A characterization of the consistent Hoare powerdomains over dcpos

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Abstract

It has been shown that for a dcpo $P$, the Scott closure of $\Gamma_c(P)$ in $\Gamma(P)$ is a consistent Hoare powerdomain of $P$, where $\Gamma_c(P)$ is the family of nonempty, consistent and Scott closed subsets of $P$, and $\Gamma(P)$ is the collection of all nonempty Scott closed subsets of $P$. In this paper, by introducing the notion of a $\bigvee$-existing set, we present a direct characterization of the consistent Hoare powerdomain: the set of all $\bigvee$-existing Scott closed subsets of a dcpo $P$ is exactly the consistent Hoare powerdomain of $P$. We also introduce the concept of an $F$-Scott closed set over each dcpo-$\lor\uparrow$-semilattice. We prove that the Scott closed set lattice of a dcpo $P$ is isomorphic to the family of all $F$-Scott closed sets of $P$'s consistent Hoare powerdomain.

Keywords: consistent Hoare powerdomain, $\bigvee$-existing set, dcpo-$\lor\uparrow$-semilattice, $F$-Scott closed set

1. Introduction

The Hoare powerdomain plays an important role in modeling the programming semantics of nondeterminism, which is analogous to the powerset construction (see for example [3, 4, 5, 7, 8, 9]). The Hoare power domain over a dcpo (directed complete poset) is the free inflationary semilattice where the inflationary operator is exactly the binary join operator. The standard construction of a Hoare powerdomain consists all nonempty Scott closed subsets $\Gamma(P)$ of the dcpo $P$, order given by the inclusion relation. In [10], Yuan and Kou introduced a new type of powerdomain, called the consistent Hoare powerdomain. The new powerdomain over a dcpo is a free algebra where the inflationary operator delivers joins only for consistent pairs. They provided a concrete way of constructing the consistent Hoare powerdomain over a domain (continuous dcpo): the family of all nonempty relatively consistent Scott closed subsets of the domain is such a powerdomain. Follows from the work, there is a natural problem: whether every dcpo has a consistent Hoare powerdomain. Geng and Kou [1] gave an affirmative answer to the question by showing that the Scott closure $\Gamma_c(P)$ of $\Gamma_c(P)$ in $\Gamma(P)$ is a consistent Hoare powerdomain over the dcpo $P$, where $\Gamma_c(P)$ is the collection of all nonempty consistent Scott closed subsets of $P$. One question arises naturally: can we give a direct characterization of the consistent Hoare powerdomain over every dcpo $P$? That is to say, what type of Scott closed subsets of $P$ is exactly the consistent Hoare powerdomain? This paper is mainly set to answer this question.

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The consistent Hoare powerdomain can be viewed as a type of completion that embeds every dcpo into a dcpo which is also complete with respect to consistent pair joins. In [11], Zhao and Fan introduced a type of dcpo-completion embedding each poset into a dcpo, called D-completion. They proved that the Scott closed set lattice of a poset is order isomorphic to that of the D-completion. It is also the case for the sobrification, i.e., the topology lattice \( \mathcal{O}(X) \) of a \( T_0 \)-space \( X \) is isomorphic to the topology lattice \( \mathcal{O}(X^s) \) of the sobrification \( X^s \) of \( X \) (see [2]). In this paper, we introduce a type of closed sets on every dcpo-\( \vee \)-semilattice, called F-Scott closed sets. We prove that the Scott closed set lattice of a dcpo is isomorphic to the family of all F-Scott closed subsets of its consistent Hoare powerdomain. It is known that two sober dcpos are isomorphic if and only if their Scott set lattices are isomorphic (see [6, 12]). Consequently, the consistent Hoare powerdomains for sober dcpos are uniquely determined up to isomorphism.

2. Preliminaries

This section is an introduction to the concepts and results about the consistent Hoare powerdomains over domains and dcpos. For more details, refer to [1, 2, 10]. In this paper, the order relation on each family of sets is the set-theoretic inclusion relation.

A nonempty subset \( D \) of a poset \( P \) is called directed if every pair has an upper bound in \( D \). If every directed \( D \subseteq P \) has a supremum \( \bigvee D \), then \( P \) is called a dcpo. For \( A \subseteq P \), let \( \downarrow A = \{ x \in P : x \leq a \text{ for some } a \in A \} \), \( \downarrow x = \{ x \} \) where \( x \in P \). If \( A = \downarrow A \), then \( A \) is called a lower set. For \( x, y \in P \), we say \( x \) is way below \( y \), written \( x \ll y \), if \( y \leq \bigvee D \) implies \( x \in \downarrow D \) for any directed \( D \subseteq P \). Let \( \downarrow x = \{ y \in P : y \ll x \} \), and \( \downarrow A = \bigcup \{ \downarrow a : a \in A \} \). A dcpo \( P \) is called a domain if each element \( x \) is the directed join of \( \downarrow x \). A subset \( A \) of a dcpo \( P \) is Scott closed if it is a lower set and closed under directed sups. Let \( \text{cl}(A) \) denote the Scott closure of \( A \).

**Definition 2.1.** [10] A consistent inflationary semilattice is a dcpo \( P \) with a Scott continuous binary partial operator \( \bigvee \) defined only for consistent pairs of points that satisfies three equations for commutativity \( x \bigvee y = y \bigvee x \), associativity \( x \bigvee (y \bigvee z) = (x \bigvee y) \bigvee z \) and idempotency \( x \bigvee x = x \) together with the inequality \( x \leq x \bigvee y \) for \( x, y, z \in P \). The free consistent inflationary semilattice over a dcpo \( P \) is called the consistent Hoare powerdomain over \( P \) and denoted by \( H_c(P) \).

For a consistent inflationary semilattice, the operator \( \bigvee \) coincides with \( \vee \), the join operator defined only for consistent pairs. That is to say, a consistent inflationary semilattice is exactly a dcpo that is a \( \vee \)-semilattice, also called a dcpo-\( \vee \)-semilattice. A Scott closed subset \( A \) of a domain \( L \) is called relatively consistent if \( A \) is the Scott closure of the directed union \( \bigcup \{ \downarrow F : F \in \mathcal{F}_C(A) \} \), where

\[
\mathcal{F}_C(A) = \{ F : F \text{ is a nonempty finite consistent subset of } L \text{ and } F \subseteq \downarrow A \}.
\]

The set of all nonempty relatively consistent Scott closed subsets of \( L \) is denoted by \( R\Gamma C(L) \).

**Theorem 2.2.** [10] Let \( L \) be a domain. The consistent Hoare powerdomain \( H_c(P) \) over \( L \) can be realized as \( R\Gamma C(L) \) where \( \bigvee = \bigcup \). The embedding \( j \) of \( L \) into \( R\Gamma C(L) \) is given by \( j(x) = \downarrow x \) for \( x \in L \).

Let \( \Gamma_c(P) \) denote the family of all nonempty, consistent and Scott closed subsets of a dcpo \( P \). And let \( \Gamma_c(P) \) be the Scott closure of \( \Gamma_c(P) \) in \( \Gamma(P) \). The family \( \Gamma_c(P) \) was introduced by Geng and Kou [11], in order to declare that the consistent Hoare powerdomain over every dcpo exists.
Lemma 2.3. Let $P$ be a dcpo and $L$ is a dcpo-$\lor$-semilattice. If $f : P \to L$ is a Scott continuous function, then for any $A \in \Gamma_c(P)$, $\lor f(A)$ exists in $L$.

Theorem 2.4. Let $P$ be a dcpo. Then the consistent Hoare powerdomain $H_c(P)$ over $P$ can be realized as the $\Gamma_c(P)$, where $A \lor B = A \cup B$ whenever $A, B$ are consistent in $\Gamma_c(P)$. The embedding $j$ of $P$ into $\Gamma_c(P)$ is given by $j(x) = \downarrow x$ for any $x \in P$.

3. A direct characterization of the consistent Hoare powerdomain

Definition 3.1. A subset $A$ of a dcpo $P$ is called $\lor$-existing if for any continuous function $f : P \to L$ mapping into a dcpo-$\lor$-semilattice $L$, $\lor f(A)$ always exists in $L$.

For any directed subset $D$ of $P$, $f(D)$ is directed in $L$ since continuous functions are monotone, and then $\lor f(D)$ exists in the dcpo-$\lor$-semilattice $L$. Hence every directed subset is $\lor$-existing. Similarly, every consistent finite subset is also $\lor$-existing. Notice that an empty set is not $\lor$-existing because a dcpo-$\lor$-semilattice $L$ may not have a least element.

Proposition 3.2. Let $f : P \to L$ be a continuous function from dcpo $L$ to dcpo-$\lor$-semilattice $L$, and $A \subseteq P$. Then

1. $\lor f(A)$ exists $\iff \lor f(cl(A))$ exists $\Rightarrow \lor f(A) = \lor f(cl(A))$.
2. $A$ is $\lor$-existing if and only if $cl(A)$ is $\lor$-existing.

Proof. (1) Assume that $x$ is an upper bound of $f(A)$. Then $f(A) \subseteq \downarrow x$ and $A \subseteq f^{-1}(\downarrow x) \in \Gamma(P)$. Hence $cl(A) \subseteq f^{-1}(\downarrow x)$, and then $f(cl(A)) \subseteq \downarrow x$, i.e., $x$ is an upper bound of $f(cl(A))$. The converse is clearly true. Thus $\lor f(A)$ exists iff $\lor f(cl(A))$ exists, and both imply that $\lor f(A) = \lor f(cl(A))$.

(2) It is straightforward from (1) and Definition 3.1.

For any dcpo $P$, we write $P^\lor = \{A \subseteq P : A$ is $\lor$-existing and Scott closed$\}$. By Lemma 2.3, we immediately have that $\Gamma_c(P) \subseteq P^\lor$. We shall show that the converse inclusion is also true, and then a characterization of the consistent Hoare powerdomain is obtained.

Definition 3.3. A subset $A$ of a dcpo-$\lor$-semilattice $L$ is called $F$-Scott closed if it is Scott closed and for any consistent nonempty finite set $F \subseteq A, \forall F \in A$. We write $\Gamma_F(L)$, called the $F$-Scott closure system on $L$, for the set of all $F$-Scott closed subsets of $L$, and let $cl_F$ denote the corresponding closure operator. A function $f : L \to M$ between dcpo-$\lor$-semilattices is called $F$-Scott continuous if $f$ is continuous with respect to the $F$-Scott closure systems.

Proposition 3.4. (1) A function $f : L \to M$ between dcpo-$\lor$-semilattices is a dcpo-$\lor$-semilattice homomorphism iff $f$ is $F$-Scott continuous.

2. If a nonempty subset $A$ of a dcpo-$\lor$-semilattice $L$ is consistent, then $cl_F(A) = \downarrow \lor A$.

Proof. (1) Suppose that $f$ is a dcpo-$\lor$-semilattice homomorphism. Let $A \subseteq M$ be $F$-Scott closed. We shall show that $f^{-1}(A) \subseteq L$ is also $F$-Scott closed. For any consistent nonempty finite $F \subseteq f^{-1}(A)$, we have $f(F)$ is also nonempty finite and consistent since $f$ is monotone. Then $f(\lor F) = \lor f(F) \in A$ and then $\lor F$ is in $f^{-1}(A)$. Similarly, $f^{-1}(A)$ is also closed with respect to directed joins. Thus $f$ is $F$-Scott continuous.
Conversely, let $f$ be $F$-Scott continuous. Clearly, $f$ is monotone. Let $F$ be a consistent nonempty finite subset of $L$. Then $\bigvee f(F) \leq f(\bigvee F)$. The set $\downarrow \bigvee f(F)$ is $F$-Scott closed in $M$. Then $f^{-1}(\downarrow \bigvee f(F))$ is also $F$-Scott closed and $F \subseteq f^{-1}(\downarrow \bigvee f(F))$. Hence $\bigvee F \in f^{-1}(\downarrow \bigvee f(F))$ and then $f(\bigvee F) \leq \bigvee f(F)$. Thus $f(\bigvee F) = \bigvee f(F)$. Analogously, $f(\bigvee D) = \bigvee f(D)$ for all directed $D \subseteq L$. Therefore, $f$ is a dcpo-$\vee\uparrow$-semilattice homomorphism.

(2) Let $C \subseteq L$ be any $F$-Scott closed set with $A \subseteq C$. Notice that each subset of $A$ is also consistent. Then $D := \{\bigvee F : F \subseteq A \text{ is nonempty and finite}\}$ is a directed subset of $C$, and hence $\bigvee D = \bigvee A \in C$. Moreover, every $F$-Scott closed set is a lower set. Then $\downarrow \bigvee A \subseteq C$, and thus $\text{cl}_F(A) = \downarrow \bigvee A$.

Definition 3.5. Let $L$ be a dcpo-$\vee\uparrow$-semilattice. A subset $A \subseteq L$ is called $F$-$\bigvee$-existing if for any dcpo-$\vee\uparrow$-semilattice homomorphism $f : L \to M$ mapping into a dcpo-$\vee\uparrow$-semilattice $M$, $\bigvee f(A)$ exists in $M$.

Lemma 3.6. Let $L$ be a dcpo-$\vee\uparrow$-semilattice. A subset $A \subseteq L$ is $F$-$\bigvee$-existing iff $\text{cl}_F(A)$ is $F$-$\bigvee$-existing.

Proof. The process is similar to that of Proposition 3.2. Notice that every principal ideal $\downarrow x$ is $F$-Scott closed.

Lemma 3.7. If a subset $A$ of a dcpo-$\vee\uparrow$-semilattice $L$ is $F$-Scott closed and $F$-$\bigvee$-existing, then $A = \downarrow a$ for some $a \in L$.

Proof. The identity function $id : L \to L$ is a dcpo-$\vee\uparrow$-semilattice homomorphism. We have that $\bigvee id(A) = \bigvee A$ exists in $L$. Then every nonempty finite subset $F \subseteq A$ is consistent, and hence $\bigvee F$ exists. Since $A$ is $F$-Scott closed, we obtain that $D = \{\bigvee F : F \subseteq A \text{ is nonempty and finite}\}$ is a directed subset of $A$, and then $\bigvee D = \bigvee A \in A$. Thus $A = \downarrow \bigvee A$, which completes the proof.

Lemma 3.8. A subset $A$ of a dcpo $P$ is $\bigvee$-existing iff $j(A)$ is an $F$-$\bigvee$-existing subset of $\overline{\Gamma_c(P)}$. (Notice that $j$ is the function in Theorem 2.4)

Proof. Suppose that $A \subseteq P$ is $\bigvee$-existing. Let $f : \overline{\Gamma_c(P)} \to L$ be any dcpo-$\vee\uparrow$-semilattice homomorphism mapping into a dcpo-$\vee\uparrow$-semilattice $L$. Then $f \circ j$ is a Scott continuous function from $P$ to $L$, and hence $\bigvee f \circ j(A) = \bigvee f(j(A))$ exists in $L$. Thus $j(A)$ is an $F$-$\bigvee$-existing subset of $\overline{\Gamma_c(P)}$.

Now assume that $j(A)$ is an $F$-$\bigvee$-existing subset of $\overline{\Gamma_c(P)}$. Let $f : P \to L$ be a Scott continuous mapping into a dcpo-$\vee\uparrow$-semilattice $L$. Then by Theorem 2.4 there exists a unique dcpo-$\vee\uparrow$-semilattice homomorphism $\tilde{f} : \overline{\Gamma_c(P)} \to L$ such that $f = \tilde{f} \circ j$. Then $\bigvee \tilde{f}(j(A)) = \bigvee f(A)$ exists. Thus $A$ is a $\bigvee$-existing subset of $P$.

We now come to the characterization of the consistent Hoare powerdomains by $\bigvee$-existing Scott closed sets:

Theorem 3.9. Let $P$ be a dcpo and $A \subseteq P$. Then $A \in \overline{\Gamma_c(P)}$ iff $A$ is Scott closed and $\bigvee$-existing, i.e., $\overline{\Gamma_c(P)} = P^\vee$. 


Proof. By Lemma 2.3, we have that $\Gamma_c(P) \subseteq P^\vee$. Now suppose that $A \in P^\vee$. Then, by Lemma 3.5, the set $j(A)$ is an $F$-$\lor$-existing subset of $\Gamma_c(P)$. And by Lemma 3.6, the F-Scott closed set $cl_F(j(A))$ is also $F$-$\lor$-existing. Then $cl_F(j(A))$ has a supremum which is in itself by Lemma 3.7. Since the order on $\bigcup A$ is closed set proves the claim. By Lemma 2.3, we have that $F$ is consistent in $\bigcup A$. Thus $A \in \Gamma_c(P)$ because $\Gamma_c(P)$ is a lower set of $\Gamma(P)$, which completes the proof. 

**Theorem 3.10.** Let $P$ be a dcpo. The family of Scott closed sets $\Gamma_0(P) = \Gamma(P) \bigcup \{0\}$ is order isomorphic to the F-Scott closure system $\Gamma_F(\Gamma_c(P))$ on the consistent Hoare powerdomain $H_c(P)$, i.e., $\Gamma_0(P) \cong \Gamma_F(\Gamma_c(P))$.

**Proof.** Define a function $\eta : \Gamma_0(P) \to \Gamma_c(P)$ by $\eta(A) = cl_F(j(A))$, where $cl_F(j(A))$ is the F-Scott closure of $j(A) = \{a : a \in A\}$ in $\Gamma_c(P)$. We shall show that $\eta$ is an order isomorphism. Obviously, $\eta$ is monotone.

Firstly, we prove that $\eta$ is injective. Suppose that $A, B \in \Gamma_0(P)$ with $A \neq B$. Without loss of generality, we may assume that there is $b \in B \setminus A$. Let $C = \{C \in \Gamma_c(P) : C \subseteq A\}$. Clearly, $j(A) \subseteq C$ and $\nabla b \notin C$. We claim that $C$ is F-Scott closed in $\Gamma_c(P)$. Then $cl_F(j(A)) \subseteq C$, and hence $\nabla b \notin cl_F(j(A))$. But $\nabla b \in j(B) \subseteq cl_F(j(B))$. Thus $\eta(A) \neq \eta(B)$, i.e., $\eta$ is injective. Indeed, let $D \subseteq C$ be directed, then $\bigvee D$, the supremum of $D$ in $\Gamma_c(P)$, is the Scott closure of $\bigcup D$ in $P$, and hence $\bigvee D \subseteq A$ since each element of $D$ is contained in $A$, i.e., $\bigvee D \in C$. And if $C_1, C_2 \in C$, then $C_1 \cup C_2 \subseteq A$, thus $C_1 \vee C_2 \in C$ (notice that $\Gamma_c(P)$ is closed under finite unions), which proves the claim.

We next prove that $\eta$ is surjective. Suppose that $A$ is F-Scott closed in $\Gamma_c(P)$. We claim that $\bigcup A$ is Scott closed in $P$. Indeed, if $D \subseteq \bigcup A$ is directed, then $j(D) \subseteq A$ is directed, and hence $\bigvee j(D) = \nabla \bigvee D \in A$, i.e., $\bigvee D \in \bigcup A$. We shall show that $\eta(\bigcup A) = A$. Since $j(\bigcup A) \subseteq A$, we have $\eta(\bigcup A) = cl_F(j(\bigcup A)) \subseteq A$. Conversely, assume that $A \in A$. Then $j(A)$ is consistent in $\Gamma_c(P)$. By Proposition 3.4(2), we have $\bigvee j(A) = A \in cl_F(j(A)) \subseteq cl_F(j(\bigcup A))$. Thus $\eta(\bigcup A) = cl_F(j(\bigcup A)) = A$, i.e., $\eta$ is surjective, which completes the proof.

A dcpo is called **sober** if every Scott closed irreducible set is of the form $\nabla x$. It is known that for sober dcpos $P_1$ and $P_2$, $\Gamma_0(P_1) \cong \Gamma_0(P_2)$ iff $P_1 \cong P_2$ (see [12] for more results about uniqueness of dcpos based on the Scott closed set lattices). In particularly, the Scott topology of every quasicontinuous domain is sober (each domain is a quasicontinuous domain, see [2]) for the detailed definition of a quasicontinuous domain). By applying the above theorem, we immediately have that for sober dcpos, the consistent Hoare powerdomains are uniquely determined up to isomorphism:

**Corollary 3.11.** If dcpos $P_1$ and $P_2$ are sober, then $H_c(P_1) \cong H_c(P_2)$ iff $P_1 \cong P_2$.

**Proof.** By Theorem 3.10, $H_c(P_1) \cong H_c(P_2)$ implies $\Gamma_F(H_c(P_1)) \cong \Gamma_F(H_c(P_2))$ iff $\Gamma_0(P_1) \cong \Gamma_0(P_2)$ iff $P_1 \cong P_2$ implies $H_c(P_1) \cong H_c(P_2)$.

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