Chern-Simons Gauge Invariance and Boundary Conformal Fields

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Abstract

The topological non-Abelian Chern-Simons theory with a boundary is shown to require a scalar field companion in order to preserve overall gauge-invariance both in the 3 dimensional manifold, as well as on its boundary. This scalar field, originally being a pure-gauge Chern-Simons excitation that indicates the most general topological gauge structure in the bulk, is found to be crucial for dynamic scalar excitation on the 2 dimensional generic boundary manifold, the latter naturally representing a conformal mode. This gauge/CFT correspondence is general, for a compact gauge group.

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1 Introduction

Topological gauge theories have generated wide-spread interest for quite some time. Especially in 3 dimensions, such a theory appear most naturally through Abelian $[A \wedge dA]$ or non-Abelian $[Tr (A \wedge dA + \frac{2}{3} A \wedge A \wedge A)]$ Chern-Simons (CS) terms [1], $A$ being a one-form. Both the versions implement a gauge-invariant mass devoid of any symmetry-breaking mechanism, that further determines the spin of the gauge field [1, 5, 8, 9]. The geometrical origin of the non-Abelian CS term has been well-known to be the self-dual (anti-self dual) Yang-Mills action in 4 dimensions, viz.,

$$T_r \int F^2 := T_r \int F \wedge F \equiv Tr \int d \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right),$$

with $F = dA + [A, A] = \pm ^* F$ being the field-strength two-form for $A$, so that the above bracket contains the CS descendant in 3 dimensional space-time [10]. This CS term explicitly breaks parity and unlike the usual Maxwell term ($F \wedge ^* F$), does not effect any local dynamics, as is expected from the default topology. The physical origin of such a term is attributed to interactions with both fermions [2, 3, 5, 7, 8, 9, 11, 12] as well as bosons [13, 14] and these contributions are protected beyond 1-loop [15].

In addition to being the effective theory for a host of physical phenomena such as quantum hall effect [16], topological insulators [17] and fractional statistics [18, 19] etc., the CS theory has been linked deeply with planar gravity [20, 21, 22] and string theory [23, 24]. More specifically, the CS theory is connected to two dimensional ‘rational’ conformal field theories (CFTs), having a finite number of conformal blocks, which are in one-to-one correspondence with the CS Hilbert space [22, 25, 26, 27, 28]. All these approaches consider pure CS theory on a 3 dimensional space-time of the form $\Sigma \times \mathbb{R}$, where $\Sigma$ is a compact Riemann surface.

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1Given the large number of important publications in these areas, we are referring to comprehensive reviews/books.
In absence of a boundary, the CS theory Hilbert space has one-to-one correspondence with the conformal blocks of a CFT defined on $\Sigma$. However, if $\Sigma$ has a boundary $\partial \Sigma$, the corresponding Hilbert space is infinite dimensional, with a chiral current algebra representation of the CFT on $\partial \Sigma \times \mathbb{R}$ [22] [25] [28]. In the present work, we seek to obtain a boundary conformal dynamics, starting from a more relaxed condition, namely that of a CS theory defined in a generic 3 dimensional Riemannian manifold $M$ having a general 2 dimensional boundary $\partial M$. If the space is of the form $M = \Sigma \times \mathbb{R}$, with boundary $\partial \Sigma$ of $\Sigma$, then $\partial M$ in our case will be $\partial \Sigma \times \mathbb{R}$. The dynamical fields on the boundary are found to have representation in the bulk, as a part of the generalized Euclidean CS action, completely justified by the corresponding topological (CS) hierarchy [1] [10]. Overall gauge-invariance plays the pivotal role in this process, with the scalar mode responsible for conformal dynamics on the boundary, additionally identified as a pure-gauge solution corresponding to the same non-Abelian gauge group. The boundary contains a non-dynamic, massive gauge excitation equivalent to chiral anomaly, for intrinsically flat manifolds like $\partial M = T^2$. However, for suitable gauge choice on the boundary, a variety of $\partial Ms$ (including the erstwhile $\partial M = \partial \Sigma \times \mathbb{R}$ case) lead to conformal dynamics.

In the following, we consider the non-Abelian Euclidean CS action extended with a pure-gauge term in Section 2. Section 3 deals with over-all gauge invariance of the system, including the boundary. Simple conformal structures are obtained on the boundary in Section 4, with various forms for manifolds with different topologies. Finally, we conclude.

2 Non-Abelian Chern-Simons Theory

We consider a general Riemannian 3-manifold $M$, not necessarily of the $\Sigma \times \mathbb{R}$ form, having a 2 dimensional boundary $\partial M$. Therein, a pure non-Abelian CS theory is defined through the action,

$$I^M_{CS} = \kappa \int_M T_r \left[ A \wedge dA + \frac{2}{3} g A \wedge A \wedge A \right], \quad (1)$$

which exists essentially on the topology class of that manifold, thereby not affected by local geometry, so to say the integrand is spread over the manifold. Herein, the one- and differential two-forms are defined respectively as,

$$A := A_{\mu}(x)dx^\mu \quad \text{and} \quad dA = \partial_{\mu}A_{\nu}dx^\mu \wedge dx^\nu, \quad \mu, \nu = 1, 2, 3. \quad (2)$$

The very form of the action in Eq. (1) makes it topological, as for defining it on $M$ with a genuine curvature, there is no dependence of the non-trivial metric $g_{\mu\nu}$ (geometry) is present. The trace in Eq. (1) is over the generators of the gauge group, associated with the corresponding fields $A_\mu := A_\mu^A T^A$.

From a topological point-of-view, $I^M_{CS}$ belongs to a hierarchy of topological gauge structures across manifolds with different dimensionality. The present action is the CS-descendant of a topological gauge Lagrangian, living in a 4 dimensional manifold $\bar{M}$ [10],

$$I^\bar{M}_D = \frac{\kappa}{2} \int_{\bar{M}} \bar{T}_r \hat{F} \wedge \hat{F} \equiv \kappa T_r \int_{\bar{M}} d \left[ A \wedge dA + \frac{2}{3} g A \wedge A \wedge A \right]$$

$$= \kappa T_r \int_{\text{Boundary}} \left[ A \wedge dA + \frac{2}{3} g A \wedge A \wedge A \right], \quad (3)$$

where $\hat{F}$ is the gauge field strength in 4 dimensions. The last equality above follows from Stokes theorem. It is to be noticed that the second term above is topological in nature, as a two-form is wedged with itself, unlike the corresponding Maxwell term $(\hat{F} \wedge * \hat{F})$. Being a total derivative, this leads to the CS Lagrangian on the 3 dimensional boundary of the 4 dimensional space by Stokes theorem. However the most general CS descendant could be taken as,

$$I^\bar{M}_{CS} = \kappa T_r \int_{\bar{M}} \left[ A \wedge dA + \frac{2}{3} g A \wedge A \wedge A - \frac{2}{3} g d\Theta \wedge d\Theta \wedge d\Theta \right], \quad (4)$$

where $\Theta(x) = \Theta^A(x)T^A$ is a non-Abelian scalar field such that $d\Theta$ transform as a gauge field. Such a term can only exist in 3 dimensions, as in $\bar{M}$ a term like,

$$T_r \int_{\bar{M}} d \left[ d\Theta \wedge d\Theta \wedge d\Theta \right],$$
vanishes identically. This kind of gauge field is found to be the source of conformal fields on the boundary of \( M \). We start with the general CS action in Eq. 4 on a 3-manifold \( M \), which is not a boundary to any 4-manifold, but is bounded itself by a 2 dimensional manifold \( \partial M \). In principle one can add any number of scalar fields to the above Lagrangian. However that does not lead to anything new except increased analytical complexity, especially regarding the present work.

Important aspects of such a generalization to the CS theory can be derived from the corresponding dynamics, given by the equations of motion (EOMs) for the two disjoint sectors:

\[
\begin{align*}
\epsilon^{\alpha\mu\nu} \left[ \text{Tr} \left( T^A T^B \right) \partial_\mu A^B_\nu + ig \text{Tr} \left( T^A T^B T^C \right) A^B_\mu A^C_\nu \right] &= 0, \\
\epsilon^{\alpha\mu\nu} \text{Tr} \left( T^A T^B T^C \right) \partial_\alpha \left( \partial_\mu \Theta^B \partial_\nu \Theta^C \right) &= 0.
\end{align*}
\]

(5)

The \( A \)-sector is covariantly non-dynamical, being first order in derivatives, as is well-known. It can at most lead to Galilean dynamics only in the presence of interactions [29]. For a given gauge group, one can always adopt the normalization \( \text{Tr} \left( T^A T^B \right) \propto \delta^{AB} \) and only the totally anti-symmetric part of \( \text{Tr} \left( T^A T^B T^C \right) \) survives the absolute anti-symmetry in the covariant Greek indexes. Therefore, the first of the above equation reduces to,

\[
\epsilon^{\alpha\mu\nu} F_{\mu\nu} = 0,
\]

(6)

in the adjoint representation with a pure-gauge solution \( A = -\left( i/g \right) U^{-1} dU \), so that it satisfies an EOM exactly same as that for \( \Theta \) above. This justifies considering \( d\Theta \) to be a fundamental variable which is a pure gauge configuration. Such a configuration can undergo further gauge transformations. Moreover, the gauge transformation acts in the space of gauge parameters, independent of the coordinate manifold \( M \) wherein the partial derivative is defined and \( \Theta(x) \) transforms like a local scalar. Thus, the scalar sector is identified as an off-shell pure gauge CS theory, with \( d\Theta \) transforming identically under a generic gauge group defined by \( U \) as \( A \):

\[
A \rightarrow U^{-1} A U - \frac{i}{g} U^{-1} dU, \quad d\Theta \rightarrow U^{-1} d\Theta U - \frac{i}{g} U^{-1} dU
\]

(7)

This aspect is of crucial importance, given that the non-Abelian CS action is not gauge-invariant in presence of a finite boundary, as gauge transformation of the former yields a total derivative leading to a boundary term.

2.1 Possible physical origin

By its structure, the extended CS Lagrangian can physically be attributed to the massive matter part of a generalized planar QCD Lagrangian [30, 31],

\[
\mathcal{L}_{\text{Mat}} = \Psi_\alpha \left[ i\delta_{\alpha\beta} \gamma^\mu \partial_\mu - m\delta_{\alpha\beta} - g\gamma^\mu T^C_{\alpha\beta} A^C_\mu \right. \\
\left. 0 \right] \Psi_\beta,
\]

(8)

where \( \Psi_\alpha = \left( \psi^a_\alpha \psi^b_\alpha \right) \), with mass \( m \) of two component \( (a, b) \) fermion field \( \Psi_\alpha \), and coupling strength \( g \), defined on \( M \). Here, \( (\alpha, \beta) \) denote the color species. In 3-space, the sign of \( m \) also doubles as that of the fermionic spin [4, 5, 8, 9, 32] fixed for fermion and anti-fermion, and therefore \( A \) and \( d\Theta \) couple to a particle-antiparticle pair. This is a crucial fact, as will be seen in the next section.

\footnote{Here, to be able to consistently define fermions, \( M \) needs to be without \textit{intrinsic} curvature, as fermions cannot be considered in a curved space-time.}
In the effective gauge sector, apart from the usual wave-function renormalization, the presence of fermionic mass leads exactly to the CS contribution of Eq. [4]. The CS coefficient has the explicit expression:

\[ \kappa = \left( \frac{g^2}{8\pi} \right) \left( m/|m| \right), \quad \text{(9)} \]

doubles as the topological gauge mass whose sign determines the gauge spin \([4, 5, 8, 9]\). However, this identification is restricted for \(M\) being geometrically flat and the dynamic gauge (Maxwell) sector in the effective theory will change the dynamics from that in Eqs. [3]. In the following, we consider the primary theory to be that in Eq. [4] and forgo the identification of Eq. [9].

### 3 Complete Gauge Invariance with a Boundary

Under generic gauge transformation designated by \(U\), common to both the gauge sectors on \(M\), the two gauge fields transform as in Eq. [7]. This changes the parent action of Eq. [4] as,

\[ \mathcal{I}_{\text{CS}}^M \to \mathcal{I}_{\text{CS}}^M + i \frac{\kappa}{g} T_r \int_M d\left[ (A - d\Theta) \right], \quad \text{(10)} \]

The second term is the total derivative mentioned earlier, that does not vanish in presence of a finite boundary \(\partial M\), and therefore, the system loses gauge-invariance. In absence of the scalar field, along with the above boundary contribution (without the \(d\Theta\) term), a pure non-Abelian CS theory like that in Eq. [1] yields one more term of the form,

\[ -\frac{\kappa}{3g^2} T_r \left( dU^{-1} \wedge dU^{-1} \wedge dU^{-1} \right), \quad \text{(11)} \]

under the same gauge transformation. This leads to a non-zero topological contribution (winding number) for large gauge transformations at infinity \([10, 32]\) that changes the action by an integral phase, keeping the generating functional \(\sim (\exp i I)\) of the system unchanged. However, in the present system with a finite \(\partial M\), this creates a serious problem. As the off-shell gauge field \(A\) does not go to pure gauge there, there is no mapping of the 3-space \(M\) into \(S^3\), thereby violating off-shell gauge-invariance even in the bulk itself.

However, in the present model of Eq. [4] the equivalent contribution from the \(d\Theta\) sector exactly cancels-out, preserving gauge-invariance in the bulk. This is the most crucial aspect of the present model, and validates the presence of a second gauge field. In geometrically flat \(M\), this convenient aspect is attributed to the negative sign in front of the last term in Eq. [4] originating from the opposite sign of fermion masses in Eq. [8]. Thus, the interaction of the gauge-fields with a fermion-antifermion doublet is a necessity here. However, this does not require the second gauge field to be a pure-gauge one. This additional condition is necessary in order to have complete gauge-invariance of the system including the boundary also.

The aforementioned gauge non-invariance of Eq. [10] leaves a boundary contribution,

\[ i \frac{\kappa}{g} T_r \int_{\partial M} dU^{-1} \wedge (a - d\theta), \quad \text{(12)} \]

following Stokes theorem, on \(\partial M\). This is a general property of any CS action, even an Abelian one \([33]\). The fields (including the gauge element) on \(\partial M\) are designated differently than their bulk counterparts, as the boundary manifold may be different, both in geometry and topology. Also, the reduction of dimensions changes the coordinate dependence in general, which can non-trivially effect the field configurations. Specifically \(d\theta\) is considered as the corresponding boundary counterpart of \(d\Theta\). Accordingly, the gauge group is re-labeled as \(u\), with the gauge-group generators \((T^A_s)\) remaining the same. Same is true for the constants \(g\) and \(\kappa\).

The only way the overall gauge-invariance can be regained is by postulating an action of the form,

\[ \mathcal{I}_{\partial M}^M = \kappa T_r \int_{\partial M} d\theta \wedge a, \quad \text{(13)} \]

on the boundary manifold \(\partial M\). Under a ‘new’ set of gauge transformations, \(\delta\theta = 0\)

\[ \delta M_{\text{eff}} \to \mathcal{I}_{\text{eff}}^M + \kappa T_r \int_{\partial M} d\theta \wedge a, \]

\[ \text{as } \mathcal{I}_{\text{eff}}^M \text{ is the complete effective action including the boundary contributions.} \]

\[ \mathcal{I}_{\text{eff}}^M \text{ yields a Maxwell-like contribution, which will dominate for small coupling } g \ll 1, \text{ which is required to attain an effective theory, except in the long-wavelength limit } [4, 5, 12]. \]

\[ \text{3 Even for } \mathcal{L}_{\text{eff}} \text{ being the complete Lagrangian, though only in the } A\text{-sector, the wave-function renormalization in the effective theory yields a Maxwell-like contribution. It will dominate for small coupling } g \ll 1, \text{ which is required to attain an effective theory, except in the long-wavelength limit } [4, 5, 12]. \]
\[ \rho \rightarrow u^{-1} \rho u - \frac{i}{g} u^{-1} du; \quad \rho = (a, d\theta), \]  

(14)
on $\partial M$, the boundary action transforms as,

\[ T_B^{\partial M} \rightarrow T_B^{\partial M} - i \frac{K}{g} T_r \int_{\partial M} duu^{-1} \wedge \left(a - d\theta - \frac{i}{g} duu^{-1}\right). \]  

(15)
The last integrand is the 2 dimensional analogue of that in Eq. 11 that identically vanishes in two dimensions, confirming the absence of inherent topology therein [10]. Then the rest of the terms exactly cancel the boundary contribution from the bulk in Eq. 12. Therefore, for a 3 dimensional system with a finite boundary, the complete gauge-invariant CS action is,

\[ T^{M+\partial M} = \int_M L^{\partial M} + \int_{\partial M} L^\partial_B = \kappa \int_M T_r \left[A \wedge dA + \frac{i}{3} g A \wedge A \wedge A + \frac{2}{3} g d\Theta \wedge d\Theta \wedge d\Theta\right] \]  

(16)

following respective transformation laws of Eq.s 7 and 14. The corresponding gauge invariant EOMs for the bulk sector are already given in Eq. 5 leading to non-dynamic pure-gauge conditions, satisfied by both $A$ and $d\Theta$. On the boundary, the corresponding equations are,

\[ da = 0 \quad \text{and} \quad \epsilon^{\mu \nu} \partial_\nu \theta = 0, \]  

(17)

leading to trivial, no-dynamic solutions.

The above results show why a field of the form $d\Theta$, that transforms as a gauge field under the same gauge group as $A$, was required in the bulk at the first place. Even in its absence, the standard pure CS theory of Eq. 1 would yield a boundary term $i(gK) T_r \left( duu^{-1} \wedge a \right)$ that cannot be compensated by any boundary Lagrangian. The additional winding-number-like term of Eq. 11 would always have remained. This has to do with the non-Abelian nature of the gauge fields, as for the Abelian case, no such field in the bulk is required [34]. However, the non-Abelian generalization naturally admits a pure-gauge field of the form $d\Theta$ as a part of the topological descendant, as seen from Eq. 4. All these requirements cannot simultaneously be fulfilled by, say, a second CS term. The present Lagrangian of Eq. 16 is the only one that provides overall gauge-independence over the whole of $M \oplus \partial M$.

From the EOMs Eq. 17, $a$ and $d\theta$ are of chiral nature. Indeed, the EOM of Eq. 6 essentially represented a chiral current, and one would expect this property to be reflected in the boundary through overall gauge invariance. We will see similar structure to be retained once we extend the boundary theory in order to incorporate dynamics therein.

4 Gauge-Invariant Dynamics on the Boundary

We intend to extend the gauge-invariant theory of Eq. 16 to include boundary dynamics. This is motivated by the earlier observed scalar KdV solitons on such a boundary [35, 36] which appeared on the boundary as dual to planar gravity [37]. However, from a pure CS theory point-of-view, such a boundary dynamics has not been obtained for the non-Abelian case. The simplest dynamic extension to the boundary Lagrangian in Eq. 13 can be taken as [33],

\[ L^\partial_B \rightarrow L^\partial_B = T_r \left[(d\theta - a) \wedge \star (d\theta - a) + \kappa d\theta \wedge a\right]. \]  

(18)
The first term, containing the wedge with a Hodge dual is not topological, unlike the terms considered till now. However, this term is form-invariant over the 2-manifold owing to the fact that a 2-manifold is conformally flat and the second term is also form-invariant as it is topological by construction. Thus, $L^\partial_B$ in Eq. 18 is 'spread over' the manifold in a form-invariant manner and it is therefore a possible Lagrangian on
\(\partial M\). It is a valid Lagrangian density in 2-space also because it lives on the corresponding topology. Further, it is invariant under the gauge transformation of Eq. [32] and clearly leads to dynamics. Though \(a\) is still not dynamic, it contributes critically through the equations of motions, which have the final forms,

\[da = 0 \quad \text{and} \quad \partial^2 \theta = \partial_\alpha a^\alpha.\]  

(19)

From the first equation, \(a\) now represents the chiral mode, whereas the second represents covariant dynamics of the massless boundary scalar mode \(\theta\), which becomes free in the covariant Lorentz gauge for \(a\). However, such a naïve gauge-fixing is not straightforward as \(\theta\) also transforms under the same gauge group and so as the fields in the bulk. This topic will be considered later. In the present case, such a choice is too trivial as from the first equation above, \(a\) is curl-less too. This already hints at the possibility of constructing a linear combination of \(\theta\) and \(a\), that can represent a free mass-less mode on \(\partial M\), which is a conformal structure.

Essentially, topological terms in the Lagrangian do not contribute to the dynamics of the system by their own, owing to their essential non-locality. This becomes apparent in the Hamiltonian formalism, which is implicitly covariant, wherein the topological terms do not appear. It is a straightforward evaluation through the Legendre transformation that the Hamiltonian corresponding to pure CS-type Lagrangian in Eq. [1] identically vanishes \([22]\). However, this requires a-priori on-shell gauge-fixing. Following Ref. \([22]\), on considering the Minkowskian counterpart of the present system with \(\partial M\) being a 1+1 dimensional manifold, the Weyl gauge \(a_0 = 0\) can be adopted. The Hamiltonian corresponding to the dynamical Lagrangian of Eq. [18] will be non-vanishing. As mentioned above, the intracity of gauge-fixing in the present case means an overall gauge-fixing that manifests simultaneously in the \(\theta\) sector and also in the bulk fields. Importantly, the boundary Hamiltonian will represent a dynamics independent of the bulk, as \(\partial M\) is not connected \(M\) dynamically. In the Weyl gauge, \(a_0\) essentially is a Lagrange multiplier for the Gauss law constraint of the system, vanishing upon the Legendre transformation. The Hamiltonian then obtained as,

\[\mathcal{H}_B^{\partial M} = \frac{1}{2} T \left[ \dot{\theta}^2 - (\partial_1 \theta - a_1)^2 \right], \quad \theta := \partial_0 \theta,\]  

(20)

which does not contain any topological term. This result is quite general as it is independent of specifics of both \(M\) and \(\partial M\). Although no explicit off-shell conformal structure appear until now, the presence of a dynamic mass-less scalar in the Hamiltonian tempts in that direction, obstructed only by the interacting gauge field.

To obtain a conformal structure, specific reduction of the variables, based on the particular topological structure of the system is in order. It is known \([22, 24, 25, 26, 28]\) that the boundary gauge field can be resolved in longitudinal (L) and transverse (T) components as \(a = a_L + a_T\). For \(\partial M\), any vector can be cast in the form \([38]\),

\[a = d\eta + d\xi, \quad \eta, \xi \in \mathbb{R}\]  

(21)

which owes to the fact that there are only two directions. This ‘splitting’ of \(a\) is generic to the particular \(\partial M\), so to say there is no parameter taking care of any other \(\partial M\) diffeomorphic to the present one. Any real shift in the ‘longitudinal’ scalar \(\eta\) corresponds to gauge transformation of \(a\) \([32]\). For \(\partial M\) being \(T^2\), the decomposition has the generic form,

\[A = dz \chi + \Gamma(z; \tau)\alpha; \quad \mathcal{A} = a_0 + ia_1, \quad z = x_0 + ix_1, \quad \chi = \eta - i\xi.\]  

(22)

The local function \(\Gamma(z; \tau)\) depends on the topological parameter \(\tau\), labeling \(\partial M\). A small gauge transformation (e. g. \(SU(N)\)-type) effects exclusively the longitudinal part: \(\eta \rightarrow \eta + \lambda, \lambda \in \mathbb{R}\), whereas a large gauge transformation shifts the constant parameter \(\lambda\) \([32]\). However, an orthogonal set of coordinates \((x_0, x_1)\) can always be constructed on the 2-manifold \(\partial M\) whether it is \(R^2, T^2\) or \(S^2\). Therefore the resolution in Eq. [21] is quite valid.

As already had been mentioned, although a gauge field has no degrees of freedom in 2-space, the same gauge constraint is spread over the whole field space defined in the bulk too. On attributing the gauge fixing exclusively to \(\partial M\), choosing a particular \(u\) (or \(U\)) amounts for reduction of two degrees of freedom,

\[u = \alpha(t).\]  

(23)

In a 3-manifold resolvable as \(T^2 \times \mathbb{R}\), \(\alpha = \alpha(t)\). In our case, with \(\partial M = T^2\), it is a constant.
leaving-out a single physical one, as $\theta$ is a scalar and $a$ is a 2-vector. We will find this to be the case exactly. On substituting Eq. (21) in Eq. (15) the boundary Lagrangian takes the form,

$$L_0^M = T_r \left[ d\theta \wedge *d\theta + d\eta \wedge *d\eta = d\xi \wedge *d\xi + 2d\theta \wedge *d(\eta + \kappa \xi) \\
+ d\theta \wedge d(\xi + \kappa \eta) - d\eta \wedge d\xi \right].$$  \hspace{1cm} (23)

The kinetic part in $\xi$ represents negative-norm states, exactly canceling-out the positive-norm $\eta$ states. Thus, there is no dynamical degree of freedom for the erstwhile $a$ field. This re-confirms the above inference regarding the constrained nature of the system. All three fields satisfy massless Klein-Gordon equation,

$$ (d^*d)\zeta = 0, \quad \zeta = \theta, \eta, \xi. $$  \hspace{1cm} (24)

This re-confirms the prior gauge-fixing again. Simply stated, implementation of Eq. (21) reduces topological terms in Eq. (23) to total derivatives, with no contribution as there is no boundary of a boundary. In both representations, we have only one dynamical field ($\theta$), with interactions, as the dynamical contributions of the other two cancel-out in the latter case. In the next, it will be shown to be resolved into a free field theory.

### 4.1 The Conformal Theory

Even after non-trivial gauge-fixing, the boundary theory of Eq. (15) is left with interactions and topological terms, preventing a conformal structure. The topological terms in Eq. (23) can be removed at the level of the Lagrangian itself (off-shell) by performing a rotation in the space of field-variables defined on $\partial M$. This is a general property of topological models like CS theory in the 3 dimension and BF theory in 4 dimensions.\footnote{This point is not explicitly demonstrated in most literature. However, it can indirectly be seen in effective treatment of the said theories, leading to apparent non-local mass terms. In the present case though, no such non-locality arise, owing to the absence of the usual Maxwell dynamics at the tree level. Indeed, the gauge field is non-dynamical in the present case.}

For analytical brevity, we opt for the specific rotation accompanied with a gauge choice:

$$d\tilde{\theta} := d\theta + \kappa^* a \quad \text{and} \quad a := d\tilde{\eta},$$ \hspace{1cm} (25)

at the level of Eq. (15) that corresponds to a unit functional Jacobian. The second expression amounts to a pure gauge form, like that in case of $d\theta$ ($d\tilde{\theta}$), as $a$ satisfies $da = 0$ on-shell (Eq.s (16)). For $\partial M = T^2$ this can correspond to a large gauge choice ($X, a = 0$)\footnote{This is ensured by the vanishing of the trace of respective energy-momentum tensor $T_{\mu\nu} = \gamma \frac{\delta L}{\delta g^{\mu\nu}} - g_{\mu\nu} L$.}, as discussed before, requiring judicious choice of coordinates. Essentially the system now becomes on-shell with overall gauge invariance of the action being retained as before. As a result, the re-arranged Lagrangian is,

$$L_0^M = T_r \left[ d\left(\tilde{\theta} - \tilde{\eta}\right) \wedge *d\left(\tilde{\theta} - \tilde{\eta}\right) + \kappa^2 d\tilde{\eta} \wedge *d\tilde{\eta} \right] = T_r \left[ d\tilde{\theta} \wedge *d\tilde{\theta} + \kappa^2 d\tilde{\eta} \wedge *d\tilde{\eta} \right].$$ \hspace{1cm} (26)

The above Lagrangian contains two free massless scalar modes $\tilde{\theta} = \tilde{\theta} - \tilde{\eta}$ and $\tilde{\eta}$, representing conformal modes in a 2 manifold $\partial M$ like $\partial M$. However, their obvious linear dependence leads to only one independent dynamical mode, which is evident from the individual EOMs:

$$\partial^2 \tilde{\theta} = \partial^2 \tilde{\eta} \quad \text{and} \quad \partial^2 \tilde{\theta} = (1 + \kappa^2) \partial^2 \tilde{\eta} \quad \left(\partial^2 = \partial_\mu \partial^\mu\right),$$ \hspace{1cm} (27)

respectively. Both relate d’Alembertians of the two fields, leading to the trivial solutions that both d’Alembertians vanish. However, this result cannot be obtained unless one equation of motion is substituted in the other, containing the same variables. This justifies essentially one degree of freedom in the system as before, a signature of its constrained nature. This exactly in accordance with $d\tilde{\theta}$ having one degree of freedom whereas $a_\mu$ has none.

The manifestation of conformal dynamics in 2 space is ensured by local scale (Weyl) invariance. The later is ensured by the vanishing of the trace of respective energy-momentum tensor $T_{\mu\nu} = \gamma \frac{\delta L}{\delta g^{\mu\nu}} - g_{\mu\nu} L$.\footnote{This is ensured by the vanishing of the trace of respective energy-momentum tensor $T_{\mu\nu} = \gamma \frac{\delta L}{\delta g^{\mu\nu}} - g_{\mu\nu} L$.}
For the boundary Lagrangian of Eqs. 18 and 23, the respective traces,

\[ T^\mu_\mu = 2\kappa T, d\theta \wedge a \quad \text{and} \quad T^\mu_\mu = 2T, [d\theta \wedge d(\xi + \kappa\eta) - d\eta \wedge d\xi], \]

(29)

turn out to be non-zero. However, they are exclusively of the topological nature whereas the energy-momentum tensor itself is the generator of local translations. Moreover, in the second expressions, the integral over \( \partial M \) do vanish as it is a total derivative. As physical observables are essentially the integrals of these ‘densities’, this leads to a vanishing result. However for conformal symmetry to exist, which is of local nature, the trace density itself must vanish. This ambiguity is expectedly resolved with the Lagrangian representation of Eq. 25 exactly leading to \( T^\mu_\mu = \frac{\Lambda}{\kappa} \). Strictly speaking, this result ensures local scale or Weyl invariance of the theory \[41\], which means conformal invariance in a 2-manifold \[39, 40\].

A conformal structure can be obtained, however, without subjecting the resolution in Eq. 21 in general. For a generic boundary 2-manifold \( \partial M \), the first of Eqs. 25 is always valid, leading to the rotated Lagrangian,

\[ L^{BM}_B = \left[ (d\bar{\theta} - a) \wedge * (d\bar{\theta} - a) + \kappa^2 a \wedge * a \right]. \]

(30)

The corresponding energy-momentum tensor is again traceless, ensuring Weyl symmetry. The field equations essentially yield a mass-less scalar \( \bar{\theta} \), even without any gauge-fixing, with the \( a \) sector being non-dynamic. Most importantly, although the theory is not conformal except in \( \mathbb{R}^2 \) or in \( T^2 \) following large gauge transformation, it is Weyl-invariant on a generic 2 dimensional \( \partial M \).

Eq. 30 embodies an interesting observation. The second term here is essentially a gauge mass term, and on inherently flat manifolds like \( \mathbb{R}^2 \) and \( T^2 \), appears as the celebrated chiral anomaly contribution \[35, 42\]. This identification implies \( \kappa^2 = e^2/\pi \), \( e \) being the coupling of \( a \) with some integrated-out massless fermions on \( \partial M \). But since the gauge group on \( M \oplus \partial M \) is uniquely parameterized by \( g, \epsilon = g \). Therefore, the interpretation of effective chiral anomaly is valid only if \( \kappa^2 = g^2/\pi \). However, this is not possible if the CS term in \( M \) is induced from a system like that in Eq. 8 as it would impose \( \kappa \sim g^2 \) \[30, 4, 5, 8, 9\], yielding a contradiction.

On the other hand, this still leaves room for extension to \( L^{BM}_B \) by a massless boundary fermion (\( \sigma \)) sector \( \bar{\sigma} (i\gamma \cdot d - g\gamma \cdot a) \sigma \). When integrated-out, this sector manifests in a parametric shift: \( \kappa^2 \rightarrow \kappa^2 + g^2/\pi \) in Eq. 30 due to chiral anomaly \[35, 42\]. It is to be noted that these results are exclusive to intrinsically flat manifolds, as fermions cannot be defined in presence of curvature.

The present theory can be made manifestly conformal (i. e., free scalar one) at the effective theory level by integrating-out the non-dynamic gauge field. Up to a constant normalization, this results in complete Bosonization as,

\[ L^{BM}_B = \left[ 5 + \frac{\kappa^2}{1 + \kappa^2} \right] T, d\bar{\theta} \wedge * d\bar{\theta}, \]

(31)

which is conformal by construction on the 2-manifold \( \partial M \). As Eq. 30 is valid in a generic \( \partial M \), so as the above conformal structure. This result is off-shell and effects in wave-function renormalization of the scalar field due to the original interaction. However, the conformality of the full boundary theory is apparent through the corresponding traceless energy-momentum tensor at an off-shell level.

5 Conclusions

To summarize the results, we have generalized the pure non-Abelian CS descendant to a generalized CS descendant. Necessarily being of pure-gauge nature, this extension is decoupled from the usual gauge sector in \( M \) and thus, does not effect the characteristic CS dynamics of references \[21, 22, 23, 24, 25, 26\], including the latter’s correspondence with 3 dimensional gravity. However, in presence of a finite boundary, overall gauge-invariance necessarily couple the two fields in a topological manner on the boundary \( \partial M \). In general, this allows for a dynamic extension on \( \partial M \), leaving only one degree of freedom, that encompasses \( M \oplus \partial M \) and the whole of the field space.

\[ ^{6}\text{The energy-momentum tensor for free scalar field is not traceless in general, except in 2-space, allowing the latter case to be only example of conformal field theory with free scalar field.} \]
Upon a rotation in field-space on $\partial M$, the free scalar mode separates from the non-dynamic gauge sector, irrespective of the geometric and topological nature of the prior. This makes the theory explicitly Weyl-invariant and the dynamic scalar mode is identified as a conformal one on the boundary 2 manifold, and necessarily demands the boundary gauge field $a$ also to be of pure-gauge form. For a generic $a$, conformal dynamics is obtained on intrinsically flat boundary manifolds with additional interacting massless fermions, with the non-dynamic sector effectively attributed to chiral anomaly. Indeed, the effective theory obtained by integrating the non-dynamical fields out, is a free scalar one.

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**References**

[1] S.-S. Chern and J. Simons, *Characteristic forms and geometric invariants*, Ann. Math. **99** 18 (1974).
[2] J. F. Schonfeld, *A mass term for three-dimensional gauge fields*, Nucl. Phys. B **185** (1981) 157.
[3] I. Affleck, J. Harvey and E. Witten, *Instantons and (super-) symmetry breaking in (2+1) dimensions*, Nucl. Phys. B **206** (1982) 413.
[4] S. Deser, R. Jackiw and S. templeton, *Three-Dimensional Massive Gauge Theories*, Phys. Rev. Lett. **48** (1982) 975.
[5] S. Deser, R. Jackiw and S. templeton, *Topologically massive gauge theories*, Ann. Phys. **140** (1982) 372.
[6] A. Redlich, *Gauge Noninvariance and Parity Nonconservation of Three-Dimensional Fermions*, Phys. Rev. Lett. **52**, 18 (1984).
[7] A. Redlich, *Parity violation and gauge noninvariance of the effective gauge field action in three dimensions*, Phys. Rev. D **29**, 2366 (1984).
[8] D. Boyanovsky, R. Blankenbecler and R. Yahalom, *Physical origin of topological mass in 2 + 1 dimensions*, Nucl. Phys. B **270** (1986) 483.
[9] B. Binegar, *Relativistic field theories in three dimensions*, J. Math. Phys. **23** (1982) 1511.
[10] *Current Algebra and Anomalies*, S. B. Treiman, R. Jackiw, B. Zumino and E. Witten, Princeton University Press, Princeton, 1985.
[11] T. W. Appelquist, M. Bowick, D. Karabali, and L. C. R. Wijewardhana, *Spontaneous chiral-symmetry breaking in three-dimensional QED*, Phys. Rev. D **33** (1986) 3704.
[12] C. R. Hagen, *A new gauge theory without an elementary photon*, Ann. Phys. **157** (1984) 342.
[13] R. Pisarski and S. Rao, *Topologically Massive Chromodynamics in the Perturbative Regime*, Phys. Rev. D **32**, 2081 (1985).
[14] C. R. Hagen, P. Panigrahi, and S. Ramaswamy, *Still More Corrections to the Topological Mass*, Phys. Rev. Lett. **61** (1988) 389.
[15] S. Coleman and B. Hill, *No more corrections to the topological mass term in QED$_3$*, Phys. Lett. B **159** (1985) 184.
[16] M. Stone, *Quantum Hall Effect* (World Scientific, Singapore, 1992).
[17] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
[18] F. Wilczek, *Fractional Statistics and Anyon Superconductivity*, (World Scientific, Singapore, 1990).
[19] R. Iengo and K. Lechner, *Anyon Quantum Mechanics and Chern-Simons theory*, Phys. Rep. **213**, 179 (1992).
[20] A. M. Polyakov, Mod. Phys. Lett. **2**, 893 (1987).
[21] E. Witten, *Topological Quantum Field Theory*, Commun. Math. Phys. **117**, 353 (1988).
[22] E. Witten, *Quantum Field Theory and the Jones Polynomial*, Commun. Math. Phys. **121**, 351 (1989).
[23] E. Witten, *Chern-Simons Theory as a String Theory*, Prog. Math. **133**, 637 (1995).
[24] M. Marino, *Chern-Simons Theory and Topological Strings*, Rev. Mod. Phys. **77**, 675 (2005).
[25] S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg, Remarks on the Canonical Quantization of the Chern-Simons-Witten Theory, Nucl. Phys. B 326, 108 (1989).
[26] J. Labastida and A. Ramallo, Chern-Simons Theory and Conformal Blocks, Phys. Lett. B 228, 214 (1989).
[27] A. Polychronakos, Abelian Chern-Simons Theories in 2+1 Dimensions, Ann. Phys. 203, 231 (1990).
[28] M. Bos and V. P. Nair, Coherent State Quantization of Chern-Simon Theory, Int. J. Mod. Phys. A 5, 959 (1990).
[29] C. R. Hagen, Phys. Rev. D 31, 848 (1985).
[30] K. S. Babu, A. Das and P. K. Panigrahi, Phys. Rev. D 36, 3725 (1987).
[31] K. Nakamura et. al., J. Phys. G 37, 075021 (2010) 114.
[32] G. V. Dunne, Lectures on Topological Aspects of Lower Dimensional Systems: 1998 Les Hauches Summer School, arXiv:hep-th/9902115
[33] K. Abhinav, V. M. Vyas and P. K. Panigrahi, Pramana-journal of physics 85, 1023 (2015).
[34] P. K. Panigrahi, V. M. Vyas and T. Shreecharan, arXiv:0901.1034[cond-mat.supr-con]; V. M. Vyas and P. K. Panigrahi, arXiv:1107.5521[cond-mat.supr-con].
[35] A. M. Polyakov, Mod. Phys. Lett. 2, 893 (1987); V. G. Knizhnik, A. M. Polyakov and A. B. Zamolodchikov, Mod. Phys. Lett. A 3, 819 (1988); A. H. Chamseddine and M. Reuter, Nucl. Phys. B 317, 757 (1989); A. M. Polyakov, Int. J. Mod. Phys. A 5, 833 (1990).
[36] A Das, W-J Huang and S Roy, Int. J. Mod. Phys. 7, 3447 (1992).
[37] E. Witten, Comm. Math. Phys. 117, 353 (1988); ibid., Nucl. Phys. B 311, 46 (1988/1989).
[38] A. Das, Field Theory: A Path Integral Approach, (World Scientific, Singapore, 2006).
[39] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. B 241, 333 (1984).
[40] P. Ginsparg, Applied Conformal Field Theory, Les Houches, Session XLIX, 1988, Champs, Fields, Strings and Critical Phenomena, ed. by E. Brézin and J. Zinn-Justin, Elsevier Science Publishers B.V. (1989), arXiv:hep-th/9108028
[41] C. Callan, S. Coleman and R. Jackiw, Ann. Phys. 59, 42 (1970).
[42] K. Fujikawa and H. Suzuki, Path Integrals and Quantum Anomalies, Clarendon Press, Oxford, (2004).