Generalized Integral Fluctuation Relation with Feedback Control for Diffusion Processes

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Abstract We extend a generalized integral fluctuation relation in diffusion processes that we obtained previously to the situation with feedback control. The general relation not only covers existing results but also predicts other unnoticed fluctuation relations. In addition, we find that its explanation of time-reversal automatically emerges in the derivation. This interesting observation leads into an alternative inequality about the entropy-like quantity with an improved lower bound. Two feedback-controlled Brownian models are used to verify the result.

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1 Introduction

In the past two decades, an important progress of nonequilibrium physics is the discovery of a variety of fluctuation relations. [1–14] These exact relations about the statistics of entropy production or dissipated work have greatly deepened our understanding of the second law of thermodynamics and nonequilibrium physics of small systems. Recently, these fluctuation relations were also extended into other fields, e.g., the thermodynamics of information-processing systems. [15–16] For instance, Sagawa and Ueda [17] obtained a generalized Jarzynski equality with feedback control,

\[
\exp(-\beta W_{\text{diss}} - I) = 1, \tag{1}
\]

where \( \beta \) is the inverse temperature of heat bath, \( W_{\text{diss}} \) is the dissipated work, [7] and \( I \) is the mutual information. This intriguing work equality leads into an inequality,

\[
\beta(I) \geq -\left\langle I \right\rangle, \tag{2}
\]

which agrees with the second law of information thermodynamics. [18] The validity of these relations has been verified by the single colloidal particle experiment. [19] Very recently, Abreu and Seifert [20] extended Sagawa–Ueda’s equality (1) into genuine nonequilibrium processes in the master equation systems. Analogous equalities and inequalities with feedback control were obtained for the excess entropy production and total entropy production. Other more general relations were also reported. [21–23]

As we mentioned at the beginning, in classical systems there are various fluctuation relations. The results of Sagawa and Ueda, [17] and Abreu and Seifert [20] have implied the plausibility of introducing feedback control into those relations. Moreover, we note that in the past few years there were quiet a few efforts of unifying the fluctuation relations. [24–29] Hence, it shall be more desirable if these various relations with feedback control could be as well unified under a single formula. Here we show that the unification does exist for the continuous diffusion processes. Instead of starting from the conventional detailed fluctuation theorem, [17,21–22,30–31] our theory is based upon the generalized fluctuation integral relation (GIFR) [27,29] that we proposed earlier. Its advantage is that the GIFR does not need to explicitly define time-reversal. Hence, it is capable to unify the various fluctuation relations relying on distinct time-reversals, e.g., that of the Hatano–Sasa equality. [9,14]

The organization of the work is as follows. In Sec. 2 we briefly review the GIFR in the diffusion processes. In Sec. 3 we extend it by including feedback control. In Sec. 4 we first show that the generalized relation not only covers the existing results but also gives other previously unnoticed equalities. Then we point out its time reversal explanation and obtain an improved inequality about the entropy-like quantity. In the end, we use two feedback-controlled Brownian particles to verify this result. Section 5 is a summary of this work. Appendix A extends the GIFR with feedback control in the diffusion processes into the continuous time master equations with discrete states. Appendix B explains the Bochkov–Kuzovlev equality (BKE) in overdamped Brownian motion from the point of view of the GIFR.
2 Overview of GIFR in Diffusion Processes

Consider a general N-dimension (N-d) stochastic system with variables \( x = \{ x_i \}, i = 1, \ldots, N \). The dynamics of the system is described by the stochastic differential equations (SDE)\[32–33\]

\[
d x_i(t) = A_i(x,t) \, dt + \sqrt{B(x,t)} \, dW(t),
\]

during time interval \((t_0, t_f)\), where \( dW \) is an N-d Wiener process. \( A = \{ A_i \} \) denotes an N-d drift vector, and \( \sqrt{B} \) is the square root of an \( N \times N \) positive definite and symmetric diffusion matrix \( B \). In the work we follow the Ito's convention for the SDE. Rather than directly solving (3), one usually converts it into the evolving equation of the probability density function (pdf) \( p(x,t) \) of the system, i.e., the Fokker–Planck equation, \( \partial_t p = \mathcal{L}(x,t)p \). The Fokker–Planck operator is

\[
\mathcal{L}(x,t) = -\partial_{x_i} A_i(x,t) + \frac{1}{2} \partial_{x_i} \partial_{x_j} B_{ij}(x,t),
\]

where the Einstein’s summation convention is used. We proved that the \( x \)-integral,

\[
\int dx_0 p(x,t') R(t_f|x,x') = \int dx_0 p(x,t') \left( e^{-\int_{t_f}^{t_0} J[\theta,S](x(r),\tau) \, dr} O(x(t_f)) \right)\bigg|_{\theta_0} = (O)_{\theta_0},
\]

is a \( t' \)-invariant \((t_0 \leq t' \leq t_f)\)[27,29] where \( p(x,t') \) is an arbitrary pdf, \( O(x) \) is an arbitrary function, and the average \( \langle \cdot \rangle \) is over all stochastic trajectories starting from \( x \) at time \( t' \) and following (3). In the above equation we defined

\[
J[\theta,S] = \theta^{-1} [\mathcal{L} - \partial_{x_i} S_i + 2\theta^{-1} S_i B_{ij}^{-1} S_j] + 2\theta^{-1} S_i B_{ij}^{-1} (x_1 - A_i),
\]

where the dot denotes the time derivative \( d/d\tau \), \( S = \{ S_i \} \) is an \( N \)-d vector field which is either zero at the boundary of the system or is periodic if the system is periodic. Particularly, if we let \( t' = t_0 \), \( t_f \) implies an identity

\[
\langle e^{-\int_{t_f}^{t_0} J[\theta,S](x(r),\tau) \, dr} O(x(t_0)) \rangle_{\theta_0} = (O)_{\theta_0},
\]

where the subscripts \( \theta_0 \) and \( \theta_f \) indicate that the averages are done over \( p(x,t_0) \) and \( p(x,t_f) \), respectively. Since (7) is a general mathematic identity and can cover several important fluctuation relations[6–9,11] by choosing specific \( \theta \) and \( S \), we named it the GIFR.

The function \( R(t_f|x,x') \) has an interesting time-reversal explanation. Assume the variables \( x \) of the stochastic system to be even or odd according to their rules under time reversal: \( x_i \rightarrow x_i \) is even and \( x_i \rightarrow -x_i \) is odd; in abbreviation \( x_i \rightarrow \tilde{x}_i = \varepsilon_i x_i \) with \( \varepsilon_i = \pm 1 \). We found\[27,29\]

\[
R(t_f|x,x') p(x,t') = (O)_{\theta_0} g(\tilde{x},s),
\]

where \( s + t' = t_f \), \( g(x,s) \) is the pdf of the time-reversed Fokker–Planck equation, whose drift force and diffusion matrix are

\[
\tilde{A}_i(x,s) = -\varepsilon_i A_i(\tilde{x},t') + \frac{2\varepsilon_i}{\theta}(\tilde{x},t'),
\]

\[
\tilde{B}_{ij}(x,s) = \varepsilon_i \varepsilon_j B_{ij}(\tilde{x},t'),
\]

respectively, and its initial condition is \( p(x; t_f) = (O)_{\theta_f} \)[34]. Note that there is no summation over the index \( i \) here.

The time-reversal explanation leads into two important consequences. First, the generalized detailed balance relation was established\[26,29\]

\[
R(x_2,t_2|x_1,t_1) p(x_1,t_1) = G(\tilde{x},s_1) g(s_2) p(x_2,t_2),
\]

where \( G(x_1,s_1|x_2,s_2) \) is a specific \( R(t_2|x_1,t_1) \) in (5) with \( O(x) = \delta(x - x_2) \), and \( G(x_1,s_1|x_2,s_2) \) is the transition probability of the time-reversed process from earlier time \( s_2 = t_f - t_2 \) to the later time \( s_1 = t_f - t_1 \). Second, applying the Jensen’s inequality to (8) with \( O(x) = 1 \), we obtain an inequality

\[
\langle \int_{t_0}^{t_f} J[\theta,S](x(r),\tau) \, dr \rangle_{\theta_0} \geq D \langle g(x,t_0) \rangle_{\theta_0},
\]

where the right-hand side is the relative entropy between the two pdfs, which is always nonnegative.\[35\] We name the left hand side of (12) the entropy-like quantity. We did not present this result in our previous work. Several specific cases can be found in [36].

3 GIFR with Feedback Control

Rather than considering very general diffusion processes, in the remaining sections we focus on those that the time-dependence of their drift vectors and diffusion matrices only comes from a set of external control parameters \( \lambda \). During the whole process, we assume that there are \( M \) measurements and feedback loops applied on (3) at discrete times \( t_k, k = 1, \ldots, M \)[30–31]. From \( t_0 \) to \( t_1 \), the system evolves under the predetermined protocol \( \lambda_0 \). In the following time, at each \( t_k \), a physical observable is measured and its outcome \( y_k \) is obtained with pdf \( P(y_k|x_k) \). During time interval \( (t_k, t_{k+1}) \) the parameters \( \lambda_k \) vary according to a continuously protocol that is uniquely determined by previous outcomes \( \mu_k = \{ y_1, \ldots, y_k \} \) up to time \( t_k \). We explicitly indicate this point by notation \( \lambda_k(\mu_k) \). Additionally, we let \( t_M+1 = t_f \). After completion of one process, we have a protocol \( \Lambda = \{ \lambda_1(\mu_1), \ldots, \lambda_M(\mu_M) \} \). With these notations, we extend (8) into the case with feedback control:

\[
x_0 \langle e^{-\sum_{k=0}^{M} \int_{t_k}^{t_{k+1}} J[\theta,S](x(r),\tau,\lambda_k) \, dr - \int_{t_k}^{t_{k+1}} O(x(t_r)) \rangle_{\theta_0}} = \langle (O)_{\theta_0} \rangle_{\lambda_0(t_0,\mu_0)},
\]

where the averages are done over all possible outcomes and trajectories, \( \theta \) and \( S \) may or may not be functions of the control parameters, and the trajectory-dependent terms \( J \) and \( I \)[30] are

\[
J = \ln \prod_{k=1}^{M} \frac{g(x(t_k), t_k, \lambda_k^{\mu_{k-1}})}{g(x(t_k), t_k, \lambda_k^{\mu_k})},
\]

\[
I = \ln \prod_{k=1}^{M} \frac{P(y_k|x(t_k))}{P(y_k|\mu_{k-1})},
\]

where \( J \) and \( I \) are the new \( J \) and \( I \) with feedback control. Considering a protocol \( \Lambda = \{ \lambda_1(\mu_1), \ldots, \lambda_M(\mu_M) \} \), the time-evolving probability of the system is governed by the Fokker–Planck equation, whose drift force and diffusion matrix are
respective. We have denoted $P(y_1) = P(y_1 | \mu_0)$. On the right-hand side of (13) $q^a(\bar{x}_t, t_f)$ is the pdf of the time-reserved process (7) at time $t_f$ under the specific time-reversed protocol $\hat{\Lambda} = \{\hat{\lambda}^M(\mu_M), \ldots, \hat{\lambda}^0\}$ and $\hat{\lambda}^k(\mu_k) = \lambda^k_{\tau_f-\tau}(\mu_k)$. The $J$-term (14) arises from possible discontinuity of the control parameters at the measurement times. Such type of protocols was often used in modelling.\cite{16,20,31,37,38} Obviously, if we integrate the two sides of (13) on $x_0$, we have the GIFR with feedback control:

$$
\int \left( \prod_{k=M}^{1} dx_k dy_k \right) R_{\lambda^M}(t_f | x_M, t_M) \prod_{k=M}^{1} P(y_k | \mu_{k-1}) R_{\lambda^1_{k-1}}(x_k, t_k | x_{k-1}, t_{k-1}) \frac{\rho(x_k, t_k, \lambda^k_{\tau})}{\rho(x_k, t_k, \lambda^1_{\tau-1})},
$$

where the subscript $\lambda^k_{\tau}$ indicates the external parameters during the time interval $(t_k, t_{k+1})$. Substituting (8) and (11) into the above equation, we have

$$
\frac{1}{\rho(x_0, t_0)} \int \left( \prod_{k=1}^{M} dx_k dy_k \right) \prod_{k=1}^{M} P(y_k | \mu_{k-1}) G_{\lambda^k_{k-1}}(\bar{x}_{k-1}, s_{k-1} | \bar{x}_k, s_k) q^M_{\lambda^k}(\bar{x}_M, s_M) (O)_{\varphi_f} = \frac{1}{\rho(x_0, t_0)} \int \prod_{k=1}^{M} dy_k \prod_{k=1}^{M} P(y_k | \mu_{k-1}) \int d\bar{x}_k G_{\lambda^k_{k-1}}(\bar{x}_{k-1}, s_{k-1} | \bar{x}_k, s_k) q^M_{\lambda^k}(\bar{x}_M, s_M) (O)_{\varphi_f} = \frac{1}{\rho(x_0, t_0)} \int \prod_{k=1}^{M} dy_k P(\mu_{k}) (O)_{\varphi_f} q^M_{\lambda}(\bar{x}_0, t_0),
$$

where $s_k + t_k = t_f$, and the subscript $\lambda^k_{\tau}$ indicates that the transition probability of the reversed protocol is under the control of the reversed protocol.

There are two simple observations in the above proof. First, the $I$-term (15) is indispensable if one wants to rewrite (13) as (17). Second, extending the fluctuation relations with feedback control essentially depends on the validity of the relations themselves. Finally, there is a highly analogous GIFR in the continuous time master equation with discrete states.\cite{33} Its extension of feedback control can be easily carried out as what we did here; see the details in A.

4 Results and Discussion

4.1 Existing Special Cases

The very general (16) in fact presents an infinite number of equalities due to almost arbitrary choice of $g$ and $S$. However, it is nontrivial to reveal some of them of physical interest. We first show that (16) can cover several known fluctuation relations with feedback control.\cite{17,20} Below we set $O(x) = 1$ for simplicity. The first example is for the diffusion process that starts from the steady state $\rho_{ss}(x, \lambda^0_0)$. If the system has instantaneous steady-state solution $\rho_{ss}(x, \lambda^k_0)$ at any fixed parameters $\lambda^k_0$, we may choose the solution as $g$ and $S = 0$. Then the GIFR (16) reduces into

$$
\langle e^{- \sum_{k=0}^{M-1} f_{\tau_k}^{k+1} \delta_{a \ln \rho_{ss}(x(\tau), \lambda^k_0)} d\tau - J-I} \rangle_{\rho_0} = 1.
$$

This is the generalized Jarzynski–Hatano–Sasa equality with feedback control. Abreu and Seifert first presented the equality in the discrete master equations.\cite{20} Specifically, if the steady-state solution is just the instantaneous thermal equilibrium state, $\rho_{eq}(x, \lambda^k_0)$, (1) may be further simplified into the work equality (1) of Sagawa and Ueda.\cite{17} Note that under this circumstance if the control parameters jump at the measurement times, the $J$-term represents the amount of energy input by the external controller into the system. In Subsec. 4.4(ii), we use a Brownian model to illustrate this point.

The second example is to choose $g$ to be the pdf $\rho(x, t, \lambda^k_0)$ of the system and $S_t$ to be its irreversible current,

$$
\mathcal{J}^g_t(\rho) = \mathcal{A}^g_t \rho - \frac{1}{2} \partial_{x_1} (B_{ii}(\rho)),
$$

where $\mathcal{A}^g_t$ is the irreversible component of the drift force.\cite{29,32,33} Then the GIFR (16) reduces into

$$
\langle e^{- \sum_{k=0}^{M-1} f_{\tau_k}^{k+1} \mathcal{J}^g(\rho)(x(\tau), \lambda^k_0) d\tau - J-I} \rangle_{\rho_0} = 1.
$$

We have shown that, for the nonequilibrium processes with predetermined external parameters, the integrand (6) with these specific $g$ and $S$ is the rate of the total entropy production.\cite{20} Therefore, the equality (21) is nothing but its extension of taking feedback control into account.\cite{20,23} The reader is reminded that the $J$-term here is generally zero, since the pdf of the system is continuous in time even if the control parameters have finite jumps. However, this is no longer true for the parameters being $\delta$-functions, e.g., an underdamped Brownian model in Subsec. 4.4(ii). Contrary to the equality (19), which requires the existence of the instantaneous steady-state solutions, (21) holds for general diffusion processes.
4.2 Generalized Bochkov–Kuzovlev Equality with Feedback Control

In addition to the above known results in the literature, the GIFR (16) predicts a previously unnoticed equality in the dynamic perturbation problem.\[53,40\] Consider a perturbed system with the Fokker–Planck operator L = L₀ + L_c(t), where L₀ denotes the time-independent operator of the free system, and L_c(t) is the perturbation in which the control parameters are involved. Before applying the perturbation, the system is assumed to be at the thermal state ρ₀(x). Choosing q to be the equilibrium pdf and setting S = 0, we obtain

\[
\left\langle e^{-\sum_{k=0}^{M} \int_{t_k}^{t_{k+1}} \rho_0 \cdot L_c(\rho_0)(x(\tau), \lambda_c^\tau) d\tau - I} \right\rangle_{\rho_0} = 1. \tag{22}
\]

Obviously, the J-term here exactly vanishes. We name (22) the generalized BKE with feedback control. The physical relevance of above equality may be clearly explained by the 1-d underdamped Brownian particle:

\[
dq = \frac{p}{m} dt, \tag{23}
\]

\[
fp = -\partial_x H_0 dt + \lambda^\tau dt - \gamma \frac{p}{m} dt + \sqrt{\frac{2 \gamma}{\beta}} dW, \tag{24}
\]

where x = (q,p) is the coordinate of the particle in its phase space, γ is the friction coefficient, the dynamic force λ^\tau represents the control parameter, and H_0 = p^2/2m + U(x) is the Hamiltonian of the free system. Note that the force may be nonconservative.\[5\] Given these notations, we immediately have ρ₀ ∝ e^{-βH_0(x)} and L_c = -λ^\tau p. Substituting them into (22), we have

\[
\left\langle e^{-\beta \sum_{k=0}^{M} \int_{t_k}^{t_{k+1}} \lambda^\tau dq - I} \right\rangle_{\rho_0} = 1. \tag{25}
\]

We see that the integral therein is the work done by the external force λ^\tau on the system. In the absence of the feedback control, (25) is the canonical BKE.\[6,41-42\] On the other hand, unexpectedly, for the 1-d overdamped Brownian particle, i.e., (24) with vanishing momentum terms, (22) leads into an alternative equality

\[
\left\langle e^{-\beta \sum_{k=0}^{M} \int_{t_k}^{t_{k+1}} \gamma^{-1} \lambda^\tau \partial_x U(x(\tau)) d\tau - I} \right\rangle_{\rho_0} = 1, \tag{26}
\]

where ρ₀ ∝ e^{-βU(x)} and L_c = -γ^{-1}λ^\tau \partial_x. We do not clearly understand the physical meaning of the above equality, though the integral indeed possesses the dimension of work. Even though we must emphasize that for the overdamped particle the BKE (25) is still true but it does not originate from (22). We leave the discussion in Subsec. 4.2.

4.3 Time Reversal Explanation

Horowitz and Vaikuntanathan,\[30\] and Sagawa and Ueda,\[31\] have discussed the time reversal explanation of (1) in great details. The equality was thought to be the consequence of the detailed trajectory fluctuation theorem with discrete feedback control. This theorem is about the ratio of the joint probability of observing trajectory and protocol in the forward process and the joint probability of observing reversed trajectory and protocol in the time-reversed process.\[36\] Different from the forward protocols determined by the feedback measurements, the reversed protocols are randomly drawn from the known probability distribution of the protocols recorded in the forward process.\[17,30\] Interestingly, this result automatically emerges in (13): its right-hand side is the statistical average of the pdf q_λ(x₀, t_f) of the time reversed process with the specific reversed protocol Λ over the pdf P(μ, M); there are no measurements at all. We must emphasize that (13) is valid for very general diffusion processes. Particularly, the content of the time-reversal is far broader than that considered in previous works.\[30-31\]

In addition to the interest in the concept, the time-reversal explanation of (13) is also useful. We can obtain an alternative inequality with feedback control:

\[
\sum_{k=0}^{M} \int_{t_k}^{t_{k+1}} J[\rho, S](x(\tau), \lambda, \lambda^\tau) dq + \langle J \rangle_{\rho_0} \geq D[p(x, 0)||\langle q_\lambda(x, t_f) \rangle_{\mu, M}] - (I)_{\rho_0}. \tag{27}
\]

It is worth pointing out that, the direct application of the Jensen’s inequality to (16) leads into another very similar inequality, where the nonnegative D-term above is absent. Compared with the lower bound set by - (I)_{\rho_0} alone, e.g., (2),\[17,20\] (27) presents a stronger one on the entropy-like quantity. This inequality is the other central result of this work. In the next section, we will concretely illustrate it by two Brownian models with single feedback control.

4.4 Examples

(i) Overdamped Brownian Particle

The model of the overdamped Brownian particle in [39] is a good example to show the effect of the discontinuous control parameters and to verify the inequality (27). At time 0 the particle is at thermally equilibrium state ρ₀(x, 0) in a harmonic potential x^2/2. By measuring position x of the particle with outcome x_m, which follows a Gaussian pdf P(x_m|x) = N_\mu_0(x_m), the center of the potential is moved according to a protocol λ(t|x_m) and reaches the fixed λ_f at the terminal time t_f. The equation of motion of the particle is simply

\[
dx = -[x - \lambda(t|x_m)] dt + \sqrt{2} dW. \tag{28}
\]

Note that all physical quantities are dimensionless. Abreu and Seifert\[39\] found that work can be extracted from the single heat bath if the protocol follows the optimal one:

\[
\lambda^\tau(x_m) = \lambda_f - b(x_m) t_f + \left( t + 1 \right) b(x_m), \tag{29}
\]

where b(x_m) = x_m/(1 + y_m^2). Obviously, the optimal protocol has two jumps at time 0 and t_f. Since we are interested in the work, we choose q to be

\[
\rho_{eq}(x, \lambda^\tau(x_m)) = \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( x - \lambda^\tau(x_m) \right)^2 \right], \tag{30}
\]
and \( S = 0 \) for (27). Its left-hand side is then written as
\[
- \int dx_m P(x_m) \int dx \int_0^{t_f} \lambda^*_m(x_m)(x(\tau) - \lambda^*_m(x_m)) d\tau \rho(x, 0|x_m) \\
+ \int dx_m P(x_m) \int dx \left[ \frac{1}{2} (x - \lambda^*_m(x_m))^2 - \frac{1}{2} x^2 \right] \rho(x, 0|x_m) \\
+ \int dx_m P(x_m) \int dx \rho(x, t|x_m) \left[ \frac{1}{2} (x - \lambda^*_m(x_m))^2 - \frac{1}{2} (x - \lambda^*_m(x_m))^2 \right],
\]
where \( \rho(x, t|x_m) \) is the pdf of the Brownian particle at time \( t \) under the protocol \( \lambda^*_m(x_m) \), and the initial condition is defined by
\[
P(x_m) \rho(x, 0|x_m) = P(x_m|x) \rho_{eq}(x, 0).
\]
We clearly see that (31) is the total mean work \( \langle W^\star \rangle \) done on the particle,[43] the first equation is the work done by the continuous part of the protocol, and the latter two equations are the energy input due to the two jumps at the beginning and ending times. One may check that (31) is equivalent to Eq. (32) in [39], e.g., by directly substituting (29) into (31) and obtaining
\[
\langle W^\star \rangle = \frac{\lambda^2}{t_f + 2} - \frac{1}{2(t_f + 2)} 1 + \rho_{eq}^2.
\]
On the other hand, given the optimal protocol, the \( D \)-term on the right-hand side of (27) is exactly
\[
D = -\frac{1}{2} \ln \left( \frac{1 + \frac{1}{1 + K}}{1 + K} + \frac{1}{2} \tau \right) + \frac{1}{2} \frac{2 \lambda^2}{K} \left( \frac{t_f \tau}{t_f + 2} \right)^2 > 0,
\]
where \( K = (1 + \rho_{eq}^2)(1 + 2/t_f)^2 \). The mutual information \( \langle I \rangle \) is \( \ln(1 + \rho_{eq}^2)/2 \).[39] In Fig. 1(a) we show (27) at different \( t_f \) for typical \( \rho_{eq} \) and \( \lambda_f \); see the lines therein. We find that the presence of the \( D \)-term can indeed improve the lower bound on the mean work, especially at shorter \( t_f \) in which considerable dissipation occurs. In order to check the correctness of the analytical expressions, we also perform the Langevian simulation under the same parameters; see the crosses in the same figure. We see that the theory and simulation excellently agree with each other.

In the above model we have assumed that the friction coefficient is constant. However, in many real situations it may vary with position,[44–47] e.g., Brownian particle near a wall. In this case, the stochastic term in (28) is multiplicative noise[32] rather than the previously additive noise. The position-dependent friction coefficient \( \gamma(x) \) has significant influence on the dynamics of the particle: (28) has to be appropriately revised[48–50] as
\[
\gamma(x) dx = -[x - \alpha^*(t(x_m))] dt - \partial_x \gamma(x)/\gamma(x) dx + \sqrt{2 \gamma(x)} dW.
\]
This equation of motion generally has no analytical solutions. According to our previous discussion, however, (19) and corresponding (27) still hold since (35) has the instantaneous equilibrium solution \( \rho_{eq}(x, \alpha^*(x_m)) \). To il-

![Fig. 1](image_url)

**Fig. 1** (Color online) (a) The solid, dashed, and dotted line are the mean total work \( \langle W^\star \rangle \) (33), \( D - \langle I \rangle_{eq} \), and \( -\langle I \rangle_{eq} \), respectively, where \( D \) is (34). The cross symbols are the results of the simulation. The parameters are \( \rho_{eq} = 0.4 \) and \( \lambda_f = 2 \). The open symbols are the simulation data for the case with the position-dependent friction coefficient, where \( c = 0.5 \). Inset: the cross and open symbols are the left-hand side of (19) obtained by the simulation; they are for the cases with the constant and position-dependent friction coefficients, respectively. (b) The solid, dashed, and dotted lines are the mean total work \( W_{HK} \) (38), \( D - \langle I \rangle_{eq} \), and \( -\langle I \rangle_{eq} \), respectively, where \( D \) is (38). The x-axis is \( \rho_{eq} \).
illustrate this claim, we assume $\gamma(x) = 1 + cx^2$ \cite{45,49} and simulate the mean work and the $D$-term; see the empty squares and circles in Fig. 1(a). We find that introduction of the position-dependent friction coefficient does not significantly change their features under the optimal protocol (29) and the given parameters. Additionally, we also note that in the current model the numerical verification of the equality (19) is more difficult than the verification of the inequality; see the inset in Fig. 1(a).

(ii) Underdamped Brownian Particle

Here we use an underdamped Brownian particle (23) and (24) with feedback control to verify the inequality (27) from the BKE. The potential is assumed to be harmonic, $U(q) = q^2/2$. At the beginning the particle is at thermal equilibrium. After measuring velocity $v$ of the particle with outcome $v_m$ at time 0, we immediately apply an impulse force

$$\lambda(t) = -\alpha v_m \delta(t-0),$$

the strength $\alpha v_m$ of which depends on the measured velocity. We assume that the measurement bears a Gaussian error $P(v_m|v) = N(\alpha, \nu_m)$. Obviously, the measurement is exact, i.e., $\nu_m = 0$, the force will make the velocity $v$ of the particle jumping to $(1-\alpha)v$. The average work done on the particle is

$$\langle W_{\text{work}} \rangle = \frac{1 + y_m^2}{2} \left( \alpha - \frac{1}{1 + y_m^2} \right)^2 = \frac{1}{2(1 + y_m^2)}.\quad (37)$$

The work may be negative and especially has a minimal. Hence, there exists an optimal force protocol $\lambda^*_t(v_m)$ with $\alpha^* = 1/(1 + y_m^2)$ that we can extract the maximum work from the single heat bath. Additionally, the $D$-term here has a simple expression:

$$D = \frac{1}{2} \ln[1 + (1 + y_m^2)\alpha^2] - \frac{(1 + y_m^2)\alpha^2}{2[1 + (1 + y_m^2)\alpha^2]} > 0.\quad (38)$$

The mutual information is the same with that in the preceding example. We show the inequality (27) for $\alpha = 1.5$ and $\alpha^*$ at different $y_m$ in Fig. 1(b).

5 Conclusion

In this work, we presented the GIFR with feedback control for general diffusion processes. This general relation not only reobtains the existing fluctuation relations with feedback control but also predicts other unnoticed relations. Moreover, we derived an alternative inequality about the entropy-like quantity with an improved lower bound. Two Brownian particle models are demonstrated to verify the claim. Our discussion clearly shows that, given any integral fluctuation relation in classical stochastic Markovian systems, one can always obtain its counterpart when the feedback control is taken into account. In our opinion, hence, it is not very essential to derive these relations one by one again in future.

Appendix A: GIFR with Feedback Control in Master Equations

Although the extension is straightforward, considering that very different notations are involved in the master equation, for reader’s convenience we first give a review of the GIFR\cite{28,51} in this distinctive system. We assume that the master equation has the following form:

$$\frac{dp_n(t)}{dt} = [H(t)p(t)]_n,\quad (A1)$$

where $n$ is the state index, the $N$-d column vector $p(t) = (p_1, ..., p_N)^T$ is the probability of the system at individual states at time $t$, and the matrix element $(H)_{mn} = H_{mn}(t) > 0 (m \neq n)$ is the rate and $(H)_{nn} = -\sum_{m \neq n} H_{mn}(t)$. Given an arbitrary normalized positive column vector $\varrho(t) = (\varrho_1, ..., \varrho_N)^T$ and an $N \times N$ matrix $A$ whose elements $(A)_{mn} = A_{mn} (m \neq n)$ satisfy the conditions of $H_{mn}\varrho_n + A_{mn} > 0$ and $A_{nn} = -\sum_{m \neq n} A_{mn}$, we found that there is a GIFR\cite{28}

$$\sum_{n=1}^{N} \varrho_n(t_0) R_n(t_0) = \sum_{n=1}^{N} \varrho_n(t_0)^n \left( e^{-\sum_{\tau} J[\varrho,A](x(\tau),\tau)} d\tau O_x(t_0) \right) = (O_{\varrho}),\quad (A2)$$

where $O = (O_1, ..., O_N)$ is an arbitrary $N$-d vector, $n \langle \cdot \rangle_{\varrho}$ is statistical average over all trajectories starting from the state $n$ at time $t_0$, and $\langle \cdot \rangle_{\varrho}$ indicates an averages over $\varrho(t)$. The integrand in (A2) is

$$J[\varrho,A](x(\tau),\tau) = \varrho^{-1}_{x(\tau)} [-\partial_\tau \varrho + H \varrho + A] x(\tau) + Q[\varrho_{x(\tau)} A],\quad (A3)$$

where the $N$-d vector $1 = (1, ..., 1)^T$, and

$$Q[B] = -B x(\tau)x(\tau) - \ln\left[ 1 + \frac{B x(\tau) x(\tau)}{H x(\tau) x(\tau)} \right] \times \sum_{i=1}^{K} \delta(\tau - \tau_i),\quad (A4)$$

$x(\tau)$ is the discrete state of the system at time $\tau$, $x(\tau^-)$ and $x(\tau^+)$ represent the states just before and after a jump occurring at time $\tau$, respectively, and we assumed the jumps occur $K$ times for a trajectory. Analogous to the GIFR (7), the GIFR (A2) can as well cover several known fluctuation relations in the literature by selecting specific $\varrho$ and $A$\cite{28} (A2) has a time-reversal explanation:

$$\varrho_n(t') R_n(t') = (O_{\varrho})_{\varrho} \varrho_n(s),\quad (A5)$$

where $q(s)$ is the $N$-d probability vector of the time-reversed master equation,

$$\frac{dq_n(s)}{ds} = [\tilde{H}(s)q(s)]_n,\quad (A6)$$

and the elements of the matrix $\tilde{H}(s)$ are

$$\tilde{H}_{nm}(s) = \varrho^{-1}_{n}(t') [H_{mn}(t') \varrho_n(t') + A_{mn}(t')],\quad (A7)$$
for \( m \neq n \) and \( \tilde{H}_{mm}(s) = -2 \sum_{n \neq m} \tilde{H}_{nm}(s) \). Because the master equation (A1) is Markovian and possesses equations like (8) and (11), repeating the same derivations as those in the main text we then obtain the GIFFR with feedback control in the master equation:

\[
\left\langle e^{-\sum_{k=0}^{M} J[k+1] J[t_k] \delta[H(x(t))] - J'[O(t)]} \right\rangle = \langle \omega \rangle \rho_M ,
\]

where

\[
J = \ln \frac{\rho(x(t_k)) \lambda(t_k - 1)}{\rho(x(t_k)) \lambda(t_k) ,}
\]

and the I-term is still (5). Here we have assumed that the rates \( H_{mm}(t) \) depend on time only through the external control parameters \( \lambda \) and feedback control is completely the same with that in the diffusion processes; see Sec. 3.

**Appendix B: BKE of Overdamped Brownian Particle**

Here we only discuss the fluctuation relation without feedback control. Specifically choosing \( \varrho = \rho_0 \) and \( S = \rho_0 \lambda(t)/\gamma \) and substituting them into the GIFFR (7), we can obtain the canonical BKE[6] for the overdamped Brownian particle, where we used the notation \( \lambda(t) \) instead of \( \lambda_k \). On the basis of (9) and (10), we find that the BKE and (26) have distinctive time reversal interpretations. For the former the time-reversed SDE is the same with the forward one except that the dynamic force is changed into \( \lambda(t_f - t) \), whereas for the latter the direction of the force is also reversed, namely, \( -\lambda(t_f - t) \). This point can be explicitly checked for the Brownian particle in the harmonic potential \( U(x) = x^2/2 \). After a simple calculation, we find that before integrating on \( x_0 \) the left-hand side of (26) is

\[
x_0 \left\langle e^{-\int_0^{t_f} \lambda(t) x(t) \, dt} \right\rangle = \exp \left[ -x_0 \int_0^{t_f} \lambda(t) e^{-\tau} \, dt - \frac{1}{2} \left( \int_0^{t_f} \lambda(t) e^{-\tau} \right)^2 \right] .
\]

Multiplying (A10) by the thermal state \( \rho(x_0, 0) \propto e^{-U(x_0)} \), we obtain

\[
x_0 \left\langle e^{-\int_0^{t_f} \lambda(t) x(t) \, dt} \right\rangle \rho(x_0, 0) = N_{x_0} \left[ -\int_0^{t_f} \lambda(t) e^{-\tau} \, dt, 1 \right] ,
\]

which is just the pdf \( \rho(x_0, t_f) \) of the time-reversed process with the dynamic force \( -\lambda(t_f - t) \). On the other hand, interestingly, one may check that the time reversal explanation of the BKE for the underdamped Brownian particle, (23) and (24), agrees with that of the BKE for the overdamped Brownian particle. These little surprising characteristics of the overdamped Brownian particle are due to lack of the degree of freedom of velocity.

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