Condensation and thermalization of an easy-plane ferromagnet in a spinor Bose gas

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Bose–Einstein condensates are an ideal platform to explore dynamical phenomena emerging in the many-body limit, such as the build-up of long-range coherence, superfluidity or spontaneous symmetry breaking. Here we study the thermalization dynamics of an easy-plane ferromagnet employing a homogeneous one-dimensional spinor Bose gas. We demonstrate the dynamic emergence of effective long-range coherence for the spin field and verify spin-superfluidity by experimentally testing Landau’s criterion. We reveal the structure of one massive and two massless emerging modes—a consequence of explicit and spontaneous symmetry breaking, respectively. Our experiments allow us to observe the thermalization of an easy-plane ferromagnetic Bose gas. The relevant momentum-resolved observables are in agreement with a thermal prediction obtained from an underlying microscopic model within the Bogoliubov approximation. Our methods and results are a step towards a quantitative understanding of condensation dynamics in large magnetic spin systems and the study of the role of entanglement and topological excitations for their thermalization.

In recent years, analogue quantum simulators with ultracold atoms have allowed for unprecedented insights by implementing building blocks of complex condensed-matter systems12. This opens up new possibilities for studying pressing questions concerning quantum many-body dynamics and thermalization3–11. When probing these phenomena in macroscopic systems, it is often the case that either the timescales are too short or the control to extract information is not available, such that direct observation of the dynamical processes is not possible.

In Bose–Einstein condensates (BECs)12, the macroscopic occupation of the ground state, together with a spontaneously broken symmetry, manifests itself in a globally well-defined phase of the complex-valued order parameter in each realization. This phase can be probed experimentally by interferometric measurements, as has been demonstrated with different platforms13–15. In an easy-plane ferromagnetic system, the order parameter is characterized by a well-defined magnitude in the transversal plane, and all orientations in the plane are equally likely. Theoretically, this is due to spatial anisotropy, breaking the full rotational SO(3) symmetric part of the Hamiltonian down to a transversal SO(2) symmetry. In condensed-matter physics, prototype models include the XXZ model16, which recently has also been realized with ultracold atoms in lattice systems17 and Rydberg atoms18,19.

We realize a spinor BEC of 87Rb with easy-plane ferromagnetic properties20,21 in a quasi-one-dimensional (1D) box trap22 (Fig. 1a). This consists of three internal states, labelled by their magnetic quantum number $m \in \{0, \pm 1\}$. The system features rotationally invariant ferromagnetic spin–spin interactions described by $\hat{H}_s = c_1 \int dV \hat{F}^2 / 2$, where $c_1 < 0$ is the spin–spin interaction constant and $\hat{F}$ denotes the spin operator (see Methods for details). A quadratic Zeeman shift $q$ induced by the magnetic field plays the role of the isotropy-breaking term; it shifts the energy of the $m = \pm 1$ levels (Fig. 1b) and is explicitly given by $\hat{H}_q = q \int dV (\hat{N}_{+1} + \hat{N}_{-1})$. We adjust $q$ by using off-resonant microwave dressing23 such that the mean-field ground state exhibits easy-plane ferromagnetic properties ($0 < q < 2 n |c_1|$), where $n$ is the atomic density, and our initial conditions restrict the dynamics to the spatially averaged longitudinal ($z$-) spin being zero. In addition to...
Goldstone mode and a Higgs mode related to the excitation of the orientation field, that is, taking nearly continuous values (Fig. 1b), which we identify using spatially resolved joint measurements based on positive operator valued measures (POVMs).

The capability to extract the relevant order-parameter field allows us to study the build-up of effective long-range coherence in a time- and space-resolved fashion; accessing the full structure factor of the observables defining the Hamiltonian is the key to faithfully witnessing thermalization. We experimentally examine the order parameter, which is the transversal spin degree of freedom, by acquiring many realizations of the complex-valued field $F_\perp(y) = F_\perp(y) + iF_\parallel(y)$ using spatially resolved joint measurements based on positive operator valued measures (POVMs). We obtain a value for the transversal spin $F_\perp(y) = |F_\perp| e^{i\phi_\perp}$ with length $|F_\perp|$ and orientation in the plane $\phi_\perp$. The position $y$ along the long axis of the cloud is discretized by our imaging resolution; in each typical imaging volume we infer the spin from an average over ~500 atoms that are described by a spin field, that is, taking nearly continuous values (Fig. 1d), which we identify as the macroscopic order-parameter field describing the spin condensation.

For studying the condensation dynamics, we initialize the system far from equilibrium without well-defined spin length, and with fluctuations solely in the plane. We visualize the emergence of a spin ($F_\perp$) field by evaluating the histogram of $F_\perp$, taking into account all spatial positions and realizations (Fig. 1e).

To test for eventual spin condensation, we characterize the coherence properties of the transversal spin by evaluating first- and second-order coherence functions with $g_1(x, y) \propto \langle F_\perp(x) F_\perp(y) \rangle$ and $g_2(x, y) = \langle F_\perp(x) F_\perp(y) \rangle$, respectively (see Methods for details). In contrast to earlier experiments observing the emergence of long-range coherence in one-component BECs, we do not rely on spatial interference, as we access the relevant spin field directly by joint measurements entailing interference in the internal degrees of freedom. We find that coherence is built up dynamically and the system finally features long-range order, that is non-zero $g_1(x, y)$ over the whole extent of the atomic cloud (Fig. 2a). At the same time, the spin length fluctuations, quantified by $g_2(x, y)$ (Fig. 2b), settle close to unity at zero distance, as expected for a weakly interacting BEC.

To characterize the final state, we first test for superfluidity of the spin as well as the density. In the spirit of Landau, we drag a field by evaluating the histogram of $F_\perp$, taking into account all spatial positions and realizations (Fig. 1e). After 5 s, which corresponds to $\sim 10^4$ times the typical timescale of the spin interaction energy $\hbar/\langle n|\langle \gamma \rangle|\rangle$, the spin is still far from equilibrium and shows large fluctuations in orientation and length. After 30 s ($\sim 60 \times t_c$) of evolution we find that the fluctuations settle around a well-defined spin length $|F_\perp|$ and the phase $\phi_\perp$ becomes well-defined over the whole sample; that is, effective long-range order emerges. This is expected for a thermal state incorporating spontaneous symmetry breaking in the transversal spin degree of freedom and can be intuitively grasped by looking at the underlying Mexican-hat-like free-energy potential (Fig. 1d and ref. ).

![Figure 1](https://example.com/fig1.png)

**Fig. 1** Homogeneous spinor Bose gas and easy-plane ferromagnetic properties. a. We realize a homogeneous spinor BEC of $^{87}$Rb in a box-like trapping potential by a combination of an elongated attractive potential (red) and two repulsive end caps (green; see Methods for details). The total density (grey shading) is flat over the extent of the cloud. b. Level structure of the $F = 1$ hyperfine manifold. We control the offset energy between the $m$-states by microwave dressing (blue shading) such that the system features easy-plane ferromagnetic properties in its ground state. c. The spatial degree of freedom is continuous, but, in the analysis is discretized by the finite pixel size of the camera and the imaging resolution (blue shading) such that the system features easy-plane and two repulsive end caps (green; see Methods for details). The total density far from equilibrium without well-defined spin length, and with fluctuations solely in the plane. We visualize the emergence of a spin ($F_\perp$) field by evaluating the histogram of $F_\perp$, taking into account all spatial positions and realizations (Fig. 1e). After 5 s, which corresponds to $10^4$ times the typical timescale of the spin interaction energy $\hbar/(n|\langle \gamma \rangle|\langle \gamma \rangle|\rangle)$, the spin is still far from equilibrium and shows large fluctuations in orientation and length. After 30 s ($60 \times t_c$) of evolution we find that the fluctuations settle around a well-defined spin length $|F_\perp|$ and the phase $\phi_\perp$ becomes well-defined over the whole sample; that is, effective long-range order emerges. This is expected for a thermal state incorporating spontaneous symmetry breaking in the transversal spin degree of freedom and can be intuitively grasped by looking at the underlying Mexican-hat-like free-energy potential (Fig. 1d and ref. ).

**Fig. 2** Emergence of effective long-range coherence and superfluidity. a. Absolute value of first-order coherence $|g_1(x, y = 0)|$ of transversal spin $F_\perp$; the reference position (y = 0) is chosen at the left edge of the cloud with system size $L = 74 \pm 0.25$ mm. We observe a build-up of effective long-range order; that is, for long times the system features non-zero coherence over its whole size. Inset: 2D coherence function $|g_2(x, y)|$ after 27 s evolution time. For long times we find the correlations to be translation-invariant. b. Second-order coherence of the transversal spin showing the evolution and character of spin-length fluctuations. c. Superfluid properties of the spin condensate: s.d. along the cloud of spin length (purple) and density (grey) for different speeds v of the local perturbation. The rapid increase at finite speed indicates superfluid properties of spin and density. Insets: representative single realizations of the spin length and total density in the different regimes.
well-localized obstacle (see Methods for details) coupling to density and spin through the BEC\(^{39-40}\) and measure the response of the system. We quantify the response by evaluating the mean standard deviation (s.d.) of the total density and the transversal spin length along the cloud. The breakdown of superfluidity is signalled by a rapid increase of the response at a non-zero critical velocity. We find two different critical velocities for spin and density (Fig. 2c). Although the spin shows superfluidity up to \(v \approx 3 \times 10^{-2} \text{ mm s}^{-1}\), the density tolerates a moving barrier for up to ten times faster speeds. This is consistent with the interaction strengths and the corresponding stiffness of the degrees of freedom.

In the following we address the underlying structure in more detail. With two spontaneously broken symmetries—the U(1) symmetry of the total density and the SO(2) symmetry of the spin orientation—we anticipate two Goldstone-like modes with linear dispersions in the infrared. The different energy scales of density and spin interactions are reflected in two associated sound speeds; these are theoretically expected to differ by more than an order of magnitude, which is consistent with the observed critical velocities. Additionally, the symmetry explicitly broken by \(\mathcal{K}_s + \mathcal{K}_q\) leads to a Higgs-like gapped mode (Fig. 3a). Compared to two-component BECs, we find an additional mode due to the increased number of degrees of freedom\(^{41,42}\).

Experimentally, we probe the three different modes by applying local perturbations (Fig. 3 and Methods present details). After the perturbation we observe and analyse the temporal evolution to learn about the underlying structure of the dispersion relations. First, we probe the linear mode associated with the spin orientation by imprinting a spatially varying orientation \(\Phi\) pattern onto the thermalized state. The probing scheme is based on our capabilities to combine global and local radiofrequency spin rotations with fixed relative phases. The initially imprinted Gaussian wavepacket splits into two wavepackets travelling with velocities of \(\pm v_{\text{c}}\), a clear indication for a linear dispersion relation. To access the density degree of freedom we imprint a Gaussian-shaped perturbation. To ease its realization we prepare an elongated spin initial ensemble. To trigger it we prepare an elongated spin initial
interaction and temperature.

independent single-component condensates at corresponding density, $n_c$, of the easy-plane ferromagnetic phase \(46,47\). Interestingly, the single theoretical estimates for the critical temperature for the emergence of the order parameter.

The structure factor (Fig. 4) as well as the local fluctuations (Extended Data Fig. 2) of all observables are consistently described using a thermal prediction. The latter is obtained for the spinor Bose gas with contact interactions within the Bogoliubov approximation using a single temperature for all three quasiparticle modes. We thus conclude that the system has evolved to a thermal state within experimentally accessible timescales. The found temperature is $T = 252 \pm 54 \text{ Hz}$ ($\approx 3 \text{ nK}$) and thus a factor approximately five times smaller than the density–density interaction ($\langle n_{c0} \rangle = 252 \pm 54 \text{ Hz}$) and more than one order of magnitude larger than the spin–spin interaction energy scale ($\langle n_{c1} \rangle = 1.17 \pm 0.25 \text{ Hz}$). The temperature is consistent with theoretical estimates for the critical temperature for the emergence of the easy-plane ferromagnetic phase \(46,47\). Interestingly, the single spin components also feature high fluctuations in comparison with the three independent single-component condensates at corresponding density, interaction and temperature.

We repeat our measurement close to the phase boundary at $q = 0$ and find structures beyond thermal Bogoliubov theory in that case (Extended Data Fig. 3). The observed enhanced fluctuations can be associated with long-lived nonlinear excitations of the spin superfluid, which are energetically less suppressed for lower $q$. These excitations can delay equilibration, as previously discussed for polar core vortices in two dimensions \(48\).

In conclusion, the high degree of control allows us to experimentally observe the thermalization process of an easy-plane ferromagnet. This sets the foundations for studies in quantum field settings addressing the microscopic processes for thermalization, as well as its absence due, for example, to long-lived topological defects. The robust generation of a spin superfluid is a prerequisite for spin Josephson junctions where finite temperature effects and spin–density separation can now be studied on a new quantitative level due to the direct access to the order parameter.

Online content
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**Methods**

**Experimental details**

We prepare a spinor BEC of $^{87}$Rb in a quasi-1D trapping geometry. Details concerning the preparation and readout of the transversal spin are available in refs. [24,25,31,50].

Here we employ a box-like trapping potential. We use a weakly focused red-detuned laser beam creating a quasi-1D trapping potential with $\omega_x = 2 \pi \times 1.7 \text{ Hz}$ and $\omega_z = 2 \pi \times 170 \text{ Hz}$, corresponding to a transversal harmonic oscillator length of $\sqrt{\hbar/M\omega_x} \approx 0.8 \mu\text{m}$, with $M$ the atomic mass. Repulsive potential walls are created by two blue-detuned laser beams, resulting in a trapping volume of adjustable size around the centre of the harmonic trap. The longitudinal harmonic potential is in good approximation constant over the employed sizes and thus effectively leads to a 1D box-like confinement for the atomic cloud.

For the measurements of the thermalized state shown in Fig. 4 we utilize a box with dimensions of ~100 $\mu\text{m}$. The position of the walls fluctuates ~0.4 $\mu\text{m}$ from realization to realization. The final characterized state has 65,000 atoms (initially starting with 160,000 atoms).

**Initial conditions.** For detailed observation of the emergence of coherence we prepare the atoms in the state $|F, m\rangle = |1, 0\rangle$, the so-called polar state. To allow for thermalization for shorter times, we initially prepare a coherent spin state with maximal length. For this we apply a $\pi/2$-rf rotation with the atoms initially prepared in the state $|F, m\rangle = |1, -1\rangle$. As a reference noise level (grey diamonds in Fig. 4 and Extended Data Fig. 3) for the thermalized structure factor, we prepare this coherent spin state by performing the rotation after holding the atoms in $|1, -1\rangle$ for 30 $\text{s}$.

**Readout.** After evolution time $t$, we image the atomic densities using spatially resolved absorption imaging. Employing a Stern–Gerlach magnetic field gradient followed by a short TOF (2 $\text{ms}$), we are able to image the atomic densities of all eight magnetic sublevels of the electronic ground state. Additional coherent microwave and radiofrequency manipulations before the imaging allow us to map the two spin projections, $F_x$ and $F_y$, of the transversal spin onto measurable densities. With this we infer the complex-valued transversal spin $F_y(y) = F_y(y) + iF_x(y) = |F_y(y)|e^{-i\phi_y(y)}$ as a function of position $y$. The position $y$ is the centre of a spatial bin that contains ~500 atoms and has a spatial extension of ~1.2 $\mu\text{m}$ along the cloud (we bin three adjacent camera pixels where each pixel corresponds to 420 nm in the atom plane).

For the measurement of the density fluctuations $|N_{\text{out}}|^2(k)$ we take in situ images without spin resolution (without Stern–Gerlach separation). This is important, because any free propagation will transform phase fluctuations to density fluctuations, leading to a strongly enhanced structure factor [24]. It is important to note that the observed increase (by a factor of 2) in the fluctuations compared to the spin coherent state can be a result of only one particle per $k$-mode. For the spin observables and the single densities we checked that the enhanced fluctuations due to the TOF are negligible.

**Local perturbation**

To access the superfluid properties of the spin and density degrees of freedom we use a localized perturbation (root mean square (r.m.s.) width of ~5 $\mu\text{m}$) that we drag through the thermalized system. Specifically, we use a blue-detuned, steerable laser beam (760 nm), the position of which is controlled by an acousto–optical deflector. Using a linear frequency ramp we implement a sweep over the cloud with fixed velocity, which we change over two orders of magnitude. The density is probed after 35 $\text{s}$ and the spin after 20 $\text{s}$ of evolution time. The ramp duration for the lowest speed is ~18 $\text{s}$.

**Local perturbation of the Larmor phase.** We use a combination of global and local radiofrequency rotations (see ref. [32] for details on local radiofrequency rotations). A first global $\pi/2$-rf rotation around the $x$ axis maps the $z$ axis onto the $y$ axis. Using a local rotation with a well-defined phase with respect to the global rotation, we perform a rotation with variable angle around the $y$ axis. Because of the performed mapping, this effectively leads to rotation around the $z$ axis in the original coordinate system. At time $\Delta t = 210 \mu\text{s}$ after the first global radiofrequency pulse, we apply a global radiofrequency $\pi$-pulse followed by a second global radiofrequency $\pi/2$-pulse after another time delay of $\Delta t$, where all pulses rotate around the same axis. This constitutes a spin echo sequence that additionally executes a full 2 $\pi$ spin rotation, which ensures that the global rotation pulses do not excite the system. The last $\pi/2$-pulse maps the local rotation axis back to the $z$ axis in the original system. The perturbation has an approximate Gaussian shape with an r.m.s. width of ~5 $\mu\text{m}$ according to the shape of the used laser beam.

**Local perturbation of the total density.** We reduce the total density locally by ~5% by shinning a blue-detuned laser beam (760 nm) onto the centre of the cloud. We adiabatically ramp up the potential in 100 $\mu\text{s}$ such that we get no further excitations in the density. After the ramp, the potential is instantaneously switched off to generate the wavepacket.

**Local perturbation of the transversal spin length $|F_y\rangle$.** We induce a local density reduction by applying the same blue-detuned laser beam. During the evolution time of 30 $\text{s}$ we let the system thermalize subject to the local density reduction. This effectively leads to a spatially dependent mean-field ground-state spin length. We linearly ramp down the potential over 50 $\text{ms}$. This implements an adiabatic ramp for the total density and a rapid switch off for the spin.

For experimentally accessing the gap we excite the $k = 0$ mode of the spin length by changing the phase of the $m = 0$ component (spiner phase) globally. For this we use two microwave $\pi$-pulses between $|1, 0\rangle$ and $|2, 0\rangle$, where the second pulse is phase-shifted by $\Delta\phi$. We record the subsequent oscillations of the $m = 0$ population and fit a sinusoidal function to extract the frequency. The theoretical prediction for the gap, $\Delta$, deduced from the oscillation, and the $m = 0$ ground-state population, $n_{0\text{g}}$, in the easy-plane ferromagnetic phase is given by [31]

$$
\Delta = \sqrt{4n^2c_1^2 - q^2} \text{ and } n_{0\text{g}} = \frac{1}{2} - \frac{q}{4nc_1},
$$

(1)

Assuming $n_{1\text{g}} = n_{-1\text{g}}$, these formulae also hold true for $0 < q < 2nc_1$ (dashed lines, Extended Data Fig. 1).

**Experimental coherence functions and structure factor**

In every experimental realization $(i)$ we measure atomic densities from which we infer single-shot realizations $O^{(i)}$ of different observables $O$. The quantum expectation value is approximated by averaging over many realizations as

$$
O = \langle \hat{O} \rangle = \frac{1}{N_i} \sum_{i=1}^{N_i} O^{(i)},
$$

(2)

where $N_i$ is the number of realizations.

The coherence functions of the transversal spin are explicitly given by

$$
g_1(x, y) = \frac{\langle \hat{F}_y^\dagger(x)\hat{F}_y(y) \rangle}{\sqrt{\left(\langle \hat{F}_y^\dagger(x)\hat{F}_y(x) \rangle\langle \hat{F}_y^\dagger(y)\hat{F}_y(y) \rangle\right)}}
$$

(3)

and

$$
g_2(x, y) = \frac{\langle \hat{F}_y^\dagger(x)\hat{F}_x^\dagger(y)\hat{F}_x(x)\hat{F}_y(y) \rangle}{\langle \hat{F}_y^\dagger(y)\hat{F}_y(y) \rangle \langle \hat{F}_x^\dagger(x)\hat{F}_x(x) \rangle}.
$$

(4)
For the inferred single-shot results of the transversal spin $F_\perp(x)$, the $^1$ is
treated as the complex conjugate.

The structure factors as a function of the spatial momentum $k$
are defined as

$$
\langle \hat{O}(k) \rangle^2 = \langle \hat{O}(k) \rangle = \frac{1}{N_{\text{tot}}} \sum_{\xi} \sum_{m} \text{DFT}_{x=\xi} (\hat{O}(0) - \langle \hat{O}(0) \rangle)^2,
$$

(5)

where DFT$_{x=\xi}$ is the discrete Fourier transform, and $k = 1/\lambda$ is the spatial
momentum. All structure factors are normalized by the mean total atom number $N_{\text{tot}}$ to
obtain an atom-number-independent measure for the fluctuations and allow comparison between theory and experiment.

For the total density structure factor, a value of one corresponds to the
atomic shot-noise level.

**Bogoliubov transformations in the easy-plane ferromagnetic phase**

We explicitly derive the Bogoliubov transformations in the easy-plane ferromagnetic phase ($0 < q/(n|c_j|) < 2$). Here we set $\hbar = 1$.

In terms of total density and spin operators

$$
N(x) = \sum_{m=1}^{N} \hat{n}_m(x) = \sum_{m} \hat{\psi}^\dagger_m(x) \hat{\psi}_m(x),
$$

$$
F_{\perp}(x) = \sum_{m,m'\perp} \hat{\psi}^\dagger_m(x) f^\perp m_{m'\perp} \hat{\psi}_m(x)
$$

with the spin-1 matrices

$$
\begin{align*}
 f^x &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & f^y &= \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, & f^z &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\end{align*}
$$

(7)

the system Hamiltonian reads

$$
\mathcal{H} = \int d^3x \sum_{m} \hat{\psi}^\dagger_m(x) \left( \frac{\nabla^2}{2M} + q m^2 \right) \hat{\psi}_m(x) + \frac{\epsilon_0}{2} N(x) + \sum_{m,m'\perp} \hat{F}_{\perp}^\dagger m_{m'\perp} \hat{F}_{\perp} m_{m'\perp}
$$

(8)

With momentum-space creation and annihilation operators

$$
\hat{a}_{k_m}^\dagger = \frac{1}{\sqrt{V}} \int d^3x \hat{\psi}^\dagger_m(x) e^{ik\cdot x}, \quad \hat{a}_{k_m} = \frac{1}{\sqrt{V}} \int d^3x \hat{\psi}_m(x) e^{-ik\cdot x},
$$

(9)

the Hamiltonian in the number-conserving Bogoliubov approximation becomes\(^{22}\)

$$
\mathcal{H}_{\text{B}} = E_0 + \sum_{k_{m}\perp} \left( \epsilon_k + qm^2 - \mu \right) \hat{a}_{k_m}^\dagger \hat{a}_{k_m} + \sum_{j,j',m,m'\perp} \Gamma_{j',j,m,m'} \hat{c}_{j_m}^\dagger \hat{c}_{j_{m'}} \hat{a}_{k_{m'}}^\dagger \hat{a}_{k_m}
$$

(10)

$$
+ \sum_{j,j',m,m'\perp} \frac{\epsilon_j - \epsilon_{j'}}{2} \left( \hat{c}_{j_m}^\dagger \hat{c}_{j_{m'}}^\dagger \hat{a}_{k_{m'}}^\dagger \hat{a}_{k_m} + \hat{c}_{j_m} \hat{c}_{j_{m'}} \hat{a}_{k_{m'}} \hat{a}_{k_m} \right)
$$

with $\epsilon_k = k^2/(2M)$, atom mass $M$, total atom number $N$ and $\hat{a}_{k_m}^\dagger = \hat{a}_{k_m}^\dagger \hat{a}_{k_m}$. The spinor $\left( c_{j_m} \right)$ specifies the normalized condensate configuration and we set $\hat{a}_{k_{m}=0} = \sqrt{N_{\text{cond}}}$. $\Gamma_{j',j,m,m'}$ denotes density and spin interactions

$$
\Gamma_{j',j,m,m'} \equiv \epsilon_j \delta_{j',j} \delta_{m,m'} + c_1 \sum_{m\perp} \left( \hat{f}_{j'm}_{m'\perp} \right). \quad \text{with} \quad \hat{f}_{j'm}_{m'\perp} \equiv \hat{c}_{j_m}^\dagger \hat{c}_{j_{m'}}^\dagger \hat{a}_{k_{m'}}^\dagger \hat{a}_{k_m}.
$$

(11)

The ground state energy is given by

$$
E_0 \equiv N \sum_{m} \nabla m^2 \left( \epsilon_k + \frac{N}{2V} \sum_{j,j',m,m'} \Gamma_{j',j,m,m'} \hat{c}_{j_m} \hat{c}_{j_{m'}} \hat{a}_{k_{m'}} \hat{a}_{k_m} \right)
$$

(12)

the chemical potential reads

$$
\mu \equiv \sum_{m} \nabla m^2 \left( \epsilon_k + \frac{2N}{2V} \sum_{j,j',m,m'} \Gamma_{j',j,m,m'} \hat{c}_{j_m} \hat{c}_{j_{m'}} \hat{a}_{k_{m'}} \hat{a}_{k_m} \right)
$$

(13)

We set $\sin \theta = \sqrt{1 - q/(4n|c_j|)}$, such that the mean-field ground state reads $\xi = (\sin \theta \sqrt{2}, \cos \theta, \sin \theta \sqrt{2})$ (ref. 11). The initial orthogonal transformation, given by

$$
\begin{pmatrix}
\hat{a}_{k,1} \\
\hat{a}_{k,0} \\
\hat{a}_{k,-1}
\end{pmatrix} = \begin{pmatrix}
\sin \theta \sqrt{2} & \sin \theta \sqrt{2} & \cos \theta \\
\cos \theta \sqrt{2} & -\cos \theta \sqrt{2} & \sin \theta \\
1 & 0 & -1 \end{pmatrix} \begin{pmatrix}
\hat{a}_{k,1} \zeta_{k} \\
\hat{a}_{k,0} \zeta_{k} \\
\hat{a}_{k,-1} \zeta_{k}
\end{pmatrix} \equiv A(\theta) \begin{pmatrix}
\hat{a}_{k,1} \zeta_{k} \\
\hat{a}_{k,0} \zeta_{k} \\
\hat{a}_{k,-1} \zeta_{k}
\end{pmatrix}
$$

(14)

leads to a description of the system in terms of longitudinal and transversal spin fluctuations.

The longitudinal ($z$) spin fluctuations can be diagonalized using the Bogoliubov transformation\(^{25}\)

$$
\hat{b}_{k,m} = u_{k,m} \hat{a}_{k,m} + v_{k,m} \hat{a}_{k,m}^\dagger,
$$

(15)

where

$$
u_{k,m} \equiv \sqrt{\epsilon_k + q/2 + E_{k,m} \hbar^2}, \quad \mu_{k,m} \equiv \sqrt{\epsilon_k + q/2 - E_{k,m} \hbar^2}
$$

(16)

with the dispersion

$$
E_{k,m} = \sqrt{\epsilon_k^2 + n(c_0 - c_1) \epsilon_k + 2n^2c_1(c_1 - c_q) \pm E_{k}}.
$$

(17)

To diagonalize transversal spin fluctuations we follow the diagonalization procedure outlined in ref. 12. We obtain mode energies $\pm E_{k}$, and $\pm E_{k}$ as in ref. 23, explicitly given by

$$
E_{k,m} = \sqrt{\epsilon_k^2 + n(c_0 - c_1) \epsilon_k + 2n^2c_1(c_1 - c_q) \pm E_{k}}.
$$

(18)

with

$$
E_{k} = \left[ \left( n^2(c_0 + 3c_1)^2 + 4n^2c_q(c_0 + 2c_1) \right) \epsilon_k^2 - 4n^2c_1(c_0 + 3c_1)(c_1 - c_q) \epsilon_k 
\right]^{1/2}.
$$

(19)

Defining

$$
h_{00} \equiv n(c_0 + c_1 - c_q), \quad h_{01} \equiv q \sin(2\theta)/2,
$$

(20)

$$
h_{11} \equiv -2n c_q, \quad h_{21} \equiv n c_q
$$

and

$$
u_{k,m} = \left( \frac{2E_{k} + 4\epsilon_k^2 + 2\epsilon_k (2E_{k} + h_{00} + 2h_{11} - h_{21}) + (h_{11} - h_{21}) (2E_{k} + h_{11} + h_{21}))}{(4\epsilon_k^2 + 2\epsilon_k (h_{11} - h_{00}) h_{21} + h_{21} (2E_{k} + h_{11} - h_{21}))} \right)^{1/2}
$$

(21)
we find Bogoliubov transformation matrices (in the parametrization of ref.13)

\[
U_{k,00} = \begin{pmatrix}
        \langle \hat{a}_{k,0} | \hat{a}_{k,0} \rangle & 0 \\
        0 & \langle \hat{a}_{k,0} | \hat{a}_{k,0} \rangle
\end{pmatrix} A(\theta),
\]

\[
V_{k,00} = \begin{pmatrix}
        \langle \hat{a}_{k,0} | \hat{a}_{k,0} \rangle & 0 \\
        0 & \langle \hat{a}_{k,0} | \hat{a}_{k,0} \rangle
\end{pmatrix} A(\theta).
\]

These fulfill the identities

\[
U_{k,00}U_{k,00}^\dagger - V_{k,00}V_{k,00}^\dagger = 1,
\]

\[
U_{k,00}^\dagger U_{k,00} = V_{k,00}^\dagger V_{k,00} = 0,
\]

as required for the transformations to preserve canonical commutation relations. This requirement is not fulfilled for the transformations given in ref.13.

The complete transformation matrices diagonalizing the Bogoliubov Hamiltonian (10) read

\[
U_k = \begin{pmatrix}
        U_{k,00} & 0 \\
        0 & U_{k,0}\end{pmatrix} A(\theta),
\]

\[
V_k = \begin{pmatrix}
        V_{k,00} & 0 \\
        0 & V_{k,0}\end{pmatrix} A(\theta).
\]

**Thermal structure factors from Bogoliubov theory**

We provide analytical computations of thermal structure factors in the spinor Bose gas from Bogoliubov theory. We are interested in correlators of the form

\[
\langle \hat{C}^\dagger (x) \hat{C}(y) \rangle_{\beta,s}
\]

for a composite field \(\hat{C}(x)\) given by

\[
\hat{C}(x) = \sum_{m,m'=-1}^1 \psi_{m'}^\dagger(x) c_{mm'} \psi_m(x)
\]

with \(c_{mm'}\) a 3 \times 3 matrix corresponding to the type of spectrum under investigation; \(\hat{C} = \hat{c}^\dagger \hat{V}^\dagger \hat{c} \hat{V} \) leads to the transversal magnetization spectrum, \(\hat{c} = \hat{c}^\dagger = \hat{V} \) describes the spectrum of magnetization in the z direction, \(\hat{V} = \text{diag}(1, 1, 1)\) describes the total density spectrum. In equation (29) \(\langle \hat{C}^\dagger \hat{C} \rangle_{\beta,s}\) indicates the thermal expectation value at inverse temperature \(\beta = 1/(k_B T)\) (\(k_B\) is the Boltzmann constant) with symmetrically (Weyl-) ordered arguments. We compare with symmetrically ordered predictions, because expectation values of experimental observables are inferred from realizations of observables \(\hat{C}^\dagger \hat{C}\) given by polynomials of complex numbers (cf. ref. 33) for a similar normal-ordered computation. Fourier-transforming equation (29) with respect to the relative coordinate \(x - y\), we obtain the structure factor

\[
\langle \hat{C}^\dagger (x) \hat{C}(y) \rangle_{\beta,s} = \int d(x - y) \langle \hat{C}^\dagger (x) \hat{C}(y) \rangle_{\beta,s} e^{-\beta(x-y)}
\]

\[
= \frac{1}{V} \sum_{m,m',n,n'} c_{mm'} \sum_{p,q} \left( \langle \hat{a}_{p,k,m}^\dagger \hat{a}_{q,k,m} \rangle_{\beta,s} \right)_{p,q},
\]

To the total density structure factors a photon shot-noise level of 0.6 after normalization is added.

In the Bogoliubov approximation, equation (32) simplifies as follows. We replace zero modes of creation and annihilation operators by numbers, \(a_{p,m} = \sqrt{N_{\text{cond}}} c_{m}\), where \(N_{\text{cond}}\) is the total number of condensed atoms. Contributions from fluctuating modes \(a_{p|p} = 0\), are computed via Wick’s theorem. Symmetrically ordered propagators are defined as

\[
G_{1 \text{cond}}^1(k) \equiv \langle \hat{a}_{k,m} \hat{a}^\dagger_{k,m} \rangle_{\beta,s},
\]

\[
G_{2 \text{cond}}^2(k) \equiv \langle \hat{a}_{k,m} \hat{a}^\dagger_{k,m} \rangle_{\beta,s},
\]

\[
G_{1 \text{cond}}^1(k) \equiv \langle \hat{a}_{k,m} \hat{a}^\dagger_{k,m} \rangle_{\beta,s},
\]

\[
G_{1 \text{cond}}^2(k) \equiv \langle \hat{a}_{k,m} \hat{a}^\dagger_{k,m} \rangle_{\beta,s},
\]

\[
G_{2 \text{cond}}^1(k) \equiv \langle \hat{a}_{k,m} \hat{a}^\dagger_{k,m} \rangle_{\beta,s},
\]

\[
G_{2 \text{cond}}^2(k) \equiv \langle \hat{a}_{k,m} \hat{a}^\dagger_{k,m} \rangle_{\beta,s},
\]

The first two of these refer to normal propagators, and the second two are anomalous propagators. Any normal propagator evaluated for non-diagonal momenta such as \(\langle \hat{a}^\dagger_{k,m} \hat{a}_{k',m'} \rangle_{\beta,s} \) for \(k' \neq k\) and any anomalous propagator evaluated for non-anti-diagonal momenta such as \(\langle \hat{a}^\dagger_{k,m} \hat{a}_{k'-m} \rangle_{\beta,s} \) for \(k' \neq k\) vanishes to zero. We then find for \(k = 0\):

\[
\langle \hat{C}^\dagger (0) \hat{C}(0) \rangle_{\beta,s} = \frac{N_{\text{cond}}(N_{\text{cond}} - 1)}{V} \sum_{m,m',n,n'} c_{mm'} c_{m'n'} \langle \hat{C}_m^\dagger C_{m'} \rangle_{\beta,s} + O(N_{\text{cond}}).
\]

and for non-zero modes \(k \neq 0\):

\[
\langle \hat{C}^\dagger (k) \hat{C}(k) \rangle_{\beta,s} = \frac{N_{\text{cond}}}{V} \sum_{m,m',n,n'} c_{mm'} c_{m'n'} \langle \hat{C}_m^\dagger C_{m'} \rangle_{\beta,s} + O(1).
\]
The propagators can now efficiently be computed from the tensor product
\[
\begin{pmatrix}
G^2_{\text{mm}}(k) \\
G^1_{\text{mm}}(k) \\
G^2_{\text{nn}}(-k) \\
G^1_{\text{nn}}(-k)
\end{pmatrix} = \begin{pmatrix}
\hat{a}_{k,m} \\
\hat{a}_{-k,m} \\
\hat{a}_{-k,m} \\
\hat{a}_{k,m}
\end{pmatrix} \otimes \begin{pmatrix}
\hat{a}_{-k,m}^\dagger \\
\hat{a}_{k,m}^\dagger \\
\hat{a}_{k,m}^\dagger \\
\hat{a}_{-k,m}^\dagger
\end{pmatrix}_{\beta,s} \tag{39}
\]
\[
= \sum_{j,s \in \{\uparrow, \downarrow\}} \begin{pmatrix}
U'_{k,j} & -V'_{k,j} \\
V'_{k,j} & U''_{k,j}
\end{pmatrix} \begin{pmatrix}
\hat{b}_{k,j} \\
\hat{b}_{-k,j}
\end{pmatrix}_{\beta,s} \begin{pmatrix}
\hat{b}_{k,j}^\dagger \\
\hat{b}_{-k,j}^\dagger
\end{pmatrix}_{\beta,s} \tag{40}
\]

The Bogoliubov quasiparticle modes $\hat{b}_{k,j}$ are occupied thermally:
\[
\langle \hat{b}_{k,j}^\dagger \hat{b}_{k,j} \rangle_p = \delta_p \eta_p(\mathbf{E}_{k,j}), \quad \langle \hat{b}_{k,j}^\dagger \hat{b}_{-k,j} \rangle_p = \delta_p (\eta_p(\mathbf{E}_{k,j}) + 1), \tag{41}
\]
with the Bose–Einstein distribution $\eta_p(\mathbf{E}_{k,j}) = 1/(\exp(\mathbf{E}_{k,j}) - 1)$. Anomalous propagators of $\hat{b}_{k,j}$ modes are zero. Insertion of equation (41) into (40) and using that $U'_{k,j}$ and $U''_{k,j}$ only depend on $|k|$ leads to
\[
G^1_{\text{mm}}(k) = \sum_{j \in \{\uparrow, \downarrow\} \in \{\text{even}, \text{odd}\}} \begin{pmatrix}
U'_{k,j} & -V'_{k,j} \\
V'_{k,j} & U''_{k,j}
\end{pmatrix} \begin{pmatrix}
\hat{b}_{k,j} \\
\hat{b}_{-k,j}
\end{pmatrix}_{\beta,s} \begin{pmatrix}
\hat{b}_{k,j}^\dagger \\
\hat{b}_{-k,j}^\dagger
\end{pmatrix}_{\beta,s} \tag{42}
\]
\[
G^2_{\text{mm}}(k) = \sum_{j \in \{\uparrow, \downarrow\} \in \{\text{even}, \text{odd}\}} \begin{pmatrix}
V'_{k,j} & U''_{k,j} \\
U'_{k,j} & -V'_{k,j}
\end{pmatrix} \begin{pmatrix}
\hat{b}_{k,j} \\
\hat{b}_{-k,j}
\end{pmatrix}_{\beta,s} \begin{pmatrix}
\hat{b}_{k,j}^\dagger \\
\hat{b}_{-k,j}^\dagger
\end{pmatrix}_{\beta,s} \tag{43}
\]
\[
G^1_{\text{nn}}(k) = -\sum_{j \in \{\uparrow, \downarrow\} \in \{\text{even}, \text{odd}\}} \begin{pmatrix}
U'_{k,j} & -V'_{k,j} \\
V'_{k,j} & U''_{k,j}
\end{pmatrix} \begin{pmatrix}
\hat{b}_{k,j} \\
\hat{b}_{-k,j}
\end{pmatrix}_{\beta,s} \begin{pmatrix}
\hat{b}_{k,j}^\dagger \\
\hat{b}_{-k,j}^\dagger
\end{pmatrix}_{\beta,s} \tag{44}
\]
\[
G^2_{\text{nn}}(k) = -\sum_{j \in \{\uparrow, \downarrow\} \in \{\text{even}, \text{odd}\}} \begin{pmatrix}
V'_{k,j} & U''_{k,j} \\
U'_{k,j} & -V'_{k,j}
\end{pmatrix} \begin{pmatrix}
\hat{b}_{k,j} \\
\hat{b}_{-k,j}
\end{pmatrix}_{\beta,s} \begin{pmatrix}
\hat{b}_{k,j}^\dagger \\
\hat{b}_{-k,j}^\dagger
\end{pmatrix}_{\beta,s} \tag{45}
\]

With these expressions, thermal structure factors can be readily computed from equation (36).

**Fitting thermal Bogoliubov theory structure factors**

Using a least-squares fitting procedure and Gibbs sampling, systematic as well as statistical uncertainties on the optimal set of parameters are estimated. Given experimental structure factors $S_{\text{exp}}(k) = \mathcal{C}(k)\hat{C}(k)$ with $\mathcal{C} \in \mathbb{C} = \{N, N, ..., F, F\}$ we determine an optimal set of parameters $T, q, nc_1$ by minimizing
\[
\chi^2(T, q, nc_1; k_{\text{max}}) = \sum_{C \in \mathcal{C}} \sum_{k} \frac{(S_{\text{exp}}(k) - S_{\text{Bog}}(k; T, q, nc_1))^2}{\Delta S_{\text{exp}}(k)^2}, \tag{46}
\]
with $S_{\text{Bog}}(k; T, q, nc_1) = \langle \hat{C}(k)\hat{C}(k)\rangle_{\text{Bog}}(k; T, q, nc_1)$ being the Bogoliubov theory structure factor computed for parameters $T, q, nc_1$, and $\Delta S_{\text{exp}}(k)$ the s.d. of $S_{\text{exp}}(k)$ computed from experimental realizations. Technical correlations of the absorption imaging are described by real-space signals convoluted with a Gaussian of r.m.s. width $w = 5.0 \, \mu\text{m}$, taken into account by the multiplication of momentum-space structure factors with a Gaussian of width $2\pi/w$ (ref. 14). Throughout the fitting procedure we set $c_0/c_1 = -216$ in accordance with ref. 26. In the definition of $\chi^2$ we did not include the structure factor of the total density.

We define a distribution of parameters for specific $k_{\text{max}}$ as
\[
W(T, q, nc_1; k_{\text{max}}) \sim \exp(-\chi^2(T, q, nc_1; k_{\text{max}})/2). \tag{47}
\]

We exploit Gibbs sampling to draw $i = 1, ..., 100$ approximately i.i.d. samples $(T^{(i)}(k_{\text{max}}), q^{(i)}(k_{\text{max}}), nc_1^{(i)}(k_{\text{max}}))$ from $W(T, q, nc_1; k_{\text{max}})$, which only requires computing conditional distributions normalized individually. For each $k_{\text{max}}$ we compute their mean $(\overline{T}(k_{\text{max}}), \overline{q}(k_{\text{max}}), \overline{nc_1}(k_{\text{max}}))$. We repeat this for five values of $k_{\text{max}}$ evenly spaced between $0.1 \times 2\pi$ ($\mu\text{m})^{-1}$ and $0.2 \times 2\pi$ ($\mu\text{m})^{-1}$. Collecting all $5 \times 100$ samples in a single array, we take the mean values $\overline{T}, \overline{q}, \overline{nc_1}$ of all samples as final parameter estimates, and their distances to the boundaries of 68% confidence intervals as corresponding error estimates. Errors include systematic fit uncertainties. We obtain the final fit parameters
\[
T = (57.4 \pm 2.9) \, \text{Hz}, \quad q = (0.30 \pm 0.08) \, \text{Hz}, \quad nc_1 = (-1.17 \pm 0.25) \, \text{Hz}, \tag{48}
\]
such that $nc_0 = (252 \pm 54) \, \text{Hz}$ and $q/(nc_1) = (0.26 \pm 0.09)$.

**Drawing Bogoliubov theory samples**

With Bogoliubov theory being quadratic in fluctuating field creation and annihilation operators, it is fully described by zero modes and a suitable covariance matrix of fluctuations. The latter can be constructed from the propagators $G_{\text{mm}}(k)$.

Given a 1D real-space lattice $[-N, ..., N] \times a$ with lattice spacing $a = 1/(2N)$, the corresponding momentum-space lattice reads $[-N, ..., N] \times \pi/(N a)$. With $\Delta z = z_1 - z_2 \in [-2N, ..., 2N] \times \pi/(N a)$ we compute real-space propagators via
\[
G_{\text{mm}}(\Delta z) = \frac{1}{\pi} \sum_{p=-N}^{N} \sum_{s=-N}^{N} \exp\left(-\frac{2\pi ip\Delta z}{2N+1}\right). \tag{49}
\]

We assemble these into magnetic sublevel-specific covariance matrices
\[
\begin{pmatrix}
\psi_{\text{m}}(-N) \\
\vdots \\
\psi_{\text{m}}(N)
\end{pmatrix} \sim \begin{pmatrix}
\mathbf{Cov}_{\text{mm}} & \psi_{\text{m}}^{(1)}(-N) & \psi_{\text{m}}^{(2)}(-N) & \cdots & \psi_{\text{m}}^{(N)}(-N) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\psi_{\text{m}}^{(N)}(-N) & \psi_{\text{m}}^{(N)}(N) & \psi_{\text{m}}^{(N)}(2N) & \cdots & \psi_{\text{m}}^{(N)}(2N)
\end{pmatrix}_{\beta,s}. \tag{50}
\]
having exploited spatial homogeneity. We decompose complex field operators into real components, \( \psi(x) = \frac{1}{\sqrt{2}} (\psi_1(x) + i\psi_2(x)) \), translating into the unitary transformation

\[
\begin{pmatrix}
\psi_{m,1}(-N) \\
\vdots \\
\psi_{m,1}(N) \\
\psi_{m,2}(-N) \\
\vdots \\
\psi_{m,2}(N)
\end{pmatrix} = A
\begin{pmatrix}
\psi_m(-N) \\
\vdots \\
\psi_m(N) \\
\psi'_m(-N) \\
\vdots \\
\psi'_m(N)
\end{pmatrix},
\]

\( A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \) \( \text{(S2)} \)

with \( 2(N+1) \times 2(N+1) \) dimensional identity matrix. We define the final covariance matrix of the theory as

\[
\text{Cov} = \begin{pmatrix}
A \text{Cov}_{+,+} A^T & A \text{Cov}_{+,0} A^T & A \text{Cov}_{+,1} A^T \\
A \text{Cov}_{0,+} A^T & A \text{Cov}_{0,0} A^T & A \text{Cov}_{0,1} A^T \\
A \text{Cov}_{1,+} A^T & A \text{Cov}_{1,0} A^T & A \text{Cov}_{1,1} A^T
\end{pmatrix}.
\]

\( \text{(S3)} \)

Finally, we sample \( i = 1, \ldots, N_{\text{sample}} \) field realizations

\[
\psi^{(i)} = \begin{pmatrix}
\psi^{(i)}_{m,1} \\
\psi^{(i)}_{m,0} \\
\psi^{(i)}_{m,2}
\end{pmatrix}, \quad \psi^{(i)}_m = (\psi^{(i)}_{m,1}(-N), \ldots, \psi^{(i)}_{m,1}(N)),
\]

\( \psi^{(i)}_{m,2}(-N), \ldots, \psi^{(i)}_{m,2}(N)^T \).

from the multivariate Gaussian distribution with zero mean vector and covariance matrix Cov. This corresponds to samples from the Wigner distribution of the symmetrically ordered Bogoliubov theory of fluctuating modes at inverse temperature \( \beta \). We sample fields in position space instead of momentum space, because in momentum space, having decomposed the operators \( \hat{a}_{k,m} \) into \( \hat{a}_{k,m} = (\hat{a}_{k,m,1} + i\hat{a}_{k,m,2})/\sqrt{2} \), the components \( \hat{a}_{k,m} \) need to satisfy \( \hat{a}^\dagger_{k,m,j} = \hat{a}_{-k,-m,j} \) such that samples of individual momentum modes cannot be drawn independently. From the fluctuating realizations \( \psi^{(i)} \) we can compute realizations of the individual spin sublevel fields in real space:

\[
\psi^{(i)}_m(x) = \frac{1}{\sqrt{2}} [\psi^{(i)}_{m,1}(x) + i\psi^{(i)}_{m,2}(x)] + \sqrt{n_{\text{cond}}} \psi^{(i)}_m,
\]

\( 55 \)

where \( n_{\text{cond}} \) is the condensate density. We explicitly checked that for increasing sample numbers, structure factors computed from samples \( \psi^{(i)}_m(x) \) converge towards their expectations \( c^{(i)}_{mn}(k) \).

The composite operator histograms displayed in Extended Data Fig. 2 are computed from composite profiles of individual realizations given by \( \sum_{m, n=1}^{+1} \psi^{(i)}_m(x)^* \psi^{(i)}_{n,m}(x) \sqrt{n_{\text{cond}}^n} \). With matrices \( c \) as denoted in the section ‘Thermal structure factors from Bogoliubov theory’.

**Data availability**

Data that support the plots within this paper and other findings of this study are available from the corresponding author upon reasonable request. Source data are provided with this paper.

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**Author contributions**

M.P., S.L. and H.S. took the measurement data. M.P., D.S., S.L., H.S. and M.K.O. discussed the measurement results and analysed the data. D.S., M.P. and J.B. elaborated the theoretical framework. All authors contributed to the discussion of the results and the writing of the manuscript.

**Competing interests**

The authors declare no competing interests.

**Additional information**

Extended data is available for this paper at https://doi.org/10.1038/s41567-022-01779-6.

**Supplementary information**

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Extended Data Fig. 1 | Measurement of the gap by observation of temporal oscillations of the $k = 0$ mode. a. We measure the gap of the quadratic spin mode by a global rotation of the spinor phase. We record the resulting oscillations of the fractional $m = 0$ population as a function of evolution time after the rotation. We fit a sinusoidal function (solid line) to infer the frequency. b. Extracted oscillation frequency (diamonds) and mean value of the $m = 0$ population (circles). We compare to theoretical expectations for the easy-plane phase (solid lines; see Methods equation 1). The dashed line extrapolates the expectations to $q < 0$ under the assumption of equal populations of $m = \pm 1$. For the theory curves we use $n_c = 1.3\text{Hz}$. 
Extended Data Fig. 2 | Histograms of local observables in the thermalized state. Histograms obtained from evaluating the local observations of the experimental data presented in Fig. 4 (green bars). Here, each local observable is normalized to the square-root of the local mean of the total atom number. On top we display theoretical estimates from 1000 samples generated according to thermal Bogoliubov theory with parameters as in Fig. 4 (grey line; grey band indicates 68% confidence interval including statistical and systematic uncertainties). The mean value of each histogram is subtracted. For details on the sampling procedure see Methods.
Extended Data Fig. 3 | Structure factor close to $q = 0$. We show experimental power spectra of different spin and density degrees of freedom close to $q = 0$ (green diamonds). The grey diamonds represent the fluctuations of a coherent spin state with comparable atom numbers. We compare to thermal Bogoliubov theory predictions for the same parameters as displayed in Fig. 4 but with $q = 0$ (green line; grey band indicates 68% confidence interval of statistical and systematic uncertainties). Experimentally, we find that for momenta in the range of 0.02 $\mu$m$^{-1}$ to 0.1 $\mu$m$^{-1}$ the fluctuations are higher than for the thermal predictions for all observables (except the transversal spin $F_\perp$). The length scale of these fluctuations is in accordance with observable localized long-lived non-linear excitations which are not present in the thermalized data of Fig. 4.