Remarks on statistical aspects of safety analysis of complex systems

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Abstract

We analyze safety problems of complex systems using the methods of mathematical statistics for testing the output variables of a code simulating the operation of the system under consideration when the input variables are uncertain. We have defined a black box model of the code and derived formulas to calculate the number of runs needed for a given confidence level to achieve a preassigned measure of safety. In order to show the capabilities of different statistical methods, firstly we have investigated one output variable with unknown and known distribution functions. The general conclusion has been that the different methods do not bring about large differences in the number of runs needed to ensure a given level of safety. Analyzing the case of several statistically dependent output variables we have arrived at the conclusion that the testing of the variables separately may lead to false, safety related decisions with unforeseen consequences. We have advised two methods: the sign test and the tolerance interval methods for testing more then one mutually dependent output variables.

List of key words: safety analysis, black box model, best estimate, Bayesian method, quantile test, confidence interval, sign test, tolerance interval.
1 Introduction

There are two approaches to safety analysis of large complex systems. Since
the analysis has to demonstrate safety of the operation under the investigated
circumstances, we may scrutinize a not too realistic but rather unfavorable
situation saying that if that situation is safe then any real situation must be
on the safe side. This approach we call conservatism.

An alternative approach may attempt to investigate the real situation
and show that no limit violation can occur. In this case the calculated values
should be increased by the possible error when compared with the safety
limit \[1\]. That approach is called best estimate which is not a very fortunate
but generally accepted name.

In \textit{conservative analysis}, the first problem is in the selection of the case
to be studied. It identifies an overt attempt to bound the actual expected
state hence it should estimate also the consequences of model uncertainties.
How do we know if a given situation is more conservative than is the other?
It is often impossible to foresee the outcome of a non-linear process. Another
problem may be the interplay between approximations. It may happen that
either of two approximations leads to conservatism but their simultaneous
presence does not. The conservative approach has been prevalent for a long
time, although today rather the best estimate methods are in the focus.

The main difficulty with \textit{best estimate calculation} is in the complexity of
the phenomena involved. (A new material phase may appear, at a given tem-
perature chemical reactions may take place producing new material proper-
ties, and also producing or removing heat, the process dynamics is nonlinear
etc.) In spite of the problem’s complexity, a best estimate method attempts
to solve the equations describing the involved physical processes as accurately
as our knowledge permits. From \textit{licensing viewpoint}, several key parameters
should be selected and compared to the acceptance criteria.

Best estimate methods are accompanied by an \textit{uncertainty analysis} to
learn the uncertainty band of the response \[2\]. The purpose of the uncertainty
evaluation is to provide assurance that the selected parameters at least with
probability 95% or more will be in the \textit{acceptance region} or will not exceed
their acceptance level.

The present work is dealing with the \textit{code uncertainty} only, which is rather
important constituent of the total uncertainty. We assume the modeled pro-
cedure to start from a known initial state. All the physical quantities in the
model we sort as \textit{input, output, and latent data}. By definition, a datum is
input if its domain is known along with a distribution function associating a probability with any admitted value. In a model, there are several constants, which are considered either as input or latent data. Input, when the given constant is looked upon as a variable in a given range and a probability is allotted to every possible value. Distinction between input and latent data is a matter of engineering judgement. The nature of the distribution may depend on the determination of the constant. Latent, when we refrain from analyzing the uncertainties of the constant, temporarily we take it as a fixed number. A datum not falling into the input or latent category is called output.

The paper is organized as follows. In Section 2 we define a simple black box model linking the output variables to input variables, while in Section 3 we analyze possibilities and limitations of several well-known statistical tests for one output variable with unknown and known cumulative distribution function. Special attention is paid to the application of a slightly new variant of the tolerance interval method. In Section 4 we deal with the case of several not independent output variables by using the advantages of order statistics, and, finally the conclusions are summarized in Sections 5 and 6.

The present work focuses on deriving criteria for safe operation when the output variables are fluctuating as a result of randomness of input variables, and intends to give some help in practical applications. In the sequel we follow the notation used in the classical handbook of statistics by M.G. Kendall and A. Stuart [3].

2 Black box model

Let us consider a system as complex as a nuclear power plant, or an oil refinery plant for instance. Assume we have a model describing that system, and that model enables us to calculate physical parameters characterizing the system at arbitrary instant \( t \). Let \( n \) be the number of technologically important variables. In the frame of the model, the operation of the system is considered safe if all calculated variables belong to a given set of intervals

\[
\mathcal{V}_T = \left\{ \left[ L_T^{(j)}, U_T^{(j)} \right], \quad j = 1, \ldots, n \right\}
\]

determined by the technology.

In order not to be set back by the complexity of the problem, we suggest a simple black box model, in which output variables are linked to input
variables. That link can be a computer code that transforms vector $\vec{x} \in \mathcal{X}$, the input variables, into a vector $\vec{y}(t) \in \mathcal{Y}$, the output variables. Here $\mathcal{X}$ and $\mathcal{Y}$ are sets of all possible values of $\vec{x}$ and $\vec{y}(t)$, respectively. In general, the dimension of $\vec{x}$, i.e. the number of input variables is not the same as the dimension of $\vec{y}(t)$, i.e. the number of output variables. Every data that enters into the model is treated as an input variable, hence we do not distinguish parameters. The model is an explicit relationship between input $\vec{x}$ and output $\vec{y}$:

$$\vec{y}(t) \Leftarrow \hat{C}(t) \vec{x},$$

(1)

where $\hat{C}(t)$ is a nonlinear operator that maps

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{into} \quad \vec{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}.$$ 

In practical cases the link between input and output is very complex hence there is no reason to anticipate an analytical relationship like $\vec{y}(t) = \vec{f}(\vec{x}, t)$. In the sequel $\hat{C}(t)$ is assumed to be deterministic, in other words once the input has been fixed, we obtain the same output within the computation accuracy for each run. At the same time, if the input vector fluctuates according to distribution laws simulating possible variations of the technology, or, reflecting uncertainty of some parameters of the model then the output parameters also fluctuate in repeated runs.

We present an illustration of how random input may influence an output variable, see Fig. 1. We used the thermohydraulic code ATHLET [4] to generate several output variables for a simple experimental setup, but in Fig. 1 we presented only one output variable as function of time for three independent runs. It is obvious that in this case the above given criterion for safe operation of the system needs to be changed because there is no guarantee that a new run after a successful run will also be successful.

We call a state $\vec{x}_0$ nominal, if all the input parameters take their respective expectation value, i.e. $\vec{x}_0 = \mathbf{E}\{\vec{x}\}$. We can perform a calculation in the nominal state to get the corresponding output $\vec{y}_0 = \hat{C}(t)\vec{x}_0$. Usually the state $\vec{x}_0$ is called safe if $\vec{y}_0$ is in the safety envelope $\mathcal{V}_T$. However, we need a more stringent definition: state $\vec{x}$ is called safe if $\vec{y}$ is in the safety envelope $\mathcal{V}_T$ for every $\vec{x} \in \mathcal{X}$.
Figure 1: Influence of the random input on the time dependence of one output variable in three independent runs.

Here we should make three remarks. (i) $\mathcal{X}$ may be an infinite interval when at least one of the input variables is of normal distribution. In practical calculations such variables are confined to a finite interval by engineering judgement. (ii) We check that statement by a given, finite number of calculations with input from $\mathcal{X}$. If there is a value outside the safety envelop $\mathcal{V}_T$ the state $\vec{x}$ is unsafe independently of the fact that the nominal state $\vec{x}_0$ may be safe. (iii) Even if every calculated output is safe, there is a probability that the state is actually unsafe.

Fixing time $t$ after $N$ runs we obtain $N$ randomly varying output vectors $\{\vec{y}_1(t), \vec{y}_2(t), \ldots, \vec{y}_N(t)\}$ which carries information on the fluctuating input and the code properties. In the next Section we are considering only one output variable with continuous cumulative distribution function $G(y) = \int_{-\infty}^{y} g(u) \, du$, and the time $t$ is taken as fixed and its notation is omitted.

3 One output variable

3.1 Old Bayesian method

If we carry out $N$ runs with fluctuating input, then we obtain a sample $S_N = \{y_1, y_2, \ldots, y_N\}$ of the random variable $y$ at a fixed time point. Through
technological considerations, we define a fix acceptance and a fix rejection interval to variable $y$. Let the acceptance interval be $\mathcal{H}_a = [L_T, U_T]$, and $\mathcal{H}_r = [L_T, U_T] = (-\infty, L_T) \cup (U_T, +\infty)$ the rejection interval. \(^1\)

The probability

$$\mathcal{P}\{y \in \mathcal{H}_a\} = \int_{\mathcal{H}_a} g(u) \, du = w$$

that an observed $y$ lays in the acceptance interval is not known. Knowing however that $k$ elements of the sample $S_N$ are in the acceptance interval $\mathcal{H}_a$, then utilizing Bayes’ theorem, without knowing $g(u)$, we can claim that

$$\beta(\omega|N, k) =$$

$$= \frac{\int_0^1 u^k (1 - u)^{N-k} \, du}{\int_0^1 u^k (1 - u)^{N-k} \, du} = \sum_{j=0}^{k} \binom{N+1}{j} (1 - \omega)^j \omega^{N+1-j} = \beta(\omega|N, k) \quad (2)$$

is the probability that the unknown acceptance probability $w$ is greater than a prescribed $\omega$. The proof of the mentioned theorem is available in textbooks\(^2\) hence we omit it here. We wish to point out the expression

$$\beta(\omega|N, 0) = 1 - \omega^{N+1}, \quad (3)$$

which shows convincingly that even when the whole sample $S_N$ consists of elements to be accepted, we can state only that $w \geq \omega$ with probability $1 - \omega^{N+1}$. If one element in the sample $S_N$ is in the rejection interval, then we have

$$\beta(\omega|N, 1) = 1 - \omega^{N+1} - (N + 1) (1 - \omega)\omega^N. \quad (4)$$

Using (2), one can easily determine the allowed number of rejections in a sample of $N$ elements so that the unknown probability of the acceptance $w$ to be larger than the prescribed limit $\omega$ with a given probability $\beta(\omega|N, k) \geq \alpha$. It can be agreed on that a system is safe if it is almost certain ($0 << \alpha \leq 1$) that the unknown probability of the acceptance $w$ is larger than a prescribed $\omega$.

\(^1\)In many practically important cases $L_T = -\infty$, and so $\mathcal{H}_a = (-\infty, U_T]$ and $\mathcal{H}_r = (U_T, +\infty)$.

\(^2\)Pál. L.: Fundamentals of probability Theory and Statistics, vol. I.-II., 109-113, Budapest, Akadémiai Kiadó, Budapest (1995), in Hungarian.
Table I. Number of observations $N$ at which $w \geq \omega$ with probability $\beta(\omega|N, k) \geq \alpha$ for several values of $\alpha$, $\omega$, and the number of rejected values $N - k$.

| $\alpha$ | $\omega$ | $N - k = 0$ | $N - k = 1$ | $N - k = 2$ |
|----------|----------|-------------|-------------|-------------|
| 0.90     | 0.90     | 21          | 31          | 51          |
|          | 0.95     | 44          | 75          | 104         |
|          | 0.99     | 228         | 387         | 530         |
| 0.95     | 0.90     | 27          | 45          | 60          |
|          | 0.95     | 57          | 92          | 123         |
|          | 0.99     | 297         | 472         | 626         |
| 0.99     | 0.90     | 43          | 63          | 80          |
|          | 0.95     | 89          | 129         | 164         |
|          | 0.99     | 457         | 660         | 836         |

For example, we read out from Table I that if all the 297 observed values were acceptable, i.e. there was not a single value to be rejected, then, larger than 95% is the probability that $w \geq 0.99$, i.e. the proportion of rejected observations in any sample will be not larger than 0.01. The more observations we have, with the higher probability we can state that the investigated system is safe, and the higher is the lower level $\omega$ for the unknown acceptance probability $w$.

### 3.2 Distribution free confidence interval for quantile

Assume again the cumulative distribution function $G(y)$ of the output variable $y$ to be unknown but continuous and strictly increasing. Denote by $Q_\gamma$ the $\gamma$-quantile of $G(y)$, i.e the value satisfying the equation

$$G(Q_\gamma) = \int_{-\infty}^{Q_\gamma} dG(y) = \gamma.$$ 

Clearly, the interval $(-\infty, Q_\gamma]$ covers the proportion $\gamma$ of the distribution $G(y)$. Since $G(y)$ is continuous and strictly increasing \(^3\) one can write

$$Q_\gamma = G^{-1}(\gamma).$$

It is to mention that the point estimate of $Q_\gamma$ is that element of the ordered sample the index $k$ of which is the nearest integer to $N\gamma$.

\(^3\)If $G(y)$ is a continuous and not decreasing function, then $Q_\gamma = \inf\{y : G(y) \geq \gamma\}$.
3.2.1 Two-tailed test

Carrying out \( N \) independent runs, we get a sample \( S_N = \{y_1, \ldots, y_N\} \). Arrange the sample elements in increasing order, \(^4\) and denote by \( y(k) \) the \( k \)th of ordered elements; hence we have

\[
y(1) < y(2) < \cdots < y(r) < \cdots < y(s) < \cdots < y(N),
\]

and by definition \( y(0) = -\infty \), while \( y(N + 1) = +\infty \). As known the joint density function of random variables

\[
z(r) = G[y(r)] \quad \text{and} \quad z(s) = G[y(s)],
\]

where \( r \) and \( s > r \) are positive integers from \( \{1, 2, \ldots, N\} \) is given by

\[
g_{r,s}(u, v) = \frac{u^{r-1} (v-u)^{s-r-1} (1-v)^{N-s}}{B(r, s-r) B(s, N-s+1)},
\]

\[0 \leq u \leq v \leq 1.\]

Here \( B(j, k) \) is the Euler beta function.

**Theorem 1** If \( r \) and \( s \) positive integers satisfying the inequality \( 0 < r < (N + 1)\gamma < s \leq N \), then the random interval \( [y(r), y(s)] \) covers the unknown \( \gamma \)-quantile \( Q_\gamma \) with probability

\[
\beta = \mathcal{P}\{y(r) \leq Q_\gamma \leq y(s)\} =
\]

\[= I(1 - \gamma, N - s + 1, s) - I(1 - \gamma, N - r + 1, r)\] \hspace{1cm} (5)

where

\[I(c, j, k) = \frac{B(c, j, k)}{B(j, k)}\]

is the regularized incomplete beta function for non-singular cases.

The proof of the theorem is simple and it can be find in the Appendix I. One can see that the confidence level \( \beta \) for the the random interval \( [y(r), y(s)] \) does not depend on \( G(y) \), in other words, the confidence interval for the unknown \( Q_\gamma \) is distribution free.

Clearly, there are many different confidence intervals covering \( Q_\gamma \) with a prescribed probability \( \beta \). We have to chose the shortest interval by using the following procedure:

\(^4\)The probability that equal values occur is zero.
• from the ordered sample determine the integer \( q = [(N + 1)\gamma] \) due to the point estimate \( \tilde{Q}_\gamma = y(q) \) of the \( \gamma \)-quantile \( Q_\gamma \),

• calculate the confidence level \( \beta \) step by step for intervals defined by integer pairs \([r_j, s_k]\) where \( r_j = q - j \), \( j = 1, 2, \ldots, q - 1 \) and \( s_k = q + k \), \( k = 1, 2, \ldots, N - k \), respectively, until the prescribed value of \( \beta \) is reached provided that it is possible at the sample size \( N \) that we have,

• if the prescribed \( \beta \) could not be reached, then the sample size should have been increased.

When the confidence interval \([y(r_j), y(s_k)]\) covering the \( \gamma \)-quantile of the unknown distribution \( G(y) \) at prescribed confidence level \( \beta \) is a part of the interval \([L_T, U_T]\) defined by technology, then the system can be qualified safe at level \((\beta|\gamma)\).

Table II. Confidence levels \( \beta \) for confidence intervals covering the unknown quantile \( Q_{0.9} \) in the case of sample size \( N = 100 \). (The point estimate of \( Q_{0.9} \) is equal to \( \tilde{Q}_{0.9} = y(90) \)).

| \( r \backslash s \) | 95   | 96   | 97   | 98   | 99   | 100  |
|-----------------|------|------|------|------|------|------|
| 89              | 0.6455 | 0.6793 | 0.6952 | 0.7011 | 0.7027 | 0.7030 |
| 88              | 0.7442 | 0.7781 | 0.7940 | 0.7999 | 0.8015 | 0.8018 |
| 87              | 0.8185 | 0.8524 | 0.8683 | 0.8742 | 0.8758 | 0.8761 |
| 86              | 0.8699 | 0.9037 | 0.9196 | 0.9255 | 0.9271 | 0.9274 |
| 85              | 0.9025 | 0.9364 | 0.9523 | 0.9582 | 0.9598 | 0.9601 |
| 84              | 0.9218 | 0.9557 | 0.9716 | 0.9775 | 0.9791 | 0.9794 |
| 83              | 0.9324 | 0.9663 | 0.9822 | 0.9880 | 0.9897 | 0.9900 |
| 82              | 0.9378 | 0.9717 | 0.9876 | 0.9935 | 0.9951 | 0.9954 |
| 81              | 0.9404 | 0.9743 | 0.9902 | 0.9961 | 0.9977 | 0.9980 |
| 80              | 0.9416 | 0.9755 | 0.9914 | 0.9972 | 0.9989 | 0.9992 |

In Table II we see that, for example, the confidence interval \([y(85), y(97)]\) defined by elements \( y(85) \) and \( y(97) \) of the ordered sample of size \( N = 100 \) covers the quantile \( Q_{0.9} \) of the unknown distribution of the output variable \( y \) with probability (on confidence level) \( \beta = 0.9523 \). In other words, having \( N = 100 \) observations for the output variable \( y \) we can state with probability \( \beta = 0.9523 \) that \( y[85] < Q_{0.9} < y[97] \), i.e. the upper limit of the interval \((-\infty, Q_{0.9}]\) containing 90% of the unknown distribution \( G(y) \) is covered by
\([y(85), y(97)]\) on confidence level \(\beta = 0.9523\). If \([y(85), y(97)] \subseteq [L_T, U_T]\), then the system is safe, but only on the level \((0.9523|0.9)\).

When we need stronger criteria of safety, then we have to find confidence intervals covering quantiles \(Q_{0.95}\) or \(Q_{0.99}\) with probability near the unity. As seen in Tables III and IV the sample size \(N\) should be greatly increased. For example, if we would like to construct a confidence interval for the quantile \(Q_{0.99}\) at the level of \(\beta = 0.9467\) we need sample with \(N \approx 700\) elements. The production of such a large sample for even one output variable of complex systems is very expensive, and at the same time, there is no guarantee that the relation \([y(r), y(s)] \subseteq [L_T, U_T]\) will be always satisfied, especially when the distribution is asymmetric.

**Table III.** Confidence levels \(\beta\) for confidence intervals covering the \(Q_{0.95}\) unknown quantile in the case of sample size \(N = 150\). (The point estimate of \(Q_{0.95}\) is equal to \(\hat{Q}_{0.95} = y(143)\)).

| \(r\) | 144 | 1456 | 146 | 147 | 148 | 149 | 150 |
|-------|-----|------|-----|-----|-----|-----|-----|
| 142   | 0.2909 | 0.4293 | 0.5382 | 0.6090 | 0.6456 | 0.6597 | 0.6633 |
| 141   | 0.4080 | 0.5464 | 0.6553 | 0.7261 | 0.7627 | 0.7768 | 0.7804 |
| 140   | 0.4949 | 0.6333 | 0.7422 | 0.8130 | 0.8496 | 0.8637 | 0.8673 |
| 139   | 0.5531 | 0.6916 | 0.8004 | 0.8712 | 0.9078 | 0.9219 | 0.9255 |
| 138   | 0.5886 | 0.7270 | 0.8359 | 0.9067 | 0.9433 | 0.9574 | 0.9610 |
| 137   | 0.6084 | 0.7469 | 0.8557 | 0.9265 | 0.9632 | 0.9773 | 0.9809 |
| 136   | 0.6186 | 0.7571 | 0.8659 | 0.9368 | 0.9734 | 0.9875 | 0.9911 |

**Table IV.** Confidence levels \(\beta\) for confidence intervals covering the \(Q_{0.99}\) unknown quantile in the case of sample size \(N = 700\). (The point estimate of \(Q_{0.99}\) is equal to \(\hat{Q}_{0.99} = y(694)\)).

| \(r\) | 694 | 695 | 696 | 697 | 698 | 699 | 700 |
|-------|-----|-----|-----|-----|-----|-----|-----|
| 692   | 0.2808 | 0.4303 | 0.5581 | 0.6490 | 0.7007 | 0.7226 | 0.7280 |
| 691   | 0.3826 | 0.5321 | 0.6509 | 0.7508 | 0.8024 | 0.8244 | 0.8306 |
| 690   | 0.4536 | 0.6031 | 0.7309 | 0.8218 | 0.8735 | 0.8954 | 0.9017 |
| 689   | 0.4986 | 0.6481 | 0.7759 | 0.8668 | 0.9185 | 0.9405 | 0.9467 |
| 688   | 0.5247 | 0.6742 | 0.8020 | 0.8929 | 0.9446 | 0.9666 | 0.9728 |
| 687   | 0.5387 | 0.6882 | 0.8160 | 0.9069 | 0.9585 | 0.9805 | 0.9867 |
| 686   | 0.5456 | 0.6951 | 0.8229 | 0.9138 | 0.9655 | 0.9874 | 0.9936 |
3.2.2 One-tailed test

In order to declare that a system is operating safely on a given level, in many practical cases it seems to be enough to know that the value of a properly selected output variable $y$ with probability near 1 is smaller than the value $U_T$ prescribed by technology. In this case we should determine that element $y(s)$ of the ordered sample which, with probability $\beta$, is larger than the quantile $Q_\gamma$ of the unknown distribution $G(y)$ of the output variable $y$. It means that the random interval $(-\infty, y(s)]$ covers the proportion larger than $\gamma$ of the unknown distribution $G(y)$ of output variable $y$ with probability

$$\beta = \mathcal{P}\{y(s) > Q_\gamma\}.\]

In order to determine this probability we should substitute $r = 0$ into Eq. (5), since according to our definition $y(0) = -\infty$. We obtain that

$$\beta = I(1 - \gamma, N - s + 1, s) = \sum_{j=0}^{s-1} \binom{N}{j} \gamma^j (1 - \gamma)^{N-j}, \quad (6)$$

where $I(c, j, k)$ is the regularized incomplete beta function for non-singular cases. \(^5\) If $y(s)$ is smaller than $U_T$, then we can state: the system is safe at the level $(\beta|\gamma)$.

If $s = N$, i.e. if the largest element of the sample is chosen as upper limit of the random interval, then one obtains the well-known formula:

$$\beta = 1 - \gamma^N. \quad (7)$$

\(^5\)This equation can be easily derived directly. It is obvious that

$$\beta = \mathcal{P}\{y(s) > Q_\gamma\} = \mathcal{P}\{y(s) > G^{-1}(\gamma)\} = \mathcal{P}\{G[y(s)] > \gamma\},$$

and since the probability density function of the random variable $z(s) = G[y(s)]$ is nothing else than

$$g_s(u) = \frac{u^{s-1} (1 - u)^{N-s}}{B(s, N - s + 1)},$$

so we can write immediately that

$$\beta = \frac{1}{B(s, N - s + 1)} \int_{\gamma}^{1} u^{s-1} (1 - u)^{N-s} \, du = I(1 - \gamma, N - s + 1, s) =$$

$$= 1 - I(\gamma, s, N - s + 1),$$

and this nothing else than (6).
Since in the engineering practice one can find misinterpretations it is not superfluous to underline the just proven notion of this formula: \( \beta \) is the probability that the largest value \( y(N) \) of a sample consisting of \( N \) observations is greater than the \( \gamma \) quantile of the unknown distribution of the output variable \( y \). This statement can be formulated also as follows: \( \beta \) is the probability that the interval \( (-\infty, y(N)] \) covers the proportion larger than \( \gamma \) of the unknown distribution \( G(y) \) of the output variable \( y \).

If \( s = N - 1 \), i.e. if the \( (N-1) \)-th element of the ordered sample is chosen as upper limit, then we get from (6) the following formula:

\[
\beta = 1 - \gamma N - N(1 - \gamma) \gamma^{N-1},
\]

the notion of which is obvious. Clearly, when \( \beta \) and \( \gamma \) are fixed, and the second largest element of the sample is chosen for upper limit, then the sample size \( N \) needed to reach the level \( (\beta|\gamma) \) is obviously greater than if the largest element would have been chosen. For example, let the certainty level \( 0.95 \mid 0.95 \), then if the largest element is chosen, the sample size should be \( N_0 = 58 \), \(^6\) while if the second largest one is applied, the sample size has to be \( N_1 = 93 \). However, it is at all not certain that \( y^{(93)}(92) \leq y^{(58)}(58) \). (The superscript denotes the sample size.)

![Figure 2: Dependence of the probability \( \beta \) on \( \gamma = G(Q_p) \) at three values of \( s \).](image)

\(^6\)The root of Eq. \( 0.95^N - 0.05 = 0 \) is \( N \approx 58.404 \), and we are using the rounded value \( N = 58 \). In engineering practice the value \( N = 59 \) is accepted.
Figure 2 shows the dependence of the probability $\beta$ on $\gamma$ when $N = 100$ and $s = 100, 99$ and 98. One can see the sharp decrease of $\beta$ when the quantile-level $\gamma$ approaches the unity.

**Table V.** Sample sizes $N_0, N_1, \ldots, N_6$ for finding elements $y(s), \ s = N_0, N_1 - 1, \ldots, N_6 - 6$ to be larger than quantiles $Q_{0.90}, Q_{0.95}$ and $Q_{0.99}$ of the unknown distribution of the output variable $y$ with prescribed probabilities $\beta = 0.90, 0.95$ and 0.99, respectively.

| $\gamma \backslash \beta$ | 0.90 | 0.95 | 0.99 | $s$     |
|-------------------------|------|------|------|---------|
|                          | 22   | 28   | 44   | $N_0 - 0$ |
|                          | 37   | 46   | 64   | $N_1 - 1$ |
|                          | 52   | 61   | 81   | $N_2 - 2$ |
| 0.90                    | 65   | 75   | 97   | $N_3 - 3$ |
|                          | 78   | 89   | 112  | $N_4 - 4$ |
|                          | 91   | 102  | 127  | $N_5 - 5$ |
|                          | 103  | 115  | 141  | $N_6 - 6$ |
|                          | 45   | 58   | 90   | $N_0 - 0$ |
|                          | 76   | 93   | 130  | $N_1 - 1$ |
|                          | 105  | 124  | 165  | $N_2 - 2$ |
| 0.95                    | 132  | 153  | 197  | $N_3 - 3$ |
|                          | 158  | 180  | 228  | $N_4 - 4$ |
|                          | 183  | 207  | 258  | $N_5 - 5$ |
|                          | 206  | 234  | 287  | $N_6 - 6$ |
|                          | 229  | 298  | 458  | $N_0 - 0$ |
|                          | 388  | 473  | 661  | $N_1 - 1$ |
|                          | 531  | 627  | 837  | $N_2 - 2$ |
| 0.99                    | 666  | 773  | 1001 | $N_3 - 3$ |
|                          | 797  | 913  | 1157 | $N_4 - 4$ |
|                          | 925  | 1049 | 1307 | $N_5 - 5$ |
|                          | 1051 | 1181 | 1453 | $N_6 - 6$ |

By fixing the values $\beta$ and $\gamma$ we may calculate sample sizes $N_0, N_1, \ldots, N_k$ which are needed for finding elements $y(s), \ s = N_0, N_1 - 1, \ldots, N_k - k$ such to be larger than the $\gamma$-quantile of the unknown distribution of the output variable $y$ with prescribed probability $\beta$. We can see in Table V that for example the largest element in a sample of size $N = 58$ with probability $\beta = 0.95$ is greater than the quantile $Q_{0.95}$ of the unknown distribution. If $N = 234$, then this statement is true for the element $y(227)$.

### 3.2.3 Illustrations

In order to get a deeper insight into the properties of the just outlined method, we choose the lognormal distribution with parameters $m$ and $d$.
as the "unknown" distribution $G(y)$. We note that this distribution arises when many independent random variables are combined in a multiplicative fashion. The density function

$$g(y) = \frac{1}{\sqrt{2\pi} dy} \exp \left\{ -\frac{1}{2} \left( \frac{\log y - m}{d} \right)^2 \right\}, \quad y \geq 0$$

Figure 3: Lognormal density function with parameter values $m = 2.0, 2.5$ and $d = 0.5$. The vertical arrows indicate the quantile $Q_{0.95}$.

can be seen in Fig. 3 when $m = 2.0, 2.5$ and $d = 0.5$. The arrows show the quantiles $Q_{0.95} \approx 16.8$ ($m = 2$) and $Q_{0.95} \approx 27.7$ ($m = 2.5$).

By using Monte Carlo simulation let us generate now four sample of size $N = 100$ corresponding to lognormal distribution with parameters $(m = 2.5, d = 0.5)$, and denote by $A, B, C$ and $D$ these samples. Calculate the point estimates of 0.95-quantiles for each of the samples, and determine the shortest two-tailed confidence intervals which cover with probability $0.95$ the "unknown" quantile $Q_{0.95}$. In the present case we know that $Q_{0.95} \approx 16.8$ ($m = 2$) and $Q_{0.95} \approx 27.7$ ($m = 2.5$).

In Fig. 4 the confidence intervals are shown by vertical straight lines. Obviously, these intervals are random variables, hence fluctuate from sample to sample. In the presented example the sample $D$ is the most unfavorable, because in this case we can state only that the "unknown" quantile $Q_{0.95}$ is covered by the interval $[23.29, 53.05]$ with probability larger than $\beta = 0.95$. 

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Table VI. Confidence intervals $[y(r), y(s)]$ covering the "unknown" quantile $Q_{0.95}$ with probability 0.95.

|       | A   | B   | C   | D   |
|-------|-----|-----|-----|-----|
| $y(r)$| 22.66 | 25.21 | 22.48 | 23.29 |
| $Q_{0.95}$ | 27.73 | 27.73 | 27.73 | 27.73 |
| $y(s)$ | 33.25 | 38.28 | 35.88 | 53.05 |
| $(r, s)$ | (91, 100) | (91, 100) | (91, 100) | (91, 100) |

Figure 4: Two-sided confidence intervals denoted by vertical straight lines for samples $A, B, C$ and $D$. The intervals are calculated to be covered the true value of the quantile $Q_{0.95}$ with probability larger than $\beta = 0.95$. The density function is lognormal with parameters $m = 2.5$ and $d = 0.5$. The vertical dashed lines are indicating the true value of the quantile $Q_{0.95}$.

If the upper limit $U_T$ determined by technology would be $U_T = 40$, then only three ($A, B, C$) of four samples could be regarded safe at the level $(0.95|0.95)$, however, sample $D$, which is certainly a “rare event”, would decrease the weight of our statement.

As mentioned, in many cases it is enough to know only the element $y(s), \ s = N, N - 1, \ldots, N - k$ of the ordered sample of size $N$ for which the
The equation

\[ \mathcal{P}\{-\infty \leq Q_\gamma < y(s)\} = \mathcal{P}\{y(s) > Q_\gamma\} = \beta \]

is valid. The test based on the interval \((-\infty, y(s)]\) is called one tailed test. First, determine the sample size \(N\) at which the largest element of the sample \(y(N)\) with probability \(\beta\) is greater than the quantile \(Q_\gamma\) of the unknown distribution \(G(y)\) of the output variable \(y\). If \(\beta = 0.95\) and \(\gamma = 0.95\), then the largest element has to be chosen out of a sample containing \(N = 58\) elements. Produce a sample of size \(N = 58\) simulating the lognormal distribution with parameters \(m = 2.5, \ d = 0.5\), and call it basic sample, denoted by \(y(b)\). Then, repeat randomly the sample production \(n\)-times, and denote by \(y^{(1)}, y^{(2)}, \ldots, y^{(n)}\) the series of samples. We are interested in the largest elements \(y^{(j)}(58), \ j = 1, \ldots, n\) of samples \(y^{(j)}, \ j = 1, \ldots, n\).

**Figure 5:** Largest elements of 1000 samples of size \(N = 58\). The horizontal line corresponds to the largest element of the basic sample of size \(N = 58\). This element is equal to \(y^{(b)}(58) \approx 44.99\).

Fig. 5 shows the largest elements of \(n = 1000\) randomly produced, independent samples of size \(N = 58\). The minimal value of the largest elements is 22.62, while the maximal value is 132.27. One can observe that 224 largest elements exceed the value \(y^{(b)}(58) \approx 44.99\) which is the largest element of
the basic sample. However, this surprisingly great number is in full agreement with the statement that the interval \([0, y^{(b)}(58)]\) covers the "unknown" 0.95-quantile with probability at least 0.95.

In order to show this, let us introduce the random variable \(\xi_n(Q_{\gamma})\) which gives the number of largest elements being greater than the quantile \(Q_{\gamma}\) in \(y^{(j)}, \ j = 1, \ldots, n\) independent samples of size \(N\). Since the probability that the largest element in a given sample is greater than \(Q_{\gamma}\) is nothing else than \(1 - \gamma N\), hence, we conclude that

\[
P\{\xi_n(Q_{\gamma}) = k\} = \binom{n}{k} (1 - \gamma N)^k \gamma^N (n-k).
\]

From this we obtain immediately that

\[
\mathbb{E}\{\xi_n(Q_{\gamma})\} = n(1 - \gamma N) \quad \text{and} \quad \mathbb{D}\{\xi_n(Q_{\gamma})\} = \sqrt{n \gamma N (1 - \gamma N)}.
\]

As known, if \(n\) and \(k\) are sufficiently large, then the distribution of the random variable

\[
\chi_n(Q_{\gamma}) = \frac{\xi_n(Q_{\gamma}) - \mathbb{E}\{\xi_n(Q_{\gamma})\}}{\mathbb{D}\{\xi_n(Q_{\gamma})\}}
\]

is approximately standard normal, hence we can write that

\[
w = P\{|\chi_n(Q_{\gamma})| \leq \lambda\} = P\{\mathbb{E}\{\xi_n(Q_{\gamma})\} - \lambda \mathbb{D}\{\xi_n(Q_{\gamma})\} \leq \xi_n(Q_{\gamma}) \leq \mathbb{E}\{\xi_n(Q_{\gamma})\} + \lambda \mathbb{D}\{\xi_n(Q_{\gamma})\}\},
\]

where \(\lambda\) is the root of Eq.

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-u^2/2} du = \frac{1 + w}{2}.
\]

It means that the inequality

\[
\mathbb{E}\{\xi_n(Q_{\gamma})\} - \lambda \mathbb{D}\{\xi_n(Q_{\gamma})\} \leq \xi_n(Q_{\gamma}) \leq \mathbb{E}\{\xi_n(Q_{\gamma})\} + \lambda \mathbb{D}\{\xi_n(Q_{\gamma})\}
\]

is valid with probability \(w\).

If \(n = 1000, \ N = 58, \ \gamma = 0.95\) and \(w = 0.95\), then we obtain the values \(\mathbb{E}\{\xi_n(Q_{\gamma})\} = 950, \ \mathbb{D}\{\xi_n(Q_{\gamma})\} \approx 6.96\) and \(\lambda \approx 1.96\), hence we can state with probability 0.95 that

\[936 < \xi_{1000}(Q_{0.95}) < 964.\]
If we count the number of largest elements \( y^{(j)}(58), \ j = 1, \ldots, 1000 \) exceeding \( Q_{0.95} \) that we know in this example \( (Q_{0.95} \approx 27,728) \), we obtain the value 949 that is indeed inside of the interval \([936, 964]\).

In spite of this ”nice” agreement we have to underline that the requirement of safety, for instance, at the level \((0.95|0.95)\) does not exclude the appearance of ”rare events” such as exceeding the technological limit \( U_T \). Therefore, we advice stronger requirements of safety the fulfillment of which, of course, is much more expensive.

### 3.3 Method based on sign test

Assume again the cumulative distribution function \( G(y) \) of the output variable \( y \) to be continuous but unknown. Let \( S_N = \{y_1, \ldots, y_N\} \) be a sample containing the values of \( N \) observations. Define the function

\[
\Delta(x) = \begin{cases} 
1, & \text{if } x > 0, \\
0, & \text{if } x < 0,
\end{cases}
\]

and introduce the statistical function

\[
z_N = \sum_{j=1}^{N} \Delta(U_T - y_j). \quad (9)
\]

which gives the number of sample elements smaller than \( U_T \). Criteria based on this statistical function are used to be named sign criteria because \( z_N \) counts only the positive differences \( U_T - y_j, \ j = 1, \ldots, N \). Since we assumed that \( G(y) \) is continuous, hence the probability of the event \( \{U_T - y = 0\} \) is zero.

Obvious that \( z_N \) has binomial distribution since \( z_N \) is nothing else than the sum of \( N \) independent random variables with values either 0 or 1. By using the notation

\[
P\{\Delta(U_T - y) = 1\} = P\{y \leq U_T\} = p, \quad (10)
\]

we can write

\[
P\{z_N = j\} = \binom{N}{j} p^j (1 - p)^{N-j}, \quad (11)
\]

\[j = 0, 1, \ldots, N.\]
The task is very simple. Assume that we have a sample of size $N$ and for this sample $z_N = k \leq N$. We should determine a confidence interval $[\gamma_L(k), \gamma_U(k)]$ which covers the value $p$ with a prescribed probability $\beta$. The unknown $p$ defined by (10) is nothing else than the probability that the output variable $y$ is not larger than the technological limit $U_T$. When the lower confidence limit $\gamma_L(k)$ is near the unity, then, since $\gamma_L(k) < p$, we can state at least with probability $\beta$ that the chance of finding the output variable $y$ smaller than $U_T$ is also near the unity, and so the system operation can be regarded safe at the level $[\beta | \gamma_L(k)]$.

3.3.1 Approximate calculation

If the sample size $N > 50$, then the random variable

$$\frac{k - Np}{\sqrt{Np(1-p)}} = \zeta_k$$

has approximately standard normal distribution, where $k$ is the number of sample elements not larger than $U_T$. Let $\beta$ be the confidence level, then we can write that

$$P\{\left|\zeta_k\right| \leq u_\beta\} = P\left\{\left|\frac{k - Np}{\sqrt{Np(1-p)}}\right| \leq u_\beta\right\} = 2\Phi(u_\beta) - 1 = \beta,$$

where $\Phi(x)$ is the standard normal distribution function. This equation can be rewritten in the following form:

$$P\{\left|\zeta_k\right| \leq u_\beta\} = P\{\zeta_k^2 \leq u_\beta^2\} =$$

$$= P\{(N + u_\beta^2)(p - \gamma_L)(p - \gamma_U) \leq 0\} = \beta,$$  \hspace{0.5cm} (12)

where

$$\gamma_L = \gamma_L(k, u_\beta) = \frac{1}{N + u_\beta^2} \left[ k + \frac{1}{2}u_\beta^2 - u_\beta \sqrt{k(1-k/N) + u_\beta^2/4} \right],$$ \hspace{0.5cm} (13)

and

$$\gamma_U = \gamma_U(k, u_\beta) = \frac{1}{N + u_\beta^2} \left[ k + \frac{1}{2}u_\beta^2 + u_\beta \sqrt{k(1-k/N) + u_\beta^2/4} \right].$$ \hspace{0.5cm} (14)

\^The following elementary considerations can be found in any textbook for statistics, e.g. [6].
It is obvious that \([p - \gamma_L(k, u_\beta)][p - \gamma_U(k, u_\beta)] \leq 0\) is fulfilled only, if

\[\gamma_L(k, u_\beta) \leq p \leq \gamma_U(k, u_\beta),\]

and therefore

\[\mathcal{P}\{\mathcal{C}_k \leq u_\beta\} = \mathcal{P}\{\gamma_L(k, u_\beta) \leq p \leq \gamma_U(k, u_\beta)\} = \beta\]

(15)

where \(u_\beta\) is the root of Eq.

\[\Phi(u_\beta) = \frac{1}{2}(1 + \beta).\]

This equation shows clearly that the interval \([\gamma_L(k, u_\beta), \gamma_U(k, u_\beta)]\) covers the unknown \(p\) with probability \(\beta\).

In many cases we do not need the restriction due to the upper confidence limit. We want to know only the probability of the event \(\{\gamma_L(k, v_\beta) \leq p\}\). Since \(\mathcal{C}_k\) at fixed \(k\) is a decreasing function of \(p\), the events \(\{\mathcal{C}_k \leq v_\beta\}\) and \(\{\gamma_L(k, v_\beta) \leq p\}\) are equivalent, and so we can write

\[\mathcal{P}\{\mathcal{C}_k \leq v_\beta\} = \mathcal{P}\{\gamma_L(k, v_\beta) \leq p\} = \Phi(v_\beta) = \beta.\]

(16)

Consequently, the operation of a system can be regarded safe if the parameter \(p\) for all output variables is covered by \([\gamma_L(k, v_\beta), 1]\) with a prescribed probability \(\beta\), provided that \(\gamma_L(k, v_\beta)\) is near the unity.  

For the sake of simpler notation in the sequel \(\gamma_L(k, v_\beta)\) and \(\gamma_U(k, u_\beta)\) will be denoted by \(\gamma_L\) and \(\gamma_U\), respectively.

The event \(\{y \leq U_T\}\) belonging to the acceptance region of the sample space will be called success. Now, let us calculate the number of successes \(k\) needed in a sample of size \(N\) to ensure a fixed confidence level \(\beta\) and a given lower confidence limit \(\gamma_L\).

**Table VII.** Numbers of sample elements \(k\) in samples of size \(N = 100(10)200\) needed for the acceptance on level \(\beta = \gamma_L = 0.95\). (The approximate formula (13) has been used for calculations.)

| \(N\)  | 100  | 110  | 120  | 130  | 140  | 150  | 160  | 170  | 180  | 190  | 200  |
|--------|------|------|------|------|------|------|------|------|------|------|------|
| \(k\)  | 99   | 108  | 118  | 128  | 137  | 147  | 157  | 166  | 176  | 185  | 195  |

\(^8\)It is obvious that \(\gamma_L(k, v_\beta) \geq \gamma_L(k, u_\beta)\).
In Table VII we see the numbers of successes needed in samples of size $N = 100(10)200$ in order to reach the level $\beta = \gamma_L = 0.95$. The requirement is quite severe: if the sample size $N = 100$ one should have $k = 99$ successes!

For illustration of the method the approximate $\gamma_L < p$ values have been calculated at confidence levels $\beta = 0.90(0.01)0.99$ when the sample size $N = 100$ and the number of successes $k = 90(1)100$. The results are shown in Table VIII. It can be seen, for example, that if the event $\{y \geq U_T\}$ occurs only once, then it can be stated with probability $\beta = 0.95$ that $\gamma_L = 0.9564 < p$. It means that the appearance of "dangerous" events $\{y \geq U_T\}$ is not excluded even if the level of acceptence is better than $(0.95|0.9564)$.

**Table VIII.** Approximate $\gamma_L < p$ values calculated at confidence levels $\beta = 0.90(0.01)0.99$ for numbers of success $k = 90(1)100$. Sample size $N = 100$.

| $k$ \ $\beta$ | 0.90 | 0.91 | 0.92 | 0.93 | 0.94 | 0.95 |
|----------------|------|------|------|------|------|------|
| 90             | 0.8549 | 0.85245 | 0.8498 | 0.8469 | 0.8435 | 0.8396 |
| 91             | 0.8664 | 0.8640 | 0.8615 | 0.8586 | 0.8553 | 0.8515 |
| 92             | 0.8781 | 0.8758 | 0.8733 | 0.8704 | 0.8672 | 0.8635 |
| 93             | 0.8899 | 0.8877 | 0.8852 | 0.8825 | 0.8794 | 0.8757 |
| 94             | 0.9019 | 0.8997 | 0.8974 | 0.8947 | 0.8917 | 0.8882 |
| 95             | 0.9141 | 0.9120 | 0.9097 | 0.9072 | 0.9043 | 0.9008 |
| 96             | 0.9266 | 0.9246 | 0.9224 | 0.9200 | 0.9171 | 0.9138 |
| 97             | 0.9394 | 0.9376 | 0.9355 | 0.9331 | 0.9304 | 0.9273 |
| 98             | 0.9528 | 0.9511 | 0.9491 | 0.9469 | 0.9444 | 0.9414 |
| 99             | 0.9672 | 0.9655 | 0.9637 | 0.9617 | 0.9593 | 0.9564 |
| 100            | 0.9838 | 0.9823 | 0.9806 | 0.9787 | 0.9764 | 0.9737 |

| $k$ \ $\beta$ | 0.96 | 0.97 | 0.982 | 0.99 |
|----------------|------|------|--------|------|
| 90             | 0.8350 | 0.8292 | 0.8213 | 0.8085 |
| 91             | 0.8470 | 0.8413 | 0.8335 | 0.8208 |
| 92             | 0.8591 | 0.8535 | 0.8458 | 0.8333 |
| 93             | 0.8714 | 0.8659 | 0.8584 | 0.8460 |
| 94             | 0.8839 | 0.8786 | 0.8712 | 0.8591 |
| 95             | 0.8967 | 0.8915 | 0.8843 | 0.8724 |
| 96             | 0.9099 | 0.9048 | 0.8978 | 0.8861 |
| 97             | 0.9235 | 0.9186 | 0.9117 | 0.9003 |
| 98             | 0.9377 | 0.9330 | 0.9264 | 0.9152 |
| 99             | 0.9529 | 0.9484 | 0.9420 | 0.9311 |
| 100            | 0.9703 | 0.9658 | 0.9505 | 0.9487 |
3.3.2 Exact calculation

When the sample size $N$ is smaller than 50 we cannot apply the asymptotically valid normal distribution. For the exact calculation of confidence limits we used a slightly new version of the method proposed by Clopper and Pearson [7].

The probability of finding at least $k$ successes from $N$ observations is nothing else than

$$S_k^{(N)}(p) = \sum_{j=0}^{k} \binom{N}{j} p^j (1 - p)^{N-j},$$  \hspace{1cm} (17)

where

$$p = \mathcal{P}\{y \leq U_T\}.$$  

As known, this formula can be written in the form:

$$S_k^{(N)}(p) = \frac{N!}{k! (N - k - 1)!} \int_p^1 u^k (1 - u)^{N-k-1} \, du =$$

$$= \frac{N!}{k! (N - k - 1)!} \int_0^{1-p} (1 - v)^k v^{N-k-1} \, dv,$$  \hspace{1cm} (18)

and it is obvious, that $S_k^{(N)}(p)$ is a continuous monotone decreasing function of $p$, since

$$\frac{dS_k^{(N)}(p)}{dp} = -\frac{N!}{k! (N - k - 1)!} p^k (1 - p)^{N-k-1} < 0.$$  

Taking into account that

$$S_k^{(N)}(p) = \begin{cases} 1, & \text{if } p = 0, \\ 0, & \text{if } p = 1, \end{cases}$$

it is evident that $S_k^{(N)}(p)$ assumes any values in the interval $[0, 1]$ only once. Consequently, a $p = p_\delta$ value can be determined so that

$$S_k^{(N)}(p_\delta) = \delta, \quad \forall \, 0 < \delta < 1.$$
Since $S_k^{(N)}(p)$ is a monotone decreasing function, if $p > p_\delta$, then

$$S_k^{(N)}(p) < S_k^{(N)}(p_\delta) = \delta.$$ 

![Graph showing the dependence of the upper and lower confidence limits on the number of successes $k$ at confidence level $\beta = cl = 0.95$ in cases of sample size $N = 50$ and 100, respectively.]

Figure 6: Dependence of the upper and the lower confidence limits on the number of successes $k$ at confidence level $\beta = cl = 0.95$ in cases of sample size $N = 50$ and 100, respectively.
Clearly, the function

$$R_k^{(N)}(p) = 1 - S_{k-1}^{(N)}(p) = \sum_{j=k}^{N} \binom{N}{j} p^j (1 - p)^{N-j},$$  \hspace{1cm} (19)$$

will satisfy the inequality

$$R_k^{(N)}(p) < R_k^{(N)}(p_\delta) = \delta, \quad \text{if} \quad p < p_\delta.$$

Fixing the confidence level $\beta$ one can obtain the upper confidence limit $\gamma_U$ for the unknown parameter $p$ from $S_k^{(N)}(\gamma_U) \leq \frac{1}{2}(1 - \beta)$, while the lower confidence limit $\gamma_L$ is determined by $R_k^{(N)}(\gamma_L) \leq \frac{1}{2}(1 - \beta)$. Now one can formulate the statement that the random interval $[\gamma_L, \gamma_U]$ covers the unknown parameter $p$ with probability $\beta$.

Figure 7: Dependence of the the lower confidence limit on the number of successes $k$ on three confidence levels $\beta = cl = 0.90, 0.95, 0.99$ when the sample size $N = 100$.

For the sake of illustration Fig. 6 shows the dependence of the upper and the lower confidence limits on the number of successes $k$ on confidence level $\beta = 0.95$ in cases of sample size $N = 50$ and 100, respectively. For example, if $k = 98$, i.e. two observations out of $N = 100$ are failed, then we can state with probability 0.95 that the unknown $p$ is covered by the interval $[0.9296, 0.9975]$.

As mentioned already in many practical situations it suffices to know that the interval $[\gamma_L, 1]$ calculated from the sample of $N$ observations covers the
chance of success \( p = \mathcal{P}\{y \leq U_T\} \) with prescribed probability \( \beta \). Fig. 7 shows the dependence of the the lower confidence limit \( \gamma_L \) on the number of successes \( k \) at three confidence levels \( \beta = c_l = 0.90, 0.95, 0.99 \) when the sample size \( N = 100 \).

**Table IX.** Lower confidence limits at three levels when the number of successes \( k = 90(1)100 \). Sample size \( N = 100 \).

| \( \beta \) | 90  | 91  | 92  | 93  | 94  | 95  |
|----------------|-----|-----|-----|-----|-----|-----|
| 0.90           | 0.8501 | 0.8616 | 0.8733 | 0.8850 | 0.8970 | 0.9092 |
| 0.95           | 0.8362 | 0.8482 | 0.8602 | 0.8725 | 0.8850 | 0.8977 |
| 0.99           | 0.8086 | 0.8212 | 0.8340 | 0.8471 | 0.8604 | 0.8741 |

| \( \beta \) | 96  | 97  | 98  | 99  | 100 |
|----------------|-----|-----|-----|-----|-----|
| 0.90           | 0.9216 | 0.9344 | 0.9476 | 0.9616 | 0.9772 |
| 0.95           | 0.9108 | 0.9242 | 0.9383 | 0.9534 | **0.9704** |
| 0.99           | 0.8882 | 0.9030 | 0.9185 | 0.9354 | 0.9549 |

Table IX contains the \( \gamma_L \) values plotted in Fig. 7 for the mostly used confidence levels provided that the sample size \( N = 100 \). It is remarkable that even in that case when \( k = 100 \), i.e. when all elements of a sample can be found in the acceptance interval we can state with probability \( \beta = 0.95 \) only that the unknown \( p \) value is covered by the interval \( [0.9704, 1] \), or simply, but not precisely: the \( p \) is larger than 0.97 with probability 0.95 One can imagine a number of cases where this statement is not enough to declare: the operation of the analyzed system can be regarded safe.

### 3.4 Tolerance interval method

Assume again that we have \( N \) independent values \( y_1, \ldots, y_N \) of the output variable \( y \). Let \( \gamma \) and \( \beta \) be positive numbers not larger than 1. Now, we wish to answer the following question: *On the basis of a sample \( S_N = \{y_1, \ldots, y_N\} \) can we state that a fraction larger than \( \gamma \) of the distribution \( G(y) \) lays with probability \( \beta \) in an interval \( [L, U] \subseteq [L_T, U_T] \)?*

In order to answer this question, let us construct from the sample \( S_N \) two random functions \( L = L(y_1, \ldots, y_N) \) and \( U = U(y_1, \ldots, y_N) \), called *tolerance limits*, such that

\[
\mathcal{P}\{ \int_L^U dG(y) > \gamma \} = \beta. \quad (20)
\]
We remark that
\[ \int_L^U dG(y) = A(y_1, \ldots, y_N) \] (21)
is a random variable, sometimes called probability content, which measures the proportion of the distribution included in the random interval \([L, U]\). Probability \(\beta\) bears the name confidence level. For safe operation it is advisable to specify the probability content \(\gamma\) and the confidence level \(\beta\) as large as possible in the interval \((0, 1)\).

Having fixed \(\beta\) and \(\gamma\), from definitions of \(L(y_1, \ldots, y_N)\) and \(U(y_1, \ldots, y_N)\) it becomes possible to determine the number of runs \(N\). Carrying out \(N\) runs, we get a sample \(\{y_1, \ldots, y_N\}\), from which we can calculate an appropriate tolerance interval \([L, U]\). If that interval lies in \([L_T, U_T]\) we declare the operation safe. \(^9\) This program can be easily realized when the distribution \(G(y)\) is known and normal, however, in subsection 3.4.1 the problem of distribution free tolerance interval will be discussed.

### 3.4.1 Distribution free tolerance limits

To solve the problem of setting tolerance limits when nothing is known about the cumulative distribution function \(G(y)\) except that it is continuous, seems to be not an easy task. Exploiting advantages of the order statistics, Wilks \(^8\) was the first who found a satisfactory solution to the problem and somewhat later Robbins \(^10\) published a nice proof that distribution free tolerance limits can be given only by means of order statistics.

It is evident that in the order statistics we are unable to exploit the total amount of information which is present in the sample when the distribution function \(G(y)\) is unknown. Consequently, with \(\gamma\) and \(\beta\) given, we anticipate either a wider tolerance interval around the sample mean or a larger sample size to achieve the same tolerance interval as in the case of known \(G(y)\). Not going into details, we give here a well-known theorem, which is useful in uncertainty and sensitivity analysis of codes.

**Theorem 2** Let \(y_1, \ldots, y_N\) be \(N\) independent observations of the random output \(y\). Suppose that nothing is known about the distribution function \(G(y)\) except that it is continuous. \(^{10}\) Arrange the values of \(y_1, \ldots, y_N\) in increas-

\(^9\) Many authors have discussed the problem of setting tolerance limits for a distribution on the basis of an observed sample. The pioneering work was done by S. S. Wilks \(^8\) and by A. Wald \(^9\).

\(^{10}\) It can be shown that the one-sided continuity only is needed.
ing order, and denote by \( y(k) \) the \( k \)-th of these ordered values; hence in particular
\[
y(1) = \min_{1 \leq k \leq N} y_k, \quad y(N) = \max_{1 \leq k \leq N} y_k,
\]
and by definition \( y(0) = -\infty \), while \( y(N + 1) = +\infty \). In this case for some positive \( \gamma < 1 \) and \( \beta < 1 \) there can be constructed two random function \( L(y_1, \ldots, y_N) \) and \( U(y_1, \ldots, y_N) \), called tolerance limit, such that the probability that
\[
\int_L^U dG(y) > \gamma
\]
holds is equal to
\[
\beta = 1 - I(\gamma, s - r, N - s + r + 1) = \sum_{j=0}^{s-r-1} \binom{N}{j} \gamma^j (1-\gamma)^{N-j}, \tag{22}
\]
where
\[
I(\gamma, j, k) = \int_0^\gamma \frac{u^{j-1} (1-u)^{k-1}}{B(j,k)} du, \quad B(j,k) = \frac{(j-1)! (k-1)!}{(j+k-1)!}, \tag{23}
\]
\[
0 \leq r < s \leq N, \quad \text{and} \quad L = y(r), \quad U = y(s).
\]

The proof of Theorem 2, which is a simplified version of Wald’s proof, is given in Appendix II.

The selection of tolerance limits \( L = y(1) \) and \( U = y(N) \) appears to be expedient in many cases. Substituting \( r = 1 \) and \( s = N \) in Eq. \( 22 \), we get for the \textit{two-sided tolerance interval} the expression
\[
\beta = 1 - \gamma^N - N(1-\gamma) \gamma^{N-1}. \tag{24}
\]
Often we are interested solely in the upper tolerance limit \( U = y(N) \) and we call the interval \( [y(0), y(N)] \) \textit{one-sided tolerance} interval. Now \( r = 0 \) and \( s = N \), therefore
\[
\beta = 1 - \gamma^N. \tag{25}
\]
When the lower limit is of interest, we select \( [y(1), y(N+1)] \) and this is also a one-sided tolerance interval. Substituting \( r=1 \) and \( s=N+1 \) into expression \( 22 \), we obtain \( 25 \) again. \footnote{The probability that equal values occur is zero.}
Finally, we make two remarks. Two outputs are considered the same if their difference is smaller than the round-off error. Therefore the probability that two runs yield the same output is very small but not zero. The second remark is that expressions (24)-(25) may appear as a relationship between two probabilities $\beta$ and $\gamma$. However, $\gamma$ is not a probability, which can be seen from the nonsensical interpretation for $\gamma$ from any of the mentioned expressions. In Table X. we compiled the probability content $\gamma$ of the tolerance interval $[y(1), y(N)]$ for $\beta = 0.9, 0.95, 0.99$ and $N = 10(10)100(25)300$.

If we are interested in a tolerance interval $[L, U]$ which includes larger than $\gamma = 0.953$ proportion of the distribution of the output with probability $\beta = 0.95$, then we should make 100 runs, see Table X. and select the lowest output as $L$ and the largest as $U$. If $U$ is smaller than the technological limit $U_T$, then the system is safe at the level $\gamma = 0.953$, $\beta = 0.95$. This means that additional runs may produce an output exceeding $U$ but this portion of runs is not larger than 4.7% of the total number of runs. However, these rare output values may be greater than the technological limit $U_T$. Evidently, if $U$ is larger than $U_T$, the system must be declared unsafe.

**Table X.** $\gamma$ values of tolerance interval $[y(1), y(N)]$ for $\beta = 0.9, 0.95, 0.99$ and $N = 10(10)100(25)300$.

| N     | $\gamma$ values          |
|-------|---------------------------|
|       | $\beta = 0.90$ | $\beta = 0.95$ | $\beta = 0.99$ |
| 10    | 0.66315 | 0.60584 | 0.49565 |
| 20    | 0.81904 | 0.78389 | 0.71127 |
| 30    | 0.87643 | 0.85141 | 0.79845 |
| 40    | 0.90620 | 0.88682 | 0.84528 |
| 50    | 0.92443 | 0.90860 | 0.87448 |
| 60    | 0.93671 | 0.92336 | 0.89442 |
| 70    | 0.94557 | 0.93402 | 0.90890 |
| 80    | 0.95225 | 0.94207 | 0.91989 |
| 90    | 0.95747 | 0.94837 | 0.92851 |
| 100   | 0.96166 | 0.95344 | 0.93554 |
| 125   | 0.96924 | 0.96262 | 0.94813 |
| 150   | 0.97432 | 0.96877 | 0.95658 |
| 175   | 0.97796 | 0.97318 | 0.96268 |
| 200   | 0.98069 | 0.97650 | 0.96736 |
| 225   | 0.98282 | 0.97909 | 0.97087 |
| 250   | 0.98453 | 0.98118 | 0.97375 |
| 275   | 0.98593 | 0.98287 | 0.97618 |
| 300   | 0.98710 | 0.98429 | 0.97809 |

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Figure 8: The dependence of the probability $\beta = pr$ on the number of runs $N$ at probability contents $\gamma = pc = 0.8, 0.9, 0.95, 0.98, 0.99$.

To get some insight into relation (24) we present the probabilities $\beta$ versus $N$ for six $\gamma$ values, see Fig. 8. With increasing number of runs, each interpolated curve reaches saturation, and $\beta$ tends to unity as $N$ tends to infinity. The smaller is the $\gamma$ value the sooner comes the saturation, because small $\gamma$ means that only a small fraction of the calculated output is required to fall into the given interval.

Small $\gamma$ value is not acceptable in safety analysis for small $\gamma$ means that a large portion of output values may fall outside the tolerance interval. Practically we need $\gamma > 0.95$. For example, if we wish the tolerance interval $[L,U]$ to include larger than $\gamma = 0.98$ proportion of the output values with probability $\beta = 0.95$, we need approximately 235 runs in order to get the proper $L$ and $U$. In spite of the large number runs the probability content $\gamma = 0.98$ is far from being completely satisfactory. To achieve a better probability
content, say \( \gamma = 0.99 \) with probability \( \beta = 0.95 \), we need 473 runs, which is practically hard to realize.

### 3.4.2 Known cumulative distribution function

Let us assume the cumulative distribution function \( G(y) \) to be known. However, one should emphasize that there are situations where it would be particularly dangerous to make unwarranted assumptions about the exact shape of distribution \( G(y) \). In general, the attempt to get an explicit expression for \( \beta \) by means of expression (20) would fail. There is however one exception, when \( G(y) \) is of normal distribution \( N(m, \sigma) \) then exact formula can be obtained for \( \beta \).\(^{12}\)

We shall denote by \( \tilde{y}_N \) the sample estimate of the expectation value \( m \) and by \( \tilde{\sigma}_N^2 \) that of the variance \( \sigma^2 \), i.e.

\[
\tilde{y}_N = \frac{1}{N} \sum_{k=1}^{N} y_k, \quad \text{and} \quad \tilde{\sigma}_N^2 = \frac{1}{N-1} \sum_{k=1}^{N} (y_k - \tilde{y}_N)^2.
\]

(26)

Let us construct two random variables, viz.

\[
L = L(y_1, \ldots, y_N; \lambda) = \tilde{y}_N - \lambda \tilde{\sigma}_N \quad \text{and} \quad U = U(y_1, \ldots, y_N; \lambda) = \tilde{y}_N + \lambda \tilde{\sigma}_N,
\]

where the parameter \( \lambda \) scales the length of the interval \([L, U]\). Denote by \( A(\tilde{y}_N, \lambda \tilde{\sigma}_N) \) the proportion of the output distribution included between the limits \( L(y_1, \ldots, y_N; \lambda) = \tilde{y}_N - \lambda \tilde{\sigma}_N \) and \( U(y_1, \ldots, y_N; \lambda) = \tilde{y}_N + \lambda \tilde{\sigma}_N \), i.e.

\[
A(\tilde{y}_N, \lambda \tilde{\sigma}_N) = \int_{L}^{U} g(y) \, dy = \frac{1}{\sqrt{2\pi\sigma}} \int_{L}^{U} \exp\left[-\frac{(y-m)^2}{2\sigma^2}\right] \, dy.
\]

(27)

Introducing new variable \( z = (y-m) / \sigma \) we obtain

\[
A(m + \sigma \hat{z}_N; \lambda \tilde{\sigma}_N) = \rho(\hat{z}_N, \tilde{s}_N) = \frac{1}{\sqrt{2\pi}} \int_{\epsilon}^{u_N} e^{-z^2/2} \, dz,
\]

(28)

where

\[
\hat{z}_N = \frac{\tilde{y}_N - m}{\sigma} \quad \text{and} \quad \tilde{s}_N = \frac{\tilde{\sigma}_N}{\sigma}.
\]

\(^{12}\)It is worth mentioning that if output variable \( y \) is a sum of a large number of small, statistically independent random variable, then its distribution is almost normal. Now we discuss the case when the output variable \( y \) is of normal distribution.
while 
\[ \ell_N = z_N - \lambda \bar{s}_N \quad \text{and} \quad u_N = z_N + \lambda \bar{s}_N. \]

We stress again that \( \rho(z_N, s_N) \) is a random variable because in expression (28) the limits of the integral are random variables.

**Theorem 3** For any given positive value of \( \lambda \) the probability that \( \rho > \gamma \), where \( 0 < \gamma < 1 \) is expressed by

\[ W(\lambda, \gamma, N) = 1 - \sqrt{\frac{N}{2\pi}} \int_{-\infty}^{\infty} K_{N-1} \left( (N-1) \left( \frac{q(\mu, \gamma)}{\lambda} \right)^2 \right) e^{-N\mu^2/2} d\mu, \]  

(29)

where \( K_{N-1}[\cdots] \) is the \( \chi^2 \) distribution with \( (N-1) \)-degrees of freedom and \( q(\mu, \gamma) \) is the solution of the equation

\[ \frac{1}{\sqrt{2\pi}} \int_{-q}^{\mu+q} e^{-x^2/2} dx = \gamma. \]  

(30)

The value \( \lambda \) determining the tolerance interval\(^{13}\) at a preassigned probability content \( \gamma \) and a preassigned significance level \( \beta \) in the case of \( N \) runs can be calculated from the equation

\[ W(\lambda, \gamma, N) = \beta, \]  

(31)

and it is independent of unknown parameters \( m \) and \( \sigma \) of the distribution function \( G(y) \). The equation (31) has exactly one root in \( \lambda \), since \( W(\lambda, \gamma, N) \) is a strictly increasing function of \( \lambda \).

Proof of Theorem 3 is given in Appendix III, since the mathematical details are not relevant to the aim of the present work. However, it is worth mentioning that an approximate tolerance interval can be derived when \( N \) is large (e.g. \( N > 50 \)).

\(^{13}\)If one-sided tolerance interval with upper limit is needed, then Eq. (30) should be replaced by

\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\mu+q} e^{-x^2/2} dx = \gamma. \]
Theorem 4  The approximate two-sided tolerance interval is given by

\[ [\tilde{y}_N - \lambda_a(\gamma, \beta) \tilde{\sigma}_N, \tilde{y}_N + \lambda_a(\gamma, \beta) \tilde{\sigma}_N], \]

where

\[ \lambda_a(\gamma, \beta) = \sqrt{\frac{N - 1}{Q_{N-1}(1-\beta)}} q(1/\sqrt{N}, \gamma). \]  (32)

Here \( Q_{N-1}(1-\beta) \) is \((1-\beta)\)-percentile of the \(\chi^2\) distribution with \((N-1)\) degree of freedom and \(q(1/\sqrt{N}, \gamma)\) is the root of the equation

\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{1/\sqrt{N}+q} e^{-z^2/2} \, dz = \gamma. \]  (33)

The \(\lambda_a\) for the approximate one-sided tolerance interval with upper limit can be calculated in the same way, but Eq. (33) has to be replaced by

\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{1/\sqrt{N}-q} e^{-z^2/2} \, dz = \gamma. \]

Proof of Theorem 4 is given in Appendix IV.

Table XI. \(\lambda\) values of two-sided tolerance intervals for the number of runs \(N=50(5)100\)

| \(N\) | \(\beta = 0.90\) | \(\beta = 0.95\) | \(\beta = 0.99\) |
|-------|-----------------|-----------------|-----------------|
| \(N\) | 0.90 | 0.95 | 0.99 | 0.90 | 0.95 | 0.99 | 0.90 | 0.95 | 0.99 |
| 50    | 1.916 | 2.284 | 3.001 | 1.996 | 2.379 | 3.126 | 2.162 | 2.576 | 3.385 |
| 55    | 1.901 | 2.265 | 2.976 | 1.976 | 2.354 | 3.093 | 2.130 | 2.538 | 3.335 |
| 60    | 1.887 | 2.248 | 2.956 | 1.958 | 2.333 | 3.066 | 2.103 | 2.506 | 3.293 |
| 65    | 1.875 | 2.234 | 2.936 | 1.943 | 2.315 | 3.042 | 2.080 | 2.478 | 3.257 |
| 70    | 1.865 | 2.222 | 2.920 | 1.929 | 2.299 | 3.021 | 2.060 | 2.454 | 3.225 |
| 75    | 1.856 | 2.211 | 2.906 | 1.917 | 2.285 | 3.002 | 2.042 | 2.433 | 3.197 |
| 80    | 1.848 | 2.202 | 2.894 | 1.907 | 2.272 | 2.986 | 2.026 | 2.414 | 3.173 |
| 85    | 1.841 | 2.193 | 2.882 | 1.897 | 2.261 | 2.971 | 2.012 | 2.397 | 3.150 |
| 90    | 1.834 | 2.185 | 2.872 | 1.889 | 2.251 | 2.958 | 1.999 | 2.382 | 3.130 |
| 95    | 1.828 | 2.178 | 2.862 | 1.881 | 2.241 | 2.945 | 1.987 | 2.368 | 3.112 |
| 100   | 1.822 | 2.172 | 2.854 | 1.874 | 2.233 | 2.934 | 1.977 | 2.355 | 3.096 |

In order to give an impression of \(\lambda\) values (i.e. of the tolerance interval around the sample mean of the output variable), Table XI. contains the \(\lambda\)
values\textsuperscript{14} associated with often used $\gamma$ and $\beta$ for the sample sizes $N=50(5)100$. One can see that at $N=100$ the tolerance interval which includes 95\% of the distribution with 95\% probability is given by

$$[ar{y}_{100} - 2.23\bar{s}_{100}, \bar{y}_{100} + 2.23\bar{s}_{100}] .$$

If that interval\textsuperscript{15} lies within $[L_T, U_T]$ then the system is safe on level $\gamma = 0.95$ and $\beta = 0.95$.

Fig. 9 shows convincingly the interrelations between the basic characteristics of the tolerance intervals for a normal distribution. As expected the confidence level $\beta$ increases with increasing sample size $N$ provided that the coverage $pc = \gamma$ and the interval parameter $ip = \lambda$ are fixed.

However, if the fixed coverage $\gamma$ exceeds a critical value $\gamma_{crt} \approx 0.98758$ when $\lambda = ip = 2.5$, then one can observe an "anomalous" behavior of the dependence $\beta$ on $N$, as shown in Fig. 10. It is seen that the probability $\beta$ of finding the proportion $\gamma > \gamma_{crt}$ of the distribution $G(y)$ in the interval $(\bar{z}_N - \lambda \bar{s}_N, \bar{z}_N + \lambda \bar{s}_N)$ decreases with increasing sample size $N > N_{crt}$, where $N_{crt}$ depends on both $\lambda$ and $\gamma$. The explanation is straightforward:

\textsuperscript{14}More detailed tables can be found in [11].

\textsuperscript{15}If one-sided tolerance interval with upper limit is needed, then $\lambda = 2.23$ has to be replaced by $\lambda = 1.75$!
Figure 10: Dependence of the confidence level $\beta$ on the sample size $N$ at probability contents larger than the critical value $\gamma_{crt} \approx 0.98758$ and provided the interval parameter $\lambda = ip = 2.5$ is fixed.

Figure 11: Dependence of the confidence level $\beta$ on the interval parameter $\lambda = ip$ at probability content $\gamma = pc = 0.95$ for three sample sizes $N = 40, 50, 60$.

Since
\[
\lim_{N \to \infty} \bar{z}_N \xrightarrow{p} 0 \quad \text{and} \quad \lim_{N \to \infty} \bar{s}_N \xrightarrow{p} 1,
\]
Figure 12: Dependence of the confidence level $\beta$ on the interval parameter $\lambda = ip$ at probability content $\gamma = pc = 0.99$ higher than the critical value for three sample sizes $N = 40, 50, 60$.

It is evident that

$$\lim_{N \to \infty} \rho (\tilde{z}_N, \tilde{s}_N) \overset{p}{=} \gamma_{crt},$$

where $\gamma_{crt} = \frac{1}{\sqrt{2\pi}} \int_{-\lambda}^{+\lambda} e^{-x^2/2} dx$,

consequently, if $\gamma = \gamma_{crt} + \delta$, where $0 < \delta < 1 - \gamma_{crt}$, then

$$\lim_{N \to \infty} P\{|\rho (\tilde{z}_N, \tilde{s}_N) - \gamma_{crt}| > \delta\} = 0,$$

i.e. $P\{|\rho (\tilde{z}_N, \tilde{s}_N) - \gamma_{crt}| > \delta\}$ is a monotonously decreasing function of $N > N_{crt}$. It is easy to show that $^{16}$

$$P\{|\rho (\tilde{z}_N, \tilde{s}_N) - \gamma_{crt}| > \delta\} > P\{\rho (\tilde{z}_N, \tilde{s}_N) > \gamma_{crt} + \delta\} = \beta,$$

$^{16}$Introducing the notations:

$$\{\rho (\tilde{z}_N, \tilde{s}_N) \leq \gamma_{crt} - \delta\} = A_N^{(-)}$$

and

$$\{\rho (\tilde{z}_N, \tilde{s}_N) > \gamma_{crt} + \delta\} = A_N^{(+)},$$

and taking into account that $A_N^{(+)} \cap A_N^{(-)} = \emptyset$, we can write that

$$P\{|\rho (\tilde{z}_N, \tilde{s}_N) - \gamma_{crt}| > \delta\} = P\{A_N^{(+)} \cup A_N^{(-)}\} > P\{\rho (\tilde{z}_N, \tilde{s}_N) > \gamma_{crt} + \delta\}.$$
and so one can state that $\beta$ decreases with increasing $N > N_{\text{crt}}$ if $\gamma > \gamma_{\text{crt}}$ provided $\lambda$ is fixed.

It is not superfluous to know how does the confidence level $\beta$ depend on the interval parameter $\lambda$ at a fixed probability content (coverage) $\gamma$ and at a given sample size $N$. Fig. 11 shows this dependence at $\gamma = pc = 0.95$ for three sample sizes $N = 40, 50, 60$. What we see completely corresponds to our expectations, however, as seen in Fig. 12, the character of $\beta$ vs. $\lambda$ curves is radically changing. The explanation is the same as in the case of Fig. 10.

4 Several output variables

Now we assume the output to comprise $n$ variables. Let these variables be $y_1, \ldots, y_n$. If they are statistically completely independent\(^{17}\) we can apply the results of previous Sections, otherwise we need new considerations. Let $G(y_1, \ldots, y_n)$ be the unknown joint cumulative distribution function of the output variables, furthermore, let

\[
\mathbf{S}_N = \begin{pmatrix}
y_{11} & y_{12} & \cdots & y_{1N} \\
y_{21} & y_{22} & \cdots & y_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
y_{n1} & y_{n2} & \cdots & y_{nN}
\end{pmatrix}
\]

be the sample matrix obtained in $N >> 2n$ independent observations (runs). Introducing the $n$-components vector

\[
\bar{y}_k = \begin{pmatrix}
y_{1k} \\
y_{2k} \\
\vdots \\
y_{nk}
\end{pmatrix},
\]

the sample matrix can be written in the form:

\[
\mathbf{S}_N = \{\bar{y}_1, \ldots, \bar{y}_N\}.
\]

By using proper statistical methods for testing the sample matrix we can make useful probabilistic statement about the safety of system operation.

\(^{17}\)There are many fairly good statistical tests to prove the independence of random variables.
First, we will show how to generalize the method of \textit{sign test for several output variables}, and then we will deal with the problem of setting \textit{tolerance limits for more than one random variable}.

\subsection{Sign test}

For the sake of simplicity we are going to deal with two output variables \( y_1 \) and \( y_2 \) provided their joint distribution function \( G(y_1, y_2) \) is unknown, but continuous at least from right (or from left) in both variables. Let us accept that the system operation can be declared safe if the requirement \( \{ y_1 < U^{(1)}_T, y_2 < U^{(2)}_T \} \) is realized with probability

\begin{equation}
    p_{12} = \mathcal{P}\{ y_1 < U^{(1)}_T, y_2 < U^{(2)}_T \}
\end{equation}

near the unity. Here \( U^{(1)}_T \) and \( U^{(2)}_T \) are the limit values defined by technology, and they define the \textit{acceptance region} of the \((y_1, y_2)\) plane. Since the \( p_{12} \) is unknown, the task is to construct from the sample a confidence interval \([\gamma^{(1,2)}_L, \gamma^{(1,2)}_U]\) which covers the \( p_{12} \) with probability \( \beta_{12} \). In most of the cases it is sufficient to calculate the \( \gamma^{(1,2)}_L \) only and to use the interval \([\gamma^{(1,2)}_L, 1]\) as confidence interval. Let the 2-components vectors

\[ \vec{y}_k = \begin{pmatrix} y_{1k} \\ y_{2k} \end{pmatrix}, \quad k = 1, \ldots, N \]

be elements of a sample \( \mathcal{S}_N \) obtained by \( N \) independent observations. One should emphasize that \( \vec{y}_j \) and \( \vec{y}_k \) are independent if \( j \neq k \), but the components of a given sample vector are not.

In order to use a terminology as simple as possible, the event \( \{ y_1 < U^{(1)}_T, y_2 < U^{(2)}_T \} \) will be called success. Define now the function

\[ \Delta \left( U^{(1)}_T - y_{1k} \right) \Delta \left( U^{(2)}_T - y_{2k} \right) = \begin{cases} 1, & \text{if } y_{1k} < U^{(1)}_T \text{ and } y_{2k} < U^{(2)}_T, \\ 0, & \text{otherwise}, \end{cases} \]

and introduce the statistical function

\begin{equation}
    z^{(1,2)}_N = \sum_{k=1}^{N} \Delta \left( U^{(1)}_T - y_{1k} \right) \Delta \left( U^{(2)}_T - y_{2k} \right),
\end{equation}

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which gives the number of successes in a sample of size \( N \). Since \( z^{(1,2)}_N \) is the sum of \( N \) independent random variables with values either 1 or 0, it is obvious that \( z^{(1,2)}_N \) is of binomial distribution. By using the notation

\[
\mathcal{P}\{\Delta \left(U_T^{(1)} - y_1\right) \Delta \left(U_T^{(2)} - y_2\right) = 1\} = \\
= \mathcal{P}\{y_1 < U_T^{(1)}, y_2 < U_T^{(2)}\} = p_{12},
\]

we can write

\[
\mathcal{P}\{z^{(1,2)}_N = k\} = \binom{N}{k} p_{12}^k (1 - p_{12})^{N-k}, \quad \forall \ k = 0, 1, \ldots, N,
\]

and this is the point where we can use from the results of Subsection 3.3.

Now, we would like to make a trivial but important amendment. Define two statistical functions:

\[
z^{(1)}_N = \sum_{i=1}^{N} \Delta \left(U_T^{(1)} - y_{1i}\right) \quad \text{and} \quad z^{(2)}_N = \sum_{j=1}^{N} \Delta \left(U_T^{(2)} - y_{2j}\right).
\]

Clearly, \( z^{(1)}_N \) and \( z^{(2)}_N \) are not independent, but both of them are sum of \( N \) independent random variables with values either 1 or 0, consequently one can write

\[
\mathcal{P}\{z^{(1)}_N = i\} = \binom{N}{i} p_{1}^i (1 - p_{1})^{N-i}
\]

and

\[
\mathcal{P}\{z^{(2)}_N = j\} = \binom{N}{j} p_{2}^j (1 - p_{2})^{N-j},
\]

\( i, j = 1, \ldots, N, \)

where

\[ p_{\ell} = \mathcal{P}\{y_{\ell} < U_T^{(\ell)}\} = \mathcal{P}\{\Delta \left(U_T^{(\ell)} - y_{\ell}\right) = 1\}, \]

\( \ell = 1, 2, \)

are unknown probabilities. By using the samples \( S^{(1)}_N = \{y_{1i}, \ i = 1, \ldots, N\} \) and \( S^{(2)}_N = \{y_{2j}, \ j = 1, \ldots, N\} \) separately with help of the method described in Subsection 3.3 we can construct two random intervals \([\gamma^{(1)}_L, 1]\) and \([\gamma^{(2)}_L, 1]\) covering \( p_1 \) as well as \( p_2 \) with probabilities \( \beta_1 \) and \( \beta_2 \), respectively.
Obviously it could be occurred that the levels \((\beta_1|\gamma_{L}^{(1)})\) and \((\beta_2|\gamma_{L}^{(2)})\) support the statement that the samples \(S_{N}^{(1)}\) and \(S_{N}^{(2)}\) separately do not contradict to the requirement of safe operation, however, from this one cannot conclude that the operation of the system is safe on a preassigned level for variables \(y_1\) and \(y_2\) tested jointly. The reason is clear: the output variables \(y_1\) and \(y_2\) are not independent, and in this case we have to know weather the value \(p_{12} = \mathcal{P}\{y_1 < U_T^{(1)}, y_2 < U_T^{(2)}\}\) is covered by the interval \([\gamma_{L}^{(1,2)}, 1]\) with a preassigned probability \(\beta_{12}\). Clearly, \(\gamma_{L}^{(1,2)} \leq \min\{\gamma_{L}^{(1)}, \gamma_{L}^{(2)}\}\), therefore \(\gamma_{L}^{(1)}\) and \(\gamma_{L}^{(2)}\) do not contain sufficient information to declare that the operation of the system is safe. The procedure should be as follows: firstly test the hypothesis that the output variables \(y_1\) and \(y_2\) are dependent, and if this is the case, estimate the probability of the event \(\{y_1 < U_T^{(1)}, y_2 < U_T^{(2)}\}\), and not the events \(\{y_1 < U_T^{(1)}\}\) and \(\{y_2 < U_T^{(2)}\}\) separately.

Finally, we would like to note that the generalization of the sign test for \(n > 2\) output variables is straightforward: we have to use the statistical function

\[
z_{N}^{(1,\ldots,n)} = \sum_{k=1}^{N} \prod_{j=1}^{n} \Delta \left( U_{T}^{(j)} - y_{jk} \right),
\]

in order to obtain the sum of \(N\) independent random variables, and then the further steps will be the same as they were in Subsection 3.3.

4.1.1 Illustration

Now we want to present an example to show how the sign test method is working. By using Monte Carlo simulation we have generated two samples \(a\) and \(b\). Both are consisting of \(N = 100\) value pairs due to the population of a bivariate normal distribution with parameters \(m_1 = m_2 = 0\) and \(\sigma_1 = \sigma_2 = 1\), but the correlation coefficient is \(C = 0.1\) in \(a\), while \(C = 0.7\) in \(b\).

One can see in Fig. 13 that in the sample \(a\) four, while in the \(b\) two observations out of \(N = 100\) can be found in rejection region.

From Table IX. one can read that in the case of sample \(a\) the interval \([0.9108, 1]\) covers the parameter \(p_{12}\) with probability \(\beta_{12} = 0.95\), at the same time both \(p_1\) and \(p_2\) are covered by the interval \([0.9383, 1]\) with \(\beta_1 = \beta_2 = 0.95\). The level \((0.9383|0.95)\) is not "very good", but better than \((0.9108|0.95)\), however, in the decision about the safety one should take evidently into account the level calculated for the parameter \(p_{12}\), and not those calculated separately for \(p_1\) and \(p_2\).
Figure 13: Sample vectors denoted by points in the sample plane \((y_1, y_2)\) and the acceptance region defined by technological requirements. Upper figure (sample a) and lower figure (sample b) refer to correlation coefficients \(C = 0.1\) and \(C = 0.7\), respectively.

Testing the sample b which shows a strong correlation between the variables \(y_1\) and \(y_2\), we find that the confidence interval \([0.9383, 1]\) covers the parameter \(p_{12}\) with probability \(\beta_{12} = 0.95\). Consequently, we can state with probability 0.95 that the chance of the event \(\{y_1 < U_T^{(1)}, y_2 < U_T^{(2)}\}\) is higher...
than the value 0.9383, i.e. we are able to declare that the operation of the system is safe on the level (0.95|0.9383) only. The parameters $p_1$ and $p_2$ are covered by intervals $[0.9534, 1]$ and $[0.9383, 1]$, respectively, with the prescribed probability $\beta_1 = \beta_2 = 0.95$, however, these values are not informative for the safety of the system.

This simple example shows convincingly that the tests performed separately on output variables which are depending on one another could bring about false decision concerning the safety of the system operation.

4.2 Tolerance region

The problem of setting tolerance limits for output variables $y_1, \ldots, y_n$ can be formulated as follows. Assume that the unknown joint distribution function $G(y_1, \ldots, y_n)$ is absolute continuous, i.e. it has a joint density function $g(y_1, \ldots, y_n)$. For some given positive values $\gamma < 1$ and $\beta < 1$ we have to construct $n$ pairs of random variables $L_j(y_1, \ldots, y_n)$ and $U_j(y_1, \ldots, y_n)$ $j = 1, \ldots, n$ such that the probability that

$$\int_{L_1}^{U_1} \ldots \int_{L_n}^{U_n} g(y_1, \ldots, y_n) dy_1 \ldots dy_n > \gamma,$$  

(37)

holds is equal to $\beta$. A natural extension of the procedure applied previously to the one variable case would seem the right selection. Unfortunately that choice does not provide the required solution since the probability of the inequality (37) depends on the unknown joint density function $g(y_1, \ldots, y_n)$. Our task is to find a reasonable procedure such that the probability $\beta$ is independent of $g(y_1, \ldots, y_n)$. It can be shown that such a procedure exists but its uniqueness has not been proven yet.

Since the distribution function $G(y_1, \ldots, y_n)$ is continuous, we can state that no two elements of the sample matrix $S_N$ are equal. The sequence of rows in the sample matrix $S_N$ can be arbitrary, reflecting the fact that we number the output variables arbitrarily.

Let us choose the first row of the sample matrix, and arrange its elements in order of increasing magnitude $y_1(1), y_1(2), \ldots, y_1(N)$. Select from these $y_1(r_1)$ as $L_1$ and $y_1(s_1) > y_1(r_1)$ as $U_1$. Let $i_1, i_2, \ldots, i_{s_1-r_1-1}$ stand for the original column indices of elements $y_1(r_1+1), y_1(r_1+2), \ldots, y_1(s_1-1)$. In the next step, choose the second row, the $N$ observed values of the output variable $y_2$ and arrange the part $y_{2i_1}, y_{2i_2}, \ldots, y_{2i_{s_1-r_1-1}}$ of its elements in
increasing order to obtain \( y_2(1) < y_2(2) < \cdots < y_2(s_1 - r_1 - 1) \). From among these, \( y_2(r_2) \) and \( y_2(s_2) > y_2(r_2) \) are selected for \( L_2 \) and \( U_2 \) and evidently \( r_2 \geq r_1, \ s_2 \leq s_1 - r_1 - 1 \). We continue this imbedding procedure to the last row of the sample matrix and define a \( n \)-dimensional volume 18

\[
\mathcal{V}_n = \{ [L_1, U_1] \times [L_2, U_2] \times \cdots \times [L_n, U_n] \},
\]

where

\[
L_j = y_j(r_j), \quad U_j = y_j(s_j),
\]

and

\[
r_j \geq r_{j-1} \geq \cdots \geq r_1,
\]

while

\[
r_j < s_j \leq s_{j-1} - r_{j-1} - 1, \quad \forall \ j = 2, \ldots, n.
\]

**Theorem 5** In the case of \( n \geq 2 \) dependent output variables with continuous joint distribution function \( G(y_1, \ldots, y_n) \) it is possible to construct \( n \)-pairs of random intervals \([L_j, U_j], \ j = 1, \ldots, n\) such that the probability of the inequality

\[
\int_{L_1}^{U_1} \cdots \int_{L_n}^{U_n} g(y_1, \ldots, y_n) \ dy_1 \cdots dy_n > \gamma,
\]

is free of \( g(y_1, \ldots, y_n) \) and is given by

\[
\Pr \left\{ \int_{L_1}^{U_1} \cdots \int_{L_n}^{U_n} g(y_1, \ldots, y_n) \ dy_1 \cdots dy_n > \gamma \right\} =
\]

\[
= 1 - I(\gamma, s_n - r_n, N - s_n + r_n + 1) = \beta. \tag{38}
\]

Here function \( I(\cdots) \) is the regularized incomplete beta-function and

\[
s_n \leq s_{n-1} - r_{n-1} - 1 \leq s_1 - \sum_{j=1}^{n-1} (r_j + 1) \quad \text{and} \quad r_n \geq r_{n-1} \geq \cdots \geq r_1. \tag{39}
\]

Proof of Theorem 5 is given in Appendix V.

---

18This \( n \)-dimensional volume is the tolerance region which is nothing else than a subspace of an \( n \)-dimensional Euclidian space.
4.2.1 Illustrations

In several practical applications the choice \( r_1 = r_2 = \cdots = r_n = 1 \) and \( s_n = N - 2(n - 1) \) can be advised, hence the confidence level \( \beta \) for a two-sided tolerance region is given by

\[
\beta = 1 - I (\gamma, N - 2n + 1, 2n) = \sum_{j=0}^{N-2n} \binom{N}{j} \gamma^j (1 - \gamma)^{N-j}.
\]

(40)

The structure of expression (40) is remarkably similar to that of expression (22), which refers to the one output variable case. Furthermore, if the lower limits \( L_j = -\infty, \quad \forall \ j = 1, \ldots, n \), i.e. if

\[
r_1 = r_2 = \cdots = r_n = 0 \quad \text{and} \quad s_n = N - n + 1,
\]

then one obtains the confidence level

\[
\beta = 1 - I (\gamma, N - n + 1, n) = \sum_{j=0}^{N-n} \binom{N}{j} \gamma^j (1 - \gamma)^{N-j}
\]

(41)

for one-sided tolerance region.

In many practical cases it is sufficient to use one-sided tolerance regions (limited from above). If \( n = 2 \), i.e. if two mutually dependent output variables \( y_1 \) and \( y_2 \) are tested, then from (41) one obtains

\[
\beta = 1 - \gamma^N - N(1 - \gamma)\gamma^{N-1}
\]

(42)

which is exactly the same as (24) derived for the two-sided tolerance interval for one output variable. Here, it is worthwhile to cite two sentences from [13]. 

"There are several ways to interpret even a simple mathematical formula. The problem under consideration decides which interpretation we need. Notwithstanding, we should carefully prove the appropriateness of the interpretation chosen."

Perhaps, it is not superfluous to show how to determine the two-dimensional one-sided tolerance region for output variables \( y_1 \) and \( y_2 \). First, calculate from (41) the number of observations \( N \) needed for the preassigned safety level \( (\beta | \gamma) \). Secondly, create the sample

\[
\mathcal{S}_N = \left( \begin{array}{c} y_{11}, y_{12}, \cdots, y_{1N} \\ y_{21}, y_{22}, \cdots, y_{2N} \end{array} \right)
\]
and arrange the elements of the first row in increasing order. We obtain the matrix
\[
\begin{pmatrix}
y_1(1), & y_1(2), & \ldots, & y_1(N) \\
y_2(1), & y_2(2), & \ldots, & y_2(N)
\end{pmatrix},
\]
and choose the element \(y_1(N)\) as upper limit for \(y_1\), i.e. \(U_1 = y_1(N)\). Thirdly, search the largest element in the series \(y_2(1), y_2(2), \ldots, y_2(N)\) that gives the upper limit for \(y_2\), i.e. \(U_2 = \max_{1 \leq j \leq N-1} y_{2j}\), and finally, construct the region \([-\infty, U_1] \times [-\infty, U_2]\) which is tolerance region of variables \((y_1\) and \(y_2)\).

Clearly, if \(U_1 < U_1^{(1)}\) and \(U_2 < U_2^{(2)}\), then we can state that the operation of the system is safe on the level \((\beta|\gamma)\) for the jointly tested two variables \((y_1\) and \(y_2)\).

In order to compare the number of runs needed to determine two-sided tolerance regions at a given \((\beta|\gamma)\) level for \(n = 1, 2,\) and 3 mutually dependent output variables with unknown distributions, we compiled Table XII. In order to achieve the usual safety level \((0.95|0.95)\) we need \(N = 153\) observations in the case of two and \(N = 207\) observations in the case of three output variables. The number of observations (runs) needed to meet stringent requirement, e.g. with three output variables the level \((\gamma = 0.98|\beta = 0.98)\) we need \(N = 598\) runs. Therefore it seems to be inevitable to seek methods with lower computational demands.

Table XII. Number of runs needed to determine the two-sided tolerance region for \(n = 1, 2, 3\) output variables at listed \(\gamma, \beta\) values.

| \(\beta|\gamma\) | 0.95 | 0.96 | 0.97 | 0.98 | 0.99 | \(n\) |
|------------------|------|------|------|------|------|------|
| 0.95             | 93   | 117  | 156  | 235  | 473  | 1    |
|                  | 153  | 191  | 256  | 385  | 773  | 2    |
|                  | 207  | 260  | 348  | 523  | 1049 | 3    |
| 0.96             | 98   | 123  | 165  | 249  | 499  | 1    |
|                  | 159  | 200  | 267  | 402  | 806  | 2    |
|                  | 215  | 269  | 360  | 542  | 1086 | 3    |
| 0.97             | 105  | 132  | 176  | 266  | 533  | 1    |
|                  | 167  | 210  | 281  | 422  | 848  | 2    |
|                  | 224  | 281  | 376  | 565  | 1134 | 3    |
| 0.98             | 114  | 143  | 192  | 289  | 581  | 1    |
|                  | 179  | 224  | 300  | 451  | 905  | 2    |
|                  | 237  | 297  | 397  | 598  | 1199 | 3    |
| 0.99             | 130  | 163  | 218  | 329  | 661  | 1    |
|                  | 197  | 248  | 331  | 499  | 1001 | 2    |
|                  | 258  | 324  | 433  | 651  | 1307 | 3    |

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In order to provide some insight, let us consider the following example. We have two output variables \( y_1 \) and \( y_2 \), their the joint distribution function is known:

\[
g(y_1, y_2) = \frac{1}{2\pi \sqrt{1 - C^2}} \exp \left[ -\frac{1}{2(1 - C^2)} \left( y_1^2 - 2Cy_1y_2 + y_2^2 \right) \right], \tag{43}
\]

where \(|C| \leq 1\) is the correlation coefficient of variables \( y_1 \) and \( y_2 \). We are interested in the relationship between the significance level \( \beta \) and probability content of a given two dimensional region \([L, U] = [L_1, U_1] \times [L_2, U_2]\) at the number of runs \( N = 50(50)200\).

### Table XIII. Levels of significance \( \beta \) of two-sided tolerance regions for two output variables at listed \( \gamma \) and \( N \) values.

| \( N \) \( \backslash \) \( \gamma \) | 0.95  | 0.96  | 0.97  | 0.98  | \( C = 0.1 \) | \( C = 0.9 \) | \( \text{DF} \) |
|---|---|---|---|---|---|---|---|
| 50 | 0.8831 | 0.7547 | 0.5351 | 0.2376 | \( C = 0.1 \) | \( C = 0.9 \) | \( \text{DF} \) |
|    | 0.9433 | 0.8775 | 0.7442 | 0.4970 | \( C = 0.9 \) | \( \text{DF} \) |    |
|    | 0.2396 | 0.1391 | 0.0628 | 0.0178 | \( \text{DF} \) |    |    |
| 100| 0.9109 | 0.8836 | 0.6297 | 0.2121 | \( C = 0.1 \) | \( C = 0.9 \) | \( \text{DF} \) |
|    | 0.9911 | 0.9590 | 0.8488 | 0.4970 | \( C = 0.9 \) | \( \text{DF} \) |    |
|    | 0.7422 | 0.5705 | 0.3528 | 0.1410 | \( \text{DF} \) |    |    |
| 150| 0.9933 | 0.9443 | 0.6871 | 0.1894 | \( C = 0.1 \) | \( C = 0.9 \) | \( \text{DF} \) |
|    | 0.9981 | 0.9869 | 0.9044 | 0.5554 | \( C = 0.9 \) | \( \text{DF} \) |    |
|    | 0.9452 | 0.8542 | 0.6616 | 0.3528 | \( \text{DF} \) |    |    |
| 200| 0.9986 | 0.9683 | 0.7380 | 0.1612 | \( C = 0.1 \) | \( C = 0.9 \) | \( \text{DF} \) |
|    | 0.9998 | 0.9955 | 0.9414 | 0.5779 | \( C = 0.9 \) | \( \text{DF} \) |    |
|    | 0.9910 | 0.9605 | 0.8528 | 0.5685 | \( \text{DF} \) |    |    |

Now we can proceed in two ways. The first way is to fix the fraction of the samples to fall into the given interval \([L, U]\), and to determine the associated probability \( \beta \), from Eq. (41), these numbers are in row \( \text{DF} \) (referring to Distribution Free). The second way is to use the known joint distribution function, calculate the estimates of variances \( \bar{\sigma}_i \) for \( i = 1, 2 \) from \( N \) runs and define the interval \([L_i, U_i] = [-2.5\bar{\sigma}_i, +2.5\bar{\sigma}_i]\). From \( 10^5 \) random cases we estimated the \( \beta \) value, see Table XIV. These values are given for two correlation coefficient in rows \( C = 0.1 \) and \( C = 0.9 \).

As we see in Table XIII, the order statistics gives lower \( \beta \) values, in most of the cases, compared to those obtained by using the density function.
This indicates a considerable gain from a known distribution function of output variables.

In order to visualize the dependence of the confidence level \( \beta \) on probability content \( \gamma < \gamma_{cr} \) three curves are shown in Fig. 14 when \( N = 100 \). The two upper curves correspond to the known bivariate normal distribution of \( y_1 \) and \( y_2 \) with \( \lambda = 2.5 \), while the curve denoted by DF refers to the distribution free case.

5 Safety Inference

The purpose of performing safety analysis is to assure that the designed equipment can be operated safely. It is a self-understanding premise that by altering input data randomly within their prescribed distribution all the states will be either safe or unsafe. If both safe and unsafe states would occur, the entire range under consideration should be regarded as unsafe.

\[^{19}\text{Without going into details, we mention only there exists a critical value } \gamma_{cr}(C) \text{ such that for } \gamma > \gamma_{cr}(C) \text{ for all } N_1 < N_2 \text{ we have } \beta(\gamma, N_1) > \beta(\gamma, N_2). \text{ The critical value } \gamma_{cr}(C) \text{ is defined by the integral } \int_{\lambda}^{\lambda} \int_{\lambda}^{\lambda} g(y_1, y_2, C) \, dy_1 \, dy_2, \text{ similarly to that proved in one-dimensional case. See sub-subsection 3.4.2! The decrease of } \beta \text{ with increasing } N \text{ can be so large for } N > 100 \text{ that the } \beta \text{ becomes smaller than the value obtained from the order statistics. This is the case for some } \beta \text{ values with } C = 0.1 \text{ in Table XIII, when } N > 100 \text{ and } \gamma > \gamma_{cr}(0.1) = 0.9753.\]

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Our approach has severe consequences on every statement concerning safety. The present section assesses those consequences. The first consequence is that we can not speak of safety of a given state, rather we can speak of the probability of a given state to be safe. Assume for the sake of simplicity that the model is not chaotic around the nominal state \((\vec{x}_0, \vec{y}_0)\) where \(\vec{y}_0 = \hat{C}\vec{x}_0\). The input variable(s) may take values in a given range, that range is mapped into a range of the output variables. From some other considerations, which thus far have not been regarded as part of safety analysis, we get information on the probability distribution of the input variable. And, we select a range into which a large portion, say more than 90%, of the possible area lies with a given high, say 95%, probability. Consequently, we conclude that:

1. It is insufficient to show that the nominal state is safe because there may be probable inputs, which are unsafe. Therefore, when the calculations are carried out exclusively in the nominal state, safety analysis should demonstrate the estimated error to be realistic.

2. Another possibility is that safety analysis should show that images of all \(x\) points in the vicinity of \(x_0\) are safe. In this case we get rid of the uncertainty caused by input uncertainty.

3. Assumptions or knowledge of probability distributions of the input do have an impact on safety issues, therefore they must not be treated separately. Here two problems occur. Engineering input data are usually not accompanied by probability distributions and input variables, which are actually internal in the given calculational model, may influence the output error decisively. Such internal inputs are usually obtained from a fitting but that procedure usually gives no information on the probability distribution of fitted parameters (although theory and technique are known).

4. Even if every \(\hat{C}\vec{x}\) is safe for a given interval, there is a slight chance that some input(s) may be associated with unsafe output(s). Those chances can be read out from Table XI. for normally distributed output and from Tables IX. and X. for a single arbitrarily distributed output variable, as well as from Table XII. for two and three arbitrarily distributed output variables.
5. Safety is described by random variables therefore we can make only statistical assertions. Any claim concerning safety is associated with \((\beta, \gamma)\) and the assumptions on the probability distribution(s) of the input variables. An alarming example is given in sub-subsection 3.2.3 where we see that 22.4\% of the rejected calculations result in larger maxima than in the basic sample.

6. Biased probability density functions seem to be extremely dangerous. The simple problem of determining a quantile (see Fig. 4) may lead to large differences. Based on the presented examples it seems desirable to treat certain class of distributions of the output variables with special care.

7. Safety analysis should make it clear that every output interval lies inside the safety envelope. The safety is not unconditional but the values \(\beta\) and \(\gamma\) characterize that ”level” of safety. When any of inequalities \(y_k(N) \geq U_T^{(k)},\) \(k = 1, \ldots, n\) would be observed, then the system operation could hardly be declared safe. This clearly indicates that safety is not deterministic, as treated by many, but random.

Since the general consideration results in loss of a large amount of information, efforts should be made to chose a reasonable test for the estimation of the probability distribution of the output variable(s). To this end specific safety analysis models should be analyzed individually. All these caveats necessitate a reconsideration of safety issues.

6 Conclusions

The object of our investigation has been a complex system (e.g. a computer code) that we treated as a black box: From a well-defined input set the system (code) produces a well-defined output set. Both sets have metrics; we can speak of distance between two input sets or between two output sets. The computer code simulating the complex system is a map \(\hat{C} : \mathcal{X} \rightarrow \mathcal{Y}\), where \(\mathcal{X}\) is the input set and \(\mathcal{Y}\) is the output set. When analyzing given equipment, we have a nominal input \(x_0\) but the actual input might as well be anywhere in \(\mathcal{X}\), hence the input is a random vector. The probability distribution of input components is usually derived from diverse engineering considerations. In that setting, we have to predict the statistical behavior of the output vector;
and, we have to specify a safety envelope into which the actual output falls with high probability. The exact statements are formulated as theorems in Sections 3 and 4, while the conclusions are summarized as follows.

1. The nominal state \( \left( \bar{x}_0, \hat{C} \bar{x}_0 \right) \) is determined from the expectation value of the input and from the associated output. The investigation of the nominal state alone is insufficient to declare the system operation to be safe.

2. When the distribution of output is not known, four methods, namely the Bayesian, the percentile and the sign test as well as the tolerance interval methods are proposed for testing the output data. The statistical statements which can be obtained by these methods do not differ significantly from one another. As expected, since the distribution is unknown, only a fraction of the information present in the output can be utilized. As a result, more runs are needed or lower probabilities are achieved.

3. When the output is of normal distribution, Theorem 3 determines an interval \([L, U]\) around the estimated mean value of the output into which a larger than a prescribed fraction \(\gamma\) of the distribution falls with preassigned probability \(\beta\). The limits \(L\) and \(U\) are determined by the sample estimate of the standard deviation and by a positive factor \(\lambda\). For the mostly used \(N, \beta, \gamma\) values the \(\lambda\) factors are given in Table XI. Our results are in accordance with Ref. \([12]\) and \([11]\).

4. When the output consists of more than one statistically not independent quantities, the portion of information content that can be utilized rapidly decreases with the number of simultaneously tested output variables. That manifests again in a larger number of runs or in lower probabilities. This is true for both the sign test and tolerance interval methods. The results are given in Tables IX. and XII. It is worth noting that our results comply with the results given in Ref. \([12]\) only when the output variables are independent. To achieve identical \((\beta, \gamma)\) level for statistically dependent outputs we need a larger number of runs than given in Ref. \([12]\).

\[20\text{The present work was initiated by a remark stating that a limited number of runs suffices to determine the safety envelop.}\]
All these observations may influence, in safety analysis, the application of best estimate methods, and underline the opinion that any realistic modeling and simulation of complex systems must include the probabilistic features of the system and the environment.

I Appendix. Proof of Theorem 1.

Obvious, if $G(y)$ is continuous strictly increasing function of $y$, then

$$\mathcal{P}\{y(r) \leq Q \gamma \leq y(s)\} = \mathcal{P}\{y(r) \leq G^{-1}(\gamma) \leq y(s)\} = \mathcal{P}\{z(r) \leq \gamma \leq z(s),\}$$

which is nothing else than

$$\mathcal{P}\{z(r) \leq \gamma \leq z(s),\} = \mathcal{P}\{z(r) \leq \gamma, z(s) \geq \gamma,\}.$$

Introducing the notations $\mathcal{A} = \{z(s) \leq \gamma\}$ and $\overline{\mathcal{A}} = \{z(s) \geq \gamma\}$, we can write that

$$\mathcal{P}\{z(r) \leq \gamma\} = \mathcal{P}\{z(r) \leq \gamma, \mathcal{A} + \overline{\mathcal{A}}\}$$

$$= \mathcal{P}\{z(r) \leq \gamma, z(s) \leq \gamma\} + \mathcal{P}\{z(r) \leq \gamma, z(s) \geq \gamma\},$$

and from this we obtain

$$\mathcal{P}\{y(r) \leq Q \gamma \leq y(s)\} = \mathcal{P}\{z(r) \leq \gamma, z(s) \geq \gamma\} =$$

$$= \mathcal{P}\{z(r) \leq \gamma\} - \mathcal{P}\{z(r) \leq \gamma, z(s) \leq \gamma\}. \tag{I-a}$$

By using the well known expression

$$\mathcal{P}\{u \leq z(r) \leq u + du, v \leq z(s) \leq v + dv\} = g_{r,s}(u, v) \, du \, dv =$$

$$= \frac{u^{r-1} (v - u)^{s-r-1} (1 - v)^{N-s}}{B(r, s - r) \, B(s, N - s + 1)} \, du \, dv, \tag{I-b}$$

$$0 \leq u \leq v \leq 1,$$

we obtain that

$$\beta = \int_0^\gamma \int_0^1 g_{r,s}(u, v) \, du \, dv - \int_0^\gamma \int_0^\gamma g_{r,s}(u, v) \, du \, dv. \tag{I-c}$$
The first integral:

\[ Y_1 = \int_{0}^{\gamma} \int_{0}^{1} g_{r,s}(u, v) \, du \, dv = \int_{0}^{\gamma} g_r(u) \, du, \]

where

\[ g_r(u) = \frac{u^{r-1} (1 - u)^{N-r}}{B(r, N-r+1)}, \]

hence

\[ Y_1 = I(\gamma, r, N - r + 1). \]

Taking into account that \( v \geq u \) the second integral is nothing else than

\[ Y_2 = \int_{0}^{\gamma} \int_{0}^{\gamma} g_{r,s}(u, v) \, du \, dv = \]

\[ = C_{r,s} \int_{0}^{\gamma} dv \int_{0}^{v} u^{r-1} (v - u)^{s-r-1} (1 - v)^{N-s} \, du, \]

where

\[ C_{r,s} = \frac{1}{B(r, s-r) B(s, N-s+1)}. \quad (I-d) \]

By performing the transformations

\[ u = t_1 t_2 \quad \text{and} \quad v = t_2 \]

the integral \( Y_2 \) can be easily calculated. Since

\[ J = \begin{vmatrix} \frac{\partial u}{\partial t_1} & \frac{\partial u}{\partial t_2} \\ \frac{\partial v}{\partial t_1} & \frac{\partial v}{\partial t_2} \end{vmatrix} = t_2, \]

we find that

\[ Y_2 = C_{r,s} \int_{0}^{\gamma} t_2 \, dt_2 \int_{0}^{1} (t_1 t_2)^{r-1} (t_2 - t_1 t_2)^{s-r-1} (1 - t_2)^{N-s} \, dt_1 = \]

\[ = C_{r,s} \int_{0}^{1} t_1^{r-1} (1 - t_1)^{s-r-1} \, dt_1 \int_{0}^{\gamma} t_2^{s-1} (1 - t_2)^{N-s} \, dt_2. \]

Replacing \( C_{r,s} \) by (I-d) one obtains that

\[ Y_2 = I(\gamma, s, N - s + 1). \]
By using the well known identity \( I(c, a, b) = 1 - I(1 - c, b, a) \), finally we have

\[
\beta = \mathcal{P}\{ y(r) \leq Q_\gamma \leq y(s) \} = Y_1 - Y_2 = \\
I(1 - \gamma, N - s + 1, s) - I(1 - \gamma, N - r + 1, r),
\]

and so the Theorem 1 is proven.

II Appendix. Proof of Theorem 2.

The derivation of Eq. (22) is based on the following observation. Let

\[
G(y) = \int_{-\infty}^{y} g(t) \, dt
\]

be the unknown but continuous cumulative distribution function of the output variable \( y \). Let \( y_1, \ldots, y_N \) be its independently observed values. Arrange the sample elements \( y_k, \quad k = 1, \ldots, N \) in increasing order, and denote by \( y(k) \) the \( k \)th element of the ordered sample. Introduce the random variables \( z(k) = G[y(k)], \quad k = 1, \ldots, N \) which are not independent \([6]\). According to (I-b) the bivariate density function of \( z(s) \) and \( z(r), \quad r < s \) is given by

\[
g_{(r,s)}^{(N)}(u, v) = \frac{N!}{(r - 1)! (s - r + 1)! (N - s)!} u^{r-1} (v - u)^{s-r+1} v^{N-s}, \quad 0 \leq u \leq v \leq 1.
\]

In order to determine the probability

\[
\mathcal{P}\left\{ \int_{y(r)}^{y(s)} dG(y) > \gamma \right\} = \\
= \mathcal{P}\{ G[y(s)] - G[y(r)] > \gamma \} = \mathcal{P}\{ z(s) - z(r) > \gamma \},
\]

we need the probability

\[
\mathcal{P}\{ t \leq z(s) - z(r) \leq t + dt \} = w_{(r,s)}^{(N)}(t) \, dt = \int_{0}^{1-t} g_{(r,s)}^{(N)}(u, u + t) \, du \, dt.
\]

(II-a)
Substituting $g_{r,s}^{(N)}(u,v)$ into (II-a), the integration in (II-a) can be carried out:

$$w_{(r,s)}^{(N)}(t) = \frac{1}{B(r, s-r) B(s, N-s+1)} t^{s-r+1} \int_{0}^{1-t} u^{r-1} (1 - u)^{N-s} du,$$

(II-b)

where $B(j, k)$ is the Euler beta function. Taking this expression into account, we get

$$P \{ z(s) - z(r) > \gamma \} = \int_{\gamma}^{1} w_{(r,s)}^{(N)}(t) \, dt,$$

after integration we obtain

$$P \{ z(s) - z(r) > \gamma \} = 1 - \int_{0}^{\gamma} \frac{t^{s-r-1} (1 - t)^{N-s-r}}{B(s-r, N-s+r+1)} \, dt.$$

In other words,

$$\beta = 1 - I(\gamma, s-r, N-s+r+1),$$

(II-c)

as stated. Q.E.D.

**III Appendix. Proof of Theorem 3.**

The proof of the Theorem 3 is based on a few well known relations of mathematical statistics. By the definition of conditional probability,

$$P\{\rho(\tilde{z}, \tilde{s}) > \gamma \} = \int_{-\infty}^{+\infty} P\{\rho(\tilde{z}, \tilde{s}) > \gamma | \tilde{z} = \mu \} \, dP\{\tilde{z} \leq \mu\},$$

(III-a)

where

$$dP\{\tilde{z} \leq \mu\} = dP \left\{ \frac{1}{N} \sum_{n=1}^{N} \frac{y_n - m}{\sigma} \leq \mu \right\} = \sqrt{\frac{N}{2\pi}} e^{-N\mu^2/2} \, d\mu.$$

Since

$$P\{\rho(\tilde{z}, \tilde{s}) > \gamma | \tilde{z} = \mu \} = P\{\rho(\mu, \tilde{s}) > \gamma \},$$

we have

$$P\{\rho(\tilde{z}, \tilde{s}) > \gamma \} = \sqrt{\frac{N}{2\pi}} \int_{-\infty}^{+\infty} P\{\rho(\mu, \tilde{s}) > \gamma \} \, e^{-N\mu^2/2} \, d\mu.$$  

(III-b)
where
\[ \rho(\mu, \tilde{s}) = \frac{1}{\sqrt{2\pi}} \int_{\mu-\lambda\tilde{s}}^{\mu+\lambda\tilde{s}} e^{-z^2/2} \, dz \]
is random variable. Let us define the function
\[ r(\mu, \lambda s) = \frac{1}{\sqrt{2\pi}} \int_{\mu-\lambda s}^{\mu+\lambda s} e^{-z^2/2} \, dz \]
for real \( s \). If \( \mu \) and \( \lambda \) are fixed, then \( r(\mu, \lambda s) \) is strictly monotonously increasing function of \( s \), therefore the equation
\[ \frac{1}{\sqrt{2\pi}} \int_{\mu-\lambda s}^{\mu+\lambda s} e^{-z^2/2} \, dz = \gamma \]
has only one root in \( s \). It is clear that \( \lambda s \) is independent of \( \lambda \), hence we may write \( \lambda s = q(\mu, \gamma) \) and \( q \) is obtained from
\[ \frac{1}{\sqrt{2\pi}} \int_{-\lambda q}^{\lambda q} e^{-z^2/2} \, dz = \gamma. \]
It follows from the property of \( r(\mu, \lambda s) \) that the probability of \( \rho(\mu, \tilde{s}) > \gamma \) equals to the probability of \( \tilde{s} > s_r \), i.e.
\[ P\{\rho(\mu, \tilde{s}) > \gamma\} = P\{\tilde{s} > s_r\} = P\left\{\tilde{s} > \frac{q(\mu, \gamma)}{\lambda}\right\}. \quad \text{(III-c)} \]
By using this relations we can write that
\[ P\left\{\tilde{s} > \frac{q(\mu, \gamma)}{\lambda}\right\} = P\left\{\tilde{s}^2 > \frac{[q(\mu, \gamma)]^2}{\lambda^2}\right\} = P\left\{\tilde{\sigma}^2 > \frac{[q(\mu, \gamma)]^2}{\lambda^2}\right\}, \]
and taking into account that the random variable
\[ (N-1) \frac{\tilde{\sigma}^2}{\sigma^2} = \sum_{n=0}^{N} \left( \frac{y_n - \bar{y}}{\sigma} \right)^2 \]
is of \( \chi^2 \) distribution with \( (N-1) \) degree of freedom [6], we get
\[ P\{\rho(\mu, \tilde{s}) > \gamma\} = 1 - K_{N-1} \left\{ (N-1) \frac{[q(\mu, \gamma)]^2}{\lambda^2} \right\}, \quad \text{(III-d)} \]
where
\[ K_N(x) = \frac{1}{2\Gamma(N/2)} \int_0^x \left( \frac{u}{2} \right)^{(N-2)/2} e^{-u/2} \, du. \]
Substituting (III-d) into (III-d) we get the theorem proven. Q.E.D.
Appendix. Proof of Theorem 4.

Before setting out for the proof of Theorem 4, we set forth the following notation. Let

\[ H(\lambda, \gamma | \mu) = \mathcal{P} \{ \rho(\mu, \tilde{s}) > \gamma \}. \]

We need

**Lemma 1** It can be shown that

\[ H(\lambda, \gamma | 1/\sqrt{N}) - W(\lambda, \gamma, N) = O(N^{-2}). \] (IV-a)

where

\[ W(\lambda, \gamma, N) = \sqrt{\frac{N}{2\pi}} \int_{-\infty}^{+\infty} H(\lambda, \gamma | \mu) e^{-\mu^2/2} d\mu. \] (IV-b)

**Proof of Lemma 1.** The expression

\[ H(\lambda, \gamma | \mu) = \mathcal{P} \left\{ \frac{1}{\sqrt{2\pi}} \int_{\mu-\lambda \tilde{s}}^{\mu+\lambda \tilde{s}} e^{-z^2/2} dz > \gamma \right\} \] (IV-c)

is an even function of \( \mu \) and can be developed into Taylor series around \( \mu = 0 \) as

\[ H(\lambda, \gamma | \mu) = \sum_{n=0}^{\infty} \left[ \frac{\partial^{2n} H(\lambda, \gamma | \mu)}{\partial \mu^{2n}} \right]_{\mu=0} \frac{\mu^{2n}}{(2n)!}. \] (IV-d)

Substituting (IV-d) into (IV-c) we obtain

\[ W(\lambda, \gamma, \mu) = \sum_{n=0}^{\infty} \left[ \frac{\partial^{2n} H(\lambda, \gamma | \mu)}{\partial \mu^{2n}} \right]_{\mu=0} \frac{(2n-1)!!}{(2n)! N^n} \]

\[ = H(\lambda, \gamma | 0) + \left[ \frac{\partial^2 H(\lambda, \gamma | \mu)}{\partial \mu^2} \right]_{\mu=0} \frac{1}{2N} + O(N^{-1/2}), \] (IV-e)

and replacing \( \mu \) by \( 1/\sqrt{N} \) in (IV-d), we have

\[ H(\lambda, \gamma | 1/\sqrt{N}) = H(\lambda, \gamma | 0) + \left[ \frac{\partial^2 H(\lambda, \gamma | \mu)}{\partial \mu^2} \right]_{\mu=0} \frac{1}{2N} + O(N^{-1/2}). \] (IV-f)
From Eqs. (IV-f) and (IV-e) follows that (IV-a) is true. Consequently, we have the following approximate equation

\[ H(\lambda, \gamma | 1/\sqrt{N}) = 1 - K_{N-1} \left[ (N - 1) \frac{[q(N^{-1/2}, \gamma)]^2}{\lambda^2} \right] \approx \beta, \]

where the argument of \( K_{N-1}[\cdots] \) is nothing else than the \((1 - \beta)\) percentile of \( \chi^2 \) distribution with \((N - 1)\) degree of freedom. Introducing the notation

\[ Q_{N-1}(1 - \beta) = (N - 1) \left[ \frac{[q(N^{-1/2}, \gamma)]^2}{\lambda^2} \right], \]

we find that

\[ \lambda \approx \lambda_a(\gamma, \beta) = \sqrt{\frac{N - 1}{Q_{N-1}(1 - \beta)}} q(1/\sqrt{N}, \gamma). \]  

(IV-g)

This completes the proof of Theorem 2. Q.E.D.

V Appendix. Proof of Theorem 5.

The proof of Theorem 5 is given in two steps. In the first step we show that the Theorem holds for \( n = 2 \), and then we generalize the claim for \( n > 2 \).

\textit{Step 1.} We assume that the unknown joint distribution function of two output variables \( y_1 \) and \( y_2 \) is given by

\[ G(y_1, y_2) = \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} g(t_1, t_2) \, dt_1 \, dt_2, \]

and denote by

\[ g_1(y_1) = \int_{-\infty}^{+\infty} g(y_1, t_2) \, dt_2 \]

the density function of the output variable \( y_1 \). Let us consider the following random variable

\[ A_2 = A_2(L_1, U_1, L_2, U_2) = \int_{L_1}^{U_1} \int_{L_2}^{U_2} g(y_1, y_2) \, dy_1 dy_2, \]  

(V-a)

where the boundaries of the integration are random variables. The limits were discussed in Section 4.2.2. \( A_2 \) can be expressed almost surely as

\[ A_2(L_1, U_1, L_2, U_2) = C_2(U_2|L_1, U_1) A_1(L_1, U_1). \]  

(V-b)
Here
\[ \mathcal{A}_1(L_1, U_1) = \int_{L_1}^{U_1} g_1(y_1) \, dy_1 \]
and
\[ C_2(L_2, U_2|L_1, U_1) = \int_{L_2}^{U_2} \phi_2(y_2|L_1, U_1) \, dy_2. \]
where
\[ \phi_2(y_2|L_1, U_1) = \frac{\int_{L_1}^{U_1} g(y_1, y_2) \, dy_1}{\int_{-\infty}^{\infty} dy_2 \int_{L_1}^{U_1} g(y_1, y_2) \, dy_1} = \frac{\int_{L_1}^{U_1} g(y_1, y_2) \, dy_1}{\mathcal{A}_1(L_1, U_1)} \]
is the random density of variable \( y_2 \) under the condition that \( y_1 \) lies in \( [L_1, U_1] \).
Since \( \mathcal{A}_1(L_1, U_1) = G_1[y_1(s_1)] - G_1[y_1(r_1)] \), using relation (IV-b), we find that
\[ P \{ t_1 \leq \mathcal{A}_1(L_1, U_1) \leq t_1 + dt_1 \} = \frac{t_1^{s_1-r_1-1} (1-t_1)^{N-s_1+r_1}}{B(s_1 - r_1, N - s_1 + r_1 + 1)} \, dt_1 = k_{(r_1,s_1)}^{(N)}(t_1) \, dt_1. \]

To obtain the density function of \( C_2(L_2, U_2|L_1, U_1) \), we define the random probability measure
\[ G(t|L_1, U_1) = \int_{-\infty}^{t} \phi_2(y_2|L_1, U_1) \, dy_2, \]
with which we can express \( C_2 \) as
\[ C_2(L_2, U_2|L_1, U_1) = G[y_2(s_2)|L_1, U_1] - G[y_2(r_2)|L_1, U_1], \]
where \( r_1 < r_2 < \cdots < s_2 < s_1 \). Finally we get
\[ P \{ t_2 \leq C_2(L_2, U_2|L_1, U_1) \leq t_2 + dt_2 \} = \frac{t_2^{s_2-r_2-1} (1-t_2)^{s_1-r_1-1-s_2+r_2}}{B(s_2 - r_2, s_1 - r_1 - s_2 + r_2)} \, dt_2 = k_{(r_2,s_2)}^{(s_1-r_1-1)}(t_2) \, dt_2. \]

Note that expression (V-g) contains neither \( L_1 \) nor \( U_1 \), therefore, distribution of random variable \( C_2 \) is independent of \( L_1 \) and \( U_1 \). Consequently, the joint density distribution of \( \mathcal{A}_1 \) and \( C_2 \) is the product of (V-f) and (V-g). We still need the density function of the random variable \( \mathcal{A}_2 \). Exploiting the independence of \( C_2 \) and \( \mathcal{A}_1 \), we get
\[ \mathcal{P} \{ t \leq A_2(L_1, U_1, L_2, U_2) \leq t + dt \} = \]
\[ = \int_t^1 \frac{1}{x} k^{(N)}_{(r_1, s_1)}(x) \ell^{(s_1-r_1-1)}(t/x) dx \, dt = w_{A_2}(t) \, dt. \]  
(V-h)

Substituting here (V-f) and (V-g) and performing the indicated calculations we obtain:

\[ w_{A_2}(t) = \frac{t^{s_2-r_2-1} (1-t)^{N-s_2+r_2}}{B(s_2-r_2, N-s_2+r_2+1)}. \]  
(V-i)

From this, immediately follows

\[ \mathcal{P} \{ A_2(L_1, U_1, L_2, U_2) > \gamma \} =
\]
\[ = 1 - \frac{B(\gamma, s_2-r_2, N-s_2+r_2+1)}{B(s_2-r_2, N-s_2+r_2+1)} = 1 - I(\gamma, s_2-r_2, N-s_2+r_2+1). \]

This completes Step 1.

**Step 2.** Now we generalize the above result for \( n > 2 \). Let us assume that the unknown joint probability distribution of the output variables \( y_1, \ldots, y_n \) is given by

\[ G(y_1, \ldots, y_n) = \int_{-\infty}^{y_n} \cdots \int_{-\infty}^{y_1} g(v_1, \ldots, v_n) \, dv_1 \cdots dv_n. \]

Our task is to derive the probability distribution of the random variable

\[ A_p(L_1, U_1, \ldots, L_n, U_n) = \int_{L_n}^{U_n} \cdots \int_{L_1}^{U_1} g(y_1, \ldots, y_n) \, dy_1 \cdots dy_n, \]

which is an \( n \)-fold integral over the \( n \) output variables. We introduce an intermediate term, in which an \( i \)-fold definite integral over the first \( i \) variables is involved, and the rest of the variables are integrated over the \([ -\infty, +\infty ]\) range:

\[ A_i(L_1, U_1, \ldots, L_i, U_i) =
\]
\[ = \int_{-\infty}^{+\infty} dy_n \cdots \int_{-\infty}^{+\infty} dy_{i+1} \int_{L_i}^{U_i} dy_i \cdots \int_{L_1}^{U_1} dy_1 \, g(y_1, \ldots, y_n), \]

and

\[ \phi_i(y_i | L_1, U_1, \ldots, L_{i-1}, U_{i-1}) = \frac{1}{A_{i-1}} \int_{L_{i-1}}^{U_{i-1}} dy_{i-1} \cdots \int_{L_1}^{U_1} dy_1 \, g(y_1, \ldots, y_n).\]
which is the random density of the variable \( y_i \) under the condition that 
\( L_j \leq y_j \leq U_j, \ j = 1, \ldots, i - 1 \). As we did in (V-d), we introduce a random 
probability measure associated with the condition that the first \((i - 1)\) output 
variables lie in the interval assigned to them by \([L_j, U_j], \ j = 1, \ldots, i - 1\):

\[
C_i = C_i (L_i, U_i|L_1, U_1, \ldots, L_{i-1}, U_{i-1}) = \int_{L_i}^{U_i} \phi_i (y_i|L_1, U_1, \ldots, L_{i-1}, U_{i-1}) \, dy_i.
\]

The above defined \( A_i \)'s obey the recursion

\[
A_{i+1} = C_{i+1} A_i.
\]

Lemma 2 The probability of finding \( A_i \) in the interval \([t_i, t_i + dt_i]\) is given
by

\[
P \{ t_i \leq A_i \leq t_i + dt_i \} = \frac{t_i^{s_i - r_i - 1} (1 - t_i)^{N - s_i + r_i}}{B(s_i - r_i, N - s_i + r_i + 1)} \, dt_i.
\]

Proof of Lemma 2. Eq. (V-k) is certainly true for \( i = 1, 2 \) because

\[
A_1 = A_0 C_1 = C_1 = \int_{L_1}^{U_1} g(y) \, dy
\]

and

\[
A_2 = C_2 A_1 = C_2 (L_2, U_2|L_1, U_1) A_1 (L_1, U_1) = \int_{L_2}^{U_2} \int_{L_1}^{U_1} g(y_1, y_2) \, dy_1 \, dy_2.
\]

Now we assume that (V-k) is true for \( i = j \) and show that it is true also for 
\( i = j + 1 \). Note that \( A_j \) and \( C_{j+1} \) are statistically independent because

\[
P \{ t_{j+1} \leq C_{j+1} \leq t_{j+1} + dt_{j+1} \} = \frac{t_{j+1}^{s_{j+1} - r_{j+1} - 1} (1 - t_{j+1})^{s_j - r_j - 1 - s_{j+1} + r_{j+1}}}{B(s_{j+1} - r_{j+1}, s_j - r_j - s_{j+1} + r_{j+1})} \, dt_{j+1}
\]

does not involve the quantities \( L_j, U_j, \ldots, L_1, U_1 \), which occur in \( A_j \). The 
joint density function of \( A_j \) and \( C_{j+1} \) takes the form of the joint density 
function of \( A_1 \) and \( C_2 \) in the case of \( n = 2 \). Hence the density function of

\[
A_j C_{j+1} = A_{j+1}
\]
is obtainable from Eq. (V-1) by substituting $r_{j+1}$ for $r_2$ and $s_{j+1}$ for $s_2$, i.e.

$$
P \{ t_{j+1} \leq A_{j+1} \leq t_{j+1} + dt_{j+1} \} = \frac{t_{j+1}^{s_{j+1} + r_{j+1} - 1} (1 - t_{j+1})^{N - s_{j+1} + r_{j+1} + 1}}{B(s_{j+1} - r_{j+1}, N - s_{j+1} + r_{j+1} + 1)} dt_{j+1}.
$$

Hence, Eq. (V-3) is proven for $i = 1, 2, \ldots, n$. This completes Step 2. Furthermore, the density function of $A_n$ is given by

$$
P \{ t \leq A_n \leq t + dt \} = w_{A_n}(t) dt = \frac{t^{s_n - r_n - 1} (1 - t)^{N - s_n + r_n}}{B(s_n - r_n, N - s_n + r_n + 1)} dt.$$

It is interesting to note that the density function of $A_n$ does not depend on the integers $r_1, s_1, \ldots, r_{n-1}, s_{n-1}$. This completes the proof of Theorem 4. Q.E.D.

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