Time Optimal Synthesis for Left–Invariant Control Systems on $SO(3)$

Ugo Boscain,
SISSA-ISAS, Via Beirut 2-4, 34014 Trieste, Italy
Yacine Chitour
Université Paris XI, Département de Mathématiques, F-91405 Orsay, France

Abstract Consider the control system $(\Sigma)$ given by $\dot{x} = x(f + ug)$, where $x \in SO(3)$, $|u| \leq 1$ and $f, g \in so(3)$ define two perpendicular left-invariant vector fields normalized so that $\|f\| = \cos(\alpha)$ and $\|g\| = \sin(\alpha)$, $\alpha \in [0, \pi/4]$. In this paper, we provide an upper bound and a lower bound for $N(\alpha)$, the maximum number of switchings for time-optimal trajectories of $(\Sigma)$. More precisely, we show that $N_S(\alpha) \leq N(\alpha) \leq N_S(\alpha) + 4$, where $N_S(\alpha)$ is a suitable integer function of $\alpha$ such that $N_S(\alpha) \sim \alpha \to \pi/(4\alpha)$.

The result is obtained by studying the time optimal synthesis of a projected control problem on $\mathbb{R}P^2$, where the projection is defined by an appropriate Hopf fibration. Finally, we study the projected control problem on the unit sphere $S^2$. It exhibits interesting features which will be partly rigorously derived and partially described by numerical simulations.

1 Introduction

Let $(\Sigma)$ be the control system given by:

$$\dot{x} = x(f + ug),$$

where $x \in SO(3)$, $|u| \leq 1$ and $f, g \in so(3)$ give rise to two non zero perpendicular left-invariant vector fields on $SO(3)$. In this paper, we consider the following problem: given any pair of points $x_1, x_2$ of $SO(3)$, find a trajectory of $(1)$ steering $x_1$ to $x_2$ in minimum time. That issue is known as the problem of determining the time optimal synthesis (TOS) for $(\Sigma)$. The strategy to determine a TOS usually consists in two steps:

1. Reduction procedure: it is based on the Pontryagin Maximum Principle (PMP) which is a first-order necessary condition for optimality. Roughly speaking, the PMP reduces the candidates for time optimality to the so called extremals, which are solutions of a pseudo-Hamiltonian system. This reduction procedure may be refined using higher order conditions, such as Clebsch-Legendre conditions, higher-order maximum principle, envelopes, conjugate points, index theory (cf. for instance [2, 3, 5, 12, 18, 19, 21, 26, 27, 28, 30, 32, 33, 34, 35]);

2. Selection procedure: it consists of selecting the time optimal trajectories among the extremals that passed the test of Step 1. (see for instance [8, 12, 16, 25]).

Step 1. is already non trivial and, in general, the second one is extremely difficult: if the state space is two-dimensional, the problem of determining the TOS for single-input control systems is now well-understood [9, 11, 12, 14, 15, 23, 24, 32, 33]. However, for higher dimensions, very few examples of complete TOS for a non linear control system are available (see for instance [29]). Intermediate issues were thus deeply investigated: determining estimates for the number of switchings of optimal trajectories, describing the local structure of optimal trajectories, finding families of trajectories sufficient for optimality, cf. [6, 13, 19, 22, 26, 28, 36], etc.

For the control system $(\Sigma)$, we normalize the two perpendicular vector fields induced by $f$ and $g$ in such a way that $\|f\| = \cos(\alpha)$, $\|g\| = \sin(\alpha)$, with $\alpha \in [0, \pi/2]$ (for the precise meaning of “perpendicular” and of the symbol $\|\|$, we refer to Section 2.1). Defining $X_+ := f + g$ and $X_- := f - g$, we have $\|X_+\| = \|X_-\| = 1$ and $\alpha$ is the angle between $f$ and $X_+$.

By a standard argument (see Section 2 below), one can show that every time optimal trajectory is a finite concatenation of bang arcs (i.e. $u \equiv \pm 1$) or singular arcs ($u = 0$) and thus, the Fuller phenomenon (i.e.
existence of a trajectory of a control system joining two points in (finite) minimum time, with an infinite number of switchings, cf. [20, 37]) never occurs. (A switching time – or simply a switching – along an extremal is a time $t_0$ so that the control $u$ is not constant in any open neighborhood of $t_0$.) Moreover, one can easily show that, the supremum $N(\alpha)$ of the number of switchings over all time optimal trajectories of $(\Sigma)$, is finite.

By using the index theory developed by Agrachev, it is proved in [3] that

$$N(\alpha) \leq N_A := \left\lfloor \frac{\pi}{\alpha} \right\rfloor,$$

(2)

where $[\cdot]$ stands for the integer part. That result was not only an indirect indication that $N(\alpha)$ would tend to infinity as $\alpha$ tends to zero, but it also provided a hint on the asymptotic of $N(\alpha)$ as $\alpha$ tends to zero.

A related line of work regards the study of the distributional version of $(\Sigma)$, which is the driftless control system given by $\dot{x} = x(u_1f_1 + u_2f_2)$, $|u_1|, |u_2| \leq 1$ and $f_1, f_2 \in so(3)$ linearly independent. Indeed, assuming that $\|f_1\| = \|f_2\|$, Sussmann and Tang ([36]) showed that time optimal trajectories have at most four switchings and they provided a finitely parametrized family of trajectories sufficient for optimality. That result was extended to the general case ($f_1$ and $f_2$ just linearly independent, cf [13]): time optimal trajectories have at most five switchings. For both works, the elimination from optimality of extremals with respectively five or six bangs relies on the envelope theory developed in the context of control theory by Sussmann (cf. [35]).

At the light of the previous results, there was strong evidence for two radically situations as $\alpha$ tends to zero: for $(\Sigma)$, $N(\alpha)$ is expected to go to infinity, as for the distributional control system, there exists a universal bound on the number of switchings. The main result of the present paper confirms that difference, i.e. $N(\alpha)$ tends to infinity as $\alpha$ tends to zero.

More precisely, we complete the inequality (2) as follows:

**Theorem 1** Let $(\Sigma)$ be the control system defined in (1) with $f, g$ perpendicular so that $\|f\| = \cos(\alpha)$ and $\|g\| = \sin(\alpha)$, $\alpha \in [0, \pi/4]$. Then, if $N(\alpha)$ is the maximum number of switchings along a time-optimal trajectory of $(\Sigma)$, we have

$$N_S(\alpha) \leq N(\alpha) \leq N_S(\alpha) + 4,$$

where $N_S(\alpha) := 2 \left\lfloor \frac{\pi}{8\alpha} \right\rfloor - 2 \left\lfloor \frac{\pi}{8\alpha} \right\rfloor - \frac{\pi}{4\alpha}$.

(3)

The above theorem improves (2) in two ways: i) for $\alpha$ small, it (essentially) divides the upper bound of $N(\alpha)$ by four with respect to (2); ii) it provides a lower bound of $N(\alpha)$ differing from the upper bound by a constant.

The lower bound is in fact our main contribution and, to get it, one must prove the existence of time optimal trajectories of $(\Sigma)$ admitting at least a number of switchings equal to that lower bound. Our strategy consists of projecting the control problem onto another $(\Sigma)_S$ defined next. First, let $\mathbb{R}P^2$ be the two-dimensional real projective space (i.e. the two-dimensional manifold made of the directions of $\mathbb{R}^3$) and fix a point $x_0 \in SO(3)$. Consider the Hopf fibration $\Pi : SO(3) \rightarrow \mathbb{R}P^2$ defined by $\text{Ker}(\Pi(x_0)) = \text{Span}\{x_0f\}$, which means, roughly speaking, that $\Pi$ annihilates the drift term $f$ at $x_0$. Then, we project $(\Sigma)$ by $\Pi$ and obtain a single-input $SO(3)$-equivariant control system $(\Sigma)_S$ on $\mathbb{R}P^2$ given by $\dot{y} = y(f_S + u_S)$, with $f_S = d\Pi(f)$ and $g_S = d\Pi(g)$, that is locally controllable. We then consider the minimum time problem of connecting $\Pi(x_0)$ to any other point of $\mathbb{R}P^2$.

In fact, we study a slightly different time optimal problem by lifting $(\Sigma)_S$ to the unit sphere $S^2$. By an abuse of notation, we still denote by $(\Sigma)_S$ the control system obtained in that way. Hence $\Pi(x_0)$ is identified with the north pole and $\mathbb{R}P^2$ is identified with $\mathcal{M}$, the subset of the sphere made of the union of $\mathcal{M}$, the (open) top hemisphere of $S^2$, together with half of the equator.

The time optimal problem consists now of connecting, in minimum time, the north pole with any point of $\mathcal{M}$. Thanks to the suitable choice of the Hopf projection and since $\alpha$ belongs to the interval $[0, \pi/4]$, all extremals of the projected problem are bang-bang (i.e. they are a finite concatenation of trajectories corresponding to controls $+1$ or $-1$). Let $N_S(\alpha)$ be the supremum of the number of switchings for time optimal trajectories on $(\Sigma)_S$ starting at the north pole and ending in $\mathcal{M}$ (such trajectories of $(\Sigma)_S$ are actually entirely contained in $\mathcal{M}$, see Lemma 7).

The use of the Hopf fibration $\Pi$ is motivated by two facts: first, every time optimal trajectory for the time optimal problem on $(\Sigma)_S$ staying in $\mathcal{M}$ is the projection by $\Pi$ of a time optimal trajectory for the time optimal problem on $(\Sigma)$ and thus, $N_S(\alpha) \leq N(\alpha)$. Taking full advantage of the theory developed in [12], we will actually compute exactly $N_S(\alpha)$ as given in (3). Second, using the fact that the fiber above $\Pi(x_0)$ is the
support of a singular arc (for this problem singular arcs are integral curves of the drift $xf$), we show that every regular bang-bang trajectory with at least $N_S(\alpha) + 5$ cannot be optimal and thus, the upper bound.

It is then clear, by now, that the most delicate part of the argument relies on the exact determination of $N_S(\alpha)$. This is done by studying the time optimal synthesis, (TOS for short) for the time optimal problem on $(\Sigma)_S$. Such a TOS is usually constructed, following the theory developed in [9, 11, 12, 14, 15, 23, 24, 32, 33], recursively on the number of extremals arcs, and by checking at each step whether they are optimal or not. For the problem on $\mathbb{R}P^2$, we are not able to complete all the steps of the above construction, which would imply as a byproduct the existence of the TOS. In particular, we cannot show the optimality of all the extremals (i.e. the trajectories candidate for time optimality), but, from their study, we can demonstrate enough partial results in order compute $N_S(\alpha)$ precisely and thus to conclude the proof of Theorem 1.

The complete time optimal synthesis is then studied numerically (actually on the whole $S^2$) and is showed in the top of Fig. 11. In particular, due to the compactness of $S^2$, one of the main issues is to understand the singularities developed by the minimum time wave front as it approaches to the south pole. We provide numerical simulations that describe the evolution of the extremal front. As $\alpha \to 0$, these numerical simulations suggest the emergence of three cyclically alternating patterns of optimal synthesis, each of them depending on an arithmetic property of $\alpha$.

The balance of the paper is organized as follows: Section 2 collects basic facts relative to the time-optimal trajectories of $(\Sigma)$; in Section 3, the Hopf fibration is described and the proof of Theorem 1 is provided, assuming some facts about the time-optimal synthesis of $(\Sigma)_S$, whose arguments are deferred to the next section. In particular we use the expression of $N_S(\alpha)$ and the relation between the length of interior bang arcs for the problem on $\mathbb{R}P^2$. The construction of the time-optimal synthesis of $(\Sigma)_S$ is investigated in Section 4, where an exact computation of $N_S(\alpha)$ is established. We conclude the section with two remarks, the first one explaining the relation between the TOS on the sphere and the TOS of a controlled linear pendulum, the second one establishing a link with an optimal control problem on $SU(2)$. Finally, in Section 5, we provide the results of the numerical simulations completing the study of the time optimal synthesis (in particular of the possible behaviors in a neighborhood of the south pole) and we propose some open problems stated as conjectures.

2 Statement of the Problem and Properties of Optimal Trajectories

2.1 Basic Facts

In this paper, we consider the control $(\Sigma)$ given by (1), where $x \in SO(3)$, $|u| \leq 1$ and $f, g \in so(3)$. An admissible control $u$ is a measurable function $u : [a, b] \to [-1, 1]$, where $a, b$ depend (in general) on $u$, cf. [18]. A trajectory $\gamma$ of $(\Sigma)$ is an absolutely continuous curve $\gamma : J \to SO(3)$, where $J = [a, b]$ is a compact segment of $\mathbb{R}$ such that there exists an admissible control $u$ for which $\dot{\gamma}(t) = \gamma(t)(f + u(t)g)$ holds a.e. in $J$. We then say that $(\gamma, u)$, defined as before, is an admissible pair for $(\Sigma)$.

**Definition 1** A trajectory $\gamma$ of $(\Sigma)$, defined on $[a, b]$, is time optimal if, for every trajectory $\gamma'$ of $(\Sigma)$ defined on $[a', b']$ with $\gamma(a) = \gamma'(a')$ and $\gamma(b) = \gamma'(b')$, we have $b - a \leq b' - a'$.

The Lie algebra $(so(3), [,])$ is isomorphic to the Lie algebra $(\mathbb{R}^3, \times)$, where $\times$ denotes the vector product. This isomorphism is realized by the map:

$$\phi_L : so(3) \to \mathbb{R}^3$$

$$\phi_L \left( \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \right) := \begin{pmatrix} b \\ a \\ c \end{pmatrix},$$

and provides an inner product on $so(3)$ given by $\langle z_1, z_2 \rangle := \langle \phi_L(z_1), \phi_L(z_2) \rangle$, where $z_1, z_2 \in so(3)$. The symbol $\langle \cdot, \cdot \rangle$, in the right–hand side of the above equation, stands for the Euclidean inner product of $\mathbb{R}^3$. With this definition, it follows that: $\langle z_1, z_2 \rangle := -\frac{1}{2} Tr(z_1 z_2)$. In other words, this scalar product is the opposite of the Killing form on $so(3)$. In the following, $\|z\| := \sqrt{z, z}$ and $Id$ is the $3 \times 3$ identity matrix. We will sometimes consider the $2 \times 2$ matrix corresponding to the planar rotation of angle $\beta$ and we use $R_\beta$ to denote it.
In this paper, we will assume that \( f \) and \( g \) are perpendicular and normalized so that \( \|f\| = \cos(\alpha) \) and \( \|g\| = \sin(\alpha), \alpha \in [0, \pi/2] \). Here, we adopt the following notation used throughout the paper, \( c_\alpha := \cos(\alpha), \)
\( c_\alpha^2 := \cos^2(\alpha), s_\alpha := \sin(\alpha) \) and \( s_\alpha^2 := \sin^2(\alpha) \). We define \( h := [f, g] = fg - gf \) and
\[
X_+ := f + g, \quad X_- := f - g.
\]
Note that \( \|X_+\| = 1, \) with \( \varepsilon = +, - \). For a vector field \( z \in \mathfrak{so}(3) \), we use \( e^{tz} \) to denote the flow of \( z \), acting on the right, so that \( t \mapsto pe^{tz} \in SO(3) \) is the integral curve of \( z \) starting at \( p \) at time 0. Since \( z \) is linear, we have 
\[
e^{tz} = \sum_{n=0}^{\infty} \frac{(tz)^n}{n!}.
\]
We use \( \text{ad}_z \) to denote the operator \( w \mapsto [z, w] \), acting on vector fields. If \( z, w \) are vector fields, then \( e^{t \text{ad}z}(w) := e^{tz}we^{-tz} \). The Lie bracket relations between \( f, g, h \) are
\[
[f, g] = h, \quad [g, h] = s_\alpha^2f, \quad [h, f] = c_\alpha^2g.
\]
From them, one deduces the following classical relations that will be useful later:
\[
e^{t \text{ad}x}(f) = (c_\alpha^2 + s_\alpha^2 \cos(t))f + \varepsilon c_\alpha^2(1 - \cos(t))g - \varepsilon \sin(t)h, \quad (5)
\]
\[
e^{t \text{ad}x}(g) = \varepsilon s_\alpha^2(1 - \cos(t))f + (s_\alpha^2 + c_\alpha^2 \cos(t))g + \sin(t)h, \quad (6)
\]
\[
e^{t \text{ad}x}(h) = \varepsilon s_\alpha^2 \sin(t)f - c_\alpha^2 \sin(t)g + \cos(t)h, \quad (7)
\]
\[
e^{t \text{ad}x}(X_\varepsilon) = (\cos(2\alpha) + 2s_\alpha^2 \cos(t))f + \varepsilon(\cos(2\alpha) - 2s_\alpha^2 \cos(t))g - 2\varepsilon \sin(t)h, \quad (8)
\]
\[
e^{tX_\varepsilon} = \text{Id} + \sin(t)X_\varepsilon + (1 - \cos(t))X_\varepsilon^2, \quad (9)
\]
\[
e^{t\text{adj}}f = \cos(tc_\alpha)g + \frac{\sin(tc_\alpha)}{c_\alpha}h. \quad (10)
\]

2.2 Existence of Optimal Trajectories

A control system is complete if, for every measurable control function \( u : [a, b] \to [-1, 1] \) and every initial state \( p \), there exists a trajectory \( \gamma \) corresponding to \( u \), which is defined on the whole interval \([a, b]\) and satisfies \( \gamma(a) = p \). Since \( SO(3) \) is compact and the function \( F(x, u) := x(f + ug) \) is regular enough, the system (1) is complete. Note that \( (f, g) \) satisfies the Strong Bracket Generating Condition (cf. [31]) and the set of velocities \( V(x) := \{x(f + ug), \quad u \in [-1, 1]\} \) is compact and convex. Then, (cf. for instance [36]):

**Proposition 1** For each pair of points \( p \) and \( q \) belonging to \( SO(3) \), there exists a time optimal trajectory joining \( p \) to \( q \).

2.3 Pontryagin Maximum Principle and Switching Functions

We next state the Pontryagin Maximum Principle (PMP) (cf. [26]) for our minimum time problem on \( SO(3) \). Define the following maps called respectively Hamiltonian and minimized Hamiltonian:
\[
\mathcal{H} : T^*SO(3) \times [-1, 1] \to \mathbb{R}, \quad \mathcal{H}(p, x, u) := < p, x(f + ug) >, \quad (11)
\]
\[
H : T^*SO(3) \to \mathbb{R}, \quad H(p, x) := \min_{v \in [-1, 1]} \mathcal{H}(p, x, v). \quad (12)
\]
The PMP asserts that, if \( \gamma : [a, b] \to SO(3) \) is a time optimal trajectory corresponding to a control \( u : [a, b] \to [-1, 1] \), then there exists a nontrivial field of covectors along \( \gamma \), that is an absolutely continuous function \( \lambda : [a, b] \to \mathbb{R} \) never vanishing and a constant \( \lambda_0 \geq 0 \) such that, for a.e. \( t \in \text{Dom}(\gamma) \), we have:
\begin{enumerate}
  \item \( \dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial x}(\lambda(t), \gamma(t), u(t)) = -\lambda(t)(f + u(t)g), \)
  \item \( \mathcal{H}(\gamma(t), \lambda(t), u(t)) + \lambda_0 = 0, \)
  \item \( \mathcal{H}(\gamma(t), \lambda(t), u(t)) = H(\gamma(t), \lambda(t)). \)
\end{enumerate}
Remark 1 The PMP is just a necessary condition for optimality. A trajectory \( \gamma \) (resp. a couple \((\gamma, \lambda)\)) satisfying the conditions given by the PMP is said to be an extremal (resp. an extremal pair). An extremal corresponding to \( \lambda_0 = 0 \), said to be an abnormal extremal, otherwise we call it a normal extremal. For a normal extremal, we can always normalize \( \lambda_0 = 1 \), and we do this all along the paper. Notice that in general an extremal corresponds to more than one covector. For this reason, usually, one distinguishes between abnormal extremal that are strict (i.e. they correspond only to covectors satisfying \( \lambda_0 = 0 \)) and abnormal extremal that are non-strict (i.e. they correspond to covectors with \( \lambda_0 = 0 \) and to covectors with \( \lambda_0 \neq 0 \)).

A control \( u : [a, b] \rightarrow [-1, 1] \) is said to be bang-bang if \( u(t) \in \{-1, 1\} \) a.e. in \([a, b] \). Moreover, if \( u(t) \in \{-1, 1\} \) and \( u(t) \) is constant for almost every \( t \in [a, b] \), then \( u \) is called a bang control. A switching time of \( u \) is a time \( t \in [a, b] \) such that, for each for every \( \varepsilon > 0 \), \( u \) is not bang on \((t - \varepsilon, t + \varepsilon) \cap [a, b] \). A control with a finite number of switchings is called regular bang-bang. A trajectory of \( \Sigma \) is a bang trajectory, bang-bang trajectory, regular bang-bang trajectory respectively, if it corresponds to a bang control, bang-bang control, regular bang-bang control respectively. The switching functions, associated to an extremal pair \((\gamma, \lambda)\), are the three “components” of the covector \( \lambda(t) \) on the basis \( \{f, g, h\} \) transported to the point \( \gamma(t) \). More precisely:

Definition 2 (switching functions) Let \( \Phi_i(x, p) \) \((i = 1, 2, 3)\) be the Hamiltonian functions corresponding respectively to the vector fields \( f, g, h \) (cf. [18]). i.e. \( \Phi_1(x, p) := < p, xf >, \Phi_2(x, p) := < p, xg >, \Phi_3(x, p) := < p, xh > \) and \((\gamma, \lambda)\) be an extremal. The switching functions associated to \((\gamma, \lambda)\) are the evaluations of \( \Phi_i(x, p) \) along the extremal i.e.:

\[
\begin{align*}
\varphi_1(t) & := \Phi_1(\gamma(t), \lambda(t)) = < \lambda(t), \gamma(t)f >, \\
\varphi_2(t) & := \Phi_2(\gamma(t), \lambda(t)) = < \lambda(t), \gamma(t)g >, \\
\varphi_3(t) & := \Phi_3(\gamma(t), \lambda(t)) = < \lambda(t), \gamma(t)h >.
\end{align*}
\]

Remark 2 Notice that the \( \varphi_i \)'s are at least continuous and since \( \lambda \) never vanishes, the three switching functions cannot be all zero at the same time \( t \). Moreover, using the switching functions, ii) of PMP reads:

\[
\mathcal{H}(\lambda(t), \gamma(t), u(t)) = \varphi_1(t) + u(t)\varphi_2(t) + \lambda_0 = 0 \quad \text{a.e.}
\]

The switching functions are important because they determine where the controls may switch. In fact, using the PMP, one easily gets:

Proposition 2 A necessary condition for a time \( t \) to be a switching is that \( \varphi_2(t) = 0 \). Therefore, on any interval where \( \varphi_2 \) has no zeroes (respectively finitely many zeroes), the corresponding control is bang (respectively bang-bang). In particular, \( \varphi_2 > 0 \) (resp \( \varphi_2 < 0 \)) on \([a, b]\) implies \( u = -1 \) (resp. \( u = +1 \)) a.e. on \([a, b]\). On the other hand, if \( \varphi_2 \) has a zero at \( t \) and \( \varphi_2(t) \) exists and is different from zero, then \( t \) is an isolated switching.

As a corollary, it holds a.e. along an extremal trajectory that:

\[
u(t)\varphi_2(t) = -|\varphi_2(t)|.
\]

An extremal trajectory \( \gamma \) of \( \Sigma \) defined on \([c, d]\) is said to be singular if the switching function \( \varphi_2 \) vanishes on \([c, d]\). To compute the control corresponding to a singular trajectory, one should compute the derivatives of the \( \varphi_i \)'s. Using the Lie bracket relations between \( f, g, h \), one gets the system of differential equations (called the adjoint system) satisfied a.e.:

\[
\begin{align*}
\dot{\varphi}_1 &= -u\varphi_3, \\
\dot{\varphi}_2 &= \varphi_3, \\
\dot{\varphi}_3 &= s^2 u\varphi_1 - c^2 \varphi_2.
\end{align*}
\]

From Eqs. (15) and (19), one immediately gets that \( \varphi_2 \) is at least a \( C^1 \) function. Moreover, if \( \gamma \) is singular in \([a, b]\), then \( \varphi_2 = 0 \) and, from Eq. (19), we get \( \varphi_3 = 0 \) a.e.. From (16) (cf. PMP ii)), we get that \( \varphi_1 \equiv -1 \) a.e. on \([a, b]\). From (20) we get \( u = 0 \) a.e. i.e.:

Proposition 3 For the minimum time problem for \((\Sigma)\), singular trajectories are integral curves of the drift, i.e. they correspond to a control a.e. vanishing.
In the sequel, we will use the following convention. The letter \( B \) refers to a bang trajectory and the letter \( S \) refers to a singular extremal trajectory. A concatenation of bang and singular trajectories will be labeled by the corresponding letter sequence, written in order from left to right. Sometimes, we will use a subscript to indicate the time duration of a trajectory so that we use \( B_t \) to refer to a bang trajectory defined on an interval of length \( t \) and, similarly, \( S_t \) for a singular trajectory defined on an interval of length \( t \).

If we fix \( u \in [-1, 1] \), then the integral curves of \( x(f + u\gamma) \) are periodic. In particular, the integral curves of \( xX_\gamma \) are periodic with period \( 2\pi \) while the integral curves of the drift \( xf \) are periodic with period \( 2\pi/c_\alpha \). This means:

**Proposition 4** If \( \gamma \) is an extremal trajectory of type \( B_t \) (resp. \( S_t \)), then \( t < 2\pi \) (resp. \( t < 2\pi/c_\alpha \)).

There are two quantities that are remain constant along an extremal trajectory. The first one comes from the fact that the minimized Hamiltonian \( H \) is constant along the extremal pairs \( (\gamma, \lambda) \) (cf. (16) and 17):

\[
I_1 := -\varphi_1(t) + |\varphi_2(t)| = \lambda_0, \tag{21}
\]

with \( \lambda_0 \) equal to zero or one (cf. Remark 1). The second conserved quantity is:

\[
I_2 := c_\alpha^2 \varphi_2^2 + s_\alpha^2 \varphi_1^2 + \varphi_3^2 = K^2, \quad \text{for some } K \in \mathbb{R}. \tag{22}
\]

**Remark 3** Equations (18), (19), (20) are Hamiltonian equations on the dual of \( so(3) \), with respect to the canonical Poisson structure induced by the brackets of \( f, g, h \in so(3) \), and corresponding to the left-invariant Hamiltonian (11). The conserved quantity \( I_2 \) is the Casimir function (see for instance [1]).

There is a geometric interpretation of the above equations. Let \( (\gamma, \lambda) \) be a normal extremal lift of the time-optimal control problem. Then, the adjoint vector \( \lambda \) with coordinates \( (\varphi_i)_{i=1,2,3} \), lies in the intersection of the region defined by Eq. (21) and the ellipsoid defined by Eq. (22).

### 2.4 Classification of optimal trajectories

In this section, we investigate the structure of time optimal trajectories by analyzing the extremal flow defined in (18)-(20), subject to (21) and (22). First we study abnormal extremals (we prove that they are regular bang-bang and we establish a relation between the interior bang times). Then we study normal extremals that are bang bang (again we find relation between the interior bang times). Finally we study optimal trajectories containing a singular arc. The results presented in this section are well-known, and some of them already contained in [3, 4], although in many cases without proof. To have a self-contained paper, we provide an argument for all of them.

#### 2.4.1 Abnormal Extremals

The following proposition describes the switching behavior of abnormal extremals.

**Proposition 5** Let \( \gamma \) be an abnormal extremal. Then, it is regular bang-bang and the time duration between two consecutive switchings is always equal to \( \pi \). In other words, \( \gamma \) is of kind \( B_{\pi}B_{\pi}...B_{\pi}B_{\pi} \) with \( t \leq \pi \).

**Proof of Proposition 5** By definition, \( \lambda_0 = 0 \). Then Eq. 16 becomes

\[
\varphi_1(t) = -u(t)\varphi_2(t), \quad \text{for a.e. } t \in Dom(\gamma). \tag{23}
\]

If \( \gamma \) is singular on some interval \([c,d]\), then \( \varphi_2 \equiv 0 \) and from (19) \( \varphi_3 \equiv 0 \) on \([c,d]\). Eq. (23) gives \( \varphi_1 \equiv 0 \), contradicting the non triviality of \( \lambda \) (cf. Remark 2). Then \( \gamma \) cannot contain a singular arc. Therefore, \( u^2 = 1 \) a.e. \( t \in Dom(\gamma) \).

From Eqs. (20) and (23), we get a.e. \( \varphi_3(t) = (-c_\alpha^2 u(t)^2 - s_\alpha^2)\varphi_2(t) = -\varphi_2(t) \). This means that, in the \((\varphi_3, \varphi_2)\) plane, the vector \( z(t) := (\varphi_3(t), \varphi_2(t)) \) rotates with angular velocity equal to one (cf. Eq. (19)). This implies \( \gamma \) is a regular bang-bang trajectory and the time duration between two consecutive switchings along \( \gamma \) is always equal to \( \pi \).
2.4.2 Normal Bang-Bang Extremals

Let \( \gamma \) be a bang-bang trajectory starting at \( p_0 \) and ending at \( p_0 e^{(t_0 X_+)} e^{(t_1 X_-)} e^{(t_2 X_+)} e^{(t_3 X_-)} \). The case in which the first bang is of kind \( X_- \) is similar. We have \( \varphi_2(t_0) = \varphi_2(t_0 + t_1) = \varphi_2(t_0 + t_1 + t_2) = 0 \) which implies:
\[
< \lambda(t_0 + t_1), p_2 e^{-t_1 \text{ad} X_-}(g) >=< \lambda(t_0 + t_1), p_2 g >=< \lambda(t_0 + t_1), p_2 e^{t_2 \text{ad} X_+}(g) >= 0,
\]
where \( p_2 = p_0 e^{(t_0 X_+)} e^{(t_1 X_-)} \).

We need the following definition. If \( z_1, z_2, z_3 \) are (possibly time-varying) vector fields of \( SO(3) \), the application \( g \mapsto q(z_1 \wedge z_2 \wedge z_3) \) is the field of 3-vectors associated to the \( z_i \)'s, where \( q(z_1 \wedge z_2 \wedge z_3) \) is an element of \( \Lambda^3 T_q SO(3) \), the 3-fold exterior power of \( T_q SO(3) \). We now rewrite Eq. (24) by using fields of 3-vectors. We obtain:
\[
g \wedge e^{-t_1 \text{ad} X_-}(g) \wedge e^{t_2 \text{ad} X_+}(g) = 0.
\]

Thanks to (6), Eq. (25) is equivalent to \( r(t_1, t_2) f \wedge g \wedge h = 0 \) for an appropriate real-valued function \( r \). After computations, we get \( r(t_1, t_2) = \sin(\frac{t_1 - t_2}{2}) \). This implies that \( t_1 = t_2 = t_3 \).

Similarly to what we did in the proof of Proposition 5, consider now a time optimal trajectory of the form \( BB_T B \), where \( B_T \) is a nontrivial interior bang arc associated to a normal extremal and \( T \in [0, 2\pi[ \). From (20) and (16) (with \( \lambda_0 = 1 \)), we get \( \varphi_3 = - (\varphi_2 + s_\alpha^2 u) \). Using (19), this means that the vector \( z = (\varphi_3, \varphi_2 + s_\alpha^2)^T \) satisfies the differential equation
\[
\dot{z} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z, \quad t \in [0, T],
\]
with boundary conditions (the switching conditions imply \( \varphi_2(0) = \varphi_2(T) = 0 \), \( z(0) = (\varphi_3(0), us_\alpha^2) \) and \( z(T) = (\varphi_3(T), us_\alpha^2) \). Using \( \varphi_2(0) = 0 \) and the fact that \( u > 0 \) (resp. \( u < 0 \)) implies \( 0 < \varphi_2(0) = \varphi_3(0) \) (resp. \( 0 < \varphi_2(0) = \varphi_3(0) \)), one easily gets \( \tan(T/2) = \varphi_3(0)/(u s_\alpha^2) < 0 \). It follows that \( T \in (\pi, 2\pi) \). In summary, we proved that

**Proposition 6** Let \( \gamma \) be a bang-bang normal extremal. Then the time duration \( T \) along an interior bang arc is the same for all interior bang arcs and verifies \( \pi < T < 2\pi \).

**Remark 4** From Propositions 5 and 6, we get that, for an extremal bang-bang trajectory (normal or abnormal), the time duration \( T \) along an interior bang arc is the same for all interior bang arcs and verifies \( \pi \leq T < 2\pi \).

2.4.3 Optimal Trajectories Containing a Singular Arc

The purpose of this paragraph is to describe the structure of time optimal trajectories containing singular arcs.

**Proposition 7** Let \( \gamma \) be a time optimal trajectory containing a singular arc. Then \( \gamma \) is of the type \( B_t S_s B_{t'} \), with \( s \leq \frac{\pi}{c_\alpha} \) if \( t > 0 \) or \( t' > 0 \) and \( s < 2\frac{\pi}{c_\alpha} \) otherwise.

**Proof of Proposition 7** Let \( \gamma \) be a time optimal trajectory containing a singular arc \( S_t, t > 0 \). From Proposition 3, we know that \( t < 2\pi/c_\alpha \).

Assume now that \( \gamma \) contains a singular arc and a nontrivial interior bang arc. Then, we may assume that \( \gamma \) contains a piece of the type \( S_t B_t \) or \( B_t S_s \) (say the first), with \( B_t \) a complete bang arc. Then we have \( \varphi_2(s) = \varphi_3(s) = \varphi_2(s + t) = 0 \). This translates to: \( g \wedge h \wedge e^{t \text{ad} X_+}(g) = 0 \). Using (6), it implies that \( \cos(t) = 1 \), i.e. \( t = 2\pi \). This contradicts the time optimality of \( \gamma \). Finally, from (10), we get \( e^{t \text{ad} f}(g) = -g \). From this, we deduce for \( t \geq 0 \), \( e^{\frac{t}{c_\alpha} f} e^{t X_+} = e^{X_+} e^{\frac{t}{c_\alpha} f} \). Therefore, for \( t, s \geq 0 \), we have \( e^{s f} e^{t X_+} = e^{(s - \frac{t}{c_\alpha}) f} e^{X_+} e^{\frac{s}{c_\alpha} f} \). Then, if \( s > \frac{\pi}{c_\alpha} \) and \( t > 0 \) and taking into account what precedes, \( e^{s f} e^{t X_+} \) cannot be optimal.

2.5 Uniform Bound on the Number of Switchings for Time Optimal Trajectories

For \( \alpha \in [0, \pi/2[ \), let \( N(\alpha) \) be the supremum of the number of switchings of any time optimal trajectory on \( SO(3) \). Thanks to the left-invariance of the control system (1), we may assume that the supremum is taken over any time optimal trajectory starting at \( Id \). In this paragraph, we prove the following:

**Proposition 8** For \( \alpha \in [0, \pi/2[ \), \( N(\alpha) \) is finite (and thus achieved).
Proof of Proposition 8 Let us first prove that:

Claim every optimal trajectory of (Σ) is a finite concatenation of bang and singular arcs.

Let γ : [a, b] → SO(3) be a time optimal trajectory of (Σ). Let S be the set of zeroes of φ2 such that, if t ∈ S, then φ2 does not vanish identically in some neighborhood of t. Clearly, S is the set of times t such that γ(t) is the junction of two bang arcs or the the junction of a singular arc and a bang arc. The conclusion follows if S is finite. Reasoning by contradiction, S must have a limit point ̃t. Moreover ̃t ∈ S, otherwise φ2 would vanish identically in a neighborhood of ̃t, contradicting the fact that ̃t is a limit point of S. Note also that φ2 is continuous in an open (in [a, b]) neighborhood N of ̃t2 (see Remark 2). By definition of S and ̃t, there exists a sequence (tn) in N converging to ̃t such that φ2(tn) ≠ 0. Pick n large enough, so that, if [tn, tn+1] is the maximal subinterval containing tn with φ2 ≠ 0 on (tn, tn+1), then [tn, tn+1] ⊂ N. Clearly, φ2(tn) = φ2(tn+1) = 0, γ is a bang arc on [tn, tn+1] and tn+1−tn tends to zero as n goes to infinity since tn tends to ̃t. But, by Proposition 6, tn+1−tn ≥ π for n large enough. So we reached a contradiction and S is finite. The claim is proved. □

To finish the proof of Proposition 8, it remains to show that the (finite) number of switchings for any time optimal trajectory is uniformly bounded over SO(3). The argument goes by contradiction: there would exist, then, a sequence of regular bang-bang time optimal trajectories Bs1Bs2...BsNBs1, where sn, tn < 2π, π < Tn < 2π and the number of switchings mn goes to infinity as n goes to infinity. Therefore, there exists a sequence of points (xn) of SO(3) such that, the minimum time τn needed to connect Id to x0 by a trajectory of (Σ), goes to infinity as n goes to infinity.

To reach a contradiction, it is enough to show that there exists a time T so that, for every point x ∈ SO(3), there exists a trajectory γ of (Σ) connecting Id to x with T(γ) ≤ T. By a compactness argument and thanks to the SO(3)-invariance of (Σ), that would result from the following fact: there exists ̃t > 0 and an open neighborhood U ⊂ SO(3) of Id such that every point x ∈ U can be reached from Id in time less or equal than ̃t. The latter simply results from the facts that (Σ) has the accessibility property and eIf is periodic. □

Remark 5 Since the degree of non-holonomy of the distribution generated by (f, g) is equal to two, then, by standard controllability arguments, one can quantitatively relate the size of U and ̃t as follows: U contains a ball of radius 1 2 C for some positive constant C. Therefore, N(α) can be bounded above by S C, for some positive constant C′.

3 The Hopf Fibration and Proof of Theorem 1

3.1 The Hopf Projection

In this paragraph, we describe explicitly the Hopf projection from SO(3) to RP2. This projection provides SO(3) with a structure of fiber bundle with base RP2 and fiber S1. In the sequel, we use the identification of RP2 with S2 \ ∼, where ~ is the antipodal map, that is RP2 is the set of rows (y1, y2, y3), y2 2 = 1, where (y1, y2, y3) ~ (−y1, −y2, −y3). In the sequel, RP2 is identified with the subset H, made of the (open) top hemisphere together with half of the equator. Fix a point y0 ∈ RP2. The Hopf projection is defined as:

\[ \Pi : SO(3) \rightarrow RP^2, \]
\[ x \mapsto y = y_0 x, \]

where y0x is the standard matrix product. Then, any left-invariant vector field V : x ↦ xv on SO(3), v ∈ so(3), is transformed by dΠ into the (left-equivariant) vector field V_S = dΠ(V) : y ↦ yv. In the following, we call respectively, the control systems ̇x = x(f + ug), x ∈ SO(3) and ̇y = g(f + ug), y ∈ RP2, the control systems upstairs and downstairs. As explained next, it is crucial to choose y0 so that the drift term at the initial point, Id f, vanishes downstairs, so we require that Π(Id) = y0. Indeed, if it is the case, then: i) from the point y0, we have local controllability (see next section) and this greatly helps in the construction of the optimal synthesis; ii) if a trajectory upstairs starts with a singular arc (that is, with u = 0, i.e. it is an integral curve of the drift f), then its projection is a point. That suitable choice of y0 is made possible by the following normalizations:

\[ f = c_α \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \] and \[ g = s_α \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \]
\[ y_0 = (0, 0, 1). \]
For the Proof of Lemma 1, every subarc of a time optimal trajectory is also time optimal, we may assume that connecting the north pole to any point of \( N \), be the initial and terminal points of \( \gamma \). We now prove separately the two inequalities of Theorem 1.

\[ \gamma \]

\[ ii) \]

\[ i) \]

\[ ii) \]

\[ iii) \]

\[ \text{The above formula means that } N_S(\alpha) \text{ can take the values } N_0(\alpha), N_0(\alpha) + 1 \text{ or } N_0(\alpha) + 2; \]

\[ \text{They are projections, through the Hopf map } \Pi, \text{ of time optimal trajectories of } (\Sigma) \text{ starting at } Id. \]

\[ \text{Proof of Lemma 1} \]

For the proof of i), see Proposition 10 and Proposition 11. For the proof of iii), see Lemma 7 and Lemma 8. For the proof of ii) see Proposition 13.

We now prove separately the two inequalities of Theorem 1.

\[ \text{Proof of the inequality } N_S(\alpha) \leq N(\alpha). \]

From iii) of Lemma 1, every time optimal trajectory of \((\Sigma)_S\) connecting the north pole to any point of \( N \) is the projection by \( \Pi \) of a time optimal trajectory of \((\Sigma)\) with the same time duration (in particular of the time optimal trajectory connecting the two fibers). Therefore, \( N_S(\alpha) \leq N(\alpha) \).

\[ \text{Proof of the inequality } N(\alpha) \leq N_S(\alpha) + 4. \]

We refer to Fig.1. Consider a time optimal trajectory \( \gamma \) of \((\Sigma)\) containing \( N(\alpha) \) switchings. With no loss of generality, we may assume that \( N(\alpha) > 2 \). By Propositions 6 and 7, we deduce that \( \gamma \) is regular bang-bang and is of the type \( B_s B_{v(s)} \cdots B_{v(t)} B_t \), with \( s, t \geq 0, \pi \leq T \leq 2\pi \). Since every subarc of a time optimal trajectory is also time optimal, we may assume that \( s = t = 0 \). Let \( Id \) and \( x_1 \) be the initial and terminal points of \( \gamma \) and consider \( \gamma_S \), a time optimal trajectory for \((\Sigma)_S\) connecting II(\(Id\)) and II(\(x_1\)). From i) of Lemma 1, \( \gamma_S \) is of the type \( B_s B_{v(s')} \cdots B_{v(t')} B_t \), with \( s' \leq \pi, t' < v(s') \) and \( m \) interior bangs. We thus have \( m \leq N_S(\alpha) - 1 \). We now build, from \( \gamma_S \), a suboptimal trajectory connecting \( Id \) and \( x_1 \).
as follows: we can lift $\gamma_S$ to $SO(3)$ to an admissible trajectory $\tilde{\gamma}_S$ of $(\Sigma)$ connecting $x_0$ and $x_1$, with $x_0$ in the fiber of $\Pi(Id)$. It is also clear that $\gamma_S$ and $\tilde{\gamma}_S$ have same time durations. By construction of the fiber of $\Pi(Id)$, we get that $x_0 = e^{t''f}$ with $t'' \leq 2\pi$. Finally, the curve $\tilde{\gamma}$ obtained as the concatenation of $e^{t''f}$ and $\tilde{\gamma}_S$ is an admissible trajectory of $(\Sigma)$ connecting $Id$ and $x_1$. Its time duration is equal to

$$T(\tilde{\gamma}) = t'' + T(\tilde{\gamma}_S) = t'' + T(\gamma_S) = t'' + mv(s') + s' + t',$$

with $m \leq N_S(\alpha) - 1$. Since $\gamma$ is time optimal, we have $T(\gamma) \leq T(\tilde{\gamma})$, which implies that

$$\left(N(\alpha) - 1\right)T \leq t'' + mv(s') + s' + t'.$$

Using all the estimates on $T, s', t'$, we deduce that

$$\left(N(\alpha) - 1\right)\pi < N_S(\alpha)V(\alpha) + 3\pi,$$

where $V(\alpha) := \max_{s' \in [0,\pi]} v(s')$, $t'' < 2\pi$,

from which we have

$$N(\alpha) - N_S(\alpha) < N_S(\alpha)\frac{V(\alpha) - \pi}{\pi} + 4.$$  \hspace{1cm} (29)

Set $r(\alpha) := N_S(\alpha)\frac{V(\alpha) - \pi}{\pi}$. A simple computation shows that $r(\alpha) = N_S(\alpha)\frac{2}{\pi} \arcsin(\tan^2(\alpha))$. Using (28), it is easy to see that $r(\alpha) \in [0, 1]$ on $[0, \pi/4]$. Since $N(\alpha)$ and $N_S(\alpha)$ are integers, we get $N(\alpha) - N_S(\alpha) \leq 4$. \hfill $\Box$

### 4 The Time Optimal Synthesis Downstairs

In this section, to compute $N_S(\alpha)$, we study the time optimal synthesis for the problem downstairs (26), starting from the point $y_0$.

**Definition 3** A time optimal synthesis for the problem downstairs (26), starting from the point $y_0$ is a family of time optimal trajectories $\Gamma = \{\gamma_y : [0, b_y] \to \mathbb{R}P^2, y \in \mathbb{R}P^2 : \gamma_y(0) = y_0, \gamma_y(b_y) = y\}$.

For that purpose, we use the theory of optimal syntheses on 2-D manifolds developed by Sussmann, Bressan, Piccoli and the first author in [9, 10, 11, 14, 15, 23, 24, 32, 33] and recently rewritten in [12]. The core of
the Theory consist of an explicit algorithmic construction (by induction on the number of switchings) of the optimal synthesis.

Note that the previous theory uses a more elaborated concept of synthesis, namely that of regular synthesis (see for instance [8, 12, 16, 25] and cf. Section 4.1.1). In the following, in order to compute \( N_S(\alpha) \) we just need to follow the steps of the algorithmic construction mentioned above, without requiring the existence of a regular synthesis. In the sequel, by time optimal synthesis, we refer to one in the sense of Definition 3, whose existence is simply guaranteed by Proposition 1.

Consider a two dimensional smooth manifold \( M \) and the problem of computing the time optimal synthesis from a fixed point \( y_0 \in M \) for the control system:

\[
\dot{y} = F(y) + uG(y), \quad y \in M, \quad |u| \leq 1,
\]

where \( F \) and \( G \) are \( C^\infty \) vector fields. We introduce three functions:

\[
\begin{align*}
\Delta_A(y) & := \text{Det}(F(y), G(y)) = F_1(y)G_2(y) - F_2(y)G_1(y), \\
\Delta_B(y) & := \text{Det}(G(y), [F,G](y)) = G_1(y)[F,G]_2(y) - G_2(y)[F,G]_1(y), \\
f_S(y) & := -\Delta_B(y)/\Delta_A(y).
\end{align*}
\]

The sets \( \Delta_A^{-1}(0), \Delta_B^{-1}(0) \) of zeroes of \( \Delta_A, \Delta_B \) are respectively the set of points where \( F \) and \( G \) are parallel, and the set of points where \( G \) is parallel to \([F,G]\). These loci are fundamental in the construction of the optimal synthesis. In fact, assuming that they are smooth embedded one dimensional submanifold of \( M \) we have the following:

- in each connected region of \( M \setminus (\Delta_A^{-1}(0) \cup \Delta_B^{-1}(0)) \), every extremal trajectory is bang-bang with at most one switching. Moreover, if the trajectory is switching, then the the value of the control switches from \(-1\) to \(+1\) if \( f_S > 0 \) and from \(+1\) to \(-1\) if \( f_S < 0 \);
- the support of singular trajectories (that are trajectories for which the switching function identically vanishes, see Definition 4 below) is always contained in the set \( \Delta_B^{-1}(0) \);
- a trajectory not switching on the set of zeroes of \( G \) is an abnormal extremal (i.e. a trajectory with vanishing Hamiltonian) if and only if it switches on the locus \( \Delta_A^{-1}(0) \).

Then the synthesis is built recursively on the number of switchings of extremal trajectories, canceling at each step the non optimal trajectories (see [12], Chapter 1).

Remark 6 As we will see later (see Proposition 10), the condition that \( \alpha < \pi/4 \) guarantees that there are no singular trajectories for the problem downstairs.

4.1 Basic Definitions and Facts on Optimal Synthesis on 2-d Manifolds

Consider the minimum time problem for the control system (30). In this Section, we recall some key facts for the construction of time optimal synthesis following [12].

The first ingredient is, as usual, the PMP, that, on a two dimensional manifold, has exactly the same form as described in Section 2.3 but with the following change of notation: \( x \in SO(3) \rightarrow y \in M, \lambda(t) \in T_\gamma(t)SO(3) \rightarrow \lambda(t) \in T_\gamma(t)M \). As for the problem upstairs, switchings are described by the switching function:

Definition 4 (Switching Function) Let \( (\gamma, \lambda) \) be an extremal pair. The corresponding switching function is defined as \( \phi(t) := < \lambda(t), G(\gamma(t)) > \).

Again, \( \phi \) is at least continuously differentiable \( (\dot{\phi}(t) = < \lambda(t), [F,G](\gamma(t)) > \), cf. discussion in (19), and it determines the switching rule, according to Proposition 2 with the change of notation \( \varphi_2 \rightarrow \phi \). Again, an extremal trajectory \( \gamma \), defined on \([a,b]\), is called singular if \( \phi \equiv 0 \) in \([a,b]\). The following three Lemmas illustrate the role of the two functions defined in (31), (32). The proofs can be found in [9, 12, 24].
Lemma 2 Let $\gamma$ be an extremal trajectory that is singular in $[a, b] \subset \text{Dom}(\gamma)$. Then $\gamma|_{[a,b]}$ is associated to the so called singular control $\varphi(\gamma(t))$, where:

$$
\varphi(y) = -\frac{\nabla A(y) \cdot F(y)}{\nabla B(y) \cdot G(y)},
$$

with $\Delta_A$ and $\Delta_B$ defined in Eqs. (31) and (32). Moreover, on $\text{Supp}(\gamma)$, $\varphi(y)$ is always well-defined and its absolute value is less than or equal to one. Finally $\text{Supp}(\gamma|_{[a,b]}) \subset \Delta_A^{-1}(0)$.

Lemma 3 Let $\gamma$ be an extremal bang-bang trajectory for the control problem (30), $t_0 \in \text{Dom}(\gamma)$ be a time such that $\phi(t_0) = 0$ and $G(\gamma(t_0)) \neq 0$. Then, the following conditions are equivalent: i) $\gamma$ is an abnormal extremal; ii) $\gamma(t_0) \in \Delta_A^{-1}(0)$; iii) $\gamma(t) \in \Delta_A^{-1}(0)$, for every time $t \in \text{Dom}(\gamma)$ such that $\phi(t) = 0$.

The following lemma describes what happens when $\Delta_A$ and $\Delta_B$ are different from zero.

Lemma 4 Let $\Omega \subset M$ be an open set such that $\Omega \cap (\Delta_A^{-1}(0) \cup \Delta_B^{-1}(0)) = \emptyset$. Then all connected components of $\text{Supp}(\gamma) \cap \Omega$, where $\gamma$ is an extremal trajectory of (30), are bang-bang with at most one switching. Moreover, if $f_S > 0$ throughout $\Omega$, then $\gamma|_{\Omega}$ is associated to a constant control equal to $+1$ or $-1$ or has a switching from $-1$ to $+1$. If $f_S < 0$ throughout $\Omega$, then $\gamma|_{\Omega}$ is associated to a constant control equal to $+1$ or $-1$ or has a switching from $+1$ to $-1$.

Definition 5 Let $\gamma^+ : [0, \tau] \rightarrow M$ (resp. $\gamma^- : [0, \tau] \rightarrow M$) be the trajectory of (30) starting at $y_0$ and corresponding to the constant control $u \equiv 1$ (resp. $u \equiv -1$). For $t \in [0, \tau]$, let $\Gamma^+(t)$ (resp. $\Gamma^-(t)$) be the support of the curve $\gamma^+|_{[0,t]}$ (resp. $\gamma^+|_{[0,t]}$).

Under the assumption $F(y_0) = 0$ and $\Delta_B(y_0) \neq 0$, the next lemma (for a proof, see for instance [12, 26]), describes the shape of the optimal synthesis in a neighborhood of $y_0$. That local behavior of the optimal synthesis remains actually the same as long as $\Gamma^+(t)$ and $\Gamma^-(t)$ do not intersect $\Delta_B^{-1}(0)$ and $\Delta_A^{-1}(0)$ (except of course at $y_0$).

Lemma 5 Consider the control system (30). Assume that $F(y_0) = 0$ and $\Delta_B(y_0) \neq 0$. Let $\Omega$ be an open neighborhood of $y_0$ such that, $\Omega \cap \Delta_B^{-1}(0) = \emptyset$ and $\Omega \cap \Delta_A^{-1}(0)$ is an embedded one-dimensional submanifold of $\Omega$. Let $\gamma^+ : [0, \tau] \rightarrow M$ (resp. $\gamma^- : [0, \tau] \rightarrow M$) be the trajectory of (30) starting at $y_0$ and corresponding to the constant control $u \equiv 1$ (resp. $u \equiv -1$). Then, for every $t_+, t_- \in [0, \tau]$ such that: (a) $\Gamma^+(t_+) \cap \Gamma^-(t_-) \subset \Omega$, (b) $\Gamma^+(t_+) \cap \Delta_A^{-1}(0) = \Gamma^-(t_-) \cap \Delta_A^{-1}(0) = \{y_0\}$, (c) $\Gamma^+(t_+) \cap \Gamma^-(t_-) = \{y_0\}$, we have the following. There exists an open neighborhood $U$ of $\Gamma^+(t_+) \cup \Gamma^-(t_-)$ contained in $\Omega$ such that, for every $y \in U$, there exists a unique extremal trajectory of (30) of the type $B_2B_1$ contained in $U$, which is time optimal and steers $y_0$ to $y$. In particular, the system (30) is controllable in $U$ and $\gamma^+$ (resp. $\gamma^-$) is time optimal up to $t_+$ (resp. $t_-$), see Fig. 2 A.
Finally, we need one more lemma, related to Lemma 3, and whose hypothesis are illustrated in Fig.2 B:

**Lemma 6** Consider the control system (30). Assume that i) \( F(y_0) = 0, \Delta_B(y_0) \neq 0 \), ii) there exists \( \bar{t}_+ > 0 \) such that \( \Gamma^+(\bar{t}_+) \cap \Delta_A^{-1}(0) = \{y_0, \gamma^+(\bar{t}_+)\} \), iii) there exists \( \varepsilon > 0 \) such that \( \Gamma^+(\bar{t}_+ + \varepsilon) \cap \Delta_A^{-1}(0) = 0 \). Then \( \gamma^+ \) is extremal exactly up to time \( \bar{t}_+ \). Moreover, any extremal trajectory \( \gamma \) defined on \([0, T]\) with \( T > \bar{t}_+ \) and coinciding with \( \gamma^+ \) on \([0, \bar{t}_+]\), switches at \( \bar{t}_+ \) to the constant control \( u \equiv -1 \) and thus \( \gamma \) is an abnormal extremal (cf. Lemma 3). A similar statement holds for \( \gamma^- \).

**Remark 7** Under the hypotheses of Lemma 6, one can prove that the abnormal extremal \( \gamma \) restricted to an interval \([0, \bar{T}]\), is a non-strict abnormal extremal if \( \bar{T} < \bar{t}_+ \), while it becomes a strict abnormal extremal if \( \bar{T} \geq \bar{t}_+ \) (cf. Section 2.3). In other words, \( \gamma \) becomes a strict abnormal extremal after the first switching. These fact are analyzed in details in [11] and [12] (see Chapter 4, and in particular Section 4.3, where strict abnormal extremals are called Non Trivial Abnormal Extremals).

### 4.1.1 Frame Curves and Frame Points

In this paragraph, we briefly recall, for sake of completeness, the main results of the theory developed in [23, 24] (see also [12]). That material is only used here and in Section 5, where some numerical simulations and conjectures are presented. In [23, 24] (see also [12]), it was proved that the control system (30), under generic conditions on \( F \) and \( G \) (with the additional assumption \( F(y_0) = 0 \) admits a time optimal regular synthesis in finite time \( T \), starting from \( y_0 \). By generic conditions, we mean conditions verified on an open and dense subset of the set of \( C^\infty \) vector fields endowed with the \( C^1 \) topology (see [12], formula 2.6 pp. 39). More precisely, let \( \mathcal{R}(T) \) be the reachable set in time \( T > 0 \) given by:

\[
\mathcal{R}(T) := \{y \in M : \exists b_y \in [0, T] \text{ and a trajectory } \gamma_y : [0, b_y] \to M \text{ of (30) such that } \gamma_y(0) = y_0, \gamma_y(b_y) = y\}.
\]

Then a time optimal regular synthesis is defined by: i) a family of time optimal trajectories \( \Gamma = \{\gamma_y : [0, b_y] \to M, y \in \mathcal{R}(T) : \gamma_y(0) = y_0, \gamma_y(b_y) = y\} \) such that if \( \gamma_y \in \Gamma \) and \( \bar{y} = \gamma_{b(t)} \) for some \( t \in [0, b_y] \), then \( \gamma_y = \gamma_{b(t)} |_{[0, t]} \); ii) a stratification of \( \mathcal{R}(T) \) (roughly speaking a partition of \( \mathcal{R}(T) \) in manifolds of different dimensions, see [12], Definition 27, p.56) such that the optimal trajectories of \( \Gamma \) can be obtained from a feedback \( u(y) \) satisfying:

- on strata of dimension 2, \( u(y) = \pm 1 \),
- on strata of dimension 1, called frame curves (FC for short), \( u(y) = \pm 1 \) or \( u(y) = \varphi(y) \), where \( \varphi(y) \) is defined by (34).

The strata of dimension 0 are called frame points (FP). Every FP is an intersection of two FCs. In [24] (see also [12]), it is provided a complete classification of all types of FPs and FCs, under generic conditions. All the possible FCs are:

- FCs of kind \( Y \) (resp. \( X \)), corresponding to subsets of the trajectories \( \gamma^+ \) (resp. \( \gamma^- \)) defined as the trajectory exiting \( y_0 \) with constant control +1 (resp. constant control –1);
- FCs of kind \( C \), called switching curves, i.e. curves made of switching points;
- FCs of kind \( S \), i.e. singular trajectories;
- FCs of kind \( K \), called overlaps and reached optimally by two trajectories coming from different directions;
- FCs which are arcs of optimal trajectories starting at FPs. These trajectories “transport” special information.

The FCs of kind \( Y, C, S, K \) are depicted in Fig. 3. There are eighteen topological equivalence classes of FPs. A detailed description can be found in [10, 12, 24].
Remark 8 The proof of the existence of a regular synthesis is made by means of a constructive algorithm (working recursively on the number of switchings), that builds explicitly the optimal trajectories (see [12], Section 2.5 p.56). We stress the fact that the existence of a regular synthesis cannot be guaranteed before the complete execution of the algorithm. Since for our systm (26), we do not reach the end of that construction, we cannot conclude that such a regular synthesis exists. However, we conjecture that last fact (see also Section 5).

4.2 The Problem Downstairs

In this section, we apply the theory recalled in Section 4.1 to the control system (26) on $S^2$ in order to compute $N_S(\alpha)$, the maximum number of switchings for time optimal trajectories connecting the north pole to any point of $\mathcal{NH}$. First we need some notations.

Definition 6 Set:

$$X_S^+(y) = F_S(y) + G_S(y) = X^+y = X^+ \times y = \begin{pmatrix} 0 & -c_\alpha & 0 \\ c_\alpha & 0 & -s_\alpha \\ 0 & s_\alpha & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -c_\alpha y_2 \\ c_\alpha y_1 - s_\alpha y_3 \\ s_\alpha y_2 \end{pmatrix},$$

$$X_S^-(y) = F_S(y) - G_S(y) = X^-y = X^- \times y = \begin{pmatrix} 0 & -c_\alpha & 0 \\ c_\alpha & 0 & s_\alpha \\ 0 & -s_\alpha & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -c_\alpha y_2 \\ c_\alpha y_1 + s_\alpha y_3 \\ -s_\alpha y_2 \end{pmatrix}.$$

Let $\gamma : [t_1, t_2] \to S^2$ be a trajectory of (30). If $\gamma$ corresponds to the constant control +1 (resp. -1) in $[t_1, t_2]$, we say that $\gamma|_{[t_1, t_2]}$ is a $X^+$-trajectory (resp. $X^-$-trajectory). Moreover, we call $\gamma^\pm$ the trajectories exiting the point $x_0$ with respectively, constant control +1 and −1. Let $t_\Delta^\pm$ be the last times for which $\gamma^\pm$ are optimal. We define $\gamma^\pm_{op} := \gamma^\pm|_{[0, t_\Delta^\pm]}$. If $\gamma_1 : [a, b] \to S^2$ and $\gamma_2 : [b, c] \to S^2$ are trajectories of (26) such that $\gamma_1(b) = \gamma_2(b)$, then the concatenation $\gamma_2 \ast \gamma_1$ is the trajectory:

$$(\gamma_2 \ast \gamma_1)(t) := \begin{cases} \gamma_1(t) \text{ for } t \in [a, b], \\ \gamma_2(t) \text{ for } t \in [b, c]. \end{cases}$$

Notice that, in the notation $\gamma_2 \ast \gamma_1$, $\gamma_1$ comes first.

The first quantities to be computed are $\Delta_A^{-1}(0), \Delta_B^{-1}(0)$ and the sign of $f_S$. Referring to Fig.5, we have for the system (26):

$$\Delta_A^{-1}(0) = \{(y_1, y_2, y_3)^T \in S^2 : y_2 = 0\},$$

$$\Delta_B^{-1}(0) = \{(y_1, y_2, y_3)^T \in S^2 : y_3 = 0\},$$

$$f_S(y) > 0, \ \forall \ y \in \{(y_1, y_2, y_3)^T \in S^2 : y_2y_3 > 0\},$$

$$f_S(y) < 0, \ \forall \ y \in \{(y_1, y_2, y_3)^T \in S^2 : y_2y_3 < 0\}.$$  

(35)
The set $\Delta_B^{-1}(0)$ is called the equator and $\Delta_A^{-1}(0)$ the meridian. Moreover, let $\mathcal{N}$ be the (open) top hemisphere, i.e. the set of points $(y_1, y_2, y_3)^T$ so that $y_3 > 0$ and (see Fig.4):

\[ \mathcal{N}^+ = \{ y \in \mathcal{N} : y_2 > 0 \}, \quad \mathcal{M}^+ = \{ y \in \mathcal{N} : y_1 > 0, y_2 = 0 \}, \quad \mathcal{E}^+ = \{ y \in S^2 : y_2 < 0, y_3 = 0 \}. \]

Similarly

\[ \mathcal{N}^- = \{ y \in \mathcal{N} : y_2 < 0 \}, \quad \mathcal{M}^- = \{ y \in \mathcal{N} : y_1 < 0, y_2 = 0 \}, \quad \mathcal{E}^- = \{ y \in S^2 : y_2 > 0, y_3 = 0 \}. \]

We also parametrize points $y$ of the meridian by the oriented angle between $\overrightarrow{y_0y}$ and $\overrightarrow{y}$. We use $P(\xi)$, $\xi \in [-\pi, \pi]$, to denote the point of the meridian defined by the angle $\xi$. Then $P(0) = y_0$ and $P(\alpha)$ (resp. $P(-\alpha)$) is the center of rotation in the north hemisphere of $X^+_S$ (resp. $X^-_S$). We also have that $\gamma^+$ (resp. $\gamma^-$), up to time $\pi$, is a half-circle with diameter $[y_0, P(2\alpha)]$ (resp. $[y_0, P(-2\alpha)]$), see Fig. 4.

From Lemma 4, it follows:

**Proposition 9** Let $\gamma : [0, T] \to S^2$, $\gamma(0) = y_0$ be an optimal trajectory for the control system (26). Then:

- $\gamma$ has at most a $X^+ * X^-$ switching in $\mathcal{N}^-$, that is, if $\text{Supp}(\gamma|_{[a, b]}) \subset \mathcal{N}^-$, then $\gamma|_{[a, b]}$ corresponds to one of the three following controls:
  
  1. $u = +1$ in $[a, b]$,
  2. $u = -1$ in $[a, b]$,
  3. there exists $c \in [a, b]$, such that $u = -1$ in $[a, c]$ and $u = +1$ in $[c, b]$;

- $\gamma$ has at most an $X^-_S * X^+_S$ switching in $\mathcal{N}^+$;

- $\gamma$ has at most an $X^-_S * X^+_S$ switching in the region $\{ x \in S^2 : y_2 > 0, y_3 < 0 \}$;

- $\gamma$ has at most an $X^+_S * X^-_S$ switching in the region $\{ x \in S^2 : y_2 < 0, y_3 < 0 \}$.

In Fig. 5, the integral curves of $F_S, G_S, X^+_S, X^-_S$ and the loci $\Delta_A^{-1}(0), \Delta_B^{-1}(0)$ are depicted. Moreover, the allowed switchings are indicated.

**Remark 9** Notice that, in $\mathcal{N}^+$ (resp. $\mathcal{N}^-$), $X^+_S$ points on the right (resp. on the left) of $X^-_S$, while, on the meridian, $X^+_S$ and $X^-_S$ are parallel (see Fig.4). More precisely, $X^+_S$ and $X^-_S$ point in the same direction on $\{ P(\xi), \xi \in [\alpha, \pi - \alpha[ \cup ] - \pi + \alpha, -\alpha[\}$ and in opposite directions on $\{ P(\xi), \xi \in [\alpha, \pi - \alpha[ \cup ]\pi - \alpha, \pi] \cup [-\pi, -\pi + \alpha[\}$. 

\[ \begin{align*}
    \text{Figure 4:} \\
    \end{align*} \]
4.2.1 Two properties of extremal trajectories

The following two propositions are essential in the construction of the optimal synthesis.

**Proposition 10** Every time optimal trajectory of (26), starting at the north pole, is regular bang-bang.

**Proof of Proposition 10** Since $\alpha < \frac{\pi}{4}$, by taking into account Lemmas 5 and 6, the curves $\gamma^+$ and $\gamma^-$ defined in Definition 6, do not intersect the equator and are time optimal until the first time they meet the meridian, i.e. exactly up to time $\pi$. Moreover, since singular arcs are contained in the equator and, thanks to Lemma 6, any time optimal trajectory $\gamma$ of (26), with at least one switching, is of the form $B_sB_t\ldots$, with $s \in [0, \pi]$ and $t > 0$. Finally, since $\gamma$ is the projection of a time optimal trajectory $\hat{\gamma}$ of (1), then the latter is also of the type $B_sB_t\ldots$. Therefore, by Proposition 7, $\hat{\gamma}$ cannot contain any singular arc, and so $\gamma$. □

**Proposition 11** Let $\gamma : [0, T] \to S^2$ be a time optimal trajectory for the control system (26) of the type $B_sB_t\ldots$. Then, all time durations of interior bang arcs are equal to $v(s)$, where:

$$v(s) := \pi + 2 \arctan \left( \frac{s}{c_s + \cot^2(\alpha)} \right).$$  \hspace{1cm} (36)

**Proof of Proposition 11** Consider $\hat{\gamma} : [0, T] \to SO(3)$, an optimal trajectory that projects on $\gamma$ through the Hopf fibration $\Pi$. Thanks to Proposition 6 (see also Remark 4), we have $\hat{\gamma} = B_sB_tB_t\ldots$, where $t_1 \in [\pi, 2\pi]$. Moreover, since that curve projects on a time optimal trajectory for (26), we will establish a relation between $s$ and $t_1$. We start from the relations $\varphi_2(s) = \varphi_2(s + t_1) = 0$, which can be written

$$< \lambda(s), G_S(\gamma(s)) > = < \lambda(s + t_1), G_S(\gamma(s + t_1)) > = 0. \hspace{1cm} \text{(37)}$$

Recall that $\lambda(s) = \lambda(0)e^{-sX_s}$, $\lambda(s + t_1) = \lambda(0)e^{-sX_s}e^{-t_1X_{-s}}$ and $\gamma(s) = e^{sX_s}\gamma(0)$, $\gamma(s + t_1) = e^{t_1X_{-s}}e^{sX_s}\gamma(0)$. Since $\gamma$ is nontrivial, then $\lambda(0)$ is a nonzero line vector of $\mathbb{R}^3$. Moreover, $\gamma(0) = y_0 = (0, 0, 1)^T$. Eq. (37) can be written as:

$$\lambda(0)e^{-sX_s}(g \times e^{sX_s}\gamma(0)) = 0, \hspace{0.5cm} \lambda(0)e^{-sX_s}e^{-t_1X_{-s}}(g \times e^{t_1X_{-s}}e^{sX_s}\gamma(0)) = 0.$$

The previous equations can be transformed to:

$$\det(e^{sX_s}\lambda(0)^T, g, e^{sX_s}\gamma(0)) = 0, \hspace{0.5cm} \det(e^{t_1X_{-s}}e^{sX_s}\lambda(0)^T, g, e^{t_1X_{-s}}e^{sX_s}\gamma(0)) = 0,$$

and then to:

$$\det(e^{sX_s}\lambda(0)^T, g, e^{sX_s}\gamma(0)) = 0, \hspace{0.5cm} \det(e^{sX_s}\lambda(0)^T, e^{-t_1X_{-s}}g, e^{sX_s}\gamma(0)) = 0.$$
Since $e^{xs} \lambda(0)^T$ is not zero, we deduce that:

$$\det(g, e^{xs} \gamma(0), e^{-t_1 X} g) = 0. \quad (38)$$

We end up with the relation:

$$-s^2 \alpha \cos (s - t_1/2) = c^2 \alpha \cos (t_1/2). \quad (39)$$

Taking into account that $\pi \leq t_1 < 2\pi$, we can simplify the previous equation to get (36).

4.3 Construction of the time optimal synthesis

In this section, we present, step by step, the construction of the TOS for (26). Since we will not complete that construction, we only provide here the steps for which the outcome is justified by a rigorous argument. For the other steps of the construction, we refer to the last section where we propose conjectures on their outcomes, which are supported by numerical simulation.

Step 1 By Lemmas 5 and 6, for every $\varepsilon > 0$, there exists an open neighborhood $U$ of $\Gamma^+(\pi - \varepsilon) \cup \Gamma^-(\pi - \varepsilon)$ (recall Definition 5) where the time optimal synthesis is described in Fig. 6 A. Moreover, $t_{op}^+ = t_{op}^- = \pi$ (recall Definition 6);

Step 2 taking into account the analysis of Sections 4.1 and 4.2, the time optimal trajectories for the problem downstairs are described by the following:

**Proposition 12** Every time optimal trajectory for the system (26), starting from the north pole, is contained in the following two sets of extremals, which are parametrized by the length of the first bang arc, the one of the last bang arc and the number of arcs:

$$\Xi^+(s, t) = \underbrace{e^{X^+ s} e^{X^+ v(s)} \cdots e^{X^+ v(s)} e^{X^+ y_0}}_{m \text{ terms}}, \quad (40)$$

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The switching curves (SC for short), associated to the set of extremals given in (40) and (41), are

\[ \Xi^-(s, t) = e^X_{S'} t e X_{S''} v(s) \cdots e^X_{S'} v(s) e X_{S''} y_0, \quad m' \text{ terms} \]

where \( s \in [0, \pi], t \in [0, v(s)] \), the number of bang arcs (\( m \) and \( m' \) respectively) is an integer and

(-) \( \epsilon = +1 \) (resp. \( \epsilon = -1 \)), if \( m \) is odd (resp. even),

(-) \( \epsilon' = +1 \) (resp. \( \epsilon' = -1 \)), if \( m' \) is even (resp. odd).

Step 3 Let \( A^+ \) and \( A^- \) be the two extremal trajectories starting resp. with controls \( u \equiv 1 \) and \( u \equiv -1 \), and

switching after time \( \pi \), i.e. corresponding resp. to \( \Xi^+(\pi, \cdot) \) and \( \Xi^-(\pi, \cdot) \). These two curves are abnormal extremals and their respective first bang arcs coincide with \( \gamma_{\text{op}}^+ \) and \( \gamma_{\text{op}}^- \). As explained in Remark 7, these two curves become strict abnormal extremals after time \( \pi \).

To describe them, consider, for \( \epsilon = \pm \) and \( 0 \leq k \leq \tilde{k} \) (\( \tilde{k} \) defined below), the half-circles \( L^\epsilon_k \subset \text{Clos}(\mathcal{N}^\epsilon) \), whose centers lie on \( 0 \bar{P}(\epsilon \alpha) \) and passing through the points \( P_k^{\epsilon-} \) and \( P_k^{\epsilon+} \), where

\[ P_n^+ := P(2n\alpha), \quad P_n^- := P(-2n\alpha), \]

for the integers \( n \) so that \( 2n\alpha \leq \frac{\pi}{2} + 2\alpha \). Note that \( \frac{\pi}{2} + 2\alpha < \pi \) for \( \alpha < \frac{\pi}{2} \) and, in fact, the last \( P_n^+ \) belongs to the bottom-half hemisphere, i.e. \( n \leq \tilde{k} \) where \( \tilde{k} := 2 + \left[ \frac{3\pi}{4\alpha} \right] \). It is easy to see that \( A^+ \) intersects the top half-meridian according to the following ordered sequence of points: \( y_0, P_1^+, P_2^+, P_3^+, \ldots \). Similarly, \( A^- \) intersects the top half-meridian at \( y_0, P_1^-, P_2^-, P_3^-, \ldots \). Moreover, let \( y_{eq}^+ \) and \( y_{eq}^- \) be the antipodal points of the equator which are the respective first intersections of \( A^+ \) and \( A^- \) with the equator. Note that they are reached at the same time \( T_{eq} \). Finally, consider the open subset of the top-hemisphere bounded below by the equator and obtained by removing the supports of \( A^+ \) and \( A^- \) up to time \( T_{eq} \), i.e. all the \( L^\epsilon_k \). That set is the disconnected union of the two “snake-shaped” simply connected regions \( S^+ \) and \( S^- \) (defined so that each \( S^\pm \) contains the center of rotation of \( X_{S}^\pm \)). Clearly \( S^+ \) and \( S^- \) are made of open segments of the meridian and open simply connected regions \( D^\epsilon_k \subset \mathcal{N}^\epsilon \) defined as follows. For \( k = 0 \), \( D^\epsilon_0 \) is delimited by \( \text{Supp}(\gamma_{\text{op}}) \) and the segment \( [P_0, P_1^\epsilon] \) and, for \( k \geq 1 \), \( D^\epsilon_k \) is delimited by \( L_{k-1}^\epsilon \) on the top, \( L_k^\epsilon \) on the bottom and by the segments \( [P_{k-1}^{\epsilon+}, P_k^{\epsilon+}] \) and \( [P_k^{\epsilon-}, P_{k+1}^{\epsilon-}] \) on the sides, see Fig. 6 B.

In the sequel, if \( A, B \) are two subsets of points of \( S^\epsilon \), we say that \( A \) is above \( B \) (or equivalently \( B \) is below \( A \)) if \( A \subset D_{k'}^\epsilon \) and \( B \subset D_{k''}^\epsilon \) with \( k < k' \), for some \( \epsilon, \epsilon'' \).

Step 4 The switching curves (SC for short), associated to the set of extremals given in (40) and (41), are defined as follows: they can be divided in two families, \( (C^\epsilon_k)^+ \) and \( (C^\epsilon_k)^- \). If \( \epsilon = \pm, 1 \leq k \leq N_0 - 1 \) and \( s \in [0, \pi] \), then

\[ C_k^\epsilon(s) = e^X_{s^\epsilon} v(s) e X_{s^\epsilon-} y_0, \quad C_{k+1}^\epsilon(s) = e^X_{s^\epsilon} v(s) C_{k+1}^\epsilon(s). \]

The boundary points of \( C^\epsilon_k \) are \( C_0^\epsilon(0) = P_0^\epsilon \) and \( C_k^\epsilon(\pi) = P_{k+1}^\epsilon \). By using Proposition 9 and since \( v(s) \geq \pi \), the support of \( C^\epsilon_k \) is contained in the subset of \( \text{Clos}(\mathcal{N}^\epsilon) \), delimited by the half-circle centered on \( 0 P(\alpha(2k + 1)) \) and passing through the points \( P_k^\epsilon, P_{k+1}^\epsilon \), and the segment of the meridian \( [P_k^\epsilon, P_{k+1}^\epsilon] \), see Fig 6 B. In particular, a (SC) with boundary points in the top-hemisphere is entirely contained in the top-hemisphere and the intersection of its support with the top-meridian reduces to its boundary points (see Lemma 3).

We next describe the shape of the first (SC) intersecting the equator. By symmetry, we may assume \( \epsilon = + \). We claim that its intersection with the equator reduces to the point \( P(\pi) = (1, 0, 0)^T \). Indeed, by the switching rules established in Proposition 9, the (SC) intersecting the equator is contained in \( \{ y \in S^2 : y_2 \leq 0, y_3 \geq 0 \} \cup \{ y \in S^2 : y_2 \geq 0, y_3 \leq 0 \} \). Taking into account the regularity of the (SC) and the values of its boundary points, the claim is proved, see Fig. 7 A.
4.4 Computation of $N_S(\alpha)$

In the previous section, we provided detailed informations about extremal trajectories and switching curves but we did not show that every extremal of (40) and (41) is in fact time optimal. Anyway, a rigorous derivation of $N_S(\alpha)$ is possible with the available knowledge of time optimal trajectories combined with the subsequent lemmas.

**Lemma 7** Every time optimal trajectory $\gamma$ starting at $y_0$ intersects the equator at most once.

**Proof of Lemma 7** we argue by contradiction. There would exist two distinct points of the equator $q_i, q_f$ so that $\gamma(t_i) = q_i, \gamma(t_f) = q_f$ and $\gamma|_{(t_i, t_f)}$ is entirely contained in the (closed) bottom hemisphere. Let $\gamma_{\text{sing}}$ be the integral curve of $F_S$ (contained in the equator) connecting $q_i$ to $q_f$. Consider now the region of the bottom hemisphere bounded by $\gamma_{\text{sing}}$ and $\gamma|_{(t_i, t_f)}$. Taking into account, first, the relative positions of $X^+ \epsilon S, X^- \epsilon S, F_S$ and $G_S$ along the equator and, second, the sign of $f_S$ in the bottom hemisphere, one can check that $T(\gamma_{\text{sing}}) \leq T(\gamma|_{(t_i, t_f)})$. The argument is similar to that of [32] (see also [12]) and is based on the use of Stokes theorem. Since time optimal trajectories starting at $y_0$ do not contain a singular arc, it follows that $\gamma$ cannot be time optimal. We reached a contradiction. ■

**Lemma 8** Every time optimal trajectory $\gamma$, starting at $y_0$ and remaining in $\mathcal{N} \mathcal{H}$, is the projection of a time optimal trajectory of $(\Sigma)$ starting at $\text{Id}$.

**Proof of Lemma 8** From the definition of the Hopf fibration, every trajectory $\gamma$ of $(\Sigma) S$, starting at $y_0$, associated to an admissible control $u$ and staying in $\mathcal{N} \mathcal{H}$, is the projection of the trajectory $\bar{\gamma}$ of $(\Sigma)$ starting at $\text{Id}$ with the same control $u$. In particular, $\gamma$ and $\bar{\gamma}$ have same time duration. It is clear that, if $\gamma$ is time optimal, then $\bar{\gamma}$ is also time optimal. ■

**Lemma 9** Recall that $S^e \subset \mathcal{N} \mathcal{H}$. With the notations above, pick any point $y$ in the region $S^e$ and let $\gamma_y$ be a time optimal trajectory connecting the north pole $y_0$ to $y$. If $s \in [0, \pi]$ is the time duration of the first bang arc and $T(y)$ the total time duration of $\gamma_y$, then $\gamma_y|_{(s, T(y))}$ is entirely contained in $S^e$.

**Proof of Lemma 9** By the switching rules of Proposition 9, along every time optimal trajectory contained in $\mathcal{N} \mathcal{H}$, the control must switch from $\epsilon$ to $-\epsilon$, when arriving at a switching curve $C^e_k$. In addition, the time optimal trajectory switches from being an arc of circle (integral curve of $X^e_k$) to another arc of circle of bigger radius (integral curve of $X^{\epsilon \epsilon}_S$). After rectification of the flow of $X^e_k$, (i.e. the one entering the (SC) $C^e_k$), then, by taking into account Remark 9, one gets the situation depicted in Fig. 7 B. By contradiction, we assume that there exists a time optimal trajectory $\gamma$ with time duration $T$ and first bang arc time duration $s < T$ such that $\gamma$ connects $y_0$ to $y \in S^e$ and $\gamma|_{(s, T(y))}$ exits from $S^e$. Let $t'$ be the smallest
time (in \([0,T]\)) so that \(\gamma|_{(t',T]}\) is entirely contained in \(S^\varepsilon\). Then \(\gamma(t')\) belongs to \(\text{Supp}(\mathcal{A}^+_{[0,T_{eq}]} \cup \text{Supp}(\mathcal{A}^-_{[0,T_{eq}]}))\) (see step 3 of Section 4.3 for the definition of \(T_{eq}\)). If \(\gamma(t')\) is on the (top)-meridian, then it has to switch so that the interior bang time duration is constant, equal to \(\pi\). Therefore \(\gamma|_{(t',T]}\) will never re-enter \(S^\pm\). We thus deduce that \(\gamma(t')\) is not on the meridian and, with no loss of generality, we will assume that \(\gamma(t')\) belongs to the \((1\text{-dim.})\) interior of some \(L^k_+\), \(k \geq 1\).

Now we make the following two claims:

**Claim 1** With the notations above, there exist \(t'' < t' < t'''\) such that

\[
\gamma|_{(t',t'')} \subset D^\varepsilon_k \subset S^\varepsilon \quad \text{and} \quad \gamma|_{(t'',t')} \subset D^{\varepsilon_{k+1}}_k \subset S^{-\varepsilon}, \quad \text{for some } \varepsilon' \in \{-, +\},
\]

i.e. \(\gamma\) passes (backward in time) from \(S^\varepsilon\) to \(S^{-\varepsilon}\) at time \(t'\) by going “down”.

Proof of Claim 1: it is clear that there exists a neighborhood \(U\) of \(t'\) so that \(\gamma|_U\) is an integral curve of \(X^-\varepsilon\). Thanks to Remark 9 and to the argument above, \(\gamma|_U\) intersects \(\text{Int}(L^k_+)\) transversally (see figure 8) in such a way that \(\gamma\), run backward in time, goes from \(D^+_k\) to \(D^{+}_{k+1}\). Claim 1 is proved.

Now, by definition of \(t'\), \(\gamma(t') \in \mathcal{A}^+_{[0,T_{eq}]}\). Let \(\gamma_{ab}\) be the restriction of \(\mathcal{A}^\varepsilon\) between \(y_0\) and \(\gamma(t')\). Consider \(\tilde{\gamma}\), the concatenation of \(\gamma_{ab}\) and \(\gamma|_{(t',T]}\). The conclusion of Lemma 9 will follow if one can show that the time duration \(T'\) of \(\tilde{\gamma}\) is less than \(T\), the time duration of \(\gamma\). This, in turn, amounts to show that \(T'\), the time duration of \(\gamma_{ab}\) is less than \(t'\), the time duration of \(\gamma|_{[0,t']}\). This is the object of the next Claim.

**Claim 2** With the notations above, we have \(T' < t'\).

Proof of Claim 2: The trajectory \(\gamma\), run backward in time from \(t'\), is an \(X^-\varepsilon\)-integral curve until it hits a (SC) in some \(D^k_L \in \mathcal{N}^\varepsilon\), for some integer \(L \geq k + 1\), at a point \(C^L_k(s)\), \(s \in [0,\pi]\). One can easily conclude that the only possibility is \(L = k + 1\). By Claim 1, a time optimal trajectory can pass (backward in time) from \(S^\varepsilon\) to \(S^{-\varepsilon}\) only by going down, i.e. by passing from some \(D^k_L\) to \(D^{k+1}_L\). Therefore, by an elementary counting argument, one gets

\[
t' = s + t_s + (k + 1)v(s),
\]
where $t_s$ is the time needed to go from $\gamma(t')$ to $C^r_{k+1}(\bar{s})$. On the other hand,

$$T' = (K + 1)\pi - \tilde{t},$$

where $\tilde{t} \in [0, \pi]$ is the time needed to $A^*$ to go from $\gamma(t')$ to $P^r_{k+1}$. Since $v(\bar{s}) > \pi$, then $T' < t'$. The proof of Lemma 9 is finished.

**Remark 10** Coupled with the proof of Claim 2, a simple continuity argument implies that $A^+$ and $A^-$ are time optimal trajectories in the top hemisphere.

Gathering all the information on time optimal trajectories, we are now able to compute $N_S(\alpha)$.

**Proposition 13** For $\alpha \in [0, \pi/4]$, we have

$$N_S(\alpha) := 2 \left[ \frac{\pi}{8\alpha} \right] - 2 \left[ \frac{\pi}{8\alpha} \right] - \frac{\pi}{4\alpha}.$$  \hspace{1cm} (42)

**Proof of Proposition 13** Let $y \in S^+$ and $\gamma$, a time-optimal trajectory connecting $y_0$ to $y$. The point $y$ belongs to some $D^r_k$, $k \leq N_0$ and, by Lemma 9, $\gamma$ remains in $S^+$. Since the function $v$ takes values in $[\pi, \pi + \pi/2]$, it is easy to see, from Eqs. (40) and (41) and Remark 10 that $\gamma$, run backward in time, will go through the ordered sequence of regions $D^r_k$, $D^r_{k-1}$, $D^r_{k-2}$, etc, until hitting one of the two curves $\gamma^+_k$ or $\gamma^-_k$. Moreover, in each of the region $D^r_k$, $\gamma$ will switch exactly once, thanks to Proposition 9. Therefore, the number of times, where an optimal trajectory $\gamma$ starting at $y_0$ switches, is exactly equal to the number of times $\gamma$ crosses the subset of the meridian contained in $NH$. The same conclusion holds for points belonging to $S^-$, $A^+$ and $A^-$. By a systematic examination of all the possibles cases, we end up with (42). Note that $N_S(\alpha)$ is the number of switchings for a time optimal trajectory ending on the equator. 

### 4.5 Geometric Remarks

#### 4.5.1 Relations with the Linear Pendulum

In a fixed neighborhood of the north pole, the control system on the sphere (26) behaves, when $\alpha > 0$ is small enough, as a controlled linear pendulum. More precisely, let us consider the stereographic projection of the sphere from the south pole $(0, 0, -1)$ on $V$, the tangent plane to the sphere at the north pole. If $y_1, y_2, y_3$ are the coordinates of the three dimensional Euclidean space where the sphere is embedded, a system of coordinates on $V$ is $(y_1, y_2)$, see Fig. 9. Let $x_0^+$ and $x_0^-$ be the projections of the equilibrium points of $X^+_{\gamma}$ and $X^-_{\gamma}$ in $NH$.

An alternative way of parametrizing this problem (instead of fixing the radius of the sphere and varying the axes of rotations) consists of fixing the points $x_0^+ = (1, 0)^T$, $x_0^- = (-1, 0)^T$ and varying the radius $r$ of the sphere. The relation between $\alpha$ and $r$ is $\tan(\alpha) = 1/r$. The range $\alpha \in [0, \pi/4]$ becomes $r \in [1, \infty]$ and $\alpha \to 0$ corresponds to $r \to \infty$. In $V$, fix a ball $B(0, r_0)$ of radius $r_0 > 0$ centered in the origin, and consider the stereographic projection of the integral curves of $X^+_{\gamma}$ and $X^-_{\gamma}$. For $r \to \infty$, they become circles centered at the points $x_0^\pm$. Then, one easily sees that, in $B(0, r_0)$, the limit system (and the associated synthesis) corresponds to a controlled linear pendulum (with the associated synthesis) of equation: $\dot{y}_1 = -y_2$, $\dot{y}_2 = y_1 - u$, $|u| \leq 1$. Notice that $\lim_{\alpha \to \infty} v(s) = \pi$, that is exactly the time duration of interior bang arcs for the linear pendulum.

#### 4.5.2 The time optimal problem on $SU(2)$

The optimal control problem on $NH$ is the projection (by a Hopf fibration) of an optimal control problem on $SO(3)$. Similarly, the corresponding problem on the whole sphere $S^2$ is the projection (by an appropriate Hopf fibration) of an optimal control problem on $SU(2)$. Indeed, $SU(2)$ is the universal (double) covering of $SO(3)$ and they have the same Lie algebra $so(3)$. The existence of that double covering justifies, by a factor two, the difference between our bound and the bound (2), on the maximal number of switchings for the control problem on $SO(3)$. Indeed, the index theory developed by Agrachev and Gamkrelidze in [3, 5] provides a bound on the number of switchings by proving that a certain extremal is not optimal because it loses local optimality working at the Lie algebraic level. This is why the upper bound in (2) corresponds (essentially) to a control problem on $SU(2)$, and thus, after projection, on a control problem on the whole sphere $S^2$, and not just on $NH$. The other factor two, of the difference between our bound and the bound given in (2), comes from the fact that in [3] the index of the second variation was estimated up to an additive factor 1 (see [3], p.275).
5 Conclusion and Open Problems

In the previous Section, we derived a set of properties of the optimal synthesis that were sufficient to compute the maximum number of switchings of a time optimal trajectory joining $y_0$ to any point of the north hemisphere. This enabled us to provide a precise estimate for $N(\alpha)$, $\alpha \in [0, \pi/4]$. However, the following questions remain unsolved:

**Question 1** are all the extremal trajectories (40) and (41) optimal in the north hemisphere?

The answer to this question depends on the answer to the next question:

**Question 1’** in the north hemisphere, are the switching curves $C^\varepsilon_k(s)$, $s \in [0, \pi]$, locally optimal? (The points $s = 0, s = \pi$ are not included since we already know that the two abnormal extremal $A^\pm$ are optimal in $N_H$.)

Roughly speaking we say that a switching curve is locally optimal if it never “reflects” the trajectories. More precisely, we have the following definition (clarified by Fig.10).

**Definition 7** Consider a smooth switching curve $C$ between two smooth vector field $Y_1$ and $Y_2$ on a smooth two dimensional manifold. Let $C(s)$ be a smooth parametrization of $C$. We say that $C$ is **locally optimal** if, for every $s \in \text{Dom}(C)$, we have

$$\dot{C}(s) \neq \alpha_1 Y_1(C(s)) + \alpha_2 Y_2(C(s)), \text{ for every } \alpha_1, \alpha_2 \text{ s.t. } \alpha_1 \alpha_2 \geq 0.$$

(43)

The points of a switching curve on which relation (43) is not satisfied are usually called “conjugate points”.

**Remark 11** Notice that, if all the switching curves are locally optimal in the north hemisphere, it follows that the set of extremals (40) and (41) (restricted to $N_H$) is an optimal synthesis for the problem (26) on $\mathbb{RP}^2$. In this case, on $\mathbb{RP}^2$, the extremals (40) and (41) lose global optimality before losing local optimality.

**Question 2** If the answer to Question 1’ is yes, what about the same question for the optimal control problem on $S^2$? More precisely, one would like to understand how the extremal trajectories (40) and (41) are going to lose optimality in a neighborhood of the south pole (i.e. if the loss of optimality is local or just global).

**Question 3** What is the shape of the optimal synthesis in a neighborhood of the south pole?
In this section, we present the results of some numerical simulations which provide some hints regarding the above questions. More precisely:

- There is strong numerical evidence for a positive answer to Question 1’. This means that the switching curves in the north hemisphere never reflect trajectories. In other words, situations like those considered in the proof of Lemma 9, (cf. Fig.8) are not possible.

- As concerns Question 2, we conjecture the following:

  C1 The curves $C_\varepsilon(k)$, $s \in [0, \pi]$ are locally optimal if and only if $X^+_S(C_\varepsilon(0)) = \alpha_1 X^-_S(C_\varepsilon(0))$ and $X^+_S(C_\varepsilon(\pi)) = \alpha_2 X^-_S(C_\varepsilon(\pi))$ with $\alpha_1, \alpha_2 \geq 0$ but not both vanishing.

  This condition is verified if and only if: $k \leq \left[\frac{\pi}{\alpha_2} \right] - 1$, which simply follows from Remark 9.

  Set $N_A := \left[\frac{\pi}{2\alpha} \right]$. Analyzing the evolution of the minimum time wave front in a neighborhood of the south-pole, it is reasonable to conjecture that:

  C2 For $T \leq (N_A - 1)\pi$, the synthesis built above is optimal. Every $x \in S^2$ is reached in time $T \leq (N_A + 1)\pi$. Every optimal trajectory has at most $N_A$ switchings and there exists an optimal trajectory having $N_A - 1$ switchings.

  In the top of Fig. 11, the optimal synthesis is plotted.

- Regarding Question 3, numerical simulations suggest that the shape of the optimal synthesis for time $T > (N_A - 1)\pi$, depends on the remainder:

  $$r := \pi - 2\alpha N_A = \pi - 2\alpha \left[\frac{\pi}{2\alpha} \right].$$

  Notice that $r$ belongs to the interval $[0, 2\alpha]$. More precisely, we conjecture the following:

  C3 For $\alpha \in [0, \pi/4]$, there exist two positive numbers $\alpha_1$ and $\alpha_2$ such that $0 < \alpha_1 < \alpha < \alpha_2 < 2\alpha$ and:

  CASE A: $r \in [\alpha_2, 2\alpha]$. The switching curve starting at $P^+_N$ glues to an overlap curve that passes through the origin (see the bottom of Fig. 11, Case A).

  CASE B: $r \in [\alpha_1, \alpha_2]$. An overlap curve starts exactly at $P^+_N$ and passes through the origin.

  CASE C: $r \in [0, \alpha_1]$. The situation is more complicated and it is depicted in the bottom of Fig. 11, Case C.

  For $r = 0$, the situation is the same as in CASE A, but for the switching curve starting at $P^+_N - 1$.

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Figure 11: The Time Optimal Synthesis on the Sphere (top) and Optimal Synthesis in a neighborhood of the south pole, Case A and Case C (bottom)
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