Axial Symmetry and Bound States of Particles with Anomalous Magnetic Moment

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Considered bound states of fermions with an anomalous magnetic moments in the field of charged cylinder. Also obtained second order equations for axially symmetric $Z_0(r)$-boson field, radial axially symmetric magnetic field, obtained second order equations for case of bound states of fermions in the radial electric and magnetic fields.
1. Introduction

As known, the dynamics of the neutral fermions with anomalous magnetic moments are described by Dirac equation with non-minimal couplings of neutral fermions with electromagnetic field:

\[ (\hat{k} - m + \mu(\vec{\Sigma}\vec{B} - i\vec{\alpha}\vec{E}) + iq(\vec{E}\vec{B})\gamma_5)\psi(k) = 0, \]

where \( \mu \) is anomalous magnetic moment, \( \frac{1}{2}\vec{\Sigma} \) is spin operator, and defined by formula (21,21) of [1]. The last term described by lagrangian:

\[ L = iq(\vec{E}\vec{H})\bar{\psi}\gamma_5\psi \]

in Dirac equations is obtained in [3] and depends only on \( r \) in case of monopole (for monopoles see e.g. [4] and references therein) which have both electric and magnetic fields.

In this article we consider cylindrically symmetric bound states and resonances of particles with anomalous magnetic moments.

It is of interest to consider several special cases in particular:

1) Particle energy levels in pure electric axial field case \( \vec{E} = (E_r(r), 0, 0) \) which created e.g. by homogeneously charged cylinder.

In accordance with [6] we have:

\[ E_r = \frac{2\sigma}{r} \quad \text{at} \quad r > R \]  

(3)

and

\[ E_r = \frac{2\sigma r}{R^2} \quad \text{at} \quad r < R \]  

(4)

where \( \sigma \) is density of charge of the unit of length of the cylinder.
Also we consider second order equations for axially symmetric $Z_0(r)$-boson field, radial axially symmetric magnetic field obtain second order equations for case of bound states of fermions in the radial electric and magnetic fields \[5\], and also obtain second order equations for case of pseudoscalars described by last term in Dirac equation (1).

Radial axially symmetric electric field

In ref.\[8\] has been obtained the system of equations for radial functions (see below formulas (20)-(23) in which we add also radial magnetic field besides radial electric field). Below will be presented the second order equations which obtained after excluding two of four radial functions:

\[
\left(\frac{1}{r} \frac{d}{dr} + \frac{d}{dr}\right) + \frac{\epsilon^2 - m^2 - p_z^2}{r} - 4\pi \rho - \left(\frac{l}{r} - \mu E\right)^2 f_1(r) + 2i\mu Ep_z f_2(r) = 0 \quad (5)
\]

\[
\left(\frac{1}{r} \frac{d}{dr} + \frac{d}{dr}\right) + \frac{\epsilon^2 - m^2 - p_z^2}{r} - 4\pi \rho - \left(\frac{(l+1)}{r} + \mu E\right)^2 f_2(r) - 2i\mu Ep_z f_1(r) = 0 \quad (6)
\]

The presence of the $i$ mean that one of the radial functions must be purely imagine.

Analogous system of equations obtained if we esclude $\phi$ and consider $\chi$ as $\chi = (e^{i\phi} f_3(r), e^{i(l+1)\phi} f_4(r))$:

\[
\left(\frac{1}{r} \frac{d}{dr} + \frac{d}{dr}\right) + \frac{\epsilon^2 - m^2 - p_z^2}{r} + 4\pi \rho - \left(\frac{l}{r} - \mu E\right)^2 f_3(r) + 2i\mu Ep_z f_4(r) = 0 \quad (7)
\]

\[
\left(\frac{1}{r} \frac{d}{dr} + \frac{d}{dr}\right) + \frac{\epsilon^2 - m^2 - p_z^2}{r} + 4\pi \rho - \left(\frac{(l+1)}{r} + \mu E\right)^2 f_4(r) - 2i\mu Ep_z f_3(r) = 0 \quad (8)
\]

In case of electric field created by charged line or charged cylinder at $r > a$ we obtain:

\[
\left(\frac{1}{r} \frac{d}{dr} + \frac{d}{dr}\right) + \frac{\epsilon^2 - m^2 - p_z^2}{r^2} - \frac{(l - 2\mu \sigma)^2}{r^2} f_1(r) + \frac{4i\mu \sigma p_z}{r} f_2(r) = 0 \quad (9)
\]
\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} + \epsilon^2 - m^2 - p_z^2 - \frac{(l + 1 + 2\mu\sigma)^2}{r^2} \right) f_2(r) - \frac{4i\mu\sigma p_z}{r} f_1(r) = 0 \quad (10)
\]

Inside homogeneously charged cylinder we obtain:

\[
(\Omega - \frac{(l)^2}{r^2} + \frac{4\mu^2\sigma^2 r^2}{a^4} + \frac{4\mu\sigma l}{a^2}) f_1(r) + \frac{4i\mu\sigma p_z}{r} f_2(r) = 0 \quad (11)
\]

\[
(\Omega - \frac{(l + 1)^2}{r^2} + \frac{4\mu^2\sigma^2 r^2}{a^4} + \frac{4\mu\sigma (l + 1)}{a^2}) f_2(r) - \frac{4i\mu\sigma p_z}{r} f_1(r) = 0 \quad (12)
\]

where \( \Omega = \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} + \epsilon^2 - m^2 - p_z^2 \right) \) we see that at \( p_z = 0 \) both equations decouples and every of them is the same as for 2-dimensional harmonic oscillator and in accordance with [10] we have the following energy levels:

\[
\epsilon^2 = m^2 + \frac{4\mu\sigma}{a^2} + \frac{8|\mu\sigma|}{a^2} (n_r + \frac{l + |l| + 1}{2}) \quad (13)
\]

Analogously for second equation in result for energy levels we must replace:

\[
l \rightarrow (l + 1)
\]

The particle is localized at distances \( r_H \sim (\frac{4\mu\sigma}{a^2})^{-\frac{1}{2}} \) or smaller. Of course our consideration is available only if size of the particle localization is essentially smaller than the radius of the cylinder \( r_H << a \). The wave functions at large distances are suppressed by exponent \( e^{-\frac{r^2}{r_H}} \).

Above was considered energy levels of neutrons in case infinite radius of the cylinder. It is of interest to consider also case of the finite radius of the cylinder.

For this purpose we must find wave function inside and outside of the cylinder and in this case energy levels of neutrons will be defined from condition:

\[
\frac{R'_{r,a}(a)}{R_{r,a}(a)} = \frac{R'_{r,a}(a)}{R_{r,a}(a)}. \quad (14)
\]

4
Inside cylinder as we seen above we obtain equation which is equivalent to the 2-dimensional harmonic oscillator and radial wave functions are expresses through degenerate hypergeometric function:

$$R_{r<a}(r) = C_1 x^n e^{-\frac{x}{2}} F\left(\frac{l+|l|+1}{2} - \frac{\epsilon^2 - m^2 - 4\pi\mu\rho}{2m\omega}, |l| + 1, x\right)$$  (15)

where $\omega = \frac{4|\mu\sigma|}{m\omega}$, $x = m\omega r^2$. Outside of the cylinder radial wave functions are expresses through Macdonald’s function:

$$R_{r<a}(r) = C_2 K_{|l+2\mu\sigma|} (\sqrt{|\epsilon^2 - m^2| r})$$  (16)

Below we consider case of the charged line.

In the limit:

$$\mu\sigma << 1$$  (17)

we obtain Coulomb-like spectrum for energy levels of neutrons in the field of charged line.

Indeed, if $l$ is not equal to the $0,-1$ we can put in equations (5), (6) $l - 2\mu\sigma \approx l$ in the limit $\mu\sigma << 1$. At $l = 0,-1$ must be

$$\frac{\mu^2\sigma^2}{r_B} << \frac{\mu\sigma p_z}{r_B}$$  (18)

where $r_B = \frac{1}{\mu\sigma p_z}$ is Bohr radius (see below). Substituting $r_B$ in (18) we obtain again the condition $\mu\sigma << 1$. Thus, in the limit $\mu\sigma << 1$ at $l = 0,-1$ term $= (0 + \mu\sigma)^2 r^{-2}$ must be neglected, and we obtain for all $l$ the following equations:

$$\left(-\frac{1}{r} \frac{d}{dr} \frac{d}{dr} + e^2 - m^2 - p_z^2 + \frac{l^2}{r^2}\right) f_1(r) + \frac{4\mu\sigma p_z}{r} i f_2(r) = 0$$  (19)

$$\left(-\frac{1}{r} \frac{d}{dr} \frac{d}{dr} + e^2 - m^2 - p_z^2 + \frac{(l+1)^2}{r^2}\right) i f_2(r) + \frac{4\mu\sigma p_z}{r} f_1(r) = 0$$  (20)
This equations are similar to equations obtained in [7] where has been consider non-relativistic neutrons energy levels in the magnetic field $\vec{H} = (0, H_\phi(r) = 2I/r, 0)$ if we making the following replacement:

$$I \rightarrow \frac{\sigma p_z}{m} \quad (21)$$

Thus, energy levels is also same and in accordance with result of [7] and substitution (21) are defines by quantum number $n$:

$$\epsilon_n^2 = m^2 + p_z^2 - \frac{\mu^2 \sigma^2 p_z^2}{n^2} \quad (22)$$

It must be noted, our consideration is relativistic, only assumption $\mu \sigma << 1$ has been used. In the near future we will present the solution where $\mu \sigma$ is not small.

Radial Axially Symmetric Magnetic + Radial Axially Symmetric electric field and bound states of neutral fermions with an anomalous magnetic moments

In this paper we consider also bound states of neutral fermions with an anomalous magnetic moments in radial axially symmetric magnetic field $\vec{H} = \left( \frac{\sqrt{x^2 + y^2}}{x^2 + y^2} H(r), \frac{\sqrt{x^2 + y^2}}{x^2 + y^2} H(r), 0 \right)$ which analogously to the above considered electric field in case of cylinder has the following form:

$$H_r(r) = \frac{2\sigma m}{r} \quad at \quad r > R \quad (23)$$

and

$$H_r(r) = \frac{2\sigma_m r}{R^2} \quad at \quad r < R \quad (24)$$

where $\sigma_m$ is density of magnetic charge of the unit of length of the cylinder.

$$(\epsilon - m)f_1 + \mu H f_2 - p_3 f_3 - i\left(\frac{d}{dr} + \frac{l}{r} - \mu E\right)f_1 = 0 \quad (25)$$
\[ \mu H f_1 + (\epsilon - m) f_2 - i \left( \frac{d}{dr} \frac{l - 1}{r} - \mu E \right) f_3 + p_3 f_4 = 0 \] (26)

\[ p_3 f_1 + \left( \frac{d}{dr} + \frac{1}{r} + \mu E \right) f_2 - (\epsilon + m) f_3 + \mu H f_4 = 0 \] (27)

\[ - i \left( \frac{d}{dr} - \frac{l - 1}{r} + \mu E \right) f_1 - p_3 f_2 + \mu H f_3 - (\epsilon + m) f_4 = 0 \] (28)

It is interesting to notice that equations for radial magnetic field is similar to equations for magnetic field \( \vec{H} = (0, H_\phi(r), 0) \) considered in [8].

In non-relativistic approximation we have the following system of equations (instead Dirac equation has been used Pauli equation with non-relativistic spinor \( \phi = (f_1(r)e^{i(l-1)\phi}, f_2(r)e^{il\phi}) \)):

\[ \frac{1}{2m} \Omega_1(l - 1) f_1(r) + \mu H(r) f_2(r) = 0 \] (29)

\[ \frac{1}{2m} \Omega_1(l) f_2(r) + \mu H(r) f_1(r) = 0 \] (30)

This equations are similar to equations obtained in [8] where has been consider non-relativistic neutrons energy levels in the magnetic field \( \vec{H} = (0, H_\phi(r) = 2I/r, 0) \). It is seen from the following replacement in above derived equations (),():

\[ f_2 \rightarrow -if_2, \sigma_m \rightarrow I \] (31)

Thus, energy levels is also same and in accordance with result of [8] are defines by quantum number \( n \):

\[ E_n = -\frac{(\mu \sigma_m)^2 m}{2n^2} \] (32)

**Fermions with anomalus magnetic moments in magnetic field**

\( \vec{H} = (0, 0, H_z(r) = H(r)) \)

In component form \( (\psi^T = (f_1(r)e^{i(l-1)\phi}, f_2(r)e^{il\phi}, f_3(r)e^{i(l-1)\phi}, f_4(r)e^{il\phi})) \) the equations has been obtained in [8]:

\[ (\epsilon - m + \mu H)f_1 - p_3 f_3 - p_4 f_4 = 0 \] (33)
\[(\epsilon - m - \mu H)f_2 - p_+ f_3 + p_3 f_4 = 0\]  
(34)

\[p_3 f_1 + p_- f_2 + (-\epsilon - m + \mu H) f_3 = 0\]  
(35)

\[p_+ f_1 - p_3 f_2 + (-\epsilon - m - \mu H) f_4 = 0\]  
(36)

where

\[p_- = -i \left( \frac{d}{dr} + \frac{l}{r} \right)\]  
(37)

\[p_+ = -i \left( \frac{d}{dr} - \frac{l - 1}{r} \right)\]  
(38)

Excluding two of four components we obtain the system of two second order equations:

\[i(2\mu H(r)\left( \frac{d}{dr} - \frac{l - 1}{r} \right) + 4\pi \mu j\phi(r))f_1(r) + ((m + \mu H(r))^2 + \Omega_1(l))f_4(r) = 0\]  
(39)

\[i(2\mu H(r)\left( \frac{d}{dr} + \frac{l}{r} \right) + 4\pi \mu j\phi(r))f_4(r) + ((m - \mu H(r))^2 + \Omega_1(l - 1))f_1(r) = 0\]  
(40)

where \(\Omega_1(l) = -\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} + \frac{l^2}{r} + p_z^2 + m^2 - \epsilon^2\), \(\vec{j} = (0, j\phi(r), 0)\) is density of current.

**Majorana neutrinos**

In this paper we consider second order equations for Majorana neutrinos \((g_V = 0)\). For this purpose it is necessary writing Dirac equation:

\[\left(\hat{k} - g_A \hat{Z}\gamma_5 - m\right)\psi(k) = 0,\]  
(41)

in component form and excluding one of the component.
The final result for equations of the second order is following:

\[ T(\kappa)f_1 + g_A(Z'_0(r)f_2 + 2Z_0(r)f'_2(r) + \frac{1-\kappa}{r}f_2(r)) = 0 \]  \hspace{1cm} (42)

\[ T(\kappa-1)f_2 - 2g_AZ_0(r)(f'_1(r) + \frac{1+\kappa}{r}f_1(r)) = 0 \]  \hspace{1cm} (43)

where

\[ T(\kappa) = \epsilon^2 + \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} - \frac{\kappa(\kappa+1)}{r^2} - g_A^2Z'_0(r) - m^2, \]  \hspace{1cm} (44)

\[ \kappa = l(l+1) - j(j+1) - \frac{1}{4}, \]  \hspace{1cm} (45)

We see that term \( g_A^2Z'_0(r) \) is always repulsive. In cylindrical symmetry case choosing two-component spinor \( \phi \) as \( \phi = (f_1(r)e^{il\phi}, f_2(r)e^{i(l+1)}) \) we obtain the following system of the two second order equations:

\[ (P(l) - 2g_AZ_0(r)p_z)f_1 + g_A(2Z_0(r)(\frac{d}{dr} + \frac{1+l}{r}) + Z'_0(r))if_2(r) = 0 \]  \hspace{1cm} (46)

\[ (P(l+1) + 2g_AZ_0(r)p_z)if_2 - g_A(2Z_0(r)(\frac{d}{dr} - \frac{1}{r}) + Z'_0(r))f_1(r) = 0 \]  \hspace{1cm} (47)

where \( P(l) = \epsilon^2 + \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} - \frac{\kappa^2}{r^2} - p_z^2 - g_A^2Z'_0 - m^2. \)

As known, anomalous magnetic moments of the Majorana neutrino is equal to the zero. Thus, in case of Majorana neutrino the attraction is possible only via interaction (2) in radial electric and magnetic fields of monopole (see [5] for details). The second order equations in this case (i.e. if only interaction (2) presented) are following:

\[ (\epsilon^2 + \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} - \frac{\kappa(\kappa+1)}{r^2} - A^2 - m^2)f_1 + Af_2(r) = 0 \]  \hspace{1cm} (48)

\[ (\epsilon^2 + \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} - \frac{\kappa(\kappa-1)}{r^2} - A^2 - m^2)if_2 + Af_1(r) = 0, \]  \hspace{1cm} (49)
where \( A = iqE(r)H(r) \). We see that term \( A^2(r) \sim r^{-8} \) is always repulsive, dominate at small distances and prevent fall down on the center.

In cylindrical symmetry case we have:

\[
(\epsilon^2 + \frac{1}{r} \frac{d}{dr} r \frac{d}{dr} \frac{l^2}{r^2} - p_z^2 - A^2 - m^2)f_1 + Aif_2(r) = 0,
\]

(50)

\[
(\epsilon^2 + \frac{1}{r} \frac{d}{dr} r \frac{d}{dr} \frac{(l + 1)^2}{r^2} - p_z^2 - A^2 - m^2)f_1 + Aif_2(r) = 0,
\]

(51)

**Bound states of fermions with anomalous magnetic moments in radial electric and magnetic field: second order equations**

Although it is possible to exclude to of four radial functions from system of equations of the first order (see (12)-(16) in [5]) which defines energy levels of the neutral fermion with anomalous magnetic moment, much more convenient in order to obtain the second order equations to start from Dirac equation in component form:

\[
(\epsilon - m + \mu \sigma \vec{H})\phi - \sigma(\vec{p} + i\mu \vec{E})\chi = 0
\]

(52)

\[
(-\epsilon - m + \mu \sigma \vec{H})\chi + \sigma(\vec{p} - i\mu \vec{E})\phi = 0
\]

(53)

Excluding e.g. component \( \chi \) and presenting \( \phi \) as linear
combinations of spherical spinors with different $P$-parity (analogously \[9\], because in studied case $P$-parity violation take place) \[1]\:

$$\phi^T = f_1(r)\Omega_{jlM}(\vec{n}) + (-1)^{\frac{1+l+l'}{2}} f_2(r)\Omega_{jl'M}(\vec{n})$$

(54)

we obtain the following system of the second order equations:

$$2m\mu H f_2 - T_+(\kappa) f_1 + Q(\kappa)(a_+ S_+(-\kappa)i f_2 - b_+ S_+(-\kappa) f_1)$$

(55)

$$2m\mu H f_1 - T_+(-\kappa)i f_2 + Q(-\kappa)(a_+ S_+(-\kappa) f_1 - b_+ S_+(-\kappa)i f_2)$$

(56)

Analogously, excluding $\phi$ and presenting $\chi$ as:

$$\chi = g_1(r)\Omega_{jlM}(\vec{n}) + (-1)^{\frac{1+l+l'}{2}} g_2(r)\Omega_{jl'M}(\vec{n})$$

(57)

we obtain the system for $g_{1,2}$:

$$2m\mu H g_2 - T_-(-\kappa)g_1 - Q(\kappa)(a_- S_-(-\kappa)ig_2 - b_- S_-(-\kappa) g_1)$$

(58)

$$2m\mu H g_1 - T_-(-\kappa)ig_2 - Q(-\kappa)(a_- S_-(-\kappa) g_1 - b_- S_-(-\kappa)ig_2)$$

(59)

where:

$$Q(\kappa) = 4\pi\mu\rho_m - \frac{2\mu H(r)}{r}(1 \pm \kappa),$$

(60)

$$S_{\pm}(\kappa) = \frac{d}{dr} + \frac{1 \pm \kappa}{r} \pm \mu E,$$

(61)

\[1\]it must be stressed that besides $P$-parity violation also presented $T$-parity violation (term $\vec{\Sigma}rH$ in Dirac equation is $P$- and $T$-parity violating) and purely imagine character of the two of four radial function in equations (12)-(16) of the \[5\]is connected with this circumstances. Also, in (55),(56) e.g. $f_2$ must be purely imagine, $f_1$ must be real.
\[ T_{\pm}(\kappa) = \epsilon^2 - m^2 + \frac{1}{r^2} \frac{d}{dr} \frac{d}{dr} \frac{\kappa(\kappa + 1)}{r^2} - \mu^2 (E^2 + H^2) \pm 4\mu \pi \rho \mp \frac{2\mu E}{r} (1 \pm \kappa) \] (62)

\[ a_{\pm} = \frac{\epsilon \pm m}{(\epsilon \pm m)^2 - \mu^2 H^2} \] (63)

\[ b_{\pm} = \frac{\pm \mu H}{(\epsilon \pm m)^2 - \mu^2 H^2} \] (64)

During derivation of this formulas we take into account that \( \text{div}\vec{E} = 4\pi \rho, \text{div}\vec{H} = 4\pi \rho_m \).

In nonrelativistic limit terms which are proportional to the \( a_{\pm}, b_{\pm} \) are small and may be neglected.

We see, that terms \((\mu E(r))^2 \sim r^{-4}\) \( (\mu H(r))^2 \sim r^{-4}\) are repulsive and prevent fall down on the center.

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