Instanton condensation in field strength formulated QCD

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Abstract

Field strength formulated Yang-Mills theory is confronted with the traditional formulation in terms of gauge fields. It is shown that both formulations yield the same semiclassics, in particular the same instanton physics. However, at the tree level the field strength approach is superior because it already includes a good deal of quantum fluctuations of the standard formulation. These quantum fluctuations break the scale invariance of classical QCD and give rise to an instanton interaction and this causes the instantons to condense and form a homogeneous instanton solid. Such the instanton solids show up in the field strength approach as homogeneous (constant up to gauge transformations) vacuum solutions. A new class of SU(N) instantons is presented which are not embeddings of SU(N-1) instantons but have non-trivial SU(N) color structure and carry winding number \( n = N(N^2 - 1)/6 \). These instantons generate (after condensation) the lowest action solutions of the field strength approach. The statistical weight (entropy) of different homogeneous solutions for SU(3) is numerically estimated by Parisi’s stochastic quantization method. Finally, we compare instanton induced quark condensation with the condensation of quarks in the homogeneous field strength solutions. Our investigations show that the homogeneous vacuum of the field strength approach simulates in an efficient way a condensate of instantons.

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1 Introduction

Quantumchromodynamics (QCD) is widely accepted as the theory of strong interactions. Our belief in QCD is mainly based on two facts: first, it provides a successful description at high energies where asymptotic freedom has been experimentally verified. Secondly, because it embodies the approximate chiral symmetry seen in the low-energy hadron spectrum.

The low-energy sector of QCD is not fully understood yet, but it is well established that the perturbative (continuum) QCD vacuum becomes unstable at low energies \[1\]. At low energies, the effective coupling constant is large, and non-perturbative methods are required. The only rigorous approach to the strong coupling regime of QCD are lattice simulations based on Wilson’s lattice formulation of Yang-Mills (YM) theory \[2\]. However, due to renormalization, the lattice approach...
does not coincide a priori with the continuum YM-theory. It is only in the weak coupling limit that the renormalized lattice theory exhibits the correct scaling behavior of perturbative YM-theory [3]. Furthermore, the lattice approach requires very time consuming numerical work yet the gained physical insight is rather meager. Hence, alternative non-perturbative approaches to YM-theories would be very welcome and in the past several proposes have been made: for example the approach of Pagels [4], the instanton gas approach of Callan, Dashen and Cross [5], the instanton liquid model proposed by Shuryak [6] and later elaborated in refs. [8] and [9].

Instantons are generally believed to play an important role in the QCD ground state and in particular, in the confinement mechanism. Furthermore instantons (in both the dilute gas and instanton liquid picture) may possibly trigger spontaneous breaking of chiral symmetry and perhaps offer an explanation of the $U_A(1)$ problem [10].

Recently, a somewhat different approach to low-energy QCD has been proposed where the YM sector is entirely described in terms of the field strength [13, 14], resulting in an effective tensor theory. This so called field strength approach (FSA) offers a very concise description of the QCD vacuum - at the ‘classical’ (tree) level it is given by a homogeneous condensate of gluons [14, 16]. When quantum fluctuations are included [17], such homogeneous condensates become unstable, and tend to break into domains [18] of constant color electric and color magnetic field, reminiscent of the Copenhagen vacuum [19]. Furthermore the FSA results in an effective tensor theory, which already exhibits an anomalous breaking of scale invariance at the tree level. This is because the corresponding effective action of the field strength contains an explicit energy scale, which means that the action cannot tolerate the usual instantons as stationary points, since instantons contain a free scale parameter. One might wonder how the familiar instantons manage to escape in the FSA.

In this paper we confront the FSA with the more traditional instanton picture of the QCD vacuum. We will find that the constant chromo-electric and chromo-magnetic field configurations of the FSA represent a coherent superposition of condensed instantons. Individual instantons cease to exist as classical solutions in the FSA since the effective action already includes a good deal of the quantum fluctuations of the original gluon field. It is these quantum fluctuations which trigger the condensation of the instantons.

The remainder of the paper is as follows: In the next section we briefly review the field strength formulation of YM-theory and compare the semiclassical description based on instantons in the field strength formulation with that of the standard formulation. Section 3 presents a special class of SU(N) instantons which are not embeddings of the SU(2) instanton but have non-trivial SU(N) color structure. In section 4 we confront the FSA with the instanton picture. We find that in the field strength formulation the SU(N) instantons condense and form homogeneous
instanton solids which appear as constant classical FSA solutions. We then show that these instanton solids form the lowest action solutions of the FSA in section 5. We also give an estimate of the statistical weight of classical FSA solutions with higher actions by using Parisi’s stochastic quantization method. Section 6 compares quark condensation in an instanton background and in the homogeneous FSA solutions. A short summary and some concluding remarks are given in the final section.

2 Semi-classical approximation to Yang-Mills theory

In the following we concentrate on the gluon sector with the quarks considered as spectators. The generating functional for gluonic Green’s functions of QCD is defined by

\[ Z[j] = \int \mathcal{D}A \exp\left(-\frac{1}{4g^2} \int d^4x \ F^a_{\mu\nu}[A]F^a_{\mu\nu}[A] + \int d^4x \ j^a_\mu A^a_\mu \right), \tag{1} \]

where \( j^a_\mu \) is the color octet quark current which acts like an external source in the gluon sector, and

\[ F^a_{\mu\nu}[A] = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu \tag{2} \]

is the field strength of the gauge field \( A^a_\mu \). The integration over gauge field degrees of freedom cannot be performed exactly and one has to resort to approximations. At high energies, a perturbative expansion in terms of the coupling strength is appropriate. However, at low energies the coupling constant is expected to be large and non-perturbative methods are required. One possibility is to resort to a semi-classical description, expanding (1) around the non-trivial gauge field configurations which are known to exist and referred to as instantons. This yields the instanton picture extensively discussed in the literature [20]. We will return to this approach in section 2.1.

A different approach to low energy QCD results if one reformulates (1) in terms of the field strength [13, 14]. For this purpose we introduce the field strength as independent variable by inserting the identity

\[ 1 = \int \mathcal{D}G \delta(G - F[A]) = \int \mathcal{D}G \mathcal{D}T \ \exp\left[+\frac{i}{2} \int d^4x \ T^a_\mu (G^a_\mu - F^a_\mu[A]) \right] \tag{3} \]

\[ ^1 \text{We discard explicitly gauge fixing, which will not alter the following considerations. It can be implemented afterwards by restricting the class of instantons.} \]
into the generating functional $Z[j]$. The path integral (4) then becomes

$$Z[j] = \int \mathcal{D}A \mathcal{D}G \mathcal{D}T \exp(-\int \mathrm{d}^4x \, L)$$

$$L = \frac{1}{4g^2} G^a_{\mu\nu} G^a_{\mu\nu} - \frac{i}{2} G^a_{\mu\nu} T^a_{\mu\nu} - \frac{i}{2} A^b_{\mu} \tilde{T}^{bc}_{\mu\nu} A^c_{\nu} - (i\partial_{\mu} T^a_{\mu\nu} + j^a_{\nu}) A^a_{\nu} ,$$

where $\tilde{T}^{ab}_{\mu\nu} = f^{abc} T^c_{\mu\nu}$. In the following we will assume $\det \tilde{T} \neq 0$ since configurations with $\det \tilde{T} = 0$ are statistically suppressed. Following [14], integration over the gauge field $A^a_{\mu}$ and field strength $G^a_{\mu\nu}$ yields

$$Z[j] = \int \mathcal{D}T \mathcal{G}[j] \det^{-1/2} \tilde{T} \exp[-\int \mathrm{d}^4x \, L_T]$$

$$L_T = \frac{g^2}{4} T^a_{\mu\nu} T^a_{\mu\nu} - \frac{i}{2} \partial_{\sigma} T^a_{\mu\nu} (\tilde{T}^{-1})^{ab}_{\mu\nu} \partial_{\tau} T^b_{\sigma\nu} ,$$

$$\mathcal{G}[j] = \exp[j^a_{\mu} (\tilde{T}^{-1})^{ab}_{\mu\nu} \partial_{\sigma} T^b_{\omega\nu} - \frac{i}{2} j^a_{\mu} (\tilde{T}^{-1})^{ab}_{\mu\nu} j^b_{\nu}] .$$

Equations (6, 7) are an exact reformulation of the gluonic generating functional (1) as a path integral over the conjugate field $T^a_{\mu\nu}$. Note that the integral over $T^a_{\mu\nu}$ in (6) can no longer be evaluated in closed form. However, recent investigations show that a semiclassical treatment is appropriate (see e. g. [23, 24, 25, 26]).

### 2.1 Instanton picture

In the standard continuum approach to the YM ground state a semiclassical analysis of the path integral over the gauge field $A^a_{\mu}$ is performed. Below we demonstrate that the same semiclassical approximation is obtained in the field strength formulation, as was recently shown [27]. Extremizing the YM action

$$S_{YM} = \frac{1}{4g^2} \int \mathrm{d}^4x \, F^a_{\mu\nu}[A] F^a_{\mu\nu}[A]$$

yields the classical equation of motion:

$$\partial_{\nu} F^a_{\nu \mu}[A] + f^{abc} A^b_{\nu} F^c_{\nu \mu}[A] = 0 .$$

Non-trivial finite action solutions to this equation are usually referred as instantons. These instanton configurations also show up in the field strength formulation [13, 27]. Variation of the exponent in (6) with respect to the conjugate field $T^a_{\mu\nu}$ yields the classical equation of motion

$$g^2 T^a_{\mu\nu} + i F^a_{\mu\nu}[V](x) = 0 ,$$
where \( F_{\mu\nu}^a[V] \) denotes the field strength (2) of the vector field
\[
V^a_\mu[T] = (\hat{T}^{-1})_{\mu\nu}^{ab}\partial_\omega T^b_{\nu\omega}.
\] (11)

This quantity transforms under changes of gauge in the same manner as the original gauge potential \( A_\mu \) and has only a linear coupling to the quark current, also in common with \( A_\mu \). It should therefore be understood as the counterpart of the gauge potential in the field strength formulation. Equation (10) is solved by the field strength of an instanton \( A^a_{\text{inst}} \)
\[
T^a_{\mu\nu} = -\frac{i}{g^2} F^a_{\mu\nu}[A_{\text{inst}}] \] (12)
with the corresponding induced vector field (11)
\[
V^a_\mu[T] = V^a_\mu[-\frac{i}{g^2} F^a_{\mu\nu}[A_{\text{inst}}]] = V^a_\mu[F^a_{\mu\nu}[A_{\text{inst}}]] = A_\mu(x)_{\text{inst}}
\] (13)
which coincides with the instanton potential.

The classical solutions contribute to the generating functional \( Z[j] \) with a weight factor given by the exponent of the classical instanton action times the integral over small fluctuations. The classical instanton action is the same in both approaches (1) and (6) [27]. In fact, the tensor action in (6) can be written as
\[
\int d^4x L_T = \int d^4x \left\{ \frac{g^2}{4} T^2 + \frac{i}{2} T^a_{\mu\nu} F^a_{\mu\nu}[V] \right\},
\] (14)
where we have used
\[
\int d^4x \partial_\omega T^a_{\mu\nu}(\hat{T}^{-1})_{\mu\nu}^{ab}\partial_\sigma T^b_{\nu\sigma} = -\int d^4x T^a_{\mu\nu} F^a_{\mu\nu}[V],
\] (15)
Inserting (12) into (15) gives the field strength action of an instanton i. e.,
\[
\int d^4x L_T = \int d^4x \frac{1}{4g^2} F^a_{\mu\nu}[V] F^a_{\mu\nu}[V],
\] (16)
which agrees with the standard YM action (8).

It remains to compare the weight factors obtained from the integration over the modes of the fluctuations. In the standard formulation this gives rise to a weight factor
\[
Q(\text{Det}' \frac{\delta^2 S}{\delta A^a_\mu \delta A^a_\nu})^{-1/2},
\] (17)
where the prime indicates that the zero modes are to be excluded from the determinant. The zero mode contribution is denoted by \( Q \). In the field strength formulation the weight factor is given by
\[
Q_{FS}(\text{Det}' \hat{T})^{-1/2} (\text{Det}' \frac{\delta^2 S_T}{\delta T^a_{\mu\nu} \delta T^b_{\mu\nu}})^{-1/2}.
\] (18)
The second factor is here already present at the tree level whereas the first and the last factor arise from the integration of fluctuations around classical instanton field strength configurations. It has been shown recently [27] that both determinants (17) and (18) are equal (up to an irrelevant constant). Hence the field strength formulation reproduces the correct semiclassical description of the standard formulation of YM theory. The approaches differ, however, at tree level, where the field strength formulation yields a pre-exponential weight factor, $\text{Det}^{-1/2} \hat{T}$, and an additional quark current-current interaction, see (6), (7). This weight factor arises from the integration over the gauge fields in the field strength formulation and so already includes some of the effects of the fluctuations of the gauge field around an instanton configuration. It should also be noted that the additional quark interaction in (6) and (7) has a similar structure to the effective quark interaction which arises in a correlated instanton anti-instanton gas [5, 6]. One therefore expects the field strength formulation to already describe the dominant quantum and correlation effects of the instantons at tree level. We will provide further evidence for this later. This fact makes the field strength formulation very attractive for studying the low energy sector of YM theories.

2.2 Effective tensor theory (FSA)

The field strength formulation of non-abelian YM-theories (8,9) offers a concise non-perturbative treatment if one includes the weight factor, $\text{Det}^{-1/2} \hat{T}$, into the action. This yields the so-called field strength approach (FSA) to YM-theories [14]. After renormalization, the generating functional of the field strength approach takes the form [14]

$$ Z[j] = \int \mathcal{D}T \mathcal{G}[j] \exp\left[-\int d^4x L_{\text{FSA}}\right], $$

where the effective Lagrangian is given by

$$ L_{\text{FSA}} = g^2 T - \frac{i}{2} \partial_\omega T^{a}_{\omega\mu} (\hat{T}^{-1})^{ab}_{\mu\nu} \partial_\sigma T^{b}_{a\sigma\nu} + \frac{\mu^4}{2} \text{tr} \ln \hat{T}(x) $$

and the source term $\mathcal{G}[j]$ (7) remains unchanged. The renormalization procedure induces the energy scale $\mu$ at tree level reflecting the presence of the scale anomaly [28]. The physical value of $\mu$ has to be fixed by a renormalization condition. For instance, in the classical approximation, $\mu$ is related to the non-perturbative gluon condensate [14]. The classical equation of motion of the tensor field then becomes (cf. eq. (10))

$$ g^2 T^{a}_{\mu\nu} + \mu^4 f^{abc} (\hat{T}^{-1})^{ab}_{\mu\nu} + i F^{a}_{\mu\nu}[V] = 0, $$

Due to the presence of the explicit energy scale $\mu$ one might expect that instantons with arbitrary size no longer exist. In fact, the effective action (20) no longer has
instanton solutions at all. Instead the non-trivial stationary points are given in a certain gauge by space-time constant tensor field configurations \(^{14}\). These homogeneous classical solutions appear at the expense of the instantons. One is tempted to interpret these constant solutions as a conglomerate of condensed instantons. In the following sections we give further support to this interpretation. In fact we demonstrate that the constant SU(N) field strength solutions of minimal classical action have the same color and Lorentz structure as certain SU(N) instantons. First of all we describe these SU(N) instanton solutions.

3 SU(N) instanton solutions

There is a general method based on works of Atiyah et al. \(^{21, 22}\) which, in principle, allows one to find all (anti-) self-dual instanton solutions, although in practice it might be difficult to obtain explicit representations. In order to compare the FSA with the more traditional instanton physics we will be interested in a certain class of instantons for which we need explicit representations satisfying simple algebraic relations. For this purpose it is more convenient to follow a different route which immediately provides the explicit representation of the instantons. Actually, we only became aware of the work of Atiyah et al. after we had already found the instantons relevant in the context of the FSA.

3.1 Instantons in Schwinger-Fock gauge

Instantons are finite action extrema of the classical Euclidean Yang-Mills action \(^{[1]}\) which solve the classical equations of motion:

\[
\partial_{\mu} F^a_{\mu \nu} = \tilde{F}^{ab} A^b_{\mu} , \quad \tilde{F}^{ab}_{\mu \nu} = f^{abc} F^c_{\mu \nu} .
\]

Any finite action gauge potential \(A^a_{\mu}(x)\), in particular the instantons, must become pure gauge \(U \partial_{\mu} U^\dagger\) as \(x^2\) tends to infinity. They can be classified by the so called topological charge

\[
n[F] = \frac{1}{16 \pi^2} \int d^4 x \ tr(F_{\mu \nu} F^*_{\nu \mu})
\]

which is an integer (the Pontryagin index). The Bianchi-identity

\[
\partial_{\mu} F^a_{\mu \nu} = (\tilde{F}^*)_{\mu \nu}^{ab} A^b_{\mu} , \quad F^a_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \alpha \beta} F^a_{\alpha \beta} ,
\]

which holds for arbitrary gauge fields, implies that any gauge field which has an (anti-) self dual field strength (i.e. \(F^a_{\alpha \beta} = (-) \tilde{F}^a_{\alpha \beta}\)), represents an instanton solution to the classical YM equation \(^{22}\), in agreement with the standard variational result \(^{20}\).
For a given gauge potential $A_{\mu}^a(x)$ however, we do not know whether its field strength $F_{\mu\nu}^a$ is self-dual until we actually calculate it. Thus in the search for instantons, it is much more convenient to consider the field strength as an independent dynamical variable instead of the gauge potential itself. This can be accomplished in the Fock-Schwinger gauge

$$x_{\mu}A_{\mu}^a(x) = 0 .$$

In this gauge, the gauge potential can be entirely expressed in terms of the field strength i.e.,

$$A_{\mu}^a(x) = -\int_0^1 d\alpha \, \alpha \, F_{\mu\nu}^a(\alpha x) \, x_\nu .$$

and the equation of motion (22) becomes an equation for $F_{\mu\nu}^a$ i.e.,

$$\partial_\mu F_{\mu\nu}^a(x) = -\tilde{F}^{ab}_{\nu\lambda}(x) \int_0^1 d\alpha \, \alpha \, F_{\lambda\sigma}^b x_\sigma .$$

This equation is equivalent to the original equation of motion only for those $F_{\mu\nu}^a$ which are the field strengths to some gauge potential. For an arbitrary $F_{\mu\nu}^a$ equation (26) yields a gauge potential $A_{\mu}^a[A']$ whose field strength $F_{\mu\nu}^a[A'[F']]$ differs in general from the initial $F_{\mu\nu}^a$. We must therefore supplement (27) with the constraint

$$F_{\mu\nu}^a[A[F]] = F_{\mu\nu}^a .$$

With this constraint, (27) is completely equivalent to the classical YM equation of motion (22) that we started with.

We now solve (27) with the isotropic ansatz

$$F_{\mu\nu}^a = G_{\mu\nu}^a \psi(x^2) ,$$

where $G_{\mu\nu}^a$ is a constant, (anti-) self-dual matrix antisymmetric under the exchange of $\mu$ and $\nu$. For this ansatz the gauge field (26) becomes

$$A_{\mu}^a = -G_{\mu\nu}^a x_\nu \frac{1}{2x^2} \int_0^{x^2} du \, \psi(u) = -G_{\mu\nu}^a x_\nu \Phi(x^2) .$$

and the equation of motion (27) reduces to

$$-2G_{\mu\nu}^a x_\nu \psi' + \tilde{G}^{ab}_{\mu\lambda} G_{\nu\omega}^b x_\omega \Phi(x^2) \psi(x^2) = 0 .$$

This equation makes it easy to separate the color and Lorentz structure from the space-time dependence, reducing (31) (uniquely up to unimportant rescaling of $G \to G/\beta$ and $\Phi \to \beta \Phi$ with an arbitrary constant $\beta$) to

$$\tilde{G}^{ab}_{\mu\lambda} G_{\nu\omega}^b x_\omega \Phi(x^2) \psi(x^2) = 0 .$$

and

$$\psi'(x^2) = \Phi(x^2) \psi(x^2) .$$

9
In order to solve the matrix equation (32) it is convenient to expand the antisymmetric color matrices $G_{a \mu \nu}^i$ in terms of 't Hooft’s $\eta$-symbols i.e.,

$$G_{a \mu \nu}^i = G_{a \mu \nu}^i \eta_{\mu \nu}^i + \bar{G}_{a \mu \nu}^i \bar{\eta}_{\mu \nu}^i.$$  \hspace{1cm} (34)

The $\eta^i$ and $\bar{\eta}^i$, ($i = 1, 2, 3$) are self-dual and anti self-dual space-time tensors, which generate the $SU(2) \times SU(2) \sim SO(4)$ symmetry group of Euclidean space, satisfying

$$[\eta^i, \eta^k] = -2 \epsilon^{ikl} \eta^l, \quad [\bar{\eta}^i, \eta^k] = -2 \epsilon^{ikl} \bar{\eta}^l, \quad [\eta^i, \bar{\eta}^k] = 0,$$  \hspace{1cm} (35)

$$\{\eta^i, \eta^k\} = -2 \delta^{ik}, \quad \{\bar{\eta}^i, \bar{\eta}^k\} = -2 \delta^{ik}, \quad \text{tr}\{\eta^i \bar{\eta}^k\} = 0.$$  \hspace{1cm} (36)

For the moment let us confine ourselves to self-dual configurations i.e., $\bar{G}_{i}^a = 0$.

Using (35) the relation (32) can be cast in the form

$$G_{a \mu \nu}^i = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} f^{abc} G_{b \mu}^k G_{c \nu}^l x \Phi^2 = G_{i}^a \psi$$

Using the relation (37) of the coefficients $G_{i}^a$, we can reduce these to ordinary differential equations i.e.,

$$x^2 \Phi' + 2 \Phi + \frac{1}{2} x^2 \Phi^2 = \psi$$

$$\Phi' - \frac{1}{2} \Phi^2 = 0.$$  \hspace{1cm} (38)

(39)

These equations are solved by

$$\Phi(x^2) = -\frac{2}{x^2 + \rho^2}, \quad \psi(x^2) = -\frac{4 \rho^2}{(x^2 + \rho^2)^2}$$  \hspace{1cm} (40)

where $\rho$ is an arbitrary parameter. It is not surprising that these solutions also satisfy (33), which follows from the classical YM equation (22) within the gauge (24) and with the ansatz (29). This is merely a consequence of the general fact that any (anti-) self-dual field configuration satisfies the classical YM equations. In fact,
the solutions have precisely the space-time dependence of the gauge potential and the field strength of the standard Polyakov-t’Hooft instanton.

What remains to be done is to solve the algebraic equation which defines the color and Lorentz structure of the instantons. Let \( t^a \) denote the generators of the gauge group \( G \) satisfying \([t^a, t^b] = if^{abc}t^c\) in the fundamental representation. Defining

\[
G_i = G_i^a t^a, \quad \bar{G}_i = \bar{G}_i^a t^a
\]

the algebraic equation can be rewritten as

\[
[G_j, G_k] = i\epsilon_{ijk}G_l,
\]

Since the \( \epsilon_{ijk} \) are the structure constants of the SU(2) group, any realization \( G_i \) of the SU(2) algebra yields an instanton solution.

For field configurations with radial dependence the winding number and the corresponding action become

\[
n = \frac{1}{3} G_i^a G_i^a = \frac{2}{3} \text{tr} G_i G_i, \quad S = \frac{8\pi^2}{3g^2} G_i^a G_i^a = \frac{4\pi^2}{g^2} n.
\]

Noting that instanton solutions \( G_i \) are representations of the SU(2) spin algebra for some spin \( s \), the quantity \( G_i G_i \) represents the quadratic casimir operator of the SU(2) group, i.e.,

\[
G_i G_i = s(s+1) 1_G =: c(G) 1_G \quad \text{with} \quad \text{tr} 1_G =: d(G).
\]

11

3.2 Explicit construction of SU(N) instantons

Let us now construct explicit solutions to which define the color structure of the instantons. In the fundamental representation of SU(N) the \( t^a \) form a complete basis for the hermitean \( N \times N \) matrices. Furthermore, the lowest dimensional irreducible spin \( s \) representation of SU(2) is realized by hermitean \((2s+1) \times (2s+1)\) matrices. Hence, the N-dimensional \( s = (N - 1)/2 \) spin representation always provides an instanton \( G_i \) with non-trivial SU(N) color structure. We refer to this instanton.
as the instanton of maximum spin (for given $N$). From (45) it follows that this instanton carries winding number

$$n = \frac{2}{3} N s(s + 1) = \frac{2}{3} N \frac{N - 1}{2} \frac{N + 1}{2} = \frac{1}{6} N (N^2 - 1) .$$

(46)

Additional instantons will usually arise from $N$-dimensional realizations of lower spin $s < (N - 1)/2$ representations. In particular, all embeddings of SU($N - 1$) instantons in SU($N$) obviously represent SU($N$) instanton configurations.

For the gauge group SU(2) any representation $t^a$ of the color generators naturally provides an instanton by identifying the color group with the spin group, i.e.,

$$G_i^a = \delta_i^a \quad \text{ (identity map) .}$$

(47)

Eq. (29,34) then yields the standard Polyakov-'t Hooft instanton ($G_{\mu\nu}^a = \eta_{\mu\nu}^a$). This is the instanton with maximal spin for SU(2). Obviously, in this case, there are (up to global color and Lorentz transformations) no mappings of the SU(2) color group into the SU(2) spin algebra other than the identity map and therefore no other instantons of the type (29) exist.

In the case of a SU(3) gauge group the $G_i$ (41) are hermitian $3 \times 3$ matrices. The SU(3) embedding of the SU(2) instanton forms a SU(3) instanton configuration

$$G_i^a = \delta_i^a , \quad a = 1, 2, 3 \quad G_i^b = 0 , \quad b = 4, \ldots, 8 .$$

(48)

This is the known SU(3) instanton and is the basis of existing instanton models of the QCD vacuum [20]. This instanton corresponds to the $s = 1/2$ representation of the spin group and is illustrated in figure (5). There also exists another non-trivial SU(3) instanton, corresponding to the $s = 1$ representation of the spin group, i.e.,

$$G_i^7 = -G_i^5 = G_i^2 = 2 .$$

(50)

This solution thus has a non-trivial SU(3) color structure and from (45) it carries winding number $n = 4$. Since in three dimensions (besides the trivial $s = 0$) only the $s = 1/2$ and $s = 1$ representations of SU(2) can be realized there are no further self-dual SU(3) instantons of type (29).

For a SU(4) gauge group the embeddings of the SU(3) instantons discussed above are trivially SU(4) instanton solutions. A SU(4) instanton with non-trivial SU(4)
color structure again arises from the identification of the $G_i$'s with the maximal spin $s = 3/2$ representation of the SU(2) spin group in $N = 4$ dimensions. According to (46) this solution carries winding number $n = 10$. Two further SU(4) solutions are provided by the two $s = 1/2$ representations of SU(2) in four dimensions given by the 't Hooft’s symbols

$$G_k = -\frac{i}{2}\eta^k \quad \text{or} \quad G_k = -\frac{i}{2}\bar{\eta}^k$$

when the $\eta^k_{\mu\nu}, \bar{\eta}^k_{\mu\nu}$ are now matrices in color space. These instantons correspond to the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of the SO(4)$\sim$SU(2)$\times$SU(2) subgroup of SU(4). From (44) and (45) we see that these instantons carry winding number $n = 2$.

Of course, to any self-dual instanton solution with winding number $n$ found above there exists an anti self-dual instanton with winding number $-n$ constructed by exchanging $G_i$ and $\bar{G}_i$.

4 Confrontation of the FSA with the instanton picture

In this section we demonstrate, using the special class of instantons constructed in the previous section, that the homogeneous field strength vacuum solutions of the FSA represent solids of condensed instantons. First, we show that two instantons with equal orientation possess an attractive interaction. Second, we proof that only constant solutions of the classical equation of motion exist in the FSA. We then argue that superpositions of instantons with uniform orientation, averaged over positions and sizes, form a homogeneous, constant field strength vacuum solution.

4.1 Estimate of the instanton interaction

In order to investigate the effect of quantum fluctuations on the instantons we need to examine the effect of the pre-exponential factor $\text{Det}^{-1/2}\hat{T}$ in (4) and (7), treating it as a weight factor for instanton configurations. This determinant is divergent and needs regularization and renormalization. We adopt a simplified renormalization procedure motivated by physical arguments, and which turns out to be sufficient at our level of approximation. In particular, for instanton configurations $T^a_{\mu\nu} = -iF^a_{\mu\nu}$, the renormalized determinant yields the same $\rho$-dependence as the standard fluctuation determinant calculated by t’Hooft [11] (see also [12]). This $\rho$-dependence implies that instantons with large radii $\rho$ compared to the fundamental (renormalization) scale $1/\mu$ are preferred. This fact does not change if one considers two instantons and their binary interaction as we shall see.
We now investigate the interaction of two separated instantons within the FSA by comparing their tree level effective action to that of two isolated instantons. We confine ourselves to the simplest case of two maximum spin instantons with the same radius and orientation. We find that for medium radii the potential contains short-range repulsive and medium-range attractive parts, while for large instanton radii the potential becomes purely attractive. Since instantons with large radii are preferred, instantons will fuse and form a solid.

As discussed in section 2.1, once the fluctuations of the $T$-field are included, the full instanton determinant of the standard formulation is recovered. In this case there is no need to regularize $\text{Det}^{-1/2}\hat{T}$ separately. However, once this term is renormalized, it already contains quantum effects which are physically relevant at low energies. In particular, it already describes the scale anomaly as we shall demonstrate shortly.

We start from the generating functional for Green’s functions \( \mathcal{G} \), where we now include the weight factor $\text{Det}^{-1/2}\hat{T}$ in the action. Adopting lattice regularization we obtain

$$\text{Det}^{-1/2}\hat{T} = \exp\left\{-\frac{1}{2} \sum_i \text{tr} \ln \hat{T}(x_i)\right\} = \exp\left\{-\frac{1}{2} \int d^4x \, \mu_0(x) \text{tr} \ln \hat{T}(x)/\mu^2\right\},$$

(52)

where $\mu$ is a reference scale and $\mu_0(x)$ is a divergent measure function reflecting the fact that the sum over space-time points has been replaced by a Riemann integral. For equally spaced lattice points the regulator is $\mu_0(x) = 1/a^4$ with $a$ the lattice spacing. The ambiguity of the space-time dependence of the regulator function $\mu_0(x)$ is due to the fact that one is dealing with the determinant of a purely local operator. However, such ambiguity can be removed by including fluctuations in the field strengths which induce non-local terms. At our level of approximation it is sufficient to remove this ambiguity on physical grounds. We demand that first the correct anomaly is generated by scale transformations, and second that the regularization preserves the individual particle picture of instantons. As an initial estimate we follow the renormalization adopted in the FSA [14]. One finds that $\mu_0$ is related, in the classical approximation, to the gluon condensate in the vacuum. As our renormalization condition, we retain the relation between $\mu_0$ and the field strength for space-time dependent instanton configurations and absorb the divergence of $\mu_0$ by a rescaling of the field strength. We therefore generalize to space-time dependent field configurations by setting the regulator function as

$$\mu_0^A(x) = \omega F^{a\mu}_\nu(x) F^{a\mu}_\nu(x),$$

(53)

where the constant $\omega$ must be chosen to produce the correct anomalous behavior under scale transformations. The choice (53) means that each instanton contributes the same amount to the scale anomaly. The averaging process over the instanton
ensemble can now be performed in a trivial manner. It is clear that the independent particle picture is preserved during renormalization which is performed in the standard way by rescaling $T^a_{\mu \nu} \rightarrow T^a_{\mu \nu} / Z_T$ corresponding to space-time independent counter terms.

Following the FSA, we include the determinant (52) in the action and the generating functional for Green’s function then becomes

$$Z = \int \mathcal{D}T \exp[-S]$$

$$S[V] = \int d^4x \left\{ \frac{g^2}{4} T^2 + \frac{i}{2} T^a_{\mu \nu} F^a_{\mu \nu}[V] + \frac{\mu_0^4(x)}{2} \text{tr} \ln \hat{T} / \mu^2 \right\},$$

where we have used the relation (14). $F^a_{\mu \nu}$ is the field strength functional (2) and $V^a[T]$ is the counterpart of the gauge potential in the FSA, defined by equation (11).

We confine ourselves to the investigation of instantons of the type (30) whose field strength is given by (29,40). As shown in section 2.1, these instanton configurations minimize the first two terms of the action in (54), giving rise to the classical equation of motion (10). For any solution to this equation, in particular for the instantons of type (30), the effective action becomes

$$S[V] = \int d^4x \left\{ \frac{1}{4g^2} F^a_{\mu \nu}[V] F^a_{\mu \nu}[V] + \frac{\mu_0^4(x)}{2} \text{tr} \ln \hat{F}[V] / \mu^2 \right\}$$

and is a function of collective variables characterizing the instanton medium (e.g., the instanton radius or the average distance of two instantons). Substituting $\mu_0(x)$ from (53), the renormalized action becomes

$$S[V] = \int d^4x \left\{ \frac{1}{4g^2} F^a_{\mu \nu}[V] F^a_{\mu \nu}[V] + \frac{\omega}{2} F^2 \text{tr} \ln \hat{F}[V] / \mu^2 \right\}$$

The dependence of this expression on the arbitrary scale $\mu$ can be removed by appealing to renormalization group arguments, letting $g$ depend on $\mu$. From $dS/d \ln \mu = 0$ we obtain the $\beta$-function

$$\beta(g) = -8(N^2 - 1) \omega g^3,$$

which, in fact, agrees with the asymptotic behavior of $\beta(g)$ in perturbative QCD. So if the preexponential factor $\text{Det}^{-1/2} \hat{T}$ is the dominant part of the determinant arising from fluctuations around an instanton, we may set $\omega \approx \beta_0 / 8(N^2 - 1)$ with $\beta_0 = 11N / 48\pi^2$ for a pure SU(N) YM theory [34]. This choice of $\omega$ also yields the correct scale anomaly. This can easily be seen by examining the behavior of $S$ under a scale transformation, i.e.,

$$\delta S = -4(N^2 - 1) \omega \int d^4x F^2(x).$$

\footnote{Note that $F$ does not depend on the coupling strength.}
Here, the left hand side represents the trace of the energy-momentum tensor $E_\mu^\mu$, which sets the fundamental energy scale in YM theories. The choice $\omega = \beta_0/8(N^2 - 1)$ then gives

$$E_\mu^\mu = -\frac{\beta_0}{2} F^a_{\mu\nu} F^a_{\mu\nu},$$

which is the correct result, first obtained by Collins et al. [28].

Inserting the one instanton solution (29) into this equation we obtain the action as a function of the instanton radius $\rho$, i. e.,

$$S_\rho = G^2 \int d^4x \left\{ \frac{1}{4g^2} + \omega^2 \ln \det \hat{G} \right\} \psi^2 + 2\omega(N^2 - 1)\psi^2 \ln \psi/\mu^2 \right\},$$

where $G^2 = G^a_{\mu\nu} G^a_{\mu\nu}$ with the matrix $G^a_{\mu\nu}$ defined in (29). A straightforward calculation yields the following $\rho$-dependence

$$S_\rho = \frac{8\pi^2 G^2}{3} \left[ \frac{1}{4g^2} - \frac{\beta_0}{2} \ln(\rho\mu) \right],$$

where we have inserted the explicit value for $\omega$ and irrelevant numerical constants have been ignored. Equation (61) gives the same $\rho$-dependence of the statistical weight as obtained in the standard approach if one includes the full contribution from the Gaussian fluctuations around an instanton solution in the action [11, 12]. The difference, however, is that we have obtained this result using physical arguments implying that $\beta_0$ is undetermined at this level. Note that large instanton radii are preferred here since the coefficient of the logarithm in (61) is negative.

Up to now, we have only considered the weight factor $\text{Det}^{-1/2}\hat{T}$. If we add the contribution from the fluctuations of the field $T^a_{\mu\nu}$ to the effective action, we expect that the only change will be in the value of $\omega$. This is because we then recover the standard instanton determinant calculated by t’Hooft [11] which has precisely the $\rho$-dependence given in (61). This fact also implies $\text{Det}^{-1/2}\hat{T}$ must already contain the dominant quantum fluctuations relevant to instantons, and thus the tree level field strength formulation gives the same $\rho$ dependence of the statistical weight as the full semiclassical description in the standard YM approach. In view of this, it is worthwhile to investigate the tree level instanton interaction in field strength formulated QCD. For this purpose we compare the effective action (56) for two maximum spin instantons localized at $x = 0$ and $x = r$ and with the same orientation, i. e.,

$$F^a_{\mu\nu}^{(2)} = G^a_{\mu\nu} [\psi(x - r) + \psi(x)].$$

Note that due to the presence of the explicit scale $\mu$, the zero mode corresponding to scale transformations is absent, implying that there are no further contributions from treating this zero mode.
with the action of two independent instantons, giving
\[
S_I := S[F^{(2)}] - 2S[F].
\]

The first term in (56) yields a repulsive interaction when the shape of the instantons is kept fixed. Adding the second term, arising from the integration over the gauge fields \( A_a^\mu \), and thus including the quantum effects of the standard approach, the potential becomes short-range repulsive and medium-range attractive for \( \rho < \rho_c \) but purely attractive for \( \rho \geq \rho_c \). The instanton medium is thus infra-red unstable. The dependence of \( S_I \) on the instanton distance in terms of the instanton radius is shown in fig. (1) for different values \( \rho \mu \) at fixed coupling strength \( \alpha = g^2(\mu) / 4\pi = 0.4 \). Fig. (2) investigates the critical radius \( \rho_c \) in terms of the coupling \( g(\mu) \). Assuming \( \alpha = 0.4 \) at a renormalization point \( \mu \approx 1 \text{ GeV} \) as suggested by perturbative QCD the critical instanton radius is \( \rho_c \approx 0.3 \text{ fm} \). It is seen that a medium of small instantons \((\rho \mu < 1.48)\) is stable for all values of \( g(\mu) \). On the other hand, there are always large instantons (for fixed \( g(\mu) \)) which exceed the critical size \( \rho_c \) and trigger a phase transition from an instanton gas to an homogeneous instanton medium.

From instanton liquid calculations [7], it is known that the the vacuum distribution of the instanton radius is sharply peaked. We have checked that for instantons with size larger than some critical value there is no stabilization mechanism preventing the classical instanton vacuum from collapsing when \((\text{Det} \hat{T})^{-1/2} \) is included in the classical approximation. There are two major differences in our investigations as compared with instanton liquid simulations; first, in instanton liquid models quarks, play an important role in the stabilization of the instanton medium, and we have neglected the influence of quarks in our considerations so far. Second, in instanton liquid models one usually includes only the instanton-anti-instanton interaction, which is already present at the tree level, while the instanton-instanton interaction, which arises only from quantum fluctuations (and quarks), is discarded. In our case, order \( h \)-effects yield the attractive instanton interaction causing the infra-red instability. Finally note that if one sticks to large but finite volume \( L^4 \), implying that the instanton radius cannot exceed \( L \), it is always possible to choose a renormalization point at which the instanton medium is stable.

4.2 Constant FSA solutions versus instantons

In section 2.1 we saw that field strength formulated YM theories of the form presented in [13, 27], with the weight factor \( \text{Det}^{-1/2} \hat{T} \) excluded from the effective action, have the same instanton solutions as exist in the standard approach. We now investigate whether the field strength approach (FSA) to YM-theories in the form proposed in [14], where the weight factor \( \text{Det}^{-1/2} \hat{T} \) is included in the action still
allows for instanton-type solutions of the classical equation of motion (21). Below we show that in the FSA, individual localized instantons do not exist, as one might already expect from the presence of an energy scale $\mu$ in the effective action. For the SU(2) gauge group, the absence of the standard instantons has already been proven in [14]. For simplicity we restrict ourselves to instantons of the type discussed in the previous section and use the ansatz (29), i.e.,

$$g^2 T^a_{\mu\nu}(x) = -i G^a_{\mu\nu} \psi(x^2) ,$$  

(64)

where $G^a_{\mu\nu}$ is a constant (anti-) self-dual matrix satisfying the relation (32). A surprising consequence of the property (32) of the instanton matrix $G^a_{\mu\nu}$ is that it implies (see appendix A)

$$G^a_{\mu\nu} = \kappa f^{abc} (\hat{G}^{-1})^{bc}_{\mu\nu}$$  

(65)

where $\kappa$ is some constant depending on the underlying gauge group and the dimension of space-time. Setting

$$T^a_{\mu\nu} = i \frac{\mu^2}{g\sqrt{\kappa}} G^a_{\mu\nu}$$  

(66)

(67) becomes precisely the classical equation of motion of the FSA (21) for space-time independent field configurations. Therefore, any instanton of the type considered in section 3 generates a homogeneous solution (66) of the FSA with the same color and Lorentz structure. This is a major result of our paper.

Let us now go further and show that in the FSA these instantons only generate (up to gauge rotations) space-time constant solutions. For the configurations (64) the equation of motion (21) becomes

$$G^a_{\mu\nu} \left[ \psi - \frac{\mu^4 g^2}{\kappa \psi} \frac{1}{\psi} \right] - F^a_{\mu\nu}[V] = 0 .$$  

(67)

This equation only allows for constant $\psi(x^2)$. To see this explicitly we take the covariant derivative $D^b_{\mu} = \delta^b_{\mu} + f^{bca} V^c_{\mu}$ of (67), giving

$$G^b_{\mu\nu} \partial_{\mu} \left[ \psi - \frac{\mu^4 g^2}{\kappa \psi} \frac{1}{\psi} \right] + f^{bca} V^c_{\mu} G^a_{\mu\nu} \left[ \psi - \frac{\mu^4 g^2}{\kappa \psi} \frac{1}{\psi} \right] - D^b_{\mu} F^b_{\mu\nu}[V] = 0 .$$  

(68)

By assumption $G^a_{\mu\nu}$ is (anti-) self-dual and by (67) the same must be true for $F^a_{\mu\nu}[V]$. The last term then vanishes by the Bianchi identity $D^b_{\mu} F_{\mu\nu}^b[V] = 0$. Note, that any solution of (67) must also satisfy the reduced equation (68), although the converse is not true. Inserting the explicit form of $V^a_{\mu}$ (11) into (68) and using (32), (37) may be simplified to

$$G^a_{\mu\nu} x_{\omega} (\ln \psi) \frac{1}{\psi} = 0 .$$  

(69)

This equation can only be satisfied by constant $\psi$. 

18
We have thus shown that when $\text{Det}^{-1/2}\hat{T}$ is included in the effective tensor action, the instantons disappear and instead constant, homogeneous solutions with the same color and Lorentz tensor structure emerge. In view of the results of the previous subsection we may thus conclude that the homogeneous vacuum solutions of the FSA represent coherent superpositions of uniformly oriented instantons.

Note that the condition (32) on the matrix $G^a_{\mu\nu}$ of the instanton, is more restrictive than the equations of motion of the field strength approach (21). Accordingly, the field strength approach allows further solutions than those having the color and Lorentz structure of the maximum spin instanton, discussed above. For example, (21) allows solutions of mixed duality (see section 5). In the instanton approach such configurations would correspond to a mixed solid like medium of self-dual and antiself-dual instantons, with different orientations and different localizations, but which is found to be homogeneous after averaging over all instanton positions and radii (but keeping color and Lorentz structure fixed).

We conclude that the two instanton potential induced by the additional weight factor $\text{Det}^{-1/2}\hat{T}$ in field strength formulated YM theory at tree level is responsible for the condensation of instantons, forming the of homogeneous field strength vacuum seen in the FSA.

5 Instanton solids

We have just seen that the $\text{tr}\ln\hat{T}$ term gives rise to a substantial instanton interaction and hence this term deserves a non-perturbative treatment. One should therefore include it in the action and thus in the definition of the stationary point. But this is precisely the definition the field strength approach (FSA) of ref. [14] with the action (21).

We now go on to examine the spectrum of homogeneous solutions to the classical equations of motion (21) of the FSA. For constant $T^a_{\mu\nu}$ configurations these reduce to

$$g^2T^a_{\mu\nu} + \mu^4 f^{abc} (\hat{T}^{-1})^{bc}_{\mu\nu} = 0,$$

which have the purely imaginary solutions [14]

$$ig^2T^a_{\mu\nu} = G^a_{\mu\nu}.$$  \hspace{1cm} (71)

These solutions describe a homogeneous vacuum with real field strength $G^a_{\mu\nu}$ which minimizes the effective potential [14]

$$\Gamma(G) = \frac{1}{4g^2}G^a_{\mu\nu}G^a_{\mu\nu} - \frac{\mu^4}{2}\text{tr}\ln(\hat{G}).$$  \hspace{1cm} (72)

Also note that in view of (10) the instanton solutions carry real field strength $G^a_{\mu\nu}(x) = (ig^2T_{\mu\nu})^2 = F^a_{\mu\nu}[V] F^a_{\mu\nu}[V].$
From the equation of motion it follows that the constant solutions satisfy the relation
\[ G^a_{\mu\nu} \tilde{G}^a_{\mu\nu} = 4 \left( G^a_i G^a_i + \tilde{G}^a_i \tilde{G}^a_i \right) = 4(N^2 - 1) \mu^4 g^2. \]  
(73)

If we consider the expansion coefficients \( G^a_i \) and \( \tilde{G}^a_i \) as cartesian coordinates of \( R^{6(N^2-1)} \), then the classical solutions are given by points on the unit sphere \( S^{6(N^2-1)-1} \).

In ref. [14] the various classical solutions have been classified according to the topological quantum number
\[ m = \frac{1}{2} (N_+ - N_-), \]  
(74)
where \( N_\pm \) are the number of positive and negative eigenvalues of the matrix \( \hat{G} \). Solutions with different \( m \) are separated by infinite energy barriers, since \( \text{tr} \ln \hat{G} \) diverges when one of the eigenvalues crosses zero.

## 5.1 Lowest action solution of the FSA

It was shown in section 4.1.2 that any instanton solution of the type discussed in section 3 provides a homogeneous (anti-) self-dual solution to the equation of motion (21) of the FSA. Surprisingly, it is just the maximal spin \( s = (N - 1)/2 \) instanton of the SU(N) YM-theory that provides the lowest action solution of the FSA. More precisely, in SU(2) (in four space-time dimensions) it has been shown there exist only (up to gauge and Lorentz transformations) six independent constant solutions, which are all degenerate. Four of them are (anti-) self-dual solutions arising from the maximal spin \( s = 1/2 \) SU(2) instantons. These solutions carry the topological quantum number (74) \( m = \pm 2 \). There are two further solutions which are not (anti-) self-dual, carry \( m = 0 \), and are only accidentally degenerate with the self-dual solutions. The lowest action solution for SU(3), the maximal spin instanton \( s = 1 \), which has \( m = 4 \), is non-degenerate (up to symmetry transformations).

In order to get a better understanding of the homogeneous solutions, generated by instantons, of the FSA equation of motion we investigate the structure of the matrix \( \hat{G} \). For self-dual field configurations we have
\[ \hat{G}_{\mu\nu}^{ab} = \hat{G}_i^{ab} \eta_{i\mu\nu} \quad \hat{G}_i^{ab} = f^{abc} G_i^c. \]  
(75)

For the classical FSA solutions which are generated by instantons, the \( G_k \) [41] defined in the fundamental representation of the SU(N) gauge group, form an SU(2) algebra (see eq. (42)) in some spin \( s \) representation. The same must be true for the matrices \( -i \hat{G}_k \) which are members of the adjoint representation of SU(N). These form an (in general reducible) SU(2) representation, i.e.,
\[ L_k := -i \hat{G}_k \quad [L_k, L_i] = i\epsilon_{kim} L_m, \]  
(76)
with different spin $L \neq s$. Furthermore, the $S^k := \frac{i}{2} \eta^k$ form the four dimensional $s = 1/2$ representation of SU(2). Hence for these solutions the matrix $\hat{G}$ can be represented entirely in terms of SU(2) generators, i.e.,

$$\hat{G}^{ab}_{\mu\nu} = 2 \tilde{L}_{ab} \tilde{S}_{\mu\nu}.$$  \hspace{1cm} (77)

Due to the absence of anti self-dual components, i.e. of the spin $\tilde{S}^k = \frac{i}{2} \tilde{\eta}^k$ all eigenvalues of $\hat{G}$ are (at least) two-fold degenerate. Introducing the grand spin $\vec{K} = \vec{L} + \vec{S}$, the matrix $\hat{G}$ becomes

$$\hat{G} = \vec{K}^2 - \vec{L}^2 - \vec{S}^2$$  \hspace{1cm} (78)

and its eigenvalues are

$$\lambda = k(k + 1) - l(l + 1) - s(s + 1),$$  \hspace{1cm} (79)

where $k = l \pm \frac{1}{2}$. If $\vec{L}$ is an irreducible representation of SU(2) the eigenvalues $\lambda$ are $(2k + 1)(2\bar{s} + 1)$ degenerate ($\bar{s} = 1/2$). If $\vec{L}$ decays in $\nu$ irreducible SU(2) representations, the degree of degeneracy is $(2\bar{s} + 1) \sum_{\rho=1}^{\nu} (2k_{(\rho)} + 1)$. Since the adjoint representation of SU(N) is $(N^2 - 1)$-dimensional, we have the constraint

$$2 \sum_{\rho=1}^{\nu} \sum_{k_{(\rho)}=l_{(\rho)} \pm 1/2} (2k_{(\nu)} + 1) = 4 \sum_{\rho=1}^{\nu} (2L_{(\rho)} + 1) = 4 (N^2 - 1).$$  \hspace{1cm} (80)

Furthermore there are

$$N_+ = 4 \sum_{\rho=1}^{\nu} (L_{(\rho)} + 1)$$  \hspace{1cm} (81)

positive eigenvalues ($k = l + 1/2$) and

$$N_- = 4 \sum_{\rho=1}^{\nu} L_{(\rho)}$$  \hspace{1cm} (82)

negative eigenvalues ($k = l - 1/2$) of $\hat{G}$, so that the topological quantum number (74) becomes

$$m = \frac{1}{2} (N_+ - N_-) = 2 \nu.$$  \hspace{1cm} (83)

Since the number of irreducible representations is restricted by (80) their is a maximum topological quantum number for each SU(N) group. In the case of SU(2) the adjoint representation is 3-dimensional and can accommodate only a single spin representation ($L = 1/2$ or $L = 1$, the latter is realized for the maximum spin solution). Therefore $\nu = 1$ and the maximal quantum number is $m = 2$, in agreement with the explicit solutions [14]. In the case of SU(3) the adjoint representation is eight
dimensional and, for the maximum spin solution, may be decomposed into \( L(1) = 1 \) and \( L(2) = 2 \) irreducible SU(2) representations implying \( m \leq 4 \), in agreement with numerical simulations (see section 5.2). It seems that this property is not restricted to the instanton like solutions, since up to now no configuration \( \hat{G} \) with \( m > 4 \) has been found numerically by choosing \( G_{\mu\nu}^a \) randomly.

## 5.2 Statistical weight of the constant vacuum solutions

In this section we investigate the weight with which each classical solution contributes to the path integral. We argue that the completely self-dual FSA solutions discussed in the previous subsection, should be statistically suppressed compared to the solutions with mixed duality. The reason is that the self-dual solutions can be interpreted as a coherent superposition of maximum spin instantons with unique orientation, whilst the mixed duality solutions represent superpositions of instantons and anti-instantons.

In the interesting case of an SU(3) gauge group, many classical solutions are known to exist. It is therefore of great practical interest to investigate which are the most important ones dominating the quark interaction (see (7)). For this purpose we assume that the inclusion of the quarks modifies the gluonic sector only slightly, a fact which seems to be supported by the QCD sum rules [15].

Each classical solution contributes a weight factor \( \exp[-\Gamma(G_c)/\mu^4] \) with \( \Gamma(G) \) defined by (72). Upon rescaling \( T_{\mu\nu}^a \to T_{\mu\nu}^a/g \) one can explicitly check, using (71) and (72), that this weight factor is independent of \( g \) and \( \mu \) and therefore renormalization group invariant. A second weight factor arises from the integration over the Gaussian fluctuations in the tensor field around a classical solution. These weight factors have been evaluated for SU(2) in three dimensions [23]. For SU(3) in four dimensions however, the evaluation of these weight factors for all classical vacuum solutions is not feasible. We therefore take an alternative approach, motivated by the stochastical quantization method. This approach is not only feasible for SU(3) but also allows a deeper insight into the phase space structure of the FSA.

We generate a series of field strengths by the iteration

\[
G_{\mu\nu}^a(n+1) = G_{\mu\nu}^a(n) - \tau \frac{\partial \Gamma}{\partial G_{\mu\nu}^a}[G^{(n)}] + \chi_{\mu\nu}^a(n),
\]

where \( \chi_{\mu\nu}^a(n) \) are random numbers with a Gaussian distribution. From Parisi’s work [32] it is well known that the \( G^{(n)} \) are distributed according \( \exp[-\Gamma(G)] \) if the width of the \( \chi \)-distribution, \( w \) and \( \tau \) are related by \( w = \sqrt{4\tau} \). Putting \( \chi_{\mu\nu}^a(n) = 0 \), the iteration (84) reduces to a fixed point equation and the \( G_{\mu\nu}^a(n) \) converge to the
classical value $G^a_{\mu \nu}$ in the course of the iteration:

$$G^{a(n+1)}_{\mu \nu} = G^{a(n)}_{\mu \nu} - \tau \frac{\partial \Gamma}{\partial G^a_{\mu \nu}} [G^{(n)}], \quad (\tau \leq 1). \quad (85)$$

In order to go beyond the classical approximation, we decompose the $G$-space that is integrated over in the generating functional for gluonic Green's functions

$$Z_G = \int \mathcal{D}G \exp[-\Gamma(G)] \quad (86)$$

into domains $D_i$. Each domain is characterized by a single classical solution and all the iterations (85) with a starting point inside such a domain converge to this configuration. We therefore write

$$Z_G = \sum_i \int_{\{D_i\}} \mathcal{D}G \exp[-\Gamma(G)] , \quad (87)$$

where $\{D_i\}$ indicates that the integration extends only over the domain $D_i$. Performing a classical approximation in each domain we obtain

$$Z_G \simeq \sum_i \exp[-\Gamma(G^{(i)}_c)] \int_{\{D_i\}} \mathcal{D}G . \quad (88)$$

This implies that the additional weight factor is given by the volume of a domain corresponding to an attractive fixed point (classical solution). These domains possess a highly non-trivial, non-continuous structure. However, their volumes can be estimated by randomly choosing a starting configuration $G^a_{\mu \nu}$ and observing which solution the iteration (85) converges to. The result of 100000 sweeps is presented in table 5.2. The first column gives the classical action of a particular solution and the second column gives the quantum number $m$ defined by (74). The third column lists the number of events a particular solution attained within 100000 sweeps. Also given is the effective action including the statistical weight factor:

$$S_{eff} = \sum_i [\Gamma(G^{(i)}_c) - \ln \int_{\{D_i\}} \mathcal{D}G]. \quad (89)$$

The spectrum of solutions classified by the effective action and $m$-sector quantum number is shown in figure (3). The classical homogeneous field strength solution corresponding to the maximal spin $s = 1$ self-dual SU(3) instanton (50) carries $m = 4$. This solution is the only self-dual SU(3) field strength solution found so far and has the lowest classical action $\Gamma(G_c) = 1.3041 \, [16]$. However, it is strongly statistically suppressed as expected since the configurations of mixed duality can be formed in many more ways by superposing instantons and anti-instantons. The solution with the lowest effective action is an $m = 0$ configuration with $S_{eff} = 2.1836$.
So far, we have mainly confined ourselves to the description of the ground state of pure Yang-Mills theory. We now turn to the investigation of the quarks in this gluonic vacuum. The quark ground state directly influences the low energy hadron physics. It is generally believed that the $SU(n)_L \times SU(n)_R$ chiral (flavour) symmetry is spontaneously broken to the diagonal $SU(n)_V$ flavour group by quark-condensation. A similar mechanism of spontaneous symmetry breaking was first proposed by Nambu and Jona-Lasinio in a model with a chiral invariant four fermion contact interaction [30]. A more realistic model, based on QCD, was proposed by Callan et al. [3]. In this model a four-fermion interaction is induced by instanton

| $S_c$ | m-sector | events | $S_{eff}$ |
|-------|----------|--------|----------|
| 1.3041 | 4        | 15     | 8.3943   |
| 2.1836 | 0        | 18003  | 2.1836   |
| 2.2714 | 2        | 6911   | 3.2288   |
| 2.4751 | 1        | 7463   | 3.3556   |
| 2.5421 | 2        | 433    | 6.2697   |
| 2.6382 | 0        | 16377  | 2.7779   |
| 2.6736 | 0        | 9290   | 7.1434   |
| 2.7402 | 0        | 9090   | 3.4236   |
| 2.7422 | 1        | 4414   | 4.1480   |
| 2.7662 | 0        | 6116   | 3.8458   |
| 2.7946 | 0        | 3435   | 4.4511   |
| 2.8039 | 1        | 10927  | 3.3032   |
| 2.8076 | 1        | 2736   | 4.6916   |
| 2.9978 | 0        | 1606   | 5.4146   |
| 3.0118 | 2        | 66     | 8.6204   |
| 3.0285 | 0        | 122    | 8.0228   |
| 3.0302 | 0        | 61     | 8.7176   |
| 3.1539 | 1        | 1545   | 5.6094   |
| 3.1834 | 0        | 1347   | 5.7760   |
| 3.2335 | 2        | 43     | 9.2706   |

found in [14]. The solution is eightfold degenerated and four of the orientations of the chromo-electric and magnetic fields are shown in figure (4). The remaining four configurations are obtained from figure (4) by reflecting the magnetic field $B^a \to -B^a$.
anti-instanton interactions which leads to quark condensates of the right order of magnitude.

In this section we first investigate the quarks in an isolated instanton background and take the average over instanton positions and sizes. The results obtained are then compared with those of the field strength approach, where the quarks move in a vacuum of constant field strength representing a condensate of instantons. We argue that the averaging process over the instanton ensemble yields the same gap equation for the quark condensates as in the field strength approach where the averaging is automatically performed by including the weight factor in the action (see section 2.2).

6.1 Quarks in an isolated instanton field

We wish to investigate the quark theory obtained in (6). To do so we assume that instantons of the pure Yang-Mills theory dominate the $T$-functional integral and treat the additional pre-exponential factor $\text{Det}^{-1/2} \hat{F}$ as weight factor for each classical configuration. Identifying the external source with the quark color octet current $j_\mu^a = \bar{q} \gamma_\mu t^a q$, the generating functional of the quark theory (6,7) in the lowest order stationary phase approximation $T_{\mu \nu}^a = -\frac{1}{g^2} F_{\mu \nu}^a [V]$ (12) becomes

$$ W = \int Dq \, D\bar{q} \, \text{Det}^{-1/2} \hat{F} [V] \, \mathcal{G} [j] \exp[-\int d^4 x \, \bar{q} (i\partial \! / \! - m) q] \quad (90) $$

$$ \mathcal{G} [j] = \exp\left[\int d^4 x \, \{ j_\mu^a v_\mu^a + \frac{g^2}{2} \hat{j}_\mu^a(x)(\hat{F}^{-1}[V](x))_{\mu \nu}^{ab} \hat{j}_\nu^b(x) \} \right]. \quad (91) $$

Since the field strength of an instanton vanishes asymptotically, the matrix $\hat{F}^{-1}[V](x)$ diverges for large $x$, implying that non-zero color currents $j_\mu^a \neq 0$ are localized near the instanton position. To get a feel for the quark ground state we investigate the quarks in a one instanton background field, i.e.,

$$ F_{\mu \nu}^a [V] = G_{\mu \nu}^a \frac{4 \rho^2}{(x^2 + \rho^2)^2} = G_{\mu \nu}^a \eta_{\mu \nu}^a \psi(x^2). \quad (92) $$

The use of Feynman’s variational principle $\langle e^F \rangle \geq e^{\langle F \rangle}$ where the angle brackets denote averaging over the collective coordinates of the instanton, implies that the bilinear term in the quark current is more important than the linear term since $\langle j_\mu^a \rangle = 0$. As an initial estimate we skip the linear term in $j_\mu^a$. This should in fact be a good approximation if one calculates static ground state properties like condensates, but will probably fail if dynamical properties like meson masses are required. Adopting the usual bosonization procedure, introducing mesonic fields,
we insert the identity
\[ 1 = \int \mathcal{D} \Omega \delta[\Omega_{ik}^{ab} - iq_i^a(x)\bar{q}_k^b(x)] \]
\[ \sim \int \mathcal{D} \Omega \mathcal{D} \Sigma \exp[-i \int d^4x \text{tr} \Omega - i \bar{q} \Sigma] \]
into (90) and integrate over the quark fields. The classical equations of motion for the \( \Sigma \) and \( \Omega \) fields are the Dyson-Schwinger equations. In the strong coupling approximation to the quark propagator [24, 25] they are
\[ i \Sigma(x) = \mu_0(x)^4 \Omega^{-1}(x), \]
\[ i \Sigma(x) - m = g^2 \gamma^a \Omega(x) \gamma^b (\hat{F}^{-1})_{\mu \nu}^{ab}(x) \]
where \( \mu_0(x) \) is the space-time dependent regulator (introduced in section 4.1). For large current masses \( m \) the solution of this system of equations is
\[ \Omega(x) = \frac{\mu_0^4(x)}{m}. \]
In order not to spoil the independent pseudo-particle picture (compare section 4.1) we impose the asymptotic relation between the quark- and gluon-condensate [33],
\[ -i \langle \bar{q} q \rangle = \frac{1}{12} \langle \bar{q} \bar{q} \rangle \alpha_s \pi \], even with one instanton. This yields with \( -i \langle \bar{q} q \rangle = \text{tr} \Omega \) and, using (96),
\[ \mu_0^4(x) = \frac{1}{144} \frac{1}{4 \pi^2} F^2. \]
Up to an irrelevant constant, this is the same relation between the local regulator \( \mu_0^4(x) \) and the field strength of the instanton \( F_{\mu \nu}^a(x) \) found in section 4.1 (see eq.(53)). Again the divergences of \( \mu_0^4(x) \) can then be absorbed by a field renormalization.
In the chiral limit \( (m = 0) \) an explicit solution of (96) and (97) in the one instanton background, with \( F_{\mu \nu}^a \) given by (92), is
\[ \Omega(x) = \Omega_0 \frac{\mu_0^4(x)^2}{g} \sqrt{\psi(x^2)} \]
where the constant matrix \( \Omega_0 \) satisfies the algebraic equation:
\[ \Omega_0^{-1} - \gamma^a \Omega_0 \gamma^b (\hat{G}^{-1})_{\mu \nu}^{ab}(x) = 0, \]
with \( G_{\mu \nu} \) being the instanton matrix in (29). The contribution of the quarks localized in a single instanton (92) to the total condensate is
\[ \int d^4x \langle \bar{q} q \rangle_I = \frac{\sqrt{G^2}}{24 \pi g} \text{tr} \Omega_0 \int d^4x \psi^{3/2}(x^2). \]
The subscript \( I \) indicates that it is only the contribution of one instanton that is considered. A straightforward evaluation of the integral for the profile function \((92)\) yields

\[- \langle im\bar{q}q \rangle_I = \frac{\sqrt{G^2 \pi}}{6} \frac{m}{g} \rho \text{tr} \Omega_0 \]  

(101)

The ratio of the quark- and gluon-condensate for the one instanton configuration is thus

\[- \frac{\langle im\bar{q}q \rangle}{\langle \alpha_s \pi GG \rangle} = \frac{\pi}{4 \sqrt{G^2}} \frac{m}{g} \rho \text{tr} \Omega_0 . \]  

(102)

Since we have preserved the independent instanton picture during renormalization the total condensates are obtained by multiplying by the instanton density. This implies that the ratio \((102)\) already displays the total value. However, the ratio is divergent if there is no stabilization of the instanton at a certain radius. This indicates that the quarks might play an important role in stabilizing the instanton medium. Since the quark condensate is a renormalization group invariant we observe that the renormalized current mass scales with \(g\), which is the approximate behavior known from perturbative QCD.

6.2 Quark condensation in the field strength approach

As discussed in section 5, the field strength approach provides a non-trivial gluonic vacuum with a non-vanishing gluon condensate. Furthermore, the non-vanishing classical field strength mediates a Nambu-Jona-Lasinio type of quark interaction. This quark interaction has been investigated in the literature for the gauge group \(SU(2)\) analytically in ref. [23] and for \(SU(3)\) numerically in refs. [24, 26]. Using the gluonic configuration with the lowest effective action from section 5 (shown in figure (4)), the quantitative results for the condensates agree with the experimental values [24]. Here we briefly review the mechanism leading to quark condensation in the field strength approach in order to compare it with the quark condensation in the instanton approach discussed in the previous section.

The generating functional for quark Green’s functions in the FSA is obtained from \((19, 20)\) and \((7)\). Assuming that the gluonic integral is dominated by a classical configuration \(g^2 T^a_{\mu\nu} = -i G^a_{\mu\nu}\), which are constant in a certain color and Lorentz frame, the Euclidean generating functional becomes

\[W = \int \mathcal{D}q \mathcal{D}\bar{q} \exp\left[- \int d^4 x \left\{ \bar{q}(i\partial + im)q - \frac{g^2}{2} j^a_{\mu} (\hat{G}^{-1})^{ab} j^b_{\nu} \right\} \right], \]  

(103)

where we have used the fact that \(V^a_\mu = 0\) for (in this frame) homogeneous field strength configurations. Note that an average over all gauge and Lorentz equivalent
classical configurations is understood in \((103)\). We apply the same bosonization procedure as in the previous section, introducing the condensate variable \(\Omega\), and meson fields \(\Sigma\). The classical condensate \(\langle i : \bar{q}q : \rangle = \text{tr}\Omega_c\) in the zero momentum approximation (for details see \([24]\)) can be calculated from the gap-equation, i.e.,

\[
\mu^4\Omega_c^{-1} - t^a\gamma_\mu\Omega_c t^b\gamma_\nu g^2(\hat{G}_c^{-1})^{ab}_{\mu\nu} = 0
\]

(104)

where \(\mu\) is a momentum cutoff, rendered finite by a renormalization of the quark fields. This equation is the counterpart of \((99)\) of the instanton picture in the field strength formulation and obviously exhibits the same algebraic structure. However, the renormalization procedure in the field-strength approach is greatly simplified, because we only have constant field strength configurations and so no averaging process is required. We only need to absorb the divergence of the momentum cutoff by redefining the quark fields. For numerical details we refer the reader to \([24, 25]\).

7 Concluding remarks

An alternative formulation of YM theories can be made in terms of field strengths. Recently it was shown \([27]\) that this field strength formulation yields the same semi-classical description as the conventional formulation in terms of gauge fields. Not only the same classical solutions (instantons) are obtained, but also the quantum fluctuations around the instantons give the same result in both approaches. The field strength formulation is, however, superior to the standard approach at the tree level since the corresponding quantum transition amplitude already includes quantum effects. In the standard approach these will only show up beyond the tree level. These quantum corrections give rise to an instanton-instanton interaction and are contained in a functional determinant arising from integrating out the gauge potential in the field strength formulation. In the FSA this functional determinant is included into an effective action of the field strengths. This effective action does not admit explicit instantons as stationary points but rather homogeneous field strength configurations. We have presented a class of SU(N) instantons, the maximum spin instantons, which give rise to the vacuum solution of lowest classical action in the FSA. These constant field strength solutions mediate a quark interaction which leads to quark condensation very similarly to the instanton induced quark interaction proposed by Callan et al.

We have further shown that there is an intriguing connection between the FSA and the more traditional instanton physics. In particular our investigations have revealed that the homogeneous vacuum of the FSA can be interpreted as a solid of condensed instantons aligned in color and Lorentz space. In the instanton liquid approach, fermions (and in principle, quantum fluctuations of the gauge field too)
give rise to strong instanton correlations leading to a liquid like structure of the instanton vacuum. Recent investigations [35], where shape variations of instantons are included, indicate that the strongly correlated instantons might lose their identity tending to form a homogeneous ground state. This is precisely what one finds in the FSA, where one obtains homogeneous field strength configurations as a stationary solutions. Thus if sufficient correlations are included, the instanton picture might lead to a similar vacuum as the FSA. However, the FSA provides a more efficient description of the non-perturbative vacuum than the instanton approach. While in the latter case higher order correlations have to be included, in the FSA a strongly correlated vacuum is already obtained at tree level, which already includes a good deal of of the higher order quantum fluctuations of the conventional formulation.

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A Instanton induced classical field strength solutions

We have to show that the instanton matrix $G^a_{\mu\nu}$ defined by either (32), (37) or (42) also satisfies

$$G^a_{\mu\nu} = \kappa f^{abc} (\hat{G}^{-1})_{bc}^{\mu\nu}$$

(105)

where $\kappa$ is some constant. The crucial step is a calculation of the inverse of $\hat{G}^{ab}_{\mu\nu} = f^{abc} G^c_{\mu\nu}$. For this end we note that for instanton solutions the matrices

$$L_k = -i \hat{G}_k = G^a_k T^a, \quad \text{and} \quad (T^a)^{bc} = -i f^{abc}$$

(106)

define an SU(2) algebra as we have already seen in section 5.1, i.e.,

$$[L_k, L_l] = i \epsilon_{klm} L_m.$$  

(107)

For SU($N > 2$) gauge groups, this SU(2) representation is, however, reducible. But since the matrix $\hat{G}$ is assumed to be regular, the $\hat{G}_k$ must be built up entirely from spin $L^{(k)} \neq 0, (k = 1, 2, \ldots)$ irreducible representations. In particular, the $-i \hat{G}_k$ can be expressed in the form

$$-i \hat{G}_k = \begin{pmatrix}
L^{(1)}_k \\
L^{(2)}_k \\
\vdots \\
L^{(r)}_k
\end{pmatrix}$$

(108)
where the $L_k^{(i)}$ denote the $(2L_k^{(i)} + 1)$-dimensional irreducible representations. Since the $\hat{G}_k$ have dimension $(N^2 - 1)$ we must have

$$\sum_{p=1}^{r} (2L_k^{(p)} + 1) = N^2 - 1.$$  

(109)

Furthermore, since the $\hat{G}_k$ are antisymmetric matrices, only antisymmetric spin representations can enter (108). For the gauge group SU(2) (where only the maximum spin instanton exist) the $\hat{G}_k$ are three dimensional and must thus be given by the spin $L = 1$ irreducible representation, i.e.,

$$(\hat{G}_k)^{ab} = \epsilon^{kab}.$$  

(110)

For SU(3) the matrices $\hat{G}_k$ are eight dimensional and can be built up either from the two four dimensional antisymmetric spin $s = 1/2$ representations ($\eta^k$ or $\bar{\eta}^k$) or from an $L = 1$ and an $L = 2$ representation. For the maximum spin $s = (N - 1)/2 = 1$ instanton, it turns out that the latter case is realized. In order to find the inverse of $\hat{G}$ we calculate

$$\hat{G}^a_{\mu\alpha} \hat{G}^c_{\alpha\nu} = \hat{G}^a_{\mu\alpha} \hat{G}^c_{\alpha\nu} \left( \frac{1}{2} \{ \eta^i, \eta^j \} + \frac{1}{2} [\eta^i, \eta^j] \right) = M^{ab} \delta_{\mu\nu} + \hat{G}^{ab}_{\mu\nu},$$  

(111)

where we have used the algebra of the 't Hooft symbols (35) and the instanton properties (37, 107). The matrix $M$ is the SU(2) Casimir operator in the adjoint representation of SU(N), i.e.,

$$M := L_k L_k,$$  

(112)

which is symmetric by construction. From the representation (108) it is clear that this matrix is given by the direct sum of the quadratic Casimir operators of the irreducible SU(2) representations $L_k^{(i)}$

$$M = \begin{pmatrix} L^{(1)}(L^{(1)} + 1) & 1_{2L^{(1)} + 1} \\ L^{(2)}(L^{(2)} + 1) & 1_{2L^{(2)} + 1} \\ \vdots & \vdots \end{pmatrix}. $$  

(113)

The matrix $M$ is hence regular and commutes with $\hat{G}$ from (108). We thus find that the inverse of $\hat{G}$ is given by

$$(\hat{G}^{-1})^{ab}_{\mu\nu} = (M^{-1})^{ac} [\hat{G}^{cb}_{\mu\nu} - \delta^{cb} \delta_{\mu\nu}].$$  

(114)

For the FSA equation of motion, we only require the projection of $\hat{G}^{-1}$ onto the SU(N) adjoint representation:

$$f^{ca}(\hat{G}^{-1})^{ab}_{\mu\nu} = -\text{tr}\{T^c M^{-1} i\hat{G}_I \} \eta^l_{\mu\nu}. $$  

(115)
Note that there is no contribution from the unit matrix on the right hand side of (114) since the matrix $M$ is symmetric. If the SU(2) representation $\hat{G}_k$ is irreducible the Casimir operator $M$ (as well as its inverse) is proportional to the unit matrix. In this case, (115) directly implies that the equations of motion of the FSA (105) are satisfied. Straightforward algebraic manipulation shows that

$$M^{ab} G^b_l = -f^{a}{}^{m}{}^{e} G_{k}^{e} f^{m}{}^{l}{}^{d} G_{k}^{d} G^{b}_l = 2 G^a_l = L(L+1) G^a_l \quad \text{for} \quad l = 1 \ldots 3 ,$$

(116)

which implies that the $L_k$ always contain a spin $L = 1$ representation. The adjoint representation of SU(2) is 3-dimensional and hence irreducible. However, (116) also implies that the representation $L_k$ is reducible for SU($N > 2$) if we insist that the inverse matrix $\hat{G}^{-1}$ exists ($\det M \neq 0$). In fact as we have already seen for SU(3), $L_k$ must be given by the direct product of spin $L = 1$ and $L = 2$ representations. From (113) we find for the inverse

$$M^{-1} = \begin{pmatrix}
\frac{1}{L(1)(L(1)+1)} & 1_{2L(1)+1} \\
\frac{1}{L(2)(L(2)+1)} & 1_{2L(2)+1} \\
\vdots & \vdots
\end{pmatrix} ,$$

(117)

which is also diagonal. Since the matrices $\hat{G}_k$ form an SU(2) algebra the non-zero coefficients $G^a_l$ can be chosen to form an orthogonal $3 \times 3$ matrix. Defining $G^a_i = b_{ia}$ for the three values for which $G^a_i \neq 0$ we express the generators

$$T^c = (b^{-1})_{ck} L_k = b_{kc} L_k = G^c_k L_k .$$

(118)

For these 3 generators we then find

$$-\text{tr}(T^c M^{-1} i \hat{G}_l) = G^c_k \text{tr}(L_k M^{-1} L_l) .$$

(119)

Decomposing $L_i$ into the $\nu$ irreducible representations and using (117) gives

$$-\text{tr}(T^c M^{-1} i \hat{G}_l) = G^c_k \delta_{lk} \sum_{\rho=1}^{\nu} (2L(\rho)+1) = G^c_l \text{ const.} ,$$

(120)

which provides the desired result.

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