Approximating Continuous Functions by ReLU Nets of Minimal Width

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Abstract. This article concerns the expressive power of depth in deep feed-forward neural nets with ReLU activations. Specifically, we answer the following question: for a fixed \( d \geq 1 \), what is the minimal width \( w \) so that neural nets with ReLU activations, input dimension \( d \), hidden layer widths at most \( w \), and arbitrary depth can approximate any continuous function of \( d \) variables arbitrarily well. It turns out that this minimal width is exactly equal to \( d + 1 \). That is, if all the hidden layer widths are bounded by \( d \), then even in the infinite depth limit, ReLU nets can only express a very limited class of functions. On the other hand, we show that any continuous function on the \( d \)-dimensional unit cube can be approximated to arbitrary precision by ReLU nets in which all hidden layers have width exactly \( d + 1 \). Our construction gives quantitative depth estimates for such an approximation.

1. Introduction

Over the past several years, artificial neural networks, especially deep networks, have become the state of the art in a wide variety of machine learning tasks. These tasks include important benchmark problems in machine vision (\cite{KSH12}) and machine translation (\cite{SVL14, WSC16}) as well as superhuman performance at games such as Go \cite{SHM16}. Despite these varied and striking successes, a theory of why neural nets provide such good approximations to interesting functions and can be effectively trained is only beginning to take shape.

While non-linear activations help neural nets express a wide variety of functions, repeated non-linearities can also “garble” the signal, leading to a loss of mutual information between the input and the activations at various hidden layers. Such an information theoretic point of view on neural nets has recently been systematically taken up in the work of Tishby with Shwartz-Ziv, Moshkovitz, and Zaslavsky \cite{SZT17, MT17, TZ15}. In the present article, we answer a basic information theoretic question about neural nets. Namely, for each \( d \geq 1 \), what is the minimal width \( w_{\text{min}}(d) \) so that neural nets whose hidden layers have width at least \( w_{\text{min}}(d) \) and arbitrary depth can approximate arbitrarily well any continuous function of \( d \) variables? We treat only neural nets with a popular and particularly simple activation function called rectified linear units, defined

\[
\text{ReLU}(t) := \max\{0, t\}.
\]

It have been known since the 1980’s (e.g. the work of Cybenko \cite{Cyb89} and Hornik-Stinchcombe-White \cite{HSW89}) that feed-forward neural nets with a single hidden-forward neural nets with a single hidden layer can approximate essentially any function if the hidden layer is allowed to be arbitrarily wide. Such results hold for a wide variety of activations, including ReLU.
However, part of the recent renaissance in neural nets, is the empirical observation that deep neural nets tend to achieve greater expressivity per parameter than their shallow cousins. There are now a number of rigorous results about this so-called expressive power of depth \cite{ABMM16, MLP16, LTR17, MP16, PLR16, RPK16, RT17, Tel15, Tel16, Tel17, Yar16}. We refer the reader to §3 in \cite{Han17} for a discussion of the relationships between some of these articles.

The main result of this article shows a sharp transition in the representational power of deep feed-forward neural nets with ReLU activations as a function of the widths of their hidden layers. To state it, we need some notation. We say that $N$ is a feed-forward neural net with ReLU activations, input dimension $d_{in}$, output dimension $d_{out}$, and widths $d_{in} = d_1, d_2, \ldots, d_k, d_{k+1} = d_{out}$ (a ReLU net for short) if it is a function of the form

$$\text{ReLU} \circ A_k \circ \text{ReLU} \circ A_{k-1} \circ \cdots \circ \text{ReLU} \circ A_1,$$

where $A_i : \mathbb{R}^{d_i} \to \mathbb{R}^{d_{i+1}}$ are affine transformations and for any $m \geq 1$

$$\text{ReLU} (x_1, \ldots, x_m) = (\max\{0, x_1\}, \ldots, \max\{0, x_m\}).$$

The integers $d_2, \ldots, d_k$ are said to be the widths of the hidden layers of $N$, and the integer $k$ is the depth of $N$. Notice that for fixed $d_1, \ldots, d_{k+1}$, the family of neural nets is a finite dimensional family of non-linear functions parameterized by the affine transformations $A_i$. Our main result determines $\omega_{\min}(d)$, defined to be the minimal value of $w$ such that for every positive continuous function $f : [0, 1]^d \to \mathbb{R}$ and every $\varepsilon > 0$ there is a ReLU net $N$ with input dimension $d$, hidden layer widths at most $w$, and output dimension 1 that $\varepsilon$-approximates $f$:

$$\sup_{x \in [0, 1]^d} |f(x) - N(x)| \leq \varepsilon.$$

**Theorem 1.** For every $d \geq 1$, $\omega_{\min}(d) = d + 1$.

Proving the upper bound $\omega_{\min}(d) \leq d+1$ in Theorem 1 requires a novel construction by which any continuous function of $d$ variables can be approximated to arbitrary precision by a ReLU net with width $d+1$ and depth depending on its modulus of continuity $\omega_f$. In fact, we will show that if $K \subseteq \mathbb{R}^d$ is any compact set and $f : K \to \mathbb{R}$ is continuous, then there exists a ReLU net $N$ with hidden layer width $d+1$ and depth $O(\text{diam}(K)/\omega_f(\varepsilon))^{d+1}$ that $\varepsilon$-approximates $f$ on $K$:

$$\sup_{x \in K} |f(x) - N(x)| \leq \varepsilon.$$

We refer the reader to Proposition 3 for the precise statement. The construction is carried out in \S2.

In contrast, the lower bound $\omega_{\min}(d) \geq d + 1$ requires constructing, for every $d \geq 1$, a continuous positive function $f : [0, 1]^d \to \mathbb{R}$ and a constant $\eta > 0$ so that any width $d$ ReLU net $N$ must satisfy

$$\sup_{x \in [0, 1]^d} |f(x) - f_N(x)| > \eta.$$

Our construction in \S3 shows that it is enough to take $f$ to simply be a quadratic function. Our construction in fact shows that any function that takes different values
in the center of \([0,1]^d\) and its vertices cannot be arbitrarily well-approximated by a ReLU net of width \(d\).

Before proceeding to the proof of Theorem 1 we make two remarks. First, the neural nets we consider here are not allowed to have skip (e.g. residual) connections, popularized in the ResNets introduced by He-Zhang-Ren-Sun in [HZRS16] and in the Highway Nets introduced by Srivastava-Greff-Schidhuber in [SGS15]. A skip connection allows the input to a given hidden layer to be an affine function of the all the outputs of all the previous hidden layers, instead of just the one preceding it. If one allows skip connections, then a ReLU net whose hidden layers have width 1 can already approximate any continuous function if the net is allowed to be arbitrarily deep. The reason is that any feed-forward neural net with one hidden layer of width \(k\) can be converted into a neural net with \(k\) hidden layers, each of width 1, that computes the same function. The construction is simply to “turn the hidden layer on its’ side.” That is, each neuron in the single hidden layer in the original shallow net becomes its own hidden layer. The input to the net is connected to the single neural in every new hidden layer, which is in turn connected to the output. In this construction, each hidden layer is connected only to the input and output. In the language of Veit-Wilber-Belongie [VWB16], the resulting ResNet implements an ensemble of paths of length 1. Second, it is tempting to generalize Theorem 1 to arbitrary piecewise linear activations. However, it seems that such a generalization is not straightforward, even for activations of the form \(\sigma(t) = \max\{\ell_1(t), \ell_2(t)\}\), where \(\ell_1, \ell_2\) are two affine functions with different slopes.

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2. Proof of the Upper Bound in Theorem 1

Fix \(\varepsilon > 0\), \(d \geq 1\), a compact set \(K \subseteq \mathbb{R}^d\), and a non-negative continuous function \(f : K \to \mathbb{R}\). In this section, we prove that there exists a ReLU net \(N\) with input dimension \(d\), hidden layer width \(d + 1\), and output dimension 1 such that
\[
\|f - f_N\|_{C^0(K)} = \sup_{x \in K} |f(x) - f_N(x)| \leq \varepsilon.
\]
(3)
We will use the following definition.

Definition 1. A function \(g : \mathbb{R}^d \to \mathbb{R}\) is a max-min string of length \(k\) on \(d\) variables if there exist affine functions \(\ell_1, \ldots, \ell_k : \mathbb{R}^d \to \mathbb{R}\) such that
\[
g = \sigma_{k-1}(\ell_k, \sigma_{k-2}(\ell_{k-1}, \ldots, \sigma_2(\ell_3, \sigma_1(\ell_1, \ell_2))\cdots),
\]
where each \(\sigma_i\) is either a max or a min.

The statement (3) follows immediately from the following two propositions.

Proposition 2. For every max-min string \(g\) on \(d\) variables with length \(k\) and every compact \(K \subseteq \mathbb{R}^d\), there exists a ReLU net with input dimension \(d\), hidden layer width \(d + 1\), output dimension 1, and depth \(k\) that computes \(x \mapsto g(x)\) for every \(x \in K\).
Proposition 3. For every compact \( K \subseteq \mathbb{R}^d \), any non-negative continuous \( f : K \to \mathbb{R}_+ \) and each \( \varepsilon > 0 \) there exists a non-negative max-min strings \( g \) on \( d \) variables with length
\[
\left( O(\text{diam}(K)) \right)^{d+1} \frac{1}{\omega_f(\varepsilon)}
\]
for which
\[
\|f - g\|_{C^0(K)} \leq \varepsilon.
\]

Proposition 2 is essentially Lemma 4 in [Han17]. We include a short proof for the reader’s convenience in §2.1. Proposition 3 appears to be new, however, and is the main technical result in the present article. It is proved in §2.2. It is related in spirit to results in the literature (e.g. [Sch12, Prop. 2.2.2.]) that express a continuous piecewise affine \( h : K \to \mathbb{R} \) on a convex domain as
\[
\max_{1 \leq i \leq N} \min_{1 \leq j \leq M(i)} \{ \ell_{1,i}, \ldots, \ell_{M(i),i} \}, \quad \ell_{j,i} : K \to \mathbb{R} \text{ affine}.
\]
Nonetheless, Proposition 3 is of a rather different nature since we are allowed to take only max and min of two affine functions at a time.

2.1. Proof of Proposition 2. We may assume without loss of generality that \( K \) is contained in the positive orthant:
\[
K \subseteq \mathbb{R}_+^d = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \middle| x_i \geq 0, \quad 1 \leq i \leq d \right\}
\]
since we can always shift the input to a neural net by a fixed vector. Moreover, fix a non-negative max-min string
\[
g = \sigma_{k-1}(\ell_k, \sigma_{k-2}(\ell_{k-1}, \ldots, \sigma_2(\ell_3, \sigma_1(\ell_1, \ell_2)) \ldots)).
\]
Note that for any constant \( C \), the function \( g + C \) is also a max-min string whose affine tranformations are \( \ell_i + C \). Since we may subtract an arbitrary constant in the output of the last layer in a ReLU net, we may additionally assume that each \( \ell_i \) is non-negative on \( K \). With these reductions, we construct the neural net that computes \( g(x) \) for every \( x \in K \). For all \( j = 2, \ldots, k \) define affine tranformations \( A_j : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1} \) by
\[
A_j(x, y) = \begin{cases} 
A_j(x, y) = (x, y - \ell_j(x)), & \text{if } \sigma_{j-1} = \max \\
A_j(x, y) = (x, -y + \ell_j(x)), & \text{if } \sigma_{j-1} = \min.
\end{cases}
\]
Further, set
\[
A_1(x) = (x, \ell_1(x)), \quad x \in \mathbb{R}^d.
\]
Write \( H_1 := \text{ReLU} \circ A_1 \) and
\[
H_j := A_j \circ \text{ReLU} \circ A_j^{-1}, \quad j = 2, \ldots, k.
\]
The image of \( K \) under \( H_0 \) is the graph of \( \ell_1 \), and the image of the graph of any function \( g : K \to \mathbb{R} \) under \( H_j \) is the graph of \( \sigma_{j-1}(\ell_j, g) \). Hence, the image of \( K \) under the ReLU net
\[
\text{ReLU} \circ H_k \circ \cdots \circ H_1
\]
is the graph of \( g \). Note that the final ReLU is trivial since \( g \) is non-negative. Appending a final layer \( (x_1, \ldots, x_{d+1}) \mapsto \text{ReLU}(x_{d+1}) \) yields the desired net. \( \square \)
2.2. **Proof of Proposition** 3. Note that if $g$ is a max-min string on $d$ variables, then so is $g(x - x_0)$ for any $x_0 \in \mathbb{R}^d$. Using also that every compact set is contained in a ball shows that we may assume without loss of generality that $K$ is a ball $B_r$ of radius $r$ centered at the origin.

Fix a continuous function $f : B_r \to \mathbb{R}_+$. We first explain how to uniformly approximate $f$ by max-min strings in the model case when we seek to approximate $f$ on an arbitrary finite subset of $\mathbb{R}^d$.

**Proposition 4.** Let $S \subseteq \mathbb{R}^d$ be a finite set. Then any function $f : S \to \mathbb{R}_+$ can be computed exactly by a max-min string.

**Proof.** We prove the proposition by induction on $|S|$. If $S = \{s\}$, then the max-min string $\max\{f(s), 0\}$ agrees with $f$ on $S$. Suppose now that $|S| \geq 2$. The idea is to consider the convex hull $\hat{S}$ of the points in $S$ and “repeatedly cut off a corner.” Let $s_0 \in S$ be a vertex of the convex hull of $S$. By the inductive hypothesis, there is a max-min string $g$ on $d$ variables that agrees with $f$ on $S \setminus \{s_0\}$. Moreover, for every $t > 0$, we can find an affine function $\ell$ with $\ell(s_0) = 0$ and $\ell(s) \geq t$ for $s \in S \setminus \{s_0\}$. Taking $t$ large, define the max-min string

$$\tilde{g} = \max(\min(g, f(s_0) + \ell), f(s_0) - \ell).$$

By construction, $\tilde{g}(s_0) = f(s_0)$. Further, because $t$ is large, $\tilde{g}(s) = f(s)$ for $s \in S \setminus \{s_0\}$. Hence $\tilde{g}$ and $f$ agree on $S$, completing the proof. □

We carry out the same proof idea for continuous functions on $\mathbb{R}^d$. We focus on the construction for $d = 2$ and then explain the minor modification needed for $d \geq 3$. Before getting into the details, we emphasize the main difference between the discrete case treated in Proposition 4 above and the continuous case below. The issue is that now when we cut off a corner from the convex hull of the set where we have $\varepsilon$-approximated the function $f$, we have to approximate $f$ correctly on the entire piece we cut off, not just at a single vertex. To get an $\varepsilon$-approximation, we need our corner piece to have diameter $O(\omega_f(\varepsilon))$ so that the variation of $f$ on the piece is $O(\varepsilon)$ (recall that $\omega_f(\varepsilon)$ is the modulus of continuity). That is, we can only cut off small-diameter pieces at a time. Thus, to build an approximation to $f$ on ball of radius $R$ from an approximation to $f$ on a ball of radius $r < R$, we have to slowly add small pieces to $B_r$ in all directions until the resulting set grows to contain $B_R$. Our precise construction repeatedly uses the following observation. We state the observation for $d = 2$ and explain below its extension to $d \geq 3$.

**Lemma 5.** Fix $\varepsilon > 0$ and a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$. Write $\omega_f$ for the modulus of continuity of $f$. Suppose $K \subseteq \mathbb{R}^2$ and $\triangle ABC$ is an triangle with

$$\text{diam}(\triangle ABC) \leq \omega_f(\varepsilon)$$

such that $K$ is contained in the infinite planar sector $\angle BAC$. Then if there exists a max-min string $g$ with

$$\sup_{x \in K} |f(x) - g(x)| \leq \varepsilon,$$

then there also exists a max-min string $\tilde{g}$ with

$$\sup_{x \in K \cup \triangle ABC} |f(x) - g(x)| \leq \varepsilon.$$
Proof. Let \( g \) be a max-min string that \( \varepsilon \)-approximates \( f \) on \( K \). Let \( \ell \) be the affine function with \( \ell(A) = 0 \), \( \ell(B) = \ell(C) = \varepsilon \). As in Proposition 4, define
\[
\ell_-(x) := f(A) - \ell(x), \quad \ell_+(x) := f(A) + \ell(x)
\]
and consider the max-min string
\[
\hat{g} = \max(\ell_-, \min(\ell_+, g)).
\]
Next, by the definition of \( \omega_f \), we have
\[
|\hat{g}(x) - f(x)| \leq \varepsilon, \quad x \in ABC.
\]
We now show that this estimate continues to hold for \( x \in K \) as well. We claim that on \( K \cup \triangle ABC \) we have
\[
\ell_- - \varepsilon \leq f \leq \ell_+ + \varepsilon
\]
(4)
These inequalities follow from the fact that the absolute values of the slopes of \( \ell_\pm \) on the rays \( AB \) and \( AC \) are bounded below by \( \frac{2\omega_f(\varepsilon)}{\varepsilon} \). The idea is that in \( \triangle ABC \) the deviation is at most \( \varepsilon \) while outside the large slope of \( \ell \) dominates any changes in \( f \).
More precisely, we note that for any \( x \in K \) we have
\[
|f(x) - f(A)| \leq \max\left(\varepsilon, \frac{2\varepsilon}{\omega_f(\varepsilon)}|x - A|\right).
\]
The first case covers \(|x - A| \leq \omega_f(\varepsilon)\) while the second covers \(|x - A| > \omega_f(\varepsilon)\) by chaining triangle inequalities together (the use of the triangle inequality might lose a factor of 2 if \(|x - A|\) is only slightly bigger than \( \omega_f(\varepsilon) \)). For \( x \in \triangle ABC \) the upper bound \(|f(x) - f(A)| \leq \varepsilon\) establishes Equation 4. For \( x \in K \setminus \triangle ABC \) there is a unique point \( x' \) where segments \( Ax \) and \( BC \) intersect. We have
\[
|A - x'| \leq \max(|A - B|, |A - C|) \leq \omega_f(\varepsilon)
\]
implying that the magnitudes of the slopes of \( \ell_\pm \) on line \( Ax \) are at least \( \frac{2\varepsilon}{\omega_f(\varepsilon)} \). Hence the other case of the upper bound on \(|f(x) - f(A)|\) establishes Equation 4 in this case. These estimates imply that on \( K \)
\[
f - \varepsilon = \min(f, f - \varepsilon) \leq \min(\ell_+, g) \leq \max(\ell_-, \min(\ell_+, g)) = \hat{g}
\]
and
\[
\hat{g} - \varepsilon \leq f \leq \hat{g} + \varepsilon.
\]
Therefore, \( \hat{g} - \varepsilon \leq f \leq \hat{g} + \varepsilon \), as desired. \( \square \)

We now turn to the details of the proof of Proposition 4. We will explain how to approximate our fixed continuous function \( f \) by a max-min string on a ball radius \( R > 0 \) centered at the origin. As above, the constant max-min string \( f(0) \) is an \( \varepsilon \)-approximation to \( f \) on the small ball \( B_{\omega_f(\varepsilon)}(0) \). We will use Lemma 5 to show that we can approximate \( f \) on successively larger and larger balls. Observe that if \( r \leq \omega_f(\varepsilon) \) then
\[
\|f - f(0)\|_{C^0(B_r)} \leq \varepsilon.
\]
To prove that we can approximate \( f \) on larger balls, suppose \( g \) is a max-min string on \( d \) variables that approximates \( f \) to within \( \varepsilon \) on the ball \( B_r(0) \) with \( r \geq \varepsilon \). We use
Figure 1. To extend an \( \varepsilon \)-approximation of \( f \) on the inner disk of radius \( R \) to the outer disk of radius \( R' = R + \frac{\varepsilon^2}{c R} \), we proceed in steps. Each step, we draw triangle \( X'Z'Y' \) as shown and apply Lemma 5 to extend our approximation to a larger region. Because the outer circle \( B_{R'}(P) \) is contained in sector \( X'Z'Y' \), we do not lose any area contained in \( B_{R'}(P) \) when applying Lemma 5.

Lemma 5 to construct a new max-min string \( \hat{g} \) which uniformly \( \varepsilon \)-approximates \( f \) on a ball of slightly larger radius

\[
R_{r,\varepsilon} := r + \frac{\omega f(\varepsilon)^2}{r}.
\]

Using this procedure repeatedly allows us to increase \( r \) without bound and will complete the proof. Our approach is illustrated in Figures 1 and 2.

We begin with the construction when \( d = 2 \) and will explain the simple modification for \( d \geq 3 \) below. For each \( R' > r \) and any two sufficiently close points \( X, Y \) on the boundary of \( B_r \), let \( X', Y' \) be the intersections of line \( XY \) with the boundary circle of \( B_{R'}(P) \). Also, denote by \( Z' \) be the intersection of the tangents to \( B_{R'} \) through \( X', Y' \) (see Figure 1). Then \( B_r \) is contained in the planar sector \( \angle X'Z'Y' \), and the diameter of \( \triangle X'Z'Y' \) can be made arbitrarily small by taking \( R' \) close to \( r \) and \( X \) close to \( Y \).
In Figure 2, after applying Lemma 5, the region on which we approximated $f$ has grown to include the shaded circular sector $X_0P_0Y_0$. (This is just because it is contained in the union of the two shaded regions in Figure 1.) Since $d(X,Y) \approx \varepsilon$, this means that applying Lemma 5 to $O(R^2/\varepsilon)$ rotated configurations of this form extends the region of $\varepsilon$-approximation from $B_R(P)$ to $B_{R'}(P)$.

In particular, for every $r \geq \omega_f(\varepsilon)$, if we take

$$R' = R_{r,\varepsilon} = r + \frac{\omega_f(\varepsilon)^2}{10r}, \quad |X - Y| = \omega_f(\varepsilon) \left( \frac{1}{20} - \frac{\omega_f(\varepsilon)^2}{100r^2} \right)^{1/2},$$

then

$$|X' - Y'| = \omega_f(\varepsilon).$$

Write $Q'$ for the midpoint of $X'Y'$. By the similarity of $\triangle Q'Y'Z'$ and $PY'Q'$, we have

$$\frac{\omega_f(\varepsilon)}{|Z'Y'|} = 2 \sqrt{1 - \frac{1}{4} \left( \frac{R}{\omega_f(\varepsilon)} \right)^2} \geq \sqrt{3} > 1.$$

Thus,

$$\text{diam}(\triangle X'Z'Y') = |X' - Y'| = \omega_f(\varepsilon).$$

Lemma 5 therefore shows that there exists a max-min string $g'$ that uniformly $\varepsilon$ approximates $f$ on $K' = \triangle X'Y'Z' \cup B_r$. Notice that $K'$ contains the circular sector of $B_{R_{r,\varepsilon}}$. 
cut out by the rays $OX$ and $OY$. Finally, consider a $\delta$-net $\{p_i\}$ on the circumference of $B_r$ with

$$\delta = \frac{\omega_f(\varepsilon)}{2} \left( \frac{1}{20} - \frac{\omega_f(\varepsilon)^2}{100r^2} \right)^{1/2}.$$ 

The size of this net is $O(r/\omega_f(\varepsilon)^{-1})$. Applying Lemma 5 $O(r/\omega_f(\varepsilon)^{-1})$ times and repeating the above argument with $(X,Y) = (p_i,p_{i+1})$ completes the proof of the upperbound in Theorem 1 when $d = 2$.

The argument when $d \geq 3$ is essentially the same. The idea is to take the diagrams depicted and rotate them around the axis $PZ$. Lemma 5 extends to higher dimensions with the triangle $\triangle ABC$ replaced by the tip of a cone with the same diameter requirement. Such a cone is obtained by rotating $X'Z'Y'$ in Figures 1 and 2. The rest of the argument then carries over verbatim.

Now we analyze the efficiency of this procedure. First, to complete a single radius increment requires covering the boundary of $B_r$ with balls of radius $O(\omega_f(\varepsilon))$. It is standard that in $\mathbb{R}^d$, this requires

$$\left( \frac{O(r)}{\omega_f(\varepsilon)} \right)^{d-1}$$

balls. We get one extra max and min in the max-min string we build to approximate $f$ for each such ball. Thus, at a cost of $\left( \frac{O(r)}{\omega_f(\varepsilon)} \right)^{d-1}$ many maxes and mins, the radius on which we approximate $f$ increases

$$r \mapsto R_{r,\varepsilon} = r + \frac{\omega_f(\varepsilon)^2}{10r}.$$ 

Hence, if we fix $R > \omega_f(\varepsilon)$, then for every $\omega_f(\varepsilon) \leq r \leq R$, we have

$$R_{r,\varepsilon} - r \geq \frac{\omega_f(\varepsilon)^2}{10R}$$

and to obtain an approximation of $f$ on $B_R$, by a max-min we need to extend the approximation of $f$ from a small ball to a larger ball at most $10R^2/\omega_f(\varepsilon)^2$ times. The number of maxes and mins required for each extension is $\left( \frac{O(R)}{\omega_f(\varepsilon)} \right)^{d-1}$. Hence, the length of the max-min string we construct to approximate $f$ on $B_R$ is

$$\left( \frac{O(R)}{\omega_f(\varepsilon)} \right)^{d+1},$$

as claimed. \hfill \Box

3. Proof of the Lower Bound in Theorem 1

The purpose of this section is the prove that for every $d \geq 1$, there exists $\eta = \eta(d) > 0$ so that the function

$$f(x) := \sum_{i=1}^{d} \left( x_i - \frac{1}{2} \right)^2$$

(5)
satisfies the following property. For any ReLU net \( N \) with input dimension \( d \), hidden layer width \( d \), and output dimension 1, we have 

\[
\| f - f_N \|_{C^0} \geq \eta.
\]

Fix \( d \geq 1 \), and consider a width \( d \) ReLU net

\[
N := \text{ReLU} \circ A_n \circ \cdots \circ \text{ReLU} \circ A_1,
\]

where the \( A_i \)'s are affine and \( A_1 \) maps \( \mathbb{R}^d \) to \( \mathbb{R} \), while for \( 1 \leq i \leq n - 1 \), the transformations \( A_i \) map \( \mathbb{R}^d \) to \( \mathbb{R}^d \). The following Lemma makes clear that the functions

\[
f_j(x) = \text{ReLU} \circ A_n \circ \cdots \circ \text{ReLU} \circ A_1(x)
\]

computed by the first \( j \) hidden layers of \( N \) are of a rather special form.

**Lemma 6.** For each \( j \geq 1 \) there exists a convex subset \( S_j \subseteq [0,1]^d \) with the following properties:

1. \( f_j \) is affine on \( S_j \)
2. For every \( 1 \leq j \leq n \) and each \( x \in [0,1]^d \setminus S_j \) there exists a piecewise linear path \( \gamma_x^j : [0,1] \to [0,1]^d \) such that

\[
\gamma_x^j(0) = x, \quad \gamma_x^j(1) \in \partial S_j,
\]

and \( f_j \) restricted to \( \gamma_x^j \) is constant.

**Proof.** Suppose \( S \subseteq \mathbb{R}^d \) is convex. Then \( A(S) \) is convex for every affine \( A : \mathbb{R}^d \to \mathbb{R}^d \). Moreover, set

\[
S' := S \cap \{(x_1, \ldots, x_d) \mid x_i \geq 0\}.
\]

Then \( S' \subseteq S \) is convex, ReLU restricted to \( S' \) is affine (in fact the identity), and for every \( x \in S \setminus S' \) there exists a piecewise linear path from \( x \) to the boundary of \( S' \) obtained tracing action of ReLU on \( x \). On this path, the ReLU is constant by construction. Repeatedly using these observations completes the proof. \( \Box \)

We now complete the proof of the lower bound in Theorem 1. Recall the definition (5) of \( f \), and note that

\[
T := \left\{ x \in \mathbb{R}^d \mid f(x) \leq f\left(\frac{1}{4},0,\ldots,0\right) \right\}
\]

is a ball of radius \( \sqrt{\frac{1}{16} + \frac{d-1}{4}} \) centered at \( \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) \). Hence,

\[
[0,1]^d \setminus T = \bigsqcup_v T_v
\]

is a disjoint union made up of \( 2^d \) sets such that \( v \in T_v \) and \( T_v \) is contained in a ball of radius \( 1/4 \) around \( v \). Fix any ReLU net \( N \) with input dimension \( d \), hidden layer width \( d \), and output dimension 1 such that

\[
\sup_{x \in T} |f(x) - f_N(x)|_{C^0} \leq \frac{3}{16}.
\]

Since \( f(0,\ldots,0) - f(1/4,0,\ldots,0) = 3/16 \), property (2) of Lemma 6 guarantees that for each vertex \( v \) of \([0,1]^d\), the path \( \gamma_v^0 \) cannot leave \( T_v \). Hence, since it is convex, the
set $S_n$ from Lemma 6 must contain the cube $C$ of side-length $1/4$ around $(\frac{1}{2}, \ldots, \frac{1}{2})$. Setting
\[
\eta = \eta(d) := \min \left\{ \frac{3}{16}, \inf_{L_{\text{affine}}} \| f - L \|_{C^0(C)} \right\},
\]
completes the proof.

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