Semiclassical model for a memory dephasing channel

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Abstract. We study a dephasing channel with memory, described by a Hamiltonian model in which the system-environment interaction is described by a stochastic process. We propose a useful way to describe the channel uses correlations. Moreover, we give a general expression for the coherences decay factors as a function of the number of channel uses and of the stochastic process power spectrum. We also study the impact of memory on the three qubit code, showing that correlations among channel uses affect very little the code performance.

1. Introduction

State transfer between different units of a quantum computer or entanglement distribution between two parties require quantum communication channels \([1,2]\). They are quantum systems transferring quantum information: the proper quantity to characterize the channel performance is the quantum capacity, defined as the maximum number of qubits that can be reliably transmitted per channel use \([3]\).

Quantum channels are often thought as memoryless, implying that the effect of the channel on each information carrier is always described by the same map \(\mathcal{E}\). In other terms there is no memory in the interaction between carriers and the environmental degrees of freedom physically describing the channel. In this case the quantum operation for \(N\) channel uses is given by \(\mathcal{E}_N = \mathcal{E}^\otimes N\). However, in several physically relevant situations this is not a realistic assumption. Memory effects appear when the characteristic time scales for the environment dynamics are comparable or longer than the time between consecutive channel uses. For instance, solid state implementations, which are the most promising for their scalability and integrability, suffer from low frequency noise \([4]\). In optical fibers, memory effects may appear due to slow birefringence fluctuations \([5]\). This introduces correlation among uses, then \(\mathcal{E}_N \neq \mathcal{E}^\otimes N\), this kind of channels being referred in the literature as memory channels \([6, 7, 8]\).
A very interesting question, raised for the first time in Ref. [9], is whether memory can enhance the transmission capacity of a quantum channel. Recently we have considered a channel subject to dephasing noise described by a Markov chain, showing that the quantum capacity increases with respect to memoryless limit [10] (see also [11, 12]). Furthermore based on theoretical arguments and numerical simulations, we have conjectured that the enhancement of the quantum capacity also takes place for a dephasing quantum environment modelled by a bosonic bath [10].

This issue is also relevant for the performance of Quantum Error-Correcting Codes (QECCs). Since quantum capacity is the maximum rate of reliable quantum information transmission, it puts an upper bound to the asymptotic rate achievable by any QECC. On the other hand, realistic QECCs necessarily work on a finite number of channel uses. Moreover, present day experimental implementations [13, 14] are based on very few channel uses. Previous studies have investigated the impact of correlations on the performance of QECCs [15, 16]. Depending on the chosen model, correlations may have positive or negative impact on QECCs. In a previous paper [17] we have shown, for a Markovian dephasing channel, that also low values of memory, for which the quantum capacity does not change appreciably, can have a detrimental impact on the three-qubit code performance.

In this paper we describe a dephasing channel by a Hamiltonian where the system environment interaction is modelled by a stochastic process. Then we discuss the three-qubit code error performance in presence of channel correlations.

2. Channel Model

We suppose that information is carried by qubits that transit across a communication channel, modelled as an environment determining pure dephasing of the qubits. The environment acts as a stochastic drive $\xi(t)$ on the system and the Hamiltonian describing the transmission of $N$ qubits through the channel reads

$$\mathcal{H}(t) = -\frac{\lambda}{2} \xi(t) \sum_{k=1}^{N} \sigma_z^{(k)} f_k(t).$$  \hspace{1cm} (1)

The $k$–th qubit is coupled to the environment via its Pauli operator $\sigma_z^{(k)}$, with coupling strength $\lambda$. The functions $f_k(t) = u(t - t_k) - u(t - t_k - \tau_p)$, where $u(t)$ is the unit step function [18], switch the coupling on and off. Here $\tau_p$ is the time each carrier takes to cross the channel; $\tau \equiv t_{k+1} - t_k$ is the time interval that separates two consecutive qubits entering the channel. Only when the $k$-th qubit is inside the channel the function $f_k = 1$. We assume $\xi(t)$ is a stationary and Gaussian stochastic process [19] with zero average value, characterized by its autocorrelation function $C(\tau)$.

To deal with this problem, we first consider the time evolution of the system for a given realization $\xi(t)$ of the stochastic process, and then we perform an average over all
possible realizations. The $N$-qubit time evolution operator for a given realization is

$$U_\xi(t) = \bigotimes_{k=1}^N \exp(-i\sigma_z^{(k)}\phi_k), \quad (2)$$

where $\phi_k$ is the phase acquired by the $k$-th qubit coherences after the qubit crossed the channel:

$$\phi_k = \frac{\lambda}{2} \int_{t_k}^{t_{k+1}} \xi(t') \, dt'.$$

Time evolution is conveniently described in the factorized basis $\{|j\rangle \equiv |j_1, \ldots, j_N\rangle, j_1, \ldots, j_N = 0, 1\}$, where $\{|j_k\rangle\}$ are eigenvectors of $\sigma_z^{(k)}$. Let $\rho^Q = \sum_{j,l} a_{jl} |j\rangle \langle l|$ be the initial state of the $N$-qubit system; the final state $\rho^{Q'}$ after all $N$ qubits crossed the channel is given by

$$\rho^{Q'} = U_\xi(t) \rho^Q U_\xi^\dagger(t) = \sum_{j,l} a_{jl} \exp\left(2i \sum_{k=1}^N s_k \phi_k\right) |j\rangle \langle l|, \quad (4)$$

where $s_k \equiv l_k - j_k = \frac{1}{2}[(1)j_k - (1)^{l_k}]$. By averaging over the stochastic process we finally obtain

$$\rho^{Q'} = \langle \rho^{Q'} \rangle = \sum_{j,l} a_{jl} \left\langle \exp\left(2i \sum_{k=1}^N s_k \phi_k\right) \right\rangle |j\rangle \langle l|, \quad (5)$$

which is a quantum operation for the $N$-qubits system: $\rho^{Q'} = \mathcal{E}_N(\rho^Q)$. It is possible to show that the quantity $\sum_{k=1}^N s_k \phi_k$ is itself a Gaussian variable, therefore

$$\left\langle \exp\left(2i \sum_{k=1}^N s_k \phi_k\right) \right\rangle = \exp\left(-2 \sum_{k,k'=1}^N s_k s_{k'} \langle \phi_k \phi_{k'} \rangle\right). \quad (6)$$

We call this quantity the $(j, l)$-coherence decay factor, since it is just the damping experienced by the $(j, l)$ system coherence:

$$D_{jl} \equiv \frac{|j\rangle \langle j| \rho^{Q'} |l\rangle \langle l|}{|j\rangle \langle j| \rho^Q |l\rangle \langle l|} = \exp\left(-2 \sum_{k,k'=1}^N s_k s_{k'} \langle \phi_k \phi_{k'} \rangle\right). \quad (7)$$

Next, by using the stationarity of $\xi(t)$, we calculate

$$\langle \phi_k \phi_{k'} \rangle = \left\langle \frac{\lambda^2}{4} \int_{t_k}^{t_k + \tau_p} dt_1 \xi(t_1) \int_{t_{k'}}^{t_{k'} + \tau_p} dt_2 \xi(t_2) \right\rangle = \frac{\lambda^2}{4} \int_{t_k}^{t_k + \tau_p} dt_1 \int_{t_{k'}}^{t_{k'} + \tau_p} dt_2 C(t_1 - t_2). \quad (8)$$

Since the autocorrelation function can be expressed in terms of the power spectral density $S(\omega) = \int d\tau e^{i\omega \tau} C(\tau)$ of $\xi(t)$ we obtain

$$\langle \phi_k \phi_{k'} \rangle = \lambda^2 \int_{0}^{\infty} \frac{d\omega}{2\pi} S(\omega) \frac{1 - \cos(\omega \tau_p)}{\omega^2} \cos[\omega(k - k') \tau], \quad (9)$$

\[\dagger\] Given two Gaussian variables $x$ and $y$ with arbitrary variances and some degree of correlation, the two variables $z_\pm = x \pm y$ are again Gaussian, as one can find by deriving the $(z_+, z_-)$ mixed density function from the one of $(x, y)$. Each phase $\phi_k$ is a time integral of a Gaussian stochastic process, so it can be view as the limit of a sum of Gaussian variables, so in its turn it is Gaussian.
and the final form of the coherences decay factor follows:

\[ D_{jl} = \exp \left(- \lambda^2 \int_0^\infty \frac{d\omega}{\pi} S(\omega) \frac{1 - \cos(\omega \tau_p)}{\omega^2} \sum_{k,k'=1}^N (l_k - j_k) (l_{k'} - j_{k'}) \cos[\omega (k - k') \tau] \right) \]  

(10)

This result is identical to the coherences decay due to a dephasing channel modelled by a set of quantum harmonic oscillators [10].

The expression (10) can be put in a nice and useful form. To this end we define \( \mu_{kk'} \) as the correlation coefficient [19] between the phases \( \phi_k \) and \( \phi_{k'} \):

\[ \mu_{kk'} = \frac{\langle \phi_k \phi_{k'} \rangle}{\sqrt{\langle \phi_k^2 \rangle \langle \phi_{k'}^2 \rangle}} \]  

(11)

where we set \( \eta^2 = \langle \phi_k^2 \rangle \); in fact, thanks to stationarity of the process, the quantity \( \langle \phi_k^2 \rangle \) does not depend on \( k \) (see equation (9)). The coherence decay factor (7) can be rewritten as

\[ D_{jl} = \exp \left(- 2\eta^2 \sum_{k,k'=1}^N s_k s_{k'} \mu_{kk'} \right). \]  

(12)

Now we observe that \( g \equiv \exp(-2\eta^2) \) is just the damping experienced by single qubit coherences for one channel use \( (N = 1) \). We finally write

\[ D_{jl} = g \sum_{k,k'=1}^N s_k s_{k'} \mu_{kk'} = g \left( \sum_{k=1}^N s_k^2 + 2 \sum_{k'=1,k>k'}^N s_k s_{k'} \mu_{k-k'} \right). \]  

(13)

where we have defined \( \mu_{k-k'} = \mu_{kk'} \). In fact the stationarity of \( \xi(t) \) implies that \( \mu_{kk'} \) depends only on \( |k - k'| \). The quantity \( \mu_{k-k'} \) is a measure of the degree of the correlation between the channel uses \( k \) and \( k' \).

3. Three-Qubit Code performance

As a measure of the quantum information transmission reliability we use the entanglement fidelity [20]. To define this quantity we look at the system \( Q \) as a part of a larger quantum system \( RQ \), initially in a pure entangled state \( |\psi_RQ\rangle \). The initial density operator of the system \( Q \) is then obtained from that of \( RQ \) by a partial trace over the reference system \( R \): \( \rho_Q = \text{Tr}_R[|\psi_RQ\rangle\langle\psi_RQ|] \). The system \( Q \) is sent through the channel, while \( R \) remains ideally isolated from any environment, being \( \rho^{RQ'} \) the final state of \( RQ \) after the transmission. Entanglement fidelity is just the fidelity between the initial and the final state of \( RQ \):

\[ F_e = \langle \psi_RQ | \rho^{RQ'} | \psi_RQ \rangle. \]  

(14)

First we consider a single use of the channel described by Hamiltonian [11]. We suppose to feed the channel with a quantum source [21] described by the density operator \( \rho^Q = \frac{1}{2} \mathbb{1} \). The entanglement fidelity is [22]:

\[ F_e = \langle \psi^Q | \rho^Q | \psi^Q \rangle = \frac{1 + g}{2}, \]  

(15)
where $I_R$ is the identity operator and $E^Q = E_1$. This case is relevant in quantum information field as it takes place when two communication parties try to share a Bell state: the party that initially possesses the pair sends one half of it through the quantum channel $E$. $F_e$ is the fidelity between the actually shared pair and the original one; it means that a Bell measurement on $\rho^{RQ'}$ able to distinguish the ideally shared state from the other states of the Bell basis fails with error probability $P_e = 1 - F_e$. From (15) it follows that the error probability for a single channel use is $\frac{1 - g^2}{2}$; in what follows we identify this quantity by $\epsilon$.

Now we suppose to use the Three-Qubit Code (TQC) \cite{[1,2]} to send $\rho^Q$. The system’s state is encoded by using two ancillary systems $A$ and $B$. The system and the ancillary qubits are encoded by means of a set of quantum operations that we resume as $C^{QAB}$ (stages a, b, c in figure 1) and then transmitted in $N = 3$ uses of channel (1). Then the receiver performs the decoding $D^{QAB}$ (stages e, f, g, h in figure 1) on the system $QAB$. After tracing out $AB$, he obtains the final, generally mixed state of system $RQ$:

$$\rho^{RQ'}_{TQC} = \text{tr}_{AB}[I_R \otimes D^{QAB} \circ E^{QAB} \circ C^{QAB}(|\psi^{RQAB}\rangle\langle\psi^{RQAB}|)]$$

where $|\psi^{RQAB}\rangle = |\psi^R\rangle \otimes |00^{AB}\rangle$. Entanglement fidelity $F_e^{(TQC)} = \langle\psi^R \mid \rho^{RQ}_{TQC} \mid \psi^R \rangle$ just gives the probability that the code is successful. The merit of this code is that it drastically reduces - in absence of use correlations, i.e. for $E^{QAB} = E_1^{\otimes 3}$ - the transmission error probability from $\epsilon$ to $P_e^{(TQC)} = 1 - F_e^{(TQC)} \simeq 3\epsilon^2$.

![Figure 1. Scheme of a three qubit code \cite{[13}](image)

Now we investigate the effects of channel correlations on the performance of a TQC. After some involved calculations it comes out that

$$F_e^{(TQC,m)} = \frac{1}{2} + \frac{3}{4}g - \frac{11}{16}g^3[2\mu_{QA} - 2\mu_{QB} - 2\mu_{AB} + g^{2\mu_{QA} + 2\mu_{QB} - 2\mu_{AB}} + g^{2\mu_{QA} - 2\mu_{QB} - 2\mu_{AB}} + g^{2\mu_{QA} + 2\mu_{QB} + 2\mu_{AB}}].$$

By observing that $\mu_{QA} = \mu_{AB} = \mu_1$ and $\mu_{QB} = \mu_2$ we can rewrite equation (17) as:

$$F_e^{(TQC,m)} = \frac{1}{2} + \frac{3}{4}g - \frac{11}{16}g^3[2g^{2\mu_2} + g^{2\mu_1} + g^{2\mu_2 + 2\mu_1}].$$
An interesting result can be obtained by considering the case of a small error probability $\epsilon \ll 1$. In this regime we can take the series expansion of (18) near $\epsilon = 0$:

$$F_e(TQC,m) \approx 1 - (3 + 4\mu_1^2 + 2\mu_2^2)\epsilon^2.$$  

(19)

This expression tells us that even though memory lowers the fidelity, this worsening is always slight and absolutely negligible when $\mu_1, \mu_2 \ll 1$. Moreover, it highlights that channel correlations - inside the Hamiltonian model (1) - permit the TQC to maintain its error probability $P_e(TQC,m) = 1 - F_e(TQC,m)$ of the order of $\epsilon^2$. However, one has to take care that in the case perfect memory the code error triplicates from $3\epsilon^2$ to $9\epsilon^2$.

These results are very similar to the ones by Clemens et al. [16]. However, we discuss time rather than space correlations and average with respect to stochastic processes.

Rather than choosing a particular autocorrelation function $C(\tau)$ for $\xi(t)$ and then trying to carry out a specific relation between it and the TQC error probability, we make some general considerations about the impact of correlations on the code error probability. For the sake of simplicity we assume $\mu_1$ and $\mu_2$ having positive values (we do not consider anti-correlation cases). While the range of $\mu_1$ is $[0, 1]$, we can argue that $\mu_2 \leq \mu_1$ since one expects that the phase correlation does not increase when increasing the channel uses distance; furthermore it can be proved that it must be $\mu_2 \geq \tilde{\mu}_2 \equiv 2\mu_1^2 - 1$. Studying the first derivatives of $P_e(TQC,m)$ as a function of $\mu_1$ and $\mu_2$ it turns out that $P_e(TQC,m)$ is monotonical with respect to $\mu_1$ (error grows with $\mu_1$), but not with respect to $\mu_2$. It indeed can displays a minimum at $\mu_{2,\text{opt}} \equiv -0.25\log_2[(g^{4\mu_1} + g^{-4\mu_1})/2]$, but its presence is substantially irrelevant, and one can reasonably say that the code error probability is also increasing with respect to $\mu_2$. Thus to characterize the $P_e(TQC,m)$ behaviour, we plot it as a function $\mu_1$, using $\mu_2 \in \{\max(0, \tilde{\mu}_2), \mu_1\}$ as parameter. As it is showed in figure 2, in which we set $\epsilon = 10^{-3}$, the TQC error probability weakly depends on $\mu_1$, and the $\mu_2$ allowable values affect very little it. In the same figure we also plot the error probability for a two-qubit code [22] encoding a qubit into the subspace spanned by $\{|01\rangle, |10\rangle\}$: the performance of this last code is always worse than the TQC ones, unless in the case $\mu_1 \rightarrow 1$, for which the coding subspace becomes decoherence-free.

The TQC exhibits the same kind of behaviour showed in figure 2 as $\epsilon$ changes. In figure 3 we plot $P_e(TQC,m)$ as a function of $\epsilon$ for $\mu_2 = \mu_1 = 1$, the case in which the code shows the worst performance; we do not plot the cases of low correlations ($\mu_1 \leq 0.1$) since the correspondent curves are practically indistinguishable from the memoryless ones. We also compare $P_e(TQC,m)$ with the error probability of the two-qubit code [22]: to produce good results this last one requires very high degrees of correlation between successive channel uses.

In conclusion we find that a Hamiltonian formulation of a memory dephasing channel shows that the three qubit code is robust against channel correlations.

§ One can see it by considering the average of $[\phi_2 + a(\phi_1 + \phi_3)]^2$ where $a$ is a real variable: it is a quadratic form in $a$ and by imposing that it must always be positive we obtain the desired condition on $\mu_2$ [19].
Figure 2. Plot of code error probability as a function of $\mu_1$. The dotted and the dashed grey lines represent respectively the single channel use error probability ($\epsilon = 10^{-3}$) and TQC error probability for the memoryless channel ($P_{e}^{(TQC)}$). The error probabilities for the three qubit code in presence of correlations ($P_{e}^{(TQC,m)}$) are represented by black curves: solid curve refers to $\mu_2 = \max(0, \bar{\mu}_2)$ and the dotted one to $\mu_2 = \mu_1$. There is also displayed (solid gray curve) the error probability for a simple two-qubit code encoding a qubit into the subspace spanned by $\{|01\rangle, |10\rangle\}$.

Significantly different results emerge if we describe channel correlations inside a Markovian model [17]: In this latter case memory restores the $\epsilon$--dependence in the code error probability, thus drastically reducing the code performance.

Figure 3. Plot of code error probability as a function of the single channel use error probability $\epsilon$. The dashed grey line represents the TQC error probability in the memoryless case ($P_{e}^{(TQC)}$). For the error probabilities of the three qubit code in presence of correlations ($P_{e}^{(TQC,m)}$) we plot the worst case ($\mu_2 = \mu_1 = 1$) by a dotted black curve. There is also displayed (solid gray curve) the error probability for a simple two-qubit code encoding a qubit into the subspace spanned by $\{|01\rangle, |10\rangle\}$ for $\mu_1 = 0.99$ (triangles down).
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