CONSTRUCTION OF BOLTZMANN AND MCKEAN VLASOV TYPE FLOWS (THE SEWING LEMMA APPROACH)

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Abstract. We are concerned with a mixture of Boltzmann and McKean-Vlasov type equations, this means (in probabilistic terms) equations with coefficients depending on the law of the solution itself, and driven by a Poisson point measure with the intensity depending also on the law of the solution. Both the analytical Boltzmann equation and the probabilistic interpretation initiated by Tanaka [40, 41] have intensively been discussed in the literature for specific models related to the behavior of gas molecules. In this paper, we consider general abstract coefficients that may include mean field effects and then we discuss the link with specific models as well. In contrast with the usual approach in which integral equations are used in order to state the problem, we employ here a new formulation of the problem in terms of flows of self-maps on the space of probability measure endowed with the Wasserstein distance. This point of view already appeared in the framework of rough differential equations. Our results concern existence and uniqueness of the solution, but we also prove that the "flow solution" is a solution of the classical integral weak equation and admits a probabilistic interpretation. Moreover, we obtain stability results and regularity with respect to the time for such solutions. Finally we prove the convergence of empirical measures based on particle systems to the solution of our problem, and we obtain the rate of convergence. We discuss as examples the homogeneous and the inhomogeneous Boltzmann (Enskog) equation with hard potentials.

1. Introduction

In this paper we consider a mixture of Boltzmann and McKean-Vlasov type equations defined as follows. Let $\mathcal{P}_1(\mathbb{R}^d)$ denote the space of probability measures on $\mathbb{R}^d$ with a finite first moment. We consider $\rho \in \mathcal{P}_1(\mathbb{R}^d)$, an abstract measurable space $(E, \mu)$ and three coefficients $b : \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$, $c : \mathbb{R}^d \times E \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ and $\gamma : \mathbb{R}^d \times E \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}_+$ that verify some linear growth and some Lipschitz continuity hypothesis (see Assumption (A) for precise statements) and we associate the following weak equation on $f_{s,t} \in \mathcal{P}_1(\mathbb{R}^d)$, $0 \leq s \leq t$:

$$\forall \varphi \in C^1_b(\mathbb{R}^d), \int_{\mathbb{R}^d} \varphi(x)f_{s,t}(dx) = \int_{\mathbb{R}^d} \varphi(x)\rho(dx) + \int_s^t \int_{\mathbb{R}^d} \langle b(x, f_{s,r}), \nabla \varphi(x) \rangle f_{s,r}(dx)dr \quad + \int_s^t \int_{\mathbb{R}^d \times \mathbb{R}^d} f_{s,r}(dx)f_{s,r}(dv) \int_E (\varphi(x + c(v, z, x, f_{s,r})) - \varphi(x)) \gamma(v, z, x, f_{s,r})\mu(dz)dr. \quad (1.1)$$

Here, $C^1_b(\mathbb{R}^d)$ denotes the set of bounded $C^1$ functions with bounded gradient. Given a fixed $s \geq 0$, a solution of this equation is a family $f_{s,t}(dx) \in \mathcal{P}_1(\mathbb{R}^d)$, $t \geq s$, which verify (1.1) for every test
function \( \varphi \). The term \( \langle b(x, f_{s,r}), \nabla \varphi(x) \rangle \) is a classical transport when \( b \) only depends on \( x \). Here, we add the possibility of a mean-field interaction with other particles. The last term corresponds to the collisions in the Boltzmann equation: a particle with velocity \( x \) is struck by another particle of velocity \( v \), which generates a change of velocity described by the function \( c \) and the jump measure \( \mu(dz) \). The intensity of collisions is described by the function \( \gamma \). Note that in the classical Boltzmann equation, this intensity indeed depends on the velocity density \( f_{s,r} \), while the change of velocity \( c \) only depends on \( (x, z, v) \). In Equation (1.1), the struck particles and the striking particles are of same type and have the same velocity density. If one assumed that the striking particles were different from the struck ones and followed a given velocity density \( \mu \), one would replace in the last term of (1.1) the solution \( f_{s,r}(dv) \) by a \( g_{s,r}(dv) \), which would lead to a McKean-Vlasov type equation. Moreover, when the coefficients \( b, c, \gamma \) do not depend on the solution \( f_{s,r} \) and for specific choices of \( b, c, \gamma \), this equation covers variants of the Boltzmann equation (see Villani \cite{42} and Alexandre \cite{2} for the mathematical approach and Cercignani \cite{13} for a presentation of the physical background). In the case of the homogeneous Boltzmann equation the particles are (and remain) uniformly distributed in space, so their positions do not appear as variables in the equation. Then, \( x \in \mathbb{R}^d \) represents the velocity of the typical particle. In this case, the drift coefficient is simply \( b = 0 \). In the case of the inhomogeneous Boltzmann equation also known as the Enskog equation (see Arkeryd \cite{3}), the positions of the particles matter. One works then on \( \mathbb{R}^{2d} \), \( (x^1, ..., x^d) \) is the position and \( (x^{d+1}, ..., x^{2d}) \) represents the velocity of the typical particle. Then, the drift coefficient will be \( b'(x) = x^{i+d}, i = 1, ..., d \) and \( b'(x) = 0, i = d + 1, ..., 2d \). This is one motivation for considering a general drift term in our abstract formulation.

The probabilistic approach to this type of Boltzmann equation has been initiated by Tanaka in \cite{40},\cite{41} followed by many others (see \cite{6},\cite{15},\cite{23},\cite{25} for example). One takes \( f_{s,t}(dx), t \geq s \) to be the solution of the equation (1.1) and constructs a Poisson point measure \( N_f \) with state space \( \mathbb{R}^d \times E \times \mathbb{R}_+ \) and with intensity measure \( f_{s,r}(dv)\mu(dz)1_{\mathbb{R}_+}(u)dv1_{\mathbb{R}_+(s,\infty)}(r)dr \). Then, one associates the stochastic equation

\[
X_{s,t} = X + \int_s^t b(X_{s,r}, f_{s,r})dr + \int_s^t \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} c(v, z, X_{s,r} - , f_{s,r} - )1_{\{u \leq \gamma(v, z, X_{s,r} - , f_{s,r} - )\}}N_f(dv, dz, du, dr).
\]

Here, the initial value \( X \) is a random variable with law \( \rho \) which is independent of the Poisson measure \( N_f \). Under suitable hypothesis (for specific coefficients) one proves that the stochastic equation (1.2) has a unique solution and moreover, the law of \( X_{s,t} \) is \( f_{s,t}(dx) \). In this sense, (1.2) is a probabilistic interpretation of (1.1) and \( (X_{s,t}) \) is called the "Boltzmann process" (see \cite{20} for example).

In the present paper, we give an alternative formulation of the problem presented above. We first recall the definition of the Wasserstein distance \( W_1 \) on the space \( \mathcal{P}_1(\mathbb{R}^d) \):

\[
\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d), \quad W_1(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| \pi(dx, dy) = \sup_{\mathcal{L}(f) \leq 1} \left| \int_{\mathbb{R}^d} f(x)\mu(dx) - \int_{\mathbb{R}^d} f(x)\nu(dx) \right|,
\]

where \( \mathcal{L}(f) \) is the Lipschitz constant of \( f \).
where $\Pi(\mu, \nu)$ is the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals $\mu$ and $\nu$, and $L(f) := \sup_{x \neq y} \frac{|f(y) - f(x)|}{|y - x|}$ is the Lipschitz constant of $f$. The second equality is a classical consequence of Kantorovich duality, see e.g. Remark 6.5 [43]. We also introduce $\mathcal{E}_0(\mathcal{P}_1(\mathbb{R}^d))$, the metric space of the maps $\Theta : \mathcal{P}_1(\mathbb{R}^d) \to \mathcal{P}_1(\mathbb{R}^d)$ such that $\sup_{\rho \in \mathcal{P}_1(\mathbb{R}^d)} \int_{\mathbb{R}^d} |x| \rho(dx)/\int_{\mathbb{R}^d} \rho(dx) < \infty$, endowed with the distance

$$d_s(\Theta, \Theta) := \sup_{\rho \in \mathcal{P}_1(\mathbb{R}^d)} W_1(\Theta(\rho), \Theta(\rho)).$$

This is a complete metric space (see Lemma B.1).

Then we construct $\Theta_{s,t} \in \mathcal{E}_0(\mathcal{P}_1(\mathbb{R}^d))$ in the following way. Given $\rho \in \mathcal{P}_1(\mathbb{R}^d)$, we construct a Poisson point measure $\hat{N}_\rho$ with state space $\mathbb{R}^d \times E \times \mathbb{R}^+$ and with intensity measure

$$\hat{N}_\rho(dv, dz, du, dr) := \rho(dv)\mu(dz)1_{E}1_{\{s, \infty\}}(r)dr.$$

Moreover, we take a random variable $X$ with law $\mu$ which is independent of the Poisson measure $\hat{N}_\rho$ and we define

$$X_{s,t}(\rho) = X + b(X, \rho)(t - s) + \int_s^t \int_{\mathbb{R}^d \times E \times \mathbb{R}^+} c(v, z, X, \rho) 1_{\{u \leq \gamma(v, z, X, \rho)\}} N_\rho(dv, dz, du, dr).$$

Clearly $X_{s,t}(\rho)$ is the one step Euler scheme for the stochastic equation (1.2). We define $\Theta_{s,t}(\rho)$ to be the probability distribution of $X_{s,t}(\rho)$:

$$\Theta_{s,t}(\rho)(dv) := \mathbb{P}(X_{s,t}(\rho) \in dv).$$

Under suitable assumptions, we get that $\Theta_{s,t}$ indeed belongs to $\mathcal{E}_0(\mathcal{P}_1(\mathbb{R}^d))$.

**Definition 1.1.** A family of maps $\theta_{s,t} \in \mathcal{E}_0(\mathcal{P}_1(\mathbb{R}^d))$ with $0 \leq s < t$ is a flow if

$$\theta_{s,t} = \theta_{r,t} \circ \theta_{s,r}, \text{ for every } 0 \leq s < r < t.$$

It is a stationary flow if $\theta_{s,t} = \theta_{0, t-s}$.

Our problem is stated as follows: find a flow of maps such that

$$d_s(\theta_{s,t}, \Theta_{s,t}) \leq C(t - s)^2.$$

We call this $\theta$ a "flow solution" of the equation associated to the coefficients $b, c, \gamma$ and to the measure $\mu$. It turns out that under suitable hypotheses, a flow solution exists and is unique. Moreover the flow solution is a weak solution of Equation (1.1) and admits the stochastic representation (1.2).

This special way to characterize the solution of an equation by means of the distance in short time to the one step Euler scheme first appears, to our knowledge, in the paper [14] of Davie in the framework of rough differential equations. Then, Bailleul in [4] and [5] coupled this idea with the concept of flows. These ideas appeared in the framework of rough path integration initiated by Lyons in his seminal paper [36] (we refer to Friz and Victoir [31] and to Friz and Hairer [30] for a complete and friendly presentation of this topic). It is worth to mention that a central instrument in the rough path theory is the so called "sewing lemma" introduced by Feyel and De la Pradelle in [17],[18] and in the same time, independently, by Gubinelli in [32]. This point of view has been recently developed thoroughly by Brault and Lejay in a series of papers [9, 11, 10]. This is a generic and efficient way to treat the convergence of Euler type schemes. In the present paper, we give a general abstract version of this lemma which plays a crucial part in our approach. In our framework
this lemma states as follows. We consider an abstract family of maps \( \Theta_{s,t} \) which has the following two properties. First, we assume the Lipschitz continuity property
\[
d^*(\Theta_{s,t} \circ U, \Theta_{s,t} \circ \tilde{U}) \leq C e^{C(t-s)} d^*(U, \tilde{U}), \quad \forall U, \tilde{U} \in E_0(P_1(\mathbb{R}^d)).
\]
Moreover, notice that \( \Theta_{s,t} \) has not the flow property, although we expect this property to be true for \( \theta_{s,t} \). However, we assume that it has almost this property in the following asymptotic (small time) sense: we assume that for every \( s < u < t \),
\[
d_e(\theta_{s,t}, \theta_{u,t} \circ \Theta_{s,u}) \leq C (t-s)^2.
\]
This is the "sewing property". These two properties essentially allows to construct by the sewing lemma the flow \( \theta_{s,t} \) which satisfies (1.4) as the limit in \( d_e \) of the Euler schemes based on \( \Theta_{s,t} \). More precisely for a partition \( \mathcal{P} = \{ s = s_0 < \ldots < s_n = t \} \) one defines the Euler scheme \( \Theta^P_{s,t} = \Theta_{s_{n-1},s_n} \circ \ldots \circ \Theta_{s_0,s_1} \) and constructs \( \theta_{s,t} \) as a limit as \( \max_i = 1, \ldots, n s_i - s_{i-1} =: |\mathcal{P}| \to 0 \) of such Euler schemes. Besides, the following error estimate holds:
\[
d_e(\Theta^P_{s,t}, \theta_{s,t}) \leq C |\mathcal{P}| (t-s).
\]
Section 2 presents the abstract framework that allows us to prove Lemma 2.1, a generalized sewing lemma that give the existence and the uniqueness of a flow satisfying (1.4). A pleasant feature is that the uniqueness of the flow is quite easy to obtain, since it essentially has to match with the limit of Euler schemes. Thus, the flow provides one notable solution of the weak equation (1.1): this is the one which is obtained as the limit of Euler schemes. In Section 3, we present the framework of our study (i.e. jump type equations) and our main assumptions. We then use this sewing lemma to prove in Theorem 3.5 that the flow \( \theta_{s,t} \) defined as the solution of (1.4) exists and is unique. We further obtain the estimate (1.7). Besides, we prove that the flow solution \( \theta_{s,t} \) constructed in Theorem 3.5 is a weak solution of equation (1.1) and admits a probabilistic interpretation (see equation (1.2)). Then, in Section 4 we give a numerical approximation scheme for \( \theta_{s,t}(\rho) \) based on a particle system. We obtain in Theorem 4.1 the convergence of the law of any particle towards the flow solution, and give a rate of convergence that is interesting for practical applications. We also obtain a propagation of chaos result for the Wasserstein distance. Last, Section 5 applies the general results to the homogeneous Boltzmann equation and the non homogeneous Boltzmann (Enskog) equation. The problem of the uniqueness of the homogeneous Boltzmann equation has been studied in several papers by Fournier [20], Desvillettes and Mouhot [16], Fournier and Mouhot [26]. The study of the Enskog equation is up to our knowledge much more recent: we mention here contributions concerning existence, uniqueness and particle system approximations by Albeverio, Rüdiger and Sundar [1] and Friesen, Rüdiger and Sundar [27, 29]. Here, for technical reasons, we only deal with truncated coefficients. The interesting problem of analysing the convergence of the equation with truncated coefficients towards the general equation is not related to our approach based on the sewing lemma and is thus beyond the scope of this paper. We show that the assumptions of Theorem 3.5 are satisfied, which enables to define the flow \( \theta_{s,t} \) that is a weak solution of (1.1) and admits a probabilistic representation (1.2). Interestingly, our approach enables us to study equations that combine interactions of Boltzmann type and mean field interactions of McKean-Vlasov type. To illustrate this, we introduce an alternative equation to the Enskog equation, where we replace the space localization function by a mean-field interaction: collisions are more frequent when the typical particle is in a region with a high density of particles. Such a problem enters as well in our framework, and we thus obtain the same results for the flow given by this equation.
2. Abstract sewing lemma

We consider an abstract set \( V \), and we denote by \( \mathcal{E}(V) \) the space of the maps \( \varphi : V \to V \). Here and in the rest of the paper, we use the multiplicative notation for composition, so that

\[
\varphi \psi(v) := \varphi(\psi(v)).
\]

Thus, \( \mathcal{E}(V) \) is a monoid and we consider \( \mathcal{E}_0(V) \subset \mathcal{E}(V) \) a submonoid of maps (i.e. \( \text{Id} \in \mathcal{E}_0(V) \) and \( \varphi, \psi \in \mathcal{E}_0(V) \implies \varphi \psi \in \mathcal{E}_0(V) \)). We assume that there is a distance \( d_* \) on \( \mathcal{E}_0(V) \) such that

\[
(\mathcal{E}_0(V), d_*) \text{ is a complete metric space. We assume besides that}
\]

\[
\forall \varphi, \psi \in \mathcal{E}_0(V), \quad d_*(\varphi U, \psi U) \leq C(U) d_*(\varphi, \psi),
\]

and moreover that we can pick the constant \( C(U) \) locally uniformly in the following sense:

\[
\forall R > 0, \exists \tilde{C}_R \in \mathbb{R}_+, \forall U, \varphi, \psi \in \mathcal{E}_0(V), \quad d_*(U, \text{Id}) \leq R \implies d_*(\varphi U, \psi U) \leq \tilde{C}_R d_*(\varphi, \psi).
\]

Thanks to (2.1), we get that \( \varphi_n \to \varphi \implies \varphi_n U \to \varphi U \) for any \( U \in \mathcal{E}_0 \), and (2.2) ensures that this convergence is uniform on bounded sets.

We now consider a time horizon \( T > 0 \) which will be fixed in the following and a family of maps \( \Theta_{s,t} \in \mathcal{E}_0(V) \) for \( 0 \leq s \leq t \leq T \), such that \( \Theta_{s,t} = \text{Id} \) for \( s = t \) and

\[
(H_0) \quad D^\Theta(T) := \sup_{0 \leq s \leq t \leq T} d_*(\Theta_{s,t}, \text{Id}) < \infty.
\]

For a partition \( P = \{s = s_0 < ... < s_r = t\} \) of the interval \([s, t] \subset [0, T]\) we define the corresponding scheme

\[
\Theta_{s,t}^P := \Theta_{s_{r-1}, s_r} \cdots \Theta_{s_0, s_1} \in \mathcal{E}_0(V).
\]

More generally, for \( s < t \) and a partition \( P = \{s_0 < ... < s_r \} \) such that \( s = s_i \) and \( t = s_j \) with \( 0 \leq i < j \leq r \), we define

\[
\Theta_{s,t}^P := \Theta_{s_{j-1}, s_j} \cdots \Theta_{s_i, s_{i+1}}.
\]

For \( s \in (0, T) \), we define

\[
(2.3) \quad \mathcal{E}_s^\Theta = \cup_{r \in [0, s]} \{ \Theta_{r,s}^P : P = \{r = r_0 < ... < r_k = s\} \text{ a partition of } [r, s] \subset \mathcal{E}_0(V) \}.
\]

We assume:

- (Lipschitz property) There exists \( C_{\text{lip}} \) such that for any \( 0 \leq s \leq t < T \) and \( U, \tilde{U} \in \mathcal{E}_0(V) \),

\[
(H_1) \quad d_*(\Theta_{s,t}^P U, \Theta_{s,t}^P \tilde{U}) \leq C_{\text{lip}} d_*(U, \tilde{U}).
\]

- (Sewing property) There exists \( C_{\text{sew}} \) and \( \beta > 1 \) such that for any \( 0 \leq s < u < t < T \), \( U \in \mathcal{E}_s^\Theta \)

\[
(H_2) \quad d_*(\Theta_{s,u} U, \Theta_{u,t} \Theta_{s,u} U) \leq C_{\text{sew}} (t - s)^\beta.
\]

We stress that the constants \( C_{\text{lip}} \) and \( C_{\text{sew}} \), for \( T \) being fixed, do not depend on \( (s, u, t) \) and on \( (U, \tilde{U}) \). A family of maps \( \Theta_{s,t} \) that verifies the hypotheses \((H_0), (H_1) \) and \( (H_2) \) will be called an "almost-flow". In this general framework the "sewing lemma" can be stated as follows.

\[1\]We use here the terminology of Brault and Lejay [9] while Bailleul [4] calls this an approximate flow.
Lemma 2.1. (Sewing lemma) Suppose that \((H_0), (H_1)\) and \((H_2)\) hold. Then, there exists \(\theta_{s,t} \in \mathcal{E}_0(V)\), \(0 \leq s \leq t \leq T\), which is a flow (see Definition 1.1) and satisfies
\[
d_s(\theta_{s,t}, \Theta_{s,t}) \leq 2^\beta C_{\text{lip}} C_{\text{sew}} \zeta(\beta)(t - s)^\beta,
\]
with \(\zeta(\beta) = \sum_{n=1}^\infty \frac{1}{n^\beta}\). Moreover, it satisfies the Lipschitz property
\[
d_s(\theta_{s,t} U, \theta_{s,t} \bar{U}) \leq C_{\text{lip}} d_s(U, \bar{U}) \quad \text{for } U, \bar{U} \in \mathcal{E}_0(V),
\]
Besides, we have the approximation estimate
\[
d_s(\Theta_{s,t}^P, \theta_{s,t}) \leq 2^\beta C_{\text{lip}}^2 C_{\text{sew}} \zeta(\beta)(t - s) |\mathcal{P}|^{\beta - 1} \quad \text{for } \mathcal{P} \text{ partition of } [s, t],
\]
with \(|\mathcal{P}| := \max_{i=0,...,r-1} (s_{i+1} - s_i)\).

Furthermore, this is the unique flow such that \(d_s(\theta_{s,t}, \Theta_{s,t}) \leq C(t - s) h(t - s)\) for some constant \(C > 0\) and nondecreasing function \(h : \mathbb{R}^+ \to \mathbb{R}^+\) such that \(\lim_{t \to 0} h(t) = 0\).

Note that, contrary to other versions of the sewing lemma (see [4, 9]), Lemma 2.1 is stated without any assumption on the space \(V\): all the requirements are made at the level of functions. Our proof of this lemma is directly inspired from the one given by Friz and Hairer [30, Lemma 4.2], but we are here considering flows instead of integrals.

*Proof.* We first prove that for any \(U \in \mathcal{E}_o\),
\[
d_s(\Theta_{s,t}^P U, \Theta_{s,t} U) \leq 2^\beta C_{\text{lip}} C_{\text{sew}} \zeta(\beta)(t - s)^\beta.
\]
We consider \(\mathcal{P} = \{s = s_0 < ... < s_r = t\}\) and prove (2.7) by iteration on \(r\). For \(r = 1\), the inequality is obvious. Let \(r \geq 2\). For a fixed \(i \in \{1, ..., r - 1\}\), we denote by \(\mathcal{P}_i\) the partition in which we have canceled \(s_i\). Then, we have
\[
d_s(\Theta_{s,t}^P U, \Theta_{s,t} U) = d_s(\Theta_{s_i, s_{i+1}}^P ZU, \Theta_{s_i, s_{i+1}}^P Z'U)
\]
with
\[
Y = \Theta_{s_i, s_{i-1}}^P, \quad Z = \Theta_{s_i, s_{i+1}} \Theta_{s_{i-1}, s_i} Y \quad \text{and} \quad Z' = \Theta_{s_i, s_{i+1}} \Theta_{s_{i-1}, s_i} Y.
\]
Using \((H_1)\) and \((H_2)\) next we obtain
\[
d_s(\Theta_{s,t}^P U, \Theta_{s,t}^P U) \leq C_{\text{lip}} d_s(ZU, Z'U) = C_{\text{lip}} d_s(\Theta_{s_{i-1}, s_{i+1}} YU, \Theta_{s_{i+1}, s_i} YU) \leq C_{\text{lip}} C_{\text{sew}} (s_{i+1} - s_i)^\beta.
\]
We give now the sewing argument. We choose \(i_0 \in \{1, ..., r - 1\}\) such that
\[
s_{i_0+1} - s_{i_0-1} \leq \frac{2}{r - 1} (t - s).
\]
Such an \(i_0\) exists, otherwise we would have \(2(t - s) \geq \sum_{i=1}^{r-1} (s_{i+1} - s_{i-1}) > 2(t - s)\). Using the inequality (2.8) for this \(i_0\) we obtain
\[
d_s(\Theta_{s,t}^P U, \Theta_{s,t}^P u) \leq \frac{2^\beta C_{\text{lip}} C_{\text{sew}}}{(r - 1)^\beta} (t - s)^\beta.
\]
We iterate this procedure up to the trivial partition \(\{s < t\}\), and we obtain (2.7).
We are now in position to define $\theta_{s,t}$ as the limit of $\Theta^{s,t}_P$ when $|P|$ goes to zero. Since $(E_0(V),d_*)$ is complete, it is sufficient to check the Cauchy criterion:

$$\lim_{|P|\to 0} d_*(\Theta^{s,t}_P,\Theta^{s,t}_P) = 0.$$ 

Let $P \cup \bar{P}$ denote the partition of $[s,t]$ obtained by merging both partitions. Since $d_*(\Theta^{s,t}_P,\Theta^{s,t}_{\bar{P}}) \leq d_*(\Theta^{s,t}_{P,\bar{P}},\Theta^{s,t}_{P,\bar{P}}) + d_*(\Theta^{s,t}_{P,\bar{P}},\Theta^{s,t}_{\bar{P},\bar{P}})$, we may assume without loss of generality that $\bar{P}$ is a refinement of the partition $P$. Thus, we can write $P = \{s = s_0 < \ldots < s_r = t\}$ and $\bar{P} = \bigcup_{i=1}^r P^i$, where $P^i$ is a partition of $[s_{i-1}, s_i]$. We now introduce for $l \in \{0, \ldots, r\}$ the partition $\bar{P}_l$ of $[s,t]$ defined by

$$\bar{P}_0 := P \text{ and } \bar{P}_l := \left(\bigcup_{i=1}^l P^i\right) \cup P \text{ for } l \geq 1.$$ 

So, $\bar{P}_l$ is the partition in which we refine the intervals $[s_{i-1}, s_i]$, $i = 1, \ldots, l$, according to $\bar{P}$ but we do not refine the intervals $[s_{i-1}, s_i]$, $i = l + 1, \ldots, r$ (we keep them unchanged, as they are in $P$). Thus, we have $\bar{P}_0 = P$ and $\bar{P}_r = \bar{P}$, and we obtain by using the triangle inequality:

$$d_*(\Theta^{s,t}_P,\Theta^{s,t}_{\bar{P}}) \leq \sum_{l=0}^{r-1} d_*(\Theta^{s,t}_{P,l},\Theta^{s,t}_{\bar{P},l}).$$

We note $\varphi_l = \Theta^{s,t}_{s_{l+1},t}, \psi_l = \Theta^{s,s_l}_s$ and have

$$d_*(\Theta^{s,t}_{P,l},\Theta^{s,t}_{\bar{P},l}) = d_*(\varphi_l,\Theta^{s,t}_{s_{l+1},t},\psi_l,\Theta^{s,t}_{s_{l+1},t}).$$

Using first $(H_1)$ and then (2.7), we obtain

$$d_*(\Theta^{s,t}_{P,l},\Theta^{s,t}_{\bar{P},l}) \leq C_{\text{lip}} d_*(\Theta^{s,s_{l+1},t}_{s_{l+1},t},\Theta^{s,t}_{s_{l+1},t}) \leq C(s_{l+1} - s_{l-1})^\beta,$$

with $C = 2^3 C_{\text{lip}}^2 C_{\text{sew}} \zeta(\beta)$. This leads to

$$d_*(\Theta^{s,t}_P,\Theta^{s,t}_{\bar{P}}) \leq C \sum_{l=0}^{r-1} (s_{l+1} - s_{l-1})^\beta \leq C(t-s)|P|^{\beta-1} \to 0 \text{ as } |P| \to 0.$$ 

This shows the existence of $\theta_{s,t}$ together with (2.6). We then get easily (2.5) : by sending $|P| \to 0$, we get the Lipschitz property of $\theta_{s,t}$ from $(H_1)$. Moreover, we get (2.4) $d_*(\theta_{s,t}U,\Theta^{s,t}_s) \leq 2^3 C_{\text{lip}} C_{\text{sew}} \zeta(\beta)(t-s)^\beta$ from (2.7) when $|P| \to 0$, and we simply take $U = Id.$

We now prove the flow property. Let $s, u, t$ be such that $0 \leq s < u < t \leq T$, $P_1$ and $P_2$ be respectively a partition of $[s, u]$ and $[u, t]$. We have by using the triangle inequality, $(H_1)$ and (2.1)

$$d_*(\Theta^{P_2}_{u,t} \Theta^{P_1}_{s,u}, \theta_{u,t} \theta_{s,u}) \leq d_*(\Theta^{P_2}_{u,t} \Theta^{P_1}_{s,u}, \Theta^{P_2}_{u,t} \theta_{s,u}) + d_*(\Theta^{P_2}_{u,t} \theta_{s,u}, \theta_{u,t} \theta_{s,u})$$

$$\leq C_{\text{lip}} d_*(\Theta^{P_2}_{s,u}, \Theta^{P_1}_{s,u}) + C(\theta_{s,u}) d_*(\Theta^{P_2}_{u,t}, \theta_{u,t} \theta_{s,u}) \to 0,$$

as $|P_1| \vee |P_2| \to 0$. The concatenation $P_1 \cup P_2$ is a partition of $[s, t]$ and thus $d_*(\Theta^{P_2}_{u,t} \Theta^{P_1}_{s,u}, \theta_{s,t}) \to 0$. We get $d_*(\theta_{s,t}, \theta_{u,t} \theta_{s,u}) = 0$ and so $\theta_{s,t} = \theta_{u,t} \theta_{s,u}$.

We finally prove uniqueness. Let $\bar{\theta}_{s,t} \in E_0(V)$, $0 \leq s \leq t \leq T$, be a family satisfying the flow property $\bar{\theta}_{s,t} = \bar{\theta}_{u,t} \theta_{s,u}$, $d_*(\bar{\theta}_{s,t} U, \Theta^{s,t}_s) \leq C(t-s)^\beta$ for any $U \in E_s^{\Theta}$. We consider a partition
\( \mathcal{P} = \{s = s_0 < ... < s_r = t\} \) and have by using first the flow property and the triangle inequality, second the Lipschitz property:

\[
\begin{align*}
    d_s(\tilde{\theta}_{s,t}, \Theta^p_{s,t}) & \leq \sum_{i=0}^{r-1} d_s(\Theta^p_{s_i}, \tilde{\theta}_{s,s_i}, \Theta^p_{s_{i+1}}, \tilde{\theta}_{s_{i+1}}) \\
    & \leq C \sum_{i=0}^{r-1} d_s(\Theta_{s_i,s_{i+1}}, \tilde{\theta}_{s_i}, \tilde{\theta}_{s_{i+1}}).
\end{align*}
\]

Now, we observe that \( d_s(\tilde{\theta}_{s,s_i}, I_d) \leq C T^\beta + d_s(\Theta_{s,s_i}, I_d) \leq C T^\beta + D(T) := R(T) \) by using (H_0). Thanks to the uniform bound (2.2), we get \( d_s(\Theta_{s_i,s_{i+1}}, \tilde{\theta}_{s_i}, \tilde{\theta}_{s_{i+1}}) \leq \tilde{C} R(T) d_s(\Theta_{s_i,s_{i+1}}, \tilde{\theta}_{s_i,s_{i+1}}) \) and thus

\[
    d_s(\tilde{\theta}_{s,t}, \Theta^p_{s,t}) \leq C^2 \tilde{C} R(T) \sum_{i=0}^{r-1} (s_{i+1} - s_i) h(s_{i+1} - s_i) \leq C^2 \tilde{C} R(T) (t - s) h(|\mathcal{P}|).
\]

This yields to \( \theta_{s,t} = \tilde{\theta}_{s,t} \) by taking \(|\mathcal{P}| \to 0. \)

**Remark 2.2.** The hypothesis (H_2) may be weakened by replacing \((t-s)^\beta \) by \((t-s)(1 + |\ln(t-s)|)^{-\rho} \) for some \( \rho > 1. \) The proof is exactly the same by using the fact that the series \( \sum_n \frac{1}{(\ln n)^\rho} \) converges iff \( \rho > 1. \) But in this case, the estimates in (2.4) and (2.6) are less explicit. So, we keep (H_2) which is verified in our framework of Section 3 with \( \beta = 2. \)

**Remark 2.3.** We observe from the proof of Lemma 2.1 that the uniform bound (2.2) on the distance and Hypothesis (H_0) are only needed for the uniqueness result of Lemma 2.1.

Lemma 2.1 gives a general tool to analyse the existence and uniqueness of equations like (1.1). From the probabilistic representation (1.2), one may construct a Euler-type approximation scheme that will generate an almost flow on probability measures. If the required assumptions are fulfilled, the sewing lemma 2.1 gives then the natural candidate for the solution of (1.1). We will use and detail this approach in Section 3.

We now precise how we will use the sewing lemma in the rest of the paper. We first recall that \( \mathcal{P}_1(\mathbb{R}^d) \) is the space of probability measures \( \nu \) on \( \mathbb{R}^d \) such that \( \int |x| d\nu(x) < \infty, \) and \( W_1 \) is the 1-Wasserstein distance on \( \mathcal{P}_1(\mathbb{R}^d) \) defined by

\[
    W_1(\mu, \nu) := \inf_\pi \int_{\mathbb{R}^d} |x - y| \pi(dx, dy)
\]

with the infimum taken over all the probability measures \( \pi \) on \( \mathbb{R}^d \times \mathbb{R}^d \) with marginals \( \mu \) and \( \nu. \) We will work with maps \( \Theta : \mathcal{P}_1(\mathbb{R}^d) \to \mathcal{P}_1(\mathbb{R}^d) \) and we denote by \( \mathcal{E}(\mathcal{P}_1(\mathbb{R}^d)) \) the space of these maps and

\[
    \mathcal{E}_0(\mathcal{P}_1(\mathbb{R}^d)) := \left\{ \Theta \in \mathcal{E}(\mathcal{P}_1(\mathbb{R}^d)) : \sup_{\rho \in \mathcal{P}_1(\mathbb{R}^d)} \int \frac{|v|}{\Theta(\rho)(dv)} < \infty \right\}.
\]

On this space we define the distance

\[
(2.9) \quad d_*(\Theta, \overline{\Theta}) := \sup_{\rho \in \mathcal{P}_1(\mathbb{R}^d)} \frac{W_1(\Theta(\rho), \overline{\Theta(\rho)})}{1 + \int_{\mathbb{R}^d} |v| \rho(dv)}.
\]
We are precisely in the framework presented in Appendix B with $V = \mathcal{P}_1(\mathbb{R}^d)$ endowed with the distance $W_1$ and $\nu_0 = \delta_0$ is the Dirac mass at 0, so that $W_1(\rho, \delta_0) = \int_{\mathbb{R}^d} |v| \rho(dv)$. It is well known that $(\mathcal{P}_1(\mathbb{R}^d), W_1)$ is a complete metric space (see e.g. Bolley [8]) and we get from Lemma B.1 that $(\mathcal{E}(\mathcal{P}_1(\mathbb{R}^d)), d_\mu)$ is a complete space that satisfies (2.2), so we can apply the results of Appendix B to get the next result.

**Proposition 2.4.** Let $T > 0$ and $\Theta_{s,t} \in \mathcal{E}_0(\mathcal{P}_1(\mathbb{R}^d))$ for $0 \leq s \leq t \leq T$ such that $\Theta_{t,t} = \text{Id}$. Property $\text{(H}_1\text{)}$ holds for the distance (2.9) if, and only if,

$$\forall s, t \in [0, T] \text{ with } s \leq t, \forall \rho_1, \rho_2 \in \mathcal{P}_1(\mathbb{R}^d), \ W_1(\Theta_{s,t}^\mu(\rho_1), \Theta_{s,t}^\mu(\rho_2)) \leq C_{lip}W_1(\rho_1, \rho_2).$$

Property $\text{(H}_2\text{)}$ holds for the distance (2.9) if there exists $C \in \mathbb{R}_+$ such that for $\rho \in \mathcal{P}_1(\mathbb{R}^d)$,

$$\sup_{0 \leq r \leq s \leq T} W_1(\Theta_{r,s}^\mu(\rho), \rho) \leq C \left(1 + \int_{\mathbb{R}^d} |v| \rho(dv)\right)$$

and, for $0 \leq s \leq u \leq t \leq T$,

$$W_1(\Theta_{s,t}(\rho), \Theta_{u,t}(\Theta_{s,u}(\rho))) \leq C \left(1 + \int_{\mathbb{R}^d} |v| \rho(dv)\right) (t-s)^\beta.$$

Note that quite similar conditions are considered by Bailleul [4, Theorem 1], but the result is stated on a Banach space while we are working here on a complete metric space. Brault and Lejay [9] state, in a general framework, different versions of the sewing lemma for different set of assumptions. The sewing lemma under assumptions (2.10), (2.11) and (2.12) can essentially be deduced from [9, Theorems 4.2 and 4.5], but we prefer to give here a short self-contained presentation.

3. **Jump type equations**

3.1. **Framework and assumptions.** We consider a measurable space $(E, \mu)$ and three functions $b : \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$, $c : \mathbb{R}^d \times E \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ and $\gamma : \mathbb{R}^d \times E \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}_+$ (we give below the precise hypotheses on these functions). We denote

$$Q(v, z, u, x, \nu) := c(v, z, x, \nu)1_{\{u \leq \gamma(v, z, x, \nu)\}}, \text{ for } u \geq 0.$$

Then, for a probability measure $\rho \in \mathcal{P}_1(\mathbb{R}^d)$ we take $X \in L^1(\Omega)$ with law $\mathcal{L}(X) = \rho$ and we define

$$X_{s,t}(X) = X + b(X, \rho)(t-s) + \int_s^t \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} Q(v, z, u, X, \rho) N_{\rho}(dv, dz, du, dr).$$

Here, $N_{\rho}$ is a Poisson point measure independent of $X$ with intensity measure

$$\tilde{N}_{\rho}(dv, dz, du, dr) = \rho(dv)\mu(dz)dudr.$$

We stress that the law of $X$ appears in the intensity of the point process. Moreover, we define for $s \leq t$ the almost-flow

$$\Theta_{s,t}(\rho) := \mathcal{L}(X_{s,t}(X)).$$

Here, and through the paper, $\mathcal{L}(X)$ the probability law of the random variable $X$. So $\Theta_{s,t}(\rho)$ is the law of the solution which has initial value with distribution $\rho$. Our aim is to construct the flow corresponding to the almost flow $\Theta_{s,t}$ by using the sewing lemma 2.1.

Before going on, we specify the hypotheses that we require for the coefficients. We make the three following assumptions:
• The drift coefficient $b$ is globally Lipschitz continuous: we assume that
\begin{equation}
(A_1) \quad \exists L_b \in \mathbb{R}_+^*, \quad |b(x, \nu) - b(y, \rho)| \leq L_b(|x - y| + W_1(\nu, \rho))
\end{equation}

• For every $(v, x) \in \mathbb{R}^d \times \mathbb{R}^d$ there exists a function $Q_{v, x} : \mathbb{R}^d \times E \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ such that for every $v, x, v', x' \in \mathbb{R}^d, \rho \in \mathcal{P}_1(\mathbb{R}^d)$ and for every $\varphi \in C^1_b(\mathbb{R}^d)$, $\varphi(Q(v', z, u, x', \rho))$ is $\mu(dz)du$-integrable iff $\varphi(Q_{v, x}(v', z, u, x', \rho))$ is $\mu(dz)du$-integrable, and we have
\begin{equation}
(A_2) \quad \int_{E \times \mathbb{R}_+} \varphi(Q(v', z, u, x', \rho))\mu(dz)du = \int_{E \times \mathbb{R}_+} \varphi(Q_{v, x}(v', z, u, x', \rho))\mu(dz)du
\end{equation}
in this case. We assume that $(v, x, v', z, u, x', \rho) \to Q_{v, x}(v', z, u, x', \rho)$ is jointly measurable.

• We assume that $\int |Q(0, z, u, 0, \delta_0)|\mu(dz)du < \infty$ and that there exists a constant $L_\mu(c, \gamma)$ such that for every $v_1, x_1, v_2, x_2 \in \mathbb{R}^d$ and $\rho_1, \rho_2 \in \mathcal{P}_1(\mathbb{R}^d)$
\begin{equation}
(A_3) \quad \int_{E \times \mathbb{R}_+} |Q(v_1, z, u, x_1, \rho_1) - Q_{v_1, x_1}(v_2, z, u, x_2, \rho_2)|\mu(dz)du
\leq L_\mu(c, \gamma)(|x_1 - x_2| + |v_1 - v_2| + W_1(\rho_1, \rho_2)).
\end{equation}

We will simply say that $(A)$ is satisfied when these three Assumptions $(A_1)$, $(A_2)$, and $(A_3)$ are fulfilled.

**Remark 3.1.** The identity in law $(A_2)$ as well as the (pseudo) Lipschitz condition $(A_3)$ may look surprising at first sight. It enables to upper bound the difference between the jump contributions $\int_{E \times \mathbb{R}_+} \varphi(Q(v, z, u, x, \rho))\mu(dz)du$ and $\int_{E \times \mathbb{R}_+} \varphi(Q(v', z, u, x', \rho'))\mu(dz)du$ from two different starting points $(v, x, \rho)$ and $(v', x', \rho')$ when $\varphi$ is Lipschitz continuous. In some cases such as the two-dimensional Boltzmann equation, $Q$ satisfies the standard Lipschitz property and one may simply take $Q_{v, x} = Q$. For some other interesting models such as the three-dimensional Boltzmann equation, we need to use a non-trivial transformation $Q_{v, x}$ satisfying $(A_2)$. The difficulty in this case comes from the fact that there is no smooth parametrisation of the unit sphere in $\mathbb{R}^3$, and Tanaka [40] has been able to get around this difficulty by using such transformation.

From $(A_1)$ and $(A_3)$, we easily deduce the following sublinear growth estimates for any $x, v \in \mathbb{R}^d$ and $\rho \in \mathcal{P}_1(\mathbb{R}^d)$:
\begin{equation}
(b, x, v, \rho) \leq |b(0, \delta_0)| + L_b(|x| + W_1(\rho, \delta_0)) = |b(0, \delta_0)| + L_b \left(|x| + \int_{\mathbb{R}^d} |x|\rho(dx)\right),
\end{equation}
\begin{equation}
\int_{E \times \mathbb{R}_+} |Q(v, z, u, x, \rho)|\mu(dz)du \leq C_\mu(c, \gamma)(1 + |v| + |x| + W_1(\rho, \delta_0)),
\end{equation}
with $C_\mu(c, \gamma) = L_\mu(c, \gamma) \vee (\int |Q(0, z, u, 0, \delta_0)|\mu(dz)du)$. In particular $(3.5)$ and $(3.6)$ implies
\begin{equation}
E(|X_{s,t}(X) - X|) \leq \left[|b(0, \delta_0)| + C_\mu(c, \gamma) + (2L_b + 3C_\mu(c, \gamma))\int |v|\rho(dv)\right](t - s).
\end{equation}
This ensures that $\Theta_{s,t}(\rho) \in \mathcal{P}_1(\mathbb{R}^d)$ for $\rho \in \mathcal{P}_1(\mathbb{R}^d)$ and that $\Theta_{s,t} \in \mathcal{E}_0(\mathcal{P}_1(\mathbb{R}^d))$. 

3.2. Preliminary results. We give now a stability result for the one step Euler scheme which is a key ingredient with our approach.

**Lemma 3.2.** Let us assume that the coefficients $b$ and $Q$ satisfy Assumption (A) with constants $L_b$ and $L_\mu(c, \gamma)$. For $i = 1, 2$, we consider a $\mathbb{R}^d$-valued random variable $Z_i \in L^1$ and a family of probability measures $f_{s,t,i}(dv) \in \mathcal{P}_1(\mathbb{R}^d)$ for $0 \leq s \leq t \leq T$ such that $[s,T] \ni t \mapsto f_{s,t,i}(dv)$ is continuous in Wasserstein distance. Let $N_{fi}$ be a Poisson point processes independent of $Z^i$ with intensity measure $f_{s,s,i}(dv)\mu(dz)1_{\mathbb{R}_+}(u)dudt$ and let $(X_{s,t,i}, t \geq s)$ be defined by

$$X_{s,t,i}(Z^i, \rho^i) = Z^i + \int_s^t b(Z^i, \rho^i)dr + \int_s^t \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+} Q(v, z, u, Z^i, \rho^i)N_{fi}(dv, dz, du, dr),$$

where $\rho^i \in \mathcal{P}_1(\mathbb{R}^d)$. Then, we have:

$$W_1(\mathcal{L}(X_{s,t,1}(Z^1, \rho^1)), \mathcal{L}(X_{s,t,2}(Z^2, \rho^2))) \leq W_1(\mathcal{L}(Z^1), \mathcal{L}(Z^2)) + L_\mu(c, \gamma) \int_s^t W_1(f_{s,r,1}, f_{s,r,2})dr \quad (3.8)$$

$$+ (L_b + L_\mu(c, \gamma))(W_1(\rho^1, \rho^2) + W_1(\mathcal{L}(Z^1), \mathcal{L}(Z^2)))(t-s). \quad (3.9)$$

**Proof.** We first recall the following useful lemma.

**Lemma 3.3.** There exists a measurable map $\psi : [0,1] \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ such that

$$\forall f \in \mathcal{P}_1(\mathbb{R}^d), \forall \varphi : \mathbb{R}^d \to \mathbb{R} \text{ bounded measurable}, \int_0^1 \varphi(\psi(u,f))du = \int_{\mathbb{R}^d} \varphi(x)f(dx).$$

This result is stated in [12] (p. 391, Lemma 5.29) in a $L^2$ framework, but their proof works the same in our setting.

Let $\pi_0^Z \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ be an optimal coupling for $W_1$ of $\mathcal{L}(Z^1)$ and $\mathcal{L}(Z^2)$, that is

$$W_1(\mathcal{L}(Z^1), \mathcal{L}(Z^2)) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |z_1 - z_2| \pi_0^Z(dz_1, dz_2).$$

Then, we construct a random variable $\mathcal{Z} = (\mathcal{Z}^1, \mathcal{Z}^2)$ such that $\mathcal{Z} \sim \pi_0^Z$. We assume here that the probability space is atomless, so that we can construct $\mathcal{Z}$ and more generally any sequence of independent random variables, see e.g. [19, Proposition A.27]. By definition of $\mathcal{Z}$, we will have

$$\mathbb{E}(|\mathcal{Z}^1 - \mathcal{Z}^2|) = W_1(\mathcal{L}(Z^1), \mathcal{L}(Z^2)). \quad (3.10)$$

Moreover, for every $s \leq t$ we consider a probability measure $\pi^f_{s,t}(dv_1, dv_2)$ on $\mathbb{R}^d \times \mathbb{R}^d$ which is an optimal $W_1$-coupling between $f_{s,t,1}(dv_1)$ and $f_{s,t,2}(dv_2)$, and we construct $\tau^f_{s,t}(w) = (\tau^f_{s,t,1}(w), \tau^f_{s,t,2}(w))$ which represents $\pi^f_{s,t}$ in the sense of Lemma 3.3, this means

$$\int_0^1 \varphi(\tau^f_{s,t,1}(w))dw = \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(v_1, v_2)\pi^f_{s,t}(dv_1, dv_2).$$

In particular, we have

$$\int_0^1 |\tau^f_{s,t,1}(w) - \tau^f_{s,t,2}(w)|dw = \int_{\mathbb{R}^d \times \mathbb{R}^d} |v_1 - v_2| \pi^f_{s,t}(dv_1, dv_2) = W_1(f^1_{s,t}, f^2_{s,t}). \quad (3.11)$$

Since $t \mapsto f^i_{s,t}$ is continuous in Wasserstein distance (hence measurable), we may construct $\tau^f_{s,t}(w)$ to be jointly measurable in $(s,t,w)$ by using Corollary 5.22 [43].
Now, we consider $N(dw, dz, du, dr)$ a Poisson point measure on $[0, 1] \times E \times \mathbb{R}_+ \times \mathbb{R}_+$, with intensity measure $dw \mu(dz)du dr$. We stress that $Z^1$ and $Z^2$ are independent of the Poisson point measure $N$. We then consider the equation
\[
x_{s,t}^i(Z^i, \rho^i) = Z^i + \int_s^t b(Z^i, \rho^i) dr + \int_s^t \int_{[0,1] \times E \times \mathbb{R}_+} Q(\tau_{s,t}^f(w), z, u, Z^i, \rho^i) N(dw, dz, du, dr)
\]
and we notice that the law of $x_{s,t}^i(Z^i, \rho^i)$ leads to (3.9).

We start by proving the sewing and Lipschitz properties for the almost-flow $\Theta_{s,t}$ defined in (3.4).

**Lemma 3.4.** Suppose that (A) holds. Then, for every $0 \leq s < u < t \leq T$ and for every $\rho \in \mathcal{P}_1(\mathbb{R}^d)$
\[
W_1(\Theta_{s,t}(\rho), \Theta_{u,t}(\Theta_{s,u}(\rho))) \leq \tilde{C}_{\text{sew}}(\rho)(t - s)^2
\]
with
\[
\tilde{C}_{\text{sew}}(\rho) = (L_b + 2L_\mu(c, \gamma)) \left[ |b(0)| + C_\mu(c, \gamma) + (L_b + 2C_\mu(c, \gamma)) \int |v| \rho(dv) \right]
\]

3.3. **Flows of measures.** We go on and prove the sewing and Lipschitz properties for the almost-flow $\Theta_{s,t}$ defined in (3.4).
Moreover, for every \( \rho, \xi \in \mathcal{P}_1(\mathbb{R}^d) \)
\begin{equation}
W_1(\Theta_{s,t}(\rho), \Theta_{s,t}(\xi)) \leq C_{lip}(T) W_1(\rho, \xi) \quad \text{with}
\end{equation}
\begin{equation}
C_{lip}(T) = 1 + (2L_b + 3L_\mu(c, \gamma)) T.
\end{equation}

Proof. The estimate (3.14) is a direct consequence of the estimate (3.9) obtained in Lemma 3.2. Let us prove (3.12). We take \( X \) a random variable with law \( \rho \) and we consider \( X_{s,t}(X) \) defined in (3.2). So, by definition, \( \Theta_{s,t}(\rho) = \mathcal{L}(X_{s,t}(X)) \). Moreover, we take \( u \in (s,t) \) and we denote \( Y = X_{s,u}(X) \). Then we write
\[
X_{s,t}(X) = Y + b(X, \rho)(t - u) + \int_u^t \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} Q(v, z, u, X, \rho) N_\rho(dv, dz, du, dr).
\]
We also denote \( \overline{\rho} = \mathcal{L}(Y) = \Theta_{s,u}(\rho) \) and we define
\[
Y_{u,t}(Y) = Y + b(Y, \overline{\rho})(t - u) + \int_u^t \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} Q(v, z, u, Y, \rho) N_{\overline{\rho}}(dv, dz, du, dr).
\]
We notice that the law of \( Y_{u,t}(Y) \) is \( \Theta_{u,t}(\overline{\rho}) = \Theta_{u,t}(\Theta_{s,u}(\rho)) \). Using again (3.9) on \([u,t]\), it follows that
\[
W_1(\Theta_{s,t}(\rho), \Theta_{u,t}(\Theta_{s,u}(\rho))) = W_1(\mathcal{L}(X_{s,t}(X)), \mathcal{L}(Y_{u,t}(Y)))
\leq (L_b + 2L_\mu(c, \gamma))(t - u) (W_1(\mathcal{L}(X), \mathcal{L}(Y)) + W_1(\rho, \overline{\rho}))
= 2(L_b + 2L_\mu(c, \gamma))(t - u) W_1(\rho, \overline{\rho}).
\]
By (3.7), we get
\[
W_1(\rho, \overline{\rho}) = W_1(\mathcal{L}(X), \mathcal{L}(Y)) \leq \mathbb{E}(|X - Y|) = \mathbb{E}(|X - X_{s,u}(X)|) \leq C(u - s)
\]
with \( C \) given in (3.7). So (3.12) is proved. \(\square\)

We are now able to use the abstract sewing lemma in order to construct the solution of our problem.

**Theorem 3.5.** Suppose that (A) holds true. Then, there exists a unique flow \( \theta_{s,t} \in E_0(\mathcal{P}_1(\mathbb{R}^d)) \) with \( 0 \leq s < t \leq T \) such that
\begin{equation}
\exists C > 0, \quad d_*(\theta_{s,t}, \Theta_{s,t}) \leq C(t - s)^2.
\end{equation}
Moreover, there exists a constant \( C_2 > 0 \) such that for every partition \( \mathcal{P} \) of \([s,t]\),
\begin{equation}
d_*(\theta_{s,t}, \Theta_{s,t}^{\mathcal{P}}) \leq C_2(t - s) |\mathcal{P}|.
\end{equation}
Besides, \( \theta \) is stationary in the sense that \( \theta_{s,t} = \theta_{0,t-s} \) and the map \( t \mapsto \theta_{s,t} \) is Lipschitz continuous for \( d_* \), i.e.
\begin{equation}
d_*(\theta_{s,t}, \theta_{s,t'}) \leq C|t' - t|.
\end{equation}
Last, we have the following stability result:
\begin{equation}
\forall 0 \leq s \leq t \leq T, \rho, \xi \in \mathcal{P}_1(\mathbb{R}^d), \quad W_1(\theta_{s,t}(\rho), \theta_{s,t}(\xi)) \leq \exp\left[(2L_b + 3L_\mu(c, \gamma))(t - s)\right] W_1(\rho, \xi).
\end{equation}
Proof. We will use the sewing lemma 2.1 and check the different assumptions. \((H_b)\) is a straightforward consequence of \((3.7)\). From \((3.9)\), we get that

\[
W_1(\Theta_{s,t}(\rho), \Theta_{s,t}(\xi)) \leq [1 + (L_b + L_\mu(c,\gamma))(t-s)]W_1(\rho, \xi) \leq \exp((2L_b + 3L_\mu(c,\gamma))(t-s))W_1(\rho, \xi).
\]

By iterating, this gives for any partition \(P\) of \([s,t]\)

\[
W_1(\Theta^P_{s,t}(\rho), \Theta^P_{s,t}(\xi)) \leq \exp((2L_b + 3L_\mu(c,\gamma))(t-s))W_1(\rho, \xi).
\]

\((3.20)\) Property \((H_1)\) thus holds by using \((2.10)\).

We now check \((H_2)\) and will use the sufficient conditions \((2.11)\) and \((2.12)\). We get from \((3.12)\):

\[
W_1(\Theta_{s,t}\Theta^P_{s,t}(\rho), \Theta_{u,t}\Theta^P_{s,u}(\rho)) \leq (L_b + 2L_\mu(c,\gamma))(s-t)^2\left([b(0)] + C_\mu(c,\gamma) + (L_b + 2C_\mu(c,\gamma))\int_{\mathbb{R}^d} |v|\Theta^P_{r,s}(\rho)(dv)\right)
\]

\[
\text{We now observe that for } \mathcal{P}' = \{r = r_0 < \cdots < r_m = s\}, \text{ we have by } (3.5), (3.6) \text{ and } (3.7)
\]

\[
\int_{\mathbb{R}^d} |v|\Theta^P_{r,s}(\rho)(dv) \leq (1 + (2L_b + 3C_\mu(c,\gamma))(r_{i+1} - r_i))\int_{\mathbb{R}^d} |v|\Theta^P_{r_i,r_{i+1}}(\rho)(dv)
\]

\[
+ (r_{i+1} - r_i)[|b(0,\delta_0)| + C_\mu(c,\gamma)].
\]

By iterating this inequality, we get

\[
\int_{\mathbb{R}^d} |v|\Theta^P_{r,s}(\rho)(dv) \leq \exp((2L_b + 3C_\mu(c,\gamma))T)\left(\int_{\mathbb{R}^d} |v|\rho(dv) + [|b(0,\delta_0)| + C_\mu(c,\gamma)]T\right).
\]

Therefore, \((H_2)\) holds with \(\beta = 2\) by applying Proposition 2.4. We can thus apply Lemma 2.1 to get the existence and uniqueness of the flow \(\theta\). Then, we easily get \((3.19)\) from \((3.20)\) and the stationarity is a simple consequence of the obvious fact that \(\Theta_{s,t} = \Theta_{0,t-s}\).

To prove the Lipschitz property \((3.18)\), we consider \(t' \in [s,t]\) and get by iterating the upper bound \((3.7)\)

\[
W_1(\Theta^P_{s,t}(\rho), \Theta^P_{s,t'}(\rho)) \leq \left([b(0)] + C_\mu(c,\gamma) + (L_b + 2C_\mu(c,\gamma))\max_{u \in \mathcal{P}} \int_{\mathbb{R}^d} |v|\Theta^P_{s,u}(\rho)(dv)\right) |t - t'|,
\]

for any partition \(\mathcal{P}\) of \([0,T]\) such that \(t' \in \mathcal{P}\), since \(\Theta^P_{s,t}(\rho)\) can be obtained from \(\Theta^P_{s,t'}(\rho)\) by applying the Euler scheme. Using \((3.21)\), we get \(d_s(\Theta^P_{s,t}(\rho), \Theta^P_{s,t'}(\rho)) \leq C|t' - t|\), and we conclude by sending \(|\mathcal{P}| \to 0\).

\[\square\]

3.4. The weak equation. Theorem 3.5 provides a unique solution in terms of flows. Now we prove that this solution solves an integral equation, in the weak sense. Namely, for every \(s \geq 0\) and \(\rho \in \mathcal{P}(\mathbb{R}^d)\) we associate the weak equation

\[
\int_{\mathbb{R}^d} \varphi(x)\theta_{s,t}(\rho)(dx) = \int_{\mathbb{R}^d} \varphi(x)\rho(dx) + \int_s^t \int_{\mathbb{R}^d} \langle b(x, \theta_{s,r}(\rho)), \nabla \varphi(x) \rangle \theta_{s,r}(\rho)(dx)dr
\]

\[
+ \int_s^t \int_{\mathbb{R}^d} \Lambda_{\varphi}(v, x, \theta_{s,r}(\rho))\theta_{s,r}(\rho)(dx)\theta_{s,r}(\rho)(dv)dr, \varphi \in C^1_b(\mathbb{R}^d),
\]
where
\[
\begin{align*}
\Lambda_\varphi(v, x, \rho) &= \int_{E \times \mathbb{R}_+} (\varphi(x + Q(v, z, u, \rho)) - \varphi(x)) \mu(dz) du \\
&= \int_{E \times \mathbb{R}_+} \langle \nabla \varphi(x + \lambda Q(v, z, u, \rho)), Q(v, z, u, \rho) \rangle \mu(dz) du
\end{align*}
\]

Here, we have written equation (3.22) with a double indexed family of probability measures \((\theta_{s,t}(dx), 0 \leq s \leq t)\). This is to make a direct link with the flow constructed by the sewing lemma that is naturally double indexed. In the literature, one usually rather considers the following equation for a family of probability measures \((f_t, t \geq 0)\)

\[
\begin{align*}
\int_{\mathbb{R}^d} \varphi(x) f_t(\rho)(dx) &= \int_{\mathbb{R}^d} \varphi(x) \rho(dx) + \int_0^t \int_{\mathbb{R}^d} (b(x, f_r(\rho)), \nabla \varphi(x)) f_r(\rho)(dx) dr \\
&\quad + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \Lambda_\varphi(v, x, f_r(\rho)) f_r(\rho)(dx) f_r(\rho)(dv) dr.
\end{align*}
\]

The link between (3.22) and (3.25) is clear. If \(\theta_{s,t}\) solves (3.22), then \(\theta_{0,t}\) solves (3.25). Conversely, if \(f_t\) solves (3.25), then \(f_{t-s}\) solves (3.22).

We need the following preliminary lemma.

**Lemma 3.6.** Assume that \((\text{A})\) holds. If \(\varphi \in C_0^1(\mathbb{R}^d)\), then we have

\[
|\Lambda_\varphi(v, x, \rho)| \leq C \mu(c, \gamma) \|\nabla \varphi\|_\infty \left(1 + |v| + |x| + \int_{\mathbb{R}^d} |z| \rho(dz)\right)
\]

and

\[
|\Lambda_\varphi(v, x, \rho) - \Lambda_\varphi(v', x, \rho')| \leq L \mu(c, \gamma) \|\nabla \varphi\|_\infty (|v - v'| + W_1(\rho, \rho')),
\]

where \(\forall v \in \mathbb{R}^d, \rho \in \mathcal{P}_1(\mathbb{R}^d), x \mapsto \Lambda_\varphi(v, x, \rho)\) is continuous.

**Proof.** We get the first bound by using (3.23), \(|\varphi(x+Q(v, z, u, \rho)) - \varphi(x)| \leq \|\nabla \varphi\|_\infty |Q(v, z, u, \rho)|\)

and (3.6). From \((\text{A}_2)\) we have \(\Lambda_\varphi(v', x, \rho') = \int_{E \times \mathbb{R}_+} (\varphi(x + Q(v', z, u, \rho')) - \varphi(x)) \mu(dz) du\) and thus

\[
|\Lambda_\varphi(v, x, \rho) - \Lambda_\varphi(v', x, \rho')| \leq \|\nabla \varphi\|_\infty \int_{E \times \mathbb{R}_+} |Q(v, z, u, \rho) - Q(v', z, u, \rho')| \mu(dz) du
\]

\[
\leq L \mu(c, \gamma) \|\nabla \varphi\|_\infty (|v - v'| + W_1(\rho, \rho')),
\]

by using \((\text{A}_3)\).

We now prove (3.28). Let \((x_n)_{n \in \mathbb{N}}\) be a sequence in \(\mathbb{R}^d\) such that \(x_n \to x\). We write:

\[
\begin{align*}
\Lambda_\varphi(v, x_n, \rho) &= \int_{E \times \mathbb{R}_+} (\varphi(x_n + Q(v, z, u, v, \rho)) - \varphi(x_n)) \mu(dz) du \\
&= \int_{E \times \mathbb{R}_+} (\varphi(x_n + Q(v, z, u, v, \rho)) - \varphi(x_n + Q_{v, x_n}(v, z, u, v, \rho))) \mu(dz) du \\
&\quad + \int_{E \times \mathbb{R}_+} (\varphi(x_n + Q_{v, x_n}(v, z, u, v, \rho)) - \varphi(x_n)) \mu(dz) du.
\end{align*}
\]
By \((A_4)\), the first integral is upper bounded by \(\|\nabla \varphi\|_{L_\infty}(c, \gamma)|x - x_n| \to 0\). By \((A_2)\), the second integral is equal to
\[
\int_{E \times \mathbb{R}_+} (\varphi(x_n + Q(v, z, u, \rho)) - \varphi(x_n)) \mu(\mathrm{d}z) \mu(\mathrm{d}u).
\]
We have \(\varphi(x_n + Q(v, z, u, \rho)) - \varphi(x_n) \to \varphi(x + Q(v, z, u, \rho)) - \varphi(x)\) and \(|\varphi(x_n + Q(v, z, u, \rho)) - \varphi(x_n)| \leq \|\nabla \varphi\|_{L_\infty} |Q(v, z, u, \rho)|\) that is \(\mu(\mathrm{d}z) \mu(\mathrm{d}u)\)-integrable. The dominated convergence theorem gives then \(\Lambda_\varphi(x_n, \rho) \to \Lambda_\varphi(x, \rho)\).

**Theorem 3.7.** Suppose that \((A)\) holds. Then \(\theta_{s,t}\), the flow given by Theorem 3.5, satisfy Equation (3.22).

The proof of Theorem 3.7 consists in writing the weak equation associated to the Euler scheme, see Equation (3.30) below. We know from Theorem 3.5 that it converges to the flow, and we have to justify the convergence of each term as the time step goes to 0.

**Proof.** Let us consider \(\rho \in \mathcal{P}_1(\mathbb{R}^d)\) and the Euler scheme \(\Theta_{s,s_k}^P(\rho)\) associated to the partition \(\mathcal{P} = \{s_k = s + \frac{k(t-s)}{n} : k = 0, \ldots, n\}\). For \(r \in [s_k, s_{k+1})\) we denote \(\tau(r) = s_k\) and associate the stochastic equation

\[
X_{s,r}^P = X + \int_s^r b(X_{s,r}^P, \Theta_{s,\tau(r)}^P(\rho)) \mathrm{d}r + \int_s^r \int_{\mathbb{R}^d \times \mathbb{R}^d} Q(v, z, u, X_{s,r}^P, \Theta_{s,\tau(r)}^P(\rho)) \mu(\mathrm{d}v) \mu(\mathrm{d}z) \mu(\mathrm{d}u) \mu(\mathrm{d}r'),
\]

where \(\Lambda_{s,t}^P\) is a Poisson point measure of intensity \(\Theta_{s,\tau(r)}^P(\rho)(\mathrm{d}v) \mu(\mathrm{d}z) \mu(\mathrm{d}u) \mu(\mathrm{d}r')\) and \(\mathcal{L}(X) = \rho\). One has \(\Theta_{s,s_k}^P(\rho)(\mathrm{d}x) = \mathcal{L}(X_{s,s_k}^P)\). Then, using Itô’s formula in order to compute \(\mathbb{E}(\varphi(X_{s,r}^P))\) for \(\varphi \in C^1_b(\mathbb{R}^d)\) we obtain

\[
\int_{\mathbb{R}^d} \varphi(x) \Theta_{s,t}^P(\rho)(\mathrm{d}x) = \int_{\mathbb{R}^d} \varphi(x) \rho(\mathrm{d}x) + \int_s^r \int_{\mathbb{R}^d} \langle b(x, \Theta_{s,\tau(r)}^P(\rho)), \nabla \varphi(x) \rangle \Theta_{s,\tau(r)}^P(\rho)(\mathrm{d}x) \mathrm{d}r' + \int_s^r \int_{\mathbb{R}^d \times \mathbb{R}^d} \Lambda_{s,t}^P(v, x, \Theta_{s,\tau(r)}^P(\rho)) \Theta_{s,\tau(r)}^P(\rho)(\mathrm{d}v) \Theta_{s,\tau(r)}^P(\rho)(\mathrm{d}x) \mathrm{d}r'.
\]

From (3.7), we get easily that \(\mathbb{E}(|X_{s,r}^P - X_{s,\tau(r)}^P|) \leq C/n\) for some constant \(C \in \mathbb{R}_+\). Besides, we have \(d_s(\theta_{s,r}, \Theta_{s,t}^P) \leq C(r - s)/n\) by Theorem 3.5. From \((A_1)\) and Lemma 3.6, this leads to

\[
\int_{\mathbb{R}^d} \varphi(x) \Theta_{s,t}^P(\rho)(\mathrm{d}x) = \int_{\mathbb{R}^d} \varphi(x) \rho(\mathrm{d}x) + \int_s^r \int_{\mathbb{R}^d} \langle b(x, \theta_{s,r'}(\rho)), \nabla \varphi(x) \rangle \Theta_{s,\tau(r')}^P(\rho)(\mathrm{d}x) \mathrm{d}r' + \int_s^r \int_{\mathbb{R}^d \times \mathbb{R}^d} \Lambda_{s,t}^P(v, x, \theta_{s,r'}(\rho)) \Theta_{s,\tau(r')}^P(\rho)(\mathrm{d}v) \Theta_{s,\tau(r')}^P(\rho)(\mathrm{d}x) \mathrm{d}r' + R_n,
\]

with \(|R_n| \leq C/n\). Now, let us recall (see e.g. [43, Theorem 6.9]) that \(W_1(\rho_n, \rho_\infty) \to 0\) if, and only if, \(\int_{\mathbb{R}^d} f(x) \rho_n(\mathrm{d}x) \to \int_{\mathbb{R}^d} f(x) \rho_\infty(\mathrm{d}x)\) for any continuous function \(f : \mathbb{R}^d \to \mathbb{R}\) with sublinear growth (i.e. such that \(\sup_{x, f(x)} \leq C(1 + |x|)\) for some constant \(C > 0\)). From (3.26) and (3.28) (resp. (3.5) and \((A_1)\)), the map \((v, x) \mapsto \Lambda_{\varphi}(v, x, \theta_{s,r}(\rho))\) (resp. \(x \mapsto \langle b(x, \theta_{s,r}(\rho)), \nabla \varphi(x) \rangle\)) is, for any \(r'\), continuous with sublinear growth since \(\varphi \in C^1_b(\mathbb{R}^d)\). Since \(W_1(\Theta_{s,\tau(r')}^P(\rho), \theta_{s,r'}(\rho)) \to 0\)
Proof. of (3.31). Skorohod space and that any converging subsequence leads to a solution of the Martingale problem marginal laws by Theorem 3.5. We show by classical arguments that it gives a tight sequence in the

\[ \Theta_{s,\tau(r')}^p(\rho) \otimes \Theta_{s,\tau(r')}^p(\rho, \theta_{s,r'}(\rho) \otimes \theta_{s,r'}(\rho)) \to 0, \]

this gives the pointwise convergence for any \( r' \):

\[
\int_{\mathbb{R}^d} \langle b(x, \theta_{s,r'}(\rho)), \nabla \varphi(x) \rangle \Theta_{s,\tau(r')}^p(\rho)(dx) \to \int_{\mathbb{R}^d} \langle b(x, \theta_{s,r'}(\rho)), \nabla \varphi(x) \rangle \theta_{s,r'}(\rho)(dx),
\]

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \Lambda_{\varphi}(v, x, \theta_{s,r'}(\rho)) \Theta_{s,\tau(r')}^p(\rho)(dx) \Theta_{s,\tau(r')}^p(\rho)(dv) \to \int_{\mathbb{R}^d \times \mathbb{R}^d} \Lambda_{\varphi}(v, x, \theta_{s,r'}(\rho)) \theta_{s,r'}(\rho)(dx) \theta_{s,r'}(\rho)(dv).
\]

From the standard uniform bounds on the first moment, we can then get the claim by the dominated convergence theorem.

\[ \square \]

3.5. Probabilistic representation. In this subsection, we are interested in the existence and uniqueness of a process \((X_r, r \geq 0)\) on a probability space such that

\[ X_t = X + \int_0^t b(X_r, \mathcal{L}(X_r))dr + \int_0^t Q(v, z, u, X_{r-}, \mathcal{L}(X_{r-}))N(dv, dz, du, dr), \]

where \( N \) is a Poisson point measure with intensity \( \mathcal{L}(X_{r-})(dv)\mu(dz)du dr \). If such a process exists, we call it “Boltzmann process”. We notice that the sublinear growth properties (3.5) and (3.6) gives \( \sup_{t \in [0,T]} \mathbb{E}[|X_t|] < \infty \) for any \( T > 0 \) and then

\[ \forall T > 0, \exists C_T > 0, \forall t \in [0,T], \mathbb{E}[|X_{t+h} - X_t|] \leq C_T h, \]

which yields to \( \mathcal{L}(X_t) = \mathcal{L}(X_t) \) for any \( t \geq 0 \). The existence of a solution to (3.31) is classically related to Martingale problems, see Horowitz and Karandikar [33] in the context of the Boltzmann equation. Our goal here is to underline some relations between the flow \( \theta \) introduced by Theorem 3.5 and Equation (3.31).

From It\'o’s Formula, it is clear that a Boltzmann process leads to a solution of the weak equation (3.25). In Theorem 3.7, we have proved under suitable conditions that the flow constructed by Theorem 3.5 is a weak solution of (3.25). Here, we prove that there exists a Boltzmann process that has the marginal laws given by the flow \( \theta_{0,t} \).

**Theorem 3.8.** Suppose that (A) holds. Then, there exists a process \( X \) that satisfies (3.31) such that \( \mathcal{L}(X_t) = \theta_{0,t}(\rho) \), i.e. there exists a probability space endowed with a Poisson point measure \( N(dv, dz, du, dr) \) with intensity \( \theta_{0,t}(\rho)(dv)\mu(dz)du dr \) on which it exists a process \( X \) satisfying (3.31) and \( \mathcal{L}(X_t) = \theta_{0,t}(\rho) \), \( t \geq 0 \).

To prove this result, we consider the Euler scheme for which we know the convergence of the marginal laws by Theorem 3.5. We show by classical arguments that it gives a tight sequence in the Skorohod space and that any converging subsequence leads to a solution of the Martingale problem of (3.31).

**Proof.** Let \( X \sim \rho \) with \( \rho \in \mathcal{P}_1(\mathbb{R}^d) \). We consider the time grid \( t_k = \frac{k}{n} \), \( k \in \mathbb{N} \) and we denote \( \tau(t) = \frac{k}{n} \) for \( \frac{k}{n} \leq t < \frac{k+1}{n} \). For \( t \in \mathbb{R} \), we denote

\[ \Theta_{n,t}^p = \Theta_{\lfloor nt\rfloor/n, t}^1 \Theta_{\lfloor nt\rfloor-1/n, \lfloor nt\rfloor/n-1}^1 \cdots \Theta_{0,1/n}, \]

\[ \Theta_{n,t}^p = \Theta_{\lfloor nt\rfloor/n, t}^1 \Theta_{\lfloor nt\rfloor-1/n, \lfloor nt\rfloor/n-1}^1 \cdots \Theta_{0,1/n}, \]
with $\Theta_{s,t}$ defined by (3.4), so that $\Theta^n_{0,t} = \Theta^n_{0,t}$ with the partition $P = \{t_0 < \cdots < t_{[nt]} \leq t\}$. Then, we define the corresponding Euler scheme

$$X^n_t = X + \int_0^t b(X^n_{\tau(r)}(\rho))\,dr + \int_0^t \int_{\mathbb{R}^d \times \mathbb{E} \times \mathbb{R}^+} Q(v, z, X^n_{\tau(r)}, \Theta^n_{0,\tau(r)}(\rho))N_{\Theta^n}(dv, dz, du, dr),$$

where $N_{\Theta^n}$ is a Poisson point measure with compensator $\Theta^n_{0,\tau(r)}(\rho)(dv)\mu(dz)du\,dr$ that is independent of $X$. By construction, we have $\mathcal{L}(X^n_t) = \Theta^n_{0,t}(\rho)$ for all $t \geq 0$. Theorem 3.5 gives that there exists a flow $\theta_{s,t}$ corresponding to $\Theta_{s,t}$, and we have

$$d_s(\theta_{0,t}, \Theta^n_{0,t}) \leq \frac{C_2t}{n}.$$  

We first write the martingale problem associated with $X^n$. For $\varphi \in C^1_b(\mathbb{R}^d)$, we define

$$M^n_{\varphi}(t) := \varphi(X^n_t) - \varphi(X) - I^n_t - J^n_t$$

with

$$I^n_t = \int_0^t \int_{\mathbb{R}^d} \tilde{\Lambda}_{\varphi}(v, X^n_{\tau(r)}, \Theta^n_{0,\tau(r)}(\rho))\Theta^n_{0,\tau(r)}(\rho)(dv)\,dr$$

$$J^n_t = \int_0^t \int_{\mathbb{R}^d} \left\langle b(X^n_{\tau(r)}, \Theta^n_{0,\tau(r)}(\rho)), \nabla \varphi(X^n_{\tau(r)}) \right\rangle \,dr,$$

with $\tilde{\Lambda}_{\varphi}(v, x, \tilde{x}, \rho) = \int_{\mathbb{E} \times \mathbb{R}^+} (\varphi(x + Q(v, z, u, \tilde{x}, \rho)) - \varphi(x))\mu(dz)du$. This is a martingale, and we have for every $0 \leq s_1 < \cdots < s_m < t < t'$ and every $\psi_j \in C^0_b(\mathbb{R}^d)$

$$\mathbb{E} \left( \prod_{j=1}^m \psi_j(X^n_{\tau(s_j)})M^n_{\varphi}(\tau(t')) \right) = \mathbb{E} \left( \prod_{j=1}^m \psi_j(X^n_{\tau(t_j)})M^n_{\varphi}(\tau(t)) \right).$$

We now analyse the convergence when $n \to \infty$ and denote $P_n$ the probability measure on the Skorohod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ produced by the law of $X^n$. We easily check Aldous’ criterion: from (3.5) and (3.6), we get a uniform bound on the first moment

$$\forall T > 0, \mathbb{E} \left[ \sup_{n \geq 1} \sup_{t \in [0, T]} |X^n_t| \right] < \infty$$

and then

$$\forall T > 0, \exists C_T \in \mathbb{R}_+, \forall h \in [0, 1], \mathbb{E}\left[ \sup_{t \leq T} |X^n_{t+h} - X^n_t| \right] \leq C_T h.$$

Thus, we obtain that the sequence $(P_n)_{n \in \mathbb{N}}$ is tight. Let $P$ be any limit point of this sequence. Up to consider a subsequence, we may assume that $(P_n)_{n \in \mathbb{N}}$ weakly converges to $P$. We denote by $X_t$ the canonical projections on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$. We define

$$M_{\varphi}(t) := \varphi(X_t) - \varphi(X) - J_t - I_t$$
with
\[
I_t = \int_0^t \int_{\mathbb{R}^d} \Lambda_\varphi(v, X_r, \theta_{0,r}(\rho))\theta_{0,r}(\rho)(dv)dr,
\]
\[
J_t = \int_0^t \int_{\mathbb{R}^d} \langle b(X_r, \theta_{0,r}(\rho)), \nabla \varphi(X_r) \rangle dr.
\]

We now prove that \( \mathbb{E} \left( \prod_{j=1}^m \psi_j(X^n_{\tau(s_j)})M^n_\varphi(t') \right) \rightarrow \mathbb{E}_P \left( \prod_{j=1}^m \psi_j(X_s)M_\varphi(t') \right) \), where \( \mathbb{E}_P \) denotes the integration on \( \mathcal{D}(\mathbb{R}_+, \mathbb{R}^d) \) with respect to \( P \). This then gives from (3.37)
\[
(3.41) \quad \mathbb{E} \left( \prod_{j=1}^m \psi_j(X_s)M_\varphi(t) \right) = \mathbb{E} \left( \prod_{j=1}^m \psi_j(X_s)M_\varphi(t) \right).
\]

We define the intermediary terms
\[
\tilde{I}_t^n = \int_0^t \int_{\mathbb{R}^d} \Lambda_\varphi(v, X^n_r, \Theta^n_{0,\tau(r)}(\rho))\Theta^n_{0,\tau(r)}(\rho)(dv)dr,
\]
\[
\hat{I}_t^n = \int_0^t \int_{\mathbb{R}^d} \Lambda_\varphi(v, X^n_r, \theta_{0,r}(\rho))\theta_{0,r}(\rho)(dv)dr \quad \text{and} \quad \hat{J}_t^n = \int_0^t \int_{\mathbb{R}^d} \langle b(X^n_r, \theta_{0,r}(\rho)), \nabla \varphi(X^n_r) \rangle dr.
\]

We first notice that
\[
|\tilde{\Lambda}_\varphi(v, x, \bar{x}, \rho) - \Lambda_\varphi(v, x, \rho)| = \left| \int_{E \times \mathbb{R}_+} (\varphi(x + Q_{v,x}(v, z, u, \bar{x}, \rho)) - \varphi(x + Q_{v,x}(v, z, u, \bar{x}, \rho)))\mu(dz)du \right|
\]
\[
\leq \| \nabla \varphi \| \infty L_\mu(c, \gamma)|\bar{x} - x|
\]
by using first (A2) and then (A3). We get \( \mathbb{E}||I^n_t - \hat{I}_t^n|| \leq C/n \rightarrow 0 \) by (3.39). From the Lipschitz property of \( b \) and \( \Lambda_\varphi \) (Lemma 3.6), we get
\[
|I^n_t - \hat{I}_t^n| + |J^n_t - \hat{J}_t^n| \leq C \int_0^t W_1(\theta_{0,r}(\rho), \Theta^n_{0,\tau(r)}(\rho))dr \rightarrow 0,
\]
by (3.35) and (3.39). Thus, it is sufficient to check the convergence of
\[
\mathbb{E} \left( \prod_{j=1}^m \psi_j(X^n_{\tau(s_j)})|\varphi(X^n_{\tau(t')} - \varphi(X) - \hat{I}^n_{\tau(t')} - \hat{J}^n_{\tau(t')}) \right).
\]
From (3.26), \( \Lambda_\varphi \) has a sublinear growth and is continuous with respect to \( x \). Therefore, \( \mathcal{D}(\mathbb{R}_+, \mathbb{R}^d) \ni \bar{x} \mapsto \int_0^t \int_{\mathbb{R}^d} \Lambda_\varphi(v, \bar{x}(r), \theta_{0,r}(\rho))\theta_{0,r}(\rho)(dv)dr \) is continuous and bounded by \( C(1 + \sup_{r \in [0,t]} |\bar{x}(r)| + \sup_{r \in [0,t]} \int_{\mathbb{R}^d} |z\theta_{0,r}(\rho)(dz)) \) for some \( C \in \mathbb{R}^*_+ \). Since \( \mathbb{E} \left[ \sup_{n \geq 1} \sup_{r \in [0,t]} \left| X^n_{\tau(r)} \right| \right] < \infty \) by (3.38) and \( P_n \) weakly converges to \( P \), this gives the desired convergence for \( s_1, \ldots, s_m, t, t' \in [0, T] \setminus \mathcal{D} \), where \( \mathcal{D} \) is an at most countable subset of \( (0, T) \) (see Billingsley [7, p. 138]). Last, from the right continuity under \( P \), we get that (3.41) holds for any \( 0 < s_1 < \cdots < s_m < t < t' \), which shows that \( P \) is a solution of the Martingale Problem, i.e. is such that for any \( \varphi \in C^1_b(\mathbb{R}^d) \), \( M_\varphi(t) \) defined by (3.40) is a Martingale. Besides, let us notice that for all \( t \geq 0 \), \( \mathcal{L}(X_t) = \theta_{0,t}(\rho) \) by using (3.35).

The classical theory of martingale problems allows to obtain now the Equation (3.31). Let us be more explicit. Let us denote by \( \mu^X \) the random point measure associated to the jumps
of $X_t$. Then, Theorem 2.42 in [35] guarantees that, as a solution of the martingale problem, $X$ is a semimartingale with characteristics $B_r = b(X_r, \theta_0,\nu)\) and $\nu$ defined by $\nu((0,t) \times A) = \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^+} 1_A(Q(v, z, X_r, \theta_0,\nu)) \mu(dz) du \theta_0,\nu(dv) dr$. Then, by Theorem 2.34 in [35] one has

$$X_t = X + \int_0^t b(X_r, \theta_0,\nu) dr + \int_0^t y \mu^X(dr, dy)$$

and the compensator of $\mu^X$ is $\nu$ (in [35] a truncation function $h$ appears, here we take the truncation function $h(x) = 0$, which is possible because we work in the framework of finite variation $\int |x| \theta_0,\nu(dx) < \infty$). Then, using the representation given in [34, Theorem 7.4, p. 93], one may construct a probability space and a Poisson point measure $N_\theta$ of compensator $\mu(dz) du \theta_0,\nu(dv) dr$ such that the process

$$\overline{X}_t = X_0 + \int_0^t b(\overline{X}_r, \theta_0,\nu) dr + \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^+} Q(v, z, \overline{X}_{r-}, \theta_0,\nu)) N_\theta(dr, dz, du, dv)$$

has the same law as $X$. Since $\mathcal{L}(\overline{X}_t) = \mathcal{L}(X_t) = \theta_{0,t}(\nu)$ is continuous with respect to $t$, this produces a solution of (3.31).

Theorem 3.8 gives the existence of a Boltzmann process (3.31) such that $\mathcal{L}(X_t) = \theta_{0,t}$. It would be interesting to have a uniqueness result and get for example that the marginal laws of any process $X$ of (3.31) satisfy the following Lipschitz assumption:

$$\int_{\mathbb{R}^d} |Q(v, z, x_1, \rho_1) - Q(v, z, x_2, \rho_2)| \mu(dz) du \leq L_\mu(c, \gamma)(|x_1 - x_2| + W_1(\rho_1, \rho_2)).$$

Then, any process $(X_t, t \geq 0)$ that satisfies (3.31) is such that $\mathcal{L}(X_t) = \theta_{0,t}(\nu)$ for all $t \geq 0$.

To prove Proposition 3.9, we would like to compare directly $\mathcal{L}(X_t)$ with the law of $\Theta^n_{0,t}(\nu)$ defined by (3.33), since we know from Theorem 3.5 that it converges to the flow. This is unfortunately not obvious: the Poisson measure associated to $X$ involves the marginal laws of $X$, and it is not clear how to construct from it a scheme with law $\Theta^n_{0,t}(\nu)$. To get around this difficulty, we introduce an intermediary process $X^n$, see (3.43), and show that both $W_1(\mathcal{L}(X_t), \mathcal{L}(X^n_t))$ and $W_1(\mathcal{L}(X^n_t), \Theta^n_{0,t}(\nu))$ converge to 0.

**Proof.** Let $X$ be a solution of (3.31). We denote $f_t = \mathcal{L}(X_t)$. There exists a Poisson point measure $N$ with intensity $f_t(dv)\mu(dz)du dr$ such that

$$X_t = X + \int_0^t b(X_r, f_r) dr + \int_0^t Q(v, z, X_r, f_r) N(dv, dz, du, dr).$$

As for the preceding proof, we consider the time grid $t_k = \frac{k}{n}$, $k \in \mathbb{N}$ and denote $\tau(t) = \frac{k}{n}$ for $\frac{k}{n} \leq t < \frac{k+1}{n}$. We define the process $X^n$ by:

$$X^n_t = X + \int_0^t b(X^n_{\tau(r)}, f_{\tau(r)}) dr + \int_0^t Q(v, z, X^n_{\tau(r)}, f_{\tau(r)}) N(dv, dz, du, dr).$$
We have
\[ |X_t - X^n_t| \leq \int_0^t |b(X_r, f_r) - b(X^n_r, f_r)| dr + \int_0^t |Q(v, z, u, X_{r-}, f_r) - Q(v, z, u, X^n_{r-}, f_r)| N(dv, dz, du, dr). \]

By using (A1) and (3.42), we get
\[ \mathbb{E}[|X_t - X^n_t|] \leq \int_0^t (L_b + L_\mu(c, \gamma)) \left( \mathbb{E}[|X_r - X^n_r|] + W_1(f_r, f_r) \right) dr. \]

Now, we observe that \( W_1(f_r, f_{r-}) \leq \frac{C_T}{n} \) for \( r \in [0, T] \) by using (3.32). Similarly, we observe from the sublinear growth properties (3.5) that for any \( T > 0, \mathbb{E} \left[ \sup_{n \geq 1} \sup_{t \in [0, T]} |X^n_t| \right] < \infty \) and then
\[ \exists C_T \in \mathbb{R}^+, \forall h \in [0, 1], \mathbb{E}[\sup_{t \leq T} |X^n_{t+h} - X^n_t|] \leq C_TH. \]

We therefore get for any \( T > 0 \) the existence of a constant \( C_T \) such that \( \mathbb{E}[|X_t - X^n_t|] \leq \int_0^t (L_b + L_\mu(c, \gamma)) \mathbb{E}[|X_r - X^n_r|] dr + \frac{C_T}{n}, \) and then
\[ \mathbb{E}[|X_t - X^n_t|] \leq \frac{C_T}{n} \exp((L_b + L_\mu(c, \gamma))t), \ t \in [0, T], \]
by Gronwall lemma. This gives \( W_1(\mathcal{L}(X^n_t), f_t) \rightarrow 0. \)

On the other hand, we get by using Lemma 3.2 that
\[ W_1(\mathcal{L}(X^n_{t+k+1}), \Theta^n_{0, t_k+1}(\rho)) \leq W_1(\mathcal{L}(X^n_{t_k}), \Theta^n_{0, t_k}(\rho)) \left( 1 + \frac{2L_\mu(c, \gamma) + L_b}{n} \right) + \left( 1 + \frac{3L_\mu(c, \gamma) + 2L_b}{n} \right). \]

where \( \Theta^n_{0, t} \) is defined by (3.33). For \( t_{k+1} \leq T \) and \( t \in [t_k, t_{k+1}] \), we have
\[ W_1(f_t, \Theta^n_{0, t_k}(\rho)) \leq W_1(f_t, f_{t_k}) + W_1(f_{t_k}, \mathcal{L}(X^n_{t_k})) + W_1(\mathcal{L}(X^n_{t_k}), \Theta^n_{0, t_k}(\rho)) \leq \frac{C_T}{n} + W_1(\mathcal{L}(X^n_{t_k}), \Theta^n_{0, t_k}(\rho)) \]
for some constant \( C_T \) by using (3.32) and (3.45). Therefore, we get for \( t_{k+1} \leq T \) that
\[ W_1(\mathcal{L}(X^n_{t_{k+1}}), \Theta^n_{0, t_{k+1}}(\rho)) \leq W_1(\mathcal{L}(X^n_{t_k}), \Theta^n_{0, t_k}(\rho)) \left( 1 + \frac{3L_\mu(c, \gamma) + 2L_b}{n} \right) + \frac{C_T}{n^2}, \]
for some constant \( C_T > 0. \) Since \( \mathcal{L}(X^n_0) = \Theta^n_{0, 0}(\rho) = \rho \), we get for \( t_k \in [0, T]: \)
\[ W_1(\mathcal{L}(X^n_{t_k}), \Theta^n_{0, t_k}(\rho)) \leq \frac{C_T}{n^2} \left( 1 + \frac{3L_\mu(c, \gamma) + 2L_b}{n} \right)^k - 1 \leq \frac{C_T}{(3L_\mu(c, \gamma) + 2L_b)n} \exp((3L_\mu(c, \gamma) + 2L_b)T). \]
Since \( T > 0 \) is arbitrary, we obtain by using this bound together with (3.44) and Theorem 3.5 that \( W_1(\mathcal{L}(X^n_t), \theta_{0, t}(\rho)) \rightarrow 0 \) for any \( t \geq 0. \) This shows that \( f_t = \theta_{0, t}(\rho) \) since we already have proven that \( W_1(\mathcal{L}(X^n_t), f_t) \rightarrow 0. \) \( \square \)
Remark 3.10. To prove Proposition 3.9 without the Lipschitz condition (3.42) with the same arguments, it would be natural to consider the following process

\begin{equation}
\dot{X}_t^n = X + \int_0^t b(\tilde{X}^n_{\tau(r)}, f_{\tau(r)})dr + \int_0^t Q_{v,X_{\tau(r)} - (v, z, u, \tilde{X}^n_{\tau(r)}), f_{\tau(r)})N(dv, dz, du, dr).
\end{equation}

instead of (3.43). However, it is not clear if it is indeed well defined since \(Q_{v,X_{\tau(r)} - (v, z, u, \tilde{X}^n_{\tau(r)}), f_{\tau(r)})\) is no longer frozen on each time step. Besides, for the same reason, we can no longer use Lemma 3.2 to analyse \(W_1(\mathcal{L}(\tilde{X}_{t_k}), \Theta^n_{0,t_k}(\rho))\).

4. Particle system approximation

Section 3 has shown that the sewing lemma is a powerful tool to analyse nonlinear equations of Boltzmann and McKean Vlasov type. Besides this, it gives also a constructive way to obtain the solution of such equations as the limit of iterated almost flows. However, generally, the almost flow (3.4) cannot be implemented in practice on a computer without approximating the measure \(\rho\). In this section, we approximate this measure by an empirical measure and use then an interacting particle system to approximate the almost flow. The goal of this section is to analyse the error between the limit flow and the iterated approximated almost flow in function of the number of particles and time steps. This is stated in Theorem 4.1 below, which is the main result of this section.

Particle systems have been used for a long time to show existence results on nonlinear SDE of McKean-Vlasov and Boltzmann type, see e.g. Sznitman [39] or Méliard [37]. Formally, the interacting particle system associated to Equation (3.31) can be written as follows:

\[ X^i_t = X^i_0 + \int_0^t b(X^i_{\tau(r)}, \frac{1}{N} \sum_{j=1}^N \delta_{X^j_{\tau(r)}}) dr + \int_0^t \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} Q(v, z, u, X^i_{\tau(r)}, \frac{1}{N} \sum_{j=1}^N \delta_{X^j_{\tau(r)}}) N^i(dv, dz, du, dr), \]

where \(N^i(dv, dz, du, dr), \ i = 1, \ldots, N,\) are independent Poisson point measure with intensity \(\left( \frac{1}{N} \sum_{j=1}^N \delta_{X^j_{\tau(r)}}(dv) \right) \mu(dz)du dr.\) In this section, we do not discuss this interacting particle system itself, but we focus on its time discretization. We are rather interested in the approximation of operator \(\Theta_{t,n}\) defined by (3.4) and of the corresponding Euler scheme.

Through this section, we will work with the space \(\mathcal{P}(\mathcal{P}_1(\mathbb{R}^d))\) of the probability measures on \(\mathcal{P}_1(\mathbb{R}^d)\) (with the Borel \(\sigma\) field associated to the distance \(W_1\)). We denote by \(\mathcal{P}_1(\mathcal{P}_1(\mathbb{R}^d))\) the space of probability measures \(\eta \in \mathcal{P}(\mathcal{P}_1(\mathbb{R}^d))\) such that

\[ \int_{\mathcal{P}_1(\mathbb{R}^d)} W_1(\mu, \delta_0) \eta(d\mu) < \infty. \]

On \(\mathcal{P}_1(\mathcal{P}_1(\mathbb{R}^d))\), we take the Wasserstein distance

\[ W_1(\eta_1, \eta_2) := \inf_{\pi \in \Pi(\eta_1, \eta_2)} \int_{\mathcal{P}_1(\mathbb{R}^d) \times \mathcal{P}_1(\mathbb{R}^d)} W_1(\mu, \nu) \pi(d\mu, d\nu) \]

\[ = \sup_{L(\Phi) \leq 1} \left| \int_{\mathcal{P}_1(\mathbb{R}^d)} \Phi(\mu) \eta_1(d\mu) - \int_{\mathcal{P}_1(\mathbb{R}^d)} \Phi(\mu) \eta_2(d\mu) \right| \]

where \(\Pi(\eta_1, \eta_2)\) is the set of probability measures on \(\mathcal{P}_1(\mathbb{R}^d) \times \mathcal{P}_1(\mathbb{R}^d)\) with marginals \(\eta_1\) and \(\eta_2\) and \(L(\Phi)\) is the Lipschitz constant of \(\Phi\), so that \(|\Phi(\mu) - \Phi(\nu)| \leq L(\Phi) W_1(\mu, \nu)\).

Before going on,
we list some basic properties of $W_1$ which will be used in the following. First we notice that $\Pi(\eta, \delta_\mu) = \{\eta \otimes \delta_\mu\}$ (the product probability of $\eta$ and $\delta_\mu$ is the only probability measure on the product space which has the marginals $\eta$ and $\delta_\mu$). As an immediate consequence, we have

$W_1(\eta, \delta_\mu) = \int_{P_1(\mathbb{R}^d)} W_1(\nu, \mu) \eta(d\nu)$

and $W_1(\delta_\mu, \delta_\nu) = W_1(\mu, \nu)$. Another fact, used in the following, is that for every $\eta \in P_1(P_1(\mathbb{R}^d))$ and $\mu_0 \in P_1(\mathbb{R}^d)$

$\left| \int_{P_1(\mathbb{R}^d)} f(x) \nu(dx) - \int_{\mathbb{R}^d} f(x) \mu_0(dx) \right| \eta(d\nu) \leq L(f) \int_{P_1(\mathbb{R}^d)} W_1(\nu, \mu_0) \eta(d\nu)$

$= L(f) W_1(\eta, \delta_{\mu_0})$.

The main object in this section is a random vector $X = (X_1, ..., X^N)$, $X^i \in \mathbb{R}^d$, $i = 1, ..., N$, where the dimension $N$ is given (fixed). We assume that $\mathbb{E}(|X^i|) < \infty$ and we associate the (random) empirical measure on $\mathbb{R}^d$

$\hat{\rho}(X)(dv) := \frac{1}{N} \sum_{i=1}^N \delta_{X^i}(dv)$.

Notice that $\hat{\rho}(X)$ is a random variable with values in $P_1(\mathbb{R}^d)$ so the law $\mathcal{L}(\hat{\rho}(X)) \in \mathcal{P}(P_1(\mathbb{R}^d))$ and, for every $\Phi : P_1(\mathbb{R}^d) \to \mathbb{R}_+$

$\int_{P_1(\mathbb{R}^d)} \Phi(\mu) \mathcal{L}(\hat{\rho}(X))(d\mu) = \mathbb{E}(\Phi(\hat{\rho}(X)))$.

In particular, we have

$W_1(\mathcal{L}(\hat{\rho}(X)), \mathcal{L}(\hat{\rho}(Y))) = \sup_{L(\Phi) \leq 1} |\mathbb{E}(\Phi(\hat{\rho}(X))) - \mathbb{E}(\Phi(\hat{\rho}(Y)))|$

$\leq \mathbb{E}(W_1(\hat{\rho}(X), \hat{\rho}(Y)))$

$\leq \frac{1}{N} \sum_{i=1}^N \mathbb{E}(|X^i - Y^i|)$.

This also proves by taking $Y^i = 0$ that $\mathcal{L}(\hat{\rho}(X)) \in P_1(P_1(\mathbb{R}^d))$.

In the following we consider an initial vector $X_0$ and will assume that the components $X_0^1, ..., X_0^N$ are identically distributed and we denote $\rho \in P_1(\mathbb{R}^d)$ the common law: $X_0^i \sim \rho, i = 1, ..., N$. We consider the uniform grid $s = s_0 < ... < s_n = t, s_i = s + \frac{i}{n}(t - s)$ and we construct two sequences $X_k$ and $\overline{X}_k$, $k = 0, 1, ..., n$ in the following way. We start with $\overline{X}_0 = X_0$. The sequence $X_k$ (respectively $\overline{X}_k$) is constructed by using the empirical measures $\hat{\rho}(X_k)$ (respectively the measure $\mathcal{L}(\hat{\rho}^k)$).
\(\Theta^n_{s,s_k}(\rho) := \Theta_{s_{k-1},s_k} \ldots \Theta_{s,s_1}(\rho)\) with \(\Theta_{s,t}(\rho)\) defined by (3.4), and we define by recurrence:

\[
\begin{align*}
X^i_{k+1} &= X^i_k + b(X^i_k, \hat{\rho}(X_k))(s_k + 1 - s_k) \\
&\quad + \int_{s_k}^{s_{k+1}} \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} Q(v, z, u, X^i_k, \hat{\rho}(X_k))^N_{\hat{\rho}(X_k)}(dv, dz, du, dr),
\end{align*}
\]

\[
\begin{align*}
\bar{X}^i_{k+1} &= \bar{X}^i_k + b(\bar{X}^i_k, \Theta^n_{s,s_k}(\rho))(s_k + 1 - s_k) \\
&\quad + \int_{s_k}^{s_{k+1}} \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} Q(v, z, u, \bar{X}^i_k, \Theta^n_{s,s_k}(\rho))^N_{\Theta^n_{s,s_k}(\rho)}(dv, dz, du, dr),
\end{align*}
\]

where \(N^i_{\hat{\rho}(X_k)}(dv, dz, du, dr)\), \(i = 1, \ldots, N\) (resp. \(N^i_{\Theta^n_{s,s_k}(\rho)}(dv, dz, du, dr)\)) are Poisson point measures that are independent each other conditionally to \(X_k\) (resp. \(\bar{X}_k\)) with intensity \(\hat{\rho}(X_k)(dv)\mu(dz)du\) \(dudr\) (resp. \(\Theta^n_{s,s_k}(\rho)(dv)\mu(dz)du\) \(dudr\)). Let us observe that the common law of \(\bar{X}^i_k\), \(i = 1, \ldots, N\), is

\[
\mathcal{L}(\bar{X}^i_k) = \Theta^n_{s,s_k}(\rho).
\]

Note that \(X^i_k, \ldots, X^N_k\) are independent.

**Theorem 4.1.** Assume that (A) holds true and \(X^i_0, i = 1, \ldots, N\) are independent and of law \(\rho \in \mathcal{P}_1(\mathbb{R}^d)\). We assume that \(M_q = \left(\int_{\mathbb{R}^d} |x|^q \rho(dx)\right)^{1/q} < \infty\) with \(q > \frac{d}{d-1} \wedge 2\). We define

\[
V_N = 1_{d=1}N^{-1/2} + 1_{d=2}N^{-1/2} \log(1 + N) + 1_{d \geq 3}N^{-1/d}.
\]

Then there exists a constant \(C\) depending on \(d, q, (b,c,\gamma)\) and \((t-s)\) such that for every Lipschitz function \(f : \mathbb{R}^d \to \mathbb{R}\) with \(L(f) \leq 1\)

\[
\mathbb{E}\left(\left|\frac{1}{N} \sum_{i=1}^{N} f(X^n_i) - \int_{\mathbb{R}^d} f(x)\Theta^n_{s,s_n}(\rho)(dx)\right|\right) \leq CM_qV_N.
\]

Besides, we have the propagation of chaos in Wasserstein distance:

\[
W_1(\mathcal{L}(X^1_n, \ldots, X^n_n), \Theta^n_{s,s_n}(\rho)(dx) \otimes \cdots \otimes \Theta^n_{s,s_n}(\rho)(dx)) \leq mCM_qV_N \to 0.
\]

Furthermore, if \(\theta_{s,t}\) denotes the flow given by Theorem 3.5, we have

\[
\mathbb{E}\left(\left|\frac{1}{N} \sum_{i=1}^{N} f(X^n_i) - \int_{\mathbb{R}^d} f(x)\theta_{s,t}(\rho)(dx)\right|\right) \leq CM_qV_N + \frac{C}{n}.
\]

Let us stress here that the estimates (4.6) and (4.7) in function of the number of particles \(N\) directly come from the estimates of [22, Theorem 1] on the distance between a probability measure and its associated empirical measure. The principle of the proof of Theorem 4.1 is to use the argument of Lindeberg, i.e. to introduce particle systems where (4.4) is used on the first \(\kappa\) time steps and (4.3) is used one the \(n - \kappa\) time steps, for \(\kappa \in \{0, \ldots, n\}\). We then analyse the difference between two successive particle systems.

To do so, we introduce an intermediary sequence \(\bar{X}^i_k\) defined as follows. On the first time step, we define

\[
\bar{X}^i_1 = X^i_0 + b(X^i_0, \rho)(s_1 - s_0) + \int_{s_0}^{s_1} \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} Q(v, z, u, X^i_0, \rho)^N_{\rho}(dv, dz, du, dr),
\]
so that $\overline{X}_1^i = \overline{X}_1$. Then, for the next time steps, we define for $k \geq 1$,

$$\overline{X}_{k+1}^i = \overline{X}_k^i + b(\overline{X}_k^i, \overline{\rho}(\overline{X}_k))(s_{k+1} - s_k) + \int_{s_k}^{s_{k+1}} \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} Q(v, z, u, \overline{X}_k^i, \overline{\rho}(\overline{X}_k))N^i(\overline{\rho}(\overline{X}_k)) \, dv \, dz \, du \, dr,$$

where $N^i(\overline{\rho}(\overline{X}_k)) \, (dv, dz, du, dr)$ is a Poisson process with intensity $\overline{\rho}(\overline{X}_k)(dv)\mu(dz)du \, dr$. We stress that for $k \geq 1$, the intensity of the Poisson point measures is, as for $X_{k+1}^i$, the empirical measure of the vector constructed in the previous step. However, since $X_1 \neq \overline{X}_1$, the two chains are different.

**Lemma 4.2.** Let assumption (A) hold. We assume that the components of $X_0 = (X_0^1, ..., X_0^N)$ have the common distribution $\rho \in \mathcal{P}_1(\mathbb{R}^d)$. Then, we have

$$W_1(\mathcal{L}(\overline{\rho}(X_0)), \mathcal{L}(\overline{\rho}(\overline{X}_n))) \leq \frac{e^{2L(t-s)}L(t-s)}{n} W_1(\mathcal{L}(\overline{\rho}(X_0)), \delta_\rho),$$

$$W_1\left(\mathcal{L}(X_n^1, ..., X_n^m), \mathcal{L}(\overline{X}_n^1, ..., \overline{X}_n^m)\right) \leq m \frac{e^{2L(t-s)}L(t-s)}{n} W_1(\mathcal{L}(\overline{\rho}(X_0)), \delta_\rho),$$

with $L = L_0 + 2L(c, \gamma)$.

**Proof. Step 1** We first construct by recurrence the sequences $x_k, \overline{x}_k, k = 1, ..., n$ in the following way. We take $\tau_0(dv, d\sigma)$ to be the optimal coupling of $\rho$ and of $\overline{\rho}(X_0)$ and we take $\tau_0 : [0, 1] \to \mathbb{R}^d \times \mathbb{R}^d$ that represents $\tau_0$. In particular, if $\tau_0 = (\tau_0^1, \tau_0^2)$ then $\tau_0^1$ represents $\overline{\rho}(X_0)$ and $\tau_0^2$ represents $\rho$. We note that the optimality of $\tau_0$ gives $W_1(\rho, \overline{\rho}(X_0)) = \int_0^1 |\tau_0^2(u) - \tau_0^1(u)| \, du$ and thus

$$W_1(\delta_\rho, \mathcal{L}(\overline{\rho}(X_0))) = \mathbb{E}[W_1(\rho, \overline{\rho}(X_0))] = \mathbb{E} \left[ \int_0^1 |\tau_0^2(u) - \tau_0^1(u)| \, du \right].$$

Then we define

$x_1^i = X_0^i + b(X_0^i, \overline{\rho}(X_0))(s_1 - s_0) + \int_{s_0}^{s_1} \int_{[0,1] \times E \times \mathbb{R}_+} Q((\tau_0^1)^1(w), z, u, X_0^i, \overline{\rho}(X_0))N^i(dv, dz, du, dr),$

$\overline{x}_1^i = X_0^i + b(X_0^i, \rho)(s_1 - s_0) + \int_{s_0}^{s_1} \int_{[0,1] \times E \times \mathbb{R}_+} Q((\tau_0^2)^1(w), z, u, X_0^i, \rho)N^i(dv, dz, du, dr),$

where $N^i$ is a Poisson process with intensity $1_{[0,1]}(w)dv \mu(dz)du \, dr$. We also assume that the Poisson point measures $N^i, i = 1, ..., N$ are independent. Notice that $x_1$ has the same law as $X_1$ and $\overline{x}_1$ has the same law as $\overline{X}_1$ by (A2).

Then, for $k \geq 1$, if $x_k, \overline{x}_k$ are given, we construct $x_{k+1}, \overline{x}_{k+1}$ as follows. We consider $\pi_k(dv, d\sigma)$ an optimal coupling of $\overline{\rho}(x_k)$ and $\overline{\rho}(\overline{x}_k)$ and we take $\tau_k : [0, 1] \to \mathbb{R}^d \times \mathbb{R}^d$ that represents $\tau_k$. Then we define

$x_{k+1}^i = x_k^i + b(x_k^i, \overline{\rho}(x_k))(s_{k+1} - s_k) + \int_{s_k}^{s_{k+1}} \int_{[0,1] \times E \times \mathbb{R}_+} Q(\tau_k^1(w), z, u, x_k^i, \overline{\rho}(x_k))N^i(dv, dz, du, dr),$

$\overline{x}_{k+1}^i = \overline{x}_k^i + b(\overline{x}_k^i, \overline{\rho}(\overline{x}_k))(s_{k+1} - s_k) + \int_{s_k}^{s_{k+1}} \int_{[0,1] \times E \times \mathbb{R}_+} Q(\tau_k^2(w), z, u, \overline{x}_k^i, \overline{\rho}(\overline{x}_k))N^i(dv, dz, du, dr)$

Using again (A2), we get by induction on $k$ that $x_k$ has the same law as $X_k$ and $\overline{x}_k$ has the same law as $\overline{X}_k$. 


Step 2. Suppose that $k \geq 1$. We use now Assumptions (A₁) and (A₃) to get

\[
\mathbb{E} |x_{k+1}^i - \overline{x}_{k+1}^i| \leq \mathbb{E} |x_k^i - \overline{x}_k^i| + (L_b + L_\mu(c, \gamma))(\mathbb{E} |x_k^i - \overline{x}_k^i| + W_1(\hat{\rho}(x_k), \hat{\rho}(\overline{x}_k))(s_{k+1} - s_k) + L_\mu(c, \gamma)\mathbb{E} \int_{s_k}^{s_{k+1}} \int_0^1 |\tau_k^1(\omega) - \tau_k^2(w)| \, dw \, ds.
\]

Since

\[
\int_0^1 |\tau_k^1(\omega) - \tau_k^2(w)| \, dw = W_1(\hat{\rho}(x_k), \hat{\rho}(\overline{x}_k)) \leq \frac{1}{N} \sum_{j=1}^{N} |x_k^j - \overline{x}_k^j|,
\]

we obtain

\[
(4.10) \quad \mathbb{E} |x_{k+1}^i - \overline{x}_{k+1}^i| \leq \mathbb{E} |x_k^i - \overline{x}_k^i| [1 + (L_b + L_\mu(c, \gamma))(s_{k+1} - s_k)] + (L_b + 2L_\mu(c, \gamma))(s_{k+1} - s_k) \frac{1}{N} \sum_{j=1}^{N} \mathbb{E} |x_k^j - \overline{x}_k^j|.
\]

Summing over $i = 1, ..., N$, we get

\[
\frac{1}{N} \sum_{i=1}^{N} \mathbb{E} |x_{k+1}^i - \overline{x}_{k+1}^i| \leq \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} |x_k^i - \overline{x}_k^i| [1 + 2(L_b + 2L_\mu(c, \gamma))(s_{k+1} - s_k)].
\]

Using this inequality, we get by recurrence

\[
\frac{1}{N} \sum_{i=1}^{N} \mathbb{E} |x_{n}^i - \overline{x}_{n}^i| \leq \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} |x_1^i - \overline{x}_1^i| \left(1 + \frac{2(L_b + 2L_\mu(c, \gamma))}{n} (s - t)\right)^{n-1}.
\]

Step 3. We go now from $x_1, \overline{x}_1$ to $X_0$. We have

\[
\mathbb{E} |x_1^i - \overline{x}_1^i| \leq L_\mu(c, \gamma) \int_s^{s_1} \mathbb{E} \int_0^1 |\tau_0^1(\omega) - \tau_0^2(w)| \, dw \, ds + (L_b + L_\mu(c, \gamma))W_1(\rho, \hat{X}_0)(s_1 - s) = (L_b + 2L_\mu(c, \gamma))(s_1 - s)W_1(\hat{\delta}_\rho, \mathcal{L}(\hat{\rho}(X_0)))
\]

by using (4.9) for the last equality, so that we get by summing over $i = 1, ..., N$,

\[
\frac{1}{N} \sum_{i=1}^{N} \mathbb{E} |x_1^i - \overline{x}_1^i| \leq (L_b + 2L_\mu(c, \gamma)) \frac{t - s}{n} W_1(\hat{\delta}_\rho, \mathcal{L}(\hat{\rho}(X_0))).
\]

We combine with the previous inequality and we obtain

\[
\frac{1}{N} \sum_{i=1}^{N} \mathbb{E} |x_n^i - \overline{x}_n^i| \leq W_1(\hat{\delta}_\rho, \mathcal{L}(\hat{\rho}(X_0))) \frac{(L_b + 2L_\mu(c, \gamma))(t - s)}{n} \left(1 + \frac{L_b + 2L_\mu(c, \gamma)}{n} (t - s)\right)^{n-1} \leq \frac{e^{(L_b + 2L_\mu(c, \gamma))(t - s)}(L_b + 2L_\mu(c, \gamma))(t - s)}{n} W_1(\hat{\delta}_\rho, \mathcal{L}(\hat{\rho}(X_0))).
\]

We notice that the law of $(x_n^i, \overline{x}_n^i)_i$ is invariant up to a permutation on the $i$'s. In particular, we have $\mathbb{E} |x_n^i - \overline{x}_n^i| = \mathbb{E} |x_n^1 - \overline{x}_n^1|$, and therefore

\[
\sum_{i=1}^{m} \mathbb{E} |x_n^i - \overline{x}_n^i| \leq m \frac{e^{(L_b + 2L_\mu(c, \gamma))(t - s)}(L_b + 2L_\mu(c, \gamma))(t - s)}{n} W_1(\hat{\delta}_\rho, \mathcal{L}(\hat{\rho}(X_0))).
\]
Step 4 Since the law of $X_n$ coincides with the law of $x_n$ it follows that $\mathcal{L}(\hat{\rho}(X_n)) = \mathcal{L}(\hat{\rho}(x_n))$ and $\mathcal{L}(X_1^n, \ldots, X_m^n) = \mathcal{L}(x_1^n, \ldots, x_m^n)$. The same is true for $\bar{X}_n$ and $\bar{x}_n$. So, we have by (4.2)
\[
W_1 \left( \mathcal{L}(\hat{\rho}(X_n)), \mathcal{L}(\hat{\rho}(\bar{X}_n)) \right) = W_1(\mathcal{L}(\hat{\rho}(x_n)), \mathcal{L}(\hat{\rho}(\bar{x}_n))
\leq \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \left| x_n^i - \bar{x}_n^i \right|
\leq \frac{e^{2(L_b+2L_{\mu}(c, \gamma))(t-s)}(L_b + 2L_{\mu}(c, \gamma))(t-s)}{n} W_1(\delta_{\rho}, \mathcal{L}(\hat{\rho}(X_0))).
\]
We get the other inequality by using $W_1(\mathcal{L}(X_1^n, \ldots, X_m^n), \mathcal{L}(\bar{X}_1^n, \ldots, \bar{X}_m^n)) \leq \mathbb{E} \left( \sum_{i=1}^{m} \left| x_n^i - \bar{x}_n^i \right| \right)$. □

Proof of Theorem 4.1. We use the argument of Lindeberg. In order to pass from the sequence $X_k$ to the sequence $\bar{X}_k$, we construct intermediary sequences as follows. Given $\kappa \in \{0, \ldots, n-1\}$ we define $X_{\kappa,k} = \bar{X}_k$ for $k \leq \kappa$ and, for $k \geq \kappa$ we define $X_{\kappa,k}$ by the recurrence formula (4.3). So the construction of $k \leq \kappa$ employs the intensity measure based on the common law $\rho(\bar{X}_k)$ while for $k > \kappa$ we use the empirical measure. In particular, $X_{\kappa,k}^i, i = 1, \ldots, N$ are independent and have the common distribution $\Theta_{s,s,\kappa}^n(\rho)$. Then we write
\[
W_1(\mathcal{L}(\hat{\rho}(X_\kappa)), \mathcal{L}(\hat{\rho}(\bar{X}_\kappa))) \leq \sum_{\kappa=0}^{n-1} W_1(\mathcal{L}(\hat{\rho}(X_{\kappa,n})), \mathcal{L}(\hat{\rho}(X_{\kappa+1,n}))).
\]
Let us compare the sequences $X_{\kappa,k}$ and $X_{\kappa+1,k}$. Both sequences start with $\bar{X}_{\kappa}$ at time $s_n$ and then, in the following step, $\rho(\bar{X}_\kappa)$ is used to produce $X_{\kappa+1,n}$ and the empirical measure $\hat{\rho}(\bar{X}_\kappa)$ is used to produce $X_{\kappa+1,k+1}$. Afterwards, for $\kappa \geq \kappa + 1$ both sequences use their corresponding empirical measure. This is exactly the framework of Lemma 4.2, so we get
\[
W_1(\mathcal{L}(\hat{\rho}(X_\kappa)), \mathcal{L}(\hat{\rho}(\bar{X}_\kappa))) \leq \frac{e^{2(L_b+2L_{\mu}(c, \gamma))(t-s)}L_{\mu}(c, \gamma)(t-s)}{n} \times W_1(\delta_{\Theta_{s,s,\kappa}^n(\rho)}, \mathcal{L}(\hat{\rho}(X_\kappa))),
\]
and summing over $\kappa$ we obtain
\[
W_1(\mathcal{L}(\hat{\rho}(X_n)), \mathcal{L}(\hat{\rho}(\bar{X}_n))) \leq \frac{e^{2(L_b+2L_{\mu}(c, \gamma))(t-s)}L_{\mu}(c, \gamma)(t-s)}{n} \times \sum_{\kappa=0}^{n-1} W_1(\delta_{\Theta_{s,s,\kappa}^n(\rho)}, \mathcal{L}(\hat{\rho}(X_\kappa))).
\]
(4.11) \[
W_1(\mathcal{L}(\hat{\rho}(X_n)), \mathcal{L}(\hat{\rho}(\bar{X}_n))) \leq \frac{e^{2(L_b+2L_{\mu}(c, \gamma))(t-s)}L_{\mu}(c, \gamma)(t-s)}{n} \times \sum_{\kappa=0}^{n-1} W_1(\delta_{\Theta_{s,s,\kappa}^n(\rho)}, \mathcal{L}(\hat{\rho}(X_\kappa))).
\]
It is well known that the moments of order $q$ are preserved by the Euler scheme, thanks to (3.5) and (3.6). We can therefore use Theorem 1 of the article [22] by Fournier and Guillin and get $\mathcal{W}_1(\delta_{\Theta_{s,s,\kappa}^n(\rho)}, \mathcal{L}(\hat{\rho}(X_\kappa))) \leq \overline{C} q V_N$, leading to
\[
W_1(\mathcal{L}(\hat{\rho}(X_n)), \mathcal{L}(\hat{\rho}(\bar{X}_n))) \leq C q V_N.
\]
Now using (4.1) with $\eta = \mathcal{L}(\hat{\rho}(\bar{X}_n))$ and $\mu_0 = \Theta_{s,s,\kappa}^n(\rho)$ we get, for every $f$ with $L(f) \leq 1$
\[
\mathbb{E} \left( \left| \frac{1}{N} \sum_{i=1}^{N} f(X_n^i) - \int_{\mathbb{R}^d} f(x) \Theta_{s,s,\kappa}^n(\rho)(dx) \right| \right) \leq W_1(\mathcal{L}(\hat{\rho}(X_n)), \delta_{\Theta_{s,s,\kappa}^n(\rho)}) \leq C q V_N.
\]
Then, (4.7) is a consequence of (3.17).
Last, the propagation of chaos follows by the same arguments, since we have from Lemma 4.2
\[
W_1 \left( \mathcal{L}(X_{k,n}^1, \ldots, X_{k,n}^m), \mathcal{L}(X_{k+1,n}^1, \ldots, X_{k+1,n}^m) \right) \\
\leq m \frac{e^{(L_b + 2L_n)c(\gamma)(t-s)} L_\mu(c, \gamma)(t-s)}{n} \times W_1(\delta_{\hat{\rho}_{s,n}(\rho)}, \mathcal{L}(\hat{\rho}(X_n))).
\]

\[\square\]

**Approximating particles system and algorithm**

We now discuss briefly the problem of sampling the system of particles defined by (4.3). To do so, we will assume that:
\[(4.12) \quad \mu(E) < \infty \quad \text{and} \quad |\gamma(v, z, x)| \leq \Gamma, \forall v, x \in \mathbb{R}^d, z \in E.\]

The approximation in a more general framework requires then to use some truncation procedures and to quantify the corresponding error.

When (4.12) holds, the solution of (4.3) is constructed in an explicit way as follows. Let us assume that the values of \((X_k^i, i \in \{1, \ldots, N\})\) have been obtained: we explain how to construct then \((X_{k+1}^i, i \in \{1, \ldots, N\})\). We take \(T_{i, \ell} \in \mathbb{N}\) to be the jump times of a Poisson process of intensity \(\mu(E) \times \Gamma\) and we take \(Z_{i, \ell} \sim \frac{1}{\mu(E)} \mu(dz)\), \(U_{i, \ell} \sim 1_{[0,1]}(u)du\) and \(\varepsilon_{i, \ell}\) uniformly distributed on \(\{1, \ldots, N\}\). For each \(i = 1, \ldots, N\) this set of random variables are independent. Then one computes explicitly
\[X_{k+1}^i = X_k^i + b(X_k^i, \hat{\rho}(X_k)) \frac{s - t}{n} + \sum_{s_k \leq T_{i, \ell} < s_{k+1}} Q(X_k^i, Z_{i, \ell}^i, U_{i, \ell}, X_k^i, \hat{\rho}(X_k)),\]
which gives the desired particle system that satisfies (4.7).

5. The Boltzmann equation

5.1. The homogeneous Boltzmann equation. We consider the following more specific set of coefficients, which corresponds to the Boltzmann equation with hard potential. We take \(a \in (0, 1)\) and we define
\[(5.1) \quad \gamma(v, x) := |v - x|^a.\]

Moreover we take \(b : \mathbb{R}^d \to \mathbb{R}^d\) that is Lipschitz continuous (and thus satisfies (A1)) and \(c : \mathbb{R}^d \times E \times \mathbb{R}^d \to \mathbb{R}^d\) that verifies the following hypothesis. We assume that for every \((v, x) \in \mathbb{R}^d \times \mathbb{R}^d\) there exists a function \(c_{v,x} : \mathbb{R}^d \times E \times \mathbb{R}^d \to \mathbb{R}\) such that for every \(v', x' \in \mathbb{R}^d\) and \(v \in C_b^1(\mathbb{R}^d)\)
\[(5.2) \quad \int_E \varphi(c(v', z, x')) \mu(dz) = \int_E \varphi(c_{v,x}(v', z, x')) \mu(dz).\]

Notice that, since \(\gamma\) does not depend on \(z\), this guarantees that \(Q(v, z, u, x) := c(v, z, x) 1_{\{u < \gamma(v, x)\}}\) verifies (A2). Then, we assume that there exists some function \(\alpha : E \to \mathbb{R}_+\) such that \(\int_E \alpha(z) \mu(dz) < \infty\).
\[(5.3) \quad |c(v, z, x)| \leq \alpha(z) |v - x| \quad \text{and} \quad |c(v, z, x) - c_{v,x}(v', z, x')| \leq \alpha(z)(|v - v'| + |x - x'|).\]

Notice that \(Q\) may not satisfy the sublinear growth property (3.6) because (5.3) only ensures that \(\int_{E \times \mathbb{R}_+} |Q(v, z, u, x)| \mu(dz) du = \gamma(v, x) \int_E c(v, z, x) \mu(dz) \leq \int_E \alpha(z) \mu(dz)|v - x|^{1+a}\). Thus (A3) may not hold. So, our results does not apply directly for these coefficients. In order to fit our framework,
we will use a truncation procedure. For $\Gamma \geq 1$ we define $H_{\Gamma}(v) = v \times \frac{|v| \cdot \sqrt{\Gamma}}{|v|}$ and we notice that $|H_{\Gamma}(v)| \leq \Gamma$ and $|H_{\Gamma}(v) - H_{\Gamma}(w)| \leq |v - w|$. Then we define

$$
\gamma_{\Gamma}(v, x) = \gamma(H_{\Gamma}(v), H_{\Gamma}(x)) = |H_{\Gamma}(v) - H_{\Gamma}(x)|^a,
$$

(5.4) 

$$
c_{\Gamma}(v, z, x) = c(H_{\Gamma}(v), z, H_{\Gamma}(x)), \quad c_{\Gamma}(v, x')(v', z, x') = c_{H_{\Gamma}(v), H_{\Gamma}(x)}(H_{\Gamma}(v'), z, H_{\Gamma}(x')),
$$

$$
Q_{\Gamma}(v, z, u, x) = Q(H_{\Gamma}(v), z, u, H_{\Gamma}(x)), \quad Q_{\Gamma}(v', z, x') = 1_{v < v'} c_{\Gamma}(v, x)(v', z, x').
$$

**Lemma 5.1.** Let $b : \mathbb{R}^d \to \mathbb{R}^d$ be Lipschitz continuous and assume (5.2) and (5.3). Then, the triplet $(b, c, \gamma)$ satisfies (A) with $L_{\mu}(c_{\Gamma}, \gamma_{\Gamma}) = 6\Gamma^{a} \int E \alpha(z) dz$.

Besides, for every $\Gamma \geq 1$, the triplet $(b, c_{\Gamma}, \gamma_{\Gamma})$ satisfies (A) with $L_{\mu}(c_{\Gamma}, \gamma_{\Gamma}) = 6\Gamma^{a} \int E \alpha(z) dz$.

The proof of this Lemma is postponed to Appendix A. Thanks to this result, we can then apply Theorem 3.5 to construct a flow $\theta^F_{s,t}(\rho)$. By Theorem 3.7, this flows solves the weak equation (3.22) associated with $\gamma_{\Gamma}$ and $Q_{\Gamma}$. Besides, by Theorem 3.8 there exists a probabilistic representation of this solution. The natural question is then to know if $\theta^F_{s,t}(\rho)$ converges when $\Gamma \to \infty$. This would produce a flow that would be a natural candidate for the solution of the Boltzmann equation. We leave this issue for further research.

**The 3D Boltzmann equation with hard potential.** We now precise the coefficients which appear in the homogeneous Boltzmann equation in dimension three. We follow the parametrization introduced in [24] and [20]. For this equation, the space $E$ is $E = [0, \pi] \times [0, 2\pi]$, we note $z = (\zeta, \varphi)$ and the measure $\mu$ is defined by $\mu(dz) = \zeta^{-(1+\nu)} d\zeta d\varphi$, for some $\nu \in (0, 1)$. The coefficient $\gamma$ is given by (5.1). We now define $c$. Given a vector $X \in \mathbb{R}^3 \setminus \{0\}$, one may construct $I(X), J(X) \in \mathbb{R}^3$ such that $X \mapsto (I(X), J(X))$ is measurable and $(\frac{X}{|X|}, \frac{I(X)}{|X|}, \frac{J(X)}{|X|})$ is an orthonormal basis in $\mathbb{R}^3$. We define the function $\Delta(X, \varphi) = \cos(\varphi)I(X) + \sin(\varphi)J(X)$ and then

$$
c(v, (\zeta, \varphi)) = -\frac{1 - \cos \zeta}{2} (v - x) + \frac{\sin \zeta}{2} \Delta(v - x, \varphi).
$$

The specific difficulty in this framework is that $c$ does not satisfy the standard Lipschitz continuity property. It has been circumvented by Tanaka in [40] (see also Lemma 2.6 in [24]) who proves that one may construct a measurable function $\eta : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, 2\pi]$ such that

$$
|c(v, (\zeta, \varphi), x) - c(v', (\zeta, \varphi) + \eta(v' - x', v - x)), x')| \leq 2\zeta(|v - v'| + |x - x'|).
$$

This means that Hypothesis (5.3) holds with $c_{v,x}(v', (\zeta, \varphi), x') = c(v', (\zeta, \varphi + \eta(v' - x', v - x)), x')$. This function also satisfies (5.2):

$$
\int_{0}^{\pi} \frac{d\zeta}{\zeta^{1+\nu}} \int_{0}^{2\pi} f(x + c(v, (\zeta, \varphi), x)) d\varphi = \int_{0}^{\pi} \frac{d\zeta}{\zeta^{1+\nu}} \int_{0}^{2\pi} f(x + c(v, (\zeta, \varphi + \eta(v - x, v - x)), x)) d\varphi,
$$

when these integrals are well-defined, since for every $v, x \in \mathbb{R}^3$ and $\zeta \in (0, \pi)$ the function $\varphi \mapsto f(x + c(v, (\zeta, \varphi), x))$ is $2\pi$-periodic. We are therefore indeed in the framework of Lemma 5.1. Many results exist on the existence and uniqueness of solutions of the Boltzmann equation, and we refer to the introduction [21] for a recent overview of the topic. In particular, Fournier and Mischler [25] give an existence and uniqueness result with exponential moments. Our approach does not improve these results, but gives a simple tool to get existence and uniqueness results that can be applied to a larger family of equations involving both mean field and Boltzmann type interactions.
5.2. The Boltzmann-Enskog equation. In this section we consider the non homogeneous Boltzmann equation called Enskog equation which has been discussed in [3]. The study of this equation has been initiated in [38], and more recent contributions concerning existence, uniqueness, probabilistic interpretation and particle system approximations are given in [1], [27] and [29]. We consider a model in which $\mathcal{X} = (X^1, \ldots, X^d) \in \mathbb{R}^d$ with $d = 3$ represents the position of the "typical particle" and $X = (X^1, \ldots, X^d) \in \mathbb{R}^d$ is its velocity. In all this subsection, letters with bar will refer to positions, and bold letters $\mathbf{X} = (\bar{X}, X)$ will denote the couple position-velocity. Then, the position follows the dynamics given by the velocity:

$$\bar{X}_{s,t} = \bar{X}_0 + \int_s^t \bar{X}_{s,r} dr,$$

where $\bar{X}_0$ is an integrable random variable. As for the velocity, $X_{s,t}$ follows the equation

$$X_{s,t} = X_0 + \int_s^t \int_{(\mathbb{R}^d \times \mathbb{R}^d) \times E \times \mathbb{R}_+} \gamma(v, z, X_{s,r}) \times \beta(\bar{v}, \bar{X}_{s,r}) N(dv, dz, du, dr)$$

where $v = (\bar{v}, v), \gamma(v, x) = |x - v|^{a}, \beta \in C^1_b(\mathbb{R}^d \times \mathbb{R}^d)$ and $N$ is a Poisson point measure of intensity

$$\theta_{s,r}(dv) \mu(dz) du dr \quad \text{with} \quad \theta_{s,r}(dv) = \mathbb{P}( (\bar{X}_{s,r}, X_{s,r}) \in dv ).$$

Here, as in the case of the homogeneous equation, $E = [0, \pi] \times [0, 2\pi]$ and $z = (\zeta, \varphi)$ and the measure $\mu(dz) = \zeta^{-(1+a)} d\zeta d\varphi, a \in (0, 1)$.

We look at this dynamics as a system in dimension $2d$ (typically with $d = 3$). We denote $x = (\bar{x}, x), v = (\bar{v}, v)$ and $\mathbf{X} = (\bar{X}, X)$. The drift is then given by

$$b(x) := \begin{cases} x^i & \text{for } i = 1, \ldots, d, \\ 0 & \text{for } i = d + 1, \ldots, 2d, \end{cases}$$

and the collision kernel (cross section) is

$$c(v, z, x) := c(v, z, x)\beta(\bar{v}, \bar{x}),$$

where $c$ is defined as in the Boltzmann equation by (5.5) with $z = (\zeta, \varphi) \in E$, and

$$\gamma(x, v) := |v - x|^{a},$$

with $a \in (0, 1)$. The equation (3.22) associated to these coefficients is the $(d$-dimensional) Enskog equation. In the particular case $\beta(\bar{v}, \bar{x}) = 1$, we recover the case of the homogeneous Boltzmann equation and $\bar{X}$ is just the time-integral of the process. The specificity of the inhomogeneous case is illustrated by the following example. Let us take $R > 0$, $i_R$ be a regularized version of the indicator function $1_{x < R}$ and define $\beta_R(\bar{v}, \bar{x}) = i_R(|\bar{x} - \bar{v}|)$. Then, the coefficient $\beta_R(\bar{v}, \bar{x})$ means that only the particles which are closer to the distance $R$ may collide.

We now define the truncated coefficients $c(\Gamma)$ and $\gamma(\Gamma)$ for $\Gamma > 0$. We still denote, for $v \in \mathbb{R}^d$, $H_\Gamma(v) = v \times \frac{|v|}{|v|},$ and we define $c(\Gamma)(v, z, x) = c(v, z, x)\beta(\bar{v}, \bar{x})$ with $\beta(\bar{v}, \bar{x}) = \beta(H_\Gamma(\bar{v}), H_\Gamma(\bar{x}))$ and $\gamma(\Gamma)(x, v) = |H_\Gamma(v) - H_\Gamma(x)|^a$.

**Lemma 5.2.** Assume that $\beta \in C^1_b(\mathbb{R}^d \times \mathbb{R}^d)$ and that (5.2) and (5.3) hold. Then, for every $\Gamma \geq 1$, the triplet $(b, c(\Gamma), \gamma(\Gamma))$ satisfies (A) with $L_\mu(c(\Gamma), \gamma(\Gamma)) = C \Gamma^{a+1}$ for some constant $C > 0$. 

The proof is postponed to Appendix A. As for the Boltzmann equation, this lemma allows by
Theorem 3.5 to construct the flow, and then by Theorems 3.7 and 3.8, a weak solution and a
probabilistic representation for the Enskog-Boltzmann equation with truncated coefficients. The
convergence when $\Gamma \to \infty$ remains an open problem. Up to our knowledge, there are still few
results on the Boltzmann-Enskog equation. Friesen et al. [27] obtain existence of solutions by the
mean of particle systems, and they have obtained very recently in [28] uniqueness results under
exponential moments. Here, we obtain existence and uniqueness with truncated coefficients.

5.3. A Boltzmann equation with a mean field interaction on the position. The fact that
we are able with our approach to mix easily Boltzmann and McKean-Vlasov interactions gives
more flexibility to model the behaviour of particles. Here, we give a very simple example that is
derived from the Boltzmann-Enskog equation discussed above. In this equation, interactions are
made both for the position and the velocity through a Poisson point measure. More precisely, when
derived from the Boltzmann-Enskog equation.

We now define for $\Gamma$ the probability distribution of $(\bar{x}, \rho)$ and $\theta(s,r)$ to get a probabilistic representation associated
5.3. A Boltzmann equation with a mean field interaction on the position. The fact that
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results on the Boltzmann-Enskog equation. Friesen et al. [27] obtain existence of solutions by the
mean of particle systems, and they have obtained very recently in [28] uniqueness results under
exponential moments. Here, we obtain existence and uniqueness with truncated coefficients.
to this flow. Again, one needs further assumptions to justify that this produces, when \( \Gamma \to \infty \) a solution to (5.9). This is left for further research. Up to our knowledge, there are no study in the literature of Equation (5.9). This example illustrates the strength of our approach: it enables to obtain existence and uniqueness result for a large variety of equations involving mean field and Boltzmann type interactions.

Appendix A. Technical Proofs

Proof of Lemma 5.1. It is easy to check that \( c_\Gamma \) also satisfies (5.2) and (5.3). Therefore (A_2) holds for \( (b, c_\Gamma, \Gamma) \). We now check (A_3). We have

\[
\int_{E \times \mathbb{R}_+} |Q_\Gamma(v_1, z, u, x_1) - Q_{\Gamma,(v_1,x_1)}(v_2, z, u, x_2)| \mu(dz)du \leq A + B, \text{ with}
\]

\[
A = \int_{E \times \mathbb{R}_+} |c_\Gamma(v_1, z, x_1) - c_{\Gamma,(v_1,x_1)}(v_2, z, x_2)| 1_{u < \gamma_\Gamma(v_2,x_2)} \mu(dz)du,
\]

\[
B = \int_{E \times \mathbb{R}_+} |c_\Gamma(v_1, z, x_1)(1_{u < \gamma_\Gamma(v_1,x_1)} - 1_{u < \gamma_\Gamma(v_2,x_2)})| \mu(dz)du.
\]

From (5.3) and \(|\gamma_\Gamma(v, x)| \leq 2 \Gamma^a\), we get \( A \leq 2 \Gamma^a \int_E \alpha(z) \mu(dz)(|v_1 - v_2| + |x_1 - x_2|)\). We have

\[
(A.1) \quad B \leq \int_E |\gamma_\Gamma(v_1, x_1) - \gamma_\Gamma(v_2, x_2)| |c_\Gamma(v_1, z, x_1)| \mu(dz),
\]

and we denote

\[
X = |H_\Gamma(v_1) - H_\Gamma(x_1)| \quad \text{and} \quad Y = |H_\Gamma(v_2) - H_\Gamma(x_2)|
\]

We now use the following basic inequality:

\[
\forall x, y > 0, (x + y)|x^a - y^a| \leq (x^a + y^a)|x - y|.
\]

This is easily checked by taking for example \( x > y \), expanding and observing that \( yx^a - xy^a = (xy)^a(y^{1-a} - x^{1-a}) < 0 \) since \( a \in (0, 1) \). We then get

\[
X(X^a - Y^a) \leq (X + Y)(X^a - Y^a) \leq (X^a + Y^a)|X - Y| \leq 4 \Gamma^a |X - Y|
\]

\[
\leq 4 \Gamma^a (|H_\Gamma(v_1) - H_\Gamma(v_2)| + |H_\Gamma(x_1) - H_\Gamma(x_2)|)
\]

\[
\leq 4 \Gamma^a (|v_1 - v_2| + |x_1 - x_2|).
\]

We also have

\[
|c_\Gamma(v_1, z, x_1)| \leq \alpha(z) |H_\Gamma(v_1) - H_\Gamma(x_1)| = \alpha(z)X
\]

so we get

\[
(A.2) \quad B \leq X(X^a - Y^a) \times \int_E \alpha(z) \mu(dz) \leq 4 \Gamma^a \int_E \alpha(z) \mu(dz)(|v_1 - v_2| + |x_1 - x_2|).
\]

Proof of Lemma 5.2. Using that \( \beta \) is bounded, we have \(|c_\Gamma(v, z, x)| \leq \|\beta\|_\infty \alpha(z) |v - x|\). Besides, we have

\[
|c_\Gamma(v, z, x) \beta_\Gamma(\bar{v}, \bar{x}) - c_{\Gamma,(v,x)}(v', z, x') \beta_{\Gamma}(\bar{v}', \bar{x}')|
\]

\[
\leq |c_\Gamma(v, z, x)||\beta_\Gamma(\bar{v}, \bar{x}) - \beta_{\Gamma}(\bar{v}', \bar{x}')| + |\beta_{\Gamma}(\bar{v}', \bar{x}')||c_{\Gamma}(v, z, x) - c_{\Gamma}(v', z, x')|
\]

\[
\leq 2 \alpha(z) \Gamma L(\beta)(|\bar{v} - \bar{v}'| + |\bar{x} - \bar{x}'|) + \alpha(z) \|\beta\|_\infty (|v - v'| + |x - x'|),
\]

\(\square\)
where \( L(\beta) \) is the Lipschitz constant of \( \beta \). Thus, if we define \( c_{v,x}(v', z, x') = c_{v,x}(v', z, x') \beta(v', x') \), \( c \) satisfies equations (5.2) and (5.3) with \( \alpha(z) = (2 \Gamma L(\beta) + \|\beta\|_\infty) \alpha(z) \). We then get the claim by applying the same arguments as for Lemma 5.1.

Proof of Lemma 5.3. We first observe that \( \|p_R\|_\infty = (2 \pi R^2)^{-d/2} \) and that \( p_R \) is Lipschitz with constant \( \|p_R\|_\infty/R \). Thus, the proof of Lemma 5.1 can be easily adapted. In particular, the term “B” is now

\[
B = \int_{E \times \mathbb{R}^+} |c_T(v_1, z, x_1)|_{u \leq \gamma_T(v_1, x_1, \rho_1)} - c_T(v_2, z, x_2)|_{u \leq \gamma_T(v_2, x_2, \rho_2)} |\mu(\alpha)du
\]

\[
\leq \int_{E \times \mathbb{R}^+} |c_T(v_1, z, x_1)|\gamma_T(v_1, x_1, \rho_1) - \gamma_T(v_2, (x_1, x_2), \rho_1) |\mu(\alpha)du
\]

\[
+ \int_{E \times \mathbb{R}^+} |c_T(v_1, z, x_1)|\gamma_T(v_2, (x_1, x_2), \rho_1) - \gamma_T(v_2, (x_2, \rho_2)) |\mu(\alpha)du.
\]

The first term can be bounded by \( 4 ||p_R||_\infty \Gamma^a \int E \alpha(\|v_1 - v_2\| + |x_1 - x_2|) \) as in Lemma (5.1) while the second term is bounded by

\[
2\Gamma \times 2\Gamma \times \left| \int_{\mathbb{R}^d} p_R(\bar{x}_1 - \bar{x})\rho_1(\bar{x})\rho_2(\bar{x}) \right|
\]

\[
\leq \frac{4\Gamma^{a+1} ||p_R||_\infty}{R} (|\bar{x}_1 - \bar{x}_2| + W_1(\rho_1, \rho_2)).
\]

The other properties are easy to check. \( \Box \)

APPENDIX B. GENERIC APPLICATION OF THE SEWING LEMMA

We present in this appendix a typical setting that falls into the framework of Lemma 2.1. We consider a complete metric space \( (V, d) \) and \( v_0 \) be a fixed element of \( V \). We define

\[
\mathcal{E}_0(V) = \{ \Theta \in \mathcal{E}(V) : \sup_{v \in V} \frac{d(v_0, \Theta(v))}{1 + d(v_0, v)} < \infty \},
\]

the set of maps with sublinear growth. We endow \( \mathcal{E}_0(V) \) with the following distance

\[
d_*(\Theta, \Theta) = \sup_{v \in V} \frac{d(\Theta(v), \Theta(v))}{1 + d(v_0, v)}, \Theta, \Theta \in \mathcal{E}_0(V).
\]

It is clear from the definition of \( \mathcal{E}_0(V) \) that \( d_*(\Theta, \Theta) < \infty \) and the distance properties of \( d_* \) are obviously inherited from those of \( d \). We remark also that if \( \Theta_1 \in \mathcal{E}_0(V) \) and \( \Theta_2 \in \mathcal{E}(V) \) is such that \( d_*(\Theta_1, \Theta_2) < \infty \), then \( \Theta_2 \in \mathcal{E}_0(V) \). Besides, we check also easily that \( Id \in \mathcal{E}_0(V) \) and \( (\mathcal{E}_0(V), \circ) \) is a monoid: for \( \Theta_1, \Theta_2 \in \mathcal{E}_0(V) \),

\[
\sup_{v \in V} \frac{d(v_0, \Theta_2(\Theta_1(v)))}{1 + d(v_0, v)} \leq \sup_{v \in V} \frac{d(v_0, \Theta_2(\Theta_1(v)))}{1 + d(v_0, \Theta_1(v))} \sup_{v \in V} \frac{1 + d(v_0, \Theta_1(v))}{1 + d(v_0, v)} < \infty.
\]

Note that this type of distance is considered by Brault and Lejay [9, Notation 1.3] who present refined existence and uniqueness result based on the sewing lemma argument.

**Lemma B.1.** \( (\mathcal{E}_0(V), d_*) \) defined by (B.1) and (B.2) is a complete metric space. Besides, (2.2) holds.
Proof. Let $\theta_n \in \mathcal{E}_0(V)$ be a sequence such that $\sup_{p,q \geq n} d_*(\theta_p, \theta_q) \to 0$. Then, for any $v \in V$, there exists $\theta_{\infty}(v) \in V$ such that $d(\theta_n(v), \theta_{\infty}(v)) \to 0$ since $(V, d)$ is complete. Therefore, we have $d(\theta_n(v), \theta_{\infty}(v)) \leq \sup_{q \geq n} d(\theta_n(v), \theta_q(v))$, which gives $d_*(\theta_n, \theta_{\infty}) \leq \sup_{q \geq n} d_*(\theta_n, \theta_q) \to 0$.

We now consider $U, \varphi, \psi \in \mathcal{E}_0(V)$ and have

$$d_*(\varphi U, \psi U) = \sup_{v \in V} \frac{d(\varphi(U(v)), \psi(U(v)))}{1 + d(v_0, U(v))} \leq C(U) d_*(\varphi, \psi),$$

with $C(U) = \sup_{v \in V} \frac{1 + d(v_0, U(v))}{1 + d(v_0, v)} < \infty$ since $U \in \mathcal{E}_0(V)$. Using that $d(v_0, U(v)) \leq d(v_0, v) + d(v, U(v))$, we get $C(U) \leq 1 + d_*(U, Id)$ and we therefore obtain (2.2).

Last, we discuss in this setting the properties ((H1) and (H2)). We first have:

$$\text{(H1)} \iff \forall v_1, v_2 \in V, d(\Theta_{s,t}^P(v_1), \Theta_{s,t}^P(v_2)) \leq C_{tp} d(v_1, v_2).$$

To get the direct implication, we take $U(v) = v_1$, $\hat{U}(v) = v_2$ and observe that $d_*(U, \hat{U}) = \sup_{v \in V} \frac{d(v_1, v_2)}{1 + d(v_0, v)} = d(v_1, v_2)$. The other implication is clear from (B.2). Besides, we have

$$\exists \tilde{C}_{\text{sew}} : V \to \mathbb{R}_+, \frac{d(\Theta_{s,t}, \Theta_{s,u}(v))}{1 + d(v_0, v)} \leq \tilde{C}_{\text{sew}}(v)(t - s)^\beta$$

and observe that

$$\sup_{v \in V} \frac{\tilde{C}_{\text{sew}}(v)}{1 + d(v_0, v)} < \infty,$$

(B.4) and $\tilde{D} := \sup_{0 \leq s \leq T, U \in \mathcal{E}_0^q} d_*(U, Id) < \infty \implies \text{(H2)}$.

In fact, we then have by (2.2)

$$d_*(\Theta_{s,t} U, \Theta_{u,t}\Theta_{s,u} U) \leq C_D \sup_{v \in V} \frac{\tilde{C}_{\text{sew}}(v)}{1 + d(v_0, v)}(t - s)^\beta,$$

which gives (H2) with $C_{\text{sew}} = C_D \sup_{v \in V} \frac{\tilde{C}_{\text{sew}}(v)}{1 + d(v_0, v)}$.

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