Nested Refinements for Dynamic Languages

Ravi Chugh  Patrick M. Rondon  Ranjit Jhala
University of California, San Diego
{rchugh, prondon, jhala}@cs.ucsd.edu

Abstract

Programs written in dynamic languages make heavy use of features — run-time type tests, value-indexed dictionaries, polymorphism, and higher-order functions — that are beyond the reach of type systems that employ either purely syntactic or purely semantic reasoning. We present a core calculus, System D, that merges these two modes of reasoning into a single powerful mechanism of nested refinement types wherein the typing relation is itself a predicate in the refinement logic. System D coordinates SMT-based logical implication and syntactic subtyping to automatically typecheck sophisticated dynamic language programs. By coupling nested refinements with McCarthy’s theory of finite maps, System D can precisely reason about the interaction of higher-order functions, polymorphism, and dictionaries. The addition of type predicates to the refinement logic creates a circularity that leads to unique technical challenges in the metatheory, which we solve with a novel stratification approach that we use to prove the soundness of System D.

1. Introduction

So-called dynamic languages like JavaScript, Python, Racket, and Ruby are popular as they allow developers to quickly put together scripts without having to appease a static type system. However, these scripts quickly grow into substantial code bases that would be much easier to maintain, refactor, evolve and compile, if only they could be corralled within a suitable static type system.

The convenience of dynamic languages comes from their support of features like run-time type testing, value-indexed finite maps (i.e., dictionaries), and duck typing, a form of polymorphism where functions operate over any dictionary with the appropriate keys. As the empirical study in [13] shows, programs written in dynamic languages make heavy use of these features, and their safety relies on invariants which can only be established by sophisticated reasoning about the flow of control, the run-time types of values, and the contents of data structures like dictionaries.

The following code snippet, adapted from the popular Dojo Javascript framework [31], illustrates common dynamic features:

```javascript
let onto callbacks f obj =
if f = null then
new List(obj, callbacks)
else
let cb = if tag f = "Str" then obj[f] else f in
new List(fun () -> cb obj, callbacks)
```

The function `onto` is used to register callback functions to be called after the DOM and required library modules have finished loading. The author of `onto` went to great pains to make it extremely flexible in the kinds of arguments it takes. If the `obj` parameter is provided but `f` is not, then `obj` is the function to be called after loading. Otherwise, both `f` and `obj` are provided, and either: (a) `f` is a string, `obj` is a dictionary, and the (function) value corresponding to key `f` in `obj` is called with `obj` as a parameter after loading; or (b) `f` is a function which is called with `obj` as a parameter after loading. To verify the safety of this program, and dynamic code in general, a type system must reason about dynamic type tests, control flow, higher-order functions, and heterogeneous, value-indexed dictionaries.

Current type systems are not expressive enough to support the full spectrum of reasoning required for dynamic languages. Syntactic systems use advanced type-theoretic constructs like structural types [3], row types [13], intersection types [12], and union types [13, 22] to track invariants of individual values. Unfortunately, such techniques cannot reason about value-dependent relationships between program variables, as is required, for example, to determine the specific types of the variables `f` and `obj` in `onto`. Semantic systems like [3] support such reasoning by using logical predicates to describe invariants over program variables. Unfortunately, such systems require a clear (syntactic) distinction between complex values that are typed with arrows, type variables etc., and base values that are typed with predicates [11, 18, 27]. Hence, they cannot support the interaction of complex values and value-indexed dictionaries that is ubiquitous in dynamic code, for example in `onto`, which can take as a parameter a dictionary containing a function value.

Our Approach. We present System D, a core calculus that supports fully automatic checking of dynamic idioms. In System D all values are described uniformly by formulas drawn from a decidable, quantifier-free refinement logic. Our first key insight is that to reason precisely about complex values (e.g. higher-order functions) nested deeply inside structures (e.g. dictionaries), we require a single new mechanism called nested refinements wherein syntactic types (resp. the typing relation) may be nested as special type terms (resp. type predicates) inside the refinement logic. Formally, the refinement logic is extended with atomic formulas of the form $x :: U$ where $U$ is a type term, "$::" (read "has type") is a binary, uninterpreted predicate in the refinement logic, and where the formula states that the value $x$ "has the type" described by the term $U$. This unifying insight allows to us to express the invariants in idiomatic dynamic code like `onto` — including the interaction between higher-order functions and dictionaries — while staying within the boundaries of decidability.

Expressiveness. The nested refinement logic underlying System D can express complex invariants between base values and richer values. For example, we may disjoin two tag-equality predicates

$$\{ \nu | \text{tag}(\nu) = \text{"Int"} \lor \text{tag}(\nu) = \text{"Str"} \}$$

to type a value $\nu$ that is either an integer or a string; we can then track control flow involving the dynamic type tag-lookup function `tag` to ensure that the value is safely used at either more specific type. To describe values like the argument $f$ of the `onto` function we combine tag-equality predicates with the type predicate. We can give $f$ the type

$$\{ \nu | \nu = \text{null} \lor \text{tag}(\nu) = \text{"Str"} \lor \nu :: \text{Top} \rightarrow \text{Top} \}$$
where Top is an abbreviation for \{ν | \text{true}\}, which is a type that describes all values. Notice the uniformity — the types nested within this refinement formula are themselves refinement types.

Our second key insight is that dictionaries are finite maps, and so we can precisely type dictionaries with refinement formulas drawn from the (decidable) theory of finite maps [20, 21]. In particular, McCarthy’s two operators — \text{sel}(x, a), which corresponds to the contents of the map \text{at} the address \text{a}, and \text{upd}(x, a, v), which corresponds to the new map obtained by updating \text{x} at the address \text{a} with the value \text{v} — are precisely what we need to describe reads from and updates to dictionaries. For example, we can write

\{ν | \text{tag}(ν) = \text{“Dict”} \land \text{tag}(\text{sel}(ν, y)) = \text{“Int”}\}

to type dictionaries \text{ν} that have (at least) an integer field \text{y}, where \text{y} is a program variable that dynamically stores the key with which to index the dictionary. Even better, since we have nested function types into the refinement logic, we can precisely specify, for the first time, combinations of dictionaries and functions. For example, we can write the following type for obj

\{ν | \text{tag}(\text{ν}) = \text{“Str”} \Rightarrow \text{sel}(ν, \text{ν}) :: \text{Top} \Rightarrow \text{Top}\}

to describe the second portion of the onto specification, all while staying within a decidable refinement logic. In a similar manner, we show how nested refinements support polymorphism, datatypes, and even a form of bounded quantification.

Subtyping. The huge leap in expressiveness yielded by nesting types inside refinements is accompanied by some unique technical challenges. The first challenge is that because we nest complex types (e.g. arrows) as uninterpreted terms in the logic, subtyping (e.g. between arrows) cannot be carried out solely via the usual syntactic decomposition into SMT queries [11, 27]. (A higher-order refinement logic would solve this problem, but that would preclude algorithmic checking; we choose the uninterpreted route precisely to relieve the SMT solver of higher-order reasoning!) We surmount this challenge with a novel decomposition mechanism where subtyping between types, syntactic type terms, and refinement formulas are defined inter-dependently, thereby using the logical structure of the refinement formulas to divide the labor of subtyping between the SMT solver for ground predicates (e.g. equality, uninterpreted functions, arithmetic, maps, etc.) and classical syntactic rules for type terms (e.g. arrows, type variables, datatypes, etc.).

Soundness. The second challenge is that the inter-dependency between the refinement logic and the type system renders the standard proof techniques for (refinement) type soundness inapplicable. In particular, we illustrate how uninterpreted type predicates break the usual substitution property and how nesting makes it difficult to define a type system that is well-defined and enjoys this property. We meet this challenge with a new proof technique: we define an infinite family of increasingly precise systems and prove soundness of the family, of which System D is a member, thus establishing the soundness of System D.

Contributions. To sum up, we make the following contributions:

- We define an algorithmic version of the type system with local type inference that we implement in a prototype checker. Thus, by carefully orchestrating the interplay between syntactic and SMT-based subtyping, the nested refinement types of System D enable, for the first time, the automatic static checking of features found in idiomatically dynamic code.

2. Overview

We start with a series of examples that give an overview of our approach. First, we show how by encoding types using logical refinements, System D can reason about control flow and relationships between program variables. Next, we demonstrate how nested refinements enable precise reasoning about values of complex types. After that, we illustrate how System D uses refinements over the theory of finite maps to analyze value-indexed dictionaries. We conclude by showing how these features combine to analyze the sophisticated invariants in idiomatically dynamic code.

Notation. We use the following abbreviations for brevity.

\text{Top}(x) \equiv \text{true}

\text{Int}(x) \equiv \text{tag}(x) = \text{“Int”}

\text{Bool}(x) \equiv \text{tag}(x) = \text{“Bool”}

\text{Str}(x) \equiv \text{tag}(x) = \text{“Str”}

\text{Dict}(x) \equiv \text{tag}(x) = \text{“Dict”}

\text{IorB}(x) \equiv \text{Int}(x) \lor \text{Bool}(x)

We abuse notation to use the above as abbreviations for refinement types; for each of the unary abbreviations \text{T} defined above, an occurrence \text{without} the parameter denotes the refinement type \{ν | \text{T}(ν)\}. For example, we write \text{Int} as an abbreviation for \{ν | \text{tag}(ν) = \text{“Int”}\}. Recall that function values are also described by refinement formulas (containing type predicates). We often write arrows outside refinements to abbreviate the following:

\text{x : T}_1 \rightarrow \text{T}_2 \equiv \{ν | ν :: x : \text{T}_1 \rightarrow \text{T}_2\}

We write \text{T}_1 \rightarrow \text{T}_2 when the return type \text{T}_2 does not refer to \text{x}.

2.1 Simple Refinements

To warm up, we show how System D describes all types through refinement formulas, and how, by using an SMT solver to discharge the subtyping (implication) queries, System D makes short work of value- and control flow-sensitive reasoning [13, 14].

Ad-Hoc Unions. Our first example illustrates the simplest dynamic idiom: programs which operate on ad-hoc unions. The function \text{negate} takes an integer or boolean and returns its negation:

\text{let negate =}
\begin{verbatim}
if tag x = "Int" then 0 - x else not x
\end{verbatim}

In System D we can ascribe to this function the type

\text{negate :: IorB \rightarrow IorB}

which states that the function accepts an integer or boolean argument and returns either an integer or boolean result.

To establish this, System D uses the standard means of reasoning about control flow in refinement-based systems [26], namely strengthening the environment with the guard predicate when processing the then-branch of an if-expression and the negation of the guard predicate for the else-branch. Thus, in the then-branch, the environment contains the assumption that \text{tag}(x) = \text{“Int”}, which allows System D to verify that the expression \text{0 - x} is well-typed. The return value has the type \{ν | \text{tag}(ν) = \text{“Int”} \land ν = \text{0 - x}\}. This type is a subtype of \text{IorB} as the SMT solver can prove that
tag(ν) = “Int” and ν = 0 − x implies tag(ν) = “Int” ∨ tag(ν) = “Bool”. Thus, the return value of the then-branch is deduced to have type IorB.

On the other hand, in the else-branch, the environment contains the assumption ¬tag(x) = “Int”). By combining this with the assumption about the type of negate’s input, tag(x) = “Int” ∨ tag(x) = “Bool”, the SMT solver can determine that tag(x) = “Bool”. This allows our system to type check the call to

\[ \text{not} :: \text{Bool} \rightarrow \text{Bool}, \]

which establishes that the value returned in the else branch has type IorB. Thus, our system determines that both branches return a value of type IorB, and thus that negate meets its specification.

**Dependent Unions.** System D’s use of refinements and SMT solvers enable expressive relational specifications that go beyond previous techniques [13, 34]. While negate takes and returns ad-hoc unions, there is a relationship between its input and output: the output is an integer (resp. boolean) if the input is an integer (resp. boolean). We represent this in System D as

\[ \text{negate} :: x : \text{IorB} \rightarrow \{ ν \mid \text{tag}(ν) = \text{tag}(x) \} \]

That is, the refinement for the output states that its tag is the same as the tag of the input. This function is checked through exactly the same analysis as before; the tag test ensures that the environment in the then- (resp. else-) branch implies that x and the returned value are both Int (resp. Bool). That is, in both cases, the output value has the same tag as the input.

### 2.2 Nested Refinements

So far, we have seen how old-fashioned refinement types (where the predicates refine base values [5, 15, 23, 27]) can be used to check ad-hoc unions over base values. However, a type system for dynamic languages must be able to express invariants about values and function types with equal ease. We accomplish this in System D by adding types (resp. the typing relation) to the refinement logic as nested type terms (resp. type predicates).

However, nesting raises a rather tricky problem: with the typing relation included in the refinement logic, subtyping can no longer be carried out entirely via SMT implication queries [5]. We solve this problem with a new subtyping rule that extracts type terms from refinements to enable syntactic subtyping for nested types.

Consider the function maybeApply which takes an integer x and a value f which is either null or a function over integers:

\[
\text{let} \_ \_ = \text{maybeApply} \ 42 \ \text{negate}
\]

At the call to maybeApply we must show that the actuals are subtypes of the formals, i.e. that the two subtyping relationships

\[
\Gamma_1 \vdash \{ ν \mid ν = 42 \} \sqsubseteq \text{Int}
\]

\[
\Gamma_1 \vdash \{ ν \mid ν = \text{null} \} \sqsubseteq \{ ν \mid ν = \text{null} \} \sqsubseteq \nu : U_1
\]

hold, where \( \Gamma_1 = \text{negate} : \{ ν \mid ν : U_0 \} \), maybeApply \( \cdots \) and \( U_0 = x : \text{IorB} \rightarrow \{ ν \mid \text{tag}(ν) = \text{tag}(x) \} \). Alas, while the SMT solver can make short work of the first obligation, it cannot be used to discharge the second via implication: the “real” types that must be checked for subsumption, namely, \( U_0 \) and \( U_1 \), are embedded as totally unrelated terms in the refinement logic!

Once again, extraction rides to the rescue. We show that all subtyping checks of the form \( \Gamma \vdash \{ ν \mid p \} \sqsubseteq \{ ν \mid q \} \) can be reduced to a finite number of sub-goals of the form:

\[ (\text{"type predicate-free"}) \quad \Gamma_0 \Rightarrow p' \quad \text{or} \quad (\text{"type predicate"}) \quad \Gamma_0 \Rightarrow x : U \]

The former kind of goal has no type predicates and can be directly discharged via SMT. For the latter, we use extraction to find the finitely many type terms \( U \) that flow to \( p' \). (If there are none, the check fails.) For each \( U \), we use syntactic subtyping to verify that the corresponding type is subsumed by (the type corresponding to) \( U \) under \( \Gamma_0 \).

In our example, the goal reduces to proving either

\[ \Gamma_0 \Rightarrow ν = \text{null} \quad \text{or} \quad \Gamma_0 \Rightarrow ν : U_1 \]

where \( \Gamma_0 = \Gamma_1, ν = \text{negate} \). The former implication contains no type predicates, so we attempt to prove it by querying the SMT solver. The solver tells us that the query is not valid, so we turn to the latter implication. The extraction procedure uses the SMT solver to deduce that, under \( \Gamma_0 \), the type term \( U_0 \) flows into \( ν \). Thus, all that remains is to retrieve the definition of \( U_0 \) and \( U_1 \) and check

\[ \Gamma_0 \vdash x : \text{IorB} \rightarrow \{ ν \mid \text{tag}(ν) = \text{tag}(x) \} \sqsubseteq \text{Int} \rightarrow \text{Int} \]

which follows via standard syntactic refinement subtyping [13], thereby checking the client’s call. Thus, by carefully interleaving SMT implication and syntactic subtyping, System D enables, for the first time, the nesting of rich types within refinements.

### 2.3 Dictionaries

Next, we show how nested refinements allow System D to precisely check programs that manipulate dynamic dictionaries. In essence, we demonstrate how structural subtyping can be done via nested
reduction formulas over the theory of finite maps \cite{SAP,OD}. We introduce several abbreviations for dictionaries.

\[
\begin{align*}
Sel(x, y, z) & \equiv \text{has}(x, y) \land sel(x, y) = z \\
Fld(x, y, \text{Int}) & \equiv \text{Dict}(x) \land Str(y) \land \text{has}(x, y) \land \text{Int}(sel(x, y)) \\
Fld(x, y, U) & \equiv \text{Dict}(x) \land Str(y) \land \text{has}(x, y) \land sel(x, y) : U
\end{align*}
\]

The last abbreviation states that the type of a field is a syntactic type term \(U\) (e.g. an arrow).

**Dynamic Lookup.** SMT-based structural subtyping allows System D to support the common idiom of dynamic field lookup and update, where the field name is a value computed at run-time. Consider the following function:

```plaintext
let getCount t c = if has t c then toInt (t[c]) else 0
```

The function `getCount` uses the primitive operation

\[
\begin{align*}
\text{has} & :: \text{Dict} \to k : \text{Str} \to \{\nu | \text{Bool}(\nu) \land \nu = \text{true} \iff \text{has}(d, k)\} \\
\text{sel} & :: \text{Dict} \to k : \text{Str} \to \{\nu | \nu = \text{sel}(d, k)\}
\end{align*}
\]

to check whether the key \(c\) exists in \(t\). The refinement for the input \(d\) expresses the precondition that \(d\) is a dictionary, while the refinement for the key \(k\) expresses the precondition that \(k\) is a string. The refinement of the output expresses the postcondition that the result is a boolean value which is true if and only if \(d\) has a binding for the key \(k\), expressed in our refinements using \(\text{has}(d, k)\), a predicate in the theory of maps that is true if and only if there is a binding for key \(k\) in the map \(d\).

The `dictionary lookup` \(t[c]\) is desugared to \(get t c\) where the primitive operation `get` has the type

\[
\begin{align*}
\text{get} & :: d : \text{Dict} \to k : \text{Str} \to \{\nu | \text{Str}(\nu) \land \text{has}(d, k)\} \to \{\nu | \nu = \text{get}(d, k)\}
\end{align*}
\]

and \(\text{sel}(d, k)\) is an operator in the theory of maps that returns the binding for key \(k\) in the map \(d\). The refinement for the key \(k\) expresses the precondition that it is a string value in the domain of the dictionary \(d\). Similarly, the refinement for the output asserts the postcondition that the value is the same as the contents of the map at the given key.

The function `getCount` first tests the dictionary \(t\) has a binding for the key \(c\); if so, it is read and its contents are converted to an integer using the function `toInt`, of type \(\text{Top} \to \text{Int}\). Note that the if-guard strengthens the environment under which the lookup appears with the fact \(\text{has}(t, c)\), ensuring the safety of the lookup. If \(t\) does not contain the key \(c\), the default value \(0\) is returned. Both branches are thus verified to have type \(\text{Int}\), so System D verifies that `getCount` has the type `getCount :: \text{Dict} \to \text{Str} \to \text{Int}`.

**Dynamic Update.** Dually, to allow dynamic updates, System D includes a primitive

\[
\begin{align*}
\text{set} & :: d : \text{Dict} \to k : \text{Str} \to x : \text{Top} \\
& \to \{\nu | \text{EqMod}(\nu, d, k) \land \text{Sel}(\nu, k, x)\}
\end{align*}
\]

where \(\text{EqMod}(d_1, d_2, k)\) abbreviates a predicate that stipulates that \(d_1\) is identical to \(d_2\) at all keys \(except\) \(k\). Thus, the `set` primitive returns a dictionary that is identical to \(d\) everywhere \(except\) that it maps the key \(k\) to \(x\). The following illustrates how `set` can be used to update (or extend) a dictionary:

```plaintext
let incCount t c = let newcount = 1 + getCount t c in let res = set t c newcount in res
```

We give the function `incCount` the type

\[
d : \text{Dict} \to c : \text{Str} \to \{\nu | \text{EqMod}(\nu, d, c) \land \text{Fld}(\nu, c, \text{Int})\}
\]

The output type of `getCount` allows System D to conclude that `newcount :: \text{Int}`. From the type of `set`, System D deduces

\[
\text{res} :: \{\nu | \text{EqMod}(\nu, t, c) \land \text{Sel}(\nu, c, \text{newcount})\}
\]

which is a subtype of the output type of `incCount`. Next, consider

```plaintext
let d0 = {"files" = 42 } 
let d1 = incCount d0 "dirs" 
```

System D verifies that

\[
\begin{align*}
d0 & :: \{\nu | \text{Fld}(\nu, "files", \text{Int})\} \\
d1 & :: \{\nu | \text{Fld}(\nu, "files", \text{Int}) \land \text{Fld}(\nu, "dirs", \text{Int})\}
\end{align*}
\]

and, hence, the field lookups return \(\text{Ints}\) that can be safely added.

### 2.4 Type Constructors

Next, we use nesting and extraction to enrich System D with data structures, thereby allowing for very expressive specifications. In general, System D supports arbitrary user-defined datatypes, but to keep the current discussion simple, let us consider a single type constructor `List[T]` for representing unbounded sequences of \(T\)-values. Informally, an expression of type `List[T]` is either a special `null` value or a dictionary with a “hd” key of type \(T\) and a “tl” key of type `List[T]`. As for arrows, we use the following notation to write list types outside of refinements.

\[
\text{List}[T] \equiv \{\nu | \nu :: \text{List}[T]\}
\]

**Recursive Traversal.** Consider a textbook recursive function that takes a list of arbitrary values and concatenates the strings:

```plaintext
let rec concat sep xs = if xs = null then "" else let hd = xs["hd"] in let tl = xs["tl"] in if tag hd != "Str" then concat sep tl else if tl != null then hd ^ sep ^ concat sep tl else hd
```

We ascribe the function the type `concat :: \text{Str} \to \text{List}[\text{Top}] \to \text{Str}`. The null test ensures the safety of the “hd” and “tl” accesses and the tag test ensures the safety of the string concatenation using the techniques described above.

**Nested Ad-Hoc Unions.** We can now define ad-hoc unions over constructed types by simply nesting `List[]` as a type term in the refinement logic. The following illustrates a common Python idiom when an argument is either a single value or a list of values:

```plaintext
let runTest cmd fail_codes = let status = syscall cmd in if tag fail_codes = "Int" then not (status = fail_codes) else not (listMem status fail_codes)
```

Here, `listMem :: \text{Top} \to \text{List}[\text{Top} \to \text{Bool}]` and `syscall :: \text{Str} \to \text{Int}`. The input \(cmd\) is a string, and `fail_codes` is either a single integer or a list of integer failure codes. Because we nest `List[]` as a type term in our logic, we can use the same kind of type extraction reasoning as we did for `maybeApply` to ascribe `runTest` the type

\[
\text{runTest} :: \text{Str} \to \{\nu | \text{Int}(\nu) \lor \nu :: \text{List}[\text{Int}]\} \to \text{Bool}
\]

### 2.5 Parametric Polymorphism

Similarly, we can add parametric polymorphism to System D by simply treating type variables \(A, B, \text{etc.}\) as (uninterpreted) type
terms in the logic. As before, we use the following notation to write type variables outside of refinements.

\[ A \equiv \forall \nu. \nu :: A \]

**Generic Containers.** We can compose the type constructors in the ways we all know and love. Here is list `map` in System D:

```ocaml
let rec map f xs =
  if xs = null then null
  else new List(f xs["hd"], map f xs["tl"])
```

(Of course, pattern matching would improve matters, but we are merely trying to demonstrate how much can be — and is! — achieved with dictionaries.) By combining extraction with the reasoning used for `concat`, it is easy to check that

\[ \text{map} :: \forall A, B. (A \rightarrow B) \rightarrow \text{List}[A] \rightarrow \text{List}[B] \]

Note that type abstractions are automatically inserted where a function is ascribed a polymorphic type.

**Predicate Functions.** Consider the list filter function:

```ocaml
let rec filter f xs =
  if xs = null then null
  else if not (f xs["hd"])
    then filter f (xs["tl"])
  else new List(xs["hd"], filter f xs["tl"])
```

In System D, we can ascribe `filter` the type

\[ \forall A, B. (x : A \rightarrow \nu. \nu = \text{true} \Rightarrow x :: B) \rightarrow \text{List}[A] \rightarrow \text{List}[B], \]

Note that the return type of the predicate, \( f \), tells us what type is satisfied by values \( x \) for which \( f \) returns \text{true}, and the return type of `filter` states that the items `filter` returns all have the type implied by the predicate \( f \). Thus, the general mechanism of nested refinements subsumes the kind of reasoning performed by specialized techniques like latent predicates [14].

**Bounded Quantification.** Nested refinements enable a form of bounded quantification. Consider the function

```ocaml
let dispatch d f = d[f] d
```

The function `dispatch` works for any dictionary \( d \) of type \( A \) that has a key \( f \) bound to a function that maps values of type \( A \) to values of type \( B \). We can specify this via the dependent signature

\[ \forall A, B. d :: \{\nu. \text{Dict}(\nu) \land \nu :: A \} \rightarrow \{\nu. \text{Fld}(d, \nu, A \rightarrow B) \} \rightarrow B \]

Note that there is no need for explicit type bounds; all that is required is the conjunction of nested refinements.

### 2.6 All Together Now

With the tools we've developed in this section, System D is now capable of type checking sophisticated code from the wild. The original source code for the following can be found in Appendix C.

**Unions, Generic Dispatch, and Polymorphism.** We now have everything we need to type the motivating example from the introduction, `map`, which combined multiple dynamic idioms: dynamic fields, tag-tests, and the dependency between nested dictionary functions and their arguments. Nested refinements let us formalize the flexible interface for `onto` given in the introduction:

\[ \forall A. \text{callbacks} :: \text{List}[\text{Top} \rightarrow \text{Top}] \]

\[
\begin{align*}
  & \rightarrow f :: \{\nu. \nu = \text{null} \lor \text{Str}(\nu) \lor \nu :: A \rightarrow \text{Top} \} \\
  & \rightarrow \text{obj} :: \{\nu. \nu :: A \land (f = \text{null} \Rightarrow \nu :: \text{Top} \rightarrow \text{Top}) \}
  \land (\text{Str}(f) \Rightarrow \text{Fld}(\nu, f, A \rightarrow \text{Top}))) \\
  & \rightarrow \text{List}[\text{Top} \rightarrow \text{Top}] 
\end{align*}
\]

Using reasoning similar to that used in the previous examples, System D checks that `onto` enjoys the above type, where the specification for `obj` is enabled by the kind of bounded quantification described earlier.

**Reflection.** Finally, to round off the overview, we present one last example that shows how all the features presented combine to allow System D to statically type programs that introspect on the contents of dictionaries. The function `toXML` shown below is adapted from the Python 3.2 standard library’s `plistlib.py`:

```ocaml
let rec toXML x =
  if tag x = "Bool" then
    if x then element "true" null
    else element "false" null
  else if tag x = "Int" then
    element "integer" (intToStr x)
  else if tag x = "Str" then
    element "string" x
  else if tag x = "Dict" then
    let ks = keys x in
    let vs = map (\{v| Str(v) and has(x, v)\} Str
      (fun k -> element "key" k \* toXML x[k])) ks in
    "<data>" \* concat "n" vs "</data>"
  else element "function" null
```

The function takes an arbitrary value and renders it as an XML string, and illustrates several idiomatic uses of dynamic features. If we give the auxiliary function `intToStr` the type `Int \rightarrow Str` and `element` the type `Str \rightarrow \nu. \nu = \text{null} \lor \text{Str}(\nu) \rightarrow \text{Str},` we can verify that

\[ \text{toXML} :: \text{Top} \rightarrow \text{Str} \]

Of especial interest is the dynamic field lookup \( x[k] \) used in the function passed to `map` to recursively convert each binding of the dictionary to XML. The primitive operation `keys` has the type

\[ \text{keys} :: d : \text{Dict} \rightarrow \text{List}[[\nu. \text{Str}(\nu) \land \text{has}(d, \nu)] \]

that is, it returns a list of string keys that belong to the input dictionary. Thus, `ks` has type `List[[\nu. \text{Str}(\nu) \land \text{has}(x, \nu)]]`, which enables the call to map to typecheck, since the body of the argument is checked in an environment where \( k \mapsto \{\nu. \text{Str}(\nu) \land \text{has}(x, \nu)\} \), which is the type that `A` is instantiated with. This binding suffices to prove the safety of the dynamic field access. The control flow reasoning described previously uses the tag tests guarding the other cases to prove each of them safe.

### 3. Syntax and Semantics

We begin with the syntax and evaluation semantics of System D. Figure 1 shows the syntax of values, expressions, and types.

**Values.** Values \( w \) include variables constants, functions, type functions, dictionaries, and records created by type constructors. The set of constants \( c \) include base values like integer, boolean, and string constants, the empty dictionary \( \{\} \), and \text{null}. Logical values \( lw \) are all values and applications of primitive function symbols \( F \), such as addition \( + \) and dictionary selection `sel`, to logical values. The constant `tag` allows introspection on the type tag of a value at run-time. For example,

\[ \text{tag}(3) \equiv \text{"Int"} \quad \text{tag(true)} \equiv \text{"Bool"} \]

\[ \text{tag}(\{\text{"Int"}\}) \equiv \text{"Str"} \quad \text{tag(\text{\lambda} x. c)} \equiv \text{"Fun"} \]

\[ \text{tag}(\{\} \text{) \equiv \text{"Dict"} \quad \text{tag(\text{\lambda} A. c)} \equiv \text{"TFunc"} \]

**Dictionaries.** A dictionary \( w_1 \leftrightarrow \{w_2 \mapsto w_3\} \) extends the dictionary \( w_1 \) with the binding from string \( w_2 \) to value \( w_3 \). For example,
the dictionary mapping “x” to 3 and “y” to true is written
\[
\{ \} \{ “x” \mapsto 3 \} \{ “y” \mapsto \text{true} \}.
\]
The set of constants also includes operators for extending dictionaries and accessing their fields. The function get is used to access dictionary fields and is defined
\[
\begin{align*}
get (w ++ \{ “x” \mapsto w_x \}) & \quad “x” \doteq w_x \\
get (w ++ \{ “y” \mapsto w_y \}) & \quad “y” \doteq \text{get } w “x”
\end{align*}
\]
The function has tests for the presence of a field and is defined
\[
\begin{align*}
\text{has (} w ++ \{ “y” \mapsto w_y \} & \quad “x” \doteq \text{has } w “x” \\
\text{has (} w ++ \{ “x” \mapsto w_x \} & \quad “x” \doteq \text{true} \\
\text{has (} {} & \quad “x” \doteq \text{false}
\end{align*}
\]
The function set updates the value bound to a key and is defined
\[
\text{set } d k w \doteq d ++ \{ k \mapsto w \}
\]

**Expressions.** The set of expressions e consists of values, function applications, type instantiations, if-then-else expressions, and let-bindings. We use an A-normal presentation so that we need only define substitution of values (not arbitrary expressions) into types.

**Types.** We stratify types into monomorphic types T and polymorphic type schemes ∀A. S. In System D, a type T is a refinement type of the form [ν | p], where p is a refinement formula, and is read “ν such that p.” The values of this type are all values w such that the formula p[w/ν] “is true.” What this means, formally, is core to our approach and will be considered in detail in Section 5.

**Refinement Formulas.** The language of refinement formulas includes predicates P, such as the equality predicate and dictionary predicates has and sel, and the usual logical connectives. For example, the type of integers is \{ν | tag(ν) = “Int”\}, which we abbreviate to Int. The type of positive integers is \{ν | tag(ν) = “Int” \∧ ν > 0\}

and the type of dictionaries with an integer field “f” is

\{ν | tag(ν) = “Dict” \∧ has(ν, “f”) \∧ tag sel(ν, “f”) = “Int”\}

We refer to the binder ν in refinement types as “the value variable.”

**Nesting: Type Predicates and Terms.** To express the types of values like functions and dictionaries containing functions, System D permits types to be nested within refinement formulas. Formally, the language of refinement formulas includes a form, lww :: U, called a type predicate, where U is a type term. The type term \(x : T_1 \rightarrow T_2\) describes values that have a dependent function type, i.e., functions that accept arguments w of type T1 and return values of type T2[w/x], where x is bound in T2. We write T1 \rightarrow T2 when x does not appear in T2. Type terms A, B, etc., correspond to type parameters to polymorphic functions. The type term Null corresponds to the type of the constant value null. The type term C[T] corresponds to records constructed with the C type constructor instantiated with the sequence of type arguments T. For example, the type of the (integer) successor function is

\{ν | ν : x : Int \rightarrow \{ν | tag(ν) = “Int” \∧ ν = x + 1\}\},

dictionaries where the value at key “f” maps Int to Int have type

\{ν | tag(ν) = “Dict” \∧ has(ν, “f”) \∧ tag sel(ν, “f”) : Int \rightarrow Int\}

and the constructed record List(1, null) can be assigned the type \{ν | ν :: List[IInt]\}.

**Datatype Definitions.** A datatype definition of C defines a named, possibly recursive type. A datatype definition includes a sequence \\(\theta A\) of type parameters A paired with variance annotations \(\theta\). A variance annotation is either + (covariant), - (contravariant), or = (bivariant). The rest of the definition specifies a sequence \(f : T\) of field names and their types. The types of the fields may refer to the type parameters of the declaration. A well-formedness check, which will be described in Section 4, ensures that occurrences of type parameters in the field types respect their declared variance annotations. By convention, we will use the subscript i to index into the sequence \(\theta A\) and j for \(f : T\). For example, \(\theta_i\) refers to the variance annotation of the \(i^{th}\) type parameter, and \(f_j\) refers to the name of the \(j^{th}\) field.

**Programs.** A program is a sequence of datatype definitions \(\overline{td}\) followed by an expression e. Requiring all datatype definitions to appear first simplifies the subsequent presentation.

**Semantics.** The small-step operational semantics of System D is standard for a call-by-value, polymorphic lambda calculus; we provide the formal definition in Appendix A. Following standard practice, the semantics is parametrized by a function \(\delta\) that assigns meaning to primitive functions c, including dictionary operations like has, get, and set.
4. Type Checking

In this section, we present the System D type system, comprising several well-formedness relations, an expression typing relation, and, at the heart of our approach, a novel subtyping relation which discharges obligations involving nested refinements through a combination of syntactic and semantic, SMT-based reasoning. We first define environments for type checking.

Environments. Type environments $\Gamma$ are of the form

$$\Gamma ::= \emptyset \mid \Gamma, x:S \mid \Gamma, A \mid \Gamma, p$$

where bindings either record the derived type $S$ for a variable $x$, a type variable $A$ introduced in the scope of a type function, or a formula $p$ that is recorded to track the control flow along branches of an if-expression. A type definition environment $\Psi$ records the definition of each constructor type $C$. As type definitions appear at the beginning of a program, we assume for clarity that $\Psi$ is fixed and globally visible, and elide it from the judgments. In the sequel, we assume that $\Psi$ contains at least the definition

$$\text{type } \text{List}[:A] \{ \text{"hd"} : \{ \nu \mid \nu :: A \}; \text{"tl"} : \{ \nu \mid \nu :: \text{List}[A] \} \}.$$

4.1 Well-formedness

Figure 2 defines the well-formedness relations.

Formulas, Types and Environments. We require that types be well-formed within the current type environment, which means that formulas used in types are boolean propositions and mention only variables that are currently in scope. By convention, we assume that variables used as binders throughout the program are distinct and different from the special value variable $\nu$, which is reserved for types. Therefore, $\nu$ is never bound in $\Gamma$. When checking the well-formedness of a refinement formula $p$, we substitute a fresh variable $x$ for $\nu$ and check that $p[x/\nu]$ is well-formed in the environment extended with $x:Top$, to the environment, where $\text{Top} = \{ \nu | \nu:: \text{true} \}$. We use fresh variables to prevent duplicate bindings of $\nu$.

Note that the well-formedness of formulas does not depend on type checking; all that is needed is the ability to syntactically distinguish between terms and propositions. Checking that formulas are well-formed is straightforward; the important point is that a variable $x$ may be used only if it is bound in $\Gamma$.

Datatype Definitions. To check that a datatype definition is well-formed, we first check that the types of the fields are well-formed in an environment containing the declared type parameters. Then, to enable a sound subtyping rule for constructed types in the sequel, we check that the declared variance annotations are respected within the type definition. For this, we use a procedure $\text{VarianceOk}$ (defined in Appendix A) that recursively walks formulas to record whether type variables occur in positive or negative positions within the types of the fields.

4.2 Expression Typing

The expression typing judgment $\Gamma \vdash e :: S$, defined in Figure 3 verifies that expression $e$ has type scheme $S$ in environment $\Gamma$. We highlight the important aspects of the typing rules.

Constants. Each primitive constant $c$ has a type, denoted by $ty(c)$, that is used by $\text{T-Const}$. Basic values like integers, booleans, etc. are given singleton types stating that their value equals the corresponding constant in the refinement logic. For example:

- $1 :: \{ \nu | \nu = 1 \}$
- $\text{true} :: \{ \nu | \nu = \text{true} \}$
- $\text{false} :: \{ \nu | \nu = \text{false} \}$

Arithmetic and boolean operations have types that reflect their semantics. Equality on base values is defined in the standard way, while equality on function values is physical equality.

- $+ :: x: \text{Int} \rightarrow y: \text{Int} \rightarrow \{ \nu | \nu = x + y \}$
- $\text{not} :: x: \text{Bool} \rightarrow \{ \nu | \nu = \text{true} \leftrightarrow \nu = \text{false} \}$
- $= :: x: \text{Top} \rightarrow y: \text{Top} \rightarrow \{ \nu | \text{Bool}(\nu) \land \nu = \text{true} \leftrightarrow \nu = \text{false} \}$
- $\text{fix} :: \forall A. (A \rightarrow A) \rightarrow A$
- $\text{tag} :: x: \text{Top} \rightarrow \{ \nu | \nu = \text{tag}(x) \}$

The constant $\text{fix}$ is used to encode recursion, and the type for the tag-test operation uses an axiomatized function in the logic.

The operations on dictionaries are given refinement types over the theory of finite maps.

- $\{ \} :: \{ \nu | \nu = \text{empty} \}$
- $\text{has} :: d: \text{Dict} \rightarrow k: \text{Str} \rightarrow \{ \nu | \text{Bool}(\nu) \land \nu = \text{true} \leftrightarrow \nu = \text{has}(d,k) \}$
- $\text{get} :: d: \text{Dict} \rightarrow k: \{ \nu | \text{Str}(\nu) \land \text{has}(d,k) \} \rightarrow \{ \nu | \nu = \text{get}(d,k) \}$
- $\text{set} :: d: \text{Dict} \rightarrow k: \text{Str} \rightarrow x: \text{Top} \rightarrow \{ \nu | \text{EqMod}(\nu, d, k) \land \text{has}(d,k) \land \text{set}(d,k) = x \}$
- $\text{keys} :: d: \text{Dict} \rightarrow \text{List}[\{ \nu | \text{Str}(\nu) \land \text{has}(d,\nu) \}]$

In the theory of finite maps, the operator $\text{dom}(d)$ denotes the domain of the map $d$, and $\text{restrict}(d, y)$ restricts $d$ to the set of keys $y$. (These primitives can all be reduced to McCarthy’s select and update operators [23, 31]; we define these in Appendix A.) Thus, we define $\text{empty}$ as a special constant such that $\text{dom}(\text{empty}) = \emptyset$. The refinements for the other operators use $\text{has}(d,k)$, which abbreviates $k \in \text{dom}(d)$, and $\text{EqMod}(d_1, d_2, a)$, which abbreviates

$$\text{restrict}(d_1, \text{dom}(d_1) \setminus \{a\}) = \text{restrict}(d_2, \text{dom}(d_2) \setminus \{a\})$$

The predicate $\text{has}(d,k)$ checks that a key $k$ is defined in a map $d$, and is used as a precondition for $\text{set}$. The predicate $\text{EqMod}(d_1, d_2, k)$ states that the dictionaries $d_1$ and $d_2$ are identical except at the key $k$. This is useful for dictionary updates where
Type Checking

\[ \Gamma \vdash e :: S \]

- \[ \Gamma \vdash e :: ty(c) \qquad [\text{T-CONST}] \]
- \[ \Gamma(x) = T \quad \Gamma \vdash x :: \{ \nu | \nu = x \} \qquad [\text{T-VAR}] \]
- \[ \Gamma(x) = \forall A. S \quad \Gamma \vdash x :: \forall A. S \qquad [\text{T-VARPOLY}] \]
- \[ \Gamma \vdash w_1 :: \text{Dict} \quad \Gamma \vdash w_2 :: \text{Str} \quad \Gamma \vdash w_3 :: S \qquad [\text{T-EXTEND}] \]
- \[ \Gamma \vdash w_1 +\{w_2 \mapsto w_3\} :: \{ \nu | \nu = w_1 +\{w_2 \mapsto w_3\}\} \]
- \[ \Gamma \vdash w :: \text{Bool} \quad \Gamma, w = \text{true} \vdash e_1 :: S \quad \Gamma, w = \text{false} \vdash e_2 :: S \qquad [\text{T-BF}] \]
- \[ \Gamma \vdash T_1 \quad \Gamma, x; T_1 \vdash e :: T_2 \qquad [\text{T-FUN}] \]
- \[ \Gamma \vdash w_1 :: \{ \nu | \nu :: x; T_1 \rightarrow T_{12} \} \quad \Gamma \vdash w_2 :: T_{11} \quad [\text{T-APP}] \]
- \[ \Gamma \vdash w_1 w_2 :: T_{12}[w_2/x] \quad A \notin \Gamma \quad \Gamma, A :: e :: S \quad [\text{T-TFUND}] \]
- \[ \Gamma \vdash T \quad \Gamma \vdash w :: \forall A. S \quad [\text{T-TAPP}] \]
- \[ \Gamma \vdash w :: [T] :: \text{Inst}(S, \overline{A}, T) \quad [\text{T-FOLD}] \]
- \[ \forall i, \Gamma \vdash T_i \quad \Psi(C) = [\emptyset A][T_i] \quad [\text{T-FOLD}] \]
- \[ \forall j, \Gamma, w_1 :: \text{Inst}(T_j, \overline{A}, T) \quad [\text{T-FOLD}] \]
- \[ \Gamma \vdash e :: \{ \nu | \nu :: C(T_i) \} \quad [\text{T-UNFOLD}] \]
- \[ \Gamma \vdash e :: \{ \nu | \nu :: [\text{Fold}(C, T_i, w)] \} \quad [\text{T-UNFOLD}] \]
- \[ \Gamma \vdash S_1 \quad \Gamma \vdash e_1 :: S_1 \quad \Gamma, x; S_1 \vdash e_2 :: S_2 \quad \Gamma \vdash S_2 \quad [\text{T-LET}] \]
- \[ \Gamma \vdash e :: S' \quad \Gamma \vdash S' \subseteq S \quad \Gamma \vdash S \quad [\text{T-SUB}] \]

**Figure 3.** Type checking for System D

we do not know the exact value being stored, but do know some abstraction thereof, e.g. its type. For example, in incCounter (from section 2) we do not know what value is stored in the count field c, only that it is an integer. Thus, we say that the new dictionary is the same as the old except at c, where the binding is an integer. A more direct approach would be to use an existentially quantified variable to represent the stored value and say that the resulting dictionary is the original dictionary updated to contain this quantified value. Unfortunately, that would take the formulas outside the decidable quantifier-free fragment of the logic, thereby precluding SMT-based logical subtyping.

**Standard Rules.** We briefly identify several typing rules that are standard for lambda calculi with dependent refinements. T-VAR and T-VARPOLY assign types to variable expressions x. If x is bound to a (monomorphic) refinement type in \( \Gamma \), then T-VAR assigns x the singleton type that says that the expression x evaluates to the same value as the variable x. T-FUN assigns the type scheme S to an if-expression if the condition w is a boolean-valued expression, the then-branch expression \( e_1 \) has type scheme \( S \) under the assumption that \( w \) evaluates to true, and the else-branch expression \( e_2 \) has type scheme \( S \) under the assumption that \( w \) evaluates to false. The T-APP rule is standard, but notice that the arrow type of \( w_1 \) is nested inside a refinement type. In T-LET, the type scheme \( S_2 \) must be well-formed in \( \Gamma \), which prevents the variable \( x \) from escaping its scope. T-SUB allows expression e to be used with type S if e has type \( S' \) and \( S' \) is a subtype of S.

**Type Instantiation.** The T-TAPP rules use the procedure Inst to instantiate a type variable with a (monomorphic) type. Inst is defined recursively on formulas, type terms, and types, where the only non-trivial case involves type predicates with type variables:

\[ \text{Inst}(lw :: A, A \mathbin{\{ \nu | p \}}) = p[lw/\nu] \]

We write \( \text{Inst}(S, \overline{A}, T) \) to mean the result of applying Inst to S with the type variables and type arguments in succession.

**Fold and Unfold.** The T-FOLD rule is used for records of data created with the datatype constructor C and type arguments. The rule succeeds if the argument \( w_j \) provided for each field \( f_j \) has the required type \( T_j' \) after instantiating all type parameters \( \overline{A} \) with the type arguments \( T_i \). If these conditions are satisfied, the formula returned by Fold\((C, \overline{T}, \overline{w})\), defined as

\[ \nu \neq \text{null} \land \text{tag}(\nu) = \text{“Dict”} 
\land \nu :: C(T_i) 
\land (\land_j \text{sel}(\nu, f_j) = w_j) \]

records that the value is non-null, that the values stored in the fields are precisely the values used to construct the record, and that the value has a type corresponding to the specific constructor used to create the value. T-UNFOLD exposes the fields of non-null constructed data as a dictionary, using Unfold\((C, T_i)\), defined as

\[ \nu \neq \text{null} \Rightarrow (\text{tag}(\nu) = \text{“Dict”} 
\land \nu :: C(T_i) 
\land (\land_j \text{sel}(\nu, f_j))) \]

where \( \Psi(C) = [\emptyset A][T_i] \), \( \{[\nu | p][lw] \neq p[lw/\nu] \), and for all \( j, T_j'' = \text{Inst}(T_j, \overline{A}, T) \). For example, Unfold\((List, \text{Int})\) is

\[ \nu \neq \text{null} \Rightarrow (\text{tag}(\nu) = \text{“Dict”} 
\land \text{tag}(\nu, \text{“hd”}) = \text{“Int”} 
\land \text{sel}(\nu, \text{“tl”}) :: \text{List}[\text{Int}]) \]

**4.3 Subtyping**

In traditional refinement type systems, there is a two-level hierarchy between types and refinements that allows a syntax-directed reduction of subtyping obligations to SMT implications. In contrast, System D’s refinements include uninterpreted type predicates that are beyond the scope of (first-order) SMT solvers.

Let us consider the problem of establishing the subtyping judgment \( \Gamma \vdash \{ \nu | p_1 \} \subseteq \{ \nu | p_2 \} \). We cannot use the SMT query

\[ [\Gamma] \land p_1 \Rightarrow p_2 \]

as the presence of (uninterpreted) type-predicates may conservatively render the implication invalid. Instead, our strategy is to massage the refinements into a normal form that makes it easy to factor the implication in 2 into a collection of subgoals whose consequences are either simple (non-type) predicates or type predicates. The former can be established via SMT and the latter by recursively invoking syntactic subtyping. Next, we show how this strategy is realized by the rules in Figure 4.

**Step 1: Split query into subgoals.** We start by converting \( p_2 \) into a normalized conjunction \( \land_j (q_j \Rightarrow r_i) \). Each conjunct, or clause, \( q_j \Rightarrow r_i \) is normalized such that its consequent is a disjunction of type predicates. We use the symbol \( \Rightarrow \) instead of the usual implication arrow \( \Rightarrow \) to emphasize the normal structure of each

\[ \land_j (q_j \Rightarrow r_j) \]

where the \( q_j \land r_j \) are each normalized (now simple) type predicates.
Subtyping

\[ \Gamma \vdash S_1 \subseteq S_2 \]

\[ x \text{ fresh} \quad p'_1 = p_1[x/v] \quad p'_2 = p_2[x/v] \]

Normalization of \( p_2' \):

\[ \wedge_i (q_i \Rightarrow r_i) \quad \forall i, \Gamma, \quad p'_1 \vdash q_i \Rightarrow r_i \]

\[ \Gamma \vdash \{ \nu | p_1 \} \subseteq \{ \nu | p_2 \} \quad [S-MONO] \]

\[ \Gamma \vdash S_1 \subseteq S_2 \]

\[ \Gamma \vdash \forall A. \quad S_1 \subseteq \forall A. \quad S_2 \quad [S-POLY] \]

Clause Implication

\[ \Gamma \vdash q \Rightarrow r \]

Valid(\( \Gamma \) \& \( q \Rightarrow r \) ) \quad [C-VALID] \]

\[ \exists j. \quad \text{Valid}(\( \Gamma \) \& \( q \Rightarrow lw_j : U \) ) \quad \Gamma, \quad q \vdash U \lll U_j \quad [C-IMP\,SYN] \]

Syntactic Subtyping

\[ \Gamma \vdash T_{21} \subseteq T_{11} \quad \Gamma, \quad x : T_{21} \vdash T_{12} \subseteq T_{22} \quad [U-ARROW] \]

\[ \Gamma \vdash A :< A \quad [U-VAR] \quad \Gamma \vdash \text{Null} :< C[T] \quad [U-NULL] \]

\[ \Psi(C) = \emptyset \quad \text{for } \forall i, \theta_i \in \{ +, - \} \quad \text{then } \Gamma \vdash T_i \subseteq T_{2i} \]

\[ \forall i, \theta_i \in \{ +, - \} \quad \text{then } \Gamma \vdash T_{2i} \subseteq T_{i} \quad [U-DATATYPE] \]

\[ \Gamma \vdash C[T_{i}] :< C[T_{2i}] \]

Figure 4. Subtyping for System D

Clause. By splitting \( p_2 \) into its normalized clauses, rule S-MONO reduces the goal \( \square \) to the equivalent collection of subgoals

\[ \forall i. \quad \Gamma, \quad p_1 \vdash q_i \Rightarrow r_i \]

Step 2: Discharge subgoals. The normalization ensures that the consequent of each subgoal above is a disjunction of type predicates. When the disjunction of a clause is empty, the subgoal is

\[ \text{("type predicate-free")} \quad \Gamma, \quad p_1 \vdash q_i \Rightarrow \text{false} \]

which rule C-VALID handles by SMT. Otherwise, the subgoal is

\[ \text{("type predicate")} \quad \Gamma, \quad p_1 \vdash q_i \Rightarrow lw_j : U_j \]

which rule C-IMP\,SYN handles via type extraction followed by an invocation of syntactic subtyping. In particular, the rule tries to establish one of the disjuncts \( lw_j : U_j \), by searching for a type term \( U \) that occurs in \( \Gamma \) that 1) flows to \( lw_j \), i.e. for which we can deduce via SMT that

\[ \square \quad \Gamma \vdash \forall A. \quad lw_j : U \]

\[ \square \quad \text{is valid and, 2) is a syntactic subtype of } U_j \text{ in an appropriately strengthened environment (written } \Gamma, \quad p_1, \quad q_i \vdash U \lll U_j \text{). The rules U-DATATYPE and U-ARROW establish syntactic (re)typing) subtyping, by (recursively) establishing that subtyping holds for the matching components } [U] \quad [U-A-RROW]. \]

Because syntactic subtyping recursively refers to subtyping, the S-MONO rule uses fresh variables to avoid duplicate bindings of \( \nu \) in the environment.

Formula Normalization. Procedure Normalize converts a formula \( p \) into a conjunction of clauses \( \wedge_i (q_i \Rightarrow r_i) \) as described above. The conversion is carried out by translating \( p \) to conjunctive normal form (CNF), and then for each CNF clause, rearranging literals and adding negations as necessary. For example,

\[ \text{Normalize}(\nu = \text{null}) = \neg (\nu = \text{null}) \Rightarrow \text{false} \]

\[ \text{Normalize}(\nu = \text{null} \lor \nu : U) = \neg (\nu = \text{null}) \lor \nu : U \]

Formula Implication. In each SMT implication query \( \Gamma \wedge \theta \Rightarrow q \), the operator \( [\cdot] \) describes the embedding of environments and types into the logic as follows:

\[ \Gamma \wedge \theta \Rightarrow q \]

\[ \Gamma, \quad x : T \Rightarrow [\Gamma, \quad x : T \wedge p \Rightarrow q \Rightarrow T[x/v]] \]

Recap. Recall that our goal is to typecheck programs which use value-indexed dictionaries which may contain functions as values. On the one hand, the theory of finite maps allows us to use logical refinements to express and verify complex invariants about the contents of dictionaries. On the other, without resorting to higher-order logic, such theories cannot express that a dictionary maps a key to a value of function type.

To resolve this tension, we introduced the novel concept of nested refinements, where types are nested into the logic as uninterpreted terms and the typing relation is nested as an uninterpreted predicate. The logical validity queries arising in typechecking are discharged by rearranging the formula in question into an implication between a purely logical formula and a disjunction of type predicates. This implication is discharged using a novel combination of logical queries, discharged by an SMT solver, and syntactic subtyping. This approach enables the efficient, automatic type checking of sophisticated dynamic language programs that manipulate complex data, including dictionaries which map keys to function values.

5. Soundness

At this point in the proceedings, it is customary to make a claim about the soundness of the type system by asserting that it enjoys the standard preservation and progress properties. Unfortunately, the presence of nested refinements means this route is unavailable to us, as the usual substitution property does not hold! Next, we describe why substitution is problematic and define a stratified system System D* for which we establish the preservation and progress properties. The soundness of System D follows, as it is a special case of the stratified System D*.

5.1 The Problems

The key insight in System D is that we can use uninterpreted functions to nest types inside refinements, thereby unlocking the door to expressive SMT-based reasoning for dynamic languages. However, this very strength precludes the usual substitution lemma upon which preservation proofs rest.

Substitution. The standard substitution property requires that if \( x : S, \Gamma \vdash e : S' \) and \( \nu : S \), then \( \Gamma[w/x] \vdash e[w/x] : S'[w/x] \). The following snippet shows why System D lacks this property:

\[ \text{let foo f = 0 in foo (fun x -> x + 1)} \]

Suppose that we ascribe to foo the type

\[ \text{foo : f : (Int \rightarrow Int) \rightarrow \{ \nu \mid f : Int \rightarrow Int \}} \]

The return type of the function states that its argument \( f \) is a function from integers to integers and does not impose any constraints on the return value itself. To check that foo does indeed have this type, by T-F, the following judgment must be derivable:

\[ f : Int \rightarrow Int \vdash 0 \lll \{ \nu \mid f : Int \rightarrow Int \} \quad (3) \]
By T-CONST, T-SUB, S-MONO and C-VALID the judgment reduces to the implication

\[ true \land f :: \text{Int} \rightarrow \text{Int} \land [t y(0)][0/\nu] \Rightarrow f :: \text{Int} \rightarrow \text{Int} \]

which is trivially valid, thereby deriving \([3]\), and showing that \(\text{foo}\) does indeed have the ascribed type.

Next, consider the call to \(\text{foo}\). By T-APP, the result has type

\[ \{\nu\} (\text{fun } x \rightarrow x + 1) :: \text{Int} \rightarrow \text{Int} \]

The expression \(\text{foo} (\text{fun } x \rightarrow x + 1)\) evaluates in one step to 0. Thus, if the substitution property is to hold, 0 should also have the above type. In other words, System D must be able to derive

\[ \vdash 0 :: \{\nu\} (\text{fun } x \rightarrow x + 1) :: \text{Int} \rightarrow \text{Int} \]

By T-CONST, T-SUB, S-MONO, and C-VALID, the judgment reduces to the implication

\[ true \land [t y(0)][0/\nu] \Rightarrow (\text{fun } x \rightarrow x + 1) :: \text{Int} \rightarrow \text{Int} \]

(4) which is invalid as type predicates are uninterpreted in our refinement logic! Thus, the call to \(\text{foo}\) and the reduced value do not have the same type in System D, which illustrates the crux of the problem: the C-VALID rule is not closed under substitution.

**Circularity.** Thus, it is clear that the substitution lemma will require that we define an interpretation for type predicates. As a first attempt, we can define an interpretation \(I\) that interprets type predicates involving arrows as:

\[ I[\lambda x. e] = \text{fun } x :: x:T_1 \rightarrow T_2 \]

Next, let us replace C-VALID with the following rule that restricts the antecedent to the interpretation above:

\[ I[\lambda x. e] = \text{fun } x :: x:T_1 \rightarrow T_2 \quad \text{iff} \quad x:T_1 \vdash e :: T_2 \]

Notice that the new rule requires the implication be valid in the particular interpretation \(I\) instead of in all interpretations. This allows the logic to “hook back” into the type system to derive types for closed lambda expressions, thereby discharging the problematic implication query in \([4]\). While the rule solves the problem with substitution, it does not take us safely to the shore — it introduces a circular dependence between the typing judgments and the interpretation \(I\). Since our refinement logic includes negation, the type system corresponding the set of rules outlined earlier combined with C-VALID-INTERPRETED is not necessarily well-defined.

5.2 The Solution: Stratified System D

Thus, to prove soundness, we require a well-founded means of interpreting type predicates. We achieve this by stratifying the interpretations and type derivations, requiring that type derivations at each level refer to interpretations at the same level, and that interpretations at each level refer to derivations at strictly lower levels. Next, we formalize this intuition and state the important lemmas and theorems. The full proofs may be found in Appendix A.

Formally, we make the following changes. First, we index typing judgments \((\Gamma_n)\) and interpretations \((\mathcal{I}_n)\) with a natural number \(n\). We call these the level-\(n\) judgments and interpretations, respectively. Second, we allow level-\(n\) judgments to use the rule

\[ \mathcal{I}_n \models [\Gamma] \land p \Rightarrow q \quad \text{[C-VALID-N]} \]

and the level-\(n\) interpretations to use lower-level type derivations:

\[ \mathcal{I}_n \models \lambda x. e :: x:T_1 \rightarrow T_2 \quad \text{iff} \quad x:T_1 \vdash_{n-1} e :: T_2. \]

Finally, we write

\[ \Gamma \vdash_{n} e :: S \quad \text{iff} \quad \exists n. \Gamma \vdash_{n} e :: S. \]

The derivations in System D* consist of the derivations at all levels. The following “lifting” lemma states that the derivations at each level include the derivations at all lower levels:

**Lemma (Lifting Derivations),**

1. If \(\Gamma \vdash e :: S\), then \(\Gamma \vdash_{n} e :: S\).
2. If \(\Gamma \vdash_{n} e :: S\), then \(\Gamma \vdash_{n+1} e :: S\).

The first clause holds since the original System D derivations cannot use the C-VALID-N rule, i.e. \(\Gamma \vdash e :: S\) exactly when \(\Gamma \vdash_{0} e :: S\). The second clause follows from the definitions of \(\Gamma_n\) and \(\mathcal{I}_n\). Stratification snaps the circularity knot and enables the proof of the following stratified substitution lemma:

**Lemma (Stratified Substitution),**

If \(x:S, \Gamma \vdash_{n} e :: S'\) and \(\vdash_{n} w :: S\), then \(\Gamma[w/x] \vdash_{n+1} e:w/x :: S'[w/x]\).

The proof of the above depends on the following lemma, which captures the connection between our typing rules and the logical interpretation of formulas in our refinement logic:

**Lemma (Satisfiable Typing),**

If \(\vdash_{n} w :: T\), then \(\vdash_{n+1} [T]/w/x\).

Stratified substitution enables the following preservation result:

**Theorem (Stratified Preservation),**

If \(\vdash_{n} e :: S\), and \(e \rightsquigarrow e'\) then \(\vdash_{n+1} e' :: S\).

From this, and a separate progress result, we establish the type soundness of System D*:

**Theorem (System D* Type Soundness),**

If \(\vdash_{n} e :: S\), and \(e \rightsquigarrow e'\) then \(\vdash_{n+1} e' :: S\).

By coupling this with Lifting, we obtain the soundness of System D as a corollary.

6. Algorithmic Typing

Having established the expressiveness and soundness of System D, we establish its practicality by implementing a type checker and applying it to several interesting examples. The declarative rules for type checking System D programs, shown in section 4, are not syntactic and thus unsuitable for implementation. We highlight the problematic rules and sketch an algorithmic version of the type system that also performs local type inference \([25]\).

The algorithmic system is sound with respect to the declarative one and, modulo a restriction to ensure that subtyping terminates, is as precise. Our prototype implementation \([1]\) verifies all of the examples in this paper and in \([3]\), using Z3 \([8]\) to discharge SMT obligations. A more detailed discussion of the algorithmic system may be found in Appendix B.

6.1 Algorithmic Subtyping

Nearly all of the declarative subtyping rules presented in Figure 4 are non-overlapping and directed by the structure of the judgment being derived. The sole exception is C-IMP\ SYN, whose first premise requires us to synthesize a type term \(U\) such that the SMT solver can prove \(lw_j :: U\) for some \(j\), where \(U\) is used in the second premise. We note that, since type predicates are uninterpreted, the only type terms \(U\) that can satisfy this criterion must come from the environment \(\Gamma\). Thus, we define a procedure MustFlow(\(\Gamma, T\)) that uses the SMT solver to compute the set of type terms \(U',\) out of all possible type terms mentioned in \(\Gamma\), such that for all values \(x, x:T\) implies that \(x :: U'\). To implement C-IMP\ SYN, we call MustFlow(\(\Gamma, \{\nu | \nu = lw_j\}\)) to compute the set \(U\) of type terms that might be needed by the second premise. Since the declarative rule cannot possibly refer to a type term \(U\) that is not in \(\Gamma\), this
strategy guarantees that $U \in \mathcal{U}$ and, thus, does not forfeit precision.

**Ensuring Termination.** An important concern remains: because we extract type terms from the environment and recursively invoke the subtyping relation on them, we do not have the usual guarantee that subtyping is recursively invoked on strictly syntactically smaller terms, and thus it is not clear whether subtyping checks will terminate. Indeed, they may not! Appendix B presents an example obligation that, although unlikely to appear in practice, leads to non-termination when subtyping is implemented directly. The crux of the matter is that an inner subtyping obligation may be isomorphic to an outer one, triggering an infinitely repeating derivation. Fortunately, we can cut the loop as follows: along any branch of a subtyping derivation, we allow a type term to be returned by MustFlow at most once. Since there are only finitely many type terms in the environment, this is enough to ensure termination. The price we pay is that algorithmic subtyping is not complete with respect to declarative subtyping; we have not found and do not expect this to be a problem in practice.

### 6.2 Bidirectional Type Checking

We extend the syntax of System D with optional type annotations for binding constructs and constructed data, and, following work on local type inference [23], we define a bidirectional type checking algorithm. In the remainder of this section, we highlight the novel aspects of our bidirectional type system.

**Function Applications.** To typecheck an application $w_1 \, w_2$, we must synthesize a type $T_1$ for the function $w_1$ and use type extraction to convert $T_1$ to a syntactic arrow. Since the procedure MustFlow can return an arbitrary number of type terms, we must decide how to proceed in the event that $T_1$ can be extracted to multiple different arrow types. To avoid the need for backtracking in the type checker, and to provide a semantics that is simple for the programmer to understand, we synthesize a type for $w_1$ only if there is exactly one syntactic arrow that is applicable to the given argument $w_2$.

**Remaining Rules.** We will now briefly summarize some of the other algorithmic rules presented in Appendix B. Uses of T-SUB can be factored into other typing rules. However, uses of T-UNFOLD cannot, since we cannot syntactically predict where it is needed. Since we do not have pattern matching to determine exactly when to unfold type definitions, as in languages like ML, we eagerly unfold type definitions to anticipate all situations in which unfolding might be required. For let-expressions, to handle the fact that synthesized types might refer to variables that are about to go out of scope, making them ill-formed, we use several simple heuristics to eliminate occurrences of local variables. In all of the examples we have tested, the annotations provided on top-level let-bindings are sufficient to allow synthesizing well-formed types for all unannotated inner let-expressions. Precise types are of the examples we have tested, the annotations provided on top-level let-bindings are sufficient to allow synthesizing well-formed local type inference will use the type of $w_1$, $w_2$. Finally, although the techniques in [23] would allow us to, for simplicity we do not attempt to synthesize parameters to type functions.

**Soundness.** We write $\Gamma \vdash e \triangleleft S$ for the algorithmic type checking judgment, which verifies $e$ against the given type $S$, and $\Gamma \vdash e \gg S$ for the algorithmic type synthesis judgment, which produces a type $S$ for expression $e$. Each of the techniques employed in this section are sound with respect to the declarative system, so we can show the following property, where we use a procedure erase to remove type annotations from functions, let-bindings, and constructed data because the syntax of the declarative system does not permit them:

**Proposition (Sound Algorithmic Typing).** If $\Gamma \vdash e \gg S$ or $\Gamma \vdash e \triangleleft S$, then $\Gamma \vdash$ erase$(e) \gg S$.

### 7. Related Work

In this section, we highlight related approaches to statically verifying features of dynamic languages. For a thorough introduction to contract-based and other hybrid approaches, see [10, 18, 30].

**Dynamic Unions and Control Flow.** Among the earliest attempts at mixing static and dynamic typing was adding the special type dynamic to a statically-typed language like ML [8]. In this approach, an arbitrary value can be injected into dynamic, and a typecase construct allows inspecting its precise type at run-time. However, one cannot guarantee that a particular dynamic value is one of a subset of types (cf. negate from section 2). Several researchers have used union types and tag-test sensitive control-flow analyses to support such idioms. Most recently, $\lambda_{TR}$ [24] and $\lambda_S$ [13] feature values of (untagged) union types that can be used at more precise types based on control flow. In the former, each expression is assigned two propositional formulas that hold when the expression evaluates to either true or false; these propositions are strengthened by recording the guard of an if-expression in the typing environment when typing its branches. Typechecking proceeds by solving propositional constraints to compute, for each value at each program point, the set of tags it may correspond to. The latter shows how a similar strategy can be developed in an imperative setting, by coupling a type system with a data flow analysis. However, both systems are limited to ad-hoc unions over basic and function values. In contrast, System D shows how, by pushing all the information about the value (resp. reasoning about flow) into expressive, but decidable refinement predicates (resp. into SMT solvers), one can statically reason about significantly richer idioms (related tags, dynamic dictionaries, polymorphism, etc.).

**Records and Objects.** There is a large body of work on type systems for objects [17, 24]. Several early advances incorporate records into ML [26], but the use of records in these systems are unfortunately unlikely to be flexible enough for dynamic dictionaries. In particular, record types cannot be joined when they disagree on the type of a common field, which is crucially enabled by the use of the theory of finite maps in our setting. Recent work includes type systems for JavaScript and Ruby. [4] presents a rich type system and inference algorithm for JavaScript, which uses row-types and width subtyping to model dictionaries (objects). The system does not support unions, and uses fixed field names. This issue is addressed in [3], which models dictionaries using row types labeled by singletons indexed by string constants, and depth subtyping. A recent proposal [55] incorporates an initialization phase during which object types can be updated. However, these systems preclude truly dynamic dictionaries, which require dependent types, and moreover lack the control flow analysis required to support ad-hoc unions. DRuby [12] is a powerful type system designed to support Ruby code that mixes intersections, unions, classes, and parametric polymorphism. DRuby supports “duck typing,” by converting from nominal to structural types appropriately. However, it does not support ad-hoc unions or dynamic dictionary accesses.
Dependent Types and SMT Solvers. The observation that ad-hoc unions can be checked via dependent types is not new. \[5\] develops a dependent type system called guarded types that is used to describe records and ad-hoc unions in legacy Cobol programs that make extensive use of tag-tests, where the “tag” is simply the first few bytes of a structure. \[16\] presents an SMT-based system for statically inferring dependent types that verify the safety of ad-hoc unions in legacy C programs. \[7\] describes how type-checking and property verification are two sides of the same coin for C objects. \[5\] uses refinement types to formalize similar ideas in the context of Dminor, a first-order functional data description language with fixed-key records and run-time tag-tests. The authors show how unions and intersections can be expressed in refinements (and even collections, via recursive functions), and hence how SMT solvers can wholly discharge all subtyping obligations. However, the above techniques apply only to first-order languages, with static keys and dictionaries over base values.

Combining Decision Procedures. Our approach of combining logical reasoning by SMT solvers and syntactic reasoning by subtyping is reminiscent of work on combining decision procedures \[22, 29\]. However, such techniques require the theories being combined to be disjoint; since our logic includes type terms which themselves contain arbitrary terms, our theory of syntactic types cannot be separated from the other theories in our system, so these techniques cannot be directly applied.

8. Conclusions and Future Work
We have shown how, by nesting type predicates within refinement formulas and carefully interleaving syntactic- and SMT-based subtyping, System D can statically type check dynamic programs that manipulate dictionaries, polymorphic higher-order functions and containers. Thus, we believe that System D can be a foundation for two distinct avenues of research: the addition of heterogeneous dictionaries to static languages like C#, Java, OCaml and Haskell, or dually, the addition of expressive static typing to dynamic languages like Clover, JavaScript, Racket, and Ruby.

We anticipate several concrete lines of work that are needed to realize the above goals. First, we need to add support for references and imperative update, features common to most popular dynamic languages. Since every dictionary operation in an imperative language goes through a reference, we will need to extend the type system with flow-sensitive analyses, as in \[28\] and \[13\], to precisely track the values stored in reference cells at each program point. Furthermore, to precisely track updates to dictionaries in the imperative setting, we will likely need to introduce some flow-sensitivity to the type system itself, adopting strong update techniques as in \[14\] and \[15\]. Second, our system treats strings as atomic constants. Instead, it should be possible to incorporate modern decision procedures for strings \[15\] to support logical operations on keys, which would give even more precise support for reflective metaprogramming. Third, we plan to extend our local inference techniques to automatically derive polymorphic instantiations \[25\] and use Liquid Types \[24\] to globally infer refinement types. Finally, for dynamic languages, it would be useful to incorporate some form of staged analysis to support dynamic code generation \[3\].

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A. Metatheory

This section deals with the formal properties of System D*. First, we provide some definitions that were omitted from the presentation of System D in Sections 3 and 4. Next, we provide the complete definitions of stratified System D*. Finally, we specify the assumptions and definitions specific to our refinement logic, and present the details of the proof. Compared to the proof outline in Section 5, we prove the progress and preservation parts of System D**. Type Soundness together, rather than with separate progress and Stratified Preservation theorems.

A.1 Additional System D Definitions

A.1.1 Operational Semantics

The small-step operational semantics of System D expressions is parametrized on a function $\delta$ that defines the behavior of constants $c$ that are functions. Dictionary operations like has, get, and set are factored into the $\delta$ function. As terms are in A-normal form, there is a single congruence rule, E-COMPAT.

$$
\text{E-Delta} \quad \frac{\text{if } \delta(c, w) \text{ is defined}}{c \ w \Rightarrow \delta(c, w)}
$$

$$
\text{E-App} \quad \frac{(\lambda x. \ c) \ w \Rightarrow c[w/x]}{x = w \text{ in } e \Rightarrow e[w/x]} \quad \text{E-Let}
$$

$$
\text{E-ItTrue} \quad \frac{\text{if } \text{true} \text{ then } e_1 \text{ else } e_2 \Rightarrow e_1} {\text{if } \text{false} \text{ then } e_1 \text{ else } e_2 \Rightarrow e_2} \quad \text{E-ItFalse}
$$

$$
\text{E-COMPAT} \quad \frac{e_1 \Rightarrow e'_1 \quad e_2 \Rightarrow e'_2} {x = e_1 \text{ in } e_2 \Rightarrow x = e'_1 \text{ in } e'_2}
$$

A.1.2 Well-formedness

We briefly supplement our discussion in Section 4.

Refinement Types. The well-formedness of formulas does not depend on type checking; all that is needed is the ability to syntactically distinguish between terms and propositions. We omit the straightforward rules for well-formed values. The important point is that a variable $x$ may be used only if it is (bound) in $\Gamma$. Since our refinement logic is unsorted, all logical predicate and function symbols must be defined for all values in any model of the logic. Thus, ill-typed expressions like $\text{true} \implies \text{false}$ may evaluate to nonstandard "error" values in such models. This means that, for example, $\{\nu \mid \nu > 0\}$ is not the same as $\{\nu \mid \text{tag}(\nu) = \text{"Int"} \land \nu > 0\}$ since the former may also include non-integer values. Such values never arise at run-time, as the types of our primitive operations and constants guarantee that they only consume and produce standard, non-error values.

Datatype Definitions. To enable a sound subtyping rule for constructed types in the sequel, we check that the declared variance annotations are respected within the type definition. The VarianceOk predicate is defined as

$$
\text{VarianceOk}(A, T) \iff \bigcup_j \text{Poles}(A, T_j) \subseteq \{\ast\}
$$

where Poles is a helper procedure that recursively walks formulas, type terms, and types to record where type variables occur within the types of the fields. Poles$(A, +, T)$ computes a subset of $\{\ast, \ast\}$ that includes $\ast$ (resp. $\ast\$) if $A$ occurs in at least one positive (resp. negative) position inside $T$. For each type variable, these polarities are computed across all field types in the definition and then checked against its variance annotation. After successfully checking that a type definition is well-formed, it is added to the globally available type definition environment $\Psi$. For example, when checking the well-formedness of the type term $C[\Gamma]$, we make sure that $C$ is defined by testing for its presence in $\Psi$.

$$
\neg \theta = \begin{cases} - & \text{if } \theta = + \\ + & \text{if } \theta = - \end{cases}
$$

$$
Poles(A, \theta, \nu) = \text{Poles}(A, \theta, P)
$$

A.2 Stratified System D*

The complete definition of the System D* typing and subtyping relations in Figures 5 and 6. The only differences compared to the base system are that all typing and subtyping derivations are now indexed with an integer $n$, and the clause implication relation contains the new C-VALID-N rule. The well-formedness relations remain unchanged.

A.3 Definitions and Assumptions

We often use the following abbreviations for types and substitution into types.

$$
\{p\} \equiv \{\nu \mid p\}
$$

$$
p(lw) \equiv p[lw/\nu]
$$

$$
[T](lw) \equiv [T](lw/\nu)
$$

Proposition (Refinement Logic). The refinement logic underlying the type system at level zero is the quantifier-free fragment of first-order logic with equality and the decidable theories listed below. Logical terms of a universal sum sort called Val include integers, booleans, strings, and dictionaries (finite maps from strings to values). Expressions, formulas and type terms can be encoded in the logic as uninterpreted constructed terms. Function and type function terms are pairs of formal parameters and expression terms.

- (Theory: Uninterpreted Functions)
- (Theory: Linear Arithmetic)
- (Theory: Dictionaries)
We use the following axiomatization of dictionaries that can be reduced to the theory of finite maps [20].

\( \forall w. \)

- \( \neg \text{has}(\text{empty}, w) \)
- \( \text{has}(w_1, w_2, w_3) \)
- \( \text{has}(w_1 ++ \{w_2 \mapsto w_3\}, w_2) \)
- \( \text{sel}(w_1 ++ \{w_2 \mapsto w_3\}, w_2) = w_3 \)
- \( \text{EqMod}(w_1 ++ \{w_2 \mapsto w_3\}, w_1, w_2) \)

\( \forall w_1, w_2, x, y. \)

- \( \text{EqMod}(w_1, w_2, x) \land x \neq y \Rightarrow (\text{has}(w_1, y) \Rightarrow \text{has}(w_2, y)) \)
- \( \text{EqMod}(w_1, w_2, x) \land x \neq y \Rightarrow (\text{sel}(w_1, y) = \text{sel}(w_2, y)) \)

- (Assumption: Tag Function)
We assume the presence of a unary function symbol tag that maps values to strings.

- (Fact: Validity)
We write Valid \( (p) \) to mean that, as usual, \( p \) is satisfiable in all interpretations. In the C-Valid rule, we appeal to a decision procedure to check whether Valid \( (p) \).

- (Assumption: Boolean Values)
We assume Valid \( (\text{tag}(w) = \text{"Bool"}) \) iff \( w \in \{\text{true}, \text{false}\} \).

- (Fact: Free Variable Substitution)
If \( \nu \) appears free in \( p \) and \( q \), then \( p \Rightarrow q \) implies \( p[w/x] \Rightarrow q[w/x] \) for all \( w \).

- (Fact: Uninterpreted Predicate Substitution)
If \( P \) is an uninterpreted predicate symbol in \( p \) and \( q \), then \( p \Rightarrow q \) implies \( p[P' / P] \Rightarrow q[P' / P] \) for all \( P' \).

\( \text{Assumption} \) (Constant Types). For every constant \( c \in \text{Dom}(ty) \), the following properties hold.
1. (Well-formed). \( \vdash ty(c) \).
2. (Normal).
   \( \text{ty}(c) = \{ \nu \mid \nu = c \land p \} \) where either
   \( p = \text{true} \) or
   \( p = \nu :: x : T_1 \rightarrow T_2. \)

3. (App).
   \( \text{ty}(c) = \{ \nu \mid \nu = c \land \nu :: x : T_1 \rightarrow T_2 \} \)
   then for all \( w' \) and \( n \) such that \( \Gamma_n w' :: T_1 \)
   \( \delta(c, w') \) is defined and \( \Gamma_n \delta(c, w') :: T_2[w'/x]. \)

4. (Valid).
   \( \text{Valid}(\text{ty}(c)[\epsilon/c].) \)
   In other words, we add these to the initial
   typing environment.

**Definition** (Type Predicate Interpretation). The System D Interpretation at level \( n \) interprets type predicates as follows.

- \( \Gamma_n \models w :: x : T_11 \rightarrow T_{12} \) iff \( \Gamma_n \vdash x : T_11 \rightarrow T_{12} \) and either:
  1. \( w = \lambda x. e \) and \( x : T_{11} \Gamma_n x \vdash e :: T_{12}; \) or
  2. \( w = c, \)
     \( \text{ty}(c) = \{ \nu \mid \nu = c \land \nu :: x : T_0 \rightarrow T_{02} \}, \)
     and \( \Gamma_n x : T_0 \vdash T_{02} :: x : T_{11} \ightarrow T_{12}. \)
- \( \Gamma_n \models w :: A \) never.
- \( \Gamma_n \models w :: \text{Null} \) iff \( w = \text{null}. \)
- \( \Gamma_n \models w :: C|T \) iff \( \Gamma_n \vdash T, \Psi(C) = [\overline{A}]/[T|T], \) and either:
  1. \( w = \text{null}; \) or
  2. \( w = C|\overline{0} \) and \( \text{for all } j, \Gamma_n w_j :: \text{Inst}(T_j, A, T). \)

**Assumption** (Datatype Representation). This assumption ensures that the implementation treats constructed data just like ordinary
dictionaries. Let \( \Psi(C) = [\overline{A}]/[T|T]. \)

\[
\text{If } \Gamma_n \models w :: C|T, \quad \text{then } \Gamma_n \models \text{tag}(w) = \text{"Dict"}
\]
and \( \Gamma_n \models \land_j \text{Inst}(T_j, A, T)(\text{sel}(w, f_j)). \)

### A.4 Formal Properties

To reduce clutter, we elide the well-formedness requirements of all
expressions, formulas, types, type terms, typing environments, and
type definitions mentioned in the lemmas and theorems that follow.

**1 Lemma** (Inversion).

1. If \( \Gamma_n \vdash x : T_{11} \rightarrow T_{12} < :: x : T_{21} \rightarrow T_{22} \),
   then \( \Gamma_n \vdash x :: T_{21} \subseteq T_{11} \) and \( \Gamma, x : T_{21} \vdash T_{12} \subseteq T_{22}. \)
2. \( \Gamma_n \vdash \lambda x. e :: S, \)
   then \( \Gamma_n \vdash \lambda x. e :: S \quad \text{where} \quad \{ \nu \mid \nu = x \land \nu :: x : T_1 \rightarrow T_2 \} \)

**Proof.** By induction. Note that we have only listed the properties we will need to use. \( \square \)

**2 Lemma** (Reflexive Subtyping).

1. \( \Gamma_n \vdash p \Rightarrow p \)
2. \( \Gamma_n U < :: U \)
3. \( \Gamma_n S \subseteq S \)

**Proof.** By mutual induction. \( \square \)

**3 Lemma** (Transitive Subtyping).

1. If \( \Gamma_n p \Rightarrow q \) and \( \Gamma_n q \Rightarrow r, \)
   then \( \Gamma_n p \Rightarrow r. \)
2. If \( \Gamma_n U_1 < :: U_2 \) and \( \Gamma_n U_2 < :: U_3, \)
   then \( \Gamma_n U_1 < :: U_3. \)

3. If \( \Gamma_n S_1 \subseteq S_2 \) and \( \Gamma_n S_2 \subseteq S_3, \) then \( \Gamma_n S_1 \subseteq S_3. \)

**Proof.** By mutual induction.

**4 Lemma** (Narrowing). Suppose \( \Gamma_n S \subseteq S'. \)

1. If \( \Gamma_n x : S' \vdash p \Rightarrow q, \)
   then \( \Gamma_n x : S \vdash p \Rightarrow q. \)
2. If \( \Gamma_n x : S' \vdash U_1 < :: U_2, \)
   then \( \Gamma_n x : S \vdash U_1 < :: U_2. \)
3. If \( \Gamma_n x : S' \vdash S_1 \subseteq S_2, \)
   then \( \Gamma_n x : S \vdash S_1 \subseteq S_2. \)
4. If \( \Gamma_n x : S' \vdash e :: S_1, \)
   then \( \Gamma_n x : S \vdash e :: S_1. \)

**Proof.** By mutual induction.

**5 Lemma** (Weakening). Suppose \( \Gamma = \Gamma_1, \Gamma_2 \) and \( \Gamma' \) is such that \( \vdash \Gamma' \) and either

\[
\Gamma' = \Gamma_1, x : S, \Gamma_2 \quad \text{or} \quad \Gamma' = \Gamma_1, p, \Gamma_2 \quad \text{or} \quad \Gamma' = \Gamma_1, A, \Gamma_2.
\]

1. If \( \Gamma_n q \Rightarrow q, \) then \( \Gamma_n q \Rightarrow q. \)
2. If \( \Gamma_n U_1 < :: U_2, \) then \( \Gamma_n U_1 < :: U_2. \)
3. If \( \Gamma_n S_1 \subseteq S_2, \) then \( \Gamma_n S_1 \subseteq S_2. \)
4. If \( \Gamma_n e :: S, \) then \( \Gamma_n e :: S. \)

**Proof.** By mutual induction.

**6 Lemma** (Free Variables in Subtyping). Recall that the variable \( \nu \) can appear free in the formulas, type terms, and types mentioned in the
outputs of the following derivations. Suppose \( lw \) is a closed, well-formed value.

1. If \( \Gamma_n p \Rightarrow q, \) then \( \Gamma_n p [lw/v] \Rightarrow q[lw/v]. \)
2. If \( \Gamma_n U_1 < :: U_2, \) then \( \Gamma_n U_1 [lw/v] < :: U_2[lw/v]. \)
3. If \( \Gamma_n S_1 \subseteq S_2, \) then \( \Gamma_n S_1 [lw/v] \subseteq S_2[lw/v]. \)

**Proof.** By mutual induction. The premise of the C-VALID-N case is \( \Gamma_n \models \{ \Gamma \} \land p \Rightarrow q. \) Since \( \nu \) appears free in the implication,
by Free Variable Substitution, \( \Gamma_n \models \{ \Gamma \} \land p [lw/v] \Rightarrow q[lw/v]. \)
Thus, by C-VALID-N, \( \Gamma_n \models p [lw/v] \Rightarrow q[lw/v]. \) The rest of the proof is a straightforward induction. \( \square \)

**7 Lemma** (Sound Variance).

1. Suppose \( \Gamma_n T_1 \subseteq T_2. \)
   (a) If \( B \) appears only positively in \( T, \)
   then \( \Gamma_n \text{Inst}(T, B, T_1) \subseteq \text{Inst}(T, B, T_2). \)
   (b) If \( B \) appears only positively in \( p, \)
   then \( \Gamma_n \{ \text{Inst}(p, B, T_1) \} \subseteq \{ \text{Inst}(p, B, T_2) \}. \)
2. Suppose \( \Gamma_n T_1 \subseteq T_2. \)
   (a) If \( B \) appears only negatively in \( T, \)
   then \( \Gamma_n \text{Inst}(T, B, T_2) \subseteq \text{Inst}(T, B, T_1). \)
   (b) If \( B \) appears only negatively in \( p, \)
   then \( \Gamma_n \{ \text{Inst}(p, B, T_2) \} \subseteq \{ \text{Inst}(p, B, T_1) \}. \)
3. Suppose \( \Gamma_n T_1 \subseteq T_2 \) and \( \Gamma_n T_2 \subseteq T_1. \)
   (a) Then \( \Gamma_n \text{Inst}(T, B, T_2) \subseteq \text{Inst}(T, B, T_1) \)
   and \( \Gamma_n \text{Inst}(T, B, T_1) \subseteq \text{Inst}(T, B, T_2). \)
   (b) Then \( \Gamma_n \{ \text{Inst}(p, B, T_2) \} \subseteq \{ \text{Inst}(p, B, T_1) \} \)
   and \( \Gamma_n \{ \text{Inst}(p, B, T_1) \} \subseteq \{ \text{Inst}(p, B, T_2) \}. \)

**Proof.** The proofs of (1) and (2) are by mutual induction on types and formulas. The proof of (3) is a stand-alone induction on types and formulas.

**Proof of (1a).**

Let \( T = \{ \nu \mid p \}. \)

The goal follows by IH (1b), since
\[
\text{Inst}(\{ \nu \mid p \}, B, T_1) = \{ \nu \mid \text{Inst}(p, B, T_1) \}
\]
and
\[
\text{Inst}(\{ \nu \mid p \}, B, T_2) = \{ \nu \mid \text{Inst}(p, B, T_2) \}.
\]
Proof of (1b).

Case: $p = lw$.

Trivial, since $\text{Inst}(p, B, T_1) = \text{Inst}(p, B, T_2) = p$.

Cases: $p = q_1 \land q_2$, $p = q_1 \lor q_2$.

By IH (1b), C-VALID, and S-MONO.

Case: $p = \neg q$.

By IH (2b), C-VALID, and S-MONO.

Case: $p = lw :: U$.

Subcase: $U = B$.

By definition, $\text{Inst}(p, B, T_1) = [T_1](lw)$.

By definition, $\text{Inst}(p, B, T_2) = [T_2](lw)$.

By Free Variables in Subtyping, $\vdash^n T_1[lw/\nu] \subseteq T_2[lw/\nu]$.

That is, $\vdash^n \{[T_1][lw]\} \subseteq \{[T_2][lw]\}$.

Subcase: $U = x : S_1 \rightarrow S_2$.

(Not that we are using $S_1$ and $S_2$ for types.)

Let $S_{11} = \text{Inst}(S_1, B, T_1)$ and $S_{12} = \text{Inst}(S_1, B, T_2)$.

Let $S_{21} = \text{Inst}(S_2, B, T_1)$ and $S_{22} = \text{Inst}(S_2, B, T_2)$.

Since $B$ appears only pos in $T$, it appears only neg in $S_1$.

By the well-formedness of the type definition and the decision of Poles.

By IH (2a), $\vdash^n S_{12} \sqsubseteq S_{11}$.

Since $B$ appears only pos in $T$, it appears only pos in $S_2$.

By IH (1a), $\vdash^n S_{21} \sqsubseteq S_{22}$.

By Weakening, $x : S_{21} \vdash^n S_{21} \sqsubseteq S_{22}$.

By U-ARROW, $\vdash^n x : S_{11} \rightarrow S_{21} < : x : S_{12} \rightarrow S_{22}$.

By C-VALID,

$p :: x : S_{11} \rightarrow S_{21} \vdash^n \text{true} \Rightarrow p :: x : S_{11} \rightarrow S_{21}$.

By C-IMPSYN,

$\vdash^n p :: x : S_{11} \rightarrow S_{21} \Rightarrow p :: x : S_{12} \rightarrow S_{22}$.

By S-MONO,

$\vdash^n \{p :: x : S_{11} \rightarrow S_{21}\} \sqsubseteq \{p :: x : S_{12} \rightarrow S_{22}\}$.

That is, $\vdash^n \text{Inst}(T, B, T_1) \sqsubseteq \text{Inst}(T, B, T_2)$.

Subcase: $U = C[S]$, where $\Psi(C) = [\theta|A|\{\cdots\}$.

(Not that we are using $S$ for a type.)

Subsubcase: $\theta = \ast$.

Let $S_1 = \text{Inst}(S, B, T_1)$ and $S_2 = \text{Inst}(S, B, T_2)$.

Since $B$ appears only pos in $T$, it appears only pos in $S$.

By IH (1a), $\vdash^n S_1 \sqsubseteq S_2$.

By U-DATATYPE, $\vdash^n C[S_1] < : C[S_2]$.

By C-VALID, C-IMPSYN, and S-MONO,

$\vdash^n \{p :: C[S_1]\} \sqsubseteq \{p :: C[S_2]\}$.

Subsubcase: $\theta = \ast$.

Similar.

Subsubcase: $\theta = -$.

Since $B$ appears only pos in $T$, it cannot appear in $S$.

Thus, $\text{Inst}(T, B, T_1) = \text{Inst}(T, B, T_2) = T$.

Proof of (2a) and (2b). Similar.

Proof of (3). Straightforward induction.

8 Lemma (Lifting).

1. If $\Gamma \vdash_n p \Rightarrow q$, then $\Gamma \vdash_{n+1} p \Rightarrow q$.

2. If $\Gamma \vdash_n U_1 \ll U_2$, then $\Gamma \vdash_{n+1} U_1 \ll U_2$.

3. If $\Gamma \vdash_n S_1 \sqsubseteq S_2$, then $\Gamma \vdash_{n+1} S_1 \sqsubseteq S_2$.

4. If $\Gamma \vdash_n e :: S$, then $\Gamma \vdash_{n+1} e :: S$.

5. If $\mathcal{I}_n \models p$, then $\mathcal{I}_{n+1} \models p$.

Furthermore, for each of the first four properties, the size of the output derivation is the same size as the original.

Proof. By mutual induction. In the C-VALID-N case of (1), the conclusion follows by C-VALID-N after applying IH (5). The type predicate case for (5) follows from IH (4).

9 Lemma (Strengthening). Suppose $\mathcal{I}_n \models p$.

1. If $p, \Gamma \vdash_n q_1 \Rightarrow q_2$, then $\Gamma \vdash_n q_1 \Rightarrow q_2$.

2. If $p, \Gamma \vdash_n U_1 \ll U_2$, then $\Gamma \vdash_n U_1 \ll U_2$.

3. If $p, \Gamma \vdash_n S_1 \sqsubseteq S_2$, then $\Gamma \vdash_n S_1 \sqsubseteq S_2$.

4. If $p, \Gamma \vdash_n e :: S$, then $\Gamma \vdash_n e :: S$.

Furthermore, for each property, the size of the output derivation is the same size as the original.

Proof. By mutual induction.

Proof of (1).

Case: C-VALID-N.

$\mathcal{I}_n \models [p, \Gamma] \land q_1 \Rightarrow q_2$.

By expanding the embedding, $\mathcal{I}_n \models p \land [\Gamma] \land q_1 \Rightarrow q_2$.

Thus, $\mathcal{I}_n \models p \Rightarrow [\Gamma] \land q_1 \Rightarrow q_2$.

Because of the assumption, $\mathcal{I}_n \models [\Gamma] \land q_1 \Rightarrow q_2$.

By C-VALID-N, $\Gamma \vdash_n q_1 \Rightarrow q_2$.

By Validity, $\mathcal{I}_n \models [p, \Gamma] \land q_1 \Rightarrow q_2$.

The rest of the reasoning in this case follows the previous case.

$\exists j. \text{Valid}([p, \Gamma] \land q \Rightarrow lw_j :: U)$.

Case: C-IMPSYN.

$p, \Gamma \vdash_n q \Rightarrow \forall i lw_i :: U_i$.

By C-VALID, $\Gamma \vdash_n q \Rightarrow lw_j :: U_j$.

By IH (2), $q \vdash_n U_i < : U_j$.

By C-IMPSYN, $\Gamma \vdash_n q \Rightarrow \forall i lw_i :: U_i$.

Proof of (2), (3), and (4). Straightforward induction.

The following lemma intuitively captures the relationship between the type system and the underlying refinement logic: if a closed value $w$ can be given the type $T$ with a derivation at level $n$, then the formula $[T](w)$ is true in the System D Interpretation at level $n+1$. This property plays a crucial role in the proof of Value Substitution. Notice that nothing is said about values that are assigned polypotypes.

Because the following lemma works only with the empty environment, the Strengthening lemma is helpful for proving the C-IMPSYN and S-MONO cases, which have premises that use non-empty environments.

10 Main Lemma (Satisfying Typing).

1. If $\Gamma \vdash_n p \Rightarrow q$, then $\mathcal{I}_{n+1} \models p \Rightarrow q$.

2. If $\Gamma \vdash_n U_1 \ll U_2$, then $\mathcal{I}_{n+1} \models U_1 \Rightarrow U_2$.

3. If $\Gamma \vdash_n \{\nu \mid p\} \subseteq \{\nu \mid q\}$, then $\mathcal{I}_{n+1} \models p \Rightarrow q$.

4. If $\Gamma \vdash_n w :: T$, then $\mathcal{I}_{n+1} \models [T](w)$.

In the first three properties, the variable $\nu$ appears free in the implication. Thus, they are implicitly quantified over all values.
Proof. By mutual induction on the size of derivations, not by structural induction. The reason for this induction principle is that in the C-IMPSYN and S-MONO cases, subderivations are manipulated by Lifting and Strengthening (which preserve derivation size) before appealing to the induction hypothesis.

Proof of (1).

\textbf{Case: C-VALID.}

\[
\vdash \forall p \ p \Rightarrow q \\
\]
By Validity, \(\models \text{true} \land p \Rightarrow q\), and thus, \(\models p \Rightarrow q\).

\textbf{Case: C-VALID-N.}

\[
\exists j. \models \text{true} \land p \Rightarrow \omega_{ij} :: U_i \\
\]
By Validity and Lifting, \(\models p \Rightarrow q\).

\textbf{Case: C-IMPSYN.}

We assume \(\models p \Rightarrow q\) and will prove \(\models \omega_i :: U_i\).

By Validity (1), \(\vdash \omega_i :: U_i\).

By IH (1), \(\models \omega_i :: U_i\).

By Transitive Subtyping, \(\vdash \omega_i :: U_i\).

The last derivation is the same size as \(\vdash U < U_i\) since Lifting and Strengthening preserve derivation size.

Thus, we can apply the induction hypothesis.

By IH (2), \(\models \nu :: U \Rightarrow \nu :: U_i\).

By Type Predicate Interpretation, there are two cases.

\textbf{Subcase: \(\nu = \lambda x. e\).}

\[
\vdash \lambda x. e :: U_i \\
\]
By Type Predicate Interpretation, \(\vdash \lambda x. e :: U_i\).

\textbf{Subcase: \(\nu = \text{Inst} (T_j, A, T_i)\).}

\[
\vdash \text{Inst} (T_j, A, T_i) :: T_i \\
\]
By Type Predicate Interpretation, \(\vdash \text{Inst} (T_j, A, T_i) :: T_i\).

Proof of (2).

Case: U-ARROW.

\[
\vdash \forall x :: T_1 \rightarrow T_2 :: x :: T_1 \rightarrow T_2 \\
\]
Let \(U_1 = x :: T_1 \rightarrow T_2\) and \(U_2 = x :: T_1 \rightarrow T_2\).

We assume \(\models \nu :: U_i\) and will prove \(\models \nu :: U_i\).

By Type Predicate Interpretation, \(\vdash \lambda x. e :: U_i\).

\textbf{Subcase: \(\nu = \lambda x. e\).}

\[
\vdash \lambda x. e :: U_i \\
\]
By Lifting, \(\vdash \lambda x. e :: T_i\).

By Narrowing, \(\vdash \lambda x. e :: T_i\).

By T-SUB, \(\vdash \lambda x. e :: T_i\).

Thus, by Type Predicate Interpretation, \(\vdash \lambda x. e :: U_i\).

\textbf{Subcase: \(\nu = \text{Inst} (T_j, A, T_i)\).}

\[
\vdash \text{Inst} (T_j, A, T_i) :: U_i \\
\]
By Type Predicate Interpretation, \(\vdash \text{Inst} (T_j, A, T_i) :: U_i\).

\textbf{Case: U-VAR.}

Trivial.

\textbf{Case: U-NULL.}

By Type Predicate Interpretation.

\[
\Psi (C) = \begin{cases} \theta A \{ f : T \} & \text{if } \theta \in \{ +, \} \text{ then } \forall i. \exists j. \text{sel} (C [\overline{w}], f_j) = w_j \\ \forall i. \exists j. \text{sel} (C [\overline{w}], f_j) = w_j & \text{if } \theta \in \{ -, \} \text{ then } \forall i. \exists j. \text{sel} (C [\overline{w}], f_j) = w_j \\ \end{cases} \\
\]
\textbf{Case: U-DATATYPE.}

\[
\vdash \forall n :: T_i < : C [T_i] \rightarrow C [T_i] \\
\]
We consider the special case when there is exactly one type parameter \(A\) with variance annotation \(\theta\). The type actuals are, therefore, labeled \(T_1\) and \(T_2\). The reasoning extends to an arbitrary number of type parameters by a strong induction on the length of the sequence.

Subcase: \(\theta = +\).

Consider an arbitrary \(w_0\) such that \(\models w_0 :: C [T_i]\).

By Type Predicate Interpretation, there are two cases.

In one case, \(w_0 = \text{null}\), and trivially \(\models w_0 :: C [T_i]\).

In the other case, \(w_0 = C [\overline{w}]\) and \(\forall j. \forall n. w_j :: \text{Inst} (T_j, A, T_i)\).

By well-formedness of the type definition, \(A\) appears only positively in every \(T_j\).

By Type Predicate Interpretation, \(\vdash \exists j. \text{Inst} (T_j, A, T_i) :: U_i\).

By T-SUB, \(\vdash \text{Inst} (T_j, A, T_i) :: U_i\).

By Type Predicate Interpretation, \(\vdash \exists j. \text{Inst} (T_j, A, T_i) :: U_i\).

Subcase: \(\theta = -\). Similar, using Type Predicate Interpretation.

Subcase: \(\theta = \ast\). Similar, using Type Predicate Interpretation.

Proof of (3). Only the rule for monotypes applies.

\[
\text{fresh } p' :: p [x / \nu] q' :: q [x / \nu] \quad \forall (q_1, q_2) \in \text{Normalize} (q'). \forall p' \vdash q_1 :: q_2 \\
\]
\textbf{Case: S-MONO.}

\[
\vdash \forall \nu :: p [x / \nu] \sqsubseteq \forall \nu :: q [x / \nu] \\
\]
Note that the alpha-renaming preserves satisfiability.

So we assume \(\models \nu :: p'\) and then prove \(\models \nu :: q'\).

By Strengthening on each premise, \(\vdash q_1 :: q_2\).

Each of these derivations has the same size as the original.

Thus, by IH (1) on each, \(\vdash q_i :: q_2\).

Thus, \(\vdash \nu :: p [x / \nu] \sqsubseteq \nu :: q [x / \nu]\).

Proof of (4). We only need to consider the rules that can derive a monotype \(T\) for a value \(w\) in the empty environment.

Case: T-CONST.

By Constant Types (Valid).

Case: T-EXTEND.

Trivially, since \(\nu :: \text{Inst} (T_j, A, T_i) :: \forall \nu :: U\).

\[
U = x :: T_1 \rightarrow T_2 \\
\]
\textbf{Case: T-FUN.}

By Type Predicate Interpretation, \(\vdash \lambda x :: U :: e :: U\).

Furthermore, by Validity, \(\vdash \lambda x :: e :: U :: e :: x :: U\).

\[
\Psi (C) = \begin{cases} \theta A \{ f : T \} & \text{if } \theta \in \{ +, \} \\ \forall i. \exists j. \text{sel} (C [\overline{w}], f_j) = w_j & \text{if } \theta \in \{ -, \} \\ \end{cases} \\
\]
\textbf{Case: T-FOLD.}

\[
\vdash \forall \nu :: \text{Fold} (C, T, \overline{w}) :: \forall \nu :: C [T] \\
\]
We consider each of the components of the formula from Fold.

By Validity, \(\vdash \nu :: \overline{w} \neq \text{null}\).

By Type Predicate Interpretation, \(\vdash \nu :: C [\overline{w}] :: C [\overline{T}]\).

By Datatype Representation, \(\vdash \nu :: \text{tag} (C [\overline{w}]) :: \text{Biect}\).

\textbf{Case: U-FOLD.}

\[
\vdash \forall u :: \nu :: \text{Fold} (C, T, \overline{w}) :: \nu :: C [\overline{T}] \\
\]
By IH (3), \(\vdash \forall u :: C [\overline{T}]\).

The goal follows by Type Predicate Interpretation and Datatype Representation.

\[
\vdash \forall u :: T' :: \forall u :: T' :: C [T] \\
\]
\textbf{Case: T-SUB.}

\[
\vdash \forall u :: T' :: \forall u :: T' :: T \\
\]
In the following lemma we lift substitution to judgments in the obvious way. For example, we write \( (\Gamma \vdash e :: S)[w/x] \) to mean \( \Gamma[e/x] \vdash e[w/x] :: S[w/x] \).

11 Main Lemma (Stratified Value Substitution). Let \( \vdash n \) w :: S.

1. If \( \vdash x, S, \Gamma, \vdash p \Rightarrow q \), then \( (\Gamma[n/x] + 1) \vdash p \Rightarrow q \).
2. If \( x, S, \Gamma \vdash U_1 < U_2 \), then \( (\Gamma[n/x] + 1) \vdash U_1 < U_2 \).
3. If \( x, S, \Gamma \vdash S_1 < S_2 \), then \( (\Gamma[n/x] + 1) \vdash S_1 < S_2 \).
4. If \( x, S, \Gamma \vdash e :: S' \), then \( (\Gamma[n/x] + 1) \vdash e :: S' \).

Proof. By mutual induction. In the C-VALID and C-VALID-N cases, we will distinguish between whether S is a monotype or a polymorphic type scheme. In all other cases, this difference will not affect the reasoning. The T-VAR case is interesting because singleton types must be preserved after substitution.

Proof of (1). Recall that we use the notation \( p(x) \) to mean \( p[x/v] \) and \( [T](x) \) to mean \( T[x/v] \). Furthermore, we lift this to \( [\Gamma](x) \) in the obvious way.

\[ I_n = [x:S, \Gamma] \]
\[ x:S, \Gamma \vdash p \Rightarrow q \]

Case: C-VALID-N.

Subcase: \( S = T \).

Thus, \( I_n = [T] \) \( \vdash [\Gamma](x) \) \( \vdash p \Rightarrow q \).

By Substitution, \( I_n = [T] \) \( \vdash [\Gamma](x) \) \( \vdash p \Rightarrow q \).

By Lifting, \( I_n = [\Gamma[w/x]] \) \( \vdash w[x] \Rightarrow q[w/x] \).

By Satisfactory Typing, \( I_n = [\Gamma[w/x]] \) \( \vdash w[x] \Rightarrow q[w/x] \).

By C-VALID-N, \( [w/x] \) \( \vdash p[w/x] \Rightarrow q[w/x] \).

Valid(\( [x:S, \Gamma] \)

\[ x:S, \Gamma \vdash p \Rightarrow q \]

Case: C-VALID.

Subcase: \( S = T \).

Thus, Valid(\( [\Gamma](x) \) \( \vdash p \Rightarrow q \)).

By Substitution, \( I_n = [\Gamma[w/x]] \) \( \vdash p[w/x] \Rightarrow q[w/x] \).

By Lifting, \( I_n = [\Gamma[w/x]] \) \( \vdash p[w/x] \Rightarrow q[w/x] \).

Case: C-IMP SYN.

\[ x:S, \Gamma \vdash p \Rightarrow \forall x w[j]: U \]

By C-VALID, \( x:S, \Gamma \vdash p \Rightarrow \forall x U \).

By IH (1), \( \vdash [\Gamma[w/x]] \) \( \vdash p[w/x] \Rightarrow U[w/x] \).

By IH (2), \( \vdash [\Gamma[w/x]] \) \( \vdash p[w/x] \Rightarrow U[w/x] \).

By C-IMP SYN, \( [\Gamma[w/x]] \) \( \vdash p[w/x] \Rightarrow (\forall x U) \).

Proof of (3). Straightforward induction.
Note that in this case, $e = \lambda y. e_0$.

By IH (4), $\Gamma'[w/x], y: T_1[w/x] \vdash_{n+1} e_0[w/x] :: T_2[w/x]$.

By T-FUN,

$\Gamma'[w/x], y: T_1[w/x] \vdash_{n+1} e[w/x] :: \{ \nu \equiv e[w/x] \wedge \nu' : U[w/x] \}$.

Thus, $\Gamma'[w/x], y: T_1[w/x] \vdash_{n+1} e[w/x] :: \{ \nu = e \wedge \nu' : U[w/x] \}$.

$x: S, \Gamma \vdash_{n+1} w_1 :: \{ \nu : x : T_{11} \rightarrow T_{12} \}$

$x: S, \Gamma \vdash_{n+1} w_2 :: T_{11}$

Case: T-APP.

Let $\Gamma' \equiv \Gamma'[w/x], y: T_1[w/x], w_1' = w_1[w/x], w_3 = w_2[w/x]$, $T_{11}' = T_{11}[w/x], T_{12} = T_{12}[w/x]$.

By IH (4), $\Gamma \vdash_{n+1} w_1' :: \{ \nu : y : T_{11} \rightarrow T_{12} \}$.

Thus, $\Gamma' \vdash_{n+1} w_1' :: \{ \nu : \nu' : T_{11} \rightarrow T_{12} \}$.

By IH (4), $\Gamma \vdash_{n+1} w_2 :: T_{11}$.

By T-APP, $\Gamma' \vdash_{n+1} w_1' w_2 :: T_{12}[w_3/y]$.

Thus, $\Gamma \vdash_{n+1} w_1' w_2 :: T_{12}[w_3/y]$.

We now expand $T_{12}[w_3/y]$ to $T_{12}[w/x][w_3/y][w/x]$.

Since $w$ and $w_2$ are closed values, and $x$ and $y$ are distinct, this is the same as $T_{12}[w_3/y][w/x][w/x]$.

Furthermore, this is $(T_{12}[w_3/y])(w/x)$.

Finally, we note that $w_1' w_2' = (w_1 w_2)[w/x]$.

Thus, the derivation from T-APP does indeed satisfy the goal.

Case: T-SUB.

By IH (4), $\Gamma'[w/x] \vdash_{n+1} e'[w/x] :: S'[w/x]$.

By IH (3), $\Gamma'[w/x] \vdash_{n+1} S''[w/x] :: S'[w/x]$.

By T-SUB, $\Gamma'[w/x] \vdash_{n+1} e \equiv S''[w/x]$.

Cases: T-LET, T-IF, T-FUN, T-TAPP, T-EXTEND.

By IH on the premises and original rule to conclude.

Cases: T-FOLD, T-UNFOLD.

By IH on the premises and original rule to conclude.

Proof of (3). By induction on the derivation. We consider only the rules that can derive a polytype.

Case: T-FUN.

Immediate.

Case: T-TAPP.

Impossible, since $w$ is a value.

Case: T-VARPOLY.

Impossible, since the environment is empty.

Case: T-SUB.

By Canonical Forms, there are two cases.

Subcase: $w = true$.

By E-IFTRUE, $e' = e_1$.

Valid(true = true), so by Strengthening, $\vdash_{n+1} e_1 :: S$.

By Lifting, $\vdash_{n+1} e_1 :: S$.

Subcase: $w = false$.

By E-IFFalse, $e' = e_2$.

Valid(false = false), so by Strengthening, $\vdash_{n+1} e_2 :: S$.

By Lifting, $\vdash_{n+1} e_2 :: S$.

Case: T-APP.

By Canonical Forms, there are two cases.

Subcase: $w_1 :: Bool$.

By Satisfiable Typing, $\vdash_{n+1} w_1 :: True$.

By Canonical Forms, there are two cases.

Subcase: $w_1 :: T_{11}$.

By Value Substitution, $\vdash_{n+1} e_0[w_2/x] :: T_{12}[w_2/x]$.

This concludes the subcase, since by E-APP, $e' = e_0[w_2/x]$.

Subcase: $w_1 = c$.

Since $\vdash_{n+1} w_2 :: T_{11}$, we are also given that $\delta(c, w_2)$ is defined and $\vdash_{n+1} \delta(c, w_2) :: T_{12}[w_2/x]$.

By Lifting, $\vdash_{n+1} \delta(c, w_2) :: T_{12}[w_2/x]$.

This concludes the subcase, since by E-Delta, $e' = \delta(c, w_2)$.
\( \vdash_{n} w' :: \forall A. S' \)

**Case: T-TApp.**

\[ \vdash_{n} w'[T] :: \text{Inst}(S', A, T) \]

By Canonical Forms, \( w' = \lambda A. e_0 \) and \( A \vdash_{n} e_0 :: S' \).

By Type Substitution, \( A \vdash_{n} e_0 :: \text{Inst}(S', A, T) \).

By Lifting, \( A \vdash_{n+1} e_0 :: \text{Inst}(S', A, T) \).

This concludes the case, since by T-TApp, \( e' = e_0 \).

\[
\begin{array}{c}
\vdash S_1 \\
\vdash_{n} e_1 :: S_1 \\
\vdash_{n} e_2 :: S_2 \\
\vdash_{n} \text{let } x = e_1 \text{ in } e_2 :: S_2
\end{array}
\]

**Case: T-LET.**

By the IH, there are two cases.

Subcase: \( e_1 \) is a value \( w \).

By E-LET, \( e' = e_2[w/x] \).

By Value Substitution, \( A \vdash_{n+1} e_2[w/x] :: S_2[w/x] \).

Since \( A \vdash_{n} S_2, x \) does not appear free in \( S_2 \), so \( S_2[w/x] = S_2 \).

Subcase: \( e_1 \leftrightarrow e'_1 \) and \( A \vdash_{n+1} e'_1 :: S \).

By E-COMPAT, \( e' = \text{let } x = e'_1 \text{ in } e_2 \).

By Lifting, \( x : S \vdash_{n+1} e_2 :: S_2 \).

By T-LET, \( A \vdash_{n+1} \text{let } x = e'_1 \text{ in } e_2 :: S_2 \).

**Case: T-SUB.**

\[
\begin{array}{c}
\vdash_{n} w :: S' \\
\vdash_{n} S' \sqsubseteq S
\end{array}
\]

By IH, Lifting, and T-SUB.

**15 Corollary (System D Type Soundness).**

If \( \vdash_0 e :: S \), then either \( e \) diverges or \( e \leftrightarrow^* w \) and \( \vdash_{\infty} w :: S \).

*Proof.* Follows from System D$^*$ Type Soundness.
B. Algorithmic Typing

A type checker for System D cannot directly implement the declarative type system for a couple of reasons. First, the typing rules are not syntax-directed because of T-SUB and T-UNFOLD, which can apply to any expression \( e \), and C-IMPSYN, which non-deterministically refers to a type rule \( U \). Second, the syntax of values lacks type annotations, so the premises of rules like T-FUN, T-LET, and T-IF manipulate types that cannot be inferred by the syntax of the expression being checked.

In this section, we define an algorithmic version of the type system. First, we extend the syntax of the language with optional type annotations for binding constructs and for constructed data. Next, we show how to implement the non-deterministic C-IMPSYN rule. Then, we define an algorithmic type system without the non-deterministic T-SUB and T-UNFOLD rules. To eliminate the former, we derive unique types and then add explicit subtyping checks in the typing rules that require them. To eliminate the latter, we eagerly attempt to unfold the types of bindings in anticipation of where T-UNFOLD might be needed. Furthermore, although we could require that all binding constructs and constructed data be annotated with types, this would lead to redundant and tedious type annotations. Instead, we define a bidirectional type system in the style of [25] that locally infers type annotations where possible.

B.1 Syntax

We extend the syntax of System D as follows.

\[
\begin{align*}
  w & ::= \ldots & \text{Values} \\
  & | \lambda x : T, e & \text{annotated function} \\
  & | C[T](\overline{\tau}) & \text{annotated constructed data} \\
  e & ::= \ldots & \text{Expressions} \\
  & | \text{let } x : S = e_1 \text{ in } e_2 & \text{annotated let-binding}
\end{align*}
\]

B.2 Subtyping

The algorithmic subtyping rules for System D are shown in Figure 7. The derivation rules of the algorithmic subtyping, clause implication, and syntactic subtyping relations are analogous to their counterparts in in the declarative system, except that they include an additional input \( U \), which is a set of type terms \( U \). To begin the discussion, this additional input \( U \) should be ignored, and the procedure \( \text{Extend}(\Gamma, x, S) \) can be assumed to extend a type environment in the usual way, that is, \( \Gamma, x : S \); we will return to both of these issues shortly.

Type Extraction. We now show how CA-IMPSYN implements the non-deterministic C-IMPSYN rule. First, we define the procedure \( \text{TypeTerms} \) that traverses the environment \( \Gamma \) and syntactically collect all of its type terms \( U \).

\[
\text{TypeTerms}(\Gamma, x : \{\nu | p\}) = \text{TypeTerms}(\Gamma) \cup \text{TypeTerms}(p)
\]

The interesting case for formulas is for type predicates:

\[
\text{TypeTerms}(lw :: U) = \{ U \}
\]

Notice that types contained within \( U \) are not collected, only “top-level type terms” are.

The CA-IMPSYN rule then uses the following MustFlow procedure to compute which type terms \( U \) out of all possible type terms in the environment (ignoring the “\( \cup U \)" part for now) are such that the solver can prove \( w :: U \) is true for all values \( w \) of type \( T \).

\[
\text{MustFlow}(\Gamma, T, U) = \{ U' \mid \text{Valid}(\Gamma, x : T) \Rightarrow x :: U' \}
\]

where \( U' = \text{TypeTerms}(\Gamma) \setminus U \) and \( x \) is fresh

\[
\begin{align*}
\text{Algorithmic Subtyping} & \quad \Gamma; \vdash S_1 \subseteq S_2 \\
& \quad x \text{ fresh } p'_1 = p_1[x/\nu] \quad p'_2 = p_2[x/\nu] \\
& \quad \text{Normalize}(p_2) = \wedge_i(q_i \Rightarrow r_i) \\
& \quad \forall i, \Gamma; p_i; \vdash q_i \Rightarrow r_i \\
& \quad \Gamma; \vdash \{\nu | p_1\} \subseteq \{\nu | p_2\} & \text{[SA-MONO]} \\
& \quad \Gamma; \vdash S_1 \subseteq S_2 & \text{[SA-POLY]}
\end{align*}
\]

\[
\begin{align*}
\text{Algorithmic Clause Implication} & \quad \Gamma; \vdash q \Rightarrow r \\
& \quad \exists j, \Gamma = \text{MustFlow}(\Gamma, \{\nu | \nu = lw_j\}, U) \\
& \quad \exists U \in U', \Gamma, q; U \cup U' \not\vdash U <: U_j & \text{[CA-IMPERSYN]}
\end{align*}
\]

\[
\begin{align*}
\text{Algorithmic Syntactic Subtyping} & \quad \Gamma; \vdash U_1 <: U_2 \\
& \quad \Gamma; \vdash T_{21} \subseteq T_{11} \quad \text{Extend}(\Gamma, x_1, T_{21}; U_{21} \cup U_{12} \subseteq T_{22} \vdash x_2 : T_{12} <: x_2 : T_{21} \rightarrow T_{22}) & \text{[UA-ARRROW]} \\
& \quad \Gamma; \vdash x_1 : T_{11} \rightarrow T_{12} <: x_2 : T_{21} \rightarrow T_{22} & \text{[UA-NULL]} \\
& \quad \Gamma; \vdash A <: A & \text{[UA-VAR]} \\
& \quad \Gamma; \vdash U \vdash \text{Null} <: C[T] & \text{[UA-DATATYPE]}
\end{align*}
\]

Figure 7. Algorithmic subtyping for System D

That is, CA-IMPSYN tries all type terms \( U \) that C-IMPSYN might possibly refer to.

Termination. We now turn to the question of whether algorithmic subtyping terminates. Because the subtyping, implication, and syntactic subtyping relations are mutually defined, we may worry that it is possible to construct an implication query (and hence a subtyping obligation) which is non-terminating. Indeed, a naïve approach to deciding implications over type predicates using the above strategies (without considering the \( \overline{U} \) parameters) may not terminate. In the following, we write judgments without the \( \overline{U} \) parameters to see what goes wrong when they are not considered.

Consider the environment

\[
\Gamma \vdash y :: \text{Top}, x :: \{\nu | \nu = y \land \nu :: \text{U}\}
\]

where \( U \vdash a :: \{\nu | \nu :: b :: \{\nu | \nu = y \rightarrow \text{Top}\} \rightarrow \text{Top} \) and suppose we wish to check that

\[
\Gamma \vdash \text{true} \Rightarrow y :: \{\nu | \nu = y \rightarrow \text{Top}. \quad (5)
\]

CA-VALID cannot derive this judgment, since the implication

\[
\Gamma \vdash \text{true} \Rightarrow y :: \{\nu | \nu = y \rightarrow \text{Top}
\]

is not valid. Thus, we must try to derive \( \text{Equation 5} \) by CA-IMPSYN. Type extraction derives that \( y :: \text{U} \) in \( \Gamma \), so the remaining obligation is

\[
\Gamma \vdash U <: x :: \{\nu | \nu = y \rightarrow \text{Top}
\]
Because of the contravariance of function subtyping on the left-hand side of the arrow, the following judgment must be derivable:

\[ \Gamma \vdash \{ \nu | \nu = y \} \subseteq \{ \nu | \nu : b : \{ \nu | \nu = y \} \rightarrow \text{Top} \}. \]

After SA-MONO substitutes a fresh variable, say \( \nu' \), for \( \nu \) in both types, this reduces to the clause implication obligation

\[ \Gamma, \nu' = y \vdash \text{true} \Rightarrow \nu' : b : \{ \nu | \nu = y \} \rightarrow \text{Top}. \]

Alas, this is essentially Equation 8, so we are stuck in an infinite loop! We will again extract the type \( U \) for \( y \) (alised to \( \nu' \) here) and repeat the process ad infinitum.

This situation arises because we are allowed to invoke the rule CA-IMPSYN infinitely many times. Then it must also be the case that CA-IMPSYN extracts a single type term from the environment infinitely often, since there are only finitely many in the environment. Thus, to ensure termination, we make the restriction that along any branch of a subtyping derivation, a type term may be extracted from the environment at most once. This is the purpose of the set \( U \) that is propagated through subtyping judgments; the MustFlow procedure excludes from consideration any type terms in the set \( U \) of already-used type terms. Notice that in the CA-IMPSYN rule, the results of the call to MustFlow are included in the already-used set of the syntactic subtyping judgment.

B.3 Bidirectional Type Checking

In this section, we define an algorithm for type checking programs where type annotations for binding constructs and constructed data expressions may or may not be provided. Following work on local type inference [2], our type checking algorithm is split into two mutually-dependent parts: a type synthesis relation \( \Gamma \vdash e \triangleright S \) that given an expression \( e \), a type environment \( \Gamma \), and no information about the expected type of \( e \) attempts to synthesize, or derive, a well-formed type \( S \); and a type conversion relation \( \Gamma \vdash e \triangleleft S \) that, in addition to \( e \) and \( \Gamma \), takes a type \( S \) that is required of \( e \), and checks whether or not \( e \) can indeed be given type \( S \). Thus, \( S \) is an output of a synthesis judgment but an input to a conversion judgment. We will highlight some of the more interesting cases of type checking relations after dealing with two issues.

Inconsistent Type Environments. Recall that the type extraction procedure collects the type terms \( U \) such that \( \text{Valid}(\Gamma, x : T) \Rightarrow x : U) \). If the environment \( \Gamma, x : T \) happens to be inconsistent, then all such implications will be valid. As we will see, our typing rules for function application will depend on type extraction returning exactly one syntactic arrow, which will not be the case in an inconsistent environment. This is a precision issue that we avoid by simply not performing type extraction when in an inconsistent environment. To this end, both the synthesis and conversion algorithms start off by checking whether the environment is inconsistent, and if it is, they trivially succeed.

\[
\begin{align*}
\text{TS-\text{FALSE}} & \quad \text{Valid}(\Gamma) \Rightarrow \text{false} \\
\Gamma & \vdash e \triangleright \text{false} \\
\end{align*}
\]

\[
\begin{align*}
\text{TC-\text{FALSE}} & \quad \text{Valid}(\Gamma) \Rightarrow \text{false} \\
\Gamma & \vdash e \triangleleft S \\
\end{align*}
\]

These rules are sound because when the environment is inconsistent, the underlying implications can be discharged by CA-\text{VALID} anyway.

Unfolding. Unlike T-SUB, uses of T-\text{UNFOLD} cannot be factored into other typing rules, since we cannot syntactically predict where it is needed. It is not sufficient, for example, to unfold type definitions only at uses of variables (that is, in the typing rule for variables). To demonstrate, consider the function

\[ \text{let get_hd x = get x “hd“} \]

and an attempt to assign it the type

\[ \text{get_hd} :: x : \{ \nu \neq \text{null} \land \nu : \text{List}[\text{Top}] \} \rightarrow \{ \nu = \text{sel}(x, “hd“) \}. \]

Say we unfold the type \text{List}[\text{Top}] at the use of \( x \), when it is passed to the \text{get} function. By the definition of Unfold(\text{List}, \text{Top}), we obtain

\[ x \neq \text{null} \Rightarrow (\text{tag}(x) = \{ \text{Dict} \land \text{has}(x, “hd“) \land \text{has}(x, “tl“)) \}
\]

which, together with the assumption that \( x \neq \text{null} \), allows the call to get to typecheck. Then, to check the subsequent call with argument “hd”, we require that \( \text{has}(x, “hd“) \). The unfolded formula is sufficient to prove this, but it is no longer in the environment of logical assumptions, since it was not recorded in the type environment.

Languages like ML leverage pattern matching to determine exactly when to unfold type definitions. We do not have this option, however, since our core language does not include a syntactic form for unpacking constructed data. Instead, we eagerly try to unfold type definitions every time a variable is added to the environment.

We define a procedure Extend that, in addition to extending a type environment as usual, uses type extraction to determine whether the variable has a constructed type and, if so, unfolds and records its type definition.

\[
\begin{align*}
\text{Extend}(\Gamma, x, T) & = \Gamma, x : T, \land_{\{ \Gamma \} \in U} \text{Unfold}(C, \overline{T})[x/\nu] \\
\end{align*}
\]

where \( U = \text{MustFlow}(\Gamma, \{ \nu = x \}, \emptyset) \)

\[
\begin{align*}
\text{Extend}(\Gamma, x, \forall A \cdot S) & = \Gamma, x : \forall A \cdot S \\
\end{align*}
\]

### Constants and Variables.

We now consider some of the algorithmic typing rules. For non-function values, the synthesis rules are similar to the declarative typing rules, whereas the conversion rules invoke synthesis and then call into subtyping to check the synthesized type against the goal.

\[
\begin{align*}
\text{TS-\text{CONST}} & \quad \Gamma \vdash c \triangleright S \quad \Gamma, \emptyset \vdash S' \subseteq S \\
\Gamma & \vdash c \triangleright S' \\
\text{TC-\text{CONST}} & \quad \Gamma \vdash c \triangleright S \quad \Gamma, \emptyset \vdash S' \subseteq S \\
\Gamma & \vdash c \triangleleft S \\
\text{TS-VAR} & \quad \Gamma(x) = T \\
\Gamma & \vdash x \triangleright \nu \vdash x = T \\
\text{TC-VAR} & \quad \Gamma, \emptyset \vdash x \triangleright T' \\
\Gamma & \vdash x \triangleright T \\
\end{align*}
\]

### Functions.

The synthesis rule for annotated functions is straightforward. The best we can do when the function binder \( x \) is not annotated is try to typecheck the body assuming that \( x \) has type \text{Top}.

\[
\begin{align*}
\text{Extend}(\Gamma, x, T_1) & \vdash e \triangleright T_2 \\
\Gamma & \vdash \lambda x : T_1 : e \triangleright \nu : x : T_1 \rightarrow T_2 \\
\end{align*}
\]

\[
\begin{align*}
\text{Extend}(\Gamma, x, \text{Top}) & \vdash e \triangleright T_2 \\
\Gamma & \vdash \lambda x, \text{Top} : e \triangleright \nu : x : \text{Top} \rightarrow T_2 \\
\end{align*}
\]

When checking whether a function, annotated or not, can be converted to a particular type \( T \), we require that \( T \) syntactically have the form \( \{ \nu : U \} \) where \( U \) is an arrow. This seems to be a reasonable source-level requirement, but it could be loosened if needed.

\[
\begin{align*}
\text{Extend}(\Gamma, x, T_1) & \vdash e \triangleright T_2 \\
\Gamma & \vdash \lambda x, e : \nu : x : T_1 \rightarrow T_2 \\
\end{align*}
\]

\[
\begin{align*}
\text{Extend}(\Gamma, x, T_1) & \vdash e \triangleright T_2 \\
\Gamma & \vdash \lambda x, e \triangleright \nu : x : T_1 \rightarrow T_2 \\
\end{align*}
\]
**Function Applications.** The cases for application are the most unique to our setting. To synthesize an application, we must be able to synthesize a type $T_1$ for the function $w_1$ and use type extraction to convert $T_1$ to a syntactic arrow. The procedure MustFlow can return an arbitrary number of syntactic type terms, so we must decide how to proceed in the event that $T_1$ can be extracted to multiple different arrow types. To avoid the need for backtracking in the type checker, and to provide a semantics that is simple for the programmer to understand and use, we consider an application $w_1 w_2$ to be well-typed if there is exactly one syntactic arrow that is applicable for the given argument $w_2$.

Determining what is “applicable” separates into two cases. In the case that we can synthesize a type $T_2$ for $w_2$, we use the following procedure that succeeds if there is exactly one arrow in the set $\mathcal{U}$ of type terms with a domain that is a supertype of $T_2$.

$$\text{FilterByArgTyp}(\Gamma, \mathcal{U}, T_2) =$$

$$\begin{cases} x: T_{11} \to T_{12} & \text{if } x: T_{11} \to T_{12} \text{ is the only } U \in \mathcal{U} \text{ such that } \Gamma; \emptyset \triangleright T_2 \subseteq T_{11} \\ \text{fail} & \text{otherwise} \end{cases}$$

The first synthesis rule for application uses this procedure to derive an output type for the call. (We write parentheses around the type of the function or argument can be synthesized. The second synthesis rule for application uses this procedure to either the type of the function or argument can be synthesized. The equation expression $\Gamma; \emptyset \triangleright T_2 \subseteq T_{11}$ is applicable for the given argument $w_2$ to synthesize a type $T_{12}$ for $w_1$.)

$$\text{FilterByArgTyp}(\Gamma, \mathcal{U}, T_2) =$$

$$\begin{cases} x: T_{11} \to T_{12} & \text{if } x: T_{11} \to T_{12} \text{ is the only } U \in \mathcal{U} \text{ such that } \Gamma; \emptyset \triangleright T_2 \subseteq T_{11} \\ \text{fail} & \text{otherwise} \end{cases}$$

The first synthesis rule for application uses this procedure to derive an output type for the call. (We write parentheses around the last premise, because it is not needed; it is implied by the successful FilterByArgTyp call. We include the premise in the rule for clarity.)

$$\Gamma \triangleright w_1 \triangleright T_1 \quad \Gamma \triangleright w_2 \triangleright T_2$$

$$\mathcal{U} = \text{MustFlow}(\Gamma, T_1, \emptyset)$$

$$x: T_{11} \to T_{12} = \text{FilterByArgTyp}(\Gamma, \mathcal{U}, T_2)$$

$$\Gamma \triangleright w_1 w_2 \triangleright T_{12}[w_2/x]$$

The second synthesis rule for application uses this procedure to derive an output type for the call.

$$\Gamma \triangleright w_1 \triangleright T_1$$

$$\mathcal{U} = \text{MustFlow}(\Gamma, T_1, \emptyset)$$

$$x: T_{11} \to T_{12} = \text{FilterByArgVal}(\Gamma, \mathcal{U}, w_2)$$

$$\Gamma \triangleright w_1 w_2 \triangleright T_{12}[w_2/x]$$

Type conversion for an application can proceed in two ways, if either the type of the function or argument can be synthesized. The first case, when the function type can be synthesized to an arrow, is similar to TS-APP-2 with an additional subtyping check.

$$\Gamma \triangleright w_1 \triangleright T_1$$

$$\mathcal{U} = \text{MustFlow}(\Gamma, T_1, \emptyset)$$

$$x: T_{11} \to T_{12} = \text{FilterByArgVal}(\Gamma, \mathcal{U}, w_2)$$

$$\Gamma; \emptyset \triangleright T_{12}[w_2/x] \subseteq T$$

$$\Gamma \triangleright w_1 w_2 \triangleright T$$

In the second case, when we can synthesize a type $T_2$ for the argument, we combine $T_2$ with the goal $T$ to infer a plausible arrow type for the function. Notice that we use a dummy formal parameter $x$, since we have no (reasonable) way of computing where $x$ might have appeared in $T$ before substituting $w_2$ for $x$.

$$\Gamma \triangleright w_2 \triangleright T_2 \quad x \text{ fresh}$$

$$\Gamma \triangleright w_1 \triangleright \{x: T_2 \to T\}$$

$$\Gamma \triangleright w_1 w_2 \triangleright T$$

**If-expressions.** We can synthesize a precise type for if-expressions by tracking the guard predicates in the output type. Type conversion for if-expressions is straightforward.

$$\Gamma \triangleright w < Bool$$

$$\Gamma; \emptyset \triangleright e_1 \triangleright \{\nu: p_1\}$$

$$\Gamma; \emptyset \triangleright e_2 \triangleright \{\nu: p_2\}$$

$$q \triangleq (w \Rightarrow p_1 \land w \Rightarrow p_2)$$

$$\Gamma \triangleright \text{if } w \text{ then } e_1 \text{ else } e_2 \triangleright \{\nu: q\}$$

**Let-expressions.** The rules for let-expressions share a similar structure. The choice whether to use synthesis or conversion on the equation expression $e_1$ depends on whether there is an annotation $S$ or not. The choice for the body expression $e_2$ depends on the kind of derivation for the overall let-expression. Whenever a let-binding contains an annotation $S$, we must check that it is well-formed. The synthesis rules for both kinds of let-bindings must also check that the synthesized type $T$ is well-formed in $\Gamma$, since we need to ensure that synthesized types are always well-formed in their environment.

$$\Gamma \triangleright S \quad \Gamma \triangleright e_1 \triangleright S \quad \text{Extend}(\Gamma, x, S) \triangleright e_2 \triangleright T \quad \Gamma \triangleright T$$

$$\Gamma \triangleright \text{let } x: S \triangleright e_1 \text{ in } e_2 \triangleright T$$

$$\Gamma \triangleright x: S \triangleright e_2 \triangleright T$$

$$\Gamma \triangleright \text{let } x \triangleright e_2 \triangleright T$$

$$\Gamma \triangleright e_1 \triangleright S \quad \text{Extend}(\Gamma, x, S) \triangleright e_2 \triangleright T \quad \Gamma \triangleright T$$

$$\Gamma \triangleright \text{let } x \triangleright e_1 \text{ in } e_2 \triangleright T$$

$$\Gamma \triangleright \text{let } x \triangleright e_2 \triangleright T$$

Because the syntax of System D is A-normal form, programs will contain many let-expressions. Ideally, our algorithmic type rules will deal well with bare let-expressions well to avoid an overwhelming and redundant annotation burden. The TS-LETBARE-1 rule does not, however, successfully synthesize types in common situations where we would expect it to. We will show three problematic examples and then incorporate a simple technique that supports them.

First, consider the function

```
let get_f (x:{tag(v)="Dict" \ / has(v,"f")}) =
  x "f"
```

In A-normal form, this function might be written as

```
let get_f (x:{tag(v)="Dict" \ / has(v,"f")}) =
  let a = get x in
  let b = a "f" in
  b
```
Notice that the function binder is annotated but the let-binders are not. It seems reasonable to expect that the annotation on x would be sufficient for type synthesis to derive the type

\[\text{get}_f :: x : \{\text{Dict}(v) \land \text{has}(v, "f")\} \rightarrow \{\text{sel}(x, "f")\}\]

but it does not. Consider an attempt to apply TS-LETBARE-1 for the let-expression that binds b. At that point, type synthesis can derive the type \(T = \{\nu = \text{sel}(x, "f")\}\) for the equation expression a "f". Then, in the type environment extended with \(b : T\), TS-VAR synthesizes the singleton type \(\{\nu = b\}\) for the body expression. But this type is, of course, not well-formed in the type environment without the binding for b, so the TS-LETBARE-1 rule fails. This is quite unfortunate, since the TS-VAR rule will be used extensively, and clearly there is a type that we could have used instead of \(\{\nu = b\}\), namely, the type stored for b in the environment!

As a second problematic situation, consider the following variation of the previous example.

```
let maybe_get_f (x:Dict) =
  if mem x "f" then get x "f" else 0
```

In A-normal form, this function might be written as

```
let maybe_get_f (x:Dict) =
  let a = mem x in
  let b = a "f" in
  if b then
    let c = get x in
    c "f"
  else
    0
```

Again, we have a problem applying the TC-LETBARE-1 rule to the let-expression that binds b. The type synthesized for the equation a "f" is \(T = \{\text{Bool}(\nu) \land (\nu = \text{true} \Leftrightarrow \text{has}(x, "f"))\}\). To synthesize the type of the body, the culprit this time is the TS-IF rule, which derives the type \(\{b = \text{true} \Rightarrow \nu = \text{sel}(x, "f")\ \land \ b = \text{false} \Rightarrow \nu = 0\}\) that refers to b. We observe that the type T indicates that it is a boolean flag that records the property \(\text{has}(x, "f")\), so in this case, we would like to replace the problematic body type with \(\{\text{has}(x, "f") = \text{true} \Rightarrow \nu = \text{sel}(x, "f") \land \text{has}(x, "f") = \text{false} \Rightarrow \nu = 0\}\). Furthermore, we might expect to be able to play this trick quite often, since the shape of \(T = \{p = \text{true} \Rightarrow \nu = \text{sel}(x, "f") \land p = \text{false} \Rightarrow \nu = 0\}\) for some formula p is the same as the return type of several common primitive functions, including has and =.

The third and final problematic situation that we consider originates with a small twist on the previous example.

```
let another_maybe_get_f (x:Dict) =
  let a = mem x in
  let b = a "f" in
  if b' then
    let c = get x in
    c "f"
  else
    0
```

This time, the boolean condition used in the if-expression goes through one more level of indirection, namely, the variable b'. Thus, when processing the b' let-expression, the type synthesized by TS-IF for the body expression is \(\{b' = \text{true} \Rightarrow \nu = \text{sel}(x, "f") \land b' = \text{false} \Rightarrow \nu = 0\}\) The type for b', which is \(\{\nu = b\}\), does not, however, match the special shape of boolean flags from before. The trick we can play is to simply replace b' with b, and derive \(\{b = \text{true} \Rightarrow \nu = \text{sel}(x, "f") \land b = \text{false} \Rightarrow \nu = 0\}\) for the body expression. This type is well-formed, and when considered as the body expression for the enclosing let-expression that binds b, will be further rewritten using the technique for eliminating singletons to the type \(\{\text{has}(x, "f") = \text{true} \Rightarrow \nu = \text{sel}(x, "f") \land \text{has}(x, "f") = \text{false} \Rightarrow \nu = 0\}\) We encapsulate these three simple heuristics in a procedure Elim and use it to define the following more precise synthesis rule for bare let-bindings.

\[
\begin{array}{c}
\Gamma \vdash e_1 \triangleright S \\
\text{Extend}(\Gamma, x, S) \vdash e_2 \triangleright T \\
T' = \text{Elim}(x, S, T) \\
\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 \triangleright T' \\
\end{array}
\]

[TS-LetBare-2]

The procedure Elim(x, S, T) procedure takes a variable x whose equation expression has been synthesized to type S, and the type T for the body expression, and attempts to remove occurrences of x. When the procedure succeeds, the resulting type is guaranteed to be well-formed in the environment without x. It starts by processing the top-level refinement predicate.

\[
\text{Elim}(x, S, \nu = x) = \begin{cases} 
  p & \text{if } S = \{\nu | p\} \\
  \text{fail} & \text{otherwise}
\end{cases}
\]

The first non-trivial case is for equality predicates that correspond to the singleton types synthesized by TS-VAR.

\[
\text{Elim}(x, S, \nu = x) = \begin{cases} 
  p & \text{if } S = \{\nu | p\} \\
  \text{fail} & \text{otherwise}
\end{cases}
\]

The other non-trivial case is for equality predicates that equate variables with boolean values, as the TS-IF rule does. The two cases correspond to whether S matches the canonical shape of boolean flags or whether S is a singleton type.

\[
\text{Elim}(x, S, x = \text{true}) = \\
\begin{cases} 
  p & \text{if } S = \{\text{Bool}(\nu) \land (\nu = \text{true} \Leftrightarrow p)\} \\
  \text{fail} & \text{otherwise}
\end{cases}
\]

\[
\text{Elim}(x, S, x = \text{false}) = \\
\begin{cases} 
  \neg p & \text{if } S = \{\text{Bool}(\nu) \land (\nu = \text{true} \Leftrightarrow p)\} \\
  \text{fail} & \text{otherwise}
\end{cases}
\]

The rest of the cases recursively process the formula.

\[
\text{Elim}(x, S, F(\overline{lw})) = F(\text{Elim}(x, S, \overline{lw}))
\]

\[
\text{Elim}(x, S, lw :: U) = \text{Elim}(x, S, lw) :: \text{Elim}(x, S, U)
\]

\[
\text{Elim}(x, S, p \land q) = \text{Elim}(x, S, p) \land \text{Elim}(x, S, q)
\]

As one final heuristic, we attempt to rewrite occurrences of x that do not appear in the two kinds of equality predicates that we have built support for. The following is the non-trivial case for logical values that replaces the variable x when its type is a singleton.

\[
\text{Elim}(x, S, x) = \begin{cases} 
  y & \text{if } S = \{\nu = y\} \\
  \text{fail} & \text{otherwise}
\end{cases}
\]

If variable elimination fails, we can synthesize Top as a last resort.

\[
\begin{array}{c}
\Gamma \vdash e_1 \triangleright S \\
\text{Extend}(\Gamma, x, S) \vdash e_2 \triangleright T \\
\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 \triangleright \text{Top}
\end{array}
\]

[TS-LetBare-3]

Since synthesis annotated let-expressions must also check that the output type is well-formed, we define two additional rules TS-LETANN-2 and TS-LETANN-3 that are analogous to the conversion rules.
**Constructed Data.** We briefly discuss how we infer type parameters that are omitted in constructed data expressions. We extend the syntax of type definitions as follows. For every type variable $A$ of a type definition for constructor $C$, we allow exactly one occurrence of $A$ to be marked, written $\ast A$, in the definition of $C$. When attempting to synthesize a type for unannotated constructed data, we use the positions of marked type variables to match the corresponding positions in the types of the value arguments that are used to construct the record. For simplicity, we infer omitted type parameters for constructed data only when all type parameters are omitted. Therefore, we require that either zero or all of the type parameters in a definition are marked.

For example, we update the `List` definition as follows to use the type of the “`hd`” field to infer the type parameter:

\[
\text{type } \text{List(} \ast A\text{)}\{\text{"hd"} : \{\nu : A\}; \text{"tl"} : \{\nu : \text{List(} \ast A\text{)}\}\}
\]

Therefore, if the variable $x_0$ has type `List[Int]`, then `List(1, x_0)` is well-typed; we infer the type argument `Int`, which is a supertype of $\{\nu = 1\}$. Notice that putting the marker for $A$ in the type of the “`hd`” field would lead to less successful inference, since the type of an element added to a list will often be more specific than the type of the rest of the list, and so the inferred type parameter would be too specific. For example, `List(1, x_0)` would not be well-typed, since the type $\{\nu = x_0\}$ is a subtype of `List[Int]`, but is not a subtype of `List(\{\nu = 1\})`.

**Remaining Rules.** We omit the definition of the remaining synthesis and conversion rules since they do not illuminate any new concerns. Although the techniques that we have employed so far would allow us to, we do not synthesize type instantiations.

### B.4 Soundness

We now consider how derivations in the algorithmic type system relate to derivations in the declarative type system. We use a procedure `erase` to remove type annotations from functions, let-bindings, and constructed data because the syntax of the declarative system does not permit them.

16 Proposition (Sound Algorithmic Typing).

1. If $\Gamma ; U \vdash p \Rightarrow q$, then $\Gamma \vdash p \Rightarrow q$.
2. If $\Gamma ; U \vdash U_1 <; U_2$, then $\Gamma \vdash U_1 <; U_2$.
3. If $\Gamma ; U \vdash S_1 \subseteq S_2$, then $\Gamma \vdash S_1 \subseteq S_2$.
4. If $\Gamma \vdash e >; S$, then $\Gamma \vdash \text{erase}(e) : S$.
5. If $\Gamma \vdash e <; S$, then $\Gamma \vdash \text{erase}(e) : S$.

**Proof sketch.** We consider the key aspects of the development of the algorithmic type system and provide an intuition for why they are sound. To prove that algorithmic clause implication is sound with respect to declarative clause implication, we must consider CA-ImpSYN and its use of the type extraction procedure. It is easy to see that uses of MustFlow can be converted into derivations by C-VALID, since it depends on the validity of logical implications. Proving that algorithmic subtyping and syntactic subtyping are sound with respect to their declarative counterparts goes by induction on their derivation rules, which correspond one-to-one.

To prove that type synthesis and type conversion are sound with respect to declarative typing, there are a few points to consider. The first is the initial check for an inconsistent type environment that TS-FALSE and TC-FALSE perform. It is simple to show that in the declarative system any judgment is derivable when the type environment is inconsistent. The proof is a straightforward induction, using the C-VALID rule to check that an inconsistent environment means all clause implications can be proven valid. Second, we can show that the Extend procedure, which uses type extraction to unfold type definitions, can be replaced with uses of T-Unfold. Third, we can show that in the TC-LetBare-
C. Examples
In this section, we present the original, unadapted source code corresponding to the noted examples in 1 and 2.

C.1 Introduction
The introduction references the following function from the Dojo Javascript library, version 1.6.1 [31]:

```
"base/loader/loader.js"
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d._onto = function(arr, obj, fn)
    if(!fn){
        arr.push(obj);
    }else if(fn){
        var func = (typeof fn == "string") ? obj[fn] : fn;
        arr.push(function(){ func.call(obj); });
    }
```

C.2 Overview
The toXML example is adapted from the Python 3.2 standard library:

```
class DumbXMLWriter:
    def __init__(self, file, indentLevel=0, indent="\t"):
        self.file = file
        self.stack = []
        self.indentLevel = indentLevel
        self.indent = indent

    def beginElement(self, element):
        self.stack.append(element)
        self.writeln("<%s>")
        self.indentLevel += 1

    def endElement(self, element):
        assert self.indentLevel > 0
        assert self.stack.pop() == element
        self.indentLevel -= 1
        self.writeln("</%s>")

    def simpleElement(self, element, value=None):
        if value is not None:
            value = _escape(value)
            self.writeln("<%s>%s</%s>" % (element, value, element))
        else:
            self.writeln("<%s/>")

    def writeln(self, line):
        if line:
            self.file.write(self.indentLevel * self.indent)
            self.file.write(line)
            self.file.write(b"\n")

class PlistWriter(DumbXMLWriter):
    def __init__(self, file, indentLevel=0, indent=b"\t", writeHeader=1):
        if writeHeader:
            file.write(PLISTHEADER)
        DumbXMLWriter.__init__(self, file, indentLevel, indent)
```

def writeValue(self, value):
    if isinstance(value, str):
        self.simpleElement("string", value)
    elif isinstance(value, bool):
        # must switch for bool before int, as bool is a # subclass of int...
        if value:
            self.simpleElement("true")
        else:
            self.simpleElement("false")
    elif isinstance(value, int):
        self.simpleElement("integer", "%d" % value)
    elif isinstance(value, float):
        self.simpleElement("real", repr(value))
    elif isinstance(value, dict):
        self.writeDict(value)
    elif isinstance(value, Data):
        self.writeData(value)
    elif isinstance(value, (tuple, list)):
        self.writeArray(value)
    else:
        raise TypeError("unsupported type: %s" % type(value))

def writeData(self, data):
    self.beginElement("data")
    self.indentLevel += 1
    maxlinelength = 76 - len(self.indent.replace(b"\t", b"\t" * 8))
    for line in data.asBase64(maxlinelength).split(b"\n"):
        if line:
            self.writeln(line)
        self.indentLevel += 1
    self.endElement("data")

def writeDict(self, d):
    self.beginElement("dict")
    items = sorted(d.items())
    for key, value in items:
        if not isinstance(key, str):
            raise TypeError("keys must be strings")
        self.simpleElement("key", key)
        self.writeValue(value)
    self.endElement("dict")

def writeArray(self, array):
    self.beginElement("array")
    for value in array:
        self.writeValue(value)
    self.endElement("array")