The structure sheaf of the moduli of oriented $p$-divisible groups

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Abstract

Using spectral algebraic geometry, we define a derived moduli stack of oriented $p$-divisible groups and study its structure sheaf. This stack is consequently used to prove a theorem of Lurie which predicts the existence of a certain sheaf of $\mathcal{E}_\infty$-rings $\mathcal{O}_{\text{top}}^{\text{BT}}_{p^n}$ on a formally étale site of the classical moduli stack of $p$-divisible groups. A variety of sections of this sheaf are then shown to be equivalent to known $\mathcal{E}_\infty$-rings in stable homotopy theory, and the natural symmetries on these sections also recover well-studied actions and operations.

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Introduction

One goal of spectral algebraic geometry is to reinterpret classical statements found in algebraic geometry or homotopy theory in a more natural setting. This idea has been positively utilised by Lurie to obtain an alternative definition of the $E_\infty$-ring of topological modular forms as the global sections of a certain derived moduli stack. In this article we continue this line of thought, by constructing and then utilising a spectral algebro-geometric object which has been described and used in the literature, but for which no written account is publicly available. This particular object is a derived moduli stack $M_{or}^{\ast}$, called the moduli stack of oriented $p$-divisible groups.

This article is devoted to studying the structure sheaf of $M_{or}^{\ast}$. To this end, we study the derived moduli stacks $M_{or}^{\ast}$ and $M_{un}^{\ast}$ of classical and unoriented $p$-divisible groups, respectively, which are related to $M_{or}^{\ast}$ by a derived infinitesimal thickening $\Theta: M_{or}^{\ast} \to M_{un}^{\ast}$, and a forgetful morphism $\Phi: M_{or}^{\ast} \to M_{un}^{\ast}$. Our main result is Theorem 0.0.5 (also see Theorem 1.1.4), which characterises affine sections of the sheaf of $E_\infty$-rings

$$\Theta^*\Phi^*\mathcal{O}_{or}^{\ast} = \mathcal{O}_{top}^{\ast}$$

in algebro-geometric terms. Using this theorem, we recover $E_\infty$-rings already present in homotopy theory, such as complex topological $K$-theory, Lubin–Tate theory, and topological modular and automorphic forms, all as sections of $\mathcal{O}_{top}^{\ast}$. We also recognise various restriction maps between these sections as Adams operators and known groups actions. To motivate the precise statement of Theorem 0.0.5, let us review some background material.

A ring spectrum $E$ is said to be complex oriented if it possesses a complex orientation; an element $e \in E^2(\mathbb{CP}^\infty)$ whose restriction to $E^2(\mathbb{CP}^1) \simeq \pi_0E$ is the unit element. Furthermore, we say that a ring spectrum $E$ is weakly 2-periodic if $\Sigma^2E$ is a locally free $E$-module of rank 1. The remarkable fact about complex oriented ring spectra $E$ is that the formal spectrum $\text{Spf}(E^0_{\mathbb{CP}^\infty})$ has the structure of a (1-dimensional and commutative) formal group, called the Quillen formal group associated to $E$; see [Ada74, Part II] or [EC2, §4.1]. A weakly 2-periodic complex oriented (abbreviated to complex periodic) ring spectrum $E$ is called an elliptic cohomology theory if the associated Quillen formal group is isomorphic to the formal group of an elliptic curve $E$ ([Sil86, Chapter IV]) over $E^0(\ast)$ and $\pi_kE = 0$ for odd integers $k$. Such theories were originally constructed and defined by Landweber, Ravenel, and Stong using the following Landweber exact functor theorem.

**Theorem 0.0.1.** Let $E$ be an elliptic curve over a ring $R$ such that the defining map into the moduli stack of elliptic curves, $\text{Spec} R \to M_{\text{Ell}}$, is flat, and the sheaf of invariant differentials $\omega_E$ of $E$ ([KMS86, 2.2.1.2]]) is trivial. Then there exists a multiplicative elliptic cohomology theory $E$ such that

$$\pi_*E \simeq R[u^{\pm 1}], \quad |u| = -2,$$

and whose associated Quillen formal group is isomorphic to the formal group of $E$ over $R$. 


For an older reference and survey, see \cite{Lan88}, and for a modern interpretation, see \cite{CHT} Lectures 15-16 or \cite{Mei19} §4.7-4.8.

Such cohomology theories connect stable homotopy theory with arithmetic and algebraic geometry, provide example of spectra of chromatic height 2, and also possess connections to physics and elliptic genera; see \cite{Lan88} for a survey. However, a common defect of elliptic cohomology theories obtained using the Landweber exact functor theorem is that they are only defined in the stable homotopy category, and hence only produce ring spectra that are commutative up to homotopy. A major achievement in homotopy theory of Goerss–Hopkins–Miller states that one can obtain $E_{\infty}$-structures on certain elliptic cohomology theories.

**Theorem 0.0.2.** There is an étale sheaf $\mathcal{O}_{\text{top}}$ of $E_{\infty}$-rings on the affine étale site of $\mathcal{M}_{\text{Ell}}$ such that for an object $E$: $\text{Spec } R \to \mathcal{M}_{\text{Ell}}$ of this site, the $E_{\infty}$-ring $\mathcal{E} = \mathcal{O}_{\text{top}}(E)$ has homotopy groups

$$\pi_{2k} \mathcal{E} \simeq \omega_E^{\otimes k}, \quad \pi_{2k+1} \mathcal{E} = 0, \quad \forall k \in \mathbb{Z},$$

and whose associated Quillen formal group is isomorphic to the formal group of $E$ over $R$.

A statement of this theorem can be found as \cite{EC2} Theorem 7.0.1].

This theorem has more features than just enhancing some of the homotopy commutative ring spectra produced from Theorem 0.0.1 with $E_{\infty}$-structures. As an étale sheaf, $\mathcal{O}_{\text{top}}$ is also defined on elliptic curves over Deligne–Mumford stacks such as $\mathcal{M}_{\text{Ell}}$ itself. This leads us to the definition of topological modular forms.

**Definition 0.0.3.** The $E_{\infty}$-ring $\text{TMF}$ of (periodic) topological modular forms is defined as the global sections $\mathcal{O}_{\text{top}}(\mathcal{M}_{\text{Ell}})$ of $\mathcal{O}_{\text{top}}$.

A survey on $\text{TMF}$, including variants such as $\text{tmf}$ and $\text{Tmf}$, as well as various applications can be found in \cite{Mil19} §6.

This brings us to the recent history of elliptic cohomology, and the reason for this article. In \cite{SUR}, Lurie sketched a new construction of $\mathcal{O}_{\text{top}}$ from the structure sheaf of a derived moduli stack of oriented elliptic curves, $\mathcal{M}^r_{\text{Ell}}$; see \cite{EC2} §7 for details. It is made clear in \cite{Mil19} §6.7, \cite{EC2}, and \cite{SUR} that this is not the end of the story, and that $\mathcal{O}_{\text{top}}$ is the pullback of yet another sheaf. This is precisely the point of the following theorem, which we state in more detail and prove as Theorem 2.3.3.

**Theorem 0.0.4.** Let $p$ be a prime and denote by $\mathcal{O}_{\text{top}}$ the $p$-completion of $\mathcal{O}_{\text{top}}$. Then there exists an étale sheaf $\mathcal{O}_{\text{top}}^{\text{BT}_2}$ such that for every elliptic curve $E$ whose defining map to the moduli stack of elliptic curves is étale, one has a natural equivalence of $E_{\infty}$-rings

$$\mathcal{O}_{\text{top}}(E) \simeq \mathcal{O}_{\text{top}}^{\text{BT}_2}(E[p^{\infty}]),$$

where $E[p^{\infty}]$ is the $p$-divisible group (also known as a Barsotti–Tate group) associated to the elliptic curve $E$. 

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In particular, not only will automorphisms of the elliptic curve $E$ act on $\hat{\mathcal{O}}_{\text{top}}(E)$, but automorphisms of the associated $p$-divisible group $E[p^{\infty}]$ will too. Theorem 0.0.4 is a corollary of the following more general theorem of Lurie, often called “Lurie’s theorem”, which first appeared without proof in [BL10, Theorem 8.1.4]. The goal of this article is to fill this gap in the literature by proving this theorem and to study the expected examples. We provide a more precise and general statement in Theorem 1.1.4.

**Theorem 0.0.5.** Fix a prime $p$ and a positive integer $n \geq 1$. There is an étale sheaf of $E_\infty$-rings $\mathcal{O}_{\text{top}}^{\text{BT}_p^n}$ on a particular site $\mathcal{C}_{\mathbb{Z}_p}^n$ of formal Deligne–Mumford stacks which are formally étale over the $p$-completion of the moduli stack of $p$-divisible groups of height $n$ such that its value on formal affines $\text{Spf} \, B_0$, with $p$-divisible group $G_0$, is an $E_\infty$-ring $\mathcal{E}$ such that:

1. $\mathcal{E}$ is complex periodic,
2. the groups $\pi_k \mathcal{E}$ vanish for all odd integers $k$,
3. there is a natural isomorphism of rings $\pi_0 \mathcal{E} \simeq B_0$, and
4. there is an isomorphism of formal groups $G_0^\circ \simeq \hat{G}_Q^\circ$ over $B_0$, where the former is the formal group associated to the $p$-divisible group $G_0$ ([Tat67, §2.2]) and the latter is the Quillen formal group associated to the above complex orientation.

In particular, $\pi_{2k} \mathcal{O}_{\text{top}}^{\text{BT}_p^n} \simeq \omega_{G_0}^{\otimes k}$.

The construction of the sheaf $\mathcal{O}_{\text{top}}^{\text{BT}_p^n}$ of Theorem 0.0.5 involves constructing the moduli stack $\mathcal{M}_{\text{BT}_p^n}$ of oriented $p$-divisible groups of height $n$; see [1.3.1]. In various contexts, the $E_\infty$-ring $\mathcal{E}$ of Theorem 0.0.5 recovers known $E_\infty$-rings.

**Example 0.0.6** (Topological modular forms). Given an abelian variety $X$ of dimension $g$ over a scheme $S$ and a positive integer $n \geq 1$, one can define the $n$-torsion points of $X$, denoted as $X[n]$, using the Cartesian diagram of schemes

$$
\begin{array}{ccc}
X[n] & \longrightarrow & X \\
\downarrow & & \downarrow[n] \\
S & \longrightarrow & X
\end{array}
$$

In particular, for a fixed prime $p$ the collection $X[p^{\infty}] = \{X[p^n]\}_{n \geq 1}$ is a $p$-divisible group of height $2g$ over $S$; see [Tat67, §2]. An abelian variety of dimension 1 is an elliptic curve, and in this case the above construction provides us with a map of stacks

$$[p^{\infty}] : \mathcal{M}_{\text{Ell}} \to \mathcal{M}_{\text{BT}_2^n}.$$

The classical Serre–Tate theorem states that after a base-change over $\text{Spf} \, \mathbb{Z}_p$, this map of moduli stacks is formally étale (Proposition 2.3.1), which readily implies that we can apply Theorem 0.0.5 to $[p^{\infty}]$. We then see that $\mathcal{O}_{\text{top}}^{\text{BT}_2^n}([p^{\infty}])$ is the $E_\infty$-ring TMF of $p$-completed topological modular forms, as predicted by Proposition 0.0.4; see Theorem 2.3.3.
Another classical example is that of Lubin–Tate theory; see [LT66] and [GH04, §7].

**Example 0.0.7** (Lubin–Tate theory). If $\kappa$ is a perfect field of characteristic $p$ and $\hat{G}_0$ is a $1$-dimensional formal group of finite height $n \geq 1$ over $\kappa$, then Lubin–Tate theory states that there exists a universal deformation of $\hat{G}_0$ by a formal group $\hat{G}$ over a ring $R_{\hat{G}_0}^\mathrm{LT}$ which is noncanonically isomorphic to $W(\kappa)[[v_1, \ldots, v_{n-1}]]$. The equivalence between formal groups over $p$-complete rings and $p$-divisible groups ([Tat67 (2.2)]) and the fact that $\hat{G}$ is a universal deformation show that the $p$-divisible group $G$ associated to $\hat{G}$ gives a formally étale map (Proposition 1.2.11)

$$\mathrm{Spf} \ R_{\hat{G}_0}^\mathrm{LT} \to M_{\mathcal{O}_{\hat{G}_0}}^\vee,$$

and we see that we can apply $\hat{G}_{\mathcal{O}_{\hat{G}_0}}^{\top}$ to this object; see Proposition 2.2.1. In this case $\hat{G}_{\mathcal{O}_{\hat{G}_0}}^{\top}(G)$ is the Lubin–Tate $\mathbb{E}_\infty$-ring $E_n$ of [GH04 §7] and [EC2 §5]; see Proposition 2.2.2. As a special case of this example, we also obtain the $p$-completion $\mathbf{K}_p$ of complex topological $K$-theory when $G = \mu_{p, \infty}$ is the multiplicative $p$-divisible group over $\mathbb{Z}_p$; see Proposition 2.1.7.

**Example 0.0.8** (PEL-Shimura stacks). A novel example of an $\mathbb{E}_\infty$-ring produced by Theorem 0.0.5 is the $\mathbb{E}_\infty$-ring $TAF$ of topological automorphic forms; see Proposition 2.4.4. Moduli stacks of abelian varieties of dimension greater than $1$ produce $p$-divisible groups with dimension greater than $1$, which does not fit into our theory. One can amend this problem by studying certain PEL-Shimura stacks, built from moduli stacks of abelian varieties by considering polarisations, endomorphisms, and level-structure. The resulting Deligne–Mumford stacks $X_{V,L}$ have $1$-dimensional $p$-divisible groups to which we can apply $\hat{G}_{\mathcal{O}_{\hat{G}_0}}^{\top}$; see Proposition 2.4.3. The object $\hat{G}_{\mathcal{O}_{\hat{G}_0}}^{\top}(X_{V,L})$ is then equivalent to what Behrens–Lawson denote by $TAF$ by definition; see Proposition 2.4.4.

The functorality of $\hat{G}_{\mathcal{O}_{\hat{G}_0}}^{\top}$ gives rise to many actions and operations on the $\mathbb{E}_\infty$-rings in the examples above. Using this idea, we construct Adams operations, real topological $K$-theory $\mathbf{K}_O$, stabiliser group actions, Galois group actions, and Mackey functor structures.

**Outline**

1. In the first section, we will state and prove Theorem 0.0.5 following the outline of Lurie’s construction of $\hat{G}_{\mathcal{O}_{\hat{G}_0}}^{\top}$ from [EC2 §7].

1.1 We begin by introducing a precise statement (Theorem 1.1.4) and the supporting cast of objects. This is followed by an outline of our proof, which gives a synopsis of the following three subsections.

1.2 The deformation theory of Lurie ([SAG §17-18]) manifests itself in this article as the adjective formally étale. These techniques are then used to lift classical algebraic geometry to spectral algebraic geometry; see Theorem 1.2.3.

1.3 Next, we use the orientation theory for $p$-divisible groups à la [EC2 §4]. This allows us to define a sheaf $\mathbf{O}_{\mathcal{O}_{\hat{G}_0}}^{\top}$ which takes a $p$-divisible group over a $p$-complete $\mathbb{E}_\infty$-ring and produces the universal orientation classifier; see Theorem 1.3.9.
(1.4) Finally, we define the sheaf $O_{\text{top}}^{\text{BT}_p}$ by first applying the process of §1.2 followed by the sheaf $O_{\text{or}}^{\text{BT}_p}$ of §1.3. The fact that this definition satisfies the conditions of Theorem 0.0.5 follows using the same arguments made in [EC2 §7.3].

(2) In this second section, we construct a variety of well-known $\mathbb{E}_\infty$-rings, including those of Examples 0.0.6, 0.0.7, and 0.0.8, as well as some operations and actions thereof.

(A,B) In these two appendices, we study summarise some needed facts above formal spectral Deligne–Mumford stacks and the descent theory of oriented $p$-divisible groups.

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Conventions

Now, and forever, fix a prime $p$.

Due to the sustained references to Lurie’s collection of work, we have decided to reference them in a nonstandard way, writing $[\text{CHT}]$, $[\text{EC1}]$, $[\text{EC2}]$, $[\text{EC3}]$, $[\text{HA}]$, $[\text{HTT}]$, $[\text{SAG}]$, and $[\text{SUR}]$ for the corresponding sources.

Higher categories and higher algebra

We will make free and extensive use of the language of $\infty$-categories, higher algebra, and spectral algebraic geometry, following $[\text{HTT}]$, $[\text{HA}]$, and $[\text{SAG}]$, so all categories, functors, (co)limits, Kan extensions, mapping spaces, Yoneda lemmas, etc., are all with respect to this context. In particular:

- All 1-categories $\mathcal{C}$ will not be distinguished from their nerves.
- Adjunctions will often be given as a pair of functors $(L, R)$, with all other data implicit.
- All commutative diagrams of shape a discrete category $I$ in an $\infty$-category $\mathcal{C}$ are functors of $\infty$-categories $I \to \mathcal{C}$, and hence will only commute up to the higher homotopy coherence dictated by $\mathcal{C}$, e.g., commutative squares are functors $\Delta^1 \times \Delta^1 \to \mathcal{C}$.
- All discrete commutative rings and discrete abelian groups will not be distinguished from their associated Eilenberg–Mac Lane spectra inside the $\infty$-category of spectra, $\text{Sp}$. Moreover, the smash product of spectra will be written as $\otimes$, even if the spectra involved are discrete (this obviously does not mean the output will be discrete). The same goes for completions, and in this case the $\infty$-categorical completions will be written as $(-)^I_1$ following $[\text{SAG} \, \S 7]$.
- All module categories $\text{Mod}_R$ refer to the stable $\infty$-category of $R$-modules, where $R$ is an $E_\infty$-ring. In particular, if $R$ is a discrete commutative ring, then $\text{Mod}_R$ will be the stable $\infty$-category of $R$-module spectra, and not the abelian 1-category of $R$-modules. The same holds for $\infty$-categories of quasi-coherent sheaves of $\mathcal{O}_X$-modules.
- Following $[\text{EC2}]$ (and contrary to $[\text{SAG}]$), we will write $\text{Spec} R$ for the nonconnective spectral Deligne–Mumford stack associated to an $E_\infty$-ring $R$. This is to more closely match $[\text{EC2}]$ and to highlight the functor-of-points perspective.

Sites and topoi

There will be occasion for us to discuss sheaves on $\infty$-categories with a Grothendieck topology and $\infty$-topoi; see for a comprehensive introduction $[\text{HTT} \, \S 6]$, though everything we will need can be found in $[\text{SAG} \, \S 1.3.1]$. 
Definition 0.0.9. Given an essentially small ∞-category $T$ with a Grothendieck topology $\tau$ and an ∞-category $C$, then a $C$-valued $\tau$-sheaf $F$ on $T$ is a functor $F: T^{\text{op}} \to C$ such that for all $\tau$-sieves $T^0_U \subseteq T_U$, the composite

$$((T^0_U)^\text{op})^\text{op} \to ((T_U)^\text{op})^\text{op} \to T^{\text{op}} F \to C$$

is a limit diagram inside $C$; see [SAG, Definition 1.3.1.1]. A $C$-valued sheaf on an ∞-topos $X$ is a presheaf $F: X^{\text{op}} \to C$ which preserves small limits; see [SAG, Definition 1.3.1.4]. The associated ∞-categories $\text{Shv}_C^\tau(T)$ and $\text{Shv}_C^\tau(X)$ are defined as the ∞-subcategories of $\text{Fun}(T^{\text{op}}, C)$ and $\text{Fun}(X^{\text{op}}, C)$, respectively.

We will often have occasion to switch between sheaves on sites and sheaves on topoi. In particular, [SAG, Proposition 1.3.1.7] tells us that precomposition with the Yoneda embedding and sheafification induces an equivalence of ∞-categories

$$\text{Shv}_C(\text{Shv}^\tau(T)) \xrightarrow{\sim} \text{Shv}_C^\tau(T),$$

whose inverse is given by taking right Kan extensions.

Topological rings and formal stacks

We will follow the definition of an adic $E_\infty$-ring from [EC2, Definition 0.0.11], except we will only consider the connective case.

Definition 0.0.11. An adic ring $A$ is a discrete ring with a topology defined by an $I$-adic topology for some finitely generated ideal of definition $I \subseteq A$. Morphisms between adic rings are continuous ring homomorphisms, defining a 1-category $\text{CAlg}_{\text{ad}}$ inside $\text{CAlg}^\heartsuit$. An adic $E_\infty$-ring is a connective $E_\infty$-ring $A$ such that $\pi_0 A$ is an adic ring. We define the ∞-category of adic $E_\infty$-rings as the fibre product

$$\text{CAlg}_{\text{cn}}^\heartsuit = \text{CAlg}_{\text{cn}}^\heartsuit \times_{\text{CAlg}^\heartsuit_{\text{ad}}}.$$

An adic $E_\infty$-ring $A$ is said to be complete if it is complete with respect to an ideal of definition $I$; see [SAG] Definition 7.3.1.1] and [SAG] Theorem 7.3.4.1]. An $E_\infty$-ring $R$ is local if $\pi_0 R$ is a local ring, and we call an adic $E_\infty$-ring $R$ local if the topology on $\pi_0 R$ is defined by the maximal ideal of $\pi_0 R$. We give $\text{CAlg}_{\text{ad}}$ and $\text{CAlg}_{\text{cn}}$ the usual Grothendieck topologies (fpqc, étale, etc.) via the forgetful functors to $\text{CAlg}^\heartsuit$ and $\text{CAlg}^\heartsuit_{\text{ad}}$, respectively.

The subtleties to keep in mind with the above definition is the mere existence of an ideal of definition, rather than a fixed ideal of definition, and the insistence on the finite generation of such an ideal. The geometric definitions of a formal Deligne–Mumford stack and a formal spectral Deligne–Mumford stack then follow.

Definition 0.0.12. Let $\text{Spf}: \text{CAlg}_{\text{cn}}^\heartsuit \to \infty \text{Top}_{\text{CAlg}^\heartsuit}$ be the functor described in [SAG Construction 8.1.1.10] and [SAG Proposition 8.1.2.1]. A spectrally ringed ∞-topos $X$ is said to be an affine formal spectral Deligne–Mumford stack if it lies in the essential image of $\text{Spf}$. A
formal spectral Deligne–Mumford stack is a spectrally ringed \( \infty \)-topos with a cover by affine formal spectral Deligne–Mumford stacks; see [SAG, Definition 8.1.3.1]. Let \( \mathfrak{fSpDM} \) denote the full \( \infty \)-subcategory of \( \mathcal{T}^{\text{loc}}_{\text{CAlg}} \) spanned by formal spectral Deligne–Mumford stacks. Similarly, one can defined classical formal Deligne–Mumford stacks (Definition A.3.1), and we assume all such objects are locally Noetherian.

**Definition 0.0.13.** Let \( \mathcal{X} = (\mathcal{X}, \mathcal{O}_X) \) be a formal spectral Deligne–Mumford stack. We call an object \( U \) inside \( \mathcal{X} \) affine if the spectrally ringed \( \infty \)-topos \( (\mathcal{X}/U, \mathcal{O}_X|_U) \) is equivalent to \( \text{Spf} A \) for some adic \( E_\infty \)-ring \( A \). We will also say that \( \mathcal{X} \) is locally Noetherian if for every affine object \( U \) of \( \mathcal{X} \), the \( E_\infty \)-ring \( \mathcal{O}_X(U) \) is Noetherian (in the sense of [HA, Definition 7.2.4.30]).

Note that \( \text{Spf} B \) is locally Noetherian if and only if \( B \) itself is a Noetherian \( E_\infty \)-ring; see [SAG, Proposition 8.4.2.2]. We will consistently be working over a base adic \( E_\infty \)-ring \( A \).

**Notation 0.0.14.** Let \( A \) denote some fixed complete local Noetherian adic \( E_\infty \)-ring with perfect residue field of characteristic \( p \). We will also write \( A_0 = \mathbb{Z}_p \), \( m_A = \text{maximal ideal of } A_0 \), and \( \kappa_A = \text{perfect residue field} \).

The \( A \) we expect the reader to keep in mind is the \( p \)-complete sphere, \( S_p \), with associated \( A_0 = \mathbb{Z}_p \), over which all of our examples will live. Other choices include the spherical Witt vectors of a perfect field of characteristic \( p \); see [EC2, §5.1].

**Functor of points**

If \( \mathcal{X} \) is a spectral Deligne–Mumford stack, and \( Y_0 \to \tau_{\leq 0} \mathcal{X} \) is an étale morphism into the 0th truncation of \( \mathcal{X} \), then using the fact that \( L_{Y_0/\tau_{\leq 0} \mathcal{X}} \) vanishes and the fact that each \( \tau_{\leq n} \mathcal{X} \to \tau_{\leq n+1} \mathcal{X} \) is a “square-zero thickening”, one can show through an inductive argument that there is a universal deformation \( Y \) of \( Y_0 \). In most cases, \( Y \) will not be a spectral Deligne–Mumford stack, but a formal spectral Deligne–Mumford stack. We cannot readily apply this argument to our stack of interest \( \mathcal{M}_{\text{BTP}} \) though, as it is not a spectral Deligne–Mumford stack, either formal or otherwise; see [EC2, Remark 3.2.7]. In particular, it doesn’t make sense to define \( \tau_{\leq n} \mathcal{M}_{\text{BTP}} \): in which category would this object live in? This, and indeed Lurie’s definition of \( \mathcal{M}_{\text{BTP}} \), necessitates our use of the functor-of-points perspective. This applies to everything; our classical algebraic geometry, our formal spectral algebraic geometry, our moduli problems, and our deformation theory.

**Definition 0.0.15.** Let us write

\[
\text{Aff} = \text{CAlg}^{\text{op}}, \quad \text{Aff}^{\text{cn}} = (\text{CAlg}^{\text{cn}})^{\text{op}}, \quad \text{Aff}^{\uparrow} = \left( \text{CAlg}^{\uparrow} \right)^{\text{op}},
\]

\[
\text{Aff}^{\text{cn}}_{\text{ad}} = (\text{CAlg}^{\text{cn}}_{\text{ad}})^{\text{op}}, \quad \text{Aff}^{\uparrow}_{\text{ad}} = \left( \text{CAlg}^{\uparrow}_{\text{ad}} \right)^{\text{op}}.
\]

For any \( \infty \)-category \( \mathcal{C} \), we will write \( \mathcal{P}(\mathcal{C}) \) for the presheaf category \( \text{Fun}(\mathcal{C}^{\text{op}}, S) \). We will permute superscripts with subscripts if typographically obliged to do so.
When working in $P(Aff^\vee)$ or $P(Aff^{cn})$, we will abuse notation and not distinguish between the objects representing functors and the representing functors themselves. This is justified by the commutative diagram of fully-faithful functors of $\infty$-categories

\[
\begin{array}{cccccc}
Aff^{\vee}_{\text{loc.N}} & \longrightarrow & \text{Aff}^{cn} & \leftarrow & \text{Aff}^{cn}_{\text{ad}} & \\
\downarrow & & \downarrow & & \downarrow & \\
Aff^{\vee}_{\text{ad,loc.N}} & \longrightarrow & \text{Aff}^{cn}_{\text{ad}} & \leftarrow & \text{DM}_{\text{loc.N}} & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
\text{fDM} & \longrightarrow & \text{SpDM} & \leftarrow & \text{fSpDM} & \longrightarrow \\
\end{array}
\]

(0.0.16)

where the $\text{loc.N}$ subscript denotes those full $\infty$-subcategories spanned by Noetherian or locally Noetherian objects; see Definition 0.0.13. The definitions and fully-faithfulness of the above functors is explained in Corollary A.3.5 with the exception of the functors

$\text{Aff}^{\vee}_{\text{loc.N}} \xrightarrow{\text{(a)}} \text{DM}_{\text{loc.N}} \xrightarrow{\text{(b)}} \text{fDM}$, $\text{Aff}^{cn} \xrightarrow{\text{(c)}} \text{SpDM} \xrightarrow{\text{(d)}} \text{fSpDM}$.

For these functors, fully-faithfulness can be justified as follows:

(a) is fully-faithful as this is true without the locally Noetherian hypotheses; see [SAG, Remark 1.2.3.6] and restrict to the underlying $(2,1)$-category.

(b) is fully-faithful by using part (d) below, [SAG, Remark 1.4.8.3] (which embeds $\text{DM}_{\text{loc.N}}$ into $\text{SpDM}$), Proposition A.3.4 (which embeds $\text{fDM}$ into $\text{fSpDM}$), and the fact that if the composition of two functors is fully-faithful, and the latter functor is fully-faithful, then the first functor must have been fully-faithful.

(c) is fully-faithful by making a connective (justified by [SAG, Corollary 1.4.5.3]) version of [SAG, Remark 1.4.7.1] (which states that nonconnective $E_\infty$-rings embeds into nonconnective spectral Deligne–Mumford stacks).

(d) is fully-faithful as a consequence of both $\text{SpDM}$ and $\text{fSpDM}$ being defined as full $\infty$-subcategories of $\infty\text{Top}_{\text{alg}}^{\text{loc}}$ and the fact that spectral Deligne–Mumford stacks are examples of formal spectral Deligne–Mumford stacks by [SAG, p. 628].

Similarly, we will consider most of classical algebraic geometry as living in the $(2,1)$-category $\text{Fun}(\text{CAlg}^{\vee}, S_{\leq 1})$ which we then embed inside the $\infty$-category $P(Aff^\vee)$ using the inclusion $S_{\leq 1} \rightarrow S$, which preserves limits.

Warning 0.0.17. When we consider quasi-coherent sheaves on a formal spectral Deligne–Mumford stack $\mathcal{X}$, then what we write as $\text{QCoh}(\mathcal{X})$ is what Lurie would write as $\text{QCoh}(h_{\mathcal{X}})$, in other words, we consider the formation of $\infty$-categories of quasi-coherent sheaves on formal spectral Deligne–Mumford stacks through their functors of points. By [SAG, Corollary...
8.3.4.6], we see that these two notations are equivalent as long as one restrictions to almost connective quasi-coherent sheaves on both sides. As all of our quasi-coherent sheaves of interest will be cotangent complexes, they will be almost connective by definition ([SAG] Definition 17.2.4.2]), so this should not cause any confusion.

Cotangent complexes

The functor of points perspective also controls our study of cotangent complexes. Given a natural transformation between functors \( X \to Y \) in \( \mathcal{P}(\text{Aff}^{\text{cn}}) \) which admit a cotangent complex ([SAG] Definition 17.2.4.2)), we write this cotangent complex as \( L_{X/Y} \) and consider it as an object of \( \text{QCoh}(X) \); see [SAG] §6.2. There are a few specific cases in which this definition simplifies:

1. If \( A \to B \) is a morphism of connective \( E_{\infty} \)-rings, which we consider as a morphism \( \text{Spec} \, B \to \text{Spec} \, A \) of spectral Deligne–Mumford stacks, and \( X \to Y \) is the associated transformation of functors in \( \mathcal{P}(\text{Aff}^{\text{cn}}) \), then it follows from [SAG, Corollary 17.2.5.4] that \( L_{X/Y} \) is equivalent to \( L_{\text{Spec} \, B/\text{Spec} \, A} \) (under the equivalence of categories \( \text{QCoh}(\text{Spec} \, A) \cong \text{Mod}_{A} \)).

2. If \( X \to Y \) is a morphism of spectral Deligne–Mumford stacks and \( X \to Y \) is the associated transformation of functors in \( \mathcal{P}(\text{Aff}^{\text{cn}}) \), then \( L_{X/Y} \) is equivalent to \( L_{X/X} \) by [SAG, Corollary 17.2.5.4].

3. If \( f: X \to Y \) is a morphism of formal spectral Deligne–Mumford stacks and \( F: X \to Y \) is the associated transformation of functors in \( \mathcal{P}(\text{Aff}^{\text{cn}}) \), then \( L_{X/Y} \) is equivalent to \( L_{X/Y} \) (under the equivalence of categories \( \text{QCoh}(X) \cong \text{QCoh}(X) \) by [SAG] Corollary 17.1.2.8], under the equivalence of categories \( \Theta_{X} : \text{QCoh}(X)^{\text{acn}} \to \text{QCoh}(X)^{\text{acn}} \) of [SAG] Corollary 8.3.4.6], where the superscript acn indicates full \( \infty \)-subcategories of almost connective objects. If \( X = \text{Spf} \, A \) for an adic \( E_{\infty} \)-ring \( A \), then \( L_{\text{Spf} \, A} \simeq (L_{A})^{\text{I}} \) (under the equivalence of categories \( \text{QCoh}(\text{Spf} \, A) \cong \text{Mod}^{\text{pl}}_{I} \)), where \( I \) is a finitely generated ideal of definition for the topology on \( \pi_{0} A \); see [SAG] Example 17.1.2.9].

The following standard properties of the cotangent complex of functors will be used without explicit reference:

\[ f^{*} L_{X/Y} \to L_{X} \to L_{X/Y} \]

\[ f^{*} L_{X} \to L_{X} \to L_{X/Y} , \]

where \( f^{*} \) is the completed pullback of [SAG] Remark 8.3.2.10], in \( \text{QCoh}(X) \) and \( \text{QCoh}(X) \), respectively. It is then stated in [SAG] Proposition 17.2.5.1] that we have an equivalence \( \Theta_{X}(L_{X}) \simeq L_{X} \), and similarly for \( \mathcal{D} \). We then obtain our desired statement from the natural equivalence

\[ F^{*} \circ \Theta_{\mathcal{D}} \simeq \Theta_{X} \circ f^{*} : \text{QCoh}(\mathcal{D}) \to \text{QCoh}(X) , \]

itself a consequence of the naturality of \( \Theta_{(-)} \); see [SAG] Construction 8.3.4.1] for the construction of \( \Theta_{(-)} \).
• For any map of connective $\mathbb{E}_\infty$-rings $A \to B$, we have a natural equivalence in $\text{Mod}_{\pi_0B}$

$$\pi_0 L_{B/A} \simeq \Omega^1_{\pi_0 B/\pi_0 A};$$

see [HA, Proposition 7.4.3.9].

• For composable transformations of functors $X \to Y \to Z$ in $\mathcal{P}(\text{Aff}^{cn})$, where each functor has a cotangent complex, we obtain a canonical (co) fibre sequence in $\text{QCoh}(X)$

$$L_{Y/Z} \mid_X \to L_{X/Z} \to L_{X/Y};$$

see [SAG, Proposition 17.2.5.2].

• If we have transformations $X \to Y \leftarrow Y'$ of functors inside $\mathcal{P}(\text{Aff}^{cn})$, where $L_{X/Y}$ exists, then $L_{X \times Y'Y'/Y'}$ exists and is naturally equivalent to $\pi_1^1 L_{X/Y}$; see [SAG, Remark 17.2.4.6].

Warning 0.0.18. One should note that the cotangent complexes considered in this article are not the same as those developed by André and Quillen; see [Aut20, Tag 08P5]. In particular, for an ordinary commutative ring $R$ considered as a discrete $\mathbb{E}_\infty$-ring, then $L_R$ is what some call the topological cotangent complex. For more discussion, see [SAG, §25.3].
Table of notation

| Symbol | Description |
|--------|-------------|
| $A, B, C, D, R$ | unital (possible adic) rings, either discrete commutative or $E_\infty$. |
| $\alpha_k$ | operations on the multiplicative $p$-divisible group $\mu_p^\infty$; see Definition 2.1.10. |
| $C_{A_0}$ | our site of classical formal Deligne–Mumford stacks; see Definition 1.1.6. |
| $C_A$ | our site of formal spectral Deligne–Mumford stacks; see Definition 1.1.3. |
| $(X/Y)_{dR}$ | de Rham space of a map $X \to Y$ in $\mathcal{P}(\text{Aff}^{cn})$. |
| $E_a$ | Lubin–Tate theory $E_\infty$-ring. |
| $\Gamma$ | Morava stabiliser group. |
| $G$ | $p$-divisible group over an $E_\infty$-ring. |
| $\hat{G}$ | formal group over an $E_\infty$-ring. |
| $\hat{G}_{m}$ | formal multiplicative group over an $E_\infty$-ring. |
| $\mathbb{K}, \mathbb{K}_p$ | $E_\infty$-ring of real topological $K$-theory and its $p$-completion. |
| $\mathbb{K}_U, \mathbb{K}_U^p$ | $E_\infty$-ring of complex topological $K$-theory and its $p$-completion. |
| $L_{X/Y}$ | cotangent complex of a transformation $X \to Y$ in $\mathcal{P}(\text{Aff}^{cn})$. |
| $\mathcal{M}^\odot$ | restriction of a functor $\mathcal{M}: \mathrm{CAlg}^{cn} \to \mathcal{S}$ to $\mathrm{CAlg}^\odot$. |
| $\mathcal{M}_A$ | fibre product of a functor $\mathcal{M}: \mathrm{CAlg}^{cn} \to \mathcal{S}$ with $\text{Spf} A$. |
| $\mathcal{M}_{\text{Ell}}$, $\mathcal{M}_{\text{BTr}}, \mathcal{M}_{\text{Br}_n}$ | moduli stack of strict elliptic curves. |
| $\mathcal{M}_{\text{Ell}}^\odot$, $\mathcal{M}_{\text{BTr}}^\odot$, $\mathcal{M}_{\text{Br}_n}^\odot$ | moduli stack of $p$-divisible groups, and that of height $n$. |
| $\mathcal{M}_{\text{Ell}}^\odot$, $\mathcal{M}_{\text{BTr}}^\odot$, $\mathcal{M}_{\text{Br}_n}^\odot$ | moduli stack of oriented $p$-divisible groups, and that of height $n$. |
| $\mathcal{N}^\odot_{\text{BTr}}$, $\mathcal{N}^\odot_{\text{Br}_n}$ | spectrally ringed $\infty$-topos of Construction 1.2.41. |
| $\mu_p^\infty$ | Goerss–Hopkins–Miller sheaf on the étale site of $\mathcal{M}_{\text{Ell}}^\odot$. |
| $\mathcal{O}^\text{top}$ | Goerss–Hopkins–Lurie–Miller sheaf on $C_{A_0}$; see Definition 1.4.1. |
| $\Phi, \phi$ | morphisms between (un)oriented moduli stacks; see §1.3. |
| $\psi^q$ | $q$th Adams operation (on $\mathbb{K}_U$). |
| $S, S_p$ | sphere spectrum and its $p$-completion. |
| $\mathbb{TAF}_{V,L}$ | $E_\infty$-ring of topological automorphic forms with initial data $(V, L)$. |
| $\mathbb{TMF}, \mathbb{TMF}_p$ | $E_\infty$-ring of periodic topological modular forms and its $p$-completion. |
| $\theta$ | morphisms between $C_{A_0}$ and $C_A$, or $\mathcal{M}_{\text{BTr}}^\odot$ and $\mathcal{M}_{\text{BTr}}^\odot$; see §1.2. |
| $X, Y, M$ | functors $\mathrm{CAlg}^\odot \to \mathcal{S}$ or $\mathrm{CAlg}^{cn} \to \mathcal{S}$. |
| $X, \mathcal{Y}$ | topoi, either classical or $\infty$-.. |
| $X, \mathcal{Y}$ | Deligne–Mumford stacks, either classical or spectral. |
| $\mathcal{X}, \mathcal{Y}$ | formal Deligne–Mumford stacks, either classical or spectral. |
| $\omega_E$ | sheaf of invariant differentials of elliptic curve $E$. |
| $\omega_{\hat{G}}, \omega_{\hat{G}}$ | dualising line of formal group $\hat{G}$ or $p$-divisible group $G$. |
| $\mathbb{Z}, \mathbb{Z}_p$ | integers and its $p$-completion. |
1 Lurie’s theorem

The titular theorem promises the existence of a sheaf $\mathcal{O}^\text{top}_{\text{BT}_n}$ on some site over the moduli stack of $p$-divisible groups satisfying certain properties. The idea behind the construction of $\mathcal{O}^\text{top}_{\text{BT}_n}$ has been outlined in the Introduction: given the morphisms of stacks

\[ \mathcal{M}^\text{or}_{\text{BT}_n} \xrightarrow{\Phi} \mathcal{M}_{\text{un}}^\text{or}_{\text{BT}_n} \xleftarrow{\Theta} \mathcal{M}^\heartsuit_{\text{BT}_n} \]

we set $\mathcal{O}^\text{top}_{\text{BT}_n} = \Theta^* \Phi^* \mathcal{O}^\text{or}_{\text{BT}_n}$, and check this possesses the desired properties. In §1.1 we state a precise version of “Lurie’s theorem” and give a more detailed outline of the proof, in §1.2 we construct $\Theta$ using spectral deformation theory disguised as the adjective formally étale, in §1.3 we construct $\Phi$ using the orientation theory of Lurie ([EC2]), and in §1.4 we check the above definition of $\mathcal{O}^\text{top}_{\text{BT}_n}$ satisfies the conditions of our main theorem.

1.1 Statement and proof outline

1.1.1 Statement of theorem

First, let us recall the definition of a $p$-divisible group over an $E_{\infty}$-ring.

**Definition 1.1.1.** Let $R$ be a connective $E_{\infty}$-ring. A $p$-divisible (Barsotti–Tate) group over a connective $E_{\infty}$-ring $R$ is a functor $G : \text{CAlg}_{R}^{cn} \to \text{Mod}^{cn}_{Z}$ with the following properties:

1. For every connective $E_{\infty}$-$R$-algebra $B$, the $Z$-module $G(B)[1/p]$ vanishes.

2. For every finite abelian $p$-group $M$, the functor

\[ \text{CAlg}_{R}^{cn} \to S; \quad B \mapsto \text{Map}_{\text{Mod}^{cn}_{Z}}(M, G(B)) \]

is corepresented by a finite flat $E_{\infty}$-$R$-algebra.

3. The map $p : G \to G$ is locally surjective with respect to the finite flat topology;

see [EC2 Definition 2.0.2] for this definition, and [EC1 §6] or [EC3 §2] for a wider discussion. A $p$-divisible group over a not necessarily connective $E_{\infty}$-ring $R$, is a $p$-divisible group over its connective cover. The $\infty$-category $\text{BT}^p(R)$ of $p$-divisible groups over an $E_{\infty}$-ring $R$ is the full $\infty$-subcategory of $\text{Fun}(\text{CAlg}_{R}^{cn}, \text{Mod}^{cn}_{Z})$ spanned by $p$-divisible groups. Let $\mathcal{M}_{\text{BT}^p}$ be the moduli stack of $p$-divisible groups, which is the functor inside $\mathcal{P}(\text{Aff}^{cn})$ defined on objects by sending $R$ to the $\infty$-groupoid core $\text{BT}^p(R)^{\sim}$; see [EC2 Definition 3.2.1]. We say a $p$-divisible group $G$ has height $n$ if the $E_{\infty}$-$R$-algebra corepresenting the functor

\[ \text{CAlg}_{R}^{cn} \to S; \quad B \mapsto \text{Map}_{\text{Mod}^{cn}_{Z}}(Z/pZ, G(B)) \]

is finite of rank $p^n$; see [EC1 §6.5]. Using this notion of height, we can further define an subfunctor $\mathcal{M}_{\text{BT}^p_n}$ for all $n \geq 1$ consisting of all $p$-divisible groups of height $n$. By Remark 1.3.8 this subfunctor is open.
Our main object of study in this article is the moduli stack $\mathcal{M}_{\text{BT}_p^n}$ inside the presheaf category $\mathcal{P}(\text{Aff}^{cn})$. We will also relate these spectral moduli stacks to classical stacks.

**Notation 1.1.2.** For a functor $\mathcal{M}: \text{CAlg}^{cn} \to S$, write $\mathcal{M}^\triangledown$ for its restriction to $\text{CAlg}^{\triangledown}$. For a functor $\mathcal{M}: \text{CAlg}^{cn} \to S$ and an adic $E_\infty$-ring $B$, denote by $\widehat{\mathcal{M}}_B$ the product in $\mathcal{P}(\text{Aff}^{cn})$

$$\widehat{\mathcal{M}}_B = \mathcal{M} \times \text{Spf} \, B.$$ It is clear that there is a natural equivalence $(\widehat{\mathcal{M}}_B)^\triangledown \simeq \mathcal{M}^\triangledown \times \text{Spf} \, \pi_0 B$ inside $\mathcal{P}(\text{Aff}^{\triangledown})$. The hat indicates a base-change over Spf, rather than Spec.

We can now define our sites upon which we will soon define our sheaves of $E_\infty$-rings. Adjectives used below that have not yet been defined will be discussed in §1.1.2.

**Definition 1.1.3.** Recall the conventions of Notation 0.0.14, so fix a complete local Noetherian adic $E_\infty$-ring $A$ with perfect residue field of characteristic $p$. Let

$$\mathcal{C}_A \subseteq \mathcal{P}(\text{Aff}^{cn})_{/\widehat{\mathcal{M}}_{\text{BT}_p^n, A}^\triangledown}$$

denote the full $\infty$-subcategory spanned by those objects $G_0: X_0 \to \widehat{\mathcal{M}}_{\text{BT}_p^n, A}^\triangledown$ such that $X_0$ is a locally Noetherian qcqs formal Deligne–Mumford stack whose residue fields at closed points are all perfect of characteristic $p$, the cotangent complex $L_{X_0/\widehat{\mathcal{M}}_{\text{BT}_p^n, A}^\triangledown}$ is almost perfect and 1-connective inside QCoh$(X_0)$, and $G_0$ is formally étale in $\mathcal{P}(\text{Aff}^{\triangledown})$. Let

$$\mathcal{C}_A \subseteq \mathcal{P}(\text{Aff}^{cn})_{/\widehat{\mathcal{M}}_{\text{BT}_p^n, A}^\triangledown}$$

denote the full $\infty$-subcategory spanned by those objects $G: \mathfrak{x} \to \widehat{\mathcal{M}}_{\text{BT}_p^n, A}$ such that $\mathfrak{x}$ is a locally Noetherian qcqs formal spectral Deligne–Mumford stack whose residue fields at closed points are all perfect of characteristic $p$, and $G$ is formally étale in $\mathcal{P}(\text{Aff}^{cn})$. We will endow $\mathcal{C}_A$ and $\mathcal{C}_A$ with both the fpqc and étale topologies; see [Aut20, Tag 020 K].

A simplified necessary condition for a morphism $\mathfrak{x} \to \widehat{\mathcal{M}}_{\text{BT}_p^n, A}^\triangledown$ to lie in $\mathcal{C}_A$ is discussed in Proposition 1.1.6. Our main theorem (often called “Lurie’s theorem”) can now be stated.

**Theorem 1.1.4.** Given an adic $E_\infty$-ring $A$ as in Notation 0.0.14, there is an étale hypersheaf (Definition 1.1.7) of $E_\infty$-rings $\mathcal{E}_{\text{BT}_p^n}^{\text{top}}$ on $\mathcal{C}_A$ such that for a formal affine $G_0: \text{Spf} \, B_0 \to \widehat{\mathcal{M}}_{\text{BT}_p^n}^\triangledown$ in $\mathcal{C}_A$, the $E_\infty$-ring $\mathcal{E}_{\text{BT}_p^n}^{\text{top}}(G_0) = E$ has the following properties:

1. $E$ is complex periodic.
2. The groups $\pi_k E$ vanish for all odd integers $k$.  

\footnote{A locally Noetherian and quasi-compact scheme is called a Noetherian scheme. We choose to keep these two adjectives separate though, as they play different roles in this article.} 

\footnote{Such a cotangent complex always exists as one does for $\mathfrak{x}_0$ and $\widehat{\mathcal{M}}_{\text{BT}_p^n, A_0}$ – a consequence of [SAG, Proposition 17.2.5.1] and [EC2 Proposition 3.2.2], respectively.}
3. There is a natural equivalence of rings $\pi_0 E \simeq B_0$.

4. There is a natural equivalence of formal groups $G_0 \simeq \hat{G}_E^{Q_0}$ over $B_0$, where the former is the identity component of $G_0$ and the latter is the classical Quillen formal group of $E$.

Moreover, $E$ is $m_A$-complete, and there is a natural equivalence of $B_0$-modules $\pi_2 k E \simeq \omega^{\otimes k}_{G_0}$, where $\omega^{\otimes k}_{G_0}$ is the dualising line of $G_0$; see [EC2, §4.2].

It can further be shown that for a general object $G_0 : X \to \hat{M}_{BT^n,A_0}$, the étale sheaf $G_0^{\text{et}}_{BT^n}$ is also quasi-coherent as an $O_X$-module, but this will not be applicable here.

1.1.2 Motivation for our sites

For transparency’s sake, let us give some explanation as to why each of the adjectives appearing in the definition of $C_A$ and $C_{A_0}$ do so.

(Locally Noetherian) One reason why we assume our formal spectral Deligne–Mumford stacks are locally Noetherian (Definition 0.0.13) is because completions of rings in the classical world and derived world do not necessarily agree otherwise; see [SAG, Warning 8.1.0.4]. Even solely in the world of spectral algebraic geometry there is reason to prefer such objects ([SAG, §8.4]), such as the simplicity of their truncations (Proposition A.2.1).

(Qcqs) This acronym stands for quasi-compact and quasi-separated; see Definition A.4.2. When a scheme $X$ is qcqs, then it has a Zariski cover $\text{Spec} A \to X$ (qc) and the fibre product $P = \text{Spec} A \times_X \text{Spec} A$ also a Zariski cover of $B \to P$ (qs). Eventually, we will define an étale (hyper) sheaf $\mathcal{O}_{\text{aff}}$ on the affine objects of $C_A$, and to extend this to a formal Deligne–Mumford stack $X$ inside $C_A$, one will use these adjectives; see the proof of Theorem 1.3.11. One could write this article again, with the word separated replacing the word quasi-separated and deleting all occurrences of the prefix hyper. We prefer to work in the more general setting as the formalism of hypersheaves seems no more complicated than the formalism of sheaves (as all the difficulties can be referenced to work of Lurie), the adjective qs is only defined on the underlying $\infty$-topoi (Proposition A.4.7), and our own mathematical idealism.

(Formal) One main reason we work with formal spectral Deligne–Mumford stacks (Appendix A and [SAG, §8]) is related to the application of Theorem 1.1.4 to TMF. In this case, one must appeal to the classical Serre–Tate theorem which demands that one works with schemes where $p$ is locally nilpotent, i.e., over $\text{Spf} \mathbb{Z}_p$; see Example 1.2.7. Another, somewhat disjoint reason is for deformation theoretic purposes. As stated in [EC2, Remark 3.2.7]:

“The central idea in the proof of Theorem 3.1.15 (of [EC2]) is (...) to guarantee the representability of $\mathcal{M}_{BT^n}$ in a formal neighborhood of any sufficiently nice $R$-valued point.”

As our moduli stack of interest here is $\mathcal{M}_{BT^n}$, we are forced to embrace formal spectral Deligne–Mumford stacks.
(Perfect residue fields of characteristic $p$) One of the final steps in showing our definition of $\mathcal{C}_{BT_n}$ satisfies the conditions of Theorem [1.1.4] is to reduce ourselves to the situation over the closed points of an affine objects of $C_{A_0}$. It will then be important that these residue fields are perfect (they will already be of characteristic $p$ as we are working over $\text{Spf } Z_p$) to apply some of our formal arguments; see Proposition [1.4.2].

(Formally étale over $\widetilde{M}_{BT_n}$) Again, one inspiration for Theorem [1.1.4] is the classical Serre–Tate theorem, which posits that $\widetilde{M}_{BT_n,A_0}$ is formally étale (H2) over $\widetilde{M}_{BT_n,Z_p}$. Essentially, the phrase formally étale is used in this article to control and package the necessary deformation theory.

(Cotangent complex conditions in $C_{A_0}$) These conditions are mostly technical, although one will see them explicitly come into play in Construction [1.2.38] using an existence criterion of Lurie; see Theorem [1.2.35].

Let us now discuss a simple criterion for checking if an object lies in $C_{A_0}$.

**Definition 1.1.5.** A morphism $f : X_0 \to \text{Spf } A_0$ of classical formal Deligne–Mumford stacks is locally of finite presentation if for all étale morphisms $\text{Spf } B_0 \to X_0$, the induced morphisms of adic discrete rings $A_0 \to B_0$ are of finite presentation as maps of discrete rings. We say $f$ is of finite presentation if $f$ is locally of finite presentation and quasi-compact (Definition [A.4.2]).

**Proposition 1.1.6.** Let $A$ be as in Notation [0.0.14] and $G_0 : X_0 \to \widetilde{M}_{BT_n,A_0}$ be a $p$-divisible group defined on a formal Deligne–Mumford stack $X_0$ of finite presentation over $\text{Spf } A_0$ such that the associated map into $\widetilde{M}_{BT_n,A_0}$ is formally étale. Then $G_0 : X_0 \to \widetilde{M}_{BT_n,A_0}$ lies in $C_{A_0}$.

An important point in the above proposition is that the map $G_0 : X_0 \to \text{Spf } A_0$ must be of finite presentation, not locally of finite presentation.

**Proof.** First we note that $X_0$ is locally Noetherian, qcqs, and has all residue field corresponding to closed points perfect of characteristic $p$ as the morphism $X_0 \to \text{Spf } A_0$ is of finite presentation. It remains to show that the cotangent complex in question,

$$L = L_{X_0/\widetilde{M}_{BT_n,A_0}}$$

is 1-connective and almost perfect. To see this, we consider the composition in $\mathcal{P}(\text{Aff}^{cn})$

$$X_0 \xrightarrow{G_0} \widetilde{M}_{BT_n,A_0} \xrightarrow{\pi_2} \text{Spf } A,$$

which induces the (co) fibre sequence in $\text{QCoh}(X_0)$

$$G_0^* L_{\widetilde{M}_{BT_n,A_0}/\text{Spf } A} \to L_{X_0/\text{Spf } A} \to L.$$

Indeed, for locally Noetherian one can use [Aut20, Tag 00FN], for qcqs one can use [GW10, Appendix D], and the residue fields clearly have characteristic $p$ and are perfect as finite field extensions of perfect fields are perfect, say by [Aut20, Tag 05DU].
Abbreviating the above to $G_0^* L_1 \to L_2 \to L$, we first focus on $G^* L_1$. As a quasi-coherent sheaf on a formal spectral Deligne–Mumford stack $X_0$, to see $G_0^* L_1$ is almost perfect, it suffices to see that $\eta^* G_0^* L_1$ is almost perfect inside $\text{Qcoh}(X)$ for every morphism $\eta: X \to X_0$ where $X$ is a spectral Deligne–Mumford stack; see [SAG, Theorem 8.3.5.2]. Using the base-change equivalence

$$L_1 = L_{\mathcal{M}_{\text{Btre}, A}/\text{Spf } A} \simeq \pi_1^* L_{\mathcal{M}_{\text{Btre}}},$$

it suffices to show $\eta^* G_0^* \pi_1^* L_{\mathcal{M}_{\text{Btre}}}$ is almost perfect. This is indeed true, as this can be checked locally ([SAG, Corollary 8.3.5.3]), so we may assume $X \simeq \text{Spec } R$ for a connective $E_\infty$-ring $R$, and then this follows from [EC2, Proposition 3.2.5], as for every affine spectral Deligne–Mumford stack $\text{Spec } R$ over $\text{Spf } A$, we see that $p$ is nilpotent in $\pi_0 R$. Moreover, $G_0^* L_1$ is connective if and only if $\eta^* G_0^* L_1$ is connective for all morphisms $\eta: \text{Spec } R \to X_0$, where $R$ is a connective $E_\infty$-ring; see [SAG, Example 6.2.5.7]. It follows by the same arguments made above (for the almost perfectness) that this is indeed the case, again by [EC2, Proposition 3.2.5]. Hence $G_0^* L_1$ is almost perfect and connective.

Focusing on $L_2$ now, we consider the composition $X_0 \to \text{Spf } A_0 \to \text{Spf } A$ and the induced (co) fibre sequence in $\text{Qcoh}(X_0)$

$$L_{\text{Spf } A_0/\text{Spf } A}|_{X_0} \to L_{X_0}/\text{Spf } A = L_2 \to L_{X_0}/\text{Spf } A_0.$$  \hfill (1.1.7)

By Proposition A.4.1, we see $L_{\text{Spf } A_0/\text{Spf } A}$ is almost perfect in $\text{Qcoh}(\text{Spf } A_0)$, and pullbacks of almost perfect quasi-coherent sheaves are almost perfect and quasi-coherent by [SAG, Corollary 8.4.1.6], hence the first term of (1.1.7) is almost perfect. To see the third term of (1.1.7) is almost perfect we may work locally, hence $X_0 \simeq \text{Spf } B_0$, where we can assume $B_0$ is complete. In this case we use the assumption that $A_0 \to B_0$ is of finite presentation, which implies $L_{B_0/A_0}$ is almost perfect (and also connective) in $\text{Mod } B_0$; see [HA, Theorem 7.4.3.18]. It then follows that $L_{B_0/A_0}$ is also complete with respect to an ideal of definition $J$ of $B_0$ (by [SAG, Proposition 7.3.5.7]), hence the object of $\text{Qcoh}(\text{Spf } B_0)$

$$L_{B_0/A_0} \simeq (L_{B_0/A_0})^J \simeq L_{\text{Spf } B_0/\text{Spf } A_0}$$

is almost perfect (and connective too, by [SAG, Corollary 8.2.5.4]). It follows then that $L_2$ is almost perfect, and furthermore, it is connective, as being connective is a property preserved by pullback; see [SAG, §6.2.5]. It immediately follows that $L$ is almost perfect and connective, so it remains to show that in fact $L$ is 1-connective.

This follows as our $p$-divisible group $G_0$ is nonstationary; see [EC2, Definition 3.0.8]. Indeed, if this is the case, one can then copy the argument of [EC2, Remark 3.4.4] to see $L$ is 1-connective. To see $G_0$ is nonstationary, we note that all of the residue fields of $X_0$ are perfect of characteristic $p$, so the condition of $G_0$ being nonstationary is vacuous as there are no nontrivial differentials; see [EC2, Example 3.0.10].

\[ \square \]

\footnote{Here we want to remind the reader that cotangent complexes are almost connective by definition, and this allows us to consider quasi-coherent sheaves on $X$ via the functor-of-points perspective; see Warning 0.0.17.}
The simplified hypotheses above are practical, but they do not apply to one of our favourite examples, Lubin–Tate theory, as power series rings $R[[x]]$ are simply never of finite presentation over $R$.

1.1.3 Outline of the proof of Theorem 1.1.4

**Proof.** Our proof moves in three distinct, but connected, stages.

(I) First, we move from classical algebraic geometry (in $\mathcal{P}(\text{Aff}^\circ)$) to spectral algebraic geometry (in $\mathcal{P}(\text{Aff}_{cn})$) using deformation theory, presented here through the adjective *formally étale*. Given an object $G_0 : X_0 \to \hat{M}_{BT,p,A_0}$ inside $C_{A_0}$, we consider the object $X$ inside the Cartesian diagram in $\mathcal{P}(\text{Aff}_{cn})$

$$
\begin{array}{ccc}
\hat{M}_{BT,p,A_0} & \xrightarrow{f} & \hat{M}_{BT,p,A_0} \\
\downarrow & & \downarrow \\
\tau_{\leq 0}^* X_0 & \xrightarrow{G} & \tau_{\leq 0}^* \hat{M}_{BT,p,A_0}
\end{array}
$$

where given a presheaf $Y$ in $\mathcal{P}(\text{Aff}^\circ)$ the presheaf $\tau_{\leq 0}^* Y$ in $\mathcal{P}(\text{Aff}_{cn})$ is defined as $\tau_{\leq 0}^* Y = Y(\pi_0 R)$, and $f$ is induced by the canonical truncation map $R \to \pi_0 R$ for a connective $E_\infty$-ring $R$. The assumption that $G_0$ was formally étale in $\mathcal{P}(\text{Aff}^\circ)$ implies that $X$ is what Lurie calls the de Rham space of the map $X_0 \to \hat{M}_{BT,p,A}$ and that $G$ is formally étale; see Proposition 1.2.32. Most of the adjectives defining $C_{A_0}$ then allow us to employ a powerful theorem of Lurie (Theorem 1.2.35), which then states that $X$ is represented by a formal spectral Deligne–Mumford stack, which we denote as $\hat{X}$. A little analysis then shows $G : X \to \hat{M}_{BT,p,A}$ lies in $C_A$; see Construction 1.2.38. Moreover, the assignment

$$
\Theta : C_{A_0} \to C_A, \quad (X_0, G_0) \mapsto (X, G)
$$

is an equivalence of $\infty$-categories (Theorem 1.2.39), and can be interpreted by a kind of infinitesimal deformation of spectrally ringed $\infty$-topoi

$$
\theta : \mathcal{M}_{BT,p,A_0}^\circ \to \mathcal{M}_{BT,p,A}^\text{un};
$$

see Construction 1.2.41.

(II) Next, we apply the orientation theory of $p$-divisible groups devised by Lurie in [EC2]. From this we obtain a moduli stack of oriented $p$-divisible groups $\mathcal{M}_{BT,p,A}^\text{or}$ and a map of presheaves on $p$-complete (not necessarily connective) $E_\infty$-rings

$$
\Phi : \mathcal{M}_{BT,p,A}^\text{or} \to \mathcal{M}_{BT,p,A}^\text{un};
$$

see Definition 1.3.7. The bulk of this section is formalising the construction of the spectrally ringed $\infty$-topos $\mathcal{M}_{BT,p,A}^\text{or}$ associated to $\mathcal{M}_{BT,p,A}^\text{or}$, and a morphism of spectrally ringed $\infty$-topoi $\phi : \mathcal{M}_{BT,p,A}^\text{or} \to \mathcal{M}_{BT,p,A}^\text{un}$ associated to $\Phi$; see Construction 1.3.14. The importance of this construction is captured by the fact that evaluating $\phi^* O_{BT,p,A}$ on a $p$-divisible group $G$ over a $p$-complete $E_\infty$-ring $R$ is precisely the orientation classifier of $G$; see Lemma 1.3.13.
Finally, we set $\Theta^{\text{top}}_{\text{BT}}$ as the restriction of $\theta^*(\phi_* \Theta^\text{or}_{\text{BT}})$ to $C_{A_0}$, which can also be described as first sending $(X_0, G_0)$ to $(X, G)$ using $\Theta$, and then taking the orientation classifier of $G$; see Definition 1.4.1. To check this definition of $\Theta^{\text{top}}_{\text{BT}}$ satisfies the properties described in Theorem 1.1.4, we study universal deformations over closed points with perfect residue field (Proposition 1.4.2) and apply the results of [EC2, §6].

The following three subsections carry out each of these three steps given above.

### 1.2 Formally étale natural transformations

At the heart of spectral algebraic geometry is deformation theory; see [SAG, p.1385]. The adjective **formally étale** will help us navigate between the two worlds of classical and spectral algebraic geometry using Lurie’s spectral deformation theory. More concretely, given a (nice) formally étale morphism $X_0 \to M$, where $X_0$ is a classical formal scheme, there is a universal spectral deformation of $X_0$, say $X$, such that $X_0$ can be viewed as the 0th truncation of $X$. This process allows us to lift objects in classical algebraic geometry to spectral algebraic geometry without changing the underlying classical object; see Theorem 1.2.39.

#### 1.2.1 In the category $\mathcal{P}(\text{Aff}^{\odot})$

Let us first consider formally étale maps between presheaves of discrete rings.

**Definition 1.2.1.** A natural transformation $f : X \to Y$ of functors in $\mathcal{P}(\text{Aff}^{\odot})$ is said to be **formally étale** if, for all surjective maps of discrete rings $\tilde{R} \to R$ whose kernel is square-zero, also called a square-zero extension of $R$, the natural diagram of spaces

$$
\begin{array}{ccc}
X(\tilde{R}) & \longrightarrow & X(R) \\
\downarrow & & \downarrow \\
Y(\tilde{R}) & \longrightarrow & Y(R)
\end{array}
$$

is Cartesian.

Let us mention some useful formal properties of formally étale morphisms.

**Proposition 1.2.2.** Let

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{h} & & \downarrow{g} \\
Z & \xrightarrow{q} &
\end{array}
$$

be a commutative diagram inside $\mathcal{P}(\text{Aff}^{\odot})$ such that $q$ is formally étale. Then $f$ is formally étale if and only if $h$ is formally étale. Moreover, formally étale morphisms are closed under base-change.

**Proof.** This is a consequence of the pasting law for pullbacks and standard base-change properties of Cartesian squares.
Let us now relate Definition 1.2.1 to the definitions found in classical algebraic geometry.

**Definition 1.2.3.** We say \( f : X \to Y \) between functors in \( \mathcal{P}(\text{Aff}^{\circlearrowright}) \) is affine if every ring \( R \), and every \( R \)-point \( \eta \in Y(R) \), the fibre product \( \text{Spec} R \times_Y X \) is represented by an affine scheme.

Note that maps between (functors represented by) affine Deligne–Mumford stacks in \( \mathcal{P}(\text{Aff}^{\circlearrowright}) \) are always affine, as the Yoneda embedding \( \text{Aff} \to \mathcal{P}(\text{Aff}^{\circlearrowright}) \) preserves limits and fibre products of affine schemes are affine schemes.

**Proposition 1.2.4.** Let \( f : X \to Y \) be a natural transformation of functors in \( \mathcal{P}(\text{Aff}^{\circlearrowright}) \). Then \( f \) is formally étale if and only if for every ring \( R \), every square-zero extension of rings \( \tilde{R} \to R \), and every commutative diagram of the form

\[
\begin{array}{ccc}
\text{Spec} R & \longrightarrow & X \\
\downarrow & & \downarrow f \\
\text{Spec} \tilde{R} & \longrightarrow & Y
\end{array} \tag{1.2.5}
\]

the mapping space

\[
\text{Map}_{\mathcal{P}(\text{Aff}^{\circlearrowright})/Y}(\text{Spec} \tilde{R}, X)
\]

is contractible.\(^6\) Moreover, if \( f \) is affine, then \( f \) is formally étale if and only if for every ring \( A \), and every \( A \)-point \( \eta \in Y(A) \) such that the fibre product \( \text{Spec} A \times_Y X \) is equivalent to any affine scheme \( \text{Spec} B \), the natural projection map \( A \to B \) is formally étale as a map of rings.\(^7\)

**Proof.** For all rings \( R \), all square-zero extensions \( \tilde{R} \to R \), and all commutative diagrams of the form \((1.2.5)\), consider the diagram of spaces

\[
\begin{array}{ccc}
\text{Map}_{\mathcal{P}(\text{Aff}^{\circlearrowright})/Y}(\tilde{R}, X) & \longrightarrow & \text{Map}_{/Y}(\tilde{R}, X) \\
\downarrow & & \downarrow \\
\text{Map}_{/R}(\tilde{R}, X) & \longrightarrow & \text{Map}(\tilde{R}, X) \\
\downarrow & & \downarrow \\
\text{Map}_{/R}(\tilde{R}, Y) & \longrightarrow & \text{Map}(\tilde{R}, Y) \\
\end{array}
\]

where by definition the rows and columns are fibre sequences.\(^8\) We have abbreviated the categories above to express only the over/under categories as everything is happening with respect to \( \mathcal{P}(\text{Aff}^{\circlearrowright}) \), and we suppressed the functor \( \text{Spec} \). By the Yoneda lemma, \( f \) is formally étale if and only if, for all the quantifiers mentioned above, the bottom-right square is Cartesian.

\(^6\)This condition is a formal way of saying “there exists an essentially unique map \( \text{Spec} \tilde{R} \to X \) fitting into the commutative diagram \((1.2.5)\).”

\(^7\)For the definition of a formally étale map of rings simply apply Definition 1.2.1 to the transformation (co)representing this map of rings, or see \([\text{Aut20, Tag 02HF}]\).

\(^8\)The fibres in this diagram have been taken with respect to the maps from \((1.2.5)\).
which is equivalent to the object in the top-left being contractible under these same quantifiers.

For the “moreover” statement, suppose that $f$ is affine. If $f$ is formally étale then the map $\text{Spec } B \to \text{Spec } A$ is formally étale by base-change; see Proposition 1.2.2. Conversely, suppose we are given a diagram of the form (1.2.5), then by assumption the fibre product $\text{Spec } \tilde{R} \times_X Y \simeq \text{Spec } B$ is affine and $\text{Spec } B \to \text{Spec } \tilde{R}$ is formally étale, giving us the diagram

![Diagram](image)

One then observes the following sequence of natural equivalences of spaces

$$
\text{Map}_{R//\tilde{R}}(\tilde{R}, B) \simeq \text{Map}_{R/\tilde{R}}(\tilde{R}, B) \times \text{Map}_{R/\tilde{R}}(\tilde{R}, \tilde{R}) \{\text{id}_{\tilde{R}}\} \\
\simeq \text{Map}_{R/\tilde{R}}(\tilde{R}, X) \times \text{Map}_{R/\tilde{R}}(\tilde{R}, \tilde{R}) \times \text{Map}_{R/\tilde{R}}(\tilde{R}, \tilde{R}) \{\text{id}_{\tilde{R}}\} \simeq \text{Map}_{R//Y}(\tilde{R}, X),
$$

where we have used the same abbreviations from earlier in the proof. The above space is contractible as $\text{Spec } B \to \text{Spec } \tilde{R}$ is formally étale, hence $f$ is formally étale.

Let us list some instances of formally étale morphisms found in algebraic geometry.

**Example 1.2.6 (Formally étale morphisms of schemes).** In the setting of classical algebraic geometry, we usually take the existence of a unique map $\text{Spec } \tilde{R} \to X$ (under $\text{Spec } R$ and over $Y$) as the definition of a formally étale maps of rings (or schemes); see Proposition 1.2.4. An object in $\mathcal{P}(\text{Aff}^\otimes)$ represented by a scheme factors through $\text{Fun}(\text{CAlg}^\otimes, \text{Set})$, as mapping spaces between classical schemes are discrete, and we see Proposition 1.2.4 matches precisely with that found in [Aut20, Tag 02HG].

**Example 1.2.7 (Classical Serre–Tate theorem).** The classical Serre–Tate theorem (see [CS15, p.854] for the original source, or [EC1, Theorem 7.0.1] for statement of the spectral version) states that if $\tilde{R} \to R$ is a square-zero extension of commutative rings such that $p$ nilpotent within them, then the diagram of groupoids

$$
\begin{array}{ccc}
\text{AVar}_g(\tilde{R})^\geq & \longrightarrow & \text{AVar}_g(R)^\geq \\
\downarrow^{[p^\infty]} & & \downarrow^{[p^\infty]} \\
\text{BT}_{2g}(\tilde{R})^\geq & \longrightarrow & \text{BT}_{2g}(R)^\geq
\end{array}
$$

is Cartesian. This implies the morphism of moduli stacks $[p^\infty] : \mathcal{M}_{\text{AVar}_g}^\otimes \to \mathcal{M}_{\text{BT}_{2g}^p}^\otimes$ sending an abelian variety $X$ to its associated $p$-divisible group $X[p^\infty]$ (Example 0.0.6) is formally étale.
after a base-change over Spf \( \mathbb{Z}_p \). This base-change is important, as the only time there exists a map \( \text{Spec } R \to \text{Spf } \mathbb{Z}_p \) inside \( \mathcal{P}(\text{Aff}^\circ) \) is when \( p \) is nilpotent inside \( R \), hence we only consider diagrams of the form \([1.2.8]\) when \( p \) is nilpotent in both \( R \) and \( R \). If we fail to make this base-change, then there is no guarantee \([1.2.8]\) is Cartesian.⁹

**Example 1.2.10 (Classical Lubin–Tate theorem).** Another classical example of a formally étale map in \( \mathcal{P}(\text{Aff}^\circ) \) comes from Lubin–Tate theory. The original source for this is [LT66] with respect to formal groups, but we will follow [EC2] §3 as our intended application is for \( p \)-divisible groups (see [EC2] Example 3.0.5) for a statement of the dictionary between deformations of formal and \( p \)-divisible groups). Let \( G_0 \) be a \( p \)-divisible groups of height \( 0 < n < \infty \) over a perfect field \( \kappa \) of characteristic \( p \). Then there exists a universal deformation of \( G_0 \), in other words, there exists a ring \( R_{G_0}^{\text{LT}} \), a map \( R_{G_0}^{\text{LT}} \to \kappa \), a \( p \)-divisible group \( G \) over \( R_{G_0}^{\text{LT}} \), and an equivalence of \( p \)-divisible groups \( G_\kappa \simeq G_0 \) over \( \kappa \), such that the natural evaluation map

\[
\text{Hom}_{\text{CAlg}_k}(R_{G_0}^{\text{LT}}, R) \xrightarrow{\simeq} \text{Def}_{G_0}(R)
\]

is an equivalence for all adic rings \( R \). These ideas more or less formally imply that the map into \( \mathcal{M}_{\text{BT}_p}^\circ \) defining \( G \) is formally étale:

**Proposition 1.2.11.** Given a perfect field \( \kappa \) of characteristic \( p \) and a \( p \)-divisible group \( G_0 \) over \( \kappa \) of height \( n \). Then the map induced by the universal deformation \( G: \text{Spec } R_{G_0}^{\text{LT}} \to \mathcal{M}_{\text{BT}_p}^\circ \) is formally étale. Moreover, the natural map \( \mathbf{G}: \text{Spec } R_{G_0}^{\text{LT}} \to \mathcal{M}_{\text{BT}_p, \mathbb{Z}_p}^\circ \) is also formally étale.

**Proof.** Let \( R \to R/J \) be the quotient map where \( R \) is discrete and \( J \) is a square-zero ideal.

⁹Indeed, consider the elliptic curve \( E \) over \( \mathbb{F}_3 \) defined by the equation \( y^2 = x^3 + x^2 + x + 1 \). The \( 2^k \)-torsion subgroups of \( E \) are, by [KMS5] Theorem 2.3.1, equivalent to the constant group schemes over \( \text{Spec } \mathbb{F}_3 \)

\[
E[2^k] \simeq (\mathbb{Z}/2^k\mathbb{Z})^2,
\]

hence the associated 2-divisible group is equivalent to the constant 2-divisible group

\[
E[2^\infty] \simeq (\mathbb{Q}/\mathbb{Z})^2
\]

over \( \mathbb{F}_3 \). We then define two deformations \( E_1 \) and \( E_2 \) of \( E \) over the dual numbers \( \mathbb{F}_3[\epsilon] \) (augmented over \( \mathbb{F}_3 \) by the morphism \( \epsilon \to 0 \)), by the formulae

\[
E_1: y^2 = x^3 + x^2 + x + 1 + \epsilon, \quad E_2: y^2 = x^3 + x^2 + x + 1 - \epsilon.
\]

From their definitions we see that base-changing either \( E_1 \) or \( E_2 \) to \( \mathbb{F}_3 \) through the above augmentation yields \( E \). Once more, by [KMS5] Theorem 2.3.1, we can calculate \( E_1[2^\infty] \) and \( E_2[2^\infty] \) to both be the constant 2-divisible group \( (\mathbb{Q}/\mathbb{Z})^2 \) over \( \mathbb{F}_3[\epsilon] \), and hence these 2-divisible groups also base-change to \([1.2.9]\) over \( \mathbb{F}_3 \). As a final observation, one realises that \( E_1 \) and \( E_2 \) are not equivalent as elliptic curves over \( \mathbb{F}_3[\epsilon] \), as one can calculate their \( j \)-invariants ([Shi88] §III.1),

\[
j(E_1) = \epsilon - 1 \neq \epsilon + 1 = j(E_2),
\]

hence \([2^\infty] \): \( \mathcal{M}_{\ell}^\circ \to \mathcal{M}_{\text{BT}_p}^\circ \) is not formally étale (over \( \text{Spec } \mathbb{Z} \)).
We wish to show the diagram of spaces

\[
\begin{array}{c}
\text{(Spf } R_{G_0}^L)(R) \xrightarrow{\text{l}} (\mathcal{M}_{BT_n}^\triangledown)(R) \\
\text{(Spf } R_{G_0}^L)(R/J) \xrightarrow{\text{r}} (\mathcal{M}_{BT_n}^\triangledown)(R/J)
\end{array}
\]

is Cartesian. Following [EC2, Definition 3.1.4], for an adic $E_\infty$-ring $A$, the $\infty$-category $\text{Def}_{G_0}(A)$ is defined as

\[
\text{Def}_{G_0}(A) = \colim_I \left( BT^n(A) \times_{BT^n(\pi_0 A/I)} \text{Hom}_{\text{CAlg}}(\kappa, \pi_0 A/I) \right),
\]

where the colimit is indexed over the filtered system of finitely generated ideals of definition $I \subseteq \pi_0 A$. By [EC2, Lemma 3.1.10], if $A$ is complete, then $\text{Def}_{G_0}(A)$ is a Kan complex, so in particular this is the case if $A$ is given the discrete topology. As $R_{G_0}^L$ is the universal deformation of $G_0$ one obtains for any discrete ring $A$ (considered as an adic ring with the discrete topology) an equivalence of spaces

\[
\text{Hom}_{\text{CAlg}}(R_{G_0}^L, A) \xrightarrow{\simeq} \text{Def}_{G_0}(A) = \colim_I \left( BT^n(A) \times_{BT^n(\pi_0 A/I)} \text{Hom}_{\text{CAlg}}(\kappa, \pi_0 A/I) \right),
\]

where the colimit is taken over all finitely generated nilpotent ideals $I$ inside $A$; see [EC2, Theorem 3.1.15]. As $\kappa$ is perfect, it follows that the cotangent complex $L_\kappa$ is almost perfect in $\text{Mod}_\kappa$ ([EC2, Corollary 3.3.8]), and as $\kappa$ is of characteristic $p$ we can apply [EC2, Proposition 3.4.3] to see that the natural map

\[
\text{Def}_{G_0}(A) \xrightarrow{\simeq} \colim_{I \in \text{Nil}(A)} \left( BT^n(A) \times_{BT^n(A/I)} \text{Hom}_{\text{CAlg}}(\kappa, A/I) \right),
\]

where the colimit above is indexed over all nilpotent ideals $I \subseteq A$, is an equivalence. Given a nilpotent ideal $J \subseteq A$, denote by $\text{Nil}_J(A)$ the poset of nilpotent ideals of $A$ which contain $J$. We obtain a natural inclusion functor $\text{Nil}_J(A) \to \text{Nil}(A)$. It is easy to see this functor is cofinal, as any nilpotent ideal $I$ lies within the sum $I + J$, which itself is a nilpotent ideal. Hence the natural map

\[
\colim_{I \in \text{Nil}_J(A)} \left( BT^n(A) \times_{BT^n(A/I)} \text{Hom}(\kappa, A/I) \right) \xrightarrow{\simeq} \colim_{I \in \text{Nil}(A)} \left( BT^n(A) \times_{BT^n(A/I)} \text{Hom}(\kappa, A/I) \right)
\]

is an equivalence. The map $l$ of (1.2.12) can then be written as

\[
\colim_{I \in \text{Nil}_J(R)} \left( BT^n(R) \times_{BT^n(R/I)} \text{Hom}(\kappa, R/I) \right) \xrightarrow{l} \colim_{I \in \text{Nil}_J(R)} \left( BT^n(R/J) \times_{BT^n(R/I)} \text{Hom}(\kappa, R/I) \right),
\]
where above we use the commutative algebra fact that ideals in $R/J$ correspond to ideals in $R$ containing $J$. As filtered colimits commute with finite limits, we see the fibre of $l$ is precisely

$$\text{fib}(l) \simeq \colim_{I \in \text{Nil}_J(R)} \left( \text{BT}_n^p(R) \times_{\text{BT}_n^p(R/I)} \text{Hom}(\kappa, R/I) \xrightarrow{g} \text{BT}_n^p(R/J) \times_{\text{BT}_n^p(R/I)} \text{Hom}(\kappa, R/I) \right),$$

as $\text{Nil}_J(R)$ is filtered. To simplify the above fibre term, we contemplate the diagram in $\mathcal{C}_{at_\infty}$

$$\begin{array}{cccc}
\text{BT}_n^p(R) & \times_{\text{BT}_n^p(R/I)} & \text{Hom}(\kappa, R/I) & \xrightarrow{g} & \text{BT}_n^p(R/J) \times_{\text{BT}_n^p(R/I)} \text{Hom}(\kappa, R/I) & \longrightarrow & \text{Hom}(\kappa, R/I) \\
& & \downarrow & & \downarrow & & \\
\text{BT}_n^p(R) & \xrightarrow{f} & \text{BT}_n^p(R/J) & \longrightarrow & \text{BT}_n^p(R/I) \\
\end{array}$$

The right square and outer rectangle above are Cartesian by definition, so by the pasting law for pullbacks we see the left square is also Cartesian. This implies the natural map $\text{fib}(g) \to \text{fib}(f)$ is an equivalence in $\mathcal{C}_{at_\infty}$, hence our fibre of $l$ becomes

$$\text{fib}(l) \simeq \colim_{I \in \text{Nil}_J(R)} \left( \text{fib}(\text{BT}_n^p(R) \xrightarrow{f} \text{BT}_n^p(R/J)) \right) \simeq \text{fib} \left( \text{BT}_n^p(R) \xrightarrow{f} \text{BT}_n^p(R/J) \right).$$

This fibre of $f$ is not clearly in the essential image of $S \to \mathcal{C}_{at_\infty}$, but $\text{fib}(l)$ is, so $\text{fib}(f) \simeq \text{fib}(f)^{\chi}$. As $r$ is simply $(f)^{\chi}$, we obtain a natural equivalence $\text{fib}(l) \simeq \text{fib}(r)$. As the fibres of $l$ and $r$ are naturally equivalent, we see that (1.2.12) is Cartesian.

For the “moreover” statement, we notice that formally étale morphisms are closed under base-change, and the map in the moreover statement is simply the base-change of our (already proven formally étale) map of interest by the morphism $\text{Spf} \mathbb{Z}_p \to \text{Spec} \mathbb{Z}$. Indeed, as $R_{LT}^{G_0}$ is Noetherian (as Witt vectors of perfect fields are Noetherian and the Hilbert basis theorem) and $p$-complete (as $\kappa$ has characteristic $p$), we see there is a natural equivalence

$$\text{Spf} R_{LT}^{G_0} \times \text{Spf} \mathbb{Z}_p \simeq \text{Spf} (R_{LT}^{G_0})^p \simeq \text{Spf} R_{LT}^{G_0}. \quad \square$$

1.2.2 In the category $\mathcal{P}(\text{Aff}^{cn})$

We are now in the position to make a spectral definition. See [HA §7.4] for definition of the definition of (trivial) square-zero extension of $\mathcal{E}_{\infty}$-rings, and and [SAG §17.2] for the definition of infinitesimally cohesive and nilcomplete functors in $\mathcal{P}(\text{Aff}^{cn})$ and the definition of $L_{X/Y}$.

**Definition 1.2.13.** Let $f: X \to Y$ be a natural transformation of functors in $\mathcal{P}(\text{Aff}^{cn})$. For an integer $0 \leq n \leq \infty$, we say $f$ is $n$-formally étale if for all square-zero extensions of connective $n$-truncated $\mathcal{E}_{\infty}$-rings $\tilde{R} \to R$ the natural diagram of spaces

$$\begin{array}{cccc}
X(\tilde{R}) & \longrightarrow & X(R) & \\
\downarrow & & \downarrow & \\
Y(\tilde{R}) & \longrightarrow & Y(R) \\
\end{array}$$

is Cartesian. We abbreviate $\infty$-formally étale to formally étale.
Remark 1.2.14. Notice that if \( X \to Y \) is \( n \)-formally étale, then it is also \( m \)-formally étale for all \( 0 \leq m \leq n \leq \infty \). In particular, for all \( 0 \leq n \leq \infty \), if the transformation \( X \to Y \) is \( n \)-formally étale then \( X^{\heartsuit} \to Y^{\heartsuit} \) is formally étale inside \( \mathcal{P}(\text{Aff}^{\heartsuit}) \).

Warning 1.2.15. Although the above remark tells us formally étale morphisms of connective \( E_\infty \)-rings give formally étale morphisms on 0th truncations (in \( \mathcal{P}(\text{Aff}^{\heartsuit}) \)), it is not true that the image of formally étale morphisms in \( \mathcal{P}(\text{Aff}^{\heartsuit}) \) are sent to formally étale morphisms in \( \mathcal{P}(\text{Aff}^{\text{cn}}) \) under \( \tau_*^{\leq 0} \) – it is only clear that they are sent to 0-formally étale morphisms of connective 0-truncated \( E_\infty \)-rings.

Remark 1.2.16. If \( X \to Y \) is a formally étale morphism of (locally Noetherian) classical formal Deligne–Mumford stacks inside \( \mathcal{P}(\text{Aff}^{\heartsuit}) \), then the corresponding morphism inside \( \mathcal{P}(\text{Aff}^{\text{cn}}) \) is 0-formally étale. This follows by the fully-faithfulness of \( \text{fDM} \to \text{fSpDM} \); see Proposition A.3.4.

Remark 1.2.17. Notice this deviates from Lurie’s definition of étale morphisms ([HA, Definition 7.5.0.4]) as there is no flatness assumption. However, even in \( \mathcal{P}(\text{Aff}^{\heartsuit}) \) a formally étale morphism of discrete rings need not be flat. This means there is no inherent descent theory for formally étale morphisms. We will see when these adjectives do interact; see Remark 1.2.24.

The basic properties of Proposition 1.2.2 also hold in \( \mathcal{P}(\text{Aff}^{\text{cn}}) \), with the same proof.

Proposition 1.2.18. Let \( 0 \leq n \leq \infty \) and

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{h} & & \downarrow{g} \\
Z & \xrightarrow{r} & Z
\end{array}
\]

be a commutative diagram inside \( \mathcal{P}(\text{Aff}^{\text{cn}}) \) such that \( g \) is \( n \)-formally étale. Then \( f \) is \( n \)-formally étale if and only if \( h \) is \( n \)-formally étale. Moreover, \( n \)-formally étale morphisms are closed under base-change.

We would now like alternative ways to test if a map \( X \to Y \) is formally étale in \( \mathcal{P}(\text{Aff}^{\text{cn}}) \).

Proposition 1.2.19. Let \( X \to Y \) be a natural transformation of functors in \( \mathcal{P}(\text{Aff}^{\text{cn}}) \) and \( 0 \leq n \leq \infty \).

\[ C[t^q | q \in \mathbb{Q}, q > 0] \to C \]

sending \( t \mapsto 0 \) is formally étale but not flat. One can always lift square-zero extensions of rings uniquely, as we have all square roots of \( t \) in the above ring, hence it is formally étale. To see this map is not flat, we can tensor it with the exact sequence

\[ 0 \to (t) \to C[t^q] \to C[t^q]/(t) \to 0, \]

which yields the clearly not exact sequence

\[ 0 \to C \to C \to C \to 0. \]
1. The map \( X \to Y \) is \( n \)-formally étale for finite \( n \) if and only if \( X \to Y \) is 0-formally étale and for every connective \( n \)-truncated \( \mathbf{E}_\infty \)-ring \( R \) the natural diagram of spaces

\[
\begin{array}{ccc}
X(R) & \longrightarrow & X(\pi_0 R) \\
\downarrow & & \downarrow \\
Y(R) & \longrightarrow & Y(\pi_0 R)
\end{array}
\]

is Cartesian.

2. If \( X \to Y \) is nilcomplete, then \( X \to Y \) is formally étale if and only if \( X \to Y \) is 0-formally étale and for every connective \( \mathbf{E}_\infty \)-ring \( R \) the natural diagram of spaces

\[
\begin{array}{ccc}
X(R) & \longrightarrow & X(\pi_0 R) \\
\downarrow & & \downarrow \\
Y(R) & \longrightarrow & Y(\pi_0 R)
\end{array}
\]

is Cartesian.

3. If \( X \to Y \) is infinitesimally cohesive, then \( X \to Y \) is \( n \)-formally étale if and only if for all trivial square-zero extensions of connective \( n \)-truncated \( \mathbf{E}_\infty \)-rings \( R \to R \) the natural diagram of spaces

\[
\begin{array}{ccc}
X(\widetilde{R}) & \longrightarrow & X(R) \\
\downarrow & & \downarrow \\
Y(\widetilde{R}) & \longrightarrow & Y(R)
\end{array}
\]

is Cartesian.

4. If \( X \to Y \) is infinitesimally cohesive, \( L_{X/Y} \) exists, and \( L_{X/Y} \) is connective, then \( X \to Y \) is formally étale if and only if \( L_{X/Y} \) vanishes.

If \( X \to Y \) is infinitesimally cohesive, nilcomplete, and \( L_{X/Y} \) exists and is connective, then \( X \to Y \) is \( n \)-formally étale if certain \( \text{Ext} \)-groups \( \text{Ext}_R^m(\eta^* L_{X/Y}, M) \) vanish in a range, for certain discrete objects \((R, \eta, M)\) of \( \text{Mod}_{\mathbf{E}_\infty}^X \), à la the deformation theory of [Ill71]. There is also a sharpening of part 4 above which deals with an \( n \)-connective cotangent complex \( L_{X/Y} \), which we note for the readers benefit is not equivalent to \( X \to Y \) being \( n \)-formally étale. We will not need these ideas here though.

**Proof.** Write \( f \) for the transformation \( X \to Y \) in question.

1. Suppose \( f \) is \( n \)-formally étale for a finite \( n \geq 0 \), then \( f \) is 0-formally étale by Remark [12,14] Given a connective \( n \)-truncated \( \mathbf{E}_\infty \)-ring \( R \), then for any \( 0 \leq m \leq n \) we can consider the diagram

\[
\begin{array}{ccc}
X(\tau_{\leq m+1} R) & \longrightarrow & X(\tau_{\leq m} R) & \longrightarrow & X(\pi_0 R) \\
\downarrow & & \downarrow & & \downarrow \\
Y(\tau_{\leq m+1} R) & \longrightarrow & Y(\tau_{\leq m} R) & \longrightarrow & Y(\pi_0 R)
\end{array}
\] (1.2.20)
In the above diagram, the left square is always Cartesian by virtue of $f$ being $n$-formally étale as $\tau_{\leq m+1} \vec{R} \to \tau_{\leq m} R$ is a square-zero extension of $E_\infty$-rings by [HA, Corollary 7.1.4.28]. To show the outer rectangle Cartesian we use induction. For $m = 0$ the horizontal maps in the right square are the identity, and hence the right square Cartesian, and for $m \geq 1$ the right square is Cartesian by our inductive hypotheses, hence the whole rectangle is Cartesian. Conversely, if the second condition of part 1 holds, we consider a square-zero extension of $E_\infty$-rings $\vec{R} \to R$ and the following natural diagram of spaces:

$$
\begin{array}{ccc}
X(\vec{R}) & \longrightarrow & X(\pi_0 \vec{R}) \\
\downarrow & & \downarrow \\
X(R) & \longrightarrow & X(\pi_0 R) \\
\downarrow & & \downarrow \\
Y(\vec{R}) & \longrightarrow & Y(\pi_0 \vec{R}) \\
\downarrow & & \downarrow \\
Y(R) & \longrightarrow & Y(\pi_0 R)
\end{array}
$$

The back and front faces are Cartesian by the second condition of part 1, and the rightmost face is Cartesian as the second condition of part 1 also assumes $f$ is 0-formally étale. Hence by a base-change argument, we see the leftmost square is Cartesian, and we are done.

2. Suppose $f$ is formally étale. Then for any $0 \leq m \leq n < \infty$ we can consider [12.20], within which all square and the outer rectangle are Cartesian by the same arguments made in part 1. As $X \to Y$ is nilcomplete, then for every connective $E_\infty$-ring $R$, the diagram of spaces

$$
\begin{array}{ccc}
X(R) & \longrightarrow & \lim X(\tau_{\leq n} R) \\
\downarrow & & \downarrow \\
Y(R) & \longrightarrow & \lim Y(\tau_{\leq n} R)
\end{array}
$$

is Cartesian. This, combined with [12.20], gives us the second condition of part 2. The converse holds by a similar argument.

3. If $f$ is $n$-formally étale, then logic implies the second condition holds. Conversely, let $e: \vec{R} \to R$ be a square-zero extension of a connective $E_\infty$-ring $R$ by a connective $R$-module $M$ and a derivation $d: L_R \to \Sigma M$. By definition ([HA, Definition 7.4.1.6]) $\vec{R}$ is defined by the Cartesian diagram of connective $E_\infty$-rings

$$
\begin{array}{ccc}
\vec{R} & \longrightarrow & R \\
\downarrow & \rho \downarrow & \downarrow \\
R & \longrightarrow & R \oplus M
\end{array}
$$
where the bottom-horizontal map is induced by the zero map $L_R \to \Sigma M$ and the right-vertical map is induced by the derivation $d$. This Cartesian diagram of connective $E_\infty$-rings then induces the following natural diagram of spaces:

$$
\begin{array}{cccccc}
X(\tilde{R}) & X(e) & X(R) & \downarrow & \downarrow & \downarrow \\
\downarrow & & \downarrow & & \downarrow & \downarrow \\
X(R) & X(\rho) & \tilde{X}(R \oplus M) & \to & X(R) & \to \\
\downarrow & & \downarrow & & \downarrow & \downarrow \\
Y(\tilde{R}) & Y(e) & Y(R) & \downarrow & \downarrow & \downarrow \\
\downarrow & & \downarrow & & \downarrow & \downarrow \\
Y(R) & Y(0) & Y(R \oplus M) & \to & Y(R) & \to \\
\end{array}
$$

As $X \to Y$ is infinitesimally cohesive the left cube is Cartesian; see [SAG, Proposition 17.3.6.1(3)]. By assumption the rightmost square is Cartesian, and the only rectangle in the diagram is also Cartesian as the composition $R \to R \oplus M \to R$ is equivalent to the identity, hence the left square in that same rectangle (the front face of the cube) is Cartesian. By a base-change argument \[1\] we see that the desired back square of the cube (containing $X(e)$ and $Y(e)$) is also Cartesian, and we are done.

4. On the one hand, by [SAG, Proposition 6.2.5.2(1)] and [SAG, Definition 6.2.5.3], we see that for some fixed integer $m$, an object $F$ of $\text{QCoh}(X)$ is $m$-connective if and only if for all connective $E_\infty$-rings $R$ and transformations $\eta: \text{Spec} R \to X$, the object $\eta^*F$ is $m$-connective inside $\text{QCoh}(\text{Spec} R) \simeq \text{Mod}_R$. Furthermore, if $F$ is connective and $m \geq 0$, the object $\eta^*F$ is $m$-connective if and only if the mapping space $\text{Map}_{\text{Mod}^{cn}_R}(\eta^*F, N) \simeq \text{Map}_{\text{Mod}^{cn}_R}(\tau \leq m \eta^*F, N)$ is contractible, for all connective $m$-truncated $R$-modules $N$, by the Yoneda lemma. On the other hand, the object $L_{X/Y}$ in $\text{QCoh}(X)$ exists if and only if the functor $F: \text{Mod}^{cn}_X \to S$, defined by

$$F(R, \eta, M) = \text{fib} \left( X(R \oplus M) \to X(R) \times_{Y(R)} Y(R \oplus M) \right),$$

\[1\]This base-change argument can be formalised as follows: write $I$ for the poset of nonempty subsets of $\{1, 2, 3\}$, ordered by inclusion, and use this poset to index the cube in (1.2.21) by setting $F_\emptyset = X(\tilde{R}), F_1 = X(R)$ (in the top-right), $F_2 = X(R)$ (in the centre), $F_3 = Y(\tilde{R})$, etc. As the whole cube is Cartesian we have $F_\emptyset \simeq \lim_{\emptyset \subseteq I} F_{I_0}$ and as the front face is also Cartesian we have $F_2 \simeq \lim (F_{12} \to F_{123} \leftarrow F_{23})$. These two facts, together with [MV15, Example 5.3.8] give us the following natural chain of equivalences of spaces

$$F_\emptyset \simeq \lim_{\emptyset \subseteq I} F_{I_0} \simeq \lim (F_2 \to G_{123} \leftarrow G_{13}) \simeq G_{13},$$

where $G_{123} = \lim (F_{12} \to F_{123} \leftarrow F_{23}) \simeq F_2$ and $G_{13} = \lim (F_1 \to F_{13} \leftarrow F_3)$. This shows the back face of the cube (indexed by $\emptyset, \{1\}, \{3\}, \text{and } \{1, 3\}$, is Cartesian.
is locally almost representable, meaning that we have a (locally almost; see \[SAG, Definition 17.2.3.1\]) natural equivalence for all triples \((R, \eta, M)\)

\[ F(R, \eta, M) \simeq \text{Map}_{\text{Mod}_R}(\eta^*L_{X/Y}, M), \]

where \(R\) is a connective \(E_\infty\)-ring, \(\eta: \text{Spec} R \to X\) a map in \(\mathcal{P}(\text{Aff}^\text{cn})\), and \(M\) a connective \(R\)-module. If \(L_{X/Y}\) vanishes, then we immediately see \(F(R, \eta, M)\) is contractible for all triples \((R, \eta, M)\), which by part 3 implies \(X \to Y\) is formally étale. Conversely, if \(X \to Y\) is formally étale, then \(F(R, \eta, M)\) is contractible for all triples \((R, \eta, M)\), hence the mapping space

\[ \text{Map}_{\text{Mod}_R}(\eta^*L_{X/Y}, M) \simeq F(R, \eta, M) \]

is contractible for all triples \((R, \eta, M)\). By the observations we opened this part of the proof with, we see this implies that our connective cotangent complex \(L_{X/Y}\) is \(m\)-connective for all \(m \geq 0\), and hence vanishes.

There are many examples of formally étale maps in spectral algebraic geometry.

Let us first note that all formal spectral Deligne–Mumford stacks are cohesive, nilcomplete, and absolute cotangent complexes always exist, which follows from by copying the proof of \[SAG, Corollary 17.3.8.5\] (the same statement for SpDM), as all of the references made there also apply to fSpDM.

**Example 1.2.22 (Étale morphisms of \(E_\infty\)-rings).** Let \(A \to B\) be an étale morphism of \(E_\infty\)-rings, then by \[HA, Corollary 7.5.4.5\] we know \(L_{B/A}\) vanishes, hence \(A \to B\) is also a formally étale morphism of \(E_\infty\)-rings by Proposition 1.2.19.

**Example 1.2.23 (Relatively perfect discrete \(F_p\)-algebras).** Another classic example, which will not show up explicitly in this note but is at the heart of much of the work done in \[EC2\], is that a flat relatively perfect map of discrete commutative \(F_p\)-algebras has a vanishing cotangent complex (by \[EC2, Lemma 5.2.8\]), and hence is formally étale.

**Remark 1.2.24.** In Remark 1.2.17 we noted that formally étale morphisms of connective \(E_\infty\)-rings were not necessarily flat. However, \[EC2, Proposition 3.5.5\] states that morphisms of (not necessarily connective) Noetherian \(E_\infty\)-rings with vanishing cotangent complex are flat. Combining this with Proposition 1.2.19 we see formally étale morphism of connective Noetherian \(E_\infty\)-rings are flat. From this follows (as in classical algebraic geometry, see \[Aut20, Tag 02HM\]) that formally étale morphisms of almost finite presentation between connective Noetherian \(E_\infty\)-rings are étale.

The functor \(\mathcal{M}_{\text{BT}^p}\) is cohesive, nilcomplete, and admits a cotangent complex by \[EC2, Proposition 3.2.2\]. It follows that \(\mathcal{M}_{\text{BT}^p}\) (as well as all base-changes \(\mathcal{M}_{\text{BT}^p,A}\)) also satisfy these properties; see Remark 1.3.8.

**Example 1.2.25 (Spectral Serre–Tate theorem).** It follows from the spectral Serre–Tate theorem (\[EC1, Theorem 7.0.1\]) and Proposition 1.2.19 that the map \([p^\infty]: \mathcal{M}_{\text{Ell}}^{p,S_p} \to \mathcal{M}_{\text{BT}^p_{\text{ Ell}}}^{p,S_p}\) is formally étale.
Example 1.2.26 (Spectral Lubin–Tate theory). Lurie uses his de Rham space formalism to construct a map $\text{Spf } \mathcal{R}^\text{un}_{G_0} \to \mathcal{M}_{\text{BT}}$ ([EC2 Theorem 3.4.1]) which is formally étale by [SAG Corollary 18.2.1.11(2)] and Proposition 1.2.19.

Example 1.2.27 (Formal spectral completions). Let $X$ be a spectral Deligne–Mumford stack and $K \subseteq \vert X \vert$ be a cocompact closed subset, then the natural map from the formal completion of $X$ along $K$ ([SAG Definition 8.1.6.1])

$$X_K^\wedge \to X$$

is formally étale by [SAG Example 17.1.2.10] and Proposition 1.2.19.

Example 1.2.28 (Spectral de Rham space). Given a morphism $X \to Y$ of functors in $\mathcal{P}(\text{Aff}^{\text{cn}})$, one can associate a de Rham space $(X/Y)_{dR}$ inside $\mathcal{P}(\text{Aff}^{\text{cn}})$, whose value on a connective $E_\infty$-ring is

$$(X/Y)_{dR}(R) = \underset{\text{colim}}{\text{colim}} \left( \left( \frac{Y(R)}{\pi_0 R} \times_{Y(\pi_0 R/I)} X(\pi_0 R/I) \right) \right),$$  \hspace{1cm} (1.2.29)

where the colimit is taken over all nilpotent ideals $I \subseteq \pi_0 R$, which we note is a discrete filtered system; see [1.2.29] and [SAG §18.2.1]. By [SAG Corollary 18.2.1.11(2)], the natural map $(X/Y)_{dR} \to Y$ is always nilcomplete, infinitesimally cohesive, and admits a vanishing cotangent complex, so by Proposition 1.2.19, it is formally étale.

1.2.3 An effect on de Rham spaces

Formally étale morphisms are important to us because if $X \to Y$ is 0-formally étale, then the natural map $(X/Y)_{dR} \to Y$ is $\infty$-formally étale, and $(X/Y)_{dR}$ agrees with $X$ on discrete rings.

Definition 1.2.30. Write $\iota: \text{CAlg}^{\vee} \to \text{CAlg}^{\text{cn}}$ for the inclusion (a right adjoint, inducing a left adjoint on presheaf categories), and $\tau_{\leq 0}$ for the truncation functor (a left adjoint, inducing a right adjoint on presheaf categories) $\text{CAlg}^{\text{cn}} \to \text{CAlg}^{\vee}$, and also for the composition $\iota \circ \tau_{\leq 0}$ (this should seldom cause confusion). Using this notation, for each functor $M$ in $\mathcal{P}(\text{Aff}^{\text{cn}})$ there is a natural unit map

$M \to \tau_{\leq 0}^* M$

induced by the truncation $R \to \pi_0 R$ of a connective $E_\infty$-ring $R$. One can also check that given a functor $\mathcal{M} : \text{CAlg}^{\vee} \to \mathcal{S}$, the functor $\tau_{\leq 0}^* \mathcal{M}$ is the right Kan extension in the following diagram:

$$\begin{array}{ccc}
\text{CAlg}^{\vee} & \xrightarrow{\mathcal{M}} & \mathcal{S} \\
\iota \downarrow & & \downarrow \text{Ran} \\
\text{CAlg}^{\text{cn}} & & \\
\end{array}$$

Warning 1.2.31. In [A.2] we introduce the truncation of a locally Noetherian formal spectral Deligne–Mumford stack $\tau_{\leq 0} \mathcal{X}$, and we note that this is not equivalent to $\tau_{\leq 0} \mathcal{X}$.

The following is a key proposition for us, giving us some control of de Rham spaces.
Proposition 1.2.32. Let $X \to Y$ be a natural transformation of functors in $\mathcal{P}(\text{Aff}^{cn})$ which is 0-formally étale. Then the following natural diagram of functors in $\mathcal{P}(\text{Aff}^{cn})$

\[
\begin{array}{ccc}
(X/Y)_{\text{dR}} & \xrightarrow{g} & \tau_{\leq 0}^* X \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\tau_{\leq 0}^*} & \tau_{\leq 0}^* Y
\end{array}
\]

is Cartesian, and the natural map

$X \to (X/Y)_{\text{dR}}$

induces an equivalence when evaluated on discrete $\mathbb{E}_\infty$-rings. Moreover, $(X/Y)_{\text{dR}} \to Y$ is formally étale.

Proof. Recall the value of the de Rham space $(X/Y)_{\text{dR}}$ on a connective $\mathbb{E}_\infty$-ring $R$ from (1.2.29). Writing $F_{R,I}$ for the fibre product within the colimit of (1.2.29), we place $F_{R,I}$ in to the following commutative diagram of spaces:

\[
\begin{array}{ccc}
F_{R,I} & \xrightarrow{\pi_0} & X(\pi_0 R) \\
\downarrow & & \downarrow \left(\pi_0 R \times \pi_0 R \right) \\
Y(R) & \xrightarrow{\pi_0} & Y(\pi_0 R)
\end{array}
\]

The outer rectangle is Cartesian by definition. As $X \to Y$ is 0-formally étale, this implies that the right square is also Cartesian. These two facts imply that the left square is Cartesian, so we can rewrite the $R$-points of the de Rham space in question as

\[
(X/Y)_{\text{dR}}(R) \cong \text{colim}_I Y(R) \times_{Y(\pi_0 R)} X(\pi_0 R),
\]

where the second equivalence holds as the diagram defining our colimit is filtered. This implies that the natural diagram

\[
\begin{array}{ccc}
(X/Y)_{\text{dR}} & \xrightarrow{\tau_{\leq 0}^*} & \tau_{\leq 0}^* X \\
\downarrow & & \downarrow \tau_{\leq 0}^* f \\
Y & \xrightarrow{\tau_{\leq 0}^*} & \tau_{\leq 0}^* Y
\end{array}
\]

is Cartesian in $\mathcal{P}(\text{Aff}^{cn})$. As the bottom horizontal map is an equivalence on discrete $\mathbb{E}_\infty$-rings, then this implies the top horizontal map is also an equivalence of discrete $\mathbb{E}_\infty$-rings. The natural maps

\[
(X/Y)_{\text{dR}} \to \tau_{\leq 0}^* X \leftarrow X
\]

in $\mathcal{P}(\text{Aff}^{cn})$ both induce equivalences when evaluated on discrete $\mathbb{E}_\infty$-rings. For the “moreover” statement, we use Example 1.2.28, or one could combine (1.2.33) and (1.2.34) with Proposition 1.2.19.

The following theorem of Lurie is crucial, as it allows us to identify when our de Rham spaces are represented by formal spectral Deligne–Mumford stacks.
Theorem 1.2.35 (SAG, Theorem 18.2.3.1). Let \( f : \mathcal{X} \to \mathcal{M} \) be a natural transformation of functors \( \text{CAlg}^{\text{cn}} \to \mathcal{S} \), where \( \mathcal{X} \) is a formal spectral Deligne–Mumford stack. Suppose that \( \mathcal{M} \) is nilcomplete, infinitesimally cohesive, admits a cotangent complex, and is an étale sheaf. If \( L_{\mathcal{X}/\mathcal{M}} \) is 1-connective and almost perfect, then \( (\mathcal{X}/\mathcal{M})_{\text{dR}} \) is represented by a formal thickening of \( \mathcal{X} \).

1.2.4 The case \( \mathcal{M} = \widehat{\mathcal{M}}_{\text{BT}^p,A} \)

Let us now apply the theory of formally étale natural transformations to the functor \( \widehat{\mathcal{M}}_{\text{BT}^p,A} \), where recall we are working over a fixed adic \( E_{\infty} \)-ring \( A \) as in Notation 0.0.14.

Definition 1.2.36. Let \( \mathcal{X} \) be a formal spectral Deligne–Mumford stack. A \( p \)-divisible group over \( \mathcal{X} \) is a natural transformation \( G : \mathcal{X} \to \mathcal{M}_{\text{BT}^p} \) in \( \text{P}(\text{Aff}^{\text{cn}}) \). We say \( G \) has height \( n \) if this map factors through \( \mathcal{M}_{\text{BT}^p n} \). By [EC2, Proposition 3.2.2(4)], this is equivalent to a coherent family of \( p \)-divisible groups \( G_{B_i} \) on \( \text{Spec} B_i \), where the collection \( \text{Spf} B_i \to \mathcal{X} \) form an affine étale cover of \( \mathcal{X} \).

Recall the categories \( C_{A_0} \) and \( C_A \) of Definition 1.1.3. To show these categories are equivalent, let us construct two functors between them.

Proposition 1.2.37. Let \( \iota : \text{CAlg}^{\rotate} \to \text{CAlg}^{\text{cn}} \) be the natural inclusion functor. The functor \( \iota^* : C_A \to \text{P}(\text{Aff}^{\rotate})/\widehat{\mathcal{M}}_{\text{BT}^p,A_0} \) factors through \( C_{A_0} \).

Our proof of the above proposition relies on Appendix \( \text{B} \) and the concept of hypercovers; see Definition 1.1.1. We also see that \( \mathcal{X}_0 \) is represented by \( \tau_{\leq 0} \mathcal{X} \) inside \( \text{P}(\text{Aff}^{\rotate}) \), so it follows that \( \mathcal{X}_0 \) can be realised as the 0th truncation of \( \mathcal{X} \), hence they have the same closed points. As \( \iota^* \mathcal{U}_0 \) is a Noetherian affine classical formal Deligne–Mumford stack, we also see \( \mathcal{X}_0 \) is locally Noetherian. It also follows from Remark 1.2.14 that \( \mathcal{X}_0 \to \widehat{\mathcal{M}}_{\text{BT}^p,A_0} \) is formally étale inside \( \text{P}(\text{Aff}^{\rotate}) \). To see the cotangent complex of the map \( \mathcal{X}_0 \to \widehat{\mathcal{M}}_{\text{BT}^p,A_0} \), say \( L \), is 1-connective and almost perfect inside \( \text{QCoh}(\mathcal{X}_0) \), we consider the following composition of maps in \( \text{P}(\text{Aff}^{\text{cn}}) \)

\[
\tau_{\leq 0} \mathcal{X} \simeq \mathcal{X}_0 \to \mathcal{X} \to \widehat{\mathcal{M}}_{\text{BT}^p,A_0},
\]

from which we obtain a (co) fibre sequence in \( \text{QCoh}(\mathcal{X}_0) \) of the form

\[
L_{\mathcal{X}/\widehat{\mathcal{M}}_{\text{BT}^p,A_0}}|_{\mathcal{X}_0} \to L \to L_{\mathcal{X}_0/\mathcal{X}}.
\]

By Proposition 1.2.19 the first term in the above (co) fibre sequence vanishes, so our desired conditions on \( L \) come from Proposition 1.4.1 as \( L_{\mathcal{X}_0/\mathcal{X}} \) is 1-connective and almost perfect. □
To see \( \imath^* \) is an equivalence, we will construct an explicit inverse.

**Construction 1.2.38.** Define a functor \( \Theta : \mathcal{C}_A \to \mathcal{C}_{A_0} \) as follows: given an object \( G_0 : X_0 \to \widehat{\mathcal{M}}^{\infty}_{BT_p,A_0} \) of \( \mathcal{C}_{A_0} \) we form the Cartesian square in \( \mathcal{P}(\text{Aff}^n) \)

\[
  \begin{array}{ccc}
    X & \xrightarrow{g} & \tau_{\leq 0} X \\
    \downarrow & & \downarrow \\
    \widehat{\mathcal{M}}^{\infty}_{BT_p,A} & \longrightarrow & \widehat{\mathcal{M}}^{\infty}_{BT_p,A_0}
  \end{array}
\]

Using Proposition 1.2.32 we identify \( X \) with the de Rham space \( (X_0/\widehat{\mathcal{M}}^{\infty}_{BT_p,A_0})_{\text{dR}} \). From [EC2, Proposition 3.2.2] and the hypothesis that the object \( L_{X_0/\widehat{\mathcal{M}}^{\infty}_{BT_p,A}} \) inside \( \text{Qcoh}(X_0) \) is almost perfect and 1-connective, we see \( X \) is represented by a formal thickening of \( X_0 \), denoted by \( \hat{X} \), courtesy of Theorem 1.2.35. We then set

\[
\Theta \left( G_0 : X_0 \to \widehat{\mathcal{M}}^{\infty}_{BT_p,A_0} \right) = (X_0/\widehat{\mathcal{M}}^{\infty}_{BT_p,A_0})_{\text{dR}} \simeq \left( G : \hat{X} \to \widehat{\mathcal{M}}^{\infty}_{BT_p,A} \right).
\]

To see this assignment lies in \( \mathcal{C}_A \) we note the following:

- \( \hat{X} \) is locally Noetherian, as it is a formal thickening of the locally Noetherian \( X_0 \); see [SAG, Corollary 18.2.4.4].

- \( \hat{X} \) is qcqs as a formal thickening of a qcqs formal spectral Deligne–Mumford stack is qcqs; see Proposition A.4.7.

- \( \hat{X} \) has residue fields which are perfect of characteristic \( p \) as this is true for \( X_0 \) and \( \hat{X}_0 \simeq \tau_{\leq 0} \hat{X} \) by Corollary A.2.4, hence the residue fields of \( \hat{X}_0 \) agree with \( \hat{X} \).

- \( G \) is formally étale, as \( L_{X/\widehat{\mathcal{M}}^{\infty}_{BT_p,A}} \) vanishes by Example 1.2.28 (or Proposition 1.2.32).

It also follows from [SAG, Corollary 18.2.3.3] that if \( \hat{X}_0 \simeq \text{Spf} \ B_0 \) is affine, then the image of any \( G_0 : \text{Spf} \ B_0 \to \widehat{\mathcal{M}}^{\infty}_{BT_p,A_0} \) in \( \mathcal{C}_{A_0} \) under \( \Theta \) is also affine.

Let us now prove our desired theorem of this subsection.

**Theorem 1.2.39.** The functor \( \imath^* : \mathcal{C}_A \to \mathcal{C}_{A_0} \) is an equivalence of \( \infty \)-categories, with inverse given by \( \Theta \).

For our purposes, we only need to construct a functor \( \Theta : \mathcal{C}_{A_0} \to \mathcal{C}_A \) such that \( \Theta \circ \imath \simeq \text{id} \), but the fact it is an equivalence fits nicely with the analogy with étale sites on spectral Deligne–Mumford 1-stacks; see the example of the moduli stack of elliptic curves in [EC1, Remark 2.4.2] and [EC2, §7], or the affine case in [HA, Theorem 7.5.0.6].
Proof. The composition \( C_{A_0} \xrightarrow{\Theta} C_A \xrightarrow{\iota^*} C_{A_0} \) is equivalent to the identity, which is clear by Construction 1.2.38. For the other composition, as \( x \to \tilde{M}_{BT^p_{\pi},A} \) is formally étale, then part 2 of Proposition 1.2.19 states the following diagram of spaces is Cartesian for any \( E_\infty \)-ring \( R \):

\[
\begin{array}{ccc}
X(R) & \xrightarrow{\sim} & X(\pi_0 R) \\
\downarrow & & \downarrow \\
\tilde{M}_{BT^p_{\pi},A}(R) & \xrightarrow{\sim} & \tilde{M}_{BT^p_{\pi},A}(\pi_0 R)
\end{array}
\]

Similarly, Proposition 1.2.32 shows us the following diagram of spaces is Cartesian for all \( E_\infty \)-rings \( R \):

\[
\begin{array}{ccc}
(\iota^* X/\tilde{M}_{BT^p_{\pi},A})_{\text{dR}}(R) & \xrightarrow{\sim} & X(\pi_0 R) \\
\downarrow & & \downarrow \\
\tilde{M}_{BT^p_{\pi},A}(R) & \xrightarrow{\sim} & \tilde{M}_{BT^p_{\pi},A}(\pi_0 R)
\end{array}
\]

where above we used the definition of \( \tau_{\leq 0}^* X(R) = X(\pi_0 R) \) and the fact that \( X(\pi_0 R) \simeq \iota^* X(\pi_0 R) \). From these natural Cartesian squares we obtain a natural equivalence inside \( \mathcal{P}(\text{Aff}^{cn}) \) of the form

\[
X \xrightarrow{\sim} (\iota^* X/\tilde{M}_{BT^p_{\pi},A})_{\text{dR}} = \Theta \iota^* X,
\]

and we are done. \( \square \)

Remark 1.2.40. For later use, let us also note that étale hypercovers (Definition B.1.1) in \( C_{A_0} \) are sent to étale hypercovers in \( C_A \). Indeed, writing \( \mathfrak{U}_0^\bullet \to X_0 \) for a formally étale hypercover in \( C_{A_0} \) and dropping the zeroes after applying \( \Theta \), then our claim follows immediately from base-change and the following commutative diagram in \( \mathcal{P}(\text{Aff}^{cn}) \)

\[
\begin{array}{ccc}
\mathfrak{U}^\bullet & \xrightarrow{\sim} & X & \xrightarrow{\sim} & \tilde{M}_{BT^p_{\pi},A} \\
\downarrow & & \downarrow & & \downarrow \\
\tau_{\leq 0}^* \mathfrak{U}^\bullet & \xrightarrow{\sim} & \tau_{\leq 0}^* X & \xrightarrow{\sim} & \tau_{\leq 0}^* \tilde{M}_{BT^p_{\pi},A}
\end{array}
\]

in which by Proposition 1.2.32 the right and outside, and hence all, rectangles are is Cartesian.

For both aesthetic and practical reasons, we would like to work with these ideas in the language of spectrally ringed \( \infty \)-topoi.

Construction 1.2.41. Given an adic \( E_\infty \)-ring \( A \) as in Notation 0.0.14, define two spectrally ringed \( \infty \)-topoi

\[
\begin{align*}
\mathcal{M}^\text{un}_{BT^p_{\pi},A} &= (X^\text{un}_{BT^p_{\pi}}, \mathcal{O}^\text{un}_{BT^p_{\pi}}), \\
\mathcal{M}^\bigcirc_{BT^p_{\pi},A_0} &= (X^\bigcirc_{BT^p_{\pi}}, \mathcal{O}^\bigcirc_{BT^p_{\pi}}),
\end{align*}
\]

\[
to have underlying \( \infty \)-topoi \( X^\text{un}_{BT^p_{\pi}} = \text{Shv}^{\text{ét}}(C_A) \) and \( X^\bigcirc_{BT^p_{\pi}} = \text{Shv}^{\text{ét}}(C_{A_0}) \), and structure sheaves \( \mathcal{O}^\text{un}_{BT^p_{\pi}} \) and \( \mathcal{O}^\bigcirc_{BT^p_{\pi}} \) associated to the global section functors \( \Gamma \) under the natural equivalence.
A little more explicitly, $\mathcal{O}^\text{un}_{\text{BT}_n}$ and $\mathcal{O}^\heartsuit_{\text{BT}_n}$ are the right Kan extensions in the respective diagrams

\[
\begin{array}{ccc}
\mathcal{C}_A^\text{op} & \xrightarrow{\Gamma} & \text{CAlg} \\
\downarrow & & \downarrow \\
\mathcal{P}(\mathcal{C}_A)^\text{op} & \xrightarrow{\text{Ran}=\mathcal{O}^\text{un}_{\text{BT}_n}} & \mathbf{Shv}^{\text{et}}(\mathcal{C}_A)^\text{op} \\
\downarrow & & \downarrow \\
\mathbf{Shv}^{\text{et}}(\mathcal{C}_A_0)^\text{op} & \xrightarrow{\text{Ran}=\mathcal{O}^\heartsuit_{\text{BT}_n}} & \mathbf{Shv}^{\text{et}}(\mathcal{C}_A_0)^\text{op}
\end{array}
\]

where the first vertical map is the Yoneda embedding, and the second the sheafification functor. Using $\Theta$ we obtain a morphism of spectrally ringed $\infty$-topoi

$$\mathcal{M}_{\text{BT}_n}^{\heartsuit} \xrightarrow{\theta} \mathcal{M}^\text{un}_{\text{BT}_n},$$

such that the map $\mathcal{O}^\text{un}_{\text{BT}_n} \to \theta_* \mathcal{O}^\heartsuit_{\text{BT}_n}$ is the natural truncation map.

**Warning 1.2.42.** Throughout this article we denote the moduli stack of $p$-divisible groups in $\mathcal{P}(\text{Aff}^{\text{cn}})$ by $\mathcal{M}_{\text{BT}_n}$, and the associated spectrally ringed $\infty$-topos by $\mathcal{M}^\text{un}_{\text{BT}_n}$, as seen in Construction 1.2.41. The same notational convention applies to the classical moduli stack of $p$-divisible groups $\mathcal{M}^\heartsuit_{\text{BT}_n}$ in $\mathcal{P}(\text{Aff}^{\text{cn}})$ (Definition 1.1.1 and Notation 1.1.2) and the moduli stack of oriented $p$-divisible groups $\mathcal{M}^\text{or}_{\text{BT}_n}$ in $\mathcal{P}(\text{Aff}^{p})$ (Definition 1.3.7), both with associated spectrally ringed $\infty$-topoi $\mathcal{M}^\heartsuit_{\text{BT}_n}$ and $\mathcal{M}^\text{or}_{\text{BT}_n}$, respectively; see Constructions 1.2.41 and 1.3.14. The reader should morally ignore this difference in notation, however, the technical distinctions are important to our proofs.

### 1.3 Orientations of $p$-divisible groups

The study of orientations of $p$-divisible (and formal) groups over $\mathbf{E}_\infty$-rings is the focus of [EC2]. Using Lurie’s work, we construct a “derived stack” classifying oriented $p$-divisible groups, $\mathcal{M}^\text{or}_{\text{BT}_n}$, defined on (not necessarily connective) $p$-complete $\mathbf{E}_\infty$-rings. With a little extra work we obtain a spectrally ringed $\infty$-topos $\mathcal{M}^\heartsuit_{\text{BT}_n}$ whose structure sheaf is the titular sheaf of this article; see Construction 1.3.14. Much of the technical complications in this section come from the fact that we must work between presheaves of connective $\mathbf{E}_\infty$-rings, nonconnective $\mathbf{E}_\infty$-rings, and spectrally ringed $\infty$-topoi.

Let us suggest that the reader keeps copy of [EC2] close by when reading this section.

#### 1.3.1 The moduli functor $\mathcal{M}^\text{or}_{\text{BT}_n}$

First let us recall the concept of an orientations of a formal group.

**Definition 1.3.1.** Given an $\mathbf{E}_\infty$-ring $R$, a **formal group** $\hat{G}$ over $R$ is a functor $\hat{G} : \text{CAlg}_{\mathbf{B}_{\geq 0}^{\text{cn}} R} \to \text{Mod}_\mathbb{Z}^\text{op}$ such that $\Omega^\infty \hat{G}$ is a **formal hyperplane** (in the sense of [EC2] Definition 1.5.10). A
preorientation of $\hat{G}$ is an element $e$ of $\Omega^2(\Omega^\infty \hat{G})(R)$. A preorientation $e$ is an orientation if the associated Bott map (in the sense of [EC2 Construction 4.3.7]) $\beta_e : \omega_{\hat{G}} \to \Sigma^{-2}R$ is an equivalence of $R$-modules, where $\omega_{\hat{G}}$ is the dualising line of $\hat{G}$; see [EC2 §4.2]. Denote by $\text{OrDat}(\hat{G})$ the component of $\Omega^2(\Omega^\infty \hat{G})(R)$ consisting of orientations. An orientation of a $p$-divisible group $G$ over a $p$-complete $E_\infty$-ring is an orientation of its identity component $G^\circ$ (in the sense of [EC2 Theorem 2.0.8]).

Recall that each time we associate to a functor $C \to \mathcal{C}$ at $\infty$ (resp. $C \to S$) a coCartesian (resp. left) fibration $D \to C$, or visa versa, we are using the straightening–unstraightening adjunction of [HTT Theorem 3.2.0.1] – the $\infty$-categorical analogue of a Grothendieck construction.

**Definition 1.3.2.** 1. Let $\mathcal{M}^\text{nc}_{\text{BT}^p} : \text{CAlg} \to S$ be the composite of $\tau \geq 0 : \text{CAlg} \to \text{CAlg}^{cn}$ and $\mathcal{M}^{\text{BT}_p^e} : \text{CAlg} \to S$; see [EC2, Variant 2.0.6]. Similarly, we define a functor $\mathcal{M}^\text{un}_{\text{BT}^p} : \text{CAlg} \to S$.

2. Denote by $\text{CAlg}^p$ the full $\infty$-subcategory of $\text{CAlg}$ consisting of $p$-complete $E_\infty$-rings, and write $\text{Aff}^p$ for $(\text{CAlg}^p)^\text{op}$. Let $\mathcal{M}^\text{un}_{\text{BT}^p} : \text{CAlg}^p \to S$ be the composition of $\mathcal{M}^\text{nc}_{\text{BT}_p^e}$ with the inclusion $\text{CAlg}^p \to \text{CAlg}$. Similarly, a functor $\mathcal{M}^\text{un}_{\text{BT}^e} : \text{CAlg}^p \to S$.

3. Let $R$ be a $p$-complete $E_\infty$-ring. Write $\text{OrBT}^p(R) \to \text{BT}^p(R)$ for the left fibration associated to the functor $\text{BT}^p(R) \to S$, $G \mapsto \text{OrDat}(G^\circ) = \text{OrDat}(G)$, where $G^\circ$ denotes the identity component of $G$; see [EC2 Theorem 2.0.8]. Similarly, define $\text{OrBT}_h^p(R)$.

Our goal here is to define a moduli functor $\mathcal{M}^\text{or}_{\text{BT}^p} : \text{CAlg}^p \to S$ sending $R$ to $\text{OrBT}^p(R)$, a sort of iterated Grothendieck construction. To do this formally, we will construct the associated fibration.

**Construction 1.3.3.** Let $\text{CAlg}_{\text{co}}^p$ be the full $\infty$-subcategory of $\text{CAlg}^p$ spanned by those $p$-complete complex oriented $E_\infty$-rings; see [EC2 §4.1.1]. Define the functor $\text{FGroup}(\cdot)$ as the composition $\text{CAlg}_{\text{co}}^p \xrightarrow{\text{FGroup}(\cdot)} \mathcal{G}_{\infty} \xrightarrow{(-)^\infty} S$, $R \mapsto \text{FGroup}(R) \mapsto \text{FGroup}(R)^\infty$, sending a $p$-complete $E_\infty$-ring to its associated $\infty$-category of formal groups ([EC2 Definition 1.6.1]), a functor by [EC2 Remark 1.6.4], and then take the $\infty$-groupoid core. Write $F : \text{FGroup} \to \text{CAlg}_{\text{co}}^p$ for the left fibration associated to the functor $\text{FGroup}(-)$. The functor $F$ has a section $Q$ which sends a $p$-complete complex oriented $E_\infty$-ring $R$ to its Quillen formal group $\hat{G}^\circ_R$ ([EC2 Construction 4.1.13]), which is functorial as taking the $R$-homology and then cospectrum are functorial operations. Let $\text{PreFGroup}$ be the comma category associated to the pair $(Q \downarrow \text{id}_{\text{FGroup}})$, in other words, define this category by the Cartesian diagram inside $\mathcal{G}_{\infty}$

\[ \begin{array}{ccc} \text{PreFGroup} & \rightarrow & \text{FGroup}^\Delta^l \\ \downarrow & & \downarrow (s,t) \\ \text{FGroup} & \rightarrow & \text{FGroup} \times \text{FGroup} \end{array} \]
where $\Delta^1$ is the 1-simplex, $\Delta$ is the diagonal map, and $(s, t)$ sends an arrow in $\text{FGroup}$ to its source and target. More informally, an object of $\text{PreFGroup}$ is a complex oriented $p$-complete $E_\infty$-ring $R$, a formal group $\hat{G}$ over $R$, and a morphism $\hat{G}^R \to \hat{G}$ of formal groups over $R$. By [EC2 Proposition 4.3.21], such a morphism of formal groups over $R$ is precisely the data of a preorientation of $\hat{G}$, hence the name $\text{PreFGroup}$. Let $\text{OrFGroup}$ be the full $\infty$-subcategory of $\text{PreFGroup}$ spanned by those preoriented formal groups $\hat{G}$ which are in fact oriented à la Definition 1.3.1. The composition

$$\text{OrFGroup} \to \text{PreFGroup} \to \text{FGroup}, \quad (R, \hat{G}, e) \mapsto (R, \hat{G}) \tag{1.3.5}$$

is a coCartesian fibration, which we will prove in Lemma 1.3.6, and it is the coCartesian fibration associated to the assignment $\text{OrFGroup} \to \text{FGroup}$ by the composition

$$R \mapsto \text{OrDat}(\hat{G}).$$

In fact, (1.3.5) is a left fibration by (the dual of) [HTT Proposition 2.4.2.4] as each $\text{OrDat}(\hat{G})$ is an $\infty$-groupoid. Write $G: \text{BT}^p_{\text{co}} \to \text{CAlg}_{\text{co}}^p$ for the left fibration associated to the composition

$$\text{BT}_{\text{co}}^p(-): \text{CAlg}^p_{\text{co}} \to \text{CAlg} \xrightarrow{\text{BT}_{\text{co}}^p(-)} \mathcal{C}_{\text{at}_\infty} \xrightarrow{(\cdot)^*} \mathcal{S}; \quad R \mapsto \text{BT}^p(R)^\infty.$$

The natural assignment sending a $p$-divisible group $G$ over a $p$-complete $E_\infty$-ring $R$ to its identity component ([EC2 Theorem 2.0.8]) induces a functor $(-)^*: \text{BT}_{\text{co}}^p \to \text{FGroup}$ between categories over $\text{CAlg}_{\text{co}}^p$. Define an $\infty$-category $\text{OrBT}^p$ by the Cartesian diagram in $\mathcal{C}_{\text{at}_\infty}$

$$\begin{array}{ccc}
\text{OrBT}^p & \to & \text{OrFGroup} \\
\downarrow & & \downarrow \\
\text{BT}_{\text{co}}^p & \to & \text{FGroup} \\
\end{array}$$

As (1.3.5) is a left fibration, then $\text{OrBT}^p \to \text{BT}^p_{\text{co}}$ is also a left fibration. Similarly, we define the category $(\text{BT}_{\text{co}}^p)^\infty$, which comes with a natural map $(\text{BT}_{\text{co}}^p)^\infty \to \text{BT}^p_{\text{co}}$ associated to the inclusion $\text{BT}_{\text{co}}^p(R)^\infty \to \text{BT}^p(R)^\infty$. We then define a left fibration $\text{OrBT}^p_n \to (\text{BT}_{\text{co}}^p)^\infty$ by the Cartesian diagram in $\mathcal{C}_{\text{at}_\infty}$

$$\begin{array}{ccc}
\text{OrBT}^p_n & \to & \text{OrBT}^p \\
\downarrow & & \downarrow \\
(\text{BT}_{\text{co}}^p)^\infty & \to & \text{BT}_{\text{co}}^p \\
\end{array}$$

There is still something to check to validate the above construction.

**Lemma 1.3.6.** The composition (1.3.5) is a coCartesian fibration.

---

12This is clear by computing the fibre product of categories

$$\{ (R, \hat{G}) \} \times_{\text{FGroup}} \text{OrFGroup} \simeq \text{Map}_{\text{FGroup}}(R)(\hat{G}^R, \hat{G}) \simeq \text{OrDat}(\hat{G}),$$

where the second equivalence comes from [EC2 Proposition 4.3.21].
Proof. First note that $(s, t)$ from (1.3.4) is a bifibration according to [HTT, Corollary 2.4.7.11], so in particular it is an inner fibration. This implies $\text{PreFGroup} \to \text{FGroup}$ is an inner fibration, and $\text{OrFGroup} \to \text{PreFGroup}$ is an inner fibration as a full subcategory inclusion, hence $\text{OrFGroup} \to \text{FGroup}$ is an inner fibration. Next, take a morphism $f: (R, \hat{G}) \to (R', \hat{G}')$ inside $\text{FGroup}$, which consists of: a morphism of complex oriented $p$-complete $E_\infty$-rings $g: R \to R'$ and an equivalence $g^* \hat{G} = \hat{G}_R \simeq \hat{G}$ of formal groups over $R$ ([EC2, Remark 4.3.10]). Also take some $(R, \hat{G}, e)$ in $\text{OrFGroup}$, where $e$ is an orientation of $\hat{G}$. It follows from [EC2, Remark 4.3.10] that the image of $e \in \Omega^{\infty+2} \hat{G}(R)$ in

$$\Omega^{\infty+2} g^* \hat{G}(R') \simeq \Omega^{\infty+2} \hat{G}'(R')$$

induced by $g$ is an orientation $g(e)$ of $\hat{G}'$. Using $f$ and $g(e)$, we obtain a morphism

$$h: X = (R, \hat{G}, e) \to (R', \hat{G}', g(e)) = Y$$

inside $\text{OrFGroup}$ and we have $p(h) = f$ in $\text{FGroup}$, where $p$ is the composition (1.3.5). It remains to check that $h$ is $p$-coCartesian, which by (the dual of) [HTT, Proposition 2.4.4.3] is equivalent to the condition that for every object $Z = (R'', \hat{G}'', e'')$ in $\text{OrFGroup}$, the natural square of spaces

$$\begin{array}{ccc}
\text{Map}_{\text{OrFGroup}}(Y, Z) & \longrightarrow & \text{Map}_{\text{OrFGroup}}(X, Z) \\
\downarrow^p & & \downarrow^p \\
\text{Map}_{\text{FGroup}}(p(Y), p(Z)) & \longrightarrow & \text{Map}_{\text{FGroup}}(p(X), p(Z))
\end{array}$$

is Cartesian. This is clear though, as the vertical maps are, by definition, the inclusion of the orientation preserving maps in $\text{FGroup}$, hence the natural map between the fibres of the vertical maps are always equivalences, yielding the square Cartesian.

**Definition 1.3.7.** By [EC2, Remark 4.1.3], a morphism in $\text{CAlg}^p$ sends a complex orientation to a complex orientation, hence we can define a functor $\mathcal{M}_{\text{BTp}}^p: \text{CAlg}^p \to \mathcal{S}$ on $\text{CAlg}^p_{\text{co}}$ to be the functor associated to the composition of left fibrations

$$\text{OrBT}^p \to \text{BT}_{\text{co}}^p \to \text{CAlg}_{\text{co}}^p$$

defined in Construction (1.3.3), and the empty space on objects in $\text{CAlg}^p$ without a complex orientation. To calculate the value of this functor on a $p$-complete complex oriented $E_\infty$-ring $R$, one could compute the functor associated to a composition of left fibrations as a certain Kan extension, or one could consider the object $\mathcal{C}$ in the diagram in $\mathcal{C} at_\infty$ of Cartesian squares

$$\begin{array}{ccc}
\mathcal{C} & \longrightarrow & \mathcal{D} \\
\downarrow & & \downarrow \\
\text{OrBT}^p & \longrightarrow & \text{BT}^p_{\text{co}} \overset{G}{\longrightarrow} \text{CAlg}^p_{\text{co}}
\end{array}$$

26
The object $D$ can be identified with $BT^p(R)$ from the definition of $G$ from Construction 1.3.3, and functor $C \to D$ is precisely the left fibration associated to the composition

$$BT^p(R) \to BT^p_R \xrightarrow{\text{OrDat}(-)} S, \quad G \mapsto \text{OrDat}(G),$$

and Definition 1.3.2 identifies $C$ with $\text{OrBT}^p(R)$, as promised. More informally, $\mathcal{M}^\text{or}_{BT^p}$ is the assignment

$$R \mapsto \begin{cases} \text{OrBT}^p(R) \simeq & \text{if } R \text{ has a complex orientation} \\
\emptyset & \text{if } R \text{ has no complex orientation} \end{cases}$$

Define $\mathcal{M}^\text{or}_{BT^p}$ by the Cartesian square in $\mathcal{P}(\text{Aff}^p)$

$$\begin{array}{ccc}
\mathcal{M}^\text{or}_{BT^p} & \rightarrow & \mathcal{M}^\text{or}_{BT^p} \\
\downarrow \Phi & & \downarrow \Phi \\
\mathcal{M}^\text{un}_{BT^p} & \rightarrow & \mathcal{M}^\text{un}_{BT^p}
\end{array}$$

where right $\Phi$ is the functor naturally induced by Construction 1.3.3.

Remark 1.3.8. We claim $\mathcal{M}_{BT^p} \to \mathcal{M}^\text{or}_{BT^p}$ is an open embedding. Indeed, from Definition 1.1.1 we say a $p$-divisible group $G$ over an $E_\infty$-ring $R$ has height $n$ if the commutative finite flat group scheme $G(Z/pZ)$ has rank $p^n$ over $\text{Spec } R$. Lurie’s definition of a commutative finite flat group scheme over $\text{Spec } R$ ([Lur1 Definition 6.1.1]) states that $G(Z/pZ) \simeq \text{Spec } B$ is affine and $\pi_0 R \to \pi_0 B$ realises $\pi_0 B$ as a projective $\pi_0 A$-module of finite rank equal to $p^n$. By [Aut20 Tag 00NX] this rank is locally (with respect to the Zariski topology on $|\text{Spec } R| = |\text{Spec } \pi_0 R|$) constant. In particular, if $R$ is a local connective $E_\infty$-ring then the commutative finite flat group scheme $G(Z/pZ)$ has a well-defined height, and we find that

$$\text{Spec } R \times_{\mathcal{M}_{BT^p}} \mathcal{M}^\text{or}_{BT^p} \simeq \begin{cases} \text{Spec } R & \text{ht}(G) = n \\
\emptyset & \text{ht}(G) \neq n \end{cases}.$$ 

By base-change we see that $\mathcal{M}^\text{or}_{BT^p} \to \mathcal{M}^\text{or}_{BT^p}$ is also an open embedding.

It is our goal now to try and understand universal orientations.

1.3.2 Orientation classifiers

Construction 1.3.9. Given an adic $E_\infty$-ring $A$ as in Definition 1.1.3, define a $\text{CAlg}$-valued presheaf $\mathcal{D}_{BT^n}$ on $\text{Aff}^\text{ad,p} / \mathcal{M}_{BT^n,A}$ (affine formal spectral Deligne–Mumford stacks over $\mathcal{M}_{BT^n,A}$) as the following composite:

$$\mathcal{D}_{BT^n} : \left( \text{Aff}^\text{ad,p} / \mathcal{M}_{BT^n,A} \right)^{\text{op}} \xrightarrow{a} \left( \text{Aff} / \mathcal{M}_{BT^n} \right)^{\text{op}} \xrightarrow{b} \left( \text{Aff}^{\text{un,p}} / \mathcal{M}_{BT^n} \right)^{\text{op}} \xrightarrow{c} \text{CAlg}.$$ 

The map $a$ is induced by the canonical projection along $\text{Spf } A \to \text{Spec } S$, the inverse to the map $b$ is induced by the canonical map $\text{Spf } B \to \text{Spec } B$ for a $p$-adic $E_\infty$-ring $B$ and the
fact this map is an equivalence follows from [EC2 Theorem 3.2.2(4)], and the map \( c \) is the composition of the assignments

\[
\left( \text{Spec } B \xrightarrow{G} \mathcal{M}_{\text{BT}_n^p} \right) \mapsto (G^\omega \in \text{FGroup}(B)) \mapsto \mathcal{O}_{G^\omega} \mapsto \hat{\mathcal{O}}_{G^\omega},
\]

where the first comes from [EC2 Theorem 2.0.8], the second from [EC2 Proposition 4.3.13], and the third is a \( p \)-completion.

**Definition 1.3.10.** Let \( A \) be as in Notation \( 0.0.14 \). Write \( \mathcal{C}_{\text{aff}} \) (resp. \( \mathcal{C}_A^{\text{aff}} \)) for the full \( \infty \)-subcategory of \( \mathcal{C}_{A_0} \) (resp. \( \mathcal{C}_A \)) spanned by objects \( X_0 \to \hat{\mathcal{M}}_{\text{BT}_n^p, A_0}^\omega \) such that \( X_0 \) is affine.

Our goal is to show \( \mathcal{O}_{\text{BT}_n^p} \) restricts to a sheaf on \( \mathcal{C}_A^{\text{aff}} \), and then extend this sheaf to \( \mathcal{C}_A \).

**Theorem 1.3.11.** The functor \( \mathcal{O}_{\text{BT}_n^p} : \left( \text{Aff}_{\text{ad}}^\omega \right)^{\text{op}} \to \text{CAlg} \) of Construction 1.3.9 is an \( \text{fpqc} \) hypersheaf. Moreover, the restriction of \( \mathcal{O}_{\text{BT}_n^p} \) to \( \mathcal{C}_A^{\text{aff}} \), denoted as \( \mathcal{O}_{\text{BT}_n^p}^{\text{op}} : \mathcal{C}_A^{\text{aff}} \to \text{CAlg} \).

The proof of this theorem is the only time we need to use the adjectives qcqs, and it will also be the first time we need to restrict our attention from the \( \text{fpqc} \) topology to the \( \text{étale} \) topology. First for some auxiliary statements.

**Construction 1.3.12.** Let \( \mathcal{U} = (U, \partial) \) be the spectrally ringed \( \infty \)-topos defined by \( U = \text{Shv}_{\text{fpqc}}(\text{Aff}^p) \) and with structure sheaf \( \partial \) associated to the inclusion \( \text{CAlg}^p \to \text{CAlg} \) under the equivalence \( 0.0.10 \) as done in Construction 1.2.11. Let \( \mathcal{F}_{\text{BT}_n^p}^{\text{or}} \) and \( \mathcal{F}_{\text{BT}_n^p}^{\text{un}} \) be the spectrally ringed \( \infty \)-topoi defined by

\[
\mathcal{F}_{\text{BT}_n^p}^{\text{or}} = \mathcal{U}/\mathcal{M}_{\text{BT}_n^p}^{\text{or}}, \quad \mathcal{F}_{\text{BT}_n^p}^{\text{un}} = \mathcal{U}/\mathcal{M}_{\text{BT}_n^p}^{\text{un}},
\]

using Theorem 3.2.11 to see that both \( \mathcal{M}_{\text{BT}_n^p}^{\text{or}} \) and \( \mathcal{M}_{\text{BT}_n^p}^{\text{un}} \) are \( \text{fpqc} \) (hyper)sheaves. The transformation \( \Phi \) of Definition 1.3.7 induces a morphism of spectrally ringed \( \infty \)-topoi

\[
\phi : \mathcal{F}_{\text{BT}_n^p}^{\text{or}} \to \mathcal{F}_{\text{BT}_n^p}^{\text{un}},
\]

where the map of underlying \( \infty \)-topoi is the natural map of under categories induced by \( \Phi \) and \( \phi^\#: \phi^* \mathcal{O}_{\mathcal{F}_{\text{BT}_n^p}^{\text{un}}} \simeq \mathcal{O}_{\mathcal{F}_{\text{BT}_n^p}^{\text{or}}} \to \mathcal{O}_{\mathcal{F}_{\text{BT}_n^p}^{\text{or}}} \) is the identity. As \( \mathcal{M}_{\text{BT}_n^p}^{\text{or}} \) and \( \mathcal{M}_{\text{BT}_n^p}^{\text{un}} \) are both \( \text{fpqc} \) hypersheaves, it then follows that \( \mathcal{O}_{\mathcal{F}_{\text{BT}_n^p}^{\text{un}}} \) and \( \mathcal{O}_{\mathcal{F}_{\text{BT}_n^p}^{\text{or}}} \) are both hypercomplete sheaves of \( \mathbf{E}_\infty \)-rings; see Remark 3.1.2.

Note that the objects in the above construction will only be used to aid the proof of Theorem 1.3.11.

**Lemma 1.3.13.** Given an affine object \( \text{Spec } R \) in the underlying \( \infty \)-topos of \( \mathcal{F}_{\text{BT}_n^p}^{\text{un}} \), so a \( p \)-complete \( \mathbf{E}_\infty \)-ring \( R \) and an associated \( p \)-divisible group \( G \) of height \( n \), then there is a natural equivalence of \( \mathbf{E}_\infty \)-rings

\[
\phi_* \mathcal{O}_{\mathcal{F}_{\text{BT}_n^p}^{\text{un}}} (G) \simeq \hat{\mathcal{O}}_G^\omega,
\]

where the latter is the \( p \)-completion of the orientation classifier \( \hat{\mathcal{O}}_G^\omega \).
Proof. By Construction 1.3.12 we see that $\phi_*\mathcal{O}_{\mathcal{X}_{\text{or}}}$ is given by $\mathcal{O}_{\mathcal{X}_{\text{or}}}(\phi^*)^{\text{op}}$, where $\phi^*$ is the functor

$$\phi^*: \mathcal{U}/\mathcal{M}_{\text{un}}_{\mathcal{B}_{\text{Pn}}^{n}} \to \mathcal{U}/\mathcal{M}_{\text{or}}_{\mathcal{B}_{\text{Pn}}^{n}}^{\text{un}}, \quad (F \to \mathcal{M}_{\text{B}_{\text{Pn}}^{n}}^{\text{un}}) \mapsto \left( F \times_{\mathcal{M}_{\text{B}_{\text{Pn}}^{n}}^{\text{un}}} \mathcal{M}_{\text{B}_{\text{Pn}}^{n}}^{\text{or}} \right) \overset{\pi_2}{\to} \mathcal{M}_{\text{B}_{\text{Pn}}^{n}}^{\text{or}}.$$

For an object $F$ of $\mathcal{U}/\mathcal{M}_{\text{un}}_{\mathcal{B}_{\text{Pn}}^{n}}$ represented by an affine $G: \text{Spec } B \to \mathcal{M}_{\text{B}_{\text{Pn}}^{n}}^{\text{un}}$, we now consider the Cartesian square in $\mathcal{P}($Aff$^p$)$

$$\phi^*G \to \mathcal{M}_{\text{B}_{\text{Pn}}^{n}}^{\text{or}}, \quad \text{Spec } B \to \mathcal{M}_{\text{B}_{\text{Pn}}^{n}}^{\text{un}}.$$

For a $p$-complete $E_{\infty}$-$B$-algebra $C$ one then obtains natural equivalences

$$\phi^*G(C) \simeq \{ B \to C \} \times_{\mathcal{M}_{\text{B}_{\text{Pn}}^{n}}^{\text{un}}(C)} \mathcal{M}_{\text{B}_{\text{Pn}}^{n}}^{\text{or}}(C) \simeq \text{OrDat}(G_C).$$

Using [EC2, Proposition 4.3.13] one recognises that $\text{OrDat}(G_C)$ is represented by $\mathcal{D}_{G_C}$ in $\mathcal{C}_{\text{Alg}^p_B}$, and by the $p$-completion of $\mathcal{D}_{G_C}$ in the full $\infty$-subcategory of $p$-complete $E_{\infty}$-$B$-algebras.

Proof of Theorem 1.3.11. In Construction 1.3.9 the functor $\mathcal{D}_{\mathcal{B}_{\text{Pn}}^{n}}$ is defined as the composition of maps labelled $a$, $b$, and $c$. The map $a$ clearly sends fpqc hypercovers to fpqc hypercovers, and $b$ does as well due to the fact that the fpqc topology on $\mathcal{C}_{\text{Alg}^{ad}}$ is defined through the forgetful map to $\mathcal{C}_{\text{Alg}^\text{cn}}$. It therefore suffices to show $c$ is an fpqc hypersheaf. The map $c$ can be written as the following composition,

$$\left( \text{Aff}^{\text{cn},p}_{\mathcal{M}_{\text{B}_{\text{Pn}}^{n}}^{\text{un}}} \right)^{\text{op}} \to \left( \text{Aff}^{p}_{\mathcal{M}_{\text{B}_{\text{Pn}}^{n}}^{\text{un}}} \right)^{\text{op}} \overset{c'}{\to} \mathcal{C}_{\text{Alg}}$$

where the first map is induced by the inclusion $\mathcal{C}_{\text{Alg}^p_B} \to \mathcal{C}_{\text{Alg}}$, which sends fpqc hypercovers to fpqc hypercovers, and the second map $c'$ is the same construction as $c$ from Construction 1.3.9 none of the theorems referred to there needed a connective input. It follows immediately from the definition of $c'$ above and Lemma 1.3.13 that $c'$ is equivalent to the restriction of $\phi_*\mathcal{O}_{\mathcal{X}_{\text{or}}}$ along the map

$$\text{Aff}^{p}_{\mathcal{M}_{\text{B}_{\text{Pn}}^{n}}^{\text{un}}} \to \mathcal{U}/\mathcal{M}_{\text{B}_{\text{Pn}}^{n}}^{\text{un}}$$

induced by the Yoneda embedding. It follows that $c'$ is an fpqc sheaf as $\phi_*\mathcal{O}_{\mathcal{X}_{\text{or}}}$ was an a sheaf on an $\infty$-topos defined as fpqc sheaves on a particular site. Furthermore, we can write $\phi_*\mathcal{O}_{\mathcal{X}_{\text{or}}}$ as $\mathcal{O}_{\mathcal{X}_{\text{or}}}(\phi^*)^{\text{op}}$ (as discussed in the proof of Lemma 1.3.13), and as $\phi^*$ sends hypercomplete sheaves to hypercomplete sheaves (as fibre products commute with limits), we see $\phi_*\mathcal{O}_{\mathcal{X}_{\text{or}}}$ is also a hypersheaf.
We now have our fpqc (and hence étale) hypersheaf $\mathcal{O}^\text{aff}_{BT_n}$ on $\mathcal{C}^\text{aff}_A$. To extend this to $\mathcal{C}_A$ we set $\mathcal{O}^\text{or}_{BT_n}$ to be the right Kan extension in the following diagram:

$$
\begin{array}{ccc}
(C^\text{aff}_A)^\text{op} & \xrightarrow{\mathcal{O}^\text{aff}_{BT_n}} & \mathcal{C}\text{Alg} \\
\downarrow & & \downarrow \\
\mathcal{C}^\text{op}_A & \xrightarrow{\mathcal{O}^\text{or}_{BT_n}} & \mathcal{C}\text{Alg}.
\end{array}
$$

This is easily calculable too: for some $X$ from $\mathcal{C}_A$, we can use Proposition A.4.6 to find an affine étale hypercover $U$ of $X$, and then we see from the formula of a right Kan extension as a limit that

$$
\mathcal{O}^\text{or}_{BT_n}(X) \simeq \lim \mathcal{O}^\text{aff}_{BT_n}(U).
$$

Importantly, by Proposition A.4.6, the above limit is nonempty. The transitivity of étale hypercovers implies that $\mathcal{O}^\text{or}_{BT_n}$ is an étale hypersheaf too.

**Construction 1.3.14.** Recall the notation from Construction 1.2.41. Let $\mathcal{M}^\text{or}_{BT_n,A}$ be the spectrally ringed ∞-topos $(\mathcal{X}^\text{un}_{BT_n}, \mathcal{E}^\text{or}_{BT_n})$, where $\mathcal{E}^\text{or}_{BT_n}$ is the étale sheaf associated to the functor $\mathcal{O}^\text{or}_{BT_n}$ under (0.0.10). A little more explicitly, $\mathcal{E}^\text{or}_{BT_n}$ is the right Kan extension in the diagram

$$
\begin{array}{ccc}
\mathcal{C}^\text{op}_A & \xrightarrow{\mathcal{O}^\text{or}_{BT_n}} & \mathcal{C}\text{Alg} \\
\downarrow & & \downarrow \\
\text{Shv}^\text{ét}(\mathcal{C}_A)^\text{op} & \xrightarrow{\mathcal{E}^\text{or}_{BT_n}} & \mathcal{C}\text{Alg}.
\end{array}
$$

There is a natural map $\phi: \mathcal{M}^\text{or}_{BT_n,A} \rightarrow \mathcal{M}^\text{un}_{BT_n,A}$ defined by the identity on the underlying ∞-topos, and where $\mathcal{E}^\text{un}_{BT_n} \rightarrow \mathcal{E}^\text{or}_{BT_n}$ the canonical map $B \rightarrow \hat{S}\mathcal{G}$ corresponding to the $p$-completion orientation classifier of a $p$-divisible group $G$ over an affine formal spectral Deligne–Mumford stack $\text{Spf} B$ in $\mathcal{C}^\text{aff}_A$, à la Lemma 1.3.13.

The spectrally ringed ∞-topos $\mathcal{M}^\text{or}_{BT_n,A}$ is an incarnation of the derived stack of oriented $p$-divisible groups referred to in the introduction, and has everything we need to define our sheaf of interest.

### 1.4 The sheaf $\mathcal{E}^\text{top}_{BT_n}$

The definition of $\mathcal{E}^\text{top}_{BT_n}$ mirrors Lurie’s definition of $\mathcal{E}^\text{top}$ ([EC2, §7.3]), and the proof that this definition satisfies Theorem 1.1.4 also follows this outline given by Lurie. Recall the morphisms of spectrally ringed ∞-topoi of Construction 1.2.41 and 1.3.14

$$
\mathcal{M}^\text{or}_{BT_n,A_0} \xrightarrow{\theta} \mathcal{M}^\text{un}_{BT_n,A} \xleftarrow{\phi} \mathcal{M}^\text{or}_{BT_n,A'}
$$
Definition 1.4.1. Given an adic $E_\infty$-ring $A$ as in Notation 0.0.14 let $\mathcal{O}_{\text{top}}^{\text{BT}_p}$ be the étale sheaf on $\mathcal{C}_{A_0}$ defined by restricting $\theta^* \phi^* \mathcal{O}_{\text{top}}^{\text{BT}_p}$ along the composition

$$(\mathcal{C}_{A_0})^{\text{op}} \to \mathcal{P}(\mathcal{C}_{A_0})^{\text{op}} \to \text{Shv}^\text{ét}(\mathcal{C}_{A_0})^{\text{op}} = \mathcal{X}_{\text{BT}_p}^{\text{op}}$$

of the Yoneda embedding followed by sheafification. Equivalently, following Constructions 1.2.41 and 1.3.14 this is given by the composition

$$\mathcal{C}_{A_0}^{\text{op}} \xrightarrow{\Theta} \mathcal{C}_A^{\text{op}} \xrightarrow{\mathcal{O}_{\text{top}}^{\text{BT}_p}} \text{CAlg.}$$

It follows from Remark 1.2.40 and Theorem 1.3.11 that $\mathcal{O}_{\text{top}}^{\text{BT}_p}$ is an étale hypersheaf.

The rest of this section is now dedicated to proving $\mathcal{O}_{\text{top}}^{\text{BT}_p}$ satisfies the conditions of Theorem 1.1.4.

1.4.1 Universal deformations of $p$-divisible groups

The following is the $p$-divisible group analogue of [EC2, Proposition 7.4.2], and essentially falls from a few series of reductions and the formally étale hypothesis.

Proposition 1.4.2. Let $A$ be an adic $E_\infty$-ring as in Notation 0.0.14 and consider a transformation $G : \text{Spf} B \to \mathcal{M}_{\text{BT}_p, A}$ where $B$ is a complete adic Noetherian $E_\infty$-ring and $G$ is formally étale. Fix a maximal ideal $m \subseteq \pi_0 B$ such that $\pi_0 B / m$ is perfect of characteristic $p$. Then the $p$-divisible group $G_{B_m}$ is the universal deformation of $G_\kappa$ (in the sense of [EC2, Definition 3.1.11]), where $\kappa$ is the residue field of $B_m$.

Proof. As $\kappa$ is perfect of characteristic $p$, it follows from [EC2, Example 3.0.10] that $G_\kappa$ is nonstationary, and we can further apply [EC2, Theorem 3.1.15] to obtain the spectral deformation ring $R_{G_\kappa}^\text{un} = B^\text{un}$ with a universal $p$-divisible group $G^\text{un}$. By definition $G_{B_m}$ is a deformation over $G_\kappa$ ([EC2, Definition 3.0.3]), so from the universality of $(B^\text{un}, G^\text{un})$ we obtain a canonical continuous morphism of adic $E_\infty$-rings $B^\text{un} \xrightarrow{\alpha} B_m = \widehat{B}$ inducing the identity on the shared residue field $\kappa$. By [EC2, Theorem 3.1.15] we see $B^\text{un}$ belongs to the full $\infty$-subcategory $C$ of $(\text{CAlg}_{\text{cn}}^\text{ad})/\kappa$ of complete local Noetherian adic $E_\infty$-rings such that the augmentation $\kappa$ exhibiting $\kappa$ as its residue field. To see $\alpha$ is an equivalence in this category, consider an arbitrary object $C$ of this category $\mathcal{C}$ and consider the induced map on mapping spaces

$$\text{Map}_{\text{CAlg}/\kappa}(\widehat{B}, C) \xrightarrow{\alpha^*} \text{Map}_{\text{CAlg}/\kappa}(B^\text{un}, C).$$

By writing $C$ as the limit of its truncations we are reduced to the case where $C$ is truncated, and then writing $\pi_0 C$ as a limit of Artinian rings of $\pi_0 C$ we are further reduced to the case when $\pi_0 C$ is Artinian. In this situation, when have a finite sequence of maps $C = C_m \to C_{m-1} \to \cdots \to C_1 \to C_0 = k$.

---

13Recall that our conventions demand that local adic $E_\infty$-rings have their topology determined by their maximal ideal.
where each map is a square-zero extension by an almost perfect connective module. Hence, it would suffice to show that for every \( C \to \kappa \) as above, and every square-zero extension \( \tilde{C} \to C \) of \( C \) by an almost perfect connective \( C \)-module, with \( \tilde{C} \) also in \( C \), the natural diagram of spaces

\[
\begin{array}{ccc}
\text{Map}_{C Alg_{/\kappa}}(\hat{B}, \tilde{C}) & \longrightarrow & \text{Map}_{C Alg_{/\kappa}}(B^{un}, \tilde{C}) \\
\downarrow & & \downarrow \\
\text{Map}_{C Alg_{/\kappa}}(\hat{B}, C) & \longrightarrow & \text{Map}_{C Alg_{/\kappa}}(B^{un}, C)
\end{array}
\] (1.4.3)

is Cartesian, the \( C = \kappa \) case being trivial. As \( \hat{B} \) is the completion of \( B_m \), then we can use Proposition A.1.1 to see that the map induced by the canonical completion map \( B_m \to \hat{B} \)

\[
\text{Map}_{C Alg_{/\kappa}}(\hat{B}, D) \xrightarrow{\simeq} \text{Map}_{C Alg_{/\kappa}}(B_m, D)
\]

is an equivalence for every object \( D \) inside \( C \), essentially by the completeness of \( D \) with respect to its augmentation ideal. Moreover, for every \( D \) inside \( \hat{C} \) we have the natural identifications

\[
\text{Map}_{C Alg_{/\kappa}}(B^{un}, D) \simeq \text{fib}_{B^{un} \to \kappa} (\text{Map}_{C Alg}(B^{un}, D) \to \text{Map}_{C Alg}(B^{un}, \kappa))
\]

\[
\simeq \text{fib}_{B^{un} \to \kappa} ((\text{Spf } B^{un})(D) \to (\text{Spf } B^{un})(\kappa)) \simeq \text{fib}_{G^{un}} (\text{Def}_{G^{un}}(D) \to \text{Def}_{G^{un}}(\kappa))
\]

\[
\simeq \text{Def}_{G^{un}}(D, (D \to \kappa)) \simeq \text{BT}_{G^{un}}(D) \times_{\text{BT}_{G^{un}}(\kappa)} \{G^{un}\},
\]

where the first equivalence is a categorical fact about over/under categories, the second is the identification of the \( R \)-valued points of \( \text{Spf } B^{un} \) ([SAG Lemma 8.1.2.2]), the third from universal property of spectral deformation rings ([EC2 Theorem 3.1.15]), and the fourth and fifth can be taken as two alternative definitions of \( \text{Def}_{G^{un}}(D, (D \to \kappa)) \) ([EC2 Definition 3.0.3 & Remark 3.1.6]). As both \( C \) and \( \tilde{C} \) lie in \( C \) we see (1.4.3) is equivalent to the upper-left square in the following natural diagram of spaces:

\[
\begin{array}{ccc}
\text{Map}_{C Alg_{/\kappa}}(B_m, \tilde{C}) & \longrightarrow & \text{BT}_{n}(\tilde{C}) \times_{\text{BT}_{n}(\kappa)} \{G_{\kappa}\} \longrightarrow \text{BT}_{n}(\tilde{C}) \simeq \\
\downarrow & & \downarrow \\
\text{Map}_{C Alg_{/\kappa}}(B_m, C) & \longrightarrow & \text{BT}_{n}(C) \times_{\text{BT}_{n}(\kappa)} \{G_{\kappa}\} \longrightarrow \text{BT}_{n}(C) \simeq \\
\downarrow & & \downarrow \\
\{G_{\kappa}\} & \longrightarrow & \text{BT}_{n}(\kappa) \simeq
\end{array}
\] (1.4.4)

The bottom-right square and right rectangle are both pullbacks by definition, so the upper-right square is a pullback. To see the upper rectangle is a pullback we consider the natural
The top square is Cartesian as \( \text{Spf } B \to \hat{M}_{\mathfrak{B}_{\mathcal{P}n},A} \) (and hence \( \text{Spf } B_m \to \hat{M}_{\mathfrak{B}_{\mathcal{P}n},A} \)) is formally étale, and the bottom square is trivially Cartesian. Taking the fibres of the vertical morphisms (at the given map \( B_m \to \kappa \)) we obtain the upper rectangle of (1.4.3), whence this upper rectangle is also Cartesian and we are done.

1.4.2 Proof of Theorem 1.1.4

All the tools and facts we want are now in order. The following proof follows the outline of the proof of [EC2, Theorem 7.0.1].

Proof of Theorem 1.1.4. We have an étale hypersheaf \( \Theta^{\top\text{op}}_{\mathfrak{B}_{\mathcal{P}n}} \) on \( \mathcal{C}_A \) from Definition 1.4.1. It remains to show that when restricted to objects \( G_0 : \text{Spf } B_0 \to \hat{M}_{\mathfrak{B}_{\mathcal{P}n},A_0} \) in \( \mathcal{C}^{\text{aff}}_A \), the \( \mathbb{E}_\infty \)-ring \( \mathbb{E} = \Theta^{\top\text{op}}_{\mathfrak{B}_{\mathcal{P}n}}(G_0) \) has the expected properties. Under \( \Theta \), the object \( G_0 \) as above is sent to an object

\[ G : \text{Spf } B \to \hat{M}_{\mathfrak{B}_{\mathcal{P}n},A} \]

of \( \mathcal{C}^{\text{aff}}_A \) such that \( \pi_0 B \simeq B_0 \) and \( \text{Spf } B_0 \simeq G_0 \); see Construction 1.2.38. By Construction 1.3.14 we see \( \mathbb{E} \) is the \( p \)-completion of the orientation classifier of the identity component \( G^\circ \) of \( G \), denoted by \( \mathcal{O}_{G^\circ} \). First we will argue that the \( \mathbb{E}_\infty \)-ring \( \mathcal{O}_{G^\circ} \) satisfies the desired properties 1-3, and then for \( \mathcal{E} \). By [EC2, Proposition 4.3.23] we see \( \mathcal{O}_{G^\circ} \) is complex periodic. To see conditions 2 and 3, it suffices to show the formal group \( G^\circ \) is balanced over \( B \), the natural equivalence \( \pi_0 \mathcal{O}_{G^\circ} \simeq B_0 \) being induced by \( \pi_0 \) of the unit \( B \to \mathcal{O}_{G^\circ} \); see [EC2, Definition 6.4.1].

Applying [EC2, Remark 6.4.2] (twice), we are reduced to showing that for all maximal ideals \( m \subseteq B_0 \), the identity component of the base-change of \( G \) to \( B_m^\wedge \), written as \( G_{B_m^\wedge} \), is balanced. By Proposition 1.4.2 we see \( G_{B_m^\wedge} \) is the universal deformation of \( G^\circ \), which directly implies the identity component of \( G_{B_m^\wedge} \) is balanced by [EC2, Theorem 6.4.6]. Turning our attention to \( \mathcal{E} \), from the definition of a balanced formal group and [EC2, Proposition 4.3.23], we see \( \pi_0 \mathcal{O}_{G^\circ} \simeq B_0 \) is \( m_A \)-complete and all other nonzero homotopy groups \( \pi_k \mathcal{O}_{G^\circ} \) are invertible \( B_0 \)-modules by complex periodicity, so in particular they are also \( m_A \)-complete. It follows
from [SAG, Theorem 7.3.4.1] that $O_{G^0}$ is $m_A^*$, and in particular $p_*$, complete, and hence we see $O_{G^0} \simeq E$, so $E$ satisfies properties 1-3.

For property 4, note that [EC2, Proposition 4.3.23] states that the canonical orientation of the $p$-divisible group $G_{un}$ over $E$ supplies us with an equivalence $\hat{G}_E^Q \xrightarrow{\sim} G_{un}$ between the Quillen formal group of $E$ and the identity component of $G_{un}$. In particular, this implies the classical Quillen formal group $\hat{G}_E^Q$ is isomorphic to the formal group $G_0^0$ after an extension of scalars along the unit map $B_0 \simeq \pi_0 B \to \pi_0 E$. As $G^0$ is a balanced formal group over $B$, this map is an isomorphism, giving us property 4.

The “moreover” statement summarises what we already know. Indeed, we have seen $E$ is $p$-complete, and the calculation of $\pi_2 k E$ as $\omega_{G_0}^{\otimes k}$ follows from the facts that $E$ is weakly 2-periodic, the $p$-divisible group $G_{un}$ over $E$ comes equipped with a canonical orientation and hence a chosen equivalence of locally free $E$-modules of rank 1

$$\beta: \omega_{G_{un}} \to \Sigma^{-2} E,$$

and the equivalence of $\pi_0 E \simeq B_0$-modules $\pi_0 \omega_{G_{un}} \simeq \omega_{G_0}$,

$$\pi_2 k E \simeq (\pi_2 E)^{\otimes k} \simeq (\pi_0 \omega_{G_{un}})^{\otimes k} \simeq \omega_{G_0}^{\otimes k}. \tag{\textcircled{4}}$$

**Remark 1.4.5.** Let us close this section by stating that there have, of course, been other iterations of Lurie’s theorem; see [BL10, Theorem 8.1.4] and [Mil19, §6.7]. The statements made there are certainly not aesthetically identical to our Theorem 1.1.4, however, we believe that the section to follow, detailing applications of Lurie’s theorem, justifies that all available statements of Lurie’s theorem apply to the same set of examples. In particular, as we can construct Lubin–Tate theories, TMF, and TAF, all using Theorem 1.1.4, we do not find any reason to compare all available statements in too much detail – call this a sin of sloth.

\[\text{\textsuperscript{14}}\text{Phrases such as “(locally) fibrant in the Jardine model structure” can be translated to “is an étale hypersheaf”, and compatibility with checking fibres are universal deformation spaces and the adjective “formally étale” is explained in [Mil19, Remark 6.7.5]; see Proposition 1.2.11 for a similar iteration of that idea.}\]
2 Applications of Lurie’s theorem

The main goal behind the construction of $\mathcal{O}_\text{BT}^\text{top}$ is to analyse TMF and Lubin–Tate theories without appealing to Goerss–Hopkins obstruction theory, and perhaps shed some extra light on these objects that many homotopy theorists care about. Although a vast majority of our applications of Theorem 1.1.4 listed below can be found in either [BL10], [EC2], or [Mil19, §6.7], we would like to present a unified account here.

2.1 KU and Adams operations

As our first application of Theorem 1.1.4, we would like to prove that one of the simplest $p$-divisible groups gives us an example of an $E_\infty$-ring near and dear to stable homotopy theory: complex topological $K$-theory.

2.1.1 The $E_\infty$-ring KU

To define this $E_\infty$-ring, we will follow the construction of [EC2, §6.5], which we will repeat here for the readers convenience.

Construction 2.1.1. Denote by Vect$\simeq$ the category of finite-dimensional complex vector spaces and complex linear isomorphisms. Considering this as a topologically enriched category with a symmetric monoidal structure given by the direct sum of vector bundles, the (topological) coherent nerve $N(Vect_\simeq)$ is a Kan complex with an $E_\infty$-structure. The inclusion

$$\coprod_{n\geq 0} BU(n) \to N(Vect_\simeq),$$

classified on each summand BU($n$) by the universal $n$-dimensional complex vector bundle, is an equivalence of spaces and the $E_\infty$-structure restricts to one on the domain. The group completion of this $E_\infty$-space is the zeroth space of a connective spectrum $ku$, connective complex topological $K$-theory, and the natural group completion map can be identified with the map

$$\xi: \coprod_{n\geq 0} BU(n) \simeq N(Vect_\simeq) \to \Omega^\infty ku \simeq \mathbb{Z} \times BU$$

sending each BU($n$) component to $\{n\} \times BU$ via the canonical inclusion, which represents the tautological complex vector bundle $\xi_n$ over BU($n$). Moreover, there is a multiplicative $E_\infty$-structure on $N(Vect_\simeq)$ given by the tensor product of vector bundles, which also gives the connective spectrum ku the structure of a connective $E_\infty$-ring; see [GGN15, Example 5.3(ii)].

The map $\xi$ is also morphism of $E_\infty$-spaces with respect to this multiplicative $E_\infty$-structure. By identifying $CP^\infty \simeq BU(1)$ within $N(Vect_\simeq)$, we note that $CP^\infty$ inherits the multiplicative $E_\infty$-structure. As $\xi$ restricted to $CP^\infty$ lands in the identity component of $\Omega^\infty ku$, that is $\{1\} \times BU$, we obtain a map of $E_\infty$-spaces $CP^\infty \to GL_1(ku)$, which under the $(\Sigma_+^\infty, GL_1)$-adjunction corresponds to a morphism of $E_\infty$-rings

$$\rho: \Sigma_+^\infty CP^\infty \to ku.$$
Furthermore, the inclusion
\[ \iota: S^2 \simeq \mathbb{C}P^2 \to \mathbb{C}P^\infty \]
post-composed with the unit \( \eta: \mathbb{C}P^\infty \to \Omega^\infty \Sigma^\infty \mathbb{C}P^\infty \) followed by \( \Omega^\infty \) of the inclusion into the first summand \( j: \Sigma^\infty \mathbb{C}P^\infty \to \Sigma^\infty \mathbb{C}P^\infty \oplus S \simeq \Sigma^\infty \mathbb{C}P^\infty \) gives us an element \( \beta \) inside \( \pi_2 \Sigma^\infty \mathbb{C}P^\infty \). The image of \( \beta \) under the map \( \rho \) is also called \( \beta \in \pi_2 \text{ku} \), which one can identify with the element \( [\gamma_1] - 1 \) inside \( \text{ku}(\mathbb{C}P^1) \), where \( \gamma_1 \) is the tautological line bundle over \( \mathbb{C}P^1 \), as a consequence of Proposition 2.1.2. We define the \( E_\infty \)-ring of periodic complex topological \( K \)-theory as the localisation \( \text{KU} = \text{ku} / [\beta] \); see [CC2, Proposition 4.3.17] for a discussion about localising line bundles over \( E_\infty \)-rings, and [HA] §7.2.3 for the \( E_1 \)-ring case.

**Proposition 2.1.2.** The composition
\[
\mathbb{C}P^\infty \xrightarrow{\eta} \Omega^\infty \Sigma^\infty \mathbb{C}P^\infty \xrightarrow{\Omega^\infty j} \Omega^\infty \Sigma^\infty \mathbb{C}P^\infty \xrightarrow{\Omega^\infty \rho} \Omega^\infty \text{ku},
\]
represents the class \( [\xi_1] - 1 \) in \( \text{ku}(\mathbb{C}P^\infty) \), where \( \xi_1 \) is the universal line bundle over \( \mathbb{C}P^\infty \).

Let us recall that for a spectrum \( E \) and a based space \( X \), one defines the unreduced and reduced \( E \) cohomology groups of \( X \) as the abelian groups
\[
E^0(X) = \pi_0 \text{Map}_{\text{Sp}}(\Sigma^\infty X, E), \quad \tilde{E}^0(X) = \pi_0 \text{Map}_{\text{Sp}}(\Sigma^\infty X, E) \simeq \pi_0 \text{Map}_{\Sigma^1}(X, \Omega^\infty E).
\]

Let us also state a lemma we will use regarding the \( (\Sigma^\infty_+, \text{GL}_1) \)-adjunction.

**Lemma 2.1.3.** If \( R \) is an \( E_\infty \)-ring, then we claim the composite
\[ \text{GL}_1(R)_+ \xrightarrow{\eta_+} \Omega^\infty \Sigma^\infty_+ \text{GL}_1(R) \xrightarrow{\Omega^\infty \epsilon} \Omega^\infty R \]
is equivalent to the pointed map defined by the inclusion \( \text{GL}_1(R) \to \Omega^\infty R \), where \( \epsilon: \Sigma^\infty_+ \text{GL}_1(R) \to R \) is the counit of the \( (\Sigma^\infty_+, \text{GL}_1) \)-adjunction.

In other words, precomposition of the above composition with the inclusion \( \text{GL}_1(R) \to \text{GL}_1(R)_+ \) yields the inclusion \( \text{GL}_1(R) \to \Omega^\infty R \).

**Proof.** As the \( (\Sigma^\infty_+, \text{GL}_1) \)-adjunction is the composite of the adjunctions
\[ \text{CMon}^{\text{grp inc}} \overset{\text{inc.}}{\rightleftarrows} \text{CMon}^{\Sigma^\infty_+} \overset{\Sigma^\infty_+}{\rightleftarrows} \text{CAlg}_R \]
it follows the counit \( \epsilon: \Sigma^\infty_+ \text{GL}_1(R) \to R \) factors as
\[ \Sigma^\infty_+ \text{GL}_1(R) \to \Sigma^\infty_+ \Omega^\infty R \to R, \]
the first map induced by the obvious inclusion and the second the counit of the \( (\Sigma^\infty_+, \Omega^\infty) \)-adjunction. This implies the diagram of spaces
\[
\begin{array}{ccc}
\text{GL}_1(R) & \longrightarrow & \text{GL}_1(R) \\
\downarrow & & \downarrow \\
\Omega^\infty R & \longrightarrow & (\Omega^\infty R)_+ \\
\end{array}
\begin{array}{ccc}
& & \Omega^\infty \epsilon \\
\Omega^\infty \epsilon \\
\text{GL}_1(R) & \longrightarrow & \Omega^\infty \Sigma^\infty_+ \Omega^\infty R \\
\Omega^\infty R & \longrightarrow & \Omega^\infty R \\
\end{array}
\]
commutes. The triangle identities for the \( (\Sigma^\infty_+, \Omega^\infty) \)-adjunction imply our desired result. \( \square \)
**Proof of Proposition 2.1.2.** Consider the natural commutative diagram of (based) spaces

\[
\begin{array}{ccc}
\mathbb{C}P_+ & \xrightarrow{\eta_+} & \Omega^\infty \Sigma_+ \mathbb{C}P_+ \\
\downarrow & & \downarrow \Omega^\infty \rho \\
\text{GL}_1(ku)_+ & \xrightarrow{\eta_+} & \Omega^\infty \Sigma_+ \text{GL}_1(ku) \\
\end{array}
\]

where $\eta$ is the counit of the $(\Sigma_+^\infty, \text{GL}_1)$-adjunction. By Lemma 2.1.3, we see the bottom horizontal composite is simply the inclusion $\text{GL}_1(ku) \to \Omega^\infty ku$ (as well as preserving base-points) defining the domain. This then implies the composition $\Omega^\infty \rho \circ \eta_+$ corresponds to the morphism $\xi|_{BU(1)} : \mathbb{C}P^\infty \to \Omega^\infty ku$ (as well as preserving base-points) which lands in $\{1\} \times BU$ defining the universal line bundle $\xi_1$ over $\mathbb{C}P^\infty$. By adjunction, the map $\rho : \Sigma_+^\infty \mathbb{C}P^\infty \to ku$ represents the element $[\xi_1]$ inside $ku^0$.

Writing $p : \Sigma_+^\infty \mathbb{C}P^\infty \simeq \Sigma_+^\infty \mathbb{C}P^\infty \oplus S \to \Sigma_+^\infty \mathbb{C}P^\infty$ for the natural projection, and noting that $p \circ j \simeq id_{\Sigma_+^\infty \mathbb{C}P^\infty}$, we obtain a split short exact sequence

\[
0 \to ku^0(\mathbb{C}P^\infty) \xrightarrow{p^*} ku^0(\mathbb{C}P^\infty) \xrightarrow{d} \pi_0 ku \simeq \mathbb{Z} \to 0,
\]

(2.1.4)

where the splitting to $p^*$ is $j^*$. The map $d$ sends a (virtual) vector bundle to its (virtual) dimension and has splitting induced by $q$ taking $\Sigma^\infty$ of the collapse map $\mathbb{C}P^\infty_+ \to S^0$ preserving base-points and collapsing $\mathbb{C}P^\infty$ to the other point of $S^0$. As $q^*$ is a map of rings, it follows that it is precisely described by sending $n \mapsto \epsilon_n$, where $\epsilon_n$ is the $n$-dimensional trivial bundle over $\mathbb{C}P^\infty$. From this description, we see the splitting of $p^*$ in (2.1.4) sends a virtual vector bundle $[E] - [F]$ to

\[
[E] - [F] - [\epsilon_{\dim E}] + [\epsilon_{\dim F}].
\]

As $j^*$ is the splitting of $p^*$ in (2.1.4), we see $j^*$ can be described in this way too. Writing $[\epsilon_n] = n$, we see that our desired composition then represents the class

\[
j^*(\rho) = j^*([\xi_1]) = [\xi_1] - 1.
\]

The crucial consequence of this proposition is that we obtain the usual complex orientation on $\text{KU}$ as well.

**Remark 2.1.5.** The map $j : \Sigma^\infty \mathbb{C}P^\infty \to \Sigma^\infty \mathbb{C}P^\infty$ defines a class

\[
j \in (\Sigma^\infty_+ \mathbb{C}P^\infty)^0(\mathbb{C}P^\infty).
\]

Let us also write $j \in \check{\mathcal{E}}^0(\mathbb{C}P^\infty)$ for the image of the above element under the localisation map

\[
\Sigma^\infty_+ \mathbb{C}P^\infty \to \Sigma^\infty_+ \mathbb{C}P^\infty[\beta^{-1}] = \mathcal{E}.
\]

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A complex orientation $x_{\mathcal{E}}$ can be defined as

$$x_{\mathcal{E}} = \frac{j}{\beta} \in \tilde{\varepsilon}^2(\mathbb{CP}^\infty),$$

as then we see $\iota^*(x_{\mathcal{E}}) = \beta \cdot \beta^{-1} = 1$. Furthermore, it follows that the image of $x_{\mathcal{E}}$ inside $\text{KU}$ under the map

$$\rho[\beta^{-1}]: \mathcal{E} = \Sigma^\infty_{+} \mathbb{CP}^\infty[\beta^{-1}] \to \text{ku}[\beta^{-1}] = \text{KU}$$

is a complex orientation $x_{\text{KU}}$ of $\text{KU}$, and indeed, it is the orientation we were all expecting by Proposition 2.1.2

$$x_{\text{KU}} = \rho_*(x_{\mathcal{E}}) = \frac{\xi_1 - 1}{\beta} \in \tilde{\gamma}^0(\mathbb{CP}^\infty),$$

where $\xi_1$ is the universal line bundle over $\mathbb{CP}^\infty$.

With a definition of $\text{KU}$ in hand, let us define an algebraic object to compare it to.

**Definition 2.1.6.** Let $\mu_p^{\infty}$ denote the multiplicative $p$-divisible group over $\text{Spec} \mathbb{Z}$, whose $R$-valued points (for a discrete ring $R$) are defined as

$$\mu_p^{\infty}(R) = \{ x \in R \mid x^{p^n} = 1 \}.$$

This lifts to a $p$-divisible group $\mu_p^{\infty}$ over $\text{Spec} S$ by [EC2, Proposition 2.2.11].

**Proposition 2.1.7.** The object $\mu_p^{\infty}$ over $\text{Spf} \mathbb{Z}_p$ is an object of $C^{\mathbb{Z}_p}_{\text{top}}$ (for $n = 1$). Moreover, there is a natural equivalence of $E^{\infty}$-rings

$$\Theta_{\text{BT}^1}(\mu_p^{\infty}) \simeq \text{KU}_p.$$

**Proof.** The fact that $\mu_p^{\infty}$ lies in $C^{\mathbb{Z}_p}_{\text{top}}$ follows immediately from Proposition [1.1.9] and Proposition [1.2.11]. Alternatively, one can view this as a special case of Proposition [2.2.1].

For the “moreover” statement, we follow [EC2]. First we notice that we have a natural equivalence

$$\Theta(\mu_p^{\infty}: \text{Spf} \mathbb{Z}_p \to \tilde{\mathcal{M}}_{\text{BT}^1_{\text{top}}, \mathbb{Z}_p}) \simeq (\mu_p^{\infty}: \text{Spf} S_p \to \tilde{\mathcal{M}}_{\text{BT}^1_{\text{top}}, S_p}),$$

which follows from [EC2, Corollary 3.1.19]. By [EC2, Proposition 2.2.12] we see that the identity component of $\mu_p^{\infty}$ over $\text{Spf} S_p$ is precisely the multiplicative formal group $\tilde{\mathcal{G}}_m$ over $\text{Spf} S_p$. By [EC2, Proposition 4.3.25] the preorientation classifier of $\tilde{\mathcal{G}}_m$ over $\text{Spec} S$ is $\Sigma^\infty_+ \mathbb{CP}^\infty$, and from [EC2, Corollary 4.3.27] we see that the orientation classifier of $\tilde{\mathcal{G}}_m$ over $\text{Spec} S$ is $(\Sigma^\infty_+ \mathbb{CP}^\infty)[\beta^{-1}]$, for the $\beta$ inside $\pi_2 \Sigma^\infty_+ \mathbb{CP}^\infty$ from Construction 2.1.1 see [EC2, §4.3.6]. Upon taking $p$-completions we obtain a natural equivalence

$$E_{\text{BT}^1_{\text{top}}}(\mu_p^{\infty}) \simeq (\Sigma^\infty_+ \mathbb{CP}^\infty)[\beta^{-1}] = \mathcal{E}_p.$$
Snaith’s theorem ([EC2 Theorem 6.5.1]) then states that the \((p\text{-completion of the})\) morphism \(\rho\) of Construction 2.1.1
\[
\rho \colon E_p \xrightarrow{\sim} KU_p
\]
is an equivalence of \(E_\infty\)-rings.

There is a standard trick to obtain the integral \(E_\infty\)-ring \(KU\) from \(O_{\text{top}} BT_p^1(\mu_p^\infty)\) by purely algebraic methods.

Remark 2.1.8. Consider the symmetric monoidal Schwede–Shipley equivalence of \(\infty\)-categories
\[
\text{Mod}_R \simeq \mathcal{D}(R), \quad (2.1.9)
\]
where \(R\) is a discrete commutative ring; see [HA, Theorem 7.1.2.13]. Replacing \(R\) with \(\mathbb{Q}\), we note that the \(E_\infty\)-\(\mathbb{Q}\)-algebra \(ku\), the rationalisation of \(ku\), has homotopy groups \(\pi_* ku \simeq \mathbb{Q}[\beta]\), for \(|\beta| = 2\). One can then construct a map of \(\mathbb{Q}\)-cdgas \(\Lambda_{\mathbb{Q}}[x_{2}^{\pm 1}] \to ku_{\mathbb{Q}}\) from the free \(\mathbb{Q}\)-cdga on one element in degree 2 to \(ku_{\mathbb{Q}}\), defined by the element \(\beta\). This is easily seen to be an equivalence of \(\mathbb{Q}\)-cdgas, and moreover, the obtain an equivalence upon localisations at \(x_2\)
\[
\Lambda_{\mathbb{Q}}[x_{2}^{\pm 1}] \xrightarrow{\sim} ku_{\mathbb{Q}}[\beta^{-1}] \simeq KU_{\mathbb{Q}},
\]
where \(KU_{\mathbb{Q}}\) is the rationalisation of \(KU\). We then define a morphism in \(\text{CAlg}_{\mathbb{Q}}\)
\[
\phi : \Lambda_{\mathbb{Q}}[x_{2}^{\pm 1}] \to \left( \prod_p KU_p \right)_{\mathbb{Q}}
\]
to send \(x_2\) to the \(\beta\) element in \(\pi_2 KU_p\) for each \(p\). One then obtains \(KU\) from the following Hasse Cartesian square of \(E_\infty\)-rings:
\[
\begin{array}{ccc}
KU & \to & \prod_p KU_p \\
\downarrow & & \downarrow \\
\Lambda_{\mathbb{Q}}[x_{2}^{\pm 1}] & \xrightarrow{\phi} & (\prod_p KU_p)_{\mathbb{Q}}
\end{array}
\]
where the two products are taken over all prime numbers \(p\); see [Bau14].

2.1.2 Adams operations

The next example exploits the intrinsic functorality of the sheaf \(\mathcal{O}_{BT_1}^{\text{top}}\).

Definition 2.1.10. Let \(k\) be an integer. Define an automorphism \(\alpha_k : \mu_p^\infty \to \mu_p^\infty\) by setting
\[
\mu_p^\infty(R) \simeq \{x \in R \mid x^{p^n} = 1\} \xrightarrow{x \mapsto x^k} \{x \in R \mid x^{p^n} = 1\} \simeq \mu_p^\infty(R),
\]
for every discrete ring \(R\). When \(\alpha_k\) is an equivalence of \(p\)-divisible groups over \(\text{Spf} \mathbb{Z}_p\), so that it induces an automorphism of the corresponding object in \(\text{Aff}_{/\mathcal{M}_{BT_1}^{\text{top}} \mathbb{Z}_p}}\), we write \(\alpha_k^*\) for \(\mathcal{O}_{BT_1}^{\text{top}}\) applied to this automorphism \(\alpha_k\).

Proposition 2.1.11. For integers \(k\) not divisible by \(p\), the map of \(E_\infty\)-rings \(\alpha_k^* : KU_p \to KU_p\) represents the Adams operation \(\psi^k\) on cohomology, that is, for every compact space \(X\), the

\[\text{For us a compact space is an object of } S \text{ with the homotopy type of a finite CW-complex.}\]
\[ \alpha^*_k : KU^*_p(X) \to KU^*_p(X) \]
is the classical Adams operation \( \psi^k \); see [Ati67, §3.2].

Using Remark 2.1.8 one can construct maps of \( E_{\infty} \)-rings \( \alpha^*_p : KU[1/p] \to KU[1/p] \) for every prime \( p \). To define the Adams operation \( \psi^p \) as a map of spectra, i.e., for this operation to be stable, one must invert \( p \); see [Ada74, §II.13].

Proof. The first thing we check here, and indeed the reason why we must restrict to the case when \( p \) does not divide \( k \), is that in this case \( \alpha_k \) induces an equivalence of \( p \)-divisible groups over \( \text{Spf } \mathbb{Z}_p \). This follows as \( \mu_p^\infty \) is a filtered system of finite abelian \( p \)-group schemes and \( p \) does not divide \( k \), but, for later use, let us justify this fact using the tools provided for us in [EC2]. By a classical theorem of Tate ([Tat67, §2.2] or [EC2, Theorem 2.3.2]) we see that endomorphisms of \( \mu_p^\infty \) over \( \text{Spf } \mathbb{Z}_p \) are equivalent to endomorphisms of its identity component \( \hat{\mathbb{G}}_m \) over \( \text{Spf } \mathbb{Z}_p \). Hence, it suffices to see that the map \( \alpha^*_k : \hat{\mathbb{G}}_m \to \hat{\mathbb{G}}_m \), represented by taking \( k \)th powers \( ^{\wedge} \), is an equivalence of formal groups over \( \text{Spf } \mathbb{Z}_p \). It follows from [EC2, Remark 4.2.18] that this is true if and only if the induced map on dualising lines ([EC2, Example 4.2.15])

\[ \alpha^*_k : \omega_{\hat{\mathbb{G}}_m, \mathbb{Z}_p} \to \omega_{\hat{\mathbb{G}}_m, \mathbb{Z}_p} \]
is an equivalence. In this case we see that \( \omega_{\hat{\mathbb{G}}_m, \mathbb{Z}_p} \) is the \( \mathbb{Z}_p \)-linear dual of the Lie algebra \( \text{Lie}(\hat{\mathbb{G}}_m) \) ([EC2, Example 4.2.15]), hence the map \( \alpha^*_k \) induces the derivative of \( \alpha_k \) on \( \omega_{\hat{\mathbb{G}}_m, \mathbb{Z}_p} \); this is similar to [Sil86, Corollary IV.4.3]. It follows that the map \( \alpha^*_k \) corresponds multiplication by the integer \( k \), which is an invertible map of \( \mathbb{Z}_p \)-modules if and only if \( p \) does not divide \( k \).

Let us write \( \mathcal{E} = O_{\text{et}}^{\text{top}}(\mu_p^\infty) \). Proposition 2.1.7 states that the map \( \rho : \mathcal{E} \to KU_p \) of Construction 2.1.1 is an equivalence of \( E_{\infty} \)-rings \( \rho : \mathcal{E} \to KU_p \), and Remark 2.1.5 states this equivalence sends the canonical complex orientation \( x_\mathcal{E} \) of \( \mathcal{E} \) to the usual complex orientation \( x_{KU} \) of \( KU_p \). We obtain orientations (now in the sense of Definition 1.3.1) \( e_\mathcal{E} \) and \( e_{KU} \) of the formal multiplicative group \( \hat{\mathbb{G}}_m \) over \( \mathcal{E} \) and \( KU_p \), respectively, ([EC2, Example 4.3.22]) such that \( \rho(e_\mathcal{E}) = e_{KU} \). As these orientations of \( \hat{\mathbb{G}}_m \) determine morphisms from the associated Quillen formal group to \( \hat{\mathbb{G}}_m \) ([EC2, Proposition 4.3.23]), and \( \rho \) preserves the given complex orientations defining these Quillen formal groups, we obtain the commutative diagram of (classical) formal groups over \( \mathbb{Z}_p \)

\[ \begin{array}{ccc}
\hat{G}_{\mathbb{Q}_p}^{KU_p} & \xrightarrow{\rho^*} & \hat{G}_{\mathbb{Q}_0} \\
\downarrow \rho \downarrow & & \downarrow \\
\hat{G}_{m, \mathbb{Z}_p} & \to & \hat{G}_{m, \mathbb{Z}_p}
\end{array} \]

\[ ^{\wedge} \text{When } k \text{ is negative, } x^k \text{ can be interpreted by means of the binomial expansion} \]

\[ x^k = 1 + k(x - 1) + \frac{k(k - 1)}{2!} (x - 1)^2 + \cdots . \]
where all of the morphisms are equivalences; see [EC2, Proposition 4.3.23]. The above diagram can be rewritten in terms of formal schemes over \(Spf \mathbb{Z}_p\):

\[
\begin{array}{c}
\text{Spf } \mathbb{KU}_p^0(\mathbb{CP}^\infty) \\
\quad \xrightarrow{\rho^*} \quad \text{Spf } \mathbb{E}_0^0(\mathbb{CP}^\infty) \\
\quad \xleftarrow{f} \quad \text{Spf } \mathbb{Z}_p[[t]] \xrightarrow{g} \end{array}
\]

(2.1.12)

We know exactly how \(\alpha_k\) acts on the bottom formal group, it takes \(k\)th powers, which is an operation represented by the map of rings

\[
\alpha_k^*: \mathbb{Z}_p[[t]] \xrightarrow{\chi^k} \mathbb{Z}_p[[t_1, \ldots, t_k]] \xrightarrow{\mu_k}\mathbb{Z}_p[[t]], \quad t \mapsto (t + 1)^k - 1,
\]

where the first map is the \(k\)-fold iteration of the comultiplication\(^{18}\), and the second map is the multiplication map. As the maps \(f\) and \(g\) induce maps of adic rings sending \(t\) to \(\beta x_{\mathbb{KU}}\) and \(\beta x_{\mathbb{E}}\), respectively, we then obtain the same formulae for \(\alpha_k^*\) in \(\mathbb{KU}_p^0(\mathbb{CP}^\infty)\) and \(\mathbb{E}_0^0(\mathbb{CP}^\infty)\):

\[
\alpha_k^*(\beta x_{\mathbb{KU}}) = (\beta x_{\mathbb{KU}} + 1)^k - 1, \quad \alpha_k^*(\beta x_{\mathbb{E}}) = (\beta x_{\mathbb{E}} + 1)^k - 1.
\]

As \(\beta x_{\mathbb{KU}} \in \mathbb{KU}_p^0(\mathbb{CP}^\infty)\) is represented by \([\xi_1] - 1\), we see that

\[
\alpha_k^*([\xi_1]) = \alpha_k^*(\beta x_{\mathbb{KU}}) = \alpha_k^*(\beta x_{\mathbb{KU}} + 1) = (\beta x_{\mathbb{KU}} + 1)^k - 1 + 1 = [\xi_1^k].
\]

It follows that for any compact space \(X\) and any complex line bundle \(\mathcal{L}\) over \(X\) with corresponding map \(g: X \to \mathbb{CP}^\infty\), the inherent naturality of \(\alpha_k^*\) gives us the formula

\[
\alpha_k^*([\mathcal{L}]) = \alpha_k^*([g^*\xi_1]) = g^*(\alpha_k^*(\xi_1)) = g^*[\xi_1]^k = [\mathcal{L}^\otimes k].
\]

It follows from [Ati67, Proposition 3.2.1(3)] that the operations \(\alpha_k^*\) on \(\mathbb{KU}_p^0(X)\) are the Adams operations \(\psi^k\).

Remark 2.1.13. It seems possible to prove Proposition 2.1.11 using a descent spectral sequence argument: one would make the same reduction to computing \(\alpha_k^*\) on \(\beta \in \pi_2 \mathbb{KU}_p\), and then use a descent spectral sequence of the form

\[
E_2^{s,t} \simeq \mathit{H}^t\left( \text{Spf } \mathbb{Z}_p, \omega_{G_m}^{\otimes s/2} \right) \Rightarrow \pi_{s-t} \mathbb{KU}_p,
\]

which has \(E_2\)-page concentrated in filtration \(t = 0\) by [AJL99, Corollaries 3.1.6 & 3.1.8]. We have tried to avoid any sudden detailed discourse on descent spectral sequences here.

\(^{18}\)The comultiplication on the ring \(\mathbb{Z}_p[[t]]\) representing the multiplicative formal group is given by

\[
\mathbb{Z}_p[[t]] \to \mathbb{Z}_p[[x, y]], \quad t + 1 \mapsto xy + x + y + 1 = (x + 1)(y + 1).
\]
Remark 2.1.14. It is well-known that the automorphism group of $\hat{G}_m$ over $\mathbb{Z}_p$ is isomorphic to the units $\mathbb{Z}_p^\times$ of the $p$-adic integers; see [Lub64] or [Rav04, Theorem A2.2.17]. From this one obtains Adams operations $\psi^\lambda$ for each $\lambda \in \mathbb{Z}_p^\times$, which agree with our Adams operations $\psi^k$ above when restricted to the image of the integers $\mathbb{Z} \to \mathbb{Z}_p$ which are not divisible by $p$. Using the Teichmüller character $F_p^\times \to \mathbb{Z}_p^\times$, which sends $d$ to the limit of the Cauchy sequence $\{d^{p^n}\}_{n \geq 0}$, one obtains an action of $F_p^\times \cong \mathbb{C}_p^{-1}$ on $KU_p$, at least for odd primes $p$. The $E_\infty$-ring $KU_p^{hC_{p-1}}$ is then equivalent to the $p$-complete Adams summand, corresponding to the idempotent map

$$\frac{1}{p-1} \sum_{d \in F_p^\times \subseteq \mathbb{Z}_p^\times} \psi^d : KU_p \to KU_p.$$ 

2.1.3 The $E_\infty$-ring KO

The $E_\infty$-ring KO can also be obtained through these means. The following is a carbon copy of Construction 2.1.14, replacing $C$ with $R$.

Construction 2.1.15. Denote by $\text{Vect}_R^\sim$ the category of finite-dimensional real vector spaces and real linear isomorphisms. Considering this as a topologically enriched category with symmetric monoidal structures given by the direct sum tensor product of vector bundles, the (topological) coherent nerve $N(\text{Vect}_R^\sim)$ is a commutative monoid object in the $\infty$-category of $E_\infty$-spaces. Moreover, the functor

$$c : \text{Vect}_R^\sim \to \text{Vect}_C^\sim, \quad V \mapsto V \otimes_R C$$

is symmetric monoidal with respect to both monoidal structures, hence we obtain a morphism of commutative monoid objects in $E_\infty$-spaces

$$c : N(\text{Vect}_R^\sim) \to N(\text{Vect}_C^\sim).$$

The group completion (with respect to the direct sum $E_\infty$-structure) of $N(\text{Vect}_R^\sim)$ is the zeroth space of the connective $E_\infty$-ring $ko$, connective real topological $K$-theory, and $c$ induces a morphism $ko \to ku$ of $E_\infty$-rings. There is an element $\beta_R$ inside $\pi_8 ko$, represented by an element which maps to the element $\beta^4$ inside $\pi_8 ku$; see [Ada74] §III for example. We define the $E_\infty$-ring of periodic real $K$-theory as the localisation $KO = ko[\beta_R^{-1}]$, and we notice this induces a morphism $KO \to KU$.

Definition 2.1.16. Let $\sigma$ be the automorphism of $\mu_p^\vee$ over $\text{Spf} \mathbb{Z}_p$ defined by $\sigma = \alpha_{-1}$. Write $\sigma^*$ for $\sigma^\top$ applied to the automorphism $\sigma$.

Proposition 2.1.17. The $C_2$-action on $KU_p$ given by $\sigma^*$ induces a $C_2$-action of $E_\infty$-rings on $KU$, and this agrees with the complex conjugation action of $[Ati 66]$. In particular, the natural map $E_\infty$-rings $KO \to KU$ induces an equivalence

$$KO \simeq KU^{hC_2}.$$
Proof. It is clear by the first paragraph in the proof of Proposition 2.1.11 that \( \sigma^*: \text{KU}_p \to \text{KU}_p \) exists for all primes \( p \). Moreover, from that same paragraph we see the effect of \( \alpha^* \) on \( \beta \in \pi_2 \text{KU}_p \) is given by \( \alpha^*(\beta) = k\beta \). We use this and Remark 2.1.8 to construct an automorphism \( \sigma^* \) on the \( \E\text{∞}-\text{Q} \)-algebra \( \Lambda \mathbb{Q}[x_{\pm1}^2] \) by sending \( \beta \mapsto -\beta \), such that \( \phi \) of Remark 2.1.8 is \( \mathbb{C}^2 \)-equivariant. This all means that Remark 2.1.8 endows the \( \E\text{∞}-\text{ring} \) \( \text{KU} \) with a \( \mathbb{C}^2 \)-action, which we will further denote as \( \sigma^*: \text{KU} \to \text{KU} \). Furthermore, this map sends \( \beta \) to \( -\beta \), and sends complex line bundles over compact spaces to their inverses. From this, it follows that \( \sigma^* \) induces the complex conjugation action on \( \text{KU} \)-cohomology on compact spaces. Indeed, we use the splitting principle to reduce ourselves to the case of a line bundle, and [MS74, p.168] to note that the dual of a complex line bundle is equivalent to its complex conjugate. To lift this statement from one about cohomology theories to one about the spectra that represent them, we now show there are no phantom maps of spectra \( \text{KU} \to \text{KU} \), as this is the only obstacle to the fully-faithfulness of the functor \( \text{hSp} \to \text{CohomTh} \),

where \( \text{CohomTh} \) denotes the 1-category cohomology theories on compact spaces; see [HS99, §2 & Corollary 2.15] and [CHT, Lecture 17]. As \( \text{KU} \) represents an even periodic Landweber exact cohomology theory, it follows there exists no phantom endomorphisms of \( \text{KU} \); see [CHT, Corollary 7, Lecture 17]. This implies that \( \sigma^*: \text{KU} \to \text{KU} \) is the same as the complex conjugation action on \( \text{KU} \) up to homotopy. This is then enough data to run the (homotopy fixed point spectral sequence) arguments of [HS14] and see that \( c: \text{KO} \to \text{KU} \) induces an equivalence \( \text{KO} \to \text{KU}_{\mathbb{C}^2} \).

Remark 2.1.18. A more algebro-geometric construction of \( \text{KO} \) is also possible, by realising \( \text{KO} \) as the sections of \( \mathcal{O}_{\text{top} BT_1} \) with respect to a moduli stack of one-dimensional tori and using Galois descent; see [LN14, Appendix A] or [MM15, Example 6.1].

2.2 \( E_n \) and the Morava stabiliser group action

The above example of \( \mathcal{O}_{\text{top} BT_1} \) \( (\mu_p^\infty) \simeq \text{KU}_p \) can be extended to arbitrary heights. Recall the Lubin–Tate deformation theory of Examples 0.0.7 and 1.2.10.

2.2.1 The \( E\text{∞} \)-ring \( E_n \)

Proposition 2.2.1. Let \( G_0 \) be a \( p \)-divisible group of height \( n \) over a perfect field \( \kappa \). Write \( G \) for the classical universal deformation of \( G_0 \), which is a \( p \)-divisible group over the discrete ring \( R_{G_0}^{LT} \). The object \( G \): \( \text{Spf} \ R_{G_0}^{LT} \to \tilde{M}_{\text{BT}^n_1, \mathbf{Z}_p} \) lies in \( \mathbf{C}_{\mathbf{Z}_p} \).

Proof. It is shown in Proposition 1.2.11 that the map in question is formally étale. As \( \text{Spf} \ R_{G_0}^{LT} \) is an affine formal Deligne–Mumford stack, it follows it is qcqs; see Corollary A.4.5. The adic ring \( R_{G_0}^{LT} \) is also Noetherian, as it is (noncanonically) isomorphic to \( W(\kappa)[[v_1, \ldots, v_{n-1}]] \) ([LT66]), and this ring is Noetherian as the \( (p \text{-typical}) \) Witt vectors of a field are Noetherian if and only if that field is perfect, and the Hilbert basis theorem implies \( R_{G_0}^{LT} \) is Noetherian as well. From this description it is also clear that the residue field \( \kappa \) of the closed point of \( R_{G_0}^{LT} \) is

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perfect and of characteristic \( p \). It is left to show the cotangent complex of \( \text{Spf} \ R_{G_0}^{LT} \to \hat{M}_{B^\text{top}}^p, z_p \) is almost perfect and 1-connective inside \( \text{QCoh}(\text{Spf} \ R_{G_0}^{LT}) \). As \( R_{G_0}^{LT} \simeq \pi_0 R_{G_0}^{un} \), where \( R_{G_0}^{un} \) is the spectral universal deformation ring of \([\text{EC2}, \text{Definition 3.1.11}]\), and whose existence is justified by \([\text{EC2}, \text{Theorem 3.1.15}]\), we have the commutative diagram in \( \mathcal{P}(\text{Aff}^{cn}) \)

\[
\begin{array}{ccc}
\text{Spf} \ R_{G_0}^{LT} & \xrightarrow{c} & \text{Spf} \ R_{G_0}^{un} \\
G & \downarrow & \\
\hat{M}_{B^\text{top}}^p, z_p & & \\
\end{array}
\]

The associated (co)fibre sequence in \( \text{QCoh}(\text{Spf} \ R_{G_0}^{LT}) \) is then of the form

\[
c^* L_{\text{Spf} \ R_{G_0}^{LT} / \hat{M}_{B^\{top}}^p, z_p} \to L_{\text{Spf} \ R_{G_0}^{un} / \hat{M}_{B^\text{top}}^p, z_p} \to L_{\text{Spf} \ R_{G_0}^{LT} / \text{Spf} \ R_{G_0}^{un}}.
\]

The first object vanishes as \( \text{Spf} \ R_{G_0}^{un} \) is the de Rham space of \( G \) and such cotangent complexes always vanish; see Example \[1.2.28\]. The third object is 1-connective and almost perfect as it is the cotangent complex of a truncation of a Noetherian adic \( E_\infty \)-ring; see Proposition \[A.4.1\]. It follows that the middle object, our desired cotangent complex, is 1-connective and almost perfect.

The following is an immediate consequence of \([\text{EC2}, \S 5]\).

**Proposition 2.2.2.** In the situation of Proposition \[2.2.1\], we have a natural equivalence of \( E_\infty \)-rings

\[
\theta_{B^\text{top}}^\text{top}(G): \text{Spf} \ R_{G_0}^{LT} \to \hat{M}_{B^\text{top}}^p, z_p \simeq E_n,
\]

where \( E_n = E(\hat{G}_0) \) is the \( E_\infty \)-ring of Morava \( E \)-theory (also known as Lubin-Tate theory or completed Johnson-Wilson theory).

**Proof.** Using Definition \[1.4.1\] we see \( \theta_{B^\text{top}}^\text{top}(G) \) is the orientation classifier of \( R_{G_0}^{un} \), and that is the definition of \( E_n \) given in \([\text{EC2}], \text{EC2} \text{ Construction 5.1.1} \& \text{Remark 5.1.2}]\). \( \square \)

### 2.2.2 Action of Morava stabiliser group

From the functorality of \( \theta_{B^\text{top}}^\text{top} \), we obtain an action of the automorphism group of \( G_0 \) on the \( E_\infty \)-ring \( E_n \). For chromatic homotopy theorists, this means that these \( E_n \) obtain an action by a group that has been long studied in the area; see \([\text{EC2}, \S 5]\) and \([\text{GH04}, \S 7]\)

**Definition 2.2.3.** Let \( \mathcal{F}G \) be the 1-category whose objects are pairs \((R, \hat{G})\) of a commutative ring \( R \) and a formal group \( \hat{G} \) over \( R \), and whose morphisms are pairs \((f, \alpha)\) consisting of a ring map \( f \) and an equivalence \( \alpha \) of formal groups over the codomain. Given a pair \((\kappa, G_0)\) as in Proposition \[2.2.1\] then one defines the **Morava stabiliser group** as the group of automorphisms

\[
\Gamma = \text{Aut}_{\mathcal{F}G}(G_0^0, \kappa).
\]

**Corollary 2.2.4.** The \( E_\infty \)-rings \( E_n \) of Proposition \[2.2.2\] have a natural action of the Morava stabiliser group \( \Gamma \).

A more detailed discussion of Lubin-Tate \( E_\infty \)-rings can be found in \([\text{GH04}]\) and \([\text{EC2}, \S 5]\).
2.3 TMF and level structures

The authors personal goal throughout all of this has been to understand the construction of TMF in more detail.

2.3.1 The $E_\infty$-ring TMF

Recall the morphism $[p^\infty]: \mathcal{M}_{\text{Ell}}^\odot \to \mathcal{M}_{\text{BT}_2}^\odot$ of Example 1.2.7.

**Proposition 2.3.1.** The map $[p^\infty]: \hat{\mathcal{M}}_{\text{Ell},Z_p} \to \hat{\mathcal{M}}_{\text{BT}_2,Z_p}$ lies inside $\mathcal{C}_{Z_p}$.

**Proof.** Using Proposition 1.1.6 we simply need to show that the map $[p^\infty]$ above is formally étale inside $\mathcal{P}(\text{Aff}^{\odot})$ and that $\hat{\mathcal{M}}_{\text{Ell},Z_p}$ is finitely presented over $\text{Spf } \mathbb{Z}_p$. The former follows from Example 1.2.7; a consequence of the classical Serre–Tate theorem. The latter follows from [Ols16, Theorem 13.1.2], which states that $\mathcal{M}_{\text{Ell}}^{\odot}$ is locally of finite presentation over $\text{Spec } \mathbb{Z}$, the fact that the adjective “locally of finite presentation” is stable under base-change, and the fact that $\hat{\mathcal{M}}_{\text{Ell},Z_p}$ is qcqs. Indeed, to see $\hat{\mathcal{M}}_{\text{Ell},Z_p}$ is qc, we recall that it is also well known that after inverting an integer $n$ that there is a surjective étale map $\mathcal{M}_1(n) \to \mathcal{M}_{\text{Ell}}^{\odot} \times \text{Spec } \mathbb{Z}[1/n]$ from the moduli stack of elliptic curves with exact level $n$-structure, and for $n \geq 4$ the Deligne–Mumford stacks $\mathcal{M}_1(n)$ are affine, hence $\hat{\mathcal{M}}_{\text{Ell},Z_p}$ has a finite affine étale cover for all primes $p$; see [KMSA, Theorem 3.7.1] or [Con07]. Moreover, the iterated fibre products of this cover can be checked to be affine too, as $\mathcal{M}_{\text{Ell}}^{\odot}$ is separated over $\text{Spec } \mathbb{Z}$; see [Ols16, Theorem 13.1.2] again. Hence we obtain a finite affine étale hypercover of $\hat{\mathcal{M}}_{\text{Ell},Z_p}$. By Proposition A.4.6 we see $\hat{\mathcal{M}}_{\text{Ell},Z_p}$ is qcqs.

As promised in the introduction, we should relate $\mathcal{O}_{\text{BT}_2}$ to a more classical object.

**Definition 2.3.2.** Let $\mathcal{O}_{\text{BT}_2}$ denote the Goerss–Hopkins–Miller sheaf of $E_\infty$-rings on the étale site $\text{DM}_{/\hat{\mathcal{M}}_{\text{Ell}}^{\odot}}$ of $\hat{\mathcal{M}}_{\text{Ell}}^{\odot}$; see [EC2, Theorem 7.0.1]. Denote by $\hat{\mathcal{O}}_{\text{BT}_2}$ the base-change of this sheaf over $\text{Spf } \mathbb{Z}_p$, so the (completed) pullback of $\mathcal{O}_{\text{BT}_2}$ along the projection $\hat{\mathcal{M}}_{\text{Ell},Z_p} \to \mathcal{M}_{\text{Ell}}^{\odot}$. Consider $\hat{\mathcal{O}}_{\text{BT}_2}$ as a sheaf of $E_\infty$-rings on the étale site of $\hat{\mathcal{M}}_{\text{Ell},Z_p}$, which we denote as $\text{fDM}_{/\hat{\mathcal{M}}_{\text{Ell}}^{\odot}}$.

**Theorem 2.3.3.** There is a natural equivalence of sheaves $\hat{\mathcal{O}}_{\text{BT}_2} \simeq \mathcal{O}_{\text{BT}_2} \circ [p^\infty]$. In other words, the following diagram commutes

$$
\begin{array}{ccc}
\left(\text{fDM}_{/\hat{\mathcal{M}}_{\text{Ell}}^{\odot}}\right)^{\text{op}} & \xrightarrow{\hat{\mathcal{O}}_{\text{BT}_2}} & \text{CAlg} \\
[p^\infty] \downarrow & & \downarrow \text{CAlg} \\
\mathcal{C}_{Z_p}^{\text{op}} & \xrightarrow{\hat{\mathcal{O}}_{\text{BT}_2}} & \text{CAlg} \\
\end{array}
$$

In particular, there is a natural equivalence of $E_\infty$-rings

$$
\mathcal{O}_{\text{BT}_2}^{\text{op}}([p^\infty]: \hat{\mathcal{M}}_{\text{Ell},Z_p} \to \hat{\mathcal{M}}_{\text{BT}_2,Z_p}) \simeq \text{TMF}_p.
$$
The following proof is essentially the proof of [EC2, Theorem 7.0.1].

Proof. Notice that the functor

$$[p^\infty]: \text{fDM}_{\mathcal{M}_{\text{Ell}, \mathbb{Z}_p}}^{\ell} \to \mathcal{C}_{\mathbb{Z}_p}$$

is clearly well-defined as by Proposition 2.3.1 the object $[p^\infty]: \mathcal{M}_{\text{Ell}, \mathbb{Z}_p} \to \mathcal{M}_{\text{BT}_2^p, \mathbb{Z}_p}$ lies in $\mathcal{C}_{\mathbb{Z}_p}$, hence objects in its étale site do as well. As done in the proof of [EC2, Theorem 7.0.1], we will conclude the proof by checking that for each affine object $E_0: \text{Spf} B_0 \to \mathcal{M}_{\text{Ell}, \mathbb{Z}_p}$ of fDM of $\mathcal{M}_{\text{Ell}, \mathbb{Z}_p}$, the $E_\infty$-ring $\mathcal{E} = \Theta_{\text{BT}_2^p}^{\text{top}}(E[p^\infty])$ satisfies the same defining properties as the $E_\infty$-ring $\Theta_{\text{BT}_2^p}^{\text{top}}(E)$ does from Theorem 0.0.2. That is, $\mathcal{E}$ satisfies the following:

1. $\mathcal{E}$ is complex periodic,
2. the groups $\pi_k \mathcal{E}$ vanish for all odd $k$,
3. there is a natural equivalence of rings $\pi_0 \mathcal{E} \simeq B_0$, and
4. there is a natural equivalence of formal groups $\widehat{\mathcal{E}} \simeq \widehat{G}_{E_0}$ over $B_0$.

Once one applies [EC2, Proposition 7.4.1] to see that there is a natural equivalence $\widehat{\mathcal{E}} \simeq E[p^\infty]$, one notices these properties are precisely the properties of $\Theta_{\text{BT}_2^p}^{\text{top}}$ by Theorem 1.1.4. The "in particular" statement comes straight from the definitions; see [EC2, Definition 7.0.3].

The advantage of viewing TMF through $\Theta_{\text{BT}_2^p}^{\text{top}}$ as opposed to $\Theta^{\text{top}}$, is that now separable isogenies of elliptic curves will act on TMF. For example, the $k$-fold multiplication map $[k]: E \to E$ produces elliptic Adams operations $\psi^k$: TMF$_p \to$ TMF$_p$ for integers $k$ not divisible by $p$. The author is exploring these considerations in current research.

As done in [Mil19], we can use the collection of all $p$-complete $E_\infty$-rings and a little rational information to construct integral TMF, similar to Remark 2.1.8.

Remark 2.3.4. We have an étale hypersheaf of $E_\infty$-rings

$$\prod_p \pi^*_p \left( \Theta_{\text{BT}_2^p}^{\text{top}} \circ [p^\infty] \right)$$

on the étale site Deligne–Mumford stacks over $\mathcal{M}_{\text{Ell}}^{\wedge}$ defined by the composition

$$\left( \text{DM}_{/\mathcal{M}_{\text{Ell}}^{\wedge}}^{\ell} \right)^{\text{op}} \to \prod_p \left( \text{fDM}_{/\mathcal{M}_{\text{Ell}, \mathbb{Z}_p}}^{\ell} \right)^{\text{op}} \to \prod_p \left( \mathcal{C}_{\mathbb{Z}_p} \right) \to \prod_p \mathcal{C}_{\mathbb{Z}_p} \to \prod_p \text{CAlg} \to \text{CAlg}.$$
This gives a canonical map into its rationalisation

\[ \prod_p \pi_p^* \left( \mathcal{O}^\text{top} \circ [p^\infty] \right) \to \prod_p \pi_p^* \left( \mathcal{O}^\text{top} \circ [p^\infty] \right)_Q. \]

To give rational information, we recall the symmetric monoidal equivalence of \( \infty \)-categories \((2.1.9)\). An étale hypersheaf of \( \mathbf{E}_\infty \)-rings \( \mathcal{O}^\text{top} \) is defined to send an affine \( E_0: \text{Spec} B_0 \to \mathcal{M}^\Diamond_{\text{Ell}} \) in \( \text{DM}^\text{ct}/\mathcal{M}^\Diamond_{\text{Ell}} \) to the formal rational cdga

\[ \mathcal{O}^\text{top} \left( E_0 \right)_n = \begin{cases} \omega^\otimes_{E} \otimes \mathbb{Q} & n = 2k \\ 0 & \text{else} \end{cases}, \]

where we are taking \(|k|\)-fold tensor products of the \( B_0 \)-dual if \( k < 0 \), and this is extended to qcqs Deligne–Mumford stacks which are étale over \( \mathcal{M}^\Diamond_{\text{Ell}} \) using an affine étale cover; see the proof of Theorem 1.3.11 or Proposition A.4.6. Moreover, we construct a morphism

\[ \mathcal{O}^\text{top} \to \prod_p \pi_p^* \left( \mathcal{O}^\text{top} \circ [p^\infty] \right)_Q \]

by defining it on an affine \( E_0: \text{Spec} B_0 \to \mathcal{M}^\Diamond_{\text{Ell}} \) as the morphism

\[ \omega^\otimes_{E, Q} \to \left( \prod_p \omega^\otimes_{E, \mathbb{Z}_p} \right)_Q, \]

which itself is simply the rationalisation, of the product over all primes, of the map \( \mathbb{Z} \to \mathbb{Z}_p \) tensored with \( \omega^\otimes_{E} \). One can then recover \( \mathcal{O}^\text{top} \) itself as the pullback in the following Cartesian square of sheaves on the étale site of \( \mathcal{M}^\Diamond_{\text{Ell}} \):

\[ \begin{array}{ccc} \mathcal{O}^\text{top} & \to & \prod_p \pi_p^* \left( \mathcal{O}^\text{top} \circ [p^\infty] \right) \\ \downarrow & & \downarrow \\ \mathcal{O}^\text{top} & \to & \left( \prod_p \pi_p^* \left( \mathcal{O}^\text{top} \circ [p^\infty] \right) \right)_Q \end{array} \]

Taking global sections, one obtains the Cartesian square of \( \mathbf{E}_\infty \)-rings

\[ \begin{array}{ccc} \text{TMF} & \to & \prod_p \text{TMF}_p \\ \downarrow & & \downarrow \\ \text{TMF}_Q & \to & \left( \prod_p \text{TMF}_p \right)_Q \end{array} \]

Remark 2.3.5. One may naturally ask if the \( \mathbf{E}_\infty \)-ring \( \text{TMF}_Q \), or more generally \( \text{TMF}^\text{P} \) where \( P \) is a set of prime numbers in \( \mathbb{Z} \), defined as \( \mathcal{O}^\text{top} \left( \mathcal{M}^\Diamond_{\text{Ell,Spec} \mathbb{Z}^\text{P} \text{-} 1} \right) \), is the same as
the $E_\infty$-rings $\text{TMF} \otimes S[1/P^1]$. The answer for us is yes, these two a priori different $E_\infty$-rings are equivalent, however, in general an answer may be difficult to come across. The reason why it works for TMF is the existence of a horizontal vanishing line in the descent spectral sequence associated to TMF. For a larger discussion, see [MM15, §4]. A similar remark holds for completions. This remark is does not affect this article, as we will only ever localise (or complete) and then take global sections – never in the opposite order.

2.3.2 The $E_\infty$-rings $\text{TMF}(n)$, $\text{TMF}_1(n)$, and $\text{TMF}_0(n)$

Let us also mention a few variations on TMF that one can obtain from $O_{\text{top}}$.

**Definition 2.3.6.** There exist moduli functors $\text{CAlg}^\vee \to S$ denoted as $M(n)$, $M_1(n)$, and $M_0(n)$ for each $n \geq 1$. These are defined in [KM85, Chapter 3], and they sit in a commutative diagram in $\mathcal{P}(\text{Aff}^{\mathbb{Z}})$

$$
\begin{array}{ccc}
M(n) & \longrightarrow & M_1(n) \\
& \downarrow & \\
& M_0(n) & \\
& \nearrow & M_{\text{Ell}}
\end{array}
$$

where all the transformations above are some kind of forgetful functor. Moreover, by [KM85 Theorem 3.7.1], we see that when working over $\text{Spec} \mathbb{Z}[1/n]$, all of the morphisms above are finite étale. Using these maps one then defines the $E_\infty$-rings

$$
\text{TMF}(n) = \mathcal{O}^{\text{top}}(M(n)), \quad \text{TMF}_1(n) = \mathcal{O}^{\text{top}}(M_1(n)), \quad \text{TMF}_0(n) = \mathcal{O}^{\text{top}}(M_0(n)),
$$

which are all naturally $E_\infty$-$\text{TMF}[1/n]$-algebras.

Combining Remark 2.3.4 and Definition 2.3.6 one sees that one can obtain the $E_\infty$-rings of periodic topological modular forms with level structure of (2.3.7) directly from Theorem 1.1.4 (of course one needs to complete at a prime $p$ not dividing the integer $n$). Once again, the functoriality of these constructions ensures us that the natural $\text{GL}_2(\mathbb{Z}/n\mathbb{Z})$-action on $M(n)$ over $M_{\text{Ell}} \times \text{Spec} \mathbb{Z}[1/n]$ gives us a natural equivalence of $E_\infty$-rings

$$
\text{TMF}[1/n] \xrightarrow{\sim} \text{TMF}(n)^{h\text{GL}_2(\mathbb{Z}/n\mathbb{Z})};
$$

this is explained and explored in more detail in [MM15, §7].

2.4 TAF and level structures

The final example of this section is the first published example of a new cohomology theory constructed using Theorem 1.1.4. One might have the idea to use the classical Serre–Tate theorem for dimension $g$ abelian varieties, but one thing we have hidden throughout this article so far is the fact that we are only dealing with 1-dimensional formal groups, and when the

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19By definition we know $\Omega^\infty G$ is a formal hyperplane, and by definition a formal hyperplane is equivalent to the cospectrum of a smooth coalgebra $C$. Moreover, a smooth coalgebra $C$ over an $E_\infty$-ring $R$ has a well-defined dimension; see [EC2, Definition 1.2.4]. We then define the dimension of a formal group to be the dimension of the smooth coalgebra associated to the formal hyperplane $\Omega^\infty G$. 

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dimension of abelian varieties moves to $g \geq 2$ we find ourselves in trouble. To still use this classical deformation theory and obtain a 1-dimensional formal group at the end of the day, one has to add more restrictions, which come in the form of polarisations, endomorphisms, and level-structure, leading us PEL-Shimura varieties. For a full introduction to the subject, the intended application to stable homotopy theory, and to avoid repeated referencing in this section, see [BL10].

2.4.1 The $E_\infty$-ring TAF

**Notation 2.4.1.** Fixed an integer $n \geq 1$.

- Let $F$ be a quadratic imaginary extension of $\mathbb{Q}$, such that $p$ splits as $u\pi$.
- Let $\mathcal{O}_F$ be the ring of integers of $F$.
- Let $V$ be an $F$-vector space of dimension $n$ equipped with a $\mathbb{Q}$-valued nondegenerate Hermitian alternating form of signature $(1, n-1)$.
- Let $L$ be an $\mathcal{O}_F$-lattice in $V$ such that the alternating form above takes integer values on $L$ and makes $L(p)$ self-dual.

**Definition 2.4.2.** Write $X_{V,L}$ for the formal Deligne–Mumford stack over $\text{Spf} \mathbb{Z}_p$ (of [BL10, Theorem 6.6.2] with $K^p = K^p_0$) where a point in $X_{V,L}(S)$ for a locally Noetherian formal scheme $S$ over $\text{Spf} \mathbb{Z}_p$, is a triple $(A, \lambda, i)$ where

- $A$ is an abelian scheme over $S$ of dimension $n$,
- $\lambda: A \to A^\vee$ is a polarisation (principle at $p$), with Rosati involution $\dagger$ on $\text{End}(A)(p)$, and
- $i: \mathcal{O}_F(p) \to \text{End}(A)(p)$ is an inclusion of rings satisfying $i(z) = i(z)^\dagger$.

These triples have to satisfy two conditions as well, which essentially state that these triples are locally modelled by $L$ and the alternating form on $V$; see [Mil19, §6.7].

In the situation above, the splitting $p = u\pi$ induces a splitting of the $p$-divisible group $A[p^\infty]$ as

$A[p^\infty] \simeq A[u^\infty] \oplus A[\pi^\infty]$.

The first of the conditions on the triple $(A, \lambda, i)$ that we did not write down is that the object $A[u^\infty]$ is a 1-dimensional $p$-divisible group. This implies that there is a map $[u^\infty]: X_{V,L} \to \bar{\mathcal{M}}_{\text{Ell}, \mathbb{Z}_p}$ which sends $(A, \lambda, i)$ to $A[u^\infty]$.

**Proposition 2.4.3.** Given $V$ and $L$ as in Notation 2.4.1, then the morphism $[u^\infty]: X_{V,L} \to \bar{\mathcal{M}}_{\text{Ell}, \mathbb{Z}_p}$ is an object of $\mathcal{C}_{\mathbb{Z}_p}$. 

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Proof. Proposition [1.1.6] reduces us to show that \([u^\infty]\) is formally étale inside \(\mathcal{P}(\text{Aff}^\text{cn})\) and that \(X_{V,L}\) is of finite presentation over \(\text{Spf}\mathbb{Z}_p\). The first statement, that \([u^\infty]\) is formally étale, follows straight from the definitions of a formally étale morphism and [BL10, Theorem 7.3.1], which itself is a consequence the classical Serre–Tate theorem and an analysis of \(X_{V,L}\). We now use [BL10, Corollary 7.3.3] to see \(X_{V,L}\) is of locally finite presentation over \(\text{Spf}\mathbb{Z}_p\), so it suffices to show now that \(X_{V,L}\) is qcqs. To do this, we first use [BL10, Theorem 6.6.2], which states that \(X_{V,L}\) has an étale cover by a quasi-projective scheme. As a quasi-projective formal scheme \(X\) is separated and qc (see [GW10, Appendix D] for example), we see \(X\) has a Zariski cover by an affine formal scheme \(\text{Spf}B\), meaning \(X_{V,L}\) has an étale cover by \(\text{Spf}B\). By Proposition [A.4.6] this implies \(X_{V,L}\) is qcqs.

We hope the reader will agree that the above proposition frees us of the responsibility of showing that the Lurie’s theorem of [BL10] and our Theorem 1.1.4 are equivalent. The following follows more-or-less by definition; see [BL10, §8] or [Mil19, §6.7].

Proposition 2.4.4. Given \(V, L\) as in Notation 2.4.1, one has a natural equivalence of \(E_\infty\)-rings

\[
\mathcal{O}_{\text{BT}_n}^{\text{top}} \left( X_{V,L} \left[ u^\infty \right] \right) \xrightarrow{\sim} \text{TAF}_{V,L}.
\]

2.4.2 Mackey functor structure of TAF

As mentioned in the constructions of \(KU_p\), \(E_n\), and \(\text{TMF}_p\) above, the inherent functoriality of \(\mathcal{O}_{\text{BT}_n}^{\text{top}}\) gives us objects with level structures and operations.

Example 2.4.5 (TAF with level structure). By [BL10, §6.5] we see there are versions of the moduli problems for \((V, L)\) mentioned above encoding level structures corresponding to a subgroup \(K^p\) of \(K_0^p\); see [BL10, §6.2] for a definition of these objects. By [BL10, Theorem 6.6.2], the various morphisms of Deligne–Mumford stacks \(X_{V,L}(K^p) \to X_{V,L}(K_{V,L})\), corresponding to the inclusions of subgroups \(K^p \leq K_p\), are étale, and hence every \(X_{V,L}(K_p)\) also lives in \(\mathcal{C}_{\mathbb{Z}_p}\) and one can define \(\text{TAF}_{V,L}(K_p)\).

Example 2.4.6 (Morphisms of TAF). Suppressing the pair \((V, L)\) from our notation for now, we can now define three types of functorial operations between the various \(E_\infty\)-rings \(\text{TAF}(K_p)\) following [BL10, §11]:

1. For an inclusion of (compact open) subgroups \(K^p \leq K_p\), by functoriality there exists a restriction map of \(E_\infty\)-rings

\[
\text{res}: \text{TAF}(K_p) \to \text{TAF}(K^p).
\]

2. For an element \(g\) of some larger group \(GU(A^\infty)\) ([BL10]), by functoriality there exists a conjugation action of \(E_\infty\)-rings

\[
g*: \text{TAF}(K_p) \to \text{TAF}(gK_p g^{-1}).
\]

\[\text{We believe the correct statement of [BL10, Corollary 7.3.3] is that } X_{V,L} \text{ is smooth of relative dimension } n - 1 \text{ over } \text{Spf} \mathbb{Z}_p, \text{ as opposed to over } \text{Spec} \mathbb{Z}_p.\]
3. For an inclusion of (compact open) subgroups $K^p \leq K^p$, by formal arguments ([BL10, §10]) there exists a transfer map of in the stable homotopy category $\text{hSp}$

$$\text{tr}: \text{TAF}(K^p) \to \text{TAF}(K^p).$$

4. For two subgroups $K^p, K'^p$ of $GU(A^\infty)$ which we abbreviate to $K, K' \in G$, and an element $g$ representing a double coset in $K \backslash G / K'$, we define a Hecke operator in $\text{hSp}$ as the composite

$$T[g]: \text{TAF}(K') \xrightarrow{g_\ast} \text{TAF}(gK'g^{-1}) \xrightarrow{\text{res}} \text{TAF}(K \cap gK'g^{-1}) \xrightarrow{\text{tr}} \text{TAF}(K).$$
Appendix on formal spectral Deligne–Mumford stacks

Throughout this article we have used basic properties of formal spectral Deligne–Mumford stacks that are not contained in [SAG] (at least not obviously to the author), so we have arranged this appendix to prove these statements. Every single statement below is an extension of a proof in [SAG] from SpDM to fSpDM, and the author claims no originality for the ideas below.

A.1 Completions of augmented $\infty$-rings

To open this appendix, let us prove a statement we used in the proof of Proposition 1.4.2.

**Proposition A.1.1.** Let $\kappa$ be a field, and $C$ be the full $\infty$-subcategory of $(\text{CAlg}_{\text{spdm}}^\kappa)/\kappa$ spanned by objects $\rho_A: A \to \kappa$ such that $A$ is a local Noetherian adic $E_\infty$-ring whose topology is defined by the unique maximal ideal $m_A = \ker(\pi_0(\rho_A))$. Write $C^{\text{Cpl}}$ for the full $\infty$-subcategory of $C$ spanned by those objects $A \to \kappa$ where $A$ is $m_A$-complete. Then the inclusion

$$C^{\text{Cpl}} \hookrightarrow C$$

has a left adjoint, the completion functor

$$C \to C^{\text{Cpl}}; \quad (A \to \kappa) \mapsto (A^{\wedge}_{m_A} \to k).$$

The point here is that completion is a left adjoint for a fixed finitely generated ideal ([SAG, Notation 7.3.1.5]), and when working with augmented algebras with shared residue fields there is a similar result.

**Proof.** First note that for all maps $\phi: A \to B$ in $C$, the assumption that $\phi$ is continuous in redundant. Indeed, if we do not assume $\phi$ is continuous, then the mere fact that $\phi$ is a map of $\kappa$-augmented $E_\infty$-rings, hence we have the containment $\pi_0(\phi(m_A)) \subseteq m_B$, and in particular $\phi$ is local, hence continuous. It then suffices to show that for every pairs of objects $A$ in $C$ and $B$ in $C^{\text{Cpl}}$, the natural map

$$\text{Map}_{\text{CAlg}_{/\kappa}}(A^{\wedge}_{m_A}, B) \to \text{Map}_{\text{CAlg}_{/\kappa}}(A, B), \quad (A.1.2)$$

induced by $A \to A^{\wedge}_{m_A}$, is an equivalences of spaces. This is equivalent to showing that for every map $\phi: A \to B$ inside the codomain above the fibre of (A.1.2) over $\phi$,

$$\text{Map}_{\text{CAlg}_{A/\kappa}}(A^{\wedge}_{m_A}, B),$$

is contractible. As $\pi_0(\phi(m_A)) \subseteq m_B$, we see from the fact that $B$ is $m_B$-complete that $B$ is also $m_A$-complete as an $E_\infty$-$A$-algebra. It then follows that $m_A$-completion is a left-adjoint, as by [SAG, Notation 7.3.1.5] this is true for $A$-modules and the completion functor is strong symmetric monoidal on $C$. For this latter fact, note that both $A$ and $B$ are Noetherian, so

---

\[\text{Note that if no such } \varphi \text{ exists, then (A.1.2) is the unique endomorphism of the empty set.}\]
completions are flat by [SAG Corollary 7.3.6.9], hence one can explicitly inspect that the natural map
\[ A_{\text{m}_A} \otimes_A B \to B_{\phi(m_A)}\]
is an equivalence of \( \text{E}_\infty \)-\( A \)-algebras using the Tor-spectral sequence; see [HA Proposition 7.2.1.19]. This then yields the natural equivalences of spaces
\[ \text{Map}_{\text{CAlg}_{A//\kappa}}(A_{\text{m}_A}, B) \simeq \text{Map}_{\text{CAlg}_{A//\kappa}}(A, B) \simeq \ast. \]

### A.2 Truncations

We would like to show that for locally Noetherian formal spectral Deligne–Mumford stacks, there is a well-defined truncation functor. The following is a generalisation of [SAG Proposition 1.4.6.3] to formal spectral Deligne–Mumford stacks; we will even use the same proof and notation.

**Proposition A.2.1.** Let \( \mathcal{X} = (X, \mathcal{O}_X) \) be a locally Noetherian formal spectral Deligne–Mumford stack. For each \( n \geq 0 \), the object \( \tau_{\leq n}\mathcal{X} = (X, \tau_{\leq n}\mathcal{O}_X) \) is a locally Noetherian formal spectral Deligne–Mumford stack. Moreover, for every \( (\mathcal{Y}, \mathcal{O}_\mathcal{Y}) \) inside \( \infty\text{Top}_{\text{CAlg}}^{\text{Hden}} \) if \( \mathcal{O}_\mathcal{Y} \) is connective and \( n \)-truncated, then the canonical map
\[ \text{Map}_{\infty\text{Top}_{\text{CAlg}}^{\text{Hden}}}(\mathcal{Y}, \mathcal{O}_\mathcal{Y}, \tau_{\leq n}\mathcal{X}) \to \text{Map}_{\infty\text{Top}_{\text{CAlg}}^{\text{Hden}}}(\mathcal{Y}, \mathcal{O}_\mathcal{Y}, \mathcal{X}) \]
is an equivalence.

**Proof.** The first half of the proof of [SAG Proposition 1.4.6.3] works exactly the same as in our case. That is, by copying that proof we see that for every strictly Henselian spectrally ringed \( \infty \)-topos \( (\mathcal{Y}, \mathcal{O}_\mathcal{Y}) \) which is connective and \( n \)-truncated, the canonical map
\[ \text{Map}_{\infty\text{Top}_{\text{CAlg}}^{\text{Hden}}}(\mathcal{Y}, \mathcal{O}_\mathcal{Y}, \tau_{\leq n}\mathcal{X}) \to \text{Map}_{\infty\text{Top}_{\text{CAlg}}^{\text{Hden}}}(\mathcal{Y}, \mathcal{O}_\mathcal{Y}, \mathcal{X}) \]
is an equivalence of spaces. Hence, we are left to show that \( \tau_{\leq n}\mathcal{X} = (X, \tau_{\leq n}\mathcal{O}_X) \) is a locally Noetherian formal spectral Deligne–Mumford stack. By [SAG Proposition 8.1.3.3] and [SAG Proposition 8.4.2.7] these two conditions (being a formal spectral Deligne–Mumford stack and being locally Noetherian, respectively) are local conditions, hence we may assume \( \mathcal{X} = \text{Spf} A \) for a complete Noetherian adic \( \text{E}_\infty \)-ring \( A \). Set \( B = \tau_{\leq n} A \), with the same topology as \( A \) induced by \( I \subseteq \pi_0 A \) using the isomorphism \( \pi_0 A \simeq \pi_0 B \). We claim that \( \text{Spf} B = (\mathcal{X}_{\text{Spf}} B, \mathcal{O}_{\text{Spf}} B) \) is connective and \( n \)-truncated. Indeed, connectivity follows from [SAG Proposition 8.1.1.13] and for truncatedness one can argue as follows: for affine objects \( U \) of \( \mathcal{X}_{\text{Spf}} B \) we have \( \mathcal{O}_{\text{Spf}} B(U) \simeq C_I^\wedge \) for some étale \( B \)-algebra \( C \). As \( C \) is an étale \( \text{E}_\infty \)-\( B \)-algebra, then it is almost of finite presentation, and as \( B \) is Noetherian (as a truncation of the Noetherian \( \text{E}_\infty \)-ring \( A \)), then the spectral Hilbert basis theorem ([HA Proposition 7.2.4.31]) implies that \( C \) is also Noetherian. It then follows from [SAG Corollary 7.3.6.9] that the natural map of \( \text{E}_\infty \)-\( A \)-algebras \( C \to C_I^\wedge \) is flat. As the composition
\[ B \to C \to C_I^\wedge \simeq \mathcal{O}_{\text{Spf}} B(U) \]
is flat, then we see $\mathcal{O}_{\text{Spf}B}(U)$ is $n$-truncated as $B$ is so. The $\infty$-topos $\mathcal{X}_{\text{Spf}B}$ is generated by affine objects under small colimits ([SAG, Proposition 8.1.3.7]) and the structure sheaf $\mathcal{O}_{\text{Spf}B}: X_{\text{Spf}B} \to \text{CAlg}$ preserves limits, so it follows that $\mathcal{O}_{\text{Spf}B}(X)$ is $n$-truncated for all $X \in X_{\text{Spf}B}$. By [SAG, Remark 1.3.2.6], a sheaf of $E_\infty$-rings on an $\infty$-topos is $n$-truncated if all sections are $n$-truncated. It follows that the natural map $\text{Spf}B \to \text{Spf}A$ factors as

$$\text{Spf}B \overset{\phi}{\to} \tau_{\leq n}\mathcal{X} = (\mathcal{X}, \tau_{\leq n}\mathcal{O}_\mathcal{X}) \to (\mathcal{X}, \mathcal{O}_\mathcal{X}) = \mathcal{X}.$$ Using [SAG, Remark 8.1.1.9] we see the map of underlying $\infty$-topoi induced by $\phi: A \to \tau_{\leq n}A = B$ is an equivalence,

$$\text{Shv}_{\pi_0B/I}^{\text{ét}} \simeq \text{Shv}_{A}^{\text{ad}} \xrightarrow{\phi_*} \text{Shv}_{\pi_0A}^{\text{ad}} \simeq \text{Shv}_{\pi_0B/I}^{\text{ét}},$$

where we used the notation of [SAG, Notation 8.1.1.8]. Under this map, we further see the structure sheaf of $\text{Spf}B$ is sent to the functor

$$\phi_*\mathcal{O}_{\text{Spf}B}: \text{CAlg}^\text{et}_A \to \text{CAlg}^\text{cn}_A; \quad D \mapsto (D \otimes_A B)_I^\wedge \simeq (\tau_{\leq n}D)_I^\wedge,$$

where the equivalence above comes from the fact that $A \to B$ is étale, hence flat, and a Tor-spectral sequence calculation; see [HA, Proposition 7.2.1.19]. To see $\phi$ is an equivalence, it therefore suffices to see that (A.2.2) is equivalent to $\tau_{\leq n}\mathcal{O}_{\text{Spf}A}$. This is slight variation on an argument made above. As $D$ is étale over the Noetherian $E_\infty$-ring $A$, then the spectral Hilbert basis theorem implies that $D$ is also Noetherian. It follows straight from the definition that for any integer $n \geq 0$ the $E_\infty$-ring $\tau_{\leq n}D$ is Noetherian, so the natural completion map of $E_\infty$-$A$-algebras

$$\tau_{\leq n}D \to (\tau_{\leq n}D)_I^\wedge$$

is flat. This implies that $(\tau_{\leq n}D)_I^\wedge$ is $n$-truncated. As $\tau_{\leq n}(D_I^\wedge)$ is $I$-complete by [SAG, Corollary 7.3.4.3], there is a natural equivalence of $E_\infty$-$A$-algebras

$$(\tau_{\leq n}D)_I^\wedge \simeq \tau_{\leq n}(D_I^\wedge).$$

Hence $\phi$ is an equivalence of spectrally ringed $\infty$-topoi. □

The following is a formal generalisation of [SAG, Corollary 1.4.6.4]:

**Corollary A.2.3.** For each integer $n \geq 0$, write $\mathbb{fSpDM}_{\text{loc.N}}^{\leq n}$ for the full $\infty$-subcategory of $\mathbb{fSpDM}_{\text{loc.N}}$ spanned by those $n$-truncated locally Noetherian formal spectral Deligne–Mumford stacks. The inclusion $\mathbb{fSpDM}_{\text{loc.N}}^{\leq n} \to \mathbb{fSpDM}_{\text{loc.N}}$ has a right adjoint, given on objects by

$$\mathcal{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X}) \mapsto \tau_{\leq n}\mathcal{X} = (\mathcal{X}, \tau_{\leq n}\mathcal{O}_\mathcal{X}).$$

**Proof.** This follows straight from Proposition [A.2.1] and the observation and truncations of locally Noetherian formal spectral Deligne–Mumford stacks remain locally Noetherian. □

**Corollary A.2.4.** Let $\mathcal{X}$ be a locally Noetherian formal spectral Deligne–Mumford stack. Then for any integer $n \geq 0$ the truncation $\tau_{\leq n}\mathcal{X}$ represents the same functor as $\mathcal{X}$ on $n$-truncated $E_\infty$-rings.

**Proof.** Follows straight from Proposition [A.2.1] as $\text{Spec}R$ is a connective $n$-truncated spectrally ringed $\infty$-topos when $R$ is a connective $n$-truncated $E_\infty$-ring; see [SAG, Example 1.4.6.2]. □

A3
A.3 The fully-faithful embedding $\text{fDM} \to \text{fSpDM}$

To formalise the relationship between classical formal Deligne–Mumford stacks and spectral formal Deligne–Mumford stacks, and for this relationship to behave in a way analogous to the nonformal Deligne–Mumford stack case, we need to construct a functor $\text{fDM} \to \text{fSpDM}$. Let us begin by defining these categories.

**Definition A.3.1.** Let $A$ be a discrete adic Noetherian ring with finitely generated ideal of definition $I \subseteq A$, cutting out a closed subset $V \subseteq |\text{Spec } A|$. We define the ringed topos $\text{Spf } A = (\text{Shv}_{\text{Set}}(\text{CAlg}_{\text{ad}}^\text{ét}), \mathcal{O}_{\text{Spf } A})$ as follows:

1. The topos $\text{Shv}_{\text{Set}}(\text{CAlg}_{\text{ad}}^\text{ét})$ is the full subcategory of $\text{Shv}_{\text{Set}}(\text{CAlg}_{\text{ad}}^\text{ét})$ spanned by those étale sheaves $\mathcal{F}$ such that if the space $V \times |\text{Spec } A| |\text{Spec } B|$ is empty, then $\mathcal{F}(B)$ is a point.

2. One has a sheaf $\mathcal{O}_{\text{Spec } A}$ on $\text{Shv}_{\text{Set}}(\text{CAlg}_{\text{ad}}^\text{ét})$ as in [SAG, Definition 1.2.3.1], which we complete at $I$ to obtain a sheaf $\mathcal{O}$, $\hat{\mathcal{O}}$. This sheaf factors through $\text{Shv}_{\text{Set}}(\text{CAlg}_{\text{ad}}^\text{ét})$ as clearly $\hat{\mathcal{O}}(B) \simeq B^I$ vanishes if whenever the image of $I$ generates the unit ideal of $B$.

We call $\text{Spf } A = (\text{Shv}_{\text{Set}}(\text{CAlg}_{\text{ad}}^\text{ét}), \hat{\mathcal{O}})$ the formal spectrum of $A$, leaving the dependency on the specific topology on $A$ implicit (often $A$ is local with topology defined by its unique maximal ideal). A **locally Noetherian formal Deligne–Mumford stack** is a ringed topos $X = (X, \mathcal{O}_X)$ such that $X$ has a cover $U_\alpha$ such that each ringed topos $X/ U_\alpha$ is equivalent (in the 2-category of ringed topoi of [SAG, Definition 1.2.1.1]) to $\text{Spf } A_\alpha$ for some discrete adic Noetherian ring $A_\alpha$. Write $\text{fDM}$ for the full $(2,1)$-category of $\text{Spf } A$.

The $\infty$-category of formal spectral Deligne–Mumford stacks $\text{fSpDM}$ can be defined similarly; see Definition 0.0.12 or [SAG, Definition 8.1.3.1].

As in [SAG, §8], when dealing with classical formal Deligne–Mumford stacks, we restrict ourselves to the locally Noetherian case in the definition, as opposed to the spectral case, when we only add this assumption when we need it. As mentioned in [SAG, Warning 8.1.0.4], this is due to the incompatibility between completions in the classical and derived worlds.

**Remark A.3.2.** If an adic discrete ring $A$ has a nilpotent ideal of definition, then $\text{Spf } B$ is naturally equivalent to $\text{Spec } B$ straight from the definition above. In this way we can see (Noetherian) affine Deligne–Mumford stacks as affine formal Deligne–Mumford stacks. It then also immediately follows from the definitions that $\text{DM}_{\text{loc},N}$ is a full $(2,1)$-subcategory of $\text{fDM}$.

**Construction A.3.3.** Recall [SAG, Remark 1.4.1.5], which posits the fully-faithful embedding of $\infty$-categories from classical ringed topoi to spectrally ringed $\infty$-topoi

$$\mathcal{I}_\text{op}_{\text{CAlg}^\infty} \hookrightarrow \infty\mathcal{T}_{\text{op}_{\text{CAlg}}} \quad (X, \mathcal{O}_X) \mapsto (\text{Shv}(X), \mathcal{O}).$$

In other words, it associates to a classical Grothendieck topos $X$ the associated $\infty$-topos $\text{Shv}(X)$ (this is done using [HTT, Proposition 6.4.5.7]) and by [SAG, Remark 1.3.5.6], we obtain a connective 0-truncated structure sheaf on $\text{Shv}(X)$, denoted as $\mathcal{O}$ (similar to 0.0.10). In fact, the essential image of the above embedding is spanned by the spectrally ringed $\infty$-topoi $(X, \mathcal{O}_X)$ where $X$ is 1-localic and $\mathcal{O}_X$ is connective and 0-truncated.
It is explained in \[SAG\] Remark 1.4.8.3 that the fully-faithful embedding of the above construction restricts to the full-faithful embedding \(DM \to \text{SpDM}\). Let us show that the same holds for formal Deligne–Mumford stacks.

**Proposition A.3.4.** The functor of Construction \[A.3.3\] when restricted to \(fDM\) factors through \(fSpDM\). Moreover, the essential image of this fully-faithful functor \(DM \to fSpDM\) consists of those locally Noetherian formal spectral Deligne–Mumford stacks \(X = (\mathcal{X}, \mathcal{O}_X)\) for which the \(\infty\)-topos \(\mathcal{X}\) is 1-localic (\([HTT\), Definition 6.4.5.8]) and the structure sheaf \(\mathcal{O}_X\) is 0-truncated.

**Proof.** It is not stated in \[SAG\] Remark 1.4.8.3, but it is clear that the fully-faithful functor of Construction \[A.3.3\] descends to a fully-faithful functor between (not full) subcategories of local topoi,

\[
\mathcal{T}_{\text{Top}}^{\text{loc}} \hookrightarrow \mathcal{T}_{\text{Top}}^{\text{loc}}\text{CAlg}.
\]

Indeed, we say \(X = (\mathcal{X}, \mathcal{O}_X)\) in \(\mathcal{T}_{\text{Top}}^{\text{loc}}\text{CAlg}\) is local if \(\pi_0 \mathcal{O}_X\) is local on \(\mathcal{X}\)\(^\circ\) (\[SAG\], Definition 1.4.2.1), and given \(X_0 = (\mathcal{X}_0, \mathcal{O}_{X_0})\) in \(\mathcal{T}_{\text{Top}}^{\text{loc}}\text{CAlg}\), then the ringed topos \((\mathcal{X}_0, \pi_0 \mathcal{O}_{X_0})\) is naturally equivalent to \(\mathcal{X}_0\) by \([HTT\), Proposition 6.4.5.7].

Let \(X_0 = (\mathcal{X}_0, \mathcal{O}_{X_0})\) be a classical formal Deligne–Mumford stack, and write \(X = (\mathcal{X}, \mathcal{O})\) for the image of \(X_0\) under Construction \[A.3.3\] so \(X = \text{Shv}(\mathcal{X}_0)\). By \[SAG\], Proposition 8.1.3.3], the property of being a formal spectral Deligne–Mumford stack is a local one, so it suffices to show that there exists a cover \(U_\alpha\) of \(\mathcal{X}\) such that each \(\mathcal{X}_\alpha/\mathcal{U}_\alpha\) is in \(fSpDM\). Consider a formal affine cover of \(X_0\) in \(\mathcal{T}_{\text{Top}}^{\text{loc}}\text{CAlg}\), so a collection of \(U_\alpha\) inside \(X_0\) such that \(\coprod U_\alpha \to X_0\) is an effective epimorphism and \((X_0)/U_\alpha\) is equivalent in \(\mathcal{T}_{\text{Top}}^{\text{loc}}\text{CAlg}\) to \(\text{Spf} A\). Considering \(U_\alpha\) as a discrete object \(V\) of \(\mathcal{X}\) (as in \([HTT\), Proposition 6.4.5.7], then \[SAG\], Lemma 1.4.7.7(2)) states that as \(\mathcal{X}\) is 1-localic and \(V\) is 0-truncated in \(\mathcal{X}\), then \(\mathcal{X}/V\) is 1-localic. One then notes the following natural equivalences:

\[
\mathcal{X}/V \simeq \text{Shv}((\mathcal{X}/V)^\circ) \simeq \text{Shv}((X_0)/U_\alpha) \simeq \text{Shv}(\text{Shv}^{\text{ad}}(\text{CAlg}_{\text{et}}^{\alpha})) \simeq \text{Shv}^{\text{ad}}(\text{CAlg}_{\text{et}}^{\alpha}),
\]

the first from the fact that \(\mathcal{X}/V\) is 1-localic, the second by identifying \(X_0\) as the underlying discrete objects of \(\mathcal{X}\) (and then \([HTT\), Remark 7.2.2.17], the third from the choice of \(U_\alpha\) as an affine object of \(X_0\), and the forth from the fact that affine formal spectral Deligne–Mumford stacks are 1-localic; see \[SAG\], Remark 8.1.1.9]. Furthermore, as \(\mathcal{O}\) was defined as the sheaf of connective 0-truncated \(E_\infty\)-rings on \(\mathcal{X}\) associated to the commutative ring object \(\mathcal{O}_{X_0}\) on \(X_0\), we claim that by \[SAG\], Remark 1.3.5.6] the spectrally ringed \(\infty\)-topos \(\mathcal{X}/U_\alpha\) is equivalent to \(\text{Spf} A\). To see this, one notes that \(\mathcal{O}(\text{Spf} B) = B^{\alpha}\) for some \(\text{etale}\) morphism \(\text{Spf} B \to \text{Spf} A\) in \(X_0 \subseteq \mathcal{X}\), and one also has \(\mathcal{O}_{\text{Spf} A}(\text{Spf} B) \simeq B^{\alpha}\) by \[SAG\], Construction 8.1.1.10]. The “moreover” statement follows by \[SAG\], Remark 1.4.1.5].

Combining the functor of points approach with the above, we obtain the following:
Corollary A.3.5. There is the following commuting diagram of $\infty$-categories and fully-faithful functors

\[
\begin{array}{ccc}
\text{Aff}_{\text{loc},N} & \xrightarrow{a} & \text{Aff}_{\text{ad,loc},N} \\
\downarrow{c} & & \downarrow{b} \\
\text{Aff}^\circ & \xrightarrow{f} & \text{Aff}_{\text{ad}}^\circ \\
\downarrow{d} & & \downarrow{g} \\
\text{fSpDM} & \xrightarrow{h} & \mathcal{P}(\text{Aff}^\circ)
\end{array}
\]

Warning A.3.6. One might want to place $\mathcal{P}(\text{Aff}^\circ)$ in the top-right corner of the diagram above, however, we do not see a functor $\mathcal{P}(\text{Aff}^\circ) \to \mathcal{P}(\text{Aff}^\circ)$ such that the diagram above commutes. Indeed, the right Kan extension mentioned in Definition 1.2.30 doesn’t commute with the other constructions above by inspection and a left Kan extension would not necessarily preserve sheaves. In fact, the existence of the functors $c$, $d$, and $e$ above, are all due to nontrivial theorems of Lurie, and the lack of a similar functor $\mathcal{P}(\text{Aff}^\circ) \to \mathcal{P}(\text{Aff}^\circ)$ might indicate why we restrict our attention to (formal) Deligne–Mumford stacks.

Proof of Corollary A.3.5. The functors $a$, $b$, $f$, and $g$ are all the inclusions of full $\infty$-subcategories, $c$ and $d$ are the inclusions of $\infty$-subcategories as shown by Lurie ([HA, Proposition 7.1.3.18]), $e$ is Construction A.3.3 and $h$ is the functor of points functor. The diagram commutes by inspection. To see why each functor is fully-faithful, we have:

- By definition we see that $a$, $b$, $f$, and $g$ are fully-faithful.
- By [HA, Proposition 7.1.3.18] we see $c$ and hence $d$ are fully-faithful.
- Proposition A.3.3 shows $e$ is fully-faithful.
- The fact that $h$ is fully-faithful is the content of [SAG, Theorem 8.1.5.1].

A.4 Finiteness and compactness in $\text{fSpDM}$

Next, we would like to discuss finiteness and compactness conditions on formal spectral Deligne–Mumford stacks.

Proposition A.4.1. Let $\mathcal{X}$ be a locally Noetherian formal spectral Deligne–Mumford stack. Then for any $n \geq 0$ the natural map $\tau_{\leq n} \mathcal{X} \to \mathcal{X}$ induces a $(n+1)$-connective and almost perfect cotangent complex.

Proof. As both of these conditions are local on $\mathcal{X}$, we may take $\mathcal{X} = \text{Spf} A$ for a complete Noetherian adic $E_\infty$-ring $A$ with finitely generated ideal of definition $I \subseteq \pi_0 A$. By the Hilbert basis theorem for connective $E_\infty$-rings ([HA, Proposition 7.2.4.31]) we see $\tau_{\leq n} A$ is an almost finitely presented as an $E_\infty$-$A$-algebra and the cofibre of map $A \to \tau_{\leq n} A$ is $(n+1)$-connective, hence by [HA, Corollary 7.4.3.2] and [HA, Theorem 7.4.3.18] we see $L_1 = L_{\tau_{\leq n} A/A}$ is $(n+1)$-connective and almost perfect inside $\text{Mod}_{\tau_{\leq n} A}$. It follows from [SAG, Proposition 7.3.5.7] that $L_1$ is in fact $I$-complete, hence we have a natural equivalence $L_1 \simeq L_{\text{Spf} \tau_{\leq n} A/\text{Spf} A}$ by [SAG, Definition 17.1.2.8], and we are done. \(\square\)
Definition A.4.2. A formal spectral Deligne–Mumford stack \( X = (\mathcal{X}, \mathcal{O}_X) \) is \emph{quasi-compact} (qc) if the underlying \( \infty \)-topos \( \mathcal{X} \) is quasi-compact, i.e., every cover of \( \mathcal{X} \) has a finite subcover; see [SAG, Definition A.2.0.12]. A morphism of formal spectral Deligne–Mumford stack

\[
f : X = (\mathcal{X}, \mathcal{O}_X) \to Y = (\mathcal{Y}, \mathcal{O}_Y)
\]

is \emph{quasi-compact} if for any quasi-compact object \( U \) of \( Y \), the pullback \( f^*(U) \) is quasi-compact in \( X \). A morphism of formal spectral Deligne–Mumford stacks is called \emph{quasi-separated} (qs) if the diagonal map \( \Delta : Y \to Y \times_{\mathcal{X}} Y \) is quasi-compact. We say \( X \) is qs if \( X \to \text{Spec } \mathcal{S} \) is qs.

Proposition A.4.3. Let \( A \) be an adic \( \mathbb{E}_\infty \)-ring. Then \( \text{Spf } A \) is qc.

Proof. By [SAG, Remark 8.1.1.9] we see the underlying \( \infty \)-topos of \( \text{Spf } A \) is equivalent to \( \text{Shv}_{\pi_0 A/I}^\text{ét} \) where \( I \) is a finitely generated ideal of definition for the topology on \( \pi_0 A \). As this is the same underlying \( \infty \)-topos of \( \text{Spec}(\pi_0 A/I) \), it follows from [SAG, Proposition 2.3.1.2] that \( \text{Spf } A \) is qc.

The following is a formal generalisation of a special case of [SAG, Proposition 2.3.2.1].

Proposition A.4.4. Let \( X = (\mathcal{X}, \mathcal{O}_X) \) be a formal spectral Deligne–Mumford stack. Then the following are equivalent.

1. \( X \) is qs.
2. For all qc objects \( U, V \) of \( X \), the product \( U \times V \) in \( X \) is qc.
3. For all affine objects \( U, V \) of \( X \), the product \( U \times V \) is qc.

Proof. It is clear that 1 implies 2 as \( U \times V = \Delta^*(U, V) \) inside \( \mathcal{X} \times \mathcal{X} \), and 2 also implies 1 as the quasi-compact objects of \( \mathcal{X} \times \mathcal{X} \) are all of the form \( (U, V) \) for \( U \) and \( V \) quasi-compact in \( \mathcal{X} \). Proposition A.4.3 show 2 implies 3. Conversely, for two arbitrary qc objects \( U \) and \( V \) of \( \mathcal{X} \), using the fact they are qc, there exists two effective epimorphisms \( U' \to U \) and \( V' \to V \) where \( U' \) and \( V' \) are affine. It then follows that \( U \times V \) is qc as there is an effective epimorphism

\[
U' \times V' \to U \times V
\]

from a qc object of \( \mathcal{X} \).

Corollary A.4.5. Let \( A \) be an adic \( \mathbb{E}_\infty \)-ring. Then \( \text{Spf } A \) is qcqs.

Proof. By Proposition A.4.3 we see \( \text{Spf } A \) is qc, and by Proposition A.4.4 it suffices to see that for all affine objects \( U = \text{Spf } B \) and \( V = \text{Spf } C \) inside \( \mathcal{X}_{\text{Spf } A} \), that the product \( U \times V \) in \( \mathcal{X}_{\text{Spf } A} \) is qc. This product can be recognised as the fibre product

\[
\text{Spf } B \underset{\text{Spf } A}{\times} \text{Spf } C \simeq \text{Spf } \left(B \otimes_A C \right)^I, 
\]

where \( I \) is an ideal of definition for the topology on \( \pi_0 A \), which is qc by Proposition A.4.3.
The following statement is why we care about the adjectives of Definition A.4.2; see Appendix B for some background on hyperdescent.

**Proposition A.4.6.** Let $\mathcal{X}$ be a formal spectral Deligne–Mumford stack. Then $\mathcal{X}$ is qcqs if and only if there exists an étale hypercover $\mathcal{U}_\bullet$ of $\mathcal{X}$ such that each $\mathcal{U}_n$ is an affine formal spectral Deligne–Mumford stack for every $n \geq 0$. Moreover, the same holds for classical Deligne–Mumford stacks.

We call such étale hypercovers, *finite étale affine hypercovers*.

**Proof.** First, let us assume $\mathcal{X}$ is qcqs and write $\mathcal{X} = (\mathcal{X}, O_\mathcal{X})$ and set $\mathcal{U}_{-1} = \mathcal{X}$. As a formal spectral Deligne–Mumford stack, there exists a collection of affine objects $U_\alpha$ in $\mathcal{X}$ such that $\coprod_{\alpha} U_\alpha$ cover $\mathcal{X}$. As $\mathcal{X}$ is qc, we find a finite subset $I_0 \subseteq I$ such that $\coprod_{I_0} U_\alpha$ still covers $\mathcal{X}$. As $\mathcal{X}/U_\alpha \simeq \text{Spf } A_\alpha$ for some adic $\mathbf{E}_\infty$-ring $A_\alpha$, we see the fact $\coprod_{I_0} U_\alpha$ covers $\mathcal{X}$ is equivalent to the statement that $\text{Spf } A_0 = \text{Spf } \left( \prod_{I_0} A_\alpha \right) \simeq \coprod_{I_0} \text{Spf } A_\alpha \to \mathcal{X}$

is an étale surjection (this of course uses the finiteness of $I_0$). Set $U_0 = \text{Spf } A_0$, then $U_0 \to M_0(\mathcal{U}_{\leq -1}) \simeq \mathcal{U}_{-1} = \mathcal{X}$ is the étale surjection above. Inductively, we suppose we have an $n$-truncated étale hypercover $\mathcal{U}_{\leq n}$ such that $\mathcal{U}_m \simeq \text{Spf } A_m$ for each $0 \leq m \leq n$ and the natural map

$\mathcal{U}_n \to M_n(\mathcal{U}_{\leq n-1})$

is an étale surjection. Let us further suppose that for each $m \leq n$, $M_m(\mathcal{U}_{\leq m-1})$ is qs, the base case $\mathcal{U}_{-1} \simeq \mathcal{X}$ is qs by hypothesis. Noticing that

$M_{n+1}(\mathcal{U}_{\leq n}) \simeq \mathcal{U}_n \times_{M_n(\mathcal{U}_{\leq n-1})} \mathcal{U}_n,$

we use the fact that $M_n(\mathcal{U}_{\leq n-1})$ is qs and Proposition A.4.4 (and [SAG], Proposition 8.1.7.1) to see $\text{fSpDM}$ has finite limits) to determine that $M_{n+1}(\mathcal{U}_{\leq n})$ is a quasi-compact formal spectral Deligne–Mumford stack. Choosing an affine étale surjection $\text{Spf } A_{n+1} \to M_{n+1}(\mathcal{U}_{\leq n})$, we set $\mathcal{U}_{n+1} = \text{Spf } A_{n+1}$. By construction the map $\mathcal{U}_{n+1} \to M_{n+1}(\mathcal{U}_{\leq n})$ is an étale surjection.

Conversely, if we assume that $\mathcal{X}$ has a finite affine étale hypercover $\mathcal{U}_\bullet \to \mathcal{X}$, which we write as $U_\bullet \to 1$ when considered as objects in $\mathcal{X}$. Suppose we have a cover $\{V_\alpha\}_{\alpha \in I}$ of $\mathcal{X}$, so an effective epimorphism $\coprod I V_\alpha \to 1$, then we can consider the Cartesian square inside $\mathcal{X}$ of the form

$$
\begin{array}{ccc}
W & \longrightarrow & U_0 \\
\downarrow & & \downarrow \\
\coprod I V_\alpha & \longrightarrow & 1
\end{array}
$$

All of the maps above are effective epimorphisms either by assumption or as the class of such maps is stable under pullback; see [HTT], Proposition 6.2.3.15. Products commute.
with colimits in an \(\infty\)-topos as colimits in \(\infty\)-topoi are universal\(^{22}\) hence we have a natural equivalence in \(W \simeq \coprod_i W_\alpha\) in \(\mathcal{X}\), where we write \(W_\alpha = V_\alpha \times U_0\). As \(U_0\) is quasi-compact (as an affine object; see Proposition \(\text{A.4.3}\)), we can choose a finite subset of \(I\), say \(I_0\), such that \(\coprod_{I_0} W_\alpha \to U_0\) is an effective epimorphism. We then consider the commutative diagram inside the \(\infty\)-topos \(\mathcal{X}\):

\[
\begin{array}{ccc}
\coprod_{I_0} W_\alpha & \longrightarrow & U_0 \\
\downarrow & & \downarrow \\
\coprod_{I_0} V_\alpha & \longrightarrow & 1
\end{array}
\]

The top and right maps are effective epimorphisms by assumption, and by we then see the bottom map is an effective epimorphism by \([\text{HTT}, \text{Corollary 6.2.3.12}]\), hence \(\mathcal{X}\) is qc. To see \(\mathcal{X}\) is qs, we look at the Cartesian diagram of \(\infty\)-topoi

\[
\begin{array}{ccc}
\mathcal{U}_0 & \xrightarrow{\Delta_{U_0}} & \mathcal{U}_0 \times \mathcal{U}_0 \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{\Delta_X} & \mathcal{X} \times \mathcal{X}
\end{array}
\]

As \(\mathcal{U}_0 \to \mathcal{X}\) is an étale hypercover, it follows the map \(\mathcal{U}_0 \times \mathcal{U}_0 \to \mathcal{X} \times \mathcal{X}\) is an effective epimorphism. As \(\mathcal{U}_0\) is the \(\infty\)-topos of an affine formal Deligne–Mumford stack, then by Proposition \(\text{A.4.5}\) we see \(\mathcal{U}_0\) is qs and the map \(\Delta_{\mathcal{U}_0}\) is qc. It follows from \([\text{SAG}, \text{Corollary A.2.1.5}]\) that \(\Delta_{\mathcal{X}}\) is qc, where we note that in this referenced corollary a qc morphism is called “relatively 0-coherent”. Hence, \(\mathcal{X}\), and therefore \(\mathcal{X}\), is qs.

\[\square\]

Let us now show the formal thickenings of \([\text{SAG}, \S 18.2.2]\) preserve the adjective qcqs.

**Proposition A.4.7.** Let \(\mathcal{X}_0\) be a qcqs formal spectral Deligne–Mumford stack and \(\mathcal{X}_0 \to \mathcal{X}\) is a formal thickening. Then \(\mathcal{X}\) is also qcqs.

**Proof.** As the adjective qcqs is clearly only dependent on the underlying \(\infty\)-topoi, it suffices to prove the following:

**Claim A.4.8.** If \(\mathcal{X}_0 \to \mathcal{X}\) is a formal thickening of formal spectral Deligne–Mumford stacks, then it induces an equivalence of \(\infty\)-topoi.

To see this claim, consider the reduction of a formal spectral Deligne–Mumford stack of \([\text{SAG}, \text{Proposition 8.1.4.4}]\) (which is clearly natural as a colimit). From this one obtains the following commutative diagram of formal spectral Deligne–Mumford stacks:

\[
\begin{array}{ccc}
\mathcal{X}_{0\text{red}} & \longrightarrow & \mathcal{X}_0 \\
\downarrow & & \downarrow \\
\mathcal{X}_{\text{red}} & \longrightarrow & \mathcal{X}
\end{array}
\]

\[^{22}\text{We say that colimits in a presentable }\infty\text{-category }\mathcal{C}\text{ are }\text{universal}\text{ if pullbacks commute with all small colimits; see }[\text{HTT}, \text{Definition 6.1.1.2}].\text{ This holds in an }\infty\text{-topos due to the }\infty\text{-categorical Giraud’s axioms; see }[\text{HTT}, \text{Theorem 6.1.0.6}].\]

A9
We know the natural map from the reduction of a formal spectral Deligne–Mumford stack $X$ back into $X$ is an equivalence of underlying $\infty$-topoi (by [SAG, Proposition 8.1.4.4]), and the underlying $\infty$-topoi of the reduction of a formal thickening is also an equivalence (by [SAG, Proposition 18.2.2.6]). Hence the horizontal and the left vertical maps are equivalences of underlying $\infty$-topoi, hence the right vertical map is as well.

B Hyperdescent and $\mathcal{M}_{\text{BT}^p}^{\text{or}}$

In this appendix, we discuss hyperdescent and show that the moduli functor $\mathcal{M}_{\text{BT}^p}^{\text{or}}$ satisfies fpqc hyperdescent.

B.1 Generalities of hypersheaves

Definition B.1.1 ([SAG, Definition A.5.7.1]). Let $\Delta_{a,+}$ denote the 1-category whose objects are linearly ordered sets of the form $\{n\} = \{0 < 1 < \cdots < n\}$ for $n \geq -1$, and whose morphisms are strictly increasing functions. We will omit the + when we consider the full $\infty$-subcategory with $n \geq 0$. If $\mathcal{T}$ is an $\infty$-category, we will refer to a functor $X_\bullet : \Delta_{a,+}^{\text{op}} \to \mathcal{T}$ as an augmented semisimplicial object of $\mathcal{T}$. When $\mathcal{T}$ admits finite limits, then for each $n \geq 0$, we can associate to $X_\bullet$ (an augmented semisimplicial object) the $n$th matching object $M_n(X_\bullet)$ and its associated matching map

$$X_n \to \lim_{[i] \to [n]} X_i = M_n(X_\bullet),$$

where the limit above is taken over all injective maps $[i] \hookrightarrow [n]$ such that $i < n$. Given a collection of morphisms $S$ inside $\mathcal{T}$, we say an augmented semisimplicial object $X_\bullet$ is an $S$-

hypercover (for $X_{-1} = X$) if all the natural matching maps belongs to $S$, for every $n \geq 0$. Given a Grothendieck topology $\tau$ on $\mathcal{T}$, then a presheaf of spectra $\mathcal{F}$ on $\mathcal{T}$ is called a $\tau$-hypersheaf if for all $\tau$-hypercovers $X_\bullet \to X$, the natural map

$$\mathcal{F}(X) \to \lim_{\Delta_{a,+}^{\text{op}}} \mathcal{F}(X_\bullet)$$

is an equivalence of spectra. Some useful general references for the prefix hyper in the homotopy theory of sheaves are [CM19], [DH104], and [SAG] Appendices A-D.

Our favourite examples will be when $\mathcal{T}$ is the $\infty$-category $\text{Aff}^{\text{cn}}$, $\text{Aff}$, $\mathcal{C}_{A_{10}}$, or $\mathcal{C}_A$, and $S$ is either fpqc or étale surjections. When we discuss these concepts with respect to $E_\infty$-rings, we will implicitly be talking about their opposite categories.

Given $\mathcal{T}$ and $\tau$ from Definition B.1.1, then for each $\tau$-covering family $\{C_i \to C\}$ in $\mathcal{C}$ one can associate a Čech nerve $C_\bullet$, which is a $\tau$-hypercover of $C$. It is then clear that $\tau$-hypersheaves are $\tau$-sheaves. It is also obvious that if $S \subseteq S'$ then $S'$-hypersheaves are $S$-hypersheaves. We find the following diagram of implications useful, and they will often be used implicitly:

$$\begin{array}{ccc}
\text{fpqc hypersheaf} & \longrightarrow & \text{fpqc sheaf} \\
\downarrow & & \downarrow \\
\text{étale hypersheaf} & \longrightarrow & \text{étale sheaf}
\end{array}$$

B1
Remark B.1.2. Write $\mathcal{X}$ for $\text{Shv}_{\text{Sp}}(\mathcal{T})$ where $\mathcal{T}$ is a given site with Grothendieck topology $\tau$. By [CM19] Example 2.5, given an object $F$ of $\mathcal{X}$, then $F$ is a $\tau$-hypersheaf if and only if $F$ is hypercomplete ([CM19] Definition 2.4]) as a sheaf of spectra, which is equivalent to the condition that $\Omega^\infty F$ is hypercomplete as an element of the $\infty$-topos $\text{Shv}(\mathcal{T})$ as defined in [SAG, Definition 1.3.3.4]. As hypercompleteness of sheaves of $\mathbb{E}_\infty$-rings is defined by their underlying sheaves of spectra, then we see the same statement above holds for sheaves of $\mathbb{E}_\infty$-rings. Furthermore, following [SAG, §1.3.3], it is clear that under the equivalence (0.1.0.10) hypercomplete sheaves of spectra (resp. $\mathbb{E}_\infty$-rings) on the $\infty$-topos $\text{Shv}^\tau(\mathcal{T})$ correspond exactly to these hypercomplete $\tau$-sheaves of spectra (resp. $\mathbb{E}_\infty$-rings) on the Grothendieck site $\mathcal{T}$. The reader is referred to [CM19], [HTT, §6.5.3], or [SAG, §1.3.3] for more background.

Let us now state two useful lemma regarding hypersheaves.

**Lemma B.1.3.** Let $\mathcal{T}$ be an $\infty$-category with a Grothendieck topology $\tau$, $F: \mathcal{T}^{\text{op}} \to S$ be a $\tau$-sheaf, and $G$ be an object of the pullback $\infty$-category

$$\text{Shv}_{(\mathcal{C}^{\text{at,}\infty})/\mathcal{T}}(\mathcal{T}) \times \text{Shv}_{\mathcal{C}^{\text{at,}\infty}}(\mathcal{T}) \{F\}.$$ Write $\text{Un}$ for the unstraightening functor of [HTT, §3.2]. Then the presheaf $H: \mathcal{T}^{\text{op}} \to S$ defined by $C \mapsto \text{Un}(G(C): F(C) \to S)$ is a $\tau$-sheaf. Moreover, if $F$ and $G$ are $\tau$-hypersheaves, then $H$ is a $\tau$-hypersheaf.

More informally, this says that applying a Grothendieck construction to a sheaf is a sheaf.

**Proof.** It clearly suffices to show $H': \mathcal{T}^{\text{op}} \to \mathcal{C}^{\text{at,}\infty}$ defined by $C \mapsto \text{Un}(G(C): F(C) \to S)$ is a $\tau$-sheaf, as $(\cdot)^\vee$ preserves all small limits as a right adjoint. Write $\coprod_{\alpha} C_{\alpha} \to C$ for a $\tau$-cover of an object $C$ in $\mathcal{T}$ and $|C_{\alpha}|$ as the geometric realisation of the Čech nerve of this cover. We then note the following composite of natural equivalences is equivalent to the natural map $H'(\{C_{\alpha}\}) \to \lim H'(C_{\alpha})$:

$$H'(\{C_{\alpha}\}) = \text{Un}(G(\{C_{\alpha}\}): F(\{C_{\alpha}\}) \to S) \xrightarrow{\sim} \text{Un}(\lim G(C_{\alpha}): \lim F(C_{\alpha}) \to S) \xrightarrow{\sim} \lim \text{Un}(G(C_{\alpha}): F(C_{\alpha}) \to S) = \lim H'(C_{\alpha}).$$

The first equivalence comes from the fact that $F$ and $G$ are both $\tau$-sheaves second equivalence from the fact that $\text{Un}$ is a right adjoint. The proof for $\tau$-hypersheaves is similar, with $\tau$-covers replaced with $\tau$-hypercovers. □

**Lemma B.1.4** ([SAG, Corollary D.6.3.4 & Theorem D.6.3.5]). The identity functor $\text{CAlg} \to \text{CAlg}$ is a hypercomplete $\text{CAlg}$-valued sheaf on $\text{Aff}$ (with respect to the fpqc topology). In particular, for any $\mathbb{E}_\infty$-ring $R$ and any fpqc hypercover $R^\bullet$ of $R$, the natural map

$$R \xrightarrow{\sim} \lim R^\bullet$$

is an equivalence.

Notice that if $R \to R^\bullet$ is an fpqc hypercover of an $\mathbb{E}_\infty$-ring $R$, then one notes that there are natural equivalences

$$\tau_{\geq 0} R \xrightarrow{\sim} \tau_{\geq 0} \lim R^\bullet \xrightarrow{\sim} \lim \tau_{\geq 0} R^\bullet,$$

from the above lemma and as $\tau_{\geq 0}$ commutes with limits as a right adjoint.

B2
B.2 $\mathcal{M}_{\mathbf{BT}^p}^{\text{fr}}$ is a hypersheaf

**Theorem B.2.1.** Let $R$ be an $\mathbb{E}_\infty$-ring and $n$ a positive integer. Then the functors $\mathcal{M}_{\mathbf{BT}^p}^{\text{fin}}, \mathcal{M}_{\mathbf{BT}^p}^{n}, \mathcal{M}_{\mathbf{BT}^p}^{\text{fr}}, \mathcal{M}_{\mathbf{BT}^p}^{\text{fr}} : \text{CAlg}^p \to \mathcal{S}$ are all fpqc (hence also étale) hypersheaves.

Let us first slightly generalise the result from [EC2, Proposition 3.2.2(5)] that $\mathcal{M}_{\mathbf{BT}^p}$ is an fpqc sheaf.

**Proposition B.2.2.** The functors $\mathcal{M}_{\mathbf{BT}^p}, \mathcal{M}_{\mathbf{BT}^p}^{n} : \text{CAlg}^{\text{cn}} \to \mathcal{S}$ are fpqc hypersheaves.

**Proof.** By Remark 1.3.8 it suffices to prove this for $\mathcal{M}_{\mathbf{BT}^p}$, and our proof (as well as our notation) follows along the lines of the proof of [EC2, Proposition 3.2.2(5)]. For every $\mathbb{E}_\infty$-ring $R$, we write $\text{Mod}^\text{ff}_R$ for the full $\infty$-subcategory of $\text{Mod}_R$ spanned by those finite flat $R$-modules, and define $\text{CAlg}^\text{ff}_R \subseteq \text{CAlg}_R$ similarly. Writing $\text{Ab}_\text{fin}^p$ denote the category of finite abelian $p$-groups, it follows from [EC1, Proposition 6.5.5] that $\mathcal{M}_{\mathbf{BT}^p}(R)$ is equivalent to the full $\infty$-subcategory of $\text{Fun}(\text{Ab}_\text{fin}^p, \text{CAlg}^\text{ff}_R)^\simeq$ spanned by those functors $F$ such that:

1. The functor $F$ preserves finite coproducts.

2. For every monomorphism $M' \to M$ of finite abelian $p$-groups, the induced map $F(M') \to F(M)$ is faithfully flat.

3. For every short exact sequence $0 \to M' \to M \to M'' \to 0$ of finite abelian $p$-groups, the diagram

$$\begin{array}{ccc}
F(M') & \to & F(M) \\
\downarrow & & \downarrow \\
0 & \to & F(M'')
\end{array}$$

is a pushout square in $\text{CAlg}^\text{ff}_R$.

Given a fpqc hypercover $R \to R^\bullet$ inside $\text{CAlg}^{\text{cn}}$. It follows from Lemma B.1.4 that the natural map $\text{Mod}^\text{ff}_R \xrightarrow{\sim} \lim \text{Mod}^\text{ff}_{R^\bullet}$ is an equivalence of symmetric monoidal $\infty$-categories. In particular, this implies that upon taking commutative algebra objects we see the natural map $u : \text{CAlg}^\text{ff}_R \xrightarrow{\sim} \lim \text{CAlg}^\text{ff}_{R^\bullet}$ is an equivalence of $\infty$-categories, hence also induces a natural equivalence $\text{Fun}(\text{Ab}_\text{fin}^p, \text{CAlg}^\text{ff}_R) \xrightarrow{\sim} \lim \text{Fun}(\text{Ab}_\text{fin}^p, \text{CAlg}^\text{ff}_{R^\bullet})$.

To complete the proof, it will suffice to show that a functor $F : \text{Ab}_\text{fin}^p \to \text{CAlg}_R$ satisfies conditions 1, 2, and 3 if and only if for every $n \geq 0$ the composite

$$\text{Ab}_\text{fin}^p \xrightarrow{F} \text{CAlg}_R \xrightarrow{A \mapsto A \otimes_R R^n} \text{CAlg}_{R^n}$$

B3
satisfies conditions 1, 2, and 3. As the functor $A \mapsto A \otimes_R R^n$ above preserves colimits, and due to the fact that $R \to R^n$ is faithfully flat, we see the above composite will satisfy conditions 1, 2, and 3, when $F$ does. Conversely, parts 1 and 3 hold for $F$ as $u$ is an equivalence of $\infty$-categories, and 2 follows from the following diagram of $\mathbf{E}_{\infty}^\mathrm{-}\mathbf{R}$-algebras,

$$
\begin{array}{ccc}
F(M) & \longrightarrow & F(M') \\
\downarrow & & \downarrow \\
F(M) \otimes_R R^n & \longrightarrow & F(M') \otimes_R R^n
\end{array}
$$

as $R \to R^n$ is itself faithfully flat. \qed

Proof of Theorem B.2.7. To see $\mathcal{M}_{\mathbf{B}\mathbf{T}^p}^\mathrm{un}$ is an fpqc hypersheaf, it suffices to see $\mathcal{M}_{\mathbf{B}\mathbf{T}^p}^\mathrm{nc}$ is an fpqc hypersheaf as the inclusion $\text{CAlg}^p \to \text{CAlg}$ sends fpqc hypercovers to fpqc hypercovers. To see $\mathcal{M}_{\mathbf{B}\mathbf{T}^p}^\mathrm{nc}$ is an fpqc hypersheaf, take a (nonconnective) $\mathbf{E}_{\infty}$-ring $R$ and an fpqc hypercover $R \to R^\bullet$. From Lemma B.1.3 and Proposition B.2.2, we see that the natural maps

$$\mathcal{M}_{\mathbf{B}\mathbf{T}^p}(R) = \mathcal{M}_{\mathbf{B}^p}(\tau_{\geq 0}R) \xrightarrow{\simeq} \mathcal{M}_{\mathbf{B}^p}(\lim (\tau_{\geq 0}R)^\bullet) \xrightarrow{\simeq} \lim \mathcal{M}_{\mathbf{B}^p}(\tau_{\geq 0}R)^\bullet = \mathcal{M}_{\mathbf{B}^p}(R^\bullet)
$$

are all equivalences. Hence $\mathcal{M}_{\mathbf{B}\mathbf{T}^p}^\mathrm{nc}$, and also $\mathcal{M}_{\mathbf{B}\mathbf{T}^p}^\mathrm{un}$, are fpqc hypersheaves. It follows that $\mathcal{M}_{\mathbf{B}\mathbf{T}^p}^\mathrm{un}$ is also an fpqc hypersheaf as it is an open subfunctor of $\mathcal{M}_{\mathbf{B}\mathbf{T}^p}^\mathrm{nc}$; see Remark 1.3.8.

By Lemma B.1.3 to show $\mathcal{M}_{\mathbf{B}\mathbf{T}^p}^\mathrm{un}$ is an fpqc hypersheaf, it suffices to see that the functor $\mathcal{M}_{\mathbf{B}\mathbf{T}^p}^\mathrm{un}: \text{CAlg}^p \to \mathcal{S}$ is an fpqc hypersheaf, and that the functor

$$\mathbf{B}^p \text{OrDat}(-): \text{CAlg}^p \to (\mathscr{E}_{\mathrm{at}_{\infty}})_{/\mathcal{S}}, \quad R \mapsto \left(\begin{array}{c}
\mathbf{B}^p(R)^\simeq \\
\mathbf{G}
\end{array} \mapsto \begin{array}{c}
\mathbf{S} \\
\text{OrDat}(\mathbf{G}^o)
\end{array}\right)
$$

is an fpqc hypersheaf. We’ve just seen this for $\mathcal{M}_{\mathbf{B}\mathbf{T}^p}^\mathrm{un}$, so it suffices to see that $\mathbf{B}^p \text{OrDat}(-)$ is an fpqc hypersheaf. Again, let’s write $R \to R^\bullet$ for an fpqc hypercover of $R$ in $\text{CAlg}^p$. As $\mathbf{B}^p(-)^\simeq: \text{CAlg}^n \to \mathcal{S}$ is an fpqc hypersheaf (by Proposition B.2.2), we obtain the following natural equivalence from the definition of $\text{OrDat}(\lim \mathbf{G}^\bullet)$:

$$
\left(\begin{array}{c}
\mathbf{B}^p(\lim \mathbf{R}^\bullet)^\simeq \\
\mathbf{G}
\end{array} \mapsto \begin{array}{c}
\mathbf{S} \\
\text{OrDat}(\mathbf{G}^o)
\end{array}\right) \xrightarrow{\simeq} \left(\begin{array}{c}
\lim \mathbf{B}^p(\mathbf{R}^\bullet)^\simeq \\
\mathbf{G}^\bullet
\end{array} \mapsto \begin{array}{c}
\mathbf{S} \\
\text{OrDat}(\lim \mathbf{G}^o)^\bullet
\end{array}\right).
$$

(B.2.4)

Using the characterising property of the identity component (as seen in [EC2, Theorem 2.0.8]), we take some $A \in \mathcal{E}$ (using the notation of [EC2, Theorem 2.0.8]) and obtain the following sequence of natural equivalences

$$
(\lim \mathbf{G}^\bullet)^o(A) = \text{fib}(\lim (\mathbf{G}^\bullet)(A) \to \lim (\mathbf{G}^\bullet)(A^\mathrm{red})) \simeq \lim \text{fib}(\mathbf{G}^\bullet(A) \to \mathbf{G}^\bullet(A^\mathrm{red})) = (\lim (\mathbf{G}^\bullet)^o(A)) \simeq (\lim \mathbf{G}^\bullet)^o(A).
$$

The first equivalence comes from the fact that fibres commutes with small limits and the second equivalence from the fact that limits in $\text{Fun}(\text{CAlg}^n_{\tau_{\geq 0}R}, \text{Mod}^p_Z)$ are computed levelwise, as in all functor categories. From this we see that (B.2.4) is naturally equivalent to

$$
\left(\begin{array}{c}
\lim \mathbf{B}^p(\mathbf{R}^\bullet)^\simeq \\
\mathbf{G}^\bullet
\end{array} \mapsto \begin{array}{c}
\mathbf{S} \\
\text{OrDat}(\lim (\mathbf{G}^o)^\bullet)
\end{array}\right).
$$

(B.2.5)

B4
where above we take the identity component of $G$ and then consider the diagram of formal groups associated to base-changing this formal group via the given fpqc hypercover. For a fixed pointed formal hyperplane $X$ over an $\mathbb{E}_\infty$-ring $R$, the functor

$$\text{CAlg}_R \to \mathcal{S}, \quad A \mapsto \text{OrDat}(X_A)$$

is representable by [EC2, Proposition 4.3.13], hence it commutes with small limits. In particular, this implies that the expression (B.2.5) is naturally equivalent to

$$\left( \lim_{\rightarrow} BT^p(R^\bullet) \overset{\simeq}{\rightarrow} \mathcal{S} \quad \text{OrDat}(G^\bullet) \mapsto \lim_{\rightarrow} \text{OrDat}(G^\bullet) \right) = \lim_{\rightarrow} BT^p \text{OrDat}(R^\bullet)$$

Combining everything, we obtain our desired natural equivalence

$$BT^p \text{OrDat}(R) \overset{\simeq}{\rightarrow} \lim_{\rightarrow} BT^p \text{OrDat}(R^\bullet).$$

The corresponding statement for $M^\text{or}_{BT^p}$ follows as it is a fibre product of fpqc hypersheaves. $\square$
Please note that all references to work of Jacob Lurie are categorised by title, and will not be found under Lurie. We believe this causes less confusion than the usual referencing system, but the author would also be happy for suggestions to improve this further. Nevertheless, everything can also be found on Lurie's website.

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