NOTE ON LINEAR RELATIONS IN ÉTALE K-THEORY OF CURVES

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Abstract. In this paper we investigate a local to global principle for étale K-theory of curves. More precisely, we show that the result obtained by G.Banaszak and the author in [BK13] describing the sufficient condition for the local to global principle to hold is the best possible (i.e this condition is also necessary). We also give examples of curves that fulfill the assumptions imposed on the Jacobian of the curve. Finally, we prove the dynamical version of the local to global principle for étale K-theory of a curve. The dynamical local to global principle for the groups of Mordell-Weil type has recently been considered by S.Baraniczuk in [B17]. We show that all our results remain valid for Quillen K-theory of X if the Bass and Quillen-Lichtenbaum conjectures hold true for X.

1. Introduction

The local to global type questions are of mathematical interest since the celebrated Hasse principle was proven. For the history of these type of problems in number theory and its extensions to the context of abelian varieties and linear algebraic groups see [FK17].

In [BK13] G.Banaszak and the author proved a sufficient condition for the local to global condition to hold for étale K-theory of curves. The main result of the current paper is a proof that this condition is also necessary (cf. Theorem 1.2).

To state this result let us recall the following definitions and notations from [BK13]. Let $X/F$ be a smooth, proper and geometrically irreducible curve of genus $g$ defined over a number field $F$ and let $J$ be the Jacobian of $X$.

Definition 1.1. We call a finite field extension $F'/F$ an isogeny splitting field of the Jacobian $J$ if $J$ is isogenous over $F'$ to the product $A_1^{e_1} \times \cdots \times A_t^{e_t}$ where $A_1, \ldots, A_t$ are pairwise nonisogenous, absolutely simple abelian varieties defined over $F'$.

Remark 1.1. Notice that $J$ is an abelian variety and therefore by Poincaré decomposition theorem the isogeny splitting field exists.

Let $\tilde{V}_{l,i}$ denote $T_i(A_i)(n) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$, where $T_i(A_i)(n)$ is the $n$-th twist of the $l$-adic Tate module of $A_i$ (cf. [T68]).

The main result of [BK13] is the following theorem:

Theorem 1.1. Let $X/F$ be a smooth, proper and geometrically irreducible curve of genus $g$. Let $F'$ be an isogeny splitting field of the Jacobian $J$ and assume that for the corresponding product $A_1^{e_1} \times \cdots \times A_t^{e_t}$, we have $\text{End}_{\mathbb{Q}_l} A_i = \mathbb{Z}$ for each $1 \leq i \leq t$. Let $l > 2$ be a prime

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number which is coprime to the polarisation degrees of the abelian varieties $A_i$. Let $S_l$ be a set of places of $F$ containing the places of bad reduction, the archimedean places and the primes above $l$. Let $S$ be a set of places of $F$ containing the places of bad reduction, the archimedean places and the primes above $l$. Let $\hat{P} \in K_{2n}^e(X)$ and let $\hat{\Lambda}$ be a finitely generated $\mathbb{Z}_l$-submodule of $K_{2n}^e(X)$.

If $r_v(\hat{P}) \in r_v(\hat{\Lambda})$ for almost all $v$ of $\mathcal{O}_{F,S_l}$ then $\hat{P} \in \hat{\Lambda} + K_{2n}^e(X)_{\text{tor}}$.

We prove the following theorem

**Theorem 1.2.** Let $X/F$ be a curve as in Theorem 1.1. Then the numerical bound in Theorem 1.1 is the best possible i.e if $\dim_{\mathbb{Q}_l} \tilde{V}_{1,i} < e_i$ then local to global principle of Theorem 1.1 does not hold.

**Remark 1.2.** Theorem 1.2 shows that the condition of Theorem 1.1 is also necessary.

**Remark 1.3.** Without loss of generality in the proof of Theorem 1.2 we may assume $F' = F$. From now on we assume this.

In section 2 we give some examples of Jacobians of curves satisfying the assumptions of Theorem 1.1. Sections 3, 4 and 5 contain necessary background from Galois cohomology, intermediate Jacobians and Kummer theory. In section 6 we prove our main theorem (Theorem 1.2). We also show that we can replace étale $K$-theory by Quillen theory of a curve provided Bass and Quillen-Lichtenbaum conjectures hold true. Section 7 is devoted to proof of the dynamical version of the local to global principle. This is done by checking the axioms for the dynamical local to global principle introduced in [B17]. We check these axioms for Galois cohomology (cf. Lemma 7.3) and then pass to the étale $K$-theory.

# 2. Examples

In this section we give examples of curves whose Jacobian decomposes as $A_1^{e_1} \times \ldots A_t^{e_t}$, with $\text{End}_{\mathbb{F}_l} A_i = \mathbb{Z}$. In general deciding whether the Jacobian of a curve splits is a difficult problem and has vast literature (cf. e.g. [HN65], [ES93]) However, we have the following [CO12], p.589:

**Theorem 2.1.** Any abelian variety $A$ defined over $\overline{\mathbb{Q}}$ of dimension $1 \leq g \leq 3$ is isogenous to $J(C)$, where $C$ is a curve of genus $g$.

**Remark 2.1.** Genus 1 case of Theorem 2.1 is trivial, genus 2 is [W57] Satz 2 and $g = 3$ is Theorem 4 of [OU73].

**Example 2.2.** Let $A/F$ be principally polarized abelian surface with $\text{End}_{\mathbb{F}} A_i = \mathbb{Z}$ then by Theorem 2.1 there exists a curve $X$ defined over $F' \subset \overline{\mathbb{Q}}$ such $J(X)$ is isogenous to $A$.

**Example 2.3.** Let $B = A \times E$, where $A$ is as in Example 2.2 and $E$ an elliptic curve without complex multiplication then there exists a curve $X$ such that $J(X)$ is isogenous to $B$.

A special case of Example 2.2 is given in the following (see [BPPT18] section 7.1 p. 39):

**Example 2.4.** Let $X/\mathbb{Q}$ be a smooth projective curve given by the following equation

$$(2.1) \quad X : \quad y^2 + (x^3 + x^2 + x + 1)y = -x^2 - x.$$  

Then $J(X)$ is a principally polarized abelian isotypical surface over $\mathbb{Q}$ of conductor 277.
Remark 2.5. Since the \( j \) invariant of a curve with complex multiplication is an algebraic integer (cf. [C89], Theorem 11.1), in Example 2.3 it is enough to pick an elliptic curve defined over \( \mathbb{Q} \) whose \( j \) invariant is non-integral rational number.

The following lemma guarantees that there are many simple abelian surfaces \( A \) with \( \text{End}_\mathbb{F}A = \mathbb{Z} \)

**Lemma 2.2.** ([BPP18], Lemma 4.1.1) Let \( A \) be a simple, semistable abelian surface over \( \mathbb{Q} \) with non-square conductor then \( A \) is isotypical, i.e. \( \text{End}_\mathbb{Q}A = \mathbb{Z} \).

### 3. NECESSARY RESULTS CONCERNING \( \acute{e} \)TAL E \( K \)-THEORY OF CURVES AND COHOMOLOGY

We start with the definition of continuous cohomology [J88] and [DF85].

**Definition 3.1.** Let \( Y \) be a scheme over \( \mathbb{Z} \) and let \((\mathcal{F}_m)\) be a projective system of \( \mathbb{Z}/l^m \) - \( \acute{e} \)tale sheaves \( \mathcal{F}_m \). The functor \( (\mathcal{F}_m) \to \lim \limits_{\leftarrow} H^0(Y, (\mathcal{F}_m)) \) is left exact and its \( i \)-th right derived functor is by definition the continuous cohomology group \( H^i_{\text{cts}}(Y, (\mathcal{F}_m)) \).

The \( \acute{e} \)tale \( K \)-theory spaces and groups were defined in [DF85] by W.Dwyer and E.Friedlander. In [BGK99] using the spectral sequences of [DF85]:

\[
E_2^{p,-q} = H^p_{\text{cont}}(Y, \mathbb{Z}_l(q/2)) \Rightarrow K^e_{q-p}(Y) \\
E_2^{p,-q} = H^p_{\text{et}}(Y, \mathbb{Z}/l^k(q/2)) \Rightarrow K^e_{q-p}(Y, \mathbb{Z}/l^k).
\]

it was shown that one has the following exact sequence connecting continuous cohomology of \( X \) and the Galois cohomology of the Galois group of the maximal unramified outside \( S \)l extension of \( F \)

\[
0 \to H^2(G_{S_l}; \mathbb{Z}_l(k)) \to H^{2*}_{\text{cts}}(X; \mathbb{Z}_l(k)) \to H^1(G_{S_l}; T_l(J)(k-1)) \to 0.
\]

as well as existence of the following commutative diagram

\[
\begin{align*}
K^e_{2n}(X) & \xrightarrow{r_v} K^e_{2n}(X_v) \\
H^1(G_{S_l}; T_l(J)(n)) & \xrightarrow{r_v} H^1(g_v; T_l(J_v)(n)).
\end{align*}
\]

The right hand vertical arrow in the diagram (3.2) is an isomorphism whereas the left vertical arrow is an epimorphism with finite kernel (cf. [BK13], [BGK99], [BG08]).

One has the following Dwyer-Friedlander homomorphisms, for odd prime \( l \) and \( k \geq 1 \), connecting Quillen \( K \)-theory with the \( \acute{e} \)tale \( K \)-theory with coefficients (cf. [DF85])

\[
h_{1k} : K_m(X, \mathbb{Z}/l^k \mathbb{Z}) \to K^e_m(X, \mathbb{Z}/l^k \mathbb{Z})
\]

\[
\overline{h}_{1k} : K_m(X_v, \mathbb{Z}/l^k \mathbb{Z}) \to K^e_m(X_v, \mathbb{Z}/l^k \mathbb{Z}).
\]

In the sequel we assume the following
Conjecture 3.1. *(Bass conjecture)* The Quillen $K$-theory groups $K_m(\mathcal{X})$ are finitely generated for $m > 0$.

Assuming Conjecture 3 we obtain the following equality

$$\lim_{\leftarrow} K_m(\mathcal{X}, \mathbb{Z}/l^n\mathbb{Z}) = K_m(\mathcal{X}) \otimes \mathbb{Z}_l.$$  

(3.5)

The Dwyer-Friedlander homomorphisms (3.3) and (3.4) induce the following homomorphisms

$$h : K_{2n}(\mathcal{X}) \otimes \mathbb{Z}_l \to K^\text{et}_{2n}(\mathcal{X})$$  

(3.6)

$$\overline{h} : K_{2n}(\mathcal{X}_v) \otimes \mathbb{Z}_l \to K^\text{et}_{2n}(\mathcal{X}_v)$$  

(3.7)

The maps (3.6) and (3.7) are isomorphisms if we assume that the Quillen-Lichtenbaum conjecture holds true for $\mathcal{X}$. Thus we obtain the following commutative diagram

$$\begin{array}{ccc}
K_{2n}(\mathcal{X}) & \xrightarrow{rv} & K_{2n}(\mathcal{X}_v) \\
\downarrow h' & & \downarrow \overline{h'} \\
K^\text{et}_{2n}(\mathcal{X}) & \xrightarrow{rv} & K^\text{et}_{2n}(\mathcal{X}_v)
\end{array}$$

where the map $h'$ (resp. $\overline{h'}$) is the composition of the natural homomorphism $K_{2n}(\mathcal{X}) \to K_{2n}(\mathcal{X}_v) \otimes \mathbb{Z}_l$ (resp. $K_{2n}(\mathcal{X}_v) \to K_{2n}(\mathcal{X}_v) \otimes \mathbb{Z}_l$) with $h$ (resp. $\overline{h}$).

Concatenation of diagrams (3.9) and (3.8) yields the following commutative diagram

$$\begin{array}{ccc}
K_{2n}(\mathcal{X}) & \xrightarrow{rv} & K_{2n}(\mathcal{X}_v) \\
\downarrow & & \downarrow \\
H^1(G_{S_l}; T_1(J)(n)) & \xrightarrow{rv} & H^1(g_v; T_1(J_v)(n))
\end{array}$$

with the vertical maps having finite kernels.

4. Intermediate Jacobian

Let $L$ be a finite extension of $F$. For $w \notin S_l$, let $G_w := G(\overline{L}/L_w)$ be the absolute Galois group of the completion $L_w$ of $L$ at $w$. Let $k_w$ be the residue field at $w$. Put $H^1_f(G_w; \tilde{T}_{l,i}) = i_w^{-1}H^1_f(G_w; \tilde{V}_{l,i})$. Here $i_w : H^1(G_w; \tilde{T}_{l,i}) \to H^1(G_w; \tilde{V}_{l,i})$ and $H^1_f(G_w; \tilde{V}_{l,i}) = \ker(res : H^1(G_w; \tilde{V}_{l,i}) \to H^1(I_w; \tilde{V}_{l,i}))$ where $I_w \subset G_w$ is the inertia subgroup and $res$ is the restriction map. Let

$$H^1_{f,S_l}(G_L; \tilde{T}_{l,i}) = \ker(H^1(G_L; \tilde{T}_{l,i}) \to \prod_{w \notin S_l} H^1(G_w; \tilde{T}_{l,i})/H^1_f(G_w; \tilde{T}_{l,i})).$$

(4.1)

We define the intermediate Jacobian (see [BGK05], [BGK03]):

$$J_{f,S_l}(\tilde{T}_{l,i}) = \lim_{L'/L} H^1_{f,S_l}(G_{L'}/L', \tilde{T}_{l,i}).$$

(4.2)

In [BGK05], [BGK03], in the more general situation of any free $\mathbb{Z}_l$-module of finite rank $T_l$, we made the following:
Assumptions 4.1. For any finite extension $L/F$ and any place $w \in \mathcal{O}_L$ such that $w \notin S_i$, we have $T_{i}^{Fr_w} = 0$.

Remark 4.1. We have $H^1_{f,S_i}(G_L; \tilde{T}_{i}(J)) \cong H^1(G_{L,S_i}; \tilde{T}_{i}(J))$ and the assumption [4.1] is satisfied (cf. [BGK05], p.5).

By [BGK03, Prop. 2.14] we have the following isomorphisms

\begin{equation}
J_{f,S_i}(\tilde{T}_{i}) \cong \bar{V}_{i}/\tilde{T}_{i} \cong V_i(A_i)/T_i(A_i)(n) \cong A_i[\ell^n](n).
\end{equation}

Let $\tilde{T}_i := \bigoplus_{i=1}^{t} \tilde{T}_{i,i}$. Since each $\tilde{T}_{i,i}$ is a free $\mathbb{Z}_\ell\otimes n$-module we have the following natural isomorphism:

\begin{equation}
J_{f,S_i}(\tilde{T}_i) \cong \bigoplus_{i=1}^{t} J_{f,S_i}(\tilde{T}_{i,i}).
\end{equation}

For a finite extension $L/F$ the following map corresponding to these used in the direct systems (4.2)

\begin{equation}
i^* : H^1(G_{F,S_i}, T_i(J)(n)) \rightarrow H^1(G_{L,S_i}, T_i(J)(n))
\end{equation}

is an injection. One readily verifies, using transfer in the Galois cohomology, that the map (4.5) has $l$-torsion kernel. But the $l$-torsion part of (4.5) is just

\begin{equation}
H^0(G_{F,S_i}, V_i(J)/T_i(J)(n)) \rightarrow H^0(G_{L,S_i}, V_i(J)/T_i(J)(n)),
\end{equation}

which is clearly injective. Similarly, the reduction of the map (4.5)

\begin{equation}
H^1(g_w, T_i(J_v)(n)) \rightarrow H^1(g_w, T_i(J_v)(n))
\end{equation}

is injective for any finite extension $L/F$ and any prime $w$ of $\mathcal{O}_L$ over $v$, $v \notin S_i$. Notice that by Definition [4.1] we have

\begin{equation}
H^1(G_{F,S_i}; T_i(J)(n)) \cong H^1(G_{F,S_i}; T_i(A)(n)) = \prod_{i=1}^{t} H^1(G_{F,S_i}; T_i(A_i)(n))^e_i.
\end{equation}

5. Kummer theory

Kummer theory in the context of abelian variety was developed in [R79]. In this section we collect necessary facts which will be useful in section 7.
**Definition 5.1.** Let $A$ be an abelian variety over a number field $F$. Let $\mathcal{R} = \text{End}_F A$. For $\alpha \in \mathcal{R}$ we set $F_\alpha = F(A[\alpha])$ and $G_\alpha = \text{Gal}(F_\alpha/F)$. The Kummer map

$$\psi^\alpha : A(F)/\alpha A(F) \to \text{Hom}_{G_\alpha}(G_{F_\alpha}, A[\alpha])$$

is defined as the composition

$$A(F)/\alpha A(F) \hookrightarrow H^1(F, A[\alpha]) \xrightarrow{\text{res}} H^1(F_\alpha, A[\alpha])^{G_\alpha}$$

with the first map a coboundary map for the $G_{F_\alpha}$-cohomology of the Kummer sequence

$$0 \to A[\alpha] \to A(F) \xrightarrow{\alpha} A(F) \to 0$$

and the second map restriction to $F_\alpha$.

Explicitly, the map (5.1) is given by the following formula

$$\psi^\alpha(x)(\sigma) = \sigma(x) - x_\alpha,$$

where $x_\alpha$ is a fixed ”$\alpha$-root” of $x$ i.e. an element $y \in A(F)$ such that $\alpha y = x$.

We are interested in Kummer maps where $l$ is a rational prime and $\alpha = l^k \in \mathbb{Z} \hookrightarrow \mathcal{R}$. Thus we have the the family of Kummer maps:

$$\psi_{l^k} : A(F)/l^k A(F) \to \text{Hom}_{G_{l^k}}(G_{F_{l^k}}, A[l^k]).$$

The maps (5.5) are compatible with the natural maps induced by multiplication by $l$. Therefore taking the inverse limit of both sides and twisting it with $\mathbb{Z}_l(n)$ yields the map

$$A(F) \otimes \mathbb{Z}_l(n) \to \text{Hom}(G_{F_{l^\infty}}, T_l(A)(n)),$$

where $F_{l^\infty} = \bigcup_k F_{l^k}$ and $G_{F_{l^\infty}} = \text{Gal}(\bar{F}/F_{l^\infty})$.

### 6. Proof of Theorem 1.2

**Proof.** We have the following commutative diagram involving the Mordell-Weil groups of an abelian variety and the reduction maps

$$\xymatrix{ A(F) \otimes \mathbb{Z}_l(n) \ar[r]^-{\psi_{F,l} \otimes \mathbb{Z}_l(n)} \ar[d]_-{\tau_v} & H^1(G_{F,S}; T_l(J)(n)) \ar[d]_-{\tau_v} \\
A(k_v) \otimes \mathbb{Z}_l(n) \ar[r] & H^1(g_v; T_l(J_v)(n)) },$$

where the map $\psi_{F,l} \otimes \mathbb{Z}_l(n)$ is a natural imbedding which comes from the Kummer map (5.6) ( see [BGK03] discussion on p.148 ). Our proof is a generalization of the counterexample to local - global principle for abelian varieties constructed by P. Jossen and A.Perucca in [JP10] and later extended to the context of $t$-modules in [BoK18]. Because of (4.8) and the choice of $F$ we may assume that $A = A_1^{e_1}$ where $A_1$ is a geometrically simple abelian variety. So, let $e_1 = d_1 + 1$ where $d_1 = \dim \tilde{V}_{l,1}$. Let $P_1, \ldots, P_e \in H^1(G_{F,S}; T_l(A)(n)) = \cdots$
H^1(G_{F,S_i}; T_l(A_1)(n))^{e_1}$ be points linearly independent over $\mathbb{Z}_l(n)$ which are in the image of $A(F) \otimes \mathbb{Z} \hookrightarrow A(F) \otimes \mathbb{Z}_l(n)$. Let

$$P := \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_e \end{bmatrix}, \quad \Lambda := \{ MP : M \in \text{Mat}_{e \times e}(\mathbb{Z}_l(n)), \quad \text{tr}M = 0 \}.$$  

Let $v \in \mathcal{O}_{F,S_i}$ be a prime of good reduction for $A_1$ and $\overline{P} = [\overline{P}_1, \ldots, \overline{P}_e]^T$, $\overline{P}_i \in H^1(g_v; T_l(A_v)(n))$ be the reduction mod $v$ of $P$. Notice that $P \notin \Lambda$ since $P_1, \ldots, P_e$ are $\mathbb{Z}_l(n)$-linearly independent. We will find a matrix $M \in \text{Mat}_{e \times e}(\mathbb{Z}_l(n))$ such that $\overline{P} = M\overline{P}$. This will show that $\text{red}_v P \in \text{red}_v \Lambda$. Since the group $H^1(g_v; T_l(A_v)(n))$ is finite there exist $\alpha_1, \ldots, \alpha_e \in \mathbb{Z}$ with $\alpha_i$ minimal such that

$$\begin{align*}
\alpha_1 \overline{P}_1 + m_{1,2} \overline{P}_2 + \cdots + m_{1,e} \overline{P}_e &= 0 \\
m_{2,1} \overline{P}_1 + \alpha_2 \overline{P}_2 + \cdots + m_{2,e} \overline{P}_e &= 0 \\
\cdots & \\
m_{e,1} \overline{P}_1 + m_{e,2} \overline{P}_2 + \cdots + \alpha_e \overline{P}_e &= 0.
\end{align*}$$

We will show that $D = \gcd(\alpha_1, \ldots, \alpha_e) = 1$. Assume opposite. Choose a rational prime $p$ that divides $D$. This means, by our choice of $\alpha_1, \ldots, \alpha_e$, that $p$ divides coefficients of any linear combination of points $\overline{P}_1, \ldots, \overline{P}_e \in H^1(g_v; T_l(A_v)(n))$. In particular $p$ divides the orders of $\overline{P}_i, i = 1, \ldots, e$. The $p$-torsion of $A_v(k_v)$ is generated by at most $d$-elements. Therefore the group $Y = (P_1, \ldots, P_e) \cap H^1(g_v; T_l(A_v)(n))$ is generated by fewer than $e$ elements. We may assume without loss of generality that $Y = (P_2, \ldots, P_e) \cap H^1(g_v; T_l(A_v)(n))$ Let $\alpha_1 = x_1 p$ and

$$\alpha_1 \overline{P}_1 + x_2 p \overline{P}_2 + \cdots + x_e p \overline{P}_e = 0$$

be a linear relation. Therefore

$$x_1 \overline{P}_1 + x_2 \overline{P}_2 + \cdots + x_e \overline{P}_e = T.$$  

But $T \in A(k_v)_p$ is generated by $\overline{P}_2, \ldots, \overline{P}_n$. Thus we obtain a contradiction with the minimality of $\alpha_1$.

Hence there exist $a_1, \ldots, a_e \in \mathbb{Z}$ such that

$$e = a_1 \alpha_1 + \cdots + a_e \alpha_e.$$  

Put $m_{i,i} = 1 - a_i \alpha_i$. Then $m_{1,1} + \cdots + m_{e,e} = 0$ and

$$\begin{bmatrix} m_{1,1} & \cdots & m_{1,e} \\ \cdots & \cdots & \cdots \\ m_{e,1} & \cdots & m_{e,e} \end{bmatrix} \begin{bmatrix} \overline{P}_1 \\ \cdots \\ \overline{P}_e \end{bmatrix} = \begin{bmatrix} \overline{P}_1 \\ \cdots \\ \overline{P}_e \end{bmatrix}.$$  

Therefore $\overline{P} \in \Lambda$. We view the matrix $M$ in (6.6) as the matrix with the $\mathbb{Z}_l(n)$ coefficients via the obvious map $\mathbb{Z} \rightarrow \mathbb{Z} \otimes \mathbb{Z}_l(n)$, $m \rightarrow m \otimes t^n$, where $t^n$ is the generator of $\mathbb{Z}_l(n)$. □
7. Local to Global Principle and Dynamical Systems

S. Barańczuk in \cite{B17} considers abelian groups satisfying the following two axioms.

**Assumptions 7.1.** Let $B$ be an abelian group such that there are homomorphisms $r_v : B \to B_v$ for an infinite family $v$, whose targets $B_v$ are finite abelian groups.

1. Let $l$ be a prime number, $(k_1, \ldots, k_m)$ a sequence of nonnegative integers. If $P_1, \ldots, P_m \in B$ are points linearly independent over $\mathbb{Z}$, then there is a family of primes $v$ in $F$ such that $l^{k_i} \mid \ord_v P_i$ if $k_i > 0$ and $l \nmid \ord_v P_i$ if $k_i = 0$.
2. For almost all $v$ the map $B_{\text{tors}} \to B_v$ is injective.

**Remark 7.1.** Notice assumption (1) describes the assertion of the Reduction Theorem of \cite{K1}, whereas assumption (2) is one of the assumptions in \cite{BGK03}.

Here $\ord_v P$ is the order of a reduced point $P \mod v$. Under the Assumptions\footnote{7.1} S. Barańczuk was able to prove the following dynamical version of the local to global principle

**Theorem 7.2.** (\cite{B17}) Let $\Lambda$ be a subgroup of $B$ and $P \in B$ be a point of infinite order and $\phi$ be a natural number. Then the following are equivalent:

i) For almost every $v$

\[ O_\phi(P \mod v) \cap (\Lambda \mod v) \neq \emptyset, \]

ii) $O_\phi(P) \cap (\Lambda) \neq \emptyset$.

Here $O_\phi(P) = \{ \phi^n(P) : n \geq 0 \}$. We have the following lemma

**Lemma 7.3.** Let $A = A_1^{e_1} \times \cdots \times A_t^{e_t}, i = 1, \ldots, t$ be an abelian variety such that $\End A_i = \mathbb{Z}$ for $i = 1, \ldots, t$. Let $B = H^1(G_{F,S}; T_i(A)(n)) = \prod_{i=1}^t H^1(G_{F,S}; T_i(A_i)(n))^{e_i}$, and $B_v = H^1(g_v; T_i(J_v)(n))$ for $v \in \mathcal{O}_v$ fulfill Assumption 7.1.

**Proof.** Assumption (1) of 7.1 is a specialization to $\mathcal{R}_i = \mathbb{Z}$ of Corollary 3.5 of \cite{BK13}. Assumption (2) is an assertion of Lemma 4.2.

**Theorem 7.4.** Let $X/F$ be a smooth, proper and geometrically irreducible curve of genus $g$. Let $F^v$ be an isogeny splitting field of the Jacobian $J$ and assume that for the corresponding product $A_1^{e_1} \times \cdots \times A_t^{e_t}$, we have $\End_{F^v} A_i = \mathbb{Z}$ for each $1 \leq i \leq t$. Let $l > 2$ be a prime number which is coprime to the polarisation degrees of the abelian varieties $A_i$. Let $S_i$ be a set of places of $F$ containing the places of bad reduction, the archimedean places and the primes above $l$. Let $X$ be a regular and proper model of $X$ over $\mathcal{O}_{F,S_i}$. Let $\hat{P} \in K_2^{et}(X)$ and let $\hat{\Lambda}$ be a finitely generated $\mathbb{Z}_l$-submodule of $K_2^{et}(X)$. Let $O_w = \{ w^n(P) : n \geq 0 \}$ Then the following are equivalent

1. For almost every prime $v \in \mathcal{O}_{F,S_i}$

\[ O_w(r_v(P)) \cap r_v(\Lambda) \neq \emptyset, \]

2. $O_w(P) \cap \Lambda \neq \emptyset$.

**Proof.** The proof follows from Lemma 7.3, commutativity of the diagram (3.2) and finiteness of the kernel of the left vertical map in this diagram.
Remark 7.2. If the Bass Conjecture and Quillen-Lichtenbaum Conjecture hold true for $\mathcal{X}$ then we obtain (using diagram (3.9) instead of (3.2)) the corresponding to Theorem 7.4 statement for Quillen $K$-theory of $\mathcal{X}$.

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