FROBENIUS-POINCARÉ FUNCTION AND HILBERT-KUNZ MULTIPLICITY

ALAPAN MUKHOPADHYAY

ABSTRACT. We generalize the notion of Hilbert-Kunz multiplicity of a graded triple \((M, R, I)\) in characteristic \(p > 0\) by proving that for any complex number \(y\), the limit
\[
\lim_{n \to \infty} \left( \frac{1}{p^n} \right)^{\dim(M)} \sum_{j=-\infty}^{\infty} \lambda \left( \frac{M}{I^{(p^n)}M} \right)_j e^{-iyj/p^n}
\]
exists. We prove that the limiting function in the complex variable \(y\) is entire and name this function the Frobenius-Poincaré function. We establish various properties of Frobenius-Poincaré functions including its relation with the tight closure of the defining ideal \(I\); and relate the study Frobenius-Poincaré functions to the behaviour of graded Betti numbers of \(R\) as \(n\) varies. Our description of Frobenius-Poincaré functions in dimension one and two and other examples raises questions on the structure of Frobenius-Poincaré functions in general.

1. Introduction

In this article, we introduce the Frobenius-Poincaré function of a graded pair \((R, I)\), where \(I\) is a finite co-length homogeneous ideal in the standard graded domain \(R\) over a perfect field of positive characteristic \(p\). This function is holomorphic everywhere on the complex plane and is roughly the limit of the Hilbert series of the graded \(R\)-modules \(R/I^{(p^n)}\) as \(n\) goes to infinity. The Frobenius-Poincaré function encodes the information of the Hilbert-Kunz multiplicity of the pair \((R, I)\) along with other asymptotic invariants of \((R, I)\).

To be precise, fix a pair \((R, I)\) as above. For each positive integer \(n\), consider the \(R\)-module \(R/I^{(p^n)}\), the collection of \(p^n\)-th roots of elements of \(R\) in a fixed algebraic closure of the fraction field of \(R\). There is a natural \(1/p^n\mathbb{Z}\)-grading on \(R/I^{(p^n)}\). So one can consider the Hilbert series of \(R/I^{(p^n)}\) by allowing for rational powers of the variable \(t\), namely
\[
\sum_{\nu \in 1/p^n\mathbb{Z}} \lambda((R/I^{(p^n)})_{\nu}) t^\nu.
\]

To study these Hilbert series as holomorphic functions on the complex plane, a natural approach is to replace \(t\) by \(e^{-iy}\), which facilitates taking \(p^n\)-th roots as holomorphic functions. The process described above gives a sequence \((G_n)_n\) of holomorphic functions where
\[
G_n(y) = \sum_{\nu \in 1/p^n\mathbb{Z}} \lambda((R^{1/p^n})_{\nu}) e^{-iy\nu}.
\]

Our main result, Theorem 3.1, guarantees that the sequence of functions
\[
\frac{G_n(y)}{(p^\dim(R))n}
\]
converges, as \(n\) goes to infinity, to a function \(F(y)\) which is holomorphic everywhere on the complex plane. Furthermore, this convergence is uniform on every compact subset. We call this function \(F(y)\) the Frobenius-Poincaré function associated to the pair \((R, I)\).

The Frobenius-Poincaré function can be viewed as a natural refinement of the Hilbert-Kunz multiplicity (see Definition 2.1). Indeed, for the pair \((R, I)\), the Hilbert-Kunz multiplicity is the value of the Frobenius-Poincaré function at zero. In fact, we provide an explicit formula for the coefficients of the power series expansion of
the Frobenius-Poincaré entire function around zero, from which it is apparent that each of the coefficients of this power series is an invariant generalizing the Hilbert-Kunz multiplicity (see Proposition 4.1). Even when \( R \) is a polynomial ring, there are examples of ideals with the same Hilbert-Kunz multiplicity but different Frobenius-Poincaré functions—see Example 5.10. Like the Hilbert-Kunz multiplicity itself, the Frobenius-Poincaré function of \((R, I)\) depends only on the tight closure of \( I \) in the ring \( R \), as we prove in Theorem 4.7.

The information carried by the Frobenius-Poincaré function can also be understood in terms of homological data associated to the pair \((R, I)\). For example, when \( R \) is a polynomial ring (or more generally, when \( R/I \) has finite projective dimension), we prove that the Frobenius-Poincaré function has the form

\[
Q(e^{-iy}) \frac{(iy)^d}{(iy)^{dim R}}
\]

where \( Q \) is a polynomial whose coefficients are explicitly determined by the graded Betti numbers of \( R/I \); see Proposition 5.9. More generally, for an arbitrary graded pair \((R, I)\), the Frobenius-Poincaré function of \((R, I)\) can be described in terms of the sequence of graded Betti numbers of \( R/I \):

**Theorem A:** Let \( S \to R \) be a graded Noether normalization. Set \( B^S(j, n) = \sum_{\alpha=0}^{\infty} (-1)^\alpha \lambda(Tor^S_\alpha(R/I[p^n], k)_j) \). Then the limiting function

\[
B^S(R, I)(y) = \lim_{n \to \infty} \sum_{j \in \mathbb{N}} B^S(j, n)e^{-i\alpha j/p^n}
\]

is entire. Furthermore, \( B^S(R, I)(y) \) is the Frobenius-Poincaré function of \((R, I)\) – see Theorem 5.1 and Remark 5.4.

In a slightly different direction, we show that when \( R \) is Cohen-Macaulay, the Frobenius-Poincaré entire function is a limit of a sequence of entire functions described in terms of Koszul homologies with respect to a homogeneous system of parameters for \( R \), or alternatively, Serre’s intersection numbers, suitably interpreted; see Theorem 5.11.

The Frobenius-Poincaré function of \((R, I)\) turns out to be the Fourier transform of the Hilbert-Kunz density function of \((R, I)\) introduced by Trivedi in [Tri18], as we show in Proposition 4.9. Using Fourier transform, **Theorem A** allows us to describe the higher order weak derivatives of the Hilbert-Kunz density function in terms of the sequence graded Betti numbers of \( \frac{R}{I[p^n]} \) — see Remark 5.5. Such a description is not apparent in the existing theory of Hilbert-Kunz density functions. In fact, **Theorem A** relates the question on the order of smoothness of the Hilbert-Kunz density function raised in [Tri21] to the question asking whether \( B^S(R, I)(y) \) in **Theorem A** is bounded on the real line – see Question 5.6. Although our work on Frobenius-Poincaré functions is inspired by Trivedi’s remark that considering Fourier transforms of density functions might be useful (see [Tri18, page 3]), our proof of the existence and holomorphicity of the Frobenius-Poincaré function (Theorem 3.1) is independent of [Tri18]. When \( R \) has dimension at least two and \( R \) is strongly \( F \)-regular at each point on the punctured spectrum of \( R \), the Hilbert-Kunz density function and hence the Frobenius-Poincaré function of \((R, I)\) captures the information of \( F \)-threshold of \( I \) – see Theorem 4.9 of [TW21]. Recently Trivedi has used Hilbert-Kunz density functions to partially settle two conjectures on Hilbert-Kunz multiplicities of quadric hypersurfaces posed by Yoshida and Watanabe-Yoshida—see Theorem A and Theorem B in [Tri21].

We speculate that the entire functions that are Frobenius-Poincaré functions should have a special structure reflecting that each of these is determined by the data of a finitely generated module. Any such special structure will shed more light not only on the theory of Hilbert-Kunz multiplicities but also on the behaviour of graded Betti numbers of \( \frac{R}{I[p^n]} \) as \( n \) changes. We ask whether Frobenius-Poincaré functions always have a form generalizing the expression (1) above; see Question 5.13. Question 5.13 is answered for one dimensional rings in Proposition 4.6. When \( R \) is two dimensional, Question 5.13 is answered in Theorem 6.1, where we
show that the Frobenius-Poincaré function is described by the Harder-Narasimhan filtration on a sufficiently high Frobenius pullback of the syzygy bundle of $I$ on the curve $\text{Proj}(R)$ following [Tri05] and [Bre07]. The necessary background materials on vector bundles on curves and other topics are reviewed in Section 2. Also when the ideal $I$ is generated by a homogeneous system of parameters, our computation in Proposition 4.5 answers Question 5.13 positively.

We develop the theory of Frobenius-Poincaré functions more generally for triples $(M, R, I)$, where $M$ is a finitely generated $\mathbb{Z}$-graded $R$ module; see Definition 3.2. We show that the Frobenius-Poincaré function is additive on short exact sequences in Proposition 4.4. In addition to generalizing classical additivity formulas for Hilbert series and multiplicity, Proposition 4.4 allows us to compute the Frobenius-Poincaré function of a graded ring with respect to an ideal generated by homogeneous system of parameters; see Proposition 4.5.

**Notation and Convention 1.1.** In this article, $k$ stands for a field. By a finitely generated $\mathbb{N}$-graded $k$-algebra, we mean an $\mathbb{N}$-graded commutative ring whose degree zero piece is $k$ and which is finitely generated over $k$.

For any ring $S$ containing $\mathbb{F}_p$, the Frobenius or $p$-th power endomorphism of $S$ is denoted by $F_S$. The symbol $F^n_S$ will denote the $e$-times iteration of $F_S$. We set $S^p = F^e(S) \subseteq S$. For an ideal $J \subseteq S$, $JS^p$ is the image of $J$ in $S^p$ under the $p^e$-th power map. The ideal generated by $p^e$-th power of elements of $J$ in $S$ is denoted by $J[p^e]$.

For an $S$-module $N$, we denote the Krull dimension of $N$ by $\dim_S(N)$ or $\dim(N)$ when the underlying ring $S$ is clear from the context. When $N$ has finite length, $\lambda_S(N)$ denotes the length of the $S$-module $N$. When $S = k$, simply $\lambda(N)$ will be used to denote the length.

Recall that an entire function is a function holomorphic everywhere on the complex plane (see [Ahl79], section 2.3).

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2. Background material

In this section, we recall some results, adapted to our setting, for future reference.

2.1. Hilbert-Kunz multiplicity. Hilbert-Kunz multiplicity is a multiplicity theory in positive characteristic. We refer readers to [Hun13] for a survey of this theory. In this subsection, $k$ is a field of characteristic $p > 0$.

**Definition 2.1.** Let $R$ be a finitely generated $\mathbb{N}$-graded $k$-algebra; $J$ be a homogeneous ideal such that $R/J$ has finite length. Given a finitely generated $\mathbb{Z}$-graded $R$-module $N$, the Hilbert-Kunz multiplicity of the triple $(N, R, J)$ is defined to be the following limit

$$\lim_{n \to \infty} \left(\frac{1}{p^n}\right)^{\dim(N)} \lambda_R\left(\frac{N}{J[p^n]N}\right).$$

Similarly one can define the Hilbert-Kunz multiplicity of a triple $(S, I, M)$- where $I$ is any finite co-length ideal in a Noetherian local ring $S$ and $M$ is a finitely generated $S$-module.

The existence of the limit in the Definition 2.1 was first established by Monsky (see [Mon83]).

The Hilbert-Kunz multiplicity of any local ring is at least one. Moreover, under mild hypothesis, it is exactly one if and only if the ring is regular; see Theorem 1.5 of [Kei00] and [Cra02]. These two facts suggest
that Hilbert-Kunz multiplicity is a candidate for a multiplicity theory. In general, rings with Hilbert-Kunz multiplicity closer to one are interpreted to have better singularities; see [Man04] and [GN01].

Unlike the usual Hilbert-Samuel function, the structure of the Hilbert-Kunz function $f(n) = \lambda\left(\frac{N}{n}\right)$ is rather elusive. We refer interested readers to [HMM04], [Tei02], [FT03].

2.2. Betti numbers. We review results on graded Betti numbers which we use in Section 5. References for most of these results are [Ser00], and [BH98]. Recall that $R$ is a finitely generated $\mathbb{N}$-graded $k$-algebra (see Notation and Convention 1.1).

Given a finitely generated $\mathbb{Z}$-graded $R$-module $M$, one can choose a minimal graded free resolution of $M$: this is a free resolution $(G_\bullet, d_\bullet)$ of $M$ such that each $G_n$ is a graded free $R$-module, the boundary maps preserve graded structures, and the entries of the matrices representing boundary maps are forms of positive degrees. As a consequence, $G_r \cong \bigoplus_{s \in \mathbb{Z}} R(-s)b^R_{M}(r, s)$ where $b^R_{M}(r, s) = \lambda(Tor^R_r(k, M)_s)$.

**Definition 2.2.** Let $M$ be a finitely generated $\mathbb{Z}$-graded $R$-module. The $r$-th Betti number of $M$ with respect to $R$ is the rank of the free module $G_r$ at the $r$-th spot in a minimal graded free resolution of $M$, or equivalently, the length $\lambda(Tor^R_n(k, M))$.

**Definition 2.3.** The $\mathbb{N}$-graded ring $R$ is a graded complete intersection over $k$ if $R \cong k[X_1, \ldots, X_n]_{(f_1, \ldots, f_h)}$ where each $X_i$ is homogeneous of positive degree and $f_1, \ldots, f_h$ is a regular sequence consisting of homogeneous polynomials.

We recall a special case of a result in [GUL74].

**Lemma 2.4.** Let $R$ be a graded complete intersection over $k$. Then for any finitely generated $\mathbb{Z}$-graded $R$-module $M$, there is polynomial $P_M(t) \in \mathbb{Z}[t]$ such that for all $n$, $\lambda(Tor^R_n(M, k)) \leq P_M(n)$.

**Proof.** Let $m$ be the homogeneous maximal ideal of $R$. Then $R_m$ is a local complete intersection as is meant in Corollary 4.2, [GUL74]. Since for all $n \in \mathbb{N}$, $Tor^R_n(M, k) \cong Tor^R_n(M_m, k)$, using Corollary 4.2, [GUL74], we have a polynomial $\pi(t) \in \mathbb{Z}[t]$ and $r \in \mathbb{N}$ such that, $\sum_{n=0}^{\infty} \lambda(Tor^R_n(M, k)) t^n = \frac{\pi(t)}{(1-t)^r}$. The assertion in Lemma 2.4 now follows by using the formal power series in $t$ representing $\frac{1}{(1-t)^r}$. □

**Lemma 2.5.** Let $R$ be a finitely generated $\mathbb{N}$-graded $k$-algebra. Let $M$ be a finitely generated $\mathbb{Z}$-graded $R$-module. There is a positive integer $l$ such that given any integer $s$, $b^R_{M}(r, s) := \lambda(Tor^R_r(M, k)_s) = 0$ for all $r \geq s + l$.

**Proof.** Pick a minimal free resolution $(G_\bullet, d_\bullet)$ of $M$, then $G_r \cong \bigoplus_{j \in \mathbb{Z}} R(-j)b^R_{M}(r, j)$. Since the boundary maps of $G_\bullet$ are represented by matrices whose entries are positive degree forms and have non-zero columns, $\phi(r) := \min\{j : b^R_{M}(r, j) \neq 0\}$ is a strictly increasing function of $r$. So we can choose an integer $l$ such that $\phi(l) > 0$. Again since $\phi(r)$ is strictly increasing, for all $r \geq j + l$, $\phi(r) \geq \phi(l) + j > j$. So given an integer $s$, $b^R_{M}(r, s) = 0$ for all $r \geq s + l$. □

**Lemma 2.6.** Let $R$ be a graded complete intersection over $k$ and $M$ be a finitely generated $\mathbb{Z}$-graded $R$-module. For a given integer $s$, let $B_M(s)$ denote the sum $\sum_{r \in \mathbb{N}} (-1)^r b^R_{M}(r, s)$. Then the formal Laurent series $\sum_{s \in \mathbb{Z}} B_M(s)t^s$ is absolutely convergent at every non-zero point on the open unit disk centered at the origin in $\mathbb{C}$.

**Proof.** By Lemma 2.5, there is an $l \in \mathbb{N}$ such that for any integer $s$, $|B_M(s)| \leq \sum_{j=0}^{s+l} \lambda(Tor^R_j(M, k))$. With $P_M$ the same as in Lemma 2.4, consider the polynomial

$$Q_M(s) = \sum_{j=0}^{s+l} P_M(j).$$
Thus for $s \in \mathbb{N}$, $|B_M(s)| \leq Q_M(s)$. So the radius of convergence of the power series $\sum_{s=0}^{\infty} |B_M(s)|t^s$ is at least one and the desired conclusion follows. 

2.3. Hilbert series and Hilbert-Samuel multiplicities. The references for this subsection are [BH98] and [Ser00]. Throughout, $R$ is a finitely generated $\mathbb{N}$-graded algebra over a field $k$. Recall that the Hilbert series (also called the Hilbert-Poincaré series) of a finitely generated $\mathbb{Z}$-graded $R$-module $M$ is the formal Laurent series $H_M(t) := \sum_{n \in \mathbb{Z}} \lambda(M_n)t^n$.

**Theorem 2.7.** (see Proposition 4.4.1, [BH98]) Let $M$ be a finitely generated $\mathbb{Z}$-graded $R$-module.

1. There is a Laurent polynomial $Q_M(t) \in \mathbb{Q}[t, t^{-1}]$ such that,

$$H_M(t) = \frac{Q_M(t)}{(1-t^{\delta_1}) \cdots (1-t^{\delta_{\dim(M)})}}$$

for some non-negative integers $\delta_1, \ldots, \delta_{\dim(M)}$.

2. The choice of $Q_M$ depends on the choices of $\delta_1, \ldots, \delta_{\dim(M)}$. One can choose $\delta_1, \ldots, \delta_{\dim(M)}$ to be the degrees of elements of $R/\Ann(M)$ forming a homogeneous system of parameters. 

In Proposition 2.8, we extend part of Proposition 4.1.9 of [BH98] where $R$ is assumed to be standard graded-to our setting. We use Proposition 2.8 to define Hilbert-Samuel multiplicity of a finitely generated $\mathbb{Z}$-graded module over a graded ring- where the ring is not necessarily standard graded-in part (1), Definition 2.9.

**Proposition 2.8.** Let $M$ be a finitely generated $\mathbb{Z}$-graded $R$-module of Krull dimension $d$. Denote the Poincaré series of $M$ by $H_M(t)$.

1. The limit $d! \lim_{n \to \infty} n^d (\sum_{j \leq n} \lambda(M_j))$ exists. The limit is denoted by $e_M$.

2. The limit $\lim_{t \to 1} (1-t)^d H_M(t)$ is the same as $e_M$.

**Proof of Proposition 2.8.** When $M$ has Krull dimension zero, the desired conclusion is immediate. So we assume that $M$ has a positive Krull dimension. We first prove (1).

Let $f_1, \ldots, f_d$ be a homogeneous system of parameters of $R/\Ann(M)$ of degree $\delta_1, \ldots, \delta_d$ respectively. Set $\delta$ to be the product $\delta_1 \cdots \delta_d$ and $g_j = f_j^{\delta_j}$. Then each of $g_1, \ldots, g_d$ has degree $\delta$ and these form a homogeneous system of parameters of $R/\Ann(M)$. Denote the $k$-subalgebra generated by $g_1, \ldots, g_d$ by $S$. We endow $S$ with a new $\mathbb{N}$-grading: given a natural number $n$, declare the $n$-th graded piece of $S$ to be

$$S_n := S \cap (R/\Ann(M))_{\delta n}.$$ 

From now on, by the grading on $S$ we refer to the grading defined above. Note that $S$ is a standard graded $k$-algebra. Now for each $r$, where $0 \leq r < \delta$, set

$$M^r := \bigoplus_{n \in \mathbb{Z}} M_{n \delta + r}.$$ 

Given an $r$ as above, we give $M^r$ a $\mathbb{Z}$-graded structure by declaring the $n$-th graded piece of $M^r$ to be $M_{n \delta + r}$. Then each $M^r$ is a finitely generated $\mathbb{Z}$-graded module over $S$. Since $S$ is standard graded, for each $r$, $0 \leq r < \delta$, the limit

$$\lim_{n \to \infty} \frac{1}{n^{d-r}} \lambda(M^r_n)$$

2A homogeneous system of parameters of a finitely generated $\mathbb{N}$-graded $k$-algebra $S$, is a collection of homogeneous elements $f_1, \ldots, f_{\dim(S)}$ such that $\lambda(S/f_1, \ldots, f_{\dim(S)})$ has a finite length (see [BH98], page 35).
exists (see Theorem 4.1.3, [BH98]). This implies the existence of a constant \( C \) such that \( \lambda(M_n) \leq Cn^{d-1} \) for all \( n \). So the sequence \( \frac{d!}{n^d} \left( \sum_{j \leq n} \lambda(M_j) \right) \) converges if and only if the subsequence

\[
\left( \frac{d!}{(\delta n + \delta - 1)^d} \left( \sum_{j \leq (n+1)\delta-1} \lambda(M_j) \right) \right)_n
\]

converges. Now we show that the above subsequence is convergent by computing its limit.

\[
d! \lim_{n \to \infty} \frac{1}{(\delta n + \delta - 1)^d} \left( \sum_{j \leq (n+1)\delta-1} \lambda(M_j) \right)
= \lim_{n \to \infty} \frac{n^d}{(\delta n + \delta - 1)^d} \left( \sum_{r=0}^{\delta-1} \frac{d!}{n^d} \left( \sum_{j \leq n} \lambda(M_j^r) \right) \right)
\]

Since each \( M^r \) where \( 0 \leq r \leq \delta - 1 \) is a finitely generated module over the standard graded ring \( S \), by Proposition 4.1.9 and Remark 4.1.6 of [BH98], the last limit in (2) exists and

\[
e_M = \frac{e_M^0 + \ldots + e_M^{\delta-1}}{\delta^d}
\]

For (2), note that

\[
\lim_{t \to 1} (1 - t)^d H_M(t) = \lim_{t \to 1} \sum_{r=0}^{\delta-1} (1 - t)^d H_{M^r}(t^\delta) t^r
= \lim_{t \to 1} \left( \sum_{r=0}^{\delta-1} (1 - t^\delta)^d H_{M^r}(t^\delta) t^r \right)
\]

Again, since each \( M^r \) is a finitely generated module over the standard graded ring \( S \), by Proposition 4.1.9 and Remark 4.1.6 of [BH98]

\[
e_{M^r} = \lim_{t \to 1} (1 - t)^d H_{M^r}(t).
\]

So from (4) and (3), we get

\[
\lim_{t \to 1} (1 - t)^d H_M(t) = \frac{e_M^0 + \ldots + e_M^{\delta-1}}{\delta^d} = e_M.
\]

**Definition 2.9.** Let \( M \) be a finitely generated \( \mathbb{Z} \)-graded \( R \)-module of Krull dimension \( d \).

1. The **Hilbert-Samuel multiplicity** of \( M \) is defined to be the limit

\[
d! \lim_{n \to \infty} \frac{1}{n^d} \left( \sum_{j \leq n} \lambda(M_j) \right)
\]

and denoted by \( e_M \). The limit exists by (1) Proposition 2.8.

2. Given a homogeneous ideal \( I \) of finite co-length, the **Hilbert-Samuel multiplicity of \( M \) with respect to \( I \)** is defined to be the limit:

\[
d! \lim_{n \to \infty} \frac{1}{n^d} \lambda(M/I^n M).
\]

**Proposition 2.10.** Let \( f_1, \ldots, f_d \) be a homogeneous system of parameters of \( R \) of degree \( \delta_1, \ldots, \delta_d \) respectively. Then the Hilbert-Samuel multiplicity of \( R \) with respect to \((f_1, \ldots, f_d)\) (see Definition 2.9) is \( \delta_1 \ldots \delta_d e_R \).

**Proof.** By Proposition 2.10 of [HTW11], the desired multiplicity is \( \delta_1 \ldots \delta_d \lim_{t \to 1} (1 - t)^d H_R(t) \), which by Proposition 2.8 is \( \delta_1 \ldots \delta_d e_R \).
2.4. Vector bundles on curves. In this subsection, $C$ stands for a curve, where by a curve we mean a one dimensional, irreducible smooth projective variety over an algebraically closed field; the genus of $C$ is denoted by $g$. A vector bundle on $C$ means a locally free sheaf $\mathcal{O}_C$-modules of finite constant rank. Morphisms of vector bundles are a priori morphisms of $\mathcal{O}_C$-modules. We recall some results on vector bundles on $C$ which we use in Section 6. For any unexplained terminology, readers are requested to turn to [Har97] or [Pot97].

Definition 2.11. Let $F$ be a coherent sheaf on the curve $C$.

1. The rank of $F$, denoted by $\text{rk}(F)$, is the dimension of the stalk of $F$ at the generic point of $C$ as a vector space over the function field of $C$.
2. The degree of $F$, denoted by $\text{deg}(F)$, is defined as $h^0(C, F) - h^1(C, F) - \text{rk}(F)(1 - g)$.
3. The slope of $F$, denoted by $\mu(F)$, is the ratio $\frac{\text{deg}(F)}{\text{rk}(F)}$. By convention, $\mu(F) = \infty$ if $\text{rk}(F) = 0$.

Definition 2.12. A vector bundle $E$ on $C$ is called semistable if for any nonzero coherent subsheaf $F$ of $E$, $\mu(F) \leq \mu(E)$.

Theorem 2.13. (see [HN75, Prop 1.3.9]) Let $E$ be a vector bundle on $C$. Then there exists a unique filtration:
\[ 0 = E_0 \subset E_1 \subset \ldots \subset E_t \subset E_{t+1} = E \]
such that,
1. All the quotients $E_{j+1}/E_j$ are non-zero, semistable vector bundles.
2. For all $j$, $\mu(E_j/E_{j-1}) > \mu(E_{j+1}/E_j)$.

This filtration is called the Harder-Narasimhan filtration of $E$.

Proposition 2.14. Let $0 = E_0 \subset E_1 \subset \ldots \subset E_t \subset E_{t+1} = E$ be the HN filtration on $E$. If the slope of $E_1$ is negative, $E$ cannot have a non-zero global section.

Proof. On the contrary, assume that $E$ has a non-zero global section $s$. Let $\lambda_s : \mathcal{O}_C \to E$ be the non-zero map induced by $s$. Let $b$ be the largest integer such that the composition $\mathcal{O}_C \xrightarrow{\lambda_s} E \to E/b$ is non-zero. Then $\lambda_s$ induces a non-zero map from $\mathcal{O}_C$ to $E_{b+1}/E_b$, whose image $L$ is a line bundle with a non-zero global section. So the slope of $L$ is positive. On the other hand, since $L$ is a non-zero subsheaf of the semistable sheaf $E_{b+1}/E_b$, $\mu(L) < \mu(E_{b+1}/E_b)$. Since $\mu(E_{b+1}/E_b) < \mu(E_1)$, the slope of $E_{b+1}/E_b$ is negative; so $L$ cannot have a positive slope.

Lemma 2.15. (1) For a coherent sheaf of $\mathcal{O}_C$-modules $F$ and a line bundle $L$, $\mu(F \otimes L) = \mu(F) + \text{deg}(L)$.

Here we stick to the convention that the sum of $\infty$ and a real number is $\infty$.

2. Tensor product of a semistable vector bundle and a line bundle is semistable.

3. Given a vector bundle $E$ and a line bundle $L$ on $C$, the HN filtration on $E \otimes L$ is obtained by tensoring the HN filtration on $E$ with $L$.

Proof. Because (3) follows from (1) and (2); and assertion (2) follows from (1), it suffices to prove (1). For (1), it is enough to show that
\[ \text{deg}(F \otimes L) = \text{deg}(F) + \text{rk}(F)\text{deg}(L). \]

This is clear when $\text{rk}(F) = 0$. In the general case, take the short exact sequence of sheaves $0 \to F' \to F \to F'' \to 0$, where $F'$ is the torsion subsheaf of $F$ and $F'' := F/F'$ is a vector bundle; note that the rank of $F'$ is zero. Since degree is additive over short exact sequences (see section 2.6, [Pot97]), it suffices to show (5) when $F = F''$, that is, when $F$ is locally free. In this case, $\text{deg}(F \otimes L) = \text{deg}(\text{det}(F \otimes L)) = \text{deg}(\text{det}(F) \otimes L^{\otimes \text{rk}(F)})$ (for e.g. by Theorem 2.6.9 of [Pot97]), so $\text{deg}(F \otimes L) = \text{deg}(F) + \text{rk}(F)\text{deg}(L)$ by Theorem 2.6.3, [Pot97].

In the next lemma, for a sheaf of $\mathcal{O}_C$-modules $F$, $F^\vee$ denotes the dual sheaf $\text{Hom}_{\mathcal{O}_C}(F, \mathcal{O}_C)$.

Lemma 2.16. (1) The dual of a semistable vector bundle is semistable.

2. Let $0 = E_0 \subset E_1 \ldots \subset E_t \subset E_{t+1} = E$ be the HN filtration on a vector bundle $E$. For $j$ between 0 and $t + 1$, set $K_j = \text{ker}(E^\vee \to E_{j-1}^\vee)$. Then
\[ 0 = K_0 \subset K_1 \subset \ldots \subset K_t \subset K_{t+1} = E^\vee \]
is the HN filtration on $E^\vee$. 

Proof. (1) Let $\mathcal{F}$ be semistable vector bundle and $\mathcal{G}$ be a non-zero subsheaf of $\mathcal{F}$. We show that $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$-this is clear when $\text{rk}(\mathcal{F} / \mathcal{G}) = 0$ as $\deg(\mathcal{F} / \mathcal{G}) = \deg(\mathcal{G}) + \deg(\mathcal{F} / \mathcal{G})$ and $\text{rk}(\mathcal{F} / \mathcal{G}) = \text{rk}(\mathcal{G})$. If $\text{rk}(\mathcal{F} / \mathcal{G})$ is not zero, set $\mathcal{G}'$ to be the inverse image of $(\mathcal{F} / \mathcal{G})_{\text{tor}}$: the torsion subsheaf of $\mathcal{F} / \mathcal{G}$, under the quotient map $\mathcal{F} \to \mathcal{F} / \mathcal{G}$. Then $\mathcal{G}' / \mathcal{G} \cong (\mathcal{F} / \mathcal{G})_{\text{tor}}$. So $\mathcal{G}$ and $\mathcal{G}'$ have the same rank. Since $\deg(\mathcal{G}') \geq \deg(\mathcal{G})$, it is enough to show that $\mu(\mathcal{G}') \leq \mu(\mathcal{F})$. Since $\mathcal{F} / \mathcal{G}'$ is a vector bundle, after dualizing we get an exact sequence:

$$0 \to (\mathcal{F} / \mathcal{G}')^\vee \to \mathcal{F} \to (\mathcal{G}')^\vee \to 0.$$ 

Since $\mathcal{F}$ is semistable, $\mu((\mathcal{G}')^\vee) \geq \mu(\mathcal{F})$- see section 5.3, [Pot97]. Since for a vector bundle $\mathcal{E}$, $\deg(\mathcal{E}) = -\text{deg}(\mathcal{E})$, we have $\mu(\mathcal{G}') \leq \mu(\mathcal{F})$.

(2) $K_{i+1} / K_j \cong (E_{i+1-j} / E_{i-j})^\vee$, so by (1) $(E_{i+1-j} / E_{i-j})$ is semistable. Moreover, since $\mu(K_{i+1} / K_j) = -\mu(E_{i+1-j} / E_{i-j})$, slopes of $K_{i+1} / K_j$ form a decreasing sequence. \hfill \square

Let $C$ be a curve over an algebraically closed field of positive characteristic. Let $f$ be the absolute Frobenius endomorphism of $C$. Since $C$ is smooth, $f$ is flat map (see Theorem 2.1, [Kun69]). So the pullback of the HN filtration on a given vector bundle gives a filtration of the pull back bundle by subbundles- in general this is not the HN filtration on the pullback bundle.

**Theorem 2.17.** (see Theorem 2.7 [Lan04]) Let $\mathcal{E}$ be a vector bundle on a curve $C$. Then there is an $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, the HN filtration on $(f^n)^*(\mathcal{E})$ is the pullback of the HN filtration on $(f^{n_0})^*(\mathcal{E})$ via $f^{n-n_0}$.

### 3. Existence of Frobenius-Poincaré functions

In this section, we define the Frobenius-Poincaré function associated to a given triple $(M, R, I)$, where $R$ is a finitely generated $\mathbb{N}$-graded $k$-algebra – $k$ has characteristic $p > 0$, (see Notation and Convention 1.1), $I$ is a homogeneous ideal of finite co-length and $M$ is a finitely generated $\mathbb{Z}$-graded $R$-module. In Theorem 3.1 we prove that Frobenius-Poincaré functions are entire functions.

Given $(M, R, I)$ as above and a non-negative integer $d$, define a sequence of functions $F_n(M, R, I, d)$, where for a complex number $y$,

$$F_n(M, R, I, d)(y) = \left( \frac{1}{p^n} \right)^d \sum_{j=-\infty}^{\infty} \lambda \left( \frac{M}{I^{\lceil p^n \rceil} M} \right)_j e^{-iyj/p^n}. \tag{6}$$

Since $M/I^{\lceil p^n \rceil} M$ has only finitely many non-zero graded pieces, each $F_n(M, R, I, d)$ is a polynomial in $e^{-iy/p^n}$, hence is an entire function. When the context is clear, we suppress one or more of the parameters among $M, R, I, d$ in the notation $F_n(M, R, I, d)$. Whenever there is no explicit reference to the parameter $d$ in $F_n(M, R, I, d)$, it should be understood that $d = \dim(M)$.

The goal in this section is to prove the following result:

**Theorem 3.1.** Fix a triple $(M, R, I)$, where $R$ is a finitely generated $\mathbb{N}$-graded $k$-algebra (see Notation and Convention 1.1), $I$ is a finite co-length homogeneous ideal, and $M$ is a finitely generated $\mathbb{Z}$-graded $R$-module. The sequence of functions $(F_n(M, R, I, \dim(M)))_n$, where

$$F_n(M, R, I, \dim(M))(y) = (\frac{1}{p^n})^{d(\dim(M))} \sum_{j=-\infty}^{\infty} \lambda \left( \frac{M}{I^{\lceil p^n \rceil} M} \right)_j e^{-iyj/p^n},$$

converges for every complex number $y$. Furthermore, the convergence is uniform on every compact subset of the complex plane and the limit is an entire function.

Theorem 3.1 motivates the next definition.

**Definition 3.2.** The Frobenius-Poincaré function of the triple $(M, R, I)$ is the limit of the convergent sequence of functions

$$(F_n(M, R, I, \dim(M)))_n$$

as defined in Theorem 3.1. The Frobenius-Poincaré function of the triple $(M, R, I)$ is denoted by $F(M, R, I, \dim(M))$ or alternately $F(M, R, I)$ or just $F(M)$ when the other parameters are clear from the context.
Before giving examples of Frobenius-Poincaré functions, we single out a limit computation.

**Lemma 3.3.** Given a complex number $a$, the sequence of functions $(p^n(1 - e^{-ia/p^n}))_n$ converges to the function $g(y) = iay$ and the convergence is uniform on every compact subset of the complex plane.

**Proof.** Using the power series representing $e^z$ around the origin, we get

$$p^n(1 - e^{-ia/p^n}) = iay + \sum_{j=2}^{\infty} \frac{(-iay)^j}{j!(p^n)^j}.$$

So

$$|p^n(1 - e^{-ia/p^n}) - iay| \leq \frac{1}{p^n} \sum_{j=2}^{\infty} \frac{|ay|^j}{j!} \leq \frac{1}{p^n} e^{\lambda|ay|}.$$

Since $e^{\lambda|ay|}$ is bounded on any compact subset, we get the desired uniform convergence. \(\square\)

**Example 3.4.** Consider the $\mathbb{N}$-graded polynomial ring in one variable $R = \mathbb{F}_p[X]$ where $X$ has degree $\delta \in \mathbb{N}$ and elements of $\mathbb{F}_p$ have degree zero. Take $I$ to be the ideal generated by $X$. Then, for any nonzero $y \in \mathbb{C}$,

$$F_n(R)(y) = \frac{1}{p^n} \sum_{j=0}^{p^n-1} e^{-iy\delta_j/p^n} = \frac{1 - e^{-iy}}{p^n} 1 - e^{-iy/p^n}.$$

Taking limit as $n$ goes to infinity in the above equation and using Lemma 3.3, we get that for a non-zero complex number $y$, $F(R)(y) = \frac{1 - e^{-iy}}{iay}$. Note that $\frac{1 - e^{-iy}}{iay}$ can be extended to an analytic function with value one at the origin. Since $F_n(0) = 1$ for all $n$, $F(R)$ and the analytic extension of $\frac{1 - e^{-iy}}{iay}$ are the same function.

Similar computation shows that the Frobenius-Poincaré function of the triple $(R, I, R)$ where $I$ is the ideal generated by $X$’s $1 - e^{-iy}$.

**Example 3.5.** Take $R = \mathbb{F}_p[X_1, \ldots, X_n]$ with the grading assigning degree $\delta_j$ to $X_j$ and degree zero to the elements of $\mathbb{F}_p$. Since as graded rings,

$$\frac{\mathbb{F}_p[X_1, \ldots, X_n]}{(X_1^{p^n}, X_2^{p^n}, \ldots, X_d^{p^n})} \cong \mathbb{F}_p[X_1]/(X_1^{p^n}) \otimes_{\mathbb{F}_p} \mathbb{F}_p[X_2]/(X_2^{p^n}) \otimes_{\mathbb{F}_p} \ldots \otimes_{\mathbb{F}_p} \mathbb{F}_p[X_d]/(X_d^{p^n}),$$

we have $F_n(R, R, (X_1, \ldots, X_n))(y) = F_n(\mathbb{F}_p[X_1], \mathbb{F}_p[X_1], (X_1)) \ldots F_n(\mathbb{F}_p[X_d], \mathbb{F}_p[X_d], (X_d))$. So from Example 3.4, it follows that $F(R, R, (X_1, \ldots, X_d))(y) = \prod_{j=1}^{d} \left( \frac{1 - e^{-iy}}{iay} \right)$.

**Remark 3.6.** The fractional exponents of exponentials in the definition of Frobenius-Poincaré function of a triple $(M, R, I)$ comes from the natural $\frac{1}{p^n}\mathbb{Z}$ grading on $F^n(M)$. Here $F^n(M)$ is the abelian group $M$ endowed with the $R$-module structure coming from the restriction of scalars via the $n$-th iteration of Frobenius $F^n_R : R \to R$. The $\frac{1}{p^n}\mathbb{Z}$-graded structure on $F^n(M)$ is as follows: for an integer $m$, $F^n(M)_{m/p^n} = M_m$. For example, when $R$ is a domain, the $\frac{1}{p^n}\mathbb{Z}$-grading on $F^n(R)$ described above is the one obtained by importing the natural $\frac{1}{p^n}\mathbb{Z}$-grading on $R^{1/p^n}$ via the $R$-module isomorphism $R^{1/p^n} \to F^n(R)$ given by the $p^n$-th power map.

Note that as $\frac{1}{p^n}\mathbb{Z}$-graded modules $F^n(M)_{\frac{M}{T_{p^n}(M)}} \cong F^n(M) \otimes_R R/I$. Set $\ell$ to be the degree of the field extension $k^p \subseteq k$ and $\mathbb{Z}[1/p]$ to be the ring $\mathbb{Z}$ with $p$ inverted. Alternatively $F_n$ can be expressed as

$$F_n(M, R, I)(y) = \left( \frac{1}{\ell p^n \dim(M)} \right)^n \sum_{t \in \mathbb{Z}[1/p]} \lambda_k ((F^n(M) \otimes R/I)_t) e^{-iy}.$$

That is, $F_n$ is the Hilbert series (see Section 2.3) of $F^n(M) \otimes R/I$ normalized by $\left( \frac{1}{\ell p^n \dim(M)} \right)^n$ in the ‘variable’ $e^{-iy}$. The associated Frobenius-Poincaré function is the limit of these normalized Hilbert series.
Remark 3.7. Given a field extension $k \subseteq k'$, $R \otimes_k k'$ is a finitely generated $\mathbb{N}$-graded $k'$-algebra. Note that $F_n(M, R, I, d) = F_n(M \otimes_k k', R \otimes_k k', I \otimes_k k', d)$. Thus for any complex number $y$, $(F_n(M, R, I, d)(y))_n$ converges if and only if $(F_n(M \otimes_k k', R \otimes_k k', I \otimes_k k', d)(y))_n$ converges. So in the proof of Theorem 3.1 without loss of generality we can assume that $k^p \subseteq k$ is a finite extension or even algebraically closed. Thus we can assume $k = k^p$.

The remainder of the section is dedicated to the proof of Theorem 3.1. The proof has two main steps. First, we reduce the problem to the case where $R$ is an $\mathbb{N}$-graded domain and $M = R$ as a graded module—this reduction step is achieved in Theorem 3.16. Then we show that when $R$ is a domain, $F_n(R, R, I)(u)$ is uniformly Cauchy on every compact subset of $\mathbb{C}$. Thus $F_n(R, R, I)$ converges uniformly on every compact subset of the complex plane. The analyticity of the limiting function then follows from Theorem 1 in Chapter 5 of [Ahl79]: a sequence of holomorphic functions on an open subset $U \subseteq \mathbb{C}$, which converges uniformly on every compact subset of $U$ has a holomorphic limiting function.

One of the purposes of the next result is to show that in the definition of $F_n$ in (6), it is enough to take the sum over indices $j$, where $|j|/p^n$ is bounded by a constant which is independent of $n$.

Lemma 3.8. Let $M$ be a finitely generated $\mathbb{Z}$-graded $R$-module. Given $i \in \mathbb{N}$, there exists a positive integer $C$ such that for all $n$ whenever $|j| \geq C p^n$, $(\text{Tor}_i^R(M, R/p^nR))_j$ is zero.

Proof. We claim that it is sufficient to prove existence of some $C$ such that $(\text{Tor}_i^R(M, R/p^nR))_j = 0$ for $|j| \geq C p^n$. To see this, consider a graded minimal free resolution (see section 2.2) of the $R$-module $M$:

\[ \partial_{i+1} \rightarrow F_i \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F_0 \rightarrow M \rightarrow 0. \]

As a graded module, $\text{Tor}_i^R(M, R/p^nR)$ is a subquotient of $\text{Tor}_i^R(F_1, R/p^nR) \equiv F_1/I[p^n]F_1$ and $F_i$ is a direct sum of finitely many graded modules of the form $R(l)$, so our claim follows.

Now choose homogeneous elements of positive degrees $r_1, r_2, \ldots, r_s$ of $R$ such that $R = k[r_1, \ldots, r_s]$. Let $\Delta = \max\{\deg(r_i)\}$ and $\delta = \min\{\deg(r_i)\}$. Denote the ideal of generated by homogeneous elements of degree at least $t$ by $R_{\geq t}$. Then note that for every $n \in \mathbb{N}$,

\[ R_{\geq n\Delta} \subseteq (R_{\geq \delta})^n. \]

Indeed, for each $r \in R_t$ where $t \geq n\Delta$, we can choose homogeneous elements $\lambda_1, \ldots, \lambda_s$ of $R$ such that

\[ r = \lambda_1 r_1 + \cdots + \lambda_s r_s. \]

Then each $\lambda_i \in R_{\geq t-\Delta} \subseteq R_{(n-1)\Delta}$. Since each $r_i$ is in $R_{\geq \delta}$, the claimed assertion in (7) follows by induction on $n$.

Pick $m_0$ such that $(R_{\geq \delta})^{m_0} \subseteq I$. Suppose the minimal number of homogeneous generators of $I$ is $\mu$. Using (7), we get

\[ R_{\geq m_0\mu p^n\Delta} \subseteq (R_{\geq \delta})^{m_0\mu p^n} \subseteq I[p^n] \subseteq I[p^n]. \]

Therefore, if we set $C = m_0\mu$, for $m \geq C p^n$, $(\text{Tor}_i^R(M, R/p^nR))_m = 0$. \hfill \Box

We now bound the asymptotic growths of two length functions.

Lemma 3.9. Let $(S, m)$ be Noetherian local ring containing a field of positive characteristic $p$, $J$ be an $m$-primary ideal. For any finitely generated $S$-module $N$, there exist positive constants $C_1, C_2$ such that for all $n \in \mathbb{N}$,

\[ \lambda_S(N/J[p^n]N) \leq C_1(p^n)^{\dim(N)} \quad \text{and} \quad \lambda_S(S, N, S) \leq C_2(p^n)^{\dim(N)}. \]

Proof. The assertion on the growth of $\lambda_S(N/J[p^n]N)$ is standard, for example see Lemma 3.5 of [Hum13] for a proof. For the other assertion, we present a simplified version of the argument in Lemma 7.2 of [Hum13]. Suppose that $J$ is generated by $f_1, \ldots, f_\mu$. Let $K_*$ be the Koszul complex of $f_1, f_2, \ldots, f_\mu$. Recall that the Frobenius functor $\mathfrak{F}$, from the category of $S$-modules to itself, is the scalar extension via the Frobenius $F_S : S \rightarrow S$ (see page 7, [HH93]). Let $\mathfrak{F}^n(K_*)$ stand for the complex of $S$-modules obtained by applying
$n$-th iteration of $\mathfrak{g}$ to the terms and the boundary maps of $K_*$. Then the part $\mathfrak{g}^n(K_1) \to \mathfrak{g}^n(K_0)$ of $\mathfrak{g}^n(K_*)$ can be extended to a free resolution of the $S$-module $S/J^{[p^n]}$. So $\text{Tor}_1^S(N, \mathfrak{g}^n(S/K^n))$ is isomorphic to a quotient of $H_1(N \otimes_S \mathfrak{g}^n(K_*))$. Hence $\lambda(\text{Tor}_1^S(N, \mathfrak{g}^n(S/K^n))) \leq \lambda_S(H_1(N \otimes_S \mathfrak{g}^n(K_*)))$. The conclusion follows from Theorem 6.6 of [HH93], which guarantees that there is a constant $C_2$ such that for all $n$, $\lambda_S(H_1(N \otimes_S \mathfrak{g}^n(K_*))) \leq C_2(p^n)^{\dim(S/N)}$. □

The next result bounds the growth of the sequence $(F_n(M, R, I, d))_n$ on a given compact subset of the complex plane.

**Proposition 3.10.** Let $M$ be a finitely generated $\mathbb{Z}$-graded $R$-module.

1. Given a compact subset $A \subseteq \mathbb{C}$, there exists a constant $D$ such that for all $y \in A$ and $n \in \mathbb{N}$,

$$|F_n(M, R, I, d)(y)| \leq \left(\frac{1}{p^n}\right)^{d-\dim(M)}D.$$

2. On a given compact subset of $\mathbb{C}$, when $d \geq \dim(M)$, the sequence $(F_n(M, R, I, d))_n$ is uniformly bounded and when $d > \dim(M)$, the sequence $(F_n(M, R, I, d))_n$ uniformly converges to the constant function taking value zero.

**Proof.** Assertion (2) is immediate from (1).

We now prove (1). Let $C'$ be a positive constant such that for $y \in A, |y| \leq C'$. According to Lemma 3.8, we can choose a positive constant $C$ such that for all $n \in \mathbb{N}$ and $|j| > Cp^n$, $(M/I^{[p^n]}|M)_j = 0$. For $y \in A$ and $|j| \leq Cp^n$,

$$|e^{-iyj/p^n}|^2 = e^{-iyj/p^n} e^{iyj/p^n} = e^{2\text{Re}(-iyj/p^n)} \leq e^{2|y||j|/p^n} \leq e^{2CC'}.$$

Now note that,

$$|F_n(M, d)(y)| = \left| \frac{1}{p^n} \right|^d \sum_{|j| \leq Cp^n} \lambda \left( \left( \frac{M}{I^{[p^n]}|M} \right)_j \right) e^{-iyj/p^n} \leq \left( \frac{1}{p^n} \right)^d \sum_{|j| \leq Cp^n} \lambda \left( \left( \frac{M}{I^{[p^n]}|M} \right)_j \right) |e^{-iyj/p^n}|,$$

so using (8) first and then Lemma 3.9, we get,

$$|F_n(M, d)(y)| \leq \left( \frac{1}{p^n} \right)^d \lambda \left( \frac{M}{I^{[p^n]}|M} \right) e^{CC'} \leq \left( \frac{1}{p^n} \right)^{d-\dim(M)}C_1 e^{CC'}$$

for some constant $C_1$. □

The next few results are aimed towards Theorem 3.16.

**Lemma 3.11.** Let $0 \longrightarrow K \overset{\phi_1}{\longrightarrow} M_1 \overset{\phi}{\longrightarrow} M_2 \overset{\phi_2}{\longrightarrow} C \longrightarrow 0$ be an exact sequence of finitely generated $\mathbb{Z}$-graded $R$-modules (i.e. assume the boundary maps preserve the respective gradings). Let $d$ be an integer greater than both $\dim(K)$ and $\dim(C)$.

1. Given a compact subset $A \subseteq \mathbb{C}$, there exists a constant $D$, such that for all $y \in A$ and $n \in \mathbb{N}$,

$$|F_n(M_2, R, I, d)(y) - F_n(M_1, R, I, d)(y)| \leq \frac{D}{p^n}.$$

2. The sequence of functions $(F_n(M_2, R, I, d) - F_n(M_1, R, I, d))_n$ converges to the constant function zero and the convergence is uniform on every compact subset of $\mathbb{C}$.

**Proof.** We prove assertion (1) below, assertion (2) is immediate from assertion (1).

Break the given exact sequence into two short exact sequences:

\begin{align*}
(*) & \quad 0 \longrightarrow K \overset{\phi_1}{\longrightarrow} M_1 \overset{\phi}{\longrightarrow} \text{Im}(\phi) \longrightarrow 0, \\
(**) & \quad 0 \longrightarrow \text{Im}(\phi) \longrightarrow M_2 \overset{\phi_2}{\longrightarrow} C \longrightarrow 0.
\end{align*}
Now apply $\otimes_R \frac{R}{I^{[p^n]}}$ to (\ref{eq:1}) and (\ref{eq:2}); the corresponding long exact sequences of Tor modules give the following two exact sequences of graded modules for each $n$:

(*\textsubscript{n}) \quad \xymatrix{ K \ar[r]^{\phi_1, n} & M_1 \ar[r] & \text{Im}(\phi) \ar[r] & 0, }

(**\textsubscript{n}) \quad \xymatrix{ \text{Tor}^R_1(C, \frac{R}{I^{[p^n]}}) \ar[r]^{\tau_n} & \text{Im}(\phi) \ar[r] & M_2 \ar[r] & \frac{C}{I^{[p^n]}} \ar[r] & 0. }

Using (\ref{eq:1}) and (\ref{eq:2}), for each $j \in \mathbb{Z}$, we get

\[
\lambda \left( \left( \frac{M_1}{I^{[p^n]}_n} \right)_j \right) = \lambda \left( \left( \frac{K}{I^{[p^n]}_n} \right)_j \right) + \lambda \left( \left( \frac{\text{Im}(\phi)}{I^{[p^n]}_n} \right)_j \right),
\]

\[
\lambda \left( \left( \frac{M_2}{I^{[p^n]}_n} \right)_j \right) = \lambda \left( \left( \frac{\text{Im}(\phi)}{I^{[p^n]}_n} \right)_j \right) - \lambda \left( \tau_n \left( \text{Tor}^R_1(C, \frac{R}{I^{[p^n]}}) \right)_j \right) + \lambda \left( \left( \frac{C}{I^{[p^n]}} \right)_j \right).
\]

Therefore,

\[
F_n(M_2, R, I, d)(y) - F_n(M_1, R, I, d)(y)
\]

\[
= F_n(C, R, I, d)(y) - \left( \frac{1}{p^n} \right)^d \sum_{j=\infty}^{\infty} \lambda \left( \left( \tau_n \left( \text{Tor}^R_1(C, \frac{R}{I^{[p^n]}}) \right) \right)_j \right) e^{-i\nu j/p^n}
\]

\[
- \left( \frac{1}{p^n} \right)^d \sum_{j=-\infty}^{\infty} \lambda \left( \left( \frac{K}{I^{[p^n]}} \right)_j \right) e^{-i\nu j/p^n}.
\]

By Lemma 3.8, one can choose a positive integer $C_1$ such that given any $n$ and all $m$ such that $|m| > C_1 p^n$,

\[
\left( \frac{C}{I^{[p^n]}} \right)_m = \left( \text{Tor}^R_1(C, \frac{R}{I^{[p^n]}}) \right)_m = \left( \frac{K}{I^{[p^n]}} \right)_m = 0.
\]

Since $A$ is compact, there is a constant $C_2$ such that for all $j$, where $|j| \leq C_1 p^n$ and for $y \in A$, $|e^{-i\nu j/p^n}| \leq C_2$-the argument is similar to that in (8). Using (9), we conclude that for $y \in A$,

\[
|F_n(M_2, R, I, d)(y) - F_n(M_1, R, I, d)(y)|
\]

\[
\leq \left( \frac{1}{p^n} \right)^d \sum_{|j| \leq C_1 p^n} [\lambda \left( \left( \frac{C}{I^{[p^n]}} \right)_j \right) + \lambda \left( \left( \text{Tor}^R_1(C, \frac{R}{I^{[p^n]}}) \right)_j \right) + \lambda \left( \left( \frac{K}{I^{[p^n]}} \right)_j \right)] |e^{-i\nu j/p^n}|
\]

\[
\leq C_2 \left( \frac{1}{p^n} \right)^d [\lambda \left( \left( \frac{C}{I^{[p^n]}} \right) \right) + \lambda \left( \text{Tor}^R_1(C, \frac{R}{I^{[p^n]}}) \right) + \lambda \left( \frac{K}{I^{[p^n]}} \right)] .
\]

Since both $\text{dim}(C)$ and $\text{dim}(K)$ are less than $d$, the desired result follows from Lemma 3.9. \hfill $\Box$

Recall that for an integer $h$, $M(h)$ denotes the $R$-module $M$ but with a different $\mathbb{Z}$-grading: the $n$-th graded piece of $M(h)$ is $M_{n+h}$. From now on, we use the terminology set in the next definition.

**Definition 3.12.** Whenever the sequence of complex numbers $(F_n(M, R, I, d)(y))_n$ (see (6)) converges, we set

\[
F(M, R, I, d)(y) = \lim_{n \to \infty} F_n(M, R, I, d)(y).
\]

In the case $d = \text{dim}(M)$, we set $F(M, R, I)(y) = F(M, R, I, \text{dim}(M))(y)$. Analogously we use $F(M)(y)$ when $R, I$ are clear from the context.

**Proposition 3.13.** Let $R$ be a finitely generated $\mathbb{N}$-graded $k$-algebra, $I$ be a homogeneous ideal of finite co-length and $M$ be a finitely generated $\mathbb{Z}$-graded $R$-module. Fix an integer $h$.

1. Given a compact subset $A \subseteq \mathbb{C}$, there exists a constant $D$ such that for all $y \in A$ and all $n$,

\[
|F_n(M(h), R, I, d)(y) - F_n(M, R, I, d)(y)| \leq D \frac{D}{p^n} |F_n(M, R, I, d)(y)|.
\]

2. For any complex number $y$, $(F_n(M, R, I, d)(y))_n$ converges if and only if $(F_n(M(h), R, I, d)(y))_n$ converges. When both of these converge, their limits are equal.
Proof. (1) Note that
\[F_n(M(h), R, I, d)(y) = (\frac{1}{p^n})^d \sum_{j=-\infty}^{\infty} \lambda((\frac{M}{(1/p^n)|M|})_{j+h}) e^{-iy(j+h)/p^n} e^{iy/p^n} F_n(M, R, I, d)(y).\]
Thus,
\[|F_n(M(h), R, I, d)(y) - F_n(M, R, I, d)(y)| = |e^{iy/p^n} - 1||F_n(M, R, I, d)(y)|.\]
Since \(A\) is bounded, it follows from Lemma 3.3, there is a constant \(D\) such that for \(y \in A\), \(|1 - e^{-iy/p^n}| \leq \frac{D}{p^n}\).

(2) It follows from the first assertion that whenever \((F_n(M, d)(y))_n\) converges, the sequence \(F_n(M(h), d)(y)\) also converges and the two limits coincide. The other direction follows from the observation that as a graded module \(M\) is isomorphic to \((M(h))(-h)).\]

Lemma 3.14. Let \(R \to S\) be a degree preserving finite homomorphism of finitely generated \(\mathbb{N}\)-graded \(k\)-algebras. For any finitely generated \(S\)-module \(N\) and any complex number \(y\), \((F_n(N, R, I, d)(y))_n\) converges if and only if \((F_n(N, S, IS, d)(y))_n\) converges. When both of these converge \(F_n(N, R, I, d)(y) = F_n(N, S, IS, d)(y)\).

Proof. Since the \(R\)-module structure on \(N\) comes via the restriction of scalars, for each \(n\), the two \(k\)-vector spaces \((\frac{N}{(p^n)|N|})_j\) and \((\frac{N}{(IS)|p^n|N})_j\) are isomorphic. Thus \(F_n(N, R, I, d)(y) = F_n(N, S, IS, d)(y)\) and the conclusion follows.

Note that, for a finitely generated \(\mathbb{N}\)-graded \(k\)-algebra \(R\), \(R^{p^n} = \{r^{p^n} \mid r \in R\}\) is an \(\mathbb{N}\)-graded subring of \(R\)-the \(\mathbb{N}\)-grading on \(R^{p^n}\) will refer to this grading.

Proposition 3.15. Let \(k\) be a field of characteristic \(p > 0\) such that \(k^p \subseteq k\) is finite. Let \(R\) be a finitely generated \(\mathbb{N}\)-graded \(k\)-algebra, \(I\) be a homogeneous ideal of finite co-length and \(M\) be a finitely generated \(\mathbb{Z}\)-graded \(R\)-module. Given two non-negative integers \(d, m\) and a complex number \(y\),

(1) Denote the image of \(I\) in \(R^{p^m}\) under the \(p^m\)-th power map by \(IR^{p^m}\). Then
\[F_{n+m}(M, R, I, d)(y) = \frac{1}{p^{md}[k : k^{p^n}]} F_n(M, R^{p^m}, IR^{p^m}, d)(y/p^{n+m}).\]

(2) When \((F_n(M, R, I, d)(y))_n\) converges,
\[F(M, R, I, d)(y) = \frac{1}{p^{md}[k : k^{p^n}]} F(M, R^{p^m}, IR^{p^m}, d)(y/p^{n+m}).\]

(3) If \(R\) is reduced, for all \(n\)
\[F_n(R^p, R^{p^n}, IR^{p^n}, d)(y/p) = F_n(R, R, I, d)(y).\]

Proof. Given \(n \in \mathbb{N}\),
\[F_n(M, R^{p^m}, IR^{p^m}, d)(y/p^{n+m}) = (\frac{1}{p^n})^d \sum_{j=-\infty}^{\infty} \lambda_{k^{p^m}}((\frac{M}{IR^{p^m}|p^n|})_{j}) e^{-iyj/p^{n+m}}\]
\[= (\frac{1}{p^n})^d \sum_{j=-\infty}^{\infty} \lambda_{k^{p^m}}((\frac{M}{(IR^{p^m})|p^n|})_{j}) e^{-iyj/p^{n+m}}\]
\[= p^{md}[k : k^{p^n}](\frac{1}{p^{n+m}})^d \sum_{j=-\infty}^{\infty} \lambda_k((\frac{M}{(IR^{p^m})|p^n|})_{j}) e^{-iyj/p^{n+m}}\]
\[= p^{md}[k : k^{p^n}] F_{n+m}(M, R, I, d)(y).\]

(1) and (2) follows directly from the calculation above.

Now, we verify (3). Since \(R\) is reduced, the Frobenius \(F_R : R \to R\) induces an isomorphism onto \(R^p\); it takes \(R_j\) to \((R^p)_j\). Thus for each \(n, j \in \mathbb{N}\), \(F_R\) induces an isomorphism of abelian groups from \((\frac{R_j}{(1/p^n)R_j})_j\) to \((\frac{R^p}{(1/p^n)R^p})_j\). So, \(\lambda_k((\frac{R}{(1/p^n)R})_j) = \lambda_k((\frac{R^p}{(1/p^n)R^p})_j)\). Now,
\[
F_n(R^p, R^p, IR^p, d)(y/p) = \left( \frac{1}{p^m} \right)^d \sum_{j=0}^{\infty} \lambda^j \left( \frac{R^p}{(IR^p)^{[p^n]}R^p} \right) e^{-iyj/p^n}
\]

The rightmost quantity on the above equality is \( F_n(R, R, I, d)(y) \).

**Theorem 3.16.** Let \( R \) be a finitely generated \( \mathbb{N} \)-graded \( k \)-algebra, \( I \) be a homogeneous ideal of finite co-length. Let \( M \) be a finitely generated \( \mathbb{Z} \)-graded \( R \)-module of Krull dimension \( d \) and \( Q_1, \ldots, Q_l \) be the \( d \)-dimensional minimal prime ideals in the support of \( M \). Let \( m \) be such that \( \text{nil}(R)^{[p^m]} \) is zero, where \( \text{nil}(R) \) is the nilradical of \( R \).

1. Given a compact subset \( A \subseteq \mathbb{C} \), there exists a constant \( D \) such that for all \( y \in A \),

\[
|F_{n+m}(M, R, I, d)(y) - \sum_{j=1}^{l} \lambda_{R_{Q_j}}(M_{Q_j})F_n \left( \frac{R}{Q_j}, I \right) \left( \frac{R}{Q_j}, d \right)(y) | \leq \frac{D}{p^n}.
\]

2. Given \( y \in \mathbb{C} \), whenever \( (F_n \left( \frac{R}{Q_j}, I \right) \left( \frac{R}{Q_j}, d \right)(y))_n \) is convergent for every \( j \), \( (F_n(M, R, I, d)(y))_n \) is also convergent and

\[
F(M, R, I, d)(y) = \sum_{j=1}^{l} \lambda_{R_{Q_j}}(M_{Q_j})F \left( \frac{R}{Q_j}, I \right) \left( \frac{R}{Q_j}, d \right)(y).
\]

To prove 3.16, we first establish several lemmas to handle the reduced case. The main algebraic input into the proof is the following.

**Lemma 3.17.** Let \( R \) be a reduced finitely generated \( \mathbb{N} \)-graded \( k \)-algebra and let \( Q \) be a minimal prime ideal of \( R \). Let \( U \) be the multiplicative set of homogeneous elements in \( R - Q \). Then,

1. The ideal \( QU^{-1}R \) is zero. Moreover, there is a field \( k \) such that \( U^{-1}R \) is isomorphic to either \( k \) or \( k[t, t^{-1}] \), where \( t \) is an indeterminate over \( k \).
2. Set \( r = \lambda_{R_Q}(N_Q) \). Then there exist integers \( h_1, \ldots, h_r \) and a grading preserving \( R \)-linear morphism

\[
\phi_Q : \bigoplus_{j=1}^{r} \frac{R}{Q}(-h_j) \rightarrow N,
\]

such that the map induced by \( \phi_Q \) after localizing at \( Q \) is an isomorphism.

**Proof.** (1) Any non-zero homogeneous prime ideal of \( U^{-1}R \) is the extension of a homogeneous prime ideal of \( R \) contained in \( R \setminus U \); so is contained in \( Q \). As \( Q \) is minimal, we conclude that \( U^{-1}R \) has a unique prime ideal namely \( QU^{-1}R \). Since \( R \) is reduced, so is \( U^{-1}R \). So \( QU^{-1}R = 0 \). Since \( U^{-1}R \) does not have any non-zero homogeneous prime ideal, every non-zero homogeneous element of \( R \) is a unit. Therefore \( U^{-1}R \) is isomorphic to either \( k \) or \( k[t, t^{-1}] \) for some field \( k \); see [BH98, Lemma 1.5.7].

(2) Since \( R \) is reduced and \( Q \) is a minimal prime, \( R_Q \) is a field. We produce \( r \) homogeneous elements of \( N \), each of which is annihilated by \( Q \) and their images in \( N_Q \) form an \( R_Q \)-basis of \( N_Q \). For that, start with \( r \) homogeneous elements \( m'_1, \ldots, m'_r \) such that \( \{m'_1, \ldots, m'_r\} \) is an \( R_Q \)-basis of \( N_Q \). Since by part (1) \( QU^{-1}R \) is the zero ideal and \( Q \) is finite generated, we can pick an element \( s \) in \( U \) such that \( s \) annihilates \( Q \). Now set \( n_j = sm_j \) for each \( j \). Each \( m_j \) is annihilated by \( Q \). Since \( s \) is not in \( Q \), the images of \( m_1, \ldots, m_r \) in \( N_Q \) form an \( R_Q \)-basis of \( N_Q \).

Now, set \( h_j = \text{deg}(m_j) \). Let

\[
\phi_Q : \bigoplus_{j=1}^{r} \frac{R}{Q}(-h_j) \rightarrow N
\]
be the $R$-linear map sending $1 \in R(\text{inv})$ to $m_j$. Clearly $\phi Q$ preserves gradings. Since the images of $m_1, \ldots, m_r$ form an $R_Q$-basis of $M_Q$, the map induced by $\phi Q$ after localizing at $Q$ is an isomorphism, so our desired conclusion in Lemma 3.18 follows. □

**Lemma 3.18.** Suppose that $R$ is reduced and let $P_1, P_2, \ldots, P_t$ be those among the minimal prime ideals of $R$ such that $\dim(R) = \dim(R/P_j)$. Let $N$ be a finitely generated $\mathbb{Z}$-graded $R$-module. For each $j$, where $1 \leq j \leq t$, let $r_j = \lambda_{R_p}(NP_j)$. Then there exist integers $h_j, n_j$, where $1 \leq j \leq t, 1 \leq n_j \leq r_j$ and a degree preserving $R$-linear map,

$$\phi : \bigoplus_{j=1}^{t} \bigoplus_{n_j=1}^{r_j} R \frac{F_j}{F_j(h_j, n_j)} \to N,$$

such that the $\dim_R(\text{ker}(\phi)) < \dim(R), \dim_R(\text{coker}(\phi)) < \dim(R)$.

**Proof.** Consider for each $j, 1 \leq j \leq t$, a $\phi_{P_j}$ as in assertion (2) of Lemma 3.17. Let $\phi$ be the map induced by these $\phi_{P_j}$'s. Since $P_1, \ldots, P_t$ are all distinct minimal primes, after localizing at any $P_j$, the maps induced by $\phi$ and $\phi_{P_j}$ coincide. So the map induced by $\phi$ after localizing at each $P_j$ is an isomorphism. Hence, none of the supports of kernel and cokernel of $\phi$ include any of $P_1, \ldots, P_t$. Since $P_1, \ldots, P_t$ are precisely the minimal primes of $R$ of maximal dimension, Lemma 3.18 is proved. □

**Proof of Theorem 3.16:** The second assertion follows from the first one; so we just prove the first assertion below.

By Remark 3.7 we can assume that $k^p \subseteq k$ is a finite extension. Using Lemma 3.14 we can replace $(M, R, I)$ by $(M, R_{\text{Ann}(M)}), I_{\text{Ann}(M)}).$ So we assume that $d = \dim(R)$.

First we additionally assume that $R$ is reduced and show that taking $m = 0$ works in assertion (1). By Lemma 3.14, for all $j$ and $n$,

$$F_n \left( \begin{array}{c} R \\ Q_j \\ R \\ Q_j \end{array} \right) \left( \begin{array}{c} R \\ Q_j \\ I \\ Q_j \end{array} \right), d)(y) = F_n \left( \begin{array}{c} R \\ Q_j \\ R \\ I \end{array} \right)(y, d)(y).$$

Assertion (1) follows from direct applications of Lemma 3.18, assertion (1) of Lemma 3.11 and assertion (1) of Proposition 3.13.

We now prove assertion (1) of Theorem 3.16 without assuming $R$ is reduced. We use the Frobenius endomorphism to pass to the reduced case. Pick an $m$ such that $\text{nil}(R)^{p^m} = 0$. Then the kernel of the $m$-th iteration of the Frobenius $F^m : R \to R$ is $\text{nil}(R)$; thus $R^{p^m}$ - the image of $F^{p^m}$ is reduced. Recall $R^{p^m}$ inherits the graded structure of $R$. The $d$-dimensional minimal primes of $R^{p^m}$ in the support of the $R^{p^m}$ module $M$ are precisely of $Q_1, R^{p^m}, \ldots, Q_t, R^{p^m}$ the respective images under the $p^m$-th power map. Since $R^{p^m}$ is reduced and $\frac{1}{p^m} A := \{z/p^m | z \in A\}$ is compact, we can find a $D$ such that for each $y \in A$ and all $n$,

$$(11) \quad |F_n(M, R^{p^m}, I R^{p^m})(y/p^m) - \sum_{j=1}^{l} \lambda_{R^{p^m}, R^{p^m}}((M)_{Q_j}, R^{p^m}) F_n( \left( \begin{array}{c} R^{p^m} \\ Q_j \\ R^{p^m} \end{array} \right), \left( \begin{array}{c} R^{p^m} \\ Q_j \\ I \end{array} \right), d)(y/p^m)| \leq \frac{D}{p^m}.$$

$$\text{(12)} \quad \leq \frac{D}{p^m}.$$

For each $j$, the graded ring $R^{p^m}_{Q_j}$ is isomorphic to the graded subring $(R^{p^m}_{Q_j})_{Q_j} \subseteq R_{Q_j}$; so for all $n$ and $y \in A$,

$$F_n \left( \begin{array}{c} R \\ Q_j \\ p^m \\ R \\ Q_j \\ p^m \\ I \\ Q_j \\ p^m \end{array} \right)(y) = F_n \left( \begin{array}{c} R^{p^m} \\ Q_j \\ R^{p^m} \\ Q_j \\ I \end{array} \right)(y).$$

Since $R^{p^m}_{Q_j}$ is reduced,

$$F_n \left( \begin{array}{c} R \\ Q_j \\ p^m \\ R \\ Q_j \\ I \end{array} \right)(y) = F_n \left( \begin{array}{c} R \\ Q_j \\ I \\ Q_j \\ d \end{array} \right).$$

by Proposition 3.15, (3).

Using Proposition 3.15, assertion (1), we have
\[ p^{md}[k : k^{p^n}]F_{n+m}(M, R, I, d)(y) = F_n(M, R^{p^n}, IR^{p^n}, d)(y/p^n). \]

Since for each \( j \), \( R_{Q_j} \) has Krull dimension \( d \),
\[ \lambda_{R^{p^n}_{Q_j}}((M)_{Q_j}, R^{p^n}) = [k : k^{p^n}]p^d \lambda_{R_{Q_j}}(M_{Q_j}). \]

So Equation (11) yields
\[ |F_{n+m}(M, R, I, d)(y) - \sum_{j=1}^l \lambda_{R_{Q_j}}(M_{Q_j})F_n(R_{Q_j} R_{Q_j} I_{Q_j} R_{Q_j} d)(y)| \leq \frac{D}{p^{dm}[k : k^{p^n}]} \frac{1}{p^n}, \]
proving assertion (1).

**Proof of Theorem 3.1:** Using Remark 3.7 we can assume that \( k^p \subseteq k \) is a finite extension. We argue that the sequence \( (F_n(M, R, I, \dim(M)))_n \) is uniformly Cauchy on every compact subset. By Theorem 3.16, assertion (1), we can assume that \( R \) is a domain and \( M = R \). Fix a compact subset \( A \subseteq \mathbb{C} \). Since the torsion free rank of \( R \) as an \( R^p \) module is \( p^d[k : k^p] \), we have an exact sequence of finitely generated graded \( R^p \) modules (see Lemma 3.18):
\[
0 \longrightarrow K \longrightarrow \bigoplus_{j=1}^{p^d[k : k^p]} R^p(h_j) \longrightarrow R \longrightarrow C \longrightarrow 0
\]
for some integers \( h_i \) such that both \( \dim_{R^p}(K) \) and \( \dim_{R^p}(C) \) are less than \( d \). Hence there exist constants \( D, D' \) such that for all \( n \) and for any \( y \in A \),
\[
|F_{n+1}(R, R, I)(y) - F_n(R, R, I)(y)| = \frac{1}{p^d[k : k^p]} F_n(R, R^p, IR^p)(y/p) - F_n(R, R, I)(y) |
\]
\[
\leq \frac{1}{p^d[k : k^p]} \sum_{j=1}^{p^d[k : k^p]} F_n(R^p(h_j), R^p, IR^p)(y/p) \]
\[
- F_n(R, R, I)(y) + \frac{D}{p^n} \]
\[
\leq |F_n(R^p, R^p, IR^p)(y/p)| - F_n(R, R, I)(y) | + \frac{D'}{p^n} + \frac{D}{p^n} \]
\[
= \frac{D + D'}{p^n}. \]

The first equality comes from assertion 1 of Proposition 3.15. The first inequality is a consequence of assertion (1) of Lemma 3.11. The second inequality is obtained by applying assertion (1) of Proposition 3.13 and assertion (1) of Proposition 3.10. The last equality follows from Proposition 3.15, assertion (3). Hence for \( m, n \in \mathbb{N} \) and for any \( y \in A \),
\[
|F_{n+m}(R, R, I)(y) - F_n(R, R, I)(y)| \leq (D + D') \left( \sum_{j=n}^{\infty} \frac{1}{p^j} \right) = \frac{D + D'}{p^n} \frac{p}{p - 1}. \]

Thus the sequence of entire functions \( (F_n(R, R, I)(y))_n \) is uniformly Cauchy on \( A \).

A sequence of entire functions which is uniformly Cauchy on every compact subset of \( \mathbb{C} \) converges to a entire function and the convergence is uniform on every compact subset; see Theorem 1 in Chapter 5 of [Ahl79]. This finishes the proof of Theorem 3.1. \( \square \)
4. Properties of Frobenius-Poincaré functions

This section is devoted to developing general properties of Frobenius-Poincaré functions. Some of these are analogues of properties of Hilbert-Kunz multiplicities. In Proposition 4.5 and Proposition 4.6, we use these general properties to compute Frobenius-Poincaré functions in some special cases.

Proposition 4.1. Let \( M \) be a finitely generated \( \mathbb{Z} \)-graded \( \mathbb{R} \)-module of Krull dimension \( d \). Then the power series expansion of \( F(M, R, I)(y) \) around the origin in the complex plane is given by

\[
F(M, R, I)(y) = \sum_{m=0}^{\infty} a_m y^m,
\]

where for each \( m \),

\[
a_m = (-i)^m \frac{1}{m!} \lim_{n \to \infty} \left( \frac{1}{p^n} \right)^{d+m} \sum_{j=-\infty}^{\infty} j^m \lambda\left( \frac{M}{I[p^n]M} \right).
\]

Proof. Since the sequence \( (F_n(M))_n \) converges uniformly to \( F(M) \) on the closed unit disc around zero, it follows from Lemma 3, Chapter 4 of [Ahl79] that for each \( m \), the sequence

\[
\frac{d^m}{dy^m}(F_n)(0) = (-i)^m \left( \frac{1}{p^n} \right)^{d+m} \sum_{j=-\infty}^{\infty} j^m \lambda\left( \frac{M}{I[p^n]M} \right)
\]

converges to \( \frac{d^m}{dy^m}(F)(0) \). Since \( a_m = \frac{1}{m!} \frac{d^m}{dy^m}(F)(0) \), we get the result. \( \square \)

Corollary 4.2. The Hilbert-Kunz multiplicity of the triple \((M, R, I)\) is \( F(M, R, I)(0) \).

The next result provides an associativity formula for Frobenius-Poincaré functions.

Theorem 4.3. Let \( M \) be a finitely generated \( \mathbb{Z} \)-graded \( \mathbb{R} \)-module of Krull dimension \( d \). Let \( P_1, \ldots, P_t \) be the dimension \( d \) minimal prime ideals in the support of \( M \). Then

\[
F(M, R, I, d)(y) = \sum_{j=1}^{t} \lambda_{R_{P_j}}(M_{P_j}) F\left( \frac{R}{P_j}, \frac{R}{P_j}, I \frac{R}{P_j}, d \right)(y).
\]

Proof. Follows from Theorem 3.16. \( \square \)

As a consequence of Theorem 4.3, we prove that Frobenius-Poincaré functions are additive over a short exact sequence.

Proposition 4.4. Consider a short exact sequence of finitely generated \( \mathbb{Z} \)-graded \( \mathbb{R} \)-modules where the boundary maps preserve gradings,

\[
0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.
\]

Let \( d \) be the Krull dimension of \( M \). Then \( F(M, R, I, d) = F(M', R, I, d) + F(M'', R, I, d) \).

Proof. The support of \( M \) is the union of supports of \( M' \) and \( M'' \). Since for a \( d \)-dimensional minimal prime \( Q \) in the support of \( M \), \( \lambda_{R_Q}(M_Q) = \lambda_{R_Q}(M'_Q) + \lambda_{R_Q}(M''_Q) \), the desired result follows from Theorem 4.3. \( \square \)

In Proposition 4.5 we apply Theorem 4.3 to compute the Frobenius-Poincaré function with respect to an ideal generated by a homogeneous system of parameters.

Proposition 4.5. Let \( R \) be an \( \mathbb{N} \)-graded, Noetherian ring such that \( R_0 = k \). Let \( I \) be an ideal generated by a homogeneous system of parameters of degrees \( \delta_1, \delta_2, \ldots, \delta_d \). Denote the Hilbert-Samuel multiplicity of \( R \) by \( e_R \)- see Definition 2.9. Then

\[
F(R, I)(y) = e_R \prod_{j=1}^{d} \left( \frac{1 - e^{-i\delta_j y}}{iy} \right).
\]
Proof. Suppose $f_1, \ldots, f_d$ be a homogeneous system of parameters of degrees $\delta_1, \ldots, \delta_d$ respectively, such that $I = (f_1, \ldots, f_d)$. Then the extension of rings $k[f] := k[f_1, \ldots, f_d] \rightarrow R$ is finite (see Theorem 1.5.17, [BH98]). Suppose that the generic rank of $R$ as an $k[f]$ module is $r$. Since $k[f]$ is isomorphic to the graded polynomial ring in $d$ variables where the degrees of the variables are $\delta_1, \ldots, \delta_d$, from Example 3.5 and Theorem 4.3, we have

$$F(R, I)(y) = F(R, k[f], (f_1, \ldots, f_d)k[f]) = r \prod_{j=1}^{d} \left( 1 - e^{-i\delta_j y} \right).$$

Taking limit as $y$ tends to zero in (13) and proposition 4.1, we conclude that $r$ is the Hilbert-Kunz multiplicity of the pair $(R, I)$. The Hilbert-Kunz and the Hilbert-Samuel multiplicities are the same with respect to a given ideal generated by a system of parameters (see Theorem 11.2.10, [HS06]). So using Proposition 2.10 we get that $r = \delta_1 \ldots \delta_d c_R$. □

Now we compute the Frobenius-Poincaré function of a one dimensional graded domain whose degree zero piece is an algebraically closed field. This will indeed allow us to compute the Frobenius-Poincaré function of any one dimensional graded ring by using Remark 3.7 and Theorem 4.3.

**Proposition 4.6.** Let $R$ be a one dimensional finitely generated $\mathbb{N}$-graded $k$-algebra , where $k$ is algebraically closed and $R$ is a domain. Let $I$ be a finite co-length homogeneous ideal. Let $h$ be the smallest integer such that $I$ contains a non-zero homogeneous element of degree $h$. Then

$$F(R, R, I)(y) = e_R \left( 1 - e^{-ihy} \right),$$

where $e_R$ is the Hilbert-Samuel multiplicity of $R$ (see Definition 2.9).

Proof. Let $S$ be the normalization of $R$. By Theorem 11, chapter VII, [ZS60], $S$ is an $\mathbb{N}$-graded $R$-module and by Theorem 9, chapter 5, [ZS65] $S$ is finitely generated over $k$. The generic rank of $S$ as an $R$-module is one, hence $F(S, R, I) = F(R, R, I)$. Since $k$ is algebraically closed $S_0 = k$. So by Lemma 3.14 $F(S, R, I)$ is the same as $F(S, S, IS)$. So we compute $F(S, S, IS)$. Since $S$ is an $\mathbb{N}$-graded normal $k$-algebra, by Theorem 1, section 3, Appendix III of [Ser00], $S$ is isomorphic to a graded polynomial ring in one variable. So the ideal $IS$ is a homogeneous principal ideal. By our assumption, $IS$ is generated by a degree $h$ homogeneous element $f$; note that $f$ is a homogeneous system of parameter of $S$. Thus by Proposition 4.5, $F(S, S, IS)(y) = e_S \left( 1 - e^{-ihy} \right)$. Since $S$ has generic rank one as an $R$ module $e_R = e_S$. □

Let $S$ be a ring containing a field of characteristic $p > 0$. Recall that the tight closure of an ideal $J \subseteq S$ is the ideal consisting of all $x \in S$ such that there is a $c \in S$, not in any minimal prime of $S$ such that, $cJ^p \subseteq J^{p+1}$ for all large $n$–see Definition 3.1, [HH90]). The tight closure of $J$ is denoted by $J^*$. The theory of Hilbert-Kunz multiplicity is related to the theory of tight closure: for a Noetherian local domain $S$ whose completion is also a domain and ideals $J_1 \subseteq J_2$, the corresponding Hilbert-Kunz multiplicities are the same if and only if $J_1^* = J_2^*$ - see Proposition 5.4, Theorem 5.5 of [Hun13] and Theorem 8.17, [HH90]. A similar relation between tight closure of an ideal and the corresponding Frobenius-Poincaré function is the content of the next result.

**Theorem 4.7.** Let $R$ be a finitely generated $\mathbb{N}$- graded $k$-algebra. Let $I \subseteq J$ be two finite colength homogeneous ideals of $R$

1. If $J$ is contained in $I^*$-the tight closure of $I$, $F(R, R, I) = F(R, R, J)$.
2. Suppose that all of the minimal primes of $R$ have the same dimension. If $F(R, R, I) = F(R, R, J)$, $J \subseteq I^*$.

Proof. 1) Denote the Krull dimension of $R$ by $d$. For (1), first we argue that there is a constant $D$ such that $\lambda_j((J^{p_n}))$ is bounded above by $D(p^n)^{d-1}$ for all large $n$. Since $J \subseteq I^*$, there exists a $c \in R$- not in any minimal primes of $R$ such that $cJ^{p^n} \subseteq I^{p^n}$, for all large $n^3$. Pick a set of homogeneous generators of $g_1, g_2, \ldots, g_r$ of

\[c \text{ can be chosen to be homogeneous}\]
J. Since the images of \( g_1^n, \ldots, g_k^n \) generate \( \frac{I^{[p^n]}}{(c, I^{[p^n]})} \), we get a surjection for each \( n \):

\[
\bigoplus_{j=1}^r \frac{R}{(c, I^{[p^n]})} \to \frac{I^{[p^n]}}{I^{[p^n]}}.
\]

So the length of \( \frac{I^{[p^n]}}{I^{[p^n]}} \) is bounded above by \( r\lambda((\frac{R}{(c, I^{[p^n]})})) \). Since \( c \) is not in any minimal prime of \( R \), \( \dim(\frac{R}{(c, I^{[p^n]})}) \) is at most \( d-1 \). The existence of the desired \( D \) is apparent once we use Lemma 3.9 to bound the growth of \( \lambda((\frac{R}{(c, I^{[p^n]})})) \).

Now, pick \( N_0 \in \mathbb{N} \) such that for \( j \geq N_0p^n \), \( \frac{R}{(c, I^{[p^n]})} \) = 0 for all \( n \). Given \( y \in \mathbb{C} \),

\[
|F_n(R, R, I)(y) - F_n(R, R, J)(y)| = \left( \frac{1}{p^n} \right)^d \sum_{j=0}^{\infty} \lambda((\frac{I^{[p^n]}}{I^{[p^n]}})_j) e^{-(j+1)j/p^n} \\
\leq \left( \frac{1}{p^n} \right)^d \sum_{j=0}^{\infty} \lambda((\frac{I^{[p^n]}}{I^{[p^n]}})_j) |e^{-(j+1)j/p^n}| \\
\leq \left( \frac{1}{p^n} \right)^d \lambda((\frac{I^{[p^n]}}{I^{[p^n]}})) e^{N_0|y|}.
\]

To get the last inequality, we have used that for \( j \leq N_0p^n \), \( |e^{-(j+1)j/p^n}| \leq e^{N_0|y|} \). Since

\[
\lim_{n \to \infty} \left( \frac{1}{p^n} \right)^d \lambda((\frac{I^{[p^n]}}{I^{[p^n]}})) \leq \lim_{n \to \infty} \frac{1}{p^n}D = 0,
\]

we get \( F(R, R, I)(y) = F(R, R, J)(y) \).

(2) Let \( P_1, \ldots, P_t \) be the minimal primes of \( R \). For a finite co-length homogeneous ideal \( a \), denote the Hilbert-Kunz multiplicity of the triple \((R, R, a)\) (see Definition 2.1) by \( e_{HK}(R, a) \). Since all the minimal primes of \( R \) have the same dimension, evaluating the equality in Theorem 4.3 at \( y = 0 \) and using Corollary 4.2, we get

\[
e_{HK}(R, a) = \sum_{j=1}^t e_{HK}(\frac{R}{P_j}, a\frac{R}{P_j}).
\]

Since for each \( j \), where \( 1 \leq j \leq t \), \( e_{HK}(\frac{R}{P_j}, I\frac{R}{P_j}) \geq e_{HK}(\frac{R}{P_j}, J\frac{R}{P_j}) \) and \( e_{HK}(R, I) = F(R, R, I)(0) = e_{HK}(R, J) \), using (14), we conclude that for each minimal prime \( P_j \), \( e_{HK}(\frac{R}{P_j}, I\frac{R}{P_j}) = e_{HK}(\frac{R}{P_j}, J\frac{R}{P_j}) \). From here we show that the tight closure of \( I\frac{R}{P_j} \) and \( J\frac{R}{P_j} \) in \( \frac{R}{P_j} \) are the same for any \( j \); this coupled with Theorem 1.3, (c) of [Hun96] establishes that \( I^* = J^* \). To this end, fix a minimal prime \( P_j \). First note that \( \frac{R}{P_j} \): the completion of \( \frac{R}{P_j} \) at the homogeneous maximal ideal is a domain. To see this, set \( I_n \subseteq \frac{R}{P_j} \) to be the ideal generated by forms of degree at least \( n \). Then the associated graded ring of \( \frac{R}{P_j} \) with respect to the filtration \( (I_n\frac{R}{P_j})_n \) is isomorphic to the domain \( \frac{R}{P_j} \) - so by Theorem 4.5.8, [BH98] \( \frac{R}{P_j} \) is a domain. Set \( m_j \) to be the maximal ideal of \( \frac{R}{P_j} \). Since \( \frac{R}{P_j} \) is a domain, by Theorem 5.5, [Hun13] \( (\frac{R}{P_j})^* m_j = (J^{(\frac{R}{P_j})})^* \). Since both \( I \) and \( J \) are \( m_j \)-primary, by Theorem 1.5, [Hun96], we conclude that the tight closures of \( I\frac{R}{P_j} \) and \( J\frac{R}{P_j} \) in \( \frac{R}{P_j} \) are the same. □

Next, we set to show that over a standard graded ring, our Frobenius-Poincaré functions are holomorphic Fourier transforms of Hilbert-Kunz density functions introduced in [Tri18]. We first recall a part of a result in [Tri18] that implies the existence of Hilbert-Kunz density functions.

**Theorem 4.8.** (see Theorem 1.1 and Theorem 2.19 [Tri18]) Let \( k \) be a field of characteristic \( p > 0 \), \( R \) be a standard graded \( k \)-algebra of Krull dimension \( d \geq 1 \), \( I \) be a homogeneous ideal of finite co-length. Given a finitely generated \( \mathbb{N} \)-graded \( R \)-module \( M \), consider the sequence \((g_n)_n \) of real valued functions defined on the real line where

\[
g_n(x) = \left( \frac{1}{p^n} \right)^{d-1} \lambda_k((\frac{M}{I^{[p^n]}})_x^{p^n}).
\]
Then

1. There is a compact subset of the non-negative real line containing the support of \( g_n \) for all \( n \).
2. The sequence \( (g_n) \) converges pointwise to a compactly supported function \( g \). Furthermore, when \( d \geq 2 \), the convergence is uniform and \( g \) is continuous.

The function \( g \) in Theorem 4.8 is called the Hilbert-Kunz density function associated to the triple \((M, R, I)\).

Recall that the **holomorphic Fourier transform** of a compactly supported Lebesgue integrable function \( h \) defined on the real line is the holomorphic function \( \hat{h} \) given by

\[
\hat{h}(y) = \int_{\mathbb{R}} h(x)e^{-iyx} \, dx,
\]

where the integral is a Lebesgue integral (see Chapter 2, [Rud87]).

**Proposition 4.9.** The holomorphic Fourier transform of the Hilbert-Kunz density function associated to a triple \((M, R, I)\) as in Theorem 4.8 is the Frobenius-Poincaré function \( F(M, R, I, d) \).

**Proof.** Let \( g_n \) and \( g \) be as in Theorem 4.8. We first establish the claim that there is a constant \( C \), such that for any real number \( x \) and all \( n \), \( g_n(x) \leq C \). We can assume that there is compact subset \([0, N]\) containing the support of \( g_n \) for all \( n \) (see (1), Theorem 4.8). Now given \( x \) where \( \frac{1}{p^n} \leq x \leq N \),

\[
g_n(x) = \frac{1}{(p^n)^d-1} \lambda(M_{[x^n]}) = \left( \frac{[x^n]}{p^n} \right)^{d-1} \lambda(M_{[x^n]}) \leq N^{d-1} \lambda(M_{[x^n]}) \left( \frac{[x^n]}{p^n} \right)^{d-1}.
\]

Since the function \( \frac{\lambda(M_{[x^n]})}{m^{d-1}} \) is bounded above by a constant (see Proposition 4.4.1 and Exercise 4.4.11 of [BH98]), the claim follows.

The bound on \( g_n \) allows us to use dominated convergence theorem to the sequence \( (g_n)_n \), which implies that the sequence of functions \( (g_n)_n \) converges to \( \hat{g} \) pointwise. Now we claim that the sequence \( (\hat{g}_n) \) in fact converges to \( F(M, R, I, d) \) pointwise; this would imply \( \hat{g} = F(M, R, I, d) \).

Now for a non-zero complex number \( y \),

\[
g_n(y) = \left( \frac{1}{p^n} \right)^{d-1} \int_{0}^{\infty} \lambda_k\left( \frac{M}{[x^n]} \right) e^{-iyx} \, dx
\]

\[
= \left( \frac{1}{p^n} \right)^{d-1} \sum_{j=0}^{\infty} \int_{\mathbb{R}} \lambda_k\left( \frac{M}{[x^n]} \right) e^{-iyx} \, dx
\]

\[
= \left( \frac{1}{p^n} \right)^{d-1} \sum_{j=0}^{\infty} \lambda_k\left( \frac{M}{[x^n]} \right) e^{-iyx} \left( \frac{e^{-iy(n+j)/p^n} - e^{-iyj/p^n}}{-iy} \right)
\]

\[
= \left( \frac{1}{p^n} \right)^{d-1} \sum_{j=0}^{\infty} \lambda_k\left( \frac{M}{[x^n]} \right) e^{-iyj/p^n} \left( \frac{e^{-iy/n} - 1}{-iy} \right)
\]

\[
= \left( \frac{1}{p^n} \right)^{d} \sum_{j=0}^{\infty} \lambda_k\left( \frac{M}{[x^n]} \right) e^{-iyj/p^n} \left( \frac{e^{-iy/n} - 1}{-iy} \right).
\]

So using the last line of (16) and Lemma 3.3, we get that for a non-zero complex number \( y \),

\[
\hat{g}(y) = \lim_{n \to \infty} \hat{g}_n(y) = \lim_{n \to \infty} \left( \frac{1}{p^n} \right)^{d} \sum_{j=0}^{\infty} \lambda_k\left( \frac{M}{[x^n]} \right) e^{-iyj/p^n} = F(M, R, I, d)(y).
\]

Note that,

\[
g_n(0) = \left( \frac{1}{p^n} \right)^{d} \sum_{j=0}^{\infty} \lambda\left( \frac{M}{[x^n]} \right) = \left( \frac{1}{p^n} \right)^{d} \lambda\left( \frac{M}{[x^n]} \right) = F_n(M, d)(0).
\]

Taking limit as \( n \) approaches infinity in (17) gives \( \hat{g}(0) = F(M, R, I, d)(0) \). 

\(\square\)
Remark 4.10. (1) Since a compactly supported continuous function can be recovered from its holomorphic Fourier transform (see Theorem 1.7.3, [Hor65]), the existence of Frobenius-Poincaré functions gives an alternate proof of the existence of Hilbert-Kunz density functions in dimension $d \geq 2$.

(2) One way to incorporate zero dimensional ambient rings into the theory of Hilbert-Kunz density functions could be to realize the functions $g_n$ in (15) and the resulting Hilbert-Kunz density function as compactly supported distributions (see Definition 1.3.2, [Hor65]). Here by a distribution, we mean a $\mathbb{C}$-linear map from the space of complex valued smooth functions on $\mathbb{R}$ to $\mathbb{C}$. In our case, the distribution defined by each $g_n$ sends the function $f$ to $\int f(x)g_n(x)dx$. When the ambient ring has dimension at least one, the sequence of distributions defined $(g_n)_n$ converges to the distribution defined by the corresponding Hilbert-Kunz density function; see the Remark on page 7 of [Hor65] for a precise meaning of convergence of distributions. Now suppose that $R$ has dimension zero and $M$ is a finitely generated $\mathbb{Z}$-graded $R$-module; let $(g_n)_n$ be the corresponding sequence of functions given by (15) with $d = 0$. Direct calculation shows that for a complex valued smooth function $f$, the sequence of numbers $\int f(x)g_n(x)dx$ converges to $\lambda_k(M)f(0)$. This means that the sequence of distributions defined by $(g_n)_n$ converges to the distribution $\lambda_k(M)\delta_0$ where $\delta_0$ is the distribution such that $\delta_0(f) = f(0)$. So it is reasonable to define the Hilbert-Kunz density function $g(M, R, I)$ to be the distribution $\lambda_k(M)\delta_0$. In fact, incorporating the language of Fourier transform of distributions (see section 1.7, [Hor65]), it follows that the Fourier transform of the Hilbert-Kunz density function (or distribution) is our Frobenius-Poincaré function irrespective of the dimension of the ambient ring. Going in the reverse direction, Hilbert-Kunz density function of a triple can be defined to be the unique compactly supported distribution whose Fourier transform is the corresponding Frobenius-Poincaré function.

5. Descriptions using Homological Information

In this section, we give alternate descriptions of Frobenius-Poincaré functions of $(R, R, I)$ in terms of the sequence of graded Betti numbers of $R/I^{[p^n]}$. Moreover when $R$ is Cohen-Macaulay, the Frobenius-Poincaré functions are described using the Koszul homologies of $\frac{R}{I^{[p^n]}}$ with respect to a homogeneous system of parameters of $R$. Some background material for this section on Hilbert-Samuel multiplicity, Hilbert series and graded Betti numbers is reviewed in Section 2.2 and Section 2.3.

Theorem 5.1. Let $S$ be a graded complete intersection over $k$ of Krull dimension $d$ and Hilbert-Samuel multiplicity $e_S$ (see Definition 2.9). Let $S \to R$ be a module finite $k$-algebra map to a finitely generated $\mathbb{N}$-graded $k$-algebra. Let $I \subseteq R$ be a homogeneous ideal of finite co-length and $M$ be a finitely generated $\mathbb{Z}$-graded $R$-module. Set

$$B^S(j, n) = \sum_{\alpha=0}^{\infty} (-1)^\alpha \lambda((\text{Tor}_\alpha^R(k, M/I^{[p^n]}M)_j)).$$

Then

$$\lim_{n \to \infty} (p^n)^{d-\dim M} \sum_{j=0}^{\infty} B^S(j, n) e^{-ijy/p^n}$$

admits an analytic extension to the complex plane.

(2) The Frobenius-Poincaré function $F(M, R, I)(y)$ is the same as the analytic extension of the function

$$\frac{e_S}{(iy)^d} \lim_{n \to \infty} (p^n)^{d-\dim M} \sum_{j=0}^{\infty} B^S(j, n) e^{-ijy/p^n}$$

to the complex plane.

Note that for fixed integers $j, n$ the sum in (18) is finite- see Lemma 2.5. We record some remarks and consequences related to Theorem 5.1 before proving the result.

Corollary 5.2. For a graded complete intersection $R$ over $k$ and a homogeneous ideal $I \subseteq R$ of finite co-length, the function

$$\lim_{n \to \infty} \sum_{j=0}^{\infty} B^S(j, n) e^{-ijy/p^n}$$

extends to the entire function

$$\frac{1}{e_R} \frac{1}{(iy)^{\dim(R)}} F(R, R, I)(y).$$

Remark 5.3. When applied to the triple $(R, R, m)$, where $m$ is the homogeneous maximal ideal of a graded complete intersection $R$ over $k$, Theorem 5.1 applied to the case $S = R = M$, gives a way to compare the Hilbert-Kunz multiplicity of $(R, m)$ to the Hilbert-Samuel multiplicity $e_R$. 
Remark 5.4. One way to apply Theorem 5.1 to describe the Frobenius-Poincaré function of a graded triple $(M, R, I)$ is to take $S$ to be a subring of $R$ generated by a homogeneous system of parameters. Since such an $S$ is regular, for any integer $n$, the sum defining $B^S(j, n)$ in (18) is finite and the function $\sum_{j=0}^{\infty} B^S(j, n) e^{-iyj/p^n}$ appearing in Theorem 5.1 is a polynomial in $e^{-iy/p^n}$.

Remark 5.5. In [Tri21, page 7] V. Trivedi asks whether the Hilbert-Kunz density function (see Theorem 4.8) of a $d(\geq 2)$ dimensional standard graded pair $(R, I)$ is always $(d-2)$ times differentiable and the $(d-2)$-th order derivative is continuous. We use Theorem 5.1 to reformulate Trivedi’s question and produce the candidate function for the $(d-2)$-th order derivative. Denote the restriction of $(iy)^{d-2}F(R, R, I)(y)$ to the real line by $h$. The Fourier transform of the temperate distribution (see Definition 1.7.2, 1.7.3, [Hor65]) defined by $h$ determines the $(d-2)$-th order derivative of the distribution (see Definition 1.4.1, [Hor65]) defined by the Hilbert-Kunz density function of $(R, I)$- see Remark 4.10, Theorem 1.7.3, [Hor65]. In fact using tools from analysis, one can show that Trivedi’s question has an affirmative answer if the integral $\int |h(x)| dx$ is finite. When $\int |h(x)| dx$ is finite, the Fourier transform of $h$ is in fact given by the actual function $h(y) = \int h(x)e^{-iyx} dx$ for $y \in \mathbb{R}$ and the $(d-2)$-th order derivative is the function $\frac{1}{2\pi} h(-y)$. Fix a subring $S \subseteq R$ generated by a homogeneous system of parameters of $R$. Since by Theorem 5.1 applied to the case $M = R$ (also see Remark 5.4) $h(y) = e_S \lim_{n \to \infty} \frac{\sum_{j=0}^{\infty} B^S(j, n) e^{-iyj/p^n}}{(iy)^j}$, it is natural to ask

Question 5.6. (1) Is the function $h(y) = e_S \lim_{n \to \infty} \frac{\sum_{j=0}^{\infty} B^S(j, n) e^{-iyj/p^n}}{(iy)^j}$ integrable on $\mathbb{R}$?

(2) Is function $\lim_{n \to \infty} \sum_{j=0}^{\infty} B^S(j, n) e^{-iyj/p^n}$ restricted to the real line bounded?

Note that an affirmative answer to part (2) implies an affirmative answer to part (1) of the Question 5.6.

We use a consequence of a result from [AB93]-where it is cited as a folklore- in the proof of Theorem 5.1 below.

Proposition 5.7. (see Lemma 7.ii, [AB93]) Let $R$ be a finitely generated $\mathbb{N}$-graded $k$-algebra and $M, N$ be two finitely generated $\mathbb{Z}$-graded $R$-modules. Denote the formal Laurent series $\sum_{i \in \mathbb{N}} (-1)^i H_{Tor^R_i(M, N)}(t)$ by $\chi^R(M, N)(t)$. Then

$$\chi^R(M, N)(t) = \frac{H_M(t)H_N(t)}{H_R(t)},$$

where for a finitely generated $\mathbb{Z}$-graded $R$-module $N'$, $H_{N'}(t)$ is the Hilbert series of $N'$.

Proof of Theorem 5.1: Let $\mathfrak{H}$ be the set of complex numbers with a negative imaginary part. We shall prove that on the connected open subset $\mathfrak{H}$ of the complex plane

$$\frac{e_S}{(iy)^d} \lim_{n \to \infty} (p^n)^{-d} \dim M \sum_{j=0}^{\infty} B^S(j, n) e^{-iyj/p^n}$$

defines a holomorphic function and is the same as the restriction of $F(R, R, I)$ to $\mathfrak{H}$. Since $F(R, R, I)$ is an entire function, the analytic continuity in assertion 1 and the desired equality in assertion 2 follows.

Given an integer $n$, $\chi^S(M, \frac{k}{(mp+1)m}, k)(t) = \sum_{j=-\infty}^{\infty} B^S(j, n)t^j$. So using Proposition 5.7 we get

$$H_{M, \frac{k}{mp+1}M}(t) = H_S(t)(\sum_{j=-\infty}^{\infty} B^S(j, n)t^j).$$
Now for any $y \in \mathcal{H}$, $|e^{-iy/p^n}| < 1$; so by Lemma 2.6, the series $\sum_{j \in \mathbb{Z}} B^S(j, n)(e^{-iy/p^n})^j$ converges absolutely. For $y \in \mathcal{H}$, plugging in $t = e^{-iy/p^n}$ in (19), we get

$$
\left( \frac{1}{p^n} \right)^{\dim(M)} H_{M^{p^nM}}(e^{-iy/p^n}) = \left( \frac{1}{p^n} \right)^d H_S(e^{-iy/p^n})(p^n)^{d-\dim(M)} \left( \sum_{j=-\infty}^{\infty} B^S(j, n)e^{-iyj/p^n} \right)
$$

(20)

$$
= \frac{(1 - e^{-iy/p^n})^d}{(p^n(1 - e^{-iy/p^n}))^d} H_S(e^{-iy/p^n})(p^n)^{d-\dim(M)} \left( \sum_{j=-\infty}^{\infty} B^S(j, n)e^{-iyj/p^n} \right)
$$

For a fixed $y \in \mathcal{H}$, as $n$ approaches infinity, $(1 - e^{-iy/p^n})^d H_S(e^{-iy/p^n})$ approaches $e_S$ (see Proposition 2.8) and $(p^n(1 - e^{-iy/p^n}))^d$ approaches $(iy)^d$ (see Lemma 3.3). Now taking limit as $n$ approaches infinity in (20) gives the following equality on $\mathcal{H}$:

$$
F(R, R, I)(y) = \frac{e_S}{(iy)^d} \lim_{n \to \infty} (p^n)^{d-\dim M} \sum_{j=0}^{\infty} B^S(j, n)e^{-iyj/p^n}.
$$

Since the left hand side of the last equation is holomorphic on $\mathcal{H}$, so is the right hand side; this finishes the proof.

**Remark 5.8.** Take $S = R = M$ in Theorem 5.1 and let $\mathcal{H}$ be the same as in the proof of Theorem 5.1. Although the analyticity of $\sum_{j=0}^{\infty} B(j, n)e^{-iy/p^n}$ on $\mathcal{H}$, for each $n$, follows from Lemma 2.6, the existence of the analytic extension of their limit crucially depends on Theorem 5.1 and that Frobenius-Poincaré functions are entire.

When the $R$-module $R/I$ has finite projective dimension, the line of argument in Theorem 5.1 (also see [TW22]) allows to describe $F(R, R, I)$ in terms of the graded Betti numbers of $R/I$.

**Proposition 5.9.** Let $I$ be a homogeneous ideal of the $d$ dimensional ring $R$, such that the projective dimension of the $R$-module $R/I$ is finite. Set

$$
b_{\alpha, j} = \lambda(Tor^R_{\alpha}(k, R/I)_j), \quad B(j) = \sum_{\alpha=0}^{\infty} (-1)^\alpha b_{\alpha, j}, \quad e_R = Hilbert-Samuel \ multiplicity \ of \ R.
$$

Let $b$ be the smallest integer such that $B(j) = 0$ for all $j > b$. Then for a non-zero complex number $y$, we have:

$$
F(R, R, I)(y) = \frac{\sum_{j=0}^{b} B(j)e^{-iyj}}{(iy)^d}.
$$

**Proof.** Take a minimal graded free resolution of the $R$-module $R/I$:

$$
0 \to \bigoplus_{j \in \mathbb{N}} R(-j)^{\oplus b_{d, j}} \to \cdots \to \bigoplus_{j \in \mathbb{N}} R(-j)^{\oplus b_{1, j}} \to \bigoplus_{j \in \mathbb{N}} R(-j)^{\oplus b_{0, j}} \to R/I \to 0.
$$

Then we get a minimal graded free resolution of $R/I[p^n]$ by applying $n$-th iteration of the Frobenius functor to the chosen minimal graded resolution of $R/I$ (see Theorem 1.13 of [PS73]),

$$
0 \to \bigoplus_{j \in \mathbb{N}} R(-p^n j)^{\oplus b_{d, j}} \to \cdots \to \bigoplus_{j \in \mathbb{N}} R(-p^n j)^{\oplus b_{0, j}} \to R/I[p^n] \to 0.
$$

So using the notation set in (18) in the case $S = R = M$ and the ideal $I$, we have that for any positive integer $n$, $B^R(jp^n, n) = B(j)$ and $B(m, n) = 0$ if $p^n$ does not divide $m$. So for all $n \in \mathbb{N}$, $\chi^R(R_{I[p^n]}, k)(t) =$
\[
\sum_{j \in \mathbb{N}} \mathbb{B}(j) t^j = \sum_{j=0}^{n} \mathbb{B}(j) t^{jp^n} \text{ is a polynomial in } t. \text{ So for any } n \in \mathbb{N}, \text{ using Proposition 5.7 for } M = R/I[p^n], \quad N = k \text{ we have for all } y \in \mathbb{C},
\]
\[
(21) \quad \frac{1}{p^n} h_{R/I[p^n]}(e^{-iy/p^n}) = \frac{(1 - e^{-iy/p^n})^d H_R(e^{-iy/p^n})}{(p^n(1 - e^{-iy/p^n}))^d} \left(\sum_{j=0}^{b} \mathbb{B}(j)e^{-iyj}\right).
\]

Now taking limit as \( n \) approaches infinity in (21) and using Proposition 2.8 and Lemma 3.3, we get
\[
F(R, R, I)(y) = e_{R} \frac{\sum_{j=0}^{b} \mathbb{B}(j)e^{-iyj}}{(iy)^d}.
\]

**Example 5.10.** Let \( R = k[X, Y] \) be the standard graded polynomial ring in two variables, \( I = (f, g)R \) where \( f \) and \( g \) have degree \( d_1 \) and \( d_2 \) respectively. Using Proposition 5.9, we can compute \( F(R, I)(y) \). A minimal free resolution of \( R/I \) is given by the Koszul complex of \((f, g)\):
\[
0 \to R(-d_1 - d_2) \to R(-d_1) \oplus R(-d_2) \to R \to 0.
\]
Hence we get,
\[
F(R, I)(y) = e_{R} \frac{\mathbb{B}(0) + \mathbb{B}(d_1)e^{-iyd_1} + \mathbb{B}(d_2)e^{-iyd_2} + \mathbb{B}(d_1 + d_2)e^{-iy(d_1 + d_2)}}{(iy)^2} = 1 - e^{-iyd_1} - e^{-iyd_2} + e^{-iy(d_1 + d_2)}.
\]

The Hilbert-Kunz multiplicity \( e_{HK}(R, I) = d_1d_2 \) (see for example Example 11.2.10 of [HS06]). Using this observation, we can construct finite co-length ideals \( I \) and \( J \) in \( R \) such that, \( e_{HK}(R, I) = e_{HK}(R, J) \) but \( F(R, R, I) \) and \( F(R, R, J) \) are different.

In the next result, we show that the Frobenius-Poincaré function of a Cohen-Macaulay ring can be described in terms of the sequence of Koszul homologies of \( R/p^nR \) with respect to a homogeneous system of parameters.

**Theorem 5.11.** Let \( R \) be a Cohen-Macaulay \( \mathbb{N} \)-graded ring of dimension \( d \), \( I \) be a homogeneous ideal of finite co-length of \( R \). Let \( x_1, x_2, \ldots, x_d \) be a homogeneous system of parameters of \( R \) of degree \( \delta_1, \ldots, \delta_d \) respectively. Then
\[
F(R, I)(y) = \frac{1}{\delta_1 \delta_2 \ldots \delta_d} \lim_{n \to \infty} \chi^R(R/I[p^nR]) \frac{R}{(x_1, \ldots, x_d)R} \frac{R}{(x_1, \ldots, x_d)R} (e^{-iy/p^n}),
\]
where \( \chi^R(R/I[p^nR]) \) has the same meaning as in Proposition 5.7.

**Remark 5.12.** The Laurent series \( \chi^R(M, N)(t) \) for a pair of graded modules as defined in Proposition 5.7 has been used before to define multiplicity or intersection multiplicity in different contexts; see for example [Ser00] Chapter IV, A, Theorem 1 and [Erm17]. The assertion in Theorem 5.11 should be thought of as an analogue of the results since here the Frobenius-Poincaré function and hence the Hilbert-Kunz multiplicity is expressed in terms of the limit of power series \( \chi^R(R/I[p^nR]) \).

**Proof of Theorem 5.11:** Using Proposition 5.7 we have,
\[
(23) \quad h_{R/I[p^nR]}(e^{-iy/p^n}) = \frac{R}{(x_1, \ldots, x_d)R} \frac{R}{(x_1, \ldots, x_d)R} (e^{-iy/p^n}).
\]

Since \( R \) is Cohen-Macaulay, \( x_1, \ldots, x_d \) is a regular sequence. Inducing on \( d \), one can show that,
\[
(24) \quad h_{R/I[p^nR]}(t) = (1 - t^{\delta_1})(1 - t^{\delta_2}) \ldots (1 - t^{\delta_d}) H_R(t).
\]

Using (24) in (23) we get
\[
(25) \quad \frac{1}{p^n} \chi^R(R/I[p^n]) = \frac{\chi^R(R/I[p^n])}{(p^n)^d (1 - e^{-iy\delta_1/p^n}) \ldots (1 - e^{-iy\delta_d/p^n})}.
\]
The desired assertion now follows from taking limit as $n$ approaching infinity in the last equation and using Lemma 3.3.

In each of Theorem 5.1, Proposition 5.9, Theorem 5.11, the Frobenius-Poincaré function is described as a quotient: the denominator is a power of $iy$ and the numerator is a limit of a sequence of power series or polynomials in $e^{-iy/p^n}$. In particular, in Theorem 5.11, the maximum value of $j/p^n$ that appears in $e^{-iyj/p^n}$ in the sequence of functions is bounded above by a constant independent of $n$—see Lemma 3.8. So we ask

**Question 5.13.** Let $R$ be a Cohen-Macaulay $\mathbb{N}$-graded ring of dimension $d$, $I$ is a homogeneous ideal of finite co-length. Does there exist a real number $r$ and a polynomial $Q \in \mathbb{R}[X]$ such that

$$F(R, R, I)(y) = \frac{Q(e^{-iry})}{(iy)^d}?$$

### 6. Frobenius-Poincaré functions in dimension two

We compute the Frobenius-Poincaré function of two dimensional graded rings following the work of [Bre07] and [Tri05]. In this section, $R$ stands for a normal, two dimensional, standard graded domain. We assume that $R_0 = k$ is an algebraically closed field of prime characteristic $p$. The smooth embedded curve $\text{Proj}(R)$ is denoted by $C$, $\delta_R$ stands for the Hilbert-Samuel multiplicity of $R$; alternatively $\delta_R$ is the degree of the line bundle $\mathcal{O}_C(1)$. The genus of $C$ is denoted by $g$. For a sheaf of $\mathcal{O}_C$-modules $\mathcal{F}$, $\mathcal{F}(j)$ stands for the sheaf $\mathcal{F} \otimes \mathcal{O}_C(j)$. The absolute Frobenius endomorphism of $C$ is denoted by $f$. Some background materials for this section are reviewed in Section 2.4.

**Theorem 6.1.** With notation as in the paragraph above, let $I$ be an ideal of finite co-length in $R$ generated by degree one elements $h_1, h_2, \ldots, h_r$. Consider the short exact sequence of vector bundles on $C = \text{Proj}(R)$

$$0 \to S \to \bigoplus_{j=1}^r \mathcal{O}_C \to \mathcal{O}_C(1) \to 0. \tag{25}$$

Choose $n_0$ such that the Harder-Narasimhan filtration on $f^{n_0*}(S)$ given by

$$0 = E_0 \subset E_1 \subset \ldots \subset E_t \subset E_{t+1} = f^{n_0*}S \tag{26}$$

is strong \footnote{That is the pull back of the Harder-Narasimhan filtration on $(f^{n_0})^*S$ via $f^{n-n_0}$ gives the the Harder-Narasimhan filtration on $(f^n)^*S$—see Theorem 2.14, Theorem 2.17.}. For any $1 \leq s \leq t+1$, set $\mu_s$ to be the normalized slope $\mu(E_s/E_{s-1})$ of the factor $E_s/E_{s-1}$ and set $r_s$ to be its rank $\text{rk}(E_s/E_{s-1})$ (see Definition 2.11). Then

$$F(R, I)(y) = \delta_R \frac{1 - (1 + \mu(R))e^{-iy} + \sum_{j=1}^{t+1} r_j e^{-iy(1 - \mu_j/p^n)}}{(iy)^2} \tag{27}$$

**Remark 6.2.** The two relations $\text{rk}(S) = \sum_{j=1}^{t+1} r_j$ and $\sum_{j=1}^{t+1} \mu_j r_j = -\delta_R$ imply that the numerator of the right hand side of (27) has a zero of order two at the origin. So the right hand side of (27) is holomorphic at the origin. Conversely, the holomorphicity of the Frobenius-Poincaré function and the equality (27) for non-zero complex numbers reveal the two relations.

The key steps in the proof of Theorem 6.1 are Lemma 6.3 and Lemma 6.4. The standard reference for results on sheaf cohomology used here is [Har97].

**Lemma 6.3.** For $j > 2g - 2$ and for all $n$, we have

$$\lambda\left(\frac{R}{f \mathcal{O}_C} \right)^{p^n+j} = h^1(C, (f^{n*}S)(j)) \tag{28}$$

**Proof.** Given natural numbers $n$ and $j$, first pulling back (25) via $f^n$ and then tensoring with $\mathcal{O}_C(j)$, we get a short exact sequence:

$$0 \to (f^{n*}S)(j) \to \bigoplus_{j=1}^r \mathcal{O}_C(j) \to \mathcal{O}_C(p^n+j) \to 0. \tag{29}$$
Note that, since $R$ is normal, for each $j$, the canonical inclusion $R_j \subset h^0(O_C(j))$ is an isomorphism - see Exercise 5.14 of [Har97]. Also for $j > 2g-2$, $h^1(O_C(j)) = 0$ (see Example 1.3.4 of [Har97]). So the long exact sequence of sheaf cohomologies corresponding to (29) gives (28).

**Lemma 6.4.** Fix $s$ such that $1 \leq s \leq t$. For all large $n$, if an index $j$ satisfies

\[ -p^n \frac{\mu_s}{\delta_R} + \frac{2g - 2}{\delta_R} < j < -p^n \frac{\mu_{s+1}}{\delta_R}, \]

then

\[ h^1(f^n*(S)(j)) = -p^n(\sum_{b=s}^t \mu_{b+1}r_{b+1}) + (\sum_{b=s}^t r_{b+1})(g - 1 - j\delta_R). \]

And for $j > -p^n \frac{\mu_{s+1}}{\delta_R}$,

\[ h^1(f^n*(S)(j)) = 0. \]

The only reason for choosing large $n$ is to ensure that for all $s$

\[ -p^n \frac{\mu_s}{\delta_R} + \frac{2g - 2}{\delta_R} < -p^n \frac{\mu_{s+1}}{\delta_R}. \]

**Proof.** We introduce some notation below which are used in this section.

Set $K_j = \ker((f^n* S)^{\vee} \to E_{t+1-j})$. Then by Lemma 2.16, there is a HN filtration-

\[ 0 = K_0 \subset K_1 \subset \ldots K_t \subset K_{t+1} = (f^n* S)^{\vee} \]

**Claim.** Denote the sheaf of differentials of $C$ by $\omega_C$. For $j$ as in (30), we have

\[ (1) \quad h^1(f^n*(S)(j)) = h^0(f^{n-n_0*}(K_{t+1-s}) \otimes \omega_C(-j)) \quad \text{and} \quad (2) \quad h^1(f^{n-n_0*}(K_{t+1-s}) \otimes \omega_C(-j)) = 0. \]

We defer proving the above claim until deriving Lemma 6.4 from it.

Combining (33) and the Riemann-Roch theorem on curves (see Theorem 2.6.9, [Pot97]), we get that for the range of values as in (30),

\[ h^1(f^n*(S)(j)) = h^0(f^{n-n_0*}(K_{t+1-s}) \otimes \omega_C(-j)) - h^1(f^{n-n_0*}(K_{t+1-s}) \otimes \omega_C(-j)) \]

\[ = \deg(f^{n-n_0*}(K_{t+1-s}) \otimes \omega_C(-j)) + (1 - g) \text{rk}(f^{n-n_0*}(K_{t+1-s}) \otimes \omega_C(-j)) \]

\[ = \deg(f^{n-n_0*}K_{t+1-s}) + \text{rk}(K_{t+1-s}) \cdot \deg(\omega_C(-j)) + (1 - g) \text{rk}(K_{t+1-s}) \quad \text{(using, (1), Lemma 2.15).} \]

Since $K_{t+1-s}$ is the kernel of a surjection $(f^{n_0*}S)^{\vee} \to E_{s}^{\vee}$, we have,

\[ \deg(f^{n-n_0*}K_{t+1-s}) = p^n - \deg(K_{t+1-s}) = p^n - \text{deg}(f^{n_0*}S) - \deg(E_{s}^{\vee}) \]

\[ = p^n - \text{deg}(f^{n_0*}S) + \deg(E_{s}) \]

\[ = p^n - \sum_{b=s}^t \text{deg}(E_{b+1}/E_{b}) = p^n \sum_{b=s}^t \mu_{b+1}r_{b+1}. \]

Similarly one can compute the rank of $K_{t+1-s}$.

\[ \text{rk}(K_{t+1-s}) = \text{rk}(f^{n_0*}S)^{\vee} - \text{rk}(E_{s}^{\vee}) = \text{rk}(f^{n_0*}S) - \text{rk}(E_{s}) \]

\[ = \sum_{b=s}^t \text{rk}(E_{b+1}/E_{b}) = \sum_{b=s}^t r_{b+1}. \]
Now the desired conclusion follows from combining (34), (35), (36) and noting that $\deg(\omega_C(-j)) = 2g - 2 - j\delta_R$.

**Proof of Claim:** By Serre Duality (see Corollary 7.7, Chapter III, [Har77]), $h^1(f_*(S)(j)) = h^0((f_*(S)^\vee \otimes \omega_C(-j))$. We prove (1) by showing that if the left most inequality in (30) holds, the cokernel of the inclusion

$$H^0(f^n_{n-nS}(K_{t+1-s}) \otimes \omega_C(-j)) \subseteq H^0((f^n_{nS})^\vee \otimes \omega_C(-j))$$

is zero. For this, first note that by Lemma 2.15, (3), there is a HN filtration -

$$0 \subseteq f_n^{n-nS}(K_{t+2-s}) \otimes \omega_C(-j) \subseteq \cdots \subseteq f_n^{n-nS}(K_t) \otimes \omega_C(-j) \subseteq f_n^{n-nS}(K_t+1-s) \otimes \omega_C(-j).$$

The slope of the first non-zero term in the HN filtration in (37) is $-\mu_s p^n + 2g - 2 - j\delta_R$, which is negative by (30). The desired conclusion now follows from Proposition 2.14.

Now we show that if $j$ satisfies the right most inequality in (30), then assertion (2) in the claim holds. By Serre duality $h^1(f^n_{n-nS}(K_{t+1-s}) \otimes \omega_C(-j)) = h^0((f^n_{nS})^\vee (K_{t+1-s})(j))$. By Lemma 2.15, (3), the HN filtration on $(f_n^{n-nS}E_s)$ is as given below-

$$0 \subseteq f_n^{n-nS}E_{s+1}(j) \subseteq \cdots \subseteq f_n^{n-nS}E_s(j) \subseteq f_n^{nS}E_s(j).$$

Since the slope of the first non-zero term in the above filtration is $p^n\mu_{s+1} + j\delta_R$, which negative by (30), using Proposition 2.14, we get the desired conclusion.

**Proof of Theorem 6.1:** We shall show that (27) holds for all non-zero $y$. Then by the principle of analytic continuation (see page 127, [Ahl79]), we get (27) at all points.

Fix an open subset $U$ of the complex plane whose closure is compact and the closure does not contain the origin. We fix some notations below which we use in the ongoing proof. For $1 \leq s \leq t + 1$, set

$$l_s(n) = [-\mu_s, \frac{p^n}{\delta_R}] \text{ and } u_s(n) = [-\mu_s, \frac{p^n}{\delta_R} + 2g - 2].$$

Note that

$$\lim_{n \to \infty} \frac{l_s(n)}{p^n} = \frac{u_s(n)}{p^n} = \frac{-\mu_s}{\delta_R}.$$ 

There is a sequence of functions $(g_n)_n$ such that for $y \in U$, we have

$$\sum_{j=0}^\infty \lambda((\frac{R}{I(p^n)})_j)e^{-iyj/p^n} = \sum_{j<p^n} \lambda(\frac{R}{I(p^n)})_j)e^{-iyj/p^n} + \sum_{j=2g-1}^{l_1(n)-1} \lambda(\frac{R}{I(p^n)})_{j+p^n})e^{-iy (j+p^n)/p^n}$$

$$+ \sum_{s=1}^t \sum_{j=u_s(n)}^{l_s(n)-1} \lambda((\frac{R}{I(p^n)})_{j+p^n})e^{-iy (j+p^n)/p^n} + g_n(y).$$

In Lemma 6.5, we compute limits of the terms appearing on the right hand side of (40) normalized by $(\frac{1}{p^n})^2$.

**Lemma 6.5.** For $y \in U$, we have

1. $\lim_{n \to \infty} (\frac{1}{p^n})^2 g_n(y) = 0.$
2. $\lim_{n \to \infty} (\frac{1}{p^n})^2 \sum_{j<p^n} \lambda((\frac{R}{I(p^n)})_j)e^{-iyj/p^n} = -\delta_R \frac{e^{-iy}}{\frac{1}{(iy)^2}} - \delta_R \frac{e^{-iy}}{iy}$.
3. $\lim_{n \to \infty} (\frac{1}{p^n})^2 \sum_{j=2g-1}^{l_1(n)-1} \lambda((\frac{R}{I(p^n)})_{j+p^n})e^{-iy (j+p^n)/p^n}.$

$$= -\left(\sum_{b=0}^{t_1(n)+1} \mu_{b+1} T_1 b \right) \left(\frac{e^{-iy}}{\frac{1}{(iy)^2}}\right)e^{-iy} - i\delta_R \left(\sum_{b=0}^{t_2(n)+1} \frac{d}{dy} \frac{1}{\frac{1}{(iy)^2}} \right)e^{-iy}.$$
Continuation of proof of Theorem 6.1: We establish the statement Theorem 6.1 using Lemma 6.5 before verifying Lemma 6.5. When we use Lemma 6.5 to compute \( \lim_{n \to \infty} (\frac{1}{p^n})^2 \sum_{j \in \mathbb{N}} \lambda((\frac{R}{j})_{j+p^n}) e^{-iy/(j+p^n)} \), some terms on the right hand side of 3., Lemma 6.5 cancel some terms on the right hand side of 4., Lemma 6.5. After cancelling appropriate terms we get, for \( y \) in \( U \)

\[
\lim_{n \to \infty} (\frac{1}{p^n})^2 \sum_{j \in \mathbb{N}} \lambda((\frac{R}{j})_{j}) e^{-iyj/p^n} = -\delta_R e^{-iy} - \sum_{b=0}^{t-1} \mu_{b+1}(-1)^{b+1} \frac{\mu_{b+1} - iy}{iy} \sum_{b=0}^{t-1} \mu_{b+1}(-1)^{b+1} \frac{\mu_{b+1} - iy}{iy} \delta_R(k(S)) \frac{e^{-iy}}{(iy)^2} + \frac{\delta_R}{(iy)^2}.
\]

The last line is indeed equal to the right hand side of (27).

Proof of Lemma 6.5: 1) We show that there is a constant \( C \) such that \( |g_n(y)| \leq C p^n \) on \( U \). By Lemma 6.3 and Lemma 6.4, \( \lambda((\frac{R}{j})_{j}) = 0 \) for \( l > -\mu_{t+1} \frac{p^n}{y} + p^n + 2g - 2 \). So using (40) we get an integer \( N \), such that each \( g_n \) is a sum of at most \( N \) functions of the form \( \lambda((\frac{R}{j})_{j}) e^{-iyj/p^n} \), where \( l \) is at most \( -\mu_{t+1} \frac{p^n}{y} + p^n + 2g - 2 \). We prove (1) by showing that there is a \( C' \) such that for each of these functions \( \lambda((\frac{R}{j})_{j}) e^{-iyj/p^n} \) appearing in \( g_n \), \( |\lambda((\frac{R}{j})_{j}) e^{-iyj/p^n}| \leq C' p^n \) on \( U \). For that, note since \( U \) has a compact closure, there is a constant \( C_1 \) such that, for all \( l \leq -\mu_{t+1} \frac{p^n}{y} + p^n + 2g - 2 \), \( |e^{-iyj/p^n}| \) is bounded above by \( C_1 \) on \( U \). Since \( \lambda((\frac{R}{j})_{j}) \leq \lambda(R_l) \) and there is a constant \( C_2 \) such that for all \( l \), \( \lambda(R_l) \leq C_2 l \), we are done.

\[
\lim_{n \to \infty} (\frac{1}{p^n})^2 \sum_{j < p^n} \lambda((\frac{R}{j})_{j}) e^{-iyj/p^n} = \lim_{n \to \infty} (\frac{1}{p^n})^2 \sum_{j < p^n} \lambda(R_j) e^{-iyj/p^n}
\]

\[
= \lim_{n \to \infty} (\frac{1}{p^n})^2 \sum_{j=0}^{p^n-1} \lambda(R_j) e^{-iyj/p^n} \frac{p^n(1 - e^{-iy/p^n})}{iy}
\]

\[
= \lim_{n \to \infty} \frac{1}{p^n} \sum_{j=0}^{p^n-1} \lambda(R_j) \int_{j/p^n}^{(j+1)/p^n} e^{-iyx} dx.
\]

Now consider \( (h_n)_n \) the sequence of real valued functions on \([0, 1]\) defined by \( h_n(x) = \frac{1}{p^n} \lambda(R_{[x p^n]})) \). The last line of (41) is then \( \int_0^1 h_n(x) e^{-iyx} dx \). Since \( h_n(x) \) converges to the function \( \delta_R x \) uniformly on \([0, 1]\), we have

\[
\lim_{n \to \infty} (\frac{1}{p^n})^2 \sum_{j < p^n} \lambda((\frac{R}{j})_{j}) e^{-iyj/p^n} = \delta_R \int_0^1 x e^{-iyx} dx
\]

\[
= \delta_R \int_0^{1} (e^{-iy} - \frac{1}{(iy)^2}) - \delta_R \frac{e^{-iy}}{iy}.
\]

(42)
Using Lemma 6.3 and Lemma 6.4, we get,

\[
\left( \frac{1}{p^n} \right)^2 \sum_{j=2g-1}^{l_1(n)-1} \lambda(\left( \frac{R}{[p^n]} \right)_{j+p^n}) e^{-iy(j+p^n)/p^n} = \left[ -e^{-iy/p^n} \left( \sum_{b=0}^t \mu_{b+1}r_{b+1} \right) + e^{-iy/p^n} \left( \sum_{b=0}^t \left( r_{b+1} + 1 \right) \left( \sum_{j=2g-1}^{l_1(n)-1} e^{-iyj/p^n} \right) \right) \right] - e^{-iy/p^n} \sum_{b=0}^t \mu_{b+1}r_{b+1} \delta_R \sum_{j=2g-1}^{l_1(n)-1} je^{-iyj/p^n}.
\]

\[
(43)
\]

Now (4) follows from taking limit as \( n \) approaches infinity in (43) and using (39), Lemma 3.3 we get (3).

Lemma 6.3, Lemma 6.4 and a computation as in the proof of (3) shows that for \( 1 \leq s \leq t \),

\[
\sum_{j=a_s(n)}^{l_{s+1}(n)-1} \lambda(\left( \frac{R}{[p^n]} \right)_{j+p^n}) e^{-iy(j+p^n)/p^n} = \left[ -\frac{1}{p^n} \sum_{b=s}^t \mu_{b+1}r_{b+1} + \left( \frac{1}{p^n} \right)^2 \sum_{b=s}^t \left( r_{b+1} + 1 \right) \left( \sum_{j=2g-1}^{l_1(n)-1} e^{-iyj/p^n} \right) \right] - i \frac{1}{p^n} \sum_{b=s}^t \mu_{b+1}r_{b+1} \delta_R \frac{d}{dy} \left( \frac{e^{-iyu_s(n)/p^n} - e^{-iy_{s+1}(n)/p^n}}{1 - e^{-iy/p^n}} \right) e^{-iy}.
\]

\[
(44)
\]

Now (4) follows from taking limit as \( n \) approaches infinity and arguing as in the proof of (3).

\[ \square \]

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