EQUIDISTRIBUTION OF MASS FOR RANDOM PROCESSES ON FINITE-VOLUME SPACES*

BY

TIMOTHEE BENARD

Centre for Mathematical Sciences, University of Cambridge
CB3 0WB Cambridge, UK
e-mail: tb723@cam.ac.uk

ABSTRACT

Let $G$ be a real Lie group, $\Lambda \subseteq G$ a lattice, and $X = G/\Lambda$. We fix a probability measure $\mu$ on $G$ and consider the left random walk induced on $X$. It is assumed that $\mu$ is aperiodic, has a finite first moment, spans a semisimple algebraic group without compact factors, and has two non mutually singular convolution powers. We show that for every starting point $x \in X$, the $n$-th step distribution $\mu^n \ast \delta_x$ of the walk weak-$\ast$ converges toward some homogeneous probability measure on $X$.

1. Introduction

In this note we consider a semigroup $\Gamma$ acting on a space $X$ and study the way the $\Gamma$-orbits distribute in $X$. A first approach might be to describe the orbit closures. In the case where $\Gamma$ is a diagonal group acting on a finite-volume homogeneous space, it is known that these orbit closures may be extremely irregular, with transverse sections homeomorphic to a Cantor set. However, Ratner’s theorems [7] highlight that on the opposite, the orbits of a group generated by Ad-unipotent 1-parameter flows are very regular. Benoist–Quint [3] recently proved that such regularity also occurs for the action of a semigroup $\Gamma$.

* The author has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No. 803711).

Received December 10, 2021 and in revised form March 2, 2022
whose Zariski closure is semisimple. Our goal is to complement Benoist–Quint’s result by estimating how Γ-orbits equidistribute in their closure from the perspective of random walks supported on Γ. We first recall the state of the art concerning Benoist–Quint’s theorems.

Let $G$ be a real Lie group, $\Lambda \subseteq G$ a lattice, and set $X = G/\Lambda$. We consider the action by left multiplication on $X$ of a closed subsemigroup $\Gamma \subseteq G$. Throughout the text, $\Gamma$ will always be assumed to satisfy

$$(H)\text{ The real algebraic group } \overline{\text{Ad}\Gamma} \subseteq \text{Aut}(g) \text{ generated by } \Gamma \text{ in the adjoint representation is semisimple, Zariski connected, with no compact factor.}$$

We also need a precise definition to express the regularity of orbit closures.

**Definition:** A closed subset $Y$ of $\Omega$ is **homogeneous** if its stabilizer

$$G_Y = \{ g \in G, gY = Y \}$$

acts transitively on $Y$. We add that $Y$ has **finite volume** if the action of $G_Y$ on $Y$ preserves a Borel probability measure on $Y$. Such a measure is then unique, denoted by $\nu_Y$. If the semigroup $\Gamma$ is included in $G_Y$ (and acts ergodically on $(Y, \nu_Y)$), we say that $Y$ is $\Gamma$-**invariant** (and $\Gamma$-**ergodic**).

The following result is essentially due to Benoist–Quint.

**Theorem** ([3, 1]): For every $x \in X$, the orbit closure $\overline{\Gamma.x}$ is a finite-volume homogeneous closed subset of $X$ which is $\Gamma$-invariant ergodic.

The proof also yields that random walks on $\Gamma$ equidistribute in the $\Gamma$-orbits in Cesàro average. This is due to Benoist–Quint for walks with bounded jumps, and to Bénard–de Saxcé for walks with finite first moment.

**Theorem** ([3, 1]): Let $\mu$ be a Borel probability measure on $\Gamma$ with a finite first moment and whose support spans a dense subsemigroup of $\Gamma$.

Then for every $x \in X$, we have the weak-* convergence

$$\frac{1}{n} \sum_{k=0}^{n-1} \mu_k^* \delta_x \rightarrow_{n \to +\infty} \nu_x$$

where $\nu_x := \nu_{\overline{\Gamma.x}}$ denotes the homogeneous probability measure on $\overline{\Gamma.x}$.
In this statement, $\mu^k$ denotes the $k$-fold convolution of $\mu$ and the assumption of finite first moment on $\mu$ means that
\[ \int_G \log \| \text{Ad}_g \| \, d\mu(g) < \infty. \]
Moreover, the weak-$\ast$ topology refers to the pointwise convergence topology for the space of finite Borel measures on $X$ seen as linear forms on the space of continuous bounded functions on $X$.

Benoist–Quint ask in the 10th Takagi Lectures [2, p. 29] whether the equidistribution result (1) still holds without Cesàro averages. Our goal is to answer positively under the condition that $\mu$ has two distinct powers which are not mutually singular. Note also that an aperiodicity condition is required to avoid situations where $\mu^k \ast \delta_x$ is obviously non converging; for example, the case where there exists a group $\Lambda'$ containing $\Lambda$ as a finite-index normal subgroup and such that $\mu$ is supported on a non trivial class in $\Lambda'/\Lambda$ [6]. We will say that $\mu$ is aperiodic if its support is not contained in a fibre of a non trivial continuous morphism of semigroups from $\Gamma$ to a finite (cyclic) group.

**Theorem 1** (Equidistribution of mass): Let $G$ be a real Lie group, $\Lambda \subseteq G$ a lattice, and $\Gamma \subseteq G$ a closed subsemigroup satisfying (H). Let $\mu$ be an aperiodic Borel probability measure on $\Gamma$ with a finite first moment and whose support spans a dense subsemigroup of $\Gamma$. Assume there exist $k \neq l \in \mathbb{N}$ such that $\mu^k$ is not singular with $\mu^l$.

Then for every $x \in X$, we have the weak-$\ast$ convergence
\[ \mu^n \ast \delta_x \xrightarrow{n \to +\infty} \nu_x \]
where $\nu_x := \nu_{\Gamma \cdot x}$ denotes the homogeneous probability measure on $\Gamma \cdot x$.

In particular, we obtain equidistribution of mass when $\mu$ is symmetric with countable support, or if $\mu$ is aperiodic with $\mu^k(e) > 0$ for some $k \geq 1$.

**2. Proof of Theorem 1**

Given a signed measure $m$ on a measurable space, we denote by $\|m\|$ its total variation
\[ \|m\| = \sup \sum_{i \in \mathbb{N}} |m(A_i)| \]
where the supremum is taken over all the measurable countable partitions $(A_i)_{i \in \mathbb{N}}$ of the space.
Lemma 2: Let \( H \) be a measurable group and \( \mu \) a probability measure on \( H \) such that \( \mu^k, \mu^l \) are not mutually-singular for some distinct integers \( k \neq l \in \mathbb{N} \). Then we have the convergence

\[
\| \mu^n - \mu^{n+k-l} \| \xrightarrow{n \to +\infty} 0.
\]

Proof. This result is due to Foguel [4, Theorem 1], whose proof is strongly inspired by the proof of [5, Theorem 1.1] about conservative ergodic Markov operators. The argument is a bit tricky. For completeness, we give a more intuitive explanation in the case where \( l = 0 \), i.e., \( \mu^k(e) > 0 \) for some \( k \geq 1 \).

We may assume \( k = 1 \). Decompose \( \mu \) as \( \mu = \alpha \delta_e + (1 - \alpha) \mu_0 \) where \( \alpha > 0 \) and \( \mu_0 \) is a probability measure on \( H \). Then for \( n \geq 0 \),

\[
\mu^n = (\alpha \delta_e + (1 - \alpha) \mu_0)^n = \sum_{i=0}^{n} \binom{n}{i} \alpha^{n-i} (1 - \alpha)^i \mu_0^i.
\]

Hence, using the convention \( \binom{n}{i} = 0 \) if \( i \geq n + 1 \),

\[
\| \mu^n - \mu^{n+1} \| = \left\| \sum_{i=0}^{n+1} \left( \binom{n}{i} \alpha^{n-i} (1 - \alpha)^i \right) - \left( \binom{n+1}{i} \alpha^{n+1-i} (1 - \alpha)^i \right) \mu_0^i \right\|
\]

\[
\leq \sum_{i=0}^{n+1} \left| \binom{n}{i} \alpha^{n-i} (1 - \alpha)^i - \binom{n+1}{i} \alpha^{n+1-i} (1 - \alpha)^i \right| \mu_0^i
\]

\[
= \sum_{i=0}^{n+1} \left| \mathbb{P}(S_n = i) - \mathbb{P}(S_{n+1} = i) \right|
\]

where \( S_n \) denotes the \( n \)-th step of the Markov chain on \( \mathbb{N} \) starting at the origin and with i.i.d. increments of law \( \alpha \delta_0 + (1 - \alpha) \delta_1 \). Moreover, we have

\[
\mathbb{P}(S_n = i) \geq \mathbb{P}(S_{n+1} = i)
\]

\[
\iff \frac{n!}{i!(n-i)!} \alpha^{n-i} (1 - \alpha)^i \geq \frac{(n+1)!}{i!(n + 1 - i)!} \alpha^{n+1-i} (1 - \alpha)^i
\]

\[
\iff i \leq (n + 1)(1 - \alpha) =: m_n
\]

Hence, using (2),

\[
\| \mu^n - \mu^{n+1} \| \leq \left| \mathbb{P}(S_n \leq m_n) - \mathbb{P}(S_n > m_n) \right| + \left| \mathbb{P}(S_{n+1} \leq m_n) - \mathbb{P}(S_{n+1} > m_n) \right|
\]

By the Central Limit Theorem, each term in the right hand side converges to zero as \( n \) goes to infinity, whence the result. \( \blacksquare \)
Lemma 2 has the following remarkable corollary. Recall that given an action of $H$ on a space $X$, a measure $\nu$ on $X$ is said to be $\mu$-stationary if

$$\nu = \int_H g_* \nu \, d\mu(g).$$

**Corollary 3:** Let $H$ be a topological group acting on a topological space $X$, and $\mu$ a Borel probability measure on $H$ such that $\mu^k, \mu^l$ are not mutually-singular for some distinct integers $k > l \geq 0$.

Then for every $x \in X$, all the weak-* limits of the sequence $(\mu^{(k-l)n} \ast \delta_x)_{n \geq 0}$ are $\mu^{k-l}$-stationary.

**Remark:** The weak-* limits in question are not assumed to be of mass 1, there could be the null measure for instance.

**Proof of Theorem 1.** Set $d = |k - l| \geq 1$, then let $\Gamma_d$ be the closed semigroup generated by the support of $\mu^d$ and $\Gamma^d = \{g^d, g \in \Gamma\}$. The observations that $\Gamma_d \supseteq \Gamma^d$ and $(\text{Ad}\Gamma)^d$ is Zariski dense in $\text{Ad}\Gamma$ lead to the equality

$$\text{Ad}\Gamma_d^d = \text{Ad}\Gamma^d.$$

In particular, we may apply Benoist–Quint’s theorems presented above to $\mu^d$.

Fix $x \in X$ and denote by $\nu_{d,x}$ the homogeneous probability measure on $\overline{\Gamma_d.x}$.

We first prove the weak-* convergence

$$\mu^{nd} \ast \delta_x \rightarrow \nu_{d,x}. \quad (3)$$

Let $\nu$ be a weak-* limit of $(\mu^{nd} \ast \delta_x)_{n \geq 0}$. It is enough to show that $\nu = \nu_{d,x}$. We know from [1, Theorem A] and Corollary 3 that $\nu$ is a $\mu^d$-stationary probability measure on $\overline{\Gamma_d.x}$. Moreover, for every $y \in \overline{\Gamma_d.x}$, we have by [1, Theorem C] that $\frac{1}{n} \sum_{k=0}^{n-1} \mu^{kd} \ast \delta_y$ converges to the homogeneous measure on $\overline{\Gamma_d.y}$. Hence, we just need to check that $\nu$ is concentrated on the set of $y \in \overline{\Gamma_d.x}$ such that

$$\overline{\Gamma_d.y} = \overline{\Gamma_d.x}.$$

But, by [1, Lemma 6.1], the complementary subset is included in some countable union $\bigcup_{i \in \mathbb{N}} \overline{LY_i}$ where $L = Z_G(\Gamma_d)$ denotes the centralizer of $\Gamma_d$ in $G$, and each $Y_i \subseteq \overline{\Gamma_d.x}$ is a proper $\Gamma_d$-invariant ergodic finite-volume homogeneous subspace. Noticing that $x \notin LY_i$ and applying the result of non accumulation of mass established in [1, Theorem B'], we get that $\nu(LY_i) = 0$ for each $i \in \mathbb{N}$, which finishes the proof of (3).
Now combining (3) with the fact that $\mu^n \ast \delta_x$ converges toward $\nu_x$ in Cesàro averages, we obtain

$$\nu_x = \frac{1}{d}(\nu_{d,x} + \mu \ast \nu_{d,x} + \cdots + \mu^{d-1} \ast \nu_{d,x})$$.

To conclude, it is enough to show that $\nu_{d,x}$ is $\Gamma$-invariant. This is where we use the aperiodicity assumption. We denote by $Y_i$ the support of $\mu^i \ast \nu_{d,x}$ and check that every $g \in \Gamma$ acts as a permutation of the set $\{Y_i, i \in \mathbb{Z}/d\mathbb{Z}\}$. We can assume that $g \in \text{supp } \mu$. The observation that $Y_{k+i} = (\text{supp } \mu)^k Y_i$ implies in particular that $gY_i \subseteq Y_{i+1}$. Then, $Y_i$ being $\Gamma_{d}$-invariant, we have

$$Y_i = g^dY_i \subseteq g^{d-1}Y_{i+1} \subseteq Y_i,$$

whence $gY_i = Y_{i+1}$. Finally, $\Gamma$ does permute the components $\{Y_i, i \in \mathbb{Z}/d\mathbb{Z}\}$, and the elements of $\text{supp } \mu$ all act as the same permutation. The aperiodicity condition yields that the $Y_i$’s are all equal, or in other words that $\Gamma$ stabilizes the support of $\nu_{d,x}$. By homogeneity, this means that $\nu_{d,x}$ is $\Gamma$-invariant and concludes the proof. ■

References

[1] T. Bénard and N. de Saxcè, Random walks with bounded first moment on finite-volume spaces, Geometric and Functional Analysis 32 (2022), 687–724.

[2] Y. Benoist and J.-F. Quint, Introduction to random walks on homogeneous spaces, Japanese Journal of Mathematics 7 (2012), 135–166.

[3] Y. Benoist and J.-F. Quint. Stationary measures and invariant subsets of homogeneous spaces. III, Annals of Mathematics 178 (2013), 1017–1059.

[4] S. Foguel, Iterates of a convolution on a non abelian group, Annales de l’Institut Henri Poincaré. Section B. Calcul des Probabilités et Statistique 11 (1975), 199–202.

[5] D. Ornstein and L. Sucheston, An operator theorem on $L_1$ convergence to zero with applications to Markov kernels, Annals of Mathematical Statistics 41 (1970), 1631–1639.

[6] R. Prohaska, Aspects of convergence of random walks on finite volume homogeneous spaces. https://arxiv.org/abs/1910.11639.

[7] M. Ratner, Distribution rigidity for unipotent actions on homogeneous spaces, Bulletin of the American Mathematical Society 24 (1991), 321–325.