Exponential Stabilization for Itô Stochastic Systems with Multiple Input Delays

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Abstract

In this paper, we study the stabilization problem for the Itô systems with both multiplicative noise and multiple delays which exist widely in applications such as networked control systems. Sufficient and necessary conditions are obtained for the exponential stabilization problem of Itô stochastic systems with multiple delays. On one hand, we derive the solvability of the modified Riccati equation in case of the mean-square exponential stabilization. On the other hand, the mean-square exponential stabilization is guaranteed by the solvability of a modified Riccati equation. A novel stabilizing controller is shown in the feedback form of the conditional expectation in terms of the modified algebraic Riccati equation. The main technique is to reduce the original system with multiple delays to a pseudo delay-free system.

Key words: Itô stochastic system, Multiple input delays, Stabilization, Riccati equation.

1 Introduction

The mathematical models described by delayed differential equations are ubiquitous and have wide applications in physics, engineering, communication, biology and so on [Kolmanovskii et al., 1999]. As is known, time delays usually degrade the system performance, and are the source of instability, and even lead to the occurrence of chaos phenomenon. So study on the stabilization problem of time-delay system is of great significance. Some essential progress has been made on the optimal control and stabilization problems for time delay systems, see [Richard, 2003], [Smith, 2003] and references therein. In particular, [Smith, 2003] designs a predictor-like controller which reduces the original delayed system to delay-free one. By virtue of the predictor-based technique, the problem for systems with more general delays has been studied in [Artstein, 1982]-[Manitius et al., 1979]. The linear quadratic regulation (LQR) problem for systems with multiple input delays was solved in [Zhang et al., 2006] by establishing a duality between the LQR problem and a smoothing problem. The optimal controller is presented using a Riccati equation. [Tadmor et al., 2005]-[Tadmor et al., 2005] studied the $H_\infty$ preview control problem and presented the necessary and sufficient solvability conditions in terms of a standard algebraic Riccati equation and a nonstandard $H_\infty$-like algebraic Riccati equation. The aforementioned results are only related to the deterministic system and more details are referred to the survey paper [Richard, 2003].

Considering the accuracy requirement to the system in applications, it is necessary to take the uncertainty into consideration. One of the most popular models is the stochastic differential equation motivated by Brownian motion. When the stochastic system is delay-free, [Rami et al., 2000] presents some sufficient and necessary conditions for the mean-square stabilization. There have also been many important developments when both delay and uncertainty are considered, especially the noise is multiplicative, e.g., [Cao et al., 1999], [Zhang et al., 2009], [Wang et al., 2002] and references therein. Noting that most results in the literature depend on the linear matrix inequality (LMI) to characterize the sufficient conditions for the stabilization. For instance, [Wang et al., 2002] investigated the stochastic stabilization problem for a class of bilinear continuous time-delay uncertain systems with Markovian jumping parameters. Sufficient conditions were established to guarantee the existence of desired robust controllers, which are given in terms of the solutions to a set of LMIs, or coupled quadratic matrix inequality.
ties. [Xie et al., 2000] considered a class of large-scale interconnected bilinear stochastic systems with time delays and time-varying parameter uncertainties and robust stability analysis was given in terms of a set of LMIs. In addition, some convergence theorems have been given in the literature. For example, [Mao, 1999]-[Mao, 2003] investigated the LaSalle-type asymptotic convergence theorems for the solutions of stochastic differential delay equations. More recently, some substantial progress for the optimal LQ control has been made by proposing the approach of solving the forward and backward differential/difference equations (FBBDEs). See [Zhang et al., 2015] and [Zhang et al., 2017] for details. However, the stabilization problem for Itô stochastic systems with multiple delays have not yet been completely solved. The main obstacles are that the problem is in fact infinite dimensional and the classical controller such as current feedback form only leads to sufficient conditions which may be delay-dependent.

Inspired by the work [Zhang et al., 2017], we shall study the stochastic system with multiple delays. The main contribution is two-fold. Firstly, we derive the solvability of the modified Riccati equation in case of the mean-square exponential stabilization. Secondly, we obtain that the mean-square exponential stabilization can be guaranteed by the solvability of a modified Riccati equation. A novel stabilizing controller is shown in the feedback from of the conditional expectation in terms of the modified algebraic Riccati equation. The main technique is to reduce the original system with multiple delays to a pseudo delay-free system.

The rest of the paper is formulated as follows: Section 2 illustrates the studied problem. The system is reduced to a pseudo delay-free system and the optimization problem of the reduced system are studied in Section 3. Sufficient and necessary conditions are given in Section 4 for the exponential mean-square stabilization of the system. Some concluding remarks are shown in the last section.

**Notation.** $R^n$ denotes the family of $n$-dimensional vectors; $x^t$ denotes the transpose of $x$; and a symmetric matrix $M > 0$ ($\geq 0$) is strictly positive-definite (positive semi-definite). $(\Omega, \mathcal{F}, \mathcal{P}, \mathcal{F}_t|_{t \geq 0})$ is a complete stochastic basis so that $\mathcal{F}_0$ contains all $\mathcal{P}$-null elements of $\mathcal{F}$, and the filtration is generated by the standard Brownian motion $\{w(t)|_{t \geq 0}\}$. $\hat{x}(t) = E[x(t)|\mathcal{F}_s]$ denotes the conditional expectation with respect to the filtration $\mathcal{F}_s$. We simply denote $E_x(\cdot) = E[\cdot|\mathcal{F}_t]$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in Hilbert space. The following sets are useful throughout the paper:

$$C_{[-h, 0]} = \{\varphi(t) : [-h, 0) \rightarrow R^m \text{ is continuous and } \sup_{-h \leq t < 0} \|\varphi(t)\| < \infty\},$$

$$L^2_\mathcal{F}(0, T; R^m) = \{\varphi(t)_{t \in [0, T]} \text{ is an } \mathcal{F}_t \text{ - adapted stochastic} \}.$$
(3) can then be reformulated as a standard Itô form by using \( \kappa_i(t) = \mu_i + \xi_i(t) \):

\[
dx(t) = Ax(t) dt + B_0 u(t) [\mu_0 dt + \sigma_0 dw_0(t)] + B_1 u(t - h_1) [\mu_1 dt + \sigma_1 dw_1(t)] + \cdots + B_r u(t - h_r) [\mu_r dt + \sigma_r dw_r(t)]
\]

\[
= \left( Ax(t) + \sum_{i=0}^{r} \mu_i B_i u(t - h_i) \right) dt + \sum_{i=0}^{r} \sigma_i B_i u(t - h_i) dw_i(t).
\]

This is a special case of systems (1).

We now define the stabilization and exponential stabilization for system (1).

**Definition 1** System (1) is mean-square stabilizable if there exists an \( \mathcal{F}_t \)-adapted controller \( u(t) \) in the form of (4) following Definition 2:

\[
Lx(t) + \int_{t}^{t+h_r} L(s) (s-h_i) ds
\]

where \( L \) is a constant matrix and \( L(s) \) is a time-varying matrix with compatible dimensions such that the closed-loop system satisfies

\[
\lim_{t \to \infty} E\|x(t)\|^2 = 0 \quad \text{and} \quad \lim_{t \to \infty} E\|u(t)\|^2 = 0
\]

for any \( x_0 \) and any \( \mathcal{F}_t \)-adapted controller \( u(t), t \leq h_r \).

**Definition 2** System (1) is mean-square exponentially stabilizable if there exists an \( \mathcal{F}_t \)-adapted controller \( u(t) \) in the form of (4) and a positive constant \( \alpha \) such that the closed-loop system satisfies

\[
\lim_{t \to \infty} e^{\alpha t} E\|x(t)\|^2 = 0 \quad \text{and} \quad \lim_{t \to \infty} e^{\alpha t} E\|u(t)\|^2 = 0
\]

for any \( x_0 \) and any \( \mathcal{F}_t \)-adapted controller \( u(t), t \leq h_r \).

The aim of this paper is stated as follows.

**Problem**: Find the sufficient and necessary conditions for system (1) to be exponentially stabilized by a controller in the form of (4) following Definition 2.

The outline of the solvability to **Problem** is as follows: Firstly, we convert the original stochastic system with multiple input delays into a pseudo delay-free system where the delays are involved in the Brownian motions rather than the control input. Secondly, we solve finite-horizon optimization problems with a standard cost function and a discounted cost function subject to the pseudo delay-free system in terms of modified differential Riccati equations. Finally, the sufficient and necessary conditions for the exponential stabilization are characterized by the corresponding modified algebraic Riccati equation.

### 3 Reduction of the original system into a pseudo delay-free system

We firstly transform the original system (1) into a pseudo delay-free system. To this end, we define

\[
y(t) = x(t) + \sum_{i=1}^{r} \int_{t}^{t+h_i} e^{A(t-s)} B_i u(s-h_i) ds + \sum_{i=1}^{r} \int_{t}^{t+h_i} e^{A(t-s)} \bar{B}_i u(s-h_i) dw_i(s).
\]

**Lemma 1** \( y(t) \) defined by (5) satisfies the dynamic

\[
\text{dy}(t) = \left( Ay(t) + \sum_{i=0}^{r} e^{-Ah_i} B_i u(t) \right) dt + \sum_{i=0}^{r} e^{-Ah_i} B_i u(t) dw_i(t + h_i).
\]

**Proof.** By taking Itô’s formula to \( y(t) \) and using (1), it is obtained that

\[
\text{dy}(t) = \left( Ax(t) + \sum_{i=0}^{r} B_i u(t - h_i) \right) dt + \sum_{i=0}^{r} B_i u(t - h_i) dw_i(t) + \sum_{i=1}^{r} e^{-Ah_i} B_i u(t) dt - \sum_{i=1}^{r} B_i u(t - h_i) dt + \sum_{i=1}^{r} e^{-Ah_i} \bar{B}_i u(t) dw_i(t + h_i)
\]

\[
- \sum_{i=1}^{r} B_i u(t - h_i) dw_i(t) + A \left( \sum_{i=1}^{r} e^{A(t-s)} B_i u(s-h_i) ds \right) + \sum_{i=1}^{r} e^{A(t-s)} \bar{B}_i u(s-h_i) dw_i(s) dt
\]

\[
= \left( Ay(t) + \sum_{i=0}^{r} e^{-Ah_i} B_i u(t) \right) dt + \sum_{i=0}^{r} e^{-Ah_i} B_i u(t) dw_i(t + h_i).
\]

This completes the proof.
Remark 2. Noting that there exists no delay in the control input \( u(t) \). However, the delays \( h_i, i = 1, \ldots, r \) are involved in the Brownian motions \( w_i \). Thus, we call the system (6) as a pseudo delay-free system.

Define a new \( \sigma \)-algebraic \( \mathcal{G}_t = \{ w_i(s + h_i), i = 0, 1, \ldots, r, s \leq t \} \). Then it holds that \( \mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{F}_{t+h_r} \). From (6), we have \( y(t) \) is \( \mathcal{G}_t \)-adapted. In addition, considering Definition 1 and (2), the controller \( u(t) \) is \( \mathcal{F}_t \)-adapted. For convenience of the future use, it is simply denoted that \( B = \sum_{i=0}^{r} e^{-Ah_i} B_i \).

3.1 Finite-horizon optimal control problem of pseudo delay-free system

We then study the finite-horizon optimization problem of minimizing the standard linear quadratic cost function subject to (6):

\[
J_T = E \left\{ \int_0^T \left[ y'(t)Qy(t) + u(t)'Ru(t) \right] dt + y'(T)Hy(T) \right\},
\]

(7)

where \( H \) is semi-positive definite matrix of compatible dimension.

Noting that the new state \( y(t) \) is \( \mathcal{G}_t \)-adapted rather than \( \mathcal{F}_t \)-adapted, we define the admissible control set as

\[
\mathcal{U}_{ad} = \{ u(t) \in \mathbb{L}_2^2(0, \infty; \mathbb{R}^m) : u(t) = M(t) \tilde{y}(t) \},
\]

(8)

where \( M(t) \) is time-varying matrices with compatible dimension and

\[
\tilde{y}(t) = E[y(t)|\mathcal{F}_t] = x(t) + \sum_{i=1}^{r} \int_{t-h_i}^{t} e^{A(t-s)} B_i u(s-h_i) ds.
\]

Following [Wang et al., 2013], the stochastic maximum principle can be immediately obtained.

Lemma 2 The optimal solution to minimize (7) subject to (6) satisfies

\[
0 = Ru(t) + E[B'p(t) + \sum_{i=0}^{r} \tilde{B}' e^{-A^h_i} q_i(t)|\mathcal{F}_t],
\]

(9)

where \((p(t), q(t))\) is the solution of the backward stochastic differential equation (BSDE):

\[
\begin{align*}
\left\{ 
& dp(t) = -[A'p(t) + Qy(t)] dt + \sum_{i=0}^{r} q_i(t) dw_i(t), \\
& p(T) = H x(T).
\end{align*}
\]

(10)

while \( y(t) \) obeys (6) and \( H \) is defined in (7).

Based on Lemma 2, the explicit solvability of forward and backward stochastic differential equations (6), (9) and (10) is the key to the derivation of the optimal solution. To this end, we define the modified differential Riccati equation:

\[
-\frac{d}{dt} \hat{P}(t) = \hat{P}(t)A + A' \hat{P}(t) + Q - \Pi(t, t),
\]

(11)

and

\[
P(t) = \hat{P}(t) + \int_{0}^{h_r} e^{A'\theta} \Pi(t + \theta, t + h_r) e^{A\theta} d\theta,
\]

(12)

where

\[
\Pi(t, t) = K'(t)\Omega(t)K(t),
\]

(13)

\[
\Omega(t) = R + \sum_{i=0}^{r} \tilde{B}' e^{-A^h_i} P(t) e^{-A^h_i} \tilde{B}_i,
\]

(14)

\[
K(t) = -\Omega^{-1}(t) B' \hat{P}(t),
\]

(15)

with the terminal values \( \hat{P}(T) = P(T) = H \) for \( H \) defined in (7).

Lemma 3 The equation (11)-(15) is equivalent to the following equations:

\[
-\hat{P}(t) = P(t)A + A' P(t) + Q - e^{A^h_r} \Pi(t + h_r, t + h_r) e^{A^h_r},
\]

(16)

while \( \Pi(t + h_r, t + h_r) \) is given by

\[
\Pi(t, t) = K'(t)\Omega(t)K(t),
\]

(17)

\[
\Omega(t) = R + \sum_{i=0}^{r} \tilde{B}' e^{-A^h_i} P(t) e^{-A^h_i} \tilde{B}_i,
\]

(18)

\[
K(t) = -\Omega^{-1}(t) \left[ B' P(t) - B' \int_{0}^{h_r} e^{A'\theta} \Pi(t + \theta, t + \theta) \times e^{A\theta} d\theta \right],
\]

(19)

with terminal values \( P(T) = H \) and \( \Pi(T, T + \theta) = 0 \) for \( \theta \in (0, h_r] \).

Proof. The equivalence can be established by similar discussions to Remark 5 in [Zhang et al., 2017]. So we omit it.

We now present the optimal solution of the finite-horizon linear quadratic optimal control problem by using the solution to (11)-(15).
Lemma 4 Assume that the modified Riccati equation (11)-(15) admits a solution such that the matrix $\Omega(t)$ $> 0$, then there exists a unique solution to the problem of minimizing (7) subject to the system (6) and the optimal controller is given by

$$u(t) = K(t)\hat{y}(t|t).$$  \hspace{1cm} (20)

The optimal cost is as

$$J_T^* = E\left(y'(0)P(0)y(0) - y'(0)\int_0^{h_r} \Pi(0, \theta)\hat{y}(0|\theta)d\theta\right).$$  \hspace{1cm} (21)

Proof. The proof is presented in Appendix A.

As a byproduct of Lemma 4 which is useful in the stabilization, we further state the following results.

Corollary 1 Under the same conditions in Lemma 4 and let the controller satisfy that $u(t) = 0$ for $t \in [-h_r, 0)$. Then there exists a unique solution to the problem of minimizing (7) subject to the system (6). The optimal controller is given by (20) for $t \geq 0$ and the optimal cost is as

$$J_T^* = E\left(x_0^t\hat{P}(0)x_0\right).$$  \hspace{1cm} (22)

Proof. Since $u(t) = 0$ for $t \in [-h_r, 0)$, then $y(0) = x(0)$. Thus the optimal cost becomes $J_T^* = E\left(x_0^t\hat{P}(0)x_0\right)$ from (21).

Corollary 2 Under the same conditions in Lemma 4 and let the controller satisfy that $u(t) = 0$ for $t \in [-h_r, h_r)$. Then there exists a unique solution to the problem of minimizing (7) subject to the system (6). The optimal controller is given by (20) for $t \geq h_r$ and the optimal cost is as

$$J_T^* = E\left(x_0^tP(0)x_0\right).$$  \hspace{1cm} (23)

Proof. Since $u(t) = 0$ for $t \in [0, h_r)$, then $y(t) = e^{A’t}y(0)$ for $t \in [0, h_r)$. By using $u(t) = 0$ for $t \in [-h_r, 0]$, it is obtained that $y(0) = x(0)$ from (5). Combining with the proof of Lemma 4 and (12), the result follows. So we omit the details.

Next, we consider the optimization problem with respect to the admissible control set set (8).

Lemma 5 If a given linear feedback control $u(t) = K(t)\hat{y}(t|t)$ is the unique optimal solution for the problem of minimizing $J_T$ s.t. (6), then $K(t)$ obeys the equations (16)-(19) with $\Omega(t) > 0$.

Proof. The proof is presented in Appendix B.

We now give the necessary and sufficient condition for the existence and uniqueness of the solution to the finite-horizon optimization problem.

Theorem 1 The problem of minimizing (7) subject to (6) within the admissible control set (8) has a unique solution if and only if (11)-(15) admits a solution such that the matrix $\Omega(t)$ is strictly positive definite. The optimal control is as (20) and the optimal cost is given by (21).

Proof. Combining with Lemmas 3-5, the result follows directly.

3.2 Finite-horizon optimal control problem of pseudo delay-free system with discounted cost function

In this subsection, we study the finite-horizon optimization problem of minimizing the discounted cost function subject to (6):

$$J_T^* = E\left[\int_0^T e^{-\alpha t} \left(y'(t)Qy(t) + u(t)'Ru(t)\right)dt\right].$$  \hspace{1cm} (24)

The discounted setting is popular in many areas, such as in dynamic programming, reinforcement learning, and planning algorithms for optimal control. See [LaValle, 2006], [Sutton et al., 1998] and references therein.

To solve the discounted LQR problem, we define the modified Riccati equation:

$$-\frac{d}{dt} \hat{P}_\alpha(t) = \hat{P}_\alpha(t)A + A'\hat{P}_\alpha(t) + \alpha \hat{P}_\alpha(t) + Q$$

$$-\Pi_\alpha(t, t),$$  \hspace{1cm} (25)

$$\hat{P}_\alpha(t) = \hat{P}_\alpha(t) + \int_0^{h_r} e^{(A + \hat{P}_\alpha(t))} dt \Pi_\alpha(t + \theta, t + \theta)$$

$$\times e^{(A + \hat{P}_\alpha(t))} d\theta,$$  \hspace{1cm} (26)

where

$$\Pi_\alpha(t, t) = K_\alpha(t)^*Q(t)K_\alpha(t),$$

$$\Pi_\alpha(t, t) = R + \sum_{i=0}^{r} B_i'e^{-A'\hat{P}_\alpha(t)b}P_i e^{-Ah_i} B_i,$$

$$K_\alpha(t) = \Omega^{-1}_{\alpha^*}(t)B'\hat{P}_\alpha(t),$$

with $\hat{P}_\alpha(T) = 0$ and $P_\alpha(T) = 0$. Following similar discussions to Lemma 3 and Remark 5 in [Zhang et al., 2017], the following result is in force.
Lemma 6 The equation (25)-(26) is equivalent to the following equations:

\[-\dot{P}_\alpha(t) = P_\alpha(t)A + A'P_\alpha(t) + \alpha P_\alpha(t) + Q - e^{A'h_\tau}\Pi_\alpha(t + h_\tau, t + h_\tau)e^{A'h_\tau},\]  

(27)

while \(\Pi_\alpha(t + h_\tau, t + h_\tau)\) is given by

\[\Pi_\alpha(t, t) = K'_{\alpha}(t)\Omega_\alpha(t)K_{\alpha}(t),\]

\[\Omega_\alpha(t) = R + \sum_{i=0}^{\infty} \bar{B}_i' e^{-A'h_i}P_\alpha(t)e^{-A'h_i}\bar{B}_i,\]

\[K_{\alpha}(t) = -\Omega^{-1}_\alpha(t) \left[ B'P_\alpha(t) - B' \int_{0}^{h_\tau} e^{A^\theta} \Pi_\alpha(t + \theta, t + \theta) \right.\]

\[\left. \times e^{A^\theta} d\theta \right],\]

with terminal values \(P_\alpha(T) = 0\) and \(\Pi_\alpha(T, T + \theta) = 0\) for \(\theta \in (0, h_\tau]\).

It is now in the position to give the solution to the discounted LQR problem.

Theorem 2 The problem of minimizing (24) subject to (6) within the admissible control set (8) has a unique solution if and only if (25)-(26) admits a solution such that the matrix \(\Omega_\alpha(t)\) is strictly positive definite. The optimal control is as

\[u(t) = K_{\alpha}(t)\tilde{y}(t|t),\]  

(28)

and the optimal cost is given by

\[J_T^* = E \left( y'(0)P_\alpha(0)y(0) - y'(0) \int_{0}^{h_\tau} \Pi_\alpha(0, \theta)\tilde{y}(0|\theta)d\theta \right).\]  

(29)

Proof. The proof is presented in Appendix C.

4 Solution to the Problem

Based on the above results for the finite-horizon optimization problem, we discuss the mean-square stabilization problem. Sufficient and necessary conditions are to be derived for the exponential mean-square stabilization of system (1). The key is to investigate the properties of the modified Riccati equations (11)-(15) and (25)-(26) when the time \(t\) tends to \(-\infty\). Firstly, we give the necessary condition for the mean-square stabilization for system (1).

Theorem 3 Assume that the system (1) is exponentially mean-square stabilizable in the sense of Definition 2, then the following modified algebraic Riccati equation (30)-(34) has a solution \(P \geq \tilde{P} > 0\),

\[0 = A'\tilde{P} + \tilde{P}A - \Pi(0) + I,\]  

(30)

\[P = \tilde{P} + \int_{0}^{h_\tau} e^{A^\theta} \Pi(0)e^{A^\theta} d\theta,\]  

(31)

where

\[\Pi(0) = K'\Omega K,\]

\[\Omega = I + \sum_{i=0}^{\infty} \bar{B}_i' e^{-A'h_i}P(t)e^{-A'h_i}\bar{B}_i,\]  

(33)

\[K = \Omega^{-1}B'\tilde{P}.\]  

(34)

Proof. The proof is put in Appendix D.

We then present the sufficient condition for the exponential mean-square stabilization by defining a new Lyapunov function.

Theorem 4 Assume that the following equation has a unique solution \(P_\alpha \geq \tilde{P}_\alpha > 0\),

\[0 = A'\tilde{P}_\alpha + \tilde{P}_\alpha A + \alpha P_\alpha - \Pi_\alpha(0) + I,\]  

(35)

\[P_\alpha = \tilde{P}_\alpha + \int_{0}^{h_\tau} e^{(A + \tilde{\bar{H}}^T)^\theta} \Pi_\alpha(0)e^{(A + \tilde{\bar{H}}^T)^\theta} d\theta,\]  

(36)

where

\[\Pi_\alpha(0) = K'_{\alpha}\Omega_\alpha K_{\alpha},\]

\[\Omega_\alpha = I + \sum_{i=0}^{\infty} \bar{B}_i' e^{-A'h_i}P_\alpha(t)e^{-A'h_i}\bar{B}_i,\]

(38)

\[K_{\alpha} = \Omega^{-1}_\alpha B'\tilde{P}_\alpha,\]  

(39)

then the system (1) is exponentially mean-square stable with the controller \(u(t) = K_{\alpha}\tilde{y}(t|t)\) where \(K_{\alpha}\) is given by (39).

Proof. The proof is formulated in Appendix E.

5 Conclusions

This paper studied the stabilization problem for the Itô systems with both multiplicative noise and multiple delays. Sufficient and necessary conditions have been obtained for the exponential mean-square stabilization in terms of modified Riccati equations. The main technique is to reduce the original system with multiple delays to
the pseudo delay-free one and study the finite-horizon optimization problems for the pseudo system with standard and discounted linear quadratic cost functions.

A Proof of Lemma 4

Using Lemma 3, the equations (16)-(19) admit a solution such that the matrix $\Omega(t) > 0$. Applying Itô's formula to $y'(t) [P(t)y(t) - \int_0^t \Pi(t, t + \theta) \tilde{y}(t|t+\theta)d\theta]$ and combining with the equations (16)-(19), we have

\[ d\left\{ y'(t) \left[ P(t)y(t) - \int_0^{h_r} \Pi(t, t + \theta) \tilde{y}(t|t+\theta)d\theta \right] \right\} \]

\[ = \left\{ \left( Ay(t) + \sum_{i=0}^r e^{-Ah_i} \tilde{B}_i u(t) \right)' + \sum_{i=0}^r e^{-Ah_i} \tilde{B}_i u(t) \right\}' dt \]

\[ + u'(t) \sum_{i=0}^r \tilde{B}_i e^{-Ah_i} P(t) e^{-Ah_i} \tilde{B}_i u(t) \]

\[ - y'(t) \Pi(t, t + h_r) \tilde{y}(t|t) + y'(t) \Pi(t, t) \tilde{y}(t|t) \]

\[ - y'(t) \int_0^{h_r} \partial \Pi(t, t + \theta) \tilde{y}(t|t+\theta)d\theta - y'(t) \]

\[ \times \left( A\tilde{y}(t|t) + \sum_{i=0}^r e^{-Ah_i} \tilde{B}_i u(t) \right)' dt \]

\[ + \left\{ \left( \sum_{i=0}^r e^{-Ah_i} \tilde{B}_i u(t) \right)' P(t)y(t) + y'(t) P(t) \right\} \]

\[ \times \left( \sum_{i=0}^r e^{-Ah_i} \tilde{B}_i u(t) \right) - \left( \sum_{i=0}^r e^{-Ah_i} \tilde{B}_i u(t) \right)' dt \]

\[ \times \int_0^{h_r} \Pi(t, \theta) \tilde{y}(t|\theta)d\theta \}\]

\[ \int_0^T \Pi(t, \theta) \tilde{y}(t|\theta)d\theta \}

\[ = \left[ -y'(t)Qy(t) + y'(t)\Pi(t, t)\tilde{y}(t|t) + 2u'(t)B'P(t)y(t) \right. \]

\[ + u'(t) \sum_{i=0}^r \tilde{B}_i e^{-Ah_i} P(t) e^{-Ah_i} \tilde{B}_i u(t) \]

\[ - u'(t)B' \int_0^{h_r} \Pi(t, t + \theta) \tilde{y}(t|t+\theta)d\theta \]

\[ - y'(t) \int_0^{h_r} \Pi(t, \theta) \tilde{y}(t|\theta)d\theta \}

Taking integral from 0 to $T$ on both sides of (A.1) and then taking expectation, we have

\[ J_T = E \left( y'(0) P(0)y(0) - y'(0) \int_0^{h_r} \Pi(0, \theta) \tilde{y}(0|\theta)d\theta \right) \]

\[ + E \int_0^T \left( u'(t)\Omega(t)u(t) - 2u'(t)\Omega(t)K(t)y(t) \right. \]

\[ + y'(t)\Pi(t, t)\tilde{y}(t|t) \}

\[ = E \left( y'(0) P(0)y(0) - y'(0) \int_0^{h_r} \Pi(0, \theta) \tilde{y}(0|\theta)d\theta \right) \]

\[ + E \int_0^T \left( u(t) - K(t)\tilde{y}(t|t) \right)' \Omega(t) \left( u(t) \right. \]

\[ - K(t)\tilde{y}(t|t) \}

\[ + y'(t)\Pi(t, t)\tilde{y}(t|t) \}

where the fact of $E \left\{ \left[ y(t) - \tilde{y}(t|t) \right]' \tilde{y}(t|t) \right\} = 0$ has been used in the derivation of the above equality. Note that $\Omega(t) > 0$, the optimal control exists uniquely. Furthermore, the optimal control (20) and cost function (21) follows from (A.2) directly combining with Lemma 3.

B Proof of Lemma 5

Consider the optimization problem for the controller set \( \{ u(t) : u(t) = K(t)\tilde{y}(t|t) \} \) with respect to the matrix $K(t)$. The cost function is

\[ J_T = E \left[ \int_0^T \left[ y'(t)Qy(t) + \tilde{y}'(t|t)K'(t)RK(t)\tilde{y}(t|t) \right] dt \right. \]

\[ + y'(T)HY(T) \}

\[ = tr \left[ \int_0^T \left[ QY(t) + K'(t)RK(t)\tilde{Y}(t|t) \right] dt + HY(T) \right]. \]

(B.1)

where $Y(t) = E[y(t)\tilde{y}(t)]$ and $\tilde{Y}(t|t) = E[\tilde{y}(t|t)\tilde{y}(t|t)]$. The system under the controller $u(t) = K(t)\tilde{y}(t|t)$ is
Thus, we have
\[ dy(t) = \left[ Ay(t) + BK(t) \dot{y}(t|t) \right] dt + \sum_{i=0}^{r} e^{-Ah_i} \dot{\bar{B}}_i K(t) \dot{y}(t|t) dw_i(t + h_i). \]  

In this case,
\[ d[y(t) y'(t)] = \left[ Ay(t) + BK(t) \dot{y}(t|t) \right] y'(t) dt + \sum_{i=0}^{r} e^{-Ah_i} \dot{\bar{B}}_i K(t) \dot{y}(t|t) y'(t) dw_i(t + h_i) 
+ y(t) \left[ Ay(t) + BK(t) \dot{y}(t|t) \right]' dt 
+ y(t) \sum_{i=0}^{r} \left( e^{-Ah_i} \dot{\bar{B}}_i K(t) \dot{y}(t|t) \right)' dw_i(t + h_i) 
+ \sum_{i=0}^{r} e^{-Ah_i} \dot{\bar{B}}_i K(t) \dot{y}(t|t) \dot{y}'(t|t) K'(t) \ddot{B}_i e^{-A' h_i} dt, \]

that is,
\[ \frac{d}{dt} Y(t) = AY(t) + BK(t) \dot{Y}(t|t) + Y(t) A' + \dot{Y}(t|t) 
\times K'(t) B' + \sum_{i=0}^{r} e^{-Ah_i} \dot{\bar{B}}_i K(t) \dot{y}(t|t) K'(t) 
\times \dot{B}_i e^{-A' h_i}. \]  

In addition, it is obtained that
\[ \frac{\partial}{\partial t} \dot{Y}(t|\theta) = A \dot{Y}(t|\theta) + BK(t) \dot{Y}(t|t) + \dot{Y}(t|t) A' 
+ \dot{Y}(t|t) K'(t) B', \]

thus, we have
\[ \frac{d}{dt} \int_{t}^{t+h_r} \dot{Y}(t|\theta) \Pi'(t, \theta) d\theta = \int_{t}^{t+h_r} \frac{\partial}{\partial t} \dot{Y}(t|\theta) d\theta 
- Y(t) \Pi'(t, t + h_r) - \dot{Y}(t|t) \Pi'(t, t) + \int_{t}^{t+h_r} \frac{\partial}{\partial t} \dot{Y}(t|\theta) d\theta 
\times \Pi'(t, \theta) d\theta + \int_{t}^{t+h_r} \dot{Y}(t|\theta) \frac{\partial}{\partial t} \Pi'(t, \theta) d\theta. \]  

Using the Lagrange multiplier approach, the cost function can be reformulated as follows:
\[ J_T = \int_{0}^{T} tr \left[ QY(t) + K'(t) RK(t) \dot{Y}(t|t) + [AY(t) 
+ BK(t) \dot{Y}(t|t) + Y(t) A' + \dot{Y}(t|t) K'(t) B'] 
+ \sum_{i=0}^{r} e^{-Ah_i} \dot{\bar{B}}_i K(t) \dot{Y}(t|t) K'(t) \ddot{B}_i e^{-A' h_i} - \dot{Y}(t) \right] 
\times P'(t) - \int_{t}^{t+h_r} \left[ A \dot{Y}(t|\theta) + BK(t) \dot{Y}(t|t) + \dot{Y}(t|t) A' 
+ \dot{Y}(t|t) K'(t) B' - \frac{\partial}{\partial t} \dot{Y}(t|\theta) \right] \Pi'(t, \theta) d\theta \] 
\[ + tr[H Y(T)], \]

where \( P(\cdot), \Pi(\cdot, \cdot) \) are matrix parameters with compatible dimension. By making some algebraic transformation, it is further rewritten as
\[ J_T = \int_{0}^{T} \text{tr} \left[ QY(t) + K'(t) RK(t) \dot{Y}(t|t) + [AY(t) 
+ BK(t) \dot{Y}(t|t) + Y(t) A' + \dot{Y}(t|t) K'(t) B'] 
+ \sum_{i=0}^{r} e^{-Ah_i} \dot{\bar{B}}_i K(t) \dot{Y}(t|t) K'(t) \ddot{B}_i e^{-A' h_i} \right] P'(t) 
+ Y(t) \dot{P}'(t) - \int_{t}^{t+h_r} \left[ A \dot{Y}(t|\theta) + BK(t) \dot{Y}(t|t) + \dot{Y}(t|t) A' 
+ \dot{Y}(t|t) K'(t) B' - \frac{\partial}{\partial t} \dot{Y}(t|\theta) \right] \Pi'(t, \theta) d\theta 
\times \Pi'(t, \theta) d\theta + \text{tr}[H Y(T)] - P(T) Y(T) + P(0) Y(0) 
\]
\[ = \int_{0}^{T} \text{tr} \left[ QY(t) + K'(t) RK(t) \dot{Y}(t|t) + [AY(t) 
+ BK(t) \dot{Y}(t|t) + Y(t) A' + \dot{Y}(t|t) K'(t) B'] 
+ \sum_{i=0}^{r} e^{-Ah_i} \dot{\bar{B}}_i K(t) \dot{Y}(t|t) K'(t) \ddot{B}_i e^{-A' h_i} \right] P'(t) 
\times \Pi'(t, \theta) d\theta + \text{tr}[H Y(T)] - P(T) Y(T) + \text{tr}[H Y(T)] 
\]
\[ = \int_{0}^{T} \text{tr} \left[ QY(t) + K'(t) RK(t) \dot{Y}(t|t) + [AY(t) 
+ BK(t) \dot{Y}(t|t) + Y(t) A' + \dot{Y}(t|t) K'(t) B'] \Pi'(t, \theta) d\theta 
- P(T) Y(T) + P(0) Y(0) + \int_{T}^{T+h_r} \dot{Y}(t|\theta) \Pi'(t, \theta) d\theta \right] d\theta. \]
Taking partial differential yields that

\[
0 = \frac{\partial J}{\partial Y(t)} = Q + A'P(t) + P(t)A + \dot{P}(t) - \Pi(t, t + h_r),
\]

\[
0 = \frac{\partial J}{\partial Y(t)} = -\frac{\partial}{\partial t}\Pi(t, t) - A\Pi(t, t) - \Pi(t, t)A,
\]

\[
0 = \frac{\partial J}{\partial Y(t)\theta - h} = -\frac{\partial}{\partial t}\Pi(t, \theta) - A'\Pi(t, \theta) - \Pi(t, \theta)A,
\]

\[
0 = \frac{\partial J}{\partial Y(t)\theta} = K'(t)RK(t) + K'(t)B'P(t) + P(t)BK(t)
\]
\[
+ \sum_{i=0}^{r} K'(t)B_i e^{-A'h_i}P(t)e^{-Ah_i}B_i K(t)
\]
\[
- K'(t)B' \int \Pi(t, \theta)d\theta
\]
\[
- \int t + h_r \Pi(t, \theta)d\theta BK(t) + \Pi(t, t)
\]
\[
= K'(t)\Omega(t)K(t) + K'(t)(B'P(t) - B')
\]
\[
× \int \Pi(t, \theta)d\theta + \left( P(t)B - \int \Pi(t, \theta)d\theta B \right)
\]
\[
× K(t) + \Pi(t, t),
\]

\[
0 = \frac{\partial J}{\partial K(t)} = RK(t)\dot{Y}(t) + RK(t)\dot{Y}'(t) + B'P(t)\dot{Y}(t)
\]
\[
+ B'P(t)\dot{Y}'(t) + \sum_{i=0}^{r} B_i e^{-A'h_i}P(t)e^{-Ah_i}B_i K(t)
\]
\[
× \dot{\dot{Y}}'(t) - \int \dot{\dot{Y}}'(t)\Pi(t, \theta)d\theta \dot{\dot{Y}}'(t)|t
\]
\[
+ \sum_{i=0}^{r} B_i e^{-A'h_i}P(t)e^{-Ah_i}B_i K(t)\dot{Y}(t)|t
\]
\[
- \int \dot{\dot{Y}}'(t)\Pi(t, \theta)d\theta \dot{\dot{Y}}'(t)|t
\]
\[
= \left[ \Omega(t)K(t) + B'P(t) \right] - \int B'P'(t, \theta)\Pi(t, \theta)dt \dot{\dot{Y}}'(t)|t
\]
\[
+ \left[ \Omega(t)K(t) + B'P(t) \right] - \int B'P'(t, \theta)\Pi(t, \theta)dt \dot{\dot{Y}}'(t)|t,
\]

with \( P(T) = H \) and \( \Pi(T, \theta) = 0 \). Thus, we have the following equation:

\[
-\dot{P}(t) = Q + A'P(t) + P(t)A - \Pi(t, t + h_r),
\]

\[
-\frac{\partial}{\partial t}\Pi(t, \theta) = A\Pi(t, \theta) + \Pi(t, \theta)A, \quad \Pi(T, \theta) = 0,
\]

\[
\Pi(t, t) = K'(t)\Omega(t)K(t),
\]

\[
0 = \Omega(t)K(t) + B'P(t) - \int_{t}^{t+h_r} B'\Pi(t, \theta)d\theta.
\]

Using the unique existence of the optimal controller, we have the positive definiteness of the matrix \( \Omega(t) > 0 \). Thus, (16)-(19) admits a solution with \( \Omega(t) > 0 \).

**C Proof of Theorem 2**

"Necessity" By applying similar procedures to Lemma 5, the necessity follows directly. To avoid duplication, we omit the details.

"Sufficiency" Using Lemma 6, the equation (27) admits a solution such that the matrix \( \Omega_h(t) > 0 \). Applying Itô’s formula to \( e^{yt}y(t) \left[ P_h(t)y(t) - \int_{0}^{h_r} \Pi_h(t, \theta)\dot{y}(t)|t + \theta)d\theta \right] \) and combining with the equations (25)-(26), we have

\[
d \left[ e^{yt}y(t) \left( P_h(t)y(t) - \int_{0}^{h_r} \Pi_h(t, \theta)\dot{y}(t)|t + \theta)d\theta \right) \right]
\]

\[
e^{yt} \left[ \sum_{i=0}^{r} e^{-Ah_i}B_i u(t) \right] \left[ P(t)y(t) \right]
\]

\[
+ \left( Ay(t) + \sum_{i=0}^{r} e^{-Ah_i}B_i u(t) \right)' \left[ P(t)y(t) \right]
\]

\[
- \int \Pi(t, t + \theta)\dot{y}(t)|t + \theta)d\theta + y(t)\dot{P}(t)y(t)
\]

\[
+ y'(t)\left[ Ay(t) + \sum_{i=0}^{r} e^{-Ah_i}B_i u(t) \right]
\]

\[
+ u'(t) \sum_{i=0}^{r} B_i e^{-Ah_i}P(t)e^{-Ah_i}B_i u(t)
\]

\[
- y'(t)\Pi(t, t + h_r)\dot{y}(t)|t + \sum_{i=0}^{r} e^{-Ah_i}B_i u(t)
\]

\[
y'(t) \int_{0}^{h_r} \frac{\partial}{\partial t} \Pi(t, t + \theta)\dot{y}(t)|t + \theta)d\theta
\]

\[
y'(t) \int_{0}^{h_r} \Pi(t, t + \theta) \left( y(t) + \sum_{i=0}^{r} e^{-Ah_i}B_i u(t) \right)
\]

\[
x B_i u(t) \right) dt + e^{yt} \left[ \sum_{i=0}^{r} e^{-Ah_i}B_i u(t) \right] P(t)y(t)
\]

\[
y'(t)\left[ \sum_{i=0}^{r} e^{-Ah_i}B_i u(t) \right]
\]
\[ - \left[ \sum_{i=0}^{r} e^{-Ah_i} \bar{B}_i u(t) \right] \left\{ \int_{t}^{t+h_r} \Pi(t, \theta) \hat{y}(t) d\theta \right\} dt ] dw(t) = e^{\alpha t} \left[ y(t)' Q y(t) + y(t) \Pi(t, t) \hat{y}(t) \right] + u(t) \sum_{i=0}^{r} \bar{B}_i e^{-A_h} \bar{B}_i u(t) e^{-A_h} \times \hat{B}_i u(t) - u(t)' B' \left\{ \int_{t}^{t+h_r} \Pi(t, \theta) \hat{y}(t \theta) d\theta \right\} dt \]

\[ = e^{\alpha t} \left[ y(t)' Q y(t) + y(t) \Pi(t, t) \hat{y}(t) \right] + u(t) \sum_{i=0}^{r} \bar{B}_i e^{-A_h} \bar{B}_i u(t) e^{-A_h} \times \hat{B}_i u(t) - u(t)' B' \left\{ \int_{t}^{t+h_r} \Pi(t, \theta) \hat{y}(t \theta) d\theta \right\} dt \]

Taking integral from 0 to \( T \) and then taking expectation on both sides of the above equation, we have

\[ J_T^2 = E \left[ y(t)' P_s(t) y(t) - y(t)' \int_{0}^{h_r} \Pi_s(t, \theta) \hat{y}(t \theta) d\theta \right] dt + E \int_{0}^{T} e^{\alpha t} \left( y(t)' \Omega(t) u(t) - 2u(t)' \Omega(t) K(t) y(t) + y(t)' \Pi(t, t) \hat{y}(t) \right) dt \]

\[ = E \left[ y(t)' P_s(t) y(t) - y(t)' \int_{0}^{h_r} \Pi_s(t, \theta) \hat{y}(t \theta) d\theta \right] dt + E \int_{0}^{T} e^{\alpha t} \left( u(t) - K(t) \hat{y}(t) \right)' \Omega(t) \left( u(t) - K(t) \hat{y}(t) \right) dt, \]

where the fact of \( E \left\{ [y(t) - \hat{y}(t)\theta]' \hat{y}(t) \right\} = 0 \) has been used in the derivation of the above equality. Note that \( \Omega(t) > 0 \), the optimal control exists uniquely. Furthermore, the optimal control (28) and optimal cost function (29) follows from (C.1) directly.

**D Proof of Theorem 3**

In view of Theorem 1, the fact that \( R = I > 0 \) can ensure the existence of the solution to (11)-(15) with \( \Omega(t) > 0 \). Re-denote the solution \( P(t), \hat{P}(t) \) and \( \Pi(t, t + \theta) \) of (11)-(15) as \( P_T(t) \), \( \hat{P}_T(t) \) and \( \Pi_T(t, t + \theta) \) respectively, with the terminal time \( T \) and the terminal values \( P(T) = H = 0, \hat{P}(T) = 0 \) and \( \Pi(T, T + \theta) = 0 \). We first show that \( \hat{P}_T(t) \) of (11) and \( P_T(t) \) of (12) are convergent. Based on Corollary 1, the optimal cost becomes \( J_T = E \left( x_0' \hat{P}_T(0)x_0 \right) \). Noting the time-invariance of (16)-(19) with respect to \( T \), i.e., for \( t \leq T \),

\[ P_T(t) = P_{T-t}(0), \Pi_T(t, t + \theta) = \Pi_{T-t}(0, \theta), \theta \in [0, h_T]. \]

Thus, for any \( T_1 > T > t \) and for all \( x_0 \neq 0 \), we have

\[ x_0' \hat{P}_{T_1}(t)x_0 = x_0' \hat{P}_{T_1-t}(0)x_0 = J_{T_1-t} \]

\[ \geq J_{T-t} = x_0' \hat{P}_{T-t}(0)x_0 = x_0' \hat{P}_T(t)x_0. \]

Since \( x_0 \) is arbitrary, thus \( \hat{P}_{T_1}(t) \geq \hat{P}_T(t) \). Similarly, if \( t_1 < t_2 \leq T \),

\[ x_0' \hat{P}_{T_2}(t_1)x_0 = x_0' \hat{P}_{T_2-t_1}(0)x_0 = J_{T-t_2} \]

\[ \geq J_{T-t_2} = x_0' \hat{P}_{T-t_2}(0)x_0 = x_0' \hat{P}_T(t_2)x_0. \]

That is, \( \hat{P}_T(t_1) \geq \hat{P}_T(t_2) \). Thus, \( \hat{P}_T(t) \) is monotonically increasing with respect to \( T \) and is monotonically decreasing with respect to \( t \).

We then show the uniform boundedness of \( \hat{P}_T(t) \). Since system (1) is exponentially stabilizable in the sense of Definition 2, together with (5), there exists a positive constant \( \delta \) such that

\[ e^{\alpha t} E \| y(t) \|^2 \]

\[ \leq \delta e^{\alpha t} E \| x(t) \|^2 + \sum_{i=1}^{r} \int_{t}^{t+h_i} \| e^{A(t-s)} B_i u(s - h_i) \|^2 ds \]

\[ + \sum_{i=1}^{r} \int_{t}^{t+h_i} \| e^{A(t-s)} B_i u(s - h_i) \|^2 ds \]

\[ \to 0, \quad t \to \infty, \]

where the last limit holds for \( \lim_{t \to \infty} e^{\alpha t} E \| x(t) \|^2 = 0 \) and \( \lim_{t \to \infty} e^{\alpha t} E \| u(t) \|^2 = 0 \). Together with the exponential stability of \( u(t) \), we have the boundness of the cost function \( J_T \) under the stabilizing controller. In fact, there exists a positive constant \( \mu \) such that \( e^{\alpha t} E \| y(t) \|^2 \leq \mu \| x_0 \|^2 \) and \( e^{\alpha t} E \| u(t) \|^2 \leq \mu \| x_0 \|^2 \). This further implies that there exists a positive constant \( \beta \) such that

\[ E \int_{0}^{\infty} \left( y(t)' Q y(t) + u(t)' R u(t) \right) dt \leq \beta \| x_0 \|^2. \]

Thus

\[ J_T = x_0' \hat{P}_T(0)x_0 \leq \beta \| x_0 \|^2, \]
that is, $\hat{P}_T(0)$ is uniformly bounded. Recalling the monotonicity of $\hat{P}_T(t)$, it yields that $\hat{P}_T(t)$ is convergent, i.e.,

$$\lim_{t \to -\infty} \hat{P}_T(t) = \lim_{t \to -\infty} \hat{P}_{T-1}(t) = \lim_{T \to -\infty} \hat{P}_T(0) = \hat{P},$$

where $\hat{P}$ is a constant matrix which is independent of $t$.

Consider the optimal cost (23) in Corollary 2, we have that $P_T(t)$ is monotonically increasing with respect to $T$ and is monotonically decreasing with respect to $t$. Moreover, $P_T(0)$ is uniformly bounded. The discussion is similar to that of $\hat{P}_T(t)$, so we omit the details. This implies that $P_T(t)$ is convergent, i.e.,

$$\lim_{t \to -\infty} P_T(t) = \lim_{t \to -\infty} P_{T-1}(t) = \lim_{T \to -\infty} P_T(0) = P,$$

where $P$ is a constant matrix which is independent of $t$. Let $t \to -\infty$ in the equations (11)-(15), we immediately have (30)-(34).

Secondly, we show the strictly positive definiteness of the matrix $\hat{P}$. Otherwise, there exists $z \neq 0$, such that $z'\hat{P}z = 0$. Similar to (A.1) and (A.2), by applying Itô’s formula to $\hat{y}(t)[PY(t) - \int_{0}^{h_r} \Pi(\theta)\hat{y}(t + \theta)d\theta]$ where $\Pi(\theta) = e^{A\theta}\Pi(0)e^{A\theta}$, $P$ and $\Pi(\theta)$ are as in (31)-(34), it follows that

$$E\left\{\sum_{r=0}^{\infty} \sum_{i=0}^{r} \tilde{B}_i e^{-A\tilde{h}_r} P(t)e^{-A\tilde{h}_r} \tilde{B}_i u(t)\right\} dt,$$

Let $u(t) = K\hat{y}(t)$, $t \in [0, T]$, thus

$$0 \leq E \left\{ \int_{0}^{T} [y(t)y(t) + u(t)u(t)] dt \right\}$$

$$= -E \left\{ \int_{0}^{T} \Pi(\theta)\hat{y}(t + \theta) d\theta \right\}$$

$$= -E \left\{ \sum_{r=0}^{\infty} \sum_{i=0}^{r} \tilde{B}_i e^{-A\tilde{h}_r} P(t)e^{-A\tilde{h}_r} \tilde{B}_i u(t)\right\} dt,$$

Now let $x(0) = z$ where $z$ is given as $z'\hat{P}z = 0$. Then, $x_0'\hat{P}x_0 = 0$. Thus

$$0 \leq -E \left\{ \int_{0}^{T} [y(t)y(t) + u(t)u(t)] dt \right\}$$

Further note that $\Pi(\theta) \geq 0$ and $\hat{P} \geq 0$ as shown in the above, we have

$$\leq E \left\{ \frac{y(t)}{P_T(T)} - \frac{y(t)}{P_T(T)} \int_{0}^{h_r} \Pi(\theta)\hat{y}(t + \theta) d\theta \right\}$$

$$= \frac{y(t) + \Pi(\theta)\hat{y}(t + \theta) d\theta}{0}$$

$$= E \left\{ \frac{y(T)}{P_T(T)} \right\} \geq 0,$$

where $\hat{y}(t + \theta) = y(t) - \hat{y}(t + \theta)$ and $E[y(t)\hat{y}(t + \theta)] = 0$ have been used in the above. Thus, it follows from (D.1) and (D.2) that

$$0 \leq E \left\{ \int_{0}^{T} [y(t)y(t) + u(t)u(t)] dt \right\} \leq 0.$$

This implies that

$$E[y(t)y(t)] = 0, E[u(t)u(t)] = 0, t \geq 0.$$

Then, it is obtained that $y(t) = 0$ and $u(t) = 0$, $t \geq 0$, a.s.. System (6) is thus now reduced to

$$dy(t) = Ay(t)dt, y(0) = x_0 = z \neq 0,$$

with the output $y(t) = 0$ a.s., this is a contradiction with the observability of the system $(A, I)$. Thus, the matrix $\hat{P}$ is positive definite. Together with (31)-(33), $P \geq P > 0$ follows. The proof is now completed.

E Proof of Theorem 4

We will prove that the system (5) is exponentially mean-square stabilizable under the controller $u(t) = K\hat{y}(t)$.
Define the Lyapunov function candidate as
\[ V(t, y(t)) = e^{\alpha t} E[y'(t)P_\alpha y(t) - y'(t) \int_0^{h_v} \Pi_\alpha(\theta) \cdot \hat{y}(t|t + \theta) d\theta], \quad t \geq 0, \] \tag{E.1}
where \( \Pi_\alpha(\theta) = e^{A^\theta \Pi_\alpha(0)}e^{A^\theta}. \) It is obvious that
\[ V(t, y(t)) \geq e^{\alpha t} E[y'(t)P_\alpha y(t) - y'(t) \int_0^{h_v} \Pi_\alpha(\theta) \hat{y}(t|t + \theta) d\theta] \]
\[ - \int_0^{h_v} \hat{y}(t|t + \theta) \Pi_\alpha(\theta) \hat{y}(t|t + \theta) d\theta \]
\[ = e^{\alpha t} E[y'(t)\hat{P}_\alpha y(t)] \geq 0, \tag{E.2} \]
where \( \hat{y}(t|t + \theta) = y(t) - \hat{y}(t|t + \theta), \) and \( \hat{P}_\alpha > 0 \) is used in the last equality. It is clear that \( V(t, y(t)) \to \infty \) if \( E\|y(t)\|^2 \to \infty \) from (E.2). By taking time derivative along the dynamic of the stochastic system (5) and combining with (30)-(34), we have
\[ \dot{V}(t, y(t)) = e^{\alpha t} E\left\{ y'(t) \left[ A'P_\alpha + P_\alpha A + \alpha P_\alpha - \Pi_\alpha(h_v) \right] y(t) + u(t) \sum_{i=0}^r B_i' e^{-A^i h_v} P_{\alpha}(t) e^{-A^i h_v} B_i u(t) \right\} \]
\[ + u(t) \left[ \int_{t}^{t+h_v} \hat{y}(t|\theta) d\theta - \hat{y}'(t) \right] \Pi_\alpha(\theta - t) + \Pi_\alpha(\theta - t) A + A' \Pi_\alpha(\theta - t) \]
\[ + \alpha \Pi_\alpha(\theta - t) \hat{y}(t|\theta) d\theta - u'(t) \Omega_\alpha K_\alpha y(t) \]
\[ - y'(t) K_\alpha' \Gamma_\alpha u(t) + y'(t) \Pi_\alpha(0) \hat{y}(t|t) \right\} \]
\[ = -e^{\alpha t} E[y'(t)y(t) + u(t)u'(t)] \leq 0. \tag{E.3} \]
Thus from (E.3), we know \( V(t, y(t)) \) is nonincreasing, and thus \( V(t, y(t)) \leq V(0, y(0)). \) Therefore, \( \lim_{t \to \infty} V(t, y(t)) \) exists.

Integrating on both sides of (E.3) from \( t \) to \( t + T \) yields
\[ V(t + T, y(t + T)) - V(t, y(t)) \]
\[ = - \int_{t}^{t+T} e^{\alpha s} E[y'(s)y(s) + u'(s)u(s)] ds \]
\[ = - \int_{t}^{t+T} e^{\alpha s} E[y'(s)y(s) + \hat{y}'(s|s)K_\alpha y(s|s)] ds. \]
Now we consider the following cost function,
\[ E \int_0^{t+T} e^{\alpha s} E[y'(s)y(s) + u'(s)u(s)] ds. \tag{E.4} \]
By applying Theorem 2, the optimal controller to minimize (E.4) subjected to system (5) is given as \( u^*(s) = K_\alpha(s)\hat{y}'(s|s), \) where \( K_\alpha(s) \) is given by (25)-(26) with \( Q = I, R = I. \) \( y^*(t) \) is the corresponding state trajectory. Accordingly, the optimal cost of (E.4) is given by
\[ E \int_0^{t+T} e^{\alpha s} E[y'(s)y(s) + u'^*(s)u^*(s)] ds \]
\[ = e^{\alpha t} E[y'(t)P_\alpha(0)y(t) - y'(t) \int_0^{h_v} \Pi_\alpha(0, \theta) \hat{y}(t|t + \theta) d\theta], \]
Therefore, we have
\[ V(t + T, y(t + T)) - V(t, y(t)) \]
\[ = -E \int_0^{t+T} e^{\alpha s} E[y'(s)y(s) + \hat{y}'(s|s)K_\alpha y(s|s)] ds \]
\[ \leq -E \int_0^{t+T} e^{\alpha s} E[y'^*(s)y^*(s) + u'^*(s)u^*(s)] ds \]
\[ = -e^{\alpha t} E[y'(t)P_\alpha(0)y(t) - y'(t) \int_0^{h_v} \Pi_\alpha(0, \theta) \hat{y}(t|t + \theta) d\theta]\]
\[ \leq 0. \tag{E.5} \]
Note
\[ \lim_{t \to \infty} E[V(t + T, y(t + T)) - V(t, y(t))] \]
\[ = \lim_{t \to \infty} V(t + T, y(t + T)) - \lim_{t \to \infty} V(t, y(t)) = 0, \]
it follows from (E.5) that
\[ 0 = \lim_{t \to \infty} e^{\alpha t} E[y'(t)P_\alpha(0)y(t)] \]
\[ - y'(t) \int_0^{h_v} \Pi_\alpha(0, \theta) \hat{y}(t|t + \theta) d\theta]. \]
Further, since
\[ 0 \leq E[y'(t)\hat{P}_\alpha(0)y(t)] \]
\[ \leq E[y'(t)P_\alpha(0)y(t) - y'(t) \int_0^{h_v} \Pi_\alpha(0, \theta) \hat{y}(t|t + \theta) d\theta], \]
\[ \]
it follows that
\[ \lim_{t \to \infty} e^{\alpha t} E \left[ y^r(t) \hat{P}_0(0) y(t) \right] = 0. \]  
(E.6)

Now we are in the position to show that \( \hat{P}_0(0) > 0 \). If this is not the case, there would exist \( z \neq 0 \), such that \( z' \hat{P}_0(0) z = 0 \). Consider the closed-loop system \( dy_g(t) = \left[ Ay_g(t) + BK_g y(t) \right] dt + \sum_{i=0}^{\infty} e^{-\alpha t} \tilde{B}_i K_\alpha y(t) dw_t + \alpha t \) with initial value \( y(0) = z \). Return to (21), one has
\[ E \int_0^T [y^r(t)y^*(t) + u^r(t)u^*(t)] dt = z' \hat{P}(0) z = 0. \]

Together with Assumption 1, one has
\[ y^r(t) = 0, u^r(t) = 0, t \geq 0, a.s. \]
The system (6) is thus reduced to
\[ dy^r(t) = Ay^r(t)dt, y^r(t) = 0, a.s., t \in [0,T]. \]
In view of the observability of system \((A, I)\), it yields that \( z = 0 \), which is a contradiction. Thus, \( \hat{P}_0(0) > 0 \). Together with (E.6), we have
\[ \lim_{t \to \infty} e^{\alpha t} E \| y(t) \|^2 = 0. \]
Using the fact that \( u(t) = K_\alpha \hat{y}(t)|t \), it is immediately obtained that
\[ \lim_{t \to \infty} e^{\alpha t} E \| x(t) \|^2 = 0. \]
Thus \( \lim_{t \to \infty} e^{\alpha t} E \| x(t) \|^2 = 0 \) follows from (5). The exponential mean-square stability of system (1) follows. The proof is now completed.

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