Bayesian Hierarchical Models with Conjugate Full-Conditional Distributions for Dependent Data from the Natural Exponential Family

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Abstract

We introduce a Bayesian approach for analyzing (possibly) high-dimensional dependent data that are distributed according to a member from the natural exponential family of distributions. This problem requires extensive methodological advancements, as jointly modeling high-dimensional dependent data leads to the so-called “big n problem.” The computational complexity of the “big n problem” is further exacerbated when allowing for non-Gaussian data models, as is the case here. Thus, we develop new computationally efficient distribution theory for this setting. In particular, we introduce something we call the “conjugate multivariate distribution,” which is motivated by the univariate distribution introduced in Diaconis and Ylvisaker (1979). Furthermore, we provide substantial theoretical and methodological development including: results regarding conditional distributions, an asymptotic relationship with the multivariate normal distribution, conjugate prior distributions, and full-conditional distributions for a Gibbs sampler. The results in this manuscript are extremely general, and can be adapted to many different settings. We demonstrate the proposed methodology through simulated examples and three applications based on an epidemiology dataset, a federal statistics dataset, and an environmental dataset, respectively.

Keywords: Bayesian hierarchical model; Big data; Exponential family; Markov chain Monte Carlo; Non-Gaussian; Gibbs sampler.

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1 Introduction

The multivariate normal distribution has become a fundamental tool for statisticians, as it provides a way to incorporate dependence for Gaussian and non-Gaussian data alike. Notice that many statistical models are defined hierarchically, where the joint distribution of the data, latent processes, and unknown parameters are written as the product of a data model, a latent Gaussian process model, and a parameter model (e.g., see Cressie and Wikle, 2011; Banerjee et al., 2015, among others). Jointly modeling a member from the exponential family may be seen as straightforward to some. That is, one can simply define the data model to be the appropriate member of the exponential family and define latent Gaussian processes using the hierarchical modeling framework. Models of this form are often referred to as latent Gaussian process (LGP) models; see Diggle et al. (1998), Rue et al. (2009), Cressie and Wikle (2011, Sections 4.1.2 and 7.1.5), and Holan and Wikle (2016), among others.

In the Bayesian context, LGPs can be nontrivial to implement using standard Markov chain Monte Carlo (MCMC) procedures when the dataset is high-dimensional. This is primarily because big data can lead to big parameter spaces, which allows parameters to be highly correlated. This in turn, creates a challenge for defining useful proposal distributions, tuning these proposal distributions, and assessing convergence of the Markov chain (e.g., see Rue et al. (2009) and Bradley et al. (2017a) for a discussion on convergence issues of MCMC algorithms for LGPs). In this article, our primary goal is to introduce new distribution theory that facilitates Bayesian inference of dependent non-Gaussian data. In particular, we introduce a multivariate distribution that leads to conjugate forms of the full conditional distributions within a Gibbs sampler.

We provide a multivariate extension of the class of distributions introduced by Diaconis and Ylvisaker (1979). The seminal paper by Diaconis and Ylvisaker (1979) developed the conjugate prior for distributions from the natural exponential family (EF), which leads to the well-known Poisson/gamma, binomial/beta, negative binomial/beta, and gamma/inverse-gamma hierarchical models. In this article, we develop a multivariate version of this distribution, which we call the conjugate multivariate (CM) distribution. Similar to the special cases that emerged from Diaconis and Ylvisaker (1979), we obtain Poisson/multivariate log-gamma (MLG), binomial/multivariate logit beta, negative binomial/multivariate logit beta, and gamma/multivariate negative-inverse-gamma hierarchical models. The hierarchical model that specifies the data model to be from the natural exponential family, and the latent process to be a CM distribution is referred to as a latent CM process (LCM) model. The LCM model constitutes a more general paradigm for modeling dependent data than LGPs, since the LGP is a special case of the LCM model. An important motivating feature of this more general framework is that the LCM model incorporates dependency and results in full-conditional distributions (within a Gibbs sampler) that are easy to simulate from. This allows one to avoid computationally inefficient and subjective tuning methods.

An immediate issue that arises with the introduction of the LCM model is the need to define flexible prior distributions. One goal of this article is to describe the fully conjugate Bayesian hierarchical model that has a data model that belongs to the natural exponential family. By “fully conjugate” we mean that each full conditional distribution, within a Gibbs sampler, falls in the same class of distributions of the associated process or parameter models. To derive a fully con-
jugate statistical model, we introduce the LCM analogue to the prior distributions used in Daniels and Pourahmadi (2002), Chen and Dunson (2003), and Pourahmadi et al. (2007) for covariance parameters. Additionally, extensions of the standard inverse-gamma priors for variances of a normal random variable (Gelman, 2006) are discussed in context of the LCM.

There is an added benefit of the CM distribution besides providing conjugacy in the non-Gaussian dependent data setting. Namely, LGPs are not necessarily realistic for every dataset. For example, De Oliveira (2013) shows that there are parametric limitations to the LGP paradigm for count-valued data (e.g., when spatial overdispersion is small). We support this claim by showing that if certain hyperparameters (defined in Section 2) of the CM distribution are “large” then the corresponding CM distribution gives a very good approximation to a Gaussian distribution. This indicates that if the data suggests small values of these hyperparameters, then the CM distribution should be used in place of the multivariate normal distribution.

Reduced rank methods are extremely prevalent in the more general “dependent data” setting. For example, reduced rank assumptions are crucial for principle component analysis, which has become an established technique in multivariate data analysis (e.g., see Jolliffe, 2002, Cox, 2005; Everitt and Hothorn, 2011, among others). Additionally, reduced rank models have been used to great effect within spatial and spatio-temporal settings to obtain precise predictions in a computationally efficient manner (see, e.g., Wikle and Cressie, 1999; Cressie and Johannesson, 2006; Shi and Cressie, 2007; Banerjee et al., 2008; Cressie and Johannesson, 2008; Finley et al., 2009; Katzfuss and Cressie, 2011; Cressie et al., 2010a; Kang and Cressie, 2011; Katzfuss and Cressie, 2012; Bradley et al., 2015a). Thus, an additional motivating feature of the LCM model is that it can easily be cast within the reduced rank modeling framework to obtain further computational gains. Now, the ability to specify a reduced rank LCM does not imply that the LCM can handle all types of “big data” problems. For example, we consider the “big p” problem as a type of “big n” problem (Hastie et al., 2009; Matloff, 2016). Here, our focus is on difficulties with incorporating dependence when n is larger. Specifically, inverses of $n \times n$ matrices often manifest in dependent data settings (e.g., see Sun and Li, 2012, among others). Our incorporation of reduced rank modeling allows one to avoid order $n^3$ computations needed for matrix inversion, and allows one to avoid storage of large $n \times n$ matrices.

This computationally efficient fully conjugate distribution theory could have an important impact on a number of different communities within and outside statistics. High-dimensional non-Gaussian data are pervasive in official statistics (e.g., see Bradley et al., 2017a), ecology (e.g., see Hooten et al., 2003; Wu et al., 2013, among others), climatology (e.g., see Wikle and Anderson, 2003), atmospheric sciences (e.g., see Sengupta et al., 2012), statistical genetics (e.g., see Lange et al., 2014, and the references therein), neuroscience (e.g., see Zhang et al., 2015; Castruccio et al., 2016), and many other domains. The size of modern datasets are becoming more and more high-dimensional, and the aforementioned computational difficulties with LGPs suggests that there is a growing need to develop methods that are straightforward to implement (e.g., see Bradley et al., 2016, for a discussion). Hence, the methodology presented here offers an exciting avenue that makes new applied research for modeling dependent non-Gaussian data practical for modern big datasets.

The LCM model is a type of the hierarchical generalized linear model (HGLM) from Lee and
Nelder (1996). However, the current HGLM literature specifies an LGP for the dependent data setting (Lee and Nelder, 2000, 2001). Additionally, there are other alternatives to a Gibbs sampler with Metropolis-Hastings updates; in particular, integrated nested Laplace approximations (INLA) (Rue et al., 2009) and Hamiltonian MCMC have proven to be useful tools in the literature. These approaches can be adapted to our new proposed distribution theory; however, the need to adapt INLA and Hamiltonian MCMC (Neal, 2011) to the LCM is not necessary since the full conditional distributions are straightforward to simulate from in this setting.

For Poisson counts there are a number of choices besides the LGP strategy available to incorporate dependence (e.g., see Lee and Nelder, 1974; Kotz et al., 2000; Demirhan and Hamurkaroglu, 2011, among others). For example, Wolpert and Ickstadt (1998), introduced a spatial convolution of gamma random variables, and provide a data augmentation scheme for Gibbs sampling that produces spatial predictions. The recently proposed multivariate log-gamma distribution of Bradley et al. (2017a) results in a special case of our modeling approach when the data model is Poisson, and the latent processes are distributed according to a type of CM distribution. Additionally, in more specific settings (e.g., Pareto data spatio-temporal data), conjugate distribution theory has been developed (Nieto-Barajas and Huerta, 2017).

The remainder of this article is organized as follows. In Section 2, we introduce the conjugate multivariate distribution and provide the necessary technical development for fully Bayesian inference of dependent data from the natural exponential family. Specifically, we define the CM distribution, give the specification of the LCM model, introduce conjugate prior distributions, and derive the full conditional distributions for the Gibbs sampler. We end Section 2 with a brief summary of the LCM model used in subsequent sections. Then, in Section 3 we provide a simulation study to show the performance of the LCM model compared to LGPs. Several illustrations from different subject matter areas are also presented in Section 3, which is done in an effort to demonstrate the wide-applicability of the LCM. Specifically, we provide an example analyzing an epidemiology dataset, a federal statistics dataset, and an environmental dataset. Finally, Section 5 contains discussion. For convenience of exposition, proofs of the technical results, and instructions on implementation are given in the Supplemental Appendix.

2 Distribution Theory for Dependent Data from the Natural Exponential Family

In this section, we propose methodology for Bayesian analysis of non-Gaussian dependent data from the natural exponential family. In Section 2.1, we review and develop the univariate distribution introduced in Diaconis and Ylvisaker (1979). Then, in Section 2.2, this univariate distribution is used as the rudimentary quantity to develop the CM distribution. This new multivariate distribution theory is incorporated within a Bayesian hierarchical model (i.e., the aforementioned LCM model) in Section 2.3. Prior distributions for unknown parameters of the LCM model are discussed in Sections 2.4 and 2.5. The information in Sections 2.1 through 2.5 are consolidated into a brief model summary in Section 2.6.
2.1 The Diaconis and Ylvisaker Conjugate Distribution

Suppose \( Z \) is distributed according to the natural exponential family (Diaconis and Ylvisaker, 1979; Lehmann and Casella, 1998), then

\[
f(Z|Y) = \exp \{ZY - b\psi(Y) + c(Z)\}; \quad Z \in \mathcal{Z}, Y \in \mathcal{Y},
\]

(1)

where \( f \) will be used to denote a generic probability density function/probability mass function \( (\text{pdf/pmf}) \), \( Z \in \mathcal{Z} \), \( \mathcal{Z} \) is the support of \( Z \), \( \mathcal{Y} \) is the support of \( Y \), \( b \) is possibly unknown, and both \( \psi(\cdot) \) and \( c(\cdot) \) are known real-valued functions. The function \( b\psi(Y) \) is often called the log partition function (Lehmann and Casella, 1998). It will be useful for us to discuss \( \psi(Y) \) and not \( b\psi(Y); \) hence, we refer to \( \psi(Y) \) as the “unit log partition function” because it’s coefficient is one and not \( b \). Let \( \text{EF}(Y; \psi) \) denote a shorthand for the pdf in (1). It follows from Diaconis and Ylvisaker (1979) that the conjugate prior distribution for \( Y \) is given by,

\[
f(Y|\alpha, \kappa) = K(\alpha, \kappa)\exp\{\alpha Y - \kappa\psi(Y)\}; \quad Y \in \mathcal{Y}, \frac{\alpha}{\kappa} \in \mathcal{Z}, \kappa > 0,
\]

(2)

where \( K(\alpha, \kappa) \) is a normalizing constant. Let \( \text{DY}(\alpha, \kappa; \psi) \) denote a shorthand for the pdf in (2). Here “DY” stands for “Diaconis-Ylvisaker,” and we will refer to \( Y \) as either a Diaconis-Ylvisaker random variable or a DY random variable. Diaconis and Ylvisaker (1979) proved that the pdf in (2) is proper (i.e., yields a probability measure). We also call \( \alpha \) and \( \kappa \) “DY parameters.” For examples of \( \psi \), \( \text{EF}(Y; \psi) \), and \( K(\alpha, \kappa) \) see Table 1.

Multiplying both sides of (2) by \( \exp(tY) \) and integrating, gives the moment generating function

\[
E[\exp(tY)|\alpha, \kappa] = \frac{K(\alpha, \kappa)}{K(\alpha + t, \kappa)},
\]

(3)

which exists provided that \( (\alpha + t)/\kappa \in \mathcal{Z}, \kappa > 0 \), and the corresponding values of \( K(\alpha + t, \kappa) \) and \( K(\alpha, \kappa) \) are strictly positive and finite. This gives us that the mean and variance of \( Y \) is

\[
E(Y|\alpha, \kappa) = K(\alpha, \kappa)K^{(1)}(\alpha, \kappa)
\]

(4)

\[
\text{var}(Y|\alpha, \kappa) = K(\alpha, \kappa)K^{(2)}(\alpha, \kappa) - K(\alpha, \kappa)^2K^{(1)}(\alpha, \kappa)^2,
\]

(5)

assuming that the moment generating function exists at \( t = 0 \), where

\[
K^{(1)}(\alpha, \kappa) \equiv \left[ \frac{d}{dt} \frac{1}{K(\alpha + t, \kappa)} \right]_{t=0}
\]

and

\[
K^{(2)}(\alpha, \kappa) \equiv \left[ \frac{d^2}{dt^2} \frac{1}{K(\alpha + t, \kappa)} \right]_{t=0};
\]

Finally, it is immediate from (1) and (2) that

\[
Y|Z, \alpha, \kappa \sim \text{DY}(\alpha + Z, \kappa + b; \psi).
\]

(6)

This conjugacy motivates the development of a multivariate version of the DY random variable to model dependent non-Gaussian data from the natural exponential family. Thus, in this section, we define a conjugate multivariate distribution and develop a distribution theory that will prove to be
| Data Model               | Natural Parameter | Log Partition Function (i.e., \( \psi \) and \( b \)) | Normalizing Constant | How to Simulate From the DY Distribution |
|-------------------------|-------------------|-----------------------------------------------------|----------------------|------------------------------------------|
| Gamma(\(a, k\))        |                   | \( \psi(Y) = \log\left(\frac{a}{b + \exp(-Z/k)}\right) \) | \( K(\alpha, \kappa) = \frac{1}{\Gamma(\alpha) \Gamma(k)} \) | Let \( W \sim \text{Gamma}(\kappa + 1, \alpha/\kappa) \), where \( \alpha > 0 \) and \( \kappa > 0 \). Then, \(-W \sim \text{DY}(\alpha, \kappa, \psi_2)\). |
| Binomial(\(p\))        |                   | \( \psi(Y) = \log\left(\frac{1}{1 + \exp(-p)}\right) \) | \( K(\alpha, \kappa) = \frac{\Gamma(\alpha + \kappa)}{\Gamma(\alpha) \Gamma(\kappa)} \) | Let \( W \sim \text{Beta}(\alpha, \kappa - \alpha) \), where \( \kappa > \alpha > 0 \) and “Beta(\(\alpha, \kappa - \alpha\))” is a shorthand for the beta distribution with shape parameter \( \alpha \) and scale parameter \( \kappa - \alpha \). Then, \( \log\left(\frac{W}{1-W}\right) \sim \text{DY}(\alpha, \kappa, \psi_2)\). |
| NegBinomial(\(p\))     |                   | \( \psi(Y) = \log\left(\frac{1}{1 + \exp(-p)}\right) \) | \( K(\alpha, \kappa) = \frac{\Gamma(\alpha + \kappa)}{\Gamma(\alpha) \Gamma(\kappa)} \) | Let \( W \sim \text{Beta}(\alpha, \kappa - \alpha) \), where \( \kappa > \alpha > 0 \). Then, \( \log\left(\frac{W}{1-W}\right) \sim \text{DY}(\alpha, \kappa, \psi_2)\). |
| Poisson(\(\mu\))       |                   | \( \psi(Y) = \exp(Y) \) | \( K(\alpha, \kappa) = \frac{1}{(\mu + \kappa)^\alpha \Gamma(\alpha)} \) | Let \( W \sim \text{Poisson}(\kappa) \), where \( \alpha > 0 \) and \( \kappa > 0 \). Then, \( \log(W) \sim \text{DY}(\alpha, \kappa, \psi_2)\). |
| Normal(\(\mu, \sigma\))|                   | \( \psi(Y) = Y \) | \( K(\alpha, \kappa) = \left(\frac{1}{\pi^2}\right)^{\alpha/2} \exp\left(-\frac{\kappa^2}{\mu}\right) \) | Let \( W \) be a normal random variable with mean \( \mu \) and variance \( \frac{\kappa}{\mu} \). Then, \( W \sim \text{DY}(\alpha, \kappa, \psi_2)\). |

Table 1: Univariate Distributions: The first column has the data model, the second column has the natural parameter, the third column contains quantities that define the log partition function, the fourth column has the normalizing constant, and the fifth column has instructions on how to simulate from the DY random variable with the corresponding \( \psi \). Let \( \mathbb{R}^+ = \{x : x < 0\} \).
extremely useful for fully Bayesian analysis in the dependent non-Gaussian (natural exponential family) data setting.

2.2 The Conjugate Multivariate Distribution

| Unit Log Partition Function (i.e., $\psi_j$) | CM Distribution (i.e., $f(Y|\mu, V, \alpha, \kappa)$) |
|--------------------------------------------|--------------------------------------------------|
| $\psi_1(Y) = \log \left( \frac{-1}{Y} \right)$ | $\det(V^{-1}) \left\{ \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha_i)} \right\} \exp \left[ \alpha' s - \kappa' \log \left( s^{(-1)} \right) \right] I(-s \in \mathbb{R}^n)$ |
| $\psi_2(Y) = \log(1 + \exp(Y))$ | $\det(V^{-1}) \left\{ \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha_i)} \right\} \exp \left[ \alpha' V^{-1}(Y - \mu) - \kappa' \log \left( J_{n,1} + \exp \left\{ V^{-1}(Y - \mu) \right\} \right) \right] I(Y \in \mathbb{R}^n)$ |
| $\psi_3(Y) = \exp(Y)$ | $\det(V^{-1}) \left\{ \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha_i)} \right\} \exp \left[ \alpha' V^{-1}(Y - \mu) - \kappa' \exp \left\{ V^{-1}(Y - \mu) \right\} \right] I(Y \in \mathbb{R}^n)$ |
| $\psi_4(Y) = Y^2$ | $\det(V^{-1}) \left\{ \prod_{i=1}^{n} \left( \frac{s_i}{s_i} \right)^{1/2} \right\} \exp \left\{ -(Y - \mu - \gamma') V^{-1} \Sigma^{-1} V^{-1}(Y - \mu - \gamma)/2 \right\} I(Y \in \mathbb{R}^n)$ |

Table 2: Special Cases: We list the form of the CM distribution by $\psi_j$ for $j = 1, \ldots, 4$. The first column has the unit log partition function $\psi_j$, and the second column has the form of the CM distribution with generic $V^{-1} \in \mathbb{R}^n \times \mathbb{R}^n$. Let $J_{m,g}$ denote a $m \times g$ matrix of ones, $s = (s_1, \ldots, s_n)' \equiv V^{-1}(Y - \mu)$, $\gamma = \left( \frac{\alpha_1}{2\kappa_1}, \ldots, \frac{\alpha_n}{2\kappa_n} \right)'$, and $\Sigma \equiv \text{diag} \left( \frac{1}{2\kappa_i} : i = 1, \ldots, n \right)$.

Bradley et al. (2017a) use a linear combination of independent log-gamma random variables to build their multivariate log-gamma distribution. In a similar manner we take linear combinations of DY random variables to generate a conjugate version of the DY distribution. Specifically, let the $n$-dimensional random vector $w = (w_1, \ldots, w_n)'$ consist of $n$ mutually independent DY random variables such that $w_i \sim \text{DY}(\alpha_i, \kappa_i; \psi)$ for $i = 1, \ldots, n$. Then, define $Y \equiv (Y_1, \ldots, Y_n)'$ such that

$$Y = \mu + Vw,$$

where $Y \in \mathcal{M}^n$, the matrix $V \in \mathbb{R}^n \times \mathbb{R}^n$, and $\mu \in \mathbb{R}^n$. The space $\mathcal{M}^n$ is not necessarily equal to $\mathcal{Y}^n \equiv \{ Y = (Y_1, \ldots, Y_n)' : Y_i \in \mathcal{Y}, i = 1, \ldots, n \}$; for example, if $\mathcal{Y}$ is strictly positive, we obtain a $Y$ that can have negative components since $V \in \mathbb{R}^n \times \mathbb{R}^n$. Call $Y$ in (7) a conjugate multivariate (CM) random vector. A special case of the CM random vector is the multivariate normal random vector. To see this, let $\alpha_i \equiv 0$, $\kappa_i \equiv 1/2$, and $\psi(Y) = Y^2$ for $Y \in \mathbb{R}$. Then, it follows that (7) is a multivariate normal distribution with mean $\mu$ and covariance matrix $VV'$, since the elements of $w$ consist of i.i.d. standard normal random variables. Additionally, the aforementioned MLG distribution can be written as a CM distribution when $\alpha > 0$, $\kappa > 0$, and $\psi(Y) = \exp(Y)$.

To use the CM distribution in a Bayesian context, we require its pdf, which is formally stated below.
Proposition 1: Let \( Y = \mu + Vw \), where \( Y = (Y_1, \ldots, Y_n)' \), \( \mu \in \mathbb{R}^n \), the \( n \times n \) real valued matrix \( V \) is invertible, and the \( n \)-dimensional random vector \( w = (w_1, \ldots, w_n)' \) consists of \( n \) mutually independent DY random variables such that \( w_i \sim DY(\alpha_i, \kappa_i; \psi) \) for \( i = 1, \ldots, n \).

(i) Then \( Y \) has the following pdf:

\[
f(Y|\mu, V, \alpha, \kappa) = \det(V^{-1}) \left\{ \prod_{i=1}^{n} K(\kappa_i, \alpha_i) \right\} \exp \left[ \alpha' V^{-1} (Y - \mu) - \kappa' \psi \left\{ V^{-1} (Y - \mu) \right\} \right] I(Y \in \mathcal{M}^n),
\]

where \( I(\cdot) \) is the indicator function, the \( j \)-th element of \( \psi \left\{ V^{-1} (Y - \mu) \right\} \) contains \( \psi \) evaluated at the \( j \)-th element of the \( n \)-dimensional vector \( V^{-1} (Y - \mu) \), “det” denotes the determinant function, \( \alpha \equiv (\alpha_1, \ldots, \alpha_n)' \), and \( \kappa \equiv (\kappa_1, \ldots, \kappa_n)' \).

(ii) The mean and variance of \( Y \) is given by,

\[
E(Y|\alpha, \kappa) = \mu + Vk(\alpha, \kappa)
\]

\[
\text{cov}(Y|\alpha, \kappa) = VK(\alpha, \kappa)V',
\]

where, the \( n \)-dimensional real-valued vector

\[
k(\alpha, \kappa) = \left( K(\alpha_1, \kappa_1)K^{(1)}(\alpha_1, \kappa_1), \ldots, K(\alpha_n, \kappa_n)K^{(1)}(\alpha_n, \kappa_n) \right)',
\]

and the \( n \times n \) diagonal matrix \( K(\alpha, \kappa) \equiv \text{diag} \left\{ K(\alpha_i, \kappa_i)K^{(2)}(\alpha_i, \kappa_i) - K(\alpha_i, \kappa_i)^2 K^{(1)}(\alpha_i, \kappa_i)^2 \right\} \).

The proof of Proposition 1(i) can be found in the Supplemental Appendix, and in general, let \( \text{CM}(\mu, V, \alpha, \kappa; \psi) \) be shorthand for the pdf in (8). Proposition 1(ii) follows immediately from Equations (4) and (5), and thus, Proposition 1(ii) is stated without proof.

When comparing (1), (2), and (8) we see that the univariate natural exponential family, the DY pdf, and the CM pdf share a basic structure. Specifically, all three distributions have an exponential term and an “exponential of \(-\psi\) term.” This pattern is the main reason why conjugacy exists between the distributions from the natural exponential family and the DY distribution, which we take advantage of in subsequent sections. Also, Proposition 1(ii), shows that if we restrict \( V \) (or equivalently \( V^{-1} \)) to be a lower unit triangle matrix, then the expression of the covariance matrix of \( Y \) in (9) is a type of LDL decomposition \((\text{Ravishanker and Devi, 2002})\). Hence, in subsequent sections we assume that \( V \) is lower unit triangular.

In this article, we consider CM distributions that are implied by the unit log partition function of the data model including: the gamma data model, binomial data model, negative binomial data model, the Poisson data model, and the normal data model (see Tables 1 and 3). In the univariate case, each of these special cases lead to well-known hierarchical models (i.e., gamma/inverse-gamma, (negative) binomial/beta, Poisson/log-gamma, and normal/normal models) \((\text{Diaconis and Ylvisaker, 1979})\). To delineate from the univariate setting, we shall refer to \( \text{CM}(\mu, V, \alpha, \kappa; \psi_j) \)
for \( j = 1, \ldots, 4 \) (see Table 1 for the definitions of \( \psi_1, \psi_2, \psi_3, \) and \( \psi_4 \)) as the multivariate negative-inverse-gamma distribution, multivariate logit-beta distribution, the multivariate log-gamma, and the multivariate normal distribution, respectively.

These choices of the CM distribution are themselves extremely general. For example, when \( \alpha = J_{n,1} \), we obtain an exponential/multivariate negative-inverse-gamma model. Similarly, the binomial/multivariate logit-beta model has a Bernoulli/multivariate logit-beta model as a special case, which occurs when the number of Bernoulli trials that define the binomial distribution is equal to one. Likewise, when the number of successful Bernoulli trials is equal to one, the negative binomial/multivariate logit-beta model reduces to a geometric/multivariate logit-beta specification. This creates opportunity for analyzing many different types of dependent data.

Bayesian inference not only requires the pdf of \( Y \), but also requires simulating from conditional distributions of \( Y \). Thus, we provide the technical results needed to simulate from these conditional distributions.

**Theorem 1:** Let \( Y \sim CM(\mu, V, \alpha, \kappa; \psi) \), and let \( Y = (Y_1, \ldots, Y_n)' = (Y_1', Y_2)' \), so that \( Y_1 \) is \( r \)-dimensional and \( Y_2 \) is \( (n - r) \)-dimensional. In a similar manner, partition \( V^{-1} = [H \ B] \) into an \( n \times r \) matrix \( H \) and an \( n \times (n - r) \) matrix \( B \). Also let \( \mu^* = V^{-1} \mu - Bd \) for \( d \in \mathbb{R}^{n-r} \). Then, the conditional distribution \( Y_1 | Y_2 = d, \mu^*, H, \alpha, \kappa \) is given by

\[
f(Y_1|Y_2 = d, \mu^*, H, \alpha, \kappa) = M \exp \left\{ \alpha'HY_1 - \kappa'\psi(HY_1 - \mu) \right\} I\{Y_1', d' \in \mathcal{M}^n\},
\]

where \( M \) is a strictly positive and finite normalizing constant. Let \( CM_c(\mu^*, H, \alpha, \kappa; \psi) \) be a shorthand for the pdf in (10), where the subscript “c” represents the word “conditional.”

It is not immediately clear how to simulate from a CM\(_c\) distribution since \( H \) in (10) is not square, and hence, one can not use Equation (7) directly. Thus, we require an additional result.

**Theorem 2:** Let \( w \sim CM(\mu, I_n, \alpha, \kappa; \psi) \), where \( \mu \in \mathbb{R}^n \) and \( I_n \) is an \( n \times n \) identity matrix. Assume that the function \( \psi \) is absolutely continuous, and define \( H \in \mathbb{R}^n \times \mathbb{R}^r \). Let \( \mu \) and \( \kappa \) be generated according to proper priors denoted with \( g(\mu) \) and \( g_\kappa(\kappa) \), respectively.

(i) Then the \( r \)-dimensional random vector \( q = (q_1, \ldots, q_r)' \equiv (H'H)^{-1}H'w \) has the probability density function

\[
f(Y_1|Y_2 = 0_{n-r,1}, V, \alpha, \kappa) \propto \int \exp \left\{ \alpha'HY_1 - \kappa'\psi(HY_1 - \mu) \right\} g(\mu)d\mu.
\]

where recall that \( f(q|Y_2 = 0_{n-r}, \mu, H, \alpha, \kappa) \) is the CM\(_c\) density in Equation (10), and note that we have marginalized across \( \mu \).

(ii) If \( \psi = \psi_3 \) then the \( r \)-dimensional random vector \( q = (q_1, \ldots, q_r)' \equiv (H'H)^{-1}H'w \) has the probability density function

\[
f(q|\mu = 0_{n,1}, H, \alpha) \propto \int \exp \{ \alpha'HY_1 - \kappa'\exp(HY_1) \} g_\kappa(\kappa)d\kappa,
\]
where now \( \mathbf{w} \sim \text{CM}(\mathbf{0}_n, \mathbf{I}_n, \boldsymbol{\alpha}, \boldsymbol{\kappa}; \psi) \), and we have marginalized across \( \boldsymbol{\kappa} \).

(iii) The conditional mean and covariance can be computed as

\[
E(Y_1|Y_2 = \mathbf{d}, \mathbf{V}, \boldsymbol{\alpha}, \boldsymbol{\kappa}) = (H'H)^{-1}H' \{E(\mu) + k(\boldsymbol{\alpha}, \boldsymbol{\kappa})\}
\]

\[
\text{cov}(Y_1|Y_2 = \mathbf{d}, \mathbf{V}, \boldsymbol{\alpha}, \boldsymbol{\kappa}) = (H'H)^{-1}H' \text{cov}(\mu)H(H'H)^{-1} + (H'H)^{-1}H'k(\boldsymbol{\alpha}, \boldsymbol{\kappa})H(H'H)^{-1},
\]

where the expected value \( E(\mu) \) and \( \text{cov}(\mu) \) are found with respect to \( g(\mu) \).

The proof of Theorem 2(i) is given in the Supplemental Appendix. The proof of Theorem 2(ii) follows immediately from Proposition 1(ii) and Theorem 2(i). Thus, we state Theorem 2(ii) without proof.

The integrands in (11) and (12) are proportional to a CM(0, I, \( \alpha \), \( \kappa \)), and are of the same form as the full-conditional distributions that arise in the LCM. Theorem 2 shows that it is computationally easy to simulate from a pdf proportional to CM(0, I, \( \alpha \), \( \kappa \)) provided that \( r \ll n \) and that \( \mu \) (or \( \kappa \) when \( \psi = \psi_3 \)) is marginalized. Recall that \( H \) is \( n \times r \), which implies that computing the \( r \times r \) matrix \( (H'H)^{-1} \) is computationally feasible when \( r \) is “small.” By small we mean a value such that the Gauss-Jordan elimination method for the inverse of a \( r \times r \) matrix can be computed in real-time. Furthermore, Theorem 2 shows that the \( r \)-dimensional random vector \( \mathbf{q} \) is an orthogonal projection of the \( n \)-dimensional random vector \( \mathbf{w} \) onto the column space spanned by the columns of \( H \). This provides a geometric interpretation of random vectors generated from CM(0, I, \( \alpha \), \( \kappa \)) after marginalizing \( \mu \) (or \( \kappa \) when \( \psi = \psi_3 \)).

To use the affine transformation (i.e., \( \mathbf{q} = (H'H)^{-1}H'\mathbf{w} \)) as a means to generate from a pdf proportional to CM(0, I, \( \alpha \), \( \kappa \)), one does not necessarily have to marginalize across \( \mu \) or \( \kappa \). This is because the CM distribution is proportional to the marginal distribution from an improper extension of \( \mathbf{q} \). Specifically, let \( \rho \) be an unnormalized CM distribution with mean \( \mathbf{V} \mu \) and covariance parameter \( V^{-1} = [H, \frac{1}{\sigma_2^2} \mathbf{Q}_2] \), where \( \mathbf{Q}_2 \) is the \( n \times (n-r) \) orthonormal basis for the null space of \( H \). Then,

\[
\lim_{\sigma_2 \to \infty} \rho(q_1, q_2 | c = \mathbf{V} \mu, \mathbf{V}, \boldsymbol{\alpha}, \boldsymbol{\kappa}) = \exp \{ \boldsymbol{\alpha}' \mathbf{H} \mathbf{q} - \boldsymbol{\kappa}' \psi(\mathbf{H} \mathbf{q} - \mu) \}
\]

\[
= g(q_1 | \mu, \mathbf{V}, \boldsymbol{\alpha}, \boldsymbol{\kappa})g(q_2 | \mu, \mathbf{V}, \boldsymbol{\alpha}, \boldsymbol{\kappa}),
\]

where

\[
g(q_1 | \mu, \mathbf{V}, \boldsymbol{\alpha}, \boldsymbol{\kappa}) = \exp \{ \boldsymbol{\alpha}' \mathbf{H} \mathbf{q} - \boldsymbol{\kappa}' \psi(\mathbf{H} \mathbf{q} - \mu) \} \propto f(q_1 | q_2 = 0_{n-r,1}, \mu, \mathbf{H}, \boldsymbol{\alpha}, \boldsymbol{\kappa}) \quad (13)
\]

\[
g(q_2 | \mu, \mathbf{V}, \boldsymbol{\alpha}, \boldsymbol{\kappa}) = 1. \quad (14)
\]

This implies that \( q_1 \) is independent of \( q_2 \) as \( \sigma_2 \) approaches infinity, and hence, the marginal distribution of \( q = (H'H)^{-1}H'w \) (in the limit) is given by (13). The argument in (13) and (14) is very similar to a result in Bradley et al. (2017a, cf. Theorem 2), which was clarified in the rejoinder of Bradley et al. (2017a). Although the pdf in (13) is proper (see Theorem 1), it is crucial that we recognize that \( q \) is extended by an improper \( q_2 \). This *improper extension* results in
a lack of Kolmogorov consistency (Daniell, 1919; Kolmogorov, 1933). However, proper extensions of the CM distribution are Kolmogorov consistent (see Appendix), and hence, when using process/function definition of \( Y_i \) we suggest placing a prior distribution on \( \mathbf{\mu} \) when using the transformation \((H' H)^{-1} H' \mathbf{w}\) as a means to simulate from a pdf proportional to \( CM_c \).

### 2.3 The LCM Model

The LCM model can be written as the product of the following conditional and marginal distributions:

\[
\text{Data Model: } Z_i|\beta, \eta, \xi \overset{\text{ind}}{\sim} EF(x_i' \beta + \phi_i' \eta + \xi_i; \psi_j); \quad i = 1, \ldots, n, j = 1, \ldots, 4
\]

\[
\text{Process Model 1: } \eta|V, \alpha_\eta, \kappa_\eta \sim CM_c (cJ_{n+r}, 1, M, \alpha_\eta, \kappa_\eta; \psi_k);
\]

\[
\text{Process Model 2: } \xi|\alpha_\xi, \kappa_\xi \sim CM_c (c\xi J_{2n}, 1, M_\xi, \alpha_\xi, \kappa_\xi; \psi_k);
\]

\[
\text{Parameter Model 1: } \beta|\alpha_\beta, \kappa_\beta \sim CM_c (c\beta J_{n+p}, 1, M_\beta, \alpha_\beta, \kappa_\beta; \psi_k);
\]

\[
\text{Parameter Model 2: } c|\alpha_c, \kappa_c \sim CM (0, 1, \alpha_c, \kappa_c; \psi_k);
\]

\[
\text{Parameter Model 3: } c\xi|\alpha_{c\xi}, \kappa_{c\xi} \sim CM (0, 1, \alpha_{c\xi}, \kappa_{c\xi}; \psi_k);
\]

\[
\text{Parameter Model 4: } c\beta|\alpha_{c\beta}, \kappa_{c\beta} \sim CM (0, 1, \alpha_{c\beta}, \kappa_{c\beta}; \psi_k); \quad k = 1, \ldots, 4,
\]

where \( \psi_j \) and \( \psi_k \) (for \( j, k = 1, \ldots, 4 \)) are defined in Table 1 and the elements of \( n \)-dimensional vector \( Z \equiv (Z_1, \ldots, Z_n)' \) represent data that can be reasonably modeled using a member from the natural exponential family. Additionally for each \( i \), \( x_i \) is a known \( p \)-dimensional vector of covariates, \( \beta = (\beta_1, \ldots, \beta_p)' \in \mathbb{R}^p \) is an unknown vector interpreted as fixed effects, \( \phi_i \) is a known \( r \)-dimensional real-valued vector (see Section 3 for examples), and the \( r \)-dimensional vector \( \eta = (\eta_1, \ldots, \eta_r)' \) and \( n \)-dimensional vector \( \xi \equiv (\xi_1, \ldots, \xi_n)' \) are interpreted as real-valued random effects. The hyperparameters and variance parameters are as follows: define the \((n+p)\)-dimensional vector \( \alpha_\beta = (\alpha_1, \ldots, \alpha_\beta_1, \ldots, \alpha_\beta_p)' \), the \((n+r)\)-dimensional vector \( \alpha_\eta = (\epsilon_\eta_1, \ldots, \epsilon_\eta_1, \alpha_\eta_1, \ldots, \alpha_\eta_r)' \), the \((2n)\)-dimensional vector \( \alpha_\xi = (\epsilon_\xi_1, \ldots, \epsilon_\xi_1, \alpha_\xi_1, \ldots, \alpha_\xi_n)' \), the \((n+p)\)-dimensional vector \( \kappa_\beta = (\kappa_{\beta_1}, \ldots, \kappa_{\beta_1}, \ldots, \kappa_{\beta_p})' \), the \((n+r)\)-dimensional vector \( \kappa_\eta = (\epsilon_{\eta_1}, \ldots, \epsilon_{\eta_1}, \kappa_{\eta_1}, \ldots, \kappa_{\eta_r})' \), the \( 2n \)-dimensional vector \( \kappa_\xi = (\epsilon_{\xi_1}, \ldots, \epsilon_{\xi_1}, \kappa_{\xi_1}, \ldots, \kappa_{\xi_n})' \), the \((n+p) \times p \) real-valued matrix \( M_\beta = (X', V_\beta)' \), the \((n+r) \times r \) real-valued matrix \( M = (\Phi', V)' \), the \((2n) \times n \) real-valued matrix \( M_\xi = (I_n, V_\xi)' \), \( V_\beta \in \mathbb{R}^p \times \mathbb{R}^p \), \( V \in \mathbb{R}^r \times \mathbb{R}^r \), and \( V_\xi \in \mathbb{R}^n \times \mathbb{R}^n \), where to ensure propriety (see Section 2.5 \( \alpha_{\beta, i}/\kappa_{\beta, i} \in \mathcal{Y} \), \( \alpha_{\eta, j}/\kappa_{\eta, j} \in \mathcal{Y} \), \( \alpha_{\xi, k}/\kappa_{\xi, k} \in \mathcal{Y} \), \( \kappa_{\beta, i} > 0 \), \( \kappa_{\eta, j} > 0 \), and \( \kappa_{\xi, k} > 0 \); \( i = 1, \ldots, p \), \( j = 1, \ldots, r \), \( k = 1, \ldots, n \).

The values of \( \epsilon_{\alpha} > 0 \) and \( \epsilon_{\kappa, i} > 0 \) are needed to account for the case where \( Z_i \) is equal to a boundary value on it’s support (e.g., a zero Poisson count). Other solutions to this boundary value problem exist in the Poisson setting (Bradley et al., 2017a), however we have found more consistent results using the approach in this manuscript. Additionally, the priors for \( c, c\xi \), and \( c\beta \) are introduced as a means to use Theorem 2(ii), and hence \( c, c\xi, \) and \( c\beta \) need to be marginalized. When \( j = 3 \) we use Theorem 2(iii), and a prior on the rate parameter will be introduced. Details
on hyperpriors are presented in Sections 2.4 and 2.5.

An important point argued in the Section 1 is that the LGP model is a special case of an LCM. This can now easily be seen by letting \( j = 1, \ldots, 4 \), \( k = 4 \), \( \alpha_\beta = 0_{p,1} \), \( \alpha_\eta = 0_{r,1} \), and \( \alpha_n = 0_{n,1} \). This specification yields an LGP model. In this article, we consider setting \( j = k \), since this leads to expressions of the full conditional distributions for \( \beta \), \( \eta \), and \( \xi \) that are easy to simulate from.

**Theorem 3:** Suppose the \( n \)-dimensional data vector \( Z \) is distributed according to the model in (15) with \( j = k \) and \( j = 1, \ldots, 4 \). We have the following full conditional distributions for \( \beta \), \( \eta \), and \( \xi \).

\[
\begin{align*}
    f(\beta | \cdot) &= \text{CM}_c(\mu_\beta, H_\beta, \alpha^*_\beta, \kappa^*_\beta; \psi_j) \left\{ \prod_{i=1}^n I(x'_i \beta + \phi'_i \eta + \xi_i \in \mathcal{Y}) \right\} \\
    f(\eta | \cdot) &= \text{CM}_c(\mu_\eta, H_\eta, \alpha^*_\eta, \kappa^*_\eta; \psi_j) \left\{ \prod_{i=1}^n I(x'_i \beta + \phi'_i \eta + \xi_i \in \mathcal{Y}) \right\} \\
    f(\xi | \cdot) &= \text{CM}_c(\mu_\xi, H_\xi, \alpha^*_\xi, \kappa^*_\xi; \psi_j) \left\{ \prod_{i=1}^n I(x'_i \beta + \phi'_i \eta + \xi_i \in \mathcal{Y}) \right\}, \\
    f(c | \cdot) &= \text{CM}_c(\mu_c, H_c, \alpha^*_c, \kappa^*_c; \psi_j) \left\{ I(c \in \mathcal{Y}) \right\} \\
    f(c_\xi | \cdot) &= \text{CM}_c(\mu_{c,\xi}, H_{c,\xi}, \alpha^*_{c,\xi}, \kappa^*_{c,\xi}; \psi_j) \left\{ I(c_\xi \in \mathcal{Y}) \right\} \\
    f(c_\beta | \cdot) &= \text{CM}_c(\mu_{c,\beta}, H_{c,\beta}, \alpha^*_{c,\beta}, \kappa^*_{c,\beta}; \psi_j) \left\{ I(c_\beta \in \mathcal{Y}) \right\},
\end{align*}
\]

where \( f(\beta | \cdot) \) represents the pdf of \( \beta \) given all other process variables, parameters, and the data. We define \( f(\eta | \cdot) \) and \( f(\xi | \cdot) \) in a similar manner. For ease of exposition, the definitions of the remaining unknown quantities in (17) are provided in Table 3.

In the second and third columns of Table 3, we see that elements of \( \alpha^*_{\eta} \), \( \alpha^*_{\beta} \), and \( \alpha^*_{\xi} \) are equal to \( Z + \varepsilon_\alpha J_{n,1} \). If \( \varepsilon_\alpha = 0 \) then for \( \psi_2 \) and \( \psi_3 \) we run the risk of a value of zero in the shape parameter (in these cases it is possible to obtain \( Z_i = 0 \)). Thus, the presence of \( \varepsilon_\alpha > 0 \) is important because it allows us to account for these boundary values.

For \( \psi_2 \), \( \psi_3 \), and \( \psi_4 \) we have that \( \mathcal{Y} = \mathbb{R} \), \( \mathcal{M}^{p+n+p}_p = \mathbb{R}^p \), \( \mathcal{M}^{p+n+r}_{p+r} = \mathbb{R}^r \), and \( \mathcal{M}^{2n}_n = \mathbb{R}^n \). This implies that if \( \beta \in \mathcal{M}^{p+n+p}_p = \mathbb{R}^p \), \( \eta \in \mathcal{M}^{p+n+r}_{p+r} = \mathbb{R}^r \), and \( \xi \in \mathcal{M}^{2n}_n = \mathbb{R}^n \) then \( x'_i \beta + \phi'_i \eta + \xi_i \in \mathcal{Y} = \mathbb{R} \), since for \( i = 1, \ldots, n \), \( x_i \) and \( \phi_i \) are real-valued. Consequently, for \( j = 2, 3, \) and \( 4 \), we can replace (17) with
### Quantities Needed for Gibbs Sampling

| No Boundary Adjustments | \(\psi_2\) | \(\psi_3\) |
|-------------------------|-----------|-----------|
| \(\mu_\beta = (\eta \Phi - \zeta, c_\beta J_{1,p})\)' | \(\mu_\beta = (c_\beta J_{1,1}, - \eta \Phi - \zeta, c_\beta J_{1,p})\)' | \(\mu_\beta = 0_{n+1,1}\) |
| \(\mu_\eta = (\beta X - \xi, c_\eta J_{1,1})\)' | \(\mu_\eta = (c_\eta J_{1,1}, - \beta X - \zeta, c_\eta J_{1,1})\)' | \(\mu_\eta = 0_{n+1,1}\) |
| \(\mu_\zeta = (\eta \Phi - \xi, c_\zeta J_{1,1})\)' | \(\mu_\zeta = (c_\zeta J_{1,1}, - \eta \Phi - \zeta, c_\zeta J_{1,1})\)' | \(\mu_\zeta = 0_{2n+1}\) |
| \(\mu_{i,j} = (\eta, 0_{n+1,1})'; i = 2, \ldots, r\) | \(\mu_{i,j} = (\eta, 0_{n+1,1})'; i = 2, \ldots, r\) | \(\mu_{i,j} = (\eta, 0_{2n+1})'; i = 2, \ldots, r\) |
| \(\mu_\zeta = (\eta \Phi, 0)'\) | \(\mu_\zeta = (\eta \Phi, 0)'\) | \(\mu_\zeta = (\xi, 0)'\) |
| \(H_\beta = (X, V_{\beta}^{-1}y)'\) | \(H_\beta = (X, V_{\beta}^{-1}y)'\) | \(H_\beta = (X, V_{\beta}^{-1}y)'\) |
| \(H_\eta = (\Phi, V_{\eta}^{-1}y)'\) | \(H_\eta = (\Phi, V_{\eta}^{-1}y)'\) | \(H_\eta = (\Phi, V_{\eta}^{-1}y)'\) |
| \(H_\zeta = (\Lambda, V_{\zeta}^{-1}y)'\) | \(H_\zeta = (\Lambda, V_{\zeta}^{-1}y)'\) | \(H_\zeta = (\Lambda, V_{\zeta}^{-1}y)'\) |
| \(H_{i,j} = \{\eta_{i,j}, \eta_{i,j}', C_i\}; i = 2, \ldots, r\) | \(H_{i,j} = \{\eta_{i,j}, \eta_{i,j}', C_i\}; i = 2, \ldots, r\) | \(H_{i,j} = \{\eta_{i,j}, \eta_{i,j}', C_i\}; i = 2, \ldots, r\) |
| \(H_{\alpha}^2 = -J_{n+1,1}\) | \(H_{\alpha}^2 = -J_{n+1,1}\) | \(H_{\alpha}^2 = -J_{n+1,1}\) |
| \(H_{\alpha}^2 = -J_{2n+1}\) | \(H_{\alpha}^2 = -J_{2n+1}\) | \(H_{\alpha}^2 = -J_{2n+1}\) |
| \(\alpha_{i,j} = (Z, \alpha_{i,j}, 1)\)' | \(\alpha_{i,j} = (Z, \alpha_{i,j}, 1)\)' | \(\alpha_{i,j} = (Z, \alpha_{i,j}, 1)\)' |
| \(\alpha_{i,j} = (Z, \alpha_{i,j}, 1)\)' | \(\alpha_{i,j} = (Z, \alpha_{i,j}, 1)\)' | \(\alpha_{i,j} = (Z, \alpha_{i,j}, 1)\)' |
| \(\alpha_{i,j} = (Z, \alpha_{i,j}, 1)\)' | \(\alpha_{i,j} = (Z, \alpha_{i,j}, 1)\)' | \(\alpha_{i,j} = (Z, \alpha_{i,j}, 1)\)' |
| \(\alpha_{i,j} = (\alpha_{i,j}, \alpha_{i,j})'\); \(i = 2, \ldots, r\) | \(\alpha_{i,j} = (\eta_{i,j}, \eta_{i,j} \epsilon_{i,j})'\); \(i = 2, \ldots, r\) | \(\alpha_{i,j} = (\eta_{i,j}, \eta_{i,j} \epsilon_{i,j})'\); \(i = 2, \ldots, r\) |
| \(\alpha_{i,j} = \alpha_{i,j} J_{n+1,1}\) | \(\alpha_{i,j} = \alpha_{i,j} J_{n+1,1}\) | \(\alpha_{i,j} = \alpha_{i,j} J_{n+1,1}\) |
| \(\alpha_{i,j} = \alpha_{i,j} J_{2n+1}\) | \(\alpha_{i,j} = \alpha_{i,j} J_{2n+1}\) | \(\alpha_{i,j} = \alpha_{i,j} J_{2n+1}\) |
| \(\alpha_{i,j} = \alpha_{i,j} J_{n+1,1}\) | \(\alpha_{i,j} = \alpha_{i,j} J_{n+1,1}\) | \(\alpha_{i,j} = \alpha_{i,j} J_{n+1,1}\) |
| \(\alpha_{i,j} = \alpha_{i,j} J_{2n+1}\) | \(\alpha_{i,j} = \alpha_{i,j} J_{2n+1}\) | \(\alpha_{i,j} = \alpha_{i,j} J_{2n+1}\) |
| \(\epsilon_{i,j} = (\beta, \kappa_{i,j}, \kappa_{i,j}, \kappa_{i,j})'\) | \(\epsilon_{i,j} = (\beta, \kappa_{i,j}, \kappa_{i,j}, \kappa_{i,j})'\) | \(\epsilon_{i,j} = (\beta, \kappa_{i,j}, \kappa_{i,j}, \kappa_{i,j})'\) |
| \(\epsilon_{i,j} = (\beta, \kappa_{i,j}, \kappa_{i,j}, \kappa_{i,j})'\) | \(\epsilon_{i,j} = (\beta, \kappa_{i,j}, \kappa_{i,j}, \kappa_{i,j})'\) | \(\epsilon_{i,j} = (\beta, \kappa_{i,j}, \kappa_{i,j}, \kappa_{i,j})'\) |
| \(\epsilon_{i,j} = (\beta, \kappa_{i,j}, \kappa_{i,j}, \kappa_{i,j})'\) | \(\epsilon_{i,j} = (\beta, \kappa_{i,j}, \kappa_{i,j}, \kappa_{i,j})'\) | \(\epsilon_{i,j} = (\beta, \kappa_{i,j}, \kappa_{i,j}, \kappa_{i,j})'\) |

Table 3: A comprehensive list of matrices, vectors, and constants to define the full conditional distributions in Theorem 3. If \(Z_i\) does not lay on the boundary of it’s support then use the left-hand column. The other columns should be used when \(j = k = 2\) and \(j = k = 3\) and when there exists \(Z_i\) on the boundary of it’s support (i.e., there exists an \(i\) such that \(Z_i = 0\) or \(t_i\) for \(j = 2\) and \(Z_i = 0\) for \(j = 3\)). The \(i\)-th element of \(b\) is the value of \(b\) associated with \(Z_i\) in Equation (1), where we note that this value is not the same for all \(i\). In the left-most column \(\epsilon_{\alpha} = \epsilon_{\kappa,i} \equiv 0\). In the middle column \(\epsilon_{\alpha} = \epsilon_{\kappa,1, \ldots, \epsilon_{\kappa,n}}' = b\). In the third column the elements of \(\epsilon_{\alpha} \equiv (\epsilon_{\kappa,1, \ldots, \epsilon_{\kappa,n}})' = b\).
which is advantageous because one can use Theorem 2 directly to sample from the full conditional-distributions. Furthermore, if \( p \ll n \) and \( r \ll n \) it is computationally easy to simulate from \( f(\beta | \cdot) \) and \( f(\eta | \cdot) \), since Theorem 2(i) requires one to compute \((H_\beta \beta)^{-1}\) and \((H_\eta \eta)^{-1}\) and these computations are on the order of \( p^3 \) and \( r^3 \), respectively. That is, computation is straightforward if both \( p \) and \( r \) are values such that the Gauss-Jordan elimination method for the inverse of a \( r \times r \) and \( p \times p \) matrices can be computed in real-time. Additionally, since \( H_\beta \beta \) implies a diagonal matrix for \( H_\eta \eta \) it is also straightforward to simulated from \( f(\xi | \cdot) \) using Theorem 2(i). The impact of having easy to sample full conditional distributions is more easily seen when one considers the alternative (i.e., the LGP specification of the LCM). Simulating from the analogous full-conditional distributions of an LGP is something that typically requires ad-hoc Markov chain Monte Carlo (MCMC) techniques (e.g., see the discussions in [Rue et al., 2009; Bradley et al., 2017a]), and this presents an added barrier for Bayesian methodology from the point-of-view of reproducible research. Thus, the implications of Theorem 3 are important for Bayesian analysis because it is now straightforward to update fixed and random effects within a Gibbs sampler for many more classes of non-Gaussian dependent data.

### 2.4 Prior Distributions on Covariance Parameters

A critical feature of our proposed distribution theory is the incorporation of dependence in non-Gaussian data from the exponential family. From this point-of-view it is especially important to learn about these dependencies, which are quantified by the unknown lower unit triangle matrix \( V \) (or equivalently \( V^{-1} \)). Thus, we place a prior distribution on \( V^{-1} \). Specifically, let \( V^{-1} \) be an unknown lower unit triangle matrix. That is, let \( V^{-1} \equiv \{v_{i,j}\} \), where \( v_{i,j} = 1 \) for \( j = i \), \( v_{i,j} = 0 \) for \( j > i \), and \( v_{i,j} \in \mathbb{R} \) for \( j < i \). It will be useful to organize the elements below the lower main diagonal into the \((i-1)\)-dimensional vectors \( v_i \equiv (v_{i,j}: j = 1, \ldots, i-1)' \) for \( i = 2, \ldots, n \).

We place a CM prior distribution on \( v_i \) for each \( i \). Specifically, let

\[
v_i \overset{\text{ind}}{\sim} \text{CM}(0, C_i, \alpha_i, \kappa_i; \psi); \quad i = 2, \ldots, n,
\]

where, in practice, the \((i-1) \times (i-1)\) matrix \( C_i \) is set equal to \( \sigma_i I_{i-1} \), and \( \alpha_i \), \( \kappa_i \), and \( \sigma_i \) are specified such that (18) is relatively “flat” (see Section 2.6 for more discussion). This specification leads to a conjugate full-conditional distribution within a Gibbs sampler.
Theorem 4: Suppose the n-dimensional vector $Y \sim \text{CM}_c (0_{n,1}, M, \alpha_{Y}, \kappa_{Y})$, where $M = (B'V^{-1})$, $B$ is a generic $m \times n$ real-valued matrix, $\varepsilon > 0$, $\alpha_{Y} \equiv (\alpha_{Y,1}, \ldots, \alpha_{Y,n})'$, $\kappa_{Y} \equiv (\kappa_{Y,1}, \ldots, \kappa_{Y,n})'$, $\alpha_{Y,i}/\kappa_{Y,i} \in \mathcal{Y}$, and $\kappa_{Y,i} > 0$ for $i = 1, \ldots, n$. Also assume the prior distribution in (18). Then, we have the following posterior distribution for $\{v_i\}$.

$$f(v_2, \ldots, v_n|Y, \alpha_Y, \kappa_Y) = \prod_{i=2}^{n} f(v_i|Y, \alpha_Y, \kappa_Y) = \prod_{i=2}^{n} \text{CM}_c (\mu_{Y,i}, H_{Y,i}, \alpha_{Y,i}, \kappa_{Y,i}; \psi),$$  

(19)

where $H_{Y,i} \equiv (\Sigma'_i, C'_i)'$, $\mu_{Y,i} = (Y_i, 0_{1,i-1})'$, $\Sigma'_i = (Y_j: j = 1, \ldots, i-1)'$, $\alpha_{Y,i} = (\alpha_{Y,i}, \alpha'_i)'$, and $\kappa_{Y,i} = (\kappa_{Y,i}, \kappa'_i)'$ for $i = 2, \ldots, n$.

The CM prior distribution on the modified Cholesky decomposition of the precision matrix is similar to priors considered by Daniels and Pourahmadi (2002), Chen and Dunson (2003), and Pourahmadi et al. (2007) in the Gaussian setting. In fact, when $\psi = \psi_4$ the prior distribution in (18) reduces to the prior distributions used in Daniels and Pourahmadi (2002), Chen and Dunson (2003), and Pourahmadi et al. (2007). Thus, (18) constitutes a general non-Gaussian (natural exponential family) extension of such priors on modified Cholesky decompositions of precision and covariance matrices.

There are certainly other prior distributions for $V^{-1}$ that may be more appropriate. For example, see Yang and Berger (1994) and Bradley et al. (2015b) for a Givens angle prior on covariance parameters. The Wishart and inverse Wishart are also common alternatives (e.g., see Gelman et al. 2013 for a standard reference). However, conjugacy may not always be present depending on the choice of CM distribution. Thus, in this article, we investigate the fully conjugate form of the LCM and specify the prior for $V^{-1}$ as stated in (18).

2.5 Prior Distributions on DY Parameters

Bradley et al. (2017a) showed that the multivariate log-gamma distribution they proposed can be made arbitrarily close to a multivariate normal distribution by specifying the shape and scale parameters to be large. This essentially allows one to use a LGP specification with a Poisson data model, and also use the conjugacy that arises from the MLG distribution when $j = k = 3$ in (15). This important property of the MLG distribution can be extended to the more general CM distribution.

Theorem 5: Suppose that $\psi \neq \psi_4$, and denote the first and second derivatives with $\psi'$ and $\psi''$, $0 < \psi' < \infty$, and $0 < \psi'' < \infty$. Let the n-dimensional random vector $Y \sim \text{CM}(\mu, \left(\psi''(0)/\psi'(0)\right)^{1/2}, \alpha^{1/2}V, \alpha J_{n,1}, \frac{\alpha}{\psi(0)}J_{n,1}; \psi)$. Then $Y$ converges in distribution to a multivariate normal random vector with mean $\mu$ and covariance matrix $VV'$ as $\alpha$ approaches infinity.
| Unit Log Partition Function | Form of the Prior Distribution on $\alpha$ and $\kappa$ (i.e., $f(\alpha, \kappa | \gamma, \rho)$) | Suggested Hyperparameters | Special Case of the Prior Distribution |
|---------------------------|-------------------------------------------------|--------------------------|--------------------------------------|
| $\psi_1(Y) = \log \left( -\frac{1}{Y} \right)$ | $\exp \left\{ \gamma_1 \alpha + \gamma_2 \kappa - \rho \log(\Gamma(\kappa + 1)) - \rho(\kappa + 1) \log(\alpha) \right\}$ | $\gamma_1 = -1000$ $\gamma_2 = 1000$ $\rho = 10^{-15}$ | If $\kappa$ is integer-valued then the conditional distribution of $\kappa | \alpha$ is Conway-Maxwell-Poisson with parameters $\alpha \rho \exp(\gamma_2)$ and $\rho$, and the conditional distribution of $\alpha | \kappa$ is Gamma($\kappa + 1$)$\rho + 1, -1/\gamma_1$ provided that $\gamma_1 \in \mathbb{R}$, $\gamma_1$ is negative, and $\rho \geq 0$. |
| $\psi_2(Y) = \log (1+\exp(Y))$ | $\exp \left\{ \gamma_1 \alpha + \gamma_2 \kappa + \rho \log(\Gamma(\kappa)) - \rho \log(\Gamma(\alpha)) - \rho \log(\Gamma(\kappa - \alpha)) \right\}$ | $\gamma_1 = 0$ $\gamma_2 = -1000$ $\rho = 1$ | Let $\rho = 1$, $\gamma_1 \in \mathbb{R}$, and $\gamma_2 < 0$. If $\alpha$ and $\kappa$ are integer-valued, then the conditional distribution of $(\alpha - 1) | \kappa$ is binomial with $\kappa$ number of Bernoulli trials, and probability of success $\exp(\gamma_1)/(1 + \exp(\gamma_1))$. Also, $(\kappa - \alpha - 1) | \alpha$ follows a negative binomial distribution with $\alpha + 1$ number of successful Bernoulli trials, and probability of success $\exp(\gamma_2)$. |
| $\psi_3(Y) = \exp(Y)$ | $\exp \left\{ \gamma_1 \alpha + \gamma_2 \kappa - \rho \log(\Gamma(\alpha)) - \rho(\alpha) \log(\kappa) \right\}$ | $\gamma_1 = 1$ $\gamma_2 = 10^{-15}$ $\rho = 1$ | If $\alpha$ is integer-valued then the conditional distribution of $(\alpha - 1) | \kappa$ is Conway-Maxwell-Poisson with parameters $\alpha \rho \exp(\gamma_2)$ and $\rho$, and the conditional distribution of $\kappa | \alpha$ is Gamma($\alpha \rho + 1, -1/\gamma_2$) provided that $\gamma_1 \in \mathbb{R}$, $\gamma_2$ is negative, and $\rho \geq 0$. |
| $\psi_4(Y) = Y^2$ | $\exp \left\{ \gamma_1 \alpha + \gamma_2 \kappa + \frac{\rho}{2} \log \kappa - \frac{\rho}{4} \right\} = \kappa^{\rho/2} \gamma_1 \exp(\gamma_2 \kappa) \exp(-\frac{\alpha - 2 \alpha \gamma_1}{4 \gamma_2})$ | Set $\alpha = 0$ $\gamma_1 = -\frac{1}{2}$ $\rho = 2$ | We have that $\kappa$ is distributed as Gamma($\rho/2 + 1, -1/\gamma_2$) and is independent of $\alpha$, which is distributed as normal with mean $2 \kappa \gamma_1$ and variance $2 \kappa$. The suggested hyperparameters result in an inverse-gamma prior distribution on the variance of a normal random variable with shape 2 and scale 1, which yields mean 1 and variance infinity. |

Table 4: Special Cases: We list the form of the the prior distribution in (20) by $\psi_j$ for $j = 1, \ldots, 4$. The first column has the unit log partition function, the second column has the form of the prior distributions (up to a proportionality constant), the third column gives suggested hyperparameters, and the fourth column gives special cases of the conditional distributions $\alpha | \kappa$ and $\kappa | \alpha$. 
The restriction of $\psi \neq \psi_4$ is sensible, since $\psi = \psi_4$ yields a CM exactly equal to a multivariate normal distribution. Also, Theorem 5 does not hold for the multivariate negative-inverse-gamma distribution, since $\psi_4'(0) = -\infty$.

The “best” DY parameters, for the multivariate logit-beta distribution and the MLG distribution, might not lead to something that looks Gaussian. That is, we should be able to learn whether or not the multivariate normal distribution is appropriate for latent processes of binomial and Poisson data by observing whether or not posterior replicates of the DY parameters (i.e., $\alpha$ and $\kappa$) are large (which would invoke Theorem 5). Hence, from this point-of-view, it is very important that we place prior distributions on the DY parameters.

Following the theme of the previous sections, we define conjugate priors for the DY parameters by defining a distribution with an exponential term and an exponential to the negative unit log partition function. That is, consider

$$f(\alpha, \kappa | \gamma_1, \gamma_2, \rho) \propto \exp \left[ \gamma_1 \alpha + \gamma_2 \kappa - \rho \log \left( \frac{1}{K(\alpha, \kappa)} \right) \right], \quad (20)$$

where $\gamma_1$, $\gamma_2$, and $\rho$ are hyperparameters. The parameter space for $\gamma_1$, $\gamma_2$, and $\rho$ that ensures that (20) is proper (i.e., can be normalized to define a probability measure) is an immediate consequence of a result from Diaconis and Ylvisaker (1979). In particular, from Theorem 1 of Diaconis and Ylvisaker (1979), the distribution in (20) is proper provided that $\mathcal{Y}$ is a nonempty real-valued open set, the range of $\psi$ is a nonempty real-valued open set, $\gamma_1/\rho \in \mathcal{Y}$, $\gamma_2/\rho \in \mathcal{Y}_{\psi}$, and $\rho > 0$, where $\mathcal{Y}_{\psi} \equiv \{ M : M = -\psi(Y), Y \in \mathcal{Y} \}$. For the CM distribution associated with $\psi_1$ we see that $\mathcal{Y} = \{ Y : Y < 0 \}$ and the range of $\psi$ is $\mathbb{R}$; thus, for this setting $\gamma_1 < 0$, $\gamma_2 \in \mathbb{R}$, and $\rho > 0$ results in a proper prior in (20). For the CM distributions associated with $\psi_2$, $\psi_3$, and $\psi_4$ we have that $\mathcal{Y} = \mathbb{R}$ and $\psi$ is a strictly positive; thus, for this setting $\gamma_1 \in \mathbb{R}$, $\gamma_2 < 0$, and $\rho > 0$ ensures propriety of (20).

There are many interesting special cases of the prior distribution in (20). For example, when $\alpha$ is integer-valued and $\psi = \psi_1$ then the prior in (20) has a relationship with the Conway-Maxwell-Poisson distribution (Conway and Maxwell, 1962) and the gamma distribution. These special cases (listed in Table 4 for ease of exposition) are particularly useful because they give rise to interpretations of the hyperparameters. In particular, for $\psi = \psi_1$, we have that $\rho$ can be interpreted as a dispersion parameter (in relation to the dispersion parameter of a Conway-Maxwell-Poisson distribution), $\gamma_1$ can be interpreted as a location parameter, and $\gamma_2$ can be interpreted as a scale parameter. When $\psi = \psi_2$ we have that $\gamma_1$ and $\gamma_2$ can be interpreted as functions of a proportion (i.e., the inverse logit or log of a proportion). For $\psi = \psi_3$, we have that $\rho$ can be interpreted as a dispersion parameter, $\gamma_2$ can be interpreted as a location parameter, and $\gamma_1$ can be interpreted as a scale parameter. Finally, when $\psi = \psi_4$ we have that $\gamma_1$ is interpreted as a location parameter, $\rho$ represents a shape parameter, and $\gamma_2$ represents a scale parameter.

The most familiar special case occurs when $\psi = \psi_4$ (i.e., a normal data model) and $\alpha = 0$. Namely, (20) reduces to independent gamma prior distributions on $\kappa$ with shape parameter $\rho/2 + 1$, and scale parameter $-\gamma_2$. When recognizing that $\kappa$ is equal to one-half the unknown variance of a normal random variable (See Table 1), we see that the conjugate prior distribution implies an inverse gamma distribution for the variance parameter, which is a common choice of a prior
Model 1: The LCM with Conjugate Parameter Models

Data Model: \( Z_i | \beta, \eta, \xi \overset{\text{ind}}{\sim} \text{EF}(x_i | \beta + \phi(\eta + \xi); \psi_j) \); \( i = 1, \ldots, n, j = 1, 2, 3, 4 \)

Process Model 1: \( \eta | V, \alpha, \kappa \sim \text{CM}_c(0_{r,1}, M, \alpha, \kappa; \psi_k) \)

Process Model 2: \( \xi | \alpha, \kappa \sim \text{CM}_c(0_{n,1}, M \xi, \alpha, \kappa; \psi_k) \)

Parameter Model 1: \( b | \alpha_b, \kappa_b \sim \text{CM}(0, 1, \alpha_b, \kappa_b; \psi_k) \) \( (b > 0) \)

Parameter Model 2: \( \beta | \alpha, \kappa \sim \text{CM}(0_{p,1}, M \beta, \alpha, \kappa; \psi_k) \)

Parameter Model 3: \( c | \alpha_c, \kappa_c \sim \text{CM}(0, 1, \alpha_c, \kappa_c; \psi_k) \)

Parameter Model 4: \( c_{\xi} | \alpha_{\xi,c}, \kappa_{\xi,c} \sim \text{CM}(0, 1, \alpha_{\xi,c}, \kappa_{\xi,c}; \psi_k) \)

Parameter Model 5: \( c_{\beta} | \alpha_{\beta,c}, \kappa_{\beta,c} \sim \text{CM}(0, 1, \alpha_{\beta,c}, \kappa_{\beta,c}; \psi_k) \)

Parameter Model 6: \( v_i \sim \text{CM}(0, \sigma, I_{i-1}, \alpha_v, I_{i-1}, \kappa_v, I_{i-1}, \psi_k); i = 2, \ldots, r, k = 1, 2, 3, 4 \)

Parameter Model 7: \( f(\alpha, \beta, \kappa | \gamma_{\beta,1}, \gamma_{\beta,2}, \rho) \)

\[
\propto \exp \left[ \gamma_{\beta,1} \alpha + \gamma_{\beta,2} \kappa - \rho \log \left( \frac{1}{K(\alpha, \kappa)} \right) \right],
\]

Parameter Model 8: \( f(\alpha, \kappa | \gamma_{\eta,1}, \gamma_{\eta,2}, \rho \eta) \)

\[
\propto \exp \left[ \gamma_{\eta,1} \alpha \eta, m + \gamma_{\eta,2} \kappa \eta, m - \rho \eta \log \left( \frac{1}{K(\alpha \eta, m, \kappa \eta, m)} \right) \right],
\]

Parameter Model 9: \( f(\alpha_{\xi}, \kappa_{\xi} | \gamma_{\xi,1}, \gamma_{\xi,2}, \rho \xi) \)

\[
\propto \exp \left[ \gamma_{\xi,1} \alpha_{\xi} + \gamma_{\xi,2} \kappa_{\xi} - \rho \xi \log \left( \frac{1}{K(\alpha_{\xi}, \kappa_{\xi})} \right) \right],
\]

Parameter Model 10: \( f(\alpha, \kappa | \gamma_{\nu,1}, \gamma_{\nu,2}, \rho \nu) \)

\[
\propto \exp \left[ \gamma_{\nu,1} \alpha + \gamma_{\nu,2} \kappa - \rho \nu \log \left( \frac{1}{K(\alpha, \kappa)} \right) \right].
\]

distribution on the variance parameter for normally distributed data (Gelman, 2006).

Now, consider \( q \sim \text{CM}(0_{r,1}, V_q, \alpha q J_{r,1}, \kappa q J_{r,1}, V_q \in \mathbb{R}^r \times \mathbb{R}^r \), and assume that \( (\alpha q, \kappa q) \) is distributed according to \( (20) \). Then the full conditional distribution is immediate:

\[
f(\alpha q, \kappa q | \cdot) = \exp \left[ (\gamma_1 + J_{1,r} V_q^{-1} q) \alpha q + (\gamma_2 - J_{1,r} \phi(V_q^{-1} q)) \kappa q - (\rho + r) \log \left( \frac{1}{K(\alpha q, \kappa q)} \right) \right].
\] (21)

This full conditional distribution is straightforward to simulate from using a slice sampler (Neal, 2003).
2.6 Model Summary and Specifications

Using (15), (17), and (21), we can write the joint distribution of the data, random processes, covariances, and DY parameters as the product of the conditional and marginal distributions in Model 1. In Model 1, we assume constant shape and scales (i.e., \( \alpha_{\eta, i} \equiv \alpha_{\eta} \), \( \alpha_{\beta, j} \equiv \alpha_{\beta} \), \( \alpha_{\xi, k} \equiv \alpha_{\xi} \) for all \( i, j, k \)) and the components of \( \beta \) and \( \xi \) are independent in Model 1, which is less general than what is presented in (15). In our experience, we found that it is only necessary to specify \( \eta \) to have cross-covariances. If the index on \( Z_i \) is defined over space or time, then a Kolmogorov consistency result is needed (Gelfand and Schliep, 2016), which we provide in the Supplemental Appendix. The additional assumptions in Model 1 may not be appropriate for every problem, and in this case one can easily substitute the more general specification of from (15) into Model 1. In the Supplemental Appendix, we provide step-by-step instructions on implementing a Gibbs sampler for the model in Model 1.

Parameter Model 1 in (15) is only included when \( b \) is unknown (i.e., when the data model is specified to be either the negative binomial or gamma distributions). The truncated CM distribution is chosen because it is conjugate; see details in the Supplemental Appendix. In our experience (see Section 3.3), \( b \) is difficult to learn, and the results are extremely sensitive to the choice of \( \alpha_b \) and \( \kappa_b \). Consequently, we suggest either specifying a different data model for the settings where \( b \) is unknown. Specifically, for the negative binomial setting we suggest using a Poisson distribution, and when the data is distributed as gamma we suggest taking the log transform and using a normal distribution.

To use this model in practice we need to specify the hyperparameters, \( \gamma_{\beta, 1} \), \( \gamma_{\beta, 2} \), \( \gamma_{\eta, 1} \), \( \gamma_{\eta, 2} \), \( \gamma_{\xi, 1} \), \( \gamma_{\xi, 2} \), \( \rho_{\beta} \), \( \rho_{\eta} \), and \( \rho_{\xi} \). In general, we suggest specifying the hyperparameters so that the corresponding parameter models are flat or “vague.” In particular, we use the suggested choices of hyperparameters associated with outlined in Table 4 by the appropriate CM distribution. These choices were made so that the special cases listed in the fourth column of Table 4 are “flat” or “vague.”

Another important quantity that needs to be specified are the basis functions \( \{ \phi_i \} \). This choice is very important and requires careful consideration. To illustrate the generality of our proposed model we consider three classes of basis function, each of which are demonstrated in Sections 3.2, 3.3, and 3.4, respectively. Many analyses let \( \{ \phi_i \} \) consist of known covariates (e.g., see Wilson and Reich, 2014, for a recent example). Another choice is to specify latent classes to model within-subject variability; in this setting, \( \{ \phi_i \} \) is sometimes referred to as a “random effect design matrix” (e.g., see Hodges, 2013, Chp. 1 for a discussion). Consider the example where \( g_k \subset \{ 1, \ldots, n \} \) represents the \( k \)-th group. In Section 3.2, \( g_k \) represents the \( k \)-th herd of cows, and each element in \( g_k \) represents a specific cow in the sample. Here, we shall specify \( \phi_i = (I(i \in g_1), \ldots, I(i \in g_r))' \). For spatial and time-series datasets, it is often assumed that \( \{ \phi_i \} \) consists of spatial/temporally varying functions, referred to as “basis functions.” For example, Fourier basis functions/wavelets are often used in the image analysis literature (e.g., see Donoho and Johnstone, 1994, for a classic reference). Similarly, radial basis functions, empirical orthogonal functions, and splines have been used to great effect in the spatial statistics, time-series, and spatio-temporal statistics literature (e.g., see Wahba, 1990; Bradley et al., 2016; Wikle, 2010; Bradley et al., 2017b, for a different choices...
of basis functions). Kolmogorov consistency is a fundamental property needed to use Fourier functions, radial basis functions, or any basis function with a real-valued domain. We provide a proof of Kolmogorov consistency (Daniell, 1919; Kolmogorov, 1933) for proper extensions of the CM distribution in the Supplemental Appendix.

The value of \( r \) is a feature of the observed dataset when specifying \( \{ \phi_i \} \) to be either covariates or a random effects design matrix (see Sections 3.2 and 3.3 for examples). However, when using a known class of basis functions, \( r \) must be specified. In this setting, selection criteria are often used to investigate both the sensitivity to the choice of \( r \) and how many are necessary to give reasonable predictions (e.g., see Wahba, 1990; Henao, 2009; Bradley et al., 2011, among others). Spike and slab, horseshoe priors, and SSVS (among other similar techniques) are extensions of the LGP, which one might adapt to the LCM to select covariates and basis functions (O'Hara and Sillanpaa, 2009); however, we do not consider these extensions of the LCM in this article. When spatial basis functions depend on knot locations (thin-plate splines), a common rule-of-thumb is to specify equally spaced knots over the spatial domain (e.g., see Nychka, 2001, among others). In Section 3.4, we demonstrate the use of a known kernel using a big Bernoulli dataset consisting of cloud fractions. Here, we use the same basis functions specified in (Sengupta et al., 2012), where the knots were chosen to be equally spaced.

3 Empirical Results

In Section 3.1, we use simulations to demonstrate the performance of the LCM when analyzing binomial and Poisson data. To demonstrate the wide-applicability of the CM distribution, we also give several illustrations from a variety of disciplines; namely, we analyze an epidemiology dataset (Section 3.2), a federal statistics dataset (Section 3.3), and an environmental dataset (Section 3.4). All R code and Matlab code used in these examples are provided in the Supplemental Appendix. Our computations were performed on a dual 10 core 2.8 GHz Intel Xeon E5-2680 v2 processor with 256 GB of RAM.

3.1 Simulation Studies

We compare predictions using the LGP versus predictions based on a LCM. As discussed in Section 1, the LGP is the standard approach for Bayesian analysis of dependent data, and thus, the results in this section are meant to provide one comparison of the LCM to the current state-of-the-art. It is important to emphasize that if the LGP is more appropriate than the LCM, our model will be able to identify this for some settings because of Theorem 5; that is, if the posterior replicates of the DY parameters are large then Theorem 5 suggests that the latent processes are approximately Gaussian.

The \( n \times p \) matrix \( X \equiv (x_1, \ldots, x_p)' \), the \( r \times r \) matrix \( \Phi \equiv (\phi_1, \ldots, \phi_r)' \), and the \( r \times r \) lower unit triangle matrix \( V^{-1} \) are randomly generated with \( p = 500 \) and \( r = 10 \). The choices for \( p \) and \( r \) were made to represent realistic values that one might see in practice. For example, see Matloff (2016), Chp. 2, where they consider a dataset taken from a “data exposition” provided by
the American Statistical Association’s Sections on Statistical Computing and Statistical Graphics. This example had \( n = 500,000 \) and \( p = 29 \) and was considered to be a moderate \( p \) and large \( n \) setting. Also, see Huang and Sun (2003) for a recent example where \( n = 2,153,888 \) and \( r = 60 \) is considered to be a moderate \( r \) and large \( n \) setting. Each element of the \( n \times p \) matrix \( \mathbf{X} \), the \( n \times r \) matrix \( \mathbf{\Phi} \), and the \( r \times r \) matrix \( \mathbf{V}^{-1} \) are selected from a standard normal distribution. The elements of the fixed and random effects \( \beta, \eta, \) and \( \xi \) are randomly selected from a standard normal distribution as well. Then, we define \( p_i = \frac{\exp(\mathbf{x}_i'\beta + \phi_i'\eta + \xi_i)}{1 + \exp(\mathbf{x}_i'\beta + \phi_i'\eta + \xi_i)} \) for \( i = 1, \ldots, n \). We consider two different data models in this section. In particular, we consider observations \( Z_i \) generated from a binomial distribution with sample size \( t_i \) (generated from a Poisson with mean 40) and probability of success \( p_i \) for \( i = 1, \ldots, n \). For example, consider \( n \) households, where for each household \( i \) there are \( t_i \) individuals, and let \( p_i \) represent the probability that an individual is female. Then \( Z_i \) would represent the number of women living in household \( i \). Here, one might choose \( \phi_i = (1, 0)' \) if the total income of the household is below the poverty line, and \( \phi_i = (0, 1)' \) otherwise. Similarly, we consider observations \( Z_i \) generated from a Poisson distribution with mean \( \exp(\mathbf{x}_i'\beta + \phi_i'\eta + \xi_i) \) for \( i = 1, \ldots, n \).

Using the Gibbs sampler outlined in the Supplemental Appendix we implement the LCM in Model 1 with \( j = k \) and use the appropriate data model (i.e., binomial or Poisson). We use the R-package \texttt{lme4} to implement a LGP. Default choices were used when possible in when using \texttt{lme4}. The choice of hyperparameters discussed in Section 2.6 were used. For each \( i \), denote the posterior mean of \( p_i \) with \( \hat{p}_i \), the posterior mean of \( \mu \) with \( \hat{\mu}_i \), and define the total squared prediction errors to be

\[
\sum_{i} (tp_i - t\hat{p}_i)^2 \quad \sum_{i} (\mu_i - \hat{\mu}_i)^2,
\]

used for the binomial and Poisson settings, respectively. We used a burn-in of 10,000 and generate \( B = 20,000 \) posterior replications for both data models that are considered.

We consider a large sample size of \( n = 100,000 \) and simulate \( \{Z_1, \ldots, Z_n\} \) fifty times from the binomial distribution, and simulate another fifty independent replications of \( \{Z_1, \ldots, Z_n\} \) from the Poisson distribution. In Figure 1(a,b) we plot the ratio of total squared prediction error of the LCM model and the total squared prediction error of the LGP model over the fifty independent replicates. A ratio greater than one indicates that the LCM has smaller total square prediction error. Here, we see that the ratio of the total squared prediction errors are consistently larger than one, and hence, the LCM clearly outperforms the LGP for this simulation design for both the binomial and Poisson settings. Thus, not only does the LCM lead to practical advantages (no tuning is involved) over the LGP for this example, there are also clear gains in predictive performance. Thus, this simulation suggests that the LCM model yields precise predictions, and is computationally feasible for a large dataset with moderate values for \( p \) and \( r \). Note that the high predictive performance of the LCM model occurs in a setting where we do not generate the truth
from a multivariate logit-beta distribution.

3.2 An Application to Contagious Bovine Pleuropneumonia in Ethiopian Highlands

Contagious bovine pleuropneumonia (CBPP) has been classified as a list-A disease by the World Organization for Animal Health. For this reason, Lesnoff et al. (2004) conducted an extensive study on herds of cows located within the Boji district of West Wellega, Ethiopia. They collected the incidence of CBPP among 15 herds over four time periods that span 16 months. They were interested in tracking the probability of contracting the disease as a function of time, and considered a generalized linear mixed model to assess this. Time was found to be an important fixed effect, and the herds were found to be an important random effect (Lesnoff et al., 2004). This is a small dataset consisting of 54 observations.

We fit a binomial LCM to these data, where the response is the number of cows infected and the total number cows in the heard is known (i.e., \( t_i \)). We define \( x_i \) to consist of indicators of the different time-periods. Let \( g_k \) represent the \( k \)-th herd of cows, and let each element in \( g_k \) represents a specific cow in the sample. Specify \( \phi_i = (I(i \in g_1), \ldots, I(i \in g_{r}))' \) for cows \( i = 1, \ldots, 54 \). We compare our results to a LGP fitted using a standard R-package for generalized linear models; namely, the R-package \texttt{lme4}, and using the function “glmer” (Bates et al., 2017). In Figure 2, we plot the ratio of the mean squared prediction errors (i.e., MSPE associated with LGP and the MSPE associated with the LCM). Here, the paired t-test resulted in a p-value of \( 1.27 \times 10^{-5} \), which suggests that the LCM is outperforming the GLM in this setting. However, visually Figure 1
Figure 2: The response is the ratio between mean squared prediction error using the LGP model and the mean squared prediction error of the LCM. We hold out roughly 5% of the observations. A boxplot is displayed over 50 different hold-out observations. The mean squared prediction error (MSPE) is between the predicted mean (e.g., posterior mean of $\frac{\exp\{x_i'\beta + \phi_i'\eta_i + \xi_i\}}{1 + \exp\{x_i'\beta + \phi_i'\eta_i + \xi_i\}}$) and the hold out dataset. Values greater than one (indicated by the dashed-dotted magenta line) suggest that the binomial LCM outperforms the binomial LGP.

suggest that the LCM and LGP give similar results for this example.

3.3 An Application to Count-Valued ACS Public-Use Micro-Data

The US Census Bureau has replaced the decennial census long-form with the American Community Survey (ACS), which is an ongoing survey that collects an enormous amount of information on US demographics. (To date there are over 64,000 variables published through the ACS.) The estimates published from the ACS have a unique multi-year structure. Specifically, the ACS produces 1-year and 5-year period estimates of US demographics, where 1-year period estimates are summaries (e.g., median income of a particular county) made available over populations over 65,000 and 5-year period estimates are made available for all published geographies (e.g., see Torrieri, 2007, for more information).

A difficulty with using ACS period estimates published over pre-defined geographies is that it is difficult to infer fine-level (i.e., household) information. As a result, the ACS provides a public-use micro-sample (PUMS) over public-use micro-areas (PUMAs). PUMS consists of individual and household information within each PUMA, where the location of the household within the PUMA is not released to the public. In this section, we focus on household level PUMS found within one particular PUMA; namely the PUMA that covers the metropolitan area of Tallahassee Florida (labeled as PUMA number 00701). Consider 2005-2009 PUMS estimates of the number of individuals living in a household contained within PUMA 00701. This is a fairly large dataset (for multivariate statistics) consisting of 4,537 observations. An important inferential goal, besides giving an illustration of the LCM, is to accurately predict vacant households (i.e., predict zero
people living in a household). Vacant households exhaust resources for those conducting surveys, and is of practical interest to the US Census Bureau (see, http://www.census.gov/en.html).

We would expect the number of individuals living in a household to be spatially correlated, since certain neighborhoods within Tallahassee are known to be more attractive for those with a family, and hence, have more people living within a household in these neighborhoods. However, the spatial correlation can not be leveraged, since the location within PUMAs are not publicly available. Consequently, we model the dependencies within the PUMS using a generic multivariate distribution; namely, the CM distribution. In particular, we assume that the data follows a LCM. We consider three types of LCMs: the first is a Poisson LCM (i.e., \( j = k = 3 \)), the second is a negative binomial LCM (i.e., \( j = k = 2 \)), and the third is a Poisson LGP (i.e., \( j = 3 \) and \( k = 4 \)). There are a large number of potential covariates (there are 358 in total) including fuel cost of the household, number of bedrooms in the household, and lot size, among others. For illustration, we picked a small subset of covariates using least angle regression (Efron et al., 2004), which lead to 41 covariates. We consider defining each covariate as the coefficient of the random effects (i.e., a column of \( \Psi \)) so that \( r = 41 \). Additionally, we include an intercept as a fixed effect (i.e., \( X = 1_{p+1} \)). The hyperparameters discussed in Section 2.6 were used. Convergence of the MCMC algorithm was assessed visually using trace plots, and no lack of convergence was detected.

To assess the quality of the predictions we randomly selected 216 observations (roughly 5% of the data). Using the remaining data we produce estimates of the mean number of individuals living

Table 5: A cross-tabulation of a hold-out dataset with 216 observations and the corresponding rounded predicted values (i.e., the posterior mean estimated from the Gibbs sampler). These predictions are rounded to the nearest integer, since the hold-out dataset is known to be integer-valued. The red-values indicate that the rounded predictions and the hold-out data exactly agree.
Figure 3: The response is the ratio between mean absolute difference using the LCM model and the mean absolute difference of the LGP. We hold out roughly 5% of the observations. A boxplot is displayed over 50 different hold-out observations. The mean absolute difference (MAD) is between the predicted mean (e.g., posterior mean of $t_i \exp \left\{ x_i' \beta + \phi_i' \eta_i + \xi_i \right\} / [1 + \exp \left\{ x_i' \beta + \phi_i' \eta_i + \xi_i \right\}]$) and the hold-out dataset. Values less than one suggest that the LCM outperforms the LGP. The left boxplot represents the 50 ratios of the MAD using the Poisson LCM and the MAD using the Poisson LGP. The right boxplot represents the 50 ratios of the MAD using the Negative Binomial LCM and the MAD using the Poisson LGP.

in a household for the 216 observations. As an example, see Table 1 where we display the hold-out dataset and the corresponding Poisson LCM predictions that were based on the remaining 4,321 observations. Here, we see that a majority of the rounded (to the nearest integer) predictions are exactly equal to the hold-out data. In fact, $163/226 \approx 72\%$ of the 226 hold-out dataset are exactly equal to the corresponding rounded predicted value, and the remaining 30% are within two counts of the corresponding hold-out data value. Furthermore, we are able to very accurately predict an empty household, which may have implications for sampling done by the US Census Bureau.

This hold-out study was repeated 50 times, and the results of the mean absolute difference (MAD) between the hold-out data and the rounded predictions are presented in Figure 3. Here, we fit the LGP (or a Bayesian GLM) using the R-package MCMCglmm and the function “MCMCglmm” (Hadfield, 2016), and the remaining models were fitted using the Matlab (Version 9) code in the Supplemental Materials. Here, we see that both the Poisson LCM and the Negative binomial LCM outperforms the Poisson LGP. However, the negative binomial LCM performs worse than the Poisson LCM. The pairwise p-values for a paired t-tests (using 50 MAD values as the response) are as follows: A one-sided test between the Poisson LCM and the Poisson LGP resulted in a p-value of 0.0016; a one-sided test between the negative binomial LCM and the Poisson LGP resulted in a p-value of 0.0021; and a one-sided test between the negative binomial LCM and the Poisson LCM resulted in a p-value of $9.14 \times 10^{-44}$. We found that the results for the negative binomial LCM to be sensitive to the prior on $b$ (i.e., the coefficient of the unit log-partition function); hence, we suggest using the Poisson LCM instead of the negative binomial LCM.

3.4 An Application to Moderate Resolution Imaging Spectroradiometer Cloud Data

On December 18, 1999 the National Aeronautics and Space Administration (NASA) launched
Figure 4: In the left-most panel we have a plot of the data. White locations are observed clouds and black locations are observed clear skies. The middle panel are the posterior expected value of the probability of clear skies using the Bernoulli LCM, and the right-most panel is the corresponding posterior variance of the probability of clear skies. Posterior expected values and variances were computed using a training dataset consisting of 95% of the points in the left-most plot (these points were randomly selected).

the Terra satellite, which is part of the Earth Observing System (EOS). The Moderate Resolution Imaging Spectroradiometer (MODIS) is a remote sensing instrument attached to the Terra satellite and collects information on many environmental processes. In particular, the MODIS instrument converts spectral radiances into a level-2 (i.e., 1 km × 1 km spatial resolution) cloud mask using cloud detection algorithms. These cloud detection algorithms can not perfectly identify the presence of a cloud at each 1 km × 1 km region. Sengupta et al. (2012) cast this as a big spatial data problem as, visually speaking, spatial correlations appear to be present (i.e., nearby observations tend to be more similar) and \( n = 2,748,620 \) is large.

In this article we consider fitting a Bernoulli LCM (i.e., \( j = k = 2 \) and \( b = 1 \)) to the MODIS level-2 cloud mask data from Sengupta et al. (2012). This model takes approximately one day to run using the code in the Supplemental Materials. We use the same covariates and radial basis functions and covariates in Sengupta et al. (2012). Specifically, let \( s_1, \ldots, s_n \in \mathbb{R}^2 \) represent the observed data locations (latitude/longitude) seen in the left-panel in Figure 3. Set \( \phi_j = (\phi_1(s_i), \ldots, \phi_j(s_i))^\prime \), where

\[
\phi_j(s) = \left\{ 1 - \frac{||s - g_j||}{w_j} \right\}^2 I(||s - g_j|| < w_j); \quad j = 1, \ldots, r,
\]

where \( g_j, \quad j = 1, \ldots, r, \) is the aforementioned knot points. This radial basis function is referred to as a bisquare function (Cressie and Johannesson, 2008). The knot locations are divided into three groups called “resolutions.” Then \( w_j \) is set equal to 1.5 times the shortest great arc distance between the points that are in the same resolution as \( g_j \). Sengupta et al. (2012) chose \( r = 137 \) knots to have a “quad-tree” structure (or equally-spaced structure), where the knots of the different resolution all differ from one another (e.g., see Cressie et al., 2010b, among others).

The posterior predicted value of the probability of a clear sky is given in Figure 3 along with
the posterior variance. The posterior predicted probabilities reflect the general pattern of the data. Also, the posterior variances are larger at smaller posterior predicted probabilities, which is to be expected as smaller probabilities tend to be more difficult to estimate. Thus, Figure 3 shows that it’s possible to fit an LCM to a high-dimensional spatial dataset (with small $p$) and obtain reasonable in-sample results.

We consider only a single hold-out sample in this section, since the computation times for this dataset are so demanding. Here, we hold out 5% of the observations from the left-most panel of Figure 4 and produced the posterior expected value of the probability of clear skies. We threshold the values of these posterior probabilities around the midpoint of the range to classify either clear sky or cloudy. The false positive rate is 0.22 and the false negative rate is moderately large at 0.28. We chose to compare these values to the misclassification rates using a standard binary classifier, support vector machines (SVM, Hastie et al., 2009) fitted using Matlab’s “fitsvm” function. SVM took approximately three days to run, the false positive rate is larger at 0.11, and the false negative rate is larger at 0.53. Thus, this single hold-out study suggests that the Bernoulli LCM leads to a classifier that is comparable to the current industry standard, SVM. Moreover, we are able to provide prediction uncertainty.

4 Discussion

We have introduced methodology for jointly modeling dependent non-Gaussian data within the Bayesian framework. This methodology is rooted in the development of new distribution theory for dependent data that makes Bayesian inference possible to implement using a Gibbs sampler; hence, computationally intensive and ad hoc approaches needed for tuning and specifying proposal distributions are not needed. Specifically, we propose a multivariate version of the prior distributions introduced by Diaconis and Ylvisaker (1979). Furthermore, the prior distributions similar to those used by Daniels and Pourahmadi (2002), Chen and Dunson (2003), and Pourahmadi et al. (2007) are adapted to the non-Gaussian setting.

Several theoretical results were required to derive this conjugate multivariate distribution (CM), and to develop its use for Bayesian inference of dependent data from the natural exponential family. The later is facilitated through the introduction of the latent CM (LCM) model. In particular, we show that full conditional distributions are of the same form of a conditional distribution of a CM random vector, and provide a way to simulate from this conditional distribution. Relationships between the LCM and the LGP also provide motivation for the use of the LCM. In particular, the latent Gaussian process (LGP) is a special case of the LCM. Furthermore, many types of LCMs can be well approximated by a LGP, by specifying certain parameters of a LCM to be “large.” This result shows that the LCM is not only computationally easier to implement, but is also more flexible than a LGPs.

Empirical exploration of the Poisson, binomial, Bernoulli, and negative binomial special cases were performed through simulations studies and through analyses of several datasets from a variety of disciplines. These examples indicate very small out-of-sample error when using LCM for prediction, and show gains in predictive performance over the LGP. Additionally, the LCM model is applicable for extremely large datasets (in the application we implemented the LCM on a MODIS
level-2 cloud-mask data of size 2,748,620). In the first example we considered a small dataset of binomial counts of CBPP among herds of cows. We obtained precise predictions and outperformed the LGP computed using a standard R-package. In the second real data analysis section we predict the number of individuals within a household over the US city of Tallahassee Florida, and obtain very precise estimates (in terms of hold-out error). The predictions were very accurate even though 18/226 \approx 8\% of the hold-out dataset consisted of zero counts, which is known to cause difficulties in an LGP (Lambert, 2006). In the third application, we obtain posterior predicted probabilities that reflected the pattern of data at observed locations, and a binary classifier that has misclassification rates that are comparable to support vector machines.

Although there are many settings where the LCM improves both precision and computation, there are settings where it would not be feasible to implement the LCM. In particular, we consider one choice of \psi that results in a case where \mathcal{M}_p^{n+p}, \mathcal{M}_r^{n+r}, and \mathcal{M}_n^{2n} does not guarantee that \eta \in \mathcal{Y} for each i; namely \psi_1, which is the unit log partition function of a gamma data model. In this case, the full conditional distributions in (17) are truncated CM_c distributions. Thus, in this setting the LCM is most easily implemented by doing a Gibbs sampler with component-wise updates due to the truncated support of the natural parameter. This is computationally less efficient than simply transforming the gamma data to the log scale and fitting an LGP, which can give precise predictions. Additionally, we found that the negative binomial LCM to give poorer predictive results than the Poisson LCM. Thus, we suggest using the Poisson LCM when analyzing unbounded count values instead of the negative binomial LCM.

As discussed in the Introduction, a general modeling framework for dependent data that can model non-Gaussian (natural exponential family) data as easily as Gaussian data, has important implications for applied statistics. Nevertheless, there are also many opportunities for new methodological results that are exciting, since a special case of our framework (i.e., the LGP) has been the central methodological tool used in the dependent data literature. In particular, we are interested in developing the LCM model within “more specific” dependent data settings such as time-series, spatial, spatio-temporal, and multivariate spatio-temporal arenas.
Supplemental Appendix: Bayesian Hierarchical Models with Conjugate Full-Conditional Distributions for Dependent Data from the Natural Exponential Family

Introduction

In this Supplemental Appendix, we provide the proofs of the results, propositions, and theorems stated in the main text (see Appendix A). Additionally, we provide step-by-step instructions on the Gibbs sampler for the LCM model (see Appendix B).

Appendix A: Proofs

In this appendix we provide proofs for the technical results stated in the paper.

**Proof of Proposition 1(i):** From (2) of the main text we see that the distribution of the random vector \( w \) in (7) is given by,

\[
\left( \prod_{i=1}^{n} K(\alpha_i, \kappa_i) \right) \exp \left\{ \alpha'w - \kappa'\psi(w) \right\}; \quad w \in \mathbb{R}^n.
\]

The inverse of the transform of (7) is given by \( w = V^{-1}(Y - \mu) \), and the Jacobian is given by \( |\det(V^{-1})| \). Then, by a change-of-variables (e.g., see Casella and Berger [2002]), we have that the pdf of \( Y \) is given by,

\[
\det(V^{-1}) \left( \prod_{i=1}^{n} K(\alpha_i, \kappa_i) \right) \exp \left[ \alpha'V^{-1}(Y - \mu) - \kappa'\psi \left\{ V^{-1}(Y - \mu) \right\} \right]; \quad Y \in \mathcal{M}^n.
\]

This completes the proof of Proposition 1(i).

**Proof of Theorem 1:** It follows from Proposition 1(i) that the conditional distribution is given by
\[ f(Y_1|Y_2, \mu, V, \alpha, \kappa) = \frac{[f(Y|\mu, V, \alpha, \kappa)]_{Y_2=d}}{[f(Y|\mu, V, \alpha, \kappa)dY_1]_{Y_2=d}}, \]
\[ \propto \exp \left[ \alpha' (H B) \left( \frac{Y_1}{d} \right) - \kappa' \psi \left( \frac{Y_1}{d} - V^{-1}\mu \right) \right], \]
\[ \propto \exp \left\{ \alpha' HY_1 - \kappa' \psi (HY_1 + Bd - V^{-1}\mu) \right\}, \]
\[ = \exp \left\{ \alpha' HY_1 - \kappa' \psi (HY_1 - \mu^*) \right\}; \quad Y_1 \in \mathbb{R}^n, \]

which proves the result. The normalizing constant can be found using a change of variables

\[
M = \frac{\det(V^{-1}) \{\prod_{i=1}^n K(\alpha_i, \kappa_i)\} \exp(\alpha' Bd - \alpha' V^{-1}\mu)}{[f(Y|\mu, V, \alpha, \kappa)dY_1]_{Y_2=d}} \tag{A.1}
\]

Although we do not find the expression of the integral \([f(Y|\mu, V, \alpha, \kappa)dY_1]_{Y_2=d}\), and consequently \(M\), we know that \(M\) is non-zero and finite. To see this, let \(\mathcal{M}_1 = \{Y_2 : [f(Y|\mu, V, \alpha, \kappa)dY_1]_{Y_2 = 0} = 0\}\); then, by the definition of the CM distribution for \(Y \in \mathcal{M}^n\) and \(Y_2 \in \mathcal{M}_1\)

\[
f \left( \left[ \begin{array}{c} Y_1 \\ Y_2 \end{array} \right] \mid \mu, V, \alpha, \kappa \right) > 0.
\]

Taking the integral with respect to \(Y_1\) on both sides of the inequality gives \(0 > 0\), which is a false statement. Thus, we have that \([f(Y|\mu, V, \alpha, \kappa)dY_1]_{Y_2 = d}\) is non-zero, and hence, \(M\) is finite. Similarly, let \(\mathcal{M}_2 = \{Y_2 : [f(Y|\mu, V, \alpha, \kappa)dY_1]_{Y_2 = \infty} = 0\}\) be non-empty, and let \(\mathcal{M}_2^c\) denote the set complement of \(\mathcal{M}_2\). Then, if \(w \sim \text{CM}(0_{n,1}, I_{n,1}, \alpha, \kappa)\), a change of variables within the integral (see Proposition 1)

\[
1 = \int \int f(w|\mu = 0_{n,1}, V = I_{n,1}, \alpha, \kappa) \, dw = \int f(Y|\mu, V, \alpha, \kappa) \, dY = \int \int f(Y|\mu, V, \alpha, \kappa) \, dY_1 \, dY_2
\]
\[
= \int_{\mathcal{M}_2} \int f(Y|\mu, V, \alpha, \kappa) \, dY_1 \, dY_2 + \int_{\mathcal{M}_2^c} \int f(Y|\mu, V, \alpha, \kappa) \, dY_1 \, dY_2
\]
\[
= \infty,
\]

which is a contradiction. Thus, we have that the conditional distribution of \(Y_1|Y_2, \mu, V, \alpha, \kappa\) is proper.

**Proof of Theorem 2:** Let \(Y \sim \text{CM}(V\mu, V, \alpha, \kappa; \psi)\), and partition this \(n\)-dimensional random vector so that \(Y = (Y_1', Y_2')'\), where \(Y_1\) is \(r\)-dimensional and \(Y_2\) is \((n-r)\)-dimensional. Ad-
ditionally, denote the QR decomposition of $H$ with

$$H = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 & \mathbf{0}_{n-r,n} \\ \mathbf{0}_{n-r,n} & \mathbf{0}_{n-r,n} \end{bmatrix}$$

(A.2)

where recall that the $n \times r$ matrix $Q_1$ satisfies $Q_1'Q_1 = I_r$, the $n \times (n-r)$ matrix $Q_2$ satisfies $Q_2'Q_2 = I_{n-r}$, $Q_2'Q_1 = \mathbf{0}_{n-r}$, and $R_1$ is a $r \times r$ upper triangular matrix. Then define $V$ such that

$$V^{-1} = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 & \mathbf{0}_{r,n} \\ \mathbf{0}_{r,n} & \mathbf{0}_{n-r} \end{bmatrix}.$$ 

(A.3)

Notice that

$$V = \begin{bmatrix} (H'H)^{-1}H' \\ Q_2' \end{bmatrix}.$$ 

Thus, from (7) of the main text, $Y$ can be written as

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} (H'H)^{-1}H'\mu \\ Q_2' \mu \end{bmatrix} + \begin{bmatrix} (H'H)^{-1}H'w^* \\ Q_2'w^* \end{bmatrix},$$

(A.4)

where the $n$-dimensional random vector $w^* \sim \text{CM}(\mathbf{0}_{n,1}, I_n, \alpha, \kappa; \psi)$. Multiplying both sides of (A.4) by $[I_r, \mathbf{0}_{r,n-r}]$ we have

$$Y_1 = (H'H)^{-1}H'\mu + (H'H)^{-1}H'w^* = (H'H)^{-1}H'w,$$

(A.5)

and hence the distribution associated with $(H'H)^{-1}H'w$ is the marginal distribution associated with $\text{CM}(V\mu, V, \alpha, \kappa; \psi)$. To complete the proof, we need to show that the distribution of $Y_1$ is proportional to (11) of the main text after marginalizing across $\mu$.

Theorem 2(i): Let $g$ be used to denote proper densities, $g(\mu)$ be the density for $\mu = (\mu_1, \ldots, \mu_n)'$, and let $\mu_1 = (\mu_{11}, \ldots, \mu_{n1})'$ be independent and identically distributed as $\mu$. Also, let $M(\mu_1, H, \alpha, \kappa)$ be the normalizing constant of a $\text{CM}_c(\mu_1, H, \alpha, \kappa)$. Then, for a given fixed $Y_3 \in \mathbb{R}^n$,

$$g(Y_1|Y_2 = \mathbf{0}_{n-r,1}, V, \alpha, \kappa) \propto \int g(\mu_1) \exp \left\{ \alpha'HY_1 + \alpha'Q_2Y_3 - \kappa'\psi(HY_1 - \mu) \right\} M(\mu_1, H, \alpha, \kappa) g(\mu = \mu_1 - Q_2Y_3) d\mu.$$ 

(A.6)

where $\frac{g(\mu_1)}{M(\mu_1, H, \alpha, \kappa)g(\mu = \mu_1 - Q_2Y_3)}$ is a proportionality constant and does not depend on $\mu$. Now, for
a given $Y_3 \in \mathbb{R}^n$ consider the change of variables $\mu = -Q_2 Y_3 + \mu_1$. Thus, (A.6) is equal to,
\[
\int \frac{1}{M(\mu_1, H, \alpha, \kappa)} \exp \left\{ \alpha' V^{-1} Y - \kappa' \psi (V^{-1} Y - \mu_1) \right\} g(\mu_1) d\mu_1
= g(Y_1 | Y_2 = Y_3, V, \alpha, \kappa).
\]

Hence, for every $g(Y_1 | Y_2 = 0_{n-r,1}, V, \alpha, \kappa)$, where notice that we marginalize across $\mu$. Since $g(Y_1 | Y_2 = 0_{n-r,1}, V, \alpha, \kappa)$ is not a function of $Y_3$,
\[
g(Y_1 | Y_2 = 0_{n-r,1}, V, \alpha, \kappa) = E[g(Y_1 | Y_2 = 0_{n-r,1}, V, \alpha, \kappa)]
= E[g(Y_1 | Y_2 = Y_3, V, \alpha, \kappa)]
= g(Y_1 | V, \alpha, \kappa),
\]
where the expectation is with respect to the joint $Y_3$, and $Y_3$ is assumed to follow the density $g(Y_3 | V, \alpha, \kappa)$. Thus, a composite sampling approach can be used to sample from $g(Y_1 | Y_2 = 0_{n-r,1}, V, \alpha, \kappa)$. First sample $\mu$ from $g(\mu)$ and then sample $Y_1$.

Theorem 2(ii): This proof is similar to a proof given in the rejoinder of Bradley et al. (2017a). Let $g$ be used to denote proper densities, $g_\kappa(\kappa)$ be the density for $\kappa = (\kappa_1, \ldots, \kappa_n)'$, and let $\kappa_1 = (\kappa_{11}, \ldots, \kappa_{n1})'$ be independent and identically distributed as $\kappa$. For a given $Y_3 \in \mathbb{R}^n$,
\[
g(Y_1 | Y_2 = 0_{n-r,1}, \mu, V, \alpha) \propto \int g_\kappa(\kappa_1) \left( \prod_{i=1}^{m} \kappa_i^{\alpha_i} \right) \exp \left\{ -J_{n,1} Q_2 Y_3 + \alpha' HY_1 - \kappa' \exp(HY_1) \right\} \left( \prod_{i=1}^{m} \kappa_i^{\alpha_i} \right) M(\mu, H, \alpha, \kappa_1) g_\kappa(\kappa = \exp(Q_2 Y_3 + \log(\kappa_1))) d\kappa.
\]
where
\[
\int g_\kappa(\kappa_1) \left( \prod_{i=1}^{m} \kappa_i^{\alpha_i} \right) M(\mu, H, \alpha, \kappa_1) g_\kappa(\kappa = \exp(Q_2 Y_3 + \log(\kappa_1))) d\kappa
\]
is a proportionality constant and does not depend on $\kappa$. Now, consider the change of variables $\kappa = \exp(Q_2 Y_3 + \log(\kappa_1))$. The Jacobian for a given $Y_3$ is given by $\exp(-J_{n,1} Q_2 Y_3)$. Thus, (A.8) is equal to,
\[
= \int \frac{1}{M(\mu, H, \alpha, \kappa_1)} \exp \left\{ \alpha' V^{-1} q - \kappa' \exp(V^{-1} q) \right\} g_\kappa(\kappa_1) d\kappa_1
\]
where
\[
\int \frac{1}{M(\mu, H, \alpha, \kappa_1)} \exp \left\{ \alpha' V^{-1} q - \kappa' \exp(V^{-1} q) \right\} g_\kappa(\kappa_1) d\kappa_1
\]
is a proportionality constant and does not depend on $\kappa$. Now, consider the change of variables $\kappa = \exp(Q_2 Y_3 + \log(\kappa_1))$. The Jacobian for a given $Y_3$ is given by $\exp(-J_{n,1} Q_2 Y_3)$. Thus, (A.8) is equal to,
\[
= g(Y_1 | Y_2 = 0_{n-r,1}, \mu = 0_{n,1}, V, \alpha).
\]

Hence, for every $Y_3$, $g(Y_1 | Y_2 = 0_{n-r,1}, \mu = 0_{n,1}, V, \alpha) = g(Y_1 | Y_2 = Y_3, \mu = 0_{n,1}, V, \alpha)$, where notice that we marginalize across $\kappa$. Since $g(Y_1 | Y_2 = 0_{n-r,1}, \mu = 0_{n,1}, V, \alpha)$ is not a function of $Y_3$,
\[
g(Y_1 | Y_2 = 0_{n-r,1}, \mu = 0_{n,1}, V, \alpha)
= E[g(Y_1 | Y_2 = 0_{n-r,1}, \mu = 0_{n,1}, V, \alpha)]
= E[g(Y_1 | Y_2 = Y_3, \mu = 0_{n,1}, V, \alpha)]
= g(Y_1 | \mu = 0_{n,1}, V, \alpha),
\]
where the expectation is with respect to the joint $Y_3$, and $Y_3$ is assumed to follow $g(Y_3 | \mu = \ldots$
Proof of Theorem 3: For \( i = 1, \ldots, n \), let \( Z_i | \beta, \eta, \xi_i \overset{\text{ind}}{\sim} \text{EF} \left( x_i^T \beta + \phi_i \eta + \xi_i; \psi \right) \), and let \( \beta \sim \text{CM}_c \left( c_{\beta} J_{n+p,1}, M_{\beta}, \alpha_{\beta}, \kappa_{\beta} \right) \), \( \eta \sim \text{CM}_c \left( c_{\eta} J_{n+r,1}, V, \alpha_{\eta}, \kappa_{\eta} \right) \), and \( \xi \sim \text{CM}_c \left( c_{\xi} J_{2n,1}, M_{\xi}, \alpha_{\xi}, \kappa_{\xi} \right) \). Let \( Y_i = x_i^T \beta + \phi_i \eta + \xi_i \). The data model can be written as
\[
f(Z | \cdot) \propto \exp \left\{ \sum_{i=1}^{n} Z_i Y_i - \sum_{i=1}^{n} \psi(Y_i) \right\} = \exp \left\{ Z' Y - J_{n,1}' \psi(Y) \right\},
\]

\[
= \exp \left\{ Z' \beta + Z' \Phi \eta + Z' \xi - J_{n,1}' \psi(\beta \eta + \xi) \right\},
\]

where \( Y \equiv (Y_1, \ldots, Y_n)' \), the \( n \times p \) matrix \( X \equiv (x_1, \ldots, x_n)' \), the \( n \times r \) matrix \( \Phi \equiv (\phi_1, \ldots, \phi_n)' \), and \( \propto \) denotes the "proportional to as a function of \( Z \)" symbol. It follows that
\[
f(Z | \cdot) \propto \exp \left( Z' X \beta - b J_{n,1}' \psi(\beta \eta + \xi) \right) h
\]

\[
f(Z | \cdot) \propto \exp \left( Z' \Phi \eta - b J_{n,1}' \psi(\Phi \eta + X \beta + \xi) \right) h
\]

\[
f(Z | \cdot) \propto \exp \left( Z' \xi - b J_{n,1}' \psi(\xi + X \beta + \Phi \eta) \right) h,
\]

where \( h = \{ \prod_{i=1}^{n} I(x_i^T \beta + \phi_i \eta + \xi_i \in \mathcal{Y}) \} \). From Theorem 1 we have that
\[
f(\beta | V_\beta, \alpha_\beta, \kappa_\beta) \propto \exp \left\{ \alpha'_\beta M_\beta \beta - \kappa'_\beta \psi( M_\beta \beta - c_{\beta} J_{n+p,1} ) \right\},
\]

\[
f(\eta | V, \alpha_\eta, \kappa_\eta) \propto \exp \left\{ \alpha'_\eta M_\eta \eta - \kappa'_\eta \psi( M_\eta \eta - c_{\eta} J_{n+r,1} ) \right\},
\]

\[
f(\xi | V_\xi, \alpha_\xi, \kappa_\xi) \propto \exp \left\{ \alpha'_\xi M_\xi \xi - \kappa'_\xi \psi( M_\xi \xi - c_{\xi} J_{2n,1} ) \right\}.
\]

Using (A.12), (A.15), and (A.17) we have that
\[
f(\beta | \cdot) \propto f(Z | \cdot) f(\beta | V_\beta, \alpha_\beta, \kappa_\beta, c_{\beta})
\]

\[
\propto \exp \left\{ Z' X \beta + \alpha'_\beta M_\beta \beta - \kappa'_\beta \psi( M_\beta \beta - c_{\beta} J_{n+p,1} \beta ) - b J_{n,1}' \psi( X \beta + \Phi \eta + \xi ) \right\} h
\]

\[
= \exp \left\{ Z' X \beta + e J_{n,1}' X \beta + \alpha'_{\beta,-\epsilon} V^{-1} \beta - \kappa'_\beta \psi( M_\beta \beta - c_{\beta} J_{n+p,1} \beta ) - b J_{n,1}' \psi( X \beta + \Phi \eta + \xi ) \right\} h
\]

\[
= \text{CM}_c \left\{ \mu'_\beta, H_\beta, \alpha'_\beta, \kappa'_\beta, \psi \right\} \left\{ \prod_{i=1}^{n} I(x_i^T \beta + \phi_i \eta + \xi_i \in \mathcal{Y}) \right\},
\]

where \( \alpha'_{\beta,-\epsilon} = (\alpha_{\beta,1}, \ldots, \alpha_{\beta,p})' \), and \( \mu'_\beta, H_\beta, \alpha'_\beta, \kappa'_\beta \) are defined in Table 3. The value \( \epsilon \) accommodates values of \( Z \) on the boundary. For example, the log gamma distribution the shape
and scale parameters need to be nonnegative, and for the logit beta distribution the shape parameter must be positive and less than the scale parameter. When no boundary correction is needed set \( \varepsilon = 0 \) in (A.18).

In a similar manner we can find the full conditional distributions associated with \( \eta \) and \( \xi \). Using (A.13) and (A.16) we have that

\[
f(\eta | \xi) \propto f(\xi | \eta) f(\eta | V, \alpha_\beta, \kappa_\beta, c) \]

Substituting (A.14) and (A.17) into (A.19) we have that

\[
f(\eta | \xi) \propto \exp \left\{ \eta^\prime \Phi \eta + \alpha_\eta^\prime \eta - \kappa_\eta^\prime \psi \{ M \eta - c J_{n+1,1} \} - b_{J_{n,1}} \psi (\Phi \eta + X \beta + \xi) \right\} h
\]

where \( \alpha_{\eta, -\varepsilon} = (\alpha_{\eta,1}, \ldots, \alpha_{\eta,n})^\prime \), and \( \mu_\eta, H_\eta, \alpha_\eta^*, \kappa_\eta^* \) are defined in Table 3. When no boundary correction is needed set \( \varepsilon = 0 \) in (A.19).

Using (A.14) and (A.17) we have that

\[
f(\xi | \eta) \propto f(\eta | \xi) f(\xi | V_\xi, \alpha_\xi, \kappa_\xi, c_\xi)
\]

Substituting (A.14) and (A.17) into (A.19) we have that

\[
f(\xi | \eta) \propto \exp \left\{ \xi^\prime \Phi \xi + \alpha_\xi^\prime \xi - \kappa_\xi^\prime \psi \{ M_\xi \xi - c_\xi J_{2n,1} \} - b_{J_{n,1}} \psi (\xi + X \beta + \Phi \eta) \right\} h
\]

where \( \alpha_{\xi, -\varepsilon} = (\alpha_{\xi,1}, \ldots, \alpha_{\xi,n})^\prime \), and \( \mu_\xi, H_\xi, \alpha_\xi^*, \kappa_\xi^* \) are defined in Table 3. When no boundary correction is needed set \( \varepsilon = 0 \) in (A.19).

Finally,

\[
f(c | \cdot) \propto f(c | \eta) f(\eta | V, \alpha_\beta, \kappa_\beta, c)
\]

Substituting (A.14) and (A.17) into (A.19) we have that

\[
f(c | \cdot) \propto \exp \left\{ \alpha_c c + \alpha_\eta^\prime M \eta - \kappa_\eta^\prime \psi \{ M \eta - c J_{n+1,1} \} - \kappa_c \psi (c) \right\} I(c \in \mathcal{B}) h
\]

where \( \alpha_c, \mu_c, H_c, \alpha_c^*, \kappa_c^* \) are defined in Table 3. The full-conditional distributions for \( c_\varepsilon \) and \( c_\beta \) are found in a similar way.
Proof of Theorem 4: Using induction we find that,

\[ f(Y | \mu = 0_{n,1}, M, \alpha_Y, \kappa_Y) \propto \exp \left[ \alpha_Y' V^{-1} Y - \kappa_Y' \psi \{ V^{-1} Y \} \right] \]

\[ \propto \exp \left[ \sum_{i=2}^{n} \sum_{j=1}^{i-1} \alpha_{Y,i} v_{i,j} Y_j - \sum_{i=2}^{n} \kappa_{Y,i} \psi \left( \sum_{j=1}^{i-1} Y_j v_{i,j} + Y_i \right) \right] \]

\[ = \prod_{i=2}^{n} \exp \{ \alpha_i \Sigma_i v_i - \kappa_i \psi(\Sigma_i v_i + Y_i) \} \]

where \( \Sigma_i' = (Y_j : j = 1, \ldots, i-1)' \). Thus, the full conditional distribution is given by

\[ f(v_2, \ldots, v_r | \cdot) \propto f(Y | \mu = 0_{n,1}, V, \alpha_Y, \kappa_Y) \prod_{i=2}^{n} f(v_i) \]

\[ \propto \prod_{i=2}^{n} \exp \left[ (\alpha_{Y,i}, \alpha_i') H_{Y,i} v_i - (\kappa_{Y,i}, \kappa_i') \psi \{ H_{Y,i} v_i - \mu_{Y,i} \} \right] , \]

\[ \propto \prod_{i=2}^{n} \text{CM}_c \left( \mu_{Y,i}, H_{Y,i}, (\alpha_{Y,i}, \alpha_i'), (\kappa_{Y,i}, \kappa_i')'; \psi \right) , \]

where

\[ H_{Y,i} = \begin{bmatrix} \Sigma_i & C_i \\ C_i & 0 \end{bmatrix}, \]

\[ \mu_{Y,i} = (Y_i, 0_{1,i-1})' , \]

which is the desired result.

Proof of Theorem 5: Consider the transformation \( Q = \left( \frac{\psi''(0)}{\psi'(0)} \right)^{1/2} \alpha^{1/2} W, \) where \( W \sim \text{DY} \left( \alpha, \frac{\alpha}{\psi'(0)} ; \psi \right) \). Then we have that

\[ f(Q | \alpha, \kappa) \propto \exp \left[ \left( \frac{\psi'(0)}{\psi''(0)} \right)^{1/2} \alpha^{1/2} Q - \frac{\alpha}{\psi'(0)} \psi \left\{ \left( \frac{\psi'(0)}{\psi''(0)} \right)^{1/2} \alpha^{-1/2} Q \right\} \right] , \]

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and using the Taylor Series expansion of $\psi(x)$ we have

$$f(Q|\alpha, \kappa) \propto \exp \left[ \left( \frac{\psi'(0)}{\psi''(0)} \right)^{1/2} \alpha^{1/2} Q \right]$$

$$- \frac{\alpha}{\psi'(0)} \left\{ \psi'(0) \left( \frac{\psi'(0)}{\psi''(0)} \right)^{1/2} \alpha^{-1/2} Q + \psi''(0) \left( \frac{\psi'(0)}{\psi''(0)} \right) \alpha^{-1} \frac{Q^2}{2} + O \left( \frac{\psi'(0)^{3/2}}{\psi''(0)^{3/2}} \alpha^{-3/2} Q^3 \right) \right\} ,$$

where “$O(\cdot)$” is the “Big-O” notation (e.g., see Lehmann, 1999 among others). Then, letting $\alpha$ go to infinity yields,

$$\lim_{\alpha \to \infty} f(Q|\alpha, \kappa) \propto \exp \left( - \frac{Q^2}{2} \right) \propto \text{Normal}(0,1).$$

Thus, $Q$ converges in distribution to a standard normal distribution as $\alpha$ goes to infinity. Now suppose $w = (w_1, \ldots, w_n)' \sim \text{CM} \left( 0_n, \alpha^{1/2} I_n, \alpha J_{n,1}, -\frac{\alpha}{\psi'(0)} J_{n,1}; \psi \right)$. Then it follows from the result above that $\left( \frac{\alpha}{\psi'(0)} \right)^{1/2} w$ converges to a standard multivariate Gaussian distribution. Now, define the transformation $Y = \mu + \nu (\alpha^{1/2} w)$, which follows a $\text{CM} \left( \mu, \alpha^{1/2} V, \alpha J_{n,1}, -\frac{\alpha}{\psi'(0)} J_{n,1}; \psi \right)$. It follows from Theorem 5.1.8 of Lehmann (1999), and the fact that $\frac{\alpha}{\psi'(0)} w$ converges to a standard Gaussian distribution, that $Y$ converges in distribution to a multivariate normal distribution with mean $\mu$ and covariance matrix $VV'$.

**Proof of Result 1:** In the main-text we stated that the CM distribution is Kolmogorov consistent. We now prove that result. To prove Kolmogorov consistency we need to show the following:

1. For any finite set $\{1, \ldots, n\}$ and for a generic permutation $\{i_1, \ldots, i_n\}$, we have
   $$f \{ Y_{i_1}, \ldots, Y_{i_n} \mid c, V, \alpha, \kappa \} = f \{ Y_{1}, \ldots, Y_{n} \mid c, V, \alpha, \kappa \} .$$

2. Let $\{j_1, \ldots, j_n\}$ be a generic permutation of $\{1, \ldots, n\}$ and let $m < n$. Then we have that the marginal density $f \{ Y_{j_1}, \ldots, Y_{j_m} \mid c, V, \alpha, \kappa \} = \int_{Y_{j_{m+1}}} \ldots \int_{Y_{j_n}} f \{ Y_{1}, \ldots, Y_{n} \mid c, V, \alpha, \kappa \} dY_{j_{m+1}} \ldots dY_{j_n}$ exists.

Note that the conditions of the Kolmogorov extension theorem do not require that probability density functions exist. However, from Proposition 1(i), we have an expression of the pdf of $Y$, which will be useful in our proof; hence, we can simplify the conditions of the Kolmogorov extension theorem to the setting where the joint probability density function exists.

For Item 1, define a $n \times n$ permutation matrix $\Pi$ such that $\{Y_{i_1}, \ldots, Y_{i_n}\}' \equiv Y_{\pi} = \Pi Y$. Recall that permutation matrices have the following properties: $\Pi \Pi' = \Pi' \Pi = I_n$ and $\Pi^{-1} = \Pi'$. From Equation (7) of the main text we have that,

$$Y_{\pi} = \Pi c + \Pi Vw,$$

(A.20)
where \( \mathbf{w} \) consist of mutually independent \( \mathbf{D} \mathbf{Y} \) random variables with respective shape and scale parameters organized into the \( n \)-dimensional vectors \( \mathbf{\alpha} \) and \( \mathbf{\kappa} \).

From Proposition 1(i),
\[
    f(\mathbf{Y}_\pi | \mathbf{c}, \mathbf{V}, \mathbf{\alpha}, \mathbf{\kappa})
    = \det(\mathbf{V}^{-1}) \left( \prod_{i=1}^{n} K(\alpha_i, \kappa_i) \right) \exp[\mathbf{\alpha}' \mathbf{V}^{-1} \mathbf{\Pi}'(\mathbf{Y}_\pi - \mathbf{\Pi}\mathbf{c}) - \mathbf{\kappa}' \psi(\mathbf{V}^{-1} \mathbf{\Pi}'(\mathbf{Y}_\pi - \mathbf{\Pi}\mathbf{c}))]
    = f(\mathbf{Y} | \mathbf{c}, \mathbf{V}, \mathbf{\alpha}, \mathbf{\kappa}),
\]
where the last equality holds since \( \mathbf{\Pi}' \mathbf{\Pi} = \mathbf{I}_n \) and \( \mathbf{\Pi}' \mathbf{Y}_\pi = \mathbf{\Pi}' \mathbf{Y} = \mathbf{Y} \). Thus, permutation holds.

We now need to show that the marginal distribution stays the same regardless of what the “extended” proper joint distribution is defined as. Without loss of generality (due to Item 1) set \( \mathbf{P}'_m = [I_m, 0_{m,n-m}] \) where \( 0_{m,n-m} \) is a \( m \times (n-m) \) matrix of zeros. Then define \( \mathbf{V} = [\mathbf{M}, \mathbf{C}]' \), \( \mathbf{M}' \) to be a \( m \times n \) is a real-valued matrix, \( \mathbf{C} \) to be any \( n \times (n-m) \) real-valued matrix such that \( \mathbf{V} \) is invertible, \( \mathbf{Y} \in \mathbb{R}^n \), \( \mathbf{Y} = \mathbf{c} + \mathbf{Vw} = (\mathbf{Y}_1', \mathbf{Y}_2')' \), \( \mathbf{Y}_1' \) is \( m \)-dimensional, and \( \mathbf{Y}_2' \) is \( (n-m) \)-dimensional.

The joint distribution is determined by \( \mathbf{V} = [\mathbf{M}, \mathbf{C}], \mathbf{c}, \mathbf{\alpha}, \) and \( \mathbf{\kappa} \). Thus, we need to show that joint probability density functions with different values of \( \mathbf{C} \) and \( \mathbf{c} \) results in the same marginal probability density function upon integrating the joint probability density function. Let \( \mathbf{C}_1 \) denote a generic real-valued matrix such that \( \mathbf{V}_1 = [\mathbf{M}, \mathbf{C}_1]' \) is invertible and \( \mathbf{C} \neq \mathbf{C}_1 \). Let \( \mathbf{c}_1 \in \mathbb{R}^m \). Define \( \mathbf{Y}_1' = \mathbf{c}_1 + \mathbf{V}_1 \mathbf{w} = (\mathbf{Y}_1', \mathbf{Y}_2')' \), where \( \mathbf{Y}_1' \) is \( m \)-dimensional, and \( \mathbf{Y}_2' \) is \( (n-m) \)-dimensional. Then we have that
\[
f(\mathbf{Y}_1'|\mathbf{c}_1, \mathbf{V}_1, \mathbf{\alpha}, \mathbf{\kappa}) = \int f(\mathbf{Y}_1'|\mathbf{c}_1, \mathbf{V}_1, \mathbf{\alpha}, \mathbf{\kappa})d\mathbf{q}_2^{(1)}, \tag{A.21}
\]
and a change of variables \( \mathbf{Y} = \mathbf{V}_1^{-1} \mathbf{Y}_1' - \mathbf{V}_1^{-1} \mathbf{c}_1 + \mathbf{c} \) within (A.21) gives,
\[
f(\mathbf{Y}_1'|\mathbf{c}_1, \mathbf{V}_1, \mathbf{\alpha}, \mathbf{\kappa}) = \int f(\mathbf{Y}_1'|\mathbf{c}_1, \mathbf{V}_1, \mathbf{\alpha}, \mathbf{\kappa})d\mathbf{Y}_2^{(1)} = \int f(\mathbf{Y}|\mathbf{c}, \mathbf{V}, \mathbf{\alpha}, \mathbf{\kappa})d\mathbf{Y}_2
\]
\[= f(\mathbf{Y}_1'|\mathbf{c}, \mathbf{V}, \mathbf{\alpha}, \mathbf{\kappa}).\]
This completes the proof.

**Appendix B: The Gibbs Sampler**

The Gibbs sampler associated with (20) of the main text requires one to compute certain quantities. These values are listed in Table 3. To aid the reader, we provide step-by-step instructions for implementing the Gibbs sampler associated with (20) of the main text as follows.

1. Initialize \( \mathbf{\beta}, \mathbf{\eta}, \mathbf{\xi}, \mathbf{c}, \mathbf{c}_\beta, \mathbf{c}_{\xi}, \{\mathbf{v}_i\}, \alpha_\eta, \alpha_\xi, \kappa_\eta, \) and \( \kappa_\xi \). Denote these initializations with \( \mathbf{\beta}^{[0]}, \mathbf{\eta}^{[0]}, \mathbf{\xi}^{[0]}, \mathbf{c}^{[0]}, \mathbf{c}_\beta^{[0]}, \mathbf{c}_\xi^{[0]}, \{\mathbf{v}_i^{[0]}\}, \alpha_\eta^{[0]}, \alpha_\xi^{[0]}, \kappa_\eta^{[0]}, \) and \( \kappa_\xi^{[0]} \). Set \( m = 1 \).
2. Set $\beta^{[m]}$ equal to a draw from $\text{CM}_c(\mu_\beta, H_\beta, \alpha_\beta^*, \kappa_\beta^*)$ using Theorem 2, where $\mu_\beta$, $H_\beta$, $\alpha_\beta^*$, and $\kappa_\beta^*$ are computed using Table 3 and the values of $\eta^{[m-1]}$ and $\xi^{[m-1]}$.

3. Set $c_\beta^{[m]}$ equal to a draw from $\text{CM}_c(\mu_{c,\beta}, H_{c,\beta}, \alpha_{c,\beta}^*, \kappa_{c,\beta}^*)$ using Theorem 2, where $\mu_{c,\beta}$, $H_{c,\beta}$, $\alpha_{c,\beta}^*$, and $\kappa_{c,\beta}^*$ are computed using Table 3 and the values of $\beta^{[m-1]}$. We have found that $c_\beta$ is weakly identifiable, and hence, truncating the support of the prior or using an informative prior often leads to better results.

4. Set $\eta^{[m]}$ equal to a draw from $\text{CM}_c(\mu_\eta, H_\eta, \alpha_\eta^*, \kappa_\eta^*)$ using Theorem 2, where $\mu_\eta$, $H_\eta$, $\alpha_\eta^*$, and $\kappa_\eta^*$ are computed using Table 3 and the values of $\beta^{[m]}$, $\xi^{[m-1]}$, $\alpha_\eta^{[m-1]}$, $\kappa_\eta^{[m-1]}$, and $\{v_i^{[m-1]}\}$.

5. Set $c^{[m]}$ equal to a draw from $\text{CM}_c(\mu_c, H_c, \alpha_c^*, \kappa_c^*)$ using Theorem 2, where $\mu_c$, $H_c$, $\alpha_c^*$, and $\kappa_c^*$ are computed using Table 3 and the values of $\eta^{[m-1]}$. We have found that $c$ is weakly identifiable, and hence, truncating the support of the prior or using an informative prior often leads to better results.

6. Set $\xi^{[m]}$ equal to a draw from $\text{CM}_c(\mu_\xi, H_\xi, \alpha_\xi^*, \kappa_\xi^*)$ using Theorem 2, $H_\xi$, $\alpha_\xi^*$, and $\kappa_\xi^*$ are computed using Table 3 and the values of $\eta^{[m]}$, $\beta^{[m]}$, $\alpha_\xi^{[m-1]}$, and $\kappa_\xi^{[m-1]}$.

7. Set $c_\xi^{[m]}$ equal to a draw from $\text{CM}_c(\mu_{c,\xi}, H_{c,\xi}, \alpha_{c,\xi}^*, \kappa_{c,\xi}^*)$ using Theorem 2, where $\mu_{c,\xi}$, $H_{c,\xi}$, $\alpha_{c,\xi}^*$, and $\kappa_{c,\xi}^*$ are computed using Table 3 and the values of $\xi^{[m-1]}$. We have found that $c_\xi$ is weakly identifiable, and hence, truncating the support of the prior or using an informative prior often leads to better results.

8. For $i = 2, \ldots, r$ set $v_i^{[m]}$ equal to a value generated from $\text{CM}_c(\mu_{\gamma,i}, H_{\gamma,i}, \alpha_{\gamma,i}^*, \kappa_{\gamma,i}^*)$, where $\mu_{\gamma,i}$, $H_{\gamma,i}$, $\alpha_{\gamma,i}^*$, and $\kappa_{\gamma,i}^*$ are given in Table 3, and are computed using $\eta^{[m]}$, $\alpha_\eta^{[m-1]}$, and $\kappa_\eta^{[m-1]}$.

9. Use a slice sampler (or Metropolis) to set $\alpha_\beta^{[m]}$ and $\kappa_\beta^{[m]}$ to a value generated from the pdf:

$$f(\alpha_\beta, \kappa_\beta) \propto \exp \left[ (\gamma_{\beta,1} + J_{1,p}v_{\beta}^{[m]} - c_\beta - J_{1,g} \Psi(M_{\beta}^{[m]} \beta^{[m]} - c_\delta J_{\gamma,1}) ) \kappa_\beta \right]$$

$$- (\rho_\beta + g) \log \left\{ \frac{1}{K(\alpha_\beta, \kappa_\beta)} \right\},$$

where $g = p$ if no boundary value update is needed, $g = n + p$ if $\psi = \psi_3$, and $c = 2n + p$ if $\psi = \psi_2$. 

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10. Use a slice sampler (or Metropolis) to set $\alpha_{\eta}^{[m]}$ and $\kappa_{\eta}^{[m]}$ to a value generated from the pdf:

$$f(\alpha_{\eta}, \kappa_{\eta} | \cdot) \propto \exp \left[ (\gamma_{\eta,1} + J_{1,r} V^{[m]-1} \eta^{[m]}) \alpha_{\eta} + \left\{ \gamma_{\eta,2} - J_{1.1.g} \psi(M^{[m]} \eta^{[m]} - cJ_{g,1}) \right\} \kappa_{\eta} 
- (\rho_{\eta} + g) \log \left\{ \frac{1}{K(\alpha_{\eta}, \kappa_{\eta})} \right\} \right],$$

where $g = r$ if no boundary value update is needed, $g = n + r$ if $\psi = \psi_3$, and $g = 2n + r$ if $\psi = \psi_2$.

11. Using a slice sampler (or Metropolis) to set $\alpha_{\zeta}^{[m]}$ and $\kappa_{\zeta}^{[m]}$ equal to values generated from the pdf:

$$f(\alpha_{\zeta}, \kappa_{\zeta} | \cdot) \propto \exp \left[ (\gamma_{\zeta,1} + J_{1,n} \zeta^{[m]}) \alpha_{\zeta} + \left\{ \gamma_{\zeta,2} - J_{1.2n} \psi(M_{\zeta} \zeta^{[m]} - c\zeta J_{g,1}) \right\} \kappa_{\zeta} 
- (\rho_{\zeta} + g) \log \left\{ \frac{1}{K(\alpha_{\zeta}, \kappa_{\zeta})} \right\} \right],$$

where $g = n$ if no boundary value update is needed, $g = 2n$ if $\psi = \psi_3$, and $g = 3n$ if $\psi = \psi_2$.

12. Set $m = m + 1$

13. Repeat steps 2 through 12 until convergence of the Gibbs sampler.

It is straightforward to adjust this Gibbs sampler in variety of ways to be more appropriate for a particular problem. For example, one could consider different hyperparameters, different basis functions $\{\phi_j\}$, update the shape and scale of the prior on $V^{-1}$, and assume heterogeneous DY parameters associated with $\beta$, $\eta$, and $\xi$.

It is important to note that many software packages have built in functions to simulate from beta and gamma distributions, which are needed when $j = k = 2$ and $j = k = 3$, respectively. However, it is common for the Gibbs sampler to produce small values of shape and scale parameters, which may lead to computational errors when simulating from a beta or a gamma distribution. In this setting, we simulate beta and gamma random variables using strategies outlined in Devroye (1986, pgs. 181, 182, and 419). Additionally, if the shape and scale parameters are so small (i.e., close to zero) that it is not possible to simulate the beta and gamma random variables using the techniques in Devroye (1986), we reject the proposed sample. However, after a sufficient burn-in period of the Gibbs sampler, the acceptance rate is approximately equal to one.

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