Magnetic moment interaction in the anyon superconductor

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Magnetic moment interaction is shown to play a defining role in the magnetic properties of anyon superconductors. The necessary condition for the existence of the Meissner effect is found.

The zero-temperature Meissner effect presented in the 2+1 dimensional anyon matter provoked considerable efforts in order to promote the Chern-Simons gauge theory as a hypothetical candidate for the high-$T_c$ superconductivity.

The most important points in that development are existence of the massless pole in the current correlators [1], cancellation of bare and induced Chern-Simons terms [2] and detailed calculations of effective action and thermodynamic potential for the fermions interacting with Chern-Simons and Maxwell fields [3,4,5,6].

As a convincing argument in favor of the superconducting nature of the anyon system, one can use the energetic one: energy density evaluated as a function of the net particle density, and therefore is a constant, one concludes that the $\lambda$ term simply defines the energy scale and does not lead to any new effects in magnetic properties of the system.

To be complete, we consider the relativistic version and imply the normal ordering of the fermion operators in Hamiltonian and particle number operator. In this consideration Hamiltonian (1) becomes positively defined, and the planar density of relativistic thermodynamic potential looks as follows:

$$
\Omega = \frac{k_B T}{2\pi \ell^2} \left\{ \frac{1 + \sigma \epsilon}{2} \ln(1 - \rho_0) + \sum_{n=1}^{\infty} \ln(1 - \rho_n) \right\} + \frac{k_B T}{2\pi \ell^2} \left\{ \frac{1 - \sigma \epsilon}{2} \ln(1 - \bar{\rho}_0) + \sum_{n=1}^{\infty} \ln(1 - \bar{\rho}_n) \right\},
$$

(3)

where $\rho_n$ and $\bar{\rho}_n$ are the Fermi distribution functions for particles and antiparticles, respectively,

$$
\rho_n = \left[ 1 + e^{(E_n - \mu)/k_B T} \right]^{-1},
$$

$$
\bar{\rho}_n = \left[ 1 + e^{(E_n + \mu)/k_B T} \right]^{-1}.
$$

Relativistic energies $E_n$ are given by

$$
E_n = mc^2 \left[ 1 + 2(\lambda^2/\ell^2) n \right]^{1/2},
$$

(4)

where $\lambda = \hbar/mc$ is the Compton wavelength, while the magnetic length $\ell$ and the parameter $\epsilon$ are defined by

$$
\frac{1}{\ell^2} = \frac{1}{\hbar^2} eB + eb,
$$

(5)

$$
\epsilon = \text{sgn}(eB + eb).
$$

In the previous considerations [5] the second term in (2) was completely neglected ($\sigma = 0$). Evidently, this cannot be done in (1). The effects related to magnetic moment interactions can be included dealing either with (1) or (2).

In the relativistic theory the covariant coupling leads to the magnetic interaction of $(B + b)$-type only, which in the nonrelativistic limit is reduced to (2). However, in the nonrelativistic treatment one can admit the extra contributions of the Chern-Simons magnetic fields: $H_{\sigma}^{\text{rel}} \rightarrow H_{\sigma}^{\text{rel}} + \lambda b\psi^\dagger \psi$. Taking into account that $b$ is fixed by the net particle density, and therefore is a constant, one concludes that the $\lambda$ term simply defines the energy scale and does not lead to any new effects in magnetic properties of the system.
The fact that the relativistic thermodynamic potential does not contain the $n = 0$ terms corresponding to particles (when $\sigma = -\epsilon$) or antiparticles (when $\sigma = +\epsilon$) is the consequence of the absence of these modes in the spectrum of the relativistic one-particle Hamiltonian.

The system can be described by the Helmholtz free energy

$$F(B + b, T, \mathcal{N}) = \Omega(B + b, T, \mu) + \mu \mathcal{N},$$

where the chemical potential $\mu = \mu(B + b, T, \mathcal{N})$ should be found out from the equation

$$\mathcal{N} = -\frac{\partial \Omega}{\partial \mu} \equiv n_e - n_\bar{e}.$$

Here $n_e$ and $n_\bar{e}$ are the average densities of particles and antiparticles, respectively,

$$n_e = \frac{1}{2\pi^2} \left\{ \frac{1 + \sigma \epsilon}{2} \rho_0 + \sum_{n=1}^{\infty} \rho_n \right\},$$

$$n_\bar{e} = \frac{1}{2\pi^2} \left\{ \frac{1 - \sigma \epsilon}{2} \bar{\rho}_0 + \sum_{n=1}^{\infty} \bar{\rho}_n \right\}.$$ 

Using the usual definition of the filling fraction ($\nu = 2\pi^2 \mathcal{N}$) and of magnetic length (5) we get

$$\nu = \frac{2\pi \hbar}{\epsilon e} \frac{\mathcal{N}}{B + b}$$

implying that for a given $\mathcal{N}$ the filling fraction can be used instead of $B + b$ as a one of the independent arguments in the free energy (6). In terms of the distribution functions it appears as

$$\nu = \frac{1 + \sigma \epsilon}{2} \rho_0 - \frac{1 - \sigma \epsilon}{2} \bar{\rho}_0 + \sum_{n=1}^{\infty} (\rho_n - \bar{\rho}_n).$$

The mechanism leading to the Meissner effect is based on the assumption that at some value $\nu = \nu_0$ the free energy $F(\nu, T, \mathcal{N})$ possesses a minimum. In that case, choosing the corresponding value of the Chern-Simons magnetic field to be

$$b = \frac{2\pi \hbar \mathcal{N}}{\epsilon e \nu_0},$$

we observe that this minimum is achieved at $B = 0$, while any small $B \neq 0$ costs some positive energy.

The free energy calculated for the nonrelativistic Hamiltonian (2) with completely dropped magnetic moment interaction possesses the local minima at the integer values of filling fraction [5], where the Meissner effect just takes the place.

To simplify the present account, note that due to $\rho_n(\mu) = \bar{\rho}_n(-\mu)$ we have $\Omega(\sigma, \mu) = \Omega(-\sigma, -\mu)$ reflecting the invariance of the relativistic thermodynamic potential (3) under the interchange of particles and antiparticles. Correspondingly, the free energy (6) is invariant under $\{\sigma \to -\sigma, n_e \leftrightarrow n_\bar{e}\}$. Therefore, without loss of generality we can deal with $\mu > 0$. Consequently, due to $E_n + \mu > mc^2$ one has $\bar{\rho}_n < \exp(-mc^2/k_BT)$ and assuming

$$\frac{mc^2}{k_BT} >> 1,$$

we get the antiparticle contributions to be effectively zero ($\bar{\rho}_n = 0$), implying $n_e = 0$ and $\mathcal{N} = n_e$ (note that for $m$ being of order of an electron mass $m \sim m_e$ and $T \sim 100K$ we have $mc^2/k_BT \sim 10^5$).

In what follows we present the zero-temperature analysis and comment on the changes appearing at finite temperatures in the end.

Due to $\rho_n = 0$ we have $\nu \geq 0$, and presenting the filling fraction as $\nu = \mathcal{N} + \theta$, where $0 \leq \theta \leq 1$, $\mathcal{N} = 0, 1, 2, \ldots$ we write down the zero-temperature values of the particle distribution functions as

$$\rho_n = \begin{cases} 
1 & \text{if } n < N + (1/2)(1 - \sigma \epsilon) \\
\theta & \text{if } n = N + (1/2)(1 - \sigma \epsilon) \\
0 & \text{if } n > N + (1/2)(1 - \sigma \epsilon). 
\end{cases}$$

The free energy at $T = 0$ coincides with the internal energy $U = (1/2\pi^2) \sum \rho_n E_n$. For $0 \leq \nu \leq 1$ ($\mathcal{N} = 0$) one gets

$$F = mc^2 n_e \left[ 1 + 2\pi n_e \lambda^2 (1 - \sigma \epsilon) \nu^{-1} \right]^{1/2}.$$  

To carry out the analysis for $\nu \geq 1$ we assume

$$n_e \lambda^2 << 1$$

(for $m \sim m_e$ and the typical value [5] $n_e \sim 10^{14} cm^{-2}$, $n_e \lambda^2 \sim 10^{-7}$). Further, summation over the particle contributions has the upper bound defined in (9). Up to this value of $n$ due to (11) we can use

$$\lambda^2/\ell^2 n = 2\pi n_e \lambda^2 (n/\nu) << 1,$$

and the corresponding relativistic energies (4) are effectively reduced to the nonrelativistic ones

$$E_n = mc^2 n_e + \frac{\hbar^2}{m \ell^2 n},$$

leading to the following expression of the free energy:

$$F = mc^2 n_e + \frac{\pi \hbar^2 n_e^2}{m} \left[ 1 + \frac{\theta(1 - \theta)}{(N + \theta)^2} - \frac{\sigma \epsilon}{N + \theta} \right].$$

Note that for $\sigma = +\epsilon$ the expression (13) with $N = 0$ coincides with (10). For $\sigma = -\epsilon$, the reduction of (10) to (13) with $N = 0$ becomes invalid for $\nu \sim 0$ where the condition (12) cannot be justified. However, here we
can exclude this region from our consideration, since the Meissner effect is expected to take the place near the integer values of $\nu$.

The second term in (13) is the nonrelativistic free energy, which could be obtained if one starts with Hamiltonian (2). It should be pointed out that the nonrelativistic expression (13) was obtained from the relativistic considerations using the assumptions $\frac{mc^2}{\hbar} >> k_B T$ (8) and $n_e << \lambda^{-2}$ (11), but not a direct nonrelativistic limit ($c \to \infty$).

Omitting the relativistic contribution $mc^2 n_e$, we present $F(\nu)$ for the different values of $\sigma$ in figure 1. The case $\sigma = 0$ corresponds to the nonrelativistic Hamiltonian (2) with completely neglected magnetic moment interaction, but not to the massless relativistic fermions. As one can see, the local minima occurring for $\sigma = 0$ are lost for $\sigma = \pm \epsilon \neq 0$. Therefore, the relativistic fermionic system (1) as well as the nonrelativistic one (2) with preserved magnetic moment interaction does not exhibit the Meissner effect.

![Figure 1. Free energy of the one-component system.](image)

Consider now the system containing two types of fermions with opposite values of $\sigma$. In that case the system is the combination of two subsystems with $\sigma = \pm \epsilon$ and the corresponding quantities will be distinguished by the uppercase indices $\pm$. Now, the relation (7) (with $n_{\pm}^f = 0$) should be rewritten as

$$\nu = \frac{2\pi \hbar}{e \epsilon} \frac{n_e^+ + n_e^-}{B + b} = \nu^+ + \nu^-,$$

where the partial filling fractions appear as

$$\nu^{\pm} = \frac{1 \pm 1}{2} \rho_0^{\pm} + \sum_{n=1}^{\infty} \rho_n^{\pm},$$

and, as we see, depends on $2\theta \equiv \theta^+ + \theta^-$, but not on $\theta^+ - \theta^-$. Consequently, the free energy can be considered as a function of $\nu = \nu^+ + \nu^-$ and the corresponding curve is presented in figure 2 (we have omitted the relativistic contribution $mc^2 n_e$).

![Figure 2. Free energy of the compound system.](image)

The system is assumed to be in contact with a particle reservoir which keeps the total particle density $n_e = n_e^+ + n_e^-$ fixed and guarantees the chemical equilibrium, i.e. converts the particles of one type into another and vice versa if energetically favorable. Taking into account the energy spectrum with the aforementioned asymmetry of $n = 0$ modes, one can easily derive the conditions for the chemical equilibrium between the two subsystems.

If the density $n_e$ is small enough ($2\pi \ell^2 n_e \leq 1$), then all particles will be disposed on the level with the lowest energy $E_0$, and we have

$$0 \leq \nu^+ \leq 1,$$

$$\nu^- = 0.$$
As one can see, the local minima are restored for \( \nu = 2K + 1 = 3, 5, \ldots \), and the values corresponding to these minima are given by

\[
F_{\text{min}} = \frac{\pi \hbar^2 n^2}{2m} \left[ 1 - \frac{1}{(2K + 1)^2} \right].
\]

In order to comment on the basic changes brought by the finite temperature corrections, let us first point out the main features of the zero-temperature case presented in figure 1.

(a) For \( \sigma = 0 \), the magnetization \( M = -\partial F/\partial B \) changes the sign from “+” to “−” at the integer values of \( \nu \). This implies the susceptibility \( \chi = -\partial M/\partial B \) to be positive (\( \chi > 0 \)), confirming the existence of the Meissner effect.

(b) The curve \( \sigma = 0 \) is not smooth at \( \nu = \text{integer} \), i.e. the magnetization undergoes the discontinuity, which on its term means that the susceptibility takes the infinite magnitude, implying that the Meissner effect is perfect.

(c) Taking into account the magnetic moment interaction (\( \sigma = \pm \epsilon \)), the magnetization does not change the sign at \( \nu = \text{integer} \), and one loses the perfect Meissner effect, which can be restored via the duplication of the fermionic degrees of freedom.

The nonanalyticity of zero-temperature free energy at the integer values of \( \nu \) mentioned in (b) is the result of the nonanalytical behaviour of zero-temperature chemical potential, which is a steplike function at the integer values of \( \nu \).

At finite temperatures the chemical potential, and consequently the free energy, become smooth with respect to \( \nu \). At \( T \neq 0 \) the magnetization corresponding to \( \sigma = 0 \) still changes the sign, from “+” to “−” at \( \nu = \text{integer} \), and the susceptibility is still positive leading to the Meissner effect, which however is partial since the susceptibility turns out to be finite at \( T \neq 0 \). Accounting for the magnetic moment interactions (\( \sigma = \pm \epsilon \)) at \( T \neq 0 \), one loses the partial Meissner effect, which can be restored in the framework of a duplicated system. So, at \( T \neq 0 \) one observes that only point (b) is changed.

Thus the final conclusion can be formulated as follows: the circumstance, whether the temperature is zero or finite, defines whether the Meissner effect is perfect or partial, while the magnetic moment interactions determine whether the Meissner effect (perfect or partial) can exist at all.

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