Entanglement distribution in multi-particle systems in terms of unified entropy

Yu Luo\(^1\), Fu-Gang Zhang\(^2\) & Yongming Li\(^1,2\)

We investigate the entanglement distribution in multi-particle systems in terms of unified \((q, s)\)-entropy. We find that for any tripartite mixed state, the unified \((q, s)\)-entropy entanglement of assistance follows a polygamy relation. This polygamy relation also holds in multi-particle systems. Furthermore, a generalized monogamy relation is provided for unified \((q, s)\)-entropy entanglement in the multi-qubit system.

Quantum entanglement is an important resource in quantum information theory. Different from classical correlations, this restricted shareability of entanglement in multi-particle systems is known as monogamy property. The more entanglement shared between two parties implies the less entanglement shared with the rest of the system. Monogamy property plays a crucial role in quantum cryptography: which restricts the quantity of information captured by an eavesdropper about the secret key to be extracted\(^1\)–\(^3\). Monogamy property has also been discussed in the device-independent quantum information processing\(^4\), condensed matter physics\(^5\) and black-hole physics\(^6\)–\(^7\).

The study of monogamy property has a long history. The first monogamy relation was found by Coffman et al., who considered a three-qubit system \(ABC\)\(^8\), and showed that the amount of entanglement (which is quantified by the squared concurrence) between A and B, plus the amount of entanglement between A and C, cannot be greater than the amount of entanglement between A and the pair BC. Further, Osborne et al. proved the squared concurrence follows a general monogamy inequality for the \(N\)-qubit system\(^1\). Monogamy inequalities for different entanglement measures have been noted, such as concurrence\(^9\)–\(^12\), entanglement of formation\(^13\), \(^14\), negativity\(^15\)–\(^19\), Rényi entropy entanglement\(^20\),\(^21\), and Tsallis entropy entanglement\(^22\)–\(^24\). For the other physical resources, such as discord and steering, the monogamy property of them has also been discussed\(^25\)–\(^28\).

As dual to monogamy property, polygamy property in multi-particle systems has arisen many interests by researchers\(^15\),\(^19\),\(^22\),\(^29\),\(^30\). Polygamy property was first provided by using the concurrence of assistance to quantify the distributed bipartite entanglement in multi-qubit systems\(^29\),\(^30\). Polygamy property has also considered in many entanglement measures, such as Rényi entropy\(^20\), Tsallis entropy\(^22\),\(^23\) and convex-roof extended negativity\(^19\).

Unified \((q, s)\)-entropy is an important entropic measure, which can be used in many areas of quantum information theory. In this paper, we investigate the entanglement distribution in multi-particle systems in terms of unified \((q, s)\)-entropy. We find that for any tripartite mixed state, the unified \((q, s)\)-entropy entanglement of assistance follows a polygamy relation. This polygamy relation also holds in multi-particle systems. Furthermore, a generalized monogamy relation is provided for unified \((q, s)\)-entropy entanglement in the multi-qubit system.

Results

This paper is organized as follows. In the first subsection, we recall the definition of unified \((q, s)\)-entropy and discuss the properties of unified \((q, s)\)-entropy entanglement. In the second subsection, we give our main results. We summarize our results in the third subsection.

Unified \((q, s)\)-entropy entanglement and unified \((q, s)\)-entropy entanglement of assistance.

Given a quantum state \(\rho\) in the Hilbert space \(\mathcal{H}\). The unified \((q, s)\)-entropy is defined as\(^32\)

\[
S_{q,s}(\rho) = \frac{1}{(1-q)s} [\text{Tr}(\rho^s)^q - 1]
\]  

\(^1\)College of Computer Science, Shaanxi Normal University, Xi’an, 710062, China. \(^2\)School of Mathematics and Information Science, Shaanxi Normal University, Xi’an, 710062, China. Correspondence and requests for materials should be addressed to Y.L. (email: liyongm@snnu.edu.cn)
for any \( q, s \geq 0 \) such that \( q = 1 \) and \( s = 0 \).

When \( s \) tends to 1, the unified \((q, s)\)-entropy converges to Tsallis entropy \( T_q(\rho) \)\(^{33}\)
\[
\lim_{s \to 1} S_{q,s}(\rho) = \frac{1}{1 - q} \left[ \text{Tr}(\rho^q) - 1 \right].
\] (2)

When \( s \) tends to 0, the unified \((q, s)\)-entropy converges to Rényi entropy \( R_q(\rho) \)\(^{34}\)
\[
\lim_{s \to 0} S_{q,s}(\rho) = \frac{1}{1 - q} \ln \text{Tr}(\rho^q).
\] (3)

When \( q \) tends to 1, the unified \((q, s)\)-entropy converges to von Neumann entropy \( S(\rho) \)\(^{35}\)
\[
\lim_{q \to 1} S_{q,s}(\rho) = - \text{Tr} \rho \ln \rho.
\] (4)

Because the limits exist in the case of \( q \to 1 \) and \( s \to 0 \), we will use \( q = 1 \) and \( s = 1 \) to represent the limits in this paper. Now, let’s consider the entanglement in terms of the unified \((q, s)\)-entropy. For any pure state \( |\psi\rangle_{AB} \) in the Hilbert space \( \mathcal{H}_A \otimes \mathcal{H}_B \) (it’s does not matter for the sizes of subsystem \( A \) and \( B \)), the unified \((q, s)\)-entropy entanglement is defined as\(^{36}\)
\[
E_{q,s}(|\psi\rangle_{AB}) = S_{q,s}(\rho_A)
\] (5)
for any \( q, s \geq 0 \).

For a mixed state \( \rho_{AB} \), the unified \((q, s)\)-entropy entanglement can be defined via the convex-roof extension
\[
E_{q,s}(\rho_{AB}) = \min \sum_i p_i E_{q,s}(|\psi_i\rangle_{AB})
\] (6)
where the minimum is taken over all possible ensembles \( \{p_i, |\psi_i\rangle_{AB}\} \) of \( \rho_{AB} \) with \( \sum p_i = 1 \) and \( p_i \geq 0 \). It is straightforward to verify that \( E_{q,s}(\rho_{AB}) = 0 \) if and only if \( \rho_{AB} \) is a separable state for \( q, s \geq 0 \).

When \( s \) tends to 1, the unified \((q, s)\)-entropy entanglement becomes Tsallis entanglement\(^{31}\). When \( s \) tends to 0, the unified \((q, s)\)-entropy entanglement becomes Rényi entanglement\(^{35}\). Especially, The unified \((q, s)\)-entropy entanglement becomes entanglement of formation when \( q \) tends to 1. The entanglement of formation is defined as\(^{37,38}\)
\[
E(\rho_{AB}) = \min \sum_i p_i E_i(|\psi_i\rangle_{AB})
\] (7)
where \( E_i(|\psi_i\rangle_{AB}) = - \text{Tr} \rho_i \ln \rho_i = - \text{Tr} \rho_i \ln \rho_i \) is the von Neumann entropy, the minimum is taken over all possible ensembles \( \{p_i, |\psi_i\rangle_{AB}\} \) of \( \rho_{AB} \) with \( \sum p_i = 1 \) and \( p_i \geq 0 \).

As a dual quantity to the unified \((q, s)\)-entropy entanglement, the unified \((q, s)\)-entropy entanglement of assistance \((q, s)\)-EOA) can be defined as
\[
E^a_{q,s}(\rho_{AB}) = \max \sum_i p_i E_i(|\psi_i\rangle_{AB})
\] (8)
where the maximum is taken over all possible ensembles \( \{p_i, |\psi_i\rangle_{AB}\} \) of \( \rho_{AB} \) with \( \sum p_i = 1 \) and \( p_i \geq 0 \). To understand \((q, s)\)-EOA better, consider a tripartite pure state \( |\psi\rangle_{ABC} \) shared among three parties referred to as Alice, Bob, and Charlie\(^{35}\). The entanglement supplier, Charlie, performs a measurement on his share of the tripartite state, which yields a known bipartite entangled state for Alice and Bob. Tracing over Charlie’s system yields the bipartite mixed state \( \rho_{AB} = \text{Tr}_C(|\psi\rangle_{ABC} \langle \psi|) \) shared by Alice and Bob. Charlie’s aim is to maximize entanglement for Alice and Bob, and the maximum average entanglement he can create is the \((q, s)\)-EOA.

**Concurrence and concurrence of assistance.** For any pure state \( |\psi\rangle_{AB} \) in the Hilbert space \( \mathcal{H}_A \otimes \mathcal{H}_B \), the concurrence is defined as\(^{40}\)
\[
C(\rho_{AB}) = \sqrt{2(1 - \text{Tr} \rho_A^2)},
\] (9)
where \( \rho_A = \text{Tr}_B(\rho_{AB}) \).

For a mixed state \( \rho_{AB} \), the concurrence can be defined via the convex-roof extension
\[
C(\rho_{AB}) = \min \sum_i p_i C(|\psi_i\rangle_{AB})
\] (10)
where the minimum is taken over all possible ensembles \( \{p_i, |\psi_i\rangle_{AB}\} \) of \( \rho_{AB} \) with \( \sum p_i = 1 \) and \( p_i \geq 0 \).

As a dual quantity to concurrence, the concurrence of assistance (COA) can be defined as
\[
C^a(\rho_{AB}) = \max \sum_i p_i C(|\psi_i\rangle_{AB})
\] (11)
where the maximum is taken over all possible ensembles \( \{p_i, |\psi_i\rangle_{AB}\} \) of \( \rho_{AB} \) with \( \sum p_i = 1 \) and \( p_i \geq 0 \).
Analytical formula for two-qubit states. For a two-qubit mixed state $\rho_{AB}$, concurrence and COA are known to have analytic formula\(^{30, 40}\)

$$C(\rho_{AB}) = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\},$$

$$C^s(\rho_{AB}) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4,$$

where $\lambda_i$ being the eigenvalues, in decreasing order, of matrix $\sqrt{\rho_{AB}(\sigma_y \otimes \sigma_y)\rho_{AB}^\dagger(\sigma_y \otimes \sigma_y)}$.

In ref. 40, Wootters derived an analytical formula of entanglement of formation for a two-qubit mixed state $\rho_{AB}$

$$E_f(\rho_{AB}) = h\left(1 + \frac{1 - C^2(\rho_{AB})}{2}\right),$$

where $h(x) = -x \ln x - (1 - x) \ln(1 - x)$ is the binary entropy.

In ref. 36, Kim found $E_{q_{AB}}(\rho_{AB})$ has an analytical formula for a two-qubit mixed state, which can be expressed as a function of concurrence $C_{AB}$ for $q \geq 1, 0 \leq s \leq 1$ and $qs \leq 3$

$$E_{q_s}(\rho_{AB}) = f_s\left[C(\rho_{AB})\right],$$

where the function $f_s(x)$ has the form

$$f_s(x) = \frac{[1 + \sqrt{1 - x^2}]x + (1 - x^2)^{1/2} - 2x}{(1 - q)x^{2q}} ,$$

where $0 \leq x \leq 1$.

Main Results. In this section, we will provide our main results. First, we have following result:

**Theorem 1.** For any tripartite mixed state $\rho_{ABC}$ in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, we have

$$E^a_{q_{AB}}(\rho_{ABC}) \leq E^a_{q_{A|BC}}(\rho_{A|BC}) + E^a_{q_{B|AC}}(\rho_{B|AC}) + E^a_{q_{C|AB}}(\rho_{C|AB}),$$

where $q \geq 1$ and $qs \geq 1$.

**Proof:** Let $\rho_{ABC} = \max \sum_i E_{\psi_i}(\rho_{A|BC})$ be an optimal decomposition of $E^a_{q_{AB}}(\rho_{ABC})$. That is

$$E^a_{q_{A|BC}}(\rho_{ABC}) = \max \sum_i E_{\psi_i}(\rho_{A|BC}).$$

For any bipartite pure state $|\psi\rangle_{A|BC}$, the unified $(q, s)$-entropy entanglement $E_{q_s}(|\psi\rangle_{A|BC}) = S_{q_s}(\rho_{A|BC})$. In ref. 41, Rastegin proved that for any $q \geq 1$ and $qs \geq 1$, the unified $(q, s)$-entropy is subadditive, that is

$$S_{q_s}(\rho_{A|BC}) \leq S_{q_{AB}}(\rho_{A|BC}) + S_{q_{AC}}(\rho_{A|BC}).$$

Combining Eq. (18) with Eq. (19), we have

$$E^a_{q_{AB}}(\rho_{ABC}) \leq \sum_i S_{q_{A|BC}}(\rho_{A|BC})$$

$$\leq \sum_i S_{q_{A|BC}}(\rho_{A|BC}) + \sum_i S_{q_{B|AC}}(\rho_{B|AC})$$

$$\leq E^a_{q_{A|BC}}(\rho_{A|BC}) + E^a_{q_{B|AC}}(\rho_{A|BC}) + E^a_{q_{C|AB}}(\rho_{A|BC}).$$

Thus, the proof is completed.

**Theorem 1.** Shows a simple but interesting polygamy relation of $(q, s)$-EOA in a tripartite quantum system. The upper bound of $(q, s)$-EOA of $A|BC$ can’t be greater than the sum of $(q, s)$-EOA of $B|AC$ and $(q, s)$-EOA of $C|AB$. In particular, for a tripartite pure state $|\psi\rangle_{A|BC}$, the unified $(q, s)$-entropy entanglement $E_{q_s}(|\psi\rangle_{A|BC}) \leq E_{q_{A|BC}}(\rho_{A|BC}) + E_{q_{B|AC}}(\rho_{B|AC}) + E_{q_{C|AB}}(\rho_{C|AB})$.

We also have the following corollary:

**Corollary 1.** For any mixed state $\rho_{A_1|A_2\cdots A_n}$ in the Hilbert space $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_n}$, we have

$$E^a_{q_s}(\rho_{A_1|A_2\cdots A_n}) \leq \sum_{i=1}^n E^a_{q_s}(\rho_{A_1|A_2\cdots A_{i-1}A_{i+1}\cdots A_n}).$$

where $q \geq 1$ and $qs \geq 1$.

**Corollary 1.** Shows a constrained relationship of $(q, s)$-EOA in the multi-particle system, and gives an upper bound of $(q, s)$-EOA of $A_1|A_2\cdots A_n$. In particular, for any pure state $|\psi\rangle_{A_1|A_2\cdots A_n}$, the unified $(q, s)$-entropy entanglement $E_{q_s}(|\psi\rangle_{A_1|A_2\cdots A_n}) \leq \sum_{i=1}^n E_{q_s}(|\psi\rangle_{A_1|A_{i+1}\cdots A_n})$.
Example 1: Let’s consider the general GHZ state $|\text{GHZ}\rangle = \alpha |0\rangle^\otimes n + \beta |1\rangle^\otimes n$, where $|\alpha|^2 + |\beta|^2 = 1$ and $n \geq 3$. It’s easy to show that $\sum_{i=1}^{n-2} E^q_{\psi}(\rho_{A_1\cdots A_{i-1}A_{i+1}\cdots A_n}) - E^q_{\psi}(|\text{GHZ}\rangle_{A_1\cdots A_n}) = \frac{n-2}{n-1} |\alpha|^4 + |\beta|^4 - 1 \geq 0$.

Example 2: Consider a four-qubit cluster state $|C_4\rangle = \frac{1}{\sqrt{2}}(0000 + 0011 + 1100 - 1111)$, which is a type of highly entangled state of four-qubit $^{42,43}$. The reduced states of $|C_4\rangle$ are $\rho_A = \rho_B = \rho_C = \rho_D = \frac{I}{4}$, thus $\sum_{i=1}^{n-2} E^q_{\psi}(\rho_{A_1\cdots A_{i-1}A_{i+1}\cdots A_n}) - E^q_{\psi}(|C_4\rangle) = \frac{1}{8} \left( \frac{1}{q-1}\right) - 1$ which is nonnegative for $q \geq 1$ and $q^2 \geq 1$.

We note that for any $n$-qubit mixed state $\rho_{A_1\cdots A_n}$, the polygamy relation holds:

$$E^q_{\psi}(\rho_{A|A_1\cdots A_n}) \leq \sum_{i=1}^{n} E^q_{\psi}(\rho_{A_i})$$  \hspace{1cm} (22)

for $1 \leq q \leq 2$ and $-q^2 + 4q - 3 \leq s \leq 1$.$^{44}$ Combining Eq. (17) with Eq. (22), we have

Corollary 2. For any multi-qubit mixed state $\rho_{A_1\cdots A_n}$, the following inequality holds:

$$E^q_{\psi}(\rho_{A|B|C_1\cdots C_n}) \leq E^q_{\psi}(\rho_{A|B|C_1\cdots C_n}) + E^q_{\psi}(\rho_{A|C_1\cdots C_n})$$

$$\leq 2E^q_{\psi}(\rho_{AB}) + \sum_{i=1}^{n} E^q_{\psi}(\rho_{AC_i}) + \sum_{i=1}^{n} E^q_{\psi}(\rho_{BC_i}),$$  \hspace{1cm} (23)

where $1 \leq q \leq 2$, $s = 1$. In this case, $(q, s)$-EOA becomes Tsallis entropy entanglement of assistance which has discussed in ref. 22.

Before our second main result, we have following lemma:

Lemma 1. For $q = 2$ and $\frac{1}{2} \leq s \leq 1$, the function $f_{q,s}(x)$ in Eq. (16) satisfies

$$f_{q,s}(\sqrt{x^2 + y^2}) = f_{q,s}(x) + f_{q,s}(y).$$  \hspace{1cm} (24)

Proof: For $q \geq 2$, $0 \leq s \leq 1$, and $q^2$ is even, we have $f_{q,s}(\sqrt{x^2 + y^2}) = f_{q,s}(x) + f_{q,s}(y)^{36}$. On the other hand, for $1 \leq q \leq 2$ and $0 \leq s \leq 1$, we have $f_{q,s}(\sqrt{x^2 + y^2}) \leq f_{q,s}(x) + f_{q,s}(y)^{44}$. The equality holds if and only if $q = 2$ and $\frac{1}{2} \leq s \leq 1$. This completes the proof.

Next, the following result will provide a lower bound of unified $(q, s)$-entropy entanglement of $|\psi\rangle_{AB|C_1\cdots C_n}$, with respect to the bipartition between $AB$ and $C_1\cdots C_n$.

Theorem 2. For any multi-qubit pure state $|\psi\rangle_{A_1\cdots A_n}$ in the Hilbert space, we have

$$E^q_{\psi}(|\psi\rangle_{AB|C_1\cdots C_n})$$

$$\geq \max \left[ \sum_{i=1}^{n} E^q_{\psi}(\rho_{C_i}) - E^q_{\psi}(\rho_{AB}) \right] \sum_{i=1}^{n} E^q_{\psi}(\rho_{AC_i}) - E^q_{\psi}(\rho_{AC_i}),$$  \hspace{1cm} (25)

where $q = 2$ and $\frac{1}{2} \leq s \leq 1$.

Proof: Given a multi-qubit pure state $|\psi\rangle_{AB|C_1\cdots C_n}$, the unified $(q, s)$-entropy is subadditive for any $q \geq 1$ and $q^2 \geq 1$. Thus, the following equality holds

$$S_{q,s}(\rho_{C_1\cdots C_n}) = S_{q,s}(\rho_{A})$$

$$\leq S_{q,s}(\rho_{A}) + S_{q,s}(\rho_{B})$$

$$= S_{q,s}(\rho_{C_1\cdots C_n})$$  \hspace{1cm} (26)

which implies $S_{q,s}(\rho_{C_1\cdots C_n}) - S_{q,s}(\rho_{B}) \leq S_{q,s}(\rho_{AC_1\cdots C_n})$, and similarly, $S_{q,s}(\rho_{A}) - S_{q,s}(\rho_{C_1\cdots C_n}) \leq S_{q,s}(\rho_{AC_1\cdots C_n})$.

Combine with the two equalities above, one obtain

$$|S_{q,s}(\rho_{A}) - S_{q,s}(\rho_{C_1\cdots C_n})| \leq S_{q,s}(\rho_{AC_1\cdots C_n}),$$  \hspace{1cm} (27)

From the definition of unified $(q, s)$-entropy entanglement of $|\psi\rangle_{AB|C_1\cdots C_n}$, with respect to the bipartition between $AB$ and $C_1\cdots C_n$, we have

$$E^q_{\psi}(|\psi\rangle_{AB|C_1\cdots C_n}) = S_{q,s}(\rho_{A})$$

$$\geq S_{q,s}(\rho_{A}) - S_{q,s}(\rho_{B})$$

$$= E^q_{\psi}(\rho_{ABC_1\cdots C_n}) - E^q_{\psi}(\rho_{BAC_1\cdots C_n}).$$  \hspace{1cm} (28)

Note that for any pure state $|\psi\rangle_{ABC}$ in a $2 \otimes 2 \otimes d$ system, the following equality holds$^{45,46}$

$$C^2(|\psi\rangle_{ABC}) = |C^q(\rho_{AB})|^2 + C^q(\rho_{AC}),$$  \hspace{1cm} (29)
where $\rho_{AB}$ and $\rho_{AC}$ are the reduced matrices of state $|\psi\rangle_{ABC}$ respectively. For $q = 2$ and $\frac{1}{2} \leq s \leq 1$, we have

$$E_{q,s}(\langle \psi \rangle_{ABC}) = f_{q,s}(\mathcal{C}(\langle \psi \rangle_{ABC}))$$

$$= f_{q,s}\left(\mathcal{C}(\rho_{AB}) + \mathcal{C}(\rho_{AC})\right)$$

$$= f_{q,s}\left(\mathcal{C}(\rho_{AB}) + \mathcal{C}(\rho_{AC})\right)$$

where we have used Eq. (29) in the second equality, the third equality holds is due to lemma 1. Therefore,

$$E_{q,s}(\langle \psi \rangle_{ABC}) = f_{q,s}(\mathcal{C}(\langle \psi \rangle_{ABC}))$$

$$\leq f_{q,s}\left(\sqrt{n|\mathcal{C}(\rho_{AB})|^2 + \sum_{i=1}^{n} \mathcal{C}(\rho_{AC})^2}\right)$$

$$\leq f_{q,s}(\mathcal{C}(\rho_{AB})) + f_{q,s}\left(\sum_{i=1}^{n} \mathcal{C}(\rho_{AC})^2\right).$$

(30)

Compare Eq. (30) with Eq. (31), it's easy to see that

$$E_{q,s}(\langle \psi \rangle_{ABC}) - E_{q,s}(\langle \psi \rangle_{BC}) \geq f_{q,s}(\mathcal{C}(\rho_{AC})) - f_{q,s}\left(\sum_{i=1}^{n} \mathcal{C}(\rho_{BC})^2\right).$$

(31)

We also note that

$$f_{q,s}(\mathcal{C}(\rho_{AC})) \geq f_{q,s}\left(\sum_{i=1}^{n} \mathcal{C}(\rho_{BC})^2\right)$$

$$= \sum_{i=1}^{n} f_{q,s}(\mathcal{C}(\rho_{BC}))$$

$$= \sum_{i=1}^{n} E_{q,s}(\rho_{BC}).$$

(33)

where the first equality holds is due to the monogamy of concurrence$^1$ and $f_{q,s}(x)$ is an increasing function for $q \geq 2$, $0 \leq s \leq 1$, and $q^s \leq 3^s$.

On the other hand, we have

$$f_{q,s}\left(\sum_{i=1}^{n} \mathcal{C}(\rho_{BC})^2\right) = \sum_{i=1}^{n} f_{q,s}(\mathcal{C}(\rho_{BC}))$$

$$\leq \sum_{i=1}^{n} E_{q,s}(\rho_{BC})$$

(34)

Combine Eqs (32) and (33) with Eq. (34), we have

$$E_{q,s}(\langle \psi \rangle_{ABC}) - E_{q,s}(\langle \psi \rangle_{BC}) \geq \sum_{i=1}^{n} E_{q,s}(\rho_{BC}) - E_{q,s}(\rho_{BC})$$

(35)

Putting Eq. (35) into Eq. (32), we obtain our result. Similarly, we have

$$E_{q,s}(\langle \psi \rangle_{ABC}) \geq \sum_{i=1}^{n} E_{q,s}(\rho_{BC}) - E_{q,s}(\rho_{BC})$$

(36)

Thus, the proof is completed.

Theorem 2 shows a monogamy relation for a multi-qubit pure state $|\psi\rangle_{ABC}$. The lower bound of the unified $(q, s)$-entanglement for $AB\{C_1, \ldots, C_n\}$ can't be less than the sum of the two-qubit entanglement between bipartitions of the system. In particular, if $|\psi\rangle_{ABC} = |\psi\rangle_{AC_1, \ldots, C_n} \otimes |\psi\rangle_{BC}$, the entanglement of $AB\{C_1, \ldots, C_n\}$ is equal to the entanglement of $A\{C_1, \ldots, C_n\}$. In this case, $E_{q,s}(\rho_{BC}) = 0$ for $i = 1, 2, \ldots, n$. Theorem 2 becomes $E_{q,s}(\langle \psi \rangle_{ABC}) \geq \sum_{i=1}^{n} E_{q,s}(\rho_{BC})$, which is a CKW-type monogamy relation$^{1-4}$.

Example 3: Consider a pure state $|\psi\rangle_{ABC} = \frac{1}{\sqrt{3}} (|0000\rangle + |0001\rangle)$ in the four-qubit system. For the range $q \geq 2$ and $\frac{1}{2} \leq s \leq 1$, we have $E_{q,s}(\rho_{AC}) = E_{q,s}(\rho_{AC}) = 0$, and $E_{q,s}(\rho_{AC}) = E_{q,s}(\rho_{AC}) = \frac{s}{2} (1 - \frac{s}{2})$. 
A generalized monogamy relation is provided for $s$ in the multi-qubit system. In particular, if
\[ E_{AB}'(\rho_{BC}) = E_{AB}'(\rho_{BC}) = 0 \] for $i = 1, 2$ and $E_{AB}'(\rho_{AB}) = \frac{1}{2}(1 - \frac{1}{n})$. Therefore, we can see $|\psi\rangle_{ABC:C_2}$ saturates the inequality Eq. (25).

Example 4: Finally, let’s consider a general W state $|W\rangle_{A_1A_2A_3} = a_1|00\cdots01\rangle + a_2|00\cdots10\rangle + \cdots + a_n|10\cdots00\rangle$ in the $n$-qubit system, where $\sum_i|a_i|^2 = 1$. The reduced state of subsystem $A_iA_i$ is
\[
\rho_{A_iA_i} = \begin{pmatrix}
1 - |a_{n-i}|^2 & 0 & \cdots & 0 \\
0 & |a_{n-i}|^2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & |a_n|^2
\end{pmatrix}
\]
which implies $E_{AB}'(\rho_{A_iA_i}) = \frac{1}{2}$. It’s also easy to show that the reduced state $\rho_{A_iA_i}$ is separable, where $i, j = [1, 2, \ldots, n]$. Thus $E_{AB}'(\rho_{A_iA_i}) = E_{AB}'(\rho_{A_iA_j}) = E_{AB}'(\rho_{A_iA_A}) = E_{AB}'(\rho_{A_A}) = 0$. We find that the right side of the inequality Eq. (25) is $\sum_{i=1}^{n-1} E_{AB}'(\rho_{A_iA_i}) = \sum_{i=1}^{n} E_{AB}'(\rho_{A_iA_A}) = 0$ which implies the inequality Eq. (25) holds for the general W state.

Conclusion
Unified $(q, s)$-entropy is an important generalized entropy in quantum information theory. Many entropies such as Tsallis entropy, Rényi entropy, and von Neumann entropy can be seen as a special case for unified $(q, s)$-entropy.

In this paper, we have investigated the entanglement distribution in multi-particle systems in terms of unified $(q, s)$-entanglement. We find that for any tripartite mixed state, the $(q, s)$-Eq. 4.1 follows a polygamy relation for $q \geq 1$ and $qs \geq 1$. This polygamy relation provides an upper bound for the bipartition $A|BC$, which also holds in multi-particle systems. Furthermore, for $q = 2$ and $1 \leq s \leq 1$, a generalized monogamy relation is provided for unified $(q, s)$-entanglement. This monogamy relation provides a lower bound for the bipartition $AB|C_i$ in the multi-qubit system. In particular, if $|\psi\rangle_{ABC:C_i} = |\psi\rangle_{AC:C_i} \otimes |\psi\rangle_B$, the unified monogamy relation becomes a CKW-type monogamy relation.

Both monogamy property and polygamy property are fundamental properties of multipartite entangled states. We have studied the properties above in detail, and provided a two-parameters entropy function to study the entanglement distribution. We believe our result provides a useful methodology to understand the entanglement distribution of multi-particle entanglement.

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**Author Contributions**

Y. Luo performed the calculations and wrote the main manuscript. F.-G. Zhang checked the calculations. Y. Li improved the manuscript. All authors contributed to the discussion and reviewed the manuscript.

**Additional Information**

**Competing Interests:** The authors declare that they have no competing interests.

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