Abstract

For a given positive integer \( \ell \) we show the existence of the limiting gap distribution measure for the sets of Farey fractions \( \frac{a}{q} \) of order \( Q \) with \( \ell \nmid a \), and respectively with \((q, \ell) = 1\), as \( Q \to \infty \).

1 Introduction

The set \( \mathcal{F}_Q \) of Farey fractions of order \( Q \) consists of those rational numbers \( \frac{a}{q} \in (0, 1] \) with \((a, q) = 1\) and \( q \leq Q \). The spacing statistics of the increasing sequence \( (\mathcal{F}_Q) \) of finite subsets of \((0, 1]\) have been investigated by several authors [9, 1, 7]. Recently Badziahin and Haynes considered a problem related to the distribution of gaps in the subset \( \mathcal{F}_{Q,d} \) of \( \mathcal{F}_Q \) of those fractions \( \frac{a}{q} \) with \((q, d) = 1\), where \( d \) is a fixed positive integer and \( Q \to \infty \). They proved [2] that, for each \( k \in \mathbb{N} \), the number \( N_{Q,d}(k) \) of pairs \((\frac{a}{q}, \frac{a'}{q'})\) of consecutive elements in \( \mathcal{F}_{Q,d} \) with \( a'q - aq' = k \) satisfies the asymptotic formula

\[
N_{Q,d}(k) = c(d, k)Q^2 + O_{d,k}(Q \log Q) \quad (Q \to \infty),
\]

for some positive constant \( c(d, k) \) that can be expressed using the measure of certain cylinders associated with the area-preserving transformation introduced by Cobeli, Zaharescu, and the first author in [4]. The pair correlation function of \( (\mathcal{F}_{Q,d}) \) was studied and shown to exist by Xiong and Zaharescu [11], even in the more general situation where \( d = d_Q \) is no longer constant but increases according to the rules \( d_{Q_1} \mid d_{Q_2} \) as \( Q_1 < Q_2 \) and \( d_Q \ll Q^{\log \log Q / 4} \).

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This paper is concerned with the gap distribution of the sequence of sets \((\mathcal{F}_{Q,d})\), and respectively of \((\tilde{\mathcal{F}}_{Q,\ell})\), the sequence of sets \(\tilde{\mathcal{F}}_{Q,\ell}\) of Farey fractions \(\gamma = \frac{a}{q} \in \mathcal{F}_Q\) with \(\ell \nmid a\). Our peculiar interest in \(\tilde{\mathcal{F}}_{Q,\ell}\) arises from the problem studied in [5], concerning the distribution of the free path associated to the linear flow through \((0,0)\) in \(\mathbb{R}^2\) in the small scatterer limit, in the case of circular scatterers of radius \(\varepsilon > 0\) placed at the points \((m,n)\in \mathbb{Z}^2\) with \(\ell \nmid (m-n)\). When \(\ell = 3\) this corresponds, after suitable normalization, to the situation of scatterers distributed at the vertices of a honeycomb tessellation, and the linear flow passing through the center of one of the hexagons. When \(\ell = 2\) the scatterers are placed at the vertices of a square lattice and the linear flow passes through the center of one of the squares. Arithmetic properties of the number \(\ell\) are shown to be explicitly reflected by the gap distribution of the elements of \((\tilde{\mathcal{F}}_{Q,\ell})\). The symmetry \(x \mapsto 1-x\) shows that for the purpose of studying the gap distribution of these fractions on \([0,1]\) one can replace the condition \(\ell \nmid (m-n)\) by the more esthetic one \(\ell \nmid n\).

The gap distribution (or nearest neighbor distribution) of a numerical sequence, or more generally of a sequence of finite subsets of \([0,1)\), measures the distribution of lengths of gaps between the elements of the sequence. Let \(\mathcal{A} = \{x_0 \leq x_1 \leq \ldots \leq x_N\}\) be a finite list of numbers in \([0,1)\), scaled to \(\tilde{x}_N = \frac{N x_N - x_0}{\tilde{x}_N}\) with mean spacing \(\tilde{x}_N = 1\). The gap distribution measure of \(\mathcal{A}\) is the finitely supported probability measure on \([0,\infty)\) defined by

\[
\nu_{\mathcal{A}}(\xi) = \frac{1}{N} \# \{ j \in [1, N] : \tilde{x}_j - \tilde{x}_{j-1} \leq \xi \}, \quad \xi \geq 0.
\]

If it exists, the weak limit \(\nu = \nu_{\mathcal{A}}\) of the sequence \((\nu_{\mathcal{A}_n})\) of probability measures associated with an increasing sequence \(\mathcal{A} = (\mathcal{A}_n)\) of finite lists of numbers in \([0,1)\), is called the limiting gap measure of \(\mathcal{A}\).

It is elementary (see, e.g., Lemma 1 below) that

\[
\#\tilde{\mathcal{F}}_{Q,\ell} = K_{\ell} Q^2 + O_{\ell}(Q \log Q), \quad \#\mathcal{F}_{Q,d} = K_d Q^2 + O_d(Q \log Q),
\]

where

\[
K_{\ell} = \frac{1}{2 \zeta(2)} - \frac{C(\ell)}{2\ell}, \quad K_d = \frac{C(d)}{2}, \quad \text{with} \quad C(\ell) = \frac{1}{\zeta(2)} \prod_{p | \ell} \left(1 + \frac{1}{p}\right)^{-1}.
\]

We prove the following result:

**Theorem 1.** Given positive integers \(\ell\) and \(d\), the limiting gap measures \(\tilde{\nu}_\ell\) of \((\tilde{\mathcal{F}}_{Q,\ell})\), and respectively \(\nu_d\) of \((\mathcal{F}_{Q,d})\), exist. Their densities are continuous on \([0,\infty)\) and real analytic on each component of \((0,\infty)\setminus \mathbb{N}K_{\ell}\), and respectively of \((0,\infty)\setminus \mathbb{N}K_d\).

The existence of \(\tilde{\nu}_\ell\) is proved in Section 2 and the limiting gap distribution is explicitly computed in (2.9) using tools from [4], [8] and [5]. The result on \(\nu_d\) is proved in Section 4. When \(d\) is a prime power, an explicit computation can be
done as for \( \tilde{\nu}_\ell \). In general the repartition function of \( \nu_d \) depends on the measure of some cylinders associated with the transformation \( T \) from (2.7), and on the length of strings of consecutive elements in \( \mathcal{F}_Q \) with at least one denominator relatively prime with \( d \).

The upper bound 4\( d^3 \) for \( L(d) = \min \{ L : \forall i, \forall Q, \exists j \in [0, L], (q_{i+j}, d) = 1 \} \) was found in [2], where \( q_i, \ldots, q_{i+L} \) denote the denominators of a string \( \gamma_i < \cdots < \gamma_{i+L} \) of consecutive elements in \( \mathcal{F}_Q \). Although we expect this bound to be considerably smaller, we could only improve it in a limited number of situations. In Section 3 we lower it to 4\( \omega(d)^3 \) for integers \( d \) with the property that the smallest prime divisor of \( d \) is \( \geq \omega(d) \), where \( \omega(d) \) denotes as usual the number of distinct prime factors of \( d \). The bound \( L(d) = 1 \) is trivial when \( d \) is a prime power. Employing properties of the transformation \( T^2 \) we show that \( L(d) \leq 5 \) when \( d \) is the product of two prime powers, which is sharp. Finding better bounds on \( L(d) \) when \( \omega(d) \geq 3 \) appears to be an interesting problem in combinatorial number theory.

2 The gap distribution of \( \tilde{\mathcal{F}}_{Q,\ell} \)

Let \( \mathcal{F}_{Q}^{(\ell)} = \mathcal{F}_Q \setminus \tilde{\mathcal{F}}_{Q,\ell} \) denote the set of Farey fractions \( \gamma = \frac{a}{q} \in \mathcal{F}_Q \) with \( \ell \mid a \), and let \( N_Q^{(\ell)} \) denote the cardinality of \( \mathcal{F}_{Q}^{(\ell)} \). Consider also:

\[
\mathcal{G}_Q(\xi) := \left\{ (\gamma, \gamma') : \gamma, \gamma' \text{ consecutive in } \mathcal{F}_Q, 0 < \gamma' - \gamma \leq \frac{\xi}{Q^2} \right\},
\]

\[
\mathcal{G}_Q^{(\ell)}(\xi) := \left\{ (\gamma, \gamma') : \gamma, \gamma' \text{ consecutive in } \tilde{\mathcal{F}}_{Q,\ell}, 0 < \gamma' - \gamma \leq \frac{\xi}{Q^2} \right\},
\]

\[
N_Q(\xi) := \# \mathcal{G}_Q(\xi), \quad N_Q^{(\ell)}(\xi) := \# \mathcal{G}_Q^{(\ell)}(\xi).
\]

Lemma 1. \( N_Q^{(\ell)} = \frac{C(\ell)}{2\ell} Q^2 + O_\ell(Q \log Q) \) as \( Q \to \infty \).

Proof. It is clear that

\[
N_Q^{(\ell)} = \# \mathcal{F}_{Q}^{(\ell)} = \sum_{q=1}^{Q} \sum_{a=1 \atop (\ell, q) = 1}^{q} 1.
\]

Letting \( k = \frac{q}{\ell} \) and noting that whenever \( (\ell, q) = 1 \) we have \( (k\ell, q) = 1 \) if and only if \( (k, q) = 1 \), the sum above becomes

\[
\sum_{q=1}^{Q} \sum_{k=1 \atop (k, q) = 1}^{[q/\ell]} 1.
\]
Standard Möbius summation, cf. (A.1) and (A.2), and $\sum_{q=1}^{Q} \sigma_0(q) = O(Q \log Q)$, where $\sigma_0(q) = \sum_{d|q} 1$, yield

$$
\sum_{q=1}^{Q} \frac{[q/\ell]}{Q \ell} = \sum_{q=1}^{Q} \left( \frac{\varphi(q)}{q} \cdot \frac{q}{\ell} + O(\sigma_0(q)) \right) = \frac{C(\ell)}{2\ell} Q^2 + O(\ell Q \log Q),
$$

concluding the proof.

This also establishes the first equality in (1.2) because

$$
\#\mathcal{F}_{Q,\ell} = \#\mathcal{F}_Q - \#\mathcal{F}_Q^{(\ell)} \sim \left( \frac{1}{2\zeta(2)} - \frac{C(\ell)}{2\ell} \right) Q^2.
$$

Letting $\xi > 0$ and $Q, \ell \in \mathbb{N}$ with $\ell \geq 2$, we set out to asymptotically estimate the number $N_Q^{(\ell)}(\xi)$ as $Q \to \infty$. Now if $\gamma = \frac{a}{q}$ and $\gamma' = \frac{a'}{q'}$ are consecutive elements in $\mathcal{F}_Q$ and $\gamma' \in \mathcal{F}_Q^{(\ell)}$, then $1 = a'q - aq' \equiv -aq' \pmod{\ell}$, which implies that $(a, \ell) = 1$, and thus $\gamma \notin \mathcal{F}_Q^{(\ell)}$. Similarly, if $\gamma \in \mathcal{F}_Q^{(\ell)}$, then $\gamma' \notin \mathcal{F}_Q^{(\ell)}$, and so no two consecutive elements of $\mathcal{F}_Q$ belong simultaneously to $\mathcal{F}_Q^{(\ell)}$. This means that if $\gamma < \gamma'$ are consecutive elements in $\mathcal{F}_{Q,\ell}$, then two cases can occur:

Case 1. $\gamma$ and $\gamma'$ are consecutive elements in $\mathcal{F}_Q$ and $\gamma, \gamma' \notin \mathcal{F}_Q^{(\ell)}$. In this case the number of gaps in consecutive fractions of length $\leq \frac{\xi}{Q^2}$ is equal to $N_1(Q, \xi) = N_Q(\xi) - M_1(Q, \xi) - M_2(Q, \xi)$, where $M_1(Q, \xi)$ is the number of pairs $(\gamma, \gamma') \in \mathcal{G}_Q(\xi)$ with $\gamma' \in \mathcal{F}_Q^{(\ell)}$, and $M_2(Q, \xi)$ the number of pairs $(\gamma, \gamma') \in \mathcal{G}_Q(\xi)$ with $\gamma \in \mathcal{F}_Q^{(\ell)}$.

The number $N_Q(\xi)$ is estimated employing the well-known fact that $\gamma < \gamma'$ are consecutive elements in $\mathcal{F}_Q$ if and only if $q, q' \in \{1, \ldots, Q\}$, $q + q' > Q$, and $a'q - aq' = 1$. Furthermore, $\frac{a'}{q'} - \frac{a}{q} = \frac{1}{qq'}$, and so $\frac{a'}{q'} - \frac{a}{q} \leq \frac{\xi}{Q^2}$ if and only if $qq' \geq \frac{Q^2}{\xi}$.

This establishes the equality

$$
N_Q(\xi) = \# \left\{ (q, q') \in \mathbb{N}^2 : q, q' \leq Q, q + q' > Q, (q, q') = 1, qq' \geq \frac{Q^2}{\xi} \right\}
$$

(2.1)

where $I_Q(q') = Q \cdot \left( \lceil \eta_Q(q') \rceil, 1 \right]$ and $\eta_Q(q') = \max \left\{ 1 - \frac{q' - 1}{Q}, \frac{Q}{q'} \right\}$.

Standard Möbius summation provides

$$
N_Q(\xi) = \sum_{q'=1}^{Q} \left( \frac{\varphi(q')}{q'} |I_Q(q')| + O(\sigma_0(q')) \right) = \sum_{q'=1}^{Q} \frac{\varphi(q')}{q'} |I_Q(q')| + O(Q \log Q)
$$

$$
= \frac{A(\xi)}{\zeta(2)} Q^2 + O(Q \log Q),
$$

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This proves $\delta > 0$. Now by (2.3) and (A.4), for any $(q, q') = 1$, we have $(\ell, q') = 1$. Therefore, we have to count all pairs of integers $(q, q') \in (0, Q)^2$ with $q + q' > Q$, $(q, q') = 1$, $qq' \geq Q^2$, in which $(\ell, q') = 1$, and there is an $a' \in \{1, \ldots, q'\}$ such that $a'q \equiv 1 \pmod{q'}$ and $\ell \mid a'$. As a result, after also letting $k = \frac{q'}{\ell}$, $\ell k q' \equiv 1 \pmod{q'}$, $M_1(Q, \xi)$ can be expressed as

$$M_1(Q, \xi) = \sum_{q' = 1}^{Q} \sum_{q \in I_Q(q')} \sum_{\substack{a' = 1 \\ (a', q') = 1}}^{q'} \sum_{(\ell, q') = 1}^{q'} 1 = \sum_{q' = 1}^{Q} \sum_{q \in I_Q(q')} \sum_{k \in \{0, q'/\ell\}} 1,$$

Now by (2.3) and (A.4), for any $\delta > 0$,

$$M_1(Q, \xi) = \sum_{q' = 1}^{Q} \left( \frac{\varphi(q')}{q'^2} \int_{I_Q(q') \times [0, q'/\ell]} dx dy + O_\delta(q'^{1/2+\delta}) \right)$$

$$= \frac{1}{\ell} \sum_{q' = 1}^{Q} \frac{\varphi(q')}{q'} |I_Q(q')| + O_{\ell, \delta}(Q^{3/2+\delta}).$$

Then using (A.2), we have

$$\frac{1}{\ell} \sum_{q' = 1}^{Q} \frac{\varphi(q')}{q'} |I_Q(q')| = \frac{C(\ell)}{\ell} \int_0^Q |I_Q(q')| dq' + O_{\ell}(Q \log Q)$$

$$= \frac{C(\ell)}{\ell} A(\xi) Q^2 + O_{\ell}(Q \log Q).$$

This proves $M_1(Q, \xi) \sim \frac{C(\ell)}{\ell} A(\xi) Q^2$ if $\xi > 1$. The formula for $M_2(Q, \xi)$ is analogous and we infer

$$N_1(Q, \xi) = N_Q(\xi) - M_1(Q, \xi) - M_2(Q, \xi)$$

$$= \left( \frac{1}{\xi(2)} - \frac{2C(\ell)}{\ell} \right) A(\xi) Q^2 + O_{\ell, \delta}(Q^{3/2+\delta}). \tag{2.4}$$
Case 2. There is exactly one fraction in $\mathcal{F}_Q$ between $\gamma$ and $\gamma'$ that belongs to $\mathcal{F}^{(f)}_Q$. It is more convenient to change $\gamma'$ to $\gamma''$, so we shall consider triples $\gamma < \gamma' < \gamma''$ of elements in $\mathcal{F}_Q$ with $\gamma' \in \mathcal{F}^{(f)}_Q$ and with $\gamma'' - \gamma \leq \frac{\xi}{q'}$. The equalities

$$\frac{a'' + a}{a'} = \frac{q'' + q}{q'} = K \quad \text{and} \quad \gamma'' - \gamma = \frac{K}{qq''},$$

(2.5)

involving the number

$$K = \nu_2(\gamma) = \left\lfloor \frac{Q + q}{q'} \right\rfloor,$$

called the index of the Farey fraction $\gamma = \frac{a}{q} \in \mathcal{F}_Q$, will be useful here. In particular, the inequality $\gamma'' - \gamma \leq \frac{\xi}{q'}$ enforces $K \leq \xi$. Consider the set $J_{Q,K,\xi}(q')$ of elements $q \in (Q - q', Q) \cap [Kq' - Q, (K + 1)q' - Q)$ that satisfy $\frac{K}{q'(q - q)} \leq \frac{\xi}{q'}$. This set is either empty, an interval, or the union of two intervals. The number $N_2(Q, \xi)$ of gaps of consecutive elements in $\tilde{\mathcal{F}}_Q,\ell$ of length $\leq \frac{\xi}{q'}$ that arise in this case can now be expressed, with $k$ and $\ell$ as in (2.3), as

$$N_2(Q, \xi) = \sum_{1 \leq K \leq \xi} \sum_{q' \leq Q} \sum_{q \in J_{Q,K,\xi}(q')} \sum_{a' = 1}^q \sum_{a'' = 1}^{q'} \frac{1}{a'a''},$$

(2.6)

We will employ elementary properties of the area preserving invertible transformation $T : T \to T$ defined [4] by

$$T(x, y) = (y, \kappa(x, y)y - x), \quad (x, y) \in T, \quad \text{where}$$

$$T = \{(x, y) \in (0, 1]^2 : x + y > 1\} \quad \text{and} \quad \kappa(x, y) = \left[ \frac{1 + x}{y} \right].$$

An important connection with Farey fractions is given by the equality

$$T \left( \frac{q_i}{Q}, \frac{q_{i+1}}{Q} \right) = \left( \frac{q_{i+1}}{Q}, \frac{q_{i+2}}{Q} \right).$$

(2.8)

For each $K \in \mathbb{N}$ consider the subset $T_K = \{(x, y) \in T : \kappa(x, y) = K\}$ of $T$, described by the inequalities $0 < x, y \leq 1$, $x + y > 1$, and $Ky - 1 \leq x < (K + 1)y - 1$.

Denote $V_{Q,K,\xi}(q') = |J_{Q,K,\xi}(q')|$, so $V_{Q,K,\xi}(Qu) = W_{K,\xi}(u)$, where

$$W_{K,\xi}(u) = \{v : (v, u) \in T_K \cap \{v : K \leq \xi v(Ku - v)\}\}.$$
Similar arguments as in the proof of (2.4) lead to

\[ N_2(Q, \xi) = \frac{C(\ell)}{\mathcal{T}} Q^2 \sum_{K \leqslant \xi} \int_0^1 W_{K, \xi}(u) \, du + O_{\ell, \delta, \xi}(Q^{3/2 + \delta}) \]

\[ = \frac{C(\ell)}{\mathcal{T}} Q^2 \sum_{K \leqslant \xi} A_K(\xi) + O_{\ell, \delta, \xi}(Q^{3/2 + \delta}), \]

uniformly in \( \xi \) on compact subsets of \([0, \infty), \)

where

\[ A_K(\xi) = \text{Area}(\Omega_K(\xi)), \quad \Omega_K(\xi) = \left\{ (v, u) \in \mathcal{T}_K : u \geqslant f_{K, \xi}(v) := \frac{v}{K} + \frac{1}{\xi v} \right\}. \]

Summarizing, we have shown

\[ N_Q^{(\ell)}(\xi) = G_\ell(\xi) Q^2 + O_{\ell, \delta, \xi}(Q^{3/2 + \delta}) \quad \text{(as } Q \to \infty), \]

where

\[ G_\ell(\xi) = \left( \frac{1}{\zeta(2)} - \frac{2C(\ell)}{\ell} \right) A(\xi) + \frac{C(\ell)}{\ell} \sum_{K \leqslant \xi} A_K(\xi). \]  

(2.9)

Taking also into account Lemma 1 we conclude that the gap limiting measure of \((\mathcal{F}_Q, \mathcal{K})\) exists and its distribution function is given by

\[ \tilde{F}_\ell(\xi) = \int_0^\xi d\tilde{\nu}_\ell = \frac{1}{K_\ell} G_\ell \left( \frac{\xi}{K_\ell} \right). \]

### 2.1 Explicit expressions of \( A_K(\xi) \)

#### 2.1.1 \( K = 1 \)

\( \mathcal{T}_1 \) is the triangle with vertices \((0, 1), (1, 1), \) and \((1, 2/3) \). When \( \xi \leqslant 4 \) we have \( f_{1, \xi}(v) \geqslant 1 \) for every \( v > 0 \), so \( A_1(\xi) = 0 \). When \( \xi > 4 \) we have

\[ A_1(\xi) = \int_{u_1}^{u_2} \left( 1 - \max \left\{ f_{1, \xi}(v), 1 - v, \frac{v + 1}{2} \right\} \right) dv, \]

where \( u_{1,2} = \frac{1}{2} \left( 1 \pm \sqrt{1 - \frac{4}{\xi}} \right), 0 < u_1 < u_2 < 1, \) are the solutions of \( f_{1, \xi}(v) = 1 \). When \( 4 < \xi \leqslant 8 \) we have \( f_{1, \xi}(v) \geqslant \max \left\{ 1 - v, \frac{v + 1}{3} \right\}, \) so \( A_1(\xi) \) is the area of the region defined by \( v \in [u_1, u_2] \) and \( u \in [f_{1, \xi}(v), 1] \). When \( \xi \geqslant 8 \) let \( v_{1,2} = \frac{1}{4} \left( 1 \pm \sqrt{1 - \frac{8}{\xi}} \right), v_1 < v_2, \) denote the solutions of \( f_{1, \xi}(v) = 1 - v \) and by \( v_{1,2} := 2v_{1,2} \) the solutions of \( f_{1, \xi}(w) = \frac{w + 1}{8} \). If \( 8 \leqslant \xi \leqslant 9 \), then \( 0 < u_1 < v_1 \leqslant v_2 \leqslant \frac{1}{4} < 1 \) \( w_1 \leqslant w_2 < u_2 < 1 \). In this case \( A_1(\xi) \) is the area of the region described by \( v \in [v_1, v_2] \cup [v_2, v_1] \cup [2w_2, w_2] \) and \( u \in [f_{1, \xi}(v), 1] \), \( v \in [v_1, v_2] \) and \( u \in [1 - v, 1) \), or \( v \in [w_1, w_2] \) and \( u \in [\frac{w_1 + v}{w_2}, 1 \). Finally, if \( \xi > 9 \), then \( 0 < u_1 < v_1 < w_1 < \frac{1}{4} < 1 \).
$w_2 < u_2 < 1$, and $A_1(\xi)$ is the area of the region described by $v \in [u_1, v_1] \cup [w_2, u_2]$ and $u \in [f_1, 1]$, or $v \in \left[ v_1, \frac{1}{2} \right]$ and $u \in [1 - v, 1]$, or $v \in \left[ \frac{1}{2}, w_2 \right]$ and $u \in \left[ \frac{1}{2}, 1 \right]$. A plain calculation gives

$$A_1(\xi) = \begin{cases} 0 & \text{if } 0 < \xi \leq 4 \\ \frac{1}{2} \sqrt{1 - \frac{2}{\xi} - \frac{1}{\xi} \ln \left( \frac{u_2}{u_1} \right)} - \frac{1}{2} \sqrt{1 - \frac{2}{\xi} + \frac{1}{\xi} \ln \left( \frac{w_2}{v_1} \right)} & \text{if } 4 \leq \xi \leq 8 \\ \frac{1}{2} \sqrt{1 - \frac{2}{\xi} - \frac{1}{\xi} \ln \left( \frac{u_2}{u_1} \right)} - \frac{1}{2} \sqrt{1 - \frac{2}{\xi} - \frac{1}{\xi} + \frac{1}{\xi} \ln \left( \frac{2w_2}{v_1} \right)} & \text{if } \xi \geq 9. \end{cases}$$

Figure 1: The intersection between the quadrilateral $T_K$ and the curve $u = f_{K;\xi}(v)$ when $K < \xi < \frac{K(K+1)}{K-1}$, $\frac{K(K+1)}{K-1} \leq \xi < \frac{(K+2)^2}{K}$, and respectively $\xi \geq \frac{(K+2)^2}{K}$

### 2.1.2 $K \geq 2$

Note that $f_{K;\xi}(1) = f_{K;\xi}(\frac{K}{\xi}) = \frac{1}{K} + \frac{1}{\xi}$. The situation is described by Figure 1. The solution of $f_{K;\xi}(v) = \frac{w+1}{K}$ is $v = \frac{K}{\xi}$, so the curve $u = f_{K;\xi}(v)$ intersects the upper edge of $T_K$ if and only if $K < \xi < \frac{K(K+1)}{K-1}$, in which case it does not intersect the two lower edges of $T_K$ and

$$A_K(\xi) = \int_{K/\xi}^{1} \left( \frac{v+1}{K} - f_{K;\xi}(v) \right) dv = \int_{K/\xi}^{1} \left( \frac{1}{K} - \frac{1}{\xi} v \right) dv.$$ 

The solution of $f_{K;\xi}(\frac{K}{K+2}) > 2 \frac{K}{K+2}$ is $\xi < \frac{(K+2)^2}{K}$. This shows that when $\frac{K(K+1)}{K-1} \leq \xi < \frac{(K+2)^2}{K}$ the graph of $u = f_{K;\xi}(u)$ intersects the segment $u = 1 - v$, $v \in \left[ \frac{K-1}{K+1}, \frac{K}{K+2} \right]$, exactly when $v = v_K = \frac{K}{2(K+1)} \left( 1 + \sqrt{1 - \frac{4}{\xi}(1 + \frac{1}{K})} \right)$, and the segment $u = \frac{w+1}{K+1}$, $v \in \left[ \frac{K}{K+2}, 1 \right]$, exactly at $v = w_K = \frac{K}{\xi} \left( 1 - \sqrt{1 - \frac{4}{\xi}(1 + \frac{1}{K})} \right)$, so in this
case
\[ A_K(\xi) = \text{Area}(T_K) - \int_{\nu_K}^{w_K} f_{K,\xi}(v) \, dv + \int_{\nu_K}^{K/(K+2)} (1 - v) \, dv + \int_{K/(K+2)}^{w_K} \frac{v + 1}{K + 1} \, dv. \]

Finally, when \( \xi > \frac{(K+2)^2}{K} \), the graph of \( u = f_{K,\xi}(v) \) does not intersect any of the edges of \( T_K \) and
\[ A_K(\xi) = \text{Area}(T_K). \]

In summary, a quick calculation leads to
\[ A_K(\xi) = \begin{cases} 
0 & \text{if } 0 \leq \xi \leq K \\
\frac{1}{K} - \frac{1}{\xi} - \frac{1}{\xi} \ln \left( \frac{K}{\xi} \right) & \text{if } K \leq \xi \leq \frac{K(K+1)}{K-1} \\
\frac{1}{2K} + \frac{1}{2} \ln \left( \frac{w_K}{\nu_K} \right) - \frac{w_K}{2} + \frac{w_K}{\pi(K+1)} & \text{if } \frac{K(K+1)}{K-1} \leq \xi \leq \frac{(K+2)^2}{K} \\
4 & \text{if } \xi \geq \frac{(K+2)^2}{K}. 
\end{cases} \]

Figure 2: The repartition function \( 1 - G_3(\xi) \) and the density \(-G'_3(\xi)\)

3 Consecutive elements in \( F_Q \) with denominator relatively prime to \( d \)

In this section we comment on the first two steps in the proof of (1.1) from [2].

3.1 Upper bounds on the number of consecutive Farey fractions whose denominators are not relatively prime to \( d \)

One of the key steps in the proof of (1.1) in [2] is to show that for any \( Q \) and any \( d \), any string of consecutive elements in \( F_Q \) of length \( 4d^3 \) contains at least one element whose denominator is coprime with \( d \). Next we provide two arguments which show that the upper bound \( L(d) \) should actually be much smaller than \( 4d^3 \).

Lemma 2. If \( \omega(d) \leq \min\{p \in \mathbb{P} : p \mid d\} \), then \( L(d) \leq 4\omega(d)^3 \).
Proof. We first revisit the proof of the first part of Step (i) in the proof of Theorem 1 in [2] (pp. 210–211). Suppose $Q$ and $i_1 < i_2$ are chosen such that, for every $j \in [i_1, i_2]$,

$$\max\{q_{i_1}, q_{i_2}\} \leq q_j \quad \text{and} \quad (q_j, d) > 1.$$ 

Then $(q_{i_1}, q_{i_2}) = 1$ and

$$\{q_j : i_1 < j < i_2\} \subset \{mq_{i_1} + nq_{i_2} : m, n \in \mathbb{N}, (m, n) = 1, mq_{i_1} + nq_{i_2} \leq Q\}. \quad (3.1)$$

Let $d_1 = p_1^{\nu_1} \cdots p_\omega^{\nu_\omega}$, with $p_1 < \cdots < p_\omega$ primes, be the largest divisor of $d$ which is co-prime to $q_{i_1}$. Then $\omega < \omega(d) \leq \min\{p \in \mathcal{P} : p \mid d\} \leq p_1$. Fix some integer $L$ with $\omega + 1 \leq L \leq p_1$. We claim that there exists $m_1 \in \mathbb{N}$, $m_1 \leq L$ such that $(m_1q_{i_1} + q_{i_2}, d_1) = 1$. If not, then $(\ell q_{i_1} + q_{i_2}, d_1) > 1$ for all $\ell \in \{1, \ldots, L\}$. Since $L > \omega$, the Pigeonhole Principle shows that there exist $i_0 \in \{1, \ldots, \omega\}$ and $1 \leq \ell < \ell' \leq L$ such that $p_{i_0} \mid (\ell q_{i_1} + q_{i_2})$ and $p_{i_0} \mid (\ell' q_{i_1} + q_{i_2})$, and so $p_{i_0} \mid (\ell' - \ell)q_{i_1}$. But $(p_{i_0}, q_{i_1}) = 1$, hence $L > \ell' - \ell \geq p_{i_0} \geq p_1$, which contradicts $L \leq p_1$.

So if $(m_1q_{i_1} + q_{i_2}, d) > 1$, then there exists $p$ prime with $p \mid q_{i_1}$ and $p \mid (m_1q_{i_1} + q_{i_2})$, thus contradicting $(q_{i_1}, q_{i_2}) = 1$. Hence $(m_1q_{i_1} + q_{i_2}, d) = 1$, which in turn yields $Q \leq m_1q_{i_1} + q_{i_2} \leq Lq_{i_1} + q_{i_2}$. In a similar way one has $Q \leq q_{i_1} + Lq_{i_2}$, thus (3.1) leads to

$$\{q_j : i_1 < j < i_2\} \subset \{mq_{i_1} + nq_{i_2} : 1 \leq m, n \leq L\},$$

and in particular $i_2 - i_1 \leq L^2$.

The second part of the proof proceeds ad litteram as in the proof of Step (i) [2, pp. 211–212] replacing $d$ there by $L$. \Halmos

When $d$ is the product of two prime powers the bound above can be lowered. In this case we show that $L(d) \leq 5$, which is sharp for $d = 6$ because $\frac{1}{2} < \frac{2}{3} < \frac{3}{2} < \frac{3}{4} < \frac{4}{3}$ are consecutive in $\mathcal{F}_4$. Our proof employs elementary properties of the transformation $T$ from (2.7). In particular (2.8) and the following inclusions will be useful in the proof of Lemma 3:

$$T \mathcal{T}_k \subset \mathcal{T}_k \quad \text{if} \quad k \geq 5,$$

$$T \mathcal{T}_2 \subset \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4,$$

$$T \mathcal{T}_3 \subset \mathcal{T}_3 \cup \mathcal{T}_4.$$ 

Lemma 3. If $d = p^a q^b$, then for each $i \in \{0, \ldots, \#\mathcal{F}_Q - 5\}$ there exists $j \in \{0, \ldots, 5\}$ such that $(q_{i+j}, d) = 1$, and so $L(d) \leq 5$.

Proof. We have $q_{i+2} = Kq_{i+1} - q_i$, $q_{i+3} = K'q_{i+2} - q_{i+1}$, $q_{i+4} = K''q_{i+3} - q_{i+2}$, $q_{i+5} = K'''q_{i+4} - q_{i+3}$, where $K = \kappa\left(\frac{q_i}{Q}, \frac{q_{i+1}}{Q}\right)$, $K' = \kappa\left(\frac{q_{i+1}}{Q}, \frac{q_{i+2}}{Q}\right)$, $K'' = \kappa\left(\frac{q_{i+2}}{Q}, \frac{q_{i+3}}{Q}\right)$, and $K''' = \kappa\left(\frac{q_{i+3}}{Q}, \frac{q_{i+4}}{Q}\right)$. Suppose that $(q_i, d), \ldots, (q_{i+5}, d) > 1$. Then either $p \mid (q_i, q_{i+2}, q_{i+4})$ and $q \mid (q_{i+1}, q_{i+3}, q_{i+5})$, or vice versa.

Without loss of generality we can work in the first case. The equality $q_{i+2} + q_i = Kq_{i+1}$ and $p \mid q_{i+1}$ yield $p \mid K$. Similarly we have $q \mid K'$. Assume first that $K \geq 5$. Since $\left(\frac{q_{i+3}}{Q}, \frac{q_{i+4}}{Q}\right) \in \mathcal{T}_k$ and $T \mathcal{T}_k \subset \mathcal{T}_i$ we must have $K' = 1$, which contradicts $q \geq 2$. In particular $p \geq 5$ cannot occur.
When \( p = 3 \) and \( K = 3 \), from \( TT_3 \subseteq T_1 \cup T_2 \) it follows that \( K' \in \{1, 2\} \). Since \( q \mid K' \), we infer \( q = 2 \). The region \( TT_3 \cap T_2 \) is the quadrilateral with vertices at \((1, 1), (0, 1), (1, 0), \) and \((0, 0), \) being further mapped by \( T \) into a subset of \( T_1 \cup T_2 \) whence \( K'' \in \{1, 2\} \). Again \( K'' = 1 \) leads to an immediate contradiction, while \( K'' = 2 \) yields \( q_{i+2} + q_{i+4} = 2q_{i+3} \), showing that \( p = 2 \), another contradiction.

When \( p = 2 \) and \( K < 5 \), we have \( K \in \{2, 4\} \). Assume first \( K = 2 \). As \( TT_2 \subseteq T_1 \cup T_2 \cup T_3 \cup T_4 \) and \( K' \neq 1 \) it remains that \( K' \in \{2, 3, 4\} \). Since \( q \geq 3 \) divides \( K' \), we infer \( q = 3 \). Furthermore, \( TT_3 \subseteq T_1 \cup T_2 \) and \( K'' \neq 1 \) yield \( K'' = 2 \). Employing again \( T(TT_3 \cap T_2) \subseteq T_1 \cup T_2 \), we infer \( K''' = 2 \), and so \( q_{i+3} + q_{i+5} = 2q_{i+4} \). This is again a contradiction, because 3 divides \( q_{i+3} + q_{i+5} \) and cannot divide \( 2q_{i+4} \).

Finally, assume \( K = 4 \), so \( K' \in \{1, 2\} \), which is not possible because \( q \geq 3 \) divides \( K' \).

Note that if \((p_n)\) is the sequence of primes, then none of the denominators of the fractions in \( F_{p_n} \setminus \{1\} \) are relatively prime to \( \prod_{i=1}^n p_i \). This gives the lower bound \#\( F_{p_n} - 1 \) on the size of the largest string of consecutive fractions in \( F_Q \setminus F_Q, d \) for some \( Q, d \in \mathbb{N} \) with \( \omega(d) = n \). Since \( p_n \sim n \log n \) as \( n \to \infty \) and \#\( F_Q \sim \frac{3}{\pi^2} Q^2 \) as \( Q \to \infty \), there exists \( A > 0 \) such that \#\( F_{p_n} - 1 \geq A(n \log n)^2 \). Thus any upper bound on \( L(d) \) involving only \( \omega(d) \) must be greater than \( A(\omega(d) \log \omega(d))^2 \).

### 3.2 The index and the continuant

The second step in the proof of (1.1) in [2] relies on [2, Lemma 1], which is actually exactly Remark 2.6 in [6] (see also [4, Lemma 5]), and on a result relating the \( \ell \)-index of a Farey fraction and the continuant of regular continued fractions. The \( \ell \)-index of \( \gamma_i = \frac{a_i}{q_i} \in F_Q \) is the positive integer \( \nu_\ell(\gamma_i) = a_{i+\ell-1}q_{i-1} - a_{i-1}q_{i+\ell-1} \) where \( \frac{a_{i+k}}{q_{i+k}} \) denotes the \( k \)-th successor of \( \gamma_i \) in \( F_Q \). The (regular continued fraction) continuants are defined as usual by \( K_0(\cdot) = 1, K_1(x_1) = 1, \) and

\[
K_\ell(x_1, \ldots, x_\ell) = x_\ell K_{\ell-1}(x_1, \ldots, x_{\ell-1}) + K_{\ell-2}(x_1, \ldots, x_{\ell-2}) \quad \text{if } \ell \geq 2.
\]

In [10] the identity

\[
\nu_\ell(\gamma_i) = \epsilon_\ell K_{\ell-1}(\nu_2(\gamma_i), \nu_3(\gamma_{i+1}), \ldots, (-1)^{\ell-1} \nu_2(\gamma_{i+\ell-2}))
\]  

was proved, with \( \epsilon_\ell = 1 \) if \( \ell \in \{0, 1\} \) (mod 4) and \( \epsilon_\ell = -1 \) if \( \ell \in \{2, 3\} \) (mod 4).

We give a very short proof of (3.2). We define the Farey continuants \( K_\ell^F \) by \( K_0^F(\cdot) = 1, K_1^F(x_1) = x_1, \) and

\[
K_\ell^F(x_1, \ldots, x_\ell) = x_\ell K_{\ell-1}^F(x_1, \ldots, x_{\ell-1}) - K_{\ell-2}^F(x_1, \ldots, x_{\ell-2}) \quad \text{if } \ell \geq 2.
\]

The defining equalities for \( K_\ell \) and \( K_\ell^F \) plainly yield, for all \( \ell \geq 2, \)

\[
\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} x_\ell & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} K_\ell(x_1, \ldots, x_\ell) & K_{\ell-1}(x_1, \ldots, x_{\ell-1}) \\ K_{\ell-1}(x_2, \ldots, x_\ell) & K_{\ell-2}(x_2, \ldots, x_{\ell-1}) \end{pmatrix},
\]  

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\[
\begin{pmatrix}
1 & 1 \\
-1 & 0
\end{pmatrix} \cdots \begin{pmatrix}
1 & 1 \\
-1 & 0
\end{pmatrix} = \begin{pmatrix}
K_1^F(x_1, \ldots, x_\ell) & \cdots & K_1^F(x_1, \ldots, x_{\ell-1}) \\
K_{\ell-1}^F(x_2, \ldots, x_\ell) & \cdots & K_{\ell-1}^F(x_2, \ldots, x_{\ell-1})
\end{pmatrix}.
\]

From (3.4) and the definition of \(\nu_\ell(\gamma_i)\) we now infer
\[
\nu_\ell(\gamma_i) = K_{\ell-1}^F(\nu_2(\gamma_i), \nu_2(\gamma_{i+1}), \ldots, \nu_2(\gamma_{i+\ell-2})).
\]

The equality (3.2) follows immediately from (3.3), (3.4), (3.5) and
\[
\begin{pmatrix}
x & 1 \\
-1 & 0
\end{pmatrix} \begin{pmatrix}
y & 1 \\
-1 & 0
\end{pmatrix} = -\begin{pmatrix}
x & 1 \\
1 & 1
\end{pmatrix} \begin{pmatrix}
y & 1 \\
1 & 0
\end{pmatrix}.
\]

4 The gap distribution of \(F_{Q,d}\)

Letting \(d \in \mathbb{N}\) and \(\xi > 0\), we wish to asymptotically estimate the number of pairs of consecutive elements \(\gamma < \gamma'\) in \(F_{Q,d}\) with \(\gamma' - \gamma \leq \frac{\xi}{Q}\) as \(Q \to \infty\). It is plain that
\[
\#F_{Q,d} = \sum_{q=1}^{Q} \varphi(q) = C(d) \int_0^Q q \, dq + O_d(Q \log Q) = \frac{C(d)}{2} Q^2 + O_d(Q \log Q),
\]
showing the second equality in (1.2). Denote \(N_Q = \#F_Q\) and \(\gamma_j = \frac{a_j}{q_j}\), so the number of pairs of fractions we wish to estimate is
\[
N_d(Q, \xi) = \sum_{\ell=1}^{L(d)} \sum_{k=1}^{[\xi]} \# \left\{ \begin{array}{l}
i \in [1, N_Q] : \frac{\nu_{\ell}(\gamma_i)}{q_{i-1}q_{i+\ell-1}} \leq \frac{\xi}{Q^2}, \\
(q_{i-1}, d) = (q_{i+\ell-1}, d) = 1
\end{array} \right\}
\]
\[
= \sum_{\ell=1}^{L(d)} \sum_{k=1}^{[\xi]} \# \left\{ \begin{array}{l}
i \in [1, N_Q] : \\
\nu_{\ell}(\gamma_i) = k
\end{array} \right\}.
\]

It is shown in [4, 5] that given \(i \in [1, N_Q]\) and \(k, \ell \in \mathbb{N}\) with \(\ell \geq 2\), if \(\nu_{\ell}(\gamma_i) = k\), then the \((\ell - 1)\)-tuple \((\nu_2(\gamma_i), \ldots, \nu_2(\gamma_{i+\ell-2}))\) can take on \(n(k, \ell)\) values, where \(n(k, \ell) \in \mathbb{N} \cup \{0\}\) depends only on \(k\) and \(\ell\) and not on \(i\) or \(Q\); and in [10], it is proven that \(\nu_{\ell}(\gamma_i)\) can be determined if \((\nu_2(\gamma_i), \ldots, \nu_2(\gamma_{i+\ell-2}))\) is known (cf. identity (3.2) above). Therefore, letting \((x(k, \ell, m))_{m=1}^{n(k, \ell)}\) be the \((\ell - 1)\)-tuples for which \(\nu_{\ell}(\gamma_i) = k\) whenever \(x(k, \ell, m) = (\nu_2(\gamma_i), \ldots, \nu_2(\gamma_{i+\ell-2}))\) for some \(m \in \{1, \ldots, n(k, \ell)\}\), we have
\[
N_d(Q, \xi) = \# \left\{ \begin{array}{l}
i \in [1, N_Q] : (q_{i-1}, d) = (q_i, d) = 1, \\
q_{i-1} \geq \frac{Q^2}{\xi}
\end{array} \right\}
\]
\[
+ \sum_{\ell=2}^{L(d)} \sum_{k=1}^{[\xi]} \sum_{m=1}^{n(k, \ell)} \# \left\{ \begin{array}{l}
i \in [1, N_Q] : \\
q_{i-1}q_{i+\ell-1} \geq \frac{kQ^2}{\xi}, \\
(q_{i-1}, d) = (q_{i+\ell-1}, d) = 1 \\
(q_{i}, d) > 1, \ldots, (q_{i+\ell-2}, d) > 1
\end{array} \right\}.
\]
Since $q_{j+1} = \nu_2(\gamma_j)q_j - q_{j-1}$ for $j \in [1, N_Q - 1]$, the residue classes of the denominators $q_{i-1}$, ..., $q_{j-1}$ can be determined once the residue classes of $q_{i-1}$ and $q_i$, and the $(\ell - 1)$-tuple $(\nu_2(\gamma_i), \ldots, \nu_2(\gamma_{i+\ell-2}))$ are known. Thus, there is a subset $A_{k,\ell,m} \subseteq \{1, \ldots, d\}^2$ such that when $(\nu_2(\gamma_i), \ldots, \nu_2(\gamma_{i+\ell-2})) = \nu(k, \ell, m)$, we have $(q_{j-1}, d) = (q_{j+\ell-1}, d) = 1$ and $(q_{j+\ell-1}, d) > 1$ for $1 \leq j < \ell$ if and only if $(q_{j-1}, q_i) (\text{mod } d) \in A_{k,\ell,m}$. (Note clearly that $(a, b) = 1$ for $(a, b) \in A_{k,\ell,m}$.) Furthermore, if we let $x(k, \ell, m) = (x_1(k, \ell, m), \ldots, x_{\ell-1}(k, \ell, m))$ and denote $\mathbb{Z}^2_{\text{vis}} = \{(a, b) \in \mathbb{Z}^2 : (a, b) = 1\}$, it is clear that $(\nu_2(\gamma_i), \ldots, \nu_2(\gamma_{i+\ell-2})) = \nu(k, \ell, m)$ if and only if

$$(q_{i-1}, q_i) \in Q \cdot (\mathcal{T}_{\mathcal{S}_1(k,\ell,m)} \cap T^{-1}\mathcal{T}_{\mathcal{S}_2(k,\ell,m)} \cap \cdots \cap T^{-(\ell-2)}\mathcal{T}_{\mathcal{S}_{\ell-1}(k,\ell,m)}) \cap \mathbb{Z}^2_{\text{vis}}.
$$

Now if we let $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be the canonical projections, then

$$\frac{q_{i-1}q_{i+\ell-1}}{Q^2} = \pi_1\left(\frac{q_{i-1}}{Q}, \frac{q_{i}}{Q}\right) \cdot (\pi_2 \circ T^{\ell-1})\left(\frac{q_{i-1}}{Q}, \frac{q_{i}}{Q}\right),$$

and so

$$q_{i-1}q_{i+\ell-1} \geq \frac{k Q^2}{\xi} \iff (q_{i-1}, q_i) \in Q g_T^{-1}\left[\frac{k}{\xi}, \infty\right),$$

where $g_T = \pi_1 \cdot (\pi_2 \circ T^{\ell-1})$. Now set $g_1(x, y) = xy$ and

$$\Omega_{k,\ell,m}(\xi) = \mathcal{T}_{\mathcal{S}_1(k,\ell,m)} \cap T^{-1}\mathcal{T}_{\mathcal{S}_2(k,\ell,m)} \cap \cdots \cap T^{-(\ell-2)}\mathcal{T}_{\mathcal{S}_{\ell-1}(k,\ell,m)} \cap g_T^{-1}\left[\frac{k}{\xi}, \infty\right),$$

$$\Omega_1(\xi) = \mathcal{T} \cap g_1^{-1}\left[\frac{1}{\xi}, \infty\right), \quad A_1 = \{(a, b) : a, b \in [1, d], (a, d) = (b, d) = 1\}.$$

We then have

$$N_d(Q, \xi) = \sum_{(a,b) \in A_1} \#Q \Omega_1(\xi) \cap ((a, b) + d\mathbb{Z}^2) \cap \mathbb{Z}^2_{\text{vis}}$$

$$+ \sum_{\ell = 2}^{\ell(d)} \sum_{k=1}^{\nu(k,\ell)} \sum_{m=1}^{n(k,\ell)} \sum_{(a,b) \in A_{k,\ell,m}} \#Q \Omega_{k,\ell,m}(\xi) \cap ((a, b) + d\mathbb{Z}^2) \cap \mathbb{Z}^2_{\text{vis}},$$

where we have used the fact that if $(a, b) \in Q \mathcal{T} \cap \mathbb{Z}^2_{\text{vis}}$, then there is an $i$ such that $a = q_{i-1}$ and $b = q_i$. One can prove in a similar manner to [2, Lemma 2] that for all bounded $\Omega \subseteq \mathbb{R}^2$ whose boundary can be covered by the images of finitely many Lipschitz functions from $[0, 1]$ to $\mathbb{R}^2$, and for all $A \subseteq \{1, \ldots, d\}^2$ in which $(a, d) = 1$ for all $(a, b) \in A$, we have

$$\sum_{(a,b) \in A} \#Q \Omega \cap ((a, b) + d\mathbb{Z}^2) \cap \mathbb{Z}^2_{\text{vis}} = \frac{\text{Area}(\Omega) \#A}{\zeta(2)d^2} \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p^2}\right)^{-1} Q^2 + O_d(Q \log Q)$$

as $Q \to \infty$. It is easily seen that the boundaries of $\Omega_1(\xi)$ and $\Omega_{k,\ell,m}(\xi)$ can be covered by finitely many Lipschitz functions from $[0, 1]$ to $\mathbb{R}^2$, and so we have
where
\[ C_d(\xi) = \frac{1}{\zeta(2)d^2} \prod_{p \in P} \left( 1 - \frac{1}{p^2} \right)^{-1} \]
\[ \cdot \left( \varphi(d)^2 \text{Area}(\Omega_1(\xi)) + \sum_{\ell=2}^{L(d)} \sum_{k=1}^{n(k,\ell)} \sum_{m=1}^{\text{Area}(\Omega_{k,\ell,m}(\xi))} \#A_{k,\ell,m} \right), \]
noting that \( \#A_1 = \varphi(d)^2 \).

The gap limiting measure of \((\mathcal{F}_Q,d)\) exists with distribution function given by
\[ F_d(\xi) = \int_0^\xi d\nu_d = \frac{1}{K_d} C_d \left( \frac{\xi}{K_d} \right). \]

When \(d\) is a prime power this can be expressed more explicitly as in (2.9).

### Appendix

For the convenience of the reader we collect in this appendix the asymptotic formulas used in this paper.

Assuming that \(f\) is a \(C^1\) function on the interval of integration in (A.1)-(A.3) and that \(I, J\) are intervals and \(f \in C^1(I \times J)\) in (A.4), we have
\[ \sum_{a < k \leq b \atop (k,q)=1} f(k) = \frac{\varphi(q)}{q} \int_a^b f(x) \, dx + O \left( \sigma_0(q) \left( \|f\|_\infty + T_a^b f \right) \right). \text{ (A.1)} \]
\[ \sum_{1 \leq k \leq N \atop (k,q)=1} \frac{\varphi(k)}{k} f(k) = C(\ell) \int_0^N f(x) \, dx + O_{\ell} \left( \|f\|_\infty + T_0^N f \log N \right). \text{ (A.2)} \]
\[ \sum_{1 \leq k \leq N \atop (k,q)=1} \frac{\varphi(k)}{k} f(k) = \ell C(\ell) \int_0^N f(x) \, dx + O_{\ell,\delta} \left( \|f\|_\infty + T_0^N f \right) N^\delta. \text{ (A.3)} \]
\[ \sum_{a \in I, b \in J \atop ab \equiv h \text{ (mod } q) \atop (b,q)=1} f(a,b) = \frac{\varphi(q)}{q^2} \int_I \int_J f(x,y) \, dxdy + O_{\delta} \left( T^2 \|f\|_\infty q^{1/2+\delta} (h,q)^{1/2} \right) \]
\[ + O_{\delta} \left( T^2 \|\nabla f\|_\infty q^{3/2+\delta} (h,q)^{1/2} \right) + \frac{1}{T} \|\nabla f\|_\infty |I| \cdot |J| \right). \text{ (A.4)} \]

Proofs can be found for instance in [3, Lemma 2.2], [5, Lemmas 2.1 and 2.2], and respectively in [7, Proposition A4].
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