WELL-POSEDNESS ISSUES ON THE PERIODIC MODIFIED KAWAHARA EQUATION

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Abstract. This paper is concerned with the Cauchy problem of the modified Kawahara equation posed on $\mathbb{T}$. The main result in this paper is the global well-posedness in $L^2(\mathbb{T})$. As an application of this result, we show the unconditional uniqueness in $H^s(\mathbb{T})$, $s > \frac{1}{2}$. The proof basically relies on the idea introduced in Takaoka-Tsutsumi’s works [68, 59], which weakens the non-trivial resonance in the cubic interactions (a kind of smoothing effect). This is the first low regularity (global) well-posedness result for the periodic modified Kawahara equation, as far as we know. A direct interpolation argument ensures the unconditional uniqueness in $H^s(\mathbb{T})$, $s > \frac{1}{2}$. As a byproduct, we show the weak ill-posedness below $H^{\frac{1}{2}}(\mathbb{T})$, in the sense that the flow map fails to be uniformly continuous.

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1. Introduction

1.1. Setting. A study on waves starts from an examination of a two-dimensional, irrotational flow of an incompressible ideal fluid with a free surface under the gravitational field. The fluid is bounded below by a solid bottom and above by an atmosphere of constant pressure. The upper surface is a free boundary, and the influence of the surface tension is naturally taken into account
on the free surface. The motion of the free surface is called a capillary-gravity wave, and it is called a gravity wave or a water wave in the case without the surface tension.

In the mathematical view, the waves are formulated as a free boundary problem for the incompressible, irrotational Euler equation. Rewriting the equations in an appropriate non-dimensional form, one gets two non-dimensional parameters $\delta := \frac{h}{\lambda}$ and $\varepsilon := \frac{a}{\lambda}$, where $h$, $\lambda$ and $a$ denote the water depth, the wave length, and the amplitude of the free surface, respectively, and another non-dimensional parameter $\mu$ called the Bond number, which comes from the surface tension on the free surface. The physical condition $\delta \ll 1$ characterizes the waves, which are called long waves or shallow water waves, but there are several long wave approximations according to relations between $\varepsilon$ and $\delta$. We introduce three typical long wave regimes.

(1) Shallow water wave: $\varepsilon = 1$ and $\delta \ll 1$.
(2) Korteweg-de Vries (KdV): $\varepsilon = \delta^2 \ll 1$ and $\mu \neq \frac{1}{3}$.
(3) Kawahara: $\varepsilon = \delta^4 \ll 1$ and $\mu = \frac{1}{9} + \nu \varepsilon^{\frac{4}{3}}$.

In Item (1) regime, we obtain the (so-called) shallow water equations as the limit $\delta \to 0$. It is known that the shallow water equations are analogous to one-dimensional compressible Euler equations for an isentropic flow of a gas of the adiabatic index 2, and thus its solutions generally have a singularity in finite time, even if the initial data are sufficiently smooth. Therefore, this long wave regime is used to explain breaking waves of water waves. In Item (2) regime, the following well-known, notable equation called the KdV equation has been derived from the equations for capillary-gravity waves by Korteweg and de Vries [44]:

$$\pm 2u_t + 3uu_x + \left(\frac{1}{5} - \mu\right) u_{xxx} = 0. $$

Remark that when the Bond number $\mu = \frac{1}{3}$, this equation degenerates to the inviscid Burgers equation. In connection with this critical Bond number, Hasimoto [26] derived a higher-order KdV equation of the form

$$\pm 2u_t + 3uu_x - \nu u_{xxx} + \frac{1}{45} u_{xxxxx} = 0. $$

in Item (3) regime, which is nowadays called the Kawahara equation. It is also known that when the Bond number is nearly 1 and $\nu \varepsilon^{\frac{4}{3}} \ll 1$ characterizes the waves, which are called long waves.

This paper concerns with the Cauchy problem of the modified Kawahara equation given by

$$\begin{aligned}
\partial_t v - \delta^2 \partial_x^2 v + \beta \partial_x^4 v + \gamma \partial_x v - \frac{\mu}{3} \partial_x (v^3) &= 0, \\
v(t, x) &= v_0(x) \in H^s(\mathbb{T}), \\
(t, x) &\in [0, T] \times \mathbb{T},
\end{aligned}$$

(1.1)

where $\mathbb{T} = \mathbb{R}/2\pi \mathbb{Z}$, $\beta \geq 0^2$, $\gamma \in \mathbb{R}$, $\mu = \pm 1$ and $u$ is a real-valued unknown.

The equation (1.1) can be generalized as follows:

$$\partial_t v - \delta^2 \partial_x^2 v + \beta \partial_x^4 v + \gamma \partial_x v - \frac{\mu}{3} \partial_x (v^p) = 0, \quad p = 2, 3, \ldots.$$  

(1.2)

As seen before, when $p = 2$, the equation (1.2) is the Kawahara equation, which is a higher-order Korteweg-de Vries (KdV) equation with an additional fifth-order derivative term. This type of the equation (1.2) was first found by Kakutani and Ono [34] in an analysis of magnet-acoustic waves in a cold collision free plasma. The equation (1.2) was also derived, as already mentioned above, by Hasimoto [26] as a model of capillary-gravity waves in an infinitely long canal over a flat bottom in a long wave regime when the Bond number is nearly $\frac{1}{3}$. Kawahara [39] studied this equation (1.2) numerically and observed that the equation has both oscillatory and monotone solitary wave solutions. This equation is also regarded as a singular perturbation of Korteweg-de Vries (KdV) equation. We further refer to, for instance, [3, 1, 75, 29, 64, 8, 24, 30, 66] and references therein for more background informations.

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1The equation (1.1) is reduced from the equation of the form

$$\partial_t v + \alpha \partial_x^2 v + \beta \partial_x^4 v + \gamma \partial_x v + \mu \partial_x (v^3) = 0$$

by the renormalization of $v$.

2One may extend the range of $\beta$ to negative values, but this change does not cause any difficulty in our analysis by regarding the case $|\xi| \leq \frac{6|\beta|}{\delta^2}$ as low frequency part.
The equation (1.1) admits at least three conservation laws:

\[ E[v](t) = \int v \, dx = M_0[v_0], \]

\[ M[v](t) = \frac{1}{2} \int v^2 \, dx = M[v_0] \] (1.3)

and

\[ H[v](t) = \frac{1}{2} \int (\partial_x^2 v)^2 \, dx + \frac{\beta}{2} \int (\partial_x v)^2 \, dx - \frac{\gamma}{2} \int v^2 \, dx + \frac{\mu}{12} \int v^4 \, dx = H[v_0]. \] (1.4)

The \( L^2 \) conserved quantity (1.3) will help us to extend the local solution to global one, so to attain the global well-posedness in \( L^2(\mathbb{T}) \). Moreover, the equation (1.1) can be written as the Hamiltonian equation with respect to (3.9) as follows:

\[ v_t = \partial_x \nabla_v H(v(t)) = \nabla_{\omega_{-\frac{1}{2}}} H(v(t)), \] (1.5)

where \( \nabla_v \) is the \( L^2 \) gradient and \( \omega_{-\frac{1}{2}} \) is the symplectic form in \( H^{-\frac{1}{2}} \) defined as

\[ \omega_{-\frac{1}{2}}(v,w) := \int_T v \partial_x^{-1} w \, dx, \]

for all \( u,v \in H_0^{-\frac{1}{2}} \). Indeed, a direct computation yields

\[ \omega_{-\frac{1}{2}}(w,\nabla_{\omega_{-\frac{1}{2}}} H(v(t))) := \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} H(v + \varepsilon w) \]

\[ = \int \left( \partial_x^2 v - \beta \partial_x^2 v - \gamma v + \frac{\mu}{3} v^3 \right) w \]

\[ = \int \left( -\partial_x^5 v + \beta \partial_x^3 v - \gamma \partial_x v + \frac{\mu}{3} \partial_x (v^3) \right) \partial_x^{-1} w \]

\[ = \omega_{-\frac{1}{2}}(w,\partial_x^2 v - \beta \partial_x^2 v - \gamma \partial_x v + \frac{\mu}{3} \partial_x (v^3)). \]

Such an expression by the Hamiltonian form (or regarding the flow map as a symplectomorphism on \( H^{-\frac{1}{2}} \)) enables to study some symplectic property, in particular, non-squeezing property, which is initiated (for the dispersive PDE or a non-compact operator) by Bourgain [7]. However, our well-posedness result presented below is available only up to in \( L^2 \) regularity level, thus we cannot explore it as of now. More delicate analysis (or new clever idea) will facilitate the \( H^{-\frac{1}{2}} \) global well-posedness, and so the non-squeezing analysis. We refer to [45, 16, 65, 54, 28, 42, 43, 47] for more detailed expositions of the non-squeezing property.

On the other hand, the expression (1.5) provides a convenient setting to use the spectral stability theory of [19]. We also refer to [72] for another application of the Hamiltonian form (1.5) to derive criteria for instability of small-amplitude periodic solutions of (1.5).

These conserved quantities play important roles in the study of the partial differential equations. In particular, such conserved quantities enable to treat the (nontrivial) resonant interaction in the study of the initial value problem under the periodic boundary condition. In this work, the second conserved quantity (1.3) is enough to deal with the cubic resonance, since the nonlinearity in (1.1) has only one derivative and is of the cubic form. On the other hand, an appropriate nonlinear transformation, which has a bi-continuity property, helps to kill the cubic nontrivial resonance without using the conservation law (1.3) (see Section 2 for more details, and refer to [67, 46] for similar or more complicate cases).

1.2. Different phenomena: periodic vs. non-periodic. The Cauchy problems for some dispersive equations have plenty of interesting issues under the periodic setting compared with the non-periodic problems. The first interesting (and also different from the non-periodic problem) issue is the presence of non-trivial resonances. In particular, the modified Kawahara equation (1.1) contains two non-trivial resonant terms (of the Fourier coefficient forms) in the nonlinearity such as

\[ -\mu \text{in} |\vec{v}(n)|^2 \vec{v}(n) \quad \text{and} \quad \mu \text{in} \left( \sum_{n' \in \mathbb{Z}} |\vec{v}(n')|^2 \right) \vec{v}(n), \]
whenever the Fourier variables have the following frequency relations:

\[ n_1 + n_2 + n_3 = n \quad \text{and} \quad (n_1 + n_2)(n_2 + n_3)(n_3 + n_1) = 0. \]

The latter resonance causes an uncontrollable perturbation phenomenon near the linear solution in the Sobolev space (of any regularity), while the former one is controllable perturbation (at least up to \( H^\frac{3}{2} \)). Such phenomena never happen under the non-periodic condition, since this happens on the set of frequencies \((n_1, n_2 \text{ and } n_3)\), for which elements satisfy \((n_1 + n_2)(n_2 + n_3)(n_3 + n_1) = 0\). To deal with the second resonance, one can use \( L^2 \) conservation law \((1.3)\) to make \((\sum_{n' \in \mathbb{Z}} |\hat{r}(n)|^2)\) as a constant coefficient of the first order linear term, and thus remove this resonance in the nonlinearity.

On the other hand, the first resonance is more difficult to be dealt with, precisely, one cannot make it a constant coefficient linear part of the equation unlike the second one. However, as mentioned above, this resonance does not make any trouble in the study on the well-posedness problem up to \( H^\frac{3}{2} \) regularity. To lower the regularity (in other words, to study the IVP with rougher data), it is necessary to take a more delicate analysis on this term. When studying on this term, we face on another interesting issue under the periodic setting: the lack of smoothing effect. Possible remedies to this problem are for instance, the normal form reduction method and the short time Fourier restriction norm method. In the present paper, we take the normal form mechanism to gain a smoothing effect under non-resonant interactions. A better example to capture this difference is the fifth-order modified KdV equation \([48]\).

For more detailed expositions, see Sections 2 and 4.

1.3. Main results. Before stating our main result, we introduce an well-known notion of well-posedness. The duhamel’s principle ensures that the equation \((1.1)\) is equivalent to the following integral equation

\[
v(t) = S(t)v_0 + \frac{\mu}{3} \int_0^t S(t-s)(v(s)^3)_x \, ds, \tag{1.6}
\]

where \(S(t)\) is the linear propagator associated to the linear equation \(\partial_t v - \partial^2_x v + \beta \partial^3_x v + \gamma \partial_x v = 0\), precisely defined by

\[
S(t)f = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} e^{inx} e^{it(\lambda^n + \beta n^3 - \gamma n)} \hat{f}(n).
\]

The equation

\[
\partial_t v - \partial^2_x v + \partial_x (v^3) = 0 \tag{1.7}
\]

allows the scaling invariance, that is, if \(v\) is a solution to \((1.7)\), then \(v_\lambda := \lambda^2 v(\lambda^2 t, \lambda x)\), \(\lambda > 0\), is also a solution to \((1.7)\). A straightforward calculation gives

\[
\|f \chi\|_{H^s(\mathbb{T})} = \lambda^{\frac{3}{2} + s} \|f\|_{H^s(\mathbb{T})},
\]

which says \(s_c = -\frac{3}{2}\) is (scaling) critical Sobolev index, where \(\mathbb{T}_\lambda = \mathbb{R} / 2\pi \lambda^{-1} \mathbb{Z}\).

Remark 1.1. One can not get the scaling invariance for the equation \((1.1)\) due to \(\partial^3_x v\) and \(\partial_x v\) terms. Instead, one sees that the equation \((1.1)\) allows a scaling equivalence, that is to say, if \(v\) is solution to \((1.1)\), then \(v_\lambda = \lambda^2 v(\lambda^2 t, \lambda x)\) is a solution to

\[
\partial_t v_\lambda - \partial^2_x v_\lambda + \beta \lambda^3 \partial^3_x v_\lambda + \gamma \lambda^4 \partial_x v_\lambda - \frac{\mu}{3} \partial_x (v_\lambda^3) = 0.
\]

However, even the smaller dispersive effect from \(\partial^2_x v\) and \(\partial_x v\) itself is negligible compared with one from \(\partial^3_x v\) (and thus no influence on our analysis), hence the equation \((1.1)\) follows the scaling rule observed above.

We first state well-known definition of the local well-posedness for (scaling sub-critical) IVPs (see, for instance \([11, 70]\)).

**Definition 1.1** (Local well-posedness). Let \(v_0 \in H^s(\mathbb{T})\) be given. We say that the IVP of \((1.1)\) is locally well-posed in \(H^s(\mathbb{T})\) if the following properties hold:

1. (Existence) There exist a time \(T = T(\|v_0\|_{H^s}) > 0\) and a solution \(v\) to \((1.1)\) such that \(v\) satisfies \((1.6)\) and belongs to a subset \(X^s_T\) of \(C([0, T]; H^s)\).

2. (Uniqueness) The solution is unique in \(X^s_T\).
(3) (Continuous dependence on the data) The map \( v_0 \mapsto v \) is continuous from a ball \( B \subset H^s \) to \( X^s_T \) (with the \( H^s \) topology).

Remark 1.2. We extend the notion of well-posedness presented in Definition 1.1 in (at least) three directions.

(1) (Global well-posedness) We say the IVP is \textit{global} well-posed if we can take \( T \) arbitrary large.

(2) (Unconditional uniqueness) We say that the IVP is \textit{unconditional} well-posedness if we can take \( X^s_T = C([0,T];H^s) \).

(3) (Uniform well-posedness) We say that the IVP is \textit{uniform} well-posedness if the solution map \( v_0 \mapsto v \) is uniformly continuous from a ball \( B \subset H^s \) to \( X^s_T \). Similarly, one can define the notion of Lipschitz well-posedness, \( C^k \) well-posedness, \( k = 1, 2, \ldots \), and analytic well-posedness.

Remark 1.3. Once the Picard iteration method works well on a IVP, one immediately obtain that the map is not only uniformly continuous but also real analytic (in this case, we say the problem is \textit{a semilinear} problem). On the other hand, if one cannot apply the iteration method to a IVP (due to, for example, a strong nonlinearity compared to a dispersion or the presence of non-trivial resonances), one cannot reach the uniform well-posedness. This case is referred as \textit{weakly} or \textit{mild} ill-posedness (in this case, we say the problem is \textit{a quasilinear} problem).

We are now in a position to state results established in this paper. The first theorem is to show the uniform well-posedness of (1.1).

**Theorem 1.1.** Let \( s \geq \frac{1}{2} \). Then, the Cauchy problem of (1.1) is locally (in time) well-posed in \( H^s(\mathbb{T}) \). Moreover, the uniform continuity (indeed analytic) of the flow map holds in the class

\[
\{ v_0 \in H^s : \|v_0\|_{L^2} = c \}, \quad c \geq 0 \quad \text{is fixed.} \tag{1.8}
\]

The proof is based on the standard Fourier restriction norm method (with trilinear estimates), initially introduced by Bourgain [6]. The regularity threshold \( s = \frac{1}{2} \), for which the local well-posedness of (1.1) holds, occurs due to the nontrivial resonant term \( i|n|\hat{v}(n)|^2\hat{v}(n) \) as explained in Section 1.2. To improve Theorem 1.1 below \( H^{\frac{1}{2}}(\mathbb{T}) \), it is necessary to reduce the strength of the resonance.

In [68, 59], authors introduced a way to weaken the resonance by establishing a kind of smoothing effects in the context of modified KdV equation. Let us be more precise (in the context of modified Kawahara equation (1.1)). The evolution operator given by

\[
W(t) := \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} e^{i(nx + \xi p(n) + t|n|\hat{v}_0(n)|^2)} \hat{v}(n)
\]

enables to reduce \( in|\hat{v}(n)|^2\hat{v}(n) \) by \( (|\hat{v}(n)|^2 - \hat{v}_0(n)|^2) \hat{v}(n) \). An appropriate estimate for

\[
\sup_{n \in \mathbb{Z}} \left| \text{Im} \left[ \int_0^t n^2 \sum_{N_n} \hat{v}(s,n_1)\hat{v}(s,n_2)\hat{v}(s,n_3)\hat{v}(s,-n) \, ds \right] \right|
\]

succeeds in getting a kind of smoothing effects, and so the local well-posedness below \( H^{\frac{1}{2}}(\mathbb{T}) \). However, due to a technical problem arising in the estimate of the reduced resonance in \( L^2 \), a modification of the argument in [68, 59] is needed. See Section 2 for more details. We state the main result in this paper.

**Theorem 1.2.** Let \( 0 \leq s < \frac{1}{2} \). Then, the Cauchy problem of (1.1) is locally (in time) well-posed in \( H^s(\mathbb{T}) \).

Thanks to \( L^2 \) conservation law (1.3), we extend Theorem 1.2 to the global one.

**Theorem 1.3.** The Cauchy problem of (1.1) is globally (in time) well-posed in \( L^2(\mathbb{T}) \).

In the proof of Theorem 1.2, we are not able to attain the uniform well-posedness even in the class (1.8) due to the resonance \( in|\hat{v}(n)|^2\hat{v}(n) \), that is, the flow map defined in the proof of Theorem 1.2 for \( 0 \leq s < \frac{1}{2} \) does not hold in the following sense:
Definition 1.2 (Uniform continuity of flow maps). We say that the flow map is uniformly continuous if for all $R > 0$, there exist $T > 0$ and a continuous function $\zeta$ on $[0, \infty)$ satisfying $\zeta(r) \to 0$ as $r \to 0$ such that solutions $v_1, v_2$ to (1.1) with $\|v_1(0)\|_{H^s}, \|v_2(0)\|_{H^s} \leq R$ satisfy

$$\|v_1 - v_2\|_{C^2_{T}H^s} \leq \zeta(\|v_1(0) - v_2(0)\|_{H^s}).$$

As a byproduct of Theorem 1.2, we have

Theorem 1.4. Let $0 \leq s < \frac{1}{2}$. Then, the Cauchy problem of (1.1) is weakly ill-posed in $H^s(\mathbb{T})$ in a sense of Remark 1.3. In other words, the flow map does not hold the property presented in Definition 1.2.

An interesting issue in the well-posedness theory is the unconditional uniqueness as mentioned in Remark 1.2, that is to say, the uniqueness holds in some larger spaces that contain weak solutions even in higher regularity. Such a issue was first proposed by Kato [36] in the context of Schrödinger equation. The unconditional uniqueness is referred as the uniqueness in $L_T^2 H^s$ without the restriction of any auxiliary function space (for instance $X_T^{s,rac{1}{2}}$ used in Theorem 1.2).

The unconditional well-posedness of some dispersive equations have been studied (for instance [36, 78, 20, 74, 71, 2, 50, 25, 56, 57, 51, 58] and references therein). Some of these uniqueness results employed some auxiliary function spaces (for example Strichartz spaces [71], $X^{s,b}$-type [78, 74, 56, 57]), which are designed to be large enough to contain $C_T H^s$ such that the uniqueness of the solution holds. On the other hand, a straightforward energy-type estimate via finite or infinite iteration scheme of the normal form reduction method is also available to prove the unconditional well-posedness in a certain class of $C_T H^s$ [2, 50, 25, 51, 58]. Such an argument seems more natural and elementary since any other auxiliary function spaces does not be taken.

We finally state the last result established in this work. The uniqueness in $X_T^{0,\frac{1}{2}}$ established in Theorems 1.2 and 1.3 ensures

Theorem 1.5. Let $s > \frac{1}{2}$. Then, the Cauchy problem of (1.1) is unconditionally globally well-posed in $H^s(\mathbb{T})$.

For the proof of Theorem 1.5, we show the embedding property $C_T H^s \subset X_T^{0,\frac{1}{2}}, s > \frac{1}{2}$, where the space $X_T^{0,\frac{1}{2}}$ was designed for the proof of Theorem 1.2. Thus, the uniqueness in $X_T^{0,\frac{1}{2}}$ in addition to Theorem 1.3 implies the unconditional well-posedness.

Remark 1.4. An alternative way used in [50] seems available to prove the unconditional well-posedness in $H^s(\mathbb{T}), s > \frac{1}{2}$ (also possible in $H^s(\mathbb{T})$). However, we do not take their argument for the proof of Theorem 1.5 in order to avoid abusing the normal form mechanism.

Remark 1.5. In author’s forthcoming work [49], the unconditional uniqueness issue of modified Kawahara equation (1.1) in $H^s, \frac{1}{6} \leq s \leq \frac{1}{2}$, will be addressed.

1.4. About the proof of Theorem 1.2. The proof is the standard compactness argument (or referred as energy method). After changing of suitable variables ($v \mapsto u$ by using $L^2$ conservation law and an appropriate nonlinear transformation, see Section 2), and collecting the non-resonance estimate (Lemma 3.2) and a smoothing effect (Corollary 4.1), one establishes

$$\|u\|_{X_T^{s,\frac{1}{2}}} \leq C\|u_0\|_{H^s} + C T^\theta \left(\|u_0\|_{H^s}^4 + \left(\|u_0\|_{H^s}^2 + \|u\|_{X_T^{s,\frac{1}{2}}}^4 \right)^2 + \left(1 + \|u\|_{X_T^{s,2}} + \|u\|_{X_T^{s,\frac{1}{2}}}^3 \right)\right)\|u\|_{X_T^{s,\frac{1}{2}}}.$$

This guarantees the uniform boundedness of $u$ in $X_T^{s,\frac{1}{2}}$ dependent only on the initial data, and thus the weak and strong convergence of $u_j$ to $u$ in $H^s$ can be attained via Arzelà-Ascoli compactness theorem. In addition, Lemma 2.4 enables to obtain the solution $v$ from $u$, and $L^2$ conservation law (1.3) extends the local solution $u$ on $[-T,T]$ to the global one on $\mathbb{R}$.

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3The function space $X_T^{s,b}$ is the time localization of the standard $X^{s,b}$ introduced by Bourgain [6]
1.5. **About the literatures.** Not only the modified Kawahara equation, but also the generalized Kawahara equation (including the Kawahara, \( p = 1 \) in (1.2)) have been extensively studied in several directions.

There has been a great deal of work on solitary wave solutions of the Kawahara equation in the last fifty years. Compared to the KdV solitary waves, the Kawahara solitary wave solutions exponentially decay to zero as \( x \to \infty \) analogously to KdV, while, the Kawahara solitary waves have oscillatory trails, unlike the KdV equation whose solitary waves are non-oscillating. The strong physical background of the Kawahara equation and such similarities and differences between Kawahara and the KdV equations in both the formulations, and the behavior of the solutions propound the mathematical interesting questions of this equation. We refer to, for instance, [39, 22, 1, 29, 64, 8, 35, 31, 41, 5] for more informations associated to solitary waves of (1.2) and [52, 53, 60, 72, 33].

As for the low regularity Cauchy problem associated to (1.2) (when \( p = 2 \)), Cui and Tao [18] used the Strichartz estimates to prove the local well-posedness in \( H^s(\mathbb{R}) \), \( s > \frac{1}{2} \). Then, Cui, Deng and Tao [17] has improved the previous result by taking the Fourier restriction method, that is to say the local well-posedness in \( H^s(\mathbb{R}) \), \( s > -1 \). Both local results immediately extended to the global ones in \( H^2(\mathbb{R}) \) and \( L^2(\mathbb{R}) \) thanks to (3.9) and (1.3), respectively. Later, Wang, Cui and Deng [73] further improved the local and global results, in lower regularity Sobolev spaces \( H^s(\mathbb{R}) \), \( s \geq -\frac{7}{5} \) by performing a more delicate analysis. Moreover, they used \( I\)-method introduced in [15] to establish the global well-posedness in \( H^s(\mathbb{R}) \), \( s > -\frac{1}{2} \). In [13] and [32], authors independently established the sharp bilinear estimate (by taking the Tao’s \([K;Z]\)-multiplier norm method) in the standard \( X^{s,b} \) space to prove the local well-posedness in \( H^s(\mathbb{R}) \), \( s > -\frac{7}{4} \). At the critical regularity Sobolev space \( H^{-7/4}(\mathbb{R}) \), Chen and Guo [12] proved local and global well-posedness by using Besov-type critical space and \( I\)-method. Kato [37] proved the local well-posedness for \( s \geq -2 \) by modifying \( X^{s,b} \) space and the ill-posedness for \( s < -2 \) in the sense that the flow map is discontinuous at zero. Recently, Okamoto [63] observed the norm inflation with general initial data, which implies that the flow map of the Kawahara equation is discontinuous everywhere in \( H^s(\mathbb{R}) \) with \( s < 2 \).

When \( p = 2 \), the Cauchy problem for (1.1) was studied by Jia and Huo [32] and Chen, Li, Miao and Wu [13], independently. They established the local well-posedness in \( H^s(\mathbb{R}) \), \( s > -1/4 \), by using the Fourier restriction norm method. The global well-posedness of (1.1) in \( H^s(\mathbb{R}) \), \( s > -3/22 \) was shown by Yan, Li and Yang [77] via the \( I\)-method. We also refer to [76] for the weak ill-posedness result for the modified Kawahara equation in \( H^s(\mathbb{R}) \), \( s < -\frac{1}{2} \).

Compared with above well-posedness results for the non-periodic problems, there is a few work on the Cauchy problems under the periodic boundary condition. Gorsky and Himonas [23] have first studied the higher-order KdV-type equation of the form

\[
u_t + u_{mx} + uu_x = 0, \quad m = 3, 5, 7, \ldots
\]

under the periodic boundary condition. They established the bilinear estimate in \( X^{s,\frac{2}{3}} \), \( s \geq -\frac{1}{2} \) to prove the local well-posedness in \( H^{-\frac{1}{3}}(\mathbb{T}) \). This result was improved by Hirayama [27]. He improved the bilinear estimate established in [23] in \( H^s(\mathbb{T}) \) level, \( s \geq -\frac{m}{m+1} \) to show the local well-posedness \( H^{-\frac{m}{m+1}}(\mathbb{T}) \), and this estimate was shown to be sharp in the standard \( X^{s,b} \). The global extension of this result was done by Hong and author [28] via \( I\)-method. The optimal local well-posedness result in \( H^s(\mathbb{T}) \), \( s \geq -\frac{3}{2} \), for the Kawahara equation has been established by Kato [38] by constructing a modified \( X^{s,b} \) space (motivated by Bejenaru and Tao [4]) in order to handle the strong nonlinear interactions appeared when \( s < -1 \). He also proved the \( C^s \)-ill-posedness when \( s < -\frac{3}{2} \).

1.6. **Notations.** Let \( x, y \in \mathbb{R}_+ \). We use \( \lesssim \) when \( x \leq Cy \) for some \( C > 0 \). Conventionally, \( x \sim y \) means \( x \lesssim y \) and \( y \lesssim x \). \( x \ll y \), also, denotes \( x \leq cy \) for a very small positive constant \( c > 0 \).

Let \( f \in \mathcal{S}'(\mathbb{R} \times \mathbb{T}) \) be given. \( \tilde{f} \) or \( \mathcal{F}(f) \) denotes the space-time Fourier transform of \( f \) defined by

\[
\tilde{f}(\tau, n) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^{2\pi} e^{-inx} e^{-i\tau t} f(t, x) \, dx \, dt.
\]
Then, it is known that the (space-time) inverse Fourier transform is naturally defined as

\[ f(t, x) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} e^{ixn} e^{i\tau n} \hat{f}(\tau, n) \, d\tau. \]

Moreover, we use \( \mathcal{F}_x \) (or \( \hat{\cdot} \)) and \( \mathcal{F}_t \) to denote the spatial and temporal Fourier transform, respectively.

**Organization of this paper.** The rest of the paper is organized as follows: In Section 2, we give a fundamental observation to study (1.1), and introduce (modified) Takaoka-Tsutsumi’s idea adapted to this problem. We also introduce \( X^{s,b} \) space and its properties, and provide essential lemmas for the rest of sections. In Section 3, we prove the standard trilinear estimates in \( X^{s,b} \). In section 4, we prove a smoothing property to control the reduced resonance below \( H^2 \), and thus we prove local and global well-posedness results in Section 5. In Section 6, as an application of local well-posedness in \( L^2 \), we show the unconditional uniqueness of weak solutions to (1.1) in \( H^s(\mathbb{T}) \), \( s > \frac{1}{2} \). In Appendices, we provide the proof of \( L^4 \) Strichartz estimate and a short proof of Theorem 1.4 for the sake of the reader’s convenience.

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2. Preliminaries

2.1. Setting. Taking the Fourier transform to (1.1), one has

\[ \partial_t \hat{v}(n) - ip_0(n) \hat{v}(n) = \frac{\mu i n}{3} \sum_{n = n_1 + n_2 + n_3} \hat{v}(n_1) \hat{v}(n_2) \hat{v}(n_3), \quad (2.1) \]

where

\[ p_0(n) = n^5 + \beta n^3 - \gamma n. \]

Note that the resonance function generated by \( p_0(n) \) (in the cubic interactions) is given by

\[ H_0 = H_0(n_1, n_2, n_3, n) := p_0(n) - p_0(n_1) - p_0(n_2) - p_0(n_3) \]

\[ = \frac{5}{2} (n_1 + n_2)(n_2 + n_3)(n_3 + n_1) \left( n_1^2 + n_2^2 + n_3^2 + n^2 + \frac{6}{5} \beta \right) \quad (2.2) \]

It is known from (2.2) that the non-trivial resonances appear when \( H_0 = 0 \), equivalently, \( (n_1 + n_2)(n_2 + n_3)(n_3 + n_1) = 0 \).

We split the nonlinear term in (2.1) into two parts, and hence we rewrite (2.1) as follows:

\[ \partial_t \hat{v}(n) - ip_0(n) \hat{v}(n) = - \min |\hat{v}(n)|^2 \hat{v}(n) + \mu \min \left( \sum_{n' \in \mathbb{Z}} |\hat{v}(n')|^2 \right) \hat{v}(n) \]

\[ + \frac{\mu i n}{3} \sum_{n' \in \mathbb{N}_n} \hat{v}(n_1) \hat{v}(n_2) \hat{v}(n_3), \quad (2.3) \]

where \( \mathcal{N}_n \) is the set of frequencies (with respect to the fixed frequency \( n \)), for which the relations of frequencies never generate the resonance, given by

\[ \mathcal{N}_n = \left\{ (n_1, n_2, n_3) \in \mathbb{Z}^3 : n_1 + n_2 + n_3 = n, \ (n_1 + n_2)(n_2 + n_3)(n_3 + n_1) \neq 0 \right\}. \]

We call the first two terms in the right-hand side of (2.3) (non-trivial) resonant terms and the rest non-resonant term.

**Remark 2.1.** Compared to the non-periodic problem, such resonant terms are one of enemies to study the ”low regularity” local theory of periodic dispersive equations, while the (exact) resonant phenomena can be never seen in the non-periodic dispersive equation, since the set of frequencies, which generate the resonance (in this case \( (n_1 + n_2)(n_2 + n_3)(n_3 + n_1) = 0 \)), is a measure zero set.
The $L^2$ conservation law (1.3) enables us to kill the second term in the resonant terms, so that we reduce (2.3) by
\[
\partial_t \tilde{v}(n) - ip(n)\tilde{v}(n) = -\mu in|\tilde{v}(n)|^2\tilde{v}(n) + \frac{\mu i}{3} n \sum_{N_n} \tilde{v}(n_1)\tilde{v}(n_2)\tilde{v}(n_3),
\]
where
\[
p(n) = n^3 + \beta n^3 - (\gamma + \mu 2\pi|v_0|^2)n.
\]

Remark 2.2. One can use the Gauge transform defined by
\[
\mathcal{G}[v](t) := e^{i\mu t} \int_0^t |v'(x)|^2 \, dx \, v(t),
\]
where $\frac{1}{2\pi} \int_0^{2\pi} f \, dx$, to reduce (1.1) by the renormalized (or Wick ordered) modified Kawahara equation
\[
\partial_t v - \partial_x^5 v + \beta \partial_x^3 v + \gamma \partial_x v - \mu (v^2 - \frac{1}{2} \int v^2 \, dx) \partial_x v = 0.
\]

It is well-known that the Gauge transform (2.6) is well-defined and invertible when $s \geq 0$, thanks to $L^2$ conservation law (1.3). Moreover, this reduction (2.7) is identical to (2.4) in the sense that no more dispersive smoothing effect arises in the cubic interactions.

Remark 2.3. It is clear that the resonant term $in|\tilde{v}(n)|^2\tilde{v}(n)$ has an effect on the solution in the sense that the solution oscillates rapidly so that the uniform continuity of the solution map breaks in $H^s(\mathbb{T})$, $s < \frac{1}{4}$. In fact, the estimate of this term is valid for $s \geq \frac{1}{4}$ (see Lemma 3.1 below), thus the local well-posedness of (1.1) is naturally expected to hold at this regularity.

We denote the resonant and the non-resonant terms in (2.4) by $\mathcal{N}_R(v)$ and $\mathcal{N}_{NR}(v)$, respectively, and these can be generally defined by
\[
\mathcal{N}_R(v_1, v_2, v_3) = F_x^{-1} [\mu in\tilde{v}_1(n)\tilde{v}_2(-n)\tilde{v}_3(n)]
\]
and
\[
\mathcal{N}_{NR}(v_1, v_2, v_3) = F_x^{-1} \left[ \frac{\mu i}{3} n \sum_{N_n} \tilde{v}_1(n_1)\tilde{v}_2(n_2)\tilde{v}_3(n_3) \right].
\]

The modified linear operator in (2.4) (defined by $p(n)$ in the Fourier mode) generates another cubic resonance function given by
\[
H = H(n_1, n_2, n_3, n) := p(n) - p(n_1) - p(n_2) - p(n_3)
\]
\[
= \frac{5}{2} n_1 + n_2)n_2 + n_3)(n_3 + n_1) \left( n_1^2 + n_2^2 + n_3^2 + \frac{6}{5} \beta \right) = H_0.
\]

It is noted that the resonance function (2.10) is identical to (2.2), since the first-order linear operator does not produce the dispersive effect as mentioned in Remark 2.2.

The standard Fourier restriction norm method ensures the local well-posedness of (1.1) in $H^s(\mathbb{T})$, $s \geq \frac{1}{2}$. It follows from the resonance and non-resonance estimates at such regularities, see Lemmas 3.1 and 3.2.

On the other hand, the main purpose of this paper (as seen in Section 1) is to show the well-posedness of (1.1) below $H^{s}(\mathbb{T})$. In view of Lemma 3.1 (compared to Lemma 3.2), one can see that the resonant term $\mathcal{N}_R(v)$ in (2.4) prevents the regularity threshold from going down below $\frac{1}{2}$.

Takaoka and Tsutsumi [68] introduced new idea to weaken the nonlinear perturbation of the form $in|\tilde{u}(n)|^2\tilde{u}(n)$ in the context of modified KdV equation. We briefly explain what the idea is. The evolution operator $\mathcal{V}(t)$ given by
\[
\mathcal{V}(t)v := \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} e^{inx + p(n) - \mu \int_0^t |\tilde{v}(s,n)|^2 \, ds} \tilde{v}(n)
\]
can completely remove whole non-trivial resonance $\mathcal{N}_R(v)$ in (2.4), while the nonlinear oscillation factor $e^{-i\mu \int_0^t |\tilde{v}(s,n)|^2 \, ds}$ itself is difficult to be dealt with, in particular, in the uniqueness part (see
Theorem 1.1 in [59] for the existence result), since the oscillation factor contains the solution to be estimated. Instead, by choosing the first approximation of \( \mathcal{V}(t) \) given by
\[
\mathcal{W}(t)u := \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} e^{i(nx+tp(n)+tn|\tilde{v}_0(n)|^2)}\tilde{v}(n),
\]
we further reduce (2.4) to
\[
\partial_\tau \tilde{v}(n) - i\tilde{p}(n)\tilde{v}(n) = -\mu \text{min}(|\tilde{v}(n)|^2 - |\tilde{v}_0(n)|^2)\tilde{v}(n) + \frac{\mu}{3} n \sum_{N_n} \tilde{v}(n_1)\tilde{v}(n_2)\tilde{v}(n_3),
\]
where
\[
\tilde{p}(n) = n^5 + \beta n^3 - (\gamma + \mu 2\pi||v_0||^{2} n) - \mu n|\tilde{v}_0(n)|^2.
\]
Then, authors proved (in the context of modified KdV) a kind of smoothing effect (to control the reduced resonant term \( n(|\tilde{v}(n)|^2 - |\tilde{v}_0(n)|^2) \)), and thus showed the local well-posedness below \( H^s(\mathbb{T}) \).

**Remark 2.4.** One can immediately check that the resonance function (2.10) is roughly bounded above by \( \max(|n_1|^3, |n_2|^3, |n_3|^3, |n|^3)^4 \) on the set \( N_n \). On the other hand, \( |n||\tilde{v}_0(n)|^2 \) is much less than \( |n|^3 \), when \( u_0 \in H^s(\mathbb{T}) \) for \( -1 < s \). These observations ensure that the new resonant function generated by \( \tilde{p}(n) \) shows the same effect as (2.10) in the analysis, in other words, the factor \( n|\tilde{v}_0(n)|^2 \) is negligible compared to (2.10) for \( s > -1 \).

Unfortunately, we were able to obtain \( L^4 \) Strichartz estimate in the \( X^{s,b} \) space associated to (2.12) for \( s > 0 \), in other words, the first approximation operator (2.11) seems prevent us from getting \( L^4 \)-strichartz estimate in \( L^2(\mathbb{T}) \). It becomes another enemy to obtain the global well-posedness of (1.1) in \( L^2(\mathbb{T}) \), in a sharp contrast to others [68, 59, 55, 56, 62, 47].

An alternative way to capture Takaoka-Tsutsumi’s idea is to define the nonlinear transform, which was used in author’s previous works [46, 48] in the context of the fifth-order KdV and modified KdV equations, respectively (see also [67]), in order to control non-trivial resonances. We define the nonlinear transform by
\[
u(t, x) := \mathcal{N}\mathcal{T}(u)(t, x) := \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} e^{i(nx-\mu t|\tilde{v}_0(n)|^2)}\tilde{v}(t, n).
\]
Using the nonlinear transform (2.13), one finally reduces (2.4) to
\[
\partial_\tau \tilde{u}(n) - i\tilde{p}(n)\tilde{u}(n) = -\text{min}(|\tilde{u}(n)|^2 - |\tilde{u}_0(n)|^2)\tilde{u}(n) + \frac{\mu}{3} n \sum_{N_n} e^{it\phi(u_0)}\tilde{u}(n_1)\tilde{u}(n_2)\tilde{u}(n_3),
\]
where \( \phi(u_0) = n|\tilde{u}_0(n)|^2 - n_1|\tilde{u}_0(n_1)|^2 - n_2|\tilde{u}_0(n_2)|^2 - n_3|\tilde{u}_0(n_3)|^2 \). We denote the resonant and the non-resonant terms in (2.14) by \( \mathcal{N}_R^*(u)(n) \) and \( \mathcal{N}_{NR}^*(u)(n) \), respectively.

We end this section with some remarks.

**Remark 2.5.** The oscillator \( e^{it\phi(u_0)} \) does not make any effect in our analysis in the sense of Remark 2.4. Moreover, one may remove it in all estimates, thanks to \( |e^{it\phi(u_0)}| \leq 1 \).

**Remark 2.6.** The nonlinear transform \( \mathcal{N}\mathcal{T}(u) \) in (2.13) plays the same role as the first approximation operator (2.11) in the sense that both weaken the non-trivial resonance \( \mathcal{N}_R(n) \). However, the bi-continuity property (indeed, the continuity property of the inverse transform) of (2.13) is necessary to close the argument for the well-posedness of (1.1), in order to use the nonlinear transform (2.13). See Lemma 2.4 for this analysis.

**Remark 2.7.** The key in the reduction of the non-trivial resonance in not only [68, 59], but also here, is that the (reduced) resonant term, in particular, \( n(|\tilde{u}(n)|^2 - |\tilde{u}_0(n)|^2) \), has a smoothing effect (see Corollary 4.1). Indeed, using (2.14), one has
\[
n(|\tilde{u}(n)|^2 - |\tilde{u}_0(n)|^2) = 2i\text{Im} \int_0^t \sum_{N_n} e^{is\phi(u_0)}n^2\tilde{u}(s, n_1)\tilde{u}(s, n_2)\tilde{u}(s, n_3)\tilde{u}(s, -n) \, ds.
\]
The smoothing effect occurs due to the highly non-resonant structure, stronger than the loss of regularities in (2.15).

---

4This upper bound is the weakest dispersive effect arising in the high-high-high to high interactions.
Remark 2.8. The $\mathcal{F}L^1$-smoothing estimate lose the (logarithmic) derivative compared to the $\mathcal{F}L^\infty$-smoothing effect. In other word, in [68, 59], the $\mathcal{F}L^1$-smoothing estimate to control $n(|\tilde{u}(n)|^2 - |\tilde{u}_0(n)|^2)$ has been shown for $\frac{1}{4} < s < \frac{3}{4}$, in the context of modified KdV equation, see also [55, 62]. In contrast with this, the $\mathcal{F}L^\infty$-smoothing effect (Corollary 4.1) holds even in the end point regularity, see also [47, 56]. Among other works, this observation is significant and the $\mathcal{F}L^\infty$-smoothing effect is essential in this work in the sense that we obtain the local well-posedness in $L^2$, and so global well-posedness in $L^2$. This observation may recover the lack of the well-posedness at the end point regularity in [68, 59, 55, 62].

2.2. Function spaces. We, in this section, introduce the $X^{s,b}$ space, which was first proposed by Bourgain [6] to solve the periodic NLS and generalized KdV. Later, for three decades, many mathematicians, in particular, Kenig, Ponce and Vega [40] and Tao [69], have further developed.

The Sobolev space $H^s(\mathbb{T})$, $s \in \mathbb{R}$, is known to be equipped with norm

$$\|f\|_{H^s(\mathbb{T})}^2 = \frac{1}{2\pi} \int_{\mathbb{T}} |\partial_x f(t)|^2 \, dt = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} |n|^{2s} |\hat{f}(n)|^2,$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$.

Define the $X^{s,b}$ space as the closure of Schwartz functions $S_{t,x}(\mathbb{R} \times \mathbb{T})$ under the norm

$$\|f\|_{X^{s,b}}^2 = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} |n|^{2s} |\tau - p(n)|^{2b} |\hat{f}(\tau,n)|^2 \, d\tau,$$

which is equivalent to the expression $\|e^{it\xi - i\xi|\cdot|}f(t,x)\|_{H^s_tH^b_x}$.

Let $T > 0$. The time localization of $X^{s,b}$ denoted by $X^{s,b}_T$ is given by

$$X^{s,b}_T = \left\{ f \in D'((-T,T) \times \mathbb{T}) : \|f\|_{X^{s,b}_T} < \infty \right\},$$

equipped with the norm

$$\|f\|_{X^{s,b}_T} = \inf \{ \|g\|_{X^{s,b}} : g \in X^{s,b}, g \equiv f \text{ on } (-T,T) \}.$$

For a cut-off function $\psi$ given by

$$\psi \in C^\infty_0(\mathbb{R}) \text{ such that } 0 \leq \psi \leq 1, \quad \psi \equiv 1 \text{ on } [-1,1], \quad \psi \equiv 0, \ |t| \geq 2,$$

we fix the time localized function

$$\psi_T(t) = \psi(t/T), \quad 0 < T < 1.$$

The following lemma provides the selective properties of $X^{s,b}$ space.

Lemma 2.1 (Properties of $X^{s,b}$, [21, 70]). We have

1. (Embedding) Let $s \in \mathbb{R}$ and $b > \frac{1}{2}$. Then, for any $u \in X^{s,b}(\mathbb{R} \times \mathbb{T})^5$, we have

$$\|v\|_{C^b_tL^\infty_x(\mathbb{R} \times \mathbb{T})} \lesssim_b \|v\|_{X^{s,b}}.$$

2. ($X^{s,b}$ energy estimate) Let $s \in \mathbb{R}$, $b > \frac{1}{2}$, $-\frac{1}{2} < b' < 0 < b < b' + 1$ and $0 < T \leq 1$. Then, for any functions $v \in S_{t,x}(\mathbb{R} \times \mathbb{T})$ satisfying (1.1), we have

$$\|\psi_Tv\|_{X^{s,b}_T} \lesssim_{\psi,b} T^{\frac{1}{4} - b} \|v_0\|_{H^s_x} + T^b \|N_R(v) + N_{NR}(v)\|_{H^{1/2} \times H^{1/2}}.$$

2.3. Basic estimates. This section devotes to the introduction of some lemmas, which will be essentially used for our analysis.

Lemma 2.2 ($L^4$-Strichartz estimate). For $b \geq \frac{3}{10}$, we have

$$\|f\|_{L^4_tL^8_x(\mathbb{R} \times \mathbb{T})} \lesssim \|f\|_{X^{0,b}(\mathbb{R} \times \mathbb{T})}$$

for any Schwartz function $f$ on $\mathbb{R} \times \mathbb{T}$.

Proof. The $L^4$-type estimate was first introduced by Bourgain [6], for which the local well-posedness of periodic NLS and gKdV equations have been proved. The $L^4$ estimate plays an important role to compensate for the lack of smoothing effect under the periodic setting. The proof is analogous to one in [6] and further improved in many works. We leave the proof in Appendix A. \qed

5One can extend the domain $\mathbb{R} \times \mathbb{T}$ to $\mathbb{R} \times \mathbb{Z}$, where $\mathbb{Z} = \mathbb{R}^d$ or $\mathbb{T}^d$, $d \geq 1$. 
Lemma 2.3 (Sobolev embedding). Let $2 \leq p < \infty$ and $f$ be a smooth function on $\mathbb{R} \times T$. Then for $b \geq \frac{1}{2} - \frac{1}{p}$, we have

$$
\|f\|_{L_t^p(H^b_x)} \lesssim \|f\|_{X^{s,b}}.
$$

(2.16)

When $p = \infty$, the usual Sobolev embedding ($b > \frac{1}{2}$) holds.

Proof. The proof directly follows from the Sobolev embedding with respect to the temporal variable $t$. For $S(t)f(t,x) = F^{-1}(e^{it\mu(n)}F(f, n))$, we know $\|S(-t)f\|_{H_x^s} = \|f\|_{H_x^s}$. Thus,

$$
\|f\|_{L_t^p(H_x^b)} = \|\|S(-t)f\|_{H_x^s}\|_{L_t^p} \lesssim \|S(-t)f\|_{H_x^s} = \|f\|_{X^{s,b}},
$$

which completes the proof. \qed

The following lemma shows the bi-continuity of the nonlinear transform defined as in (2.13), which guarantees the equivalence the local-well-posedness between (1.1) and (2.14).

Lemma 2.4. Let $s \geq 0$ and $0 < T < \infty$. Then, $N^T(v)$ defined as in (2.13) is bi-continuous from a ball in $C([-T,T];H^s(\mathbb{T}))$ to itself.

Proof. The proof is analogous to one in [48]. It suffices to show the continuity of $N^T^{-1}$, since the converse is similar and easier. Suppose that $u_k \in C_TH^s$ converges to $u$ in $C_TH^s$ as $k \to \infty$. The claim is to show $v_k = N^T^{-1}(u_k) \to N^T^{-1}(u) = v$ in $C_TH^s$, as $k \to \infty$.

We fix $s \geq 0$, $0 < T < \infty$ and $t \in [-T,T]$, and assume that $\|u_k\|_{L_T^\infty H^s}, \|u\|_{L_T^\infty H^s} \leq K$, for some $K > 0$. A direct calculation gives

\begin{align*}
\tilde{v}_k(n) - \tilde{v}(n) &= e^{i\mu t n} \tilde{u}_k(n) - e^{i\mu t n} \tilde{u}(n) \\
&= e^{i\mu t n} \left( e^{i\mu t n} \tilde{u}(n) - 1 \right) \tilde{u}_k(n) \\
&\quad + e^{i\mu t n} \left( e^{i\mu t n} \tilde{u}(n) - 1 \right) \tilde{u}_k(n)
\end{align*}

for $n \neq 0$, which imply

\begin{align*}
\|v_k(t) - v(t)\|_{H^s}^2 &\leq |\tilde{u}_k(0) - \tilde{u}(0)|^2 \\
&\quad + 2^{s+1} \sum_{|n| \geq 1} |n|^{2s} \left| e^{i\mu t n} \tilde{u}(n) - 1 \right|^2 |\tilde{u}_k(n)|^2 \\
&\quad + 2^{s+1} \sum_{|n| \geq 1} |n|^{2s} |\tilde{u}_k(n) - \tilde{u}(n)|^2.
\end{align*}

(2.17) (2.18) (2.19)

We remark that $\tilde{u}_{k,0}(n) = \tilde{u}_{k,0}(n)$ and $\tilde{u}_0(n) = \tilde{u}_0(n)$ for all $n \in \mathbb{Z}^6$.

The standard $\epsilon$-$\delta$ argument for the continuity of functions (in particular, $e^{i\theta}$ at $\theta = 0$) says that for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$
|e^{i\theta} - 1| < \frac{\epsilon}{2^{s+1} \cdot \sqrt{6K}},
$$

(2.20)

whenever $|\theta| < \delta$.

On the other hand, the boundedness of $H^s$-norm ensures that there exists $M > 0$ such that

$$
\sum_{|n| > M} |n|^{2s} |\tilde{u}(n)|^2 < \frac{\epsilon^2}{2^{s+1} \cdot 24}.
$$

(2.21)

Moreover, from the convergence assumption $u_k \to u$ in $C_TH^s$ as $k \to \infty$, we can choose $N \gg 1$ such that

$$
\|u_k - v\|_{L_T^\infty H^s} < \min \left( \frac{\epsilon}{2^{s+1} \cdot 3}, \frac{\delta}{2MTK} \right)
$$

(2.22)

for all $k \geq N$. In what follows, we fix $k \geq K$.

Thanks to (2.22), we control (2.17) and (2.19), that is to say

$$
2^{s+1} \sum_{|n| \geq 0} |n|^{2s} |\tilde{u}_k(n) - \tilde{u}(n)|^2 < \frac{\epsilon^2}{3}
$$

(2.23)

---

\footnote{This fact make the proof of Lemma 2.4 be easier than one in [48].}
On the other hand, a direct calculation, in addition to (2.22), gives
\[
T \left| |\hat{u}_{k,0}(n)|^2 - |\hat{u}_0(n)|^2 \right| \leq T |\hat{u}_{k,0}(n) - \hat{u}_0(n)| (|\hat{u}_{k,0}(n)| + |\hat{u}_0(n)|) \\
\leq 2TK|\hat{u}_{k,0}(n) - \hat{u}_0(n)| \leq \frac{\delta}{M},
\] (2.24)
for \( s \geq 0 \).

We divide the summation in (2.18) into
\[
\sum_{1 \leq |n| \leq M} + \sum_{|n| > M}.
\]
Over the first summation, thanks to (2.24), we have
\[
|\mu nt (|\hat{u}_{k,0}(n)|^2 - |\hat{u}_0(n)|^2)| < \delta
\]
which, in addition to (2.20), implies
\[
\left| e^{i\mu nt(|\hat{u}_{k,0}(n)|^2 - |\hat{u}_0(n)|^2)} - 1 \right|^2 < \frac{\varepsilon^2}{2^{s+1} \cdot 6K^2}.
\]
Over the second summation, a trivial bound
\[
\left| e^{i\mu nt(|\hat{u}_{k,0}(n)|^2 - |\hat{u}_0(n)|^2)} - 1 \right|^2 \leq 4,
\]
and (2.21) imply
\[
2^{s+1} \sum_{|n| > M} |n|^{2s} \left| e^{i\mu nt(|\hat{u}_{k,0}(n)|^2 - |\hat{u}_0(n)|^2)} - 1 \right|^2 |\hat{v}_k(n)|^2 < \frac{\varepsilon^2}{6}.
\]
Summing all, one has
\[
2^{s+1} \sum_{|n| \geq 1} |n|^{2s} \left| e^{i\mu nt(|\hat{u}_{k,0}(n)|^2 - |\hat{u}_0(n)|^2)} - 1 \right|^2 |\hat{v}_k(n)|^2 < \frac{\varepsilon^2}{3}, \tag{2.25}
\]
Collecting all (2.23) and (2.25), one concludes
\[
\|u_k - u\|_{H^s} < \varepsilon,
\]
which completes the proof of Lemma 2.4. \( \square \)

3. Trilinear estimates

In this section, we establish the trilinear estimates, which is the main task in the Fourier restriction norm method. We split the nonlinear estimate into two: Resonance and Non-resonance estimates.

**Lemma 3.1** (Resonance estimate). Let \( s \geq \frac{1}{2} \), \( \frac{1}{2} \leq b < 1 \) and \( 0 < \delta \leq 1 - b \). Let \( 0 \leq T \leq 1 \). Then we have
\[
\|\mathcal{N}_R(u_1, u_2, u_3)\|_{X_T^{s,b-1+s}} \lesssim \prod_{j=1}^{3} \|u_j\|_{X_T^{s,b}}. \tag{3.1}
\]
Proof. For the term \( n\hat{u}_1(n)\hat{u}_2(-n)\hat{u}_3(n) \), since
\[
\|n\hat{u}_1(n)\hat{u}_2(-n)\hat{u}_3(n)\|_{L^2} \lesssim \prod_{j=1}^{3} \|u_j\|_{H^s},
\]
for \( s \geq \frac{1}{2} \), we have from the Sobolev embedding (2.16) that
\[
\|F^{-1}_x[n\hat{u}_1(n)\hat{u}_2(-n)\hat{u}_3(n)]\|_{X_T^{s,b-1+s}} \lesssim \|F^{-1}_x[n\hat{u}_1(n)\hat{u}_2(-n)\hat{u}_3(n)]\|_{X_T^{0,0}} \lesssim \prod_{j=1}^{3} \|u_j\|_{L^2(H^s)} \lesssim \prod_{j=1}^{3} \|u\|_{X_T^{s,b}},
\]
which implies (3.1), we, thus, complete the proof. \( \square \)
In contrast to the resonance estimate, one can use the dispersive smoothing effect arising from the cubic non-resonant interactions. From the symmetry

\[ n_1 + n_2 + n_3 = n \quad \text{and} \quad \tau_1 + \tau_2 + \tau_3 = \tau, \]

we know

\[ (\tau_1 - p(n_1)) + (\tau_2 - p(n_2)) + (\tau_3 - p(n_3)) = \tau - p(n) + H. \]

Thus, we always assume that

\[ \max(|\tau_j - p(n_j)|, |\tau - p(n)|; j = 1, 2, 3) \gtrsim |H|. \] (3.2)

**Lemma 3.2** (Non-resonance estimate\(^7\)). Let \(-1/4 < s\) and \(0 \leq T \leq 1\). Then, there exist \(0 < \delta = \delta(s) \ll 1\) and \(\frac{1}{2} \leq b = b(s)\) such that the following estimate holds:

\[ \|N_{\mathcal{N}_H}(u_1, u_2, u_3)\|_{X_T^{s, b-1+\delta}} \lesssim \sum_{j=1}^3 \|u_j\|_{X_T^{s,b}}. \] (3.3)

**Proof.** Let \(L_j = \langle \tau_j - p(n_j) \rangle\) and \(L = \langle \tau - p(n) \rangle\). From the duality argument in addition to the Plancherel theorem, it suffices to show

\[ \int \sum_{n \in \mathcal{N}} \mathcal{M}_0 \hat{f}_1(n_1) \hat{f}_2(n_2) \hat{f}_3(n_3) \hat{g}(-n) \, dt \lesssim \prod_{j=1}^3 \|u_j\|_{X_T^{s,b}} \|g\|_{X_T^{0,1-b-\delta}}, \] (3.4)

where \(\mathcal{M} = \mathcal{M}(n_1, n_2, n_3, n)\) is a Fourier multiplier defined as

\[ \mathcal{M}_0(n_1, n_2, n_3, n) = |n| \langle n \rangle^{s} \langle n_1 \rangle^{-s} \langle n_2 \rangle^{-s} \langle n_3 \rangle^{-s}, \]

\[ n_* = \max(|n_1|, |n_2|, |n_3|, |n|), \quad \hat{f}_j(n_j) = \langle n_j \rangle^s \hat{u}_j(n_j) \quad \text{and} \quad \|g\|_{X_T^{0,1-b-\delta}} \lesssim 1 \text{ with } \hat{g} \geq 0. \]

**Remark 3.1.** When \(L \geq \max(L_1, L_2, L_3)\), thanks to the Sobolev embedding (Lemma 2.3) and \(L^4\) Strichartz estimate (Lemma 2.2), it suffices for (3.4) to show

\[ \int \sum_{n \in \mathcal{N}} \mathcal{M} \hat{f}_1(n_1) \hat{f}_2(n_2) \hat{f}_3(n_3) \hat{f}_4(-n) \, dt \lesssim \prod_{j=1}^4 \|f_j\|_{L_T^{4,s}} \|\mathcal{F}_3\|_{L_T^{4,s}} \|\mathcal{F}_4\|_{L_T^{4,s}}, \] (3.5)

where the multiplier \(\mathcal{M}\) is given by

\[ \mathcal{M}(n_1, n_2, n_3, n) = \frac{\langle n \rangle^{s} \langle n_1 \rangle^{-s} \langle n_2 \rangle^{-s} \langle n_3 \rangle^{-s}}{|n_1 + n_2 + n_3| \langle n_1 \rangle^{s} \langle n_2 \rangle^{s} \langle n_3 \rangle^{s} \langle n_1 \rangle^{s} \langle n_2 \rangle^{s} \langle n_3 \rangle^{s}} \] (3.6)

and \(\hat{f}_4(\tau, n) = \langle \tau - p(n) \rangle^{1-b-\delta} \hat{g}(\tau, n)\). Here, \(\hat{f}_3\) instead denotes \(\hat{f}_3(\tau, n) = \langle \tau - p(n) \rangle^{-\delta} \hat{f}_3(\tau, n)\) to estimate it in \(X_T^{0,\frac{1}{2}} \) \((\|f_3\|_{L_T^{4,s}} \lesssim \|u\|_{X_T^{s,\frac{1}{2}}})\). One can change the role of \(f_3\) into either \(f_1\) or \(f_2\) without loss of the dispersive smoothing effect \((\langle \tau_j - p(n_j) \rangle)\). On the other hand, when \(\max(L_1, L_2, L_3) \gg L, (3.4)\) follows from

\[ \int \sum_{n \in \mathcal{N}} \mathcal{M} \hat{f}_1(n_1) \hat{f}_2(n_2) \hat{f}_3(n_3) \hat{f}_4(-n) \, dt \lesssim \prod_{j=1}^4 \|f_j\|_{L_T^{4,s}} \|\mathcal{F}_3\|_{L_T^{4,s}} \|\mathcal{F}_4\|_{L_T^{4,s}}, \] (3.7)

if \(L_3 = \max(L_1, L_2, L_3)\), where, in this case, \(\hat{f}_3(\tau_3, n_3) = \langle n_3 \rangle^{s} \langle \tau_3 - p(n_3) \rangle^{b} \hat{u}_3\), \(\hat{f}_4(\tau, n) = \langle \tau - p(n) \rangle^{1-2b-\delta} \hat{g}\) and \(N_{\mathcal{N}_H} = \mathcal{M}\) is defined as in (3.6). The multiplier \(\mathcal{M}\) is still valid due to

\[ L_3^{-b} L^{-1+2b+\delta} \ll L_3^{-1+b+\delta} \lesssim |H|^{-1+b+\delta}. \]

Otherwise, the exact same computation gives

\[ \int \sum_{n \in \mathcal{N}} \mathcal{M} \hat{f}_1(n_1) \hat{f}_2(n_2) \hat{f}_3(n_3) \hat{f}_4(-n) \, dt \lesssim \|\mathcal{F}_1\|_{L_T^{4,s}} \|\mathcal{F}_2\|_{L_T^{4,s}} \prod_{j=3}^4 \|f_j\|_{L_T^{4,s}}, \] (3.8)

where, in this case \((L_2 = \max(L_1, L_2, L_3))\), \(\hat{f}_1(\tau_1, n_1) = \langle n_1 \rangle^{s} \langle \tau_1 - p(n_1) \rangle^{-\delta} \hat{u}_2\) and \(\hat{f}_2(\tau_2, n_2) = \langle n_2 \rangle^{s} \langle \tau_2 - p(n_2) \rangle^{-\delta} \hat{u}_2\), since

\[ L_2^{-b} \lesssim L_1^{-\delta} H^{-b+\delta} \lesssim L_1^{-\delta} H^{-1+b+2\delta}. \]

\(^7\)In view of the proof, one obtain Lemma 3.2 for \(N_{\mathcal{N}_H}(u_1, u_2, u_3)\) without any change, since the oscillation factor \(e^{it\phi(u_0)}\) could be removed, thanks to \(|e^{it\phi(u_0)}| \leq 1\).
One can switch the roles between \( f_1 \) and \( f_2 \), if \( L_1 = \max(L_1, L_2, L_3) \). In view of the proof below, no more assumption is needed for (3.5), (3.7) and (3.8), and thus it suffices to show (3.5) with \( \mathcal{M} \) as in (3.6).

From the definitions of \( f_j \), \( j = 1, 2, 3, 4 \), we may assume that all \( \tilde{f}_j \) are positive.

**Case I** (high \( \times \) high \( \times \) high \( \Rightarrow \) high, \( |n_1| \sim |n_2| \sim |n_3| \sim |n| \sim n_s \)). We may assume that \( 0 < |n_1 + n_2| \ll n_s \) without loss of generality, since otherwise, (3.5), (3.7) and (3.8) can be obtained with different pairs of functions in \( L_{1,x}^4 \) and \( L_{1,x}^\infty L_2^2 L_{1,x}^2 \) norms, i.e. \( (f_2, f_3) \) in \( L_{1,x}^4 \) and \( (f_1, f_4) \) in \( L_{1,x}^\infty L_2^2 L_{1,x}^2 \) or vice versa. In this case, the multiplier \( \mathcal{M} \) is bounded by

\[
\frac{1}{|n_1 + n_2|^{1 - b - 2c} n_s^{2 - 3b - 6\delta + 2s}}.
\]

For \( s > -\frac{1}{4} \), we choose \( \delta = \frac{4s + 1}{24} \). Then, for all \( \frac{1}{2} \leq b < \frac{2 + 4s - 6\delta}{3} \), we have

\[
\mathcal{M} \leq \frac{1}{|n_1 + n_2|^{3 \gamma - 8\delta + 2s}}.
\]

The change of variable \( (n' = n_1 + n_2) \) and the summation over \( n_2, n_3 \) yield

\[
\text{LHS of (3.5) } \leq \int \sum_{n', n_2, n_3 \in \mathbb{Z} \setminus \{0\}} \frac{1}{n' |n_1 - n_2|^{3 - 4b - 8\delta + 2s}} \tilde{f}_1 (n' - n_2) \tilde{f}_2 (n_2) \tilde{f}_3 (n_3) \tilde{f}_4 (-n' - n_3)
\]

\[
= \int \sum_{n' \in \mathbb{Z} \setminus \{0\}} \frac{1}{n' |n_1 - n_2|^{3 - 4b - 8\delta + 2s}} \tilde{f}_2 (n') \tilde{f}_3 \tilde{f}_4 (-n').
\]

For \( s > -\frac{1}{4} \), choose \( \delta = \min\left(\frac{4s + 1}{20}, \frac{1}{20}\right) \). Then, taking the \( \ell^\infty \)-norm at \( \tilde{f}_3 \tilde{f}_4 (-n') \) and the Cauchy-Schwarz inequality for the rest, one has

\[
\text{LHS of (3.5) } \leq \int \|f_1 f_2\|_{L^2} \|f_3 f_4\|_{L^1}
\]

\[
\leq \prod_{j=1}^2 \|f_j\|_{L_{1,x}^4} \|f_3\|_{L_{1,x}^\infty} \|f_4\|_{L_{1,x}^2},
\]

due to

\[
\sum_{n' \in \mathbb{Z} \setminus \{0\}} \frac{1}{n' |n_1 - n_2|^{3 - 4b - 8\delta + 2s}} < \infty
\]

for all \( \frac{1}{2} \leq b < \frac{5}{8} - 2\delta + \min\left(\frac{2s}{5}, 0\right) \).

The argument used in **Case I** could be applicable to the other cases. In what follows, we only point out the bound of the multiplier \( \mathcal{M} \), but omit the computation.

**Case II** (high \( \times \) high \( \times \) low \( \Rightarrow \) high, \( |n_1| \ll |n_2| \sim |n_3| \sim |n| \sim n_s \)). The choice of the minimum frequency \( n_1 \) does not lose of the generality in the proof below. In this case, \( \mathcal{M} \) is bounded by

\[
\frac{1}{|n_1 + n_2|^{5(1 - b - 2\delta) - 1 + s}},
\]

for \( s \geq 0 \), and

\[
\frac{1}{|n_1 + n_2|^{5(1 - b - 2\delta) - 1 + 2s}},
\]

for \( s < 0 \), due to \( |n_1 + n_2| \sim n_s \).

The same computation used in **Case I** is available, when \( 5(1 - b - 2\delta) - 1 + s \) for \( s \geq 0 \) and \( 5(1 - b - 2\delta) - 1 + 2s > \frac{1}{2} \) for \( s < 0 \). Thus, for \( s > -\frac{1}{2} \), we choose \( \delta = \min\left(\frac{2s + 1}{20}, \frac{1}{20}\right) \) such that (3.5) holds true for \( \frac{1}{2} \leq b < \frac{7}{16} - 2\delta + \min\left(\frac{2s}{5}, 0\right) \).

**Case III** (high \( \times \) high \( \times \) high \( \Rightarrow \) low, \( |n| \ll |n_1| \sim |n_2| \sim |n_3| \sim n_s \)). In this case, \( \mathcal{M} \) is bounded by

\[
\frac{1}{|n_1 + n_2|^{5(1 - b - 2\delta) - 1 + 2s}},
\]

for \( s \geq 0 \), and

\[
\frac{1}{|n_1 + n_2|^{5(1 - b - 2\delta) - 1 + 3s}},
\]
for $s < 0$, due to $|n_1 + n_2| \sim n_*$. For $s > -\frac{1}{3}$, choosing $\delta = \min(\frac{3s+1}{20}, \frac{1}{20})$, one shows (3.5) for $\frac{1}{2} \leq b < \frac{7}{10} - 2\delta + \min\left(\frac{20}{3}, 0\right)$ via the same computation used in Case I.

**Case IV** (high $\times$ low $\times$ low $\Rightarrow$ high, $|n_1|, |n_2| \ll |n_3| \sim |n| \sim n_*$). The choice of the maximum frequency $|n_3|$ is to ensure $0 < |n_1 + n_2| \ll n_*$, and it does not lose the generality, thanks to the same reason in Case I, where $0 < |n_1 + n_2| \ll n_*$ is to be supposed. In this case, $\mathcal{M}$ is bounded by

$$
\frac{1}{|n_1 + n_2|^{1-b-2\delta} n_*^{4(1-b-2\delta)-1}},
$$

for $s \geq 0$, and

$$
\frac{1}{|n_1 + n_2|^{1-b-2\delta} n_*^{4(1-b-2\delta)-1+2s}},
$$

for $s < 0$. For $s > -\frac{1}{4}$, by taking $\delta = \min\left(\frac{3s+1}{20}, \frac{1}{20}\right)$, we have (3.5) for all $\frac{1}{2} \leq b < \frac{7}{10} - 2\delta + \min\left(\frac{20}{3}, 0\right)$.

**Case V** (high $\times$ high $\times$ low $\Rightarrow$ low, $|n_1|, |n_3| \ll |n_2| \sim n_*$). Similarly as in Case IV, the choice of the minimum frequency $|n_3|$ is to ensure $0 < |n_1 + n_2| \ll n_*$, and hence it does not lose the generality. In this case, $\mathcal{M}$ is bounded by

$$
\frac{1}{|n_1 + n_2|^{1-b-2\delta} n_*^{4(1-b-2\delta)-1+s}},
$$

for $s \geq 0$, and

$$
\frac{1}{|n_1 + n_2|^{1-b-2\delta} n_*^{4(1-b-2\delta)-1+3s}},
$$

for $s < 0$. For $s > -\frac{1}{4}$, by taking $\delta = \min\left(\frac{3s+1}{20}, \frac{1}{20}\right)$, we have (3.5) for all $\frac{1}{2} \leq b < \frac{7}{10} - 2\delta + \min\left(\frac{20}{3}, 0\right)$.

Gathering all, for $s > -\frac{1}{4}$, we can choose $0 < \delta = \min\left(\frac{3s+1}{20}, \frac{1}{20}\right)$ such that (3.4) holds true for all $\frac{1}{2} \leq b < \frac{7}{10} + 2\delta + \min\left(\frac{20}{3}, 0\right)$.

**Remark 3.2.** For the positive regularity regime $s \geq 0$, one can choose $\delta > 0$ and $\frac{1}{2} \leq b$ independently on $s$ such that (3.3) holds true.

Using (2.14), we have

$$
\partial_t |\tilde{u}(t, n)|^2 = -2\mu \text{Im} \left[ \sum_{N_n} e^{i\phi(u_0)} \tilde{u}(t, n_1) \tilde{u}(t, n_2) \tilde{u}(t, n_3) \tilde{u}(t, -n) \right].
$$

(3.9)

An immediate corollary of Lemma 3.2, in addition to (3.9), is the following:

**Corollary 3.1.** Let $-1/4 < s$, $0 \leq T \leq 1$ and $t \in [-T, T]$. Suppose that $u$ is a real-valued smooth solution to (2.14) and $u \in X_T^{s, b}$. Then, there exists $\frac{1}{2} \leq b = b(s)$ such that the following holds true.

$$
\sum_{n \in \mathbb{Z}} (n)^{2s} |\tilde{u}(t, n)|^2 \lesssim \|u_0\|^2_{H^s} + \|u\|^2_{X_T^{s, b}}.
$$

(3.10)

4. **Smoothing effect**

4.1. **A priori bound.**

**Proposition 4.1.** Let $0 \leq s < \frac{1}{2}$ and $0 \leq T \leq 1$, $t \in [-T, T]$ and $u_0 \in C^\infty(\mathbb{T})$. Suppose that $u$ is a real-valued smooth solution to (2.14) and $u \in X_T^{s, b}$. Then the following estimate holds:

$$
\sup_{n \in \mathbb{Z}} \left| \text{Im} \left[ \int_0^t n^2 \sum_{N_n} e^{i\phi(u_0)} \tilde{u}(s, n_1) \tilde{u}(s, n_2) \tilde{u}(s, n_3) \tilde{u}(s, -n) \text{d}s \right] \right| \lesssim \|u_0\|^2_{H^s} + \left(\|u_0\|^2_{H^s} + \|u\|^2_{X_T^{s, b}}\right)^2 + \|u\|^4_{X_T^{s, b}} + \|u\|^6_{X_T^{s, b}}.
$$

(4.1)
Before proving Proposition 4.1, we introduce the projection operator $P_N$ as follows: For $N = 2^k$, $k \in \mathbb{Z}_{\geq 0}$, let

$$I_1 = [-1, 1] \quad \text{and} \quad I_N = [-2N, -N/2] \cup [N/2, 2N], \quad N \geq 2.$$ 

For a characteristic function $\chi_E$ on a set $E$, define $P_N$ by

$$\mathcal{F}_x[P_N f](n) = \chi_{I_N}(n) \tilde{f}(n).$$

We use the convention

$$P_{\leq N} = \sum_{M \leq N} P_M, \quad P_{> N} = \sum_{M > N} P_M.$$ 

Let $N \geq 1$ be given. We decompose a function $f$ into the following three pieces:

$$f = f_{\text{low}} + f_{\text{med}} + f_{\text{high}},$$

where $f_{\text{med}} = P_N f$, $f_{\text{low}} = P_{\leq N} f - f_{\text{med}}$ and $f_{\text{high}} = P_{\geq N} f - f_{\text{med}}$.

Proof of Proposition 4.1. The left-hand side of (4.1) bounded by

$$\sup_{N \geq 1} \sum_{n \in I_N} \left| \text{Im} \left[ \int_0^t n^2 \sum_{N_n} e^{i\phi(u_0)} \tilde{u}(s, n_1) \tilde{u}(s, n_2) \tilde{u}(s, n_3) \tilde{u}(s, -n) \, ds \right] \right|. \quad (4.2)$$

We deal with (4.2) by dividing into several cases.

**Case I.** $(\text{high} \times \text{high} \times \text{high} \Rightarrow \text{high})$ We consider the following term:

$$\sup_{N \geq 1} \sum_{n \in I_N} \left| \text{Im} \left[ \int_0^t n^2 \sum_{N_n} e^{i\phi(u_0)} \tilde{u}_{\text{med}}(s, n_1) \tilde{u}_{\text{med}}(s, n_2) \tilde{u}_{\text{med}}(s, n_3) \tilde{u}(s, -n) \, ds \right] \right|. \quad (4.3)$$

In order to control (4.3) in $L^2$ regularity level, it is required to use the Normal form reduction method.

Observe that\(^8\)

$$\partial_t \tilde{\varphi} := \partial_t \left( e^{-ip(n)} \tilde{u}(n) \right)$$

$$= e^{-ip(n)} \left( \partial_t \tilde{u}(n) - ip(n) \tilde{u}(n) \right)$$

$$= e^{-ip(n)} \left( -\min(|\tilde{u}(n)|^2 - |\tilde{u}_0(n)|^2) \tilde{u}(n) + \frac{\mu i}{3} n \sum_{N_n} e^{i\phi(u_0)} \tilde{u}(n_1) \tilde{u}(n_2) \tilde{u}(n_3) \right). \quad (4.4)$$

Taking the integration by parts with respect to the time variable $s$, one has

$$\int_0^t n^2 \sum_{N_n} e^{i\phi(u_0)} \tilde{u}(s, n_1) \tilde{u}(s, n_2) \tilde{u}(s, n_3) \tilde{u}(s, -n) \, ds$$

$$= \int_0^t n^2 \sum_{N_n} e^{-i\phi(H-\phi(u_0))} \tilde{\varphi}(s, n_1) \tilde{\varphi}(s, n_2) \tilde{\varphi}(s, n_3) \tilde{\varphi}(s, -n) \, ds$$

$$= -\sum_{N_n} \frac{n^2}{i(H-\phi(u_0))} e^{it\phi(u_0)} \tilde{u}(t, n_1) \tilde{u}(t, n_2) \tilde{u}(t, n_3) \tilde{u}(t, -n)$$

$$+ \sum_{N_n} \frac{n^2}{i(H-\phi(u_0))} \tilde{u}_0(n_1) \tilde{u}_0(n_2) \tilde{u}_0(n_3) \tilde{u}_0(-n)$$

$$+ \int_0^t \sum_{N_n} \frac{n^2 e^{-i\phi(H-\phi(u_0))}}{i(H-\phi(u_0))} \cdot \frac{d}{ds} \left( \tilde{\varphi}(s, n_1) \tilde{\varphi}(s, n_2) \tilde{\varphi}(s, n_3) \tilde{\varphi}(s, -n) \right) \, ds,$$

\(^8\)It is immediately known from (2.5) that $p(n)$ is the odd function, i.e. $p(-n) = -p(n)$. 
where $H$ is defined as in (2.10). Then, (4.3) is reduced as follows:

\[
(4.3) \leq \sup_{N \geq 1} \sum_{n \in I_N} \sum_{N_n} \frac{n^2}{|H - \phi(u_0)|} |\tilde{u}_{med}(t, n_1)\tilde{u}_{med}(t, n_2)\tilde{u}_{med}(t, n_3)\tilde{u}(t, -n)| \\
+ \sup_{N \geq 1} \sum_{n \in I_N} \sum_{N_n} \frac{n^2}{|H - \phi(u_0)|} |\tilde{u}_{0,med}(n_1)\tilde{u}_{0,med}(n_2)\tilde{u}_{0,med}(n_3)\tilde{u}_0(-n)| \\
+ \sup_{N \geq 1} \sum_{n \in I_N} \left| \int_0^t \sum_{N_n} \frac{n^2 e^{-i(H-H(u_0))}}{i(H-H(u_0))} \right|
\]

\[
\times \left| \frac{d}{ds} (\tilde{\varphi}_{med}(s, n_1)\tilde{\varphi}_{med}(s, n_2)\tilde{\varphi}_{med}(s, n_3)\tilde{\varphi}(s, -n)) \right| ds
\]

\[=: \Sigma_1 + \Sigma_2 + \Sigma_3.\]

From Remark 2.4, we know that $|H - \phi(u_0)| \sim |H|$, thus we drop $\phi(u_0)$ in $|H - \phi(u_0)|$ in the following analysis, for the sake of simplicity.

For $\Sigma_1$ and $\Sigma_2$, it suffices to show

\[
\sup_{N \geq 1} \sum_{n \in I_N} \left| \sum_{N_n} \frac{n^2}{H} \tilde{f}_{1,med}(n_1)\tilde{f}_{2,med}(n_2)\tilde{f}_{3,med}(n_2)\tilde{f}_{4,med}(-n_4) \right| \lesssim \prod_{j=1}^4 \|f_j\|_{H^s},
\]

for a certain regularity $s \in \mathbb{R}$.

Put $\tilde{g}_i(n) = \langle n \rangle^{s} |\tilde{f}_{i,med}(n)|$, $i = 1, 2, 3, 4$. We know from (2.10) that $|H| \gtrsim |(n_1 + n_2)(n_2 + n_3)| n_4^2$. We may assume $|n_3 + n_1| \sim N$. Since $|H|^{-1} \lesssim N^{-1}|n_1 + n_2|^{-1}$, a straightforward calculation (after the change of variable $n' = n_1 + n_2 \neq 0$) yields

\[
(4.6) \lesssim \sup_{N \geq 1} N^{-(1+4s)} \sum_{n_1, n_4 \neq 0} \frac{1}{|n'|} \tilde{g}_1(n_1)\tilde{g}_2(n' - n_1)\tilde{g}_3(n_4 - n')\tilde{g}_4(-n_4) \]

\[= \sup_{N \geq 1} N^{-(1+4s)} \sum_{0 < |n'| \leq N} \frac{1}{|n'|} \tilde{g}_1\tilde{g}_2(n')\tilde{g}_3\tilde{g}_4(-n') \]

\[\lesssim \sup_{N \geq 1} N^{-(1+4s)} \log N \|\tilde{g}_1\tilde{g}_2\|_{L^\infty} \|\tilde{g}_3\tilde{g}_4\|_{L^\infty} \]

\[\lesssim \prod_{j=1}^4 \|f_j\|_{H^s},
\]

whenever $1 + 4s > 0 \implies -1/4 < s < 1/2$, which, in addition to (3.10), implies

\[
\Sigma_1 + \Sigma_2 \lesssim \|u_0\|_{H^s}^4 + \|u(t)\|_{H^s}^4 \lesssim \|u_0\|_{H^s}^4 + \left(\|u_0\|_{H^s}^2 + \|u\|_{X^s_T, \frac{4}{7}}^2\right)^2,
\]

whenever $-1/4 < s < 1/2$.

Remark 4.1. In view of the computation (4.7), one can know that another choice of the assumption, $|n_1 + n_2| \sim N$ or $|n_2 + n_3| \sim N$, does not make a difference in the result. Hence our assumption does not lose the generality.

Remark 4.2. An analogous argument for the estimates of the boundary terms cannot be available in the uniqueness part, since Corollary 3.1 does not hold for the difference of two solutions (due to the lack of the symmetry), that is to say, the estimate

\[
\sum_{n \in \mathbb{Z}} (n)^{2s} |\tilde{w}(t, n)|^2 \leq C (\|u_1\|_{X^s_T, \frac{4}{7}}, \|u_2\|_{X^s_T, \frac{4}{7}}) \|w\|_{X^s_T, \frac{4}{7}}^2
\]

fails to hold for any $s \in \mathbb{R}$, when $u_1, u_2 \in X^s_T, \frac{4}{7}$ are solutions to (2.14) with $u_1(0) = u_2(0) = u_0$ and $w = u_1 - u_2$. However, the loss of regularity in (4.9) (see Lemma 4.1 below) is allowed in the estimate (4.7), hence we can completely circumvent "the lack of the symmetry" issue. Such an argument was used in author’s previous work [47]. See Proposition 4.2 below for more details.
For $\Sigma_3$, we take the time derivative in $\tilde{p}_{med}(s, n_1)$.

Remark that the estimates below are analogous for the case when we choose another frequency mode in which the time derivative is taken. Thus, we omit the other cases.

Using (4.4), one has

$$
\Sigma_3 \leq \sup_{N \geq 1} \sum_{n \in I_N} \int_0^T \sum_{N_n} \frac{n^2 |n_1|}{|H|} \left| \left( |\tilde{u}_{med}(n_1)|^2 - |\tilde{u}_{0,med}(n_1)|^2 \right) \tilde{u}_{med}(n_2) \tilde{u}_{med}(n_3) \tilde{u}(-n) \right| ds
\leq \sup_{N \geq 1} \sum_{n \in I_N} \int_0^T \sum_{N_n} \frac{n^2}{|H|} \left| P_N \left( \sum_{N_{n_1}} e^{i\tilde{f}(u_0)} \tilde{u}(n_{11}) \tilde{u}(n_{12}) \tilde{u}(n_{13}) \right) \tilde{u}_{med}(n_2) \tilde{u}_{med}(n_3) \tilde{u}(-n) \right| ds
= \Sigma_{3,1} + \Sigma_{3,2},
$$

where $\tilde{f}(u_0) = n_1 |\tilde{u}_0(n_1)|^2 - n_1 |\tilde{u}_0(n_{11})|^2 - n_1 |\tilde{u}_0(n_{12})|^2 - n_1 |\tilde{u}_0(n_{13})|^2$. Here the oscillation $e^{i\tilde{f}(u_0)}$ does not have any effect in the analysis.

For $\Sigma_{3,1}$, since

$$
|\tilde{u}_{med}(n_1)|^2 - |\tilde{u}_{0,med}(n_1)|^2 | \leq N^{-2s} \left( \|u_0\|_{H^s}^2 + \|u\|_{H^s}^2 \right),
$$

it suffices to deal with

$$
\sum_{n \in I_N \cup N_n} \frac{n^2 |n_1|}{|H|} \tilde{u}_{med}(n_1) \tilde{u}_{med}(n_2) \tilde{u}_{med}(n_3) \tilde{u}(-n).
$$

We assume from (3.2) that

$$
|\tau - p(n)| \gtrsim |H|.
$$

Define $g_t$ similarly as before ($\tilde{g}(u) = (n)^s |\tilde{u}_{med}(n)|$), and

$$
\tilde{h}(n) = \langle \tau - p(n) \rangle^{1/2} (n)^s |\tilde{u}(\tau, n)|,
$$

then, similarly as in (4.7), one has

$$
(4.11) \lesssim N^{-\frac{3}{2} + 4s} \sum_{n' \neq 0} \frac{1}{|n'|^{3/2} \tilde{g}(n')} \tilde{g}(-n')
\lesssim N^{-\frac{3}{2} + 4s} \|u\|_{H^s}^3 \|F^{-1}[\tau - \mu(n)]^{1/2} (n)^s \tilde{u}(n)\|_{L^2_x},
$$

Together with (4.10) and (4.14), in addition to the Sobolev embedding property (Lemma 2.3), we conclude that

$$
\Sigma_{3,1} \lesssim \left( \|u\|_{H^s}^3 \|u\|_{H^s}^3 \|u\|_{L^2_x} \right) \|F^{-1}[\tau - \mu(n)]^{1/2} (n)^s \tilde{u}(n)\|_{L^2_x}
\lesssim \left( \|u_0\|_{H^s}^3 + \|u\|_{H^s}^2 \right) \|F^{-1}[\tau - p(n)]^{1/2} (n)^s \tilde{u}(n)\|_{L^2_x}
$$

whenever $3/2 + 6s \geq 0 \Rightarrow -1/4 \leq s < 1/2$.

Remark 4.3. The estimate (4.14) above does not be affected by the choice of the maximum modulation (4.12), thus our choice, in addition to Remark 4.1, does not lose the generality.

For $\Sigma_{3,2}$, we further decompose distributed functions $\tilde{u}(n_{1,i})$ into $\tilde{u}_{low}(n_{1,i}), \tilde{u}_{med}(n_{1,i})$ and $\tilde{u}_{high}(n_{1,i}), i = 1, 2, 3$. Then, the followings are possible cases (up to the symmetry of frequencies):\(^9\)

$$
\begin{align*}
\tilde{u}_{med}(n_{11})\tilde{u}_{med}(n_{12})\tilde{u}_{med}(n_{13}), & \quad \text{(Case A)} \\
\tilde{u}_{med}(n_{11})\tilde{u}_{low}(n_{12})\tilde{u}_{med}(n_{13}), & \quad \text{(Case B-1)} \\
\tilde{u}_{med}(n_{11})\tilde{u}_{high}(n_{12})\tilde{u}_{high}(n_{13}), & \quad \text{(Case B-2)} \\
\tilde{u}_{low}(n_{11})\tilde{u}_{high}(n_{12})\tilde{u}_{high}(n_{13}), & \quad \text{(Case B-3)} \\
\tilde{u}_{med}(n_{11})\tilde{u}_{med}(n_{12})\tilde{u}_{low}(n_{13}), & \quad \text{(Case C-1)} \\
\tilde{u}_{high}(n_{11})\tilde{u}_{high}(n_{12})\tilde{u}_{high}(n_{13}). & \quad \text{(Case C-2)}
\end{align*}
$$

\(^9\)The cases A, B and C are referred as high-high-high-high, high-high-low-low and high-high-high-low, respectively.
Case A All comparable frequencies produce the new resonance in the quintic nonlinear interactions. The new resonant function defined by

$$\tilde{H} := p(n) - p(n_2) - p(n_3) - (p(n_{11}) + p(n_{12}) + p(n_{13}))$$

$$= \frac{5}{2} (n_1 + n_2)(n_2 + n_3)(n_3 + n_1) \left( n_1^2 + n_2^2 + n_3^2 + n^2 + \frac{6}{5} \beta \right)$$

$$+ \frac{5}{2} (n_1 + n_12)(n_12 + n_13)(n_13 + n_11) \left( n_{11}^2 + n_{12}^2 + n_{13}^2 + n_1^2 + \frac{6}{5} \beta \right) \tag{4.15}$$

vanishes when the frequencies satisfy, for instance,

$$n_1 = N + a, \quad n_2 = -N - a - b, \quad n_3 = N + b, \quad n_{11} = N + a + b, \quad n_{12} = -N - b, \quad n_{13} = N, \tag{4.16}$$

where $a, b \in \mathbb{Z}$ with $a \neq 0, b \neq 0$ and $a + b \neq 0$. Hence, we do not have any advantage from the dispersive smoothing effect

$$L_{\text{max}} := \max(|\tau_{1k} - p(n_{1k})|, |\tau_j - p(n_j)|, |\tau - p(n)|; k = 1, 2, 3, j = 2, 3) \gtrsim |\tilde{H}| \tag{4.17}$$

We assume $|n_1 + n_2| \sim N$. We further assume $L_{\text{max}} \neq |\tau_3 - p(n_3)|, |\tau_11 - p(n_{11})|$. In the case, we may choose $L_{\text{max}} = |\tau - p(n)|$ without loss of generality, since the following argument is to use a spare weight $(\tau - p(n))^{-2\epsilon}$ arising from the gap between $L^1_{t,x}$ and $X^6$, in order to avoid the logarithmic divergence coming from the Sobolev embedding. Let us define

$$\hat{f}(m) = (m)^\epsilon \tilde{u}_{\text{med}}(m), \quad \tilde{g}_-(m) = (\tau - p(m))^{-\epsilon} (m)^\epsilon \tilde{u}_{\text{med}}(m),$$

where $\epsilon > 0$ will be chosen later. Then it suffices to consider

$$N^{-6\epsilon} \sum_{n \in \mathcal{N}} \sum_{n_{11}, n_{12}} \frac{1}{|n_2 + n_3| n_3 + n_1} \tilde{g}_-(n_11) \hat{f}(n_{12}) \hat{f}(n_{13}) \tilde{f}(n_2) \tilde{g}_-(n_3) \tilde{g}_+(n),$$

since $|H| \gtrsim |n_1 + n_2||n_2 + n_3| n_3 + n_1|N^2$. We change the variables as follows:

$$n_2 = n_{12}, \quad n_3 = n_{11} - n_{12}, \quad n = n_{12} + n_{13} = n_{11} - n_{12},$$

which guarantees $n_1 + n_3 = n', n_2 + n_2 = n''$, and $n_2 + n_3 = n_2 + n_1 - n_{11}$. A direct calculation ensures

$$\leq N^{-6\epsilon} \sum_{n', n'', n_{11}, n_{12}} \frac{1}{|n'| n_2 + n' - n_{11} - n''}$$

$$\times \tilde{g}_-(n_{11}) \hat{f}(n_{12}) \hat{f}(n_{13}) \tilde{f}(n_2) \tilde{g}_-(n' - n_{11} - n'') \tilde{g}_+(n - n'') \tag{4.20}$$

$$= N^{-6\epsilon} \sum_{n', n'', n_{11}, n_{12}} \frac{1}{|n'| n_2 + n' - n_{11} - n''}$$

$$\times \tilde{g}_-(n_{11}) \hat{f}(n_{12}) \hat{f}(n_{13}) \tilde{f}(n_2) \tilde{g}_-(n' - n_{11} - n'') \tilde{g}_+(n - n'') \tag{4.22}$$

$$\leq N^{-6\epsilon} \sum_{n', n_{11}, n_{12}} \frac{1}{|n'|} \tilde{g}_-(n_{11}) \hat{f}(n_2) \tilde{g}_+(n - n') \left( \sum_{n''} |\hat{f}(n'')\tilde{g}_-(n' - n_{11} - n'')|^2 \right)^{\frac{1}{2}}$$

$$\leq N^{-6\epsilon} \sum_{n' \neq 0} \frac{1}{|n'|} \tilde{g}_-(n_{11}) \left( \sum_{n_{11}} |\tilde{g}_-(n_{11})|^2 \right)^{\frac{1}{2}} \left( \sum_{n_{11}, n''} |\hat{f}(n'')\tilde{g}_-(n' - n_{11} - n'')|^2 \right)^{\frac{1}{2}}$$

$$\leq N^{-6\epsilon} \left\| \tilde{g}_- \right\|_{L^2} \left\| f \right\|_{L^2} \left\| \tilde{g}_+ \right\|_{L^2}.$$
**Remark 4.4.** Another change of variables enables us to cover the rest of the modulation assumptions. Precisely, when $L_{\text{max}} = |\tau_3 - p(n_3)|$ or $|\tau_11 - p(n_{11})|$, the following change of variables ensures to estimate the Fourier coefficients at $n_3$ and $n_{11}$ modes in $L^4$ norm, and so \((4.23)\):

\[
\begin{align*}
n_2 &= n' - n_3, & n_3 &= n_3, & n &= n'' + n_{13} + n', \\
n_{11} &= n'' - n_{12}, & n_{12} &= n_{12}, & n_{13} &= n_{13}.
\end{align*}
\]

Under this change, additional weights are necessary for $\hat{u}(n)$ and $\hat{u}(n_{13})$.

**Remark 4.5.** An analogous computation ensures that the assumption $|n_1 + n_2| \sim N$ does not lose the generality. Indeed, performing (4.21) when $|n_2 + n_3| \sim N$ or

\[
\begin{align*}
n_2 &= n_2, & n_3 &= n' - n_2, & n &= n_{11} + n' + n', \\
n_{11} &= n_{11}, & n_{12} &= n_{12}, & n_{13} &= n'' - n_{12},
\end{align*}
\]

when $|n_3 + n_1| \sim N$, ensures (4.22), and hence we obtain (4.23).

**Remark 4.6.** The $FL_{\infty}$-smoothing estimate enables to achieve the local well-posedness at the endpoint regularity $s = 0$, while the $FL^1$-smoothing estimate holds on only sub-critical regularity regime $s > 0$. See [47, 56] and [68, 59, 55, 62] for the comparison. This approach is more important in this paper than others, since the local well-posedness in $L^2$ immediately ensures the global one, thanks to the conservation (1.3).

On the other hand, our proof fails to control $\Sigma_3.2$ below $L^2$, due to (4.16). See also Remark 3.2 in [59] for the similar phenomenon in the context of mKdV.

**Remark 4.7.** One can see that $\Sigma_3.2 \lesssim \|u\|_{X_{\tau}^{s,\frac{1}{2}}}^6$ holds for $s \geq 0$ not only under Case A, but also Cases B–C, since $(n)^{-s} \lesssim \min(1, N^{-s})$ holds for all cases, and the computation (4.22) does not depend on the frequency supports.

Let denote $\max(|n_{11}|, |n_{12}|, |n_{13}|)$ by $n^*$. Under Cases B–C, we have from (4.15) and (4.17) that

\[L_{\text{max}} \gtrsim |n_{11} + n_{12}| |n_{12} + n_{13}| |n_{13} + n_{11}| (n^*)^2 \gg N^3.\]

By Remark 4.7, we only consider the case when $s < 0$.

**Case B-1** It is enough to consider

\[
\sup_{N \geq 1} \sum_{n \in I_N} \sum_{N_r} \sum_{N_{12}} n^2 |n_1| \frac{|\hat{u}_{med}(n_{11})\hat{u}_{low}(n_{12})\hat{u}_{low}(n_{13})\hat{u}_{med}(n_2)\hat{u}_{med}(n_3)\hat{u}(-n)|}{|H|}.
\]

We assume that $|\tau - \mu(n)| = L_{\text{max}}$. Then, we know $(n_{1,k})^{-s} \lesssim N^{-s}$, $k = 1, 2, 3$, $|H| \gtrsim |n_1 + n_2| N^{4.10}$,

\[
L_{\text{max}} \lesssim \langle \tau_2 - p(n_2) \rangle^{-\frac{4}{7}} \langle \tau_3 - p(n_3) \rangle^{-\frac{4}{7}} \langle \tau_11 - p(n_{11}) \rangle^{-\frac{4}{7}} |n_{12} + n_{13}|^{-\frac{4}{7} + \delta} N^{-2+4\delta},
\]

for sufficiently small $\delta > 0$ to be chosen later, and $0 < |n_{12} + n_{13}| \leq N$. We use the same notations $f$, $g_-$ (but with a weight $|\tau - p(m)|^{-\frac{4}{7}}$) and $h$ as in (4.18), (4.19) and (4.13), respectively. An analogous argument as (4.22) gives

\[
N^{-2+6s+4\delta} \sum_{n,n',n''} \frac{1}{|n'\parallel n''|^{-1+\delta}} \times \hat{g}_-(n_{11}) \hat{f}(n_{12}) \hat{f}(n'' - n_{12}) \hat{g}_-(n' - n_{11} - n'') \hat{g}_-(n + n') \hat{h}(n) \approx \sup_{N \geq 1} N^{-2-6s+4\delta} \sum_{0 < |n'|, |n''| \leq N} \frac{1}{|n'| \parallel n''|^{-2-8} \hat{f}(n'' - n'') \hat{g}_-(n' - n'') \hat{g}_-(n + n') \hat{h}(n)} \lesssim \sup_{N \geq 1} N^{-2-6s+5\delta} \langle \log N \rangle \|g_2\|_{L^1} \|f\|_{L^2} \|g_2\|_{L^2} \|g_-\|_{L^2} \lesssim \|f\|_{L^2}^2 \|g_2\|_{L^2}\|g_-\|_{L^2}\|h\|_{L^2},
\]

for $-1/3 < s < 0$ and $\delta = \frac{2+4s}{10}$. Thus, Lemma 2.2 and the Sobolev embedding yield

\[
\Sigma_3.2 \lesssim \|f\|_{L^2}^2 \|g_-\|_{L^2}^\infty \|h\|_{L^2} \|h\|_{L^2} \lesssim \|u\|_{X_{\tau}^{s,\frac{1}{2}}}^6.
\]

\textsuperscript{10}We may assume that $0 < |n_1 + n_2| \leq N$ and $|n_2 + n_3| |n_3 + n_1| \geq N$, thanks to Remark 4.5.
A similar argument as in Remark 4.4 ensures that the assumption $L_{\text{max}} = |\tau - \mu(n)|$ does not lose the generality.

**Case B-2** One cannot apply (4.25) to
\[
\sup_{N \geq 1} \sum_{n \in I_N} \sum_{N_n, N_{n_1}} n^2 n_1 \frac{1}{H} \tilde{u}_{\text{med}}(n_1) \tilde{u}_{\text{high}}(n_12) \tilde{u}_{\text{high}}(n_13) \tilde{u}_{\text{med}}(n_2) \tilde{u}_{\text{med}}(n_3) \tilde{u}(-n),
\]
deceived to the logarithmic divergence appearing in the $\ell^2$-summation of $|n''|^{-\frac{1}{2}}$. In order to avoid it, we use a trick as follows: Given $-1/3 < s < 0$, let $\delta := \frac{2+6s}{12}$. We know
\[
\hat{L}_{\text{max}} \lesssim |n_{12} + n_{13}|^{-\frac{1}{2} - \delta n^* - 2+6\delta} \prod_{j=2}^{3} |\tau_j - p(n_j)|^{-\frac{5}{2}} \prod_{k=1}^{3} |\tau_{1k} - p(n_{1k})|^{-\frac{5}{2}}
\]
and
\[
(n_11)^{-s} (n_{12})^{-s} (n_{13})^{-s} n^* - 2+6\delta \lesssim N^{-2-3s+6\delta}.
\]
Then, the same argument as in (4.25) yields
\[
\text{(4.27)} \lesssim \sup_{N \geq 1} N^{-2-6s+6\delta} \sum_{0 < |n''| \leq N} \frac{1}{|n''|} \frac{1}{|n_0''|^{1/2} + \varepsilon} \hat{g}^2(n'') \hat{g}^2(n' - n'') \hat{g} h(-n')
\]
\[
\lesssim \sup_{N \geq 1} N^{-2-6s+6\delta} (\log N) ||g_2||_{L^1} ||g_2||_{L^2} ||g - h||_{L^1}
\]
\[
\lesssim ||g - h||_{L^1} ||g - h||_{L^2} ||h||_{L^2},
\]
which implies (4.26), whenever $-\frac{1}{3} < s < 0$.

An analogous argument used in **Case B-2** is still available to **Case B-3**. On the other hand, under **Case C**, one has $|\tilde{H}| \sim \langle n^* \rangle^5$ from (4.15), which is better than one in **Case B**. Hence, the same arguments used in **Case B-1** and **Case B-2** can be applied to **Case C-1** and **Case C-1**, respectively. Thus, we omit the details.

**Case II.** We deal with the high-low interactions. Thanks to the symmetry, we may assume that $|n_1| \geq |n_2| \geq |n_3|$. Moreover, it is enough to consider the high $\times$ low $\times$ low $\Rightarrow$ high interaction case, due to $n^2$ in (4.28) below (see also Remark 4.8). We first address the regularity $s > 0$. It suffices to estimate
\[
\sup_{N \geq 1} \sum_{n \in I_N} \sum_{N_n} n^2 |\tilde{u}_{\text{med}}(n_1) \tilde{u}_{\text{low}}(n_2) \tilde{u}_{\text{low}}(n_3) \tilde{u}(-n)|. \tag{4.28}
\]
We know from (2.10) that
\[
|\tilde{H}| \gtrsim |n_2 + n_3| N^4.
\]
Without loss of generality\textsuperscript{11}, we assume $|\tau - p(n)| \gtrsim |\tilde{H}|$. For
\[
L_{\text{max}} \lesssim \langle \tau_1 - p(n_1) \rangle^{-\delta} |n_2 + n_3|^{-\frac{1}{2} - \delta} N^{-2+4\delta}, \quad \delta = \frac{2s}{5}
\]
we use the notations $f$, $g_-$, and $h$ defined as in (4.18) and (4.13), respectively. For $s > 0$, we know $\langle n_j \rangle^{-s} \lesssim N^{-s}, j = 2, 3$. A straightforward computation, in addition to the change of variables $(n_2 + n_3 = n')$, gives
\[
\text{(4.28)} \lesssim \sup_{N \geq 1} N^{-4s} \sum_{n_1, n_2, n'} \frac{1}{|n'|^{1/2 - \delta}} \hat{g}_-(n_1) \hat{f}(n_2) \hat{f}(n' - n_2) \hat{h}(-n_1 - n')
\]
\[
= \sup_{N \geq 1} N^{-4s} \sum_{n' \neq 0} \frac{1}{|n'|^{1/2 - \delta}} \hat{h}(-n') \hat{f}^2(n') \tag{4.29}
\]
\[
\lesssim \sup_{N \geq 1} N^{-4s} \frac{||f^2||_{L^2} ||g_0||_{L^1}}{||g_0||_{L^2}},
\]
\textsuperscript{11}The estimate (4.29) below does not depend on the choice of the maximum modulation.
whenever \( s > 0 \). Thus, Lemma 2.2 and the Sobolev embedding ensure

\[
(4.2) \lesssim \|f\|_{L^2_{t,x}} \|g\|_{L^\infty_{t,x}} \|h\|_{L^2_{t,x}} \lesssim \|u\|^{4}_{X^0_T}.
\]

Remark 4.8. The high \( \times \) high \( \times \) low \( \Rightarrow \) low interaction case can be dealt with by (4.29), thanks to

\[
\langle n_1 \rangle^{-s} \langle n_2 \rangle^{-s} (n_1)^{-2} \lesssim N^{-2-2s},
\]

for \( s > 0 \).

Now we address the end point regularity \( s = 0 \). In this case, we cannot obtain

\[
\sup_{N \geq 1} \left| \sum_{n \in I_N} \int_0^t \sum_{N_n} e^{i s \phi(\ell_0)} n^2 \tilde{u}_{med}(n_1) \tilde{u}_{low}(n_2) \tilde{u}_{low}(n_3) \tilde{u}(-n) \, ds \right| \lesssim \|u\|^{4}_{X^0_T},
\]

due to the logarithmic divergence appearing in (4.29). To overcome it we again use the normal form reduction method. For the sake of simplicity, we do not push the regularity \( s \) below 0. Similarly as (4.5), we have

\[
\text{LHS of (4.30)} \leq \sup_{N \geq 1} \sum_{n \in I_N} \sum_{T_n} \frac{n^2}{H - \phi(\ell_0)} |\tilde{u}_{med}(t, n_1) \tilde{u}_{low}(t, n_2) \tilde{u}_{low}(t, n_3) \tilde{u}(t, -n)|
\]

\[
+ \sup_{N \geq 1} \sum_{n \in I_N} \sum_{T_n} \frac{n^2}{H - \phi(\ell_0)} |\tilde{u}_{0,med}(n_1) \tilde{u}_{0,low}(n_2) \tilde{u}_{0,low}(n_3) \tilde{u}_0(-n)|
\]

\[
+ \sup_{N \geq 1} \sum_{n \in I_N} \int_0^t \sum_{T_n} \frac{n^2 e^{-i s (H - \phi(\ell_0))}}{i (H - \phi(\ell_0))} \times \frac{d}{ds} (\tilde{\varphi}_{med}(s, n_1) \tilde{\varphi}_{low}(s, n_2) \tilde{\varphi}_{low}(s, n_3) \tilde{\varphi}(s, -n)) \, ds
\]

\[=: \Xi_1 + \Xi_2 + \Xi_3.\]

Remark that a direct computation gives

\[|H - \phi(\ell_0)| \gtrsim |n_2 + n_3| N^4,\]

which is stronger than one in Case I. Thus, \( \Xi_1 \) and \( \Xi_2 \) are controlled, similarly as the estimates of \( \Sigma_1 \) and \( \Sigma_2 \), by

\[\Xi_1 + \Xi_2 \lesssim \|u_0\|_{H^4}^4 + \|u(t)\|_{H^6}^4 \lesssim \|u_0\|_{H^4}^4 + (\|u_0\|_{H^6}^2 + \|u\|_{X^0_T}^4)^2.\]

Remark 4.2 is available to this estimate for the difference of two solutions.

Taking the time derivative to \( n_1 \) mode, one has

\[\Xi_3 \leq \sup_{N \geq 1} \sum_{n \in I_N} \int_0^t \sum_{T_n} \frac{n^2 |\ell_0|}{|H|} \left| (|\tilde{u}_{med}(n_1)|^2 - |\tilde{u}_{0,med}(n_1)|^2) \tilde{u}_{med}(n_1) \tilde{u}_{low}(n_2) \tilde{u}_{low}(n_3) \tilde{u}(-n) \right| \, ds
\]

\[+ \sup_{N \geq 1} \sum_{n \in I_N} \int_0^t \sum_{T_n} \frac{n^2 |\ell_0|}{|H|} \left| P_N \left( \sum_{n_{11} \neq n_{12}} e^{-i s \phi(\ell_0)} \tilde{u}(n_{11}) \tilde{u}(n_{12}) \tilde{u}(n_{13}) \right) \tilde{u}_{low}(n_2) \tilde{u}_{low}(n_3) \tilde{u}(-n) \right| \, ds
\]

\[=: \Xi_{3,1} + \Xi_{3,2},\]

Then, the same computation as in (4.29) (but, here \( \tilde{f}(n) = |\tilde{u}(n)| \)) yields

\[\Xi_{3,1} \lesssim \int_0^t (\|u\|_{L^2_T}^2 + \|u_0\|_{L^2_T}^2) \sup_{N \geq 1} N^{-1} \sum_{n_1, n_2, n_3 \neq 0} \frac{1}{|n'|} \tilde{f}(n_1) \tilde{f}(n_2) \tilde{f}(n_3) \, ds
\]

\[= \int_0^t (\|u\|_{L^2_T}^2 + \|u_0\|_{L^2_T}^2) \|f\|_{L^2_T}^2,
\]

which, in addition to Lemma 2.3, implies

\[\Xi_{3,1} \lesssim \left( \|u\|_{X^0_T}^4 + \|u_0\|_{L^2_T}^2 \right) \|u\|_{X^0_T}^4.
\]
On the other hand, one can split the frequency relation among \( n_{11}, n_{12} \) and \( n_{13} \) into Case A–Case C. Under the relation presented in Case B-1, one cannot have an additional smoothing effect \( (L_{\text{max}} \gtrsim |\hat{H}|) \) in \( \Xi_{3,2} \), where \( \hat{H} \) is defined as in (4.15), due to the same reason in \( \Sigma_{3,1} \) under Case A. However, (4.22) is still available for \( \Xi_{3,2} \) under Case B-1, thus we handle this case. In the other cases, the stronger \( |\hat{H}| \) and additional dispersive smoothing effects enable us to estimate \( \Xi_{4,2} \) more easily or similarly as the estimate of \( \Sigma_{3,2} \). Thus we have

\[
\Xi_{3,2} \lesssim \|u\|_{X_T^{s,\frac{3}{2}}}^6.
\]

Contributions from the time derivative taken in the other modes in \( \Xi_3 \) could be dealt with similarly or easily, due to \( |n_2|, |n_3| \ll |n_1| \sim |n| \). We skip the details. We remark that all computations established as in Case I are available for Case II, when \( s = 0 \).

The argument used in Case II always holds\(^{12}\) under the high \( \times \) high \( \times \) low \( \Rightarrow \) high\(^{13}\) interaction case, we hence obtain the same result as in Case II.

Gathering all results in Cases I, II and III, we complete the proof of (4.1). \( \square \)

As an immediate corollary, we have

**Corollary 4.1.** Let \( 0 \leq s < \frac{1}{2}, 0 \leq T \leq 1, t \in [-T, T] \) and \( u_0 \in C^\infty(\mathbb{T}) \). Suppose that \( u \) is a real-valued smooth solution to (2.14) and \( u \in X_T^{s,\frac{3}{2}}. \) Then the following estimate holds:

\[
\sup_{n \in \mathbb{Z}} |n| \left| \hat{u}(t,n) \right|^2 - \left| \hat{u}_0(n) \right|^2 \lesssim \|u\|_{H^s}^4 + \left( \|u_0\|_{H^s}^2 + \|u\|_{X_T^{s,\frac{3}{2}}}^4 \right)^2 + \|u\|_{X_T^{s,\frac{3}{2}}}^6.
\]

### 4.2. Difference of two solutions

Let \( u_1, u_2 \) be solutions to (2.14) with the same initial data \( u_1(0) = u_2(0) = u_0 \). Let \( w = u_1 - u_2 \). Then \( w \) satisfies

\[
\partial_t \hat{w}(n) - ip(n)\hat{w}(n) = - \frac{\mu_1}{2} (|\hat{u}_1(n)|^2 - |\hat{u}_0(n)|^2)\hat{w}(n) - \frac{\mu_2}{2} (|\hat{u}_2(n)|^2 - |\hat{u}_0(n)|^2)\hat{w}(n) + \frac{\mu}{3} n \sum_{N_n} e^{it\phi(u_0)} \hat{F}(u_1, u_2),
\]

where

\[
\hat{F}(u_1, u_2) = \hat{w}(n_1)\hat{u}_1(n_2)\hat{u}_1(n_3) + \hat{w}(n_2)\hat{u}_2(n_1)\hat{u}_1(n_3) + \hat{w}(n_1)\hat{u}_1(n_2)\hat{w}(n_3).
\]

Corollary 4.1 and Lemma 3.2 enable us to handle the first and third terms in the right-hand side of (4.32). Thus, it remains to control \( |n| \left| \hat{u}_1(n) \right|^2 - \left| \hat{u}_2(n) \right|^2 \) in the resonant terms. Using (3.9), one reduces to dealing with

\[
\int_0^t n^2 \sum_{N_n} e^{is\phi(u_0)} \left[ \hat{w}(n_1)\hat{u}_1(n_2)\hat{u}_1(n_3)\hat{u}_1(-n) + \hat{u}_2(n_1)\hat{w}(n_2)\hat{u}_1(n_3)\hat{u}_1(-n) + \hat{u}_2(n_1)\hat{w}_2(n_2)\hat{u}_2(n_3)\hat{w}(n_3) + \hat{w}(n_1)\hat{u}_2(n_2)\hat{w}(n_3)\hat{u}_1(-n) + \hat{u}_2(n_1)\hat{u}_2(n_2)\hat{w}_2(n_3)\hat{w}(n_3)\hat{u}_1(-n) + \hat{u}_2(n_1)\hat{w}_2(n_2)\hat{u}_2(n_3)\hat{w}(n_3) \right] ds.
\]

We, without loss of generality, choose the second term in (4.34) in order to state and prove the main proposition in this section.

**Proposition 4.2.** Let \( 0 \leq s < \frac{1}{2}, 0 \leq T \leq 1, t \in [-T, T] \) and \( u_0 \in C^\infty(\mathbb{T}) \). Suppose that \( u_1 \) and \( u_2 \) are a real-valued smooth solution to (2.14) with \( u_1(0) = u_2(0) = u_0 \) and \( u_1, u_2 \in X_T^{s,\frac{3}{2}}. \) Let \( w = u_1 - u_2 \). Then the following estimate holds:

\[
\left| \int_0^t n^2 \sum_{N_n} e^{is\phi(u_0)} \hat{w}(s, n_1)\hat{w}(s, n_2)\hat{u}_1(s, n_3)\hat{u}_1(s, -n) ds \right| \lesssim C(\|u_0\|_{H^s}, \|u_1\|_{X_T^{s,\frac{3}{2}}}, \|u_2\|_{X_T^{s,\frac{3}{2}}})\|w\|_{X_T^{s,\frac{3}{2}}}.
\]

\(^{12}\)A direct calculation for \( s > 0 \) and the normal form method for \( s = 0 \) are needed.

\(^{13}\)Remark 4.8 allows to deal with the high \( \times \) high \( \times \) high \( \Rightarrow \) low case by the same argument.
Remark 4.9. The proof of Proposition 4.2 basically follows from the proof of Proposition 4.1. The only difference is to estimate the boundary terms generated in the normal form process, as mentioned in Remark 4.2. We only point this difference out in the proof of Proposition 4.2 below.

In order to handle the difficulty arising from the lack of the symmetry, we need the following lemma:

Lemma 4.1. Let $0 \leq s < \frac{1}{2}$, $0 \leq T \leq 1$, $t \in [-T, T]$ and $u_0 \in C^\infty(T)$. Suppose that $u_1$ and $u_2$ are solutions to (2.14) on $[-T, T]$ with $u_{1,0} = u_{2,0}$, and $u, v \in X_s^{\frac{3}{2}}$. Let $w = u_1 - u_2$. Then the following estimate holds:

$$
\|w(t)\|^2_{H^{-\frac{1}{2}}} \lesssim (\|u_1\|^2_{X_s^1} + 2\|u_1\|_{X_s^1} \|u_2\|_{X_s^1} + \|u_2\|^2_{X_s^1})\|w\|^2_{X_s^1}.
$$

(4.35)

Proof. Using (4.32), a direct calculation gives

$$
\partial_t|\hat{w}(n)|^2 = -2\mu i \text{Im} [n\hat{u}_1(n)\hat{w}(-n)\hat{u}_2(n)\hat{w}(-n)]
+ 2\mu i \text{Im} \left[ n \sum_{N_n} e^{it\phi(u_0)} \hat{F}(u_1, u_2)\hat{w}(-n) \right]
=: A(t, n) + B(t, n),
$$

for $\hat{F}(u_1, u_2)$ as in (4.33). One immediately obtains

$$
\sum_{n \in \mathbb{Z}} \hat{u}_1(n)\hat{w}(-n)\hat{u}_2(n)\hat{w}(-n) \lesssim \|u_1(t)\|_{L^2} \|u_2(t)\|_{L^2} \|w(t)\|^2_{L^2}.
$$

Hence, the Hölder inequality and Lemma 2.3 yield

$$
\int_0^t \sum_{n \in \mathbb{Z}} (n)^{-1} A(s, n) \, ds \lesssim \|u_1\|_{L^1_tL^2} \|u_2\|_{L^1_tL^2} \|w\|^2_{L^1_tL^2}
\lesssim \|u_1\|_{X_s^1} \|u_2\|_{X_s^1} \|w\|^2_{X_s^1}.
$$

On the other hand, by Lemma 3.2, we have

$$
\int_0^t \sum_{n \in \mathbb{Z}} (n)^{-1} B(s, n) \, ds \lesssim \int_0^t \sum_{n \in \mathbb{Z}} B(s, n) \, ds
\lesssim (\|u_1\|^2_{X_s^1} + \|u_1\|_{X_s^1} \|u_2\|_{X_s^1} + \|u_2\|^2_{X_s^1})\|w\|^2_{X_s^1} .
$$

Collecting all, one proves (4.35).

Proof of Proposition 4.2. In view of the proof of Proposition 4.1, as mentioned again, our analysis does not rely on the symmetry of functions (or the structure of equation (2.14)), except for the estimate of the boundary term in the normal form process, in particular, an application of Corollary 3.1 in (4.8). Thus, we are going to show how to deal with this case compared to the estimates (4.7) and (4.8).

The normal form argument in addition to Remark 2.4 reduces to dealing with (see (4.5))

$$
\sup_{N \geq 1} \sum_{n \in I_N} \sum_{N_n} \frac{n^2}{|H|} |\tilde{u}_2(t, n_1)\hat{w}(t, n_2)\tilde{u}_1(t, n_3)\hat{u}_1(t, -n)|
+ \sup_{N \geq 1} \sum_{n \in I_N} \left| \int_0^t \sum_{N_n} \frac{n^2 e^{-is(H-\phi(u_0))}}{i(H-\phi(u_0))} \frac{d}{ds} \left( (e^{-isp(n_1)}\tilde{u}_2(s, n_1))(e^{-isp(n_2)}\hat{w}(s, n_2)) \times (e^{-isp(n_3)}\tilde{u}_1(s, n_3))(e^{isp(n)}\hat{u}_1(s, -n)) \right) \, ds \right|
=: \Sigma_1 + \Sigma_3.
$$

14The boundary term at $s = 0$ ($\Sigma_2$ in (4.5)) does not appear, due to $w(0, x) = 0$. 
where $H$ is defined as in (2.10), and $\tilde{u}_1$, $\tilde{u}_2$ and $\tilde{w}$ are supported in $I_N$. The estimate of $\tilde{\Sigma}_3$ is analogously dealt with as the estimate of $\Sigma_3$ in Case I in the proof of Proposition 4.1. Indeed, using (4.4) for $u_1$ and $u_2$, or

$$
\partial_t \left( e^{-ip(n)} \tilde{w}(n) \right)
= e^{-ip(n)} \left( \partial_t \tilde{w}(n) - ip(n) \tilde{w}(n) \right)
= -\frac{\mu in}{2} e^{-ip(n)} \left( \left| \tilde{u}_1(n) \right|^2 - \left| \tilde{u}_0(n) \right|^2 \right) \tilde{w}(n) + \left( \left| \tilde{u}_1(n) \right|^2 - \left| \tilde{u}_2(n) \right|^2 \right) \tilde{u}_2(n)
$$

$$
= \frac{\mu}{3} n \sum_{n_o} e^{ip(w)} \left[ \tilde{w}(n_1(n_2)\tilde{u}_1(n_3) + \tilde{u}(n_1)\tilde{w}(n_2)\tilde{u}_1(n_3) + \tilde{u}_1(n_1)\tilde{w}(n_2)\tilde{w}(n_3) \right],
$$

for $w$, one can apply the exact same arguments used in Case A–C to $\tilde{\Sigma}_3$ to obtain

$$\tilde{\Sigma}_3 \lesssim C(\|u_1\|_{X^{s,b}_T}, \|u_2\|_{X^{s,b}_T}) \|w\|_{X^{s,b}_T}. $$

Thus, it suffices to estimate $\tilde{\Sigma}_1$. Compared to (4.7), we perform an unfair distribution of derivatives to use Lemma 4.1. Let

$$
\hat{g}_1(n_1) = \|\tilde{u}_2(n_1)\|, \quad \hat{g}_2(n_2) = \langle n_2 \rangle^{-\frac{s}{2}} |\tilde{w}(n_2)|;
$$

$$
\hat{g}_3(n_3) = \|\tilde{u}_1(n_3)\|, \quad \hat{g}_4(-n) = |\tilde{u}_1(-n)|.
$$

We assume $|H|^{-1} \lesssim N^{-3}|n_1 + n_2|^{-1}$\footnote{This assumption does not lose the generality, see Remark 4.1.}. The change of variable $n' = n_1 + n_2 \neq 0$ and a direct computation yield

$$\tilde{\Sigma}_1 \lesssim \sup_{N \geq 1} N^{-\frac{3s}{2}} \sum_{n_1, n_4 \neq 0} \frac{1}{n'} \hat{g}_1(n_1) \hat{g}_2(n' - n_1) \hat{g}_3(n_4 - n') \hat{g}_4(-n_4)
$$

$$\lesssim \|u_2\|_{L^2} \|u_1\|_{X^{s,b}_T} \|w\|_{H^{-\frac{s}{2}}}. $$

Corollary 3.1 and Lemma 4.1 enable to estimate the last terms in (4.36), and hence we have

$$\tilde{\Sigma}_1 \lesssim C(\|u_0\|_{H^s}, \|u_1\|_{X^{s,b}_T}, \|u_2\|_{X^{s,b}_T}) \|w\|_{X^{s,b}_T}, $$

for $s \geq 0$. An analogous argument holds true for $\tilde{\Sigma}_1$, which can be similarly defined as in (3.31). Thus, it completes the proof of Proposition 4.2. \hfill \Box

From (4.34), one immediately has

**Corollary 4.2.** Let $0 \leq s < \frac{1}{2}$, $0 \leq T \leq 1$, $t \in [-T, T]$ and $u_0 \in C^\infty(T)$. Suppose that $u_1$ and $u_2$ are a real-valued smooth solution to (2.14) with $u_1(0) = u_2(0) = u_0$ and $u_1, u_2 \in X^{s,b}_T$. Let $w = u_1 - u_2$. Then the following estimate holds:

$$\sup_{n \in Z} |n| \left| \|\tilde{u}_1(n)\|^2 - \|\tilde{u}_2(n)\|^2 \right| \lesssim C(\|u_0\|_{H^s}, \|u_1\|_{X^{s,b}_T}, \|u_2\|_{X^{s,b}_T}) \|w\|_{X^{s,b}_T}. $$

**5. Global well-posedness in $L^2(T)$: Proofs of Theorems 1.1, 1.2 and 1.3**

**5.1. Short proof of Theorem 1.1.** Let $s \geq \frac{1}{2}$ be fixed. We recall the integral equation (1.6) associated to (1.1) in the Fourier space as follows:

$$\hat{v}(n) = e^{ip(n)} \hat{v}_0(n) + \int_0^t e^{i(t-s)p(n)} \left( \hat{N}_R(v) + \hat{N}_{NR}(v) \right) (s, n) ds, $$

for $\hat{N}_R(v)$ and $\hat{N}_{NR}(v)$ as in (2.8) and (2.9), respectively.

We denote by $\Gamma(v)$ the map defined as in (5.1) (after time localization). Then, Lemmas 2.1, 3.1 and 3.2 yield

$$\|\Gamma(v)\|_{X^{s,b}_T} \leq CT^{\frac{1}{2} - b} \|v_0\|_{H^s} + CT^b \|v\|_{X^{s,b}_T}^3
$$

and

$$\|\Gamma(v_1) - \Gamma(v_2)\|_{X^{s,b}_T} \leq CT^{\frac{1}{2} - b} \|v_0,1 - v_0,2\|_{H^s} + 2CT^b \left( \|v_1\|_{X^{s,b}_T}^2 + \|v_2\|_{X^{s,b}_T}^2 \right) \|v_1 - v_2\|_{X^{s,b}_T},$$
for some \(0 < \beta = \beta(s)\) and \(\frac{1}{2} < b = b(s)\) satisfying
\[
1 - 2b + \beta < 0.
\]
(5.2)

Remark that it is possible to choose \(b\) and \(\beta\) satisfying (5.2), see the proofs of Lemmas 3.1 and 3.2. Let \(\|u_0\|_{H^s} \leq R\), for a fixed \(R > 0\). Choosing \(T > 0\) satisfying
\[
16C^3 T^{1-2b+\beta} R^2 \leq \frac{1}{2},
\]
one can show that the map \(\Gamma\) is a contraction on the set
\[
\left\{ v \in X_T^{s,b} : \|v\|_{X_T^{s,b}} \leq 2CT^{1-b} R \right\},
\]
which completes the proof of Theorem 1.1.

5.2. Proof of Theorem 1.2. The proof of Theorem 1.2 is based on the standard energy method in addition to the bi-continuity of the nonlinear transform. We particularly follow the argument in [47]. We also refer to [68, 59, 48, 56] and references therein.

5.2.1. Existence. We first recall (2.14)
\[
\partial_t \tilde{u}(n) - ip(n) \tilde{u}(n) = -\mu in(|\tilde{u}(n)|^2 - |\tilde{u}_0(n)|^2)\tilde{u}(n) + \frac{\mu i}{3} \sum_{N_n} e^{it\phi(u_0)} \tilde{u}(n_1)\tilde{u}(n_2)\tilde{u}(n_3)
\]
\[
= \tilde{N}_R^s(u)(n) + \tilde{N}_{NR}^s(u)(n).
\]
(5.3)
The standard \(X_T^{s,b}\) analysis (after the time localization by multiplying by the smooth cutoff function, but dropping it) gives
\[
\|u\|_{X_T^{s,b}} \leq \|u_0\|_{H^s} + T^{\delta} \left( \|\tilde{N}_R(u)\|_{L_T^2 H^s} + \|\tilde{N}_{NR}^s(u)\|_{X_T^{s-b}}, s \right),
\]
for some \(0 < \delta \ll 1\). On one hand, Lemma 3.2 controls \(\|\tilde{N}_R^s(u)\|_{X_T^{s-b}}\) by
\[
\|u\|_{X_T^{s-b}} \leq C \|u_0\|_{H^s} \leq C \|u\|_{X_T^{s-b}} + \|u\|_{H^{s}}^\theta \leq \|u\|_{H^s}^{\theta} + \|u\|_{X_T^{s-b}}^\theta + \|u\|_{X_T^{s-b}}^\theta \leq \|u\|_{X_T^{s-b}}^\theta.
\]
On the other hand, a trivial estimate and Corollary 4.1 yield
\[
\|\tilde{N}_R^s(u)\|_{L_T^2 H^s} \leq \left( \sup_{t \in [-T,T]} \sup_{n \in Z} |\tilde{u}(n)| \|\tilde{u}(t,n)|^2 - |\tilde{u}_0(n)|^2 | \right) \|u\|_{L_T^2 H^s}
\]
\[
\leq T^\theta \left( \|u_0\|_{H^s}^4 + \|u_0\|_{H^s}^4 + \|u\|_{X_T^{s-b}}^{4} \right) \|u\|_{X_T^{s-b}}^\theta,
\]
for \(s \geq 0\) and for some \(\theta > 0\). Collecting all, we conclude
\[
\|u\|_{X_T^{s-b}} \leq C \|u_0\|_{H^s} + CT^{\theta} \left( \|u_0\|_{H^s}^4 + \|u_0\|_{H^s}^4 + \|u\|_{X_T^{s-b}}^{4}\right)^2
\]
\[
+ (1 + \|u\|_{X_T^{s-b}}^{3} + \|u\|_{X_T^{s-b}}^{3}) \|u\|_{X_T^{s-b}}^\theta,
\]
(5.5)
s \(\geq 0\) and for some \(C, \theta > 0\). Note that the constant \(C\) depends only on \(s\).

We fix \(0 \leq s < \frac{1}{2}\) and let \(T\) be a positive constant with \(T \leq 1\) to be determined later. Let \(u_0 \in H^s(\mathbb{T})\) be given. The density argument ensures the existence of a sequence \(\{u_{0,j}\} \subset C^\infty(\mathbb{T})\) such that \(u_{0,j} \rightarrow u_0\) in \(H^s\) as \(j \rightarrow \infty\). Choose \(K > 0\) such that
\[
\|u_{0,j}\|_{H^s}, \|u_0\|_{H^s} \leq K \quad \text{for all } j \geq 1.
\]
(5.6)
From Theorem 1.1 in addition to the energy conservation law (3.9), we have global solutions to (5.3) with initial data \(u_{0,j}\). Let
\[
X_j(T) = \|u_j\|_{X_T^{s-b}}(T > 0), \quad X_j(0) := \lim_{t \rightarrow 0^+} X_j(t).
\]
Then, (5.5) yields
\[
X_j(T) \leq CK + CT^\theta \left( K^4 + (K^2 + X_j(T)^4)^2 + (1 + X_j(T) + X_j(T)^3)X_j(T)^3 \right) X_j(T)
\]
for all $j \geq 1$. The continuity argument enables us to choose $0 < T \ll 1$ such that
\begin{equation}
X_j(t) \leq L
\end{equation}
for some $L = L(K) > 0$ and for $0 \leq t \leq T$. The choice of $T$ is independent on $j$, but dependent on $s, K$. We remark that the standard compactness argument concludes the existence of a weak limit in $X_T^{s, \frac{1}{2}}$.

To close the strong limit argument, we define the Dirichlet projection $\mathbb{P}_k$ for all positive integers $k$ by
\begin{equation*}
\mathbb{P}_k f = \frac{1}{\sqrt{2\pi}} \sum_{|n| \leq k} \hat{f}(n) e^{inx}.
\end{equation*}
Let $u_{j,k} = \mathbb{P}_k u_j$. Then $u_{j,k}$ satisfies
\begin{equation*}
\partial_t \hat{u}_{j,k}(n) - ip(n) \hat{u}_{j,k}(n) = \sum_{|n| \leq k} \left( \hat{N}_R^s(u_j)(n) + \hat{N}_{NR}(u_j)(n) \right),
\end{equation*}
with $u_{j,k}(0) = \mathbb{P}_k u_{0,j}$. Remark from (5.7) that
\begin{equation}
\|u_{j,k}\|_{X_T^{s, \frac{1}{2}}} \leq L, \quad j, k \geq 1.
\end{equation}

We are now ready to pass to the (strong) limit. Let $\varepsilon > 0$ be given. The proof of Proposition 4.1 in addition to (5.6) and (5.8) ensures
\begin{equation*}
\sum_{n \in I_N} |n| \left| \hat{u}_j(n) \right| \left| n \right|^2 \lesssim C(K, L) N^{-\gamma}, \quad N \in 2^{\mathbb{Z}_{\geq 0}},
\end{equation*}
for some $\gamma \geq 0$, which implies
\begin{equation}
\begin{split}
\| (I - \mathbb{P}_k) u_j \|_{C_T H^s} \lesssim & \sum_{|n| \geq k} \sum_{|n| < k} \left| \hat{u}_j(n) \right| \left| n \right|^2 \lesssim \sum_{|n| \geq k} |n|^2 \left| \hat{u}_j(n) \right| \left| n \right|^2 \\
\leq & C_0 \| (I - \mathbb{P}_k) u_{0,j} \|_{H^s} + k^{2s-1} C(K, L),
\end{split}
\end{equation}
for $C_0 > 0$ and where $I$ is the identity operator. On the other hand, the fact that $u_{0,j} \to u_0$, and (5.9) guarantee that there exists $M > 0$ such that for all $j \geq 1$,
\begin{equation}
\| (I - \mathbb{P}_k) u_j \|_{H^s} < \varepsilon
\end{equation}
holds true when $k > M$. Precisely, we can choose $N_0 > 0$ such that $\| u_0 - u_{0,j} \|_{H^s} < \varepsilon/(2\sqrt{2C_0})$ holds for $j > N_0$. Fix $N_0 > 0$. Then, the $H^s$-boundedness of $u_{0,j}$ ensures that for each $1 \leq j \leq N_0$, there exist $M_j > 0$, $j = 1, \ldots, N_0$ such that
\begin{equation}
\| (I - \mathbb{P}_k) u_{0,j} \|_{H^s} < \varepsilon/(\sqrt{2C_0}), \quad k > M_j, 1 \leq j \leq N_0.
\end{equation}
An analogous argument in addition to $s < \frac{1}{2}$ yields that there exists $M_0 > 0$ such that $k > M_0$ implies $\| (I - \mathbb{P}_k) u_0 \|_{H^s} < \varepsilon/(2\sqrt{2C_0})$ and $k^{2s-1} C(K, L) < \varepsilon^2/(2C_0)$. Let $M := \max(M_0, M_j : 1 \leq j \leq N_0)$ be fixed. Then, for $k > M$, we conclude $\| (I - \mathbb{P}_k) u_{0,j} \|_{H^s} \leq \frac{\varepsilon}{\sqrt{2C_0}}$ for all $j \geq 1$, thanks to (5.11) and
\begin{equation*}
\| (I - \mathbb{P}_k) u_{0,j} \|_{H^s} \leq \| u_{0,j} - u_0 \|_{H^s} + \| (I - \mathbb{P}_k) u_0 \|_{H^s},
\end{equation*}
which, in addition to (5.9), implies (5.10).

Arzelà-Ascoli compactness theorem and the diagonal argument yield that for each $\ell \geq 1$, there exists a subsequence $\{u_{j', k'}\} \subset \{u_{j,k}\}$ (denoted by $u_j$) such that
\begin{equation*}
\| \mathbb{P}_k (u_j - u_k) \|_{C([-T, T]; H^s)} \to 0, \quad j, k \to \infty,
\end{equation*}
holds. Therefore, we have a solution $u$ to (5.3) on $[-T, T]$ satisfying
\begin{equation}
\begin{split}
u \in & \ C([-T, T]; H^s) \cap X_T^{s, \frac{1}{2}}, \quad \| u \|_{X_T^{s, \frac{1}{2}}} \leq L, \quad \| u_j - u \|_{C([-T, T]; H^s)} \to 0, j \to \infty.
\end{split}
\end{equation}
5.2.2. Completion of the proof of Theorem 1.2: Uniqueness, continuity of the flow map and return to (1.1). Recall (4.32)
\[ \partial_t \hat{w}(n) - ip(n)\hat{w}(n) = \mathcal{N}_R^*(u_1, u_2, w)(n) + \mathcal{N}_N^*(u_1, u_2, w)(n), \]
for \( u_1 \) and \( u_2 \) are solutions to (5.3) satisfying (5.12) with initial data \( u_1(0) = u_2(0) = u_0 \), and \( w = u - v \). Here \( \mathcal{N}_R^*(u_1, u_2, w)(n) \) and \( \mathcal{N}_N^*(u_1, u_2, w)(n) \) are explicitly given by
\[ \mathcal{N}_R^*(u_1, u_2, w)(n) = -\mu \min \left( \left| \frac{(0)}{n} \right|^2 - |\hat{u}_0(n)|^2 \right) \hat{w}(n), \]
and
\[ \mathcal{N}_N^*(u_1, u_2, w)(n) = \frac{\mu \nu}{3} \sum_n e^{i\theta(n)} \hat{F}(u_1, u_2), \]
for \( \hat{F}(u_1, u_2) \) as in (4.33). Similarly as before, the standard \( X^{s,b} \) analysis yields
\[ \|u\|_{X_T^{s,\frac{1}{2}}} \lesssim \|u_0\|_{H^s} + T^s \left( \|\mathcal{N}_R^*(u_1, u_2, w)(n)\|_{L^2_T H^s} + \|\mathcal{N}_N^*(u_1, u_2, w)(n)\|_{X_T^{s,\frac{1}{2}}} \right). \]
Moreover, Corollaries 4.1 and 4.2 and Lemma 3.2 yield
\[ \|\mathcal{N}_R^*(u_1, u_2, w)(n)\|_{L^2_T H^s} \lesssim \left( \sup_{t \in [-T,T]} \sup_{n \in \mathbb{Z}} n \left| |\hat{u}_1(t, n)|^2 - |\hat{u}_0(n)|^2 \right| \right) \|u\|_{X_T^{s,\frac{1}{2}}} \]
\[ + \left( \sup_{t \in [-T,T]} \sup_{n \in \mathbb{Z}} n \left| |\hat{u}_1(t, n)|^2 - |\hat{u}_2(n)|^2 \right| \right) \|u_2\|_{X_T^{s,\frac{1}{2}}} \]
\[ \lesssim C_1(K, L) \|u\|_{X_T^{s,b}}, \]
and
\[ \|\mathcal{N}_N^*(u_1, u_2, w)\|_{X_T^{s,\frac{1}{2}+\frac{s}{2}}} \lesssim C_2(L) \|u\|_{X_T^{s,\frac{1}{2}}}, \]
respectively.

Collecting all and choosing \( T' > 0 \) sufficiently small (if necessary), one concludes
\[ \|w\|_{X_T^{s,\frac{1}{2}}} \leq c \|w\|_{X_T^{s,\frac{1}{2}}}, \]
for some \( 0 < c < 1 \), which implies \( w \equiv 0 \) on \([-T', T']\). The iterative process ensures the uniqueness of the solution on \([-T, T]\).

The proof of the continuous dependence of the flow map from initial data to solutions is analogous to the proof of the existence of solutions. The uniqueness of solutions ensures that all convergent subsequences in the sense of (5.12) have the same limit, thus complete this part.

Lastly, Lemma 2.4 ensures that the local-well-posedness of (5.3) implies the local well-posedness of (1.1), which completes the proof of Theorem 1.2. In Appendix B, we see that the oscillation factor \( e^{-|\mu| t |\hat{v}_0(n)|^2} \) in the nonlinear transform (or the resonance \( \mathcal{N}_R(v) \) as in (2.8)) indeed causes the weak ill-posedness, for \( 0 \leq s \leq \frac{1}{2} \).

5.3. Global well-posedness: Proof of Theorem 1.3. The global well-posedness of (1.1) immediately follows from the conservation law
\[ \int_T v^2(t, x) \, dx = \int_T v^2(0, x) \, dx. \tag{5.13} \]
The conserved quantity (5.13) ensures \( v(T) = v(0) \), where \( T > 0 \) is chosen in the local theory. Thus, we repeat the local theory on \([T, 2T]\) and further, we, then, obtain the global well-posedness.

6. Unconditional uniqueness: Proof of Theorem 1.5

The aim is to prove that \( X_T^{0,\frac{1}{2}} \) space designed as a solution space in the previous section is large enough to contain \( C([0,T]; H^s(\mathbb{T})) \), \( s > \frac{1}{2} \) such that the uniqueness in \( X_T^{0,\frac{1}{2}} \) ensures the unconditional uniqueness in \( H^s \), \( s > \frac{1}{2} \) via the interpolation argument.

We start with an essential nonlinear estimate in \( H^{-s} \).
Lemma 6.1. Let $s > \frac{1}{2}$ and $u \in C([0, T]; H^s(\mathbb{T}))$ be a solution to (5.3). Then we have
\begin{equation}
\|\mathcal{N}_R^s(u)\|_{L^\infty_T H^{s}} \lesssim \|u\|^3_{L^\infty_T H^s}
\end{equation}
and
\begin{equation}
\|\mathcal{N}_R^s(u)\|_{L^\infty_T H^{s}} \lesssim \|u\|^3_{L^\infty_T H^s}.
\end{equation}
Proof. A direct computation gives
\[ \text{LHS of (6.1)} \lesssim \|n|^{-s}(|\hat{u}(n)|^2 - |\hat{v}_0(n)|^2)\|_{L^2} \lesssim \|u\|^3_{L^\infty_T H^s(\mathbb{T})}, \]
for $s \geq \frac{1}{2}$. On the other hand, by the duality argument, one reduces the left-hand side of (6.2) as
\begin{equation}
\sum_{n, N_n} n^{1-s} e^{i\phi(n)} \hat{u}(n_1) \hat{u}(n_2) \hat{u}(n_3) \hat{g}(-n),
\end{equation}
for $g \in L^2$ with $\|g\|_{L^2} \leq 1$. Without loss of generality, we assume $|n_1| \leq |n_2| \leq |n_3|$. We split the summation over frequencies into several cases.

Case I. (high × high × high ⇒ high). We further assume that $|n_1| \sim |n_3| \sim |n|$. Then, the Cauchy-Schwarz inequality yields
\begin{equation}
\lesssim \sum_{n, n_1, n_2 \mid n_1 \sim |n_2| \sim |n|} |n|^{1-4s} \hat{f}(n_1) \hat{f}(n_2) \hat{f}(n_3) \hat{g}(-n)|
\lesssim \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} |n|^{3-8s} \right)^{\frac{1}{2}} \|f\|_{L^2}^3 \|g\|_{L^2}
\lesssim \|u\|^3_{L^\infty_T H^s},
\end{equation}
for $s > \frac{1}{2}$, where $\hat{f}(n) = \langle n \rangle^s |\hat{u}(n)|$.

Case II-a. (low × high × high ⇒ high). We assume that $|n_1| \ll |n_2| \sim |n_3| \sim |n|$. A similar argument yields
\begin{equation}
\lesssim \sum_{n_1 \in \mathbb{Z} \setminus \{0\}} \left( \sum_{n_2 \ll |n|} |n|^{2-6s} \right)^{\frac{1}{2}} |n_1|^{-s} \hat{f}(n_1) \|f\|_{L^2}^3 \|g\|_{L^2}
\lesssim \left( \sum_{n_1 \in \mathbb{Z} \setminus \{0\}} |n_1|^{3-8s} \right)^{\frac{1}{2}} \|u\|^3_{L^\infty_T H^s},
\end{equation}
for $s > \frac{1}{2}$, which implies the right-hand side of (6.2).

Case II-b. (high × high × high ⇒ low). Under the condition $|n| \ll |n_1| \sim |n_3|$, an analogous argument ensures
\begin{equation}
\lesssim \sum_{n_1, n_2 \mid n \ll |n_1| \sim |n_2|} |n|^{1-s} |n_1|^{-3s} \hat{f}(n_1) \hat{f}(n_2) \hat{f}(n_3) \hat{g}(-n)|
\lesssim \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} \left( \sum_{n_1 \ll |n|} |n_1|^{-6s} \right)^{\frac{1}{2}} |n|^{1-s} \|g(-n)\|_{L^2} \|f\|_{L^2}^3 \right)
\lesssim \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} |n|^{3-8s} \right)^{\frac{1}{2}} \|u\|^3_{L^\infty_T H^s} \|g\|_{L^2},
\end{equation}
for $s > \frac{1}{2}$, which implies the right-hand side of (6.2).
Case III-a. (low × low × high ⇒ high). We assume that \( |n_1| \leq |n_2| \ll |n_3| \sim |n| \). Since \( |n|^{1-2s} \leq |n|^2 \), The Cauchy-Schwarz inequality shows

\[
(6.3) \lesssim \sum_{|n_1| \leq |n_2| \ll |n|} \sum_{n_1,n_2} \left| n_2 \right|^{1-3s} |n_1|^{-s} \hat{f}(n_1) \hat{f}(n_2) \hat{f}(n_3) \hat{g}(-n) \\
\lesssim \sum_{n \in \mathbb{Z} \setminus \{0\}} \left( \sum_{|n_1| \leq |n_2|} |n_2|^{2-6s} \right)^{\frac{1}{2}} |n_1|^{-s} \hat{f}(n_1) \| f \|_{L^2} \| g \|_{L^2} \\
\lesssim \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} |n_1|^{-3-8s} \right)^{\frac{1}{s}} \| u \|_{L^3_t H^s},
\]

for \( s > \frac{1}{2} \), which implies the right-hand side of (6.2).

Case III-b. (low × high × high ⇒ low). Using the same argument as before with the fact \( |n_2|^{-2s} \leq \max(|n_1|, |n|)^{-2s} \), one obtains the right-hand side of (6.2). We omit the details and complete the proof.

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. Let \( T > 0 \) be given and \( u \in C([0, T]; H^s(\mathbb{T})) \), \( s > \frac{1}{2} \) be a solution to (5.3). Then, a straightforward calculation

\[
\| u \|_{X^s_T} = \| u \|_{L^2_T H^s} \leq T^{\frac{1}{2}} \| u \|_{L^\infty_T H^s} \tag{6.4}
\]

ensures that the solution \( u \) belongs to \( X^s_T \).

On the other hand, Lemma 6.1 in addition to (6.4) reveals that

\[
\mathcal{N}_T^s(u) + \mathcal{N}_{NR}^s(u) \in X^{-s,0}_T,
\]

which implies \( u \) belongs to \( X^{-s,1}_T \) thanks to (5.3).

The interpolation theorem for \( X^s_T \) with \( X^{-s,1}_T \) ensures \( u \in X^{0,\frac{s}{2}}_T \), that is, the space \( C_T H^s \) is embedded in the space \( X^{0,\frac{s}{2}}_T \), \( s > \frac{1}{2} \). Therefore, the uniqueness result in \( X^{0,\frac{s}{2}}_T \) established in the previous section (a part of Theorem 1.2) guarantees the uniqueness in \( C_T^\infty \cap H^s \), which completes the proof of Theorem 1.5.

APPENDIX A. \( L^4 \)-STRICHARZ ESTIMATES

The aim of this appendix is to provide the proof of \( L^4 \)-Strichartz estimates (Lemma 2.2) for the sake of reader’s convenience.

Proof of Lemma 2.2. The idea of proof follows the proof of Proposition 2.13 in [70] associated the Airy flow. We also refer to [6, 40, 69, 68, 61, 47, 10] for similar arguments. Let \( M \) be a dyadic number and split

\[
f = \sum_M f_M,
\]

where \( f_M \) be a modulation localized portion on the region \( M \leq \langle \tau - p(n) \rangle < 2M \). Then, we have from the Plancherel’s theorem that

\[
\sum_M M^{\frac{2}{3}} \| f_M \|_{L^4_{t,x}}^2 \leq \| f \|_{X^{0,\frac{1}{3}}_{t,x}}^2.
\]

Furthermore, by triangle inequality, we also obtain

\[
\| f \|_{L^4_{t,x}}^2 = \| f \|_{L^2_{t,x}}^2 \lesssim \sum_M \| f_M \|_{L^2_{t,x}}^2 \lesssim \sum_{m=0}^\infty \sum_M \| f_M \|_{L^2_{t,x}}^2 \lesssim \sum_{m=0}^\infty \sum_M \| f_M \|_{L^4_{t,x}}^2 \lesssim \sum_{m=0}^\infty \sum_M \| f_M \|_{L^2_{t,x}}^2,
\]

and hence it suffices to show that

\[
\| f_M f_{2^m M} \|_{L^2_{t,x}} \lesssim 2^{-\frac{4m}{3}} M^{\frac{2}{3}} \| f_M \|_{L^2_{t,x}} \| f_{2^m M} \|_{L^2_{t,x}}, \tag{A.1}
\]

since taking Cauchy-Schwarz inequality with respect to \( M \) first, and performing summation on \( m \) for \( 2^{-\frac{4m}{3}} \) yield our conclusion. Let \((\tau_1, n_1)\) and \((\tau_2, n_2)\) be the space-time Fourier variables of \( f_M \) and \( f_{2^m M} \), respectively.
We first consider \( |n| \ll (2^m M)^{\frac{1}{2}} \). We use Plancherel theorem, Minkowski inequality in \( \tau \) and Cauchy-Schwarz inequality in \( \tau_1 \) and \( n_1 \) to obtain that
\[
\| f_M f_{2^m M} \|_{L^2_{t,x}} \lesssim \left\| \sum_{n_1} \int_{\tau_1 = p(n_1) + O(M)} f_M(\tau_1, n_1) f_{2^m M}(\tau - \tau_1, n - n_1) \, d\tau_1 \right\|_{L^2_t L^2_x},
\]
\[
\lesssim M^{\frac{1}{2}} \sum_n \left\| f_M(n) \right\|_{L^2_t} \left\| f_{2^m M}(n - n_1) \right\|_{L^2_t} \lesssim (2^m M)^{\frac{1}{2}} M^{\frac{1}{2}} \| f_M \|_{L^2_{t,x}} \| f_{2^m M} \|_{L^2_{t,x}},
\]
which implies exactly (A.1).

Now, we focus on the case \( |n| \gtrsim (2^m M)^{\frac{1}{4}} \). From the Plancherel' theorem, Cauchy-Schwarz inequality and the relation between support of \( f_M \) and that of \( f_{2^m M} \), we have
\[
\| f_M f_{2^m M} \|_{L^2_{t,x}} \lesssim \sum_n \int_{\tau_1 = p(n_1) + O(M)} f_M(\tau_1, n_1) f_{2^m M}(\tau_2, n_2) \, d\tau_1 \left\| f_M \right\|_{L^2_t} \| f_{2^m M} \|_{L^2_t} \lesssim (2^m M)^{\frac{1}{4}} M^{\frac{1}{4}} \left\| f_M \right\|_{L^2_{t,x}} \| f_{2^m M} \|_{L^2_{t,x}}.
\]
Hence, it suffices to show for fixed \( n, \tau \) that
\[
\sum_{n_1 + n_2 = n} \int_{\tau_1 = \tau + \tau_2} 1 \, d\tau_1 \lesssim 2^{\frac{1}{2}m} M^{\frac{1}{2}}. \tag{A.2}
\]
To do this, we need to count the number of \( n_1 \) under the condition
\[
\tau = \tau_1 + \tau_2, \quad \tau_1 = p(n_1) + O(M), \quad \tau_2 = p(n_2) + O(2^m M).
\]
Let \( X = (n_1 - \frac{n}{2})^2 \). Then, a direct calculation gives
\[
p(n_1) + p(n - n_1) = 5n \left( X + \left( \frac{n^2}{4} + \frac{3\beta}{10} \right) \right)^2 - n^5 - \frac{3\beta n^3}{2} - (\gamma + \mu \| v_0 \|_{L^2}^2 + \frac{9\beta^2}{20} n),
\]
which, in addition to \( |n| \gtrsim (2^m M)^{\frac{1}{4}} \), implies
\[
\left( X + \left( \frac{n^2}{4} + \frac{3\beta}{10} \right) \right)^2 = \frac{\tau}{5n} + n^4 \frac{1}{20} + \frac{3\beta n^2}{10} + \frac{1}{5} (\gamma + \mu \| v_0 \|_{L^2}^2 + \frac{9\beta^2}{20}) + O((2^m M)^{\frac{1}{4}})
\]
\[
\Rightarrow \left( n_1 - \frac{n}{2} \right)^2 = C(\tau, n) + O((2^m M)^{\frac{1}{4}}).
\]
This observation says that \( n_1 \) is contained in at most two intervals of length \( O((2^m M)^{\frac{1}{4}}) \) except for \( \tau_1 = p(n_1) + O(M) \) and the summation on \( n_1 \) under above observation in the left-hand side of (A.2) yield our conclusion (A.1). \( \square \)

**Appendix B. Weak ill-posedness in \( H^s(\mathbb{T}) \), \( 0 \leq s < \frac{1}{2} \): Proof of Theorem 1.4**

As mentioned in Section 1, the proof of Theorem 1.4 closely follows Takaoka and Tzutsumi [68], initially motivated by Burq, Gérard and Tzvetkov [9] (for the Schrödinger case) and Christ, Colliander and Tao [14] (not only for the Schrödinger but also KdV cases).

We fix \( 0 \leq s < \frac{1}{2} \). Let \( K \) be a positive integer and set
\[
\theta_K(x) = \frac{e^{iKx} - e^{-iKx}}{2i\sqrt{\pi}}.
\]
We choose initial data \( v_0, v_{0,K} \) and \( v_{0,K}^* \) as follows:
\[
v_{0,K}(x) = K^{-s} \theta_K(x) \quad \text{and} \quad v_{0,K}^*(x) = K^{-s} (1 + 2\pi K^{2s-1+i\theta}) \theta_K(x) = (1 + 2\pi K^{2s-1+i\theta}) \theta_K(x),
\]
for $0 < \vartheta < 1 - 2s$. A straightforward calculation gives
\[
\|v_{0,K}\|_{H^s} \leq 1
\]
and
\[
\|v_{0,K} - v^*_{0,K}\|_{H^s} \lesssim 1 - (1 + 2\pi K^{2s-1+\vartheta})^{1/2} \to 0, \quad \text{as } K \to \infty.
\]
Taking $t_K = K^{-\vartheta}$, one has
\[
t_K|\tilde{v}_{0,K}(n)|^2 = \frac{1}{2} (\delta_{nK} + \delta_{n-K}) K^{1-2s-\vartheta} \quad \text{and} \quad t_K|\tilde{v}_{0,K}(n)|^2 = \frac{1}{2} (\delta_{nK} + \delta_{n-K}) (K^{1-2s-\vartheta} + \pi),
\]
where $\delta_{ij}$ is well-known Kronecker delta.

Theorem 1.2 ensures that there exist $T > 0$ and solutions $v_K$ and $v^*_K$ to (1.1) on $[-T,T]$. We take $K$ sufficiently large such that $t_K < T$. Set for $a \in \mathbb{C}$
\[
\Phi(K) := K^5 + \beta K^3 - \gamma K.
\]
Define
\[
\tilde{w}_K(n) = e^{i\Phi(K) - \mu n} |\tilde{v}_{0,K}(n)|^2 - e^{i\Phi(K) - \mu n} |\tilde{v}^*_{0,K}(n)|^2 \tilde{v}^*_K(n)
\]
\[= e^{i\Phi(K) t} (\tilde{u}_{0,K}(n) - \tilde{u}^*_{0,K}(n)),
\]
where $u$ is the transformed form from $v$ by $N/T$ as in (2.13). Note that a direct computation gives
\[
\|w_K(t_K)\|_{H^s}^2 = 1 - e^{i\pi} (1 + 2\pi K^{2s-1+\vartheta})^{1/2} \geq 4, \quad K \geq 1. \tag{B.1}
\]
Suppose that the uniform continuity of the flow map holds true, which implies
\[
\sup_{t \in [-T,T]} \|v_K(t) - v_K^*(t)\|_{H^s} \to 0, \quad \text{as } K \to \infty. \tag{B.2}
\]
The duhamel formula form (2.14) yields
\[
(v_K(t_K) - v_K^*(t_K), w_K(t_K))_{H^s} = \|w_K(t_K)\|_{H^s}
\]
\[+ \left( \int_0^{t_K} e^{i\Phi(K)(t_K - s)} (N^*_R(u_K) + N^*_R(u^*_K) - N^*_R(u_K) - N^*_R(u^*_K)), w_K(t_K) \right)_{H^s}, \tag{B.3}
\]
where $(\cdot, \cdot)_{H^s}$ is an usual $H^s$ inner product. The standard argument under the periodic setting, in addition to Lemma 3.2 and Proposition 4.1 with the uniform boundedness of $u$ in $X^{s,2}_T$, ensures the second term in the right-hand side of (B.3) (is bounded by $\|v_{0,K}^\theta\|_{H^s}$ for some $\theta > 0$ similarly as the right-hand side of (5.5) without the initial part term, and hence) tends to $0$ as $K \to \infty$. Using (B.1) and (B.2) in (B.3), one concludes the contradiction, which ends the proof.

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Another auxiliary space based on $\ell^2_T L^4_x$ is necessary in order to recover the lack of the embedding property ($X^{s,2} \not\hookrightarrow C_t H^s$). Such a space has a property (duality), in our case,
\[
\left\| \int_0^T e^{i(\gamma - p(s)) w(s)} ds \right\|_{H^s} \lesssim \|\langle n \rangle^{s} (\gamma - p(n))^{-1} \tilde{w} \|_{\ell^2_T L^4_x}.
\]
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