Phase-number uncertainty from Weyl commutation relations

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We derive suitable uncertainty relations for characteristics functions of phase and number variables obtained from the Weyl form of commutation relations. This is applied to finite-dimensional spin-like systems, which is the case when describing the phase difference between two field modes, as well as to the phase and number of a single-mode field. Some contradictions between the product and sums of characteristic functions are noted.

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I. INTRODUCTION

Uncertainty relations are a key item of the quantum theory. This is from fundamental reasons, but also regarding practical applications, since phase-number uncertainty relations are the heart of the quantum limits to the precision of signal detection schemes [1].

Typically, uncertainty relations are expressed in terms of variances and are derived directly from the Heisenberg form of commutation relations. However, this approach is not always useful. On the one hand, variance may not be a suitable uncertainty measure. This is specially clear regarding periodic phase-angle variables [2]. On the other hand, the phase may not admit a simple well-behaved operator description suitable to obey a Heisenberg form of commutation relations with the number operator [3–5]. This has lead to the introduction of alternative uncertainty relations [4–9], some of them involving characteristic functions [10].

In this regard, a recent work has proposed an uncertainty relation for position and momentum based on characteristic functions, which is derived directly from the Weyl form of commutation relations [11]. In this work we translate this approach to phase-number variables. Despite the problems that quantum phase encounters, a very fundamental approach admits without difficulties the Weyl form of commutation relations and has well-defined characteristic functions. Therefore, the approach in Ref. [11] is a quite interesting formulation particularly suited to phase-angle variables. We also show that this encounters fundamental ambiguities when contrasting different slightly different alternative implementations, as it also holds for other approaches [12–14].

Let us point out that the Weyl form is equivalent to say that every system state experiences a global phase shift after a cyclic transformation in the corresponding phase space. This implies that the quantum structure including uncertainty relations might be traced back to a geometric phase [15].

II. SPIN-LIKE SYSTEMS

Let us consider general systems describable in a finite-dimensional space as a spin $j$. This admits very general scenarios, including especially the phase difference between two modes of the electromagnetic field. This is because the total number of photons $N$ is compatible with the phase difference and defines finite-dimensional subspaces of dimension $N + 1$, where $N$ plays the role of the spin modulus as $j = N/2$ [10].

Let us focus on a spin component $j_3$ and the canonically conjugate phase $\phi$. To avoid periodicity problems we focus on the complex exponential of $\phi$, we shall call $E$, this is $E = e^{i\phi}$. The eigenvectors $E|m\rangle = e^{i\frac{2\pi}{N}m}|m\rangle$ can be referred to as phase states [17], being

$$|m\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} e^{-i\frac{2\pi}{N}m\tilde{m}} |m\rangle,$$

where $|m\rangle$ are the eigenstates of $j_3$, as usual $j_3|m\rangle = m|m\rangle$, and $m, \tilde{m} = -j, -j + 1, \ldots, j$. Likewise, we may define the exponential of $j_3$ as

$$F = e^{i\frac{2\pi}{N}\tilde{m}j_3}.$$  

These exponentials $E$ and $F$ are quite suited to the Weyl form of commutation relation [18]

$$E^k F^\ell = e^{-i\frac{2\pi}{N}k\ell} F^\ell E^k,$$

for any $k, \ell \in \mathbb{Z}$. It is worth noting that the Weyl form has a quite interesting meaning when expressed as

$$E^{k\ell} F^{\ell k} E^k F^\ell = e^{-i\frac{2\pi}{N}k\ell}.$$  

This represents a cyclic transformation in the form of a closed excursion over a $k \times \ell$ rectangle in the associated phase space for the problem. The result is that every system state acquires a global phase after returning to the starting point.

Following Ref. [11] we can construct the Gram matrix $G$ for the following three vectors

$$|\Psi\rangle, \quad F^{\ell} |\Psi\rangle, \quad E^{k} |\Psi\rangle,$$

where
where $|\Psi\rangle$ is an arbitrary state assumed pure for simplicity and without loss of generality, so that
\[ G = \begin{pmatrix} 1 & \Phi & \tilde{\Phi} \\ \Phi^* & 1 & \Omega \\ \tilde{\Phi}^* & \Omega^* & 1 \end{pmatrix}, \] (6)

involving the characteristic functions
\[ \Phi = \langle \Psi | F^k | \Psi \rangle, \quad \tilde{\Phi} = \langle \Psi | E^k | \Psi \rangle, \] (7)

and
\[ \Omega = \langle \Psi | F^k E^k | \Psi \rangle, \] (8)

which is the term invoking the Weyl commutator (3). From this point we can follow exactly the same steps in Ref. [11]. These involve to construct another Gram matrix after replacing $k$ by $-k$ and $\ell$ by $-\ell$, adding the two determinants, using Eq. (3) and then following some clever simple algebraic bounds. This leads to
\[ |\Phi|^2 + |\tilde{\Phi}|^2 \leq B, \] (10)

with
\[ B = 2\sqrt{2} - 1 - \cos \gamma \] (11)

where $\Theta = \Omega \Phi \tilde{\Phi}^*$. From this point we can follow exactly the same steps in Ref. [11]. These involve to construct another Gram matrix after replacing $k$ by $-k$ and $\ell$ by $-\ell$, adding the two determinants, using Eq. (3) and then following some clever simple algebraic bounds. This leads to
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Therefore, most of the analysis and results found in Ref. [11] could be translated here. Even, the limit of vanishing argument of the characteristic functions may be reproduced in the limit of very large $j$.

Besides the sums, uncertainty relations can be also formulated as the products of uncertainty estimators. In our case from Eq. (10) we can readily derive a bound for the product of characteristic functions
\[ |\Phi| |\tilde{\Phi}| \leq B/2. \] (12)

A rather interesting point is that this can lead to conclusions fully opposite to the sum relation (10). This is specially so regarding the minimum uncertainty states, as we shall clearly show by some examples below.

The smallest value for the bound $B$ is obtained for $\gamma = \pi$. In such a case, the sum of the two Gram matrices commented above leads directly to the uncertainty relation
\[ |\Phi|^2 + |\tilde{\Phi}|^2 + |\Omega|^2 \leq 1, \] (13)

where the $\Omega$ term is expressing phase-number correlations that in standard variance-based approaches is expressed by the the anti-commutator. If this correlation term is ignored we get the more plain relations:
\[ |\Phi|^2 + |\tilde{\Phi}|^2 \leq 1, \quad |\Phi| |\tilde{\Phi}| \leq 1/2. \] (14)

Let us note that these relations might be called certainty instead of uncertainty since we get upper bounds for characteristic functions, that take their maximum value when there is full certainty about the corresponding variable.

### A. Example: qubit

The most simple and illustrative example is provided by the case $j = 1/2$. The most general state is of the form $\rho = (\sigma_0 + s \cdot \sigma)/2$ where $\sigma$ are the Pauli matrices, $\sigma_0$ is the identity, and $s = \text{tr}(\rho \sigma)$ is a three-dimensional real vector with $|s| \leq 1$. We can chose the basis so that $F = \sigma_z$ and $E = \sigma_x$. The only nontrivial uncertainty relation holds for $k = \ell = 1$ so that $\gamma = \pi$, $B = 1$
\[ \Phi = s_z, \quad \tilde{\Phi} = s_x, \quad \Omega = is_x s_y s_z, \] (15)

and Eqs. (10), (13), and (12) become, respectively
\[ s_x^2 + s_z^2 \leq 1, \quad s_x^2 + s_z^2 + s_y^2 s_z^2 \leq 1, \quad |s_x s_z| \leq 1/2, \] (16)

Actually, $s_x^2 + s_z^2 \leq 1$ is a well-known duality relation expressing complementarity [19]. The minimum uncertainty both for Eqs. (10) and (13) holds for every pure state $|s| = 1$ with $s_y = 0$.

Turning our attention to the alternative product of characteristic functions in Eq. (12) we get that the minimum uncertainty states are those pure states with $s_y = 0$ and $|s_x| = |s_z| = 1/\sqrt{2}$. On the other hand, the states with $s_y = 0$ and $|s_z| = 0$ or $|s_z| = 0$ are of maximum uncertainty, contrary to the predictions of the sum relations (10) and (13).

### III. SINGLE-MODE PHASE AND NUMBER

Next we address the case of the number and phase for a single field mode. There is always the possibility of addressing this from the number and phase difference taking a suitable reference state in one of the modes [16] [20], but the direct approach has also its advantages. One of them is that it faces the fact that, roughly speaking, there is no phase operator. The exponential of the phase is not unitary, but represented instead by the one-sided unitary Susskind-Glogower operator [4] [5].

\[ E = \sum_{n=0}^\infty |n\rangle \langle n+1|, \quad E|\phi\rangle = e^{i\phi}|\phi\rangle, \] (17)

where $|n\rangle$ are the eigenstates of the number operator $\hat{n}$, and $|\phi\rangle$ are the phase states
\[ |\phi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^\infty e^{in\phi}|n\rangle. \] (18)

The lack of unitary is conveniently expressed as
\[ E^k E^{jk} = I, \quad E^{jk} E^k = I - \hat{N}_k, \] (19)
where \( \hat{\Pi}_k \) is the orthogonal projector on the subspace with less than \( k \) photons

\[
\hat{\Pi}_k = \sum_{n=0}^{k-1} |n\rangle \langle n|,
\]

and \( I \) is the identity.

This does not prevent the existence of a proper probability distribution for the phase in any field state \( \rho \). This can be defined thanks to the phase states \( |\Phi\rangle \) as \( P(\phi) = \langle \phi | \rho | \phi \rangle \), that lead to the characteristic function

\[
\hat{\Phi} = \int d\phi e^{ik\phi} P(\phi) = \text{tr} (E^k \rho),
\]

where the last equality holds because for all \( k \)

\[
E^k = \int d\phi e^{ik\phi} |\phi\rangle \langle \phi|,
\]

in spite of the fact that the phase states are not orthogonal. This is to say that the lack of unitarity is equivalent to a description of phase in terms of a positive-operator measure.

Despite the lack of unitarity of \( E^k \) there is also a suitable Weyl form of commutation relations

\[
E^k e^{i\phi\tilde{\Phi}} = e^{ik\phi} e^{i\phi\tilde{\Phi}} E^k.
\]

Since the operator \( E \) is not unitary, the change of \( k \) by \(-k\) is not trivial, so in order to follow the procedure in Ref. [11] we have to construct explicitly the two Gram matrices.

The first one for the vectors

\[
|\Psi\rangle, \quad e^{i\phi\tilde{\Phi}} |\Psi\rangle, \quad E^{\dagger k} |\Psi\rangle,
\]

is

\[
G_+ = \begin{pmatrix}
1 & \Phi & \tilde{\Phi} \\
\Phi^* & 1 & \Omega \\
\tilde{\Phi}^* & \Omega^* & 1
\end{pmatrix},
\]

with

\[
\det G_+ = 1 - |\Phi|^2 - |\tilde{\Phi}|^2 - |\Omega|^2 + \Theta + \Theta^* \geq 0,
\]

being in this case

\[
\Theta = \Omega \Phi \tilde{\Phi}^*, \quad \text{and}
\]

\[
\Omega = \langle \Psi | e^{-i\phi\tilde{\Phi}} E^{\dagger k} |\Psi\rangle.
\]

The second Gram matrix corresponds to the change of \( \phi \) by \(-\phi\) and \( k \) by \(-k\), so the three vectors are now

\[
|\Psi\rangle, \quad e^{-i\phi\tilde{\Phi}} |\Psi\rangle, \quad E^k |\Psi\rangle,
\]

leading to

\[
G_- = \begin{pmatrix}
1 & \Phi^* & \tilde{\Phi}^* \\
\Phi & 1 & e^{-ik\phi} \Omega^* \\
\tilde{\Phi} & e^{ik\phi} \Omega & 1 - \Pi_k
\end{pmatrix},
\]

and

\[
\Pi_k = \langle \Psi | \Pi_k |\Psi\rangle
\]

where Eqs. (23) and (19) have been used, being \( \Pi_k = \langle \Psi | \Pi_k |\Psi\rangle \) and the same \( \Phi \), \( \tilde{\Phi} \) and \( \Omega \) in Eqs. (27) and (28). This leads to the determinant

\[
\det G_- = 1 - |\Phi|^2 - |\tilde{\Phi}|^2 - |\Omega|^2 + e^{ik\phi} \Theta + e^{-ik\phi} \Theta^* - \Pi_k \left( 1 - |\Phi|^2 \right) \geq 0,
\]

for the same \( \Theta \) above.

At this point several routes can be followed. For definiteness from now on we will focus always in the most stringent scenario of \( k\phi = \pi \). In such a case we readily get from the sum of Eqs. (26) and (31) the following bound:

\[
|\Phi|^2 + |\tilde{\Phi}|^2 + |\Omega|^2 + \frac{\Pi_k}{2} \left( 1 - |\Phi|^2 \right) \leq 1.
\]

The lack of unitarity of the exponential of the phase reflects in the presence of the \( \Pi_k \) term. Thus, whenever this term is absent \( \Pi_k = 0 \) we recover the same expressions obtained for the spin-like systems. Otherwise, this term might be also moved to the right-hand side of the relation meaning that the nonunitarity implies a lower upper bound in accordance with the noisy nature of positive-operator measures.

For definiteness, on what follows we will consider the following forms

\[
U = |\Phi|^2 + |\tilde{\Phi}|^2 \leq 1, \quad U' = |\Phi|^2 + |\tilde{\Phi}|^2 + |\Omega|^2 \leq 1,
\]

and

\[
U'' = |\Phi|^2 + |\tilde{\Phi}|^2 + |\Omega|^2 + \frac{\Pi_k}{2} \left( 1 - |\Phi|^2 \right) \leq 1,
\]

as well as the product

\[
V = |\Phi| |\tilde{\Phi}| \leq 1/2.
\]
A. Example: phase and number states

Readily simple examples are provided by the eigenstates of $\hat{n}$ and $E$. For the number states $|n\rangle$ we get for all $\phi$, $k$, and $n$ that $|\Phi\rangle = 1$, $\Phi = 0$, and there is no effect of the $\Pi_k$ term. Thus these are minimum uncertainty states. Note that we have the opposite conclusions regarding the uncertainty product.

On the other hand, the phase states (18) do not provide a suitable example since they are not normalizable. Instead, we can use their normalized counterparts, that are also eigenstates of $E$

$$|\Psi\rangle = \sqrt{1-|\xi|^2} \sum_{n=0}^{\infty} \xi^n |n\rangle, \quad E|\xi\rangle = \xi|\xi\rangle, \quad (36)$$

with mean number of photons $\bar{n} = |\xi|^2/(1-|\xi|^2)$. These states can be suitably approached in practice via quadrature squeezed states [21].

In this case it can be readily seen that

$$\Phi = \frac{1-|\xi|^2}{1-|\xi|^2 e^{-i\phi}}, \quad \Phi = \xi^k, \quad \Pi_k = 1-|\xi|^2, \quad (37)$$

and

$$\Omega = \xi^k e^{-i k \phi} \frac{1-|\xi|^2}{1-|\xi|^2 e^{-i\phi}} = e^{-i k \phi} \Phi \Phi^*. \quad (38)$$

In Fig. 1 we have represented the combinations $U$, $U'$ and $U''$ in Eqs. (33) and (34) as functions of $|\xi|$ for $k = 1$ and $\phi = \pi$. The minima of these functions represent maximum uncertainty and hold for phase states with very small mean number of photons, i.e., $\bar{n} = 0.6$, 0.7, and 1.3, for $U$, $U'$, and $U''$, respectively. On the other hand, when $|\xi| \to 1$, this is when $\bar{n} \to \infty$, we get $\Phi \to 0$, $|\Phi| \to 1$, and $\Pi_k \to 0$, as expected for ideal phase states, becoming minimum uncertainty states.

However when considering the product $V$ in Eq. (35) again for $k = 1$ and $\phi = \pi$ it can be easily seen after Eq. (37) that when $|\xi| \to 1$ and $|\xi| \to 0$ we get maximum uncertainty $V \to 0$, while $V$ attains its maximum value (i.e., minimum uncertainty), $V = 0.3$, when $|\xi| = 0.49$, this is $\bar{n} = 0.30$. Thus we see another clear example where maximum and minimum uncertainty states exchange their roles depending on the assessment of joint uncertainty considered.

B. Example: Complex Gaussian states

States with Gaussian statistics are usually minimum uncertainty states in typical variance-based uncertainty relations. Then it is worth examining the case in which the number statistics can be approximated by a Gaussian distribution. This will work provided that the distribution is concentrated in large photon numbers and that it is smooth enough so that the number $n$ can be treated as a continuous variable. Thus let us consider a pure state $|\psi\rangle$ with

$$\langle n|\psi\rangle \simeq \left(\frac{2\bar{n}}{\pi}\right)^{1/4} \exp\left[ -\frac{(a+ib)(n-\bar{n})^2}{4} \right], \quad (39)$$

where $\bar{n}$ represents the mean number, $a$ is given by the inverse of the number variance $\Delta^2 n = 1/(4a)$, and $b$ provides phase-number correlations taking positive as well as negative values. Consistently with the above approximations we shall consider $a \ll 1$ as well as $k \ll \bar{n}$. This situation includes the Glauber coherent states $|\alpha\rangle$ for large enough mean photon numbers $\Delta^2 n = \bar{n} \gg 1$ with $b = 0$. Throughout $k \Phi = \pi$ will be assumed.

In these conditions we readily get

$$|\Phi|^2 = \exp\left(-\frac{\bar{n}^2}{4a}\right), \quad |\Phi|^2 = \exp\left(-\frac{(a^2+b^2)k^2}{a}\right), \quad (40)$$

and

$$|\Omega|^2 = |\Phi|^2 |\Phi|^2 \exp\left(-\frac{b^2 k^2}{a}\right), \quad (41)$$

with $\Pi_k \simeq 0$.

The first thing we can notice is that $b$ increases phase uncertainty. Let us begin with the simplest case $b = 0$. We can focus first on the plain sum relation $U$ in Eq. (33), which is plotted in Fig. 2 in solid line as a function of $a$. We can see that $U$ is just a function of $ak^2$ and that minimum uncertainty, this is maximum $U$, holds for $ak^2$ tending both to 0 and infinity: this is when the state tends to be phase or number state, respectively, in accordance with the above results. In between we get a maximum uncertainty state, i.e., minimum $U$, when $ak^2 \simeq \pi/2$ that correspond to $|\Phi| = |\Phi|$, this is uncertainty equally split between phase and number. Similar results are obtained for $U'$ in the same Eq. (33), as shown in Fig. 2 in dashed line.

On the other hand, the situation is quite the opposite for the certainty product $V$ in Eq. (35): we have maximum uncertainty $V \to 0$ for phase and number states $ak^2 \to 0, \infty$, while we have minimum uncertainty, this is maximum $V$, for $ak^2 = \pi/2$. This is just the opposite of the conclusion of the sum of characteristics.
For the case $b \neq 0$ in Fig. 3 we have plotted the certainty sums $U$ and $U'$ as functions of $b$ for $\Delta^2 n = 10$ and $k = 1$, showing that from $b = 0$ increasing $b$ increases uncertainty until reaching $bk^2 = \pi / 2$ where a revival of $U'$ is produced, reaching the same certainty values around $b = 0$. Regarding the product $V$ we have the same behavior of the case $b = 0$ but the minimum uncertainty state holds for $ak^2 = (\pi / 2) \sqrt{a^2 / (a^2 + b^2)}$.

C. Example: $\hat{n} + i \lambda E^\dagger$ eigenstates

Looking for states with interesting phase-number relations we may consider the eigenstates of $\hat{n} + i \lambda E^\dagger$, where $\lambda$ is a real parameter [7]:

$$\left( \hat{n} + i \lambda E^\dagger \right) |\Psi\rangle = \mu |\Psi\rangle$$

(42)

that has the following solution, for $\mu = 0$ for definiteness,

$$|\Psi\rangle = \frac{1}{\sqrt{I_0(2\lambda)}} \sum_{n=0}^{\infty} \frac{(-i\lambda)^n}{n!} |n\rangle,$$

(43)

where $I_n$ are the corresponding modified Bessel functions.

It could be interesting to apply the previous approach to these states [43], looking for the $\lambda$ that lead to minimum uncertainty. Easily we obtain:

$$|\Phi|^2 = \frac{I_0 \left( 2 \lambda e^{i\phi/2} \right)^2}{I_0^2 (2\lambda)}, \quad |\Phi|^2 = \frac{I_0^2 (2\lambda)}{I_0^2 (2\lambda)},$$

(44)

and

$$|\Omega|^2 = \frac{|I_k (2\lambda e^{i\phi/2})|^2}{I_0^2 (2\lambda)}.$$ (45)

As before, we focus on the case $k = 1$ and $\phi = \pi$. Performing the numerical computation, we obtain plots of $U$ and $U'$ in Eq. [33] similar to the ones obtained in previous cases, as we can see in Fig. 4. The maximum uncertainty states are given by the minimum value of $U$ and $U'$ for a given $k$. In the present case the values of $\lambda$ which minimizes these functions are $\lambda = 0.77$ with $\tilde{n} = 0.8$, and $\lambda = 0.88$ with $\tilde{n} = 1.1$, respectively. Here again we obtain opposite results for the certainty product $V$ in Eq. [35].

D. Example: Phase-number intermediate states

Joint uncertainty relations of two observables are often minimized by states with properties somewhat intermediate between the two observables. The simplest case is a readily coherent superposition of number and phase states of the form

$$|\Psi\rangle \propto \alpha |n\rangle + \beta |\xi\rangle.$$ (46)

We focus on the case $n > 0$ and $|\xi| \to 1$ with the idea that $|\xi|$ approaches the ideal phase states [18]. In such a case it can be seen that the normalization condition is just $|\alpha|^2 + |\beta|^2 = 1$ and that

$$\Phi \simeq |\alpha|^2 e^{i\nu n}, \quad \hat{\Phi} \simeq |\beta|^2 e^{i\phi k}, \quad \Omega \simeq 0, \quad \Pi_1 \simeq 0,$$ (47)

and we shall consider $k = 1$ and $\phi = \pi$. Thus, Eq. [32] reads

$$|\alpha|^4 + |\beta|^4 \leq 1$$ (48)

Minimum uncertainty holds just in the limiting cases $\alpha \to 0$ and $\beta \to 0$, recovering the cases of phase and number states. On the other hand maximum uncertainty holds for the intermediate state $|\alpha|^2 = |\beta|^2 = 1/2$. Clearly, the situation is reversed if we consider the product $V$ so that maximum and minimum uncertainty are exchanged.
IV. DISCUSSION AND OUTLOOK

We have successfully derived meaningful phase-number uncertainty relations from the Weyl form of commutation relations. This can be applied to study phase-number statistical properties of meaningful field states, especially intermediate states that have already demonstrated interesting properties regarding uncertainty relations [7,13,14]. Moreover, this can be a suitable tool to explore quantum metrology limits. In typical interferometry $\hat{n}$ is the generator of phase shifts. Thus the characteristic function $\Phi$ is actually expressing the distinguishability of the probe state before and after a phase shift, which should be naturally related to detection resolution. Therefore this uncertainty relations may be connected to optimized signal detection schemes [22].

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