Complete Solving of Explicit Evaluation of Gauss Sums in the Index 2 Case *

Dedicated to Professor Yuan Wang on the occasion of his 80th Birthday

Jing Yang
Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, China;
& Division of Mathematical Sciences, School of Physical and Mathematical Sciences,
Nanyang technological University, 637371, Singapore
Email: jingyang@math.tsinghua.edu.cn

Lingli Xia†
Basic Courses Department of Beijing Union University, Beijing, 100101, China
Email: lingli@buu.edu.cn

Abstract

Let $p$ be a prime number, $N$ be a positive integer such that $\gcd(N, p) = 1$, $q = p^f$ where $f$ is the multiplicative order of $p$ modulo $N$. And let $\chi$ be a primitive multiplicative character of order $N$ over finite field $\mathbb{F}_q$. This paper studies the problem of explicit evaluation of Gauss sums $G(\chi)$ in “index 2 case” (i.e. $\left\langle (\mathbb{Z}/N\mathbb{Z})^* :< p > \right\rangle = 2$). Firstly, the classification of the Gauss sums in index 2 case is presented. Then, the explicit evaluation of Gauss sums $G(\chi^\lambda)$ ($1 \leq \lambda \leq N - 1$) in index 2 case with order $N$ being general even integer (i.e. $N = 2^r \cdot N_0$ where $r, N_0$ are positive integers and $N_0 \geq 3$ is odd.) is obtained. Thus, combining with the researches before, the problem of explicit evaluation of Gauss sums in index 2 case is completely solved.

**keywords:** Gauss sum, Stickelberger’s Theorem, Stickelberger congruence, Davenport-Hasse lifting formula, Davenport-Hasse product formula

1 Introduction

Gauss sum is one of the most important and fundamental objects and tools in number theory and arithmetical geometry. The explicit evaluation of Gauss sums is an important but difficult problem, which has not only theoretical value in number theory and arithmetical geometry, but also important practical applications in computer science, information theory, combinatorics and experimental designs.

In 1801, C. F. Gauss [7] gave the first result of this problem for quadratic Gauss sums over $\mathbb{F}_p$. More exactly, he determined the sign of the quadratic Gauss sums. Let $N$ be the order of Gauss sum (the definition is in section 2). For the relatively small $N$, such as $N = 3, 4, 5, 6, 8, 12$, people also gave researches by the properties of the cyclotomic fields with relatively small degree. One can see the more details in the Chapter 4 of [3] or [8, §9.12].

In another research direction, for the Gauss sums with relatively large orders $N$, by the Galois Theory of cyclotomic fields, people have evaluated Gauss sums in some cases, such as pure Gauss sum and the ones in index 2 and 4 case. For pure Gauss sums, i.e. the case that $-1 \in< p > \subset (\mathbb{Z}/N\mathbb{Z})^*$, Stickelberger [16] gave an evaluation of Gauss sums $G(\chi)$ in 1890. (Also see [3] [11.6], [10] Thm5.16 and Lemma2.2 of this paper.)

In 1970’s–2000’s, for the Gauss sums of index 2, i.e. the case that $-1 \not\in< p > \subset (\mathbb{Z}/N\mathbb{Z})^*$ and $\left\langle (\mathbb{Z}/N\mathbb{Z})^* :< p > \right\rangle = 2$, a series explicit evaluations of Gauss sums have been given. In this case, the order $N$ of Gauss sum $G(\chi)$ has no more than 2 distinct odd primes factors. In 1970’s, R. J. McEliece [13] gave the evaluation of Gauss sums in index 2 case for $N = l$ ($l$ is odd prime) and applied this to determine the (Hamming) weight distribution...
of some irreducible cyclic codes. In 1992, M. Van Der Vlugt [17] gave the evaluation of Gauss sums in index 2 case for \( N = l_1 l_2 \) (\( l_1 \) and \( l_2 \) are distinct odd primes). Similarly, the result was applied to calculate the Hamming weight distribution of some irreducible cyclic codes. (For the details on the relationship between Hamming weight distribution of irreducible cyclic codes and Gauss sums, see \([1, 2, 11, 18, 15]\) or \([3, \S 11.7]\).) In 1997, P. Langevin [9], as generalization of \([13]\), gave the evaluation of Gauss sums in index 2 case for \( N = t^r \) (\( t \) is odd prime, \( r \geq 1 \)). One year later, O. D. Mbodj [12], as generalization of \([17]\), gave the evaluation of Gauss sums in index 2 case for \( N = l_1^r l_2^r \) (\( l_1, l_2 \) are distinct odd primes, \( r_1, r_2 \geq 1 \)). For \( N \) being power of 2, i.e. \( N = 2^t \), (\( t \geq 3 \)), P. Meijer and M. van der Vlugt [14], in 2003, evaluated the Gauss sums in index 2 case and applied the results of Gauss sums to solve the problem of calculating the number of rational points for some algebraic curves. Since 2005, K. Feng, S. Luo and J. Yang [6,19,5] have given explicit evaluation of Gauss sums in index 4 case for \( N \) being odd and power of 2.

Up till now, there is no any work to study the Gauss sums in index 2 or 4 case with \( N \) being “general even number”, i.e. \( N = t^r \cdot N_0 \) where \( r, N_0 \) are positive integers and \( N_0 \geq 3 \) is odd. In this paper, we list all the classifications of index 2 case, and give explicit evaluation of Gauss sums with \( N \) being general even number. Thus, combining with the previous papers, the problem of explicit evaluation of Gauss sums in index 2 case is completely solved.

This paper is organized as follow. Firstly, in section 2, we introduce the preliminaries we need including the definitions and several famous formulas about Gauss sums. In section 3, we present all the classifications of index 2 case. More exactly, we list six subcase A, B, C, D, E and F according to the factorization of \( N \). Then, in section 4.1, we present and prove the explicit formulas of Gauss sums \( G(\chi) \) in the later three subcases (Case D, E and F). Finally, in section 4.2, we give the evaluation of Gauss sums \( G(\chi^\lambda) \) (\( 1 \leq \lambda \leq N - 1 \)) in all the six subcases.

## 2 Preliminaries

Let \( p \) be a prime number, \( N \geq 2 \) be an integer such that \( (N, p) = 1 \). Let \( f \) be the multiplicative order of \( p \) modulo \( N \), denote by \( f = \text{ord}_N(p) \), i.e. \( f \) is the smallest positive integer such that \( p^f \equiv 1 \pmod{N} \). Take \( q = p^f \) and \( \chi \) be a primitive multiplicative character of order \( N \) over \( \mathbb{F}_q \), \( T \) be the trace map from \( \mathbb{F}_q \) to \( \mathbb{F}_p \). Then, for \( 1 \leq \lambda \leq N - 1, 1 \leq \mu \leq p - 1 \), the Gauss sum over \( \mathbb{F}_p \) is defined as

\[
G(\chi^{\lambda}, \mu) := \sum_{x \in \mathbb{F}_p} \chi(x)\zeta_p^{\mu x},
\]

(2.1)

where \( \zeta_p = \exp(2\pi i/p) \) is complex primitive \( p \)-th root of 1. When \( (\lambda, N) = 1 \), \( G(\chi^{\lambda}, \mu) \) is called Gauss sum of order \( N \). \( N \) is called the order of \( G(\chi^{\lambda}, \mu) \).

Since \( G(\chi^{\lambda}, v) = \chi^{\lambda}(v) G(\chi^{\lambda}) \), we can just consider \( G(\chi^r, 1) \), which is denoted as \( G(\chi^{\lambda}) \) for simplicity. Generally, \( G(\chi^{\lambda}) \) is belong to the ring of integers of cyclotomic field \( \mathbb{Q}(\zeta_N p) = \mathbb{Q}(\zeta_N, \zeta_p) \). As we known, the Galois group \( \text{Gal}(\mathbb{Q}(\zeta_N, \zeta_p)/\mathbb{Q}) \) is isomorphic to group \( (\mathbb{Z}/N\mathbb{Z})^* \cong (\mathbb{Z}/N\mathbb{Z})^* \times (\mathbb{Z}/p\mathbb{Z})^* \). More exactly,

\[
\text{Gal}(\mathbb{Q}(\zeta_N, \zeta_p)/\mathbb{Q}) = \{ \sigma \tau_1 | l \in (\mathbb{Z}/N\mathbb{Z})^*, t \in (\mathbb{Z}/p\mathbb{Z})^* \},
\]

where

\[
\sigma_l(\zeta_N) = \zeta_N^l, \quad \sigma_l(\zeta_p) = \zeta_p^l, \quad \tau_1(\zeta_N) = \zeta_N, \quad \tau_1(\zeta_p) = \zeta_p.
\]

The following Lemma shows several basic results for Gauss sums. For more details, we refer to \([8, \S 8.2]\), and \([10, \S 5.2]\).

**Lemma 2.1.** For \( l \in (\mathbb{Z}/N\mathbb{Z})^* \), \( l \in (\mathbb{Z}/p\mathbb{Z})^* \), let \( G(\chi) \) be Gauss sum of order \( N \) over finite field \( \mathbb{F}_q \). Then,

1. if \( \chi \) is trivial character), \( G(\chi) = -1 \); otherwise, i.e. \( \chi \neq 1, |G(\chi)| = \sqrt{q} \).
2. \( G(\chi) = G(\chi) \), where \( \chi \) denotes the complex conjugation of \( \chi \).
3. \( \sigma_l(\chi^{\lambda}) = \chi^{\lambda}(l) \chi^{\lambda}(l) \), especially, \( \sigma_p(\chi^{\lambda}) = G(\chi^{\lambda}) = G(\chi) \). So \( G(\chi) \in O_K[\zeta_p] \), where \( K \) is the decomposition field of \( p \) in \( \mathbb{Q}(\zeta_N) \), i.e. \( K \) is the fixed subfield of \( \sigma_p \) in \( \mathbb{Q}(\zeta_N) \).
4. \( G(\chi)^N \in O_K \), and \( G(\chi)^N/G(\chi)^N \in O_K \) for each positive integer \( s \) where \( O_K \) is the ring of integers in \( K \).

For the Gauss sum \( G(\chi) \) of order \( N \) over \( \mathbb{F}_p \), \( G(\chi)^N \in \mathbb{Z}[\zeta_N] = O_M \) \( (M = \mathbb{Q}(\zeta_N)) \) by Lemma 2.1.4). The Galois group

\[
G = \text{Gal}(M/\mathbb{Q}) = \{ \sigma_a | 0 \leq a \leq N - 1, (a, N) = 1 \} = \{ \sigma_a(\zeta_N) = \zeta_N^a \}
\]

is canonically isomorphic to group \( (\mathbb{Z}/N\mathbb{Z})^* \). So, \( (\mathbb{Z}/N\mathbb{Z})^* \) is often identified with \( G \). A profound result of Gauss sums was given by S. Stickerberger [16] in 1890, so called the Stickelberger’s Theorem (see \([3, \S 11.2]\) or \([8]\).}
It reveals the prime ideal decomposition of \((G^N(\chi))O_K\). We note that S. Stickelberger, actually, gave another more exact result, so called “Stickelberger Congruence” (see [3] §11.2). And in the following text, we need it to determine the sign (or unit root) ambiguities of Guass sums in some cases.

The first explicit evaluation of Gauss sums, for quadratic character \(\chi(x) = \left(\frac{x}{p}\right)\) (the Legendre symbol) of \(\mathbb{F}_p\), was given by Gauss [7]:

\[
G(\chi) = \begin{cases} 
\sqrt{n}, & \text{if } p \equiv 1 \pmod{4}; \\
\frac{i}{\sqrt{n}}, & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\] (2.2)

This result can be generalized to quadratic Gauss sums over \(\mathbb{F}_q\) for any prime-power \(q \equiv 2 \pmod{4}\) by Davenport-Hasse (Lift) Theorem (see [3] §11.5 or [10, Thm5.14]). More exactly, let \(\chi' = \chi \circ N\). Then, the corresponding quadratic Gauss sums over \(\mathbb{F}_q = \mathbb{F}_{p^f}\) are given by

\[
G(\chi') = \begin{cases} 
(-1)^{d-1} \sqrt{q}, & \text{if } p \equiv 1 \pmod{4}; \\
(-1)^{d} \sqrt{q}, & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\] (2.3)

After Gauss’s result on \(N = 2\), using arithmetic properties on field \(\mathbb{Q}(\zeta_N)\), the value of Gauss sums \(G(\chi)\) with relatively small order \(N\) have been determined explicitly. See [8] §9.12 for cubic and quartic Gauss sums \((N = 3, 4)\) and Chapter 5 of [3] for more other cases.

On the other hand, for the Gauss sums with relatively large order \(N\), by the Galois Theory of cyclotomic field, people have evaluated Gauss sums in some cases. Such as, when \((-1) = q\) \(= 2\) or \(= 3\) \(\equiv 2 \pmod{4}\), Gauss sums \(G(\chi)\) can be determined by the following result, which are called “self-conjugate” or “pure” Gauss sums.

**Lemma 2.2** (See [3] §11.6 or [10, Thm5.16]). Suppose that \(\chi\) is a multiplicative character of order \(N\) over \(\mathbb{F}_q\), \(q = p^f\). Assume that there exists an integer \(t \geq 1\) such that \(p^t \equiv -1 \pmod{m}\), with \(t\) chosen minimal. Then \(f = 2ts\) for some positive integer \(s\), and

\[
G(\chi) = \begin{cases} 
(-1)^{t-1} \sqrt{q}, & \text{if } p = 2; \\
(-1)^{t-1+(p+1)s/N} \sqrt{q}, & \text{if } p \geq 3.
\end{cases}
\]

\[\square\]

From now on we assume that \(-1 \not\equiv (p) \subset (\mathbb{Z}/N\mathbb{Z})^*\) so that \(K\) (defined in Lemma 2.13)) is an imaginary abelian field of degree \(r\), where

\[r = [(\mathbb{Z}/N\mathbb{Z})^* : <p>] = \varphi(N)/f,\]

\(\varphi(\cdot)\) is Euler function. It is called the “index \(r\) cases”. In 1970’s–2000’s, the Gauss sums in index \(r = 2\) case have been studied and evaluated explicitly in a series of papers [13, 17, 9, 12, 14]. And since 2005, the case of index \(r = 4\) has been studied in papers [6, 19, 5].

**Lemma 2.3.** Suppose that \(\chi\) is a multiplicative character of order \(N\) over \(\mathbb{F}_q\), and let \(T\) be the trace map from \(\mathbb{F}_q\) onto \(\mathbb{F}_p\). And \(K \subset \mathbb{Q}(\zeta_N)\) is the invariant subfield for \(\sigma_p\). Then

\[
G(\chi) = \left( \sum_{\substack{x \in \mathbb{F}_q \\text{Tr}(x) \equiv 1}} \chi(x) \right) \left( \sum_{y \in \mathbb{F}_p} \chi(y) \zeta_p^y \right)
\] (2.4)

and \(\sum_{\substack{x \in \mathbb{F}_q \\text{Tr}(x) = 1}} \chi(x) \in O_K\).

**Proof.** Since

\[G(\chi) = \sum_{x \in \mathbb{F}_q} \chi(x) + \sum_{a=1}^{p-1} \sum_{x \in \mathbb{F}_q} \chi(ax) \zeta_p^a\]

and \(\chi\) is nontrivial on \(\mathbb{F}_q\), we know that the first summation of the formula above is equal to zero, while the second summation is equal to the right side of (2.4). Finally, since

\[f(x) = \sum_{x \in \mathbb{F}_q} \chi(x) \in O_M, M = \mathbb{Q}(\zeta_N),\]

we have \(f(x) \in O_K\). \[\square\]
Let $\chi|_{\mathbb{F}_p}$ denote the restriction of $\chi$ onto $\mathbb{F}_p$. From Lemma 2.4 we know that

$$G(\chi) = \left( \sum_{x \in \mathbb{F}_q} \chi(x) \right) G_p(\chi).$$  

(2.5)

where $G_p(\chi) = G(\chi|_{\mathbb{F}_p})$. When the order of $\chi|_{\mathbb{F}_p}$ is relatively small, we can calculate $G_p(\chi)$ by the results of the Gauss sum with a relatively small order over $\mathbb{F}_p$. For the order of $\chi|_{\mathbb{F}_p}$, there exists the following result.

**Lemma 2.4** (see [3] Prop 11.4.1). Assume that $\chi$ is a multiplicative character of order $N$ over $\mathbb{F}_q = \mathbb{F}_{p^r}$. Then the order of the restriction of $\chi$ onto $\mathbb{F}_p$ is

$$\frac{N}{(N, p^{r-1})}$$

Particularly, when $\chi|_{\mathbb{F}_p}$ is a quadratic character, the evaluation of the Gauss sums $G(\chi)$ is directly reduced to the evaluation of the summation $\sum_{x \in \mathbb{F}_q} \chi(x) \in O_K$ by formula (2.2).

### 3 Classification of order $N$ in index 2 case

We always keep these assumptions in following text. Assume that

(I). $p$ is a prime number, $N \geq 2$, $(p, N) = 1$, the order of $p$ modulo $N$ is $f = \frac{\varphi(N)}{2}$, so that $[(\mathbb{Z}/N\mathbb{Z})^* : <p>] = 2$ and the decomposition field $K$ of $p$ in $\mathbb{Q}(\zeta_N)$ is a quadratic abelian field.

(II). $q = p^r$, $\chi$ is a multiplicative character of $\mathbb{F}_q$ with order $N$, $G(\chi)$ is the Gauss sum of order $N$ over $\mathbb{F}_q$ defined by (2.1).

(III). $-1 \not\equiv p \not\equiv q \pmod{\mathbb{Z}/N\mathbb{Z}}$, so $K$ is an imaginary field.

In this section, we will determine all the possibilities of $N$ satisfying assumptions (I),(II) and (III), and also determine the type of the corresponding imaginary quadratic subfield $K$ of $\mathbb{Q}(\zeta_N, \zeta_p)$.

Suppose that $N$ has the prime factorization

$$N = 2^{r_0}l_1^{r_1} \cdots l_s^{r_s},$$

where $s \geq 0$, $l_i$ are distinct odd primes, $r_i$ are non-negative integers $(0 \leq i \leq s)$. And suppose that any two of $\varphi(l_i^{r_i})$ $(1 \leq i \leq s)$ have no odd common prime factors. By the Chinese Remainder Theorem, we have

$$(\mathbb{Z}/N\mathbb{Z})^* \cong (\mathbb{Z}/2^{r_0}\mathbb{Z})^* \times (\mathbb{Z}/l_1^{r_1}\mathbb{Z})^* \times \cdots \times (\mathbb{Z}/l_s^{r_s}\mathbb{Z})^*.  

(3.1)$$

Let $\varphi(2^{r_0}), \varphi(l_1^{r_1}), \ldots, \varphi(l_s^{r_s})$ be the order of $p$, respectively, in each subgroup in right-side of (3.1), where $a_0, a_1, \ldots, a_s \in \mathbb{N}, a_0 \mid \varphi(2^{r_0}), a_i \mid \varphi(l_i^{r_i})$ for $1 \leq i \leq s$. Then

$$f = \frac{\varphi(N)}{2} = \frac{1}{2} \varphi(2^{r_0}) \varphi(l_1^{r_1}) \cdots \varphi(l_s^{r_s}) = \left[ \frac{\varphi(2^{r_0})}{a_0}, \frac{\varphi(l_1^{r_1})}{a_1}, \ldots, \frac{\varphi(l_s^{r_s})}{a_s} \right].$$  

(3.2)

By the Chinese Remainder Theorem again, there are the primitive roots $g_j$ modulo $l_j^{r_j}$ $(1 \leq j \leq s)$ and primitive root $g_0$ modulo $2^{r_0}$ such that

$$\left\{ \begin{array}{ll}
g_0 & \equiv 1 \pmod{l_\lambda^{r_\lambda}}, \quad \text{for } 1 \leq \lambda \leq s;
g_j & \equiv 1 \pmod{2^{r_0}}, \quad \text{for } 1 \leq j \leq s, \quad \text{and } \lambda \neq j;
g_j & \equiv g_0 g_{j_1} \cdots g_{j_{r_j}} \pmod{N}.
\end{array} \right.$$  

(3.3)

When $r_0 = 0$, $N$ is odd. So, by (3.2), $N$ has no more than 2 odd prime factors, i.e. $s \leq 2$. Then we have two subcases according to $N$ having one odd prime factor or two odd prime factors, where $g_1, g_2$ can be odd or even.

**Case A.** $N = l_1^{r_1}$, $l_1$ be odd, $r_1 \geq 1$, $p \equiv g_1^{l_1}$ (mod $l_1^{r_1}$). The assumption (III) $-1 \not\equiv p \not\equiv q$ implies $l_1 \equiv 3 \pmod{4}$, $K = \mathbb{Q}(\sqrt{-l_1})$.

**Case B.** $N = l_1^{r_1} l_2^{r_2}$, $\varphi(l_1^{r_1}) \varphi(l_2^{r_2}) = \frac{\varphi(l_1^{r_1}) \varphi(l_2^{r_2})}{a_1 a_2}$, respectively, in each subgroup in right-side of (3.1), where $a_2 \mid \varphi(2^{r_0}), a_1 \mid \varphi(l_1^{r_1})$ for $1 \leq i \leq s$. Then we have two subcases of Case B:
Case B1. $a_1a_2 = 1$, $p \equiv g_1g_2 \pmod{N}$, $-1 \not\equiv p \implies \{l_1, l_2\} \equiv \{1, 3\} \pmod{4}$. Without loss of generality, we assume that $(l_1, l_2) \equiv (3, 1) \pmod{4}$, then $< p > = < g_1^2, g_2^2 >$, $K = \mathbb{Q}(\sqrt{-l_1l_2})$.

Case B2. $a_1a_2 = 2$, and let $p \equiv g_1^2g_2 \pmod{N}$, then $(\frac{l_1+1}{2}, l_2 - 1) = 1 \implies l_1 \equiv 3 \pmod{4}$, $l_2 \equiv 1 \pmod{2}$, $< p > = < g_1^2, g_2 >$, $K = \mathbb{Q}(\sqrt{-l_1})$.

For simplicity of the following evaluation, we assume $l_1 \not\equiv 3 \pmod{4}$ and Case A and Case B2.

When $s = 0$, i.e. $N$ just has prime factor 2. We have the following subcase for $r_0 \geq 3$. (Since when $r_0 = 2$ we have $p \equiv 3 \pmod{4}$, it's self-conjugate (pure) Gauss sum, which can be determined by Theorem 2.2)

Case C. $N = 2^{r_0}$, $r_0 \geq 3$, $p \equiv 3$ or $5 \pmod{8}$, $f = 2^{r_0-2}$, $K = \mathbb{Q}(\sqrt{-2})$ or $\mathbb{Q}(\sqrt{-7})$.

When $r_0 = 1$, let $N = 2N_0$ ($N_0$ is odd). Similarly, by \[\text{(3.2)}, s < 2. Since (\mathbb{Z}/N\mathbb{Z})^* \cong (\mathbb{Z}/N_0\mathbb{Z})^*, we have two subcases, Case D and E, respectively corresponding to Case A and B, where $g_1 g_2$ must be odd.

Case D. $N = 2l_1^4$, $N = l_1^4$, $3 \not\equiv l_1 \equiv 3 \pmod{4}$, $p \equiv g_1^2 \pmod{N}$, $K = \mathbb{Q}(\sqrt{-l_1})$.

Case E. $N = 2l_1^2l_2^2$. Similar as Case B, we have two subcases of Case E according to $a_1 a_2$ equals 1 or 2:

Case E1. $a_1 a_2 = 1$, $p \equiv g_1 g_2 \pmod{N}$, $-1 \not\equiv p \implies \{l_1, l_2\} \equiv (3, 1) \pmod{4}$ $< p > = < g_1^2, g_2^2 >$, $K = \mathbb{Q}(\sqrt{-l_1l_2})$.

Case E2. $a_1 a_2 = 2$, and let $p \equiv g_1^2g_2 \pmod{N}$, $3 \not\equiv l_1 \equiv 3 \pmod{4}$, $l_2 \equiv 1 \pmod{2}$, $< p > = < g_1^2, g_2 >$, $K = \mathbb{Q}(\sqrt{-l_1})$.

When $s = 1$, $r_0 \geq 2$, we have $s = 1$, $r_0 = 2$ by \[\text{(3.2)}.

Case F. $N = 4l_1^4$, $N = 3l_1^4$. Then $a_0 a_1 \leq 2$, and we have three subcase of Case F according to the values of $a_0$ and $a_1$:

Case F1. $a_0 = a_1 = 1$, $p \equiv g_1g_2 \pmod{N}$, i.e. $p \equiv 3 \pmod{4}$ and $p \equiv g_1 \pmod{N}$, $-1 \not\equiv p \implies l_1 \equiv 1 \pmod{4}$, $< p > = < g_1, g_2 >$, $K = \mathbb{Q}(\sqrt{-l_1})$.

Case F2. $a_0 = 2$, $a_1 = 1$, $p \equiv g_1^2g_2 \pmod{N}$, i.e. $p \equiv 1 \pmod{4}$, $p \equiv g_1 \pmod{N}$, $< p > = < g_1, g_2 >$, $K = \mathbb{Q}(\sqrt{-l_1})$.

Case F3. $a_0 = 1$, $a_1 = 2$, $p \equiv 3 \pmod{4}$, $l_1 \equiv 3 \pmod{4}$, $p \equiv g_1 g_2^2 \pmod{N}$, $< p > = < g_1^2, g_2 >$, $K = \mathbb{Q}(\sqrt{-l_1})$.

4 Explicit evaluation of Gauss sums in index 2 case

4.1 Explicit evaluation of $G(\chi)$

In this section, we give explicit evaluation of Gauss sum $G(\chi)$ in each subcases (i.e. Case A, B, C, D, E, F), where the results of Case A, B and C has been shown in previous papers.

Case A.

$N = l_1^4$ ($r_1 \geq 1$), the result was given by P.Langevin [9] in 1997.

Theorem 4.1. Let $N = l_1^4$, $l_1 \equiv 3 \pmod{4}$, $l_1 > 3$, $[(\mathbb{Z}/N\mathbb{Z})^* : < p >] = 2$, i.e. $f = \frac{\varphi(N)}{2}$, $q = p^f$ and let $\chi$ be a primitive multiplicative character of order $N$ over $\mathbb{F}_q$. Then

$$G(\chi) = \frac{1}{2}p^{\frac{1}{2}(f-h_1)}(a + b\sqrt{-l_1}),$$

where $h_1 = h(\mathbb{Q}(\sqrt{-l_1}))$ is the ideal class number of field $\mathbb{Q}(\sqrt{-l_1})$ and $a, b \in \mathbb{Z}$ are determined by

$$\begin{cases} 4p^{h_1} = a^2 + l_1b^2, \\ a \equiv -2p^{\frac{1}{2}(f+h_1)} \pmod{l_1}. \end{cases} \quad (4.1)$$

Remark 1. Since

$$p^{\frac{l_1-h}{2}}/p^{\frac{h_1}{2}} = \frac{p^{\frac{l_1}{2}+h}}{p^{\frac{h_1}{2}} + p^{\frac{l_1}{2}+h}} = \left(\frac{p^{\frac{1}{2}}}{p^{\frac{h_1}{2}}}\right)^{\frac{l_1-h}{2}} = \left(g_1^{l_1-h-1/2}\right)^{-1} \equiv 1 \pmod{l_1},$$

the second equation of \[\text{(4.1)}\] implies $a \equiv -2p^{\frac{1}{2}(f+h)} \pmod{l_1}$. Therefore, equations \[\text{(4.1)}\] are equivalent to

$$\begin{cases} 4p^{h_1} = a^2 + l_1b^2, \\ a \equiv 2p^{\frac{1}{2}(f+h)} \pmod{l_1}. \end{cases} \quad (4.2)$$
From equations (4.2), one can find that the sign of \( a \) is just relational with \( l_1, p \), however, is not relational with \( r_1 \). So, we always take the principal ideal \( \mathfrak{p} \) in the following text, where \( l_1 \equiv 3 \pmod{4} \). \( \mathfrak{p} \) is a prime ideal factor of \( p \) in the integral ring of \( \mathbb{Q}(\sqrt{-l_1}) \) and integers \( a, b \) are determined by equations (4.2).

\[ \mathbf{Case \ B} \]

\[ N = l_1^2 l_2^2 \] (\( l_1, l_2 \) are different odd prime numbers, \( r_1, r_2 \geq 1 \)). In 1998, O. D. Mboij [1, 2] gave evaluation of the Gauss sums \( G(\chi) \) in Case B.

**Theorem 4.2.** Let \( N = l_1^2 l_2^2 \), \( [(\mathbb{Z}/N\mathbb{Z})^* : p > 2] = 2 \), i.e., \( f = \frac{\varphi(N)}{2} = p^f \), and let \( \chi \) be a primitive multiplicative character of order \( N \) over \( \mathbb{F}_q \). Assume that the orders of \( p \) in groups \( (\mathbb{Z}/l_1^r\mathbb{Z})^* \) and \( (\mathbb{Z}/l_2^r\mathbb{Z})^* \) are respectively \( \varphi(l_1^r) / a_0 \) and \( \varphi(l_2^r) / a_1 \). Then

(i). For Case B1, \( a_1 = a_2 = 1, (l_1, l_2) \equiv (1, 3) \pmod{4} \)

\[
G(\chi) = \frac{1}{2} p^{\frac{1}{2}(f - h_{12})} (a' + b' \sqrt{-l_1 l_2}),
\]

where \( h_{12} = h(\mathbb{Q}(\sqrt{-l_1 l_2})) \) and integers \( a', b' \) are determined by equations

\[
\begin{cases}
4p^{h_{12}} = (a')^2 + l_1 l_2 (b')^2, \\
a' = 2p^{\frac{1}{2}h_{12}} \pmod{l_1};
\end{cases}
\]

(ii). For Case B2, \( a_1 = 2, a_2 = 1, 3 \neq l_1 \equiv 3 \pmod{4} \)

\[
G(\chi) = \begin{cases}
p^\frac{f}{2}, & \text{if } \left( \frac{l_2}{l_1} \right) = 1, \\
p^\frac{f - h_1}{2} (\frac{a + b \sqrt{-l_1}}{2}), & \text{if } \left( \frac{l_2}{l_1} \right) = -1.
\end{cases}
\]

where \( h_1 = h(\mathbb{Q}(\sqrt{-l_1})) \), integers \( a, b \) are determined by equations (4.2) and \( \left( \frac{l_2}{l_1} \right) \) is Legendre Symbol modulo \( l_1 \).

\[ \mathbf{Case \ C} \]

For \( N = 2^n \) \( (r_0 \geq 3) \), P. Meijer and M. van der Vlugt [14] gave evaluation of the Gauss sums in index 2 case, in 2003. When \( N = 2^n \) \( (r_0 \geq 3) \), it is known from elementary number theory that the primes \( p \) such that \( p \equiv 3, 5 \pmod{8} \) are exactly the primes which generate subgroup of index 2 in \( (\mathbb{Z}/N\mathbb{Z})^* \). So,

**Theorem 4.3.** Let \( N = 2^n \) \( (r_0 \geq 3) \), \( \chi \) be a primitive multiplicative character of order \( N \) over \( \mathbb{F}_q \) and \( P_1 \) be a prime ideal factor of \( p \) in the integral ring of \( \mathbb{Q}(\sqrt{-N}) \). Then

(i). When \( p \equiv 3 \pmod{8} \)

\[
G(\chi) = G(\chi_{P_1}) = \varepsilon_1 i \sqrt{p} p^{\frac{n-3}{2}} (a + ib\sqrt{2}),
\]

where \( a, b \in \mathbb{Z} \) are determined by \( (a + ib\sqrt{2}) = P_1 \cap \mathcal{O}(i\sqrt{2}) \) and \( \varepsilon_1 = \pm 1 \) can be solved by Stickelberger congruence.

(ii). When \( p \equiv 5 \pmod{8} \)

\[
G(\chi) = G(\chi_{P_1}) = \varepsilon_2 p^{\frac{n-3}{2}} \sqrt{a + ib},
\]

where \( a, b \in \mathbb{Z} \) are determined by \( (a + ib) = P_1 \cap \mathbb{Z}[i], \sqrt{a + ib} \) has a positive real part, and \( \varepsilon_2 \in \{ \pm 1, \pm i \} \) can be solved by Stickelberger congruence.

\[ \mathbf{Case \ D} \]

When \( N = 2l_1^r \) = \( 2N_0 \), since \( (N, p) = 1, p \) must be odd prime. \( 3 \neq l_1 \equiv 3 \pmod{4} \), \( f = \frac{\varphi(N)}{2} \) = \( \frac{(l_1 - 1)l_1^r - 1}{2} \equiv 1 \pmod{2} \). \( q = p^f = 2N_0 \cdot 1 + 1 \pmod{2} \). \( g_1 \) is defined by (3.3) in Section 1, i.e. \( g_1 \) is odd primitive root modulo \( N_0 \). Then \( g_1 \) is also the primitive root modulo \( N \), and we can take \( p \equiv q_1^2 \pmod{N} \).

In Case D, \( \chi \) is a primitive multiplicative character of order \( N = 2l_1^r \) over \( \mathbb{F}_q \), which means that \( \chi^2 \) is the character of order \( N_0 = l_1^r \), and since \( f = \text{ord}_N(p) = \text{ord}_{N_0}(p) \), \( \chi^2 \) is primitive. By the result of Case A,

\[
G(\chi^2) = p^{\frac{f}{2} - h_1} \left( \frac{a + b \sqrt{-l_1}}{2} \right).
\]
where integers \(a, b\) are determined by equations (42). By Darvenport-Hasse product formula (3 \(\S\) 11.3)), we have that
\[
G(\chi^2) = \chi^2(2) \frac{G(\chi)G(\chi^{l_1^2+1})}{G(\chi^{l_1})} = \frac{G(\chi)G(\chi^{l_1^2+1})}{(\sqrt{p^r})^f} = \left\{ \begin{array}{ll}
\frac{G(\chi)}{(\sqrt{p^r})^f}G(\chi^2) & \text{if } \frac{l_1^2+1}{2} \in R_2 \\
\frac{G(\chi)}{(\sqrt{p^r})^f}G(\chi^2) & \text{if } \frac{l_1^2+1}{2} \in \overline{R}_2,
\end{array} \right.
\]
where \(R_2, \overline{R}_2\) denote respectively as the sets of quadratic remainder and quadratic non-remainder modulo \(l_1\). Then
\[
G(\chi) = \left\{ \begin{array}{ll}
\left(\frac{p^r}{l_1}\right) = (-1)^{\frac{l_1^2+1}{2}} \sqrt{p^r} p^{\frac{l_1-1}{2}} & \text{if } \left(\frac{l_1^2+1}{2}\right) = 1 \\
\left(\frac{p^r}{l_1}\right) = (-1)^{\frac{l_1^2+1}{2}} \sqrt{p^r} p^{\frac{l_1-1}{2}} - b_1(a + b - \sqrt{l_1})^2 & \text{if } \left(\frac{l_1^2+1}{2}\right) = -1.
\end{array} \right. \tag{4.4}
\]
Since
\[
\left(\frac{(l_1^2+1)/2}{l_1}\right) = \left(\frac{l_1^2+1}{l_1}\right) = \left(\frac{1}{l_1}\right) \left(\frac{2}{l_1}\right) = \left\{ \begin{array}{ll}
1 & \text{if } l_1 \equiv 7 \pmod{8}; \\
-1 & \text{if } l_1 \equiv 3 \pmod{8},
\end{array} \right.
\]
we have got the formula of Gauss sums \(G(\chi)\) in Case D as follows:

**Theorem 4.4 (Case D).** Let \(N = 2l_1^2, l_1 \equiv 3 \pmod{4}, l_1 > 3, [(\mathbb{Z}/N\mathbb{Z})^* : < p >] = 2\), i.e. \(f = \frac{\varphi(N)}{2}, q = p^f\), and let \(\chi\) be a primitive multiplicative character of order \(N\) over \(\mathbb{F}_q\). Then
\[
G(\chi) = \left\{ \begin{array}{ll}
\left(\frac{-1}{l_1}\right) = \sqrt{p^r} p^{\frac{l_1-1}{2}}, & \text{if } l_1 \equiv 7 \pmod{8}; \\
\left(\frac{-1}{l_1}\right) + \sqrt{p^r} p^{\frac{l_1-1}{2}} - h_1(a + b - \sqrt{l_1})^2, & \text{if } l_1 \equiv 3 \pmod{8},
\end{array} \right.
\]
where \(h_1\) is the ideal class number of \(\mathbb{Q}(\sqrt{-l_1})\) and integers \(a, b\) are determined by equations (42).

**Example 4.1.** (For Case D)

1. Let \(l_1 = 7, N = 14, f = \frac{\varphi(N)}{2} = 3\). The primitive roots modulo 7 are 2, 3, and \(\equiv g_1^2 \equiv 2, 4 \pmod{7}\). Take \(p = 11\), then \(< p >= \{1, 11, 9\}\), \(g_1 < p >= \{3, 5, 13\}\). By Theorem 4.4 we know that the Gauss sum \(G(\chi)\) of order 14 over \(\mathbb{F}_{11^2}\) is
\[
G(\chi) = -11\sqrt{-11}.
\]

2. Let \(l_1 = 11, N = 22, f = \frac{\varphi(22)}{2} = 5\). The primitive roots modulo 11 are 2, 6, 7, 8, and \(\equiv g_1^2 \equiv 3, 5, 9 \pmod{11}\). Take \(p = 3\), so the equations \(\begin{array}{ll}
a^2 + 11 \cdot b^2 = 3 \cdot 4 \\
a \equiv 2 \cdot 3^l, b \equiv 1 \pmod{11}
\end{array}\) have solutions \(\begin{array}{ll}
a = 1 \\
b = \pm 1.
\end{array}\) By Theorem 4.4 we have the Gauss sum \(G(\chi)\) of order 22 over \(\mathbb{F}_{23^2}\) and its conjugation \(\overline{G(\chi)}\) are
\[
\{G(\chi), \overline{G(\chi)}\} = \{2\sqrt{3} \left(\frac{-5 \pm \sqrt{-11}}{2}\right)\}.
\]

\[\Box\]

**Case E.**

Let \(\chi\) be a primitive multiplicative character of order \(N = 2N_0 = 2l_1^2 g_2^2\) over \(\mathbb{F}_q\), \(f = \text{ord}_N(p) = \varphi(N)/2\) (must be even). By \((\mathbb{Z}/N\mathbb{Z})^* \equiv (\mathbb{Z}/N_0\mathbb{Z})^*\), we know that \(\chi^2\) is the primitive character of order \(N_0\) over \(\mathbb{F}_q\). Then \(G(\chi^2)\) can be evaluate by Theorem 4.2 in Case B. By Davenport-Hasse product formula, we have that
\[
G(\chi^2) = \chi^2(2) \frac{G(\chi)G(\chi^{N_0+1})}{G(\chi^{N_0})},
\]
where \(\chi^{N_0}\) is quadratic character over \(\mathbb{F}_q\). By formula (2.3), \(G(\chi^{N_0}) = -(\sqrt{p^r})^f = (-1)^{\frac{l_1^2+1}{2}} p^{\frac{l_1-1}{2}}\). And since \((N_0 + 1, 2N_0) = 2, \)
\[
G(\chi^{N_0+1}) = G(\chi^{2N_0+1}) = \left\{ \begin{array}{ll}
G(\chi^2) & \text{if } \frac{N_0+1}{2} \in < p >; \\
G(\chi^2) & \text{if } \frac{N_0+1}{2} \in - < p >.
\end{array} \right.
\]
Then
\[
G(\chi) = \left\{ \begin{array}{ll}
(-1)^{\frac{l_1^2+1}{2}} \chi^{l_1^2+1}(2)p^{\frac{l_1}{2}} & \text{if } \frac{N_0+1}{2} \in < p >; \\
(-1)^{\frac{l_1^2+1}{2}} + \chi^{2N_0}(2)p^{-\frac{l_1}{2}}(G(\chi^2))^2 & \text{if } \frac{N_0+1}{2} \in - < p >.
\end{array} \right.
\]
For **Case E1** \(p \equiv g_1g_2 \pmod{N_0}, (l_1, l_2) \equiv (3, 1) \pmod{4}, 4 | f = \varphi(N_0)/2, K = \mathbb{Q}(\sqrt{-l_1l_2})\). Since \(a \in < p > \Leftrightarrow \left(\frac{a}{l_1l_2}\right) = 1, \frac{N_0}{2} \in < p > \Leftrightarrow \left(\frac{N_0-1}{l_1l_2}\right) = 1 \Leftrightarrow \left(\frac{2}{l_1l_2}\right) = 1 \Leftrightarrow l_1l_2 \equiv 7 \pmod{8};\) and
where integers \( \{ \}
\)

Therefore, by Theorem 4.2, \( G(\chi) = \)

\[
G(\chi) = \begin{cases} 
-p^2_f & 
\text{if } l_1 l_2 \equiv 7 \pmod{8}; \\
-p^2_f - h_{12}(a' + b' \frac{1 + \sqrt{-111}}{2})^2 & 
\text{if } l_1 l_2 \equiv 3 \pmod{8}, 
\end{cases}
\]

where \( h_{12} = h(Q(\sqrt{-l_1 t_2})) \) and integers \( a', b' \) are determined by equations 4.3 in Theorem 4.2

For Case E2, \( p \equiv a_1^2 t_2 \pmod{N_0} \), \( l_1 \equiv 3 \pmod{4}, l_2 \equiv 1 \pmod{2} \), \( K = Q(\sqrt{-l_1 t_2}) \), and \( f = \varphi(N)/2 \equiv 1 \pmod{8} \) if \( l_2 \equiv 1 \pmod{4} \); \( 2 \pmod{4} \) if \( l_2 \equiv 3 \pmod{4} \). Similarly, we have \( (p-1, N_0) = 1 \) and \( \frac{N}{(N, \frac{N}{p-1})} = 1 \). Then, by Theorem 4.2

\[
G(\chi) = \begin{cases} 
(-1)^{\frac{p-1}{2}} \frac{1 + \sqrt{-111}}{2}^4 & 
\text{if } l_1 l_2 \equiv 7 \pmod{8}; \\
(-1)^{\frac{p-1}{2}} \frac{1 + \sqrt{-111}}{2}^4 & 
\text{otherwise,}
\end{cases}
\]

where \( h_1 = h(Q(\sqrt{-l_1})) \) and integers \( a, b \) are determined by equations 4.1 in Theorem 4.1

To summarize, we obtain,

**Theorem 4.5.** Let \( N = 2(l_1^2 + l_2^2) \cdot \{ \mathbb{Z}/N \mathbb{Z} \} : < p > = 2 \), i.e. \( f = \frac{\varphi(N)}{2} \). Take \( q = p^f \), \( \chi \) a primitive the multiplicative character of order \( N \) over \( \mathbb{F}_q \) and \( h_{12} \) be respectively the ideal class numbers of \( Q(\sqrt{-l_1}) \) and \( Q(\sqrt{-l_1 t_2}) \). Assume that the orders of \( p \) in group \((\mathbb{Z}/l_1^2 \mathbb{Z})^* \) and \((\mathbb{Z}/l_2^2 \mathbb{Z})^* \) are respectively \( \varphi(l_1^2)/a_0 \) and \( \varphi(l_2^2)/a_1 \). Then

(i) For Case E1, \( a_1 = a_2 = 1, (l_1, l_2) \equiv (3, 1) \pmod{4} \)

\[
G(\chi) = \begin{cases} 
-p^2_f & 
\text{if } l_1 l_2 \equiv 7 \pmod{8}; \\
-p^2_f - h_{12}(a' + b' \frac{1 + \sqrt{-111}}{2})^2 & 
\text{if } l_1 l_2 \equiv 3 \pmod{8}, 
\end{cases}
\]

where integers \( a', b' \) are determined by equations 4.3 in Theorem 4.2

For Case E2, \( (a_1 = a_2 = 1, 3 \neq l_1 \equiv 3 \pmod{4}) \)

\[
G(\chi) = \begin{cases} 
(-1)^{\frac{p-1}{2}} \frac{1 + \sqrt{-111}}{2}^4 & 
\text{if } l_1 l_2 \equiv 7 \pmod{8}; \\
(-1)^{\frac{p-1}{2}} \frac{1 + \sqrt{-111}}{2}^4 & 
\text{otherwise,}
\end{cases}
\]

where integers \( a, b \) are determined by equations 4.2 in Case A.

**Example 4.2.** (For Case E1).

(1). Let \( l_1 = 5, l_2 = 3, N = 30, f = \frac{\varphi(30)}{2} = 4 \), \( h_{12} = 2 \). The minimum primitive root modulo 5 \( g_1 = 7 \), such that \( g_1 \equiv 1 \pmod{3} \), while The minimum primitive root modulo 3 \( g_2 = 11 \) such that \( g_2 \equiv 1 \pmod{5} \). Then \( p \equiv 7 \equiv \frac{30}{2} (\pmod{5}) \).

Take \( p = 17 \), by Theorem 4.5, we have that the Gauss sum \( G(\chi) \) of order 30 over \( \mathbb{F}_{17^2} \) is

\[
G(\chi) = -17^2.
\]

(2). Let \( l_1 = 5, l_2 = 7, N = 70, f = \frac{\varphi(70)}{2} = 12 \), \( h_{12} = 2 \). The minimum primitive root modulo 5 \( g_1 = 8 \) such that \( g_1 \equiv 1 \pmod{7} \), while The minimum primitive root modulo 7 \( g_2 = 26 \) such that \( g_2 \equiv 1 \pmod{5} \). Then \( p \equiv 33 \pmod{35} \).

Take \( p = 103 \), and \( a = 199, b = \pm 9 \) are solutions of equations 4.2. Then, by Theorem 4.5, we have that the Gauss sum \( G(\chi) \) of order 70 over \( \mathbb{F}_{103^{12}} \) and its conjugation are

\[
\{G(\chi), \overline{G(\chi)}\} = \left\{ -103^4 \left( \frac{199 \pm 9\sqrt{-35}}{2} \right)^2 \right\}.
\]
We claim that

Proof. By Stickelberger’s Theorem, $\varphi(N) = (l_1 - 1)L_1^{-1} \equiv 0 \pmod{4}$. Let $g_0, g_1$ defined as in Section 1. So, we consider the following cases according to the values of $a_0, a_1$.

Case F1. $a_0 = a_1 = 1 \pmod{2}$ and $p \equiv g_0 \pmod{4}$ and $p \equiv g_1 \pmod{N_0}$, $l_1 \equiv 1 \pmod{4}$, $f = \varphi(N) = (l_1 - 1)L_1^{-1} \equiv 0 \pmod{4}$. Let $K = \mathbb{Q}(\sqrt{-1})$. We note that $K \not\subseteq \mathbb{Q}(\zeta_{N_0})$.

Lemma 4.6. In Case F1, the order of $\chi|_{\mathbb{F}_p}$ is 1, i.e., $\chi|_{\mathbb{F}_p}$ is trivial.

Proof. We claim that $(p - 1, N_0) = 1$. Otherwise, $l_1 \mid p - 1 \Rightarrow p \equiv 1 \pmod{l_1} \Rightarrow p^{f_1 - 1} = 1 \pmod{l_1} = N_0$. This is contradict to $p \equiv g_1 \pmod{l_1}$.

By Lemma 2.2, the order of $\chi|_{\mathbb{F}_p}$ is $\frac{N}{\gcd(N, 2 - 1)} = \frac{4f^r_1}{(4f_1^n - p - 1)} = \frac{4}{(4, \frac{4n}{p - 1})}$.

Let $p = 2k + 1$ where $k \geq 1$ is odd integer, for $p \equiv 3 \pmod{4}$. Since $4|f$, we have that

$$\frac{q - 1}{p - 1} = \frac{p^{f_1 - 1} - 1}{p - 1} = \frac{(2k)^f + (f - 1)(2k)^{f-1} + \cdots + (f^{k-1})}{2k} \equiv \frac{f}{2}2k + f \equiv 0 \pmod{4}.$$ 

Then $(4, \frac{4n}{p - 1}) = (4, \frac{4n}{p - 1}) = 4$ and the lemma has been proved.

Let $R_2$ and $\overline{R}_2$ respectively denote the quadratic residue set and quadratic non-residue set in group $(\mathbb{Z}/N\mathbb{Z})^*$, and take

$R_2^{(1)} = \{x \in R_2 \mid x \equiv 1 \pmod{4}\}$  
$R_2^{(3)} = \{x \in R_2 \mid x \equiv 3 \pmod{4}\}$  
$\overline{R}_2^{(1)} = \{x \in \overline{R}_2 \mid x \equiv 1 \pmod{4}\}$  
$\overline{R}_2^{(3)} = \{x \in \overline{R}_2 \mid x \equiv 3 \pmod{4}\}$.

Then $< p >= R_2^{(1)} \cup \overline{R}_2^{(3)}$. We define a isomorphic mapping between $(\mathbb{Z}/N\mathbb{Z})^*$ and $(\mathbb{Z}/N_0\mathbb{Z})^*$ as

$$\Phi : (\mathbb{Z}/N\mathbb{Z})^* \rightarrow (\mathbb{Z}/N_0\mathbb{Z})^*$$  
$$\Phi^{-1} : y + \frac{1 + (-1)^{s}}{2} \leftrightarrow y.$$ 

Considering the group isomorphism

$$(\mathbb{Z}/N\mathbb{Z})^* \cong \{\pm 1\} \times (\mathbb{Z}/2N\mathbb{Z})^* \cong (\mathbb{Z}/2N\mathbb{Z})^* \cup [(\mathbb{Z}/2N\mathbb{Z})^* + 2N_0],$$

each element $s$ in $(\mathbb{Z}/N\mathbb{Z})^*$ can be viewed as

$$s = s_0 + \frac{(-1)^{s_0} + 1}{2}L_1^{-1} + 2L_1^{-1} \cdot j = s_0 + \left[\frac{(-1)^{s_0} + 1}{2} + 2j\right]l_1^{-1}$$

(4.6)

(4.6)

So, we find that, when $j = 0$, $(\mathbb{Z}/2N\mathbb{Z})^* = R_2^{(1)} \cup \overline{R}_2^{(3)}$, and when $j = 1$, $(\mathbb{Z}/2N\mathbb{Z})^* + 2N_0 = R_2^{(3)} \cup \overline{R}_2^{(1)}$. Then

$$< p >= (\mathbb{Z}/2N\mathbb{Z})^* + 2N_0.$$ 

By Stickelberger’s Theorem,

$$(G(\chi))O_K = \varphi^{l_0_0 + \sigma - 1}b_1$$

(4.6)

So,

$$b_0 = \frac{1}{N} \sum_{s \in (\mathbb{Z}/N\mathbb{Z})^*} s = \frac{1}{N} \sum_{s \in (\mathbb{Z}/2N\mathbb{Z})^*} s = \frac{1}{N} \sum_{s_0 \in (\mathbb{Z}/2N\mathbb{Z})^*} \left( s_0 + \frac{(-1)^{s_0} + 1}{2}L_1^{-1} \right)$$

$$= \frac{1}{N} \sum_{y = 0}^{L_1^{-1} - 1} \sum_{x = 1}^{L_1^{-1} - 1} \left( x + l_1 y + \frac{L_1^{-1}}{2} L_1^{-1} \right) = \frac{1}{N} \left[ \frac{l_1^{-1}}{2} + \frac{L_1^{-1}}{2} \right] = \frac{1}{N} f$$

$$b_1 = \frac{1}{N} \sum_{s \in (\mathbb{Z}/2N\mathbb{Z})^*} (s + 2L_1^{-1}) = \frac{2}{N} f.$$

9
Thus, \((G(\chi)) = \varphi \sqrt[4]{\varphi} f = p \sqrt[4]{\varphi} f\). Ideal \(\varphi f\) is principal ideal and we assume that \(\varphi f = (a + b\sqrt{-l_1})\), where \(a, b \in \mathbb{Z}\) are determined by \(a^2 + b^2l_1 = p, p \nmid b\). Then

\[
G(\chi) = \varepsilon p^\ell (a + b\sqrt{-l_1}).
\]

As we known, \(\varepsilon\) is a unit root in \(K\). Since \(l_1 \neq 1, 3, \varepsilon = \pm 1\). It means that we need to determine the sign of integer \(a\). Considering the \(l_1^r\) power of \(G(\chi)\), if we let \(\varepsilon = 1\), then

\[
G_{l_1^r}(\chi) \equiv \left\{ \begin{array}{ll}
\sum_x \chi^{l_1^r}(x)G_{l_1^r}(\chi^{l_1^r}) & \equiv -p^\ell (\mod l_1) \\
\frac{a}{p^\ell} & (\mod \sqrt{-l_1})
\end{array} \right.
\]

Therefore,

\[
a \equiv -p^\ell (\mod l_1).
\]

**Case F2** \(a_0 = 2, a_1 = 1\) \(p \equiv 9\) \(\equiv 3\) \((\mod N)\), i.e. \(p \equiv 1\) \((\mod 4)\) and \(p \equiv 3\) \((\mod N)\), \(K = \mathbb{Q}(\sqrt{-1})\).

\[f = \varphi(N)/2 = \varphi(N_0)\equiv \begin{cases} 2 \ (\mod 4) \text{ if } l_1 \equiv 3 \ (\mod 4); \\ 0 \ (\mod 4) \text{ if } l_1 \equiv 1 \ (\mod 4). \end{cases}\]

Similar as Lemma 4.6, we have \((p - 1, N_0) = 1, \ (p - 1, 4) = 4\) and the order of \(\chi|_{F_p}\) is

\[
\frac{N}{\left(N, \frac{2}{p}\right)} = \begin{cases} 2 \ (\mod 4) \text{ if } l_1 \equiv 3 \ (\mod 4); \\ 1 \ (\mod 4) \text{ if } l_1 \equiv 1 \ (\mod 4). \end{cases}
\]

If \(l_1 \equiv 1 \ (\mod 4)\), \(G(\chi) \in O_K\), by Stickelberger’s Theorem, we have

\[
(G(\chi))O_K = \varphi^{b_0 + \sigma - 1}b_1,
\]

where

\[
b_0 = \frac{1}{N} \sum_{s 
oplus \left(\mathbb{Z}/N_0\mathbb{Z}\right)^*} s = \frac{1}{N} \sum_{s \equiv 1 \ (\mod 4)} s,
\]

\[
b_1 = \frac{1}{N} \sum_{s 
oplus \left(\mathbb{Z}/N_0\mathbb{Z}\right)^*} s = \frac{1}{N} \sum_{s \equiv 3 \ (\mod 4)} s.
\]

As we known, \(b_0 + b_1 = f\). Next, we calculate \(b_0 - b_1\). Similar as Case F1, we consider

\[
\left(\mathbb{Z}/N_0\mathbb{Z}\right)^* \overset{\Phi_1}{\longrightarrow} \left(\mathbb{Z}/2N_0\mathbb{Z}\right)^* \overset{\Phi_2}{\longrightarrow} \left(\mathbb{Z}/N\mathbb{Z}\right)^*,
\]

where \(\Phi_1, \Phi_2\) are, respectively, defined by 4.3 and 4.6. For the pairs \((s_0, l_1^r - s_0) \ (s_0 \in \left(\mathbb{Z}/2N_0\mathbb{Z}\right)^*)\), we find that \(\Phi_1(s_0) + \Phi_1(l_1^r - s_0)\) are always equal to \(l_1 + l_1^r\), and \(\Phi_2(s_0) - \Phi_1(l_1^r - s_0)\equiv 0 \ (\mod 4)\). Since \(#\{s \in \left(\mathbb{Z}/2N_0\mathbb{Z}\right)^* \mid s \equiv 1 \ (\mod 4)\} = #\{s \in \left(\mathbb{Z}/2N_0\mathbb{Z}\right)^* \mid s \equiv 3 \ (\mod 4)\}\),

\[
b_0 - b_1 = \sum_{s \in \left(\mathbb{Z}/N\mathbb{Z}\right)^*} s \left(\frac{-1}{s}\right) = 0.
\]

Therefore

\[
G(\chi) = \varepsilon p^\ell,
\]

where \(\varepsilon\) is a unit root in \(O_K\), i.e. \(\varepsilon \in \{\pm 1, \pm i\} \ (i = \sqrt{-1})\), which can be determined by Stickelberger congruence [3, §11.3]. (The detailed discussing of how to determine the sign or unit root ambiguities for Gauss sums is given by [20].)

If \(l_1 \equiv 3 \ (\mod 4)\), \(G(\chi) = \left(\sum_{T(x)=1} \chi(x)\right)G(\chi|_{F_p}) = \left(\sum_{T(x)=1} \chi(x)\right)\sqrt[4]{\phi} \in O_K[\varphi_p].\) By Stickelberger’s Theorem, we have

\[
(G(\chi))_{O_M} = \varphi^{b_0 l + b_1 \sigma - 1} \ (\text{where } b_0 = \frac{p - 1}{N} \sum_{s \in \left(\mathbb{Z}/N\mathbb{Z}\right)^* \mid s \equiv 1 \ (\mod 4)} s, \ b_1 = \frac{p - 1}{N} \sum_{s \in \left(\mathbb{Z}/N\mathbb{Z}\right)^* \mid s \equiv 3 \ (\mod 4)} s).
\]
Take \( d_1 := \# \{ s \in (\mathbb{Z}/2N\mathbb{Z})^* \mid s \equiv 1 \pmod{4} \}, \ d_3 := \# \{ s \in (\mathbb{Z}/2N\mathbb{Z})^* \mid s \equiv 3 \pmod{4} \}, \) similarly by the group maps \( \Phi_1, \Phi_2, \)

\[
\begin{align*}
  b_0 &= \frac{\alpha}{N} \sum_{s \in (\mathbb{Z}/2N\mathbb{Z})^*} s + \sum_{s \equiv 2 \pmod{4}} (s + 2l_1) = \frac{\alpha}{N} [l_1 f + 2l_1 d_3]
  b_1 &= \frac{\alpha}{N} \sum_{s \equiv 1 \pmod{4}} s + \sum_{s \equiv 2 \pmod{4}} (s + 2l_1) = \frac{\alpha}{N} [l_1 f + 2l_1 d_1]
\end{align*}
\]

Lemma 4.7. In Case \( F2, \) if \( l_1 \equiv 3 \pmod{4}, \) then

\[
d_1 = \frac{\varphi(l_1)}{2} + 1, \quad d_3 = \frac{\varphi(l_1)}{2} - 1,
\]

i.e. there are \( \frac{\varphi(l_1)}{2} + 1 \) elements \( s \) in \((\mathbb{Z}/2N\mathbb{Z})^*\) such that \( s \equiv 1 \pmod{4}, \) and \( \frac{\varphi(l_1)}{2} - 1 \) elements \( s \) in \((\mathbb{Z}/2N\mathbb{Z})^*\) such that \( s \equiv 3 \pmod{4}. \)

**Proof.** \( \#(\mathbb{Z}/2N\mathbb{Z})^* = \varphi(l_1) = (l_1 - 1)l_1^{-1}. \) Suppose that

\[
S_1 = \{ \text{all the odd numbers from 1 to } 2l_1^{-1} \}.
\]

Since \( 2l_1^{-1} - 1 \equiv 2(-1)^{-1} - 1 \equiv 1 \pmod{4}, \) there are \( \frac{2l_1^{-1} + 1}{2} \) elements \( s \) in \( S_1 \) such that \( s \equiv 1 \pmod{4}, \) and \( \frac{2l_1^{-1} - 1}{2} \) elements \( s \) such that \( s \equiv 3 \pmod{4}. \)

Next, we consider \( S_2 = \{ s \in S_1 : l_1 | s \} \). \forall x \in S_2, x \) must be the form as \( x = l_1(2k - 1), \) where \( k = 1, \cdots , l_1^{-1}. \) Since

\[
x \equiv (-1)(2k-1) \equiv 2k + 1 \equiv \begin{cases} 3 & \text{if } k \text{ is odd;} \\ 1 & \text{if } k \text{ is even.} \end{cases} \pmod{4}
\]

Then there are \( \frac{l_1^{-1} - 1}{2} \) elements \( s \) in \( S_2 \) such that \( s \equiv 1 \pmod{4}, \) and \( \frac{l_1^{-1} + 1}{2} \) elements \( s \) such that \( s \equiv 3 \pmod{4}. \)

Finally, since \((\mathbb{Z}/2N\mathbb{Z})^* = S_1 \setminus S_2, \) the lemma has been proved. \( \Box \)

By Lemma[4.7]

\[
\begin{align*}
  b_0 + b_1 &= \frac{\alpha}{N} (2l_1 f + 2l_1 (d_1 + d_3)) = (p - 1)f \\
  b_0 - b_1 &= \frac{\alpha}{N} 2l_1 (d_3 - d_1) = -(p - 1) \\
  \Rightarrow \quad b_0 &= \frac{\alpha}{N} (p - 1) \\
  b_1 &= \frac{\alpha}{N} (-p - 1)
\end{align*}
\]

Then

\[
(G(\chi)) = p^{-\frac{\alpha}{N} 2^{-1} f} = p^{-\frac{\alpha}{N} 2^{-1} p} \in O_K.
\]

Assume that \( \varphi = (a + b\sqrt{-1}), \) where \( a, b \in \mathbb{Z} \) such that \( a^2 + b^2 = p, \ p \nmid b. \) Thus,

\[
G(\chi) = \varepsilon p^{-\frac{\alpha}{N} 2^{-1} (a + b\sqrt{-1})},
\]

where \( \varepsilon \in \{ \pm 1, \pm i \} \) can be determined by Stickelberger congruence. For more details, one can refer to [20].

**Case F3.** \( a_0 = 1, \ a_1 = 2, \ p \equiv g_2 \pmod{N}, \ \text{i.e.} \ p \equiv 3 \pmod{4} \) and \( p \equiv g_1^2 \pmod{N}, \ 3 \neq l_1 \equiv 3 \pmod{4}, \)

\[
K = \mathbb{Q}(\sqrt{-l_1}).
\]

Here \( f = \varphi(N)/2 = \varphi(N_0) = (l_1 - 1)l_1^{-1} \equiv 2 \pmod{4} \) and \( (p - 1, l_1^+) = 1, \) then the order of \( \chi \mid_{F_p} \) is \( \frac{N}{(N, -p)} = 1. \) By Stickelberger’s Theorem, \( (G(\chi))_{O_K} = \varphi l_1 b_{i+1}\sigma - 1, \)

\[
\begin{align*}
  b_0 &= \frac{\alpha}{N} \sum_{s \in (\mathbb{Z}/2N\mathbb{Z})^*} s = \frac{\alpha}{N} \sum_{s \equiv 2 \pmod{4}} (s_0 + (\frac{-1}{2})^{\alpha + 1}(l_1) + 2l_1 \sqrt{\varphi l_1}) \\
  b_1 &= \frac{\alpha}{N} \sum_{s \in (\mathbb{Z}/2N\mathbb{Z})^*} s = \frac{\alpha}{N} \sum_{s \equiv 2 \pmod{4}} (s_0 + (\frac{-1}{2})^{\alpha + 1}(l_1) + 2l_1 \sqrt{\varphi l_1}).
\end{align*}
\]
Then

\[ b_0 + b_1 = \frac{1}{\varphi(\ell_1^i)} \varphi(\ell_1^1), \]

\[ b_0 - b_1 = \frac{1}{\varphi(\ell_1^i)} \left[ -2h_1 \ell_1^i + \ell_1^i \sum_{s_0 \in (\mathbb{Z}/N\mathbb{Z})^*} (-1)^{s_0} \left( \frac{2\ell_1^j}{s_0} \right) \right] \]

\[ = \left\{ \begin{array}{ll}
-\frac{l_1}{2} + 2h_1 = 0 & \text{if } l_1 \equiv 7 \pmod{8}; \\
-\frac{l_1}{2} = \frac{6h_1}{l_1} = -2h_1 & \text{if } l_1 \equiv 3 \pmod{8},
\end{array} \right. \]

where the last equation, above, is by the formula on the class number of imaginary quadratic field \( \mathbb{Q}(\sqrt{-l_1}) \).

If \( l_1 \equiv 7 \pmod{8} \), \( G(\chi) = \varepsilon p^{\frac{1}{2}} \), where \( \varepsilon \in \{ \pm 1 \} \). We can determine \( \varepsilon \) by the congruences below

\[ G(\chi)(\chi) \equiv \left\{ \begin{array}{ll}
\chi(\ell_1^1) \chi(\ell_1^i) \varepsilon (\pm 1) \chi^{\frac{1}{2}} p^{\frac{1}{2}} & \pmod{l_1},
\end{array} \right. \]

Hence, \( \varepsilon = (\pm 1) \chi^{\frac{1}{2}} p^{\frac{1}{2}} \).

If \( l_1 \equiv 3 \pmod{8} \), we have \( G(\chi) = \varepsilon p^{\frac{1}{2}} - h_1 (\varphi(\ell_1^1) / \sqrt{m_1}) \) by the result in Case A, where integers \( a, b \) are determined by equations (4.2). And \( \varepsilon \) can be determined by the congruences below

\[ G(\chi)(\chi) \equiv \left\{ \begin{array}{ll}
\chi(\ell_1^1) \chi(\ell_1^i) \varepsilon (\pm 1) \chi^{\frac{1}{2}} p^{\frac{1}{2}} & \pmod{l_1},
\end{array} \right. \]

Hence, \( \varepsilon \equiv (\pm 1) \chi^{\frac{1}{2}} p^{\frac{1}{2}} \pmod{l_1} \).

In conclusion, we obtain that

**Theorem 4.8.** Let \( N = 4l_1^i \cdot \left[ (\mathbb{Z}/N\mathbb{Z})^* : \left\langle p \right\rangle = 2 \right. \), i.e. \( f = \frac{\varphi(N)}{2} \). Take \( q = p^l, \chi \) be a multiplicative character of order \( N \) over \( \mathbb{F}_q \) and \( h_1 \) be the ideal class number of \( \mathbb{Q}(\sqrt{-l_1}) \). Assume that the orders of \( p \) in group \( (\mathbb{Z}/4\mathbb{Z})^* \) and \( (\mathbb{Z}/l_1^i \mathbb{Z})^* \) are respectively \( 2/\varphi_0 \) and \( \varphi(\ell_1^1) / a_1 \). Then

(i). For Case F1, \( (a_0 = a_1 = 1, p \equiv 3 \pmod{4}, l_1 \equiv 1 \pmod{4}) \)

\[ G(\chi) = p^{\frac{1}{2}} (a' + b\sqrt{-l_1}), \]

where integers \( a', b' \) are determined by equations

\[ \left\{ \begin{array}{l}
(a')^2 + l_1(b')^2 = p^\frac{1}{2}; \\
(a) \equiv p^{\frac{1}{2}} \pmod{l_1},
\end{array} \right. \] (4.7)

(ii). For Case F2, \( ((a_0 = 2, a_1 = 1, p \equiv 1 \pmod{4})) \)

\[ G(\chi) = \left\{ \begin{array}{ll}
\varepsilon p^{\frac{1}{2}} & \text{if } l_1 \equiv 1 \pmod{4}; \\
\varepsilon \sqrt{p} p^{\frac{1}{2}} (a'' + b'' \sqrt{-l_1}) & \text{if } l_1 \equiv 3 \pmod{4},
\end{array} \right. \]

where \( \varepsilon \in \{ \pm 1, \pm i \} \) can be determined by Stickelberger congruence and integers \( |a''|, |b''| \) are determined by \( (a'')^2 + (b'')^2 = p \).

(iii). For Case F3, \( (a_0 = 1, a_1 = 2, p \equiv 3 \pmod{4}, l_1 \equiv 3 \pmod{4}) \)

\[ G(\chi) = \left\{ \begin{array}{ll}
(\pm 1) p^\frac{1}{2} & \text{if } l_1 \equiv 7 \pmod{8}; \\
-\frac{l_1}{2} p^\frac{1}{2} & \text{if } l_1 \equiv 3 \pmod{8}.
\end{array} \right. \]

where integers \( a, b \) are determined by equations (4.2).

### 4.2 Explicit evaluation of Gauss sums \( G(\chi^\lambda) \) \( (1 \leq \lambda < N) \)

In this section, we give explicit evaluation of a series Gauss sums \( G(\chi^\lambda) \) \( (1 \leq \lambda < N) \), by the results given in section 4.1. Our main mathematical tool is Davenport-Hasse lifting formula, and we omit some similar proofs for simplicity.
Case A.

Let $N = l_1^{r_1}, \ 3 \neq l_1 \equiv 3 \pmod{4}$. For $1 \leq \lambda \leq N - 1$, let $\lambda = xt_1^1 + y$ for some integer $t < r_1$, where $0 \leq x \leq l_1 - 1, 0 \leq y \leq l_1 - 1$ and $(x,y) \neq (0,0)$. If $y \neq 0, (\lambda, l_1^{r_1}) = 1$, then

$$G(\chi^\lambda) = \begin{cases} G(\chi) & \text{if } y \neq 0 \pmod{l_1}; \\ G(\overline{\chi}) & \text{if } y = 0 \pmod{l_1}. \end{cases}$$

If $y = 0, \lambda = xt_1^1$, then

$$G(\chi^\lambda) = \begin{cases} G(\chi^{t_1}) & \text{if } x \neq 0 \pmod{l_1}; \\ G(\overline{\chi^{t_1}}) & \text{if } x = 0 \pmod{l_1}. \end{cases}$$

Thus, the problem owns to the calculation of Gauss sums $G(\chi^t)$ ($0 \leq t \leq r_1 - 1$).

The order of $\chi^t$ is $N = l_1^{r_1 - 1}$ and $\text{ord}_N(p) = \frac{\sqrt{N}}{2} = f$. Let $\eta$ be the corresponding primitive character of order $\tilde{N}$ over $\mathbb{F}_{p^f}$, then by Theorem 4.1,

$$G(\eta) = p\frac{f}{2}(a+b\sqrt{-l_1})/2,$$

where integer $a, b$ are determined by equations 4.2. Then by Davenport-Hasse lifting formula, the theorem is given as follow.

Theorem 4.9 (Case A). Let $N = l_1^{r_1}, l_1 \equiv 3 \pmod{4}, l_1 > 7, \left[\mathbb{Z}/NZ\right]^* \lhd p > 2$, i.e $\text{ord}_N(p) = f = \frac{1}{2}\varphi(N)$. Take $q = p^f, \chi$ be a multiplicative character of order $N$ over $\mathbb{F}_q$ and $h_1 = h(\mathbb{Q}(\sqrt{-l_1}))$. Then, for $0 \leq t < r_1$,

$$G(\chi^{t_1}) = p^{\frac{f-1}{2}}(a+b\sqrt{-l_1})^{t_1},$$

where integers $a, b$ are determined by equations 4.2.

Case B.

Here, $N = l_1^{r_1}l_2^{r_2}$, where $l_1, l_2$ are distinct odd primes and $r_1, r_2 \geq 1$. Similarly as Case A, we only consider the Gauss sums $G(\chi^{t_1}l_1^{r_2}), G(\chi^{t_1}l_2^{r_2})$ and $G(\chi^{t_1}l_2^{r_2})$ ($0 \leq t_1 < r_1, 0 \leq t_2 < r_2$). And the result is given as follow.

Theorem 4.10 (Case B). Let $N = l_1^{r_1}l_2^{r_2}, \left[\mathbb{Z}/NZ\right]^* \lhd p > 2$, i.e. $\text{ord}_N(p) = f = \frac{1}{2}\varphi(N)$. Take $q = p^f, \chi$ be a multiplicative character of order $N$ over $\mathbb{F}_q$. Then,

1. For Case B1 ($0 \leq t_1 < r_1, 0 \leq t_2 < r_2$),

$$G(\chi^{t_1}l_1^{r_2}) = p^{\frac{f}{2}}(\varphi l_1^{t_1}) (a+b\sqrt{-l_1})^{t_1}l_1^{r_2},$$

$$G(\chi^{t_1}l_2^{r_2}) = p^\varphi,$$

$$G(\chi^{t_1}l_2^{r_2}) = -p^\varphi,$$

where $h_{12} = h(\mathbb{Q}(\sqrt{-l_1}))$ and integers $a', b'$ are determined by equations 4.3.

2. For Case B2 ($0 \leq t_1 < r_1, 0 \leq t_2 < r_2$),

$$G(\chi^{t_1}l_1^{r_2}) = \begin{cases} p^\varphi, & \text{if } \left(\frac{l_1}{l_2}\right) = 1; \\ \varphi + \varphi l_1^{t_1} (a+b\sqrt{-l_1})^{t_1}l_2^{r_2}, & \text{if } \left(\frac{l_1}{l_2}\right) = -1, \end{cases}$$

$$G(\chi^{t_1}l_2^{r_2}) = p^\varphi,$$

$$G(\chi^{t_1}l_2^{r_2}) = -p^\varphi(\varphi l_1^{t_1} (a+b\sqrt{-l_1})^{t_1}l_2^{r_2}),$$

where $h_1 = h(\mathbb{Q}(\sqrt{-l_1}))$ and integers $a, b$ are determined by equations 4.2.

Proof. Let $\chi^\lambda = \chi^{t_1}l_1^{r_2}$ ($0 \leq x \leq r_1, 0 \leq y \leq r_2$), which correspond the primitive characters $\eta$ of order $\tilde{N}$ over $\mathbb{F}_{p^f}$ with $\tilde{f} = \text{ord}_{\tilde{N}}(p)$. We list all the subcases in the following tables. Then, the theorem can be proved by Davenport-Hasse lifting formula.

(1) For Case B1,
Case C.

For $N = 2^{t}$ $(t \geq 3)$, the following results were given by [14].

Theorem 4.11. (14) Let $N = 2^{t}$ $(t \geq 3)$, $\chi$ be a multiplicative character of order $N$ over $\mathbb{F}_{q}$. Then, for $1 \leq s \leq t$, Gauss sums $G(\chi^{2^{s-t}})$ are given as follow:

(i). If $p \equiv 3 \pmod{8}$, $K = \mathbb{Q}(\sqrt{-2})$, then

\[
G(\chi) = \sqrt{-p}p^{2^{t-3}-1}(a + b\sqrt{-2}) \quad \text{for } s = t;
\]
\[
G(\chi^{2^{s-t}}) = -\sqrt{-p}p^{2^{t-3}-2^{s-t-1}}(a + b\sqrt{-2})^{2^{t-s}} \quad \text{for } 3 \leq s \leq t - 1;
\]
\[
G(\chi^{2^{t-1}}) = -p^{2^{t-3}} \quad \text{for } s = 2;
\]
\[
G(\chi^{2^{s-1}}) = \begin{cases} p & \text{if } t = 3; \\ -p^{2^{t-3}} & \text{if } t \geq 4 \end{cases} \quad \text{for } s = 1,
\]

where $a, b \in \mathbb{Z}$ are given in Theorem 4.3(i).

(ii). If $p \equiv 5 \pmod{8}$, $K = \mathbb{Q}(i)$, then

\[
G(\chi) = p^{2^{t-3}}\sqrt{a + bi}/\sqrt{p} \quad \text{for } s = t;
\]
\[
G(\chi^{2^{s-t}}) = p^{2^{t-3}-1}\sqrt{a + bi} \quad \text{for } s = t - 1;
\]
\[
G(\chi^{2^{t-1}}) = -p^{2^{t-3}}(a + bi)^{2^{t-s-1}}/p^{2^{t-s-2}} \quad \text{for } 2 \leq s \leq t - 1;
\]
\[
G(\chi^{2^{s-1}}) = -p^{2^{t-3}} \quad \text{for } s = 1;
\]

where $a, b \in \mathbb{Z}$ and $\sqrt{a + bi}$ are given in Theorem 4.3(2).

Case D.

$N = 2N_{0} = 2l_{1}l_{2}$, for the Gauss sums $G(\chi^{2^{t_{1}}})$ $(i = 0, 1; 0 \leq t_{1} \leq r_{1}, (i, t_{1}) \neq (1, r_{1}))$, we have the following results:

Theorem 4.12 (Case D). Let $N = 2N_{0} = 2l_{1}l_{2}$, $3 \not\equiv l_{1} \equiv 3 \pmod{4}$, $[(\mathbb{Z}/N\mathbb{Z})^{\ast} : <p>] = 2$, i.e. $f = \frac{\varphi(N)}{2}$. Take $q = p^{f}$ and $\chi$ be a multiplicative character of order $N$ over $\mathbb{F}_{p}$. Then, for $0 \leq t_{1} \leq r_{1} - 1$, $f = \frac{\varphi(N)}{2}$.

\[
G(\chi^{t_{1}}) = \begin{cases} (-1)\frac{1}{2}(r_{1} - t_{1}) + t_{1}p^{2^{t_{1}} - h_{1}l_{1}l_{2}}\sqrt{a + bi}/\sqrt{p} & \text{if } l_{1} \equiv 3 \pmod{8}; \\ (-1)\frac{1}{2}(r_{1} - t_{1}) + t_{1}p^{2^{t_{1}} - h_{1}l_{1}l_{2}}\sqrt{a + bi}/\sqrt{p} & \text{if } l_{1} \equiv 3 \pmod{8};
\end{cases}
\]
\[
G(\chi^{2^{t_{1}}}) = p^{\frac{1}{2}(1 - h_{1}l_{1})}(a + b\sqrt{-l_{1}});
\]
\[
G(\chi^{2^{t_{1}}}) = (-1)f^{-1}\sqrt{p^{f}} / (-1)\frac{1}{2}p^{2^{t_{1}} - l_{1}l_{2}}p^{\frac{1}{2}} \sqrt{p^{f}},
\]

where $h_{1} = h(\mathbb{Q}(\sqrt{-l_{1}}))$ and integers $a, b$ are determined by equations (2.2).

Case E.

$N = 2N_{0} = 2l_{1}l_{2}$. For the Gauss sums $G(\chi^{2^{t_{1}}l_{2}})$ $(i = 0, 1; 0 \leq x \leq r_{1}, 0 \leq y \leq r_{2}, (i, x, y) \neq (1, r_{1}, r_{2}))$, we have the following evaluations.
Theorem 4.13. Let \( N = 2N_0 = 2[l_1^*l_2^*] \), \((\mathbb{Z}/N\mathbb{Z})^* : < p > = 2\), i.e. \( f = \frac{\varphi(N)}{2} \). Take \( q = p^f \) and \( \chi \) be a multiplicative character of order \( N \) over \( \mathbb{F}_{p} \). Assume that the orders of \( p \) in groups \((\mathbb{Z}/l_1^*\mathbb{Z})^* \) and \((\mathbb{Z}/l_2^*\mathbb{Z})^* \) are respectively \( \varphi(l_1^*)/a_1 \) and \( \varphi(l_2^*)/a_2 \). Then, for \( 0 \leq l_1 < r_1 \), \( 0 \leq l_2 < r_2 \).

**Case F1:** \((a_1 = a_2 = 1, (l_1, l_2) = (3, 1) (\text{mod } 4))\)

\[
G(\chi^{l_1^*l_2^*}) = \begin{cases} -p^{f}, & \text{if } l_1l_2 \equiv 3 \pmod{8}; \\
-p^{f} - l_1^*l_2^*h_{12}\left(\frac{a+b\sqrt{-l_1}}{2}\right)2^{l_1^*l_2^*}, & \text{if } l_1l_2 \equiv 7 \pmod{8}, \end{cases}
\]

where \( h_{12} \) is the ideal class number of \( \mathbb{Q}(\sqrt{-l_1l_2}) \) and integers \( a', b' \) are determined by equations (4.3).

**Case F2:** \((a_1 = 2, a_2 = 1, 3 \neq l_1 \equiv 3 \pmod{4}, l_2 \equiv 1 \pmod{2})\)

\[
G(\chi^{l_1^*l_2^*}) = \begin{cases} \left\{ -p^{f}, \quad \text{if } l_1 \equiv 3 \pmod{8}; \\
-p^{f} - l_1^*l_2^*h_{12}\left(\frac{a+b\sqrt{-l_1}}{2}\right)2^{l_1^*l_2^*}, \quad \text{when } \left(\frac{1}{l_2}\right) \text{ and } l_1 \equiv 3 \pmod{8}; \\
\left\{ -p^{f}, \quad \text{if } l_1 \equiv 7 \pmod{8}, \end{cases}
\]

where \( h_1 \) is the ideal class number of \( \mathbb{Q}(\sqrt{-l_1}) \) and \( a, b \in \mathbb{Z} \) are determined by equations (4.2).

\[\Box\]  

**Case F.**

\( N = 4[l_1^*l_2^*] \). For the Gauss sums \( G(\chi^{2^i l_i^*}) \) \((i = 0, 1, 2; 0 \leq x \leq r_1, (i, x) \neq (2, r_1))\), we have the following evaluations.

**Theorem 4.14.** Let \( N = 4l_1^* \), \((\mathbb{Z}/N\mathbb{Z})^* : < p > = 2\), i.e. \( f = \varphi(l_1^*)/a_1 \). Take \( q = p^f \) and \( \chi \) be a multiplicative character of order \( N \) over \( \mathbb{F}_{p} \). Assume that the order of \( p \) in group \((\mathbb{Z}/l_1^*\mathbb{Z})^* \) is \( \varphi(l_1^*)/a_1 \). Then, for \( 0 \leq l_1 < r_1 \).

(1) **Case F1:** \((l_1 \equiv 1 \pmod{4}, a_1 = 1, p \equiv 3 \pmod{4}, K = \mathbb{Q}(\sqrt{-l_1})\))

\[
G(\chi^{l_1^*}) = p^{f}(a + b\sqrt{-l_1})^{l_1^*},
\]

\[
G(\chi^{2^i l_1^*}) = G(\chi^{2^{i+1} l_1^*}) = -p^{f},
\]

\[
G(\chi^{4^i l_1^*}) = p^{f},
\]

where \( a, b \in \mathbb{Z} \) are determined by equations (4.7).

(2) **Case F2:** \((l_1 \equiv 1 \pmod{2}, a_1 = 1, p \equiv 1 \pmod{4}, K = \mathbb{Q}(\sqrt{-1})\))

\[
G(\chi^{l_1^*}) = \begin{cases} ep^{f}, & \text{if } l_1 \equiv 1 \pmod{4}; \\
ep^{f} \frac{1}{\sqrt{pp^{f}(f - l_1^* - 1)}(a' + b'\sqrt{-l_1})^{l_1^*}}, & \text{if } l_1 \equiv 1 \pmod{4}, \end{cases}
\]

where \( a', b' \in \mathbb{Z} \) are determined by equations (4.7),
where \( \varepsilon \in \{ \pm 1, \pm i \} \) can be determined by Stickelberger congruence, integers \(|a'|, |b'|\) are determined by \((a')^2 + (b')^2 = p\) and by the formula (3.3) of quartic Gauss sums over \( \mathbb{F}_p \). \( \tilde{a}, b, C \in \mathbb{Z} \) are determined by \( \tilde{a}^2 + \tilde{b}^2 = p, \tilde{a} \equiv 1 \pmod{4}, C \equiv \frac{|b|}{2} \left( \left( \frac{p-1}{2} \right) ! \right) \pmod{p} \).

(3). Case F3: \( 3 \not\equiv l_1 \equiv \mp 3 \pmod{4}, a_1 = 2, p \equiv 3 \pmod{4}, K = \mathbb{Q}(\sqrt{-l_1}) \)

\[ G(x^{l_1}) = \begin{cases} (-1)^{l_1+1} p^{\frac{l_1}{2}} & \text{if } l_1 \equiv 7 \pmod{8}; \\ (-1)^{l_1+1} p^{\frac{l_1}{2} - l_1^* h_1} \left( \frac{a+b\sqrt{-l_1}}{2} \right)^{l_1^*} & \text{if } l_1 \equiv 3 \pmod{8}. \end{cases} \]

Irreducible cyclic codes and Gaussian integers are determined by equations (4.2).

References

[1] Baumert L D, McEliece R J. Weights of irreducible cyclic codes. *Inform. and Control*, 20(2): 158-175 (1972)
[2] Baumert L D, Mykkeltve J. Weight distribution of some irreducible cyclic codes. *D. S. N. Report*, 16: 128-131 (1973)
[3] Berndt B C, Evans R J, Williams K S. Gauss and Jacobi Sums. New York: J.Wiley and Sons Company, 1997
[4] Davenport H, Hasse H. Die Nullstellen der Kongruenzzetafunktionen in gewissen zyklischen Fällen. *J. Reine Angew. Math.*, 172: 151-182 (1934)
[5] Feng K Q, Yang J. The evaluation of Gauss sums for characters of 2-power order in the index 4 case. accepted by *Algebra Colloquium*
[6] Feng K Q, Yang J, Luo S X, Gauss sum of Index 4: (1) cyclic case. *Acta Math. Sin. Eng. Ser.*, 21(6): 1425-1434 (2005)
[7] Gauss C F. Disquisitiones arithmeticae (translateed by A.A.Clarke). New Haven: Yale Univ. Press, 1966
[8] Ireland K, Rosen M. A classical introduction to modern number theory. New York: Springer-Varlag, 1982
[9] Langevin P. Calculs de Certaines Sommes de Gauss. *J. of Number Theory*, 32: 59-64 (1977)
[10] Lidl R, Niederreiter H. Finite Fields. Addison-Wesley, New York, 1983
[11] MacWilliams J, Seery J. The weight distributions of some minimal cyclic codes. *IEEE Trans. Inform. Theory*, 27(6): 796-806 (1981)
[12] Mbojd O D. Quadratic Gauss Sums. *Finite Fields and their Appl.* 4: 347-361 (1998)
[13] McEliece R J. Irreducible cyclic codes and Gaussian integers. *Math. Centre Tracts.*, 55: 179-196 (1974)
[14] Meijer P, Van der Vlugt M. The evaluation of Gauss sums for characters of 2-power order. *J. of Number Theory*, 100(2): 381-395 (2003)
[15] Moisio M. Exponential Sums, Gauss Sums and Cyclic Codes: [Doc. Dissertation]. Vassa Finland: Acta Univ. Oul. A 306, 1998
[16] Stickelberger L. Über eine Verallgemeinerung von der Kreistheilung. *Math. Ann.*, 37: 321-367 (1890)
[17] Van der Vlugt M. Hasse-Davenport curves, Gauss sums and weight distributions of irreducible cyclic codes. *J. of Number Theory*, 55: 145-159 (1995)
[18] Van Lint J H, Introduction to Coding Theory. New York: Springer-Varlag, 1982
[19] Yang J, Luo S X, Feng K Q. Gauss sum of Index 4: (2) non-cyclic case. *Acta Math. Sin. Eng. Ser.*, 22(3): 833-844 (2006)
[20] Yang J, Xia L L. A note on the sign (unit root) ambiguities of Gauss sums in index 2 and 4 cases. Submitted, 2009. [http://arxiv.org/pdf/0912.1414v1](http://arxiv.org/pdf/0912.1414v1)