Sparsified Cholesky Solvers for SDD linear systems

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Abstract

We show that Laplacian and symmetric diagonally dominant (SDD) matrices can be well approximated by linear-sized sparse Cholesky factorizations. We show that these matrices have constant-factor approximations of the form $\mathbf{L} \mathbf{L}^T$, where $\mathbf{L}$ is a lower-triangular matrix with a number of nonzero entries linear in its dimension. Furthermore linear systems in $\mathbf{L}$ and $\mathbf{L}^T$ can be solved in $O(n)$ work and $O(\log n \log^2 \log n)$ depth, where $n$ is the dimension of the matrix.

We present nearly linear time algorithms that construct solvers that are almost this efficient. In doing so, we give the first nearly-linear work routine for constructing spectral vertex sparsifiers—that is, spectral approximations of Schur complements of Laplacian matrices.

1 Introduction

There have been incredible advances in the design of algorithms for solving systems of linear equations in Laplacian and symmetric, diagonally dominant (SDD) matrices. Cohen et. al. [CKM+14] have recently designed algorithms that find $\epsilon$-approximate solutions to such systems of equations in time $O(m \log^{1/2} n \log \epsilon^{-1})$, where $n$ is the dimension of the matrix and $m$ is its number of nonzero entries. Peng and Spielman [PS14] recently discovered the first parallel algorithms that require only poly-logarithmic time and nearly-linear work. In this paper, we prove that for every such matrix there is an operator that approximately solves equations in this matrix and that can be evaluated in linear work and depth $\log n (\log \log n)^2$. These operators are analogous to the LU decompositions produced by Gaussian elimination: they take longer to compute than to apply.

We present two fast parallel algorithms for finding solvers that are almost as fast. One runs in nearly linear time and polylogarithmic depth (Theorem 9.2). The algorithm presented in Theorem 9.8 has preprocessing depth $n^{o(1)}$, but is more efficient in terms of work and produces a solver whose work and depth are within a logarithmic factor of the best one we can show exists.

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A matrix $A$ is diagonally dominant if each of its diagonal entries is at least the sum of the absolute values of the off-diagonal entries in its row. The most famous symmetric, diagonally dominant matrices are the Laplacian matrices of graphs: those with non-positive off-diagonal such that every diagonal is exactly equal to the sum of the absolute values of the off-diagonal entries in its row. Laplacian and SDD matrices arise in many applications, including the solution of optimization problems such as maximum flow [CKM+11, KMPT2, LRS13, Mad13], minimum cost flow [DS08, LS13], semi-supervised learning [ZGL03], and the solution of elliptic PDEs [BHV08].

Building on the work of Vaidya [Vai90], Spielman and Teng [ST14] discovered that through the use of two constructions in graph theory—sparsifiers and low stretch spanning trees—one could design algorithms for solving such linear equations that run in nearly-linear time. Kelner et. al. [KOSZ13] construct an elementary algorithm for solving SDD systems in nearly linear time that only makes use of low stretch spanning trees. Conversely, Peng and Spielman [PS14] design an algorithm that only uses sparsifiers. The present paper builds on their approach.

The parallel algorithm of Peng and Spielman [PS14] approximates the inverse of a matrix by the sum and product of a small number of sparse matrices. The main bottleneck in their algorithm is that all of the matrices it produces have the same dimension, and that the number of these matrices depends on the condition number of the system to be solved. This leads to each matrix having an average number of nonzero entries per column that is proportional to the square of the logarithmic of the condition number, leading to work $O((m + n \log^3 \kappa) \log \epsilon^{-1})$.

Our result improves on the construction of Peng and Spielman [PS14] in a number of ways. First, the depth and work of our new algorithms are independent of the condition number of the matrix. Second, the matrices in the product that approximates the inverse are of geometrically decreasing sizes. This leads to much faster algorithms. That said, our efficient algorithms for constructing solvers and spectral vertex sparsifiers critically relies on their work.

We introduce sparsified Cholesky factorization in in Section 5 where we prove that the inverse of every SDD matrix $A$ can be approximated by an operator that can be evaluated in linear work and depth $O(\log^2 n \log \log n)$. By using this operator as a preconditioner, or by applying iterative refinement, this leads to a solver that produces $\epsilon$-approximate solutions to systems in $A$ in work $O(m \log \epsilon^{-1})$ and depth $O(\log^2 n \log \log n \log \epsilon^{-1})$, where $m$ is the number of nonzeros in $A$. We begin by eliminating a block consisting of a constant fraction of the vertices. The elimination of these vertices adds edges to the subgraph induced on the remaining vertices. We use the work of [BSS12] to sparsify the modified subgraph (Figure 2, Lemma 5.8 and Theorem 5.10). The choice of which vertices we eliminate is important. We use subset of vertices whose degrees in their induced subgraph are substantially smaller than in the original graph (see Definition 5.1 and Lemma 5.2).

In Section 6 we show how to convert this solver into a sparse approximate inverse. That is, we show that $A$ can be approximated by a product of the form $U^T D U$ where $U$ an upper-triangular matrix with $O(n)$ nonzero entries and $D$ is diagonal. While we can construct this $U$ and $D$ in polynomial time, we do not yet have a nearly linear time or low depth efficient parallel algorithm that does so.

We obtain our best existence result in Section 7 by reducing the depth of the parallel solvers by a logarithmic factor. The reduction comes from observing that the construction of Section 6 would have the desired depth if every vertex in $A$ and in the smaller graphs produced had bounded degree. While we can use sparsification to approximate an arbitrary graph by a sparse
one, the sparse one need not have bounded degree. We overcome this problem by proving that
the Laplacian of every graph can be approximated by a Schur complement of the Laplacian of
a larger graph of bounded degree (Theorem 7.2).

We then turn to the problem of computing our solvers efficiently in parallel. The first
obstacle is that we must quickly compute an approximation of a Schur complement of a set of
vertices without actually constructing the Schur complement, as it could be too large. This
is the problem we call Spectral Vertex Sparsification. It is analogous to the problem of vertex
sparsification for cut and combinatorial flow problems [LM10, Moi13]: given a subset of the
vertices we must compute a graph on those vertices that allows us to compute approximations
of electrical flows in the original graph between vertices in that subset. In contrast with cut
and combinatorial flow problems, there is a graph that allows for this computation exactly on
the subset of vertices, and it is the Schur complement in the graph Laplacian. In Section 8,
we build on the techniques of [PS14] to give an efficient algorithm for spectrally approximating
Schur complements.

The other obstacle is that we need to compute sparsifications of graphs efficiently in parallel.
We examine two ways of doing this in Section 9. The first, examined in Section 9.1, is to use
a black-box parallel algorithm for graph sparsification, such as that of Koutis [Kou14]. This
gives us our algorithm of best total depth. The second, examined in Section 9.2, employs a
recursive scheme in which we solve smaller linear systems to compute probabilities with which
we sample the edges, as in [SST14]. Following [CLM⁺14], these smaller linear systems are obtained
by crudely sub-sampling the original graph. The resulting algorithm runs in depth \( n^{o(1)} \), but
produces a faster solver. We expect that further advances in graph sparsification will result in
even better algorithms.

2 Some Related Work

Gaussian elimination solves systems of equations in a matrix \( A \) by computing lower and upper
triangular matrices \( L \) and \( U \) so that \( A = LU \). Equations in \( A \) may then be solved by solving
equations in \( L \) and \( U \), which takes time proportional to the number of nonzero entries in those
matrices. This becomes slow if \( L \) or \( U \) has many nonzero entries, with is often the case.

Cholesky factorization is the natural symmetrization of this process: it writes symmetric
matrices \( A \) as a product \( LL^T \). Incomplete Cholesky factorizations [MV77] instead approximate
\( A \) by a product of sparse matrices \( LL^T \) by strategically dropping some entries in the computation
of Cholesky factors. One can then use these approximations as preconditioners to compute highly
accurate solutions to systems in \( A \). While this is a commonly used heuristic, there have been
few general theoretical analyses of the performance of the resulting algorithms. Interestingly,
Meijerink and van der Vorst [MV77] analyze the performance of this algorithm on SDD matrices
whose underlying graph is a regular grid.

SDD linear systems have been extensively studied in scientific computing as they arise when
solving elliptic partial differential equations. Multigrid methods have proved very effective at
solving the resulting systems. Fedorenko [Fed64] gave the first multigrid method for SDD systems
on regular square grids and proved that it is an nearly-linear time algorithm. Multigrid methods
have since been used to solve many types of linear systems [Bra77, Hac85], and have been
shown to solve special systems in linear work and logarithmic depth [Nic78, Hac82] under some
smoothness assumptions. Recently, Artem and Yvan [NN12] gave the first algebraic multigrid method with a guaranteed convergence rate. However, to the best of our knowledge, a worst-case nearly-linear work bound has not been proved for any of these algorithms.

Our algorithm is motivated both by multigrid methods and incomplete Cholesky factorizations. Both exploit the fact that elimination operations in SDD matrices result in SDD matrices. That is, Schur complements of SDD matrices result in SDD matrices with fewer vertices. However, where multigrid methods eliminate a large fraction of vertices at each level, our algorithms eliminate a small but constant fraction. The main novelty of our approach is that we sparsify the resulting Schur complement. A heuristic approach to doing this was recently studied by Krishnan, Fattal, and Szeliski [KFS13].

3 Background

We will show that diagonally dominant matrix $A$ can be well-approximated by a product $U^T D U$ where $U$ is upper-triangular and sparse and $D$ is diagonal. By solving linear equations in each of these matrices, we can quickly solve a system of linear equations in $A$. We now review the notion of approximation that we require along with some of its standard properties.

For symmetric matrices $A$ and $B$, we write $A \succ B$ if $A - B$ is positive semidefinite. The ordering given by $\succ$ is called the “Loewner partial order”.

Fact 3.1. For $A$ and $B$ positive definite, $A \succ B$ if and only if $B^{-1} \succ A^{-1}$.

Fact 3.2. If $A \succ B$ and $C$ is any matrix of compatible dimension, then $CAC^T \succ CBC^T$.

We say that $A$ is an $\epsilon$-approximation of $B$, written $A \approx_\epsilon B$, if

$$ e^\epsilon B \succ A \succ e^{-\epsilon} B. $$

Observe that this relation is symmetric.

Simple arithmetic yields the following fact about compositions of approximations.

Fact 3.3. If $A \approx_\epsilon B$ and $B \approx_\delta C$, then $A \approx_{\epsilon + \delta} C$.

We say that $\tilde{x}$ is an $\epsilon$-approximate solution to the system $Ax = b$ if

$$ \| \tilde{x} - A^{-1}b \|_A \leq \epsilon \| x \|_A, $$

where

$$ \| x \|_A = (x^T A x)^{1/2}. $$

This is the notion of approximate solution typically used when analyzing preconditioned linear system solvers, and it is the notion assumed in the works we reference that use these solvers as subroutines.

Fact 3.4. If $\epsilon < 1/2$, $A \approx_\epsilon B$ and $B \tilde{x} = b$, then $\tilde{x}$ is a $2\sqrt{\epsilon}$ approximate solution to $Ax = b$.

So, if one can find a matrix $B$ that is a good approximation of $A$ and such that one can quickly solve linear equations in $B$, then one can quickly compute approximate solutions to systems of linear equations in $A$. Using methods such as iterative refinement, one can use multiple solves.
in $B$ and multiplies by $A$ to obtain arbitrarily good approximations. For example, if $B$ is a constant approximation of $A$, then for every $\epsilon < 1$, one can obtain an $\epsilon$ approximate solution of a linear system in $A$ by performing $O(\log(\epsilon^{-1}))$ solves in $B$ and multiplies by $A$ (see, for example, [PS14, Lemma 4.4]).

It is known that one can reduce the problem of solving systems of equations in SDD matrices to either the special case of Laplacian matrices or SDDM matrices—the family of SDD matrices that are nonsingular and have non-positive off diagonal entries (see, e.g. [ST14, CKM+14]). We will usually consider SDDM matrices. Every SDDM matrix $A$ can be uniquely written as a sum $L + X$ where $L$ is a Laplacian matrix and $X$ is a nonnegative diagonal matrix.

The main properties of SDDM matrices that we exploit are that they are closed under Schur complements and that they can be sparsified. The strongest known sparsifications come from the main result of [BSS12], which implies the following.

**Theorem 3.5.** For every $n$-dimensional SDDM matrix $A$ and every $\epsilon \leq 1$, there is a SDDM matrix $B$ having at most $10n/\epsilon^2$ nonzero entries that is an $\epsilon$-approximation of $A$. In particular, the number of non-zero entries in $B$ above the diagonal is at most $4.1n/\epsilon^2$.

While the matrix $B$ guaranteed to exist by this theorem may be found in polynomials time, this is not fast enough for the algorithms we desire. So, we only use Theorem 3.5 to prove existence results. We later show how to replace it with faster algorithms, at some expense in the quality of the sparsifiers we produce.

### 4 Block Cholesky Factorization

Our algorithm uses block-Cholesky factorization to eliminate a block of vertices all at once. We now review how block-Cholesky factorization works.

To begin, we remind the reader that Cholesky factorization is the natural way of performing Gaussian elimination on a symmetric matrix: by performing eliminations on rows and columns simultaneously, one preserves the symmetry of the matrix. The result of Cholesky factorization is a representation of a matrix $M$ in the form $U^T U$, where $U$ is an upper-triangular matrix. We remark that this is usually written as $LL^T$ where $L$ is lower-triangular. We have chosen to write it in terms of upper-triangular matrices so as to avoid confusion with the use of the letter $L$ for Laplacian matrices.

To produce matrices $U$ with 1s on their diagonals, and to avoid the computation of square roots, one often instead forms a factorization of the form $U^T DU$, where $D$ is a diagonal matrix. Block-Cholesky factorization forms a factorization of this form, but with $D$ being a block-diagonal matrix.

To begin, we must choose a set of rows to be eliminated. We will eliminate the same set of columns. For consistency with the notation used in the description of multigrid algorithms, we will let $F$ (for finer) be the set of rows to be eliminated. We then let $C$ (for coarse) be the remaining set of rows. In contrast with multigrid methods, we will have $|F| < |C|$. By re-arranging rows and columns, we can write $M$ in block form:

$$M = \begin{bmatrix} M_{FF} & M_{FC} \\ M_{CF} & M_{CC} \end{bmatrix}.$$
Elimination of the rows and columns in $F$ corresponds to writing

$$M = \begin{bmatrix} I & 0 \\
M_{CF}M_{FF}^{-1} & I \end{bmatrix} \begin{bmatrix} M_{FF} & 0 \\
0 & M_{CC} - M_{CF}M_{FF}^{-1}M_{FC} \end{bmatrix} \begin{bmatrix} I & M_{FF}^{-1}M_{FC} \\
0 & I \end{bmatrix}. \quad (1)$$

Note that the left and right matrices are lower and upper triangular. The matrix in the lower-right block of the middle matrix is the Schur complement of $F$ in $M$. We will refer to it often by the notation

$$Sc(M, F) \overset{\text{def}}{=} M_{CC} - M_{CF}M_{FF}^{-1}M_{FC}.$$We remark that one can solve a linear system in $Sc(M, F)$ by solving a system in $M$: one just needs to put zeros coordinates corresponding to $F$ in the right-hand-side vector. Recall that

$$\begin{bmatrix} I & 0 \\
M_{CF}M_{FF}^{-1} & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\
-M_{CF}M_{FF}^{-1} & I \end{bmatrix}. \quad (2)$$

So, if we can quickly multiply by this last matrix, and if we can quickly solve linear systems in $M_{FF}$ and in the Schur complement, then we can quickly solve systems in $M$. Algebraically, we exploit the following identity:

**Fact 4.1.**

$$M^{-1} = \begin{bmatrix} I & -M_{FF}^{-1}M_{FC} \\
0 & I \end{bmatrix} \begin{bmatrix} M_{FF}^{-1} & 0 \\
0 & Sc(M, F)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\
-M_{CF}M_{FF}^{-1} & I \end{bmatrix}. \quad (3)$$

Our algorithms depend upon the following important property of Schur complements of SDDM matrices.

**Fact 4.2.** If $M$ is a SDDM matrix and $F$ is a subset of its columns, the Schur complement $Sc(M, F)$ is also a SDDM matrix.

We now mention two other facts that we will use about the $\preceq$ order and Schur complements.

**Fact 4.3.** If $M_{FF} \preceq \tilde{M}_{FF}$, then

$$\begin{pmatrix} M_{FF} & M_{FC} \\
M_{CF} & M_{CC} \end{pmatrix} \preceq \begin{pmatrix} \tilde{M}_{FF} & M_{FC} \\
M_{CF} & M_{CC} \end{pmatrix}.$$**Fact 4.4** (Lemma B.1. from [MP13]). If $M$ and $\tilde{M}$ are positive semidefinite matrices satisfying $M \preceq \tilde{M}$, then

$$Sc(M, F) \preceq Sc(\tilde{M}, F).$$

The first idea that motivates our algorithms is that we can sparsify $M$ and $Sc(M, F)$. If $M$ is sparse, then we can quickly multiply vectors by $M_{FC}$. However, to be able to quickly apply the factorization of $M^{-1}$ given in Fact 4.1, we also need to be able to quickly apply $M_{FF}^{-1}$. If we can do that, then we can quickly solve systems in $M$ by recursively solving systems in $Sc(M, F)$.

The easiest way to find an $F$ for which we could quickly apply $M_{FF}^{-1}$ would be to choose $F$ to be a large independent set, in which case $M_{FF}$ would be diagonal. Such a set $F$ must exist.
as we can assume $M$ is sparse. However, the independent set we are guaranteed to find by the sparsity of $M$ is not big enough: if we repeatedly find large independent sets and then sparsify the resulting Schur complements, the error that accumulates could become too big. The second idea behind our algorithms is that we can find a large set $F$ for which $M_{FF}$ is well-approximated by a diagonal matrix. This will allow us to apply $M_{FF}^{-1}$ quickly. In the next section, we show that a very good choice of $F$ always exists, and that the use of such sets $F$ yields nearly-optimal algorithms for solving linear systems in $M$.

In order to make the entire algorithm efficient, we are still left with the problem of quickly computing a sparsifier of the Schur complement. In Section \[8\] we show how to quickly compute and use *Spectral Vertex Sparsifiers*, which are sparsifiers of the Schur complement. In particular, we do this by expressing the Schur complement as the sum of the Schur complements of two simpler matrices: one with a diagonal $FF$ block, and the other with a better conditioned $FF$ block. We handle the matrix with the diagonal block directly, and the matrix with the better conditioned block recursively.

5 A Polynomial Time Algorithm for Optimal Solver Chains

Our algorithms will begin by eliminating a set of vertices $F$ that is $\alpha$-strongly diagonally dominant, a concept that we now define.

**Definition 5.1.** A symmetric matrix $M$ is $\alpha$-strongly diagonally dominant if for all $i$

$$M_{ii} \geq (1 + \alpha) \sum_{j \neq i} |M_{ij}|.$$  

We say that a subset $F$ of the rows of a matrix $M$ is $\alpha$-strongly diagonally dominant if $M_{FF}$ is an $\alpha$-strongly diagonally dominant matrix.

We remark that $0$-strongly diagonal dominance coincides with the standard notion of weak diagonal dominance. In particular, Laplacian matrices are $0$-strongly diagonally dominant.

It is easy to find an $\alpha$-strongly diagonally dominant subset containing at least an $1/8(1 + \alpha)$ fraction of the rows of an SDD matrix: one need merely pick a random subset and then discard the rows that do not satisfy the condition. Pseudocode for computing such a subset is given in Figure \[11\].

**Lemma 5.2.** For every $n$-dimensional SDD matrix $M$ and every $\alpha \geq 0$, SDDSubset computes an $\alpha$-strongly diagonally dominant subset $F$ of size at least $n/(8(1 + \alpha))$ in $O(m)$ expected work and $O(\log n)$ expected depth, where $m$ is the number of nonzero entries in $M$.

**Proof.** As $F$ is a subset of $F'$,

$$\sum_{j \in F, j \neq i} |M_{ij}| \leq \sum_{j \in F', j \neq i} |M_{ij}|.$$  

So, when the algorithm does return a set $F$, it is guaranteed to be $\alpha$-strongly diagonally dominant.
$F = \text{SDDSubset}(M, \alpha)$, where $M$ is an $n$-dimensional SDD matrix.

1. Let $F'$ be a uniform random subset of $\{1, \ldots, n\}$ of size $\frac{n}{4(1+\alpha)}$.

2. Set

$$F = \left\{ i \in F' \text{ such that } \sum_{j \in F', j \neq i} |M_{ij}| \leq \frac{1}{1 + \alpha} |M_{ii}| \right\}.$$  

3. If $|F| < \frac{n}{8(1+\alpha)}$, goto Step 1.

4. Return $F$

Figure 1: Routine for Generating an $\alpha$-strongly diagonally dominant subset $F$

We now show that the probability that the algorithm finishes in each iteration is at least $1/2$. Let $A_i$ be the event that $i \in F'$ and that $i \notin F$. This only happens if $i \in F'$ and

$$\sum_{j \in F', j \neq i} |M_{ij}| > \frac{1}{1 + \alpha} |M_{ii}|.$$  

The set $F$ is exactly the set of $i \in F'$ for which $A_i$ does not hold.

Given that $i \in F'$, the probability that each other $j \neq i$ is in $F'$ is

$$\frac{1}{n-1} \left( \frac{n}{4(1+\alpha)} - 1 \right).$$

So,

$$\mathbb{E} \left[ \sum_{j \in F', j \neq i} |M_{ij}| \right] | i \in F' \leq \frac{1}{n-1} \left( \frac{n}{4(1+\alpha)} - 1 \right) \sum_{j \neq i} |M_{ij}| < \frac{1}{4(1+\alpha)} \sum_{j \neq i} |M_{ij}| \leq \frac{1}{4(1+\alpha)} |M_{ii}|,$$

as $M$ is strongly diagonally dominant. So, Markov’s inequality tells us that

$$\Pr \left[ \sum_{j \in F', j \neq i} |M_{ij}| > \frac{1}{1 + \alpha} |M_{ii}| \mid i \in F' \right] < 1/4,$$

and thus

$$\Pr [A_i] = \Pr [i \in F'] \Pr [i \notin F | i \in F'] < \frac{1}{4(1+\alpha)} \frac{1}{4} = \frac{1}{16(1+\alpha)}.$$

Again applying Markov’s inequality allows us to conclude

$$\Pr \left[ \{|i : A_i| \geq n/8(1+\alpha)\} \right] < 1/2.$$

So, with probability at least $1/2$, $|F| \geq n/8(1+\alpha)$, and the algorithm will pass the test in line 3. Thus, the expected number of iterations made by the algorithm is at most 2. The claimed bounds on the expected work and depth of the algorithm follow.  

\[ \square \]
Strongly diagonally dominant subsets are useful because linear systems involving them can be solved rapidly. Given such a set \( F \), we will construct an operator \( Z^{(k)}_{FF} \) that approximates \( M_{FF}^{-1} \), and that can be applied quickly. To motivate our construction, observe that if \( M_{FF} = X_{FF} + L_{FF} \) where \( X_{FF} \) is a nonnegative diagonal matrix and \( L_{FF} \) is a Laplacian, then

\[
M_{FF}^{-1} = X_{FF}^{-1} - X_{FF}^{-1}L_{FF}X_{FF}^{-1} + \sum_{i \geq 2} (-1)^i X_{FF}^{-1}(L_{FF}X_{FF}^{-1})^i.
\]

We will approximate this series by its first few terms:

\[
Z^{(k)}_{FF} \overset{\text{def}}{=} \sum_{i=0}^{k} X_{FF}^{-1}(-L_{FF}X_{FF}^{-1})^i.
\] (5)

In the following lemmas, we show that using \( Z_{FF} \) in place of \( M_{FF}^{-1} \) in (3) provides a good approximation of \( M_{FF}^{-1} \). We begin by pointing out that \( X_{FF} \) is much greater than \( L_{FF} \). In particular, this implies that all diagonal entries of \( X_{FF} \) are positive, so that \( X_{FF}^{-1} \) actually exists.

**Lemma 5.3.** Let \( M \) be a SDDM matrix that is \( \alpha \)-strongly diagonally dominant. Write \( M = X + L \) where \( X \) is a nonnegative diagonal matrix and \( L \) is a Laplacian. Then,

\[
X \succ \frac{\alpha}{2} L.
\]

**Proof.** Write \( L = Y - A \) where \( Y \) is diagonal and \( A \) has zero diagonal. As \( L \) is diagonally dominant, so is \( Y + A \). This implies that \( Y \succ -A \), and so \( 2Y \succ L \).

As \( M \) is \( \alpha \)-strongly diagonally dominant and the diagonal of \( M \) is \( X + Y \),

\[
((X + Y)1)_i \geq (\alpha + 1)(A1)_i.
\]

As \( L \) is a Laplacian, \( L1 = 0 \), which implies \( Y1 = A1 \) and

\[
(X1)_i \geq \alpha(A1)_i = \alpha(Y1)_i.
\]

As both \( X \) and \( Y \) are diagonal, this implies that

\[
X \succ \alpha Y \succ \frac{\alpha}{2} L.
\]

\( \square \)

We now bound the quality of approximation of the power series (5).

**Lemma 5.4.** Let \( M \) be a SDDM matrix and let \( F \) be a set of columns so that when we write \( M_{FF} = X_{FF} + L_{FF} \) with \( X_{FF} \) nonnegative diagonal and \( L_{FF} \) a Laplacian, we have \( L_{FF} \preceq \beta X_{FF} \). Then, for odd \( k \) and for \( Z^{(k)}_{FF} \) as defined in (5) we have:

\[
X_{FF} + L_{FF} \preceq (Z^{(k)}_{FF})^{-1} \preceq X_{FF} + (1 + \delta) L_{FF},
\] (6)

where

\[
\delta = \beta k \frac{1 + \beta}{1 - \beta^{k+1}}.
\]
Proof. The left-hand inequality is equivalent to the statement that all the eigenvalues of $Z_{FF}^{(k)}(X_{FF} + L_{FF})$ are at most 1 (see [BGH+06, Lemma 2.2] or [ST14, Proposition 3.3]). To see that this is the case, expand

$$Z_{FF}^{(k)}(X_{FF} + L_{FF}) = \left( \sum_{i=0}^{k} X_{FF}^{-1}(-L_{FF}X_{FF}^{-1})^{i} \right)(X_{FF} + L_{FF})$$

$$= \sum_{i=0}^{k} (-X_{FF}^{-1}L_{FF})^{i} - \sum_{i=1}^{k+1} (X_{FF}^{-1}L_{FF})^{i}$$

$$= I_{FF} - (X_{FF}^{-1}L_{FF})^{k+1}.$$

As all the eigenvalues of an even power of a matrix are nonnegative, all of the eigenvalues of this last matrix are at most 1.

Similarly, the other inequality is equivalent to the assertion that all of the eigenvalues of $Z_{FF}^{(k)}(X_{FF} + (1 + \delta)L_{FF})$ are at least one. Expanding this product yields

$$\left( \sum_{i=0}^{k} X_{FF}^{-1}(-L_{FF}X_{FF}^{-1})^{i} \right)(X_{FF} + (1 + \delta)L_{FF})$$

$$= I_{FF} - (X_{FF}^{-1}L_{FF})^{k+1} + \delta \sum_{i=0}^{k} (-1)^{i}(X_{FF}^{-1}L_{FF})^{i+1}$$

The eigenvalues of this matrix are precisely the numbers

$$1 - \lambda^{k+1} + \delta \sum_{i=0}^{k} (-1)^{i}\lambda^{i+1},$$

(7)

where $\lambda$ ranges over the eigenvalues of $X_{FF}^{-1}L_{FF}$. The assumption $L_{FF} \preceq \beta X_{FF}$ implies that the eigenvalues of $X_{FF}^{-1}L_{FF}$ are at most $\beta$, so $0 \leq \lambda \leq \beta$. We have chosen the value of $\delta$ precisely to guarantee that, under this condition on $\lambda$, the value of (7) is at least 1.

We remark that this power series is identical to the Jacobi iteration for solving linear systems.

The following lemma allows us to extend the approximation of $M_{FF}$ by the inverse of $Z_{FF}^{(k)}$ to the entire matrix $M$.

Lemma 5.5. Under the conditions of Lemma 5.4 and assuming that $0 \leq \beta \leq 1/2$,

$$M \preceq \left( \begin{array}{cc} Z_{FF}^{(k)} & M_{FC} \\ M_{CF} & M_{CC} \end{array} \right) \preceq (1 + \beta^{k})M.$$

Proof. The left-hand inequality follows immediately from Fact 1.3 and the left-hand side of (6). To prove the right-hand inequality we apply Fact 1.3 and the right-hand side of (6) to conclude

$$\left( \begin{array}{cc} Z_{FF}^{(k)} & M_{FC} \\ M_{CF} & M_{CC} \end{array} \right) \preceq \left( \begin{array}{cc} M_{FF} & M_{FC} \\ M_{CF} & M_{CC} \end{array} \right) = M + \delta \left( \begin{array}{cc} L_{FF} & 0 \\ 0 & 0 \end{array} \right).$$
As $\beta M_{FF} \geq L_{FF}$, 
\[
\begin{pmatrix}
L_{FF} & 0 \\
0 & 0
\end{pmatrix} \preceq \beta M,
\]
and so we may conclude that 
\[
M + \delta \begin{pmatrix}
L_{FF} & 0 \\
0 & 0
\end{pmatrix} \preceq M + \beta \delta M.
\]
To finish the proof, observe that for $\delta = \beta^k(1+\beta)/(1-\beta^{k+1})$ for $k \geq 1$ and $\beta \leq 1/2$, $\beta \delta \leq \beta^k$. 

We now show that we can obtain a good approximation of $M^{-1}$ by replacing $M_{FF}^{-1}$ by $Z_F^{(k)}$ in the three places in which it explicitly appears in (3), but not in the Schur complement.

**Lemma 5.6.** Let $M$ be a SDDM matrix and let $F$ be an $\alpha$-diagonally dominant set of columns for some $\alpha \geq 4$. Then, for $k$ odd and $Z^{(k)}$ as defined in (5), 
\[
\begin{pmatrix}
I & -Z_{FF}^{(k)}M_{FC} \\
0 & I
\end{pmatrix} \begin{pmatrix}
Z_{FF}^{(k)} & 0 \\
0 & Sc(M,F)^{-1}
\end{pmatrix} \begin{pmatrix}
I & 0 \\
-M_{CF}Z_{FF}^{(k)} & I
\end{pmatrix} \approx \gamma M^{-1},
\]
for $\gamma = (2/\alpha)^k$.

**Proof.** Define 
\[
\hat{M} = \begin{pmatrix}
(Z_{FF}^{(k)})^{-1} & M_{FC} \\
M_{CF} & M_{CC}
\end{pmatrix}.
\]

Lemma 5.3 tells us that $M$ satisfies the conditions of Lemma 5.4 with $\beta = 2/\alpha$. So, Lemma 5.5 implies 
\[
M \preceq \hat{M} \preceq (1+\gamma)M.
\]
By facts 4.1 and 3.1 this implies 
\[
M^{-1} \preceq \begin{pmatrix}
I & -Z_{FF}^{(k)}M_{FC} \\
0 & I
\end{pmatrix} \begin{pmatrix}
Z_{FF}^{(k)} & 0 \\
0 & Sc(\hat{M},F)^{-1}
\end{pmatrix} \begin{pmatrix}
I & 0 \\
-M_{CF}Z_{FF}^{(k)} & I
\end{pmatrix} \approx (1+\gamma)^{-1}M^{-1}.
\]
From Facts 4.4 and 3.1 we know that 
\[
Sc(M,F)^{-1} \succeq Sc(\hat{M},F)^{-1} \approx (1+\gamma)^{-1}Sc(M,F)^{-1}.
\]
When we use Fact 3.2 to substitute this inequality into the one above, we obtain 
\[
(1+\gamma)M^{-1} \preceq \begin{pmatrix}
I & -Z_{FF}^{(k)}M_{FC} \\
0 & I
\end{pmatrix} \begin{pmatrix}
Z_{FF}^{(k)} & 0 \\
0 & Sc(M,F)^{-1}
\end{pmatrix} \begin{pmatrix}
I & 0 \\
-M_{CF}Z_{FF}^{(k)} & I
\end{pmatrix} \approx (1+\gamma)^{-1}M^{-1},
\]
which implies the lemma. 

\[
\square
\]
We now use Lemma 5.6 to analyze a solver obtained by iteratively sparsifying Schur complements of strongly diagonally dominant subsets. We refer to the sequence of subsets and matrices obtained as a vertex sparsifier chain, as an approximation of a Schur complement is a spectral vertex sparsifier. In the following definition, $M^{(1)}$ is intended to be a sparse approximation of $M^{(0)}$. The sparsity of the matrices will show up in the analysis of the runtime, but not in the definition of the chain.

**Definition 5.7 (Vertex Sparsifier chain).** For any SDDM matrix $M^{(0)}$, a vertex sparsifier chain of $M^{(0)}$ with parameters $\alpha_i \geq 4$ and $1/2 \geq \epsilon_i > 0$ is a sequence of matrices and subsets $(M^{(1)}, \ldots, M^{(d)}; F_1, \ldots, F_{d-1})$ such that:

1. $M^{(1)} \approx_{\epsilon_0} M^{(0)}$,
2. $M^{(i+1)} \approx_{\epsilon_i} Sc \left( M^{(i)}, F_i \right)$,
3. $M^{(i)}_{F_i F_i}$ is $\alpha_i$-strongly diagonally dominant and
4. $M^{(d)}$ has size $O(1)$.

We present pseudocode that uses a vertex sparsifier chain to approximately solve a system of equations in $M^{(0)}$ in Figure 2. We analyze the running time and accuracy of this algorithm in Lemma 5.8.

**Lemma 5.8.** Given a vertex sparsifier chain where $M^{(i)}$ has $m_i$ non-zero entries, the algorithm $\text{ApplyChain}(M^{(1)}, \ldots, M^{(d)}, F_1, \ldots, F_{d-1}, \alpha_1 \ldots \alpha_{d-1}, \epsilon_0 \ldots \epsilon_{d-1}, b)$ corresponds to a linear operator $W$ acting on $b$ such that

\[
x^{(1)} = \text{APPLYCHAIN}(M^{(1)}, \ldots, M^{(d)}, F_1, \ldots, F_{d-1}, \alpha_1 \ldots \alpha_{d-1}, \epsilon_0 \ldots \epsilon_{d-1}, b^{(1)})
\]

1. For $i = 1, \ldots, d - 1$
   (a) let $k_i$ be the smallest odd integer greater than or equal to $\log_{\alpha_i/2} \epsilon_i^{-1}$.
   (b) $x^{(i)}_{F_i} \leftarrow Z^{(k_i)}_{F_i} b^{(i)}$, where $Z^{(k_i)}_{F_i}$ is obtained from $M^{(i)}_{F_i F_i}$ as in (5).
   (c) $b^{(i+1)} \leftarrow b^{(i)} - M^{(i)}_{C_i F_i} x^{(i)}_{F_i}$.
2. $x^{(d)} \leftarrow \left(M^{(d)}\right)^{-1} b^{(d)}$.
3. For $i = d - 1, \ldots, 1$
   (a) $x^{(i)}_{C_i} \leftarrow x^{(i+1)}$.
   (b) $x^{(i)}_{F_i} \leftarrow x^{(i)}_{F_i} - Z^{(k_i)}_{F_i F_i} M^{(i)}_{F_i C_i} x^{(i+1)}$.

Figure 2: Solver Algorithm using Vertex Sparsifier Chain
1. \[ W^{-1} \approx_{\sum_{i=0}^{d-1} 2 \epsilon_i} M^{(0)}, \]

and

2. for any vector \( b \), \textsc{ApplyChain}(\( M^{(1)}, \ldots, M^{(d)}, F_1, \ldots, F_{d-1}, \alpha_1 \ldots \alpha_d, \epsilon_0 \ldots \epsilon_d, b \)) runs in \( O \left( \sum_{i=1}^{d-1} (\log_{\alpha_i} (\epsilon_i^{-1}) \log n) \right) \) depth and \( O \left( \sum_{i=1}^{d-1} (\log_{\alpha_i} (\epsilon_i^{-1})) m_i \right) \) work.

Proof. We begin by observing that the output vector \( x^{(1)} \) is a linear transformation of the input vector \( b^{(1)} \). Let \( W^{(1)} \) be the matrix that realizes this transformation. Similarly, for \( 2 \leq i \leq d \), define \( W^{(i)} \) to be the matrix so that

\[ x^{(i)} = W^{(i)} b^{(i)}. \]

An examination of the algorithm reveals that

\[ W^{(d)} = (M^{(d)})^{-1}, \]

and

\[ W^{(i)} = \begin{bmatrix} I & -Z^{(k_i)}_{F_i, F_i, M_{F_i, C_i}} \\ 0 & I \end{bmatrix} \left[ \begin{bmatrix} Z^{(k_i)}_{F_i, F_i} & 0 \\ 0 & W^{(i+1)} \end{bmatrix} \right] \begin{bmatrix} I & 0 \\ -M_{C_i, F_i} Z^{(k_i)}_{F_i, F_i} & I \end{bmatrix}. \]

We will now prove by backwards induction on \( i \) that

\[ \left( W^{(i)} \right)^{-1} \approx_{\sum_{j=1}^{d-i} 2 \epsilon_j} M^{(i)}. \]

The base case of \( i = d \) follows from [5]. When we substitute our choice of \( k_i \) from line 1a of \textsc{ApplyChain} into Lemma 5.6, we find that

\[ \begin{bmatrix} I & -Z^{(k_i)}_{F_i, F_i, M^{(i)}_{F_i, C_i}} \\ 0 & I \end{bmatrix} \left[ \begin{bmatrix} Z^{(k_i)}_{F_i, F_i} & 0 \\ 0 & S_{C_i} \left( M^{(i)}_{F_i, F_i} \right)^{-1} \end{bmatrix} \right] \begin{bmatrix} I & 0 \\ -M^{(i)}_{C_i, F_i} Z^{(k_i)}_{F_i, F_i} & I \end{bmatrix} \approx_{\epsilon_i} \left( M^{(i)} \right)^{-1}. \]

As \( M^{(i+1)} \approx_{\epsilon_i} S_{C_i} \left( M^{(i)}_{F_i, F_i} \right), \)

\[ \begin{bmatrix} I & -Z^{(k_i)}_{F_i, F_i, M^{(i)}_{F_i, C_i}} \\ 0 & I \end{bmatrix} \left[ \begin{bmatrix} Z^{(k_i)}_{F_i, F_i} & 0 \\ 0 & \left( M^{(i+1)} \right)^{-1} \end{bmatrix} \right] \begin{bmatrix} I & 0 \\ -M^{(i)}_{C_i, F_i} Z^{(k_i)}_{F_i, F_i} & I \end{bmatrix} \approx_{\epsilon_i} \left( M^{(i)} \right)^{-1}. \]

By combining this identity with [7] and our inductive hypothesis, we obtain

\[ W^{(i)} \approx_{\sum_{j=1}^{d-i} 2 \epsilon_j} \left( M^{(i)} \right)^{-1}. \]

Finally, as \( M^{(0)} \approx_{\epsilon_0} M^{(1)}, \)

\[ W^{(1)} \approx_{\sum_{j=0}^{d-1} 2 \epsilon_j} \left( M^{(0)} \right)^{-1}. \]
To bound the work and depth of the algorithm, we observe that we do not need to construct the matrices $Z^{(k_i)}_{E_i F_i}$ explicitly. Rather, we multiply vectors by the matrices by performing $k_i$ matrix-vector products by the submatrices of $M^{(i)}$ that appear in the expression $F^{(i)}$. As each matrix-vector product can be performed in depth $O(\log n)$, the depth of the whole algorithm is bounded by $O((\log n) \sum_i k_i)$. As each matrix $M^{(i)}$ has $m_i$ non-zero entries, and the work of the $i$th iteration is dominated by the cost of multiplying by submatrices of $M^{(i)}$ $O(k_i)$ times, the total work of the algorithm is $O(\sum_{i=1}^d m_i k_i)$.

**Definition 5.9 (Work and Depth of a Vertex Sparsifier chain).** An $\epsilon$-vertex sparsifier chain of an SDDM matrix $M^{(0)}$ of depth $D$ and work $W$ is a vertex sparsifier chain of $M^{(0)}$ with parameters $\alpha_i \geq 4$ and $1/2 \geq \epsilon > 0$ that satisfies

1. $2 \sum_{i=0}^{d-1} \epsilon_i \leq \epsilon$, 
2. $\sum_{i=0}^{d-1} m_i \log \alpha_i \epsilon_i^{-1} \leq W$, where $m_i$ is the number of nonzeros in $M^{(i)}$, and 
3. $\sum_{i=0}^{d-1} (\log n) \log \alpha_i \epsilon_i^{-1} \leq D$, where $n$ is the dimension of $M^{(0)}$.

**Theorem 5.10.** Every SDDM matrix $M$ of dimension $n$ has a $1$-vertex sparsifier chain of depth $O(\log^2 n \log \log n)$ and work $O(n)$. Given such vertex sparsifier chain, for any vector $b$, we can compute an $\epsilon$ approximate solution to $M^{-1}b$ in $O(m \log(1/\epsilon))$ work and $O(\log^2 n \log \log n \log(1/\epsilon))$ depth.

**Proof.** We will show the existence of such a vertex sparsifier chain with $\alpha_i = 4$ for all $i$ and $\epsilon_i = \frac{1}{2(i+2)}$. Lemma 5.2 tells us that every SDDM matrix has a $4$-strongly diagonally dominant subset consisting of at least a $1/8(1+4) = 1/40$ fraction of its columns. By taking such a subset, we ensure that the number of vertices of $M^{(i)}$, which we define to be $n_i$, satisfies

$$n_i \leq \left(\frac{39}{40}\right)^{i-1} n.$$ 

In particular, this means that $d$, the number of matrices in the chain, will be logarithmic in $n$.

If we use Theorem 5.3 to find a matrix $M^{(1)}$ that is an $\epsilon_0$ approximation of $M^{(0)} = M$, and to find a matrix $M^{(i+1)}$ that is an $\epsilon_i$ approximation of $Sc(M^{(i)}, F_i)$, then each matrix $M^{(i)}$ will have a number of nonzero entries satisfying

$$m_i \leq O(n_i/\epsilon_i^{2}) \leq O\left(\left(\frac{39}{40}\right)^{i-1} (i+1)^4 n\right).$$

Lemma 5.8 tells us that the vertex sparsifier chain induces a linear operator that is an $\epsilon$-approximation of the inverse of $M$, where

$$\epsilon \leq 2 \sum_{i=0}^{d-1} \epsilon_i \leq 2 \sum_{i=0}^{d-1} \frac{1}{2(i+2)^2} \leq \sum_{i \geq 2} \frac{1}{i^2} \leq 1.$$

To compute the work and depth of the chain, recall that we set $k_i$ to be the smallest odd integer that is at least $\log \alpha_i/2 \epsilon_i^{-1}$, so $k_i \leq O(\log i)$. Thus, the work of the chain is at most

$$\sum_{i=1}^{d} k_i m_i \leq O\left(\sum_{i=1}^{d} \log(i) \left(\frac{39}{40}\right)^{i-1} (i+1)^4 n\right) \leq O\left(\sum_{i=1}^{d} \left(\frac{39}{40}\right)^{i-1} i^5 n\right) \leq O(n).$$

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Similarly, the depth of the chain is at most

\[
\sum_{i=1}^{d} (\log n) k_i \leq O \left( \sum_{i=1}^{d} (\log n) \log d \right) \leq O(\log^2 n \log \log n).
\]

6 Linear sized \(U^T D U\) approximations

We now show that the vertex sparsifier chains of \(M\) from the previous section can be used to construct Cholesky factorizations of matrices that are 2-approximations of \(M\). In particular, we prove that for every SDDM matrix \(M\) of dimension \(n\) there exists a diagonal matrix \(D\) and an upper-triangular matrix \(U\) having \(O(n)\) nonzero entries such that \(U^T D U\) is a 2-approximation of \(M\).

The obstacle to obtaining such a factorization is that it does not allow us to multiply a vector by \(Z^{(k_i)}_{F_i} F_i\) in many steps. Rather, we must explicitly construct the matrices \(Z^{(k_i)}_{F_i} F_i\). If we directly apply the construction suggested in the previous section, these matrices could be dense and thereby result in a matrix \(U\) with too many nonzero entries. To get around this problem, we show that we can always find strongly diagonally dominant subsets in which all the vertices have low degree. This will ensure that all of the matrices \(Z^{(k_i)}_{F_i}\) are sparse.

Lemma 6.1. For every \(n\)-dimensional SDD matrix \(M\) and every \(\alpha \geq 0\), there is an \(\alpha\)-strongly diagonally dominant subset of columns \(F\) of size at least \(\frac{n}{\log(1+\alpha)}\) such that the number of nonzeros in every column \(F\) is at most twice the average number of nonzeros in columns of \(M\).

Proof. Discard every column of \(M\) that has more than twice the average number of nonzeros per column. Then remove the corresponding rows. The remaining matrix has dimension at least \(n/2\). Use Lemma 5.2 to find an \(\alpha\)-strongly diagonally subset of the columns of this matrix. \(\Box\)

To obtain a \(U^T D U\) factorization from a vertex sparsifier chain, we employ the procedure in Figure 3.

Lemma 6.2. On input a vertex sparsifier chain of \(M\) with parameters \(\alpha_i \geq 4\) and \(\epsilon_i\), the algorithm Decompose produces matrices \(D\) and \(U\) such that

\[
U^T D U \approx_{\gamma} M,
\]

where

\[
\gamma \leq 2 \sum_{i=0}^{d-1} \epsilon_i + 4/\min_i \alpha_i.
\]

Proof. Consider the inverse of the operator \(W = W^{(1)}\) realized by the algorithm ApplyChain, and the operators \(W^{(i)}\) that appear in the proof of Lemma 5.8.

We have

\[
(W^{(i)})^{-1} = \begin{bmatrix} I & 0 & 0 \\ M_{C_iF_i} & Z^{(k_i)}_{F_iF_i} & I \end{bmatrix} \begin{bmatrix} (Z^{(k_i)}_{F_iF_i})^{-1} & 0 & 0 \\ 0 & (W^{(i+1)})^{-1} & 0 \\ 0 & 0 & I \end{bmatrix},
\]

where

\[
(Z^{(k_i)}_{F_iF_i})^{-1} = \begin{bmatrix} I \\ M_{C_iF_i} \end{bmatrix} Z^{(k_i)}_{F_iF_i} \begin{bmatrix} I \end{bmatrix}.
\]
\( (D, U) = \text{DECOMPOSE} \left( M^{(1)}, \ldots, M^{(d)}, F_1, \ldots, F_{d-1} \right) \), where each \( M^{(i)} \) is a SDDM matrix.

1. let \( k_i \) be the smallest odd integer greater than or equal to \( \log_{\alpha_i/2} \epsilon_i^{-1} \).

2. For each \( i < d \), write \( M^{(i)} = X^{(i)} + L^{(i)} \) where \( X^{(i)} \) is a positive diagonal matrix and \( L^{(i)} \) is a Laplacian.

3. Let \( X^{(d)} = I_{C_{d-1}} \) and let \( \hat{U} \) be the upper-triangular Cholesky factor of \( M^{(d)} \).

4. Let \( D \) be the diagonal matrix with \( D_{F_iF_i} = X^{(i)} \), for \( 1 \leq i < d \), and \( D_{C_{d-1}C_{d-1}} = I_{C_{d-1}} \).

5. Let \( U \) be the upper-triangular matrix with 1s on the diagonal, \( UC_{d-1}C_{d-1} = \hat{U} \), and \( U_{F_iC_i} = Z_{F_iF_i}^{(k_i)} M_{F_iC_i}^{(i)} \), for \( 1 \leq i < d \).

Figure 3: Converting a vertex sparsifier chain into \( U \) and \( D \).

and

\[
\left( W^{(d)} \right)^{-1} = M^{(d)} = \hat{U}^T \hat{U}.
\]

After expanding and multiplying the matrices in this recursive factorization, we obtain

\[
\left( W^{(1)} \right)^{-1} = U^T \begin{bmatrix} \left( Z_{F_1F_1}^{(k_1)} \right)^{-1} & \ldots & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & \ldots & \left( Z_{F_{d-1}F_{d-1}}^{(k_{d-1})} \right)^{-1} & 0 \\ 0 & \ldots & 0 & I_{C_{d-1}C_{d-1}} \end{bmatrix} U.
\]

Moreover, we know that this latter matrix is a \( 2 \sum_{i=0}^{d-1} \epsilon_i \) approximation of \( M \). It remains to determine the impact of replacing the matrix in the middle of this expression with \( D \).

It suffices to examine how well each matrix \( \left( Z_{F_iF_i}^{(k_i)} \right)^{-1} \) is approximated by \( X^{(i)} \). From Lemma 5.3 we know that \( X^{(i)} \approx (\alpha_i/2) L^{(i)} \).

Thus, we may use Lemma 5.4 with \( \beta = \alpha_i/2 \) to conclude that

\[
X^{(i)} \approx_{4/\alpha_i} \left( Z_{F_iF_i}^{(k_i)} \right)^{-1}.
\]

This implies that replacing each of the matrices \( \left( Z_{F_iF_i}^{(k_i)} \right)^{-1} \) by \( X^{(i)} \) increases the approximation factor by at most \( 4/\min_i \alpha_i \).

\[ \square \]

**Theorem 6.3.** For every \( n \)-dimensional SDDM matrix \( M \) there exists a diagonal matrix \( D \) and an upper triangular matrix \( U \) with \( O(n) \) nonzero entries so that

\[ U^T D U \approx_2 M. \]
Moreover, back and forward solves in $U$ can be performed with linear work in depth $O(\log^2 n)$. 

Proof. We choose the same parameters as were used in the proof of Theorem 5.10: $\alpha_i = 4$ for all $i$ and $\epsilon_i = 1/(2(i + 2)^2)$. Theorem 3.5 then guarantees that the average number of nonzero entries in each column of $M^{(i)}$ is at most $10i/\epsilon_i^2 = 40(i + 1)^4$. If we now apply Lemma 6.1 to find 4-diagonally dominant subsets $F_i$ of each $M^{(i)}$, we find that each such subset contains at least a $1/80$ fraction of the columns of its matrix and that each column and row of $M^{(i)}$ indexed by $F$ has at most $80(i + 1)^4$ nonzero entries. This implies that each row of $Z_{F_iF_i}^{(k_i)}M_{F_iC_i}$ has at most $(80(i + 1)^4)^{k_i+1}$ nonzero entries.

Let $n_i$ denote the dimension of $M^{(i)}$. By induction, we know that 

$$n_i \leq n \left(1 - \frac{1}{80}\right)^{i-1}.$$ 

So, the total number of nonzero entries in $U$ is at most 

$$\sum_{i=1}^{d} n_i(80(i + 1)^4)^{k_i+1} \leq \sum_{i=1}^{d} \left(1 - \frac{1}{80}\right)^{i-1}(80(i + 1)^4)^{k_i+1}.$$ 

We will show that the term multiplying $n$ in this later expression is upper bounded by a constant. To see this, note that $k_i \leq 2 + \log_4 \epsilon_i^{-1} \leq \nu \log(i + 1)$ for some constant $\nu$. So, there is some other constant $\mu$ for which 

$$(80(i + 1)^4)^{k_i+1} \leq \exp(\mu \log^2(i + 1)).$$ 

This implies that the sum is at most 

$$\sum_{i \geq 1} \exp(\mu \log^2(i + 1) - i/80),$$ 

which is bounded by a constant.

The claimed bound on the work to perform backwards and forwards substitution with $U$ is standard: these operations require work linear in the number of nonzero entries of $U$. The bound on the depth follows from the fact that the substitutions can be performed blockwise, take depth $O(\log n)$ for each block, and the number of blocks, $d$, is logarithmic in $n$. \qed

7 Existence of Linear Work and $O(\log n \log^2 \log n)$ depth Solvers

The factorizations constructed in the previous section can be evaluated in $O(\log^2 n)$ depth and $O(n)$ work. One $O(\log n)$ factor comes from the depth of the recursion and another $O(\log n)$ factor comes from the depth of matrix vector multiplication. The reason that matrix-vector multiplication can take logarithmic depth is that computing the sum of $k$ numbers takes $O(\log k)$ depth. Thus, if we can instead multiply by matrices with $k^{O(1)}$ nonzeros in each row and column, for some small $k$, we can reduce the depth of each matrix-vector multiplication to $O(\log k)$.

Although the number of non-zeros in each row of $Z_{F_iF_i}^{(k_i)}M_{F_iC_i}$ is bounded by $(80(i + 1)^4)^{k_i+1}$, the number of non-zeros per column can be high. This is because although we picked $F_i$ to be of bounded degree, many of those vertices can be adjacent to a few vertices in $C_i$. For the factorization constructed in Section 6, $k$ can be as large as $n$. In this section, we reduce this degree to $\log^{O(1)} n$ by splitting high degree vertices. This leads a factorization that can be evaluated in linear work and $O(\log n \log^2 \log n)$ depth.
7.1 Splitting High Degree Vertices

While sparsification produces graphs with few edges, it does not guarantee that every vertex has low degree. We will approximate an arbitrary graph by one of bounded degree by splitting each high degree vertex into many vertices. The edges that were originally attached to that vertex will be partitioned among the vertices into which it is split. The vertices into which it is split will then be connected by a complete graph, or an expander if the complete graph would have too high degree. The resulting bounded-degree graph has more vertices. To approximate the original graph, we take a Schur complement of the bounded-degree graph with respect to the extra vertices. We recall that one can solve a system of equations in a Schur complement of a matrix by solving one equation in the original matrix.

We begin our analysis by examining what happens when we split an individual vertex.

**Lemma 7.1.** Let $G$ be a weighted star graph with vertex set $\{v_1, \ldots, v_n, u\}$ and edges connecting $u$ to each $v_i$ with weight $w_i$. Let $\hat{G}$ be a graph with vertex set $\{v_1, \ldots, v_n, u_1, \ldots, u_k\}$ in which the vertices $\{u_1, \ldots, u_k\}$ are connected by a complete graph of edges of weight $W = \delta^{-1} \sum_i w_i$, and each vertex $v_i$ is connected to exactly one vertex $u_j$, again by an edge of weight $w_i$. Let $U = \{u_2, \ldots, u_k\}$. Then, $\text{Sc} (\hat{G}, U) \preceq G$, and in $\text{Sc} (\hat{G}, U)$ the edge between $u_1$ and $v_i$ has weight at least $w_i(1 - 2\delta)$, for every $i$.

**Proof.** We will examine the Laplacian matrices of $G$ and $\hat{G}$. Define $w_{\text{tot}} = \sum_i w_i$, so $W = w_{\text{tot}}/\delta$. Let $b$ be the vector of weights $w_1, \ldots, w_n$, and let $B$ be the diagonal matrix of $b$, so that so that

$$L_G = \begin{pmatrix} B & -b \\ -b^T & -I_{w_{\text{tot}}} \end{pmatrix}.$$ 

Similarly, let $C$ be the adjacency matrix between $v_1, \ldots, v_n$ and $u_1, \ldots, u_k$, and let $D$ be the diagonal matrix whose $j$th entry is the sum of the $w_i$ for which $v_i$ is connected to $u_j$. Then,

$$\hat{L}_G = \begin{pmatrix} B & -C \\ -C^T & D + W(kI_k - J_k) \end{pmatrix},$$

where $J_k$ is the $k \times k$ all ones matrix and $kI_k - J_k$ is the Laplacian of the complete graph on $k$ vertices.

To express the Schur complement, let $D_2$ be the submatrix of $D$ obtained by excluding its first row and column, let $C_2$ be the submatrix of $C$ excluding its first column, and let $c_1$ be the first column of $C$. Let $c_2 = C_2 1$, so $b = c_1 + c_2$. We then have that $\text{Sc} (\hat{L}_G, U)$ equals

$$\begin{pmatrix} B & -c_1 \\ -c_1^T & D(1,1) \end{pmatrix} - \begin{pmatrix} -C_2^T & -W_1^T \\ -W_1 & D_2 + W(kI_{k-1} - J_{k-1}) \end{pmatrix}^{-1} \begin{pmatrix} -C_2 & -W_1 \\ -W_1^T & D_2 + W(kI_{k-1} - J_{k-1}) \end{pmatrix}^{-1} \begin{pmatrix} C_2 & W_1 \\ W_1 & 1 \end{pmatrix}$$

\[\]
To understand this expression, we will show that it approaches $L_G$ as $\delta$ goes to zero. We first note that

$$(kI_{k-1} - J_{k-1})^{-1} = \frac{1}{k}(I_{k-1} + J_{k-1}),$$

and so

$$C_2^T(kI_{k-1} - J_{k-1})^{-1} 1 = C_2^T 1 = c_2^T.$$ 

So, the last row and column of the Schur complement agrees with $L_G$ as $\delta$ goes to zero. On the other hand, the upper-left block becomes

$$B - C_2^T (D_2 + W(kI_{k-1} - J_{k-1}))^{-1} C_2,$$

which goes to $B$ as $\delta$ goes to zero.

To bound the discrepancy in terms of $\delta$, we recall that $J = 11^T$, and so we can use the Sherman-Morrison-Woodbury formula to compute

$$(D_2 + W(kI_{k-1} - J_{k-1}))^{-1} = (D_2 + WkI_{k-1})^{-1} + \frac{(D_2 + WkI_{k-1})^{-1} WJ_{k-1} (D_2 + WkI_{k-1})^{-1}}{1 - W1^T (D_2 + WkI_{k-1})^{-1} 1}. $$

Note that $D_2 + WkI_{k-1}$ is a diagonal matrix. As all entries of $D_2$ are less than $\delta W$, every diagonal entry of this matrix is at least $(Wk(1 + \delta))^{-1}$. So, we have the entry-wise inequality

$$\frac{(D_2 + WkI_{k-1})^{-1} WJ_{k-1} (D_2 + WkI_{k-1})^{-1}}{1 - W1^T (D_2 + WkI_{k-1})^{-1} 1} \geq \frac{WJ_{k-1}/(Wk(1 + \delta))^2}{1/k} = \frac{1}{Wk(1 + \delta)^2} J_{k-1}. $$

This tells us that, entry-wise,

$$(D_2 + W(kI_{k-1} - J_{k-1}))^{-1} \geq (1 - 2\delta) \frac{1}{Wk} (I_{k-1} + J_{k-1}) = (1 - 2\delta)(W(kI_{k-1} - J_{k-1}))^{-1}. $$

The claimed bound on the entries in row and column corresponding to $u_1$ of the Schur complement now follows from the fact that they are obtained by multiplying this matrix inverse on either side by $C_2$ and $W 1$: as these are non-negative matrices, the entry-wise inequality propagates to the product.  

The following theorem states the approximation we obtain if we split all the vertices of high degree and connect the clones of each vertex by expanders.

**Theorem 7.2.** For any graph $G = (V, E)$ with $n$ vertices, $\varepsilon > 0$ and $t > 1/\varepsilon^2$, there is a graph $\tilde{G} = (V \cup S, \tilde{E})$ of maximum degree $O(t)$ such that

$$G \approx_{\varepsilon} Sc\left(\tilde{G}, S\right),$$

|S| = O\left(n/(\varepsilon^2 t)\right), and $|\tilde{E}| \leq O(n/\varepsilon^2 + n/(\varepsilon^4 t))$.

**Proof.** We first sparsify $G$ using Theorem 3.5, obtaining $\tilde{G}$ with $O(n/\varepsilon^2)$ edges such that $\tilde{G} \approx_{\varepsilon/3} G$.

Let $U$ be the set of vertices in $\tilde{G}$ of degree more than $t$. We will split each vertex in $U$ into many vertices. For each $u \in U$, let $d_u$ be its degree in $\tilde{G}$. We split $u$ into $\lceil d_u/t \rceil$ vertices, one
of which we identify with the original vertex \( u \), and the rest of which we put in \( S \). We then partition the edges that were attached to \( u \) among these \( \lceil d_u / t \rceil \) vertices, so that each is now attached to at most \( t \) of these edges. We then place a complete graph between all of the vertices derived from \( u \) in which every edge has weight equal to the sum of the weights of edges attached to \( u \), times \( 12 / \varepsilon \). That is, we apply the construction of Lemma 7.1 with \( \delta = \varepsilon / 3 \). Call the resulting graph \( G' \).

If \( \lceil d_u / t \rceil > t \), we replace that complete graph by a weighted expander of degree \( O(1 / \varepsilon^2) \) that is an \( \varepsilon / 3 \) approximation of this weighted complete graph, as guaranteed to exist by Lemma A.9. The resulting graph is \( \tilde{G} \).

To show that (10) holds, we first show that

\[
\tilde{G} \approx_{\varepsilon / 3} \text{Sc}(G', S).
\]

Lemma 7.1 tells us that \( \text{Sc}(G', S) \preceq \tilde{G} \). It also tells us that the graph looks like \( \tilde{G} \) except that it can have some extra edges and that the edges attached to vertices we split can have a slightly lower weight. If an edge is attached to just one of the split vertices, its weight can be lower by a factor of \( 2\delta = \varepsilon / 6 \). However, some edges could be attached to two of the split vertices, in which case they could have weight that is lower by a factor of \( \varepsilon / 3 \). This implies that \( (1 - \varepsilon / 3)\tilde{G} \preceq \text{Sc}(G', S) \). To prove (10), we now combine this with the factors of \( \varepsilon / 3 \) that we lose by sparsifying at the start and by replacing with expanders at the end.

It is clear that every vertex in \( \tilde{G} \) has degree at most \( t + O(1 / \varepsilon^2) \). To bound the number of edges in \( \tilde{G} \), we observe that the sum of the degrees of vertices that are split is at most \( O(n / \varepsilon^2) \), and so the number of extra vertices in \( S \) is at most \( O(n / \varepsilon^2 t) \). Our process of adding expanders at the end can create at most \( O(1 / \varepsilon^2 t) \) new edges for each of these vertices, giving a total of at most \( O(n / \varepsilon^4 t) \) new edges.

**Remark 7.3.** We do not presently know how to implement the exact construction from the above theorem in polynomial time, because it relies on the nonconstructive proof of the existence of expanders from [MSS15]. One can transform this into a polynomial time construction by instead using the explicit constructions of Ramanujan graphs [Mar88, LPS88] as described in Lemma A.8. This would, however, add the requirement \( t > 1 / \epsilon^6 \) to Theorem 7.2. While this would make Theorem 7.2 less appealing, it does not alter the statement of Theorem 7.4.

It remains to incorporate this degree reduction routine into the solver construction. Since our goal is to upper-bound the degree by \( O(\log c n) \) for some constant \( c \), we can pick \( t \) in Theorem 7.2 so that \( \epsilon^2 t \leq \log^{O(1)} n \). This leads to a negligible increase in vertex count at each step. So we can use a construction similar to Theorem 6.3 to obtain the lower depth solver algorithm.

**Theorem 7.4.** For every \( n \)-dimensional SDDM matrix \( M \) there a linear operator \( Z \) such that

\[
Z \approx_{2} M^{-1}
\]

and matrix-vector multiplications in \( Z \) can be done in linear work and \( O(\log n \log^2 \log n) \) depth. Furthermore, this operator can be obtained via a diagonal \( D \), an upper triangular matrix \( U \) with \( O(n) \) non-zero entries and a set of vertices \( \hat{V} \) such that

\[
M \approx_{2} \text{Sc}(U^T D U, \hat{V}).
\]
Proof. We will slightly modify the vertex sparsification chain from Definition 5.7. Once again, we utilize $\alpha_i = 4$ for all $i$ and $\epsilon_i = 1/2(i+2)^2$. The main difference is that instead of using spectral sparsifiers from Theorem 3.5 directly, we use Theorem 7.2 to control the degrees. Specifically we invoke it with $\epsilon_i = \epsilon_i$ and $t_i = 200\epsilon_i^{-2}$ on $Sc\left(M^{(i)}, F_i\right)$ to obtain $M^{(i+1)}$ and $S_{i+1}$ s.t.

$$Sc\left(M^{(i)}, F_i\right) \approx \epsilon_i \, Sc\left(M^{(i+1)}, S_{i+1}\right).$$

This leads to a slightly modified version of the vertex sparsifier chain. We obtain a sequence of matrices $M_1, M_2 \ldots$ and subsets $S_i$ and $F_i$ s.t.

a. $M^{(1)} \approx \epsilon_0 \, M^{(0)}$,

b. $Sc\left(M^{(i+1)}, S_{i+1}\right) \approx \epsilon_i \, Sc\left(M^{(i)}, F_i\right)$,

c. Each row and column of $M^{(i)}$ has at most $t_i$ non-zeros.

d. Each column and row of $M^{(i)}$ indexed by $F_i$ has at most $80(i+1)^4$ nonzero entries. (obtained by combining the bound on non-zeros from Theorem 7.2 with Lemma 6.1)

e. $M^{(i)}_{F_iF_i}$ is 4-strongly diagonally dominant and

f. $M^{(d)}$ has size $O(1)$.

This modified chain can be invoked in a way analogous to the vertex sparsifier chain. At each step we

1. Apply a recursively computed approximation to $(M^{(i)}_{F_iF_i})^{-1}$ on $b^{(i)}$ to obtain $x^{(i+1)}$.

2. Pad $x^{(i+1)}$ with zeros on $S_i$ to obtain $b^{(i+1)}$.

3. Repeat on level $i + 1$.

4. Restrict the solution $x^{(i+1)}$ to obtain $x^{(i+1)}$.

5. Apply a recursively computed approximation to $(M^{(i)}_{F_iF_i})^{-1}$ to $x^{(i+1)}$ to obtain $x^{(i)}$.

Let $n_i$ denote the dimension of $M^{(i)}$. Since $t_i$ was set to $200\epsilon_i^{-2}$, the increase in vertex size given by $S^{(i)}$ is at most:

$$n_{i+1} \leq n_i \left(1 - \frac{1}{80}\right) \left(1 + \frac{1}{\epsilon^2 t_i}\right) \leq n_i \left(1 - \frac{1}{400}\right)$$

By induction this gives

$$n_i \leq n \left(1 - \frac{1}{400}\right)^{i-1}.$$ 

So the total work follows in a way analogous to Theorem 5.10 and it remains to bound depth.

The constant factor reduction in vertex count gives a bound on chain length of $d = O(\log n)$. This in turn implies $t = O(\epsilon_i^{-2}) = O(\log^4 n)$. Therefore the depth of each matrix-vector multiplication by $M_{F_iC_i}$ is bounded by $O(\log \log n)$. Also, choosing $k_i$ as in Theorem 6.3 gives that
the number of non-zeros in $Z_{F_i}^{(k_i)}$ is bounded by $(\log n)^{O(\log \log n)}$, giving a depth of $O(\log^2 \log n)$ for each matrix-vector multiplication involving $Z_{F_i}$. The $O(\log n)$ bound on $d$ then gives a bound on the total depth of $O(\log n \log^2 \log n)$.

This algorithm can also be viewed as a linear operator corresponding to a $U^T D U$ factorization of a larger matrix. We will construct the operators inductively. Suppose we have $\hat{D}^{(i+1)}$, $U^{(i+1)}$, and $\hat{V}^{(i+1)}$ such that

$$M^{(i+1)} \approx 2 \sum_{i'=i+1}^d \epsilon_{i'} \, S_c \left( \left( U^{(i+1)} \right)^T \hat{D}^{(i+1)} U^{(i+1)}, \hat{V}^{(i+1)} \right).$$

An argument similar to that in the proof of Lemma 6.2 gives

$$M^{(i)} \approx \epsilon_i \begin{bmatrix} I & 0 \\ U_{F_i C_i}^T & I \end{bmatrix} \begin{bmatrix} \left( Z_{F_i}^{(k_i)} \right)^{-1} & 0 \\ 0 & S_c \left( M^{(i)}, F_i \right) \end{bmatrix} \begin{bmatrix} I & U_{F_i C_i} \\ 0 & I \end{bmatrix}.$$

Consider the entry $S_c \left( M^{(i)}, F_i \right)$. Combining condition b of the chain with the inductive hypothesis and Fact 4.4 gives

$$S_c \left( M^{(i)} \right) \approx \epsilon_i \, S_c \left( M^{(i+1)}, S_{i+1} \right) \approx \epsilon_i + \sum_{i'=i+1}^d \epsilon_{i'} \, S_c \left( S_c \left( U^{(i+1)} \right)^T \hat{D}^{(i+1)} U^{(i+1)}, \hat{V}^{(i+1)} \right), S_{i+1} \right).$$

Since the order by which we remove vertices when taking Schur complements does not matter, we can set

$$\hat{V}^{(i)} = \hat{V}^{(i+1)} \cup S_{i+1},$$

to obtain

$$S_c \left( M^{(i)} \right) \approx \epsilon_i + \sum_{i'=i+1}^d \epsilon_{i'} \, S_c \left( U^{(i+1)} \right)^T \hat{D}^{(i+1)} U^{(i+1)}, \hat{V}^{(i)} \right).$$

Block-substituting this and using Fact 4.3 then gives:

$$M^{(i)} \approx 2 \sum_{i'=i}^d \epsilon_{i'} \begin{bmatrix} I & 0 \\ U_{F_i C_i}^T & I \end{bmatrix} \begin{bmatrix} \left( Z_{F_i}^{(k_i)} \right)^{-1} & 0 \\ 0 & S_c \left( U^{(i+1)} \right)^T \hat{D}^{(i+1)} U^{(i+1)}, \hat{V}^{(i+1)} \right) \begin{bmatrix} I & U_{F_i C_i} \\ 0 & I \end{bmatrix}. $$

We will show in Lemma 7.5 that the Schur complement operation can be taken outside multiplications by $U^T$ and $U$. This allows us to rearrange the right-hand side into:

$$S_c \left( \begin{bmatrix} I & 0 & 0 \\ U_{F_i C_i}^T & I & 0 \\ 0 & 0 & I_{\hat{V}^{(i)}} \end{bmatrix} \begin{bmatrix} \left( Z_{F_i}^{(k_i)} \right)^{-1} & 0 \\ 0 & U^{(i+1)} \end{bmatrix} \begin{bmatrix} I & U_{F_i C_i} \\ 0 & I \\ 0 & 0 \end{bmatrix}, \hat{V}^{(i)} \right).$$

Hence choosing

$$\hat{D}^{(i)} = \begin{bmatrix} \left( Z_{F_i}^{(k_i)} \right)^{-1} & 0 \\ 0 & \hat{D}^{(i+1)} \end{bmatrix},$$
and
\[ U^{(i)} = \begin{bmatrix} I & 0 \\ 0 & U^{(i+1)} \end{bmatrix} \begin{bmatrix} I & U_{F,C_i} \\ 0 & I \\ 0 & 0 & I_{\tilde{\nu}(i)} \end{bmatrix} = \begin{bmatrix} I & U_{F,C_i} \\ 0 & I \\ 0 & 0 & U^{(i+1)} \end{bmatrix}, \]
gives \( M^{(i)} \approx 2 \sum_{d'=1}^{d} \epsilon_{d'} S_c \left( \left( U^{(i)} \right)^T D^{(i)} U^{(i)}, \tilde{\nu}(i) \right) \), and the inductive hypothesis holds for \( i \) as well.

We then finish the proof as in Lemma 6.2 by replacing \( \tilde{D}^{(0)} \) with a matrix \( D \) whose diagonals contain \( X^{(i)} \) instead of \( \left( Z_{F,F_i}^{(k_i)} \right)^{-1} \).

It remains to show the needed Lemma rearranging the order of taking Schur complements.

**Lemma 7.5.** Let \( P \) be an arbitrary matrix, and \( M = S_c \left( \tilde{M}, \tilde{\nu} \right) \). Then
\[ P^T M P = S_c \left( \begin{bmatrix} P & 0 \\ 0 & I_{\tilde{\nu}} \end{bmatrix}^T \tilde{M} \begin{bmatrix} P & 0 \\ 0 & I_{\tilde{\nu}} \end{bmatrix}, \tilde{\nu} \right). \]

**Proof.** Let the rows and columns of \( M \) be indexed by \( V \). It suffices to show that the matrix
\[ \left( \begin{bmatrix} P & 0 \\ 0 & I_{\tilde{\nu}} \end{bmatrix}^T \tilde{M} \begin{bmatrix} P & 0 \\ 0 & I_{\tilde{\nu}} \end{bmatrix} \right)^{-1}_{VV} \]
is the same as \( (P^T M P)^{-1} \). This matrix can be written as:
\[ \left[ P^{-1} 0 \\ 0 I_{\tilde{\nu}} \right] \tilde{M}^{-1} \left[ P^{-T} 0 \\ 0 I_{\tilde{\nu}} \right]. \]
The top left block corresponding to \( V \) gives
\[ P^{-1} \left( \tilde{M}^{-1} \right)_{VV} P^{-T}. \]
The definition of Schur complements gives \( M^{-1} = \left( \tilde{M}^{-1} \right)_{VV} \), which completes the proof.

\[ \Box \]

8 Spectral Vertex Sparsification Algorithm

In this section, we give a nearly-linear work algorithm for computing spectral vertex sparsifiers. Recall that our goal is to approximate the matrix
\[ S_c (M, F) = M_{CC} - M_{CF} M_{F,F}^{-1} M_{FC}. \]

Our algorithm approximates \( M_{F,F}^{-1} \) in a way analogous to the recent parallel solver by Peng and Spielman [PS14]. It repeatedly writes the Schur complement as the average of the Schur complements of two matrices. The \( FF \) block in one of these is diagonal, which makes its
construction easy. The other matrix is more strictly diagonally dominant than the previous one, so that after a small number of iterations we can approximate it by a diagonal matrix.

This splitting of the Schur complement is based on the following identity from [PS14]:

\[
(D - A)^{-1} = \frac{1}{2} \left[ D^{-1} + (I + D^{-1}A) \left( D - AD^{-1}A \right)^{-1} (I + AD^{-1}) \right]. \tag{11}
\]

We write \( M_{FF} = D_{FF} - A_{FF} \) where \( D_{FF} \) is diagonal and \( A_{FF} \) has zero diagonal, and apply (11) to obtain the following expression for the Schur complement.

\[
\text{Sc}(M, F) = \frac{1}{2} \left[ 2M_{CC} - M_{CF}D_{FF}^{-1}M_{FC} - M_{CF} \left( I_{FF} + D_{FF}^{-1}A_{FF} \right) \left( D_{FF} - A_{FF}D_{FF}^{-1}A_{FF} \right)^{-1} (I + A_{FF}D_{FF}^{-1}) M_{FC} \right]. \tag{12}
\]

Our key observation is that this is the average of the Schur complement of two simpler matrices. The first term is the Schur complement of:

\[
\begin{bmatrix}
D_{FF} & M_{FC} \\
M_{CF} & 0
\end{bmatrix},
\]

while the second term is the Schur complement of the matrix:

\[
\begin{bmatrix}
D_{FF} - A_{FF}D_{FF}^{-1}A_{FF} & \left( I + A_{FF}D_{FF}^{-1} \right) M_{FC} \\
M_{CF} \left( I + D_{FF}^{-1}A_{FF} \right) & 2M_{CC} - \text{diag} \left( M_{CF}D_{FF}^{-1}M_{FC} \right)
\end{bmatrix}.
\]

This leads to a recursion similar to that used in [PS14]. However, to ensure that the Schur complements of both matrices are SDDM, we move some of the diagonal from the \( CC \) block of the second matrix to the \( CC \) block of the first. To describe this precisely, we use the notation \( \text{diag}(x) \) to indicate the diagonal matrix whose entries are given by the vector \( x \). We also let \( 1 \) denote the all-ones vector. So, \( \text{diag}(1) \cdot 1 = 1 \).

**Lemma 8.1.** Let \( M \) be a SDDM matrix, and let \((F, C)\) be an arbitrary partition of its columns. Let \( M_{FF} = D_{FF} - A_{FF} \), where \( D_{FF} \) is a diagonal matrix and \( A_{FF} \) is a nonnegative matrix with zero diagonal. Define the matrices:

\[
M_1 \overset{\text{def}}{=} \begin{bmatrix}
D_{FF} & M_{FC} \\
M_{CF} & \text{diag} \left( M_{CF}D_{FF}^{-1}M_{FC}1_C \right)
\end{bmatrix}, \tag{13}
\]

and

\[
M_2 \overset{\text{def}}{=} \begin{bmatrix}
D_{FF} - A_{FF}D_{FF}^{-1}A_{FF} & \left( I + A_{FF}D_{FF}^{-1} \right) M_{FC} \\
M_{CF} \left( I + D_{FF}^{-1}A_{FF} \right) & 2M_{CC} - \text{diag} \left( M_{CF}D_{FF}^{-1}M_{FC}1_C \right)
\end{bmatrix}. \tag{14}
\]

Then \( \text{Sc}(M_1, F) \) is a Laplacian matrix, \( M_2 \) is a SDDM matrix, and

\[
\text{Sc}(M, F) = \frac{1}{2} \left( \text{Sc}(M_1, F) + \text{Sc}(M_2, F) \right). \tag{15}
\]
Proof. Equation 15 follows immediately from equation 12.

To prove that \( \text{Sc}(M_1, F) \) is a Laplacian matrix, we observe that all of its off-diagonal entries are nonpositive, and that its row-sums are zero:

\[
\text{Sc}(M_1, F) \mathbf{1}_C = \text{diag}(M_{CF}D_F^{-1}M_{FC}\mathbf{1}_C)\mathbf{1}_C - M_{CF}D_F^{-1}M_{FC}\mathbf{1}_C = \mathbf{0}_C.
\]

To prove that \( M_2 \) is a SDDM matrix, we observe that all of its off-diagonal entries are also nonpositive. For the \( FF \) block this follows from the nonnegativity of \( A_{FF} \) and \( D_{FF} \). For the \( FC \) and \( CF \) blocks it follows from the nonpositivity of \( M_{CF} \) and \( M_{FC} \). We now show that

\[
M_2 \mathbf{1} \geq M_1.
\]

This implies that \( M_2 \) is an SDDM matrix, as it implies that its row-sums are nonnegative and not exactly zero.

We first analyze the row-sums in the rows in \( F \).

\[
(M_2 \mathbf{1})_F = \begin{bmatrix} D_{FF} - A_{FF}D_{FF}^{-1}A_{FF} & (I + A_{FF}D_{FF}^{-1})M_{FC} \end{bmatrix} \begin{bmatrix} \mathbf{1}_F \\ \mathbf{1}_C \end{bmatrix}
\]

\[
\geq D_{FF} \mathbf{1}_F + M_{FC} \mathbf{1}_C - A_{FF}D_{FF}^{-1}(A_{FF} \mathbf{1}_F - M_{FC} \mathbf{1}_C)
\]

\[
= D_{FF} \mathbf{1}_F - A_{FF} \mathbf{1}_F + M_{FC} \mathbf{1}_C
\]

Before, analyzing the row-sums for rows in \( C \), we derive an inequality. As \( M \) is diagonally dominant, every entry of \( D_{FF}^{-1}(A_{FF} \mathbf{1}_F - M_{FC} \mathbf{1}_C) \) is between 0 and 1. As \( M_{FC} \) is nonpositive, this implies that

\[
M_{FC}D_{FF}^{-1}(A_{FF} \mathbf{1}_F - M_{FC} \mathbf{1}_C) \geq M_{FC} \mathbf{1}_C.
\]

Using this inequality, we obtain

\[
(M_2 \mathbf{1})_C = \begin{bmatrix} M_{CF}(I + D_{FF}^{-1}A_{FF}) & 2M_{CC} - \text{diag}(M_{CF}D_{FF}^{-1}M_{FC}\mathbf{1}_C) \end{bmatrix} \begin{bmatrix} \mathbf{1}_F \\ \mathbf{1}_C \end{bmatrix}
\]

\[
= M_{CF} \mathbf{1}_F + M_{CF}D_{FF}^{-1}A_{FF} \mathbf{1}_F + 2M_{CC} \mathbf{1}_C - \text{diag}(M_{CF}D_{FF}^{-1}M_{FC}\mathbf{1}_C)\mathbf{1}_C
\]

\[
= M_{CF} \mathbf{1}_F + M_{CF}D_{FF}^{-1}A_{FF} \mathbf{1}_F + 2M_{CC} \mathbf{1}_C - M_{CF}D_{FF}^{-1}M_{FC} \mathbf{1}_C
\]

\[
= (M_{CC} \mathbf{1}_C + M_{CF} \mathbf{1}_F) + M_{CC} \mathbf{1}_C + M_{CF}D_{FF}^{-1}(A_{FF} \mathbf{1}_F - M_{FC} \mathbf{1}_C)
\]

\[
\geq (M_{CC} \mathbf{1}_C + M_{CF} \mathbf{1}_F) + M_{CC} \mathbf{1}_C + M_{CF} \mathbf{1}_C
\]

\[
= 2(M_1)_{C}.
\]

We first discuss how to approximate the Schur complement of \( M_1 \).

Lemma 8.2. There is a procedure \textsc{ApproxSchurDiag}(\( M, (F, C), \epsilon \)) that takes a graph Laplacian matrix \( M \) with \( m \) non-zero entries, partition of variables \( (F, C) \) and returns a matrix \( \tilde{M}_{SC} \) such that:
1. $\tilde{M}_{SC}$ has $O(m\epsilon^{-4})$ non-zero entries, and
2. $\tilde{M}_{SC} \approx M_1$ where $M_1$ is defined in equation 13.

Furthermore, the procedure takes in $O(m\epsilon^{-4})$ work and $O(\log n)$ depth.

The proof is based on the observation that this graph is a sum of product demand graphs, one per vertex in $F$. These demand graphs can be formally defined as:

Definition 8.3. The product demand graph of a vector $d$, $G(d)$, is a complete weighted graph whose weight between vertices $i$ and $j$ is given by

$$w_{ij} = d_id_j.$$

In Section A we give a result on directly constructing approximations to these graphs that can be summarized as follows:

Lemma 8.4. There is a routine \textsc{WeightedExpander}(d, \epsilon) such that for any demand vector $d$ of length $n$ and a parameter $\epsilon$, \textsc{WeightedExpander}(d, \epsilon) returns in $O(n\epsilon^{-4})$ work and $O(\log n)$ depth a graph $H$ with $O(n\epsilon^{-4})$ edges such that

$$L_H \approx \epsilon L_G(d).$$

Proof. (of Lemma 8.2) Since there are no edges between vertices in $F$, the resulting graph consists of one clique among the neighbors of each vertex $u \in F$. Therefore it suffices to sparsify these separately.

It can be checked that the weight between two neighbors $v_1$ and $v_2$ in such a clique generated from vertex $u$ is $w_{u,v_1}w_{u,v_2}/d_u$. Therefore we can replace it with a weighted expander given in Lemma 8.4 above.

Now, we can invoke Lemma 8.2 on $M_1$ to compute its Schur complement, which means it remains to iterate on $M_2$. Of course, $M_2$ may be a dense matrix. Once again, we approximate it implicitly using weighted expanders. Here we also need weighted bipartite expanders:

Definition 8.5. The bipartite product demand graph of two vectors $d^A$, $d^B$, $G(d^A,d^B)$, is a weighted bipartite graph whose weight between vertices $i \in A$ and $j \in B$ is given by

$$w_{ij} = d^A_id^B_j.$$

Lemma 8.6. There is a routine \textsc{WeightedBipartiteExpander}(d^A, d^B, \epsilon) such that for any demand vectors $d^A$ and $d^B$ of total length $n$ and a parameter $\epsilon$, it returns in $O(n\epsilon^{-4})$ work and $O(\log n)$ depth a graph $H$ with $O(n\epsilon^{-4})$ edges such that

$$L_H \approx \epsilon L_{G(d^A,d^B)}.$$

Lemma 8.7. There exists a procedure \textsc{SquareSparsify} such that, \textsc{SquareSparsify}(M, (F, C), \epsilon) returns in $O(m\epsilon^{-4})$ work and $O(\log n)$ depth a matrix $\tilde{M}_2$ with $O(m\epsilon^{-4})$ non-zero entries such that $\tilde{M}_2 \approx M_2$, where $M_2$ is defined in equation 14.
Proof. The edges in this graph come from $-M_{FC}, A_{FF}D_{FF}^{-1}A_{FF}$ and $A_{FF}D_{FF}^{-1}M_{FC}$. The first is a subset, so we can keep them without increasing total size by a more than a constant factor. The later two consist of length two paths involving some $u \in F$. Therefore we can once again sum together a set of expanders, one per each $u \in F$.

The edges in $A_{FF}D_{FF}^{-1}A_{FF}$ correspond to one clique with product demands given by $A_{uv}$ for each $u \in F$, and can be approximated using the weighted expander in Lemma 8.4.

The edges in $A_{FF}D_{FF}^{-1}M_{FC}$ can be broken down by midpoint into edges of weight $\frac{A_{uvF}A_{uvC}}{d_u}$ where $v_F \in F, v_C \in C$ are neighbors of $u$. This is a bipartite demand graph, so we can replace it with the weighted bipartite expanders given in Lemma 8.6.

The total size of the expanders that we generate is $O(\deg(u) \epsilon^{-4})$. Therefore the total graph size follows from $\sum_{u \in F} \deg(u) \leq m$.

Combining these routines gives our algorithm:

\[ \tilde{L}_\text{schur}[C] = \text{APPROXSHUR} (M, (F, C), \alpha, \epsilon) \]

1. Initialize $\tilde{M}_{SC} \leftarrow 0$, $M^{(0)} \leftarrow M$, $d = \log_{1+\alpha} (7\epsilon^{-1})$

2. For $i$ from 1 to $d$ do
   a. Form $M^{(i-1)}_1$ as in Equation 13.
   b. Form $M^{(i-1)}_2$ as in Equation 14.
   c. $M_{SC} \leftarrow M_{SC} + \frac{1}{2} \text{APPROXSHURDIAG} (M^{(i-1)}_1, \frac{\epsilon}{3})$.
   d. $M^{(i)} \leftarrow \frac{1}{2} \text{SQUARESPARSIFY} (M^{(i-1)}, (F, C), \frac{\epsilon}{3})$.

3. Form $M^{(d)}_1$ by removing all off-diagonal entries in FF from $M^{(d)}$.

4. $\tilde{M}_{SC} \leftarrow \tilde{M}_{SC} + \text{APPROXSHURDIAG} (M^{(d)}_1, \frac{\epsilon}{5})$.

5. Return $\tilde{M}_{SC}$.

The bound on the number of iterations depends on the approximation error of removing off-diagonal entries from $M^{(d)}$. Here we once again use the approach introduced in [PS14] by showing that $M_2$ is more diagonally dominant than $M$.

Lemma 8.8. If $D - A$ is $\alpha$-strongly diagonally dominant and $A$ has 0s on the diagonal, then $D - AD^{-1}A$ is $((1 + \alpha)^2 - 1)$-strongly diagonally dominant.

Proof. Consider the sum of row $i$ in $AD^{-1}A$, it is

\[ \sum_j \sum_k |A_{ij}D^{-1}_{jj}A_{jk}| = \sum_j |A_{ij}| D^{-1}_{jj} \sum_k |A_{jk}| \leq (1 + \alpha)^{-1} \sum_j |A_{ij}| \]
where the inequality follows from applying the fact that $D$ is $1 + \alpha$-strongly diagonally dominant to the $j^{th}$ row. The result then follows from $\sum_j |A_{ij}| \leq (1 + \alpha)^{-1} D_{ii}$. □

This notion is also stable under spectral sparsification.

**Lemma 8.9.** If $A = X + Y$ is $\alpha$-strongly diagonally dominant, $X$ is diagonal, $Y$ is a graph Laplacian, and $Y \approx \tilde{Y}$. Then $A = X + \tilde{Y}$ is $\exp(-\epsilon) \alpha$-strongly diagonally dominant.

**Proof.** Using $Y \approx \epsilon \tilde{Y}$, we have

\[ \tilde{Y}_{ii} \leq \exp(\epsilon) Y_{ii}. \]

The fact that $A$ is $\alpha$-strongly diagonally dominant also gives $X_{ii} \geq \alpha Y_{ii}$. Combining these gives $X_{ii} \geq \exp(-\epsilon) \alpha \tilde{Y}_{ii}$, which means $X + \tilde{Y}$ is $\exp(-\epsilon) \alpha$-strongly diagonally dominant. □

**Theorem 8.10.** Suppose that $M$ is $\alpha$-strongly diagonally dominant and $0 < \epsilon < 1$, then APPROXSchur($M$, $(F, C), \alpha, \epsilon$) returns a matrix $\tilde{M}_{SC}$ with $O\left( m \left( \epsilon^{-1} \log_{\alpha} (\epsilon^{-1}) \right)^{O(\log_{\alpha} (\epsilon^{-1}))} \right)$ non-zeros such that

\[ \tilde{M}_{SC} \approx \epsilon \text{Sc}(M, F). \]

in $O\left( m \left( \epsilon^{-1} \log_{\alpha} (\epsilon^{-1}) \right)^{O(\log_{\alpha} (\epsilon^{-1}))} \right)$ work and $O\left( \log_{\alpha} (\epsilon^{-1}) \log(n) \right)$ depth.

**Proof.** Let $\tilde{M}_{SC}^{(i)}$ denote the $\tilde{M}_{SC}$ after $i$ steps of the main loop in APPROXSchur We will show by induction that at each $i$,

\[ \text{Sc}(M, F) \approx \frac{1}{3d} \tilde{M}_{SC}^{(i)} + \text{Sc}\left(M^{(i)}, F\right). \]

The base case of $i = 0$ clearly holds. For the inductive case, suppose we have the result for some $i$, then

\[ \text{Sc}(M, F) \approx \frac{1}{3d} \tilde{M}_{SC}^{(i)} + \frac{1}{2} \left( \text{Sc}\left(M_1^{(i)}, F\right) + \text{Sc}\left(M_2^{(i)}, F\right) \right). \]

Lemma 8.2 gives

\[ \tilde{M}_{SC}^{(i+1)} = \tilde{M}_{SC}^{(i)} + \frac{1}{2} \text{APPROXSchurDiag}\left(M_1^{(i)}, (F, C), \frac{\epsilon}{3d} \right) \approx \frac{1}{3d} \tilde{M}_{SC}^{(i)} + \frac{1}{2} \text{Sc}\left(M_1^{(i)}, F\right), \quad (16) \]

while Lemma 8.7 gives

\[ M^{(i+1)} \approx \frac{1}{3d} \frac{1}{2} M_2^{(i)}, \]

which combined with the preservation of Loewner ordering from Fact 4.4 gives

\[ \text{Sc}\left(M^{(i+1)}, F\right) \approx \frac{1}{3d} \frac{1}{2} \text{Sc}\left(M_2^{(i)}, F\right). \quad (17) \]

Combining these two bounds (16) and (17) then gives:

\[ \tilde{M}_{SC}^{(i)} + \frac{1}{2} \left( \text{Sc}\left(M_1^{(i)}, F\right) + \text{Sc}\left(M_2^{(i)}, F\right) \right) \approx \frac{1}{3d} \tilde{M}_{SC}^{(i+1)} + \text{Sc}\left(M^{(i+1)}, F\right). \]

Hence, the inductive hypothesis holds for $i + 1$ as well.
By Lemmas 8.8 and 8.9, we have that $M_{d}^{(F)}$ is $6\epsilon^{-1}$-strongly diagonally dominant at the last step. Therefore $M_{1,F}^{(d)} \approx 1/\epsilon$, $M_{F,F}^{(d)}$, and the block-substitution rule from Fact 1.3 gives

$$M_{1}^{(d)} \approx 1/\epsilon M^{(d)}.$$ 

Composing this bound with the guarantees of the iterations then gives the bound on overall error. The work of these steps, and the size of the output graph follow from Lemma 8.2 and 8.7.

In our invocations to this routine, both $\alpha$ and $\epsilon$ will be set to constants. As a result, this procedure is theoretically $O(m)$ time. For a spectral vertex sparsification algorithm for handling general graph Laplacians, $\alpha$ can be 0 and we need to invoke spectral sparsifiers to $L_{i}$ after each step. Any parallel algorithm for spectral sparsification (e.g. [ST11, SS11, OV11, Kou14]) will then lead to nearly linear work and polylog depth.

**Corollary 8.11.** Given a SDDM matrix with condition number $\kappa$, a partition of the vertices into $(F, C)$, and error $\epsilon > 0$, we can compute in $O\left(m \log^{O(1)}(n\kappa \epsilon^{-1})\right)$ work and $O\left(\log^{O(1)}(n\kappa \epsilon^{-1})\right)$ depth a matrix $\tilde{M}_{SC}$ with $O\left(n \log^{O(1)} n \epsilon^{-2}\right)$ non-zeros such that

$$\tilde{M}_{SC} \approx \epsilon Sc(M, F).$$

**Proof.** We can add $\frac{\text{Tr}(M)}{n\kappa}$ to each element on the diagonal to obtain $M' \approx \epsilon M$. Therefore it suffices to assume that $M_{F,F}$ is $1/\poly(n)\kappa$-strongly diagonally dominant.

Therefore Theorem 8.10 gives that APPROX\textsc{Schur} terminates in $d = O(\log \kappa + \log n)$ steps. If we invoke a spectral sparsification algorithm at each step, the number of non-zeros in each $M^{(i)}$ can be bounded by $O(n \log^{O(1)} n (\epsilon/d)^{-2}) = O(n \log^{O(1)} (n\kappa \epsilon^{-1}))$. The overall work bound then follows from combining this with the $\poly(\epsilon^{-1}d)$ increase in edge count at each step, and the nearly-linear work guarantees of spectral sparsification algorithms.

We remark that the setting of $\epsilon_{i} = 1/\log \kappa$ leads to a fairly large number of log factors. In the rest of this paper we only invoke spectral vertex sparsifiers with moderate values of $\epsilon_{i}$ (unless we’re at graphs that are smaller by $\poly(n)$ factors). Also, given recent developments in improved spectral sparsification algorithms [Kou14], we believe optimizing the performance of this algorithm is a question that falls beyond the scope of this paper.

### 9 Algorithmic Constructions

In this section, we gives two algorithms to compute vertex sparsifier chains, the first algorithm uses existing spectral sparsifier for graphs and the second algorithm does not. Although combining two approaches gives a better theoretical result, we do not show it because we believe there will be better spectral sparsifier algorithms for graphs soon and hybrid approaches may not be useful then.
9.1 Black Box Construction

The first construction relies on existing parallel spectral sparsifier algorithms. For concreteness, we use the parallel spectral graph sparsification algorithm given by Koutis [Kou14].

**Theorem 9.1.** Given any SDD matrix $M$ with $n$ variables and $m$ non-zeros, there is an algorithm $\text{BlackBoxSparsify}(M, \epsilon)$ outputs a SDD matrix $B$ with $O(n \log^3 n / \epsilon^2)$ non-zeros such that $M \approx B$ in $O(\log^3 n \log \alpha / \epsilon^2)$ depth and $O((m + n \log^3 n / \epsilon^2) \log^2 n / \epsilon^2)$ work where $\alpha = \frac{m}{n \log^2 n / \epsilon^2}$.

$$(M^{(1)}, M^{(2)}, \ldots; F_1, F_2, \ldots) = \text{BlackBoxConstruct}(M^{(0)})$$

1. Let $k = 1$, $M^{(1)} \leftarrow M^{(0)}$ and $F_0$ be the set of all variables.

2. While $M^{(k)}$ has more than 100 variables
   (a) $M^{(k)} \leftarrow \text{BlackBoxSparsify}(M^{(k)}, 1/(k \log^2(k + 4)))$.
   (b) Find a subset $F_k$ of size $\Omega(n(k))$ such that $M^{(k)}_{F_k F_k}$ is 4-strongly diagonally dominant.
   (c) $M^{(k+1)} \leftarrow \text{ApproxSchur}(M^{(k)}, (F_k, F_{k-1} \setminus F_k), 4, 1/(k \log^2(k + 4)))$.
   (d) $k \leftarrow k + 1$.

In the $k^{th}$ step of the algorithm, we sparsify the graph and compute an approximate Schur complement to $1/(k \log^2(k + 1))$ accuracy and this makes sure the cumulative error is upper bounded by $\sum_{k=1}^{\infty} 1/(k \log^2(k + 1))$ which is a constant.

**Theorem 9.2.** Given any SDD matrix $M^{(0)}$ with $n$ variables and $m$ non-zeros, the algorithm $\text{BlackBoxConstruct}(M^{(0)})$ returns a vertex sparsifier chain such that the linear operator $W$ corresponding to it satisfies

$$W^\dagger \approx_{O(1)} M^{(0)}.$$ 

Also, we can evaluate $Wb$ in $O(\log^2(n) \log \log n)$ depth and $O(n \log^3 n \log \log n)$ work for any vector $b$.

Furthermore, the algorithm $\text{BlackBoxConstruct}(M^{(0)})$ runs in $O(\log^6 n \log^4 \log n)$ depth and $O(m \log^2 n + n \log^5 n)$ work.

**Proof.** Let $n^{(k)}$ and $m^{(k)}$ be the number of vertices and non zero entries in matrix $M^{(k)}$. Let $s(n) = n \log^3 n$ which is the output size of $\text{BlackBoxSparsify}$ and $\epsilon(k) = 1/(k \log^2(k + 4))$ which is the accuracy of the $k$-th sparsification and approximate schur complement.

We first prove the correctness of the algorithm. The ending condition ensures $M^{(last)}$ has size $O(1)$; step (2a) and (2c) ensures $M^{(k+1)} \approx_{2 \epsilon(k)} SC(M^{(k)}, F_k)$ and step (2b) ensures $M^{(k)}_{F_k F_k}$ is 4 strongly diagonally dominant. Therefore, the chain $(M^{(1)}, \ldots; F_1, \ldots)$ is a vertex sparsifier chain. Since the cumulative error $\sum \epsilon(k) = O(1)$, Lemma 3.5 shows that the resultant operator $W$ satisfies

$$W^\dagger \approx_{O(1)} M^{(0)}.$$
Now, we upper bound the cost of evaluating $Wb$. Lemma 5.2 shows that $|F_k| = \Omega(n^{(k)})$ and hence a constant portion of variables is eliminated each iteration. Therefore, $n^{(k)} \leq c^{k-1}n$ for some $c$. Using this, Lemma 5.8 shows the depth for evaluating $Wb$ is
\[ O(\log n) \sum_{k=1}^{O(\log n)} \log(k) \log(n) = O(\log^2 n \log \log n) \]
and the work for evaluating $Wb$ is
\[ O(\log n) \sum_{k=1}^{O(\log n)} \log(k)s(c^{k-1}n)/\epsilon(k)^2). \]
Using $s(n) = n \log^3 n$ and $\epsilon(k) = 1/(k \log^2(k + 4))$, the work for evaluating is simply $O(s(n))$.

For the work and depth of the construction, Lemma 5.2 shows that it takes $O(m^{(k)})$ work and $O(\log n^{(k)})$ depth to find $F_k$ and Theorem 8.10 shows that ApproxSchur takes $O(m^{(k)}k^{O(\log k)})$ work and $O(\log n^{(k)} \log k)$ depth. Using $n^{(k)} \leq c^{k-1}n$ and $m^{(i)} = s(n^{(i)})/\epsilon(i)^2$, the total work for this algorithm excluding BlackBoxSparsify is
\[ O(\log n) \sum_{k=1}^{O(\log n)} O(s(c^{k-1}n)k^{O(\log k)})/\epsilon(k)^2) = O(s(n)). \]
Hence, the total work for BlackBoxConstruct is
\[ O(s(n)) + O(m \log^2 n) + \sum_{k=2}^{O(\log n)} O(s(n^{(k)})k^{O(\log k)} \log^2 n^{(k)}/\epsilon(k)^2). \]
Using $s(n^{(k)})$ is geometric decreasing, the total work is $O(m \log^2 n + n \log^5 n)$. We can bound the total depth similarly.

**Remark 9.3.** Given an sparsifier algorithm that takes $d(m,n)$ depth and $w(m,n)/\epsilon^2$ work to find a sparsifier of size $s(n)/\epsilon^2$, the BlackBoxConstruct roughly takes $O(\log^2 n \log \log n) + O(d(m,n) \log n)$ depth and $O(w(m,n))$ work to construct a vertex sparsifier chain and such chain has total depth $O(\log^2 n \log \log n)$ and total work $O(s(n))$.

Therefore, the work for preprocessing is roughly linear to the work needed to sparsify and the work for solving is linear to the size of sparsifier. Hence, solving Laplacian system is nearly as simple as computing sparsifier.

### 9.2 Recursive Construction

We now give a recursive construction based on the idea that solvers can be used to compute sampling probabilities [SS11]. We will describe the construction in phases, each containing $r$ iterations. Each iteration decreases the number of vertices while maintaining the density of graph. We maintain the density by the general sparsification technique introduced by [CLM+14] as follows:
Lemma 9.4 ([CLMT14]). Given $\mathcal{M}$ be a class of positive definite $n \times n$ matrices. Let $\mathcal{M}(m)$ be the set of all $B^T B \in \mathcal{M}$ such that $B$ has $m$ rows. Assume that

1. For any $B^T B \in \mathcal{M}$ and non negative diagonal matrix $D$, we have $B^T D B \in \mathcal{M}$.

2. For any matrix $B^T B \in \mathcal{M}$, we can check if every row $b$ is in $\text{im}(B^T)$ or not in depth $d_{\text{chk}}(m)$ and work $w_{\text{chk}}(m)$.

3. For any $B^T B \in \mathcal{M}(m)$, we can find an implicit representation of a matrix $W$ such that $W \approx (B^T B)^\dagger$ in depth $d_{\text{con}}(m,n)$ and work $w_{\text{con}}(m,n)$ and for any vector $b$, we can evaluate $Wb$ in depth $d_{\text{eval}}(m,n)$ and work $w_{\text{eval}}(m,n)$.

For any $k \geq 1$, $1 \geq \epsilon > 0$ and matrix $B^T B \in \mathcal{M}(m)$, the algorithm $\text{SPARSIFY}(B^T B, k, \epsilon)$ outputs an explicit matrix $C^T C \in \mathcal{M}(O(kn \log n / \epsilon^2))$ with $C^T C \approx B^T B$.

Also, this algorithm runs in $d_{\text{con}}(\frac{m}{\epsilon}, n) + O(d_{\text{eval}}(m,n) + d_{\text{chk}}(m) + \log n)$ depth and $w_{\text{con}}(\frac{m}{\epsilon}, n) + O(w_{\text{eval}}(m,n) \log n + w_{\text{chk}}(m) + m \log n)$ work.

Each call of spectral vertex sparsification increases edge density, but the $\text{SPARSIFY}$ routine allows us to reduce the density at a much faster rate. A higher reduction parameter $r$ in the algorithm $\text{RECURSIVECONSTRUCT}$, allows us to reduce cost of these recursive sparsification steps.

\[
(M^{(1)}, M^{(2)}, \ldots; F_1, F_2, \ldots) = \text{RECURSIVECONSTRUCT}_r(M^{(0)})
\]

1. $M^{(1)} \leftarrow \text{SPARSIFY}(M^{(0)}, 2^{c_2 r}, 1/4)$, $k \leftarrow 1$ and $F_0$ be the set of all variables.

2. While $M^{(k)}$ has more than $\Theta(1)^r$ vertices,

   (a) Find a subset $F_k$ of size $\Omega(n^{(k)})$ such that $M_{F_k, F_k}$ is 4-strongly diagonally dominant.

   (b) $M^{(k+1)} \leftarrow \text{APPROXSchur}(M^{(k)}, (F_k, F_{k-1} \setminus F_k), 4, (k + 8)^{-2})$.

   (c) If $k + 1 \mod r = 0$, Then

      i. $M^{(k+1)} \leftarrow \text{SPARSIFY}(M^{(k+1)}, (k + 9)^{-2}, 2^{c_2 r \log^2 (k+1)})$.

      (d) $k \leftarrow k + 1$.

The following lemma proves that the algorithm $\text{RECURSIVECONSTRUCT}_r$ produces a vertex sparsifier chain and the linear operator corresponding to the vertex sparsifier can be evaluated efficiently.

Lemma 9.5. Given a large enough constant $r$. There are universal constants $0 < c_1 < 1$ and $c_2 > 0$ such that for any SDD matrix $M^{(0)}$ with $n$ variables, the algorithm $\text{RECURSIVECONSTRUCT}_r(M^{(0)})$ returns a vertex sparsifier chain $(M^{(1)}, M^{(2)}, \ldots; F_1, F_2, \ldots)$ satisfying the following conditions

1. For all $k \geq 1$, $n^{(k)} \leq c_1^{k-1} n$ where $n^{(k)}$ are the number of variables in $M^{(k)}$.

2. Except step 1, at any moment, all intermediate matrices $M$ appears at the $k^{th}$ iteration has density

   \[
   \frac{m'}{n' \log n'} \leq 2^{3c_2 r \log^2 k}
   \]

   for $k > 1$ where $m'$ and $n'$ are the number of non-zeros and variables of $M$.  

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3. For all $k \geq 1$, $M_{F_k F_k}^{(k)}$ is 4-strongly diagonally dominant,

4. For all $k \geq 1$, $M^{(k+1)} \approx \Omega(2^{(k+8)-2} Sc(M^{(k)}, F_k))$.

Furthermore, the linear operator $W$ corresponding to the vertex sparsifier chain satisfies

$$W \approx_{1} (M^{(0)})^\dagger.$$ 

Also, we can evaluate $Wb$ in $O(\log^2 n \log \log n)$ depth and $2^{O(r \log^2 r)} n \log n$ work for any vector $b$.

**Proof.** For the assertion (1), we note that the step (2a) ensures $|F_k| = \Omega(n^{(k)})$ and hence a constant portion of variables is eliminated each iteration. This proves $n^{(k)} \leq c^{k-1} n$ for some $c$.

For the assertion (2), Theorem 8.10 shows that after the approximate Schur complement

$$m^{(k+1)} = O(m^{(k)}(k^2 \log(k+8))O(\log(k+8))) \leq 2^{O(\log^2(k+1))} m^{(k)}.$$

Hence, it shows that each iteration the density increases by at most $2^{c_2 \log^2(k+1)}$ for some constant $c_2$. After the Sparsify step in (2ci), we have

$$\frac{m^{(sr)}}{n^{(sr)} \log n^{(sr)}} \leq 2^{c_2 \log^2(sr)}.$$

Then, after $r$ iterations of APPROXSYNCH and before the sparsification of $M^{((s+1)r)}$, we have

$$\frac{m^{((s+1)r)}}{n^{((s+1)r)} \log n^{((s+1)r)}} \leq 2^{c_2 \log^2(sr)2^{c_2 \log^2(sr+1)} \cdots 2^{c_2 \log^2((s+1)r)}} \leq 2^{2^{c_2 \log^2((s+1)r)}}.$$

This proves the assertion (2).

For the assertion (3), it follows from the construction of $F_k$ in step (2a).

For the assertion (4), we note that in step (2b), we construct the approximate Schur complement $M^{(k+1)}$ such that $M^{(k+1)} \approx_{2^{(k+8)-2}} Sc(M^{(k)}, F_k)$. Therefore, we only need to check $M^{(sr)}$ for all $s$ because $M^{(sr)}$ is modified at step (2ci) after the sparsification. Note that Lemma 33 guarantees that $M^{(k)}$ changes only by $(k+8)^{-2}$ factor. Hence, in total, we have

$$M^{(k+1)} \approx_{2^{(k+8)-2}} Sc(M^{(k)}, F_k).$$

For the last claim, Lemma 5.8 shows that

$$W \approx_{1/2+4 \sum_{k} (k+8)^{-2}} (M^{(0)})^\dagger$$

and we can evaluate $Wb$ in

$$O(\sum_k \log k \log n^{(k)}) = O(\log^2 n \log \log n)$$

depth and

$$O(\sum_k 2^{c_2 r \log^2 k} n^{(k)} \log n^{(k)} \log k) = 2^{O(r \log^2 r)} n \log n$$

work. \qed
In the algorithm \textsc{RecursiveConstruct}, we call the \((sr+1)^{th}\) to the \(((s+1)r)^{th}\) iteration as the \(s^{th}\) phase. At the end of each phase, the \textsc{Sparsify} is called once. The previous lemma showed that the density of the graph at the \(k^{th}\) iteration is less than \(2^{3c_2r \log^2 k}\). This explains our choice of reduction factor \(2^{2c_2r \log^2 k}\) in the \textsc{Sparsify} algorithm as follows:

**Lemma 9.6.** Let \(n(k)\) is the number of variables of \(M^{(k)}\). From the \((sr+1)^{th}\) to the \(((s+1)r)^{th}\) iteration including the \textsc{Sparsify} call at the end, the algorithm takes
\[
2^{O(r \log^2 (sr))} n^{(sr+1)} \log^2 n^{(sr+1)}
\]
work and
\[
O(r \log (sr) \log^2 n^{(sr)}) + O(\log^2 n^{(sr+1)} \log \log n^{(sr+1)})
\]
depth and the time to construct the vertex sparsifier chain for a SDD matrix with \(n^{((s+1)r)}\) variables and \(2^{c_2r \log^2 ((s+1)r)} n^{((s+1)r)} \log n^{((s+1)r)}\) non zeros.

**Proof.** Let \(m(k)\) and \(n(k)\) be the number of non zeros and variables in \(M^{(k)}\) before the \textsc{Sparsify} call if there is. Lemma 5.2 and Theorem 8.10 shows that the depth and work of the \(k^{th}\) iteration takes \(O(m(k) + m(k+1))\) work and \(O(\log k \log n(k))\) depth. Lemma 9.5 shows that
\[
n(k) \leq c_1 k^{-1} n\text{ and } m(k) \leq 2^{3c_2r \log^2 k} n(k) \log n(k)
\]
and hence, from the \((sr+1)^{th}\) to the \(((s+1)r)^{th}\) iteration (excluding the \textsc{Sparsify} call at the end), the algorithm takes
\[
\sum_{k=sr+1}^{(s+1)r} O(m(k) + m(k+1)) \leq \sum_{k=sr+1}^{(s+1)r} 2^{O(r \log^2 k)} n(k) \log n(k)
\]
work and
\[
\sum_{k=sr+1}^{(s+1)r} O(\log k \log n(k)) \leq O(r \log (sr) \log n^{(sr)})
\]
depth.

Now, we bound the cost of the \textsc{Sparsify} call. Let \(m^*\) and \(n^*\) be the the number of non zeros and variables in \(M^{((s+1)r)}\) before the \textsc{Sparsify} call. Lemma 9.4 shows that the \textsc{Sparsify} call takes \(d_{con}\left(m^* 2^{-2c_2r \log^2 ((s+1)r)}, n^*\right) + O(d_{eval}(m^*, n^*) + d_{chk}(m^*) + \log n^*)\) depth and \(w_{con}\left(m^* 2^{-2c_2r \log^2 ((s+1)r)}, n^*\right) + O(w_{eval}(m^*, n^*) \log n^* + w_{chk}(m^*) + m^* \log n^*)\) work.

For any SDD matrix \(B^T B\), an edge \(b \in \text{im}(B^T)\) if and only if the end points of the edge is in the same connected component of the graph corresponding to \(B^T B\). Halperin and Zwick [HZ96] shows how to compute the connected components of a graph with \(m\) edges and \(n\) vertices in \(O(\log n)\) depth and \(O(m + n)\) work for the EREW PRAM model. Using this, we can check every edge in \(O(\log n)\) depth and \(O(m + n)\) work.

To construct an implicit approximate inverse for the sampled SDD matrix, we can use \textsc{RecursiveConstruct}. Lemma 9.5 showed that it takes \(O(\log^2 n^* \log \log n^*)\) depth and \(2^{O(r \log^2 r)} n^* \log n^*\) work to apply the approximate inverse once.
Hence, the total running time from the \((sr + 1)^{th}\) to the \(((s + 1)r)^{th}\) iteration including the \textsc{Sparsify} call is the time to construct the vertex sparsifier chain plus
\[
2^{O(r \log^2(sr))}n^{(sr+1)} \log^2 n^{(sr+1)}
\]
extra work and
\[
O(r \log(sr) \log n^{(sr)}) + O(\log^2 n^{(sr+1)} \log \log n^{(sr+1)})
\]
extra depth.

Note that at the end of the \(s^{th}\) phase, the time required to construct an extra vertex sparsifier chain for the \textsc{Sparsify} call is less than the remaining cost after the \(s^{th}\) phase. This is the reason why we use \(2^{O(r \log^2 k)}\) as the reduction factor for the \textsc{Sparsify} call. The following theorem takes account for the recursive call and show the total running time for the algorithm.

**Lemma 9.7.** With high probability, the algorithm \textsc{RecursiveConstruct}\(_r(M(0))\) returns a vertex sparsifier chain such that the linear operator \(W\) corresponding to it satisfies
\[
W \approx_{1} \left(M(0)\right)^{\dagger}.
\]
Assume \(r \log^2 r = o(\log n)\), we can evaluate \(Wb\) in \(O(\log^2 n \log \log n)\) depth and \(2^{O(r \log^2 r)} n \log n\) work. Also, the algorithm \textsc{RecursiveConstruct}\(_r(M(0))\) takes \(2^{O(\log n/r)}\) depth and \(m \log n + 2^{O(r \log^2 r)} n \log n\) work.

**Proof.** All result is proved in lemma 9.5 except the construction time.

To bound the construction time, we first consider the case \(M(0)\) has only \(2^{c_2 r} n \log n\) non-zeros. In that case, the algorithm skips step 1 because the matrix is already sparse. Lemma 9.6 shows that during the \(s^{th}\) phase, the \textsc{Sparsify} call requires us to construct an extra vertex sparsifier chain for a matrix with \(n^{((s+1)r)}\) variables and at most \(2^{c_2 r \log^2 ((s+1)r)} n^{((s+1)r)} \log n^{((s+1)r)}\) non-zeros. Also, we know that the \textsc{Sparsify} returns a matrix with \(n^{((s+1)r)}\) variables and \(2^{c_2 r \log^2 ((s+1)r)} n^{((s+1)r)} \log n^{((s+1)r)}\) non-zero. Hence, the cost of remaining iteration (excluding the recursion created afterward) is larger than the cost to construct the extra vertex sparsifier chain required at the \(s^{th}\) phase.

Hence, considering this recursion factor, the running time of the \(s^{th}\) phase is multiplied by a factor of \(2^s\).

Since there are \(O(\log n/r)\) phases and \(r \log^2 r = o(\log n)\), the total depth of the algorithm is
\[
\sum_{s=1}^{O(\log n/r)} 2^s \left(r \log(sr) \log^2 n^{(sr)} + \log^2 n^{(sr)} \log \log n^{(sr)}\right)
\]
\[
= 2^{O(\log n/r)} \left(r \log \log n \log^2 n^{(last)} + \log^2 n^{(last)} \log \log n^{(last)}\right)
\]
\[
= 2^{O(\log n/r)} \left(r^2 \log \log n + r^2 \log r\right)
\]
\[
= 2^{O(\log n/r)} r^2 \log \log(n)
\]
\[
= 2^{O(\log n/r)}
\]
and the total work of the algorithm is
\[
O(\log n/r) \sum_{s=1}^{O(r \log^2 (sr))} 2^s m^{(sr+1)} n \log^2 n
= 2^{O(r \log^2 r)} n \log^2 n.
\]

For general \( m \), during the first step, \textsc{Sparsify}, we need to solve a certain SDD matrix with at most \( m^{(0)} 2^{-c_2 r} \) non-zeros and \( n^{(0)} \) variables. To solve that SDD matrix, we use \textsc{RecursiveConstruct}_r to construct a vertex sparsifier chain and use the chain to solve \( O(\log(n)) \) different right hand sides. Using \( r \log^2 r = o(\log n) \), the total depth for this algorithm is
\[
O \left( \log \left( \frac{m \log n}{\log n} \right) \right) 2^{O(\log n/r)} = \log (m) 2^{O(\log n/r)} = 2^{O(\log n/r)}.
\]
and the total work of the algorithm is
\[
m \log n + 2^{O(r \log^2 r)} n \log^2 n \log \left( \frac{m}{n \log n} \right) = m \log n + 2^{O(r \log^2 r)} n \log^2 n \log \left( \frac{m}{n \log n} \right).
\]

Note that the first term dominate if \( \frac{m}{n} \geq 2^{O(r \log^2 r)} \) and hence we can simplify the term to
\[
m \log n + 2^{O(r \log^2 r)} n \log^2 n.
\]

The following theorem follows from Lemma [9.7] by setting \( r = \log \log \log n \).

**Theorem 9.8.** Given any SDD matrix \( M \) with \( n \) variables and \( m \) non-zeros. We can find an implicit block-Cholesky factorization for the matrix \( M \) in \( O(m \log n + n \log^2 + o(1) n) \) work and \( O(n^{o(1)}) \) depth such that for any vector \( b \), we can compute an \( \epsilon \) approximation solution to \( M^{-1} b \) in \( O((m + n \log^2 + o(1) n) \log(1/\epsilon)) \) work and \( O(\log^2 n \log \log n \log(1/\epsilon)) \) depth.

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A Weighted Expander Constructions

In this section, we give a linear time algorithm for computing linear sized spectral sparsifiers of complete and bipartite product demand graphs. Recall that the product demand graph with vertex set $V$ and demands $d : V \rightarrow \mathbb{R}_{>0}$ is the complete graph in which the weight of edge $(u,v)$ is the product $d_u d_v$. Similarly, the bipartite demand graph with vertex set $U \cup V$ and demands $d : U \cup V \rightarrow \mathbb{R}_{>0}$ is the complete bipartite graph on which the weight of the edge $(u,v)$ is the product $d_u d_v$. Our routines are based on reductions to the unweighted, uniform case. In particular, we

1. Split all of the high demand vertices into many vertices that all have the same demand. This demand will still be the highest.

2. Given a graph in which almost all of the vertices have the same highest demand, we
   a. drop all of the edges between vertices of lower demand,
   b. replace the complete graph between the vertices of highest demand with an expander, and
   c. replace the bipartite graph between the high and low demand vertices with a union of stars.

3. To finish, we merge back together the vertices that split off from each original vertex.
We start by showing how to construct the expanders that we need for step (2b). We state formally and analyze the rest of the algorithm for the complete case in the following two sections. We explain how to handle the bipartite case in Section A.3.

Expanders give good approximations to unweighted complete graphs, and our constructions will use the spectrally best expanders—Ramanujan graphs. These are defined in terms of the eigenvalues of their adjacency matrices. We recall that the adjacency matrix of every $d$-regular graph has eigenvalue $d$ with multiplicity 1 corresponding to the constant eigenvector. If the graph is bipartite, then it also has an eigenvalue of $-d$ corresponding to an eigenvector that takes value 1 on one side of the bipartition and $-1$ on the other side. These are called the trivial eigenvalues. A $d$-regular graph is called a Ramanujan graph if all of its non-trivial eigenvalues have absolute value at most $2\sqrt{d-1}$. Ramanujan graphs were constructed independently by Margulis [Mar88] and Lubotzky, Phillips, and Sarnak [LPS88]. The following theorem and proposition summarizes part of their results.

**Theorem A.1.** Let $p$ and $q$ be unequal primes congruent to 1 modulo 4. If $p$ is a quadratic residue modulo $q$, then there is a non-bipartite Ramanujan graph of degree $p+1$ with $q^2(q-1)/2$ vertices. If $p$ is not a quadratic residue modulo $q$, then there is a bipartite Ramanujan graph of degree $p+1$ with $q^2(q-1)$ vertices.

The construction is explicit.

**Proposition A.2.** If $p < q$, then the graph guaranteed to exist by Theorem A.1 can be constructed in parallel depth $O(\log n)$ and work $O(n)$, where $n$ is its number of vertices.

**Sketch of proof.** When $p$ is a quadratic residue modulo $q$, the graph is a Cayley graph of $\text{PSL}(2, \mathbb{Z}/q\mathbb{Z})$. In the other case, it is a Cayley graph of $\text{PGL}(2, \mathbb{Z}/q\mathbb{Z})$. In both cases, the generators are determined by the $p+1$ solutions to the equation $p = a_0^2 + a_1^2 + a_2^2 + a_3^2$ where $a_0 > 0$ is odd and $a_1, a_2,$ and $a_3$ are even. Clearly, all of the numbers $a_0, a_1, a_2$ and $a_3$ must be at most $\sqrt{p}$. So, we can compute a list of all sums $a_0^2 + a_1^2$ and all of the sums $a_2^2 + a_3^2$ with work $O(p)$, and thus a list of all $p+1$ solutions with work $O(p^2) < O(n)$.

As the construction requires arithmetic modulo $q$, it is convenient to compute the entire multiplication table modulo $q$. This takes time $O(q^2) < O(n)$. The construction also requires the computation of a square root of $-1$ modulo $q$, which may be computed from the multiplication table. Given this data, the list of edges attached to each vertex of the graph may be produced using linear work and logarithmic depth.

For our purposes, there are three obstacles to using these graphs:

1. They do not come in every degree.
2. They do not come in every number of vertices.
3. Some are bipartite and some are not.

We handle the first two issues by observing that the primes congruent to 1 modulo 4 are sufficiently dense. To address the third issue, we give a procedure to convert a non-bipartite expander into a bipartite expander, and *vice versa.*

An upper bound on the gaps between consecutive primes congruent to 1 modulo 4 can be obtained from the following theorem of Tchudakoff.
Theorem A.3 ([Tch36]). For two integers \(a\) and \(b\), let \(p_i\) be the \(i\)th prime congruent to \(a\) modulo \(b\). For every \(\epsilon > 0\),
\[
p_{i+1} - p_i \leq O(p_i^{3/4+\epsilon}).
\]

Corollary A.4. There exists an \(n_0\) so that for all \(n \geq n_0\) there is a prime congruent to 1 modulo 4 between \(n\) and \(2n\).

We now explain how we convert between bipartite and non-bipartite expander graphs. To convert a non-bipartite expander into a bipartite expander, we take its double-cover. We recall that if \(G = (V, E)\) is a graph with adjacency matrix \(A\), then its double-cover is the graph with adjacency matrix
\[
\begin{pmatrix}
0 & A \\
A^T & 0
\end{pmatrix}.
\]
It is immediate from this construction that the eigenvalues of the adjacency matrix of the double-cover are the union of the eigenvalues of \(A\) with the eigenvalues of \(-A\).

Proposition A.5. Let \(G\) be a connected, \(d\)-regular graph in which all matrix eigenvalues other than \(d\) are bounded in absolute value by \(\lambda\). Then, all non-trivial adjacency matrix eigenvalues of the double-cover of \(G\) are also bounded in absolute value by \(\lambda\).

To convert a bipartite expander into a non-bipartite expander, we will simply collapse the two vertex sets onto one another. If \(G = (U \cup V, E)\) is a bipartite graph, we specify how the vertices of \(V\) are mapped onto \(U\) by a permutation \(\pi : V \to U\). We then define the collapse of \(G\) induced by \(\pi\) to be the graph with vertex set \(U\) and edge set
\[
\{(u, \pi(v)) : (u, v) \in E\}.
\]
Note that the collapse will have self-loops at vertices \(u\) for which \((u, v) \in E\) and \(u = \pi(v)\). We assign a weight of 2 to every self loop. When a double-edge would be created, that is when \((\pi(v), \pi^{-1}(u))\) is also an edge in the graph, we give the edge a weight of 2. Thus, the collapse can be a weighted graph.

Proposition A.6. Let \(G\) be a \(d\)-regular bipartite graph with all non-trivial adjacency matrix eigenvalues bounded by \(\lambda\), and let \(H\) be a collapse of \(G\). Then, every vertex in \(H\) has weighted degree \(2d\) and all adjacency matrix eigenvalues of \(H\) other than \(d\) are bounded in absolute value by \(2\lambda\).

Proof. To prove the bound on the eigenvalues, let \(G\) have adjacency matrix
\[
\begin{pmatrix}
0 & A \\
A^T & 0
\end{pmatrix}.
\]
After possibly rearranging rows and columns, we may assume that the adjacency matrix of the collapse is given by
\[
A + A^T.
\]
Note that the self-loops, if they exist, correspond to diagonal entries of value 2. Now, let \(x\) be a unit vector orthogonal to the all-1s vector. We have
\[
x^T(A + A^T)x = \begin{pmatrix} x^T \\ x \end{pmatrix}^T \begin{pmatrix} 0 & A \\
A^T & 0
\end{pmatrix} \begin{pmatrix} x \\ x \end{pmatrix} \leq \lambda \left\| \begin{pmatrix} x \\ x \end{pmatrix} \right\|^2 \leq 2\lambda.
\]
as the vector $[x; x]$ is orthogonal to the eigenvectors of the trivial eigenvalues of the adjacency matrix of $G$. \hfill \square

We now state how bounds on the eigenvalues of the adjacency matrices of graphs lead to approximations of complete graphs and complete bipartite graphs.

**Proposition A.7.** Let $G$ be a graph with $n$ vertices, possibly with self-loops and weighted edges, such that every vertex of $G$ has weighted degree $d$ and such that all non-trivial eigenvalues of the adjacency matrix of $G$ have absolute value at most $\lambda \leq d/2$. If $G$ is not bipartite, then $(n/d)L_G$ is an $\epsilon$-approximation of $K_n$ for $\epsilon = (2 \ln 2)(\lambda)/d$. If $G$ is bipartite, then $(n/d)L_G$ is an $\epsilon$-approximation of $K_{n,n}$ for $\epsilon = (2 \ln 2)(\lambda)/d$.

**Proof.** Let $A$ be the adjacency matrix of $G$. Then,

$$L_G = dI - A.$$

In the non-bipartite case, we observe that all of the non-zero eigenvalues of $L_{K_n}$ are $n$, so for all vectors $x$ orthogonal to the constant vector,

$$x^T L_{K_n} x = nx^T x.$$

As all of the non-zero eigenvalues of $L_G$ are between $d - \lambda$ and $d + \lambda$, for all vectors $x$ orthogonal to the constant vector

$$n \left( 1 - \frac{\lambda}{d} \right) x^T x \leq x^T (n/d) L_G x \leq n \left( 1 + \frac{\lambda}{d} \right) x^T x.$$

Thus,

$$\left( 1 - \frac{\lambda}{d} \right) L_{K_n} \preceq L_G \preceq \left( 1 + \frac{\lambda}{d} \right) L_{K_n}.$$

In the bipartite case, we naturally assume that the bipartition is the same in both $G$ and $K_{n,n}$. Now, let $x$ be any vector on the vertex set of $G$. Both the graphs $K_{n,n}$ and $(n/d)G$ have Laplacian matrix eigenvalue $0$ with the constant eigenvector, and eigenvalue $2n$ with eigenvector $[1; -1]$. The other eigenvalues of the Laplacian of $K_{n,n}$ are $n$, while the other eigenvalues of the Laplacian of $(n/d)G$ are between

$$n \left( 1 - \frac{\lambda}{d} \right) \quad \text{and} \quad n \left( 1 + \frac{\lambda}{d} \right).$$

Thus,

$$\left( 1 - \frac{\lambda}{d} \right) L_{K_{n,n}} \preceq L_G \preceq \left( 1 + \frac{\lambda}{d} \right) L_{K_{n,n}}.$$

The proposition now follows from our choice of $\epsilon$, which guarantees that

$$e^{-\epsilon} \leq 1 - \lambda/d \quad \text{and} \quad 1 + \lambda/d \leq e^\epsilon,$$

provided that $\lambda/d \leq 1/2$. \hfill \square
Lemma A.8. There are algorithms that on input $n$ and $\epsilon > n^{-1/6}$ produce a graph having $O(n/\epsilon^2)$ edges that is an $O(\epsilon)$ approximation of $K_{n'}$ or $K_{n',n'}$ for some $n \leq n' \leq 8n$. These algorithms run in $O(\log n)$ depth and $O(n/\epsilon^2)$ work.

Proof. We first consider the problem of constructing an approximation of $K_{n',n'}$. By Corollary A.4, there is a constant $n_0$ so that if $n > n_0$, then there is a prime $q$ that is equivalent to 1 modulo 4 so that $q^2(q-1)$ is between $n$ and $8n$. Let $q$ be such a prime and let $n' = q^2(q-1)$. Similarly, for $\epsilon$ sufficiently small, there is a prime $p$ equivalent to 1 modulo 4 that is between $\epsilon^{-2}/2$ and $\epsilon^{-2}$. Our algorithm should construct the corresponding Ramanujan graph, as described in Theorem A.1 and Proposition A.2. If the graph is bipartite, then Proposition A.7 tells us that it provides the desired approximation of $K_{n',n'}$. If the graph is not bipartite, then we form its double cover to obtain a bipartite graph and use Proposition A.5 and Proposition A.7 to see that it provides the desired approximation of $K_{n',n'}$.

The non-bipartite case is similar, except that we require a prime $q$ so that $q^2(q-1)/2$ is between $n$ and $8n$, and we use a collapse to convert a bipartite expander to a non-bipartite one, as analyzed in Proposition A.6.

In Section 7, we just need to know that there exist graphs of low degree that are good approximations of complete graphs. We may obtain them from the recent theorem of Marcus, Spielman, and Srivastava that there exist bipartite Ramanujan graphs of every degree and number of vertices \cite{MSS15}.

Lemma A.9. For every integer $n$ and even integer $d$, there is a weighted graph on $n$ vertices of degree at most $d$ that is a $4/\sqrt{d}$ approximation of $K_n$.

Proof. The main theorem of \cite{MSS15} tells us that there is a bipartite Ramanujan graph on $2n$ vertices of degree $k$ for every $k \leq n$. By Propositions A.5 and A.7, a collapse of this graph is a weighted graph of degree at most $2k$ that is a $(4\ln 2)/\sqrt{k}$ approximation of $K_{n,n}$. The result now follows by setting $d = 2k$.

A.1 Sparsifying Complete Product Demand Graphs

Our algorithm for sparsifying complete product demand graphs begins by splitting the vertices of highest demands into many vertices. By splitting a vertex, we mean replacing it by many vertices whose demands sum to its original demand. In this way, we obtain a larger product demand graph. We observe that we can obtain a sparsifier of the original graph by sparsifying the larger graph, and then collapsing back together the vertices that were split.

Proposition A.10. Let $G$ be a product demand graph with vertex set $\{1, \ldots, n\}$ and demands $d$, and let $\hat{G} = (\hat{V}, \hat{E})$ be a product demand graph with demands $\hat{d}$. If there is a partition of $\hat{V}$ into sets $S_1, \ldots, S_n$ so that for all $i \in V$, $\sum_{j \in S_i} \hat{d}_j = d_i$, then $\hat{G}$ is a splitting of $G$ and there is a matrix $M$ so that

$$L_G = ML_{\hat{G}}M^T.$$ 

Proof. The $(i, j)$ entry of matrix $M$ is 1 if and only if $j \in S_i$. Otherwise, it is zero.

We now show that we can sparsify $G$ by sparsifying $\hat{G}$. 
Proposition A.11. Let $\hat{G}_1$ and $\hat{G}_2$ be graphs on the same vertex set $\hat{V}$ such that $\hat{G}_1 \approx_\epsilon \hat{G}_2$ for some $\epsilon$. Let $S_1, \ldots, S_n$ be a partition of $\hat{V}$, and let $G_1$ and $G_2$ be the graphs obtained by collapsing together all the vertices in each set $S_i$ and eliminating any self loops that are created. Then

$$G_1 \approx_\epsilon G_2.$$  

Proof. Let $M$ be the matrix introduced in Proposition A.10. Then,

$$L_{G_1} = ML_{\hat{G}_1}M^T \quad \text{and} \quad L_{G_2} = ML_{\hat{G}_2}M^T.$$  

The proof now follows from Fact 3.2. \qed

For distinct vertices $i$ and $j$, we let $(i, j)$ denote the graph with an edge of weight 1 between vertex $i$ and vertex $j$. If $i = j$, we let $(i, j)$ be the empty graph. With this notation, we can express the product demand graph as

$$\sum_{i<j} d_id_j(i, j) = \frac{1}{2} \sum_{i,j \in V} d_id_j(i, j).$$

This notation also allows us to precisely express our algorithm for sparsifying product demand graphs.

$G' = \text{WeightedExpander}(d, \epsilon)$

1. Let $\hat{n}$ be the least integer greater than $2n/\epsilon^2$ such that the algorithm described in Lemma A.8 produces an $\epsilon$-approximation of $K_{\hat{n}}$.

2. Let $t = \sum_k d_k / \hat{n}$.

3. Create a new product demand graph $\hat{G}$ with demand vector $\hat{d}$ by splitting each vertex $i$ into a set of $\lceil d_i/t \rceil$ vertices, $S_i$:

   (a) $\lceil d_i/t \rceil$ vertices with demand $t$.

   (b) one vertex with demand $d_i - t \lceil d_i/t \rceil$.

4. Let $H$ be a set of $\hat{n}$ vertices in $\hat{G}$ with demand $t$, and let $L$ contain the other vertices. Set $k = |L|$.

5. Partition $H$ arbitrarily into sets $V_1, \ldots, V_k$, so that $|V_i| \geq \lceil \hat{n}/k \rceil$ for all $1 \leq i \leq k$.

6. Use the algorithm described in Lemma A.8 to produce $\tilde{K}_{HH}$, an $\epsilon$-approximation of the complete graph on $H$. Set

$$\tilde{G} = t^2 \tilde{K}_{HH} + \sum_{l \in L} |H| \sum_{h \in V_l} \hat{d}_l \hat{d}_h(l, h).$$

7. Let $G'$ be the graph obtained by collapsing together all vertices in each set $S_i$.

This section and the next are devoted to the analysis of this algorithm. Given Proposition A.11, we just need to show that $G$ is a good approximation to $\hat{G}$.  

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Proposition A.12. The number of vertices in \( \hat{G} \) is at most \( n + \hat{n} \).

Proof. The number of vertices in \( \hat{G} \) is

\[
\sum_{i \in V} \left\lceil \frac{d_i}{t} \right\rceil \leq n + \sum_{i \in V} \frac{d_i}{t} = n + \hat{n}.
\]

So, \( k \leq n \) and \( \hat{n} \geq \frac{2k}{\varepsilon^2} \). That is, \(|H| \geq 2|L|/\varepsilon^2\). In the next section, we prove the lemmas that show that for these special product demand graphs \( \hat{G} \) in which almost all weights are the maximum, our algorithm produces a graph \( \tilde{G} \) that is a good approximation of \( G \).

Theorem A.13. Let \( 0 < \varepsilon < 1 \) and let \( G \) be a product demand graph with \( n \) vertices and demand vector \( d \). Given \( d \) and \( \varepsilon \) as input, \textsc{WeightedExpander} produces a graph \( G' \) with \( O(n/\varepsilon^4) \) edges that is an \( O(\varepsilon) \) approximation of \( G \). Moreover, \textsc{WeightedExpander} runs in \( O(\log n) \) depth and \( O(n/\varepsilon^4) \) work.

Proof. The number of vertices in the graph \( \hat{G} \) will be between \( n + 2n/\varepsilon^2 \) and \( n + 16n/\varepsilon^2 \). So, the algorithm described in Lemma A.8 will take \( O(\log n) \) depth and \( O(n/\varepsilon^4) \) work to produce an \( \varepsilon \) approximation of the complete graph on \( \hat{n} \) vertices. This dominates the computational cost of the algorithm.

Proposition A.11 tells us that \( G' \) approximates \( G \) at least as well as \( \hat{G} \) approximates \( \tilde{G} \). To bound how well \( G \) approximates \( \tilde{G} \), we use two lemmas that are stated in the next section. Lemma A.15 shows that

\[
\hat{G}_{HH} + \hat{G}_{LH} \approx O(\varepsilon^2) \hat{G}.
\]

Lemma A.17 shows that

\[
\hat{G}_{HH} + \hat{G}_{LH} \approx \varepsilon^4 \hat{G}_{HH} + \sum_{l \in L} \frac{|H|}{|V_l|} \sum_{h \in V_l} \hat{d}_l \hat{d}_h(l, h).
\]

And, we already know that \( t^2 \tilde{K} \) is an \( \varepsilon \)-approximation of \( \hat{G}_{HH} \). Fact 3.3 says that we can combine these three approximations to conclude that \( G \) is an \( O(\varepsilon) \)-approximation of \( \hat{G} \).

A.2 Product demand graphs with most weights maximal

In this section, we consider product demand graphs in which almost all weights are the maximum. For simplicity, we make a slight change of notation from the previous section. We drop the hats, we let \( n \) be the number of vertices in the product demand graph, and we order the demands so that

\[
d_1 \leq d_2 \leq \cdots \leq d_k \leq d_{k+1} = \cdots = d_n = 1.
\]

We let \( L = \{1, \ldots, k\} \) and \( H = \{k+1, \ldots, n\} \) be the set of low and high demand vertices, respectively. Let \( G \) be the product demand graph corresponding to \( d \), and let \( G_{LL}, G_{HH} \) and \( G_{LH} \) be the subgraphs containing the low-low, high-high and low-high edges respectively. We now show that little is lost by dropping the edges in \( G_{LL} \) when \( k \) is small.

Our analysis will make frequent use of the following Poincaré inequality:
Lemma A.14. Let \( c(u,v) \) be an edge of weight \( c \) and let \( P \) be a path from \( u \) to \( v \) consisting of edges of weights \( c_1, c_2, \ldots, c_k \). Then
\[
c(u,v) \preceq c \left( \sum c_i^{-1} \right) P.
\]

As the weights of the edges we consider in this section are determined by the demands of their vertices, we introduce the notation
\[
[i,j] = d_id_j(i,j).
\]
With this notation, we can express the product demand graph as
\[
\sum_{i<j} [i,j] = \frac{1}{2} \sum_{i,j \in V} [i,j].
\]

Lemma A.15. If \(|L| \leq |H|\), then
\[
G_{HH} + G_{LH} \approx \frac{|L|}{|H|} G.
\]

Proof. The lower bound \( G_{HH} + G_{LH} \preceq G_{HH} + G_{LH} + G_{LL} \) follows from \( G_{LL} \succeq 0 \).

Using lemma A.14 and the assumptions \( d_l \leq 1 \) for \( l \in L \) and \( d_h = 1 \) for \( h \in H \), we derive for every \( l_1, l_2 \in L \),
\[
[l_1, l_2] = \frac{1}{|H|^2} \sum_{h_1, h_2 \in H} [l_1, l_2]
\]
(by Lemma A.14)
\[
\leq \frac{1}{|H|^2} \sum_{h_1, h_2 \in H} d_{l_1}d_{l_2} \left( \frac{1}{d_{l_1}d_{h_1}} + \frac{1}{d_{h_1}d_{h_2}} + \frac{1}{d_{h_2}d_{l_2}} \right) ([l_1, h_1] + [h_1, h_2] + [h_2, l_2])
\]
\[
\leq \frac{3}{|H|^2} \sum_{h_1, h_2 \in H} ([l_1, h_1] + [h_1, h_2] + [h_2, l_2])
\]
\[
= \frac{3}{|H|} \sum_{h \in H} ([l_1, h] + [l_2, h]) + \frac{6}{|H|^2} G_{HH}.
\]

So,
\[
G_{LL} = \frac{1}{2} \sum_{l_1, l_2 \in L} [l_1, l_2]
\]
\[
\leq \frac{1}{2} \sum_{l_1, l_2} \left( \frac{3}{|H|} \sum_{h \in H} ([l_1, h] + [l_2, h]) + \frac{6}{|H|^2} G_{HH} \right)
\]
\[
= \frac{3|L|}{|H|} G_{LH} + \frac{3|L|^2}{|H|^2} G_{HH}.
\]

The assumption \(|L| \leq |H|\) then allows us to conclude
\[
G_{HH} + G_{LH} + G_{LL} \preceq \left( 1 + \frac{3|L|}{|H|} \right) (G_{HH} + G_{LH}).
\]
\[\square\]
Using a similar technique, we will show that the edges between $L$ and $H$ can be replaced by the union of a small number of stars. In particular, we will partition the vertices of $H$ into $k$ sets, and for each of these sets we will create one star connecting the vertices in that set to a corresponding vertex in $L$.

We employ the following consequence of the Poincare inequality in Lemma \ref{lem:poincare}.

**Lemma A.16.** For any $\varepsilon \leq 1$, $l \in L$ and $h_1, h_2 \in H$, $\varepsilon[h_1, l] + (1/2)[h_1, h_2] \approx \sqrt[4]{\varepsilon} \varepsilon[h_2, l] + (1/2)[h_1, h_2]$.

**Proof.** By applying Lemma \ref{lem:poincare} and recalling that $d_{h_1} = d_{h_2} = 1$ and $d_l \leq 1$, we compute

\[
[h_1, l] \leq d_{h_1}d_l \left( \frac{\sqrt{\varepsilon}}{d_{h_1}d_{h_2}} + \frac{1}{d_{h_2}d_l} \right) \left( \frac{1}{\sqrt{\varepsilon}}[h_1, h_2] + [h_2, l] \right) \leq \frac{1 + \sqrt{\varepsilon}}{\sqrt{\varepsilon}}[h_1, h_2] + (1 + \sqrt{\varepsilon})[h_2, l] \leq (1 + \sqrt{\varepsilon})[h_2, l] + \frac{2}{\sqrt{\varepsilon}}[h_1, h_2].
\]

Multiplying both sides by $\varepsilon$ and adding $(1/2)[h_1, h_2]$ then gives

\[
\varepsilon[h_1, l] + (1/2)[h_1, h_2] \leq (1 + \sqrt{\varepsilon})\varepsilon[h_2, l] + (2\sqrt{\varepsilon} + 1/2)[h_1, h_2] \leq (1 + 4\sqrt{\varepsilon}) (\varepsilon[h_2, l] + (1/2)[h_1, h_2]) \leq e^{4\sqrt{\varepsilon}} (\varepsilon[h_2, l] + (1/2)[h_1, h_2]).
\]

By symmetry, we also have

\[
\varepsilon[h_2, l] + (1/2)[h_1, h_2] \leq e^{4\sqrt{\varepsilon}} (\varepsilon[h_1, l] + (1/2)[h_1, h_2]).
\]

$\square$

**Lemma A.17.** Recall that $L = \{1, \ldots, k\}$ and let $V_1, \ldots, V_k$ be a partition of $H = \{k + 1, \ldots, n\}$ so that $|V_l| \geq s$ for all $l$. Then,

\[
G_{HH} + G_{LH} \approx_4 \sqrt{s} \sum_{l \in L} \frac{|H|}{|V_l|} \sum_{h \in V_l} [l, h].
\]

**Proof.** Observe that

\[
G_{LH} = \sum_{l \in L} \sum_{h \in H} [l, h].
\]

For each $l \in L$, $h_1 \in H$ and $h_2 \in V_l$ we apply Lemma \ref{lem:poincare} to show that

\[
\frac{1}{|V_l|}[l, h_1] + \frac{1}{2}[h_1, h_2] \approx_4 \sqrt{s} \frac{1}{|V_l|}[l, h_2] + \frac{1}{2}[h_1, h_2].
\]

Summing this approximation over all $h_2 \in V_l$ gives

\[
[l, h_1] + \sum_{h_2 \in V_l} \frac{1}{2}[h_1, h_2] \approx_4 \sqrt{s} \sum_{h_2 \in V_l} \left( \frac{1}{|V_l|}[l, h_2] + \frac{1}{2}[h_1, h_2] \right).
\]
Summing the left-hand side of this approximation over all \( l \in L \) and \( h_1 \in H \) gives

\[
\sum_{l \in L, h_1 \in H} [l, h_1] + \frac{1}{2} \sum_{h_2 \in V_l} [h_1, h_2] = \sum_{l \in L, h_1 \in H} [l, h_1] + \frac{1}{2} \sum_{h_1 \in H} \sum_{l \in L, h_2 \in V_l} [h_1, h_2] = G_{LH} + G_{HH}.
\]

On the other hand, the sum of the right-hand terms gives

\[
G_{HH} + \sum_{l \in L, h_1 \in H} \sum_{h_2 \in V_l} \frac{1}{|V_l|} [l, h_2] = G_{HH} + \sum_{l \in L} \sum_{h_2 \in V_l} \frac{|H|}{|V_l|} [l, h_2].
\]

\[\square\]

### A.3 Weighted Bipartite Expanders

This construction extends analogously to bipartite product graphs. The bipartite product demand graph of vectors \((d^A, d^B)\) is a complete bipartite graph whose weight between vertices \( i \in A \) and \( j \in B \) is given by \( w_{ij} = d_i^A d_j^B \). Without loss of generality, we will assume \( d_1^A \geq d_2^A \geq \cdots \geq d_n^A \) and \( d_1^B \geq d_2^B \geq \cdots \geq d_n^B \).

As the weights of the edges we consider in this section are determined by the demands of their vertices, we introduce the notation

\[ [i, j] = d_i^A d_j^B (i, j). \]

Our construction is based on a similar observation that if most vertices on \( A \) side have \( d_i^A \) equaling to \( d_1^A \) and most vertices on \( B \) side have \( d_i^B \) equaling to \( d_1^B \), then the uniform demand graph on these vertices dominates the graph.
\[ G' = \text{WeightedBipartiteExpander}(d^A, d^B, \epsilon) \]

1. Let \( n' = \max(n^A, n^B) \) and \( \hat{n} \) be the least integer greater than \( 2n'/\epsilon^2 \) such that the algorithm described in Lemma A.15 produces an \( \epsilon \)-approximation of \( K_{\hat{n}, \hat{n}} \).

2. Let \( t^A = \frac{\sum_i d^A_i}{\hat{n}} \) and \( t^B = \frac{\sum_i d^B_i}{\hat{n}} \).

3. Create a new bipartite demand graph \( \tilde{G} \) with demands \( \hat{d}^A \) and \( \hat{d}^B \) follows:

   (a) On the side \( A \) of the graph, for each vertex \( i \), create a subset \( S_i \) consisting of \( \left\lceil \frac{d^A_i}{t^A} \right\rceil \) vertices:

      i. \( \left\lceil \frac{d^A_i}{t^A} \right\rceil \) with demand \( t^A \).

      ii. one vertex with demand \( d^A_i - \left\lceil \frac{d^A_i}{t^A} \right\rceil \).

   (b) Let \( H^A \) contain \( \hat{n} \) vertices of \( A \) of with demand \( t^A \), and let \( L^A \) contain the rest. Set \( k^A = \left\lceil L^A \right\rceil \).

   (c) Create the side \( B \) of the graph with partition \( H^B, L^B \) and demand vector \( \hat{d}^B \) similarly.

4. Partition \( H^A \) into sets of size \( |V_i^A| \geq \left\lceil \hat{n}/k^A \right\rceil \), one corresponding to each vertex \( l \in L^A \). Partition \( V_B \) similarly.

5. Let \( \tilde{K}_{H^AH^B} \) be a bipartite expander produced by Lemma A.8 that \( \epsilon \)-approximates \( K_{\hat{n}, \hat{n}} \), identified with the vertices \( H^A \) and \( H^B \).

   Set
   \[
   \tilde{G} = t^A t^B \tilde{K} + \sum_{l \in L^A} \frac{|H^B|}{|V^B_l|} \sum_{h \in V^B_l} \hat{d}^A_l \hat{d}^B_h(l, h) + \sum_{l \in L^B} \frac{|H^A|}{|V^A_l|} \sum_{h \in V^A_l} \hat{d}^B_l \hat{d}^A_h(l, h).
   \]

6. Let \( G' \) be the graph obtained by collapsing together all vertices in each set \( S^A_i \) and \( S^B_i \).

Similarly to the nonbipartite case, the Poincare inequality show that the edges between low demand vertices can be completely omitted if there are many high demand vertices which allows the demand routes through high demand vertices.

**Lemma A.18.** Let \( G \) be the bipartite product demand graph of the demand \( (d^A_i, d^B_j) \). Let \( H^A \) a subset of vertices on \( A \) side with demand higher than the set of remaining vertices \( L^A \) on \( A \) side. Define \( H^B, L^B \) similarly. Assume that \( |L^A| \leq |H^A| \) and \( |L^B| \leq |H^B| \), then

\[
G_{H^AH^B} + G_{H^AL^B} + G_{L^AH^B} \approx 3\max\left(\frac{|L^A|}{|H^A|}, \frac{|L^B|}{|H^B|}\right) G.
\]

**Proof.** The proof is analogous to Lemma A.15, but with the upper bound modified for bipartite graphs.

For every edge \( l_A, l_B \), we embed it evenly into paths of the form \( l_A, h_B, h_A, l_B \) over all choices of \( h_A \) and \( h_B \). The support of this embedding can be calculated using Lemma A.14 and the overall accounting follows in the same manner as Lemma A.15.

\[ \square \]
It remains to show that the edges between low demand and high demand vertices can be compressed into a few edges. The proof here is also analogous to Lemma A.16. We use the Poincare inequality to show that all demands can routes through high demand vertices. The structure of the bipartite graph makes it helpful to further abstract these inequalities via the following Lemma for four edges.

**Lemma A.19.** Let $G$ be the bipartite product demand graph of the demand $(d_A^i, d_B^j)$. Given $h_A, l_A \in A$ and $h_{B,1}, h_{B,2} \in B$. Assume that $d_{h_A}^A = d_{h_{B,1}}^B = d_{h_{B,2}}^B \geq d_A^A$. For any $\epsilon < 1$, we have

$$\epsilon[l_A, h_{B,1}] + [h_A, h_{B,2}] + [h_A, h_{B,1}] \approx_{3, \epsilon} \epsilon[l_A, h_{B,2}] + [h_A, h_{B,2}] + [h_A, h_{B,1}].$$

**Proof.** Using Lemma A.14 and $d_{h_A}^A = d_{h_{B,1}}^B = d_{h_{B,2}}^B \geq d_A^A$, we have

$$[l_A, h_{B,1}] \leq d_{h_A}^A d_{h_{B,1}} + \frac{1}{d_{h_A}^A d_{h_{B,2}}} \left( \frac{\sqrt{\epsilon}}{d_{h_A}^A d_{h_{B,2}}} + \frac{\sqrt{\epsilon}}{d_{h_A}^A d_{h_{B,1}}} \right) \left( [l_A, h_{B,2}] + \frac{1}{\sqrt{\epsilon}} [h_A, h_{B,2}] + \frac{1}{\sqrt{\epsilon}} [h_A, h_{B,1}] \right).$$

$$\leq (1 + 2\sqrt{\epsilon})[l_A, h_{B,1}] + \frac{1 + 2\sqrt{\epsilon}}{\sqrt{\epsilon}} [h_A, h_{B,2}] + \frac{1 + 2\sqrt{\epsilon}}{\sqrt{\epsilon}} [h_A, h_{B,1}].$$

Therefore,

$$\epsilon[l_A, h_{B,1}] + [h_A, h_{B,2}] + [h_A, h_{B,1}] \leq (1 + 3\sqrt{\epsilon})(\epsilon[l_A, h_{B,2}] + [h_A, h_{B,2}] + [h_A, h_{B,1}]).$$

The other side is similar due to the symmetry. \hfill \Box

**Theorem A.20.** Let $0 < \epsilon < 1$ and let $G$ be a bipartite demand graph with $n$ vertices and demand vector $(d_A^i, d_B^j)$. WeightedBipartiteExpander produces a graph $G'$ with $O(n/\epsilon^4)$ edges that is an $O(\epsilon)$ approximation of $G$. Moreover, WeightedBipartiteExpander runs in $O(\log n)$ depth and $O(n/\epsilon^4)$ work.

**Proof.** The proof is analogous to Theorem A.13. After the splitting, the demands in $H^A$ are higher than the demands in $L^A$ and so is $H^B$ to $L^B$. Therefore, Lemma A.18 shows that that

$$\tilde{G}_{H^A H^B} + \tilde{G}_{H^A L^B} + \tilde{G}_{L^A H^B} \approx_{3\epsilon^2/2} \tilde{G}.$$

By a proof analogous to Lemma A.17 one can use Lemma A.19 to show that

$$\tilde{G}_{H^A H^B} + \tilde{G}_{H^A L^B} + \tilde{G}_{L^A H^B} \approx_{O(\epsilon)} \tilde{G}_{H^A H^B} + \frac{|H^B|}{|V_I^B|} \sum_{h \in V_I^B} \tilde{d}_h^B \hat{l}(l, h) + \sum_{l \in L^B} \frac{|H^A|}{|V_I^A|} \sum_{h \in V_I^A} \tilde{d}_h^A \hat{l}(l, h).$$

And, we already know that $t^B \hat{K}$ is an $\epsilon$-approximation of $\tilde{G}_{H^A H^B}$. Fact 3.3 says that we can combine these three approximations to conclude that $\tilde{G}$ is an $O(\epsilon)$-approximation of $\tilde{G}$. \hfill \Box