Inversion formulas for complex Radon transform on projective varieties and boundary value problems for systems of linear PDE.

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June 15, 2011

Abstract

Let $G \subset \mathbb{C}P^n$ be a linearly convex compact with smooth boundary, $D = \mathbb{C}P^n \setminus G$, and let $D^* \subset (\mathbb{C}P^n)^*$ be the dual domain. Then for an algebraic, not necessarily reduced, complete intersection subvariety $V$ of dimension $d$ we construct an explicit inversion formula for the complex Radon transform $R_V : H^{d,d-1}(V \cap D) \to H^{1,0}(D^*)$, and explicit formulas for solutions of an appropriate boundary value problem for the corresponding system of differential equations with constant coefficients on $D^*$.

1 Introduction.

Complex Radon-type transforms on complex projective varieties were introduced in different forms and with different purposes in the works of Fantappie [Fa1], Martineau [Mar2], Andreotti, Norguet [AN1, AN2], Eastwood, Penrose, Wells [Pe, EPW], Gindikin, Henkin, Polyakov [GH, HP1], . . . . In a recent paper [HP2] we have shown that the complex Radon transform realizes an isomorphism between the quotient-space of residual $\bar{\partial}$-cohomologies $H^{d,d-1}(V \cap D)/H^{d,d-1}(V)$ of algebraic (not necessarily reduced) $d$-dimensional locally complete intersection $V$ in a linearly concave domain $D$ of $\mathbb{C}P^n$ and the space of holomorphic solutions of the associated homogeneous system of linear differential equations with constant coefficients in the dual domain $D^* \subset (\mathbb{C}P^n)^*$.

In the present paper for an arbitrary algebraic complete intersection $V$ and a smoothly bounded linearly convex compact $G$ in $\mathbb{C}P^n$ we construct an explicit inversion formula for complex Radon transform on $V \cap D$, where $D = \mathbb{C}P^n \setminus G$. This inversion formula is based on the explicit formulas for solutions of appropriate boundary value problems for the associated with $V$ system of differential equations with constant coefficients in the dual domain $D^*$. Those formulas are motivated by the “explicit fundamental principle” of Berndtsson-Passare [BP].

To formulate the main result of the present paper we introduce the following notations. Let $(z_0, \ldots, z_n)$ and $(\xi_0, \ldots, \xi_n)$ be the homogeneous coordinates of points $z \in \mathbb{C}P^n$ and $\xi \in (\mathbb{C}P^n)^*$. Let $\langle \xi \cdot z \rangle \overset{\text{def}}{=} \sum_{k=0}^{n} \xi_k \cdot z_k$, and let $\mathbb{C}P^{n-1}_\xi$ denote the hyperplane

$$\mathbb{C}P^{n-1}_\xi = \left\{ z \in \mathbb{C}P^n : \langle \xi \cdot z \rangle = 0 \right\}.$$
Following [Mar1] and [GH] we call a domain $D \subset \mathbb{CP}^n$ linearly concave, if there exists a continuous map $D \ni z \rightarrow \xi(z) \in (\mathbb{CP}^n)^*$ satisfying

$$z \in \mathbb{CP}^{n-1}_{\xi(z)} \subset D.$$  

A compact $G \subset \mathbb{CP}^n$ is called linearly convex, if the domain $D = \mathbb{CP}^n \setminus G$ is linearly concave. The set of hyperplanes, which are contained in the linearly concave domain $D$, forms the dual domain $D^* \subset (\mathbb{CP}^n)^*$. We may assume without loss of generality that the hyperplane $\{z \in \mathbb{CP}^n : z_0 = 0\}$ is contained in $D$.

We will denote by $H(D^*, \mathcal{O}(l))$ the space of holomorphic functions of homogeneity $l$ on $D^*$. Let $\{\tilde{P}_j\}_{j=1}^m$ be homogeneous polynomials of projective coordinates, let $\{P_j = \tilde{P}_j(1, z_1, \ldots, z_n)\}$ be the corresponding polynomials of affine coordinates, and let $V \subset \mathbb{CP}^n$ be the algebraic subvariety $V = \{z \in \mathbb{CP}^n : \tilde{P}_1(z) = \cdots = \tilde{P}_m(z) = 0\}$.  

(1) From [Mar2] we obtain that for $\forall g \in H(D^*, \mathcal{O}(l-1))$ with $l < 0$ the solution $f \in H(D^*, \mathcal{O}(l))$ of the equation

$$\frac{\partial f}{\partial \xi_0} = g$$

exists and is unique, and therefore the operators

$$D_j = -\left(\frac{\partial}{\partial \xi_0}\right)^{-1} \frac{\partial}{\partial \xi_j} \quad \text{for } j = 1, \ldots, n,$$

(2) are well defined on the spaces $H(D^*, \mathcal{O}(l))$ for $l < 0$.

For a polynomial $R(u) = \sum_{|I|=0}^r R_I u_1^{i_1} \cdots u_n^{i_n}$ of degree $r$ we denote by $R(D)$ the operator

$$R(D) = \sum_{|I|=0}^r R_I \cdot D_1^{i_1} \cdots D_n^{i_n}.$$  

We denote by $\{Q^{(k)}\}_{j=1}^m$ the vector-polynomials $Q^{(k)}(\zeta, z) = \{Q_1^{(k)}(\zeta, z), \ldots, Q_n^{(k)}(\zeta, z)\}$, such that

$$P_k(\zeta) - P_k(z) = \sum_{j=1}^n (\zeta_j - z_j)Q_j^{(k)}(\zeta, z).$$

For a linearly convex compact in $\mathbb{C}^n \subset \mathbb{CP}^n$

$$G = \{z \in \mathbb{C}^n : \rho(z) \leq 0\},$$

(3) such that $D = \mathbb{CP}^n \setminus G$ is a linearly concave domain and $\rho \in C^\infty(\mathbb{CP}^n)$, we denote

$$\eta(\zeta) = (\eta_0(\zeta), \eta'(\zeta)) = (\eta_0(\zeta), \eta_1(\zeta), \ldots, \eta_n(\zeta)), \quad \text{and} \quad \eta_0(\zeta) = \sum_{j=1}^n \zeta_j \eta_j(\zeta), \quad \eta_j(\zeta) = \frac{\partial \rho}{\partial \zeta_j}(\zeta).$$

(4)
Theorem 1. Let $G$ be a linearly convex compact as in (3), $D = \mathbb{C}P^n \setminus G$, and let $V \subset \mathbb{C}P^n$ be a complete intersection algebraic subvariety as in (1). Then any function $g \in \mathcal{H}(D^*, \mathcal{O}(-1))$, satisfying the system of differential equations

$$\hat{P}_j \left( \frac{\partial}{\partial \xi} \right) g(\xi) = 0, \text{ for } j = 1, \ldots, m, \text{ and } \xi \in D^*,$$

may be represented through its values on the infinitesimal neighborhood of the set

$$\{ \xi \in D^*: \xi = \eta(\zeta) \text{ for } \zeta \in V \cap bG \}$$

by an explicit formula of Cauchy-Fantappie-Leray type:

$$g(\xi) = (-1)^{n-m-1} \frac{(n-1)!}{(2\pi i)^n(n-m-1)!} \int_{(\zeta,\mu) \in bG \times \Lambda} \frac{d\zeta}{(\xi_0 + \xi' \cdot \zeta)} \wedge \partial \left( \frac{1}{P_1(\zeta)} \right) \wedge \cdots \wedge \partial \left( \frac{1}{P_m(\zeta)} \right) \wedge \omega'_0 \left( \vartheta(\mu, \zeta, \mathcal{D}) \right) \left( \frac{\partial^{n-m-1} g}{\partial \eta_0^{n-m-1}}(\eta(\zeta)) \right),$$

(6)

where

$$\vartheta(\mu, \zeta, \mathcal{D}) = \sum_{k=1}^{m} \mu_k Q^{(k)}(\zeta, \mathcal{D}) + \left( 1 - \sum_{k=1}^{m} \mu_k \right) \eta'(\zeta)$$

$$\omega'_0(\vartheta) = \sum_{j=1}^{n} (-1)^{j-1} \partial_j \wedge_{i \neq j} d\vartheta,$$

and the integral in (6) is understood as

$$\lim_{t \to 0} \int_{T^e_\mathcal{P}(t) \times \Lambda} \frac{d\varphi_\delta(\zeta) \wedge d\zeta}{(\xi_0 + \xi' \cdot \zeta)} \wedge \omega'_0 \left( \vartheta(\mu, \zeta, \mathcal{D}) \right) \left( \frac{\partial^{n-m-1} g}{\partial \eta_0^{n-m-1}}(\eta(\zeta)) \right)$$

with an arbitrary function $\varphi_\delta \in \mathcal{E}_c(\mathbb{C}^n)$ satisfying

$$\varphi_\delta(\zeta) = \begin{cases} 1 & \text{if } \rho(\zeta) \leq 0, \\ 0 & \text{if } \rho(\zeta) > \delta, \end{cases}$$

(7)

$$\Lambda = \left\{ \mu \in \mathbb{R}^m_+: \sum_{k=1}^{m} \mu_k \leq 1 \right\},$$

$$T^e_\mathcal{P}(t) = \left\{ z \in \mathbb{C}^n : |P_k(z)| = \epsilon_k(t) \text{ for } k = 1, \ldots, m \right\},$$

(8)

and $\epsilon(t) = (\epsilon_1(t), \ldots, \epsilon_m(t))$ being an admissible path in the sense of Coleff-Herrera, i.e. an analytic map $\epsilon: [0, 1] \to \mathbb{R}^m_+$, satisfying

$$\lim_{t \to 0} \epsilon_m(t) = 0, \quad \lim_{t \to 0} \frac{\epsilon_j(t)}{\epsilon_{j+1}(t)} = 0 \text{ for } \forall \ l \in \mathbb{Z}_+.$$
Remarks.

- An earlier version of Theorem I was proved in [He] for the case of the variety \( V \) transversally intersecting \( bG \), i.e.
  \[
  d\rho \wedge dP_1 \wedge \ldots \wedge dP_m \neq 0 \text{ on } V \cap bG.
  \]

- Theorem I generalizes for the case of general boundary value problems results of Fantappie [Fa1, Fa2], Leray [L1, L2], Rigat [R] on explicit solutions of the holomorphic Cauchy (or Goursat) problems for systems of linear differential equations with constant coefficients. Important results on explicit solutions of nonstandard boundary value problems for two-dimensional linear integrable PDE were obtained by Fokas [Fo].

A corollary of Theorem I presented below is an application of the result of this theorem to the complex Radon transform. To formulate this corollary we use definitions from [HP2].

A current \( f \) in \( D \) with support in \( V \cap D \) is called a residual current \( f \in C^{n-m,n-m-1}(V \cap D) \), if

\[
f = \tilde{f} \wedge \bar{\partial} \left( \frac{1}{P_1} \right) \wedge \ldots \wedge \bar{\partial} \left( \frac{1}{P_m} \right),
\]

where \( \tilde{f} \in \mathcal{E}^{(n,n-m-1)}. \) A residual current \( f \) is called \( \bar{\partial} \)-closed (denoted \( f \in Z^{n-m,n-m-1}(V \cap D) \)) if \( \tilde{f} = \sum_{k=1}^{m} P_k \cdot \Omega_k \) with \( \Omega_k \in \mathcal{E}^{(n,n-m)}. \) We denote by \( H^{n-m,n-m-1}(V \cap D) \) the space \( Z^{n-m,n-m-1}(V \cap D)/\partial C^{n-m,n-m-2}(V \cap D) \) if \( n-m \geq 2 \) and \( H^{1,0}(V \cap D) = Z^{1,0}(V \cap D) \) if \( n-m = 1. \)

**Corollary 1.** If under the assumptions of Theorem I the coefficient \( f_0 \) of a closed holomorphic 1-form \( f = \sum_{j=0}^{n} f_j d\xi_j \) of homogeneity \((-1)\) on \( D^* \) satisfies the system of equations (5), then \( f \) is the complex Radon transform

\[
f(\xi) = R_V[\phi](\xi) = \sum_{j=0}^{n} \left( \int_{\xi \in D} \zeta_j \phi \wedge \bar{\partial} \left( \frac{1}{\zeta \cdot \xi} \right) \right) d\xi_j
\]

of a residual \( \bar{\partial} \)-cohomology class \( \phi \in H^{n-m,n-m-1}(V \cap D) \). This cohomology class corresponds by Serre-Malgrange duality to the functional \( \phi^* \in \mathcal{H}'(V \cap G) \), defined on \( \forall h \in \mathcal{H}(V \cap G) \) by the equality

\[
\langle \phi^*, h \rangle = (-1)^{m-n-1} \frac{(n-1)!}{(2\pi i)^{n-1}(n-m-1)!} \int_{bG \times \Lambda} h(\zeta) d\zeta \wedge \bar{\partial} \left( \frac{1}{P_1(\zeta)} \right) \wedge \ldots \wedge \bar{\partial} \left( \frac{1}{P_m(\zeta)} \right) \wedge \omega_0' \left( \partial(\mu, \zeta, D) \right) \left( \frac{\partial^{n-m-1} f_0}{\partial \eta^{n-m}_{0-1}}(\eta(\zeta)) \right). \quad (10)
\]

The proof of Theorem I relies on two ingredients: a version of the Martineau type inversion formula [Mar2, GH] for the Fantappie transform, given here in Proposition 3.1, and an interpolation formula for holomorphic functions from a complete intersection subvariety \( V \cap G \), not necessarily reduced, to the linearly convex domain \( G \setminus bG \). This interpolation formula (II), proved in Proposition 2.1 below, is based on the results of Weil [W], Leray [L2], Norguet [N], and Coleff-Herrera [CH].
2 Cauchy-Leray formula on pseudo-convex complete intersections.

In Proposition 2.1 below we prove a residual interpolation formula in a linearly convex domain, which can also be considered as the Cauchy-Leray formula for holomorphic functions on complete intersections. On the one hand the integral term in equality (11) of this proposition presents a new interpolation formula for holomorphic functions in linearly convex domains. On the other hand equality (11) gives a more precise version of the duality theorem of Dickenstein-Sessa and Passare (see [DS, Pa]).

**Proposition 2.1.** Let \( G \) be a linearly convex compact as in (3), and let \( \{P_k\}_{k=1}^m \) be polynomials such that the analytic set

\[
V_G = \{ z \in G : P_1(z) = \cdots = P_m(z) = 0 \}
\]

is a complete intersection in \( G \). Then for \( h \in \mathcal{H}(G) \) the following formula holds for \( z \in G \setminus bG \)

\[
h(z) = \frac{(n-1)!}{(2\pi i)^n} \left[ \lim_{\epsilon \to 0} \int_{T_{I_P}^\epsilon(t) \times \Lambda} \frac{h(\zeta)}{P_k(\zeta)} d\phi_\delta(\zeta) \wedge \omega_0 \left( \sum_{k=1}^m \mu_k Q^{(k)}(\zeta, z) \right) + \left( 1 - \sum_{k=1}^m \mu_k \right) \frac{\eta'(\zeta) \cdot (\zeta - z)}{\eta'(\zeta) \cdot (\zeta - z)} \right] + \sum_{k=1}^m h_k(z) \cdot P_k(z),
\]

where \( T_{I_P}^\epsilon(t) \) is defined in (8), \( \{\epsilon_k(\epsilon)\}_{k=1}^m \) is an admissible path, function \( \phi_\delta(\zeta) \) is a function satisfying (7), \( \eta(\zeta) \) is defined in (4), and \( h_k \in H(G) \).

**Proof.** We start from the following Weil-Leray-Norguet type integral formula

\[
h(z) = \frac{(n-1)!}{(2\pi i)^n} \left[ \sum_{0 \leq |I| \leq m} \int_{\sigma_I(t) \times \Lambda} \frac{h(\zeta)}{P_k(\zeta)} \omega_0 \left( \sum_{i \in I} \mu_i Q^{(i)}(\zeta, z) \right) P_i(\zeta) - P_i(z) \right.
\]

\[
+ \left. \left( 1 - \sum_{i \in I} \mu_i \right) \frac{\eta'(\zeta) \cdot (\zeta - z)}{\eta'(\zeta) \cdot (\zeta - z)} \right] \wedge \omega(\zeta)
\]

(12)

for a holomorphic function \( h \) on the compact

\[
U^\epsilon(t) = \left\{ z \in \mathbb{C}P^n : \rho(z) \leq 0, \ |P_k(z)| \leq \epsilon_k(t) \right\}_{k=1}^m,
\]

where

\[
\sigma_I(t) = \left\{ z \in G : \rho(z) = 0, \ |P_i(z)| = \epsilon_i(t) \right\}_{i \in I}, \ |P_k(z)| \leq \epsilon_k(t) \}
\]

and

\[
\Lambda_I = \left\{ \mu \in \mathbb{R}_+^{|I|} : \sum_{i \in I} \mu_i \leq 1 \right\}.
\]

To transform formula (12) into a residue-type formula we assume that function \( h \) is defined in

\[
G_\delta = \left\{ z \in \mathbb{C}P^n : \rho(z) \leq \delta \right\}
\]

for some \( \delta > 0 \), define

\[
T^\epsilon_I(t) = \left\{ z : 0 \leq \rho(z) \leq \delta, \ |P_i(z)| = \epsilon_i(t), \right\}_{i \in I}, \ |P_k(z)| \leq \epsilon_k(t) \}
\]

and let
and consider the chain

\[ C = \sum_{0 \leq |I| \leq m} T^*_I(t) \times \Lambda_I \]

with the boundary

\[ B = \sum_{0 \leq |I| \leq m} [\sigma^*_I(t) - \sigma^*_I(\delta, t)] \times \Lambda_I + \sum_{0 \leq |I| \leq m} T^*_I(t) \times \Gamma_I, \]

where

\[ \sigma^*_I(\delta, t) = \left\{ z \in G : \rho(z) = \delta, \{ |P_i(z)| = \epsilon_i(t) \}_{i \in I}, \{ |P_k(z)| \leq \epsilon_k(t) \}_{k \not\in I} \right\}, \]

and

\[ \Gamma_I = \left\{ \mu \in \mathbb{R}^{|I|}_+: \sum_{i \in I} \mu_i = 1 \right\}. \]

We consider a function \( \phi_{\delta} \in C^\infty(\mathbb{C}^n) \) satisfying (7) and apply the Stokes’ formula to the form

\[ h(\zeta)\phi_{\delta}(\zeta)\omega_0^0 \left( \sum_{k=1}^{m} \mu_k \frac{Q^{(k)}(\zeta, z)}{P_k(\zeta) - P_i(z)} + \left( 1 - \sum_{k=1}^{m} \mu_k \right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta), (\zeta - z) \rangle} \right) \wedge \omega(\zeta) \]

on the chain \( C \). Then, using equality \( \phi_{\delta}|_{\sigma^*_I(\delta, \zeta)} = 0 \), we obtain

\[ \sum_{0 \leq |I| \leq m} \int_{\sigma^*_I(t) \times \Lambda_I} h(\zeta)\phi_{\delta}(\zeta)\omega_0^0 \left( \sum_{i \in I} \mu_i \frac{Q^{(i)}(\zeta, z)}{P_i(\zeta) - P_i(z)} + \left( 1 - \sum_{i \in I} \mu_i \right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta), (\zeta - z) \rangle} \right) \wedge \omega(\zeta) \]

\[ = - \sum_{0 \leq |I| \leq m} \int_{T^*_I(t) \times \Gamma_I} h(\zeta)\phi_{\delta}(\zeta)\omega_0^0 \left( \sum_{i \in I} \mu_i \frac{Q^{(i)}(\zeta, z)}{P_i(\zeta) - P_i(z)} \right. \]

\[ + \left. \left( 1 - \sum_{i \in I} \mu_i \right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta), (\zeta - z) \rangle} \right) \wedge \omega(\zeta) \]

\[ + \sum_{0 \leq |I| \leq m} \int_{T^*_I(t) \times \Lambda_I} h(\zeta)d\phi_{\delta}(\zeta) \wedge \omega_0^0 \left( \sum_{i \in I} \mu_i \frac{Q^{(i)}(\zeta, z)}{P_i(\zeta) - P_i(z)} \right. \]

\[ + \left. \left( 1 - \sum_{i \in I} \mu_i \right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta), (\zeta - z) \rangle} \right) \wedge \omega(\zeta). \]

From the dimensional considerations we obtain that

\[ \int_{T^*_I(t) \times \Gamma_I} h(\zeta)\phi_{\delta}(\zeta)\omega_0^0 \left( \sum_{i \in I} \mu_i \frac{Q^{(i)}(\zeta, z)}{P_i(\zeta) - P_i(z)} + \left( 1 - \sum_{i \in I} \mu_i \right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta), (\zeta - z) \rangle} \right) \wedge \omega(\zeta) \]

\[ = \int_{T^*_I(t) \times \Gamma_I} h(\zeta)\phi_{\delta}(\zeta)\omega_0^0 \left( \sum_{i \in I} \mu_i \frac{Q^{(i)}(\zeta, z)}{P_i(\zeta) - P_i(z)} \right) \wedge \omega(\zeta) = 0, \]
and therefore the equality above can be rewritten as

\[\sum_{0 \leq |I| \leq m} \int_{\sigma^*_I(t) \times \Lambda_I} h(\zeta) \omega'_0 \left( \sum_{i \in I} \mu_i \frac{Q^{(i)}(\zeta, z)}{P_i(\zeta) - P_i(z)} + \left( 1 - \sum_{i \in I} \mu_i \right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta) \cdot (\zeta - z) \rangle} \right) \wedge \omega(\zeta)
\]

\[= \sum_{0 \leq |I| \leq m} \int_{T^*_I(t) \times \Lambda_I} h(\zeta) d\phi_\delta(\zeta) \wedge \omega'_0 \left( \sum_{i \in I} \mu_i \frac{Q^{(i)}(\zeta, z)}{P_i(\zeta) - P_i(z)} + \left( 1 - \sum_{i \in I} \mu_i \right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta) \cdot (\zeta - z) \rangle} \right) \wedge \omega(\zeta).
\]

We transform the right-hand side of the last formula for \( z \in U^c(t) \) as follows

\[\sum_{0 \leq |I| \leq m} \int_{T^*_I(t) \times \Lambda_I} h(\zeta) d\phi_\delta(\zeta) \wedge \omega'_0 \left( \sum_{i \in I} \mu_i \frac{Q^{(i)}(\zeta, z)}{P_i(\zeta) - P_i(z)} + \left( 1 - \sum_{i \in I} \mu_i \right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta) \cdot (\zeta - z) \rangle} \right) \wedge \omega(\zeta)
\]

\[= \sum_{0 \leq |I| \leq m} \int_{T^*_I(t) \times \Lambda_I} \frac{h(\zeta)}{P_{i_1}(\zeta)} d\phi_\delta(\zeta) \wedge \omega'_0 \left( \mu_{i_1} Q^{(i_1)}(\zeta, z) + \sum_{k=2}^{|I|} \mu_{i_k} \frac{Q^{(i_k)}(\zeta, z)}{P_{i_k}(\zeta) - P_{i_k}(z)} + \left( 1 - \sum_{i \in I} \mu_i \right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta) \cdot (\zeta - z) \rangle} \right) d\zeta
\]

\[+ \sum_{0 \leq |I| \leq m} \sum_{r=1}^{\infty} (P_{i_1}(z))^r \int_{T^*_I(t) \times \Lambda_I} \frac{h(\zeta)}{(P_{i_1}(\zeta))^{r+1}} d\phi_\delta(\zeta) \wedge \omega'_0 \left( \mu_{i_1} Q^{(i_1)}(\zeta, z) + \sum_{k=2}^{|I|} \mu_{i_k} \frac{Q^{(i_k)}(\zeta, z)}{P_{i_k}(\zeta) - P_{i_k}(z)} + \left( 1 - \sum_{i \in I} \mu_i \right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta) \cdot (\zeta - z) \rangle} \right) d\zeta
\]

\[= \sum_{0 \leq |I| \leq m} \int_{T^*_I(t) \times \Lambda_I} \frac{h(\zeta)}{\prod_{i \in I} P_i(\zeta)} d\phi_\delta(\zeta) \wedge \omega'_0 \left( \sum_{i \in I} \mu_i Q^{(i)}(\zeta, z) + \left( 1 - \sum_{i \in I} \mu_i \right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta) \cdot (\zeta - z) \rangle} \right) d\zeta
\]

\[+ \sum_{0 \leq |I| \leq m} \sum_{r=1}^{\infty} (P_{i_1}(z))^r \int_{T^*_I(t) \times \Lambda_I} \frac{h(\zeta)}{(P_{i_1}(\zeta))^{r+1}} d\phi_\delta(\zeta) \wedge \omega'_0 \left( \mu_{i_1} Q^{(i_1)}(\zeta, z) + \sum_{k=2}^{|I|} \mu_{i_k} \frac{Q^{(i_k)}(\zeta, z)}{P_{i_k}(\zeta) - P_{i_k}(z)} + \left( 1 - \sum_{i \in I} \mu_i \right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta) \cdot (\zeta - z) \rangle} \right) d\zeta
\]
+ \sum_{0 \leq |I| \leq m} \sum_{r=1}^{\infty} (P_{i_r}(z))^r \int_{T^r(t) \times \Lambda_I} \frac{h(\zeta)}{\prod_{k=1}^{s-1} P_{i_k}(\zeta) (P_{i_k}(\zeta))^{r+1}} d\phi_d(\zeta) \wedge \omega_0^r \left( \sum_{i \in I} \mu_i Q^{(i)}(\zeta, z) + \left( 1 - \sum_{i \in I} \mu_i \right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta) \cdot (\zeta - z) \rangle} \right) \wedge d\zeta \tag{14}

where \( s = |I| \), and “\( \cdots \)” stands for the terms of the form

\[
\sum_{0 \leq |I| \leq m} \sum_{r=1}^{\infty} (P_{j_p}(z))^r \int_{T^r(t) \times \Lambda_I} \frac{h(\zeta)}{\prod_{k=1}^{p-1} P_{i_k}(\zeta) (P_{i_k}(\zeta))^{r+1}} d\phi_d(\zeta) \wedge \omega_0^r \left( \sum_{k=p+1}^{p} \mu_i Q^{(i)}(\zeta, z) + \sum_{k=p+1}^{p} \mu_i \frac{Q^{(i)}(\zeta, z)}{P_{i_k}(\zeta) - P_{i_k}(\zeta)} + \left( 1 - \sum_{i \in I} \mu_i \right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta) \cdot (\zeta - z) \rangle} \right) \wedge d\zeta,
\]

for \( 1 < p < s \).

Denoting then

\[
g_k(z, t) = \frac{(n - 1)!}{(2\pi i)^n} \sum_{r=0}^{\infty} (P_k(z))^r \sum_{\substack{k = i_p \in I \\ 1 \leq |I| \leq m}} \int_{T^r(t) \times \Lambda_I} \frac{h(\zeta)}{\prod_{j=1}^{p-1} P_j(\zeta) (P_k(\zeta))^{r+1}} d\phi_d(\zeta) \wedge \omega_0^r \left( \sum_{k=p+1}^{p} \mu_i Q^{(i)}(\zeta, z) + \sum_{k=p+1}^{p} \mu_i \frac{Q^{(i)}(\zeta, z)}{P_{i_k}(\zeta) - P_{i_k}(\zeta)} + \left( 1 - \sum_{i \in I} \mu_i \right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta) \cdot (\zeta - z) \rangle} \right) \wedge d\zeta,
\]

and using equalities (12), (13), and (14) we obtain the following equality for \( z \in U^r(t) \)

\[
h(z) = \frac{(n - 1)!}{(2\pi i)^n} \left[ \sum_{0 \leq |I| \leq m} \int_{T^r(t) \times \Lambda_I} \frac{h(\zeta)}{\prod_{i \in I} P_i(\zeta)} d\phi_d(\zeta) \wedge \omega_0^r \left( \sum_{i \in I} \mu_i Q^{(i)}(\zeta, z) + \left( 1 - \sum_{i \in I} \mu_i \right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta) \cdot (\zeta - z) \rangle} \right) \wedge d\zeta \right] + \sum_{k=1}^{m} g_k(z, t) \cdot P_k(z). \tag{15}
\]

To transform the equality above into equality (11) we have to pass to the limit as \( t \to 0 \) in the right-hand side of (15). To prove the existence of limits of the integrals in the right-hand side of equality above when \( t \to 0 \) we use the results of Coleff and Herrera. Since all integrals in (15) are the integrals of the forms with compact support, those integrals can be reduced to the integrals of the forms over polydisks. In the proposition below we collect the statements from [CH], which are used in the completion of the proof of Proposition 2.1.

**Proposition 2.2.** Let \( D^n = \{ z \in \mathbb{C}^n : |z_j| < 1, \ j = 1, \ldots, n \} \) be a polydisk in \( \mathbb{C}^n \), \( \{ P_k \}_{k=1}^{m} \) - a set of polynomials,

\[
V = \{ z \in D^n : P_1(z) = \cdots = P_m(z) = 0 \}
\]
- an algebraic variety of pure dimension $n - m$ such that the restriction to $V$ of the projection

$$\pi : D^n \to D^{n-m},$$
defined by the formula $\pi(z_1, \ldots, z_n) = (z_{m+1}, \ldots, z_n)$ is a finite analytic covering, such that the origin is an isolated point in $\pi^{-1}(0) \cap V$. Let $z' = (z_1, \ldots, z_m)$, and $z'' = (z_{m+1}, \ldots, z_n)$. Then

(i) there exists an analytic function $g$ on $V$ such that for an arbitrary form $\alpha \in E_c^{(n,n-m)}(D^n)$ the following equality holds

$$\lim_{t \to 0} \int_{T_{[p]}(t)} \frac{\alpha(\zeta)}{P_k(\zeta)} = \lim_{\gamma \to 0} \int_{V \cap \{|g(\zeta)| > \gamma\}} \text{res}_{(p,\pi)}[\alpha](\zeta),$$

where

$$\text{res}_{(p,\pi)}[\alpha](\zeta) = \lim_{t \to 0} \int_{T_{[p]}(t)} \frac{\hat{\alpha}(\zeta)}{\prod_{k=1}^{m} P_k(\zeta)},$$

and the limit in the left-hand side of (16) exists,

(ii) the limit in the left-hand side of (16) defines a continuous linear functional on $E_c^{(n,n-m)}$,

(iii) if $\alpha$ admits a representation $\alpha = f(\zeta)\,d\bar{\zeta}'_{m+1} \wedge d\zeta_n \wedge d\zeta$, then there exist $N \in \mathbb{N}$ and meromorphic functions $\{h_I(\zeta)\}_{|I|=0}^N$ such that the equality

$$\text{res}_{(p,\pi)}[\alpha](\zeta) = \sum_{|I|=0}^N f_I(\zeta) \cdot h_I(\zeta)$$

holds, where $f_I$ are holomorphic Taylor coefficients of $f$ with respect to $\zeta'$.

(iv) under conditions of (iii) the following equality holds

$$\lim_{t \to 0} \int_{T_{[p]}(t)} \frac{\alpha(\zeta)}{\prod_{k=1}^{m} \bar{P}_k(\zeta)} = \sum_{|I|=0}^N \lim_{\gamma \to 0} \int_{V \cap \{|g(\zeta)| > \gamma\}} f_I(\zeta) \cdot h_I(\zeta).$$

Using the existence of the limit in the left-hand side of (16) we obtain the existence of the limit

$$\lim_{t \to 0} \int_{T_{[p]}(t) \times \Lambda_I} \frac{h_I(\zeta)}{\prod_{i \in I} P_i(\zeta)} \, d\phi_0(\zeta) \wedge \omega'_0\left(\sum_{i \in I} \mu_i Q^{(i)}(\zeta, z) + \left(1 - \sum_{i \in I} \mu_i\right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta) \cdot (\zeta - z) \rangle}\right) \wedge d\zeta$$

for $I = (1, \ldots, m)$.

Also, motivated by equality (16) we define for $I \subset (1, \ldots, m)$ with $|I| = r$ and $\alpha \in E_c^{(n,n-r)}(D^n)$

$$\lim_{t \to 0} \int_{T_{[p]}(t)} \frac{\alpha(\zeta)}{\prod_{i \in I} P_i(\zeta)} = \lim_{\gamma \to 0} \int_{V \cap \{|g(\zeta)| > \gamma\}} \lim_{t \to 0} \int_{T_{[p]}(t)} \frac{\hat{\alpha}(w)}{\prod_{i \in I} \bar{P}_i(w)},$$

(19)
where we use the notations from (16), and additionally
\[ \hat{T}_I^*(\zeta, t) = \left\{ w \in \pi^{-1}(\zeta_{m+1}, \ldots, \zeta_n) : \left\{ |\hat{P}_i(w)| = \epsilon_i(t) \right\}_{i \in I}, \left\{ |\hat{P}_k(w)| \leq \epsilon_k(t) \right\}_{k \notin I} \right\}. \]

Now, using formula (19), we can pass to the limit as \( t \to 0 \) in the right-hand side of (15) for the integrals from (15) with \( I \neq (1, \ldots, m) \). For such integrals we have the following lemma.

**Lemma 2.3.** For an arbitrary fixed \( z \in G \setminus bG \) the following equality holds
\[ \lim_{t \to 0} \int_{T_I(t) \times \Lambda_I} \frac{h(\zeta)}{\prod_{i \in I} P_i(\zeta)} d\phi(\zeta) \wedge \omega_0' \left( \sum_{i \in I} \mu_i Q^{(i)}(\zeta, z) + \left( 1 - \sum_{i \in I} \mu_i \right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta) \cdot (\zeta - z) \rangle} \right) \wedge d\zeta = 0, \quad (20) \]
if \( I \neq (1, \ldots, m) \).

**Proof.** To prove equality (20) we denote
\[ \omega_I(\zeta, z) = \int_{\Lambda_I} \omega_0' \left( \sum_{i \in I} \mu_i Q^{(i)}(\zeta, z) + \left( 1 - \sum_{i \in I} \mu_i \right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta) \cdot (\zeta - z) \rangle} \right) \wedge d\zeta \]
and rewrite the integral in the left-hand side of (20) as
\[ \lim_{t \to 0} \int_{T_I(t)} \frac{h(\zeta)}{\prod_{i \in I} P_i(\zeta)} d\phi(\zeta) \wedge \omega_I(\zeta, z). \]

Then using formula (19) we rewrite the last limit as
\[ \lim_{t \to 0} \int_{T_I(t)} \frac{h(\zeta)}{\prod_{i \in I} P_i(\zeta)} d\phi(\zeta) \wedge \omega_I(\zeta, z) = \lim_{\gamma \to 0} \int_{V \cap \{|g(\zeta)| > \gamma\}} \lim_{t \to 0} \left\{ \left| \hat{P}_i(w) \right| = \epsilon_i(t) \text{ for } i \in I, \left| \hat{P}_k(w) \right| \leq \epsilon_k(t) \text{ for } k \notin I \right\} \frac{h(w)}{\prod_{i \in I} P_i(w)} d\phi(w) \wedge \omega_I(w, z), \]
therefore reducing the proof of the Lemma to the proof of equality
\[ \lim_{t \to 0} \int_{\left\{ \left| \hat{P}_i(w) \right| = \epsilon_i(t) \text{ for } i \in I, \left| \hat{P}_k(w) \right| \leq \epsilon_k(t) \text{ for } k \notin I \right\}} \frac{h(w)}{\prod_{i \in I} P_i(w)} d\phi(w) \wedge \omega_I(w, z) = 0. \quad (21) \]

To prove the last equality we apply the resolution of singularities \([H]\) to the isolated point
\[ V \cap \pi^{-1}(\zeta'') = \left\{ \zeta' \in \mathbb{C}^m : \hat{P}_1(\zeta', \zeta'') = \cdots = \hat{P}_m(\zeta', \zeta'') = 0 \right\} \]
in \( \mathbb{C}^m(\zeta'') = \pi^{-1}(\zeta'') \) for a fixed \( \zeta'' = (\zeta_{m+1}, \ldots, \zeta_n) \). Then in a small enough neighborhood of the origin the lifted variety becomes a normal crossing algebraic variety of the form
\[ S = \{ u \in \mathbb{C}^m : u_1^{\alpha_1} = \cdots = u_m^{\alpha_m} = 0 \}, \]
and the limit in (21) becomes

\[
\lim_{t \to 0} \int \left\{ \left| \hat{P}_i(w) \right| = \epsilon_i(t) \text{ for } i \in I, \\
\left| \hat{P}_k(w) \right| \leq \epsilon_k(t) \text{ for } k \notin I \right\} \frac{h(w)}{\prod_{i \in I} \hat{P}_i(w)} d\phi(w) \wedge \omega_I(w, z) \\
= \lim_{t \to 0} \int \left\{ \left| u_{\alpha_i}^\alpha \right| = \epsilon_i(t) \text{ for } i \in I, \\
\left| u_{\alpha_k}^\alpha \right| \leq \epsilon_k(t) \text{ for } k \notin I \right\} \frac{h^*(u)}{\prod_{i \in I} u_{\alpha_i}^\alpha} d\phi^*(u) \wedge \omega^*_I(u, z) = 0.
\]

Using Lemma 2.3 we conclude that in passing to the limit as \( t \to 0 \) in equality (15) the only nonzero may be produced by the integral over \( T^\epsilon_I(t) \times \Lambda_I \), i.e. over \( T^\epsilon_{\{p\}}(t) \times \Lambda \). The analytic dependence on \( z \) of this limit follows from Lemma 2.4 below.

**Lemma 2.4.** Let \( D^n, V, \pi, \) and \( g \) be the same as in Proposition 2.2 and let \( T^\epsilon_{\{p\}}(t) \) be as in (8).

If \( F \in E_c^{(n, n-m)}(D^n) \) is a differential form with respect to variables \( \zeta \), with coefficients infinitely differentiable with respect to both variables \( \zeta \) and \( z \), and holomorphic with respect to variables \( z \), then

\[
R(z) = \lim_{t \to 0} \int_{T^\epsilon_{\{p\}}(t)} \frac{F(\zeta, z)}{\prod_{k=1}^m P_k(\zeta)}.
\]

is a holomorphic function.

**Proof.** Without loss of generality we may assume that \( F(\zeta, z) = f(\zeta, z) d\zeta_{m+1} \wedge d\zeta_n \wedge d\zeta \). Then, following [CH], we consider the Taylor series of \( f \) at \( \zeta \in V \) with respect to \( \zeta' \)

\[
f(w, z) \bigg|_{\pi^{-1}(\zeta''')} = \sum_{|I| + |J| = 0}^\infty f_{I, J}(\zeta, z) \cdot (w' - \zeta')^I \cdot (\bar{w}' - \bar{\zeta})^J,
\]

and using equality (18) obtain the existence of \( N \in \mathbb{N} \) and of meromorphic functions \( \{h_I(\zeta)\}_{|I|=0}^N \) such that the equality

\[
\lim_{t \to 0} \int_{T^\epsilon_{\{p\}}(t)} \frac{F(\zeta, z)}{\prod_{k=1}^m P_k(\zeta)} = \sum_{|I|=0}^N \lim_{\gamma \to 0} \int_{V \cap \{|g(\zeta)| > \gamma\}} f_I(\zeta, z) \cdot h_I(\zeta)
\]

holds.

If \( f(\zeta, z) \) is a polynomial with respect to \( z \), then the left-hand side of (22) is a polynomial as well. For an arbitrary \( f(\zeta, z) \in E_c(D^n) \) analytically depending on \( z \) we approximate it by polynomials, and then use the continuity of a residual current as a functional on \( E_c^{(n, n-m)} \), which follows from (ii) in Proposition 2.2.

Continuing with the proof of Proposition 2.1 we obtain from Lemmas 2.3 and 2.4 the following
\[
\sum_{0 \leq |I| \leq m} \lim_{t \to 0} \int_{T_I^*(t) \times \Lambda_I} \frac{h(\zeta)}{P_I(\zeta)} \prod_{i \in I} P_i(\zeta) d\phi_0(\zeta) \wedge \omega' \left( \sum_{i \in I} \mu_i Q^{(i)}(\zeta, z) \right) + \left( 1 - \sum_{i \in I} \mu_i \right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta) \cdot (\zeta - z) \rangle} \right) \wedge d\zeta
\]

\[
= \lim_{t \to 0} \int_{T_I^*(t) \times \Lambda_I} \frac{h(\zeta)}{P_I(\zeta)} \prod_{i \in I} P_i(\zeta) d\phi_0(\zeta) \wedge \omega' \left( \sum_{i \in I} \mu_i Q^{(i)}(\zeta, z) + \left( 1 - \sum_{i \in I} \mu_i \right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta) \cdot (\zeta - z) \rangle} \right) \wedge d\zeta
\]

with the right-hand side being holomorphic with respect to \(z\).

To prove the existence of coefficients \(h_k \in H(G)\) in (11), and therefore to complete the proof of Proposition 2.1 we notice that the functions

\[
u_t(z) = h(z) - \frac{(n-1)!}{(2\pi i)^n} \left[ \sum_{0 \leq |I| \leq m} \int_{T_I^*(t) \times \Lambda_I} \frac{h(\zeta)}{P_I(\zeta)} \prod_{i \in I} P_i(\zeta) d\phi_0(\zeta) \wedge \omega' \left( \sum_{i \in I} \mu_i Q^{(i)}(\zeta, z) + \left( 1 - \sum_{i \in I} \mu_i \right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta) \cdot (\zeta - z) \rangle} \right) \wedge d\zeta \right]
\]

form a family of holomorphic functions on the interior of \(G\) depending on \(t\) and converging to the holomorphic function

\[
u(z) = h(z) - \frac{(n-1)!}{(2\pi i)^n} \left[ \lim_{t \to 0} \int_{T_I^*(t) \times \Lambda_I} \frac{h(\zeta)}{P_I(\zeta)} \prod_{i \in I} P_i(\zeta) d\phi_0(\zeta) \wedge \omega' \left( \sum_{i \in I} \mu_i Q^{(i)}(\zeta, z) + \left( 1 - \sum_{i \in I} \mu_i \right) \frac{\eta'(\zeta)}{\langle \eta'(\zeta) \cdot (\zeta - z) \rangle} \right) \wedge d\zeta \right)
\]

on the interior of \(G\). Since for each \(t\) the function \(\nu_t\) defines a section of the sheaf of ideals, defined by the functions \(P_1, \ldots, P_m\) on \(G\) from H. Cartan’s Theorems (A) and (B) in [Ca] we obtain that the limit function \(u = \lim_{t \to 0} \nu_t\) admits a representation on the interior of \(G\)

\[
u(z) = \sum_{k=1}^m h_k(z) \cdot P_k(z)
\]

with \(h_k \in H(G)\).

\[\square\]

3 Proof of Theorem 1

Before proving Theorem 1 we present a version of the Martineau’s (see [Mar2]) inversion formula for the Fantappié transform from [GH], which is used in the proof.
For $f \in \mathcal{H}(D^*)$, following \[Mar2\] and \[GH\], we consider the analytic functional $\mu^f$ on the space $H(G)$ defined by the formula

$$
\mu^f(h) = \int_{bG_{-\nu}} h \cdot \Omega_f,
$$

(23)

where

$$
\Omega_f(z) = \frac{(-1)}{(2\pi)^n} \cdot \frac{\partial^n f}{\partial \eta_0^n} (\eta(z)) \omega'(\eta(z)) \bigwedge_{j=1}^n d \left( \frac{z_j}{z_0} \right),
$$

$$
G_{-\nu} = \{ z \in \mathbb{C}P^n \colon \rho(z) \leq -\nu \}, \text{ and a map } \eta : bG_{-\nu} \to (\mathbb{C}P^n)^* \text{ satisfies } \langle \eta(z) \cdot z \rangle = 0 \text{ for } z \in bG_{-\nu}.
$$

The indicatrice of Fantappiè of the functional $\mu^f$ is a holomorphic 1-form on $D^*$ defined by the formula

$$
\mathcal{F}_{\mu^f} = \sum_{k=0}^n \mu^f \left( \frac{z_k}{(\xi \cdot z)} \right) d\xi_k
$$

$$
= \frac{(-1)}{(2\pi)^n} \sum_{k=0}^n \left( \int_{bG_{-\nu}} \left( \frac{z_k}{(\xi \cdot z)} \right) \frac{\partial^n f}{\partial \eta_0^n} (\eta(z)) \omega'(\eta(z)) \bigwedge_{j=1}^n d \left( \frac{z_j}{z_0} \right) \right) d\xi_k.
$$

The most important application of the indicatrice of Fantappiè of $\mu^f$ is the inversion formula described in the proposition below.

**Proposition 3.1.** (Martineau type inversion formula. \[Mar2\], \[GH\].) Let $D \subset \mathbb{C}P^n$ be a linearly concave domain such that $D^* \subset \{ \xi_0 \neq 0 \}$, and let $f \in \mathcal{H}(D^*)$. Then the following equality holds:

$$
\mathcal{F}_{\mu^f}(\xi) = df(\xi),
$$

(24)

or

$$
\frac{(-1)}{(2\pi)^n} \int_{z \in bG_{-\nu}} \frac{z_k}{(\xi \cdot z)} \frac{\partial^n f}{\partial \eta_0^n} (\eta(z)) \omega'(\eta(z)) \bigwedge_{j=1}^n d \left( \frac{z_j}{z_0} \right) = \frac{\partial f}{\partial \xi_k}(\xi)
$$

for $k = 0, \ldots, n$, and $\xi \in D^*$.

Moreover, for $g \in \mathcal{H}(D^*, \mathcal{O}(-1))$ we have the following equality

$$
g(\xi) = \frac{(-1)}{(2\pi)^n} \int_{bG_{-\nu}} \frac{\partial^{n-1} g}{\partial \eta_0^{n-1}} (\eta(u)) \omega'(\eta(u)) \bigwedge \frac{u}{(\xi_0 + \xi' \cdot u)}
$$

(25)

where $\xi \in D^*$ and

$$
u_j = \frac{z_j}{z_0} \text{ for } j = 1, \ldots, n.
$$

To prove Theorem \[1\] we consider $g \in \mathcal{H}(D^*, \mathcal{O}(-1))$ satisfying the system of equations (5) and using equality (25) obtain the equality

$$
\frac{(-1)^{1+\deg P_k}}{(2\pi)^n} (\deg P_k)! \int_{bG_{-\nu}} \frac{P_k(u)}{(\xi_0 + \xi' \cdot u)^{1+\deg P_k}} \cdot \frac{\partial^{n-1} g}{\partial \eta_0^{n-1}} (\eta(u)) \omega'(\eta(u)) \bigwedge \frac{u}{(\xi_0 + \xi' \cdot u)}
$$

$$
= P_k \left( \frac{\partial}{\partial \xi} \right) [g](\xi) = 0.
$$
Then, using the Cauchy-Leray formula \[L2\] we obtain the density of the set of functions

\[
\left\{ \frac{1}{(\xi_0 + \xi' \cdot u)^{1+\deg P_k}} \right\}_{\xi \in D^*},
\]

in the space \(H(G)\), and therefore the equality

\[
\int_{bG_\nu} f(u) \cdot P_k(u) \cdot \frac{\partial^{n-1} g}{\partial \eta_0^{n-1}}(\eta(u)) \omega'(\eta(u)) \wedge du = 0 \tag{26}
\]

for an arbitrary \(f \in H(G)\).

Using notation \(4\) for \(\eta_0(w) = \langle \eta'(w) \cdot w \rangle\) and applying Proposition \(2.1\) we consider the function

\[
H_V(\xi, u) = \frac{(n - 1)!}{(2\pi i)^n} \lim_{t \to 0} \int_{T(p) \times \Lambda} \frac{d\phi_t(w)}{\prod_{k=1}^{m} P_k(w) \cdot (\xi_0 + \xi' \cdot w)} \wedge \omega_0' \left( \sum_{k=1}^{m} \mu_k Q^{(k)}(w, u) + \left(1 - \sum_{k=1}^{m} \mu_k\right) \frac{\eta'(w)}{(\eta_0(w) - \langle \eta'(w) \cdot u \rangle)} \right) \wedge dw, \tag{27}
\]

satisfying the equality

\[
H_V(\xi, u) = \frac{1}{(\xi_0 + \xi' \cdot u)} + \sum_{k=1}^{m} h_k(\xi, u) \cdot P_k(u)
\]

for \(u \in G\).

Using the equality above and equality \(26\) in equality \(25\) we obtain the following equality

\[
g(\xi) = \frac{(-1)}{(2\pi i)^n} \int_{bG_\nu} \frac{\partial^{n-1} g}{\partial \eta_0^{n-1}}(\eta(u)) \omega'(\eta(u)) \wedge du \frac{1}{(\xi_0 + \xi' \cdot u)} = \frac{(n - 1)!}{(2\pi i)^n} \prod_{k=1}^{m} P_k(w) \cdot (\xi_0 + \xi' \cdot w) \frac{\partial^{n-1} g}{\partial \eta_0^{n-1}}(\eta(u)) H_V(\xi, u) \omega'(\eta(u)) \wedge du,
\]

which after the substitution of expression \(27\) and the change of the order of integration becomes

\[
g(\xi) = \frac{(-1)}{(2\pi i)^n} \lim_{t \to 0} \int_{T(p) \times \Lambda} \frac{d\phi_t(w)}{\prod_{k=1}^{m} P_k(w) \cdot (\xi_0 + \xi' \cdot w)} \times \frac{(-1)}{(2\pi i)^n} \int_{bG_\nu} \frac{\partial^{n-1} g}{\partial \eta_0^{n-1}}(\eta(u)) \wedge \omega_0' \left( \sum_{k=1}^{m} \mu_k Q^{(k)}(w, u) + \left(1 - \sum_{k=1}^{m} \mu_k\right) \frac{\eta'(w)}{(\eta_0(w) - \langle \eta'(w) \cdot u \rangle)} \right) \wedge \omega'(\eta(u)) \wedge du. \tag{28}
\]
To transform formula (28) we notice that the operators $D_j$ defined in (2), satisfy the following condition

$$D_i \left( \frac{1}{\eta_0(w) - \sum_{j=1}^{n} \eta_j(w) u_j} \right) = - \left( \frac{\partial}{\partial \eta_0} \right)^{-1} \left( \frac{u_i}{(\eta_0(w) - \sum_{j=1}^{n} \eta_j(w) u_j)^2} \right) = \frac{u_i}{\eta_0(w) - \sum_{j=1}^{n} \eta_j(w) u_j}. \quad (29)$$

Then, using equality (29) we obtain that for a polynomial $Q_j^{(k)}(w, u)$ the differential operator $Q_j^{(k)}(w, D)$ satisfies the following property

$$Q_j^{(k)}(w, D) \left( \frac{1}{\eta_0(w) - \langle \eta'(w) \cdot u \rangle} \right) = \frac{Q_j^{(k)}(w, u)}{\eta_0(w) - \langle \eta'(w) \cdot u \rangle}. \quad (30)$$

Using the equality above we rewrite equality (28) as

$$g(\xi) = \frac{(n - 1)!}{(2\pi i)^n} \lim_{t \to 0} \int_{T^{(p)}(t) \times \Lambda} \frac{d\phi_\delta(w) \wedge dw}{\prod_{k=1}^{m} P_k(w) \cdot (\xi_0 + \xi^t \cdot w)} \wedge \omega_0' \left( \sum_{k=1}^{m} \mu_k Q_j^{(k)}(w, D) + \left( 1 - \sum_{k=1}^{m} \mu_k \right) \eta'(w) \right) \left( \frac{(-1)}{(2\pi i)^n} \int_{B_{\gamma} - \nu} \frac{\partial^{n-1} g(\eta(u))}{\partial \eta_0^{n-m-1}} \frac{\omega'(\eta(u)) \wedge du}{(\eta_0(w) - \langle \eta'(w) \cdot u \rangle)^{n-m}} \right)$$

$$= \frac{(n - 1)!}{(2\pi i)^n} \lim_{t \to 0} \int_{T^{(p)}(t) \times \Lambda} \frac{d\phi_\delta(w) \wedge dw}{\prod_{k=1}^{m} P_k(w) \cdot (\xi_0 + \xi^t \cdot w)} \wedge \omega_0' \left( \sum_{k=1}^{m} \mu_k Q_j^{(k)}(w, D) + \left( 1 - \sum_{k=1}^{m} \mu_k \right) \eta'(w) \right) \left( \frac{(-1)}{(2\pi i)^n} \int_{B_{\gamma} - \nu} \frac{\partial^{n-1} g(\eta(u))}{\partial \eta_0^{n-m-1}} \frac{\omega'(\eta(u)) \wedge du}{(\eta_0(w) - \langle \eta'(w) \cdot u \rangle)^{n-m}} \right)$$

$$= (-1)^{m-n-1} \frac{(n - 1)!}{(2\pi i)^n(n - m - 1)!} \lim_{t \to 0} \int_{T^{(p)}(t) \times \Lambda} \frac{d\phi_\delta(w) \wedge dw}{\prod_{k=1}^{m} P_k(w) \cdot (\xi_0 + \xi^t \cdot w)} \wedge \omega_0' \left( \sum_{k=1}^{m} \mu_k Q_j^{(k)}(w, D) + \left( 1 - \sum_{k=1}^{m} \mu_k \right) \eta'(w) \right) \left( \frac{\partial^{n-m-1} g(\eta(u))}{\partial \eta_0^{n-m-1}} \right),$$

where in the last equality we have used equality (25). \qed

**Proof of Corollary**
Let \( f_0 \) be the coefficient of a closed 1-form \( f = \sum_{j=0}^{n} f_j d\xi_j \) on \( D^* \) satisfying the system of equations (5). Then from equality (6) in Theorem 1 for the functional \( \phi^* \) defined in (10) we obtain the equality

\[
\frac{1}{2\pi i} \left\langle \phi^*, \frac{1}{\xi_0 + \xi^t w} \right\rangle.
\]

On the other hand, using the linear convexity of \( D^* \), we can find \( g \in H(D^*) \) such that \( f = dg \), and, in particular, \( f_0 = \partial g / \partial \xi_0 \). Since the function \( g \) has homogeneity 0, the following equality holds

\[
\xi_0 \partial g / \partial \xi_0 = -\sum_{j=1}^{n} \xi_j \partial g / \partial \xi_j,
\]

which leads to equality \( \langle \phi^*, 1 \rangle = f_0(1, 0, \ldots, 0) = 0 \). From the closedness of the form \( f \) we obtain the following equality

\[
\frac{1}{2\pi i} \sum_{j=0}^{n} \left\langle \phi^*, \frac{w_j}{\xi_0 + \xi^t w} \right\rangle d\xi_j = \sum_{j=0}^{n} f_j d\xi_j = f, w \in G, w_0 = 1, \xi \in D^*.
\]

Since a complete intersection \( V \) in \( \mathbb{C}P^n \) is connected (see [Ha], § III.5), Theorem 2 of [HP2] implies the existence of a residual cohomology class \( \phi \in H^{n-m, n-m-1}(V \cap D) \) such that

\[
R_V[\phi](\xi) = f(\xi) = \frac{1}{2\pi i} \sum_{j=0}^{n} \left\langle \phi^*, \frac{w_j}{\xi_0 + \xi^t w} \right\rangle d\xi_j.
\]

A representative of this cohomology class can be found explicitly. To find such a representative we consider \( \phi^* \) as the \((n-m, n-m)\)-current with support in \( V \cap G \). Condition \( \langle \phi^*, 1 \rangle = 0 \) and the connectedness of \( V \) imply by Serre-Malgrange duality (see [S], [Mal]) that there exists a current \( \tilde{\phi} \) of bidegree \((n-m, n-m-1)\) on \( V \), such that \( \partial \tilde{\phi} = \phi^* \) on \( V \). So the sought cohomology class on \( V \cap D \) can be defined as \( \phi = \tilde{\phi}|_{V \cap D} \).

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