Huygens’ Principle for the Klein-Gordon equation in the de Sitter spacetime

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Abstract

In this article we prove that the Klein-Gordon equation in the de Sitter spacetime obeys the Huygens’ principle only if the physical mass $m$ of the scalar field and the dimension $n \geq 2$ of the spatial variable are tied by the equation $m^2 = (n^2 - 1)/4$. Moreover, we define the incomplete Huygens’ principle, which is the Huygens’ principle restricted to the vanishing second initial datum, and then reveal that the massless scalar field in the de Sitter spacetime obeys the incomplete Huygens’ principle and does not obey the Huygens’ principle, for the dimensions $n = 1, 3$, only. Thus, in the de Sitter spacetime the existence of two different scalar fields (in fact, with $m = 0$ and $m^2 = (n^2 - 1)/4$), which obey incomplete Huygens’ principle, is equivalent to the condition $n = 3$ (in fact, the spatial dimension of the physical world). For $n = 3$ these two values of the mass are the endpoints of the so-called in quantum field theory the Higuchi bound. The value $m^2 = (n^2 - 1)/4$ of the physical mass allows us also to obtain complete asymptotic expansion of the solution for the large time.

Keywords: Huygens’ Principle; Klein-Gordon Equation; de Sitter spacetime; Higuchi Bound

1 Introduction and Statement of Results

In this article we prove that the Klein-Gordon equation in the de Sitter spacetime obeys the Huygens’ principle only if the physical mass $m$ of the scalar field and the dimension $n \geq 2$ of the spatial variable are tied by the equation $m^2 = (n^2 - 1)/4$. Moreover, we define the incomplete Huygens’ principle, which is the Huygens’ principle restricted to the vanishing second initial datum, and then reveal that the massless scalar field in the de Sitter spacetime obeys the incomplete Huygens’ principle and does not obey the Huygens’ principle, for the dimensions $n = 1, 3$, only.

The Klein-Gordon equation arising in relativistic physics and, in particular, general relativity and cosmology, as well as, in more recent quantum field theories, is a covariant equation that is considered in the curved pseudo-Riemannian manifolds. (See, e.g., Birrell and Davies [7], Parker and Toms [25], Weinberg [30].) Moreover, the latest astronomical observational discovery that the expansion of the universe is speeding supports the model of the expanding universe that is mathematically described by the manifold with metric tensor depending on time and spatial variables. In this paper we restrict ourselves to the manifold arising in the
The so-called de Sitter model of the universe, which is the curved manifold due to the cosmological constant.

The line element in the de Sitter spacetime has the form

\[ ds^2 = -\left(1 - \frac{r^2}{R^2}\right)c^2 dt^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \]

The Lamaitre-Robertson transformation \( r' = \frac{r}{\sqrt{1 - r^2/R^2}} e^{-ct/R}, \ t' = t + \frac{R}{2c} \ln \left(1 - \frac{r^2}{R^2}\right), \ \theta' = \theta, \ \phi' = \phi, \) leads to the following form for the line element [24 Sec.134], [28 Sec.142]:

\[ ds^2 = -c^2 dt'^2 + e^{2ct'/R} (dr'^2 + r'^2 d\theta'^2 + r'^2 \sin^2 \theta' d\phi'^2). \]

Finally, defining new space coordinates \( x', y', z', t' \) connected with \( r', \theta', \phi' \) by the usual equations connecting Cartesian coordinates and polar coordinates in a Euclidean space, [1] may be written [24 Sec.134]

\[ ds^2 = -c^2 dt'^2 + e^{2ct'/R} (dx'^2 + dy'^2 + dz'^2). \]

The new coordinates \( x', y', z', t' \) can take all values from \(-\infty \) to \( \infty \). Here \( R \) is the “radius" of the universe. In fact, the de Sitter model belongs to the family of the Friedmann-Lemaître-Robertson-Walker spacetimes (FLRW spacetimes). In the FLRW spacetime [19], one can choose coordinates so that the metric has the form \( ds^2 = -dt^2 + a^2(t) d\sigma^2 \).

The homogeneous and isotropic cosmological models possess the highest degree of symmetry that makes them more amenable to rigorous study. Among them we mention FLRW models. The simplest class of cosmological models can be obtained if we assume, additionally, that the metric of the slices of constant time is flat and that the spacetime metric can be written in the form

\[ ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \]

with an appropriate scale factor \( a(t) \). The assumption that the universe is expanding leads to the positivity of the time derivative \( \frac{da}{dt} a(t) \). A further assumption that the universe obeys the accelerated expansion suggests that the second derivative \( \frac{d^2}{dt^2} a(t) \) is positive. Under the assumption of FLRW symmetry the equation of motion in the case of positive cosmological constant \( \Lambda \) leads to the solution \( a(t) = a(0) e^{t\sqrt{\Lambda}/3} \), which produces models with exponentially accelerated expansion, which is referred to as the de Sitter model.

In quantum field theory the matter fields are described by the function \( \phi \) must satisfy equations of motion. In the case of the massive scalar field, the equation of motion is the Klein-Gordon equation generated by the metric \( g \):

\[ \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left( \sqrt{|g|} g^{jk} \frac{\partial \phi}{\partial x^j} \right) = m^2 \phi + V'(\phi). \]

In physical terms this equation describes a local self-interaction for a scalar particle. In the de Sitter universe the equation for the scalar field with mass \( m \) and potential function \( V \) written out explicitly in coordinates is

\[ \phi_{tt} + nH \phi_t - e^{-2Ht} \Delta \phi + m^2 \phi = -V'(\phi). \]

Here \( x \in \mathbb{R}^n, t \in \mathbb{R}, \) and \( \Delta \) is the Laplace operator on the flat metric, \( \Delta := \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} \), while \( H = \sqrt{\Lambda/3} \) is the Hubble constant. For the sake of simplicity, henceforth, we set \( H = 1 \). A typical example of a potential function would be \( V(\phi) = \phi^4 \).

For the solution \( \Phi \) of the Cauchy problem for the linear Klein-Gordon equation

\[ \Phi_{tt} + n\Phi_t - e^{-2Ht} \Delta \Phi + m^2 \Phi = 0, \quad \Phi(x,0) = \varphi_0(x), \quad \Phi_t(x,0) = \varphi_1(x), \]

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the following formula is obtained in [31]:

\[
\Phi(x, t) = e^{-\frac{n-1}{2}t} \varphi_0(x, \phi(t)) + e^{-\frac{n+4}{2}t} \int_0^1 \varphi_0(x, \phi(t)s) (2K_0(\phi(t)s, t) + nK_1(\phi(t)s, t)) \phi(t) \, ds \\
+ 2e^{-\frac{n}{2}t} \int_0^1 \varphi_1(x, \phi(t)s) K_1(\phi(t)s, t) \phi(t) \, ds, \quad x \in \mathbb{R}^n, \ t > 0,
\]

provided that the mass \( m \) is large, that is, \( m^2 \geq n^2/4 \). Here, \( \phi(t) := 1 - e^{-t} \) and for \( x \in \mathbb{R}^n \), the function \( \varphi(x, \phi(t)s) \) coincides with the value \( v(x, \phi(t)s) \) of the solution \( v(x, t) \) of the Cauchy problem

\[
v_{tt} - \Delta v = 0, \quad v(x, 0) = \varphi(x), \quad v_t(x, 0) = 0.
\]

To define the kernels \( K_0(z, t) \) and \( K_1(z, t) \) we introduce the following notations. First, we define a chronological future \( D_+(x_0, t_0) \) of the point (event) \((x_0, t_0)\), \( x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R} \), and a chronological past \( D_-(x_0, t_0) \) of the point (event) \((x_0, t_0)\), \( x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R} \), as follows

\[
D_{\pm}(x_0, t_0) := \left\{ (x, t) \in \mathbb{R}^{n+1}; |x - x_0| \leq \pm (e^{-t_0} - e^{-t}) \right\}.
\]

In fact, any intersection of \( D_-(x_0, t_0) \) with the hyperplane \( t = \text{const} < t_0 \) determines the so-called dependence domain for the point \((x_0, t_0)\), while the intersection of \( D_+(x_0, t_0) \) with the hyperplane \( t = \text{const} > t_0 \) is the so-called domain of influence of the point \((x_0, t_0)\). We define also the characteristic conoid (ray cone) by

\[
C_{\pm}(x_0, t_0) := \left\{ (x, t) \in \mathbb{R}^{n+1}; |x - x_0| = \pm (e^{-t_0} - e^{-t}) \right\}.
\]

Thus, the characteristic conoid \( C_+(x_0, t_0) \) (\( C_-(x_0, t_0) \)) is the surface of the chronological future \( D_+(x_0, t_0) \) (chronological past \( D_+(x_0, t_0) \)) of the point \((x_0, t_0)\).

Then, we define for \((x_0, t_0) \in \mathbb{R}^n \times \mathbb{R} \) the function

\[
E(x, t; x_0, t_0) = (4e^{-t_0-t})^{iM} \left( (e^{-t} + e^{-t_0})^2 - (x - x_0)^2 \right)^{-\frac{1}{2} - iM} F\left( \frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(e^{-t_0} - e^{-t})^2 - (x - x_0)^2}{(e^{-t_0} + e^{-t})^2 - (x - x_0)^2} \right)
\]

in \( D_+(x_0, t_0) \cup D_-(x_0, t_0) \), where \( F(a, b; c; \zeta) \) is the hypergeometric function. (For the definition of \( F(a, b; c; \zeta) \) see, e.g., [3].) Here the notation \( x^2 = x \cdot x = |x|^2 \) for \( x \in \mathbb{R}^n \) has been used. The kernels \( K_0(z, t) \) and \( K_1(z, t) \) are defined by

\[
K_0(z, t) := - \left[ \frac{\partial}{\partial b} E(z, t; 0, b) \right]_{b=0}
\]

\[
= (4e^{-t})^{iM} \left( (1 + e^{-2t})^2 - z^2 \right)^{-iM} \frac{1}{[(1 - e^{-t})^2 - z^2]\sqrt{(1 + e^{-t})^2 - z^2}} \\
\times \left[ (e^{-t} - 1 - iM(e^{-2t} - 1 - z^2)) F\left( \frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(1 - e^{-t})^2 - z^2}{(1 + e^{-t})^2 - z^2} \right) \\
+ (1 - e^{-2t} + z^2) \left( \frac{1}{2} - iM \right) F\left( - \frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(1 - e^{-t})^2 - z^2}{(1 + e^{-t})^2 - z^2} \right) \right]
\]

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and \( K_1(z, t) := E(z, t; 0, 0) \), that is, 
\[
K_1(z, t) = (4e^{-t})^{iM}(1 + e^{-t})^2 - z^2)^{-\frac{1}{2} - iM} F \left( \frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(1 - e^{-t})^2 - z^2}{(1 + e^{-t})^2 - z^2} \right), \quad 0 \leq z \leq 1 - e^{-t},
\] respectively. Here \( M = \sqrt{m^2 - n^2/4} \). The main properties of \( K_0(z, t) \) and \( K_1(z, t) \) are listed and proved in Section 3 [31].

For the case of small mass, \( m^2 \leq n^2/4 \), the similar formula is obtained in [32]. More precisely, if we denote \( M = \sqrt{n^2 - m^2} \), then for the solution \( \Phi \) of the Cauchy problem [33], there is a representation
\[
\Phi(x, t) = e^{-\frac{n-1}{2}t} \varphi_0(x, \phi(t)) + \sum_{i=1}^{n-1} 2e^{-\frac{n}{2}t} \int_0^1 \varphi_0(x, \phi(t)s) (2K_0(\phi(t)s, t; M) + nK_1(\phi(t)s, t; M)) \phi(t) ds,
\]
Here we have used the new functions \( E(x, t; x_0, t_0; M) \), \( K_0(z, t; M) \), and \( K_1(z, t; M) \), which can be obtained by the analytic continuation of the functions \( E(x, t; x_0, t_0) \), \( K_0(z, t) \), and \( K_1(z, t) \), respectively, to the complex domain. First we define the function
\[
E(x, t; x_0, t_0; M) = 4^{-M} e^{M(t_0+t)} \left( (e^{-t} + e^{-t_0})^2 - (x - x_0)^2 \right)^{-\frac{1}{2} + M} F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-t_0} - e^{-t})^2 - (x - x_0)^2}{(e^{-t_0} + e^{-t})^2 - (x - x_0)^2} \right).
\]
Hence, it is related to the function \( E(x, t; x_0, t_0) \) of [3] as follows:
\[
E(x, t; x_0, t_0) = E(x, t; x_0, t_0; -iM).
\]
Next we define also new kernels \( K_0(z, t; M) \) and \( K_1(z, t; M) \) by
\[
K_0(z, t; M) := -\left[ \frac{\partial}{\partial b} E(z, t; 0, b; M) \right]_{b=0} 
= 4^{-M} e^{iM} \left( (1 + e^{-t})^2 - z^2 \right)^M \frac{1}{(1 - e^{-t})^2 - z^2} \sqrt{(1 + e^{-t})^2 - z^2}
\times \left[ (e^{-t} - 1 + M(e^{-2t} - 1 - z^2)) F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(1 - e^{-t})^2 - z^2}{(1 + e^{-t})^2 - z^2} \right) \right]
\]
and \( K_1(z, t; M) := E(z, t; 0, 0; M) \), that is,
\[
K_1(z, t; M) = 4^{-M} e^{Mt} ((1 + e^{-t})^2 - z^2)^{-\frac{1}{2} + M} F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(1 - e^{-t})^2 - z^2}{(1 + e^{-t})^2 - z^2} \right), \quad 0 \leq z \leq 1 - e^{-t},
\]
respectively. In fact, \(E(x, t;x_0, t_0;M)\) coincides with \(E(x, t;x_0, t_0)\) if we replace \(M\) with \(iM\), that is, it is an analytic continuation of the function \(E(x, t;x_0, t_0)\) to the complex plane \(M \in \mathbb{C}\). The same statement is true for the functions \(K_0(z, t; M)\) and \(K_1(z, t; M)\).

The expressions (4) and (9) can be regarded as the integral transforms applied to the solution of (5). (See for details [33].) According to [33], the fundamental solutions (the leading and advanced Green functions) of the operator have the similar representations.

Suppose now that we are looking for the simplest possible kernels \(K_0(z, t; M)\) and \(K_1(z, t; M)\) of the integral transforms. Surprisingly that perspective shades a light on the quantum field theory in the de Sitter universe and reveals a new unexpected link between the Higuchi bound [20] and the Huygens’ principle.

Indeed, in the hierarchy of the hypergeometric functions the simplest one is the constant, \(F(0, 0; 1; \zeta) = 1\). The parameter \(M\) leading to such function \(F(0, 0; 1; \zeta) = 1 \) is \(M = \frac{1}{2}\), and, consequently, \(m^2 = \frac{3}{4}\).

The next simple function of that hierarchy is a linear function. That function \(F(a, b; 1; \zeta)\) has the parameters \(a = b = -1\) and coincides with the polynomial \(1 + \zeta\). The parameter \(M\) leading to such function \(F(-1, -1; 1; \zeta) = 1 + \zeta\) is \(M = \frac{3}{2}\), and, consequently, \(m^2 = \frac{5}{4}\).

In the case of \(n = 3\) the only real masses, which simplify the kernels, that is, make \(F\) polynomial, are given by \(M = \frac{1}{2}\) and \(M = \frac{3}{2}\). For the square of the physical mass \(m^2\) they are \(m^2 = 2\) and \(m = 0\), respectively. These are exactly the endpoints of the interval (0, 2) that, in the case of \(n = 3\), is known as the so-called Higuchi bound [20]. In the physical variables it is the interval \((0, 2\Lambda/3)\).

It turns out that the interval \((0, \sqrt{2})\) plays significant role in the linear quantum field theory [20], in completely different context than the explicit representation of the solutions of the Cauchy problem. More precisely, the Higuchi bound [20], [10], [3], [8], [11] arises in the quantization of free massive fields with the spin-2 in the de Sitter spacetime with \(n = 3\). It is the forbidden mass range for spin-2 field theory in de Sitter spacetime because of the appearance of negative norm states. Thus, the point \(m = \sqrt{2}\) is exceptional for the spin-2 field theory in de Sitter spacetime. In particular, for massive spin-2 fields, it is known [10], [20] that the norm of the helicity zero mode changes sign across the line \(m^2 = 2\). The region \(m^2 < 2\) is therefore unitarily forbidden. It is noted in [2] that all canonically normalized helicity –0, ±1, ±2 modes of massive graviton on the de Sitter universe satisfy Klein-Gordon equation for a massive scalar field with the same effective mass. Then, it is known (see, e.g., [9]) that, if \(m^2 = 2\), the action is invariant under the gauge transformation, and that invariance already suggests that there exists some discontinuity in the theory at \(m^2 = 2\).

In the case of \(n \in \mathbb{N}\) we obtain for the physical mass several points, \(m^2 = \frac{n^2}{4} - \left(\frac{1}{2} + k\right)^2\), \(k = 0, 1, \ldots, \left\lfloor \frac{n-1}{2}\right\rfloor\), which make \(F\) polynomial. We will call these points knot points for the mass of the equation. For \(n = 1\) only the massless field \(m = 0\) has knot point.

The explicit representation formulas allows us to prove in Section 4 that the largest knot point, and, in particular, the right endpoint of the Higuchi bound if \(n = 3\), is the only value of the mass of the particle which produces scalar field that obeys the Huygens’ principle. Recall (see, e.g., [18]) that a hyperbolic equation is said to satisfy Huygens’ principle if the solution vanishes at all points which cannot be reached from the support of initial data by a null geodesic, that is, there is no tail. The tails are important within cosmological context. (See, e.g., [12], [16], [13] and references therein.)

An exemplar equation satisfying Huygens’ principle is the wave equation in \(n + 1\) dimensional Minkowski spacetime for odd \(n \geq 3\). According to Hadamard’s conjecture (see, e.g.,
This is the only (modulo transformations of coordinates and unknown function) Huygensian linear second-order hyperbolic equation. There exists an extensive literature on the Huygens’ principle in the 4-dimensional spacetime of constant curvature (see e.g. [13], [27] and references therein).

In the present article we have a new proof of the following theorem.

**Theorem 1** The value \( m = \sqrt{n^2 - 1/2} \) is the only value of the physical mass \( m \), such that the solutions of the equation

\[
\Phi_{tt} + n\Phi_t - e^{-2t} \Delta \Phi + m^2 \Phi = 0,
\]

(13)

obey the Huygens’ principle, whenever the wave equation in the Minkowski spacetime does, that is, \( n \geq 3 \) is an odd number.

Even if the equation is not Huygensian (not tail-free for some admissible data), one might nevertheless be interested in data that produce tail-free solution. Such data are prescribed in the following definition which is hinted by the string equation.

**Definition 2** We say that the equation obeys the incomplete Huygens’ principle with respect to the first initial datum, if the solution with the second datum \( \varphi_1 = 0 \) vanishes at all points which cannot be reached from the support of initial data by a null geodesic.

If the equation obeys the Huygens’ principle, then it obeys also the incomplete Huygens’ principle with respect to the first initial datum. However, the equation in the de Sitter spacetime shows that the converse is not true.

**Theorem 3** Suppose that equation (13) does not obey the Huygens’ principle. Then, it obeys the incomplete Huygens’ principle with respect to the first initial datum, if and only if the equation is massless, \( m = 0 \), and either \( n = 1 \) or \( n = 3 \).

We have to point out that for the classical string equation, the Huygens’ principle is not valid, but the D’Alembert formula shows that incomplete Huygens’ principle with respect to the first initial datum is fulfilled. By combining Theorem 1 and Theorem 3 we arrive at the following interesting conclusion.

**Corollary 4** Assume that the equations \( \Phi_{tt} + n\Phi_t - c_1^2 e^{-2t} \Delta \Phi + m_1^2 \Phi = 0 \) and \( \Phi_{tt} + n\Phi_t - c_2^2 e^{-2t} \Delta \Phi + m_2^2 \Phi = 0 \), where \( c_1, c_2 \) are positive numbers, obey the incomplete Huygens’ principle. Then they describe the fields with different mass, \( m_1 \neq m_2 \), (in fact, \( \sqrt{n^2 - 1/2} \) and 0) if and only if the dimension \( n \) of the spatial variable \( x \) is 3.

Thus, in the de Sitter spacetime the existence of two different scalar fields (in fact, with \( m = 0 \) and \( m^2 = (n^2 - 1)/4 \)), which obey incomplete Huygens’ principle, is equivalent to the condition \( n = 3 \). The dimension \( n = 3 \) of the last corollary agrees with the experimental data.

This paper is organized as follows. In Section 2 we define the incomplete Huygens’ principle. Then, in Theorem 3, we give description of the class of equations which obey that principle. The proofs of Theorem 1 and Theorem 3 are given in Section 4. For the value \( m = \sqrt{n^2 - 1/2} \) of the physical mass \( m \), the representation formula allows us also to derive a complete asymptotic expansion of the solution for the large time; that is done in Subsection 4.1.
2 The left knot point

The equation (13) is strictly hyperbolic. That implies the well-posedness of the Cauchy problem for equation of (13) in the various function spaces. The coefficient of the equation is an analytic function and, consequently, the Holmgren’s theorem implies local uniqueness in the space of distributions. Moreover, the speed of propagation is finite, namely, it is equal to $e^{-t}$ for every $t \in \mathbb{R}$. The second-order strictly hyperbolic equation (13) possesses two fundamental solutions resolving the Cauchy problem. They can be written microlocally in terms of the Fourier integral operators [21], which give a complete description of the wave front sets of the solutions. The distance between two characteristic roots $\lambda_1(t, \xi)$ and $\lambda_2(t, \xi)$ of the equation (13) is $|\lambda_1(t, \xi) - \lambda_2(t, \xi)| = e^{-t} |\xi|$, $t \in \mathbb{R}$, $\xi \in \mathbb{R}^n$. It tends to zero as $t$ approaches $\infty$. Thus, the operator is not uniformly strictly hyperbolic. Moreover, this equation possesses the so-called horizon. More precisely, any signal emitted from the spatial point $x_0 \in \mathbb{R}^n$ at time $t_0 \in \mathbb{R}$ remains inside the ball $|x - x_0| < e^{-t_0}$ for all time $t \in (t_0, \infty)$.

By means of the representation theorems for the solution of Cauchy’s problem we obtain in Section 4 a necessary and sufficient condition for the validity of Huygens’ principle for the field equation (13). Huygens’ principle plays an important role also in quantum field theory in the curved spacetime. According to [23] the support of the commutator-or the anticommutator-distribution, respectively, lies on the null-cone if and only if Huygens’ principle holds for the corresponding wave equation.

Huygens’ principle (or, more precisely, its “minor premise” due to Hadamard) states that the support of the fundamental solution of a given hyperbolic equation belongs to the surface of the characteristic conoid. In other words, the field equations (13) satisfy Huygens’ principle if and only if the solutions have no tail. Such domains in physical spacetime wherein the fundamental distribution solution vanishes identically are referred to as lacunas of hyperbolic operators [3]. For the equation (13) the complementary set of the characteristic conoid consists of two open connected components. The fact that the exterior component is a lacuna proves the finiteness of the wave propagation velocity. On the other hand, the existence of an inner lacuna, i.e. one that contains time-like curves in the spacetime, is a very specific property which is intrinsic for a quite exceptional class of hyperbolic operators [14, 17, 18].

Consider now the knot points for the physical mass, $m^2 = \frac{n^2}{4} - (\frac{1}{2} + k)^2$, $k = 0, 1, \ldots, \left[\frac{n-1}{2}\right]$. For $n = 1$ only the massless field $m = 0$ has knot point, while for $n = 3$ there are two knot points. The knot points are linked to the Huygens’ principle via intrinsic properties of the hypergeometric function. In fact, there are some polynomials in the hierarchy of the hypergeometric functions $F(a, b; c; \zeta)$. In particular, if $k \in \mathbb{N}$, then

$$F(-k, -k; 1; z) = \sum_{l=0}^{k} \left( \frac{k(k-1) \cdots (k+1-l)}{k!} \right)^2 z^l.$$ 

For the corresponding $M$ we obtain $M = k + \frac{1}{2}$, $k = 0, 1, \ldots, \left[\frac{n-1}{2}\right]$. If $n$ is odd and $k = \frac{n-1}{2}$, then we have $M = \frac{n}{2}$. Furthermore, for $M = \frac{n}{2}$ after simplifications, we obtain

$$E \left( x, t; x_0, t_0; \frac{3}{2} \right) = \frac{1}{8} e^{\frac{3}{2}(t_0+t)} \left( e^{-t} + e^{-t_0} \right)^2 (x - x_0)^2 \left( 1, -1, 1; \frac{(e^{-t_0} - e^{-t})^2 - (x-x_0)^2}{(e^{-t_0} + e^{-t})^2 - (x-x_0)^2} \right)$$

$$= \frac{1}{4} e^{\frac{3}{2}(t_0+t)} \left( e^{-2t_0} + e^{-2t} - (x-x_0)^2 \right)$$
while

\[ K_0 \left( z, t; \frac{3}{2} \right) = - \left[ \frac{\partial}{\partial b} E(z, t; 0, b; M) \right]_{b=0} = \frac{1}{8} e^{\frac{3}{2}t} \left[ 3(z^2 - e^{-2t}) + 1 \right] \]

and

\[ K_1 \left( z, t; \frac{3}{2} \right) = \frac{1}{4} e^{\frac{3}{2}t} (1 + e^{-2t} - z^2). \]

For \( M = \frac{3}{2} \), from (9) we derive the following representation for the solution

\[
\Phi(x, t) = e^{-\frac{n-1}{2}t} v_{\varphi_0}(x, \phi(t)) + \frac{1}{4} e^{-\frac{n-1}{2}t} \int_0^1 v_{\varphi_0}(x, \phi(t)s) \times \left( 3((\phi(t)s)^2 - e^{-2t}) + 1 + n (1 + e^{-2t} - (\phi(t)s)^2) \right) \phi(t) ds + \frac{1}{2} e^{-\frac{n-1}{2}t} \int_0^{\phi(t)} v_{\varphi_1}(x, s) (1 + e^{-2t} - s^2) ds, \quad x \in \mathbb{R}^n, \quad t > 0.
\]

It can be rewritten as follows

\[
\Phi(x, t) = e^{-t} v_{\varphi_0}(x, \phi(t)) + \int_0^{\phi(t)} v_{\varphi_0}(x, s) ds + \frac{1}{2} \int_0^{\phi(t)} v_{\varphi_1}(x, s) (1 + e^{-2t} - s^2) ds, \quad x \in \mathbb{R}^n, \quad t > 0.
\]

In particular, for \( n = 3 \), consequently \( m = 0 \), we obtain

\[
\Phi(x, t) = e^{-t} v_{\varphi_0}(x, \phi(t)) + \int_0^{\phi(t)} v_{\varphi_0}(x, s) ds + \frac{1}{2} \int_0^{\phi(t)} v_{\varphi_1}(x, s) (1 + e^{-2t} - s^2) ds, \quad x \in \mathbb{R}^n, \quad t > 0.
\]

Now, if we denote \( V_\varphi \) the solution of the problem

\[
V_{tt} - \Delta V = 0, \quad V(x, 0) = 0, \quad V_t(x, 0) = \varphi(x),
\]

then \( v_\varphi(x, t) = \frac{\partial}{\partial t} V_\varphi(x, t) \), and

\[
\Phi(x, t) = e^{-t} v_{\varphi_0}(x, \phi(t)) + V_{\varphi_0}(x, \phi(t)) + \frac{1}{2} (1 + e^{-2t}) V_{\varphi_1}(x, \phi(t)) - \frac{1}{2} \int_0^{\phi(t)} v_{\varphi_1}(x, s)s^2 ds.
\]

Hence,

\[
\Phi(x, t) = e^{-t} v_{\varphi_0}(x, \phi(t)) + V_{\varphi_0}(x, \phi(t)) + \frac{1}{2} (1 + e^{-2t}) V_{\varphi_1}(x, \phi(t)) - \frac{1}{2} V_{\varphi_1}(x, \phi(t)) \phi^2(t) + \int_0^{\phi(t)} V_{\varphi_1}(x, s) s ds
\]
implies
\[ \Phi(x, t) = e^{-t} v \varphi_0(x, \phi(t)) + V \varphi_0(x, \phi(t)) + e^{-t} V \varphi_1(x, \phi(t)) + \int_0^{\phi(t)} V \varphi_1(x, s) s \, ds. \]

Thus, the sufficiency part of Theorem 3 in the case of \( n = 3 \) is proven.

Consider now the case of \( n = 1 \) and \( M = \sqrt{\frac{1}{4} - m^2} \). There is only one knot point for such \( n \) and \( M \). Then we set \( M = \frac{1}{2} \), consequently \( m = 0 \), and obtain
\[
\Phi(x, t) = v \varphi_0(x, \phi(t)) + e^{-\frac{1}{2} t} \int_0^1 v \varphi_0(x, \phi(t) s) \left( 2K_0 \left( \phi(t) s, t; \frac{1}{2} \right) + K_1 \left( \phi(t) s, t; \frac{1}{2} \right) \right) \phi(t) \, ds
+ 2e^{-\frac{1}{2} t} \int_0^1 v \varphi_1(x, \phi(t) s) K_1 \left( \phi(t) s, t; \frac{1}{2} \right) \phi(t) \, ds.
\]

That is, the solution for the massless equation is given as follows
\[
\Phi(x, t) = \frac{1}{2} (\varphi_0(x - \phi(t)) + \varphi_0(x + \phi(t))) + \frac{1}{2} \int_0^{\phi(t)} (\varphi_1(x - s) + \varphi_1(x + s)) \, ds.
\]

It also satisfies the incomplete Huygens’ principle. The sufficiency part of Theorem 3 is proven.

### 3 Equation with the source term

Consider the linear part of the scalar equation
\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - e^{-2t} \Delta u - M^2 u &= -e^\frac{2t}{2} V'(e^{-\frac{2t}{2}} u),
\end{align*}
\]
with \( M \geq 0 \). The equation \([15]\) includes two important cases. The first one is the Higgs boson equation, which has \( V'(\phi) = \lambda \phi^3 \) and \( M^2 = \mu m^2 + n^2/4 \) with \( \lambda > 0 \) and \( \mu > 0 \), while \( n = 3 \). The second case is for the small physical mass, that is \( 0 \leq m \leq \frac{n}{2} \). For the last range of the mass we have \( M^2 = \frac{n^2}{4} - m^2 \).

To prove the existence of the local and global solutions of the Cauchy problem for the equation \([15]\) the useful tools are the representation formula for the solution of the linear equation with the source term and some decay estimates for the norms of solution. We provide now the first one to complete the list of the representation formulas. The solution \( u = u(x, t) \) to the Cauchy problem
\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - e^{-2t} \Delta u - M^2 u &= f, \quad u(x, 0) = 0, \quad u_t(x, 0) = 0,
\end{align*}
\]
with \( f \in C^\infty(\mathbb{R}^{n+1}) \) and with vanishing initial data is given in \([32]\) by the next expression
\[
\begin{align*}
u(x, t) &= 2 \int_0^t db \int_0^{e^{-b} - e^{-t}} dr v(x, r; b) E(r, t; 0, b; M),
\end{align*}
\]
where the function \( v(x, t; b) \) is a solution to the Cauchy problem for the wave equation:
\[
\begin{align*}
\frac{\partial^2 v}{\partial t^2} - \Delta v &= 0, \quad v(x, 0; b) = f(x, b), \quad v_t(x, 0; b) = 0.
\end{align*}
\]
The solution $u = u(x,t)$ to the Cauchy problem

$$u_{tt} - e^{-2t} \Delta u - M^2 u = 0, \quad u(x,0) = \varphi_0(x), \quad u_t(x,0) = \varphi_1(x),$$

with $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$, $n \geq 2$, can be represented (see [32]) as follows:

$$u(x,t) = e^{\frac{t}{2}} v_{\varphi_0}(x, \phi(t)) + 2 \int_0^1 v_{\varphi_0}(x, \phi(t)s) K_0(\phi(t)s, t; M) \phi(t) \, ds$$

$$+ 2 \int_0^1 v_{\varphi_1}(x, \phi(t)s) K_1(\phi(t)s, t; M) \phi(t) \, ds, \quad x \in \mathbb{R}^n, \ t > 0,$$

where $\phi(t) := 1 - e^{-t}$. Here, for $\varphi \in C_0^\infty(\mathbb{R}^n)$ and for $x \in \mathbb{R}^n$, the function $v_{\varphi}(x, \phi(t)s)$ coincides with the value $v(x, \phi(t))s$ of the solution $v(x,t)$ of the Cauchy problem (1).

Thus, for the solution $\Phi$ of the the Cauchy problem

$$\Phi_{tt} + n\Phi_t - e^{-2t} \Delta \Phi + m^2 \Phi = f, \quad \Phi(x,0) = 0, \quad \Phi_t(x,0) = 0,$$

(18)

due to the relation $u = e^{\frac{x}{2}} \Phi$, we obtain with $f \in C^\infty(\mathbb{R}^{n+1})$ and with vanishing initial data the next expression

$$\Phi(x,t) = 2e^{-\frac{a}{2}t} \int_0^t \int_0^t \int_0^s e^{\frac{r}{2} - e^{-t}} \, dr \, e^{\frac{r}{2}b} v(x,r;b) E(r,t;0,b;M),$$

(19)

where the function $v(x,t;b)$ is a solution to the Cauchy problem for the wave equation (17).

In fact, the representation formulas of this section have been used in [34] to establish sign-changing properties of the global in time solutions of the Higgs boson equation.

### 4 The right knot point. Proof of theorems

Here we set $M = 1/2$, that is, $m^2 = (n^2 - 1)/4$, which simplifies the hypergeometric functions, as well as, the kernels $K_0(z,t;M)$ and $K_1(z,t;M)$. In that case we have

$$E\left( x, t; x_0, t_0; \frac{1}{2} \right) = \frac{1}{2} e^{\frac{t}{2}(t_0 + t)}, \quad E\left( z, t; 0, b; \frac{1}{2} \right) = \frac{1}{2} e^{\frac{t}{2}(b+t)},$$

while

$$K_0\left( z, t; \frac{1}{2} \right) = -\frac{1}{4} e^{\frac{t}{2}}, \quad K_1\left( z, t; \frac{1}{2} \right) = \frac{1}{2} e^{\frac{t}{2}}.$$

For the solution (19) of the problem (18) with the source term it follows

$$\Phi(x,t) = e^{-\frac{n-1}{2}t} \int_0^t e^{\frac{n+1}{2}b} \int_0^{e^{-b} - e^{-t}} v(x,r;b) \, dr,$$

where the function $v(x,r;b)$ is defined by (17). In order to get rid of one integration in the last formula, we denote $V_t(x,t;b)$ the solution of the problem

$$V_{tt} - \Delta V = 0, \quad V(x,0) = 0, \quad V_t(x,0) = f(x,b),$$

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then
\[ v(x, t; b) = \frac{\partial}{\partial t} V_f(x, t; b). \]

Hence,
\[
\Phi(x, t) = e^{-\frac{n-1}{2}t} \int_0^t e^{\frac{n+1}{2}b} V_f(x, e^{-b} - e^{-t}; b) \, db.
\]

Further, due to (9) we have for the solution \( \Phi \) of the equation without source term the following representation
\[
\Phi(x, t) = e^{-\frac{n-1}{2}t} v_{\varphi_0}(x, 1 - e^{-t}) + \frac{n-1}{2} e^{-\frac{n-1}{2}t} \int_0^{1-e^{-t}} v_{\varphi_1}(x, s) \, ds
\]
\[ + e^{-\frac{n-1}{2}t} v_{\varphi_2}(x, s) \, ds, \quad x \in \mathbb{R}^n, \quad t > 0, \]

where the functions \( v_{\varphi_0} \) and \( v_{\varphi_1} \) are defined by (5). Now, if we denote \( V_\varphi \) the solution of the problem (14), then \( v_\varphi(x, t) = \frac{\partial}{\partial t} V_\varphi(x, t) \), and
\[
\Phi(x, t) = e^{-\frac{n-1}{2}t} v_{\varphi_0}(x, 1 - e^{-t}) + \frac{n-1}{2} e^{-\frac{n-1}{2}t} V_\varphi(x, 1 - e^{-t})
\]
\[ + e^{-\frac{n-1}{2}t} V_\varphi(x, 1 - e^{-t}), \quad x \in \mathbb{R}^n, \quad t > 0, \]

or, equivalently,
\[
\Phi(x, t) = e^{-\frac{n-1}{2}t} \left( \frac{\partial V_{\varphi_0}}{\partial t} \right)(x, 1 - e^{-t}) + \frac{n-1}{2} e^{-\frac{n-1}{2}t} V_{\varphi_0}(x, 1 - e^{-t})
\]
\[ + e^{-\frac{n-1}{2}t} V_{\varphi_1}(x, 1 - e^{-t}), \quad x \in \mathbb{R}^n, \quad t > 0. \]

Thus, we have proven the sufficiency part of Theorem 1!

Although the representation formulas make the proof of the necessity part very clear and straightforward, we provide details of the proof in order to reveal the path that connects the Huygens’ principle with the values of mass \( m \) and the dimension \( n \). We consider the case of small mass, \( m \leq n/2 \), since the relation between \( E(x, t; x_0, t_0; M) \) and \( E(x, t; x_0, t_0) \) (analytic continuation) shows the way how it can be proved that, for the large mass \( m > n/2 \) the Huygens’ principle is not valid.

In order to prove the necessity of the conditions \( m = \sqrt{n^2 - 1}/2 \) and \( n \) is odd, we set \( M \neq \frac{1}{2}, \varphi_0 = 0 \) and consider the solution (9) of the Cauchy problem with the radial initial datum \( \varphi_1 = \varphi_1(r) \), \( \text{supp} \varphi_1 \subset \{ x \in \mathbb{R}^n; |x| \leq 1 - \varepsilon \} \), \( \varepsilon \in (0, 1) \):

\[
\Phi(x, t) = 2e^{-\frac{n-1}{2}t} \int_0^1 v_{\varphi_1}(x, \phi(t)s)K_1(\phi(t)s, t; M)\phi(t) \, ds
\]
\[ = 2e^{-\frac{n-1}{2}t} \int_0^{\phi(t)} \frac{\partial}{\partial s} V_{\varphi_1}(x, s)K_1(s, t; M) \, ds
\]
\[ = 2e^{-\frac{n-1}{2}t} V_{\varphi_1}(x, \phi(t))K_1(s, \phi(t); M) - 2e^{-\frac{n-1}{2}t} \int_0^{\phi(t)} V_{\varphi_1}(x, s)\frac{\partial}{\partial s} K_1(s, t; M) \, ds,
\]

where \( V_{\varphi} \) is the solution of the problem (14) and \( v_{\varphi}(x, t) = \frac{\partial}{\partial t} V_{\varphi}(x, t) \).
According to the well-known formula (see, e.g., [26]), we have initial data by null geodesic. The intersection of the support of \( \varphi_1 \) with the characteristic conoid \( C_-(0, t) \) is empty, and, consequently, the contribution of the integral to the solution is crucial for the validity of Huygens' principle. We consider the value of the solution at the spatial origin \( x = 0 \):

\[
\Phi(0, t) = -2e^{-\frac{t}{2}} \int_0^{\phi(t)} V_{\varphi_1}(0, s) \frac{\partial}{\partial s} K_1(s, t; M) \, ds \quad \text{for large } t.
\]

According to the well-known formula (see, e.g., [26]), we have

\[
V_{\varphi_1}(0, t) = \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{n-3} \frac{t^{n-2}}{\omega_{n-1} c_0^{(n)}} \int_{S^{n-1}} \varphi_1(ty) \, ds_y
\]

\[
= \left( \int_{S^{n-1}} ds_y \right) \frac{1}{\omega_{n-1} c_0^{(n)}} \frac{1}{t} \frac{\partial}{\partial t} \left( \frac{n-3}{2} t^{n-2} \varphi_1(t) \right)
\]

\[
= \frac{1}{c_0^{(n)}} \frac{1}{t} \frac{\partial}{\partial t} \left( \frac{n-3}{2} t^{n-2} \varphi_1(t) \right),
\]

where \( c_0^{(n)} = 1 \cdot 3 \cdot \ldots \cdot (n - 2) \) if \( n \geq 3 \) is odd. Consequently, for large \( t \)

\[
\Phi(0, t) = -2e^{-\frac{t}{2}} \int_0^{\phi(t)} \left[ \frac{1}{c_0^{(n)}} \left( \frac{1}{s} \frac{\partial}{\partial s} \right)^{n-3} s^{n-2} \varphi_1(s) \right] \frac{\partial}{\partial s} K_1(s, t; M) \, ds
\]

\[
= -2 \frac{1}{c_0^{(n)}} e^{-\frac{t}{2}} \int_0^{1-e} \left[ \frac{1}{s} \frac{\partial}{\partial s} \right]^{n-3} s^{n-2} \varphi_1(s) \frac{\partial}{\partial s} K_1(s, t; M) \, ds.
\]

We evaluate the derivative \( \frac{\partial}{\partial s} K_1(s, t; M) \):

\[
4M e^{-Mt} \frac{\partial}{\partial s} K_1(s, t; M)
\]

\[
= 2 \left( \frac{1}{2} - M \right) s \left( (1 - e^{-t})^2 - s^2 \right)^{-\frac{1}{2}} \cdot F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(1 - e^{-t})^2 - s^2}{(1 + e^{-t})^2 - s^2} \right)
\]

\[
- \left( \frac{1}{2} - M \right)^2 \left( (1 - e^{-t})^2 - s^2 \right)^{-\frac{1}{2}} F \left( \frac{3}{2} - M; \frac{3}{2} - M; 2; \frac{(1 - e^{-t})^2 - s^2}{(1 + e^{-t})^2 - s^2} \right)
\]

\[
\times F \left( \frac{3}{2} - M; \frac{3}{2} - M; 2; \frac{(1 - e^{-t})^2 - s^2}{(1 + e^{-t})^2 - s^2} \right)
\]

\[
= - \left( \frac{1}{2} - M \right) \frac{8e^{3t}}{(1 + 2e^t + e^{2t} (1 - s^2))^2}
\]

\[
\times F \left( \frac{3}{2} - M; \frac{3}{2} - M; 2; \frac{(1 - e^{-t})^2 - s^2}{(1 + e^{-t})^2 - s^2} \right)
\]

Then, for the positive \( M \) we have

\[
\lim_{t \to \infty} e^{-t} F \left( \frac{3}{2} - M; \frac{3}{2} - M; 2; \frac{(1 - e^{-t})^2 - s^2}{(1 + e^{-t})^2 - s^2} \right) = 0,
\]

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while for \( M = 0 \) we obtain
\[
\lim_{z \to 1^-} (1 - z) F \left( \frac{3}{2}, \frac{3}{2}; 2; z \right) = \frac{4}{\pi},
\]
and, consequently,
\[
\lim_{t \to \infty} \frac{8e^{3t}}{(1 + 2e^t + e^{2t} (1 - s^2))^2} F \left( \frac{3}{2}, \frac{3}{2}; 2; \frac{(1 - e^{-t})^2 - s^2}{(1 + e^{-t})^2 - s^2} \right) = \frac{8}{\pi (1 - s^2)},
\]
uniformly with respect to \( s \in [0, 1 - \varepsilon] \).

According to Subsection 2.1.3 [5] if \( \Re(c - a - b) > 0 \), then \( F(a, b; c; 1) = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \), where \( \Gamma \) is the gamma-function. For the positive \( M \) such that \( M \neq \frac{1}{2} \), that implies
\[
\lim_{t \to +\infty} F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(1 - e^{-t})^2 - s^2}{(1 + e^{-t})^2 - s^2} \right) = \frac{\Gamma(2M)}{(\Gamma(\frac{1}{2} + M))^2}.
\]
Hence, for \( M > 0 \) it follows (See 15.3.6 of Ch.15 [1] and [5].)
\[
\lim_{t \to +\infty} 4^M e^{-Mt} \frac{\partial}{\partial s} K_1(s, t; M)
= \lim_{t \to +\infty} s \left( (1 - e^{-t})^2 - s^2 \right)^{-\frac{1}{2}+M} \left( \frac{1}{2} - M \right)
\times \left\{ 2 \left( (1 - e^{-t})^2 - s^2 \right)^{-1} F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(1 - e^{-t})^2 - s^2}{(1 + e^{-t})^2 - s^2} \right) \right. \\
\left. - \left( \frac{1}{2} - M \right) \frac{8e^{3t}}{(1 + 2e^t + e^{2t} (1 - s^2))^2} F \left( \frac{3}{2} - M, \frac{3}{2} - M; 2; \frac{(1 - e^{-t})^2 - s^2}{(1 + e^{-t})^2 - s^2} \right) \right\}
= \left( \frac{1}{2} - M \right) 2 \frac{\Gamma(2M)}{(\Gamma(\frac{1}{2} + M))^2} s \left( 1 - s^2 \right)^{-\frac{1}{2}+M},
\]
uniformly with respect to \( s \in [0, 1 - \varepsilon] \). Hence, for the positive \( M \) one can write
\[
\lim_{t \to +\infty} 4^M e^{-Mt} \frac{\partial}{\partial s} K_1(s, t; M) = -2 \left( M - \frac{1}{2} \right) \frac{\Gamma(2M)}{(\Gamma(\frac{1}{2} + M))^2} s (1 - s^2)^{M-\frac{1}{2}}.
\]
The last equation implies that derivative is a sign preserving function in \((0, 1)\). For \( M > 0 \), \( M \neq \frac{1}{2} \), the derivative vanishes for all \( s \in (0, 1) \) if and only if \( M = \frac{1}{2} \), that is,
\[
\lim_{t \to +\infty} 4^M e^{-Mt} \frac{\partial}{\partial s} K_1(s, t; M) \neq 0 \quad \text{for all} \quad s \in (0, 1 - \varepsilon).
\]
In particular,
\[
\lim_{t \to +\infty} \int_0^{1-\varepsilon} \left[ \left( \frac{1}{s} \frac{\partial}{\partial s} \right)^{\frac{n-3}{2}} s^{n-2} \varphi_1(s) \right] 4^M e^{-Mt} \frac{\partial}{\partial s} K_1(s, t; M) \, ds \\
= \int_0^{1-\varepsilon} \left[ \left( \frac{1}{s} \frac{\partial}{\partial s} \right)^{\frac{n-3}{2}} s^{n-2} \varphi_1(s) \right] \lim_{t \to +\infty} 4^M e^{-Mt} \frac{\partial}{\partial s} K_1(s, t; M) \, ds \\
= -2 \left( M - \frac{1}{2} \right) \frac{\Gamma(2M)}{(\Gamma(\frac{1}{2} + M))^2} \int_0^{1-\varepsilon} \left[ \left( \frac{1}{s} \frac{\partial}{\partial s} \right)^{\frac{n-3}{2}} s^{n-2} \varphi_1(s) \right] s (1 - s^2)^{M-\frac{1}{2}} \, ds.
\]
Consequently,
\[ \lim_{t \to +\infty} \int_0^{\phi(t)} V_{\varphi_1}(0, s) 4^M e^{-Mt} \frac{\partial}{\partial s} K_1(s, t; M) \, ds \]
\[ = -2 \left( M - \frac{1}{2} \right) \frac{\Gamma(2M)}{(\Gamma(\frac{1}{2} + M))^2 c_0^M} \int_0^1 s(1 - s^2)^{M-\frac{3}{2}} \left( \frac{1}{s} \frac{\partial}{\partial s} \right)^{\frac{n-3}{2}} s^{-n-2} \varphi_1(s) \, ds . \]
Hence,
\[ 4^{-M} e^{Mt} e^{-\frac{\pi t}{2}} \int_0^{\phi(t)} V_{\varphi_1}(0, s) 4^M e^{-Mt} \frac{\partial}{\partial s} K_1(s, t; M) \, ds \]
\[ = 4^{-M} e^{Mt} e^{-\frac{\pi t}{2}} \left\{ \int_0^{\phi(t)} V_{\varphi_1}(0, s) 4^M e^{-Mt} \frac{\partial}{\partial s} K_1(s, t; M) \, ds \right\} \]
\[ - \lim_{t \to +\infty} \int_0^{\phi(t)} V_{\varphi_1}(0, s) 4^M e^{-Mt} \frac{\partial}{\partial s} K_1(s, t; M) \, ds \]
\[ + 4^{-M} e^{Mt} e^{-\frac{\pi t}{2}} \left\{ \lim_{t \to +\infty} \int_0^{\phi(t)} V_{\varphi_1}(0, s) 4^M e^{-Mt} \frac{\partial}{\partial s} K_1(s, t; M) \, ds \right\} \]
\[ = 4^{-M} e^{Mt} e^{-\frac{\pi t}{2}} \left\{ o(1) + \lim_{t \to +\infty} \int_0^{\phi(t)} V_{\varphi_1}(0, s) 4^M e^{-Mt} \frac{\partial}{\partial s} K_1(s, t; M) \, ds \right\} , \]
where \( o(1) \to 0 \) as \( t \to \infty \). Finally,
\[ e^{-\frac{\pi t}{2}} \int_0^{\phi(t)} V_{\varphi_1}(0, s) \frac{\partial}{\partial s} K_1(s, t; M) \, ds = 4^{-M} e^{Mt} e^{-\frac{\pi t}{2}} \left\{ o(1) \right. \]
\[ -2 \left( M - \frac{1}{2} \right) \frac{\Gamma(2M)}{(\Gamma(\frac{1}{2} + M))^2 c_0^M} \int_0^1 s(1 - s^2)^{M-\frac{3}{2}} \left( \frac{1}{s} \frac{\partial}{\partial s} \right)^{\frac{n-3}{2}} s^{-n-2} \varphi_1(s) \, ds \right\} . \]
The last equation shows that for positive \( M \), \( M \neq 1/2 \), the value \( \Phi(0, t) \) of the solution \( \Phi = \Phi(x, t) \) for large \( t \) depends on the values of the initial data inside of the characteristic conoid.

The case of \( M = 0 \) can be discussed in similar way if we take into account \( [20] \), the support of the function \( V_{\varphi_1}(0, s) \), and
\[ \lim_{t \to +\infty} \ln \frac{1}{1 - \left( \frac{(1-e^{-t})^2 - s^2}{(1+e^{-t})^2 - s^2} \right)} F \left( \frac{1}{2}; 1; \frac{(1-e^{-t})^2 - s^2}{(1+e^{-t})^2 - s^2} \right) = -\frac{1}{\pi} . \]
If \( n \) is even, then the violation of the Huygens’ principle is inherited from the Minkowski spacetime through the representation formula. We skip the details of the proof of that case. Theorem 1 is proven. \( \square \)

**Proof of Theorem 3** The arguments have been used in the proof of Theorem 1 help us to prove also Theorem 3. In order to exclude the equations that obey the Huygens’ principle, we set \( M \neq \frac{1}{2} \). Then we consider odd \( n \), set \( \varphi_1 = 0 \) and choose the radial function \( \varphi_0 = \varphi_0(r) \),
supp $\varphi_0 \subset \{ x \in \mathbb{R}^n; \ |x| < 1 - \varepsilon \}$, $\varepsilon \in (0, 1)$. The solution (9) of the Cauchy problem is the following function

$$
\Phi(x, t) = e^{-\frac{M}{2}t}v_{\varphi_0}(x, \phi(t)) + e^{-\frac{M}{2}t}\int_0^{\phi(t)}v_{\varphi_0}(x, s)(2K_0(s, t; M) + nK_1(s, t; M)) \, ds, \quad x \in \mathbb{R}^n, \ t > 0.
$$

To complete the proof of theorem it remains to find the principal term of the asymptotic of the derivative

$$
\frac{\partial}{\partial s}(2K_0(s, t; M) + nK_1(s, t; M)) = 2\frac{\partial}{\partial s}K_0(s, t; M) + n\frac{\partial}{\partial s}K_1(s, t; M)
$$

for large $t$ and for $s \in [0, 1 - \varepsilon]$ on the support of $v_{\varphi_0}(0, \cdot)$. The second term of the right-hand side of the derivative is already discussed above. We evaluate the first term:

$$
\frac{\partial}{\partial s}K_0(s, t; M) = 4^{-M}e^{tM}\frac{\partial}{\partial s}\left\{(1 + e^{-t})^2 - s^2\right\}^{\frac{3}{2}+M}
$$

On the other hand, for the positive $M$ the equation

$$
F\left(-\frac{1}{2} - M, \frac{1}{2} - M; 1; 1\right) = \frac{4M}{1 + 2M}F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; 1\right)
$$

implies

$$
\lim_{t \to +\infty} 2^{1+2M}e^{-Mt}\frac{\partial}{\partial s}K_0(s, t; M)
$$

$$
= \lim_{t \to +\infty} \frac{1}{(1 - s^2)^2}e^{-4ts}(1 - s^2)^{-\frac{3}{2}+M}
$$

$$
\times \left\{-(1 + 2M)e^{4t}(-7 + 6s^2 + s^4)F\left(-\frac{1}{2} - M, \frac{1}{2} - M; 1; 1\right)
$$

$$
+ 2(M + 2M^2 + e^{3t}(4M^2 - 1)(1 + s^2)
$$

$$
+ e^{4t}(1 + 2M)(-1 + s^2)(3 + M + Ms^2) - e^{2t}(1 + 2M)(-3 + 2Ms^2))
$$

$$
\times F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; 1\right)\right\}
$$

=$$
(1 + 2M)\lim_{t \to +\infty} \frac{1}{(1 - s^2)^2}e^{-4ts}(1 - s^2)^{-\frac{3}{2}+M}
$$

$$
\times \left\{-e^{4t}(-7 + 6s^2 + s^4)F\left(-\frac{1}{2} - M, \frac{1}{2} - M; 1; 1\right)
$$

$$
+ 2e^{4t}(-1 + s^2)(3 + M + Ms^2)F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; 1\right)\right\}.$$

15
It follows
\[
\lim_{t \to +\infty} 2^{1+2M} e^{-Mt} \frac{\partial}{\partial s} K_0(s, t; M) = s \left(1 - s^2\right)^{-\frac{n}{2} + M} \left\{ (7 - 6s^2 - s^4) 4M + 2(1 + 2M)(s^2 - 1) \left(3 + M + Ms^2\right) \right\} \\
\times F\left(\frac{1}{2} - M, \frac{1}{2} - M, 1, 1\right).
\]

Consequently,
\[
\lim_{t \to +\infty} 2^{1+2M} e^{-Mt} \frac{\partial}{\partial s} (2K_0(s, t; M) + nK_1(s, t; M)) = 2s \left(1 - s^2\right)^{-\frac{n}{2} + M} \left\{ (7 - 6s^2 - s^4) 4M + 2(1 + 2M)(s^2 - 1) \left(3 + M + Ms^2\right) \right\} \\
\times F\left(\frac{1}{2} - M, \frac{1}{2} - M, 1, 1\right) \\
-4n \left(M - \frac{1}{2}\right) s(1 - s^2)^{M - \frac{3}{2}} F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; 1\right) \\
= (1 - s^2)^{-\frac{n}{2} + M} s \left\{ (7 + s^2) 8M - 4(1 + 2M) \left(3 + M + Ms^2\right) - 4n \left(M - \frac{1}{2}\right) (1 - s^2) \right\} \\
\times F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; 1\right) \\
= -8 \left(1 - s^2\right)^{-\frac{n}{2} + M} s \left(M - \frac{1}{2}\right) \left(s^2 \left(M - \frac{n}{2}\right) + M + \frac{n}{2} - 3\right) F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; 1\right).
\]
The factor
\[
\left(M - \frac{1}{2}\right) \left(s^2 \left(M - \frac{n}{2}\right) + M + \frac{n}{2} - 3\right)
\]
with \(M \neq \frac{1}{2}\) identically vanishes only if \(M = n/2\) and \(n = 3\). The rest of the proof is a repetition of the one has been done above. The case of large mass, \(m \geq n/2\) can be checked similarly. Theorem 3 is proven. \(\square\)

4.1 Asymptotic expansions of solutions at infinite time

In this subsection we present the large time asymptotic analysis of the solution of the equation, which obeys the Huygens’ principle. More precisely, we derive the complete asymptotic expansion. In fact, this analysis was started in the previous subsection.

Concerning asymptotic expansion of the solution for the large time, we mention here two recent articles on linear equations on the asymptotically de Sitter spacetimes. Vasy [29] exhibited the well-posedness of the Cauchy problem and showed that on such spaces, the solution of the Klein-Gordon equation without source term and with smooth Cauchy data has an asymptotic expansion at infinity. It is also shown in [29] that the solutions of the wave equation exhibit scattering. Baskin [4] constructed parametrix for the forward fundamental solution of the wave and Klein-Gordon equations on asymptotically de Sitter spaces without caustics and used this parametrix to obtain asymptotic expansions (principal term) for the solutions of the equation with some class of source terms. (For more references on the asymptotically de Sitter spaces, see the bibliography in [4], [29].)
For \( \varphi_1 \in C^\infty_0(\mathbb{R}^n) \) the formula for the solution \( V(x, t) \) of the Cauchy problem (13) is well-known. (See, e.g., [26].) It can be written for odd and even \( n \) separately as follows. We have

\[
V_\varphi(x, t) := \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \frac{t^{n-2}}{\omega_{n-1} c_0^{(n)}} \int_{S^{n-1}} \varphi(x + ty) \, dS_y,
\]

where \( c_0^{(n)} = 1 \cdot 3 \cdots (n - 2) \) if \( n \geq 3 \) is odd. For \( x \in \mathbb{R}^n \), and even \( n \), we have

\[
V_\varphi(x, t) := \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \frac{2 t^{n-1}}{\omega_{n-1} c_0^{(n)}} \int_{B_1^n(0)} \frac{1}{\sqrt{1 - |y|^2}} \varphi(x + ty) \, dV_y,
\]

where \( c_0^{(n)} = 1 \cdot 3 \cdots (n - 1) \). Similarly, for \( \varphi_0 \in C^\infty_0(\mathbb{R}^n) \) and for \( x \in \mathbb{R}^n \), if \( n \) is odd, the formula for the solution \( u(x, t) \) of the Cauchy problem

\[
u_{tt} - \Delta u = 0, \quad u(x, 0) = \varphi_0(x), \quad u_t(x, 0) = 0,
\]

implies

\[
v_\varphi(x, t) := \frac{\partial}{\partial t} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \frac{t^{n-2}}{\omega_{n-1} c_0^{(n)}} \int_{S^{n-1}} \varphi(x + ty) \, dS_y.
\]

In the case of \( x \in \mathbb{R}^n \) and even \( n \), we have

\[
v_\varphi(x, t) := \frac{\partial}{\partial t} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \frac{2 t^{n-1}}{\omega_{n-1} c_0^{(n)}} \int_{B_1^n(0)} \frac{1}{\sqrt{1 - |y|^2}} \varphi(x + ty) \, dV_y.
\]

The constant \( \omega_{n-1} \) is the area of the unit sphere \( S^{n-1} \subset \mathbb{R}^n \). In particular,

\[
v_\varphi(x, 1) = \begin{cases} \left[ \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \frac{t^{n-2}}{\omega_{n-1} c_0^{(n)}} \int_{S^{n-1}} \varphi(x + ty) \, dS_y \right]_{t=1} & \text{if } n \text{ is odd}, \\
\left[ \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \frac{2 t^{n-1}}{\omega_{n-1} c_0^{(n)}} \int_{B_1^n(0)} \frac{1}{\sqrt{1 - |y|^2}} \varphi(x + ty) \, dV_y \right]_{t=1} & \text{if } n \text{ is even},
\end{cases}
\]

and

\[
V_\varphi(x, 1) = \begin{cases} \left[ \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \frac{t^{n-2}}{\omega_{n-1} c_0^{(n)}} \int_{S^{n-1}} \varphi(x + ty) \, dS_y \right]_{t=1} & \text{if } n \text{ is odd}, \\
\left[ \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \frac{2 t^{n-1}}{\omega_{n-1} c_0^{(n)}} \int_{B_1^n(0)} \frac{1}{\sqrt{1 - |y|^2}} \varphi(x + ty) \, dV_y \right]_{t=1} & \text{if } n \text{ is even}.
\end{cases}
\]

Denote

\[
v_\varphi(x) := v_\varphi(x, 1), \quad V_\varphi(x) := V_\varphi(x, 1).
\]

In order to write complete asymptotic expansion of the solutions, we define the functions

\[
V_\varphi^{(k)}(x) = \frac{(-1)^k}{k!} \left[ \left( \frac{\partial}{\partial t} \right)^k V_\varphi(x, t) \right]_{t=1} \in C^\infty_0(\mathbb{R}^n), \quad k = 1, 2, \ldots.
\]
Then, for every integer $N \geq 1$ we have
\[
V_{\varphi}(x, 1 - e^{-t}) = \sum_{k=0}^{N-1} V_{\varphi}^{(k)}(x)e^{-kt} + R_{V_{\varphi},N}(x, t), \quad R_{V_{\varphi},N} \in C^\infty,
\]
where the remainder $R_{V_{\varphi},N}$ satisfies the inequality
\[
|R_{V_{\varphi},N}(x, t)| \leq C(\varphi)e^{-Nt} \quad \text{for all } \ x \in \mathbb{R}^n \text{ and all } \ t \in [0, \infty),
\]
with some constant $C(\varphi)$. Moreover, the support of the remainder $R_{V_{\varphi},N}$ is in the cylinder
\[
\text{supp } R_{V_{\varphi},N} \subseteq \{ x \in \mathbb{R}^n ; \text{dist}(x, \text{supp } \varphi) \leq 1 \} \times [0, \infty).
\]

Analogously, we define
\[
v_{\varphi}^{(k)}(x) = \frac{(-1)^k}{k!} \left[ \left( \frac{\partial}{\partial t} \right)^k v_{\varphi}(x, t) \right]_{t=1} \in C_0^\infty(\mathbb{R}^n), \quad k = 1, 2, \ldots,
\]
and the remainder $R_{v_{\varphi},N}$,
\[
v_{\varphi}(x, 1 - e^{-t}) = \sum_{k=0}^{N-1} v_{\varphi}^{(k)}(x)e^{-kt} + R_{v_{\varphi},N}(x, t), \quad R_{v_{\varphi},N} \in C^\infty,
\]
such that
\[
|R_{v_{\varphi},N}(x, t)| \leq C(\varphi)e^{-Nt} \quad \text{for all } \ x \in \mathbb{R}^n \text{ and all } \ t \in [0, \infty).
\]

Further, we introduce a polynomial in $z \in \mathbb{C}$ with the smooth in $x \in \mathbb{R}^n$ coefficients as follows:
\[
\Phi_{\text{asypt}}^{(N)}(x, z) = z^{\frac{n-1}{2}} \left[ \sum_{k=0}^{N-1} v_{\varphi_0}^{(k)}(x)z^k + \frac{n-1}{2} \sum_{k=0}^{N-1} V_{\varphi_0}^{(k)}(x)z^k \right] + z^{\frac{n-1}{2}} \sum_{k=0}^{N-1} V_{\varphi_1}^{(k)}(x)z^k.
\]
Then we write the next asymptotic expansion
\[
\Phi(x, t) = \Phi_{\text{asypt}}^{(N)}(x, e^{-t}) + O(e^{-Nt - \frac{n+1}{2}t})
\]
for large $t$ uniformly for all $x \in \mathbb{R}^n$. Thus, we have proven the next theorem.

**Theorem 5** Suppose that $m = \sqrt{n^2 - 1/2}$. Then, for every positive integer $N$ the solution of the Cauchy problem for the equation (13) with the initial values $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$ has the following asymptotic expansion at infinity:
\[
\Phi(x, t) \sim \Phi_{\text{asypt}}^{(N)}(x, e^{-t}),
\]
in the sense that for every positive integer $N$ the following estimate is valid,
\[
\|\Phi(x, t) - \Phi_{\text{asypt}}^{(N)}(x, e^{-t})\|_{L^\infty(\mathbb{R}^n)} \leq C(\varphi_0, \varphi_1)e^{-Nt - \frac{n+1}{2}t} \quad \text{for large } t.
\]
Remark 6: If we take into account the relation $v_\varphi(x,t) = \frac{\partial}{\partial t} V_\varphi(x,t)$, then

$$v_\varphi^{(k)}(x) = -(k+1)V_\varphi^{(k+1)}(x),$$

and, consequently, the function $\Phi^{(N)}_{asympt}(x,z)$ can be rewritten as follows:

$$\Phi^{(N)}_{asympt}(x,z) = z^{n/2} \left( \sum_{k=0}^{N-1} (-1)(k+1)V_\varphi^{(k+1)}(x) z^k + \frac{n-1}{2} \sum_{k=0}^{N-1} V_\varphi^{(k)}(x) z^k \right) + z^{n/2} \sum_{k=0}^{N-1} V_\varphi^{(k)}(x) z^k.$$

In the forthcoming paper we will derive a similar result for the remaining values of the mass $m \in [0, \infty)$, that is, for the equation, which does not obey the Huygens’ principle.

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