NONCOMPACT QUANTUM ALGEBRA $u_q(2, 1)$: POSITIVE DISCRETE SERIES OF IRREDUCIBLE REPRESENTATIONS

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Abstract  The structure positive of unitary irreducible representations of the non-compact $u_q(2, 1)$ quantum algebra that are related to a positive discrete series is examined. With the aid of projection operators for the $su_q(2)$ subalgebra, a $q$-analogue of the Gel’fand–Graev formulas is derived in the basis corresponding to the reduction $u_q(2, 1) \to su_q(2) \times u(1)$. Projection operators for the $su_q(1, 1)$ subalgebra are employed to study the same representations for the reduction $u_q(2, 1) \to u(1) \times su_q(1, 1)$. The matrix elements of the generators of the $u_q(2, 1)$ algebra are computed in this new basis. A general analytic expression for an element of the transformation bracket $\langle U | T \rangle_q$ between the bases associated with above two reductions (the elements of this matrix are referred to as $q$-Weyl coefficients) is obtained for a general case where the deformation parameter $q$ is not equal to a root of unity. It is shown explicitly that, apart from a phase, $q$-Weyl coefficients coincide with the $q$-Racah coefficients for the $su_q(2)$ quantum algebra.

1. Introduction

It is well known that the group theory methods are widely used in the theory of nucleus. They form the basis of nuclear spectroscopy and of various nuclear models, including the shell model, models dealing with collective degrees of freedom, and the interacting boson model. Since group theory or algebraic models usually admit an analytic solution,

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they are employed to study various properties of nuclear systems in particular, some of their asymptotic properties. For example, the popular Elliott model, based on $SU(3)$ symmetry, was successfully employed by Belyaev and his colleagues [1] to analyze the asymptotic properties of the generalized density matrix. The discovery of quantum algebras and groups that was made more than 20 years by mathematical physicists of Leningrad school [2]–[4] gave new impetus to the development of algebraic methods in theoretical physics, in particular, to searches for applications of the representations of quantum groups and algebras in physics. For example, the construction of $q$ analogs of various nuclear models became a new field of research in theoretical nuclear physics. The point is that quantum algebras involve an additional variable parameter, the deformation parameter $q$. This renders models based on quantum algebras more adaptable and extends their potential in describing physical systems (see, for example, the study of Raychev et al [5] and the review article of Bonatsos and Daskaloyannis [6]) which is devoted to applications of quantum algebras in the theoretical nuclear physics). However, the searches for physical applications of quantum algebras must be preceded by a detailed investigations of their irreducible representations. In this connection, the structure of unitary irreducible representations of the compact $u_q(3)$ algebra was examined in details in [7]–[16]. In our opinion it is important to extend these results on the noncompact $u_q(2,1)$ quantum algebra. The classical algebra $u(2,1)$ describes the dynamical symmetry of a two-dimensional harmonic oscillator and of some other physical systems. In view of this, a comprehensive analysis of unitary irreducible representations of its quantum analogs may be helpful in constructing respective physical models. In the present study, we restrict our consideration to the unitary irreducible representations associated with a positive discrete series.

Unitary irreducible representations of conventional (nondeformed) $u(n,m)$ algebras were studied by Gel’fand and Graev [7] (see also [8]), who showed, among other things, that the unitary irreducible representations of the $u(2,1)$ algebra can be divided into three discrete series. The series of the highest weight unitary irreducible representations or a negative discrete series consists of representations such that each includes the highest weight vector $|H\rangle$ that is, a vector annihilated by any raising generator of the algebra.

A positive discrete series is the series of representations such that each includes the lowest weight vector $|L\rangle$ that is, a vector annihilated by any lowering generator. There is yet another series, that is referred to as an intermediate one and which is formed by unitary irreducible representations having neither the highest weight vector $|H\rangle$ nor the
lowest weight vector $|L\rangle$. For this reason, this series deserves a dedicated consideration.

Gel’fand and Graev [17] presented explicit expressions for the matrix elements of generators associated with the above representations that is, the matrix elements of the generators $A_{ik}$ of the $u(n,m)$ algebra in the basis corresponding to the following reduction of this algebra to the chain of subalgebras:

$$u(n,m) \rightarrow u(n,m-1) \rightarrow \ldots \rightarrow u(n) \rightarrow \ldots \rightarrow u(2) \rightarrow u(1), n \geq m. \quad (1)$$

However, these authors did not give a regular procedure for deriving the expressions that they quoted in [17]. For the $u(n,1)$, these formulas were derived in [19]–[21], but there is no derivation of such formulas for the general case of the $u(n,m)$ algebras. In this study, we extend the approach proposed by Vilenkin in [22] for the case of the $u(2,1)$ classical algebra and examine the structure of its unitary irreducible representations associated with the positive series. These results obtained in this way are readily generalized to the case of negative discrete series. The intermediate discrete series of unitary irreducible representations will be considered in a separate paper. As in [7]–[16], we assume that the deformation parameter $q$ is specified by an arbitrary positive number and define $q$-numbers and $q$-factorials as follows:

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad (2)$$

$$[n]! = [n][n-1]\ldots[2][1], \quad [0]! = 1. \quad (3)$$

Below, we employ brackets to denote $q$-numbers, enclose the signatures of unitary irreducible representations in Dirac brackets, and reserve parentheses for the weight of a vector, for example, the symbol $|\langle f_1 f_2 f_3 \rangle (m_1 m_2 m_3)\rangle$ stands for a basis vector of a weight $(m_1 m_2 m_3)$ in the unitary irreducible representation $D^{\langle f_1 f_2 f_3 \rangle} = D^{[f]}$.

### 2. Positive discrete series of unitary irreducible representations

The $u(2,1)$ algebra is known to involve nine generators $A_{ik}$ ($i, k = 1, 2, 3$) satisfying the same commutation relations that the corresponding generators of the compact $u(3)$ classical Lie algebra. However, properties of the $u(2,1)$ generators under Hermitian conjugations differ from those of the $u(3)$ generators. The “compact” generators $A_{11}, A_{22}, A_{33}, A_{12}$ and $A_{21}$ of the $u(2,1)$ algebra have the same Hermitian properties, as the $u(3)$ generators,

$$A_{ik}^+ = A_{ki}. \quad (4)$$
whereas the “noncompact” generators $A_{13}$, $A_{23}$, $A_{31}$ and $A_{32}$ satisfy the relations

$$A_{13}^+ = -A_{31},$$  \hspace{1cm} (5)

$$A_{23}^+ = -A_{32}. \hspace{1cm} (6)$$

The minus sign in formulas (5) and (6) generates a fundamental distinction between the structure of any unitary irreducible representation of the $u(2,1)$ algebra and the structure of the corresponding unitary irreducible representation of the $u(3)$ algebra: all unitary irreducible representations of the compact $u(3)$ algebra are finite-dimensional, whereas all unitary irreducible representations of the noncompact $u(2,1)$ algebra (with the exception of the trivial identity representation) are infinite-dimensional. The noncompact $u_q(2,1)$ quantum algebra is also specified by nine generators $A_{ik}$ ($i,k = 1,2,3$) satisfying the same commutation relations as the generators of the $u_q(3)$ compact quantum algebra. The explicit expressions for these commutators can be found in [7].

As to their properties with respect to Hermitian conjugation, those in (4) and (6) remain valid, whereas, in view of the relations

$$A_{13}^+ = \tilde{A}_{31} = A_{32}A_{21} - qA_{21}A_{32} \neq A_{31},$$  \hspace{1cm} (7)

$$A_{31}^+ = \tilde{A}_{13} = A_{12}A_{23} - q^{-1}A_{23}A_{12} \neq A_{13},$$  \hspace{1cm} (8)

for the $u_q(3)$ algebra, that in (5) must be replaced by

$$A_{13}^+ = -\tilde{A}_{31}. \hspace{1cm} (9)$$

With the aid of (7) and (8), this relation can be recast into either of the following two equivalent forms:

$$A_{13}^+ = -A_{31} + (q - q^{-1})A_{21}A_{12},$$  \hspace{1cm} (10)

or

$$A_{13}^+ = -q^2A_{31} + (1 - q^2)A_{32}A_{21}. \hspace{1cm} (11)$$

For the $u_q(2,1)$ algebra, we will consider the unitary irreducible representation $D^{(f)}$ of the lowest weight $(f) = (f_1 f_2 f_3)$: that is, we assume that, in the space of this representation, there is the lowest weight vector $|L\rangle$ that satisfies the relations

$$A_{ii}|L\rangle = f_i|L\rangle, \quad (i = 1,2,3) \hspace{1cm} (12)$$

annihilated by a pair of lowering generators

$$A_{31}|L\rangle = 0 \text{ and } A_{32}|L\rangle = 0. \hspace{1cm} (13)$$
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Also it is annihilated by one raising generator

\[ A_{12} |L\rangle = 0. \]  

(14)

It is assumed that this vector is normalized by the relation

\[ \langle L | L \rangle = 1. \]  

(15)

All the other basis vectors \(|X\rangle\) of this unitary irreducible representation can be derived by applying the generators \( A_{13}, A_{23} \) and \( A_{21} \) to \(|L\rangle\)

\[ |X\rangle = A_{21}^k A_{23} A_{13}^l |H\rangle. \]  

(16)

In order to construct a basis of any unitary irreducible representation of the \( u_q(2,1) \) algebra, it is necessary to specify a chain of subalgebras, and this can be done, as it is well known, in three ways. The first way is to use the \( U \)-spin subalgebra involving the generators \( A_{11}, A_{12}, A_{21}, \) and \( A_{22} \), in which case the respective reduction is

\[ u_q(2,1) \rightarrow u_q(2) \rightarrow u_q(1). \]  

(17)

One can also use the generators \( A_{22}, A_{23}, A_{32}, \) and \( A_{33} \) forming the basis of the \( T \)-spin subalgebra or the generators \( A_{11}, A_{13}, A_{31}, \) and \( A_{33} \) generating the \( V \)-spin subalgebra. Either of these two subalgebras correspond to the reduction

\[ u_q(2,1) \rightarrow u_q(1,1) \rightarrow u_q(1). \]  

(18)

In this study, we restrict our consideration to the case of \( U \) and \( T \)-spin bases.

3. Basis vectors and matrix elements of the generators in the basis associated with \( U \)-spin reduction

First, we consider that the generators \( A_{11}, A_{12}, A_{21}, \) and \( A_{22} \) form a basis of the \( U \)-spin algebra, which is a compact subalgebra of the noncompact quantum \( u_q(2,1) \) algebra, the generators

\[ U_+ = A_{12}, \quad U_- = A_{21}, \quad U_0 = \frac{1}{2}(A_{11} - A_{22}) \]  

(19)

generating the compact \( su_q(2) \) subalgebra.

In the case of \( U \)-spin reduction, the basis of an unitary irreducible representation of the \( u_q(2,1) \) algebra can be derived in the same way as the basis of \( u_q(3) \) algebra [11]:

\[ |f_1 f_2 f_3 m_3 U M_U\rangle_q = \frac{1}{N(kt)N(UM_U)} A_{21}^{U-M_U} P U^{k} A_{23} A_{13}^l |H\rangle \]  

(20)
where
\[ m_3 = f_3 - k - \ell, \]  \hspace{1cm} (21)
\[ U = \frac{1}{2}(f_1 - f_2 - k + \ell), \]  \hspace{1cm} (22)
\[ M_U = \frac{1}{2}(m_1 - m_2), \quad -U \leq M_U \leq U, \]  \hspace{1cm} (23)
\[ P^U = \sum_{r=0}^{\infty} (-1)^r \frac{[2U + 1]!}{[r]![2U + r + 1]!} A_{12} A_{21}, \]  \hspace{1cm} (24)
is the projection operator for the \( su_q(2) \) algebra [23],
\[ N(U M_U) = \sqrt{\frac{[2U]![U - M_U]!}{[U + M_U]!}}, \]  \hspace{1cm} (25)
\( N(k\ell) \) are normalization factors, and
\[ |L\rangle = |\langle f_3 U_L U_L\rangle f_1 - f_2\rangle, \quad U_L = \frac{1}{2}(f_1 - f_2) \]  \hspace{1cm} (26)
is the lowest weight vector. The main distinction between the \( u_q(2, 1) \) and \( u_q(3) \) algebras lies in the normalization factor \( N(k\ell) \). In the Appendix, it is shown that, in the latter case, the square of the normalization factor has the form:
\[ N^2(k\ell) = \frac{[k]![\ell]![f_1 - f_2 - k + \ell + 1]![f_1 - f_2]!}{[f_1 - f_2 - k]![f_1 - f_2 + \ell + 1]!} \]
\[ \times \frac{[f_2 - f_3 + k - 2]![f_1 - f_3 + \ell - 1]!}{[f_1 - f_3 - 1]![f_2 - f_3 - 2]!}. \]  \hspace{1cm} (27)
Here, we impose the conventional requirement that the arguments of all \( q \)-factorials be nonnegative integers. This requirement ensures that the square of the norm of basis vectors is positive. It also follows that a nonzero vector exists only under the conditions from
\[ f_1 \geq f_2, \]
\[ f_1 - f_3 \geq 1, \]  \hspace{1cm} (28)
\[ f_2 - f_3 \geq 2. \]  \hspace{1cm} (29)
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$0 \leq k \leq f_1 - f_2$.

At the same time, no condition is imposed on the exponent $\ell$ ($\ell = 0, 1, 2, ...$), with the result that, in the case of a $U$-basis, an unitary irreducible representation of the $u_q(2,1)$ algebra is infinite-dimensional.

In [17], each basis vector of the lowest weight unitary irreducible representation was characterized by the scheme

$$\begin{pmatrix} m_{13} & m_{23} & m_{33} \\ m_{12} & m_{22} \\ m_{11} \end{pmatrix},$$

(30)

where the integers $m_{ij}$ satisfy the conditions:

$$m_{13} \geq m_{23} \geq m_{33} \geq 0,$$

(31)

$$m_{12} \geq m_{13} + 1 \geq m_{22} \geq m_{23} + 1,$$

(32)

$$m_{12} \geq m_{11} \geq m_{22}.$$  

(33)

The numbers in the first row in (30) represent the signature of a unitary irreducible representation of the $u_q(2,1)$ algebra. They are related to the components of the lowest weight by the equations

$$f_1 = m_{13} + 1,$$

(34)

$$f_2 = m_{23} + 1,$$

(35)

$$f_3 = m_{33} - 2.$$  

(36)

The numbers in the second row in (30) represent the signature of a unitary irreducible representation of the $u_q(2)$ subalgebra. In our notation,

$$m_{12} = f_1 + \ell,$$

(37)

$$m_{22} = f_2 + k.$$  

(38)

The number in the third row is

$$m_{11} = U + M_U + m_{22}.$$  

(39)

From the condition $f_1 \geq f_2$, it follows

$$m_{13} \geq m_{23}.$$  

(40)
The condition \( f_2 - f_3 - 2 \geq 0 \) means that

\[ m_{23} \geq m_{33} - 1. \]  

(41)

Combining these conditions, we obtain

\[ m_{13} \geq m_{23} \geq m_{33} - 1. \]  

(42)

At the same time, the condition \( 0 \leq k \leq f_1 - f_2 \) is equivalent to the constraints

\[ m_{13} + 1 \geq m_{22} \geq m_{23} + 1. \]  

(43)

With regard for the allowed values of the exponent \( \ell \), \( \ell = 0, 1, 2, \ldots \), we derive

\[ m_{12} \leq m_{13} + 1. \]  

(44)

The condition \(-U \leq M_U \leq U\) leads to the constraints

\[ m_{12} \geq m_{11} \geq m_{22}. \]  

(45)

A comparison of formulas (42)–(45) with (31)–(33) demonstrates that our constraints on the structure of basis vectors are identical to the constraints on the values of \( m_{ij} \) in the Gel’fand–Graev schemes, with the only exception that, in our case, there exists a unitary irreducible representation for which \( m_{23} = m_{33} - 1 \). This means that there are unitary irreducible representations corresponding to the Gel’fand–Graev signature,

\[ \{m_{13}m_{23}m_{33}\} = \{m_{13}, m_{33} - 1, m_{33}\}, \]  

(46)

which are beyond the standard constraints (31). The existence of such nonstandard discrete series of unitary representations of the \( u(2, 1) \) algebra was indicated in [20], and [21]. The \( u_q(2, 1) \) algebra has analogous special series of unitary irreducible representations.

Further, it should be noted that at \( f_1 = f_2 \), in which case \( k = 0 \), the condition that the norm \( N^2(0\ell) \) is positive requires fulfillment of inequality

\[ f_3 - f_2 - 1 + \ell > 0 \]  

(47)

for all values of \( \ell \), including \( \ell = 1 \). Therefore, the lowest weights corresponding to \( f_1 - f_3 > 0 \) are allowed at \( f_1 = f_2 \) (that is, at \( m_{13} = m_{33} \)). Therefore, there is an additional nonstandard series of the lowest weight unitary irreducible representations such that condition (31) is violated for them. Those are characterized by Gel’fand–Graev signatures \( \{m_{23} - 2, m_{23} - 2, m_{23}\} \).

Let us now consider the matrix elements of generators in the \( U \) basis. In the basis specified by (20), the weight generators \( A_{ii} \ ((i = 1, 2, 3)) \)
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naturally have a diagonal form, their matrix elements being given by

$$m_1 = f_1 + \ell - (U - M_U),$$

$$m_2 = f_2 + k + (U - M_U),$$

$$m_3 = f_3 - k - \ell,$$

where

$$m_1 + m_2 + m_3 = f_1 + f_2 + f_3.$$  \hfill (51)

The action of the generators $A_{12} = U_+$ and $A_{21} = U_-$ are well known from the theory of angular momenta:

$$U_+ |\langle f \rangle m_3 U M_U \rangle_q = \sqrt{|U + M_U||U ± M_U + 1|} |\langle f \rangle m_3 U M_U ± 1\rangle_q.$$  \hfill (52)

The matrix elements of the generators $A_{13}, A_{23}, A_{31}, A_{32}$ are given by the $q$-analogs of Gel’fand–Graev formulas

$$A_{ij} |\langle f \rangle m_3 U M_U \rangle = \sum_{U'} a_{ij} (m_3' U' M_{U'}) |\langle f \rangle m_3' U' M_{U'} \rangle,$$  \hfill (53)

where

$$U' = U ± 1/2, \quad M_{U'} = M_U ± 1/2$$  \hfill (54)

and

$$a_{ij} (m_3' U' M_{U'}) = q \langle m_3' U' M_{U'} | A_{ij} |m_3 U M_U \rangle_q.$$  \hfill (55)

The list of these matrix elements is given in the Table 1; their derivation is given in [24].

4. **Basis vectors and matrix elements of the generators in the basis associated with $T$-spin reduction.**

Let us consider the structure of the unitary irreducible representations $D^{(J)}$ of the lowest weight $(f_1 f_2 f_3)$ in the case of the reduction

$$u_q(2, 1) \rightarrow u_q(1, 1)$$  \hfill (56)

of the $u_q(2, 1)$ algebra to the $u_q(1, 1)$ subalgebra specified by generators $A_{22}, A_{23}, A_{32},$ and $A_{33},$ or to the $su_q(2)$ subalgebra of a noncompact $T$-spin, the generators in latter case being

$$T_+ = A_{23}, \quad T_- = A_{32}, \quad T_0 = \frac{1}{2} (A_{22} - A_{33}).$$  \hfill (57)
Table 1. Matrix elements of the generators of the noncompact $u_q(2,1)$ algebra for the unitary irreducible representation $D^{(1)+}$ associated with the positive discrete series ($U$-basis used here was derived from the lowest weight vector $|L\rangle$).

| Expression | Value |
|------------|-------|
| $a_{13}\left(m_3 - 1, U + \frac{1}{2}, M_U + \frac{1}{2}\right)$ | $q^{-U+M_U} \left[ \frac{[\ell+1][f_1-f_3+\ell][2U+k+2][U+M_U+1]}{[2U+1][2U+2]} \right]^{1/2}$ |
| $a_{23}\left(m_3 - 1, U + \frac{1}{2}, M_U - \frac{1}{2}\right)$ | $\left[ \frac{[\ell+1][f_1-f_3+\ell][2U+k+2][U-M_U+1]}{[2U+1][2U+2]} \right]^{1/2}$ |
| $a_{13}\left(m_3 - 1, U - \frac{1}{2}, M_U + \frac{1}{2}\right)$ | $-q^{U+M_U+1} \left[ \frac{[k+1][f_2-f_3+k-1][2U-\ell][U-M_U]}{[2U][2U+1]} \right]^{1/2}$ |
| $a_{23}\left(m_3 - 1, U - \frac{1}{2}, M_U - \frac{1}{2}\right)$ | $\left[ \frac{[k+1][f_2-f_3+k-1][2U-\ell][U+M_U]}{[2U][2U+1]} \right]^{1/2}$ |
| $a_{31}\left(m_3 + 1, U - \frac{1}{2}, M_U - \frac{1}{2}\right)$ | $-q^{U-M_U} \left[ \frac{[\ell][f_1-f_3+\ell-1][2U+k+1][U+M_U]}{[2U][2U+1]} \right]^{1/2}$ |
| $a_{32}\left(m_3 + 1, U - \frac{1}{2}, M_U + \frac{1}{2}\right)$ | $- \left[ \frac{[\ell][f_1-f_3+\ell-1][2U+k+1][U-M_U]}{[2U][2U+1]} \right]^{1/2}$ |
| $a_{31}\left(m_3 + 1, U + \frac{1}{2}, M_U - \frac{1}{2}\right)$ | $q^{-U-M_U-1} \left[ \frac{[k][f_2-f_3+k-2][2U-\ell+1][U-M_U+1]}{[2U+1][2U+2]} \right]^{1/2}$ |
| $a_{32}\left(m_3 + 1, U + \frac{1}{2}, M_U + \frac{1}{2}\right)$ | $- \left[ \frac{[k][f_2-f_3+k-2][2U-\ell+1][U+M_U+1]}{[2U+1][2U+2]} \right]^{1/2}$ |
We note that the condition (1) imposed in [17] on a chain of subalgebras is not satisfied in formulas (56). For this reason, the results obtained in [17] are not valid in the case of the $T$-spin basis even for classical $u(2,1)$ algebra, not to mention its deformation $u_q(2,1)$.

Before proceeding to discuss the $u_q(2,1)$ algebra as a whole, it is reasonable to recall general information about the $su_q(1,1)$ subalgebra and its unitary irreducible representations. The generators of the $su_q(1,1)$ subalgebra satisfy the well-known commutation relations

\[
[T_0, T_+] = T_+, \quad (58)
\]

\[
[T_0, T_-] = -T_-, \quad (59)
\]

\[
[T_+, T_-] = [2T_0]. \quad (60)
\]

Under Hermitian conjugation, they transform as follows:

\[
T_0^+ = T_0, \quad (61)
\]

\[
T_-^+ = -T_. \quad (62)
\]

The unitary irreducible representations $D^T$ of the positive discrete series are infinite-dimensional; the respective $T$-spin is given by

\[
T = -\frac{1}{2}, \ 0, \ \frac{1}{2}, \ 1, \ \frac{3}{2}, \ 2, \ \ldots \quad (63)
\]

The $T$-spin projection $M$ (or the weight of a vector) is an eigenvalue of the operator of the $T$-spin projection $T_0$, takes the positive values:

\[
M = T + 1, \ T + 2, \ \ldots \quad (64)
\]

the lowest weight being $T + 1$. We assume that the lowest weight vector $|H\rangle = |T, T + 1\rangle$ is known and that it satisfies the requirements

\[
T_-|L\rangle = 0, \quad (65)
\]

\[
T_0|L\rangle = (T + 1)|L\rangle, \quad (66)
\]

and the normalization condition

\[
\langle L|L\rangle = 1. \quad (67)
\]

The basis vectors of a higher weight can be obtained from the lowest weight vector by the formula

\[
|TM\rangle = \frac{1}{N(TM)} T_+^{M-T-1} |TT + 1\rangle. \quad (68)
\]
The square norm of a vector is derived in this way has a form \( (x = M - T - 1) \)

\[
N^2(TM) = (-1)^x \langle L | A_{23}^x A_{23}^x | L \rangle = [x][2T + x + 1]N^2(T, M + 1)
\]

\[
= \frac{[-T + M - 1][T + M]}{[2T + 1]!}.
\]

(69)

It can be seen that the condition \( N^2(TM) > 0 \) imposes no constraints on \( x = 0, 1, 2, \ldots \); therefore, the unitary irreducible representation is infinite-dimensional. Nonzero matrix elements of the generators in the basis specified (68) are given by

\[
\langle TM | T_0 | TM \rangle = M,
\]

(70)

\[
a_{23} = \langle TM + 1 | A_{23} | TM \rangle = \{[-T + M][T + M + 1]\}^{1/2},
\]

(71)

\[
a_{32} = \langle TM - 1 | A_{32} | TM \rangle = -\{[T + M][-T + M - 1]\}^{1/2}.
\]

(72)

The Casimir operator for the \( su_q(1, 1) \) algebra has the same form as for the \( su_q(2) \) algebra:

\[
C_2(su_q(1, 1)) = T_- T_+ + [T_0 + 1/2]^2.
\]

(73)

All vectors in (64) are the eigenvectors of this operator and correspond to the same eigenvalue:

\[
C_2(su_q(1, 1))|TM\rangle = [T + 1/2]^2|TM\rangle.
\]

(74)

We also need the extremal projection operator \( P^T = P^T_{T+1,T+1} \) for the discrete series of the lowest weight unitary irreducible representation. As in the case of the \( su_q(2) \) algebra, we seek the expression for the extremal projection operator in the form of series:

\[
P^T = \sum_{r=0} C_r T_+^r T_-^r.
\]

(75)

In what follows, we apply this projection operator only to those vectors \( |T + 1\rangle \) that have a specific weight \( M = T + 1 \), but which, in general, do not have a specific value of \( T \)-spin that is, to vectors that are represented by linear combinations of

\[
|T + 1\rangle = \sum_{T'} |T', T + 1\rangle.
\]

(76)
In contrast to the case of $su_q(2)$ algebra, however, the sum over $T'$ is finite in the case under study, because the inequality $T' \leq M' - 1$ must hold for the basis vectors $|T'M'\rangle$ of the positive discrete series. In the case (76) it means that $T' \leq T$. Hence the variable $T'$ in the sum (76) runs through the values from $T_{\text{min}} = -1/2$ or $0$ up to $T$, depending on whether $T$ is an integer or a half-integer. By applying the operator in (75) to the vectors in (76), can show that only a finite number of terms in (75) make a non-vanishing contribution, namely, those that satisfy $T + 1 - r \leq 1$ or $1/2$ (that is, $r \leq T$ or $T + 1/2$). Hence, the terms in (75) that involve higher powers $r$ can be disregarded.

The projection operator $P_T$ satisfies the equations

$$T_- P_T = 0, \quad (77)$$

$$P_T |T, T + 1\rangle = |T, T + 1\rangle. \quad (78)$$

From (77) it follows that the coefficients $C_r$ satisfy the recursion relation

$$C_{r-1} + [r][-2T + r - 1]C_r = 0. \quad (79)$$

From this relation, we obtain

$$C_r = C_0 \frac{[2T - r]!}{[r]![2T]!}, \quad r \leq 2T. \quad (80)$$

From the condition (78), it follows that

$$C_0 = 1. \quad (81)$$

At $r = 2T + 1$, relation (80) is meaningless, but we have shown above that we do need the coefficients $C_r$ for $r > T$ or $T + 1/2$. Thus, it is sufficient, for our purposes, to use the simple projection operator

$$P_T = \sum_{r=0}^{r=2T} \frac{[2T - r]!}{[r]![2T]!} T^r T^r. \quad (82)$$

A projection operator of a more general form can be represented as

$$P^T_{MM'} = \frac{(-1)^{T-M'-1}}{N(TM)N(TM')} T^{-T+M-1} P^T P^T_{+}^{T+M'}. \quad (83)$$

As a matter of fact, Vilenkin [22] used similar projection operators (of course, only for $q = 1$) long ago to derive the harmonic projections of polynomials depending on $n$ Cartesian variables.
Let us present yet another relation helpful for subsequent computations

\[ P^T_{T+1,M} P^T_{M,T+1} = \frac{(-1)^{T+M+1}}{N^2(TM)} P^T T^M - T^{-1} T^M - T^{-1} P^T = P^T. \]  

(84)

From this equation, it follows

\[ P^T T^M - T^{-1} T^M - T^{-1} P^T = (-1)^{M-T-1} N^2(TM) P^T. \]  

(85)

We now return to a consideration of the lowest weight unitary irreducible representations of the \( \mathfrak{u}_q(2,1) \) algebra. As in the case of the \( \mathfrak{u}_q(3) \) algebra, the basis vectors of the unitary irreducible representation \( D^{(f)} \) of the \( \mathfrak{u}_q(2,1) \) algebra that correspond to the lowest weight \( (f) = (f_1f_2f_3) \) will be represented in a form

\[ \langle f \rangle m_1 T M \rangle_q = \frac{1}{N(sp)N(TM_T)} A^{M-T-1}_2 A^{p}_3 A^{T}_1 \langle H \rangle, \]  

(86)

where

\[ T = \frac{1}{2}(f_2 - f_3 + p + s - 2), \]  

(87)

\[ M = T + 1, T + 2, \ldots. \]  

(88)

The normalization factor \( N(TM_T) \) is determined by formula (69), while the projection operator \( P^T \) is given by (82). The normalization factor \( N(sp) \) for the vectors of \( T \)-spin basis is calculated by a method similar to that used for the norm of the vectors of the \( U \)-spin basis described in the Appendix (see also [24]). The square of this norm is

\[ N^2(sp) = \frac{[s]![p]![f_1 - f_2]![f_1 - f_3 + s - 1]!}{[f_1 - f_2 - p]![f_1 - f_3 - 1]!} \times \frac{[f_2 - f_3 + s - 2]![f_2 - f_3 + p - 2]!}{[f_2 - f_3 - 2]![f_2 - f_3 + p + s - 2]!}. \]  

(89)

From the analysis of the norm of the basis vectors, we derive the conditions

\[ f_1 \geq f_2, \]  

(90)

\[ f_2 - f_3 - 2 \geq 0, \]  

(91)

\[ 0 \leq p \leq f_1 - f_2. \]  

(92)
There is no constraint on the exponent $s$; that is, $s = 0, 1, 2, \ldots$ Since the number of values of the projections $M$ is infinite, this means that the representations under study are infinite-dimensional. The constraints in (91) and (92) on the signature of unitary irreducible representations are identical to those obtained for the $U$-spin basis. For this reason, the classification of the standard and nonstandard discrete series for the $T$-basis remains unchanged, as might have been expected.

Let us now proceed to discuss the matrix elements of generators. For the weight generators $A_{ij}$ in the $T$-spin basis, only diagonal matrix elements do not vanish. They are given by

$$a_{11}(m_1 T M) = a_{11}(m_1 m_2 m_3) = m_1 = f_1 - p + s,$$

$$a_{22}(m_1 T M) = a_{22}(m_1 m_2 m_3) = m_2 = f_2 + p - T + M - 1,$$

$$a_{33}(m_1 T M) = a_{33}(m_1 m_2 m_3) = m_3 = f_3 - s + T - M + 1.$$

The matrix elements of the generators $A_{23} = T_+$ and $A_{32} = T_-$ can be determined by formulas (71) and (72).

The remaining four non-diagonal generators act on the $T$-basis vectors as follows

$$a_{ij}(m_1' T' M') = q^{\langle m_1' T' M' | A_{ij} | m_1 T M \rangle}.$$

where

$$T' = T \pm 1/2, \quad M' = M \pm 1/2,$$

and

$$a_{ij}(m_1' T' M') = q^{\langle m_1' T' M' | A_{ij} | m_1 T M \rangle}.$$

These matrix elements are presented in Table 2 (the derivation of these expressions is given in \[24\].

5. Weyl coefficients $\langle U | T \rangle_q$ for the positive discrete series of unitary irreducible representations of the $u_q(2, 1)$ quantum algebra

By definition, the Weyl coefficient $\langle U | T \rangle_q$ for an irreducible representation $\langle f \rangle$ of the $u_q(2, 1)$ quantum algebra has a form

$$\langle U | T \rangle_q = q^{\langle \{ f \} | m_3 U M_U | \{ f \} | m_1 T M_T \rangle_q}$$

$$= \frac{(-1)^{k+\ell}}{N(k\ell) N_U U M_U N(s p) N(T M)} q^{\langle L | A_{31}^2 A_{52}^3 P_U A_{12}^4 A_{23}^5 P_T A_{21}^6 A_{31}^7 L \rangle_q},$$

(99)
Table 2. Matrix elements of the generators of the noncompact $u_q(2,1)$ quantum algebra for the unitary irreducible representation $D^{(T,+)}$ of the positive discrete series ($T$-spin basis used here was constructed with the aid of the lowest weight vector $|L\rangle$).

| $a_{12}$ | $a_{13}$ | $a_{21}$ | $a_{31}$ |
|---------|---------|---------|---------|
| $(m_1 + 1, T + \frac{1}{2}, M - \frac{1}{2})$ | $[s + 1][f_1 - f_3 + s][2T - p + 1][-T + M - 1]^{1/2}$ | $q^{T-M+1} [s + 1][f_1 - f_3 + s][2T - p + 1][T + M + 1]^{1/2}$ | $[p][f_1 - f_2 - p + 1][2T - s][T + M]^{1/2}$ | $[-q][f_1 - f_2 - p + 1][2T - s][-T + M]^{1/2}$ |
| $(m_1 + 1, T + \frac{1}{2}, M + \frac{1}{2})$ | $[2T][2T + 1]$ | $[2T][2T + 1]$ | $[2T][2T + 1]$ | $[2T][2T + 1]$ |
| $(m_1 + 1, T - \frac{1}{2}, M - \frac{1}{2})$ | $[s][f_1 - f_3 + s - 1][2T - p][-T + M]^{1/2}$ | $-q^{T+M-1} [s][f_1 - f_3 + s - 1][2T - p][T + M]^{1/2}$ | $[p + 1][f_1 - f_2 - p][2T - s + 1][T + M + 1]^{1/2}$ | $[-q][f_1 - f_2 - p][2T - s + 1][-T + M - 1]^{1/2}$ |
| $(m_1 + 1, T - \frac{1}{2}, M + \frac{1}{2})$ | $[2T][2T + 1]$ | $[2T][2T + 1]$ | $[2T][2T + 1]$ | $[2T][2T + 1]$ |
where $|L\rangle$ is the lowest weight vector of the irreducible representation $\langle f \rangle$:

$$ a = U - MU, \quad b = -T + M - 1, \quad (100) $$

and the normalization factors $N(k\ell), N(UMU), N(sp)$, and $N(TM)$ and the projection operators $P^U$ and $P^T$ were defined in the foregoing. Since the weight in the left hand side of Eq. (99) for the matrix element is equal to the weight of the right hand side, we conclude that the parameters $k$ and $\ell$ are related to $s$ and $p$ by the equations

$$ U - MU = p - s + \ell, \quad (101) $$

$$ T - M + 1 = s - \ell - k. \quad (102) $$

The computation of the above matrix element is performed by making use of the commutation relations between the generators raised to a power. The scheme of the computations is identical to that in the case of the $u_q(3)$ algebra [25]. Taking into account the explicit form of projection operator $P^U$, we arrive at

$$ B = q\langle L | A_{13}^{\ell} A_{32}^k P^U A_{21}^{r} A_{12}^a A_{23}^b A^T A_{21}^p A_{13}^s | L \rangle_q $$

$$ = \sum_r (-1)^r \frac{[2U + 1]!}{[r]![2U + r + 1]!} q\langle L | A_{31}^{\ell} A_{32}^k A_{21}^r A_{12}^a A_{23}^b A^T A_{21}^p A_{13}^s | L \rangle_q. \quad (103) $$

With the aid of commutation relations, we transfer the operator $A_{21}^r$ in the matrix element to the left until it appears immediately after vector $|L\rangle$ and consider that $\langle L | A_{21} = 0$. The expression for $B$ then takes the form

$$ B = \sum_r \frac{[2U + 1]![k]!}{[r]![k - r]![2U + r + 1]!} B_1, \quad (104) $$

where

$$ B_1 = q\langle L | A_{31}^{\ell+r} A_{12}^{r+a} A_{32}^{k-r} A_{23}^b A^T A_{21}^p A_{13}^s | L \rangle_q. \quad (105) $$

To compute the matrix element $B_1$, the generators $A_{32}$ must be transferred to the right until they appear immediately before the projection operator $P^T$, whereupon the equation $A_{32} P^T = 0$ is taken into account. As a result the matrix element $B_1$ reduces to the expression

$$ B_1 = \frac{[k]![b]!}{[k - r]![b - k + r]!} \prod_t [f_3 - f_2 - a + b - k - \ell - r - t] B_2, \quad (106) $$

where

$$ B_2 = q\langle L | A_{31}^{\ell+r} A_{12}^{a+r} A_{23}^{b-k+r} P^T A_{21}^p A_{13}^s | L \rangle_q. \quad (107) $$
Considering that, in the case of $T$-spin basis,

$$2T = f_2 - f_3 + p + s - 2$$

(108)

and that all factors in the product $\prod_t$ are negative, we recast the product into the form

$$\prod_t [f_3 - f_2 - a + b - k - \ell - r - t] = (-1)^{k-r} \frac{[f_2 - f_3 + p + \ell + k - 1]!}{[f_2 - f_3 + p + \ell + r - 1]!}$$

(109)

Further, we transfer the generators $A_{23}^{b-k+r}$ in the expression for the matrix element $B_2$ to the left until they appear immediately after the vector $\langle L \rangle$, which annihilates them, and consider that $\langle L | A_{21}^p = \delta_{y,0} \langle L \rangle$. As a result we have

$$B_2 = \frac{[a+r]!}{[a-b+k]!} q \langle L | A_{31}^{\ell+r} A_{13}^{b-k+r} A_{12}^{a-b+k} p T A_{21}^p A_{13}^r | L \rangle_q.$$  

(110)

The commutation of generators $A_{31}^{\ell+r}$ and $A_{13}^{b-k+r}$ whereupon the condition $\langle L | A_{13}^r = \delta_{z,0} \langle L \rangle$ is taken into account, gives the ultimate expression for the matrix element $B_2$:

$$B_2 = (-1)^{\ell+r+s} \frac{[a+r][\ell+r][f_1 - f_3 + \ell + r - 1]!}{[p][s][f_1 - f_3 + s - 1]!}$$

$$\times q \langle L | A_{31}^{\ell+r} A_{13}^{b-k+r} A_{12}^{a-b+k} p T A_{21}^p A_{13}^r | L \rangle_q$$

$$= (-1)^{\ell+r} \frac{[a+r][\ell+r][f_1 - f_3 + \ell + r - 1]!}{[p][s][f_1 - f_3 + s - 1]!} N^2(sp).$$

(111)

Combining the above results, we reduce the expression for the Weyl coefficients $\langle U | T \rangle_q$ of the form

$$\langle U | T \rangle_q = \left\{ \frac{[2U + 1][2T + 1][k]![-T - 1 + M][U + M_U][T + M]!}{[p][s][\ell][U - M_U][f_1 - f_3 + s - 1]!} \times \frac{[f_1 - f_2 - k][f_1 - f_2 + \ell + 1][f_2 - f_3 + s - 2][f_2 - f_3 + p - 2]!}{[f_1 - f_3 - k - 2][f_1 - f_3 + \ell - 1]} \right\}^{1/2}$$

$$\times \sum_r (-1)^r \frac{[U - M_U + r][\ell + r][f_1 - f_3 + \ell + r - 1]!}{[r][2U + r + 1][k - r][\ell - s + r][f_2 - f_3 + p + \ell + r - 1]!}.$$  

(112)
The substitution \( r = k - n \) allows to rewrite the last formula as follows

\[
\langle U|T\rangle_q = \left\{ \frac{[2U + 1][2T + 1][k]![-T + M - 1]![U + M_U]![T + M]!}{[s]![p]![\ell]![U - M_U]![f_1 - f_3 + s - 1]!} \right. \\
\times \frac{[f_1 - f_2 - k]![f_1 - f_2 + \ell + 1]![f_2 - f_3 + s - 2]![f_2 - f_3 + p - 2]!}{[f_1 - f_2 - p]![f_2 - f_3 + k - 2]![f_1 - f_3 + \ell - 1]!} \right\}^{1/2} \\
\times \frac{(-1)^{k+n}[U - M_U + k - n]![\ell + k - n]!}{[n]![k - n]![2U + 1 + k - n]![\ell - s + k - n]!} \\
\times \frac{[f_1 - f_3 + \ell + k - n - 1]!}{[f_2 - f_3 + p + \ell + k - n - 1]!}.
\]

(113)

6. **Relation between the \( q \)**-Weyl coefficients for the \( u_q(2, 1) \) quantum algebra and \( q \)-Racah coefficients for the \( su_q(2) \) quantum algebra.

The explicit expression for the \( q \)-Weyl coefficient for the \( u_q(3) \) algebra was obtained in [25], and its relation to the Racah coefficient for the \( su_q(2) \) quantum algebra was established there. Here, we show that the expression (113) for the \( q \)-Weyl coefficient for the \( u_q(2, 1) \) quantum algebra can also be related to the \( q \)-Racah coefficients for the \( su_q(2) \) quantum algebra. Our consideration is based on one of five general formulas in [25] for the \( q \)-Racah coefficient for the \( su_q(2) \) quantum algebra [namely, formula (5.31) in [25]]:

\[
U_q(abed; cf) = (-1)^{a+d-c-f} \left\{ \frac{[2c + 1][2f + 1][a + b + c + 1]!}{[a + e + f + 1]![c + d + e + 1]!} \right. \\
\times \frac{[b + d + f + 1]![a - b + c]![-a + b + c]![a + e - f]![b - d + f]!}{[a + b - c]![a - e + f]![b + d - f]![c + d - e]![c - d + e]!} \\
\times \frac{[-b + d + f]![-c + d + e]!}{[-a + e + f]!} \right\}^{1/2} \\
\times \frac{(-1)^n[2b - n]![b + c - e + f - n]!}{[n]![-a + b + c - n]![b - d + f - n]![a + b + c + 1 - n]!} \\
\times \frac{1}{[b + d + f + 1 - n]!}.
\]

(114)

A comparison of the expression (114) with formula (113) for the Weyl coefficient for the positive discrete series of the representations of \( u_q(2, 1) \)
reveals the summands in the two formulas coincide, provided
\[ a = T = \frac{1}{2}(f_2 - f_3 + p + s - 2), \quad b = j_3 = \frac{1}{2}(\ell + k), \]
\[ c = j_2 = \frac{1}{2}(f_2 - f_3 + p - s + \ell + k - 2), \quad \text{(115)} \]
\[ d = U = \frac{1}{2}(f_1 - f_2 + \ell - k), \quad e = j_1 = \frac{1}{2}(f_1 - f_3 - p + s - 2), \]
\[ f = j = \frac{1}{2}(f_1 - f_2). \quad \text{(116)} \]

Further, the substitution of parameters \(a, b, c, d, e\) and \(f\) from the formulas (115) and (116) gives the relation between the \(q\)-Weyl coefficient (113) for the \(u_q(2,1)\) quantum algebra and \(q\)-Racah coefficient for the \(su_q(2)\) quantum algebra,
\[
\langle U | T \rangle_q = (-1)^s \sqrt{\frac{[2U + 1][2T + 1]}{[2j_2 + 1][2j + 1]}} U(Tj_3j_1U; j_2j)q
\]
\[ = (-1)^k U(j_1j_2jj_3; UT)q. \quad \text{(117)} \]

7. Conclusion

In this study, the projection operators for the \(su_q(2)\) subalgebra have used to explore the positive discrete series of unitary irreducible representations of the noncompact \(u_q(2,1)\) quantum algebra. The \(q\)-analog of the Gel’fand–Graev formulas has been derived in the basis associated with the reduction \(u_q(2,1) \to su_q(2) \times u(1)\). It seems that the reduction \(u_q(2,1) \to u(1) \times su_q(1,1)\) for the discrete series of the lowest weight representations has been considered for the first time in the present study. With the aid of the projection operator for the \(su_q(1,1)\) subalgebra, we constructed the basis of the representation for this reduction and calculated the matrix elements of the generators. We have obtained analytic expressions for the elements of the transformation brackets \(\langle U | T \rangle_q\) relating the \(U\)-spin and \(T\)-spin bases of the lowest weight irreducible representations. By the analogy with \(q\)-Weyl coefficients for the \(u_q(3)\) algebra [25], they can be called the \(q\)-Weyl coefficients for the noncompact \(u_q(2,1)\) algebra. It has been explicitly shown that these \(q\)-Weyl coefficients are equivalent (apart from phase factor) to specific \(q\)-Racah coefficient for the \(u_q(2)\) algebra or are proportional to the \(q\)-6\(j\) symbol for the \(su_q(2)\) algebra. The negative discrete series was discussed by us in [26]. The intermediate discrete series requires a dedicated investigation, and this will be done in our further publication.
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Appendix: Normalization of the $U$-spin basis vectors of the $u_q(2, 1)$ algebra (positive discrete series).

The structure of the $U$-basis vectors is described by formulas (19)–(26).

Here, we use the transformation properties of the “noncompact” generators under Hermitian conjugation and the properties of projection operator $P^U$:

\[ (P^U)^+ = P^U, \]
\[ (P^U)^2 = P^U, \]
\[ A_{12}^{U-MU} A_{21}^{U-MU} P^U = N^2(U_{MU}) P^U. \]  

(A.1)
(A.2)
(A.3)

With allowance for these formulas, the square of the norm $N^2(k\ell)$ takes a form:

\[ N^2(k\ell) = (-1)^{k+\ell} \langle L| A_{31}^U A_{32}^U A_{23}^U A_{13}^U |L \rangle, \]

(A.4)

where

\[ \tilde{A}_{31} = A_{32} A_{21} - q A_{23} A_{32}. \]  

(A.5)

Since, by definition, the relation

\[ A_{31} = A_{32} A_{21} - q^{-1} A_{21} A_{32} \]

(A.6)

holds we can represent the generator $\tilde{A}_{31}$ in the form

\[ \tilde{A}_{31} = A_{31} - (q - q^{-1}) A_{21} A_{32}. \]

(A.7)

From the relations

\[ A_{12} P^U = P^U A_{21} = 0 \]

(A.8)

it follows that

\[ N^2(k\ell) = (-1)^{k+\ell} \langle L| A_{12}^U A_{13}^U A_{23}^U A_{31}^U |L \rangle. \]

(A.9)

A straightforward computation of $N^2(k\ell)$ by transferring of lowering generators to the lowest vector $|L\rangle$ is rather cumbersome. In view of this, we will try to construct a recursion relation between the expressions for $N^2(k\ell)$ for various values of $k$ and $\ell$, bearing in mind that

\[ \langle L| P^U |L\rangle = \langle L| L \rangle = 1. \]

(A.10)

We begin by establishing a relation between $N^2(k\ell)$ and $N^2(k-1, \ell)$. In the expression for $N^2(k\ell)$, we replace, for this purpose, $A_{32}^U$ by $A_{32}^{U+1/2} A_{32}$. This is legitimate because, in (A.9), the projection operator $P^{U+1/2}$ taken in the combination is equivalent to the identity operator. Indeed, we have

\[ P^{U+1/2} A_{32}^U P^U = \sum_r (-1)^r \frac{[2U + 2]!}{[r]![2U + r + 2]!} A_{12}^U A_{13}^U A_{23}^U A_{32}^U P^U = A_{32}^U P^U, \]

(A.11)

since the generators $A_{12}^U$ and $A_{32}$ commute and since $A_{12}^U P^U = \delta_{00} P^U$. We now consider the application of the generator $A_{32}$ to the projection operator $P^U$:

\[ A_{32} P^U = \sum_r (-1)^r \frac{[2U + 1]!}{[r]![2U + r + 1]!} A_{32} A_{21}^U A_{12}^U = \]

\[ = \sum_r (-1)^r \frac{[2U + 1]!}{[r]![2U + r + 1]!} (q^{-r} A_{21}^U A_{32} + [r] A_{23} A_{32}) A_{12}^U. \]

(A.12)
Thus, the square of the norm,
\[ N_P \]

of this operator on the projection operator \( P \) to a power. In view of the relation \( P^{U+1/2}A_{21} = (A_{12}P^{U+1/2})^k = 0 \), the application of this operator on the projection operator \( P^{U+1/2} \) from the left yields
\[ P^{U+1/2}A_{32}P^{U} = P^{U+1/2} \left( A_{32} - \frac{A_{31}A_{12}}{[2U + 2]} \right), \]  
where
\[ [2U + 2] = [f_1 - f_2 - k + \ell + 2]. \]  
As a result, the square of the norm becomes
\[ N^2(k\ell) = (-1)^{k+\ell} \langle L | A_{31}A_{32}^{-1}P^{U+1/2} \left( A_{32} - \frac{A_{32}A_{21}}{[f_1 - f_2 - k + \ell + 2]} \right) A_{23}A_{13} | L \rangle. \]
Commuting the generators \( A_{32} \) and \( A_{23}^{-1} \), we arrive at
\[ A_{32}A_{23}A_{13} | L > = [k] [f_3 - f_2 - k - \ell + 1] A_{23}^{-1}A_{13} | L > . \]
and
\[ A_{31}A_{12}A_{23}A_{13} | L > = [k] A_{31}A_{23}^{-1}A_{13}^{-1} | L > . \]
The commutation of the generators \( A_{32} \) and \( A_{12}^{-1} \) makes it possible to derive the relation
\[ A_{32}A_{13} = \left( A_{13}^{-1} - [\ell] q^{\ell-1} A_{13}^{-1} A_{12} q^{-(A_{22} - A_{21}^{-1})} \right). \]
Transferring the generator \( A_{31} \) to the right until it appears immediately in front of the lowest weight \( |L\rangle \), which annihilates it, we obtain
\[ P^{U+1/2}A_{32}A_{12}A_{23}^{-1}A_{13} | L > = [k] [\ell + 1] [f_3 - f_1 - \ell] P^{U+1/2}A_{23}^{-1}A_{13} | L > ; \]
therefore, we have
\[ P^{U+1/2}A_{32}P^{U} A_{23}^{-1}A_{13} | L > = [k] [f_3 - f_2 - k - \ell + 1] - [\ell + 1] [f_3 - f_1 - \ell] P^{U+1/2}A_{23}^{-1}A_{13} | L >. \]
Thus, the square of the norm, \( N^2(k\ell) \), takes the form
\[ N^2(k\ell) = (-1)^{k+\ell-1} \frac{[f_3 - f_2 - k + 1][f_2 - f_3 + k - 2]}{[f_1 - f_2 - k + \ell + 2]} \times \langle L | A_{31}A_{32}^{-1}P^{U+1/2}A_{23}^{-1}A_{13} | L \rangle. \]
In other words we have derived a recursion relation between \( N^2(k\ell) \) and \( N^2(k-1, \ell) \),
\[ N^2(k\ell) = \frac{[f_3 - f_2 - k + 1][f_2 - f_3 + k - 2]}{[f_1 - f_2 - k + \ell + 2]} N^2(k-1, \ell). \]
The recursion relation
\[ N^2(0\ell) = (-1)^{\ell} \langle H | A_{31}P^{U}A_{13} | H \rangle = [\ell] [f_3 - f_1 + \ell - 1] N^2(0, \ell - 1) \]
Noncompact quantum algebra $u_q(2,1)$ can be obtained in a similar way.

Using these recursion relations, we arrive at an ultimate expression for the square of the norm in (A.9):

$$N^2(k\ell) = \frac{[k]![\ell]![f_1 - f_2]![f_1 - f_2 - k + \ell + 1]![f_2 - f_3 + k - 2]![f_1 - f_3 + \ell - 1]!}{[f_1 - f_2 - k]![f_1 - f_2 + \ell + 1]![f_1 - f_3 - 1]![f_2 - f_3 - 2]!}.$$  

(A.23)

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