A Self-Replicating Single-Shape Tiling Technique for the Design of Highly Modular Planar Phased Arrays—The Case of L-Shaped Rep-Tiles

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Abstract—The design of irregular planar phased arrays (PAs) characterized by a highly modular architecture is addressed. By exploiting the property of self-replicating tile shapes, also known as rep-tiles, the arising array layouts consist of tiles having different sizes, but equal shapes, all being generated by assembling a finite number of smaller and congruent copies of a single elementary building block. Toward this end, a deterministic optimization strategy is used so that the arising rep-tile arrangement of the planar PA is an optimal tradeoff between complexity, costs, and fitting of user-defined requirements on the radiated power pattern while guaranteeing the complete overlay of the array aperture. As a representative instance, such a synthesis method is applied to tile rectangular apertures with L-shaped tromino tiles. A set of representative results is reported for validation purposes but also to point out the possibility/effectiveness of the proposed approach, unlike state-of-the-art tiling techniques, to reliably handle large-size array apertures.

Index Terms—Irregular tiling, phased array (PA) antenna, planar array, rep-tiles.

I. INTRODUCTION

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THE interest toward innovative phased array (PA) antennas and related technologies has constantly grown over the last years owing to many possible applications as, for instance, millimeter-wave communications [1], [2] and sensing [3]; automotive [4] and drone [5] radars; directional heating systems [6]; and microwave imaging [7], [8], just to mention a few. The capability of quickly reconfiguring the radiation pattern and its features by controlling the amplitude and/or the phase coefficients of the excitations of the array elements makes PAs a very attractive and promising technology. However, regularly spaced digital PAs, where the radiating elements are arranged on a uniform lattice and each element is equipped with an independent transmit/receive module (TRM) and a digital channel, are too expensive and complex for standard applications [9], [10].

Unconventional architectures [11], such as thinned [12], [13], sparse [14], [15], and clustered [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35] arrays, have been recently considered to yield a suitable compromise between performance and costs. In such a framework, tiled PAs (i.e., clustered PAs composed of one or few elementary building blocks) are gaining more and more attention thanks to their modularity that makes the whole antenna system, including the feeding network and the baseband layer, easier to fabricate and maintain, as well as controllable with a reduced number of TRMs or digital channels by using a single output/input for all the radiating elements clustered in a tile [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35].

Nevertheless, two main issues/needs arise when dealing with clustered PA structures. First, the clusters have to be large enough to allow one a nonnegligible cost saving (i.e., reduction of TRMs or digital channels), while the whole clustered PA must still afford a desired radiation pattern. In addition, the tile shapes must permit the design of irregular...
clustering configurations, which are required to avoid periodic quantization errors that generally imply the presence of high secondary lobes and/or grating lobes in the radiated pattern [36]. To properly tackle these drawbacks, tiling methods using either single [22], [23], [24], [25], [26], [27], [28], [29], [30] or multiple [31], [32], [33], [34], [35] tile shapes have been proposed. The PAs’ design has been generally cast as the optimization of user-defined cost-function terms related to the desired pattern performance indexes (e.g., the side lobe level (SLL), the beamwidth, and the directivity) or quantifying the mismatch with a reference mask by also guaranteeing full coverage of the whole antenna aperture. For instance, optimization techniques based on the height-function coding strategy [25], [27], [28], evolutionary algorithms [25], [27], [28], [32], compressive-sensing [31], [33], convex-relaxation programming [30], random approaches [35], and exhaustive search [24] have been successfully applied to tile a finite area (i.e., the array aperture) with preselected tile shapes. However, such design methods have been usually developed for specific tile geometries, or they turn out to be computationally expensive, thus requiring the development of a novel deterministic tiling method.

This article proposes a novel and alternative methodology for the design of tiled PAs featuring high modularity instead of exploiting multiple different-shape tiles. It is based on the use and related properties of the self-replicating tiles, also called rep-tiles, to perform the irregular clustering of the array aperture. More specifically, the aperture is tiled with tiles having different sizes, but equal shapes, all being generated by assembling a finite number of smaller and congruent copies of a single elementary building block. Toward this end, a deterministic optimization strategy is used so that the arising rep-tile arrangement of the planar PA is an optimal tradeoff between complexity, costs, and fitting of user-defined requirements on the radiated power pattern while guaranteeing the complete overlay of the array aperture. As a representative instance, such a synthesis method is applied to tile rectangular apertures with L-shaped tromino tiles.

To the best of our knowledge, the main novelties of the proposed technique over the state-of-the-art array tiling comprise the following:

1) the exploitation of the concept of self-replicating tiles for the design of planar PAs;
2) the development of a novel deterministic tiling method to exploit the rep-tiling property for synthesizing clustered arrays with an optimal tradeoff between architecture complexity and the fulfillment of user-defined radiation targets;
3) the theoretical analysis of the properties of L-shape tromino rep-tiles and their engineering exploitation for the application of the proposed tiling method.

The rest of this article is organized as follows. The mathematical formulation of the design of highly modular planar PAs with self-replicating tiles is reported in Section II, while the proposed deterministic synthesis method is described in Section III and customized to L-shape tromino rep-tiles, as well. Section IV is devoted to the numerical validation against both ideal and real antenna models, including mutual-coupling effects, as well. Eventually, some conclusions and final remarks are drawn (see Section V).

II. MATHEMATICAL FORMULATION

Let us consider a planar PA covering a rectangular aperture $A$ and composed of $M \times N$ radiating elements arranged on a regular lattice with interelement spacing $d_x$ and $d_y$ along the $x$- and $y$-axes, respectively (see Fig. 1). The power pattern radiated in far-field by such a PA is given by $P(u, v) = |e(u, v) \times A(u, v)|^2$, where $e(u, v)$ is the embedded/active element pattern [37], which is assumed, without loss of generality, but only for the sake of notation simplicity, identical for all antennas, while $A(u, v)$ is the array factor that depends on the array architecture at hand as detailed in the following, with $u (u \triangleq \sin \theta \cos \phi)$ and $v (v \triangleq \sin \theta \sin \phi)$ being the direction cosines ($0 \text{[deg]} \leq \theta \leq 90 \text{[deg]}, 0 \text{[deg]} \leq \phi \leq 360 \text{[deg]}$).

The surface $A$ is partitioned into $Q$ ($Q < M \times N$) tiles, $\varsigma = \{\varsigma_q; q = 1, \ldots, Q\}$, each belonging to the rep-tile [38], [39] alphabet $\Sigma$ (i.e., $\varsigma_q \in \Sigma$). More specifically, the alphabet $\Sigma$ is composed of $B \times R$ elements (i.e., $\Sigma = \{\sigma^{(b, r)}; r = 1, \ldots, R; b = 1, \ldots, B\}$), with $B$ and $R$ being the number of admissible rep-tile orientations/flips and the maximum rep-tile order/level, respectively.

Let the smallest building block of the rep-tile alphabet $\Sigma$, namely, the generating rep-tile of order $r = 1$ with $bth (1 \leq b \leq B)$ orientation, $\sigma^{(r, b)}_{f=1}$, be a cluster of $I^{(r)}_{f=1}$ radiating elements. The rep-tile element of $\Sigma$ of order $(r + 1)th (r = 1, \ldots, R - 1)$ and orientation $bth (1 \leq b \leq B)$ is then yielded by the union of $S$ tiles of the previous $rth (r = 1, \ldots, R - 1)$ order

$$\sigma^{(r+1, b)} = \bigcup_{s=1}^{S} \sigma^{(r, b_s)}$$

(1)

where $S$ is a function of the rep-tile geometry at hand (e.g., $L$-shape tromino $\rightarrow S = 4$; see Fig. 2(b)) [39] and $b_s$, $1 \leq b_s \leq B$ is the orientation of the $s$th $(s = 1, \ldots, S)$ generating tile of $rth$ order $(r = 1, \ldots, R - 1)$. Therefore, a larger tile is yielded by assembling a finite number of $S$ smaller and congruent copies with the same shape but rotated in one of $B$ possible positions [40] (see Fig. 2(b)).

Moreover, the number of array elements belonging to a rep-tile of the $rth (r = 2, \ldots, R)$ order is given by

$$I^{(r)} = S^{r-1} \times I^{(1)}.$$  

(2)
Thus, the aperture $A$ of the array turns out to be tiled by multiple tiles with different sizes but all having the same (even if scaled) shape [e.g., L-shape; see Fig. 2(a)].

The array elements grouped into the $q$th ($q = 1, \ldots, Q$) tile, $\zeta_q$, share a unique input/output port, which is connected to the feeding network through a common TRM and/or a digital channel that provides a controllable complex weight $w_q$ (see Fig. 1) of amplitude $\alpha_q$ and phase $\beta_q$ (i.e., $w_q = \alpha_q e^{j\beta_q}$). Accordingly, the antenna array factor is given by

$$A(u, v) = \sum_{q=1}^{Q} w_q S_q^{(r,b)}(u, v) e^{j(kx_0 + y_0 v)}$$

(3)

where $(x_q, y_q)$ is the center of $\zeta_q$ within the aperture $A$ and $k$ is the wavenumber ($k = 2\pi/\lambda$), with $\lambda$ being the wavelength at the working frequency $f$. Moreover, $S_q^{(r,b)}$ is the space factor of the $q$th ($q = 1, \ldots, Q$) tile whose expression is

$$S_q^{(r,b)}(u, v) = \sum_{i=1}^{I_q} e^{j(kx_{i}^{(r,b)} u + y_{i}^{(r,b)} v)}$$

(4)

where $(x_{i}^{(r,b)}, y_{i}^{(r,b)})$ is the center of the $i$th ($i = 1, \ldots, I_q^{(r)})$ radiating element of the alphabet entry, namely, $\sigma_q^{(r,b)} = \Phi(\zeta_q)$, associated with $\zeta_q$ ($\zeta_q \in \Sigma$).

$^1$The operator $\Phi$ is called "associative operator" since it defines the correspondence between the $q$th tile and the corresponding letter $\sigma_q^{(r,b)}$ of the alphabet $\Sigma$ (i.e., $\zeta_q \in \Sigma$).

In order to define the array synthesis problem at hand, let us code the tiling configuration of the $M \times N$ elements into $Q$ rep-tiles, $\zeta = \{\zeta_q; q = 1, \ldots, Q\}$, with the clustering vector $c$ of dimension $M \times N$ whose $(m, n)$th ($m = 1, \ldots, M; n = 1, \ldots, N$) entry, $c_{mn}$, is given by $c_{mn} = \delta_{c_{mn}q}$, where $\delta_{c_{mn}q} = 1(0)$ if the $(m, n)$th radiating element belongs (does not belong) to the $q$th rep-tile, $\zeta_q$ (see Fig. 3). Moreover, the amplitude/phase coefficients of the $Q$-sized set of rep-tiles, $\zeta = \{\zeta_q; q = 1, \ldots, Q\}$, coded into the tiling vector $c$, are chosen according to the excitation matching strategy [16], [17] as follows:

$$\alpha_q = \frac{1}{I_q} \sum_{m=1}^{M} \sum_{n=1}^{N} \alpha^{ref}_{mn} \delta_{cmn}$$

(5)

and

$$\beta_q = \frac{1}{I_q} \sum_{m=1}^{M} \sum_{n=1}^{N} \beta^{ref}_{mn} \delta_{cmn}.$$  

(6)

Such a choice allows one to yield the optimal setup of the weights of the tiled array, $w = \{w_q; q = 1, \ldots, Q\}$, in the least-squares sense, since they have the minimum distance with respect to the "reference" excitations, $w^{ref} = \{w^{ref}_{mn}; m = 1, \ldots, M; n = 1, \ldots, N\}$, of a fully populated array of size $M \times N$ with a dedicated TRM for each $(m, n)$th array element ($w^{ref}_{mn} = \alpha^{ref}_{mn} e^{j\beta^{ref}_{mn}}$) to afford an optimal power pattern, $P^{ref}(u, v)$, or to optimally approximate a user-defined power pattern mask, $\Psi(u, v)$ in the visible range $\Omega = \{(u, v) : u^2 + v^2 < 1\}$.

Accordingly, the rep-tile array (RTA) synthesis problem can be formulated as follows.

Given a planar PA of $M \times N$ radiating elements arranged on a rectangular lattice, determine the optimal clustering of the array elements, $c_{opt}$, corresponding to the set of rep-tiles, $\zeta_{opt} = \{\zeta_q^{opt}; q = 1, \ldots, Q^{opt}\}$, belonging to the alphabet $\Sigma$ (i.e., $\zeta_q^{opt} \in \Sigma$), and the subarray weights, $w_{opt}$ ($w_{opt} = \{w_{opt}^{ref} = \alpha^{opt}_{MN} e^{j\beta^{opt}_{MN}}; q = 1, \ldots, Q^{opt}\}$), according to (5) and (6), so that $Q^{opt}$ is the minimum number of rep-tiles covering the aperture $A$, and the user-defined power upper mask $\Psi(u, v)$ is matched by minimizing the mask matching index

$$\Gamma(c) = \int_{\Omega} [P(u, v; c) - \Psi(u, v)]$$

$$\times \Phi_P \{P(u, v; c) - \Psi(u, v)\} dudv$$

$$\times \frac{1}{\int_{\Omega} \Psi(u, v) dudv}$$

(7)
by means of the following procedural steps. 

III. RTA Design Method (RTAM) 

Under the hypothesis that the array aperture \( A \) is fully tileable with \( R \)-th level rep-tiles, the RTAM is implemented by means of the following procedural steps. 

1) \textbf{Step 1 (Initial \( R \)-th Level Tiling):} Depending on the cardinality \( T^{(R)} \) of the problem at hand, either apply Algorithm-X [43], [44] to generate the whole set of \( T^{(R)} \)-complete clusterings of \( A \) with \( R \)-th order rep-tiles only \( \{ \tilde{c}_t \}; t = 1, \ldots, T^{(R)} \) and select the optimal configuration \( \tilde{c}_{opt} \) of \( T \) tiles (\( \hat{Q} \equiv ((M \times N)/T^{(R)}) \)) as

\[
\hat{c}_{opt} = \arg \min_{t=1, \ldots, T^{(R)}} \left[ \Gamma(\tilde{c}_t) \right] 
\]

or increase the tile order (i.e., \( R \leftarrow R+1 \)) to reduce the cardinality of the solution space \( (T^{(R+1)} = T^{(R)}) \) if the aperture can be still fully partitioned, or exploit a global optimization method [22] to sample the \( T^{(R)} \)-size solution space for retrieving the global optimum of the cost function \( \Gamma(\hat{c}_t) \), \( \hat{c}_{best} \), and set \( \tilde{c}_{opt} = \hat{c}_{best} \). Set the initial \( (h = 0, h \) being the iteration index) configuration to that yielded in Step 1 \( (\tilde{c}_{opt})_{h=0} \) and composed of \( R \)-th order rep-tiles only, \( \sigma_q^{(R,b), h=0} = \sigma_q^{(R,b), q = 1, \ldots, Q^{(b)}, h=0} \), being \( Q^{(b)}_{h=0} = \hat{Q} \).

2) \textbf{Step 2 (Rep-Tile Optimization):} Increase the iteration index \( h \leftarrow h + 1 \), set the rep-tile level to \( R = R \), and apply the following loop:

a) \textbf{Step 2.1 (Rep-Tile Selection):} For each \( q \) \((q = 1, \ldots, Q^{(b)}) \) tile of the \( h \)-th clustering, \( \sigma_q^{(b)} \), compute the substitution tiling metric (STM)

\[
\xi_q^{(b)} = \sum_{m=1}^{M} \sum_{n=1}^{N} |R_{wm, wq} - w_{qm}^{(b)}| + jI_{wm, wq}^{(b)} \delta_{wm, q} \]

\[
= \sum_{m=1}^{M} \sum_{n=1}^{N} \left| R_{wm, wq}^{(ref)} - w_{qm}^{(b)} \right| + jI_{wm, wq}^{(ref)} - w_{qm}^{(b)} \delta_{wm, q} \]  

(9)

to quantify the mismatch between the reference, \( w_{ref} \), and the current, \( w^{(b)} \) \((w^{(b)} = \{w_{qm}^{(b)} = \sigma_q^{(b)} e^{jI_q} ; q = 1, \ldots, Q^{(b)} \}) \) weight sets, with \( R \{\} \) and \( I \{\} \) being the real part and the imaginary one, respectively. Among the \( Q^{(b)} \) rep-tiles of \( \sigma_q^{(b)} \), select the \( \hat{q} \)-th one of \( R \) \((2 \leq r \leq R) \) order with maximum STM value, \( \xi_q^{(b)} \)

\[
\hat{q} = \arg \max_{q=1, \ldots, Q^{(b)}} \left\{ \xi_q^{(b)} \mid \Re \{\sigma_q^{(r,b)} \} \geq \frac{2}{\hat{Q}} \right\} 
\]

(10)

with \( \Re \{\} \) being the operator that returns the rep-tile order of its argument.

b) \textbf{Step 2.2 (Rep-Tile Split):} Subdivide the \( \hat{q} \)-th tile of the current array clustering, \( \sigma_q^{(b)} \), into \( S \) smaller tiles of the \((r-1)\)-th order \( (\sigma_q^{(r,b)} = \bigcup_{s=1}^{S} \sigma_q^{(r-1,b,s)}) \) to generate a new tiling, \( \sigma_q^{(b+1)} \), of the aperture into \( Q^{(b+1)} = Q^{(b)} + (S-1) \) tiles.

c) \textbf{Step 2.4 (Excitations Update):} Compute the amplitude, \( \alpha^{(b+1)} = \{\alpha_q^{(b+1)} ; q = 1, \ldots, Q^{(b+1)} \} \), the phase, \( \beta^{(b+1)} = \{\beta_q^{(b+1)} ; q = 1, \ldots, Q^{(b+1)} \} \), and coefficients of the \( Q^{(b+1)} \) tiles of \( e^{(b+1)} \) according to (5) and (6), respectively.

d) \textbf{Step 2.5 (Convergence Check):} If the power pattern radiated by the tiled array, \( e^{(b+1)} \), perfectly fulfills the radiation mask (i.e., \( \Gamma(e^{(b+1)}) = 0 \)) or the current number of clusters, \( Q^{(b+1)} \), exceeds the user-defined maximum, \( Q_{max} \) \((i.e., Q^{(b+1)} \geq Q_{max} \) subject to \( Q_{max} \leq ((M \times N)/I^{(R)}) \)), stop the procedure and set the convergence iteration \( H \) to the current one (i.e., \( H = h+1 \)) and the optimal tiling to the current rep-tiles arrangement, \( c_{opt}^{(b+1)} \). Otherwise, update the iteration index \( h \leftarrow h + 1 \), and go to Step 2.1.

It is worth noticing that, besides the convergence solution \( c_{opt} \), the iterative loop of Step 2 gives, at each \( h \)-th \((h = 1, \ldots, H) \) iteration, a compromise solution between feeding network complexity (i.e., \( Q \rightarrow \) number of TRMs) and fulfillment of the pattern-mask constraint (i.e., \( \Gamma(e^{(b+1)}) \)) so that, finally, a “Pareto” front of \( H \) layouts can be defined and profitably exploited by the array designer.

A. Case of L-Shaped Rep-Tiles

The RTAM is customized hereinafter to L-tromino tiles, which are rep-tiles \([38], [39] \) with \( I^{(R)} = 3 \) and \( S = 4 \) [see Fig. 2(a)], as a representative example of self-replicating tile geometries [see Fig. 2(b)].

1) \textbf{Tileability Theorem:} The RTAM requires the fulfillment of the complete tileability of the array aperture \( A \) with the largest \((i.e., \) highest order \( R) \) tiles of the alphabet \( \Sigma \) \((i.e., \{\sigma_q^{(R,b)} ; b = 1, \ldots, B\}) \) since the highest order tiles can be certainly subdivided into smaller ones thanks to the self-replicating property of the rep-tiles. The covering theorem for an exact \((i.e., complete)\) rectangular tessellation with \( R \)-th order L-tromino tiles [45] reads as follows.

L-Tromino Covering Theorem: Let \( A \) a rectangular aperture of \( M \times N \) elements/pixels, with \( M \) and \( N \) being integer numbers such that \( 2 \times I^{(R)} \leq M \leq N \), where \( I^{(R)} = \sqrt{(I^{(R)})^3} \) is the side length of the \( R \)-th order L-tromino tile. Then, \( A \) can be fully covered with L-tromino tiles of order \( R \) if and only if one of the following conditions holds true. 

1) \( \widetilde{M} = 3 \) and \( \widetilde{N} \) is even \([e.g., \text{Fig. 4(a)}]\),

2) \( \widetilde{M} \neq 3 \) and \( \{(M \times N)/I^{(R)}\} = 0 \), with \([\cdot]\) being the modulo operation [e.g., Fig. 4(c) and (e)]

where \( \widetilde{M} \equiv (M/I^{(R)}) \) and \( \widetilde{N} = (N/I^{(R)}) \). Otherwise, \( A \) is not fully tileable with \( R \)-th order L-tromino tiles [e.g., Fig. 4(b), (d), and (f)].

2) \textbf{Problem Cardinality:} According to [46], the number of tilings, \( T^{(R)} \), which exactly/fullly cover an aperture \( A \) of \( M \times N \) pixels/elements with \( R \)-th order L-trominoes, is equal to

\[
T^{(R)} = \frac{2^M}{\prod_{p=1}^{\tilde{H}}} \]  

(11)
TABLE I
Illustrative Example—Number of L-tromino Tilings of Rth Order (i.e., $T$ Value)

| $N$ | $T$ |
|-----|-----|
| 2   | 0   |
| 3   | 2   |
| 4   | 0   |
| 5   | 4   |
| 6   | 4   |
| 7   | 0   |
| 8   | 8   |
| 9   | 8   |

$M = 32$ and $N = 48$ is odd, (c) $M = 36$ and $N = 24$, (d) $M = 28$ and $N = 24$, (e) $M = 32$ and $N = 28$, (f) $M = 32$ and $N = 32$.

Fig. 4. Illustrative example ($R = 3$, $B = 4$, and $I^{(1)}_{r=1} = 3$ for $L$-tromino rep-tiles). Sketch of (a), (c), and (e) tileable and (b), (d), and (f) nontileable apertures: (a) $M = 12$ and $N = 16$, (b) $M = 12$ and $N = 20$, (c) $M = 24$ and $N = 8$, (d) $M = 20$ and $N = 16$, (e) $M = 36$ and $N = 24$, and (f) $M = 32$ and $N = 28$.

It is worth pointing out that a priori knowing the value of $T^{(R)}$ (11) or, at least, its order of magnitude, is very important information for the array designer since it enables the choice of an exhaustive enumerative strategy, which guarantees the retrieval of the optimal solution in Step 1 of the RTAM in Section III since evaluating all $T^{(R)}$ exact tiling configurations, or it forces to reformulate the problem as an optimization one when the cardinality of the solution space turns out prohibitive for an exhaustive search.

3) RTAM Customization: For illustrative purposes, let us consider the application of the RTAM ($H = 3$ iterations) to a planar array of $M \times N = 8 \times 16$ elements (see Fig. 5) by using the alphabet $\Sigma$ of $L$-tromino rep-tiles in Fig. 2(a) being $R = 3$ and $B = 4$.

Starting ($h = 0$) from the initial clustering of $R$th order $L$-tromino tiles found at Step 1 (see Section III), $\hat{e}_{\text{opt}}$ in Fig. 5(a) where the normalized values of the STM of the $Q^{(h)}$ results are $4$ tiles (i.e., $\{\xi_q(h); q = 1, \ldots, 4\}$) are indicated, as well, the $I^{(3)}$-size $(I^{(3)}_{R=3} = 48)$ tile with maximum STM (i.e., $\hat{e}_{\text{opt}} = 1$) [see Fig. 5(a)] is chosen for division according to (10) of Step 2.1 (i.e., $\hat{q} = 2$) to yield the new grid arrangement $e^{(1)}$ shown in Fig. 5(b) of $(Q^{(3)} = 7$ rep-tiles, namely, $Q^{(1)}_R = 3$ rep-tiles of order $R$, while $S = 4$ rep-tiles of order $R - 1 (Q^{(1)}_{R - 1} = 5)$, being $Q^{(h)} = \sum_{r=1}^R Q^{(h)}_r$. Afterward, other two tiles including $I^{(2)}(I^{(2)} = 12)$ [see Fig. 5(c) and $I^{(3)}$ [see Fig. 5(e)] array elements are the candidates for division (10) at the iteration $h = 2$ and $h = H$, respectively, $e^{(2)}$ of $Q^{(2)} = 10 (Q^{(2)}_R = Q^{(2)}_{R - 1} = 3$ and $Q^{(2)}_{R - 2} = S$) rep-tiles [see Fig. 5(d)], and $e^{(3)}$ of $Q^{(3)} = 13 (Q^{(3)}_R = 2$, $Q^{(3)}_{R - 1} = 7$, and $Q^{(3)}_{R - 2} = S$) rep-tiles [see Fig. 5(f)] being the generated array clusternings.

IV. Numerical Results

This section is devoted to the analysis of the behavior and the assessment of the performance of the proposed RTAM.

The first example deals with an array comprising $M \times N = 8 \times 12$ isotropic [i.e., $e(u, v) = 1$] radiating elements with uniform half-wavelength ($d_e = d_r = (\lambda/2)$) interelement spacing. The reference excitations have been computed with a convex programming (CP)-based approach [47], mathematically formulated as in Appendix II to fit the upper bound power mask $\Psi(u, v)$ in Fig. 6(a), which is centered in $(u_0, v_0) = (0, 0)$, and it is characterized by a rectangular mainlobe region of dimensions $BW_r = 0.50$ and $BW_e = 0.76$ along the
iteration: (a) $h$ (e) rep-tiled array layout and (b), (d), and (f) rep-tile splitting at the RTAM reference excitations, \{\hat{u}, \hat{v}\} maps of (a) power pattern mask corresponding power pattern, affording a sidelobe level of \{\hat{w}\}_{\text{ref}} = \{\alpha_{\text{ref}}\}; m = 1, \ldots, M, n = 1, \ldots, N\}. Color maps of (a) power pattern mask $\Psi(u, v)$ and (b) amplitude distribution of the reference excitations, $w_{\text{ref}} = a_{\text{ref}}$; $m = 1, \ldots, M, n = 1, \ldots, N$, while the corresponding power pattern, affording a sidelobe level of SLL = −25 [dB] and matching the mask $\Psi(u, v)$, is plotted along the principal planes [i.e., $u = 0$; see Fig. 7(b) and $v = 0$; see Fig. 7(c)].

By considering the $R = 2$th level alphabet of L-tromino rep-tiles with $I^{(R)} = 12$ elements, the $R$-order tileability of the aperture $A$ has been successfully checked against the theorem in Section III-A. Indeed, the second condition of the L-tromino covering theorem holds true since $I^{(R)} = 2$. The RTAM has been then executed by setting the maximum partitioning of the array to $Q_{\text{max}} = 14$ tiles. At Step 1 ($h = 0$), $T^{(R)} = 18$ (see Table 1—$\hat{M} = 4$ and $\hat{N} = 6$), different tilings of the aperture with $I^{(R)}$-sized L-trominoes have been generated with Algorithm-X [43], [44], and the optimal $R$th order tiling, $\mathbf{c}_{\text{opt}}$, which minimizes the mask matching (8), is shown in Fig. 8(a), where the color level representation of the subarray excitations, $\hat{w}_{\text{opt}} \in \{\hat{w}_{\text{opt}} = \hat{a}_{\text{opt}} e^{j\hat{\phi}_{\text{opt}}}; q = 1, \ldots, \hat{Q}\}$, is given as well. Afterward ($h = 1$), the STM is computed (9) for each qth ($q = 1, \ldots, \hat{Q}$; $\hat{Q} = 8$) tile of $\mathbf{c}_{\text{opt}}$, $\xi^{(1)} = \{\hat{q}\}_{h=1}$ [see Fig. 8(b)], and the tile with the maximum value (i.e., $\hat{q} = 7$) has been selected (10) and divided into $S = 4$ rep-tiles of level ($R - 1$) = 3 elements to yield the new array tiling $\mathbf{c}^{(h)}_{\text{opt}}$ in Fig. 8(c) ($Q^{(h)} = 11$). Once the $Q^{(1)}$ tiles of $\mathbf{c}^{(1)}$ [see Fig. 8(d)] have been ranked (10) according to their STM values ($\hat{q} = 2$) [see Fig. 8(d)], a further iteration ($h = 2$) of the RTAM has been completed by deriving the tiled arrangement, $\mathbf{c}^{(h)}_{\text{opt}}$ in Fig. 8(e) ($Q^{(h)} = 14$ being $Q^{(h)} = 11$ and $Q^{(h)}_{\text{opt}} = 8$). The iterative procedure has been stopped (Step 2.5) after $H = 2$ iterations since the convergence condition on the maximum number of subarrays has been reached (i.e., $Q^{(h)} = Q_{\text{max}}$), and the final clustered architecture has been set to the current one ($\mathbf{c}_{\text{opt}} = \mathbf{c}^{(2)}$). This latter affords a power pattern [see Fig. 7(b) and (c)] with a mask matching index equal to $\Gamma_{\text{opt}} = 7.31 \times 10^{-4}$ $\Gamma_{\text{opt}} = \Gamma(\mathbf{c}_{\text{opt}})$.

To assess the computational efficiency and the reliability of the RTAM to find the optimal tiling of the aperture $A$, that is (in other words), to prove that the solution $\mathbf{c}_{\text{opt}}$ is the global optimum subject to the constraint $Q \leq Q_{\text{max}}$, Algorithm-X has been used [43], [44] in an exhaustive way to generate all clustered layouts with $I^{(R-1)}$, and $I^{(R)}$-sized ($R = 2$) L-tromino tiles, each having $B = 4$ possible rotations. Once the whole set of $T \approx 3.0 \times 10^7$ complete
tilings has been generated, the mask matching index (7) of the 
$T_{Q_{\text{max}}}$ = 6248 partitions with $Q \leq Q_{\text{max}}$ tiles (i.e., $Q_H = 6$ and $Q_{R-1} = 8$ being $Q = \sum_{r=1}^{R} Q_r$) has been computed to
determine the optimal solution of the problem at hand, $\Gamma_{\text{opt}}$. The
mask matching value of the $T_{Q_{\text{max}}}$ solutions, sorted from
the worst to the best, is reported in Fig. 7(a) and compared to
that from the RTAM at the convergence, $\Gamma_{\text{opt}}$. As it can be also
inferred by the plots of the power patterns in Fig. 7(b) and (c),
the optimal solution found by Algorithm-X exhaustive search
coincides with the RTAM one (i.e., $\Gamma_{\text{opt}} \equiv \Gamma_{\text{opt}}$) although
this latter has been yielded saving the 98.75% of the CPU
computational costs (i.e., $\Delta_{RTAM} < 1$ [min] versus $\Delta_{X} \approx
80$ [min]) on a 1.6 GHz PC with 8 GB of RAM. Such an
important computational advantage is expected to further grow
with the size of array aperture $A$ since the only generation of
the complete tilings would result unfeasible by Algorithm-X,
while the proposed RTAM is still effective as demonstrated in
the following examples.

The second example is aimed at pointing out the advantages
of using the $L$-tromino rep-tiles instead of the square ones.
Indeed, the square geometry belongs to the rep-tiles family
since it can be represented as the union of $S = 4$ smaller
copies with halved edge length. Such a comparison has
been performed on a benchmark array of $M \times N = 24 \times 24$ isotropic
elements spaced by $d_x = d_y = (\lambda/2)$, while the reference
excitations [see Fig. 9(b)] have been still synthesized with the
CP subject to the pattern constraints [i.e., $SLL_1 = -25$ [dB], $SLL_2 = -30$ [dB], $BW_u = BW_v = 0.274$, and $(\theta_0, \phi_0) = (0, 0)$] coded by the mask $\Psi(u, v)$ in Fig. 9(a).
Moreover, the maximum tiles’ order has been set to $R = 3$ so that$L$-trominoes with $I_{L}^{R-2} = 3$, $I_{L}^{R-1} = 12$, and $I_{L}^{R} = 48$
elements and squares with $I_{S}^{R-2} = 4$, $I_{S}^{R-1} = 16$, and $I_{S}^{R} = 64$ elements have been used for tiling the array
aperture $A$. Furthermore, $Q_{\text{max}}$ has been set to $Q_{\text{max}} = 120$ to
to have a TRM reduction of about $\Delta_{T_{\text{RM}}} \approx 79\%$ ($\Delta_{T_{\text{RM}}} =
\Delta_{T_{\text{RM}}} - Q_{\text{max}}$) being $\Delta_{T_{\text{RM}}} = 1 - (Q/(M \times N))$.

Inasmuch as the $R$-order tileability condition holds true,
the RTAM has been applied with both $L$-shaped and square
tiles. Fig. 10 shows the behavior of the mask matching versus
the number of subarrays, $Q$, of the $H = 36$ layouts
generated during the RTAM loop. One can notice that the
$L$-shaped arrangement always outperforms the corresponding
(i.e., $Q_{L} = Q_{S}$) square one, more and more as the heavier the clustering ratio is (i.e., $\Delta_{T_{\text{RM}}} \rightarrow 100\%$). Table II summarizes
the results of the comparative study by reporting the mask
matching index along with the key pattern descriptors (i.e.,
the SLL, the directivity, $D$, the half-power beamwidth along azimuth, $HPBW_{\phi, z}$, and the elevation, $HPBW_{\phi, y}$, planes) for the set of selected layouts with $Q = \{39, 81, 120\}$ rep-
tiles. For illustrative purposes, the tilings synthesized at the $h = 23$th iteration of the RTAM with $Q_{S} = 81$ tiles are shown in Fig. 11(a) and (b), while the corresponding power patterns along the $v = 0$ and the $u = 0$ cuts are plotted in Fig. 11(c) and (d), respectively. In order to further compare
the behavior of the $L$-shaped and squared tiles, the effect on
the SLL when scanning the beam around the pointing direction

Fig. 8. Numerical validation ($M = 8$, $N = 12$, $\Psi(u, v) = 1$,
$d_x = d_y = (\lambda/2)$, and $(\theta_0, \phi_0) = (0, 0)$) $\rightarrow (\theta_0, \phi_0) = (0, 0)$ [deg];
$Q_{\text{max}} = 14$, $R = 2$, $I_{L}^{R-1} = 12$, and $I_{L}^{R} = 3$). Color maps of (a), (c),
and (e) amplitude distribution of the clustered excitations, ($w_{\text{ref}}^{(h)} = a_{\text{ref}}^{(h)} e^{j\phi_{\text{ref}}^{(h)}} ;
q = 1, \ldots , Q_{\text{max}}$), and (b) and (d) STML value within the array aperture at the
RTAM iterations: (a) and (b) $h = 0 (Q_{S} = 8)$, (c) and (d) $h = 1 (Q_{S} = 11)$,
and (e) $h = H (H = 2) (Q_{S} = Q_{\text{max}})$.

Fig. 9. Numerical validation ($M = 24$, $N = 24$, $\Psi(u, v) = 1$,
$d_x = d_y = (\lambda/2)$, and $(\theta_0, \phi_0) = (0, 0)$) $\rightarrow (\theta_0, \phi_0) = (0, 0)$ [deg]). Color
maps of (a) power pattern mask $\Psi(u, v)$ and (b) amplitude distribution of the
reference excitations, ($w_{\text{ref}}^{(h)} = a_{\text{ref}}^{(h)} ; m = 1, \ldots , M, n = 1, \ldots , N$).

Fig. 10. Numerical validation ($M = 24$, $N = 24$, $\Psi(u, v) = 1$,
$d_x = d_y = (\lambda/2)$, and $(\theta_0, \phi_0) = (0, 0)$) $\rightarrow (\theta_0, \phi_0) = (0, 0)$ [deg];
$Q_{\text{max}} = 120$, $R = 3$, $I_{L}^{R-1} = 48$, $I_{S}^{R-1} = 12$, $I_{L}^{R} = 3$, $I_{S}^{R} = 64$,
$I_{S}^{R-1} = 16$, and $I_{S}^{R-2} = 4$). Plot of the mask matching value, $\Gamma_{S}$, of the
tiled array configuration, $e^{(h)}$ [i.e., $\Gamma_{S} \equiv \Gamma_{S}$], synthesized at the $h$th
($h = 0, \ldots , H; H = 36$) RTAM loop versus its number of subarrays, $Q_{S}$.
is analyzed for the two arrays of Fig. 11(a) and (b). Toward this end, the subarray phases \( \{ \beta_m; q = 1, \ldots, Q \} \) have been set according to (6) starting from the reference values

\[
\beta_m^{\text{ref}} = -k(x_m u_x + y_m v_y)
\]

(12)

\( m = 1, \ldots, M; n = 1, \ldots, N \), where \( u_x \triangleq \sin(\theta_0 + \theta_i) \cos(\phi_0 + \phi_i) \) and \( v_y \triangleq \sin(\theta_0 + \theta_i) \sin(\phi_0 + \phi_i) \). The color maps shown in Fig. 11 clearly highlight the superior performance of the array with L-shaped rep-tiles [see Fig. 11(e)] compared to the array with square rep-tiles [see Fig. 11(f)] of achieving lower SLL values and better \( \phi \)-invariant behavior.

The third experiment refers to a wider rectangular aperture \( A \times N = 24 \times 36 \) elements. Because of the array size, the use of an exhaustive tiling approach becomes almost impractical.

### Table II

| \( Q \) | \( \Gamma \) | SLL [dB] | \( D \) [dB] | \( \text{HPBW}_{\text{max}} \) [deg] | \( \text{HPBW}_{\text{el}} \) [deg] |
|---|---|---|---|---|---|
| Reference | 576 | \( 7.01 \times 10^{-3} \) | -24.99 | 31.58 | 5.12 | 5.11 |
| L-trominos | 39 | \( 5.53 \times 10^{-3} \) | -19.31 | 31.65 | 4.97 | 4.97 |
| Squares | 39 | \( 1.07 \times 10^{-4} \) | -22.55 | 31.61 | 5.02 | 5.02 |
| L-trominos | 81 | \( 1.77 \times 10^{-4} \) | -23.54 | 31.60 | 5.05 | 5.05 |
| Squares | 81 | \( 4.08 \times 10^{-5} \) | -24.49 | 31.59 | 5.06 | 5.07 |
| L-trominos | 120 | \( 8.11 \times 10^{-6} \) | -24.17 | 31.60 | 5.08 | 5.08 |
| Squares | 120 | \( 2.23 \times 10^{-6} \) | -24.10 | 31.58 | 5.09 | 5.09 |

### Fig. 12

Numerical validation \( (M = 24, N = 36, e(u, v) = 1, d_i = d_i = \lambda/2, \) and \( (u_0, v_0) = (0, 0) \rightarrow (\theta_0, \phi_0) = (0, 0) \) [deg]; \( Q_{\max} = 120, R = 3, I_L^{(R)} = 48, I_L^{(R-1)} = 12, I_L^{(R-2)} = 3, I_L^{(R-3)} = 64, I_L^{(R-4)} = 16, I_L^{(R-5)} = 4, h = 60, \) and \( Q^{(h)}|_{h=60} = 81 \). Plots of (a) power pattern mask \( \Psi(u, v) \) and (b) normalized power pattern radiated by (c) amplitude distribution of the reference excitations, \( [u_m^{\text{ref}} = e^{\text{j} \theta_m^0}]; m = 1, \ldots, M, n = 1, \ldots, N \).
computationally unfeasible. Otherwise, the RTAM has been efficiently applied starting from the CP-synthesized reference array in Fig. 12(c), which affords the power pattern of Fig. 12(b) that fulfills the pattern mask $\Psi(u, v)$ featuring the following descriptors: $SLL_1 = -25$ [dB], $SLL_2 = -30$ [dB], $BW_u = 0.164$, and $BW_v = 0.274$ [see Fig. 12(a)]. In more detail, the RTAM has been executed with a rep-tiles alphabet of order $R = 3$ ($I^{(R)} = 4$, $I^{(R-1)} = 6$, and $N = 9$) so that the second condition of the covering theorem is fulfilled, while the choice of rep-tiles of higher order (e.g., $R = 4 \rightarrow I^{(R)} = 192$, $I^{(R-1)} = 8$, $M = 3$, and $N = 5$) would not have guaranteed the complete tileability of $A$. Moreover, the minimum TRM saving has been fixed to $\Delta_{min}^{TRM} \approx 69\%$ (i.e., $Q_{max} = 270$).

The RTAM has been initialized (Step 1) with the $T^{(R)}(t) = 4312$ (i.e., $M = 6$ and $N = 9$; see Table I) uniform tilings of $\tilde{Q} = 18$ $I^{(R)}$-sized $L$-tromino rep-tiles generated by Algorithm-X. The one, $\tilde{C}_{opt}$, with minimum mismatch (i.e., $\tilde{F}_{opt} = \Gamma^{(0)}|_{h=0} = 7.33 \times 10^{-5}$ being $\tilde{F}_{opt} \triangleq \Gamma(\tilde{C}_{opt})$; see Table III and Fig. 13) is shown in Fig. 14(a), the corresponding normalized power pattern at the RTAM iterations: (a) and (c) $h = 0$ ($Q^{(h)} = 18$), (b) and (f) $h = 10$ ($Q^{(h)} = 48$), (c) and (g) $h = 44$ ($Q^{(h)} = 150$), and (d) and (h) $h = H$ ($H = 84$) ($Q^{(h)} = Q_{max}$).

Fig. 14. Numerical validation ($M = 24$, $N = 36$, $e(u, v) = 1$, $d_u = d_v = (\lambda/2)$, and $(\theta_0, \phi_0) = (0, 0) \rightarrow (\theta_0, \phi_0) = (0, 0)$ [deg]; $Q_{max} = 270$, $R = 3$, $I^{(R-2)} = 48$, $I^{(R-1)} = 12$, and $I^{(R)} = 3$). Plots of (a)–(d) amplitude distribution of the clustered excitations, $w_q^{(h)} = w_q^{(h)}, q = 1, \ldots, Q^{(h)}$, and (e)–(h) corresponding normalized power pattern at the RTAM iterations: (a) and (c) $h = 0$ ($Q^{(h)} = 18$), (b) and (f) $h = 10$ ($Q^{(h)} = 48$), (c) and (g) $h = 44$ ($Q^{(h)} = 150$), and (d) and (h) $h = H$ ($H = 84$) ($Q^{(h)} = Q_{max}$).

Differently, the tiled configuration with $Q = 48$ clusters further reduces the architectural complexity up to $Q_{TRM} \approx 94.4\%$, but it gets worse in terms of mask matching ($\Gamma = 7.33 \times 10^{-5}$; see Table III) and sidelobe level ($SLL|_{Q=48} = -21.33$; see Table III). As expected, it turns out that the more the subarrays, the smaller the value of the mask matching index is at the cost of a higher complexity of the feeding network (e.g., $Q = 270$, $\Gamma = 7.17 \times 10^{-6}$, and $SLL = -24.70$ [dB]; see Table III). However, it is worth noting from Fig. 13 that the mask-mismatch variations become almost negligible, from a practical viewpoint, when $Q > 150$, since the SLL improves less than 0.1 [dB] as also confirmed by the comparison of the power patterns along the $v = 0$ and $u = 0$ planes in Fig. 15(a) and (b), respectively.

In order to validate the RTAM with arrays having nonzero reference phase excitations, the next experiment addresses the design of the same size array of the previous test case ($M \times N = 24 \times 36$), but fitting a mask $\Psi(u, v)$ steered toward the direction $(\theta_0, \phi_0) = (0.0755, 0.0436)$ (i.e., $(\theta_0, \phi_0) = (5, 30)$ [deg]) having $SLL_1 = -20$ [dB], $SLL_2 = -25$ [dB], $BW_u = 0.125$, and $BW_v = 0.2$ [see Fig. 16(a)]. Fig. 16(b) shows the reference pattern radiated by the CP-optimized amplitude and phase coefficients in Fig. 16(c) and (d), respectively.

The behavior of the mask matching index versus the number of clusters, $Q$, of the $H = 84$ tiled arrangements iteratively
Numerical validation

Fig. 16. Numerical validation ($M = 24$, $N = 36$, $e(a, v) = 1$, $d_1 = d_2 = (\lambda/2)$, and $(u_0, v_0) = (0, 0)$ [deg]; $Q_{\text{max}} = 270$, $R = 3$, $I^{(R)} = 48$, $I^{(R-1)} = 12$. Plots of the normalized power patterns along (a) $v = 0$ (i.e., $\phi = 0$ [deg]) and (b) $u = 0$ (i.e., $\phi = 90$ [deg]) planes.

(b)

Fig. 15. Numerical validation ($M = 24$, $N = 36$, $e(a, v) = 1$, $d_1 = d_2 = (\lambda/2)$, and $(u_0, v_0) = (0, 0)$ [deg]; $Q_{\text{max}} = 270$, $R = 3$, $I^{(R)} = 48$, $I^{(R-1)} = 12$, and $I^{(R-2)} = 3$). Plots of the normalized power pattern radiated by (c) amplitude and (d) phase distributions of the RTAM-synthesized excitations, $\{w_{\text{ref}}^{(m)}(u, v)\}$, for $m = 1, \ldots, M$, $n = 1, \ldots, N$.

generated by the RTAM is shown in Fig. 17, while the pattern features of the layouts having $Q = \{18, 48, 150, 270\}$ are given in Table IV, as well. For comparative purposes and analogously to the previous (nonsteered) case, the configuration with $Q = 150$ subarrays is analyzed. Fig. 18 indicates the aperture tiling within the color maps of the corresponding subarray amplitude [see Fig. 18(a)] and phase [see Fig. 18(b)] coefficients. Unlike the clustered layouts synthesized for the broadside-steered arrays [e.g., Figs. 11(a) and (b) and 14(c)], where wider tiles are placed at the center of the array, the central part of the aperture is here covered by smaller tiles [see Fig. 18(a) and (b)] to better match the complex reference excitations. As for the radiated power pattern, the plots along the main planes in Fig. 18(c) and (d) show nonnegligible

TABLE IV

| $Q$   | $\Gamma$ | $SLL$ [dB] | $D$ [dB] | $HPBW_{ax}$ [deg] | $HPBW_{el}$ [deg] |
|-------|----------|------------|---------|-------------------|-------------------|
| 18    | 7.73 x 10^{-4} | -15.91     | 33.10   | 3.18              | 4.03              |
| 48    | 2.85 x 10^{-4} | -15.91     | 33.10   | 3.18              | 4.03              |
| 150   | 2.32 x 10^{-4} | -19.30     | 33.53   | 3.18              | 4.70              |
| 270   | 1.52 x 10^{-6} | -19.71     | 33.71   | 3.18              | 4.77              |

Fig. 17. Numerical validation ($M = 24$, $N = 36$, $e(a, v) = 1$, $d_1 = d_2 = (\lambda/2)$, and $(u_0, v_0) = (0.0755, 0.0436)$ [deg]; $Q_{\text{max}} = 270$, $R = 3$, $I^{(R)} = 48$, $I^{(R-1)} = 12$, and $I^{(R-2)} = 3$). Plot of the mask matching value, $\Gamma^{(h)}$, of the tiled array configuration, $e^{(h)}$ (i.e., $\Gamma^{(h)} \Delta \Gamma^{(h')}$), synthesized at the $h$th ($h = 0, \ldots, H = 90$) RTAM loop versus its number of subarrays, $Q^{(h)}$. 

Fig. 18. Numerical validation ($M = 24$, $N = 36$, $e(a, v) = 1$, $d_1 = d_2 = (\lambda/2)$, and $(u_0, v_0) = (0.0755, 0.0436)$ [deg]; $Q_{\text{max}} = 270$, $R = 3$, $I^{(R)} = 48$, $I^{(R-1)} = 12$, and $I^{(R-2)} = 3$)—PATTERN FEATURES AND MASK MATCHING INDEX

References

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violations mainly in the mask region with $SLL_2 = -25$ [dB], while the mask matching is satisfactory near the main beam with an SLL violation of only 0.7 [dB] (see Table IV).

Next, the reliability of the RTAM-optimized tiled array when scanning the beam around the pointing direction has been investigated. Fig. 19(a) shows the color map of the SLL when scanning the beam around the pointing direction $(\theta_0, \phi_0) = (5, 30)$ [deg] within the cone $0 \leq \phi_s < 360$ [deg] and $\theta_0 \leq 30$ [deg]. It turns out that $SLL_{\text{max}} = -14.5$ [dB] and $SLL_{\text{avg}} = -18.27$ [dB] for the beam scan region $\{\theta_s \leq 5$ [deg], $0 \leq \phi \leq 360$ [deg]\}, with $SLL_{\text{max}}$ and $SLL_{\text{avg}}$ being maximum and the average SLL, respectively. Analogously, the variations of the directivities are reported in Fig. 19(b). The values of the maximum, minimum, and average directivities achieved within the same scan region are equal to $D_{\text{max}} = 33.84$ [dBi], $D_{\text{min}} = 32.67$ [dBi], and $D_{\text{avg}} = 33.39$ [dBi].

The last experiment is concerned with the asymmetric pattern mask of Fig. 20(a), which is centered at broadside with beamwidths $BW_a = 0.080$ and $BW_v = 0.125$, and it is characterized by three sidelobe regions (i.e., $SLL_1 = -20$ [dB], $SLL_2 = -25$ [dB], and $SLL_3 = -30$ [dB]). Such a mask is fulfilled by the power pattern in Fig. 20(b) radiated by a fully populated array of $M \times N$ elements with excitations in Fig. 20(c) and (d).

As a representative example, Fig. 21 pictorially describes the compromise/intermediate solution, synthesized after $h = 44$ iterations by the RTAM, having $Q^{(b)}_{h=44} = 150$ subarrays. As expected, the rep-tiles layout presents the smaller tiles in the regions of the array aperture where there are the most relevant variations in the spatial distribution of the phase and the amplitude of the reference excitations [see Fig. 21(a) and (b)]. As a result, the corresponding pattern faithfully fits the mask constraint along the $v = 0$ cut [see Fig. 21(c)], while there are two sidelobes in the $SLL_3 = -30$ [dB] sidelobe suppression region of the $u = 0$ cut that exceeds the target bound of about 2 [dB] [see Fig. 21(d) and Table V].

V. CONCLUSION

In this work, the design of highly modular planar PAs has been addressed by exploiting the self-replicating property of rep-tiles. An innovative deterministic method has been proposed to synthesize irregular rep-tiles layouts fitting user-defined power pattern requirements, which are mathematically...
coded in suitable pattern-masks, by jointly minimizing the number of clusters to fully cover the array aperture.

From the numerical assessment, the following outcomes and conclusions can be drawn.

1) The proposed RTAM is able of converging toward optimal compromise solutions of the constrained synthesis problem at hand by also enabling a nonnegligible saving of the computational cost with respect to state-of-the-art exhaustive approaches.

2) Since the iterative RTAM progressively generates multiple clustered array layouts, which are different tradeoffs between the number of clusters $Q$ and the fulfillment of the user-defined pattern requirements, it implicitly provides to the user, besides the optimal-matching arrangement at the convergence, a Pareto front of compromise solutions.

3) Thanks to the deterministic recursive subdivision of wider rep-tiles into smaller ones in correspondence with larger variations of the amplitude/phase distribution of the reference excitations, the RTAM is an effective and reliable tool for the design of arbitrary-large/dense arrays.

4) $L$-tromino rep-tiles turn out to be more effective than the square ones since they allow one to better fit the user-defined power pattern mask with fewer subarrays.

Future research activities, beyond the scope of this work, will include the investigation of different rep-tile families and aperture geometries among those of interest for modern PA applications.

APPENDIX I

The matrix $G_{\hat{M}}$ ($\hat{M} \geq 1$) is recursively computed as follows:

$$G_{\hat{M}} = \begin{bmatrix} S_{\hat{M}-1} & H_{\hat{M}-1} \\ G_{\hat{M}-1} & S_{\hat{M}-1} \end{bmatrix}$$

where

$$S_{\hat{M}} = \begin{bmatrix} Z_{\hat{M}-1} & G_{\hat{M}-1} \\ Z_{\hat{M}-1} & Z_{\hat{M}-1} \end{bmatrix}$$

and

$$H_{\hat{M}} = \begin{bmatrix} G_{\hat{M}-1} & 2 \times S_{\hat{M}-1} \\ Z_{\hat{M}-1} & 2 \times G_{\hat{M}-1} \end{bmatrix}$$

with $Z_{\hat{M}}$ being the $2\hat{M} \times 2\hat{M}$ null matrix and setting $G_{\hat{M}}|_{\hat{M}=0} = 1$, $S_{\hat{M}}|_{\hat{M}=0} = 0$, and $H_{\hat{M}}|_{\hat{M}=0} = 0$.

APPENDIX II

The synthesis of the reference excitations $w^{ref}$ is addressed through the minimization of the following function:

$$\min_{R\{w^{ref}\}, Z\{w^{ref}\}} \left\{ \int_0^{2\pi} \int_0^{\pi} \mathcal{P}^{ref}(\theta, \phi) \sin \theta d\theta d\phi \right\}$$

subject to the constraints

$$\mathcal{P}^{ref}(u_0, v_0) = 1$$

$$\mathcal{P}^{ref}(u_k, v_k) \leq \Psi(u_k, v_k)$$

where the user-defined power upper mask $\Psi(u, v)$ is sampled over a set of $K$ angular directions, $(u_k, v_k), k = 1, \ldots, K$, is uniformly distributed across the sidelobe region, and considering the invisible range (i.e., the angles such that $u^2 + v^2 > 1$) to avoid the presence of grating lobes in case the beam scanning is required. It is worth pointing out that the minimization of (16), jointly satisfying the constraint (17), allows maximizing the peak directivity of the pattern along the pointing direction $D = D(u_0, v_0)$ since it is defined as

$$D(u_0, v_0) = \frac{\mathcal{P}^{ref}(u_0, v_0) \sin \theta d\theta d\phi}{\int_0^{2\pi} \int_0^{\pi} \mathcal{P}^{ref}(\theta, \phi) \sin \theta d\theta d\phi}. \tag{19}$$

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ANSELMI et al.: SELF-REPLICATING SINGLE-SHAPE TILING TECHNIQUE FOR THE DESIGN OF HIGHLY MODULAR PLANAR PAs 3347
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