ON A VARIANT OF TARTAR’S FIRST COMMUTATION LEMMA

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ABSTRACT. We prove a variant of Tartar’s first commutation lemma involving multiplier operators with symbols not necessarily defined on a manifold of codimension one.

1. Introduction

At the beginning of 90’s, L.Tartar [14] and P.Gerard [3] independently introduced the $H$-measures (microlocal defect measures). The $H$-measures appeared to be very powerful tool in many fields of mathematics and physics (see randomly chosen [11 5 6 7 9 12 13 16]). They are given by the following theorem:

**Theorem 1.** [14] If $(u_n) = ((u_1^n, \ldots, u_r^n))$, is a sequence in $L^2(\mathbb{R}^d; \mathbb{R}^r)$ such that $u_n \rightharpoonup 0$ in $L^2(\mathbb{R}^d; \mathbb{R}^r)$, then there exists its subsequence $(u_{n'})$ and a positive definite matrix of complex Radon measures $\mu = \{\mu^{ij}\}_{i,j=1,\ldots,d}$ on $\mathbb{R}^d \times S^{d-1}$ such that for all $\varphi_1, \varphi_2 \in C_c(\mathbb{R}^d)$ and $\psi \in C(S^{d-1})$:

$$
\lim_{n' \to \infty} \int_{\mathbb{R}^d} \langle \varphi_1 u_{n'}(x) \rangle_A \varphi_2(x) \psi(x) dx = \langle \mu^{ij}, \varphi_1 \varphi_2 \psi \rangle
$$

where $A_\psi$ is a multiplier operator with the symbol $\psi \in C^\kappa(S^{d-1})$.

The complex matrix Radon measure $\{\mu^{ij}\}_{i,j=1,\ldots,d}$ defined in the previous theorem we call the $H$-measure corresponding to the subsequence $(u_{n'}) \in L^2(\mathbb{R}^d; \mathbb{R}^r)$.

The $H$-measures describe a loss of strong $L^2$ precompactness for the corresponding sequence $(u_n) \in L^2(\mathbb{R}^d; \mathbb{R}^r)$. In order to describe loss of $L^p_{loc}$ precompactness for a sequence $(u_n) \in L^p_{loc}(\mathbb{R}^d)$, $p > 1$, we have extended the notion of the $H$-measures in [8] as follows.

**Theorem 2.** [8] Assume that $(u_n)$ is a sequence in $L^p_{loc}(\mathbb{R}^d)$, $p > 1$, such that $u_n \rightharpoonup 0$, $n \to \infty$, in $L^p_{loc}(\mathbb{R}^d)$, $\beta > 0$. Assume that $(v_n)$ is a bounded sequence in $L^\infty(\mathbb{R}^d)$.

Then, there exist subsequences $(u_{n'})$ and $(v_{n'})$ of the sequences $(u_n)$ and $(v_n)$, respectively, such that there exists a complex valued distribution $\mu \in \mathcal{D}'(\mathbb{R}^d \times S^{d-1})$, such that for every $\varphi_1, \varphi_2 \in C_c(\mathbb{R}^d)$ and $\psi \in C^\kappa(S^{d-1})$, $\kappa > d/2$, $\kappa \in \mathbb{N}$:

$$
\lim_{n' \to \infty} \int_{\mathbb{R}^d} \langle \varphi_1 u_{n'}(x) \rangle_{A_\psi} \varphi_2(x) \psi(x) dx = \langle \mu, \varphi_1 \varphi_2 \psi \rangle
$$

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where $A_\psi : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is a multiplier operator with the symbol $\psi \in C^\infty(S^{d-1})$.

The first commutation lemma was one of the key points in the proof of Theorem 1 as well as Theorem 2 (with a simple modification; see [8, Lemma 14]). It is stated as follows:

**Lemma 3.** [8, Lemma 14] (First commutation lemma) Let $a \in C(S^{d-1})$ and $b \in C_0(\mathbb{R}^d)$. Let $A$ be a multiplier operator with the symbol $a$, and $B$ be an operator of multiplication given by the formulae:

$$F(Au)(\xi) = a\left(\frac{\xi}{|\xi|}\right)F(u)(\xi) \quad a.e. \ \xi \in \mathbb{R}^d,$$

$$Bu(x) = b(x)u(x) \quad a.e. \ x \in \mathbb{R}^d,$$

where $cal F$ is the Fourier transform. Then $C = AB - BA$ is a compact operator from $L^p(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$, $p > 1$.

The proof of the lemma heavily relies on the fact that the function $a$ is actually defined on the unit sphere. Recently, two new variants of the $H$-measures were introduced – the parabolic $H$-measures [2] and ultra-parabolic $H$-measures [12]. In both cases a variant of the first commutation lemma is needed, and in both cases its proof is based on the fact that a symbol $a$ of appropriate multiplier $A$ is defined on a smooth, bounded, simply connected manifold of codimension one.

In order to motivate our variant of the first commutation lemma, notice that from the proof of Theorem 1 (see [14]), it follows that we need to "commute" $A(\varphi_2u_n)$ by $\varphi_2A(u_n)$, where $(u_n)$ is the sequence bounded in $L^2(\Omega)$, which was exactly done in Lemma 3. Similarly, in Theorem 2 (see [8]), we need to "commute" $A(\varphi_2v_n)$ by $\varphi_2A(v_n\chi_{\text{supp}\varphi_2})$, where $\chi_V$ is the characteristic function of the set $V$, and $(v_n)$ is the sequence bounded in $L^\infty(\Omega)$. Therefore, it is enough to prove that the commutator $C$ is compact operator from $L^\infty_0(\Omega)$ into $L^p_{\text{loc}}(\Omega), p > 1$. We state:

**Lemma 4.** Let $a \in C^n(\mathbb{R}^d), \kappa = \lfloor d/2 \rfloor + 1$, and $b \in C_0(\mathbb{R}^d)$. Suppose that for some constant $k > 0$ and for any real number $r > 0$

$$\int_{\frac{r}{2} \leq ||\xi|| \leq r} |D^\alpha_a(\xi)|^2 d\xi \leq k^2r^{d-2n(\alpha)} \quad (3)$$

holds for every $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ satisfying $n(\alpha) = \sum_{i=1}^d \alpha_i \leq \kappa$.

Let $A$ be a multiplier operator with the symbol $a$, and $B$ be an operator of multiplication given by the formulae:

$$F(Au)(\xi) = a(\xi)F(u)(\xi) \quad a.e. \ \xi \in \mathbb{R}^d,$$

$$Bu(x) = b(x)u(x) \quad a.e. \ x \in \mathbb{R}^d,$$

where $F$ is the Fourier transform. Then $C = AB - BA$ is a compact operator from $L^\infty_0(\mathbb{R}^d)$ into $L^{p_0}_{\text{loc}}(\mathbb{R}^d)$ for every $1 < p_0 < \infty$.

**Remark 5.** We hope that the lemma could serve for defining variants of the $H$-measures adapted to equations which change type (such as non-strictly parabolic equations).
\textbf{Proof:} On the first step notice that \( a \) satisfies conditions of the Hörmander-Mikhlin theorem (see [11] [4]). Therefore, for every \( p > 1 \) there exists a constant \( k_p \) such that \( \|A\|_{L^p \to L^p} \leq k_p \). Thus,
\[
\|C\| \leq 2k_p\|b\|_{L^\infty(\mathbb{R}^d)}.
\]

Then, notice that we can assume \( b \in C_0^1(\mathbb{R}^d) \). Indeed, if we assume merely \( b \in C_0(\mathbb{R}^d) \) then we can uniformly approach the function \( b \) by a sequence \( (b_n) \in C_0^1(\mathbb{R}^d) \). The corresponding sequence of commutators \( C_n = AB_n - B_nA \), where \( B_n(u) = b_nu \), converges in norm toward \( C \). So, if we prove that \( C_n \) are compact for each \( n \), the same will hold for \( C \) as well.

Then, fix a real non-negative function \( \psi \) with a compact support and total mass one. Take the characteristic function \( \chi_{B(0,2)} \) of the ball \( B(0,2) \subset \mathbb{R}^d \) and denote:
\[
\chi(x) = \chi_{B(0,2)} \ast \frac{1}{\varepsilon^d} \Pi_{i=1}^d \omega(x_i/\varepsilon)
\]
for an \( \varepsilon > 0 \) small enough so that we have \( \chi(x) = 1 \) for \( x \in B(0,1) \), and \( (1-\chi) \equiv 1 \) out of the ball \( B(0,3) \).

Next, notice that \( A = A_{a\chi} + A_{a(1-\chi)} \), where \( A_{a\chi} \) is a multiplier operator with the symbol \( a\chi \), and \( A_{a(1-\chi)} \) is a multiplier operator with the symbol \( a(1-\chi) \). Accordingly,
\[
C = AB - BA = A_{a\chi}B - BA_{a\chi} + A_{a(1-\chi)}B - BA_{a(1-\chi)} = C_{a\chi} + C_{a(1-\chi)},
\]
where \( C_{a\chi} = A_{a\chi}B - BA_{a\chi} \) and \( C_{a(1-\chi)} = A_{a(1-\chi)}B - BA_{a(1-\chi)} \).

First, consider the commutator \( C_{a\chi} \). Notice that since \( a\chi_{B(0,2)} \) has a compact support, the multiplier \( A_{a\chi} \) is actually the convolution operator with the kernel \( \psi = \mathcal{F}(a\chi) \in L^2(\mathbb{R}^d) \), where \( \mathcal{F} \) is the inverse of the Fourier transform \( \mathcal{F} \):
\[
A_{a\chi}(u) = \psi \ast u, \ u \in L^\infty(\mathbb{R}^d).
\]
Therefore, we can state that
\[
C_{a\chi}u(x) = \int_{\mathbb{R}^d} (b(x) - b(y)) \psi(x - y)u(y)dy,
\]
is a compact operator from \( L^\infty(\mathbb{R}^d) \) into \( L^{p_0}_{\text{loc}}(\mathbb{R}^d) \), \( p_0 \geq 2 \).

Indeed, take an arbitrary bounded sequence \( (u_n) \in L^\infty(\mathbb{R}^d) \) such that \( u_n \to 0 \) weak-* in \( L^\infty(\mathbb{R}^d) \) and \( \text{supp}u_n \subset \tilde{V} \subset \subset \mathbb{R}^d \), for a relatively compact set \( \tilde{V} \). In order to prove that \( C_{a\chi} \) is compact, it is enough to prove that \( C_{a\chi}u_n \) strongly converges to zero in \( L^{p_0}_{\text{loc}}(\mathbb{R}^d) \), \( p_0 > 1 \).

Since \( \psi \in L^2(\mathbb{R}^d) \) we also have \( \psi \in L^1_{\text{loc}}(\mathbb{R}^d) \). Thus, it holds for every fixed \( x \in \mathbb{R}^d \)
\[
C_{a\chi}u_n(x) = \int_{\mathbb{R}^d} (b(x) - b(y)) \psi(x - y)u_n(y)dy \to 0, \ n \to 0.
\]

Next, since the sequence \( (u_n) \) has compact support, we also have:
\[
|C_{a\chi}u_n(x)| \leq \tilde{C},
\]
for a constant \( \tilde{C} \) depending on the support of the sequence \( (u_n) \) as well as \( L^2 \) norm of the kernel \( \psi \).
Combining (5) and (9) with the Lebesgue dominated convergence theorem, we see that for an arbitrary relatively compact \( V \subset \mathbb{R}^d \) and every \( p_0 > 0 \), it holds:

\[
\int_V |C_{a\lambda} u_n(x)|^{p_0} \, dx \to 0, \quad n \to \infty,
\]

proving (7).

In order to prove that \( C_{a(1-\chi)} \) is compact, we need more subtle arguments basically involving techniques from the proof of the Hormander-Mikhlin theorem from e.g. [11].

So, let \( \Theta \) be a non-negative infinitely differentiable function supported by \( \{ \xi \in \mathbb{R}^n : \frac{1}{2} \leq \| \xi \| \leq 2 \} \). Also let \( \theta(\xi) = \Theta(\xi) / \sum_{j=-\infty}^{\infty} \Theta(2^{-j} \xi) \), \( \theta(0) := 0 \). Then, \( \theta \) is non-negative, it is supported by \( \{ \xi \in \mathbb{R}^n : \frac{1}{2} \leq \| \xi \| \leq 2 \} \), it is infinitely differentiable, and is such that if \( \xi \neq 0 \), then \( \sum_{j=-\infty}^{\infty} \theta(2^{-j} \xi) = 1 \).

Now, let \( a_j(\xi) = a(\xi)(1 - \chi(\xi))\theta(2^{-j} \xi), \ j \geq 0 \). Then, \( a_j \) has support in the set \( \{ \xi \in \mathbb{R}^n : 2^{-j-1} \leq \| \xi \| \leq 2^{-j+1} \}, \ j \geq 0 \), and \( a(\xi)(1 - \chi(\xi)) = \sum_{j=0}^{\infty} a_j(\xi) \). Furthermore, it holds for any \( p \in \{ 1, \ldots, d \}, \ a_p \in \mathcal{N}_0 \) that

\[
\frac{\partial^{\alpha_p}}{\partial^{\alpha_p} \xi_p} (a(\xi)(1 - \chi(\xi))\theta(2^{-j} \xi)) = \sum_{l=0}^{\alpha_p} \binom{\alpha_p}{l} \frac{\partial^l a(\xi)}{\partial^l \xi_p} \frac{\partial^{\alpha_p-l}(1 - \chi(\xi))\theta(2^{-j} \xi)}{\partial^{\alpha_p-l} \xi},
\]

so that, with suitable bounded functions \( a_{\beta \gamma}, \ \beta + \gamma = \alpha, \ \alpha \in \mathcal{N}_0^d \), we have \( D^\alpha_\xi a_j(\xi) = \sum_{\beta+\gamma=\alpha} a_{\beta \gamma} 2^{-jn(\beta)} D^\gamma_\xi a(\xi) \).

From here, on applying hypothesis (3) with \( r = 2^j \), it follows from the Minkowski inequality that

\[
\int_{\mathbb{R}^d} |D^\alpha_\xi a_j(\xi)|^2 d\xi \leq \rho_0^2 \sum_{\beta+\gamma=\alpha} a_{\beta \gamma} 2^{-jn(\beta)} \int_{2^{-j-1} \leq \| \xi \| \leq 2^{-j+1}} |D^\gamma_\xi a(\xi)|^2 d\xi \tag{11}
\]

\[
\leq C k 2^{jd-2n(\alpha)},
\]

where \( C \) is a constant independent on \( k \).

Denote by \( \tilde{a}_j = \mathcal{F}(a_j(x), x \in \mathbb{R}^d \), the inverse Fourier transform of the function \( a_j \). From [11], the Cauchy-Schwartz inequality, Planchar’s theorem and the well known properties of the Fourier transform, for every \( s > 0 \) it holds (see also the proof of [11] Theorem 7.5.13):

\[
\int_{\| x \| > s} |\tilde{a}_j(x)| \, dx \leq \left( \int_{\| x \| \geq s} \| x \|^{-2\kappa} \, dx \right)^{1/2} \left( \int_{\| x \| \geq s} \| \tilde{a}_j(x) \|^{2\kappa} \, dx \right)^{1/2} \tag{12}
\]

\[
\leq \left( \frac{2^{d-1} s^{d-2\kappa}}{2\kappa - d} \right)^{1/2} \left( \frac{d\kappa}{2\kappa - d} \sum_{i=1}^{d} \int_{\mathbb{R}^d} |x_i|^{2\kappa} |\tilde{a}_j(x)|^2 \, dx \right)^{1/2}
\]

\[
= \left( \frac{2^{d-1} s^{d-2\kappa}}{2\kappa - d} \right)^{1/2} \left( \frac{d\kappa}{2\kappa - d} \sum_{i=1}^{d} \int_{\mathbb{R}^d} |D^\xi_{\xi_i} a_j(\xi)|^2 \, d\xi \right)^{1/2}
\]

\[
\leq C_1 k (2^j s)^{(d-s\kappa)},
\]

where \( C_1 \) depends only on the functions \( \theta \) and \( \chi \).
Next, consider $\bar{A}_n(x) = \sum_{j=0}^n \bar{a}_j(x), \ x \in \mathbb{R}^d$. For an arbitrary fixed $s > 0$, the series $\sum_{j=0}^n \bar{a}_j(x)$ is absolutely convergent in $L^1(\mathbb{R}^d \setminus B(0,s))$. Indeed,

$$\|\bar{A}_n(x)\|_{L^1(\mathbb{R}^d \setminus B(0,s))} \leq \sum_{j=0}^n \|\bar{a}_j\|_{L^1(\mathbb{R}^d \setminus B(0,s))} \leq C_1s(\frac{1}{d}-\kappa) \sum_{j=0}^n 2^{j(\frac{1}{d}-\kappa)} \leq C_3 < \infty,$$

for a constant $C_3 > 0$, since $\frac{1}{d} - \kappa < 0$.

Thus, for every $s > 0$ there exists $\bar{A}_s \in L^1(\mathbb{R}^d \setminus B(0,s))$ such that

$$\sum_{j=0}^\infty \bar{a}_j(x) = \bar{A}_s(x), \ x \in \mathbb{R}^d \setminus B(0,s).$$

Furthermore, for an odd $d$, from (12) we have

$$\int \|x\| \cdot |\bar{A}_n(x)| \, dx \to 0, \ s \to 0,$$

while for an even $d$ conclusion (15) follows from:

$$\int \|x\| \cdot |\bar{A}_n(x)| \, dx \leq \left( \int \|x\| \leq 1 - 2\kappa \right)^{1/2} \left( \int \|x\|^{2\kappa - 2} |\bar{a}_j(x)|^2 \, dx \right)^{1/4} \times \left( \int \|x\|^2 |\bar{a}_j(x)|^2 \, dx \right)^{1/4} \leq C_4s^{1/2}.$$  

Now, take the convolution operator:

$$A_n(u) = \bar{A}_n \ast u, \ u \in L^\infty(\mathbb{R}^d).$$

and consider the commutator $C_n = A_n B - B A_n$. It holds:

$$C_n(u)(x) = - \int_{\mathbb{R}^d} \bar{A}_n(x - y)(b(x) - b(y))u(y) \, dy.$$

Given a fixed $s > 0$, rewrite $C_n(u)$ in the following way:

$$C_n(u)(x) = \int_{\mathbb{R}^d} \bar{A}_n(x - y)(b(x) - b(y))u(y) \, dy$$

$$= \int_{\|x - y\| > s} \bar{A}_n(x - y)(b(x) - b(y))u(y) \, dy + \int_{\|x - y\| \leq s} \bar{A}_n(x - y)(b(x) - b(y))u(y) \, dy.$$

From here, combining $b \in C_0^\infty(\mathbb{R}^d)$ with (14) and (15), we conclude for an arbitrary relatively compact $V \subset \subset \mathbb{R}^d$:

$$\lim_{n \to \infty} \|C_n(u)(x)\|_{L^p(V)} \leq \|\int_{\|x - y\| > s} \bar{A}_n(x - y)(b(x) - b(y))u(y) \, dy\|_{L^p(V)} + o_s(1),$$

where

$$o_s(1) = \int_{\|x - y\| \leq s} \bar{A}_n(x - y)(b(x) - b(y))u(y) \, dy$$

$$= \int_{\|x - y\| \leq s} \|x - y\| \bar{A}_n(x - y) \frac{(b(x) - b(y))}{\|x - y\|}u(y) \, dy \to 0, \ \varepsilon \to 0.$$
Furthermore, arguing as for (7), we infer that the operator
\[ u \mapsto \int_{\|x - y\| > s} A_n(x - y)(b(x) - b(y))u(y)dy \]
(17)

is a compact operator from \( L^\infty_0(\mathbb{R}^d) \) to \( L^{p_0}_0(\mathbb{R}^d) \) for an arbitrary \( p_0 > 1 \).

Next, notice that we have for any \( u \in L^2 \):
\[
\| (C_n - C_{\phi(1-\chi)})(u) \|_{L^2} \\
\leq \| (A_n - A_{\phi(1-\chi)})(bu) \|_{L^2} + \| b \|_\infty \| (A_n - A_{\phi(1-\chi)})(u) \|_{L^2} \\
= \| (A_n - a(1-\chi))F(bu) \|_{L^2} + \| b \|_\infty \| (A_n - a(1-\chi))F(u) \|_{L^2} \to 0, \ n \to \infty,
\]
and from here and the Fatou lemma:
\[
\| C_{\phi(1-\chi)}(u) \|_{L^{p_0}(V)} \leq \limsup_{n \to \infty} \left( \int_{\|x - y\| > s} A_n(x - y)(b(x) - b(y))u(y)dy \right) \\
+ \int_{\|x - y\| < s} |x - y|A_n(x - y)\frac{(b(x) - b(y))}{|x - y|}u(y)dy\|_{L^{p_0}(V)} \\
\leq \| \int_{\|x - y\| \geq s} A_n(x - y)(b(x) - b(y))u(y)dy\|_{L^{p_0}(V)} + o_s(1),
\]
where \( o_s(1) \) denotes a quantity tending to zero as \( s \to 0 \), and appears here due to (18). From here and (17), it follows that \( C_{\phi(1-\chi)} \) is a compact operator since it can be estimated by a sum of a compact operator, and an operator bounded by an arbitrary small constant.

Thus, we see that \( C \) can be represented as the sum of two compact operators \( C_{\chi} \) and \( C_{\phi(1-\chi)} \), which means that \( C \) is a compact operator itself.

This concludes the proof.

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