On the gauge fixing of 1 Killing field reductions of canonical gravity: the case of asymptotically flat induced 2-geometry

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ABSTRACT We consider 1 spacelike Killing vector field reductions of 4-d vacuum general relativity. We restrict attention to cases in which the manifold of orbits of the Killing field is $R^3$. The reduced Einstein equations are equivalent to those for Lorentzian 3-d gravity coupled to an SO(2,1) nonlinear sigma model on this manifold. We examine the theory in terms of a Hamiltonian formulation obtained via a 2+1 split of the 3-d manifold. We restrict attention to geometries which are asymptotically flat in a 2-d sense defined recently. We attempt to pass to a reduced Hamiltonian description in terms of the true degrees of freedom of the theory via gauge fixing conditions of 2-d conformal flatness and maximal slicing. We explicitly solve the diffeomorphism constraints and relate the Hamiltonian constraint to the prescribed negative curvature equation in $R^2$ studied by mathematicians. We partially address issues of existence and/or uniqueness of solutions to the various elliptic partial differential equations encountered.
I Introduction

To develop technical tools for, as well as to address conceptual questions which arise in, the efforts to build a quantum theory of 4-d gravity, it is of use to study simple models which retain some of the features of the full theory. An interesting model which captures some of the diffeomorphism invariance as well as the nonlinear field theoretic character of full gravity, is the midisuperspace of 1 spacelike Killing vector field (henceforth referred to as 1kvf) reductions of 4-d vacuum general relativity.

We believe that it is essential to study the classical behaviour of a system before trying to discover the underlying quantum theory. In this paper we study some features of the classical theory of 1 kvf reductions of vacuum general relativity.

It is known that the Einstein equations for 4-d spacetimes with 1kvf are equivalent to the equations describing 3-d gravity suitably coupled to 2 scalar fields [1]. The 3-d manifold is the manifold of orbits of the kvf and the scalar fields are related to the norm and twist of the kvf. In the case in which the 3-d spacetime admits a 2+1 split, one can construct a Hamiltonian framework for the system. Moncrief has studied the case in which the 2-manifold in the 2+1 split is compact in [2]. In this paper we try to generalize his work to the noncompact 2 manifold case, when the 2-geometry is asymptotically flat. We use the notion of asymptotic flatness developed in [3]. We try to gauge fix the theory and develop a reduced Hamiltonian description for the true degrees of freedom along the lines of [2]. Apart from the general reasons outlined above, there is a more model specific reason for this work which we outline below.

Recently a Hamiltonian framework for asymptotically flat 2+1 gravity coupled to smooth matter fields was developed in [3]. The generator of time translations at spatial infinity was identified as the energy of the system and it was shown (as in the point particle case [4] which dealt with non smooth matter fields) that this energy was bounded from above. There have been claims [5] that the perturbative quantum theory of 2+1 gravity coupled to scalar fields is renormalizable. A natural question to ask, is whether the boundedness of the energy has anything to do with this good ultraviolet behaviour. The treatment in [3] did not explicitly take into account the

1
upper bound on energy (Also it is not clear to us whether \[5\] dealt with a single scalar field or with an order 1/N expansion).

Hence, we would like to study the classical theory of 1kvf reductions from a 2+1 perspective, in such a way as to bring to the forefront, the boundedness of the Hamiltonian, so that quantization attempts could deal more directly with this issue. So much for motivation.

Let us briefly summarize the results of this paper. We use and slightly extend the Hamiltonian framework developed in \[3\]. Viewing our midisuperpace as 2+1 gravity coupled to matter, we impose the gauge fixing conditions of 2-d conformal flatness (note that we deal only with the case of the 2-manifold being \(\mathbb{R}^2\)) and maximal slicing. This allows us to solve the diffeomorphism constraints and the Hamiltonian constraint becomes an elliptic partial differential equation for the conformal factor, much as in \[2\]. From \[3\], it is apparent that the asymptotic behaviour of the conformal factor determines the true Hamiltonian of the system. The partial differential equation for the conformal factor has been studied by mathematicians (see for eg. \[6\]) and we quote some of their results on existence of solutions. These results have intriguing connections with the upperboundedness of the Hamiltonian. The final picture is of the matter fields (which describe the true degrees of freedom) being evolved in time by the true time independent Hamiltonian of the system.

The one unattractive result of this analysis is that the lapse and shift fields diverge as spatial infinity is approached. One may question as to whether such foliations are acceptable from a spacetime point of view. However, it should be noted that the Hamiltonian framework does admit evolutions generated by such lapses and shifts. We shall say more about this later.

The layout of this paper is as follows. Section 2 is devoted to a slight extension of the Hamiltonian framework of \[3\] and the assertions made above, about the lapse and shift are proved here. In section 3, we describe the equations governing the midisuperspace and display the gauge fixing conditions. In section 4 we deal with the diffeomorphism constraint and in section 5 with the Hamiltonian constraint. Section 6 deals with the propagation of the gauge conditions and the associated elliptic pdes.
for the lapse and shift. Section 7 deals with the reduced Hamiltonian description and also describes a vague idea related to perturbative quantization. Section 8 contains conclusions and open questions.

We shall set $k = c = 1$ in this paper, $k$ being the gravitational constant in 2+1 dimensions.

II The Hamiltonian framework for asymptotically flat 2+1 gravity

We use the same notation as in [3]. Although the main part of the paper deals with the case of the spatial 2 manifold $\Sigma$ being $R^2$, for this section we shall only require that $\Sigma$ be noncompact with ($\Sigma$ - a compact set) diffeomorphic to ($R^2$ - a compact set). We shall use slightly more general boundary conditions than in [3]. Due to the similarity of analysis here and in [3], we shall be brief and only highlight the new results obtained here as a result of choosing more general boundary conditions.

The 3-d spacetime has topology $\Sigma \times R$; the phase space variables are the 2-metric $q_{ab}$ and its conjugate momentum $P^{ab}$. We fix a flat metric $e_{ab}$ in the asymptotic region of $\Sigma$ diffeomorphic to $R^2$. $(r, \theta)$ are its polar coordinates ($0 \leq \theta \leq 2\pi$) and spatial infinity is approached as $r \to \infty$. The asymptotic behaviour of $q_{ab}$ is

$$q_{ab} = r^{-\beta}[e_{ab} + O(1/r^\epsilon)] \quad \epsilon > 0$$

(1)

In [3] we had fixed $\epsilon = 1$, but here we allow $\epsilon$ to be arbitrarily small. Since $P^{ab}$ coordinatizes cotangent vectors to the space of 2-metrics $q_{ab}$,

$$P[\delta q] = \int_\Sigma d^2x P^{ab} \delta q_{ab}$$

(2)

with

$$\delta q_{ab} \sim -\delta\beta(\ln r)r^{-\beta}[e_{ab} + O(1/r^\epsilon)] + r^{-\beta}O(1/r^\epsilon),$$

(3)

should be well defined. This fixes the behaviour of $P^{ab}$ near spatial infinity to be

$$P^{ab}e_{ab} \sim r^{\beta - 2 - \delta} \quad [P^{ab} - 1/2Pq^{ab}] \sim r^{\beta - 2}$$

(4)

where $\delta > 0$ and can be arbitrarily small. (As in [3], $P = P^{ab}q_{ab}$.)
A few useful fall offs induced by those already mentioned above are
\[ \sqrt{q} \sim r^{-\beta} \quad \sqrt{q} R \sim r^{-(2+\epsilon)}, \epsilon > 0 \]  
where \( R \) is the scalar curvature of \( q_{ab} \).

Note that we assume, as in [3], that our matter fields are of compact support.

**II.1 The diffeomorphism constraints**

Given a shift \( N^a \) on \( \Sigma \), the smeared diffeomorphism constraints can be written as:
\[ C_N = -2 \int_\Sigma d^2 x \quad N^a D_c (P^{cd} q_{da}) + \text{manner terms} \]  
With our assumptions on the matter fields, the integral involving matter fields is well-defined and will play no role in the discussion of this section. We will therefore focus just on the gravitational part, i.e., the first term on the right hand side of (6), which we will refer to as \( C_{geo}^N \). It can be verified, following an analysis similar to that in [3], that for \( N^a \sim r^{1-\alpha} \), \( \alpha > 0 \):
(a) \( C_{geo}^N \) is well defined
(b) \( C_{geo}^N \) is functionally differentiable on the phase space.
(c) \( C_{geo}^N \) generates infinitesimal diffeomorphisms on \((q_{ab}, P_{ab})\) which do not take \((q_{ab}, P_{ab})\) out of the phase space i.e. the new \((q_{ab}, P_{ab})\) also respect the asymptotic conditions.

Hence, from a constraint theory viewpoint, these ‘exploding’ diffeomorphisms must be considered as gauge, since they are generated by first class constraints. Thus, the picture is in sharp contrast to 3+1 dimensions where the diffeomorphisms considered as gauge are all trivial at infinity. We see here, that what seems reasonable from a constraint theory viewpoint may not be so from a spacetime viewpoint.

**II.2 The Hamiltonian constraint**

Given a lapse function \( N \) on \( \Sigma \), we can write the smeared constraint function as:
\[ C_N = - \int_\Sigma d^2 x N \left[ \sqrt{g} R - \frac{1}{\sqrt{g}} (P^{ab} P_{ab} - P^2) \right] + \text{matter terms} \]  
\[ = C_{geo}^N + C_{N}^{\text{matter}} \]
Again, matter terms will play no role in our discussion. We note the following with regard to existence of $C_N^{\text{geo}}$:

(a) For $(N \to \text{constant})$ near spatial infinity, the integral is well defined only when $\beta < 2$. For $\beta > 2$, as in [3], the contribution due to the kinetic terms diverges; the potential term (see equation (5)) always falls off faster than $1/r^2$ and poses no problem.

(b) For $\beta < 2$ even if $N \to \ln r$ as $r \to \infty$, $C_N^{\text{geo}}$ is well defined!

Let us now turn to differentiability of $C_N^{\text{geo}}$. The kinetic terms pose no problem. We examine only the potential term. It can be checked that

$$\delta \int_{\Sigma} d^2 x N \sqrt{q}R = \int_{\Sigma} d^2 x \sqrt{q}(-D_a D_b N + D_c D^c N q_{ab}) \delta q^{ab} + \int_{r=\infty} d\theta \sqrt{c}[N v_a + (D_a N)q^{bd} \delta q_{bd} - D_c N \delta q_{ac}] r^a, \quad (9)$$

where,

$$v_a = D^b \delta q_{ab} - D_a (q^{bd} \delta q_{bd}), \quad (10)$$

$r^a$ is the unit normal to the circle at spatial infinity and $\sqrt{c}$ is the determinant of the induced metric, $c_{ab}$, on this circle. The integrals are to be understood in the sense of [3].

Now consider the following asymptotic behaviour for $N$.

(1) If $N \sim 1/r^\eta, \quad \eta > 0$, it is easy to check that the surface term vanishes.

(2) If $N \sim N_\infty + O(1/r^\eta), \quad \eta > 0, N_\infty$ being a constant, as in [3] the surface term does not vanish and we need to add a term $\delta \beta 2\pi N_\infty$ to the constraint functional to make the resulting functional differentiable on phase space. This leads, exactly as in [3], to the identification of $\beta/8$ with the total energy of the system.

(3) If $N \sim \ln r + O(1/r^\eta), \quad \eta > 0$, it is straightforward to verify that the surface term $\text{vanishes}$! This is due to the fact that both $N$ and $q^{ac} \delta q_{ab}$ diverge as $\ln r$ near spatial infinity. Again, this is in sharp contrast to what happens in 3+1 dimensions. Once again a strictly constraint systems viewpoint would identify such motions as gauge, even though it seems strange from a spacetime viewpoint.
The evolution generated by the constraint functionals on $q_{ab}, P^{ab}$ is given below:

$$\dot{q}_{ab} = 2N \frac{1}{\sqrt{q}} (P_{ab} - P q_{ab}) + L_{\vec{N}} q_{ab}$$  \hspace{1cm} (11)$$

$$\dot{P}_{ab} = \sqrt{q} [D^a D^b N - q^{ab} D^c D_c N] + \frac{q_{ab}}{\sqrt{q}} N [P^{ef} P_{ef} - P^2]$$

$$- \frac{2N}{\sqrt{q}} [P^{ac} P^b_c - P P_{ab}] + L_{\vec{N}} P_{ab} + \text{matter terms}$$ \hspace{1cm} (12)$$

Note that for infinitesimal evolutions generated by the (first class) Hamiltonian constraint smeared with

$$N \sim \ln r + O(1/r^n), \quad \eta > 0, \quad r \to \infty,$$ \hspace{1cm} (13)$$

the phase space variables $(q_{ab}(t), P^{ab}(t))$ do respect the asymptotic conditions and and these infinitesimal evolutions must be identified as gauge from a constraint system viewpoint.

For a lapse with asymptotic behaviour $N \to a \ln r + b + O(1/r^n)$, $a, b$ being constants, the above evolution equations are generated by the true Hamiltonian functional

$$H = 2\pi b \beta + \int_{\Sigma} NC d^2 x \approx 2\pi b \beta$$ \hspace{1cm} (14)$$

and again, infinitesimal evolutions preserve the boundary conditions. These evolutions are not identified with gauge. Note that in [3], there was an inconsistency, in that the evolution generated by the true Hamiltonian did not necessarily preserve the boundary conditions on $q_{ab}$. By not fixing $\epsilon = 1$ in equation (1), we have rectified this inconsistency and also shown that the upper bound on the energy still exists.

### III The action and constraints for the 1kvf midisuperspace

The discussion in the remainder of the paper could, in principle apply to arbitrary matter couplings of compact support, but we restrict ourselves to those which come
from the 1kvf reduction of 4-d general relativity (see [2]). We shall, from now on, restrict our considerations to the case where the spatial manifold, \( \Sigma \), is \( \mathbb{R}^2 \). The action \( S \) and the constraints \( C_a \) and \( C \) are

\[
S = \int dt \left[ \int_{\mathbb{R}^2} d^2x \left[ P^{ab} \dot{q}_{ab} + s \dot{\gamma} + v \dot{\omega} - NC - N^a C_a \right] - 2\pi \beta \right] \tag{15}
\]

\[
C_a = -2D_b P^b_a + s \gamma, a + v \omega, a \tag{16}
\]

\[
C = \sqrt{q} \left[ -R + 2q^{ab} \gamma_a \gamma, b + \frac{1}{2} e^{-4\gamma} q^{ab} \omega_a \omega, b \right]
\]

\[
+ \frac{1}{\sqrt{q}} [P_{ab} P^{ab} - P^2 + \frac{1}{8} s^2 + \frac{1}{2} e^{4\gamma} v^2] \tag{17}
\]

Here \( \gamma, \omega \) are the scalar fields obtained as a result of the kvf reduction and \( s, v \) are their conjugate momenta.

As mentioned earlier, all the matter fields and momenta have compact support. The asymptotic behaviours of \((q_{ab}, P^{ab})\) as well as the shift \( N^a \) have been given in section 2. The lapse

\[
N \sim a \ln r + 1 + O(1/r^\eta), \ \eta > 0 \tag{18}
\]

near infinity.

### III.1 The gauge fixing conditions

We follow Moncrief's [2] ideas for gauge fixing by adopting the York procedure [7]. The system has 2-d diffeomorphism invariance as well as time reparameterizations of the type generated by the Hamiltonian constraint. But in contrast to [2], we do have a sense of time at infinity. Thus we do not need to deparameterize the theory as in [2], but can proceed to gauge fix the system completely. We do this by choosing foliations of the 3-d spacetime by maximal slices i.e.

\[
P = 0 \tag{19}
\]

and demanding that the dynamical 2- metric be conformal to a fixed flat metric \( h_{ab} \) (this is reasonable, since the 2-manifold is \( \mathbb{R}^2 \)):

\[
q_{ab} = e^{2\lambda} h_{ab} \tag{20}
\]
As mentioned earlier we do not know the conditions under which 3-d spacetimes admit maximal slices. We now examine how the asymptotic conditions interact with the gauge fixing conditions (19) and (20).

Clearly, \( P = 0 \) is admitted by our boundary conditions. With regard to the conformal flatness condition we impose

\[
  h_{ab} \to e_{ab} \quad r \to \infty
\]

This implies the asymptotic behaviour

\[
  \lambda \to -\frac{\beta}{2} \ln r + O(1/r^{\epsilon}) \quad \epsilon > 0
\]

Note that there is no nonvanishing constant piece in \( \lambda \) near infinity. This will be important for later considerations.

We fix cartesian coordinates \((x^1, x^2)\), associated with \( h_{ab} \) and demand that they agree with those of \( e_{ab} \) near infinity. This will be important for considerations in section 6. We shall work with these coordinates in future calculations. We shall denote by \( i, j, = 1, 2 \) the corresponding cartesian components of tensors. All densities will be evaluated using these cartesian coordinates. Thus, for example, \( \sqrt{h} = 1 \).

We denote the derivative operator compatible with \( h_{ab} \) by \( \partial_a \).

All abstract indices \( a, b, \ldots \) will be raised and lowered by \( q_{ab} \), unless otherwise mentioned.

We decompose \( P^{ab} \) as in [2], in accordance with the York procedure:

\[
  P^{ab} = \frac{1}{2} P_{q_{ab}} + e^{-2\lambda} \sqrt{q} (D^b Y^a + D^a Y^b - q^{ab} D^c Y^c) - B^{ab}
\]

where \( B^{ab} \) denotes the transverse traceless part of \( P^{ab} \).

**Claim:** \( B^{ab} = 0 \)

**Proof:** Since \( B^{ab} \) is a transverse traceless symmetric tensor density of weight 1:

\[
  D_a B^a_b = \partial_a B^a_b = 0
\]

\[
  \Rightarrow \partial_a (B^a_b X^b_i) = 0
\]
where $X^b_i = \left(\frac{\partial}{\partial x^i}\right)^b$ is the translational Killing vector field of $h_{ab}$ in the $i^{th}$ direction. Define

$$B_i^a := B^a_b X^b_i \quad w_{ib} := n_{ab} B_i^a$$

(26)

where $n_{ab}$ is the Levi Civita antisymmetric tensor density of weight -1.

$$\Rightarrow \partial_a w_{ib} - \partial_b w_{ia} = 0 \Rightarrow w_{ia} = \partial_a w_i$$

(27)

where $w_i$ are scalar functions and we have used equation (25) and the fact that the 2-manifold is $\mathbb{R}^2$. Using the tracefree and symmetric properties of $B^{ab}$ we get

$$\partial_1 w_2 = \partial_2 w_1 \quad , \quad \partial_2 w_2 = -\partial_1 w_1$$

(28)

$$\Rightarrow \Delta w_i = 0$$

(29)

where $\Delta$ is the flat space Laplacian operator. From the boundary conditions on $P^{ab}$,

$$B_b^a \sim 1/r^2 \Rightarrow \partial_i w_j \sim 1/r \Rightarrow w_j \sim c_j$$

(30)

where $c_j$ are constants. The associated Dirichlet problem for equation (29) has a unique solution $w_i = c_i$ and hence $B_b^a = 0$

**IV The diffeomorphism constraints**

The diffeomorphism constraints are of the form

$$D_a P_b^a = M_b, \text{ where } M_b = \frac{1}{2} (s \partial_b \gamma + v \partial_b \omega)$$

(31)

There is an integrability condition for this equation as shown below.

Note that since $P_b^a$ is a symmetric traceless tensor density of weight 1,

$$D_a P_b^a = \partial_a P_b^a$$

(32)

Using the same notation as in the proof of the claim above, we have

$$\int_{R^2} (\partial_a P_b^a) X^b_i d^2x = \int_{r=\infty} P_b^a X^b_i f_a r d\theta - \int_{R^2} (\partial_a X^b_i) P_b^a d^2x$$

(33)
where \( \mathbf{r}_a \) is the unit radial normal in the flat metric \( h_{ab} \) to the circle at spatial infinity. The second term on the right hand side above, vanishes since \( X^b_i \) are Killing vectors of \( h_{ab} \). The first term on the right hand side vanishes because \( P^i_j \sim 1/r^2 \) near infinity. Thus, the integrability conditions for the diffeomorphism constraints are

\[
\int_{\mathbb{R}^2} M_i d^2x = 0
\]  

(34)

Using the gauge conditions and \( B^{ab} = 0 \), we substitute the York decomposition (23), into the diffeomorphism constraints and obtain

\[
\Delta(Y^j h_{ij}) = M_i
\]  

(35)

Note that \( P^a_b \) is unchanged if one adds a conformal kvf of \( q_{ab} \) (or equivalently, of \( h_{ab} \)) to \( Y^a \). Since \( P^a_b \sim 1/r^2 \) near infinity and we are not concerned about ambiguities in \( Y^a \) stemming from addition of conformal kvfs of \( h_{ab} \), we look for solutions to (35) such that \( Y^i \to 0 \) as \( r \to \infty \). Again, the associated Dirichlet problem has a unique solution

\[
Y^i h_{ij} = \frac{1}{2\pi} \int_{\mathbb{R}^2} M_i(y) \ln |\mathbf{x} - \mathbf{y}| d^2y
\]  

(36)

where the integrability condition (34), ensures that \( Y^i h_{ij} \sim 1/r \) near infinity and we define \((|\mathbf{x} - \mathbf{y}|)^2 := h_{ij}(x^i - y^i)(x^j - y^j)\).

Thus \( P^i_j \) is uniquely given by

\[
P^i_j = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{d^2y}{|\mathbf{x} - \mathbf{y}|^2} [M_j(y)(x^i - y^i) + M_k(y)(x^l - y^l)h_{lj}h^{ki}]
\]

\[
-\delta^j_i M_k(y)(x^k - y^k)]
\]  

(37)

Here \( h_{ij} = \delta_{ij} \) (the Kronecker delta function) and \( h^{ij} \) denotes the inverse to \( h_{ij} \). It can be checked that the above expression for \( P^i_j \) satisfies the relevant asymptotic behaviour by virtue of the integrability condition (34). Thus, we have solved the diffeomorphism constraints.
V The Hamiltonian constraint

The Hamiltonian constraint becomes an elliptic nonlinear partial differential equation for \( \lambda \)

\[
\Delta \lambda + 2\pi g(x) + f(x)e^{-2\lambda} = 0
\]

(38)

with (note that \( h^{ab} \) below, denotes the flat contravariant metric)

\[
2\pi g(x) = 2h^{ab}(\gamma_a\gamma_b + \frac{e^{-4\gamma}}{2}\omega_{a}\omega_{b}), \quad g(x) \geq 0
\]

(39)

\[
f(x) = P^a_b P^b_a + \frac{1}{8}s^2 + \frac{e^{4\gamma}}{2}v^2 \quad f(x) \geq 0
\]

(40)

We choose to analyze the above equation for the two exhaustive cases of \( f(x) = 0 \) and \( f(x) \) not identically zero.

V.1 \( f(x) = 0 \)

This happens if and only if the matter momenta vanish i.e. \( s = v = 0 \). The equation then reduces to

\[
\Delta \lambda = -2\pi g(x)
\]

(41)

For \( \lambda \) satisfying (22), a solution to this equation is

\[
\lambda_1 = -\int_{R^2} g(y) \ln |\vec{x} - \vec{y}| d^2y
\]

(42)

This has asymptotic behaviour

\[
\lambda \sim -(\int_{R^2} g(y)d^2y) \ln r + O(1/r) \Rightarrow \beta := \beta_1 = 2 \int_{R^2} g(y)d^2y
\]

(43)

Note that for large enough scalar field strengths, \( \beta > 2 \) is possible. Such initial data would not be acceptable. Further, note that (42) is the unique solution to (41) satisfying (22). To see this, let \( \lambda_2 \neq \lambda_1 \) be a solution to (41) with, in obvious notation, \( \beta_2 \) not necessarily equal to \( \beta_1 \). An application of the maximum principle [8] to the Dirichlet problem associated with \( (\lambda_2 - \lambda_1) \) shows that \( \lambda_1 = \lambda_2 \) is the only possible well behaved solution to (41).

Thus the energy, \( \beta_1 \), is completely determined by the Hamiltonian constraint equation.
V.2 \( f(x) \) not identically vanishing

We proceed to transform the equation (38) to a desired form. Let \( \lambda = \lambda_0 + \lambda_1 \) where

\[
\lambda_1 = -\int_{\mathbb{R}^2} g(y) \ln |\vec{x} - \vec{y}| \, d^2 y \quad \Rightarrow \Delta \lambda_1 = -2\pi g(x) \quad (44)
\]

As before

\[
\lambda_1 \sim -(\int_{\mathbb{R}^2} g(y) d^2 y) \ln r + O(1/r) \quad (45)
\]

and we define

\[
\beta_1 := 2\int_{\mathbb{R}^2} g(y) d^2 y \quad \beta_1 \geq 0 \quad (46)
\]

Substituting this into (38), we get

\[
\Delta \lambda_0 + e^{-2\lambda_0} f_1(x) = 0 \quad f_1(x) = f(x)e^{-2\lambda_1} \quad (47)
\]

Finally, let \( u = -\lambda_0 \). Then

\[
\Delta u + K(x)e^{2u} = 0, \quad K(x) = -f_1(x) \quad (48)
\]

This is the desired form of the equation. Thus, we have put

\[
u = -(\lambda + \int_{\mathbb{R}^2} g(y) \ln |\vec{x} - \vec{y}| d^2 y) \quad (49)
\]

\[
K(x) = -f(x) \exp (2\int_{\mathbb{R}^2} g(y) \ln |\vec{x} - \vec{y}| d^2 y) \quad (50)
\]

From (22), (49) and (50) and the asymptotic behaviour of \( P_a^b \) and the fact that the matter fields are of compact support, the asymptotic behaviours of \( u \) and \( K(x) \) are:

\[
u \sim \frac{(\beta - \beta_1)}{2} \ln r + O(1/r^\epsilon) \quad \epsilon > 0 \quad (51)
\]

\[
K(x) \sim 1/r^{l_1 - \beta_1} \text{to nontrivial leading order with } l_1 \geq 4 \quad (52)
\]

Hence, we are looking for solutions to (48) with asymptotic behaviour (51) for \( K \leq 0 \), \( K \) not vanishing identically, with asymptotic behaviour of \( K \) given in (52).

This is exactly the form of the equation for “Prescribed Negative Curvature in \( \mathbb{R}^{2n} \)” examined in [6]. We quote the content of the main theorem of this paper below:
Statement A: If $-Cr^{-l} \leq K \leq 0$ where $C > 0$, $l > 2$ and $K(x_0) < 0$ where $x_0$ denotes a point in $R^2$, then there exist $C^2$ solutions to (48) with

$$u = \alpha \ln |\vec{x} - \vec{x}_0| + u_{\infty} + O(r^\gamma) \quad \text{as } r \to \infty$$

(53)

with $u_{\infty}$ being a constant, for every $\alpha \epsilon (0, \frac{l-2}{2})$ and every $\gamma > \max(-1, 2 - l + 2\alpha)$.

Moreover, if $K \leq -C_0 r^{-l}$ for $r > r_0$ (where $C > C_0 > 0$) and $\alpha \geq \frac{l-2}{2}$, then (48) admits no solution satisfying

$$u = \alpha \ln r + O(1) \quad r \to \infty$$

(54)

Further, from [9], we have

Statement B: If $K \leq 0$ on $R^2$ and $K \leq -\frac{C_0}{r^2}$ for $r \geq r_0$, $(C_0, r_0 > 0)$, then (48) admits no solution.

We make the following remarks:

(1) By integrating both sides of (48) and using Stokes theorem we conclude that solutions to (48) satisfying (51) must have $\beta > \beta_1$. It follows that $\beta = 0$ corresponds uniquely to the 2+1 vacuum case (all matter fields and momenta vanish). It also follows that $\beta > 0$. Note that $K = 0$ slices were used in a similar way to prove positivity of energy by Henneaux in [10].

(2) (a) Given that a solution to (48) exists with a prescribed value of $\beta$, we do not know anything about uniqueness of this solution since we are dealing with unbounded domains. (b) Moreover, we are not interested in prescribed values of $\beta$, instead, we would like $\beta$ to be determined by the equation (48), as in the $f = 0$ case.

(3) The parameter $\alpha$ in statement A, is to be identified with $\frac{\beta - \beta_1}{2}$ and $l$ in statements A and B, with $l_1 - \beta_1$. Then A states that for $l_1 > \beta_1 + 2$,

(i) solutions to (48) exist for every $\beta$ in the open interval $(\beta_1, l_1 - 2)$.
(ii) if $\beta > l_1 - 2$ and the angular dependence of $K$ does not attenuate the leading order term in such a way as to violate the inequality on $K$ in A, then no solutions exist with behaviour (54).

Further, B states that with a similar assumption on angular dependence of $K$, that no solutions exist to (48) for $l_1 \leq \beta_1 + 2$. 

13
Note that $l_1$ is determined, essentially, by the “multipole moments” of the source term $M_i(y)$. Hence a whole range of energies seem to be possible for the same initial data! This is unphysical. The resolution to this is that in (22) there is no constant piece. Note that in the proof of A in [6], the value of $u_\infty$ is not independent of $\alpha$. As a logical consequence of remark (2)(b) we are led to make the following conjecture:

**Conjecture 1:** Among the pairs $(\alpha, u_\infty)$ in A, if a pair $(\alpha_0, 0)$ exists, then it is unique (i.e. $u_\infty = 0$ singles out the particular value of $\alpha = \alpha_0$).

If this conjecture is true, then the equation (48) itself determines the energy $\beta = 2\alpha_0 + \beta_1$. (In this way, by using physical intuition we could make more mathematical conjectures about the equation).

Note that (48) has the following property: If (48) has a solution $u_1(x)$ for $K = K_1(x)$ then $u_2(x) = u_1(x/c)$ is a solution to (48) with $K = \frac{K_1(x/c)}{c^2}$. We do not have a clear idea as to how to use this property, but simply note that $u_{2\infty} = u_{1\infty} - 2\alpha_1 \ln c, \ \alpha_1 = \alpha_2$ in obvious notation.

There is no reason to expect $\alpha_0$ in the conjecture above to be such that $\beta < 2$ and we must discard all initial data which would violate $\beta < 2$. However, note that in the “generic” case, $l_1 = 4$ and A says that $\beta < 2$!

\section*{VI \ Preservation of the gauge fixing under evolution}

Preserving the condition $P = 0$, leads to the following equation for the lapse, just as in [2]:

$$\Delta N = pN \quad p := e^{-2\lambda}[P_a^a P_b^b + \frac{s^2}{8} + \frac{e^{4\gamma}}{2}v^2]$$

Preservation of the conformal flatness condition is equivalent to demanding that $\sqrt{h}h^{ab}$ be preserved and that $\lambda$ be determined by the Hamiltonian constraint at each instant of time (see discussion in [2]). This leads to the following equation for the shift:

$$\sqrt{h}(\partial_a N^b + h^{ch}h_{da}\partial_c N^d - \delta^b_a\partial_c N^c) = NP^{bc}h_{ca}$$
Since transverse traceless tensors vanishing at spatial infinity, vanish everywhere on $R^2$ using an argument similar to that of the lemma in section 3, one can obtain an equation equivalent to the above equation by taking its divergence to get:

$$\Delta N^a = -\partial_b (2Ne^{-2\lambda}h^{bc} P_a^c)$$  \hspace{1cm} (57)

In what follows, we assume the required smoothness on functions so that we can apply results from standard elliptic theory.

VI.1 The equation for the lapse

Near infinity $p \sim 1/r^{4-\beta}$. Since we are interested only in the case $\beta < 2$,

$$p \sim 1/r^{2+\epsilon} \quad \epsilon > 0$$ \hspace{1cm} (58)

where $\epsilon$ can be arbitrarily small.

To keep as much of a spacetime interpretation as possible, we would like $N > 0$. For the case $p = 0$ the only strictly positive solution which satisfies boundary conditions on the lapse from section 2, is $N = a$ where $a$ is a positive constant whose value we can fix to unity.

From now on, we only concentrate on the case in which $p$ does not vanish identically. Assuming $N > 0$ and noting that $p \geq 0$, we integrate both sides of the lapse equation over $R^2$ to get

$$\oint_{r \to \infty} \frac{\partial N}{\partial r} rd\theta = \int_{R^2} Npd^2x \neq 0$$ \hspace{1cm} (59)

Thus $N$ must diverge at least as $\ln r$ as infinity is approached! But fortunately our framework allows such lapses (pun not intended!). Since we expect the evolution to be generated by the true Hamiltonian functional of the system, we look for solutions to the lapse equation with asymptotic behaviour

$$N = a + b \ln r + O(1/r^\epsilon), \quad \epsilon, a, b > 0$$ \hspace{1cm} (60)

Note that the condition $a > 0$, comes from demanding that the “non gauge” part of the evolution be generated in the forward time direction (this will be discussed more,
later). Using the linearity of the equation we can fix $a = 1$. So we look for solutions to the lapse equation with asymptotic behaviour

$$N = 1 + b \ln r + O(1/r^\epsilon), \quad b, \epsilon > 0,$$

(61)

We note the following:

(1) If a solution exists with the required asymptotic behaviour, then by the maximum principle, since it is strictly positive asymptotically, it is strictly positive everywhere.

(2) The solution to the lapse equation satisfying (61) is unique if it exists. We show this in two steps

(i) Let there be 2 distinct solutions $N_1, N_2$ with the same value of $b$. Then an application of the maximum principle to their difference implies uniqueness.

(ii) Let there be 2 solutions $N_1, N_2$ with $b_1 \neq b_2$ in obvious notation. Then a linear combination of $N_1$ and $N_2$ exists (call it $N_3$) which approaches a positive constant near infinity and which is a solution to (55), due to the linearity of the equation. By the maximum principle $N_3 > 0$ everywhere. But we have shown that if $N_3 > 0$ everywhere then $N_3$ must diverge at infinity. Hence $b_1 = b_2$ and by (i), $N_3 = 0$ everywhere.

(3) Using the method of sub and supersolutions we have made partial progress on the question of existence of solutions to the lapse equation. We describe the details in the appendix. The result is that for weak enough matter fields and momenta a solution does exist to (55) with required asymptotic behaviour. For arbitrary initial data, we have been able to show that a solution exists with asymptotic behaviour such that as $r \to \infty$

$$\delta \ln r - a_1 < N < \delta \ln r + a_2, \quad \delta, a_1, a_2 > 0$$

(62)

where $\delta, a_1$ and $a_2$ have values given in the appendix.

### VI.2 The shift equation

Let

$$f^{ba} := -2N e^{-2\lambda} h^{bc} P^a_c \quad f^{ri} = f^{ba}(dr)_b(dx)_a$$

(63)
Note that
\[ f^{ri} \sim 1/r^\epsilon \quad \epsilon = (2 - \beta) > 0 \tag{64} \]

Since \(0 \leq \beta < 2\), \(0 < \epsilon \leq 2\).

We first examine the case in which \(\epsilon < 2\). We define a solution \(u_i^R(x)\) to (57) for \((r = |\vec{x}|) < R\) in a ball of radius \(R\) centred on the origin and denoted by \(B_R(0)\):
\[ 2\pi u_i^R(x) = \left[ \int_{B_R(0)} f^{ri}(\vec{x}) d\theta d\bar{r} - \oint_{r=R} f^{ri}(\vec{x})(\ln \bar{r})\bar{r} d\theta \right] + \int_{B_R(0)} \frac{\partial f^{ki}(\bar{x})}{\partial \bar{x}^k} \ln |\vec{x} - \bar{x}| d^2\bar{x} \tag{65} \]

A straightforward, but lengthy analysis shows:

1. For fixed finite \(r\) and every finite \(R\) such that \(R > 1\) and \(R > (3/2)r\),
\[ |u_i^R(x)| < M(r, R) \]
where \(M\) is a finite number depending on \(R\) and \(r\) (we could try and remove the restriction \(R > (3/2)r\), but since it suffices to deal with such \(R\) in our subsequent arguments, we have not attempted to do so).

2. For every \(r\), there exists an \(R_0\) such that for every \(R > R_0\),
\[ |u_i^R(x)| < M(r, R_0) \]
Moreover, for \(R_0 \to \infty\), \(M(R_0, r) \to f(r)\) such that \(f(r)\) is finite for every finite \(r\) and as \(r \to \infty\), the behaviour of \(f(r)\) to leading order is
\[ f(r) \sim r^{1-\epsilon} + C_\epsilon \quad \text{for } \epsilon \neq 1 \tag{66} \]
\[ \sim (\ln r)^2 \quad \text{for } \epsilon = 1 \tag{67} \]
where \(C_\epsilon\) is a finite constant depending on \(\epsilon\). Also, \(2 > (\epsilon = 2 - \beta) > 0\).

Using standard arguments which involve the construction of appropriate sequences of solutions from the above 1 parameter set and which invoke the Arzela-Ascoli theorem (for this theorem, see, for example [13]), we are guaranteed the existence of a solution to (57) with asymptotic behaviour, at worst, as in (66) and (67). Note that the norm of the shift evaluated with the metric \(q_{ab}\) goes to zero near infinity.
Uniqueness of this solution up to addition of a constant, \( c \), follows from the fact that the Laplace equation in 2 dimensions admits no solutions with sublinear growth near infinity except the constant solutions.

For the case, \( \epsilon = 2 \), the only possible data is that for vacuum 2+1 gravity (see remark 1, section 5.2). In this case \( f^{ab} = 0 \) and the shift equation admits only constant solutions.

The nonuniqueness up to addition of a constant is also present in the compact case. Here, it comes about because we have a freedom to specify the coordinate system on each slice up to a translational isometry of \( h_{ab} \) (note that we do not have a similar ‘rotational’ freedom because rotations are generated by the conserved angular momentum (see [3]) and not by the diffeomorphism constraints). We can fix this nonuniqueness, as in [4] by imposing an appropriate condition on the shift, say that the ‘constant’ part of the shift near infinity vanish.

### VI.3 Discussion

We are forced to use lapses and shifts which diverge at spatial infinity. It is questionable whether we accept such behaviour from a spacetime viewpoint.

In 3+1 dimensions, spacetime intuition and the constraint theory interpretations agree nicely. Evolutions which do not move the spatial manifold with respect to the fixed structure at infinity are interpreted as gauge from the constraint systems point of view and this is natural from a spacetime viewpoint. In the 2+1 dimensional case studied here, we have divergent motions in space-time, of the points of the spatial manifold near spatial infinity. Yet, these motions are interpreted as gauge (if \( a = 0 \) in (60)) from a constraint theory viewpoint. We do not understand, in any deep way, how this comes about. But we emphasize that viewed purely as a constrained dynamical system, the formalism developed is self consistent. We do admit, however, that we do not know if a rigorous treatment of the evolution equations using appropriate function spaces (such as in [11] for the 3+1 case) would show an inconsistency in the framework developed in section 2.

With regard to the permissible behaviour of the lapse, there may be an even more
dramatic clash of spacetime interpretation and constraint theory viewpoint, than that alluded to above. If the lapse equation (55) admits solutions with asymptotic behaviour as in (60) but with $a < 0$, $b > 0$, the “non gauge” part of the evolution is backward in time. But since $b > 0$, the “gauge” part of the evolution dominates this completely and the 2-slices are still pushed forward in time!

We believe that the conformal flatness gauge fixing is the most natural one to impose since, not only is the 2 manifold $R^2$, but in addition, this gauge fixing interacts well with the boundary conditions on the 2-metric, which are also conformally flat (to leading order). The maximal slicing condition simplifies the solving of the constraints and ensures that $K(x) \leq 0$ in (48), so that we can use the results of [6, 9]. It also simplifies part of the problems one faces in the attempt to define a reduced Hamiltonian description (see section 7). Unfortunately, the corresponding spacetime picture is not very appealing, although there seems to be something to be understood here.

VII The reduced Hamiltonian description

To go to a reduced Hamiltonian description, we will assume that:

(i) the Hamiltonian constraint can be solved uniquely for $\lambda$ with $\beta$ determined by (48).

(ii) the lapse equation (55) has a solution with the asymptotic behaviour in (61).

If (i) and (ii) are true, we can eliminate the constraints by expressing the gravitational variables in terms of the matter variables (the latter parameterize the true degrees of freedom of the theory). Using the gauge fixing conditions and eliminating the constraints from the action (15), we obtain the reduced action:

$$S_{\text{red}} = \int dt \left[ \int_{R^2} d^2x \, s \dot{\gamma} + v \dot{\omega} \right] - 2\pi \beta (68)$$

where $\beta$ is to be understood as a functional of the matter fields and momenta. The reduced Hamiltonian is then,

$$H_{\text{red}} = 2\pi \beta[\gamma, \omega, s, v] (69)$$
Note that unlike in [2], this is a time independent Hamiltonian. Thus, given initial data satisfying the integrability condition (34), we have reduced the system (assuming (i) and (ii) above) to that of two matter fields whose time evolution is determined by a non-explicit Hamiltonian. (One of the reasons we worked on $R^2$ is that we could explicitly solve the diffeomorphism constraints.)

This suggests that if we could set up some sort of formal perturbation scheme to solve (38), we could try to use machinery from perturbative quantum field theory to examine the quantum theory. One usually applies perturbative quantum field theory to systems in which one cannot solve the field equations nonperturbatively, but for which the Hamiltonian is known explicitly in terms of the fields. Here one has the additional complication that the Hamiltonian is not explicit but (maybe) can also be written as a formal perturbation series. So given the perturbation expansion of the Hamiltonian in terms of the matter fields, one would expect a more complicated combinatorics in calculating transition amplitudes. The perturbation expansion would have to be supplemented with the condition that the total energy be bounded. How one might do this is an interesting question whose answer may lead to a sensible perturbative quantum field theory. (Note that (34) would also have to be imposed).

Admittedly, these remarks are of a vague and speculative character.

In any event, the work in this paper has lead to a form of the classical theory such that any attempt at quantization must face, head on, the issue of upper boundedness of the energy.

**VIII Conclusions and open issues**

We have worked towards a reduced Hamiltonian description of 1 spacelike Killing vector field reductions of 4 dimensional vacuum general relativity. The reduced Hamiltonian is not explicit, but determined by the solution to (48). This equation has been studied by mathematicians in the guise of the equation for prescribed negative curvature in $R^2$. There is presumably more in the mathematics literature than [6, 9], which could lead to a better understanding of (48). The physical interpretation of
(48) led us to a mathematical conjecture (see section 5) which (hopefully) can be validated in the future.

There is also an open question with regard to existence of solutions to the lapse equation. Since this equation is linear, it is hoped that the question can be answered by workers more mathematically knowledgeable than this author.

Balancing the good news that some properties of (48) are known to mathematicians, is the bad news regarding the spacetime interpretation of this work. It may be that one can choose better gauge conditions (although it has been argued in section 6 that the ones chosen in this paper seem to be the simplest) so that no divergent lapses or shifts are encountered. One possibility is to turn for inspiration to the cylindrical waves analysed in [12].

The cylindrical wave spacetimes may be viewed as 1 (z-directional, translational) kvf reductions with an additional rotational kvf (see discussion in [3]). Thus the system is equivalent to rotationally symmetric 2+1 gravity coupled to a single rotationally symmetric scalar field (in this midisuperpace the twist of the translational kvf vanishes and the single scalar field is related to the norm of the translational kvf). The gauge conditions used in [12] are not the same as those used in this work. Our gauge conditions particularized to the rotationally symmetric case lead to a Hamiltonian constraint which seems to be difficult to solve in closed form. Thus our gauge conditions are not adapted to the rotational symmetry, in contrast to the ones used in [12]. In fact the ones used in [12] permit an exact and complete closed form solution to the equations.

Note, however, that there is an additional subtlety when one compares our work with that in [12]. Because the asymptotic conditions we use are different from those in [12], so are the permitted variations of the various fields in phase space. Thus we use a (subtly) different symplectic structure as compared to [12], the difference coming from what we can and cannot hold fixed at infinity. It would clarify our gauge fixing conditions if maximal slices (in the 2-d sense) could be explicitly constructed in the cylindrical wave spacetimes.

One could also try and generalize the cylindrical wave boundary conditions and
gauge fixings to the general 1 (translational) kvf case. We plan to look at this issue in the future.

Finally, we restricted our considerations to $\Sigma = R^2$ mainly for simplicity and so that we could explicitly solve the diffeomorphism constraints. We hope that some progress can be made towards a quantum theory along the (extremely vague) lines sketched out in section 7.

**Appendix**

**A. Subsolution to the lapse equation**

Given $B > 1$,

$$p(x) \leq \frac{A}{(B + r^\delta)^{2+\frac{2}{3}}}$$  \hspace{1cm} (70)

for suitable $A, \delta > 0$ and $\delta < 1$. Consider the two cases:

(i) $\frac{2A}{B^2\delta^2} \leq 1$: This corresponds to “small” initial data. In this case a subsolution is

$$N_{\text{sub}} = \ln(B + r^\delta) + 1 - e^{-1}$$  \hspace{1cm} (71)

where $e$ is Euler’s constant. Note that the maximum value of $\ln r / r$ for $r > 0$ is $1 - e^{-1}$. Using this it is straightforward to check that

$$\Delta N_{\text{sub}} > q(x)N_{\text{sub}}$$  \hspace{1cm} (72)

as required with

$$N_{\text{sub}} \sim \delta \ln r + 1 - e^{-1}$$  \hspace{1cm} (73)

as infinity is approached.

(ii) $\frac{2A}{B^2\delta^2} > 1$: This corresponds to the generic case. We choose

$$N_{\text{sub}} = \begin{cases} 
\ln(B + r^\delta) - \ln(B + R^\delta) & \text{for } r > R \\
0 & \text{for } r \leq R 
\end{cases}$$  \hspace{1cm} (74)

where $R$ is large enough that for $r > R$,

$$\Delta N_{\text{sub}} = \frac{B\delta^2 r^{\delta - 2}}{2(B + r^\delta)^2} \geq \frac{A}{(B + r^\delta)^{2+\frac{2}{3}}} \ln(B + r^\delta)$$  \hspace{1cm} (76)

Thus asymptotically

$$N_{\text{sub}} \sim \delta \ln r - \ln(B + R^\delta)$$  \hspace{1cm} (77)
B. Supersolution to the lapse equation

We concentrate only on the $p \neq 0$ case. Then there exists a ball of radius $\epsilon$ around a point $x_0$ denoted by $B_\epsilon(x_0)$ such that for $x$ in $B_\epsilon(x_0)$, $p(x) \geq k > 0$. Define $\phi(x)$ to be a smooth function with support on $B_\epsilon(x_0)$, such that $0 \leq \phi < (k/2\pi)$ and $\phi$ does not vanish identically. Define

$$ y_0(x) = \int_{B_\epsilon(x_0)} \phi(\bar{x}) \ln |\bar{x} - \bar{x}| d^2\bar{x} \quad (78) $$

$$ y_1(x) = y_0(x) - \inf_{B_\epsilon+1(x_0)} + 2 \quad (79) $$

Then $\Delta y_1(x) \leq q(x)y_1(x)$ and $y_1(x)$ is a supersolution with asymptotic behaviour

$$ y_1(x) \sim \int_{B_\epsilon(x_0)} \phi(\bar{x}) d^2\bar{x} \ln r + 2 - \inf_{B_\epsilon+1(x_0)} \quad (80) $$

where $\inf_{B_\epsilon+1(x_0)}$ refers to the infimum of $y_0(x)$ in the ball of radius $(\epsilon + 1)$ centred at the point $x_0$. If $y_1(x)$ is a supersolution to the linear lapse equation, then so is $y_2 = y_1 + e, \ d, e > 0$. Hence, for case (i) we can choose $d, e$ in such a way that $N_{sup}$ has asymptotic behaviour

$$ N_{sup} \sim \delta \ln r + a_2 \quad a_2 > 1 - e^{-1} \quad (81) $$

By standard arguments in the method of sub and super solution using the maximum principle and the Ascoli-Arzela theorem [13], we are guaranteed existence of a solution, $N$, to the lapse equation with asymptotic behaviour such that

$$ \delta \ln r + (1 - e^{-1}) \leq N \leq \delta \ln r + a_2 \quad (82) $$

For case (ii), using the same standard arguments, we can show existence of a solution with asymptotic behaviour such that

$$ \delta \ln r - \ln(B + R^2) \leq N \leq \delta \ln r + a_2 \quad (83) $$

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