Riemann surfaces and the Galois correspondence

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To the memory of my mother

Abstract

In this paper we introduce a space with some additional topologies using filter bases and renew the definition of Riemann surfaces of algebraic functions. We then present a Galois correspondence between these Riemann surfaces and their deck transformation groups. We also extend the monodromy theorem to the case that the global analytic function possesses singularities, which can be non-isolated.

1 Introduction

Let us recall two points lying in algebra and complex analysis respectively. At first, it is known that the Galois correspondence theorem, which is called the fundamental theorem of Galois theory, is one of the most important results in modern algebra (refer to [9], [14], [21] and [25], etc). The Galois correspondence and related issues have been extended to a number of cases, see e.g. [3], [6], [10], [18], [19], [21], [20], [28] and [29]. Then, we look at algebraic functions and their Riemann surfaces. Algebraic functions are studied in both function theory and algebraic geometry. To deal with the trouble of multivaluedness of functions Riemann designed the “Riemann surface”, which is a source of some modern mathematical branches. The theory of Riemann surfaces also provides a model for developments in many research areas in mathematics. There is a lot of literature in Riemann surfaces and algebraic functions, see [1], [2], [5], [7], [10], [12], [13], [17], [20], [22], [23], [27], [28], [29] and [30], etc.

Originally, a Riemann surface may be regarded as a covering space (surface) of the (extended) complex plane (or a part of it). We may consider the Galois correspondence in the case of covering spaces. In fact, there is a correspondence in covering spaces similar to the Galois correspondence in the classical Galois theory, see [11, 13d], [15] and [16] (in [15] and [16] finite ramified coverings over Riemann surfaces were also considered). If we observe [10] Theorem (8.12)], we may expect the Galois correspondence occurs between Riemann surfaces of algebraic functions and covering transformation groups, even in general infinite cases. However, branch points become a key problem.

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In order to deal with branch points and multivaluedness of functions we employ filter bases. This idea is inspired by [31] and [32] and develops therefrom. About the notions of filters and filter bases, which were originated by H. Cartan, we refer to [1] §6 and §7 of Chapter I] and [8] Chapter X]. Using filter bases we introduce a space with some additional topologies, which we call a universal topological space (see Section 2). By using neighborhoods in this space we can deal with branch points in a natural manner, which enables us to carry out algebraic operations of functions and germs freely. To this end, we study presheaves on a universal topological space in Section 2. In addition, we introduce some more general notions for the extension of the monodromy theorem.

In Section 3, we present a new version of the definition of Riemann surfaces of algebraic functions, where the notions of harmonious equivalence and up-harmonious equivalence are introduced. We also consider analytic continuations in more extensive senses and extend the monodromy theorem to the case that the global analytic function possesses singularities, which can be non-isolated.

Finally, in Section 4 we present a Galois correspondence between Riemann surfaces of algebraic functions (algebraic Riemann surfaces, see Subsection 3.4) and their deck transformation groups (Theorems [1.1] and [4.10], which may be regarded as a geometric version of the classical Galois correspondence.

2 A universal topological space

2.1 A perfect filterbase structure system and a universal topological space

We recall that a nonempty family $\mathcal{B}$ of subsets of a nonempty set $X$ is a filter base precisely if $\mathcal{B}$ does not contain the empty set and the intersection of any two sets of $\mathcal{B}$ contains a set of $\mathcal{B}$, see [1] §6.3 of Chapter I] and [8] Definition (2.1) in Chapter X]. Suppose $\mathfrak{B}$ is a family of filter bases in the nonempty set $X$. Two filter bases $\mathcal{B}_1$ and $\mathcal{B}_2$ in $\mathfrak{B}$ are said to be equivalent, denoted $\mathcal{B}_1 \sim \mathcal{B}_2$, if $\mathcal{B}_1 \Vdash \mathcal{B}_2$ and $\mathcal{B}_2 \Vdash \mathcal{B}_1$, where “$\Vdash$” means “be subordinate to” (i.e. $\mathcal{B}_1 \Vdash \mathcal{B}_2$ precisely if for each $B_2 \in \mathcal{B}_2$ there exists $B_1 \in \mathcal{B}_1$ such that $B_1 \subseteq B_2$, see [8] Definition (2.4) in Chapter X]). It is obvious that this really is an equivalence relation in $\mathfrak{B}$. The equivalence class of $\mathcal{B}$ is denoted $\tilde{\mathcal{B}}$.

Suppose $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are two families of filter bases in a (nonempty) set $X$. If for each $B_2 \in \mathfrak{B}_2$ there exists $B_1 \in \mathfrak{B}_1$ such that $B_1 \Vdash B_2$ (resp. $\mathcal{B}_1 \sim \mathcal{B}_2$) then we say that $\mathfrak{B}_1$ is a (resp. an exact) refinement of $\mathfrak{B}_2$. If $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are (resp. exact) refinements of one another then we say that they are compatible (resp. equivalent), denoted $\mathfrak{B}_1 \sim \mathfrak{B}_2$ (resp. $\mathfrak{B}_1 \sim \mathfrak{B}_2$). Obviously both the exact refinement relation and the refinement relation of filterbase families are preorders and they are also partial orders if “$=$” means “$\sim$” or “$\sim$”. Both the compatibility and the equivalence are equivalence relations and the equivalence implies the compatibility.

Suppose $(X, \mathcal{T})$ is a (nonempty) topological space ($\mathcal{T}$ is the topology on $X$)
and $\mathcal{B}$ is a family of filter bases consisting of open subsets of $X$. Suppose for each $x \in X$ there is precisely one filter base $B \in \mathcal{B}$ such that $B \to x$ (“$\to$” means “converge to”, i.e. $B \to x$ precisely if for any neighborhood $U$ of $x$ there exists $B \subseteq U$, see [8, Definition (2.3) in Chapter X]), or if there is another filter base $B' \in \mathcal{B}$ such that $B' \to x$ then $B' \sim B$. If $B \in \mathcal{B}$ does not converge to any points in $X$, then we consider that it converges to some “ideal points”. Generally, it may be allowed that some filter bases (in $\mathcal{B}$) converging in $(X, T)$ also converge to ideal points (of course, the Hausdorff condition will remove this case). Here we consider that $B' \sim B$ converges to the same ideal point(s) precisely if $B' \sim B$. We also assume that if $B \in \mathcal{B}$, $B \to x \in X$, then $x /\in B$ for any $B \in \mathcal{B}$. We call the filterbase family $\mathcal{B}$ satisfying the above conditions a perfect filterbase structure system on $(X, T)$.

Suppose $\mathcal{B}$ is a perfect filterbase structure system on $X$ and $I$ denotes the set of all ideal points of $\mathcal{B}$. Let $\hat{X} := X \cup I$. Let
\[
\hat{B}(x) := \{ \hat{B} = B \cup \{ x \} : B \in \mathcal{B}, \text{ where } B \in \mathcal{B} \text{ and } B \to x \}
\]
for $x \in \hat{X}$ and
\[
\hat{B} := \bigcup_{x \in \hat{X}} \hat{B}(x).
\]
Noticing every filter base $B \in \mathcal{B}$ consists of open subsets of $X$, it is easy to verify that $\hat{B}$ is a basis for some topology on $\hat{X}$. So we obtain a topology on $\hat{X}$ determined by $\hat{B}$. We call this new topology the filterbase topology or partial topology on $\hat{X}$ determined by $\mathcal{B}$, denoted $\hat{T}$. Set $\hat{B} \in \mathcal{B}(x)$ is called a basic (open) partial neighborhood of $x$, $\hat{B}(x)$ a basic (open) partial neighborhood basis at $x$ ($x \in \hat{X}$) and $\hat{B}$ a basic (open) partial neighborhood basis on $\hat{X}$.

Let $\hat{B}(x)$ be an open neighborhood basis at $x$ in $(X, T)$ for $x \in X$ and $\hat{B}(x) := \hat{B}(x)$ (or an open neighborhood basis at $x$ in $(\hat{X}, \hat{T})$) for $x \in I$. Let
\[
\hat{B} := \bigcup_{x \in \hat{X}} \hat{B}(x).
\]
Then it is easy to verify that $\hat{B}$ is a basis for some topology on (set) $\hat{X}$. Again we obtain a topology on $\hat{X}$ determined by $\hat{B}$, which is called the essential topology on $\hat{X}$ and denoted $\hat{T}$. We call $\hat{B}$ (resp. $\hat{B}(x)$) a basic (open) essential neighborhood basis (resp. at $x$).

Generally, choose a subset $A$ of $\hat{X}$ containing the ideal point set $I$, which we call a partial point set (of $\hat{X}$). Let $\hat{B}_A(x)$ be an open neighborhood basis at $x$ in $(X, T)$ for $x \in X \setminus A$ and $\hat{B}_A(x) := \hat{B}(x)$ for $x \in A$. Then
\[
\hat{B}_A := \bigcup_{x \in \hat{X}} \hat{B}_A(x)
\]
is also a basis for some topology on (set) $\hat{X}$. In this way we obtain a topology on $\hat{X}$, determined by $\hat{B}_A$, which we call the mixed topology on $\hat{X}$ with the partial
point set $A$ or the $A$-mixed topology on $X$, denoted $\hat{T}_A$, and $\hat{B}_A$ (resp. $\hat{B}_A(x)$) is called an $A$-mixed neighborhood basis (resp. at $x$).

The space $\hat{X}$, equipped with a filterbase topology $\hat{T}$, an essential topology $\hat{Y}$ and some mixed topologies $\hat{F}(A)$, is called a universal topological space on $(X, \mathcal{F})$ determined by $\mathfrak{B}$. The universal topological space $\hat{X}$ on $(X, \mathcal{F})$ is also denoted $(X, \mathcal{F}, \hat{X}, \hat{T})$. The set $\hat{I}$ is also called the ideal point set of $\hat{X}$.

We also remark here that if $A = X$ then $\hat{T}(A) = \hat{T}$ and if $A = I$ then $\hat{T}(A) = \hat{T}$. We also remark that for the topological space $(X, \mathcal{F})$ we can obtain a number of filterbase topologies by different perfect filterbase structure systems on $X$ (see e.g. Subsection 2.5). Suppose $\hat{T}_1$ and $\hat{T}_2$ are filterbase topologies determined by two perfect filterbase structure systems $\mathfrak{B}_1$ and $\mathfrak{B}_2$ on $(X, \mathcal{F})$, respectively. If $\mathfrak{B}_1 \sim \mathfrak{B}_2$ then it follows that $\hat{T}_1 = \hat{T}_2$ (under some assumption on ideal points) and generally the two mixed topologies $\hat{T}_1(A)$ and $\hat{T}_2(A)$ are the same. Therefore, the universal topological spaces $(X, \mathcal{F}, \hat{X}, \hat{T}_1)$ and $(X, \mathcal{F}, \hat{X}, \hat{T}_2)$ we obtain here are the same.

If the topological space $(\hat{X}, \hat{F})$ is Hausdorff, then we say that the universal topological space $\hat{X}$ is Hausdorff. Thus that $(X, \mathcal{F}, \hat{X}, \hat{F})$ is Hausdorff implies that $(\hat{X}, \hat{F}(A))$ is Hausdorff and specially both $(X, \mathcal{F})$ and $(\hat{X}, \hat{F})$ are also Hausdorff.

Let

$$\mathfrak{B}(x) := \{B : B \in \mathfrak{B}, \text{ where } B \in \mathfrak{B} \text{ and } B \to x\}$$

for $x \in \hat{X}$. Clearly $\mathfrak{B}(x)$ is a filter base. We call $\mathfrak{B}(x)$ a basic (open) punctured partial neighborhood basis at $x$ and set $B \in \mathfrak{B}(x)$ a basic (open) punctured partial neighborhood of $x$ ($x \in X$). In the paper, we may also use the terminology deleted to replace the “punctured”. Let

$$\mathfrak{B} := \bigcup_{x \in \hat{X}} \mathfrak{B}(x),$$

which is called a basic (open) punctured partial neighborhood basis on $\hat{X}$.

For a partial point set $A$ ($I \subseteq A \subseteq \hat{X}$), let $\mathfrak{B}_A(x)$ be an open punctured neighborhood basis at $x$ in $(X, \mathcal{F})$ for $x \in X \setminus A$ and $\mathfrak{B}_A(x) := \mathfrak{B}(x)$ for $x \in A$. Let

$$\mathfrak{B}_A := \bigcup_{x \in \hat{X}} \mathfrak{B}_A(x),$$

which is called an $A$-mixed punctured neighborhood basis on $\hat{X}$.

Suppose $(X, \mathcal{F}, \hat{X}, \hat{F})$ is a universal topological space determined by a perfect filterbase structure system $\mathfrak{B}$ and $Y$ is a subset of $\hat{X}$. Suppose $I_0 \subseteq X$, $I_1 := I \cup I_0$ ($I$ is the ideal point set of $\hat{X}$), $I' := I_1 \cap \hat{Y}$ and $Y := \hat{Y} \setminus I' \neq \emptyset$. We call $(Y, \mathcal{F}', \hat{Y}, \hat{F}')$ a universal topological subspace of $(X, \mathcal{F}, \hat{X}, \hat{F})$, where $\mathcal{F}' := \mathcal{F} \cap Y$ and the partial topology $\hat{F}' := \hat{F} \cap \hat{Y}$ are induced topologies, the $A'$-mixed topologies $\hat{F}'(A')$ of $\hat{Y}$ are just the induced topologies $\hat{F}(A_1) \cap \hat{Y}$, where $I_1 \subseteq A_1 \subseteq \hat{X}$, $A' = A_1 \cap \hat{Y}$ and $\hat{F}(A_1)$ is the $A_1$-mixed topology of $\hat{X}$, and specially the essential topology $\hat{F}'$ of $\hat{Y}$ is the induced topology $\hat{F}(I_1) \cap \hat{Y}$. The set $I'$ is called the ideal point set of $\hat{Y}$. 
If $B \to x \in \hat{X}$ ($B \in \mathcal{B}$), then we denote $B$ by $B(x)$. Let $\mathcal{B}_Y(x) := B(x) \cap Y = \{B \cap Y : B \in \mathcal{B}(x)\}$ for $x \in \hat{X}$ and $\mathcal{B}_Y := \{\mathcal{B}_Y(y) : y \in \hat{Y}\}$. Usually we assume that $B \cap Y \neq \emptyset$ for any $B \in \mathcal{B}(y)$, $y \in Y$ (hence $\mathcal{B}_Y(y)$ is a filter base) and one of the following two assumptions holds:

1. $(\hat{X}, \mathcal{T})$ is Hausdorff;
2. $\mathcal{B}_Y(y) \sim B(y)$ for each $y \in \hat{Y}$.

Then $\mathcal{B}_Y$ is a perfect filterbase structure system on $(Y, \mathcal{T}_Y)$ ($\mathcal{T}_Y := \mathcal{T} \cap Y$ is the induced topology $\mathcal{T}'$), called the induced perfect filterbase structure system by $\mathcal{B}$ on $Y$. Thus, we obtain a universal topological space $(Y', \mathcal{T}_Y; \hat{Y}, \mathcal{T}_Y)$ determined by $\mathcal{B}_Y$, which is just the universal topological subspace $(Y', \mathcal{T}_Y; \hat{Y}, \mathcal{T}')$ (defined above) of $(X, \mathcal{T}; \hat{X}, \mathcal{T})$ (cf. the proof of Theorem 2.1(4) below). If $\hat{Y}$ is an open subset of $(\hat{X}, \mathcal{T}(I_1))$, then the subspace $(Y, \mathcal{T}'; \hat{Y}, \mathcal{T}')$ is called an open set in $(X, \mathcal{T}; \hat{X}, \mathcal{T})$.

### 2.2 Partial continuity, essential continuity and (exact) continuity

Suppose $(X, \mathcal{T}; \hat{X}, \mathcal{T})$ and $(Y, \mathcal{T}'; \hat{Y}, \mathcal{T}')$ are two universal topological spaces. Let $\hat{f}$ be a mapping from $\hat{Y}$ to $\hat{X}$. The mapping $\hat{f}$ is said to be partially continuous (resp. at a point $y \in \hat{Y}$) if $\hat{f} : (\hat{Y}, \mathcal{T}') \to (X, \mathcal{T})$ is continuous (resp. at $y$). If $\hat{f} : (\hat{Y}, \mathcal{T}') \to (X, \mathcal{T})$ is continuous (resp. at $y \in \hat{Y}$), where $\mathcal{T}$ and $\mathcal{T}'$ are the essential topologies of $\hat{X}$ and $\hat{Y}$ respectively, then we say that $\hat{f} : \hat{Y} \to \hat{X}$ is essentially continuous (resp. at $y$). Now suppose $\hat{f}$ is partially continuous and $\hat{f}(Y) \subseteq X$. If $\hat{f}|_Y : (Y, \mathcal{T}) \to (X, \mathcal{T})$ is also continuous then we say that $\hat{f} : \hat{Y} \to \hat{X}$ is (exactly) continuous. Evidently the (exact) continuity of $\hat{f}$ implies the essential continuity of $\hat{f}$.

Suppose $\hat{f} : \hat{Y} \to \hat{X}$ is a bijective. It is called an essential (resp. a partial) homeomorphism if $\hat{f} : (\hat{Y}, \mathcal{T}') \to (\hat{X}, \mathcal{T})$ (resp. $\hat{f} : (\hat{Y}, \mathcal{T}') \to (\hat{X}, \mathcal{T})$) is a homeomorphism. It is called an (exact) homeomorphism if $\hat{f}(Y) = X$ and both $\hat{f} : (\hat{Y}, \mathcal{T}') \to (\hat{X}, \mathcal{T})$ and $\hat{f}|_Y : (Y, \mathcal{T}) \to (X, \mathcal{T})$ are homeomorphisms. The (exact) homeomorphism obviously implies the essential homeomorphism.

As for local homeomorphisms, we may consider partial localness, essential localness and exact localness respectively. Then we may further locally consider partial, essential and exact homeomorphisms, respectively. In this paper we define essential local homeomorphisms and (exact) local homeomorphisms as follows. Suppose $\hat{f} : \hat{Y} \to \hat{X}$ is a mapping. It is called an essential local homeomorphism if $\hat{f} : (\hat{Y}, \mathcal{T}) \to (\hat{X}, \mathcal{T})$ is a local homeomorphism ($\mathcal{T}$ and $\mathcal{T}'$ are essential topologies on $\hat{X}$ and $\hat{Y}$ respectively). It is called an (exact) local homeomorphism if for each point $y \in \hat{Y}$ there exists an open essential neighborhood $\hat{N}$ of $y$ (i.e. an open neighborhood of $y$ in $(\hat{Y}, \mathcal{T})$) such that $\hat{f}(\hat{N})$ is an open essential neighborhood of $\hat{f}(y)$ and $\hat{f}|_Y : \hat{N} \to \hat{f}(\hat{N})$ is an exact (i.e. partial and essential) homeomorphism. Obviously, the (exact) local homeomorphism implies the essential local homeomorphism.
Let $\hat{X}$, $\hat{Y}$ and $\hat{Z}$ be universal topological spaces. It is evident that if $\hat{f} : \hat{Y} \to \hat{X}$ and $\hat{g} : \hat{Z} \to \hat{Y}$ are (exactly) (resp. partially, essentially) continuous then $\hat{f} \circ \hat{g} : \hat{Z} \to \hat{X}$ is also (exactly) (resp. partially, essentially) continuous.

### 2.3 Covering maps and deck transformations

Suppose $(X, \mathcal{T}; \hat{X}, \hat{\mathcal{T}})$, $(Y, \mathcal{T}'; \hat{Y}, \hat{\mathcal{T}}')$ and $(Z, \mathcal{T}''; \hat{Z}, \hat{\mathcal{T}}'')$ are universal topological spaces. Suppose $\hat{p} : \hat{Y} \to \hat{X}$ and $\hat{q} : \hat{Z} \to \hat{X}$ are essentially continuous for every $x \in \hat{X}$, the set $\hat{p}^{-1}(x)$ is called the fiber of $\hat{p}$ over $x$. A mapping $f : \hat{Z} \to \hat{Y}$ is called fiber-preserving if $\hat{p} \circ f = \hat{q}$.

Suppose for every $x \in X \setminus \hat{p}(\hat{Y} \setminus Y)$ there exists $U \in \mathcal{T}(x)$, where $\mathcal{T}(x)$ denotes the set of all open neighborhoods (i.e. the open neighborhood system) of $x \in X$ in $(X, \mathcal{T})$, and for every point $\hat{x} \in (\hat{X} \setminus X) \cup \hat{p}(\hat{Y} \setminus Y)$ there exists $U \in \mathcal{T}(\hat{x})$ (the partial neighborhood system at $\hat{x}$, i.e. the open neighborhood system at $\hat{x}$ in $(\hat{X}, \mathcal{T})$) such that $\hat{p}^{-1}(U) = \bigcup_{j \in J} V_j$ and $\hat{p}^{-1}(U) = \bigcup_{k \in K} \hat{V}_k$, where $V_j \in \mathcal{T}'$ ($j \in J$) are disjoint and $\hat{V}_k \in \mathcal{T}''$ ($k \in K$) are disjoint. If all the mappings $\hat{p}|_{V_j} : V_j \to U$ ($j \in J$) and $\hat{p}|_{\hat{V}_k} : \hat{V}_k \to \hat{U}$ ($k \in K$) are essential homeomorphisms, then $\hat{p} : \hat{Y} \to \hat{X}$ is called an essential covering map. If all the mappings $\hat{p}|_{V_j} (j \in J)$ and $\hat{p}|_{\hat{V}_k} (k \in K)$ are exact homeomorphisms, then $\hat{p} : \hat{Y} \to \hat{X}$ is called a (or an exact) covering map.

Let $\hat{p} : \hat{Y} \to \hat{X}$ be an exact (resp. essential) covering map. A fiber-preserving (exact) (resp. essential) homeomorphism $\hat{f} : \hat{Y} \to \hat{X}$ is called a (or an exact) (resp. an essential) deck transformation of $\hat{p} : \hat{Y} \to \hat{X}$. We denote the set of all (exact) deck transformations of $\hat{p} : \hat{Y} \to \hat{X}$ by Deck($\hat{Y} \to \hat{X}$) or Deck($\hat{Y}/\hat{X}$). Then Deck($\hat{Y}/\hat{X}$) forms a group with operation the composition of mappings.

### 2.4 A universal topological space derived by a presheaf

Suppose $(X, \mathcal{T}; \hat{X}, \hat{\mathcal{T}})$ is a universal topological space with a basic open (resp. punctured) partial neighborhood basis $\mathcal{B}$ (resp. $\mathcal{B}$). Suppose $(\mathcal{F}, \rho)$, where $\mathcal{F} = (\mathcal{F}(U))_{U \in \mathcal{T}}$, is a presheaf of some algebraic system on $(X, \mathcal{T})$ (refer to [10, §6]). Denote $f|_V := \rho^U_V(f)$ for $U, V \in \mathcal{T}, V \subseteq U$ and $f \in F(U)$.

For $\hat{U} \in \hat{\mathcal{T}}$, $\hat{U} \neq \emptyset$, let $U$ be the interior of $\hat{U} \setminus \mathcal{I}$ in $(X, \mathcal{T})$ ($\mathcal{I}$ is the ideal point set). Such an open set $U$ in $(X, \mathcal{T})$ is not empty, called the body of $\hat{U}$ and denoted $\hat{U}^o$. If $\hat{U} \in \hat{\mathcal{T}}(x)$ ($x \in \hat{X}$), then the interior of $\hat{U} \setminus \mathcal{I}\{x\}$ in $(X, \mathcal{T})$ is also a nonempty open set, which we call the (T-)open punctured partial neighborhood of $x$ corresponding to $\hat{U}$. Denote the set of all $\mathcal{T}$-open punctured partial neighborhoods of $x$ by $\hat{\mathcal{T}}^o(x)$ and let $\hat{\mathcal{T}}^o := \bigcup_{x \in \hat{X}} \hat{\mathcal{T}}^o(x)$.
there exists $x$ for that this really is an equivalence relation. The set $\Delta X$ of all equivalence classes is called the germ of $X$, in $(x)$ of all equivalence classes is called the punctured germ $X$, in $(x)$ of all equivalence classes is called the punctured partial stalk of $F$ at the point $x \in \hat{X}$ and the equivalence class of $g \in F(\hat{X}(x))$ is called the punctured partial germ of $g$ at $x$, denoted $\langle g \rangle_x$.

For $x \in X$ we denote the (usual) germ of $f \in F(U)$ at $x$ by $[f]_x$. For $g \in F(\hat{X}(x))$, if there exist $U \in \mathcal{T}(x)$ (the open neighborhood system at the point $x$ in $(X, \mathcal{T})$), $f \in F(U)$ and $V \in \hat{X}(x)$ such that $f|_V = g|_V$, then $x$ is called a usual point of $\langle g \rangle_x$ or $g$, a usual element at $x$ corresponding to $g$, $\langle g \rangle_x$ a punctured partial germ and $g$ usual at $x$. Denote $\langle f \rangle_x := \langle f|_V \rangle_x$. Then $\langle f \rangle_x = \langle g \rangle_x$. In this case we say that $\langle g \rangle_x$ and $[f]_x$ are equivalent, denoted $\langle g \rangle_x \sim [f]_x$.

By the condition of a perfect filterbase structure system on $X$ we know that any $U \in \mathcal{T}$ is not a singleton. Now assume $(X, \mathcal{T})$ is a $T_1$ space. For $x \in X$ let $\hat{\mathcal{T}}(x) := \{U \setminus \{x\} : U \in \mathcal{T}, x \in U\}$, which is the punctured open neighborhood system at the point $x$ in $(X, \mathcal{T})$, and

$$\hat{\mathcal{T}} := \bigcup_{x \in \hat{X}} \hat{\mathcal{T}}(x).$$

Then $\hat{\mathcal{T}} \subseteq \mathcal{T}$. Let

$$F(\hat{\mathcal{T}}(x)) := \bigcup_{U \in \hat{\mathcal{T}}(x)} F(U).$$

In $F(\hat{\mathcal{T}}(x))$, two elements $f_1 \in F(U_1)$ and $f_2 \in F(U_2)$ ($U_1, U_2 \in \hat{\mathcal{T}}(x)$) are said to be equivalent at $x$, denoted $f_1 \sim_x f_2$, if there exists $U \in \hat{\mathcal{T}}(x)$ with $U \subseteq U_1 \cap U_2$ such that $f_1|_U = f_2|_U$. Easily we see that this is an equivalence relation. The set

$$\hat{F}_x := F(\hat{\mathcal{T}}(x)) / \sim_x$$

of all equivalence classes is called the punctured stalk of $F$ at $x \in \hat{X}$ and the equivalence class of $f \in F(\hat{\mathcal{T}}(x))$ is called the punctured germ of $f$ at $x$, denoted $[f]_x$. For $f \in F(\hat{\mathcal{T}}(x))$ $(x \in X)$, letting $f \in F(U_0)$ ($U_0 \in \hat{\mathcal{T}}(x)$), if there exist $U \in \mathcal{T}(x)$, $h \in F(U)$ and $V \in \hat{\mathcal{T}}(x)$ such that $V \subseteq U_0 \cap U$ and $h|_V = f|_V$, then
x is called a full point of f or $[f]^\circ_x$, h a full element at x corresponding to f, \([f]^\circ_x\) full and f full at x. Denote $[h]^\circ_x := [h|_V]^\circ_x$. Then $[h]^\circ_x = [f]^\circ_x$. In this case we say that $[f]^\circ_x$ and $[h]^\circ_x$ are equivalent, denoted $[f]^\circ_x \sim [h]^\circ_x$.

For $g \in \mathcal{F}(\hat{T}(\circ)(x))$ (\(x \in X\)), letting $g \in \mathcal{F}(V_0)$ (\(V_0 \in \hat{T}(\circ)(x)\)), if there exist \(U \in \hat{T}(x)\), \(f \in \mathcal{F}(U)\) and \(V \in \hat{T}(\circ)(x)\) such that \(V \subseteq V_0 \cap U\) and \(f|_V = g|_V\), then \(x\) is called an unbranched point or a complete point of \(\langle g \rangle_x\) or \(g\), f a complete element at x corresponding to \(g\), \(\langle g \rangle_x\) unbranched or complete and \(g\) unbranched or complete at x. Denote \(\langle f \rangle_x := \langle f|_V \rangle_x\). Then \(\langle f \rangle_x = \langle g \rangle_x\). In this case we say that \(\langle g \rangle_x\) and \([f]^\circ_x\) are equivalent, denoted \(\langle g \rangle_x \sim [f]^\circ_x\). Obviously, the usualness of \(\langle g \rangle_x\) implies its completeness.

**Remark 1.** If \(\hat{\mathcal{B}}(x)\) is a basis for \(\hat{T}(x)\) at \(x \in X\), then the punctured partial germ at \(x\) and the punctured germ at \(x\) are just the same.

If \(\mathcal{F}\) is a presheaf of fields (resp. rings, vector spaces, etc), then the punctured partial stalk \(\hat{\mathcal{F}}_x(x \in \hat{X})\) and the punctured stalk \(\mathcal{F}_x(x \in \hat{X})\) with the operation defined on punctured partial germs and punctured germs respectively, by means of the operation defined on representatives, are both fields (resp. rings, vector spaces, etc).

Define
\[
\hat{\mathcal{S}} := \bigcup_{x \in \hat{X}} \hat{\mathcal{F}}_x
\]

and
\[
\hat{\mathcal{S}}^\circ := \bigcup_{x \in \hat{X}} \hat{\mathcal{F}}^\circ_x,
\]

which are the disjoint unions of all the punctured partial stalks over \(\hat{X}\) and all the punctured stalks over \(X\), respectively. Let
\[
\hat{\rho} : \hat{\mathcal{S}} \to \hat{X} \quad \text{and} \quad \hat{\rho} : \hat{\mathcal{S}}^\circ \to X
\]

be the projections, i.e. \(\hat{\rho}(\langle g \rangle_x) = x\) for \(\langle g \rangle_x \in \hat{\mathcal{S}} (x \in \hat{X})\) and \(\hat{\rho}([f]^\circ_x) = x\) for \([f]^\circ_x \in \hat{\mathcal{S}} (x \in X)\).

Let \(\mathcal{F}^\circ_x\) be the set of all the complete punctured partial germs in \(\hat{\mathcal{F}}_x(x \in \hat{X})\), which is called the complete (or unbranched) punctured partial stalk of \(\mathcal{F}\) at \(x\). Let
\[
\mathcal{S}^\circ := \bigcup_{x \in X} \mathcal{F}^\circ_x,
\]

which is the disjoint union of all the complete punctured partial stalks over \(X\), and \(p := \hat{\rho}|_{\mathcal{S}^\circ}\). Then
\[
p : \mathcal{S}^\circ \to X
\]

is also a projection.

For nonempty \(\hat{U} \in \hat{T}\) with body \(\hat{U}^\circ \in T\) and \(f \in \mathcal{F}(\hat{U}^\circ)\) we denote
\[
\langle \hat{U}, f \rangle := \{\langle f \rangle_x : x \in \hat{U}\}.
\]
Define 
\[ \hat{\mathcal{N}}(\mathcal{F}) := \{ \langle \hat{U}, f \rangle : 0 \neq \hat{U} \in \hat{T}, f \in \mathcal{F}(\hat{U}^\circ) \} \]
and 
\[ \hat{\mathcal{N}}(\mathcal{F}(\hat{\mathcal{B}})) := \{ \langle \hat{B}, f \rangle : \hat{B} \in \hat{\mathcal{B}}, f \in \mathcal{F}(\hat{B}^\circ) \}, \]
where \( \hat{B}^\circ \in \mathcal{B} \) is the nonempty body of \( \hat{B} \). For \( f \in \mathcal{F}(B) \), where \( B \in \mathcal{B}(x) \) \( (x \in \hat{X}) \), define
\[ \hat{\mathcal{N}}_f(\mathcal{F}(\hat{\mathcal{B}}))(\langle f \rangle_x) := \{ \langle \hat{U}, f|_{\hat{U} \setminus \{x\}} \rangle : \hat{U} \in \hat{\mathcal{B}}(x), \hat{U} \subseteq \hat{B} \}, \]
where \( \hat{B} = B \cup \{x\} \in \hat{\mathcal{B}}(x) \). Then it is easy to verify that \( \hat{\mathcal{N}}_f(\mathcal{F}(\hat{\mathcal{B}}))(\langle f \rangle_x) \) is a filter base and
\[ \hat{\mathcal{N}}(\mathcal{F}(\hat{\mathcal{B}})) = \bigcup \{ \hat{\mathcal{N}}_f(\mathcal{F}(\hat{\mathcal{B}}))(\langle f \rangle_x) : f \in \mathcal{F}(B), B \in \mathcal{B}(x), x \in \hat{X} \}. \]
For nonempty \( U \in \mathcal{T} \), if there exists a point \( a \in U \) such that \( f \in \mathcal{F}(U \setminus \{a\}) \), then we denote \( f \in \hat{\mathcal{F}}(U) \), or generally we may define
\[ \hat{\mathcal{F}}(U) := \{ f : f \in \mathcal{F}(U \setminus C), \text{ where } C \text{ is a discrete point set in } U \}. \quad (2.1) \]
For \( f \in \hat{\mathcal{F}}(U) \) we denote
\[ \langle U, f \rangle := \{ \langle f \rangle_x : x \in U \}. \]
Define
\[ \mathcal{N}^0(\mathcal{F}) := \{ \langle U, f \rangle : \emptyset \neq U \in \mathcal{T}, f \in \hat{\mathcal{F}}(U) \} \]
and
\[ \mathcal{N}(\mathcal{F}(\mathcal{B})) := \{ \langle B, f \rangle : B \in \mathcal{B}, f \in \mathcal{F}(B) \}. \]
For \( f \in \mathcal{F}(B) \), where \( B \in \mathcal{B}(x) \) \( (x \in \hat{X}) \), define
\[ \mathcal{N}_f(\mathcal{F}(\mathcal{B}))(\langle f \rangle_x) := \{ \langle U, f|_U \rangle : U \in \mathcal{B}(x), U \subseteq B \} \]
and
\[ \mathfrak{M}(\mathcal{F}(\mathcal{B})) := \{ \mathcal{N}_f(\mathcal{F}(\mathcal{B}))(\langle f \rangle_x) : f \in \mathcal{F}(B), B \in \mathcal{B}(x), x \in \hat{X} \}. \quad (2.2) \]
Then \( \mathcal{N}_f(\mathcal{F}(\mathcal{B}))(\langle f \rangle_x) \) is a filter base and
\[ \mathcal{N}(\mathcal{F}(\mathcal{B})) = \bigcup \{ \mathcal{N} : \mathcal{N} \in \mathfrak{M}(\mathcal{F}(\mathcal{B})) \}. \]

If for a common complete point \( x \in X \) of \( g_1 \in \mathcal{F}(V_1) \) and \( g_2 \in \mathcal{F}(V_2) \), where \( V_1, V_2 \in \hat{\mathcal{T}}^\circ(x) \), letting \( f_j|_{V_j \cap U_j} = g_j|_{V_j \cap U_j} \), where \( f_j \in \mathcal{F}(U_j) \), \( U_j \in \hat{T}(x) \) \( (j = 1, 2) \), the equality \( \langle g_1 \rangle_x = \langle g_2 \rangle_x \) always implies \([f_1]_x^\circ = [f_2]_x^\circ \), then the presheaf \( \mathcal{F} \) is called consistent at \( x \) (on \( (X, \mathcal{T}; \hat{X}, \hat{T}) \)). The presheaf is called consistent (on \( (X, \mathcal{T}; \hat{X}, \hat{T}) \)) if it is consistent at all the complete points.
For a partial point set $A$ ($I \subseteq A \subseteq \hat{X}$) let $\hat{F}_x(A) := \hat{F}_x$ (the punctured partial stalk at $x$) for $x \in A$ and $\hat{F}_x(A) := \hat{F}_x^\circ$ (the complete punctured partial stalk at $x$) for $x \in X \setminus A$. Denote 

$$\hat{\mathcal{F}}(A) := \bigcup_{x \in \hat{X}} \hat{F}_x(A)$$

and define 

$$\hat{N}(\hat{F}(\hat{\mathcal{B}}_A)) := \{\langle V, f \rangle : V \in \hat{\mathcal{B}}_A(x), f \in \mathcal{F}(V \setminus \{x\}), x \in \hat{X}\},$$

where $\hat{\mathcal{B}}_A(x)$ is an $A$-mixed neighborhood basis at $x \in \hat{X}$.

**Theorem 2.1.** Suppose $(X, \mathcal{T}; \hat{X}, \hat{\mathcal{T}})$ is a universal topological space with a basic open (resp. punctured) partial neighborhood basis $\hat{\mathcal{B}}$ (resp. $\mathcal{B}$). Suppose $\mathcal{F}$ is a presheaf on $(X, \mathcal{T})$. In (2), (3) and (4) below we also suppose $(X, \mathcal{T})$ is a $T_1$ space and $\mathcal{F}$ is consistent. Then

1. $\hat{N}(\hat{F}(\hat{\mathcal{B}}))$ is a basis for a topology on $\hat{\mathcal{F}}$, and so is $\hat{N}(\mathcal{F})$ for the same topology, denoted $\hat{\mathcal{T}}(\mathcal{F})$, and the projection

$$\hat{p} : (\hat{\mathcal{F}}, \hat{\mathcal{T}}(\mathcal{F})) \longrightarrow (\hat{X}, \hat{\mathcal{T}})$$

is a local homeomorphism.

2. $\mathcal{N}^\circ(\mathcal{F})$ is a basis for a topology on $\mathcal{F}^\circ$, denoted $\mathcal{T}^\circ(\mathcal{F})$, and the projection

$$p : (\mathcal{F}^\circ, \mathcal{T}^\circ(\mathcal{F})) \longrightarrow (X, \mathcal{T})$$

is a local homeomorphism.

3. For a partial point set $A$ ($I \subseteq A \subseteq \hat{X}$), $\hat{N}(\hat{F}(\hat{\mathcal{B}}_A))$ is a basis for a topology on $\hat{\mathcal{F}}(A)$, denoted $\hat{\mathcal{T}}_A(\mathcal{F})$, and the projection

$$\hat{p}_A : (\hat{\mathcal{F}}(A), \hat{\mathcal{T}}_A(\mathcal{F})) \longrightarrow (\hat{X}, \hat{\mathcal{T}}(A))$$

is a local homeomorphism.

4. $\mathcal{N}(\mathcal{F}(\mathcal{B}))$ is a perfect filterbase structure system on $(\mathcal{F}^\circ, \mathcal{T}^\circ(\mathcal{F}))$ and under some obvious assumption $(\mathcal{F}^\circ, \mathcal{T}^\circ(\mathcal{F}); \mathcal{F}, \mathcal{T}(\mathcal{F}))$ is a universal topological space determined by $\mathcal{N}(\mathcal{F}(\mathcal{B}))$ with the basic (resp. punctured) partial neighborhood basis $\mathcal{N}(\hat{\mathcal{F}}(\hat{\mathcal{B}}))$ (resp. $\hat{\mathcal{N}}(\hat{\mathcal{F}}(\hat{\mathcal{B}})))$. Moreover, the projection

$$\hat{p} : \hat{\mathcal{F}} \longrightarrow \hat{X}$$

is an exact local homeomorphism and hence an essential local homeomorphism.

**Proof.** (1) Obviously we have

$$\hat{\mathcal{F}} = \bigcup \{\langle \hat{B}, f \rangle : \langle \hat{B}, f \rangle \in \hat{N}(\hat{\mathcal{F}}(\hat{\mathcal{B}}))\}.$$

For $\alpha_x \in \langle \hat{B}_1, f_1 \rangle \cap \langle \hat{B}_2, f_2 \rangle$, where $\alpha_x$ denotes a punctured partial germ at $x \in \hat{X}$ and $\langle \hat{B}_j, f_j \rangle \in \hat{N}(\hat{\mathcal{F}}(\hat{\mathcal{B}}))$ $(j = 1, 2)$, there exists $\langle \hat{B}, f \rangle \in \hat{N}(\mathcal{F}(\mathcal{B}))$ such that
The mapping \( \hat{x} \) is a homeomorphism.

Therefore, \( \hat{x} \) is evidently a homeomorphism.

(2) It is easy to see that

\[
\hat{\mathcal{X}} = \bigcup \{ (U, f) : (U, f) \in \mathcal{N}(\mathcal{F}) \}.
\]

For \( \alpha_x \in \langle \hat{B}, f \rangle \subseteq \langle \hat{B}_1, f_1 \rangle \cap \langle \hat{B}_2, f_2 \rangle \). This follows from that \( f_1|_B = f_2|_B =: f \) for some basic punctured partial neighborhood \( B \in \mathcal{B}(x) \) satisfying \( \hat{B} = B \cup \{ x \} \subseteq \hat{B}_1 \cap \hat{B}_2 \).

Therefore, \( \hat{\mathcal{N}}(\mathcal{F}(\hat{B})) \) is a basis for a topology on \( \hat{\mathcal{X}} \). Easiliy we can also show that \( \hat{\mathcal{N}}(\mathcal{F}) \) is a basis for the same topology.

For \( \alpha_x \in \hat{\mathcal{X}} \) (\( x \in \hat{X} \)), there is \( B \in \mathcal{B}(x) \) and \( f \in \mathcal{F}(B) \) such that \( \alpha_x = \langle f \rangle_x \).

The mapping

\[
p|_{\langle \hat{B}, f \rangle} : \langle \hat{B}, f \rangle \longrightarrow \hat{B},
\]

where \( \hat{B} = B \cup \{ x \} \), is evidently a homeomorphism.

(3) It is evident that

\[
\hat{\mathcal{X}}(A) = \bigcup \{ (V, f) : (V, f) \in \hat{\mathcal{N}}(\mathcal{F}(\hat{B}_A)) \}.
\]

To prove \( \hat{\mathcal{N}}(\mathcal{F}(\hat{B}_A)) \) is a topological basis we need to show this in three cases. We now consider the case that \( \alpha_x \in \langle \hat{B}, f_1 \rangle \cap \langle U, f_2 \rangle \), where \( \alpha_x \) denotes a punctured partial germ at \( x \in X \), \( x \in U \) and \( \langle U, f_1 \rangle \in \mathcal{N}(\mathcal{F})(j = 1, 2) \) we have \( \langle f_1 \rangle_x = \langle f_2 \rangle_x \). By the consistency of \( \mathcal{F} \) it follows \( [f_1]^0 = [f_2]^0 \). Thus there is \( U \in \mathcal{T}(x) \) satisfying \( U \subseteq U_1 \cap U_2 \) and \( f|_{U_1 \setminus \{ x \}} = f_2|_{U_0 \setminus \{ x \}} \).

Let \( f = f_1|_{U_0 \setminus \{ x \}} \). Then we have \( \alpha_x = \langle f \rangle_x \in \langle U, f \rangle \subseteq \langle U_1, f_1 \rangle \cap \langle U_2, f_2 \rangle \).

Therefore \( \mathcal{N}(\mathcal{F}) \) is a basis for a topology on \( \hat{\mathcal{X}} \). For \( \alpha \in \hat{\mathcal{X}} \) there is \( \hat{f} \in \hat{\mathcal{F}}(U) \), where \( U \in \mathcal{T}(x) \) (\( x \in X \)), such that \( \alpha = \langle f \rangle_x \).

The mapping

\[
p|_{\langle U, f \rangle} : \langle U, f \rangle \longrightarrow U
\]

is a homeomorphism.

(4) For \( \alpha \in \hat{\mathcal{X}} \) there is \( x \in \hat{X} \) and \( f \in \mathcal{F}(B) \), where \( B \in \mathcal{B}(x) \), such that \( \alpha = \langle f \rangle_x \).

Thus

\[
\hat{\mathcal{N}}(\mathcal{F}(\hat{B}))(\langle f \rangle_x) \longrightarrow \alpha
\]
in topology \( \hat{T}(F) \) and if \( \alpha \in \hat{\mathcal{F}}^0 \) then the above limit also holds in topology \( \mathcal{T}^0(F) \).

Now let \( \alpha = \langle f \rangle_x \in \hat{\mathcal{F}}^0 \) and assume

\[
\mathcal{N}_\alpha(F(B))((g)_x) \rightarrow \alpha
\]

in topology \( \mathcal{T}^0(F) \), where \( f \in F(U) \), \( U \in \hat{T}(x) \), \( g \in F(B) \), \( B \in B(x) \) and \( x \in X \). Then there exists \( V \in B(x) \), \( V \subseteq B \), such that

\[
\langle V, g|_V \rangle \subseteq \langle U, f \rangle.
\]

Hence \( V \subseteq U \) and \( \langle g \rangle_y = \langle f \rangle_y \) for each \( y \in V \). So \( \langle V, g|_V \rangle = \langle V, f|_V \rangle \). This implies that if both \( \mathcal{N}_\alpha(F(B))((g)_x) \) and \( \mathcal{N}_\beta(F(B))((h)_x) \) converge to \( \alpha \) in topology \( \mathcal{T}^0(F) \) then they are equivalent to one another. Similarly we see that if both \( \mathcal{N}_\alpha(F(B))((g)_x) \) and \( \mathcal{N}_\beta(F(B))((h)_x) \) \( (x \in \hat{X}) \) converge to \( \alpha \in \hat{\mathcal{F}}^0 \) in topology \( \hat{T}(F) \) then they are also equivalent to one another. Assume the ideal points are just the incomplete punctured partial germs. Then \( \hat{\mathcal{N}}(F(B)) \) is a perfect filterbase structure system on \( (\hat{\mathcal{F}}^0, \mathcal{T}^0(F)) \) and \( (\hat{\mathcal{F}}^0, \mathcal{T}^0(F); \hat{\mathcal{F}}, \hat{T}(F)) \) is a universal topological space determined by \( \hat{\mathcal{N}}(F(B)) \) with the basic (resp. punctured) partial neighborhood basis \( \hat{\mathcal{N}}(F(B)) \) (resp. \( \hat{\mathcal{N}}(F(B)) \)).

Let \( \alpha_a \in \hat{\mathcal{F}}^0 \) \((a \in X)\) and \( \alpha_b \in \hat{\mathcal{F}} \) \((b \in \hat{X})\). Then there exist \( f \in \hat{F}(U) \) and \( g \in F(B) \), where \( U \in \hat{T}(a) \) and \( B \in B(b) \), such that \( \alpha_a = \langle f \rangle_a \) and \( \alpha_b = \langle g \rangle_b \). It is easy to see that both

\[
\hat{p}|_{\langle U, f \rangle} : \langle U, f \rangle \rightarrow U
\]

and

\[
\hat{p}|_{\langle \hat{B}, g \rangle} : \langle \hat{B}, g \rangle \rightarrow \hat{B}
\]

are (exact) homeomorphisms, where \( \hat{B} = B \cup \{ b \} \).

We call the space \((\hat{\mathcal{F}}^0, \mathcal{T}^0(F); \hat{\mathcal{F}}, \hat{T}(F))\) in the above theorem the derived universal topological space over \((X, \hat{T}, \hat{X}, \hat{T})\) by \( F \).

A \( T_1 \) space \((X, T)\) is called strongly locally connected if for any \( x \in X \) and \( U \in T(x) \) there exists \( V \in T(x) \) such that \( V \subseteq U \) and \( V \setminus \{ x \} \) is a domain (nonempty connected open set). A universal topological space \((X, T; \hat{X}, \hat{T})\) is called locally connected if \((X, T)\) is strongly locally connected and there exists a punctured partial neighborhood basis \( B_1 \) on \( \hat{X} \) satisfying every set \( B \) in \( B_1 \) is connected in \((X, T)\) (we say \( B_1 \) is connected). Here \( B_1 \) may be different from the basic punctured partial neighborhood basis \( B \) of \( \hat{X} \).

The uniqueness condition on a presheaf \( F \) on a universal topological space \((X, T; \hat{X}, \hat{T})\) means the following one: For every domain \( Y \) in \((X, T)\), given any \( f, g \in F(Y) \) and any \( a \in \hat{X} \) satisfying there exists \( B \in B(a) \) with \( B \subseteq Y \), the equality \( (f)_a = (g)_a \) always implies \( f = g \).

**Theorem 2.2.** Suppose \((X, T; \hat{X}, \hat{T})\) is locally connected Hausdorff universal topological space and \( F \) is a presheaf on \((X, T)\). If \( F \) satisfies the uniqueness condition on \( \hat{X} \), then both \((\hat{\mathcal{F}}^0, \mathcal{T}^0(F))\) and \((\hat{\mathcal{F}}, \hat{T}(F))\) are Hausdorff spaces, furthermore, \((\hat{\mathcal{F}}^0, \mathcal{T}^0(F); \hat{\mathcal{F}}, \hat{T}(F))\) is a Hausdorff universal topological space.
Proof. Since the local connectedness of the Hausdorff space \((X, \mathcal{T}; \hat{X}, \hat{\mathcal{T}})\) and the uniqueness condition on \(\mathcal{F}\) imply that \(\mathcal{F}\) is consistent on \(\hat{X}\), by Theorem 2.1 we know that \((\mathfrak{F}^0, \mathfrak{T}^0(\mathcal{F}); \mathfrak{F}, \hat{\mathcal{T}}(\mathcal{F}))\) is a universal topological space. In the following we prove the spaces are Hausdorff.

At first, suppose \(\langle f \rangle_x \neq \langle g \rangle_y\), where \(x, y \in \hat{X}\), \(f \in \mathcal{F}(B_1)\), \(g \in \mathcal{F}(B_2)\), \(B_1 \in \mathcal{B}(x)\) and \(B_2 \in \mathcal{B}(y)\) (\(\mathcal{B}\) is a punctured partial neighborhood basis). If \(x \neq y\), then there exist \(\hat{U}_1 \in \hat{\mathcal{B}}(x)\) and \(\hat{U}_2 \in \hat{\mathcal{B}}(y)\) such that \(\hat{U}_1 \subseteq B_1 \cup \{x\}\), \(\hat{U}_2 \subseteq B_2 \cup \{y\}\) and \(\hat{U}_1 \cap \hat{U}_2 = \emptyset\) (\(\hat{\mathcal{B}}\) is the partial neighborhood basis corresponding to \(\mathcal{B}\)). Clearly we have
\[
\langle \hat{U}_1, f \rangle \cap \langle \hat{U}_2, g \rangle = \emptyset.
\]

If \(x = y\), then by the local connectedness of \((X, \mathcal{T}; \hat{X}, \hat{\mathcal{T}})\) we may assume \(\mathcal{B}\) is connected and there exists \(B \in \mathcal{B}(x)\) such that \(B \subseteq B_1 \cap B_2\). By the uniqueness condition we have
\[
\langle \hat{B}, f|_B \rangle \cap \langle \hat{B}, g|_B \rangle = \emptyset
\]
(\(\hat{B} = B \cup \{x\}\)), since otherwise it follows that there exists \(a \in \hat{B}\) such that \(\langle f \rangle_a = \langle g \rangle_a\), which implies \(f|_B = g|_B\), so \(\langle f \rangle_x = \langle g \rangle_y\) \((x = y)\), a contradiction. By the reasoning above we see that \((\mathfrak{F}, \hat{\mathcal{T}}(\mathcal{F}))\) is Hausdorff.

Next, suppose \(\langle f \rangle_x \neq \langle g \rangle_y\), where \(x, y \in X\), \(f \in \mathcal{F}(U_1 \setminus \{x\})\) \((U_1 \in \mathcal{T}(x))\) and \(g \in \mathcal{F}(U_2 \setminus \{y\})\) \((U_2 \in \mathcal{T}(y))\). If \(x \neq y\), then there exist \(V_1 \in \mathcal{T}(x)\) and \(V_2 \in \mathcal{T}(y)\) such that \(V_1 \subseteq U_1\), \(V_2 \subseteq U_2\) and \(V_1 \cap V_2 = \emptyset\). Therefore
\[
\langle V_1, f|_{V_1 \setminus \{x\}} \rangle \cap \langle V_2, g|_{V_2 \setminus \{y\}} \rangle = \emptyset.
\]
If \(x = y\), then by the local connectedness of \((X, \mathcal{T}; \hat{X}, \hat{\mathcal{T}})\), there exists \(V \in \mathcal{T}(x)\) such that \(V \subseteq V_1 \cap V_2\) and \(V \setminus \{x\} \neq \emptyset\) is a domain in \((X, \mathcal{T})\). By the uniqueness condition we have
\[
\langle V, f|_{V \setminus \{x\}} \rangle \cap \langle V, g|_{V \setminus \{x\}} \rangle = \emptyset.
\]
This follows from that otherwise there exists \(a \in V\) such that \(\langle f \rangle_a = \langle g \rangle_a\), which implies \(f|_{V \setminus \{x\}} = g|_{V \setminus \{x\}}\), hence \(\langle f \rangle_x = \langle g \rangle_y\) \((x = y)\), a contradiction. Consequently, \((\mathfrak{F}^0, \mathfrak{T}^0(\mathcal{F}); \mathfrak{F}, \hat{\mathcal{T}}(\mathcal{F}))\) is Hausdorff.

To prove \((\mathfrak{F}^0, \mathfrak{T}^0(\mathcal{F}); \mathfrak{F}, \hat{\mathcal{T}}(\mathcal{F}))\) is Hausdorff, we assume \(x \in X\), \(y \in \hat{X}\), \(x \neq y\) and \(\langle f \rangle_x \neq \langle g \rangle_y\), where \(f \in \mathcal{F}(U)\) \((U \in \mathcal{T}(x))\) and \(g \in \mathcal{F}(B)\) \((B \in \mathcal{B}(y))\) is a punctured neighborhood basis). Then there exist \(U_1 \in \mathcal{T}(x)\) and \(\hat{B}_1 \in \hat{\mathcal{B}}(y)\) such that \(U_1 \subseteq U \cup \{x\}\), \(\hat{B}_1 \subseteq B \cup \{y\}\) and \(U_1 \cap \hat{B}_1 = \emptyset\). Thus we have
\[
\langle U_1, f \rangle \cap \langle \hat{B}_1, g \rangle = \emptyset.
\]

Recall \(\mathfrak{F}\) is the disjoint union of all the punctured stalks over \(X\). Denote
\[
[U, f] \circ := \{[f]_x^\circ : x \in U\}
\]
for \(U \in \mathcal{T}\) and \(f \in \mathfrak{F}(U)\), and define
\[
\mathfrak{N}(\mathcal{F}) := \{[U, f] \circ : U \in \mathcal{T}, f \in \mathfrak{F}(U)\}.
\]
As a special case of Theorems 2.1 and 2.2 (refer to Remark 11) we have
Theorem 2.3. Suppose $(X, T)$ is a strongly locally connected Hausdorff topological space and $F$ is a presheaf on $(X, T)$ which satisfies the uniqueness condition. Then $\mathcal{N}(F)$ is a basis for a topology on $\hat{\mathfrak{X}}$, denoted $\hat{T}(F)$. Moreover, $(\hat{\mathfrak{X}}, \hat{T}(F))$ is a Hausdorff space and the projection

$$\hat{p}: (\hat{\mathfrak{X}}, \hat{T}(F)) \longrightarrow (X, T)$$

is a local homeomorphism.

Here the uniqueness condition on a presheaf $F$ on a $T_1$ space $(X, T)$ means: For every domain $Y$ in $X$, the equality $[f]_a^a = [g]_a^a$, where $f, g \in F(Y)$ and $a$ is any point in $X$ satisfying $Y \setminus \{a\}$ is a punctured neighborhood of $a$, always implies $f = g$.

We can also directly prove Theorem 2.3 similarly to [10, Theorems (6.8) and (6.10)].

2.5 A porous universal topological space

Suppose $X$ is a topological space. Let $U$ be an open set and $E$ a set in $X$. If for any nonempty domain $D \subseteq U$, the interior of $D \setminus E$ is a nonempty subdomain of $D$ and for any open subset $G$ of $U$ we have $G \subseteq G \setminus E$ (the closure of $G \setminus E$), then we say that $E$ is quasi-discrete in $U$ and $U \setminus E$ is porous corresponding to $U$.

Here we also attach the following conditions: (1) if $E_1, E_2$ and $E$ are quasi-discrete in $U$ then $E_1 \cup E_2$ and all subsets of $E$ are quasi-discrete in $U$; (2) every discrete subset of $X$ is quasi-discrete.

If $(X, \mathcal{T}; \hat{X}, \hat{\mathcal{T}})$ is a Hausdorff universal topological space (i.e. $(\hat{X}, \hat{\mathcal{T}})$ is Hausdorff) and $(X, \mathcal{T})$ is $T_3$ (Hausdorff and regular), then we say $\hat{X}$ is $T_3$. Suppose $(X, \mathcal{T}; \hat{X}, \hat{\mathcal{T}})$ is a $T_3$ universal topological space with a basic punctured partial neighborhood basis $B$ determined by a perfect filterbase structure system $\mathfrak{B}$. For $B \in \mathfrak{B}$, denote

$$B^\circ := \{B \setminus E : B \in \mathfrak{B} \text{ and } E \text{ is closed and quasi-discrete in } B\}$$

and

$$\mathfrak{B}^\circ := \{B^\circ : B \in \mathfrak{B}\}.$$ 

Then easily we see that $\mathfrak{B}^\circ$ is a filter base, which we call a porous filter base corresponding to $\mathfrak{B}$ or $\mathfrak{B}$. Assume $\mathfrak{B}^\circ \to x$ precisely if $B \to x$ for an ideal point $x$ in $\hat{X}$. Then $\mathfrak{B}^\circ$ is a perfect filterbase structure system on $(X, \mathcal{T})$, which we call a perfect porous filterbase structure system corresponding to $\mathfrak{B}$. Let $\mathfrak{B}^\circ(x)$ (resp. $\mathfrak{B}^\circ(x)$) be a basic (resp. punctured) partial neighborhood basis at $x \in \hat{X}$ corresponding to $\mathfrak{B}^\circ$. Let

$$\hat{\mathfrak{B}}^\circ := \bigcup_{x \in \hat{X}} \hat{\mathfrak{B}}^\circ(x)$$

and

$$\mathfrak{B}^\circ := \bigcup_{x \in X} \mathfrak{B}^\circ(x),$$

and
which are called the basic porous partial neighborhood basis and basic punctured porous partial neighborhood basis for \((X, \mathcal{T}, \hat{X}, \hat{\mathcal{T}})\), respectively.

Suppose \(\hat{\mathcal{T}}^{po}\) is the filterbase topology determined by \(\mathcal{B}^{po}\). We call \(\hat{\mathcal{T}}^{po}\) the porous filterbase topology or porous partial topology on \(\hat{X}\) determined by \(\mathcal{B}\) and \((\hat{X}, \hat{\mathcal{T}}^{po})\) the porous partial topological space corresponding to \((\hat{X}, \hat{\mathcal{T}})\). Let \(\hat{T}^{po}(x)\) denote the set of all punctured open partial neighborhoods of \(x \in \hat{X}\) in \((\hat{X}, \hat{\mathcal{T}}^{po})\)

\[
\hat{T}^{po} := \bigcup_{x \in \hat{X}} \hat{T}^{po}(x).
\]

We call \(\hat{T}^{po}\) (resp. \(\hat{T}^{po}(x)\)) the punctured porous open partial neighborhood system (resp. at \(x\)) on \(\hat{X}\).

Similarly by the perfect porous filterbase structure system

\[
\check{T}^{po} := \{\mathcal{B}^{po} : \mathcal{B} \in \check{\mathcal{T}}\}
\]  

in \(X\), where \(\check{\mathcal{T}} := \{\check{T}(x) : x \in X\}\) \(\check{T}(x)\) is the set of all punctured open neighborhoods of \(x \in X\) in \((X, \check{\mathcal{T}})\), we obtain a filterbase topology, denoted \(\check{T}\), which is called the porous topology on \(X\). The space \((X, \check{T})\) is called the porous topological space corresponding to \((X, \check{\mathcal{T}})\). Let \(\check{T}(x)\) denote the set of all punctured open neighborhoods of \(x \in X\) in \((X, \check{\mathcal{T}})\) and let

\[
\check{T} := \bigcup_{x \in X} \check{T}(x).
\]

We call \(\check{T}\) (resp. \(\check{T}(x)\)) the punctured porous open neighborhood system (resp. at \(x\)) on \((X, \check{\mathcal{T}})\).

For a partial point set \(A\) \((I \subseteq A \subseteq \hat{X})\), let \(\hat{\mathcal{B}}^{po}_A(x)\) (resp. \(\mathcal{B}^{po}_A(x)\)) be a (resp. punctured) porous open neighborhood basis at \(x\) in \((X, \mathcal{T})\) for \(x \in X \setminus A\) and \(\mathcal{B}^{po}_A(x) := \mathcal{B}^{po}(x)\) (resp. \(\mathcal{B}^{po}_A(x) := \mathcal{B}^{po}(x)\)) for \(x \in A\). Then \(\mathcal{B}^{po}_A(x)\) and \(\mathcal{B}^{po}(x)\) are filter bases. Let

\[
\hat{\mathcal{B}}^{po}_A := \bigcup_{x \in \hat{X}} \hat{\mathcal{B}}^{po}_A(x) \quad \text{and} \quad \mathcal{B}^{po}_A := \bigcup_{x \in X} \mathcal{B}^{po}_A(x).
\]

Then \(\hat{\mathcal{B}}^{po}_A\) is a basis for some topology on \((\text{set}) \hat{X}\). The topology on \(\hat{X}\) determined by \(\hat{\mathcal{B}}^{po}_A\) is called the mixed porous topology on \(\hat{X}\) with the partial point set \(A\) or the A-mixed porous topology on \(\hat{X}\) determined by \(\mathcal{B}_A\), denoted \(\hat{\mathcal{T}}^{po}(A)\). \(\mathcal{B}^{po}_A\) (resp. \(\mathcal{B}^{po}_A(x)\)) is called an A-mixed punctured porous neighborhood basis (resp. at \(x\)) on \(\hat{X}\). Denote \(\hat{\mathcal{T}}^{po} := \hat{\mathcal{T}}^{po}(I)\), which is called the porous essential topology on \(\hat{X}\). We can also define \(\mathcal{B}^{po}(x)\) \((x \in \hat{X}\) and \(\mathcal{B}^{po}\) in an obvious way to get the porous essential topology \(\check{\mathcal{T}}^{po}\).

Since \(\hat{X}\) is assumed to be \(T_3\), we easily see that \(\mathcal{B}^{po}\) is a perfect filterbase structure system on \((X, \check{\mathcal{T}})\) and \((X, \check{\mathcal{T}}; \hat{X}, \hat{\mathcal{T}}^{po})\) is a universal topological space.
Let \( \hat{\mathfrak{B}}^\times \) (with a basic (resp. punctured) open neighborhood basis \( \hat{\mathfrak{B}}^\times \) (resp. \( \mathfrak{B}^\times \)), which we call the porous universal topological space corresponding to \((X, \mathcal{T}; \hat{X}, \mathcal{T})\). Clearly, \( \hat{\mathfrak{T}}^\times (A) \) is a mixed topology of \((X, \mathcal{T}; \hat{X}, \mathcal{T})\). Specially, \( \hat{\mathfrak{T}}^\times \) is the essential topology of \((X, \mathcal{T}; \hat{X}, \mathcal{T})\).

Suppose \( \mathcal{F} \) is a presheaf of some algebraic system on \((X, \mathcal{T})\). For \( U \in \mathcal{T} \) let

\[
\hat{\mathcal{F}}(U) := \{ f \in \mathcal{F}(U \setminus E) : E \text{ is a quasi-discrete closed subset of } U \}.
\]

Let \( \hat{\mathcal{F}}_x^\times \) be the punctured partial stalk of \( \mathcal{F} \) at \( x \in \hat{X} \) in \((\hat{X}, \hat{\mathfrak{T}}^\times)\), called the punctured porous partial stalk of \( \mathcal{F} \) at \( x \in \hat{X} \) in \((\hat{X}, \hat{\mathcal{T}}^\times)\). Define

\[
\hat{\mathfrak{B}}^\times := \bigcup_{x \in \hat{X}} \hat{\mathcal{F}}_x^\times.
\]

Let \( g \in \hat{\mathcal{F}}(V) \) \((V \in \hat{\mathcal{T}}^\times(x), x \in \hat{X})\) (i.e. \( g \in \mathcal{F}(\hat{\mathfrak{T}}^\times(x)), \hat{\mathfrak{T}}^\times(x) \) is the punctured porous open partial neighborhood system at \( x \)). We call the punctured partial germ of \( g \) at \( x \) in \((\hat{X}, \hat{\mathfrak{T}}^\times)\) the punctured porous partial germ of \( g \) at \( x \) in \((\hat{X}, \hat{\mathcal{T}}^\times)\), denoted \( \langle g \rangle^\times_x \). For \( U \in \mathcal{T}, x \in U \) and \( f \in \hat{\mathcal{F}}(U) \) denote \( \langle f \rangle^\times_x := \langle f|_V \rangle^\times_x \), where \( V \in \hat{\mathfrak{T}}^\times(x), V \subseteq U \setminus E \) for some quasi-discrete closed subset \( E \) of \( U \) and \( f \in \mathcal{F}(U \setminus E) \), and denote

\[
\langle U, f \rangle^\times := \{ \langle f \rangle^\times_x : x \in U \}.
\]

For nonempty \( \hat{U} \in \hat{\mathcal{T}} \) with body \( \hat{U}^\circ \) and \( f \in \hat{\mathcal{F}}(\hat{U}^\circ) \) denote

\[
\langle \hat{U}, f \rangle^\times := \{ \langle f \rangle^\times_x : x \in \hat{U} \}.
\]

Define

\[
\hat{N}^\times(\mathcal{F}) := \{ \langle \hat{U}, f \rangle^\times : \emptyset \neq \hat{U} \in \hat{\mathcal{T}}, f \in \hat{\mathcal{F}}(\hat{U}^\circ) \}
\]

and

\[
\hat{N}^\times(\mathcal{F}(\hat{\mathfrak{B}})) := \{ \langle \hat{B}, f \rangle^\times : \hat{B} \in \hat{\mathcal{B}}, f \in \hat{\mathcal{F}}(\hat{\mathfrak{B}}^\times) \},
\]

where \( \hat{U}^\circ \in \mathcal{T} \) and \( \hat{B}^\circ \in \mathcal{B} \) are the bodies of \( \hat{U} \) and \( \hat{B} \), respectively. For \( f \in \hat{\mathcal{F}}(\mathfrak{B}) \), where \( \mathfrak{B} \in \mathcal{F}(x) \) \((x \in \hat{X})\), define

\[
\hat{N}^\times_f(\mathcal{F}(\hat{\mathfrak{B}}))(\langle f \rangle^\times_x) := \{ \langle \hat{U}, f|_{\hat{U}(\hat{x})} \rangle^\times : \hat{U} \in \hat{\mathcal{B}}(x), \hat{U} \subseteq \hat{B} \},
\]

where \( \hat{B} = B \cup \{ x \} \in \hat{\mathcal{B}}(x) \). Then \( \hat{N}^\times_f(\mathcal{F}(\hat{\mathfrak{B}}))(\langle f \rangle^\times_x) \) is a filter base and

\[
\hat{N}^\times(\mathcal{F}(\hat{\mathfrak{B}})) = \bigcup_{f \in \hat{\mathcal{F}}(\mathfrak{B})} \{ \hat{N}^\times_f(\mathcal{F}(\hat{\mathfrak{B}}))(\langle f \rangle^\times_x) : f \in \hat{\mathcal{F}}(\mathfrak{B}), B \in \mathcal{B}(x), x \in \hat{X} \}.
\]

Define

\[
\hat{N}^\times(\mathcal{F}) := \{ \langle U, f \rangle^\times : \emptyset \neq U \in \mathcal{T}, f \in \hat{\mathcal{F}}(U) \}
\]

and

\[
\hat{N}^\times(\mathcal{F}(\mathfrak{B})) := \{ \langle B, f \rangle^\times : B \in \mathcal{B}, f \in \hat{\mathcal{F}}(\mathfrak{B}) \}.
\]
For \( f \in \mathcal{F}_x^\text{po}(B) \), where \( B \in \mathcal{B}(x) \) (\( x \in \hat{X} \)), define
\[
\mathcal{N}_f^\text{po}(\mathcal{F}(\mathcal{B}))(\langle f \rangle_x^\text{po}) := \{ (U, f|_U)^\text{po} : U \in \mathcal{B}(x), \, U \subseteq B \}
\]
and
\[
\mathcal{N}^\text{po}(\mathcal{F}(\mathcal{B})) := \{ \mathcal{N}_f^\text{po}(\mathcal{F}(\mathcal{B}))(\langle f \rangle_x^\text{po}) : f \in \mathcal{F}_x^\text{po}(B), \, B \in \mathcal{B}(x), \, x \in \hat{X} \}.
\]
Then \( \mathcal{N}_f^\text{po}(\mathcal{F}(\mathcal{B}))(\langle f \rangle_x^\text{po}) \) is a filter base and
\[
\mathcal{N}^\text{po}(\mathcal{F}(\mathcal{B})) = \bigcup\{ \mathcal{N} : \mathcal{N} \in \mathcal{N}^\text{po}(\mathcal{F}(\mathcal{B})) \}.
\]

Suppose \( \mathcal{F} \) is a presheaf on a \( T_1 \) space \((X, \mathcal{T})\). Similarly to the punctured porous partial stalk and the punctured porous partial germ we can define the \textit{punctured porous stalk} of \( \mathcal{F} \) at \( x \in X \) in \((X, \mathcal{T})\), denoted \( \hat{\mathcal{F}}_x \), and the \textit{punctured porous germ} of \( f \in \mathcal{F}(U) \) (\( U \in \mathcal{T} \)) at \( x \in X \), denoted \([f]_x^\text{op}\), which are the punctured porous partial stalk and the punctured porous partial germ corresponding to the perfect porous filterbase structure system \( \mathcal{T}^\text{po} \) (see (2.3)), respectively. Let
\[
\hat{\mathcal{F}} := \bigcup_{x \in X} \hat{\mathcal{F}}_x.
\]

If for \( g \in \mathcal{F}(\hat{\mathcal{F}}^\text{po}(x)) \) (\( x \in X \)) there exists \( f \in \mathcal{F}(\mathcal{F}(x)) \) such that \( \langle f \rangle_x^\text{po} = \langle g \rangle_x^\text{po} \) (\( \mathcal{F}(x) \) is the punctured porous open neighborhood system at \( x \)), then \( x \) is called a \textit{porously complete point} of \( \langle g \rangle_x^\text{po} \) or \( g \), \( f \) a \textit{porously complete element} corresponding to \( g \) at \( x \), \( \langle g \rangle_x^\text{po} \) \textit{porously complete} and \( g \) \textit{porously complete} at \( x \) (here we may also use the terminology \textit{unbranched} to replace “complete”). In this case, we say that \( \langle g \rangle_x^\text{po} \) and \([f]_x^\text{op}\) are \textit{equivalent}, denoted \( \langle g \rangle_x^\text{po} \sim [f]_x^\text{op} \).

Suppose \( g_1, g_2 \in \mathcal{F}(\hat{\mathcal{F}}^\text{po}(x)) \) and \( x \in X \) is a common porously complete point of \( g_1 \) and \( g_2 \). Let \( f_1 \) and \( f_2 \) be porously complete elements corresponding to \( g_1 \) and \( g_2 \) at \( x \), respectively. If \( \langle g_1 \rangle_x^\text{po} = \langle g_2 \rangle_x^\text{po} \) always implies \([f_1]_x^\text{op} = [f_2]_x^\text{op}\), then the presheaf \( \mathcal{F} \) is called \textit{porously consistent} at \( x \) (on \( \hat{X} \)). If \( \mathcal{F} \) is porously consistent at all the porously complete points then we say that \( \mathcal{F} \) is \textit{porously consistent} (on \( X \)).

Let
\[
\hat{p} : \hat{\mathcal{F}}^\text{po} \rightarrow \hat{X} \quad \text{and} \quad \hat{p} : \hat{\mathcal{F}}^\text{po} \rightarrow X
\]
be the projections, i.e. \( \hat{p}(\langle g \rangle_x^\text{po}) = x \) for \( \langle g \rangle_x^\text{po} \in \hat{\mathcal{F}}^\text{po}(x) \in \hat{X} \) and \( \hat{p}([f]_x^\text{op}) = x \) for \([f]_x^\text{op} \in \hat{\mathcal{F}}^\text{po}(x) \in \hat{X} \). Let \( \mathcal{F}_x^\text{op} \) be the set of all complete punctured porous partial germs in \( \hat{\mathcal{F}}_x^\text{po}(x \in X) \), which is called the \textit{complete} (or \textit{unbranched}) \textit{punctured porous partial stalk} of \( \mathcal{F} \) at \( x \). Let
\[
\hat{\mathcal{F}}^\text{po} := \bigcup_{x \in X} \mathcal{F}_x^\text{op},
\]
and \( p := \hat{p}|_{\hat{\mathcal{F}}^\text{po}} \). Then
\[
p : \hat{\mathcal{F}}^\text{po} \rightarrow X
\]
is also a projection.

For a partial point set \( A \) let \( \hat{F}^{\text{po}}_x(A) := F^{\text{po}}_x \) for \( x \in X \setminus A \) and \( \hat{F}^{\text{po}}_x(A) := \hat{F}^{\text{po}}_x \) for \( x \in A \). Define
\[
\hat{\mathcal{F}}^{\text{po}}(A) := \bigcup_{x \in X} \hat{F}^{\text{po}}_x(A)
\]
and
\[
\hat{\mathcal{N}}^{\text{po}}(\mathcal{F}(\hat{\mathcal{B}}_A)) := \bigcup_{x \in X} \{(V, f)^{\text{po}} : V \in \hat{\mathcal{B}}_A(x), f \in \mathcal{F}(V \setminus \{x\})\}.
\]

We can obtain the following results corresponding to Theorems 2.1, 2.2 and 2.3 by similar reasoning to the proofs of the theorems, respectively.

**Theorem 2.1’.** Suppose \( (X, T; \hat{X}, \hat{T}) \) is a \( T_3 \) universal topological space with a basic open (resp. punctured) partial neighborhood basis \( \hat{\mathcal{B}} \) (resp. \( \mathcal{B} \)). Suppose \( \mathcal{F} \) is a presheaf on \( (X, T) \). In (2), (3) and (4) below, we also suppose \( \mathcal{F} \) is porously consistent. Then

1. \( \hat{\mathcal{N}}^{\text{po}}(\mathcal{F}(\hat{\mathcal{B}})) \) is a basis for a topology on \( \hat{\mathcal{F}}^{\text{po}} \), and so is \( \hat{\mathcal{N}}^{\text{po}}(\mathcal{F}) \) for the same topology, denoted \( \hat{\mathcal{F}}^{\text{po}}(\mathcal{F}) \), and the projection
   \[
   \hat{p} : (\hat{\mathcal{F}}^{\text{po}}, \hat{\mathcal{F}}^{\text{po}}(\mathcal{F})) \rightarrow (\hat{X}, \hat{T})
   \]
   is a local homeomorphism.

2. \( \mathcal{N}^{\text{po}}(\mathcal{F}) \) is a basis for a topology on \( \mathcal{F}^{\text{po}} \), denoted \( \mathcal{F}^{\text{po}}(\mathcal{F}) \), and the projection
   \[
   p : (\mathcal{F}^{\text{po}}, \mathcal{F}^{\text{po}}(\mathcal{F})) \rightarrow (X, T)
   \]
   is a local homeomorphism.

3. For a partial point set \( A (1 \subseteq A \subseteq \hat{X}) \), \( \hat{\mathcal{N}}^{\text{po}}(\mathcal{F}(\hat{\mathcal{B}}_A)) \) is a basis for a topology on \( \hat{\mathcal{F}}^{\text{po}}(A) \), denoted \( \hat{\mathcal{F}}^{\text{po}}_A(\mathcal{F}) \), and the projection
   \[
   \hat{p}_A : (\hat{\mathcal{F}}^{\text{po}}(A), \hat{\mathcal{F}}^{\text{po}}_A(\mathcal{F})) \rightarrow (\hat{X}, \hat{T}(A))
   \]
   is a local homeomorphism.

4. \( \mathcal{N}^{\text{po}}(\mathcal{F}(\mathcal{B})) \) is a perfect filterbase structure system on \( (\mathcal{F}^{\text{po}}, \mathcal{F}^{\text{po}}(\mathcal{F})) \) and under some obvious assumption \( (\mathcal{F}^{\text{po}}, \mathcal{F}^{\text{po}}(\mathcal{F}); \hat{\mathcal{F}}^{\text{po}}, \hat{\mathcal{F}}^{\text{po}}(\mathcal{F})) \) is a universal topological space determined by \( \mathcal{N}^{\text{po}}(\mathcal{F}(\mathcal{B})) \) with the basic (resp. punctured) partial neighborhood basis \( \mathcal{N}^{\text{po}}(\mathcal{F}(\mathcal{B})) \) (resp. \( \mathcal{N}^{\text{po}}(\mathcal{F}(\mathcal{B})) \)). Moreover, the projection
   \[
   \hat{p} : \hat{\mathcal{F}}^{\text{po}} \rightarrow \hat{X}
   \]
   is an exact local homeomorphism and hence an essential local homeomorphism. \( \square \)

We call the space \( (\mathcal{F}^{\text{po}}, \mathcal{F}^{\text{po}}(\mathcal{F}); \hat{\mathcal{F}}^{\text{po}}, \hat{\mathcal{F}}^{\text{po}}(\mathcal{F})) \) in the above theorem the derived porous universal topological space over \( (X, T; \hat{X}, \hat{T}) \) by \( \mathcal{F} \).

Suppose \( \mathcal{F} \) is a presheaf on \( (X, T) \) and \( Y \) is an open set in \( X \). Let \( f, g \in \mathcal{F}(Y) \). If there exists a quasi-discrete closed subset \( E \) of \( Y \) such that \( f|_{Y \setminus E} = g|_{Y \setminus E} \), then
we say that \( f \) is porously equal to \( g \), denoted \( f \equiv g \) (if \( f, g \in \hat{\mathcal{F}}(Y) \) (see (2.1)) and \( f|_{\mathcal{V}\setminus E} = g|_{\mathcal{V}\setminus E} \), where \( E \) is a discrete subset of \( Y \), then we say that \( f \) is permissibly equal to \( g \), denoted \( f \doteq g \)\). The presheaf \( \mathcal{F} \) on \( (X, \mathcal{T}) \) is said to satisfy the porous uniqueness condition on a universal topological space \((X, \mathcal{T}; \hat{X}, \hat{\mathcal{T}})\) if for every domain \( Y \) in \((X, \mathcal{T})\), given any \( f, g \in \hat{\mathcal{F}}(Y) \) and any \( a \in \hat{X} \) satisfying there exists \( B \in \mathcal{B}(a) \) with \( B \subseteq Y \), the equality \( \langle f \rangle_a = \langle g \rangle_a \) always implies \( f \equiv g \).

**Theorem 2.2′.** Suppose \((X, \mathcal{T}; \hat{X}, \hat{\mathcal{T}})\) is a locally connected \( T_3 \) universal topological space and \( \mathcal{F} \) is a presheaf on \((X, \mathcal{T})\). If \( \mathcal{F} \) satisfies the porous uniqueness condition on \( \hat{X} \), then both \((\hat{\mathcal{S}}_p, \mathcal{T}^p(\mathcal{F}))\) and \((\hat{\mathcal{S}}_{op}, \hat{\mathcal{T}}^p(\mathcal{F}))\) are Hausdorff spaces, furthermore, \((\hat{\mathcal{S}}_p, \mathcal{T}^p(\mathcal{F}); \hat{\mathcal{S}}_{op}, \hat{\mathcal{T}}^p(\mathcal{F}))\) is a \( T_3 \) universal topological space. \( \square \)

For nonempty \( U \in \mathcal{T} \) and \( f \in \hat{\mathcal{F}}(U) \) we denote
\[
[U, f]^p := \{ [f]_x^p : x \in U \}
\]
and define
\[
\mathcal{N}(\mathcal{F}) := \{ [U, f]^p : \emptyset \neq U \in \mathcal{T}, f \in \hat{\mathcal{F}}(U) \}.
\]

**Theorem 2.3′.** Suppose \((X, \mathcal{T})\) is a strongly locally connected \( T_3 \) topological space and \( \mathcal{F} \) is a presheaf on \((X, \mathcal{T})\) which satisfies the porous uniqueness condition. Then \( \hat{\mathcal{N}}(\mathcal{F}) \) is a basis for a topology on \( \hat{\mathcal{S}}_p \), denoted \( \hat{\mathcal{T}}(\mathcal{F}) \). Moreover, \( (\hat{\mathcal{S}}_p, \hat{\mathcal{T}}(\mathcal{F})) \) is a \( T_3 \) space and the projection
\[
\hat{p} : (\hat{\mathcal{S}}_p, \hat{\mathcal{T}}(\mathcal{F})) \longrightarrow (X, \mathcal{T})
\]
is a local homeomorphism. \( \square \)

The porous uniqueness condition in Theorem 2.3′ means: For every domain \( Y \) in \((X, \mathcal{T})\), the equality \( [f]_a^p = [g]_a^p \), where \( f, g \in \hat{\mathcal{F}}(Y) \) and \( a \) is any point in \( X \) satisfying \( Y \setminus \{a\} \) is a punctured neighborhood of \( a \), always implies \( f \equiv g \).

In the following we present a general case. Suppose \((X, \mathcal{T}; \hat{X}, \hat{\mathcal{T}})\) is a \( T_3 \) universal topological space and \( \mathcal{P} \) is a subset of \( \hat{X} \). Let \( U \) be an open set in \((X, \mathcal{T})\) and \( E \) a set in \( X \). If \( E \) is a quasi-discrete in \( U \) and for \( x \in X \setminus P \) there exists a punctured neighborhood \( \hat{N}(x) \) of \( x \) in \((\hat{X}, \hat{\mathcal{T}})\) such that \( \hat{N}(x) \cap U \subseteq U \setminus E \), then we say that \( E \) is \( \mathcal{P} \)-quasi-discrete in \( U \) and \( U \setminus E \) is \( \mathcal{P} \)-porous corresponding to \( U \). We call \( P \) a porous point set.

If we use \( \mathcal{P} \)-quasi-discrete (resp. \( \mathcal{P} \)-porous) sets to replace quasi-discrete (resp. porous) sets in the preceding part of this subsection, then we can get corresponding notions and results. What we need to do is just to replace “quasi”, “porous(ly)”, “po” and “op” by “\( \mathcal{P} \)-quasi”, “\( \mathcal{P} \)-porous(ly)”, “\( \mathcal{P} \)” and “\( \circ \mathcal{P} \)”, respectively. For instance, we have a \( \mathcal{P} \)-porous filter base
\[
\mathcal{B}^p := \{ B \setminus E : B \in \mathcal{B} \text{ and } E \text{ is closed and } \mathcal{P} \text{-quasi-discrete in } B \},
\]
a perfect $P$-porous filterbase structure system (in $X$ corresponding to a perfect filterbase structure system $\mathfrak{B}$)

$$\mathfrak{B}^p := \{ B^p : B \in \mathfrak{B} \},$$

the basic (resp. punctured) $P$-porous partial neighborhood basis $\mathfrak{B}^p$ (resp. $\mathfrak{B}_p$) (on $(X, T; \hat{X}, \hat{T})$), the $P$-porous filterbase topology $\hat{T}^p$, the $P$-porous universal topological space $(X, \hat{T}; \hat{X}, \hat{T}^p)$ (corresponding to $(X, T; \hat{X}, \hat{T})$), and so on. We list some other notations as follows: $\mathfrak{B}^p(x), \mathfrak{B}_p(x), \hat{T}_p(x), \mathfrak{T}_p(x), \hat{T}_p(x), \mathfrak{B}_p(x), \mathfrak{B}_p(x), \hat{T}_p(x)$, $\mathfrak{B}_p(x), \hat{T}_p(x), \hat{T}_p(x), \mathfrak{T}_p(x), \hat{T}_p(x), \hat{T}_p(x), \mathfrak{T}_p(x), \hat{T}_p(x)$, respectively.

In the same. Specially if $\varphi = \psi$, and $\varphi$ and $\psi$ are chosen from $\{ [f]_x, [f]_x^p, \langle f \rangle_x, \langle f \rangle_x^p \}$ and $\{ [g]_x, [g]_x^p, \langle g \rangle_x, \langle g \rangle_x^p \}$, respectively. Specifcally if $\varphi$ and $\psi$ are the same kind of germs then $\varphi \sim \psi$ means $\varphi = \psi$. Some of the equivalences have been defined in the preceding paragraphs and here as another example we define $[f]_x^p \sim [g]_x^p$ as follows: Suppose $f \in \mathfrak{F}(\hat{T}^p(x))$ and $g \in \mathfrak{F}(\hat{T}^p(x))$. If there exist $V \in \mathfrak{T}^p(x)$ and a quasi-discrete closed subset $E$ of $V$ such that $f|_{V \setminus E} = g|_{V \setminus E}$ and thus $\langle f \rangle_x^p := \langle f \rangle_x|_{V \setminus E} = \langle g \rangle_x^p$, then we say that $[f]_x^p$ and $[g]_x^p$ are equivalent. If necessary, we may regard two equivalent germs as the same.

3 \quad Algebraic functions and Riemann surfaces

3.1 \quad A basic Riemann surface

Suppose $X$ is a Riemann surface (in the usual sense, see e.g. [7], [10] and [30], which we will call a traditional Riemann surface later) and $T$ is its topology. Now we choose a perfect filterbase structure system $\mathfrak{B}$, whose elements consist of domains (usually simply connected ones), and then obtain a universal topological space $(X, T; \hat{X}, \hat{T})$, which we call a basic Riemann surface. Here we also assume $X$ is Hausdorff (i.e. $(X, T)$ is Hausdorff).

We recall the notion of a universal topological subspace defined in the end of Subsection 2.1. Suppose $\hat{Y} \subseteq \hat{X}$ and $I_0 \subseteq X \cap \hat{Y}$ satisfy that $I_0$ is closed
and discrete in $(X, \mathcal{J})$, $\hat{Y}$ is an open subset of $(\hat{X}, \mathcal{J}(I \cup I_0))$ ($I$ is the ideal point set of the basic Riemann surface $\hat{X}$ and $\mathcal{J}(I \cup I_0)$ is the $(I \cup I_0)$-mixed topology of $\hat{X}$) and $Y := (\hat{Y} \cap X) \setminus I_0 \neq \emptyset$. Then $Y$ is an open subset of $(X, \mathcal{J})$ and the space $(\hat{Y}, \mathcal{J}(I \cup I_0)|_{\hat{Y}})$ is Hausdorff ($\mathcal{J}(I \cup I_0)|_{\hat{Y}}$ is the induced topology). If $Y$ is connected then $Y$ is a traditional Riemann surface. Let $\mathcal{B}_Y = \{B_Y(y) : y \in \hat{Y}\}$ be the induced perfect filterbase structure system by $\mathcal{B}$ on $Y$. Then $B_Y(y) \sim \mathcal{B}(y)$ for each $y \in Y$ and the universal topological subspace $(Y, \mathcal{J}; \hat{Y}, \mathcal{J})$ is the universal topological space determined by $\mathcal{B}_Y$. If $Y$ is connected then $(Y, \mathcal{J}; \hat{Y}, \mathcal{J})$ is also a basic Riemann surface, which we call a basic (Riemann) subsurface of the basic Riemann surface $(X, \mathcal{J}; \hat{X}, \mathcal{J})$. This kind of subsurface is similar to a domain in a traditional Riemann surface. So generally the propositions which hold on a basic Riemann surface are also true on “domains” in a basic Riemann surface.

3.2 Analytic continuation

Suppose $(X, \mathcal{J}; \hat{X}, \hat{\mathcal{J}})$ is a basic Riemann surface and $\hat{\mathcal{J}}(A)$ is a mixed topology of $\hat{X}$, where $I \subseteq A \subseteq \hat{X}$ ($I$ is the ideal point set of $\hat{X}$). Let $(\mathcal{H}, \rho)$ denote the sheaf of holomorphic functions on $(X, \mathcal{J})$. Suppose $u : [0, 1] \to \hat{X}$ is a curve in $(\hat{X}, \hat{\mathcal{J}}(A))$ (i.e. $u : [0, 1] \to (X, \mathcal{J}(A))$ is continuous), which is called an $A$-curve in $X$, and $a = u(0)$, $b = u(1)$. If $\hat{T}(A) = \hat{T}$ (the essential topology of $\hat{X}$) then $u$ is called an essential curve in $X$. Obviously, $A$-curves are essential curves.

Let $\mathcal{H}_x$ denote the set of all kinds of germs of $\mathcal{H}$ at $x \in \hat{X}$, i.e. $\mathcal{H}_x := \mathcal{H}_x \cup \mathcal{H}_x^\circ \cup \mathcal{H}_x^0 \cup \mathcal{H}_x^\rho$ for $x \in \hat{X}$ and $\mathcal{H}_x^p := \mathcal{H}_x^0 \cup \mathcal{H}_x^\rho$ for $x \in I$. Suppose $P \subseteq \hat{X}$. Let $\mathcal{H}_x^p(A) := \mathcal{H}_x^p$ for $x \in X \setminus A$ and $\mathcal{H}_x^0(A) := \mathcal{H}_x^0$ for $x \in A$, where $\mathcal{H}_x^p := \mathcal{H}_x^p$ for $x \in X \setminus P$, $\mathcal{H}_x^p := \mathcal{H}_x^0$ for $x \in X \setminus P$, $\mathcal{H}_x := \mathcal{H}_x$ for $x \in \hat{X} \setminus P$, and $\mathcal{H}_x := \mathcal{H}_x^p$ for $x \in P$. Suppose $\psi_a \in \mathcal{H}_a$ and $\psi_b \in \mathcal{H}_b$. If there exists a family $\{\psi_t \in \mathcal{H}_a^p(A) : t \in [0, 1]\}$ such that $\psi_a \sim \psi_t$, $\psi_b \sim \psi_{b(t)}$, and for every $s \in [0, 1]$, there exists a neighborhood $T \subseteq [0, 1]$ of $s$, a domain $U$ in $(\hat{X}, \hat{T}(A))$ with $u(T) \subseteq U$ and $f \in \mathcal{H}(U)$ ($\mathcal{H}(U) := \{f \in \mathcal{H}(U \setminus E) : E$ is a $P$-quasi-discrete closed subset of $U\}$) satisfying $\langle f \rangle^p_{a(t)} = \psi_t$ for every $t \in T$, where $\langle f \rangle^p_{a(t)} := \langle f \rangle^p_{a(t)}$ for $x \in P$ and $\langle f \rangle^p_{x(t)} := \langle f \rangle_{x(t)}$ for $x \in \hat{X} \setminus P$, then we say that $\varphi_b$ is a $P$-porous (analytic) continuation of $\varphi_a$ along $u$ in $(\hat{X}, \hat{T}(A))$ or that $\varphi_b$ is a $P$-porous $A$-analytic continuation (or $P$-porous $A$-continuation) of $\varphi_a$ along $u$. Corresponding to $P = \emptyset$ (resp. $P = \hat{X}$) the continuation is called an (resp. a porous) $A$-analytic continuation (or $A$-continuation). The (resp. $P$-porous, porous) $I$-continuation is also called an (resp. a $P$-porous, a porous) essential (analytic) continuation. If the partial point set $I = \emptyset$ then essential curves and essential analytic continuations are curves and analytic continuations in the usual sense, respectively. It is obvious that the ($P$-porous) $A$-continuation implies the porous $A$-continuation.

Easily we see that $\varphi_b$ is a $P$-porous $A$-continuation of $\varphi_a$ along $u$ if and only if the following holds: There exist a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0, 1]$, domains $U_j$ in $(\hat{X}, \hat{T}(A))$ with $u([t_{j-1}, t_j]) \subseteq U_j$ and $f_j \in \mathcal{H}(U_j \setminus E_j)$ ($E_j$ is a $P$-quasi-discrete and closed in $U_j$, $j = 1, 2, \ldots, n$) such that $\langle f_j \rangle_{a(t_j)} \sim \varphi_a$, \ldots, $\langle f_n \rangle_{a(t_n)} \sim \varphi_a$.\ldots
\begin{equation}
(f_n)_n^p \sim \varphi_b \quad \text{and} \quad f_j|_{V_j \setminus (E_j \cup E_{j+1})} = f_{j+1}|_{V_j \setminus (E_j \cup E_{j+1})} \quad (j = 1, 2, \ldots, n-1), \quad \text{where } V_j \text{ is the connected component of } U_j \cap U_{j+1} \text{ containing } u(t_j).
\end{equation}

Let
\begin{equation}
\hat{\mathcal{N}}^p(A) := \bigcup_{x \in \hat{X}} \hat{\mathcal{H}}^p_x(A)
\end{equation}
and
\begin{equation}
\hat{\mathcal{N}}^p(\mathcal{H}(\hat{\mathcal{B}}_A)) := \bigcup_{x \in \hat{X}} \{(V, f)^p : V \in \hat{\mathcal{B}}_A(x), f \in \mathcal{H}(V \setminus \{x\})\},
\end{equation}
where \(\langle V, f \rangle^p := \{(f)^p_x : x \in V\}\). Then by Theorem 2.1′(3) (see Remark 2), \(\hat{\mathcal{N}}^p(\mathcal{H}(\hat{\mathcal{B}}_A))\) is a basis for a topology \(\hat{\mathcal{N}}^p(A)\) on \(\hat{\mathcal{Y}}(A)\). Similarly to [10, Lemma (7.2)], by the definition of \(\mathcal{P}\)-porous \(A\)-continuation and Theorem 2.1′(3) we have

**Lemma 3.1.** Suppose \((X, \mathcal{T}; \hat{X}, \hat{\mathcal{T}})\) is a basic Riemann surface and \(A\) is a partial point set \((I \subseteq A \subseteq \hat{X})\). Suppose \(u : [0, 1] \to \hat{X}\) is an \(A\)-curve with \(a = u(0)\) and \(b = u(1)\). Suppose \(P \subseteq \hat{X}, \varphi_a \in \mathcal{H}_a\) and \(\varphi_b \in \mathcal{H}_b\). Then \(\varphi_b\) is a \(\mathcal{P}\)-porous \(A\)-continuation of \(\varphi_a\) along \(u\) if and only if there exists a lifting \(\hat{u} : [0, 1] \to (\hat{\mathcal{Y}}(A), \hat{\mathcal{Y}}(\mathcal{H}))\) of the \(A\)-curve \(u\) (with respect to the projection \(\hat{p}_A\)) such that \(\hat{u}(0) \sim \varphi_a\) and \(\hat{u}(1) \sim \varphi_b\). \(\square\)

By Theorems 2.1′ and 2.2′, Lemma 3.1 and [10, Theorem (4.10)] we obtain

**Theorem 3.2** (Monodromy Theorem). Suppose \((X, \mathcal{T}; \hat{X}, \hat{\mathcal{T}})\) is a basic Riemann surface and \(\mathcal{T}(A)\) is a mixed topology of \(\hat{X}\) \((I \subseteq A \subseteq \hat{X})\). Let \(a, b \in \hat{X}\). Suppose \(u_0\) and \(u_1\) are homotopic \(A\)-curves from \(a\) to \(b\) and \(u_s (0 \leq s \leq 1)\) is a deformation from \(u_0\) to \(u_1\) in \((\hat{X}, \mathcal{T}(A))\). Let \(\varphi_a \in \mathcal{H}_a\) and let \(\varphi_a^{(0)}, \varphi_a^{(1)} \in \mathcal{H}_b\) be the porous \(A\)-continuations of \(\varphi_a\) along \(u_0\) and \(u_1\), respectively. If \(\varphi_a\) admits a porous \(\mathcal{P}\)-continuation along every curve \(u_s\), then \(\varphi_a^{(0)} \sim \varphi_a^{(1)}\). \(\square\)

We consider a traditional Riemann surface \((X, \mathcal{T})\) as a basic Riemann surface \((X, \mathcal{T}; X, \mathcal{T})\) (i.e. the ideal point set \(I = \emptyset\)). Then Theorem 3.2 implies

**Corollary 1.** Suppose \((X, \mathcal{T})\) is a (traditional) Riemann surface. Let \(a, b \in X\). Suppose \(u_0, u_1\) are homotopic curves from \(a\) to \(b\) and \(u_s (0 \leq s \leq 1)\) is a deformation from \(u_0\) to \(u_1\). Let \(\varphi_a \in \mathcal{H}_a := \mathcal{H}_a \cup \mathcal{H}_a^\circ\) and let \(\varphi_a^{(0)}, \varphi_a^{(1)} \in \mathcal{H}_b\) be the porous continuations of \(\varphi_a\) along \(u_0\) and \(u_1\), respectively. If \(\varphi_a\) admits a porous continuation along every curve \(u_s\), then \(\varphi_a^{(0)} \sim \varphi_a^{(1)}\). \(\square\)

For a traditional Riemann surface \((X, \mathcal{T})\), suppose \(S\) is a quasi-discrete closed subset of \((X, \mathcal{T})\). We call \(F \in \mathcal{H}(X \setminus S)\) an analytic function in \(X\) with the singularities \(S\). If \(a \in S\) is a removable singularity of \(F\) then we also say \(F\) is analytic at \(a\). By Corollary 1 we have

**Corollary 2.** Suppose \((X, \mathcal{T})\) is a simply connected (traditional) Riemann surface. Suppose \(S\) is a quasi-discrete closed subset of \(X\). Let \(a \in X\) and \(\varphi_a \in \mathcal{H}_a = \mathcal{H}_a \cup \mathcal{H}_a^\circ\). If \(\varphi_a\) admits a porous continuation along every curve \(u\) starting
at a to some \( \varphi_b \in \overline{F}_b \) (\( b \in X \) is the end point of \( u \)) and \( \varphi_b \) is equivalent to a usual germ for every \( b \in X \setminus S \), then there exists a unique analytic function \( F \) in \( X \) with the singularities \( S \) such that \( [F]_a \sim \varphi_a \).

**Proof.** Define \( F(x) := \varphi_x(x) \) for \( x \in X \setminus S \), where \( \varphi_x(x) := f(x) \) if \( \varphi_x \sim [f]_x \) for some usual element \( f \) at \( x \in X \).

\[ \]

### 3.3 Harmonious equivalences and up-harmonious equivalences

Let \( X \) and \( Y \) be sets and let \( \lambda : Y \to X \), \( y \mapsto x \) be a surjection. We consider pairs \((X, x)\) and \((Y, y)\), where \( x \in X \) (resp. \( y \in Y \)) is a variable which traverses all elements in \( X \) (resp. \( Y \)). We say that \((X, x)\) is up-harmoniously equivalent to \((Y, y)\) modulo \( \lambda \), and \( \lambda \) is called an up-harmonious mapping. If \( \lambda \) is bijective then \((X, x)\) and \((Y, y)\) are said to be harmoniously equivalent (with one another) modulo \( \lambda \), where \( \lambda \) is called a harmonious mapping. Later, we simply use the terminology (up-)harmonious to replace “(up-)harmoniously equivalent”.

Suppose \( X \) and \( Y \) are topological spaces. If further the surjection \( \lambda : Y \to X \) is continuous then we say that \((X, x)\) is continuously up-harmonious with \((Y, y)\) modulo \( \lambda \), where \( \lambda \) is called a continuously up-harmonious mapping and if \( \lambda \) is a homeomorphism then \((X, x)\) and \((Y, y)\) are said to be continuously harmonious (with one another) modulo \( \lambda \), where \( \lambda \) is called a continuously harmonious mapping.

In the case that \( X \) and \( Y \) are traditional Riemann surfaces and the surjection \( \lambda : Y \to X \) is analytic, we say that \((X, x)\) is analytically up-harmonious with \((Y, y)\) modulo \( \lambda \), where \( \lambda \) is called an analytically up-harmonious mapping. If \( \lambda \) is biholomorphic then \((X, x)\) and \((Y, y)\) are said to be analytically harmonious (with one another) modulo \( \lambda \), where \( \lambda \) is called an analytically harmonious mapping.

Now suppose \((X, \mathcal{T}; \hat{X}, \mathcal{T})\) and \((Y, \mathcal{T}'; \hat{Y}, \mathcal{T}')\) are two basic Riemann surfaces and \( \hat{\lambda} : \hat{Y} \to \hat{X} \) is a surjection satisfying \( \hat{\lambda}(Y) \subseteq \hat{X} \) and \((\hat{\lambda}(Y), \mathcal{T}|_{\hat{\lambda}(Y)}; \hat{X}, \mathcal{T})\) is a subsurface of \((X, \mathcal{T}; \hat{X}, \mathcal{T})\). We say that \((X, \mathcal{T}; \hat{X}, \mathcal{T}; x)\) (simply denoted \((\hat{X}, x)\), \( x \in \hat{X} \)) is analytically up-harmonious with \((Y, \mathcal{T}'; \hat{Y}, \mathcal{T}'; y)\) \( (y \in \hat{Y}) \) modulo \( \hat{\lambda} \), denoted \((\hat{X}, x) \overset{\hat{\lambda}}{\leftrightarrow} (\hat{Y}, y) \), if \((\hat{\lambda}(Y), x)\) is analytically up-harmonious with \((Y, y)\) modulo \( \hat{\lambda}|_Y \) and \(((\hat{X}, \mathcal{T}); x)\) is continuously up-harmonious with \(((\hat{Y}, \mathcal{T}'; y)\) modulo \( \hat{\lambda} \), where \( \hat{\lambda} \) is called an analytically up-harmonious mapping. Here we use the notation “\((\hat{X}, x) \overset{\hat{\lambda}}{\leftrightarrow} (\hat{Y}, y)\)” (or “\((\hat{Y}, y) \overset{\hat{\lambda}}{\leftrightarrow} (\hat{X}, x)\)”) rather than “\((\hat{Y}, y) \overset{\hat{\lambda}}{\leftrightarrow} (\hat{X}, x))\)” because we consider that \((\hat{X}, x)\) may be “pasted into” \((\hat{Y}, y)\) by \( \hat{\lambda} \). Usually we assume \( \hat{\lambda}|_Y : Y \to \hat{\lambda}(Y) \) is an unbranched covering. If moreover \( \hat{\lambda}|_Y : Y \to X \) is biholomorphic and \( \hat{\lambda} : \hat{Y} \to \hat{X} \) is homeomorphic, then we say that \((\hat{X}, x)\) and \((\hat{Y}, y)\) are analytically harmonious (with one another) modulo \( \hat{\lambda} \), denoted \((\hat{X}, x) \overset{\hat{\lambda}}{\leftrightarrow} (\hat{Y}, y)\), where \( \hat{\lambda} \) is called an analytically harmonious mapping.

Consider a family \( \mathcal{X} \) of pairs \((\hat{X}, x)\), where \( \hat{X} \) is a basic Riemann surface and \( x \in \hat{X} \). Suppose \( \Lambda \) is a set consisting of some analytic up-harmonious mappings, which satisfies that \( \text{id}_{\hat{X}} \in \Lambda \) for all pairs \((\hat{X}, x) \in \mathcal{X} \) (here \( \text{id}_A \) denotes the identity
mapping from set $A$ to itself) and that if $(\hat{X}, x) \overset{\hat{\lambda}_1}{\leftrightarrow} (\hat{Y}, y), (\hat{Y}, y) \overset{\hat{\lambda}_2}{\leftrightarrow} (\hat{Z}, z)$ and $\hat{\lambda}_1, \hat{\lambda}_2 \in \hat{A}$. Then we call $\hat{A}$ an analytically up-harmonious relation in $\mathfrak{X}$. If for two pairs $(\hat{X}, x), (\hat{Y}, y) \in \mathfrak{X}$ there is $\hat{\lambda} \in \hat{A}$ such that $(\hat{X}, x) \overset{\hat{\lambda}}{\leftrightarrow} (\hat{Y}, y)$, then we say that $(\hat{X}, x)$ is analytically up-harmonious with $(\hat{Y}, y) \bmod \hat{A}$, denoted $(\hat{X}, x) \overset{\hat{\lambda}}{\leftrightarrow} (\hat{Y}, y)$. Let

$$\hat{A}_0 := \{\hat{\lambda} \in \hat{A} : \hat{\lambda} : \hat{Y} \to \hat{X} \text{ is homeomorphic,} \}$$

$$\hat{\lambda}|_{\hat{Y}} : Y \to X \text{ is biholomorphic and } \hat{\lambda}^{-1} \in \hat{A}\}.$$ 

If $(\hat{X}, x) \overset{\hat{\lambda}}{\leftrightarrow} (\hat{Y}, y)$ for some $\hat{\lambda} \in \hat{A}_0$ then we say that $(\hat{X}, x)$ and $(\hat{Y}, y)$ are analytically harmonious (with one another) modulo $\hat{A}_0$, denoted $(\hat{X}, x) \overset{\hat{\lambda}}{\leftrightarrow} (\hat{Y}, y)$, where $\hat{A}_0$ is called an analytically harmonious (equivalence) relation. We may attach additional conditions to the analytically (up-)harmonious relation (in the next subsection we add the base-preserving condition).

For $\mathbf{Y} \subseteq \mathfrak{X}$ we define the analytically up-harmonious class (of $\mathbf{Y}$)

$$\hat{\mathbf{Y}} := \{ (\hat{X}, x) \in \mathfrak{X} : \text{there exists } (\hat{Y}, y) \in \mathbf{Y} \text{ such that } (\hat{X}, x) \overset{\hat{\lambda}}{\leftrightarrow} (\hat{Y}, y) \}.$$ 

If there is an element $(\hat{Y}, y) \in \hat{\mathbf{Y}}$ such that for all $(\hat{X}, x) \in \hat{\mathbf{Y}}$ we have $(\hat{X}, x) \overset{\hat{\lambda}}{\leftrightarrow} (\hat{Y}, y)$, then we call $(\hat{Y}, y)$ a holographic element of $\hat{\mathbf{Y}}$.

When we emphasize the surface $\hat{X}$ in the pair $(\hat{X}, x)$, we write $\hat{X}(x)$ instead of $(\hat{X}, x)$. We can define similar notions to the above for sets, topological spaces and traditional Riemann surfaces, respectively.

### 3.4 Algebraic Riemann surfaces

Suppose $(X, \mathfrak{T}; \hat{X}, \hat{\mathfrak{T}})$ is a universal topological space and $\mathfrak{I}$ is the set of all ideal points. Suppose there is a partition $\mathfrak{I} = \bigcup_{j \in J} \mathfrak{I}_j$ of $\mathfrak{I}$, where $\mathfrak{I}_j = \{ \hat{x}_{jk} : k = 1, \ldots, k_j \}$ $(k_j (j \in J)$ are positive integers), and a topological space $(\hat{X}, \hat{\mathfrak{T}})$ such that $\hat{X} = X \cup \mathfrak{I}$, where $X \cap \mathfrak{I} = \emptyset$, $\mathfrak{I} = \{ \hat{x}_j : j \in J \}$, $\hat{\mathfrak{T}}|_\mathfrak{I} = \mathfrak{T}$ and for every neighborhood $\mathfrak{N}(\hat{x}_j)$ of $\hat{x}_j$ in $(\hat{X}, \hat{\mathfrak{T}})$ $(j \in J)$ there exist punctured partial neighborhoods $\mathfrak{N}^\circ(\hat{x}_{jk})$ of $\hat{x}_{jk}$ such that $\mathfrak{N}^\circ(\hat{x}_{jk}) \subseteq \mathfrak{N}(\hat{x}_j)$ for $k = 1, \ldots, k_j$. Then we say that $(\hat{X}, \hat{\mathfrak{T}})$ is a tied space corresponding to $(X, \mathfrak{T}; \hat{X}, \hat{\mathfrak{T}})$ and $(X, \mathfrak{T}; \hat{X}, \hat{\mathfrak{T}})$ is an untied space corresponding to $(\hat{X}, \hat{\mathfrak{T}})$. In this case we may write $(X, \mathfrak{T}; \hat{X}, \hat{\mathfrak{T}})$ for $(X, \mathfrak{T}; \hat{X}, \hat{\mathfrak{T}})$, or simply $(\hat{X}, \hat{\mathfrak{T}})$. Denote $(\hat{X}, \hat{\mathfrak{T}})$ by $\hat{X}$, which we call a universal topological space with a tied space.

Suppose $(X, \mathfrak{T}; \hat{X}, \hat{\mathfrak{T}})$ is a basic Riemann surface and a mapping $f : (\hat{X}, \hat{\mathfrak{T}}) \to \hat{\mathbb{C}}$ is continuous, where $\hat{\mathfrak{T}}$ is the essential topology of $\hat{X}$ and $\hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \}$ denotes the extended complex plane. If $f|_X$ is meromorphic then we call $f$ (essentially) para-meromorphic on $\hat{X}$. If $f(X) \subseteq \mathbb{C}$ and $f|_X$ is holomorphic then we call $f$ (essentially) para-holomorphic on $\hat{X}$. If there exists a traditional Riemann surface $(\hat{X}, \hat{\mathfrak{T}})$, which is a tied space corresponding to $\hat{X}$ (we always assume $\hat{X}$ has a uniform complex structure with $X$), and a meromorphic (resp. holomorphic)
function \( \hat{f} \) on \( \hat{X} \) such that \( \hat{f}|_{\hat{X}} = \hat{f}|_{X} \), then we call \( \hat{f} := (\hat{f}, \bar{f}) \) or \( \hat{f} \) (essentially) meromorphic (resp. (essentially) holomorphic) on \( X = (\hat{X}, \hat{X}) \) or \( \hat{X} \). Define \( f|_{\hat{X}} := \hat{f} \) and \( \bar{f}|_{X} := \hat{f} \). We denote the set of all para-meromorphic (resp. para-holomorphic) functions on \( \hat{X} \) by \( \mathcal{M}(\hat{X}) \) (resp. \( \mathcal{H}(\hat{X}) \)) and the set of all meromorphic (resp. holomorphic) functions on \( \hat{X} \) by \( \mathcal{M}(\hat{X}) \) (resp. \( \mathcal{H}(\hat{X}) \)). Denote \( \mathcal{M}(\hat{X}) := \{f|_{\hat{X}} : f \in \mathcal{M}(X)\} \) and \( \mathcal{H}(\hat{X}) := \{f|_{X} : f \in \mathcal{H}(X)\} \). For \( \hat{f} = (\hat{f}, \bar{f}) \), \( \hat{g} = (\hat{g}, \bar{g}) \in \mathcal{M}(\hat{X}) \) we define \( \hat{f} + \hat{g} := (\hat{f} + \hat{g}, \bar{f} + \bar{g}) \) and \( \hat{f} \cdot \hat{g} := (\hat{f} \cdot \hat{g}, \bar{f} \cdot \bar{g}) \).

Suppose \((X, \mathcal{T}; \hat{X}, \hat{\mathcal{T}})\) and \((Y, \mathcal{T}'; \hat{Y}, \hat{\mathcal{T}}')\) are basic Riemann surfaces and a mapping \( \hat{f} : (\hat{Y}, \hat{\mathcal{T}}') \to (\hat{X}, \hat{\mathcal{T}}) \) is continuous, where \( \mathcal{T} \) and \( \mathcal{T}' \) are the essential topologies of \( X \) and \( \hat{Y} \) respectively. If \( f(Y) \subseteq X \) and \( f|_{Y} : Y \to X \) is holomorphic then we call \( \hat{f} : \hat{Y} \to \hat{X} \) essentially para-holomorphic. If \( \hat{f} : \hat{Y} \to \hat{X} \) is essentially para-holomorphic and partially continuous, then we call \( \hat{f} \) (exactly) para-holomorphic. A mapping \( \hat{f} : \hat{Y} \to \hat{X} \) is called (exactly) (resp. essentially) para-biholomorphic if it is bijective and both \( \hat{f} : \hat{Y} \to \hat{X} \) and \( \hat{f}^{-1} : \hat{X} \to \hat{Y} \) are (exactly) (resp. essentially) para-holomorphic.

Suppose \( f : \hat{Y} \to \hat{X} \) is an essentially para-holomorphic mapping. If there exist Riemann surfaces \((\hat{X}, \hat{\mathcal{T}})\) and \((\hat{Y}, \hat{\mathcal{T}}')\), which are tied spaces corresponding to \( \hat{X} \) and \( \hat{Y} \), respectively, and a holomorphic mapping \( \hat{f} : \hat{Y} \to \hat{X} \) such that \( \hat{f}|_{Y} = f|_{Y} \), then we say that \( \hat{f} = (\hat{f}, \bar{f}) \) is essentially holomorphic from \( \hat{Y} \) to \( \hat{X} \) and \( f : Y \to X \) is essentially holomorphic. Define \( \hat{f}|_{\hat{Y}} := \hat{f} \) and \( f|_{Y} := f \). If \( \hat{f} : \hat{Y} \to \hat{X} \) is essentially holomorphic and partially continuous, then we call \( \hat{f} \) and \( \hat{f} \) (exactly) holomorphic. A mapping \( \hat{f} = (f, \bar{f}) : \hat{Y} \to \hat{X} \) is called (exactly) (resp. essentially) biholomorphic if it is bijective (i.e. both \( \hat{f} \) and \( f \) are bijective) and both \( \hat{f} : \hat{Y} \to \hat{X} \) and \( f^{-1} := (\hat{f}^{-1}, \bar{f}^{-1}) : \hat{X} \to \hat{Y} \) are (exactly) (resp. essentially) holomorphic. If \( \hat{f} : \hat{Y} \to \hat{X} \) is (exactly) (resp. essentially) biholomorphic then we also say \( \hat{f} : \hat{Y} \to \hat{X} \) is (exactly) (resp. essentially) biholomorphic.

Denote the set of all (resp. essentially) para-holomorphic mappings from \( \hat{Y} \) to \( \hat{X} \) by \( \mathcal{H}(\hat{Y} \to \hat{X}) \) (resp. \( \mathcal{H}(\hat{Y} \to \hat{X}) \)). Denote the set of all (resp. essentially) holomorphic mappings from \( Y \) to \( X \) by \( \mathcal{H}(Y \to X) \) (resp. \( \mathcal{H}(Y \to X) \)). Denote \( \mathcal{H}(\hat{Y} \to \hat{X}) := \{f|_{\hat{Y}} : f \in \mathcal{H}(\hat{Y} \to \hat{X})\} \) and \( \mathcal{H}(\hat{Y} \to \hat{X}) := \{f|_{\hat{X}} : f \in \mathcal{H}(\hat{Y} \to \hat{X})\} \). It is easy to see that \( \hat{f} \in \mathcal{H}(\hat{Y} \to \hat{X}) \) and \( \hat{g} \in \mathcal{H}(\hat{Z} \to \hat{Y}) \) imply \( \hat{f} \circ \hat{g} \in \mathcal{H}(\hat{Z} \to X) \), that \( \hat{f} \in \mathcal{H}(\hat{Y} \to \hat{X}) \) and \( \hat{g} \in \mathcal{H}(\hat{Z} \to \hat{Y}) \) imply \( \hat{f} \circ \hat{g} \in \mathcal{H}(\hat{Z} \to X) \), that \( \hat{f} \in \mathcal{H}(\hat{Y} \to \hat{X}) \) and \( \hat{g} \in \mathcal{H}(\hat{Z} \to \hat{Y}) \) imply \( \hat{f} \circ \hat{g} \in \mathcal{H}(\hat{Z} \to X) \) (\( \hat{f} \circ \hat{g} := (\hat{f} \circ \hat{g}, \bar{f} \circ \bar{g}) \)). Specially, we know that \( f \in \mathcal{M}(\hat{X}) \) (resp. \( \mathcal{H}(\hat{X}) \)) and \( \hat{g} \in \mathcal{H}(\hat{Y} \to \hat{X}) \) imply \( \hat{f} \circ \hat{g} \in \mathcal{M}(\hat{Y}) \) (resp. \( \mathcal{H}(\hat{Y}) \)) and that \( f \in \mathcal{M}(\hat{X}) \) (resp. \( \mathcal{H}(\hat{X}) \)) and \( \hat{g} \in \mathcal{H}(\hat{Y} \to \hat{X}) \) imply \( \hat{f} \circ \hat{g} \in \mathcal{M}(\hat{Y}) \) (resp. \( \mathcal{H}(\hat{Y}) \)).

**Remark 3.** \( \mathcal{H}(\hat{X}), \mathcal{H}(\hat{X}) \) and \( \mathcal{H}(\hat{X}) \) are rings; \( \mathcal{M}(\hat{X}) \) and \( \mathcal{M}(\hat{X}) \) are fields.

Suppose \((X, \mathcal{T}; \hat{X}, \hat{\mathcal{T}})\) and \((Y, \mathcal{T}'; \hat{Y}, \hat{\mathcal{T}}')\) are basic Riemann surfaces and \( \hat{f} : \hat{Y} \to \hat{X} \) is an essentially para-holomorphic mapping. Let \( \hat{f}^* : \mathcal{M}(\hat{X}) \to \mathcal{M}(\hat{Y}) \) be defined by \( \hat{f}^*(\hat{\phi}) := \hat{\phi} \circ \hat{f} \).
for $\varphi \in \tilde{M}(\hat{X})$. Then

$$\hat{f}^*|_{\tilde{H}(\hat{X})} : \tilde{H}(\hat{X}) \rightarrow \tilde{H}(\hat{Y})$$

is a ring homomorphism. If $\hat{f} \in \tilde{H}(\hat{Y} \rightarrow \hat{X})$ then

$$\hat{f}^* : \tilde{M}(\hat{X}) \rightarrow \tilde{M}(\hat{Y}),$$

defined by $\hat{f}^*(\varphi) := \varphi \circ \hat{f}$, and

$$\hat{f}^*|_{\tilde{H}(\hat{X})} : \tilde{H}(\hat{X}) \rightarrow \tilde{H}(\hat{Y})$$

are ring homomorphisms.

Suppose $(R, T_0; \hat{R}, \hat{T}_0)$ is a basic Riemann surface with $\hat{R} = R$ (equal as sets, i.e. the ideal point set $I = \emptyset$ and so $\tilde{R} = R$). Suppose $(X, T; \hat{X}, \hat{T})$ is a basic Riemann surface and $(\hat{X}, \hat{T})$ is a traditional Riemann surface which is a tied space corresponding to $\hat{X}$. Suppose $\bar{\lambda} = (\hat{\lambda}, \hat{T}) : \hat{X} \rightarrow \hat{R}$ is a holomorphic mapping such that $(\hat{R}, \hat{T}) \leftrightarrow (\hat{X}, \hat{T}) (\hat{R} = (\hat{R}, \hat{R})$ and $\hat{X} = (\hat{X}, \hat{X})$). Then

$$\bar{\lambda}^* : \tilde{M}(\hat{R}) \rightarrow \tilde{M}(\hat{X})$$

is a ring monomorphism. Define

$$\hat{h} \cdot \hat{g} := (\bar{\lambda}^* \hat{h}) \cdot \hat{g}$$

for $\hat{h} \in \tilde{M}(\hat{R})$ and $\hat{g} \in \tilde{M}(\hat{X})$. Then $\tilde{M}(\hat{X})$ is a vector space over $\tilde{M}(\hat{R})$. We also consider $\hat{h} \in \tilde{M}(\hat{R})$ as $\bar{\lambda}^* \hat{h} = \hat{h} \circ \bar{\lambda}$ and then consider $\tilde{M}(\hat{R})$ as a subfield of the field $\tilde{M}(\hat{X})$. So $\tilde{M}(\hat{R})$ is a subfield of $\tilde{M}(\hat{X})$ and $\tilde{M}(\hat{X})$. Consequently and similarly, the punctured partial stalk $\tilde{M}_{\hat{X}, x}$ of the sheaf $\tilde{M}$ of meromorphic functions on $\hat{X}$ at $x \in \hat{X}$ is also a vector space over $\tilde{M}(\hat{R})$ by defining

$$h \cdot \langle \hat{g} \rangle_x := \langle (h \circ \bar{\lambda}) \cdot \hat{g} \rangle_x$$

for $h \in \tilde{M}(\hat{R})$ and $\langle \hat{g} \rangle_x \in \tilde{M}_{\hat{X}, x}$, and we may also consider $\tilde{M}(\hat{R})$ as a subfield of $\tilde{M}_{\hat{X}, x}$.

Let $\hat{f} : \hat{Y} \rightarrow \hat{X}$ be para-holomorphic. Suppose for $y \in \hat{Y}$ with $\hat{f}(y) = x \in \hat{X}$ there exists a punctured partial neighborhood $\hat{N}^\circ(y)$ of $y$ such that $x \notin \hat{f}(\hat{N}^\circ(y))$. Then the mapping

$$\hat{f}^* : \tilde{M}_{\hat{X}, x} \rightarrow \tilde{M}_{\hat{Y}, y},$$

defined by $\hat{f}^*(\langle \hat{g} \rangle_y) := \langle \hat{g} \circ \hat{f} \rangle_y$, is a ring monomorphism. If further for every partial neighborhood $\hat{N}(y)$ of $y$ which is open in $(\hat{Y}, \hat{T})$ there exists a punctured partial neighborhood $\hat{N}_1(x)$ of $x$ which is open in $(\hat{X}, \hat{T})$ such that $\hat{f}|_{\hat{N}(y)} : \hat{N}(y) \rightarrow \hat{N}_1(x)$ is para-biholomorphic, then $\hat{f}^*$ is an isomorphism. Let

$$\hat{f}_*: \tilde{M}_{\hat{Y}, y} \rightarrow \tilde{M}_{\hat{X}, x}$$

be the inverse of $\hat{f}^*$.
Suppose \( \hat{p} \in \mathcal{H}(\hat{Y} \to \hat{X}) \), \( \bar{p} \in \mathcal{H}(\bar{Y} \to \bar{X}) \) (the set of all holomorphic mappings from \( \bar{Y} \) to \( \bar{X} \)), \( \hat{P}(t) = \hat{c}_0 t^n + \hat{c}_1 t^{n-1} + \cdots + \hat{c}_n \in \mathcal{M}(\hat{X})[t] \) and \( \bar{P}(t) = \bar{c}_0 t^n + \bar{c}_1 t^{n-1} + \cdots + \bar{c}_n \in \mathcal{M}(\bar{X})[t] \), where \( \mathcal{M}(\hat{X}) \) denotes the set of all meromorphic functions on \( \hat{X} \). Define

\[
(p^* \hat{P})(t) := (\hat{p}^* \hat{c}_0) t^n + (\hat{p}^* \hat{c}_1) t^{n-1} + \cdots + (\hat{p}^* \hat{c}_n),
\]
and

\[
(p^* \bar{P})(t) := (\bar{p}^* \bar{c}_0) t^n + (\bar{p}^* \bar{c}_1) t^{n-1} + \cdots + (\bar{p}^* \bar{c}_n),
\]

which are in \( \mathcal{M}(\hat{Y})[t] \) and \( \mathcal{M}(\bar{Y})[t] \), respectively. Suppose \( \hat{p} = (\hat{p}, \bar{p}) \in \mathcal{H}(\hat{Y} \to \hat{X}) \) and \( \bar{P}(t) = \bar{c}_0 t^n + \bar{c}_1 t^{n-1} + \cdots + \bar{c}_n \in \mathcal{M}(\bar{X})[t] \), where we may write \( \bar{P}(t) = (\bar{P}(t), \bar{P}(t)) \) and denote \( \bar{P}|_{\bar{Y}} := \bar{P} \) and \( \bar{P}|_{\bar{Y}} := \bar{P} \). Define

\[
(p^* \hat{P})(t) := (\hat{p}^* \hat{P})(t) + (p^* \bar{P})(t),
\]
i.e.

\[
(p^* \hat{P})(t) := ((\hat{p}^* \hat{P})(t), (p^* \bar{P})(t)),
\]

which is in \( \mathcal{M}(\hat{Y})[t] \). If \( \hat{p} : \hat{Y} \to \hat{X} \) is a (resp. an essentially) holomorphic \( n \)-sheeted covering map and \( \bar{p} : \bar{Y} \to \bar{X} \) is a branched holomorphic \( n \)-sheeted covering map, then we say that \( \hat{p} : \hat{Y} \to \hat{X} \) is a (or an exactly) (resp. an essentially) holomorphic \( n \)-sheeted covering map.

Remark 4. Suppose \( \hat{f} := (\hat{f}, \bar{f}) \) is meromorphic on \( \hat{Y} = (\hat{Y}, \hat{Y}) \). Then \( (\hat{p}^* \hat{P})(\hat{f}) = 0 \) if and only if \( (\bar{p}^* \bar{P})(\bar{f}) = 0 \). In this case, \( (\hat{p}^* \hat{P})(\hat{f}) = (\hat{p}^* \hat{P})(\hat{f}), (p^* \bar{P})(\bar{f}) = 0 \).

**Theorem 3.3.** Suppose \( (R, \mathcal{I}_0; \hat{R}, \hat{\mathcal{I}}_0) \) is a basic Riemann surface determined by a perfect filterbase structure system \( \mathfrak{B} \) whose elements consist of simply connected domains with \( \hat{R} = R \) and \( P(t) \in \mathcal{M}(R)[t] \) is an irreducible monic polynomial of degree \( n \) (here \( P(t) = \hat{P}(t) = P(t) \)). Then there exists a basic Riemann surface \( \hat{S} = (S, \mathcal{I}; \hat{S}, \hat{\mathcal{I}}; \hat{S}, \hat{\mathcal{I}}) \), a holomorphic \( n \)-sheeted covering map \( \hat{p} : \hat{S} \to \hat{R} \) and a meromorphic function \( \hat{F} \in \mathcal{M}(\hat{S}) \) such that \( (\hat{p}^* \hat{P})(\hat{F}) = 0 \). We call \( \hat{F} \) a basic algebraic function over \( \hat{R} \) (or \( R \) or \( R \)) with domain \( \hat{S} = (\hat{S}, \hat{S}) \), denoted by \( (\hat{S}, \hat{p}, \hat{F}) \). If \( (\hat{Z}, \hat{q}, \hat{G}) \) has the corresponding properties, then there exists exactly one fiber-preserving biholomorphic mapping \( \hat{\sigma} : \hat{Z} \to \hat{S} \) (i.e. \( \hat{p} \circ \hat{\sigma} = \hat{q} \)) such that \( \hat{G} = \hat{\sigma}^* \hat{F} \).

**Proof.** Let \( \mathcal{H} \) be the sheaf of holomorphic functions on \( R \). Then by Theorems 2.1 and 2.2 we obtain a Hausdorff universal topological space \( (\hat{S}, \mathcal{I}(\hat{\mathcal{H}}); \hat{S}, \hat{\mathcal{I}}(\hat{\mathcal{H}})) \). Let

\[
\hat{S} := \{ \hat{\varphi} \in \hat{\mathcal{S}} : P(\hat{\varphi}) = 0 \}
\]
and

\[
S := \hat{S} \cap \mathcal{S}.
\]
Let

\[
\mathcal{J} := \mathcal{I}(\mathcal{H})|_S := \mathcal{I}(\mathcal{H}) \cap S
\]
and
\[ \hat{T} := \hat{T}(\mathcal{H})|_{\hat{S}} := \hat{T}(\mathcal{H}) \cap \hat{S}. \]

Then \((S, T; \hat{S}, \hat{T})\) is a Hausdorff universal topological space determined by the perfect filterbase structure system
\[ \mathfrak{N}(\mathcal{H}(\mathcal{B}))|_{S} := \{ N : N \in \mathfrak{N}(\mathcal{H}(\mathcal{B})) \text{ and every } N \in \mathfrak{N} \text{ is a subset of } S \} \]
induced by \(\mathfrak{N}(\mathcal{H}(\mathcal{B}))\) (see (2.2)) on \(S\). Let
\[ \hat{p} : \hat{S} \to \hat{R} \]
be the projection. Then evidently we see that \(\hat{p}\) is an (exact) \(n\)-sheeted covering map. It is also evident to see that \(\hat{p} \circ \hat{\sigma} = \hat{p} \circ \hat{\sigma}\) (resp. \(\hat{p} \circ \hat{\sigma} = \hat{p} \circ \hat{\sigma}\)) is bijective and \(\hat{\sigma}\) is biholomorphic. Obviously we have \(\hat{G} = \hat{\sigma}^* \hat{F}\) and the mapping \(\hat{\sigma}\) is uniquely determined by this relation.

By reasoning similar to [10] Theorem (8.9) we can obtain a traditional Riemann surface \(\hat{S}\), which is a tied space corresponding to \(\hat{S}\), and \(\hat{F} \in \mathcal{M}(\hat{S})\) with \((\hat{p}^* \hat{P})(\hat{F}) = 0\) and \(\hat{F}|_{\hat{S}} = \hat{F}|_{\hat{S}}\), where the projection \(\hat{p} : \hat{S} \to \hat{R}\) is a branched holomorphic \(n\)-sheeted covering map with \(\hat{p}|_{\hat{S}} = \hat{p}|_{\hat{S}}\). Hence \(\hat{F} := (\hat{F}, \hat{F}) \in \mathcal{M}(\hat{S})\) and \(\hat{p} := (\hat{p}, \hat{p}) \in \mathcal{H}(\hat{S} \to \hat{R})\), where \(\hat{S} = (\hat{S}, \hat{S})\).

For \(z \in \hat{Z}\) let \(\hat{q}(z) = r\) and \(\hat{\varphi} := \hat{q}^* \hat{G}_z\), where \(\hat{G} = \hat{G}|_{\hat{Z}}\) and \(\hat{G}_z\) denotes the partial germ of \(\hat{G}\) at \(z\). Then \(\hat{P}(\hat{\varphi}) = 0\). Hence \(\hat{\varphi} \in \hat{S}\) and \(\hat{p}(\hat{\varphi}) = r\). Define \(\hat{\sigma} : \hat{Z} \to \hat{Y}\) by \(\hat{\sigma}(z) = \hat{\varphi}(z) \in \hat{Z}\). Then \(\hat{\sigma}\) is fiber-preserving and \(\hat{G} = \hat{\sigma}^* \hat{F}\). Easily we see that \(\hat{\sigma}\) is continuous. According to the reasoning in [10] Theorem (8.9)], \(\hat{\sigma}|_{\hat{Z}}\) can be extended to a fiber-preserving biholomorphic mapping \(\hat{\sigma} : \hat{Z} \to \hat{Y}\) such that \(\hat{\sigma}|_{\hat{Z}} = \hat{\sigma}|_{\hat{Z}}\) and \(\hat{G} = \hat{\sigma}^* \hat{F}\). Let \(\hat{\sigma} = (\hat{\sigma}, \hat{\sigma})\). It is also easy to see that \(\hat{\sigma}\) is bijective and \(\hat{\sigma}^{-1}\) is continuous. Therefore, \(\hat{\sigma}\) is biholomorphic. Obviously we have \(\hat{G} = \hat{\sigma}^* \hat{F}\) and the mapping \(\hat{\sigma}\) is uniquely determined by this relation. \(\square\)

We call \(\hat{Z} = (\hat{Z}, \hat{Z})\) (or \(\hat{Z}\)) in Theorem 3.3 a basic algebraic Riemann surface over \(\hat{R}\) (or \(R\)) and \(\hat{S} = (\hat{S}, \hat{S})\) (or \(\hat{S}\)) in the proof of Theorem 3.3 the original basic algebraic Riemann surface over \(\hat{R}\) (or \(R\)) (determined by \(\hat{P}(t)\)). We call \(\hat{G}\) (or \(\hat{G}\)) (resp. \(\hat{F}\) (or \(\hat{F}\))) in (resp. the proof of) Theorem 3.3 a (resp. the original) basic function on \(\hat{Z}\) (or \(\hat{Z}\)) (resp. on \(\hat{S}\) (or \(\hat{S}\))). We call \(\hat{R}\) a base surface. The holomorphic covering maps \(\hat{q}\) and \(\hat{p}\) in Theorem 3.3 and its proof are called canonical or natural projections.

Suppose \(\hat{Z}_1\) and \(\hat{Z}_2\) are basic algebraic Riemann surfaces over a base surface \(\hat{R}\) and suppose \(\hat{p}_1\) and \(\hat{p}_2\) are canonical projections from \(\hat{Z}_1\) and \(\hat{Z}_2\) to \(\hat{R} = (\hat{R}, \hat{R})\), respectively. Then a mapping \(\hat{\lambda} = (\hat{\lambda}, \hat{\lambda}) : \hat{Z}_2 \to \hat{Z}_1\) (resp. \(\hat{\lambda}, \hat{\lambda}\)) satisfying \(\hat{p}_1 \circ \hat{\lambda} = \hat{p}_2\) (resp. \(\hat{p}_1 \circ \hat{\lambda} = \hat{p}_2\), \(\hat{p}_1 \circ \hat{\lambda} = \hat{p}_2\)) is said to be base-preserving.

Remark 5. The topological space \((\hat{Z}, \hat{T}')\) in Theorem 3.3 where \(\hat{T}'\) is the essential topology of \(\hat{Z}\), is connected and path-connected. Therefore, by Lemma 3.1 we see that any two punctured partial germs in the original basic algebraic Riemann surface \(\hat{S}\) are analytic continuations along some curve in \(R\) from one another.
Suppose \( (R, \mathcal{T}_0; \hat{R}, \hat{\mathcal{T}}_0) \) is a base surface with \( \hat{R} = R \). Let \( \hat{\varphi} \in \hat{\mathcal{H}} \) be a punctured partial germ satisfying \( P(\hat{\varphi}) = 0 \), where \( P(t) \in \mathcal{M}(R)[t] \) is an irreducible monic polynomial of degree \( n \) (called the minimal polynomial of \( \hat{\varphi} \) in \( \mathcal{M}(R) \) or in \( \hat{R} \)). Here \( \hat{\varphi} \) is called an algebraic punctured partial germ of degree \( n \) on \( \hat{R} \). We call basic algebraic Riemann surfaces and the original basic algebraic Riemann surface determined by \( P(t) \) basic algebraic Riemann surfaces and the original basic algebraic Riemann surface determined by \( \hat{\varphi} \), respectively.

By [10] Theorem (8.12)] (refer to its proof) we have

**Lemma 3.4.** Suppose \( \hat{\mathcal{S}} = (\hat{\mathcal{S}}, \hat{S}) \) is an original basic algebraic Riemann surface over a base surface \( \hat{R} \) and \( F \) is the original basic function on \( \hat{S} \). If \( \hat{f} \) is a meromorphic function on \( \hat{S} \), then there exists a polynomial \( Q(t) \in \mathcal{M}(\hat{R})[t] \) \( (\hat{R} = (\hat{R}, \hat{\mathcal{R}})) \) such that \( \hat{f} = (\hat{p}^*Q)(\hat{F}) \), where \( \hat{p} : \hat{S} \to \hat{R} \) is the canonical projection, i.e. \( \hat{f} = \hat{Q}(\hat{F}) \).

Let \( \mathcal{A} \) (resp. \( \mathcal{A}^0 \)) denote the set of all (resp. original) basic algebraic Riemann surfaces over \( \hat{R} \). We consider a pair \( (\hat{Z}, \zeta) \), where \( \hat{Z} = (\hat{Z}, \hat{Z}) \in \mathcal{A} \) and \( \zeta \) is a variable in \( \hat{Z} \). Let

\[
\mathcal{A}(\zeta) := \{ (\hat{Z}, \zeta) : \hat{Z} = (\hat{Z}, \hat{Z}) \in \mathcal{A} \text{ and } \zeta \text{ is a variable in } \hat{Z} \}
\]

and

\[
\mathcal{A}^0(\xi) := \{ (\hat{S}, \xi) : \hat{S} = (\hat{S}, \hat{S}) \in \mathcal{A}^0 \text{ and } \xi \text{ is a variable in } \hat{S} \}.
\]

Let \( (\hat{S}_1, \xi_1), (\hat{S}_2, \xi_2) \in \mathcal{A}^0(\xi) \). We say that \( (\hat{S}_1, \xi_1) \) is directly up-harmonious with \( (\hat{S}_2, \xi_2) \) if there is a polynomial \( Q(t) \in \mathcal{M}(\hat{R})[t] \) such that \( \xi_1 = Q(\xi_2) \) and \( \hat{S}_1 = Q(\hat{S}_2) \), denoted \( (\hat{S}_1, \xi_1) \overset{Q}{\leftrightarrow} (\hat{S}_2, \xi_2) \). If \( (\hat{S}_1, \xi_1) \) and \( (\hat{S}_2, \xi_2) \) are directly up-harmonious with one another then we say that they are directly harmonious (with one another), denoted \( (\hat{S}_1, \xi_1) \leftrightarrow (\hat{S}_2, \xi_2) \). Let \( (\hat{Z}_1, \zeta_1), (\hat{Z}_2, \zeta_2) \in \mathcal{A}(\zeta) \). If there exists a holomorphic base-preserving surjection \( \lambda : \hat{Z}_2 \to \hat{Z}_1 \) with \( \lambda(\zeta_2) = \zeta_1 \) then we say that \( (\hat{Z}_1, \zeta_1) \) is analytically up-harmonious with \( (\hat{Z}_2, \zeta_2) \) modulo \( \hat{\lambda} \), denoted \( (\hat{Z}_1, \zeta_1) \overset{\hat{\lambda}}{\leftrightarrow} (\hat{Z}_2, \zeta_2) \), where \( \hat{\lambda} \) and \( \hat{\lambda} \) are called analytically up-harmonious mappings. If further \( \hat{\lambda} \) is biholomorphic, then we say that \( (\hat{Z}_1, \zeta_1) \) and \( (\hat{Z}_2, \zeta_2) \) are analytically harmonious (with one another) modulo \( \hat{\lambda} \), denoted \( (\hat{Z}_1, \zeta_1) \leftrightarrow (\hat{Z}_2, \zeta_2) \), where \( \hat{\lambda} \) and \( \hat{\lambda} \) are called analytically harmonious mappings.

**Proposition 3.5.** Let \( (\hat{S}_1, \xi_1), (\hat{S}_2, \xi_2) \in \mathcal{A}^0(\xi) \). Then \( (\hat{S}_1, \xi_1) \) is analytically up-harmonious with \( (\hat{S}_2, \xi_2) \) modulo \( \hat{\lambda} \) if and only if \( (\hat{S}_1, \xi_1) \) is directly up-harmonious with \( (\hat{S}_2, \xi_2) \). In this case, \( \hat{\lambda} : \hat{S}_2 \to \hat{S}_1 \) is a holomorphic covering map (i.e. \( \hat{\lambda} : \hat{S}_2 \to \hat{S}_1 \) is an exact covering map and \( \hat{\lambda} : \hat{S}_2 \to \hat{S}_1 \) is a holomorphic branched covering map).

**Proof.** Suppose \( \hat{\lambda} = (\hat{\lambda}, \hat{\lambda}) \) is the analytically up-harmonious mapping from \( \hat{S}_2 = (\hat{S}_2, \hat{S}_2) \) to \( \hat{S}_1 = (\hat{S}_1, \hat{S}_1) \). Suppose \( \xi_1 \in \hat{S}_1, \xi_2 \in \hat{S}_2 \) and \( \xi_1 = \lambda(\xi_2) \). Then there exists a punctured partial neighborhood \( V \) of \( r = \hat{p}_1(\xi_1) \) in \( \hat{R} \) (if necessary we
will shrink \( V \), where \( \hat{p}_1 = (\hat{p}_1, \bar{p}_1) \) is the canonical projection from \( \hat{S}_1 \) to \( \hat{R} \), and \( f_1, f_2 \in \mathcal{H}(V) \) such that \( \xi_1 = \langle f_1 \rangle_r \) and \( \xi_2 = \langle f_2 \rangle_r \). Suppose \( \hat{F}_1 = (\hat{F}_1, \bar{F}_1) \) and \( \hat{F}_2 = (\hat{F}_2, \bar{F}_2) \) are the original basic functions on \( \hat{S}_1 \) and \( \hat{S}_2 \), respectively. Then by Lemma 3.4, there exists a polynomial \( Q(t) \in \mathcal{M}(\hat{R})[t] \) such that \( \hat{F}_1 \circ \lambda = ((\hat{p}_1 \circ \bar{\lambda})^* Q)(\bar{F}_2) \). Since \( \bar{\lambda} \) is base-preserving and partially continuous, we have \( \lambda((f_2)_r) = ((\hat{p}_1 \circ \bar{\lambda})^* Q)(\bar{F}_2)((f_2)_r) = (Q(f_2))(r') \).

Then we get

\[
\xi_1 = \langle f_1 \rangle_r = \langle Q(f_2) \rangle_r = Q(\xi_2).
\]

Suppose there is a polynomial \( Q(t) \in \mathcal{M}(\hat{R})[t] \) such that \( \xi_1 = Q(\xi_2) \), where \( \xi_1 \) and \( \xi_2 \) travel around the whole \( \hat{S}_1 \) and \( \hat{S}_2 \), respectively and correspondingly. Suppose \( \bar{\lambda} : \hat{S}_2 \rightarrow \hat{S}_1 \) is defined by \( \lambda(\xi_2) := Q(\xi_2) \) for \( \xi_2 \in \hat{S}_2 \), \( \bar{\lambda}(\xi_2) := Q(\xi_2) \) for \( \xi_2 \in S_2 = \hat{S}_2 \cap \hat{S}_1 \) and for \( \xi_2 \in \hat{S}_2 \setminus S_2 \) we continuously continue \( \bar{\lambda} \) on \( \hat{S}_2 \). Then \( \lambda : \hat{S}_2 \rightarrow \hat{S}_1 \) is an exact covering map and \( \bar{\lambda} : \hat{S}_2 \rightarrow \hat{S}_1 \) is a proper holomorphic map (cf. [10] Theorems (8.4) and (8.9)).

From Proposition 3.5 it follows

**Corollary.** Let \( (\hat{S}_1, \xi_1), (\hat{S}_2, \xi_2) \in \mathcal{A}_0(\xi) \). Then \( (\hat{S}_1, \xi_1) \) and \( (\hat{S}_2, \xi_2) \) are analytically harmonious if and only if they are directly harmonious.

Suppose \( \hat{R} \) is a base surface. Fix a point \( r_0 \) in \( \hat{R} \), which we call a *base point* in \( \hat{R} \). Let \( \hat{\varphi} \in \mathcal{H}_{r_0} \) be an algebraic punctured partial germ at \( r_0 \) on \( \hat{R} \) and let \( \hat{S} \) be the original basic algebraic Riemann surface determined by \( \hat{\varphi} \). In order to make the difference, we put a “label” \( \hat{\varphi} \) on \( \hat{S} \) and call \( (\hat{\varphi}; \hat{S}) \) an original basic algebraic Riemann surface with label (or tag) \( \hat{\varphi} \), denoted by \( \hat{S} \). We say \( \hat{\varphi} \) is the (resp. a) *natural label* of \( \hat{S} \) (resp. \( \hat{S} \)). The original basic function \( \hat{F} \) (or \( \bar{F} \) on \( \hat{S} \)) is also considered as the original basic function on \( \hat{S} \). Let \( \hat{S}_1 = (\hat{\varphi}_1; \hat{S}_1) \) and \( \hat{S}_2 = (\hat{\varphi}_2; \hat{S}_2) \) be original basic algebraic Riemann surfaces with natural labels. Then we consider that \( (\hat{S}_1, \xi_1) \cong (\hat{S}_2, \xi_2) \) precisely if \( \hat{\varphi}_1 = \hat{\lambda}(\hat{\varphi}_2) \) (refer to Lemma 3.6 below).

**Remark 6.** By Lemma 3.6 (refer to Remark 5), we can continue the label \( \hat{\varphi} \) along curves in \((\hat{R}, \mathcal{T}(A))\), where \( A \) is the set of branch points of the minimal polynomial of \( \hat{\varphi} \) in \( \mathcal{M}(\hat{R})[t] \), to get \( \hat{S} \).

In order to give a label to \( \hat{Z} \in \mathcal{A} \), we now introduce a punctured partial germ in a system of “equivalent presheaves”. Suppose \( \mathcal{X} \) is a family of basic Riemann surfaces that are analytically harmonious with one another and \( \bar{\lambda}_0 \) is the analytically harmonic relation. Suppose \( (X, \mathcal{F}; X, \mathcal{F}) \in \mathcal{X} \) and \( Y \) is a traditional Riemann surface. Denote by \( \mathcal{H}(U \rightarrow Y) \) the set of all holomorphic mappings from \( U \) to \( Y \), where \( U \) is an open set in \((X, \mathcal{F})\). Let \( \mathcal{H}_{X,Y} = (\mathcal{H}(U \rightarrow Y))_{U \in \mathcal{F}} \) be the family consisting of all holomorphic mappings from \( U \) to \( Y \) for all \( U \in \mathcal{F} \). It is a sheaf of sets from \( X \) to \( Y \). Denote by \( \mathcal{H}_{X,Y} \) the system of all sheaves \( \mathcal{H}_{X,Y} \) for \((X, \mathcal{F}; X, \mathcal{F}) \in \mathcal{X} \).
Let $a_0 \in \tilde{X}_0$, where $\tilde{X}_0 \in \mathfrak{X}$. Denote
\[ \tilde{a} := \{ \hat{\lambda}(a_0) : \hat{\lambda} \in \hat{A}_0 \text{ is an analytically harmonious mapping from } \tilde{X}_0 \text{ to } \tilde{X} \in \mathfrak{X} \}. \]

Let
\[ H_{\tilde{a}} := \bigcup_{\hat{\lambda} \in \hat{A}_0} \bigcup_{U \in \mathcal{T}(\hat{\lambda}(a_0))} \mathcal{H}(U \to Y), \]
which is a disjoint union (we may also use $\mathcal{H}(\tilde{T}(\tilde{a}) \to Y)$ to represent $H_{\tilde{a}}$).

In $H_{\tilde{a}}$, two elements (mappings) $f_1 \in \mathcal{H}(U_1 \to Y)$ and $f_2 \in \mathcal{H}(U_2 \to Y)$ ($U_1 \in \tilde{T}(\hat{\lambda}_1(a_0))$ and $U_2 \in \tilde{T}(\hat{\lambda}_2(a_0))$, $\hat{\lambda}_1, \hat{\lambda}_2 \in \hat{A}_0$) are said to be equivalent, denoted $f_1 \sim f_2$, if there exists $U \in \tilde{T}(\hat{\lambda}(a_0))$ ($\hat{\lambda} \in \hat{A}_0$) with $\hat{\lambda}^{-1}(U) \subseteq \hat{\lambda}_1^{-1}(U_1) \cap \hat{\lambda}_2^{-1}(U_2)$ such that $f_1 \circ \hat{\lambda}_1|_{\hat{\lambda}^{-1}(U)} = f_2 \circ \hat{\lambda}_2|_{\hat{\lambda}^{-1}(U)}$. It is easy to see that this really is an equivalence relation. Denote
\[ H_{\tilde{a}} := H_{\tilde{a}} / \sim_{\tilde{a}}, \]
which is the set of all equivalence classes and is called the punctured partial stalk of the sheaf system $H_X \tilde{a}$ at $\tilde{a}$ or $\hat{\lambda}(a_0)$. Suppose $f \in H(V \to Y)$, where $V \supseteq U$, $V \in \mathcal{I}$ and $U \in \tilde{T}(\hat{\lambda}(a_0))$. The equivalence class of $f|_U \in H(U \to Y)$ modulo $\sim_{\tilde{a}}$ is called the punctured partial germ of $f$ at $\tilde{a}$ or $\hat{\lambda}(a_0)$, denoted $\langle f \rangle_{\tilde{a}}$ or $\langle f \rangle_{\hat{\lambda}(a_0)}$.

Suppose $\mathfrak{A}$ is the set consisting of all algebraic punctured partial germs at the base point $r_0$ on $\hat{R}$. Then $\mathfrak{A}$ is a field. Let $\bar{\mathfrak{A}}_0$ denote the set of all original basic algebraic Riemann surfaces with labels over $\hat{R}$ determined by germs in $\mathfrak{A}$. Given a subset $\mathcal{S}$ of $\mathfrak{A}$, let $\mathcal{S}$ be the subfield $\mathcal{M}_{r_0}(\mathcal{S})$ of $\mathfrak{A}$ generated by $\mathcal{S}$ and $\mathcal{M}_{r_0} = \{ \langle f \rangle_{r_0} : f \in \mathcal{M}(\hat{R}) \}$, and then let $\bar{\mathcal{S}}_0$ be the set of all original basic algebraic Riemann surfaces with labels determined by germs in $\mathcal{S}$. Let $\hat{\Lambda}$ and $\hat{\Lambda}_0$ denote the analytically harmonious relation and the directly harmonious relation, respectively. Suppose
\[ \bar{\mathcal{S}}_0 = \bigcup_{j \in J} L_j^0 \]
is the partition of $\bar{\mathcal{S}}_0$ by $\hat{\Lambda}_0$. We call $L_j^0 \ (j \in J)$ original level surfaces or original levels. Suppose $\bar{\mathcal{L}}_j^0$ is the directly harmonious relation in $L_j^0 \ (j \in J)$. Suppose $\hat{Z} = (\tilde{Z}, \tilde{Z})$ is a basic algebraic Riemann surface that is analytically harmonious with $\tilde{S} \in L_j^0$ and $\hat{\sigma} = (\hat{\sigma}, \hat{\sigma})$ is the analytically harmonious mapping from $\tilde{S}$ to $\tilde{Z}$. Then $\hat{\sigma} \in H(\tilde{S} \to \tilde{Z})$. Let $\mathfrak{X} = L_j^0$ and let $\langle \hat{\sigma} \rangle_{\hat{\phi}}$ denote the punctured partial germ $\hat{\sigma}_{\hat{\phi}}$. We now put label $\langle \hat{\sigma} \rangle_{\hat{\phi}}$ on $\tilde{Z}$ and call $(\langle \hat{\sigma} \rangle_{\hat{\phi}}, \tilde{Z})$ a basic algebraic Riemann surface with label (or tag) $\langle \hat{\sigma} \rangle_{\hat{\phi}}$, denoted by $\tilde{Z}$. We say $\langle \hat{\sigma} \rangle_{\hat{\phi}}$ is the (resp. a) given label of $\tilde{Z}$ (resp. $\tilde{Z}$). It is worth noting that if $\tilde{Z}$ is an original basic algebraic Riemann surface with natural label $\hat{\phi}$ then the given label of $\tilde{Z}$ is just (id)$_{\hat{\phi}}$, which is uniform with its natural label $\hat{\phi}$.

Let $\mathcal{A}$ denote the set of all basic algebraic Riemann surfaces with labels over $\hat{R}$. Let $\hat{A} = (\hat{A}, \hat{A})$ denote the analytically up-harmonious relation in $\hat{A}$ determined
by the direct up-harmonious relation \( \hat{A}^0 = (\hat{A}, \hat{A}^0) \) in \( \hat{A}^0 \), which is defined as follows: For \( \tilde{Z}_1, \tilde{Z}_2 \in \tilde{A}, \tilde{Z}_1(\zeta_1) \mapsto \tilde{Z}_2(\zeta_2) \) (\( \hat{l} \in \hat{A} \)) if and only if there exist \( \hat{S}_1, \hat{S}_2 \in \hat{A}^0 \) such that \( \tilde{Z}_1(\zeta_1) \mapsto \hat{S}_1(\zeta_1), \tilde{Z}_2(\zeta_2) \mapsto \hat{S}_2(\zeta_2), \hat{S}_1(\zeta_1) \mapsto \hat{S}_2(\zeta_2) \) (\( \hat{\lambda}_1, \hat{\lambda}_2 \in \hat{\Lambda}, \hat{\lambda}_0 \in \hat{A}^0 \) and \( \tilde{Z}(\zeta) \) denotes \((\tilde{Z}, \zeta)\)) and \( \lambda = \lambda_1 \circ \lambda_0 \circ \lambda_2^{-1} \), where \( \lambda \) is called an analytically up-harmonious relation. We may also denote \( \tilde{Z}_1(\zeta_1) \mapsto \hat{S}_2(\zeta_2) \). Meanwhile, \( \tilde{Z}_1 \mapsto \tilde{Z}_2 \) also means that the label \( (\lambda_2)_{\hat{\phi}_2} \) of \( \tilde{Z}_2 \) is mapped to the label \( (\hat{\lambda}_1)_{\hat{\phi}_1} \) of \( \tilde{Z}_1 \) by \( \lambda = (\hat{\lambda}, \lambda) \) \((\hat{\phi}_1, \hat{\phi}_2) \). Here if \( \lambda : \tilde{Z}_2 \rightarrow \tilde{Z}_1 \) is not biholomorphic, then we say \( \tilde{Z}_2 \) is strictly under \( \tilde{Z}_1 \) or \( \tilde{Z}_1 \) is strictly under \( \tilde{Z}_2 \). If \( \lambda : \tilde{Z}_2 \rightarrow \tilde{Z}_1 \) is biholomorphic, then we say \( \tilde{Z}_2 \) is equivalent to \( \tilde{Z}_1 \) modulo \( \lambda \) or \( \hat{\lambda} \), denoted \( \tilde{Z}_1 \leftrightarrow \tilde{Z}_2 \) (i.e. \( \tilde{Z}_2(\zeta_2) \) is harmonious with \( \tilde{Z}_1(\zeta_1) \)).

Suppose

\[
L_j := \{ \tilde{Z} : \tilde{Z} \in \tilde{A} \text{ is analytically harmonious with some } \hat{S} \in \hat{L}_j^0 \} \quad (j \in J),
\]

which are called level surfaces or levels. Let

\[
\tilde{Z} := \bigcup_{j \in J} L_j.
\]

Then \( \tilde{Z} \) is the analytically up-harmonious class of \( \hat{S}^0 \) in \( \hat{A} \). We call \( \tilde{Z} \) the algebraic Riemann surface (over \( \hat{R} \)) determined by \( \hat{S} \) and \( \hat{S}^0 \) the original algebraic Riemann surface corresponding to \( \tilde{Z} \), denoted by \( \hat{Z}_0 \). We call \( \hat{S} \) in the above the natural label set or the natural label field of \( \tilde{Z} \) or \( \hat{Z}_0 \), denoted by \( \hat{L}(\tilde{Z}) \) or \( \hat{L}(\hat{Z}_0) \).

Remark 7. If \( \hat{S} = \{ \hat{\phi} \} \), where \( \hat{\phi} \) is an algebraic punctured partial germ at the base point \( r_0 \), then instead of a (traditional) Riemann surface (determined by \( \hat{\phi} \)) we consider the analytically up-harmonious class \( \tilde{Z} \) determined by \( \hat{S} \), which has a geographic element the basic Riemann surface that is determined by \( \hat{\phi} \), as our (algebraic) Riemann surface, which we call the (algebraic) Riemann surface determined by \( \hat{\phi} \).

Remark 8. Generally, the above definition means that we consider an algebraic Riemann surface as a system of basic algebraic Riemann surfaces organized by the analytically up-harmonious relation with the aid of labels.

Remark 9. Suppose \( \tilde{Z} \) is an algebraic Riemann surfaces over a base surface \( \hat{R} \) with base point \( r_0 \). Then we may consider that all \( \hat{Z} \), for \( \hat{Z} = (\hat{Z}, \zeta) \) and \( \hat{Z} = ((\hat{\phi})_{\hat{\phi}}; \hat{Z}) \in \tilde{Z} \), form a “coordinate system” in \( \tilde{Z} \) and all \( \hat{Z} \) together show the topological and complex structure of \( \tilde{Z} \), where the base point \( r_0 \) may be regarded as an “origin of coordinates”.

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Lemma 3.6. Suppose \( \tilde{\varphi}, \tilde{\varphi}_1 \in A \) and \( \hat{\varphi} = \hat{P}(\tilde{\varphi}_1) \) for some \( \hat{P}(t) \in \mathcal{M}(\hat{R})[t] \).

Suppose \( \tilde{S} \) and \( \tilde{S}_1 \) are two original basic algebraic Riemann surfaces over the base surface \( \hat{R} \) determined by \( \hat{\varphi} \) and \( \hat{\varphi}_1 \) with labels \( \hat{\varphi} \) and \( \hat{\varphi}_1 \), respectively. Then there is an up-harmonious mapping \( \hat{\psi}_1 : \tilde{S}_1 \rightarrow \tilde{S} \), which is defined by \( \hat{\psi}(\psi_1) = \hat{P}(\tilde{\psi}_1) \) for \( \hat{\psi}_1 \in \tilde{S}_1 \) and \( \hat{P}(\tilde{\psi}_1) = \hat{P}(\tilde{\psi}_1) \) for \( \hat{\psi}_1 \in \tilde{S}_1 \) and \( \tilde{S}_1 = \tilde{S}_1 \cap \tilde{S}_1 \) (\( \tilde{S}_1 = (\tilde{S}_1, \tilde{S}_1) \)), such that \( \hat{\psi} \circ \hat{\psi}_1 = \hat{P}(\tilde{F}_1) \), where \( \hat{F} \) and \( \hat{F}_1 \) are the original basic functions on \( \tilde{S} \) and \( \tilde{S}_1 \) respectively and \( \hat{P} = (\hat{P}, \hat{P}) \). \( \tilde{P} = \hat{P} \). \( \square \)

The (directly) up-harmonious mapping \( \hat{\psi}_1 : \tilde{S}_1 \rightarrow \tilde{S} \) in Lemma 3.6 is said to be corresponding to \( \hat{\varphi} = \hat{P}(\tilde{\varphi}_1) \) or determined by \( \hat{P}(t) \). By Proposition 3.5 we see that this \( \hat{\psi}_1 \) is a holomorphic covering map. We can also see that generally an analytically up-harmonious mapping \( \lambda = (\hat{\lambda}, \tilde{\lambda}) : \tilde{Z}_1 \rightarrow \tilde{Z} \) is a holomorphic covering map, which means that \( \lambda : \tilde{Z}_1 \rightarrow \tilde{Z} \) is a covering map and \( \tilde{\lambda} : Z_1 \rightarrow Z \) is a branched holomorphic covering map.

Suppose \( \hat{X} \) and \( \hat{Y} \) are connected universal topological spaces (i.e. \( \hat{X}, \hat{Y} \) and \( \hat{Y}, \hat{Y}' \) are connected) and \( \hat{\psi} : \hat{Y} \rightarrow \hat{X} \) is a covering map. The covering is called Galois if for every pair of points \( y_1, y_2 \in \hat{Y} \) with \( \hat{\psi}(y_1) = \hat{\psi}(y_2) \) there exists a deck transformation \( \hat{\sigma} : \hat{Y} \rightarrow \hat{Y} \) such that \( \hat{\sigma}(y_1) = y_2 \). Suppose \( \hat{X} = (\hat{X}, \hat{\nabla}) \) and \( \hat{Y} = (\hat{Y}, \hat{\nabla}) \) are universal topological spaces with tied spaces and \( \hat{\psi} : \hat{Y} \rightarrow \hat{X} \) is a covering map, which means \( \hat{\psi} : \hat{Y} \rightarrow \hat{X} \) is a covering map, where \( \hat{\psi} = (\hat{\psi}, \hat{\psi}) \). The covering \( \hat{\psi} : \hat{Y} \rightarrow \hat{X} \) is called Galois if \( \hat{\psi} : \hat{Y} \rightarrow \hat{X} \) is Galois.

Suppose \( \hat{R} \) is a base surface with base point \( r_0 \in \hat{R} \). Suppose \( \hat{Y} \) is an algebraic Riemann surface over \( \hat{R} \) and \( \hat{A} \) is the analytically up-harmonious relation in \( \hat{Y} \). For \( \hat{Y} \in \hat{Y} \) we denote mapping \( \hat{f} : \hat{Y} \rightarrow (\hat{C}, \hat{\nabla}) \) (i.e. \( \hat{f} : \hat{Y} \rightarrow \hat{C} \) and \( \hat{f} : \hat{C} \rightarrow \hat{Y} \) are mappings) by \( \hat{f} : \hat{Y} \rightarrow \hat{C} \), called a (complex) function on \( \hat{Y} \). Let \( \hat{f}_1 : \hat{Y}_1 \rightarrow \hat{C} \) and \( \hat{f}_2 : \hat{Y}_2 \rightarrow \hat{C} \) be functions, where \( \hat{Y}_1, \hat{Y}_2 \in \hat{Y} \). If there exists \( \hat{\lambda} \in \hat{A} \) such that \( \hat{Y}_1 \overset{\hat{\lambda}}{\rightarrow} \hat{Y}_2 \) and \( \hat{f}_2 = \hat{f}_1 \circ \hat{\lambda} \), then we say that \( \hat{f}_2, \hat{f}_1 \) is over \( \hat{f}_1, \hat{f}_1 \). If \( \hat{f}_2, \hat{f}_1 \) is over \( \hat{f}_1, \hat{f}_1 \) or \( \hat{f}_2, \hat{f}_1 \) is over \( \hat{f}_1, \hat{f}_1 \), then we say that \( \hat{f}_2, \hat{f}_1 \) and \( \hat{f}_2, \hat{f}_1 \) are directly compatible. If there exists a chain of functions \( \hat{g}_1, \hat{g}_2, \ldots, \hat{g}_n \) such that \( \hat{g}_1, \hat{g}_2, \ldots, \hat{g}_n, \hat{g}_n \) is a (complex) function on \( \hat{Y} \), \( \hat{f}_1 = \hat{f}_1, \hat{f}_1 = \hat{f}_1, \ldots, \hat{f}_1 \) are directly compatible. It is plain that the compatibility relation is an equivalence relation. We call the equivalence class of \( \hat{f} \) a (complex) function on \( \hat{Y} \), denoted \( \hat{f} \) or \( \hat{f} : \hat{Y} \rightarrow \hat{C} \). We call \( \hat{f} \) an element of \( \hat{f} \) an expression element of \( \hat{f} \), where \( \hat{f} \) is called an expression function and \( \hat{Y} \) an expression domain. Let \( \hat{f}|_{\hat{Y'}} := \hat{f} \), called a restriction of \( \hat{f} \) on \( \hat{Y} \). If partial elements of \( \hat{f} \) (as a set) are omitted, then we still use it to denote the same function.

For functions \( \hat{f}_1 \) on \( \hat{Y}_1 \) and \( \hat{f}_2 \) on \( \hat{Y}_2 \), where \( \hat{Y}_1, \hat{Y}_2 \in \hat{Y} \), by the following lemma we know that \( \hat{f}_1 \) and \( \hat{f}_2 \) are compatible precisely if there exists \( \hat{Y}_0 \in \hat{Y} \) such that \( \hat{Y}_1 \overset{\hat{f}_1}{\rightarrow} \hat{Y}_0, \hat{Y}_2 \overset{\hat{f}_2}{\rightarrow} \hat{Y}_0 \) for \( \hat{f}_1, \hat{f}_2 \in \hat{A} \) and \( \hat{f}_1 \circ \hat{f}_1 = \hat{f}_2 \circ \hat{f}_2 \).

Lemma 3.7. Let \( \tilde{S}_1, \tilde{S}_2 \in \tilde{S}^0 \). Then there exists \( \tilde{S}_0 \in \tilde{S}^0 \) and \( \tilde{q}_1, \tilde{q}_2 \in \tilde{A}^0 \) (\( \tilde{A}^0 \) is the direct up-harmonious relation in \( \tilde{A}^0 \)) such that \( \tilde{S}_1 \overset{\tilde{q}_1}{\rightarrow} \tilde{S}_0 \) and \( \tilde{S}_2 \overset{\tilde{q}_2}{\rightarrow} \tilde{S}_0 \).
Proof. Suppose $\tilde{S}_j = (\tilde{\varphi}_j, \tilde{S}_j) \ (j = 1, 2)$ and $S_0 = \mathcal{M}_{r_0}(\tilde{\varphi}_1, \tilde{\varphi}_2)$ (the field generated by $\tilde{\varphi}_1$, $\tilde{\varphi}_2$ and $\mathcal{M}_{r_0}$), where $r_0$ is the base point in the base surface $\tilde{R}$. Then there exists $\tilde{\varphi}_0 \in S_0$ such that $S_0 = \mathcal{M}_{r_0}(\tilde{\varphi}_0)$. Hence, there exist polynomials $\tilde{Q}_1(t)$ and $\tilde{Q}_2(t)$ in $\mathcal{M}([\tilde{R}][t])$ such that $\tilde{\varphi}_1 = \tilde{Q}_1(\tilde{\varphi}_0)$ and $\tilde{\varphi}_2 = \tilde{Q}_2(\tilde{\varphi}_0)$. Let $\tilde{S}_0$ be the original algebraic Riemann surface determined by $\tilde{\varphi}_0$ with label $\tilde{\varphi}_0$. Then $\tilde{S}_1 \leftrightarrow \tilde{S}_0$ and $\tilde{S}_2 \leftrightarrow \tilde{S}_0$ by Lemma 3.6 where $\tilde{q}_1$ and $\tilde{q}_2$ are determined by $\tilde{Q}_1$ and $\tilde{Q}_2$ respectively. □

Let $\tilde{f} \in \mathcal{M}(\tilde{Y})$, where $\tilde{Y} \in \tilde{Y}$. Then the equivalence class $\tilde{f}$ of $(\tilde{f}, \tilde{Y})$ is called a meromorphic function on $\tilde{Y}$. If $\tilde{f} \in \mathcal{H}(\tilde{Y})$ then $\tilde{f}$ is called a holomorphic function on $\tilde{Y}$. Denote the set of all meromorphic (resp. holomorphic) functions on $\tilde{Y}$ by $\mathcal{M}(\tilde{Y})$ (resp. $\mathcal{H}(\tilde{Y})$). Then $\mathcal{M}(\tilde{Y})$ is a field and $\mathcal{H}(\tilde{Y})$ is a ring by means of the operation defined on representatives. $\mathcal{M}(\tilde{Y})$ is also a vector space over $\mathcal{M}(\tilde{R})$ under the scalar multiplication that $h \cdot \tilde{f} := \tilde{g}$ for $\tilde{h} \in \mathcal{M}(\tilde{R})$ and $\tilde{f} \in \mathcal{M}(\tilde{Y})$, where $\tilde{f}$ is determined by $\tilde{f} \in \mathcal{M}(\tilde{Y})$, $\tilde{\varphi} : \tilde{Y} \rightarrow \tilde{R}$ is the canonical projection and $\tilde{g} \in \mathcal{M}(\tilde{Y})$ is determined by $\tilde{g} = (h \circ \tilde{\varphi}) \cdot \tilde{f}$. Moreover, we may consider $\mathcal{M}(\tilde{R})$ as a subfield of $\mathcal{M}(\tilde{Y})$ by the monomorphism $\gamma : \mathcal{M}(\tilde{R}) \rightarrow \mathcal{M}(\tilde{Y})$, defined by $\gamma(\tilde{h}) := \tilde{h}$, where $\tilde{h} \in \mathcal{M}(\tilde{Y})$ is determined by $\tilde{h} \in \mathcal{M}(\tilde{R})$.

Suppose $Y$ and $Z$ are algebraic Riemann surfaces (over base surfaces $\hat{R}_1$ and $\hat{R}_2$, respectively). Let $\sigma_j : Y_j \rightarrow \hat{Z}_j$ be mappings ($Y_j \in Y$ and $\hat{Z}_j \in \hat{Z}$, $j = 1, 2$). If there exist $\lambda, \mu \in \hat{A}$ such that $\hat{Y}_1 \overset{\lambda}{\sim} \hat{Y}_2$, $\hat{\lambda}_1 \overset{\mu}{\sim} \hat{\lambda}_2$ and $\mu \circ \hat{\sigma}_2 = \hat{\sigma}_1 \circ \hat{\lambda}_2$ then we say $\hat{\sigma}_2$ is over $\hat{\sigma}_1$ or $\hat{\sigma}_1$ is under $\hat{\sigma}_2$; if moreover $\hat{\sigma}_2 \circ \hat{\lambda}_1 = \hat{\mu}_1^{-1} \circ \hat{\sigma}_1$ (i.e. $\hat{\sigma}_2(\hat{\lambda}_1^{-1}(y_1)) = \hat{\mu}_1^{-1}(\hat{\sigma}_1(y_1))$ for any $y_1 \in Y_1$), then we say $\hat{\sigma}_2$ is exactly over $\hat{\sigma}_1$ or $\hat{\sigma}_1$ is exactly under $\hat{\sigma}_2$. If $\hat{\sigma}_1$ is (resp. exactly) over $\hat{\sigma}_2$ or $\hat{\sigma}_2$ is (resp. exactly) over $\hat{\sigma}_1$, then we say $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are directly (resp. directly and exactly) compatible. If in the above $\lambda$ and $\mu$ are biholomorphic, i.e. $\hat{Y}_1 \overset{\lambda}{\leftrightarrow} \hat{Y}_2$ and $\hat{Z}_1 \overset{\mu}{\leftrightarrow} \hat{Z}_2$, then we say $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are equivalent, denoted $\hat{\sigma}_1 \sim \hat{\sigma}_2$. This is clearly an equivalence relation.

Remark 10. Evidently, $\hat{\sigma}_2 \circ \hat{\lambda}_1 = \hat{\mu}_1^{-1} \circ \hat{\sigma}_1$ implies $\mu \circ \hat{\sigma}_2 = \hat{\sigma}_1 \circ \hat{\lambda}$. If $\hat{\sigma}_1$ is injective and $\hat{\sigma}_2$ is surjective, then $\hat{\sigma}_2 \circ \hat{\lambda}_1 = \hat{\mu}_1^{-1} \circ \hat{\sigma}_1$ precisely if $\mu \circ \hat{\sigma}_2 = \hat{\sigma}_1 \circ \hat{\lambda}$.

For a mapping $\hat{\sigma} : \hat{Y} \rightarrow \hat{Z}$ we denote its domain $\text{dom}(\hat{\sigma}) := \hat{Y}$ and its codomain $\text{codom}(\hat{\sigma}) := \hat{Z}$. Suppose $\hat{\sigma}$ is a set of some mappings. Suppose $\hat{Y}$ and $\hat{Z}$ are two algebraic Riemann surfaces. If for two mappings $\hat{\sigma}_1$ and $\hat{\sigma}_2$, there exists $\hat{\sigma}_0 \in \hat{\sigma}$ over $\hat{\sigma}_1$ and $\hat{\sigma}_2$ and, moreover, that $\text{dom}(\hat{\sigma}_2)$ is over (resp. under) $\text{dom}(\hat{\sigma}_1)$ implies that $\hat{\sigma}_2$ is over (resp. under) $\hat{\sigma}_1$, then we say $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are compatible in $\hat{\sigma}$. If any two mappings $\hat{\sigma}_1$, $\hat{\sigma}_2 \in \hat{\sigma}$ are compatible in $\hat{\sigma}$ then we say $\hat{\sigma}$ is compatible (similarly we have the notion of exact compatibility of $\hat{\sigma}$). Suppose $\hat{\sigma}$ is compatible and satisfies the following two conditions:

1. There exists $\hat{\sigma}_0 \in \hat{\sigma}$ such that for any $\hat{Y} \in \hat{Y}$ over $\text{dom}(\hat{\sigma}_0)$ there exists $\hat{\sigma} \in \hat{\sigma}$ with $\text{dom}(\hat{\sigma}) \leftrightarrow \hat{\sigma}_0$.

2. For any $\hat{Z} \in \hat{Z}$, there exists $\hat{\sigma}' \in \hat{\sigma}$ with $\text{codom}(\hat{\sigma}')$ over $\hat{Z}$.

Then we say $\hat{\sigma}$ is a mapping from $\hat{Y}$ to $\hat{Z}$. We call $(\hat{\sigma}, \hat{Y})$ an expression element of $\hat{\sigma}$, where $\hat{\sigma}$ is called an expression mapping and $\hat{Y}$ an expression domain. Denote
\( \tilde{\sigma} |_{\tilde{Y}} := \sigma \), called the \textit{restriction} of \( \sigma \) on \( \tilde{Y} \). Specially, a function \( \tilde{f} : \tilde{Y} \to \tilde{C} \) is also a mapping.

Suppose \( \tilde{\sigma} \) and \( \tilde{\tau} \) are mappings from \( \tilde{Y} \) to \( \tilde{Z} \). If every \( \sigma \in \tilde{\sigma} \) over some \( \tilde{\sigma}_0 \in \tilde{\sigma} \) is compatible with every \( \tau \in \tilde{\tau} \) over some \( \tilde{\tau}_0 \in \tilde{\tau} \) both in \( \tilde{\sigma} \) and \( \tilde{\tau} \), then we say \( \tilde{\sigma} \) and \( \tilde{\tau} \) are \textit{equal}, denoted \( \tilde{\sigma} \equiv \tilde{\tau} \) (it is probable that as sets \( \tilde{\sigma} \) and \( \tilde{\tau} \) are not equal). This is equivalent to that there exists \( \tilde{Y}_0 \in \tilde{Y} \) such that for every \( \sigma \in \tilde{\sigma} \) with \( \text{dom}(\tilde{\sigma}) \) over \( Y_0 \) there exists \( \sigma \in \tilde{\sigma} \) such that \( \sigma \sim \sigma \) and for every \( \tau \in \tilde{\tau} \) with \( \text{dom}(\tilde{\tau}) \) over \( Y_0 \) there exists \( \sigma \in \tilde{\sigma} \) such that \( \sigma \sim \tau \). In fact, if necessary we may assume \( \tilde{\sigma} \) contains any mapping \( \tilde{\sigma} : \tilde{Y} \to \tilde{Z} \) that is under any \( \tilde{\sigma}_1 \in \tilde{\sigma} \), where \( \tilde{Y} \in \tilde{Y} \) and \( \tilde{Z} \in \tilde{Z} \).

Suppose \( \tilde{Z} \) and \( \tilde{W} \) are algebraic Riemann surfaces over a base surface \( \tilde{R} \) with base point \( r_0 \). If \( \tilde{Z} \subseteq \tilde{W} \) as sets, then we say \( \tilde{W} \) is \textit{over} \( \tilde{Z} \) or \( \tilde{Z} \) is \textit{under} \( \tilde{W} \), denoted \( \tilde{W} \supseteq \tilde{Z} \) or \( \tilde{Z} \subseteq \tilde{W} \). Suppose \( \tilde{\tau} : \tilde{X} \to \tilde{Y}_1 \) and \( \tilde{\sigma} : \tilde{Y}_2 \to \tilde{Z} \) are mappings, where \( \tilde{Y}_1 \subset \tilde{Y}_2 \). Let \( \tilde{\sigma} \circ \tilde{\tau} \) be the set of all mappings \( \tilde{\sigma} \circ \tilde{\lambda} \circ \tilde{\tau} \) for all possible \( \tilde{\tau} : \tilde{X} \to \tilde{Y}_1 \) in \( \tilde{\tau} \) and the corresponding \( \tilde{\sigma} : \tilde{Y}_2 \to \tilde{Z} \) in \( \tilde{\sigma} \) with \( \tilde{Y}_2 \to \tilde{Y}_1 \). If \( \tilde{\sigma} \circ \tilde{\tau} \) satisfies the condition (2) of a mapping (when \( \tilde{Y}_1 = \tilde{Y}_2 \) this condition is satisfied naturally), then it is a mapping from \( \tilde{X} \to \tilde{Z} \), which we call the \textit{composition} of \( \tilde{\sigma} \) and \( \tilde{\tau} \). It is easy to see that \( \tilde{\sigma}_1 \equiv \tilde{\sigma}_2 \) and \( \tilde{\tau}_1 \equiv \tilde{\tau}_2 \) imply \( \tilde{\sigma}_1 \circ \tilde{\tau}_1 \equiv \tilde{\sigma}_2 \circ \tilde{\tau}_2 \).

Remark 11. If \( \tilde{\sigma}_2 \) is over \( \tilde{\sigma}_1 \), then \( \tilde{\sigma}_2 \) is surjective implies \( \tilde{\sigma}_1 \) is surjective. If \( \tilde{\sigma}_2 \) is exactly over \( \tilde{\sigma}_1 \), then \( \tilde{\sigma}_2 \) is injective implies \( \tilde{\sigma}_1 \) is injective.

Suppose \( \tilde{\sigma} : \tilde{Y} \to \tilde{Z} \) is a mapping. If every \( \sigma \in \tilde{\sigma} \) over some \( \tilde{\sigma}_0 \in \tilde{\sigma} \) is surjective then we say \( \tilde{\sigma} \) is \textit{surjective}. We say \( \tilde{\sigma} \) is \textit{injective} if there exists \( \tilde{\sigma}_0 \in \tilde{\sigma} \) such that the following conditions are satisfied:

1. For any \( \tilde{Z} \in \tilde{Z} \) over \( \text{codom}(\tilde{\sigma}_0) \), there exists \( \tilde{\sigma}' \in \tilde{\sigma} \) with \( \text{codom}(\tilde{\sigma}') \leftrightarrow \tilde{Z} \);
2. For \( \tilde{\sigma}_1, \tilde{\sigma}_2 \in \tilde{\sigma} \) over \( \tilde{\sigma}_0 \), that \( \text{codom}(\tilde{\sigma}_2) \) is over \( \text{codom}(\tilde{\sigma}_1) \) implies that \( \tilde{\sigma}_2 \) is over \( \tilde{\sigma}_1 \);
3. Every \( \sigma \in \tilde{\sigma} \) over \( \tilde{\sigma}_0 \) is injective.

We say \( \tilde{\sigma} \) is \textit{bijective} if it is both surjective and injective. Suppose \( \tilde{\sigma} \) is a bijection. Let \( \tilde{\sigma}_0 \in \tilde{\sigma} \) be the one in the conditions for \( \tilde{\sigma} \) being bijective. Denote

\[
\tilde{\sigma}^{-1} := \{ \tilde{\sigma}^{-1} : \tilde{\sigma} \in \tilde{\sigma} \}.
\]

Then \( \tilde{\sigma}^{-1} \) is a mapping from \( \tilde{Z} \) to \( \tilde{Y} \), called the \textit{inverse} of \( \tilde{\sigma} \). Evidently, \( \tilde{\sigma} \equiv \tilde{\tau} \) implies \( \tilde{\sigma}^{-1} \equiv \tilde{\tau}^{-1} \). We also have \( \tilde{\sigma}^{-1} \circ \tilde{\sigma} \equiv \text{id}_{\tilde{Y}} \) and \( \tilde{\sigma} \circ \tilde{\sigma}^{-1} \equiv \text{id}_{\tilde{Z}} \), where \( \text{id}_{\tilde{Y}} := \{ \text{id}_{\tilde{Y}} : \tilde{Y} \in \tilde{Y} \} \), called the \textit{identical mapping} on \( \tilde{Y} \) (clearly, \( \text{id}_{\tilde{Y}} \circ \tilde{\tau} \equiv \tilde{\tau} \) and \( \tilde{\tau} \circ \text{id}_{\tilde{Y}} \equiv \tilde{\tau} \) for mappings \( \tilde{\tau} : \tilde{X} \to \tilde{Y} \) and \( \tilde{\sigma} : \tilde{Y} \to \tilde{Z} \)). Conversely, suppose \( \tilde{\sigma} : \tilde{Y} \to \tilde{Z} \) is a mapping and there exists a mapping \( \tilde{\tau} : \tilde{Z} \to \tilde{Y} \) such that \( \tilde{\tau} \circ \tilde{\sigma} \equiv \text{id}_{\tilde{Y}} \) and \( \tilde{\sigma} \circ \tilde{\tau} \equiv \text{id}_{\tilde{Z}} \). Then \( \tilde{\sigma} \) is a bijection and \( \tilde{\sigma}^{-1} \equiv \tilde{\tau} \).
Let \( \hat{\sigma} \) be a mapping from \( \hat{Y} \) to \( \hat{Z} \). If every \( \hat{\sigma} \in \hat{\sigma} \) over some \( \hat{\sigma}_0 \in \hat{\sigma} \) is (resp. essentially) continuous (i.e. \( \hat{\sigma} \) and \( \hat{\sigma} \) are (resp. essentially) continuous, where \( \hat{\sigma} = (\hat{\sigma}, \hat{\sigma}) \) then \( \hat{\sigma} \) is said to be (exactly) (resp. essentially) continuous. If \( \hat{\sigma} \) is bijective and both \( \hat{\sigma} \) and \( \hat{\sigma}^{-1} \) are (resp. essentially) continuous then \( \hat{\sigma} \) is said to be (exactly) (resp. essentially) homeomorphic. If every \( \hat{\sigma} \in \hat{\sigma} \) over some \( \hat{\sigma}_0 \in \hat{\sigma} \) is (resp. essentially) holomorphic then \( \hat{\sigma}_0 = (\hat{\sigma}, \hat{\sigma}) \) (resp. essentially) holomorphic (analytic). If \( \hat{\sigma} \) is bijective and both \( \hat{\sigma} \) and \( \hat{\sigma}^{-1} \) are (resp. essentially) holomorphic then \( \hat{\sigma} \) is said to be (exactly) (resp. essentially) biholomorphic. In fact, it is sufficient for a (resp. an essential) homeomorphism \( \hat{\sigma} \) being (resp. essentially) biholomorphic that \( \hat{\sigma} \) is (resp. essentially) holomorphic.

Suppose \( \hat{X}, \hat{Y} \) and \( \hat{Z} \) are algebraic Riemann surfaces. Suppose \( \hat{p} : \hat{Y} \to \hat{X} \) and \( \hat{q} : \hat{Z} \to \hat{X} \) are (resp. essentially) continuous maps. A mapping \( \hat{\sigma} : \hat{Y} \to \hat{Z} \) is called fiber-preserving (over \( \hat{X} \)) if \( \hat{p} = \hat{q} \circ \hat{\sigma} \). A mapping \( \hat{p} : \hat{Y} \to \hat{X} \) is called a (or an exact) (resp. an essential) covering map if it is a surjection and every \( \hat{p} \in \hat{p} \) over some \( \hat{p}_0 \in \hat{p} \) is a (resp. an essential) covering map.

Suppose \( \hat{p} : \hat{Y} \to \hat{X} \) is a (resp. an essential) covering map. We call a fiber-preserving (resp. essential) homeomorphism \( \hat{\sigma} : \hat{Y} \to \hat{Y} \) a (or an exact) (resp. an essential) covering transformation or a (or an exact) (resp. an essential) deck transformation of \( \hat{p} \). Obviously, the set of all deck transformations of \( \hat{p} \) forms a group under the composition of mappings, denoted \( \text{Deck}(\hat{Y} \to \hat{Y}) \) or \( \text{Deck}(\hat{Y}/\hat{X}) \).

The (resp. essential) covering \( \hat{p} : \hat{Y} \to \hat{X} \) is called \( \text{Galois} \) if for any \( \hat{p} \in \hat{p} \) there is \( \hat{q} \in \hat{p} \) over \( \hat{p} \) such that \( \hat{q} \) is \( \text{Galois} \). It is easy to see that if the (resp. essential) covering map \( \hat{p} : \hat{Y} \to \hat{X} \) is (resp. essentially) holomorphic then the (resp. essential) deck transformations \( \hat{\sigma} \) are (resp. essentially) biholomorphic.

Suppose \( \hat{Z} \) and \( \hat{W} \) are algebraic Riemann surfaces over a base surface \( \hat{R} \) with base point \( r_0 \) and \( \hat{Z} \subseteq \hat{W} \). Suppose \( \hat{Z} \) is determined by a subfield \( S \) of \( A \) and \( \hat{W} \subseteq \hat{W} \) is an original algebraic Riemann surface with natural label \( \hat{\psi} \). Let \( S' = S \cap S_{\hat{R}}(\hat{\psi}) \), where \( S_{\hat{R}} = \{ (\hat{f})_{\hat{r}_0} : \hat{f} \in S_{\hat{R}} \} \). Then there exists \( \hat{\phi} \in S' \) and \( \hat{P}(t) \in S_{\hat{R}}(\hat{\phi}) \) such that \( \hat{S}' = S_{\hat{R}}(\hat{\phi}) \) and \( \hat{\phi} = \hat{P}(\hat{\psi}) \). Suppose \( \hat{Z} \) is an original algebraic Riemann surface determined by \( \hat{\phi} \) with label \( \hat{\psi} \). Then the direct up-harmonious mapping \( \hat{p} : \hat{W} \to \hat{Z} \) determined by \( \hat{P}(t) \) (see Lemma 3.3) is maximal, which means if there exists an analytically up-harmonious mapping \( \hat{q} : \hat{W} \to \hat{Z}' \) for \( \hat{Z}' \subseteq \hat{Z} \) over \( \hat{Z} \) then \( \hat{q} \sim \hat{p} \). Let \( \hat{p} \) be the set of all \( \hat{p} \) defined above. Then \( \hat{p} \) is a holomorphic covering map (see Proposition 3.5), called the natural covering map from \( \hat{W} \) to \( \hat{Z} \). Suppose \( \hat{X}, \hat{Y} \) and \( \hat{Z} \) are algebraic Riemann Surfaces satisfying \( \hat{X} \subseteq \hat{Z} \subseteq \hat{Y} \). If \( \hat{p} : \hat{Y} \to \hat{X}, \hat{q}_1 : \hat{Y} \to \hat{Z} \) and \( \hat{q}_2 : \hat{Z} \to \hat{X} \) are the natural covering maps, then it follows that \( \hat{q}_2 \circ \hat{q}_1 \) is maximal.

Remark 12. If the concerned algebraic Riemann surfaces have holographic elements, then we may define the mappings between them by means of the holographic elements. In general, we may replace these algebraic Riemann surfaces by their holographic elements, respectively.

Remark 13. Some of the notions in Subsections 3.3 and 3.4 are based on the consideration of the following example: If we are going to find the sum of two algebraic functions \( \sqrt{z} \) and \( \sqrt{z} \), we may calculate \( \omega_1(\sqrt{z})^2 + \omega_2(\sqrt{z})^2 \) on the
Riemann surface of algebraic function \( \sqrt{z} \) instead, where \( \omega_1^2 = \omega_2^3 = 1 \).

4 Algebraic Riemann surfaces and the Galois correspondence

Let \( \hat{R} \) be a base surface (a basic Riemann surface with \( \hat{R} = R \)) with base point \( r_0 \in \hat{R} \). Suppose \( \hat{X} \) and \( \hat{Y} \) are algebraic Riemann Surfaces over \( \hat{R} \) and \( \hat{Y} \) is over \( \hat{X} \). Suppose the natural covering map \( \pi : \hat{Y} \to \hat{X} \) is Galois (then we say \( \hat{Y} \) is *Galois* over \( \hat{X} \) or \( \hat{Y}/\hat{X} \) is *Galois*). In this section Deck(\( \hat{Y}/\hat{X} \)) always means Deck(\( \hat{Y} \xrightarrow{\pi} \hat{X} \)). Suppose \( \hat{Z} \) is an algebraic Riemann Surface satisfying \( \hat{X} \leq \hat{Z} \leq \hat{Y} \), which we call an *intermediate* (algebraic) Riemann Surface of \( \hat{Y}/\hat{X} \).

Let \( D := \text{Deck}(\hat{Y}/\hat{X}) \). Then \( E = \text{Deck}(\hat{Y}/\hat{Z}) \) is a subgroup of \( D \). Denote

\[
[\hat{Z} : \hat{X}] := [\mathcal{M}(\hat{Z}) : \mathcal{M}(\hat{X})]
\]

(the degree of field extension \( \mathcal{M}(\hat{Z})/\mathcal{M}(\hat{X}) \)), called the (covering) order of \( \hat{Z} \) over \( \hat{X} \) (or of \( \hat{Z}/\hat{X} \)). If \([\hat{Z} : \hat{X}] \) is finite then we say \( \hat{Z} \) is *finite* over \( \hat{X} \) or \( \hat{Z}/\hat{X} \) is *finite*.

Denote

\[
\text{Int}(\hat{Y}/\hat{X}) := \{ \hat{Z} : \hat{X} \leq \hat{Z} \leq \hat{Y} \},
\]

\[
\text{FG}(\hat{Y}/\hat{X}) := \{ \hat{Z} \in \text{Int}(\hat{Y}/\hat{X}) : \hat{Z}/\hat{X} \text{ is finite Galois} \}
\]

and

\[
\mathcal{E} := \{ E = \text{Deck}(\hat{Y}/\hat{Z}) : \hat{Z} \in \text{FG}(\hat{Y}/\hat{X}) \}.
\]

We define a topology on \( D \) similar to the Krull topology on a Galois group as follows:

\[
\mathcal{K}_D := \{ T : T = \emptyset \text{ or } T = \bigcup_j \hat{\tau}_j E_j \text{ for some } \hat{\tau}_j \in D \text{ and } E_j \in \mathcal{E} \}
\]

(cf. [21 Definition 17.5]). By Lemma 4.5 below (refer to Lemma 4.7 below) and similar reasoning to that in [21 Section 17] we see that \( \mathcal{K}_D \) is really a topology on \( D \). We also call this topology the *Krull topology* on \( D \). We always assume that \( D \) is equipped with the Krull topology later. Denote

\[
\mathcal{C} := \{ C : C \text{ is a closed subgroup of } D \}.
\]

If \( \hat{Y} \) is finite over \( \hat{X} \), then \( \mathcal{C} \) is consisting of all subgroups of \( D \) (refer to Lemma 4.7 below and [21 Example 17.9]).

**Theorem 4.1** (Galois Correspondence on Algebraic Riemann Surfaces). *Suppose \( \hat{X} \) and \( \hat{Y} \) are algebraic Riemann Surfaces and \( \hat{Y} \) is Galois over \( \hat{X} \). Let \( D = \text{Deck}(\hat{Y}/\hat{X}) \). Then the mapping*

\[
\Delta : \text{Int}(\hat{Y}/\hat{X}) \to \mathcal{C}
\]

\[
\hat{Z} \mapsto \text{Deck}(\hat{Y}/\hat{Z})
\]
is a bijection, which gives an inclusion reversing correspondence and whose inverse mapping is \( \Gamma \) given in (1.10) below. Moreover, letting \( E = \Delta(Z) \), we have

(1) The following statements are equivalent:

(i) \([D : E]\) (the index of \( E \) in \( D \)) is finite;

(ii) \([Z : X]\) is finite;

(iii) \( E \) is open in \( D \).

On this condition it is true that \([D : E] = [\tilde{Z} : \tilde{X}]\).

(2) \( \tilde{Z} \) is Galois over \( \tilde{X} \) if and only if \( E \) is normal in \( D \). On this condition, there is a group isomorphism

\[
\text{Deck}(\tilde{Z}/\tilde{X}) \cong D/E,
\]

which is also a homeomorphism as the quotient group \( D/E \) is given the quotient topology.

In order to obtain Theorem 4.1 we give some preliminary results.

Suppose \( \tilde{Y} \) is an algebraic Riemann surface over a base surface \( \tilde{R} \) with base point \( r_0 \) and \( Y^0 \) is the corresponding original algebraic Riemann surface. Suppose \( \tilde{f} \in \mathcal{M}(\tilde{Y}) \). Then there exists \( \tilde{S}_1 = (\tilde{\phi}_1; \tilde{S}_1) \in \tilde{Y}^0 \) with the original basic function \( \tilde{F}_1 \) and a polynomial \( \tilde{P}_1(t) = (\tilde{P}_1(t), \tilde{P}_1(t)) \in \mathcal{M}(\tilde{R})[t] \) such that \( \tilde{f}|_{\tilde{S}_1} = \tilde{P}_1(\tilde{F}_1) \) by Lemma 3.4. If there exists another \( \tilde{S}_2 = (\tilde{\phi}_2; \tilde{S}_2) \in \tilde{Y}^0 \) with the original basic function \( \tilde{F}_2 \) and a polynomial \( \tilde{P}_2(t) = (\tilde{P}_2(t), \tilde{P}_2(t)) \in \mathcal{M}(\tilde{R})[t] \) such that \( \tilde{f}|_{\tilde{S}_2} = \tilde{P}_2(\tilde{F}_2) \), then by Lemma 3.7 we can take a common covering \( \tilde{S}_0 = (\tilde{\phi}_0; \tilde{S}_0) \in \tilde{Y}^0 \) of \( \tilde{S}_1 \) and \( \tilde{S}_2 \) with \( \tilde{S}_1 \overset{\tilde{\phi}_0}{\rightarrow} \tilde{S}_0, \tilde{S}_2 \overset{\tilde{\phi}_0}{\rightarrow} \tilde{S}_0 \), \( \tilde{Q}_1(\tilde{\phi}_0) = \tilde{\phi}_1 \) and \( \tilde{Q}_2(\tilde{\phi}_0) = \tilde{\phi}_2 \), where \( \tilde{Q}_j(t) = (\tilde{Q}_j(t), \tilde{Q}_j(t)) \in \mathcal{M}(\tilde{R})[t] \) and \( \tilde{q}_j \) is determined by \( \tilde{Q}_j \) \( (j = 1, 2) \). Thus \( F_1 \circ q_1 = Q_1(F_0) \) and \( F_2 \circ q_2 = Q_2(F_0) \) by Lemma 3.6 where \( F_0 \) is the original basic function on \( \tilde{S}_0 \). Hence

\[
\tilde{f}|_{\tilde{S}_0} = \tilde{f}|_{\tilde{S}_1} \circ \tilde{q}_1 = \tilde{P}_1(\tilde{F}_1) \circ \tilde{q}_1 = \tilde{P}_1(\tilde{F}_1 \circ \tilde{q}_1) = \tilde{P}_1(\tilde{Q}_1(\tilde{F}_0)),
\]

and similarly

\[
\tilde{f}|_{\tilde{S}_0} = \tilde{P}_2(\tilde{Q}_2(\tilde{F}_0)),
\]

which imply \( \tilde{P}_1(\tilde{Q}_1(\tilde{F}_0)) = \tilde{P}_2(\tilde{Q}_2(\tilde{F}_0)) \). Therefore,

\[
\tilde{P}_1(\tilde{Q}_1(\tilde{\phi}_0)) = \tilde{P}_2(\tilde{Q}_2(\tilde{\phi}_0)),
\]

i.e. \( \tilde{P}_1(\tilde{\phi}_1) = \tilde{P}_2(\tilde{\phi}_2) \). So we can give the following definition: We call a (resp. the original) basic algebraic Riemann surface determined by \( \tilde{P}_1(\tilde{\phi}_1) \) with label (resp. with natural label \( \tilde{P}_1(\tilde{\phi}_1) \)) a (resp. the original) basic domain of \( \tilde{f} \). For the basic domain of \( \tilde{f} \), the label means that corresponding to the natural label \( \tilde{P}_1(\tilde{\phi}_1) \). We denote the original basic domain of \( \tilde{f} \) by \( \text{obdom}(\tilde{f}) \). We call the (resp. original) level surface determined by \( \tilde{P}_1(\tilde{\phi}_1) \) (i.e. containing \( \text{obdom}(\tilde{f}) \)) the (resp. original) level domain, denoted by \( \mathcal{L}(\tilde{f}) \) (resp. \( \mathcal{L}^0(\tilde{f}) \)). We call the (resp. original) algebraic Riemann surface determined by \( \tilde{P}_1(\tilde{\phi}_1) \) the (resp. original) natural domain of \( \tilde{f} \), denoted by \( \text{Ndom}(\tilde{f}) \) (resp. \( \text{oNdom}(\tilde{f}) \)).

By the reasoning and the definition in the above and Lemma 3.6, we can deduce
Lemma 4.2. Suppose $\tilde{Y}$ is an algebraic Riemann surface and $\tilde{f} \in \mathcal{M}(\tilde{Y})$. Then $\text{Ndom}(\tilde{f}) \leq Y$ and every element in $\mathcal{L}(\tilde{f})$, i.e. every basic domain of $\tilde{f}$, is a holographic element of $\text{Ndom}(\tilde{f})$. For $\tilde{S} \in \tilde{Y}$ and the original basic function $\hat{F}$ on $\tilde{S}$ we have $\text{obdom}(\hat{F}) = \tilde{S}$, where $\hat{F} \in \mathcal{M}(\hat{Y})$ is determined by $\tilde{F}$. If $\tilde{S} = (\tilde{\varphi}; \tilde{S}) = \text{obdom}(\tilde{f})$, $\tilde{S}_1 = (\tilde{\varphi}_1; \tilde{S}_1) \in \tilde{Y}$ is an expression domain of $\tilde{f}$ and $\tilde{F}_1$ is the original basic function on $\tilde{S}_1$, then $\tilde{S}$ is an expression domain of $\tilde{f}$ and $\tilde{F} := \tilde{f}|_\tilde{S}$ is the original basic function on $\tilde{S}$, and there exists a directly up-harmonious mapping $\tilde{\pi}_1 : \tilde{S}_1 \to \tilde{S}$ (i.e. $\tilde{S} \overset{\pi_1}{\twoheadrightarrow} \tilde{S}_1$) and a polynomial $\hat{P}_1(t) = (\hat{P}_1(t), \hat{P}_1(t)) \in \mathcal{M}(\hat{R})[t]$ such that $\hat{F} \circ \tilde{\pi}_1 = \hat{P}_1(\tilde{F}_1) = \tilde{f}|_{\tilde{S}_1}$ and $\tilde{\varphi} = \hat{P}_1(\tilde{\varphi}_1)$. □

Suppose $\tilde{Y}$ is an algebraic Riemann surface over the base surface $\hat{R}$. We will consider algebraic Riemann surfaces under $\hat{Y}$. Suppose $\tilde{Z} \leq \tilde{Y}$. Let

$$\mathcal{M}_{\hat{Y}}(\tilde{Z}) := \{\tilde{f} \in \mathcal{M}(\hat{Y}) : \text{there exists } \tilde{Z} \in \tilde{Z} \text{ and } \hat{g} \in \mathcal{M}(\tilde{Z}) \text{ such that } \hat{g} \in \tilde{f}\}.$$ 

If there is no confusion, we will write $\mathcal{M}(\tilde{Z})$ instead of $\mathcal{M}_{\hat{Y}}(\tilde{Z})$. Actually, we may consider $\mathcal{M}(\tilde{Z})$ in the usual sense as $\mathcal{M}_{\hat{Y}}(\tilde{Z})$.

Lemma 4.3. Suppose $\tilde{X} \leq \tilde{Y}$ and $\text{Int}(K/N)$ denotes the set of all intermediate fields of $K/N$, where $N = \mathcal{M}(\tilde{X})$ and $K = \mathcal{M}(\hat{Y})$. Then the mapping

$$\mathcal{M} : \text{Int}(\tilde{Y}/\tilde{X}) \to \text{Int}(K/N)$$

$$\tilde{Z} \mapsto L = \mathcal{M}(\tilde{Z})$$

is a partial order preserving bijection, whose inverse mapping is $\mathcal{R}$ given in (4.1) below.

Proof. Suppose $L \in \text{Int}(K/N)$. Define

$$\mathcal{R}(L) := \bigcup \{\mathcal{L}(\tilde{f}) : \tilde{f} \in L\}.$$ 

We see that $\mathcal{R}(L) \in \text{Int}(\tilde{Y}/\tilde{X})$, since it is determined by

$$\mathcal{L}(L) := \{\tilde{\varphi} \in \mathcal{A} : \tilde{\varphi} \text{ is the natural label of obdom}(\tilde{f}) \text{ for } \tilde{f} \in L\},$$

which is an intermediate field of $\mathcal{L}(\hat{Y})/\mathcal{L}(\tilde{X})$ ($\mathcal{L}(\tilde{X})$ and $\mathcal{L}(\hat{Y})$ are the natural label fields of $\tilde{X}$ and $\hat{Y}$ respectively) by Lemma 4.2. It is easy to see that the mapping

$$\mathcal{R} : \text{Int}(K/N) \to \text{Int}(\tilde{Y}/\tilde{X})$$

$$L \mapsto \tilde{Z} = \mathcal{R}(L)$$

is the inverse of $\mathcal{M}$ by Lemma 4.2. The preserving of partial order by $\mathcal{M}$ is obvious. □

Suppose $\tilde{Y}_0$ is the original algebraic Riemann surface corresponding to $\tilde{Y}$. Suppose $\tilde{X}$, $\tilde{Z} \in \tilde{Y}_0$ and $\tilde{Z}$ is over $\tilde{X}$. Let $\hat{X}$ and $\tilde{Z}$ denote the level surfaces containing $\tilde{X}$ and $\tilde{Z}$, respectively. Let $N_0 = \mathcal{M}(\tilde{X})$, $L_0 = \mathcal{M}(\tilde{Z})$, $N = \mathcal{M}(\hat{X}) := \mathcal{M}(\hat{Y})$.
\( \mathcal{M}(\tilde{X}) \), where \( \tilde{X} \) is the algebraic Riemann surface with holographic element \( \hat{X} \), and \( L = \mathcal{M}(\tilde{Z}) \). Let \( \hat{\pi} : \tilde{Z} \to \hat{X} \) be the directly up-harmonious mapping, which is a holomorphic covering map (called the natural covering map). Then \( \hat{\pi}^* : N_0 \to L_0 \) is a monomorphism of fields and there exists an isomorphism \( \gamma : L_0 \to L \), i.e. \( L_0 \cong L \), which can be defined by \( \hat{f} \mapsto f \), where \( f \in \mathcal{M}(\tilde{Z}) \) is determined by \( \hat{f} \in \mathcal{M}(\tilde{Z}) \). (\( \hat{f} \) may be written \( (\hat{f})^\gamma \)), such that \( \hat{\pi}^*(N_0) \cong N \ (\gamma_1 = \gamma|_{\hat{\pi}^*(N_0)} \). Let \( \hat{F} \) be the original basic function on \( \tilde{Z} \) and let \( \hat{P}_0(t) \in N_0[t] \) be the minimal polynomial of \( \hat{F} \) over \( \hat{\pi}^*(N_0) \), i.e. the monic irreducible polynomial in \( N_0[t] \) satisfying \( (\hat{\pi}^*\hat{P}_0)(\hat{F}) = 0 \). We call \( \hat{P}_0(t) \) the minimal polynomial of \( \tilde{Z} \) over \( \hat{X} \).

For the \( n \)-sheeted natural covering map \( \hat{\pi} : \tilde{Z} \to \hat{X} \), suppose \( B \) is the set of branch points of \( \tilde{\pi} : \tilde{Z} \to \hat{X} \) (\( \tilde{\pi} = (\hat{\pi}, \tilde{\pi}) \)) and \( U \subseteq \hat{X} \setminus B \) \( (\hat{X} = (\hat{X}, \tilde{X})) \) is a non-empty open set such that \( \tilde{\pi}^{-1}(U) \) is the disjoint union of open sets \( V_1, \ldots, V_n \) and \( \tilde{\pi}|_{V_j} : V_j \to U \) is biholomorphic \( (j = 1, \ldots, n) \). Let \( \hat{\tau}_j = \hat{\tau}_j : U \to V_j \) be the inverse mapping of \( \tilde{\pi}|_{V_j} \). Let \( \hat{f}_j = \hat{f} \circ \hat{\tau}_j \), where \( \hat{f} \in \mathcal{M}(\tilde{Z}) \) and \( \hat{\tau}_j = (\hat{\tau}_j, \hat{\tau}_j) \). We consider the polynomial

\[
\hat{Q}_f(t) = \prod_{j=1}^n (t - \hat{f}_j) = t^n + \hat{c}_1 t^{n-1} + \cdots + \hat{c}_n,
\]

where \( \hat{c}_j = (-1)^j s_j(\hat{f}_1, \ldots, \hat{f}_n) \) \( (s_j \) denotes the \( j \)-th elementary symmetric function in \( n \) variables, \( j = 1, \ldots, n) \). Similarly to \( [10, \S 8.1, \S 8.2 \text{ and } \S 8.3] \) we can deduce that \( \hat{Q}_f(t) \) is just the minimal polynomial \( \hat{P}_0(t) \) of \( \hat{F} \) over \( \hat{\pi}^*(N_0) \). So we get

\[
\deg \hat{P}_0 = n, \tag{4.2}
\]

where \( \deg \hat{P}_0 \) denotes the degree of \( \hat{P}_0(t) \).

Let \( G_\sigma(\hat{f}) := \hat{f} \circ \sigma^{-1} \) for \( \sigma \in \text{Deck}(\tilde{Z}/\hat{X}) \) and \( \hat{f} \in L_0 = \mathcal{M}(\tilde{Z}) \). Then \( G_\sigma \in \text{Gal}(L_0/\hat{\pi}^*(N_0)) \). Define

\[
G(\sigma) := G_\sigma
\]

for \( \sigma \in \text{Deck}(\tilde{Z}/\hat{X}) \). Let \( \hat{\alpha}(\hat{f}) = \gamma(\alpha(\hat{f})) \) for \( \alpha \in \text{Gal}(L_0/\hat{\pi}^*(N_0)) \) and \( \hat{f} \in L_0 \), where \( \hat{f} = \gamma(\hat{f}) \). Then \( \hat{\alpha} \in \text{Gal}(L/N) \). Define

\[
\beta(\alpha) := \hat{\alpha}
\]

for \( \alpha \in \text{Gal}(L_0/\hat{\pi}^*(N_0)) \). Similarly to \( [10, \text{Theorem } (8.12)] \) we have

**Lemma 4.4.** Suppose \( \hat{Y}_0 \) is an original algebraic Riemann surface over a base surface \( \hat{R} \), \( \hat{X} \), \( \hat{Z} \in \hat{Y}_0 \) and \( \tilde{Z} \) is over \( \hat{X} \). Let \( N_0 = \mathcal{M}(\hat{X}), L_0 = \mathcal{M}(\tilde{Z}), N = \mathcal{M}(\tilde{X}) := \mathcal{M}(\hat{X}) \) and \( L = \mathcal{M}(\tilde{Z}) \). Suppose \( \tilde{\pi} : \tilde{Z} \to \hat{X} \) is the \( n \)-sheeted natural covering map and \( \hat{P}_0(t) \in N_0[t] \) is the minimal polynomial of \( \tilde{Z} \) over \( \hat{X} \). Then

1. \( [L : N] = [L_0 : \hat{\pi}^*(N_0)] = \deg \hat{P}_0 = n \) and \( L \cong L_0 \cong N_0[t]/(\hat{P}_0(t)); \)
2. \( \text{Deck}(\tilde{Z}/\hat{X}) \cong \text{Gal}(L_0/\hat{\pi}^*(N_0)) \cong \text{Gal}(L/N); \)
3. The natural covering \( \hat{\pi} : \tilde{Z} \to \hat{X} \) is Galois precisely if the field extension \( L/N \) (or \( L_0/\hat{\pi}^*(N_0) \)) is Galois.
Proof. Noticing (4.2), by Lemma 3.4 we can easily see that (1) is true. For (2), we show that $\mathcal{G}$ is surjective as follows.

Suppose $\alpha \in \text{Gal}(L_0/\hat{\pi}^*(N_0))$ and $\tilde{F}$ is the original basic function on $\hat{Z}$. Then $P(\alpha(\tilde{F})) = \alpha(P(\tilde{F})) = 0$, where $P(t) \in \mathcal{M}(\hat{R})[t]$ is the minimal polynomial of $\tilde{F}$ over $\mathcal{M}(\hat{R})$. By Theorem 3.3, there exists a base-preserving biholomorphic mapping $\hat{\sigma} : \hat{Z} \to \hat{Z}$ such that $\mathcal{G}_\hat{\sigma}(\tilde{F}) = \hat{F} \circ \hat{\sigma}^{-1} = \alpha(\tilde{F})$. For $\tilde{f} \in \mathcal{M}(\hat{Z})$ there exists $\hat{Q}(\tilde{t}) \in \mathcal{M}(\hat{R})[\tilde{t}]$ such that $\hat{f} = \hat{Q}(\hat{F})$ by Lemma 3.4. Hence,

\[ \mathcal{G}_\hat{\sigma}(\tilde{f}) = \hat{f} \circ \hat{\sigma}^{-1} = \hat{Q}(\hat{F}) \circ \hat{\sigma}^{-1} = \hat{Q}(\hat{F} \circ \hat{\sigma}^{-1}) = \hat{Q}(\alpha(\tilde{F})) = \alpha(\hat{Q}(\hat{F})) = \alpha(\hat{f}). \]

Specially for the original basic function $\tilde{F}_0$ on $\hat{X}$, since $\tilde{F}_0 \circ \hat{\pi} \in \hat{\pi}^*(N_0)$, we have

\[ \tilde{F}_0 \circ \hat{\pi} \circ \hat{\sigma}^{-1} = \alpha(\tilde{F}_0 \circ \hat{\pi}) = \tilde{F}_0 \circ \hat{\pi}. \]

So it follows $\hat{\pi} \circ \hat{\sigma}^{-1} = \hat{\pi}$, which means $\hat{\sigma} \in \text{Deck}(\hat{Z}/\hat{X})$.

Assertion (3) follows from assertions (1) and (2) and that $L/N$ is Galois precisely if $|\text{Gal}(L/N)| = [L : N]$ (see [21, Corollary 2.16]). \hfill \square

Remark 14. Because of the isomorphism $\beta$ we may use $\alpha$ instead of $\tilde{\alpha}$.

In the following we will write “=” instead of “$\equiv$” for the equality of mappings on algebraic Riemann surfaces.

Lemma 4.5. Suppose $\hat{Z} \subseteq \hat{W}$ ($\hat{Z}, \hat{W} \in \text{Int}(\hat{Y}/\hat{X})$). Let $L = \mathcal{M}(\hat{Z})$ and $M = \mathcal{M}(\hat{W})$. For $\hat{\sigma} \in \text{Deck}(\hat{W}/\hat{Z})$ define

\[ \mathcal{G}_\hat{\sigma}(\tilde{f}) := \tilde{f} \circ \tilde{\sigma}^{-1}, \]

where $\tilde{f} \in M$. Then $\mathcal{G}_\hat{\sigma} \in \text{Gal}(M/L)$ and the mapping

\[ \mathcal{G} = \mathcal{G}_{\hat{W}/\hat{Z}} : \text{Deck}(\hat{W}/\hat{Z}) \to \text{Gal}(M/L) \]

\[ \hat{\sigma} \mapsto \mathcal{G}_\hat{\sigma} \]

is a group isomorphism. Moreover, $\hat{W}/\hat{Z}$ is Galois if and only if $M/L$ is Galois.

Proof. At first, $\mathcal{G}_\hat{\sigma} \in \text{Gal}(M/L)$ since $\tilde{f} \circ \tilde{\sigma} = \tilde{f}$ for $\tilde{f} \in L = \mathcal{M}(\hat{Z})$, where $\tilde{\pi} : \hat{W} \to \hat{Z}$ is the natural covering map. For $\hat{\sigma}, \hat{\tau} \in \text{Deck}(\hat{W}/\hat{Z})$ we have

\[ \mathcal{G}(\hat{\tau} \circ \hat{\sigma}) = \mathcal{G}_{\hat{\sigma} \circ \hat{\tau}} = \mathcal{G}_\hat{\sigma} \circ \mathcal{G}_\hat{\tau} = \mathcal{G}(\hat{\sigma}) \circ \mathcal{G}(\hat{\tau}), \]

since

\[ \mathcal{G}_{\hat{\sigma} \circ \hat{\tau}}(\tilde{f}) = \tilde{f} \circ (\hat{\sigma} \circ \hat{\tau})^{-1} = \tilde{f} \circ \hat{\tau}^{-1} \circ \hat{\sigma}^{-1} = (\mathcal{G}_\hat{\sigma} \circ \mathcal{G}_\hat{\tau})(\tilde{f}) \]

for $\tilde{f} \in M$. Hence, $\mathcal{G}$ is a group homomorphism.

(i) Suppose $\mathcal{G}(\hat{\sigma}) = \mathcal{G}_\hat{\sigma} = \text{id}_M$ for $\hat{\sigma} \in \text{Deck}(\hat{W}/\hat{Z})$. Then for every $\tilde{f} \in M$ we have $\mathcal{G}_\hat{\sigma}(\tilde{f}) = \tilde{f}$, i.e. $\hat{f} \circ \hat{\sigma}^{-1} = \hat{f}$. For $\mathcal{S} = (\hat{\varphi}, \hat{S}) \in \hat{W}_0$ (the original algebraic Riemann surface corresponding to $\hat{W}$) over some $S_0 \in \hat{W}_0$, let $\tilde{F} \in M$ be determined by the original basic function $\tilde{F}$ on $\mathcal{S}$. Then $\hat{S} = \text{obdom}(\tilde{F})$ by
Lemma 4.2 Since \( \tilde{F} \circ \tilde{\sigma}^{-1} = \tilde{F} \), i.e., \( \tilde{F} \circ \tilde{\sigma} = \tilde{F} \), there exists \( \tilde{\sigma} : \tilde{W}_1 \to \tilde{W}_2 \) in \( \tilde{\sigma} \), where \( \tilde{S} \not\leftrightarrow \tilde{W}_2 \) and \( \tilde{W}_1 \) is also an expression domain of \( \tilde{F} \) in \( \tilde{W} \), such that

\[
\tilde{F}|_{\tilde{W}_2} \circ \tilde{\sigma} = \tilde{F}|_{\tilde{W}_1}. \tag{4.3}
\]

By Lemma 4.2 there exists an analytically up-harmonious mapping \( \tilde{\pi}_1 : \tilde{W}_1 \to \tilde{S} \) such that

\[
\tilde{F}|_{\tilde{W}_1} = \tilde{F} \circ \tilde{\pi}_1. \tag{4.4}
\]

By (4.3) and (4.4) it follows \( \tilde{F} \circ \hat{\mu} \circ \tilde{\sigma} = \tilde{F} \circ \tilde{\pi}_1 \). So we get \( \hat{\mu} \circ \tilde{\sigma} = \tilde{\pi}_1 \), i.e. \( \hat{\mu} \circ \tilde{\sigma} = \tilde{\pi}_1 \), which mean \( \tilde{S} \not\leftrightarrow \tilde{W}_1 \) (since \( \hat{\mu} \) and \( \tilde{\sigma} \) are biholomorphic) and \( \sigma \sim \id_\tilde{S} \). Therefore, \( \tilde{\sigma} = \id_{\tilde{W}} \) and then \( \mathcal{G} \) is injective.

(ii) Now we show that \( \mathcal{G} \) is surjective. Suppose \( \alpha \in \Gal(M/L) \). Suppose \( \tilde{S}_1 = (\tilde{\varphi} ; \tilde{S}) \in \tilde{W}_0 \) and \( \tilde{F} \) is the original basic function on \( \tilde{S}_1 \). Let \( \tilde{F} \in M \) be determined by \( \tilde{F} \). Suppose \( \tilde{S}' = (\tilde{\varphi}' ; \tilde{S}') = \obdom(\alpha(\tilde{F})) \). Then \( \tilde{S}' \in \tilde{W}_0 \) by Lemma 4.2 Since

\[
\hat{P}(\alpha(\tilde{F})|_{\tilde{S}'}) = \hat{P}(\alpha(\tilde{F}))|_{\tilde{S}'} = \alpha(\hat{P}(\tilde{F}))|_{\tilde{S}'} = 0,
\]

where \( \hat{P}(t) \in \mathcal{M}(\tilde{R})[t] \) is the minimal polynomial of \( \tilde{F} \) over \( \mathcal{M}(\tilde{R}) \), there exists a base-preserving biholomorphic mapping \( \check{\sigma} = \check{\sigma}(\alpha, \tilde{S}) : \tilde{S} \to \tilde{S}' \) such that

\[
\check{F} \circ \check{\sigma} = \tilde{F}, \tag{4.5}
\]

where \( \check{F} = \alpha(\tilde{F})|_{\tilde{S}'} \), by Theorem 3.3 (cf. [10, Theorem (8.9)]). Let

\[
\check{\sigma} = \check{\sigma}_\alpha := \{ \check{\sigma}(\alpha, \tilde{S}) : \tilde{S} \in \tilde{W}_0 \}.
\]

Given \( \tilde{S}' \in \tilde{W}_0 \), suppose \( \tilde{F}' \) is the original basic function on \( \tilde{S}' \) and \( \check{F}' \in M \) is determined by \( \check{F}' \), i.e. \( \check{F}'|_{\tilde{S}'} = \check{F}' \). Let \( \check{F} = \check{\alpha}^{-1}(\check{F}') \), \( \check{S} = \obdom(\check{F}) \) and \( \check{F} = \check{F}|_{\tilde{S}} \). Then similarly to the above we can deduce that there is a unique mapping \( \check{\sigma}(\alpha, \check{S}) : \check{S} \to \check{S}' \) in \( \check{\sigma} \) (satisfying (4.3)) corresponding to \( \check{S}' \) and \( \alpha \). Moreover, we can deduce that \( \check{\sigma}(\alpha^{-1}, \check{S}') : \check{S}' \to \check{S} \) is just the inverse of \( \check{\sigma}(\alpha, \check{S}) \).

Remark 15. In the above definition of \( \check{\sigma} \), we may assume the expression domain \( \check{S}' \) of \( \alpha(\tilde{F}) \) is replaced by \( \check{S} \) if necessary provided that \( \check{S}' \not\leftrightarrow \check{S} \) for \( \check{\lambda} \in \check{\Lambda} \) (the analytically harmonious relation). To see that this assumption is reasonable, we consider different \( \check{S} \) and \( \check{S}' \), which are analytically harmonious modulo \( \check{\lambda} : \check{S} \to \check{S}' \). Then by Theorem 3.3 there are base-preserving biholomorphic mappings \( \check{\sigma} : \check{S} \to \check{S}' \) and \( \check{\sigma}_0 : \check{S} \to \check{S} \) such that \( \alpha(\tilde{F})|_{\tilde{S}'} = \tilde{F} \circ \check{\sigma}^{-1} \) and \( \alpha(\check{F})|_{\check{S}} = \check{F} \circ \check{\sigma}_0^{-1} \), where \( \check{F} \) is the original basic function on \( \check{S} \) and \( \check{F}|_{\check{S}} = \check{F} \). Therefore,

\[
\check{F} \circ \check{\sigma}_0^{-1} = \alpha(\tilde{F})|_{\check{S}} = \alpha(\check{F})|_{\check{S}} \circ \check{\lambda} = \tilde{F} \circ \check{\sigma}^{-1} \circ \check{\lambda}.
\]

We deduce \( \check{\sigma}_0 = \check{\lambda}^{-1} \circ \check{\sigma} \), i.e. \( \check{\sigma}_0 \sim \check{\sigma} \), since \( \check{F} \) is the original basic function on \( \check{S} \).

Similarly, \( \check{S}' \) may also be replaced by another \( \check{S}'_1 \), provided that \( \check{S}' \leftrightarrow \check{S}'_1 \).
Remark 16. In fact, by (4.5) we can deduce that $\dot{S} = \dot{S}'$ ($\dot{F}' = \alpha(\dot{F})|_{\dot{S}_0}$ is the original basic function on $\dot{S}'$ by Lemma 4.2) since $\dot{S}' = \text{obdom}(\alpha(\dot{F}))$ and $\dot{\sigma}(\alpha, \dot{S}) : \dot{S} \to \dot{S}'$ is an identical mapping. But generally $\dot{S} \neq \dot{S}'$ (for $\alpha \neq \text{id}$) and so $\dot{\sigma}(\alpha, \dot{S}) : S \to \dot{S}'$ is not any identical mapping essentially.

Suppose $\ddot{S}_1, \ddot{S}_2 \in \ddot{W}^0$. Suppose $\ddot{F}_1$ and $\ddot{F}_2$ are the original basic functions on $\ddot{S}_1$ and $\ddot{S}_2$, respectively. If there is a direct up-harmonious mapping $\dot{p} : \ddot{S}_2 \to \ddot{S}_1$, then by Lemma 3.4 there exists $\dot{P}(t) \in \mathcal{M}(\ddot{R})[t]$ such that

$$\dot{F}_1 \circ \dot{p} = \dot{P}(\ddot{F}_2). \quad (4.6)$$

Suppose $\ddot{F}_1$ and $\ddot{F}_2$ in $M$ are determined by $\ddot{F}_1$ and $\ddot{F}_2$ respectively, i.e. $\ddot{F}_1|_{\ddot{S}_1} = \ddot{F}_1$ and $\ddot{F}_2|_{\ddot{S}_2} = \ddot{F}_2$. Then

$$\dot{P}(\ddot{F}_2)|_{\ddot{S}_2} = \dot{P}(\ddot{F}_2|_{\ddot{S}_2}) = \dot{P}(\ddot{F}_2) = \ddot{F}_1|_{\ddot{S}_1} \circ \dot{p}$$

by (1.6), so that $\dot{P}(\ddot{F}_2) = \ddot{F}_1$. Hence $\alpha(\ddot{F}_1) = \alpha(\dot{P}(\ddot{F}_2)) = \dot{P}(\alpha(\ddot{F}_2))$. Let $\ddot{F}_j' = \alpha(\ddot{F}_j) \ (j = 1, 2)$. Then

$$\ddot{F}_1' = \dot{P}(\ddot{F}_2'). \quad (4.7)$$

Let $\dot{\sigma}_j : \ddot{S}_j \to \ddot{S}'_j \ (j = 1, 2)$ be base-preserving biholomorphic mappings such that

$$\ddot{F}_j'|_{\ddot{S}_j'} = \ddot{F}_j \circ \dot{\sigma}_j^{-1}, \quad (4.8)$$

where $\ddot{S}_j' = (\phi_j'; \ddot{S}_j') = \text{obdom}(\ddot{F}_j')$.

Suppose $\ddot{S}_0' = (\phi_0'; \ddot{S}_0') \in \ddot{W}$ is an original algebraic Riemann surface over $\ddot{S}_0'$ and $\ddot{S}_2'$, and $\ddot{F}_0'$ is the original basic function on $\ddot{S}_0'$ (refer to Lemma 3.7). Then by Lemma 3.4 there exists $\ddot{Q}_j(t) \in \mathcal{M}(\ddot{R})[t]$ such that $\ddot{F}_j'|_{\ddot{S}_j'} \circ \ddot{\pi}_j = \ddot{F}_j'|_{\ddot{S}_0'} = \ddot{Q}_j(\ddot{F}_0') \ (j = 1, 2)$, where $\ddot{\pi}_j : \ddot{S}_0' \to \ddot{S}_j' \ (j = 1, 2)$ are directly up-harmonious mappings. Therefore,

$$\ddot{Q}_1(\ddot{F}_0') = \ddot{F}_1'|_{\ddot{S}_0'} = \dot{P}(\ddot{F}_2)|_{\ddot{S}_0'} = \dot{P}(\ddot{F}_2'|_{\ddot{S}_0'}) = \dot{P}(\ddot{Q}_2(\ddot{F}_0'))$$

by (4.7) and $\ddot{\sigma}_j' = \ddot{Q}_j(\ddot{\phi}_0') \ (j = 1, 2)$. Consequently $\dot{Q}_1(\ddot{\phi}_0') = \dot{P}(\ddot{Q}_2(\ddot{\phi}_0'))$, i.e.

$$\ddot{\phi}_1' = \dot{P}(\ddot{\phi}_2').$$

Suppose $\ddot{p}' : \ddot{S}_2' \to \ddot{S}_1'$ is the corresponding directly up-harmonious mapping (see Lemma 3.6). Then

$$\ddot{F}_1|_{\ddot{S}_1'} \circ \ddot{p}' = \ddot{F}_1'|_{\ddot{S}_2'} = \dot{P}(\ddot{F}_2)|_{\ddot{S}_2'} = \dot{P}(\ddot{F}_2'|_{\ddot{S}_2'}) = \dot{P}(\ddot{F}_2 \circ \dot{\sigma}_2^{-1}) = \dot{P}(\ddot{F}_2) \circ \dot{\sigma}_2^{-1}$$

by (4.7) and (4.8), i.e. $\ddot{F}_1 \circ \dot{\sigma}_1^{-1} \circ \ddot{p}' = \ddot{F}_1 \circ \dot{p} \circ \dot{\sigma}_2^{-1}$ by (1.6) and (4.8). Since $\ddot{F}_1$ is the original basic function on $\ddot{S}_1$, we get

$$\dot{\sigma}_1 \circ \dot{p} = \ddot{p}' \circ \dot{\sigma}_2. \quad (4.9)$$
Therefore, \( \hat{\sigma}_1 \) and \( \hat{\sigma}_2 \) are directly compatible. In fact, \( \hat{\sigma}_1 \) and \( \hat{\sigma}_2 \) are exactly and directly compatible since \( (1.9) \) we can deduce \( (\hat{\sigma}_2)^{-1} \circ \hat{\sigma}_1 = \hat{\sigma}_2 \circ \hat{\sigma}_1^{-1} \) (see Remark \( 10 \)). By the reasoning above we can see that \( \hat{\sigma} \) is compatible and satisfies the two conditions of a mapping. Hence \( \hat{\sigma} \) is a mapping from \( \hat{W} \) to \( \hat{W} \).

We can also see that if \( \text{codom}(\hat{\sigma}_2) \) is over \( \text{codom}(\hat{\sigma}_1) \) then \( \hat{\sigma}_2 \) is exactly over \( \hat{\sigma}_1 \). Therefore, it is easy to know that \( \hat{\sigma} \) is a biholomorphic transformation on \( \hat{W} \).

For \( \hat{f} \in M \), by \( (1.3) \) we have

\[
G_\tilde{\sigma} (\hat{f}) = \tilde{f} \circ \hat{\sigma}^{-1} = (\tilde{f}|_{\tilde{S}} \circ \hat{\sigma}^{-1})^\sim = (\alpha(\tilde{f})|_{\tilde{S}})^\sim = \alpha(\hat{f}),
\]

where \( \tilde{S} = \text{obdom}(\hat{f}) \), \( \hat{\sigma} = \hat{\sigma}(\alpha, \tilde{S}) \in \tilde{\sigma} \), \( \tilde{S}' = \text{obdom}(\alpha(\tilde{f})) \) and \( (\hat{f})^\sim \) denotes \( \hat{g} = \gamma(\hat{g}) \), which is determined by \( \hat{g} \).

Suppose \( \tilde{\pi} : \hat{W} \to \hat{Z} \) is the natural covering map and \( \tilde{\pi} \in \pi \) is the natural covering map from \( \hat{W} \in \hat{W}^0 \) to \( \hat{Z} \in \hat{Z}^0 \) (thus \( \hat{Z} \in \hat{W}^0 \)). In the above reasoning, let \( \hat{S}_1 = \hat{Z} \) and \( \hat{S}_2 = \hat{W} \). Since \( \alpha|_L = \text{id}_L \), letting \( \tilde{F}_1 \in L \) be determined by the original basic function \( \tilde{F}_1 \) on \( \hat{S}_1 \), then \( \alpha(\tilde{F}_1) = \tilde{F}_1 \) and \( \hat{S}_1' = \text{obdom}(\alpha(\tilde{F}_1)) = \hat{S}_1 \) by Lemma \( 4.2 \). By \( (1.3) \), letting \( \hat{F} = \tilde{F}_1 \) and \( \hat{F}' = \alpha(\tilde{F}_1)|_{\hat{S}_1'} = \hat{F}_1 \), we have \( \hat{\sigma}_1 = \text{id}_{\hat{S}_1} \).

Then we obtain a base-preserving biholomorphic mapping \( \hat{\sigma} = \hat{\sigma}_2 : \hat{W} \to \hat{W}' \) \( (\hat{W}' = \hat{S}_2') \) such that \( \tilde{\pi}' \circ \hat{\sigma} = \tilde{\pi} \) (see \( (1.9) \)), where \( \tilde{\pi}' : \hat{W}' \to \hat{Z} \) is a maximal natural covering map in \( \tilde{\pi} \), by the preceding reasoning in the proof (in fact, \( \hat{\sigma}(\alpha, \hat{W}) : \hat{W} \to \hat{W}' = \hat{W} \) is an identical mapping by Remark \( 16 \). Easily we see that \( \tilde{\pi} \circ \hat{\sigma} = \tilde{\pi} \), which means \( \tilde{\sigma} \in \text{Deck}(\hat{W}/\hat{Z}) \). Therefore, \( G \) is surjective.

(iii) In the following we prove that \( \hat{W}/\hat{Z} \) is Galois if and only if \( M/L \) is Galois.

Now we assume the natural covering map \( \tilde{\pi} : \hat{W} \to \hat{Z} \) is Galois. For \( \hat{f} \in M \) we will prove that the minimal polynomial \( \text{min}(L, \hat{f}) \) of \( \hat{f} \) over \( L \) splits in \( M \).

Suppose \( \text{min}(L, \hat{f}) = t^n + c_1 t^{n-1} + \cdots + c_n \). Suppose \( L_0 = \mathcal{M}(\hat{R})(\tilde{c}_1, \ldots, \tilde{c}_n) \) and \( L_0 \leq L_1 \leq L \) for fields \( L_1 \) and \( L_2 \), by \( L_1 \leq L_2 \) we mean \( L_1 \) is a subfield of \( L_2 \). Then \( \text{min}(L_1, \hat{f}) = \text{min}(L, \hat{f}) \) and there exists \( \tilde{g} \in L_0 \) such that \( L_0 = \mathcal{M}(\hat{R})(\tilde{g}) \). Suppose \( M_0 = L_0(\hat{f}) \).

Then there exists \( \tilde{h} \in M_0 \) such that \( M_0 = \mathcal{M}(\hat{R})(\tilde{h}) \).

Let \( \tilde{W}_0 = \text{obdom}(\tilde{h}) \) and \( \tilde{Z}_0 = \text{obdom}(\tilde{g}) \). Then \( \tilde{Z}_0 \in \tilde{Z} \) and \( \tilde{W}_0 \in \tilde{W} \) by Lemma \( 4.2 \).

Since \( \tilde{\pi} : \hat{W} \to \hat{Z} \) is Galois, there exists a Galois covering map \( \tilde{\pi}_1 : \tilde{W}_1 \to \tilde{Z}_1 \) in \( \tilde{\pi} \) with \( \tilde{Z}_0 \leq \tilde{Z}_1 \) and \( \tilde{W}_0 \leq \tilde{W}_1 \). Therefore \( M_1/L_1 \), where \( M_1 = \mathcal{M}(\tilde{W}_1) \) and \( L_1 = \mathcal{M}(\tilde{Z}_1) \), is Galois by Lemma \( 4.3(3) \), which means \( \text{min}(L, \hat{f}) = \text{min}(L_1, \hat{f}) \) splits in \( M_1 \), hence in \( M \).

At last, assume \( M/L \) is Galois. For a natural covering map \( \pi_0 : \hat{W}_0 \to \hat{Z}_0 \) in \( \pi \), where \( \hat{W}_0 \in \hat{W} \) and \( \hat{Z}_0 \in \hat{Z} \), let \( M_0 = \mathcal{M}(\hat{W}_0) \) and \( L_0 = \mathcal{M}(\hat{Z}_0) \). Then there exists \( M_1 \leq M \) such that \( M_1/M_0 \) is a finite extension and \( M_1/L_0 \) is a Galois extension, where \( M_0 = \mathcal{M}(\hat{R})(\hat{f}_1), \text{min}(L, \hat{f}_1) = t^n + c_1 t^{n-1} + \cdots + c_n \) and \( L_0 = L_0(\tilde{c}_1, \ldots, \tilde{c}_n) \), since \( [M_0 : \mathcal{M}(\hat{R})] \) is finite and \( M/L \) is Galois. In fact, we may let \( M_1 = L_0(\hat{f}_1, \ldots, \hat{f}_n) \), where \( \hat{f}_1, \ldots, \hat{f}_n \) are all the roots of \( \text{min}(L, \hat{f}_1) \) in \( M \). Suppose \( M_1 = \mathcal{M}(\hat{R})(\tilde{g}) \) for some \( \tilde{g} \in M_1 \) and \( \tilde{W}_1 = \text{obdom}(\tilde{g}) \). Take a natural covering map \( \tilde{\pi}_1 : \tilde{W}_1 \to \tilde{Z}_1 \) in \( \pi \) over \( \pi_0 : \hat{W}_0 \to \hat{Z}_0 \), where \( \tilde{Z}_1 \in \tilde{Z} \), and let \( L_1 = \mathcal{M}(\tilde{Z}_1) \). Then by the definition of natural covering maps of algebraic Riemann surfaces (the definition before Remark \( 12 \)) and by Lemma \( 4.3 \) we have
Lemma 4.4(3). Thus \( E \) is Galois by Lemma 4.4(3).

Suppose \( \tilde{X} \) and \( \tilde{Y} \) are algebraic Riemann surfaces and \( \tilde{Y} \) is Galois over \( \tilde{X} \). Suppose \( \tilde{Z} \in \text{Int}(\tilde{Y}/\tilde{X}) \) and \( \tilde{\sigma} : \tilde{Y} \to \tilde{Y}' \) is a mapping, where \( \tilde{Y}' \) is some algebraic Riemann surface. Define

\[ \tilde{\sigma}|_{\tilde{Z}} := \{ \tilde{\sigma}|_{\tilde{Z}} : \tilde{Z} \in \tilde{Z} \}. \]

If \( \tilde{\sigma}|_{\tilde{Z}} : \tilde{Z} \to \tilde{Z}' \) is still a mapping, where \( \tilde{Z}' \) is some algebraic Riemann surface under \( \tilde{Y}' \), then we call \( \tilde{\sigma}|_{\tilde{Z}} \) a restriction of \( \tilde{\sigma} \) to \( \tilde{Z} \).

By Lemma 4.6 and [21, Theorem 3.28] and by the reasoning in part (ii) of the proof of Lemma 4.5 we deduce

Lemma 4.6. Suppose \( \tilde{Y} \) is Galois over \( \tilde{X} \) and \( \tilde{Z} \in \text{Int}(\tilde{Y}/\tilde{X}) \). If \( \tilde{Z}/\tilde{X} \) is Galois and \( \tilde{\sigma} \in \text{Deck}(\tilde{Y}/\tilde{X}) \), then \( \tilde{\sigma}|_{\tilde{Z}} \in \text{Deck}(\tilde{Z}/\tilde{X}) \) and for \( \tilde{\tau} \in \text{Deck}(\tilde{Z}/\tilde{X}) \) there is a \( \tilde{\sigma} \in \text{Deck}(\tilde{Y}/\tilde{X}) \) with \( \tilde{\sigma}|_{\tilde{Z}} = \tilde{\tau} \).

We assume \( G = \text{Gal}(\mathcal{M}(\tilde{Y})/\mathcal{M}(\tilde{X})) \) possesses the Krull topology (see [21, Definition 17.5]). Recall that \( D = \text{Deck}(\tilde{Y}/\tilde{X}) \) has been given a similar topological structure in the beginning of this section. By Lemma 4.5 we deduce

Lemma 4.7. Suppose \( \tilde{X} \leq \tilde{Z} \leq \tilde{Y} \). Let \( K = \mathcal{M}(\tilde{Y}), N = \mathcal{M}(\tilde{X}) \) and \( L = \mathcal{M}(\tilde{Z}) \). Then \( G = G_{\tilde{Y}/\tilde{X}} \) is an isomorphism and a homeomorphism from \( D = \text{Deck}(\tilde{Y}/\tilde{X}) \) to \( G = \text{Gal}(K/N) \) and \( G|_E \) is an isomorphism from \( E = \text{Deck}(\tilde{Y}/\tilde{Z}) \) to \( H = \text{Gal}(K/L) \).

By Lemma 4.7 and [21, Theorem 17.6] we have

Proposition 4.8. Suppose \( D = \text{Deck}(\tilde{Y}/\tilde{X}) \) and \( \mathcal{K}_D \) is its Krull topology. Then the topological space \( (D, \mathcal{K}_D) \) is Hausdorff, compact and totally disconnected.

For \( H \leq G \) let \( \mathcal{F}(H) \) denote the fixed field of \( H \). Define

\[ \Gamma := \mathcal{R} \circ \mathcal{F} \circ G, \]

where \( \mathcal{R} \) and \( G = G_{\tilde{Y}/\tilde{X}} \) are defined in Lemmas 4.3 and 4.5 respectively.

Lemma 4.9. For \( E \leq D = \text{Deck}(\tilde{Y}/\tilde{X}) \) we have the closure

\[ \overline{E} = \text{Deck}(\tilde{Y}/\Gamma(E)). \]

Thus \( E \) is closed precisely if \( E = \text{Deck}(\tilde{Y}/\Gamma(E)) \).

Proof. Denote \( H = G(E) \). Suppose \( L = \mathcal{F}(H) \) and \( \tilde{Z} = \mathcal{R}(L) \). Then \( \mathcal{M}(\tilde{Z}) = L \) by Lemma 4.3 and

\[ \tilde{Z} = (\mathcal{R} \circ \mathcal{F})(H) = (\mathcal{R} \circ \mathcal{F} \circ G)(E) = \Gamma(E). \]

Therefore, by Lemma 4.7 and [21, Theorem 17.7] it follows that

\[ \overline{E} = G^{-1}(\overline{H}) = G^{-1}(\text{Gal}(K/\mathcal{F}(H))) = G^{-1}(\text{Gal}(K/L)) = \text{Deck}(\tilde{Y}/\tilde{Z}) = \text{Deck}(\tilde{Y}/\Gamma(E)), \]

where \( K = \mathcal{M}(\tilde{Y}) \).\qed
Proof of Theorem 4.1. Suppose $\tilde{Z} \in \text{Int}(\tilde{Y}/\tilde{X})$ and $E = \text{Deck}(\tilde{Y}/\tilde{Z})$. Let $K = \mathcal{M}(\tilde{Y})$, $N = \mathcal{M}(\tilde{X})$ and $L = \mathcal{M}(\tilde{Z})$. Then $\tilde{Z} = \mathcal{R}(L)$ by Lemma 4.3 and $K/N$ is Galois by Lemma 4.5. Hence, $K/L$ is Galois. Let $H = G(E)$. Then we have $H = \text{Gal}(K/L)$ by Lemma 4.5. Hence by [21, Lemma 2.9(6)] it follows $H = \text{Gal}(K/F(H))$, which means that $H$ is closed by [21, Theorem 17.7] and so is $E$ by Lemma 4.7. By Lemma 4.5 and the fundamental theorem of infinite Galois theory (Krull’s, see [21, Theorem 17.8]) (or [21, Definition 2.15], since $K/L$ is Galois) we have

$$(\Gamma \circ \Delta)(\tilde{Z}) = (\mathcal{R} \circ \mathcal{F} \circ G)(\text{Deck}(\tilde{Y}/\tilde{Z})) = (\mathcal{R} \circ \mathcal{F})(\text{Gal}(K/L)) = \mathcal{R}(L) = \tilde{Z}.$$  

On the other hand, for a closed subgroup $E$ of $D = \text{Deck}(\tilde{Y}/\tilde{X})$, let $\tilde{Z} = \Gamma(E)$. Then $\tilde{Z} \in \text{Int}(\tilde{Y}/\tilde{X})$ by Lemmas 4.3 and 4.7 and [21, Theorem 17.8] and by Lemma 4.9 we have $E = \text{Deck}(\tilde{Y}/\tilde{Z}) = \Delta(\tilde{Z})$. Therefore,

$$(\Delta \circ \Gamma)(E) = \Delta(\tilde{Z}) = E.$$  

By Lemma 4.7 we know that $[D : E] = [G : H]$ and that $E$ is open in $D$ if and only if $H$ is open in $G$. Thus, by [21, Theorem 17.8], assertion (1) in Theorem 4.1 follows since $[\tilde{Z} : \tilde{X}] = [L : N]$, where $L = \mathcal{M}(\tilde{Z})$ and $N = \mathcal{M}(\tilde{X})$.

By Lemmas 4.5 and 4.7 we have the following isomorphisms

$$\text{Deck}(\tilde{Z}/\tilde{X}) \cong \text{Gal}(L/N) \quad \text{and} \quad D/E \cong G/H$$

when $E$ is normal in $D$, which are also homeomorphisms. Hence, assertion (2) in Theorem 4.1 follows by Lemmas 4.5 and 4.7 and [21, Theorem 17.8].

According to Lemma 4.6 and noticing that when $\tilde{\sigma} \in \text{Deck}(\tilde{Y}/\tilde{X})$ and $\tilde{Z}/\tilde{X}$ is Galois we have $\tilde{\sigma}|_{\tilde{Z}} = \text{id}_{\tilde{Z}}$ if and only if $\tilde{\pi} \circ \tilde{\sigma} = \tilde{\pi}$ (refer to part (ii) of the proof of Lemma 4.5), where $\tilde{\pi} : \tilde{Y} \to \tilde{Z}$ is the natural covering map, we can also deduce that if $\tilde{Z}/\tilde{X}$ is Galois then the mapping from $D/E$ to $\text{Deck}(\tilde{Z}/\tilde{X})$ defined by $\tilde{\sigma} \mapsto \tilde{\sigma}|_{\tilde{Z}}$ is both an isomorphism and a homeomorphism by similar reasoning to the proof of [21, Theorem 17.8] and by Proposition 4.8.  

For the finite case, as a corollary of Theorem 4.1, we have

**Theorem 4.10** (Galois Correspondence on Algebraic Riemann Surfaces in the Finite Case). Suppose $\tilde{X}$ and $\tilde{Y}$ are algebraic Riemann surfaces and $\tilde{Y}$ is a finite Galois covering of $\tilde{X}$. Let $D = \text{Deck}(\tilde{Y}/\tilde{X})$. Then the mapping

$$\Delta : \text{Int}(\tilde{Y}/\tilde{X}) \to \mathfrak{C}$$

$$\tilde{Z} \mapsto \text{Deck}(\tilde{Y}/\tilde{Z})$$

is a bijection, which gives an inclusion reversing correspondence and whose inverse mapping is $\Gamma$ given in (1.10). Moreover, if $E = \Delta(\tilde{Z})$ then

(1) $[\tilde{Y} : \tilde{Z}] = |E|$ (the order of $E$) and $[\tilde{Z} : \tilde{X}] = [D : E]$;

(2) $\tilde{Z}$ is Galois over $\tilde{X}$ if and only if $E$ is normal in $D$, and on this condition we have a group isomorphism

$$\text{Deck}(\tilde{Z}/\tilde{X}) \cong D/E.$$  

□
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