SOBOLEV SPACES AND TRACE THEOREM ON THE SIERPINSKI GASKET

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Abstract. On the Sierpinski gasket $SG$, we consider Sobolev spaces $L^2_\sigma(SG)$ associated with the standard Laplacian $\Delta$ with order $\sigma \geq 0$. When $\sigma \in \mathbb{Z}^+$, $L^2_\sigma(SG)$ consists of functions equipped with $L^2$ norms of the function itself and its Laplacians up to $\sigma$ order; when $\sigma \in \mathbb{Z}^+$, we fill up the gaps between integer orders by using complex interpolation. Let $L^2_{\alpha,D}(SG) = (I - \Delta_D)^{-\alpha}L^2(SG)$ where $\Delta_D$ is the Dirichlet Laplacian associated with $\Delta$. Let $\{p_n\}_{n \geq 0}$ be a collection of countably many points located along one of the symmetrical axes of $SG$. We make a full characterization of the trace spaces of $L^2_\sigma(SG)$ and $L^2_{\alpha,D}(SG)$ to $\{p_n\}_{n \geq 0}$. Using this, we get a full description of the relationship between $L^2_\sigma(SG)$ and $L^2_{\alpha,D}(SG)$ to $\{p_n\}_{n \geq 0}$. Using this, we get a full description of the relationship between $L^2_\sigma(SG)$ and $L^2_{\alpha,D}(SG)$ for $\sigma \geq 0$. The result indicates that when $\sigma - \frac{\log 3}{\log 5} \in \mathbb{Z}^+$, $L^2_{\alpha,D}(SG)$ is not closed in $L^2_\sigma(SG)$ and has an infinite codimension. Otherwise, $L^2_{\alpha,D}(SG)$ is closed in $L^2_\sigma(SG)$ with a finite codimension. Similar result holds for the Neumann case. Another consequence of the trace result is that the Sobolev spaces $L^2_\sigma(SG)$ are stable under complex interpolation for $\sigma \geq 0$ although they are defined by piecewise interpolation between integer orders.

1. Introduction

In classical case, Sobolev spaces play an important role in the study of functional analysis and partial differential equations. They are spaces of functions equipped with $L^p$ norms of the function itself and its weak derivatives up a to given order. There are various of ways to extend Sobolev spaces to fractional orders, among which, the complex interpolation method is a useful tool. As an important topic in the study of Sobolev spaces, the trace spaces onto subdomains have been extensively investigated from various viewpoints.

The goal of this paper is to understand analogous Sobolev spaces when the underline space is a fractal. We will consider certain trace problem, using which, we will uncover some interesting characterizations of Sobolev spaces on fractals that were never realized before.

We restrict attention to a prototype of the fractals, the Sierpinski gasket $SG$, which is generated by the iterated function system (i.f.s.) consisting of

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contractions \( F_i : x \rightarrow \frac{1}{2}(x + q_i), i = 0, 1, 2 \), such that

\[
SG = \bigcup_{i=0}^{2} F_iSG,
\]

where \( q_i \)'s are the vertices of an equilateral triangle in \( \mathbb{R}^2 \), see Figure 1. It is well-known that there is a well developed theory of Laplacians on \( SG \), which may be obtained indirectly by developing probabilistic processes analogous to Brownian motion, or directly by taking renormalized limits of graph Laplacians, see [6, 12, 10] and [1, 2, 9, 11, 13, 15, 22] for various extensions. Both the approaches are closely related to Dirichlet forms. The domain of Dirichlet forms and domain of associated Laplacians can be regarded as certain Sobolev type spaces, since the Laplacians are defined in a weak sense and the Dirichlet forms serve as \( \int \nabla u \cdot \nabla v dx \) in \( \mathbb{R}^n \) case. We can extend the definition of general Sobolev spaces of fractional orders in a routine way as in the classical case. For simplicity, let \( \mu \) be the normalized Hausdorff measure on \( SG \), i.e., \( \mu(SG) = 1 \) and \( \mu = \frac{1}{3} \sum_{i=0}^{2} \mu \circ F_i^{-1} \). We choose the Laplacian \( \Delta \) and the Dirichlet form \( (\mathcal{E}, \text{dom}\mathcal{E}) \) associated with \( \mu \) to be self-similar and to satisfy the \( D_3 \)-symmetry as \( SG \), call them the standard Laplacian and the standard Dirichlet form on \( SG \). Recall that in [21], Strichartz has made a fundamental description of Sobolev spaces \( L^p \sigma(SG) \) associated with \( \Delta \) with real order \( \sigma \geq 0 \) and \( 1 \leq p \leq \infty \), which opens the door of the study of this topic. Some Sobolev embedding theorems and some incomplete results on complex interpolation are established in [21].

**Definition 1.1.** For \( k \in \mathbb{Z}^+ \), define the Sobolev space \( L^2_k(SG) \) as

\[
L^2_k(SG) = \{ u \in L^2(SG) : \Delta^j u \in L^2(SG) \text{ for all } j \leq k \}
\]

with the norm of \( u \) given by

\[
\|u\|_{L^2_k(SG)} = \sum_{j=0}^{k} \|\Delta^j u\|_{L^2(SG)}.
\]

For \( 0 < \theta < 1, k \in \mathbb{Z}^+ \), define the Sobolev space \( L^2_{k+\theta}(SG) \) to be

\[
L^2_{k+\theta}(SG) = [L^2_k(SG), L^2_{k+1}(SG)]_{\theta},
\]
the complex interpolation space.

Analogously, we define \( L^2_{k,D}(SG) \) to be the Sobolev space by additionally requiring that each \( \Delta^j u \) satisfies the Dirichlet boundary condition for \( j < k \) in the above definition for \( k \in \mathbb{Z}^+ \) and using the complex interpolation for general \( \sigma \). Similarly we have \( L^2_{\sigma,N}(SG) \) for the Neumann case.

In [21], in terms of Laplacian eigenfunction expansions, Strichartz introduced an equivalent definition of \( L^2_{\sigma,D}(SG) \) and \( L^2_{\sigma,N}(SG) \) by virtue of Bessel or Riesz type potentials, and easily showed that they are stable under complex interpolation. In other words, for any \( 0 < \sigma_0 \leq \sigma_1 < \infty, 0 < \theta < 1 \), the complex interpolation space \([L^2_{\sigma_0,D}(SG), L^2_{\sigma_1,D}(SG)]_\theta \) identifies with \( L^2_{\sigma,D}(SG) \) with \( \sigma = (1-\theta)\sigma_0 + \theta\sigma_1 \), and similarly for the Neumann case.

Because of the boundary conditions, the above three types of spaces are not identical. However, when \( k \in \mathbb{Z}^+ \), they only differ by a finite dimensional space due to the finiteness of \#\( V_0 \). To be precise, for \( k \in \mathbb{Z}^+ \), let \( H_k \) denote the collection of \((k+1)\)-multiharmonic functions, the solutions of \( \Delta^{k+1} h = 0 \), which is of \( 3(k+1) \)-dimension. In particular, functions in \( H_0 \) are called harmonic functions. We write \( H_{-1} = \{0\} \) for the sake of uniformity. The multiharmonic functions are analogies of polynomials on the unit interval.

See [17], [19] and [20] for more details and related theory. We have the following proposition.

**Proposition 1.2.** Let \( k \in \mathbb{Z}^+ \). Then
(a) \( L^2_k(SG) = L^2_{k,D}(SG) \oplus H_{k-1} \);
(b) \( L^2_k(SG) = L^2_{k,N}(SG) \oplus H_{k-1}' \), with \( H_{-1}' = \{0\} \) and for \( k \geq 0 \), \( H_k' = \{ h \in H_{k+1} : \sum_{i=0}^2 h(q_i) = 0 \text{ and } \Delta^{(k+1)} h \equiv c \text{ for some constant } c \} \).  

**Proof.** (a) is simple.
(b) We only need to show that there is a unique \( h \in H_{k-1}' \) with any given Neumann boundary condition

\[
\partial_n \Delta^j h(q_i) = a_{i,j}, \forall 0 \leq i \leq 2, 0 \leq j < k,
\]

so that for any \( u \in L^2_k(SG) \), there exists a unique decomposition \( u = u_N + h \), with \( u_N \in L^2_{k,N}(SG) \) and \( h \in H_{k-1}' \). This can be shown by induction.

First, it is true for \( k = 1 \). We need to find a unique \( h \in H_0' \) such that \( \partial_n h(q_i) = a_{i,0} \) for any prescribed values \( a_{i,0} \)'s. For this purpose, we construct \( h = -Gc + h_0 \), where \( G \) is the Green’s operator, which satisfies that \( -\Delta (Gf) = f \) and \( (Gf)|_{V_0} = 0 \) for any \( f \in L^2(SG) \), \( c = \sum_{i=0}^2 a_{i,0} \) and \( h_0 \) is a properly chosen harmonic function, see [11] or [21] for discussions on Green’s operator. Note that \( \partial_n h(q_i) = \frac{1}{3} c + 2h_0(q_i) - h_0(q_{i-1}) - h_0(q_{i+1})(\text{cyclic notation } q_{i+3} = q_i) \). The uniqueness follows from counting the dimension.

Next, assume that the claim is true for \( k \), we prove it for \( k+1 \). We need to find a unique \( h \in H_k' \) such that \( \partial_n \Delta^j h(q_i) = a_{i,j} \) for any prescribed values \( a_{i,j} \), with \( 0 \leq i \leq 2, 0 \leq j < k + 1 \). Following the above argument, let \( \tilde{h} \in H_0' \) be a function such that \( \partial_n \tilde{h}(q_i) = a_{i,k}, i = 0, 1, 2 \). Then we construct
$h \in H_k'$ to be $h = (-G)^k \tilde{h} + h_0$ where $h_0$ is a properly chosen function in $H_{k-1}$. The uniqueness still follows from counting the dimension. □

Naturally, we have following questions.

**Question 1.** Whether the Sobolev spaces $L^2_\sigma(\mathcal{SG})$ are stable under complex interpolation?

**Question 2.** What is the relationship between $L^2_\sigma(\mathcal{SG})$ and $L^2_{\sigma,D}(\mathcal{SG})$ or $L^2_{\sigma,N}(\mathcal{SG})$ for real $\sigma \geq 0$?

We will answer these two questions in Section 6. The answer for Question 1 is Yes, while the answer for Question 2 is somewhat complicated which depends on the choice of $\sigma$. As we expect, we will then illustrate that $L^2_{1/2}(\mathcal{SG}) = \text{dom} \mathcal{E}$ with the norm of $u$ given by $\|u\|_{L^2_{1/2}(\mathcal{SG})} = \|u\|_{L^2(\mathcal{SG})} + \mathcal{E}^{1/2}(u)$.

![Figure 2. The middle line of $\mathcal{SG}$.](image)

Somewhat interesting, both the proofs of the above two questions rely on the exact description of the trace spaces of $L^2_\sigma(\mathcal{SG})$ onto the middle line of $\mathcal{SG}$, which was tentatively studied for $L^2_1(\mathcal{SG})$ in [14]. Let $p_0 = F_1 q_2$, $p_n = F_0^n p_0$ for $n \geq 1$, and write $X = \{p_n\}_{n \geq 0} \cup \{q_0\}$. Then $X$ consists of the countable many points in $\mathcal{SG}$ located on the line which passes through $q_0$ and $p_0$, see Figure 2. We will consider the trace spaces of $L^2_\sigma(\mathcal{SG})$ onto $X$. Since $q_0$ is the only accumulation point in $X$, it is reasonable to consider the trace space of $L^2_\sigma(\mathcal{SG})$ onto $\{p_n\}_{n \geq 0}$ instead. Of course, comparing with the trace theorem of Sobolev spaces in classical case, this trace problem has its own important interest. See [8] for a fascinating result by Jonsson, where the trace space of $\text{dom} \mathcal{E}$ onto the triangle that confines $\mathcal{SG}$ was characterized as a Besov space. Also, see [7] and [18] for other boundary values problems on subdomains of $\mathcal{SG}$.

**Definition 1.3.** For a function $u$ on $\mathcal{SG}$, define the restriction of $u$ onto $\{p_n\}_{n \geq 0}$ by $R u = (R_s u, R_a u)$, where

$$R_s u = \{u(p_n)\}_{n \geq 0} \quad \text{and} \quad R_a u = \{\partial_n^\gamma u(p_n)\}_{n \geq 0}.$$
Note that $R_su$ is valid for a continuous function $u$, and $R_au$ is valid for a function $u$ whose normal derivatives along $\{p_n\}_{n \geq 0}$ are well-defined. Throughout this paper, for a function $u$, for $n \geq 0$, we will write

$$\alpha_n = u(p_n) \text{ and } \eta_n = \partial_n^+ u(p_n),$$

and write

$$R_su = \alpha := \{\alpha_n\}_{n \geq 0}, R_au = \eta := \{\eta_n\}_{n \geq 0} \text{ and } Ru = (\alpha, \beta),$$

for simplicity.

**Question 3.** What are the exact trace spaces $RL^2_\sigma(SG)$ for real $\sigma > 0$?

Note that to ensure the restriction map $R$ to be well-defined, we would require that $\sigma$ not to be too small. We will make a full description of $R_sL^2_\sigma(SG)$ when $\sigma > \frac{\log 3}{2\log 5}$, and $R_aL^2_\sigma(SG)$ when $\sigma > 1 - \frac{\log 3}{2\log 5}$, and point out that $\frac{\log 3}{2\log 5}$ and $1 - \frac{\log 3}{2\log 5}$ are both critical orders. We remark that when $\sigma = \frac{1}{2}$ or 1, the trace space of $R_sL^2_{1/2}(SG) = R_s\text{dom}E$ and $RL^2_1(SG)$ was computed by Li and Strichartz in [14], which can be regarded as a starting point of the answer of Question 3.

All the above mentioned Sobolev type spaces can be extended to the $L^p$ setting. In our consideration, we focus on $L^2$ setting for simplicity, but most of the results are obviously valid for general $L^p$ setting.

The organization of this paper is as follows. In Section 2 we give necessary preliminaries, most of which were introduced in [11, 22]. From Section 3 to Section 5, we focus on answering Question 3. Firstly, in Section 3, we collect some useful facts about the multiharmonic functions on $SG$, and describe certain spaces of sequences of real numbers that will be used later. Secondly, in Section 4, we solve the trace problem for $L^2_\sigma(SG)$ with $\sigma \in \mathbb{N}$. Thirdly, in Section 5, first we write the trace spaces in a consistent manner for all $\sigma \in \mathbb{N}$, then by using interpolation, extend the trace result to real $\sigma \geq \frac{1}{2}$ for $R_sL^2_\sigma(SG)$, and $\sigma \geq 1$ for $R_aL^2_\sigma(SG)$. Next by using the atomic decomposition of $L^2_\sigma(SG)$ developed in [4] by the authors, we extend the trace result to low order case. In Section 6, we turn back to answer Question 1 and Question 2 as applications of the trace theorems. Finally, in Section 7, we extend the standard Laplacian to a two-parameter family of non-uniform $D_3$-symmetric self-similar Laplacians on $SG$, which are naturally obtained by regarding $SG$ as an invariant set of a 9-map i.f.s. by iterating $\{F_i\}_{i=0}^2$ twice. These Laplacians were introduced in [4] which contains a one-parameter subfamily that admit spectral decimation. We will briefly discuss how the trace theorem looks in this general setting.
2. Preliminaries

First, we introduce some necessary notations. Standard references are the books [11] of Kigami and [22] of Strichartz. Write $W_0 = \{ \emptyset \}$, and for $m \geq 1$, $W_m = \{0, 1, 2\}^m$. For $w = w_1 w_2 \cdots w_m$, we denote $F_w = F_{w_1} \circ F_{w_2} \cdots \circ F_{w_m}$ and call $F_w \mathcal{S}G$ a $m$-cell of $\mathcal{S}G$. Call $V_0 = \{q_0, q_1, q_2\}$ the boundary of $\mathcal{S}G$, and let $V_m = \bigcup_{w \in W_m} F_w V_0$, the vertices of level $m$. Denote $V_* = \bigcup_{m \geq 0} V_m$. Note that $V_*$ is dense in $\mathcal{S}G$. Let $\mu$ be the normalized Hausdorff measure on $\mathcal{S}G$, i.e., $\mu(\mathcal{S}G) = 1$ and $\mu = \frac{1}{3} \sum_{i=0}^2 \mu \circ F_i^{-1}$. For functions defined on $\mathcal{S}G$, we define a resistance form $(\mathcal{E}, \text{dom}\mathcal{E})$ by

$$\mathcal{E}(u) = \lim_{m \to \infty} \frac{5}{3} \left( \sum_{w \in W_m} (u(F_w q_i) - u(F_w q_j))^2 \right)$$

and $\text{dom}\mathcal{E} = \{u : \mathcal{E}(u) < \infty\}$. Using the polarization identity

$$\mathcal{E}(u, v) = \frac{1}{4} (\mathcal{E}(u + v) - \mathcal{E}(u - v)),$$

the form can be extended to a bilinear form. The self-similar property of $(\mathcal{E}, \text{dom}\mathcal{E})$ gives that

$$\mathcal{E}(u, v) = \left( \frac{5}{3} \right)^m \mathcal{E}(u \circ F_w, v \circ F_w) \quad \forall u, v \in \text{dom}\mathcal{E}.$$

It is direct to verify that $\text{dom}\mathcal{E}$ is dense both in $C(\mathcal{S}G)$ and in $L^2(\mathcal{S}G)$, and thus $(\mathcal{E}, \text{dom}\mathcal{E})$ turns out to be a Dirichlet form on $\mathcal{S}G$.

Then we define the standard Laplacian $\Delta$ using the weak formulation. Say $u \in \text{dom}\Delta$ and $\Delta u = f$ if $u \in \text{dom}\mathcal{E}$, $f$ is continuous, and

$$\mathcal{E}(u, v) = -\int_{\mathcal{S}G} f v d\mu, \quad \forall v \in \text{dom}_0\mathcal{E}$$

(2.1)

where $\text{dom}_0\mathcal{E} = \{u \in \text{dom}\mathcal{E} : u|_{V_0} = 0\}$. The Laplacian $\Delta$ satisfies the scaling property that

$$\Delta(u \circ F_w) = \frac{1}{5^m} (\Delta u) \circ F_w$$

for any $u \in \text{dom}\Delta$ and $w \in W_m$. We will have similar definition of $\text{dom}_{L^p}\Delta$ when we specify that $\Delta u$ is a $L^p$ function.

The normal derivative of a function $u$ at a boundary point $q_i$ is defined to be

$$\partial_n u(q_i) = \lim_{m \to \infty} \left( \frac{5}{3} \right)^m (2 u(q_i) - u(F_{q_i}^m(q_i - 1)) - u(F_{q_i}^m(q_i + 1)))$$

(cyclic notation $q_{i+3} = q_i$) providing it exists. The normal derivative may be localized to any vertex in $V_*$. For $x = F_w q_i$ with $w \in W_m$, $i = 0, 1, 2$, write $\partial_n^u u(x)$ the normal derivative of $u$ at $x$ within $F_w \mathcal{S}G$ to be $\left( \frac{5}{3} \right)^m \partial_n^u (u \circ F_w)(q_i)$. We will use the notations $\partial_n^u$, $\partial_n^\nu$ and $\partial_n^\rho$ to represent normal derivatives of certain directions. Note that, for functions in $\text{dom}\Delta$, the normal derivatives exists at any vertex in $V_*$ and sum up to 0 at each vertex in $V_* \setminus V_0$. This
is called the matching condition. Using the normal derivatives, the formula (2.1) extends to the Gauss-Green’s formula,

\[ \mathcal{E}(u, v) = -\int_{SG} (\Delta u) v \, d\mu + \sum_{i=0}^{2} \partial_n u(q_i) v(q_i) \]

for any \( u \in \text{dom} \Delta \) and \( v \in \text{dom} \mathcal{E} \).

For \( k \in \mathbb{Z}^+ \), the following proposition provides an equivalent norm of the space \( L^2_k(SG) \).

**Proposition 2.1.** For \( k \in \mathbb{Z}^+ \), \( \|u\|_{L^2_k(SG)} \asymp \|u\|_{L^2(SG)} + \|\Delta^k u\|_{L^2(SG)} \) for any function \( u \in L^2_k(SG) \).

**Proof.** Let \( G \) be the Green’s operator. For any \( u \in L^2_k(SG) \), we write

\[ u = (-G)^k \Delta^k u + (u - (-G)^k \Delta^k u) \]

Then for \( j \leq k \), we have

\[ \|\Delta^j ((-G)^k \Delta^k u)\|_{L^2(SG)} = \|(-G)^{k-j} \Delta^k u\|_{L^2(SG)} \lesssim \|\Delta^k u\|_{L^2(SG)} \]

This gives that \( \|(-G)^k \Delta^k u\|_{L^2_k(SG)} \lesssim \|\Delta^k u\|_{L^2(SG)} \). On the other hand,

\[ \|u - (-G)^k \Delta^k u\|_{L^2_k(SG)} \asymp \|u - (-G)^k \Delta^k u\|_{L^2_k(SG)} \]

\[ \leq \|u\|_{L^2(SG)} + \|(-G)^k \Delta^k u\|_{L^2(SG)} \]

\[ \lesssim \|u\|_{L^2(SG)} + \|\Delta^k u\|_{L^2(SG)} \]

where the first estimate comes from the fact that \( u - (-G)^k \Delta^k u \) is a \( k \)-multiharmonic function. Combining the above estimates, we get the proposition. \( \square \)

Throughout the following context, we call a function \( u \) on \( SG \) symmetric or antisymmetric if it is symmetric or antisymmetric with respect to the reflection that fixes \( q_0 \) and interchanges \( q_1 \) and \( q_2 \). We will always use the notation \( f \lesssim g \) if there is a constant \( C > 0 \) such that \( f \leq Cg \), and write \( f \asymp g \) if \( f \lesssim g \) and \( g \lesssim f \).

### 3. Lemmas

In this section, we introduce some important notations and tools to study the trace problem. Throughout this section, we will always write \( \alpha = \{\alpha_n\}_{n \geq 0} \) and \( \eta = \{\eta_n\}_{n \geq 0} \) to represent sequences of real numbers.

We start from the restriction images of multiharmonic functions onto \( \{p_n\}_{n \geq 0} \). Recall that in [17], to develop a local theory of functions at a single boundary point, for \( k \in \mathbb{N} \), a basis of \( \mathcal{H}_{k-1} \) analogous to the monomials \( x^j / j! \) on the unit interval, was described and investigated on \( SG \).
Definition 3.1. For \( i = 0, 1, 2 \) and \( j \geq 0 \), we define \( h_{i,j} \) to be a multiharmonic function on \( SG \) satisfying

\[
\begin{align*}
\Delta^l h_{i,j}(q_0) &= \delta_0 \delta_{lj}, \\
\partial_n \Delta^l h_{i,j}(q_0) &= \delta_{i1} \delta_{lj}, \\
\partial_T \Delta^l h_{i,j}(q_0) &= \delta_{22} \delta_{lj},
\end{align*}
\]

and call it a monomial on \( SG \).

Note that for \( k \in \mathbb{N} \), these \( h_{i,j} \)'s with \( j < k \) form a basis of \( \mathcal{H}_{k-1} \) with dimension \( 3k \), related by the identity \( \Delta h_{i,j} = h_{i,j-1} \). It is easy to check that \( \{h_{0,j}\} \) and \( \{h_{1,j}\} \) are symmetric, while \( \{h_{2,j}\} \) are antisymmetric. Moreover, for \( j \geq 0 \), we have

\[
h_{0,j} \circ F_0 = 5^{-j} h_{0,j}, \quad h_{1,j} \circ F_0 = 3 \cdot 5^{-j-1} h_{1,j}, \quad h_{2,j} \circ F_0 = 5^{-j-1} h_{2,j}.
\]

Thus it is easy to see that \( R_s h_{i,j} \) and \( R_\alpha h_{i,j} \) are geometric sequences.

Lemma 3.2. Let \( k \in \mathbb{N} \). We have at most \( 2k \) positive ratios \( \{\kappa_i\}_i \) taking values in \( \{5^{-j}, 3 \cdot 5^{-j-1}\}_{j=0}^{k-1} \), and at most \( k \) ratios \( \{\iota_i\}_i \) taking values in \( \{3^{-1} 5^{-j}\}_{j=0}^{k-1} \), such that

(a). \( R_s \mathcal{H}_{k-1} = \{ \alpha : \alpha = \sum_i A_i \kappa_i^n, \forall n \geq 0 \text{ with } A_i \in \mathbb{R} \} \);

(b). \( R_\alpha \mathcal{H}_{k-1} = \{ \eta : \eta = \sum_i B_i \iota_i^n, \forall n \geq 0 \text{ with } B_i \in \mathbb{R} \} \);

(c). \( R \mathcal{H}_{k-1} = \{ (\alpha, \beta) : \alpha \in R_s \mathcal{H}_{k-1}, \beta \in R_\alpha \mathcal{H}_{k-1} \} \).

Proof. (a). Since \( \mathcal{H}_{k-1} = \text{span}\{h_{i,j} : 0 \leq i \leq 2, 0 \leq j < k \} \), we have

\[ R_s \mathcal{H}_{k-1} = \text{span}\{R_s h_{i,j} : 0 \leq i \leq 2, 0 \leq j < k \}, \]

noticing that \( h_{2,j} \)'s are antisymmetric. So (a) follows immediately. The proof for (b),(c) are the same. \( \square \)

For \( k \in \mathbb{N} \), we denote \( s(k) \) and \( a(k) \) the cardinalities of \( \{\kappa_i\}_i \) in \( \{5^{-j}, 3 \cdot 5^{-j-1}\}_{j=0}^{k-1} \) and \( \{\iota_i\}_i \) in \( \{3^{-1} 5^{-j}\}_{j=0}^{k-1} \) in Lemma 3.2 respectively. We arrange \( \{\kappa_i\}_{i=0}^{s(k)-1} \) and \( \{\iota_i\}_{i=0}^{a(k)-1} \) in decreasing order. Obviously, the values of \( \kappa_i \)'s and \( \iota_i \)'s are independent of \( k \).

Remark. We conjecture that \( R h_{i,j} \neq 0 \) for all monomials in \( \{h_{i,j}\} \), in which case we have \( s(k) = 2k, a(k) = k, \{\kappa_i\}_{i=0}^{s(k)-1} = \{5^{-j}, 3 \cdot 5^{-j-1}\}_{j=0}^{k-1} \), and \( \{\iota_i\}_{i=0}^{a(k)-1} = \{3^{-1} 5^{-j}\}_{j=0}^{k-1} \). It is clear that \( \kappa_0 = 1, \kappa_1 = \frac{3}{2}, \iota_0 = \frac{1}{3} \).

Definition 3.3. Call the above defined \( \kappa_i \)'s and \( \iota_i \)'s jump points.

The collection of jump points will play a key role in the description of the trace spaces \( RL^2_k(SG) \).

The following lemma will be useful when extend the trace of multiharmonic functions to general functions in \( L^2_k(SG) \) with \( k \in \mathbb{N} \).

Lemma 3.4. Let \( k \geq 1, l \geq 0 \). Let \( \{x_i\}_{i=0}^l \) be \( l+1 \) distinct points in \( V_s \setminus V_0 \), and \( \{a_i\}_{i=0}^l \) be \( l+1 \) real numbers.
(a) Assume \( \sum_{i=0}^{l} a_i h(x_i) = 0 \) holds for any \( h \in \mathcal{H}_{k-1} \). Then there is a function \( \phi_k \in C(\mathcal{S}\mathcal{G}) \) such that
\[
\sum_{i=0}^{l} a_i u(x_i) = \int_{\mathcal{S}\mathcal{G}} \phi_k(x) \Delta^k u(x) d\mu(x),
\]
for any \( u \in L^2_{k}(\mathcal{S}\mathcal{G}) \).

(b) Assume \( \sum_{i=0}^{l} a_i \partial_n h(x_i) = 0 \) holds for any \( h \in \mathcal{H}_{k-1} \), where each \( \partial_n h(x_i) \) means a fixed choice of normal derivative of \( h \) at \( x_i \). Then there is a function \( \tilde{\phi}_k \in L^\infty(\mathcal{S}\mathcal{G}) \) such that
\[
\sum_{i=0}^{l} a_i \partial_n u(x_i) = \int_{\mathcal{S}\mathcal{G}} \tilde{\phi}_k(x) \Delta^k u(x) d\mu(x),
\]
for any \( u \in L^2_{k}(\mathcal{S}\mathcal{G}) \).

Proof. (a). First, there is a unique function \( \phi^{(k-1)} \) \( \in \text{dom}_0 \mathcal{E} \) such that
\[
\mathcal{E}(u, \phi^{(k-1)}) = -\sum_{i=0}^{l} a_i u(x_i),
\]
holds for any function \( u \in \text{dom}_0 \mathcal{E} \), since the right side is a bounded linear functional on \( \text{dom}_0 \mathcal{E} \).

Next, we inductively define a sequence of functions \( \phi^{(k-j)} \) \( j = 2, \ldots, k \) by \( \phi^{(k-j)} = (-G)\phi^{(k-j+1)} \), where \( G \) is the Green’s operator. Take \( \phi_k = \phi^{(0)} \). Then for any \( u \in L^2_{k,D}(\mathcal{S}\mathcal{G}) \), by repeatedly using the Gauss-Green’s formula, we have
\[
\sum_{i=0}^{l} a_i u(x_i) = -\mathcal{E}(u, \phi^{(k-1)}) = \int_{\mathcal{S}\mathcal{G}} \phi^{(k-1)}(x) \Delta u(x) d\mu(x)
\]
\[
= \int_{\mathcal{S}\mathcal{G}} \phi^{(k-2)}(x) \Delta^2 u(x) d\mu(x) = \cdots = \int_{\mathcal{S}\mathcal{G}} \phi_k(x) \Delta^k u(x) d\mu(x),
\]
where we use the Dirichlet boundary condition of \( \phi^{(k-j)} \) and \( \Delta^j u \) in each step.

Noticing that the equality
\[
\sum_{i=0}^{l} a_i h(x_i) = \int_{\mathcal{S}\mathcal{G}} \phi_k(x) \Delta^k h(x) d\mu(x) = 0
\]
also holds for any \( h \in \mathcal{H}_{k-1} \), we get (a) proved.

(b). For each \( x_i \), we can find a word \( w_i \in W_{*} \) such that \( x_i \) is a boundary vertex of \( F_{w_i,\mathcal{S}\mathcal{G}} \), and for any function \( u \), we have
\[
\partial_n u(x_i) = \left( \frac{5}{3} \right)^{|w_i|} (2u(x_i) - u(y_{i,1}) - u(y_{i,2})) + \int_{F_{w_i,\mathcal{S}\mathcal{G}}} h_i(x) \Delta u(x) d\mu(x),
\]
where $y_{i,1}, y_{i,2}$ are the two other boundary vertices of $F_w \mathcal{SG}$, and $h_i$ is a function harmonic and supported in $F_w \mathcal{SG}$, with $h_i(y_{i,1}) = h_i(y_{i,2}) = 0$ and $h_i(x_i) = 1$.

Thus we can rewrite $\sum_{i=0}^{l} a_i \partial_n u(x_i)$ to be

$$\sum_{i=0}^{l} a_i \partial_n u(x_i) = \sum_{i=0}^{l} a_i(\frac{5}{3}|u_i|)(2u(x_i) - u(y_{i,1}) - u(y_{i,2})) + \int_{\mathcal{SG}} \tilde{\phi}(x) \Delta u(x) d\mu(x),$$

(3.2)

with $\tilde{\phi} = \sum_{i=0}^{l} a_i h_i$.

By the assumption, we see that

$$\sum_{i=0}^{l} a_i \partial_n h(x_i) = \sum_{i=0}^{l} a_i(\frac{5}{3}|u_i|)(2h(x_i) - h(y_{i,1}) - h(y_{i,1})) = 0$$

holds for any $h \in \mathcal{H}_0$. So by using part (a), there is a function $\bar{\phi} \in C(\mathcal{SG})$ such that

$$\sum_{i=0}^{l} a_i(\frac{5}{3}|u_i|)(2u(x_i) - u(y_{i,1}) - u(y_{i,1})) = \int_{\mathcal{SG}} \bar{\phi}(x) \Delta u(x) d\mu(x)$$

holds for any $u \in L^2_1(\mathcal{SG})$. Combining the above equality with (3.2), we get that

$$\sum_{i=0}^{l} a_i \partial_n u(x_i) = \int_{\mathcal{SG}} \tilde{\phi}(k-1)(x) \Delta u(x) d\mu(x)$$

holds for any $u \in L^2_1(\mathcal{SG})$ with $\tilde{\phi}(k-1) = \tilde{\phi} + \bar{\phi}$.

The remaining proof is the same as that of part (a). The desired function $\tilde{\phi}_k = (-G)^{k-1}\tilde{\phi}(k-1)$.

Inspired by Lemma 3.4, below we will define a class of difference operators $A_l$'s and $\Theta_l$'s on sequences of real numbers, which are related to the jump points $\kappa_i$'s and $\iota_i$'s.

For $l \geq 1$, we write functions $\xi_l(x)$ and $\zeta_l(x)$ as

$$\begin{cases}
\xi_l(x) = \Pi_{i=0}^{l-1}(x - \kappa_i), \\
\zeta_l(x) = \Pi_{i=0}^{l-1}(x - \iota_i).
\end{cases}$$

Expanding the products, we write

$$\begin{cases}
\xi_l(x) = \sum_{i=0}^{l} a_{l,i} x^i, \\
\zeta_l(x) = \Pi_{i=0}^{l} b_{l,i} x^i.
\end{cases}$$

It is easy to check that $a_{l,0} = (-1)^l \prod_{i=0}^{l-1} \kappa_i$, $a_{l,l} = 1$, and $a_{l,i} = a_{l-1,i-1} - \kappa_{l-1} a_{l-1,i}, \forall 0 < i < l$,

for any $l \geq 1$, and similarly for $b_{l,i}$ with $\kappa_i$ replaced by $\iota_i$. 
Given sequences $\alpha = \{\alpha_n\}_{n \geq 0}$ and $\eta = \{\eta_n\}_{n \geq 0}$, we define two new sequences $A_l(\alpha)$ and $\Theta_l(\eta)$ by

\[
\begin{aligned}
A_l(\alpha) &= \{\sum_{i=0}^l a_i \alpha_{n+i}\}_{n \geq 0}, \\
\Theta_l(\eta) &= \{\sum_{i=0}^l b_i \eta_{n+i}\}_{n \geq 0}.
\end{aligned}
\]

Then clearly,

\[
\begin{aligned}
A_l(\alpha) &= \{(A_{l-1}(\alpha))_{n+1} - \kappa_{l-1} (A_{l-1}(\alpha))_n\}_{n \geq 0}, \\
\Theta_l(\eta) &= \{((\Theta_{l-1}(\eta))_{n+1} - \iota_{l-1} (\Theta_{l-1}(\eta))_n\}_{n \geq 0}.
\end{aligned}
\]

For the sake of consistency, we write $A_0(\alpha) = \alpha$ and $\Theta_0(\eta) = \eta$.

Using the difference operators $A_l$ and $\Theta_l$ defined above, we restate Lemma 3.2 in the following way.

**Lemma 3.2*. For $k \in \mathbb{N}$, we have $R_s H_{k-1} = \ker A_{s(k)}$ and $R_a H_{k-1} = \ker \Theta_{a(k)}$.**

As a consequence of Lemma 3.2* and Lemma 3.4, we have the following lemma.

**Lemma 3.5. For $k \in \mathbb{N}$, $u \in L^2_k(\mathcal{SG})$ with $Ru = (\alpha, \eta)$. We have**

(a). there exists a symmetric function $\varphi_{s,k} \in C(\mathcal{SG})$ independent of $u$ such that

\[
(A_{s(k)}(\alpha))_n = 3^n \cdot 5^{-kn} \int_{F_0^k \mathcal{SG}} \varphi_{s,k} \circ F_0^{-n}(x) \Delta^k u(x) d\mu(x);
\]

(b). there exists an antisymmetric function $\varphi_{a,k} \in L^\infty(\mathcal{SG})$ independent of $u$ such that

\[
(\Theta_{a(k)}(\eta))_n = 5^{-(k-1)n} \int_{F_0^k \mathcal{SG}} \varphi_{a,k} \circ F_0^{-n}(x) \Delta^k u(x) d\mu(x).
\]

**Proof.** The case $n = 0$ is a direct consequence of Lemma 3.2* and Lemma 3.4. For $n \geq 1$, we only need to use scaling. For the symmetricity of $\varphi_{s,k}$, we only need to notice that $R_s u = \{0\}_{n \geq 0}$ for any antisymmetric $u \in L^2_k(\mathcal{SG})$. The case for $\varphi_{a,k}$ is similar. \(\square\)

Next, let’s look at some estimates of $l^2$ norms of sequences related with $A_l$ and $\Theta_l$. In the following, Lemma 3.6, 3.7 and 3.8 deal with three different cases.

**Lemma 3.6. Let $l \geq 1$, $t > 0$, $\alpha, \eta$ be two sequences.**

(a). If $t < \kappa_{l-1}^{-1}$, then

\[
\|t^n (A_{l-1}(\alpha))_n\|_{l^2} \times \|t^n (A_l(\alpha))_n\|_{l^2} + \|(A_{l-1}(\alpha))_0\|_1.
\]

(b). If $t < \iota_{l-1}^{-1}$, then

\[
\|t^n (\Theta_{l-1}(\eta))_n\|_{l^2} \times \|t^n (\Theta_l(\eta))_n\|_{l^2} + \|(\Theta_{l-1}(\eta))_0\|.
\]
Proof. (a) Clearly, by (3.3), we have
\[
\|t^n(A_l(\alpha))_n\|_{l^2} \lesssim \|t^n(A_{l-1}(\alpha))_n\|_{l^2},
\]
which gives one direction of the estimate. Next, by using (3.3), we have
\[
\kappa_{l-1}^{-n}(A_{l-1}(\alpha))_n = (A_{l-1}(\alpha))_0 + \sum_{i=0}^{n-1} \kappa_{l-1}^{-i-1}(A_l(\alpha))_i.
\]
Let \(\tilde{t} = t\kappa_{l-1} < 1\). Then by using Minkowski inequality, we have
\[
\|t^n(A_{l-1}(\alpha))_n\|_{l^2} = \|\tilde{t}^n(A_{l-1}(\alpha))_0 + \tilde{t}^n \sum_{i=0}^{n-1} \kappa_{l-1}^{-i-1}(A_l(\alpha))_i\|_{l^2},
\]
\[
\lesssim |(A_{l-1}(\alpha))_0| + \|\sum_{i=1}^{n} \tilde{t}^{n-i} \kappa_{l-1}^{-i-1}(A_l(\alpha))_{n-i}\|_{l^2}
\]
\[
\lesssim |(A_{l-1}(\alpha))_0| + \sum_{i=1}^{\infty} \tilde{t}^{\kappa_{l-1}^{-1}}\|t^n(A_l(\alpha))_n\|_{l^2},
\]
which gives the other direction.

The proof of part (b) is the same. \(\square\)

Lemma 3.7. Let \(l \geq 1, t > 0, \alpha, \eta\) be two sequences.

(a). If \(t > \kappa_{l-1}^{-1}\) and \(\|t^n(A_l(\alpha))_n\|_{l^2} < \infty\), we have
\[
\alpha_n = \sum_{i=0}^{l-1} A_i \kappa_i^n + \alpha'_n,
\]
with \(A_i \in \mathbb{R}\), satisfying that
\[
\sum_{i=0}^{l-1} |A_i| + \|t^n\alpha'_n\|_{l^2} \asymp \|t^n(A_l(\alpha))_n\|_{l^2} + \sum_{i=0}^{l-1} |\alpha_i|.
\]

(b). If \(t > \iota_{l-1}^{-1}\) and \(\|t^n(\Theta_l(\eta))_n\|_{l^2} < \infty\), we have
\[
\eta_n = \sum_{i=0}^{l-1} B_i \iota_i^n + \eta'_n,
\]
with \(B_i \in \mathbb{R}\), satisfying that
\[
\sum_{i=0}^{l-1} |B_i| + \|t^n\eta'_n\|_{l^2} \asymp \|t^n(\Theta_l(\eta))_n\|_{l^2} + \sum_{i=0}^{l-1} |\eta_i|.
\]

Proof. (a). We prove the lemma by induction.
If \( l = 1 \), we have
\[
\alpha_n = \kappa_0 \alpha_{n-1} + (A_1(\alpha))_{n-1} = \kappa_0^2 \alpha_{n-2} + (A_1(\alpha))_{n-1} + \kappa_0 (A_1(\alpha))_{n-2} = \cdots = \kappa_0^n \alpha_0 + \sum_{i=0}^{n-1} \kappa_0^{n-i-1} (A_1(\alpha))_i,
\]
and thus
\[
kappa_0^{-n} \alpha_n = \alpha_0 + \sum_{i=0}^{n-1} \kappa_0^{-i-1} (A_1(\alpha))_i.
\]

Since \( t > \kappa_0^{-1} \) by the assumption of the lemma, the series \( \sum_{i=0}^{\infty} \kappa_0^{-i-1} (A_1(\alpha))_i \) converges, and so \( \kappa_0^{-n} \alpha_n \) converges as \( n \to \infty \). Let \( A_0 = \lim_{n \to \infty} \kappa_0^{-n} \alpha_n \) and \( \alpha_n = \alpha_n - A_0 \kappa_0^n \). We have
\[
\|t^n \alpha'_n\|_2 = \|t^n \sum_{i=0}^{\infty} \kappa_0^{-i-1} (A_1(\alpha))_i\|_2 = \|t^n \sum_{i=0}^{\infty} \kappa_0^{-i-1} (A_1(\alpha))_{n+i}\|_2 \leq \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} (t^{-1} \kappa_0^{-1})^j \|t^n (A_1(\alpha))_n\|_2,
\]
where we use Minkowski inequality in the last inequality. This shows that \( |A_0| + \|t^n \alpha'_n\|_2 \lesssim \|t^n (A_1(\alpha))_n\|_2 + |\alpha_0| \).

For the other direction, it is obvious that \( |\alpha_0| \leq |A_0| + |\alpha_0'| \) and
\[
\|t^n (A_1(\alpha))_n\|_2 = \|t^n (\alpha_{n+1} + \alpha_0 \alpha_n)\|_2 = \|t^n (\alpha_{n+1} - \alpha_0 \alpha_n)\|_2 \lesssim \|t^n \alpha'_n\|_2.
\]
Thus we have proved the \( l = 1 \) case of part (a).

Next, let’s look at the \( l > 1 \) case. Similar as above, we have
\[
(A_{l-1}(\alpha))_n = \kappa_{l-1}^{l-1} (A_{l-1}(\alpha))_0 + \sum_{i=0}^{n-1} \kappa_{l-1}^{n-i-1} (A_l(\alpha))_i.
\]
Let \( \tilde{A}_{l-1} = (A_{l-1}(\alpha))_0 + \sum_{i=0}^{\infty} \kappa_{l-1}^{-i-1} (A_l(\alpha))_i \), and \( A_{l-1} = \tilde{A}_{l-1} (\kappa_{l-1} - \kappa_{l-2})^{-1} \). It is easy to check that
\[
(A_{l-1} (\{A_{l-1} \kappa_{l-1}^i \}_{i \geq 0})_n = \tilde{A}_{l-1} \kappa_{l-1}^n.
\]
For \( n \geq 0 \), let \( \tilde{\alpha}_n = \alpha_n - A_{l-1} \kappa_{l-1}^n \) and write \( \tilde{\alpha} = \{\tilde{\alpha}_n\}_{n \geq 0} \). Then directly we have
\[
A_{l-1}(\alpha) = A_{l-1}(\tilde{\alpha}) + \{\tilde{A}_{l-1} \kappa_{l-1}^n\}_{n \geq 0}.
\]
By a similar argument as we did for the \( l = 1 \) case, noticing that \( |A_{l-1}| \asymp |\tilde{A}_{l-1}| \), we get
\[
\|t^n (A_1(\alpha))_n\|_2 + \|\{A_{l-1}(\alpha)\}_0\|_2 \asymp |A_{l-1}| + \|t^n (A_{l-1}(\tilde{\alpha}))_n\|_2.
\]
On the other hand, using the induction hypothesis, for \( n \geq 0 \), we could write 
\[
\tilde{\alpha}_n = \sum_{i=0}^{l-2} A_i \kappa_i^n + \alpha'_n \quad \text{with} \quad A_i \in \mathbb{R}
\] and 
\[
\sum_{i=0}^{l-2} |A_i| + \|t^n \alpha'_n\|_{l^2} \asymp \|t^n (A_{l-1}(\tilde{\alpha}))_n\|_{l^2} + \sum_{i=0}^{l-2} |\tilde{\alpha}_i|.
\]
Combining the above two estimates, we then have 
\[
\|t^n (A_l(\alpha))_n\|_{l^2} + \sum_{i=0}^{l-1} |\alpha_i| \asymp \|t^n (A_{l-1}(\tilde{\alpha}))_n\|_{l^2} + (A_{l-1})_0 + \sum_{i=0}^{l-2} |\alpha_i|
\]
\[
\asymp \|t^n (A_{l-1}(\tilde{\alpha}))_n\|_{l^2} + \sum_{i=0}^{l-2} |\tilde{\alpha}_i| + (A_{l-1})_0
\]
\[
\asymp \|t^n \alpha'_n\|_{l^2} + \sum_{i=0}^{l-1} |A_i|,
\]
Thus we have proved the \( l > 1 \) case of part (a).

The proof of part (b) is the same. \( \square \)

Lemma 3.8. Let \( l \geq 1, t > 0, \alpha, \eta \) be two sequences.
(a). If \( t = \kappa_{l-1} \) and \( \|t^n (A_l(\alpha))_n\|_{l^2} < \infty \), then we have 
\[
\alpha_n = \sum_{i=0}^{l-2} A_i \kappa_i^n + \alpha'_n,
\]
with \( A_i \in \mathbb{R} \), satisfying that 
\[
\sum_{i=0}^{l-2} |A_i| + \|t^n (\alpha'_{n+1} - \kappa_{l-1} \alpha'_n)\|_{l^2} \asymp \|t^n (A_l(\alpha))_n\|_{l^2} + \sum_{i=0}^{l-1} |\alpha_i|.
\]
(b). If \( t = \iota_{l-1}^{-1} \) and \( \|t^n (\Theta_l(\eta))_n\|_{l^2} < \infty \), then we have 
\[
\eta_n = \sum_{i=0}^{l-2} B_i \iota_i^n + \eta'_n,
\]
with \( B_i \in \mathbb{R} \), satisfying that 
\[
\sum_{i=0}^{l-2} |B_i| + \|t^n (\eta'_{n+1} - \iota_{l-1} \eta'_n)\|_{l^2} \asymp \|t^n (\Theta_l(\eta))_n\|_{l^2} + \sum_{i=0}^{l-1} |\eta_i|.
\]
Proof. (a). It is easy to find that 
\[
A_l(\alpha) = A_{l-1}(\{\alpha_{n+1} - \kappa_{l-1} \alpha_n\}_{n \geq 0}).
\]
Since \( \{\kappa_i\}_i \) is strictly decreasing, we have \( t > \kappa_{l-2}^{-1} \). Applying Lemma 3.7 we write
\[
\alpha_{n+1} - \kappa_{l-1} \alpha_n = \sum_{i=0}^{l-2} \tilde{A}_i \kappa_i^n + \tilde{\alpha}'_n,
\]
with
\[
\sum_{i=0}^{l-2} |\tilde{A}_i| + \|t^n \tilde{\alpha}'_n\|_{l^2} \asymp \|t^n (A_l(\alpha))\|_{l^2} + \sum_{i=0}^{l-1} |\alpha_i|.
\]
(3.4)
Then
\[
\kappa_{l-1}^{-n} \alpha_n = \alpha_0 + \sum_{i=0}^{l-2} \sum_{j=0}^{n-1} \kappa_{l-1}^{-j} \kappa_i^n \tilde{A}_i + \sum_{j=0}^{n-1} \kappa_{l-1}^{-j} \alpha'_j
\]
\[
= \sum_{i=0}^{l-2} \frac{\tilde{A}_i}{\kappa_{l-1} \kappa_i - 1} \kappa_{l-1}^{-n} \kappa_i^n + (\alpha_0 - \sum_{i=0}^{l-2} \frac{\kappa_{l-1}^{-1} \tilde{A}_i}{\kappa_{l-1} \kappa_i - 1} + \sum_{j=0}^{n-1} \kappa_{l-1}^{-j} \alpha'_j)
\]
\[
= \sum_{i=0}^{l-2} A_i \kappa_{l-1}^{-n} \kappa_i^n + \kappa_{l-1}^{-n} \alpha_n,
\]
with \( A_i = \frac{\kappa_{l-1}^{-1} \tilde{A}_i}{\kappa_{l-1} \kappa_i - 1} \) and \( \alpha'_n = \kappa_{l-1}^{-n} (\alpha_0 - \sum_{i=0}^{l-2} \frac{\kappa_{l-1}^{-1} \tilde{A}_i}{\kappa_{l-1} \kappa_i - 1} + \sum_{j=0}^{n-1} \kappa_{l-1}^{-j} \alpha'_j) \).
Thus we could write
\[
\alpha_n = \sum_{i=0}^{l-2} A_i \kappa_i^n + \alpha'_n,
\]
and clearly \( \tilde{\alpha}'_n = \alpha'_{n+1} - \kappa_{l-1} \alpha'_n \).
Since \( \sum_{i=0}^{l-2} |\tilde{A}_i| \asymp \sum_{i=0}^{l-2} |A_i| \), the estimate follows immediately from (3.4).
The proof of part (b) is the same. \( \square \)

Before ending this section, we introduce a class of normed spaces of sequences.

**Definition 3.9.** Let \( t > 0 \), \( l \geq 0 \). Define
\[
\ell^2(t, A_l) = \left\{ \alpha : \{t^n (A_l(\alpha))\}_n \in l^2 \right\},
\]
and
\[
\ell^2(t, \Theta_l) = \left\{ \eta : \{t^n (\Theta_l(\eta))\}_n \in l^2 \right\},
\]
with the norms given by
\[
\|\alpha\|_{\ell^2(t, A_l)} = \|t^n (A_l(\alpha))\|_{l^2} + \sum_{i=0}^{l-1} |\alpha_i|,
\]
and
\[
\|\eta\|_{\ell^2(t, \Theta_l)} = \|t^n (\Theta_l(\eta))\|_{l^2} + \sum_{i=0}^{l-1} |\eta_i|.
\]
In particular, when \( l = 0, \ t = 1 \), the space \( l^2(1, A_0) \) and \( l^2(1, \Theta_0) \) are identical, and equal to \( l^2 \).

As a consequence of Lemma 3.6, we have the following proposition.

**Proposition 3.10.** For \( t > 0 \), let \( l_a(t) = \max\{l : t \geq \kappa_{l-1}^{-1}\} \) and \( l_a(t) = \max\{l : t \geq \kappa_{l-1}^{-1}\} \), then we have

\[
I^2(t, \Lambda_i) = I^2(t, \Lambda_{l_a(t)}) \quad \text{with} \quad \|\cdot\|_{I^2(t, \Lambda_i)} \asymp \|\cdot\|_{I^2(t, \Lambda_{l_a(t)})}
\]

for any \( l \geq l_a(t) \), and

\[
I^2(t, \Theta_i) = I^2(t, \Theta_{l_a(t)}) \quad \text{with} \quad \|\cdot\|_{I^2(t, \Theta_i)} \asymp \|\cdot\|_{I^2(t, \Theta_{l_a(t)})}
\]

for any \( l \geq l_a(t) \).

4. Trace theorem for \( L^2_k(S\Gamma) \) with \( k \in \mathbb{N} \)

In this section, we study the trace space of \( L^2_k(S\Gamma) \) under the restriction map \( R \) in case of \( \sigma \in \mathbb{N} \). We begin with the following estimates.

**Proposition 4.1.** For \( k \in \mathbb{N} \), let \( u \in L^2_k(S\Gamma) \) and \( Ru = (\alpha, \eta) \). We have

\[
\alpha \in L^2(5^{k-3/2}, A_{s(k)}) \quad \text{and} \quad \eta \in L^2(5^{k-3/2}, \Theta_{a(k)}).
\]

Moreover,

(a). \( \| (5^{k-3/2})^n (A_{s(k)}(\alpha)) \|_2 \leq \| \Delta^k u \|_{L^2(S\Gamma)} \)

(b). \( \| (5^{k-3/2})^n (\Theta_{a(k)}(\eta)) \|_2 \leq \| \Delta^k u \|_{L^2(S\Gamma)} \)

**Proof.** (a). By Lemma 3.5, we have

\[
(A_{s(k)}(\alpha))_n = 3^n 5^{-kn} \int_{F_0^n S\Gamma} \varphi_{s,k} \circ F_0^{-n}(x) \Delta^k u(x) \, d\mu(x).
\]

For \( l \geq 0 \), let \( Z_l = F_0^l F_1 S\Gamma \cup F_0^l F_2 S\Gamma \). Then applying Minkowski inequality and Cauchy-Schwarz inequality, we get

\[
\| (5^{k-3/2})^n (A_{s(k)}(\alpha)) \|_2 \leq \| 3^{n/2} \int_{F_0^n S\Gamma} |\varphi_{s,k}(F_0^{-n} x)| \cdot |\Delta^k u(x)| \, d\mu(x) \|_2
\]

\[
= \| \sum_{l=n}^{\infty} 3^{n/2} |\varphi_{s,k}(F_0^{-n} x)| \cdot |\Delta^k u(x)| \, d\mu(x) \|_2
\]

\[
\leq \| \sum_{l=n}^{\infty} 3^{n/2} |\Delta^k u(x)| \, d\mu(x) \|_2
\]

\[
\leq \| \sum_{l=n}^{\infty} 3^{-l} \| \Delta^k u \|_{L^2(Z_l)} \|_2
\]

\[
= \| \sum_{l=0}^{\infty} 3^{-l/2} \| \Delta^k u \|_{L^2(Z_{l+n})} \|_2
\]

\[
\leq \sum_{l=0}^{\infty} 3^{-l/2} \| \Delta^k u \|_{L^2(Z_{l+n})} \|_2
\]
Proposition 4.1 can be extended to general $L^p_k(\mathcal{S}G)$ setting for $1 \leq p \leq \infty$, with the estimates that

\[
\| \Delta^k u \|_{L^2(\mathcal{S}G)} \lesssim \| \Delta^k u \|_{L^p(\mathcal{S}G)}.
\]

The proof of part (b) is the same, we omit it. \qed

Remark. Proposition 4.1 can be extended to general $L^p_k(\mathcal{S}G)$ setting for $1 \leq p \leq \infty$, with the estimates that

\[
\begin{align*}
&\left\| (5^k3^{-1/p})^{n} (A_{s(k)}(\alpha)) \right\|_{L^p} \leq \| \Delta^k u \|_{L^p(\mathcal{S}G)}, \\
&\left\| (5^{k-1}3^{-1/p})^{n} (\Theta_{a(k)}(\eta)) \right\|_{L^p} \leq \| \Delta^k u \|_{L^p(\mathcal{S}G)},
\end{align*}
\]

by using Holder inequality instead of Cauchy-Schwarz inequality in the above proof.

Definition 4.2. For $k \in \mathbb{N}$, we define the following trace spaces.

(a). Let $\mathcal{T}^s_{2,k} = l^2(5^k3^{-1/2}, A_{s(k)})$ with the norm $\| \cdot \|_{\mathcal{T}^s_{2,k}} = \| \cdot \|_{l^2(5^k3^{-1/2}, A_{s(k)})}$.

(b). Let $\mathcal{T}^a_{2,k} = l^2(5^{k-1}3^{1/2}, \Theta_{a(k)})$ with the norm $\| \cdot \|_{\mathcal{T}^a_{2,k}} = \| \cdot \|_{l^2(5^{k-1}3^{1/2}, \Theta_{a(k)})}$.

(c). Let $\mathcal{T}_{2,k} = \{ (\alpha, \eta) : \alpha \in \mathcal{T}^s_{2,k}, \eta \in \mathcal{T}^a_{2,k} \}$, with the norm $\| (\alpha, \eta) \|_{\mathcal{T}_{2,k}} = \| \alpha \|_{\mathcal{T}^s_{2,k}} + \| \eta \|_{\mathcal{T}^a_{2,k}}$.

By Lemma 3.7 and Proposition 3.10 we have

Proposition 4.3. For $k \in \mathbb{N}$, we have

(a). $\mathcal{T}^s_{2,k} = \{ \alpha : \alpha_n = \sum_{i=0}^{s(k)-1} A_i \kappa_i^n + \alpha'_n, \{ (5^k3^{-1/2})^n \alpha'_n \}_{n \geq 0} \in l^2 \}$, with $\| \alpha \|_{\mathcal{T}^s_{2,k}} \leq \sum_{i=0}^{s(k)-1} |A_i| + \| (5^k3^{-1/2})^n \alpha'_n \|_{l^2}$.

(b). $\mathcal{T}^a_{2,k} = \{ \eta : \eta_n = \sum_{i=0}^{l_i(5^{k-1}3^{1/2})-1} B_i \iota_i^n + \eta'_n, \{ (5^{k-1}3^{1/2})^n \eta'_n \}_{n \geq 0} \in l^2 \}$, with $\| \eta \|_{\mathcal{T}^a_{2,k}} \leq \sum_{i=0}^{l_i(5^{k-1}3^{1/2})-1} |B_i| + \| (5^{k-1}3^{1/2})^n \eta'_n \|_{l^2}$.

Proof. For part (a), we only need to notice that $5^k3^{-1/2} > \kappa_{s(k)}^{-1}$ since $\kappa_{s(k)} \geq 3 \cdot 5^{-k}$. For part (b), we only need to notice that $5^{k-1}3^{1/2} \notin \{ \iota_i \}$ and it may happen that $5^{k-1}3^{1/2} < \iota_{a(k)}^{-1}$. \qed

Remark. There are two distinct descriptions of the space $\mathcal{T}_{2,k}$ as above. The first one is useful in the interpolation that we will use later, while the second one is more concrete and will play an important role in Section 6.

We want to show that the restriction map $R : L^2_k(\mathcal{S}G) \rightarrow \mathcal{T}_{2,k}$ is a bounded surjection, for each $k \in \mathbb{N}$. It is equivalent to prove that the maps $R_s : L^2_k(\mathcal{S}G) \rightarrow \mathcal{T}^s_{2,k}$ and $R_a : L^2_k(\mathcal{S}G) \rightarrow \mathcal{T}^a_{2,k}$ are bounded and subjective respectively.

First, for the boundedness, we have

Lemma 4.4. For $k \in \mathbb{N}$, the restriction map $R : L^2_k(\mathcal{S}G) \rightarrow \mathcal{T}_{2,k}$ is bounded.

Proof. By Proposition 4.1 and Definition 4.2, it is enough to verify that all $\alpha_i$ with $i \leq s(k)-1$ and all $\eta_i$ with $i \leq a(k)-1$ are bounded by $\| u \|_{L^2_k(\mathcal{S}G)}$. 

We write \( u = h - G\Delta u \) with some harmonic function \( h \), where \( G \) is the Green’s operator. Then
\[
\|u\|_{L^\infty(SG)} \leq \|h\|_{L^\infty(SG)} + \|G\Delta u\|_{L^\infty(SG)} \\
\lesssim \|h\|_{L^2(SG)} + \|\Delta u\|_{L^2(SG)} \\
\leq \|u\|_{L^2(SG)} + \|G\Delta u\|_{L^2(SG)} + \|\Delta u\|_{L^2(SG)} \lesssim \|u\|_{L^2_k(SG)}.
\]
Thus \( \alpha_i \)'s are bounded for \( i \leq s(k) - 1 \).

For \( \eta_i \)'s, recall that
\[
\eta_i = \partial^{i-1}_{n^*}u(p_i) = \left(\frac{5}{3}\right)^{i-1}(2u(p_i) - u(F^i_0q_1) - u(F^{i+1}_0q_1)) \\
+ \int_{F^0_0F^0_1SG} h_2 \circ F^{-1}_{0}F^{-i}_0(x)\Delta u(x)d\mu(x),
\]
where \( h_2 \) is the harmonic function with boundary values \( h_2(q_0) = h_2(q_1) = 0 \) and \( h_2(q_2) = 1 \). Then \( \eta_i \)'s are bounded for \( i \leq a(k) - 1 \).

Next, for \( k \in \mathbb{N} \), to illustrate that the map \( R : L^2_k(SG) \to \mathcal{T}_{2,k} \) is surjective, we need an extension map \( E_k : \mathcal{T}_{2,k} \to L^2_k(SG) \).

**Lemma 4.5.** Let \( k \in \mathbb{N} \). We have
(a). there exists a symmetric function \( f_k \in L^2_k(SG) \) with
\[
\begin{cases}
supp f_k \subset F^0_0SG, \\
supp(\Delta^k f_k) \subset F^0_0(F_1SG \cup F_2SG),
\end{cases}
\]
such that
\[ A_{s(k)}(R\alpha f_k) = \{\delta_{0n}\}_{n \geq 0}, \]
where \( \delta_{ij} \) is the Kronecker delta, i.e., \( \delta_{ij} = 1 \) if \( i = j \) and \( \delta_{ij} = 0 \) if \( i \neq j \).

(b). there exists an antisymmetric function \( g_k \in L^2_k(SG) \) with
\[
\begin{cases}
supp g_k \subset F^0_0SG, \\
supp(\Delta^k g_k) \subset F^0_0(F_1SG \cup F_2SG),
\end{cases}
\]
such that
\[ \Theta_{a(k)}(R\alpha g_k) = \{\delta_{0n}\}_{n \geq 0}. \]

**Proof.** (a). There is a unique sequence \( \alpha \) such that
\[ A_{s(k)}(\alpha) = \{\delta_{0n}\}_{n \geq 0} \text{ and } \alpha_n = 0, \forall 0 \leq n \leq s(k) - 1, \]

since for each \( n \geq s(k) \), \( \alpha_n \) is uniquely determined by \( (A_{s(k)}(\alpha))_{n-s(k)} \) and \( \alpha_{n-j}, 1 \leq j \leq s(k) \).

By Lemma 3.2*, we can find a symmetric \( k \)-multiharmonic function \( h \) such that \( h(p_n) = \alpha_n, \forall n \geq s(k) + 1 \). Let \( f_k|_{F^0_0} = h|_{F^0_0SG} \) and \( f_k|_{SG} = h|_{F^0_0SG} \) so that \( f_k \) is a function on \( SG \).
Since we need \( f_k \) to be symmetric, we only need to look at \( f_k \) on \( F_0^{s(k)} \). For simplicity, we write \( \tilde{f}_k = f_k|_{F_0^{s(k)}} \). It is sufficient to require that

\[
\tilde{f}_k \circ F_1^{-1} \circ F_0^{-s(k)} \in L_k^2(SG),
\]

satisfying the boundary condition

\[
\begin{aligned}
\partial_n^\pm \Delta_j \tilde{f}_k(F_0^{s(k)}F_1q_0) + \partial_n^\pm \Delta_j h(F_0^{s(k)}F_1q_0) &= 0, &\forall 0 \leq j < k, \\
\Delta_j \tilde{f}_k(F_0^{s(k)}F_1q_0) &= \Delta_j h(F_0^{s(k)}F_1q_0), &\forall 0 \leq j < k,
\end{aligned}
\]

for the point \( F_0^{s(k)}F_1q_0 \), and

\[
\partial_n^\pm \Delta_j \tilde{f}_k(F_0^{s(k)}F_1q_1) = \Delta_j \tilde{f}_k(F_0^{s(k)}F_1q_1) = 0, &\forall 0 \leq j < k,
\]

for the point \( F_0^{s(k)}F_1q_1 \), and

\[
\begin{aligned}
\partial_n^\pm \Delta_j \tilde{f}_k(p_{s(k)}) &= 0, &\forall 0 \leq j < k, \\
\Delta_j \tilde{f}_k(p_{s(k)}) &= \alpha_{s(k)} \delta_{j0}, &\forall 0 \leq j < k,
\end{aligned}
\]

for the point \( p_{s(k)} \). The existence of such \( \tilde{f}_k \) is due to the fact that we can always construct a function in \( L_k^2(SG) \) with any prescribed boundary values and boundary normal derivatives. Then using the matching condition, \( f_k \) is well-defined in \( L_k^2(SG) \) and obviously satisfies the required properties.

The proof of part (b) is essentially the same. \( \square \)

**Definition 4.6.** Let \( k \in \mathbb{N} \). Define the extension maps \( E_k^s, E_k^a, E_k \) as following.

(a) \( E_k^s : T_{2,k}^s \to L_k^2(SG) \) is defined as

\[
E_k^s(\alpha) = h_s + \sum_{n=0}^{\infty} (A_{s(k)}(\alpha))_n f_k \circ F_0^{-n},
\]

where \( h_s \) is a symmetric \( k \)-multiharmonic function satisfying \( h_s(F_0^{n}p_0) = \alpha_n, \forall 0 \leq n \leq s(k) - 1 \), which depends linearly on \( (\alpha_0, \cdots, \alpha_{s(k)-1}) \).

(b) \( E_k^a : T_{2,k}^a \to L_k^2(SG) \) is defined as

\[
E_k^a(\eta) = h_a + \sum_{n=0}^{\infty} (\Theta_{a(k)}(\eta))_n g_k \circ F_0^{-n},
\]

where \( h_a \) is an antisymmetric \( k \)-multiharmonic function satisfying \( h_a(F_0^{n}p_0) = \eta_n, \forall 0 \leq n \leq a(k) - 1 \), which depends linearly on \( (\eta_0, \cdots, \eta_{a(k)-1}) \).

(c) \( E_k : T_{2,k} \to L_k^2(SG) \) is defined as \( E_k(\alpha, \eta) = E_k^s(\alpha) + E_k^a(\eta) \).

**Lemma 4.7.** For \( k \in \mathbb{N} \), the extension maps \( E_k^s, E_k^a \) and \( E_k \) defined in Definition 4.6 are all bounded linear maps. In addition, \( RE_k(\alpha, \eta) = (\alpha, \eta) \), \( \forall (\alpha, \eta) \in T_{2,k} \).
Proof. We prove the lemma for $E^s_k$. First, we show that $E^s_k$ is bounded. It is obvious that

$$\|E^s_k \alpha\|_{L^2(SG)} \leq \|h_s\|_{L^2(SG)} + \sum_{n=0}^\infty \| (A_{s(k)}(\alpha))_n f_k \circ F_0^{-n}\|_{L^2(SG)}$$

$$\lesssim \|\alpha\|_{T_{2,k}} + \sum_{n=0}^\infty 3^{-n/2} |(A_{s(k)}(\alpha))_n|$$

$$\lesssim \|\alpha\|_{T_{2,k}}.$$ 

On the other hand, we have

$$\|\Delta^k (E^s_k \alpha)\|_{L^2(SG)} = \\| \sum_{n=0}^\infty (A_{s(k)}(\alpha))_n \Delta^k (f_k \circ F_0^{-n})\|_{L^2(SG)}$$

$$= \left( \sum_{n=0}^\infty \|5^{kn} (A_{s(k)}(\alpha))_n (\Delta^k f_k) \circ F_0^{-n}\|_{L^2(SG)}^2 \right)^{1/2}$$

$$= \left( \sum_{n=0}^\infty (5^{kn}3^{-n/2}(A_{s(k)}(\alpha))_n)^2 \|\Delta^k f_k\|_{L^2(SG)}^2 \right)^{1/2}$$

$$\lesssim \|\alpha\|_{T_{2,k}},$$

where the third equality holds because $\Delta^k f_k$ is supported in $F_0^{s(k)}F_1SG \cup F_0^{s(k)}F_2SG$. Combining the above two estimates, by using Proposition 2.1, we get that $E^s_k$ is a bounded map from $T_{2,k}^{s,k}$ to $L^2_{k}^{s,k}(SG)$.

Furthermore, for any $\alpha \in T_{2,k}^{s,k}$, we can easily check that $A_{s(k)}(\{E^s_k(\alpha)(p_n)\}_{n\geq 0}) = A_{s(k)}(\alpha)$, and $\alpha_n = E^s_k(\alpha)(p_n), \forall 0 \leq n \leq s(k) - 1$. Thus $E^s_k(\alpha)(p_n) = \alpha_n, \forall n \geq 0.$

The boundedness for $E^a_k$ is the same. The result for $E_k$ is just a combination of that of $E^s_k$ and $E^a_k$. \hfill \Box

Now we have

**Theorem 4.8.** The restriction map $R : L^2_k(SG) \to T_{2,k}$ is bounded and surjective.

**Proof.** This follows from Lemma 4.4 and Lemma 4.7. \hfill \Box

Before ending this section, we would like to point out an important observation of the extension maps $E_k, k \in \mathbb{N}$, which will be used later.

**Proposition 4.9.** Let $k > i \geq 1$, we can naturally extend the extension map $E^s_k$ to a bounded linear extension map, still denoted by $E^s_k$, from $T_{2,k-i}^{s,k-i}$ to $L^2_{k-i}(SG)$, which is independent of $i$. In addition, $R_*E^s_k(\alpha) = \alpha, \forall \alpha \in T_{2,k-i}^{s,k-i}$. The $E^a_k$ case is similar.
Proof. We only prove the \( E_k^s \) case, while the \( E_k^0 \) case is essentially the same. Let \( \alpha \in \mathcal{T}_{2,k-1}^s \). We denote \( A_{s(k)}(\alpha) = \{ c_n \}_{n \geq 0}, \) and define

\[
E_k^s(\alpha) = h_s + \sum_{n=0}^{\infty} c_n f_k \circ F_0^{-n},
\]

where \( h_s \) is a symmetric \( k \)-multiharmonic function satisfying \( h_s(p_n) = x_n, \forall 0 \leq n \leq s(k) - 1 \). Obviously, this extends the original map \( E_k^s \). Moreover, we have the estimate that

\[
\| \Delta^{k-i} (\sum_{j=0}^{\infty} c_j f_k \circ F_0^{-j}) \|_{L^2(SG)} = \| \Delta^{k-i} (\sum_{j=0}^{\infty} c_j f_k \circ F_0^{-j}) \|_{L^2(Z_n)} \leq \sum_{j=0}^{n-s(k)} (5^{k-i} j 3^{-n/2} |c_j|)_{L^2} \\
\leq \sum_{j=0}^{n} 3^{-j/2} (5^{k-i} j 3^{-n/2} |c_n-j|)_{L^2} \\
\leq \sum_{j=0}^{\infty} 3^{-j/2} (5^{k-i} j 3^{-n/2} |c_n|)_{L^2} \leq \| \alpha \|_{\mathcal{T}_{2,k-1}^s},
\]

where \( Z_n = F_0^n F_1 SG \cup F_0^n F_2 SG, \) \( n \geq 0 \), and we use \( \| f_k \|_{L^\infty} < \infty \) for the second inequality, and Proposition 3.10 for the last inequality. On the other hand, as in the proof of Lemma 4.7, it is clear that \( \| \sum_{j=0}^{\infty} c_j f_k \circ F_0^{-j} \|_{L^2(SG)} + \| h_s \|_{L^2_{k-1}(SG)} \leq \| \alpha \|_{\mathcal{T}_{2,k-1}^s}. \) This completes the proof. \( \Box \)

5. Trace theorem for \( L^2_\sigma(SG) \) with real \( \sigma \geq 0 \)

In this section, first we will use interpolation to give a full description of \( R_\sigma L^2_\sigma(SG) \) for \( \sigma \geq \frac{1}{2} \), and \( R_\sigma L^2_\sigma(SG) \) for \( \sigma \geq 1 \). Then by virtue of the atomic decomposition of \( L^2_\sigma(SG) \)(see [4]), we extend the trace result to \( \sigma > \frac{\log \log \sigma}{2 \log \sigma} \). We need to extend the definition to general real \( \sigma \). Now we extend the definition to general real \( \sigma \). Note that \( l_s(5^{k-3/2}) = s(k) \) since \( \kappa_{s(k)-1} \geq 3 \cdot 5^{-k} \) and \( \kappa_{s(k)} \leq 5^{-k} \). So we have

\[
\mathcal{T}_{2,k}^s = l^2(5^{k-3/2}, A_{l_s(5^{k-3/2})}).
\]

On the other hand, note that \( l_a(5^{k-13/2}) \leq a(k) \) since \( \iota_{a(k)} \leq 3^{-15-k} \). Then by Proposition 3.10, we have

\[
\mathcal{T}_{2,k}^a = l^2(5^{k-13/2}, \Theta_{l_a(5^{k-13/2})}).
\]
Basing on the above observations, we introduce the following trace spaces for real \( \sigma \).

**Definition 5.1.** (a). For \( \sigma > \frac{\log 3}{2 \log 5} \), define \( T_{2,\sigma}^a = \ell^2(5^{\sigma}3^{-1/2}, \Lambda_{i}(5^{\sigma}3^{-1/2}) \) with the same norm.

(b). For \( \sigma > 1 - \frac{\log 3}{2 \log 5} \), define \( T_{2,\sigma}^a = \ell^2(5^{\sigma-13/2}, \Theta_{i}(5^{\sigma-13/2}) \) with the same norm.

(c). For \( \sigma > 1 - \frac{\log 3}{2 \log 5} \), define
\[
T_{2,\sigma} = \{(\alpha, \eta) : \alpha \in T_{2,\sigma}^a, \eta \in T_{2,\sigma}^a\},
\]
with the norm
\[
\| (\alpha, \eta) \|_{T_{2,\sigma}} = \| \alpha \|_{T_{2,\sigma}^a} + \| \eta \|_{T_{2,\sigma}^a}.
\]

By using Lemma 3.7 and Lemma 3.8 we have a detailed characterization of these spaces.

**Theorem 5.2.** (a). Let \( \sigma > \frac{\log 3}{2 \log 5} \). If \( 5^{\sigma}3^{-1/2} \notin \{\kappa_i^{-1}\}^\infty_{i=0} \), then
\[
T_{2,\sigma}^a = \{ \alpha : \alpha = \sum_{\kappa_i>5^{-3/2}} A_i \kappa_i^\sigma + \alpha'_i, \{\left(5^{\sigma-1/2}\right)^n \alpha'_i\} \in \ell^2, A_i \in \mathbb{R}\},
\]
and we have \( \| \alpha \|_{T_{2,\sigma}^a} \leq \sum_{i=0}^{l_i(5^{\sigma-1/2})} |A_i| + \left\| \left(5^{\sigma-1/2}\right)^n \alpha'_i \right\|_{\ell^2} \).

If \( 5^{\sigma-3/2} \in \{\kappa_i^{-1}\}^\infty_{i=0} \), then
\[
T_{2,\sigma}^a = \{ \alpha : \alpha = \sum_{\kappa_i>5^{-3/2}} A_i \kappa_i^\sigma + \alpha'_i, \{\left(5^{\sigma-1/2}\right)^n \alpha'_i\} \in \ell^2, A_i \in \mathbb{R}\},
\]
and we have \( \| \alpha \|_{T_{2,\sigma}^a} \leq \sum_{i=0}^{l_i(5^{\sigma-3/2})} |A_i| + \left\| \left(5^{\sigma-3/2}\right)^n \alpha'_i \right\|_{\ell^2} \).

(b). Let \( \sigma > 1 - \frac{\log 3}{2 \log 5} \). If \( 5^{\sigma-3/2} \notin \{\ell_i^{-1}\}^\infty_{i=0} \), then
\[
T_{2,\sigma}^a = \{ \eta : \eta = \sum_{\ell_i>5^{-3/2}+1} B_i \ell_i^\eta + \eta'_i, \{\left(5^{\sigma-3/2}\right)^n \eta'_i\} \in \ell^2, B_i \in \mathbb{R}\},
\]
and we have \( \| \eta \|_{T_{2,\sigma}^a} \leq \sum_{i=0}^{l_i(5^{\sigma-3/2})} |B_i| + \left\| \left(5^{\sigma-3/2}\right)^n \eta'_i \right\|_{\ell^2} \).

If \( 5^{\sigma-3/2} \in \{\ell_i^{-1}\}^\infty_{i=0} \), then
\[
T_{2,\sigma}^a = \{ \eta : \eta = \sum_{\ell_i>5^{-3/2}+1} B_i \ell_i^\eta + \eta'_i, \{\left(5^{\sigma-3/2}\right)^n \eta'_i\} \in \ell^2, B_i \in \mathbb{R}\},
\]
and we have \( \| \eta \|_{T_{2,\sigma}^a} \leq \sum_{i=0}^{l_i(5^{\sigma-3/2})} |B_i| + \left\| \left(5^{\sigma-3/2}\right)^n \eta'_i \right\|_{\ell^2} \).

Now we turn to the general trace theorem for real \( \sigma \).

**Lemma 5.3.** (a). Let \( \frac{\log 3}{2 \log 5} < \sigma_0 \leq \sigma_1 < \infty \). Then the complex interpolation space \([T_{2,\sigma_0}^a, T_{2,\sigma_1}^a]_\theta = T_{2,\sigma}^a\) with \( \theta = (1-\theta)\sigma_0 + \theta \sigma_1 \) for \( \theta \in [0,1] \).

(b). Let \( 1 - \frac{\log 3}{2 \log 5} < \sigma_0 \leq \sigma_1 < \infty \). Then the complex interpolation space \([T_{2,\sigma_0}^a, T_{2,\sigma_1}^a]_\theta = T_{2,\sigma}^a\) with \( \theta = (1-\theta)\sigma_0 + \theta \sigma_1 \) for \( \theta \in [0,1] \).

(c). Let \( 1 - \frac{\log 3}{2 \log 5} < \sigma_0 \leq \sigma_1 < \infty \). Then the complex interpolation space \([T_{2,\sigma_0}, T_{2,\sigma_1}]_\theta = T_{2,\sigma}^a\) with \( \theta = (1-\theta)\sigma_0 + \theta \sigma_1 \) for \( \theta \in [0,1] \).
Proof. (a). By Proposition 3.10 we have
\[ T_{2,\sigma_0} = l^2(5^{\sigma_0}3^{-1/2}, A_{l_4(5^{\sigma_0}3^{-1/2})}) = l^2(5^{\sigma_0}3^{-1/2}, A_{l_4(5^{\sigma_1}3^{-1/2})}), \]
as \( l_s(5^{\sigma_1}3^{-1/2}) \geq l_s(5^{\sigma_0}3^{-1/2}) \). As a result, we have
\[
\left[ T_{2,\sigma_0}, T_{2,\sigma_1} \right] \theta = \left[ l^2(5^{\sigma_0}3^{-1/2}, A_{l_4(5^{\sigma_1}3^{-1/2})}), l^2(5^{\sigma_1}3^{-1/2}, A_{l_4(5^{\sigma_1}3^{-1/2})}) \right] \theta \\
= l^2(5^{\sigma_0}3^{-1/2}, A_{l_4(5^{\sigma_1}3^{-1/2})}) \\
= l^2(5^{\sigma_0}3^{-1/2}, A_{l_4(5^{\sigma_1}3^{-1/2})}) = T_{2,\sigma}.
\]
(b). The proof is the same as (a).
(c). It is a combination of (a) and (b).

Using Lemma 5.3, Theorem 4.8 and Proposition 4.9 by interpolation, we immediately obtain the following trace theorem.

**Theorem 5.4.** For \( \sigma \geq 1 \), the restriction map \( R : L^2_{1/2}(SG) \to T_{2,\sigma} \) is bounded and surjective.

In the next section, we will prove that \( L^2_{1/2}(SG) = \text{dom} \mathcal{E} \), with the norm given by \( \|u\|_{L^2_{1/2}(SG)} = \|u\|_{L^2(SG)} + \mathcal{E}^{1/2}(u) \), see Corollary 6.5 whose proof essentially relies on Theorem 5.4. Using this, we can extend the trace result to \( \frac{1}{2} \leq \sigma \leq 1 \).

**Theorem 5.5.** For \( \frac{1}{2} \leq \sigma \leq 1 \), the restriction map \( R_s : L^2_{\sigma}(SG) \to T_{2,\sigma}^s \) is bounded and surjective.

**Proof.** As an immediate consequence of the energy estimate of harmonic functions on the left half domain of \( SG \) as showed in [14], we have
\[ R_s L^2_{1/2}(SG) = R_s(\text{dom} \mathcal{E}) = l^2(5^{1/2}3^{-1/2}, A_1). \]
Applying interpolation, we can immediately see that \( R_s : L^2_{\sigma}(SG) \to T_{2,\sigma}^s \) is bounded for \( \frac{1}{2} \leq \sigma \leq 1 \). It remains to show that \( R_s \) is surjective.

Recall that the extension map \( E^s_1 : T_{2,1}^s \to L^2_{1}(SG) \) is defined by
\[ E^s_1(\alpha) = h_s + \sum_{j=0}^{\infty} \left( A_{s(1)}(\alpha) \right)_j f_1 \circ F_0^{-j}, \]
for \( \alpha \in T_{2,1}^s \), where \( f_1 \) is the same function in Lemma 4.5 \( h_s \) is a symmetric harmonic function satisfying \( h_s(p_0) = \alpha_0, h_s(p_1) = \alpha_1 \). By Lemma 4.7 it is a bounded map and \( R_s E^s_1(\alpha) = \alpha, \forall \alpha \in T_{2,1}^s \). Note that \( f_1 \) is harmonic in \( F_0^{s(1)+1} SG \), which implies that \( \mathcal{E}_{Z_n}(f_1) \lesssim (\frac{3}{5})^n \) for \( Z_n = F_0^n F_1 SG \cup F_0^n F_2 SG, n \geq 0. \)
Extending the map $E_1^s$ to $T_{2,1/2}^s$ by the same definition, for any $\alpha \in T_{2,1/2}^s$, we have

$$E^{1/2}(E_1^s\alpha) = E^{1/2}\left(h_s + \sum_{j=0}^{\infty} (A_{s(1)}(\alpha))_j f_1 \circ F_0^{-j}\right)$$

$$\leq E^{1/2}(h_s) + E^{1/2}\left(\sum_{j=0}^{\infty} (A_{s(1)}(\alpha))_j f_1 \circ F_0^{-j}\right)$$

$$\lesssim \|\alpha\|_{T_{2,1/2}^s} + \left\|E^{1/2}\left(\sum_{j=0}^{n} (A_{s(1)}(\alpha))_j f_1 \circ F_0^{-j}\right)\right\|_{L^2}$$

$$= \|\alpha\|_{T_{2,1/2}^s} + \left\|E^{1/2}\left(\sum_{j=0}^{n} (A_{s(1)}(\alpha))_j f_1 \circ F_0^{-j}\right)\right\|_{L^2}$$

$$\lesssim \|\alpha\|_{T_{2,1/2}^s} + \left\|\sum_{j=0}^{n} \left(\frac{3}{5}\right)^{n/2-j} (A_{s(1)}(\alpha))_j \right\|_{L^2}$$

$$\lesssim \|\alpha\|_{T_{2,1/2}^s} + \left\|\sum_{j=0}^{n} \left(\frac{3}{5}\right)^{n/2-j} (A_{s(1)}(\alpha))_j \right\|_{L^2}$$

This shows that the map $E_1^s$ can be extended to be a bounded map from $T_{2,1/2}^s$ to $L^2(\mathcal{G})$. Obviously, $R_sE_1^s(\alpha) = \alpha, \forall \alpha \in T_{2,1/2}^s$.

Then by using interpolation, we have that the restriction map $R_s$ is surjective for $\frac{1}{2} \leq \sigma \leq 1$.

We should point out that this is not the end of the story. In [21] (Conjecture 3.14), Strichartz conjectured that for general $p > 0$, the Sobolev space $L^p(\mathcal{G})$ can be embedded into the Hölder-Zygmund space $\Lambda^{\frac{\log 5}{\log 5/3} - \frac{d}{p \log 5}} = \frac{\log 3}{\log 5/3}$. This will be verified for $p = 2$ in a sequential paper [4] of the authors, where we study the Besov type characterizations of $L^p_s(K)$ for general p.c.f. self-similar fractals $K$. See Theorem 6.10 and 7.7 in [4], and also Lemma 4.2 in [21]. Thus the restriction map $R_s$ is reasonable on $L^2_s(\mathcal{G})$ for $\sigma > \frac{\log 3}{2 \log 5}$. This matches the fact that $T_{2,\sigma}^s$ consists of sequences with a finite limit only if $\sigma > \frac{\log 3}{2 \log 5}$.

Nevertheless, according to Theorem 2.7 in [21], for a function $u$ in $\Lambda_t$ with $t > 1$, its normal derivatives exist at all vertex points in $\mathcal{G}$. So for functions in $L^p_s(\mathcal{G})$ with $\log 5/\log 3 - \frac{d}{2 \log 3/3} > 1$, the normal derivatives along $\{p_n\}_{n \geq 0}$ should always exist. This also matches the fact that $T_{2,\sigma}^s$ consists of sequences converging to 0 if $\sigma > 1 - \frac{\log 3}{2 \log 5}$.

The following is a final extension of the trace theorem.
Theorem 5.6. (a) For \( \sigma > \frac{\log 3}{2\log 5} \), the restriction map \( R_\sigma : L^2(\mathcal{S}G) \to \mathcal{T}^\sigma_{2,\sigma} \) is bounded and surjective.

(b) For \( \sigma > 1 - \frac{\log 3}{2\log 5} \), the restriction map \( R_\sigma : L^2(\mathcal{S}G) \to \mathcal{T}^\sigma_{2,\sigma} \) is bounded and surjective.

It suffices to prove this theorem for low orders. The proof relies on the atomic decomposition of \( L^2(\mathcal{S}G) \) (see Theorem 7.11 in [4]), which can be stated as follows.

Proposition 5.7. (a). For \( \frac{\log 3}{2\log 5} < \sigma < 1 - \frac{\log 3}{2\log 5} \), we have

\[
L^2_\sigma(\mathcal{S}G) = \{ u = h + \sum_{l=1}^{\infty} \sum_{x \in V_l \setminus V_{l-1}} c_x \varphi_x^{(l)} : \sum_{l=1}^{\infty} (5^{2\sigma})^{-1} \sum_{x \in V_l \setminus V_{l-1}} |c_x|^2 < \infty \},
\]

with estimation of the norm

\[
\|u\|_{L^2_\sigma(\mathcal{S}G)} \asymp (\|h\|_{L^2_\sigma(\mathcal{S}G)}^2 + \sum_{l=1}^{\infty} (5^{2\sigma})^{-1} \sum_{x \in V_l \setminus V_{l-1}} |c_x|^2)^{1/2},
\]

where \( h \in \mathcal{H}_0 \) and \( \varphi_x^{(l)} \) is a tent function, which takes value \( \varphi_x^{(l)}(y) = \delta_{x,y}, \forall y \in V_l \), and is harmonic in \( \mathcal{S}G \setminus V_l \).

(b) Let \( \tilde{K}_l = \{ \tilde{f} = \sum_{w \in V_l} c_w \chi_{F_w \mathcal{S}G} : c_w \in \mathbb{R}, \int_{F_w \mathcal{S}G} \tilde{f} d\mu = 0, \forall w' \in W_{l-1} \} \) be the space of level-\( l \) Haar functions. In other words, \( \tilde{K}_l \) consists of functions that are constants on each \( l \)-cell, and takes average 0 on each \( (l-1) \)-cell. For \( 1 - \frac{\log 3}{2\log 5} < \sigma < 1 + \frac{\log 3}{2\log 5} \), we have

\[
L^2_\sigma(\mathcal{S}G) = \{ u = h + Gc + \sum_{l=1}^{\infty} G \tilde{f}_l : \sum_{l=1}^{\infty} 5^{2l(\sigma-1)} \| \tilde{f}_l \|_{L^2(\mathcal{S}G)}^2 < \infty \},
\]

with estimation of the norm

\[
\|u\|_{L^2_\sigma(\mathcal{S}G)} \asymp (\|h\|_{L^2_\sigma(\mathcal{S}G)}^2 + |c|^2 + \sum_{l=1}^{\infty} 5^{2l(\sigma-1)} \| \tilde{f}_l \|_{L^2(\mathcal{S}G)}^2)^{1/2},
\]

where \( h \in \mathcal{H}_0, c \) is a constant, \( \tilde{f}_l \in \tilde{K}_l \) and \( G \) is the Green’s operator.

Proof of Theorem 5.6. (a). Due to Theorem 5.5, we only need to prove part (a) for \( \frac{\log 3}{2\log 5} < \sigma < \frac{1}{2} \). We will use the characterization of the space \( L^2_\sigma(\mathcal{S}G) \) in Proposition 5.7 (a). Note that for such \( \sigma, \mathcal{T}^\sigma_{2,\sigma} = l^2(5^{2\sigma-3/2}, \Lambda_1) \) since \( l_\sigma(5^{2\sigma-3/2}) = 1 \).

By a direct computation, for any \( u = h + \sum_{l=1}^{\infty} \sum_{x \in V_l \setminus V_{l-1}} c_x \varphi_x^{(l)} \in L^2_\sigma(\mathcal{S}G) \), we have

\[
u(p_n) = c_{p_n} + \frac{2}{5} \left( \frac{3}{5} \right)^n (h(q_1) + h(q_2) - \frac{4}{5} \left( \frac{3}{5} \right)^n h(q_0) + \sum_{l=1}^{n} \frac{2}{5} \left( \frac{3}{5} \right)^{n-l} (c_{F_{\sigma}q_0} + c_{F_{\sigma}q_2})).
\]
Let $\alpha_n = u(p_n)$ and $\alpha'_n = u(p_n) - u(q_0)$. Then clearly

$$
\| (5^3 - 1/2)^n \alpha'_n \|_{l^2} \lesssim \| (5^3 - 1/2)^n c_{p_n} \|_{l^2} + h(q_0) \| (5^3 - 1/2)^n (5^n/2) \|^2 \\
+ \| (5^3 - 1/2)^n \sum_{l=0}^{n} (5^{l-3/2})^{n-l} (c_{F_0(q_1)} + c_{F_0(q_1)}) \|^2 \\
\lesssim \| (5^3 - 1/2)^n c_{p_n} \|_{l^2} + |h(q_0)| \\
+ \| \sum_{l=0}^{n} (5^{l-3/2})^{n-l} (5^3 - 1/2)^n (c_{F_0(q_1)} + c_{F_0(q_1)}) \|^2 \\
\lesssim \| u \|_{L^2(SG)},
$$

where we write $c_{q_1} = h(q_1), c_{q_2} = h(q_2)$ for convenience, and use Minkowski inequality in the last but one inequality. By Theorem 5.2, the above estimate immediately shows that $R_s : L^2(SG) \to \mathcal{T}_{2,\sigma}^s$ is a bounded map.

To show the surjectivity, it suffices to construct a function $u \in L^2(SG)$ with $R_s u = \alpha$ for each $\alpha \in \mathcal{T}_{2,\sigma}^s$. Note that for $\alpha \in \mathcal{T}_{2,\sigma}^s$, we can always write $\alpha_n = A + \alpha'_n$ with $A \in \mathbb{R}$ and $\{5^n - 1/2 \alpha'_n\}_{n \geq 0} \in l^2$. Define

$$
u = A + \sum_{n=0}^{\infty} \alpha'_n \varphi^{(n+1)}_{p_n},$$

then clearly $u \in L^2(SG)$ by Proposition 5.7 (a) and $R_s u = \alpha$.

(b). To prove (b), we need following two claims.

Claim 1: For any $l \geq 1$, any $\tilde{f}_l \in \tilde{K}_l$ and $\varphi \in \text{dom} \mathcal{E}$, we have

$$
\left| \int_{SG} \varphi(x) \tilde{f}_l(x) d\mu(x) \right| \lesssim 5^{-l/2} \mathcal{E}^{1/2}(\varphi) \| \tilde{f}_l \|_{L^2(SG)}.
$$
Proof of Claim 1. For convenience, write $\tilde{f}_l = \sum_{w \in W_l} a_w \chi_{F_{w}SG}$, and denote $A_w(\varphi) = 3^{w|} \int_{F_{w}SG} \varphi d\mu$ for each $w \in W$. Then, we have the estimate

$$\left| \int_{SG} \varphi(x) \tilde{f}_l(x) d\mu(x) \right| = \left| \sum_{w \in W_{l-1}} \sum_{i=0}^{2} \int_{F_{w}SG} \varphi(x) \tilde{f}_i(x) d\mu(x) \right|$$

$$= 3^{-l} \left| \sum_{w \in W_{l-1}} \sum_{i=0}^{2} a_{wi} A_{wi}(\varphi) d\mu \right|$$

$$= 3^{-l} \left| \sum_{w \in W_{l-1}} \left( a_{w0}(A_{w0}(\varphi) - A_{w2}(\varphi)) + a_{w1}(A_{w1}(\varphi) - A_{w2}(\varphi)) \right) \right|$$

$$\leq 3^{-l} \left( \sum_{w \in W_{l-1}} (a_{w0}^2 + a_{w1}^2) \right)^{1/2}$$

$$\cdot \left\{ \sum_{w \in W_{l-1}} \left( (A_{w0}(\varphi) - A_{w2}(\varphi))^2 + (A_{w1}(\varphi) - A_{w2}(\varphi))^2 \right) \right\}^{1/2}$$

$$\lesssim 3^{-l} \cdot 3^{l/2} \|	ilde{f}_l\|_{L^2(SG)} \cdot (\frac{3}{5})^{l/2} \epsilon^{1/2}(\varphi) = 5^{-l/2} \epsilon^{1/2}(\varphi) \|	ilde{f}_l\|_{L^2(SG)},$$

where the third equality follows from $\sum_{i=0}^{2} a_{wi} = 0, \forall w \in W_{l-1}$. \hfill \Box

Claim 2: For any $N \in \mathbb{N}$ and $u = h + Gc + \sum_{l=1}^{N} G\tilde{f}_l$, we have

$$\left| \left( \Theta_1(Rau) \right)_n \right| \lesssim \sum_{l=n}^{N} \left( \frac{5}{3} \right)^{n/2} 5^{-l/2} \|	ilde{f}_l\|_{L^2(SG)}.$$

Proof of Claim 2. By using Lemma \ref{lemma3.5} in the case $k = 1$, we have

$$(\Theta_1(Rau))_n = -\sum_{l=n}^{N} \int_{F_0^n SG} \varphi_{a,1} \circ F_0^{-n}(x) \tilde{f}_l(x) d\mu(x),$$

where $\varphi_{a,1}$ is the same function in Lemma \ref{lemma3.5} which is antisymmetric. From the proof of Lemma \ref{lemma3.5}, $\varphi_{a,1}$ has finite energy on each 2-cell in $SG$. Then using Claim 1, by scaling, we have the desired result. \hfill \Box

By Theorem \ref{thm5.4} it is enough to prove (b) for $1 - \frac{\log 3}{2 \log 5} < \sigma < 1$. Note that for such $\sigma$, $T_{2,\sigma}^a = l^2(5^\sigma - 3^{1/2}, \Theta_0)$ since $l_a(5^\sigma - 3^{1/2}) = 0$. Then by Proposition \ref{prop3.10} $T_{2,\sigma}^a = l^2(5^\sigma - 3^{1/2}, \Theta_1).$
For any $u = h + Gc + \sum_{l=1}^{N} Gf_l$, where $h \in \mathcal{H}_0$, $c \in \mathbb{R}$, $N \in \mathbb{N}$ and $f_l \in \tilde{K}_l$. By using Claim 2, we have

$$
\| (5^{\sigma - 3/2})^n (\Theta_1(R_a u))_n \|_{L^2} \lesssim \| (5^{\sigma - 3/2})^n \sum_{l=n}^{N} \left( \frac{5}{3} \right)^{n/2} 5^{-l/2} \| \tilde{f}_l \|_{L^2(\mathcal{S}G)} \|_{L^2} \\
= \| 5^n(\sigma - 1/2) \sum_{l=n}^{N} 5^{-l/2} \| \tilde{f}_l \|_{L^2(\mathcal{S}G)} \|_{L^2} \\
= \| 5^n(\sigma - 1/2) \sum_{l=0}^{N-n} 5^{(l+n)(1/2-\sigma)} \| \tilde{f}_{l+n} \|_{L^2(\mathcal{S}G)} \|_{L^2} \\
\lesssim \| u \|_{L_3^2(\mathcal{S}G)},
$$

where the last inequality comes from Proposition 5.7(b).

In addition, since

$$
\partial_n^+ u(p_0) = \frac{5}{3} \left( 2u(p_0) - u(q_1) - u(F_0 q_1) \right) + \int_{F_1 \mathcal{S}G} h_2 \circ F_1^{-1}(x) \Delta u(x) d\mu(x),
$$

where $h_2$ is the harmonic function with boundary value $h_2(q_2) = 1$ and $h_2(q_0) = h_2(q_1) = 0$, by using Claim 1 again, we have the estimate that

$$
|\partial_n^+ u(p_0)| \lesssim \| u \|_{L^\infty(\mathcal{S}G)} + \sum_{l=1}^{N} 5^{-l/2} \| \tilde{f}_l \|_{L^2(\mathcal{S}G)} \lesssim \| u \|_{L_3^2(\mathcal{S}G)}.
$$

Combining the above estimates, we see that $\| R_a u \|_{T_2^a} \lesssim \| u \|_{L_3^2(\mathcal{S}G)}$ for any $u$ of the form $u = h + Gc + \sum_{l=1}^{N} Gf_l$. Since such functions are dense in $L_3^2(\mathcal{S}G)$ by Proposition 5.7 we have proved $R_a : L_3^2(\mathcal{S}G) \to T_2^a$ is bounded.

To show the surjectivity, we need to construct a function $u \in L_3^2(\mathcal{S}G)$ such that $R_a u = \eta$ for each $\eta \in T_2^a$. Let $f$ be a function in $L_3^2(\mathcal{S}G)$ satisfying that

\[
\begin{cases}
\text{supp} f \in F_1 \mathcal{S}G \cup F_2 \mathcal{S}G, \\
f(q_1) = f(q_2) = \partial_n f(q_1) = \partial_n f(q_2) = 0, \partial_n^+ f(p_0) = 1.
\end{cases}
\]

Then for $\eta \in T_2^a$, we will have

$$
E \eta := \sum_{n=0}^{\infty} \left( \frac{5}{3} \right)^n \eta_n f \circ F_0^{-n} \in L_3^2(\mathcal{S}G),
$$

and $R_a(E \eta) = \eta$. In fact, one can easily check that the map $E$ is bounded from $l^2(5^{-3/2}, \Theta_0)$ to $L^2(\mathcal{S}G)$ and from $l^2(3^{1/2}, \Theta_0)$ to $L_3^2(\mathcal{S}G)$, then by using interpolation, it is also bounded from $l^2(5^{\sigma - 3/2}, \Theta_0)$ to $L_3^2(\mathcal{S}G)$. $\square$
6. Question 1 and Question 2

In this section, we will answer Question 1 and Question 2.

Firstly, we will briefly characterize the trace spaces of \( L^2_{\sigma,D}(SG) \) and \( L^2_{\sigma,N}(SG) \). Let

\[
\{\kappa_j\}_{j \geq 0} = \{\kappa_j\}_{j \geq 0} \cap \{3 \cdot 5^{-j-1}\}_{j \geq 0}, \quad \{\kappa_j, N\}_{j \geq 0} = \{\kappa_j\}_{j \geq 0} \cap \{5^{-j}\}_{j \geq 0},
\]

arranged in decreasing order. For \( l \geq 0 \), let \( A_{l,D} \) and \( A_{l,N} \) be the difference operators on sequences defined in a similar manner as \( A_l \) introduced in Section 3, with \( \{\kappa_j\}_{j \geq 0} \) replaced by \( \{\kappa_j, D\}_{j \geq 0} \) and \( \{\kappa_j, N\}_{j \geq 0} \) respectively. For \( t > 0 \), write \( \ell_{s,D}(t) = \max\{l : t \geq \kappa_{l-1,D}\} \) and \( \ell_{s,N}(t) = \max\{l : t \geq \kappa_{l-1,N}\} \).

We then have the following theorem.

**Theorem 6.1.** For \( \sigma \geq 1 \), let \( T_{2,\sigma,D}^s = R_{s,L^2_{\sigma,D}(SG)} \) and \( T_{2,\sigma,N}^s = R_{s,L^2_{\sigma,N}(SG)} \).

Then we have

\[
\left\{ \begin{array}{l}
T_{2,\sigma,D}^s = l^2(5^\sigma 3^{-1/2}, A_{l,s,D}(5^\sigma 3^{-1/2}), D), \\
T_{2,\sigma,N}^s = l^2(5^\sigma 3^{-1/2}, A_{l,s,N}(5^\sigma 3^{-1/2}), N).
\end{array} \right.
\]

The proof of Theorem 6.1 is very similar to that of Theorem 5.4. We need the following lemmas, which are analogs of Lemma 3.2* and Lemma 3.4.

**Lemma 6.2.** (a). Let

\[ \mathcal{H}_{k-1,D}(0) = \{ h \in \mathcal{H}_{k-1} : \Delta^j h(q_0) = 0, \forall 0 \leq j < k \}. \]

Then

\[ R_{s,\mathcal{H}_{k-1,D}(0)} = \{ \alpha : \alpha_n = \sum_{i=0}^{l_{s,D}(5^k 3^{-1/2})-1} A_i \kappa_{i,D}^n, A_i \in \mathbb{R} \} \cap \ker A_{l,s,D}(5^k 3^{-1/2}), D. \]

(b). Let

\[ \mathcal{H}_{k-1,N}(0) = \{ h \in \mathcal{H}_{k-1} : \partial h \Delta^j h(q_0) = 0, \forall 0 \leq j < k \}. \]

Then

\[ R_{s,\mathcal{H}_{k-1,N}(0)} = \{ \alpha : \alpha_n = \sum_{i=0}^{l_{s,N}(5^k 3^{-1/2})-1} A_i \kappa_{i,N}^n, A_i \in \mathbb{R} \} \cap \ker A_{l,s,N}(5^k 3^{-1/2}), N. \]

**Proof.** It is enough to see that

\[ \mathcal{H}_{k-1,D}(0) = \text{span}\{ h_{1,j}, h_{2,j} \}_{j=0}^{k-1}, \]

and

\[ \mathcal{H}_{k-1,N}(0) = \text{span}\{ h_{0,j}, h_{2,j} \}_{j=0}^{k-1}, \]

where \( h_{i,j} \) are monomials given in Definition 3.1. The restriction of each \( h_{i,j} \) onto \( \{p_n\}_{n \geq 0} \) is a geometric sequence, where the restriction of \( h_{1,j} \) has the decreasing ratio \( 3 \cdot 5^{-j} \), and the restriction of \( h_{0,j} \) has the decreasing ratio \( 5^{-j} \), while \( h_{2,j} \) does not contribute to the trace space. \( \square \)
Lemma 6.3. Let $k \geq 1$, $l \geq 0$. Let $\{x_i\}_{i=0}^l$ be $l+1$ distinct points in $SG \setminus V_0$, and $\{a_i\}_{i=0}^l$ be $l+1$ real numbers.

(a) Assume $\sum_{i=0}^l a_i h(x_i) = 0$ holds for any $h \in H_{k-1,D(0)}$. Then there is a function $\phi_{k,D} \in C(SG)$ such that

$$\sum_{i=0}^l a_i f(x_i) = \int_{SG} \phi_{k,D}(x) \Delta^k f(x) d\mu(x),$$

for any $f \in L^2_k(SG)$ with boundary condition $\Delta^j f(q_0) = 0$, $0 \leq j < k$.

(b) Assume $\sum_{i=0}^l a_i h(x_i) = 0$ holds for any $h \in H_{k-1,N(0)}$. Then there is a function $\phi_{k,N} \in C(SG)$ such that

$$\sum_{i=0}^l a_i f(x_i) = \int_{SG} \phi_{k,N}(x) \Delta^k f(x) d\mu(x),$$

for any $f \in L^2_k(SG)$ with boundary condition $\partial_n \Delta^j f(q_0) = 0$, $0 \leq j < k$.

Proof. (a). First, there is a unique function $\phi^{(k-1)} \in dom_0 \mathcal{E}$ such that

$$\mathcal{E}(f, \phi^{(k-1)}) = - \sum_{i=0}^l a_i f(x_i),$$

holds for any function $f \in dom_0 \mathcal{E}$, since the right side is a bounded linear functional on $dom_0 \mathcal{E}$. By the assumption, we then have,

$$\mathcal{E}(f, \phi^{(k-1)}) = - \sum_{i=0}^l a_i f(x_i), \forall f \in dom \mathcal{E} \text{ with } f(q_0) = 0.$$

Next, we inductively define a sequence of functions $\phi^{(k-2)}, \phi^{(k-3)}, \ldots, \phi^{(0)} = \phi_{k,D}$ by

$$
\begin{align*}
\Delta \phi^{(k-j)} &= \phi^{(k-j+1)}, \\
\phi^{(k-j)}(0) &= 0, \quad \partial_n \phi^{(k-j)}(q_1) = \partial_n \phi^{(k-j)}(q_2) = 0, \quad \forall 2 \leq j \leq k.
\end{align*}
$$

Then for any $f \in L^2_k(SG)$ with boundary condition

$$\Delta^j f(q_0) = 0, \quad \partial_n \Delta^j f(q_1) = \partial_n \Delta^j f(q_2) = 0, \quad \forall 0 \leq j < k,$$

by repeatedly using the Gauss-Green’s formula, we have

$$\sum_{i=0}^l a_i f(x_i) = -\mathcal{E}(f, \phi^{(k-1)}) = \int_{SG} \phi^{(k-1)}(x) \Delta f(x) d\mu(x)$$

$$= \int_{SG} \phi^{(k-2)}(x) \Delta^2 f(x) d\mu(x) = \cdots = \int_{SG} \phi_{k,D}(x) \Delta^k f(x) d\mu(x),$$

where we use the boundary condition of $\phi^{(k-j)}$ and $\Delta^j f$ in each step.
Noticing that the equality
\[ \sum_{i=0}^{t} a_i h(x_i) = \int_{S^0} \phi_{k,D}(x) \Delta^k h(x) \mu(dx) = 0 \]
also holds for any \( h \in \mathcal{H}_{k-1,D}(0) \), we get (a) proved.

(b). There is a unique \( \tilde{\phi}^{(k-1)} \in \text{dom} \mathcal{E} \) with boundary condition \( \tilde{\phi}^{(k-1)}(q_1) = \tilde{\phi}^{(k-1)}(q_2) = 0 \) such that
\[ \mathcal{E}(f, \tilde{\phi}^{(k-1)}) = -\sum_{i=0}^{t} a_i f(x_i), \forall f \in \text{dom} \mathcal{E} \text{ satisfying } f(q_1) = f(q_2) = 0. \]
Then clearly \( \mathcal{E}(f, \tilde{\phi}^{(k-1)}) = -\sum_{i=0}^{t} a_i f(x_i), \forall f \in \text{dom} \mathcal{E} \) by the assumption. The rest of the proof is essentially the same as (a). \( \square \)

**Proof of Theorem 6.1.** We only look at the Dirichlet case since the Neumann case is the same. Let \( k \geq 1 \). For any \( h \in \mathcal{H}_{k-1,D}(0) \), by Lemma 6.2 we have that \( A_{l,s,D}(5^{k-3/2},D)(Rsh) = \{0\}_{n \geq 0} \). Then using Lemma 6.3 and by scaling, we have
\[
(A_{l,s,D}(5^{k-3/2},D)(Rsf))_n = 5^{-kn}3^n \int_{F_0^n(SG)} \Delta^k f(x) \phi_{l,D} \circ F_0^{-n}(x) d\mu(x), \forall n \geq 0,
\]
for any \( f \in L_k^2(SG) \) with boundary condition \( \Delta^j f(q_0) = 0, 0 \leq j < k \). The rest of the proof is essentially the same as that of Theorem 5.4, noticing that \( R_s L_k^2(SG) = R_s \{f \in L_k^2(SG) : \Delta^j f(q_0) = 0, 0 \leq j < k \} \). \( \square \)

To have a clear view of these trace spaces, we show the trace spaces for \( 1 \leq \sigma \leq 2 \) with a concrete description,

\[
\mathcal{T}_{2,\sigma,D}^s = \begin{cases} 
\{ \alpha : \alpha_n = A_1(\frac{3}{5})^n + \alpha'_n \text{ with } A_1 \in \mathbb{R}, \{5^{\sigma n}3^{-n/2}\alpha'_n\} \in l^2 \}, \\
\{ \alpha : \alpha_n = A_1(\frac{3}{5})^n + \alpha'_n \text{ with } A_1 \in \mathbb{R}, \{(\frac{25}{3})^n(\alpha_{n+1} - \frac{3}{25}\alpha'_n)\} \in l^2 \}, & \text{if } 1 \leq \sigma < 2 - \frac{\log 3}{2\log 5}, \\
\{ \alpha : \alpha_n = A_1(\frac{3}{5})^n + A_2(\frac{3}{25})^n + \alpha'_n \text{ with } A_1, A_2 \in \mathbb{R}, \{5^{\sigma n}3^{-n/2}\alpha'_n\} \in l^2 \}, & \text{if } 2 - \frac{\log 3}{2\log 5} < \sigma \leq 2,
\end{cases}
\]

and

\[
\mathcal{T}_{2,\sigma,N}^s = \begin{cases} 
\{ \alpha : \alpha_n = A_1 + \alpha'_n \text{ with } A_1 \in \mathbb{R}, \{5^{\sigma n}3^{-n/2}\alpha'_n\} \in l^2 \}, & \text{if } 1 \leq \sigma < 1 + \frac{\log 3}{2\log 5}, \\
\{ \alpha : \alpha_n = A_1 + \alpha'_n \text{ with } A_1 \in \mathbb{R}, \{5^n(\alpha_{n+1} - \frac{1}{5}\alpha'_n)\} \in l^2 \}, & \text{if } \sigma = 1 + \frac{\log 3}{2\log 5}, \\
\{ \alpha : \alpha_n = A_1 + A_2(\frac{1}{5})^n + \alpha'_n \text{ with } A_1, A_2 \in \mathbb{R}, \{5^{\sigma n}3^{-n/2}\alpha'_n\} \in l^2 \}, & \text{if } 1 + \frac{\log 3}{2\log 5} < \sigma \leq 2.
\end{cases}
\]
Comparing with $\mathcal{T}^s_{2,\sigma}$, we can see that $\dim(\mathcal{T}^s_{2,\sigma}/\mathcal{T}^s_{2,\sigma,D}) = \infty$ at $\sigma = 1 + \frac{\log 3}{2\log 5}$, and also $\dim(\mathcal{T}^s_{2,\sigma}/\mathcal{T}^s_{2,\sigma,N}) = \infty$ at $\sigma = 2 - \frac{\log 3}{2\log 5}$. Basing on this observation, we have the following theorem, which answers Question 2.

**Theorem 6.4.** (a). If $k - 1 + \frac{\log 3}{2\log 5} < \sigma < k + \frac{\log 3}{2\log 5}$ for some $k \geq 0$, then $L^2_{\sigma,D}(SG)$ is closed in $L^2_\sigma(SG)$. In addition, $L^2_\sigma(SG) = L^2_{\sigma,D}(SG) \oplus H_{k-1}$, where we set $H_{-1} = \{0\}$ for uniformity;

If $\sigma = k + \frac{\log 3}{2\log 5}$ for some $k \geq 0$, then $L^2_{\sigma,D}(SG)$ is not closed in $L^2_\sigma(SG)$, and $\text{cl}(L^2_{\sigma,D}(SG))/L^2_{\sigma,D}(SG)$ is of infinite dimension. In addition, $L^2_\sigma(SG) = \text{cl}(L^2_{\sigma,D}(SG)) \oplus H_{k-1}$.

(b). If $k - \frac{\log 3}{2\log 5} < \sigma < k + 1 - \frac{\log 3}{2\log 5}$ for some $k \geq 0$, then $L^2_{\sigma,N}(SG)$ is closed in $L^2_\sigma(SG)$. In addition, $L^2_\sigma(SG) = L^2_{\sigma,N}(SG) \oplus H_{k-1}$, where $H_{k-1}$ is defined in Proposition 1.2.

If $\sigma = k + 1 - \frac{\log 3}{2\log 5}$, $k \geq 0$, then $L^2_{\sigma,N}(SG)$ is not closed in $L^2_\sigma(SG)$, and $\text{cl}(L^2_{\sigma,N}(SG))/L^2_{\sigma,N}(SG)$ is of infinite dimension. In addition, $L^2_\sigma(SG) = \text{cl}(L^2_{\sigma,N}(SG)) \oplus H_{k-1}$.

**Remark.** Theorem 6.4 is an analog to the classical theorem concerning the relationship between Sobolev spaces $H^s(\Omega), H^s_0(\Omega)$ and $H^s_{00}(\Omega)$, for a domain $\Omega$ in $\mathbb{R}^n$. Readers please refer to Chapter 1, Section 11 in the book [16] by J.L. Lions and E. Magenes.

**Proof of Theorem 6.4.** The key idea is to split each Sobolev space into two parts, the kernel of the restriction map, and the image of the extension map, which is isomorphic to the trace space. Here we consider the restrictions onto the middle lines of the three 1-cells, as shown in Figure 3.

![Figure 3. The middle lines of the three 1-cells.](image)

For a function $f$ on $SG$, we define the restriction of $f$ onto the three middle lines to be

$$\tilde{R}sf = (\{f(p_{n+1})\}_{n \geq 0}, \{f(F^{n+1}_1F^*_0q_2)\}_{n \geq 0}, \{f(F^{n+1}_2F^*_1q_0)\}_{n \geq 0}).$$

Then clearly by the previous results, for $\sigma \geq 1$, we have the trace space $\tilde{T}^s_{2,\sigma} := \tilde{R}_sL^2_\sigma(SG) = (\mathcal{T}^s_{2,\sigma})^3$, and similarly $\tilde{T}^s_{2,\sigma,D} := \tilde{R}_sL^2_{\sigma,D}(SG) = (\mathcal{T}^s_{2,\sigma,D})^3$. 

and $\tilde{T}_{2,\sigma,N} := \tilde{R}_sL_{2,\sigma,N}(SG) = (T_{2,\sigma,N})^3$, where for a normed space $L$, we use $L^3$ to denote $L \times L \times L$.

Let $\tilde{E}_2 : \tilde{T}_{2,\sigma}^s \to L_2^2(SG)$ be the extension map induced by the three extension maps $E_2^s$ associated to each of the three middle lines with suitable rotations. By Proposition 4.9, the map $\tilde{E}_2^s$ can naturally be extended to a bounded map, still denoted by $\tilde{E}_2$, from $\tilde{T}_{2,1}^s$ to $L_1^2(SG)$, and $\tilde{R}_s\tilde{E}_2^s = id$.

We only prove part (a) of the theorem since part (b) is the same. First, we will prove (a) when $1 \leq \sigma \leq 2$. We have the following claims.

Claim 1: Let $f \in L_1^2(SG), k = 1, 2$. Then $f \in L_{k,D}^2(SG)$ if and only if $\tilde{R}_sf \in \tilde{T}_{2,k,D}^s$.

Claim 2 can be shown by the codimension counting. Notice that for $k = 1, 2$, the codimension of $\tilde{T}_{2,k,D}$ as a subspace of $\tilde{T}_{2,k}^s$ is $3k$, while the codimension of $L_{k,D}^2(SG)$ as a subspace of $L_k^2(SG)$ is also $3k$.

Claim 2: For $1 \leq \sigma \leq 2$, $\tilde{E}_2^s : \tilde{T}_{2,\sigma,D}^s \to L_{2,D}^2(SG)$ is bounded.

Claim 2 is true for $\sigma = 1, 2$ by Claim 1, and for $1 < \sigma < 2$ by using complex interpolation.

For $1 \leq \sigma \leq 2$, define $\ker_{\sigma} \tilde{R}_s = \{ f \in L_2^2(SG) : \tilde{R}_sf = 0 \}$.

Claim 3: For $1 \leq \sigma \leq 2$, $L_2^2(SG) = \tilde{E}_2^s\tilde{T}_{2,\sigma}^s + \ker_{\sigma} \tilde{R}_s$, and $L_{2,D}^2(SG) = \tilde{E}_2^s\tilde{T}_{2,\sigma,D}^s + \ker_{\sigma} \tilde{R}_s$.

Proof of Claim 3. Define $P = 1 - \tilde{E}_2^s\tilde{R}_s$. Obviously $\tilde{R}_sP = \tilde{R}_s(1 - \tilde{E}_2^s\tilde{R}_s) = 0$, and for any $f \in \ker_{\sigma} \tilde{R}_s$, $Pf = (1 - \tilde{E}_2^s\tilde{R}_s)f = f$. So

$$\ker_{\sigma} \tilde{R}_s = P\ker_{\sigma} \tilde{R}_s \subset PL_{2,D}^2(SG) \subset \ker_{\sigma} \tilde{R}_s,$$

which gives that $PL_{2,D}^2(SG) = \ker_{\sigma} \tilde{R}_s$. In addition, by complex interpolation for $1 \leq \sigma \leq 2$, using Claim 1, we have $P : L_2^2(SG) \to L_{2,D}^2(SG)$ is bounded. Thus $\ker_{\sigma} \tilde{R}_s = \{ f \in L_{2,D}^2(SG) : \tilde{R}_sf = 0 \}$. The desired result follows by Claim 2 and the fact that $\tilde{R}_sL_{2,D}^2(SG) = \tilde{T}_{2,\sigma,D}^s$.

By Claim 3, we see that if $\sigma = 1 + \frac{\log 3}{2\log 5}$, $L_{2,D}^2(SG)$ is not closed in $L_2^2(SG)$, and the dimension of $\text{cl}(L_{2,D}^2(SG)) / L_{2,D}^2(SG)$ is infinity; while if $\sigma \neq 1 + \frac{\log 3}{2\log 5}$, $L_{2,D}^2(SG)$ is closed in $L_2^2(SG)$. In addition, we have

Claim 4: Let $f \in L_2^2(SG), 1 \leq \sigma \leq 2$. Then $f \in L_{2,D}^2(SG)$ if and only if $\tilde{R}_sf \in \tilde{T}_{2,D}^s$.

Using Claim 4, we can see that $H_0 \cap L_{2,D}^2(SG) = \{ 0 \}$ for $1 \leq \sigma < 1 + \frac{\log 3}{2\log 5}$, $H_0 \cap \text{cl}(L_{2,D}^2(SG)) = \{ 0 \}$ for $\sigma = 1 + \frac{\log 3}{2\log 5}$ and $H_1 \cap L_{2,D}^2(SG) = \{ 0 \}$ for $1 + \frac{\log 3}{2\log 5} < \sigma \leq 2$. The result for $1 \leq \sigma \leq 2$ follows from counting the co-dimension.

For $0 \leq \sigma \leq 1$, we only need to notice that $L_{\sigma}^2(SG) = \Delta L_{\sigma+1}^2(SG)$ and the kernel of $\Delta$ is $H_0$.

For $1 + k \leq \sigma \leq 2 + k$, $k \geq 0$, we only need to notice that $L_{\sigma}^2(SG) = H_{k-1} \oplus G^k(L_{\sigma-k}^2(SG))$, where $G$ is the Green’s operator, while $L_{\sigma,D}^2(SG) =$
Corollary 6.5. $L^2_{1/2}(SG) = \text{dom}\mathcal{E}$.

Proof. It is clear that $\text{dom}\mathcal{E} = L^2_{1/2,N}(SG)$. By Theorem 6.4, we can easily check that $L^2_{1/2}(SG) = L^2_{1/2,N}(SG)$ since $0 < \frac{1}{2} < 1 - \frac{\log 3}{2\log 5}$.

Before answering Question 1, we will state the following observation.

Lemma 6.6. $\kappa_{j,N} = (\frac{1}{2})^{j-1}, \forall j \geq 0$.

Proof. We only need to show that $h_{0,j}(p_0) \neq 0, j \geq 0$. Obviously, $h_{0,0}(p_0) = 1$. From Example 7.1 in [3], we have

$$h_{0,j}(q_1) = \frac{4}{5^j-5} \sum_{i=1}^{j-1} h_{0,i}(q_1) h_{0,j-i}(q_1), \quad \forall j \geq 2,$$

$$h_{0,j}(p_0) + (1 + \frac{1}{5^j}) h_{0,j}(q_1) = \frac{4}{5^j} \sum_{i=0}^{j} h_{0,i}(q_1) h_{0,j-i}(q_1), \quad \forall j \geq 1.$$

Using this, we can find that

$$h_{0,j}(p_0) = \frac{2}{5^j} h_{0,j}(q_1), \forall j \geq 1.$$

On the other hand, by Theorem 2.7 in [17], we know that $h_{0,j}(q_1) > 0, \forall j \geq 1$. This gives the desired result.

Theorem 6.7. For $\sigma \geq 0$, the Sobolev spaces $L^2_{\sigma}(SG)$ are stable under complex interpolation.

Proof. It suffices to show that $[L^2_{k_1}(SG), L^2_{k_2}(SG)]_{\theta} = L^2_{k_2}(SG)$ with $\theta = \frac{k_2-k_1}{k_2-k_1}$ for any integers $k_1 \leq k_2 \leq k_3$.

First, we assume $k_1 \geq 1$.

It is obvious that

$$[L^2_{k_1}(SG), L^2_{k_2}(SG)]_{\theta} = [L^2_{k_1,D}(SG) \oplus \mathcal{H}_{k_1-1} \oplus L^2_{k_2,D}(SG) \oplus \mathcal{H}_{k_3-1}]_{\theta}$$

$$= [L^2_{k_1,D}(SG) + \mathcal{H}_{k_3-1}, L^2_{k_2,D}(SG) + \mathcal{H}_{k_3-1}]_{\theta}$$

$$\supset L^2_{k_2,D}(SG) + \mathcal{H}_{k_3-1} = L^2_{k_2}(SG).$$

It remains to prove the other direction. Let $P = 1 - \bar{E}_{k_3}^s \bar{R}_s$. By a same proof of Claim 3 in the proof of Theorem 6.4, for any $\sigma \geq 1$, we have $PL^2_{\sigma}(SG) = \ker_{\sigma} \bar{R}_s \subset L^2_{\sigma,D}(SG)$. As a consequence, we have

$$P([L^2_{k_1}(SG), L^2_{k_2}(SG)]_{\theta}) \subset [PL^2_{k_1}(SG), PL^2_{k_2}(SG)]_{\theta}$$

$$= [\ker_{k_1} \bar{R}_s, \ker_{k_3} \bar{R}_s] \subset L^2_{k_2,D}(SG) \subset L^2_{k_2}(SG).$$

In addition,

$$(1 - P)([L^2_{k_1}(SG), L^2_{k_2}(SG)]_{\theta}) \subset [\bar{E}_{k_3}^2 \bar{T}_{k_1}^s, \bar{E}_{k_3}^2 \bar{T}_{k_2}^s]_{\theta}$$

$$= \bar{E}_{k_3}^2 \bar{T}_{k,1}^s \subset \bar{E}_{k_3}^2 \bar{T}_{k,2}^s \subset L^2_{k_2}(SG),$$

where the last equality is due to the injectivity of $\bar{E}_{k_3}^2$. Thus the other direction follows.
Next for \( k_1 = 0 \), we claim that

\[
[L^2_{k_1}(\mathcal{G}), L^2_{k_3}(\mathcal{G})]_\theta = \delta L^2_{k_1+1}(\mathcal{G}), \Delta L^2_{k_3+1}(\mathcal{G})]_\theta
\]

\[
= \delta ([L^2_{k_1+1}(\mathcal{G}), L^2_{k_3+1}(\mathcal{G})]_\theta)
\]

\[
= \delta L^2_{k_2+1}(\mathcal{G}) = L^2_{k_2}(\mathcal{G}),
\]

where the second equality holds since we have the Green’s operator \( G \) such that \(-\Delta \circ G = \text{id}\).

\( \square \)

7. The \( \mathcal{S} \mathcal{G} \) with 9-map i.f.s.

In this section, we will consider a family of non-uniform \( D_3 \)-symmetric self-similar Laplacians on \( \mathcal{S} \mathcal{G} \) and investigate the corresponding trace problem.

In [5], a two-parameter family of Laplacians on \( \mathcal{S} \mathcal{G} \), which are symmetric and self-similar for a 9-map i.f.s. were introduced. By iterating the i.f.s. \( \{F_i\}_{i=0}^2 \) twice, we get a new i.f.s. consisting of 9 maps, \( \{F_{ij}\}_{i,j=0,1,2} \), with \( F_{ij} = F_i \circ F_j \). The Sierpinski gasket \( \mathcal{S} \mathcal{G} \) can still be generated as the attractor of this new i.f.s.

This new i.f.s. allows us to define a non-uniform, \( D_3 \)-symmetric self-similar measure on \( \mathcal{S} \mathcal{G} \). To be specific, we require the measure \( \mu \) so that \( \mu(F_i \mathcal{S} \mathcal{G}) = \mu(F_j \mathcal{S} \mathcal{G}) \) for all \( 0 \leq i, j \leq 2 \), and \( \mu(F_{i\theta} \mathcal{S} \mathcal{G}) = \mu(F_{j\theta} \mathcal{S} \mathcal{G}) \) for all \( i \neq k, j \neq l \) with \( i, k, j, l \in \{0, 1, 2\} \). Without loss of generality, set \( \mu(\mathcal{S} \mathcal{G}) = 1 \), and for \( m \geq 0 \), \( w \in W_{2m} \), write \( \mu_w = \mu(F_w \mathcal{S} \mathcal{G}) \). Then \( \mu_w = \mu_{w_1w_2w_3w_4} \cdots \mu_{w_{2m-1}w_{2m}} \), with

\[
\mu_{00} = \mu_{11} = \mu_{22} \text{ and } \mu_{01} = \mu_{10} = \mu_{20} = \mu_{02} = \mu_{12} = \mu_{21},
\]

satisfying

\[
3\mu_{00} + 6\mu_{01} = 1. \tag{7.5}
\]

There is only one free parameter \( 0 < \mu_{00} < 1 \) for the measure \( \mu \).

In addition to the measure \( \mu \), we need a \( D_3 \)-symmetric self-similar resistance form \( (\mathcal{E}, \text{dom}\mathcal{E}) \) on \( \mathcal{S} \mathcal{G} \). To maintain the \( D_3 \)-symmetry, we need the resistances of edges of the outer 1-cells to be equal and the same for the inner 1-cells. Denote them by \( r_{00}, r_{01} \) respectively. Please see Figure 4 for the resistance form on the level-1 graph of \( \mathcal{S} \mathcal{G} \) with 9-map i.f.s., and its restriction to the level-0 graph. Here \( r_{00} = \frac{6r(2+r)}{15+26r+9r^2} \), \( r_{01} = \frac{6(2+r)}{15+26r+9r^2} \), and \( 0 < r < \infty \) is a free parameter.

For \( m \geq 0 \), \( w \in W_{2m} \), write \( r_w = r_{w_1w_2}r_{w_3w_4} \cdots r_{w_{2m-1}w_{2m}} \), with \( r_{ii} = r_{00} \) for \( i = 1, 2 \), and \( r_{ij} = r_{01} \) for all \( i \neq j \in \{0, 1, 2\} \). Analogous to the 3-map i.f.s. case, the resistance form \( (\mathcal{E}, \text{dom}\mathcal{E}) \) is defined by

\[
\mathcal{E}(u) = \lim_{m \to \infty} \sum_{w \in W_{2m}} r_w^{-1} \sum_{i \neq j} (u(F_w q_i) - u(F_w q_j))^2
\]

and \( \text{dom}\mathcal{E} = \{ u : \mathcal{E}(u) < \infty \} \). We will also need the associated bilinear form

\[
\mathcal{E}(u, v) = \frac{1}{4} (\mathcal{E}(u+v) - \mathcal{E}(u-v))
\]
for \( u, v \in \text{dom} E \).

Using the measure \( \mu \) and resistance form \((E, \text{dom} E)\) described above, we then could define a two-parameter \( D_3 \)-symmetric Laplacian by the weak formulation. Say \( u \in \text{dom} \Delta \) with \( \Delta u = f \) if

\[
\mathcal{E}(u, v) = - \int_{SG} fv d\mu
\]

holds for any \( v \in \text{dom}_0 E = \{v : v \in \text{dom} E, v|_{V_0} = 0\} \).

The family of two-parameter \( D_3 \)-symmetric Laplacians defined in the above way extends the standard Laplacian we used in the previous sections. The recipe for dealing with the trace problem of Sobolev spaces associated with the standard Laplacian is still valid in this generalized setting. The results for \( L^2_\sigma \) type spaces can be generalized to \( L^p_\sigma \) type spaces for \( 1 \leq p \leq \infty \) when \( \sigma \geq 1 \) without difficulty. To avoid repetition, we only present the trace space of \( L^p_1(SG) \) associated with a generalized Laplacian and leave the full description to the readers.

For \( 1 \leq p \leq \infty \), say \( u \in L^p_1(SG) \) if \( u \) is continuous and \( \Delta u \in L^p(SG) \), where \( \Delta \) is a two-parameter generalized Laplacian. Define the norms on \( L^p_1(SG) \) by

\[
\|u\|_{L^p_1(SG)} = \|u\|_{L^p(SG)} + \|\Delta u\|_{L^p(SG)}.
\]

For a function \( u \in L^p_1(SG) \), we still denote \( R_s u = \{u(p_n)\}_{n \geq 0} \), \( R_a u = \{\partial^\alpha u(p_n)\}_{n \geq 0} \), and \( Ru = (R_s u, R_a u) \).

First, let’s look at the trace of space of harmonic functions, still denote by \( \mathcal{H}_0 \). By direct computation, we have

**Proposition 7.1.** (a) \( R_s \mathcal{H}_0 = \{\alpha = \{\alpha_n\}_{n \geq 0} : \alpha_n = A_1 + A_2 s_n r_0^{n/2}, A_1, A_2 \in \mathbb{R}\} \), where

\[
s_n = \begin{cases} 
\frac{17 + 20r + 6r^2}{15 + 25r + 9r^2}, & \text{if } n \text{ is even,} \\
\frac{4 + 14r + 6r^2}{15 + 25r + 9r^2} \cdot r_0^{n-1/2}, & \text{if } n \text{ is odd,}
\end{cases}
\]

where \( r_0 = \frac{6r(2+r)}{15 + 25r + 9r^2} \).
(b). $R_a \mathcal{H}_0 = \{ \eta = \{ \eta_n \}_{n \geq 0} : \eta_n = B a_n (\frac{1}{1 + 3r})^{n/2}, B \in \mathbb{R} \}$, where

$$a_n = \begin{cases} 2, & \text{if } n \text{ is even,} \\ \frac{1 + r}{2 + r} (6 + 3r)^{1/2}, & \text{if } n \text{ is odd.} \end{cases}$$

(c). $R \mathcal{H}_0 = R_a \mathcal{H}_0 \times R_a \mathcal{H}_0$.

Next, by following the similar argument in Section 4, noticing the remark below Proposition 4.1 and using Proposition 7.1, we get the following trace results.

For convenience, for $t > 0$, we write $l^2(t) := \{ \alpha : \{ t^n \alpha_n \}_{n \geq 0} \in l^2 \}$.

**Theorem 7.2.** (a). For $1 < p \leq \infty$, we have

$$\mathcal{R}_a \mathcal{L}_1^p(\mathcal{S}G) = \mathcal{R}_a \mathcal{H}_0 \oplus l^2(\mu_{00}^{1-p});$$

for $p = 1$, we have

$$\mathcal{R}_a \mathcal{L}_1^p(\mathcal{S}G) = \{ \alpha : \alpha_n = A_1 + \alpha'_n, A_1 \in \mathbb{R}, \{ r_{00}^{-1/2} s_{n+1}^{-1} \alpha'_n - s_n^{-1} \alpha_n \} \in l^2(r_{00}^{-1/2}) \}.$$

(b). For $\mu_{00}^{1-p} > (6 + 3r)^{1/2}$, we have

$$\mathcal{R}_a \mathcal{L}_1^p(\mathcal{S}G) = \mathcal{R}_a \mathcal{H}_0 \oplus l^2(\mu_{00}^{1-p});$$

for $\mu_{00}^{1-p} = (6 + 3r)^{1/2}$, we have

$$\mathcal{R}_a \mathcal{L}_1^p(\mathcal{S}G) = \{ \eta : \{ (6 + 3r)^{1/2} a_{n+1}^{-1} \eta_{n+1} - a_n^{-1} \eta_n \}_{n \geq 0} \in l^2(\mu_{00}^{1-p}) \};$$

for $\mu_{00}^{1-p} < (6 + 3r)^{1/2}$, we have

$$\mathcal{R}_a \mathcal{L}_1^p(\mathcal{S}G) = l^2(\mu_{00}^{1-p}).$$

(c). $\mathcal{R} \mathcal{L}_1^p(\mathcal{S}G) = \mathcal{R}_a \mathcal{L}_1^p(\mathcal{S}G) \times \mathcal{R}_a \mathcal{L}_1^p(\mathcal{S}G)$.

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