On the Sample Complexity of Compressed Counting

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Abstract

Compressed Counting (CC)[25], based on maximally skewed stable random projections, was recently proposed for estimating the \( \alpha \)th frequency moments of data streams. When \( \Delta = |1 - \alpha| \to 0 \), [25] provided an algorithm based on the geometric mean estimator and proved that the sample complexity was essentially \( O\left(\frac{1}{\varepsilon^2}\right) \), which was a large improvement compared to the previously known \( O\left(\frac{1}{\varepsilon}\right) \) bound. The case \( \Delta = |1 - \alpha| \to 0 \) is extremely useful for estimating Shannon entropy of data streams.

In this study, we provide a very simple algorithm based on the sample minimum estimator and prove that, when \( \Delta = 1 - \alpha \to 0^+ \), it suffices to let the sample size \( k \) be

\[
k \geq \frac{\log \frac{1}{\delta} - \log \left( \frac{1}{\varepsilon^2} + \frac{1}{2\Delta \log 1 + \varepsilon} + \frac{1}{2\Delta \log 1 + 2\log (1 + \varepsilon)} + O(\Delta) \right)}{\log \frac{1}{\Delta} - \log \frac{1}{2} + 1}
\]

so that, with probability at least \( 1 - \delta \), the estimated \( \alpha \)th frequency moments will be within a \( 1 + \epsilon \) factor of the truth. For example, when \( \varepsilon = 10^{-3}, \delta = 10^{-10}, \) and \( \Delta = 10^{-5} \), the required sample size is merely \( k \geq 51 \).

1 Introduction

The problem of “scaling up for high dimensional data and high speed data streams” is among the “ten challenging problems in data mining research”[36]. This paper is devoted to estimating entropy of data streams. Mining data streams[19][4][1][29] in (e.g.,) 100 TB scale databases has become an important area of research, e.g.,[10][1], as network data can easily reach that scale[36]. Search engines are a typical source of data streams[4].

Consider the Turnstile stream model[29]. The input stream \( a_t = (i_t, I_t), i_t \in [1, D] \) arriving sequentially describes the underlying signal \( A \), meaning

\[
A_t[i_t] = A_{t-1}[i_t] + I_t,
\]

where the increment \( I_t \) can be either positive (insertion) or negative (deletion). Restricting \( A_t[i] \geq 0 \) results in the strict-Turnstile model, which suffices for describing almost all natural phenomena. This study focuses on the strict-Turnstile model and studies efficient algorithms for estimating the \( \alpha \)th frequency moments of data streams

\[
F_\alpha = \sum_{i=1}^{D} A_t[i]^\alpha.
\]

We are particularly interested in the case of \( \alpha \to 1 \), which is very important for estimating Shannon entropy.

\footnote{Extended abstract, submitted on July 6, 2009.}
1.1 Entropy

A very useful (e.g., in Web and networks\cite{12} \cite{23} \cite{37} \cite{27} and neural computations\cite{30}) summary statistic is the Shannon entropy

\[
H = - \sum_{i=1}^{D} A_t[i] \log \frac{A_t[i]}{F(1)}
\]

(3)

Various generalizations of the Shannon entropy have been proposed. The Rényi entropy\cite{31}, denoted by \(H_\alpha\), and the Tsallis entropy\cite{18} \cite{33}, denoted by \(T_\alpha\), are respectively defined as

\[
H_\alpha = \frac{1}{1 - \alpha} \log \left( \sum_{i=1}^{D} A_t[i]^\alpha \right), \quad T_\alpha = \frac{1}{\alpha - 1} \left( 1 - \frac{F(\alpha)}{F(1)} \right).
\]

(4)

As \(\alpha \to 1\), both Rényi entropy and Tsallis entropy converge to Shannon entropy: \(\lim_{\alpha \to 1} H_\alpha = \lim_{\alpha \to 1} T_\alpha = H\). Thus, both Rényi entropy and Tsallis entropy can be computed from the \(\alpha\)th frequency moment; and one can approximate Shannon entropy from either \(H_\alpha\) or \(T_\alpha\) by letting \(\alpha \approx 1\). Several studies\cite{37} \cite{17} \cite{16} used this idea to approximate Shannon entropy, all of which relied on efficient algorithms for estimating the \(\alpha\)th frequency moments (2) near \(\alpha = 1\). In fact, one can numerically verify that the \(\alpha\) values proposed in \cite{17} \cite{16} are extremely close to 1, e.g., \(\Delta = |1 - \alpha| \leq 10^{-4}\).

Therefore, efficient algorithms for estimating \(F(\alpha)\) near \(\alpha = 1\) is critical for estimating Shannon entropy.

1.2 Sample Applications of Shannon Entropy

1.2.1 Real-Time Network Anomaly Detection

Network traffic is a typical example of high-rate data streams. An effective and reliable measurement of network traffic in real-time is crucial for anomaly detection and network diagnosis; and one such measurement metric is Shannon entropy\cite{12} \cite{22} \cite{35} \cite{7} \cite{23} \cite{37}. The Turnstile data stream model (1) is naturally suitable for describing network traffic, especially when the goal is to characterize the statistical distribution of the traffic. In its empirical form, a statistical distribution is described by histograms, \(A_t[i], i = 1\) to \(D\). It is possible that \(D = 2^{64}\) (IPV6) if one is interested in measuring the traffic streams of unique source or destination.

The Distributed Denial of Service (DDoS) attack is a representative example of network anomalies. A DDoS attack attempts to make computers unavailable to intended users, either by forcing users to reset the computers or by exhausting the resources of service-hosting sites. For example, hackers may maliciously saturate the victim machines by sending many external communication requests. DDoS attacks typically target sites such as banks, credit card payment gateways, or military sites.

A DDoS attack changes the statistical distribution of network traffic. Therefore, a common practice to detect an attack is to monitor the network traffic using certain summary statistics. Since Shannon entropy is a well-suited for characterizing a distribution, a popular detection method is to measure the time-history of entropy and alarm anomalies when the entropy becomes abnormal\cite{12} \cite{23}.

Entropy measurements do not have to be “perfect” for detecting attacks. It is however crucial that the algorithm should be computationally efficient at low memory cost, because the traffic data generated by large high-speed networks are enormous and transient (e.g., 1 Gbits/second). Algorithms should be real-time and one-pass, as the traffic data will not be stored\cite{4}. Many algorithms have been proposed for “sampling” the traffic data and estimating entropy over data streams\cite{23} \cite{37} \cite{6} \cite{15} \cite{3} \cite{8} \cite{17} \cite{16}.

1.2.2 Entropy of Query Logs in Web Search

The recent work\cite{27} was devoted to estimating the Shannon entropy of MSN search logs, to help answer some basic problems in Web search, such as, how big is the web?

The search logs can be viewed as data streams, and \cite{27} analyzed several “snapshots” of a sample of MSN search logs. The sample used in \cite{27} contained 10 million \(<\text{Query, URL, IP}>\) triples; each triple corresponded to a click from a particular IP address on a particular URL for a particular query. \cite{27} drew their important conclusions on this (hopefully) representative sample. Alternatively, one could apply data stream algorithms such as CC on the whole history of MSN (or other search engines).
1.2.3 Entropy in Neural Computations

A workshop in NIPS’03 was denoted to entropy estimation, owing to the wide-spread use of Shannon entropy in Neural Computations[30]. (http://www.menem.com/~ilya/pages/NIPS03) For example, one application of entropy is to study the underlying structure of spike trains.

1.3 Previous Algorithms for Estimating Frequency Moments

The problem of approximating $F_\alpha$ has been very heavily studied in theoretical computer science and databases, since the pioneering work of [2], which studied $\alpha = 0, 2$, and $\alpha > 2$. [11, 20, 24] provided improved algorithms for $0 < \alpha \leq 2$. [21] provided algorithms for $\alpha > 2$ to achieve the lower bounds proved by [32, 5, 34]. [14] suggested using even more space to trade for some speedup in the processing time.

Note that the first moment (i.e., the sum), $F_1$, can be computed easily with a simple counter[28, 13, 2]. This important property was recently somewhat captured by the method of Compressed Counting (CC) [25], which was based on the maximally-skewed stable random projections. [25] proved that, in the neighborhood of $\alpha = 1$, the sample complexity is essentially $O\left(\frac{1}{\epsilon}\right)$, which was a large improvement over the well-known $O\left(\frac{1}{\epsilon^2}\right)$ bound[34, 20, 24]. This means the required sample size using CC should be $O\left(\frac{1}{\epsilon}\right)$ in order to ensure that the estimated $\alpha$th frequency moment will be within a $1 \pm \epsilon$ factor of the truth, with high probability.

The sample complexity bound of $O\left(\frac{1}{\epsilon}\right)$ for CC is unsatisfactory, not just for theoretical reasons. From a practical point of view, $1/\epsilon$ can be too large to be practical, especially for entropy estimation. For example, one can numerically verify that the required $\epsilon$ values in [17, 16] for entropy estimation are very small. Very recently, without providing any theoretical complexity bounds, [26] proposed an empirically improved (and quite sophisticated) algorithm for CC. Because the algorithm in [26] is quite complex, its theoretical analysis was difficult.

This study proposes a very simple algorithm, which also allows us to analyze its sample complexity. The complexity is essentially $O\left(\log(1/\Delta) - \log(1/\epsilon)\right)$, when $\Delta = 1 - \alpha \to 0$.

2 The Proposed Algorithm and Main Theoretical Results

We consider the strict-Turnstile model [1]. Conceptually, we multiply the data stream vector $A_t \in \mathbb{R}^{1 \times D}$ by a random projection matrix $R \in \mathbb{R}^{D \times k}$. The resultant vector $X = A_t \times R \in \mathbb{R}^{k \times 1}$ is only of length $k$. More specifically, the entries of the projected vector $X$ are

$$x_j = [A_t \times R]_j = \sum_{i=1}^{D} r_{ij} A_t[i], \quad j = 1, 2, ..., k$$

$r_{ij}$’s are random variables generated by

$$r_{ij} = \frac{\sin(\alpha v_{ij})}{\left|\sin(v_{ij})\right|^{1/\alpha}} \left[\frac{\sin(v_{ij}\Delta)}{w_{ij}}\right]^{\frac{1}{\Delta}}, \quad \Delta = 1 - \alpha > 0,$$

where $v_{ij} \sim \text{uniform}(0, \pi)$ (i.i.d.) and $w_{ij} \sim \exp(1)$ (i.i.d.), an exponential distribution with mean 1.

Of course, in data stream computations, the matrix $R$ is never fully materialized. The standard procedure in data stream computations is to generate entries of $R$ on-demand [20]. In other words, whenever an stream element $a_t = (t_t, I_t)$ arrives, one updates entries of $X$ as

$$x_j \leftarrow x_j + I_t r_{ij}, \quad j = 1, 2, ..., k.$$ 

The proposed algorithm is to take the sample minimum:

$$\hat{F}_\alpha_{\text{min}} = \left[\min \{x_j, j = 1, 2, ..., k\}\right]^\alpha.$$

While this estimator is extremely simple, it has nice theoretical properties.
Theorem 1 As $\Delta = 1 - \alpha \rightarrow 0+$, for any fixed $\epsilon > 0$,
\[
\Pr \left( \hat{F}_{(\alpha),\min} \geq (1 + \epsilon)F(\alpha) \right) \leq \exp \left( k \log \frac{1}{2} \left[ \Delta + \frac{\Delta}{\log(1 + \epsilon)} + \frac{\Delta}{\Delta \log \Delta + \log(1 + \epsilon)} + O(\Delta^2) \right] \right) (7)
\]
Therefore, it suffices to let the sample size
\[
k \geq \log \frac{1}{\delta} - \log \left( \frac{1}{2} + \frac{1}{\Delta \log(1 + \epsilon)} + \frac{1}{2 \Delta \log \Delta + \log(1 + \epsilon)} + O(\Delta) \right)
\]
so that with probability at least $1 - \delta$, $\hat{F}_{(\alpha),\min}$ is within a $1 + \epsilon$ factor of $F(\alpha)$.

The proof is deferred to Section 4.2, which will also demonstrate that the right tail bound (7) can be slightly improved by essentially removing the $\Delta \log \Delta$ term in (7).

To help verify the results in Theorem 1, Figure 1 plots the right tail bounds (7) for $\Delta = 10^{-4}$ ($k = 1, 2, 3$) and $\Delta = 10^{-6}$ ($k = 1$ only), together with the simulated tail probabilities. We can see that the tail probabilities decrease very rapidly. In fact, it is even difficult to simulate the tail probabilities if $k > 3$ or $\Delta < 10^{-6}$.

Theorem 1 indicates that required sample size $k$ can be very small. For example, if we let $\epsilon = 10^{-3}$, $\delta = 10^{-10}$, and $\Delta = 10^{-5}$, then according to (8), the required sample size is merely $k \geq 5.1$.

Note that Theorem 1 is just for the sample complexity. To obtain the space complexity, we must consider an multiplicative factor of $\log \sum_{s=1}^{t} |I_s|$. In addition, we must store $r_{ij}$ with a sufficient accuracy. In Section 3 Lemma 1 shows that $\log r_{ij} = O(\Delta \log \Delta)$, which can be represented using $O(\log 1/\Delta)$ bits. Therefore, the required storage space would be the sample complexity (8) multiplied by a factor of $O \left( \log \sum_{s=1}^{t} |I_s| \right) + O(\log 1/\Delta)$. 

Figure 1: Right tail bound (7) for selected $\Delta$ and $k$, together with the simulated tail probabilities.
Theorem 2 presents the left tail bound.

**Theorem 2** For any $0 < \epsilon < 1$, $\alpha < 1$, and $\Delta = 1 - \alpha$,

$$\Pr \left( \tilde{F}(\alpha)_{\min} \leq (1 - \epsilon) F(\alpha) \right) \leq k \exp \left( -\frac{\Delta \alpha^{1/\Delta - 1}}{(1 - \epsilon)^{1/\Delta}} \right).$$  \tag{9}

The proof is deferred to Section 4.1

The left bound (9) approaches zero extremely fast. For example, when $\Delta = 10^{-6}$ and $\epsilon = 10^{-4}$, $\frac{\Delta \alpha^{1/\Delta - 1}}{(1 - \epsilon)^{1/\Delta}} \approx 10^{37}$; and hence $k$ does not really matter for the left bound. In a sense, the left bound will be used merely for the sanity check and one can determine the sample size mainly from the right bound in Theorem 1.

### 3 Preparation for the Proofs of the Main Results

We start with reviewing maximally-skewed stable distributions, because our formulation somewhat differs from the standard formulation.

#### 3.1 Maximally-Skewed Stable Distribution

The standard procedure for sampling from skewed stable distributions is based on the Chambers-Mallows-Stuck method. To generate a sample from $S(\alpha, \beta = 1, 1)$, i.e., $\alpha$-stable, maximally-skewed ($\beta = 1$), with unit scale, one first generates an exponential random variable with mean 1, $W \sim \exp(1)$, and a uniform random variable $U \sim \text{uniform } \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$, then,

$$Z' = \frac{\sin (\alpha(U + \rho))}{\cos U \cos (\rho \alpha)} \left[ \frac{\cos (U - \alpha(U + \rho))}{W} \right]^{1 - \frac{1}{\alpha}} \sim S(\alpha, \beta = 1, 1),$$

where $\rho = \frac{\pi}{2}$ when $\alpha < 1$ and $\rho = \frac{\pi}{2} \frac{2 - \alpha}{\alpha}$ when $\alpha > 1$.

For convenience, we will use

$$Z = Z' \cos^{1/\alpha} (\rho \alpha) \sim S(\alpha, \beta = 1, \cos (\rho \alpha)).$$

In this study, we will only consider $\alpha = 1 - \Delta < 1$, i.e., $\rho = \frac{\pi}{2}$. After simplification, we obtain

$$Z = \frac{\sin (\alpha V)}{\sin V} \left[ \frac{\sin (V \Delta)}{W} \right]^{\frac{\Delta}{\alpha}},$$

where $V = \frac{\pi}{2} + U \sim \text{uniform } (0, \pi)$. This explains (5).

Lemma 1 shows $\log Z = O \left( \frac{\Delta \log 1}{\Delta} \right)$, which can be accurately represented using $O \left( \log 1/\Delta \right)$ bits. The proof is omitted since it is straightforward.

**Lemma 1** For any given $V \neq 0$, and $W \neq 0$, as $\Delta \to 0$,

$$Z = 1 + O \left( \frac{\Delta \log 1}{\Delta} \right), \quad \text{i.e.,} \quad \log Z = O \left( \frac{\Delta \log 1}{\Delta} \right).$$

#### 3.2 Random Projections and the Sample Minimum Estimator

Let $X = A_t \times R$, where entries of $R$ are i.i.d. samples of $S(\alpha, \beta = 1, \cos \left( \frac{\pi}{2} \alpha \right))$. Then by properties of stable distributions, entries of $X$ are

$$x_j = [A_t \times R]_j = \sum_{i=1}^{D} r_{i,j} A_t[i] \sim S(\alpha, \beta = 1, \cos \left( \frac{\pi}{2} \alpha \right) F(\alpha)),

where $F(\alpha) = \sum_{i=1}^{D} A_t[i]^\alpha$ as defined in (4).

The proposed estimator of $F(\alpha)$ is based on the sample minimum:

$$\tilde{F}(\alpha)_{\min} = \left[ \min \{x_j, j = 1, 2, ..., k \} \right]^\alpha$$
3.3 Density Function

Lemma 2 Suppose a random variable $Z \sim S \left( \alpha < 1, \beta = 1, \cos \left( \frac{\pi}{2} \alpha \right) \right)$, then the cumulative density function is

$$F_Z(t) = \Pr(Z \leq t) = \frac{1}{\pi} \int_{0}^{\pi} \exp \left( -\frac{[\sin (\alpha \theta)]^{\alpha/\Delta}}{t^{\alpha/\Delta} [\sin \theta]^{1/\Delta}} \sin (\theta \Delta) \right) d\theta, \quad \Delta = 1 - \alpha.$$  

Proof:

$$\Pr(Z \geq t) = \Pr \left( \frac{\sin (\alpha V)}{[\sin V]^{1/\alpha}} \left[ \frac{\sin (V \Delta)}{W} \right]^{\Delta} \geq t \right)$$

$$= \Pr \left( W \leq \frac{[\sin (\alpha V)]^{\alpha/\Delta}}{t^{\alpha/\Delta} [\sin V]^{1/\Delta}} \sin (V \Delta) \right)$$

$$= E \left( \Pr \left( W \leq \frac{[\sin (\alpha V)]^{\alpha/\Delta}}{t^{\alpha/\Delta} [\sin V]^{1/\Delta}} \sin (V \Delta) \mid V \right) \right)$$

$$= 1 - E \left( \exp \left( -\frac{[\sin (\alpha V)]^{\alpha/\Delta}}{t^{\alpha/\Delta} [\sin V]^{1/\Delta}} \sin (V \Delta) \right) \right)$$

$$= 1 - \frac{1}{\pi} \int_{0}^{\pi} \exp \left( -\frac{[\sin (\alpha \theta)]^{\alpha/\Delta}}{t^{\alpha/\Delta} [\sin \theta]^{1/\Delta}} \sin (\theta \Delta) \right) d\theta \square$$

For $\theta \in (0, \pi)$, let

$$g(\theta; \Delta) = \frac{[\sin (\alpha \theta)]^{\alpha/\Delta}}{[\sin \theta]^{1/\Delta}} \sin (\theta \Delta),$$

Lemma 3 includes some properties of $g(\theta; \Delta)$, which will be useful for proving our main results in Theorem 1 and Theorem 2.

Lemma 3 Assume $\Delta = 1 - \alpha < 0.5$, then $g(\theta; \Delta)$ is monotonically increasing in $(0, \pi)$, with

$$\lim_{\theta \to 0^+} g(\theta; \Delta) = \Delta^{1/\Delta - 1}.$$  

Moreover, $g(\theta; \Delta)$ is a convex function of $\theta$.

4 Proofs of Theorem 1 and Theorem 2

We first prove the left bound in Theorem 2.

4.1 Proof of Theorem 2

Recall the sample minimum estimator is

$$\hat{F}_{(\alpha), \text{min}} = \min \{ x_j, j = 1, 2, \ldots, k \}^{\alpha}, \quad x_j \sim S \left( \alpha < 1, \beta = 1, \cos \left( \frac{\pi}{2} \alpha \right) \right).$$

Using the density function provided in Lemma 2 and properties of $g(\theta; \Delta) = \frac{[\sin (\alpha \theta)]^{\alpha/\Delta}}{[\sin \theta]^{1/\Delta}} \sin (\theta \Delta)$ proved in
Lemma 3, we obtain

\[
\Pr(\hat{F}(\alpha), \min \leq (1 - \epsilon)F(\alpha)) \\
\leq k \times \Pr(x_1^\alpha / F(\alpha) \leq (1 - \epsilon)) \\
= k \frac{1}{\pi} \int_0^\pi \exp \left( -\frac{[\sin(\alpha\theta)]^{\alpha/\Delta}}{(1 - \epsilon)^{1/\Delta} \sin(\theta\Delta)} \right) d\theta \\
< k \frac{1}{\pi} \int_0^\pi \exp \left( -\lim_{\theta \to 0^+} g(\theta; \Delta) \right) d\theta \\
= k \frac{1}{\pi} \int_0^\pi \exp \left( -\frac{\Delta^\alpha (1 - 1/\Delta)}{(1 - \epsilon)^{1/\Delta}} \right) d\theta \\
= k \exp \left( -\frac{\Delta^\alpha (1 - 1/\Delta)}{(1 - \epsilon)^{1/\Delta}} \right).
\]

### 4.2 Proof of Theorem 1

Using the density function provided in Lemma 2, we can obtain

\[
\Pr(\hat{F}(\alpha), \min \geq (1 + \epsilon)F(\alpha)) \\
= \Pr(\hat{F}(\alpha), \min / F(\alpha) \geq (1 + \epsilon)) \\
= \prod_{j=1}^k \Pr(x_j / F(\alpha)^{1/\alpha} \geq (1 + \epsilon)^{1/\alpha}) \\
= \left[ 1 - \frac{1}{\pi} \int_0^\pi \exp \left( -\frac{[\sin(\alpha\theta)]^{\alpha/\Delta}}{(1 + \epsilon)^{1/\Delta} \sin(\theta\Delta)} \right) d\theta \right]^k \\
= \exp \left( k \log \left[ 1 - \frac{1}{\pi} \int_0^\pi \exp \left( -\frac{g(\theta; \Delta)}{(1 + \epsilon)^{1/\Delta}} \right) d\theta \right] \right)
\]

We proceed the proof as follows:

1. Using the fact that \(e^{-x} \geq \max\{0, 1 - x\}\), we obtain

\[
\Pr(\hat{F}(\alpha), \min \geq (1 + \epsilon)F(\alpha)) \leq \exp \left( k \log \left[ 1 - \frac{1}{\pi} \int_0^{\theta_0} 1 - \frac{g(\theta; \Delta)}{(1 + \epsilon)^{1/\Delta}} d\theta \right] \right)
\]

where \(\theta_0\) is the solution to

\[
1 = \frac{g(\theta; \Delta)}{(1 + \epsilon)^{1/\Delta}}
\]

2. We prove a more general result to solve for

\[
\Delta^\gamma = \frac{g(\theta; \Delta)}{(1 + \epsilon)^{1/\Delta}}.
\]

We show the asymptotic expression for \(\theta\) is, as \(\Delta \to 0\),

\[
\theta = \pi - \frac{\Delta}{\Delta + \gamma \Delta \log \Delta + \log(1 + \epsilon) + \Delta \log \left( \frac{1}{\gamma \Delta \log(1 + \epsilon) + 1} \right) + O(\Delta^2)}
\]

(12)
3. We approximate the integral \( \int_{0}^{\theta_0} 1 - \frac{g(\theta; \Delta)}{1 + \epsilon} d\theta \) by the trapezoid rule. Because \( g(\theta; \Delta) \) is a convex function of \( \theta \) as proved in Lemma 3, we know this approximation still leads to an upper bound we are after.

4. To apply the trapezoid rule, it turns out that it suffices to use only one interior point, \( \theta = \theta_1 \), in addition to the two end points, \( \theta = 0 = \theta_\infty \) and \( \theta = \theta_0, \theta_1 \) is the solution to \( \Delta = \frac{g(\theta_1; \Delta)}{1 + \epsilon} \).

5. We can slightly improve the bound by using more points when applying the trapezoid rule, for example, \( \theta = \frac{\theta_1}{2} \), in addition to \( \theta_0, \theta_1, \text{ and } \theta_\infty \).

We defer the proof of (12) to Appendix B. Assuming (12) holds, we have

\[
\Pr \left( \hat{F}_{(\alpha), \min} \geq (1 + \epsilon) F_{(\alpha)} \right) = \exp \left( k \log \left[ 1 - \frac{1}{\pi} \int_{0}^{\pi} \exp \left( - \frac{g(\theta; \Delta)}{1 + \epsilon} \right) d\theta \right] \right) \\
\leq \exp \left( k \log \left[ 1 - \frac{1}{\pi} \int_{0}^{\theta_0} 1 - \frac{g(\theta; \Delta)}{1 + \epsilon} d\theta \right] \right) \\
\leq \exp \left( k \log \left[ 1 - \frac{1}{\pi} \left[ \theta_1 - \frac{1}{2} \theta_1 \Delta + \frac{1}{2} (1 - \Delta)(\theta_0 - \theta_1) \right] \right] \right) \\
= \exp \left( k \log \left[ 1 - \frac{1}{2\pi} \left[ \theta_0 + \theta_1 - \Delta \theta_0 \right] \right] \right)
\]

\[
2 - \frac{1}{\pi} [\theta_0 + \theta_1 - \Delta \theta_0] = \frac{1}{1 + \log \Delta + \frac{1}{\Delta} \log(1 + \epsilon) + \log \left( \frac{1}{\Delta \log \Delta \log(1 + \epsilon)} + 1 \right) + O(\Delta)} \\
+ \frac{1}{1 + \frac{1}{\Delta} \log(1 + \epsilon) + \log \left( \frac{1}{\log(1 + \epsilon)} + 1 \right) + O(\Delta)} \\
+ \frac{\Delta}{1 + \frac{1}{\Delta} \log(1 + \epsilon) + \log \left( \frac{1}{\log(1 + \epsilon)} + 1 \right) + O(\Delta)} \\
\]

Therefore, if we require

\[
\Pr \left( \hat{F}_{(\alpha), \min} \geq (1 + \epsilon) F_{(\alpha)} \right) \leq \exp \left( k \log \left[ \Delta + \frac{\Delta}{\log(1 + \epsilon)} + \frac{\Delta}{\Delta \log \Delta + \log(1 + \epsilon)} + O(\Delta^2) \right] \right) \\
\leq \delta,
\]

we obtain our main result, the sample complexity bound,

\[
k \geq \frac{\log \frac{1}{\delta}}{\log \frac{1}{\Delta} - \log \left( \frac{1}{2} + \frac{1}{2 \log(1 + \epsilon)} + \frac{1}{2 \Delta \log \Delta + \log(1 + \epsilon)} + O(\Delta) \right)}.
\]
It turns out, the term $\Delta \log \Delta$ can be almost removed, by using one additional interior point when applying the trapezoid rule. Note that $|\Delta \log \Delta|$ is almost as small as $\Delta$, but we do not want to simply ignore this term.

Using two interior points, $\theta_t$ and $\theta_i$, where $0 < t < 1$, we obtain

$$\Pr \left( \hat{F}(\alpha),_{\min} \geq (1 + \epsilon)F(\alpha) \right)$$

$$= \exp \left( \frac{k \log \int_0^{\theta_t} \exp \left( - \frac{[\sin (\alpha \theta)]^{\alpha/\Delta}}{(1 + \epsilon)^{1/\Delta} \sin (\theta \Delta)} \sin (\theta \Delta) \right) \, d\theta \right)$$

$$\leq \exp \left( \frac{k \log \left[ 1 - \frac{1}{\pi} \int_0^{\theta_0} \frac{[\sin (\alpha \theta)]^{\alpha/\Delta}}{(1 + \epsilon)^{1/\Delta} \sin (\theta \Delta)} \sin (\theta \Delta) \, d\theta \right]}{1} \right)$$

$$= \exp \left( \frac{k \log \left[ 1 - \frac{1}{2\pi} \left[ \theta_0 + \theta_t - \Delta \theta_t - \Delta^t \theta_0 + \Delta^t \theta_1 \right] \right]}{1} \right)$$

$$= \frac{1}{1 + t \log \Delta + \frac{1}{\Delta} \log (1 + \epsilon) + \log \left( \frac{1}{\Delta \log \Delta + \log (1 + \epsilon)} + 1 \right)} + O(\Delta)$$

$$+ \frac{1}{1 + \frac{1}{\Delta} \log (1 + \epsilon) + \log \left( \frac{1}{\log(1 + \epsilon)} + 1 \right)} + O(\Delta)$$

$$+ \frac{t \log \Delta + \frac{1}{\Delta} \log (1 + \epsilon) + \log \left( \frac{1}{\Delta \log \Delta + \log (1 + \epsilon)} + 1 \right)}{1 + t \log \Delta + \frac{1}{\Delta} \log (1 + \epsilon) + \log \left( \frac{1}{\log(1 + \epsilon)} + 1 \right)} + O(\Delta)$$

$$+ \frac{\Delta^t \frac{1}{\Delta} \log (1 + \epsilon) + \log \left( \frac{1}{\log(1 + \epsilon)} + 1 \right)}{1 + \frac{1}{\Delta} \log (1 + \epsilon) + \log \left( \frac{1}{\log(1 + \epsilon)} + 1 \right)} + O(\Delta)$$

$$- \frac{\Delta^t \frac{1}{\Delta} \log (1 + \epsilon) + \log \left( \frac{1}{\Delta \log \Delta + \log (1 + \epsilon)} + 1 \right)}{1 + \frac{1}{\Delta} \log (1 + \epsilon) + \log \left( \frac{1}{\log(1 + \epsilon)} + 1 \right)} + O(\Delta)$$

$$= \Delta + \frac{\Delta}{\log (1 + \epsilon)} + \frac{\Delta}{t \Delta \log \Delta + \log (1 + \epsilon)} + O(\Delta^2)$$

Note that, if we choose $t$ to be too small (too close to 0), then $(-\Delta^t \theta_0 + \Delta^t \theta_1)$ will be larger than $O(\Delta^2)$ and can not be ignored. Therefore, although we can minimize the impact of the term $\Delta \log \Delta$ to a very large extent, it can not be entirely removed, theoretically speaking.

5 Conclusion

Real-world data are often dynamic and can be modeled as data streams. Measuring summary statistics of data streams such as the Shannon entropy has become an important task in many applications, for example, detecting anomaly events in large-scale networks. One line of active research is to approximate the Shannon entropy using the $\alpha$th frequency moments of the stream with $\alpha$ extremely close to 1.

Efficiently approximating the $\alpha$th frequency moments of data streams has been very heavily studied in theoretical computer science and databases. When $0 < \alpha \leq 2$, it is well-known that efficient $O(1/\epsilon^2)$-space algorithms
exist, for example, *symmetric stable random projections*\(^\text{20, 24}\), which however are impractical for estimating Shannon entropy using \(\alpha\) extremely close to 1. Recently, \(^\text{25}\) provided an algorithm to achieve the \(O(1/\varepsilon)\) bound in the neighborhood of \(\alpha = 1\), based on the idea of *maximally-skewed stable random projections* (also called *Compressed Counting (CC)*). The \(O(1/\varepsilon)\) bound, although a very large improvement over the previous \(O(1/\varepsilon^2)\) bound, is still impractical.

This study proposes a new algorithm for CC based on the *sample minimum*, which is simple, practical, and still has very nice theoretical properties. Using this algorithm, we have proved that the sample complexity is essentially \(O\left(\frac{1}{\log 1/(1-\alpha) - \log 1/\varepsilon}\right)\) as \(\alpha \to 1^-\). This is a very large improvement over the previous \(O(1/\varepsilon)\) bound and may impact the practice.

### A Proof of Lemma 3

For \(\theta \in (0, \pi)\), let

\[
g(\theta; \Delta) = \left[\frac{\sin(\alpha \theta)}{\sin(\theta)}\right]^{\alpha/\Delta} \sin(\theta \Delta).
\]

It is easy to show that, as \(\theta \to 0^+\),

\[
\lim_{\theta \to 0^+} g(\theta, \Delta) = \lim_{\theta \to 0^+} \left[\frac{\sin(\alpha \theta)}{\sin(\theta)}\right]^{\alpha/\Delta} \sin(\theta \Delta)
= \lim_{\theta \to 0^+} \left(\frac{\sin(\alpha \theta)}{\sin(\theta)}\right)^{1/\Delta} \frac{\sin(\theta \Delta)}{\sin(\alpha \theta)}
= \alpha^{1/\Delta} \frac{\Delta}{\alpha} = \Delta \alpha^{1/\Delta - 1}.
\]

The proof of the monotonicity of \(g(\theta, \Delta)\) is omitted, because it is can be inferred from the proof of the convexity.

To show \(g(\theta, \Delta)\) is a convex function \(\theta\), it suffices to show it is log-convex. Since

\[
g(\theta, \Delta) = \sin(\theta \Delta) \left[\frac{\sin(\alpha \theta)}{\sin(\theta)}\right]^{\alpha/\Delta} = \frac{\sin(\theta \Delta)}{\sin(\alpha \theta)} \left[\left(\frac{\sin(\alpha \theta)}{\sin(\theta)}\right)^{1/\Delta}\right]^{\alpha/\Delta}
\]

it suffices to show that both \(\frac{\sin(\alpha \theta)}{\sin(\theta)\Delta}\) and \(\left[\frac{\sin(\alpha \theta)}{\sin(\theta)}\right]^{1/\Delta}\) are log-convex.

\[
\frac{\partial^2 \log \sin(\theta \Delta) - \log \sin(\alpha \theta)}{\partial \theta^2} = \frac{\cos(\theta \Delta)}{\sin(\theta \Delta)^2} \Delta - \frac{\cos(\alpha \theta)}{\sin(\alpha \theta)\Delta}
\]

\[
\frac{\partial^2 \log \sin(\theta \Delta) - \log \sin(\alpha \theta)}{\partial \theta^2} = -\frac{\Delta^2}{\sin^2(\theta \Delta)} + \frac{\alpha^2}{\sin^2(\alpha \theta)} = \left(\frac{\alpha}{\sin(\alpha \theta)} - \frac{\Delta}{\sin(\theta \Delta)}\right) \left(\frac{\alpha}{\sin(\alpha \theta)} + \frac{\Delta}{\sin(\theta \Delta)}\right)
\]

\[
\frac{\partial \alpha \sin(\theta \Delta) - \Delta \sin(\alpha \theta)}{\partial \theta} = \Delta \alpha (\cos(\theta \Delta) - \cos(\alpha \theta)) \geq 0 \quad \text{(because } \Delta < 0.5\text{)}
\]

Therefore, \(\alpha \sin(\theta \Delta) - \Delta \sin(\alpha \theta) \geq 0\) and \(\frac{\sin(\alpha \theta)}{\sin(\theta)\Delta}\) is convex.

\[
\frac{\partial \log \sin(\alpha \theta) - \log \sin(\theta)}{\partial \theta} = \frac{\cos(\theta \Delta)}{\sin(\theta \Delta)} \alpha - \frac{\cos(\alpha \theta)}{\sin(\alpha \theta)}
\]
\[
\frac{\partial^2 \log \sin(\alpha \theta) - \log(\sin(\theta))}{\partial \theta^2} = -\frac{\alpha^2}{\sin^2(\alpha \theta)} + \frac{1}{\sin^2(\theta)} = \left(\frac{1}{\sin(\theta)} - \frac{\alpha}{\sin(\alpha \theta)}\right) \left(\frac{1}{\sin(\theta)} + \frac{\alpha}{\sin(\alpha \theta)}\right)
\]

\[
\frac{\partial \sin(\alpha \theta) - \alpha \sin(\theta)}{\partial \theta} = \alpha (\cos(\alpha \theta) - \cos(\theta)) \geq 0 \quad \text{(because } \alpha = 1 - \Delta > 0.5)\]

Therefore, we have proved the convexity of \(g(\theta; \Delta)\).

**B Proof of Equation (12)**

\(\theta, \gamma\) is the solution to

\[
\Delta^\gamma = \frac{[\sin(\alpha \theta)]^{\alpha/\Delta}}{(1 + \epsilon)^{1/\Delta} \sin(\theta \Delta)} \sin(\theta \Delta),
\]

Equivalently,

\[
\gamma \log \Delta + \frac{1}{\Delta} \log(1 + \epsilon) + \frac{1}{\Delta} \log \sin \theta = \frac{1 - \Delta}{\Delta} \log \sin(\theta - \Delta \theta) + \log \sin(\Delta \theta)
\]

\[\Leftarrow\]

\[
\gamma \log \Delta + \frac{1}{\Delta} \log(1 + \epsilon) + \log \left(\frac{\sin(\theta - \Delta \theta)}{\sin(\Delta \theta)}\right) = \frac{1}{\Delta} \log \left(\frac{\cos(\theta - \Delta \theta)}{\sin(\theta)}\right).
\]

We apply Taylor expansions,

\[
\gamma \log \Delta + \frac{1}{\Delta} \log(1 + \epsilon) + \log \left(\frac{-\sin \theta \theta \cos \theta + \Delta \theta \sin \theta}{3 \cos \theta} + 1 + ...\right) + \log(-\cos \theta) = -\frac{\Delta \theta^2}{2} - \frac{\theta \cos \theta}{\sin \theta} - \frac{\Delta \theta^2 \cos^2 \theta}{2 \sin^2 \theta} + ...
\]

to obtain

\[
\gamma \log \Delta + \frac{1}{\Delta} \log(1 + \epsilon) + \log \left(1 + O(\Delta) + 1\right) + O(\Delta^2) = \frac{-\theta \cos \theta}{\sin \theta} + O(\Delta)
\]

where we have replaced \(\log(-\cos \theta)\) with \(O(\Delta^2)\) (as \(\Delta \to 0\)). This fact can be later verified.

Let \(T = -\frac{\theta \cos \theta}{\sin \theta}\), \(C = \gamma \log \Delta + \frac{1}{\Delta} \log(1 + \epsilon)\). This requires us to solve a fixed point equation:

\[
T = C + \log \left(1 + O(\Delta) + 1\right) + O(\Delta).
\]

We resort to an iterative method. Starting with \(T^{(0)} = 1\),

\[
T^{(1)} = C + \log \left(1 + O(\Delta) + 1\right) + O(\Delta) = C - \log(\Delta) + O(\Delta).
\]

\[
T^{(2)} = C + \log \left(1 + O(\Delta) + 1\right) + O(\Delta) = C + \log \left(\frac{1}{(\gamma - 1) \Delta \log(\Delta + O(\Delta)) + (1 + \epsilon) + O(\Delta^2)}\right) + O(\Delta) + 1 + O(\Delta)
\]

\[
= C + \log \left(\frac{1}{(\gamma - 1) \Delta \log(\Delta + O(\Delta) + (1 + \epsilon) + O(\Delta^2)) + O(\Delta) + 1} + O(\Delta) + 1\right) + O(\Delta)
\]

\[
= C + \log \left(\frac{1}{(\gamma - 1) \Delta \log(\Delta + O(\Delta) + (1 + \epsilon) + O(\Delta^2)) + O(\Delta)} + O(\Delta) + 1\right) + O(\Delta)
\]

\[\Leftarrow\]

\frac{\partial \sin(\theta \Delta)}{\partial \theta} - \alpha \sin(\theta \Delta) \frac{\partial \theta}{\partial \theta} \geq 0 \quad \text{(because } \alpha = 1 - \Delta > 0.5)\]

Therefore, we have proved the convexity of \(g(\theta; \Delta)\).
\[ T^{(3)} = C + \log \left( \frac{1}{\Delta} \left( C + \log \frac{1 + (\gamma - 1) \Delta \log \Delta + \log(1 + \epsilon) + O(\Delta^2)}{\Delta + \log(1 + \epsilon) + O(\Delta^2)} \right) + O(\Delta) \right) + O(\Delta) \]

\[ = C + \log \left( \frac{1}{\gamma \Delta \log \Delta + \log(1 + \epsilon) + O(\Delta)} \right) + O(\Delta) + 1 \] + O(\Delta)

\[ = C + \log \left( \frac{1 + \gamma \Delta \log \Delta + \log(1 + \epsilon) + O(\Delta)}{\gamma \Delta \log \Delta + \log(1 + \epsilon) + O(\Delta)} \right) + O(\Delta) \]

At this point, we have reached an equilibrium. Therefore, we know

\[ T = \gamma \log \Delta + \frac{1}{\Delta} \log(1 + \epsilon) + \log \left( \frac{1 + \gamma \Delta \log \Delta + \log(1 + \epsilon)}{\gamma \Delta \log \Delta + \log(1 + \epsilon)} \right) + O(\Delta) \]

Note that

\[ T = -\frac{\theta \cos \theta}{\sin \theta} = \frac{\theta \cos (\pi - \theta)}{\sin (\pi - \theta)} = \theta \left( \frac{1}{\pi - \theta} - \frac{\pi - \theta}{3} + O(\pi - \theta)^3 \right) = \frac{\theta}{\pi - \theta} + O(\pi - \theta) \]

Thus, assuming \( O(\pi - \theta_r) = O(\Delta) \) (which can be verified), we obtain

\[ \theta_\gamma = \pi \frac{\gamma \Delta \log \Delta + \frac{1}{\Delta} \log(1 + \epsilon) + \log \left( \frac{1}{\gamma \Delta \log \Delta + \log(1 + \epsilon) + 1} \right)}{1 + \gamma \log \Delta + \frac{1}{\Delta} \log(1 + \epsilon) + \log \left( \frac{1}{\gamma \Delta \log \Delta + \log(1 + \epsilon) + 1} \right) + O(\Delta^2)} + O(\Delta) \]

\[ = \pi \frac{\gamma \Delta \log \Delta + \log(1 + \epsilon) + \Delta \log \left( \frac{1}{\gamma \Delta \log \Delta + \log(1 + \epsilon) + 1} \right) + O(\Delta^2)}{\gamma \Delta \log \Delta + \log(1 + \epsilon) + O(\Delta^2)} + O(\Delta^2) \]

To complete the proof, we must verify \( O(\pi - \theta_r) = O(\Delta) \) and \( \log(-\cos(\theta_r)) = O(\Delta^2) \). Indeed,

\[ O(\pi - \theta_r) = \pi \frac{\Delta}{\Delta + \gamma \Delta \log \Delta + \log(1 + \epsilon) + O(\Delta^2)} = O(\Delta) \]

\[ \log(-\cos(\theta_r)) = \log(\cos(\pi - \theta_r)) = \log(\cos(O(\Delta))) = \log \left( 1 - \frac{O(\Delta^2)}{2} \right) = O(\Delta^2) \]

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