Verma modules over the generalized Heisenberg-Virasoro algebra

Ran Shen*, Yucai Su†

*Department of Mathematics, Shanghai Jiao Tong University
Shanghai 200240, China
†Department of Mathematics, University of Science and Technology of China
Hefei 230026, China
Email: ranshen@sjtu.edu.cn, ycsu@ustc.edu.cn

Abstract. For any additive subgroup $G$ of an arbitrary field $F$ of characteristic zero, there corresponds a generalized Heisenberg-Virasoro algebra $\mathcal{L}[G]$. Given a total order of $G$ compatible with its group structure, and any $h, h_I, c, c_I, c_{LI} \in F$, a Verma module $\tilde{M}(h, h_I, c, c_I, c_{LI})$ over $\mathcal{L}[G]$ is defined. In this note, the irreducibility of Verma modules $\tilde{M}(h, h_I, c, c_I, c_{LI})$ is completely determined.

Key Words: The generalized Heisenberg-Virasoro algebra, Verma modules

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1. Introduction

Let $F$ be a field of characteristic 0. The well-known twisted Heisenberg-Virasoro algebra is the Lie algebra $\mathcal{L} := \mathcal{L}[\mathbb{Z}]$ with an $F$-basis $\{L_m, I_m, C, C_I, C_{LI} \mid m \in \mathbb{Z}\}$ subject to the following relations (e.g., [ACKP, B])

$$[L_n, L_m] = (n - m)L_{n+m} + \delta_{n,-m}\frac{n^3 - n}{12}C,$$
$$[L_n, I_m] = -mI_{n+m} - \delta_{n,-m}(n^2 + n)C_{LI},$$
$$[I_n, I_m] = n\delta_{n,-m}C_I,$$
$$[\mathcal{L}, C] = [\mathcal{L}, C_{LI}] = [\mathcal{L}, C_I] = 0.$$ 

This Lie algebra is the universal central extension of the Lie algebra of differential operators on a circle of order at most one, which contains an infinite-dimensional Heisenberg subalgebra and the Virasoro subalgebra. The natural action of the Virasoro subalgebra on the Heisenberg subalgebra is twisted with a 2-cocycle. The structure and representation theory for the twisted Heisenberg-Virasoro algebra has been well developed (e.g., [ACKP, B, FO, JJ, SJ]). The structure of the irreducible highest weight modules for the twisted Heisenberg-Virasoro algebra are determined in [ACKP, B].

By replacing the index group $\mathbb{Z}$ by an arbitrary subgroup $G$ of the base field $F$, it is

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natural to introduce the so-called generalized Heisenberg-Virasoro algebra $\mathcal{L}[G]$ (cf. Definition 2.1, see e.g., [XLT, LJ]). This is the Lie algebra which is the 3-dimensional universal central extension of the Lie algebra of generalized differential operators of order at least one. The Harish-Chandra modules of intermediate series over generalized Heisenberg-Virasoro algebra $\mathcal{L}[G]$ are discussed in [LJ].

Given any total order of $G$ compatible with its group structure, and given any $h, h_I, c, c_I, c_{LI} \in \mathbb{F}$, there corresponds a Verma module $\widetilde{M}(h, h_I, c, c_I, c_{LI})$ over $\mathcal{L}[G]$. Due to the fact that the representations of generalized Heisenberg-Virasoro algebras are closely related to the representation theory of toroidal Lie algebras as well as some problems in mathematical physics (e.g., [ACKP, FO, JJ]) and the Verma modules play the crucial role in the representation theory, it is very natural to consider the Verma modules over the generalized Heisenberg-Virasoro algebras. In this note, we completely determine the irreducibility of Verma modules $\widetilde{M}(h, h_I, c, c_I, c_{LI})$ over $\mathcal{L}[G]$ for arbitrary $G$. Namely, if $G$ does not contain a minimal positive element with respect to the total order, then the Verma module $\widetilde{M}(h, h_I, c, c_I, c_{LI})$ is irreducible if and only if $(c_I, c_{LI}) \neq (0, 0)$; in case if $G$ contains the minimal positive element $a$, then the Verma module $\widetilde{M}(h, h_I, c, c_I, c_{LI})$ is irreducible if and only if the $\mathcal{L}[Za]$-module generated by a fixed highest weight generator is irreducible over the twisted Heisenberg-Virasoro algebra $\mathcal{L}[Za]$ (cf. Theorem 3.1).

2. Generalized Heisenberg-Virasoro algebras

Let $U := U(\mathcal{L})$ be the universal enveloping algebra of the twisted Heisenberg-Virasoro algebra $\mathcal{L}$. For any $h, h_I, c, c_I, c_{LI} \in \mathbb{F}$, denote by $I(h, h_I, c, c_I, c_{LI})$ the left ideal of $U$ generated by the elements

$$\{L_i, I_j \mid i, j > 0\} \cup \{L_0 - h \cdot 1, I_0 - h_I \cdot 1, C - c \cdot 1, C_I - c_I \cdot 1, C_{LI} - c_{LI} \cdot 1\}.$$

The Verma module with highest weight $(h, h_I, c, c_I, c_{LI})$ over $\mathcal{L}$ is defined as

$$M(h, h_I, c, c_I, c_{LI}) := U/I(h, h_I, c, c_I, c_{LI}),$$

which is a highest weight module with a basis consisting of all vectors of the form

$$I_{-p_1}I_{-p_2} \cdots I_{-p_s}L_{-j_1}L_{-j_2} \cdots L_{-j_k}v_h, \quad (2.1)$$

where $s, k \in \mathbb{N} \cup \{0\}$, $p_r, j_i \in \mathbb{N}$ and $0 < p_1 \leq p_2 \leq \cdots \leq p_s$, $0 < j_1 \leq j_2 \leq \cdots \leq j_k$.

**Definition 2.1** Let $G \subseteq \mathbb{F}$ be an additive subgroup. The generalized Heisenberg-Virasoro algebra $\tilde{\mathcal{L}} := \mathcal{L}[G]$ is a Lie algebra with $\mathbb{F}$-basis $\{L_\mu, I_\mu, C, I, C_{LI} \mid \mu \in G\}$ subject
to the following relations [XLT, LJ]

\[
\begin{align*}
[L_\mu, L_\nu] &= (\mu - \nu)L_{\mu + \nu} + \delta_{\mu, -\nu} \frac{\mu^3 - \mu}{12} C, \\
[L_\mu, I_\nu] &= -\nu I_{\mu + \nu} - \delta_{\mu, -\nu}(\mu^2 + \mu) C_{LI}, \\
[I_\mu, I_\nu] &= \mu \delta_{\mu, -\nu} C, \\
[\tilde{L}, C] &= [\tilde{L}, C_{LI}] = [\tilde{L}, C_I] = 0.
\end{align*}
\]

For any \( x \in G^* := G \setminus \{0\} \), obviously, \( \mathbb{Z} x \subseteq G \). Let \( \mathcal{L}[\mathbb{Z} x] \) be the \( \mathbb{F} \)-subspace of \( \tilde{L} \) spanned by \( \{ L_{ix}, I_{ix}, C, C_I, C_{LI} | i \in \mathbb{Z} \} \). It is clear that \( \mathcal{L}[\mathbb{Z} x] \) is a Lie algebra isomorphic to the twisted Heisenberg-Virasoro algebra \( \mathcal{L} \). Precisely, we have

**Lemma 2.2** The map

\[
\theta : \mathcal{L} \rightarrow \mathcal{L}[\mathbb{Z} x]
\]

\[
\begin{align*}
L_i &\mapsto x^{-1} L_{ix} + \delta_{i,0} \frac{x - x^{-1}}{24} C, \\
I_i &\mapsto x^{-1} I_{ix} + \delta_{i,0}(1 - x^{-1}) C_{LI}, \\
C &\mapsto x C, \\
C_I &\mapsto x^{-1} C_I, \\
C_{LI} &\mapsto C_{LI},
\end{align*}
\]

for \( i \in \mathbb{Z} \), extends uniquely to a Lie algebra isomorphism between \( \mathcal{L} \) and \( \mathcal{L}[\mathbb{Z} x] \).

**Proof.** This follows from straightforward verifications. \( \square \)

Throughout this note, we fix a total order “\( \succ \)" on \( G \) compatible with its group structure, namely, \( x \succ y \) implies \( x + z \succ y + z \) for any \( z \in G \). Denote

\[ G_+ := \{ x \in G \mid x \succ 0 \}, \quad G_- := \{ x \in G \mid x \prec 0 \}. \]

Then \( G = G_+ \cup \{0\} \cup G_- \).

For an \( \tilde{L} \)-module \( V \) and \( \lambda, h_I, c, c_I, c_{LI} \in \mathbb{F} \), denote by

\[ V_{\lambda,h_I,c,c_I,c_{LI}} := \{ v \in V | L_0 v = \lambda v, \ I_0 v = h_I v, \ C v = c v, \ C_I v = c_I v, \ C_{LI} v = c_{LI} v \}, \]

the weight space of \( V \). We shall simply write \( V_\lambda \) instead of \( V_{\lambda,h_I,c,c_I,c_{LI}} \). Define

\[ \text{supp}(V) := \{ \lambda \in \mathbb{F} | V_\lambda \neq 0 \}, \]

called the weight set (or the support) of \( V \). For any \( h, h_I, c, c_I, c_{LI} \in \mathbb{F} \), let \( \tilde{M}(h, h_I, c, c_I, c_{LI}) \) be the Verma module for \( \tilde{L} \), which is defined by using the order “\( \succ \)" and the same fashion.
as that for $\mathcal{L}$ at the beginning of this section. Then $I_0, C, C_I, C_{LI}$ acts as $h_I, c, c_I, c_{LI}$ respectively on $\widetilde{M}(h,h_I,c,c_I,c_{LI})$ and

$$\text{supp}(\widetilde{M}(h,h_I,c,c_I,c_{LI})) = h + G_+.$$

For any $x \in G_+$, let

$$\widetilde{M}_x(h,h_I,c,c_I,c_{LI}) = U(\mathcal{L}[\mathbb{Z}x])v_h,$$

be the $\mathcal{L}[\mathbb{Z}x]$-submodule of $\widetilde{M}(h,h_I,c,c_I,c_{LI})$ generated by a fixed highest weight generator $v_h$. Note that the subgroup $\mathbb{Z}x$ is also a “totally ordered abelian group”, inheriting the order “$>$” from $G$. It is easy to see that

$$ax \succ bx \iff a > b \text{ for } a, b \in \mathbb{Z}.$$

As a result, we have

**Corollary 2.3** As an $\mathcal{L}$-module, we have

$$\widetilde{M}_x(h,h_I,c,c_I,c_{LI}) \cong M(x^{-1}h + \frac{x - x^{-1}}{24}c, x^{-1}h_I + (1 - x^{-1})c_{LI}, xc, x^{-1}c_I, c_{LI}).$$

**Proof.** This is clear by Lemma 2.2. \qed

### 3. The main result

Recall that $(G, \succ)$ is a totally ordered abelian group. Denote

$$B(x) = \{y \in G \mid 0 \prec y \prec x\} \text{ for } x \in G_+.$$

The order “$\succ$” is called *dense* if $\sharp B(x) = \infty$ for all $x \in G_+$; *discrete* if there exists some $a \in G_+$ such that $B(a) = \emptyset$, in this case $a$ is called the *minimal positive element* of $G$.

For convenience, we denote

$$I_{-j} := I_{-j_1}I_{-j_2} \cdots I_{-j_k} \text{ for } 0 \prec j_1 \preceq j_2 \preceq \cdots \preceq j_k, \ j = (j_1, j_2, \ldots, j_k),$$

$$I_{-p} := I_{-p_s}I_{-p_{s-1}} \cdots I_{-p_1} \text{ for } 0 \prec p_s \preceq \cdots \preceq p_2 \preceq p_1, \ p = (p_s, \ldots, p_2, p_1).$$

Then $U(\mathcal{L}_{-})$ has a basis

$$\{I_{-p}L_{-j} \mid \text{ for all } j, p \text{ as in (3.1) and (3.2)}\}.$$  \hspace{1cm} (3.3)

Denote by $|j|$ the number of components in $j$. Then $|j| = k$ in (3.1) and $|p| = s$ in (3.2).

The main result in this note is following.

**Theorem 3.1** Let $h, h_I, c, c_I, c_{LI} \in \mathbb{F}$.

1. With respect to a dense order “$\succ$” of $G$, the Verma module $\widetilde{M}(h,h_I,c,c_I,c_{LI})$ is an irreducible $\mathcal{L}[G]$-module if and only if $(c_I, c_{LI}) \neq (0,0)$.
(2) With respect to a discrete order “⪰” of G with minimal positive element a, the Verma module \( \tilde{M}(h, h_I, c, c_I, c_{LI}) \) is an irreducible \( \mathcal{L}[G] \)-module if and only if \( \tilde{M}_a(h, h_I, c, c_I, c_{LI}) \) (cf. (2.2)) is an irreducible \( \mathcal{L}[\mathbb{Z}a] \)-module.

**Remark 3.2** Suppose \( c_I = c_{LI} = 0 \) in case of Theorem 3.1(1). Since

\[ \tilde{I} := \text{span}_F \{ I_\mu, C_I, C_{LI} \mid \mu \in G \}, \]

is an ideal of \( \tilde{L} \), the Verma module \( V := \tilde{M}(h, h_I, c, 0, 0) \) over \( \tilde{L} \) has a proper submodule \( U(\tilde{I})V \) such that the quotient module \( W := V/U(\tilde{I})V \) is simply the Verma module over the generalized Virasoro algebra \( \text{Vir}[G] := \text{span}_F \{ L_\mu, C \mid \mu \in G \} \cong \tilde{L}/\tilde{I} \), whose irreducibility is completely determined in [HWZ]. Also note that the irreducibility of a Verma module over the twisted Heisenberg-Virasoro algebra \( L \) is completely determined in [B]. Thus, essentially the above theorem has in fact determined the structure of all Verma modules over \( \tilde{L} \).

**Proof of Theorem 3.1.** (1) Suppose the order “⪰” of G is dense. Let \( v_h \) be a fixed highest weight generator in \( \tilde{M}(h, h_I, c, c_I, c_{LI}) \) of weight \( h \). Let \( u_0 \notin F v_h \) be any given weight vector in \( V := \tilde{M}(h, h_I, c, 0, 0) \).

**Claim 1:** There exists a weight vector \( u \in U(\mathcal{L}[G])u_0 \) of weight \( \lambda \) such that

\[ u = \sum_p a_p I_{-p} v_h \] (a finite sum) for some \( a_p \in F^* = F\{0\} \). (3.4)

For each \( m \in \mathbb{N} \), set

\[ V_m := \sum_{p,j:|j| \leq m} \mathbb{F} I_{-p} L_{-j} v_h. \] (3.5)

It is clear that

\[ L_x V_m \subseteq V_m, \ I_x V_m \subseteq V_m \] for \( x \in G_+ \).

We can write \( u_0 \) as (cf. (2.1) and (3.3))

\[ u_0 = \sum_{p,j} a_{pj} I_{-p} L_{-j} v_h \] for some \( a_{pj} \in F^* \).

Let \( r := \max \{|j| \mid a_{pj} \neq 0\} \). If \( r = 0 \), then the claim holds clearly. We assume \( r \geq 1 \), and write

\[ u_0 \equiv u_0' \pmod{V_{r-1}}, \] where \( u_0' = \sum_{p,j:|j|=r} a_{pj} I_{-p} L_{-j} v_h. \) (3.6)

Let \( x \in G_+ \) such that (cf. (3.2) for notation \( p_l \))

\[ x < \min\{j_1 \mid a_{pj} \neq 0\} \] and \( \{x, j_1 - x \mid a_{pj} \neq 0\} \cap \{p_l \mid a_{pj} \neq 0, \forall l\} = \emptyset. \]
Then

\[ I_xu'_0 = \sum_{p,j:|j|=r} x_{apj}I_p \left( \sum_{i=1}^{r} L_{-j_1} \cdots L_{-j_{i-1}}x_{-j_i}L_{-j_{i+1}} \cdots L_{-j_r} \right) v_h. \]

If any

\[ xa_{pj}I_p L_{-j_1} \cdots L_{-j_{i-1}}x_{-j_i}L_{-j_{i+1}} \cdots L_{-j_r} \]

and

\[ xa_{p'j'}I_{p'}L_{-j'_1} \cdots L_{-j'_{i-1}}x_{-j'_i}L_{-j'_{i+1}} \cdots L_{-j'_r}, \]

for \( 1 \leq i, s \leq r \), are linear dependent, it is not difficult to see that \( p = p' \) and \( j = j' \). Hence

\[ 0 \neq u_1 := I_xu'_0 \in V_{r-1}. \]

Similarly, let \( u_1 \equiv u'_1 \) (mod \( V_{r-2} \)) as in (3.6), then \( u'_1 \neq 0 \). For \( k = 2, \cdots, r \). We define recursively and prove by induction that,

\[ u_k := I_xu_{k-1} \in V_{r-k}, \; u_k \equiv u'_k \pmod{V_{r-k-1}}, \; u'_k \neq 0. \]

Letting \( k = r \), we get that \( 0 \neq u_r \in V_0 \). Our claim follows.

Now let \( u \) be as in (3.4). Set \( P := \{ p \mid a_p \neq 0 \} \neq \emptyset \). We define the total order “\( \succ \)” on \( P \) as follows: For any \( p, p' \in P \), if \( k := |p| > l := |p'| \), we set \( p'_i = 0 \) for \( i = l + 1, \cdots, k \). Then

\[ p \succ p' \iff \exists s \text{ with } 1 \leq s \leq k \text{ such that } p_s \succ p'_s \text{ and } p_t = p'_t \text{ for } t < s. \quad (3.7) \]

Let

\[ q := (q_{k_0}, \cdots, q_2, q_1), \quad 0 < q_{k_0} \leq \cdots \leq q_1, \]

be the unique maximal element in \( P \). Then

Case 1: If \( c_I \neq 0 \), then by the simple calculations

\[ bv_h = I_qu \in U(\mathcal{L}[G])u_0 \text{ for some } b \in \mathbb{F}^*. \]

Case 2: Suppose \( c_I = 0, \; c_{LI} \neq 0 \). Let \( y \in G_+ \) such that

\[ \{ x \in G \mid q_1 - y < x < q_2 \} \cap \{ p_1, p_2 \mid p \in P \} = \emptyset. \]

Then

\[ u' := L_{q_1-y}u = a'I_{-z}v_h \text{ for some } a' \in \mathbb{F}^*, \]

where

\[ z = (z_{k_0}, \cdots, z_2, z_1), \quad 0 < z_{k_0} \leq \cdots \leq z_2 \leq z_1, \quad \text{and} \]

\[ \{ z_i \mid i = 1, 2, \cdots, k_0 \} = \{ q_{k_0}, \cdots, q_3, q_2, y \}. \]

(i) If \( \{ z_i \mid i = 1, 2, \cdots, k_0 \} \cap \{ h_I/c_{LI} - 1 \} = \emptyset \), then

\[ b'v_h = L_zu' \in U(\mathcal{L}[G])u_0 \neq 0, \text{ where } b' = \prod_{i=1}^{k_0} z_i(h_I - (z_i + 1)c_{LI}) \in \mathbb{F}^*. \]
(ii) If there exists some \( z_i = h_I/c_{LI} - 1 \) with \( 1 \leq i \leq k_0 \). We assume
\[
\{z_i\} \cap \{z_k \mid 1 \leq k \leq k_0, k \neq i\} = \emptyset.
\]
Otherwise, we only need to recurse the following proof. Let
\[
w := L_{z_{i-1}} \cdots L_{z_{2}} L_{z_{1}} u' = a'' I_{z_{k_0}} I_{z_{k_0-1}} \cdots I_{z_i} v_h \neq 0 \text{ for some } a'' \in \mathbb{F}^*.
\]
Take \( x' \in G_+ \) such that \( z_i - x' > z_k, i < k \leq k_0 \). Then
\[
w' := L_{z_i-x'} w = \overline{a} I_{z_{k_0}} I_{z_{k_0-1}} \cdots I_{z_{i+1}} I_{x'} v_h \neq 0 \text{ for some } \overline{a} \in \mathbb{F}^*,
\]
and \( \{z_{k_0}, z_{k_0-1}, \ldots, z_{i+1}, x'\} \cap \{h_I/c_{LI} - 1\} = \emptyset \). This becomes case (i) if we take \( u' \) to be \( w' \).

Therefore, \( v_h \in U(\mathcal{L}[G]) u_0 \) in any case. Hence \( \widetilde{M}(h, h_I, c_I, c_{LI}) \) is irreducible.

(2) Suppose the order "\( \succ \)" of \( G \) is discrete with the minimal positive element \( a \). Then \( Za \subseteq G \). For any \( x \in G \), we write \( x \succ Za \) if \( x \succ na \) for all \( n \in \mathbb{Z} \). Let
\[
H_+ := \{x \in G \mid x \succ Za\}, \quad H_- = -H_+.
\]
It is not difficult to see that
\[
G = Za \cup H_+ \cup H_-.
\]
(3.8)
Then one can see that
\[
\mathcal{L}[H_+]\widetilde{M}_a(h, h_I, c, c_I, c_{LI}) = 0 \text{ (recall (2.2))}.
\]
Since
\[
\widetilde{M}(h, h_I, c, c_I, c_{LI}) \cong U(\mathcal{L}[G]) \otimes_{U(\mathcal{L}[Za]+\mathcal{L}[H_+]}) \widetilde{M}_a(h, h_I, c_I, c_{LI}),
\]
it follows that the irreducibility of \( \mathcal{L}[G] \)-module \( \widetilde{M}(h, h_I, c_I, c_{LI}) \) imply the irreducibility of \( \mathcal{L}[Za] \)-module \( \widetilde{M}_a(h, h_I, c, c_{LI}) \).

Conversely, suppose \( \widetilde{M}_a(h, h_I, c_I, c_{LI}) \) is an irreducible \( \mathcal{L}[Za] \)-module. Let \( u_0 \notin \mathbb{F} v_h \) be any weight vector in \( \widetilde{M}(h, h_I, c_I, c_{LI}) \). We want to prove
\[
U(\mathcal{L}[G]) u_0 \cap \widetilde{M}_a(h, h_I, c, c_I, c_{LI}) \neq \{0\}, \quad (3.9)
\]
from which the irreducibility of \( \widetilde{M}(h, h_I, c, c_I, c_{LI}) \) as \( \mathcal{L}[G] \)-module follows immediately.

Case 1: \( c_I \neq 0 \). We can write \( u_0 \) as (cf. (3.8))
\[
u_0 \equiv \sum_{p'j'p_j \in H_+, p, j, j' \in \mathbb{Z} + a, |j'| + |j| = r} a_{p'j'p_j} I_{-p'} L_{-j'} I_p L_{-j} v_h \text{ (mod } V_{r-1}) \text{ for some } a_{p'j'p_j} \in \mathbb{F}^*,
\]
where $V_{r-1}$, $r$ are defined as in (3.5) and (3.6). Let (cf. notation (3.2))

$$P' = \{ pp' = (p_s, \cdots, p_2, p_1', \cdots, p'_2, p'_1) | 0 < p_s \leq \cdots \leq p_2 \leq p_1 < p'_1 \leq \cdots \leq p'_2 \leq p'_1, \ a_{p'j'pj} \neq 0 \}.$$ 

If $P' \neq \emptyset$, we define the total order “$>$” on $P'$ as in (3.7). Let $q^0$ be the maximal element in $P'$. Then

$$u'_0 := I_{q^0} u_0 = \sum_{j' \in H_+, j' \in \mathbb{Z}_+ a, |j'|+|j|=r} a_{j'j} L_{-j} L_{-j} v_h \ (\text{mod } V_{r-1}) \text{ for some } a_{j'j} \in \mathbb{F}^*.$$ 

If $P' = \emptyset$, then $u_0$ has the form of $u'_0$ naturally. By the proof of [HWZ, Theorem 3.1], there exists a weight vector $0 \neq u \in U(\mathcal{L}[G]) u_0 \cap \widetilde{M}_a(h, h_I, c, c_I, c_{LI})$, which gives (3.9) as required.

Case 2: $c_I = 0$. We can write

$$u_0 = \sum_{p'_1, j'_1 \in H_+, p_s, j_r \in \mathbb{Z}_+ a} b_{p'_j'pj'j} I_{-p'} L_{-j} I_{-p} L_{-j} v_h \text{ for some } b_{p'_j'pj} \in \mathbb{F}^*.$$ 

If $J := \{ j' | b_{p'_j'pj} \neq 0 \} \neq \emptyset$, we set $j(0) := \min \{ j'_1 | b_{p'_j'pj} \neq 0 \}$. Then there exists some $m \in \mathbb{N}$ such that

$$\{ j'_1 - \varepsilon | b_{p'_j'pj} \neq 0 \} \cap \{ p'_1 | b_{p'_j'pj} \neq 0, \forall j \} = \emptyset, \text{ where } \varepsilon = j(0) - m.$$ 

Let $n_0 = \max \{ |j'| | b_{p'_j'pj} \neq 0 \}$, then

$$u' := I_{n_0} u_0 = \sum_{p'_j \in H_+, p_s, j_r \in \mathbb{Z}_+ a} b'_{p'_j} I_{-p'} I_{-p} L_{-j} v_h \neq 0 \text{ for some } b'_{p'_j} \in \mathbb{F}^*,$$

by the proof of Claim 1. If $J = \emptyset$, then $u_0$ has the form of $u'$ naturally. Let

$$Q := \{ p' | b'_{p'j} \neq 0, |p'| = t \}, \text{ where } t = \min \{ |p'| | b'_{p'j} \neq 0 \}.$$ 

If $t = 0$, the theorem holds clearly since $u'$ is a weight vector. We assume $t \geq 1$. Then $Q \neq \emptyset$. Again, we define the total order “$<$” on $Q$ as in (3.7). Let

$$q' := (q'_1, q'_2, \cdots, q'_t), \ 0 < q'_1 \leq q'_2 \leq \cdots \leq q'_t,$$

be the unique minimum element in $Q$. For $m \in \mathbb{N}$, set

$$V' = \sum_{p'_j \in H_+, p_s, j_r \in \mathbb{Z}_+ a, |p'| \geq m} \mathbb{F} I_{-p'} I_{-p} L_{-j} v_h.$$ 

Then

$$u' = \sum_{p'_j \in H_+, p_s, j_r \in \mathbb{Z}_+ a, |p'| = t} b'_{p'_j} I_{-p'} I_{-p} L_{-j} v_h \ (\text{mod } V'_{t+1}).$$
We have
\[ u(1) := L_{q'_1}u' \equiv \sum_{p^{(1)}(1) \in H_+} b^{(1)}_{p^{(1)}(1)p_j} I_{-p^{(1)}(1)}I_{-a}I_{-p}L_{-j}v_h \pmod{V'_t} \]
for some \( b^{(1)}_{p^{(1)}(1)p_j} \in \mathbb{F}^* \). Define \( Q^{(1)} = \{ p^{(1)} | b^{(1)}_{p^{(1)}(1)p_j} \neq 0 \} \), \( q^{(1)} = (q'_1, q'_2, \ldots, q'_t) \). By our assumption and the commutator relations for \( \mathcal{L}[G] \), we see that \( b^{(1)}_{q^{(1)}(1)p_j} \neq 0 \), hence \( Q^{(1)} \neq \emptyset \).

Moreover, \( q^{(1)} \) is the unique minimum element in \( Q^{(1)} \).

Now for \( s = 2, 3, \ldots, t \), we define recursively and prove by induction that

(i) Let \( u(s) := L_{q'_s}u(s-1) \). Then
\[ u(s) \equiv \sum_{p^{(s)}(s) \in H_+} b^{(s)}_{p^{(s)}(s)p_j} I_{-p^{(s)}(s)}I_{-a}I_{-p}L_{-j}v_h \pmod{V'_{t-s+1}} \]
for some \( b^{(s)}_{p^{(s)}(s)p_j} \in \mathbb{F}^* \).

(ii) Let \( Q^{(s)} = \{ p^{(s)} | b^{(s)}_{p^{(s)}(s)p_j} \neq 0 \} \neq \emptyset \). Moreover, \( q^{(s)} = (q'_{s+1}, q'_{s+2}, \ldots, q'_t) \) is the unique minimum element in \( Q^{(s)} \).

Now letting \( s = t \) and noting that \( u(t) \) is a weight vector, we get that \( 0 \neq u(t) \in U(\mathcal{L}[G])u_0 \cap \widetilde{M}_a(h, h_I, c, c_I, c_{LI}) \), which gives (3.9) as required. □

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