Thermal Pions and Isospin Chemical Potential Effects

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Abstract

The density corrections, in terms of the isospin chemical potential \( \mu_I \), to the mass of the pions are investigated in the framework of the \( SU(2) \) low energy effective chiral invariant lagrangian. As a function of temperature and \( \mu_I = 0 \), the mass remains quite stable, starting to grow for very high values of \( T \), confirming previous results. However, the dependence for a non-vanishing chemical potential turns out to be much more dramatic. In particular, there are interesting corrections to the mass when both effects (temperature and chemical potential) are simultaneously present. At zero temperature the \( \pi^\pm \) should condensate when \( \mu_I = \mp m_\pi \). This is not longer valid anymore at finite \( T \). The mass of the \( \pi_0 \) acquires also a non trivial dependence on \( \mu_I \) at finite \( T \).

Pions play a special role in the dynamics of hot hadronic matter since they are the lightest hadrons. Therefore, it is quite important to understand not only the temperature dependence of the pion’s Green functions but also their behavior as function of density. The dependence of the pion mass (and width) on temperature \( m_\pi(T) \) has been studied in a variety of frameworks, such as thermal QCD-Sum Rules [1], Chiral Perturbation Theory (low temperature expansion) [2], the Linear Sigma Model [3], the Mean Field Approximation [4], the Virial Expansion [5]. In fact the properties of pion propagation at finite temperature have been calculated at two loops in the frame of chiral perturbation theory [6]. There seems to be a reasonable agreement that \( m_\pi(T) \) is essentially independent of \( T \), except possibly near the critical temperature \( T_c \) where \( m_\pi(T) \) increases with \( T \).

Let us proceed in the frame of the \( SU(2) \) chiral perturbation theory. The most general chiral invariant expression for a QCD-extended lagrangian, under the presence of external hermitian-matrix auxiliary fields has the form

\[
\mathcal{L}_{QCD}(s,p,v,\mu,a) = \mathcal{L}_{QCD}^0 + \bar{q}\gamma^\mu(v_\mu + \gamma_5 a_\mu)q - \bar{q}(s - i\gamma_5 p)q,
\]

(1)

where \( v_\mu, a_\mu, s \) and \( p \) are vectorial, axial, scalar and pseudoscalar fields. The vector current is given by

\[
J_\mu^a = \bar{q}\gamma_\mu T^a q
\]

(2)

When \( v, a, p = 0 \) and \( s = M \), being \( M \) the mass matrix, we obtain the usual QCD Lagrangian. The effective action with finite isospin chemical potential is given by

\[
\mathcal{L}_{QCD}^I = \mathcal{L}_{QCD}(M,0,0,0) + \mu^a u^\mu J_\mu^a
\]

(3)

where \( \mu^a = (0,0,\mu_I) \) is the third isospin component, \( \mu = \mu^a \tau^a \) and \( u_\mu \) is the 4-velocity between the observer and the thermal heat bath. This is required to describe in a covariant way this system, where the Lorentz invariance is broken since the thermal heat bath represents a privileged frame of reference.

Proceeding in the same way in the low-energy description, where only the pion degrees of freedom are relevant, let us consider the most general chiral invariant lagrangian. This lagrangian is ordered according to a series in powers of the external momentum. We will start with the \( \mathcal{O}(p^2) \) chiral lagrangian

\[
\mathcal{L}_2 = \frac{f^2}{4} Tr \left[ (D_\mu U)^\dagger D^\mu U + U^\dagger \chi + \chi^\dagger U \right]
\]

(4)
with
\[
D_\mu U = \partial_\mu U - i[v_\mu, U] - i\{a_\mu, U\}
\]
\[
\chi = 2B_0(s + ip)
\]
\[
U = e^{i2\pi^*\tau^*/f}
\]
(5)

\(B_0\) in the previous equation is an arbitrary constant which will be fixed when the mass is identified setting \((m_u + m_d)B_0 = m^2\). The most general \(O(p^4)\) chiral lagrangian has the form \(\mathcal{L}_4 = \sum \alpha_i \mathcal{L}_{(\alpha_i)}\) being the SU(3) lagrangian related to the SU(2) one \([2, 3]\). We will work with the general SU(3) \(O(p^4)\) lagrangian. The effective action with finite chemical potential in terms of pion degrees of freedom has the same form as eq.3, where the different external fields are known from loop corrections, up to the fourth order in the fields, respectively, and \(\partial_\tau \equiv \partial + i\mu_\tau u\). This definition of the covariant derivative is natural according to our previous comments, since we know \([3, 4]\) that the chemical potential is introduced as the zero component of an external "gauge" field. In the previous expression, \(|\pi|\) means \(\pi^+\pi^- = \pi^*\), and \(|\partial_\tau \pi|^2\) will be neglected because it only shifts in a small quantity the neutral pion mass and we are interested in the thermal and density evolution of the masses.

For renormalizing with counterterms we introduce the following decomposition
\[
\mathcal{L}_{eff} = \mathcal{L}_{2,2}^I + \mathcal{L}_{2,4}^I + \mathcal{L}_{4,2}^I
\]
(8)
where the \(r\) index denote the lagrangian with renormalized fields. These changes are related to higher loop corrections.

Setting \(\pi_0 = \sqrt{Z_0}\pi^*_0\) and \(\pi_{\pm} = \sqrt{Z_{\pm}}\pi_{\pm}^*\) in \(\mathcal{L}_{2,2}\), we have
\[
\delta \mathcal{L} = \frac{1}{2} \delta Z_0 \left[ (|\partial_\tau \pi_0^*|^2 - m^2(\pi_0^*)^2) \right] + \delta Z_{\pm} \left[ (|\partial_\tau \pi_{\pm}^*|^2 - m^2|\pi_{\pm}^*|^2) \right]
\]
(9)
with \(\delta Z_i = Z_i - 1\).

First, let us consider the thermal and density corrections to the propagator, the two-point function pion correlator. Since our calculation will be at the one loop level, we do not need the full formalism of thermo field dynamics, including thermal ghosts and, therefore, matrix propagators. The propagator in momentum space \(D(k; T, \mu_\tau, m^2) = D(k; \mu_\tau, m^2) + D_T(k, \mu_\tau, m^2)\) for charged pions at the tree level will be given by an extension, for a non-vanishing chemical potential, of the well known Dolan-Jackiw propagators for scalar fields \([3]\). Note that since there is no chemical potential associated to the neutral pion, the thermal propagator \(D_0\) will be the usual one \(D_0(k; T, \mu_\tau, m^2) = D(k, T, 0, m^2)\) with
\[
D(k; \mu_\tau, m^2) = \frac{i}{k^2 + 2m^2 + i\epsilon}
\]
\(D_T(k; \mu_\tau, m^2) = 2\pi n_B(|k \cdot u|)\delta(k^2 - m^2)(10)\)
where \( k_\perp \equiv k \mp \mu_1 u \) and \( n_B(x) = (e^{x/T} - 1)^{-1} \) is the Bose-Einstein factor.

We will use the \( MS \)-scheme, and we renormalize as usual at \( T = 0 \), since the thermal corrections are finite. The self energy for charged pions including the counterterms has the form

\[
\Sigma(p) = [A - \delta Z_\pi] p_\perp^2 - [A' - \delta Z_\pi] m^2 + A'' u \cdot p_+ \quad (11)
\]

Our prescription to fix the counterterm \( \delta Z_\pi \) is to impose that \( \Sigma \) does not depend on \( p_\perp^2 \), so, \( \delta Z_\pi = A \). In this way, the inverse propagator will take the form \( i[D(p; \mu_1, m^2 + \Sigma)]^{-1} = p^2 + C p_0 + C' \) in the frame where the heat bath is at rest \((u = (1, 0, 0, 0))\).

The \( C' \) term will give two solutions to \( i[D(p_0, \mathbf{p} = 0; \mu_1, m^2 + \Sigma)]^{-1} = 0 \) that we identify as \( m_{\pi^+} \) and \( m_{\pi^-} \). The \( \alpha_i = \alpha_i^*(A) - \frac{1}{24\pi^2} \left[ \frac{2}{3} - \ln 4\pi + \gamma - 1 \right] \) absorb the divergences. The \( \gamma_i \) terms are tabulated \( \cite{10} \). The well known result for \( T = \mu_1 = 0 \)

\[
m_\pi^2 = m^2 \left[ 1 - g + \frac{32\pi^2}{\Lambda^2} \left( a_+ - b_+ \right) + \ln \frac{m_\pi^2}{\Lambda^2} \right] \quad (12)
\]

is identified with the physical mass with \( \Lambda \) a scale factor. \( g = m_\pi^2 / 32\pi^2 f_\pi^2 \) is the perturbative term that fixes the scale of energies in the theory (for energies below \( 32\pi^2 f_\pi^2 \)) so we neglect the \( O(g^2) \) factors. This allows us to set \( m \approx m_\pi \) in all radiative corrections (and also \( f \approx f_\pi \)). The procedure is the same for \( m_{\pi^0} \).

It is important to remark that radiative corrections will leave a dependence on the chemical potential for the pion mass only for finite values of temperature. In a strict sense, this procedure does not allow us to say nothing new for an eventual chemical potential dependence of the masses at \( T = 0 \) (cold matter) which is already included in \( L_\pi \). In this case, \( T = 0 \), we have to follow the usual procedure, \( \cite{11} \), of computing the minimum of the effective potential in \( L_\pi \) when the chemical potential is taken into account, without considering radiative corrections. This enables to identify a phase structure where a non trivial vacuum appears for higher values of \( \mu_1 \), characterized by the appearance of a condensate \( \langle \pi^- \rangle \). The opposite occurs for negative values of the chemical potential, where the vacuum state is a condensate \( \langle \pi^+ \rangle \) for \( |\mu_1| \approx m_\pi \). At \( T = 0 \) when \( \mu_1 = m_\pi \), the mass of \( \pi^- \) vanishes.

For finite \( T \) and \( \mu_1 \), we find the following expression for the masses

\[
m_{\pi^\pm}(T, \mu_1) = m_\pi \left[ 1 + 2g I(0) \pm \frac{\mu_1}{m_\pi} - 8g T \right] \quad (13)
\]

with

\[
I(\mu_1) = \int \frac{d^4 k}{(2\pi m_\pi)^2} n_B(|k_0 + \mu_1|) \delta(k^2 - m_\pi^2) \quad (14)
\]

Note that our convention for the chemical potential is contrary to the one adopted in the paper by Kogut and Toublan \( \cite{16} \), who extended previous results by Son and Stephanov \( \cite{15} \).

If the chemical potential of the charged pions vanishes, i.e for symmetric matter, at finite \( T \) we get the well known result for \( m_\pi(T) \) due to chiral perturbation theory \( \cite{4} \), see also \( \cite{8} \). However, due to radiative corrections to the neutral pion propagator, its mass will acquire a non trivial chemical potential dependence for finite values of temperature. In the approach where the minimum of the effective potential is calculated (for finite \( \mu_1 \) and \( T = 0 \), the mass of the neutral pion remains constant.

We show in Fig.\( \ref{fig1} \) a tridimensional picture for the behavior of the mass of the neutral pion. Note that when \( \mu_1 = 0 \), \( m_{\pi^0}(T) = m_{\pi^\pm}(T) \).

From Fig.\( \ref{fig2} \) we see that at zero temperature, we agree with the usual prediction, \( m_\pi^+ = m_\pi + \mu_1 \). In fact, at zero temperature the \( \pi^+ \) should condensate when \( \mu_1 = -m_\pi \) (the inverse situation occurs for \( \pi^- \)). Now, this situation changes if temperature starts to grow. The condensation point disappear at \( \mu_1 = -m_\pi \); in \( \mu_1 = m_\pi \) the mass start to decrease . For small \( T \) (for example inside an neutron star), we cannot see such effect.

In order to explore the region where \( |\mu_1| > m_\pi \), associated to a new phase where the condensates occur, we need to redefine our fields as fluctuations around the configuration corresponding to a minima
of the effective potential in $\mathcal{L}_2$.

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