On the stability phenomenon of the Navier-Stokes type equations for elliptic complexes

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ABSTRACT
Let $X$ be a Riemannian $n$-dimensional smooth closed manifold, $n \geq 2$, $E_i$ be smooth vector bundles over $X$ and $\{A_i, E_i\}$ be an elliptic differential complex of linear first order operators. We consider the operator equations, induced by the Navier-Stokes type equations associated with $\{A_i, E_i\}$ on the scale of anisotropic Hölder spaces over the layer $X \times [0, T]$ with finite time $T > 0$. Using the properties of the differentials $A_i$ and parabolic operators over this scale of spaces, we reduce the equations to a nonlinear Fredholm operator equation of the form $(I + K)u = f$, where $K$ is a compact continuous operator. It appears that the Fréchet derivative $(I + K)'$ is continuously invertible at every point of each Banach space under consideration and the map $(I + K)$ is open and injective in the space.

1. Introduction
Thanks in large part to S.L. Sobolev, the modern theory of partial differential equations is based on the Functional Analysis and Embedding Theorems for various Banach spaces. In the present paper we illustrate this thesis on the Navier-Stokes type equations for elliptic complexes over Hölder spaces.

The problem of describing the dynamics of an incompressible viscous fluid is of great importance in applications. Despite enormous efforts of many mathematicians, the existence theorem for classical solutions was proved for the two-dimensional case only (see, for instance, [1–6] among essential contributions).

In this paper we focus attention on the stability phenomenon for the Navier-Stokes Equations discovered by O.A. Ladyzhenskaya. Namely, she proved (see [5, Theorems 10 and 11]) that if for rather regular data there is a sufficiently regular unique solution to the Navier-Stokes equations then for all sufficiently small perturbations of the data there are unique solutions of the same regularity. For infinitely smooth data and solutions this phenomenon was indicated in [7] in the case of zero exterior forces. Recently, the phenomenon was verified for the Navier-Stokes equations in the scale of anisotropic Hölder spaces over the layer $\mathbb{R}^n \times [0, T]$, weighted at infinity with respect to the space variables, see [8].
We want to investigate the stability property in the context of Navier-Stokes type equations associated with elliptic differential complexes, see [9]. Namely, let $\mathcal{X}$ be a Riemannian $n$-dimensional smooth closed manifold with metric $g$ and let $E^i$ be smooth vector bundles over $\mathcal{X}$. Let $C^\infty_{E^0}(\mathcal{X})$ denote the space of all smooth sections of the bundle $E^0$. Consider an elliptic complex

$$
0 \longrightarrow C^\infty_{E^0}(\mathcal{X}) \xrightarrow{A^0} C^\infty_{E^1}(\mathcal{X}) \xrightarrow{A^1} \cdots \xrightarrow{A^{N-1}} C^\infty_{E^N}(\mathcal{X}) \longrightarrow 0
$$

of first order differential operators $A^i$ on $\mathcal{X}$, see for instance, [10] or [11, §10.4.3]. This means that $A^{i+1} \circ A^i \equiv 0$ and the Laplacians $\Delta^i = (A^i)^* A^i + A^{i-1} (A^{i-1})^*$ are second order strongly elliptic operators at each step $i$, $0 \leq i \leq N$, where $(A^i)^*$ is the formal adjoint differential operator of $A^i$; here we tacitly assume that $A_i = 0$ for $i < 0$ or $i > N-1$.

If the variable $t$ enters a section of the bundle $E^i$ as a parameter, then we easily may define the operator $\partial_t$ acting on sections of the induced bundle $E^i(t)$ over the cylinder $\mathcal{X} \times [0, +\infty)$. Then the second order operators $L^i_\mu = \partial_t + \mu \Delta^i$ are parabolic on $\mathcal{X} \times [0, +\infty)$ for each positive number $\mu$, see, for instance, [12].

Fixing two bilinear mappings $\mathcal{M}_{i,j}$, satisfying

$$
\mathcal{M}_{i,1,x} : E^{i+1}_x \otimes E^i_x \rightarrow E^i_x, \quad \mathcal{M}_{i,2,x} : E^i_x \otimes E^i_x \rightarrow E^{i-1}_x,
$$

at each point $x \in \mathcal{X}$, we set for a differentiable section $v$ of the bundle $E^i$:

$$
\mathcal{N}^i(v)(x) = \mathcal{M}_{i,1,x}((A^i v)(x), v(x)) + A^{i-1} \mathcal{M}_{i,2}(v(x), v(x)).
$$

In this paper we consider the initial problem over the cylinder $\mathcal{X}_T = \mathcal{X} \times [0, T]$ with a finite time $T > 0$: given section $f$ of the induced bundle $E^i(t)$ and section $v_0$ of the bundle $E^i$, find a section $v$ of the induced bundle $E^i(t)$ and a section $p$ of the induced bundle $E^{i-1}(t)$ such that

$$
L^i_\mu v + \mathcal{N}^i(v) + A^{i-1} p = f \quad \text{in } \mathcal{X} \times (0, T),
$$

$$
(A^{i-1})^* v = 0, \quad (A^{i-2})^* p = 0 \quad \text{in } \mathcal{X} \times [0, T],
$$

$$
v(x, 0) = v_0 \quad \text{in } \mathcal{X}.
$$

We note that the gradient operator $\nabla$, the rotation operator rot and divergence operator div represent the operators $d^i$ included in the de Rham complex and acting as differentials between the bundles $\Lambda^i$ and $\Lambda^{i+1}$ of exterior differential forms over $\mathbb{R}^3$ of degrees $i$ and $i+1$, $i = 0, 1, 2$, respectively. Let us express the standard non-linearity $\mathcal{N}^1(v) = v \cdot \nabla v$ in the so-called Lamb form (see [13, §15]):

$$
\mathcal{N}^1(v) = v \cdot \nabla v = (\text{rot } v) \otimes v + \nabla |v|^2/2.
$$

Taking the 3-dimensional torus $\mathbb{T}^3$ as $\mathcal{X}$, the de Rham complex $\{d^i, \Lambda^i\}_{i=0}^2$ over it and choosing $i = 1$ and $\mathcal{N}^1(v) = v \cdot \nabla v$, we may treat (4) as the initial problem for the Navier-Stokes equations for an incompressible fluid in the so-called periodic setting, with the dynamical viscosity $\mu$ of the fluid under consideration, the density vector of outer forces $f$, the initial velocity $v_0$ and the search-for velocity vector field $v$ and the pressure $p$ of the flow, see, for instance, [14]; for $i = 1$ we have $(d^{i-1})^* \equiv 0$ and the 'pressure' $p$ is not the subject for an additional equation in this case.
The first paper to consider the Navier-Stokes equations on Riemannian manifolds is the classical paper [15] (see also [16–18] for the development of the story, where the authors deal precisely with the issue of non-uniqueness for the Navier-Stokes equations on manifolds). However, we do not discuss here relations of (4) to Hydrodynamics and thus, under the imposed restrictions, we obtain the uniqueness of solutions to (4). Then we reduce (4) to a nonlinear Fredholm operator equation of the form \((I + K_i)u = f\) on a scale of anisotropic Hölder type Banach spaces over the cylinder \(\mathcal{X} \times [0, T]\), where \(K_i\) is a compact continuous operator. It appears that the Fréchet derivative \((I + K_i)'\) is continuously invertible at every point of each Banach space under consideration and the map \(I + K_i\) is open and injective.

2. The anisotropic Hölder spaces

We begin with the definition of proper function spaces.

Since \(\mathcal{X}\) is a compact closed Riemannian manifold, choosing a volume form \(dx\) on \(\mathcal{X}\) and a Riemannian metric \((\cdot, \cdot)_{x,i}\) in the fibres of \(E\), we equip each bundle \(E\) with a smooth bundle homomorphism \(*_i : E^i \to E^{i*}\) defined by \(\langle *_i u, v \rangle_{x,i} = \langle v, u \rangle_{x,i}\) for \(u, v \in E^i_x\), and the space \(C^\infty_{E^i}(\mathcal{X})\) with the unitary structure

\[
(u, v)_i = \int_{\mathcal{X}} (u, v)_{x,i} \, dx
\]

giving rise to the Hilbert space \(L^2_{E^i}(\mathcal{X})\) with the norm \(\|u\|_i = \sqrt{(u, u)_i}\).

As usual, we say that \((A^i)^*\) is the formal adjoint differential operator for \(A^i\) if, for all \(u \in C^\infty_{E^i}(\mathcal{X})\) and \(v \in C^\infty_{E^{i+1}}(\mathcal{X})\),

\[
(A^i u, v)_{i+1} = (u, (A^i)^* v)_i.
\]

Besides, the Riemannian metric \(g\) defines a natural metric structure on \(\mathcal{X}\). Thus, for any smooth vector bundle \(E\) over \(\mathcal{X}\) equipped with a metric \(h\) and compatible connection \(\nabla\) we may introduce the spaces of \(s\) times continuously differentiable sections of \(E\) for \(s \in \mathbb{Z}_+\) and the Hölder spaces \(Cs^{\lambda}_{E}(\mathcal{X})\) with \(0 < \lambda < 1\), see, for instance, [11, Ch. 10]. These are known to be Banach spaces with the norms:

\[
\|u\|_{Cs^{\lambda}_{E}(\mathcal{X})} = \|u\|_{Cs^{0}_{E}(\mathcal{X})} + \lambda \sum_{j=0}^{s} \langle \nabla^j u \rangle_{\lambda, \mathcal{X}, E},
\]

where

\[
\|u\|_{Cs^{0}_{E}(\mathcal{X})} = \sum_{j=0}^{s} \sup_{x \in \mathcal{X}} |\nabla^j u(x)|, \quad \langle u \rangle_{\lambda, \mathcal{X}, E} = \sup_{x, y \in \mathcal{X}, x \neq y, d(x, y) \leq d_0} \frac{|u(x) - u(y)|}{d^\lambda(x, y)}.
\]

Here \(d(x, y)\) is the geodesic distance between points \(x, y \in \mathcal{X}\), \(d_0 = \min (1, d X)\), \(d X\) is the injectivity radius of \(\mathcal{X}\), providing that the points \(x, y\) can be connected by a unique minimal
geodesic $\Gamma_{x,y}$, and, for each $\zeta \in E_x$, $\eta \in E_y$,

$$|\eta - \zeta| = |\zeta - T_{x,y}\eta|_x = |\eta - T_{y,x}\zeta|_y$$

with a $\nabla$-parallel transport $T_{x,y} : E_y \to E_x$ along the geodesic $\Gamma_{x,y}$. In this way the space $C^{s,\lambda}_E(\mathcal{X})$ is independent of the metrics $g$, $h$ and the connection $\nabla$ as a set of sections, see [11, Theorem 10.2.36]. It is also known that these Banach spaces admits the standard embedding theorems.

**Theorem 2.1:** Suppose that $s, s' \in \mathbb{Z}_+$, and $\lambda, \lambda' \in [0,1)$. If $s + \lambda \geq s' + \lambda'$ then the space $C^{s,\lambda}_E(\mathcal{X})$ is embedded continuously into the space $C^{s',\lambda'}_E(\mathcal{X})$. Moreover, the embedding is compact if $s + \lambda > s' + \lambda'$.

Let us introduce the anisotropic Hölder spaces over $\mathcal{X}_T$ adopted to the parabolic theory see, for instance, [5, 19]. Namely, for $s \in \mathbb{Z}_+$ and $\lambda \in [0, 1)$, $\gamma \in [0, 1)$ let $C^{2s,\lambda,s,\gamma}_E(\mathcal{X}_T)$ be the space of sections of the induced bundle $E(t)$ over $\mathcal{X}_T$ with continuous partial derivatives $\nabla_x^m \partial_t^j u$, for $m + 2j \leq 2s$ and the finite norm

$$\|u\|_{C^{2s,\lambda,s,\gamma}_E(\mathcal{X}_T)} = \sum_{m+2j\leq 2s} \sup_{t\in[0,T]} \|\partial_t^j \nabla_x^m u(\cdot, t)\|_{C^{0,\lambda}_E(\mathcal{X})} + \gamma \sum_{m+2j\leq 2s} \langle \nabla_x^m \partial_t^j u \rangle_{\gamma, \mathcal{X}_T,E}.$$ 

where, for $\gamma > 0$,

$$\langle u \rangle_{\gamma, \mathcal{X}_T,E} = \sup_{t',t''\in[0,T]} \frac{\|u(\cdot, t') - u(\cdot, t'')\|_{C^{0,\lambda}_E(\mathcal{X})}}{|t' - t''|^\gamma}.$$ 

We also need a function space whose structure goes slightly beyond the scale of function spaces $C^{2s,\lambda,s,\gamma}_E(\mathcal{X}_T)$. Namely, given any $k \in \mathbb{Z}_+$, we denote by $C^{2s+k,\lambda,s,\gamma}_E(\mathcal{X}_T)$ the space of all continuous functions $u$ on $\mathcal{X}_T$ with $\nabla_x^l u$ belonging to $C^{2s+k,\lambda,s,\gamma}_E(\mathcal{X}_T)$ for all $l \in \mathbb{Z}_+$ satisfying $0 \leq l \leq k$. This is a Banach space with the norm

$$\|u\|_{C^{2s+k,\lambda,s,\gamma}_E(\mathcal{X}_T)} = \sum_{l=0}^k \|\nabla_x^l u\|_{C^{2s+k,\lambda,s,\gamma}_E(\mathcal{X}_T)}.$$ 

As it is customary in the parabolic theory, we use these spaces for $\gamma = 0$ and $\gamma = \frac{1}{2}$, only. The following embedding theorem is rather expectable.

**Theorem 2.2:** Let $k, s, s' \in \mathbb{Z}_+$, $\lambda, \lambda' \in [0,1)$. If $s + \lambda \geq s' + \lambda'$ then the space $C^{2s+k,\lambda,s,\lambda/2}_E(\mathcal{X}_T)$ is embedded continuously into $C^{2s'+k,\lambda',s',\lambda'/2}_E(\mathcal{X}_T)$. The embedding is compact if $s + \lambda > s' + \lambda'$.

We also need the following expectable lemmata.

**Lemma 2.3:** Suppose that $s, k \in \mathbb{Z}_+$ and $\lambda \in [0,1)$. Then it follows that
any differential operator of order \( k' \leq k \) on \( \mathcal{X} \) acting between vector bundles \( E \) and \( F \) maps \( C^{2s+k,\lambda,s,\lambda/2}_{E} (\mathcal{T}) \) continuously into \( C^{2s+k-k',\lambda,\lambda,\lambda/2}_{F} (\mathcal{T}) \)

(2) if \( 0 \leq j \leq s \) then the operator \( \partial_{i}^{j} \) maps \( C^{2s+k,\lambda,s,\lambda/2}_{E} (\mathcal{T}) \) continuously into \( C^{2s-j+k,\lambda,\lambda,\lambda/2}_{E} (\mathcal{T}) \);

(3) the operator \( \Delta^{i} \) maps \( C^{2(s+1)+k,\lambda,s+1,\lambda/2}_{E} (\mathcal{T}) \) continuously into \( C^{2s+k,\lambda,s,\lambda/2}_{E} (\mathcal{T}) \).

In the sequel we will always assume that there are constants \( c_{ij}(\mathcal{M}) \) such that

\[
|\mathcal{M}_{i,1,x}(v, u)| \leq c_{i,1}(\mathcal{M})|u||v|, \quad |\mathcal{M}_{i,2,x}(w, u)| \leq c_{i,2}(\mathcal{M})|u||w|
\]

for all \( x \in \mathcal{X} \), all \( v \in E_{x}^{i+1} \) and \( u, w \in E_{x}^{i} \).

**Lemma 2.4:** Let \( s, k \in \mathbb{Z}_{+} \) and \( \lambda \in [0, 1) \). If (6) holds then forms (2) induce continuous bilinear operators

\[
\mathcal{M}_{i,j}(v, u) \equiv C^{2s+k,\lambda,s,\lambda/2}_{E} (\mathcal{T}) \times C^{2s+k,\lambda,s,\lambda/2}_{E} (\mathcal{T}) \rightarrow C^{2s+k,\lambda,s,\lambda/2}_{E} (\mathcal{T}),
\]

satisfying

\[
\|\mathcal{M}_{i,j}(v, u)\|_{C^{2s+k,\lambda,s,\lambda/2}_{E} (\mathcal{T})} \leq c \|u\|_{C^{2s+k,\lambda,s,\lambda/2}_{E} (\mathcal{T})} \|v\|_{C^{2s+k,\lambda,s,\lambda/2}_{E} (\mathcal{T})}
\]

with a constant \( c > 0 \) independent of \( u \) and \( v \).

**Proof:** Indeed, since \( \mathcal{M}_{i,j,x} \) are bilinear forms,

\[
\mathcal{M}_{i,j}(v, u) - \mathcal{M}_{i,j}(v^{(0)}, u^{(0)}) = \mathcal{M}_{i,j}(v - v^{(0)}, u^{(0)}) + \mathcal{M}_{i,j}(v^{(0)}, u - u^{(0)}) + \mathcal{M}_{i,j}(v - v^{(0)}, u - u^{(0)}),
\]

at each points \( v, v^{(0)}, u, u^{(0)} \) of the Banach spaces under consideration. Then Lemma 2.3 implies that (7) are continuous operators because of (6), (8).

### 3. Elliptic complexes over the Hölder spaces

The behaviour of the elliptic complexes on the Hölder scale is well known (see [[10, Ch. 2, §2.2], [11, §10.4.3]]). Namely, consider the bounded linear operator

\[
\Delta^{i} : C^{s+2,\lambda}_{E} (\mathcal{X}) \rightarrow C^{s,\lambda}_{E} (\mathcal{X})
\]

induced by the Laplacian \( \Delta^{i} \). Let \( \mathcal{H}^{i} \) stand for the so-called ‘harmonic space’ of the complex (1), i.e.

\[
\mathcal{H}^{i} = \{ u \in C^{\infty}_{E} (\mathcal{X}) : A^{i}u = 0 \text{ and } (A^{i-1})*u = 0 \text{ in } \mathcal{X} \}.
\]

Denote by \( \Pi^{i} \) the orthogonal projection from \( L^{2}_{E} (\mathcal{X}) \) onto \( \mathcal{H}^{i} \).

**Theorem 3.1:** Let \( 0 \leq i \leq N, s \in \mathbb{Z}_{+}, 0 < \lambda < 1 \). Then the operator (9) is Fredholm:

1. the kernel of the operator (9) equals to the finite-dimensional space \( \mathcal{H}^{i} \);
2. given \( v \in C^{s,\lambda}_{E} (\mathcal{X}) \) there is a form \( u \in C^{s+2,\lambda}_{E} (\mathcal{X}) \) such that \( \Delta^{i}u = v \) if and only if \( (v, h)_{i} = 0 \) for all \( h \in \mathcal{H}^{i} \);
(3) there exists a pseudo-differential operator \( \varphi^i \) on \( \mathcal{X} \) such that the operator
\[
\varphi^i : C^{s,\lambda}_{E^i}(\mathcal{X}) \to C^{s+2,\lambda}_{E^i}(\mathcal{X}),
\]
induced by \( \varphi^i \), is bounded linear and with the identity \( I \) we have
\[
A^i \Pi^i = 0, \quad (A^{-1})^* \Pi^i = 0, \quad \Pi^{i+1} A^i = 0, \quad \Pi^{i-1} (A^{-1})^*.
\]
(10) \[ \varphi^i \Delta^i = I - \Pi^i \quad \text{on} \quad C^{s+2,\lambda}_{E^i}(\mathcal{X}), \quad \Delta^i \varphi^i = I - \Pi^i \quad \text{on} \quad C^{s,\lambda}_{E^i}(\mathcal{X}). \]

**Proof:** For \( C^\infty \)-smooth sections see, for instance, [10, Theorem 2.2.2] or [11, Theorem 10.4.29]. For the extension to the Hölder spaces we refer to the standard procedure using apriori estimates for elliptic operators, see, for instance, [20, Ch. 4–6] or [11, Theorem § 10.3, 10.4].

Next, for a differential operator \( A \) acting on sections of the vector bundle \( E \) over \( \mathcal{X} \), we denote by \( C^{s,\lambda}_{E}(\mathcal{X}) \cap S_A \) the space of all the sections \( u \in C^{s,\lambda}_{E}(\mathcal{X}) \) satisfying \( Au = 0 \) in the sense of distributions in \( \mathcal{X} \). This space is obviously a closed subspace of \( C^{s,\lambda}_{E}(\mathcal{X}) \) and so this is a Banach space under the induced norm.

**Corollary 3.2:** Let \( s \in \mathbb{Z}_+ \), \( 0 < \lambda < 1 \). The differential complex (1) induces continuous linear operators
\[
A^i \oplus (A^{-1})^* : C^{s+1,\lambda}_{E^i}(\mathcal{X}) \to C^{s,\lambda}_{E^{i+1}}(\mathcal{X}) \cap S_{A^{i+1}} \times C^{s,\lambda}_{E^{i-1}}(\mathcal{X}) \cap S_{(A^{-2})^*}. \tag{11}
\]
These operators are Fredholm. More precisely,

1. the kernel of (11) coincides with the finite-dimensional space \( \mathcal{H}^i \);
2. the (closed) range of operator (11) consists of all pairs \( (f, g) \in C^{s,\lambda}_{E^{i+1}}(\mathcal{X}) \cap S_{A^{i+1}} \times (C^{s,\lambda}_{E^{i-1}}(\mathcal{X}) \cap S_{(A^{-2})^*}, \) satisfying for all \( \hat{h} \in \mathcal{H}^{i-1} \) and all \( h \in \mathcal{H}^{i+1} \)
\[
(f, h)_{i+1} + (g, \hat{h})_{i-1} = 0.
\]

**Proof:** It follows from Theorem 3.1 that the kernel of operator (11), coincides with the space \( \mathcal{H}^i \). Moreover Theorem 3.1 implies the following simple lemma.

**Lemma 3.3:** Let \( s \in \mathbb{Z}_+ \), \( \lambda \in (0, 1) \). The pseudo-differential operators \( \Phi_i = (A^i)^* \varphi^{i+1}, \)
\( \Phi^i = A^i \varphi^i \) on \( \mathcal{X} \) induce continuous maps
\[
\Phi_i : C^{s,\lambda}_{E^{i+1}}(\mathcal{X}) \to C^{s+1,\lambda}_{E^i}(\mathcal{X}) \cap S_{(A^{i-1})^*}, \quad \Phi^i : C^{s,\lambda}_{E^i}(\mathcal{X}) \to C^{s+1,\lambda}_{E^{i+1}}(\mathcal{X}) \cap S_{A^{i+1}},
\]
satisfying
\[
\Phi_i \Pi^{i+1} = 0, \quad \Pi^i \Phi_i = 0, \quad \Phi^i \Pi^i = 0, \quad \Pi^{i+1} \Phi^i = 0, \tag{12}
\]
\[
\Phi_i A^i u + A^{i-1} \Phi_{i-1} u = u - \Pi^i u, \quad \text{if} \ u \in C^{s,\lambda}_{E^i}(\mathcal{X}), \quad A^i u \in C^{s,\lambda}_{E^{i+1}}(\mathcal{X}), \tag{13}
\]
\[
\Phi_{i-1} (A^{i-1})^* v + (A^i)^* \Phi^i v = v - \Pi^i v, \quad \text{if} \ v \in C^{s,\lambda}_{E^i}(\mathcal{X}), \quad (A^{i-1})^* v \in C^{s,\lambda}_{E^{i-1}}(\mathcal{X}). \tag{14}
\]
Proof: Indeed, by definition of the complex (1),
\[
A^i \Delta^i = A^i (A^i)^* A^i = \Delta^{i+1} A^i, \quad (A^{i-1})^* \Delta^i = (A^{i-1})^* A^{i-1} (A^{i-1})^* = \Delta^{i-1} (A^{i-1})^*.
\]
Thus, we conclude that, on the sections with sufficient differentiability,
\[
A^i \varphi^i = \varphi^{i+1} A^i, \quad (A^{i-1})^* \varphi^i = \varphi^{i-1} (A^{i-1})^*,
\]
and the statement follows from Theorem 3.1.

This lemma proves the statement on the range of the operator (11).

This corollary just reflects the well-known fact that the space \( \mathcal{H}^i \) represents \( i \)th cohomologies of complex (1) over the scale \( C^{s,\lambda}_E (\mathcal{X}) \), see \([10, \text{Ch. 2}]\).

Now we start to study complex (1) over the scale \( C^{2s+k,\lambda,s,\gamma}_E (\mathcal{X}_T) \). As elliptic operators are not fully consistent with the parabolic dilation principle on \( \mathcal{X}_T \), we should expect some loss of regularity of solutions to the elliptic system
\[
A^i u = f, \quad (A^{i-1})^* u = g,
\]
on this scale of function spaces.

Again, for a differential operator \( A \) acting on sections of the induced vector bundle \( E(t) \) over \( \mathcal{X} \), we write \( C^{2s+k,\lambda,s,\gamma}_E (\mathcal{X}_T) \cap \mathcal{S}_A \) for the space of all sections of \( E \) over \( \mathcal{X} \) of the class \( C^{2s+k,\lambda,s,\gamma}_E (\mathcal{X}_T) \) satisfying in the sense of distributions
\[
Au(\cdot, t) = 0 \text{ in } \mathcal{X} \quad \text{for all fixed } t \in [0, T].
\]
This space is obviously a closed subspace of \( C^{2s+k,\lambda,s,\gamma}_E (\mathcal{X}_T) \), and so it is a Banach space under the induced norm.

Actually, we want to extend Corollary 3.2 to the operator \( A^i \oplus (A^{i-1})^* \) on the anisotropic scale \( C^{2s+k,\lambda,s,\lambda/2}_E (\mathcal{X}_T) \). Similarly to the scale \( C^{s,\lambda}_E (\mathcal{X}) \), we use the potentials \( \Phi_i, \hat{\Phi}_i \) on sections of the induced bundles over \( \mathcal{X}_T \). The variable \( t \) enters into the potentials \( (\Phi_i f)(x, t), (\hat{\Phi}_i g)(x, t) \) as a parameter and the pair \( (x, t) \) is assumed to be in the layer \( \mathcal{X}_T \). However, the elements of the space \( C^{2s+k,\lambda,s,\lambda/2}_E (\mathcal{X}_T) \) have additional smoothness with respect to \( t \) that can not be improved by the potentials \( \Phi_i, \hat{\Phi}_i \). To avoid this difficulty, we introduce \( C^{2s+k,\lambda,s,\lambda/2}_E (\mathcal{X}_T) \cap \mathcal{D}_A \) to be the space of all sections \( u \) from \( C^{2s+k,\lambda,s,\lambda/2}_E (\mathcal{X}_T) \) with the property that \( A^i u \in C^{2s+k,\lambda,s,\lambda/2}_E (\mathcal{X}_T) \). We endow this space with the so-called graph norm
\[
\|u\|_{C^{2s+k,\lambda,s,\lambda/2}_E (\mathcal{X}_T)} + \|A^i u\|_{C^{2s+k,\lambda,s,\lambda/2}_E (\mathcal{X}_T)}.
\]

Lemma 3.4: Suppose that \( k, s \in \mathbb{Z}_+, 0 < \lambda < 1 \). Then the spaces
\[
C^{2s+k,\lambda,s,\lambda/2}_E (\mathcal{X}_T) \cap \mathcal{D}_A, \quad C^{2s+k,\lambda,s,\lambda/2}_E (\mathcal{X}_T) \cap \mathcal{D}_A^{(A^i)^*}, \quad C^{2s+k,\lambda,s,\lambda/2}_E (\mathcal{X}_T) \cap \mathcal{D}_A \oplus (A^{i-1})^*
\]
are Banach spaces and the operator \( A^i \oplus (A^{i-1})^* \) maps boundedly as
\[ C_{E_i}^{2s+k,\lambda,\nu/2}(\mathcal{X}_T) \cap D_{A^i(A^{-1})^*} \rightarrow C_{E_{i+1}}^{2s+k,\lambda,\nu/2}(\mathcal{X}_T) \cap S_{A^{i+1}} \times C_{E_{i-1}}^{2s+k,\lambda,\nu/2}(\mathcal{X}_T) \cap S_{(A^{-i-1})^*}. \]

**Proof:** If \( \{u_v\} \) is a Cauchy sequence in \( C_{E_i}^{2s+k,\lambda,\nu/2}(\mathcal{X}_T) \cap D_{A^i} \), then it is a Cauchy sequence in the space \( C_{E_i}^{2s+k,\lambda,\nu/2}(\mathcal{X}_T) \) and \( \{A^iu_v\} \) is a Cauchy sequence in the space \( C_{E_{i+1}}^{2s+k,\lambda,\nu/2}(\mathcal{X}_T) \). As the spaces are complete we conclude that the sequence \( \{u_v\} \) converges in \( C_{E_i}^{2s+k,\lambda,\nu/2}(\mathcal{X}_T) \) to an element \( u \) and the sequence \( \{A^iu\} \) converges in \( C_{E_{i+1}}^{2s+k,\lambda,\nu/2}(\mathcal{X}_T) \) to an element \( f \). Obviously, \( A^iu = f \) is fulfilled in the sense of distributions and thus \( A^{i+1}f = 0 \) in the sense of distributions because of \( A^{i+1} \circ A^i \equiv 0 \). Hence, \( u \) belongs to \( C_{E_i}^{2s+k,\lambda,\nu/2}(\mathcal{X}_T) \cap D_{A^i} \) and it is the limit of the sequence \( \{u_v\} \) in this space. Thus, we have proved that the space \( C_{E_i}^{2s+k,\lambda,\nu/2}(\mathcal{X}_T) \cap D_{A^i} \) is a Banach space. The proof for the other two spaces is similar. Moreover, by the very definition of the space, the operator \( A^i \oplus (A^{i-1})^* \) induces a continuous linear operator as is shown in (17). \[ \blacksquare \]

Now, let \( C^{s,\nu}([0, T], \mathcal{H}^i) \) stand for the set of sections of the induced bundle \( E^i(t) \) over \( \mathcal{X}_T \) of the form

\[ u(x, t) = \sum_{q=1}^{\dim(\mathcal{H}^i)} c_q(t)b_q(x), \]

where \( c_q \in C^{s,\nu}[0, T] \) and \( \{b_q\}_{q=1}^{\dim(\mathcal{H}^i)} \) is an \( L^2_{E_i}(\mathcal{X}) \)-orthonormal basis in \( \mathcal{H}^i \).

**Corollary 3.5:** Suppose that \( k, s \in \mathbb{Z}_+, 0 < \lambda < 1 \). Operator (17) has a closed range consisting of all pairs \((f, g) \in C_{E_{i+1}}^{2s+k,\lambda,\nu/2}(\mathcal{X}_T) \cap S_{A^{i+1}} \times C_{E_{i-1}}^{2s+k,\lambda,\nu/2}(\mathcal{X}_T) \cap S_{(A^{-i-1})^*} \), satisfying for all \( h \in \mathcal{H}^{i+1} \), all \( \hat{h} \in \mathcal{H}^{i-1} \) and all \( t \in [0, T] \)

\[ (f(\cdot, t), h)_{i+1} + (g(\cdot, t), \hat{h})_{i-1} = 0. \]

The kernel of operator (17) equals to \( C^{s,\lambda/2}([0, T], \mathcal{H}^i) \).

**Proof:** We begin with the following lemma.

**Lemma 3.6:** The pseudo-differential operator \( \Pi^i \) induces continuous maps

\[ \Pi^i : C_{E_i}^{2s+k,\lambda,\nu/2}(\mathcal{X}_T) \cap D_{A^i} \rightarrow C_{E_i}^{2s+k',\lambda,\nu/2}(\mathcal{X}_T) \cap D_{A^i} \]

for any \( k' \in \mathbb{N} \). The range of operator (19) coincides with \( C^{s,\lambda/2}([0, T], \mathcal{H}^i) \).
Proof: Indeed, as the space $C^{2s+k,\lambda,s,\lambda/2}_{E^t}(X_T)$ is continuously embedded in $L^2_{E^t}(X)$ we see that

$$
(P_i^t u)(x, t) = \sum_{q=1}^{\dim(H^i)} (u(\cdot, t), b_q)_{i} b_q(x)
$$

for each $u \in C^{2s+k,\lambda,s,\lambda/2}_{E^t}(X_T)$. Set $c_q(t) = (u(\cdot, t), b_q)_{i}$. Then

$$
\|\left(\frac{d^ic_q(t)}{dt^i}\right)\|_{C^{0,0}[0,T]} = \sup_{t \in [0,T]} \left|\left(\frac{d^ic_q(t)}{dt^i}\right)\right| \leq C \|u\|_{C^{2s+k,\lambda,s,\lambda/2}_{E^t}(X_T)}
$$

for each $0 \leq j \leq s$, i.e. $\Pi^i u \in C^{s,0}([0, T], H^i)$. Moreover,

$$
\langle\left(\frac{d^ic_q(t)}{dt^i}\right)\rangle_{2s,0}[0,T] = \langle\left(\frac{d^ic_q(t)}{dt^i}\right)\rangle_{2s,0} \leq C \|u\|_{C^{2s+k,\lambda,s,\lambda/2}_{E^t}(X_T)}
$$

for each $0 \leq j \leq s$, i.e. $\Pi^i u \in C^{s,\lambda/2}([0, T], H^i)$.

If the section $v$ belongs to $C^{s,\lambda/2}([0, T], H^i)$ then, obviously, $\Pi^i v = v$. Moreover,

$$
\|v\|_{C^{2s+k,\lambda,s,0}_{E^t}(X_T)} \leq c \sum_{q=1}^{\dim(H^i)} \|c_q(t)\|_{C^{0,0}([0,T])} \|b_q\|_{C^{2s+k,\lambda,s,0}_{E^t}(X_T)}
$$

with a constant $c > 0$ independent of $v$, i.e. $v \in C^{2s+k,\lambda,s,0}_{E^t}(X_T)$. Moreover, as $c_q \in C^{s,\lambda/2}([0, T])$ then, for all $j \leq s$

$$
\sup_{t', t'' \in [0,T] \atop t' \neq t''} \frac{\|\partial^j_{t'} v(\cdot, t') - \partial^j_{t''} v(\cdot, t'')\|_{C^{2s+\lambda/2+k,0}_{E^t}(X)}}{|t' - t''|^\lambda/2}
$$

$$
\leq \sum_{q=1}^{\dim(H^i)} \sup_{t \in [0,T]} \langle\left(\frac{d^ic_q(t)}{dt^i}\right)\rangle_{2s,0}[0,T] \|b_q\|_{C^{2s+\lambda/2+k,\lambda,0}_{E^t}(X_T)}
$$

i.e. $v \in C^{2s+k',\lambda,s,\lambda/2}_{E^t}(X_T)$. This proves that the range of operator (19) coincides with the space $C^{s,\lambda/2}([0, T], H^i)$.

As $\Pi^i$ is a bounded linear operator from $L^2_{E^t}(X)$ into the finite-dimensional space $H^i \subset C^\infty(X)$ we see that, for any $s' \in \mathbb{N}$, $0 \leq \lambda' < 1$,

$$
\sup_{t \in [0,T]} \|\partial^j_{t'} [\Pi^i u(\cdot, t)]\|_{C^{s',\lambda'}_{E^t}(X)} = \sup_{t \in [0,T]} \|\Pi^i \partial^j_{t'} u(\cdot, t)\|_{C^{s',\lambda'}_{E^t}(X)}
$$

$$
\leq c_1 \sup_{t \in [0,T]} \|\Pi^i \partial^j_{t'} u(\cdot, t)\|_{L^2_{E^t}(X)} \leq c_2 \sup_{t \in [0,T]} \|\partial^j_{t'} u(\cdot, t)\|_{L^2_{E^t}(X)}
$$

$$
\leq c_3 \sup_{t \in [0,T]} \|\partial^j_{t'} u(\cdot, t)\|_{C^{2(s-j)+\lambda,k}_{E^t}(X)}
$$

for all $0 \leq j \leq s$ and $u \in C^{2s+k,\lambda,s,\lambda/2}_{E^t}(X_T)$ where the existence of the constant $c_1$ is granted by the well known property of finite-dimensional spaces, the constant $c_2$ is granted by the
continuity of the projection $\Pi^i$ on $L^2_E(\mathcal{X})$ and the constant $c_3$ is granted by the continuity of the embedding $C^{2(s-j)+k,0}_E(\mathcal{X}) \to L^2_E(\mathcal{X})$.

Similarly, for any $k' \in \mathbb{N}$,

$$\sup_{t',t'' \in [0,T]} \frac{\|\Pi^i \partial^j_t u(\cdot, t') - \Pi^i \partial^j_t u(\cdot, t'')\|_{C^{2(s-j)+k',0}_E(\mathcal{X})}}{|t' - t''|^\lambda/2} \leq c_1 \sup_{t',t'' \in [0,T]} \frac{\|\Pi^i(\partial^j_t u(\cdot, t') - \partial^j_t u(\cdot, t''))\|_{L^2_E(\mathcal{X})}}{|t' - t''|^\lambda/2} \leq c_2 \sup_{t',t'' \in [0,T]} \frac{\|\partial^j_t u(\cdot, t') - \partial^j_t u(\cdot, t'')\|_{C^{2(s-j)+k,0}_E(\mathcal{X})}}{|t' - t''|^\lambda/2}$$

where the existence of the constant $c_1$ is granted by the well known property of finite-dimensional spaces, the constant $c_2$ is granted by the continuity of the projection $\Pi^i$ and the continuity of the embedding $C^{2(s-j)+k',0}_E(\mathcal{X}) \to L^2_E(\mathcal{X})$.

Now, if $u$ belongs to the kernel of operator (17) then

$$A^i u(\cdot, t) = 0 \quad \text{and} \quad (A^{i-1})^* u(\cdot, t) = 0 \quad \text{for each} \ t \in [0, T]. \quad (21)$$

This is equivalent to the fact that $\Pi^i u(\cdot, t) = u(\cdot, t)$ for each $t \in [0, T]$. Hence the kernel of operator (17) equals to $C^{s,\lambda/2}([0, T], \mathcal{H}^i)$.

**Lemma 3.7:** The pseudo-differential operator $\Phi_i$ induces continuous maps

$$\Phi_i : C^{2s+k,\lambda,s,\lambda/2}_E(\mathcal{X}_T) \to C^{2s+k,\lambda,s,\lambda/2}_E(\mathcal{X}_T) \cap S_{(A^{i-1})^*}, \quad (22)$$

$$\Phi_i : C^{2s+k,\lambda,s,\lambda/2}_E(\mathcal{X}_T) \cap D_{A^{i+1}} \to C^{2s+k,\lambda,s,\lambda/2}_E(\mathcal{X}_T) \cap D_{A^i} \cap S_{(A^{i-1})^*}, \quad (23)$$

satisfying (12), (13) on the space $C^{2s+k,\lambda,s,\lambda/2}_E(\mathcal{X}_T) \cap D_{A^i}$.

**Proof:** The use of Lemma 3.3 yields

$$\sup_{t \in [0,T]} \|\partial^j_t \Phi_i f(\cdot, t)\|_{C^{2(s-j)+k+1,\lambda}_E(\mathcal{X})} = \sup_{t \in [0,T]} \|\Phi_i \partial^j_t f(\cdot, t)\|_{C^{2(s-j)+k,\lambda}_E(\mathcal{X})} \leq c \sup_{t \in [0,T]} \|\partial^j_t f(\cdot, t)\|_{C^{2(s-j)+k,\lambda}_E(\mathcal{X})}$$

for all $0 \leq j \leq s$ and $f \in C^{2s+k,\lambda,s,\lambda/2}_E(\mathcal{X}_T)$. Besides,

$$\sup_{t',t'' \in [0,T]} \frac{\|\Phi_i \partial^j_t f(\cdot, t') - \Phi_i \partial^j_t f(\cdot, t'')\|_{C^{2(s-j)+k,0}_E(\mathcal{X})}}{|t' - t''|^\lambda/2}$$
Lemma 3.9: The pseudo-differential operator $\Phi_i$ induces continuous maps

$$\Phi_i : C^{2s+k,\lambda,s,1/2}_E(\mathcal{X}_T) \to C^{2s+k,\lambda,s,1/2}_E(\mathcal{X}_T) \cap \mathcal{S}_{A^{i+1}},$$

$$\Phi_i : C^{2s+k,\lambda,s,1/2}_E(\mathcal{X}_T) \cap \mathcal{D}_{(A^{i-1})^*} \to C^{2s+k,\lambda,s,1/2}_E(\mathcal{X}_T) \cap \mathcal{D}_{(A^{i-1})^*} \cap \mathcal{S}_{A^{i+1}},$$

satisfying (12), (14) on the space $C^{2s+k,\lambda,s,1/2}_E(\mathcal{X}_T) \cap \mathcal{D}_{(A^{i-1})^*}$.

Proof: Similar to the proof of Lemma 3.8.

The statement on the range of the operator (17) follow because formulas (13), (14) are still valid on the spaces $C^{2s+k,\lambda,s,1/2}_E(\mathcal{X}_T) \cap \mathcal{D}_{A^i}$ and $C^{2s+k,\lambda,s,1/2}_E(\mathcal{X}_T) \cap \mathcal{D}_{(A^{i-1})^*}$, respectively.

Now we are ready to define the Leray-Helmholtz type projection onto the spaces $C^{s,\lambda}_E(\mathcal{X}) \cap \mathcal{S}_{(A^{i-1})^*}$ and $C^{2s+k,\lambda,s,1/2}_E(\mathcal{X}_T) \cap \mathcal{D}_{A^i} \cap \mathcal{S}_{(A^{i-1})^*}$.

Lemma 3.9: Let $s, k \in \mathbb{Z}_+, 0 < \lambda < 1$. The pseudo-differential operator $\pi^i = (A^i)^* A^i \psi^i + \Pi^i$ on $\mathcal{X}$ induces continuous surjective maps

$$\pi^i : C^{s,\lambda}_E(\mathcal{X}) \to C^{s,\lambda}_E(\mathcal{X}) \cap \mathcal{S}_{(A^{i-1})^*},$$

$$\pi^i : C^{2s+k,\lambda,s,1/2}_E(\mathcal{X}_T) \cap \mathcal{D}_{A^i} \to C^{2s+k,\lambda,s,1/2}_E(\mathcal{X}_T) \cap \mathcal{D}_{A^i} \cap \mathcal{S}_{(A^{i-1})^*},$$

$$\pi^i : C^{2s+k,1,\lambda,s,1/2}_E(\mathcal{X}_T) \to C^{2s+k,1,\lambda,s,1/2}_E(\mathcal{X}_T) \cap \mathcal{D}_{A^i} \cap \mathcal{S}_{(A^{i-1})^*}.$$
satisfying
\[ \pi^i \circ \pi^i u = \pi^i u, \quad (\pi^i u, u)_i = (u, \pi^i u)_i, \quad (\pi^i u, (I - \pi^i)u)_i = 0, \] (29)
for all \( u \in C^s_{E^i} (\mathcal{X}) \) or \( u \in C^{2s+k, s, \lambda/2}_{E^i} (\mathcal{X}_T) \cap D_{A^i} \).

**Proof:** Using (15), (16) we see that
\[ \pi^i = \Phi_i A^i + \Pi^i \circ C^{2s+k, s, \lambda/2}_{E^i} (\mathcal{X}_T) \cap D_{A^i}. \] (30)
Therefore the continuity of operators (26), (27), (28) follows from Lemmata 3.3, 3.7, formula (10) and the property of complex (1): \((A^i)^* \circ (A^i)^* \equiv 0\).

The surjectivity of the operators follows from formula (14). Indeed, if \( v \in C^s_{E^i} (\mathcal{X}) \cap S_{(A^i)^*} \) then (14) and Lemma 3.3 imply
\[ v = (A^i)^* \Phi^i v + \Pi^i v = (A^i)^* A^i \varphi^i v + \Pi^i v. \]
If \( v \in C^{2s+k, s, \lambda, \frac{1}{2}}_{E^i} (\mathcal{X}_T) \cap D_{A^i} \cap S_{(A^i)^*} \) then, using (14) and Corollary 3.5, we again conclude that
\[ v = (A^i)^* A^i \varphi^i v + \Pi^i v = \pi^i v. \]
Next, using (10), (15), (16), we see that
\[ \pi^i \circ \pi^i u = ((A^i)^* A^i \varphi^i + \Pi^i) \circ ((A^i)^* A^i \varphi^i + \Pi^i) u \]
\[ = ((A^i)^* A^i (A^i)^* \varphi^i + \Pi^i) u = \pi^i u \]
for all \( u \in C^s_{E^i} (\mathcal{X}) \) or \( u \in C^{2s+k, s, \lambda, \frac{1}{2}}_{E^i} (\mathcal{X}_T) \cap D_{A^i} \).

Finally, we recall that \( \pi^i u \in S_{(A^i)^*} \) and then, by (10),
\[ (\pi^i u, u)_i = (\pi^i u, \pi^i u + A^i(A^i)^* \varphi^i)_i \]
\[ = (\pi^i u, \pi^i u)_i = (u, \pi^i u)_i, \]
\[ (\pi^i u, (I - \pi^i)u)_i = (\pi^i u, u)_i - (\pi^i u, \pi^i u)_i = 0 \]
for all \( u \in C^s_{E^i} (\mathcal{X}) \) or \( u \in C^{2s+k, s, \lambda, \frac{1}{2}}_{E^i} (\mathcal{X}_T) \cap D_{A^i} \).

We finish the section with two important lemmata specifying the action of the non-linear operator \( \mathcal{N}^i \) and its linearizations.

**Lemma 3.10:** Let \( s, k \in \mathbb{N}, 0 < \lambda < 1 \). The bilinear forms (3) induce continuous non-linear operators
\[ \mathcal{N}^i : C^{2s+k, s, \lambda, \frac{1}{2}}_{E^i} (\mathcal{X}_T) \cap D_{A^i} \rightarrow C^{2s+k-1, s, \frac{1}{2}}_{E^i} (\mathcal{X}_T) \cap D_{A^i}, \] (31)
\[ \pi^i \mathcal{N}^i : C^{2s+k, s, \lambda, \frac{1}{2}}_{E^i} (\mathcal{X}_T) \cap D_{A^i} \rightarrow C^{2s+k-1, s, \frac{1}{2}}_{E^i} (\mathcal{X}_T) \cap D_{A^i}. \] (32)
Proof: By the definition of $\pi^i$, we see that
\[(A^{i-1}v, \pi^i u)_i = (v, (A^{i-1})^* \pi^i u)_{i-1} = 0\]for all $v \in C^{1,\lambda,0,0}_{E_i^{-1}}(X_T)$, $u \in C^{1,\lambda,0,0}_{E_i}(X_T)$, i.e.
\[\pi^i N^i(u) = \pi^i M_{i,1}(A^i u, u) \quad (34)\]
for all $u \in C^{2s+k,\lambda,s,\lambda/2}_{E_i}(X_T)$. Hence Lemma 2.4 and formulas (6), (8) imply that operators $N^i$ and $\pi^i N^i$ map $C^{2s+k,\lambda,s,\lambda/2}_{E_i}(X_T) \cap D_{A^i}$ continuously into $C^{2s+k-1,\lambda,s,\lambda/2}_{E_i}(X_T)$.

Finally, as $A^i \circ A^{i-1} = 0$, then, according to Lemmata 2.4, 3.7 and (30),
\[A^i N^i(u) = A^i M_{i,1}(A^i u, u) \in C^{2s+k-1,\lambda,s,\lambda/2}_{E_i}(X_T) \cap D_{A^{i+1}},\]
\[A^i \pi^i N^i(u) = A^i \Phi_i A^i M_{i,1}(A^i u, u) \in C^{2s+k-1,\lambda,s,\lambda/2}_{E_i}(X_T),\]
if $u \in C^{2s+k,\lambda,s,\lambda/2}_{E_i}(X_T) \cap D_{A^i}$. Thus, the continuity of operators (31), (32) follows again from Lemma 2.4 and formulas (6), (8). ■

Lemma 3.11: Let $s, k \in \mathbb{N}$, $0 < \lambda < 1$. The Fréchet derivatives $(N^i)'_{|u(0)}$ and $(\pi^i N^i)'_{|u(0)}$ of continuous operators (31), (32) at a point $u(0) \in C^{2s+k,\lambda,s,\lambda/2}_{E_i}(X_T) \cap D_{A^i}$ equal to
\[(N^i)'_{|u(0)} v = M_{i,1}(A^i v, u(0)) + M_{i,1}(A^i u(0), v) + A^{i-1}(M_{i,2}(v, u(0)) + M_{i,2}(u(0), v)),\]
\[(\pi^i N^i)'_{|u(0)} v = \pi^i (M_{i,1}(A^i v, u(1)) + M_{i,1}(A^i u(1), v)),\]
respectively; these are the bounded linear operators
\[\begin{align*}
(N^i)'_{|u(0)} : C^{2s+k,\lambda,s,\lambda/2}_{E_i}(X_T) \cap D_{A^i} &\to C^{2s+k-1,\lambda,s,\lambda/2}_{E_i}(X_T) \cap D_{A^i}, \\
\pi^i (N^i)'_{|u(0)} : C^{2s+k,\lambda,s,\lambda/2}_{E_i}(X_T) \cap D_{A^i} &\to C^{2s+k-1,\lambda,s,\lambda/2}_{E_i}(X_T) \cap D_{A^i}.
\end{align*}\] (35)

Proof: It follows from (8) that
\[N^i(u) - N^i(u(0)) = M_{i,1}(A^i(u - u(0)), u(0)) + M_{i,1}(A^i u(0), u - u(0)) + A^{i-1}(M_{i,2}(u - u(0), u(0)) + M_{i,2}(u(0), u - u(0)) + M_{i,2}(u - u(0), u - u(0)))\]
if $u, u(0) \in C^{2s+k,\lambda,s,\lambda/2}_{E_i}(X_T) \cap D_{A^i}$. Then the statement follows from Lemmata 2.4, 3.7 and 3.10 because $M_{i,j}$ are bilinear forms,
\[A^i \left( N^i(u) - N^i(u(0)) \right) = A^i \left( M_{i,1}(A^i(u - u(0)), u(0)) + M_{i,1}(A^i u(0), u - u(0)) + M_{i,1}(A^i(u - u(0)), u - u(0)) \right)\]
and, according to formula (30) and Lemmata 2.4, 3.7,
\[
A^i\left(\mathcal{M}_{i,1}(A^i v, u^{(0)}) + \mathcal{M}_{i,1}(A^i u^{(0)}, v)\right) \in C^{2s+k-1,\lambda,s,\lambda/2}_{E_i}(\mathcal{X}_T) \cap D_{A^{i+1}}
\]
\[
A^i\pi^i\left(\mathcal{M}_{i,1}(A^i v, u^{(0)}) + \mathcal{M}_{i,1}(A^i u^{(0)}, v)\right)
= A^i\Phi^i A^i\left(\mathcal{M}_{i,1}(A^i v, u^{(0)}) + \mathcal{M}_{i,1}(A^i u^{(0)}, v)\right) \in C^{2s+k-1,\lambda,s,\lambda/2}_{E_i}(\mathcal{X}_T) \cap D_{A^{i+1}},
\]
if \(v, u, u^{(0)} \in C^{2s+k,\lambda,s,\lambda/2}_{E_i}(\mathcal{X}_T) \cap D_{A^i}\).

\section{4. Parabolic operators over the Hölder spaces}

As usual, we denote by \(\gamma_0 u\) the restriction of a continuous function \(u\) on the layer \(\mathcal{X}_T\) to the set \(\{t = t_0\}\) in \(\mathcal{X}_T\), where \(t_0 \in [0, T]\). The following lemma is obvious.

**Lemma 4.1:** Let \(s, k \in \mathbb{Z}_+\) and \(\lambda \in [0, 1)\). The restriction \(\gamma_0\) induces a bounded linear operator \(\gamma_0 : C^{2s+k,\lambda,s,\lambda/2}_{E_i}(\mathcal{X}_T) \to C^{2s+k,\lambda}_{E_i}(\mathcal{X}).\)

Consider the Cauchy problem: given a section \(u_0\) of the bundle \(E^i\) over \(\mathcal{X}\), find a section \(u\) of the induced bundle \(E^i(t)\) over \(\mathcal{X}_T\), such that
\[
L^i_\mu u(x, t) = 0 \quad \text{for} \quad (x, t) \in \mathcal{X} \times (0, T),
\]
\[
(\gamma_0 u)(x) = u_0(x) \quad \text{for} \quad x \in \mathcal{X},
\]
where \(L^i_\mu = \partial_t + \mu \Delta^i\) is an evolution operator on the semiaxis \(t > 0\) with \(\mu > 0\).

If \(\mu > 0\) then it is easily seen that the endomorphism of \(E^i_x\) given by the principal symbol \(-\mu (\sigma^2(\Delta^i)(x, \xi))\) of the operator \(L^i_\mu\) at any nonzero vector \(\xi\) of the cotangent bundle \(T_x^*\mathcal{X}\) has only real eigenvalues strictly less than zero. Therefore, the operator \(L^i_\mu\) is actually one of the well-known types of partial differential operators, called parabolic, which enjoy a behaviour essentially like that of the classical equation of the heat conduction. For instance, one has as a smoothing operator the fundamental solution \(t \mapsto \psi^i_\mu(x, y, t) \in C^\infty(\mathcal{X} \times \mathcal{X}, E^i \otimes E^{i,\ast})\) which generates the solution of the Cauchy problem (36) for any \(u_0 \in C^\infty_{E_i}(\mathcal{X})\),
\[
(\psi^i_\mu(x, t)) = \int_{\mathcal{X}} (u_0, \ast^{-1} \psi^i_\mu(x, \cdot, t))_y dy.
\]

The fundamental solution \(\psi^i_\mu\) is here unique by virtue of the compactness of \(\mathcal{X}\).

We recall briefly the Hilbert-Levi procedure for constructing the fundamental solutions for parabolic operators, and state certain basic estimates for them, only to the extent which we shall need later. For the details we refer the reader to [12,21] and elsewhere.

We can cover \(\mathcal{X}\) by a finite number of coordinate patches \(U\) such that over each \(U\) the bundle \(E^i\) is trivial, i.e. the restriction \(E^i|_U\) is isomorphic to the trivial bundle \(U \times \mathbb{C}^{k_i}\), where \(k_i\) is the fibre dimension of \(E^i\). Let \(x = (x^1, \ldots, x^n)\) be local coordinates in \(U\). Then any section of \(E^i\) over \(U\) can be regarded as a \(k_i\)-column of complex-valued functions
of coordinates $x$. Under this identification, for any $u \in C^\infty_E(U)$, the $k_i$-column $\Delta_i^j u$ of functions in $U$ is represented by

$$\Delta_i^j u = \sum_{|\alpha| \leq 2} \Delta^j_{\alpha}(x) \partial^{\alpha} u,$$

where $\Delta^j_{\alpha}(x)$ are $(k_i \times k_i)$-matrices of $C^\infty$ functions of $x$, by $\partial^{\alpha}$ is meant the derivative $(\partial/\partial x^1)^{\alpha_1} \ldots (\partial/\partial x^n)^{\alpha_n}$. So, the principal symbol of $\Delta^j_i$ is given by

$$\sigma^2(\Delta^j_i)(x, \xi) = -\sum_{|\alpha| = 2} \Delta^j_{\alpha}(x) \xi^{\alpha}$$

for $(x, \xi) \in U \times \mathbb{R}^n$, where $\xi^{\alpha} = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$. As already mentioned, the map $\sigma^2(\Delta^j_i)(x, \xi)$ has only strictly negative eigenvalues for each nonzero vector $\xi$ and hence the matrix

$$P^j_U(x, y, t) = \frac{1}{(2\pi)^n} \int \exp \left( t\mu \sigma^2(\Delta^j_i)(x, \xi) + \sqrt{-1} (x - y, \xi) I \right) \, d\xi$$

is well defined for all $x, y \in U$ and $t > 0$; here $I$ stands for the unity matrix of type $k_i \times k_i$.

It will be seen below that $P^j_U$ describes the principal part of the singularity for the fundamental solution of the parabolic operator $L^j_\mu$ in $U$. For that reason $P^j_U$ is said to be a local parametrix for this operator. Letting now

$$\sum U \phi^2_U \equiv 1$$

be a quadratic partition of unity on $\mathcal{X}$ subordinate to a covering $\{U\}$, and considering the expression $\phi_U(x) P^j_U(x, y, t) \phi_U(y)$ as a section of $E^j \otimes (E^j)^*$ with support in $U \times U$, we define a global parametrix for $L^j_\mu$ by

$$P^j(x, y, t) = \sum_U \phi_U(x) P^j_U(x, y, t) \phi_U(y).$$

The fundamental solution $\psi^j_\mu(x, y, t)$ of $L^j_\mu$ will then be given as the unique solution of the integral equation of Volterra type

$$\psi^j_\mu(x, y, t) = P^j(x, y, t) - \int_0^t d\tau \int_{\mathcal{X}} (L^j_\mu \psi^j_\mu(\cdot, y, \tau') \ast \psi^j_\mu(x, \cdot, t - \tau'))dz,$$

the operator $L^j_\mu$ acting on $z$. Since the singularities of $P^j$ and $L^j_\mu P^j$ are relatively weak, we can solve the integral equation by the standard method of successive approximation. The kernel $\psi^j_\mu(x, y, t)$ obtained in this way is easily verified to be of class $C^\infty$, when $t > 0$, and,

$$u(x, t) = \int_{\mathcal{X}} (u_0, \ast \psi^j_\mu(x, \cdot, t))_y \, dy$$

satisfies $L^j_\mu u = 0$ for $t > 0$ and $u(x, 0) = u_0(x)$ for all $x \in \mathcal{X}$. 
As for the estimates, note that if we define, relatively to a Riemannian metric \( ds \) on \( X \), a distance function

\[
    d(x, y) = \inf_{\hat{xy}} \int_{\hat{xy}} ds
\]

for \( x, y \in X \), where \( \hat{xy} \) is a path from \( x \) to \( y \), then

\[
    |\partial^\alpha_x \partial^\beta_y \psi_\mu(x, y, t)| \leq c \frac{1}{t^{(n+|\alpha|+|\beta|)/2}} \exp \left( -c' \frac{(d(x, y))^2}{t} \right)
\]

locally on \( X' \) and for each entry of the matrix. Also, as can be inferred and in fact easily proved from (37) and (38), we get

\[
    |\psi_\mu^i(x, y, t) - P^i_U(x, y, t)| \leq c_1 \frac{1}{t^{(n-1)/2}} \exp \left( -c' \frac{(d(x, y))^2}{t} \right)
\]

uniformly on each compact subset of \( U \times U \), showing that locally \( p_U \) yields a first approximation to the fundamental solution \( \psi_\mu^i(x, y, t) \).

**Lemma 4.2:** Suppose that \( \mu > 0 \) and \( u \in C^\infty_E(X_T) \). Then, for all \((x, t) \in X_T\),

\[
    u(x, t) = (\psi_\mu^{(i,0)}u(\cdot, 0))(x, t) + \int_0^t dt' \int_X (L^i_\mu u(\cdot, t'), *^{-1} \psi_\mu^i(x, \cdot, t - t')) y \, dy.
\]

**Proof:** As is well known, the Cauchy problem

\[
    \begin{align*}
    L^i_\mu u(x, t) &= f(x, t) \quad \text{for } (x, t) \in X \times (0, T), \\
    (\gamma_0 u)(x) &= u_0(x) \quad \text{for } x \in X,
    \end{align*}
\]

has a unique solution in \( C^\infty_E(X_T) \) for all smooth data \( f \) in \( X_T \) and \( u_0 \) on \( X \). By the above, the first term on the right-hand side of (39) is a solution of the Cauchy problem with \( f = 0 \) and \( u_0 = u(\cdot, 0) \). Hence, we shall have established the lemma if we prove that the second term is a solution of the Cauchy problem with \( f = L^i_\mu u \) and \( u_0 = 0 \). To this end, we rewrite the second summand on the right-hand side of (39) in the form

\[
    \int_0^t \psi_\mu^{(i,0)}(L^i_\mu u(\cdot, t'))(x, t - t') \, dt'
\]

for \((x, t) \in X_T\). Obviously, the initial value at \( t = 0 \) of this integral vanishes, and so it remains to calculate its image by \( L^i_\mu \). Thus, we get

\[
    L^i_\mu \int_0^t \psi_\mu^{(i,0)}(L^i_\mu u(\cdot, t'))(x, t - t') \, dt' = \psi_\mu^i(L^i_\mu u(\cdot, t))(x, 0) + \int_0^t L^i_\mu \psi_\mu^i(L^i_\mu u(\cdot, t'))(x, t - t') \, dt' = L^i_\mu u(x, t),
\]

as desired. ■
For \( f \in C^{\infty}_{E}(\mathcal{X}_T) \), \( u_0 \in C^{\infty}_{E}(\mathcal{X}) \) we set

\[
(\Psi^{(i,v)}_{\mu} f)(x, t) = \int_0^t \Psi^{(i,in)}_{\mu} (f(\cdot, t')) (x, t - t') \, dt',
\]

\[
(\Psi_{\mu}^{(i)} (f, u_0))(x, t) = (\Psi^{(i,v)}_{\mu} f)(x, t) + (\Psi^{(i,in)}_{\mu} u_0)(x, t).
\]

We can extend the action of the operator \( \Psi_{\mu}^{i} \) to the scale \( C^{2s+k,\lambda,s,\lambda/2}_{E}(\mathcal{X}_T) \).

**Lemma 4.3:** Problem (40) has at most one solution in \( C^{2\lambda,1,\lambda/2}_{E}(\mathcal{X}_T) \).

**Proof:** Cf. Theorem 16 in [22, Ch. 1, §9].

The solution of the Cauchy problem in Hölder spaces is recovered from the data by means of the Green formula.

**Lemma 4.4:** Assume that \( \mu > 0 \). Then, for each function \( u \in C^{2\lambda,1,\lambda/2}_{E}(\mathcal{X}_T) \), it follows that \( u = \Psi_{\mu}^{i} (L^{1}_{\mu} u, \gamma_{0} u) \).

**Proof:** See ibid.

**Lemma 4.5:** Let \( \mu > 0 \), \( s \) be a positive integer, \( k \in \mathbb{Z}_+ \) and \( 0 < \lambda < 1 \). The parabolic potentials \( \Psi^{(i,v)}_{\mu} \) and \( \Psi^{(i,in)}_{\mu} \) induce bounded linear operators

\[
\Psi^{(i,v)}_{\mu} : C^{2(s-1)+k,\lambda,s-1,\lambda/2}_{E}(\mathcal{X}_T) \to C^{2s+k,\lambda,s,\lambda/2}_{E}(\mathcal{X}_T),
\]

\[
\Psi^{(i,in)}_{\mu} : C^{2s+k,\lambda}_{E}(\mathcal{X}) \to C^{2s+k,\lambda,s,\lambda/2}_{E}(\mathcal{X}_T) \cap S^{1}_{L^{1}_{\mu}}.
\]

**Proof:** See ibid.

The next result describes the crucial property of the volume parabolic potential \( \Psi_{\mu}^{i} \) which we need in the sequel.

**Theorem 4.6:** Let \( \mu > 0 \), \( s \in \mathbb{N} \), \( k \in \mathbb{Z}_+ \) and \( 0 < \lambda < 1 \). For any data \( f \in C^{2(s-1)+k,\lambda,s-1,\lambda/2}_{E}(\mathcal{X}_T) \) and \( u_0 \in C^{2s+k,\lambda}_{E}(\mathcal{X}) \) the potential \( \Psi_{\mu}^{i} (f, u_0) \) is the unique solution to problem (40) in the space \( C^{2s+k,\lambda,s,\lambda/2}_{E}(\mathcal{X}_T) \).

**Proof:** See ibid.
5. The linearised Navier-Stokes type equations

Now we begin to study the operators related to a linearization of the Navier-Stokes equations. For this purpose, consider the linear initial problem:

\[ \partial_t u + \mu \Delta u + V_0 u + A^{-1} p = f \quad \text{in } \mathcal{X} \times (0, T), \]
\[ (A^{-1})^* u = 0, \quad (A^{-2})^* p = 0 \quad \text{in } \mathcal{X} \times [0, T], \]
\[ u(x, 0) = u_0 \quad \text{in } \mathcal{X}, \]
where (cf. Lemma 3.11)

\[ V_0^j u = \mathcal{M}_{i,j}(A^j u^{(0)}, u) + \mathcal{M}_{i,j}(A^j u, u^{(0)}) + (A^{i-1})^* \left( \mathcal{M}_{i,j}(u, u^{(0)}) + \mathcal{M}_{i,j}(u^{(0)}, u) \right) \]

is the linear first order term in \( \mathcal{X}_T \) with a fixed sections \( u^{(0)} \) of the bundle \( E'(t) \).

**Theorem 5.1:** Suppose that (6) holds. If \( u^{(0)} \in C_{E_i}^{0,0,0} (\mathcal{X}_T) \cap D_{A_i} \) then for any pair \( (u, p) \) satisfying (41) with \( f = 0 \) and \( u_0 = 0 \),

\[ u \in C_{E_i}^{2,0,0} (\mathcal{X}_T) \cap S_{(A^{-1})^*}, \quad p \in C_{E_i}^{0,0,0} (\mathcal{X}_T) \cap D_{A^{-1}}, \]

the sections \( u \) and \( A^{-1} p \) are identically zero.

**Proof:** One may follow [1] or the proofs of Theorem 3.4 for \( n = 3 \) of [23], Theorem 6.1 below, proving the uniqueness result with the use of integration by parts. \( \square \)

**Lemma 5.2:** Let \( s, k \in \mathbb{N} \) and \( 0 < \lambda < 1 \). If \( u^{(0)} \in C_{E_i}^{2s+k\lambda,s\lambda/2} (\mathcal{X}_T) \cap D_{A_i} \) then the Leray-Helmholtz projection \( \pi^i \), the Fréchet derivative \( (\mathcal{N}^i)'|_{u^{(0)}} \), and the fundamental solution \( \Psi^i_{\mu} \)

induce linear bounded operators

\[ \Psi^{(iv)}_{\mu} \pi^i : C_{E_i}^{2(s-1)+k\lambda,s-1,\lambda/2} (\mathcal{X}_T) \cap D_{A_i} \rightarrow C_{E_i}^{2s+k\lambda,s\lambda/2} (\mathcal{X}_T) \cap D_{A_i} \cap S_{(A^{-1})^*}, \]
\[ \Psi^{(in)}_{\mu} : C_{E_i}^{2s+k+1,\lambda} (\mathcal{X}) \cap S_{(A^{-1})^*} \rightarrow C_{E_i}^{2s+k\lambda,s\lambda/2} (\mathcal{X}_T) \cap D_{A_i} \cap S_{(A^{-1})^*}, \]
\[ \Psi^{(iv)}_{\mu} (\mathcal{N}^i)'|_{u^{(0)}} : C_{E_i}^{2s+k\lambda,s\lambda/2} (\mathcal{X}_T) \cap D_{A_i} \rightarrow C_{E_i}^{2(s-1)+k-1,\lambda,s+1,\lambda/2} (\mathcal{X}_T) \cap D_{A_i} \cap S_{(A^{-1})^*}. \]

**Proof:** According to Lemmata 3.9, 3.11, 4.5, and the Embedding Theorem 2.2, for each \( u^{(0)} \in C_{E_i}^{2s+k\lambda,s\lambda/2} (\mathcal{X}_T) \cap D_{A_i} \) we have bounded linear operators:

\[ \Psi^{(iv)}_{\mu} \pi^i : C_{E_i}^{2(s-1)+k\lambda,s-1,\lambda/2} (\mathcal{X}_T) \cap D_{A_i} \rightarrow C_{E_i}^{2s+k\lambda,s\lambda/2} (\mathcal{X}_T) \cap D_{A_i}, \]
\[ \Psi^{(in)}_{\mu} : C_{E_i}^{2s+k+1,\lambda} (\mathcal{X}) \rightarrow C_{E_i}^{2s+k\lambda,s\lambda/2} (\mathcal{X}_T) \cap D_{A_i}, \]
\[ \Psi^{(iv)}_{\mu} (\mathcal{N}^i)'|_{u^{(0)}} : C_{E_i}^{2s+k\lambda,s\lambda/2} (\mathcal{X}_T) \cap D_{A_i} \rightarrow C_{E_i}^{2(s-1)+k-1,\lambda,s+1,\lambda/2} (\mathcal{X}_T) \cap D_{A_i}. \]

On the other hand,

\[ (A^{-1})^i L^i_{\mu} = L^i_{\mu} (A^{-1})^*, \quad \Pi^i L^i_{\mu} = L^i_{\mu} \Pi^i, \quad \Phi^i L_{i+1}^i = L^i_{\mu} \Phi^i, \]
\[(A^{i-1})^*\gamma_0 = \gamma_0 (A^{i-1})^*, \quad \Pi^i\gamma_0 = \gamma_0 \Pi^i, \quad \Phi_i \gamma_0 = \gamma_0 \Phi_i\]

by the very construction and then
\[\pi^iL^i_\mu = L^i_\mu \pi^i, \quad \pi^i\gamma_0 = \gamma_0 \pi^i.\]

Hence \(\pi^i\Psi^i(\mu) = \Psi^i(\mu) \pi^i\) because \(\Psi^i(\mu)\) represents the inverse operator for \((L^i_\mu, \gamma_0)\) on the scale \(C_{E_1}^{2s+k,\lambda,s,\lambda/2}(\mathcal{X}_T)\cap D_{A^i}.\) In particular,
\[\pi^i\Psi^i(i,\nu) = \Psi^i(i,\nu) \pi^i, \quad \pi^i\Psi^i(i,\nu) = \Psi^i(i,\nu) \pi^i,\]

that proves the continuity of the operators (43), (44) and (45).

According to Uniqueness Theorem 5.1, the ‘pressure’ \(p\) is defined by (41) up to the finite-dimensional space \(\mathcal{H}^{i-1}.\) In order to achieve the uniqueness we will look for \(p\) being \(L^2_{E_1}([0, T])\)-orthogonal to \(\mathcal{H}^{i-1},\) i.e.
\[\Pi^{i-1}p(\cdot, t) = 0 \quad \text{for each } t \in [0, T].\]

Let \(C_{E_1}^{2s+k,\lambda,s,\lambda/2}(\mathcal{X}_T) \subset C^{s,\lambda/2}([0, T], \mathcal{H}^{i-1})\) stand for all sections from \(C_{E_1}^{2s+k,\lambda,s,\lambda/2}(\mathcal{X}_T),\) satisfying (46). Then we introduce the Banach spaces:
\[
\begin{align*}
\mathcal{B}_{i,1}^{k,s,\lambda} & = C_{E_1}^{2s+k,\lambda,s,\lambda/2}(\mathcal{X}_T) \cap D_{A^i} \cap S_{(A^i)^*} \\
& \times C_{E_1}^{(s-1)+k,\lambda,s,\lambda/2}([0, T], \mathcal{H}^{i-1}) \cap S_{(A^{i-1})^*}, \\
\end{align*}
\]
\[
\begin{align*}
\mathcal{B}_{i,2}^{k,s,\lambda} & = C_{E_1}^{(s-1)+k,\lambda,s,\lambda/2}(\mathcal{X}_T) \cap D_{A^i} \cap C_{E_1}^{2s+k+1,\lambda}(\mathcal{X}) \cap S_{(A^{i-1})^*}. \\
\end{align*}
\]

The following lemma is just an adaptation to the particular function spaces of the standard reduction of the linearised Navier-Stokes type equation to a pseudo-differential equation that does not involve the ‘pressure’ \(p.\)

**Lemma 5.3:** Assume that \(s, k \in \mathbb{N}\) and \(u^{(0)} \in C_{E_1}^{2s+k,\lambda,s,\lambda/2}(\mathcal{X}_T) \cap D_{A^i}.\) Let moreover \((f, u_0)\) be an arbitrary pair of \(\mathcal{B}_{i,2}^{k,s,\lambda}.\) Then there is a solution \((u, p)\) of class \(\mathcal{B}_{i,1}^{k,s,\lambda}\) to problem (41) if and only if there is a solution \(v \in C_{E_1}^{2s+k,\lambda,s,\lambda/2}(\mathcal{X}_T) \cap D_{A^i} \cap S_{(A^{i-1})^*}\) to the pseudo-differential equation
\[v + \Psi^i(i,\nu) \pi^i (N^i)^{i}_{|\nu(0)}v = F\] (49)

with data \(F = \Psi^i(i,f, u_0) \in C_{E_1}^{2s+k,\lambda,s,\lambda/2}(\mathcal{X}_T) \cap D_{A^i} \cap S_{(A^{i-1})^*}.\) Besides,
\[u = v, \quad p = \Phi_{i-1}(I - \pi^i)(f - (N^i)^{i}_{|\nu(0)}v).\] (50)

**Proof:** First of all, we note that \(\pi^i V^i_0 = \pi^i (N^i)^{i}_{|\nu(0)}\) in this case, see Lemma 3.11.

If \((u, p)\) is a solution to problem (41) from class (47) then, applying the bounded linear operator \(\Psi^i(i,\nu) \pi^i\) to (41) we see that the section \(u = v\) is a solution to (49) with data \(F = \Psi^i(i,f, u_0) \in C_{E_1}^{2s+k,\lambda,s,\lambda/2}(\mathcal{X}_T) \cap D_{A^i} \cap S_{(A^{i-1})^*}.\)
Let $v \in C^{2s+k,\lambda,s,L/2}_{E}(X_T) \cap D_{A^i} \cap S_{(A^{i-1})^*}$ be a solution to (49) with data $F = \Psi_{\mu}(\pi^i f, u_0) \in C^{2s+k,\lambda,s,L/2}_{E}(X_T) \cap D_{A^i} \cap S_{(A^{i-1})^*}$. Then by Lemma 4.4 we conclude that
\begin{equation}
L_{\mu}^i v + \pi^i (\mathcal{N}^i)'_{|t(0)} v = \pi^i f \quad \text{in } X \times (0, T),
A^{i-1})^* v = 0 \quad \text{in } X \times [0, T],
\gamma_0 v = u_0 \quad \text{on } X.
\end{equation}

Set $(u, p)$ as in (50). As $(A^{i-1})^*(A^i)^* \equiv 0$, we have $\pi^i (A^i)^* = (A^i)^*$ and then
\[
(I - \pi^i) \left( f - (\mathcal{N}^i)'_{|t(0)} v \right), (A^i)^* w \right)_i = 0
\]
for all $w \in C^{\infty}_{E+1}(X)$, i.e.
\[
(I - \pi^i) \left( f - (\mathcal{N}^i)'_{|t(0)} v \right) \in C^{2s-1+k,\lambda,s-1,L/2}_{E}(X_T) \cap D_{A^i} \cap S_{A^i}.
\]
Moreover, since $\mathcal{H}^i \subset C^{2s-1+k,\lambda,s-1,L/2}_{E}(X_T) \cap S_{(A^{i-1})^*}$, formula (29) implies that
\[
\Pi^i (I - \pi^i) \left( f - (\mathcal{N}^i)'_{|t(0)} v \right) = 0.
\]

Then, according to Lemma 3.7 and Corollary 3.5, the section $p$ belongs to the space $C^{2s-1+k,\lambda,s-1,L/2}_{E}(X_T) \cap D_{A^{i-1}} \cap \mathcal{H}^{i-1}$ and satisfies
\begin{equation}
A^{i-1} p = (I - \pi^i) \left( f - (\mathcal{N}^i)'_{|t(0)} v \right), \quad (A^{i-2})^* p = 0.
\end{equation}

Thus, adding (52) to (51), we conclude that the pair $(u, p)$ is a solution to (41) from space (47) that was to be proved.

Now we note that the scale of Hölder spaces $C^{2s+k,\lambda,s,L/2}_{E}(X_T)$ is coherent with the elliptic and parabolic linear theories but it does not take into account the structure of the non-linearity $\mathcal{N}^i$. We are going to slightly modify the scale $C^{2s+k,\lambda,s,L/2}_{E}(X_T)$ by introducing an additional Hölder exponent $\lambda'$ in order to gain some 'smoothness' in $t$. Namely, for $s, k \in \mathbb{Z}_{\geq 0}$ and $0 < \lambda < \lambda' < 1$, we introduce
\begin{equation}
C^{k,s(s,\lambda,\lambda')}_{E}(X_T) := C^{2s+k+1,s,\lambda,\lambda'/2}_{E}(X_T) \cap C^{2s+k,s,\lambda',\lambda'/2}_{E}(X_T)
\end{equation}

When given the norm
\[
\|u\|_{C^{k,s(s,\lambda,\lambda')}_{E}(X_T)} := \|u\|_{C^{2s+k+1,s,\lambda,\lambda'/2}_{E}(X_T)} + \|u\|_{C^{2s+k,s,\lambda',\lambda'/2}_{E}(X_T)}.
\]
this is obviously a Banach space. To a certain extent these spaces are similar to those with two-norm convergence which are of the key importance for ill-posed problems.

**Corollary 5.4:** Let $s, k \in \mathbb{N}$, $0 < \lambda < \lambda' < 1$. The embedding is compact:
\[
C^{k,s(s,\lambda,\lambda')}_{E}(X_T) \hookrightarrow C^{k+1,s(s-1,\lambda,\lambda')}_{E}(X_T).
\]
A. PARFENOV AND A. SHLAPUNOV

Proof: By Theorem 2.2, we have (1) the space \( C_E^{2s+k+1,\lambda,s,\lambda'/2}(\mathcal{X}_T) \) is embedded compactly into the space \( C_E^{2(s-1)+k+1,\lambda',s-1,\lambda'/2}(\mathcal{X}_T) \) since \( s + \lambda > s - 1 + \lambda' \); (2) the space \( C_E^{2s+k,\lambda,s,\lambda'/2}(\mathcal{X}_T) \) is embedded compactly into \( C_E^{2s+k,\lambda,s,\lambda'/2}(\mathcal{X}_T) \) for \( 0 < \lambda < \lambda' \); (3) the space \( C_E^{2s+k,\lambda,s,\lambda'/2}(\mathcal{X}_T) \) is embedded continuously into the space \( C_E^{2(s-1)+k+2,\lambda,s-1,\lambda'/2}(\mathcal{X}_T) \).

Hence it follows that if \( S \) is a bounded set in the space \( \mathcal{E}_E^{k,s,\lambda}(\mathcal{X}_T) \) given by (53) then any sequence from \( S \) has a subsequence converging in the space

\[
\mathcal{E}_E^{k+1,s(s-1,\lambda,\lambda')}(\mathcal{X}_T) := C_E^{2(s-1)+k+2,\lambda,s-1,\lambda'/2}(\mathcal{X}_T) \cap C_E^{2(s-1)+k+1,\lambda,s,\lambda'/2}(\mathcal{X}_T),
\]
as desired.

\[\square\]

Remark 5.5: As the space \( \mathcal{E}_E^{k,s(\lambda,\lambda')}(\mathcal{X}_T) \) is defined as an intersection of the spaces from the scale \( C_E^{2s+k,\lambda,s,\lambda'/2}(\mathcal{X}_T) \), we conclude that all the results on the continuity for elliptic and parabolic potentials are still valid for it, too.

Thus, we denote by \( \mathcal{E}_E^{k,s(\lambda,\lambda')}(\mathcal{X}_T) \cap \mathcal{D}_{A^i} \) the Banach space

\[
\mathcal{E}_E^{k,s(\lambda,\lambda')}(\mathcal{X}_T) := \left( C_E^{2s+k+1,\lambda,s,\lambda'/2}(\mathcal{X}_T) \cap \mathcal{D}_{A^i} \right) \cap \left( C_E^{2s+k+2,\lambda,s,\lambda'/2}(\mathcal{X}_T) \cap \mathcal{D}_{A^i} \right)
\]

with the norm

\[
\|u\|_{\mathcal{E}_E^{k,s(\lambda,\lambda')}(\mathcal{X}_T) \cap \mathcal{D}_{A^i}} := \|u\|_{C_E^{2s+k+1,\lambda,s,\lambda'/2}(\mathcal{X}_T) \cap \mathcal{D}_{A^i}} + \|u\|_{C_E^{2s+k+2,\lambda,s,\lambda'/2}(\mathcal{X}_T) \cap \mathcal{D}_{A^i}}.
\]

Lemma 5.6: Let \( s,k \in \mathbb{N} \) and \( 0 < \lambda < \lambda' < 1 \). If \( u(0) \in \mathcal{E}_E^{k,s(\lambda,\lambda')}(\mathcal{X}_T) \cap \mathcal{D}_{A^i} \) then the Leray-Helmholtz projection \( \pi^i \), the Fréchet derivative \( (\mathcal{N}^i)'_{u(0)} \) and the fundamental solution \( \Psi_\mu^{(i,v)} \) induce linear bounded operators

\[
\psi_{\mu,uv}^i : \mathcal{E}_E^{k,s(\lambda,\lambda')}(\mathcal{X}_T) \cap \mathcal{D}_{A^i} \to \mathcal{E}_E^{k,s(\lambda,\lambda')}(\mathcal{X}_T) \cap \mathcal{D}_{A^i} \cap \mathcal{S}(A^{i-1}), \tag{54}
\]

\[
\psi_{\mu,un}^i : C_E^{2s+k+2,\lambda}(\mathcal{X}) \cap \mathcal{S}(A^{i-1}) \to \mathcal{E}_E^{k,s(\lambda,\lambda')}(\mathcal{X}_T) \cap \mathcal{D}_{A^i} \cap \mathcal{S}(A^{i-1}), \tag{55}
\]

and linear compact operators

\[
(\mathcal{N}^i)'_{u(0)} : \mathcal{E}_E^{k,s(\lambda,\lambda')}(\mathcal{X}_T) \cap \mathcal{D}_{A^i} \to \mathcal{E}_E^{k,s(\lambda,\lambda')}(\mathcal{X}_T) \cap \mathcal{D}_{A^i}. \tag{56}
\]

\[
\psi_{\mu,un}^i(\mathcal{N}^i)'_{u(0)} : \mathcal{E}_E^{k,s(\lambda,\lambda')}(\mathcal{X}_T) \cap \mathcal{D}_{A^i} \to \mathcal{E}_E^{k,s(\lambda,\lambda')}(\mathcal{X}_T) \cap \mathcal{D}_{A^i} \cap \mathcal{S}(A^{i-1}), \tag{57}
\]

Proof: As operators (35), (43) and (45) are bounded and linear, then operators (54), (56), (57) are bounded, too (see Remark 5.5). The compactness of operators (56) and (57) follows from the continuity of operators (35), (45) and the compact embedding described in Corollary 5.4.

Next, according to Lemma 4.5, the operator \( \psi_{\mu,un}^i(\mathcal{N}^i)'_{u(0)} \) maps the space \( C_E^{2s+k+2,\lambda}(\mathcal{X}) \) continuously into \( C_E^{2s+k+2,\lambda,s,\lambda'/2}(\mathcal{X}_T) \) and Theorem 2.2 provides the continuous embedding of the last space into \( C_E^{2s+k+1,\lambda,s,\lambda'/2}(\mathcal{X}_T) \cap \mathcal{D}_{A^i} \). On the other hand, by Theorem 2.1, the space

\[
\]
$C^{2s+k+2,\lambda}(\mathcal{X})$ is continuously embedded into $C^{2s+k+1,\lambda'}(\mathcal{X})$ and Lemma 4.5 implies that the operator $\Psi_{\mu}^{(i,iv)}$ maps the space $C^{2s+k+1,\lambda'}(\mathcal{X})$ continuously into $C^{2s+k+1,\lambda',s,\lambda'/2}(\mathcal{X}_T)$. Finally, it follows from Theorem 2.2 that $C^{2s+k+1,\lambda',s,\lambda'/2}(\mathcal{X}_T)$ is continuously embedded into $C^{2s+k,\lambda',s,\lambda'/2}(\mathcal{X}) \cap \mathcal{D}_{A^i}$. Hence the operator (55) is bounded, too.

**Theorem 5.7:** Let $s, k \in \mathbb{N}$, $0 < \lambda < \lambda' < 1$, and $u^{(0)} \in C^{k,s(s,\lambda,\lambda')}_E(\mathcal{X}_T) \cap \mathcal{D}_{A^i}$. Then the operator is continuously invertible:

$$I + \Psi_{\mu}^{(i,iv)} \pi^i V_0^i : C^{k,s(s,\lambda,\lambda')}_E(\mathcal{X}_T) \cap \mathcal{D}_{A^i} \cap \mathcal{S}_{(A^{i-1})^*} \rightarrow C^{k,s(s,\lambda,\lambda')}_E(\mathcal{X}_T) \cap \mathcal{D}_{A^i} \cap \mathcal{S}_{(A^{i-1})^*}.$$  

(58)

**Proof:** First we observe by Lemma 5.6 that the operator $(I + \Psi_{\mu}^{(i,iv)} \pi^i V_0^i)$ is a continuous selfmapping of $C^{k,s(s,\lambda,\lambda')}_E(\mathcal{X}_T) \cap \mathcal{D}_{A^i} \cap \mathcal{S}_{(A^{i-1})^*}$. Our next goal is to show that this mapping is one-to-one.

**Lemma 5.8:** Let $s, k \in \mathbb{N}$, $u^{(0)} \in C^{2s+k,\lambda,s,\lambda'/2}_E(\mathcal{X}_T) \cap \mathcal{D}_{A^i}$. If $u \in C^{2s+k,\lambda,s,\lambda'/2}(\mathcal{X}_T) \cap \mathcal{D}_{A^i} \cap \mathcal{S}_{(A^{i-1})^*}$ satisfies $(I + \Psi_{\mu}^{(i,iv)} \pi^i V_0^i)(u) = 0$, then it is identically zero.

**Proof:** Indeed, using Lemma 5.3 we deduce that the pair $(u, p)$, with the entry $p$ given by (50), belongs to the space $B_{i1}^{k,s,\lambda}$. It is a solution to the linearised Navier-Stokes Equation (41) with the zero data $f$ and $u_0$. Finally, according to the Uniqueness Theorem 5.1, we see that $v \equiv 0$.

Let us finish the proof of Theorem 5.7. According to Lemma 5.6, the operator (58) is Fredholm and its index equals to zero. Then the statement of the corollary follows from Lemma 5.8 and Fredholm theorems.

Let us introduce the Banach spaces:

$$\mathfrak{B}_{i1}^{k,s,\lambda,\lambda'} = C^{k,s(s,\lambda,\lambda')}_E(\mathcal{X}_T) \cap \mathcal{D}_{A^i} \cap \mathcal{S}_{(A^{i-1})^*} \times C^{k,s(s-1,\lambda,\lambda')}_E(\mathcal{X}_T) \cap \mathcal{D}_{A^{i-1}} \cap \mathcal{S}_{(A^{i-2})^*} \oplus C^{s,\lambda'/2}(0, T, \mathcal{H}^{i-1})$$  

(59)

$$\mathfrak{B}_{i2}^{k,s,\lambda,\lambda'} = \left(C^{k,s(s-1,\lambda,\lambda')}_E(\mathcal{X}_T) \cap \mathcal{D}_{A^i}\right) \times \left(C^{2s+k+2,\lambda}_E(\mathcal{X}) \cap \mathcal{S}_{(A^{i-1})^*}\right).$$  

(60)

By the very definition, the space $\mathfrak{B}_{i,j}^{k,s,\lambda,\lambda'}$ is continuously embedded into $\mathfrak{B}_{ij}^{k,s,\lambda}$.

**Corollary 5.9:** Assume that $s, k \in \mathbb{N}$, $0 < \lambda < \lambda' < 1$ and $u^{(0)} \in C^{k,s(s,\lambda,\lambda')}_E(\mathcal{X}_T) \cap \mathcal{D}_{A^i}$. Then (41) induces bounded linear continuously invertible operator $\mathfrak{U}_{i_{lin}}$ between the Banach spaces $\mathfrak{B}_{i1}^{k,s,\lambda,\lambda'}$ and $\mathfrak{B}_{i2}^{k,s,\lambda,\lambda'}$. 
Theorem 6.1: Suppose that (6) holds. Then for each pair \((f, u_0) \in C_{E_{i+1}}^{0,0,0}(X_T) \times C_{E_{i+1}}^{0,0}(X) \cap S_{A_{i+1}}^{*}\) nonlinear Navier-Stokes type equations (4) have at most one solution in the space \(C_{E_{i+1}}^{0,0,0}(X_T) \times C_{E_{i+1}}^{0,0,0}(X) \cap S_{A_{i+1}}^{*}\) satisfying (46).

Proof: Again, one may follow the original paper [1] or the proof of Theorem 3.4 for \(n = 3\) in [23], showing the uniqueness result by integration by parts.
Indeed, let \((u', p')\) and \((u'', p'')\) be any two solutions to (4) from the declared function space. Then for the difference \((u, p) = (u' - u'', p' - p'')\) we get

\[
L^i \mu u + A^{i-1} p = N^i(u'') - N^i(u').
\]  

(64)

As \(u \in C_{E_1}^{2,0,1,0}(X_T) \cap S_{(A^{i-1})^*}, p \in C_{E_1}^{0,0,0}(X_T) \cap D_{A^{i-1}},\) the sections \(u, A^i u, \partial_i u, L^i \mu u\) and \(A^{i-1} p\) are square integrable over all of \(X\) for each fixed \(t \in [0, T]\). Furthermore, the integrals \(\langle N^i(u'), u \rangle_i\) and \(\langle N^i(u''), u \rangle_i\) converge, for both \(N^i(u')\) and \(N^i(u'')\) are of class \(C_{E_1}^{0,0,0,0}(X_T)\) (see Lemmata 2.4 and 2.3). Then

\[
\partial_t \|u(\cdot, t)\|^2_i = 2(\partial_i u, u)_i, 
\]  

(65)

and, since \((A^{i-1})^* u = 0\) we easily obtain,

\[
(L^i \mu u + A^{i-1} p, u)_i = \frac{1}{2} \partial_t \|u(\cdot, t)\|^2_i + \mu \|A^i u(\cdot, t)\|^2_{i+1}. 
\]  

(66)

Therefore (64), (65) and (66) imply

\[
\frac{1}{2} \partial_t \|u(\cdot, t)\|^2_i + \mu \|A^i u(\cdot, t)\|^2_{i+1} = \langle N^i(u''), u \rangle_i - \langle N^i(u'), u \rangle_i. 
\]

As \((A^{i-1})^* u = 0\), a trivial verification shows that

\[
(A^{i-1} \mathcal{M}_{i,2}(u', u'), u)_i = (A^{i-1} \mathcal{M}_{i,2}(u'', u''), u)_i = 0. 
\]

Then, as the form \(\mathcal{M}_{i,1}\) is bilinear, we obtain

\[
\langle N^i(u'), u \rangle_i - \langle N^i(u''), u \rangle_i = (\mathcal{M}_{i,1}(A^i u', u'), u)_i - (\mathcal{M}_{i,1}(A^i u'', u''), u)_i \\
= (\mathcal{M}_{i,1}(A^i u', u'), u)_i + (\mathcal{M}_{i,1}(A^i u'', u), u)_i, 
\]

and so, for all \(t \in [0, T]\),

\[
\partial_t \|u(\cdot, t)\|^2_i + 2\mu \|A^i u(\cdot, t)\|^2_{i+1} = 2(\mathcal{M}_{i,1}(A^i u', u'), u)_i + 2(\mathcal{M}_{i,1}(A^i u'', u), u)_i. 
\]

As \(u', u''\) and \(u\) are of class \(C_{E_1}^{2,0,1,0}(X_T)\), by property (6) we have

\[
2|\langle \mathcal{M}_{i,1}(A^i u', u'), u \rangle_i| \leq 2c(\mathcal{M}) \|A^i u\|_{i+1} \|u\|_i \|u'\|_{C_{E_1}^{0,0,0}(X_T)} \\
\leq 2\mu \|A^i u\|^2_{i+1} + \frac{c(\mathcal{M})^2}{2\mu} \|u\|^2_i \|u'\|^2_{C_{E_1}^{0,0,0}(X_T)}, \\
|\langle \mathcal{M}_{i,1}(A^i u'', u), u \rangle_i| \leq c(\mathcal{M}) \|A^i u''\|_{C_{E_1+1}^{0,0,0}(X_T)} \|u\|^2_i, 
\]
whence for all \( t \in [0, T] \) we have

\[
\partial_t \|u(\cdot, t)\|_i^2 \leq \left( \frac{c(M)^2}{2 \mu}\|u\|^2_{C \pi_{t0}^{0,0,0}(X_T)} + c(M)\|A^i u''\|^2_{C \pi_{t0}^{0,0,0}(X_T)} \right) \|u(\cdot, t)\|_i^2.
\]

Now we note that from the inequality \( x'(t) \leq z(t)x(t) \) for all \( t \) in some interval of the real axis it follows that

\[
\frac{d}{dt}(e^{-Z(t)}x(t)) \leq 0,
\]

where \( Z \) is a primitive function for \( z \). Therefore, since \( Z(t) = 2ct \) is a primitive for the function \( z(t) = 2c \), we conclude that, for all \( t \in (0, T) \),

\[
\frac{d}{dt}(e^{-2ct}\|u(\cdot, t)\|^2_i) \leq 0.
\]

Pick any \( t \in (0, T) \). Then

\[
\int_0^t \frac{d}{ds}(e^{-2cs}\|u(\cdot, s)\|^2_i) \, ds = e^{-2ct}\|u(\cdot, t)\|^2_i - \|u(\cdot, 0)\|^2_i
\]

because \( u(x, 0) = 0 \) for all \( x \in \mathcal{X} \). Thus, \( u \equiv 0 \) because

\[
\|u(\cdot, t)\|^2_i \leq 0 \quad \text{for all } t \in [0, T].
\]

It follows that \( A^{i-1}p(\cdot, t) \equiv 0 \) for each \( t \in [0, T] \). As \( p \) is \( L^2(E) \)-orthogonal to \( H^{i-1} \) we see that \( p \equiv 0 \), i.e. the solutions \( (u', p') \) and \( (u'', p'') \) coincide, as desired.

The nonlinear Navier-Stokes equations can be reduced to a non-linear pseudo-differential Fredholm type equation in much the same way as the corresponding linearised equations. We proceed with an explicit description.

**Lemma 6.2:** Let \( s, k \in \mathbb{N} \) and \( 0 < \lambda < \lambda' < 1 \). The Leray-Helmholtz projection \( \pi^i \) and the fundamental solution \( \Psi^i \) induce nonlinear continuous operators

\[
\psi_{\mu}^{(i,v)} \pi^i \mathcal{N}^i : C^{2s+k,\lambda,s,\lambda'/2}(X_T) \cap D_{A^i} \to C^{2(s+1)+k-s-1,\lambda,s+1,\lambda'/2}(X_T) \cap D_{A^i} \cap S_{(A^i)}
\]

and nonlinear continuous compact operators

\[
\psi_{\mu}^{(i,v)} \pi^i \mathcal{N}^i : C^{k,s(s-1,\lambda,\lambda')}(X_T) \cap D_{A^i} \to C^{k,s(s,\lambda,\lambda')}(X_T) \cap D_{A^i} \cap S_{(A^i)}
\]

**Proof:** The continuity of operators (67), (69) follows from Lemmata 3.10 and 4.5. Then the compactness of operators (68) and (69) follows from the continuity of operators (67), (31) and the compact embedding described in Corollary 5.4.

The following lemma is just an adaptation to the particular function spaces of the standard reduction of the Navier-Stokes types equation to a pseudo-differential equation that does not involve the ‘pressure’ \( p \).
Lemma 6.3: Suppose that \( s, k \in \mathbb{N} \) and \( 0 < \lambda < \lambda' < 1 \). Let moreover \((f, u_0)\) be an arbitrary pair of \( B_{1,2}^{k,s,\lambda} \), defined by (48). Then there is a solution \((u, p)\) of class \( B_{1,2}^{k,s,\lambda} \), defined by (47), to problem (4) if and only if there is a solution \( v \in C_{E_i}^{2s+k,\lambda,s,\lambda/2} (X_T) \cap D_{A_i} \cap S_{(A^{i-1})^*} \) to the pseudo-differential equation
\[
v + \Psi^{(i,v)}(v) \pi^i N^i v = F
\]
with data \( F = \Psi^i(f, u_0) \in C_{E_i}^{2s+k,\lambda,s,\lambda/2} (X_T) \cap D_{A_i} \cap S_{(A^{i-1})^*} \). Besides,
\[
u = v, \quad p = \Phi_{i-1}(I - \pi^i) \left( f - N^i v \right).
\]

Proof: It is similar to the proof of Lemma 5.3.

We are already in a position to state an open mapping theorem for (70).

Theorem 6.4: Let \( s, k \in \mathbb{N}, 0 < \lambda < \lambda' < 1 \). Then the mapping is continuous, Fredholm, injective and open:

\[
I + \Psi^{(i,v)}(v) \pi^i N^i : \mathfrak{C}^{k,s(\lambda,\lambda')}_{E_i} (X_T) \cap D_{A_i} \cap S_{(A^{i-1})^*} \to \mathfrak{C}^{k,s(\lambda,\lambda')}_{E_i} (X_T) \cap D_{A_i} \cap S_{(A^{i-1})^*}.
\]

Proof: First we note that Lemma 6.2 implies that the operator \( \Psi^{(i,v)}(v) \pi^i N^i \) maps the space \( \mathfrak{C}^{k,s(\lambda,\lambda')}_{E_i} (X_T) \cap D_{A_i} \cap S_{(A^{i-1})^*} \) continuously and compactly into itself. In particular, the mapping (72) is continuous. We now turn to the one-to-one property of the mapping (72).

Lemma 6.5: Let \( s, k \in \mathbb{N} \) and \( 0 < \lambda < 1 \). If \( F \in C_{E_i}^{2s+k,\lambda,\lambda,\lambda/2} (X_T) \cap D_{A_i} \cap S_{(A^{i-1})^*} \) then (70) has no more than one solution in \( C_{E_i}^{2s+k,\lambda,\lambda,\lambda/2} (X_T) \cap D_{A_i} \cap S_{(A^{i-1})^*} \).

Proof: Suppose that \( v', v'' \in C_{E_i}^{2s+k,\lambda,\lambda,\lambda/2} (X_T) \cap D_{A_i} \cap S_{(A^{i-1})^*} \) are two solutions to (70). Using Lemma 4.4 we conclude that both \( v' \) and \( v'' \) are solutions to
\[
L^i v + \pi^i N^i v = L^i F \quad \text{in} \quad X \times (0, T),
\]
\[
(A^{i-1})^* w = 0 \quad \text{in} \quad X \times [0, T],
\]
\[
\gamma_0 w(x) = F(x, 0) \quad \text{on} \quad X.
\]

Now Lemma 6.3 implies that the pair \((v', p')\) and \((v'', p'')\) are solutions to (4) with \( f = L^i F, u_0(x) = F(x, 0), \) \( x \in X \) where \( u' = v', u'' = v'' \) and
\[
p' = \Phi_{i-1}(I - \pi^i) \left( f - N^i v' \right), \quad p'' = \Phi_{i-1}(I - \pi^i) \left( f - N^i v'' \right).
\]

Thus, by the uniqueness of Theorem 6.1, we get \( v' = v'', p' = p'' \).

Lemma 6.5 implies immediately that the mapping in (72) is actually one-to-one. Now Theorem 5.7 shows that the Fréchet derivative \((I + \Psi^{(i,v)}(v) \pi^i N^i)'_{|u(0)}\) of the mapping \((I + \Psi^{(i,v)}(v) \pi^i N^i)\) at an arbitrary point \( u(0) \in \mathfrak{C}^{k,s(\lambda,\lambda')}_{E_i} (X_T) \cap D_{A_i} \cap S_{(A^{i-1})^*} \) is a continuously
invertible selfmapping of the space $C^{k,s}_{E_i}(X_T) \cap D_{A_i} \cap S_{(A_i^{-1})^*}$. In particular, the mapping (72) is a Fredholm one. Then the openness of (72) follows from the Implicit Mapping Theorem in Banach spaces, see for instance Theorem 5.2.3 of [25, p. 101].

When combined with Lemma 6.3, Theorem 6.4 implies that the Navier-Stokes equations induce an open mapping in the function spaces under consideration.

**Corollary 6.6:** Let $s, k \in \mathbb{N}, 0 < \lambda < \lambda' < 1$. Then (4) induces a continuous nonlinear open injective mapping $\mathfrak{A}^i$ from $\mathfrak{B}^{k,s,\lambda,\lambda'}_{l,1}$ to $\mathfrak{B}^{k,s,\lambda,\lambda'}_{l,2}$, see (59), (60).

**Proof:** As operators (61), (62), (63) (68) are continuous we conclude that the mapping $\mathfrak{A}^i$, induced by (4), maps $\mathfrak{B}^{k,s,\lambda,\lambda'}_{l,1}$ continuously into the space $\mathfrak{B}^{k,s,\lambda,\lambda'}_{l,2}$.

It is left to prove that for any pair $(u^{(0)}, p^{(0)})$ from $\mathfrak{B}^{k,s,\lambda,\lambda'}_{l,1}$, there is $\delta > 0$ with the property that for all data $(f, u_0)$ from $\mathfrak{B}^{k,s,\lambda,\lambda'}_{l,2}$, satisfying the estimate

$$
\| (f, u_0) - \mathfrak{A}^i(u^{(0)}, p^{(0)}) \|_{\mathfrak{B}^{k,s,\lambda,\lambda'}_{l,2}} < \delta,
$$

nonlinear equations (4) have a unique solution $(u, p)$ in $\mathfrak{B}^{k,s,\lambda,\lambda'}_{l,1}$.

By Lemma 5.6, the parabolic potential $\Psi^i$ induces the bounded linear operator

$$
C^{k,s}_{E_i}(X_T) \cap D_{A_i} \times C^{2s+k+2,\lambda}_{E_i}(X') \rightarrow C^{k,s}_{E_i}(X_T) \cap D_{A_i},
$$

(74)

We now apply Lemma 6.3 to see that $u^{(0)}$ is a solution to the operator equation

$$(I + \Psi^i_\mu \pi^i N^i)u^{(0)} = g_0^{(0)},$$

with the right-hand side $g_0^{(0)} := \Psi^i_\mu ((I^i_\mu + + \pi^i N^i)u^{(0)}, \gamma_0 u^{(0)})$, belonging to the space $C^{k,s}_{E_i}(X_T) \cap D_{A_i} \cap S_{(A_i^{-1})^*}$. Set $g_0 = \Psi^i_\mu (\pi^i f, u_0)$ which belongs to $C^{k,s}_{E_i}(X_T) \cap D_{A_i} \cap S_{(A_i^{-1})^*}$, by Lemmata 4.5 and 5.2.

An immediate calculation shows that

$$
\| g_0 - g_0^{(0)} \|_{C^{k,s}_{E_i}(X_T) \cap D_{A_i}} \leq \| \Psi^i_\mu \| \| (f, u_0) - \mathfrak{A}^i(u^{(0)}, p^{(0)}) \|_{\mathfrak{B}^{k,s,\lambda,\lambda'}_{l,2}}
$$

where $\| \Psi^i_\mu \|$ is the norm of bounded linear operator (74).

If $\delta > 0$ in the estimate (73) is small enough, then Theorem 6.4 shows that there is a unique solution $v \in C^{k,s}_{E_i}(X_T) \cap D_{A_i} \cap S_{(A_i^{-1})^*}$ to the operator equation $v + \Psi^i_\mu \pi^i N^i v = g_0$. The pair $u = v, p = \Phi_{i-1}(I - \pi^i)(f - N^i u)$ belonging to $\mathfrak{B}^{k,s,\lambda,\lambda'}_{l,1}$ satisfies nonlinear equations (4), which is due to Lemma 6.3. Finally, the uniqueness of the solution $(u, p)$, follows from Theorem 6.1.

As we noted in the introduction, the main motivative example of this paper is concerned with the Navier-Stokes type Equations associated with the de Rham complex $\{d^i, \Lambda^i\}$ over the manifold $X$, see, for instance, [10, § 1.2.6], where

$$
M_{1,1}(w, u) = \ast(\ast w \wedge u), \quad M_{1,2}(v, u) = \ast(u \wedge \ast v)/2, \quad (75)
$$

with $\ast$ being the $\ast$-Hodge operator acting on exterior differential forms. In this way, taking the 3-dimensional torus $\mathbb{T}^3$ as $X$ and identifying 1-differential forms on $\mathbb{T}^3$ with periodic
vector fields over $\mathbb{R}^3$, Corollary 6.6 is an Open Mapping Theorem for the Navier-Stokes Equations in the so-called periodic setting, see [14, Ch. 1, §2, §3] because in this situation we have

$$A^0 = d^0 = \nabla, \quad A^1 = d^1 = \text{rot}, \quad (A^0)^* = (d^0)^* = -\text{div}, \quad (A^{-1})^* = (d^{-1})^* = 0$$

and $\mathcal{N}^1(v)$, defined by (3) and (75), gives precisely (5), see [8].

It is worth to note that no apriori estimates for solutions to the non-linear Navier-Stokes type equations were used to achieve the stability property – it is sufficient to use the standard estimates for solutions to linear elliptic and parabolic equations in a uniqueness class for the original non-linear problem that fits for any dimension $n \geq 2$. However, for the existence of even weak solutions to (4) one should assume that the bilinear forms $M_{i,1}$ have additional properties, see [5,9,23] for the properties of the so-called trilinear form. This means that the stability property is only a first step toward an Existence Theorem for regular solutions to (4).

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