DYNAMICAL GENERATION OF THE MASS GAP IN QCD

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We have unambiguously established the dynamical source of the mass scale parameter (the mass gap) responsible for the large scale structure of the true QCD vacuum. At the microscopic, Lagrangian level it is the nonlinear fundamental four-gluon interaction. At the level of the corresponding equation of motion for the full gluon propagator, it is all the skeleton loop contributions into the gluon self-energy, which contain the four-gluon vertices. The key role of the four-gluon interaction is determined by the fact that this interaction survives when all the gluon momenta involved go to zero, while the three-gluon vertex vanishes in this limit. The mass gap and the corresponding infrared singularities are "hidden" in these terms, and they show up explicitly when the gluon momentum $q$ goes to zero. The general iteration solution (i.e., when the relevant skeleton loop integrals have to be iterated) for the full gluon propagator unavoidably becomes the exact sum of the two terms. The first term is the Laurent expansion in the inverse powers of the gluon momentum squared, starting necessarily from the simplest one $1/(q^2)^2$. Each severe (i.e., more singular than $1/q^2$) power-type IR singularity is accompanied by the corresponding powers of the mass gap. The standard second term is always as much singular as $1/q^2$, otherwise remaining undetermined. The inevitable existence of the first term makes just the principal difference between non-Abelian QCD and Abelian QED. Moreover, the infrared renormalization program of the theory leads to the gluon confinement criterion in the gauge-invariant way.

I. INTRODUCTION

Quantum Chromodynamics (QCD) \[1,2\] is widely accepted as a realistic, quantum gauge field theory of strong interactions not only at the fundamental (microscopic) quark-gluon level but at the hadronic (macroscopic) level as well. However, to fulfill this role it should explain why colored gluons and quarks can not be experimentally detected, while all hadrons are the color-singlet states (color confinement problem). There are also two surprising facts about QCD, being in close relation with each other and color confinement. The first fact is that we still don’t know exactly the interaction between quarks and gluons. The second fact is that QCD Lagrangian does not explicitly contain the mass scale parameter (in what follows the mass gap, for simplicity), which is necessary in order to calculate from first principles such truly nonperturbative (NP) physical observables as decay constants, masses of particles, etc. The reason of these important problems is, of course, the complicated quantum-dynamical and topological structure of the QCD ground state. $\Lambda_{QCD}$ is responsible for the nontrivial perturbative (PT) dynamics there (asymptotic freedom (AF) \[1,2\]). However, if QCD itself is a confining theory, then a characteristic scale is very likely to exist. It should be directly responsible for the large scale structure of the true QCD vacuum.

The main purpose of this Letter is just to show how the mass gap responsible for the NP dynamics may explicitly appear in QCD. This especially becomes imperative after the Jaffe’s and Witten’s description of the Millennium Prize Problem \[3\]. The propagation of gluons is one of the main dynamical effects in the true QCD vacuum. The gluon Green’s function is (Euclidean signature here and everywhere below)

$$D_{\mu\nu}(q) = i \left\{ T_{\mu\nu}(q)d(q^2,\xi) + \xi L_{\mu\nu}(q) \right\} \frac{1}{q^2},$$

where $\xi$ is the gauge fixing parameter ($\xi = 0$ - Landau gauge and $\xi = 1$ - Feynman gauge) and $T_{\mu\nu}(q) = \delta_{\mu\nu} - q_{\mu}q_{\nu}/q^2 = \delta_{\mu\nu} - L_{\mu\nu}(q)$. Evidently, $T_{\mu\nu}(q)$ is the transverse (physical) component of the full gluon propagator, while $L_{\mu\nu}(q)$ is its longitudinal (unphysical) one. The free gluon propagator is obtained by setting simply the full gluon form factor $d(q^2,\xi) = 1$ in Eq. (1.1), i.e., $D_{\mu\nu}^0(q) = i \left\{ T_{\mu\nu}(q) + \xi L_{\mu\nu}(q) \right\} (1/q^2)$. The main tool of our investigation is the so-called Schwinger-Dyson (SD) equation of motion (see below) for the full gluon propagator (1.1), since its solution(s) reflect the quantum-dynamical structure of the true QCD ground state.
II. GLUON SD EQUATION

The general structure of the SD equation for the full gluon propagator [1 2] can be written down symbolically as follows (for our purposes it is more convenient to consider the SD equation for the full gluon propagator and not for its inverse):

\[ D(q) = D^0(q) - D^0(q)T_q(q)D(q) - D^0(q)T_{gh}(q)D(q) + D^0(q)T_g[D](q)D(q). \]  
(2.1)

Here and in some places below, we omit the dependence on the Dirac indices, for simplicity. \( T_q(q) \) and \( T_{gh}(q) \) describe the quark and ghost skeleton loop contributions into the gluon propagator (gluon self-energy). They do not contain the full gluon propagators by themselves. A pure gluon contribution \( T_g[D](q) \) into the gluon self-energy is a sum of four pure gluon skeleton loops, and consequently they contain explicitly the full gluon propagators. Precisely this makes the gluon SD equation highly nonlinear (NL), and this is one of the reasons why it cannot be solved exactly. However, its linear part, which contains only ghost and quark skeleton loops, can be summed up as usual, so Eq. (2.1) becomes

\[ D(q) = \tilde{D}^0(q) + \tilde{D}^0(q)T_g[D](q)D(q) = \tilde{D}^0(q) + D^N L(q), \]  
(2.2)

with \( \tilde{D}^0(q) \) being a modified free gluon propagator,

\[ \tilde{D}^0(q) = \frac{D^0(q)}{1 + [T_q(q) + T_{gh}(q)]D^0(q)}, \]  
(2.3)

where

\[ T_q(q) = -g^2 \int \frac{id^4p}{(2\pi)^4} Tr[\gamma_\mu S(p - q)\Gamma_\mu(p - q)S(p)], \]  
(2.4)

\[ T_{gh}(q) = g^2 \int \frac{id^4k}{(2\pi)^4} k_\mu G(k)G(k - q)\Gamma_\mu(k - q, q). \]  
(2.5)

Let us present now explicitly the NL pure gluon part, which was symbolically denoted as \( T_g[D](q) \) in the gluon SD Eqs. (2.1) and (2.2). It is

\[ T_g[D](q) = \frac{1}{2} T_1(q) + \frac{1}{2} T_2(q) + \frac{1}{6} T_3(q), \]  
(2.6)

where the so-called constant tadpole term is \( T_3 = g^2 \int (id^4q_1/(2\pi)^4)T_4^0D(q_1) \). All other skeleton loop integrals are given explicitly below as follows \((q - q_1 + q_2 - q_3 = 0)\):

\[ T_1(q) = g^2 \int \frac{id^4q_1}{(2\pi)^4} T_3^0(q, -q_1, q_1 - q)T_3(-q, q_1, q - q_1)D(q_1)D(q - q_1), \]  
(2.7)

\[ T_2(q) = g^4 \int \frac{id^4q_1}{(2\pi)^4} \int \frac{id^4q_2}{(2\pi)^4} T_4^0T_3(-q, q_1, q_2 - q_3)T_3(-q, q_1, q_3 - q_2)D(q_1)D(-q_2)D(q_3)D(q_3 - q_2), \]  
(2.8)

\[ T'_2(q) = g^4 \int \frac{id^4q_1}{(2\pi)^4} \int \frac{id^4q_2}{(2\pi)^4} T_4^0T_4(-q, q_1, -q_2, q_3)D(q_1)D(-q_2)D(q_3). \]  
(2.9)

Evidently, Eq. (2.7) describes skeleton one-loop contribution into the gluon self-energy due to the 3-gluon couplings, while skeleton two-loop integrals describe the above-mentioned contributions which are due to the combination of the 3- and 4-gluon couplings in Eq. (2.8) and the 4-gluon couplings only in Eq. (2.9).
A. General iteration solution.

The general iteration solution (i.e., when the skeleton loop integrals are to be iterated) of the gluon SD equation (2.2) looks like

\[
D(q) = \tilde{D}^0(q) + \tilde{D}^0(q)T_g[D(q)D(q) = \tilde{D}^0(q) + \tilde{D}^0(q)\tilde{D}^0(q) + \tilde{D}^0(q)\tilde{D}^0(q) + \tilde{D}^0(q)D^{(1)}(q)|\tilde{D}^0(q) + D^{(1)}(q)| + \ldots, \\
\]

with \(D^{(0)}(q) \equiv \tilde{D}^0(q)\) and we will use Eq. (2.3) for the modified free gluon propagator. Evidently, it is nothing but the skeleton loops expansion. If one knows how to sum up this expansion, so he knows the full gluon propagator and vice-versa, i.e., any solution to the gluon SD equation should be compatible with this expansion.

III. REGULARIZATION IN THE LINEAR PART

Due to AF [1] all the skeleton loop integrals as well as those which will appear in the formal iteration solution (2.10) are divergent. Thus, the general problem of their regularization arises. Let us start from the quark and ghost skeleton loop integrals (2.4) and (2.5), respectively. It is easy to see that the quark skeleton loop integral (2.4) does (2.10) are divergent. Thus, the general problem of their regularization arises. Let us start from the quark and ghost skeleton loop integrals (2.4) and (2.5), respectively.

At first sight an additional singularity at very small values of the skeleton loop variable will appear in Eq. (2.5) at \(q = 0\) limit. This means that its decomposition into the independent tensor structures can be written down as follows:

\[
T_{q}(q) \equiv T_{\mu\nu}(q) = \delta_{\mu\nu}q^{2}T_{q}^{(1)}(q^{2}) + q_{\mu}q_{\nu}T_{q}^{(2)}(q^{2}),
\]

where both invariant functions \(T_{q}^{(n)}(q^{2})\) at \(n = 1, 2\) are dimensionless with a regular behavior at zero. If the above-mentioned subtraction is assumed, then these invariant functions are, in general, represented by the finite integrals. Due to the definition \(q_{\mu}q_{\nu} = q^{2}L_{\mu\nu}\), instead of the independent structures \(\delta_{\mu\nu}\) and \(q_{\mu}q_{\nu}\) in Eq. (3.1) and below, one can use \(T_{\mu\nu}\) and \(L_{\mu\nu}\) as the independent structures with their own invariant functions.

At first sight an additional singularity at very small values of the skeleton loop variable will appear in Eq. (2.5) at \(q = 0\) because of the second ghost propagator. However, this is not the case, since the ghost-gluon vertex \(G_{gh}(k, 0)\) is the linear function of its argument and the combination \(k_{\mu}k_{\nu}\) will cancel this additional singularity. So, as in previous case we can regularize this contribution by making the corresponding subtraction, namely \(T_{gh}^{(n)}(q^{2}) = T_{gh}(q^{2}) - T_{gh}(0)\).

This again means that its decomposition into the independent tensor structures can be written down similar to the previous case as

\[
T_{gh}(q) \equiv T_{\mu\nu}^{gh}(q) = \delta_{\mu\nu}q^{2}T_{gh}^{(1)}(q^{2}) + q_{\mu}q_{\nu}T_{gh}^{(2)}(q^{2}).
\]

The both invariant functions \(T_{gh}^{(n)}(q^{2})\) at \(n = 1, 2\) are dimensionless with a regular behavior at zero. If the above-mentioned subtraction is assumed, then these invariant functions are, in general, represented by the finite integrals. From the relations (3.1) and (3.2) it follows that the quark and ghost skeleton loop contributions are of the order \(q^{2}\) always, i.e., \(T_{q}(q) = O(q^{2})\) and \(T_{gh}(q) = O(q^{2})\). Let us emphasize that just because of this the ghosts cancel unphysical (longitudinal) degrees of freedom of gauge bosons at every order of the PT and thus going beyond PT, i.e., this role, though being kinematical, is general one.

Taking into account the tensor structures of the free gluon propagator and these integrals, from Eq. (2.3) one obtains

\[
\tilde{D}^0(q) = D^0(q)A(q^{2}),
\]

where \(A(q^{2}) = 1/(1 + T(q^{2}))\), and \(T(q^{2})\) evidently describes the ghost and quark skeleton loop contributions. It is regular at zero due to the above-mentioned smooth behavior of the corresponding skeleton loop integrals (2.4) and (2.5) at small external gluon momentum \(q\). Since \(A(q^{2})\) is finite at zero, the infrared (IR) singularity of the linear part of the full gluon propagator is completely determined by the power-type exact IR singularity of the free gluon
propagator, i.e., \( \tilde{D}^0(q) = A(0)D^0(q) \), \( q^2 \to 0 \). We are especially interested in the structure of the full gluon propagator in the IR region, so the exact result (3.3) will be used as an input in the direct general iteration solution of the gluon SD equation (2.10). This form of the gluon SD equation makes it possible to take into account automatically ghost and quark degrees of freedom in all orders of the linear part in the gauge-invariant way. However, the dressing of the full gluon propagator due to the quark and ghost skeleton loop contributions only (modified free gluon propagator) cannot drastically change its behavior in the deep IR from the behavior of free gluon propagator. Thus, for the nontrivial dressing (which can substantially change the structure of the full gluon propagator in the deep IR) one should look into the NL part of the gluon SD equation (2.2).

IV. REGULARIZATION IN THE NL PART

Let us begin the investigation of the regularization of the skeleton loop integrals which enter the NL part of the gluon SD equation (2.2) with the skeleton loop integral (2.7), which contains the 3-gluon coupling only. With \( D = \tilde{D}^0 \) (i.e., after the first iteration in the gluon equation (2.10)), an additional singularity due to \( \tilde{D}^0(-q_1) = \tilde{D}^0(q_1) \) will appear in the exact \( q = 0 \) limit. It is worth reminding once more that \( \tilde{D}^0 \) has an IR singularity of the free gluon propagator, for sure (see Eq. (3.3)). However, the 3-gluon vertices from the numerator, being the linear functions of their arguments, will cancel this additional IR singularity, just as in the case of ghosts. Thus, the decomposition of this integral into the independent tensor structures again will be determined by the relation similar to Eq. (3.2) with its own invariant functions, of course. In other words, its regularization by the corresponding subtraction at the zero point \( q = 0 \) is again relevant in this case. Thus, on general ground one has

\[
T_1(q) = T_{1\mu\nu}(q) = \delta_{\mu\nu}q^2T_1^{(1)}(q^2) + q_\mu q_\nu T_1^{(2)}(q^2).
\]

Again the both invariant functions \( T_1^{(n)}(q^2) \) at \( n = 1, 2 \) are dimensionless with a regular behavior at zero. If the above-mentioned subtraction is assumed, then these invariant functions are, in general, represented by the finite integrals. From this relation it follows that, similar to the quark and ghost skeleton loop contributions, the three-gluon one-loop skeleton integral is of the order \( q^2 \) as well, i.e., \( T_1(q) = O(q^2) \).

It is instructive to start the investigation of the regularization of the two-loop skeleton integrals from the two-loop term (2.9), which contains the 4-gluon coupling only. After the first iteration in the gluon equation (2.10) and at the zero point \( q = 0 \) it becomes

\[
T_2(q) = g^4 \int \frac{id^4q_1}{(2\pi)^4} \int \frac{id^4q_2}{(2\pi)^4} T_4(0,q_1,-q_2,-q_1+q_2)\tilde{D}^0(q_1)\tilde{D}^0(-q_2)\tilde{D}^0(-q_1+q_2).
\]

This skeleton integral possesses very distinctive and important feature. An additional singularities will appear due to \( \tilde{D}^0(-q_1+q_2) \) in the integration over the very small values of the loop variables \( q_1 \) and \( q_2 \). The important observation, however, is that they cannot be cancelled by the corresponding terms from the numerator, since the full 4-gluon vertex, when all the gluon momenta involved go to zero, will be effectively reduced to the corresponding point-like one, which does not depend on the gluon momenta involved at all, and thus is finite. The straightforward \( q = 0 \) limit is certainly dangerous in this case. To regularize the initial skeleton integral (2.9) at \( D = \tilde{D}^0 \) by the corresponding subtraction, i.e., to define \( T_2^R(q) = T_2(q) - T_2(0) \), is not the case now. The problem is that by this procedure we will remove not only the ultraviolet (UV) divergences (not interesting for us), but the IR singularities with respect to \( q^2 \) as well, which are of the great interest, since we are very interested in the explicit IR structure of the full gluon propagator. Evidently, the subtraction at any safe small Euclidean point \( q^2 = -\mu^2 \) will cause the same problem, namely the total loss of information on the deep IR structure of the full gluon propagator. Much more sophisticated method is needed to investigate the region of all the small gluon momenta involved, i.e., to establish the functional dependence of the loop integral (2.9) on small \( q^2 \). Let us also make once thing perfectly clear. Due to the above-mentioned singular structure, it implicitly contains the corresponding mass scale parameter (the above-mentioned mass gap), i.e., it is hidden in the initial skeleton integral (2.9) at nonzero \( q \). The mass gap and an additional IR singularities will show up explicitly when \( q \) goes to zero (see below). The physical meaning of a mass gap is, in general, a scale responsible for the NP dynamics in the IR region.

Again an additional singularities in the integration over the very small values of the loop variables \( q_1 \) and \( q_2 \) due to \( \tilde{D}^0(-q_1+q_2) \) and \( \tilde{D}^0(-q_1) \) will appear in the two-loop skeleton integral (2.8) at \( q = 0 \). The full 3-gluon vertices, when all the gluon momenta involved go to zero, will be effectively reduced to the corresponding point-like ones, which linearly depend on the gluon momenta involved. However, their product in the numerator might be not enough to
cancel the above-mentioned additional IR singularities. So, this skeleton integral can be source of an additional IR singularities with respect to \( q^2 \) and hence of the mass gap.

From the above-discussed it clearly follows that in order to track down correctly and completely all the IR singularities which are to appear in the skeleton loop integrals (2.8) and (2.9) in the \( q = 0 \) limit, one needs the point-like counterparts of the gluon couplings but all the independent combinations of them. To achieve this goal the skeleton loop integrals (2.8) and (2.9) should be equivalently replaced by an infinite series of terms where all the NL gluon interactions are to be represent by the corresponding point-like counterparts. In this case there is no need in the information from the Slavnov-Taylor identities for the corresponding full 3- and 4-gluon vertices, which enter the above-mentioned skeleton loop integrals. Such kind of the expansion is known as the skeleton loop expansion or interpretation, let us represent the last integral as a sum of four terms, namely

\[
T^2(q) = i g^4 \int dq_1 \int dq_2 \frac{A(q_1^2)A(q_2^2)A((q - q_1 + q_2)^2)}{q_1^2 q_2^2(q - q_1 + q_2)^2},
\]

(4.3)

In order to introduce explicitly the above-mentioned hidden mass gap at the level of the separate diagram (contribution), let us represent the last integral as a sum of four terms, namely \( T^2(q^2) = \sum_{n=1}^{n=4} T^2(n)(q^2) \), where

\[
T^2(1)(q^2) = i g^4 \int_0^\Delta^2 dq_1 \int_0^\Delta^2 dq_2 \frac{A(q_1^2)A(q_2^2)A((q - q_1 + q_2)^2)}{q_1^2 q_2^2(q - q_1 + q_2)^2},
\]

(4.4)

\[
T^2(2)(q^2) = i g^4 \int_\Delta^2 dq_1 \int_0^\Delta^2 dq_2 \frac{A(q_1^2)A(q_2^2)A((q - q_1 + q_2)^2)}{q_1^2 q_2^2(q - q_1 + q_2)^2},
\]

(4.5)

\[
T^2(3)(q^2) = i g^4 \int_0^\Delta^2 dq_1 \int_\Delta^2 dq_2 \frac{A(q_1^2)A(q_2^2)A((q - q_1 + q_2)^2)}{q_1^2 q_2^2(q - q_1 + q_2)^2},
\]

(4.6)

\[
T^2(4)(q^2) = i g^4 \int_\Delta^2 dq_1 \int_\Delta^2 dq_2 \frac{A(q_1^2)A(q_2^2)A((q - q_1 + q_2)^2)}{q_1^2 q_2^2(q - q_1 + q_2)^2},
\]

(4.7)

and where not loosing generality we introduced the common mass gap squared \( \Delta^2 \) for both loop variables \( q_1^2 \) and \( q_2^2 \). The integration over angular variables is assumed. A few remarks are in order. In the integrals (4.5), (4.6) and (4.7) the mass gap \( \Delta^2 \) can be formally considered as an IR cut-off. The last integral (4.7) has no IR singularities at all, and so it can be regularized as usual by the corresponding subtraction, which makes it \( O(q^2) \), as explained above. On the other hand, since we are not interested in the UV structure of the full gluon propagator, let us regularize the integrals (4.5) and (4.6) at the upper limit by hand, i.e., the introduction of an auxiliary UV cut-off is implicitly assumed (see below).

We are especially interested in the case when the external gluon momentum \( q \) is small (let us remind that in Euclidean metrics small \( q^2 \) means the smallness of all its component and vice-versa). However, in Eq. (4.4) we can
formally consider the variables \( q_1 \) and \( q_2 \) as much smaller than the small gluon momentum \( q \), i.e., to approximate \( q \approx \delta_1 q_1, q \approx \delta_2 q_2 \), so that \( q - q_1 + q_2 \approx q(1 + \delta) \), where \( \delta = \delta_2 - \delta_1 \). To leading order in \( \delta \), one obtains

\[
T_2^{(1)}(q^2) = -i g^2 \frac{A(q^2)}{q^2} \int_0^{\Delta^2} dq_1^2 \int_0^{\Delta^2} dq_2^2 A(q_1^2) A(q_2^2),
\]

where all the finite numbers after the trivial integration over angular variables will be included into the numerical factors below, for simplicity. Since \( q^2 \) is small, we can replace the dimensionless function \( A(q^2) \) by its Taylor expansion, \( A(q^2) = A(0) + a_1(q^2/\Delta^2) + O(q^4) \). Introducing further dimensionless variables \( q_i^2 = x_i \Delta^2 \) and \( q_3^2 = x_2 \Delta^2 \), one finally obtains

\[
T_2^{(1)}(q^2) = -i[(\Delta^4/q^2)c_2 + \Delta^2 c_2' + O(q^2)]g^4,
\]

where

\[
c_2 = A(0) \int_0^1 dx_1 A(x_1) \int_0^1 dx_2 A(x_2) \\
c_2' = a_1 \int_0^1 dx_1 A(x_1) \int_0^1 dx_2 A(x_2)
\]

In Eq. (4.5) it makes sense to approximate \( q \approx \delta_2 q_1 \), \( q \approx \delta_4 q_1 \), since \( q_1 \) is much bigger than \( q_2 \) and \( q \), so that \( q - q_1 + q_2 \approx q_1(1 + \delta) \), where \( \delta = \delta_4 - \delta_3 \). To leading order in \( \delta \) and omitting some algebra, one finally obtains

\[
T_2^{(2)}(q^2) = -i[\Delta^2 c_2(\lambda) + O(q^2)]g^4,
\]

where

\[
c_2(\lambda) = \int_0^\lambda (dx_1/x_1) A^2(x_1) \int_0^1 dx_2 A(x_2),
\]

and here and below \( \lambda \) is the above-mentioned dimensionless UV cut-off.

In Eq. (4.6) it makes sense to approximate \( q_1 \approx \delta_5 q_2, q \approx \delta_6 q_2 \), since now \( q_2 \) is much bigger than \( q_1 \) and \( q \), so that \( q - q_1 + q_2 \approx q_2(1 + \delta) \), where \( \delta = \delta_5 + \delta_6 \). To leading order in \( \delta \) and similar to the previous case, one obtains

\[
T_2^{(3)}(q^2) = -i[\Delta^2 c_2'(\lambda) + O(q^2)]g^4,
\]

where

\[
c_2'(\lambda) = \int_1^\lambda (dx_2/x_2) A^2(x_2) \int_0^1 dx_1 A(x_1),
\]

The last term (4.7) is left unchanged, since all loop variables are big, and it is of the \( O(q^2) \) order, as mentioned above. The both integrals \( c_2(\lambda) \) and \( c_2'(\lambda) \) are logarithmically divergent, if one neglects the contribution from the quark and ghost skeleton loops at all (in this case \( A(x) = 1 \), by definition).

Summing up all the terms, one obtains

\[
T_2(q) = -i \left[ \frac{\Delta^4}{q^2} c_2 + \Delta^2 (c_2' + c_2(\lambda) + c_2'(\lambda)) + O(q^2) \right]g^4,
\]

The last integral (4.7) is hidden in terms \( O(q^2) \). Here the characteristic mass scale parameter \( \Delta^2 \) is responsible for the nontrivial dynamics in the IR domain. Let us also emphasize that the limit \( \lambda \to \infty \) should be taken at the final stage. So, the integral (4.3) is divergent in the exact \( q = 0 \) limit, indeed. In other words, these singularities with
respect to the external gluon momentum $q$ will show up explicitly if and only if it goes to zero. The constant tadpole term produces the only contribution at the order $g^2$ as follows: $T_1 = -i \Delta^2 c_1(\lambda) g^2$, where $c_1(\lambda) = \int_0^\lambda dx_1 A(x_1)$.

Let us emphasize that such kind of the expansion (4.15) for the initial integral (4.3) can be postulated on the general ground, not performing the explicit estimate above. What is all that matters is the hidden existence of the mass gap in the integral (4.3), its singular structure with respect to $q^2$ when it goes to zero and its regularization at the upper limit by introducing an UV cut-off $\lambda$. Then on the dimensional ground only one can, in general, write down as follows:

$$ T_2(q) = -i[(\Delta^4/q^2) c_2(\lambda, A) + \Delta^2 c'_2(\lambda, A) + O(q^2)] g^4, $$

(4.16)

where the $A$-factor in the arguments of the momentum-independent coefficients indicates their dependence on quark and ghost degrees of freedom integrated out (i.e., numerical dependence). As we have already seen some of these $A$-factors might be finite and some others divergent, depending on $\lambda$. Let us also note that the $A$-factor may explicitly depend on the coupling constant squared $g^2$ as well as on the gauge fixing parameter $\xi$, i.e., $A = A(\xi, g^2)$. As a functions of $q^2$ these degrees of freedom contribute into the terms of the $O(q^2)$ only.

At the NL $g^4$ order, there is a number of the additional diagrams, which, however, contain the three-gluon vertices along with the four-gluon ones (see skeleton loop integral (2.8)) plus the tadpole diagrams. Their contributions can be formally given by the estimates similar to the estimate (4.16) with different coefficients, so there is, in general, no cancellation at this order (let us emphasize that the two-loop contribution (4.3) is unique nontrivial one which contains the 4-gluon couplings only). In more complicated cases of the multi-loop diagrams more severe IR divergences, accompanied by the proper powers of the mass gap, will appear. We have done calculations in the Feynman gauge, for simplicity, but it is clear that such kind of calculations can be done in any covariant gauge $\xi$. Taking into account these estimates (however, the functional dependence on $q^2$ and hence on the mass gap is exactly fixed), the contribution from the NL part can be generalized as follows:

$$ T_g[\hat{D}^0](q) = \Delta^2 \sum_{n=0}^\infty (\Delta^2/q^2)^n c_n(\lambda, \xi, A, g^2) + O(q^2), $$

(4.17)

where $c_n(\lambda, \xi, A, g^2) = \sum_{m=0}^\infty c_{n,m}(\lambda, \xi, A) g^{2m}$. These series indicate that each skeleton loop integral contributes into the each term in this expansion (it is worth reminding that any skeleton loop integral is formally an infinite series in powers of $g^2$, assigned to the point-like gluon vertices). The explicit expressions for the momentum-independent coefficients $c_{n,m}$ and $a_{k,m}$ (see Eq. (5.1) below) are not important, only the explicit dependence on the mass gap and hence the functional dependence on $q^2$ is all that matters at this stage.

V. EXACT STRUCTURE OF THE FULL GLUON PROPAGATOR

Evidently, using the generalized expansion (4.17) for $T_g[\hat{D}^0](q)$, and multiplying it from both sides by $\hat{D}^0(q)$, one can find the first iteration $D^{(1)}(q)$, and on its account one will find the second iteration $D^{(2)}(q)$, and so on. Omitting all the really tedious algebra and restoring the tensor structure, the general iteration solution of the gluon SD equation (2.10) for the full gluon propagator can be algebraically (i.e., exactly) decomposed as the sum of the two principally different terms, namely

$$ D_{\mu\nu}(q) = D_{\mu\nu}^{INP}(q, \Delta^2) + D_{\mu\nu}^{PT}(q) = i T_{\mu\nu}(q) \Delta^2 (\Delta^2/q^2)^k \sum_{m=0}^\infty a_{k,m}(\lambda, \xi, A) g^{2m} + i \left[T_{\mu\nu}(q) \int_{m=0}^\infty a_{m}(q^2, \xi) g^{2m} + \xi L_{\mu\nu}(q) \right] \frac{1}{q^2}, $$

(5.1)

The superscript "INP" stands for the intrinsically NP part of the full gluon propagator. It reflects the presence of inevitable severe (for definition see below) IR singularities and the fact that this part vanishes when the mass gap goes to zero, while the PT part survives. Also this part depends only on the transverse (physical) degrees of freedom of gauge bosons. As discovered above, the momentum-independent coefficients $a_{k,m}(\lambda, \xi, A)$ include the information about quark and ghost degrees of freedom in all orders of linear part numerically, and some of them can be UV divergent (in fact they are residues at poles). As functions quark and ghost degrees of freedom can contribute into
the PT part only of the full gluon propagator $D^{\text{PT}}(q)$, which is of the order $O(q^{-2})$ up to a possible PT logarithm improvements at very large $q^2$. It includes the transverse and longitudinal components, since the latter one is always of the order $O(q^{-2})$. It remains undetermined, since the dependence of the dimensionless functions $a_m(q^2, \xi)$ on $q^2$ cannot be fixed on general ground, like it has been done in the INP part. Formal infinite series over $m$ in both terms of the full gluon propagator are not the PT series in powers of the small coupling constant squared (within our approach the strength of the coupling constant remains arbitrary). These series show that the skeleton loop integrals have been iterated, which formally are an infinite series of all the relevant contributions. In addition, in the first term these series simply show that each skeleton loop integral (apart from the 3-gluon coupling skeleton contribution (2.7)) invokes each severe IR singularity (and hence the mass gap) in the full gluon propagator. In Refs. \[4, 5\] we came to the same structure (5.1) but in a rather different way.

We distinguish between the two terms in the full gluon propagator (5.1) not by the strength of the coupling constant, but rather by the character of the IR singularities. The power-type severe IR singularity is defined as more singular than the power-type IR singularity of the free gluon propagator, i.e., more singular than $1/q^2$. The PT IR singularity is defined as much singular as $1/q^2$ always. That is why the longitudinal component of the full gluon propagator is included into its PT part. The INP part is nothing else but the corresponding Laurent expansion in powers of severe IR singularities, accompanied by the corresponding powers of the mass gap. The INP part of the full gluon propagator starts necessarily from the simplest $(1/q^2)^2$ one, possible in four-dimensional QCD \[4, 5\]. Its exact structure inevitably stems from the general iteration solution of the gluon SD equation (2.10) for the full gluon propagator.

The unavoidable presence of the first term in Eq. (5.1) makes the principal distinction between non-Abelian QCD and Abelian QED, where such kind of term in the full photon propagator is certainly absent (in the former theory there is direct coupling between massless gluons, while in the latter one there is no direct coupling between massless photons). Precisely this term violets the cluster properties of the Wightman functions \[7\], and thus validates the Strocchi theorem \[8\], which allows for such IR singular behavior of the full gluon propagator. Thus, the dressing of the full gluon propagator due to the NL part in the gluon SD equation drastically changes its behavior in the deep IR, indeed.

Thus, the true QCD vacuum is really beset with severe IR singularities. They should be summarized (accumulated) into the full gluon propagator and effectively correctly described by its structure in the deep IR domain, exactly represent by its INP part. The second step is to assign a mathematical meaning to the integrals, where such kind of severe (or equivalently the NP) IR singularities will explicitly appear, i.e., to define them correctly in the IR region \[4, 5\]. Just this IR violent behavior makes QCD as a whole an IR unstable theory, and therefore it has no IR stable fixed point, indeed \[4, 5\], which means that QCD itself might be a confining theory without involving some extra degrees of freedom (see below).

A. Discussion

The important feature of the general iteration solution is that the skeleton loop integrals are to be iterated. This means that there are no assumptions and approximations made, since all the relevant contributions have been taken into account. Moreover, the decomposition (5.1) of the full gluon propagator into the two terms is exact. It is nothing else but the one unknown function (the full gluon propagator) is represent as the sum of the two functions, one of which remains unknown (the PT part). Evidently, this can be done algebraically (i.e., exactly, for example nothing else but the one unknown function (the full gluon propagator) is represent as the sum of the two functions, into account. Moreover, the decomposition (5.1) of the full gluon propagator into the two terms is exact. It is means that there are no assumptions and approximations made, since all the relevant contributions have been taken into account. Further, the infinite series over $m$ in both terms of the full gluon propagator are not the PT series in powers of the small coupling constant squared (within our approach the strength of the coupling constant remains arbitrary). These series show that the skeleton loop integrals have been iterated, which formally are an infinite series of all the relevant contributions. In addition, in the first term these series simply show that each skeleton loop integral (apart from the 3-gluon coupling skeleton contribution (2.7)) invokes each severe IR singularity (and hence the mass gap) in the full gluon propagator. In Refs. \[4, 5\] we came to the same structure (5.1) but in a rather after way.

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A. Discussion

The important feature of the general iteration solution is that the skeleton loop integrals are to be iterated. This means that there are no assumptions and approximations made, since all the relevant contributions have been taken into account. Moreover, the decomposition (5.1) of the full gluon propagator into the two terms is exact. It is nothing else but the one unknown function (the full gluon propagator) is represent as the sum of the two functions, one of which remains unknown (the PT part). Evidently, this can be done algebraically (i.e., exactly, for example symbolically as follows: $D = D^{\text{PT}} + D^{\text{PT}} = D^{\text{INP}} + D^{\text{PT}} = D - D^{\text{INP}} + D^{\text{INP}} = D^{\text{INP}} + D^{\text{PT}}$). Due to the NL dynamics of the QCD ground state we were able to establish exactly the functional structure (i.e., the dependence on $q^2$ and hence on the mass gap squared) of the first term $D^{\text{INP}}(q)$. Let us emphasize that its Laurent expansion form is necessary, since it allows one to put each NP power-type IR singularity (which are independent distributions) under firm mathematical control \[4, 5\]. Both terms are valid in the whole energy/momentum range, i.e., they are not asymptotics. At the same time, we achieved the exact separation between the two terms responsible for the NP (dominating in the IR) and PT (dominating in the UV) dynamics in the true QCD vacuum.

Though the coefficients of the Laurent expansion may explicitly depend on the gauge fixing parameter $\xi$, the zero momentum modes enhancement (ZMME) effect itself (represented in the INP part) does not depend on it, i.e., at any $\xi$ this effect takes place. In this sense it is gauge invariant. This is very similar to AF. It is well known that the exponent, which determines the logarithmic deviation of the full gluon propagator from the free one in the UV region ($q^2 \gg \Lambda^2_{QCD}$), explicitly depends on the gauge fixing parameter. At the same time, AF itself does not depend on it, i.e., it takes place at any $\xi$.

The QCD Lagrangian does not contain a mass gap. However, we discovered that the mass scale parameter responsible for the NP dynamics in the IR region should exist in the true QCD ground state. At the level of the gluon SD equation it is hidden in the skeleton loop contributions into the gluon self-energy, which depend on the four-gluon vertices. At the fundamental quark-gluon (i.e., Lagrangian) level the dynamical source of a mass gap
(and hence of severe IR singularities) is the four-gluon interaction, determining thus its key role. This interaction survives when all the gluon momenta involved go to zero, while the three-gluon interaction vanishes in this limit ($T^3_g(0,0,0) \neq 0$, $T^3_3(0,0) = 0$).

B. A necessary generalization

We already know that for the 3-gluon coupling skeleton loop integral (2.7) the exact $q = 0$ limit is smooth. As explained above, it produces the contribution which is of the $O(q^3)$ order always, see the relation (4.1). This means that it will contribute into the PT part of the full gluon propagator only, since the structure of the computations. So, it follows that each NP IR singularity scales as $1/\epsilon$ where $\epsilon = 0$. Thus, the tensor decomposition of the NL part $T^g_\mu[D](q) \equiv T^g_{\mu\nu}[D](q)$ is necessarily to be generalized as follows:

$$T^g_{\mu\nu}[D](q) = \delta_{\mu\nu}\left[\frac{\Delta^4}{q^2}L_9^{(1)}(q^2) + \Delta^2L_9^{(2)}(q^2) + q^2T_9^{(3)}(q^2)\right] + q_{\mu\nu}\left[\frac{\Delta^2}{q^2}L_9^{(4)}(q^2) + T_9^{(5)}(q^2)\right],$$

in complete agreement with the generalized expansion (4.17). Here $T_9^{(n)}(q^2)$ at $n = 3, 5$ are invariant dimensionless functions. They are regular functions of $q^2$, i.e., they can be represent by the corresponding Taylor expansions, but possessing AF at infinity, and depending thus on $\Lambda_{QCD}$ in this limit. They are saturated by all the skeleton loop integrals apart from the tadpole term. At the same time, the invariant dimensionless functions $L_9^{(n)}(q^2)$ at $n = 1, 2, 4$ are to be represent by the corresponding Laurent expansions, namely $L_9^{(1,2,4)}(q^2) \equiv L_9^{(1,2,4)}(q^2, \Delta^2) = \sum_{k=0}^{\infty}(\Delta^2/q^2)^kB_k^{(1,2,4)}(\lambda, \xi, A, g^2)$, where the quantities $b_k^{(1,2,4)}(\lambda, \xi, A, g^2)$ by themselves are expansions in the coupling constant squared (see above). These invariant functions are to be saturated by all the skeleton loop integrals containing the 4-gluon coupling. Let us emphasize the inevitable appearance of the mass gap $\Delta^2$. It characterizes the nontrivial dynamics in the IR region. When the mass gap is zero then this decomposition takes the standard form like in QED, where the electron skeleton loop integral contributes only (see relation like the relation (3.1)). The generalization (5.2) makes the explicit dependence on the mass gap of the full gluon propagator perfectly clear. It is due to the direct interaction between massless gluons only (mainly to the 4-gluon coupling).

VI. IR RENORMALIZATION AND GLUON CONFINEMENT

A. IR renormalization.

The NP power-type IR singularities represent a rather broad and important class of functions with algebraic singularities. They regularization should be done within the theory of distributions \[6\], complemented by the dimensional regularization (DR) method \[10\]. The crucial observation is that the regularization of these singularities does not depend on their powers \[11\] \[12\], namely

$$(q^2)^{-2-k} = \frac{1}{\epsilon}\left[a(k)[\delta^4(q)]^{(k)} + O(\epsilon)\right], \quad \epsilon \to 0^+,$$

where $a(k)$ is a finite constant depending only on $k$ and $[\delta^4(q)]^{(k)}$ represents the kth derivative of the $\delta$-function. Here $\epsilon$ is the IR regularization parameter, introduced within the DR method \[10\], and which should go to zero at the end of the computations. So, it follows that each NP IR singularity scales as $1/\epsilon$ as $\epsilon$ goes to zero. This regularization expansion takes place only in four-dimensional QCD with Euclidean signature. In other dimensions and signature it is more complicated \[4\] \[5\].

In the presence of such severe IR singularities all the quantities should depend, in principle, on $\epsilon$. Thus, the general IR renormalization program is needed in order to express all the quantities in terms of their IR renormalized versions. For this purpose it is convenient to rewrite the INP part of the full gluon propagator as follows:

$$D^{INP}(q, \Delta^2) = \sum_{k=0}^{\infty}(\Delta^2)^{k+1}(q^2)^{-2-k}a_k(\lambda, \xi, A, g^2),$$

(6.2)
where we again suppressed the tensor indices and

\[ a_k(\lambda, \xi, A, g^2) = \sum_{m=0}^{\infty} a_{k,m}(\lambda, \xi, A)g^{2m}. \quad (6.3) \]

Let us introduce further the following relations:

\[ \Delta^2 = X(\epsilon)\bar{\Delta}^2, \]
\[ a_k(\lambda, \xi, A, g^2) = Z_k(\epsilon)\bar{a}_k(\lambda, \xi, A, g^2), \quad (6.4) \]

where and below all quantities with bar are the IR renormalized, i.e., they exist as \( \epsilon \) goes to zero, by definition, while \( X(\epsilon) \) and \( Z_k(\epsilon) \) are the corresponding IR multiplicative renormalization (IRMR) constants. Substituting further these relations into the Laurent expansion (6.2), in terms of the IR renormalized quantities it then becomes

\[ D^{INP}(q, \bar{\Delta}^2) = \sum_{k=0}^{\infty}(\bar{\Delta}^2)^{k+1}(q^2)^{-2-k}\bar{a}_k(\lambda, \xi, A, g^2)X^{k+1}(\epsilon)Z_k(\epsilon). \quad (6.5) \]

## B. Gluon confinement.

Due to the distribution nature of the NP IR singularities, which appear in the full gluon propagator, the two different cases should be distinguished.

I. If there is an explicit integration over the gluon momentum, then from the dimensional regularization (6.1) and Eq. (6.5), it finally follows

\[ D^{INP}(q, \bar{\Delta}^2) = \sum_{k=0}^{\infty}(\bar{\Delta}^2)^{k+1}(q^2)^{-2-k}\bar{a}_k(\lambda, \xi, A, g^2)X^{k+1}(\epsilon)Z_k(\epsilon). \quad (6.6) \]

provided the INP part to be the IR finite from the very beginning, i.e., its IRMR constant will not depend on \( \epsilon \) at all as it goes to zero. For this we should put

\[ X^{k+1}(\epsilon)Z_k(\epsilon) = \epsilon\bar{B}_k(\epsilon), \quad k = 0, 1, 2, 3..., \quad \epsilon \to 0^+, \quad (6.7) \]

then the cancellation with respect to \( \epsilon \) will be guaranteed term by term (each NP IR singularity is completely independent distribution) in the Laurent skeleton loop expansion (6.2), that is dimensionally regularized and IR renormalized, Eq. (6.6).

II. If there is no explicit integration over the gluon momentum, then the functions \( (q^2)^{-2-k} \) in the Laurent skeleton loops expansion (6.5) cannot be treated as the distributions, i.e., there is no scaling as \( 1/\epsilon \). The INP part of the full gluon propagator, expressed in the IR renormalized terms, in this case disappears as \( \epsilon \), namely

\[ D^{INP}(q, \bar{\Delta}^2) = \epsilon \sum_{k=0}^{\infty}(\bar{\Delta}^2)^{k+1}(q^2)^{-2-k}\bar{a}_k(\lambda, \xi, A, g^2)\bar{B}_k(\epsilon) \sim \epsilon, \quad \epsilon \to 0^+. \quad (6.8) \]

This means that any amplitude for any number of soft-gluon emissions (no integration over their momenta) will vanish in the IR limit in our picture. In other words, there are no transverse gluons in the IR, i.e., at large distances (small momenta) there is no possibility to observe physical gluons experimentally as free particles. So, color gluons can never be isolated. This behavior can be treated as the gluon confinement criterion. Evidently, it does not depend explicitly on the gauge choice in the full gluon propagator, i.e., it is gauge-invariant. It is also general one, since even going beyond the gluon sector nothing can invalidate it. For the first time it has been derived in Ref. [4] (see Ref. [5] as well).
VII. CONCLUSIONS

The physical meaning of our mass gap is the scale directly responsible for the NP dynamics in the true QCD ground state just as $\Lambda_{QCD}^2$ is responsible for the nontrivial PT dynamics there. In this way the former determines the deviation of the full gluon propagator from the free one in the IR, while the latter makes this in the UV. Thus, the full gluon propagator (1.1) multiplied by $q^2$ is power-type enhanced in the IR and logarithmically weakened in the UV. The coupling constant squared (which a priori is not small) itself plays no role in the presence of the mass gap, so its dynamical origin goes beyond the PT. This is also a direct evidence for the "dimensional transmutation" [1,2], which occurs whenever a massless theory acquires a mass scale parameter dynamically. This is especially important, since there is none in the QCD Lagrangian. Let us emphasize that though the mass gap has been introduced by hand in the separate diagram, nevertheless it survives after summing up an infinite number of the relevant contributions (diagrams) and performing the IR renormalization program. Our mass gap provides gluon confinement (6.8), that is, there must be no transverse gluons at large distances. At the same time, severe IR singular structure of the full gluon propagator (5.1) and the fact that all the orders of the coupling constant squared contribute into the mass gap (skeleton loops expansion) explain why the interaction in our picture is strong but short-ranged (6.6) (the $\delta$-function and its derivatives).

The ghost and quark degrees of freedom play no any role in the dynamical generation of the mass gap within this approach. Only the NL interaction of massless gluons is important. Our mass gap appears in the NL part of the Yang-Mills sector of full QCD, however, its relation to the mass gap introduced by Jaffe and Witten in Ref. [3] is still to be understood [11], though there is a great similarity between them. It is worth also emphasizing that our mass gap and the Jaffe-Witten (JW) mass gap [3] cannot be interpreted as the gluon mass, i.e., they always remain massless within our approach.

If quantum Yang-Mills with compact simple gauge group $G = SU(3)$ exists on $\mathbb{R}^4$, then it exhibits a mass gap in the sense discussed by Jaffe and Witten, indeed. Moreover, it confines gluons as well. Color confinement of gluons is the IR renormalization gauge-invariant effect within our approach. Just the fundamental NL four-gluon interaction makes the full gluon propagator so singular in the IR. This requires the introduction of a mass gap, i.e., it arises from the quartic gluon potential (Feynman [3] has also arrived at the same conclusion but on a different basis). If AF (coming from the second term in Eq. (5.1)) is mainly due to the 3-gluon coupling, color confinement of gluons (coming from the first term in Eq. (5.1)) is then mainly due to the 4-gluon coupling. Evidently, in order to take correctly into account all the NP IR singularities, we need the point-like NL interactions between massless gluons but all the different combinations of them (skeleton loops expansion).

Concluding, a few remarks are in order. The mass gap discussed here is necessarily a "bare" one, i.e., it is not yet UV renormalized. The UV renormalization program is needed which is beyond the scope of this Letter. However, it worth emphasizing that we need to start from the unrenormalized loop integrals, anyway (Sects. 3 and 4).

The first NP IR singularity which should be investigated is the famous $(q^2)^{-2}$ term in Eq. (6.3) with the $\delta$-type IR regularization (see Eqs. (6.1) and (6.4)). Just this behavior of the full gluon propagator in different gauges has been obtained and investigated, for example in Refs. [13,14,15,16,17,18] (and references therein). The general iteration solution (taking into account an infinite number of the relevant loops) inevitably leads to the severely IR singular gluon propagator (the IR enhanced gluon propagator soften by the PT logarithm at one-loop has been investigated in Ref. [18]). However, this behavior correctly connects to the PT solution AF requires. The problem is that in the deep asymptotic limit (regime) $q^2 \to \infty$ the INP part of the full gluon propagator (5.1) will be totally suppressed and the second PT term becomes dominant (let us remind that the both terms are valid in the whole energy/momentum range, i.e., they are not asymptotics from the very beginning). Just this term is responsible for the logarithmic deviation of the full gluon propagator from the free one in the asymptotic regime required by AF, as repeatedly mentioned above. It was not our goal here to fix the PT gluon form-factor $d^{PT}(q^2,\xi) = \sum_{m=0}^{\infty} a_m(q^2,\xi)g^{2m}$ (to find it exactly is a formidable task, anyway). How to find it in the asymptotic regime is well known procedure (see, for example Refs. [1,2,3]).

The SD system of equations is highly nonlinear one. It is well known that for such kind of systems the number of the solutions is not fixed. It may have several solutions of the different nature. As underlined in Ref. [1], the deep IR asymptotics of the full gluon propagator can be of the two types only: the smooth, see recent paper [21] (and references therein, for example Ref. [21]) and the singular. From the general point of view thus it follows that, at least, the two independent solutions (with different behavior in the IR) to the gluon SD equation should certainly exist. The behavior of all the possible solutions in the asymptotic regime is to be fixed by AF, as emphasized above. Moreover, to derive a closed set of equations the truncations/approximations are inevitable. Different truncations/approximations necessarily lead to qualitatively different solutions. That is why the singular and smooth in the IR solutions for the gluon propagator should be considered on equal footing. They do not contradict to each other, especially we do not know the real IR boundary condition(s) in QCD. In this Letter it is explained how the severely IR singular gluon propagator leads to color confinement of gluons in the gauge-invariant way, taking into account the distribution nature...
of severe IR singularities. However, this might be somehow possible for the smooth gluon propagator as well. Support from HAS-JINR Scientific Collaboration Fund (P. Levai) is to be acknowledged.

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