delta$. Introduce the set

$$\Theta(K) = \{ \theta \in \mathbb{R}^n : I_\ast(A_1(K), \theta) = I_\ast(A_0(K), \theta) \}.$$  

(14)

In what follows, denote for brevity $I_\ast = I_\ast(A_1(K), \theta)$ and $I^\ast = I_\ast(A_0(K), \theta)$.

Remember that the quantities $I_\ast$ and $I^\ast$ depend on $\theta$. The constants $A_0(K) = A_0(K, M_1, \delta)$ and $A_1(K) = A_1(K, M_1, \delta)$ (not depending on $\theta$) exist in view of Theorem 1. This is the main mission of Theorem 1. In essence, these constants are defined to be those which make the room between $I_\ast$ and $I^\ast$ (where $I_\ast = I_\ast(A_1(K), \theta) \subseteq I_\ast(A_0(K), \theta) = I^\ast$) as small as possible uniformly over $\theta \in \mathbb{R}^n$. Hence, imposing $I_\ast = I^\ast$ determines a subset of $\mathbb{R}^n$ for which we can provide simultaneous control for oversizing and undersizing of $\tilde{I}$ by Theorem 1. This means $\Theta(K)$ describes a set of signals with distinctive active and inactive coordinates: if $\theta \in \Theta(K)$, no indifference zone is allowed so that the active coordinates are well defined as $I_\ast = I_\ast(A_1(K), \theta)$.

The above definition (14) is still somewhat implicit, on the other hand it generalizes the traditional strong signal requirement. Indeed, in view of property (c) from Remark 7, if $\theta_i^2 \geq A_1(K)\sigma^2 \log \left( \frac{n}{\delta_i} \right)$ for all $i \in S(\theta)$, then $I_\ast = I_\ast(A_1(K), \theta) = S(\theta) = I_\ast(A_0(K), \theta) = I^\ast$ (in fact, $I_\ast(A, \theta) = S(\theta)$ for all $A \in [0, A_1(K)]$), so that $\theta \in \Theta(K)$.

3.3. Results on active set recovery and multiple testing

Now we establish the control of all the introduced quality measures (FPR, NDR, the Hamming rate, etc.) for the proposed active set selector $\hat{I}$. As consequence, we derive that the procedure $\hat{I}$ matches (in a certain sense) the lower bound results from the previous section, establishing the optimality of the proposed procedure.

**Theorem 2.** Let $\eta_\ast$ and $\tilde{I} = \tilde{I}(K)$ be defined by (10); $I^\ast = I_\ast(A_0(K))$, $I_\ast = I_\ast(A_1(K))$ and $\Theta(K)$ be defined by (14) for sufficiently large $K > 0$, and let $\eta_\ast = \eta_\ast$. Then there exist
constants $H_1, H_2, \alpha_2, \alpha_3 > 0$ such that, uniformly in $\theta \in \mathbb{R}^n$,

\[
\frac{\hat{M} \cap \hat{I}}{n-|\hat{I}|} \leq H_1 \frac{n}{|\hat{I}|} - \alpha_2, \tag{15}
\]

\[
\frac{\hat{I} \cap \hat{I}}{|\hat{I}|} \leq H_2 \frac{n}{|\hat{I}|} - \alpha_3. \tag{16}
\]

Moreover, there exist $H_3, \alpha_4 > 0$ such that, uniformly in $\theta \in \Theta(K),$

\[
R_H(\hat{I}, I_*) = E_{\theta} |\hat{\eta} - \eta_*| \leq E_{\theta} (|\hat{I} \cap I_*| + |I_* \setminus \hat{I}|) \leq H_3 n \left( \frac{n}{|I_*|} \right)^{-\alpha_4} \leq H_3 n \left( \frac{n}{|I_*|} \right)^{-\alpha_4} \tag{17}
\]

From the above theorem, the next corollary follows immediately. It describes control of k-FWER, k-FWNR, the probability of wrong discovery, and the so called almost full recovery (relation (18) below).

**Corollary 1.** Uniformly in $\theta \in \mathbb{R}^n$,

\[
k\text{-FWER}(\hat{I}, I^*) = P_{\theta}(|\hat{I} \cap I^*| \geq k) \leq H_k(n - I^*) \left( \frac{n}{|\hat{I}|} \right)^{-\alpha_2}, \tag{18}
\]

\[
k\text{-FWNR}(\hat{I}, I_*) = P_{\theta}(|I_* \setminus \hat{I}| \geq k) \leq H_k I_* \left( \frac{n}{|\hat{I}|} \right)^{-\alpha_3}.
\]

Uniformly in $\theta \in \Theta(K)$, $P_{\theta}(\hat{I} \neq I_*) \leq H_3 n \left( \frac{n}{|I_*|} \right)^{-\alpha_4}$ and

\[
R_H(\hat{I}, I_*) = \frac{1}{|I_*|} E_{\theta} (|\hat{I} \cap I_*| + |I_* \setminus \hat{I}|) \leq H_3 \left( \frac{n}{|I_*|} \right)^{-\alpha_4 - 1}. \tag{19}
\]

The following result establishes the control of FDR(\hat{I}) and FNR(\hat{I}).

**Theorem 3.** With the same notation as in Theorem 2, there exist constants $H_5, H_6, \alpha_5, \alpha_6 > 0$ such that, uniformly in $\theta \in \Theta(K),$

\[
FDR(\hat{I}, I_*) = E_{\theta} \frac{|\hat{I} \cap I_*|}{|\hat{I}|} \leq H_5 \left( \frac{n}{|I_*|} \right)^{-\alpha_5}, \tag{19}
\]

\[
FNR(\hat{I}, I_*) = E_{\theta} \frac{|I_* \setminus \hat{I}|}{n - |\hat{I}|} \leq H_6 \left( \frac{n}{|I_*|} \right)^{-\alpha_6}. \tag{20}
\]

Theorems 2 and 3 imply the next corollary.

**Corollary 2.** For some constants $H_7, \alpha_7 > 0$, uniformly in $\theta \in \Theta(K),$

\[
MTR_l(\hat{I}, I_*) \leq H_7 \left( \frac{n}{|I_*|} \right)^{-\alpha_7}, \quad l = 1, \ldots, 4. \tag{21}
\]

**Remark 10.** In view of Remark 2, the results of Theorems 1, 2 and Corollaries 1 and 2 hold also uniformly over all the measures $P_{\theta}$ satisfying Condition (A1).

**Remark 11.** Notice that all the powers $\alpha_i$'s in the above theorems and corollaries depend only on $K$ (also via $A_0 = A_0(K)$ and $A_1 = A_1(K)$) and the constants from Condition (A1). Basically, the strong signal condition is reflected by the power $\alpha$'s: the stronger the signal, the bigger the $\alpha$. 
Notice that FPR and NDP are controlled uniformly in $\theta \in \mathbb{R}^n$, whereas all the other quantities only in $\theta \in \Theta(K)$. As we already mentioned, the uniform control of either just Type I error or just Type II error is not much of a value, because this can always be achieved. It is a combination of the two types errors that one should try to control. The most natural choices of such combinations are the Hamming risk and the MTR’s, studied in the present paper. Another possible direction in obtaining interesting results is simultaneous control of Type I error (say, FDR) and some estimation risk (or posterior convergence rate in case of Bayesian approach). Such a route is investigated in [10]. We should mention that $\hat{I}$ could also be derived as a result of empirical Bayes approach with appropriately chosen prior, and similar results could be derived on optimal estimation and posterior convergence rate.

Let us finally discuss possible asymptotic regimes. First, we note that asymptotics $n \to \infty$ is not well defined, unless we describe how the true signal $\theta \in \mathbb{R}^n$ itself evolves with $n$. Assume that $\theta \in \mathbb{R}^n$ evolves with $n \in \mathbb{N}$ in such a way that $|S(\theta)| \leq p$ for some fixed $p \in [0, 1)$. Then from the definition (11) of active coordinates $I_\ast$, (12) and (13), it follows that $\frac{|I_\ast|}{n} \leq p$, but it could also $\frac{|I_\ast|}{n} \to 0$. This basically means that the signal is not getting “less sparse”, in fact it can become “more sparse”, making all the three criterions closer to zero. On the other hand, what can happen in this situation is that the signal is “getting lost” by spreading it over the bigger amount of coordinates. If, under growing dimension, we want the signal still to contain a certain portion of active coordinates, we need to make those coordinates more prominent, i.e., to strengthen the strong signal condition. The active coordinates should be increasing in magnitude when dimension is growing. This can also be attained by decreasing $\sigma^2$. Indeed, another observation is that if $\sigma^2 \to 0$, then $A \to \infty$ in the definition (11) of active coordinates $I_\ast$. One can interpret this as if the strong signal condition becomes more and more stringent. This in turn leads to $\alpha_4 \to \infty$ in (26).

### 3.4. Quantifying uncertainty for the variable selector $\hat{\eta}$

Here we construct confidence ball $B(\hat{\eta}, \hat{r})$ with optimal properties. Let $B(\hat{\eta}, \hat{r}) = \{\eta \in \{0, 1\}^n : |\hat{\eta} - \eta| \leq \hat{r}\}$, $\hat{\eta} = \hat{\eta}(K)$ and $\hat{I} = \hat{I}(K)$ be given by (9). Define

$$\hat{r} = \hat{r}(\hat{I}) = n\left(\frac{\hat{I}}{|I|^{1/2}}\right)^{-\alpha_4'},$$

for some $\alpha_4'$ such that $0 < \alpha_4' < \alpha_4$, with $\alpha_4$ from Theorem 2.

The following theorem describes the coverage and size properties of the confidence ball based on $\hat{\eta}$ and $\hat{r}$.

**Theorem 4.** With the same notation as in Theorem 2, let $\hat{r}$ be defined by (21) and $r_\ast = r_\ast(\theta) = n\left(\frac{\hat{I}}{|I|^{1/2}}\right)^{-\alpha_4'}$. Then there exist constants $M_1, H_7, H_8, \alpha_8, \alpha_9$ such that, uniformly in $\theta \in \Theta(K)$,

$$P_{\theta}(\eta_\ast \notin B(\hat{\eta}, \hat{r})) \leq H_7\left(\frac{n}{|I|^{1/2}}\right)^{-\alpha_8},$$
\[ \mathbb{P}_\theta(\hat{r} \geq M_1^r r_* ) \leq H_8(\frac{n}{|H|^{1/2}})^{-\alpha_0}. \]

According to the UQ-framework \((3)\), we have \(\Theta_{\text{cov}} = \Theta_{\text{size}} = \Theta(K)\) for some \(K > 0\), \(r(\eta_*(\theta)) = r_* (\theta) = n(\frac{n}{|H|^{1/2}})^{-\alpha_1^4}\). Notice that, according to \((26)\), the radius \(\hat{r}\) is \textit{optimal}, in a weak sense as it is up to the constant \(\alpha_1^4 < \alpha_4\).

As we mentioned in Section 1.4, typically the so called \textit{deceptiveness} issue emerges in UQ problems. But in this case, interestingly, there is no deceptiveness issue as such for our UQ problem. A heuristic explanation is as follows: the problem of active set recovery is already more difficult than the UQ-problem in a sense that solving the former problem implies solving the latter. Basically, the condition \(\theta \in \Theta(K)\) for the parameter to have distinct active coordinates implies also that the parameter is non-deceptive.

4. Discussion: weak optimality, phase transition

4.1. Lower bounds

Define \(\Theta_s(a) = \{ \theta \in \ell_0[s] : |\theta_i| \geq a, i \in S(\theta) \}\). Let \(\Theta_s^+(a)\) be the version of \(\Theta_s(a)\) when we put \(\theta_i \geq a\) instead of \(|\theta_i| \geq a\) in the definitions. Clearly, \(\Theta_s^+(a) \subseteq \Theta_s(a)\). For \(\theta \in \Theta_s(a)\), the traditional active set is \(I_s(\theta) = S(\theta)\). To ensure strict separation from the inactive set, one typically imposes \(a \geq \bar{a}_n > 0\) for appropriate \(\bar{a}_n\).

The minimax lower bound over the class \(\Theta_s(a)\) for the problem of the recovery of the active set \(I_s(\theta) = S(\theta)\) in the Hamming risk for the normal means model was derived by [8]. Precisely, under the normality assumption \(\xi_i \overset{\text{ind}}{\sim} N(0, 1)\), Theorem 2.2 from [8] states: for any \(s < n\), \(s' \in (0, s]\),

\[ r_H(\Theta_s^+(a)) \triangleq \inf_{\eta} \sup_{\theta \in \Theta_s^+(a)} \mathbb{E}_\theta |\tilde{\eta} - \eta_{{S}(\theta)}| \geq s' \Phi^+(s, a) - 4s' \exp \left\{ - \frac{(s-s')^2}{2s} \right\}, \]

where \(\Phi^+(s, a) = (\frac{n}{s} - 1) \Phi \left( -\frac{a}{\sigma} - \frac{a}{\alpha} \log \left( \frac{n}{s} - 1 \right) \right) + \Phi \left( -\frac{a}{2\sigma} + \frac{a}{\alpha} \log \left( \frac{n}{s} - 1 \right) \right)\). If \(a^2 \leq 2\sigma^2 \log \left( \frac{n}{s} - 1 \right)\) then by taking \(s' = s/2\) in the above display we get

\[ r_H(\Theta_s^+(a)) \geq \frac{4}{5} \Phi(0) - 2se^{-s/8} = s \left( \frac{1}{4} - 2e^{-s/8} \right) > 0.085s \tag{22} \]

for \(s \geq 20\). Expectedly, if \(a^2 \leq 2\sigma^2 \log \left( \frac{n}{s} - 1 \right)\), it is impossible to achieve even consistency, so there is no point in considering this case. On the other hand, if \(a^2 > 2\sigma^2 \log \left( \frac{n}{s} - 1 \right)\) and \(n/s \geq 2.7\), then

\[ \Phi \left( -\frac{a}{\sigma} - \frac{a}{\alpha} \log \left( \frac{n}{s} - 1 \right) \right) \geq \Phi(-a/\sigma) \geq (2/\pi)^{1/2} e^{-4a^2/\sigma^2}. \]

Assuming further \(\frac{a^2}{\sigma^2} \leq s\) (implying \(s \geq \log n\)) and taking again \(s' = s/2\),

\[ r_H(\Theta_s^+(a)) \geq \frac{n-s}{2} \Phi \left( -\frac{a}{2\sigma} - \frac{a}{\alpha} \log \left( \frac{n}{s} - 1 \right) \right) - 2se^{-s/8} \]

\[ \geq C_1 (n-s) e^{-C_2 a^2/\sigma^2} - C_3 e^{-C_2 s} \geq C_5(n-s) e^{-C_6 a^2/\sigma^2}. \tag{23} \]
Assume that $a^2/\sigma^2 = A \log(\frac{c n}{s})$ for some $A > 2$ and $\log n \lesssim s \leq n/2.7$. Then, under the normality assumption $\xi_i \sim N(0,1)$, it follows from (23) that for some $c_1, c_2 > 0$ (depending only on $A$)

$$\inf \sup_{I, \theta \in \Theta_s(a)} E_\theta \left( |\hat{I} \setminus S(\theta)| + |S(\theta) \setminus \hat{I}| \right) = \inf \sup_{I, \theta \in \Theta_s(a)} E_\theta |\eta_I - \eta_{S(\theta)}|$$

$$= r_H(\Theta_s(a)) \geq r_H(\Theta_s^+(a)) \geq c_1 n(n/s)^{-c_2}. \quad (24)$$

In view of (22) and (24), even in the simplest normal model with $\theta \in \Theta_s(a)$, we need a strong signal condition $a \geq \bar{a}_n = A \sigma^2 \log(\frac{c n}{s})$ with $A > 2$, just to avoid inconsistency in recovering the active set $S(\theta)$. According to the terminology from [8], if $r_H(\Theta_s(a)) \to 0$ as $n \to \infty$, the exact recovery of the active set $S(\theta)$ takes place; and almost full recovery occurs if $r_H(\Theta_s(a))/s \to 0$ as $n \to \infty$ (assuming that $s > 0$). The above lower bound (24) reveals some sort of phase transition. Indeed, the almost full recovery can occur if $s \ll n$ and the constant $A$ is sufficiently large, so that $c_2 > 1$, or, if $s \asymp n$ (but $s/n \leq c < 1$) and $c_2 \to \infty$. The exact recovery is more difficult to fulfill, it can only occur if $n(n/s)^{-c_2} \to 0$ as $n \to \infty$. This is determined by the combination of two factors, the constant $c_2$ and the order of parameter $s$. The parameter $s$ describes the sparsity of the signal $\theta$ and the constant $c_2$ depends on $A$ which expresses the signal strength.

In view of property (c) from Remark 7, it follows that $\Theta_s(a(K)) \subseteq \Theta(K)$ for $a(K) = A_1(K)\sigma^2 \log(\frac{c n}{s})$. Property (b) from Remark 7 implies also that $I_s = I_s(A_1(K), \theta) \subseteq S(\theta)$ for all $\theta \in \mathbb{R}^n$. The last two facts and (24) allow us to derive the following lower bound:

$$\inf \sup_{I, \theta \in \Theta_s(a(K))} E_\theta \frac{|\hat{I} \setminus I_s| + |I_s \setminus \hat{I}|}{n(|I_s|/n)^{c_2}} \geq \inf \sup_{I, \theta \in \Theta_s(a(K))} E_\theta \frac{|\hat{I} \setminus S(\theta)| + |S(\theta) \setminus \hat{I}|}{n(|S(\theta)|/n)^{c_2}} = \frac{r_H(\Theta_s(a(K)))}{n(s/n)^{c_2}} \geq c_1, \quad (25)$$

where the distribution $\mathbb{P}_\theta$ is taken to be the product normal as in (2), $\log n \lesssim s \leq n/2.7$ and $A_1(K) > 2$. The normalizing factor for the Hamming risk is thus $n(|I_s|/n)^{c_2}$.

Recall that the bound (25) is in the regime $A_1(K) > 2$. Otherwise (i.e., when $A_1(K) \leq 2$), we have by (22) that

$$\inf \sup_{I, \theta \in \Theta_s(a(K))} E_\theta \frac{|\hat{I} \setminus I_s| + |I_s \setminus \hat{I}|}{|I_s|} \geq 0.085.$$

**Remark 12.** Notice that the measure $\mathbb{P}_\theta$ in the above lower bounds is the product normal measure (2). However, the same lower bounds trivially hold when, instead of $\sup_{\theta \in \Theta(K)}$, we take $\sup_{\mathbb{P}_\theta \in \mathcal{P}}$, where

$$\mathcal{P} = \{\mathbb{P}_\theta \in \mathcal{P}_A : \theta \in \Theta(K)\}, \quad \mathcal{P}_A = \{\mathbb{P}_\theta : \mathbb{P}_\theta \text{ satisfies } (A1)\}.$$

This is because this normal measure is one of many that satisfy (A1). Note that the constants $A_1(K), A_0(K)$ are then defined uniformly over $\mathcal{P}_A$. 


Remark 13. Similarly to (25), it is easy to derive from (24) that
\[ \inf_l \sup_{\theta \in \Theta(K)} \frac{\text{MTR}(\hat{I}, I_*)}{(|I_*|/n)^2} \geq c_1, \quad l = 1, \ldots, 4. \]
But we conjecture that the right normalizing factor for the minimax MTR’s should be \((|I_*|/n)^2 - 1\), this cannot be derived from (24).

4.2. Phase transition

Relating the lower bound (25) with the results of the previous section, we claim that the selector \(\hat{I}\) (and the corresponding \(\hat{\eta}\)) is optimal in the following weak sense:
\[
c_1 \leq \inf_l \sup_{\theta \in \Theta(K)} \frac{R_H(\hat{I}, I_*)}{n(|I_*|/n)^2}, \quad \sup_{\theta \in \Theta(K)} \frac{R_H(\hat{I}, I_*)}{n((|I_*|/1)/n)^{\alpha_4}} \leq H_3. \tag{26}
\]
In view of Remarks 12, the above relations hold also uniformly over all the measures \(P_\theta\) satisfying Condition (A1).

Admittedly, the optimality in (26) is very weak as it up to constants \(c_2, \alpha_4\), which differ in general. But this is the best we could achieve under the general robust setting of this paper. Constant \(c_2\) in the lower bound is established for the normal submodel (2) of our more general model (1), whereas \(\alpha_4\) is obtained uniformly for the general model (1) under Condition (A1) (thus determined by the constants from Condition (A1)).

Remark 14. We should emphasize that if we want to match of upper and lower bounds we would need to specify the error distribution (or severely restrict the choice). This problem seemingly interesting and challenging does not align with the main focus of the present paper, the robust setting.

However, even these relatively loose lower and upper bounds in (26) can demonstrate some sort of phase transition phenomenon, also in the general setting (1). Precisely, the minimax Hamming \(r_H\) risk (and MTR) can be either close to zero or not, depending on the combination of the signal sparsity \(|I_*|\) and signal magnitude (how strong the signal is), reflected by the constants \(c_2, \alpha_4\). The dependence of the normalizing factor on sparsity \(|I_*|\) is only through the ratio \(n/|I_*|\). The “informativeness” of the model (how “bad” the noise is) plays a role as well in that it determines how large \(K\) must be in the set \(\Theta(K)\) for the upper bound in (26) to hold, which depends on the constants from Condition (A1).

5. Proofs of the theorems

Proof of Theorem 1. For any \(a, b \in \mathbb{R}\), \((a + b)^2 \leq 2a^2 + 2b^2\), hence also \(-(a + b)^2 \leq -a^2/2 + b^2\). Using these elementary inequalities, the definition (9) of \(I(K)\), we derive that, for any \(I, I_0 \subseteq [n],\)
\[
P_\theta(\hat{I}(K) = I) \leq P_\theta\left( \sum_{i \in I^c} X_i^2 + K \sigma^2 \ell(|I|) \leq \sum_{i \in I_0^c} X_i^2 + K \sigma^2 \ell(|I_0|) \right)
\]
\[ \mathbb{P}_\theta \left( \sum_{i \in I \setminus I_0} X_i^2 - \sum_{i \in I_0 \setminus I} X_i^2 \geq K[\ell(|I|) - \ell(|I_0|)] \right) \leq \mathbb{P}_\theta \left( \sum_{i \in I \setminus I_0} \left( \frac{2\sigma_i^2}{\sigma^2} + \sum_{i \in I_0 \setminus I} \left( \frac{\theta_i^2}{2\sigma_i^2} - \xi_i^2 \right) \right) \geq K[\ell(|I|) - \ell(|I_0|)] \right) \]

\[ = \mathbb{P}_\theta \left( \sum_{i \in I \setminus I_0} 2\xi_i^2 + \sum_{i \in I_0 \setminus I} \xi_i^2 \geq \sum_{i \in I_0 \setminus I} \frac{\theta_i^2}{2\sigma_i^2} - \sum_{i \in I \setminus I_0} \frac{2\theta_i^2}{\sigma_i^2} + K[\ell(|I|) - \ell(|I_0|)] \right). \]

In particular, for any \( I, I_0 \) such that \( I_0 \subseteq I \), we have

\[ \mathbb{P}_\theta(\bar{I}(K) = I) \leq \mathbb{P}_\theta \left( \sum_{i \in I \setminus I_0} \xi_i^2 \geq \sum_{i \in I_0 \setminus I} \frac{\theta_i^2}{2\sigma_i^2} + K[\ell(|I|) - \ell(|I_0|)] \right), \quad (27) \]

and for any \( I, I_0 \) such that \( I \subseteq I_0 \), we have

\[ \mathbb{P}_\theta(\bar{I}(K) = I) \leq \mathbb{P}_\theta \left( \sum_{i \in I \setminus I_0} \xi_i^2 \geq \sum_{i \in I_0 \setminus I} \frac{\theta_i^2}{2\sigma_i^2} + K[\ell(|I|) - \ell(|I_0|)] \right). \quad (28) \]

Now we prove (i). For brevity, denote for now \( I_* = I_*|A_0| = I_*|A_0, \theta \). If \( A_0|I\setminus I_*| \log \left( \frac{qn}{|I\setminus I_*|} \right) \) would hold for some \( I \subseteq [n] \), then

\[ r_{A_0}^2(I \cup I_*, \theta) = \sum_{i \notin I \cup I_*} \frac{\theta_i^2}{\sigma_i^2} + A_0\sigma^2 |I \cup I_*| \log \left( \frac{qn}{|I \cup I_*|} \right) \leq \sum_{i \notin I \cup I_*} \frac{\theta_i^2}{\sigma_i^2} + A_0\sigma^2 |I \setminus I_*| \log \left( \frac{qn}{|I \setminus I_*|} \right) + A_0\sigma^2 |I_*| \log \left( \frac{qn}{|I_*|} \right) \]

\[ < \sum_{i \notin I \cup I_*} \frac{\theta_i^2}{\sigma_i^2} + \sum_{i \in I \setminus I_*} \frac{\theta_i^2}{\sigma_i^2} + A_0\sigma^2 |I_*| \log \left( \frac{qn}{|I_*|} \right) \]

\[ = \sum_{i \notin I_*} \frac{\theta_i^2}{\sigma_i^2} + A_0\sigma^2 |I_*| \log \left( \frac{qn}{|I_*|} \right) = r_{A_0}^2(\theta), \]

which contradicts the definition (11) of \( I_* = I_*|A_0, \theta \). Hence,

\[ \sum_{i \in I \setminus I_*} \frac{\theta_i^2}{\sigma_i^2} \leq A_0|I \setminus I_*| \log \left( \frac{qn}{|I \setminus I_*|} \right) \leq A_0|I| \log \left( \frac{qn}{|I|} \right) = A_0 \ell(|I|). \]

Using this and (27) with \( I_0 = I_* \cap I \) (so that \( I \setminus I_0 = I \setminus I_* \)) yields

\[ \mathbb{P}_\theta(\bar{I}(K_0) = I) \leq \mathbb{P}_\theta \left( \sum_{i \in I \setminus I_0} \xi_i^2 \geq \left( \frac{K_0}{2} - A_0 \right) \ell(|I|) - \frac{K_0}{2} \ell(|I_0|) \right) \]

\[ \leq \mathbb{P}_\theta \left( \sum_{i \in I} \xi_i^2 \geq \left( \frac{K_0}{2} - A_0 \right) \ell(|I|) - \frac{K_0}{2} \ell(|I_*|) \right). \]
Let \( \mathcal{J} = \{ I \subseteq [n] : \ell(|I|) \geq M_0 \ell(|I_*(A_0)|) + M \} \), with \( M_0 = K_0/(2C_0) \) where \( C_0 = K_0/2 - A_0 - M_\xi > 0 \), which holds for any \( K_0 > 2(A_0 + M_\xi) \). The last display and (8) imply that for any \( \theta \in \mathbb{R}^n \)

\[
\mathbb{P}_\theta(\tilde{I}(K_0) \in \mathcal{J}) = \sum_{I \in \mathcal{J}} \mathbb{P}_\theta(\tilde{I}(K_0) = I) \\
\leq \sum_{I \in \mathcal{J}} \mathbb{P}_\theta \left( \sum_{i \in I} \xi_i^2 \geq \left( \frac{K_0}{2} - A_0 \right) \ell(|I|) - \frac{K_0}{2} \ell(|I_*(A_0)|) \right) \\
= \sum_{I \in \mathcal{J}} \mathbb{P}_\theta \left( \sum_{i \in I} \xi_i^2 \geq M_\xi \ell(|I|) + C_0 \ell(|I|) - \frac{K_0}{2} \ell(|I_*(A_0)|) \right) \\
\leq \sum_{I \subseteq [n]} \mathbb{P}_\theta \left( \sum_{i \in I} \xi_i^2 \geq M_\xi \ell(|I|) + C_0 M \right) \leq H_0 e^{-\alpha M},
\]

with \( \alpha_0 = C_0 \alpha_\xi \), thus ensuring the result of the assertion (i) for any \( \theta \in \mathbb{R}^n \).

To prove (i'), let \( M'_1 = 4(M_0 + 1) \). If \( |I| \geq M'_1 |I_*| + M \) and \( |I_*| \leq qn/(M'_1)^2 \),

\[
\ell(|I|) = |I| \log \left( \frac{qn}{M_1 |I_*|} \right) \geq \frac{1}{2} \ell(|I|) + \frac{1}{2} |I| \geq \frac{M'_1}{2} |I_*| \log \left( \frac{qn}{M_1 |I_*|} \right) + \frac{M}{2} \\
\geq \frac{M'_1}{4} |I_*| \log \left( \frac{qn}{M_1} \right) + \frac{M}{2} = (M_0 + 1) \ell(|I_*|) + \frac{M}{2}.
\]

Hence, for \( |I_*| \leq qn/M_1^2 \), by using (i),

\[
\mathbb{P}_\theta(|\tilde{I}(K_0)| \geq M'_1 |I_*| + M) \leq \mathbb{P}_\theta(\ell(|\tilde{I}(K_0)|) \geq (M_0 + 1) \ell(|I_*|) + \frac{M}{2}) \\
\leq H_0 e^{-\alpha_0 \ell(|I_*|) - \alpha_0 M/2}.
\]

If \( |I_*| > qn/(M'_1)^2 \) and \( M''_1 = (M'_1)^2/q \), then we trivially obtain

\[
\mathbb{P}_\theta(|\tilde{I}| \geq M''_1 |I_*| + M) \leq \mathbb{P}_\theta(|\tilde{I}| \geq M''_1 |I_*|) \leq \mathbb{P}_\theta(|\tilde{I}| > n) = 0.
\]

Hence the choice \( M_1 = \max\{ M'_1, M''_1 \} \) ensures the second relation (i').

Next, we prove the assertion (ii). For the rest of the proof, denote for brevity \( I_* = I_*(A_1) \).

Define \( \mathcal{T} = \{ I \in \mathcal{I} : \ell(|I|) \leq \delta \ell(|I_*|) - M \}, \delta \in [0, 1] \). If \( I \in \mathcal{T} \), then \( \ell(|I|) \leq \delta \ell(|I_*|) \leq \ell(|I_*|) \), implying that \( |I| \leq \delta |I_*| \), as \( \ell(x) \) is increasing for \( x \in [0, n] \). Hence, for any \( I \in \mathcal{T} \),

\[
\ell(|I_* \cup I|) = |I_* \cup I| \log \left( \frac{qn}{|I_* \cup I|} \right) \leq |I_*| \log \left( \frac{qn}{|I_* \cup I|} \right) + |I| \log \left( \frac{qn}{|I|} \right) \\
\leq (1 + \delta) |I_*| \log \left( \frac{qn}{|I_*|} \right) - M = (1 + \delta) \ell(|I_*|) - M. \tag{29}
\]

Next, by (13) and the fact that \( |I| \leq \delta |I_*| \), we obtain that for any \( I \in \mathcal{T} \),

\[
\sum_{i \in I_* \setminus I} \frac{\sigma_i^2}{\sigma} \geq |I_* \setminus I| A_1 \log \left( \frac{qn}{|I_* \setminus I|} \right) \geq A_1 (1 - \delta) |I_*| \log \left( \frac{qn}{|I_*|} \right) \geq \frac{A_1 (1 - \delta)}{2} \ell(|I_*|), \tag{30}
\]

and
Denote for brevity $C_{A_1} = \frac{A_1(1-\delta)}{4} - K_1(1+\delta)$. Using the relation (28) with $I_0 = I_* \cup I$ (so that $I_0 \setminus I = I_* \setminus I$), the relations (29), (30), and (8), we derive that

$$\mathbb{P}_\theta(\tilde{I}(K_1) \in T) = \sum_{I \in T} \mathbb{P}_\theta(\tilde{I}(K_1) = I)$$

$$\leq \sum_{I \in T} \mathbb{P}_\theta \left( \sum_{i \in I_* \setminus I} \xi_i^2 \geq \sum_{i \in I_* \setminus I} \frac{\theta_i^2}{2\sigma_i^2} + K_1(\ell(|I|) - \ell(|I_* \cup I|)) \right)$$

$$\leq \sum_{I \in T} \mathbb{P}_\theta \left( \sum_{i \in I_* \setminus I} \xi_i^2 \geq \left( \frac{A_1(1-\delta)}{4} - K_1(1+\delta) \right) \ell(|I_*|) + K_1[\ell(|I|) + M] \right)$$

$$= \sum_{I \in T} \mathbb{P}_\theta \left( \sum_{i \in I_* \setminus I} \xi_i^2 \geq C_{A_1} \ell(|I_*|) + K_1 \ell(|I|) + K_1 M \right)$$

$$\leq \sum_{I \subseteq [n]} \mathbb{P}_\theta \left( \sum_{i \in I_* \setminus I} \xi_i^2 \geq M_\xi \ell(|I_*|) + (C_{A_1} - M_\xi) \ell(|I_*|) + K_1 M \right)$$

$$\leq \sum_{I \subseteq [n]} \mathbb{P}_\theta \left( \sum_{i \in I_* \setminus I} \xi_i^2 \geq M_\xi \ell(|I_*|) + (C_{A_1} - M_\xi) \ell(|I_*|) + K_1 M \right)$$

$$\leq H_0 e^{-\alpha_1 M - \alpha_1 (C_{A_1} - M_\xi) \ell(|I_*|)} = H_0 e^{-\alpha_1 M - \alpha'_1 \ell(|I_*|)},$$

where $\alpha_1 = \alpha_1 K_1$, $\alpha'_1 = \alpha_1 (C_{A_1} - M_\xi)$ and $A_1$ is assumed to be so large that $C_{A_1} > M_\xi$. This proves (ii).

Finally, we establish (iii). Define $T' = \{ I \in T : |I| \leq \delta |I_*| \}, \delta \in [0, 1)$. The relations (30), (29) are still valid for any $I \in T'$. As before, we obtain (31) (with $M = 0$ and $T'$ instead of $T$), which we now continue as follows:

$$\mathbb{P}_\theta(\tilde{I}(K_1) \in T') \leq \sum_{I \in T'} \mathbb{P}_\theta \left( \sum_{i \in I_* \setminus I} \xi_i^2 \geq C_{A_1} \ell(|I_*|) + K_1 \ell(|I|) \right)$$

$$\leq \sum_{I \in T'} \mathbb{P}_\theta \left( \sum_{i \in I_* \setminus I} \xi_i^2 \geq M_\xi \ell(|I_*|) + (C_{A_1} - M_\xi) \ell(|I_*|) \right)$$

$$= \sum_{I \subseteq [n]} \mathbb{P}_\theta \left( \sum_{i \in I_* \setminus I} \xi_i^2 \geq M_\xi \ell(|I_*|) + (C_{A_1} - M_\xi) \ell(|I_*|) \right)$$

$$\leq H_0 e^{-\alpha'_1 \ell(|I_*|)},$$

with $\alpha'_1 = \alpha_1 (C_{A_1} - M_\xi)$, $A_1$ is assumed to be so large that $C_{A_1} > M_\xi$.

**Proof of Theorem 2.** First we prove (15). Consider the case $|I^*| \geq 1$. Let $B = \{ |\tilde{I}| > M_1 |I^*| \}$. Recalling that $I^* = I_*(A_0)$, by (13), $\theta_i^2 \leq A_0 \sigma_i^2 \log \left( \frac{2n}{|I^*|} \right)$ for all $i \in (I^*)^c$. Using this, Condition (A1), the definition (10), property (i') of Theorem 1, we have that for $I^* = I_*(A_0)$ and $\tilde{I} = \tilde{I}(K)$

$$\text{FPR}(\tilde{I}, I^*) = E_\theta \frac{|\tilde{I}(I^*)^c|}{n-|\tilde{I}|} = \frac{1}{n-|\tilde{I}|} \sum_{i \in (I^*)^c} E_\theta \eta(X_i)(1_B + 1_{B^c})$$
The relations (32) and (33) establish (15).

Next, we proof the assertion (16). If $I_s = \emptyset$, the claim follows, assume $|I_s| > 1$. Denote $B_\delta = \{|\hat{I}| \leq \delta|I_s|\}$. Using Condition (A1), the definition (10), (13), property (ii') of Theorem 1, and the fact that $(a + b)^2 \geq 2a^2 - 2b^2$ for any $a, b \in \mathbb{R}$, we have that for $I_s = I_s(A_1)$ and $\hat{I} = \hat{I}(K)$

$$\text{NDR}(\hat{I}, I_s) = \frac{1}{|I_s|} E_\theta |I_s| |\hat{I}| = \frac{1}{|I_s|} \sum_{i \in I_s} E_\theta (1 - \hat{\eta}(X_i))$$

$$= \frac{1}{|I_s|} \sum_{i \in I_s} \mathbb{P}_\theta (|\theta_i + \sigma \xi_i| < \sigma [K \log(nM_{|I_s|})]^{1/2})$$

$$\leq \frac{1}{|I_s|} \sum_{i \in I_s} \mathbb{P}_\theta \left( \frac{\theta_i^2}{2} - 2\sigma^2 \xi_i^2 < \sigma^2 K \log(nM_{|I_s|}) \right)$$

$$\leq \frac{1}{|I_s|} \sum_{i \in I_s} \mathbb{P}_\theta (\xi_i^2 > \frac{\alpha_0}{\delta} \log(nM_{|I_s|}) - \frac{K}{\delta} \log(nM_{|I_s|}), B_\delta)$$

$$\leq H_0 e^{-\alpha_0 \xi(|I_s|)} + \frac{1}{|I_s|} \sum_{i \in I_s} \mathbb{P}_\theta (\xi_i^2 > \frac{\alpha_0}{\delta} - \frac{K}{\delta} \log(nM_{|I_s|}) + \frac{K}{\delta} \log \delta)$$

$$\leq H_0 e^{-\alpha_0 \xi(|I_s|)} + C_2 \xi_{|I_s|}^{-\alpha_3} \leq H_2 \xi_{|I_s|}^{-\alpha_3},$$

(34) for sufficiently large $A_1$ (such that $\frac{A_1}{6} - \frac{K}{\delta} > M\xi$ and the property (ii') of Theorem 1 can
be applied), where \( \alpha' = \alpha \xi (\frac{A_3}{8} - \frac{K}{2} - M \xi) \) and \( C_2 = H \xi e^{-\alpha \xi K \log \sqrt{\xi}} \). The relation (16) is proved.

Since \( I^* = I_*(A_0(K)) = I_*(A_1(K)) = I_* \) for \( \theta \in \Theta(K) \), the assertion (17) follows from the relations (32) and (34): uniformly in \( \theta \in \Theta(K) \),

\[
E_\theta (|\hat{I} \setminus I_*| + |I_* \setminus \hat{I}|) \leq (n - |I_*|) H_1(\frac{n}{|I_*| - 1})^{-\alpha_2} + |I_*| H_2(\frac{n}{|I_*| - 1})^{-\alpha_3} \leq H_3 n(\frac{n}{|I_*| - 1})^{-\alpha_4}. \]

\[ \square \]

**Proof of Theorem 3.** First, we proof assertion (19). Introduce the event \( B = \{|\hat{I}| < \delta |I_*|\} \).

We argue along the same lines as in (32) and (33) for the two cases \( |I_*| > 0 \) and \( |I_*| = 0 \).

For the case \( |I_*| > 0 \), we use (ii') of Theorem 1, (15) of Theorem 2 and the fact that \( I^* = I_* \) for \( \theta \in \Theta(K) \) to derive

\[
\text{FDR}(\hat{I}) = E_\theta \left[ \frac{|\hat{I} \setminus I_*|}{|\hat{I}|} \right] = E_\theta \left[ \frac{|\hat{I} \setminus I_*|}{|\hat{I}|} (1_B + 1_{\bar{B}}) \right] \leq \frac{1}{\delta |I_*|} E_\theta \left[ |\hat{I}| I^* \right] + P_\delta (B) \leq \frac{1}{\delta |I_*|} H_1 n \left( \frac{n}{|I_*| - 1} \right)^{-\alpha_2} + H_0 e^{-\alpha_4 |I_*|} \leq H_5 \left( \frac{n}{|I_*|} \right)^{-\alpha_6}.
\]

The case \( |I_*| = 0 \) is handled as follows. If \( |\hat{I}| = 0 \), the claim holds. Assume \( |\hat{I}| \geq 1 \), then

\[
\text{FDR}(\hat{I}) \leq E_\theta \left[ |\hat{I} \setminus I_*| \right] \leq H_1 n^{-\alpha_2} = H_1 n^{-(\alpha_2 - 1)}.
\]

Next, we prove assertion (20). Introduce the event \( B = \{|\hat{I}| > M_1 |I^*|\} \). Consider two cases: the case \( |I^*| \geq n/(2M_1) \) and the case \( |I^*| < n/(2M_1) \). Suppose \( |I^*| \geq n/(2M_1) \), then FNR(\( \hat{I} \)) = \( E_\theta \left[ \frac{|\hat{I} \setminus I^*|}{n - |\hat{I}|} \right] \leq 2M_1 |I^*| n = 2M_1 |I_*| n \) and (20) holds, as \( I^* = I_* \) for \( \theta \in \Theta(K) \).

Now suppose \( |I^*| < n/(2M_1) \). Then using the same reasoning as in (34), Theorem 1 and the fact that \( I^* = I_* \) for \( \theta \in \Theta(K) \), we again obtain (20):

\[
\text{FNR}(\hat{I}) = E_\theta \left[ \frac{|I_* \setminus \hat{I}|}{n - |I|} \right] = E_\theta \left[ \frac{|I_* \setminus \hat{I}|}{n - |\hat{I}|} (1_B + 1_{\bar{B}}) \right] \leq \frac{1}{n - M_1 |I_*|} E_\theta \left[ |I_* \setminus \hat{I}| \right] + P_\delta (B) \leq H_6 \left( \frac{n}{|I_*|} \right)^{-\alpha_7}. \]

\[ \square \]

**Proof of Theorem 4.** We first establish the coverage property. Recall that \( \hat{r} = \hat{r}(\hat{I}) = n(\frac{n}{|I|})^{-\alpha_4} \) and by (17) of Theorem 2,

\[
\sup_{\theta \in \Theta(K)} E_\theta (\hat{r} - \eta) \leq H_3 n(\frac{n}{|I|})^{-\alpha_4}. \tag{35}
\]

Denote \( B = \{|\hat{I}| \leq \delta |I_*|\} \). Now, by using the Markov inequality, property (ii') of Theorem 1 and (35), we obtain, uniformly in \( \theta \in \Theta(K) \),

\[
\mathbb{P}_\theta (\eta \notin B(\hat{r}, \hat{r})) = \mathbb{P}_\theta (|\eta - \hat{r}| > \hat{r}, B_\delta) + \mathbb{P}_\theta (B_\delta) \leq \mathbb{P}_\theta (B_\delta) + \mathbb{P}_\theta (|\eta - \hat{r}| > n(\frac{n}{|I_*|})^{-\alpha_4})
\]
\[ \leq H_0 e^{-\alpha_4 |I^*|} + H_3 \delta \alpha_4 \left( \frac{n}{|\pi \cap \mathcal{V}|} \right)^{-\alpha_4} \leq H_7 \left( \frac{n}{|\pi \cap \mathcal{V}|} \right)^{-\alpha_7}, \]

which proves the coverage property.

It remains to prove the size property. Let \( B = \{|I| \geq M_1 |I^*|\} \). For any \( M'_1 > M'_1 \lor 1 \), we have that, uniformly in \( \theta \in \Theta(K) \) (so that \( I^* = I_\theta \)),

\[
P_\theta(\hat{r} \geq M'_1 r_*, B^c) \leq P_\theta \left( n \left( \frac{n}{|\pi \cap \mathcal{V}|} \right)^{-\alpha_4} \geq M'_1 n \left( \frac{n}{|\pi \cap \mathcal{V}|} \right)^{-\alpha'_4} \right) = 0.
\]

Using this, the property \((i')\) of Theorem 1 and the fact that \( I^* = I_\theta \) for \( \theta \in \theta(K) \), we derive that, uniformly in \( \theta \in \Theta(K) \),

\[
P_\theta(\hat{r} \geq M'_1 r_*) \leq P_\theta(B) + P_\theta(\hat{r} \geq M'_1 r_*, B^c) = P_\theta(B) \leq H_0 e^{-\alpha_4 |I^*|} \leq H_8 \left( \frac{n}{|\pi \cap \mathcal{V}|} \right)^{-\alpha_8},
\]

yielding the size property. \(\square\)

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