SOME OPEN BOOK EMBEDDINGS OF HIGHER DIMENSIONAL SMOOTH AND CONTACT MANIFOLDS

ARJIT NATH AND KULDEEP SAHA

Abstract. We prove some open book embedding results in the smooth and contact category. We prove an open book version of the Haefliger–Hirsch embedding theorem by showing that every $k$-connected closed manifold $M^n$ has an open book embedding in $S^{2n-k} = Ob(D^{2n-k-1}, id)$. We also give various examples of contact open book embeddings of a contact manifold $(M^{2n+1}, \xi)$ in $(S^{4n+1}, \xi_{std})$. Finally, we prove a smooth open book embedding result for Weinstein/Stein fillable contact manifolds and briefly discuss its relation to the fillability problem.

1. Introduction

Embedding of manifolds is an old and much studied problem in topology. The simplest target manifold to embed a manifold is the Euclidean space $\mathbb{R}^N$ or the Euclidean sphere $S^N$. The celebrated Whitney embedding theorem [Wh] tells that every such $n$-manifold $M^n$ can be embedded in $S^{2n}$. The work of Massey [Ma], Haefliger–Hirsch [HH] and Hirsch [Hi] says that for orientable $n$-manifolds the minimal dimension of embedding can be further reduced to give embedding in $S^{2n-1}$. In [HH], Haefliger and Hirsch generalized this theorem for $k$-connected manifolds and produced embeddings in $S^{2n-k-1}$.

Embedding questions can also be studied with extra geometric structures on the manifolds. One of the first geometric embedding problem, the $C^1$-isometric embedding problem, was studied by John Nash [N]. This inspired the study of embedding problems for manifolds preserving other given geometric structures. In the present article, we talk about embedding problems preserving smooth open book decompositions and contact open book decompositions of closed oriented manifolds.

An open book, roughly speaking, is a decomposition of a manifold into a co-dimension 2 submanifold and a mapping torus (see section 2 for details). The existence of an open book on a manifold follows from the works of Tamura [Ta], Winkelnkemper [Wi], Lawson [La] and others. It is known that all odd dimensional manifolds admit open book decomposition. For $4m$-dimensional manifolds, the index is assumed to be zero for existence of an open book.

An open book embedding is an embedding such that an open book decomposition on the target manifold induces an open book decomposition on the embedded manifold. For a formal definition of open book embedding see Definition 2.4.

In [Wi], Winkelnkemper gave one the first construction of open book decompositions on simply connected manifolds. Combining the open book decomposition of Winkelnkemper [Wi] and embedding results of Haefliger and Hirsch [HH] we get the following open book version of the Haefliger–Hirsch theorem.

**Theorem 1.1.** Let $n \geq 7$ and $0 \leq k < \frac{1}{2}(n-4)$. Then, every $k$-connected closed manifold $M^n$ has an open book embedding in $S^{2n-k} = Ob(D^{2n-k-1}, id)$.

Here $Ob(D^{2n-k-1}, id)$ denotes the trivial open book decomposition of $S^{2n-k}$ with page $D^{2n-k-1}$ and identity monodromy. See section 2.1 for the related definitions.

By the work of Giroux [Gi], any contact manifold can be seen as an open book decomposition with symplectic structure on its pages and a symplectomorphism as monodromy. The first results in contact

1991 Mathematics Subject Classification. Primary: 53D10. Secondary: 53D15, 57R17.

Key words and phrases. embedding, open book.
open book embedding were obtained by Mori [Mo] and Torres [To], using techniques from approximate holomorphic geometry.

Using Gromov’s h-principle type results for isosymplectic immersions and embeddings, we prove the following open book embedding result in the contact category.

**Theorem 1.2.** Let \((V^{2n}, \partial V^{2n}, d\lambda_V)\) be a simply connected exact symplectic manifold. If \((V^{2n}, \partial V^{2n}, d\lambda_V)\) has a proper isosymplectic embedding in \((D^{4n}, d\lambda_0, id)\) then \(\mathcal{Ob}(V^{2n}, d\lambda_V, \phi)\) contact open book embeds in \((S^{4n+1}, \xi_{std}) = \mathcal{Ob}(D^{4n}, d\lambda_0, id)\) for \(n \geq 2\).

Here, \(d\lambda_0\) denotes the standard symplectic form on the 4n-ball.

As an application of Theorem 1.2, we find large classes of contact 2n + 1-manifolds that admit contact open book embedding in the trivial open book of \((S^{4n+1}, \xi_{std})\). Note that Kasuya [Ka] has proved that every 2-connected contact \((2n + 1)\)-manifold admits isocontact embedding in \((S^{4n+1}, \xi_{std})\).

First, we prove the following corollary.

**Corollary 1.3.** Let \((W^{2n}, d\lambda_W)\) be a simply connected sub-critical Weinstein domain. Then \(\mathcal{Ob}(W^{2n}, d\lambda_W, \phi_W)\) contact open book embeds in \((S^{4n+1}, \xi_{std}) = \mathcal{Ob}(D^{4n}, d\lambda_0, id)\), for \(n \geq 3\).

Another class of contact manifolds admitting contact open book embedding is the following.

**Corollary 1.4.** Let \((W^{2n}, d\lambda_W)\) be a simply connected Weinstein domain which has the homotopy type of a CW complex that has no nonzero even dimensional cells. Then \(\mathcal{Ob}(W^{2n}, d\lambda_W, \phi_W)\) contact open book embeds in \((S^{4n+1}, \xi_{std}) = \mathcal{Ob}(D^{4n}, d\lambda_0, id)\), for \(n \geq 3\).

**Example 1.1.** Say \(n = 2n + 1\) and \((W^{2n}, d\lambda_W)\) is obtained by plumbing odd dimensional unit cotangent bundles \(DT^*S^{2n+1}\) with their canonical symplectic forms. Then for any relative symplectomorphism \(\phi\) of \((W, d\lambda_0)\), \(\mathcal{Ob}(W^{2n}, d\lambda_W, \phi)\) open book embeds in \(\mathcal{Ob}(D^{4n}, d\lambda_0, id)\).

We also point out another class of examples of \(2n + 1\)-dimensional contact manifolds that contact open book embed in \((S^{4n+1}, \xi_{std})\). To state our results we need the following definition.

**Definition 1.5.** Consider the canonical symplectic structure \(d\lambda_M\) on the cotangent bundle of a manifold \(M\). We call the contact open book \(\mathcal{Ob}(V^{2n}, \omega, \phi)\) of type-1, if it satisfies the following properties.

1. \((V^{2n}, \omega)\) be an Weinstein domain symplectomorphic to \((DT^*M_1 \# DT^*M_2 \# \ldots \# DT^*N_1 \# \ldots \# DT^*N_q, d\lambda_{M_1} \# \ldots \# d\lambda_{M_q} \# d\lambda_{N_1} \# \ldots \# d\lambda_{N_q})\).

Here, \(M_i, s\) and \(N_j, s\) are either n-sphere \(S^n\) or a closed n-dimensional manifold.

2. The monodromy \(\phi\) is generated by Dehn-Seidel twists along the \(S^n\)s among \(M_i, s\) and \(N_j, s\).

Here \(#_s\) and \(#_g\) denote boundary connected sum and plumbing, respectively.

For the notion of symplectic boundary connected sum and plumbing we refer to [Ge] and [Ko].

**Theorem 1.6.** If \((M^{2n+1}, \xi)\) is a contact manifold supported by an open book of type-1, then \((M^{2n+1}, \xi)\) has a contact open book embedding in \((S^{4n+1}, \xi_{std})\).

The proof of Theorem 1.6 is essentially the same as that of Theorem 1.4 in [S]. We will only give an outline of it with the slight modifications needed.

Finally we prove a smooth open book embedding result for manifolds which bound an Weinstein domain.

**Theorem 1.7.** Let \(M^{2n+1}\) be a closed manifold that bounds a Weinstein domain \((W^{2n+2}, d\lambda_W)\). Then, \(M^{2n+1}\) admits open book embedding in \(\mathcal{Ob}(D^{2(\frac{n+1}{2})+5}, id) = S^{2(\frac{n+1}{2})+6}\).

The proof of Theorem 1.7 relies on the existence of an abstract Weinstein Lefschetz fibration (see section 2.7 on a Weinstein domain (up to deformation equivalence) due to the work of Giroux and Pardon [GP]).

Theorem 1.7 can be used to provide a topological obstruction to the existence of Weinstein or Stein filling of almost contact manifolds of sufficiently high dimension. We discuss such examples in section 6.1.
We would like to remark that recent times have seen a lot of progress in the problem of smooth and contact open book embedding and isoscontact embedding. In particular, much progress has been made on the question of co-dimension 2 isoscontact embedding due to the works of Kasuya [Ka2], Etnyre and Furukawa [EF], Etnyre and Lekili [EL] and Pancholi and Pandit [PP]. Recently, the existence and uniqueness questions for co-dimension 2 isoscontact embedding has been completely answered by the works of Casals, Pancholi and Presas [CPP], Casals and Etnyre [CE] and Honda and Huang [HoH]. Some explicit constructions of smooth open book embedding and isoscontact embedding were also provided in the works of [EF],[EL],[PPS] and [S2]. While much is known regarding explicit isoscontact or contact open book embeddings for 3-manifolds, similar problems for higher dimensional contact manifolds are yet to be studied in full generality.

1.1. Acknowledgment. The authors thank Dishant M. Pancholi for his suggestion to study the open book embedding problem for smooth and contact manifolds. The first author is supported by the CSIR, India (Fellowship Ref. no. 09/084(0688)/2016-EMR-I).

2. PRELIMINARY

2.1. Open books. An open book is a decomposition of a manifold into a co-dimension 2 submanifold and a fibration over $S^1$.

**Definition 2.1** (Open book decomposition). An open book decomposition of a closed oriented manifold $M$ consists of a co-dimension 2 oriented submanifold $B$ with a trivial normal bundle in $M$ and a locally trivial fibration $\pi : M \setminus B \to S^1$ such that $\pi^{-1}(\theta)$ is an interior of a co-dimension 1 submanifold $N_\theta$ and $\partial N_\theta = B$, for all $\theta$. The submanifold $B$ is called the binding and $N_0$ is called a page of the open book. Denote the open book decomposition of $M$ by $(M, Ob(B, \pi))$ or sometimes simply by $Ob(B, \pi)$.

**Example 2.1.** $S^n$ admits an open book decomposition with page $D^{n-1}$ and monodromy the identity map. We call this open book, $Ob(D^{n-1}, id)$, the trivial open book of $S^n$.

There is another notion of an abstract open book that is equivalent to the above definition and in many cases is easier to work with.

**Definition 2.2.** Let $\Sigma$ be as before and let $\phi$ be a diffeomorphism of $\Sigma$ that is identity on a collar neighborhood of $\partial \Sigma$. An abstract open book decomposition of $M$ is a pair $(\Sigma, \phi)$ such that $M$ is diffeomorphic to $MT(\Sigma, \phi) \cup_{id} \partial \Sigma \times D^2$

where $id$ denotes the identity mapping of $\partial \Sigma \times S^1$.

Here, $MT(\Sigma, \phi) = \Sigma \times [0,1] / \sim$ is the mapping torus, where $\sim$ is the equivalence relation identifying $(x,0)$ with $(\phi(x),1)$. We will denote such an abstract open book by $Ob(\Sigma, \phi)$.

Two abstract open books $Ob(\Sigma_1, \phi_1)$ and $Ob(\Sigma_2, \phi_2)$ are equivalent if there is a diffeomorphism $h : \Sigma_1 \to \Sigma_2$ such that $h \circ \phi_2 = \phi_1 \circ h$. It is a well known fact that an abstract open book decomposition of $M$, up to equivalence, gives an open book decomposition of $M$ up to diffeomorphism and vice versa. The boundary of the page $\partial \Sigma$ gives the binding in the first definition and a fiber of the fibration $\pi : M \setminus B \to S^1$ gives the page in the second definition. Hence, we will not distinguish between open books and abstract open books.

For more details on open books, see [E] and [Ge].

Note that the isotopy class of $\phi$ uniquely determines $M$, up to diffeomorphisms. The map $\phi$ is called the monodromy of the open book. The manifold obtained by identifying the boundary of $MT(\Sigma, \phi)$ with the boundary of $\partial \Sigma \times D^2$, as described in the Definition 2.2, will be denoted by $Ob(\Sigma, \phi)$.

For open books, there is an important symmetry property. If $\Sigma$ is as above and $\phi_1$ and $\phi_2$ are two diffeomorphisms of $\Sigma$ that restrict to identity near boundary, then $Ob(V, \phi_1 \circ \phi_2) \cong Ob(V, \phi_2 \circ \phi_1)$.

**Example 2.2.** $S^n$ admits an open book decomposition with pages $D^{n-1}$ and monodromy the identity map of $D^{n-1}$. We call this open book the trivial open book of $S^n$. 


Example 2.3. $S^3 \times S^2$ admits an open book decomposition with pages the unit disk bundle of $T^*S^2$ and monodromy the mapping class of the identity map of the unit disk bundle of $T^*S^2$. We call this open book decomposition of $S^3 \times S^2$ the standard open book decomposition of $S^3 \times S^2$.

Definition 2.3 (Flexible embedding). A proper embedding $f : (\Sigma^n, \partial \Sigma^n) \to (V^{n+k}, \partial V^{n+k})$ is called flexible if for every diffeomorphism $\phi$ of $(\Sigma, \partial \Sigma)$ there is an isotopy $\Phi_t$ ($t \in [0,1]$) of $(V^{n+k}, \partial V^{n+k})$ such that $\Phi_0 = \text{id}$ and $\Phi_1 \circ f = f \circ \phi$. Here $\partial \Sigma$ and $\partial V$ could be empty sets.

Example 2.4. By [Wu], any two embeddings of a closed manifold $M^n$ in $\mathbb{R}^{2m+1}$ are isotopic. Let $f : M \to \mathbb{R}^{2m+1}$ be an embedding and let $\phi \in \text{Diff}_\omega(M)$. Then by [Wu], $f$ and $f \circ \phi$ are isotopic. Using the isotopy extension theorem one can then extend this isotopy to an ambient isotopy $\Phi_t$ of $\mathbb{R}^{2m+1}$ such that $\Phi_0 = \text{id}$ and $\Phi_1 \circ f = f \circ \phi$. Therefore, any embedding of $M^n$ in $\mathbb{R}^{2m+1}$ is flexible.

Now we define the notion of an open book embedding.

Let $M^n$ and $V^N$ be two manifolds admitting open book decompositions. Assume $N \geq n + 1$.

Definition 2.4 (Open book embedding). $M^n$ has an open book embedding in $V^N$ if there is an open book $\text{Ob}(\Sigma_M^{n-1}, \phi_M)$ of $M$ and an open book $\text{Ob}(\Sigma_V^{N-1}, \phi_V)$ of $V$ such that the following conditions hold:

1. there exists a proper embedding $f : (\Sigma_M, \partial \Sigma_M) \to (\Sigma_V, \partial \Sigma_V)$,
2. $\phi_V \circ f = f \circ \phi_M$.

We also say that $M^n$ open book embeds in $V^N$ with respect to the open book $\text{Ob}(\Sigma_V, \phi_V)$.

Example 2.5. $S^n \cong \text{Ob}(D^{n-1}, \text{id})$ canonically open book embeds in $S^n \cong \text{Ob}(D^{N-1}, \text{id})$ for $N - n \geq 1$.

Example 2.6. By Example 2.4, every manifold $M^n$, admitting open book decomposition, open book embeds in $S^{2n}$ with the trivial open book $\text{Ob}(D^{2n-1}, \text{id})$.

2.2. Winkelnkemper open books for closed manifolds. In [Wi], Winkelnkemper showed existence of open book decomposition for simply connected manifolds of dimension $\geq 7$. The prototype for Winkelnkemper’s proof, as Winkelnkemper himself mentions in [Wi], is the following example.

Example 2.7. Let $V$ be any compact manifold with $\partial V \neq \emptyset$. Consider the quotient space $W = \frac{V \times [0,1]}{(x,t) \sim (x,1)}$. Let $N \subset \partial W$ be the image of $\partial V \times [0,1]$ under the quotient map. $N$ divides $\partial W$ into two parts: $\partial_{W_1}W$ and $\partial_{W_2}W$. If $h$ is a diffeomorphism of the triple $(\partial W, \partial_{W_1}W, \partial_{W_2}W)$ then $W_1 \cup_h W$ has an open book decomposition with binding $N$. See Figure 1.

Let us quickly review Winkelnkemper’s proof. As in [Wi], we assume that $M^n$ is a simply connected closed $n$-manifold and $n = 2k + 1$ for $k \geq 3$.

Take a minimal handle decomposition of $M^n$. Let $W_1$ denote the handlebody constructed by attaching handles of dimension up to $k$. Let $W_2$ denote the handlebody constructed by attaching such handles of dimension up to $k$ of the dual handle decomposition. Then, we know that $M^n = W_1 \cup_\partial W_2$. Where, $W_1$ and $W_2$ are attached along their common boundary $E = W_1 \cap W_2 = \partial W_1 = \partial W_2$ via some diffeomorphism. There
exist $k$-dimensional sub-complexes $K_1 \subset W_1$ and $K_2 \subset W_2$ whose inclusions are homotopy equivalences. The following lemma is the main step in Winkelnkemper’s proof.

**Lemma 2.5** (Winkelnkemper). There exists a $k$-complex $K \subset E \subset \partial W_1 = \partial W_2$ such that both inclusions $K \subset W_j$ ($j = 1, 2$) are homotopy equivalences.

Now, let $V$ be a regular neighborhood of $K$ in $E$. $V$-will be the candidate for page and $\partial V$ will be the binding. Since $K$ has co-dimension $\geq 3$, $V$ and $\partial V$ are simply connected. Take a collar $\partial V \times I$ of $V$ and regard $W_1$ and $W_2$ as a relative cobordism between $V$ and the closure of the complement of $V \cup \partial V \times I$ in $E$. The assertion then implies that both $W_1$ and $W_2$ are relative $h$-cobordisms. Thus, by the relative $h$-cobordism theorem $W_1 = V \times I = W_2$ and by contracting the collar neighborhood $\partial V \times I$ we get the desired open book with binding $\partial V$ and page $V$. We call this open book the Winkelnkemper open book.

Note that if $M^{2k+1}$ is an $l$-connected manifold, then according to the above construction we get an open book with an $l$-connected page. This is because we can then start with a handlebody decomposition $M^{2k+1} = W_1 \cup W_2$ such that each $W_i$ has no handles up to dimension $l$. Therefore, the following holds.

**Lemma 2.6.** If $M^{2k+1}$ is an $l$-connected manifold, then the Winkelnkemper open book has $l$-connected pages.

### 2.3. Embedding and isotopy of manifolds in Euclidean space

Let $M^n$ be a closed $k$-connected manifold. Let $M_0$ denote $M \setminus \{pt\}$. In [HH], Haefliger and Hirsch proved the following generalizations of Whitney’s embedding theorem [Wh] and Wu’s isotopy theorem [Wu].

**Theorem 2.7** (Theorem 2.1, [HH]). Assume $0 \leq k < \frac{1}{2}(n - 4)$. If $M_0$ can be immersed in $\mathbb{R}^{2n-k-1}$ with a normal vector field, then $M^n$ can be embedded in $\mathbb{R}^{2n-k-1}$.

**Theorem 2.8** (Theorem 2.2, [HH]). Assume $0 \leq k \leq \frac{1}{2}(n - 4)$. If $M^n$ is orientable there is a $1 - 1$ correspondence between the isotopy classes of embeddings of $M$ in $\mathbb{R}^{2n-k}$ and the regular homotopy classes of immersions of $M_0$ in $\mathbb{R}^{2n-k}$ with a normal vector field.

The proof of Theorem 2.7 and Theorem 2.8 uses the following theorems of Hirsch and Haefliger [HH].

**Theorem 2.9** (Theorem 3.1 [HH]). Let $M^n$ be a $k$-connected manifold.

1. If $m \geq 2n - k - 1$, then $M_0$ can be immersed in $\mathbb{R}^m$, and any immersion is regularly homotopic to an embedding.

2. If $m \geq 2n - k$, any two embeddings $f$ and $g$ of $M_0$ in $\mathbb{R}^m$ are regularly homotopic. If $G$ is a regular homotopy connecting $f$ and $g$, there is a regular homotopy $G_1$ of $G$ such that $G_0 = G$, $G_1$ is a homotopy, and for each $t$, $G_1$ connects $f$ to $g$.

**Theorem 2.10** (Theorem 3.2, [HH]). Let $X$ be an $m$-manifold and $E$ an open $n$-disk.

1. Suppose $2m \geq 3(n + 1)$ and $X$ is $(2n - m + 1)$-connected. Let $g : E \to X$ be a proper map whose restriction to the complement of some compact set is an embedding. Then there is a homotopy, fixed outside of a compact set, which deforms $g$ into an embedding.

2. Suppose $2m \geq 3(n + 1)$ and $X$ is $(2n - m + 2)$-connected. Let $g_0, g_1 : E \to X$ be proper embeddings which are connected by a homotopy fixed outside of a compact set. Then $g_0$ and $g_1$ are also connected by an isotopy $g_t$, fixed outside of a compact set.

By the Smale-Hirsch immersion theory $M_0$ has an immersion in $\mathbb{R}^{2n-k-1}$ with a normal vector field. Using statement (1) of Theorem 2.9, one regularly homotopes this immersion to an embedding $f$. Now, using the normal vector field, the embedding is extended to a map $g_0 : M \to \mathbb{R}^{2n-k-1}$ such that $g_0$ coincides with $f$ outside a small neighborhood of the deleted point. Then using statement (2) of Theorem 2.10, $g_0$ is homotoped to an embedding $g$ with the homotopy identity outside a neighborhood of the deleted point. This shows the main steps in the proof of Theorem 2.7.

For Theorem 2.8, a similar approach is taken. By statement (2) of Theorem 2.9, we can start with two regularly homotopic embeddings $f_0, f_1$ of $M_0$ in $\mathbb{R}^{2n-k}$ with normal vector fields. Then one finds two homotopic maps $h_0$ and $h_1 : M \to \mathbb{R}^{2n-k}$ extending $f_0$ and $f_1$, respectively. Statement (1) of Theorem 2.10
allows to assume that $h_0$ and $h_1$ are embeddings and therefore statement (2) of Theorem 2.10 implies that $h_0$ and $h_1$ are isotopic as embeddings and the isotopy is fixed outside a neighborhood of the deleted point.

The proof of Theorem 2.7 and Theorem 2.8 also holds for proper embeddings of a manifold $(V^n, \partial V^n)$ in an Euclidean disk $(D^N, \partial D^N)$. We will be using this relative versions of Theorem 2.7 and Theorem 2.8 in our proofs.

2.4. **Contact manifold and isocontact embedding.** A contact manifold is an odd dimensional smooth manifold $M^{2n+1}$, together with a maximally non-integrable co-dimension 1 distribution $\xi \subset TM$. A contact form $\alpha$ representing $\xi$ is a local 1-form on $M$ such that $\xi = \text{Ker}(\alpha)$. The contact condition is equivalent to saying that $\alpha \wedge (d\alpha)^n$ is a volume form. The 2-form $d\alpha$ then induces a conformal symplectic structure on $\xi$. If the line bundle $TM/\xi$ over $M$ is trivial, then the contact structure is said to be co-orientable. For a co-orientable contact structure $\xi$, one can define a contact form $\alpha$ representing $\xi$ on all of $M$. We will only consider co-orientable contact structures on closed, orientable manifolds. We will denote a manifold $M$ together with a contact structure $\xi$ by $(M, \xi)$ and use $\xi_{\text{std}}$ to denote the standard contact structure on an odd dimensional Euclidean space $\mathbb{R}^{2n+1}$, given by $\text{Ker}\{dz + \sum_{i=1}^{n+1} x_i dy_i\}$. The dimension will be understood from the notation $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$.

Two contact manifolds $(M_1, \xi_1)$ and $(M_2, \xi_2)$ are equivalent if there is a diffeomorphism $h$ between them such that $Dh(\xi_1) = \xi_2$. We say, the two contact manifolds are contactomorphic to each other. For more details on contact manifolds see [Ge].

We now define the notion of embedding in the category of contact manifolds.

**Definition 2.11** (Isocontact embedding). $(M^{2n+1}, \xi)$ admits an isocontact embedding in $(V^{2N+1}, \eta)$, if there is an embedding $\iota : M \hookrightarrow V$ such that for all $p$ in $M$, $D\iota(T_p M) \cap \eta|_{T_p M} = D\iota(\xi_p)$. A manifold $M^{2n+1}$ contact embeds in $(V^{2N+1}, \eta)$ if there exists a contact structure $\xi_0$ on $M^{2n+1}$ such that $(M, \xi_0)$ has an isocontact embedding in $(V^{2N+1}, \eta)$.

It follows from the definition that if $\alpha$ is a contact form representing $\xi$ and $\beta$ is a contact form representing $\eta$, then $\iota^* \beta = h \cdot \alpha$ for some positive function $h$ on $M$. $D\iota(\xi)$ is a conformal symplectic sub-bundle of $(\eta|_{\iota(M)}, d\beta)$.

Gromov [Gr] reduced the existence of an isocontact embedding of a contact manifold $(M^{2n+1}, \xi)$ in a contact manifold $(V^{2N+1}, \eta)$, for $N \geq n+2$, to a problem in obstruction theory. In particular, Gromov proved the following contact analog of Whitney’s embedding theorem.

**Theorem 2.12** (Gromov, [Gr]). Every contact manifold $(M^{2n+1}, \xi)$ has an isocontact embedding in $(\mathbb{R}^{4n+3}, \xi_{\text{std}})$.

**Remark 2.13.** When the embedding co-dimension is $\leq \dim(M) - 1$, there is a natural topological obstruction to contact embedding. It has the following description. If $\iota : (M^{2n+1}, \xi) \hookrightarrow (\mathbb{R}^{2N+1}, \eta)$ is a contact embedding, then the normal bundle $\nu(\iota) = \iota^* (\eta)/\xi$ has an induced complex structure on it. So we have the following relation of total Chern classes.

\[ c(\xi \oplus \nu(\iota)) = c(\iota^* (\eta)) = 1 \]

Let $c_j(\xi)$ denote the $j^{\text{th}}$ order cohomology class in $(1 + c_1(\xi) + c_2(\xi) + \ldots + c_n(\xi))^{-1}$. Since the Euler class of the normal bundle of an embedding in $\mathbb{R}^{2N+1}$ is zero,

\[ c_{N-n}(\nu(\iota)) = 0 \iff c_{N-n}(\xi) = 0 \]

This gives a condition on the Chern classes of $\xi$. Thus, for isocontact embedding of co-dimension $\leq \dim(M) - 1$, one has to restrict the problem on the contact structures whose Chern classes satisfy this condition. For example, a necessary condition to isocontact embed a 3-manifold $(M, \xi)$ in $(\mathbb{R}^5, \xi_{\text{std}})$ is $c_1(\xi) = 0$.

2.5. **Contact open book and embedding.** We start with a discussion of the Thurston–Winkelnkemper construction [TW] of contact open book decomposition.

Let $(V, \partial V, d\alpha)$ be an exact symplectic manifold which has a collar symplectomorphic to $((-1, 0] \times \partial V, d(c^t \cdot \alpha))$, where $t \in (-1, 0]$. The Liouville vector field $Y$ for $d\alpha$ is defined by $i_Y d\alpha = \alpha$. Near boundary this
vector field looks like $\frac{\partial}{\partial t}$ and is transverse to $\partial V$ pointing outwards. The 1-form $e^t \cdot \alpha$ induces a contact structure on $\partial V$. Let $\phi$ be a symplectomorphism of $(V, d\alpha)$ that is identity in a collar of the boundary. The following lemma, due to Giroux, shows that we can assume $\phi^* \alpha - \alpha$ to be exact.

**Lemma 2.14** (Giroux). *The symplectomorphism $\phi$ of $(V, d\alpha)$ is isotopic, via symplectomorphisms which are identity near $\partial V$, to a symplectomorphism $\phi_1$ such that $\phi_1^* \alpha - \alpha$ is exact.*

For a proof of the above lemma see [Ko].

Let $\phi^* \alpha - \alpha = dh$. Here $h : V \to \mathbb{R}$ is a function well defined up to addition by constants. Note that $dt + \alpha$ is a contact form on $\mathbb{R} \times V$, where the $t$-co-ordinate is along $\mathbb{R}$. Take the mapping torus $\mathcal{M} T(V, \phi)$ defined by the following map.

\[
\Delta : (\mathbb{R} \times V, dt + \alpha) \longrightarrow (\mathbb{R} \times V, dt + \alpha) \\
(t, x) \longmapsto (t - h, \phi(x))
\]

The contact form $dt + \alpha$ then descends to a contact form $\lambda$ on $\mathcal{M} T(V, \phi)$. Since $\phi$ is identity near $\partial V$, a neighborhood of the boundary of $\mathcal{M} T(V, \phi)$ looks like $(-\frac{1}{2}, 0) \times \partial V \times S^1$ with the contact form $e^r \cdot \alpha |_\partial V + dt$. Denote the annulus $\{ z \in \mathbb{C} | r < |z| < R \}$ by $A(r, R)$. Define $\Phi$ as follows.

\[
\Phi : \partial V \times A(\frac{1}{2}, 1) \longrightarrow (\frac{1}{2}, 0) \times \partial V \times S^1 \\
(v, re^{it}) \longmapsto (\frac{1}{2} - r, v, t)
\]

Using $\Phi$, we can glue $\mathcal{M} T(V, \phi)$ and $\partial V \times D^2$ along a neighborhood of their boundary such that under $\Phi$, $\lambda$ pulls back to $(e^{1-r} \cdot \alpha |_\partial V + dt)$ on $V \times A(\frac{1}{2}, 1)$. We can extend this to a contact form $\beta = h_1(r) \cdot \alpha |_\partial V + h_2(r) \cdot dt$, on the interior of $\partial V \times D^2$ by the real functions $h_1$ and $h_2$ (see Figure 1) from $[0, 1)$ to get a contact structure on $W^{2n+1} = \mathcal{M} T(V, \phi) \cup_{\partial V} \partial V \times D^2$, where $\beta$ is a globally defined contact form that coincides with $dt + \alpha$ on $\mathcal{M} T(V, \phi)$ and with $\alpha + r^2 dt$ near $\partial V$. We will denote the resulting contact manifold $(W^{2n+1}, \beta)$ as $\mathcal{O} b(V, d\alpha, \phi)$.

Although the contact manifold $\mathcal{O} b(V, d\alpha, \phi)$ clearly depends on the monodromy $\phi$, there is an important symmetry property. If $(V, d\alpha)$ is as above and $\phi_1$ and $\phi_2$ are two symplectomorphisms then $\mathcal{O} b(V, d\alpha, \phi_1 \circ \phi_2) \cong \mathcal{O} b(V, d\alpha, \phi_2 \circ \phi_1)$.

**Definition 2.15** (Contact open book). $\mathcal{O} b(V, d\alpha, \phi)$ is called a contact open book with page $(V, d\alpha)$ and binding $(\partial V, \alpha)$. Given a contact manifold $M$ with a contact form $\beta$, if one can find an open book $\mathcal{O} b(V_{M}, d\alpha_{M}, \phi_{M})$ that is contactomorphic to $(M, \beta)$, then one says that $\mathcal{O} b(V_{M}, \alpha_{M}, \phi_{M})$ is an open book decomposition of $M$ supporting the contact form $\beta$. 

![Figure 2. Functions for the contact form near binding](image-url)
If a contact manifold \((M, \xi)\) has a contact form \(\beta\) representing \(\xi\) such that it has a supporting open book, then we say that \((M, \xi)\) has a supporting open book. We will write \((M, \xi) = \text{Ob}(V_M, d\alpha_M; \phi_M)\) to say that \((M, \xi)\) is supported by the open book with page \((V_M, d\alpha_M)\) and monodromy \(\phi_M\). Giroux [Gi] has proved that every contact manifold has a supporting open book.

**Definition 2.16** (contact open book embedding). \((M_1, \xi_1)\) contact open book embeds in \((M_2, \xi_2)\) if there exist a contact open book of \((M_1, \xi_1)\), \(\text{Ob}(\Sigma_1, d\alpha_1, \phi_1)\) and a contact open book of \((M_2, \xi_2)\), \(\text{Ob}(\Sigma_2, d\alpha_2, \phi_2)\) such that the following conditions hold.

1. There exists a proper symplectic embedding \(g: (\Sigma_1, d\alpha_1) \to (\Sigma_2, d\alpha_2)\),
2. \(g \circ \phi_1 = \phi_2 \circ g\).

Note, the above definition implies that the mapping torus \(\mathcal{MT}(\Sigma_1, \phi_1)\) contact embeds in the mapping torus \(\mathcal{MT}(\Sigma_2, \phi_2)\). Since \(g|_{\partial \Sigma_1}\) pulls back the contact form \(\alpha_2\) to \(\alpha_1\), we can extend this embedding to a contact embedding \(\mathcal{G}\) of \(\text{Ob}(\Sigma_1, d\alpha_1, \phi_1)\) to \(\text{Ob}(\Sigma_2, d\alpha_2, \phi_2)\) such that the restriction of \(\text{Ob}(\Sigma_2, d\alpha_2, \phi_2)\) on the image of \(\mathcal{G}\) gives the contact open book \(\text{Ob}(\Sigma_1, d\alpha_1, \phi_1)\).

### 2.6. Dehn-Seidel twist. Consider the symplectic structure on the cotangent bundle \((T^*S^n, d\lambda_{\text{can}})\). Here, \(\lambda_{\text{can}}\) is the canonical 1-form on \(T^*S^n\). In local coordinates \((x_1, x_2, ..., x_{n+1}, y_1, y_2, ..., y_{n+1})\), \(\lambda_{\text{can}}\) is given by the form \(\sum y_i dx_i\). Here, we regard \(T^*S^n\) as a submanifold of \(\mathbb{R}^{2n+2} \cong \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}\). A point \((\vec{x}, \vec{y}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}\), represents a point in \(T^*S^n\) if and only if it satisfies the relations: \(|\vec{x}| = 1\) and \(\vec{x} \cdot \vec{y} = 0\). Here, \(\vec{y} = (y_1, ..., y_{n+1})\) and \(\vec{x} = (x_1, ..., x_{n+1})\).

Let \(\sigma_1 : T^*S^n \to T^*S^n\) be a map defined as follows.

\[
\sigma_1(\vec{x}, \vec{y}) = \begin{pmatrix}
\cos t & |\vec{y}|^{-1}\sin t \\
-|\vec{y}|\sin t & \cos t
\end{pmatrix}
\begin{pmatrix}
\vec{x} \\
\vec{y}
\end{pmatrix}
\]

For \(k \in \mathbb{Z}_{>0}\), let \(g_k : [0, \infty) \to \mathbb{R}\) be a smooth function that satisfies the following properties.

1. \(g_k(0) = k\pi\) and \(g_k'(0) < 0\).
2. \(g_k(|\vec{y}|)\) decreases to 0 at \(p_0\) and then remains 0 for all \(\vec{y}\) with \(|\vec{y}| > p_0\). See Figure 3.

Now we can define the positive \(k\)-fold Dehn-Seidel twist as follows.

\[
\tau_k(\vec{x}, \vec{y}) = \begin{cases}
\sigma_{g_k(|\vec{y}|)}(\vec{x}, \vec{y}) & \text{for } \vec{y} \neq 0 \\
-Id & \text{for } \vec{y} = 0
\end{cases}
\]

The Dehn-Seidel twist is a proper symplectomorphism of \(T^*S^n\). From Figure 3, we see that \(\tau_k\) has compact support. Therefore, choosing \(p_0\) properly, \(\tau_k\) can be defined on the unit disk bundle \((DT^*S^n, d\lambda_{\text{can}})\), such that it is identity near boundary. In fact, we can choose the support as small as we wish without affecting
the symplectic isotopy class of the resulting $\tau_k$. More precisely, let $g_k^1$ and $g_k^2$ be two functions similar to $g_k$ as above. Say, $g_k^1$ has support $p_1$ and $g_k^2$ has support $p_2$. Then $\sigma_{g_k^1(|i|)} + (1 - \tau) g_k^2(|i|)$ gives a symplectic isotopy between $\sigma_{g_k^1(|i|)}$ and $\sigma_{g_k^2(|i|)}$.

Similarly, for $k < 0$, we can define the negative $k$-fold Dehn-Seidel twist. For $k = 0$, $\tau_0$ is defined to be the identity map of $DT^* S^n$. Sometimes we may say just Dehn twist instead of Dehn-Seidel twist.

2.6.1. **The standard contact open book of $(S^{2n+1}, \xi_{std})$.** An important open book decomposition of $(S^{2n+1}, \xi_{std})$ is given with page $(DT^* S^n, d\lambda^{2n}_{an})$ and monodromy a positive Dehn-Seidel twist $\tau_n$. We call this open book the standard open book of $S^{2n+1}$.

2.7. **Lefschetz fibrations on Weinstein domains.** Here we briefly recall the notion of an abstract Weinstein Lefschetz fibration defined Giroux and Pardon in [GP]. We formally define what Weinstein manifolds and domains and avoid the technical details of it as we will not use them in our proof. For details on Weinstein manifolds/domains we refer to [CE].

Recall that a 1-form $\lambda$ on a manifold $V$ such that $\omega = d\lambda$ is symplectic, is called a Liouville form and the vector field $X$ such that $i_X \omega = \lambda$, is called the Liouville field of $\lambda$. An exact symplectic manifold means a pair $(V, \lambda)$ where $\lambda$ is a Liouville form. Equivalently, an exact symplectic manifold can be described by a triple $(V, \omega, X)$ with $i_X \omega = \lambda$.

**Definition 2.17** (Liouville cobordism). A Liouville cobordism is an exact symplectic manifold $(V, \omega, X)$ such that $X$ points outward along $\partial_+ V$ and points inward along $\partial_- V$.

**Definition 2.18** (Weinstein manifolds and Weinstein domains). A Weinstein manifold $(W, \omega, X, \phi)$ is a symplectic manifold $(W, \omega)$ with a complete Liouville vector field $X$ which is gradient-like for an exhausting Morse function $\phi : W \to \mathbb{R}$. A Weinstein cobordism $(W_1, \omega, X, \phi)$ is a Liouville cobordism $(W, \omega, X)$ whose Liouville field $X$ is gradient-like for a Morse function $\phi : W_1 \to \mathbb{R}$ which is constant on the boundary. A Weinstein cobordism with $\partial_- W_1 = \emptyset$ is called an Weinstein domain.

**Definition 2.19** (Deformation equivalence). Two Weinstein domains $(W_1, J_1, \phi_1)$ and $(W_2, J_2, \phi_2)$ are called deformation equivalent if there is a diffeomorphism $h : W_1 \to W_2$ such that the Weinstein structures $(J_1, \phi_1)$ and $(h^* J_2, h^* \phi_2)$ on $W_1$ are homotopic.

**Definition 2.20** (Abstract Weinstein Lefschetz fibration). An abstract Weinstein Lefschetz fibration is a tuple $W = (W_0; L_1, \ldots, L_m)$ consisting of a Weinstein domain $W_0$ (the central fiber) along with a finite sequence of exact parameterized Lagrangian spheres $L_1, \ldots, L_m \subset W_0$ (the vanishing cycles).

Given an abstract Weinstein Lefschetz fibration $W = (W_0; L_1, \ldots, L_m)$, one can construct a Weinstein domain $W^{2n+2}$ (its total space) by attaching critical Weinstein handles to the stabilization $W_0 \times D_2$ along Legendrians $L_j \subset W_0 \times S^1 \subset \partial (W_0 \times D^2)$ near $\frac{2j}{m} \in S^1$, obtained by lifting the exact Lagrangians $L_j$.

So the abstract Weinstein Lefschetz fibration is a sort of singular fiber bundle over $D^2$ with Weinstein pages and where the monodromy is given by composing the Dehn-Seidel twists along the Lagrangian spheres.

Giroux and Pardon proved that every Weinstein domain can be deformed to an abstract Weinstein Lefschetz fibration.

**Theorem 2.21** (Giroux–Pardon,[GP]). Let $W$ be a Weinstein domain. There exists an abstract Weinstein Lefschetz fibration $\tilde{W} = (W_0; L_1, \ldots, L_m)$ whose total space (which we again denote by $\tilde{W}$) is deformation equivalent to $W$.

2.8. **$h$-principle for isosymplectic immersions.**

**Definition 2.22** (Isosymplectic immersion). An isosymplectic immersion (or embedding) $h : (M^{2n}, \omega_M) \to (W^{2n}, \omega_W)$ is an immersion (or embedding) such that $h^* (\omega_W) = \omega_M$. 
Definition 2.23 (Isosymplectic homomorphism). An isosymplectic homomorphism is a map $F : (TM^{2n}, \omega_M) \to (TV^{2N}, \omega_W)$ such that $F$ restricts to a vector space monomorphism on $(T_p M, \omega_M)$ for all $p \in M$ and $F^*(\omega_W) = \omega_M$.

It is clear that the existence of an isosymplectic homomorphism is a necessary condition for existence of an isosymplectic immersion. Gromov [Gr] proved the following $h$-principle for isosymplectic immersions to show that to some extent it is also sufficient.

Theorem 2.24 (Isosymplectic immersion theorem [Gr]). Let $(W, \omega_W)$ and $(V, \omega_V)$ be symplectic manifolds of dimensions $2N$ and $2n$ respectively. Suppose a continuous map $f_0 : V \to W$ satisfies the cohomological condition $f_0^*[\omega_W] = [\omega_V]$, and that $f_0$ is covered by an isosymplectic homomorphism $F : TV \to TW$. If $V$ is closed and $2N \geq 2n + 2$, then there exists a homotopy $f_t : V \to W$ such that $f_1$ is an isosymplectic immersion and the differential $DF_1$ is homotopic to $F$ through isosymplectic homomorphisms.

The $h$-principle for isosymplectic immersion is also true for the relative and the parametric version (see Theorem 16.4.3 in [EM]).

Following is Gromov’s $h$-principle for isosymplectic embedding.

Theorem 2.25 (Isosymplectic embedding [Gr]). Let $(V, \omega_V)$ and $(W, \omega_W)$ be symplectic manifolds of dimension $n = 2m$ and $q = 2l$ respectively. Suppose that an embedding $f_0 : V \to W$ satisfies the cohomological condition $f_0^*[\omega_W] = [\omega_V]$, and the differential $F_0 = df_0$ is homotopic via a homotopy of monomorphisms $F_t : TV \to TW$, covering $f_0$, to an isosymplectic homomorphism $F_1 : TV \to TW$.

1. (Open case) If $n \leq q - 2$ and the manifold $V$ is open then there exists an isotopy $f_t : V \to W$ such that the embedding $f_1 : V \to W$ is isosymplectic and the differential $df_1$ is homotopic to $F_1$ through isosymplectic homomorphisms.
2. (Closed case) If $n \leq q - 4$ then the above isotopy $f_t$ exists even if $V$ is closed. Moreover, one can choose the isotopy $f_t$ to be arbitrarily $C^0$-close to $f_0$.

The above $h$-principle is also true for the relative and parametric case. In particular, if a family of proper isosymplectic immersions $f_t : (V, \partial V, d\alpha_V) \to (W, \partial W, d\lambda_W)$ (satisfying the cohomology condition: $[f_t^*d\lambda_W - d\alpha_V] = 0$ in $H^2(V, \partial V, \mathbb{Z})$) is regularly homotopic to a family of embedding $e_t$, depending smoothly on $t$, then by Theorem 2.25, $f_t$ can be isotoped to a family of proper isosymplectic embeddings $f_t^*$.

Obstructions to isosymplectic homomorphism: Consider a symplectic vector space $(X, \omega)$. Let $J$ be an $\omega$-compatible almost complex structure (i.e., $\omega(Ju, Jv) = \omega(u, v)$ and $\omega(u, Ju) > 0$ for all $u, v \in X \setminus \{0\}$). If $Y$ is a symplectic subspace of $(X, \omega)$, then $Y$ has to be a $J$-subspace of $(X, J)$ and vice-versa. An almost complex structure $J_\xi$ on the contact hyperplane bundle $\xi = Ker\{\alpha\}$ is called $\xi$–compatible, if it is compatible with the conformal symplectic structure on $\xi$ induced by $d\alpha$. An isosymplectic homomorphism takes $(TV, \omega_V)$ to a symplectic sub-bundle of $(TW, \omega_W)$. If $J_W$ is an $\omega_W$–compatible almost complex structure, then the isosymplectic homomorphism takes $(TV, \omega_V)$ to a $J_W$-sub-bundle of $(TW, \omega_W)$. When $W = D^{2N}$ and $\omega_W = \omega_0$, finding an isosymplectic homomorphism from $(TV, \omega_V)$ to $(TW, \omega_W)$ is equivalent.
to finding a $U(n)$-equivariant map from the complex $n$-frame bundle associated to $(TV, \omega_V)$ to $St^C(N, n)$. In other words, finding an isosymplectic homomorphism is equivalent to the existence of a section of the associated $St^C(N, n)$-bundle of $(TV, \omega_V)$. Here, $St^C(N, n)$ denotes the complex Stiefel manifold. Thus, $(TV, \omega_V)$ has an isosymplectic homomorphism in $(TD^{2n}, \omega_0)$ if and only if all the obstruction classes in $H^i(V, \partial V; \pi_1St^C(N, n))$ vanish for $1 \leq i \leq 2n$.

Two isosymplectic homomorphisms are homotopic via isosymplectic homomorphisms if the corresponding sections of the associated $St^C(N, n)$-bundle of $(TV, \omega_V)$. Such homotopy obstructions lie in $H^i(V, \partial V; \pi_iSt^C(N, n))$, for $1 \leq i \leq 2n$.

3. Open book embedding of $k$-connected manifolds in $S^{2n-k}$

We now prove Theorem 1.1. We want to use a relative version of Theorem 2.7 and Theorem 2.8, which can be stated as follows.

Lemma 3.1. Let $(V^n, \partial V^n)$ be a $k$-connected manifold. Let $V_0$ denote the manifold obtained by removing an $n$-disk from the interior of $V$.

1. If $V_0$ can be immersed in $\mathbb{R}^{2n-k-1}$ with a normal vector field, then $V^n$ admits a proper embedding in $(D^{2n-k}, \partial D^{2n-k})$.
2. There is a 1-1 correspondence between the isotopy classes of proper embeddings of $V$ in $D^{2n-k}$ and the regular homotopy classes of proper immersions of $V_0$ in $(D^{2n-k}, \partial D^{2n-k})$.
3. Any two proper embeddings $f$ and $g$ of $V^n$ in $D^{2n-k+1}$ are isotopic.

The proof of Lemma 3.1 will become easy once we observe how the proofs of Theorem 2.7 and Theorem 2.8 works. We now review the main steps behind these proofs from [HH].

3.1. Proofs of Theorem 2.7 and Theorem 2.8.

Proof of Theorem 2.7: Let $f_0 : M^n_0 \to \mathbb{R}^{2n-k-1}$ be an immersion with a normal vector field $v$. By Theorem 2.9, $f$ is regularly homotopic to an embedding. So we may assume $f_0$ to be an embedding. Let $D_i$ be an embedded closed disk of radius $i$ around a point $x_0 \in M^n$ for $i = 1, 2$. Let $M_i = M \setminus \text{int}(D_i)$. The claim is that $f(\partial M_1)$ is an $(n-1)$-sphere homotopic to zero in $X = \mathbb{R}^{2n-k-1} \setminus f(M_2)$.

Consider an $\epsilon$-tubular neighborhood of $f(M_1)$ in $\mathbb{R}^{2n-k-1}$. Let $\lambda : M \to [0, \epsilon]$ be a smooth function that has value $\epsilon$ on $M_2$ and is $0$ on $D_1$. Then $f(\partial M_1)$ bounds the image of $M_1$ under the map $f_\nu : M \to \mathbb{R}^{2n-k-1}$ given by $x \mapsto f(x) + \lambda(x)v(x)$. Thus, $f(\partial M_1)$ is null-homologous in $X$.

By Poincare and Alexander duality $H_i(X) \cong H_{n+k+i+2-n}(M)$ (with $\mathbb{Z}$-coefficient). Since $M$ is $k$-connected, $H_i(X) = 0$ for $i \leq n-2$. Thus, by Hurewicz isomorphism between $\pi_{n-1}(X)$ and $H_{n-1}(X)$ we get that that $f(\partial M_1)$ is null-homotopic in $X$.

One can therefore, extend the map $f|M_1$ to a map $g : M \to \mathbb{R}^{2n-k-1}$ such that $g(M_2) \cap g(\text{int}(D_2)) = \emptyset$. An application of Theorem 2.10 then leads to an embedding of $D_2$ in $X$ relative to boundary. Together with the embedding $f|M_2$, this gives an embedding of $M$ in $\mathbb{R}^{2n-k-1}$.

Next we discuss the proof of Theorem 2.8. We shall continue with the notations used above.

Proof of Theorem 2.8: First one shows that Given an embedding $f : M \to \mathbb{R}^{2n-k}$, one can associate a normal vector field $v$ on $f(M_0)$ such that $f_\nu(M)$ is null-homologous in $X = \mathbb{R}^{2n-k-1} \setminus f(M_2)$. Here, $f_\nu : M \to \mathbb{R}^{2n-k-1}$ is an embedding which is equal to $f(x) + \lambda(x)v(x)$ for $x \in M_1$ and equal to $f(x)$ for $x \in D_1$.

An argument similar to the previous proof shows that $X$ is $(n-1)$-connected and $\pi_{n}(x) = H_{n}(X) = H^{n-k-1}(M_2)$. If $v_1$ and $v_2$ are two normal (to $f(M_2)$) vector fields, then the difference class $d(v_1, v_2) \in H^{n-k-1}(M_2)$ corresponds to the homology class $[f_\nu(M)] - [f_\nu(M)] \in H_{n}(X)$. Also, the homotopy classes of normal vector fields on $f(M_0)$ are in $1-1$ correspondence with $H^{n-k-1}(M_0) = H_{n}(X)$. there is one and only one normal vector field $V$, up to is homologous to zero in $X$. Therefore, up to homotopy, there is only one vector field such that $f_\nu(M)$ is null-homologous in $X$.
The next step is to show that this correspondence is $1 - 1$. We skip the proof of surjectivity and focus only on the proof of injectivity.

Let $f_1, f_2 : M \to \mathbb{R}^{2n-k-1}$ be two embeddings with corresponding vector fields $v_1$ and $v_2$, respectively. If $(f_1|_{M_0}, v_1)$ and $(f_2|_{M_0}, v_2)$ are regularly homotopic, then by Theorem 2.9, they can be assumed to be isotopic.

Let $h_t$ be an isotopy joining $f_1|_{M_0}$ and $f_2|_{M_0}$, and let $u_t$, be a normal vector field of $h_t(M_0)$ joining $v_1$ and $v_2$. An isotopy of $h_0(M_0)$ can be extended to an isotopy of $\mathbb{R}^{2n-k-1}$. So one can assume that $f_1$ and $f_2$ agree on $M_1$ and that $v_1 = v_2$. Now observe that $f_{v_1}(M)$ and $f_{v_2}(M)$ are homologous in $X$. Therefore, by similar arguments as before $f_{v_1}(M)$ and $f_{v_2}(M)$ are homotopic in $X$. Thus, $f_1|_{D_1}$ and $f_2|_{D_1}$ are homotopic in $X$. Therefore, by Theorem 2.10 they are isotopic in $X$ by an isotopy fixed on a neighborhood of $\partial D_1$. Hence, $f_1$ and $f_2$ are isotopic.

Proof of Lemma 3.1. (1) First we embed $V$ in the boundary $(2n-k-1)$-sphere of $D^{2n-k}$ by Theorem 2.7. Let $f_0$ be the embedding. Now define a smooth function $\psi : V \to [0, \varepsilon]$ such that $\psi$ is zero in a neighborhood of $\partial V$. Let $R$ be the radial unit vector field on $D^{2n-k}$ pointing inward from boundary. Let $\lambda_t$ denote the time $t$ flow of the vector field $\psi R$. $\lambda_t \circ f_0$ then gives the required embedding. Informally, we just pushed $f_0(V)$ into the interior of $D^{2n-k}$ relative to its boundary.

Let $g_1, g_2$ be two proper embeddings of $V$ in $D^{n-k}$ such that $g_1$ and $g_2$ agree on a neighborhood of $\partial V$. We assume that $g_1$ and $g_2$ are transverse to the boundary $\partial D^{2n-k}$. Recall the embedding $f_0$ from the proof of (1). Then $\lambda_1 = f_0 \cup g_1$ and $\lambda_2 = f_0 \cup g_2$ are embeddings of the double, $D(V)$, of $V$. Let $v_1$ and $v_2$ be two corresponding normal vector fields on $\lambda_1(D(V))$ and $\lambda_2(D(V))$. Since $\lambda_1 = \lambda_2$ in a neighborhood of $\partial V$, we may assume that $v_1 = v_2$ in a neighborhood $N(\partial V)$ of $\partial V$.

If $g_1$ and $g_2$ are regularly homotopic relative to boundary with homotopic normal vector fields then by Theorem 2.9 ($\lambda_1|_{D(V)\setminus v_1}$) and ($\lambda_2|_{D(V)\setminus v_2}$) can be assumed to be isotopic. Moreover, this isotopy can be fixed near $f_0(V) \cup g_1(N(\partial V))$. We can proceed as in the proof of Theorem 2.8 to show that $\lambda_1$ and $\lambda_2$ are isotopic in $D^{2n-k}$ via an isotopy $\Phi_t$. Note that during the construction of this isotopy we have not moved $f_0(V) \cup g_1(N(\partial V))$. Therefore, $\Phi_t|_{D(V)}$ gives us the required isotopy between $g_1$ and $g_2$.

The surjectivity case can also be dealt with similarly. For details see [HH].

(3) We observe what happens to the proof of Theorem 2.8 when we look at an embedding $f$ of $M^n$ in $\mathbb{R}^{2n-k+1}$, instead of $\mathbb{R}^{2n-k}$. Let $Y = \mathbb{R}^{2n-k+1} \setminus f(M)$. One then gets that $Y$ is $n$-connected and the homotopy classes of normal vector fields on $M_0$ are in $1 - 1$ correspondence with $H^{n-k}(M_0) = 0$. Let $f_1, f_2$ be two such embeddings. By Smale-Hirsch immersion theory the obstruction to a regular
homotopy between \((f_1|_{M_0}, v_1)\) and \((f_2|_{M_0}, v_2)\) lies in the group \(H^i(M_0; \pi_i(St(2n-k+1, n+1)))\) for \(1 \leq i \leq n\). Since \(M_0\) is homotopic to its \((n-k-1)\)-skeleton and \(St(2n-k+1, n+1)\) is \((n-k-1)\)-connected, these obstructions vanish. Therefore, we can assume \(f_1\) and \(f_2\) to be isotopic on \(M_0\). Since \(H_4(Y) = 0\), both \(f_n(M)\) and \(f_{n+1}(M)\) are null-homologous in \(Y\). Arguing as in Theorem 2.8, we then see that \(f_1|_{D_1}\) and \(f_2|_{D_1}\) are homotopic relative to boundary. Then by Theorem 2.10 we see that \(f_1\) and \(f_2\) are isotopic.

**Proof of Theorem 1.1.** Say, \(M^n = \text{Ob}(V^{n-1}, \phi)\), where the open book is obtained by the Winkelnkemper construction. Since \(M\) is \(k\)-connected, by Lemma 2.6 \(V^{n-1}\) is \(k\)-connected. Using (1) of Lemma 3.1, take a proper embedding \(f: (V^{n-1}, \partial V^{n-1}) \rightarrow (D^{2n-k-1}, \partial D^{2n-k-1})\). By (3) of Lemma 3.1, \(f\) and \(f \circ \phi\) are isotopic embeddings of \((V^{n-1}, \partial V^{n-1})\) in \((D^{2n-k-1}, \partial D^{2n-k-1})\). By the isotopy extension theorem, there exists an isotopy \(\Phi_t\) of \((D^{2n-k-1}, \partial D^{2n-k-1})\) for \(t \in [0, 1]\), such that \(\Phi_0 = id\) and \(\Phi_1 \circ f = f \circ \phi\). Moreover, \(\Phi_t\) can be arranged to be identity near boundary. Hence, \(\text{Ob}(V^{n-1}, \phi)\) open book embeds in \(\text{Ob}(D^{2n-k-1}, \Phi_1) \cong \text{Ob}(D^{2n-k-1}, id) \cong S^{2n-k}\).

**Remark 3.2.** Note that for manifolds which admit open book decomposition with identity monodromy, one can get open book embedding in \(S^{2n-k-1} \cong \text{Ob}(D^{2n-2k-2}, id)\). If \(N^{n-1}\) is a manifold with boundary, then \(\partial(N \times D^2)\) admits such an open book with page \(N^{n-1}\).

## 4. Contact Open Book Embedding in \((S^{4n+1}, \xi_{std})\)

We find a suitable embedding of \(V^{2n}\) in \(D^{4n}\) to apply the \(h\)-principle for isosymplectic immersion. We recall that an embedding \(g: (M^{2n+1}, \ker\{\alpha\} = \xi) \rightarrow (N^{2n+1}, \ker\{\beta\} = \eta)\) is called an isocontact embedding if \(g^*(\beta) = h \cdot \alpha\), where \(h\) is a positive function on \(M^{2n+1}\).

**Lemma 4.1.** Given an isocontact embedding \(g: (M^{2n+1}, \ker\{\alpha\} = \xi) \rightarrow (N^{2n+1}, \ker\{\beta\} = \eta)\), there exist a constant \(C > 0\) and a positive function \(\tilde{h}\) on \(M\) such that the following holds.

1. \(g^*(C \cdot \beta) = \tilde{h} \cdot \alpha\).
2. There is a homotopy of contact 1-forms \(\alpha_t\) joining \(\alpha_0 = \alpha\) and \(\alpha_1 = \tilde{h} \cdot \alpha\).

**Proof.** There is a positive function \(h\) on \(M^{2n+1}\) such that \(g^*(\beta) = h \cdot \alpha\). Since \(M^{2n+1}\) is compact, there \(C_0 > 0\) such that \(h(x) \geq C_0\) for all \(x \in M\). Thus, there exists a constant \(C\) large enough so that \(C \cdot h > 1\). Define \(\tilde{h} = C \cdot h\). Then \(g^*(C \cdot \beta) = C \cdot h \cdot \alpha = \tilde{h} \cdot \alpha\). Now define \(\alpha_t = [1 + (\tilde{h} - 1)t] \cdot \alpha\) for \(t \in [0, 1]\). Since \(\tilde{h} > 1\), \(\alpha_t\) is a path of contact forms joining \(\alpha\) and \(\tilde{h} \cdot \alpha\).

**Lemma 4.2.** Let \(f: (V^{2n}, \partial V^{2n}, d\lambda) \rightarrow (D^{4n}, \partial D^{4n}, d\lambda_0)\) be a proper symplectic embedding. We can assume that \(f\) satisfies the cohomology condition: \([f^*d\lambda_W - d\lambda_V] = 0\) in \(H^2(V, \partial V; \mathbb{Z})\).

**Proof.** By Theorem 2.12, \((\partial V, \ker\{\lambda\})\) has an isocontact embedding in \((S^{4n-1}, \ker\{\lambda_0\})\). Moreover, since \(St(4n-1, 2n-1)\) is \((2n-1)\)-connected by Wu’s theorem [Wu] and Gromov’s h-principle for isocontact embedding (12.31, [EM]), any two such isocontact embeddings of \(\partial V^{2n}\) in \(S^{4n-1}\) are isotopic. Moreover, if \(h\) is such an isocontact embedding then using Lemma 4.1 we can choose a new representative \(C \cdot \lambda_0\) of \(\ker\{\lambda_0\}\) and isotope \(\lambda\) to some \(\tilde{h} \cdot \lambda\), so that \(g^*(C \cdot \beta) = \tilde{h} \cdot \alpha\). So, without loss of generality we can assume that in a collar neighborhood \(\partial V \times (0, 1)\) of \(\partial V\) in \(V\), \(f\) restricts to a map \(g: V \times (0, 1) \rightarrow \partial D^{4n} \times (0, 1)\) such that \(g(-, t): (\partial V \times \{t\}, \ker\{\lambda\}) \rightarrow (\partial D^{4n} \times \{t\}, \ker\{\lambda_0\})\) is an isocontact embedding for all \(t \in [0, 1]\) and \(g(-, 1)^*(\lambda_0) = \lambda\). So, the cohomology condition is satisfied, i.e., \([f^*d\lambda_W - d\lambda_V] = 0\) in \(H^2(V, \partial V; \mathbb{Z})\).

The next lemma is essentially Proposition 4 in [Au], adapted to our setting. For a proof we refer to [S].

**Lemma 4.3.** Let \((V, \partial V, d\lambda_V)\) and \((W, \partial W, d\lambda_W)\) be two exact symplectic manifolds with convex boundaries of dimension \(2m\) and \(2m + 2s\) respectively. Let \(\psi_i: (V, \partial V) \rightarrow (W, \partial W)\) be a family of proper symplectic embeddings. There is a symplectic isotopy \(\Psi_t\) of \((W, \partial W, d\lambda_W)\) such that \(\Psi_0 = Id\) and \(\Psi_1 \circ \psi_0(V) = \psi_1(V)\).
Proof of Theorem 1.2. Let $f^s$ and $f^e \circ \phi$ be two proper isosymplectic embeddings of $(V^{2n}, \partial V^{2n}, d\lambda_V)$ in $(D^{4n}, \partial D^{4n}, d\lambda_0)$. Since $V^{2n}$ is simply connected, by Lemma 3.1, there exists a family of embeddings $e_t$ for $t \in [0, 1]$ such that $e_0 = f^s$ and $e_1 = f^e \circ \phi$. Since $S^{2n}(2n, n)$ is $2n$-connected, $D_{e_0}$ and $D_{e_1}$ are homotopic via a continuous family of isosymplectic homomorphisms $F_t$ for $t \in [0, 1]$. Applying the parametric relative version of the h-principle for isosymplectic immersion (Theorem 2.24), we get a family of isosymplectic immersions $f^e_t$ joining $e_0$ and $e_1$, such that there is continuous family of formal homotopy $H^r_t$ ($r \in [0, 1]$) connecting $H^0_t = D_{f^e_0}$ and $H^1_t = D_{f^e_1}$. Since each $f^e_t$ is regularly homotopic to the embedding $e_0$ and hence to $e_1$, there exists continuous family $G^r_t$ ($r \in [0, 1]$) of formal homotopy between $G^0_t = D_{e_0}$ and $G^1_t = D_{f^e_1}$. Define $(H \circ G)^r_t$ as follows.

$$(H \circ G)^r_t = \begin{cases} G^2_t & \text{for } 0 \leq r \leq \frac{1}{2} \\ H^{2r-1}_t & \text{for } \frac{1}{2} \leq r \leq 1 \end{cases}$$

Thus, $(H \circ G)^r_t$ gives a continuous family of formal homotopy between $D_{e_0}$ and $F_t$. Moreover, we can choose the formal homotopy to vary smoothly with $t$. So, we have a family of embedding $e_t$ satisfying the cohomology conditions for symplectic forms (because $e_0$ and $e_1$ satisfy) and $F_t$ is a family of isosymplectic homomorphism covering $e_t$ such that $D_{e_t}$ is formally homotopic to $F_t$ and the homotopy depends continuously on $t$. Thus, by the parametric relative version of Theorem 2.25, we can isotope each $e_t$ to an isosymplectic embedding depending continuously on $t$. Therefore, we have shown that $e_0 = f^s$ and $e_1 = f^e \circ \phi$ are isotopic as isosymplectic embeddings. By Lemma 4.3, there is a symplectic isotopy $\Psi_t$ of $(D^{4n}, \partial D^{4n}, d\lambda_0)$ such that $\Psi_0 = Id$ and $\Psi_1 \circ f^e(V) = f^e \circ \phi(V)$. Hence, $Ob(V^{2n}, d\lambda_V, \phi)$ admits a contact open book embedding in $Ob(D^{4n}, d\lambda_0, id) = (S^{2n+1}, \xi_{std})$.

\[\Box\]

5. Applications of Theorem 1.2

Theorem 1.2 says that if one can find a simply connected Weinstein domain $(W^{2n}, d\lambda_W)$ that admits proper symplectic embedding in $(D^{4n}, d\lambda_0)$, then $Ob(W^{2n}, d\lambda_W, \phi)$ contact open book embeds in $Ob(D^{4n}, d\lambda_0, id)$. Therefore, to apply Theorem 1.2 we want to find a sizable class of simply connected 2n-Weinstein domains that admit proper symplectic embedding in the standard 4n-ball.

Proof of corollary 1.3. By Cieliebak [C], any sub-critical Weinstein domain is split. Thus, $(W^{2n}, d\lambda_W)$ is isomorphic to some $(W^{2n-2}_0 \times D^2, d\lambda_W \bigoplus dx \wedge dy)$, where $(W^{2n-2}_0, d\lambda_{W_0})$ is a Weinstein domain. Now, by Gromov’s h-principle for isosymplectic embedding, $(W^{2n-2}_0, d\lambda_{W_0})$ has a proper isosymplectic embedding in $(D^{4n-2}, d\lambda_0^{2n-1})$. Therefore, $(W^{2n}, d\lambda_W) = (W^{2n-2}_0 \times D^2, d\lambda_W \bigoplus dx \wedge dy)$ has a proper isosymplectic embedding in $(D^{4n}, d\lambda_0^{2n})$. Hence, by Theorem 1.2 $Ob(W^{2n}, d\lambda_W, \phi_W)$ contact open book embed in $Ob(D^{4n}, d\lambda_0, id)$.

By Gromov’s h-principle for symplectic immersion, there exists a proper isosymplectic immersion $f^s$ of $(V^{2n}, d\lambda_V)$ in $(D^{4n}, d\lambda_0)$. Moreover, any such immersion will have only finitely many intersection points. According to Whitney [Wh], this intersection number $I(f^s) = \frac{1}{2}(c_r(\nu(f^s)), [V])$, where $c_r(\nu(f^s)) \in H^{2n}(V; \partial V; \mathbb{Z})$ denotes the relative Euler class of the normal bundle of immersion $\nu(f^s)$, and $[V] \in H^{2n}(V; \partial V; \mathbb{Z})$ is the fundamental class. Since an isosymplectic immersion induces a complex structure on the normal bundle, $c_r(\nu(f^s))$ is same as the relative Chern class $c^s_r(\nu(f^s))$ and therefore $I(f^s) = \frac{1}{2}(c^s_r(\nu(f^s)), [V])$. Moreover, by Whitney [Wh], $f^s$ is regularly homotopic to an embedding if and only if $I(f^s) = 0$. Thus, we have the following.

Lemma 5.1. Let $g : (V^{2n}, d\lambda_V) \to (D^{4n}, d\lambda_0)$ be a proper isosymplectic immersion with double points in the interior of $V^{2n}$. Then, the intersection number $I(g) = \frac{1}{2}(c^s_r(\nu(g)), [V])$. Moreover, $I(g) = 0$ if and only if $g$ can be isotoped to an isosymplectic embedding.

Recall that if $(E, F, \pi)$ is an $n$-dimensional complex vector bundle, then its $n$th Segre class $s_n(E)$ is the $n$th order term in the expansion of $(1 + c_1(E) + c_2(E) + ... + c_n(E))^{-1}$. We prove the following open book embedding result in the contact category.
Lemma 5.2. Let $(V^{2n}, d\lambda_V)$ be an Weinstein domain with a compatible almost complex structure $J_V$. Assume that $(V^{2n}, d\lambda_V)$ satisfies the following properties.

1. The restriction of $(TV, J_V)$ on $\partial V$ is trivial as a complex vector bundle.
2. The $n^{th}$ relative Segre class of $(V^{2n}, \partial V)$, $s_n^r(V, \partial V)$ vanishes.

Then $(V^{2n}, d\lambda_V)$ admits a proper isosymplectic embedding in $(D^{4n}, d\lambda_0)$, for $n \geq 3$.

Proof. The proof is essentially an application of a version of the Whitney sum formula for relative characteristic classes. It was first developed by Kervaire in [Ker].

Let $f_s$ be a proper isosymplectic immersion of $(V^{2n}, d\lambda_V)$ in $(D^{4n}, d\lambda_0)$. Note that the $i$-th Chern class is a characteristic class in the group $H^i(V^{2n}; \pi_{2n-1}ST^c(n, n-i+1))$, which is an obstruction to the existence of a section of the associated $ST^c(n, n-i+1)$-bundle of $(TV, J_V)$. Since the restriction $(TV, J_V)$ on $\partial V$ is a trivial complex bundle, there is also no obstruction to the existence of a section of that associated $ST^c(n, n-i+1)$-bundle. Therefore, we can talk about relative Chern classes of $(TV, J_V)$. Moreover, all the relative Chern classes of $(TD^{4n}, d\lambda_0, J_0)$ vanish. Therefore, the relative version of the Whitney sum formula gives the following for the normal bundle of immersion $\nu(f_s)$.

$$(1 + c_1^r(V, \partial V) + \cdots + c_n^r(V, \partial V))(1 + c_1^r(\nu(f_s)) + \cdots + c_n^r(\nu(f_s))) = 1 \implies c_n^r(\nu(f_s)) = s_n^r(V, \partial V) = 0.$$  

Now by Lemma 5.1, $f_s$ can be isotoped to a proper isosymplectic embedding of $(V, d\lambda_V)$ in $(D^{4n}, d\lambda_0)$. \hfill $\square$

Proof of Corollary 1.4. The obstruction to finding a section of the complex vector bundle $(TW, d\lambda_W, J_W)$ lies in $H^i(W; \pi_{2k-1}U(n)) = H^i(W; \pi_{2k}U)$ (since $n \geq 3 \implies n < 2n - 2$). By Bott periodicity, $\pi_{2k}U = 0$ for $k$ even and there are no nonzero even cells. So $(TW, d\lambda_W, J_W)$ is trivial and therefore, the conditions of Lemma 5.2 are satisfied. Thus, $(W^{4m+2}, d\lambda_W)$ admits proper isosymplectic embedding in the standard symplectic $(8m + 4)$-ball. Hence, by Theorem 1.2, $\operatorname{Ob}(W^{2n}, d\lambda_W, \phi)$ open book embeds in $\operatorname{Ob}(D^{4n}, d\lambda_0, \text{id})$ for every relative symplectomorphism $\phi$ of $(W, d\lambda_W)$. \hfill $\square$

Remark 5.3. In the proof of Corollary 1.4, the main point was to show the triviality of the almost complex structure on the tangent bundle of $V$. Therefore, the result will hold true for any parallelizable Stein manifold. Otto Forster [F] gave a characterization of such manifolds. In particular, he showed that every Stein manifold $V^{2n} \subset \mathbb{C}^{n+1}$ is parallelizable.

5.1. Proof of Theorem 1.6. We now give an outline of the proof of Theorem 1.6 which is just a higher co-dimensional version of Theorem 1.4 in [S]. For details we refer to [S].

We start by recalling a well-known fact that an embedding of $M^n$ in $V^{n+k}$ induces a proper symplectic embedding of $T^*M^n$ into $T^*V^{n+k}$ with the canonical symplectic forms on the cotangent bundles. Since, by Whitney, every closed manifold $M^n$ embeds in $S^{2n}$, every unit disk cotangent bundle $DT^*M^n$ admits a proper symplectic embedding in $DT^*S^{2n}$. Moreover, recall that the Euler class of the normal bundle of an embedding in Euclidean space is zero. Therefore, the normal bundle of an embedding $f : M^n \to S^{2n}$ has a sub-bundle isomorphic to $M \times \mathbb{R}$. Thus, we get an induced symplectic embedding of $(DT^*M, d\lambda_M)$ in $(DT^*S^{2n}, d\lambda_{S^{2n}})$, such that the symplectic normal bundle $\nu(f)$ of this embedding has a symplectic sub-bundle isomorphic to $(DT^*M \times T^*\mathbb{R}^1, d\lambda_M \oplus dx \wedge dy)$.

We can now define a symplectic isotopy $\phi_t : DT^*M \to DT^*S^{2n} |_M \cong DT^*M \times T^*\mathbb{R}^1 \oplus N(M)$ by sending $(p, v, (0, 0)) \to (p, v, (0, t \cdot g(|v|)), 0)$. Here, $N(M)$ is the quotient symplectic bundle of $\nu(f)$ by the $(T^*\mathbb{R}^1, dx \wedge dy)$-bundle over $M$. Note that the image of $\phi_t$ is disjoint from the zero section of $DT^*S^{2n}$. Thus, one can perform a Dehn-Seidel twist $\tau_{2n}$ along the core of $DT^*S^{2n}$ without affecting the image of $\phi_t$. One can then again use the symplectic isotopy $\phi_{t-\epsilon}$ to bring $\phi_t(DT^*M)$ back to its original position, i.e. to $\phi_0(DT^*M)$. This will give us a contact open book embedding of $\operatorname{Ob}(DT^*M^n, \text{id})$ in $\operatorname{Ob}(DT^*S^{2n}, \tau_{2n}) \cong (S^{4n+1}, \xi_{std})$. Note that all these isotopies can be assumed to be ambient in $DT^*S^{2n}$ by Lemma 4.3.

When, $M^n = S^n$ and $f$ is the standard embedding of $S^n$ in $S^{2n}$, an $k$-fold Dehn-Seidel twist $\tau_{2n}^k$ on $DT^*S^{2n}$ induces a $k$-fold Dehn-Seidel twist $\tau_{2n}^k$ on $DT^*S^n$. One can then push $DT^*S^n$ away from the zero section of $DT^*S^{2n}$ and apply $(l - k)$-fold Dehn-Seidel twist in a small enough neighborhood of the zero
Now consider the
This induces another embedding of
$DT$ in $(\mathbb{R}^4,\omega)$. Let $\Phi_t$ be a change of symplectic structure by doing sufficient number of positive stabilizations.

By the Lagrangian neighborhood Theorem, $L\subset (W^{2n},\lambda_{\omega})$ has a neighborhood symplectomorphic to $(DT^*S^n,\lambda_n)$. This gives a smooth embedding of $(DT^*S^n,\lambda_n)$ in $S^{2n}$. Let $h_1$ denote this embedding.

As discussed in the proof of Theorem 1.6, one has a standard proper isosymplectic embedding of $(DT^*S^n,\lambda_n)$ in $(DT^*S^n+k,\lambda_n+k)$ for all $k \geq 0$. Since $Ob(DT^*S^n+k,\tau_{n+k})$ is the standard open book of $S^{2n+2k+1}$, this also gives an embedding of $DT^*S^n$ in $S^{2n+2k+1}$. Note that there exists a flow $\Phi_t$ on $S^{2n+2k+1}$ whose time 1 map $\Phi_1$ maps its page $DT^*S^{n+k}$ to itself by applying a Dehn-Seidel twist along the zero section of $DT^*S^{n+k}$.

This induces another embedding of $DT^*S^n$ in $S^{2n} = D^{N+1}$. Let $h_2$ denote this embedding.

So we have two embeddings, $h_1$ and $h_2$, of $DT^*S^n$ in $S^{2n}$. By Theorem 3.1, $h_1$ and $h_2$ are isotopic in $S^{2n}$. Now consider the $N+1$-ball $D^{N+1}$ and consider $h_1$ and $h_2$ as embeddings inside $S^{2n} = D^{N+1}$. Let $h_{t+1}$ be an isotopy between $h_1$ and $h_2$ in $S^{2n}$ for $t \in [0,1]$. Let $S^{2n} \times [-\epsilon,3\epsilon]$ denote a collar neighborhood of $\partial D^{N+1}$. We assume that $h_1$ embeds $DT^*S^n$ in the 0-level sphere $S^{2n} \times \{0\}$. Define $H:S^{2n} \times [0,\epsilon] \times [0,1] \to S^{2n} \times [0,\epsilon]$ by $H(x,t,s) = h_{t+1}(x)$. Multiplying by a cut-off function we can get an isotopy $G_s$ of $D^{N+1}$ supported in $S^{2n} \times [0,\frac{\epsilon}{2}]$, such that $G_0 = id$ and $G_1$ takes $h_1(DT^*S^n))$ to $h_2(DT^*S^n))$.

6. Smooth open book embedding of Weinstein fillable contact manifolds

We now prove Theorem 1.7. The following lemma is our main ingredient.

**Lemma 6.1.** Let $L \subset (W^{2n},\lambda_{\omega})$ be a Lagrangian $n$-sphere and let $\tau_L$ denote the Dehn-Seidel twist along $L$. Then, $(W^{2n},\tau_L)$ admits a flexible embedding in $(D^{2\left[\frac{n}{2}\right]}+2,\lambda)$. 

**Proof.** By Cieliebak–Eliashberg [CE], every Weinstein domain is deformation equivalent to a Stein domain. On the other hand, Eliashberg and Gromov [EG] have shown that every Stein manifold of complex dimension $n$ admits a proper holomorphic embedding in $(\mathbb{C}^{n+1},\omega)$ of their pages also contact open books in $(S^{4n+1},\xi_{std})$. The supporting open book of $(S^{4n+1},\xi_{std})$ will then be changed by doing sufficient number of positive stabilizations.

Apart from the higher co-dimension of embedding, the proof of Theorem 1.6 follows the proof of Theorem 1.4 in [S] verbatim. Note that all these results are also implicit in the work of Casals and Murphy [CM].

![Figure 6](attachment:image.png)

**Figure 6.** Using the isotopy $\phi_t$, we push $DT^*S^0$ away from the zero section $S^1_1$ (denoted by red circles) of $DT^*S^1$. Then we apply Dehn twist along $S^1_0$, with support in the shaded region.
Recall that the embedding $h_2$ of $DT^*S^n$ in $S^N \times \{\frac{3\epsilon}{2}\}$ actually gives a proper symplectic embedding of $DT^*S^n$ in the page of the open book $Ob(DT^*S^N, \tau_{S^N})$, such that a Dehn-Seidel twist on the page $DT^*S^{\frac{N-1}{2}}$ induces a Dehn-Seidel twist on $h_2(DT^*S^n)$.

Now, let $\Psi_t$ be the isotopy of $S^N$ such that $\Psi_0 = id$ and $\Psi_1$ realizes the Dehn-Seidel twist on the page of its open book $Ob(DT^*S^N, \tau_{S^N})$. Using the isotopy $\Psi_t$, we can construct an isotopy $\Gamma_s$, $(0 \leq s \leq 1)$ of $S^N \times [-1, 1]$ that satisfies the following:

1. $\Gamma_0 = id$.
2. $\Gamma_1$ restricted to $S^N \times \{0\}$ is $\Psi_1$.

$\Gamma_s$ is defined as follows:

$$\Gamma_s(x, t) = \begin{cases} 
\Psi_s(1-t)(x) & \text{if } t \geq 0 \\
\Psi_s(t+1)(x) & \text{if } t \leq 0 
\end{cases}$$

Composing with a linear diffeomorphism $\alpha^{-1} : [\frac{3\epsilon}{2}, 2\epsilon] \to [-1, 1]$, we can get an isotopy $\hat{\Gamma}_s$ of the identity diffeomorphism of $D^{N+1}$, supported on $S^N \times [\frac{3\epsilon}{2}, 2\epsilon] \subset D^{N+1}$. Moreover, $\hat{\Gamma}_1$ induces a Dehn-Seidel twist on the page of our open book at the level $S^N \times \{\alpha(0)\}$.

We can now piece together the processes to finish the proof.

We first embed $W^{2n}$ in $S^N \times \{0\} \subset D^{N+1}$. Then apply $G_s$ to isotope a symplectic neighborhood of $L \subset W$, say $N(L)$, into the page of the standard open book of $S^N \times \{\epsilon\}$. We then further push the neighborhood of $L$ in $S^N \times \{\epsilon\}$ to the level $S^N \times \{\alpha(0)\}$. Apply $\hat{\Gamma}_s$ to induce a Dehn-Seidel twist on this pushed neighborhood of $L$. Now bring $N(L)$ back to its previous embedding in $S^N \times \{\epsilon\}$. Finally, bring $N(L)$ back to its staring embedding in $S^N \times \{0\}$ by the isotopy $G_{1-s}$.

This process will induce a Dehn-Seidel twist on $N(L)$ by an isotopy of the identity diffeomorphism of $D^{N+1}$. 

---

**Figure 7.** Schematic diagram of the isotopy of $D^{N+1}$ in the collar region $S^N \times [0, 3\epsilon]$. Here, we see $D^{N+1}$ as the complement of the interior disk $D^{N+1}_i$ in $S^{N+1} = \mathbb{R}^{N+1} \cup \{\infty\}$. The curved arc denotes the image of $\partial N(L)$ under the isotopy $G_s$. The shaded region denotes the support of the isotopy $\Gamma_s$. 

- The interior disk $D^{N+1}_i$ is the complement of $D^{N+1}$. 
- The shaded region is the support of the isotopy $\Gamma_s$. 
- The isotopy $G_s$ is defined on $S^N \times [0, 3\epsilon]$. 
- The boundary of $N(L)$ is denoted by the curved arc. 
- The isotopy $\Gamma_s$ is defined as follows:
  \[
  \Gamma_s(x, t) = \begin{cases} 
  \Psi_s(1-t)(x) & \text{if } t \geq 0 \\
  \Psi_s(t+1)(x) & \text{if } t \leq 0 
  \end{cases}
  \]
Proof of Theorem 1.7. Say $M^{2n+1}$ bounds an Weinstein domain $(V^{2n+2}, \omega_V)$. By Theorem 2.21 [GP], $(V^{2n+2}, \omega_V)$ is deformation equivalent to an abstract Weinstein Lefschetz fibration $LF(W^{2n}, d\lambda_W; (L_1, L_2, \ldots, L_m))$. This induces an open book decomposition on $M^{2n+1}$ with page $W$ and monodromy $\phi = \tau_{L_1} \circ \tau_{L_2} \circ \cdots \circ \tau_{L_m}$. Identify the $S^1$-interval of the mapping torus of this open book with $[0, 1]/0 \sim 1$. Divide $[0, 1]$ into subintervals $I_j = \left[\frac{j}{m+1}, \frac{j+1}{m+1}\right]$ for $j = 0, 1, \ldots, m$.

Let $N = 2\left\lfloor \frac{3n}{2} \right\rfloor + 4$. On the interval $I_j$, we apply Lemma 6.1 for the vanishing cycle $L_j$ to obtain embedding of the mapping cylinder of $\tau_{L_j} : W \to W$ in the mapping cylinder of the identity map (up to isotopy) of $D^{N+1}$, for $0 \leq j \leq m - 1$. We then glue these mapping cylinders together in cyclic order to obtain embedding of the mapping torus of $\phi$ in $D^{N+1} \times S^1$. This gives us an open book embedding of $M^{2n+1}$ in $S^{N+2} = Ob(D^{N+1}, id)$.

Remark 6.2. One may consider a higher dimensional analogue of the 4-dimensional achiral Lefschetz fibration, which is like an abstract Weinstein Lefschetz fibration, but it may also have, in its monodromy, negative Dehn-Seidel twists along the vanishing cycles. Such an achiral abstract Weinstein Lefschetz fibration will naturally induce a contact open book structure on its boundary and in this case also the proof of Theorem 1.7 goes through. It will be interesting to know whether one can strengthen Theorem 1.7 to get a contact open book embedding.

6.1. Relation of Theorem 1.7 to fillability of contact manifolds. If an almost contact manifold $M^{2n+1}$ does not embed in $\mathbb{R}^{2(\lfloor \frac{3n}{2} \rfloor + 1)}$, then by Theorem 1.7, it can not have a Weinstein or Stein filling. This can be used to show that certain almost contact manifolds does not admit any Weinstein/Stein fillable contact structure. For example, consider the manifold $S^1 \times CP^n$. This manifold admits almost contact structure. Now, it was shown by Sanderson and Schwarzenger [SS] that $CP^n$ cannot be immersed in $\mathbb{R}^{4n-2a(n)+\epsilon}$, where $a(n)$ is the number of 1s in the binary expression of $n$ and $\epsilon \in \{0, 1, -1\}$. In particular, $\epsilon = 0$ for $n$ even and $a(n) \equiv 1 \pmod{4}$. Take $n = 16$. Then $a(n) = 1$ and $\epsilon = 0$. Thus, $CP^{16}$ does not immerse in $\mathbb{R}^{62}$. Therefore, $S^1 \times CP^{16}$ cannot embed in $S^{62}$. Since $2\left\lfloor \frac{3 \times 16}{2} \right\rfloor + 6 = 54 < 62$, $S^1 \times CP^{16}$ does not admit any Weinstein/Stein fillable contact structure. Similarly, one can show that $S^3 \times CP^{32}$ does not embed in $126 > 2\left\lfloor \frac{3 \times 53}{2} \right\rfloor + 6 = 105$. Therefore, $S^3 \times CP^{32}$ cannot admit any Weinstein/Stein fillable contact structure.

6.2. Embedding of Stein/Weinstein fillable manifolds. By a theorem of Eliashberg and Gromov [EG], every complex $n$-dimensional Stein manifold has a proper holomorphic embedding in $\mathbb{C}^N$ for $N = \left\lceil \frac{3n}{2} \right\rceil + 2$. It is therefore natural to ask what can we say about the optimal dimension of contact open book embedding of a Stein fillable contact manifold of dimension $2n + 1$ in some trivial open book. We end our discussion with the following question.

Question 6.3. What is the minimal $k$ such that every Stein fillable contact manifold of dimension $2n + 1$ admits contact open book embedding in $(S^{2(\lfloor \frac{3n}{2} \rfloor + k)}; \xi_{std})$?

References

[Au] D. Auroux, Asymptotically homomorph families of symplectic submanifolds. Geom. Funct. Anal. 7 (1997), 971-995.
[C] K. Cieliebak, subcritical Stein manifolds are split, arXiv:math/0204351v1.
[E] J. B. Etnyre, Lectures on open book decomposition and contact structures, arXiv:math/0409042.
[CE] K. Cieliebak, Y. Eliashberg, From Stein to Weinstein and Back, AMS Colloquium Publication volume 59.
[CM] R. Casals, E. Murphy, Contact topology from the loose viewpoint Proceedings of 22nd Gökova Geometry-Topology Conference, 81-115.
[CPP] R. Casals, D. Pancholi, F. Presas, The Legendrian Whitney Trick, arXiv:1908.04828.
[EF] J. B. Etnyre, R. Furukawa, Braided embeddings of contact 3-manifolds in the standard contact 5-sphere, Journal of Topology, Volume 10, Issue 2, June 2017, 412-446.
[EG] Y. Eliashberg, M. Gromov, Embeddings of Stein manifolds of dimension $n$ into the affine space of dimension $3n/2 + 1$, Annals of Mathematics, 136 (1992), 123-135.
[EL] J. Etnyre and Y. Lekili, Embedding all contact 3–manifolds in a fixed contact 5–manifold, *Journal of the London Mathematical Society*, Volume 99, Issue 1, February 2019, 52-68.

[EM] Y. Eliashberg, N. Mishachev, Introduction to the h-principle, *Graduate Studies in Mathematics* 48 (2002), AMS.

[F] Otto Forster, Some remarks on parallelizable Stein manifolds. *Bull. Amer. Math. Soc.* Volume 73, Number 5 (1967), 712-716.

[Ge] H. Geiges, An Introduction to Contact Topology, *Cambridge Studies in Advanced Mathematics* 109.

[Gr] M. Gromov, Partial Differential Relations, *Ergeb. Math. Grenzgeb.* 9 (1986), Springer–Verlag.

[Gi] E. Giroux, Géométrie de contact: de la dimension trois vers les dimensions supérieures, *Proceedings of the ICM, Beijing* 2002, vol. 2, 405–414.

[GP] E. Giroux, J. Pardon, Existence of Lefschetz fibrations on Stein and Weinstein domains, *arXiv:1411.6176v3 [math.SG]*.

[Hi] M. W. Hirsch, Immersions of Manifolds, *Transactions of the American Mathematical Society* Vol. 93, No. 2(Nov.,1959), 242-276.

[HH] A. Haefliger, M. Hirsch, On the existence and classification of differentiable embeddings, *Topology*, vol. 2, (1963), 129–135.

[HoH] K. Honda, Y. Huang, Convex hypersurface theory in contact topology, *arXiv:1907.06025*.

[Ka] N. Kasuya, On contact embeddings of contact manifolds in the odd dimensional Euclidan spaces, *International Journal of Mathematics* vol. 26, No. 7 (2015) 1550045.

[Ka2] N. Kasuya, An obstruction for co-dimension two contact embeddings in the odd dimensional Euclidean spaces, *J. Math. Soc. Japan* Volume 68, Number 2 (2016), 737-743.

[Ker] M. A. Kervaire, Relative Characteristic Classes, *American Journal of Mathematics*, Vol. 79, No. 3 (Jul., 1957), pp. 517-558

[Ko] O. van Koert, Lecture notes on stabilization of contact open books *arXiv:1012.4359v1* (2010).

[La] T. Lawson, Open book decomposition for odd dimensional manifolds, *Topology*, vol 17, (1979), 189–192.

[Ma] W. S. Massey, On the Stiefel-Whitney Classes of a Manifold. *American Journal of Math.* Vol. 82, No. 1 (Jan., 1960), pp. 92-102.

[Mo] A. Mori, Global models of contact forms. *J. Math. Sci. Univ. Tokyo*, vol. 11(4), 447454, (2004).

[N] John Nash, $C^1$-isometric imbeddings$, Annals of Mathematics$, 60 (3): 383396.

[PP] D. M. Pancholi, S. Pandit, Iso-contact embeddings of manifolds in co-dimension 2, *arXiv:1808.04059* (2018).

[PPS] D. M. Pancholi, S. Pandit, K. Saha. Embedding of 3-manifolds via open books, *arXiv:1806.09784v2 [math.GT]* (2019).

[S] Saha, K. On Open Book Embedding of Contact Manifolds in the Standard Contact Sphere. *Canadian Math. Bulletin*, 1-16 (2019), doi:10.4153/S0008439519000808.

[S2] Saha, K. Contact and isocontact embedding of $\pi$-manifolds. to appear in *Math. Proc. Indian Academy of Sciences*. (arXiv:2005.10135v1 [math.SG])

[SS] B. J. Sanderson and R. Schwarz, Non-immersion theorems for differential manifolds. *Proc. Camb. Phil. Soc.* 59 (1963) 312-322.

[Ta] I. Tamura, Spinable structures on differentiable manifolds, *Proc. Japan Acad.* vol. 48, (1972), 293–296.

[To] D. Martínez-Torres, Contact embeddings in standard contact spheres via approximately holomorphic geometry, *J. Math. Sci. Univ. Tokyo*, vol. 18 (2), (2011), 139–154.

[TW] W. Thurston, H Winkelnkemper, On the existence of contact forms, *Proceedings of AMS* vol. 52.

[Wh] H. Whitney, The self-intersections of a smooth n–manifold in 2n–space, *Ann. of Math.*, vol. 45(2), (1944), 200–246.

[W] H. Winkelnkemper, Manifolds as open books, *Bull. Amer. Math. Soc.*, vol. 7, (1973), 45–51.

[Wu] W. T. Wu, On the isotopy of $C^\infty$-manifolds of dimension n in Euclidean (2n+1)-space. *Sci. Record* (N.S.) 2 271-275 (1958).

Indian Institute of Technology Madras
*E-mail address*: arijit2357@gmail.com

Indian Institute of Science Education and Research Bhopal
*E-mail address*: kuldeep.saha@gmail.com