Classifying spaces of compact Lie groups that are $p$–compact for all prime numbers

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We consider a problem on the conditions of a compact Lie group $G$ that the loop space of the $p$–completed classifying space be a $p$–compact group for a set of primes. In particular, we discuss the classifying spaces $BG$ that are $p$–compact for all primes when the groups are certain subgroups of simple Lie groups. A survey of the $p$–compactness of $BG$ for a single prime is included.

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A $p$–compact group (see Dwyer–Wilkerson [8]) is a loop space $X$ such that $X$ is $F_p$–finite and that its classifying space $BX$ is $F_p$–complete (see Andersen–Grodal–Møller–Viruel [2] and Dwyer–Wilkerson [11]). We recall that the $p$–completion of a compact Lie group $G$ is a $p$–compact group if $\pi_0(G)$ is a $p$–group. Next, if $C(\rho)$ denotes the centralizer of a group homomorphism $\rho$ from a $p$–toral group to a compact Lie group, according to [8, Theorem 6.1], the loop space of the $p$–completion $\Omega(BC(\rho))^\wedge$ is a $p$–compact group.

In a previous article [19], the classifying space $BG$ is said to be $p$–compact if $\Omega(BG)^\wedge_p$ is a $p$–compact group. There are some results for a special case. A survey is given in Section 1. It is well-known that, if $\Sigma_3$ denotes the symmetric group of order 6, then $B\Sigma_3$ is not 3–compact. In fact, for a finite group $G$, the classifying space $BG$ is $p$–compact if and only if $G$ is $p$–nilpotent. Moreover, we will see that $BG$ is $p$–compact toral (see Ishiguro [20]) if and only if the compact Lie group $G$ is $p$–nilpotent (see Henn [14]). For the general case, we have no group theoretical characterization, though a few necessary conditions are available. This problem is also discussed in the theory of $p$–local groups (see Broto, Levi and Oliver [6, 7]) from a different point of view.

We consider the $p$–compactness of $BG$ for a set of primes. Let $\Pi$ denote the set of all primes. For a non–empty subset $\mathbb{P}$ of $\Pi$, we say that $BG$ is $\mathbb{P}$–compact if this space is $p$–compact for any $p \in \mathbb{P}$. If $G$ is connected, then $\Omega(BG)^\wedge_p \simeq G_p^\wedge$ for any prime $p$, and hence $BG$ is $\Pi$–compact. The connectivity condition, however, is not necessary. For instance, the classifying space of each orthogonal group $O(n)$ is also $\Pi$–compact. Since $\pi_0(O(n)) = \mathbb{Z}/2$ is a 2–group, $BO(n)$ is 2–compact, and for any odd prime $p$,
the $p$–equivalences $BO(2m) \cong_p BO(2m + 1) \cong_p BSO(2m + 1)$ tell us that $BO(n)$ is $\Pi$–compact.

Next let $\mathbb{P}(BG)$ denote the set of primes $p$ such that $BG$ is $p$–compact. In [20] the author has determined $\mathbb{P}(BG)$ when $G$ is the normalizer $NT$ of a maximal torus $T$ of a connected compact simple Lie group $K$ with Weyl group $W(K)$. Namely

$$\mathbb{P}(BNT) = \begin{cases} 
\Pi & \text{if } W(K) \text{ is a 2–group,} \\
\{p \in \Pi \mid |W(K)| \not\equiv 0 \mod p\} & \text{otherwise.}
\end{cases}$$

Other examples are given by a subgroup $H \cong SU(3) \times \mathbb{Z}/2$ of the exceptional Lie group $G_2$ and its quotient group $\Gamma_2 = H/\langle \mathbb{Z}/3 \rangle$.

A result of [19] implies that $\mathbb{P}(BH) = \Pi$ and $\mathbb{P}(B\Gamma_2) = \Pi - \{3\}$.

In this paper we explore some necessary and sufficient conditions for a compact Lie group to be $\Pi$–compact. First we consider a special case. We say that $BG$ is $\mathbb{P}$–compact toral if for each $p \in \mathbb{P}$ the loop space $\Omega(BG)_p^\wedge$ is expressed as an extension of a $p$–compact torus $T_p^\wedge$ by a finite $p$–group $\pi$ so that there is a fibration $(BT)_p^\wedge \longrightarrow (BG)_p^\wedge \longrightarrow B\pi$. Obviously, if $BG$ is $\mathbb{P}$–compact toral, the space is $\mathbb{P}$–compact. A necessary and sufficient condition that $BG$ be $p$–compact toral is given in [20]. As an application, we obtain the following:

**Theorem 1** Suppose $G$ is a compact Lie group, and $G_0$ denotes its connected component with the identity. Then $BG$ is $\Pi$–compact toral if and only if the following two conditions hold:

(a) $G_0$ is a torus $T$, and the group $G/G_0 = \pi_0 G$ is nilpotent.

(b) $T$ is a central subgroup of $G$.

For a torus $T$ and a finite nilpotent group $\gamma$, the product group $G = T \times \gamma$ satisfies conditions (a) and (b). Thus $BG$ is $\Pi$–compact toral. Proposition 2.2 will show, however, that a group $G$ with $BG$ being $\Pi$–compact toral need not be a product group.
Next we ask if \( BH \) is \( \mathbb{P} \)-compact when \( H \) is a subgroup of a simple Lie group \( G \). For \( \mathbb{P} = \Pi \), the following result determines certain types of \((G, H_0)\) where \( H_0 \) is the connected component of the identity. We have seen the cases of \((G, H) = (G, NT)\) when \( W(G) = NT/T \) is a 2-group, and of \((G, H) = (G_2, SU(3) \times \mathbb{Z}/2)\) which is considered as a case with \((G, H_0) = (G_2, A_2)\). Recall that the Lie algebra of \( SU(n+1) \) is simple of type \( A_n \), and the Lie group \( SU(3) \) is of \( A_2 \)-type (see Bourbaki [4]).

**Theorem 2** Suppose a connected compact Lie group \( G \) is simple. Suppose also that \( H \) is a proper closed subgroup of \( G \) with \( \text{rank}(H_0) = \text{rank}(G) \), and that the map \( BH \to BG \) induced by the inclusion is \( p \)-equivalent for some \( p \). Then the following hold:

(a) If the space \( BH \) is \( \Pi \)-compact, \((G, H_0)\) is one of the following types:

\[
(G, H_0) = \begin{cases} 
(G, T_G) & \text{for } G = A_1 \text{ or } B_2 (= C_2) \\
(B_n, D_n) & \\
(C_2, A_1 \times A_1) & \\
(G_2, A_2) & 
\end{cases}
\]

where \( T_G \) is the maximal torus of \( G \).

(b) For any odd prime \( p \), all above types are realizable. Namely, there are \( G \) and \( H \) of types as above such that \( BH \to BG \) is \( \Pi \)-compact, together with the \( p \)-equivalent map \( BH \to BG \). When \( p = 2 \), any such pair \((G, H)\) is not realizable.

We make a remark about covering groups. Note that if \( \alpha \to \tilde{G} \to G \) is a finite covering, then \( \alpha \) is a central subgroup of \( \tilde{G} \). For a central extension \( \alpha \to \tilde{G} \to G \) and a subgroup \( H \) of \( G \), we consider the following commutative diagram:

\[
\begin{array}{ccc}
\alpha & \to & \tilde{G} \\
\| & & \uparrow \\
\alpha & \to & \tilde{H} \\
\end{array}
\]

Obviously the vertical map \( H \to G \) is the inclusion, and \( \tilde{H} \) is the induced subgroup of \( \tilde{G} \). We will show that the pair \((G, H)\) satisfies the conditions of **Theorem 2** if and only if its cover \((\tilde{G}, \tilde{H})\) satisfies those of **Theorem 2**. Examples of the type \((G, H_0) = (B_n, D_n)\), for instance, can be given by \((SO(2n+1), O(2n))\) and the double cover \((\text{Spin}(2n+1), \text{Pin}(2n))\).

For the case \((G, H_0) = (G_2, A_2)\), we have seen that \( H \) has a finite normal subgroup \( \mathbb{Z}/3 \), and that for its quotient group \( \Gamma_2 \) the classifying space \( B\Gamma_2 \) is \( p \)-compact if and only if \( p \neq 3 \). So \( \mathbb{P}(B\Gamma_2) \neq \Pi \). The following result shows that this is the only case. Namely, if \( \Gamma \) is such a quotient group for \((G, H_0) \neq (G_2, A_2)\), then \( \mathbb{P}(B\Gamma) = \Pi \).
**Theorem 3** Let \((G, H)\) be a pair of compact Lie groups as in Theorem 2. For a finite normal subgroup \(\nu\) of \(H\), let \(\Gamma\) denote the quotient group \(H/\nu\). If \((G, H_0) \neq (G_2, A_2)\), then \(B\Gamma\) is \(\Pi\)–compact.

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1 A survey of the \(p\)–compactness of \(BG\)

We summarize work of earlier articles [19, 20] together with some basic results, in order to introduce the problem of \(p\)–compactness. For a compact Lie group \(G\), the classifying space \(BG\) is \(p\)–compact if and only if \(\Omega(BG)^\wedge_p\) is \(F\)–finite. So it is a mod \(p\) finite \(H\)–space. The space \(B\Sigma_3\) is not \(p\)–compact for \(p = 3\). We notice that \(\Omega(B\Sigma_3)^\wedge_p\) is not a mod 3 finite \(H\)–space, since the degree of the first non–zero homotopy group of \(\Omega(B\Sigma_3)^\wedge_p\) is not odd. Actually there is a fibration \(\Omega(B\Sigma_3)^\wedge_p \rightarrow (S^3)^\wedge_p \rightarrow (S^3)^\wedge_p\) (see Bousfield and Kan [5]).

First we consider whether \(BG\) is \(p\)–compact toral, as a special case. When \(G\) is finite, this is the same as asking if \(BG\) is \(p\)–compact. Note that, for a finite group \(\pi\), the classifying space \(B\pi\) is an Eilenberg–MacLane space \(K(\pi, 1)\). Since \((BT)^\wedge_p\) is also Eilenberg–MacLane, for \(BG\) being \(p\)–compact toral, the \(n\)–th homotopy groups of \((BG)^\wedge_p\) are zero for \(n \geq 3\). A converse to this fact is the following.

**Theorem 1.1** [20, Theorem 1] Suppose \(G\) is a compact Lie group, and \(X\) is a \(p\)–compact group. Then we have the following:

(i) If there is a positive integer \(k\) such that \(\pi_n((BG)^\wedge_p) = 0\) for any \(n \geq k\), then \(BG\) is \(p\)–compact toral.

(ii) If there is a positive integer \(k\) such that \(\pi_n(BX) = 0\) for any \(n \geq k\), then \(X\) is a \(p\)–compact toral group.

This theorem is also a consequence of work of Grodal [12, 13]

A finite group \(\gamma\) is \(p\)–nilpotent if and only if \(\gamma\) is expressed as the semidirect product \(\nu \rtimes \gamma_p\), where \(\nu\) is the subgroup generated by all elements of order prime to \(p\), and where \(\gamma_p\) is the \(p\)–Sylow subgroup. The group \(\Sigma_3\) is \(p\)–nilpotent if and only if \(p \neq 3\). Recall that a fibration of connected spaces \(F\rightarrow E\rightarrow B\) is said to be preserved by the \(p\)–completion if \(F^\wedge_p\rightarrow E^\wedge_p\rightarrow B^\wedge_p\) is again a fibration. When \(\pi_0(G)\) is a \(p\)–group, a result of Bousfield and Kan [5] implies that the fibration \(BG_0\rightarrow BG\rightarrow B\pi_0G\) is preserved by the \(p\)–completion, and \(BG\) is \(p\)–compact.

We have the following necessary and sufficient conditions that \(BG\) be \(p\)–compact toral.

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Theorem 1.2  [20, Theorem 2] Suppose \( G \) is a compact Lie group, and \( G_0 \) is the connected component with the identity. Then \( BG \) is \( p \)-compact toral if and only if the following conditions hold:

(a) \( G_0 \) is a torus \( T \) and \( G/G_0 = \pi_0 G \) is \( p \)-nilpotent.

(b) The fibration \( BT \longrightarrow BG \longrightarrow B\pi_0 G \) is preserved by the \( p \)-completion.

Moreover, the \( p \)-completed fibration \( (BT)_p^\wedge \longrightarrow (BG)_p^\wedge \longrightarrow (B\pi_0 G)_p^\wedge \) splits if and only if \( T \) is a central subgroup of \( G \).

Next we consider the general case. What are the conditions that \( BG \) be \( p \)-compact? For example, for the normalizer \( NT \) of a maximal torus \( T \) of a connected compact Lie group \( K \), it is well-known that \( (BNT)_p^\wedge \cong (BK)_p^\wedge \) if \( p \) does not divide the order of the Weyl group \( W(K) \). This means that \( BNT \) is \( p \)-compact for such \( p \). Using the following result, we can show the converse.

Proposition 1.3  [20, Proposition 3.1] If \( BG \) is \( p \)-compact, then the following hold:

(a) \( \pi_0 G \) is \( p \)-nilpotent.

(b) \( \pi_1((BG)_p^\wedge) \) is isomorphic to a \( p \)-Sylow subgroup of \( \pi_0 G \).

The necessary condition of this proposition is not sufficient, even though the rational cohomology of \( (BG)_p^\wedge \) is assumed to be expressed as a ring of invariants under the action of a group generated by pseudoreflections.

Theorem 1.4  [19, Theorem 1] Let \( G = \Gamma_2 \), the quotient group of a subgroup \( SU(3) \times \mathbb{Z}/2 \) of the exceptional Lie group \( G_2 \). For \( p \) = 3, the following hold:

(1) \( \pi_0 G \) is \( p \)-nilpotent and \( \pi_1((BG)_p^\wedge) \) is isomorphic to a \( p \)-Sylow subgroup of \( \pi_0 G \).

(2) \( (BG)_p^\wedge \) is rationally equivalent to \( (BG_2)_p^\wedge \).

(3) \( BG \) is not \( p \)-compact.

We discuss invariant rings and some properties of \( B\Gamma_2 \) and \( BG_2 \) at \( p = 3 \). Suppose \( G \) is a compact connected Lie group. The Weyl group \( W(G) \) acts on its maximal torus \( T^n \), and the integral representation \( W(G) \longrightarrow GL(n, \mathbb{Z}) \) is obtained (see Dwyer and Wilkerson \[9, 10\]). It is well-known that \( K(BG) \cong K(BT^n)_{W(G)}^{W(G)} \) and \( H^*(BG; \mathbb{F}_p) \cong H^*(BT^n; \mathbb{F}_p)^{W(G)}_{W(G)} \) for large \( p \). Let \( W(G)^* \) denote the dual representation of \( W(G) \).

Although the mod 3 reductions of the integral representations of \( W(G_2) \) and \( W(G_2)^* \) are not equivalent, there is \( \psi \in GL(2, \mathbb{Z}) \) such that \( \psi W(G_2) \psi^{-1} = W(G_2)^* \) \[19, Lemma 3\]. Consequently, \( K(BT^2; \mathbb{Z}_3)\)\(^{W(G_2)} \cong K(BT^2; \mathbb{Z}_3)\)\(^{W(G_2)^*} \). Since \( K(B\Gamma_2; \mathbb{Z}_3) \cong K(BT^2; \mathbb{Z}_3)\)\(^{W(G_2)^*} \), we have the following result.
Theorem 1.5  [19, Theorem 3]  Let $\Gamma_2$ be the compact Lie group as in Theorem 1.4. Then the following hold:

(1) The 3–adic K-theory $K(B\Gamma_2; \mathbb{Z}_3)$ is isomorphic to $K(BG_2; \mathbb{Z}_3)$ as a $\lambda$–ring.

(2) Let $\Gamma$ be a compact Lie group such that $\Gamma_0 = PU(3)$ and the order of $\pi_0(\Gamma)$ is not divisible by 3. Then any map from $(B\Gamma)_3^{\wedge}$ to $(BG_2)_3^{\wedge}$ is null homotopic. In particular $[(B\Gamma_2)_3^{\wedge}, (BG_2)_3^{\wedge}] = 0$.

We recall that if a connected compact Lie group $G$ is simple, the following results hold:

(1) For any prime $p$, the space $(BG)_p^{\wedge}$ has no nontrivial retracts (see Ishiguro [15]).

(2) Assume $|W(G)| \equiv 0 \bmod p$. If a self-map $(BG)_p^{\wedge} \longrightarrow (BG)^p$ is not null homotopic, it is a homotopy equivalence (see Möller [22]).

(3) Assume $|W(G)| \equiv 0 \bmod p$, and let $K$ be a compact Lie group. If a map $f : (BG)_p^{\wedge} \longrightarrow (BK)_p^{\wedge}$ is trivial in mod $p$ cohomology, then $f$ is null homotopic (see Ishiguro [16]).

Replacing $G$ by $\Gamma_2$ at $p = 3$, we will see that (3) still holds. On the other hand it is not known if (1) and (2) hold, though on the level of K-theory they do.

2 $\Pi$–compact toral groups

Recall that a finite group $\gamma$ is $p$–nilpotent if and only if $\gamma$ is expressed as the semidirect product $\nu \rtimes \gamma_p$, where the normal $p$–complement $\nu$ is the subgroup generated by all elements of order prime to $p$, and where $\gamma_p$ is the $p$–Sylow subgroup. For such a group $\gamma$, we see $(B\gamma)_p^{\wedge} \simeq B\gamma_p$. For a finite group $G$, one can show that $\mathbb{P}(BG) = \{p \in \Pi \mid G$ is $p$–nilpotent$\}$. Consequently, if $G = \Sigma_n$, the symmetric group on $n$ letters, then $\mathbb{P}(B\Sigma_2) = \Pi$, $\mathbb{P}(B\Sigma_3) = \Pi - \{3\}$, and $\mathbb{P}(B\Sigma_n) = \{p \in \Pi \mid p > n\}$ for $n \geq 4$.

In [14], Henn provides a generalized definition of $p$–nilpotence for compact Lie groups. A compact Lie group $G$ is $p$–nilpotent if and only if the connected component of the identity, $G_0$, is a torus; the finite group $\pi_0G$ is $p$-nilpotent, and the conjugation action of the normal $p$–complement is trivial on $T$. We note that such a $p$–nilpotent group need not be semidirect product.
Let $\gamma = \pi_0 G$. Then, from the inclusion $\gamma_p \to \gamma$, a subgroup $G_p$ of $G$ is obtained as follows:

$$
\begin{array}{ccc}
T & \to & G \\
\downarrow & & \downarrow \\
T & \to & G_p
\end{array}
$$

A result of Henn [14] shows $(BG)_p^\wedge \simeq (BG_p)_p^\wedge$ if and only if the compact Lie group $G$ is $p$–nilpotent.

**Lemma 2.1** A classifying space $BG$ is $p$–compact toral if and only if the compact Lie group $G$ is $p$–nilpotent.

**Proof** If $BG$ is $p$–compact toral, we see from [20, Theorem 2] that the fibration $BT \to BG \to B\pi_0 G$ is preserved by the $p$–completion. Let $\pi = \pi_0 G$. Then we obtain the following commutative diagram:

$$
\begin{array}{ccc}
(BT)_p^\wedge & \to & (BG)_p^\wedge \\
\downarrow & & \downarrow \\
(BT)_p^\wedge & \to & (BG_p)_p^\wedge
\end{array}
$$

By [20, Theorem 2], the finite group $\pi$ is $p$–nilpotent, so the map $(B\pi_p)_p^\wedge \to (B\pi)_p^\wedge$ is homotopy equivalent. Thus $(BG)_p^\wedge \simeq (BG_p)_p^\wedge$, and hence the result of [14] implies that $G$ is $p$–nilpotent. Conversely, if $G$ is $p$–nilpotent, then the following commutative diagram

$$
\begin{array}{ccc}
BT & \to & BG \\
\downarrow & & \downarrow \\
BT & \to & BG_p
\end{array}
$$

tells us that $BT \to BG \to B\pi$ is $p$–equivalent to the fibration

$(BT)_p^\wedge \to (BG)_p^\wedge \to (B\pi)_p^\wedge$.

From [20, Theorem 2], we see that $BG$ is $p$–compact toral.

**Proof of Theorem 1** First suppose $BG$ is $\Pi$–compact toral. Lemma 2.1 implies that $G_0$ is a torus $T$ and $G/G_0 = \pi_0 G$ is $p$–nilpotent for any $p$. According to [20, Lemma 2.1], the group $\pi_0 G$ must be nilpotent. We notice that for each $p$ the normal $p$–complement of $\pi_0 G$ acts trivially on $T$. Thus $\pi_0 G$ itself acts trivially on $T$, and $T$ is a central subgroup of $G$. Conversely, assume that conditions (a) and (b) hold. According to [14, Proposition 1.3], we see that $G$ is $p$–nilpotent for any $p$. Therefore $BG$ is $\Pi$–compact toral.
We will show that a group which satisfies conditions (a) and (b) of Theorem 1 need not be a product group. For instance, consider the quaternion group $Q_8$ in $SU(2)$. Recall that the group can be presented as $Q_8 = \langle x, y \mid x^4 = 1, x^2 = y^2, yxy^{-1} = x^{-1} \rangle$. Let $\rho : Q_8 \rightarrow U(2)$ be a faithful representation given by the following:

$$
\rho(x) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
$$

Let $S$ denote the center of the unitary group $U(2)$ and let $G$ be the subgroup of $U(2)$ generated by $\rho(Q_8)$ and $S$. Then we obtain the group extension $S \rightarrow G \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Since $S \cong S^1$, this group $G$ satisfies conditions (a) and (b). On the other hand, we see that the non–abelian group $G$ can not be a product group. This result can be generalized as follows:

**Proposition 2.2** Suppose $\rho : \pi \rightarrow U(n)$ is a faithful irreducible representation for a non–abelian finite nilpotent group $\pi$. Let $S$ be the center of the unitary group $U(n)$ and let $G$ be the subgroup of $U(n)$ generated by $\rho(\pi)$ and $S$ with group extension $S \rightarrow G \rightarrow \pi_0 G$. Then this extension does not split, and $G$ satisfies conditions (a) and (b) of Theorem 1.

**Proof** First we show that $G$ satisfies conditions (a) and (b) of Theorem 1. Since $\pi$ is nilpotent, so is the finite group $\pi_0 G \cong G/S$. Recall that the center of the unitary group $U(n)$ consists of scalar matrices, and is isomorphic to $S^1$. Thus we obtain the desired result.

Next we show that the group extension $S \rightarrow G \rightarrow \pi_0 G$ does not split. If this extension did split, then we would have $G \cong S \rtimes \pi_0 G$. Since the action of $\pi_0 G$ on the center $S$ is trivial, it follows that $G$ is isomorphic to the product group $S \times \pi_0 G$. Let $Z(\pi)$ denote the center of $\pi$. Since the representation $\rho : \pi \rightarrow U(n)$ is irreducible and faithful, Schur’s Lemma implies $S \cap \rho(\pi) = Z(\rho(\pi)) \cong Z(\pi)$. Thus we obtain the following commutative diagram:

$$
\begin{array}{ccc}
S & \longrightarrow & G \\
\uparrow & & \uparrow \\
Z(\pi) & \longrightarrow & \pi_0 G
\end{array}
$$

Regarding $\pi$ as a subgroup of $G = S \times \pi_0 G$, an element $y \in \pi$ can be written as $y = (s, x)$ for $s \in S$ and $x \in \pi_0 G$. Notice that $\pi_0 G$ is nilpotent and this group has a non–trivial center, since $\pi$ is non–abelian. The map $q : \pi \rightarrow \pi_0 G$ is an epimorphism. Consequently we can find an element $y_0 = (s_0, x_0)$ where $s_0 \in S$ and $x_0$ is a non–identity element of $Z(\pi_0 G)$. This means that $y_0$ is contained in $Z(\pi)$, though $q(y_0)$ is a non–identity element. This contradiction completes the proof.  

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3 $\Pi$–compact subgroups of simple Lie groups

We will need the following results to prove Theorem 2.

**Lemma 3.1** Let $K$ be a compact Lie group, and let $G$ be a connected compact Lie group. If $(BK)^{\wedge}_p \simeq (BG)^{\wedge}_p$ for some $p$, we have a group extension as follows:

$$1 \longrightarrow W(K_0) \longrightarrow W(G) \longrightarrow \pi_0 K \longrightarrow 1$$

**Proof** It is well–known that $H^*((BG)^{\wedge}_p; \mathbb{Q}) = H^*((BT_G)^{\wedge}_p; \mathbb{Q})^{W(G)}$, and since $(BK)^{\wedge}_p \simeq (BG)^{\wedge}_p$, it follows that $H^*((BG)^{\wedge}_p; \mathbb{Q}) = H^*((BK)^{\wedge}_p; \mathbb{Q})$. Notice that $H^*((BK)^{\wedge}_p; \mathbb{Q}) = H^*((BK_0)^{\wedge}_p; \mathbb{Q})^{\pi_0 K} = (H^*((BT_K_0)^{\wedge}_p; \mathbb{Q})^{W(K_0)})^{\pi_0 K}$. Galois theory for the invariant rings (see Smith [23]) tells us that $W(K_0)$ is a normal subgroup of $W(G)$ and that the quotient group $W(G)/W(K_0)$ is isomorphic to $\pi_0 K$. This completes the proof. \(\Box\)

**Lemma 3.2** For a compact Lie group $K$, suppose the loop space of the $p$–completion $\Omega(BK)^{\wedge}_p$ is a connected $p$–compact group. Then $p$ doesn’t divide the order of $\pi_0 K$.

**Proof** Since $BK$ is $p$–compact, $\pi_0 K$ is $p$–nilpotent. So, if $\pi$ denotes a $p$–Sylow subgroup of $\pi_0 K$, then $(B\pi_0 K)^{\wedge}_p \simeq B\pi$. Notice that $(BK)^{\wedge}_p$ is 1–connected. Hence the map $(BK)^{\wedge}_p \longrightarrow (B\pi_0 K)^{\wedge}_p$ induced from the epimorphism $K \longrightarrow \pi_0 K$ is a null map. Consequently the $p$–Sylow subgroup $\pi$ must be trivial. \(\Box\)

For $K = NT$, the normalizer of a maximal torus $T$ of a connected compact simple Lie group, the converse of Lemma 3.2 is true, though it doesn’t hold in general. Note that $\pi_0 \Gamma_2 = \mathbb{Z}/2$ and that $B\Gamma_2$ is not 3–compact [19].

**Proof of Theorem 2** (1) Since $(BH)^{\wedge}_p \simeq (BG)^{\wedge}_p$ for some $p$, Lemma 3.1 says that the Weyl group $W(H_0)$ is a normal subgroup of $W(G)$. First we show that $W(H_0) \neq W(G)$. If $W(H_0) = W(G)$, the inclusion $H_0 \longrightarrow G$ induces the isomorphism $H^*(BH_0; \mathbb{Q}) \cong H^*(BG; \mathbb{Q})$, since rank$(H_0) = \mathrm{rank}(G)$. Hence $BH_0 \simeq BG$. Consequently if $\tilde{H}_0$ and $\tilde{G}$ denote the universal covering groups of $H_0$ and $G$ respectively, then $\tilde{H}_0 \cong \tilde{G}$. The maps $BH_0 \longrightarrow BH_0$ and $BG \longrightarrow BG$ are rational equivalences. According to [18, Lemma 2.2], we would see that $H_0 = H = G$. Since $H$ must be a proper subgroup of $G$, we obtain the desired result.
We now see that $W(H_0)$ is a proper normal subgroup of $W(G)$. If $W(H_0)$ is a nontrivial group, a result of Asano [3] implies that $(G, H_0)$ is one of the following types:

$$\begin{aligned}
(G, H_0) = \left\{ (B_n, D_n) , (C_n, A_1 \times \cdots \times A_1) , (G_2, A_2) , (F_4, D_4) \right\}
\end{aligned}$$

According to [20, Lemma 2.1 and Proposition 3.1], we notice $(2)$ We first show that, for any odd prime $p$,

$$G = C \times Z$$

Thus the action of $BNT$ is given by $a \times \cdots \times A_1$.

For $(B_n, D_n)$, take $(G, H) = (SO(3), O(2))$. Since $\pi_0(O(2)) = \mathbb{Z}/2$ and $BO(2) \simeq BSO(3)$ for odd prime $p$, the space $BO(2)$ is $\Pi$–compact. In the case $G = B_2$, take $(G, H) = (G, NT_G)$ for $G = \text{Spin}(5)$. Then $\pi_0 H$ is a $2$–group and $BNT_G \simeq BG$ for odd prime $p$, and hence $BNT_G$ is $\Pi$–compact.

In the case of $(C_2, A_1 \times A_1)$, take $G = Sp(2)$ and $H = (Sp(1) \times Sp(1)) \times \mathbb{Z}/2\langle a \rangle$ where $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in Sp(2)$. For complex numbers $z$ and $w$, we see that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} w & 0 \\ 0 & z \end{pmatrix}.$$ 

Thus the action of $\mathbb{Z}/2\langle a \rangle$ is given by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We note that

$$W(Sp(2)) = D_8 = \left\{ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$
Consequently $\pi_0 H$ is a 2–group and $BH \cong_p BG$ for odd prime $p$, and hence $BH$ is $\Pi$–compact. Finally, for $(G_2, A_2)$, as mentioned in the introduction, take $G = G_2$ and $H = SU(3) \times \mathbb{Z}/2$. Then $BH$ is $\Pi$–compact.

It remains to consider the case $p = 2$. Note that $|W(G)/W(H_0)|$ for each of such $(G, H_0)$’s is a power of 2. Lemma 3.1 implies that the finite group $\pi_0 H$ must be a 2–group. Lemma 3.2 says that $|\pi_0 H|$ is not divisible by 2, since $(BH)_{\wedge}^p \cong (BG)_{\wedge}^p$. Thus $H$ is connected, and hence $H = G$. This completes the proof. \hfill \Box

Any proper closed subgroup of $G$ which includes the normalizer $NT$ satisfies the assumption of Theorem 2. So, this theorem shows, once again, that almost all $BNT$ are not $\Pi$–compact [20]. Furthermore, for any connected compact Lie group $G$, it is well-known that $(BNT)_{\wedge}^p \cong (BG)_{\wedge}^p$ if $p$ does not divide the order of the Weyl group $W(G)$, hence $BNT$ is $p$–compact for such $p$. The converse is shown in [20].

Lemma 3.3 Let $\alpha \longrightarrow \tilde{G} \longrightarrow G$ be a central extension of compact Lie groups. Then $BG$ is $p$–compact if and only if $B\tilde{G}$ is $p$–compact.

Proof First assume that $BG$ is $p$–compact. Since $\alpha \longrightarrow \tilde{G} \longrightarrow G$ is a central extension, the fibration $B\alpha \longrightarrow B\tilde{G} \longrightarrow BG$ is principal. Thus we obtain a fibration $B\tilde{G} \longrightarrow BG \longrightarrow K(\alpha, 2)$. The base space is 1–connected, so the fibration is preserved by the $p$–completion, and hence we obtain the fibration

$$(B\alpha)^{\wedge}_p \longrightarrow (B\tilde{G})^{\wedge}_p \longrightarrow (BG)^{\wedge}_p.$$ 

Since the loop spaces $\Omega(B\alpha)^{\wedge}_p$ and $\Omega(BG)^{\wedge}_p$ are $F_p$–finite, so is $\Omega(B\tilde{G})^{\wedge}_p$. Thus $B\tilde{G}$ is $p$–compact.

Conversely we assume that $B\tilde{G}$ is $p$–compact. Consider the fibration

$$\Omega(BG)^{\wedge}_p \longrightarrow (B\alpha)^{\wedge}_p \longrightarrow (BG)^{\wedge}_p.$$ 

Since the map $(B\alpha)^{\wedge}_p \longrightarrow (B\tilde{G})^{\wedge}_p$ is induced from the inclusion $\alpha \hookrightarrow \tilde{G}$, it is a monomorphism of $p$–compact groups. Hence its homotopy fiber $\Omega(BG)^{\wedge}_p$ is $F_p$–finite, and therefore $BG$ is $p$–compact. \hfill \Box

Corollary 3.4 Let $\alpha \longrightarrow \tilde{G} \longrightarrow G$ be a central extension of compact Lie groups, and let $H$ be a subgroup of $G$ so that there is the commutative diagram:

$$\begin{array}{ccc}
\alpha & \longrightarrow & \tilde{G} \\
\downarrow & & \downarrow \\
\alpha & \longrightarrow & \tilde{H} \\
\downarrow & & \downarrow \\
\alpha & \longrightarrow & H
\end{array}$$

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Then the pair \((G, H)\) satisfies the conditions of Theorem 2 if and only if so does the pair \((\tilde{G}, \tilde{H})\).

**Proof** Lemma 3.3 implies that \(BH\) is \(\Pi\)–compact if and only if \(B\tilde{H}\) is \(\Pi\)–compact. It is clear that \(\text{rank}(H_0) = \text{rank}(G)\) if and only if \(\text{rank}(\tilde{H}_0) = \text{rank}(\tilde{G})\). Finally we see \((BH)_p^0 \simeq (BG)_p^0\) if and only if \((B\tilde{H})_p^0 \simeq (B\tilde{G})_p^0\) from the following commutative diagram of fibrations:

\[
\begin{array}{ccc}
(B\alpha)_p^0 & \longrightarrow & (B\tilde{G})_p^0 \\
\| & & \| \\
(B\alpha)_p^0 & \longrightarrow & (B\tilde{H})_p^0
\end{array}
\]

This completes the proof. \(\square\)

**Lemma 3.5** Let \(M \longrightarrow K \longrightarrow L\) be a short exact sequence of groups. If \(\nu\) is a normal subgroup of \(K\), the kernel \(\nu'\) of the composition \(\nu \longrightarrow K \longrightarrow L\) is a normal subgroup of \(M\).

**Proof** We consider the following commutative diagram:

\[
\begin{array}{ccc}
\nu' & \longrightarrow & M \\
\downarrow & & \downarrow \ \\
\nu & \longrightarrow & K \\
\downarrow & & \downarrow q \\
q(\nu) & \longrightarrow & L
\end{array}
\]

For \(x \in \nu'\) and \(m \in M\), it follows that

\[
q(mxm^{-1}) = q(m)q(x)q(m^{-1}) = q(m)q(m)^{-1} = e
\]

Thus \(mxm^{-1} \in \ker q\). Since \(\nu' \subset \nu\), \(M \subset K\), and \(\nu \triangleleft K\), we see that \(mxm^{-1} \in \nu\). So \(mxm^{-1} \in \ker q \cap \nu = \nu'\), and therefore \(\nu' \triangleleft M\). \(\square\)

**Proof of Theorem 3** First suppose \((G, H_0) = (B_n, D_n)\) or \((C_2, A_1 \times A_1)\). Let \(\nu'\) be the kernel of the composition \(\nu \longrightarrow H \longrightarrow \pi_0 H\). Consider the following commutative
Lemma 3.5 says that $\nu' \triangleleft H_0$. Since $\nu'$ is a finite normal subgroup of $H_0$, it is a finite 2–group. As we have seen in the proof of Theorem 2, $\pi_0 H = W(G)/W(H_0)$ is a 2–group, and hence so is $q(\nu)$. Consequently $\nu$ is a 2–group.

Now consider the following commutative diagram:

\[
\begin{array}{ccc}
\nu' & \longrightarrow & H_0 \\
\downarrow & & \downarrow \\
\nu & \longrightarrow & H \\
\downarrow & & \downarrow \\
q(\nu) & \longrightarrow & \pi_0 H
\end{array}
\]

Since $\pi_0 \Gamma$ is a 2–group, the fibration $B\Gamma_0 \longrightarrow B\Gamma \longrightarrow B\pi_0 \Gamma$ is preserved by the 2–completion (see Bousfield and Kan [5]). Hence $B\Gamma$ is 2–compact. Next, for odd prime $p$, we see that $(B\Gamma)_p^\wedge \simeq (BH)_p^\wedge$, since $\nu$ is a 2–group. We see also that $G$ has no odd torsion and $H^*(BH;\mathbb{F}_p) = H^*(BH_0;\mathbb{F}_p)^{\pi_0 H} \cong H^*(BG;\mathbb{F}_p)$. Consequently the space $(B\Gamma)_p^\wedge$ is homotopy equivalent to $(BG)_p^\wedge$. Therefore $B\Gamma$ is $\Pi$–compact.

It remains to consider the case $(G, H_0) = (G, T_G)$ for $G = A_1$ or $G = B_2 (= C_2)$. Since $H_0 = T_G$ and $H_0 \triangleleft H$, we see that $H$ is a subgroup of the normalizer $NT_G$. Consider the following commutative diagram:

\[
\begin{array}{ccc}
T_G & \longrightarrow & NT_G \\
\downarrow & \uparrow & \downarrow \\
\downarrow & \uparrow & \downarrow \\
T_G & \longrightarrow & H \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\pi_0 H & & \pi_0 H
\end{array}
\]

Since the map $BH \longrightarrow BG$ is $p$–equivalent for some $p$, it follows that $\pi_0 H = W(G)$. Consequently $H = NT_G$.

If $\nu$ is a finite normal subgroup of $NT_G$, then $B\nu$ is contained in the kernel of the map $(BG)_p^\wedge \simeq (BNT_G)_p^\wedge \longrightarrow (B\Gamma)_p^\wedge$. Since $G$ is simple and $G \neq G_2$, according to [16, 17], the group $\nu$ is included in the center of $G$. Thus $\nu$ is a 2–group. Therefore
\((B\Gamma)_p^\wedge \simeq (BNT_G)_p^\wedge \simeq (BG)_p^\wedge\) for odd prime \(p\), and hence \(B\Gamma\) is \(p\)-compact for such \(p\).

Finally we note that \(W(G)\) is a 2–group, and hence \(B\Gamma\) is 2–compact.

We will discuss a few more results. Basically we have been looking at three Lie groups \(H_0 \subset H \subset G\). The following shows a property of the (non–connected) middle group \(H\).

**Proposition 3.6** Suppose \(G\) is a connected compact Lie group, and \(H\) is a proper closed subgroup of \(G\) with \(\text{rank}(H_0) = \text{rank}(G)\). If the order of \(\pi_0H\) is divisible by a prime \(p\), so is the order of \(W(G)/W(H_0)\).

**Proof** Assuming \(|W(G)/W(H_0)| \not\equiv 0 \mod p\), we will show \(\pi_0H \not\equiv 0 \mod p\). Notice that we have the following commutative diagram

\[
\begin{array}{c}
T \\
| \\
| \uparrow \downarrow \\
N_GT & \longrightarrow & W(G) \\
| \downarrow \\
T & \longrightarrow & N_{H_0}T & \longrightarrow & W(H_0),
\end{array}
\]

where the vertical maps are injective, since \(\text{rank}(H_0) = \text{rank}(G)\). We recall, from Jackowski, McClure and Oliver [21], that the Sylow theorem for compact Lie groups \(G\) holds. Namely \(G\) contains maximal \(p\)-toral subgroups, and all of which are conjugate to \(N_pT\), where \(N_p(T)/T\) is a \(p\)-Sylow subgroup of \(N(T)/T = W(G)\).

Suppose \(K\) is a \(p\)-toral subgroup of \(H\). Since \(|W(G)/W(H_0)| \not\equiv 0 \mod p\), we see that \(K\) is a subgroup of \(H_0\) up to conjugate. Consequently, the composite map \(K \hookrightarrow H \longrightarrow \pi_0H\) must be homotopy equivalent to a null map. Since \(H \longrightarrow \pi_0H\) is surjective, the \(p\)-part of \(\pi_0H\) is trivial.

For each pair mentioned in the part (a) of Theorem 2, we note that \(|W(G)/W(H_0)|\) is a power of 2. **Proposition 3.6** says, for instance, that \(\pi_0H\) is a 2–group for any \((G, H)\) such that \(|W(G)/W(H_0)|\) is a power of 2. As an application, one can show that if \(H\) is a non–connected proper closed subgroup of \(SO(3)\) with \(H_0 = SO(2)\), then \(H\) is isomorphic to \(O(2)\). A proof may use the fact that \(H\) is 2–toral, and that a maximal 2–toral subgroup in \(H\) is 2–stubborn [21]. A 2–compact version of this result also holds. Suppose \(X\) is a 2–compact group such that there are two monomorphisms of 2–compact groups \(BSO(2)^\wedge_2 \longrightarrow BX\) and \(BX \longrightarrow BSO(3)^\wedge_2\). Then, along the line of a similar argument, one can also show that \(BX\) is homotopy equivalent to \(BO(2)^\wedge_2\) if \(X\) is not connected. In the case of \(X\) being connected, the classifying space \(BX\) is either \(BSO(2)^\wedge_2\) or \(BSO(3)^\wedge_2\).
Theorem 2, Lie groups of type \((C_2, A_1 \times A_1)\) has been discussed. An example is given by \(Sp(1) \times Sp(1) \subset (Sp(1) \times Sp(1)) \rtimes \mathbb{Z}/2 \subset Sp(2)\). The middle group can be regarded as the wreath product \(Sp(1) \rtimes \Sigma_n\) for \(n = 2\). We ask for what \(n\) and \(p\) its classifying space is \(p\)-compact. Note that \(Sp(1) \rtimes \Sigma_n\) is a proper closed subgroup of \(Sp(n)\).

**Proposition 3.7** Let \(\Gamma(n)\) denote the wreath product \(Sp(1) \rtimes \Sigma_n\). Then

\[
P(B\Gamma(n)) = \begin{cases} 
\Pi & \text{if } n = 2 \\
\{p \in \Pi \mid p > n\} & \text{if } n \geq 3
\end{cases}
\]

**Proof** When \(n = 2\), the desired result has been shown in our proof of the part (b) of Theorem 2. Recall from [20] that if \(B\Gamma(n)\) is \(p\)-compact, then \(\pi_0 B\Gamma(n) = \Sigma_n\) must be \(p\)-nilpotent. For \(n \geq 4\), it follows that \(\Sigma_n\) is \(p\)-nilpotent if and only if \(p > n\). Since the group \(\Gamma(n)\) includes the normalizer of a maximal torus of \(Sp(n)\), we see \(B\Gamma(n) \simeq_p BSp(n)\) if \(p > n\). Thus \(P(B\Gamma(n)) = \{p \in \Pi \mid p > n\}\) for \(n \geq 4\).

For \(n = 3\), note that \(\Sigma_3\) is \(p\)-nilpotent if and only if \(p \neq 3\). So it remains to prove that \(B\Gamma(3)\) is not \(2\)-compact. We consider a subgroup \(H\) of \(\Gamma(3)\) which makes the following diagram commutative:

\[
\begin{array}{cccc}
\prod^3 Sp(1) & \longrightarrow & \prod^3 Sp(1) & \longrightarrow * \\
\downarrow & & \downarrow & \\
H & \longrightarrow & \Gamma(3) & \longrightarrow \mathbb{Z}/2 \\
\downarrow & & \downarrow & \\
\mathbb{Z}/3 & \longrightarrow & \Sigma_3 & \longrightarrow \mathbb{Z}/2
\end{array}
\]

The fibration \(BH \longrightarrow B\Gamma(3) \longrightarrow B\mathbb{Z}/2\) is preserved by the completion at \(p = 2\). Hence, if \(B\Gamma(3)\) were \(2\)-compact, the space \(\Omega(BH)_{\mathbb{Q}_2}\) would be a connected \(2\)-compact group so that the cohomology \(H^*(BH; \mathbb{Q}_2)\) should be a polynomial ring, (see Dwyer and Wilkerson [8, Theorem 9.7]). Though \(H^*(B\prod^3 Sp(1); \mathbb{Q}_2)\) is a polynomial ring, its invariant ring \(H^*(BH; \mathbb{Q}_2) = H^*(B\prod^3 Sp(1); \mathbb{Q}_2)_{\mathbb{Z}/3}\) is not a polynomial ring, since the group \(\mathbb{Z}/3\) is not generated by reflections. This contradiction completes the proof.

For \((G, H) = (Sp(n), Sp(1) \rtimes \Sigma_n)\), we note that \((G, H_0)\) is a type of \((C_n, A_1 \times \cdots \times A_1)\). This is one of the cases that the Weyl group \(W(H_0)\) is a normal subgroup of \(W(G)\).
(see Asano [3]) discussed in our proof of the part (a) of Theorem 2. Finally we talk about the only remaining case \((G, H_0) = (F_4, D_4)\). An example is given by \(\text{Spin}(8) \subset \text{Spin}(8) \rtimes \Sigma_3 \subset F_4\). Let \(\Gamma\) denote the middle group \(\text{Spin}(8) \rtimes \Sigma_3\). Then we can show that \(\mathbb{P}(B\Gamma) = \{p \in \Pi \mid p > 3\}\). To show that \(B\Gamma\) is not 2–compact, one might use the fact, (see Adams [1, Theorem 14.2]), that \(W(F_4) = W(\text{Spin}(8)) \rtimes \Sigma_3\), and that its subgroup \(W(\text{Spin}(8)) \rtimes \mathbb{Z}/3\) is not a reflection group.

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