Extracting $|V_{ub}|$ from the inclusive charmless semileptonic branching ratio of $b$ hadrons

Changhao Jin

*School of Physics, University of Melbourne*
*Parkville, Victoria 3052, Australia*

Abstract

We calculate the inclusive charmless semileptonic decay width of the $B$ meson using a QCD-based approach. This approach is able to account for both dynamic and kinematic effects of nonperturbative QCD. The charmless semileptonic decay width is found to be enhanced by long-distance strong interactions by $+(7 \pm 5)\%$ with respect to the free quark decay width. Using the resulting theoretical value for the charmless semileptonic decay width, an extraction of $|V_{ub}|$ is made from the measured lifetime and inclusive charmless semileptonic branching ratio of $b$ hadrons.
The ALEPH \cite{1} and L3 \cite{2} Collaborations have recently reported the first measurements of the inclusive charmless semileptonic branching ratio of beauty hadrons at LEP. Their measurements have been performed exploiting the different kinematic properties differentiating the rare $b \to X_u \ell \nu$ decay from the dominant $b \to X_c \ell \nu$ decay (the symbol $\ell$ indicates either an electron or a muon). The inclusive charmless semileptonic branching ratio of beauty hadrons is measured to be

$$Br(b \to X_u \ell \nu) = (1.73 \pm 0.55 \pm 0.55) \times 10^{-3} \quad \text{(ALEPH)},$$

$$3.3 \pm 1.0 \pm 1.7 \times 10^{-3} \quad \text{(L3)},$$

where in each collaboration the first error is statistical and the second systematic.

The measured inclusive charmless semileptonic branching ratio combined with the measured beauty hadron lifetime can be used to determine the Cabibbo-Kobayashi-Maskawa matrix element $|V_{ub}|$, provided the corresponding inclusive charmless semileptonic decay width except for $|V_{ub}|$ is theoretically predicted. This determination of $|V_{ub}|$ would have a smaller theoretical error than a determination of it from a partial charmless semileptonic decay width, since, generally speaking, the latter would be more sensitive to the nonperturbative QCD effects. The purpose of this work is to calculate the inclusive charmless semileptonic decay width of the $B$ meson and estimate the theoretical uncertainty in this calculation, and then using the result to extract $|V_{ub}|$.

The inclusive charmless semileptonic decay of the $B$ meson is induced by the underlying $b \to u$ weak transition. However, experimental data are collected from decay processes at the hadron level, which is connected by the nonperturbative QCD interaction to the underlying quark-level transition. Therefore, in order to calculate the semileptonic decay width, the nonperturbative QCD effect has to be accounted for. A theoretical treatment of the nonperturbative QCD effect has been developed \cite{3–6} on the basis of the light-cone expansion and the heavy quark effective theory (HQET). This approach is from first principles and the nonperturbative QCD effect can be computed in a systematic way, leading to conceptual and practical improvements over the phenomenological models of inclusive decays such as the DIS-like parton model \cite{7} and the ACCMM model \cite{8}. Consequently, model independent predictions and a control of theoretical uncertainties become possible.

In this approach the nonperturbative QCD effects in inclusive semileptonic $B$ decays are ascribed to the distribution function $f(\xi)$, characterizing the matrix element of the light-cone bilocal $b$ quark operator between the $B$ meson states. The distribution function $f(\xi)$ can be interpreted as the probability of finding a $b$ quark with momentum $\xi P$ inside the $B$ meson with momentum $P$. Nonperturbative QCD contributions to the inclusive semileptonic decay of the $B$ meson consist of the dynamic and kinematic components. A crucial observation which emerges from this approach is that both dynamic and kinematic effects of nonperturbative QCD must be taken into account \cite{4}. The latter results in the extension of phase space from the quark level to the hadron level, shown in Fig. 1 for the $b \to u$ decay, and turns out to constitute a large part of the nonperturbative QCD contribution to the decay width, which the heavy quark expansion approach \cite{9} fails to take into account.

The inclusive charmless semileptonic decay width of the $B$ meson can be expressed as a convolution of the soft nonperturbative distribution function with a hard perturbative decay rate $\Gamma_{b \to u\ell\nu}$:
FIG. 1. Phase space for the $b \to u$ inclusive semileptonic decay. The interior of the solid curve is the hadron level phase space (the changeableness of the mass of the final hadronic state is not shown explicitly). The interior of the dashed curve is the quark level phase space.

\[
\Gamma(B \to X_u\ell\nu) = \int_0^1 d\xi f(\xi)\Gamma_{b \to u\ell\nu}(\xi P, \alpha_s) = \frac{G_F^2 M_B^5 |V_{ub}|^2}{192\pi^3} \int_0^1 d\xi f(\xi)\xi^5[1 + \sum_{n=1}^{\infty} c_n(\frac{\alpha_s}{\pi})^n],
\]

where $M_B$ denotes the $B$ meson mass. We have neglected the masses of leptons and the $u$ quark. The interplay between nonperturbative and perturbative QCD effects has been accounted for since the separation of perturbative and nonperturbative effects cannot be done in a clear-cut way. Thus the decay width is expressed in terms of the $B$ meson mass rather than the $b$ quark mass $m_b$. Equation (2) includes the leading twist contribution; higher-twist contributions are expected to be suppressed by powers of $M_B$ (or by Sudakov-exponentials).

In the $f(\xi) \to \delta(\xi - m_b/M_B)$ limit, inclusive decay rates are given by free quark decay. The first term in Eq. (2) corresponds to the Born-term for free quark decay. The remaining terms in Eq. (2) contain the perturbative QCD corrections, present formally as a series in powers of $\alpha_s/\pi$. The perturbative QCD corrections in Eq. (2) are known to first order in $\alpha_s$ with the short-distance coefficient [10]

\[
c_1 = -\frac{2}{3}(\pi^2 - \frac{25}{4}).
\]

Only the parts of the higher-order perturbative corrections originated from the running of the strong coupling, the so-called Brodsky-Lepage-Mackenzie (BLM) corrections [11], were calculated to order $\alpha_s^2$ [12] and to all orders [13]. The BLM corrections are presumably
dominant and may provide an excellent approximation to the full higher-order corrections. In our calculation, we include the second-order BLM correction with the coefficient \[ c_2^{\text{BLM}} = -3.22\beta_0, \] where \( \beta_0 = 11 - 2n_f/3 \) is the first coefficient of the QCD \( \beta \)-function. The second-order BLM correction appears to be rather large. However, this does not mean that the overall \( \alpha_s^2 \) correction to the charmless semileptonic decay width is sizeable and a breakdown of perturbation theory. It was shown \[ [13,14] \] that the large value of the second-order BLM correction is solely due to the use of the pole quark mass in the perturbative calculation. The perturbation series is better behaved when the \( B \) decay rate is written by replacing the \( b \) quark pole mass with the \( \Upsilon \) mass \[ [15] \].

In order to calculate the charmless semileptonic decay width of the \( B \) meson, we need to know the distribution function \( f(\xi) \). Due to current conservation, it is exactly normalized to unity with a support \( 0 \leq \xi \leq 1 \). Two additional sum rules for the distribution function can be obtained by using the operator product expansion and the HQET method \[ [9] \]. These two sum rules determine the mean value \( \mu \) and the variance \( \sigma^2 \) of the distribution function. The mean value is the location of the “center of mass” of the distribution function and the variance is a measure of the square of its width, which specify the basic form of the distribution function. The sum rules read \[ [3–6] \]:

\[
\mu \equiv \int_0^1 d\xi f(\xi) = \frac{m_b}{M_B} \left( 1 - \frac{\lambda_1 + 3\lambda_2}{2m_b^2} \right),
\]

\[
\sigma^2 \equiv \int_0^1 d\xi (\xi - \mu)^2 f(\xi) = \frac{m_b^2}{M_B^2} \left[ -\frac{\lambda_1}{3m_b^2} - \left( \frac{\lambda_1 + 3\lambda_2}{2m_b^2} \right)^2 \right],
\]

where

\[
\lambda_1 = \frac{1}{2M_B} < B|\bar{h}(iD)²h|B >,
\]

\[
\lambda_2 = \frac{1}{12M_B} < B|\bar{h}g_sG_{\alpha\beta}\sigma^{\alpha\beta}h|B >,
\]

are the HQET parameters, which parametrize the nonperturbative QCD effects on a variety of phenomena. The parameter \( \lambda_2 \) can be extracted from hadron spectroscopy:

\[
\lambda_2 = \frac{1}{4}(M_B^2 - M_{B^*}^2) = 0.12 \text{ GeV}^2.
\]

The parameter \( \lambda_1 \) suffers from large uncertainty. Fortunately, from the numerical analysis to be discussed below we find that the result for the charmless semileptonic decay width of the \( B \) meson is insensitive to the variation of \( \lambda_1 \). For our calculation we adopt the value of it from QCD sum rules \[ [16] \]:

\[
\lambda_1 = -(0.5 \pm 0.2) \text{ GeV}^2.
\]
These properties constrain the distribution function considerably.

Nonperturbative QCD methods such as lattice simulation could help determine further the form of the distribution function. The distribution function could also be extracted directly from experiment [7]. Since these are as yet not done, we perform the calculations using the parametrization [4] of the distribution function

\[
f(\xi) = N \frac{\xi(1 - \xi)^\alpha}{[(\xi - a)^2 + b^2]^{\beta/2}} \theta(\xi) \theta(1 - \xi),
\]

where \(a, b, \alpha,\) and \(\beta\) are four parameters, which are constrained by the sum rules (5) and (6), and \(N\) is the normalization constant.

The distribution function \(f(\xi)\) seems to us not unrelated to the shape function introduced in [18] by resumming the heavy quark expansion. It would be interesting to discuss the relation of our approach to theirs. Their expressions for the decay rate are very similar, although somewhat different, to ours obtained in the framework of light-cone expansion. We can reproduce the formulas of [18] by assuming \(\xi = m_b/M_B\). In that case, the scaling variable would be fixed to the definite value, instead of distributing in its entire range from 0 to 1 with the probability \(f(\xi)\). The predictions of these theoretical approaches are subjected to experimental tests.

Including both the nonperturbative and perturbative QCD contributions in the coherent way as described above, we are able to calculate the inclusive charmless semileptonic decay width of the \(B\) meson using Eq. (2). The decay width is evaluated and the theoretical uncertainties are estimated as follows.

- We study the variation of the decay width with respect to the mean value and the variance of the distribution function setting \(\alpha = \beta = 1\) in Eq. (11). Actually, this amounts to the study of the decay width as functions of \(m_b\) and \(\lambda_1\), since, essentially, the mean value of the distribution function is determined by the \(b\) quark mass and its variance is determined by \(\lambda_1\) according to the sum rules in Eqs. (5) and (6). The results are shown in Fig. 2, varying the \(b\) quark pole mass in the range

\[
m_b = 4.9 \pm 0.15 \text{ GeV},
\]

and \(\lambda_1\) in the range of (10). The variation of \(m_b\) leads to an uncertainty of 15% in the decay width if other parameters are kept fixed. A small uncertainty of 4% in the decay width results from the variation of \(\lambda_1\). In other words, the charmless decay width displays a strong dependence on the mean value of the distribution function of the \(b\) quark inside the \(B\) meson, but is insensitive to the variance of the distribution function.

- We examine the further sensitivity of the decay width to the form of the distribution function when keeping the mean value and variance of it fixed, by varying the values of the two additional parameters \(\alpha\) and \(\beta\) in the parametrization (11). We find that the variation of the decay width is typically at the level less than 1% if the form of the distribution function is changed but with the same mean value and variance, hence such a change has a negligible impact on the theoretical uncertainty.
FIG. 2. Dependence of the charmless semileptonic decay width on the theoretical input parameters \( m_b, \lambda_1, \) and \( \mu_r \). The solid (dashed) curves are for the renormalization scale \( \mu_r = m_b/2 \) (\( \mu_r = 2m_b \)). The curves with solid dots, boxes, triangles correspond to \( m_b = 4.75, 4.9, 5.05 \) GeV, respectively.

- We estimate the uncertainty due to the truncation of the perturbative series in Eq. (2) by varying the renormalization scale between \( m_b/2 \) and \( 2m_b \). We also show in Fig. 2 the variation of the decay width due to the change in the renormalization scale. We observe that an uncertainty of 15\% in the decay width stems from the renormalization scale dependence, in agreement with the estimate obtained in [13] based on a different method.

This analysis implies that at present the theoretical error has two main sources: the value of \( m_b \) (or equivalently, the mean value of the distribution function) and the renormalization scale dependence. The functional form of the distribution function, which is not completely known, is likely to cause some model dependence. However, the inclusive charmless semileptonic decay width of the \( B \) meson calculated in this approach is nearly model-independent since it is only sensitive to the mean value of the distribution function, which is known from HQET, but insensitive to the detailed form of the distribution function. It is not surprising that inclusive enough quantities like the total decay width are less sensitive to nonperturbative QCD effects. Finally, adding all the uncertainties in quadrature we find

\[
\Gamma(B \to X_u \ell \nu) = (76 \pm 16)|V_{ub}|^2 \text{ps}^{-1}. \tag{13}
\]

To see the impact of the nonperturbative QCD effect on the decay width, in Fig. 3 we compare the decay widths calculated in our approach, the free quark decay model, and the heavy quark expansion approach. The result in our approach shows that the nonperturbative QCD contributions enhance the decay width by \(+ (7 \pm 5)\% \) with respect to the free quark
FIG. 3. Charmless semileptonic decay width as a function of $\lambda_1$ calculated in our approach (solid curve), the free quark decay model (dotted curve) and the heavy quark expansion approach (dashed curve). We take $m_b = 4.9$ GeV and $\alpha_s = 0.2$.

decay width, in contrast to the result obtained using the heavy quark expansion approach [9] (see also [14]) where a reduction of the free quark decay width by $-(5 \pm 2)\%$ is found. The enhancement is mainly due to the extension of phase space from the quark level to the hadron level, indicating the importance of including the kinematic nonperturbative QCD effect. This result is in accord with the finding of a similar calculation of the inclusive $b \to c$ semileptonic decay width in [4].

The resulting theoretical value for the charmless semileptonic decay width of the $B$ meson in Eq. (13) can be used to determine $|V_{ub}|$. Assuming that the average charmless semileptonic decay width of $b$ hadrons is the same as that of the $B$ meson, using the average $b$ hadron lifetime $\tau_b = 1.564 \pm 0.014$ ps [19] and the inclusive charmless semileptonic branching ratio of $b$ hadrons given in (1) measured by the ALEPH and L3 Collaborations, we extract

$$|V_{ub}| = (3.81 \pm 0.86 \pm 0.40) \times 10^{-3} \quad (\text{ALEPH}),$$
$$5.27 \pm 1.60 \pm 0.55) \times 10^{-3} \quad (\text{L3}),$$

where in each case the first error is experimental and the second theoretical.

In conclusion, we have extracted $|V_{ub}|$ from the inclusive charmless semileptonic branching ratio of $b$ hadrons measured recently by the ALEPH and L3 Collaborations. The inclusive charmless semileptonic decay width of the $B$ meson has been calculated in a QCD-based approach with theoretical uncertainties under control. This approach is able to include both dynamic and kinematic components of the nonperturbative QCD effects. We have shown that including the latter is crucial, such that the charmless decay width is enhanced, which leads to a decrease in the extracted value for $|V_{ub}|$. It is encouraging that the nearly model-independent determination of $|V_{ub}|$ from the inclusive charmless semileptonic branching ratio...
of $b$ hadrons is consistent with those determined by measuring the lepton energy spectrum above the endpoint of the inclusive $b \to c \ell \nu$ spectrum or by measuring the branching ratios for the exclusive decays $B \to \pi \ell \nu$ and $B \to \rho \ell \nu$ \cite{19}.

Currently, the error bar associated with $|V_{ub}|$ determined from the inclusive charmless semileptonic branching ratio of $b$ hadrons is not predominantly theoretical in origin. Both experimental and theoretical efforts are needed to reduce the error. We observe that the uncertainty from perturbative QCD is already one of the major theoretical limiting factors, comparable in size to another one, i.e., the uncertainty from nonperturbative QCD. To reduce the error due to the perturbative QCD correction, a complete calculation to order $\alpha_s^2$ is necessary. Moreover, to reduce the theoretical uncertainty from nonperturbative QCD requires more accurate knowledge of the mean value of the distribution function (or the $b$ quark mass). Higher-twist contributions, which are not included in our calculation, also deserve investigation, which may still yield numerically sizable effects of order $\Lambda_{QCD}/M_B \sim 5\%$.

In order to extract $|V_{ub}|$, the overwhelming “background” of $b \to c$ transitions necessitates the introduction of experimental cuts if statistical errors are to be minimized. The hadronic invariant mass spectrum appears to be a promising observable for discriminating $b \to u \ell \nu$ signal from $b \to c$ background \cite{20}. We believe that in the future a measurement of the differential decay rate $d\Gamma(B \to X_u \ell \nu)/d\xi_u$ will be very useful for a high precision $|V_{ub}|$ determination with a minimum overall (theoretical and experimental) error. It has been shown \cite{17} that this differential decay rate is proportional to the nonperturbative distribution function. The measurement of the decay distribution with respect to the scaling variable $\xi_u$ is of particular significance in a range of issues. They include \cite{17}:

- The $B \to X_u \ell \nu$ decay can be differentiated from the $B \to X_c \ell \nu$ decay even more efficiently than the cut on the hadronic invariant mass.

- A model-independent determination of $|V_{ub}|$ can be obtained by avoiding hadronic complications.

- Improved measurement of the inclusive charmless semileptonic branching ratio of $b$ hadrons can be achieved.

- The nonperturbative distribution function can be extracted directly.

It remains to be seen whether such a measurement is feasible.

**ACKNOWLEDGMENTS**

This work was supported by the Australian Research Council.
REFERENCES

[1] ALEPH Collaboration, R. Barate et al., CERN-EP/98-067 (1998).
[2] L3 Collaboration, M. Acciarri et al., Phys. Lett. B 436 (1998) 174.
[3] C.H. Jin and E.A. Paschos, in Proceedings of the International Symposium on Heavy Flavor and Electroweak Theory, Beijing, China, 1995, edited by C.H. Chang and C.S. Huang (World Scientific, Singapore, 1996), p. 132; hep-ph/9504373.
[4] C.H. Jin, Phys. Rev. D 56 (1997) 2928.
[5] C.H. Jin and E.A. Paschos, Eur. Phys. J. C 1 (1998) 523.
[6] C.H. Jin, Phys. Rev. D 56 (1997) 7267.
[7] A. Bareiss and E.A. Paschos, Nucl. Phys. B327 (1989) 353; C.H. Jin, W.F. Palmer, and E.A. Paschos, Phys. Lett. B 329 (1994) 364.
[8] G. Altarelli, N. Cabibbo, G. Corbo, L. Maiani, and G. Martinelli, Nucl. Phys. B208 (1982) 365.
[9] J. Chay, H. Georgi, and B. Grinstein, Phys. Lett. B 247 (1990) 399; I.I. Bigi, N.G. Uraltsev, and A.I. Vainshtein, Phys. Lett. B 293 (1992) 430; 297 (1993) 477(E); I.I. Bigi, M.A. Shifman, N.G. Uraltsev, and A.I. Vainshtein, Phys. Rev. Lett. 71 (1993) 496; A.V. Manohar and M.B. Wise, Phys. Rev. D 49 (1994) 1310; B. Blok, L. Koyrakh, M.A. Shifman, and A.I. Vainshtein, Phys. Rev. D 49 (1994) 3356; 50 (1994) 3572(E).
[10] N. Cabibbo and L. Maiani, Phys. Lett. 79B (1978) 109.
[11] S.J. Brodsky, G.P. Lepage, and P.B. Mackenzie, Phys. Rev. D 28 (1983) 228.
[12] M. Luke, M.J. Savage, and M.B. Wise, Phys. Lett. B 343 (1995) 329; 345 (1995) 301.
[13] P. Ball, M. Beneke, and V.M. Braun, Phys. Rev. D 52 (1995) 3929.
[14] N.G. Uraltsev, Int. J. Mod. Phys. A 11 (1996) 515; I. Bigi, M. Shifman, and N. Uraltsev, Annu. Rev. Nucl. Part. Sci., 47 (1997) 591.
[15] A.H. Hoang, Z. Ligeti, and A.V. Manohar, hep-ph/9809423.
[16] P. Ball and V. Braun, Phys. Rev. D 49 (1994) 2472.
[17] C.H. Jin, hep-ph/9804249, hep-ph/9808313.
[18] M. Neubert, Phys. Rev. D 49 (1994) 3392; 49 (1994) 4623; T. Mannel and M. Neubert, Phys. Rev. D 50 (1994) 2037; I.I. Bigi, M.A. Shifman, N.G. Uraltsev, and A.I. Vainshtein, Int. J. Mod. Phys. A 9 (1994) 2467.
[19] Particle Data Group, C. Caso et al., Eur. Phys. J. C 3 (1998) 1.
[20] A.F. Falk, Z. Ligeti, and M.B. Wise, Phys. Lett. B 406 (1997) 225; R.D. Dikeman and N. Uraltsev, Nucl. Phys. B509 (1998) 378; I. Bigi, R.D. Dikeman, and N. Uraltsev, Eur. Phys. J. C 4 (1998) 453; C.H. Jin, Phys. Rev. D 57 (1998) 6851.