A-HYPERGEOMETRIC SYSTEMS AND RELATIVE COHOMOLOGY

TSUNG-JU LEE AND DINGXIN ZHANG

Abstract. We investigate the space of solutions to certain $A$-hypergeometric $\mathcal{D}$-modules, which were defined and studied by Gelfand, Kapranov, and Zelevinsky. We show that the solution space can be identified with certain relative cohomology group of the toric variety determined by $A$, which generalizes the results of Huang, Lian, Yau, and Zhu. As a corollary, we also prove the existence of rank one points for Calabi–Yau complete intersections in toric varieties.

Contents

0. Introduction 1
1. $A$-hypergeometric systems 2
2. Functors on $\mathcal{D}$-modules 7
3. Fourier–Laplace transform and Dwork cohomology 8
4. The transition lemma 12
5. Proof of the main theorem 13
References 16

0. Introduction

The purpose of this note is to give a cohomological interpretation of the space of solutions to certain $A$-hypergeometric systems defined and studied by Gelfand–Kapranov–Zelevinsky, which generalizes a result in Huang–Lian–Yau–Zhu. Precisely, the theorem of Huang et. al. works for the $A$-hypergeometric systems associated with Calabi–Yau hypersurfaces $Y_a$, where $a$ are parameters living in a vector space $V^\vee$, in a smooth projective toric variety $X \subset \mathbb{P}(V^\vee)$ of dimension $n$. The solution spaces of the $A$-hypergeometric system with parameter $(-1,0,\cdots,0)$ near $a$ are identified with the relative homology group $H_n(X - Y_a, (X - Y_a) \cap D)$, where $D$ is the union of all toric divisors.

In this note, we relax both the Calabi–Yau condition and the smoothness assumption. Also we generalize their result to the situation of complete intersections in toric varieties. See §1.4 for the precise statement.

This generalization is desirable, because most toric varieties one meets in applications and computations are singular. For example, the hypotheses of the theorem of Huang et. al. are not satisfied by the “mirror projective space”, i.e., the projective...
toric variety of dimension $n$ defined by the convex hull of
\[ e_i = (\delta_{1i}, \ldots, \delta_{ni}) \text{ and } (-1, \ldots, -1). \]
Thus, one already needs the full strength of (1.6) to relate the solutions to the $A$-hypergeometric system and the cohomological objects attached to the so-called “mirror Calabi–Yau spaces”, the simplest type of Calabi–Yau spaces.

The proof of the main result uses an algebraic analogue of Dwork cohomology, and an alternative characterization of the $A$-hypergeometric system due to Reichelt [12]. The case of complete intersections are done by using a Mayer–Vietoris argument.

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1. $A$-HYPERGEOMETRIC SYSTEMS

1.1. We recall the definition of $A$-hypergeometric system for the reader’s sake. To this end we fix the following notation. Fix an integer $r > 0$.

- Let $V_i = \mathbb{C}^{N_i}$ be complex vector spaces of dimension $N_i$, $i = 1, \ldots, r$. Set $N = N_1 + \cdots + N_r$ and $V = V_1 \times \cdots \times V_r$.
- Let $x_{i1}, \ldots, x_{iN_i}$ be a fixed coordinate system on the dual vector space $V_i^\vee$. Set $\partial_{i,j} = \partial/\partial x_{i,j}$, $1 \leq j \leq N_i$.
- Let $\{s_1, \ldots, s_r, t_1, \ldots, t_n\} = s_j, t_i \in \mathbb{C}^*$ be an algebraic torus of dimension $s + n$. We will write this torus as a product $(\mathbb{C}^*)^r \times T$ with dim $T = n$.
- Let $A$ be an $(r + n) \times N$ matrix with integral entries. We shall write $A = [A_1, \ldots, A_r]$, where $A_i$ is an $(r + n) \times N_i$ matrix. We write $A_i$ into columns
\[ A_i = \begin{pmatrix} a_{i1} & \cdots & a_{iN_i} \\ \mu_{i1} & \cdots & \mu_{iN_i} \end{pmatrix} \]
where $a_{ij} \in \mathbb{Z}^r$ and $\mu_{ij} \in \mathbb{Z}^{r+n}$. We also assume that rank$(A) = r + n$.
- Let $\tau_i : (\mathbb{C}^*)^r \times T \to V_i$ be a map defined by the submatrix $A_i$
\[ (s, t) \mapsto (s^{a_{i1}}t^{\mu_{i1}}, \ldots, s^{a_{iN_i}}t^{\mu_{iN_i}}) \]
and $\bar{\tau}_i$ be the composition $(\mathbb{C}^*)^r \times T \to V_i \to \mathbb{P}V_i$. Let $\tau = (\tau_1, \ldots, \tau_r)$ and $\bar{\tau} = (\bar{\tau}_1, \ldots, \bar{\tau}_r)$. We assume that $\tau$ is injective.
- For our purposes we tacitly assume $a_{ij} = (\delta_{i1}, \ldots, \delta_{ij}, \tau)$ for $j = 1, \ldots, N_i$. Here $\delta_{i,j}$ is the Kronecker delta. Under the hypothesis, the matrix $A$ is homogeneous.
- Let $X'$ be the closure of the image of $\bar{\tau}$. $X'$ is a (possibly non-normal) toric variety with maximal torus $T' = \bar{\tau}(T)$. For each $a = (a_1, \ldots, a_r) \in V$, we denote by $Y'_a$, the subvarieties in $X'$ defined by $a_i$, $Y'_a := \cup Y'_a$, and $U'_a := X' - Y'_a$. 
• Let $X \to X'$ be a toric resolution of singularities. Denote by $U_a$ and $Y_a$ the preimage of $U'_a$ and $Y'_a$ inside $X$ respectively.

Given $\beta \in \mathbb{C}^{r+n}$, the $A$-hypergeometric ideal $I_{A,\beta}$ is the left ideal of the Weyl algebra $\mathcal{D} = \mathbb{C}[x, \partial]$ on the dual vector space $V^\vee$ generated by the following two types of operators

• The “box operators”: $\partial^{\nu_+} - \partial^{\nu_-}$, where $\nu_\pm \in \mathbb{Z}^N_{\geq 0}$ satisfy $A \nu_+ = A \nu_-$. Here for $m \in \mathbb{Z}^N_{\geq 0}$ we write $\partial^m = \partial_{1,1}^{m_1} \cdots \partial_{r,N_r}^{m_r}$.

• The “Euler operators”: $E_i - \beta_i$, where $E_i = \sum_{j} (\mu_{i,j}, e_l) x_{i,j} \partial_{i,j}$. Here $e_l = (\delta_{1,j}, \ldots, \delta_{i,n+r}) \in \mathbb{Z}^{r+n}$.

The $A$-hypergeometric system $M_{A,\beta}$ is the cyclic $\mathcal{D}$-module $\mathcal{D}/I_{A,\beta}$. As shown by Gelfand et al. [5], (under our hypothesis on $a_{i,j}$) and Adolphson [1] (in general), $M_{A,\beta}$ is a holonomic $\mathcal{D}$-module.

**Remark 1.2.** The $\mathcal{D}$-module $M_{A,\beta}$ also arises from the study of Calabi–Yau complete intersections in toric varieties. Let $X$ be an $n$-dimensional smooth projective Fano toric variety defined by a fan $\Theta$ in $\mathbb{R}^n$, and $t_1, \ldots, t_n$ be coordinates on the maximal torus $T$. Let $\Theta(1)$ denote the set of 1-cones in $\Theta$ and $D_\rho$ denote the $T$-invariant Weil divisor defined by $\rho \in \Theta(1)$. We assume that there is a nef partition for $\Theta$, namely $\Theta(1) = \cup_{i=1}^r \Theta_i$ and each $D_i := \sum_{\rho \in \Theta_i} D_\rho$ is nef. Finally, let $\Delta_i$ be the divisor polytope of $D_i$.

Let $\mathcal{L}_Y := \mathcal{O}_X(D_i)$. By assumption, each $\mathcal{L}_Y$ determines a morphism $X \to \mathbb{P}(V_i)$ with $V_i = H^0(X, \mathcal{L}_Y)$. Put $V = V_1 \times \cdots \times V_r$. A generic element $\sigma = (\sigma_1, \ldots, \sigma_r) \in V$ determines a smooth complete intersection Calabi–Yau variety $Y_\sigma = \cap_{i=1}^r Y_{\sigma_i}$ in $X$. We can write

$$(1.1) \quad \sigma_i = \sum_{w_{i,j} \in \Delta_i \cap \mathbb{Z}^n} c_{i,j} t^{w_{i,j}}.$$ 

These data, together with $a_{i,j} = (\delta_{i,1}, \ldots, \delta_{i,r})^t$, $1 \leq i \leq r$ and $1 \leq j \leq N_i := \dim V_i$, gives the matrix $A$ in §1 which is automatically of full rank.

The $A$-hypergeometric system constructed above consists of Picard–Fuchs equations for the corresponding Calabi–Yau complete intersections $Y_\sigma$ in $X$.

Indeed, on $T$, a holomorphic top form for $Y_\sigma$ can be written as

$$(1.2) \quad \text{Res} \left( \frac{1}{\sigma_1 \cdots \sigma_r} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} \right).$$ 

Here $\text{Res} : \Omega^r_X(\cup_{i=1}^r Y_{\sigma_i}) \to \Omega^{n-r}_X$ is the “multi-residue map”. Fixing a reference fiber $Y_{\star}$, we obtain the period integrals

$$(1.3) \quad \int_C \text{Res} \left( \frac{1}{\sigma_1 \cdots \sigma_r} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} \right)$$

for $C \in H_{n-r}(Y_{\star}, \mathbb{Z})$. From the residue theorem we deduce that

$$(1.4) \quad \int_C \beta = \int_C \frac{1}{\sigma_1 \cdots \sigma_r} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}$$

for some suitable lifting $\tilde{C} \in H_n(X \setminus \cup_{i=1}^r Y_{\sigma_i}, \mathbb{Z})$. One can check directly (cf. [6][11]) that (1.3) is a solution to $M_{A,\beta}$ with

$$\beta = (-1, \ldots, -1, 0, \ldots, 0) \in \mathbb{C}^r \times \mathbb{C}^n.$$
Our main result is that, under the hypothesis 1.3, if

(1.5) \[ \beta = (-1, \ldots, -1, 0, \ldots, 0) \in \mathbb{C}^r \times \mathbb{C}^n, \]

then for any toric resolution \( X \to X'[r] \) and any \( a \in V^\vee \), we have

(1.6) \[ \text{Sol}^0(M_{A,\beta}, \hat{\mathcal{O}}_{V^\vee,a}) = H_a(U_a, U_a \cap D) = H_a(U'_a, U'_a \cap D'). \]

Here \( \text{Sol}^0 \) is the classical solution functor of \( \mathcal{D} \)-modules. The left hand side of the above displayed equality consists of all formal power series solutions of the \( A \)-hypergeometric system around a point \( a \).

As we have remarked in the introduction, when \( X' \) itself is smooth, \( r = 1 \) and \( Y_a \) are Calabi–Yau, (1.6) was proved by Huang et. al. [8] using the general theory of [9]. The said theory, as we understood, requires the smoothness hypothesis in a crucial way.

An Huang has informed us, in a private communication, that he can prove (1.6) for mirror quintics. Related to our result is Jie Zhou’s work [14]. He gives an explicit description of the solutions to the \( A \)-hypergeometric system associated with the matrix

\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 2 & -1 & -1 \\
0 & -1 & 2 & -1
\end{bmatrix},
\]

using relative homology classes on the Hesse pencil of elliptic curves.

**1.5. Application.** Fix a polytope \( \Delta \) in \( \mathbb{Z}^n \) with 0 in its interior. Let \( \Delta_i, 1 \leq i \leq r, \) be polytopes such that 0 \( \in \Delta_i \) for all \( i \) and \( \Delta = \Delta_1 + \cdots + \Delta_r. \) The integral points in \( \Delta_i \) define an integral matrix \( A. \) We retain the notation in §1.1 and let \( \beta = (-1, \ldots, -1, 0, \ldots, 0) \in \mathbb{C}^r \times \mathbb{C}^n. \)

**Theorem 1.6 (Existence of rank 1 points).** There exists a point \( a \in V^\vee \) such that \( \text{Sol}^0(M_{A,\beta}, \hat{\mathcal{O}}_{V^\vee,a}) \) is of rank one.

**Proof.** We choose \( a \) to be the section corresponding to the lattice points \((0, \ldots, 0), \) which exists by assumption 0 \( \in \Delta_i. \) The hypersurface \( Y_a \) is just the union of all toric divisors and \( U_a = T. \) The assertion follows since \( H_a(T) \) is of rank one.

The case when \( r = 1 \) was essentially proved by Hosono, Lian, and Yau in [6] using another approach.

**1.7. Remark on the injectivity of \( \tau. \)** When \( \beta = (-1, \ldots, -1, 0, \ldots, 0) \in \mathbb{C}^r \times \mathbb{C}^n, \) any integral change of bases on \( T \) do not affect the \( \mathcal{D} \)-module \( M_{A,\beta}. \) Namely for an \( (r + n) \times (r + n) \) matrix \( R \) with

\[
R = \begin{bmatrix}
I_r & 0 \\
0 & B
\end{bmatrix}, \quad B \in \text{GL}_n(\mathbb{Q}),
\]

we have \( M_{A,\beta} = M_{RA,\beta}. \)

We say that an \( (r + n) \times N \) matrix \( A \) has property (\( \ast \)) if

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1In fact, we only need to assume that \( X \) to be a smooth algebraic variety containing \( T \) as an open dense subset, and the morphism \( X \to X' \) restricts to the identity on \( T. \)
For each $i$, there exists $1 \leq j \leq N_i$ such that $w_{i,j} = (0, \dots, 0)^t$.

For a torus inclusion $(\mathbb{C}^*)^n \to (\mathbb{C}^*)^N$ defined by an $n \times N$ matrix $C$, we have a dual surjection

$$\mathbb{Z}^N \simeq \text{Hom}_\mathbb{Z}((\mathbb{C}^*)^N, \mathbb{C}^*) \to \text{Hom}_\mathbb{Z}((\mathbb{C}^*)^n, \mathbb{C}^*) \simeq \mathbb{Z}^n,$$

which, in terms of standard coordinates, is given by $C$. Hence the column vectors of $C$ generate $\mathbb{Z}^n$ as a $\mathbb{Z}$-module. We thus deduce that

**Proposition 1.8.** Assume $A$ has rank $(r + n)$ and has property $(\ast)$. Let $\tau$ be the morphism defined by $A$ and $\tau|_T$ be the restriction on $T$. Let $B \in \text{GL}_n(\mathbb{Q})$ such that $\{B^{-1}e_1, \ldots, B^{-1}e_n\}$, $e_j = (\delta_{1,j}, \ldots, \delta_{n,j})$, is an integral basis for the image torus $T' := \text{im}(\tau|_T)$. Then for $R$ defined as above, the columns of $RA$ generate $\mathbb{Z}^n$ as a $\mathbb{Z}$-module.

**Example.** Consider the Dwork family in $\mathbb{P}^2$:

$$x_1z_1^3 + x_2z_2^3 + x_3z_3^3 + x_4z_1z_2z_3 = 0, \ [z_1 : z_2 : z_3] \in \mathbb{P}^2.$$

Let $\beta = (-1,0,0,0)$ as before. The corresponding $A$ matrix in the GKZ system is

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -1 \end{bmatrix}. $$

The columns of $A$ do not generate $\mathbb{Z}^3$. However, we can perform row operations on the last two rows of $A$ to get

$$A' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}. $$

The matrices $B$ and $B^{-1}$ in this case are

$$B = \begin{bmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{bmatrix}, \ B^{-1} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

and we have

$$\begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

From the discussion above,

$$\text{Sol}^0(M_{A,\beta}, \hat{O}_{V \vee, a}) = \text{Sol}^0(M_{A',\beta}, \hat{O}_{V \vee, a}) = H_n(U_a, U_a \cap D)$$

with $X$ being a resolution of mirror $\mathbb{P}^2$, $U_a = X \setminus Y_a$.

**1.9. Independence of relative homology.** As we have been asked several times, we should explain the trivial fact that the relative homology groups displayed above are independent of the choice of the resolution, i.e., the second equality in (1.6). In order to be consistent with what follows, we choose to use the language of sheaves to show this.
Let $W$ be a (locally quasi-compact, Hausdorff) topological space (in our case $W = U_a \cap T$). Assume that there is a commutative diagram

$$
\begin{array}{ccc}
W & \xrightarrow{j_1} & Z_1 \\
\downarrow & & \downarrow \varphi \\
W & \xrightarrow{j_2} & Z_2
\end{array}
$$

of (locally quasi-compact, Hausdorff) topological spaces, in which $j_1$ and $j_2$ are open embeddings and $\varphi$ is proper. Then we know

$$H_m(Z_i, Z_i - W) = R^{-m}\Gamma_c(Z_i, R_{j_i*}\omega_W), \quad i = 1, 2,$$

where $\omega_W$ is the dualizing complex of $W$ (this can be also served as the definition of the relative homology groups). But then

$$R\Gamma_c(Z_2, R_{\varphi*}R_{j_1*}\omega_W) = R\Gamma_c(Z_2, R\varphi_!R_{j_1*}\omega_W) \quad \text{by the commutativity}$$

$$= R\Gamma_c(Z_2, R\varphi_!R_{j_1*}\omega_W) \quad \text{by the properness of } \varphi$$

$$= R\Gamma_c(Z_1, R_{j_1*}\omega_W).$$

This proves that the relative homology groups are independent of the choice of the $Z_i$’s. Taking $Z_i$ to be $U_a$ and $U'_a$ respectively proves the independence of relative homology groups in (1.6).

1.10. Ingredients of the proof. The proof of (1.6) contains two ingredients. The first ingredient is a comparison result between the extraordinary Gauss–Manin system and the $A$-hypergeometric system due to Reichelt [12], based on the results of Walther and Schulze [13]. Reichelt’s result says, in the notation we set up in 1.1 above, that

$$\text{(1.7)} \quad \text{FT}(\pi_0(\mathcal{C}^r \times T)) = M_{A, \beta}$$

(here FT stands for the Fourier–Laplace transform of $\mathcal{D}$-modules) whenever $\beta$ is “semi-nonresonant” in the sense of Matsumi Saito (cf. [12 Proposition 1.14]), i.e.,

$$\beta \not\in \bigcup_F (\mathbb{Z}^{r+n} \cap \mathbb{Q}_{>0}A) + \mathbb{C}F,$$

where the union is taken over all the faces $F$ of $A$. Recall that a face $F$ of $A$ is a subset of columns of $A$ that minimizing some linear functional on the cone generated by $A$. $\mathbb{Q}_{>0}A$ denotes the $\mathbb{Q}_{>0}$-span of the columns of $A$ and $\mathbb{C}F$ the $\mathbb{C}$-span of $F$.

As we are interested in the case when $\beta = (-1, \ldots, -1, 0, \ldots, 0) \in \mathbb{C}^r \times \mathbb{C}^n$, the $A$-hypergeometric system is clearly semi-nonresonant since $\beta$ and the pyramids are not in the same halfspace, and the hypothesis 1.3 makes sure that $\beta$ does fall in the positive translations of the linear spans of the facets of $A$ either.

The second ingredient, albeit formal in nature, is a “transition lemma” that relates the Fourier–Laplace transform (1.7) to some actual cohomological gadgets. The latter can be computed via the Dwork cohomology. Although the idea is simple, the book-keeping is not that straightforward.
2. Functors on $\mathcal{D}$-modules

2.1. To fix the notation used throughout this note, in what follows, we will recall the notion of algebraic $\mathcal{D}$-modules and functors on them. Our main reference is [7] and references therein.

Let $X$ be a smooth projective variety and $\mathcal{D}_X$ be the sheaf of differential operators on $X$. A $\mathcal{D}_X$-module on $X$, or briefly a $\mathcal{D}$-module, if the context is clear, is always a left $\mathcal{D}_X$-module. Let $D^b(\mathcal{D}_X)$ be the bounded derived categories of $\mathcal{D}$-modules over $X$. The subscripts $qc$, $h$, and $rh$ stand for the bounded derived category of quasi-coherent, holonomic, and regular holonomic $\mathcal{D}$-modules. One can define the duality functor $\mathbb{D}_X$ on $D^b(\mathcal{D}_X)$. Let $f : X \to Y$ be a morphism between smooth varieties. One can define the following functors

- For a complex $M \in D^b(\mathcal{D}_X)$, let $f_+(M) := Rf_*(\text{dR}_{X/Y}(M))$, where $\text{dR}_{X/Y}$ is the relative de Rham functor.
- For a complex $N \in D^b(\mathcal{D}_Y)$, let $f^!N := f^*[\dim X - \dim Y]$, where $f^*$ is the derived pullback on the category of quasi-coherent $\mathcal{O}_Y$-modules.

However, $\mathcal{D}$ is behaved well only on $D^b_h(\mathcal{D})$. We thus restrict ourselves to work on $D^b_h(\mathcal{D})$ instead of whole $D^b(\mathcal{D})$. All the functors $\mathcal{D}$, $f_+$ and $f^!$ preserve the holonomicity. We put

- $f^+ := \mathcal{D}_X f^! \mathcal{D}_Y$, and
- $f_! := \mathcal{D}_Y f^+ \mathcal{D}_X$.

$f^+$ is the left adjoint of $f_+$ and $f_!$ is the left adjoint of $f^!$.

When $f$ is a smooth morphism, or more generally non-characteristic with respect to a coherent $\mathcal{D}$-module $\mathcal{M}$, we have $f^* \mathcal{M} = f^! \mathcal{M} [\dim Y - \dim X] = f^+ \mathcal{M} [\dim X - \dim Y]$. Finally, given a fibred diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{g} & Y,
\end{array}
$$

with all varieties are smooth, then we have

$$g^! f_+ = f_! g^+.$$

Let $S \subset X$ be a (possibly singular) subscheme of $X$ and $\mathcal{I}_S$ be the corresponding ideal sheaf. For a $\mathcal{O}_X$-module $\mathcal{F}$ on $X$, we define

$$\Gamma_{[S]}(\mathcal{F}) := \lim_k \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X / \mathcal{I}_S^k, \mathcal{F}).$$

For a quasi-coherent $\mathcal{D}_X$-module $\mathcal{M}$, $\Gamma_{[S]}(\mathcal{M})$ inherits a $\mathcal{D}_X$-module structure and we can consider the derived functor $R\Gamma_{[S]} : D^b_{qc}(\mathcal{D}_X) \to D^b_{qc}(\mathcal{D}_X)$. Let $j : X \setminus S \to X$ be the open embedding. For $\mathcal{M} \in D^b_{qc}(\mathcal{D}_X)$ we have the distinguished triangle

$$R\Gamma_{[S]}(\mathcal{M}) \to \mathcal{M} \to j_* j^! \mathcal{M} \to .$$

If $\mathcal{M}$ is holonomic, it follows that $R\Gamma_{[S]}(\mathcal{M})$ is also holonomic.

Let $i : S \to X$ be the closed embedding. In case $S$ is smooth, $R\Gamma_{[S]}(\mathcal{M}) \cong i_* i^! \mathcal{M}$ and the distinguished triangle (2.1) becomes

$$i_* i^! \mathcal{M} \to \mathcal{M} \to j_* j^! \mathcal{M} \to .$$
The proofs of these results can be found in [3].

3. Fourier–Laplace transform and Dwork cohomology

In this section, we recall the Dwork complex and explain how it relates to the relative homology groups.

**Definition 3.1.** Let $\gamma : Z \to \mathbb{A}^1$ be a morphism between smooth algebraic varieties. We define the exponential $\mathcal{D}$-module on $Z$ to be

$$\exp(\gamma) := \gamma^*(\mathcal{D}_{\mathbb{A}^1}/(\partial_t - 1)) = \gamma!(\mathcal{D}_{\mathbb{A}^1}/(\partial_t - 1))[1 - \dim Z].$$

This is a holonomic $\mathcal{D}$-module on $Z$.

**3.2.** Let $X$ be an algebraic variety and $\pi : E \to X$ be a rank $r$ vector bundle. Let $\sigma : X \to E^\vee$ be a section of the dual bundle and $S$ be the reduced zero scheme of $\sigma$. We consider the following commutative diagram, which will be used frequently throughout this note.

$$
\begin{array}{ccc}
X & \xrightarrow{i} & E^\vee \\
\downarrow{\iota} & & \leftarrow{pr_2} \\
S & \xleftarrow{i} & X \\
\end{array}
\quad \begin{array}{ccc}
E \times_X E^\vee & \xrightarrow{\gamma} & \mathbb{A}^1 \\
\downarrow{\varepsilon} & & \\
E & \xrightarrow{F} & \\
\end{array}
$$

In this diagram, $\iota$ is the zero section embedding, $\gamma$ is the natural pairing, $i$ is a closed embedding, $\varepsilon$ is the pullback of $\sigma$, and $F = \gamma \circ \varepsilon$.

**Definition 3.3.** Adapted the notation in §3.2, for a holomonic complex $M \in D^b_{\mathcal{H}}(\mathcal{D})$, we define the Dwork complex of $M$, denoted by $Dw(M)$, to be

$$Dw(M) := \pi_+ (\pi^! M \otimes \exp(F))[-r].$$

**Theorem 3.4.** In Situation §3.2, we have

$$Dw(M) \cong R\Gamma_S[M][r] \cong R\Gamma_S(\mathcal{O}_X) \otimes M[r].$$

The theorem was obtained by many people in various situations: by Katz [10] when $X$ is affine and $E$ is the trivial line bundle, by Adolphson–Sperber [2] when $X$ is the affine space but $E$ can have higher rank, and by Dimca–Maaref–Sabbah–Saito [4] when $X$ and $E$ are both general.

We recommend the readers to consult Baldassarri–D’Agnolo’s paper [3] for a proof of the theorem. The argument they used is completely formal but somehow reveals the mystery of the complex $Dw(M)$ considered by Dwork.

Let us first show how to use the Dwork complex to compute the cohomology of the complement of a section of $E$.

**Lemma 3.5.** Let the notation be as in §3.2. Let $\zeta : X \to E$ be the zero section. Let $\beta : E^0 = E - \zeta(X) \to E$ be the inclusion of the complement of the zero section. Let $\rho : X - \Sigma \to X$ be the open immersion. Then there is a natural and functorial isomorphism

$$\rho_+ \rho^! M[r - 1] \cong \pi_+ (\beta_! \beta^* M \otimes \exp(F))$$

for any complex of holonomic $\mathcal{D}$-modules on $X$. 
Proof. For reader’s convenience, we review the proof briefly. On $E$ there is a distinguished triangle

$$\beta_! \beta^* M \to \pi^* M \to \beta_! \pi^* M \to .$$

(3.3)

As $\pi^* = \pi^+[r] = \pi^![-r]$, we have

$$\beta_! \pi^* M = \beta_! \pi^! M[r] = M[r]$$

Tensoring (3.3) with $exp(F)$ and applying the pushforward functor $\pi_+$ yield the distinguished triangle

$$\star \to \text{Dw}(M) = i_! i^! M[r] \to M[r] \to .$$

(3.4)

Thereby the term $\star$ must be $\rho_+ \rho^! M[r - 1]$. □

The Dwork cohomology computes the cohomology of the complement of a section of $E$. However, as we have mentioned in the introduction, we are forced to deal with a slightly different situation.

### 3.6. Let $X$ be a proper smooth variety, $U$ be an open subset and $\Sigma = X \setminus U$ be its complement. Assume that $\Sigma$ is a simple normal crossing divisor. Namely $\Sigma = \bigcup_{i=1}^{r} \Sigma_i$, with each $\Sigma_i$ being smooth, and all of their arbitrary intersections being smooth. Let $I \subset \{1, \ldots, r\}$. Put $\Sigma_I = \bigcap_{i \in I} \Sigma_i$ and $\iota_I : \Sigma_I \to X$. Let $j : U \to X$ and $i : \Sigma \to X$ be inclusion maps. For $M \in D^b_{qc}(\mathcal{D}X)$, there is a distinguished triangle

$$\to R\Gamma_{\Sigma}(M) \to M \to j_+ j^! M \to .$$

(3.5)

and its dual

$$\to j_! j^! M \to M \to D_X R\Gamma_{\Sigma}(\mathcal{D}X M) \to .$$

(3.6)

We made the following observation.

**Lemma 3.7.** In situation 3.6 with $M = \mathcal{O}_X$, there is a quasi-isomorphism

$$\mathcal{D}_X R\Gamma_{\Sigma}(\mathcal{D}_X \mathcal{O}_X) \cong \left[ \bigoplus_{|I| = 1} \iota_i^! \mathcal{O}_X \to \bigoplus_{|I| = 2} \iota_i^! \mathcal{O}_X \to \cdots \to \bigoplus_{|I| = r} \iota_i^! \mathcal{O}_X \right].$$

(3.7)

The $\mathcal{D}$-modules on the right hand side are all considered as $\mathcal{D}_X$-modules via the pushforward functors $\iota_I^!$.

**Proof.** We prove this via the (covariant) Riemann–Hilbert correspondence. Under the Riemann–Hilbert correspondence, the left two terms in (3.6) with $M = \mathcal{O}_X$ corresponds to

$$j_! j^! \mathcal{C}_X \to \mathcal{C}_X.$$  

(3.8)

We infer from the uniqueness of the mapping cones that the Riemann–Hilbert partner of $\mathcal{D}_X R\Gamma_{\Sigma}(\mathcal{D}_X \mathcal{O}_X)$ is $i_! i^{-1} \mathcal{C}_X = i_! C_{\Sigma}$. On this side, we have the Mayer–Vietoris resolution for simple normal crossing varieties:

$$C_{\Sigma} \to \left[ \bigoplus_{|I| = 1} C_{\Sigma I} \to \bigoplus_{|I| = 2} C_{\Sigma I} \to \cdots \to \bigoplus_{|I| = r} C_{\Sigma I} \right].$$

(3.9)

and the conclusion follows. □
Since $R\Gamma_{\Sigma}(M) = R\Gamma_{\Sigma}(O_X) \otimes M$, we have

**Corollary 3.8.**

$$\mathbb{D}_X R\Gamma_{\Sigma}(\mathbb{D}_X M) \cong (\mathbb{D}_X R\Gamma_{\Sigma}(O_X)) \otimes M.$$  

**Proof.** Note that

$$\mathbb{D}_X R\Gamma_{\Sigma}(\mathbb{D}_X M) = R\text{Hom}(R\Gamma_{\Sigma}(\mathbb{D}_X M), D_X)$$

$$\cong R\text{Hom}(R\Gamma_{\Sigma}(O_X) \otimes \mathbb{D}_X M, D_X)$$

$$\cong R\text{Hom}(R\Gamma_{\Sigma}(O_X), R\text{Hom}(\mathbb{D}_X M, D_X))$$

$$\cong R\text{Hom}(R\Gamma_{\Sigma}(O_X), \mathcal{M})$$

$$\cong \mathbb{D}_X R\Gamma_{\Sigma}(O_X) \otimes \mathcal{M}.$$  

□

As an application of the Dwork complex, we can compute the cohomology of simple normal crossing varieties.

**3.9.** Let $X$ be an algebraic variety and $L_i, 1 \leq i \leq r$, be line bundles on $X$. Put $E = L_1 \times_X \cdots \times_X L_r$ and $\pi : E \to X$ be the projection map. Let $L_i^\vee$ be the line bundle $L_i$ with the zero section removed. Let $\Sigma_i$ and $E_i^\vee$ be their dual bundles.

Let $\Sigma_i$ be the subset of $E$ with the $i$-th coordinate to be zero. For simplicity, let $I \subset \{1, \ldots, r\}$ be a subset, $\bar{I} = \{1, \ldots, r\} \setminus I$ be the complement and $\Sigma_I = \bigcap_{i \in I} \Sigma_i$. We have a closed embedding $\iota_I : \Sigma_I \to E$. Given a section $\sigma : X \to E^\vee$, for each $I$, we can draw the following diagram as in §3.2.

Note that if $\sigma = (\sigma_1, \ldots, \sigma_r)$, then the induced section $\sigma_I := \iota_I^\vee \circ \sigma$ is $(\sigma_j)_{j \in I}$. Let $F = \gamma \circ \varepsilon$ and $\varepsilon_I = (\text{id} \times \iota_I^\vee) \circ (\text{id} \times (\sigma \circ \pi_I))$.

Let $Y_I := \cap_{i \in I} \{\sigma_i = 0\}$ and $i_I : Y_I \to X$ be the inclusion. We further assume that each $\sigma_i$ defines a smooth subvariety in $X$ and so are all their arbitrary intersections. Namely, $\cup Y_I$ is a simple normal crossing variety.

We are now in the situation §3.6, in which Lemma 3.7 is applicable. Let

$$\mathcal{A}^* = \bigoplus_{|I|=1} i_I^+ O_E \to \bigoplus_{|I|=2} i_I^+ O_E \to \cdots \to \bigoplus_{|I|=r} i_I^+ O_E$$

be the resulting complex. Then
Proposition 3.10. In Situation §3.9, for a complex of holonomic $\mathcal{D}_X$-modules $\mathcal{M}$, the complex $\pi_+ (\mathcal{A}^\bullet \otimes \pi^* \mathcal{M} \otimes \exp(F))$ is quasi-isomorphic to

\[
\bigg( \bigoplus_{|I|=1} i_I^! \mathcal{O}_X \rightarrow \bigoplus_{|I|=2} i_I^! \mathcal{O}_X \rightarrow \cdots \rightarrow \bigoplus_{|I|=r} i_I^! \mathcal{O}_X \bigg) \otimes \mathcal{M}[r].
\]

Each term in the complex displayed is understood as a $\mathcal{D}$-module on $X$ via the pushforward and $i_I$ is understood as the identity map $X \to X$ if $I = \{1, \cdots, r\}$.

Proof. Via the projection formula, it suffices to prove this statement for $\mathcal{M} = \mathcal{O}_X$. Since $\pi$ is affine, the pushforward functor $\pi^*$ is exact. Let us concentrate on $\Sigma_I$.

Note that $F \circ i_I = \gamma_I \circ \varepsilon_I$, which is also the Fourier kernel associated to the induced section $\sigma_I$.

As $\pi^* = \pi^+[r]$, $\pi_I^* = \pi_I^+[r - |I|]$ and $\pi_I = \pi \circ i_I$, we have

\[
\pi_+((\iota_I)_+ i_I^+ \mathcal{O}_E \otimes \exp(F)) = \pi_+(\iota_I^+ i_I^! \mathcal{O}_E \otimes \iota_I^* \exp(F)) = \pi_+(\iota_I^+ i_I^! \mathcal{O}_X \otimes \exp(\gamma_I \circ \varepsilon_I))
\]

As $\pi^* = \pi^+[r]$, $\pi_I^* = \pi_I^+[r - |I|]$ and $\pi_I = \pi \circ i_I$, we have

\[
(3.11) = \pi_+((\iota_I)_+ i_I^! \mathcal{O}_X \otimes \exp(\gamma_I \circ \varepsilon_I))[|I|] = i_I^+ i_I^! \mathcal{O}_X[r].
\]

By Theorem 3.4. This also shows that the complex asked in the proposition can be calculated term by term. This proves the Proposition. \square

3.11. Relative cohomology. We review the mechanism of computing relative cohomology in terms of the language of $\mathcal{D}$-modules (via the Riemann–Hilbert correspondence).

Let $X$ be a smooth, proper, algebraic variety. Let $b : T \to X$ be an affine open immersion with a complement divisor $D$, possibly singular. Let $Y$ be a Cartier divisor on $X$ with complement $U$. Assume that $Y$ is smooth. Let $Y_T = Y \cap T$, $Y_D = Y \cap D$, etc. Let $\mathcal{M}$ be a $\mathcal{D}$-module on $X$. We form the following diagram

\[
\begin{array}{ccc}
Y_T & \xrightarrow{i_T} & T & \xleftarrow{\rho_T} & U_T \\
\downarrow{a} & & \downarrow{b} & & \downarrow{c} \\
Y & \xrightarrow{i} & X & \xleftarrow{\rho} & U \\
\downarrow{a'} & & \downarrow{a'} & & \downarrow{a'} \\
Y_D & \xrightarrow{i_D} & D & \xleftarrow{\rho_D} & U_D \\
\end{array}
\]

There is a $\mathcal{D}$-module that “computes” the relative cohomology of $\rho^! \mathcal{M}$ on $U$ with respect to $U_D$ (in the sense that it corresponds to the complex whose cohomology is relative cohomology, via the Riemann–Hilbert correspondence): this is $\mathcal{O}_C(\rho^! \mathcal{M})$.

Let us explain this if the reader is not familiar to this formalism. In fact if $\rho^! \mathcal{M}$ corresponds to a perverse sheaf $F$ under the covariant Riemann–Hilbert correspondence, then there is a distinguished triangle

\[
c_0 c^! F \to F \to w_* w^* F \to
\]
(taking $R\Gamma(U, ?)$ recovers the usual long exact sequence of sheaf cohomology groups).

The corresponding version of the above distinguished triangle in the language of $\mathcal{D}$-modules is

$$cc^!(\rho^!M) \to \rho^!M \to w^+_+(\rho^!M) \to .$$

Manipulating with base change functors we have

$$cc^! \rho^!M = c_1\rho_!b^!M = \rho^!b^!b^!M.$$

We can put the discussion above in the relative setting. Assume that all schemes in the diagram (3.13) are schemes over a smooth $\mathbb{C}$-variety $B$ and let $p : X \to B$ be the structure morphism. Then the “variation” of the relative cohomology groups (up to a shift) of the pair $(U, U \cap D)$ is computed by the complex

$$Rp_*(X, Rp_*\rho^!b^!1_\mathbb{C}([\dim T])) = RH(p_+\rho_+\rho^!b^!0_T),$$

where RH stands for the covariant Riemann–Hilbert correspondence. Recall that its Verdier dual is the derived solution complex: $(\text{Verdier duality}) \circ RH = \text{Sol}$.

4. The transition lemma

In this section, we will explain how the Dwork complex relates to the $A$-hypergeometric system, i.e., the Fourier–Laplace transform. To this end, we fix the notation and conventions used in the rest of this section.

4.1. Setting. Let $X$ be a proper algebraic variety of pure dimensional $n$. Let $\mathcal{L}, \ldots, \mathcal{L}_r$ be invertible sheaves on $X$.

- Assume that each $\mathcal{L}_i$ is generated by its global sections. Put $V_i = H^0(X, \mathcal{L}_i^\vee)$. We have morphisms $X \to \mathbb{P}(V_i)$ and their product $X \to \prod_{i=1}^r \mathbb{P}(V_i)$. Let $V = V_1 \times \cdots \times V_r$.

- We will regard each $V_i$ as an algebraic variety, which is $\text{Spec}(\text{Sym}^*(V_i^\vee))$.

- Let $\mathbb{L}_i = \text{Spec}_X(\text{Sym}^*(\mathcal{L}_i^\vee))$ be the geometric line bundle of $\mathcal{L}_i$. This means the sheaf of sections of $\mathbb{L}_i$ is $\mathcal{L}_i$. For each $i$, we denote by $\pi_i$ the projection morphism $\pi_i : \mathbb{L}_i \to X$, $\mathbb{L} = \mathbb{L}_1 \times_X \cdots \times_X \mathbb{L}_r$ and $\pi : \mathbb{L} \to X$.

- For each $i$, we have a morphism $b_i : \mathbb{L}_i \to V_i$, which contracts the zero section to the origin. Let $b : \mathbb{L} \to V$ be their product.

- Let $\mathcal{L}_i^\vee = \text{Spec}_X(\text{Sym}^*(\mathcal{L}_i))$, the dual bundle of $\mathcal{L}_i$. Again this means the sheaf of sections of $\mathcal{L}_i^\vee$ is $\mathcal{L}_i$.

- Let $\mathbb{L}_i^\vee$ be the space of $\mathbb{L}_i$ with its zero section removed. We denote their product $\mathbb{L}_1^\vee \times_X \cdots \times_X \mathbb{L}_r^\vee$ by $\mathbb{L}^\vee$. Let $\theta_i : \mathbb{L}_i^\vee \to \mathbb{L}_i$ be the open inclusion and $\theta : \mathbb{L}^\vee \to \mathbb{L}$ be their product.

- Let $\mathbb{L}$ be the pullback of $\mathbb{L}$ via the natural projection $X \times V^\vee \to X$. $\mathbb{L}^\vee$ is defined in the same manner. Let $\tilde{\sigma} : X \times V^\vee \to \mathbb{L}^\vee$ be the universal section. We will write $\tilde{\sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_r)$. $Y_i = \{\tilde{\sigma}_i = 0\}$, $1 \leq i \leq r$, are the universal hypersurface determined by $\tilde{\sigma}$. Note that $\bigcup_{i=1}^r Y_i$ is a simple normal crossing divisor.

- All the morphisms obtained via the pullback along $X \times V^\vee \to X$ are denoted by the same symbol with a ‘tilde’. For instance, the map $\mathbb{L} \to X \times V^\vee$ will be denoted by $\tilde{\pi}$ according to our convention.

- Put $i = b \circ \theta : \mathbb{L}^\vee \to V$. 

These data form the following commutative diagram (cf. §3.2).

\[
\begin{array}{ccc}
\tilde{L} & \xleftarrow{pr_1} \tilde{L} \times_{X \times V^\vee} \tilde{L} & \xrightarrow{\gamma} A^1 \\
\sigma & & F \\
X \times V^\vee & \xleftarrow{\varepsilon} \tilde{L}
\end{array}
\]

In this diagram,
- \(\varepsilon\) is the pullback of \(\sigma\),
- \(\gamma\) is the canonical dual pairing, and
- \(F = \gamma \circ \varepsilon\).

**Lemma 4.2** (Transition lemma). Let notation be as in §4.1. Let \(\mathcal{M}\) be a holonomic complex of \(\mathcal{D}\)-modules on \(L^\circ\). Then

\[
FT(\iota_! \mathcal{M}) = pr_{V^\vee+}(pr_{\tilde{L}}^1 \theta_! \mathcal{M} \otimes \exp(F))[- \dim V].
\]

Here \(pr_{V^\vee} : L \times V^\vee \to V^\vee\) and \(pr_L : L \times V^\vee \to L\) are projections.

**Proof.** Look at the following commutative diagram

\[
\begin{array}{ccc}
\tilde{L}^\circ & \xrightarrow{\bar{\delta}} \tilde{L} \\
V^\vee \times \tilde{L}^\circ & \xrightarrow{\bar{\delta}} V^\vee \times L & \xrightarrow{\text{Id} \times b} V^\vee \times V \xrightarrow{\text{can}} A^1 \\
L^\circ & \xrightarrow{\theta} L & \xrightarrow{b} V
\end{array}
\]

Since \(b\) is proper, \(b_+ = b\). Thus by the base change theorem, and the commutativity of the entire lower rectangle, we have

\[
pr_{V^\vee+}^1 \theta_! \mathcal{M} = pr_{V^\vee+}^1 b_+ \theta_! \mathcal{M} = (\text{Id} \times b)_+ pr_{\tilde{L}}^1 \theta_! \mathcal{M}.
\]

Hence

\[
FT(\iota_! \mathcal{M}) = pr_{V^\vee+}(pr_{\tilde{L}}^1 \theta_! \mathcal{M} \otimes \exp \text{can})[- \dim V] = (pr_{V^\vee+} \circ (\text{Id} \times b))_+ (pr_{\tilde{L}}^1 \theta_! \mathcal{M} \otimes \exp \text{can})[- \dim V] = pr_{V^\vee+}(pr_{\tilde{L}}^1 \theta_! \mathcal{M} \otimes \exp \text{can})[- \dim V],
\]

as claimed. \(\square\)

5. **Proof of the main theorem**

We will prove our main theorem in this section.

Given an \((r + n) \times N\) integral matrix \(A\), the columns of \(A\) give rise to a morphism \(\tau : (\mathbb{C}^*)^r \times T \to \prod_{i=1}^r PV_i\) as in §1.1. Let \(X' \subset \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_r\) be the closure of the image. It is a possibly singular toric variety. Choose a toric desingularization \(X \to X'\). Each \(X \to \mathbb{P}V_i\) determines a base point free line bundle \(L_i^\vee\). We are now in situation §4.1. Let us retain the notation there.
We have a sequence of maps.

\[ (\mathbb{C}^*)^r \times T \xrightarrow{j} \mathbb{L}^o \xrightarrow{\theta} \mathbb{L} \xrightarrow{b} V \]

On the one hand, owing to our hypothesis on \( A \), Reichelt’s result in [12, Proposition 1.14] implies

\[ \text{FT}(\pi\mathcal{O}_{(\mathbb{C}^*)^r \times T}) = M_{A,\beta}. \]

On the other hand, applying the transition lemma to \( M = j!\mathcal{O}_{(\mathbb{C}^*)^r \times T} \), we obtain

\[ \text{FT}(\iota!j!\mathcal{O}_{(\mathbb{C}^*)^r \times T}) = \text{pr}_V^{-} \left( \text{pr}_L^{-} \theta \right)^{-} \mathcal{O}_{(\mathbb{C}^*)^r \times T} \boxtimes \exp(F) \mid - \dim V \right]. \]

To proceed, we consider the following commutative diagram.

\[
\begin{array}{ccc}
(\mathbb{C}^*)^r \times T & \xrightarrow{\jmath^!} & V \\
\downarrow \text{pr}_2 & & \\
(\mathbb{C}^*)^r \times T & \xrightarrow{\jmath^!} & \mathbb{L}^o \\
\downarrow \text{pr}_L & & \downarrow \text{pr}_L \\
(\mathbb{C}^*)^r \times T & \xrightarrow{\jmath^!} & \mathbb{L}^o \\
\end{array}
\]

Note the all vertical maps are smooth. We have \( \text{pr}_L^+ = \text{pr}_L^{-} \left[ - \dim V \right] = \text{pr}_L^{+} \left[ \dim V \right] \).

An iterated application of the projection formula to this diagram yields

\[ \text{pr}_L^+ \theta j!\mathcal{O}_{(\mathbb{C}^*)^r \times T} \left[ \dim V \right] \]

\[ = \tilde{\theta} \text{pr}_L^+ j!\mathcal{O}_{(\mathbb{C}^*)^r \times T} \left[ \dim V \right] \]

\[ = \tilde{\theta} \text{pr}_L^+ j!\mathcal{O}_{(\mathbb{C}^*)^r \times T} \left[ \dim V \right] \]

\[ = \tilde{\theta} j!\mathcal{O}_{(\mathbb{C}^*)^r \times T} \left[ \tilde{\theta} \jmath!\mathcal{O}_{(\mathbb{C}^*)^r \times T} \right] \]

To compare this with the objects on \( X \), we identify \( ((\mathbb{C}^*)^r \times T) \times V^\vee \) with \( \mathbb{L}^o \mid_T \) and look at the following commutative diagram.

\[
\begin{array}{ccc}
\mathbb{L}^o \mid_T & \xrightarrow{\tilde{\theta}^r} & \mathbb{L} \mid_T \\
\downarrow \jmath & & \downarrow \beta \\
\mathbb{L}^o & \xrightarrow{\theta} & \mathbb{L} \\
\end{array}
\]

From the trivial fact that \( \mathcal{O}_{\mathbb{L}^o \mid_T} = (\tilde{\theta} T)^+ \mathcal{O}_{\mathbb{L}^o \mid_T} = (\tilde{\theta} T)^+ (\tilde{\pi} T)^+ \mathcal{O}_{T \mid r} \), via projection formula, the last quantity in (5.3) can be transformed into

\[ \tilde{\theta} j!\mathcal{O}_{((\mathbb{C}^*)^r \times T) \times V^\vee} \]

\[ = \tilde{\theta} \jmath!\mathcal{O}_{\mathbb{L}^o \mid_T} \]

\[ = \tilde{\theta} \jmath! (\tilde{\theta} T)^+ (\tilde{\pi} T)^+ \mathcal{O}_{T \times V^\vee} \mid r \]

\[ = \tilde{\theta} \tilde{\theta}^+ \tilde{\pi}^+ \mathcal{O}_{T \times V^\vee} \mid r \]

\[ = \tilde{\theta} \tilde{\theta}^+ \tilde{\pi}^+ \mathcal{O}_{T \times V^\vee} \mid - r \].
Plugging the displayed equation above and (5.3) into (5.2) yields
\[
\text{FT}(u\tilde{j}\mathcal{O}_C r_T) = \text{pr}_{V^+}(\text{pr}_1^*\theta\mathcal{O}_C r_T \otimes \exp(F))[-\dim V] \\
= \text{pr}_{V^+}(\tilde{\theta}\tilde{\theta}^\dagger\tilde{\pi}\mathcal{O}_{T\times V^\vee} \otimes \exp(F))[-r].
\]

Let \( \Sigma = \mathbb{L} \setminus \tilde{\mathbb{L}}^\circ \) and \( \tilde{\theta} : \tilde{\mathbb{L}}^\circ \to \tilde{\mathbb{L}} \) be the open immersion. We are then in the situation §3.9 with
- \( X = X \times V^\vee \),
- \( E = \tilde{\mathbb{L}}_i \),
- \( W = \tilde{\mathbb{L}}^\circ, \Sigma = \tilde{\Sigma} \)
and we retain the notation there.

For any \( M \in D^b_{qc}(\mathcal{D}_X \mathcal{O}_{V^\vee}) \), we have a distinguished triangle
\[
\to \tilde{\theta}\tilde{\pi}^!M \to \tilde{\pi}^!M \to (\mathcal{D}_{\Sigma}^r \mathcal{O}_C) \otimes \tilde{\pi}^!M \to .
\]

Tensorizing (5.6) with \( \exp(F) \) and applying \( \tilde{\pi}^+ \), we obtain
\[
\to \tilde{\pi}^+(\tilde{\theta}\tilde{\pi}^!M \otimes \exp(F)) \to \tilde{\pi}^+(\tilde{\pi}^!M \otimes \exp(F)) \\
\to \tilde{\pi}^+(\mathcal{D}_{\Sigma}^r \mathcal{O}_C) \otimes \tilde{\pi}^!M \otimes \exp(F)) \to \tilde{\pi}^+(\tilde{\theta}\tilde{\pi}^!M \otimes \exp(F))[1] \to .
\]

In the displayed equation, the first term is what we want, the second one can be computed via the Dwork complex, and the third one is computed by Proposition 3.10 up to a shift.

To be precise, take \( M = \beta_1^r \mathcal{O}_T \mathcal{O}_{V^\vee} \) in (5.6). We have, by Proposition 3.10
\[
\tilde{\pi}^+(\mathcal{D}_{\Sigma}^r \mathcal{O}_C) \otimes \tilde{\pi}^!\beta_1^r \mathcal{O}_{T\times V^\vee} \otimes \exp(F)) \\
\cong \bigoplus_{i=1}^{r} i^!_i \mathcal{O}_{X\times V^\vee} \to \bigoplus_{i=2}^{r} i^!_i \mathcal{O}_{X\times V^\vee} \to \cdots \to \bigoplus_{i=r}^{r} i^!_i \mathcal{O}_{X\times V^\vee} \otimes \beta_1^r \mathcal{O}_{T\times V^\vee} [2r].
\]

Let \( i : Y_1 \cap \cdots \cap Y_r \to X \times V^\vee \) be the inclusion. The second term in (5.7) is isomorphic to \( i^!_+ i^!_1 \beta_1^r \mathcal{O}_{T\times V^\vee} [2r] \cong i^!_+ i^!_1 \mathcal{O}_{X\times V^\vee} \otimes \beta_1^r \mathcal{O}_{T\times V^\vee} [2r] \).

For simplicity, put
\[
(5.8) \quad \mathcal{B}^* := \bigoplus_{|i| = 1}^{r} i^!_i \mathcal{O}_{X\times V^\vee} \to \bigoplus_{|i| = 2}^{r} i^!_i \mathcal{O}_{X\times V^\vee} \to \cdots \to \bigoplus_{|i| = r}^{r} i^!_i \mathcal{O}_{X\times V^\vee} , \\
Y = Y_1 \cup \cdots \cup Y_r \text{ and } U = X \times V^\vee \setminus Y.
\]

We look at the morphism \( i^!_+ i^!_1 \mathcal{O}_{X\times V^\vee} \to \mathcal{B}^* \) first. The mapping cone of this morphism is equal to \( \rho^*_U \mathcal{O}_U [-r-1] \), where \( \rho : U \to X \times V^\vee \) is the inclusion. Indeed, the Riemann–Hilbert partner of \( i^!_+ i^!_1 \mathcal{O}_{X\times V^\vee} \) is \( C_{Y_1 \cap \cdots \cap Y_r} \), while the Riemann–Hilbert partner of \( \mathcal{B}^* \) is
\[
\bigoplus_{|i| = r-1}^{L} C_{Y_j} \to \bigoplus_{|i| = r-2}^{L} C_{Y_j} \to \cdots \to \bigoplus_{|i| = 1}^{L} C_{Y_j} \to C_X .
\]

where \( J \subset \{1, 2, \cdots, r\} \). Hence, by Mayer–Vietoris exact sequence, the Riemann–Hilbert partner of the mapping cone of \( i^!_+ i^!_1 \mathcal{O}_{X\times V^\vee} \to \mathcal{B}^* \) is \( \rho^*_U \mathcal{C}_U [-r-1] \) and hence the mapping cone itself is \( \rho^*_U \mathcal{O}_U [-r-1] \).

To summarize, we have shown that
• From transition lemma,
  \[ M_{A,\beta} = \text{FT}(i_t \ast \mathcal{O}(\mathcal{C})_{V \times T}) = \text{pr}_{V^\vee} (\hat{\theta} \hat{\pi} \beta \mathcal{O}_{T \times V^\vee} \otimes \exp(F))[-r]. \]

• \( \hat{\pi}_+ (\hat{\theta} \hat{\pi} \beta \mathcal{O}_{T \times V^\vee} \otimes \exp(F)) \) is isomorphic to
  \[ \rho_+ \mathcal{O}_U \otimes \beta \mathcal{O}_{T \times V^\vee}[r] \cong \rho_+ \beta \mathcal{O}_{T \times V^\vee}[r]. \]

Note that the shift comes from \( 2r + (-r - 1) + 1 = r. \)

Hence \( M_{A,\beta} \cong \rho_+ \beta \mathcal{O}_{T \times V^\vee}. \)

As we have discussed in §3.11, the Riemann–Hilbert partner of the \( D \)-module \( \text{pr}_{V^\vee} \rho_+ \beta \mathcal{O}_{T \times V^\vee} \) is \( R \text{pr}_{V^\vee} (\beta \mathbb{C}[n]|_U). \)

For each \( a \in V \), let \( i_a \) be the closed immersion. By base change formula, we have
  \[ i_a^! R\text{pr}_{V^\vee} (\beta \mathbb{C}[n]|_U) = R\Gamma(U_a, U_a \cap D)[n]. \]

Applying the Verdier duality and taking the zeroth cohomology of the displayed complex, we have
  \[ \text{Sol}^0(M_{A,\beta}, \hat{\mathcal{O}}_{V^\vee,a}) = H_n(U_a, U_a \cap D). \]

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E-mail address: tjlee@cmsa.fas.harvard.edu

E-mail address: dingxin@cmsa.fas.harvard.edu
