Improved Bounds and New Techniques for Davenport–Schinzel Sequences and Their Generalizations

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Abstract. We present several new results regarding \( \lambda_s(n) \), the maximum length of a Davenport–Schinzel sequence of order \( s \) on \( n \) distinct symbols.

First, we prove that \( \lambda_s(n) \leq n \cdot 2^{1/2\alpha(n)} + O(\alpha(n)^{-1}) \) for \( s \geq 4 \) even, and \( \lambda_s(n) \leq n \cdot 2^{1/\alpha(n)} \log_2 \alpha(n) + O(\alpha(n)) \) for \( s \geq 3 \) odd, where \( t = \lfloor (s - 2)/2 \rfloor \), and \( \alpha(n) \) denotes the inverse Ackermann function. The previous upper bounds, by Agarwal et al. [1989], had a leading coefficient of 1 instead of \( 1/t! \) in the exponent. The bounds for even \( s \) are now tight up to lower-order terms in the exponent. These new bounds result from a small improvement on the technique of Agarwal et al.

More importantly, we also present a new technique for deriving upper bounds for \( \lambda_s(n) \). This new technique is very similar to the one we applied to the problem of stabbing interval chains [Alon et al. 2008]. With this new technique we: (1) re-derive the upper bound of \( \lambda_3(n) \leq 2n\alpha(n) + O(n, \sqrt{n}) \) (first shown by Klazar [1999]); (2) re-derive our own new upper bounds for general \( s \); and (3) obtain improved upper bounds for the generalized Davenport–Schinzel sequences considered by Adamec et al. [1992].

Regarding lower bounds, we show that \( \lambda_3(n) \geq 2n\alpha(n) - O(n) \) (the previous lower bound (Sharir and Agarwal, 1995) had a coefficient of \( \frac{1}{2} \), so the coefficient 2 is tight. We also present a simpler variant of the construction of Agarwal et al. [1989] that achieves the known lower bounds of \( \lambda_s(n) \geq n \cdot 2^{1/\alpha(n)^{-1}} - O(\alpha(n)^{-1}) \) for \( s \geq 4 \) even.

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1. Introduction

Given a sequence $S$, denote by $|S|$ the length of $S$, and by $\|S\|$ the number of distinct symbols in $S$. If $u$ is another sequence, we write $u \subset S$ if $S$ contains a subsequence $u'$ (not necessarily contiguous) which is isomorphic to $u$ (i.e., $u'$ can be made equal to $u$ by a one-to-one renaming of its symbols). In this case, we say that $S$ contains $u$ or that $u$ is contained in $S$. Otherwise, we write $u \not\subset S$ and we say that $S$ is $u$-free.

For example, $S = abcdabc$ contains $u = abab$, but it is $v$-free for $v = abba$.

A sequence $S = a_1a_2a_3\ldots$ is called $r$-sparse if $a_i \neq a_j$ whenever $1 \leq |j-i| \leq r-1$. In other words, $S$ is $r$-sparse if every interval in $S$ of length at most $r$ contains only distinct symbols.

A Davenport–Schinzel sequence of order $s$, for $s \geq 1$, is a sequence that is $2$-sparse (i.e., contains no adjacent repeated symbols) and is $u$-free for $u = ababab\ldots$ of length $s+2$. In other words, a Davenport–Schinzel sequence of order $s$ does not contain any alternation $a\ldots b\ldots a\ldots b\ldots$ of length $s+2$ for any pair of symbols $a, b$.

Let $\lambda_s(n)$ denote the maximum length of a Davenport–Schinzel sequence of order $s$ on $n$ distinct symbols ($\lambda_s(n)$ is finite for all $s$ and $n$). We always take $s$ to be fixed, and consider $\lambda_s(n)$ as a function of $n$.

These sequences are named after Davenport and Schinzel [1965], who were the first to study them. The main motivation for Davenport–Schinzel sequences is the complexity of the lower envelope of a set of curves in the plane. However, Davenport–Schinzel sequences have a large number of applications in computational and combinatorial geometry; the book [Sharir and Agarwal 1995] is entirely devoted to this topic. Given the prominent role these sequences play in computational geometry, it is of great interest to derive tight asymptotic bounds for $\lambda_s(n)$. This goal is quite challenging, given the complicated form of the known bounds (see below). There has been little progress in the problem for nearly 20 years.

The bounds $\lambda_1(n) = n$ (no $aba$) and $\lambda_2(n) = 2n-1$ (no $abab$) are quite easy to obtain. But for $s \geq 3$ the problem becomes much more complicated—it turns out that $\lambda_s(n)$ is slightly superlinear in $n$.

Hart and Sharir [1986] (see also Sharir and Agarwal [1995]) showed that $\lambda_3(n) = \Theta(n \alpha(n))$, where $\alpha(n)$ denotes the inverse Ackermann function. (For the upper bound see also Sharir [1987] and Klazar [1999], and for the lower bound, see also Wiernik and Sharir [1988], Komjáth [1988], and Shor [1990].)

The tightest known bounds for $\lambda_3(n)$ are

$$\frac{1}{2} n \alpha(n) - O(n) \leq \lambda_3(n) \leq 2n \alpha(n) + O(n \sqrt{\alpha(n)}).$$ (1)

The lower bound is due to Sharir and Agarwal [1995] (based on the construction by Wiernik and Sharir [1988]). The upper bound is due to Klazar [1999]. Klazar [2002] asks whether $\lim_{n \to \infty} \lambda_3(n)/(n \alpha(n))$ exists.

The current upper and lower bounds for $\lambda_s(n)$ for general $s$ were established by Agarwal et al. [1989] (see also Sharir and Agarwal [1995]), and are as follows: Let
For odd \( s \geq 5 \), the asymptotically best lower bounds known are obtained by \( \lambda_s(n) \geq \lambda_{s-1}(n) \).

The book by Sharir and Agarwal [1995] contains a complete derivation of the current upper and lower bounds for \( \lambda_s(n) \) for all \( s \).

In 2008 the author, together with Alon, Kaplan, Sharir, and Smorodinsky, conjectured that:

**CONJECTURE 1.1 [ALON ET AL. 2008].** The true bounds for \( \lambda_s(n) \) are

\[
\lambda_s(n) = \begin{cases} 
  n \cdot 2^{(1/t)\alpha(n/2) + O(\alpha(n/2))} & s \geq 4 \text{ even;} \\
  n \cdot 2^{(1/t)\alpha(n/2) + O(\alpha(n/2))} \log_2 \alpha(n) + O(\alpha(n/2)) & s \geq 3 \text{ odd;}
\end{cases}
\]

where \( t = \lfloor (s - 2)/2 \rfloor \).

This conjecture is based on some surprisingly similar tight bounds that they obtained for an unrelated problem called stabbing interval chains with \( j \)-tuples.

### 1.1. Generalized Davenport–Schinzel Sequences

Adamec et al. [1992] considered a generalization of Davenport–Schinzel sequences, in which the forbidden pattern is not limited to \( abab \ldots \), but can be an arbitrary sequence.

Let \( u \) (the forbidden pattern) be a sequence with \( \|u\| = r \) distinct symbols and length \( |u| = s \). Then, we denote by \( Ex_u(n) \) the maximum length of an \( r \)-sparse, \( u \)-free sequence on \( n \) distinct symbols. The standard Davenport–Schinzel sequences are obtained by taking \( r = 2 \) and \( u = abab \ldots \) of length \( s + 2 \).

The requirement of \( r \)-sparsity is necessary, since an \( (r - 1) \)-sparse, \( u \)-free sequence can be arbitrarily long. The requirement of \( r \)-sparsity, however, ensures that \( Ex_u(n) \) is finite.

Generalized Davenport–Schinzel sequences have found several applications in discrete mathematics. Valtr [1999] used generalized Davenport–Schinzel sequences to obtain bounds for some Turán-type problems for geometric graphs. Alon and Friedgut [2000] used them to derive an almost-tight upper bound for the so-called Stanley–Wilf conjecture (the conjecture was later proved by Marcus and Tardos [2004] by a different technique). For more information see the surveys by Klazar [2002] and by Valtr [1999]. More recently, Pettie [2008] used generalized Davenport–Schinzel sequences to improve Sundar’s [1992] near-linear upper bound for the deque conjecture for splay trees.

### 1.2. Formation-Free Sequences

Klazar [1992] developed a general technique for bounding \( Ex_u(n) \) in terms of only \( r = \|u\| \) and \( s = |u| \). His technique is based on considering what we call formation-free sequences (our name). Given integers \( r \) and \( s \), an \((r, s)\)-formation is a sequence of \( s \) permutations on \( r \) symbols. For example, \( abcd dcab dcab cdab dabc \) is a \((4, 5)\)-formation. An \((r, s)\)-formation-free sequence is a sequence which is \( r \)-sparse and does not contain any \((r, s)\)-formation as a subsequence.
Denote by $F_{r,s}(n)$ the length of the longest possible $(r, s)$-formation-free sequence on $n$ distinct symbols. Let $u$ be a sequence with $\|u\| = r$ and $|u| = s$. Since $u$ is trivially contained in every $(r, s)$-formation, it follows that $Ex_u(n) \leq F_{r,s}(n)$.

Klazar made a slight improvement to this observation, by noting that if $r \geq 2$, then $u$ is contained in every $(r, s - 1)$-formation, and thus,

$$Ex_u(n) \leq F_{r,s-1}(n) \quad \text{for } r \geq 2.$$  
(3)

(The case $r = 1$ is not interesting in any case.) Klazar proved the bound

$$F_{r,s}(n) \leq n \cdot 2^{O(\alpha(n)^{r-1})},$$

where the $O$ notation hides constants that depend on $r$ and $s$. Together with (3), this implies that

$$Ex_u(n) \leq n \cdot 2^{O(\alpha(n)^{r-1})}.$$

1.3. OUR RESULTS. In this article, we present several new results.

First, we make a small improvement on the argument of Agarwal et al. [1989] and prove:

**THEOREM 1.2.** Let $s \geq 3$ be fixed, and let $t = \lfloor (s - 2)/2 \rfloor$. Then

$$\lambda_s(n) \leq \begin{cases} n \cdot 2^{(1/t)\alpha(n)^r + O(\alpha(n)^{r-1})}, & \text{if } s \text{ is even;} \\ n \cdot 2^{(1/t)\alpha(n)^r \log_2 \alpha(n) + O(\alpha(n)^r)}, & \text{if } s \text{ is odd}. \end{cases}$$

Thus, the upper bounds for $\lambda_s(n)$ are now in line with Conjecture 1.1, and for $s$ even they are also tight up to lower-order terms in the exponent.

More importantly, we also present a new technique for deriving upper bounds for $\lambda_s(n)$. Our new technique is based on some recurrences very similar to those used by Alon et al. [2008], for the problem of stabbing interval chains with $j$-tuples.

With our new technique we re-derive Klazar’s upper bound (1) for $\lambda_3(n)$, as well as our new bounds in Theorem 1.2 for $\lambda_s(n)$, $s \geq 4$. We also apply our technique to formation-free sequences, proving that:

**THEOREM 1.3.** For $s \geq 4$ we have

$$F_{r,s}(n) \leq \begin{cases} n \cdot 2^{(1/t)\alpha(n)^r + O(\alpha(n)^{r-1})}, & \text{if } s \text{ is odd;} \\ n \cdot 2^{(1/t)\alpha(n)^r \log_2 \alpha(n) + O(\alpha(n)^r)}, & \text{if } s \text{ is even}; \end{cases}$$

where $t = \lfloor (s - 3)/2 \rfloor$. (The $O$ notation hides factors dependent on $r$ and $s$.)

As an aside, we improve on Klazar’s bound (3):

**LEMMA 1.4.** Let $u$ be a sequence with $\|u\| = r$, $|u| = s$. Then, $Ex_u(n) \leq F_{r,s-r+1}(n)$.

This, together with Theorem 1.3, yields:\

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\[^{1}\text{Klazar [1992] himself speculated that it should be possible to achieve roughly } Ex_u(n) \leq n \cdot 2^{O(\alpha(n)^{r/2})}.\]
THEOREM 1.5. Let u be a sequence with \( \|u\| = r \), \( |u| = s \), and \( s \geq r + 3 \). Let \( t = \lfloor (s - r - 2)/2 \rfloor \). Then,

\[
Ex_u(n) \leq \begin{cases} 
  n \cdot 2^{\left(1/!\right)a(n)^t+O(\alpha(n)^{-1})}, & s - r \text{ even}; \\
  n \cdot 2^{\left(1/!\right)a(n)^t \log_2 \alpha(n)+O(\alpha(n)^t)}, & s - r \text{ odd}.
\end{cases}
\]

Note that Theorem 1.5 is a generalization of Theorem 1.2: Taking \( r = 2 \) and \( u = abab \ldots \) of length \( s + 2 \) yields the theorem once again.

Regarding lower bounds, we prove:

THEOREM 1.6. \( \lambda_3(n) \geq 2n\alpha(n) - O(n) \).

COROLLARY 1.7. \( \lim_{n \to \infty} \lambda_3(n)/(\alpha(n)) = 2 \).

Finally, we present a simpler variant of the construction of Agarwal et al. [1989], which achieves the lower bounds (2) for \( s \geq 4 \) even.

1.4. THE ACKERMANN FUNCTION AND ITS INVERSE. Let us define (our version of) the Ackermann function and its inverse.

The Ackermann hierarchy is a sequence of functions \( A_k(n) \), for \( k = 1, 2, 3, \ldots \) and \( n \geq 0 \), where \( A_1(n) = 2n \), and for \( k \geq 2 \) we let \( A_k(n) = A_{k-1}(1) \). (Here \( f^{(n)} \) denotes the \( n \)-fold composition of \( f \).) Alternatively, the definition of \( A_k(n) \) for \( k \geq 2 \) can be written recursively as

\[
A_k(n) = \begin{cases} 
  1, & \text{if } n = 0; \\
  A_{k-1}(A_k(n-1)), & \text{otherwise}.
\end{cases}
\]

We have \( A_2(n) = 2^n \), and \( A_3(n) = 2^{2^{2^{2\ldots}}} \) is a “tower” of \( n \) twos. Each function in this hierarchy grows much faster than the preceding one. Namely, for every fixed \( k \) and \( c \) we have \( A_{k+1}(n) \geq A_k^{(c)}(n) \) for all large enough \( n \).

Notice that \( A_0(1) = 2 \) and \( A_1(2) = 4 \) for all \( k \), but \( A_k(3) \) already grows very rapidly with \( k \). We define the Ackermann function as \( A(n) = A_3(n) \). Thus, \( A(n) = 6, 8, 16, 65536, \ldots \) for \( n = 1, 2, 3, 4, \ldots \).

For every fixed \( k \) we have \( A(n) \geq A_k(n) \) for all large enough \( n \). It is also easy to verify that

\[
A(n) = A_{n-2}(A(n-1)), \quad \text{for } n \geq 3. \tag{5}
\]

We then define the slow-growing inverses of these rapidly-growing functions as

\[
\alpha_k(x) = \min \{ n \mid A_k(n) \geq x \}, \tag{6}
\]

\[
\alpha(x) = \min \{ n \mid A(n) \geq x \}, \tag{7}
\]

for all real \( x \geq 0 \).

\[\text{The Ackermann function is usually defined by “diagonalizing” the hierarchy, letting } A'(n) = A_n(n). \text{ This does not make any asymptotic difference, since } A'(n-2) \leq A(n) \leq A'(n-1) \text{ for } n \geq 5. \text{ (There are several other definitions of the Ackermann hierarchy and function in the literature, all of which exhibit equivalent rates of growth.) We prefer the above definition because, first, “diagonalization” is unnecessary, and second, the corresponding definition (9) of } \alpha(x) \text{ comes out simpler. For other references where } \alpha(x) \text{ is defined as in (9) see [Pettie 2008] and [Seidel 2006, slide 85].}\]
Alternatively, and equivalently, we can define these inverse functions directly without making reference to $A_k$ and $A$. We define the inverse Ackermann hierarchy by

$$\alpha_1(x) = \left\lceil \frac{x}{2} \right\rceil$$

and

$$\alpha_k(x) = \begin{cases} 0, & \text{if } x \leq 1; \\ 1 + \alpha_k(\alpha_{k-1}(x)), & \text{otherwise}; \end{cases}$$

for $k \geq 2$. In other words, for each $k \geq 2$, $\alpha_k(x)$ denotes the number of times we must apply $\alpha_{k-1}$, starting from $x$, until we reach a value not larger than 1. Thus, $\alpha_2(x) = \lceil \log_2 x \rceil$, and $\alpha_3(x) = \log^* x$. Finally, we define the inverse Ackermann function by

$$\alpha(x) = \min\{k \mid \alpha_k(x) \leq 3\}.$$ 

It is an easy exercise (only slightly tedious) to prove by induction that the above two definitions of $\alpha_k$ and $\alpha$ are exactly equivalent.

1.5. ORGANIZATION OF THIS ARTICLE. Sections 2 to 5 contain our upper-bound results. In Section 2, we show how Theorem 1.2 reduces to bounding a function denoted $\psi_s(m, n)$. In Section 3, we improve the technique of Agarwal et al. [1989] for bounding $\psi_s(m, n)$. In Section 4 we present an alternative technique, which yields the same improved bounds for $\psi_s(m, n)$.

Section 5 addresses formation-free sequences. We first prove Lemma 1.4, and then we extend our new technique to formation-free sequences, proving Theorem 1.3.

Sections 6 to 7 contain our lower-bound results. Section 6 presents our construction for $\lambda_3(n)$ that proves Theorem 1.6. Section 7 contains our simplified construction of Davenport–Schinzel sequences of even order $s \geq 4$.

Appendices A to C contain some technical calculations.

For completeness, we provide proofs in this article of most of the previous results we rely on.

2. Upper Bounds for Davenport–Schinzel Sequences

The upper bounds for $\lambda_s(n)$ are obtained by considering a function with an additional parameter $m$:

**Definition 2.1.** Let $m$, $n$, and $s$ be positive integers. Then $\psi_s(m, n)$ denotes the maximum length of a Davenport–Schinzel sequence of order $s$ on $n$ distinct symbols that can be partitioned into $m$ or fewer contiguous blocks, where each block contains only distinct symbols.

The relation between $\lambda_s(n)$ and $\psi_s(m, n)$ is as follows:

**Lemma 2.2 [Agarwal et al. 1989].** Let $\varphi_{s-2}(n)$ be a nondecreasing function in $n$ such that $\lambda_{s-2}(n) \leq n\varphi_{s-2}(n)$ for all $n$. Then,

$$\lambda_s(n) \leq \varphi_{s-2}(n)(\psi_s(2n, n) + 2n).$$

**Proof.** Let $S$ be a Davenport–Schinzel sequence of order $s$ on $n$ symbols with maximum length $\lambda_s(n)$. Partition $S$ greedily from left to right into blocks $S_1, S_2, \ldots, S_m$, such that each $S_j$ is a sequence of order $s - 2$; in other words, when scanning $S$ from left to right, start a new block $S_{i+1}$ only if an additional symbol would cause...
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$S_i$ to contain an alternation of length $s$. We claim that $m$, the number of blocks, is at most $2n$.

Indeed, consider some block $S_i$ for $i < m$. Since $S_i$ was extended maximally to the right, it must contain an alternation $abab\ldots$ of length $s - 1$, which is extended to length $s$ by the first symbol of $S_{i+1}$ (which is either $a$ or $b$, depending on the parity of $s$). But then, we cannot have both $b$ appearing in a previous $S_j$, $j < i$, and $b$ or $a$ (depending on the parity of $s$) appearing in a subsequent $S_j$, $j > i$, because then $S$ would contain a forbidden alternation of length $s + 2$.

Hence, each block $S_i$ (including the last one $S_m$) contains either the first occurrence or the last occurrence of at least one symbol. Thus, $m \leq 2n$.

Let $n_i = \|S_i\|$. Then,

$$\lambda_3(n) = |S| = \sum_{i=1}^{m} |S_i| \leq \sum_{i=1}^{m} \lambda_{s-2}(n_i) \leq \sum_{i=1}^{m} n_i \varphi_{s-2}(n_i) \leq \varphi_{s-2}(n) \sum_{i=1}^{m} n_i.$$

Let us now bound $\sum n_i$. Construct a subsequence $S'$ of $S$ by taking, for each block $S_i$, just the first occurrence of each symbol in $S_i$. Note that $S'$ has length $|S'| = \sum n_i$ and, being a subsequence of $S$, it contains no alternation of length $s + 2$. Furthermore, $S'$ is decomposable into $m$ blocks of distinct symbols $S'_1, \ldots, S'_m$. However, $S'$ might contain adjacent equal symbols at the interface between blocks, but by removing at most one symbol from each block $S'_i$, we can obtain a sequence $S''$ with no adjacent equal symbols. Therefore, $|S''| \leq \psi_s(m, n)$, and so $|S'| \leq \psi_s(m, n) + m$. Since $m \leq 2n$, we conclude that

$$\lambda_3(n) \leq \varphi_{s-2}(n)(\psi_s(2n, n) + 2n).$$

In particular, for $s = 3$ we have $\lambda_3(n) \leq \psi_3(2n, n) + 2n$ (by taking $\varphi_1(n) = 1$, since $\lambda_1(n) = n$). Actually, for $s = 3$, we have $\lambda_3(n) = \psi_3(2n, n)$ [Hart and Sharir 1986].

The main issue, then, is to bound $\psi_s(m, n)$. We present two different techniques for bounding $\psi_s(m, n)$. The first one is a minor modification of the technique of Agarwal et al. [1989]. The second one is our new technique. Both techniques yield the following bounds:

**Lemma 2.3.** For $s = 3$, we have

$$\psi_3(m, n) = O(km\omega_k(m) + kn) \quad \text{for all } k.$$

In general, for every fixed $s \geq 3$, we have

$$\psi_s(m, n) \leq C_{s,k}(m\omega_k(m)^{s-2} + n) \quad \text{for all } k,$$

for some constants $C_{s,k}$ of the form

$$C_{s,k} = \begin{cases} 2^{(1/2i)!}k^{\epsilon}O(k^{t-1}), & s \text{ even;} \\
2^{(1/2i)!}k^{\log_2 k + O(k^t)}, & s \text{ odd;} 
\end{cases}$$

where $t = [(s - 2)/2]$.

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3This greedy left-to-right approach is in fact optimal—it yields a partition of $S$ into the minimum possible number of blocks of specified order $r < s$. 

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(Equivalent bounds for $\psi_3(m, n)$ and $\psi_4(m, n)$ were previously derived by Hart and Sharir [1986], and Agarwal et al. [1989], respectively. For $s \geq 5$ these are improvements over [Agarwal et al. 1989], which for $s \geq 6$ yield improved bounds for $\lambda_s(n).$)

From Lemmas 2.2 and 2.3, it follows that $\lambda_s(n) = o(n\alpha(n))$ for every fixed $\ell.$ Just take $k = \ell + 1$ in Lemma 2.3, bounding $\varphi_{s-2}(n)$ in Lemma 2.2 by induction. Here the magnitude of the constants $C_{s,k}$ is irrelevant.

But we can also derive a tighter bound for $\lambda_s(n),$ namely Theorem 1.2, if we let $k$ grow very slowly with $m;$ for this the dependence of $C_{s,k}$ on $k$ is significant:

\textbf{Proof of Theorem 1.2.} Take $k = o(m)$ in Lemma 2.3 (recalling that $\alpha_{o(m)}(m) \leq 3$ by definition), and substitute into Lemma 2.2. For $s = 3, 4$ we get $\lambda_3(n) = O(n\alpha(n))$, $\lambda_4(n) = O(n \cdot 2^{o(n)})$ (by taking $\varphi_1(n) = 1$, $\varphi_2(n) = 2$). For $s \geq 5$, we bound $\lambda_{s-2}(n)$ by induction on $s$, and we substitute the resulting bound for $\varphi_{s-2}(n)$ into (10). We obtain the desired bounds (the factor $\varphi_{s-2}(n)$ only affects lower-order terms in the exponent). \hfill \square

3. Bounding $\psi_s(m, n)$—Improving the Known Technique

In this section we prove Lemma 2.3 by making a small improvement on the technique of Agarwal et al. [1989]. The main ingredient in the proof is the following complicated-looking recurrence relation. This is a small modification of the recurrence in Agarwal et al. [1989] (and more complicated).

\textbf{Recurrence 3.1.} Let $m, n \geq 1$ and $b \leq m$ be integers, and let

$$m = m_1 + m_2 + \cdots + m_b,$$

be a partition of $m$ into $b$ nonnegative integers. Then, there exists a partition of $n$ into $b + 1$ nonnegative integers

$$n = n_1 + n_2 + \cdots + n_b + n^*,$$

and there exist nonnegative integers $n_1^*, n_2^*, \ldots, n_b^* \leq n^*$ satisfying

$$n_1^* + n_2^* + \cdots + n_b^* \leq \psi_2(b, n^*) + b,$$

such that

$$\psi_s(m, n) \leq 2\psi_{s-1}(m, n^*) + 4m + \sum_{i=1}^{b} (\psi_{s-2}(m_i, n_i^*) + \psi_s(m_i, n_i)).$$

Here it is appropriate to repeat Matoušek’s advice [2002, p. 179] to first study the proof below and then try to understand the statement of the recurrence.

\textbf{Proof.} Let $S$ be a maximum-length Davenport–Schinzel sequence of order $s$ that is partitionable into $m$ blocks $S_1, \ldots, S_m$, each of distinct symbols. Thus, $|S| = \psi_s(m, n).$ Group the blocks $S_1, \ldots, S_m$ into $b$ layers $L_1, L_2, \ldots, L_b$ from left to right, by letting each layer $L_i$ contain $m_i$ consecutive blocks.

We partition the alphabet of $S$ into two sets of symbols. The \textit{local} symbols are those that appear in only one layer, and the \textit{global} symbols are those that appear in two or more layers. Let $n_l$ be the number of symbols local to layer $L_i$, for $1 \leq i \leq b$, and let $n^*$ be the number of global symbols. Equation (11) follows.
For each layer $L_i$, let $n^*_i$ denote the number of global symbols that appear in $L_i$. Trivially, $n^*_i \leq n^*$ for all $i$. To see that (12) holds, build a subsequence $S'$ of $S$ by taking, for each layer $L_i$ and each global symbol $a$ in $L_i$, just the first occurrence of $a$ within $L_i$. The sequence $S'$, being a subsequence of $S$, does not contain any alternation of length $s + 2$. Furthermore, $S'$ consists of $b$ blocks of distinct symbols, corresponding to the $b$ layers of $S$.

However, $S'$ might contain pairs of adjacent equal symbols at the interface between blocks. But there are at most $b - 1$ such pairs of symbols, and by deleting one symbol from each pair, we finally obtain a Davenport–Schinzel sequence. Bound (12) follows.

Each occurrence of a global symbol $a$ in a layer $L_i$ is classified into starting, middle, or ending, as follows: If $a$ does not appear in any previous layer $L_j$, $j < i$, we say that $a$ is a starting symbol for $L_i$. Similarly, if $a$ does not appear in any subsequent layer $L_j$, $j > i$, then $a$ is an ending symbol for $L_i$. If $a$ appears both before and after $L_i$, then $a$ is a middle symbol for $L_i$.

Decompose $S$ into four sequences $T_1, T_2, T_3, T_4$ (not necessarily contiguous), as follows: Let $T_1$ contain all occurrences of the local symbols of $S$. Let $T_2$ contain all occurrences of the starting global symbols in all the layers of $S$; similarly, let $T_3$ contain all occurrences of the middle global symbols, and let $T_4$ contain all occurrences of the ending global symbols in all the layers of $S$. Thus, $|T_1| + \cdots + |T_4| = \psi_s(m, n)$. Each sequence $T_1, \ldots, T_4$ inherits from $S$ the partition into $b$ layers, in which the $i$-th layer is further partitioned into $m_i$ blocks.

Each of the sequences $T_1, \ldots, T_4$ might contain pairs of adjacent equal symbols, but these can only occur at the interface between adjacent blocks. Hence, by removing at most $m - 1$ symbols from each sequence, we obtain sequences $T'_1, \ldots, T'_4$ with no adjacent equal symbols. Thus, $\psi_s(m, n) \leq |T'_1| + \cdots + |T'_4| + 4m$. We now bound each of $|T'_1|, \ldots, |T'_4|$ individually.

Let us first consider $T'_1$. The $i$-th layer in $T'_1$ is a Davenport–Schinzel sequence of order $s$ on $n_i$ symbols, and it consists of $m_i$ blocks, each of distinct symbols. Thus

$$|T'_1| \leq \sum_{i=1}^{b} \psi_s(m_i, n_i).$$

Next consider $T'_2$. We claim that each layer in $T'_2$ is a Davenport–Schinzel sequence of order $s - 1$. Indeed, suppose for a contradiction that some layer in $T'_2$ contains an alternation $abab \ldots$ of length $s + 1$. Then, since $a$ and $b$ are starting symbols for this layer, they must both appear in $S$ in some subsequent layer, and so $S$ would contain an alternation of length $s + 2$, a contradiction.

Furthermore, since each global symbol is a starting symbol for exactly one layer, the layers in $T'_2$ have pairwise disjoint sets of symbols, so all of $T'_2$ is a Davenport–Schinzel sequence of order $s - 1$. A similar argument applies for $T'_4$. Thus,

$$|T'_2|, |T'_4| \leq \psi_{s-1}(m, n^*).$$

Finally, consider $T'_3$. Each layer in $T'_3$ is composed of middle global symbols, which appear in $S$ in both previous and subsequent layers. Therefore, no layer in $T'_3$ can contain an alternation of length $s$, or else $S$ would contain an alternation of length $s + 2$. Thus, each layer in $T'_3$ is a Davenport–Schinzel sequence of order $s - 2$. (However, the whole $T'_i$ is not necessarily of order $s - 2$.) Since the $i$-th layer in $T'_3$ contains $n_i^*$ different symbols and is partitioned into $m_i$ blocks, each of
The reason for our choice of applying \( \log_2 \) = Ackermann hierarchy. Logarithmic bound, and then we use induction to go all the way down the inverse orders to obtain successively better upper bounds on \( \sum_{i=1}^{b} \).

Next, we bound the term \( \psi_{s-2}(m_i, n_i^s) \). We proceed by induction on \( s \), and if, for \( s = 2 \), then \( \psi_2(m_i, n_i) \leq 2n - 1 \), and the claim holds. So let \( s \geq 3 \).

For each \( s \) we proceed by induction on \( m \). If \( m \leq m_0(s) \), then \( \psi_s(m, n) \leq m_0(s)n \leq Q_{s,2}n \), and we are done. So assume \( m > m_0(s) \).

We apply Recurrence 3.1 with \( b = 2 \). Let \( m_1 = \lfloor m/2 \rfloor \) and \( m_2 = \lceil m/2 \rceil \), so \( m_1 + m_2 = m \). Let us bound each term in the right-hand side of (13) separately.

The term \( 2\psi_{s-1}(m, n^s) \) is bounded, by induction on \( s \), by

\[
2\psi_{s-1}(m, n^s) \leq 2P_{s-1,2} m(\log_2 m)^{s-3} + 2Q_{s-1,2} n^s.
\]

Next, we bound the term \( \sum_{i=1}^{2} \psi_{s-2}(m_i, n_i^s) \). Using again induction on \( s \), and applying \( \log_2 m_i \leq \log_2 m \), we get

\[
\sum_{i=1}^{2} \psi_{s-2}(m_i, n_i^s) \leq P_{s-2,2} m(\log_2 m)^{s-4} + Q_{s-2,2}(n_1^s + n_2^s).
\]

Remark 3.2. Our key improvement over the method of Agarwal, Sharir, and Shor lies in the bound for \( |T_2'| \). They noted that each layer in \( T_2' \) is a sequence of order \( s - 2 \), but they did not use the fact that the blocks in each layer have distinct symbols. In addition, they did not introduce the variables \( n_i^s \).

3.1. Applying the Recurrence Relation. We apply Recurrence 3.1 repeatedly to obtain successively better upper bounds on \( \psi_s(m, n) \). We first obtain a polylogarithmic bound, and then we use induction to go all the way down the inverse Ackermann hierarchy.

For \( s \geq 3 \), let \( m_0(s) \) be a large enough constant (depending only on \( s \)) such that

\[
m \geq 2 + 2^{\lceil \log_2 m \rceil - 2} \quad \text{for all } m \geq m_0(s).
\]

Define integers \( P_{s,2}, Q_{s,2} \) for \( s \geq 1 \) by

\[
P_{1,2} = P_{2,2} = 0, \quad Q_{1,2} = 1, \quad Q_{2,2} = 2,
\]

and, for \( s \geq 3 \),

\[
P_{s,2} = 4P_{s-1,2} + 2P_{s-2,2} + 2Q_{s-1,2} + 8, \quad Q_{s,2} = \max \{m_0(s), 2Q_{s-1,2} + 2Q_{s-2,2}\}.
\]

The reason for our choice of \( m_0(s) \) will become apparent later on, in the proof of Lemma 3.4. (Also recall that we take \( s \) to be a constant, so the growth of \( P_{s,2}, Q_{s,2} \) in \( s \) is irrelevant for us.)

Our polylogarithmic bound is as follows:

Lemma 3.3. For all \( m, n, \) and \( s \), we have

\[
\psi_s(m, n) \leq P_{s,2} m(\log_2 m)^{s-2} + Q_{s,2} n.
\]

Proof. We proceed by induction on \( s \). If \( s = 1 \), then \( \psi_1(m, n) \leq n \), and if \( s = 2 \), then \( \psi_2(m, n) \leq 2n - 1 \), and the claim holds. So let \( s \geq 3 \).

For each \( s \) we proceed by induction on \( m \). If \( m \leq m_0(s) \), then \( \psi_s(m, n) \leq m_0(s)n \leq Q_{s,2}n \), and we are done. So assume \( m > m_0(s) \).

We apply Recurrence 3.1 with \( b = 2 \). Let \( m_1 = \lfloor m/2 \rfloor \) and \( m_2 = \lceil m/2 \rceil \), so \( m_1 + m_2 = m \). Let us bound each term in the right-hand side of (13) separately.

The term \( 2\psi_{s-1}(m, n^s) \) is bounded, by induction on \( s \), by

\[
2\psi_{s-1}(m, n^s) \leq 2P_{s-1,2} m(\log_2 m)^{s-3} + 2Q_{s-1,2} n^s.
\]
Now, applying (12), we bound $n_i^* + n_s^*$ loosely by $n_i^* + n_s^* \leq \psi_i(2, n^*) + 2 \leq 2n^* + m$. Thus, being again very loose, we get

$$\sum_{i=1}^{2} \psi_{s-2}(m_i, n_i^*) \leq m(\log_2 m)^{t-3}(P_{s-2,2} + Q_{s-2,2}) + 2Q_{s-2,2}n^*.$$  

Next we bound the term $\sum_{i=1}^{2} \psi_i(m_i, n_i)$, using induction on $m$. Applying $\log_2 m_i \leq \log_2 m - \frac{1}{3}$, which is true for $m \geq 3$, and using the fact that $(x - \frac{1}{2})^{t-2} \leq x^{t-2} - \frac{1}{2}x^{t-3}$ for all $x \geq \frac{1}{2}$, we get

$$\sum_{i=1}^{2} \psi_i(m_i, n_i) \leq \sum_{i=1}^{2} (P_{s,2}m_i(\log_2 m_i)^{t-2} + Q_{s,2}n_i)$$

$$\leq P_{s,2}m(\log_2 m)^{t-3} - \frac{1}{2}P_{s,2}m(\log_2 m)^{t-3} + Q_{s,2}(n - n^*).$$

Finally, we bound $4m$ (very loosely for $s \geq 4$) by $4m(\log_2 m)^{t-3}$. Putting everything together, we get

$$\psi_s(m, n) \leq P_{s,2}m(\log_2 m)^{t-2} + Q_{s,2}n$$

$$+ m(\log_2 m)^{t-3} \left(2P_{s-1,2} + P_{s-2,2} + Q_{s-2,2} + 4 - \frac{1}{2}P_{s,2} \right)$$

$$+ (2Q_{s-1,2} + 2Q_{s-2,2} - Q_{s,2})n^*.$$  

By the definition of $P_{s,2}$ and $Q_{s,2}$ in (16), the last two lines are nonpositive, so

$$\psi_s(m, n) \leq P_{s,2}m(\log_2 m)^{t-2} + Q_{s,2}n. \quad \square$$

We are now ready to go all the way down the inverse Ackermann hierarchy. Define integers $P_{s,k}, Q_{s,k}$ for $k \geq 3, s \geq 1$ by

$$P_{1,k} = P_{2,k} = 0, \quad Q_{1,k} = 1, \quad Q_{2,k} = 2,$$

and, for $s \geq 3$,

$$P_{s,k} = Q_{s-2,k}(1 + P_{s,k-1}) + 2d_s P_{s-1,k} + d'_s P_{s-2,k} + 4,$$

$$Q_{s,k} = Q_{s-2,k}Q_{s,k-1} + 2Q_{s-1,k}, \quad \text{(17)}$$

for some constants $d_s$ and $d'_s$ to be specified later, with $P_{s,2}, Q_{s,2}$ as in (15), (16). These quantities $P_{s,k}, Q_{s,k}$ will give rise to $C_{s,k}$ of Lemma 2.3.

**LEMMA 3.4.** For every $s$, there exists a constant $c_s$ such that

$$\psi_s(m, n) \leq P_{s,k}m(\alpha_k(m) + c_s)^{t-2} + Q_{s,k}n \quad \text{for all integers } n, m, s, \text{ and } k.$$  

The proof is similar to the proof of Lemma 3.3, though more complex, since we proceed by induction on $k$ for each $s$. Before delving into the actual details, we give a brief sketch of the proof. For the purposes of this sketch, denote the right-hand side of (18) by $\Gamma_{s,k}(m, n)$. Now refer to Eq. (13) in Recurrence 3.1.

The proof proceeds as follows. We bound the term $\psi_{s-1}(m, n^*)$ by $\Gamma_{s-1,k}(m, n^*)$. We bound the terms $\psi_{s-2}(m_i, n^*_i)$ by $\Gamma_{s-2,k}(m_i, n^*_i)$; this produces the term $Q_{s-2,k} \sum n_i^*$, on which we apply (12). We bound the resulting term $\psi_s(b, n^*)$ by
\[ \Gamma_{s,k-1}(b, n^*) \] (here is where we use induction on \( k \)). Finally, we bound the terms \( \psi_s(m_i, n_i) \) by \( \Gamma_{s,k}(m_i, n_i) \) by induction on \( m \) (since \( m_i < m \) for every \( i \)).

**Proof of Lemma 3.4.** By induction on \( s \). As before, the claim is easily established for \( s = 1, 2 \), so assume \( s \geq 3 \) is fixed.

For each \( s \), we proceed by induction on \( k \). If \( k = 2 \), then the claim reduces to Lemma 3.3, so assume \( k \geq 3 \).

By our induction assumption on \( s \), we have

\[
\begin{align*}
\psi_{s-1}(m, n) &\leq P_{s-1,k} m(\alpha_k(m) + c_{s-1})^{s-3} + Q_{s-1,k} n, \\
\psi_{s-2}(m, n) &\leq P_{s-2,k} m(\alpha_k(m) + c_{s-2})^{s-4} + Q_{s-2,k} n,
\end{align*}
\]

for all \( m \) and \( n \).

Here, it is convenient to work with a slight variant \( \tilde{\alpha}_k(x) \) of the inverse Ackermann hierarchy. Define \( \tilde{\alpha}_k(x) \) for \( k \geq 2, x \geq 0 \) by \( \tilde{\alpha}_2(x) = \alpha_2(x) = \lceil \log_2 x \rceil \), and for \( k \geq 3 \) by the recurrence

\[
\tilde{\alpha}_k(x) = \begin{cases} 
1, & \text{if } x \leq m_0(s); \\
1 + \tilde{\alpha}_k(1 + 2\tilde{\alpha}_{k-1}(x)^{s-2}), & \text{otherwise};
\end{cases}
\]

with \( m_0(s) \) as given in (14). (Compare (20) to the definition (8) of \( \alpha_k(x) \); our choice of \( m_0(s) \) guarantees that \( \tilde{\alpha}_k(x) \) is well-defined for all \( k \) and \( x \).)

The functions \( \tilde{\alpha}_k(x) \) are almost equivalent to the usual inverse Ackermann functions \( \alpha_k(x) \). In fact, there exists a constant \( c_s \), depending only on \( s \), such that \( |\tilde{\alpha}_k(x) - \alpha_k(x)| \leq c_s \) for all \( k \) and \( x \). See Appendix B of Alon et al. [2008] for a general technique for proving bounds of this type (or see Appendix C in this article).

We will show that

\[ \psi_s(m, n) \leq P_{s,k} m\tilde{\alpha}_k(m)^{s-2} + Q_{s,k} n \]

for all \( n, m, \) and \( k \). We will do this by induction on \( k \), and for each \( k \) by induction on \( m \). Then our claim will follow.

If \( m \leq m_0(s) \), then \( \psi_s(m, n) \leq m_0(s)n \leq Q_{s,2} n \leq Q_{s,k} n \), and we are done. So assume \( m > m_0(s) \).

We want to translate the bounds (19) into bounds involving \( \tilde{\alpha}_k \). Since \( \alpha_k(m) \leq \tilde{\alpha}_k(m) + c_k \) and \( \tilde{\alpha}_k(m) \geq 1 \), it follows (being somewhat slack) that there exist multiplicative constants \( d_s, d'_s \) such that

\[
\begin{align*}
\psi_{s-1}(m, n) &\leq d_s P_{s-1,k} m\tilde{\alpha}_k(m)^{s-3} + Q_{s-1,k} n, \\
\psi_{s-2}(m, n) &\leq d'_s P_{s-2,k} m\tilde{\alpha}_k(m)^{s-4} + Q_{s-2,k} n,
\end{align*}
\]

for all \( n \) and \( m \).

Assume by induction on \( k \) that (21) holds for \( k - 1 \). Choose

\[
b = \left\lfloor \frac{m}{\tilde{\alpha}_{k-1}(m)^{s-2}} \right\rfloor.
\]

Let \( m_i = \lfloor m/b \rfloor \) or \( \lceil m/b \rceil \) for each \( i \), such that \( \sum m_i = m \). We claim that

\[
m_i \leq 1 + 2\tilde{\alpha}_{k-1}(m)^{s-2}, \quad \text{for all } 1 \leq i \leq b.
\]

\[ \text{Journal of the ACM, Vol. 57, No. 3, Article 17, Publication date: March 2010.} \]
Indeed, by our choice of \( m_0(s) \) as given in (14), we have \( \tilde{\alpha}_{k-1}(m)^{s^2} \leq [\log_2 m]^{s^2} \leq m/2 \) for all \( m \geq m_0(s) \). Thus,

\[
m_i \leq 1 + \frac{m}{b} \leq 1 + \frac{m}{m/\tilde{\alpha}_{k-1}(m)^{s^2} - 1} = 1 + \frac{m\tilde{\alpha}_{k-1}(m)^{s^2}}{m - \tilde{\alpha}_{k-1}(m)^{s^2}} \leq 1 + \frac{m\tilde{\alpha}_{k-1}(m)^{s^2}}{m - m/2} = 1 + 2\tilde{\alpha}_{k-1}(m)^{s^2}.
\]

Let us bound each term in the right-hand side of (13). We first bound the term \( 2\psi_{s-1}(m, n^*) \) using (22), and we obtain

\[
2\psi_{s-1}(m, n^*) \leq 2d_i P_{s-1, k} m\tilde{\alpha}_{k}(m)^{s^3} + 2Q_{s-1, k} n^*.
\]

Next, we bound \( \sum_{i=1}^{b} \psi_{s-2}(m_i, n_i^*) \) using (23). Observing that \( \tilde{\alpha}_{k}(m_i) \leq \tilde{\alpha}_{k}(m) \),

\[
\sum_{i=1}^{b} \psi_{s-2}(m_i, n_i^*) \leq \sum_{i=1}^{b} \left(d'_i P_{s-2, k} m_i\tilde{\alpha}_{k}(m_i)^{s^4} + Q_{s-2, k} n_i^* \right)
\]

\[
\leq d'_i P_{s-2, k} m\tilde{\alpha}_{k}(m)^{s^4} + Q_{s-2, k} \sum_{i=1}^{b} n_i^*.
\]

Now we apply (12), and we bound \( \psi_s(b, n^*) \) by (21) with \( k - 1 \) in place of \( k \).

\[
\sum_{i=1}^{b} n_i^* \leq \psi_{s}(b, n^*) + b \leq P_{s, k-1} b\tilde{\alpha}_{k-1}(b)^{s^2} + Q_{s, k-1} n^* + b.
\]

By our choice of \( b \) in (24), we have \( \tilde{\alpha}_{k-1}(b)^{s^2} \leq \tilde{\alpha}_{k-1}(m)^{s^2} \leq m/b \), so, being somewhat slack,

\[
\sum_{i=1}^{b} n_i^* \leq P_{s, k-1} m + Q_{s, k-1} n^* + m \leq m\tilde{\alpha}_{k}(m)^{s^3}(1 + P_{s, k-1}) + Q_{s, k-1} n^*.
\]

Substituting this into (26), and being slack again, we get

\[
\sum_{i=1}^{b} \psi_{s-2}(m_i, n_i^*) \leq m\tilde{\alpha}_{k}(m)^{s^3} (d'_i P_{s-2, k} + Q_{s-2, k} (1 + P_{s, k-1})) + Q_{s-2, k} Q_{s, k-1} n^*.
\]

Next we bound \( \sum_{i=1}^{b} \psi_s(m_i, n_i) \), applying (21) by induction on \( m \) (since \( m_i < m \)):

\[
\sum_{i=1}^{b} \psi_s(m_i, n_i) \leq \sum_{i=1}^{b} \left(P_{s, k} m_i\tilde{\alpha}_{k}(m_i)^{s^2} + Q_{s, k} n_i \right).
\]

But by (25) and (20),

\[
\tilde{\alpha}_{k}(m) \leq \tilde{\alpha}_{k}(1 + 2\tilde{\alpha}_{k-1}(m)^{s^2}) = \tilde{\alpha}_{k}(m) - 1.
\]
Further, we have \((x - 1)^{s-2} \leq x^{s-2} - x^{s-3}\) for all \(x \geq 1\). Therefore,
\[
\sum_{i=1}^{b} \psi_{s}(m_i, n_i) \leq P_{s,k} m \left( \widehat{\alpha}_k(m)^{s-2} - \widehat{\alpha}_k(m)^{s-3} \right) + Q_{s,k}(n - n^*) .
\]

Finally, we bound \(4m\) very loosely by \(4m\hat{\alpha}_k(m)^{s-3}\). Putting everything together, we get
\[
\psi_{s}(m, n) \leq P_{s,k} m \hat{\alpha}_k(m)^{s-2} + Q_{s,k} n
+ m\hat{\alpha}_k(m)^{s-3} (2d_s P_{s-1,k} + d'_s P_{s-2,k} + Q_{s-2,k} (1 + P_{s,k-1}) + 4 - P_{s,k})
+ (2Q_{s-1,k} + Q_{s-2,k} Q_{s,k-1} - Q_{s,k}) n^* .
\]

By the definition of \(P_{s,k}\) and \(Q_{s,k}\) in (17), the last two lines equal zero, and we get
\[
\psi_{s}(m, n) \leq P_{s,k} m \hat{\alpha}_k(m)^{s-2} + Q_{s,k} n .
\]

All that remains is to analyze the asymptotic growth of \(P_{s,k}\), \(Q_{s,k}\) in \(k\) for fixed \(s\). We have
\[
P_{3,k}, Q_{3,k} = \Theta(k), \quad P_{4,k}, Q_{4,k} = \Theta(2^k),
\]
and, in general, letting \(t = \lfloor (s - 2)/2 \rfloor\),
\[
P_{s,k}, Q_{s,k} = \begin{cases} 2^{(1/t)k^t + O(k^{t-1})}, & s \geq 4 \text{ even;} \\ 2^{(1/t)k^t \log_2 k + O(k^{t})}, & s \geq 3 \text{ odd} \end{cases}
\]
(see Appendix B for the proof). Thus, Lemma 3.4 is equivalent to Lemma 2.3.

Remark 3.5. The investment we made in using a more complicated recurrence (Recurrence 3.1 instead of the one used by Agarwal et al. [1989]) paid off in Lemma 3.4. Besides being tighter, the lemma also has a simpler form. The corresponding bound in [Agarwal et al. 1989] is of the form
\[
\psi_{s}(m, n) \leq F_{s,k}(n) \cdot m\alpha_k(m) + G_{s,k}(n) \cdot n ,
\]
where \(F_{s,k}(n)\) and \(G_{s,k}(n)\) are functions of \(\alpha(n)\). Our constants \(P_{s,k}, Q_{s,k}\), in contrast, do not depend on \(n\).

4. A New Technique for Bounding \(\psi_{s}(m, n)\)

We now present an alternative technique for bounding \(\psi_{s}(m, n)\). Our new technique is based on a variant of Davenport–Schinzel sequences, in which we turn the problem around, in a sense. We call our variant sequences almost-DS sequences.

An almost-DS sequence of order \(s\) with multiplicity \(k\) and \(m\) blocks (or an ADS\(_k\)(m)-sequence, for short) is a sequence that satisfies the following properties:

— It is a concatenation of \(m\) blocks, each block containing only distinct symbols.
— Each symbol appears at least \(k\) times (in different blocks, so we must have \(m \geq k\) for there to be any symbols at all).
— The sequence contains no alternation \(abab\ldots\) of length \(s + 2\).
Note that we do allow repetitions at the interface between adjacent blocks (this simplifies matters). This is why these are almost Davenport–Schinzel sequences.

We now pose a different problem: We ask for maximizing the number of distinct symbols. Let $\Psi_s(m,n)$ denote the maximum number of distinct symbols in an ADS$_k^s(m)$-sequence. (Note that $\Psi_s^k(m) = 0$ for $m < k$.)

The connection between $\Psi_s(m, n)$ and $\Pi_k^s(m)$ is based on the following lemma:

**Lemma 4.1.** For all $s, n, m,$ and $k$ we have $\Psi_s(m, n) \leq k(\Pi_k^s(m) + n)$.

**Proof.** Let $S$ be a maximum-length Davenport–Schinzel sequence of order $s$ on $n$ distinct symbols that is partitionable into $m$ blocks, each of distinct symbols. Thus, $|S| = \Psi_s(m, n)$. Let $k \geq 1$ be a parameter.

We transform $S$ into another sequence $S'$ in which every symbol appears exactly $k$ times as follows: For each symbol $a$, group the occurrences of $a$ in $S$ from left to right into “clusters” of size $k$, deleting the last remaining $k - 1$ occurrences of $a$. Make the occurrences of $a$ in different clusters different, by replacing each $a$ in the $i$th cluster by a new symbol $a_i$.

We deleted at most $kn$ symbols from $S$, so $|S'| \geq |S| - kn$. On the other hand, $S'$ is clearly an ADS$_k^s(m)$-sequence (the symbol deletions might have created repetitions at the interface between blocks, but these are permitted in almost-DS sequences; on the other hand, the symbol replacements do not introduce any forbidden alternations). Thus, $S'$ contains at most $\Pi_k^s(m)$ distinct symbols. Since each symbol appears exactly $k$ times, we have $|S'| \leq k \cdot \Pi_k^s(m)$. The claim follows.

Thus, the problem of bounding $\Psi_s(m, n)$ reduces to bounding $\Pi_k^s(m)$.

### 4.1. Bounding the Number of Symbols in ADS Sequences

We first derive some basic results: For every $s \geq 1$, if we take $k \leq s$, then $\Pi_k^s(m) = \infty$, but if we take $k = s + 1$ then $\Pi_k^s(m)$ is already finite.

**Lemma 4.2.** For all $s \geq 1, m \geq s$, we have $\Pi_k^s(m) = \infty$.

**Proof.** Take the sequence

$$abc \ldots cba abc \ldots$$

with $s$ blocks, with arbitrarily many symbols in each block. Each symbol appears $s$ times, and the maximum alternation is of length $s + 1$.

**Lemma 4.3.** We have $\Pi_2^s(m) = m - 1$.

**Proof.** Let $S$ be an ADS$_2^s(m)$-sequence. Since $S$ cannot contain an alternation $aba$, each symbol must have all its occurrences contiguous. Given that $S$ contains $m$ blocks, the sequence that maximizes the number of distinct symbols is

$$1 12 3 \ldots (m - 2)(m - 1) (m - 1),$$

with $m - 1$ distinct symbols.

**Lemma 4.4.** For all $s \geq 2$ we have $\Pi_{s+1}^s(m) \leq \binom{m-2}{s-1} = O(m^{s-1})$.

---

4 A similar argument has been used by Sundar [1992, Lemma 9] for a different problem.
PROOF. Suppose for a contradiction that there exists an \( \text{ADS}_{s+1}^t(m) \)-sequence \( S \) with \( n = 1 + \binom{n-2}{s-1} \) distinct symbols. Thus, each symbol appears in at least \( s+1 \) out of \( m \) different blocks. For each symbol \( a \), consider the \( s-1 \) “internal” occurrences of \( a \), meaning, all occurrences except the first and the last. These internal occurrences can fall in any of the \( m-2 \) “internal” blocks of \( S \) (excluding the first and last blocks).

By our choice of \( n \), there must be two symbols \( a, b \) whose internal occurrences fall in the same \( s-1 \) out of \( m-2 \) internal blocks. These internal occurrences create an alternation of length at least \( s+1 \) (in the best case, they form the subsequence \( ab ab \ldots \)). Since both \( a \) and \( b \) also appear before and after this subsequence, \( S \) contains an alternation of length \( s+2 \), a contradiction.

We now bound \( \Pi_k^s(m) \) by deriving recurrences and solving them, in a manner almost entirely analogous to Alon et al. [2008]. We begin with the following recurrence and corollary, which are analogous to Lemma 3.2 in Alon et al. [2008]:

**Recurrence 4.5.** For every \( s \geq 3 \) and every \( k \) and \( m \) we have

\[
\Pi_{2k-1}^s(2m) \leq 2\Pi_{2k-1}^s(m) + 2\Pi_{k-1}^{s-1}(m).
\]

**Proof.** Given an \( \text{ADS}_{2k-1}^s(2m) \)-sequence \( S \), partition the \( 2m \) blocks of \( S \) into a “left half” and a “right half” of \( m \) blocks each. The symbols of \( S \) fall into four categories:

—Symbols that appear only in the left half. Taking just these symbols produces an \( \text{ADS}_{2k-1}^s(m) \)-sequence, so there are at most \( \Pi_{2k-1}^s(m) \) such symbols.

—Symbols that appear only in the right half. There are also at most \( \Pi_{2k-1}^s(m) \) such symbols.

—Symbols that appear in both halves, but appear at least \( k \) times in the left half. Taking just these symbols, and just their left-half occurrences, produces an \( \text{ADS}_{k-1}^{s-1}(m) \)-sequence \( S' \). (An alternation \( abab \ldots \) of length \( s+1 \) in \( S' \) would be extended to length \( s+2 \) by an \( a \) or \( b \) that appears in the right half.) Thus, there are at most \( \Pi_{k-1}^{s-1}(m) \) of these symbols.

—Symbols that appear in both halves, but appear at least \( k \) times in the right half. There are also at most \( \Pi_{k-1}^{s-1}(m) \) such symbols. 

**Corollary 4.6.** For every fixed \( s \geq 2 \), if we let \( k = 2^{s-1} + 1 \), then

\[
\Pi_k^s(m) = O(m (\log m)^{s-2})
\]

(where the constant implicit in the \( O \) notation might depend on \( s \)).

**Proof.** Apply Recurrence 4.5 using induction on \( s \), using Lemma 4.4 as base case for \( s = 2 \).

The following recurrence and corollary for \( \Pi_k^3(m) \) are analogous to Recurrence 3.3 and Lemma 3.5 in Alon et al. [2008]:

**Recurrence 4.7.** Let \( t \) be an integer parameter, with \( t \leq \sqrt{m} \). Then,

\[
\Pi_k^3(m) \leq \left( 1 + \frac{m}{t} \right) \Pi_k^3(t) + \Pi_{k-2}^3\left( 1 + \frac{m}{t} \right) + 3m.
\]
We classify the symbols of \( S \) into different types. A symbol is \textit{local} for layer \( L_i \) if it only appears in \( L_i \). Taking just the symbols local to \( L_i \) produces an ADS\(_{t}(t)\)-sequence. Therefore, the number of local symbols is at most \( \Pi_{t}(t) \) per layer, or at most \( b \Pi_{t}(t) \leq (1 + \frac{m}{t})\Pi_{t}(t) \) altogether.

Symbols that appear in at least two layers are called \textit{global symbols}.

Call a global symbol \textit{middle-concentrated} for layer \( L_i \) if it makes its first appearance in \( L_i \), and it appears at least three times in \( L_i \). Given a layer \( L_i \), take just the middle-concentrated symbols for \( L_i \), and just their occurrences within \( L_i \). The resulting sequence \( S'_i \) cannot contain an alternation \( aba \), or else \( S \) would contain \( ababa \). Therefore, \( S'_i \) is an ADS\(_{t}(t)\)-sequence, so by Lemma 4.4 it has at most \( t - 2 \) different symbols. Thus, there are at most \( b(t - 2) \leq (1 + \frac{m}{t})(t - 2) \leq m \) left-concentrated symbols altogether (since \( t \leq \sqrt{m} \)).

Similarly, there are at most \( m \) right-concentrated symbols.

Next, call a global symbol \textit{middle-concentrated} for layer \( L_i \) if it appears at least twice in \( L_i \), and it also appears before \( L_i \) and after \( L_i \).

Given \( L_i \), take just the middle-concentrated symbols for \( L_i \), and just their occurrences within \( L_i \). The resulting sequence \( S''_i \) cannot contain an alternation \( aba \), so \( S''_i \) is an ADS\(_{t}(t)\)-sequence, and so by Lemma 4.3 it contains at most \( t - 1 \) different symbols. Therefore, there are at most \( b(t - 1) \leq m \) middle-concentrated symbols. (Note that we might have counted the same middle-concentrated symbol more than once.)

Finally, take all the global symbols we have not accounted for so far—the \textit{scattered symbols}. Each of these symbols must appear in at least \( k - 2 \) different layers. Build a subsequence of \( S \) by taking just the scattered symbols, and for each scattered symbol, just one occurrence per layer. Each layer becomes a block, and no new forbidden alternation can arise. Hence, we get an ADS\(_{k-2}(b)\)-sequence, which can have at most \( \Pi_{k-2}^{3}(1 + \frac{m}{b}) \) different symbols. \( \square \)

**Corollary 4.8.** There exists an absolute constant \( c \) such that, for every \( k \geq 2 \), we have

\[
\Pi_{2k+1}^{3}(m) \leq cma_k(m) \quad \text{for all } m.
\]

**Proof.** Let \( m_0 \) be a constant large enough that

\[
m \geq 1 + 9[\log_2 m]^2 \quad \text{for all } m \geq m_0.
\]

We will work with a slight variant of the inverse Ackermann function. For this proof, let \( \tilde{\alpha}_2(x) \), \( k \geq 2 \), be given by \( \tilde{\alpha}_2(x) = \alpha_2(x) = [\log_2 x] \), and, for \( k \geq 3 \), by the recurrence

\[
\tilde{\alpha}_k(x) = \begin{cases} 
1, & \text{if } x \leq m_0; \\
1 + \tilde{\alpha}_k(3\tilde{\alpha}_{k-1}(x)), & \text{otherwise}.
\end{cases}
\]

Note that \( \tilde{\alpha}_k(x) \) is well defined by our choice of \( m_0 \). Furthermore, there exists a constant \( c_0 \) such that \( |\tilde{\alpha}_k(x) - \alpha_k(x)| \leq c_0 \) for all \( k \) and \( x \) (see Appendix B of Alon et al. [2008], or Appendix C in this article).
We will prove by induction on \( k \geq 2 \) that
\[
\Pi^3_{2k+1}(m) \leq c_1 m \tilde{\alpha}(m) \quad \text{for all } m,
\]
for some absolute constant \( c_1 \); this implies our claim. By Corollary 4.6, we have \( \Pi^3_k(m) = O(m \log m) \), so the base case \( k = 2 \) follows by choosing \( c_1 \) sufficiently large. We choose \( c_1 \) large enough so that it also satisfies \( c_1 \geq \frac{1}{2} \) and that \( c_1 \geq \Pi^3_k(m)/m \) for all \( m \leq m_0 \).

Now, let \( k \geq 3 \), and assume the bound holds for \( k - 1 \). To establish the bound for \( k \), first let \( m \leq m_0 \). Then we have
\[
\Pi^3_{2k-1}(1 + \frac{m}{t}) \leq \Pi^3_{2k-1}(\frac{2m}{t}) \leq \frac{2c_1 m}{t} \tilde{\alpha}_{k-1}(\frac{2m}{t}) \leq \frac{2c_1 m}{t} \tilde{\alpha}_{k-1}(m) = \frac{2c_1 m}{3}.
\]
Substituting into Recurrence 4.7, and letting \( \Pi^3_{2k+1}(m) = mg(m) \),
\[
g(m) \leq g(t) + \frac{\Pi^3_{2k+1}(t)}{m} + \frac{2c_1}{3} + 3
\]
\[
\leq g(t) + \frac{2c_1}{3} + 4 \quad \text{(since, by Lemma 4.4, } \Pi^3_{2k+1}(t) \leq t^2 \leq m)\]
\[
\leq g(t) + c_1 \quad \text{(since } c_1 \geq 12).\]
Since \( \tilde{\alpha}_k(t) = \tilde{\alpha}_k(m) - 1 \), it follows by induction on \( m \) (with base case \( m \leq m_0 \)) that
\[
g(m) \leq c_1 \tilde{\alpha}_k(m) \quad \text{for all } m.
\]
Therefore,
\[
\Pi^3_{2k+1}(m) \leq c_1 m \tilde{\alpha}_k(m) \quad \text{for all } m. \quad \square
\]

The bound for \( \psi_3(m, n) \) in Lemma 2.3 now follows from Corollary 4.8 and Lemma 4.1.

4.2. Obtaining Klazar’s Improved Upper Bound for \( \lambda_3(n) \). Klazar’s tighter upper bound (1) for \( \lambda_3(n) \) follows by using the following relation between \( \lambda_3(n) \) and \( \psi_3(m, n) \), instead of Lemma 2.2:

**Lemma 4.9** [KLAZAR 1999]. We have \( \lambda_3(n) \leq \psi_3(1 + 2n/\ell, n) + 3n\ell, \) where \( \ell \leq n \) is a free parameter.

(Klazar actually proved this relation under a stricter definition of \( \psi_3(m, n) \).) For completeness, we prove Lemma 4.9 in Appendix A.

**Corollary 4.10.** \( \lambda_3(n) \leq 2n \alpha(n) + O(n \sqrt{\alpha(n)}) \).
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**Proof.** Taking $s = 3$ and $k = 2\alpha(m) + 1$ in Lemma 4.1, and bounding $\Pi^3_{2\alpha(m)+1}(m)$ by Corollary 4.8, we get

$$\psi_3(m, n) \leq (2\alpha(m) + 1)(\alpha(m)(m) + n) = 2n\alpha(m) + n + O(m\alpha(m)).$$

We now apply Lemma 4.9 with $\ell = \sqrt{\alpha(n)}$. □

**4.3. Bounding $\Pi^s_1(m)$ for General $s$.** The following recurrence and corollary for $\Pi^s_1(m)$ are analogous to Recurrence 3.6 and Lemma 3.8 in Alon et al. [2008]:

**Recurrence 4.11.** Let $s \geq 3$ be fixed. Let $k_1, k_2, k_3$ be integers, and put $k = \ell k_1 + 2k_1 - 3k_2 - k_3 + 2$. Then,

$$\Pi^s_k(m) \leq \left(1 + \frac{m}{t}\right) (\Pi^s_{k_1}(t) + 2\Pi^s_{k_2}(t) + \Pi^s_{k_3}(1 + \frac{m}{t})),
$$

where $t$ is a free parameter.

**Proof.** Take a sequence $S$ that maximizes $\Pi^s_k(m)$. Again partition the $m$ blocks of $S$ into $b = \lfloor m/t \rfloor \leq 1 + m/t$ layers $L_1, \ldots, L_b$, with at most $t$ blocks per layer.

We again classify the symbols of $S$ into local (if the symbol appears in only one layer), or global. As before, there are at most $(1 + \frac{m}{t})\Pi^s_{k_1}(t)$ local symbols.

And we again classify the global symbols into left-concentrated, right-concentrated, middle-concentrated, and scattered. This time we do this as follows:

A global symbol is left-concentrated for layer $L_i$ if its first $k_1$ occurrences fall in $L_i$. The overall number of left-concentrated symbols is at most $(1 + \frac{m}{t})\Pi^s_{k_1}(t)$.

Right-concentrated symbols are defined and handled analogously.

A global symbol is middle-concentrated for layer $L_i$ if it appears at least $k_2$ times in $L_i$, and it also appears before $L_i$ and after $L_i$. There are at most $(1 + \frac{m}{t})\Pi^s_{k_2}(t)$ middle-concentrated symbols altogether.

Finally, a global symbol is scattered if it appears in at least $k_3$ different layers.

Taking just these symbols, and for each symbol, just one occurrence per layer, we obtain an ADS$^s_k(b)$-sequence. Thus, there are at most $\Pi^s_{k_3}(b) \leq \Pi^s_{k_3}(1 + \frac{m}{t})$ scattered symbols.

All that remains is to show that we did not miss any global symbol. Suppose a global symbol is neither left-, middle-, nor right-concentrated, nor scattered. Then the symbol appears at most $2(k_1 - 1) + (k_3 - 3)(k_2 - 1) = k - 1$ times in $S$, a contradiction. □

The only significant difference between Recurrence 4.11 above and Recurrence 3.6 in Alon et al. [2008] lies in the formula for $k$ in terms of $k_1, k_2,$ and $k_3$. (The formula there is $k = k_2k_3 + 2k_1 - 2k_2$.) But in both cases, we get the same asymptotic behavior:

**Corollary 4.12.** Define $R_s(d)$ for $s \geq 1, d \geq 2$ by $R_1(d) = 2$, $R_2(d) = 3$, and for $s \geq 3$ by

$$R_s(2) = 2^{s-1} + 1,
R_s(d) = R_s(d-1)R_{s-2}(d) + 2R_{s-1}(d) - 3R_{s-2}(d)
- R_s(d-1) + 2, \text{ for } d \geq 3.$$
Then, for every $s \geq 2$ and $d \geq 2$, if $k \geq R_s(d)$, then
\[ \Pi_k^1(m) \leq cm\alpha_d(m)^{s-2} \quad \text{for all } m. \]

Here $c = c(s)$ is a constant that depends only on $s$.

PROOF. By induction on $s$, and on $d$ for each $s$. (Recall that $\Pi_k^1(m)$ is non-increasing in $k$ for fixed $s$ and $m$.) The base case $s = 2$ is given by Lemma 4.4. For $s = 3$ we have $R_3(d) = 2d + 1$, and the claim is equivalent to Corollary 4.8. Therefore, let $s \geq 4$ be fixed, and assume the claim holds for $s' < s$.

Let $m_0 = m_0(s)$ be a constant large enough so that\footnote{The dependence of $m_0$ on $s$ here could be greatly improved with a slightly more careful analysis.}
\[ m \geq 1 + 12'[\log_2 m]^s \quad \text{for all } m \geq m_0. \]

We again work with a slight variant of the inverse Ackermann function. For this proof, define $\widehat{a}_d(x)$, $d \geq 2$, by $\widehat{a}_2(x) = a_2(x) = [\log_2 x]$, and for $d \geq 3$ by the recurrence
\[ \widehat{a}_d(x) = \begin{cases} 1, & \text{if } x \leq m_0; \\ 1 + \widehat{a}_d(12\widehat{a}_{d-1}(x)^{s-2}), & \text{otherwise}. \end{cases} \]

The functions $\widehat{a}_d(x)$ are well defined by our choice of $m_0$. And as before, there exists a constant $c_0$ (depending only on $s$) such that $|\widehat{a}_d(x) - a_d(x)| \leq c_0$ for all $d$ and $x$.

We will show, by induction on $d$, that there exists a constant $c_1$ (depending only on $s$) such that, for all $d \geq 2$ and all $m$, we have
\[ \Pi_k^1(m) \leq c_1m\widehat{a}_d(m)^{s-2} \quad \text{for } k \geq R_s(d). \quad (28) \]

This is easily seen to imply the claim.

The base case $d = 2$ follows from Corollary 4.6, provided $c_1$ is chosen large enough. Further, by induction on $s$ we know there exist constants $c_2, c_3$ (depending on $s$), such that
\[ \Pi_k^{s-1}(m) \leq c_2m\widehat{a}_d(m)^{s-3} \quad \text{for } k \geq R_{s-1}(d), \]
\[ \Pi_k^{s-2}(m) \leq c_3m\widehat{a}_d(m)^{s-4} \quad \text{for } k \geq R_{s-2}(d), \]
for all $d \geq 3$ and all $m$. We choose $c_1$ large enough so that it also satisfies $c_1 \geq 6c_2$, $c_1 \geq 6c_3$, and
\[ c_1 \geq \Pi_{R_s,3}(m)/m, \quad \text{for all } m \leq m_0. \quad (29) \]

Now, let $d \geq 3$, and suppose (28) holds for $d - 1$. To establish (28) for $d$, assume first that $m \leq m_0$. Then, by (29), for all $k \geq R_s(d)$ we have
\[ \Pi_k^1(m) \leq \Pi_k^{R_s,3}(m) \leq c_1m = c_1m\widehat{a}_d(m)^{s-2}. \]

Thus, let $m > m_0$. Apply Recurrence 4.11 with the following parameters:
\[ k_1 = R_{s-1}(d), \quad k_2 = R_{s-2}(d), \quad k_3 = R_s(d-1), \quad k = R_s(d), \quad t = 12\widehat{a}_{d-1}(m)^{s-2}. \]
The last three terms in Recurrence 4.11 can be bounded as follows.

\[ 2\Pi_{k_1}^{s-1}(t) \leq 2c_2t\tilde{\alpha}_d(t)^{t-3} \leq \frac{c_1}{3}t\tilde{\alpha}_d(m)^{t-3}, \]
\[ \Pi_{k_2}^{s-2}(t) \leq c_3t\tilde{\alpha}_d(t)^{t-4} \leq \frac{c_1}{6}t\tilde{\alpha}_d(m)^{t-3}, \]
\[ \Pi_{k_3}^t \left(1 + \frac{m}{t}\right) \leq \Pi_{k_3}^t \left(\frac{2m}{t}\right) \leq \frac{2c_1m}{t}\tilde{\alpha}_d(m)^{t-2} = \frac{c_1}{6}m \leq \frac{c_1}{6}m\tilde{\alpha}_d(m)^{t-3}. \]

Substituting into Recurrence 4.11 we get
\[ \Pi_{k}^t(m) \leq \frac{m}{t}\Pi_{k}^t(t) + \frac{2c_1}{3}m\tilde{\alpha}_d(m)^{t-3} + \Pi_{k}^t(t) + \frac{c_1}{2}t\tilde{\alpha}_d(m)^{t-3}. \]

But, by Lemma 4.4, we have \[ \Pi_{k}^t(t) \leq t^{s-1} \leq (12[\log_2 m]^{t-2})^{s-1}, \] which is at most \[ m \] for \[ m > m_0 \] by our choice of \[ m_0. \] In turn, \[ m \] is at most \[ c_1m/6, \] since \[ c_1 \geq 6. \]

Similarly, we have \[ \frac{c_1}{2}t\tilde{\alpha}_d(m)^{t-3} \leq \frac{c_1}{6}m \] for \[ m > m_0 \] by our choice of \[ m_0. \] Thus,
\[ \Pi_{k}^t(m) \leq \frac{m}{t}\Pi_{k}^t(t) + c_1m\tilde{\alpha}_d(m)^{t-3}. \]

Letting \[ \Pi_{k}^t(m) = mg(m), \] we get
\[ g(m) \leq g(t) + c_1\tilde{\alpha}_d(m)^{t-3}. \]

Since \[ \tilde{\alpha}_d(t) = \tilde{\alpha}_d(m) - 1, \] it follows by induction on \[ m \] that
\[ g(m) \leq c_1\tilde{\alpha}_d(m)^{t-2} \quad \text{for all } m. \]
(The base case \[ m \leq m_0 \] follows from (29), and for the induction on \[ m \] we apply \[ (\tilde{\alpha}_d(m) - 1)^{t-2} \leq (\tilde{\alpha}_d(m) - 1)\tilde{\alpha}_d(m)^{t-3}. \] Therefore,
\[ \Pi_{k}^t(m) \leq c_1m\tilde{\alpha}_d(m)^{t-2} \quad \text{for all } m. \]

Let us now study the asymptotic growth of \[ R_s(d) \] for fixed \[ s. \] We have
\[ R_3(d) = 2d + 1, \quad R_4(d) = 5 \cdot 2^d - 4d - 3. \]

In general, letting \[ t = \lfloor (s - 2)/2 \rfloor, \] we have
\[ R_s(d) = \begin{cases} 2\log_{(2^d)}O(d^{s-1}), & s \text{ even;} \\
2\log_{(2^d)}O(d^{s-1}), & s \text{ odd.} \end{cases} \]
(see Appendix B again).

Lemma 2.3 now follows from Lemma 4.1 by applying Corollary 4.12 with \[ k = R_s(d). \]

5. Bounding Formation-Free Sequences

We now deal with the generalizations of Davenport–Schinzel sequences described in the Introduction. Recall that the first step in bounding \( Ex_u(n) \) is Lemma 1.4, which claims that \( Ex_u(n) \leq Fr_{s-r+1}(n) \), where \( r = ||u|| \) and \( s = |u| \).

PROOF OF LEMMA 1.4. Suppose \( u = u_1u_2\ldots u_s \), where \( 1 \leq u_i \leq r \) for each \( i. \) We can assume that the symbols in \( u \) make their first appearances in the order \( 1, 2, \ldots, r. \).
Let \( s' = s - r + 1 \), and let \( \ell = \ell_1 \ell_2 \cdots \ell_{s'} \) be an arbitrary \((r, s')\)-formation, where each \( \ell_j \) is a permutation of \( \{1, \ldots, r\} \). We want to show that \( u \subseteq \ell \).

Define a partition \( u = B_1 B_2 \cdots B_{s'} \) of \( u \) into \( s' \) blocks as follows: First let each symbol of \( u \) constitute its own block of length 1. Then, for each \( 2 \leq j \leq s' \), merge the block that contains the first occurrence of \( j \) in \( u \) with the block containing the immediately preceding symbol. The number of blocks goes down from \( s \) to \( s' \).

Here is an example of a sequence thus partitioned:

\[
    u = \{1\}[1][1][12][134][2][4][1][25][5].
\]

Clearly, each block \( B_j \) is an increasing sequence.

Now we are going to define a permutation \( \sigma \) on \( \{1, \ldots, r\} \) such that, for each block \( B_j \) with \( 1 \leq j \leq s' \), its image \( \sigma(B_j) \) is a subsequence of \( \ell_j \). We do this by examining the blocks from right to left, and by defining \( \sigma \) in the order \( \sigma(r), \sigma(r-1), \ldots, \sigma(1) \).

Note that blocks of length 1 can be safely ignored.

Suppose we have already dealt with blocks \( B_{s'}, B_{s'-1}, \ldots, B_{j+1} \), and that now is the turn of block \( B_j \), where \( |B_j| > 1 \). Let \( k \) be the last symbol in \( B_j \). The symbols preceding \( k \) in \( B_j \) are \( k-1, k-2, \ldots \), up to the second symbol of \( B_j \). All these symbols make their first appearance in \( u \) in \( B_j \). Call these the "new" symbols of \( B_j \).

Suppose we have already assigned values to \( \sigma(k+1), \ldots, \sigma(r) \) in such a way that, no matter how we assign \( \sigma(1), \ldots, \sigma(k) \), the images \( \sigma(B_{j+1}), \ldots, \sigma(B_{s'}) \) will always be subsequences of \( \ell_{j+1}, \ldots, \ell_{s'} \), respectively.

Now consider the symbols of \( \ell_j \). Call a symbol of \( \ell_j \) "free" if it has not yet been assigned as image \( \sigma(i) \) to any symbol \( i \), for \( k + 1 \leq i \leq r \).

We scan \( \ell_j \) from right to left, considering only its free symbols, and we assign in a greedy fashion these free symbols as images \( \sigma(k), \sigma(k-1), \ldots \) to \( k, k-1, \ldots \) (the "new" symbols of \( B_j \)).

After we are done with these assignments, the only symbol of \( B_j \) which has not been assigned an image is the first symbol of \( B_j \)—call it \( B_j \). But no matter how we define \( \sigma(B_j) \) later on, we will always have that \( \sigma(B_j) \) is a subsequence of \( \ell_j \) (because of our greedy approach).

At the end, the assignment \( \sigma(1) \) of 1 will be forced.

For example, with \( u \) as in (31), suppose that

\[
    \ell = \ell_1 \ell_2 32514 35421 \ell_5 \ell_6 \ell_7 35142 \ell_9
\]

(where \( \ell_1, \ell_2, \ell_5, \ell_6, \ell_7, \ell_9 \) do not matter). Then, our algorithm will assign \( \sigma(5) = 2, \sigma(4) = 1, \sigma(3) = 4, \sigma(2) = 5 \), and finally \( \sigma(1) = 3 \). Then, the sequence

\[
    \sigma(u) = [3][3][35][341][5][1][3][52][2]
\]

is a subsequence of \( \ell \), as desired. \( \square \)

**Remark 5.1.** Lemma 1.4 is not the last word in finding sequences in formations. For example, consider the sequence \( u = abcabca \). Lemma 1.4 states that \( u \) is contained in every \((3,5)\)-formation, but in fact \( u \) is contained in every \((3,4)\)-formation:

Let \( \ell = \ell_1 \ell_2 \ell_3 \ell_4 \) be a \((3,4)\)-formation. Suppose \( \ell_1 = abc \). Then, if \( u \) itself is not a subsequence of \( \ell \), then \( \ell_2 \) must have \( b \) before \( a \), \( \ell_3 \) must have \( c \) before \( b \), and \( \ell_4 \) must have \( a \) before \( c \). But then \( \ell \) contains the subsequence \( cbacbca \).

5.1. **Bounding the Length of Formation-Free Sequences.** Thus, the problem of bounding \( \text{Ex}_u(n) \) reduces to that of bounding \( \text{Ex}_{\ell}(n) \). For completeness,
we start by reproducing some simple bounds from Klazar [1992]. We first prove that $F_{r,s}(n)$ is finite.

**Lemma 5.2 [KLAZAR 1992].** We have $F_{r,s}(n) \leq sn'$ for $n \geq r$.

**Proof.** Let $S$ be an $(r, s)$-formation-free sequence on $n$ distinct symbols. Partition $S$ from left to right into blocks of length $r$. Note that each block contains $r$ distinct symbols. Suppose we had $1 + (s - 1)\binom{n}{r}$ complete blocks. Then, by the pigeonhole principle, there would exist $s$ blocks that have the same set of $r$ symbols. Such a set of $s$ blocks would be an $(r, s)$-formation. Contradiction.

Therefore, we must have

$$|S| < r \left(1 + (s - 1)\binom{n}{r}\right) \leq rs\binom{n}{r} \leq sn'. \quad \Box$$

It is also easy to get linear bounds for $F_{r,2}(n)$ and $F_{r,3}(n)$:

**Lemma 5.3 [KLAZAR 1992].** We have $F_{r,2}(n) \leq rn$ and $F_{r,3}(n) \leq 2rn$.

**Proof.** Let $S$ be an $r$-sparse sequence on $n$ distinct symbols. Again partition $S$ from left to right into blocks of length $r$ (the last block might be shorter).

If $S$ contains no $(r, 2)$-formation, then every block must contain the first occurrence of a symbol, and if $S$ contains no $(r, 3)$-formation, then every block must contain the first or last occurrence of a symbol. Thus, there are at most $n$ blocks in the first case, and at most $2n$ blocks in the second case. \(\square\)

**Lemma 5.4 [KLAZAR 1992].** Let $S = S_1 S_2 \cdots S_m$ be a sequence that is a concatenation of $m$ blocks, where each block $S_i$ contains only distinct symbols. Then, $S$ can be made $r$-sparse by deleting at most $(r - 1)(m - 1)$ symbols.

**Proof.** Build an $r$-sparse subsequence $S'$ of $S$ in a greedy fashion, by scanning $S$ from left to right and adding a symbol from $S$ to $S'$ only if it does not equal any of the last $r - 1$ symbols currently in $S'$. In this way, we will skip at most $r - 1$ symbols of each block $S_i$, except for the first block $S_1$, which we will take entirely. \(\square\)

Next, we make a definition analogous to Definition 2.1:

**Definition 5.5.** Given integers $r$, $s$, $m$, and $n$, we denote by $\psi_{r,s}^\prime(m, n)$ the length of the longest $r$-sparse, $(r, s)$-formation-free sequence on $n$ distinct symbols that can be partitioned into $m$ or fewer blocks, each block containing only distinct symbols.

**Remark 5.6.** The reader need not be intimidated (more than necessary) by the double subscript $r$, $s$ in $\psi_{r,s}^\prime(m, n)$. We are never going to use induction on $r$, only on $s$. Thus, $r$ can be assumed to be fixed throughout our analysis.

The following lemma (analogous to Lemma 2.2) relates $F_{r,s}(n)$ to $\psi_{r,s}^\prime(m, n)$.

**Lemma 5.7.** Given fixed integers $r$ and $s$, let $\varphi_{r,s-2}(n)$ be a nondecreasing function of $n$ such that $F_{r,s-2}(n) \leq n\varphi_{r,s-2}(n)$ for all $n$. Then,

$$F_{r,s}(n) \leq 2n + \varphi_{r,s-2}(n)(2(r - 1)n + \psi_{r,s}^\prime(2n, n)).$$

(This constitutes a minor improvement over Klazar [1992], since Klazar related $F_{r,s}(n)$ to $\varphi_{r,s-1}(n)$.)
Let $S$ be a maximum-length $(r, s)$-formation-free sequence on $n$ symbols. Thus, $|S| = F_{r, s}(n)$. Partition $S$ from left to right into subsequences as follows:

Let $S_1$ be the longest prefix of $S$ that is $(r, s - 2)$-formation-free. Let $x_1$ be the symbol following $S_1$ in $S$. Thus $S_1x_1$ contains an $(r, s - 2)$-formation. Let $S_2$ be the longest segment of $S$ after $x_1$ which is $(r, s - 2)$-formation-free, let $x_2$ be the symbol following $S_2$ in $S$, and so on.

We obtain a partition $S = S_1x_1S_2x_2 \cdots x_{m-1}S_m[x_m]$, where each $S_i$ is a subsequence and each $x_i$ is a symbol ($x_m$ might or might not be present).

Each subsequence $S_i x_i$ must contain either the first or the last occurrence of some symbol, for otherwise $S$ would contain an $(r, s)$-formation. Thus, $m \leq 2n$.

Let $n_i = \|S_i\|$. Then

$$F_{r, s}(n) = |S| \leq m + \sum_{i=1}^{m} |S_i| \leq m + \sum_{i=1}^{m} F_{r, s-2}(n_i)$$

$$\leq 2n + \sum_{i=1}^{m} n_i \varphi_{r, s-2}(n_i) \leq 2n + \varphi_{r, s-2}(n) \sum_{i=1}^{m} n_i.$$ 

So we just have to bound $\sum n_i$. Construct a subsequence $S'$ of $S$ by taking, for each subsequence $S_i$ in the above partition of $S$, just the first occurrence of each symbol in $S_i$. Thus, $|S'| = \sum n_i$. Next, using Lemma 5.4, “$r$-sparsify” $S'$ and obtain a sequence $S''$ with $|S''| \geq |S'| - (r - 1)(m - 1)$.

Since $S''$ is a subsequence of $S$, it contains no $(r, s)$-formation. Further, $S''$ is $r$-sparse and partitionable into $m$ blocks of distinct symbols. Therefore, $|S''| \leq \psi_{r, s}(m, n)$, and so

$$\sum_{i=1}^{m} n_i = |S'| \leq (r - 1)m + |S''| \leq 2(r - 1)n + \psi_{r, s}(2n, n).$$

The claim follows. $\square$

We now apply our “almost-DS” technique to formation-free sequences. For this, we introduce and analyze “almost-formation-free” sequences. The analysis closely parallels the analysis of almost-DS sequences.

5.2. ALMOST-FORMATION-FREE SEQUENCES. If $S$ is a sequence, we say that $S$ is an AFF$_{r, s, k}(m)$ sequence if $S$ contains no $(r, s)$-formation, can be partitioned into $m$ of fewer blocks, each composed of distinct symbols, and each symbol appears at least $k$ times (in $k$ different blocks).

Note that we do not require $r$-sparsity; this is the reason for calling $S$ “almost” formation-free.

Let $\Pi'_{r, s, k}(m)$ be the maximum possible number of distinct symbols in an AFF$_{r, s, k}(m)$ sequence.

We first show the connection between AFF sequences and $\psi'_{r, s}(m, n)$, and then we derive upper bounds for $\Pi'_{r, s, k}(m)$.

**Lemma 5.8.** For all $s \geq 2$ and $k$, we have $\psi'_{r, s}(m, n) \leq k(\Pi'_{r, s, k}(m) + n)$.

**Proof.** Let $S$ be a maximum-length $r$-sparse, $(r, s)$-formation-free sequence on $n$ distinct symbols, partitionable into $m$ blocks. Thus, $|S| = \psi'_{r, s}(m, n)$. Let $k \geq 1$ be the specified parameter.
Transform $S$ into another sequence $S'$ in which each symbol appears exactly $k$ times as follows. For each symbol $a$, group the occurrences of $a$ from left to right into “clusters” of size $k$, discarding the $< k$ occurrences left at the end. Replace each $a$ in the $i$th cluster by a new symbol $a_i$.

If $s \geq 2$, then this does not introduce any $(r, s)$-formations. (PROOF: Call two symbols $a$ and $b$ disjoint if every occurrence of $a$ lies before every occurrence of $b$ or vice-versa. Note that if $a$ and $b$ are disjoint, they cannot belong to the same $(r, s)$-formation for $s \geq 2$. Thus, if $S'$ contains an $(r, s)$-formation, that formation was already present in $S$.)

We deleted at most $kn$ symbols from $S$, and the result $S'$ is an AFF$_{r,s,k}(m)$ sequence ($S'$ is not necessarily $r$-sparse, but this is fine for an AFF sequence). Therefore, $S'$ contains at most $\Pi'_{r,s,k}(m)$ symbols, each one appearing exactly $k$ times, so it has length at most $k \cdot \Pi'_{r,s,k}(m)$. The claim follows. \[ \]

**LEMMA 5.9.** For every $r \geq 2$ we have $\Pi'_{r,2,2}(m) = (r - 1)(m - 1)$.

**PROOF.** For the upper bound, consider $m - 1$ “separators” between the $m$ blocks. We say that a symbol $a$ “contributes” to all the separators between the first two occurrences of $a$. Thus each symbol contributes to at least one separator. If there were $1 + (r - 1)(m - 1)$ symbols, then there would exist a separator with at least $r$ contributions, which would lead to the existence of an $(r, 2)$-formation.

For the lower bound, create $m$ blocks, and create $n = (r - 1)(m - 1)$ different symbols partitioned into $m - 1$ sets $A_1, \ldots, A_{m-1}$ of $r - 1$ symbols each. Make two copies of each $A_i$, and put one copy at the end of block $i$ and one copy at the beginning of block $i + 1$. We get a sequence with the desired properties. \[ \]

**LEMMA 5.10.** For every fixed $r \geq 2$ and $s \geq 3$, we have $\Pi'_{r,s,s}(m) \leq (r - 1)(s - 2) = O(m^{s-2})$.

**PROOF.** Suppose for a contradiction that there is an AFF$_{r,s,s}(m)$ sequence with $1 + (r - 1)(s - 2)$ distinct symbols. Consider the $s - 2$ middle occurrences of each symbol. They fall on $s - 2$ out of $m - 2$ different blocks. Therefore, by the pigeonhole principle, there exist $s$ symbols whose $s - 2$ middle occurrences all fall in the same $s - 2$ blocks. This leads to the existence of an $(r, s)$-formation in the given sequence. Contradiction. \[ \]

**RECURRENCE 5.11.** We have

$$\Pi'_{r,s,2k-1}(2m) \leq 2\Pi'_{r,s,2k-1}(m) + 2\Pi'_{r,s-1,k}(m).$$

The proof is exactly parallel to that of Recurrence 4.5.

**COROLLARY 5.12.** For fixed $r \geq 2$ and $s \geq 3$, if we let $k = 2^{r-2} + 1$, then

$$\Pi'_{r,s,k}(m) = O(m \log m)^{s-3}$$

(where the constant implicit in the $O$ notation might depend on $r$ and $s$).

**RECURRENCE 5.13.** Let $r \geq 2$ and $s \geq 3$ be fixed. Let $k_0, k_1, k_2, k_3$, and $k$ be integers satisfying $k = k_0 k_1 + 2k_1 - 3k_2 - k_3 + 2$. Then,

$$\Pi'_{r,s,k}(m) \leq \left(1 + \frac{m}{t}\right)\left(\Pi'_{r,s,k}(t) + 2\Pi'_{r,s-1,k}(t) + \Pi'_{r,s-2,k}(t)\right) + \Pi'_{r,s,k_1} \left(1 + \frac{m}{t}\right)$$

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where \( t \) is a free parameter.

The proof exactly parallels that of Recurrence 4.11. The corollary is almost the same as Corollary 4.12; there is just a shift of 1 in the index \( s \):

**Corollary 5.14.** Let \( R_s(d) \) be the sequences defined in Corollary 4.12. Then, for every \( s \geq 3 \) and \( d \geq 2 \), if \( k \geq R_{s-1}(d) \) then

\[
\Pi'_{r,s,k}(m) \leq cm\alpha_d(m)^{s-3}
\]

for all \( m \geq k \).

Here, \( c = c(r, s) \) is a constant that depends only on \( r \) and \( s \).

Combining Corollary 5.14 with Lemma 5.8, we obtain:

**Corollary 5.15.** Let \( s \geq 4 \). Then, for all \( r, m, \) and \( n \) we have

\[
\psi'_{r,s}(m, n) \leq C_{r,s,d}(m\alpha_d(m)^{s-3} + n)
\]

for some constants \( C_{r,s,d} \) of the form

\[
C_{r,s,d} = \begin{cases} 
2^{(1/t)d}O(d^{-1}), & \text{if } s \text{ odd;} \\
2^{(1/t)d}\log_2d\pm O(d^d), & \text{if } s \text{ even.}
\end{cases}
\]

where \( t = [(s-3)/2] \).

We can finally prove our upper bounds for \( F_{r,s}(n) \).

**Proof of Theorem 1.3.** Take \( d = \alpha(n) \) in Corollary 5.15, then substitute into Lemma 5.7, bounding \( \varphi_{r,s-2}(n) \) by induction on \( s \). Use the base cases \( F_{r,2}(n), F_{r,3}(n) = O(n) \) (by Lemma 5.3). (As before, \( \varphi_{r,s-2}(n) \) contributes only to lower-order terms in the exponent.) \( \square \)

6. The Lower Bound Construction for \( s = 3 \)

The rest of this article deals with lower bounds for Davenport–Schinzel sequences.

In this section we prove Theorem 1.6 by constructing, for every \( n \), a Davenport–Schinzel sequence of order 3 on \( n \) distinct symbols with length at least \( 2n\alpha(n) - O(n) \).

For this purpose, we first define a two-dimensional array of sequences \( Z_d(m) \), for \( d, m \geq 1 \), with the following properties:

—Each symbol in \( Z_d(m) \) appears exactly \( 2d + 1 \) times.
—\( Z_d(m) \) contains no forbidden alternation \( ababa \). (We do not preclude the presence of adjacent repeated symbols in \( Z_d(m) \).)
—\( Z_d(m) \) is partitioned into blocks, where each block contains only distinct symbols. Some of the blocks in \( Z_d(m) \) are special blocks. Each symbol in \( Z_d(m) \) makes its first and last occurrences in special blocks. Furthermore, the special blocks are entirely composed of first and last occurrences of symbols (there might be both first and last occurrences in the same special block). Moreover, each special block in \( Z_d(m) \) has length exactly \( m \).
—For \( d \geq 2 \), each special block is surrounded by regular blocks on both sides, and no regular block is surrounded by special blocks on both sides. For the former property, we place empty regular blocks at the beginning and end of \( Z_d(m) \), for \( d \geq 2 \).
In what follows, we enclose regular blocks by ( )'s, and special blocks by [ ]'s.

The base cases of the construction are as follows: For \( d = 1 \), we let
\[
Z_1(m) = [12 \ldots m](m \ldots 21)[12 \ldots m].
\]

\( Z_1(m) \) contains three blocks of length \( m \); the first and last ones are special blocks.
Note that each symbol appears exactly three times, as required. Also note that \( Z_1(m) \)
contains no alternation \( ababa \).

For \( m = 1 \) and \( d \geq 2 \) we let
\[
Z_d(1) = (&[1])(1)(1) \ldots (1)[1](1),
\]
with \( 2d + 1 \) ones. Each symbol constitutes its own block; the first and the last
nonempty blocks are special. Note that these special blocks have length 1, as required.
At the beginning and end there are regular blocks of length zero.

Denote by \( S_d(m) \) the number of special blocks in \( Z_d(m) \).

The recursive construction. For \( d, m \geq 2 \), we construct \( Z_d(m) \) recursively as
follows. Let \( Z' = Z_d(m-1) \). Let \( f = S_d(m-1) \) be the number of special blocks
in \( Z' \), and let \( Z^* = Z_{d-1}(f) \). Thus, the special blocks in \( Z^* \) have length \( f \).
Let \( g = S_{d-1}(f) \) be the number of special blocks in \( Z^* \).

Create \( g \) copies of \( Z' \), each copy using “fresh” symbols which do not occur in
\( Z^* \) nor in any preceding copy of \( Z' \). Thus, we have one copy of \( Z' \) for each
special block in \( Z^* \). Furthermore, each special block in \( Z^* \) has as many symbols as there
are special blocks in the corresponding copy of \( Z' \).

Let \( C_i \) be the \( i \)th special block in \( Z^* \), and let \( Z'_i \) be the \( i \)th copy of \( Z' \). Let \( a \) be
the \( i \)th symbol in \( C_i \), and let \( D_i \) be the \( i \)th special block in \( Z'_i \). We duplicate \( a \) into
\( aa \), and we insert the \( aa \) into \( Z'_i \) as follows:

If the \( a \) in \( C_i \) is the first \( a \) in \( Z^* \), then the first of the two \( a \)'s falls at the end of \( D_i \)
and the second \( a \) falls at the beginning of the block after \( D_i \). Otherwise, if the \( a \) in
\( C_i \) is the last \( a \) in \( Z^* \), then the first of the two \( a \)'s falls at the end of the block before
\( D_i \) and the second \( a \) falls at the beginning of \( D_i \). (Recall that \( D_i \) is surrounded by
regular blocks in \( Z'_i \).

Since no regular block in \( Z'_i \) is surrounded by special blocks on both sides, it
follows that no block in \( Z'_i \) receives more than one symbol from \( Z^* \). Thus, even
after the insertions, no block in \( Z'_i \) has repeated symbols.

After these insertions, at the place in \( Z^* \) where the block \( C_i \) used to be there
is now a hole. We insert \( Z'_i \) (with its extra symbols) into this hole. After doing
this for all special blocks \( C_i \) in \( Z^* \), we obtain the desired sequence \( Z_d(m) \).
See Figure 1.

It is easy to check that every symbol in \( Z_d(m) \) has multiplicity \( 2d + 1 \): The
symbols of the copies of \( Z' \) already had multiplicity \( 2d + 1 \), and the symbols of \( Z^* \)
had their multiplicity increased from \( 2d - 1 \) to \( 2d + 1 \).

It is also clear that each symbol makes its first and last occurrences in special
blocks, that the special blocks in \( Z_d(m) \) contain only first and last occurrences, and
that their length increased from \( m - 1 \) to \( m \). Furthermore, every special block is
surrounded by regular blocks on both sides, and no regular block is surrounded
by special blocks on both sides. And \( Z_d(m) \) contains empty regular blocks at the
beginning and at the end.
Fig. 1. Construction of $Z_d(m)$ from $Z^*$ and many copies of $Z'$. Two special blocks of $Z^*$ are depicted. In the left one, the symbol 1 makes its last occurrence, and symbols 2, 3 make their first occurrence. In the right block, symbols 2 and 4 make their last occurrence, and symbol 5 makes its first occurrence.

No ababa. Let us now verify that $Z_d(m)$ contains no alternation ababa of length 5. Assume by induction that this is true for the component sequences $Z'$ and $Z^*$.

Suppose for a contradiction that $Z_d(m)$ contains an alternation ababa. The symbols $a$ and $b$ cannot come from the same copy of $Z'$, by induction, and they cannot come from different copies of $Z'$, since they would not alternate at all.

Further, $a$ and $b$ cannot both come from $Z^*$: By the induction assumption, $Z^*$ contains no forbidden alternation. And the duplications of symbols $a \rightarrow aa$ cannot create a forbidden alternation, since the two $a$'s end up being adjacent in $Z_d(m)$.

Next, suppose that $a$ comes from a copy of $Z'$ and $b$ comes from $Z^*$. Then this copy of $Z'$ received two nonadjacent $b$'s. But this is impossible by construction: Our copy of $Z'$ received symbols from a single special block of $Z^*$, which contained at most one $b$. This $b$ was duplicated into two adjacent copies $bb$.

Finally, suppose that $a$ comes from $Z^*$ and $b$ comes from a copy of $Z'$. Then this copy of $Z'$ received an $a$ that is neither the first nor the last $a$ in $Z^*$. This is also a contradiction.

Remark 6.1. The above construction shares some similarities with an earlier construction by Komjáth [1988].

6.1. ANALYSIS. Recall that $S_d(m)$ denotes the number of special blocks in $Z_d(m)$. We define a few other quantities related to $Z_d(m)$:

—$N_d(m) = \|Z_d(m)\|$ denotes the number of distinct symbols in $Z_d(m)$.
—$L_d(m) = |L_d(m)|$ denotes the length of $Z_d(m)$.
—$M_d(m)$ denotes the total number of blocks (regular and special) in $Z_d(m)$.
—We let $X_d(m) = M_d(m)/S_d(m)$. Thus, $X_d(m)^{-1}$ is the fraction of blocks in $Z_d(m)$ that are special.
—We let $V_d(m) = L_d(m)/M_d(m)$ denote the average block length in $Z_d(m)$.

Note that

$$N_d(m) = \frac{1}{2}mS_d(m) \quad \text{(32)}$$

$$L_d(m) = (2d + 1)N_d(m) = \left(d + \frac{1}{2}\right)mS_d(m). \quad \text{(33)}$$
Equation (32) follows from the fact that each symbol appears in two special blocks, and each special block contains \( m \) symbols. Equation (33) follows from the fact that each symbol appears \( 2d + 1 \) times in \( Z_d(m) \).

Theorem 1.6 follows from the following facts:

**Lemma 6.2.** The quantity \( N_d(m) \) experiences Ackermann-like growth. Specifically, there exists a small absolute constant \( c \) such that

\[
A_d(m) \leq N_d(m) \leq A_d(m + c)
\]

for all \( d \geq 3 \) and all \( m \geq 2 \).

We also have \( X_d(m) \leq 2d + 1 \) and \( V_d(m) \geq m/2 \) for all \( d \) and all \( m \).

Let us first see how this lemma implies Theorem 1.6.

**Proof of Theorem 1.6.** Diagonalize by taking the sequences \( Z^*_d = Z_d(d) \) for \( d = 1, 2, 3, \ldots \). Let \( N^*_d = N_d(d) \), \( L^*_d = L_d(d) \), and \( V^*_d = V_d(d) \).

By (34) and (5) we have \( N^*_d \leq A_d(d + c) \leq A_d(A(d + 1)) = A(d + 2) \). Thus,

\[
A(d) < N^*_d \leq A(d + 2)
\]

for all \( d \geq 4 \). Thus, by (7),

\[
\alpha(N^*_d) - 2 \leq d < \alpha(N^*_d)
\]

for \( d \geq 4 \), and so, by (33),

\[
L^*_d \geq 2N^*_d \cdot \alpha(N^*_d) - O(N^*_d).
\]

The sequences \( Z^*_d \) are not necessarily Davenport–Schinzel sequences, since they might have adjacent repeated symbols. Therefore, create sequences \( Z'_d \) by removing adjacent repetitions from \( Z^*_d \). Since we delete at most one symbol per block, the length of \( Z'_d \) decreases by at most a \( 1/V^*_d \) fraction. But by Lemma 6.2, this ratio tends to zero with \( d \) (this is why we diagonalized). Specifically, the length of \( Z'_d \) is

\[
L'_d \geq L^*_d \left( 1 - \frac{1}{V^*_d} \right) \geq L^*_d \left( 1 - \frac{2}{d} \right) \geq 2N^*_d \cdot \alpha(N^*_d) - O(N^*_d). \quad (37)
\]

We have just proven that \( \lambda_3(n) \geq 2n\alpha(n) - O(n) \) for \( n \) of the form \( n = N^*_d \). We just have to interpolate to intermediate values of \( n \). Given \( n \), let \( d = d(n) \) be the unique integer such that

\[
N^*_d < N^*_{d+1} \leq n < N^*_d.
\]

It follows, by applying (36) twice, that

\[
\alpha(n) \leq \alpha(N^*_{d+2}) \leq d + 4 < \alpha(N^*_d) + 4. \quad (38)
\]

Also, by the rapid growth of \( N^*_d \) in \( d \), we certainly have

\[
N^*_d \leq \sqrt{N^*_{d+1}} \leq \sqrt{n} \quad (39)
\]

for \( d \geq 4 \).

We now concatenate many copies of \( Z'_d \) with disjoint sets of symbols, making sure we do not have more than \( n \) distinct symbols altogether. Specifically, we let \( t = \lfloor n/N^*_d \rfloor \), and we let \( Z''(n) \) be a concatenation of \( t \) copies of \( Z'_d \) with disjoint sets of symbols.
By (37), (38), and (39), it follows that the length of $Z''(n)$ is

$$L''(n) = tL_d' \geq \left(\frac{n}{N_d^*} - 1\right) \left(2N_d^* \cdot \alpha(N_d^*) - O(N_d^*)\right) = 2n\alpha(n) - O(n).$$

Since $\lambda_3(n) \geq L''(n)$, the bound follows.

All that remains is to prove Lemma 6.2.

**Proof of Lemma 6.2.** The quantity $S_d(m)$ is given recursively by

$$S_1(m) = 2;$$
$$S_d(1) = 2;$$
$$S_d(m) = fg = S_d(m - 1)S_{d-1}(S_d(m - 1)), \quad \text{for } d, m \geq 2. \quad (40)$$

In particular, we have $S_2(m) = 2^m = A_2(m)$, and $S_d(2) = 2^d$.

It is not hard to show (see Appendix C) that there exists a small constant $c_0$ such that

$$A_d(m) \leq S_d(m) \leq A_d(m + c_0) \quad (41)$$

for all $d \geq 2$ and all $m$. Then, by (32) we have, for $d \geq 3, m \geq 2$,

$$S_d(m) \leq N_d(m) \leq S_d(m)^2 \leq S_d(m + 1),$$

so (34) follows with $c = c_0 + 1$.

Regarding $M_d(m)$, we have

$$M_1(m) = 3;$$
$$M_d(1) = 2d + 3, \quad \text{for } d \geq 2,$$

counting the empty blocks at the ends of $Z_d(1)$. And for $d, m \geq 2$, we have

$$M_d(m) = gM_d(m - 1) + M_{d-1}(f) - g$$
$$= S_{d-1}(S_d(m - 1))(M_d(m - 1) - 1) + M_{d-1}(S_d(m - 1)) \quad (42)$$

(since the $g$ special blocks of $Z^*$ disappear). In particular, we have $M_2(m) = 2^{m+2} - 1$, and $M_d(2) = 2^{d+1}d - 1$.

Let us now examine $X_d(m) = M_d(m)/S_d(m)$. We have

$$X_1(m) = 3/2;$$
$$X_2(m) = 4 - 2^{-m},$$
$$X_d(1) = d + 3/2, \quad \text{for } d \geq 2,$$
$$X_d(2) = 2d - 2^{-d}, \quad \text{for } d \geq 2.$$ 

In general, dividing (42) by (40),

$$X_d(m) = X_d(m - 1) + \frac{X_{d-1}(S_d(m - 1)) - 1}{S_d(m - 1)}. \quad (43)$$

We now prove by induction that $X_d(m) \leq 2d + 1$ for all $d$ and $m$. The claim has been verified for $d \leq 2$ and for $m \leq 2$, so assume $d, m \geq 3$. By (43) and using
induction on \( d \), we have

\[
X_d(m) \leq X_d(m - 1) + \frac{2d - 2}{S_d(m - 1)},
\]

so

\[
X_d(m) \leq X_d(2) + (2d - 2) \sum_{m=2}^{\infty} S_d(m)^{-1} = 2d - 2^{-d} + (2d - 2) \sum_{m=2}^{\infty} S_d(m)^{-1}.
\]

It is easily checked that, for \( d \geq 3 \),

\[
\sum_{m=2}^{\infty} S_d(m)^{-1} \leq 2S_d(2)^{-1} = 2^{1-d} \leq \frac{1}{2d - 2}.
\]

It follows that \( X_d(m) \leq 2d + 1 \), as desired.

Finally, let us consider \( V_d(m) \). By (33) we have

\[
V_d(m) = \frac{L_d(m)}{M_d(m)} = \left( d + \frac{1}{2} \right) \frac{m}{X_d(m)} \geq \frac{m}{2}.
\]

**Remark 6.3.** The coefficient 2 in our bound for \( \lambda_3(n) \) comes from the fact that each symbol appears roughly \( 2d \) times in \( Z_d(m) \). In previous constructions [Wiernik and Sharir 1988; Kolljáth 1988; Sharir and Agarwal 1995] each symbol appears only \( d \pm O(1) \) times in the equivalent sequence. Sharir and Agarwal [1995] lost an additional factor of 2 in the interpolation step; we avoided this loss in the proof of Theorem 1.6 by letting \( Z''(n) \) consist of many copies of \( Z''_d \), instead of using \( Z''_{d+1} \) (which would have been a more obvious choice).

### 6.2. LOWER BOUND FOR THE NUMBER OF SYMBOLS IN ALMOST-DS SEQUENCES OF ORDER 3

In Section 4, we introduced the notion of almost-DS sequences. We derived an upper bound on the maximum number \( \Pi_3^3(m) \) of distinct symbols of an ADS\_3\^k\_m-sequence, and we used this upper bound to bound \( \lambda_3(n) \).

But the problem of ADS\_3\^k\_m-sequences is interesting in its own right, so one might naturally wonder about matching lower bounds for \( \Pi_3^3(m) \).

It turns out that the construction \( Z_d(m) \) described in this section also provides a roughly-matching lower bound for \( \Pi_3^3(x) \). We just have to change our point of view: Instead of taking a diagonal (namely, \( Z_d(d) \)), we take the rows of the construction (meaning, \( Z_d(m) \) for fixed \( d \)).

**Lemma 6.4.** For every fixed \( d \geq 2 \), we have

\[
\Pi_{2d+1}^3(x) = \Omega \left( \frac{1}{d} x \alpha_d(x) \right).
\]

**Proof.** For every \( m \geq 1 \), the sequence \( Z_d(m) \) is an ADS\_3\^{2d+1}\_x\_m-sequence for \( x_m = M_d(m) \). Let \( n_m = N_d(m) \) be the number of distinct symbols in \( Z_d(m) \).

By the definition of \( X_d(m) \), and by applying Lemma 6.2 and then (41), we have

\[
x_m = M_d(m) = X_d(m)S_d(m) \leq (2d + 1)S_d(m) \\
\leq (2d + 1)A_d(m + c_0) \leq A_d(m + c_0 + 1). \tag{44}
\]

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Thus, by (6), we have $m \geq \alpha_d(x_m) - c_0 - 1$. Therefore, by (33), applying Lemma 6.2 again, and applying (44),

$$n_m = N_d(m) = \frac{L_d(m)}{2d + 1} = \frac{V_d(m)M_d(m)}{2d + 1} \geq \frac{mM_d(m)}{4d + 2} = \Omega\left(\frac{1}{d^{x_m}x_m\alpha_d(x_m)}\right).$$

We interpolate to intermediate values of $x$ (for $x_m \leq x < x_{m+1}$) as we did above, in Section 6.1. □

Thus, for odd $d$ the bounds for $\Pi_3^d(m)$ are quite tight (they leave a multiplicative gap of $O(d)$). For even $d$ the bounds are not so tight—they are obtained by applying $\Pi_{r+1}^d(m) \leq \Pi_r^d(m) \leq \Pi_{r-1}^d(m)$.

Lemma 6.4 automatically yields a lower bound for $\Pi_1^r(m)$, $4$, $k(m)$: A sequence that does not contain $ababa$ cannot contain an $(r, 4)$-formation for any $r \geq 2$; further, as Adamec et al. [1992] showed, an $r$-sparse, $u$-free sequence can be made $r'$-sparse for $r' > r$ at the cost of shrinking the sequence by at most a constant factor.

7. The Lower-Bound Construction for $s \geq 4$ Even

In this section we present a construction that achieves the lower bounds (2). This is a simpler variant of the construction of Agarwal et al. [1989] that achieves the same bounds.

We first construct a family of sequences $S_s^k(m)$ for $s \geq 2$ even, $k \geq 0$, and $m \geq 1$. For all $s \geq 4$, $m \geq 2$, the sequences $S_s^k(m)$ are Davenport–Schinzel sequences of order $s$.

The sequences $S_s^k(m)$ are highly regular; they satisfy the following properties:

—$S_s^k(m)$ is a concatenation of blocks of length $m$, where each block contains $m$ distinct symbols. (For $s = 2$ or $m = 1$, there are adjacent repeated symbols at the interface between blocks, but only in these cases.)

—$S_s^k(m)$ does not contain any forbidden alternation $abab \cdots$ of length $s + 2$, for any distinct symbols $a \neq b$. Thus, for $s \geq 4$, $m \geq 2$, the sequence $S_s^k(m)$ is a Davenport–Schinzel sequence of order $s$.

—All symbols in $S_s^k(m)$ occur with the same multiplicity $\mu_s(k)$, which depends only on $s$ and $k$. Further, for $s \geq 4$, each symbol in $S_s^k(m)$ makes all its appearances in the same position within the blocks, and no two symbols $a$, $b$ appear together in more than one block.

7.1. THE CONSTRUCTION. For $s = 2$, the sequences $S_s^2(m)$ are given (independently of $k$) by

$$S_2^2(m) = 12 \cdots m m \cdots 21.$$ 

$S_2^2(m)$ consists of two blocks of length $m$, and each symbol occurs with multiplicity $\mu_2(k) = 2$. Clearly, $S_2^2(m)$ contains no forbidden alternation $abab$.

The construction for general $s \geq 4$ is as follows. For $k = 0$, we let $S_s^0(m)$ consist of a single block of length $m$:

$$S_s^0(m) = 12 \cdots m.$$ 

Thus, $\mu_s(0) = 1$. 

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For general $k \geq 1$, we proceed as follows. The sequence $S_s^k(1)$ consists of

$$\mu_s(k) = \mu_{s-2}(k-1)\mu_s(k-1)$$

(46)
copies of the symbol 1, each forming by itself a block of length one. Equation (46), together with the bounding cases $\mu_2(k) = 2$ and $\mu_s(0) = 1$ for $s \geq 4$, gives the recursive definition of $\mu_s(k)$.

For $m \geq 2$, the sequence $S_s^k(m)$ is constructed inductively on the lexicographic order of the triples $(s, k, m)$, using three previously created sequences as components.

The first sequence is $S' = S_s^k(m-1)$; note that $S'$ contains blocks of length $m - 1$. Let $f$ be the number of blocks in $S'$.

The second sequence is $\overline{S} = S_s^{k-1}(f)$. Thus, $\overline{S}$ contains blocks of length $f$. Let $g = \|\overline{S}\|$ be the number of distinct symbols in $\overline{S}$.

The third and final sequence is $S^* = S_{k-1}^{s-1}(g)$. Thus, $S^*$ contains blocks of length $g$.

Transform the sequence $S^*$ into a sequence $\hat{S}^*$ by replacing each block in $S^*$ by a copy of $\overline{S}$ with the same set of $g$ symbols, making their first appearances in the same order as in the replaced block. Note that $\hat{S}^*$ contains blocks of length $f$. Further, the multiplicity of each symbol in $\hat{S}^*$ is the product of the symbol multiplicities in $\overline{S}$ and $S^*$; by induction this multiplicity equals

$$\mu_{s-2}(k-1)\mu_s(k-1) = \mu_s(k).$$

Let $h$ be the number of blocks in $\hat{S}^*$.

Now, create $h$ copies of $S'$, each copy using “fresh” symbols that do not occur in $\hat{S}^*$ nor in any preceding copy of $S'$, and concatenate them into a sequence $S''$. Note that $S''$ contains $fh$ blocks of length $m - 1$, while $|\hat{S}^*| = fh$.

Insert each symbol of $\hat{S}^*$ in order at the end of each block of $S''$. Thus, each component sequence $S'$ in $S''$, containing $f$ blocks, receives the $f$ distinct symbols of a block in $\hat{S}^*$. The resulting sequence is the desired $S_s^k(m)$. Note that it contains blocks of length $m$, and, by induction and construction, each symbol in it has multiplicity $\mu_s(k)$. See Figure 2.
Letting \( t = s/2 - 1 \), we have
\[
\mu_s(k) = 2^{\binom{s}{2}} - 2^{(1/2)k - O(k^{1/2})},
\]  
(47)
if we take \( s \) to be a constant.

7.2. CORRECTNESS OF THE CONSTRUCTION. We now prove that, for \( s \geq 4 \), \( m \geq 2 \), the sequences \( S_s^k(m) \) are indeed Davenport–Schinzel sequences of order \( s \).

Let us first recall some important properties of the construction:

—The last symbol in each block of \( S_s^k(m) \) comes from \( \hat{S}^s \) (which has the same set of symbols as \( S^s \)), while every other symbol in \( S_s^k(m) \) comes from a copy of \( S' \).

—The copies of \( S' \) have pairwise disjoint sets of symbols, which are also disjoint from the set of symbols of \( \hat{S}^s \).

—When merging \( S'' \) and \( \hat{S}^s \) to form \( S_s^k(m) \), each copy of \( S' \) in \( S'' \) receives the \( f \) distinct symbols of a block of \( \hat{S}^s \).

The following lemma is easily proven by induction using the above properties:

**Lemma 7.1.** The sequence \( S_s^k(m) \) satisfies the following properties:

1. For \( s \geq 4 \), each symbol in the sequence makes all its appearances in the same position within the blocks.
2. For \( s \geq 4 \), \( m \geq 2 \), there are no adjacent repeated symbols.
3. For \( s \geq 4 \), no two symbols of \( S_s^k(m) \) appear together in more than one block.

For each symbol \( a \) in \( S_s^k(m) \), call the depth of \( a \) the position within the blocks in which \( a \) always appears in \( S_s^k(m) \). This notion is well defined by the above lemma. Thus, the symbols that come from copies of \( S' \) have depth between 1 and \( m - 1 \), while the symbols that come from \( \hat{S}^s \) have depth \( m \).

The following Lemma is also pretty straightforward:

**Lemma 7.2.** Symbols at different depths in \( S_s^k(m) \) make alternations of length at most 5.

**Proof.** By induction. The claim is clearly true if \( s = 2 \), \( k = 0 \), or \( m = 1 \). Thus, let \( s \geq 4 \), \( k \geq 1 \), and \( m \geq 2 \). Let \( a \) and \( b \) be two symbols at different depths in \( S_s^k(m) \).

If both \( a \) and \( b \) have depth at most \( m - 1 \), then they either come from the same copy of \( S' \), in which case the claim follows by induction, or else they come from different copies of \( S' \), in which case they do not alternate at all.

Thus, suppose one symbol, say \( a \), has depth \( m \) (so it comes from \( \hat{S}^s \)), while the other symbol, \( b \), has depth at most \( m - 1 \) (so it comes from a copy of \( S' \)).

The copy of \( S' \) to which \( b \) belongs receives at most one \( a \) from \( \hat{S}^s \). In the worst case, this \( a \) is surrounded by \( b \)'s from our copy of \( S' \), and this copy of \( S' \) is in turn surrounded by other \( a \)'s from \( \hat{S}^s \). Thus, the longest alternation we can get is \( ababa \).

The main issue is to show that \( S_s^k(m) \) contains no forbidden alternating subsequence of length \( s + 2 \). For this, we prove by induction that \( S_s^k(m) \) satisfies a stronger property.
LEMMA 7.3. The sequence $S_k^*(m)$ satisfies the following properties:

(1) $S_k^*(m)$ contains no forbidden alternation $abab\cdots$ of length $s + 2$.

(2) Furthermore, if each block $B$ in $S_k^*(m)$ is replaced by a sequence $T(B)$ on the same set of symbols as $B$, such that $T(B)$ contains no alternation $abab\cdots$ of length $s$, and such that the symbols in $T(B)$ make their first appearances in the same order as they did in $B$, then the resulting sequence still contains no forbidden alternation of length $s + 2$.

PROOF. Again by induction. Both properties clearly hold if $s = 2, k = 0$, or $m = 1$, so let $s \geq 4, k \geq 1$, and $m \geq 2$.

Assume by induction that Properties 1 and 2 hold for the sequences $S', S^*$, and $S^*$ from which $S_k^*(m)$ is built. We want to show that these properties hold for $S_k^*(m)$ itself.

We start with Property 1. Suppose for a contradiction that $S_k^*(m)$ contains a forbidden alternation $abab\cdots$ or $baba\cdots$ of length $s + 2$. By Lemma 7.2, $a$ and $b$ must have the same depth (since $s + 2 \geq 6$).

If $a$ and $b$ have depth at most $m - 1$, then they must belong to the same copy of $S'$, or else they would not alternate at all. But this contradicts our inductive assumption on $S'$.

And if $a$ and $b$ have depth $m$ and come from $\tilde{S}^*$, then $\tilde{S}^*$ itself contains a forbidden alternation. But $\tilde{S}^*$ is obtained from $S^*$ via block replacements, exactly as described in Property 2. Thus, the inductive assumption on $S^*$ is contradicted.

In conclusion, $S_k^*(m)$ cannot contain an alternation of length $s + 2$, so it satisfies Property 1.

Now we show that $S_k^*(m)$ satisfies Property 2. Suppose for a contradiction that, after performing a certain set of block replacements in $S_k^*(m)$, we do get an alternation $abab\cdots$ or $baba\cdots$ of length $s + 2$.

For this to happen, $a$ and $b$ must have appeared together in some block $B$ of $S_k^*(m)$. (By Lemma 7.1, they do not appear together in more than one block.) Say that $a$ appeared before $b$ in this block. This block was replaced, in the worst case, by a sequence containing an alternation $abab\cdots$ of length $s - 1$. (Without loss of generality we may assume the alternation starts with an $a$, since the block replacement preserves the order of first appearances of the symbols.)

This alternation is extended to length $s + 2$ by at least three more instances of $a$ and $b$ before or after the block $B$, according to one of four possible cases, as depicted in Figure 3 (left).

To see why none of these cases can occur, consider again where the symbols $a$ and $b$ came from. If $a$ and $b$ came from the same copy of $S'$, then the same block replacement in $S'$ would also have generated a forbidden alternation of length $s + 2$. This contradicts our inductive assumption for $S'$.

Further, $a$ and $b$ could not have come from different copies of $S'$, since then they would not lie together in the same block (and they would not alternate at all). For a similar reason, they cannot both come from $\tilde{S}^*$.

Thus, one symbol—specifically, $a$—must originate from a copy of $S'$, and the other one—namely, $b$—must originate from $\tilde{S}^*$. But all the other instances of $a$ in $S_k^*(m)$, to the left or right of our block $B$, also come from the same copy of $S'$. A case analysis shows that in each of the four cases shown in Figure 3 (left), this copy of $S'$ received two copies of $b$ from $\tilde{S}^*$. (In cases (i) and (ii) there are two $b$'s
FIG. 3. The left figure shows the case of \( s \) even. For a forbidden alternation to occur, a pair of symbols \( a, b \) in a common block must be replaced by an alternation of length at most \( s - 1 \), and extended to length \( s + 2 \) by at least three more symbols \( a, b \), according to one of four possible cases. In each case we get a contradiction. The right figure shows the case of \( s \) odd. Here the argument fails, because case (ii) fails to yield a contradiction.

surrounded by \( a \)'s, and in cases (iii) and (iv) there is a \( b \) surrounded by \( a \)'s, plus another \( b \) lying in the same block as an \( a \).) This is impossible according to our construction.  

Remark 7.4. Unfortunately, the above argument depends crucially on \( s \) being even. If we try to make the same argument with \( s \) odd, we get the four cases illustrated in Figure 3 (right), and in case (ii) we fail to get a contradiction—we cannot find two instances of \( b \) sent to the same copy of \( S' \).

7.3. ANALYSIS. Given a fixed even number \( s \geq 4 \), take the sequences \( S_s(k) \), for \( k = 0, 1, 2, \ldots \). These are Davenport–Schinzel sequences of order \( s \), in which the multiplicity of the symbols, \( \mu_s(k) \), goes to infinity. Thus, the length of these sequences grows superlinearly in the number of symbols. We want to derive the exact relation between these two quantities. For this purpose, we derive an upper bound on the number of distinct symbols in \( S_s(k) \).

Let \( N_s(k) = |S_s(k)| \) denote the number of distinct symbols in \( S_s(k) \), and let \( F_s(k) \) be the number of blocks in \( S_s(k) \). Then,

\[
|S_s(k)| = \mu_s(k) N_s(k) = m F_s(k). \tag{48}
\]

The quantities \( N_s(k) \) are initialized by

\[
N_s^2(m) = m; \\
N_s^0(m) = m; \\
N_s^1(1) = 1.
\]

To get a recurrence relation for the general case, we analyze the recursive construction of \( S_s(k) \). Using the notation there, we have

\[
f = F_s^s(m - 1); \\
g = N_s^{s-2}(f); \\
h = F_s^s(k-1)(g) \cdot F_s^{s-2}(f); \\
N_s^k(m) = N_s^{k-1}(g) + h \cdot N_s^k(m - 1).
\]
Thus, applying (48) three times and then (46),
\[ N_k^s(m) = N_{k-1}^s(g) + F_k^s(k) \cdot F_{k-1}^{s-2}(f) \cdot N_k^s(m-1) \]
\[ = N_{k-1}^s(g) + \frac{\mu_s(k-1)N_{k-1}^s(g)}{g} \cdot \frac{\mu_{s-1}(k-1) \cdot g}{f} \cdot \frac{(m-1) \cdot f}{\mu_s(k)} \]
\[ = m \cdot N_{k-1}^s(g) \]
\[ = m \cdot N_{k-1}^s(N_k^{s-2}(F_k^s(m-1))). \]

Since \( \mu_s(k) \leq 2^s \) and \( m \geq 1 \), by (48), we have
\[ F_k^s(m) \leq 2^s N_k^s(m), \quad (49) \]
so
\[ N_k^s(m) \leq m \cdot N_{k-1}^s(N_k^{s-2}(2^s N_k^s(m-1))). \]

We now simplify the analysis by getting rid of the dependence on \( s \) in the last inequality. For this, we define an Ackermann-like hierarchy of functions \( \hat{A}_k(m) \) for \( k \geq 0, m \geq 1 \), by
\[ \hat{A}_0(m) = m; \]
and
\[ \hat{A}_k(m) = \begin{cases} 1, & \text{if } m = 1; \\ m \cdot \hat{A}_{k-1}\left(\hat{A}_{k-1}\left(2^s \hat{A}_k(m-1)\right)\right), & \text{otherwise}; \end{cases} \]
for \( k \geq 1 \) (compare to (4)). It follows by induction that
\[ N_k^s(m) \leq \hat{A}_k(m) \quad (50) \]
for all \( s, k, \) and \( m \). In Appendix C, we prove that
\[ \hat{A}_k(m) \leq A_{k+1}(2m + 4) \quad \text{for all } k \geq 2 \text{ and all } m. \quad (51) \]

Now let us come back to the sequences with which we started this discussion. Let \( T_k = S_k^s(2) \) for \( k = 0, 1, 2, \ldots \), and let \( n_k = \|T_k\| \). Then, applying (50), (51), and (5),
\[ n_k = N_k^s(2) \leq \hat{A}_k(2) \leq A_{k+1}(8) \leq A_{k+1}(A(k + 2)) = A(k + 3). \]
Therefore, \( k \geq \alpha(n_k) - 3 \). Substituting into (48) applying (47), and letting \( t = s/2 - 1, \)
\[ |T_k| = n_k \cdot \mu_s(k) \geq n_k \cdot \mu_s(\alpha(n_k) - 3) \geq n_k \cdot 2^{(1/2)^{\alpha(n_k) - 3 - 3}}. \]

We have thus achieved the desired lower bound on \( \lambda_s(n) \) for \( n \) of the form \( n = n_k \).
As in Section 6, interpolating to intermediate values of \( n \) (for \( n_k \leq n < n_{k+1} \)) is straightforward, and we obtain the desired bound for all \( n \).

7.4. ADVANTAGES OVER THE PREVIOUS CONSTRUCTION. The construction we just presented follows the same basic idea as the previous construction of Agarwal et al. [1989], but it has the following advantages:
In our construction, each block is just a sequence of \( m \) distinct symbols. In the previous construction, each block (there called a “fan”) is of the form 12\( \cdots \) \( m \) \( \cdots \) 21.

In our construction, all symbols have the same exact multiplicity. This greatly simplifies calculations.

In our construction, there are no adjacent repeated symbols at the interface between blocks. (Removing these adjacent repetitions in the previous construction does not present any serious problem, but they constitute a small aesthetic blemish.)

The previous construction involves some “tiny” duplications of symbols, which our construction does not have. These duplications are not the cause of the asymptotic growth (and indeed, our construction works fine without them). This is a potential source of confusion, especially since these “tiny” duplications are also present in the lower-bound construction for order-3 sequences, and in that case they are critical.

### 7.5. LOWER BOUNDS FOR THE NUMBER OF SYMBOLS IN ALMOST-DS SEQUENCES OF EVEN ORDER \( s \geq 4 \)

As was the case with the construction of order 3, the construction described in this section yields lower bounds for \( \Pi_s^k(m) \), for \( s \geq 4 \) even. Again, the idea is to look at the rows of the construction, namely at \( S^s_k(m) \) for fixed \( s \) and \( k \).

**Lemma 7.5.** For every fixed even \( s \geq 4 \) and every \( k \geq 4 \), we have

\[
\Pi_\mu^s(x) \geq x \alpha_k(x) \text{ for all large enough } x,
\]

for some \( \mu \) asymptotically of the form

\[
\mu \geq 2^{(1/t)k^t - O(k^{t-1})},
\]

where \( t = s/2 - 1 \). Moreover, these lower bounds can be achieved by actual Davenport–Schinzel sequences.

The proof is similar to the proof of Lemma 6.4, though somewhat simpler, since the blocks in \( S^s_k(m) \) have uniform length. We omit the details.

As before, Lemma 7.5 automatically yields lower bounds for \( \Pi_{r,s,k}^r(m) \) for odd \( s \geq 5 \).

It is an open problem whether the lower bounds for the case \( s = 3 \) shown above (Section 6.2) can be achieved with actual Davenport–Schinzel sequences (without adjacent repeated symbols), as was the case here.

### 8. Conclusion and Open Problems

The bounds for \( \lambda_s(n) \) are now tight for every even \( s \). Unfortunately, for odd \( s \geq 5 \) the problem is still not completely solved. We believe the new upper bounds for odd \( s \) are the true bounds, simply by analogy to the interval-chain bounds. But the construction that gives the lower bounds does not seem to work when \( s \) is odd.

Are there other problems that, like interval chains and almost-DS sequences, satisfy recurrences like Recurrence 4.7 and Recurrence 4.11? If so, it would be interesting to find more examples of such problems.
The reason we can unambiguously talk about the coefficient that multiplies $\alpha(n)$ (e.g., in Theorems 1.2 and 1.6), despite the fact that there are several different versions of $\alpha(n)$ in the literature, is that all these versions differ from one another by at most an additive constant. Thus, the coefficient multiplying $\alpha(n)$ is not affected. On the other hand, one cannot talk about the leading coefficient in $\lambda_4(n) = \Theta\left(n \cdot 2^{\alpha(n)}\right)$, for example, unless a standard definition of $\alpha(n)$ is agreed upon.

Can our lower-bound construction for $\lambda_3(n)$ (Section 6) be realized as the lower envelope of segments in the plane? If so, it would yield a factor-of-2 improvement for this problem as well.

Appendices

A. Proof of Klazar’s Lemma 4.9

For completeness, we include here the proof of Klazar’s Lemma 4.9. Recall that the claim is that $\lambda_3(n) \leq \psi_3(1 + 2n/\ell, n) + 3n\ell$, where $\ell \leq n$ is a free parameter.

**PROOF OF LEMMA 4.9.** Let $S$ be a maximum-length Davenport–Schinzel sequence of order 3 on $n$ distinct symbols. Thus, $|S| = \lambda_3(n)$. Call an occurrence of a symbol $a$ in $S$ a terminal occurrence if it is the first or last occurrence of $a$ in $S$.

Partition $S$ into blocks $S = S_1 S_2 S_3 \cdots S_m$, where each $S_i$ starts with a terminal occurrence and contains exactly $\ell$ terminal occurrences (except for $S_m$, which might contain fewer terminal occurrences). Since $S$ contains $2n$ terminal occurrences, the number of blocks is $m = \lceil 2n/\ell \rceil \leq 1 + 2n/\ell$.

For every block $S_i$ and every symbol $a$, let $n_i(a)$ be the number of occurrences of $a$ in $S_i$. Recall that these occurrences must be nonadjacent. If $S_i$ contains the first or last occurrence of $a$ in $S$, we say that $a$ is terminal in $S_i$; otherwise, $a$ is nonterminal in $S_i$.

Let $\Lambda_i$ be the set of symbols that appear in $S_i$. Let $\Lambda_i'$ be the subset of these symbols which are terminal in $S_i$, and let $\Lambda_i''$ be the subset of those which are nonterminal. Clearly,

$$|S_i| = \|S_i\| + \sum_{a \in \Lambda_i} (n_i(a) - 1).$$

We claim that $n_i(a) \leq \ell$ for all $a \in \Lambda_i$. Indeed, suppose for a contradiction that $n_i(a) \geq \ell + 1$ for some $a \in \Lambda_i$. Then the occurrences of $a$ in $S_i$ define $\ell$ interior-disjoint, nonempty intervals. But $S_i$ contains at most $\ell$ terminal occurrences of symbols, one of which is the first symbol of $S_i$. Therefore, one of the above-mentioned intervals must be free of terminal occurrences, and so it contains a symbol $b$ which also appears both before and after the interval. Thus, $S$ contains $babab$, which is a contradiction.

For a similar reason, $S_i$ cannot contain the pattern $aba$ for any $a, b \in \Lambda_i''$. Therefore, the nonterminal symbols in $S_i$ do not intermingle at all (meaning, for every $a, b \in \Lambda_i''$, all occurrences of $a$ appear before all occurrences of $b$ or vice versa). Therefore, the symbols which are nonterminal in $S_i$ define $\sum_{a \in \Lambda_i''} (n_i(a) - 1)$ interior-disjoint, nonempty intervals of the form $a \ldots a$ in $S_i$. On the other hand, the number of such intervals cannot be larger than $\ell - 1$ (by an argument similar
to the one above). Therefore,

\[ |S_i| = \|S_i\| + \sum_{a \in \Lambda_i}(n_i(a) - 1) + \sum_{a \in \Lambda_i}(n_i(a) - 1) \]

\[ \leq \|S_i\| + (\ell - 1)|\Lambda_i'| + (\ell - 1) \]

\[ \leq \|S_i\| + \ell(\ell - 1) + (\ell - 1) = \|S_i\| + \ell^2 - 1. \]

Now, define a subsequence \( S' \) of \( S \) by taking just the first occurrence of each symbol in each \( S_i \). Then, \( |S'| = \sum_{i=1}^m \|S_i\| \), and \( S' \) is composed of \( m \) blocks, each of distinct symbols. \( S' \) might still contain adjacent repeated symbols at the interface between blocks, but these can be eliminated by deleting at most \( m - 1 \leq 2n/\ell \) symbols. We get a Davenport–Schinzel sequence \( S'' \) which satisfies \( |S''| \leq \psi_3(m, n) \), and thus

\[ \lambda_3(n) = |S| = \sum_{i=1}^m |S_i| \leq m(\ell^2 - 1) + \sum_{i=1}^m \|S_i\| \]

\[ \leq (1 + 2n/\ell)(\ell^2 - 1) + \psi_3(m, n) + 2n/\ell \]

\[ \leq \psi_3(1 + 2n/\ell, n) + 3n\ell. \]

**B. On the Asymptotic Growth of Some Recurrent Quantities**

A recurrent feature in this article are two-parameter quantities given roughly by \( C_{s,k} \approx C_{s-2,k}C_{s,k-1} \), with base cases \( C_{3,k} = \Theta(k) \) and \( C_{4,k} = \Theta(2^k) \). (Specifically, we have the quantities \( P_{s,k} \) and \( Q_{s,k} \) in Section 3, and \( R_s(d) \) in Section 4. See also \( \mu_s(k) \) in Section 7. There are also similar quantities in Alon et al. [2008].) In this appendix we give a generic analysis of the asymptotic growth of such quantities (as a function of \( k \) for \( s \) fixed).

**Lemma B.1.** Let \( C_{s,k} \) be positive quantities given recursively, for \( s \geq 3, k \geq 1 \), by

\[ C_{3,k} = \Theta(k); \]

\[ C_{4,k} = \Theta(2^k); \]

\[ C_{s,k} = (C_{s-2,k} + a)C_{s,k-1} + a'C_{s-1,k} + a''C_{s-2,k} + a''', \quad \text{for } s \geq 5, k \geq 2; \]

for some implicit constants for \( C_{3,k} \) and \( C_{4,k} \), some constants \( a, \ldots, a''' \) (which might depend on \( s \)), and some initial conditions specifying \( C_{s,1} \).

Then for every fixed \( s \geq 3, C_{s,k} \) has upper and lower bounds of the form

\[ C_{s,k} = \begin{cases} 2^{(1/t)k \pm O(k^{-1})}, & s \text{ even;} \\ 2^{(1/t)k' \log k \pm O(k'}), & s \text{ odd;}
\end{cases} \]

where \( t = [(s - 2)/2] \).

**Proof.** Let \( s \geq 5 \), and assume by induction that \( C_{s-1,k}, C_{s-2,k} \) have the claimed growth in \( k \). The constant \( a \) is “swallowed up” asymptotically by \( C_{s-2,k} \), so, by a simple transformation, we may assume that \( a = 0 \).
Let $R_{s,k} = a'C_{s-1,k} + a''C_{s-2,k} + a'''$. Then,

$$C_{s,k} = \sum_{i=2}^{k} R_{s,i} \prod_{j=i+1}^{k} C_{s-2,j} + C_{s,1} \prod_{j=2}^{k} C_{s-2,j}.$$ 

We bound each term in the right-hand-side by substituting the assumed growth rates for $C_{s-1,k}$ and $C_{s-2,k}$, and bounding the resulting sums in the exponent by integrals. The calculations are fairly routine but tedious; they show that the last term in the right-hand-side dominates, and has the form

$$C_{s,1} \prod_{j=2}^{k} C_{s-2,j} = \begin{cases} 
2^{\frac{1}{t} t' \log t' + O(1)} \quad & s \text{ even}; \\
2^{\frac{1}{t} t' \log t' + O(1)} \quad & s \text{ odd};
\end{cases}$$

and the claim follows.

C. Comparing Ackermann-Like Functions

In this appendix, we present a general technique for proving that variants of the Ackermann hierarchy exhibit equivalent rates of growth. We first give the lemma on which the technique is based, and then we illustrate the technique by proving that the function $\hat{A}_k(m)$ of Section 7.3 satisfies

$$\hat{A}_k(m) \leq A_{k+1}(2m + 4).$$

This is basically the same technique as in Appendix B of Alon et al. [2008], but rephrased so as to deal with rapidly growing functions instead of their slowly growing inverses. Our technique here is also slightly more general than the one in Alon et al. [2008].

We consider the following general setting. Suppose $F(n)$ and $G(n)$ are nondecreasing functions that satisfy $F(n), G(n) > n$ for all $n$. Define functions $F^\circ(n), G^\circ(n)$ by $F^\circ(n) = F^{(n)}(F_0), G^\circ(n) = G^{(n)}(G_0)$, with some initial conditions $F_0, G_0$. (Recall that $f^{(n)}$ denotes the $n$-fold composition of $f$.)

We want to prove that $F^\circ(n) \leq G^\circ(dn + c)$ for some constants $d$ and $c$. The following lemma gives a sufficient condition for this.

**LEMMA C.1.** Let $F(n), G(n), F^\circ(n), G^\circ(n)$ be functions as given above. Suppose there exists an integer $d$ and a function $\delta(n)$ such that

$$n \leq \delta(n),$$

$$\delta(F(n)) \leq G^{(d)}(\delta(n)),$$

for all $n \geq 1$. Then, $F^\circ(n) \leq G^\circ(dn + c)$ for a constant $c$ large enough that

$$\delta(F_0) \leq G^\circ(c).$$

**PROOF.** Applying (52), then (53) $n$ times, and then (54),

$$F^\circ(n) = F^{(n)}(F_0) \leq \delta(F^{(n)}(F_0)) \leq G^{(dn)}(\delta(F_0)) \leq G^{(dn)}(G^\circ(c)) = G^{(dn+c)}(G_0) = G^\circ(dn + c).$$

Now let us apply this technique to the task at hand.
LEMMA C.2. Let \( \hat{A}_k(m) \) be given by
\[
\hat{A}_0(m) = m, \quad \text{for } m \geq 1;
\]
and
\[
\hat{A}_k(m) = \begin{cases} 
1, & \text{if } m = 1; \\
m \cdot \hat{A}_{k-1} \left( 2^{2^k} \hat{A}_k(m-1) \right), & \text{otherwise};
\end{cases}
\]
for \( k \geq 1 \). Then \( \hat{A}_k(m) \leq A_{k+1}(2m + 4) \) for all \( k \geq 2 \) and all \( m \).

PROOF. We start by noting that
\[
\hat{A}_1(m) = 2^{2m-2}m! \leq 2^{m^2}.
\]
Unfortunately the recurrence (55) does not fit the general setting of Lemma C.1 because of the factor \( m \) in it. But it is not hard to show that
\[
\hat{A}_k(m) \leq \hat{A}_{k-1} \left( 2^{2^k} \hat{A}_k(m-1) \right)^2
\]
for \( m \geq 2 \), so we will use this recurrence instead (the penalty we pay is minimal). We are going to apply Lemma C.1 with \( d = 2 \), with
\[
F(m) = \hat{A}_{k-1} \left( 2^{2^k}m \right),
\]
\[
G(m) = A_k(m),
\]
and with the initial conditions \( F_0 = G_0 = 1 \). Thus,
\[
F^\circ(m) \geq \hat{A}_k(m),
\]
\[
G^\circ(m) = A_{k+1}(m).
\]

Let us start with the case \( k = 2 \). In this case, we have, by (56),
\[
F(m) = \hat{A}_1(16m) \leq 2^{512m^2 + 1},
\]
\[
G(m) = 2m.
\]
Then an appropriate choice of \( \delta \) is \( \delta(m) = 600m^3 \), since
\[
\delta(F(m)) \leq \delta \left( 2^{512m^2 + 1} \right) = 600 \cdot 2^{3 \cdot 2^{512m^2 + 1}} \leq 2^{600m^3} = G(G(\delta(m)))
\]
for all \( m \geq 1 \), and so \( \delta \) satisfies (53). Further, it is enough to take \( c = 4 \) in (54), since
\[
G^\circ(4) = 2^{2^2} \geq 515 = \delta(F_0).
\]
We conclude that \( \hat{A}_2(m) \leq A_3(2m + 4) \).

Now we deal with the general case \( k \geq 3 \). Suppose by induction that \( \hat{A}_{k-1}(m) \leq A_k(2m + 4) \). Substituting this into (57),
\[
F(m) \leq A_k \left( 2A_k \left( 2^{2^k}m + 4 \right) + 4 \right)^2.
\]
Now it is easy to see that taking \( \delta(m) = 2^{2^k}m + 5 \) guarantees that
\[
\delta(F(m)) \leq A_k(A_k(\delta(m))) = G(G(\delta(m)))
\]
for all \( m \geq 1 \). Furthermore, we have

\[
G^\circ(4) = A_{k+1}(4) > 2^{2k+1} + 5 = \delta(F_0).
\]

We conclude that \( \hat{A}_k(m) \leq A_{k+1}(2m + 4) \), as desired. \( \square \)

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