Universal decay of classical Loschmidt echo of neutrally stable but mixing dynamics

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We provide analytical and numerical evidence that classical mixing systems which lack exponential sensitivity on initial conditions, exhibit universal decay of Loschmidt echo which turns out to be a function of a single scaled time variable $\delta^2/\tau$, where $\delta$ is the strength of perturbation. The role of dynamical instability and entropy production is discussed.

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Fidelity, or Loschmidt echo, is defined as the overlap of two time evolving states which, starting from the same initial condition, evolve under two slightly different Hamiltonians. It is therefore an important quantity which measures the stability of the motion under systems perturbations. The recent interest in the behaviour of fidelity\cite{2,3,4,5,6,7,8,9,10,11} has been largely motivated by a possible use in quantifying stability of perturbations. The recent interest in the behaviour of fidelity decay as correlation functions. On the other hand, for dynamical correlation functions decaying as $t^{-3/2}$. It can be argued that the triangle map possesses the essential features of bounce maps of polygonal billiards and 1d hardpoint gases\cite{13,14}, namely parabolic stability in combination with decaying dynamical correlations, and as such represents a paradigmatic model for a larger class of systems.

The classical fidelity $F_5(n)$ can be written as an overlap of two phase space densities propagated by the original map $T$ and the perturbed map $T_\delta = T \circ g_\delta$ where $g_\delta(z) = z + \delta a(z)$ is some near-identity area-preserving map parametrized by a vector field $a(z)$:

$$F_\delta(n) = \frac{\int d^2z \rho(T^{-n}(z)) \rho(T_\delta^{-n}(z))}{\int d^2\rho(z)}.$$  \hfill (2)

We can make our discussion even more general by taking the perturbation explicitly time-dependent. Let the perturbed map $T_{\delta,n}$ explicitly depend on iteration time, namely we consider the following class of perturbed triangle maps, $z_{n+1} = T_{\delta,n}(z_n)$

$$\bar{y}_{n+1} = \bar{y}_n + \alpha \sgn x_n + \beta \quad \mod 2, \quad x_{n+1} = x_n + y_{n+1} \quad \mod 2, \quad (1)$$

where $\sgn x = \pm 1$ is the sign of $x$ and $\alpha, \beta$ are two parameters. Previous investigations have shown that \cite{14} (see also \cite{16} for some rigorous results on \cite{14}): for rational values of $\alpha, \beta$ the system is pseudo-integrable, as the dynamics is confined on invariant curves. If $\alpha = 0$ and $\beta$ is irrational, the dynamics is (uniquely) ergodic, but not mixing, while for incommensurate irrational values of $\alpha, \beta$ the dynamics is ergodic and mixing with dynamical correlation functions decaying as $t^{-3/2}$. It can be argued that the triangle map possesses the essential features of bounce maps of polygonal billiards and 1d hardpoint gases\cite{13,14}, namely parabolic stability in combination with decaying dynamical correlations, and as such represents a paradigmatic model for a larger class of systems.

We will assume that the force function $f_n$ has vanishing time-average for almost any initial condition. Let us further assume that the initial density $\rho(z)$ is a characteristic function over some set $\mathcal{A}$ of typical diameter $\omega$ with $\delta \ll \omega \ll 1$. Then a pair of orbits $z_n$ and $\bar{z}_n$ starting from the same point $z_0 = z_0$ in $\mathcal{A}$, contribute to \cite{24} until they hit the opposite sides of the discontinuity, at $x = 0, 1 \quad \mod 2$. The fidelity at time $n$ is then simply the probability that the pair of orbits does not hit the cut up to $n$-th iterate. Assuming ergodicity of the map...
The initial cell, after time \( t \), we write the model (12), while the dashed line has slope 

can be related, as \( n \to \infty \), to the integrated correlation function. Since, for large \( n \): 

This expression is valid until \( F_\delta(n) \) remains close to 1, that is up to time \( n < n^* = \sigma^{-1/5} |\delta|^{-2/5} \).

In fig. 1a we show the behavior of \( 1 - F_\delta(n) \) for short times \( n < n^* \) and compare with the theoretical formula (9) with \( \sigma = 3.29 \pm 0.01 \) as computed from numerical simulation of correlation function \( C(n) \). As for perturbation we choose a simple shift in the parameter \( \alpha \), so the force reads \( f(x) = \text{sgn} x \) and is, in this case, not explicitly time dependent. Yet it is pseudorandom and one can see that, as \( \delta \) decreases, the numerical curves approach the theoretical expression (4).

Notice that according to eq. (5), the average distance between two orbits increases as \( \propto n^{3/2} \). On the other hand, the distance between two initially close orbits of the same map increases only linearly with time. This is nicely confirmed by the numerical simulations of fig. 2.

For larger times, \( n > n^* \), higher order terms in the expansion of \( F_\delta(n) \) contribute, so temporal correlations among
Δx_n become important. We are here unable to derive exact theoretical predictions for the fidelity decay in this regime. However, numerical results in fig. 1b show that, for large times, fidelity decays exponentially F_δ(n) = \exp(-\gamma n) with exponent γ = C|δ|^{2/5}. We also checked that the transition time between the two regimes of decay, scales as \δ^{-2/5}. In conclusion, extensive and accurate numerical results provide clear evidence that fidelity depends on the single scaling variable \tau = \delta^{2/5} n.

In the following, we show that this scaling behavior can be derived analytically for sufficiently small δ. The only assumption is correlation decay with a finite characteristic time-scale \nu_{mix}, i.e. \langle f_n f_{n'} \rangle practically vanish for |n - n'| > \nu_{mix}. Let us divide the time-span n into \nu := n/m blocks of m steps each, such that \nu_{mix} \ll m \ll n, and make a scaling argument. The local variation of \Delta x_n, namely \Delta x_{n+1} = \Delta x_n = \delta \sum_{\nu=0}^n f_{\nu} \sim \delta \sqrt{n} is much smaller than the mean value \langle |\Delta x_n| \rangle \sim \delta n^{3/2}. Thus we approximate the product (3) within each block labelled by \nu = 1, \ldots, ν as (1 - \langle |\Delta x_{\nu-1}| \rangle)^m \approx 1 - m \langle |\Delta x_{\nu-1}| \rangle. Therefore

\[
F_\delta(n) \approx \prod_{\nu=1}^n (1 - m \langle |\Delta x_{\nu-1}| \rangle). \tag{10}
\]

Next we define the normalized block-averaged forces

\[
\xi_\nu = \frac{1}{\sigma m} \sum_{k=0}^{m-1} f_{\nu-1+k} \tag{11}
\]

which are normalized, and uncorrelated, \langle \xi_\nu \xi_\mu \rangle = \delta_{\nu\mu} since \nu \gg \nu_{mix}. Using Eq. (10) we can write \Delta x_{\nu-1} \approx \delta \sum_{\nu=1}^\nu (\nu-\mu) m \sum_{k=0}^{m-1} f_{\nu-1+k} = \delta m^{3/2} \langle \xi_\nu \rangle. If, in addition to the rescaled time \nu = n/m, we define a rescaled perturbation \epsilon = \delta / m^{5/2} then we can write Eq. (10) as

\[
\Phi_{\epsilon}(\nu) = \left( \prod_{\nu=1}^\nu \left( 1 - \epsilon \sum_{\mu=0}^{\nu-1} \langle \xi_\nu \xi_\mu \rangle \right) \right) \tag{12}
\]

The derived relation \( F_\delta(n) = \Phi_{\epsilon m^{1/2} n^{5/2}}(n/m) \) does not depend on \( m \) (for large enough \( m \)), and therefore fidelity should be a function of the scaling variable \( \tau = |\delta|^{2/5} n \) only.

Notice that due to the central limit theorem, since \( m \gg \nu_{mix}, \xi_\nu \) can be simply treated as uncorrelated, normalized, Gaussian stochastic variables. We have actually computed the universal function \( \phi(\epsilon^{2/5} n) = \Phi_{\epsilon}(\nu) \) by means of Monte-carlo integration, and checked that it is practically insensitive to \( \epsilon \), for \( \epsilon < 10^{-4} \). As it is seen in fig. 1b, the numerical data for the triangle map agree with the theoretical expression (12), namely \phi(\delta^{2/5} \sigma^{1/5} n) which is plotted as a full curve.

The two regimes of fidelity decay described above are illustrated in fig. 3 by the image at the echo time of an initial uniform phase space distribution over some set \( A \). Notice that the linear-response regime (9) is valid until
In conclusion, we have discussed the parametric stability, as characterized by classical fidelity or Loschmidt echo, of an important class of dynamical systems where neutral stability is coexisting with dynamical mixing. As a paradigmatic example of this class of systems we have considered the triangle map. By means of analytic calculations and numerical simulations we have derived two universal regimes of fidelity decay, both being characterized by a universal scaled time variable $|\delta|^{2/5} t$. This interesting dynamical behavior is supported also by a power-logarithmic behavior of the coarse-grained entropy.

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In particular, contrary to the case of exponentially unstable systems, in this case the rate of fidelity decay depends on the perturbation strength. This feature is shared by quantum systems in which exponential instability is absent as well. One may wonder if this behavior is reflected also in some other, perhaps even more fundamental dynamical property of the map. In order to explore this question, we have computed the entropy production for the map. As the extensive computation of Kolmogorov-Sinai dynamical entropies seemed too expensive for reaching any conclusive results, we have decided to compute the dynamical evolution of the coarse-grained statistical entropy $S_n = -\sum_j p_n^{(j)} \ln p_n^{(j)}$. To this end we divide the phase space in $N \times N$ equal cells, and consider an initial ensemble of points uniformly distributed over one cell. The probability $p_n^{(j)}$ is defined as the fraction of orbits which, after $n$ time steps, fall in the cell of label $j$. For a chaotic system with dynamical entropy $h$, one expects $S_n = n \lambda + \text{const}$, whereas for ergodic-only (non-mixing) dynamics one expects $S_n \sim \ln n$, for sufficiently large $N$. Our numerical results for the triangle map (fig. 4) show instead that $S_n - S_1 = |\ln n|^\lambda$ with the exponent $\lambda = 3/2$. Furthermore, as shown in the inset of fig.4, for the triangle map with $\alpha = 0$ numerical results give, quite accurately, $S(n) = \ln n$ (with no prefactor or additional constant).

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![FIG. 4: The time evolution of the coarse-grained entropy for the triangle map, computed by taking $5 \times 10^7$ points initially distributed randomly in one cell of a $N \times N$ phase-space grid with $N = 700$. We plot $\ln(S(n) - S(1))$ versus $\ln n$. The straight line has slope $3/2$.](http://www.dm.unibo.it/fismat/pub/CPco010404.pdf)