GAP THEOREMS FOR ENDS OF SMOOTH METRIC MEASURE SPACES

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Abstract. In this paper, we establish two gap theorems for ends of smooth metric measure space \((M^n, g, e^{-f} dv)\) with the Bakry-Émery Ricci tensor \(\text{Ric}_f \geq -(n-1)\) in a geodesic ball \(B_o(R)\) with radius \(R\) and center \(o \in M^n\). When \(\text{Ric}_f \geq 0\) and \(f\) has some degeneration outside \(B_o(R)\), we show that there exists an \(\epsilon = \epsilon(n, \sup_{B_o(1)} |f|)\) such that such a space has at most two ends if \(R \leq \epsilon\). When \(\text{Ric}_f \geq \frac{1}{2}\) and \(f(x) \leq \frac{1}{4} d^2(x, B_o(R)) + c\) for some constant \(c > 0\) outside \(B_o(R)\), we can also get the same gap conclusion.

1. Introduction and main results

The Cheeger-Gromoll’s splitting theorem [4] states that if a complete Riemannian manifold \((M^n, g)\) with nonnegative Ricci curvature contains a line, then \(M^n\) is isometric to \(N \times \mathbb{R}\) with the product metric, where \(N\) is a Riemannian manifold with the Ricci curvature \(\text{Ric}(N) \geq 0\). As a consequence, any manifold with nonnegative Ricci curvature has at most two ends. In [2], Cai studied a complete manifold \(M^n\) with \(\text{Ric} \geq -(n-1)K\) for some constant \(K \geq 0\) in a geodesic ball \(B_o(R)\) with radius \(R\) and center \(o \in M^n\) and \(\text{Ric} \geq 0\) outside \(B_o(R)\). He proved that the number of ends of such a manifold is finite and can be estimated from above explicitly; see also Li and Tam [7] for an independent proof by a different method. Later, Cai, Colding and Yang [3] gave a gap theorem for this class of manifolds, which states that there exists an \(\epsilon(n)\) such that such a manifold has at most two ends if \(KR \leq \epsilon(n)\). In this paper we will extend the Cai-Colding-Yang result and get two gap theorems on smooth metric measure spaces with the Bakry-Émery Ricci tensor. Our results may be useful for understanding the topological information of smooth metric measure spaces.

Recall that a complete smooth metric measure space (for short, SMMS) is a triple \((M, g, e^{-f} dv)\), where \((M, g)\) is an \(n\)-dimensional complete Riemannian manifold, \(dv\) is the volume element of metric \(g\), \(f\) is a smooth potential function on \(M\), and \(e^{-f} dv\) is called the weighted volume element. On \((M, g, e^{-f} dv)\), given a constant \(m > 0\), Bakry and Émery [1] introduced the \(m\)-Bakry-Émery Ricci tensor

\[ \text{Ric}_f^m := \text{Ric} + \text{Hess} f - \frac{d f \otimes d f}{m}, \]

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where Ric is the Ricci tensor of \((M, g)\) and Hess is the Hessian with respect to the metric \(g\). When \(m = \infty\), we have the \((\infty)\)-Bakry-Émery Ricci tensor

\[
\text{Ric}_f = \text{Ric}_f^\infty.
\]

If \(\text{Ric}_f = \lambda g\) for some \(\lambda \in \mathbb{R}\), then \((M, g, e^{-f}dv)\) is called the gradient Ricci soliton, which is a generalization of the Einstein manifold. A Ricci soliton is called shrinking, steady or expanding, if \(\lambda > 0\), \(\lambda = 0\), or \(\lambda < 0\), respectively. Gradient Ricci soliton often arises as a limit of dilations of singularities in the Ricci flow and it plays a fundamental role in the Ricci flow \([6]\) and Perelman’s resolutions of the Poincaré Conjecture \([17, 18, 19]\).

The Bakry-Émery Ricci tensor \(\text{Ric}_f\) is linked with the \(f\)-Laplacian \(\Delta_f := \Delta - \nabla f \cdot \nabla\) via the generalized Bochner formula

\[
\Delta_f |\nabla u|^2 = 2 |\text{Hess} u|^2 + 2g(\nabla u, \nabla \Delta_f u) + 2\text{Ric}_f(\nabla u, \nabla u)
\]

for \(u \in C^\infty(M)\). It plays an important role in the comparison geometry of SMMSs; see \([20]\). The Bakry-Émery Ricci tensor is also related to the probability theory and optimal transport. We refer the reader to \([10, 11, 20]\) for further details.

Lichnerowicz \([8]\), and Wei and Wylie \([20]\) independently extended the classical Cheeger-Gromoll splitting theorem to a SMMS. It states that if \((M, g, e^{-f}dv)\) with \(\text{Ric}_f \geq 0\) and bounded \(f\) contains a line, then \(M = N \times \mathbb{R}\). Fang, Li and Zhang \([3]\) showed that the above splitting result remains true for only an upper bound on \(f\). Lim \([9]\) observed that the splitting result holds if \(\nabla f \to 0\) at infinity. In \([15]\) Munteanu and Wang proved a splitting result when \(\text{Ric}_f\) has a positive lower bound and \(f\) satisfies certain quadratic growth of distance function. Recently, G. Wu \([21]\) obtained splitting results for the gradient Ricci soliton when some integral of the Ricci curvature along a line is nonnegative. From these results, we see that the above mentioned manifolds all have at most two ends.

Besides, Wei and Wylie \([20]\) proved that any SMMS with \(\text{Ric}_f > 0\) for some bounded \(f\) has only one end. The second author \([22]\) studied a SMMS with \(\text{Ric}_f \geq 0\) outside a compact set and proved that the number of ends of such a manifold is finite if \(f\) has at most sublinear growth outside the compact set.

Inspired by the gap theorem of manifolds \([3]\) and the number estimate for ends of SMMSs \([22]\), in this paper we first give a gap theorem for ends of a smooth metric measure space when \(\text{Ric}_f \geq 0\) and \(f\) has some degeneration outside a compact set.

**Theorem 1.1.** Let \((M, g, e^{-f}dv)\) be an \(n\)-dimensional complete smooth metric measure space. Fix a point \(o \in M\) and \(0 < R < 1\). Suppose \(\text{Ric}_f \geq -(n - 1)K\) for some constant \(K \geq 0\) in the geodesic ball \(B_o(R)\); outside \(B_o(R)\), suppose \(\text{Ric}_f \geq 0\) and

\[
\lim_{r \to \infty} \frac{1}{r^{n/2}} \int_0^r f(\sigma(t))dt \leq 0
\]

on any ray \(\sigma\), where \(r\) is the distance function starting from \(\sigma(0)\). There exists a constant \(\epsilon(n, A)\) depending only on \(n\) and \(A\), where \(A := \sup_{x \in B_o(1)} |f(x)|\), such that if \(\sqrt{KR} \leq \epsilon(n, A)\), then \((M, g, e^{-f}dv)\) has at most two ends.

**Remark 1.2.** There exist many examples satisfying Theorem 1.1. Let \(M = S^{n-1} \times \mathbb{R}\) \((n \geq 3)\) with the standard metric and \(f(x, t) = f(t) = 1 + \int_0^t ds \int_0^s \xi(\tau)d\tau\) for
So if Ric by applying the metric surgery techniques to manifold $M$, then $\xi(t) \geq -\alpha > 0$ and $\delta > 0$ for some constant $Ricci$ soliton because the potential $f$ grows quadratically outside a compact set. More precisely, after a suitable scaling $f''(t) = \xi(t)$. So $Ric_f \geq -(n-1)$ and $Ric_f \geq 0$ outside $B_o(R)$. Since $f(t) \leq C|t|^{2-n} + C$ for $|t| \gg 1$, $f(0) = 1$ and $A = sup_{x \in B_o(1)}|f(x)| \geq 1$, they satisfy the conditions in Theorem 1.1 for $K = 1$ and $M$ has two ends. Another example is that: as in [3], by applying the metric surgery techniques to manifold $M = S^1 \times \mathbb{R} \times \mathbb{S}^{n-2}$, $n \geq 4$, one can get an $n$-dimensional complete manifold $M$ of infinite homotopy type with exactly two ends and with $Ric \geq -\delta$ and with $Ric \geq 0$ outside a small ball. Let $f(x_1, t, x_2) = -t$ on $M$, and it satisfies (1.1). Then $Ric_f \geq -\delta$ and $Ric_f \geq 0$ outside the small ball.

**Remark 1.3.** If $f$ grows sublinearly, then (1.1) automatically holds. If $\nabla f \to 0$ at infinity, (1.1) still holds due to Lim [9]. If $f$ is constant, the theorem recovers the Cai-Colding-Yang result [3]. Theorem 1.1 is obvious suitable to gradient steady Ricci soliton because the potential $f$ of steady gradient Ricci solitons is negative linear outside a compact set. It is an interesting question if one can weaken the assumption of $f$ such that it is suitable to the quadratic growth of $f$ on gradient shrinking Ricci solitons.

**Remark 1.4.** When we say that $E$ is an end of the manifold $M$ we mean that it is an end with respect to some compact subset of the manifold. If $R_1 \leq R_2$, then the number of ends with respect to $B_o(R_1)$ is less than the number of ends with respect to $B_o(R_2)$. So we assume the radius $R < 1$ in the theorem seems to be sensible.

We can apply a similar argument to get a gap theorem under the conditions of $Ric^n_f$ (without any assumption on $f$). It states that when $Ric^n_f \geq -(n-1)K$ in $B_o(R)$ and $Ric^n_f \geq 0$ outside $B_o(R)$, there exists an $\epsilon = \epsilon(n + m)$ depending only on $n + m$ such that $M$ has at most two ends if $\sqrt{K}R \leq \epsilon$.

Furthermore, inspired by the Mantegazza-Wang splitting theorem [15], we may give another gap theorem for ends when $Ric_f$ has a positive lower bound and $f$ grows quadratically outside a compact set. More precisely, after a suitable scaling of the metric, we may in fact assume $Ric_f \geq -(n-1)$ in a ball and get that

**Theorem 1.5.** Let $(M, g, e^{-f}dv)$ be an $n$-dimensional complete smooth metric measure space. Fix a point $o \in M$ and $0 < R < 1$. Suppose $Ric_f \geq -(n-1)$ in the geodesic ball $B_o(R)$; outside $B_o(R)$, suppose $Ric_f \geq \frac{1}{R}$ and $f(x) \leq \frac{1}{2R}d^2(x, B_o(R)) + c$ for some constant $c > 0$. There exists a constant $\epsilon(n, A)$ depending only on $n$ and $A$, where $A := sup_{x \in B_o(1)}|f(x)|$, such that if $$ R \leq \epsilon(n, A), $$

then $(M, g, e^{-f}dv)$ has at most two ends.

**Remark 1.6.** We give an example satisfying Theorem 1.5. Let $M = S^{n-1} \times \mathbb{R}$ ($n \geq 3$) with the standard metric. For $0 < R < 1$, let $f(x, t) = f(t) = 1 + a + bt + \int_0^t ds \int_0^\infty \eta(t)ds$ for $(x, t) \in S^{n-1} \times \mathbb{R}$, where $a, b$ are chosen satisfying $f(R) = 1, f'(R) = 0$. Here $\eta(t)$ is a smooth even function on $\mathbb{R}$ such that $\eta(t) = -(n-1), t \in [0, \frac{R}{2}], \eta(t) \geq -(n-1), t \in (\frac{R}{2}, R)$ and $\eta(t) = \frac{1}{2}, t \in [R, \infty)$. One easily checks that $f$ is smooth, even and $f''(t) = \eta(t)$. So $Ric_f \geq -(n-1)$ and $Ric_f \geq \frac{1}{R}$ outside $B_o(R)$. Since $f(t) = \frac{1}{4}(|t| - R)^2 + 1$ for $|t| \geq R, f(R) = 1$ and $A \geq 1$, they satisfy the conditions of Theorem 1.5 and $M$ has two ends.
Remark 1.7. If the assumption $R \leq e(n, A)$ in Theorem 1.5 is removed, we can show that the number of ends for such a space is finite. Moreover, we can provide an explicit upper bound for the number; see Appendix of the paper.

We would like to point out that Munteanu and Wang systematically studied the number of ends on gradient Ricci solitons. In [13] they proved that any nontrivial steady gradient Ricci soliton has only one end. In [14] they showed that the expanding gradient Ricci soliton $\text{Ric}_f = -\frac{1}{2}$ with scalar curvature $S \geq -\frac{n-1}{2}$ has at most two ends. They also considered a similar problem for SMMS with $\text{Ric}_f \geq -\frac{1}{2}$. In [16] they proved that any shrinking Kähler gradient Ricci soliton has only one end. Recently, Munteanu, Schulze and Wang [12] showed that the number of ends is finite on shrinking gradient Ricci soliton when the scalar curvature satisfies certain scalar curvature integral at infinity.

The proof of our theorems uses the argument of Cai-Colding-Yang [3] and it relies on a Wei-Wylie’s weighted Laplacian comparison [20] and geometric inequalities for two different ends (see Lemmas 2.8 and 2.11), which are derived by locally analyzing splitting theorems. We would like to point out that Cai-Colding-Yang’s proof depends on a delicate constructional function $G(r)$, which satisfies certain Laplacian equation with the Dirichlet boundary condition. In our case, the function $G(r)$ constructed in Proposition 3.1 does not satisfy the $f$-Laplacian equation, but it is sufficient to deduce our desired results.

The paper is organized as follows. In Section 2, we give some basic concepts and results on SMMSs. In Section 3, we apply the weighted Laplacian comparison and geometric inequalities for ends in Section 2 to prove our theorems. In Appendix, we give an upper bound for the number of ends of a class of SMMSs.

2. Preliminary

In this section, we introduce some results about SMMSs, which will be used in the proof of our results. We first recall a weighted Laplacian comparison due to Wei and Wylie [20].

**Lemma 2.1.** Let $(M, g, e^{-f}dv)$ be an $n$-dimensional complete smooth metric measure space with a base point $o \in M$. If $\text{Ric}_f \geq -(n-1)K$ for some constant $K > 0$, then

$$\Delta f(r) \leq (n+4A-1)\sqrt{K} \coth(\sqrt{K}r)$$

along any minimal geodesic segment $r$ from $o$, where $A = A(o, r) = \sup_{x \in B_{o}(r)} |f(x)|$.

We also have a weighted volume comparison of Wei and Wylie [20]. The weighted volume is denoted by $V_f(B_x(R)) := \int_{B_x(R)} e^{-f}dv.$

**Lemma 2.2.** Let $(M, g, e^{-f}dv)$ be an $n$-dimensional complete smooth metric measure space. If $\text{Ric}_f \geq -(n-1)K$ for some constant $K > 0$, then

$$\frac{V_f(B_x(r_2))}{V_f(B_x(r_1))} \leq \frac{\int_{r_1}^{r_2} (\sinh^{n-1+4A} \sqrt{K} t)dt}{\int_{r_1}^{r_2} (\sinh^{n-1+4A} \sqrt{K} t)dt}$$

for any $x \in M$ and $0 < r_1 < r_2$, where $A = A(x, r_2) = \sup_{y \in B_x(r_2)} |f(y)|$.

Then we recall some definitions of geometric quantities such as line, ray, end and asymptotic ray on Riemannian manifolds.
Definition 2.3. On a complete Riemannian manifold \((M, g)\), we say that a geodesic \(\gamma: (-\infty, +\infty) \to M\) is called line if
\[
d(\gamma(s), \gamma(t)) = |s - t|
\]
for all \(s\) and \(t\). We say that a geodesic \(\gamma: [0, +\infty) \to M\) is called ray if
\[
d(\gamma(0), \gamma(t)) = t
\]
for all \(t > 0\). As we all known if \(M\) is complete noncompact, it must contain a ray.

Definition 2.4. On a manifold \(M\) with a base point \(o \in M\), two rays \(\gamma_1\) and \(\gamma_2\) starting at \(o\) are called cofinal if for any \(R > 0\) and \(t > R\), \(\gamma_1(t)\) and \(\gamma_2(t)\) lie in the same component of \(M \setminus B_o(R)\). An equivalent class of cofinal rays is called an end of \(M\). In this paper we let \([\gamma]\) be the class of the ray \(\gamma\).

One readily checks that this definition is independence of the base point \(o\) and the complete metric on manifold \(M\). Thus, the number of ends is a topological invariant of \(M\).

Next we recall the definition of the Busemann function and its properties on a complete SMMS \((M, g, e^{-\int f dv})\). The Busemann function associated to each ray \(\gamma \subset M\) is defined by
\[
b_\gamma(x) := \lim_{t \to \infty} (d(x, \gamma(t)) - t).
\]
By the triangle inequality, we know that \(b_\gamma(x)\) is Lipschitz continuous with Lipschitz constant 1 and hence it is differential almost everywhere. At the points where \(b_\gamma\) is not smooth we interpret the \(f\)-Laplacian in the following sense of barriers.

Definition 2.5. A lower barrier for a continuous function \(h\) at the point \(p \in M\) is a \(C^2\) function \(h_p\), defined in a neighborhood \(U\) of \(p\), such that
\[
h_p(p) = h \quad \text{and} \quad h_p(x) \leq h(x), \quad x \in U.
\]
A continuous function \(h\) on \(M\) satisfies \(\Delta_f h \geq a\) at \(p\) in the barrier sense, if for every \(\epsilon > 0\), there exists a lower barrier function \(h_{p, \epsilon}\) at \(p\) such that \(\Delta_f h_{p, \epsilon} \geq a - \epsilon\). A continuous function \(h\) satisfies \(\Delta_f h \leq a\) in the barrier sense is similarly defined.

Definition 2.6. For a fixed point \(p \in M\), let \(\alpha(t)\) be a minimal geodesic from \(p\) to ray \(\gamma(t)\). As \(t \to \infty\), \(\alpha(t)\) has a convergent subsequence which converges to a ray at \(p\). Such a ray is called an asymptotic ray to \(\gamma(t)\) at \(p\).

For a line \(\gamma\) in \(M\), there exist rays \(\gamma^+: [0, \infty) \to M\) by \(\gamma^+(t) = \gamma(t)\) and \(\gamma^-: [0, \infty) \to M\) by \(\gamma^-(t) = \gamma(-t)\). Similar to the above procedure, we can let \(b_\gamma^+(x)\) (or \(b_\gamma^-\), respectively) be the associated Busemann function of \(\gamma^+(\text{or} \gamma^-\), respectively).

Next we will introduce two geometric inequalities for two different ends under two types of curvature assumptions, which are important in the proof of Theorems 1.1 and 1.5. On one hand, recall that Fang, Li and Zhang [5] proved a Cheeger-Gromoll splitting theorem when \(\text{Ric}_f \geq 0\) and \(f\) satisfy some degeneration condition. As in [22], we can easily apply the Fang-Li-Zhang arguments locally and get that

Lemma 2.7. Let \(N\) be the \(\delta\)-tubular neighborhood of a line \(\gamma\) on \((M, g, e^{-\int f dv})\). Suppose that from every point \(p\) in \(N\), there are asymptotic rays to \(\gamma^\pm\) such that \(\text{Ric}_f \geq 0\) and [11] on both asymptotic rays. Then through every point in \(N\), there exists a line \(\alpha\) such that
\[
b_\gamma^+(\alpha^+(t)) = t, \quad b_\gamma^-(\alpha^-(t)) = t.
\]
Checking the previous proof, we easily see that (2.2) in fact holds for all \(d\). Suppose that from every point \(p\), there exist two different ends of \(M\). In Proposition 3.1, we only proved (2.2) when \(\text{Ric}_f \geq 0\) and \(1\) holds on \(\gamma_p\), from [5] we get that Busemann functions \(b^\pm\) satisfy \(|\nabla b^\pm| = 1\) and \(\text{Hess} b^\pm = 0\) on \(\gamma_p\). Here the restriction of \(b^\pm\) to \(\gamma_p\) is a linear function with derivative 1. So we can reparameterize \(\gamma\).

As in the proof of Lemma 3.3 in [2], we are able to apply Lemma 2.7 to prove the following property about ends. We refer the reader to Lemma 3.1 and Proposition 3.2 in [22] for the detailed discussion.

**Lemma 2.8.** Under the same assumptions of Theorem 1.7, \(M\) cannot admit a line \(\gamma\) satisfying \(d(\gamma(t), B_o(R)) \geq |t| + 2R\) for all \(t\). Moreover, if \([\gamma_1]\) and \([\gamma_2]\) are two different ends of \(M^n\), then for any \(t_1, t_2 \geq 0\),

\[
d(\gamma(1), \gamma(2)) > t_1 + t_2 - 6R.
\]

**Remark 2.9.** In Proposition 3.2 in [22], we only proved (2.2) when \(t_1, t_2 \geq 3R\). Checking the previous proof, we easily see that (2.2) in fact holds for all \(t_1, t_2 \geq 0\).

On the other hand, recall that Munteanu and Wang [15] proved another Cheeger-Gromoll type splitting theorem when \(\text{Ric}_f \geq \frac{1}{2}\) and \(f\) has certain quadratic growth. As in the preceding argument, we easily get the following result by analyzing the Munteanu-Wang’s proof locally.

**Lemma 2.10.** Let \(N\) be the \(\delta\)-tubular neighborhood of a line \(\gamma\) on \((M, g, e^{-f} dv)\). Suppose that from every point \(p\) in \(N\), there are asymptotic rays to \(\gamma^\pm\) such that \(\text{Ric}_f \geq \frac{1}{2}\) and \(f(x) \leq \frac{1}{4}d^2(x, B_o(1)) + c\) for some constant \(c > 0\) on both asymptotic rays. Then through every point in \(N\), there exists a line \(\alpha\) such that

\[
b^\pm(\alpha^n(t)) = t, \quad b^\pm(\alpha^-(t)) = t.
\]

Using Lemma 2.10, we can get a geometric inequality about two different ends along the above similar argument.

**Lemma 2.11.** Under the same assumptions of Theorem 1.5, \(M\) cannot admit a line \(\gamma\) satisfying \(d(\gamma(t), B_o(R)) \geq |t| + 2R\) for all \(t\). Moreover, if \([\gamma_1]\) and \([\gamma_2]\) are two different ends of \(M^n\), then for any \(t_1, t_2 \geq 0\),

\[
d(\gamma(1), \gamma(2)) > t_1 + t_2 - 6R.
\]

3. Gap theorems

In this section we will prove Theorems 1.1 and 1.5. We start with an important proposition, which will be used in our proof of theorems. For the convenient discussion, we assume that \((M, g, e^{-f} dv)\) has \(\text{Ric}_f \geq -(n-1)\) by scaling the metric.

**Proposition 3.1.** Let \((M, g, e^{-f} dv)\) be an \(n\)-dimensional complete smooth metric measure space with a base point \(o \in M\). Assume that \(\text{Ric}_f \geq -(n-1)\) and let \(A := \sup_{x \in B_o(1)} |f(x)|\). There exist an \(\epsilon = \epsilon(n, A)\) and a \(\delta = \delta(n, A)\) such that

\[
u(x) < 2 - 2\tilde{\delta} - 12\epsilon
\]

for all \(x \in S_o(1 - \delta) := \{x \in M | d(x, o) = 1 - \delta\}\) if \(u : M^n \to \mathbb{R}\) is a continuous function satisfying the following properties:
(i) \( u(o) = 0 \),
(ii) \( u \geq -6\epsilon \),
(iii) \( \sup_{x \neq y} |u(x) - u(y)|/d(x, y) \leq 2 \),
(iv) \( \Delta f u \leq 2(n + 4A - 1) \) in the barrier sense.

**Proof of Proposition 3.1.** Let \( H(r) := 2r + G(r) \), where \( G(r) \) is defined as

\[
G(r) := 2(n + 4A - 1) \int_r^1 \int_t^1 \left( \frac{\sinh s}{\sinh t} \right)^{n+4A-1} dsdt
\]

and \( A := \sup_{x \in B_r(1)} |f(x)| \). We remark that in general \( G(d(x, o)) \) does not satisfy the \( f \)-Laplacian equation \( \Delta f G = 2(n + 4A - 1) \). But we observe that \( G(1) = 0 \),

\[
G'(r) = -2(n + 4A - 1) \int_r^1 \left( \frac{\sinh s}{\sinh r} \right)^{n+4A-1} ds,
\]

\( G'(1) = 0 \) and \( G'(r) \leq 0 \). So \( H(1) = 2 \) and \( H'(r) > 0 \) when \( r \to 1 \). Therefore there exists a real constant \( c \) such that \( c \in (0, 1) \) and \( H(c) < 2 \). Now we choose \( \delta = \delta(n, A) \) and \( \epsilon = \epsilon(n, A) \) such that

\[
0 < \delta < \frac{1}{6} \min \{2 - H(c), 1 - c\}
\]

and

\[
0 < \epsilon < \frac{1}{12} \min \{G(1 - \delta), 2 - H(c) - 2\delta\}.
\]

Consider function \( v(y) := u(y) - G(d(x, y)) \) on the annulus \( B_x(1) \setminus B_x(c) \). By the weighted Laplacian comparison (Lemma 2.1) and \( G'(r) \leq 0 \), we compute that

\[
\Delta f G = G''(r) |\nabla v|^2 + G'(r) \Delta f r
\geq G''(r) + G'(r) [(n + 4A - 1) \coth r]
= 2(n + 4A - 1),
\]

where we used the definition of \( G(r) \) in the last equality. This implies that

\[
\Delta f v \leq 0
\]

in the barrier sense by combining the assumption (iv). By the maximum principle, \( v \) achieves its minimum on the boundary of the annulus \( B_x(1) \setminus B_x(c) \). By (3.1), we know that \( o \) is an interior point of the domain \( B_x(1) \setminus B_x(c) \). By (3.2) and the assumption (i), we see that

\[
v(o) = u(o) - G(d(o, x)) = -G(1 - \delta) < -6\epsilon.
\]

Therefore there exists a point \( z \) on the boundary of the annulus such that

\[
v(z) \leq v(o) < -6\epsilon.
\]

On the other hand, on the sphere \( S_x(1) \), by the assumption (ii), we get that

\[
v = u - G(1) = u \geq -6\epsilon,
\]

where we used \( G(1) = 0 \). This implies that \( z \in S_x(c) \). Combining this with the assumption (iii), the definitions of \( H(r) \) and \( v \), and (3.2), we finally get

\[
u(x) \leq u(z) + 2c = v(z) + H(c) < 2 - 2\delta - 12\epsilon
\]

and the result follows. \( \square \)

We now apply Proposition 3.1 to prove Theorem 1.1 by following the argument of Cai-Colding-Yang [3].
Proof of Theorem 1.1. When $K = 0$, the theorem easily follows by the Fang-Li-Zhang splitting theorem [5]. Now let $(M, g, e^{-f}dv)$ be as Theorem 1.1 with $K = 1$. Let $\epsilon = \epsilon(n, A)$ be as Proposition 3.1. We only need to show that when $R \leq \epsilon(n, A)$, $M^n$ has at most two ends. Suppose the conclusion is not true. That is, there exists three different ends, denoted by $[\gamma_1]$, $[\gamma_2]$ and $[\gamma_3]$. We consider the function $u(x) := b_{\gamma_1}(x) + b_{\gamma_2}(x)$.

We claim that $u(x)$ satisfies four conditions of Proposition 3.1. Indeed, (i) and (iii) are obvious. By (2.2) and the triangle inequality,

\[
\begin{align*}
\Delta f u(x) &= \Delta f d(x, \gamma_1(\infty)) + \Delta f d(x, \gamma_2(\infty)) \\
&\leq 2(n + 4A - 1) \lim_{r \to \infty} \coth r \\
&= 2(n + 4A - 1),
\end{align*}
\]

which implies that $u$ satisfies (iv). Therefore, by Proposition 3.1 we conclude that

\[
u(\gamma_3(1 - \delta)) < 2 - 2\delta - 12\epsilon.
\]

On the other hand, by Lemma 2.8, for any $t > 0$,

\[
u(\gamma_3(1 - \delta)) \geq 2t - 12R.
\]

In particular, letting $t = 1 - \delta$, it follows that

\[
u(\gamma_3(1 - \delta)) \geq 2(1 - \delta) - 12R \geq 2 - 2\delta - 12\epsilon.
\]

This contradicts (3.3) and hence completes the proof. \hfill \Box

Finally we give an explanation how to prove Theorem 1.5.

Sketch proof of Theorem 1.5. We can apply Proposition 3.1 and Lemma 2.11 to prove Theorem 1.5. In fact the argument in this case is exactly the same as the proof of Theorem 1.1. Here we omit the details. \hfill \Box

4. Appendix

In this part we will give a number estimate for ends of a class of SMMSs. The weight $f$ allows to be certain quadratic growth of distance function, which improves the growth of $f$ in [22].

**Theorem 4.1.** Let $(M, g, e^{-f}dv)$ be an $n$-dimensional complete smooth metric measure space. Fix a point $o \in M$ and $R > 0$. Suppose $\text{Ric}_f \geq -(n-1)$ in the geodesic ball $B_o(R)$ and $\text{Ric}_f \geq \frac{1}{2}$ outside $B_o(R)$. If $f(x) \leq \frac{1}{4}d^2(x, B_o(R)) + c$ for some constant $c > 0$ on $M$, then

\[
N_R(M) \leq \frac{2(n + 4A)}{n + 4A - 1} \cdot \frac{\epsilon(n+4A-1)R}{R^{n+4A}}
\]
where \( N_R(M) \) is the number of ends of \( M \) with respect to \( B_o(R) \), and \( A := A(R) = \sup_{x \in B_o(25/2 R)} |f(x)| \).

The proof follows by Lemmas 2.11 and 2.2 by using the arguments of [2, 22]. We include it for the readers’ convenience.

**Proof of Theorem 4.1.** For any a point \( o \in M \), let \( \gamma_1, \gamma_2, ..., \gamma_k \) be \( k \) rays with \( k \) different ends starting from the base point \( o \). Then we only need to give an upper bound of the number \( k \).

For a fixed \( R > 0 \), consider the sphere \( S_o(4R) \) and let \( \{p_j\} \) be a maximal set of points on \( S_o(4R) \) such that the balls \( B_{p_j}(R/2) \) are disjoint each other. Clearly, the balls \( B_{p_j}(R) \) cover \( S_o(4R) \). Since the set \( \{\gamma_i(4R), i = 1, 2, ..., k\} \) is contained in \( S_o(4R) \), each \( \gamma_i(4R) \) is contained in some \( B_{p_j}(R) \). From Lemma 2.11 with \( t = 4R \), we know that each ball \( B_{p_j}(R) \) contains at most one \( \gamma_i(4R) \), and hence the number of balls is not less than \( k \). Therefore, to estimate an upper bound of \( k \), it suffices to bound the number of balls \( B_{p_j}(R/2) \).

By the weighted volume comparison (2.1), using a fact that \( B_{p_j}(R/2) \subset B_o(9_2 R) \subset B_{p_j}(17_2 R) \), we have

\[
V_f(B_{p_j}(\frac{17}{2} R)) \leq \int_0^{\frac{17}{2} R} \left( \sinh^{n+4\bar{A}-1} t \right) dt \frac{V_f(B_{p_j}(R/2))}{\int_0^{R/2} \left( \sinh^{n+4\bar{A}-1} t \right) dt},
\]

where \( \bar{A} = \sup_{x \in B_{p_j}(\frac{17}{2} R)} |f(x)| \). Therefore, the number of balls \( B_{p_j}(R/2) \) is no more than

\[
\int_0^{\frac{17}{2} R} \left( \sinh^{n+4A-1} t \right) dt \frac{V_f(B_{p_j}(R/2))}{\int_0^{R/2} \left( \sinh^{n+4A-1} t \right) dt},
\]

where \( A = \sup_{x \in B_o(\frac{25}{2} R)} |f(x)| \) because \( B_{p_j}(\frac{17}{2} R) \subset B_o(\frac{25}{2} R) \). Notice that

\[
\int_0^{\frac{17}{2} R} \left( \sinh^{n+4A-1} t \right) dt \leq 2(n+4A) e^{\frac{17}{2}(n+4A-1)R} \frac{e^{\frac{17}{2}(n+4A-1)R}}{n+4A-1} \frac{e^{\frac{17}{2}(n+4A-1)R}}{R^{n+4A}}
\]

and the upper estimate follows.

\[\square\]

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