Mixed Penalty with Gradient, Gradient Projection and Frank Wolfe Methods for Solving Nonlinear Hyperbolic Optimal Control State Constraints

J A Al-Hawasy 1 and E H Al-Rawdanee 1.
1Mustansiriyah University, College of Science, Department of Mathematics, Baghdad

jhawassy17@uomustanriyah.edu.iq

Abstract. This paper is focused on studying the numerical solution (NUSO) for the discrete classical optimal control problem (DISCOPCP) ruled by a nonlinear hyperbolic boundary value problem (NHYBVP) with state constraints (SCONs). When the discrete classical control (DISCC) is given, the existence and uniqueness theorem for the discrete classical solution of the discrete weak form (DISWF) is proved. The proof for the existence theorem of the discrete classical optimal control (DISCOPC) and the necessary and sufficient conditions (NECOs and SUCOs) of the problem are given. Moreover, the DISCOPCP is found numerically from the Galerkin finite element method (GFE) for variable space and implicit finite difference scheme (IFD) for time variable (GFEIFDM) to find the NUSO of the DISWF and then the DISCOPC is found from solving the optimization problem (OPTP) (the minimum of discrete cost functional (DISCF)) by using the mixed Penalty method with the Gradient method (PGMTH), the Gradient projection method (PGPMTH) and the Frank Wolfe method (PFWMTH). Inside these three methods, the Armijo step option (ASO) is used to get a better direction of the optimal search. Finally, illustrative example for the problem is given to exam the accuracy and efficiency of these methods.

1. Introduction

Many numerical methods (NUMTHs) for solving optimal control problems (OPCPs) which are ruled by ordinary differential equations (ODEs) and partial differential equations (PDEs) subject to SCONs are studied and are currently interest in different disciplines for example electroshock therapy [1], wastewater treatment [2], air traffic flow [3], medicine [4], physics [5], chemical [6], engineering [7], and biology [8].

Many researchers were interested in studying the NUSO of OPCPs ruling by various type of PDEs as in [9] who is studied a linear elliptic PDE (LEPDE) with constant coefficient (CCO)), whereas [10] and [11] are discussed nonlinear elliptic PDEs (NLEPDES) with CCO, [10] is studied NLEPDE with variable coefficient (VCO), [12] and [13] are studied a linear parabolic and hyperbolic PDE with CCO, and [14] is studied a nonlinear parabolic PDE with VCO.

These studies encourage us to study the NUSO of the DISCCCP ruling by NHYBVP with SCONs. The CCOPCP is discretized by using the GFEIFDM (the GFE for variable space and the IFD for time variable), with a suitable assumptions. When the DISCC is given, the existence and uniqueness theorem for the DISWF is proved. The proof of existence theorem for the DISCOPC, the NECOs and SUCOs of the problem are given. The DISCOPC problem is found numerically from using the GFEIFDM to find the NUSO of the DISWF and then solving the OPTP (the minimum of the DISCF by using the PGMTH,
the PGPMTH and the PFWMTH to find the DISCOPC. Inside these three methods, the ASO is used separately to get better direction of the optimal search. Finally, illustrative example for the problem is given to show the performance of each of these methods.

2. Description of the CCOPCP

Let $K \subset \mathbb{R}^2$ be a bounded and open region with boundary $\partial K$, and let $E = [0, T]$, $0 < T < \infty$ be a time variable, $\rho = K \times E$ and $\partial \rho = \partial K \times [0, T]$. The CCOPCP ruling by the following NLHYBVP is:

\begin{align}
(1) & \quad \Psi_{tt} + \Delta \Psi = g(\tilde{x}, t, \Psi) + \omega - \omega_d, \text{ in } \rho, \tilde{x} = (y, z) \\
(2) & \quad \Psi(\tilde{x}, t) = 0, \text{ in } \partial \rho \\
(3) & \quad \Psi(t, 0) = \Psi^0(\tilde{x}), \text{ in } K \\
(4) & \quad \Psi_t(t, 0) = \Psi^1(\tilde{x}), \text{ in } K
\end{align}

where the state is denoted by $\Psi = \Psi_{\omega}(\tilde{x}, t) \in H^1_0(\rho)$ and its corresponding CCC is denoted by $\omega = \omega(\tilde{x}, t) \in L^2(\rho)$. $\omega_d = \omega_d(\tilde{x}, t) \in L^2(\rho)$ is the desired control (DS), and $g = g(\tilde{x}, t, \Psi) \in L^2(\rho)$ is a continuous w.r.t. $\rho$ and satisfies:

\begin{align}
(5) & \quad Min G_0(\omega) = \int_0^T \frac{1}{2} (\Psi - \Psi_d)^2 + \frac{\alpha}{2} (\omega - \omega_d)^2 \, d\tilde{x} \, dt \\
\end{align}

where $\Psi_d = \Psi_d(\tilde{x}, t)$ is the desired states (DS).

The SCONs are

\begin{align}
(6) & \quad G_1(\omega) = \int_0^T [g_1(\tilde{x}, t, \Psi, \omega)] \, d\tilde{x} \, dt = 0 \\
(7) & \quad G_2(\omega) = \int_0^T [g_2(\tilde{x}, \Psi, \omega)] \, d\tilde{x} \, dt \leq 0
\end{align}

The set of admissible CCCs is

\begin{align}
(8) & \quad W_{ad} = \{ \omega \in W_{ad} | G_1(\omega) = 0, G_2(\omega) \leq 0 \} \\
\end{align}

The CCOPCP is to find $\omega \in W_{ad}$ which minimize (5).

Now, the WF of problem (1-4) is obtained with $\Psi \in H^1_0(K)$

\begin{align}
(9) & \quad \langle \Psi_{tt}, \varphi \rangle + B(t, \Psi, \varphi) = (g(t, \Psi, \varphi), \varphi)_K + (\omega, \varphi)_K - (\omega_d, \varphi)_K, \forall \varphi \in H^1_0(K) \\
(10) & \quad (\Psi(0), \varphi)_K = (\Psi^0, \varphi)_K \\
(11) & \quad (\Psi_t(0), \varphi)_K = (\Psi^1, \varphi)_K
\end{align}

where $\Psi^0 \in S, \Psi^1 \in L^2(K)$ and $B(t, \Psi, \varphi) = (\nabla \Psi, \nabla \varphi)_K$ is a bilinear form.

2.1. Assumptions:

(i) $|B(t, \Psi, \varphi)| \leq c_1 \parallel \Psi \parallel_1 \parallel \varphi \parallel_1, \forall \Psi, \varphi \in H^1_0(K), t \in E$ and for some $c_1$

(ii) $|B(t, \Psi, \varphi)| \geq c_2 \parallel \Psi \parallel_1^2, \forall \Psi \in H^1_0(K), t \in E$ and for some $c_2$

(iii) Suppose that the function $g$ is defined on $Z \times E \times \mathbb{R}$ continuous w.r.t. $\psi$ and satisfies:

\begin{align}
|g(\tilde{x}, t, \psi)| \leq H(\tilde{x}) + \alpha \parallel \psi \parallel, \text{ for } (\tilde{x}, t) \in \rho, H(\tilde{x}) = H(\tilde{x}, t) \in L^2(K) \text{ and } \alpha \geq 0.
\end{align}

(iv) $|g(\tilde{x}, t, \Psi_1) - g(\tilde{x}, t, \Psi_2)| \leq C \parallel \Psi_1 - \Psi_2 \parallel$, for $\Psi_1, \Psi_2 \in \mathbb{R}$ and $C$ denotes the LIPC.

(v) The CF is of Carathéodory type (CART), and satisfies

\begin{align}
\frac{1}{2} (\Psi - \Psi_d(t))^2 + \frac{\alpha}{2} (\omega - \omega_d(t))^2 \leq \gamma(\tilde{x}) + \theta(\psi)^2, \text{ where } \\
\gamma(\tilde{x}) = \gamma(\tilde{x}, t) \in L^2(K) \text{ and } \theta \geq 0, (\tilde{x}, t) \in \rho.
\end{align}

The above assumptions are also used in its discrete form.

Assume $\Psi_t = \zeta$, then (9-11) is written as

\begin{align}
(12) & \quad \langle \zeta_t, \varphi \rangle + B(t, \Psi, \varphi) = (g(t, \Psi, \varphi), \varphi)_K + (\omega, \varphi)_K - (\omega_d, \varphi)_K, \forall \varphi \in S \\
(13) & \quad (\Psi_t, \varphi)_K = (\zeta, \varphi) \\
(14) & \quad (\Psi(0), \varphi) = (\Psi^0, \varphi), \text{ in } K \\
(15) & \quad (\zeta(0), \varphi) = (\Psi^1, \varphi), \text{ in } K
\end{align}

Under the assumptions 2.1 the existence of a unique solution of (12-15) and the existence of a CCOPC with $W_{ad} \neq \emptyset$, are proved in [15].
The adjoint state $\eta_\omega = \eta$ (for the state equations(SEs)) is given by
\begin{align}
(16) \quad (\eta_{tt}, \varphi)_K + B(t, \eta, \varphi) = & \eta \, g_{\psi}(\tilde{x}, t, \psi) + (\psi - \psi_d(t), \varphi)_K \forall \varphi \in S \\
(17) \quad (\eta, \varphi)_K = (\eta_t, \varphi)_K = 0
\end{align}
where $\eta \in S$. Then the Fréchet derivative (FRD) of CF is given [15] by
\begin{align}
(18) \quad (DG_0(\omega), \delta \omega) = \left( H_\omega(t, \psi, \eta, \omega), \varepsilon \Delta \omega \right)_K = \left( \eta + \sigma(\omega - \omega_d(t)), \varepsilon \Delta \omega \right)_K
\end{align}
where $\delta \omega, \omega \in W_\omega, \Delta \omega = \delta \omega - \omega$.

Then from the NECOs for the CCOPC when $\omega \in W_\omega$ is a CCOPC, there exist multipliers(MURS)
\begin{align}
\lambda_1 \in \mathbb{R}, \quad \sum_{l=0}^2 \lambda_l = 1 \text{ with } \lambda_0 \geq 0, \text{ and } \lambda_2 \geq 0, [15] \text{ s.t.}
(19) \quad \sum_{l=0}^2 \lambda_l (DG_0(\omega), \delta \omega - \omega) \geq 0, \forall \delta \omega, \omega \in W_\omega
(20) \quad \text{and } \lambda_1 [G_i(\omega) - e_2] = 0 \text{ for } l = 2
\end{align}
are equivalent with
\begin{align}
(21) \quad \left( \eta + \sigma(\omega - \omega_d(t)) \right) + g_\omega(t, \psi, \omega, \omega) = \min_{\delta \omega \in W_\omega} \left( \eta + \sigma(\omega - \omega_d(t)) \right) + g_\omega(t, \psi, \omega, \omega)
\end{align}

3. Description of the DISCOPC

This part deals with the DISCOPC which is obtained by employing the GFEIFDM, so consider $B(., ., t)$ is independent of $t$, the region $K$ can be divided into subregions (a polyhedron), i.e. $\forall m_1 \in \mathbb{Z}^+$, let $Z_1^i (i = 1, 2, ..., n)$, $n = \bar{m}^2$, where $\bar{m} = m_1 - 1$ be an admissible regular triangulation of $K$ i.e. $K = \bigcup_{i=1}^{m_1} Z_i^j$, let $y_i, z_i, \forall i = 0, 1, ..., m_1$ be points in polyhedron $K = [0, 1] \times [0, 1]$ and are of equal length $h = \frac{1}{m_1}$, $y_i = ih$ and $z_i = ih, \forall i = 0, 1, ..., m_1$. Second, $\forall m \in \mathbb{Z}^+$, let $E^+_j = \left[ t_j^e, t_j^m \right]$ be a subdivision of the interval $E$ and $\forall j = 0, 1, ..., m - 1$, where each interval has an equal length ($\Delta t = \frac{1}{m}$). Let $S_n \subset S = H_0^1(K)$ be the space of continuous piecewise affine (CPWA) mapping in $K$

The set of DISCCs $W_{\omega}^S$ is $W_{\omega}^S = \{ \tilde{w} = \tilde{w}^S \in W_\omega | \tilde{w}(\alpha, t) = \tilde{w}_{ij} \in U^S \text{ in } \rho_{ij} \}$ with $\rho_{ij} = Z_i^j \times E_j^+$.

The discrete SCONs (DISSCONs) are
\begin{align}
(22) \quad | G_i^0(\omega^S) | \leq e_1^S \text{ and } G_i^S(\omega^S) \leq e_2^S, \text{ where}
\end{align}
\begin{align}
(23) \quad G_i^S(\omega^S) = \Delta t \sum_{j=0}^{m-1} \int_K \frac{1}{2} (\psi_{j+1}^S - \psi_{j}^S)^2 + \frac{\varepsilon}{2} (\omega_{j+1}^S - \omega_d(t_{j}^S))^2 d\tilde{x}
\end{align}
The set of all admissible DISCCs for the DISCOPC is defined by
\begin{align}
W_{ad}^S = \{ \omega^S \in W_{\omega}^S | | G_i^0(\omega^S) | \leq e_1^S, | G_i^S(\omega^S) | \leq e_2^S \}
\end{align}
Hence, the DISCOPC is to find $\tilde{w}^S \in W_{ad}^S$, s.t. $G_i^0(\tilde{w}^S) = \min_{\omega^S \in W_{ad}^S} G_i^0(\omega^S)$
\begin{align}
(24) \quad \text{Now, for each } \varphi \in S_n, \text{ and for } j = 0, 1, ..., m - 1, \text{ the DISWF of (12-15) is}
\end{align}
\begin{align}
(24A) \quad (\zeta_j^+(t_{j+1}^+ - \zeta_j^+, \varphi)_K + \Delta t B(\psi_{j+1}^+, \varphi)_K = \Delta t (g(\psi_{j+1}^+, \varphi), \varphi)_K + \Delta t (\omega_d(t_{j+1}^+), \varphi)_K - \Delta t (\omega_d(t_{j}^+), \varphi)_K
\end{align}
\begin{align}
(24B) \quad (\psi_{j+1}^+ - \psi_j^+, \varphi)_K = \Delta t (\zeta_j^+, \varphi)_K
\end{align}
\begin{align}
(24C) \quad (\psi_{j,0}^+, \varphi)_K = (\psi_{0,0}^+, \varphi)_K
\end{align}
\begin{align}
(24D) \quad (\zeta_0^+, \varphi)_K = (\psi_0^+, \varphi)_K
\end{align}
where $\psi_j^+ = \psi(t_j^+), \zeta_j^+ = \zeta(t_j^+) \in S_n, \forall j = 0, 1, ..., m$, $\psi^+ \in S_n$ and $\psi^1 \in L^2(K)$ are given.

3.1. Assumptions:
(i) Suppose $g$ is defined on $Z_i^j \times E_j^+ \times \mathbb{R}$ (for $i = 1, 2, ..., n$) continuous w.r.t. $\psi_j^+$ and satisfies:
\begin{align}
|g(\tilde{x}, t_j^+, \psi_{j+1})| \leq H_j(\tilde{x}) + |\psi_{j+1}|, \quad \forall j = 0, 1, ..., m - 1, H_j(\tilde{x}) = H(\tilde{x}, t_j^+) \in L^2(K) \text{ and } \alpha \geq 0.
\end{align}
The following assumptions are needed to study the existence of the DISCOPCP.

4.1 Theorem: In addition to assumptions (3.1, 4.1), suppose that $W_{ad}^s \neq \emptyset$ is compact, $G_0^s(\omega^s)$ and $g_2^s$ are convex w.r.t. $\psi^s$ and $\omega^s$, and $g_1^s$ is independent on $\omega^s$, then there exists a DISCOPC.

Proof: From the hypotheses on $W_{ad}^s$ there exists $\omega^s \in W_{ad}^s$, for which $G_0^s(\omega^s) \leq \varepsilon_1^s$ and $G_2^s(\omega^s) \leq \varepsilon_2^s$. But from the Lemmas 4.1 and 4.2, we get that $G_0^s(\omega^s)$ is a continuous functional on the compact set $W_{ad}^s$, thus there exists $\bar{\omega}^s \in W_{ad}^s$, s.t. $G_0^s(\bar{\omega}^s) = \min_{\omega^s \in W_{ad}^s} G_0^s(\omega^s)$. \qed

5. The NECOs for the DISCOPCP

The following assumptions are dealt with find the NECOs for the DISCOPCP.

5.1 Assumptions:

(i) The derivative of the DISCf w.r.t. $\psi_j^s$ and $\omega_j^s$ is of CART, and ($\forall j = 0,1,\ldots, m$) satisfies:

$$\left| \psi_j^s - \psi_a^s (t_j^s) \right| \leq \gamma_j^s (x) + \theta_{

(ii) $g_1^s$, $g_\omega^s$, and $g_{\psi^s}^s$ are of CART defined on $Z^s \times E^s \times \mathbb{R} \times W_{ad}^s$, and satisfies ($\forall j = 0,1,\ldots, m$):

$$\left| g_{\psi^s}^s (\bar{x}, t_j^s, \psi_j^s, \omega_j^s) \right| \leq C_j^s (\bar{x}, t_j^s) + \theta_{jl}^s | \psi_j^s |, \forall \bar{x} \in K,$$

where $C_j^s (\bar{x}, t_j^s)$ is $L^2(K)$ and $\theta_{jl}^s \geq 0$.

5.1 Theorem: Dropping the index $l$ in $g_1^s$, for each $l = 1,2$, assume that the DISCF (23) is given and the discrete adjoint weak form (DISAJWF) (for the SEs) $\eta_{m,j}^s = \eta^s = (\eta_0^s, \eta_1^s, \ldots, \eta_{m-1}^s)$ is given (for $j = m - 1, m - 2, \ldots, 0$) by

$$\eta_{j+1}^s - \eta_j^s = \Delta t$$

where $\Delta t$ is sufficiently small.
where \( \eta_j^p = \eta(t_j^p), \phi_j^p = \phi(t_j^p) \in S_n \) (\( \forall j = 0,1, ..., m \)). Then the FRD of \( G \) is given by
\[
(DG^s(\omega^S), \bar{\omega}^S - \omega^s) = \lim_{\varepsilon \to 0} G(\omega^s + \varepsilon \Delta \omega^s) - G(\omega^s) = \Delta t \sum_{j=0}^{m-1} (H_{\omega_j^p}(t_j^p, \psi_j^{p+1}, \eta_j^p, \omega_j^p), \varepsilon \Delta \omega_j^p)_K
\]
where \( \bar{\omega}_j^S, \omega_j^S \in W_{\omega_j^p}, \Delta \omega_j^S = \bar{\omega}_j^S - \omega_j^S \) for \( j = 0, 1, ..., m \).

Proof: Suppose that \( \psi_j^S = \psi_j^p + \Delta \phi_j^p, \psi_j^p \) are two solutions of (24a), setting \( \varphi = \eta_j^p \), using the FRD on the RHS of the obtained equation, summing over \( j \) (for \( j = 0 \) to \( j = m - 1 \)) and then applying the inequality of Minkowiski, to get
\[
\Delta t \sum_{j=0}^{m-1} \left( g \psi_j^{p+1} (t_j^p, \psi_j^{p+1}, \omega_j^p) \right)_K = \Delta t \sum_{j=0}^{m-1} \left( g \psi_j^{p+1} (t_j^p, \psi_j^{p+1}, \omega_j^p) \right)_K + \Delta t \sum_{j=0}^{m-1} \left( g \psi_j^{p+1} (t_j^p, \psi_j^{p+1}, \omega_j^p) \right)_K + Q_1(\varepsilon) \| \Delta \omega^S \|_p
\]
By setting \( \varphi = \Delta \phi_j^p \psi_j^{p+1} \) in (25), and summing over \( j \) (for \( j = 0 \) to \( j = m - 1 \)), to get
\[
\Delta t \sum_{j=0}^{m-1} \left( \eta_j^p g \psi_j^{p+1} (t_j^p, \psi_j^{p+1}, \omega_j^p) \right)_K
\]
Subtract (29) from (30), to obtain
\[
\Delta t \sum_{j=0}^{m-1} \left( \Delta \phi_j^p \Delta \psi_j^{p+1} \right)_K = \Delta t \sum_{j=0}^{m-1} \left( g \psi_j^{p+1} (t_j^p, \psi_j^{p+1}, \omega_j^p) \right)_K + Q_1(\varepsilon) \| \Delta \omega^S \|_p
\]
(32) Since \( \Delta t \sum_{j=0}^{m-1} \left( \Delta \phi_j^p \Delta \psi_j^{p+1} \right)_K = \int_0^T \left( \Delta \phi^S \right)_K dt \)
and
(32b) \( \Delta t \sum_{j=0}^{m-1} \left( \Delta \phi_j^p \Delta \psi_j^{p+1} \right)_K = \int_0^T \left( \Delta \phi^S \right)_K dt \)
Then \( \int_0^T \left( \Delta \phi^S \right)_K dt = \int_0^T \left( \Delta \phi^S \right)_K dt \)
Using (33) in (31), gives
\[
\Delta t \sum_{j=0}^{m-1} \left( \psi_j^{p+1} - \psi_d (t_j^{p+1}) + \Delta t \sum_{j=0}^{m-1} \left( g \psi_j^{p+1} (t_j^p, \psi_j^{p+1}, \omega_j^p) \right)_K + Q_1(\varepsilon) \| \Delta \omega^S \|_p
\]
On the other hand, since the FRD of \( G \) exists, i.e. with \( Q_1(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \)
\[
G^S(\omega^s + \varepsilon \Delta \omega^s) = G^S(\omega^s) = \Delta t \sum_{j=0}^{m-1} \left( \psi_j^{p+1} - \psi_d (t_j^{p+1}) + \Delta t \sum_{j=0}^{m-1} \left( g \psi_j^{p+1} (t_j^p, \psi_j^{p+1}, \omega_j^p) \right)_K + Q_2(\varepsilon) \| \Delta \omega^S \|_p
\]
Substituting (34) in (35), yields to with \( Q_3(\varepsilon) = Q_1(\varepsilon) + Q_2(\varepsilon) \to 0 \), as \( \varepsilon \to 0 \)
\[ G^s(\omega^s + \varepsilon \Delta \omega^s) - G^s(\omega^s) = \Delta t \sum_{j=0}^{m-1} \left( \varepsilon \alpha_j^s - \omega_d(t_j^s) \right) + g_{s\omega_j^s}^s(t_j^s, \psi_{j+1}, \omega_j^s), \varepsilon \Delta \omega_j^s \right)_K + \\
\varepsilon \Delta t \sum_{j=0}^{m-1} \left( \Delta \omega_j^s, \eta_j^s \right)_K + Q_3(\varepsilon) \| \Delta \omega^s \|_\rho \]

Finally, taking the limit when \( \varepsilon \to 0 \) the FRD of the functional \( G \) is

\[ (DG^s(\omega^s), \bar{\omega}^s - \omega^s) = \Delta t \sum_{j=0}^{m-1} \left( \eta_j^s + \alpha(\omega_j^s - \omega_d(t_j^s)) + g_{s\omega_j^s}^s(t_j^s, \psi_{j+1}, \omega_j^s), \varepsilon \Delta \omega_j^s \right)_K \]

5.1 Lemma: The operator \( \omega^s \to \eta^s = \eta^s_{\omega^s} \) is continuous.

Proof: similar to the proof of lemma 4.1.

5.2 Lemma: The operator \( \omega^s \to DG^s(\omega^s) \) is continuous w.r.t. \( \omega^s \) for each \( l = 0,1,2 \).

Proof: From the assumptions on the DISCOP and \( g_i^s (\forall l = 1,2) \), Lemmas 4.1, 5.1 and Proposition (1.2) [17]. Then \( DG^s(\omega^s) \) is continuous w.r.t. \( \omega^s (\forall l = 0,1,2) \)

5.2 Theorem: If \( W^s_{\omega^s} \) is convex, \( \omega^s \in W^s_{\omega^s} \) is a DISCOPC of the considered DISCOPCP, then there are MURS \( \lambda_i^s \in \mathbb{R} (l = 0,1,2) \) satisfy \( \sum_{i=0}^{2} | \lambda_i^s | = 1 \) with \( \lambda_0^s \geq 0 \), and \( \lambda_2^s \geq 0 \), s.t.

\[ \sum_{i=0}^{2} \lambda_i^s(\Delta G^s(\omega^s), \bar{\omega}^s - \omega^s) \geq 0, \forall \bar{\omega}^s, \omega^s \in W^s_{\omega^s} \]

and \( \lambda_i^s \{ G^s(\omega^s) - \varepsilon \bar{\omega}^s \} = 0 \) for \( l = 2 \)

where \( \eta_j^s = \sum_{i=0}^{2} \lambda_i^s \eta_j^s \) in the definition of \( H_j^s = \sum_{i=0}^{2} H_{i\omega_j^s} \)

If \( W^s_{\omega^s} = 0 \), \( \bar{\omega}^s = \bar{\omega}^s_{\omega^s} \in U^s, j = 0,1,\ldots, m - 1 \) with \( U^s \in \mathbb{R} \), then the (37) and (38) are equivalent \( \forall j, 0 \leq j \leq m - 1 \), and \( \forall i = 1,2,\ldots, n \) with

\[ \left( \eta_j^s + \alpha(\omega_j^s - \omega_d(t_j^s)) + g_{s\omega_j^s}^s(t_j^s, \psi_{j+1}, \omega_j^s), \omega_j^s \right)_T \]

Proof: the functionals \( G_i^s(\omega^s) (l = 0,1,2) \) are continuous w.r.t. \( \omega^s \) on \( L^2(\rho) \) (from lemma (4.2)), and the functional \( DG^s(\omega^s) = (DG_i^s(\omega^s), \bar{\omega}^s - \omega^s) (l = 0,1,2) \) are continuous w.r.t. \( \omega^s \) and \( \bar{\omega}^s \), and are linear w.r.t. \( \bar{\omega}^s - \omega^s \) (from lemma (5.2), with using assumption (4.1)), then the functionals \( G_i^s(\omega^s) \) (for each \( l = 0,1,2 \)) are M-differentiable \( \forall M \in \mathbb{Z}^+ \). Then by theorem (Kuhn-Tucker Lagrange MURS), there are "MURS" \( \lambda_i^s \in \mathbb{R} (l = 0,1,2) \), \( \sum_{i=0}^{2} | \lambda_i^s | = 1 \) with \( \lambda_0^s \geq 0 \), and \( \lambda_2^s \geq 0 \), s.t. (37) and (38) are obtain. Now, by using theorem 4.1, then for each \( \bar{\omega}^s, \omega^s \in W^s_{\omega^s} \) inequality (37) becomes:

\[ \Delta t \sum_{j=0}^{m-1} \int_K \left[ \lambda_0^s\eta_0^s + \lambda_2^s \alpha(\omega_j^s - \omega_d(t_j^s)) + \sum_{i=1}^{2} \lambda_i^s \eta_i^s \right] + \left( \sum_{i=1}^{2} \lambda_i^s \right) \frac{d^2 \psi_{i\omega_j^s}(t_j^s, \psi_{j+1}, \omega_j^s)}{dt^2} \] \( d \bar{x} \geq 0 \)

Put \( \eta_j^s = \sum_{i=0}^{2} \lambda_i^s \eta_i^s + \alpha(\omega_j^s - \omega_d(t_j^s)) + g_{s\omega_j^s}^s(t_j^s, \psi_{j+1}, \omega_j^s) \) \( + \sum_{i=1}^{2} \lambda_i^s \omega_j^s \), then

\[ \Delta t \sum_{j=0}^{m-1} \int_K \left[ \eta_j^s + \alpha(\omega_j^s - \omega_d(t_j^s)) + \left( g_{s\omega_j^s}^s(t_j^s, \psi_{j+1}, \omega_j^s) \right) \right] \right) d \bar{x} \geq 0 \]

for each \( \bar{\omega}^s, \omega^s \in W^s_{\omega^s} \)

To show (40) is equivalent to (39). Let \( \bar{\omega}_j^s = \omega_j^s \), for all \( j \) except once(say \( v \)) , then from (40) we have

\[ \left( \eta_v^s + \alpha(\omega_v^s - \omega_d(t_v^s)) + g_{s\omega_v^s}(t_v^s, \psi_{v+1}, \omega_v^s), \omega_v^s \right) = \min_{\omega_v^s \in W^s_{\omega^s}} \left( \eta_v^s + \alpha(\omega_v^s - \omega_d(t_v^s)) + g_{s\omega_v^s}(t_v^s, \psi_{v+1}, \omega_v^s), \omega_v^s \right) \]

but \( v \) is arbitrary, then \( \forall j = 0,1,\ldots, m - 1 \), we get
\[
\left( \eta^s_j + \sigma \left( \omega^s_j - \omega_d(t^s_j) \right) + g^s_{\omega^s}(t^s_j, \psi^s_j, \omega^s_j) \right) \\
= \min_{\omega_j \in \Omega_j} \left( \eta^s_j + \sigma \left( \omega^s_j - \omega_d(t^s_j) \right) + g^s_{\omega^s}(t^s_j, \psi^s_j, \omega^s_j) \right)
\]

So (39) is obtained \( \forall j = 0, 1, ..., m - 1, i = 1, ..., n \). The proof of (39) is equivalent to (40) is easy.

6. The SUCOs for the DISCOPCP

6.1. Theorem:

Consider the assumptions (3.1, 4.1, 5.1), \( W_0^s \) is convex, \((\tilde{x}, t^s_j, \psi^s_j) \) & \( \tilde{g}^s(x, t^s_j, \psi^s_j, \omega^s_j) \) are affine w.r.t. \( \psi^s_j \) & \( (\psi^s_j, \omega^s_j) \), \( \forall (\tilde{x}, t^s_j)(\forall j = 0, 1, ..., m - 1) \), \( G_0^s, \tilde{g}^s_1(\tilde{x}, t^s_j, \psi^s_j, \omega^s_j) \) is convex w.r.t. \( \psi^s_j, \omega^s_j \) & \( \forall (\tilde{x}, t^s_j) \) and \( \forall j = 0, 1, ..., m - 1 \). Then the NECOs in theorem (5.2) with \( \lambda_0^s > 0 \) are also sufficient with \( \lambda_0^s > 0 \).

Proof: From assumptions (3.1, 4.1 & 5.1) and Lemma 4.2, the functional \( G_i^s(\omega^s_j) \) is M-local continuous at each \( \omega^s_j \in W_0^s \). Also from the assumptions and Theorem 5.1, the functional \( G_i^s(\omega^s_j) \) is M-differentiable at each \( \omega^s_j \in W_0^s \), \( \forall s \), and since \( W_0^s \subset L^2(\rho) \) and \( L^2(\rho) \) is open, then

\[ DG_i^s(\omega^s_j, \omega^s_j - \omega^s) = DG_i^s(\omega^s_j)(\omega^s_j - \omega^s), \text{ for } i = 0, 1, 2 \]

Assume \( \omega^s_j \in W_{ad}^s \) with \( \omega^s_j \) satisfied (37-38), i.e. there are MURS \( \lambda_i^s \in \mathbb{R} (\forall l = 0, 1, 2) \), \( \sum_{l=0}^{2} | \lambda_i^s | = 1 \) with \( \lambda_0^s \geq 0 \), and \( \lambda_2^s \geq 0 \), s.t.

\[ \Delta t \sum_{j=0}^{m-1} [\eta^s_j + \lambda_0^s \sigma(\omega^s_j - \omega_d(t^s_j))] + g^s_{\omega^s}(t^s_j, \psi^s_j, \omega^s_j)](\omega^s_j - \omega^s) \geq 0, \forall \omega^s_j, \omega^s_j \in W_0^s, \text{ and} \]

\[ \lambda_2^s \left[ G_i^s(\omega^s_j) - \epsilon_i^s \right] = 0 \]

Let

\[ G_i^s(\omega^s_j) = \sum_{l=0}^{2} \lambda_i^s G_i^s(\omega^s_j), \]

with \( G_i^s(\omega^s_j) = G_0^s(\omega^s_j), G_1^s(\omega^s_j) = G_i^s(\omega^s_j) - \epsilon_i^s \), and \( G_2^s(\omega^s_j) = G_2^s(\omega^s_j) - \epsilon_2^s \)

Now, since \( DG_i^s(\omega^s_j) = DG_i^s(\omega^s_j), \forall \lambda = 0, 1, 2 \), and \( j = 0, 1, ..., m - 1 \).

\[ DG_i^s(\omega^s_j) \Delta \omega^s_j = \sum_{l=0}^{2} \lambda_i^s DG_i^s(\omega^s_j) \Delta \omega^s_j \]

\[ = \Delta t \sum_{j=0}^{m-1} \int_{K} [\lambda_0^s \eta^s_j + \lambda_0^s \sigma(\omega^s_j - \omega_d(t^s_j))] + \left( \sum_{i=1}^{l} \lambda_i^s \eta_j^s \right) + \left( \sum_{i=1}^{l} \lambda_i^s \tilde{g}_{l,i}(t^s_j, \psi^s_j, \omega^s_j) \right)](\omega^s_j - \omega^s) d\tilde{x} \]

\[ = \Delta t \sum_{j=0}^{m-1} \left[ \eta_j^s + \sigma(\omega^s_j - \omega_d(t^s_j)) + g^s_{\omega^s}(t^s_j, \psi^s_j, \omega^s_j)](\omega^s_j - \omega^s) \right] \]

\[ = \Delta t \sum_{j=0}^{m-1} H^s_j(t^s_j, \psi^s_j, \omega^s_j, \omega^s_j) \Delta \omega^s_j \geq 0, \text{ where} \]

\[ \eta^s_j = \sum_{l=0}^{2} \lambda_i^s \eta_j^s \text{ and } \sigma(\omega^s_j - \omega_d(t^s_j)) + g^s_{\omega^s}(t^s_j, \psi^s_j, \omega^s_j) = \lambda_0^s \sigma(\omega^s_j - \omega_d(t^s_j)) + \sum_{l=0}^{2} \lambda_i^s \tilde{g}_{l,i}(t^s_j) \]

Then \( DG_i^s(\omega^s_j) \Delta \omega^s_j \geq 0 \)

Now, let \( \omega^s \) and \( \tilde{\omega}^s \) are given DISCCs, and

\[ g_i(\tilde{x}, t^s_j, \psi^s_j) = g_i(\tilde{x})\psi^s_j + g_2(\tilde{x}), \psi^s = \psi^s_{\omega^s} \text{ and } \tilde{\psi}^s = \tilde{\psi}^s_{\omega^s} \text{ are their corresponding solutions,} \]

i.e. \( (\tilde{\psi}^s_{j+1}, \phi_i)_K + (\Delta t)^2 B(\psi^s_{j+1}, \phi_i)_K = (\psi^s_j, \phi_i)_K + (\Delta t)^2 (\omega_d(t^s_j), \phi_i)_K + \]

\[ (\Delta t)^2 (g_i(\tilde{x})\psi^s_j + g_2(\tilde{x}), \phi_i)_K + (\Delta t)^2 (\omega^s(t^s_j), \phi_i)_K - (\Delta t)^2 (\omega_d(t^s_j), \phi_i)_K \] (42a)

\[ (\psi^s_{j+1} - \psi^s_j, \phi_i)_K = \Delta t (\tilde{c}_{j+1}, \phi_i)_K \] (42b)

\[ (\phi_i^0), (\phi_i)_K = (\psi^0, \phi_i)_K \] (42c)

\[ (\tilde{\psi}^s_{j+1}, \phi_i)_K = (\Delta t)^2 B(\psi^s_{j+1}, \phi_i)_K = (\tilde{\psi}^s_j, \phi_i)_K + \Delta t (\tilde{c}_{j+1}, \phi_i)_K + \]


\[(\Delta t)^2 (g_1(x)\tilde{v}_{t+1}^s + g_2(x), \varphi_t) + (\Delta t)^2 (\omega_d(t_j^s), \varphi_t) - (\Delta t)^2 (\omega_d(t_j^s), \varphi_t)\]
\[(\psi_{j+1}^s - \psi_j^s, \varphi_t) = (\Delta t)^2 B ((\alpha\psi_{j+1}^s + (1 - \alpha)\psi_j^s, \varphi_t) + (\Delta t)^2 B (\alpha\psi_{j+1}^s + (1 - \alpha)\psi_j^s, \varphi_t)\]
\[(\psi_j^s, \varphi_t) = (\psi_j^s, \varphi_t)\]
\[\tilde{v}_{j+1}^s = \tilde{v}_j^s + \Delta t (\tilde{c}_{j+1}^s, \varphi_t)
\[\tilde{v}_j^s, \varphi_t) = (\psi_j^s, \varphi_t)\]

By multiplying (42a, 42b, 42c, 42d) by \(\alpha \in [0, 1]\), and (43a, 43b, 43c, 43d) by \((1 - \alpha)\), then adding the equations that obtained from the pairs (42a)(43a), (42b)(43b), (42c)(43c), (42d)(43d), we get
\[(\alpha\psi_{j+1}^s + (1 - \alpha)\psi_j^s, \varphi_t) + (\Delta t)^2 B (\alpha\psi_{j+1}^s + (1 - \alpha)\psi_j^s, \varphi_t)\]
\[= (\alpha\psi_{j+1}^s + (1 - \alpha)\psi_j^s, \varphi_t) + \Delta t (\tilde{c}_{j+1}^s, \varphi_t) + (\Delta t)^2 (\tilde{g}_1(x)(\alpha\psi_{j+1}^s + (1 - \alpha)\psi_j^s, \varphi_t) + (\alpha\psi_{j+1}^s + (1 - \alpha)\psi_j^s, \varphi_t) + (\Delta t)^2 (\tilde{g}_2(x)(\alpha\psi_{j+1}^s + (1 - \alpha)\psi_j^s, \varphi_t) + (\alpha\psi_{j+1}^s + (1 - \alpha)\psi_j^s, \varphi_t)\]

Now, if \(\omega^s \xrightarrow{\alpha \omega^s} \psi_s^s\) is convex- linear w.r.t. \((\psi_j^s, \omega_j^s), \forall x \in K\). Then therefore \(g_2^s(x, t_j^s, \psi_j^s, \omega_j^s)\) is convex-linear w.r.t. \(\psi_j^s, \omega_j^s, \forall (x, t_j^s), \forall j \in \{0, 1, \ldots, m - 1\}\), and the MURS are defined by
\[G_2^s(\omega_j^s) = \sum_{i=0}^{\omega_j^s} (\lambda_2^s \tilde{G}_{x}^s(\omega_j^s) + \lambda_2^s \tilde{G}_{x}^s(\omega_j^s) \leq \lambda_2^s \tilde{G}_{x}^s(\omega_j^s) + \lambda_2^s \tilde{G}_{x}^s(\omega_j^s) \leq \lambda_2^s \tilde{G}_{x}^s(\omega_j^s) + \lambda_2^s \tilde{G}_{x}^s(\omega_j^s) \leq \lambda_2^s \tilde{G}_{x}^s(\omega_j^s), \forall \omega_j^s \in W_2^s\]
\[G^s(\psi_j^s) \leq \min_{\omega_j^s \in W_2^s} G^s(\omega_j^s), \forall \omega_j^s \in W_2^s,\text{ then } G^s(\psi_j^s) = \max_{\omega_j^s \in W_2^s} G^s(\omega_j^s), \text{ i.e. } \omega_j^s \text{ is a DISCOCS.}\]

7. Optimization methods
The following algorithm describes the PGMTH, PFWMTH and PGPMTH with ASO.

7.1. Algorithm
Let \(b, c \in (0, 1), \{\delta_k^j\}\) be a sequence, with \(\delta_k^j \in (0, \infty), \delta_k^j \in (0, 1)\) (for each integer \(k, j\), \(\mu > 0\), \(\omega^0 \in U\) be an ICT, and a sequence \(\{\varepsilon_k\}\) for each \(k\), with \(\varepsilon_k > 0\) and \(\varepsilon_k \to 0\) as \(k \to \infty\).

**Step 1:** Set: \(j = 0, k = 1, j_{k-1} = j_0 = 0\).

**Step 2:** Solve DISWF (DISAWF) by using the GFE to get \(\psi^s(\eta^s)\), and then calculate \(G_1(\omega^j, l = 0, 1, 2)\).

**Step 3:** Calculate the penalized discrete function
\[G_{G_2}(\omega^j) = G_0(\omega^j) + \frac{1}{2} M_{1k} \left[G_1(\omega^j)\right]^2 + \frac{1}{2} M_{2k} \left[\max(0, G_2(\omega^j))\right]^2, \text{ where } M_k \to \infty \text{ as } k \to \infty\]

**Step 4:** Find \(D\) \(G_{G_2}(\omega^j) = D_G(\omega^j) + M_{1k} G_1(\omega^j) G_1(\omega^j) + M_{2k} \max(0, G_2(\omega^j)) G_2(\omega^j)\), and the MURS are defined by \(M_{1k} = M_{2k} = 2k \max(0, G_2(\omega^j))\)

**Step 5:** Find a direction point (new control) \(u^* \in U\) (i.e. a direction \(u^* - \omega^j\)), by using the following methods separately:

(a) **PGMTH:** Find \(u^* \in U, \text{ s.t. } u^* = \omega^j - \frac{1}{\mu} D G_{G_2}(\omega^j)\)
(b) PFWMTH: Find \( u^j \in U \) , s.t. \( (DGG_k(\omega^j), u^j - \omega^j) = \min_{u \in U} (DGG_k(\omega^j), u - \omega^j) \)

(c) PGPMTH : Find \( u^j \in U \) , s.t.
\[
\xi^j = (DGG_k(\omega^j), u^j - \omega^j) + \frac{\mu}{2} \| u^j - \omega^j \|^2 = \min_{u \in U} (DGG_k(\omega^j), u - \omega^j) + \frac{\mu}{2} \| u - \omega^j \|^2
\]

**Step 6:** Calculate \( \xi^j = (DGG_k(\omega^j), u^j - \omega^j) \) (in the GMTH \( \xi^j = -\frac{1}{\mu} \| DGG_k(\omega^j) \|^2 \))

If \( |\xi^j| \leq \varepsilon \), set \( J_k = J, \omega_k = \omega^j, u_k = u^j \), \( J = J + 1 \), and go to the step 2.

**Step 7:** Choose \( \delta^j \) by using the following method:

**ASO:** Assume an initial value \( \delta_k^j \in [0, +\infty) \) (or \( \delta_k^j \in [0, 1] \)). If \( \delta_k^j \) satisfies the equality

\[
X_k^j(\delta_k^j) = G_G_k \left( \omega^j + \delta_k^j(u^j - \omega^j) \right) - G_G_k(\omega^j) \leq \delta_k^j b_k^j
\]

We set \( \delta_k^j := \delta_k^j/c \), and choose for \( \delta_k^j \) the last \( \delta_k^j \in (0, \infty) \) that satisfies the above inequality. If not obtained, we set \( \delta_k^j := \delta_k^j/c \), and choose for \( \delta_k^j \) the first \( \delta_k^j \in (0, \infty) \) (or \( \delta_k^j \in (0, 1] \) in PGMTH) that satisfies the above inequality.

**Step 8:** Set \( \omega^{j+1} = \omega^j + \delta_k^j(u^j - \omega^j), J := J + 1 \) and we go to step 2.

8. Numerical examples

In this section, some illustrative example is given to exam the activity of the methods which are described in this paper. The GFE is used in step (2) of algorithm (9.1) to find the DISCO \( \psi^s(\eta^s) \), with \( n = 9, m = 20 \), \( (\Delta t = \frac{1}{20}) \), the parameters are taken the value \( b = c = 0.5 \), and \( \mu = 0.5 \) in the PGMTH, PGPMTH and PFWMTH.

8.1. Example:

Consider the following CCOPCP governed by the NLHYBVP
\[
\psi_{tt} - \Delta \psi = 6t(e^x - 1)(x - 1) + 6tye^x(y - 1) + 3tye^x(y - 1)(z - 1) + \sin(\psi) - \sin(\psi_d) + \omega - \omega_d, \text{ in } \rho, \text{ where } \rho = K \times E, \tilde{x} = (y, z), E = [0, 1], \text{ and } K = [0, 1] \times [0, 1]
\]

with the BCO \( \psi(\tilde{x}, 0) = 0 \), in \( \partial \rho = \partial K \times [0, T] \).

And ICOs \( \psi(\tilde{x}, 0) = 0 \), in \( K \), \( \psi(\tilde{x}, 0) = 3(y(1 - y)(1 - z)(1 - e^z), \text{ in } K \)

The control constraint is \( U = [-0.6, 1.2], \text{ the CF is as given in (5) with } \omega = 1, \psi_d(\tilde{x}, t) = 3y(1 - y)(1 - z)(1 - e^z), \forall (\tilde{x}, t) \in \rho, \text{ and } \omega_d(\tilde{x}, t) \)

\[
= \begin{cases} 
0.25 & \text{for } 0 \leq t < 0.5 \\
-0.5 & \text{for } 0.5 \leq t \leq 1
\end{cases}
\]

with \( \omega_0(\tilde{x}, t) = 0.9 - t \sin(t), \forall (\tilde{x}, t) \in \rho, \text{ with } \epsilon_k = 0.1e-03 \) and

\[
G_1(\omega) = \int_\rho \left| \psi(\tilde{x}, t) + 0.06 \right| d\tilde{x} dt = 0
\]

The following results according to each of the following optimization methods are obtained:

(I) In the PGMTH: the OPCT and CORS are obtained after \( J = 41 \) iterations and are shown in the figures 1 and 2 at \( t=0.5 \), while the value of the DISCF is \( G_0(\omega^s) = 3.0554e-04 \), the \( G_1(\omega^s) = -3.1926e-08 \), with the value of \( \xi^s = 1.4864e-04 \)
Figure 1. DISCOPC at $t=0.5$

In the PFWMTH: the OPCT and CORS are obtained after $J = 53$ iterations and are shown in the figures 3, 4 at $t=0.5$, while the value of the DISCF is $G_0(\omega^*)=4.6719e-04$, $G_1(\omega^*)=-3.6563e-03$, and $\xi^*=-3.9e-04$

Figure 2. Corresponding optimal state at $t=0.5$

Figure 3. DISCOPC at $t=0.5$

Figure 4. Corresponding optimal state at $t=0.5$

(III) In the PGPMTH with ASO: the OPCT and CORS are obtained after $J = 36$ iterations and are shown in the figure 5, 6 at $t=0.5$ and $t=0.9$ respectively, while the value of the DISCF is $G_0(\omega^*)=3.04498e-04$, the SCON is $G_1(\omega^*)=-6.7326e-06$, and the value of $\xi^*=-1.3512e-04$

Figure 5. DISCOPC at $t=0.5$

Figure 6. Corresponding optimal state at $t=0.5$

9. Conclusions

I. The existence and uniqueness theorem for the DISWF is proved, when the DISCC is given.

II. The proof of existence theorem for the DISCOPC, the NECOs and SUCOs of the problem are given.

III. The DISCOPC for the DISCOPCP governing by the NLHYBVP is found through applying the GFEIFDM and the three gradients methods; the PGPMTH, PGMTH and the PFWMTH. The GFEIFDM (with step length of the space variable $h=0.1$ and step length of the time variable $\Delta t=0.05$) is utilized to find the NUSO of DISWF as well as the DISAJWF. While each of the PGPMTH, PGMTH and the PFWMTH (with the ARSO is employed to improve the NUSO of the DISCOC) are utilized to minimize the DISCF with the STCONs in order to find the DISCOPC (with the parameters values $b=0.5$, $c=0.5$ and $\mu=0.5$). We conclude from the results of the example that:

a) The GFEIFDM is an appropriate and fast method to solve the DISWF and DISAJWF.

b) The results which are obtained by applying the PGPMTH and PGMTH with the ASO are better than those are obtained by using the PFWMTH with the ASO.
References

[1] Bendahmane M, Nagaiah C, Comte É and Ainseb B 2016 A 3D Boundary Optimal Control for the Bidomain Bath System Modeling the Thoracic Shock Therapy for Cardiac Defibrillation (Journal of Mathematical Analysis and Applications) 437 (2).

[2] Martinez A., rodriiguez C and vázquez M 2000 Theoretical and Numerical Analysis of an Optimal Control Problem Related to Wastewater Treatment ( SIAMJ Control Optim.) Vol 38, No 5, P 1534-1553.

[3] Strub I and Bayen A 2006 Optimal Control of Air traffic Networks Using Continuous Flow Model (AIAA Conference on Guidance, Control and Dynamics, Keystone, Colorado) Vol 3, P 1700-1710.

[4] Ng K and Rohanin A 2012 Numerical Solution for PDE- Constrained Optimization Problem in Cardiac Electrophysiology (International Journal of Computer Applications) Vol 44, No 12, P 11-15.

[5] Tereshko D 2016 Discrete Optimization of Unsteady Fluid Flows (CEUR Workshop Proc.) 1623, Pp 293-302.

[6] Maidi A and Corriou J 2012 Optimal Control of Nonlinear Chemical Processes Using the Variational Iteration Method (IFAC Symposium Advanced Control of Chemical Processes) Vol 45, Issue 15, P 898-903.

[7] Munteanu I, Bratuć A, Cutululis N and Ceanga E 2008 Optimal Control of Wind Energy Systems Towards a Global Approach ( Springer-Advance in Industrial Control).

[8] Lenhart S. and Workman J 2007 Optimal Control Applied to Biological Model (Mathematical and Computational Biology).

[9] Bahaa G. and Khidr S 2019 Numerical Solution for Optimal Control Problem Governed by Elliptic System on Lipschitz Domain (Journal of Taibah University for Science) Vol 13, Issue 1, P 41-48.

[10] Lubysev F and Manapova A 2019 An Approximation of Problems of Optimal Control on the Coefficients of Elliptic Convection Diffusion Equation with an Imperfect Contact Matching Condition (Journal SVMO) Vol. 21, No 2, P 187-214.

[11] Chryssoverghi I 2007 Mixed Discretization Optimization Methods for Nonlinear Elliptic Optimal Control (LNCS. Springer- Heidelberg) Vol 4310, P 287-295.

[12] Mohammadi B and Pironeau O 2001 Applied Shape Optimization for Fluids (New York, the Clarendon Press, Oxford University Press).

[13] Guliyev H and Nazarova V 2014 An Optimal Control Problem for System Hyperbolic Differential Equation with Constant Coefficient (Georgian Mathematical Journal) Vol 22, Issue 2.

[14] Chryssoverghi I, Coletos J and Kokkinis B 2009 Classical and Relaxed Optimization Methods for Nonlinear Parabolic Optimal Control Problem (LNCS. Springer- Heidelberg) Vol 5910, P 247-255.

[15] Chryssoverghi I and Al-Hawasy J 2010 The Continuous Classical Optimal Control Problem of a Semilinear Parabolic Equation (Journal of Kerbala University) Vol 8, No3.

[16] Al-Rawdanee E H and Al-Hawasy J.A 2019 Mixed Methods for Solving Classical Optimal Control Governing by Nonlinear Hyperbolic Boundary Value Problem (IEEE-1STCAS).

[17] Chryssoverghi I. and Bacoopoulos A 1993 Approximation and Relaxed Nonlinear Parabolic Optimal Control Problems (Journal of Optimization Theory and Applications) Vol 77, No 1.

[18] Chryssoverghi I and Al-Hawasy J 2004 Discrete Approximation of Semilinear Parabolic Optimal Control Problems (1st IC-SCCE, Athens-Greece).