Renormalization flow for unrooted forests on a triangular lattice

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Abstract

We compute in small temperature expansion the two-loop renormalization constants and the three-loop coefficient of the $\beta$-function, that is the first non-universal term, for the $\sigma$-model with $O(N)$ invariance on the triangular lattice at $N = -1$. The partition function of the corresponding Grassmann theory is, for negative temperature, the generating function of unrooted forests on such a lattice, where the temperature acts as a chemical potential for the number of trees in the forest. To evaluate Feynman diagrams we extend the coordinate space method to the triangular lattice.
1 Introduction

Results concerning with graph theory [1–4], that is properties of a set of points which refer simply to the notion of adjacency, are of interest in a variety of fields, ranging from pure mathematics to statistical physics and find an enormous amount of applications in natural sciences besides physics like in biology or in theoretical information science.

Detailed properties of a graph can be derived from the study of the partition function of a $q$-state Potts model [5–7] with variables defined on its sites. Indeed this function is strictly related with the Tutte polynomial of the graph [8–10] and, for example, the generating polynomial of spanning trees or unrooted forests on the graph can be recovered by taking the limit $q \to 0$.

A classical result in algebraic graph theory is Kirchhoff’s matrix-tree theorem [11] which expresses the generating polynomials of spanning trees and rooted spanning forests on a given graph as determinants associated to the graph’s Laplacian matrix. For recent applications see for example [12,13]. It is quite natural to rewrite these determinants as Gaussian integrals over Grassmann variables.

Recently [14] it has been shown that the solution of other combinatorial problems on a graph can be represented in terms of Grassmann integrals, even though non-Gaussian. In particular, the generating polynomial of unrooted spanning forests on the graph is simply written adding to a Gaussian term a suitable four-fermion term. Interestingly, the same partition function can be obtained, order by order in perturbation theory, by considering an anti-ferromagnetic non-linear $\sigma$-model with $O(N)$ invariance in the limit in which $N \to -1$. These representations are very convenient to study the cases in which the graph is an infinite regular lattice, because the whole machinery of Statistical Field Theory becomes available. For example, Renormalized Perturbation Expansion can be used, Renormalization Group notions can be applied and one sees that on two dimensional lattices these models are asymptotically free [15–18]. The same mapping has been used at the transition at negative tree fugacity which corresponds to the Potts antiferromagnetic critical point [19–21].

In this paper we will concentrate on the triangular lattice and, in particular, we are interested in the evaluation of the so-called $\beta$-function. We have computed the three-loop coefficient which is the first non-universal term, which, in contrast with the square lattice, was yet unknown. A direct practical relevance of this coefficient comes from a recent study of the zeroes in the complex plane of the partition function of the Potts model by means of the numerical evaluation of a transfer matrix in a strip [22]. The locus of zeroes converges to a pair of complex-conjugate curves with horizontal asymptote, but the convergence is very slow in a region of large $\text{Re}(w)$. It turns out that the shape of this curve can be deduced perturbatively (in $1/w$) from the expression of the $\beta$-function, thus in the region where the errors are larger.
2 Unrooted forests

Let \( G = (V, E) \) be a finite undirected graph with vertex set \( V \) and edge set \( E \). Associate to each edge \( e \) a weight \( w_e \), which can be a real or complex number or, more generally, a formal algebraic variable. For \( i \neq j \), let \( w_{ij} = w_{ji} \) be the sum of \( w_e \) over all edges \( e \) that connect \( i \) to \( j \). The (weighted) Laplacian matrix \( L \) for the graph \( G \) is then defined by

\[
L_{ij} = \begin{cases} 
-w_{ij} & \text{for } i \neq j, \\
\sum_{k \neq i} w_{ik} & \text{for } i = j.
\end{cases}
\] (1)

This is a symmetric matrix with all row and column sums equal to zero.

Since \( L \) annihilates the vector with all entries 1, its determinant is zero. Kirchhoff’s matrix-tree theorem [11] and its generalizations [23–26] express determinants of square submatrices of \( L \) as generating polynomials of spanning trees or rooted spanning forests in \( G \). For any set of vertices \( \{i_1, \ldots, i_r\} \) of \( V \), let \( L(i_1, \ldots, i_r) \) be the matrix obtained from \( L \) by deleting the rows and columns \( i_1, \ldots, i_r \). Then Kirchhoff’s theorem states that \( \det L(i) \) is independent of \( i \) and equals

\[
\det L(i) = \sum_{T \in \mathcal{T}} \prod_{e \in T} w_e ,
\] (2)

where the sum runs over all spanning trees \( T \) in \( G \). (We recall that a subgraph of \( G \) is called a tree if it is connected and contains no cycles, and is called spanning if its vertex set is exactly \( V \).) The \( i \)-independence of \( \det L(i) \) expresses, in electrical-circuit language, that it is physically irrelevant which vertex \( i \) is chosen to be “ground”. There are many different proofs of Kirchhoff’s formula (2); one simple proof is based on the Cauchy–Binet theorem in matrix theory (see e.g. [2]).

The “principal-minors matrix-tree theorem” reads

\[
\det L(i_1, \ldots, i_r) = \sum_{F \in \mathcal{F}(i_1, \ldots, i_r)} \prod_{e \in F} w_e ,
\] (3)

where the sum runs over all spanning forests \( F \) in \( G \) composed of \( r \) disjoint trees, each of which contains exactly one of the “root” vertices \( i_1, \ldots, i_r \). This theorem can easily be derived by applying Kirchhoff’s theorem (2) to the graph in which the vertices \( i_1, \ldots, i_r \) are contracted to a single vertex, while it has theorem (2) as a special case \( r = 1 \), through the bijection between unrooted spanning trees and spanning trees rooted on a given fixed vertex.

Let us now introduce, at each vertex \( i \in V \), a pair of Grassmann variables \( \psi_i, \bar{\psi}_i \). All of these variables are nilpotent (\( \psi_i^2 = \bar{\psi}_i^2 = 0 \)), anticommute, and obey the usual rules for Grassmann integration. Writing

\[
\mathcal{D}(\psi, \bar{\psi}) := \prod_{i \in V} d\psi_i d\bar{\psi}_i ,
\] (4)

we have, for any matrix \( A \),

\[
\int \mathcal{D}(\psi, \bar{\psi}) e^{\bar{\psi} A \psi} = \det A
\] (5)
and more generally
\[
\int \mathcal{D}(\psi, \bar{\psi}) \left( \prod_{\alpha=1}^{r} \bar{\psi}_{i_\alpha} \psi_{i_\alpha} \right) e^{\bar{\psi} L \psi} = \det A(i_1, \ldots, i_r). \tag{6}
\]

These formulae allow us to rewrite the matrix-tree theorems in Grassmann form; for instance, (2) becomes
\[
\int \mathcal{D}(\psi, \bar{\psi}) \bar{\psi}_i \psi_i e^{\bar{\psi} L \psi} = \sum_{T \in T} \prod_{e \in T} w_e. \tag{7}
\]

while (3) becomes
\[
\int \mathcal{D}(\psi, \bar{\psi}) \left( \prod_{\alpha=1}^{r} \bar{\psi}_{i_\alpha} \psi_{i_\alpha} \right) e^{\bar{\psi} L \psi} = \sum_{F \in \mathcal{F}(i_1, \ldots, i_r)} \prod_{e \in F} w_e \tag{8}
\]

which is to say
\[
\int \mathcal{D}(\psi, \bar{\psi}) \exp \left[ \bar{\psi} L \psi + t \sum_i \bar{\psi}_i \psi_i \right] = \sum_{F \in \mathcal{F}} \prod_{F = (F_1, \ldots, F_\ell)} \prod_{e \in F} w_e. \tag{9}
\]

This formula represents vertex-weighted spanning forests as a massive fermionic free field \([2, 27]\).

More generally, it has been shown in \([14]\) that
\[
\int \mathcal{D}(\psi, \bar{\psi}) \exp \left[ \bar{\psi} L \psi + t \sum_i \bar{\psi}_i \psi_i + u \sum_{\langle ij \rangle} \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j \right] = \sum_{F \in \mathcal{F}} \left( \prod_{i=1}^{\ell} (t |V_{F_i}| + u |E_{F_i}|) \right) \prod_{e \in F} w_e \tag{10}
\]

where the sum runs over spanning forests \(F\) in \(G\) with components \(F_1, \ldots, F_\ell\); here \(|V_{F_i}|\) and \(|E_{F_i}|\) are, respectively, the numbers of vertices and edges in the tree \(F_i\). We remark that the four-fermion term \(u \sum_{\langle ij \rangle} \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j\) can equivalently be written, using nilpotency of the Grassmann variables, as \(- (u/2) \sum_{i,j} \bar{\psi}_i \psi_i L_{ij} \bar{\psi}_j \psi_j\). More interestingly, since \(|V_{F_i}| - |E_{F_i}| = 1\) for each tree \(F_i\), we can take \(u = -t\) and obtain the generating function of unrooted spanning forests with a weight \(t\) for each component.

3 Relation with the lattice \(\sigma\)-Models.

Recall that the \(N\)-vector model consists of spins \(\sigma_i \in \mathbb{R}^N, |\sigma_i| = 1\), located at the sites \(i \in V\), with Boltzmann weight \(e^{-\mathcal{H}}\) where
\[
\mathcal{H} = -T^{-1} \sum_{\langle ij \rangle} w_{ij} (\sigma_i \cdot \sigma_j - 1) \tag{11}
\]
and $T$ is the temperature.

Low-temperature perturbation theory is obtained by writing

$$\sigma_i = (\sqrt{1 - T \pi_i^2}, T^{1/2} \pi_i)$$

(12)

with $\pi_i \in \mathbb{R}^{N-1}$ and expanding in powers of $\pi$. Taking into account the Jacobian, the Boltzmann weight is $e^{-\mathcal{H}'}$ where

$$\mathcal{H}' = \mathcal{H} + \frac{1}{2} \sum_i \log(1 - T \pi_i^2)$$

(13)

$$= \frac{1}{2} \sum_{i,j} L_{ij} \pi_i \cdot \pi_j - \frac{T}{2} \sum_i \pi_i^2 - \frac{T}{4} \sum_{(ij)} w_{ij} \pi_i^2 \pi_j^2 + O(\pi_i^4, \pi_j^4).$$

(14)

When $N = -1$, the bosonic field $\pi$ has $-2$ components, and so, at least in perturbation theory, it can be replaced by a fermion pair $\psi, \bar{\psi}$ if we make the substitution

$$\pi_i \cdot \pi_j \rightarrow \psi_i \bar{\psi}_j - \bar{\psi}_i \psi_j.$$  

(15)

Higher powers of $\pi_i^2$ vanish due to the nilpotence of the Grassmann fields, and we obtain the model (10) if we identify

$$t = -u = -T.$$  

(16)

Note the reversed sign of the coupling: the spanning-forest model with positive weights ($t > 0$) corresponds to the antiferromagnetic $N$-vector model ($T < 0$).

In the case of a regular unweighted graph of order $q$, that is all the vertices are connected to other $q$ vertices, we shall take

$$w_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are connected} \\ 0 & \text{otherwise.} \end{cases}$$

(17)

and the corresponding Laplacian

$$L_{ij} = \begin{cases} -1 & \text{if } i \neq j \text{ and } i \text{ and } j \text{ are connected} \\ 0 & \text{if } i \neq j \text{ and } i \text{ and } j \text{ are not connected} \\ q & \text{if } i = j. \end{cases}$$

(18)

This is the case of a regular periodic lattice in $d$ dimensions. If we take unit lattice spacing, vertices connected to a given site correspond to sites at unit distance, so that, if $\hat{e}_k$ is a lattice direction, $f$ a lattice function, and $x$ a lattice site, the lattice derivatives are defined as

$$\nabla_k f(x) := f(x + \hat{e}_k) - f(x)$$

(19)

$$\nabla^*_k f(x) := f(x) - f(x - \hat{e}_k)$$

(20)
The Laplacian can be written as

\[(Lf)(x) := -\sum_{k=1}^{q} \nabla_k f(x) \quad (21)\]

and when, like in the square and triangular lattice, \(q\) is even and to each lattice direction corresponds an inverse lattice direction, that is \(\hat{e}_{k+q/2} = -\hat{e}_k\), we can restrict the sum to positive directions

\[-(Lf)(x) = \sum_{k=1}^{q/2} (\nabla_k - \nabla_k^*) f(x) = \sum_{k=1}^{q/2} \nabla_k \nabla_k^* f(x) = \sum_{k=1}^{q/2} \nabla_k^* \nabla_k f(x) \quad (22)\]

\[= \sum_{k=1}^{q/2} \left[ f(x + \hat{e}_k) + f(x - \hat{e}_k) - 2f(x) \right] \quad (23)\]

and, in the lattice scalar product

\[(f, g) = \sum_x f(x)g(x) \quad (24)\]

we have

\[(g, Lf) = -\sum_{k=1}^{q/2} (g, \nabla_k \nabla_k^* f) = \sum_{k=1}^{q/2} (\nabla_k g, \nabla_k f) = \sum_{k=1}^{q/2} (\nabla_k^* g, \nabla_k^* f). \quad (25)\]

### 4 The calculus of the \(\beta\)-function

We follow a procedure which already found several applications [28–32] for the square lattice. For a lattice theory, i.e. a theory regularized by introducing a discretization of the coordinates space, in principle the \(\beta\)-function can be found by a direct computation on the lattice, which also provides a regularization. However, our lattice \(\sigma\)-model has a natural continuum counterpart, with the widely investigated action

\[S(\pi, h) = \beta \int d^2x \left[ \frac{1}{2} (\partial_\mu \pi(x))^2 + \frac{1}{2} \frac{(\pi(x) \cdot \partial_\mu \pi(x))^2}{1 - \pi^2(x)} - h \sqrt{1 - \pi^2(x)} \right] , \quad (26)\]

where we have introduced an external magnetic field \(h\) which explicitly breaks the \(O(N)\)-invariance. In particular Brézin and Hikami [33] already performed the renormalization up to three loops in dimensional regularization.

A general theorem of Renormalization states that the \(n\)-loop \(\beta\)-function within a certain regularization scheme can be deduced from the knowledge of the \(\beta\)-function in any other scheme, at the same perturbative order, and of the renormalization constants in the desired scheme, up to order \(n - 1\). So, a possible procedure, which we will indeed follow in this work,
is to relate the $\beta$-function on the square and triangular lattice to the continuum results of Brézin and Hikami via the calculation of the two renormalization constants of the non-linear $\sigma$-model, denoted by $Z_1$ and $Z_2$.

More in detail, in our case we have to compare our lattice theory with the continuum theory renormalized in [33] using $\overline{MS}$-scheme (Minimal Subtraction modified) and in dimensional regularization. The starting point is the relation for the $n$-point 1-particle-irreducible (1PI) correlation functions

$$\Gamma^{(n)}_{\text{latt}}(p_1, \cdots, p_n; \beta, h; 1/a) = Z_2^{n/2} \Gamma^{(n)}_{\overline{MS}}(p_1, \cdots, p_n; Z_1^{-1}\beta, Z_1 Z_2^{-1/2} h; \mu)$$

(27)

where $a$ and $\mu$ are respectively the lattice spacing and the scale of renormalization for the continuum, while $p_1, \ldots, p_n$ are the external momenta. Here we consider the lattice theory (denoted by subscript $\text{latt}$) as a regularization of the continuum theory renormalized at the scale $1/a$ and we compare it with the continuum theory renormalized in the $\overline{MS}$-scheme (denoted by subscript $\overline{MS}$) at the scale $\mu$ to determine the finite constants $Z_1(\beta, \mu a)$ and $Z_2(\beta, \mu a)$. Both the regularized theories satisfy a Renormalization Group equation:

$$\frac{d}{d\mu} \Gamma^{(n)}_{\overline{MS}} = 0; \quad -\frac{d}{da} \Gamma^{(n)}_{\text{latt}} = 0;$$

(28)

where we added a minus sign for the lattice equation, because when $a \to 0$ we are making a RG flux toward short distances behaviour, that has the reversed sign respect to the $\mu \to \infty$ limit made for the continuum theory. For the lattice theory

$$0 = -a \frac{d}{da} \Gamma^{(n)}_{\text{latt}} = \left[-a \frac{\partial}{\partial a} + W^{\text{latt}}(\beta) \frac{\partial}{\partial \beta^{-1}} - \frac{n}{2} \gamma^{\text{latt}}(\beta) + \left(\frac{1}{2} \gamma^{\text{latt}}(\beta) + \beta W^{\text{latt}}(\beta)\right) h \frac{\partial}{\partial h}\right] \Gamma^{(n)}_{\text{latt}},$$

(29)

and analogously for the $\overline{MS}$-theory by using $W^{\overline{MS}}(\beta)$ and $\gamma^{\overline{MS}}(\beta)$ (in order to avoid confusion with the coupling constant, and in agreement with the literature on the subject, we denote the $\beta$-function as $W(\beta)$). By using the condition [27], we are able to join together the $\beta$ and $\gamma$-function on the lattice to those in $\overline{MS}$-scheme. Indeed we find

$$W^{\overline{MS}}(Z_1^{-1}\beta) = \left(Z_1 + \frac{1}{\beta} \frac{\partial Z_1}{\partial \beta^{-1}}\right) W^{\text{latt}}(\beta)$$

(30)

$$\gamma^{\overline{MS}}(Z_1^{-1}\beta) = \gamma^{\text{latt}}(\beta) - \frac{1}{Z_2} \frac{\partial Z_2}{\partial \beta^{-1}} W^{\text{latt}}(\beta)$$

(31)

The first of them is the important relation that allows us to express the coefficients of the $\beta$-function on the lattice in terms of the coefficients of the continuum theory.

Given the $\beta$-function for the non-linear $\sigma$-model with $N$ the number of vector components, we expand it in power of the coupling constant $1/\beta$ in a generic scheme of regularization

$$W^{\text{scheme}}(\beta) = -\frac{w_0}{\beta^2} - \frac{w_1}{\beta^3} - \frac{w_2^{\text{scheme}}}{\beta^4} + O(\beta^{-5});$$

(32)
the first two coefficients have not the superscript scheme because they are universal, they come from the calculation respectively at one and two loops (the term from order zero vanishes in two dimensions); explicitly they are given by

\[ w_0 = \frac{N - 2}{2\pi}, \quad w_1 = \frac{N - 2}{(2\pi)^2}; \]  

(33)

all the other terms are scheme-dependent; the \( w_n^{\text{scheme}} \) coefficient is associated with \( 1/\beta^{n+2} \) term of series expansion and correspond to a computation at \( (n + 1) \) loops. We report here the known results in \( \overline{MS} \)-scheme (see [33], or [29, 30] for other references)

\[ w_2^{\overline{MS}} = \frac{1}{4} \frac{N^2 - 4}{(2\pi)^3}. \]  

(34)

We also expand in \( 1/\beta \) the two renormalization constants

\[ Z_1 = Z_1^{(0)} + \frac{Z_1^{(1)}}{\beta} + \frac{Z_1^{(2)}}{\beta^2} + O(\beta^{-3}) \]  

(35)

\[ Z_2 = Z_2^{(0)} + \frac{Z_2^{(1)}}{\beta} + \frac{Z_2^{(2)}}{\beta^2} + O(\beta^{-3}) \]  

(36)

With the above conventions on the series expansions, now we look at (30) and we rewrite it as:

\[ \begin{aligned} W_{\text{latt}}(\beta) &= W_{\overline{MS}}(Z_1^{-1}\beta) \\ &= \frac{1}{Z_1 + \frac{1}{\beta} \frac{\partial Z_1}{\partial \beta}} \end{aligned} \]  

(37)

from this equation it can be seen that the coefficient of order \( n \) of the expansion of \( W_{\text{latt}} \) (i.e. \( w^{\text{latt}}_{n-1} \)) can be evaluated as long as one knows the coefficients of \( W_{\overline{MS}} \) up the same order (i.e. \( w_1^{\overline{MS}}, w_2^{\overline{MS}}, \ldots, w_{n-2}^{\overline{MS}} \)) and performs the computation on the lattice of the constants \( Z_1 \) and \( Z_2 \) up order \( n - 1 \). So we can argue the general result:

\[ w^{\text{latt}}_{n-1} = w^{\text{latt}}_{(n-\text{loop})} = F\left( \{ w_i^{\overline{MS}} \}_{i=\{0,1,...,n-1\}}; \{ Z_j^{(i)} \}_{j=\{0,1,...,n-1\}} \right). \]  

(38)

For example, for the first scheme-dependent coefficient \( w_2^{\text{latt}} \), from (37) we find

\[ w_2^{\text{latt}} = w_0 \left( (Z_1^{(1)})^2 - Z_1^{(2)} \right) + w_1 Z_1^{(1)} + w_2^{\overline{MS}}. \]  

(39)

### 5 Evaluation of the constants of renormalization

In order to obtain the perturbative expansion of the constants \( Z_1 \) and \( Z_2 \), we use relation (27) for the two-point function 1PI. We proceed as follows: we compute \( \Gamma^{(2)}_{\text{latt}} \) at \( n - 1 \) loops

\[ \text{To be precise, only the constant } Z_1 \text{ is required. The expansion for } Z_2 \text{ however comes out as a side result of the computation.} \]
and, from the knowledge of $\Gamma^{(2)}_{\text{MS}}$ at the same order, and the requirement of validity of (27), we find $Z_1$ and $Z_2$ at $n-1$ loops.

For the continuum theory we consider the expansion

$$\Gamma^{(2)}_{\text{MS}}(p, \beta, h; \mu) = -\beta (p^2 + h) + \Pi^{(0)}_{\text{MS}}(p, h; \mu) + \frac{\Pi^{(1)}_{\text{MS}}(p, h; \mu)}{\beta} + \ldots ;$$

we report the already known two-loop results [29, 30] in the case of $N = -1$

$$\Pi^{(0)}_{\text{MS}}(p, h; \mu) = \frac{1}{4\pi} (p^2 - h) \log \frac{h}{\mu^2}$$

$$\Pi^{(1)}_{\text{MS}}(p, h; \mu) = \frac{1}{16 \pi^2} \left( \log^2 \frac{h}{\mu^2} + 8 \log \frac{h}{\mu^2} - 3 + 12 (2\pi)^2 R \right) p^2 - \frac{1}{8\pi^2} \left( \log^2 \frac{h}{\mu^2} + \log \frac{h}{\mu^2} \right) h$$

where $R$ is an integral defined as

$$R := \lim_{h \to 0} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} dq_y \frac{1}{(p^2 + h)(q^2 + h)((p + q)^2 + h)} \right]$$

$$= \frac{1}{24 \pi^2} \psi'(\frac{1}{3}) - \frac{1}{36} : (42)$$

with $\psi(z) = d \log \Gamma(z)/dz$, but it appears only in intermediate stages of the computation and cancels out in any of the results. Therefore

$$Z_2 \frac{\Gamma^{(2)}_{\text{MS}}(p, Z_1^{1/2}, Z_1 Z_2^{1/2}; h; \mu)}{Z_2} = -\beta (p^2 + h) + \frac{1}{4\pi} (p^2 - h) \log \frac{h}{\mu^2} + \left( Z_1^{(1)} - Z_2^{(1)} \right) p^2 - \frac{1}{2} Z_2^{(1)} h$$

$$+ \frac{1}{\beta} \left[ \frac{1}{16 \pi^2} \left( \log^2 \frac{h}{\mu^2} + 8 \log \frac{h}{\mu^2} - 3 + 12 (2\pi)^2 R \right) p^2 - \frac{1}{8\pi^2} \left( \log^2 \frac{h}{\mu^2} + \log \frac{h}{\mu^2} \right) h \right] + \left( Z_1^{(2)} - Z_2^{(2)} + Z_1^{(1)} Z_2^{(1)} - \left( Z_1^{(1)} \right)^2 \right) p^2 - \frac{1}{2} Z_2^{(1)} h$$

$$+ \frac{1}{4\pi} \left( Z_1^{(2)} \right)^2 - \frac{1}{2} Z_2^{(2)} + \frac{1}{8\pi} \log \frac{h}{\mu^2} - \frac{Z_2^{(1)}}{4\pi} \log \frac{h}{\mu^2} - \frac{Z_2^{(1)}}{4\pi} \frac{Z_2^{(1)}}{8\pi} h \right].$$

6 The triangular lattice

On a triangular lattice each site has 6 neighbours. It is convenient to introduce a redundant basis of three vectors $e_{(i)}$, as shown in figure 1 such that $\sum_i e_{(i)} = 0$, $e_i \cdot e_i = 1$, and if $i \neq j$ then $e_i \cdot e_j = -\frac{1}{2}$. 9
Figure 1: Left: the cartesian basis and the redundant basis on the triangular lattice. Right: Brillouin zone in momentum space. The rhombus or the hexagon are equivalent choices, as the pairs of triangles denoted with $♭$ and $♯$ are related resp. by periodicity in $k_1$ and $k_2$. While the hexagon corresponds to the direct construction of the reciprocal lattice, the rhombus is computationally convenient, as it is a product of one-dimensional intervals.

Lattice sites are labelled by three integers $\{n_i\}$, with $x = \sum_i n_i e_{(i)}$. Because of redundancy, a constant can be added to the $n_i$'s without changing $x$, i.e. there is an equivalence relation

$$(n_1, n_2, n_3) \sim (n_1 + m, n_2 + m, n_3 + m).$$

A representative of each class is chosen, for example, by fixing $n_3 = 0$, as

$$(n_1, n_2, n_3) \sim (n_1 - n_3, n_2 - n_3, 0).$$

Remark that

$$x \cdot x = \frac{3}{2} \left[ \sum_i n_i^2 - \frac{1}{3} \left( \sum_i n_i \right)^2 \right].$$

Similarly, the conjugate quantity $k = \frac{2}{3} \sum_i k_i e_{(i)}$ is characterized by the three numbers $k_i$, such that $\sum_i k_i = 0$. The factor $\frac{2}{3}$ is introduced to have

$$k \cdot x = \sum_i k_i n_i.$$

As a consequence $|k_i| < \pi$ and

$$k \cdot k = \frac{2}{3} \sum_i k_i^2$$

so that the domain for $k$ is a hexagon of side $2\pi/\sqrt{3}$, or equivalently a $(\pi/6)$-angle rhombus of sides $2\pi$ (cfr. figure 1).

Now we can introduce the Fourier transform $\tilde{f}(k)$ for a function $f$ on the triangular lattice

$$\tilde{f}(k) = \sum_{\text{sites}} e^{-ik \cdot x} f(x).$$
which is such that
\[ f(x) = \frac{\int_{\text{hexagon}} d^2 k \, e^{i k \cdot x} \tilde{f}(k)}{\int_{\text{hexagon}} d^2 k} \] (50)

By specializing this general formula to the gauge (45) we get
\[ f(x) = \int \frac{\pi}{-\pi} \frac{dk_1}{2\pi} \int \frac{\pi}{-\pi} \frac{dk_2}{2\pi} e^{i[k_1(n_1-n_3)+k_2(n_2-n_3)]} \tilde{f}(k_1, k_2, -k_1 - k_2) \] (51)

where we substituted \( k_3 = -k_1 - k_2 \) and we kept into account the angle of \( 2\pi/3 \) between the vectors \( e_{(1)} \) and \( e_{(2)} \) in the integration measure. Remark that the volume of the elementary cell generated by \( e_{(1)} \) and \( e_{(2)} \) will pop out once more in the continuum limit, indeed
\[ \sum_{\text{sites}} \rightarrow \frac{2}{\sqrt{3}} \int d^2 x \] (52)
\[ \int \frac{\pi}{-\pi} \frac{dk_1}{2\pi} \int \frac{\pi}{-\pi} \frac{dk_2}{2\pi} \rightarrow \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} = \sqrt{3} \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \int_{-\infty}^{\infty} \frac{dk_y}{2\pi} \] (53)

7 Tree level

To compare with (26) let us change the normalization of the Grassmann fields to get for the free part of the action on the triangular lattice
\[ -\sum_{\text{sites}} \beta_t \left\{ \sum_i \bar{\psi}(x) [2\psi(x) - \psi(x + e_i) - \psi(x - e_i)] + h_t \bar{\psi}(x) \psi(x) \right\} \] (54)

which becomes by Fourier transform
\[ -\int \frac{\pi}{-\pi} \frac{dk_1}{2\pi} \int \frac{\pi}{-\pi} \frac{dk_2}{2\pi} \beta_t \bar{\psi}(k) \left[ \hat{k}^2 + h_t \right] \psi(k) \] (55)

where
\[ \hat{k}^2 := \sum_i \hat{k}_i^2 := \sum_i \left[ 2 \sin \left( \frac{k_i}{2} \right) \right]^2 = \sum_i (2 - 2 \cos k_i) \] (56)

By using \( \hat{k}^2 \approx \frac{3}{2} k^2 \) and (53) this becomes in the continuum limit
\[ -\frac{2}{\sqrt{3}} \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \int_{-\infty}^{\infty} \frac{dk_y}{2\pi} \beta_t \bar{\psi}(k) \left[ \frac{3}{2} k^2 + h_t \right] \psi(k) \] (57)

and it must be compared with the continuous expression
\[ -\int \frac{\infty}{-\infty} \frac{dk_x}{2\pi} \int_{-\infty}^{\infty} \frac{dk_y}{2\pi} \beta \bar{\psi}(k) \left[ k^2 + h \right] \psi(k) \] (58)
from which we get the identifications (see also [34])

$$\beta_t \equiv \frac{\beta}{\sqrt{3}} \quad \quad h_t \equiv \frac{3}{2} h$$

(59)

In the following it will be useful the evaluation of the integral

$$I(h_t) := \int_{-\pi}^{\pi} \frac{dp_1}{2\pi} \int_{-\pi}^{\pi} \frac{dp_2}{2\pi} \frac{1}{\vec{p}^2 + h_t}$$

(60)

in the limit of small $h_t$. Using the relation

$$\cos p_1 + \cos p_2 = 2 \cos \frac{p_1 + p_2}{2} \cos \frac{p_1 - p_2}{2}$$

(61)

we rewrite the denominator

$$\vec{p}^2 + h_t = 6 - 4 \cos \frac{p_1 + p_2}{2} \cos \frac{p_1 - p_2}{2} - 2 \cos(p_1 + p_2) + h_t$$

(62)

and then we make the change of variables $k_1 = \frac{p_1 + p_2}{2}$ and $k_2 = \frac{p_1 - p_2}{2}$; the Jacobian of the transformation is 2, but it simplifies with the factor 1/2 coming from the new area of integration; in fact $k_1$ and $k_2$ run inside the rhombus of vertices $(\pi, 0), (0, \pi), (-\pi, 0), (0, -\pi)$, the Brillouin zone, which is contained twice in the square area $[-\pi, \pi]^2$. So we obtain

$$I(h_t) = \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \int_{-\pi}^{\pi} \frac{dk_2}{2\pi} \frac{1}{6 - 4 \cos k_1 \cos k_2 - 2 \cos(2k_1) + h_t}$$

(63)

We are now able to integrate in $k_2$ using the result

$$\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{1}{\alpha + \beta \cos \theta} = \frac{1}{\sqrt{\alpha^2 - \beta^2}}$$

(64)

we have

$$I(h_t) = \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \frac{1}{2\sqrt{(3 - \cos(2k_1) + \frac{h_t}{4})^2 - 4 \cos^2 k_1}}$$

(65)

$$= \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \frac{1}{2\sqrt{\left(\frac{h_t + 6}{2} + 2 \sin^2 k_1\right)^2 - (h_t + 9)}}$$

$$= \int_{0}^{2\pi} \frac{dk_1}{2\pi} \frac{1}{2\sqrt{\left(\frac{h_t + 6}{2} - \cos k_1\right)^2 - (h_t + 9)}}$$
Finally, after the change \( \cos k_1 = x \), we can express our integral by an elliptic integral

\[
I(h_t) = \frac{1}{2\pi} \int_{-1}^{1} dx \frac{1}{\sqrt{(1-x^2)(h_t + 9 - x)(h_t + 9 - x) - \sqrt{h_t + 9 - x} + \frac{h_t + 8}{2}}}
\]

(67)

\[
= \frac{1}{2\pi} \frac{2}{6 + 2\sqrt{h_t + 9 + 3h_t + \frac{h_t^2}{4}}} K\left( \frac{4\sqrt{h_t + 9}}{6 + 2\sqrt{h_t + 9 + 3h_t + \frac{h_t^2}{4}}} \right)
\]

When \( h_t \to 0 \)

\[
I(h_t) = -\frac{1}{4\sqrt{3}\pi} \log \left( \frac{h_t}{72} \right) + O(h_t \log h_t)
\]

(68)

and therefore, because of (59)

\[
I(h_t) \approx -\frac{1}{4\sqrt{3}\pi} \log \left( \frac{h_t}{48} \right)
\]

(69)

We will also need the evaluation of the integral

\[
I_2(h_t) := \int_{-\pi}^{\pi} \frac{dp_1}{2\pi} \int_{-\pi}^{\pi} \frac{dp_2}{2\pi} \frac{1}{(p^2 + h_t)^2}
\]

(70)

in the limit of small \( h_t \). Of course

\[
I_2(h_t) = -\frac{\partial}{\partial h_t} I(h_t) = \frac{1}{4\sqrt{3}\pi h_t} + O(\log h_t)
\]

(71)

and therefore

\[
\lim_{h_t \to 0} h_t I_2(h_t) = \frac{1}{4\sqrt{3}\pi}.
\]

(72)

Of course the divergent part could be obtained by going to the continuum limit

\[
h_t \int_{-\pi}^{\pi} \frac{dp_1}{2\pi} \int_{-\pi}^{\pi} \frac{dp_2}{2\pi} \frac{1}{(p^2 + h_t)^2} \sim h_t \frac{3}{2} \int_{-\infty}^{\infty} \frac{dp_x}{2\pi} \int_{-\infty}^{\infty} \frac{dp_y}{2\pi} \frac{1}{\frac{9}{4} |p^2 + h_t|^2} \sim \frac{1}{4\sqrt{3}\pi}
\]

(73)

\footnote{From 3.148.2 of [35]

\[
\int_{-d}^{d} \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\beta, r)
\]

(66)

with \( a > b > c \geq u > d \) and \( \beta = \arcsin \sqrt{\frac{(a-c)(a-d)}{(c-d)(a-u)}} \). In our case \( a = h_t + \frac{8}{2} + \sqrt{h_t + 9} \), \( b = h_t + \frac{8}{2} - \sqrt{h_t + 9} \), \( c = u = 1 \), \( d = -1 \).

\( F(\beta, r) = \int_{0}^{\beta} \frac{d\theta}{\sqrt{1-r^2 \sin^2 \theta}} \) is the elliptic integral of the second kind, and if \( \beta = \frac{\pi}{2} \), \( F(\frac{\pi}{2}, r) = K(r) \) is called the complete integral.}
Figure 2: The Feynman diagrams for the two-point function at order 1.

Analogously

\[
\lim_{h_t \to 0} h_t \int_{-\pi}^{\pi} \frac{dp_1}{2\pi} \frac{dp_2}{2\pi} \frac{dq_1}{2\pi} \frac{dq_2}{2\pi} \frac{1}{(p^2 + h_t)(q^2 + h_t)(p + q)^2 + h_t)} \\
\sim \lim_{h_t \to 0} h \left(\frac{\sqrt{3}}{2}\right)^2 \left(\frac{2}{3}\right)^2 \int_{-\infty}^{\infty} \frac{dp_x}{2\pi} \frac{dp_y}{2\pi} \frac{dq_x}{2\pi} \frac{dq_y}{2\pi} \frac{1}{(p^2 + h)(q^2 + h)((p + q)^2 + h)} = \frac{R}{3},
\]

(74)

where \( R \) was defined in (42).

8 One-loop diagrams

The interaction terms on the triangular lattice are

\[
\int_p \bar{\psi}(p) \psi(p) - \frac{\beta_t}{2} \int_{p,q,k} \bar{\psi}(q + k) \psi(q) \hat{k}^2 \bar{\psi}(p - k) \psi(p)
\]

(75)

where we introduce the shorthand

\[
\int_k := \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \int_{-\pi}^{\pi} \frac{dk_2}{2\pi}.
\]

(76)

We wish to compute the 1PI two-point function. At one loop, two graphs contribute (fig. 2). On the triangular lattice, by defining

\[
\Delta(k) := \hat{k}^2 + h_t
\]

(77)

we get

\[
\Pi_0(p) = 1 - \int_k \frac{\hat{p} + \hat{k}^2}{\Delta(k)} = 1 - \int_k \frac{\hat{p}^2 + \hat{k}^2 - \frac{1}{2} \sum_i \hat{p}_i^2 \hat{k}_i}{\Delta(k)} \\
= 1 - \hat{p}^2 I - 1 + h_t I + \frac{1}{6} \hat{p}^2 (1 - h_t I).
\]

(78)
By going to the continuum limit, in the limit of small magnetic field

\[ \Pi_0(p) \sim \frac{\hat{p}^2}{6} - \left[ \hat{p}^2 - h_t \right] I(h_t) \]  
(79a)

\[ \rightarrow \frac{2}{\sqrt{3}} \left\{ \frac{3}{2} \frac{p^2}{6} + \frac{3}{2} [p^2 - h] \frac{1}{4\sqrt{3} \pi} \log \frac{ha^2}{48} \right\} \]  
(79b)

\[ = \frac{p^2}{2\sqrt{3}} + \frac{1}{4\pi} [p^2 - h] \frac{1}{2} \log \frac{ha^2}{48} \]  
(79c)

By comparing the two expressions we obtain the one-loop result

\[ Z_1 = 1 + \frac{3}{4\pi \beta} \log \frac{\mu^2 a^2}{48} + \frac{1}{2\sqrt{3} \beta} + \mathcal{O}\left(\frac{1}{\beta^2}\right) \]  
(80)

\[ Z_2 = 1 + \frac{2}{4\pi \beta} \log \frac{\mu^2 a^2}{48} + \mathcal{O}\left(\frac{1}{\beta^2}\right) \]  
(81)

which, of course, result to be independent from the magnetic field.

9 Two-loop diagrams

The diagrams at second order are the four ones shown in figure 3. As we expected these are the same Feynman diagrams that appear at the second order of perturbative expansion of the \( \sigma \)-model [29]. According to the Feynman rules we have to add a minus sign to the diagrams A and C: for the first one since it has a mass insertion, for the second since it has a loop. So that the expression of the second order contribution of self-energy is

\[ \Pi_1 = A - B + C - D \]  
(82)

with

\[ A = \int k \frac{\hat{p} + k^2}{\Delta(k)^2} \]  
(83)

\[ B = \int_{k,q} \frac{\hat{p} + k^2 \hat{k} + \hat{q}^2}{\Delta(q)\Delta(k)^2} \]  
(84)

\[ C = \int_{k,q} \frac{(\hat{q}^2)^2}{\Delta(p + q)\Delta(k)\Delta(k + q)} \]  
(85)

\[ D = \int_{k,q} \frac{\hat{p} + \hat{q}^2 k^2}{\Delta(q)\Delta(k - q)\Delta(p + k)} \]  
(86)
Figure 3: The Feynman diagrams for the two-point function at second order. On the top left corners, we report the identificative letters.

The first two diagrams are easy to evaluate exactly in terms of $I$ and $I_2$. We find

$$A = \hat{p}^2 \left[ -\frac{1}{6} I + I_2 + \frac{1}{6} h t I_2 \right] + I - h t I_2 \quad (87)$$

$$B = \hat{p}^2 \left[ I_2^2 - \frac{1}{2} I + I_2 + \frac{1}{3} h t I_2 - 2 h t I_2 I + \frac{1}{36} \right] + \quad (88)$$

$$+ 2 I - \frac{1}{6} - 3 h t I^2 - h t I_2 + \frac{1}{2} h t I - \frac{1}{6} h^2 I_2 + 2 h^2 I_2 I \quad (89)$$

The diagrams $C$ and $D$ are more involved. First of all remark that

$$D - C = \int \frac{\Delta(k) \Delta(k+q) \Delta(q + p)}{\Delta(q) \Delta(k+q) \Delta(p+k)} = \int \frac{\Delta(k) \Delta(p + k + q) - \Delta(k)}{\Delta(q) \Delta(k+q) \Delta(p+k)} \quad (90a)$$

$$\sim \int \frac{\Delta(k) \Delta(p + k + q) - \Delta(k)}{\Delta(q) \Delta(k+q) \Delta(p+k)} \quad (90b)$$

$$= \int \frac{\Delta(k) \Delta(p + r) - \Delta(k)}{\Delta(r - k) \Delta(r) \Delta(p+k)} \quad (90c)$$

where in (90b) we neglect terms of higher order in the small-$h$ expansion. We are interested in the first terms of the Taylor expansion for small external momentum. We get

$$D - C \sim \int \frac{1}{\Delta(r-k)} \left\{ 1 + \frac{4}{\Delta(k)} \sum_i \sin p_i \sin k_i \sum_j \sin p_j \left[ \frac{\sin k_j}{\Delta(k)} - \frac{\sin r_j}{\Delta(r)} \right] \quad (91) \right. \right.$$
We easily get

\[ \int_{k,r} \frac{1}{\Delta(r-k)} = I \]  

(92)

\[ - \int_{k,r} \frac{\Delta(k)}{\Delta(r-k)\Delta(r)} = -2I + \frac{1}{6} h_t I^2 - \frac{h_t}{3} I \]  

(93)

\[ \int_{k,r} \tilde{p}^2 = \tilde{p}^2 I^2 \]  

(94)

\[ -\frac{1}{2} \int_{k,r} \frac{1}{\Delta(r-k)\Delta(r)} \sum_i \tilde{p}_i^2 k_i^2 = \tilde{p}^2 \left( -\frac{I}{3} + \frac{1}{36} \right) \]  

(95)

We have still to compute (changing \( r \) into \(-r\))

\[ 4 \sum_{i,j} \sin p_i \sin p_j \int_{k,r} \frac{\sin k_i}{\Delta(r+k)\Delta(k)} \left[ \frac{\sin k_j}{\Delta(k)} + \frac{\sin r_j}{\Delta(r)} - \frac{\sin k_j}{\Delta(r)} \right] \]  

(96)

The tensor form of the expression above is

\[ \sum_{i,j} \sin p_i \sin p_j \Lambda_{ij} \]  

(97)

with \( \Lambda_{ij} \) symmetric under the exchange of \( i \) with \( j \), and permutation of indices 1, 2, 3, so that in general \( \Lambda_{ij} = a + b \delta_{ij} \), which substituted into the previous expression gives \( a(\sum_i \sin p_i)^2 + b \sum_i \sin^2 p_i \sim b \tilde{p}^2 + O(p^4) \) because we have that \( \sum_i p_i = 0 \). Therefore we need only the coefficient \( b \) which can be computed, for example, as

\[ \Lambda_{11} - \Lambda_{13} = 4 \int_{k,r} \frac{\sin k_1 - \sin k_3}{\Delta(r+k)\Delta(k)} \left[ \frac{\sin k_1}{\Delta(k)} + \frac{\sin r_1}{\Delta(r)} - \frac{\sin k_1}{\Delta(r)} \right] \]  

(98)

Then we get

\[ \int_{k,r} \frac{(\sin k_1 - \sin k_3) \sin k_1}{\Delta(r+k)\Delta^2(k)} = I \int_{k,r} \frac{(\sin k_1 - \sin k_3) \sin k_1}{\Delta^2(k)} \]  

\[ = I \left[ \frac{1}{2}(I - h_t I^2) - \frac{1}{12} + \frac{1}{8\sqrt{3} \pi} \right] \]  

and

\[ \int_{k,r} \frac{\sin^2 k_1}{\Delta(r+k)\Delta(k)\Delta(r)} = \frac{I^2}{3} - I \left( \frac{1}{6} - \frac{1}{2\sqrt{3} \pi} \right) - \frac{R}{9} - \frac{G}{4} \]  

(99)

\[ \int_{k,r} \frac{\sin k_1 \sin k_3}{\Delta(r+k)\Delta(k)\Delta(r)} = -\frac{I^2}{6} + \frac{I}{4\sqrt{3} \pi} + \frac{R}{18} - \frac{G}{8} - \frac{1}{144} \]  

(100)

\[ \int_{k,r} \frac{\sin k_1 \sin r_1}{\Delta(r+k)\Delta(k)\Delta(r)} = -\frac{I^2}{6} + I \left( \frac{1}{12} - \frac{1}{4\sqrt{3} \pi} \right) + \frac{R}{18} + \frac{G}{8} + \frac{L}{24} \]  

(101)

\[ \int_{k,r} \frac{\sin k_3 \sin r_1}{\Delta(r+k)\Delta(k)\Delta(r)} = \frac{I^2}{12} - \frac{I}{8\sqrt{3} \pi} - \frac{R}{36} + \frac{G}{16} - \frac{K}{16} - \frac{L}{48} + \frac{1}{288} \]  

(102)
with

\[ G := \int_{k,r} \frac{k_1 + r_1^4 [\Delta(r + k) - \Delta(k) - \Delta(r)]}{\Delta^2(r + k)\Delta(k)\Delta(r)} \] (103)

\[ K := \int_{k,r} \frac{k_1^2 k_2 k_3 r_1 k_1 + r_1}{\Delta(r + k)\Delta(k)\Delta(r)} \] (104)

\[ L := \int_{k,r} \frac{k_1 + r_1^2 r_1^2}{\Delta(r + k)\Delta(k)\Delta(r)} \] (105)

So finally we found:

\[ D - C = \bar{p}^2 \left[ I \left( \frac{1}{3} - \frac{3}{2\sqrt{3}\pi} \right) + R + \frac{3G + K + L}{4} - \frac{1}{72} \right] - I + \frac{1}{6} + h_t I^2 - \frac{h_t I}{3} \] (106)

and in conclusion

\[ \Pi_1 = \bar{p}^2 \left[ I^2 - \frac{2I}{\sqrt{3}\pi} + \frac{1}{72} + \frac{1}{24\sqrt{3}\pi} + R + \frac{3G + K + L}{4} \right] + h_t \left[ -2I^2 + I \left( \frac{1}{6} + \frac{1}{2\sqrt{3}\pi} \right) - \frac{1}{24\sqrt{3}\pi} \right] \] (107)

By comparing the two expressions [43] and (107) we obtain the two-loop result

\[ Z_1^{(2)} = \frac{9}{16\pi^2} \log^2 \frac{\mu^2 a^2}{48} + \frac{\sqrt{3}}{4\pi} \log \frac{\mu^2 a^2}{48} + \frac{3}{8\pi^2} \log \frac{\mu^2 a^2}{48} + \frac{3(3G + K + L)}{4} + \frac{3}{16\pi^2} + \frac{1}{8} \] (108)

\[ Z_2^{(2)} = \frac{5}{16\pi^2} \log^2 \frac{\mu^2 a^2}{48} + \frac{1}{4\sqrt{3}\pi} \log \frac{\mu^2 a^2}{48} \] (109)

and the three-loop result by (39)

\[ w_2^{\text{latt}} = -0.01375000819. \] (110)

By application of the coordinate-space method by Lüscher and Weisz [36] suitably modified for the triangular lattice (see appendix [A]) we have obtained the numerical determinations

\[ G = -0.025786368 \] (111a)

\[ K = -0.007632210 \] (111b)

\[ L = 0.035410394 \] (111c)

with errors smaller than the quoted digits, from which we recover the value

\[ w_2^{\text{latt}} = -0.01375000819. \] (112)
The determination of the coefficient $w_l^{\text{latt}}$ can be used, as shown in Ref. [22], to recover, for example, the phase boundary in the plane of complex temperature for the $q$-state Potts model in the limit $q \to 0$. This separatrix is, indeed, a special renormalization-group flow curve. If we call $x$ and $y$, respectively, the real and imaginary part of the complex temperature we must have therefore that

$$y(x) = y_0 \left( 1 + \frac{A_1}{x} + \frac{A_2}{x^2} + \cdots \right) \quad (113)$$

where

$$A_1 = \frac{\frac{w_1}{\sqrt{3}}}{\frac{w_0}{\sqrt{3}}} = -\frac{1}{2\pi \sqrt{3}} \quad \quad A_2 = \frac{\frac{w_l^{\text{latt}}}{\sqrt{3}}}{\frac{w_0}{\sqrt{3}}} = \frac{2\pi}{9} \frac{w_l^{\text{latt}}}{w_2^{\text{latt}}} \quad (114)$$

and $y_0$ was numerically estimated to be

$$y_0 \approx 0.394 \pm 0.004 \quad (115)$$

For numerical purposes in [22] a variant parametrization is followed, that is

$$y(x) = y_0 \exp \left[ 1 + \frac{B_1}{x - \alpha_0} + \frac{B_2}{(x - \alpha_0)^2} + \frac{B_3}{(x - \alpha_0)^3} + \cdots \right] \quad (116)$$

where comparison with (113) in the limit of large $x$ gives the relations

$$B_1 = A_1 = -\frac{1}{2\pi \sqrt{3}} \quad B_2 = A_2 - \frac{A_1^2}{2} - \alpha_0 A_1 \quad (117)$$

The parameter $\alpha_0$, and $A_i$ and therefore $B_i$ with $i \geq 2$ were not known. In [22] the authors decided to truncate (116) by setting $B_i = 0$ for $i \geq 3$ and try estimated $\alpha_0$ and $B_2$ by this ansatz by imposing the value of the function and its derivative on the last known numerical point, that is $y(0.0198) = 0.23$ and $y'(0.0198) = 0.369003$. They estimated

$$\alpha_0 \approx -0.550842 \quad (118)$$
$$B_2 \approx -0.122843 \quad (119)$$

From our calculation we get an evaluation of $B_2$

$$A_2 \approx 0.00959932 \quad (120)$$
$$B_2 \approx 0.0053776 + \frac{\alpha_0}{\frac{2\pi \sqrt{3}}{2\pi \sqrt{3}}} \quad (121)$$

so that we can use the strategy just discussed to derive $B_3$ in addition to $\alpha_0$ and $B_2$. We obtain

$$\alpha_0 \approx -0.778527 \quad (122)$$
$$B_2 \approx -0.066160 \quad (123)$$
$$B_3 \approx -0.162495 \quad (124)$$

The curve resulting from this numerical values does not differ substantially from the old one as can be seen in Fig. 4. This gives more confidence on the method and results in [22].
Figure 4: Phase boundaries for infinite strips of the triangular lattice. Numerical values from different lattice widths $L$, from 2 to 9, on gray-tone curves from left to right. Black dots reproduce the extrapolated $L \rightarrow \infty$ limiting curve in the region of negative $\text{Re} \,(1/t)$. The black dotted-dashed curve and the continuous black curve (almost indistinguishable), in the region of positive $\text{Re} \,(1/t)$, are respectively the old and new curves from the ansatz of equation (116). In the magnification on the right, we plot the discrepancy $\delta$ between the two curves along the $\text{Im} \,(1/t)$ axis, as a function of $\text{Re} \,(1/t)$: as it should, it vanishes with its first derivative, at the numerical point 0.0198 used for the extrapolation, and vanishes asymptotically because the same estimate of the asymptote $y_0$ is used; all in between, it remains of order $10^{-3}$.

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A  Lüscher-Weisz method for evaluation of lattice integrals

In the evaluation of two-dimensional lattice integrals, we used the coordinate method illustrated in the paper by Lüscher and Weisz [36], and specialized to two dimensions by
Dong-Shin Shin [31, 37], although also the momenta method proposed in the appendix C of [38] could have been used.

The main idea is the use of some basic relations for the free propagator in coordinate space (the defining Laplacian equation and a set of relations due to Vohwinkel), in order to find a recursion which, starting from the values in a certain number of sites neighbouring the origin (the fundamental lattice integrals), allows to find the whole set of free propagators in lattice sites in a large radius \( R \), in a time which scales polynomially with \( R \). As a side result, it gives a simple proof of the fact that all these values are linear combinations with rational coefficients of the fundamental lattice integrals.

Generalization of the procedure to the triangular lattice is not straightforward, and involves some delicate points. Some of them are:

- In the redundant set of variables \((p_1, p_2, p_3)\), the constraint \( \sum_i p_i = 0 \) does not allow for derivatives in a single variable: one should either perform linear combinations of derivatives where the sum of coefficients is zero (for example, \((\nabla_1 - \nabla_3) f(p_1, p_2, p_3)\)), or equivalently, perform derivation within a non-redundant choice of variables (for example, \(\nabla_1 f(p_1, p_2, -p_1 - p_2)\)).

- because of this fact, the Vohwinkel relations involve a larger number of terms, and thus it is more difficult to manipulate them in order to have a recursion relation. It will turn out that a larger strip is required for the first \( x \)-axis recursion.

- For \( G_\ell(x) \) at values of \( \ell \) larger than 1, the choice of subtraction is now not anymore easily deduced by the Taylor expansion of the exponential and the requirement of periodicity. Now we also have the requirement of gauge-invariance under \( x \to x + m(1, 1, 1) \), which forces the application of “hat” factors only to combinations of \( p_i \) where the sum of coefficients is zero.

Coming back to the point, the free subtracted propagator

\[
G(x) = \int_k \frac{e^{i k \cdot x} - 1}{\hat{k}^2} \tag{125}
\]

satisfies the Laplace equation

\[
-\Delta G(x) = \delta^{(2)}(x) . \tag{126}
\]

with the lattice operators

\[
\Delta := \sum_{i=1}^{3} \nabla_i^* \nabla_i = \sum_{i=1}^{3} (\nabla_i - \nabla_i^*) \tag{127}
\]

and we have

\[
G(0, 0) = 0 \quad G(1, 0) = -\frac{1}{6} . \tag{128}
\]
For the triangular function $H(x)$ defined as

$$H(x) = \int_k e^{ik\cdot x} \ln \hat{k}^2$$

(129)

a set of Vohwinkel relations holds

$$G(x + \hat{\mu}) - G(x - \hat{i}) - G(x + \hat{\nu}) + G(x - \hat{\nu}) = (x_\mu - x_\nu)H(x)$$

(130)

but only two of them (e.g. $(\mu, \nu) = (1, 2)$ or $(1, 3)$) are independent.

Using the previous equations and the Laplace equation (126), we are able to eliminate $H(x)$, and write a recursion relation. The one we find on a width-2 strip along the $x$ axis is given by the set of equations

$$0 = -6G(x, 1) + \sum_{\pm \mu} G((x, 1) \pm \hat{\mu})$$

(131)

$$0 = -6G(x, 0) + 2(G(x + 1, 1) + G(x, 1)) + G(x + 1, 0) + G(x - 1, 0)$$

(132)

$$0 = (G(x - 1, 1) - G(x + 1, 1)) + x(G(x, 2) - G(x, 0)) + (x - 1)(G(x + 1, 2) - G(x - 1, 0))$$

(133)

which must be solved with respect to $G(x+1, a)$, with $a = 0, 1, 2$, in order to have a consistent recursion. A new fundamental integral is required. A choice could be $G(2, 1)$, which is valued

$$G(2, 1) = \frac{1}{3} - \frac{\sqrt{3}}{\pi}$$

(134)

In a similar fashion, given the values of $G(x)$ on the width-2 strip, the function can be determined in the whole plane (a sector with $x = (n, m)$, with $n \geq 2m \geq 0$ is sufficient, because of symmetry). The Laplacian equation alone is enough to fulfill this task. So we conclude that at all values of $x$ the function $G(x)$ is in the set $\mathbb{Q} + \sqrt{3} \pi \mathbb{Q}$.

The integrals of the form

$$K_1^{2n} = \int_p \frac{p^{2n}}{p^2}$$

(135)

which involve $G(x)$ only on the real axis, are easily computed, the first values being

$$K_1^4 = -\frac{4}{3} + \frac{4\sqrt{3}}{\pi} \quad K_1^6 = 16 - \frac{24\sqrt{3}}{\pi} \quad K_1^8 = -\frac{448}{3} + \frac{288\sqrt{3}}{\pi}.$$

(136)

The next ingredient we need in order to calculate all the triangular-lattice quantities arising from our diagrammatics is the two-propagator function in coordinate space. It turns out that the proper subtraction is the following

$$G_2(x) = \int_k \frac{e^{ik\cdot x} - 1 + \frac{1}{4}(\hat{k}^2 - 2k_3^2)(x_1 - x_2)^2 + \text{cyclics}}{(\hat{k}^2)^2}$$

(137)
The triangular-lattice Laplacian relation still reads

\[- \Delta G_2(x) = G(x) \]  \hspace{1cm} (138)

while the Vohwinkel relation, still for \((\mu, \nu) = (1, 2)\) or \((1, 3)\), is

\[G_2(x + \hat{\mu}) - G_2(x - \hat{\mu}) - G_2(x + \hat{\nu}) + G_2(x - \hat{\nu}) = -(x_{\mu} - x_{\nu}) \left( G(x) + \frac{1}{4\sqrt{3} \pi} \right) \]  \hspace{1cm} (139)

(remark the presence of the corrective contribution \(1/(4\sqrt{3} \pi)\) due to regularization). At the aim of building the recursion, also in this case it turns out that, as the “support” of the relations is identical to the one of the triangular-lattice \(G(x)\) case, the independent lattice integrals still must be the ones located at the points \(x \in \{(0, 0), (1, 0), (2, 1)\}\). The first two vanish because of the subtraction, while the last one is computed analytically, with the result

\[G_2(2, 1) = \frac{1}{4\sqrt{3} \pi} \]  \hspace{1cm} (140)

and thus, as it is again a rational times \(\sqrt{3}/\pi\), still the function \(G_2(x)\) at a generic point is in the set \(\mathbb{Q} + \frac{\sqrt{3}}{\pi} \mathbb{Q}\).

The two-propagator analogues of the quantities \(K_{2n}^{12}\) are the integrals of the form

\[K_{2n}^{2(n)} = \int \frac{P_{2n}^{\mu}}{(P^2)^2} \]  \hspace{1cm} (141)

They still involve \(G_2(x)\) only on the real axis, and thus are easily computed, the first values being

\[K_2^4 = \frac{1}{3} - \frac{\sqrt{3}}{3\pi} \] \hspace{1cm} \[K_2^6 = -4 + \frac{8\sqrt{3}}{\pi} \] \hspace{1cm} \[K_2^8 = \frac{176}{3} - \frac{104\sqrt{3}}{\pi} \]  \hspace{1cm} (142)

We need also the lattice sums

\[G = \sum_{x \in \mathbb{Z}^2} \left[ (\nabla_1^* - \nabla_1)^2 G(x) \right] [G(x)]^2 \]  \hspace{1cm} (143)

\[K = 2 \sum_{x \in \mathbb{Z}^2} \left[ (\nabla_1^* - \nabla_1) (\nabla_2^* + \nabla_2) G(x) \right] \left[ (\nabla_3^* + \nabla_3) G(x) \right] G(x) \]
\[+ \sum_{x \in \mathbb{Z}^2} \left[ (\nabla_1^* - \nabla_1) (\nabla_2^* + \nabla_2) (\nabla_3^* + \nabla_3) G(x) \right] [G(x)]^2 \]  \hspace{1cm} (144)

\[L = \sum_{x \in \mathbb{Z}^2} \left[ (\nabla_1^* - \nabla_1) G(x) \right]^3 \]  \hspace{1cm} (145)

At this aim we need the full strength of coordinate method: we evaluate the subtracted propagators \(G(x)\) and \(G_2(x)\) on lattice points up to a given hexagon of side \(r\) \((\sim 25)\), exactly in terms of rationals, in negligible computational time \(\mathcal{O}(r^2)\), from which we deduce the \(\mathcal{O}(r^2)\) largest terms in the sums above, while the remaining contribution is estimated from the large-distance behaviour of the integrands. The numerical results are reported in equations (111).
Details on two loop lattice integrals

By using the identity
\[2 \sin \alpha + \sin \beta + \sin \gamma = -\hat{\alpha} \hat{\beta} \hat{\gamma},\]  \hspace{1cm} (146)
valid when \(\alpha + \beta + \gamma = 0\), in the cases \((\alpha, \beta, \gamma) = (k_1, k_2, k_3)\) or \((k_i, r_i, -k_i - r_i)\), we easily get

\[4 \int \frac{\left(\sum_i \sin k_i\right)^2}{\Delta(r + k)\Delta(k)\Delta(r)} = \int \frac{k_1^2 k_2^2 k_3^2}{\Delta(r + k)\Delta(k)\Delta(r)} = -2 I \left(1 - \frac{2}{\sqrt{3}} \pi\right) - 6 G - \frac{1}{6}\]

\[4 \int \frac{\sum_{i,j} \sin k_i \sin r_j}{\Delta(r + k)\Delta(k)\Delta(r)} = \int \frac{\hat{k}_1 \hat{r}_1 \hat{k}_2 \hat{r}_2 \hat{k}_3 \hat{r}_3}{\Delta(r + k)\Delta(k)\Delta(r)} = I \left(1 - \frac{2}{\sqrt{3}} \pi\right) + 3 G - \frac{3 K}{2} + \frac{1}{12}\]

\[4 \int \prod_{i=1,2} \frac{\sin k_i + \sin r_i - \sin (k_i + r_i)}{\Delta(r + k)\Delta(k)\Delta(r)} = \int \frac{\hat{k}_1 \hat{r}_1 \hat{k}_1 \hat{r}_1}{\Delta(r + k)\Delta(k)\Delta(r)} = -\frac{L}{2} - \frac{3 K}{2}\]

We also see that
\[-2 \sum_i \left[\sin k_i + \sin r_i - \sin (k_i + r_i)\right] = -\sum_i \hat{k}_i \hat{r}_i \hat{k}_i + r_i\]
\[= \hat{k}_1 \hat{k}_2 \hat{k}_3 + \hat{r}_1 \hat{r}_2 \hat{r}_3 - k_1 + r_k 2 + r_2 k_3 + r_3\]

therefore
\[\int \frac{\hat{k}_1 \hat{r}_1 \hat{k}_1 + r_1 \sum_i \hat{k}_i \hat{r}_i \hat{k}_i + r_i}{\Delta(r + k)\Delta(k)\Delta(r)} = \int \frac{\hat{k}_1 \hat{k}_2 \hat{k}_3 \sum_i \hat{k}_i \hat{r}_i \hat{k}_i + r_i}{\Delta(r + k)\Delta(k)\Delta(r)} = -3 K\]  \hspace{1cm} (147)

computed either using the first two lines or the last two lines of the previous block of identities.

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