The Continuum Limit of Non Compact QED

V. Azcoiti\textsuperscript{a}, G. Di Carlo\textsuperscript{b}, A. Galante\textsuperscript{c,b}, A.F. Grillo\textsuperscript{d}, V. Laliena\textsuperscript{a} and C.E. Piedrafita\textsuperscript{b}

\textsuperscript{a} Departamento de Física Teórica, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza (Spain).
\textsuperscript{b} Istituto Nazionale di Fisica Nucleare, Laboratori Nazionali di Frascati, P.O.B. 13 - Frascati 00044 (Italy).
\textsuperscript{c} Dipartimento di Fisica dell’Università dell’Aquila, L’Aquila 67100 (Italy)
\textsuperscript{d} Istituto Nazionale di Fisica Nucleare, Laboratori Nazionali del Gran Sasso, Assergi (L’Aquila) 67010 (Italy).

ABSTRACT
Since four-fermion operators in strongly coupled QED are nonperturbatively renormalizable, we analyze here the phase diagram and critical behaviour of the Gauged Nambu-Jona Lasinio model. Our mean field approximation relates the critical exponents along the continuous phase transition line with the mass dependence of the chiral condensate in the Coulomb phase of standard noncompact QED. The numerical results for noncompact QED strongly suggest non mean field exponents along the critical line.
The possible existence of a non-gaussian fixed point in strongly coupled non compact QED has been subject of extensive research during the last few years. It is well known that it has been a controversial subject for a long time. The present status of the field can be summarized in the following two points:

1. Triviality as in a logarithmically improved scalar mean field theory has been disproved by the numerical results [1].

2. Unfortunately triviality a la Nambu-Jona Lasinio, which is the most natural way to analyze triviality in a theory with strongly coupled fermion fields, can not be disproved by the present day numerical data and it seems very difficult to get an important improvement in the quality of these data in near future.

Therefore other alternative ways and new ideas are necessary in order to get some progress in this field. This is the reason why the gauged Nambu-Jona Lasinio (GNJL) model has become increasingly interesting in recent time. In fact if a non gaussian fixed point exists in non compact QED, the naive dimensional analysis does not applies. Therefore operators of dimension higher than four, which are non renormalizable in perturbation theory, could acquire anomalous dimensions and become renormalizable [2].

The lattice action for the GNJL model with noncompact gauge fields and staggered fermions reads

\[
S = \frac{\beta}{2} \sum_{n,\mu<\nu} \Theta_{\mu\nu}^2(n) + \bar{\chi} \Delta(\theta) \chi + m \bar{\chi} \chi - G \sum_{n,\mu} \bar{\chi}_n \chi_n \bar{\chi}_{n+\mu} \chi_{n+\mu} .
\] (1)

In the chiral limit, \( m = 0 \), this action is invariant under the continuous transformations

\[
\chi_n \rightarrow \chi_n e^{i\alpha(-1)^{n_1+\ldots+n_d}} , \quad \bar{\chi}_n \rightarrow \bar{\chi}_n e^{i\alpha(-1)^{n_1+\ldots+n_d}}
\] (2)

which define a continuous chiral U(1) symmetry group.

The main technical difficulty when computing vacuum averages in the GNJL model comes from the fact that the action (1) is not a bilinear of the fermion fields. The standard procedure consists in the introduction of an auxiliary vector field which allows to bilinearize the fermion action. The prize to pay for that is that we have one more field to include in the numerical simulations of this model that besides the number of free parameters \( (\beta, m, G) \), makes it difficult to analyse this model with reasonable computer resources [3].
As a first approach, we can do a standard mean field approximation which has the advantage of bilinearizing the action (1). Following the mean field technique, we do in (1) the following substitution

\[ G \sum_{n,\mu} \bar{\chi}_n \chi_{n+\mu} \chi_{n+\mu} \rightarrow 2dG \langle \bar{\chi}\chi \rangle \sum_n \bar{\chi}_n \chi_n \]

where \( d \) is the space-time dimension. The action (1) becomes in this way a bilinear in the fermion fields and the path integral over the Grassmann variables can be done by means of the Mathews-Salam formula.

The v.e.v. of the chiral condensate after the substitution of the mean field approximation (3) in the action (1) is given by

\[
\langle \bar{\chi}\chi \rangle = -\int [d\theta] e^{-\frac{1}{2} \sum_{\mu} \Theta_{\mu}(n) \det[\Delta + (m - 8G\langle \bar{\chi}\chi \rangle)I]} \frac{1}{V} tr \Delta^{-1} \frac{1}{V} \sum_{j=1}^{V/2} \frac{1}{\lambda_j^2 + 64G^2 \langle \bar{\chi}\chi \rangle^2},
\]

where the sum in (5) runs over all positive eigenvalues of the massless Dirac operator and the integration measure in the v.e.v. includes the fermionic determinant of standard noncompact QED, evaluated at the effective mass \( \bar{m} = -8G\langle \bar{\chi}\chi \rangle \).

1. The phase diagram

Equation (5) is always verified if \( \langle \bar{\chi}\chi \rangle = 0 \), and this is the only solution in the symmetric phase. In the broken phase where \( \langle \bar{\chi}\chi \rangle \neq 0 \), the v.e.v. of the chiral condensate will be given by the solution of the following equation

\[
1 = 16G \left( \frac{1}{V} \sum_{j=1}^{V/2} \frac{1}{\lambda_j^2 + 64G^2 \langle \bar{\chi}\chi \rangle^2} \right)
\]

which gives for the critical line, where the chiral condensate vanishes continuously, the following expression

\[
G_c(\beta) = \frac{1}{16\langle \sum_{j=1}^{V/2} \frac{1}{\lambda_j^2} \rangle}
\]
The existence of this critical line in the GNJL model was discovered some time ago in the continuum formulation using the quenched-ladder approximation.

In Fig. 1 we present our numerical results for the phase diagram in the $\beta, G$ plane. The critical line has been obtained by computing numerically the v.e.v. of the sum of the inverse square eigenvalues (eq. (7)), which is proportional to the chiral transverse susceptibility of the standard noncompact QED in the chiral limit. The numerical simulations were done using the MFA approach, which allows to do computations in the chiral limit. We refer the interested reader to the extended bibliography on this subject and specially to the ref. where the computation of the chiral susceptibility and the determination of the critical coupling in noncompact QED is discussed in detail.

2. The critical exponents

The phase diagram of Fig. 1 is in good qualitative agreement with the corresponding phase diagram obtained in the quenched-ladder approximation. Using this analytical approach, a line of critical points with continuously varying critical exponents was found in the intersection point of this line with the $G = 0$ axis corresponding to an essential singularity.

Later on, numerical simulations of noncompact QED disproved the essential singularity behavior, putting in evidence the limitations of the quenched-ladder approximation. Since our approach contains less approximations, we do hope to get more reliable results for the critical exponents.

In order to extract the critical exponents, we will start from the key equation of state relating the order parameter with the external magnetic field $m$ and the gauge and four fermion couplings. Using the previous notation we can write

$$\langle \bar{\chi}\chi \rangle = -2\bar{m}F(\beta, \bar{m})$$

where the right hand side in (8) is just the chiral condensate in full noncompact QED evaluated at the gauge coupling value $\beta$ and fermion mass $\bar{m}$. Concerning critical exponents the interesting physical region, as follows from the phase diagram of Fig. 1, is $\beta > \beta_c^0$ (Coulomb phase of noncompact QED).

Since we are interested in the critical region ($m \to 0, \langle \bar{\chi}\chi \rangle \to 0$), we will analyze the behavior of $F(\beta, \bar{m})$ in the $\bar{m} \to 0$ limit. In this limit we can write
\[ F(\beta, \bar{m}) = F(\beta, 0) + B\bar{m}^\omega + \ldots \quad (9) \]

The second term in (9) can contain also logarithmic contributions and \( F(\beta, 0) \) is one half of the massless transverse susceptibility in noncompact QED. Equation (9), after the substitution of \( \bar{m} \) by \( m - 8G\langle \bar{\chi}\chi \rangle \), implies the following behavior for the chiral condensate in the \( m \to 0 \) limit

\[ \langle \bar{\chi}\chi \rangle \sim m^{-\frac{1}{\omega + 1}} \quad (10) \]

and therefore the \( \omega \) and \( \delta \) exponents are related by the equation

\[ \delta = \omega + 1 \quad (11) \]

A straightforward calculation allow to compute also the magnetic \( \beta_m \) and susceptibility \( \gamma \) exponents, the final result being

\[ \beta_m = \frac{1}{\omega}, \quad \gamma = 1 \quad (12) \]

The hyperscaling relation \( \gamma = \beta_m(\delta - 1) \) is verified, as follows from (12).

The determination of the order parameter critical exponents in our mean field approach reduce therefore to the determination of the \( \omega \) exponent which controls the mass dependence of the chiral condensate in the Coulomb phase of noncompact QED. In the \( \beta \to \infty \) limit of noncompact QED, the theory is free and the chiral condensate can be analytically computed. The well known result in this case (\( \omega = 2 \) plus logarithmic corrections) implies mean field exponents for the end point of the phase transition line, with the following behavior for \( \langle \bar{\chi}\chi \rangle_{\beta=\infty, G=G^c} \)

\[ m \sim \langle \bar{\chi}\chi \rangle^3 \log \langle \bar{\chi}\chi \rangle \quad (13) \]

In the general case, the chiral condensate in the Coulomb phase of noncompact QED (\( \langle \bar{\chi}\chi \rangle_{NCQED} \)) can be parameterized as follows

\[ \langle \bar{\chi}\chi \rangle_{NCQED} = A(\beta)m + B(\beta)m^{\omega+1} + \ldots \quad (14) \]

The first contribution in (14) is linear in \( m \), as follows from the fact that the massless transverse susceptibility is finite in the Coulomb phase of noncompact QED. The next contribution can have logarithmic corrections, as happens in the \( \beta \to \infty \) limit where it becomes \( m^3 \log m \). In order to extract the \( \omega \) exponent from the numerical simulations, we can use the results for the massless chiral transverse susceptibility \[ \] to fix \( A(\beta) \) in (14) and fit
the numerical results with eq. (14). This procedure has the inconvenient that higher order contributions in (14) can induce systematic errors in the determination of \( \omega \).

A better strategy exploiting the potentialities of the MFA method is to compute vacuum expectation values of operators which can be considered as generalizations of one of the contributions to the massless nonlinear susceptibility. More precisely we have defined \( \chi_q \) by the expression

\[
\chi_q = \frac{1}{V} \left\langle \sum_{j=1}^{V/2} \frac{1}{\lambda_j^q} \right\rangle
\]

(15)

When \( q = 4 \), we get one of the contributions to the standard massless nonlinear susceptibility. In the general case we can write this vacuum expectation value as an integral over the spectral density of eigenvalues in the following way

\[
\frac{1}{V} \left\langle \sum_{j=1}^{V/2} \frac{1}{\lambda_j^q} \right\rangle = \int \frac{\rho(\lambda)}{\lambda^q} d\lambda
\]

(16)

and if the density of eigenvalues \( \rho(\lambda) \) behaves like \( \lambda^p \) near the origin, \( \chi_q \) will diverge when \( q > p + 1 \). In such a case and for lattices of finite size, we expect for \( \chi_q \) the following type of divergency with the lattice size \( L \)

\[
\chi_q \sim L^{\alpha(q-p-1)}
\]

(17)

where \( \alpha \) in (17) is some positive number.

An interesting thing to notice is that the \( p \)-exponent which controls the small \( \lambda \) behavior of the spectral density \( \rho(\lambda) \), can be related with the \( \omega \) exponent by the following equations

\[
\omega = p - 1 (p \leq 3)
\]

\[
\omega = 2 (p > 3)
\]

(18)

These relations allow to extract the \( \omega \) exponent from the finite size behavior of the generalized nonlinear susceptibility \( \chi_q \).

In Fig. 2 we have plotted our results for the inverse of the generalized nonlinear susceptibility \( \chi_q \) against the inverse lattice size for \( q \)-values running from 2 to 4 and \( \beta = 0.237 \). The solid lines in this figure correspond to a fit of all the points at any fixed \( q \) with the function
\[ \chi_q^{-1}(L) = a_q + b_q L^{-c} \]  

(19)

The results reported in this figure show how the infinite volume limit of the inverse generalized nonlinear susceptibility vanishes at large \( q \) and is different from zero at small \( q \), as expected. Fig. 3 is a plot of the extrapolated values of \( \chi_q^{-1} \) (thermodynamical limit) against \( q \). The critical value of \( q \) at which \( \chi_q^{-1} \) vanishes can be estimated from these results. Hence we get \( q_c \sim 2.5 \) at \( \beta = 0.237 \), which implies \( p \sim 1.5 \) and \( \omega \sim 0.5 \). Using now the relations (11), (12) the following results for the order parameter critical exponents can be derived

\[ \delta \sim 1.5, \quad \beta_m \sim 2, \quad \gamma = 1, \]  

(20)

values which are clearly outside the range of the mean field exponents.

Even if at first sight non mean field exponents in a mean field approach seems to be a paradox, there is no contradiction between both statements. In fact we have applied the mean field approximation to the fermion field but fluctuations of the gauge field are taken into account in our numerical simulations. In the infinite \( \beta \) limit, where the gauge field is frozen to the free field configuration, we get mean field exponents. However fluctuations of the gauge field at finite \( \beta \) seem to play a fundamental role in driving critical exponents to non mean field values.

The picture which emerges from this calculation is that the critical exponents change continuously along the critical line of Fig. 1 from their mean field values (end point of the critical line) to some non mean field values at the critical point of noncompact \( QED \). The \( \delta \) exponent approaches its mean field value (\( \delta = 3 \)) from below whereas the magnetic exponent approaches its mean field value (\( \beta_m = 0.5 \)) from above [8]. Our results for several values of the gauge coupling \( \beta \) suggest also that the value of \( \delta \) increases systematically along the critical line with increasing \( \beta \), in contrast with the magnetic exponent results which are systematically decreasing with \( \beta \).

In conclusion we do believe our qualitative picture is realistic. In fact it is hard to think that non mean field exponents in a mean field approach will become mean field exponents after removing the mean field approach.
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Figure captions

**Figure 1.** Phase diagram of the $GNJL$ model in the $\beta, G$ plane.

**Figure 2.** Logarithm of the nonlinear susceptibility against the logarithm of the lattice size for lattice sizes 4,6,8, 10 and $\beta = 0.237$.

**Figure 3.** Inverse generalized nonlinear susceptibility against the inverse lattice volume at $\beta = 0.237$.

**Figure 4.** Infinite volume limit of the generalized nonlinear susceptibility against $q$ at $\beta = 0.237$. 