Ultradistributions and The
Fractionary Schrödinger Equation *

A. L. De Paoli and M. C. Rocca
Departamento de Física, Fac. de Ciencias Exactas,
Universidad Nacional de La Plata.
C.C. 67 (1900) La Plata. Argentina.
December 15, 2009

Abstract

In this work, we generalize the results of Naber about the Fractionary Schrödinger Equation with the use of the theory of Tempered Ultradistributions. Several examples of the use of this theory are

*This work was partially supported by Consejo Nacional de Investigaciones Científicas; Argentina.
given. In particular we evaluate the Green’s function for a free parti-
cle in the general case.

PACS: 03.65.-w, 03.65.Bz, 03.65.Ca, 03.65.Db.
1 Introduction

The properties of ultradistributions (ref.[6,7]) are well adapted for their use in fractional calculus. In this respect we have shown that it is possible (ref.[2]) to define a general fractional calculus with the use of them.

Ultradistributions have the advantage of being representable by means of analytic functions. So that, in general, they are easier to work with them.

They have interesting properties. One of those properties is that Schwartz tempered distributions are canonical and continuously injected into tempered ultradistributions and as a consequence the Rigged Hilbert Space with tempered distributions is canonical and continuously included in the Rigged Hilbert Space with tempered ultradistributions.

Fractional calculus has found motivations in a growing area concerning general stochastic phenomena. These include the appearance of alternative diffusion mechanisms other than Brownian, as well as classical and quantum mechanics formalisms including dissipative forces, and therefore allowing an extension of the quantization schemes for non-conservative systems [3]. In particular it is interesting to study the fractional Schrödinger equation. Our aim is to extend a previous study [1] about this equation. Using an analytical definition of fractional derivative [2] we show here that it is possible to obtain
a general solution for the time fractional equation, for any complex value of
the derivative index. Furthermore the associated Green functions can be
evaluated in a straightforward way.

This paper is organized as follow:
In section 2 we define the fractional Schrödinger equation for all \( \nu \) complex
with the use of the fractional derivative defined via the theory of tempered
ultradistributions. In section 3 we solve this equation for the free particle and
give three examples: \( \nu = 1/2, \nu = 1 \) and \( \nu = 2 \). In section 4 we realize the
treatment of the potential well and we analyze the cases \( \nu = 1/2, \nu = 1 \) and
\( \nu = 2 \). In section 5 we study the Green fractional functions for the free parti-
cle in three cases: the retarded Green function, the advanced Green function
and the Wheeler-Green function. As an example we prove that for \( \nu = 1 \)
these functions coincide with the Green functions of usual Quantum Mech-
nics. In section 6 we discuss the results obtained in the previous sections.
Finally we have included three appendixes: a first appendix on distributions
of exponential type, a second appendix on tempered ultradistributions and
a third appendix on fractional calculus using ultradistributions.
2 The Fractional Schrödinger Equation

Our starting point in the study of the fractional Schrödinger equation is the current known Schrödinger equation:

\[ i\hbar \partial_t \psi(t, x) = -\frac{\hbar^2}{2m} \partial_x^2 \psi(t, x) + V(x)\psi(t, x) \]  

(2.1)

According to ref. [1], (2.1) can be written as:

\[ iT_p \partial_t \psi(t, x) = -\frac{L_p^2 M_p}{2m} \partial_x^2 \psi(t, x) + \frac{V(x)}{E_p} \psi(t, x) \]  

(2.2)

where \( L_p = \sqrt{\hbar G/c^3} \), \( T_p = \sqrt{\hbar G/c^3} \), \( M_p = \sqrt{\hbar c/G} \) and \( E_p = M_p c^2 \).

If we define \( N_m = m/M_p \) and \( N_v = V/E_p \) we obtain for (2.2)

\[ iT_p \partial_t \psi(t, x) = -\frac{L_p^2 M_p}{2N_m} \partial_x^2 \psi(t, x) + N_v \psi(t, x) \]  

(2.3)

By analogy with ref. [1] we define the fractional Schrödinger equation for all \( \nu \) complex as:

\[ (iT_p)^\nu \partial_t^\nu \psi(t, x) = -\frac{L_p^2}{2N_m} \partial_x^2 \psi(t, x) + N_v \psi(t, x) \]  

(2.4)

where the temporal fractional derivative is defined following ref. [2] (see Appendix III)
3 The Free Particle

From (2.4) for the free particle the fractionary equation is:

\[(i\partial_t)^\nu \psi(t, x) + \frac{L^2}{2T_\nu N_m} \partial^2_x \psi(t, x) = 0\]  \hspace{1cm} (3.1)

By the use of the Fourier transform (complex in the temporal variable and real as usual in the spatial variable) the corresponding equation is (see Appendix II and ref. [2])

\[
\left( k_0^\nu - \frac{L^2}{2T_\nu N_m} k^2 \right) \hat{\psi}(k_0, k) = b(k_0, k) \]  \hspace{1cm} (3.2)

whose solution is:

\[
\hat{\psi}(k_0, k) = \frac{b(k_0, k)}{k_0^\nu - \frac{L^2}{2T_\nu N_m} k^2} \]  \hspace{1cm} (3.3)

and in the configuration space (anti-transforming)

\[
\psi(t, x) = \oint_{\Gamma} \int_{-\infty}^{\infty} a(k_0, k) e^{-i(k_0 t + k x)} dk_0 dk \]  \hspace{1cm} (3.4)

where:

\[
a(k_0, k) = \frac{b(k_0, k)}{4\pi^2} \]

We proceed to analyze solutions of (3.4) for some typical cases in the following section.
Examples

As a first example we consider the case $\nu = 1/2$

Let $\alpha$ be given by:

$$\alpha = \frac{L_p^2}{2T_p^3 N_m} \quad (3.5)$$

From (3.4) we obtain

$$\psi(t, x) = \oint_{\Gamma} \int_{-\infty}^{\infty} \frac{a(k_0, k)}{k_0^\frac{1}{2} - \alpha k^2} e^{-i(k_0 t + k x)} dk_0 \, dk \quad (3.6)$$

or equivalently:

$$\psi(t, x) = \int_{-\infty}^{\infty} a(k) e^{-i(\alpha^2 k^4 t + k x)} \, dk + \int_{-\infty}^{0} \int_{-\infty}^{\infty} a(k_0, k) \left[ \frac{1}{(k_0 + i0)^\frac{1}{2} - \alpha k^2} - \frac{1}{(k_0 - i0)^\frac{1}{2} - \alpha k^2} \right] \, e^{-i(k_0 t + k x)} \, dk_0 \, dk \quad (3.7)$$

where:

$$a(k) = -4\pi i \alpha k^2 a(\alpha^2 k^4, k)$$

With some of algebraic calculus we obtain for (3.7):

$$\psi(t, x) = \int_{-\infty}^{\infty} a(k) e^{-i(\omega^2 t + k x)} \, dk + \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} \frac{a(k_0, k)}{k_0 + \omega^2} e^{i(k_0 t - k x)} \, dk_0 \, dk \quad (3.8)$$
with:

$$\omega = \alpha k^2$$

and where we have made the re-scaling:

$$-2i\kappa^2 a(-k_0, k) \rightarrow a(k_0, k)$$

The first term in (3.8) represent free particle on-shell propagation and the second term describes the contribution of off-shell modes.

As a second example we consider the case $\nu = 1$. In this case (3.4) takes the form:

$$\psi(t, x) = \oint \int_{\Gamma} \frac{a(k_0, k)}{k_0 - \omega} e^{-i(k_0 t + k x)} dk_0 \, dk$$

(3.9)

Evaluating the integral in the variable $k_0$ we have:

$$\psi(t, x) = \int \int_{-\infty}^{\infty} a(k) e^{-i(\omega t + k x)} dk$$

(3.10)

where $a(k) = -2\pi i a(\omega, k)$. Thus we recover the usual expression for the free-particle wave function.

Finally we consider the case $\nu = 2$. For it we have

$$\psi(t, x) = \oint \int_{\Gamma} \frac{a(k_0, k)}{k_0^2 - \omega^2} e^{-i(k_0 t + k x)} dk_0 \, dk$$

(3.11)

After to perform the integral in the variable $k_0$ we obtain from (3.11):

$$\psi(t, x) = \int \int_{-\infty}^{\infty} a(k) e^{-i(\omega t + k x)} + b^+(k) e^{i(\omega t + k x)} dk$$

(3.12)
with \( a(k) = -2\pi i a(\omega, k) \) and \( b^+(k) = -2\pi i a(-\omega, -k) \)

### 4 The Potential Well

We consider in this section the potential well. The fractionary equation for a particle confined to move within interval \( 0 \leq x \leq a \) is:

\[
(i\mathcal{T} p)^\nu \partial_t^\nu \psi(t, x) = -\frac{L_p^2}{2\mathcal{N}_m} \partial_x^2 \psi(t, x) \tag{4.1}
\]

To solve this equation we use the method of separation of variables. Thus if we write:

\[
\psi(t, x) = \psi_1(t) \psi_2(x) \tag{4.2}
\]

As is usual we obtain:

\[
\frac{(i\mathcal{T} p)^\nu \partial_t^\nu \psi_1(t)}{\psi_1(t)} = -\frac{L_p^2}{2\mathcal{N}_m} \partial_x^2 \psi_2(x) = \lambda \tag{4.3}
\]

Then we conclude that \( \psi_2(x) \) satisfies:

\[
\partial_x^2 \psi_2(x) + \frac{2\lambda \mathcal{N}_m}{L_p^2} \psi_2(x) = 0 \tag{4.4}
\]

The solution of (4.4) is the habitual one:

\[
\psi_{2n}(x) = b_n \sin \left( \frac{n\pi x}{a} \right) \tag{4.5}
\]
with:

\[ \lambda_n = \frac{1}{2N_m} \left( \frac{n\pi L_p}{a} \right)^2 \]  

(4.6)

and the boundary conditions satisfied by \( \psi_{2n}(x) \) are:

\[ \psi_{2n}(0) = \psi_{2n}(a) = 0 \]

As a consequence of (4.3), (4.5) and (4.6) the Fourier transform \( \hat{\psi}_1(k_0) \) of \( \psi_1(t) \) should be satisfy:

\[ (k_0^\gamma - \lambda_n) \hat{\psi}_{1n}(k_0) = 0 \]  

(4.7)

whose solution is:

\[ \hat{\psi}_{1n}(k_0) = \frac{c_n(k_0)}{k_0^\gamma - \lambda_n} \]  

(4.8)

Therefore the final general solution for \( \psi(t,x) \) is:

\[ \psi(t,x) = \sum_{n=1}^{\infty} \sin \left( \frac{n\pi}{a} x \right) \int \frac{a_n(k_0) e^{-ik_0 t}}{k_0^\gamma - \lambda_n} \, dk_0 \]  

(4.9)

where we have defined:

\[ a_n(k_0) = \frac{b_n c_n(k_0)}{2\pi} \]

which is an entire analytic function of \( k_0 \).
Examples

As a first example we consider the case $\nu = 1/2$. For it the solution (4.9) takes the form:

$$\psi(t, x) = \sum_{n=1}^{\infty} \sin \left( \frac{n\pi}{a} x \right) \int \frac{a_n(k_0)e^{-ik_0t}}{k_0^2 - \lambda_n} dk_0$$

or equivalently:

$$\psi(t, x) = \sum_{n=1}^{\infty} a_n \sin \left( \frac{n\pi}{a} x \right) e^{-i\lambda_n^2 t} +$$

$$\sum_{n=1}^{\infty} \sin \left( \frac{n\pi}{a} x \right) \int_{-\infty}^{0} \frac{1}{(k_0 + i0)^2 - \lambda_n} - \frac{1}{(k_0 - i0)^2 - \lambda_n} \times a_n(k_0)e^{-ik_0t} dk_0$$

(4.11)

After performing some algebra we have for (4.11) the expression:

$$\psi(t, x) = \sum_{n=1}^{\infty} a_n \sin \left( \frac{n\pi}{a} x \right) e^{-i\lambda_n^2 t} +$$

$$\sum_{n=1}^{\infty} \sin \left( \frac{n\pi}{a} x \right) \int_{0}^{\infty} \frac{a_n(k_0)}{k_0 + \lambda_n^2} e^{-ik_0t} dk_0$$

(4.12)

Analogously as before, the second term in (4.12) represents of-shell stationary modes.

As a second example we consider $\nu = 1$. In this case:

$$\psi(t, x) = \sum_{n=1}^{\infty} \sin \left( \frac{n\pi}{a} x \right) \int \frac{a_n(k_0)e^{-ik_0t}}{k_0^2 - \lambda_n} dk_0$$

(4.13)
Performing the integral in the variable $k_0$ we have:

$$
\psi(t, x) = \sum_{n=1}^{\infty} a_n \sin \left( \frac{n\pi}{a} x \right) e^{-i\lambda_n t}
$$

(4.14)

Which is the familiar general solution for the infinite well.

Finally for $\nu = 2$:

$$
\psi(t, x) = \sum_{n=1}^{\infty} \sin \left( \frac{n\pi}{a} x \right) \int \frac{a_n(k_0)e^{-ik_0 t}}{k_0^2 - \lambda_n} dk_0
$$

(4.15)

and after to compute the integral:

$$
\psi(t, x) = \sum_{n=1}^{\infty} \sin \left( \frac{n\pi}{a} x \right) \left( a_n e^{-i\sqrt{\lambda_n} t} + b_n e^{+i\sqrt{\lambda_n} t} \right)
$$

(4.16)

with $a_n = a_n(\sqrt{\lambda_n})$ and $b_n^+ = a_n(-\sqrt{\lambda_n})$

5 The Green Function for The Free Particle

As other application that shows the generality of the fractional calculus defined with the use of ultradistributions, we give the evaluation of the Green function corresponding to the free particle. Let $\beta$ be defined as:

$$
\beta^2 = \frac{L_p^2}{2\Gamma(T)N_m}
$$

(5.1)

Then $G(t - t', x - x')$ should be satisfy the equation:

$$
(i\partial_t)^\nu G(t - t', x - x') + \beta^2 \partial_x^{2\nu} G(t - t', x - x') = \delta(t - t') \delta(x - x')
$$

(5.2)
As $G$ is function of $(t - t', x - x')$ it is sufficient to consider $G$ as function of $(t, x)$:

$$(i\partial_t)^n G(t, x) + \beta^2 \partial_x^n G(t, x) = \delta(t)\delta(x) \quad (5.3)$$

For the Fourier transform $\hat{G}$ of $G$ we have:

$$(k_0^\nu - \beta^2 k^2) \hat{G}(k_0, k) = \frac{\text{Sgn}[\mathcal{J}(k_0)]}{2} + a(k_0, k) \quad (5.4)$$

where $a(k_0, k)$ is as usual a rapidly decreasing analytic entire function of the variable $k_0$. Selecting:

$$a(k_0, k) = \frac{1}{2}$$

we obtain the equation for the retarded Green function:

$$(k_0^\nu - \beta^2 k^2) \hat{G}_{\text{ret}}(k_0, k) = H[\mathcal{J}(k_0)] \quad (5.5)$$

and then:

$$G_{\text{ret}}(t, x) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{H[\mathcal{J}(k_0)]}{k_0^\nu - \beta^2 k^2} e^{-i(k_0 t + k x)} \, dk_0 \, dk \quad (5.6)$$

If we take:

$$a(k_0, k) = -\frac{1}{2}$$

we obtain the advanced Green function:

$$G_{\text{adv}}(t, x) = -\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{H[-\mathcal{J}(k_0)]}{k_0^\nu - \beta^2 k^2} e^{-i(k_0 t + k x)} \, dk_0 \, dk \quad (5.7)$$
For the Wheeler Green function (half advanced plus half retarded):

\[ G_W(t, x) = \frac{1}{2} [G_{\text{adv}}(t, x) + G_{\text{ret}}(t, x)] \]  

we have:

\[ G_W(t, x) = \frac{1}{8\pi^2} \oint \oint_{\Gamma} \text{Sgn}[\mathcal{I}(k_0)] \frac{H[\mathcal{I}(k_0)]}{k_0^2 - \beta^2 k^2} e^{-i(k_0 t + kx)} \, dk_0 \, dk \]  

**Example**

When we select \( \nu = 1 \) we obtain the usual Green functions of Quantum Mechanics. For example for \( G_{\text{ret}} \) we have:

\[ G_{\text{ret}}(t, x) = \frac{1}{4\pi^2} \oint \oint_{\Gamma} \frac{H[\mathcal{I}(k_0)]}{k_0^2 - \beta^2 k^2} e^{-i(k_0 t + kx)} \, dk_0 \, dk \]  

or equivalently:

\[ G_{\text{ret}}(t, x) = \frac{1}{4\pi^2} \int \int_{-\infty}^{\infty} \frac{1}{(k_0 + i0) - \beta^2 k^2} e^{-i(k_0 t + kx)} \, dk_0 \, dk \]  

After the evaluation of the integral in the variable \( k_0 \), \( G_{\text{ret}} \) takes the form:

\[ G_{\text{ret}}(t, x) = -\frac{i}{2\pi} H(t) \int_{-\infty}^{\infty} e^{-i(\beta^2 k^2 t + kx)} \, dk \]  

With a square’s completion (5.12) transforms into:

\[ G_{\text{ret}}(t, x) = -\frac{iH(t)}{2\pi\beta\sqrt{t}} e^{\frac{i\beta^2 x^2}{4t}} \int_{-\infty}^{\infty} e^{is^2} \, ds \]  

14
From the result of ref. [4]

\[ \int_{-\infty}^{\infty} e^{is^2} ds = \sqrt{\pi} e^{-i\frac{\pi}{4}} \]  \hspace{1cm} (5.14)

we have

\[ G_{\text{ret}}(t, x) = -iH(t) \left( \frac{m}{2\pi i\hbar} \right)^{\frac{1}{2}} e^{\frac{i m x^2}{2\hbar t}} \]  \hspace{1cm} (5.15)

Taking into account that for \( \nu = 1 \):

\[ \beta^2 = \frac{\hbar}{2m} \]

we obtain the usual form of \( G_{\text{ret}} \) (see ref. [3])

\[ G_{\text{ret}}(t - t', x - x') = -iH(t - t') \left( \frac{m}{2\pi i\hbar(t - t')} \right)^{\frac{1}{2}} e^{\frac{i m (x - x')^2}{2\hbar(t - t')}} \]  \hspace{1cm} (5.16)

With a similar calculus we have for \( G_{\text{adv}} \):

\[ G_{\text{adv}}(t - t', x - x') = iH(t' - t) \left( \frac{m}{2\pi i\hbar(t' - t)} \right)^{\frac{1}{2}} e^{\frac{i m (x - x')^2}{2\hbar(t' - t')}} \]  \hspace{1cm} (5.17)

and for \( G_{W} \):

\[ G_{W}(t - t', x - x') = -\frac{i}{2} \text{Sgn}(t - t') \left( \frac{m}{2\pi i\hbar|t - t'|} \right)^{\frac{1}{2}} e^{\frac{i m (x - x')^2}{2\hbar(t - t')}} \]  \hspace{1cm} (5.18)

6 Discussion

In a earlier paper (ref. [2] we have shown the existence of a general fractional calculus defined via tempered ultradistributions. All ultradistributions provide integrands that are analytic functions along the integration
path. These properties show that tempered ultradistributions provide an appropriate framework for applications to fractional calculus. With the use of this calculus we have generalized in the present work the results obtained by Naber (ref.[1]). We have defined the fractional Schrödinger equation for all values of the complex variable \( \nu \) and treated the cases of the free particle and the potential well. For \( \nu = 1 \) the results obtained coincide with the usual Quantum Mechanics, and the cases \( \nu = 1/2 \) and \( \nu = 2 \) have shown the appearance of extra terms, besides to those with the usual \( (\nu = 1) \) framework.

We have obtained a general expression for the Green function of the free particle and shown that for \( \nu = 1 \) this Green function coincide with the obtained in ref.[5]... For the benefit of the reader we give in this paper two Appendixes with the main characteristics of n-dimensional tempered ultradistributions and their Fourier anti-transformed distributions of the exponential type, and a third Appendix about the general fractional calculus defined via the use of tempered ultradistributions.
References

[1] M. Naber: J. of Math. Phys 45, 3339 (2004).

[2] D. G. Barci, G. Bollini, L. E. Oxman, M. C. Rocca: Int. J. of Theor. Phys. 37, 3015 (1998).

[3] F. Riewe: Phys. Rev. E 55, 3581 (1997).

[4] L. S. Gradshtein and I. M. Ryzhik: “Table of Integrals, Series, and Products”. Sixth edition, 3.322, 333 Academic Press (2000).

[5] L. Schiff: “Quantum Mechanics”, 65, McGraw-Hill Kogakusha, Ltd (1968).

[6] J. Sebastiao e Silva: Math. Ann. 136, 38 (1958).

[7] M. Hasumi: Tôhoku Math. J. 13, 94 (1961).

[8] I. M. Gel’fand and N. Ya. Vilenkin: “Generalized Functions” Vol. 4. Academic Press (1964).

[9] I. M. Gel’fand and G. E. Shilov: “Generalized Functions” Vol. 2. Academic Press (1968).

[10] L. Schwartz: “Théorie des distributions”. Hermann, Paris (1966).
7 Appendix I: Distributions of Exponential Type

For the sake of the reader we shall present a brief description of the principal properties of Tempered Ultradistributions.

Notations. The notations are almost textually taken from ref[7]. Let $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) be the real (resp. complex) n-dimensional space whose points are denoted by $x = (x_1, x_2, ..., x_n)$ (resp $z = (z_1, z_2, ..., z_n)$). We shall use the notations:

(i) $x + y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$ ; $\alpha x = (\alpha x_1, \alpha x_2, ..., \alpha x_n)$

(ii) $x \geq 0$ means $x_1 \geq 0, x_2 \geq 0, ..., x_n \geq 0$

(iii) $x \cdot y = \sum_{j=1}^{n} x_j y_j$

(iv) $|x| = \sum_{j=1}^{n} |x_j|$

Let $\mathbb{N}^n$ be the set of n-tuples of natural numbers. If $p \in \mathbb{N}^n$, then $p = (p_1, p_2, ..., p_n)$, and $p_j$ is a natural number, $1 \leq j \leq n$. $p + q$ denote $(p_1 + q_1, p_2 + q_2, ..., p_n + q_n)$ and $p \geq q$ means $p_1 \geq q_1, p_2 \geq q_2, ..., p_n \geq q_n$. $x^p$ means $x_1^{p_1} x_2^{p_2} ... x_n^{p_n}$. We shall denote by $|p| = \sum_{j=1}^{n} p_j$ and by $D^p$ we denote the differential operator $\frac{\partial^{p_1 + p_2 + ... + p_n}}{\partial x_1^{p_1} \partial x_2^{p_2} ... \partial x_n^{p_n}}$

For any natural $k$ we define $x^k = x_1^k x_2^k ... x_n^k$ and $\partial^k / \partial x^k = \partial^{nk} / \partial x_1^k \partial x_2^k ... \partial x_n^k$
The space $\mathcal{H}$ of test functions such that $e^{p|x|}D^q\phi(x)$ is bounded for any $p$ and $q$ is defined (ref.\[7\]) by means of the countably set of norms:

$$
\|\hat{\phi}\|_p = \sup_{0 \leq q \leq p} e^{p|x|} \left| D^q\hat{\phi}(x) \right|, \quad p = 0, 1, 2, \ldots \tag{7.1}
$$

According to reference\[9\] $\mathcal{H}$ is a $\mathcal{K}\{\mathcal{M}_p\}$ space with:

$$
\mathcal{M}_p(x) = e^{(p-1)|x|}, \quad p = 1, 2, \ldots \tag{7.2}
$$

$\mathcal{K}\{e^{(p-1)|x|}\}$ satisfies condition $(\mathcal{N})$ of Guelfand (ref.\[8\]). It is a countable Hilbert and nuclear space:

$$
\mathcal{K}\{e^{(p-1)|x|}\} = \mathcal{H} = \bigcap_{p=1}^{\infty} \mathcal{H}_p \tag{7.3}
$$

where $\mathcal{H}_p$ is obtained by completing $\mathcal{H}$ with the norm induced by the scalar product:

$$
<\hat{\phi}, \hat{\psi}>_p = \int_{-\infty}^{\infty} e^{2(p-1)|x|} \sum_{q=0}^{p} D^q\bar{\phi}(x)D^q\bar{\psi}(x) \, dx ; \quad p = 1, 2, \ldots \tag{7.4}
$$

where $dx = dx_1 \, dx_2 \ldots dx_n$

If we take the usual scalar product:

$$
<\hat{\phi}, \hat{\psi}> = \int_{-\infty}^{\infty} \bar{\phi}(x)\bar{\psi}(x) \, dx \tag{7.5}
$$

then $\mathcal{H}$, completed with (7.5), is the Hilbert space $\mathcal{H}$ of square integrable functions.
The space of continuous linear functionals defined on $\mathcal{H}$ is the space $\Lambda_\infty$ of the distributions of the exponential type (ref.\cite{ref7}).

The “nested space”

$$H = (\mathcal{H}, \mathcal{H}, \Lambda_\infty) \quad (7.6)$$

is a Guelfand’s triplet (or a Rigged Hilbert space \cite{ref8}).

In addition we have: $\mathcal{H} \subset \mathcal{S} \subset H \subset \mathcal{S}' \subset \Lambda_\infty$, where $\mathcal{S}$ is the Schwartz space of rapidly decreasing test functions (ref\cite{ref10}).

Any Guelfand’s triplet $\mathcal{G} = (\Phi, \mathcal{H}, \Phi')$ has the fundamental property that a linear and symmetric operator on $\Phi$, admitting an extension to a self-adjoint operator in $\mathcal{H}$, has a complete set of generalized eigen-functions in $\Phi'$ with real eigenvalues.

8 Appendix II: Tempered Ultradistributions

The Fourier transform of a function $\hat{\phi} \in \mathcal{H}$ is

$$\phi(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\hat{\phi}(x)} e^{izx} \, dx \quad (8.1)$$

$\phi(z)$ is entire analytic and rapidly decreasing on straight lines parallel to the real axis. We shall call $\mathcal{H}$ the set of all such functions.

$$\mathcal{H} = \mathcal{F}\{\mathcal{H}\} \quad (8.2)$$
It is a $\mathcal{Z}\{\mathcal{M}_p\}$ space ( ref.[9] ), countably normed and complete, with:

$$\mathcal{M}_p(z) = (1 + |z|)^p$$  \hspace{1cm} (8.3)

$\mathcal{H}$ is also a nuclear space with norms:

$$\|\phi\|_{p_n} = \sup_{z \in \mathcal{V}_n} (1 + |z|)^p |\phi(z)|$$  \hspace{1cm} (8.4)

where $\mathcal{V}_k = \{ z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n : \Im z_j \leq k, 1 \leq j \leq n \}$

We can define the usual scalar product:

$$<\phi(z), \psi(z)> = \int_{-\infty}^{\infty} \phi(z)\psi_1(z) \, dz = \int_{-\infty}^{\infty} \overline{\phi(x)}\overline{\psi(x)} \, dx$$  \hspace{1cm} (8.5)

where:

$$\psi_1(z) = \int_{-\infty}^{\infty} \hat{\psi}(x) \, e^{-iz \cdot x} \, dx$$

and $dz = dz_1 \, dz_2 ... dz_n$

By completing $\mathcal{H}$ with the norm induced by (8.5) we get the Hilbert space of square integrable functions.

The dual of $\mathcal{H}$ is the space $\mathcal{U}$ of tempered ultradistributions ( ref.[7] ). In other words, a tempered ultradistribution is a continuous linear functional defined on the space $\mathcal{H}$ of entire functions rapidly decreasing on straight lines parallel to the real axis.

The set $\mathcal{U} = (\mathcal{H}, \mathcal{U})$ is also a Guelfand’s triplet.

21
Moreover, we have: $\mathcal{H} \subset \mathcal{S} \subset \mathcal{H} \subset \mathcal{S}' \subset \mathcal{U}$.

\( \mathcal{U} \) can also be characterized in the following way (ref.\[7\]): let \( \mathcal{A}_\omega \) be the space of all functions \( F(z) \) such that:

1. \( F(z) \) is analytic for \( \{z \in \mathbb{C}^n : |\text{Im}(z_1)| > p, |\text{Im}(z_2)| > p, \ldots, |\text{Im}(z_n)| > p\} \).

2. \( F(z)/z^p \) is bounded continuous in \( \{z \in \mathbb{C}^n : |\text{Im}(z_1)| \geq p, |\text{Im}(z_2)| \geq p, \ldots, |\text{Im}(z_n)| \geq p\} \), where \( p = 0, 1, 2, \ldots \) depends on \( F(z) \).

Let \( \Pi \) be the set of all \( z \)-dependent pseudo-polynomials, \( z \in \mathbb{C}^n \). Then \( \mathcal{U} \) is the quotient space:

3. \( \mathcal{U} = \mathcal{A}_\omega / \Pi \)

By a pseudo-polynomial we understand a function of \( z \) of the form

\[ \sum_s z^j G(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \] with \( G(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \in \mathcal{A}_\omega \)

Due to these properties it is possible to represent any ultradistribution as (ref.\[7\]):

\[ F(\phi) = \langle F(z), \phi(z) \rangle = \int_{\Gamma} F(z) \phi(z) \, dz \quad (8.6) \]

\( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \ldots \Gamma_n \) where the path \( \Gamma_j \) runs parallel to the real axis from \(-\infty\) to \( \infty \) for \( \text{Im}(z_j) > \zeta, \zeta > p \) and back from \( \infty \) to \(-\infty \) for \( \text{Im}(z_j) < -\zeta, -\zeta < -p \). (\( \Gamma \) surrounds all the singularities of \( F(z) \)).

Formula (8.6) will be our fundamental representation for a tempered ul-
tradistribution. Sometimes use will be made of “Dirac formula” for ultradistributions (ref.[6]):

\[ F(z) = \frac{1}{(2\pi i)^n} \int_{-\infty}^{\infty} \frac{f(t)}{(t_1 - z_1)(t_2 - z_2)\ldots(t_n - z_n)} \, dt \]  

(8.7)

where the “density” \( f(t) \) is such that

\[ \oint_{\Gamma} F(z) \phi(z) \, dz = \int_{-\infty}^{\infty} f(t) \phi(t) \, dt \]  

(8.8)

While \( F(z) \) is analytic on \( \Gamma \), the density \( f(t) \) is in general singular, so that the r.h.s. of (8.8) should be interpreted in the sense of distribution theory.

Another important property of the analytic representation is the fact that on \( \Gamma \), \( F(z) \) is bounded by a power of \( z \) (ref.[7]):

\[ |F(z)| \leq C|z|^p \]  

(8.9)

where \( C \) and \( p \) depend on \( F \).

The representation (8.6) implies that the addition of a pseudo-polynomial \( P(z) \) to \( F(z) \) do not alter the ultradistribution:

\[ \oint_{\Gamma} (F(z) + P(z)) \phi(z) \, dz = \oint_{\Gamma} F(z) \phi(z) \, dz + \oint_{\Gamma} P(z) \phi(z) \, dz \]

But:

\[ \oint_{\Gamma} P(z) \phi(z) \, dz = 0 \]
as $P(z)\phi(z)$ is entire analytic in some of the variables $z_j$ (and rapidly decreasing),

\[ \int_{\gamma} \{ F(z) + P(z) \} \phi(z) \, dz = \int_{\gamma} F(z) \phi(z) \, dz \quad \text{(8.10)} \]

9 Appendix III: Fractional Calculus

The purpose of this section is to introduce definition of fractional derivation and integration given in ref. [6]. This definition unifies the notion of integral and derivative in one only operation. Let $\hat{f}(x)$ a distribution of exponential type and $F(\Omega)$ the complex Fourier transformed Tempered Ultradistribution. Then:

\[ F(k) = H[\Im(k)] \int_{0}^{\infty} \hat{f}(x) e^{ikx} \, dx - H[-\Im(k)] \int_{-\infty}^{0} \hat{f}(x) e^{ikx} \, dx \quad \text{(9.1)} \]

($H(x)$ is the Heaviside step function) and

\[ \hat{f}(x) = \frac{1}{2\pi} \int_{\Gamma} F(k) e^{-ikx} \, dk \quad \text{(9.2)} \]

where the contour $\Gamma$ surround all singularities of $F(k)$ and runs parallel to real axis from $-\infty$ to $\infty$ above the real axis and from $\infty$ to $-\infty$ below the real axis. According to [6] the fractional derivative of $\hat{f}(x)$ is given by

\[ \frac{d^\lambda \hat{f}(x)}{dx^\lambda} = \frac{1}{2\pi} \int_{\Gamma} (\hat{f}(x) e^{-ikx}) \, dk + \int_{\Gamma} (-ik)^\lambda a(k) e^{-ikx} \, dk \quad \text{(9.3)} \]
Where $a(k)$ is entire analytic and rapidly decreasing. If $\lambda = -1$, $d^\lambda/dx^\lambda$ is the inverse of the derivative (an integration). In this case the second term of the right side of (9.3) gives a primitive of $\hat{f}(x)$. Using Cauchy’s theorem the additional term is

$$\oint a(k) k e^{-ikx} dk = 2\pi a(0)$$  \hspace{1cm} (9.4)

Of course, an integration should give a primitive plus an arbitrary constant.

Analogously when $\lambda = -2$ (a double iterated integration) we have

$$\oint a(k) k^2 e^{-ikx} dk = \gamma + \delta x$$  \hspace{1cm} (9.5)

where $\gamma$ and $\delta$ are arbitrary constants.