Volume formula for $N$-fold reduced products

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Abstract
Let $G$ be a semisimple compact connected Lie group. An $N$-fold reduced product of $G$ is the symplectic quotient of the Hamiltonian system of the Cartesian product of $N$ coadjoint orbits of $G$ under diagonal coadjoint action of $G$. Under appropriate assumptions, it is a symplectic orbifold. Using the technique of nonabelian localization and the residue formula of Jeffrey and Kirwan, we investigate the symplectic volume of an $N$-fold reduced product of $G$. Suzuki and Takakura gave a volume formula for the $N$-fold reduced product of $SU(3)$ in [25] by using geometric quantization and the Riemann–Roch formula. We compare our volume formula with theirs and prove that our volume formula agrees with theirs in the case of triple reduced products of $SU(3)$.

Résumé
Soit $G$ un groupe de Lie compact connexe semi-simple. Un produit réduit de $G$ ($N$ fois) est la réduction symplectique du système Hamiltonien du produit de $N$ orbites coadjointes de $G$ sous l’action coadjointe diagonale de $G$. Vu certaines hypothèses, c’est un orbi-espace symplectique. Nous utilisons la localisation non-abélienne et la formule de résidus de Jeffrey et Kirwan pour étudier le volume symplectique du produit réduit de $SU(3)$. Suzuki et Takakura ont donné une formule pour le volume du produit réduit ($N$-fois) de $SU(3)$ dans [25] en se servant de la quantification géométrique et la formule de Riemann–Roch. Nous considérons leur formule par rapport à la notre. Nous démontrons que notre formule accorde avec leur formule dans le cas $N = 3$.

Keywords
Hamiltonian group actions · Coadjoint orbits · Moment map · Symplectic volume

Mathematics Subject Classification 53D

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1 Introduction

In this article, we study objects called $N$-fold reduced products. The formal definitions will be given in later chapters. Roughly speaking, these objects are reduced spaces obtained from certain Hamiltonian systems.

Given a semisimple compact connected Lie group $G$, we may consider its adjoint orbits in its Lie algebra $\mathfrak{g}$. Notice that in this article we will fix a $G$-invariant inner product on $\mathfrak{g}$ and hence we can identify adjoint orbits with coadjoint orbits, which are naturally symplectic manifolds. If we are given $N$ such adjoint orbits, we can form their Cartesian product denoted by $M$. The group acts on this product diagonally through adjoint action. This is a Hamiltonian system with moment map $\mu_G : M \to \mathfrak{g}$. An $N$-fold reduced product is the reduced space $\mu_G^{-1}(0)/G$ of this Hamiltonian system.

In general, the geometry of these reduced products are very complicated and thus difficult to study. For example, they are in general not smooth manifolds. Even with the assumptions that we are going to make, they are still in general only orbifolds. Roughly speaking, an orbifold can be thought as a space which is almost smooth with mild singularities (see [8,24]). By the Marsden–Weinstein reduction theorem, such a reduced product naturally carries a symplectic structure and we are interested in understanding its symplectic geometry.

One of the most important symplectic invariants is the symplectic volume and this is the topic we will focus on the most in this article. We will use the technique of nonabelian localization and the residue formula, developed by Jeffrey and Kirwan [19,20], to study the symplectic volume and also the intersection pairings of these reduced products.

In 2008, Suzuki and Takakura studied the symplectic volume of $N$-fold reduced products of $G = \text{SU}(3)$ in their paper [25] via Riemann–Roch. Our volume formula generalizes their volume formula in the sense that their volume formula requires more restrictive inputs. Furthermore, in the case of $N = 3$, i.e., the case of triple reduced products of $\text{SU}(3)$, we have proved that up to normalization constants, our volume formula completely agrees with theirs.

In a later paper [26], Suzuki and Takakura generalize their results on symplectic volumes of $N$-fold reduced products of $G = \text{SU}(3)$ to the case where $G$ is a general compact Lie group. We are able to obtain results for the case of general $G$ as well. Our results appear in Sect. 5 below. Our methods may be adapted to generalizing from volumes to intersection pairings. We address that subject in [18].

$N$-fold reduced products appear in the literature in other guises as well. For example, $N$-fold reduced products of $G = \text{SU}(2)$ can be identified with moduli spaces of polygons in $\mathbb{R}^3$.
with prescribed lengths of edges, which have been studied by, for example, Hausmann and Knutson [15], Kamiyama and Tezuka [22]. By Jeffrey [17, Theorem 6.6 in that paper], \( N \)-fold reduced products of \( G \) can also be identified with moduli spaces of flat \( G \)-connections on a genus 0 surface with \( N \) boundary components and the corresponding holonomies conjugate to \( \exp(\xi_i) \) for prescribed \( \xi_i \in g \) (provided that the \( \xi_i \)'s are sufficiently small).

The organization of this article is as follows.

In Sect. 2, we investigate triple reduced products of \( SU(3) \), namely, the case of \( G = SU(3) \) and \( N = 3 \). Along the way, we also introduce the notations that can be easily generalized in the later chapters. After briefly reviewing the machinery of equivariant cohomology, we describe the method of nonabelian localization. In Sect. 3, we review the residue formula of Jeffrey and Kirwan [19,20]. Then we apply these techniques to derive our volume formula. In Sect. 4, we compare our volume formula with the volume formula of Suzuki and Takakura [25] and conclude this chapter with a proof that our volume formula matches with theirs in the case of triple reduced products of \( SU(3) \).

Finally in Sect. 5 we first generalize our volume formula of triple reduced products of \( SU(3) \) to the case of \( N \)-fold reduced products of \( SU(3) \). Then we further generalize it to the case of \( N \)-fold reduced products of a general semisimple compact connected Lie group \( G \).

In a later article [18], by applying the residue formula of Jeffrey and Kirwan [19,20], we compute the intersection pairings of \( N \)-fold reduced products.

## 2 Background

In this section, we set the stage by first reviewing some basic facts about \( SU(3) \) and then constructing triple reduced products from adjoint orbits of \( SU(3) \). Along the way we set the notations that will be used throughout this article.

### 2.1 Basic facts about \( SU(3) \)

We review the following basic material.

- Let \( G = SU(3) \), the collection of all invertible complex \( 3 \times 3 \) matrices \( A \) such that \( A^* = A^{-1} \) and \( \det(A) = 1 \), where \( A^* \) denotes the conjugate transpose of \( A \).
- Let \( g = su(3) \), the Lie algebra of \( SU(3) \). The vector space \( g \) is the collection of all complex \( 3 \times 3 \) matrices \( X \) such that \( X^* = -X \) and \( \text{tr}(X) = 0 \). Notice that \( g \) is a vector space over \( \mathbb{R} \) and \( \dim_{\mathbb{R}}(g) = 8 \).
- Let \( T \) be the standard maximal torus of \( G \). In other words, \( T \) is the collection of all \( 3 \times 3 \) diagonal matrices \( \text{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{-i(\theta_1+\theta_2)}) \) such that \( \theta_1, \theta_2 \) are real.
- Let \( t \) be the Lie algebra of \( T \). Thus \( t \) is the collection of all \( 3 \times 3 \) diagonal matrices \( \text{diag}(i\theta_1, i\theta_2, -i(\theta_1+\theta_2)) \) such that \( \theta_1, \theta_2 \) are real.
- Let \( g^* := \text{Hom}_{\mathbb{R}}(g, \mathbb{R}) \) be the dual vector space of \( g \), and let \( t^* := \text{Hom}_{\mathbb{R}}(t, \mathbb{R}) \) be the dual vector space of \( t \).
- If \( V \) is a vector space over a field \( \mathbb{F} \), let \( \langle \xi, X \rangle := \xi(X) \in \mathbb{F} \) denote the natural pairing between a covector \( \xi \in V^* := \text{Hom}_{\mathbb{F}}(V, \mathbb{F}) \) and a vector \( X \in V \). In this article, \( \mathbb{F} \) is either \( \mathbb{R} \) or \( \mathbb{C} \), depending on the context.
- For all \( g \in G \), let \( c(g) : G \to G \) denote the map \( x \mapsto gxg^{-1} \).
- For all \( g \in G \), let \( \text{Ad}(g) : g \to g \) denote the differential of \( c(g) \) at the identity \( e \in G \). Thus, \( \text{Ad} : G \to \text{Aut}(g) \) is the adjoint representation of \( G \) on \( g \). The differential of \( \text{Ad} \) at the identity \( e \in G \) is denoted by \( \text{ad} \). Thus, \( \text{ad} : g \to \text{End}(g) \) is the adjoint representation of \( g \) on \( g \).
For all $g \in G$, let $K(g) : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ be defined by:
\begin{equation}
\langle K(g)\xi, X \rangle = \langle \xi, \text{Ad}(g^{-1})X \rangle
\end{equation}
for all $\xi \in \mathfrak{g}^*$, $X \in \mathfrak{g}$. Thus, $K : G \rightarrow \text{Aut}(\mathfrak{g}^*)$ is the coadjoint representation of $G$ on $\mathfrak{g}^*$. The differential of $K$ at the identity $e \in G$ is denoted by $k$. Thus, $k : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}^*)$ is the coadjoint representation of $\mathfrak{g}$ on $\mathfrak{g}^*$. Notice that for all $X \in \mathfrak{g}$,
\begin{equation}
k(X) = - \text{ad}(X)^*.
\end{equation}
For all $X, Y \in \mathfrak{g}$, we define
\begin{equation}
(X, Y) := - \text{tr}(XY).
\end{equation}
Then $(\cdot, \cdot)$ is an $\text{Ad}(G)$-invariant (or briefly, $G$-invariant) inner product on $\mathfrak{g}$. We will use this inner product to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ through the standard identification between $X \in \mathfrak{g}$ and $(X, \cdot) \in \mathfrak{g}^*$ for all $X \in \mathfrak{g}$. Since $\mathfrak{t}$ is naturally a subspace of $\mathfrak{g}$ through the inclusion $\mathfrak{t} \subset G$, the above identification induces an identification between $\mathfrak{t}$ and $\mathfrak{t}^*$. It is in this sense that we write $\mathfrak{t}^* \subset \mathfrak{g}^*$.

Suppose $\xi \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$ satisfy
\begin{equation}
\langle \xi, Y \rangle = (X, Y)
\end{equation}
for all $Y \in \mathfrak{g}$. Namely, $\xi$ corresponds to $X$ under the identification through the inner product. For all $g \in G$, $Y \in \mathfrak{g}$,
\begin{align}
\langle K(g)\xi, Y \rangle &= \langle \xi, \text{Ad}(g^{-1})Y \rangle \\
&= (X, \text{Ad}(g^{-1})Y) \\
&= (\text{Ad}(g)X, Y).
\end{align}
Thus, $K(g)\xi$ corresponds to $\text{Ad}(g)X$ for all $g \in G$ under the identification through the inner product. It is in this sense that we say the coadjoint action of $G$ on $\mathfrak{g}^*$ corresponds to the adjoint action of $G$ on $\mathfrak{g}$, and it is in this sense that we identify coadjoint orbits with adjoint orbits. In this article, we will try to focus on the adjoint version of the theory since it is often easier to carry out computations with elements in $\mathfrak{g}$, which are matrices, than with elements in $\mathfrak{g}^*$.

The Weyl group $W$ is defined by $N(T)/T$, where $N(T)$ denotes the normalizer of $T$ in $G$. Suppose $g \in N(T)$. Then $g$ is a representative of one Weyl group element $w \in W$. This element $w$ acts on $T$ through
\begin{equation}
w \cdot t = gtg^{-1}
\end{equation}
for all $t \in T$. The element $w$ acts on $\mathfrak{t}$ through
\begin{equation}
w \cdot X = \text{Ad}(g)X
\end{equation}
for all $X \in \mathfrak{t}$.

Since $G = \text{SU}(3)$, $W$ is isomorphic to the permutation group $\mathfrak{S}_3$ of 3 letters. More precisely, let
\begin{equation}
W = \{s_0, s_1, s_2, s_3, s_4, s_5\}
\end{equation}
where
\begin{equation}
s_0 \text{ corresponds to } \text{Id} \in \mathfrak{S}_3,
\end{equation}
\[ s_1 \text{ corresponds to } (1 \, 2) \in S_3, \quad (2.12) \]
\[ s_2 \text{ corresponds to } (1 \, 2 \, 3) \in S_3, \quad (2.13) \]
\[ s_3 \text{ corresponds to } (1 \, 3) \in S_3, \quad (2.14) \]
\[ s_4 \text{ corresponds to } (1 \, 3 \, 2) \in S_3, \quad (2.15) \]
\[ s_5 \text{ corresponds to } (2 \, 3) \in S_3. \quad (2.16) \]

The signature \( \text{sgn}(w) \) of a Weyl group element \( w \in W \) is defined as the signature of its corresponding element \( \sigma \in S_3 \), i.e.
\[ \text{sgn}(w) := \text{sgn}(\sigma). \quad (2.17) \]

Notice that we have chosen the subscripts \( j \) in \( s_j \) so that
\[ \text{sgn}(s_j) = (-1)^j. \quad (2.18) \]

This will be convenient for later use. We shall introduce a chosen basis for \( t \) below, and express \( W \) in terms of this basis. In (3.63) we are able to write \( W \) as \( 2 \times 2 \) matrices in terms of this basis.

- The Weyl group elements act on diagonal matrices by permuting the diagonal entries. More precisely, if \( \sigma \in S_3 \), then \( \sigma \) corresponds to a Weyl group element \( w \in W \) and
\[ w \cdot \text{diag}(a_1, a_2, a_3) = \text{diag}(a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, a_{\sigma^{-1}(3)}). \quad (2.19) \]

For example,
\[ s_2 \cdot \text{diag}(a_1, a_2, a_3) = \text{diag}(a_3, a_1, a_2). \quad (2.20) \]

If \( \sigma \in S_3 \) corresponds to a Weyl group element \( w \in W \), we define
\[ \sigma \cdot X := w \cdot X \quad (2.21) \]
for all \( X \in t \).

- Since here \( G = SU(3) \), we can identify the Weyl group \( W \) with \( S_3 \) in the above way.
- Since the inner product \( (\cdot, \cdot) \) is \( \text{Ad}(G) \)-invariant on \( g \), it is \( W \)-invariant on \( t \).
- Let \( \mathcal{L} \) denote the integral lattice in \( t \), that is,
\[ \mathcal{L} := \left\{ H \in t : e^{2\pi i H} = I \right\}. \quad (2.22) \]

Thus,
\[ \mathcal{L} = \{ m_1 H_1 + m_2 H_2 : m_1, m_2 \in \mathbb{Z} \} \quad (2.23) \]

where
\[ H_1 = \text{diag}(i, -i, 0), \quad (2.24) \]
\[ H_2 = \text{diag}(0, i, -i). \quad (2.25) \]

- A weight is an element of \( t^* \) such that it takes integer values on \( \mathcal{L} \). We will use the identification through the inner product \( (\cdot, \cdot) \) to regard weights as elements in \( t \).

Let \( \lambda_1, \lambda_2 \) be elements in \( t \) such that
\[ (\lambda_i, H_j) = \delta_{ij} \quad (2.26) \]
for all \( i, j \in \{1, 2\} \). Then, we obtain
\[ \lambda_1 = \text{diag} \left( \frac{2}{3}i, -\frac{1}{3}i, -\frac{1}{3}i \right). \quad (2.27) \]
\[ \lambda_2 = \text{diag}\left( \frac{1}{3}i, \frac{1}{3}i, -\frac{2}{3}i \right). \] (2.28)

- A weight \( \lambda \in \mathfrak{t} \) is called dominant if and only if \( (\lambda, H_j) \geq 0 \) for all \( j \). A weight \( \lambda \in \mathfrak{t} \) is called integral if and only if \( (\lambda, H_j) \in \mathbb{Z} \) for all \( j \). Thus,

\[ \Lambda_{\geq 0} := \{ n_1\lambda_1 + n_2\lambda_2 : n_1, n_2 \text{ are both nonnegative integers} \} \] (2.29)

is the collection of dominant integral weights, and

\[ \mathfrak{t}_{\geq 0} := \{ c_1\lambda_1 + c_2\lambda_2 : c_1, c_2 \geq 0 \}, \] (2.30)

\[ \mathfrak{t}_{> 0} := \{ c_1\lambda_1 + c_2\lambda_2 : c_1, c_2 > 0 \} \] (2.31)

are the closed positive Weyl chamber and the open positive Weyl chamber, respectively.

- Let

\[ \alpha_1 := 2\lambda_1 - \lambda_2 = H_1, \] (2.32)

\[ \alpha_2 := -\lambda_1 + 2\lambda_2 = H_2 \] (2.33)

be the standard simple roots for \( G = \text{SU}(3) \). Following the convention of [19], let

\[ \mathcal{R}_+ = \{ 2\pi\alpha_1, 2\pi\alpha_2, 2\pi(\alpha_1 + \alpha_2) \} \] (2.34)

denote the set of \( 2\pi \)-modified positive roots of \( \text{SU}(3) \). From now on, if we talk about roots, we mean the \( 2\pi \)-modified ones unless stated otherwise.

### 2.2 Construction of triple reduced products of SU(3)

#### 2.2.1 Set-up

Let \( \xi \in \mathfrak{g} \). Let \( O_\xi \) denote the adjoint orbit through \( \xi \). Under the identification through the inner product \((\cdot, \cdot)\), it is equivalent to consider either adjoint orbits or their corresponding coadjoint counterparts. In this article we will mainly use the adjoint setting.

It is well known that every coadjoint orbit admits a natural symplectic form, called the Kirillov–Kostant–Souriau form, or briefly the KKS form. In the adjoint setting, it is defined as follows.

Suppose \( O \) is an adjoint orbit in \( \mathfrak{g} \) and \( \xi \in O \). Then the tangent space of \( O \) at \( \xi \), \( T_\xi O \), which is naturally a subspace of \( \mathfrak{g} \), is the following collection:

\[ T_\xi O = \{ \text{ad}(X)\xi : X \in \mathfrak{g} \}. \] (2.35)

Notice that \( \text{ad}(X)\xi = [X, \xi] \) for all \( X \in \mathfrak{g} \).

**Definition 2.1** The KKS form \( \omega \) on \( O \) is defined by

\[ \omega_\xi ([X, \xi], [Y, \xi]) := (\xi, [X, Y]) \] (2.36)

for all \( \xi \in O, X, Y \in \mathfrak{g} \).

The KKS form \( \omega \) is a closed nondegenerate 2-form. Equipped with \( \omega \), \( O \) becomes a compact symplectic manifold. The group \( G \) acts on \( O \) by the adjoint action and this makes \( O \) a Hamiltonian \( G \)-space with the inclusion map \( \mu_O : O \hookrightarrow \mathfrak{g} \) as the moment map. In other words, for all \( X \in \mathfrak{g} \),

\[ d\mu^X_O = \iota_X\omega \] (2.37)
where \( \mu_X^\xi(\xi) := (\mu_O(\xi), X) = (\xi, X) \) for all \( \xi \in \mathcal{O} \) and \( X^\xi \) is the fundamental vector field on \( \mathcal{O} \) generated by \( X \) and thus \( X^\xi(\xi) = [X, \xi] \in T_\xi \mathcal{O} \) for all \( \xi \in \mathcal{O} \). Notice that
\[
([X, Y], \xi) = (X, [Y, \xi]) \tag{2.38}
\]
for all \( X, Y, \xi \in \mathfrak{g} \).

Consider 3 points \( \xi_1, \xi_2, \xi_3 \) in \( \mathfrak{g} \). We then have 3 adjoint orbits \( \mathcal{O}_{\xi_1}, \mathcal{O}_{\xi_2}, \mathcal{O}_{\xi_3} \). Let \( \omega_i \) denote the KKS form on \( \mathcal{O}_{\xi_i} \). Now consider
\[
M := \mathcal{O}_{\xi_1} \times \mathcal{O}_{\xi_2} \times \mathcal{O}_{\xi_3}.
\]
Let \( \text{pr}_i : M \to \mathcal{O}_{\xi_i} \) be the standard projection onto the \( i \)-th factor. Then the form
\[
\omega := \sum_{i=1}^{3} \text{pr}_i^* \omega_i \tag{2.39}
\]
is a symplectic form on \( M \).

### 2.2.2 Triple reduced products

Let \( G \) act on \( M \) by
\[
g \cdot (\eta_1, \eta_2, \eta_3) := (\text{Ad}(g)\eta_1, \text{Ad}(g)\eta_2, \text{Ad}(g)\eta_3) \tag{2.40}
\]
for all \( g \in G, \eta_i \in \mathcal{O}_{\xi_i} \). Then this action makes \( M \) a Hamiltonian \( G \)-space with the moment map \( \mu_G : M \to \mathfrak{g} \) such that
\[
\mu_G(\eta_1, \eta_2, \eta_3) = \sum_{i=1}^{3} \mu_{\mathcal{O}_{\xi_i}}(\eta_i) = \sum_{i=1}^{3} \eta_i \tag{2.41}
\]
for all \( \eta_i \in \mathcal{O}_{\xi_i} \).

### 2.2.3 Assumptions

In this article, we make the following assumptions about the points \( \xi_1, \xi_2, \xi_3 \):

(A1) \( M_0 := \mu_G^{-1}(0) \neq \emptyset \) and \( 0 \in \mathfrak{g} \) is a regular value of \( \mu_G \).

(A2) \( \xi_i \in \mathfrak{t}_{\geq 0} \subset \mathfrak{g} \) for all \( i \) and each \( \mathcal{O}_{\xi_i} \) is diffeomorphic to the homogeneous space \( G/\mathfrak{t} \).

**Remark** The assumption (A1) ensures that the stabilizer \( \text{Stab}_G(\vec{\eta}) \) is finite for each \( \vec{\eta} = (\eta_1, \eta_2, \eta_3) \in M_0 \). Every adjoint orbit \( \mathcal{O} \) will intersect \( \mathfrak{t}_{\geq 0} \subset \mathfrak{g} \) at exactly one point \( \xi \), so by only considering \( \xi \in \mathfrak{t}_{\geq 0} \), we still obtain every possible orbit. This explains the first part of (A2). The second part of (A2) says that the orbits we will consider are nondegenerate, that is, they are of the highest dimension possible. In fact, assuming (A2) is equivalent to assuming that \( \xi_i \in \mathfrak{t}_{\geq 0} \subset \mathfrak{g} \) for all \( i \).

Let \( M^T \) denote the set of fixed points in \( M \) under the action of \( T \subset G \). We have the following.

**Proposition 2.2** Let \( M = \mathcal{O}_{\xi_1} \times \mathcal{O}_{\xi_2} \times \mathcal{O}_{\xi_3} \) be the Cartesian product of \( N = 3 \) adjoint orbits of \( G = \text{SU}(3) \), where the \( \xi_i \) satisfy the assumptions (A1) and (A2). Then, \( M^T \) is the discrete set
\[
\{(w_1 \cdot \xi_1, w_2 \cdot \xi_2, w_3 \cdot \xi_3) : w_i \in W\}.
\]
Thus, \( |M^T| = |W|^3 \).
Proof Note that
\[ O_{\xi_i} \cap t = W \cdot \xi_i \] (2.43)
for all \( i \). The elements of \( O_{\xi_i} \cap t \) are precisely those elements in \( O_{\xi_i} \) that are fixed by the adjoint action of \( T \subset G \). Since \( \xi_i \in t_{>0} \), \( O_{\xi_i} \cap t \) is discrete and has the same cardinality as \( W \). The proposition follows immediately. \( \Box \)

Since \( G = SU(3) \), we actually have \( 6^3 \) isolated fixed points in \( M \) under the action of \( T \subset G \).

Definition 2.3 The quotient space
\[ M_{\text{red}} := M_0 / G \] (2.44)
is called a triple reduced product of \( G \). In other words, \( M_{\text{red}} \) is the Marsden–Weinstein reduction of the Hamiltonian \( G \)-space \((M, \omega, G, \mu_G)\). Sometimes we may write \( M_{\text{red}}(\vec{\xi}) \) to emphasize the dependence on the initial data \( \vec{\xi} = (\xi_1, \xi_2, \xi_3) \).

Remark In general, \( M_{\text{red}} \) is not a smooth manifold. It belongs to a type of spaces called orbifolds or \( V \)-manifolds [24]. Roughly speaking, an orbifold is almost a smooth manifold except that it has some mild singularities. At these singularities, it locally looks like \( U / \Gamma \) where \( U \) is an open subset of \( \mathbb{R}^d \) and \( \Gamma \) is a finite group of linear automorphisms of \( U \).

Fortunately, [24] tells us that on such spaces the de Rham theory and Poincaré duality work basically the same way as on smooth manifolds (see also [8]). However, to avoid such complication, we add the following assumption about the \( \xi_i \):

(A3) \( M_{\text{red}}(\vec{\xi}) \) is a smooth manifold.

This assumption is equivalent to the assumption that for each \( \vec{\eta} \in M_0 \), \( \text{Stab}_G(\vec{\eta}) = Z(G) \).

2.3 Equivariant cohomology

To study the symplectic volume \( \text{vol}^S(M_{\text{red}}) \), we are basically looking at the cohomological quantity
\[ e^{\omega_{\text{red}}}[M_{\text{red}}] = \frac{1}{i^{d/2}} e^{i \omega_{\text{red}}}[M_{\text{red}}], \] (2.45)
where \([M_{\text{red}}]\) denotes the fundamental class of \( M_{\text{red}} \), which is picked up by the orientation induced by the symplectic volume form on \( M_{\text{red}} \).

This quantity can be computed using the technique called nonabelian localization due to Jeffrey and Kirwan ([19]; see also [20]) and in particular the residue formula (Theorem 8.1 in [19] and Theorem 3.1 in [20]). To state their results, we shall first review the machinery of equivariant cohomology.

In this article, we only consider cohomology groups over \( \mathbb{C} \). Let \( K \) be a compact connected Lie group. Let \( M \) be a \( K \)-space. The equivariant cohomology of the \( K \)-space \( M \) is denoted \( H^*_K(M) \).

\[ M \times_K EK := (M \times EK) / K, \] (2.46)

We will use the Cartan model ([6,7]; see also [3]) to compute equivariant cohomology. Let \( \mathfrak{k} \) denote the Lie algebra of \( K \). A \( K \)-equivariant differential form \( \alpha \) on \( M \) can be thought of as a \( K \)-equivariant polynomial map
\[ \alpha : \mathfrak{k} \to \Omega^*(M). \] (2.47)
Let $\Omega^k_K(\mathcal{M})$ denote the collection of all $K$-equivariant differential forms on $\mathcal{M}$. In other words,

$$\Omega^k_K(\mathcal{M}) = (S(t^*) \otimes \Omega^k(\mathcal{M}))^K.$$  \hfill (2.48)

Now we return to our situation, that is, the situation of triple reduced products of $G = \text{SU}(3)$.

There is another important map called the pushforward map

$$\Pi^G_* : H^*_G(M) \to H^*_G,$$  \hfill (2.49)

which can be thought of as integration over $M$. Here we have introduced the notation $H^n_G$ to mean $H^n_G(\text{pt})$, and likewise $H^n_T$ for $H^n_T(\text{pt})$. Similarly, when we are looking at the $T$-action on $M$, the corresponding pushforward map is

$$\Pi^T_* : H^*_T(M) \to H^*_T.$$  \hfill (2.50)

Usually, when the context is clear, we will denote both $\Pi^G_*$ and $\Pi^T_*$ by the integration symbol $\int_M$ or simply $\Pi_*$.  

Recall that 0 is a regular value for the moment map $\mu_G$. By [21], the ring homomorphism

$$i_0^* : H^*_G(M) \to H^*_G(M_0)$$  \hfill (2.51)

is surjective. In addition, we have a canonical isomorphism

$$\pi_0^* : H^*(\text{M}_{\text{red}}) \to H^*_G(M_0)$$  \hfill (2.52)

induced from the map

$$\pi_0 : M_0 \times G \rightarrow M_{\text{red}}.$$  \hfill (2.53)

We have a canonical map (the Kirwan map):

$$\kappa_0 := (\pi_0^*)^{-1} \circ i_0^* : H^*_G(M) \to H^*(\text{M}_{\text{red}}).$$  \hfill (2.54)

In [19], Jeffrey and Kirwan proved a formula (Theorem 8.1 in [19]) computing the following cohomological quantity

$$\kappa_0(\eta)e^{\log[M_{\text{red}}]}$$  \hfill (2.55)

for any $\eta \in H^*_G(M)$. Also in [20], they rewrote the residue formula (Theorem 3.1 in [20]) computing the following cohomological quantity

$$\kappa_0(\eta)e^{\log[M_{\text{red}}]}$$  \hfill (2.56)

for any $\eta \in H^*_G(M)$. Both articles basically compute the same quantity.

Before we state the formula, we first fix some notations which will be convenient for later use.

The following notations can be applied to any general semisimple compact connected Lie group $G$. In this section, to avoid ambiguity, we assume $G = \text{SU}(3)$ throughout.

- Let $s$ denote the real dimension of $G$.
- Let $l$ denote the real dimension of $T$.
- Let $\varpi$ denote the product of the positive roots of $G$ (here, roots are regarded as elements in $t^*$), that is,

$$\varpi(X) = \prod_{\gamma \in \mathbb{R}^+} \gamma(X)$$  \hfill (2.57)

$$\varpi \Lambda$$ Springer
for all $X$ in $t$ or $t_C$, where $t_C$ denotes the complexification of $t$. Therefore we can regard $\varpi$ as a polynomial function on $t$ or $t_C$.

- Notice that
  
  \[ \varpi(w \cdot X) = \text{sgn}(w)\varpi(X) \quad (2.58) \]

  for all $w \in W$ and all $X$ in $t$ or $t_C$.

- The fixed $G$-invariant inner product $(\cdot, \cdot)$ induces measures on $g$ and $t$ and their corresponding complexifications, $g_C$ and $t_C$.

- Following [19], we will reserve the Greek letter $\phi$ for a variable in $g$ or $g_C$ and the Greek letter $\psi$ for a variable in $t$ or $t_C$.

- The fixed $G$-invariant inner product $(\cdot, \cdot)$ induces measures on $g$ and $t$ and their corresponding complexifications, $g_C$ and $t_C$.

- Following [19], we will reserve the Greek letter $\phi$ for a variable in $g$ or $g_C$ and the Greek letter $\psi$ for a variable in $t$ or $t_C$.

- Let $\mu_T : M \rightarrow t$ denote the composition of the moment map $\mu_G$ with the orthogonal projection from $g$ to $t$. Notice that the orthogonal projection from $g$ to $t$ corresponds to the restriction map from $g^*$ to $t^*$ under the identification through the inner product. Therefore, $\mu_T$ is a moment map for the action of $T$ on $M$.

Now we can state the residue formula of Jeffrey and Kirwan, adapted to our situation.

**Theorem 2.4** ([19, Theorem 8.1], [20, Theorem 3.1]) Let $G$ be a general semisimple compact connected Lie group. Let $M = O_{\xi_1} \times \cdots \times O_{\xi_N}$ be the Cartesian product of $N \geq 3$ adjoint orbits of $G$, where the $\xi_i$ satisfy the assumptions (A1), (A2) and (A3) outlined in Sect. 2.2.3. Then, for all $\eta \in H^*_G(M)$, we have

\[ \kappa_0(\eta)e^{i\omega_{\text{red}}}[M_{\text{red}}] = n_0C_G \text{ res}\left( \varpi^2(\psi) \sum_{F \in M^T} r_F^\eta(\psi)[d\psi] \right) \quad (2.59) \]

where $n_0$ is the number of points in the stabilizer subgroup in $G$ of a generic point in $M_0$, and the constant $C_G$ is defined by

\[ C_G := \frac{(-1)^{n_+}}{(2\pi)^{s-l} |W| \text{ vol}^R(T)}. \quad (2.60) \]

Here, $n_+$ denotes the number of positive roots, that is, $n_+ = (s-l)/2$. $\psi$ is a variable in $t_C$.

Notice that in our situation, all fixed points in $M^T$ are isolated. If $F \in M^T$, the meromorphic function $r_F^\eta$ on $t_C$ is defined by

\[ r_F^\eta(\psi) := e^{i(\mu_T(F),\psi)} \int_F i_F^*(\eta(\psi)e^{i\omega}) \quad (2.61) \]

where $i_F : F \rightarrow M$ is the inclusion and $e_F$ is the $T$-equivariant Euler class of the normal bundle to $F$ in $M$. The notation $\text{ res}$ will be defined in Sect. 3.1 below.

**Remark** Recall that for each point $\vec{\eta} \in M_0$, $\text{ Stab}_G(\vec{\eta})$ is finite. A generic point in $M_0$ is a point $\vec{\eta} \in M_0$ such that the cardinality of $\text{ Stab}_G(\vec{\eta})$ is the smallest among all points in $M_0$. Therefore, the number $n_0$ above is well defined. In particular, the above theorem does not require the action $G \circlearrowright M_0$ to be free.
Since $F$ is an isolated fixed point, we have
\[
\int_F i_F^*(\eta(\psi)e^{i\omega}) e_F(\psi) = i_F^*(\eta(\psi)e^{i\omega}) e_F(\psi)
\] (2.62)
and the quantity
\[
i_F^*(\eta(\psi)e^{i\omega})
\] (2.63)
will be a $\mathbb{C}$-valued function on $F$, that is, a complex number. We denote this complex number by
\[
\eta_F(\psi) := i_F^*(\eta(\psi)e^{i\omega}) \in \mathbb{C}.
\] (2.64)
Furthermore, since $M = O_{\xi_1} \times O_{\xi_2} \times O_{\xi_3}$, any $F \in M^T$ can be written (see Proposition 2.2) by
\[
F = (w_1 \cdot \xi_1, w_2 \cdot \xi_2, w_3 \cdot \xi_3)
\] (2.65)
for some $\tilde{w} = (w_1, w_2, w_3) \in W \times W \times W$. Then we can write
\[
F = \tilde{w} \cdot \tilde{\xi} := (w_1 \cdot \xi_1, w_2 \cdot \xi_2, w_3 \cdot \xi_3).
\] (2.66)
In this case, $e_F(\psi)$ is the complex number
\[
\text{sgn}(\tilde{w}) \sigma^3(\psi)
\] (2.67)
where
\[
\text{sgn}(\tilde{w}) := \prod_{i=1}^{3} \text{sgn}(w_i).
\] (2.68)
Recall that $\sigma(\psi) = \prod_{\gamma>0} \gamma(\psi)$ is the product of all positive roots evaluated on $\psi \in \mathfrak{t}$. In this case, we can define
\[
\text{sgn}(F) := \text{sgn}(\tilde{w}).
\] (2.69)
Here $\text{sgn}(w)$ is the signature of the Weyl group element $w$ – this is equal to its signature as a permutation. In addition, we write $\tilde{w}(F)$ to mean the $\tilde{w}$ such that $F = \tilde{w} \cdot \tilde{\xi}$.

Now, we can rewrite Eq. (2.59) as
\[
\kappa_0(\eta)e^{i\omega_{\text{red}}}[M_{\text{red}}] = n_0 C_G \, \text{res} \left( \sigma^2(\psi) \sum_{F \in M^T} e^{i(\mu_T(F),\psi)} \frac{\eta_F(\psi)}{\text{sgn}(F) \sigma^3(\psi)} [d\psi] \right).
\] (2.70)
This can be further simplified to
\[
\kappa_0(\eta)e^{i\omega_{\text{red}}}[M_{\text{red}}] = n_0 C_G \, \text{res} \left( \sum_{F \in M^T} \text{sgn}(F) \frac{\eta_F(\psi)}{\sigma(\psi)} e^{i(\mu_T(F),\psi)} [d\psi] \right).
\] (2.71)

We have the following result as a corollary of Theorem 2.4.

**Corollary 2.5** Let $M = O_{\xi_1} \times O_{\xi_2} \times O_{\xi_3}$ be the Cartesian product of $N = 3$ adjoint orbits of $G = SU(3)$, where the $\xi_i$ satisfy the assumptions (A1), (A2) and (A3) outlined in Sect. 2.2.3. Then, the symplectic volume of $M_{\text{red}}$ can be expressed as
\[
\text{vol}^S(M_{\text{red}}) = \frac{1}{i^d/2} n_0 C_G \, \text{res} \left( \sum_{\tilde{w} \in W^3} \text{sgn}(\tilde{w}) \frac{e^{i(w_1 \cdot \xi_1 + w_2 \cdot \xi_2 + w_3 \cdot \xi_3,\psi)}{\sigma(\psi)} [d\psi] \right).
\] (2.72)
Here, $d$ is the real dimension of $M_{\text{red}}$.

**Proof** To obtain the symplectic volume $\text{vol}^S(M_{\text{red}})$, we let $\eta = 1 \in H^*_G(M)$ and compute:

$$\text{vol}^S(M_{\text{red}}) = \frac{1}{i^{d/2}} e^{i0_{\text{red}}[M_{\text{red}}]} = \frac{1}{i^{d/2}} \kappa_0(1)e^{i0_{\text{red}}[M_{\text{red}}]} = \frac{1}{i^{d/2}} n_0 C_G \text{ res} \left( \sum_{F \in M^T} \text{sgn}(F) \frac{e^{i(\mu_T(F),\psi)}}{\sigma(\psi)} [d\psi] \right).$$

(2.73)

Since the fixed point set $M^T$ can be parametrized by

$$\vec{w} = (w_1, w_2, w_3) \in W^3,$$

(2.74)

the sum

$$\sum_{F \in M^T} \text{sgn}(F) \frac{e^{i(\mu_T(F),\psi)}}{\sigma(\psi)}$$

(2.75)

can be rewritten as

$$\sum_{\vec{w} \in W^3} \text{sgn}(\vec{w}) \frac{e^{i(w_1\xi_1 + w_2\xi_2 + w_3\xi_3,\psi)}}{\sigma(\psi)}.$$

(2.76)

\[\square\]

To fully understand the above formulas, we need to understand the residue res. We describe this in the next section.

**3 The residue**

In this section we shall study the residue res in detail. The main references for this section are [19,20].

**3.1 Definition of the residue**

The Fourier transform is important to the development of nonabelian localization in [19]. Following [19], we reserve the letter $z$ for a variable in $g^*$ and the letter $y$ for a variable in $t^*$.

If $f : g \to \mathbb{C}$ is a tempered distribution, we define its Fourier transform $F_G f$ on $g^*$ to be

$$(F_G f)(z) := \frac{1}{(2\pi)^{d/2}} \int_{\phi \in g} f(\phi) e^{-i(\phi,\psi)} [d\phi].$$

(3.1)

Since $f : g \to \mathbb{C}$ can be regarded as a tempered distribution on $t$ by restriction to $t$, we can also define its Fourier transform $F_T f$ on $t^*$ to be

$$(F_T f)(y) := \frac{1}{(2\pi)^{d/2}} \int_{\psi \in t} f(\psi) e^{-i(y,\psi)} [d\psi].$$

(3.2)

Proposition 8.4 in [19] (see also [16]) gives a characterization of the residue. Based on this proposition, we have the following definition:
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**Definition 3.1** (Definitions 8.5, the paragraph before Definition 8.8, and Definition 8.8 in [19]) Let \(\Lambda\) be a proper cone in \(t\). Let \(\Lambda^0\) denote the interior of \(\Lambda\). Let \(h\) be a holomorphic function on \(t - i\Lambda^0 \subset t\mathbb{C}\) such that for every compact subset \(K\) of \(t - i\Lambda^0\) there exists a constant \(C_K\) and an integer \(N_K \geq 0\) such that

\[|h(\zeta)| \leq C_K (1 + |\zeta|)^{N_K}\]

(3.3)

for all \(\zeta \in K\). Let \(\chi : g^* \to \mathbb{R}\) be a smooth invariant function with compact support and strictly positive in some neighbourhood of 0. Let \(\hat{\chi} = F_G\chi : g \to \mathbb{C}\) be its Fourier transform.

Let \(\hat{\chi}\epsilon(\phi) = \hat{\chi}(\epsilon\phi)\) so that

\[\left(F_G\hat{\chi}\epsilon\right)(z) = \chi\epsilon(z) := \frac{1}{\epsilon^s}\chi\left(\frac{z}{\epsilon}\right).\]

(3.4)

Also we assume \(\hat{\chi}(0) = 1\). Then we define

\[\operatorname{res}^{\Lambda,\chi}(h(\psi))[d\psi]) := \lim_{\epsilon \to 0^+} \frac{1}{(2\pi i)^l} \int_{\psi \in t - i\xi} \hat{\chi}\epsilon(h(\psi))[d\psi],\]

(3.5)

where \(\xi\) is any element of \(\Lambda^0\). Furthermore, in the case where \(h\) is a sum of other functions \(h_i\) and \(F_T h\) is smooth at 0 but the Fourier transforms of \(h_i\) may not be smooth at 0, we need to introduce a small generic parameter \(\rho \in t^*\) so that all the functions in this sum have Fourier transforms that are smooth at 0. More precisely, let \(\Lambda, \chi\) and \(h\) be as in the above. Let \(\rho \in t^*\) be such that the distribution \(F_T h\) is smooth on the ray \(t\rho\) for \(t \in (0, \delta)\) for some \(\delta > 0\), and suppose \(\left(F_T h\right)(t\rho)\) tends to a well defined limit as \(t \to 0^+\). Then we define

\[\operatorname{res}^{\rho,\Lambda,\chi}(h(\psi)[d\psi]) := \lim_{t \to 0^+} \operatorname{res}^{\Lambda,\chi}(h(\psi)e^{i(t\rho,\psi)}[d\psi]).\]

(3.6)

**Remark** By the paragraph after Definition 8.5 in [19], the integral in (3.5) converges and is independent of \(\xi \in \Lambda^0\). Furthermore, by Propositions 8.6, 8.7 and 8.9 in [19], the residue of a meromorphic form \(\Omega\) is independent of the choices of \(\Lambda, \chi\) and \(\rho\) if \(\Omega\) is sufficiently well behaved, as is the case for Theorem 2.4.

### 3.2 Nonabelian localization

Before we go into the computational aspects of the residue, we shall briefly sketch the proof of Theorem 2.4 by summarizing key points in [19], so that we will have a better idea about why the nonabelian localization technique in [19] works.

The first key point is the abelian localization formula:

**Theorem 3.2** (Berline and Vergne [1], [2], [5], [9], Theorem 2.1 in [19]) Let \(G\) be a general semisimple compact connected Lie group and \(M\) a symplectic manifold equipped with a Hamiltonian action of \(G\). Assume that 0 is a regular value of the moment map. If \(\sigma \in H^*_T(M)\), then

\[\left(\Pi_\sigma\sigma\right)(\psi) = \sum_{F \in \mathcal{MT}} \int_F i_F^\#(\sigma(\psi)) e_F(\psi).\]

(3.7)

A special case of Theorem 3.2 is the case when \(M\) is a product of coadjoint orbits:

**Theorem 3.3** Let \(G\) be a general semisimple compact connected Lie group and \(M = O_{\xi_1} \times \cdots \times O_{\xi_N}\) be the Cartesian product of \(N \geq 3\) adjoint orbits of \(G\), where the \(\xi_i\) satisfy the
assumptions (A1), (A2) and (A3) outlined in Sect. 2.2.3. If $\sigma \in H^*_T(M)$, then
\[(\Pi_\ast \sigma)(\psi) = \sum_{F \in MT} \int_F i_F^*(\sigma(\psi)) e_F(\psi). \tag{3.8}\]

We are interested in the case when $\sigma = \eta e^{i\bar{\omega}}$ where $\eta \in H^*_G(M)$ and
\[\tilde{\omega}(\phi) = \omega + \mu_G(\phi) \tag{3.9}\]
is the standard equivariant extension of $\omega$. Notice that here $\sigma = \eta e^{i\bar{\omega}}$ is regarded as a $T$-equivariant cohomology class through the restriction map $g^* \to \mathfrak{t}^*$. Let $r_\eta := \Pi_\ast (\eta e^{i\bar{\omega}}) \in H^*_T$, where we have defined $H^*_T := H^*_T(pt)$ as the $T$-equivariant cohomology of a point. This was (2.3). By Theorem 3.2, we have
\[r_\eta(\psi) = \sum_{F \in MT} r_\eta^F(\psi), \tag{3.10}\]
where
\[r_\eta^F(\psi) = e^{i\mu_T(F, \psi)} \int_F i_F^*(\eta(\psi)e^{i\bar{\omega}}) e_F(\psi). \tag{3.11}\]

It turns out that the Fourier transform of $\Pi_\ast (\eta e^{i\bar{\omega}})$ will be closely related to the cohomological quantity $\kappa_0(\eta)e^{i\bar{\omega}}[M_{red}]$.

Proposition 8.10 in [19] gives an explicit characterization of the residue.

Remark The rest of the proof for [19, Proposition 8.10] depends on the following important result, the normal form theorem for the symplectic structure and moment form near a regular value. The normal form theorem is due to Gotay [10], Guillemin and Sternberg [14] and Marle [23]. The statement is as follows.

**Proposition 3.4** (Proposition 5.2 in [19]) Assume 0 is a regular value of $\mu_G$. Then there is a neighbourhood
\[U \cong M_0 \times \{z \in \mathfrak{g}^* : |z| < h\} \subset M_0 \times \mathfrak{g}^* \tag{3.12}\]
where $h > 0$ is some sufficiently small number, of $M_0$ on which the symplectic form $\omega$ can be given as follows. Recall that $p_0 : M_0 \to M_{red}$ is the orbifold principal $G$-bundle. Let $\theta \in \Omega^1(M_0) \otimes \mathfrak{g}$ be a connection on this principal bundle. Recall that on $M_{red}$ there is a symplectic structure $\omega_{red}$ such that $p_0^*\omega_{red} = i^*_0\omega$. Let $\alpha$ be a 1-form on $U \subset M_0 \times \mathfrak{g}^*$ defined by
\[\alpha(p, z)(v, \xi) := \{z, \theta_p(v)\} \tag{3.13}\]
for all $p \in M_0$, $z \in \mathfrak{g}^*$ with $|z| < h$, $v \in T_p M_0$ and $\xi \in T_z \mathfrak{g}^* = \mathfrak{g}^*$. Then the symplectic form $\omega$ on $U$ is
\[\omega = p\gamma\gamma_0^* p_0^* \omega_{red} + d\alpha. \tag{3.14}\]
Moreover, the moment map $\mu_G$ on $U$ is $\mu_G(p, z) = z$.

The authors of [19] used a sequence of appropriately chosen test functions $\chi_\epsilon : \mathfrak{g}^* \to \mathbb{R}_{\geq 0}$ such that as $\epsilon \to 0$, the functions $\chi_\epsilon$ tend to the Dirac delta distribution on $\mathfrak{g}^*$. They integrated $F_G(\Pi_\ast (\eta e^{i\bar{\omega}}))$ against this sequence of test functions $\chi_\epsilon$. By invoking Proposition 3.4,
they were able to concentrate on arbitrarily small neighbourhoods of $M_0$ and obtain the estimate which eventually established the link between $F_G(\Pi_u(\eta e^{i\theta})))(0)$ and the cohomological quantity $\kappa_0(\eta)e^{i\theta}c_{M_{\text{red}}}$, finishing the proof for [19, Proposition 8.10].

We need two more properties of the residue. The first is as follows.

**Proposition 3.5** (Proposition 8.7 in [19]) Let $u : t^* \to \mathbb{C}$ be a distribution, and assume the set $\Gamma_u$ defined in Proposition 3.1 contains $-\Lambda^0$. Then, $h = FT u$ is a holomorphic function on $t - \Lambda^0$ and $h$ satisfies the hypotheses in Definition 3.1. Assume in addition that $u$ is smooth at $0$. Then $\text{res}_{\Lambda^0,\chi}(h(\psi)[d\psi])$ is independent of the test function $\chi$, and moreover,

$$\text{res}_{\Lambda^0,\chi}(h(\psi)[d\psi]) = \frac{1}{i(2\pi)^{1/2}}u(0).$$

(3.15)

The second important realization is provided by the following point due to Guillemin, Lerman, Prato and Sternberg [11–13]:

**Proposition 3.6** (Proposition 3.6 in [19])

(a) (Part (a) of Proposition 3.6 in [19]; see [11], Section 3.2 in [12], and [13]) Define

$$H_{\tilde{\beta}}(y) = \text{vol} \left\{ (s_1, \ldots, s_v) : s_i \geq 0, \ y = \sum_j s_j \beta_j \right\}$$

for some $v$-tuple $\tilde{\beta} = (\beta_1, \ldots, \beta_v)$ with $\beta_j \in t^*$ such that all $\beta_j$ lie in the interior of some half-space of $t^*$, where $v$ is a positive integer.

Here the reason for $s_i \geq 0$ is that we are working with the cone spanned by $\tilde{\beta}$. Here $\text{vol}$ denotes the standard Euclidean volume multiplied by a normalization constant which is chosen so that Eq. (3.18) below holds. Thus, $H_{\tilde{\beta}}$ is a piecewise polynomial function supported on the cone

$$C_{\tilde{\beta}} := \left\{ \sum_j s_j \beta_j : s_j \geq 0 \right\}.$$ (3.17)

Let $h(y) := H_{\tilde{\beta}}(y + \tau)$ for some $\tau \in t^*$. Then the Fourier transform of $h$ is given for $\psi$ in the complement of the union of the hyperplanes $\{ \psi \in t : \beta_j(\psi) = 0 \}$ by the formula

$$\text{FT} h(\psi) = \frac{e^{i(\tau,\psi)}}{\prod_{j=1}^v \beta_j(\psi)}.$$ (3.18)

By these considerations, $\text{res}(r_F^\eta(\psi)[d\psi])$ can be computed as

$$\text{res}^{\rho,\Lambda^0,\chi}(r_F^\eta(\psi)[d\psi]) = \lim_{t \to 0^+} \frac{1}{(2\pi)^{1/2}i} \text{FT} r_F^\eta(t\rho).$$ (3.19)

Recall that $r^\eta$ was defined above in (3.10), by Definition 3.1 and Proposition 3.5. Thus

$$\sum_{F \in M^T} \text{res}^{\rho,\Lambda^0,\chi}(\sigma(\psi)r_F^\eta(\psi)[d\psi]) = \text{res}^{\rho,\Lambda^0,\chi}(\sigma(\psi)r^\eta(\psi)[d\psi])$$

$$= \frac{1}{(2\pi)^{1/2}i} \text{FT}(\sigma^2r^\eta)(0),$$ (3.20)

providing the last link for the proof of Theorem 2.4.

Please note the following.
(1) Theorem 8.1, [19]:

\[ i \operatorname{vol}(M) = e^{i \omega_0} [M_X] = (-1)^{n^+} B \operatorname{Res} \sum_w (-1)^w e^{i < w \lambda, X>} / D(X) \]

where \( B \) is a real constant. The constant \( B \) equals

\[ \frac{1}{(2\pi)^{s-\ell}|W| \operatorname{vol}(T)}. \]

(2) [19, Proposition 8.11 (ii)]:

\[ \operatorname{Res} (e^{i \lambda(X)} \prod_j \beta_j(X)) = i^{n^+ - \ell} H_\beta \]

where \( H_\beta \) was defined in [19, Proposition 3.6 equations (3.18)]. See also [19, (8.28)], where it is stated that the residue of a function \( h \) is equal to

\[ \frac{1}{2\pi i} \int h(\psi) d\psi. \]

In the \( SU(3) \) case we have \( n^+ - \ell = 3 - 2 = 1 \). In other words, in our case there is a multiplicative factor of \( i \) in the definition of the residue.

The factor \( i \) on the right hand side comes from \( \sum (-1)^w e^{i < w \lambda, X>} \). Using the Weyl character formula this is proportional to \( i^{n^+} \) times a real number (where \( n^+ \) is the number of positive roots). In our case \( n^+ = 3 \). So when \( G = SU(3) \), we get \( i \operatorname{vol}(M) = i^{n^+} \) times a real number. So we have identified the volume.

### 3.3 Computing the residue

In this section, we will focus on computing the residue \( \text{res}(\Omega_\lambda) \) of a special class of meromorphic forms \( \Omega_\lambda \) such that

\[ \Omega_\lambda(\psi) = e^{i \lambda(\psi)[d\psi]} \prod_{j=1}^v \beta_j(\psi), \quad (3.22) \]

where \( \lambda \) is some point in \( t^* \) and \( \beta_j \) all lie in the dual cone of a proper cone \( \Lambda \) in \( t \). A proper cone is defined as an open cone such that its apex is the origin and it is properly contained in some half space. Given a proper cone \( \Lambda \subset t \), its dual cone \( \Lambda^* \) is defined to be the following collection of elements in \( t^* \):

\[ \Lambda^* := \{ \beta \in t^* : \beta(\psi) > 0 \text{ for all } \psi \in \Lambda \}. \quad (3.23) \]

In Proposition 8.11 in [19], a list of properties satisfied by the residue is given (see also Proposition 3.2 in [20]). This list of properties will be the basis for the computations in this article.

Now, we are finally ready for the computation of the symplectic volume of the triple \( (N = 3) \) reduced product \( M_{\text{red}} \), i.e., the cohomological quantity

\[ \frac{1}{i^{d/2}} e^{i \omega_{\text{red}}}[M_{\text{red}}]. \quad (3.24) \]

We have the following result.

**Theorem 3.7** The symplectic volume of the triple reduced product \( M_{\text{red}}(\vec{\xi}) \) of \( G = SU(3) \) (with the quantities \( \vec{\xi} = (\xi_1, \xi_2, \xi_3) \) satisfying (A1), (A2) and (A3) outlined in Sect. 2.2.3)
can be computed by the following explicit formula:

\[
\text{vol}^S(M_{\text{red}}(\vec{\xi})) = K \cdot \sum_{i=0}^{5} \sum_{j=0}^{5} \sum_{k=0}^{5} (-1)^{i+j+k} \cdot \max \left( \min \left( P_{ijk}^{(1)}(\vec{\xi}), P_{ijk}^{(2)}(\vec{\xi}) \right), 0 \right)
\]  

(3.25)

where \(K\) is a real constant that depends on the inner product and \(\bar{\beta}\) (although the overall formula does not depend on these choices) and

\[
P_{ijk}^{(1)}(\vec{\xi}) = \left( \frac{2}{3} \text{pr}_1 - \frac{1}{3} \text{pr}_2 \right) \left( P_{ijk}(\vec{\xi}) \right),
\]

(3.26)

\[
P_{ijk}^{(2)}(\vec{\xi}) = \left( \frac{1}{3} \text{pr}_1 + \frac{1}{3} \text{pr}_2 \right) \left( P_{ijk}(\vec{\xi}) \right),
\]

(3.27)

with

\[
P_{ijk}(\vec{\xi}) = s_i \cdot \left( \frac{\ell_1}{m_1} \right) + s_j \cdot \left( \frac{\ell_2}{m_2} \right) + s_k \cdot \left( \frac{\ell_3}{m_3} \right) \in \mathbb{R}^2.
\]

(3.28)

Here \(\text{pr}_i\) denotes the standard projection onto the \(i\)-th coordinate. Here

\[
\xi_i = (\ell_i - m_i) \cdot \Omega_1 + m_i \cdot \Omega_2 = \ell_i \cdot \Omega_1 + m_i \cdot (\Omega_2 - \Omega_1), \quad \ell_i > m_i > 0.
\]

(3.29)

Above, \(s_j\) are elements of the Weyl group of SU(3)—their representation as \(2 \times 2\) matrices is introduced in (3.63)–(3.68) below. Also the \(\Omega_j\) form a basis of the Lie algebra of the maximal torus of SU(3). They are introduced in (3.45) below.

**Proof** By Corollary 2.5, we have

\[
\frac{1}{i^{d/2}} e^{i \omega_{\text{red}} [M_{\text{red}}]} = \frac{1}{i^{d/2}} n_0 C_G \text{ res} \left( \sum_{\vec{w} \in W^3} \text{sgn}(\vec{w}) e^{i(\vec{w}_1 \cdot \vec{\xi}_1 + \vec{w}_2 \cdot \vec{\xi}_2 + \vec{w}_3 \cdot \vec{\xi}_3)} \frac{e^{i\sigma(\vec{\psi})}}{\sigma(\vec{\psi})} [d\vec{\psi}] \right).
\]

(3.30)

To compute the residue on the right hand side of the above equation, we need to first choose an open cone \(\Lambda\) in \(t\). We choose \(\Lambda = t_{>0}\), the open positive Weyl chamber. One reason that we make this choice is that we observe that (here recall that we are considering \(G = \text{SU}(3)\))

\[
\sigma(\vec{\psi}) = \prod_{j=1}^{3} \beta_j(\psi),
\]

(3.31)

where

\[
\beta_1 = (\text{diag}(2\pi i, -2\pi i, 0), \cdot),
\]

(3.32)

\[
\beta_2 = (\text{diag}(0, 2\pi i, -2\pi i), \cdot),
\]

(3.33)

\[
\beta_3 = (\text{diag}(2\pi i, 0, -2\pi i), \cdot).
\]

(3.34)

Thus \(\beta_3 = \beta_1 + \beta_2\) and the collection \(\{\beta_1, \beta_2, \beta_3\}\) is just the set of positive roots of \(G = \text{SU}(3)\). Notice that all of \(\beta_1, \beta_2, \beta_3\) lie in \(\Lambda^*\), the dual cone of \(\Lambda\).

Let \(\bar{\beta} = (\beta_1, \beta_2, \beta_3)\).

Let

\[
\bar{w} \odot \vec{\xi} := \sum_{i=1}^{N} w_i \cdot \xi_i.
\]

(3.35)
Here we are considering $N = 3$.

Let us rewrite the residue part in Eq. (3.30):

$$\text{res} \left( \sum_{\vec{w} \in W^3} \text{sgn}(\vec{w}) e^{i(w_1\xi_1 + w_2\xi_2 + w_3\xi_3, \psi)} \sigma(\psi) [d\psi] \right)$$  

(3.36)

$$= \text{res} \left( \sum_{\vec{w} \in W^3} \text{sgn}(\vec{w}) e^{i((\vec{w} \odot \vec{\xi})(\psi)) \prod_{j=1}^{3} \beta_j(\psi)} [d\psi] \right)$$  

(3.37)

$$= \sum_{\vec{w} \in W^3} \text{sgn}(\vec{w}) \left( e^{i((\vec{w} \odot \vec{\xi})(\psi)) \prod_{j=1}^{3} \beta_j(\psi)} \right)$$  

(3.38)

$$= \sum_{\vec{w} \in W^3} \text{sgn}(\vec{w}) \frac{i^3}{(2\pi i)^2} H_{\vec{\beta}}(\vec{w} \odot \vec{\xi}).$$  

(3.39)

Combining the above formula with the constant part in Eq. (3.30) and recalling that $n_+ = 3$, we have

$$C_G = -\frac{1}{(2\pi)^4 \cdot 6 \cdot \text{vol}(T)}$$

so

$$\text{vol}^S(M_{\text{red}}(\vec{\xi})) = \frac{1}{i^{d/2} n_0 C_G} \frac{i^3}{(2\pi)^2} \sum_{\vec{w} \in W^3} \text{sgn}(\vec{w}) H_{\vec{\beta}}(\vec{w} \odot \vec{\xi})$$  

$$= -3 \cdot \frac{1}{(2\pi)^6 \cdot 6 \cdot \text{vol}^R(T)} \cdot \frac{1}{(2\pi)^2} \sum_{\vec{w} \in W^3} \text{sgn}(\vec{w}) H_{\vec{\beta}}(\vec{w} \odot \vec{\xi})$$  

$$= -\frac{1}{2 \cdot (2\pi)^8 \cdot \text{vol}^R(T)} \sum_{\vec{w} \in W^3} \text{sgn}(\vec{w}) H_{\vec{\beta}}(\vec{w} \odot \vec{\xi}).$$  

(3.40)

Notice that since we are considering triple reduced products of $G = \text{SU}(3)$ here, we have

$$d = N(s - l) - 2s = 3 \cdot (8 - 2) - 2 \cdot 8 = 2,$$  

(3.41)

$$n_+ = (s - l)/2 = (8 - 2)/2 = 3,$$  

(3.42)

$$n_0 = |\text{Z(\text{SU}(3))}| = 3.$$  

(3.43)

Note that $d$ is the real dimension of the reduced space. Therefore we need to compute $H_{\vec{\beta}}(\vec{w} \odot \vec{\xi})$ to obtain an explicit formula for the symplectic volume of a triple reduced product. Since later we will compare our volume formula with the volume formula obtained by Suzuki and Takakura [25], we will parametrize things in a way similar to theirs.

Any element $\xi \in t_{>0}$ can be written as

$$\xi = (\ell - m) \cdot \frac{2\beta_1 + \beta_2}{3} + m \cdot \frac{\beta_1 + 2\beta_2}{3}$$  

(3.44)

for some $\ell > m > 0$. If $\ell$ and $m$ can be any real number, then the above formula parametrizes all $\xi \in t$.

Let

$$\Omega_1 := \frac{2\beta_1 + \beta_2}{3},$$  

(3.45)
Volume formula for $N$-fold reduced products

\[ \Omega_2 := \frac{\beta_1 + 2\beta_2}{3}. \]  \hspace{1cm} (3.46)

Then

\[ \xi = (\ell - m) \cdot \Omega_1 + m \cdot \Omega_2 = \ell \cdot \Omega_1 + m \cdot (\Omega_2 - \Omega_1). \]  \hspace{1cm} (3.47)

We fix \( \{\Omega_1, \Omega_2 - \Omega_1\} \) as the basis for \( t \) for this computation. By using this basis, it will be easier for us to compare our volume formula with the volume formula obtained by Suzuki and Takakura. Our goal here is to express \( H_\beta(\xi) \) as a function of \( \ell \) and \( m \). It is helpful to express \( H_\beta(\lambda_1 \beta_1 + \lambda_2 \beta_2) \) in terms of \( \lambda_1 \) and \( \lambda_2 \) first.

By the definition of \( H_\beta \), we have

\[ H_\beta(\lambda_1 \beta_1 + \lambda_2 \beta_2) = \text{vol} \left\{ (s_1, s_2, s_3) \in \mathbb{R}_+^3 : \sum_{j=1}^3 s_j \beta_j = \lambda_1 \beta_1 + \lambda_2 \beta_2 \right\}, \]  \hspace{1cm} (3.48)

where \( \mathbb{R}_+ \) denotes the set of all nonnegative real numbers. Therefore we want to solve the following equation:

\[ s_1 \beta_1 + s_2 \beta_2 + s_3 (\beta_1 + \beta_2) = \lambda_1 \beta_1 + \lambda_2 \beta_2. \]  \hspace{1cm} (3.49)

Notice that \( \beta_3 = \beta_1 + \beta_2 \). Collecting terms, we then have:

\[ (s_1 + s_3) \beta_1 + (s_2 + s_3) \beta_2 = \lambda_1 \beta_1 + \lambda_2 \beta_2. \]  \hspace{1cm} (3.50)

Thus we have the following linear system:

\[ s_1 + s_3 = \lambda_1, \]  \hspace{1cm} (3.51)

\[ s_2 + s_3 = \lambda_2. \]  \hspace{1cm} (3.52)

The solution set \( S \) is:

\[ S = \{ (\lambda_1 - s_3, \lambda_2 - s_3, s_3) : \lambda_1 - s_3 \geq 0, \lambda_2 - s_3 \geq 0, s_3 \geq 0 \} \]  \hspace{1cm} (3.53)

\[ = \{ (\lambda_1 - s_3, \lambda_2 - s_3, s_3) : s_3 \leq \lambda_1, s_3 \leq \lambda_2, s_3 \geq 0 \} \]  \hspace{1cm} (3.54)

\[ = \{ (\lambda_1 - s_3, \lambda_2 - s_3, s_3) : 0 \leq s_3 \leq \min(\lambda_1, \lambda_2) \} . \]  \hspace{1cm} (3.55)

Notice that \( S = \emptyset \) if \( \lambda_1 < 0 \) or \( \lambda_2 < 0 \). Therefore,

\[ H_\beta(\lambda_1 \beta_1 + \lambda_2 \beta_2) = \text{vol}(S) = \max(\min(\lambda_1, \lambda_2), 0). \]  \hspace{1cm} (3.56)

Now we compute \( H_\beta(\xi) \) for \( \xi = \ell \cdot \Omega_1 + m \cdot (\Omega_2 - \Omega_1) \). First, we rewrite \( \xi \) in the form of \( \lambda_1 \beta_1 + \lambda_2 \beta_2 \):

\[ \xi = \ell \cdot \Omega_1 + m \cdot (\Omega_2 - \Omega_1) \]  \hspace{1cm} (3.57)

\[ = (\ell - m) \cdot \Omega_1 + m \cdot \Omega_2 \]  \hspace{1cm} (3.58)

\[ = (\ell - m) \cdot \frac{2\beta_1 + \beta_2}{3} + m \cdot \frac{\beta_1 + 2\beta_2}{3} \]  \hspace{1cm} (3.59)

\[ = \frac{2\ell - m}{3} \cdot \beta_1 + \frac{\ell + m}{3} \cdot \beta_2. \]  \hspace{1cm} (3.60)

Therefore

\[ H_\beta(\xi) = C \cdot \max \left( \min \left( \frac{2\ell - m}{3}, \frac{\ell + m}{3} \right), 0 \right) \]  \hspace{1cm} (3.61)

where \( C \) is a real constant.
Now consider the triple reduced product $M_{\text{red}}(\vec{\xi})$ of $G = \text{SU}(3)$ with the quantities $\vec{\xi} = (\xi_1, \xi_2, \xi_3)$ where

$$\xi_i = (\ell_i - m_i) \cdot \Omega_1 + m_i \cdot \Omega_2 = \ell_i \cdot \Omega_1 + m_i \cdot (\Omega_2 - \Omega_1), \quad \ell_i > m_i > 0.$$  

(3.62)

Notice that each $\xi_i$ lies in the open positive Weyl chamber $t_{>0}$. Our goal is to express the symplectic volume of $M_{\text{red}}(\vec{\xi})$ in terms of $\ell_1, \ell_2, \ell_3, m_1, m_2, m_3$.

First we need to write each Weyl group element as a $2 \times 2$ matrix with respect to the basis $\{\Omega_1, \Omega_2 - \Omega_1\}$. Recall that we have enumerated the Weyl group as $\{s_0, s_1, s_2, s_3, s_4, s_5\}$ and associated the $s_j$ with elements in $S_3$ bijectively through Equations (2.11–2.16). Each $s_j$ can be regarded as a linear transformation from $t$ to itself. With respect to the basis $\{\Omega_1, \Omega_2 - \Omega_1\}$, they can be written as the following matrices:

$$s_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$  

(3.63)

$$s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$  

(3.64)

$$s_2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$  

(3.65)

$$s_3 = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix},$$  

(3.66)

$$s_4 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix},$$  

(3.67)

$$s_5 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}.$$  

(3.68)

Now, combining Eqs. (3.40) and (3.61), we obtain our volume formula. \qed

Remark Notice that the above formula is a piecewise linear function. While this function is continuous, there are places, called “walls”, where this function is not differentiable. These walls are introduced by the max and min operators. If we cross a wall, we will see that the gradient vector “jumps”. For example, a wall occurs when we cross the places where $P_{ijk}^{(1)}(\vec{\xi}) = P_{ijk}^{(2)}(\vec{\xi}) > 0$ for some $i, j, k$.

4 A result of Suzuki and Takakura

It will be interesting to compare our result with a result on symplectic volume of $N$-fold reduced products of $G = \text{SU}(3)$ by Suzuki and Takakura [25] in 2008. In this section, we will describe their result in the case when $N = 3$. The settings in their paper [25] are almost the same as ours except that their choice of inner product on $\mathfrak{g}$ is $(\cdot, \cdot)/(4\pi^2)$ where $(\cdot, \cdot)$ is our choice of inner product. Notice that this difference will not affect the symplectic volume of triple reduced products since the symplectic volume of a coadjoint orbit does not depend on a choice of inner products on $\mathfrak{g}$.

Their initial input is more restrictive than ours in the following sense. Let $\vec{\xi} = (\xi_1, \xi_2, \xi_3)$ be the quantities such that

$$\xi_i = (\ell_i - m_i) \cdot \Omega_1 + m_i \cdot \Omega_2,$$  

(4.1)
where $\ell_i > m_i > 0$ and all $\ell_i$ and $m_i$ are integers that are divisible by 3 and also

$$(\vec{w} \odot \vec{\xi}, \Omega_1) \neq 0$$

(4.2)

for all $\vec{w} \in W^3$. Note that the notation $\vec{w} \odot \vec{\xi}$ was introduced in (3.35).

**Remark** Notice that in our formula we do not require the $\ell_i$ and the $m_i$ to be integral or even rational. Thus our result is an extension of the result of Suzuki and Takakura [25].

Now, let $L = \ell_1 + \ell_2 + \ell_3$ and $M = m_1 + m_2 + m_3$. If $I$ is a subset of $\{1, 2, 3\}$,

$$\ell_I := \sum_{i \in I} \ell_i, \quad (4.3)$$

$$m_I := \sum_{i \in I} m_i. \quad (4.4)$$

If $I$ and $J$ are two disjoint subsets of $\{1, 2, 3\}$,

$$\ell_{I,J} := \ell_I + \ell_J = \sum_{i \in I \cup J} \ell_i, \quad (4.5)$$

$$m_{I,J} := m_I + m_J = \sum_{i \in I \cup J} m_i. \quad (4.6)$$

If $(I_1, \ldots, I_6)$ is a 6-tuple of subsets of $\{1, 2, 3\}$, $(I_1, \ldots, I_6)$ is called a 6-partition of $\{1, 2, 3\}$ if and only if:

$$I_1 \cup \cdots \cup I_6 = \{1, 2, 3\} \text{ and } I_j \cap I_k = \emptyset \text{ whenever } j \neq k. \quad (4.7)$$

For $N = 3$, the result of Suzuki and Takakura is the following (recall that the group $G$ here is $SU(3)$):

**Theorem 4.1** (Theorem 4.5 in [25], in the case $N = 3$) Let $\mathcal{I}_\xi$ denote the set of those 6-partitions $(I_1, \ldots, I_6)$ of $\{1, 2, 3\}$ such that

$$\ell_{I_1, I_2} + m_{I_4, I_5} < \frac{L + M}{3}, \text{ and} \quad (4.8)$$

$$\ell_{I_3, I_4} + m_{I_6, I_1} < \frac{L + M}{3}. \quad (4.9)$$

Let $\mathcal{J}_\xi$ denote the set of those 6-partitions $(I_1, \ldots, I_6)$ of $\{1, 2, 3\}$ such that

$$\ell_{I_3, I_4} + m_{I_6, I_1} > \frac{L + M}{3}, \text{ and} \quad (4.10)$$

$$\ell_{I_5, I_6} + m_{I_2, I_3} > \frac{L + M}{3}. \quad (4.11)$$

Let $A_\xi: \mathcal{I}_\xi \to \mathbb{R}$ be defined by:

$$A_\xi(I_1, \ldots, I_6) := \frac{-(-1)^{|I_1| + |I_2| + |I_5|}}{6} \left( \frac{L + M}{3} - \ell_{I_1, I_2} - m_{I_4, I_5} \right). \quad (4.12)$$

Let $B_\xi: \mathcal{J}_\xi \to \mathbb{R}$ be defined by:

$$B_\xi(I_1, \ldots, I_6) := \frac{-(-1)^{|I_1| + |I_2| + |I_5|}}{6} \left( \ell_{I_5, I_6} + m_{I_2, I_3} - \frac{L + M}{3} \right). \quad (4.13)$$

$$\text{ Springer}$$
Then, the symplectic volume of $M_{red}(\vec{\xi})$ is given by:

$$\mathcal{V}(\vec{\xi}) = \sum_{(I_1, \ldots, I_6) \in I_3^c} A_{\vec{\xi}}(I_1, \ldots, I_6) + \sum_{(I_1, \ldots, I_6) \in I_3^c} B_{\vec{\xi}}(I_1, \ldots, I_6). \quad (4.14)$$

We shall briefly explain this result.

First, 6-partitions $(I_1, \ldots, I_6)$ of $\{1, 2, 3\}$ correspond bijectively to fixed points of $T$ acting on $M$, i.e., to $M^T$, in the following way. Suzuki and Takakura have used a different enumeration $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}$ of the Weyl group $W$ of $SU(3)$ in their paper [25]. Relating their enumeration with ours, we have:

$$\sigma_1 = \text{Id} = s_0, \quad (4.15)$$
$$\sigma_2 = (2\, 3) = s_5, \quad (4.16)$$
$$\sigma_3 = (1\, 2\, 3) = s_2, \quad (4.17)$$
$$\sigma_4 = (1\, 2) = s_1, \quad (4.18)$$
$$\sigma_5 = (1\, 3\, 2) = s_4, \quad (4.19)$$
$$\sigma_6 = (1\, 3) = s_3. \quad (4.20)$$

For each $\vec{w} = (w_1, w_2, w_3) \in W^3$, let $I_j$ be defined as

$$I_j := \{ i \in \{1, 2, 3\} : w_i = \sigma_j \} \quad (4.21)$$

for $j = 1, \ldots, 6$. Then $(I_1, \ldots, I_6)$ is a 6-partition of $\{1, 2, 3\}$. For example, given $(\sigma_2, \sigma_5, \sigma_2) \in W^3$, the corresponding 6-partition of $\{1, 2, 3\}$ is

$$(\emptyset, \{1, 3\}, \emptyset, \{2\}, \emptyset). \quad (4.22)$$

In other words, $I_j$ tells us in which coordinates (in this case, the first, the second, or the third coordinate of $\vec{w}$) $\sigma_j$ appears in $\vec{w}$.

Suzuki and Takakura have observed that $\vec{w} \odot \vec{\xi}$ is the matrix

$$2\pi i \cdot \text{diag} \left( \ell_{l_1, l_2} + m_{l_4, l_5} - \frac{L + M}{3}, \ell_{l_3, l_4} + m_{l_6, l_1} - \frac{L + M}{3}, \ell_{l_5, l_6} + m_{l_2, l_3} - \frac{L + M}{3} \right). \quad (4.23)$$

We would like to determine under what conditions the above matrix is in the cone spanned by $\vec{\beta} = (\beta_1, \beta_2, \beta_3)$. The vector $\vec{w} \odot \vec{\xi}$ is in the cone spanned by $\vec{\beta}$ if and only if

$$(\vec{w} \odot \vec{\xi}, \Omega_1) > 0, \text{ and } (4.24)$$
$$(\vec{w} \odot \vec{\xi}, \Omega_2) > 0. \quad (4.25)$$

Recall that

$$\Omega_1 = \frac{2\pi i}{3} \cdot \text{diag}(2, -1, -1), \quad (4.26)$$
$$\Omega_2 = \frac{2\pi i}{3} \cdot \text{diag}(1, 1, -2). \quad (4.27)$$

Thus, $\vec{w} \odot \vec{\xi}$ is in the cone spanned by $\vec{\beta}$ if and only if

$$2\ell_{l_1, l_2} + 2m_{l_4, l_5} - \ell_{l_3, l_4} - m_{l_6, l_1} - \ell_{l_5, l_6} - m_{l_2, l_3} > 0, \text{ and } (4.28)$$
$$\ell_{l_1, l_2} + m_{l_4, l_5} + \ell_{l_3, l_4} + m_{l_6, l_1} - 2\ell_{l_5, l_6} - 2m_{l_2, l_3} > 0. \quad (4.29)$$
Thus, we have translated the condition for $I$ used in Theorem 4.1. If $\bar{\omega} \otimes \bar{\xi}$ satisfies the condition for $I$, then
\[
2\ell_{I_1, I_2} + 2m_{I_4, I_5} - \ell_{I_1, I_4} - m_{I_6, I_1} - \ell_{I_5, I_6} - m_{I_2, I_3} = 3\ell_{I_1, I_2} + 3m_{I_4, I_5} - L - M
\]
\[
= 3(\ell_{I_1, I_2} + m_{I_4, I_5} - \frac{L + M}{3}) < 0.
\]

Thus, $\bar{\omega} \otimes \bar{\xi}$ is not in the cone spanned by $\bar{\beta}$.

If we look closely at the first inequality of the condition for $I$, namely,
\[
\ell_{I_1, I_2} + m_{I_4, I_5} < \frac{L + M}{3},
\]
this means exactly
\[
(\bar{\omega} \otimes \bar{\xi}, \Omega_1) < 0.
\]

Also, the second inequality of the condition for $I$, namely,
\[
\ell_{I_3, I_4} + m_{I_6, I_1} < \frac{L + M}{3},
\]
means exactly
\[
(\bar{\omega} \otimes \bar{\xi}, \Omega_2 - \Omega_1) < 0.
\]

Thus, we have translated the condition for $I$ into the following two inequalities:
\[
(\bar{\omega} \otimes \bar{\xi}, \Omega_1) < 0, \quad \text{and} \quad (\bar{\omega} \otimes \bar{\xi}, \Omega_2 - \Omega_1) < 0.
\]

Similarly, we can translate the condition for $J$ into the following two inequalities:
\[
(\bar{\omega} \otimes \bar{\xi}, \Omega_2 - \Omega_1) > 0, \quad \text{and} \quad (\bar{\omega} \otimes \bar{\xi}, \Omega_2) < 0.
\]

Notice that if $\bar{\omega} \otimes \bar{\xi}$ satisfies either the condition for $I$ or the condition for $J$, we always have that $\bar{\omega} \otimes \bar{\xi}$ is in the cone spanned by $-\bar{\beta} = (-\beta_1, -\beta_2, -\beta_1 - \beta_2)$. In other words, only those $\bar{\omega} \otimes \bar{\xi}$ contained in the cone spanned by $-\bar{\beta}$ will contribute to the sum in the volume formula of Suzuki and Takakura. Also notice that the sign
\[
-(-1)^{|I_1|+|I_3|+|I_5|}
\]
is exactly the signature of $\tilde{w}$.

The above observations lead us to conclude that our volume formula (Theorem 3.7) and the volume formula of Suzuki and Takakura (Theorem 4.1) are very closely related.

We have the following result.

**Theorem 4.2** Under the assumptions (A1), (A2) and (A3) stated in Sect. 2.2.3, our volume formula in Theorem 3.7 agrees completely with the volume formula of Suzuki and Takakura (Theorem 4.1) for triple reduced products of $\text{SU}(3)$, provided that $K = -K' = -1/6$. Thus our volume formula extends that of [25].

**Proof** The first key observation supporting that our formula should indeed agree with theirs is that we derived our formula by using the residue formula (Theorem 2.4) with a choice of cone, namely the cone spanned by $\bar{\beta}$ and as a result, only those $\tilde{w} \odot \tilde{\xi}$ in the cone spanned by $\bar{\beta}$ will contribute to the sum in our volume formula. However, the total sum in the residue formula does not depend on the choice of cone. Therefore, we could equally well choose the cone spanned by $-\bar{\beta}$ to carry out the computations of the individual terms in the sum. Let us carry this out.

More precisely, we start from the choice of $\Lambda_+ = -t > 0$. In this way, all of $-\beta_1, -\beta_2, -\beta_3 = -\beta_1 - \beta_2$ lie in the dual cone $\Lambda_+^\ast$.

With this new choice of cone, we carry out the computation, starting from Eq. (3.36):

$$\text{res} \left( \sum_{\tilde{w} \in W^3} \text{sgn}(\tilde{w}) e^{i(\tilde{w} \odot \tilde{\xi}, \psi)} [d\psi] / \sigma(\psi) \right)$$

$$= \text{res}^{\Lambda_-} \left( \sum_{\tilde{w} \in W^3} \text{sgn}(\tilde{w}) e^{i(\tilde{w} \odot \tilde{\xi}, \psi)} [d\psi] / \prod_{j=1}^3 \beta_j(\psi) \right)$$

$$= \sum_{\tilde{w} \in W^3} \text{sgn}(\tilde{w}) \text{res}^{\Lambda_-} \left( e^{i(\tilde{w} \odot \tilde{\xi}, \psi)} [d\psi] / \prod_{j=1}^3 \beta_j(\psi) \right)$$

$$= -\sum_{\tilde{w} \in W^3} \text{sgn}(\tilde{w}) \text{res}^{\Lambda_-} \left( e^{i(\tilde{w} \odot \tilde{\xi}, \psi)} [d\psi] / \prod_{j=1}^3 (-\beta_j)(\psi) \right)$$

$$= -\sum_{\tilde{w} \in W^3} \text{sgn}(\tilde{w}) \left( \frac{1}{(2\pi i)^2} H_{(-\tilde{\beta})}(\tilde{w} \odot \tilde{\xi}) \right).$$

Now our volume formula corresponding to the cone $\Lambda_-$ can be written as

$$\text{vol}^S(M_{\text{red}}(\tilde{\xi})) = K' \sum_{\tilde{w} \in W^3} \text{sgn}(\tilde{w}) H_{(-\tilde{\beta})}(\tilde{w} \odot \tilde{\xi}),$$

where $K'$ is a constant. Notice that the constant $K$ in our volume formula corresponding to the cone $\Lambda$, i.e., Eq. (3.25), is simply $K = -K'$.

The function $H_{(-\tilde{\beta})}$ is supported in the cone spanned by $-\tilde{\beta}$. Therefore, only those $\tilde{w} \odot \tilde{\xi}$ inside this cone will contribute to the sum. This gives us the common ground to compare our volume formula (using the cone $\Lambda_-$) and the volume formula of Suzuki and Takakura.

Let us compute $H_{(-\tilde{\beta})}(\lambda_1 \cdot (-\beta_1) + \lambda_2 \cdot (-\beta_2))$ where $\lambda_1, \lambda_2 \in \mathbb{R}$.

We have:

$$H_{(-\tilde{\beta})}(\lambda_1 \cdot (-\beta_1) + \lambda_2 \cdot (-\beta_2))$$

\[\text{Springer}\]
\[
\text{vol} \left\{ (s_1, s_2, s_3) \in \mathbb{R}_+^3 : \sum_{j=1}^{3} s_j \cdot (-\beta_j) = \lambda_1 \cdot (-\beta_1) + \lambda_2 \cdot (-\beta_2) \right\}.
\]

(4.53)

Thus, we need to solve the following equation:

\[
s_1 \cdot (-\beta_1) + s_2 \cdot (-\beta_2) + s_3 \cdot (-\beta_1 - \beta_2) = \lambda_1 \cdot (-\beta_1) + \lambda_2 \cdot (-\beta_2).
\]

(4.54)

Therefore, we need to solve the following linear system:

\[
s_1 + s_3 = \lambda_1,
\]

(4.55)

\[
s_2 + s_3 = \lambda_2.
\]

(4.56)

The solution set \( S_- \) is:

\[
S_- = \{(\lambda_1, s_2, \lambda_2 - s_3, s_3) : \lambda_1 - s_3 \geq 0, \lambda_2 - s_3 \geq 0, s_3 \geq 0\}
\]

(4.57)

\[
= \{(\lambda_1 - s_3, \lambda_2 - s_3, s_3) : s_3 \leq \lambda_1, s_3 \leq \lambda_2, s_3 \geq 0\}
\]

(4.58)

\[
= \{(\lambda_1, \lambda_2 - s_3, s_3) : 0 \leq s_3 \leq \min(\lambda_1, \lambda_2)\}.
\]

(4.59)

Therefore,

\[
H_{-\tilde{\beta}}(\lambda_1 \cdot (-\beta_1) + \lambda_2 \cdot (-\beta_2)) = \text{vol}(S_-) = C \cdot \max(\min(\lambda_1, \lambda_2), 0),
\]

(4.60)

where \( C \) is the same constant as in Eq. (3.56). (In fact \( C = 1 \).

Now, let’s look at those \( \bar{w} \circ \bar{\xi} \) inside the cone spanned by \(-\tilde{\beta}\). Without loss of generality we can assume that \( \bar{\xi} \) is generic so that for all \( \bar{w} \in \mathbb{R}^3 \),

\[
(\bar{w} \circ \bar{\xi}, \Omega_1) \neq 0,
\]

(4.61)

\[
(\bar{w} \circ \bar{\xi}, \Omega_2) \neq 0,
\]

(4.62)

\[
(\bar{w} \circ \bar{\xi}, \Omega_2 - \Omega_1) \neq 0.
\]

(4.63)

Therefore, the collection of those \( \bar{w} \circ \bar{\xi} \) inside the cone spanned by \(-\tilde{\beta}\) is the disjoint union of the two sets \( \mathcal{A}_{\bar{\xi}} \) and \( \mathcal{B}_{\bar{\xi}} \), where \( \mathcal{A}_{\bar{\xi}} \) denotes the set of those \( \bar{w} \circ \bar{\xi} \) such that

\[
(\bar{w} \circ \bar{\xi}, \Omega_1) < 0, \quad \text{and}
\]

(4.64)

\[
(\bar{w} \circ \bar{\xi}, \Omega_2 - \Omega_1) < 0,
\]

(4.65)

and \( \mathcal{B}_{\bar{\xi}} \) denotes the set of those \( \bar{w} \circ \bar{\xi} \) such that

\[
(\bar{w} \circ \bar{\xi}, \Omega_2 - \Omega_1) > 0, \quad \text{and}
\]

(4.66)

\[
(\bar{w} \circ \bar{\xi}, \Omega_2) < 0.
\]

(4.67)

Notice that the above grouping is in complete agreement with the grouping by \( \mathcal{I}_{\bar{\xi}} \) and \( \mathcal{J}_{\bar{\xi}} \) in the formula of Suzuki and Takakura so that we can make a term-by-term comparison between our formula and theirs.

For each \( \bar{w} \circ \bar{\xi} \in \mathcal{A}_{\bar{\xi}} \), it is easy to see that \( \bar{w} \circ \bar{\xi} \) can be written as \( \lambda_1 \cdot (-\beta_1) + \lambda_2 \cdot (-\beta_2) \) with some \( \lambda_1, \lambda_2 \) satisfying \( 0 < \lambda_1 < \lambda_2 \). Therefore, the contribution of this \( \bar{w} \circ \bar{\xi} \) to our volume formula is

\[
K' \cdot \text{sgn}(\bar{w}) \cdot H_{-\tilde{\beta}}(\bar{w} \circ \bar{\xi}) = K' \cdot \text{sgn}(\bar{w}) \cdot \lambda_1.
\]

(4.68)

Notice that this \( \bar{w} \) will correspond to a 6-partition \( (I_1, \ldots, I_6) \) and we have

\[
\text{sgn}(\bar{w}) = -(-1)^{|I_1| + |I_3| + |I_5|}.
\]

(4.69)
Now we only need to figure out how we can express this $\lambda_1$ in terms of the $\ell_i$ and $m_i$.

First, given $a, b \in \mathbb{R}$ and

$$2\pi i \cdot \text{diag}(a, b, -a - b) = \lambda_1 \cdot (-\beta_1) + \lambda_2 \cdot (-\beta_2),$$

we want to express $\lambda_1$ and $\lambda_2$ in terms of $a, b$. This is equivalent to solving the following linear system:

$$-\lambda_1 = a, \quad \lambda_1 - \lambda_2 = b.$$ (4.71)

Thus, we have

$$\lambda_1 = -a, \quad \lambda_2 = -a - b.$$ (4.72)

Recall that $\vec{w} \odot \vec{\xi}$ is the matrix

$$2\pi i \cdot \text{diag} \left( \ell_{I_1,I_2} + m_{I_4,I_5}, \ell_{I_3,I_4} + m_{I_6,I_7} - \frac{L + M}{3}, \ell_{I_5,I_6} + m_{I_2,I_3} - \frac{L + M}{3} \right).$$ (4.73)

Hence, if $\vec{w} \odot \vec{\xi} \in A_{\vec{\xi}}$, the contribution of this $\vec{w} \odot \vec{\xi}$ to our volume formula is

$$K' \cdot \text{sgn}(\vec{w}) \cdot \lambda_1 = K' \cdot \left( -(-1)^{|I_1|+|I_3|+|I_5|} \right) \cdot \left( \frac{L + M}{3} - \ell_{I_1,I_2} - m_{I_4,I_5} \right),$$ (4.74)

which precisely matches the term

$$A_{\vec{\xi}}(I_1, \ldots, I_6) = \frac{-(-1)^{|I_1|+|I_3|+|I_5|}}{6} \left( \frac{L + M}{3} - \ell_{I_1,I_2} - m_{I_4,I_5} \right)$$ (4.75)

for the contribution of this $\vec{w} \odot \vec{\xi}$ to the volume formula of Suzuki and Takakura, provided that $K' = 1/6$.

For each $\vec{w} \odot \vec{\xi} \in B_{\vec{\xi}}$, it is easy to see that $\vec{w} \odot \vec{\xi}$ can be written as $\lambda_1 \cdot (-\beta_1) + \lambda_2 \cdot (-\beta_2)$ with some $\lambda_1, \lambda_2$ satisfying $0 < \lambda_2 < \lambda_1$. Therefore, the contribution of this $\vec{w} \odot \vec{\xi}$ to our volume formula is

$$K' \cdot \text{sgn}(\vec{w}) \cdot H_{(-\beta)}(\vec{w} \odot \vec{\xi})$$

$$= K' \cdot \text{sgn}(\vec{w}) \cdot \lambda_2$$

$$= K' \cdot \left( -(-1)^{|I_1|+|I_3|+|I_5|} \right) \cdot \left( \ell_{I_5,I_6} + m_{I_2,I_3} - \frac{L + M}{3} \right),$$ (4.76)

which precisely matches the term

$$B_{\vec{\xi}}(I_1, \ldots, I_6) = \frac{-(-1)^{|I_1|+|I_3|+|I_5|}}{6} \left( \ell_{I_5,I_6} + m_{I_2,I_3} - \frac{L + M}{3} \right)$$ (4.77)

for the contribution of this $\vec{w} \odot \vec{\xi}$ to the volume formula of Suzuki and Takakura, provided that $K' = 1/6$.

Observing that the sum in the residue formula does not depend on the choice of cone, we have proved the theorem.

\[\square\]
5 Generalizations of volume formula

In this section, we generalize some of our earlier results.

5.1 Volume formula for \( N \)-fold reduced products of SU(3)

In this section, our group \( G \) is still \( \text{SU}(3) \). As before, let \( T \) be the standard maximal torus in \( G \). Let \( W \) denote the Weyl group. Thus, \( W = \Theta_3 \).

We assume the following.

- \( N \geq 3 \) is a positive integer. Notice that previously our \( N \) was equal to 3.
- Suppose \( \vec{\xi} = (\xi_1, \ldots, \xi_N) \) is a collection of \( N \) elements in \( t_+ \) which satisfy the conditions specified in Sect. 2.
- Let \( M = O_{\xi_1} \times \cdots \times O_{\xi_N} \). This is a compact symplectic manifold with \( G \) acting diagonally on it in a Hamiltonian fashion.
- Let \( \mu_G \) and \( \mu_T \) be the moment maps of the \( G \)-action and the \( T \)-action respectively.
- Let \( M^T, M_0, M_{\text{red}} \) be defined similarly as in Sect. 2.

The following proposition is an easy generalization of Proposition 2.2.

**Proposition 5.1** Let \( G = \text{SU}(3) \) and \( M = O_{\xi_1} \times \cdots \times O_{\xi_N} \) be the Cartesian product of \( N \geq 3 \) adjoint orbits of \( G \), where the \( \xi_i \) satisfy the assumptions (A1) and (A2). Then, \( M^T \) is the discrete set

\[
\{(w_1 \cdot \xi_1, \ldots, w_N \cdot \xi_N) : w_i \in W\}. \tag{5.1}
\]

Thus, \(|M^T| = |W|^N|\).

As a result, \( M^T \) is parametrized by \( \vec{w} \in W^N \). Moreover, 6-partitions of \( \{1, \ldots, N\} \) can be defined similarly and each 6-partition \((I_1, \ldots, I_6)\) of \( \{1, \ldots, N\} \) corresponds to a unique \( \vec{w} \in W^N \) in the same way as before:

\[
I_j = \{i \in \{1, \ldots, N\} : w_i = \sigma_j\}, \tag{5.2}
\]

for \( j = 1, \ldots, 6 \). Thus, \( M^T \) can also be parametrized by all the 6-partitions of \( \{1, \ldots, N\} \).

In addition, \( \vec{w} \cdot \vec{\xi} \) and \( \vec{w} \odot \vec{\xi} \) can similarly be defined. Note that the notation \( \vec{w} \odot \vec{\xi} \) was introduced earlier in (3.35).

We have the following result, whose proof is similar to the proof of our earlier result 3.7):

**Theorem 5.2** Let \( G = \text{SU}(3) \) and \( M = O_{\xi_1} \times \cdots \times O_{\xi_N} \) be the Cartesian product of \( N \geq 3 \) adjoint orbits of \( G \), where the \( \xi_i \)’s satisfy the assumptions (A1), (A2) and (A3). Here \( s \) is the real dimension of \( G \) and \( l \) is the real dimension of \( T \). In this case \( G = \text{SU}(3) \), so \( s = 8 \) and \( l = 2 \). Therefore \( d = 6N - 16 \). Then, the symplectic volume of \( M_{\text{red}}(\vec{\xi}) \) is:

\[
\text{vol}^S(M_{\text{red}}(\vec{\xi})) = \frac{1}{\pi^{d/2} n_0 C_G} \text{res} \left( \sum_{\vec{w} \in W^N} \text{sgn}(\vec{w}) e^{i(\vec{w} \odot \vec{\xi}, \psi)} [d\psi] \right), \tag{5.3}
\]

where \( n_0 \) and \( C_G \) are as the same as in Theorem 2.4 and

\[
d = N(s - l) - 2s = (N - 2)s - Nl, \tag{5.4}
\]

where \( s \) is the real dimension of \( G \) and \( l \) is the real dimension of \( T \).
Proof When computing the symplectic volume of \( M_{\text{red}}(\tilde{\xi}) \), the only essential difference between the \( N = 3 \) case and the general \( N \geq 3 \) case lies in the equivariant Euler class \( e_{F} \) of the normal bundle of the fixed points of \( T \). In the general \( N \geq 3 \) case, for each fixed point \( F = \tilde{w} \cdot \xi \in M^{T} \),

\[
e_{F}(\psi) = \text{sgn}(\tilde{w}) \cdot \sigma^{N}(\psi).
\] (5.5)

This completes the proof. \( \square \)

As in Sect. 4, we should try to compare our formula with the general formula of Suzuki and Takakura [25]. Therefore, by the comparison argument in Sect. 4, when computing the residue, we will use the cone \( \Lambda_{\text{cone}} = \Lambda_{>0} \) instead of \( \Lambda = \Lambda_{>0} \). However, as we will see soon, the comparison for general \( N \) seems more difficult and we do not have a complete comparison for general \( N \) case.

We first write down here the general formula of Suzuki and Takakura [25] (recall that the group \( G \) here is \( SU(3) \)):

**Theorem 5.3** (Theorem 4.5 in [25]): (Compare with (4.1), the case \( N = 3 \).) Let \( N \geq 3 \) be an integer. Let

\[
\xi_{i} = (\ell_{i} - m_{i}) \cdot \Omega_{1} + m_{i} \cdot \Omega_{2},
\] (5.6)

where \( \ell_{i} > m_{i} > 0 \) are all integers divisible by 3 and

\[
(\tilde{w} \circ \xi, \Omega_{1}) \neq 0
\] (5.7)

for all \( \tilde{w} \in W^{N} \). Let

\[
L = \sum_{i=1}^{N} \ell_{i}, \quad M = \sum_{i=1}^{N} m_{i}.
\] (5.8)

Let \( \mathcal{I}_{\xi} \) denote the set of 6-partitions \( (I_{1}, \ldots, I_{6}) \) of \( \{1, \ldots, N\} \) such that

\[
\ell_{I_{1}, I_{2}} + m_{I_{4}, I_{5}} < \frac{L + M}{3},
\] (5.9)

\[
\ell_{I_{3}, I_{4}} + m_{I_{6}, I_{1}} < \frac{L + M}{3}.
\] (5.10)

Let \( \mathcal{J}_{\xi} \) denote the set of 6-partitions \( (I_{1}, \ldots, I_{6}) \) of \( \{1, \ldots, N\} \) such that

\[
\ell_{I_{3}, I_{4}} + m_{I_{6}, I_{1}} > \frac{L + M}{3},
\] (5.11)

\[
\ell_{I_{5}, I_{6}} + m_{I_{2}, I_{3}} > \frac{L + M}{3}.
\] (5.12)

Let \( A_{\xi} : \mathcal{I}_{\xi} \to \mathbb{R} \) be defined by

\[
A_{\xi}(I_{1}, \ldots, I_{6}) := \frac{(-1)^{\sum_{1}^{6}|I_{j}|}}{6(3N - 8)!} \sum_{j=0}^{N-3} \binom{3N - 8}{j} \binom{2N - 6 - j}{N - 3} \left( \frac{L + M}{3} - \ell_{I_{3}, I_{4}} - m_{I_{6}, I_{1}} \right)^{j} \left( \frac{L + M}{3} - \ell_{I_{1}, I_{2}} - m_{I_{4}, I_{5}} \right)^{3N - 8 - j}.
\] (5.13)
Let $B_\xi : J^c_\xi \to \mathbb{R}$ be defined by

$$B_\xi(I_1, \ldots, I_6) := \frac{(-1)^{|I_1|+|I_3|+|I_5|}}{6(3N-8)!} \sum_{j=0}^{N-3} \left( \binom{3N-8}{j} \binom{2N-6-j}{N-3} \right) \left( \ell_{I_3, I_4} + m_{I_6, I_1} - \frac{L + M}{3} \right)^{j} \left( \ell_{I_5, I_6} + m_{I_2, I_3} - \frac{L + M}{3} \right)^{3N-8-j}.$$  

(5.14)

Then, the symplectic volume of $M_{\text{red}}(\xi)$ is given by

$$V(\xi) = \sum_{(I_1, \ldots, I_6) \in J^c_\xi} A_\xi(I_1, \ldots, I_6) + \sum_{(I_1, \ldots, I_6) \in J^c_\xi} B_\xi(I_1, \ldots, I_6).$$  

(5.15)

We have the following result.

**Theorem 5.4** Let $G = SU(3)$ and $M = O_{\xi_1} \times \cdots \times O_{\xi_N}$ be the Cartesian product of $N \geq 3$ adjoint orbits of $G$, where the $\xi_i$'s satisfy the assumptions (A1), (A2) and (A3). Then, the symplectic volume of $M_{\text{red}}(\xi)$ is:

$$\text{vol}^S(M_{\text{red}}(\xi)) = C \cdot \sum_{\tilde{w} \in W^N} \text{sgn}(\tilde{w}) H_{(-\tilde{\beta})^N-2}(\tilde{w} \odot \tilde{\xi}),$$  

(5.16)

where $C$ is a constant.

**Proof** Recall that all of $-\beta_1, -\beta_2, -\beta_3 = -\beta_1 - \beta_2$ lie in the dual cone $\Lambda_+^*$. We have:

$$\text{res} \left( \sum_{\tilde{w} \in W^N} \text{sgn}(\tilde{w}) \frac{e^{i(\tilde{w} \odot \tilde{\xi}, \psi)} [d\psi]}{\sigma_{N-2}(\psi)} \right)$$  

(5.17)

$$= \text{res}^{\Lambda^*} \left( \sum_{\tilde{w} \in W^N} \text{sgn}(\tilde{w}) \frac{e^{i(\tilde{w} \odot \tilde{\xi}, \psi)} [d\psi]}{\sigma_{N-2}(\psi)} \right)$$  

(5.18)

$$= \sum_{\tilde{w} \in W^N} \text{sgn}(\tilde{w}) \text{res}^{\Lambda^*} \left( \frac{e^{i(\tilde{w} \odot \tilde{\xi}, \psi)} [d\psi]}{\sigma_{N-2}(\psi)} \right)$$  

(5.19)

$$= \sum_{\tilde{w} \in W^N} \text{sgn}(\tilde{w}) \text{res}^{\Lambda^*} \left( \frac{e^{i(\tilde{w} \odot \tilde{\xi}, \psi)} [d\psi]}{((-1) \cdot \prod_{j=1}^{3} (-\beta_j)(\psi))^{N-2}} \right)$$  

(5.20)

$$= (-1)^{N-2} \sum_{\tilde{w} \in W^N} \text{sgn}(\tilde{w}) \text{res}^{\Lambda^*} \left( \frac{e^{i(\tilde{w} \odot \tilde{\xi}, \psi)} [d\psi]}{((-1) \cdot \prod_{j=1}^{3} (-\beta_j)(\psi))^{N-2}} \right).$$  

(5.21)

To compute

$$\text{res}^{\Lambda^*} \left( \frac{e^{i(\tilde{w} \odot \tilde{\xi}, \psi)} [d\psi]}{((-1) \cdot \prod_{j=1}^{3} (-\beta_j)(\psi))^{N-2}} \right).$$  

(5.22)
we introduce the notation \((-\vec{\beta})^{N-2}\) to denote the following:
\[-\vec{\beta})^{N-2} := (-\beta_1, -\beta_2, -\beta_3, \ldots, -\beta_1, -\beta_2, -\beta_3), \quad (5.23)\]
where the sequence \(-\beta_1, -\beta_2, -\beta_3\) repeats itself for \(N - 2\) times.

Now we have:
\[
\text{res}^{\mathcal{A}-} \left( \frac{e^{i(\tilde{\omega} \odot \xi, \psi)}[d\psi]}{\prod_{j=1}^{3}(-\beta_j)(\psi)} \right)^{N-2} \]
\[
= \frac{i^{3(N-2)}(2\pi i)^2}{(2\pi i)^2} \cdot H_{(-\vec{\beta})^{N-2}}(\tilde{\omega} \odot \xi). \quad (5.25)
\]
This completes the proof.

\subsection*{5.2 Volume formula for general \(N\)-fold reduced products}

The method of nonabelian localization and the residue formula apply not only for \(G = \text{SU}(3)\), but also for any compact connected Lie groups. However, to apply Theorem 2.4 in our situation, namely the situation where the group \(G\) acts diagonally on the product of adjoint orbits by the adjoint action, we need to make sure that the stabilizer of any point in \(M_0 = \mu_G^{-1}(0)\) is finite. Therefore, in addition to the Lie group \(G\) being compact and connected, we assume that \(G\) is also semisimple.

We note that in [26] Suzuki and Takakura also treat volumes of \(N\)-fold reduced products of compact Lie groups \(G\). It would be interesting to verify explicitly that their results agree with ours. Our methods may be more amenable to generalization to intersection pairings, a subject we treat in [18].

Let \(T\) denote a chosen maximal torus of \(G\). Let \(g\) be the Lie algebra of \(G\) and \(t\) be the Lie algebra of \(T\). Let \(W\) denote the Weyl group \(N(T)/T\). Let \((\cdot, \cdot)\) denote a chosen \(G\)-invariant inner product on \(g\). Let \(R_+\) denote the collection of positive roots of \(G\). Let \(t_{>0}\) denote the open positive Weyl chamber.

Let \(s\) denote the real dimension of \(G\). Let \(l\) denote the real dimension of \(T\). Let \(N \geq 3\) be a positive integer.

Let \(\vec{\xi} = (\xi_1, \ldots, \xi_N)\) be an \(N\)-tuple of elements in \(g\). Let \(M = \mathcal{O}_{\xi_1} \times \cdots \times \mathcal{O}_{\xi_N}\) be the product of the corresponding adjoint orbits. Then \(G\) acts on \(M\) through the diagonal adjoint action. This is a Hamiltonian action, so we have the moment maps \(\mu_G\) and \(\mu_T\) as before. Let \(M_0 = \mu_G^{-1}(0)\). Let \(M_{\text{red}} = M_0/G\) be the reduced space, which we call an \(N\)-fold reduced product of \(G\).

The input \(\vec{\xi} = (\xi_1, \ldots, \xi_N)\) satisfies the following assumptions (as in Sect. 2):

(A1) \(\mu_G^{-1}(0) \neq \emptyset\) and 0 is a regular value for \(\mu_G\).

(A2) All \(\xi_i\)'s lie in \(t_{>0}\).

(A3) \(M_{\text{red}}(\vec{\xi})\) is a smooth manifold.

Given \(\vec{\omega} = (w_1, \ldots, w_N) \in W^N\), let
\[
\vec{\omega} \cdot \vec{\xi} = (w_1 \cdot \xi_1, \ldots, w_N \cdot \xi_N) \quad (5.26)
\]
and
\[
\vec{\omega} \odot \vec{\xi} = \sum_{i=1}^{N} w_i \cdot \xi_i, \quad (5.27)
\]
just as before. Then, we have
\[ M^T = \{ \vec{w} \cdot \vec{\xi} : \vec{w} \in W^N \}, \]
(5.28)
where \( M^T \) denotes the fixed point set of the action of \( T \) on \( M \). Notice that \( M^T \) is discrete and \(|M^T| = |W|^N\).

Let
\[ \varpi(\psi) = \prod_{\gamma \in R_+} \gamma(\psi) \]
(5.29)
for all \( \psi \in \mathfrak{t} \).

We have the following general result.

**Theorem 5.5** Let \( G \) be a general semisimple compact connected Lie group and \( M = O_{\xi_1} \times \cdots \times O_{\xi_N} \) be the Cartesian product of \( N \geq 3 \) adjoint orbits of \( G \), where the \( \xi_i \)'s satisfy the assumptions (A1), (A2) and (A3). Then, the symplectic volume of \( M_{\text{red}}(\vec{\xi}) \) is
\[ \text{vol}^S(M_{\text{red}}(\vec{\xi})) = \frac{1}{d/2} n_0 C_G \text{ res} \left( \sum_{\vec{w} \in W_N} \text{sgn}(\vec{w}) e^{i(\vec{w} \cdot \vec{\xi}, \psi)} [d\psi] \right) \]
(5.30)
where
\[ d = N(s - l) - 2s = (N - 2)s - Nl, \]
(5.31)
and \( n_0 \) is the cardinality of the stabilizer \( \text{Stab}_G(p) \) of a generic point \( p \) in \( M_0 \) and the constant \( C_G \) is defined by
\[ C_G := \frac{(-1)^{n_+}}{(2\pi)^{s-l} |W| \text{ vol}^R(T)}. \]
(5.32)
Here, \( n_+ \) is the number of positive roots, in other words, \( n_+ = (s - l)/2 \).

**Proof** At each fixed point \( F = \vec{w} \cdot \vec{\xi} \in M^T \), the \( T \)-equivariant Euler class of the normal bundle over \( F \) is
\[ e_F(\psi) = \sigma^N(\psi). \]
(5.33)
The computation of the residue is similar to that in the case \( N = 3 \) and \( G = SU(3) \). \( \square \)

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