Grid Graph Reachability

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Abstract

The reachability problem is to determine if there exists a path from one vertex to the other in a graph. Grid graphs are the class of graphs where vertices are present on the lattice points of a two-dimensional grid, and an edge can occur between a vertex and its immediate horizontal or vertical neighbor only. Asano et al. presented the first simultaneous time space bound for reachability in grid graphs by presenting an algorithm that solves the problem in polynomial time and $O(n^{1/2+\epsilon})$ space [2]. In 2018, the space bound was improved to $\tilde{O}(n^{1/3})$ by Ashida and Nakagawa [4]. In this paper, we further improve the space bound and present a polynomial time algorithm that uses $O(n^{1/4+\epsilon})$ space to solve reachability in a grid graph with $n$ vertices.

1 Introduction

The reachability problem is to determine whether there exists a path from a source vertex to a terminal vertex in a graph. This problem is of vital importance in the field of computer science. Not only is it used as a subroutine in many graph algorithms, but its study also gives insights into space bounded computations. Reachability in directed graph is complete for the class of problems solvable by a nondeterministic Turing machine in logspace. A deterministic logspace algorithm for it would show NL to be equal to L, thus solving an open question in the area of computational complexity. Reachability in undirected graph was shown to be in L by Reingold [12].

Standard graph traversal algorithms solves reachability using linear time and space. We also know of Savitch’s algorithm which can solve reachability in $O(\log^2 n)$ space, but it requires $2^{O(\log^2 n)}$ time [13]. In a survey of reachability problems and its solutions, Wigderson asked if there exists an algorithm which maintains the polynomial time bound while running in $O(n')$ space [15].

Barnes, Buss, Ruzzo, and Schieber gave the first sublinear space polynomial time algorithm for reachability in directed graph [5]. The space complexity of their algorithm is $n/2^{O(\sqrt{\log n})}$. Till date, we know of no algorithm which improves the space bound while maintaining the polynomial time for general directed graphs. However, for certain classes of directed graphs, we know of polynomial time algorithms with better space bound.

Planar graphs are an important class of directed graphs. Reachability in planar graphs belongs to a subclass of NL called unambiguous logspace UL [6]. Imai et al. [9] showed that reachability in planar graph can be solved in $O(n^{1/2+\epsilon})$ space for any $\epsilon$. Later for this space bound was improved to $\tilde{O}(n^{1/2})$ space by Asano et al. [3]. For graphs of higher genus, Chakraborty et al. gave an $\tilde{O}(n^{2/3}g^{1/3})$ space algorithm which additionally requires, as an input, an embedding of the graph on a surface of genus $g$ [7]. They also gave an $\tilde{O}(n^{2/3})$ space algorithm for $H$ minor-free graphs which requires tree decomposition of the graph as an input.
and $O(n^{1/2+\epsilon})$ space algorithm for $K_{3,3}$-free and $K_5$-free graphs. For layered planar graphs, Chakraborty and Tewari showed that for every $\epsilon > 0$ there is an $O(n^3)$ space algorithm \cite{8}. Stolee and Vinodchandran gave a polynomial time algorithm that, for any $\epsilon > 0$ solves reachability in a directed acyclic graph with $O(n^3)$ sources and embedded on the surface of genus $O(n^3)$ using $O(n^3)$ space \cite{14}. For unique-path graphs, Kannan et al. give $O(n^3)$-space and polynomial time algorithm \cite{10}.

Our concern here is with grid graphs. Grid graphs are a subclass of planar graphs. Reachability in planar graphs belongs to a subclass of NL called unambiguous logspace UL \cite{9}. Reachability in planar graphs can be reduced to reachability in grid graphs in logspace \cite{11}. Asano and Doerr presented a polynomial time algorithm that uses $O(n^{1/2+\epsilon})$ space for solving reachability in grid graphs \cite{7}. Ashida and Nakagawa presented an algorithm with improved space complexity of $O(n^{1/3})$ \cite{4}. Ashida and Nakagawa’s algorithm proceeded by first dividing the input grid graph into subgrids. It then used a gadget to transform each subgrid into a planar graph, making the whole of the resultant graph planar. Finally, it used the planar reachability algorithm of Imai et al. \cite{9} as a subroutine to get the desired space bound.

In this paper, we present an algorithm with a space complexity of $O(n^{1/4+\epsilon})$, thereby improving the bound of Ashida and Nakagawa.

**Theorem 1.** For every $\epsilon > 0$, there exists a polynomial time algorithm that decides reachability in an $n$-vertex grid graph using $O(n^{1/4+\epsilon})$ space.

Our algorithm works by first dividing the grid into subgrids. It then recursively solve each grid to get an auxiliary graph. It then solves this auxiliary graph by using a space efficient subroutine that we develop for it.

2 Preliminaries

We denote the vertex set of a graph $G$ by $V(G)$ and its edge set by $E(G)$. For a subset $U$ of $V(G)$, we denote the subgraph of $G$ induced by the vertices of $U$ as $G[U]$. For a graph $G$, we denote the set of all its connected components by $cc(G)$. For an edge $e = (u, v)$, we let $\text{tail}(e)$ be $u$ and $\text{head}(e)$ be $v$.

In a drawing of a graph on a plane, each vertex is mapped to a point of the plane, and each edge is mapped to a simple arc whose endpoints coincide with the mappings of the end vertices of the edge. Also, the interior points of the arc for an edge does not intersect with any other vertex points.

We call a graph $G$ an $N \times N$ grid graph if its vertices can be drawn on coordinates $(i, j)$ where $0 \leq i, j \leq N$ and for all edges of $G$, its end vertices are at a unit distance.

3 Auxiliary Graph

For simplicity of discussion, we begin with $N \times N$ grid graph and show that for every $\epsilon > 0$ there exists a polynomial time algorithm which has a space complexity of $O(N^{1/2+\epsilon})$. This would be enough to prove Theorem 1.

For a parameter $\alpha < 1$, we first define $\alpha$-auxiliary graph $\tilde{G}^\alpha$ of a grid graph $G$. We divide our grid graph $G$ into $N^{2\alpha}$ subgrids such that each subgrid is a $N^{1-\alpha} \times N^{1-\alpha}$ grid as shown in Figure 1. Let $G_{i,j}^\alpha$ be the graph obtained by solving reachability in the subgrid in the $i$-th row and $j$-th column. We obtain $\tilde{G}^\alpha$ by replacing each subgrid by its corresponding solved blocks. Since each of the subgrids contains $4N^{1-\alpha}$ vertices on its boundary, the total number of vertices in $\tilde{G}^\alpha$ would be at most $4N^{1+\alpha}$. An example of $\tilde{G}^\alpha$ is shown in Figure 4. For the rest of this article, for any $i$ and $j$, we will call the graph $G_{i,j}^\alpha$ as a block of $\tilde{G}^\alpha$.

Note that $\tilde{G}^\alpha$ might have parallel edges. However, each such edge will belong to a different block of $\tilde{G}^\alpha$.

Our algorithm for reachability constructs $\tilde{G}^\alpha$ by solving the $N^{1-\alpha} \times N^{1-\alpha}$ grids recursively. It then uses a polynomial time subroutine which solves $\tilde{G}^\alpha$. Note that we do not store the graph $\tilde{G}^\alpha$ explicitly, since that would require too much space. Instead, we solve a subgrid recursively everytime the subroutine queries for an edge in that subgrid of $\tilde{G}^\alpha$. 

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If two edges \( e \) and \( f \) of an auxiliary graph cross each other, then the graph would also have the edges \((\text{tail}(e), \text{head}(f))\) and \((\text{tail}(f), \text{head}(e))\).

Observation 2. If two edges \( e_1 \) and \( e_2 \) crosses a certain edge \( f \), and \( e_1 \) is closer to \( \text{tail}(f) \) than \( e_2 \), then the edge \((\text{tail}(e_1), \text{head}(e_2))\) is also present in the graph.

In this section, we formalize these observations. We first define, for a given block of \( \tilde{G}^\alpha \), a cyclic permutation on its vertices.

**Definition 1.** Let \( G \) be a grid graph and \( l = \tilde{G}^\alpha_{x,y} \) be a block of \( \tilde{G}^\alpha \) and \( v = (x, y) \) be a vertex in \( \tilde{G}^\alpha_{x,y} \) we define its counter-clockwise adjacent vertex \( c_l(v) \) as follows:

\[
c_l(v) = \begin{cases} 
(x+1, y) & \text{if } x < (x'+1)(N^{1-\alpha}) \text{ and } y'(N^{1-\alpha}) = y \\
(x, y+1) & \text{if } x = (x'+1)(N^{1-\alpha}) \text{ and } y < (y'+1)(N^{1-\alpha}) \\
(x-1, y) & \text{if } x'(N^{1-\alpha}) < x \text{ and } y = (y'+1)(N^{1-\alpha}) \\
(x, y-1) & \text{if } x'(N^{1-\alpha}) = x \text{ and } y'(N^{1-\alpha}) < y 
\end{cases}
\]

We also define \( r \)'th counter-clockwise adjacent neighbour recursively. For \( r = 0 \), \( c_{l}^0(v) = v \) and otherwise we have \( c_{l}^{r+1}(v) = c_l(c_{l}^{r}(v)) \).

We see that for a block \( l \) and a vertices \( v \) and \( w \) in it, we can write \( v \) as \( c_{l}^{p}(w) \) where \( p \) is smallest integer for which \( c_{l}^{p}(w) = v \).

**Definition 2.** Let \( G \) be a grid graph and \( l \) be a block of \( \tilde{G}^\alpha \). For two edges \( e \) and \( f \) in the block, such that \( e = (v, c_{l}^0(v)) \) and \( f = (c_{l}^p(v), c_{l}^q(v)) \). We say that edges \( e \) and \( f \) crosses each other if \( \min(q, r) < p < \max(q, r) \).

The next lemma we write follows from definitions \[1\] and \[2\].
Lemma 1. Let $G$ be a grid graph and $l$ be a block of $\tilde{G}^\alpha$. Let $w$ be an arbitrary vertex in the block $l$ and $e = (c_l^2(w), c_l^1(w))$ and $(f = c_l^2(w), c_l^1(w))$ be two edges in $l$. Then, $e$ and $f$ crosses if and only if either of the following two holds:

- $\min(p, q) < \min(r, s) < \max(p, q) < \max(r, s)$
- $\min(r, s) < \min(p, q) < \max(r, s) < \max(p, q)$

Following lemma follows due to the planarity of the underlying graph $G$ from which we have constructed $\tilde{G}^\alpha$.

Lemma 2. Let $G$ be a grid graph. For any block of $\tilde{G}^\alpha$, let $e$ and $f$ be two edges that crosses in it. Then the edges $(\text{tail}(e), \text{head}(f))$ and $(\text{tail}(f), \text{head}(e))$ are also present in that block.

We will be using the following lemmas to build our algorithm:

Definition 3. Let $G$ be a grid graph and $l$ be a block of $\tilde{G}^\alpha$. For a vertex $v$ and edges $f$, $g$ such that $f = (c_l^2(v), c_l^1(v))$ and $g = (c_l^2(v), c_l^1(v))$, we say that $f$ is closer to $v$ than $g$ if we have $\min(q, r) < \min(s, t)$.

Lemma 3. Let $G$ be a grid graph. If there are two edges $f$ and $g$ which crosses an edge $e$ in a block of $\tilde{G}^\alpha$, and $f$ is closer to the $\text{tail}(e)$ than $g$, then $(\text{tail}(f), \text{head}(g))$ also belongs to that block of $\tilde{G}^\alpha$.

Proof. Let $e = (v, c_l^2(v))$, $f = (c_l^2(v), c_l^1(v))$ and $g = (c_l^2(v), c_l^1(v))$. If $\text{tail}(f) = \text{tail}(g)$ then the lemma trivially follows. Otherwise, we have two cases to consider:

Case 1: $f$ crosses $g$.

In this case, we will have $(\text{tail}(f), \text{head}(g))$ present in $\tilde{G}^\alpha$ by Lemma 2.

Case 2: $f$ does not cross $g$.

In this case, we have $\min(q, r) < \min(s, t) < p < \max(s, t) < \max(q, r)$. Since $f$ crosses $e$, we have the edge $(c_l^2(v), c_l^1(v))$ in $\tilde{G}^\alpha$ by definition 2. This edge will cross $g$. Hence $(\text{tail}(f), \text{head}(g))$ is present in $\tilde{G}^\alpha$. □

3.2 Constructing a Pseudo Separator

Imai et al. used a separator construction to solve the reachability problem in planar graphs [9]. A separator is a small set of vertices whose removal disconnects the graph into smaller components. An essential property of a separator is that, for any two vertices, a path between the vertices must contain a separator vertex if the vertices lie in two different components with respect to the separator.

Unfortunately the graph $\tilde{G}^\alpha$ might not have a small separator. However, $\tilde{G}^\alpha$ has a different kind of separator, which we call as a PseudoSeparator (see Definition 4). A PseudoSeparator allows us to decide reachability in $\tilde{G}^\alpha$, in an efficient manner and obtain the claimed bounds.

Definition 4. Let $G$ be a grid graph and $H$ be a vertex-induced subgraph of $\tilde{G}^\alpha$ with $h$ vertices. Let $C$ be a subgraph of $H$ and $H^{(C)}$ be the subgraph of $H$ formed by removing all the vertices of $C$ and all the edges which crosses an edge of $C$. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function. We call $C$ an $f(h)$ – PseudoSeparator if the size of every connected component in $\text{cc}(H^{(C)})$ is at most $f(h)$. The size of $C$ is the total number of vertices and edges of $C$ summed together.

For a vertex-induced subgraph $H$ of $\tilde{G}^\alpha$, an $f(h)$ – PseudoSeparator is a subgraph of $H$ that has the property that, if we remove the vertices as well as all the edges which crosses one of the edges of PseudoSeparator, the graph gets disconnected into small pieces. Now, any path which connects two vertices in different components, must either contain a vertex of PseudoSeparator or must contain an edge that crosses an edge of PseudoSeparator (see Observation 3). We divide the graph using this PseudoSeparator and show an algorithm which recursively solves each subgraph and then combines their solution efficiently using the above observations.

Observation 3. Let $G$ be a grid graph and $H$ be a vertex-induced subgraph of $\tilde{G}^\alpha$. Let $C$ be a subgraph of $H$. Then the following holds:
1. For any two distinct $U_1$ and $U_2$ of $\text{cc}(H^{(C)})$, $U_1 \cap U_2 = \emptyset$.

2. $V(C) \cup (\bigcup_{U \in \text{cc}(H^{(C)})} U) = V(H)$

3. For every edge $e$ in $H$, if there exists distinct sets $U_1$ and $U_2$ in $\text{cc}(H^{(C)})$ such that one of the endpoints of $e$ is in $U$ and the other is in $U_2$, then there exists an edge $f$ in $C$ such that $e$ crosses $f$.

The subroutine would construct a PseudoSeparator using the next lemma. We discuss it in detail in section 3.2.

**Lemma 4.** Let $G$ be a grid graph and $H$ be a vertex-induced subgraph of $\tilde{G}^\alpha$ with $h$ vertices. For any constant $\beta > 0$, there exists an algorithm which takes $H$ as an input and outputs its $h^{1-\beta}$-PseudoSeparator of size $O(h^{1/2+\beta/2})$ in $O(h^{1/2+\beta/2})$-space and polynomial time.

We briefly comment on how to construct a PseudoSeparator of a vertex-induced subgraph $H$ of $\tilde{G}^\alpha$. First, we pick, in logspace, a maximal subset of edges from $H$ so that no two edges crosses. Then, we triangulate the resulting graph and use Imai et al.’s algorithm to find its separator. Call the triangulated graph as $\hat{H}$ and the separator vertices as $S$. The vertex set of PseudoSeparator of $H$ will contain all the vertices of $S$ and four additional vertices for each edge of $\hat{H}[S]$ that is not present in $H$. The edge set of PseudoSeparator of $H$ will contain all the edges of $H$ which are also in $\hat{H}[S]$ and four additional edges for each edge of $\hat{H}[S]$ that is not present in $H$.

We first show how to pick a maximal set of edges from $H$ in logspace. We call the resultant graph $\text{planar}(H)$.

**Definition 5.** Let $G$ be a grid graph and $H$ be a vertex induced subgraph of $\tilde{G}^\alpha$. We define $\text{planar}(H)$ as a subgraph of $H$. The vertex set of $\text{planar}(H)$ is same as that of $H$. For an edge $e$, let $l$ be the block to which $e$ belongs to. Let $w$ be the lowest indexed vertex in that block. Let $e = (c_i^l(w), c_j^l(w))$. We add the edge $e$ to $E(\text{planar}(H))$ if there exists no other edge $f = (c_i^q(w), c_j^q(w))$ in $H$ such that $\min(a, b) < \min(i, j) < \max(a, b) < \max(i, j)$.

**Lemma 5.** Let $G$ be a grid-graph and $H$ be a vertex-induced subgraph of $\tilde{G}^\alpha$. No two edges of $\text{planar}(H)$ crosses. For any edge $e$ in $H$ that is not in $\text{planar}(H)$, there exists another edge in $\text{planar}(H)$ that crosses it.

**Proof.** For the first part, we have, by lemma 4 that if two edges $e = (c_i^p(w), c_j^p(w))$ and $f = (c_i^q(w), c_j^q(w))$ cross, then either $\min(p, q) < \min(r, s) < \max(p, q) < \max(r, s)$ or $\min(r, s) < \min(p, q) < \max(r, s) < \max(p, q)$. Hence, by our construction of $\text{planar}(H)$, atmost one of those two can be chosen. Thus no two edges of $\text{planar}(H)$ cross.

For the second part, we will prove by contradiction. Let us assume that there exists edges in $H$ which is not in $\text{planar}(H)$ and also not crossed by an edge in $\text{planar}(H)$. We pick edge $e = (c_i^p(w), c_j^p(w))$ from such that $\min(p, q)$ of that edge is minimum. Since this edge is not present in $\text{planar}(H)$, we have by definition, an edge $f = (c_i^q(w), c_j^q(w))$ such that $\min(r, s) < \min(p, q) < \max(r, s) < \max(p, q)$. We pick the edge $f$ for which $\min(r, s)$ is minimum. Now, since this edge $f$ is not present in $\text{planar}(H)$, we have another edge $g = (c_i^l(w), c_j^l(w))$ in $\text{planar}(H)$ such that $\min(i, j) < \min(r, s) < \max(i, j) < \max(r, s)$. We pick $q$ such that $\min(i, j)$ is minimum and break ties by picking one whose $\max(i, j)$ is maximum. Now, we have the following cases:

Case 1: $i < r < j < s$ In this case, the edge $c_i^l(w), c_j^l(w)$ will be present in $H$. Since $i < p < s < q$, and $i < \min(r, s)$, this will contradict the way in which edge $f$ was chosen.

Case 2: $j < s < i < r$ In this case, the edge $(c_i^l(w), c_j^l(w))$ will be present in $H$. Since $j < p < r < q$, and $j < \min(r, s)$, this will contradict the way in which edge $f$ was chosen.

Case 3: $i < s < j < r$ In this case, the edge $(c_i^l(w), c_j^l(w))$ will be present in $H$. This edge will cross $e$ and hence not be present in $\text{planar}(H)$. Thus, we have an edge $g' = (c_i^l(w), c_j^l(w))$ such that $\min(i', j') < j < \max(i', j') < r$. We will thus have two subcases.

Subcase 3a: $i < \min(i', j')$ Here, we will have $i < \min(i', j') < j < \max(i', j')$. Hence is edge is will cross $e$ giving a contradiction.
Subcase 3b: \( \min(i', j') \leq i \) Here, this edge should have been chosen instead of \( g \) contradicting our choice of \( g \).

Case 4: \( j < r < i < s \) In this case, the edge \((c_i^1(w), c_j^1(w))\) will be present in \( H \). This edge will cross \( e \) and hence not be present in \( \text{planar}(H) \). Thus, we have an edge \( g' = (c_i^1(w), c_j^1(w)) \) such that \( \min(i', j') < i < \max(i', j') < s \). We will thus have two subcases.

Subcase 4a: \( j < \min(i', j') \) Here, we will have \( j < \min(i', j') < i < \max(i', j') \). Hence is edge will cross \( e \) giving a contradiction.

Subcase 4b: \( \min(i', j') \leq j \) Here, this edge should have been chosen instead of \( g \) contradicting our choice of \( g \).

Given \( H \) as an input, we first find \( \text{planar}(H) \) using Lemma 5. We then triangulate \( \text{planar}(H) \) by first adding edges in the boundary of each block. Let \( l \) be a block and \( v \) be a vertex in it. Let \( p \) be the smallest positive integer such that \( c_i^1(v) \) is present in \( H \). If the edge \((v, c_i^1(v))\) is not present in \( \text{planar}(H) \), we add this in \( \hat{H} \). We add the edges in such a way that every edge of \( l \) will be inside the boundary cycle. Now we triangulate the rest of the graph and add the triangulation edges to \( \hat{H} \).

We will be using the following lemma which was proved by Imai et al.

**Lemma 6.** \([\theta]\) For every \( \beta > 0 \), there exists a polynomial time algorithm which uses \( \bar{O}(h^{1/2+\beta/2}) \) space which takes as an input an \( h \)-vertex planar graph and outputs a set \( S \) of its vertices. The cardinality of \( S \) is \( O(h^{1/2+\beta/2}) \) and removal of \( S \) disconnects the graph into components of size \( O(h^{1-\beta}) \).

We will now prove a crucial lemma that will help us in our construction of PseudoSeparator.

**Lemma 7.** Let \( G \) be a triangulated planar graph and \( S \) be a subset of its vertices. For every pair of vertex \( u, v \) which belong to different components of \( G \setminus S \), there exists a cycle in \( G[S] \), such that \( u \) and \( v \) belong to different sides of this cycle.

**Proof.** Let \( C \) be a component of \( G \setminus S \) and \( C' \) be the set of vertices of \( S \) which are adjacent to at least one of the vertices of \( C \) in \( G \). Let \( F \) be the set of triangle faces of \( G \) to which at least one vertex of \( C \) belongs. We first observe that for any face \( f \) of \( F \), the vertices of \( f \) will either belong to \( C \) or \( C' \). We see that \( F \) is a region of edge-connected faces. Miller [11] proved that we could write the boundary of the region of edge-connected faces as a set of vertex-disjoint simple cycles with disjoint exteriors. These cycles will contain only the vertices of \( C' \). Hence the lemma follows.

For a subgraph \( H \) of \( G^\alpha \), we construct \( \text{psep}(H) \) in the following way. We first construct \( \hat{H} \) from \( H \). We then find a set \( S \) of vertices in \( \hat{H} \) which divides it into components of size \( O(n^{1-\beta}) \) using Lemma 7. First, we add each vertex of \( S \) to the set \( V(\text{psep}(H)) \) and each edge of \( \hat{H}(S) \) which is also in \( H \) to \( E(\text{psep}(H)) \). Then, let \( e = (v, c_i^1(v)) \) be a triangulation edge present in block \( l \) of \( \hat{H} \) whose both endpoints are in \( S \). We add the following set of at most four vertices to \( V(\text{psep}(H)) \). Let \( p_1 \) be the largest integer smaller than \( p \) such that an edge \( e_1 \) with endpoints \( v \) and \( c_i^1(v) \) exists. We add the vertex \( c_i^3(v) \) to the set \( V(\text{psep}(H)) \). Let \( p_2 \) be the smallest integer greater than 0 such that an edge \( e_2 \) with end points \( c_i^2(v) \) and \( c_i^1(v) \) exists. We add the vertex \( c_i^2(v) \) to the set \( V(\text{psep}(H)) \). Let \( p_3 \) be the largest integer such that an edge \( e_3 \) with end points \( c_i^1(v) \) and \( c_i^3(v) \) exists. We add the vertex \( c_i^1(v) \) to the set \( V(\text{psep}(H)) \). Let \( p_4 \) be the smallest integer larger than \( p \) such that an edge \( e_4 \) with end points \( v \) and \( c_i^1(v) \) exists. We add the vertex \( c_i^4(v) \) to the set \( V(\text{psep}(H)) \). We call the edges \( e_1, e_2, e_3 \) and \( e_4 \) as shadows of \( e \) and add them to \( E(\text{psep}(H)) \).

**Lemma 8.** Let \( G \) be a grid graph and \( H \) be a vertex-induced subgraph of \( G^\alpha \). The graph \( \text{psep}(H) \) is a \( h^{1-\epsilon} \)-PseudoSeparator of \( H \).

**Proof.** Let \( C = \text{psep}(H) \) and \( S \) be the set of vertices in \( \hat{H} \) which divides it into components of size \( O(n^{1-\beta}) \) using Lemma 7. We claim that any two vertices which are in different components of \( \hat{H} \setminus S \) are also in different components of \( H^{(C)} \). We prove this by contradiction. Hence, let us assume that there is an edge from \( e \) in \( H \) and two distinct connected components \( U_1 \) and \( U_2 \) of \( \hat{H} \setminus S \) such that one of the end point of \( e \) is in \( U_1 \) and the other is in \( U_2 \). This edge \( e \) was not present in \( \text{planar}(H) \). Without loss of generality,
let \( e = (v, c^i_p(v)) \), where \( v \in U \) and \( c^j_q(v) \) is not in \( U \). (we pick the edge \( e \) such that \( p \) is minimum) Due to Lemma 7, it follows that there exists a triangulation edge \( f \) such that \( f = ((c^j_q(v)), c^i_p(v)) \) and that \( e \) crosses \( f \). We orient the triangulation edge so that \( q < p < r \). Now, since \( e \) is not present in \( \text{planar}(H) \), there exists at least one edge that crosses it and is present in \( \text{planar}(H) \). We first show that for any such edge \((c^j_q(v), c^i_p(v))\), either \( q < s < p < t < r \) or \( t < q < p < r < s \). We prove this by ruling out other cases.

Case 1: \( s < q < p < r < t \). In this case, since \( g \) crosses \( e \), by Lemma 2, we have that the edge \((c^j_q(v), c^i_p(v))\) is also present in \( G \). This also crosses \( f \) and thus should have been picked instead of \( e \). Hence we have a contradiction.

Case 2: \( s = q < p < r < t \). In this case, \( e \) would cross a shadow edge of \( f \) giving us a contradiction.

Case 3: \( q < s < p < r < t \). In this case, the edge \( g \) would cross \( f \) and hence cannot be present in \( \text{planar}(H) \), giving a contradiction.

Case 4: \( s < q < p < r < t \). In this case, \( e \) would cross a shadow edge of \( f \) giving us a contradiction.

Case 5: \( s = q < p < r < t \). In this case, \( g \) would be present in \( C \) giving us a contradiction.

Case 6: \( q < s < p < r < t \). In this case, \( e \) would cross a shadow edge of \( f \) giving us a contradiction.

Case 7: \( s < q < p < r < t \). In this case, too, the edge \( g \) would cross \( f \) and hence cannot be present in \( \text{planar}(H) \), giving a contradiction.

Case 8: \( s = q < p < r < t \). In this case, \( e \) would cross a shadow edge of \( f \) giving us a contradiction.

Case 9: \( t = q < p < r < s \). In this case, \( e \) would cross a shadow edge of \( f \) giving us a contradiction.

Case 10: \( q < t < p < r < s \). In this case, the edge \( g \) would cross \( f \) and hence cannot be present in \( \text{planar}(H) \), giving a contradiction.

Case 11: \( t < q < p < r < s \). In this case, \( e \) would cross a shadow edge of \( f \) giving us a contradiction.

Case 12: \( t = q < p < r < s \). In this case, \( g \) would be present in \( C \) giving us a contradiction.

Case 13: \( q < t < p < r < s \). In this case, \( e \) would cross a shadow edge of \( f \) giving us a contradiction.

Case 14: \( t < q < p < s < r \). In this case, too, the edge \( g \) would cross \( f \) and hence cannot be present in \( \text{planar}(H) \), giving a contradiction.

Case 15: \( q = t < p < s < r \). In this case, \( e \) would cross a shadow edge of \( f \) giving us a contradiction.

Case 16: \( q < t < p < s < r \). In this case, since \( g \) crosses \( e \), by Lemma 2, we have that the edge \((v, c^j_q(v))\) is also present in \( G \). This also crosses \( f \) and thus should have been picked instead of \( e \). Hence we have a contradiction.

Let \( g = (c^j_q(v), c^i_p(v)) \) be an edge such that \( t - s \) is maximum. We thus have the following two cases:

Case 1: \( q < s < p < t < r \). In this case, the edge \((v, c^j_q(v))\) will also be present due to lemma 2. This edges also crosses \( f \). Thus there will exists an edge in \( \text{planar}(H) \) which crosses this edge. We pick edge \( g' = (c^j_q(v), c^i_p(v)) \) such that \( t - s' \) is maximum. Clearly, \( t' < q \) and \( s' > r \). Now, since the edges \( g \) and \( g' \) both crosses \( e \) and \( g' \) is closer to \( v \) than \( g \), by lemma 3, the edge \((c^j_q(v), c^i_p(v))\) will also be present in the graph. This will again cross \( f \). Now, any edge present in \( \text{planar}(H) \) that crosses this edge will contradict the way in which \( g \) or \( g' \) is chosen.

Case 2: \( t < q < p < r < s \). In this case, the edge \((c^j_q(v), c^i_p(v))\) will also be present in the graph. This will cross \( f \). Any edge present in \( \text{planar}(H) \) that also crosses this edge will contradict the way \( g \) is chosen.

### 3.3 Solving Reachability in Auxiliary Graph

Given a vertex-induced subgraph \( H \) of \( \tilde{G}^n \), we first construct its PseudoSeparator using Lemma 3. Call this PseudoSeparator as \( C \). We ensure that \( s \) and \( t \) are part of the PseudoSeparator. Let \( I_1, I_2, \ldots, I_l \) be the components received after dividing the graph using PseudoSeparator. The subroutine would perform a loop with \( |H| \) iteration and would update a set of marked vertices. Initially, it marks the vertex \( s \). After an iteration, it marks a vertex of \( C \) if there is a path from a marked vertex to \( v \) such that the internal vertices of that path all belong to only one of the component \( I_i \). Also, for each edge \( e \) of \( C \), the vertex \( v \) closest to \( \text{tail}(e) \) which satisfies the following two conditions is marked: (i) There exists an edge \( f \) which cross \( e \) and \( \text{tail}(f) = v \) and (ii) there is a path from a marked vertex to \( v \) such that the internal vertices of the path all belong to only one of the component \( I_i \).
Let $P$ be the shortest path from $s$ to $t$ in $H$. Suppose $P$ passes through the components $I_{\sigma_1}, I_{\sigma_2}, \ldots, I_{\sigma_L}$ in this order. The length of this sequence can be at most $|H|$. As the path leaves the component $I_{\sigma_i}$ and goes into $I_{\sigma_{i+1}}$, it can do in the following two ways:

1. The path exits $I_{\sigma_i}$ through a vertex of PseudoSeparator as shown in Figure 2a. In this case, our algorithm would mark the vertex $w$.

2. The path exits $I_{\sigma_i}$ through an edge $(u, v)$ whose other endpoint is in $I_{\sigma_{i+1}}$. By Observation 3, this edge will cross an edge $e$ of the PseudoSeparator. In this case, the algorithm would mark the vertex $u'$ which is closer than $u$ to $\text{tail}(e)$ and an edge $(u', v')$ crosses $e$. By Observation 2, the edge $(u', v)$ would be present in the graph.

Thus after $j$ iteration, the subroutine would traverse the fragment of the path in the component $I_{\sigma_j}$ and either mark its endpoint or a vertex which is closer to the edge $e$ of $C$ which the path crosses. Finally $t$ would be marked after $L$ iterations if and only if there is a path from $s$ to $t$ in $H$.

Our subroutine would solve reachability in a subgraph $H$ (having size $h$) of $G^\alpha$. We do not explicitly store a component of $\text{cc}(H(C))$, since it might be too large. Instead, we identify a component with the lowest indexed vertex present in it and use Reingold’s algorithm on $H(C)$ to determine if a vertex is present in that component. We require $\tilde{O}(h^{1/2+\beta/2})$ space to calculate PseudoSeparator by Lemma 4. We can potentially mark all the vertex of PseudoSeparator and for each edge of PseudoSeparator we mark at most one additional vertex. Since the size of PseudoSeparator is at most $O(h^{1/2+\beta/2})$, we require $\tilde{O}(h^{1/2+\beta/2})$ space. The algorithm recurses on a graph with $h^{1-\beta}$ vertices. Hence the depth of the recursion is $1/\beta$, which is a constant. The total space complexity would thus be $\tilde{O}(n^{1/2+\beta/2})$.

Since the graph $H$ is given implicitly in our algorithm, there is an additional polynomial overhead involved in obtaining its vertices and edges. However, the total time complexity would still remain a polynomial in the number of vertices since the recursion depth is constant.

**Lemma 9.** Let $G$ be a grid graph. For every $\beta > 0$, there is a polynomial time algorithm that takes as an input $G^\alpha$ and solves reachability in $G^\alpha$ using $\tilde{O}(n^{1/2+\beta/2})$-space where $n$ is the number of vertices in $G^\alpha$.

**Proof.** Algorithm 1 solves reachability in a subset $H$ of $G^\alpha$.

Let $p$ be a path from $s$ to $t$ in $H$. We divide the path into $p_1 \cdot p_2 \cdots \cdot p_l$. Such that all vertices of $p_i$ belong to $U \cup V(C)$ for some set $U$ of $\text{cc}(H(C))$ and either (i) $\text{head}(p_i)$ is a vertex in $V(C)$ or (ii) $\text{head}(p_i)$ and $\text{tail}(p_{i+1})$ belong to two different sets $U_1$ and $U_2$ of $\text{cc}(H(C))$ with an edge between them.

We claim that after $i$’th iteration of loop in line 5 either of the following two holds: (i) $\text{head}(p_i)$ is a vertex in $V(C)$ and it is marked. (ii) There exists an edge $f_i$ of $C$ such that the edge $(\text{head}(p_i), \text{tail}(p_{i+1}))$ crosses $f_i$ and there is an edge $e_i$ which crosses $f_i$ such that $\text{tail}(e_i)$ is closer to $\text{tail}(f_i)$ than $\text{head}(p_i)$ and $\text{mark}(f_i) = \text{tail}(e_i)$.

We prove the claim by induction. We prove the claim to be correct after the $i - 1$ iteration. If $\text{head}(p_{i-1})$ is marked after the $i - 1$ iterations, then both the edge $(\text{head}(p_i), \text{tail}(p_i))$ and the path $p_i$ lies in the subgraph $H[U \cup \{\text{head}(p_{i-1}), \text{tail}(g)\}]$ for a component $U$ of $\text{cc}(H(C))$. Hence the claim would be true. If a vertex $u$ closer to $\text{tail}(f_{i-1})$ is marked, then by lemma 1 both the edge $(u, \text{tail}(p_i))$ and the path $p_i$ will lie in the subgraph $H[U \cup \{u, \text{tail}(g)\}]$ for a component $U$ of $\text{cc}(H(C))$. Hence the claim would be true.
Input: A vertex-induced subgraph $H$ of $\tilde{G}^\alpha$ and two vertices $s$ and $t$ of $H$

Output: true if there is a path from $s$ to $t$ in $H$. false otherwise

1 if $|V(H)| \leq (\tilde{n})^{1/2}$ then
2 Use Depth-First Search to solve the problem.
3 end
4 else
5 Let $C$ be the PseudoSeparator of $H$ found using Lemma 3
6 // We require $\{s, t\} \subseteq V(C)$
7 Mark the vertex $s$;
8 for Round $i = 1$ to $|H|$ do
9 for All edges $f \in C$ do
10 Let $g$ be the edge which crosses $f$ and is closest to $\text{tail}(f)$ such that
11 $\exists u \exists U(\text{REACH}(H[U \cup \{u, \text{tail}(g)\}], u, \text{tail}(g)) = 1)$;
12 // $U \in \text{cc}(H^{(C)})$ and $u$ is a marked vertex
13 $\text{mark}(f) \leftarrow \text{tail}(g)$;
14 end
15 if $\exists u \exists U(\text{REACH}(H[U \cup \{u, v\}], u, v)$ then
16 // $U \in \text{cc}(H^{(C)})$ and $u$ is a marked vertex
17 Mark the vertex $v$;
18 end
19 end
20 if $t$ is marked then Return true ;
21 else Return false ;
22 end

Algorithm 1: AuxReach($H, s, t, \tilde{n}$)
We now measure the time and space complexity of our algorithm. Since the size of a component $U$ in $cc(H^{(C)})$ might be too large, we will not explicitly store it. We will identify a component by the lowest-index vertex present in it and use Reingold’s algorithm on $H^{(C)}$ to determine if a vertex is present in $U$. Let $S$ and $T$ denote the space and time complexity functions respectively. Consider $S(h)$ for any $h > n^{1/2}$. We require $O(h^{1/2+\beta/2})$ space to execute line 9. We can potentially mark all the vertex of $G$ we store at most one additional vertex in $mark(e)$. Since the size of $C$ is at most $O(h^{1/2+\beta/2})$, we require $O(h^{1/2+\beta/2})$ space. A call to recursion in line 10 and 14 requires $S(h^{1-\beta})$ space which can be subsequently reused. Hence the space complexity satisfies the following recurrence.

$$S(h) = \begin{cases} S(h^{1-\beta}) + O(h^{1/2+\beta/2}) & h > n^{1/2} \\ O(h^{1/2}) & \text{otherwise} \end{cases}$$

We will measure time complexity up to a polynomial factor. By a similar analysis, we get

$$T(h) = \begin{cases} p(h)T(h^{1-\beta}) + q(h) & h > n^{1/2} \\ r(h) & \text{otherwise} \end{cases}$$

where $p$, $q$ and $r$ are appropriate polynomials.

Since the graph $H$ is given implicitly in our algorithm, there is a polynomial overhead involved in obtaining its vertices and edges. This is captured in the polynomials $p(h)$, $q(h)$ and $r(h)$.

Solving the above recurrences, we get $S(n) = O(n^{1/2+\beta/2})$ and $T(n) = \text{poly}(n)$.

4 Solving Grid Graph

Let $G$ be a $N \times N$ grid graph. As mentioned in the introduction, our objective is to run Algorithm 1 on the graph $\tilde{G}^\alpha$. Since we preserve reachability in this graph, this would be enough to solve reachability in the grid graph. However, we do not have explicit access to the edges of $\tilde{G}^\alpha$. We see that, by definition, the edges of $\tilde{G}^\alpha$ can be obtained by solving the corresponding subgrid of $G$. If the subgrid is small enough, we will use a standard linear space traversal algorithm. Otherwise, if the size of the subgrid is large, we use our algorithm recursively on the subgrid. Algorithm 2 outlines this method.

```
Input: A grid graph $\tilde{G}$ and two vertices $\hat{s}$, $\hat{t}$ of $\tilde{G}$ and an integer $N$  
Output: true if there is a path from $s$ to $t$ in $G$. false otherwise
1 if $\tilde{G}$ is smaller than $N^{1/4} \times N^{1/4}$ grid then 
2 Use Depth-First Search to solve the problem.  
3 end  
4 else  
5 Use ImplicitAuxReach($\tilde{G}^\alpha$, $\hat{s}$, $\hat{t}$, $\tilde{n}$) to solve the problem; 
6 end  
```

Algorithm 2: GridReach($\tilde{G}$, $\hat{s}$, $\hat{t}$, $N$)

Let $S(N)$ be the space complexity and $T(N)$ be the time complexity of solving a gridgraph of size $N \times N$. For any $\tilde{N}$ such that $\tilde{N} > N^{1/4}$, the space required to solve the grid-graph would be $S(\tilde{N}) = S(\tilde{N}^{1-\alpha}) + O((N^{1+\alpha})^{3/2+\beta/2})$. This is so because a result to the query $(u, v) \in G$ would invoke a recursion which would require $S(\tilde{N}^{1-\alpha})$ space and the main computation of ImplicitAuxReach could be done using $O((N^{1+\alpha})^{3/2+\beta/2})$ space. Hence we get the following recurrence for space complexity.
\[ S(\hat{N}) = \begin{cases} 
S(\hat{N}^{1-\alpha}) + \tilde{O}((\hat{N}^{1+\alpha})^{1/2+\beta/2}) & \hat{N} > N^{1/4} \\
\tilde{O}(\hat{N}^{1/2}) & \text{otherwise} 
\end{cases} \]

Similarly, for appropriate polynomials \( p, q \) and \( r \), the time complexity would satisfy the following recurrence:

\[ T(\hat{N}) = \begin{cases} 
\tilde{O}((\hat{N}^{1+\alpha})^{1/2+\beta/2}) & \hat{N} > N^{1/4} \\
p(\hat{N})T(\hat{N}^{1-\alpha}) + q(\hat{N}) & \text{otherwise} 
\end{cases} \]

Solving the recurrence, we will get \( S(N) = \tilde{O}(N^{1/2+\beta/2+\alpha/2+\alpha\beta/2}) \) and \( T(N) = \text{poly}(N) \). Since \( \alpha \) and \( \beta \) are arbitrary positive constants, we can also write the space complexity as \( S(N) = O(N^{1/2+\epsilon}) \) where \( \epsilon \) is another arbitrary constant.

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