Abstract: This paper investigates a financial market where stock returns depend on a hidden Gaussian mean reverting drift process. Information on the drift is obtained from returns and expert opinions in the form of noisy signals about the current state of the drift arriving at the jump times of a homogeneous Poisson process. Drift estimates are based on Kalman filter techniques and described by the conditional mean and covariance matrix of the drift given the observations. We study the filter asymptotics for increasing arrival intensity of expert opinions and prove that the conditional mean is a consistent drift estimator, it converges in the mean-square sense to the hidden drift. Thus, in the limit as the arrival intensity goes to infinity investors have full information about the drift.

Keywords: Kalman filter, Ornstein-Uhlenbeck process, Partial information, Expert opinions, Black-Litterman model, Bayesian updating

MSC: Primary 93E11; Secondary 60F17, 60G35

1 Introduction

In this paper we investigate a hidden Gaussian model (HGM) for a financial market where asset prices follow a diffusion process with an unobservable Gaussian mean reverting drift modelled by an Ornstein-Uhlenbeck process. Such models are widely used in the study of portfolio optimization problems under partial information on the drift. There are two popular model classes for the drift, the above-mentioned HGM and the hidden Markov model (HMM) in which the drift process is a continuous-time Markov chain. For utility maximization problems under HGM we refer to Lakner [15] and Brendle [4] while HMMs are used in Rieder and Bäuerle [18], Sass and Haussmann [20]. Both models are studied in Putschögl and Sass [17]. A generalization of these approaches and further references can be found in Björk et al. [1].

For solving portfolio problems under partial information the drift has to be estimated from observable quantities such as stock returns. For the above two models, HGM and HMM, the conditional distribution of the drift process given the return observations can be described completely by finite-dimensional filter processes. This allows for efficient solutions to portfolio problems including the computation of an opt-
mal policy. For HGM and HMM finite-dimensional filters are known as the Kalman and Wonham filters, respectively, see e.g. Elliott, Aggoun and Moore [8], Liptser and Shiryaev [16].

It is well known that the drift of a diffusion process is particularly hard to estimate. Even the estimation of a constant drift would require empirical data over an extremely large time horizon, see Rogers [19, Chapter 4.2]. Therefore, in practice filters computed from historical price observations lead to drift estimates of quite poor precision since drifts tend to fluctuate randomly over time and drift effects are overshadowed by volatility. At the same time optimal investment strategies in dynamic portfolio optimization depend crucially on the drift of the underlying asset price process. For these reasons, practitioners also utilize external sources of information such as news, company reports, ratings or their own intuitive views on the future asset performance for the construction of optimal portfolio strategies. These outside sources of information are called *expert opinions*. The idea goes back to the celebrated Black-Litterman model which is an extension of the classical one-period Markowitz model, see Black and Litterman [2]. It uses Bayesian updating to improve drift estimates.

Contrary to the classical static one-period model, we consider a continuous-time model for asset prices where additional information in the form of expert opinions arrives repeatedly over time. Davis and Lleo [7] termed that approach “Black-Litterman in Continuous-Time” (BLCT). The first papers addressing BLCT are Frey et al. (2012) [9] and their follow-up paper [10]. They consider an HMM for the drift and expert opinions arriving at the jump times of a Poisson process and study the maximization of expected power utility of terminal wealth. An HGM and expert opinions arriving at fixed and known times have been investigated in Gabih et al. [11] for a market with only one risky stock, and generalized in Sass et al. [21] for markets with multiple risky stocks. Here, the authors consider maximization of logarithmic utility. Davis and Lleo [6, 7] consider BLCT for power utility maximization under an HGM and expert opinions arriving continuously over time. This allows for quite explicit solutions for the portfolio optimization problem. In [7], the authors also focus on the calibration of the model for expert opinions to real-world data.

In a recent paper Sass et al. [22] consider an HGM with expert opinions both at fixed as well as random information dates and investigate the asymptotic behavior of the filter for increasing arrival frequency of the expert opinions. They assume that a higher frequency of expert opinions is only available at the cost of accuracy. In particular, the variance of expert opinions grows linearly with the arrival frequency. This assumption reflects that it is not possible for investors to gain unlimited information in a finite time interval. Furthermore, it allows to find a certain asymptotic behavior that yields reasonable filter approximations for investors observing discrete-time expert opinions arriving with a fixed and sufficiently large intensity. The authors derive limit theorems which state that the information obtained from observing high-frequency discrete-time expert opinions is asymptotically the same as that from observing a certain diffusion process that has the same drift as the return process. The latter process can be interpreted as a continuous-time expert which permanently delivers noisy information about the drift. These so-called diffusion approximations show how the BLCT model of Davis and Lleo [6, 7] who work with continuous-time expert opinions can be obtained as a limit of BLCT models with discrete-time experts.

The present paper can be considered as a companion paper to the above mentioned work of Sass et al. [22]. However, contrary to [22] we assume that the expert’s reliability expressed by its variance remains bounded when the arrival intensity increases. For the sake of simplicity we restrict to the case of a constant variance. This leads to a different asymptotic regime corresponding to the Law of Large Numbers while the results in [22] are in the sense of Functional Central Limit Theorems.

When the arrival intensity increases, the investor receives more and more noisy signals about the current state of the drift of the same precision. It is then expected that in the limit the drift estimate is perfectly accurate and equals the actual drift, i.e., the investor has full information about the drift. While this statistical consistency of the estimator seems to be intuitively clear a rigorous proof is an open issue and will be addressed in this paper. Gabih et al. [11] and Sass et al. [21] provide such proof only for the case of fixed and known information dates. However, their results and methods cannot be applied to the present model with random information dates. Note that also the methods for the proof of the diffusion limits in [22] do not carry over to the present case of fixed expert’s reliability. To the best of our knowledge the techniques for proving convergence constitute a new contribution to the literature. Compared to [11]
and [21] we do not only give a rigorous convergence proof but we are also able to determine the rate of convergence and give explicit bounds for the estimation error.

In this paper we concentrate on the asymptotic properties of drift estimates which are based on Kalman filter techniques and described by the conditional mean and covariance matrix of the drift given the observations. We show that for increasing arrival intensity of expert opinions, the expectation of the conditional covariance goes to zero. This implies that the conditional mean is a consistent drift estimator, it converges to the hidden drift in the mean-square sense. We expect that these convergence results carry over to the value functions of portfolio optimization problems but do not include these studies in this paper. For the maximization of expected logarithmic utility, the convergence of value functions has already been proven in Sass et al. [22]. The case of power utility will be addressed in our follow-up paper [12].

The paper is organized as follows: In Section 2 we introduce the model for our financial market including expert opinions and define information regimes for investors with different sources of information. For each of those information regimes, we state the dynamics of the corresponding conditional mean and conditional covariance process in Section 3. Section 4 contains our main contributions and studies the asymptotic filter behavior for increasing arrival intensity of discrete-time expert opinions. First, Lemma 4.2 gives an estimate for the drift term in the semimartingale representation of the conditional covariance process. Based on this estimate, Theorem 4.3 shows that as the arrival intensity increases the expectation of the conditional covariance goes to zero. As a consequence, Theorem 4.6 states the mean-square convergence of the conditional mean to the hidden drift. In Section 5, we study a related problem for continuous-time expert opinions that arises in the case of diffusion approximations of discrete-time expert opinions. Section 6 illustrates the convergence results by some numerical experiments. In Appendix A we collect some auxiliary results and technical proofs needed for our main theorems.

**Notation.** Throughout this paper, we use the notation $I_d$ for the identity matrix in $\mathbb{R}^{d \times d}$. For a symmetric and positive-semidefinite matrix $A \in \mathbb{R}^{d \times d}$ we call a symmetric and positive-semidefinite matrix $B \in \mathbb{R}^{d \times d}$ the square root of $A$ if $B^2 = A$. The square root is unique and will be denoted by $A^{1/2}$.

For a vector $X$ we denote by $\|X\|$ the Euclidean norm. For a square matrix $A$ we denote by $\|A\|$ a generic matrix norm, by $\|A\|_F = \sqrt{\sum_{i,j} (A^{ij})^2}$ the Frobenius norm and by $\text{tr}(A) = \sum_i A^{ii}$ the trace of $A$.

## 2 Financial Market

### 2.1 Price Dynamics

The setting is based on Gabih et al. [11] and Sass et al. [21][22]. For a fixed date $T > 0$ representing the investment horizon, we work on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{G}, P)$, with filtration $\mathcal{G} = (\mathcal{G}_t)_{t \in [0,T]}$ satisfying the usual conditions. All processes are assumed to be $\mathcal{G}$-adapted.

We consider a market model for one risk-free bond with constant risk-free interest rate and $d$ risky securities whose return process $R = (R^1, \ldots, R^d)$ is defined by

$$dR_t = \mu_t \, dt + \sigma_R \, dW^R_t,$$

for a given $d_1$-dimensional $\mathcal{G}$-adapted Brownian motion $W^R$ with $d_1 \geq d$. The constant volatility matrix $\sigma_R \in \mathbb{R}^{d \times d_1}$ is assumed to be such that $\Sigma_R := \sigma_R \sigma_R^\top$ is positive definite. In this setting the price process $S = (S^1, \ldots, S^d)$ of the risky securities reads as

$$dS_t = \text{diag}(S_t) \, dR_t.$$

In this paper we do not only give a rigorous convergence proof but we are also able to determine the rate of convergence and give explicit bounds for the estimation error.
Note that we can write

\[
\log S_t^i = \log s_0^i + \int_0^t \mu_s^i ds + \sum_{j=1}^{d_1} \left( \sigma_{ij}^R W_t^R - \frac{1}{2} \sigma_{ij}^R \right) dt
\]

\[
= \log s_0^i + R_t^i - \frac{1}{2} \sum_{j=1}^{d_1} \sigma_{ij}^R \gamma_t^j, \quad i = 1, \ldots, d.
\]

(2.3)

So we have the equality \( G^R = \mathbb{G}^{\log S} = \mathbb{G}^S \), where for a generic process \( X \) we denote by \( \mathbb{G}^X \) the filtration generated by \( X \). This is useful since it allows to work with \( R \) instead of \( S \) in the filtering part.

The dynamics of the drift process \( \mu = (\mu_t)_{t \in [0,T]} \) in (2.1) are given by the stochastic differential equation (SDE)

\[
d\mu_t = \kappa (\overline{\mu} - \mu_t) dt + \sigma_\mu dW_t^R, \quad t \geq 0,
\]

where \( \kappa \in \mathbb{R}^{d \times d}, \sigma_\mu \in \mathbb{R}^{d \times d_2} \) and \( \overline{\mu} \in \mathbb{R}^d \) are constants such that the matrices \( \kappa \) and \( \Sigma_\mu := \sigma_\mu \sigma_\mu^\top \) are positive definite, and \( W^\mu \) is a \( d_2 \)-dimensional Brownian motion independent of \( W^R \) with \( d_2 \geq d \). Here, \( \overline{\mu} \) is the mean-reversion level, \( \kappa \) the mean-reversion speed, and \( \sigma_\mu \) describes the volatility of \( \mu \). The initial value \( \mu_0 \) is assumed to be a normally distributed random variable independent of \( W^\mu \) and \( W^R \) with mean \( \overline{\mu}_0 \in \mathbb{R}^d \) and covariance matrix \( \Sigma_0 \in \mathbb{R}^{d \times d} \) assumed to be symmetric and positive semidefinite. It is well known that SDE (2.4) has the closed-form solution

\[
\mu_t = \overline{\mu} + e^{-\kappa t} \left[ (\mu_0 - \overline{\mu}) + \int_0^t e^{\kappa s} \sigma_\mu dW_s^R \right], \quad t \geq 0.
\]

(2.5)

This is a Gaussian process and known as Ornstein-Uhlenbeck process. It has mean value and covariance function

\[
\mathbb{E}[\mu_t] = \overline{\mu} + e^{-\kappa t} (\mu_0 - \overline{\mu}) \quad \text{and}
\]

\[
\text{Cov}(\mu_s, \mu_t) = e^{-\kappa \min\{s,t\}} \left[ \Sigma_0 + \int_0^{\min\{s,t\}} e^{\kappa u} \Sigma_\mu e^{\kappa u \top} du \right] e^{-\kappa \gamma}, \quad s, t \geq 0.
\]

### 2.2 Expert Opinions

We assume that investors observe the return process \( R \) but they neither observe the factor process \( \mu \) nor the Brownian motion \( W^R \). They do however know the model parameters such as \( \sigma_R, \kappa, \overline{\mu}, \sigma_\mu \) and the distribution \( \mathcal{N}(\overline{\mu}_0, \Sigma_0) \) of the initial value \( \mu_0 \). Information about the drift \( \mu \) can be drawn from observing the returns \( R \). A special feature of our model is that investors may also have access to additional information about the drift in form of expert opinions such as news, company reports, ratings or their own intuitive views on the future asset performance. The expert opinions provide noisy signals about the current state of the drift arriving at discrete points in time \( T_k \). We model these expert opinions by a marked point process \((\overline{\tau}_k, Z_k)_{k}\), so that at \( T_k \) the investor observes the realization of a random vector \( Z_k \) whose distribution depends on the current state \( \mu_{T_k} \) of the drift process. The arrival dates \( T_k \) are modelled as jump times of a standard Poisson process with intensity \( \lambda > 0 \), independent of both the Brownian motions \( W^R, W^J \) and the initial value of the drift \( \mu_0 \), so that the timing of the information arrival does not carry any useful information about the drift. For the sake of convenience we also write \( T_0 := 0 \) although no expert opinion arrives at time \( t = 0 \).

The signals or “the expert views” at time \( T_k \) are modelled by \( \mathbb{R}^d \)-valued Gaussian random vectors \( Z_k = (Z_k^1, \ldots, Z_k^d)^\top \) with

\[
Z_k = \mu_{T_k} + \Gamma^\frac{1}{2} \varepsilon_k, \quad k = 1, 2, \ldots,
\]

(2.6)
where the matrix $\Gamma \in \mathbb{R}^{d \times d}$ is symmetric and positive definite. Further, $(\varepsilon_k)_{k \geq 1}$ is a sequence of independent standard normally distributed random vectors, i.e., $\varepsilon_k \sim \mathcal{N}(0, I_d)$. It is also independent of the Brownian motions $W^R, W^F$, the initial value $\mu_0$ and the arrival dates $(T_k)_{k \geq 1}$. That means that, given $\mu_{T_k}$, the expert opinion $Z_k$ is $\mathcal{N}(\mu_{T_k}, \Gamma)$-distributed. So, $Z_k$ can be considered as an unbiased estimate of the unknown state of the drift at time $T_k$. The matrix $\Gamma$ is a measure of the expert’s reliability. In a model with $d=1$ risky asset $\Gamma$ is just the variance of the expert’s estimate of the drift at time $T_k$: the larger $\Gamma$ the less reliable is the expert.

Note that one may also allow for relative expert views where experts give an estimate for the difference in the drift of two stocks instead of absolute views. This extension is studied in Schöttle et al. [22] where the authors show how to switch between these two models for expert opinions by means of a pick matrix.

Finally, we introduce expert opinions arriving continuously over time. This is motivated by the results of Sass et al. [22]. There the authors consider the information drawn from observing certain sequences of expert opinions and show that for a large number of expert opinions it is essentially the same as the information resulting from observing another diffusion process. The interpretation of that diffusion process is an expert providing continuous-time estimates about the state of the drift. Let this estimate be given by the diffusion process

$$dJ_t = \mu_t dt + \sigma_J dW_t^J,$$

where $W^J$ is a $d_J$-dimensional Brownian motion with $d_J \geq d$ that is independent of all other Brownian motions in the model and of the information dates $T_k$. The constant matrix $\sigma_J \in \mathbb{R}^{d \times d_J}$ is assumed to be such that $\Sigma_J := \sigma_J \sigma_J^T$ is positive definite.

### 2.3 Investor Filtration

We consider various types of investors with different levels of information. The information available to an investor is described by the investor filtration $\mathbb{F}^H = (\mathcal{F}^H_t)_{t \in [0, T]}$. Here, $H$ denotes the information regime for which we consider the cases $H = R, Z, J, F$, where

$$\begin{align*}
\mathbb{F}^R &= (\mathcal{F}^R_t)_{t \in [0, T]} \text{ with } \mathcal{F}^R_t = \sigma(R_s, s \leq t), \\
\mathbb{F}^Z &= (\mathcal{F}^Z_t)_{t \in [0, T]} \text{ with } \mathcal{F}^Z_t = \sigma(R_s, s \leq t, (T_k, Z_k), T_k \leq t), \\
\mathbb{F}^J &= (\mathcal{F}^J_t)_{t \in [0, T]} \text{ with } \mathcal{F}^J_t = \sigma(R_s, J_s, s \leq t), \\
\mathbb{F}^F &= (\mathcal{F}^F_t)_{t \in [0, T]} \text{ with } \mathcal{F}^F_t = \sigma(R_s, \mu_s, s \leq t).
\end{align*}$$

We assume that the above $\sigma$-algebras $\mathcal{F}^H_t$ are augmented by the null sets of $P$. We call the investor with filtration $\mathbb{F}^H = (\mathcal{F}^H_t)_{t \in [0, T]}$ the $H$-investor. The $R$-investor observes only the return process $R$, the $Z$-investor combines return observations with the discrete-time expert opinions $Z_k$ while the $J$-investor observes the return process together with the continuous-time expert $J$. Finally, the $F$-investor has full information and can observe the drift process $\mu$. For stochastic drift this case is not realistic, but we use it as a benchmark and in the next section it will serve as a limiting case for high-frequency expert opinions. We will denote an investor with investor filtration $\mathbb{F}^H$ as $H$-investor.

We assume that at $t = 0$ the partially informed investors start with the same initial information given by the $\sigma$-algebra $\mathcal{F}^0_0$, i.e., $\mathcal{F}^H_0 = \mathcal{F}^F_0 \subset \mathbb{F}^F_0$, $H = R, Z, J$. This initial information $\mathcal{F}^F_0$ models prior knowledge about the drift process at time $t = 0$, e.g., from observing returns or expert opinions in the past before the trading period $[0, T]$. We assume that the conditional distribution of the initial drift value $\mu_0$ given $\mathcal{F}^H_0$ is the normal distribution $\mathcal{N}(m_0, q_0)$ with mean $m_0 \in \mathbb{R}^d$ and covariance matrix $q_0 \in \mathbb{R}^{d \times d}$ assumed to be symmetric and positive semidefinite. In this setting typical examples are:

a) The investor has no information about the initial value of the drift $\mu_0$. However, he knows the model parameters, in particular the distribution $\mathcal{N}(\overline{m}_0, \overline{q}_0)$ of $\mu_0$ with given parameters $\overline{m}_0$ and $\overline{q}_0$. This corresponds to $\mathcal{F}^0_0 = \{\emptyset, \Omega\}$ and $m_0 = \overline{m}_0$, $q_0 = \overline{q}_0$.

b) The investor can fully observe the initial value of the drift $\mu_0$, which corresponds to $\mathcal{F}^0_0 = \mathcal{F}^F_0$ and $m_0 = \mu_0(\omega)$ and $q_0 = 0$. 

c) Between the above limiting cases we consider an investor who has some prior but no complete information about \( \mu_0 \) leading to \( \{ \emptyset, \Omega \} \subset \mathcal{F}^I_0 \subset \mathcal{F}^E_0 \).

### 3 Partial Information and Filtering

The trading decisions of investors are based on their knowledge about the drift process \( \mu \). While the \( F \)-investor observes the drift directly, the \( H \)-investor for \( H = R, Z, J \) has to estimate it. This leads us to a filtering problem with hidden signal process \( \mu \) and observations given by the returns \( R \) and expert opinions \( (I_k, Z_k) \) or \( J \). The filter for the drift \( \mu_t \) is its projection on the \( \mathcal{F}^H_t \)-measurable random variables described by the conditional distribution of the drift given \( \mathcal{F}^H_t \). The mean-square optimal estimator for the drift at time \( t \), given the available information is the conditional mean

\[
M^H_t := \mathbb{E}[\mu_t | \mathcal{F}^H_t].
\]

The accuracy of that estimator can be described by the conditional covariance matrix

\[
Q^H_t := \mathbb{E}[(\mu_t - M^H_t)(\mu_t - M^H_t)\top | \mathcal{F}^H_t].
\]

Since in our filtering problem the signal \( \mu \), the observations and the initial value of the filter are jointly Gaussian also the filter distribution is Gaussian and completely characterized by the conditional mean \( M^H_t \) and the conditional covariance \( Q^H_t \).

In Section 3 we will study the asymptotic behavior of the filter for the \( Z \)-investor observing expert opinions arriving more and more frequently and derive limit theorems for the filter if the arrival intensity \( \lambda \) tends to infinity. Section 5 is devoted to a related problem and considers the asymptotics of the filter processes for the \( J \)-investor with volatility \( \sigma_j \) tending to zero. These results are based on the following dynamics of the filters for \( H = R, Z, J \) which already can be found in Sass et al. [21 22].

### 3.1 \( R \)- and \( J \)-Investor

The \( R \)-investor only observes returns and has no access to additional expert opinions, the information is given by \( \mathcal{F}^R \). Then, we are in the classical case of the Kalman filter, see e.g. Liptser and Shiryaev [10], Theorem 10.3, leading to the following dynamics of \( M^R \) and \( Q^R \).

**Lemma 3.1.** For the \( R \)-investor the filter is Gaussian and the conditional distribution of the drift \( \mu_t \) given \( \mathcal{F}^R_t \) is the normal distribution \( \mathcal{N}(M^R_t, Q^R_t) \).

The conditional mean \( M^R_t \) follows the dynamics

\[
dM^R_t = \kappa(\overline{\mu} - M^R_t) \, dt + Q^R_t \Sigma^{-1}_R \left( dR_t - M^R_t \, dt \right). \tag{3.2}
\]

The dynamics of the conditional covariance \( Q^R_t \) is given by the Riccati differential equation

\[
dQ^R_t = (\Sigma_m - \kappa Q^R_t - Q^R_t \kappa \top - Q^R_t \Sigma^{-1}_R Q^R_t) \, dt. \tag{3.3}
\]

The initial values are \( M^R_0 = m_0 \) and \( Q^R_0 = q_0 \).

Note that the conditional covariance matrix \( Q^R_t \) satisfies an ordinary differential equation and is hence deterministic, whereas the conditional mean \( M^R_t \) is a stochastic process defined by an SDE driven by the return process \( R \).

Next, we consider the \( J \)-investor who observes a \( 2d \)-dimensional diffusion process with components \( R \) and \( J \). That observation process is driven by a \( (d_1 + d_3) \)-dimensional Brownian motion with components \( W^R \) and \( W^J \). Again, we can apply classical Kalman filter theory as in Liptser and Shiryaev [10] to deduce the dynamics of \( M^J \) and \( Q^J \). We also refer to Lemma 2.2 in the companion paper [22].
Lemma 3.2. For the J-investor the filter is Gaussian and the conditional distribution of the drift \( \mu_t \) given \( \mathcal{F}_t^J \) is the normal distribution \( \mathcal{N}(M_t^J, Q_t^J) \). The conditional mean \( M_t^J \) follows the dynamics

\[
dM_t^J = \kappa(\bar{\mu} - M_t^J) \, dt + Q_t^J (\Sigma_R^{-1}, \Sigma_J^{-1}) \left( \frac{dR_t - M_t^J \, dt}{dJ_t - M_t^J \, dt} \right). \tag{3.4}
\]

The dynamics of the conditional covariance \( Q_t^J \) is given by the Riccati differential equation

\[
dQ_t^J = (\Sigma_\mu - \kappa Q_t^J - Q_t^J \kappa^\top - Q_t^J (\Sigma_R^{-1} + \Sigma_J^{-1}) Q_t^J) \, dt. \tag{3.5}
\]

The initial values are \( M_0^J = m_0 \) and \( Q_0^J = q_0 \).

Note that, as in case of the R-investor, the conditional covariance \( Q_t^J \) is deterministic.

3.2 Z-Investor

Now we consider the filter for the Z-investor who combines continuous-time observations of stock returns and expert opinions received at discrete points in time.

Lemma 3.3. For the Z-investor the filter is Gaussian and the conditional distribution of the drift \( \mu_t \) given \( \mathcal{F}_t^Z \) is the normal distribution \( \mathcal{N}(M_t^Z, Q_t^Z) \).

(i) Between two information dates \( T_k \) and \( T_{k+1} \), \( k \in \mathbb{N}_0 \), the conditional mean \( M_t^Z \) satisfies SDE (3.2), i.e.,

\[
dM_t^Z = \kappa(\bar{\mu} - M_t^Z) \, dt + Q_t^Z \Sigma_R^{-1} \left( \frac{dR_t - M_t^Z \, dt}{dM_t^Z \, dt} \right) \text{ for } t \in [T_k, T_{k+1}).
\]

The conditional covariance \( Q_t^Z \) satisfies the ordinary Riccati differential equation (3.3), i.e.,

\[
dQ_t^Z = (\Sigma_\mu - \kappa Q_t^Z - Q_t^Z \kappa^\top - Q_t^Z (\Sigma_R^{-1} Q_t^Z) \) \, dt.
\]

The initial values are \( M_0^Z \) and \( Q_0^Z \), respectively, with \( M_0^Z = m_0 \) and \( Q_0^Z = q_0 \).

(ii) At the information dates \( T_k \), \( k \in \mathbb{N} \), the conditional mean and covariance \( M_{T_k}^Z \) and \( Q_{T_k}^Z \) are obtained from the corresponding values at time \( T_{k-} \) (before the arrival of the view) using the update formulas

\[
M_{T_k}^Z = \rho_k M_{T_{k-}}^Z + (I_d - \rho_k) Z_k,
\]

\[
Q_{T_k}^Z = \rho_k Q_{T_{k-}},
\]

with the update factor \( \rho_k = \Gamma(Q_{T_{k-}}^Z + \Gamma)^{-1} \).

Proof. For a detailed proof we refer to Lemma 2.3 in [21] and Lemma 2.3 in [22]. \( \square \)

Note that the dynamics of \( M^Z \) and \( Q^Z \) between information dates are the same as for the R-investor, see Lemma 3.1 The values at an information date \( T_k \) are obtained from a Bayesian update.

Recall that for the R-investor the conditional mean \( M^R \) is a diffusion process and the conditional covariance \( Q^R \) is deterministic. Contrary to that the conditional mean \( M^Z \) of the Z-investor is a jump-diffusion process and the conditional covariance \( Q^Z \) is no longer deterministic since the updates lead to jumps at the random arrival dates \( T_k \) of the expert opinions. Hence, \( Q^Z \) is a piecewise deterministic stochastic process.

3.3 Properties of the Filter

The next lemma states in mathematical terms the intuitive property that additional information from the expert opinions improves drift estimates. Since the accuracy of the filter is measured by the conditional
covariance it is expected that this quantity for the \( Z \)-investor who combines observations of returns and expert opinions is “smaller” than for the \( R \)-investor who observes returns only. Mathematically, this can be expressed by the partial ordering of symmetric matrices. For symmetric matrices \( A, B \in \mathbb{R}^{d \times d} \) we write \( A \preceq B \) if \( B - A \) is positive semidefinite. Note that \( A \preceq B \) implies that \( \|A\| \leq \|B\| \).

**Proposition 3.4.** It holds \( Q_t^Z \preceq Q_t^R \) and \( Q_t^I \preceq Q_t^R \). In particular, there exists a constant \( C_Q > 0 \) such that \( \|Q_t^I\| \leq \|Q_t^R\| \leq C_Q \) and \( \|Q_t^I\| \leq \|Q_t^R\| \leq C_Q \) for all \( t \in [0, T] \).

For the proof we refer to [22], Lemma 2.4.

### 4 Filter Asymptotics for High-Frequency Expert Opinions

In the following we consider the \( Z \)-investor and its filter for increasing arrival intensity \( \lambda \) and study the asymptotic behavior of the conditional mean and conditional covariance for \( \lambda \to \infty \). Then the average number of expert opinions per unit of time goes to infinity, i.e., the \( Z \)-investor has more and more noisy estimates of the current state of the hidden drift at his disposal. This will lead to an increasing accuracy of the drift estimator. As a consequence of the Law of Large Numbers we expect that in the limit for \( \lambda \to \infty \) the drift estimator coincides with the drift. In fact we show in Theorem 4.6 that the drift estimator given the drift estimator. As a consequence of the Law of Large Numbers we expect that in the limit for

For this purpose, it will be useful to express the dynamics of \( \tilde{M}^Z \) in a unified way that comprises both the behavior between information dates and the jumps at times \( T_k \). We therefore work with a Poisson random measure as in Cont and Tankov [5] Sec. 2.6. Let \( E = [0, T] \times \mathbb{R}^d \) and let \( U_k, k = 1, 2, \ldots \), be a sequence of independent multivariate standard Gaussian random variables on \( \mathbb{R}^d \). For any \( I \in B([0, T]) \) and \( B \in B(\mathbb{R}^d) \) let

\[
N(I \times B) = \sum_{k: T_k \in I} \mathbb{1}_{\{U_k \in B\}}
\]

denote the number of jump times in \( I \) where \( U_k \) takes a value in \( B \). Then \( N \) defines a Poisson random measure with a corresponding compensated measure \( \tilde{N}^\lambda(ds, du) = N(ds, du) - \lambda ds \varphi(u) du \), where \( \varphi \) is the multivariate standard normal density on \( \mathbb{R}^d \), see Cont and Tankov [5] Sec. 2.6.3.

The next lemma rewrites the dynamics of \( Q_t^{Z,\lambda} \) given in Lemma 3.3 and provides a semimartingale representation which is driven by the martingale \( \tilde{N}^\lambda \). For a detailed proof and further explanations we refer to Westphal [25] Prop. 8.14 and Kondakji [14] Sec. 3.1.
Lemma 4.1. The dynamics of the conditional covariance matrix $Q^{Z,\lambda}$ are given by

$$dQ_t^{Z,\lambda} = \alpha^{Z,\lambda}(Q_t^{Z,\lambda}) \, dt + \int_{\mathbb{R}^d} \gamma(Q_t^{Z,\lambda}) \, \tilde{N}^{\lambda}(ds, du), \quad Q_0^{Z,\lambda} = q_0,$$

where

$$\alpha^{Z,\lambda}(q) = \Sigma_{\mu} - \kappa q - qk^\top - q\Sigma_R^{-1}q - \lambda q (q + \Gamma)^{-1} q,$$

$$\gamma(q) = -q (q + \Gamma)^{-1} q.$$  \hspace{1cm} (4.2)

We rewrite the above $\mathbb{R}^{Z,\lambda}$-semimartingale decomposition of $Q^{Z,\lambda}$ in integral form and obtain

$$Q_t^{Z,\lambda} = A_t^{\lambda} + K_t^{\lambda} \text{ for } t \in [0, T],$$

with

$$A_t^{\lambda} := q_0 + \int_0^t \alpha^{Z,\lambda}(Q_s^{Z,\lambda}) \, ds \quad \text{and} \quad K_t^{\lambda} := \int_0^t \int_{\mathbb{R}^d} \gamma(Q_s^{Z,\lambda}) \, \tilde{N}^{\lambda}(ds, du).$$

Since by Proposition 3.4 the conditional covariance $Q_t^{Z,\lambda}$ is bounded on $[0, t]$ also $\gamma$ is bounded and the jump process $K_t^{\lambda}$ is an $F^Z$-martingale and hence $\mathbb{E}[K_t^{\lambda}] = 0$ and

$$\mathbb{E}[Q_t^{Z,\lambda}] = \mathbb{E}[A_t^{\lambda}].$$  \hspace{1cm} (4.5)

For the study of the asymptotic behavior of the conditional covariance $Q^{Z,\lambda}$ we investigate the drift of the process $A_t^{\lambda}$ which is given by the non-linear matrix-valued function $\alpha^{Z,\lambda}$. The following lemma gives an estimate of the trace of $\alpha^{Z,\lambda}(q)$ in terms of a linear function of the trace of $q$. That estimate will play a crucial role for deriving the convergence result in Theorem 4.3.

Lemma 4.2. (Properties of $\alpha^{Z,\lambda}$)

For the function $\alpha^{Z,\lambda}$ given in (4.2) there exist constants $a_\alpha, b_\alpha > 0$ independent of $\lambda$ and there exists $\lambda_0 > 0$ such that for all symmetric and positive semidefinite $q \in \mathbb{R}^{d \times d}$

$$\text{tr} \left( \alpha^{Z,\lambda}(q) \right) \leq a_\alpha - \sqrt{\lambda}b_\alpha \text{tr}(q), \quad \text{for } \lambda \geq \lambda_0.$$  \hspace{1cm} (4.6)

The above estimate holds for every $a_\alpha > \text{tr}(\Sigma_{\mu})$.

$$b_\alpha < \overline{b}_\alpha = \overline{b}_\alpha(a_\alpha) := 2 \sqrt{\frac{a_\alpha - \text{tr}(\Sigma_{\mu})}{\text{tr}(\Gamma)}},$$

$$\lambda_0 = \lambda_0(a_\alpha, \beta_\alpha) := \left( \frac{d(a_\alpha - \text{tr}(\Sigma_{\mu}))}{2 \sqrt{\text{tr}(\Gamma)(a_\alpha - \text{tr}(\Sigma_{\mu})) - b_\alpha \text{tr}(\Gamma)}} \right)^2.$$  \hspace{1cm} (4.7)

The quite technical proof is given in Appendix A.2. The following main theorem gives an upper bound for the expectation of the trace of $Q^{Z,\lambda}$ from which the convergence to zero can be deduced.

Theorem 4.3. For every $\delta \in (0, T]$ there exists $\lambda_Q > 0$ such that

$$\mathbb{E}\left[ \text{tr} \left( Q_t^{Z,\lambda} \right) \right] \leq \frac{K^Z}{\sqrt{\lambda}} \quad \text{for } \lambda \geq \lambda_Q, \quad t \in [\delta, T] \quad \text{and}$$

$$K^Z = K^Z(\delta) = \left( \text{tr}(\Gamma) \left( \text{tr}(\Sigma_{\mu}) + \text{tr}(q_0)(e\delta)^{-1} \right) \right)^{1/2}.$$  \hspace{1cm} (4.8)

where $e = \exp(1)$ denotes Euler’s number.

In particular, it holds $\mathbb{E}\left[ \text{tr} \left( Q_t^{Z,\lambda} \right) \right] \to 0$ as $\lambda \to \infty$ for all $t \in (0, T]$.  \hspace{1cm} (4.9)
Proof. Let us define the function \( g(t) := E[\text{tr}(Q_t^Z)] \) for \( t \in [0, T] \). Then using (4.4), (4.5) and the linearity of the expectation and the trace operator yields

\[
g(t) = \text{tr}(E[Q_t^Z]) = \text{tr}(q_0) + \int_0^t E[\text{tr}(\alpha Z_t^\lambda(Q_t^Z))] \, ds.
\]

Since according to Proposition 4.4 the conditional covariance \( Q_t^Z \) is bounded and piecewise continuous the function \( g \) is piecewise differentiable and for its derivative it holds \( g'(t) = E[\text{tr}(\alpha Z_t^\lambda(Q_t^Z))] \). Further we have \( g(0) = \text{tr}(q_0) \). Lemma 4.2 implies that there are constants \( a_\alpha, b_\alpha, \lambda_0 > 0 \) such that

\[
g'(t) \leq E[\alpha - \sqrt{\lambda} b_\alpha \text{tr}(Q_t^Z)] = a_\alpha - \sqrt{\lambda} b_\alpha E[\text{tr}(Q_t^Z)] = a_\alpha - \sqrt{\lambda} b_\alpha g(t) \quad \text{for} \; \lambda \geq \lambda_0.
\]

(4.10)

We now apply Gronwall’s Lemma in differential form to obtain for \( t \in [\delta, T] \) and \( \lambda \geq \lambda_0 \)

\[
g(t) \leq g(0)e^{-\sqrt{\lambda} b_\alpha t} + \frac{a_\alpha}{\sqrt{\lambda} b_\alpha} (1 - e^{-\sqrt{\lambda} b_\alpha t}) \leq \frac{1}{\sqrt{\lambda}} h(\delta, \lambda, b_\alpha + a_\alpha/b_\alpha),
\]

(4.11)

where \( h(\delta, \lambda, b_\alpha) := \text{tr}(q_0)\sqrt{\lambda} e^{-\sqrt{\lambda} b_\alpha \delta} \). Next we show how for given \( \delta \in (0, T] \) we can choose the constants \( a_\alpha, b_\alpha, \lambda_Q > 0 \) such that \( h(\delta, \lambda, b_\alpha) + a_\alpha/b_\alpha \leq K^2(\delta) \) for \( \lambda \geq \lambda_Q \) with the constant \( K^2(\delta) \) given in (4.9).

Consider for \( \lambda \geq 0 \) the function \( \lambda \mapsto f(\lambda) = h(\delta, \lambda, b_\alpha) \) for fixed \( \delta \in (0, T] \) and \( b_\alpha \in (0, b_\alpha^\delta) \), where \( b_\alpha \) is given in (4.7). The function \( f \) is non-negative, it holds \( f(0) = 0 \) and \( f(\lambda) \to 0 \) for \( \lambda \to \infty \). There is a unique maximum at \( \lambda^\ast = (b_\alpha \delta)^{-2} \) with \( f(\lambda^\ast) = (e b_\alpha \delta)^{-1} \text{tr}(q_0) \). Hence for the last term on the r.h.s. of (4.11) we obtain

\[
h(\delta, \lambda, b_\alpha + a_\alpha/b_\alpha) \leq \frac{1}{b_\alpha}(\text{tr}(q_0)(e \delta)^{-1} + a_\alpha) \quad \text{for} \; \lambda \geq \lambda_Q.
\]

(4.12)

The latter expression is decreasing in \( b_\alpha \) and the minimum on \((0, b_\alpha^\delta)\) is attained for \( b_\alpha = b_\alpha^\delta \). According to (4.7) this selection leads to \( \lambda_0 = \infty \) which is not feasible and we have to restrict to values \( b_\alpha < b_\alpha^\delta \). However, we can achieve the above mentioned minimal value by choosing \( b_\alpha = b_\alpha^\delta - \eta \) with a sufficiently small \( \eta > 0 \) and \( \lambda_Q \geq \min(\lambda_0, \lambda^\ast) \) such that

\[
h(\delta, \lambda, b_\alpha + a_\alpha/b_\alpha) = h(\delta, \lambda, b_\alpha^\delta - \eta) + a_\alpha/b_\alpha - \eta \leq \frac{1}{b_\alpha}(\text{tr}(q_0)(e \delta)^{-1} + a_\alpha) \quad \text{for} \; \lambda \geq \lambda_Q.
\]

To see this estimate we note that \( f(\lambda) \) is decreasing on \((\lambda^\ast, \infty)\) and tends to zero for \( \lambda \to \infty \). Hence \( h(\delta, \lambda, b_\alpha^\delta - \eta) \) can be made arbitrarily small by selecting \( \lambda \) large enough.

Finally, we study the dependence of the above estimate on \( a_\alpha \) and take into account the definition of \( b_\alpha \) given in (4.7), i.e. we consider the function

\[
a_\alpha \to \frac{1}{b_\alpha}(\text{tr}(q_0)(e \delta)^{-1} + a_\alpha) = \frac{\sqrt{\text{tr}(F)}}{2\sqrt{a_\alpha - \text{tr}(\Sigma_\mu)}}(\text{tr}(q_0)(e \delta)^{-1} + a_\alpha)\]

for \( a_\alpha > \text{tr}(\Sigma_\mu) \). There is a unique minimizer at \( a_\alpha^* = 2\text{tr}(\Sigma_\mu) + \text{tr}(q_0)(e \delta)^{-1} \) and the minimal value is given by \( K^2(\delta) \) defined in (4.9). This proves the first claim.

Since that inequality holds for all \( \delta \in (0, T] \), the convergence \( E[\text{tr}(Q_t^Z)] \to 0 \) for \( \lambda \to \infty \) holds for all \( t \in (0, T] \).

From the above asymptotic properties for the expectation of the trace of \( Q_t^Z \) we can easily deduce analogous results for the expectation of the norm \( \|Q_t^Z\| \) of the conditional covariance.

**Corollary 4.4.** For every \( \delta \in (0, T] \) and any matrix norm \( \| \cdot \| \) there exist constants \( C, \lambda_Q > 0 \) such that

\[
E[\|Q_t^Z\|^p] \leq \frac{C}{\sqrt{\lambda}} \quad \text{for} \; \lambda \geq \lambda_Q, \; t \in [\delta, T] \; \text{and} \; p \geq 1.
\]

(4.13)
For the Frobenius norm \( \| \cdot \|_F \) the constant \( C \) can be chosen as \( C = K_Z C_F^{-1} \) where \( K_Z \) is given in (4.9) and \( C_F \) denotes the upper bound from Proposition 3.4 for the Frobenius norm \( \| Q^{Z,\lambda} \|_F \).

In particular, it holds \( E[\| Q^{Z,\lambda}_t \|_F^p] \rightarrow 0 \) as \( \lambda \rightarrow \infty \) for all \( t \in (0, T] \).

**Proof.** For the Frobenius norm of a symmetric and positive semidefinite matrix \( A \) it holds \( \| A \|_F \leq \text{tr}(A) \) (see Lemma A.1, Inequality (A.5)). Further, Proposition 3.4 implies \( \| A \|_F \leq C_F \). Hence

\[
\| Q^{Z,\lambda}_t \|_F^p \leq C_F^{-1} \| Q^{Z,\lambda}_t \|_F \leq C_F^{-1} \text{tr}(Q^{Z,\lambda}_t)
\]

and Theorem 4.3 with inequality (4.8) proves the claim. The equivalence of matrix norms implies the assertion for other norms. \( \square \)

### 4.2 Conditional Mean

We are now in a position to state and prove a similar convergence result for the asymptotic behavior of the filter \( M_t \). The proof is based on the following identity which relates the mean-square error of the filter estimate to the conditional covariance.

**Lemma 4.5.** It holds

\[
E[\| M_t^{Z,\lambda} - \mu_t \|^2] = \text{tr}\left( E[Q^{Z,\lambda}_t] \right).
\]  
(4.14)

**Proof.** For the mean-square criterion from (4.14) it holds

\[
E[\| M_t^{Z,\lambda} - \mu_t \|^2] = E[(M_t^{Z,\lambda} - \mu_t)(M_t^{Z,\lambda} - \mu_t)^\top) = \text{tr}\left( E[(M_t^{Z,\lambda} - \mu_t)(M_t^{Z,\lambda} - \mu_t)^\top] \right).
\]  
(4.15)

For the expectation in the last term the tower law of conditional expectation and the definition of the conditional covariance in (3.1) yields

\[
E[(M_t^{Z,\lambda} - \mu_t)(M_t^{Z,\lambda} - \mu_t)^\top] = E\left[ E[(M_t^{Z,\lambda} - \mu_t)(M_t^{Z,\lambda} - \mu_t)^\top | F_t^{Z,\lambda}] \right] = E[Q^{Z,\lambda}_t].
\]

Substituting into (4.15) yields the assertion (4.14). \( \square \)

**Theorem 4.6.** Let \( K_Z, \lambda_Q \) be the constants given in Theorem 4.3. Then for every \( \delta \in (0, T] \)

\[
E[\| M_t^{Z,\lambda} - \mu_t \|^2] \leq \frac{K_Z}{\sqrt{\lambda}} \quad \text{for} \quad \lambda \geq \lambda_Q, \quad t \in [\delta, T].
\]  
(4.16)

In particular, it holds \( E[\| M_t^{Z,\lambda} - \mu_t \|^2] \rightarrow 0 \) as \( \lambda \rightarrow \infty \) for all \( t \in (0, T] \).

**Proof.** Using identity (4.14) from Lemma 4.5 and applying inequality (4.8) of Theorem 4.3 we obtain

\[
E[\| M_t^{Z,\lambda} - \mu_t \|^2] = \text{tr}\left( E[Q^{Z,\lambda}_t] \right) \leq \frac{K_Z}{\sqrt{\lambda}} \quad \text{for} \quad \lambda \geq \lambda_0, \quad t \in [\delta, T].
\]

Since the above inequality holds for all \( \delta \in (0, T] \) we finally obtain the desired convergence of the filter \( M_t^{Z,\lambda} \) for \( t \in (0, T] \) as \( \lambda \rightarrow \infty \), i.e.

\[
E[\| M_t^{Z,\lambda} - \mu_t \|^2] \rightarrow 0 \quad \text{as} \quad \lambda \rightarrow \infty.
\]  
\( \square \)
5 Filter Asymptotics for Continuous-Time Expert Opinions

In the preceding section we already mentioned that there is another asymptotic regime if the variance of the expert opinions $\Gamma$ is not independent of the arrival intensity $\lambda$ but grows linearly in $\lambda$. We now want to establish some relations to the case of constant $\Gamma$ studied above.

Suppose that $\Gamma = \Gamma^\lambda = \lambda \sigma_{J} \sigma_{J}^T$ where $\sigma_{J}$ is the volatility matrix of the continuous-time expert opinion process $dJ_t = \mu_t \, dt + \sigma_{J} \, dW_t^J$ defined in (2.7). There we introduced the $J$-investor who combines observations of stock returns with those of $J$ (instead of discrete-time expert opinions). For that setting Sass et al. [22] show that the information the $Z$-investor obtains from observing discrete-time expert opinions is asymptotically the same as the information of the $J$-investor extracting from observing the diffusion process $J$ if the model of the expert’s views $Z_k$ given in (2.6) uses standard normally distributed random variables $\varepsilon_k$ defined by the increments of $W^J$ in the form $\varepsilon_k = \sqrt{\lambda}(W^J_{k/\lambda} - W^J_{(k-1)/\lambda}), k \in \mathbb{N}$. In particular they prove the mean-square convergence of filter processes $M^Z, Q^Z$ to the corresponding filter processes $M^J, Q^J$ of the $J$-investor and also provide the corresponding error estimates. These limit theorems justify so-called approximation of the filter for high-frequency discrete-time expert opinions to fixed and sufficiently large variance $\Gamma$ of the expert stating that the filter for the $Z$-investor can be approximated by the filter of a $J$-investor with volatility matrix $\sigma_{J} = \sigma_{J}^\lambda = \frac{1}{\lambda^2} \Gamma^{1/2}$.

Motivated by the results of the preceding section where we studied the filter asymptotics of the $Z$-investor with fixed expert’s variance $\Gamma = \Gamma^\lambda$ for $\lambda \to \infty$ we now want to study the asymptotics of the associated diffusion approximations. We therefore introduce a family of diffusion processes $(J^\lambda)_{\lambda > 0}$ defined by

$$dJ^\lambda_t = \mu_t \, dt + \frac{1}{\sqrt{\lambda}} \sigma_{J} \, dW_t^J \tag{5.1}$$

with a constant matrix $\sigma_{J} \in \mathbb{R}^{d \times d_3}$ chosen such that $\Sigma_{J} := \sigma_{J} \sigma_{J}^T$ is positive definite. Then it holds $\Sigma_{J} = \Sigma_{J}^\lambda = \frac{1}{\lambda} \Sigma_{J}$. Since $= \frac{1}{\lambda} \sigma_{J} \to 0$ for $\lambda \to \infty$ the limit case is not covered by the limit theorems in [22] and the diffusion approximation degenerates. Nevertheless, from a statistical point of view there is a clear interpretation. In the limit the $J$-investor can perfectly reconstruct the hidden drift $\mu$ from observing the limiting process $J^\infty$ defined by the (deterministic) ODE $dJ^\infty_t = \mu_t \, dt$ and has thus full information on the drift.

Below we provide a precise mathematical meaning to that convergence to full information and prove the corresponding limit theorems for the filter processes which are analogues to their counterparts for high-frequency discrete-time experts in Section 4. We also provide the associated error bounds for the drift estimates of the $J$-investor. It turns out that we can benefit a lot from the techniques developed in the proofs of Section 4.

Starting point is the conditional covariance $Q^J = Q^{J, \lambda}$. Note that, contrary to the stochastic conditional covariance $Q^Z, \lambda$ of the $Z$-investor, $Q^{J, \lambda}$ is deterministic. According to Lemma 3.5 it satisfies the Riccati differential equation (3.3) which we rewrite as

$$dQ^{J, \lambda}_t = \alpha^{J, \lambda} (Q^{J, \lambda}_t) \, dt, \quad Q^{J, \lambda}_0 = q_0, \quad \text{where}$$

$$\alpha^{J, \lambda}(q) = \Sigma_{\mu} - \kappa q - q \Sigma_{R}^{-1} = q (\Sigma_{R}^{-1} + \lambda \Sigma_{J}^{-1}) q. \tag{5.3}$$

**Lemma 5.1. (Properties of $\alpha^{J, \lambda}$)**

For the function $\alpha^{J, \lambda}$ given in (5.3) there exist constants $a_\alpha, b_\alpha > 0$ independent of $\lambda$ such that for all symmetric and positive semidefinite $\alpha \in \mathbb{R}^{d \times d}$

$$\operatorname{tr}(\alpha^{J, \lambda}(q)) \leq a_\alpha - \sqrt{\lambda} b_\alpha \operatorname{tr}(q), \quad \text{for} \quad \lambda > 0. \tag{5.4}$$

The above estimate holds for

$$a_\alpha = \operatorname{tr}(\Sigma_{\mu}) + (d \, \operatorname{tr}(\Sigma_{J})) r^{-1} \quad \text{and} \quad b_\alpha = 2(d \, \operatorname{tr}(\Sigma_{J}) \sqrt{r})^{-1} \tag{5.5}$$

and every $r > 0$. 

The proof is given in Appendix [A.3] Note that contrary to the corresponding estimate for $\alpha^Z,\lambda$ given in Lemma 4.2 the above estimate for $\alpha^J,\lambda$ is valid not only for sufficiently large $\lambda \geq \lambda_0 > 0$ but for all $\lambda > 0$.

The following theorem provides an analogous result to Theorem 4.3 and gives an upper bound for the expectation of the trace of $Q^J,\lambda$ from which the convergence to zero can be deduced.

**Theorem 5.2.** For every $\delta \in (0, T]$ and $\lambda > 0$ it holds

$$\text{tr} \left( Q_t^J,\lambda \right) \leq \frac{K_J}{\sqrt{\lambda}} \quad \text{for} \quad t \in [\delta, T]$$

where $e = \exp(1)$ denotes Euler’s number.

In particular, it holds $\text{tr} \left( Q_t^J,\lambda \right) \to 0 \quad \text{as} \quad \lambda \to \infty \quad \text{for all} \quad t \in (0, T]$.

**Proof.** Let us define the function $g(t) := \text{tr}(Q_t^J,\lambda)$ for $t \in [0, T]$. Then using (5.2) and the linearity of the trace operator yields $g(t) = \text{tr}(q_0) + \int_0^t \text{tr}(\alpha^J,\lambda(Q^J,\lambda)) ds = \text{tr}(q_0) + \int_0^t g(s) ds$. Analogous to the proof of (4.12) in Theorem 4.3 i.e., applying Lemma 5.1 and Gronwall’s Lemma we obtain for $t \in [\delta, T]$ and $\lambda > 0$

$$g(t) \leq \frac{1}{\sqrt{\lambda}} b_\alpha (\text{tr}(q_0) (e^\delta - 1) + a_\alpha).$$

Recall (5.5) stating that the constants $a_\alpha, b_\alpha$ can be chosen as

$$a_\alpha = a_\alpha(r) = \text{tr}(\Sigma_\mu) + (d \text{tr}(\Sigma_\mu)r)^{-1} \quad \text{and} \quad b_\alpha = b_\alpha(r) = 2(d \text{tr}(\Sigma_\mu) r)^{-1}$$

for any $r > 0$. We now choose $r$ such that the r.h.s. of (5.8) attains its minimum. The unique minimizer is found as $r^* = (d \text{tr}(\Sigma_\mu)(\text{tr}(q_0)(e^\delta)^{-1} + \text{tr}(\Sigma_\mu)))^{-1}$ and the minimal value is given by $K_J/\sqrt{\lambda}$ with $K_J$ defined in (5.7). This proves the first claim. Since that inequality holds for all $\delta \in (0, T]$ the convergence $\text{tr}(Q^J,\lambda) \to 0$ for $\lambda \to \infty$ holds for all $t \in (0, T]$. □

As in Section 4.1 the above asymptotic properties for the trace of $Q^J,\lambda$ imply analogous results for its norm. The proof is analogous to the proof of Corollary 4.4.

**Corollary 5.3.** For every $\delta \in (0, T]$ and any matrix norm $\| \cdot \|$ there exists a constant $C > 0$ such that

$$\| Q_t^J,\lambda \|^p \leq \frac{C}{\sqrt{\lambda}} \quad \text{for} \quad \lambda > 0, \quad t \in [\delta, T] \quad \text{and} \quad p \geq 1.$$  

For the Frobenius norm $\| \cdot \|_F$ the constant $C$ can be chosen as $C = K_J C_F^{p-1}$ where $K_J$ is given in (5.7) and $C_F$ denotes the upper bound from Proposition 3.4 for the Frobenius norm $\| Q^J,\lambda \|_F$.

In particular, it holds $\| Q_t^J,\lambda \|^p \to 0$ as $\lambda \to \infty$ for all $t \in (0, T]$.

Based on the limit theorem for the conditional variance we now can state the corresponding result for the convergence of the conditional mean $M^J = M^J,\lambda$. The proof is analogous to the proof of Theorem 4.6.

**Theorem 5.4.** Let $K_J$ be the constant given in Theorem 5.2. Then for every $\delta \in (0, T]$

$$\mathbb{E} \left[ \| M_t^J,\lambda - \mu_t \|^2 \right] \leq \frac{K_J}{\sqrt{\lambda}} \quad \text{for} \quad \lambda > 0, \quad t \in [\delta, T].$$

In particular, it holds $\mathbb{E} \left[ \| M_t^J,\lambda - \mu_t \|^2 \right] \to 0$ as $\lambda \to \infty$ for all $t \in (0, T]$.

Note that contrary to the corresponding estimate for $M^Z,\lambda$ given in Theorem 4.6 the above estimate for $M^J,\lambda$ is valid not only for sufficiently large $\lambda \geq \lambda_Q > 0$ but for all $\lambda > 0$. 
6 Numerical Example

In this section we illustrate the theoretical findings of the previous sections by results of some numerical experiments. These experiments are based on a stock market model where the unobservable drift $\mu$ follows an Ornstein-Uhlenbeck process as given in [2.4] and [2.5] whereas the volatility is known and constant. For simplicity, we assume that there is only one risky asset in the market, i.e. $d = 1$. For our numerical experiments we use the model parameters given in Table 1.

The distribution of the initial value $\mu_0$ of the drift process is assumed to be the stationary distribution of the Ornstein-Uhlenbeck process, i.e., the limit of the marginal distribution of $\mu_t$ for $t \to \infty$ which is known to be Gaussian with mean $\overline{\mu}_0 = \overline{\mu}$ and variance $\overline{\sigma}_0^2 = \overline{\sigma}^2 / \overline{\kappa}$.

| Mean reversion level $\overline{\mu}$ | 0.1 | Time horizon $T$ | 1 year |
|--------------------------------------|-----|------------------|--------|
| Mean reversion speed $\kappa$       | 3   | Stock volatility $\sigma_R$ | 0.25   |
| Volatility $\sigma_\mu$             | 1   | Expert’s variance $\Gamma = \overline{\sigma}^2$ | 0.05   |
| Initial value $\mu_0$: mean $\overline{\mu}_0 = \overline{\mu}$ | 0.1 | Filter: initial values $m_0 = \overline{\mu}_0$ | 0.1    |
| variance $\overline{\sigma}_0^2 = \overline{\sigma}^2 / \overline{\kappa}$ | 0.15 | $\phi_0 = \overline{\sigma}_0$ | 0.16   |

Table 1. Model parameters for numerical experiments

The arrival dates of the expert opinions are modelled as jump times of a Poisson process with intensity $\lambda$. Then the waiting times between two information dates are exponentially distributed with parameter $\lambda$ and the investor receives until time $T$ on average $\lambda T$ expert opinions. Recall that the expert’s views are modelled by

$$Z_k = \mu T_k + \Gamma^{1/2} \epsilon_k, \quad k \in \mathbb{N},$$

where $(\epsilon_k)_{k \geq 1}$ is a sequence of independent standard normally distributed random variables.

At initial time $t = 0$ all partially informed investors have the same information about the hidden drift. For the experiment we assume that they only know the model parameters described by $\mathcal{F}_0^H = \{\emptyset, \Omega\}$. Then the initial values for the filter processes $M^H$ and $Q^H$ are the parameters of the Gaussian distribution of $\mu_0$, i.e. $m_0 = \overline{\mu}_0 = \overline{\mu}$ and $\phi_0 = \overline{\sigma}_0 = \overline{\sigma}^2 / \overline{\kappa}$, respectively.

In Figure 1 we plot the filters given by conditional mean $M^H$ and conditional variance $Q^H$ of the $R$-investor (blue), the $Z$-investor together with the associated $J$-investor against time. For the $Z$-investor we consider the arrival intensities $\lambda = 5, 20, 2000$ (yellow, orange, red). The volatility of the associated continuous-time expert opinions is chosen as $\sigma_\mu^2 = \Gamma / \lambda$. In the upper plot one can see the conditional variances $Q^R$, $Q^{Z,\lambda}$, $Q^{J,\lambda}$ and we also highlight (in green) the zero level corresponding to the limit process for $\lambda \to \infty$. The lower plot shows a realization of the unobservable drift process $\mu$ (in green) together with its estimates given by the conditional means $M^R$ (blue) and $M^{Z,\lambda}$ (yellow, orange, red). We omit plotting the paths of $M^{J,\lambda}$.

Since the filter processes for the $R$- and $Z$-investor start with the same initial value their paths are identical until the arrival of the first expert opinion leading to a filter update. This can be nicely seen for $\lambda = 5$ and also for $\lambda = 20$ while for $\lambda = 2000$ the first update is almost immediately after the initial time $t = 0$. At the information dates the updates decrease the conditional variance and lead to a jump of the conditional mean. The updates of the conditional mean typically decrease the distance of $M^{Z,\lambda}$ to the hidden drift $\mu$, of course this depends on the actual value of the expert’s view. Note that the drift estimate $M^R$ of the $R$-investor is quite poor and fluctuates just around the mean-reversion level $\overline{\mu}$. However, the expert opinions visibly improve the drift estimate.
Fig. 1. Simulation of the filter processes $Q^H$ and $M^H$. The upper subplot shows realizations of the conditional variances $Q^R$, $Q^{Z,\lambda}$ (solid) and $Q^{J,\lambda}$ (dotted) for various intensities $\lambda$. The volatility of the continuous-time expert opinions is chosen as $\sigma^2 = \Gamma/\lambda$. The lower subplot shows realizations of the corresponding conditional means $M^R$ and $M^{Z,\lambda}$ together with the path of the drift process $\mu$ (green).

After an update the conditional variance $Q^{Z,\lambda}$ increases and if the waiting time to the next information date is sufficiently large then it almost approaches the level of $Q^R$. Again, this can nicely be observed for $\lambda = 5$. During such long periods without new expert opinions the conditional mean of the $Z$-investor $M^{Z,\lambda}$ tends to move towards the path of $M^R$.

Looking at the paths of the conditional variance it can be seen that $Q^R_t$ dominates $Q^{Z,\lambda}_t$ and $Q^{J,\lambda}_t$ for all $t \in (0, T]$ which confirms the corresponding property stated in Proposition 3.4 and illustrates the fact that additional information by expert opinions leads to improved drift estimates. Note that for increasing $t$ the conditional variances $Q^R_t$ and $Q^{J,\lambda}_t$ quickly approach a constant which is the limit for $t \to \infty$. That convergence $Q^R$ has been proven in Proposition 4.6 of Gabih et al. [11] for markets with a single stock and generalized in Theorem 4.1 of Sass et al. [21] for markets with multiple stocks. The proof for $Q^{J,\lambda}$ is analogous.

Comparing the paths of the filter processes of the $Z$- and $J$-investor for increasing arrival intensity $\lambda$ it can be observed that the conditional variances $Q^{Z,\lambda}$ and $Q^{J,\lambda}$ approach zero for any $t \in (0, T]$. This fact illustrates our findings in Theorems 4.3 and 5.2. Further, with increasing $\lambda$ the path of the conditional mean $M^{Z,\lambda}$ approaches the path of the hidden drift $\mu$ which confirms the mean-square convergence stated in Theorem 4.6.

Finally, we want to examine the goodness of the upper bounds $K^Z/\sqrt{\lambda}$ and $K^J/\sqrt{\lambda}$ for the conditional variances of the $Z$- and $J$-investor given in Theorems 4.3 and 5.2, respectively. Note that in the present example with $d = 1$ stock the two constants $K^Z$, $K^J$ coincide, it holds $K^Z = K^J = \left(\Gamma[\sigma^2 + q_0(e^\delta - 1)]\right)^{1/2}$. In order to facilitate the visual comparison of the conditional variances and their upper bounds we focus on the information regime $H = J$ and rewrite the estimate (5.6) as $\sqrt{\lambda}Q^{J,\lambda}_t \leq K^J = K^J(\delta)$ for $t \in [\delta, T]$. Fig. 2 shows for $\lambda = 5, 20, 2000$ (yellow, orange, red solid lines) the conditional variances $Q^{J,\lambda}_t$ scaled by $\sqrt{\lambda}$ together with the upper bounds $K^J(\delta)$ (green) for two values of $\delta$. It can be seen that the upper bounds are quite close to the actual values on $[\delta, T]$, in particular for larger $\delta$.

We also plot realizations of $\sqrt{\lambda}Q^{Z,\lambda}_t$ for the associated $Z$-investor (dashed lines). Note that estimate (4.8) does hold for the expected variance $E[Q^{Z,\lambda}_t]$ but not for the realizations.
Fig. 2. Conditional variances scaled by $\sqrt{\lambda}$ of the $J$-investor, i.e. $\sqrt{\lambda} Q_{J,\lambda}$ for $\lambda = 5, 20, 2000$ (solid), and of the $Z$-investor $\sqrt{\lambda} Q_{Z,\lambda}$ (dotted) for $\lambda = 5, 20$. The volatility of the continuous-time expert opinions is chosen as $\sigma_{J} = \sqrt{\Gamma/\lambda}$.

The green lines represent the upper bounds $K^{J} = K^{J}(\delta)$ given in Theorem 5.2 for $\delta = \delta_{1} = 0.1$ and $\delta = \delta_{2} = 0.5$.

A Proofs

A.1 Auxiliary Results

The proof of Lemma 4.2 which is given in Appendix A.2 is based on various properties of symmetric and positive semidefinite matrices which we collect in the next lemma.

Lemma A.1. (Properties of symmetric and positive semidefinite matrices)

Let $A, B \in \mathbb{R}^{d \times d}, d \in \mathbb{N}$, symmetric and positive semidefinite matrices. Then it holds

1. $A + B$ is symmetric positive semidefinite.

2. The eigenvalues $\varphi_{i} = \varphi_{i}(A)$ of $A$ are nonnegative, and there exists an orthogonal matrix $V$ such that

$$A = V D V^{T} \quad \text{with} \quad D = \text{diag}(\varphi_{1}, \ldots, \varphi_{d}),$$

i.e., $A$ is diagonalizable.

3. If $A$ is positive definite then it is nonsingular and the inverse $A^{-1}$ is symmetric and positive definite.

4. $\varphi_{\text{min}}(A) \text{tr}(B) \leq \text{tr}(AB) \leq \varphi_{\text{max}}(A) \text{tr}(B)$

where $\varphi_{\text{min}}(A)$ and $\varphi_{\text{max}}(A)$ denote the smallest and largest eigenvalue of $A$, respectively.

5. $\left\frac{\text{tr}(B)}{\text{tr}(A^{-1})}\right \text{tr}(AB) \leq \text{tr}(A) \text{tr}(B)$

where for the first inequality $A$ is assumed to be positive definite.

6. $\text{tr}^{2}(A) \geq \text{tr}(A^{2}) \geq \frac{1}{d} \text{tr}(A)$

7. $\|A\|_{F} = \sqrt{\text{tr}(A^{2})} \leq \text{tr}(A)$

where $\|A\|_{F}$ denotes the Frobenius norm of $A$.

Proof. The first three properties are standard and we refer to Horn and Johnson [13, Chapter 7]. The proof of A.2 is given in Wang et al. [24, Lemma 1].
5. From (A.1) we have \( A = V D V^\top \) with an orthogonal matrix \( V \) and \( D = \text{diag}(\varrho_1, \ldots, \varrho_d) \). If \( A \) is positive definite then \( \varrho_{\text{min}}(A) > 0 \) and the inverse \( A^{-1} \) exists, see property 3. It holds \( \text{tr}(A) = \sum_{i=1}^{d} \varrho_i(A) \geq \varrho_{\text{max}}(A) \) and
\[
\text{tr} \left( A^{-1} \right) = \text{tr}(V D^{-1} V^\top) = \text{tr}(D^{-1}) = \sum_{i=1}^{d} \frac{1}{\varrho_i(A)} \geq \frac{1}{\varrho_{\text{min}}(A)}.
\]
The above inequalities together with (A.2) imply (A.3).

6. As above we use \( A = V D V^\top \) with an orthogonal matrix \( V \) and deduce \( A^2 = V D^2 V^\top \) and
\[
\text{tr} \left( A^2 \right) = \text{tr} \left( V^\top D^2 V \right) = \text{tr} \left( D^2 \right) = \sum_{i=1}^{d} \varrho_i^2 \geq \frac{1}{d} \left( \sum_{i=1}^{d} \varrho_i \right)^2 = \frac{1}{d} \text{tr}^2(A),
\]
where we have applied the Cauchy-Schwarz inequality. The first inequality in (A.4) follows from (A.3) with \( A = B \).

7. Let \( C = A^2 \), then \( C^{ii} = \sum_{k=1}^{d} (A^{ik})^2 \) and
\[
\text{tr}(A^2) = \text{tr}(C) = \sum_{i=1}^{d} C^{ii} = \sum_{i,k=1}^{d} (A^{ik})^2 = \|A\|_F^2,
\]
yielding the first equality. The inequality follows from (A.4).

\[\square\]

**A.2 Proof of Lemma 4.2**

For the convenience of the reader we recall the statement of Lemma 4.2:

For the function \( \alpha^{Z,\lambda} \) given in (4.2), there exist constants \( a_\alpha, b_\alpha > 0 \) independent of \( \lambda \) and there exists \( \lambda_0 > 0 \) such that for all symmetric and positive semidefinite \( q \in \mathbb{R}^{d \times d} \)
\[
\text{tr} \left( \alpha^{Z,\lambda}(q) \right) \leq a_\alpha - \sqrt{\lambda} b_\alpha \text{tr}(q), \quad \text{for} \quad \lambda \geq \lambda_0.
\]
The above estimate holds for every \( a_\alpha > \text{tr}(\Sigma_\mu), \quad a_\alpha > \text{tr}(\Sigma_\mu) \),
\[
b_\alpha < b_\alpha = b_\alpha(a_\alpha) := 2 \sqrt{a_\alpha - \text{tr}(\Sigma_\mu) \over \text{tr}(\Gamma)},
\]
\[
\lambda_0 = \lambda_0(a_\alpha, b_\alpha) := \left( \frac{d(a_\alpha - \text{tr}(\Sigma_\mu))}{2 \sqrt{\text{tr}(\Gamma)(a_\alpha - \text{tr}(\Sigma_\mu)) - b_\alpha \text{tr}(\Gamma)}} \right)^2.
\]

**Proof.** Using the definition of \( \alpha^{Z,\lambda} \) in (4.2), the linearity of \( \text{tr}(\cdot) \) and that \( q \) and \( \Sigma_R \) and therefore \( \Sigma_R^{-1} \) are symmetric positive definite, and that \( \kappa \) is positive definite we find
\[
\text{tr} \left( \alpha^{Z,\lambda}(q) \right) = \text{tr} \left( \Sigma_\mu - \kappa q - \kappa^\top q \Sigma_R^{-1} q - \lambda q \Gamma + q^{-1} q \right) \leq \text{tr} \left( \overline{\alpha}(q) \right),
\]
where \( \overline{\alpha}(q) := \Sigma_\mu - \lambda q (\Gamma + q)^{-1} q \).

The inequality follows from properties of symmetric positive definite matrices, see (A.2), (A.3) and (A.4) from which we deduce
\[
\text{tr}(\kappa q + q \kappa^\top) = \text{tr}((\kappa + \kappa^\top) q) \geq \varrho_{\text{min}}(\kappa + \kappa^\top) \text{tr}(q) \geq 0,
\]
\[
\text{tr}(q \Sigma_R^{-1} q) = \text{tr}(q \Sigma_R^{-1} q) \geq \frac{\text{tr}(q^2)}{\text{tr}(\Sigma_R)} \geq \frac{1}{d} \frac{\text{tr}(q^2)}{\text{tr}(\Sigma_R)} \geq 0.
\]
Here, \( g_{\min}(\cdot) \) denotes the the smallest eigenvalue of a positive definite symmetric matrix, which are all positive. Note that since \( \kappa \) is positive definite \( \kappa + \kappa^T \) is symmetric and positive definite. Further, \( q^2 \) is symmetric and positive semidefinite and according to property 3 of Lemma A.1 \( \Sigma^{-1} \) is symmetric and positive definite.

Inequality (A.6) implies that it suffices to prove the claim for \( \alpha^\lambda \), i.e.,

\[
\text{tr}(\alpha^\lambda(q)) \leq a_\alpha - \sqrt{\lambda b_\alpha} \text{tr}(q) \quad \text{for} \quad \lambda \geq \lambda_0. 
\]  
(A.7)

For the proof of (A.7) we set \( \varepsilon = \frac{1}{\sqrt{\lambda}} \), \( q = \varepsilon z \), \( a_\mu = \text{tr} (\Sigma_\mu) \) and consider the function \( H^\varepsilon : \mathbb{R}^{d \times d} \to \mathbb{R} \) with

\[
H^\varepsilon(z) := -\text{tr}(\alpha^{1/\varepsilon^2}(\varepsilon z)) + a_\alpha - \frac{1}{\varepsilon} b_\alpha \text{tr}(\varepsilon z) 
\]

\[
= \text{tr}(z(\Gamma + \varepsilon z)^{-1}z) - b_\alpha \text{tr}(z) + a_\alpha - a_\mu 
\]  
(A.8)

for \( a_\alpha, b_\alpha, \varepsilon_0 > 0 \) and symmetric and positive semidefinite matrices \( z \). Below we show that there exist positive constants \( a_\alpha, b_\alpha, \varepsilon_0 \) such that for all \( z \) it holds

\[
H^\varepsilon(z) \geq 0 \quad \text{for} \quad \varepsilon \leq \varepsilon_0. 
\]  
(A.9)

That inequality implies for \( z = \frac{1}{\varepsilon} q = \sqrt{\lambda} q \)

\[
0 \leq H^\varepsilon(z) = H^\varepsilon(\sqrt{\lambda} q) = -\text{tr}(\alpha^\lambda(q)) - \sqrt{\lambda} b_\alpha \text{tr}(q) + a_\alpha, 
\]

and (A.6) yields for \( \lambda \geq \lambda_0 = (\frac{1}{\varepsilon_0})^2 \)

\[
\text{tr}(\alpha^{Z_\lambda}(q)) \leq \text{tr}(\alpha^\lambda(q)) \leq a_\alpha - \sqrt{\lambda b_\alpha} \text{tr}(q), 
\]

which proves the assertion.

In the remainder of the proof we show inequality (A.9). The matrices \( z \) and \( \Gamma \) are symmetric, \( z \) is positive semidefinite and \( \Gamma \) is strictly positive definite. Then \( \Gamma + \varepsilon z \) is strictly positive definite and according to properties 1. and 3. of Lemma A.1 the matrix \( (\Gamma + \varepsilon z)^{-1} \) is symmetric and strictly positive definite. Further, \( z^2 \) is symmetric and positive semidefinite. Inequality (A.3) implies \( \text{tr}(AB) \geq \text{tr}(B)/\text{tr}(A^{-1}) \) and with \( A = (\Gamma + \varepsilon z)^{-1} \) and \( B = z^2 \) we find

\[
\text{tr}(z(\Gamma + \varepsilon z)^{-1}z) = \text{tr}(z^2(\Gamma + \varepsilon z)^{-1}) \geq \frac{\text{tr}(z^2)}{\text{tr}(\Gamma + \varepsilon z)} = \frac{\text{tr}(z^2)}{\text{tr}(\Gamma) + \varepsilon \text{tr}(z)}. 
\]

Inequality (A.4) yields \( \text{tr}(z^2) \geq \frac{1}{\psi} \text{tr}(z)^2 \), and hence we obtain

\[
\text{tr}(z(\Gamma + \varepsilon z)^{-1}z) \geq \frac{\frac{1}{\psi} \text{tr}(z)^2}{\text{tr}(\Gamma) + \varepsilon \text{tr}(z)} = \frac{\text{tr}(z)^2}{\psi + \varepsilon d \text{tr}(z)}, 
\]  
(A.10)

where \( \psi = d \text{tr}(\Gamma) \). Set \( x = \text{tr}(z) \geq 0 \) and \( g(x) = \frac{\psi^2}{\psi + \varepsilon dx} - b_\alpha x + a_\alpha - a_\mu \) then (A.10) implies

\[
H^\varepsilon(z) \geq g(x). 
\]

Now it remains to choose constants \( a_\alpha, b_\alpha, \varepsilon_0 > 0 \) such that

\[
g(x) \geq 0 \quad \text{for all} \quad x \geq 0 \quad \text{and} \quad \varepsilon \leq \varepsilon_0. 
\]  
(A.11)

Let \( a_\alpha > a_\mu = \text{tr}(\Sigma_\mu) \). Then \( \pi := g(0) = a_\alpha - a_\mu > 0 \). Since \( \psi + \varepsilon dx > 0 \) the inequality \( g(x) \geq 0 \) is equivalent to

\[
0 \leq f(x) := (\psi + \varepsilon dx)g(x) = A^x x^2 + B^x x + C, 
\]
where $A^\varepsilon = 1 - \varepsilon db_\alpha$, $B^\varepsilon = \varepsilon d\bar{\alpha} - b_\alpha \psi$, $C = \psi \bar{\alpha}$. Let $\varepsilon > 0$ be chosen so that $A^\varepsilon > 0$, then
\[
  f(x) = A^\varepsilon \left( x^2 + \frac{B^\varepsilon}{A^\varepsilon} x + \frac{C}{A^\varepsilon} \right) = A^\varepsilon \left( x - \frac{K^\varepsilon}{2} \right)^2 + D^\varepsilon
\]
with
\[
  K^\varepsilon = \frac{B^\varepsilon}{A^\varepsilon} \quad \text{and} \quad D^\varepsilon := C - \frac{1}{4} \left( \frac{B^\varepsilon}{A^\varepsilon} \right)^2 = \frac{4CA^\varepsilon - (B^\varepsilon)^2}{4A^\varepsilon}.
\]
We choose $a_\alpha > a_\mu$, i.e., $\bar{\alpha} = a_\alpha - a_\mu > 0$, then we have $f(x) \geq 0$ if $D^\varepsilon \geq 0$ or equivalently if
\[
  P(\varepsilon) := 4CA^\varepsilon - (B^\varepsilon)^2 \geq 0.
\]
$P(\varepsilon)$ is a quadratic function and it holds $P(\varepsilon) = 4\psi \bar{\alpha} - (\varepsilon d\bar{\alpha} + b_\alpha \psi)^2$, hence $P(0) = 4\psi \bar{\alpha} - b_\alpha \psi$ and $P$ is decreasing for $\varepsilon > 0$. Thus we have to require $P(0) > 0$ which gives
\[
  0 < b_\alpha \leq \bar{b}_\alpha = \bar{b}_\alpha(a_\alpha) = 2 \sqrt{\frac{a_\alpha - a_\mu}{\psi}}.
\]
Then $P(\varepsilon) \geq 0$ for $\varepsilon \in (0, \varepsilon_0]$ where $\varepsilon_0$ is the positive zero of $P$ given by
\[
  \varepsilon_0 = \varepsilon_0(a_\alpha, \beta_\alpha) = \frac{1}{d(a_\alpha - a_\mu)} \left( 2\sqrt{\psi(a_\alpha - a_\mu)} - b_\alpha \psi \right).
\]
It is not difficult to check that for $\varepsilon < \varepsilon_0$ it holds
\[
  A^\varepsilon = 1 - \varepsilon db_\alpha > \left( 1 - b_\alpha \sqrt{\frac{\psi}{a_\alpha - a_\mu}} \right)^2 \geq 0.
\]
Note that for $b_\alpha = \bar{b}_\alpha$ it holds $\varepsilon_0 = 0$ which is not feasible. Hence for $a_\alpha > a_\mu$, $b_\alpha \in (0, \bar{b}_\alpha(a_\alpha))$ and for $\varepsilon \leq \varepsilon_0 = \varepsilon_0(a_\alpha, b_\alpha)$ or equivalently $\lambda \geq \lambda_0 = 1/\varepsilon_0^2$ it holds (A.11) and therefore $H^\varepsilon(z) \geq 0$ under the conditions given in (4.7). This completes the proof. □

A.3 Proof of Lemma 5.1

For the convenience of the reader we recall the statement of Lemma 5.1

For the function $\alpha^{J,\lambda}$ given in (5.3) there exist constants $a_\alpha, b_\alpha > 0$ independent of $\lambda$ such that for all symmetric and positive semidefinite $q \in \mathbb{R}^{d \times d}$
\[
  \text{tr} \left( \alpha^{J,\lambda}(q) \right) \leq a_\alpha - \sqrt{\lambda} b_\alpha \text{tr}(q), \quad \text{for} \quad \lambda > 0.
\]
The above estimate holds for $a_\alpha = \text{tr}(\Sigma_\mu) + (d \text{tr}(\Sigma_J)) r^{-1}$ and $b_\alpha = 2(d \text{tr}(\Sigma_J)) \sqrt{\lambda} r^{-1}$ and every $r > 0$.

Proof. Using the definition of $\alpha^{J,\lambda}$ in (5.3) and the linearity of $\text{tr}(\cdot)$ we find
\[
  \text{tr} \left( \alpha^{J,\lambda}(q) \right) = \text{tr}(\Sigma_\mu) - \text{tr}(q \Sigma_\mu) + (\text{tr}(q) + q \Sigma_\lambda) - \text{tr}(q \Sigma_J) \geq \beta \text{tr}(q) \quad \text{where} \quad \beta := \varrho_\text{min}(\kappa + \kappa^T) > 0 \text{ is the smallest eigenvalue of } \kappa + \kappa^T.
\]
That matrix is symmetric and positive definite since $\kappa$ is positive definite. Using (A.3) and (A.4) we deduce
\[
  \text{tr}(q \Sigma_\mu) \geq \lambda \text{tr}(q \Sigma_J) = \lambda \text{tr}(q^2 \Sigma_J^{-1}) \geq \lambda \frac{\text{tr}(q^2)}{\text{tr}(\Sigma_J)} \geq \frac{1}{2} \frac{\text{tr}(q^2)}{\text{tr}(\Sigma_J)} = \lambda \psi \text{tr}(q^2)
\]
where $\psi := (d \text{tr}(\Sigma_J))^{-1} > 0$. Substituting the above estimates into (A.12) we obtain
\[
  \text{tr} \left( \alpha^{J,\lambda}(q) \right) \leq f(\text{tr}(q)) \quad \text{with} \quad f(x) := a_\mu - \beta x - \lambda \psi x^2, \quad x \geq 0,
\]
where we set $a_\mu = \text{tr}(\Sigma_\mu)$. The quadratic function $f$ is strictly concave, thus for any $x_0 \geq 0$ it holds $f(x) \leq f(x_0) + f'(x_0)(x - x_0)$. Choosing $x_0 = 1/\sqrt{\lambda r}$ for some $r > 0$ it follows
\[
  f(x) \leq a_\mu + \frac{\psi}{\sqrt{\lambda}} \frac{2\psi}{\sqrt{\lambda}} x = a_\alpha - \sqrt{\lambda} b_\alpha x
\]
where we used the definition of $a_\alpha, b_\alpha$ in (5.5). Substituting this estimate into (A.13) proves the claim. □
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