Magnon Landau Levels and Topological Spin Responses in Antiferromagnets

Bo Li and Alexey A. Kovalev

1Department of Physics and Astronomy and Nebraska Center for Materials and Nanoscience, University of Nebraska, Lincoln, Nebraska 68588, USA
(Dated: June 23, 2020)

We study skyrmion and vortex-antivortex crystal phases in AFM insulators. We predict the topological spin Nernst response due to formation of magnonic Landau levels. In the long wavelength limit, the Landau levels exhibit relativistic physics described by the Klein-Gordon equation. To further uncover this physics, we construct a generic model of AFM topological insulator of magnons in which a uniform fictitious flux is induced by inhomogeneous Dzyaloshinskii-Moriya interaction. Our studies show that AFM insulators exhibit rich physics associated with topological excitations.

Emergent electromagnetism [1, 2] is at the core of a multitude of fascinating physical phenomena ranging from topological Hall effect [3–9] to formation of topological magnons [10–24]. Many applications related to information storage and processing can emerge from such useful features of magnetic systems as topological protection and low-dissipation spin transport [23–26]. The need for minimizing losses due to Joule heating has shifted the focus of recent research to insulating materials lacking itinerant electrons but still capable of carrying spin currents [27].

Recently, antiferromagnets (AFM) became the focus of active research as they possess unique features associated with lack of stray fields and ultrafast dynamics in THz range [28]. Many spintronics concepts readily extend to AFM materials as is the case with spin-orbit torques [29] demonstrated experimentally in CuMnAs [30–33]. Skyrmions in AFM can be potentially stabilized by staggered fields [32–33] induced by field-like spin-orbit torques in CuMnAs and Mn2Au or by coupling to boundary magnetization in Cr2O3. AFMs are expected to exhibit interesting physics associated with vanishing topological and skyrmion Hall effects [33–35].

The topological spin Hall effect has been predicted for conducting systems [32–39]. In insulating materials, the topological spin Hall effect mediated by magnons has been studied for isolated skyrmions [11]. The topological spin Nernst effect in skyrmion crystals have not been studied in insulators. The nature of topological response in skyrmion crystals can be associated with appearance of Landau levels of magnons [12–13]. Interestingly, in AFM the Landau levels are described by relativistic Klein-Gordon equation which is reflected in the shape of steps describing the accumulation of the spin Chern number. This also suggests a realization of unconventional magnonic topological insulator which in contrast to previous proposals [19] maps to the Klein-Gordon equation in the presence of magnetic field.

In this paper, we study the magnonic topological spin Nernst response in AFM skyrmion crystals and square crystals of vortices and antivortices, which, as we show, can be both stabilized by staggered magnetic field and anisotropy. The topological spin responses can be qualitatively understood by considering Landau levels induced by a uniform magnetic flux in a generic model of AFM magnonic topological insulator. We construct a model of AFM magnonic topological insulator that in the long wavelength limit maps to the Klein-Gordon equation in the presence of uniform magnetic field. In contrast to previous proposals [19], our model does not rely on the Aharonov-Casher effect with prefactor $1/e^2$ but originates in the Dzyaloshinskii-Moriya interactions (DMI).

AFM skyrmions and stability phase diagram—We begin by considering the free energy density of a quasi-two-dimensional AFM written in a long wavelength limit:

$$\mathcal{F}[\mathbf{n}] = \frac{J}{2} (\partial_i \mathbf{n})^2 + K(\mathbf{n} \cdot \hat{\mathbf{u}})^2 - \mathcal{H}_s(\mathbf{n} \cdot \hat{\mathbf{u}}) + \mathcal{D}_j (\partial_j \mathbf{n} \times \mathbf{n})$$

where we sum over repeated index $i = x, y$, $\mathbf{n}$ is a unit vector along the Néel order, $J$ is the exchange constant, $K$ is the effective uniaxial anisotropy along the direction $\hat{u}$ (typically $\hat{u} = \hat{z}$), $\mathcal{H}_s$ is the staggered magnetic field along the direction $\hat{u}$ arising due to the spin-orbit torque or the effect of boundary magnetization [32–33], and $D_{ij} = (\mathcal{D}_{ij})_{s}$ is the DMI described by a general tensor. When DMI is induced by axially symmetric interface with a heavy metal, which is the focus of this paper, there are only two non-zero tensor coefficients $D_{12} = -D_{21} = \mathcal{D}$ [44]. The free energy density in Eq. (1) and resulting from it phase diagram can also describe other spin textures obtained from Néel skyrmions by global transformation in spin space (e.g. antiskyrmions or Bloch skyrmions) [45]. This can be seen by applying a global transformation to the spin texture followed by similar transformations on $\hat{u}$ and $D_{ij}$ [45]. The zero temperature phase diagram in Fig. 1 has been calculated using the method of Ref. [45] relying on energy minimization [46] and rescaling of unit cell. In addition to AFM-SkX phase identified in Ref. [33], we also identify AFM-SC vortex-antivortex lattice [45–51] stabilized by the inplane anisotropy. Such textures can also contain antiferromagnetic antimersions with fractional topological charge as shown in Fig. 1.
parametrized spin field into the free energy $F[m, n]$ generates a Hamiltonian, in which magnons couple to a spin texture induced emergent gauge field “a” [33, 52, 53],
$$\mathcal{H}_{\text{mag}} = \frac{1}{2} \hat{\psi}^\dagger \hat{H} \psi$$
where $\hat{H} = \hat{H}_+ \oplus \hat{H}_-$. \[ (2) \]

The term $\mathcal{H}_x$ matches with the chirality of two copies of magnons. The emergent gauge field has two contributions, $\mathbf{a} = \mathbf{a}^T + \mathbf{a}^d$, where $a_i^T = \cos \theta \partial_\phi \phi$ and $a^d = -(\mathcal{D}/\mathcal{J}) \exp(\pi L_z/2) \mathbf{n}_0$. These two parts result in emergent magnetic fields, $b^T = (\mathbf{V} \times \mathbf{a}^T)_i = -\frac{1}{2} \epsilon_{ijk} \mathbf{n}_0 \cdot (\partial_j \mathbf{n}_0 \times \partial_k \mathbf{n}_0)$, and $b^d = \mathbf{V} \times \mathbf{a}^d$.

The kinetic term of magnons can be extracted from the Berry phase Lagrangian of spins [53]. We obtain $\mathcal{L}_{\text{kin}} = iSv^\dagger S^J_0 \tau_3 \gamma_0/4$ with $S$ being the spin density. The total Lagrangian density of magnon field is block-diagonal with respect to subspace $\eta_+ = (\psi_A, \psi_B)^T$, $\eta_- = (\psi_A^*, \psi_B^*)^T$. The decoupled matrix Schrödinger equations are
$$i \chi \mathbf{S} \frac{\tau_3}{2} \partial_\eta \eta = \mathcal{H} \chi \eta_x,$$  \[ (3) \]

To understand physics associated with emergence of Landau levels, we approximate the emergent magnetic field by its spatial average which is justified for smooth enough textures. In particular, we consider $\mathbf{b} = -B\hat{z}$, with $B = |(\mathbf{V} \times \mathbf{a})| = 4\pi \rho_{\text{top}} > 0$, where $\rho_{\text{top}} = \mathbf{n}_0 \cdot (\partial_\phi \mathbf{n}_0 \times \partial_\theta \mathbf{n}_0)$ and DMI induced contribution vanishes. In the Landau gauge, $\mathbf{a}_0 = (yB, 0, 0)$, the eigenenergies are chirality degenerate, $\varepsilon_{\pm}^T = \pm \sqrt{\mathcal{J} \mathcal{A}_{\text{kin}}/2S}$ with $\mathcal{A}_0 = B(2n + 1)$, which agrees with Landau levels of the Klein-Gordon equation [55]. The wave function can be found by substituting $\xi_{\pm}^{\lambda}(\mathbf{r}) = (\alpha_1, \alpha_2)^T \xi_{\pm}^{\lambda}(\mathbf{r})$ into Hamiltonian [2], where $\xi_{\pm}^{\lambda}(\mathbf{r})$ is the known eigenfunction of $n$-th non-relativistic Landau level [56]. The number of degenerate states is determined by the total number of the magnetic flux quanta where each unit cell with topological charge one contributes two flux quanta. The magnon Landau levels result in various Hall-like responses. However, the two species of magnons with opposite chirality feel opposite magnetic flux in Eq. (2) as they are time-reversal partners of each other, which always results in vanishing thermal Hall response. On the other hand, spin and chirality current responses are nonzero.

For a nonuniform fictitious field of skyrmion lattice with basis vectors $\tilde{\mathbf{a}}_1$ and $\tilde{\mathbf{a}}_2$, the Landau-level wave functions can be linearly combined to a new periodic basis for each energy level, $\varphi_{\text{nmk}}$, which satisfies $T_{\tilde{\mathbf{a}}_1, 2}(\varphi_{\text{nmk}}) = e^{i\mathbf{R}_{\text{nnk}}} \varphi_{\text{nmk}}$ with magnetic translational operator $T_{\tilde{\mathbf{a}}_1, 2}$ satisfying $T_{\tilde{\mathbf{a}}_1, 2} T_{\tilde{\mathbf{a}}_2} = e^{iQ_1 \pi T_{\tilde{\mathbf{a}}_1} T_{\tilde{\mathbf{a}}_2}}$. The phase factor indicates that each skyrmion unit cell contains topological charge $Q$ which leads to splitting into $2|Q|$ subbands described by quantum number $m$. In this new basis, one can include perturbations to Hamiltonian due to nonuniform fictitious flux and higher order terms disregarded.
Spin Nernst effect in AFM topological insulator—A square lattice Hamiltonian of collinear FM (AFM) reads

\[ H = \sum_{ij} JS_i \cdot S_j + D_{ij}(S_i \times S_j) - \sum_i H_i S_i^0 - K(S_i^0)^2 \]

As the order parameter direction controls DMI effect on magnons, the order parameter is oriented along the y-axis to realize the Landau gauge. Above, the exchange parameter is \( J < 0 \) (\( J > 0 \)) for FM (AFM), \( H_i \) is (staggered) magnetic field, \( K \) is the magnetic anisotropy, and \( D_{ij} = D(\hat{r}) \hat{z} \times \delta_{ij} \) describes DMI with Rashba symmetry for a bond \( \delta_{ij} \). In FM case, we rewrite the exchange and DMI terms in a rotated frame with the quantization axis along the y-axis as \( J_{ij}(e^{i\beta/2}S_j^0 + e^{-i\beta/2}S_j^0)/2 + JS_j^z \) where \( J_{ij}e^{i\beta/2} = J + iD_{ij} \cdot \mathbf{n}_0 \) with \( \mathbf{n}_0 \) being the direction of the order parameter. In AMF case, we need to replace \( S_j^\mp \rightarrow S_j^\mp \), and \( S_j^\mp \rightarrow -S_j^\mp \) for one of sublattices.

To replicate the Landau gauge, we assume that bonds are along the Cartesian coordinates and the strength of DMI is nonuniform, i.e., \( D(\hat{r}) \delta/J = \tan(\delta B y) \) where \( \delta \) is the bond length (when DMI is small \( D(\hat{r})/J \approx By \)). Using the Holstein-Primakoff transformation in the limit of large \( S \), i.e., \( S_j^\mp \approx \sqrt{2S_a_i} \), \( S_j^\mp \approx \sqrt{2S_a_i} \), \( S_j^\mp \approx S = a_i^\dagger a_i \), we recover discreet realization of noninteracting magnons subjected to uniform magnetic field described by a vector potential \( \mathbf{a}_0 = (yB, 0, 0) \). In the long wavelength limit FM magnons are described by the Schrödinger equation while AFM magnons by the Klein-Gordon equation.

In what follows, we concentrate on AFM, using FM system only for comparison. The role of the chiral index \( \chi \) in Eq. [2] is played by the spin index \( s \) as the spin along the quantization axis is conserved. After the Fourier transform, the Hamiltonian for \( s_z = 1 \) becomes

\[ H_+ = \frac{1}{2} JS \sum_k \Psi_{-}^\dagger(k) H_{+}(k) \Psi_{+}(k), \]

where \( \Psi_{\pm} = (a_1(k), b_1(-k), \ldots, a_N(k), b_N(-k))^T \) is the bosonic field, and the unit cell contains \( N \) by 2 array of atoms from each sublattice of the square-lattice AFM. The Hamiltonian has a block structure

\[ H_{+}(k) = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{b} & \hat{a} \end{pmatrix}, \]

where for \( 2 \times 2 \) matrices \( \hat{a} \) and \( \hat{b} \) the nonzero elements are given by \( a_{i,j} = 4 \), \( b_{i,j} = \cos(k_x + j\phi_0) \) for \( i = j \), and \( a_{i,j} = b_{i,j} = e^{-ik_y} \) for \( i - j = 1 \) modulo 2\( N \). Here the phase factor \( \phi_0 = 2\pi p/q \) describes the strength of magnetic field, i.e., \( 2p \) is the number of flux quanta for enlarged unit cell and \( q = 2N \). For subspace \( s_z = -1 \), \( H_{-}(k) = H_{+}^T(-k) \) and \( \Psi_{-}(k) = (a_1(-k), b_1(k), \ldots, a_N(-k))^T \). The total Hamiltonian matrix can be diagonalized by a paraunitary matrix \( T_k \), i.e., \( T_k^HHT_k = \mathcal{E}_k \), where \( \mathcal{E}_k \) is a diagonal matrix.
describing eigenvalues 57. By varying strength of DMI, we can control the magnetic flux per unit cell which allows us to observe the Hofstadter butterfly in full analogy with electronic systems (see Fig. 2). Similarly to electronic systems, the exact energy bands can be found from expansion of $p/q$ into continuous fractions or from the Diophantine equation 58, 59. As can be seen from Fig. 2 the form of the Hofstadter butterfly differs from the case of nonrelativistic electrons.

The spin responses of magnons can be described with the help of the spin Berry curvature 17, 60,

$$
\Omega^{\alpha}_{mn} = i \sum_{m \neq n} (\tilde{\sigma}_3)_{nm} (\tilde{\sigma}_3)_{mn} \frac{1}{2} \left\{ \hat{v} \cdot \hat{\Sigma}^\alpha \right\}_{nm} \times \hat{v}_{mn} \left( \varepsilon_{n,k} - \varepsilon_{m,k} \right)^2, \quad (7)
$$

where we define the anticommutator $\{ \hat{v} \cdot \hat{\Sigma}^\alpha \}$, the Pauli matrix in the particle-hole space, i.e., $(\tilde{\sigma}_3)_{mn} = 1$ for particle-like states and $(\tilde{\sigma}_3)_{mn} = -1$ for hole-like states.

The magnon spin density operator along the $\alpha$-axis is given by $\Sigma^\alpha(r) = \hat{\Psi}^\dagger(r) \Sigma^\alpha \hat{\Psi}(r)$, where $\Sigma^\alpha = -\sigma_\alpha \otimes \text{Diag}(m^1, \cdots, m^q)$ with the Pauli matrix $\sigma_\alpha$ describing the particle-hole space and $m_i$ being the direction of magnetic moment at position $i$ in a unit cell of $M$ atoms 60. We consider the spin Nernst response 61,

$$
a^{\alpha xy}_{xy} = k_B/V \sum_{k,n=1}^N c_1 (g(\varepsilon_{n,k})) \Omega^{(k)}_{n} \left( \varepsilon_{n,k} \right), \quad (\alpha, \varepsilon, c_1) = (e^{c_1/T}, -1)\text{-}1 \text{ is the Bose-Einstein distribution and } c_1(x) = (1 + x) \ln(1 + x) - x \ln(x). \quad \text{Due to degeneracy, we apply Eq. (7) to each subspace } s_z = \pm 1 \text{ separately. The total spin Chern number is a sum of spin Chern numbers for each subspace, i.e., } C_{\alpha} = (1/2\pi) \int_{BZ} m^{(z)}_{n} d^2 k \text{ where } \Omega^{(z)}_{n} = \Omega^{(z)+}_{n} + \Omega^{(z)-}_{n}.
$$

To establish a connection to QHE, we study the total Berry curvature of states below certain energy, $C^{(k)}(\varepsilon) = (1/2\pi) \int_{BZ} \sum_{c_{\alpha},k} \Omega^{(k)}_{n} d^2 k$. For FM magnons, the results for the total Berry curvature and the magnon density of states (DOS) are shown in Figs. 3(a) and (b) where we choose $p = 1$ and $q = 77$ to replicate the flux produced by two skyrmions in SkX unit cell of $14 \times 22$ atoms (see Fig. 1). We observe a behavior associated with the van Hove singularity 62 of the magnon band structure. This causes a sign change in the total Berry curvature at the transition between particle- and hole-like states 63, 64.

For AFM magnons, we choose $p = 2$ and $q = 270$ to replicate AFM SkX on a lattice of $18 \times 30$ atoms. The total spin Berry curvature shown in Fig. 3(d) exhibits steps of 2 and uneven energy height even in the long wavelength limit. We observe sharp change in the spin Berry curvature at the DOS singularity in Fig. 3(c). For both FM and AFM magnons, away from DOS singularity the formation of magnon Landau levels can be described by the Onsager's quantization scheme 65, 66. We confirm this by comparing the semiclassical curve corresponding to the area enclosed by DOS with the Berry curvature curves in Fig. 3. The spin Nernst response is shown in

![Fig. 4. Spin Nernst conductivity as a function of temperature. Red curve describes the topological spin Nernst response for square lattice AFM with a unit cell of $18 \times 30$ atoms containing two skyrmions. Blue curve describes the spin Nernst response in AFM magnonic topological insulator with DMI induced fictitious flux $\Phi = \frac{q}{p} \Phi_0$ for $p = 2$ and $q = 270.

Topological spin Nernst effect in AFM—To describe magnon excitations on top of textures in Fig. 1 we use the Holstein-Primakoff transformation in a local frame 67. The resulting Hamiltonian describes noninteracting magnons and can be diagonalized using the unitary matrices $T_k$. Spectrum for the lowest bands of a lattice containing $18 \times 30$ atoms is shown in Fig. 2. We observe that the Landau levels become dispersive and that AFM chiral modes split. The total sublattice Berry curvature is shown in Fig. 3(d) where we use sublattice index instead of spin index in Eq. (7). The sublattice index in Eq. (2) and spin index in Eq. (5) can be mapped onto each other in the absence of coupling between chiral modes. We observe only qualitative agreement with Landau levels in AFM calculated earlier for $p = 2$ and $q = 270$ due to coupling of chiral modes in AFM SkX as a consequence of higher order corrections. In Fig. 3(b), we observe better agreement between Berry curvatures calculated for FM SkX (lattice of $14 \times 22$ atoms) and for Landau levels in FM with DMI induced uniform flux ($p = 1$ and $q = 77$). The sign change of the Berry curvatures in Figs. 3(b) and (d) can lead to the sign change of the topological thermal Hall and spin Nernst responses as a function of temperature. Using the spin Berry curvature, we calculate the topological spin Nernst response in Fig. 4 and confirm the sign change. As expected, the spin Nernst response in AFM SkX is suppressed compared to similar response in AFM topological insulator (see Fig. 4).

Conclusions—We have shown that AFM-SkX should exhibit a large topological spin Nernst response. The spin response is associated with the formation of dispersive Landau levels. AFM magnon Landau levels exhibit relativistic physics which in the long wavelength limit can be described by the Klein-Gordon equation. Similar physics also arises in AFM square vortex-antivortex phase. To further uncover this behavior, we have constructed a model of AFM topological insulator where
the fictitious flux is induced by inhomogeneous DMI and leads to formation of unconventional Hofstadter butterfly. Our predictions can be tested in magnetoelectrics where the staggered field can be induced by the boundary magnetization. The spin Nernst response can be potentially observed in ferromagnets, e.g., similar to TmIG.

This work was supported by the U.S. Department of Energy, Office of Science, Basic Energy Sciences, under Award No. de-sc0014189.

[1] G. E. Volovik, J. Phys. C 20, L83 (1987).
[2] G. Tatara, H. Kohno, and J. Shibata, J. Phys. Rep. 468, 213 (2008).
[3] P. Bruno, V. K. Dugaev, and M. Taillefumier, Phys. Rev. Lett. 93, 096806 (2004).
[4] A. Neubauer, C. Pfeiferer, B. Binz, A. Rosch, R. Ritz, P. G. Niklowitz, and P. Böni, Phys. Rev. Lett. 102, 186602 (2009).
[5] M. Lee, W. Kang, Y. Onose, Y. Tokura, and N. P. Ong, Phys. Rev. Lett. 102, 186601 (2009).
[6] N. Kanazawa, Y. Onose, T. Arima, D. Okuyama, K. Ohoyama, S. Wakimoto, K. Kakurai, S. Ishiwata, and Y. Tokura, Phys. Rev. Lett. 106, 156603 (2011).
[7] T. Schulz, R. Ritz, A. Bauer, M. Halder, M. Wagner, C. Franz, C. Pfeiferer, K. Everschor, M. Garst, and A. Rosch, Nature Phys. 8, 301 (2012).
[8] P. B. Ndiaye, C. A. Akosa, and A. Manchon, Phys. Rev. B 95, 064426 (2017).
[9] B. Göbel, A. Mook, J. Henk, and I. Mertig, New J. Phys. 19, 063042 (2017).
[10] A. Bogdanov and A. Hubert, J. Magn. Magn. Mater. 138, 255 (1994).
[11] U. K. Rößler, A. N. Bogdanov, and C. Pfeiferer, Nature 442, 797 (2006).
[12] S. Mühlbauer, B. Binz, F. Jonietz, C. Pfeiferer, A. Rosch, A. Neubauer, R. Georgii, and P. Böni, Science 323, 915 (2009).
[13] X. Z. Yu, Y. Onose, N. Kanazawa, J. H. Park, J. H. Han, Y. Matsui, N. Nagaosa, and Y. Tokura, Nature 465, 901 (2010).
[14] R. Shindou, R. Matsumoto, S. Murakami, and J.-i. Ohe, Phys. Rev. B 87, 174427 (2013).
[15] L. Zhang, J. Ren, J.-S. Wang, and B. Li, Phys. Rev. B 87, 144101 (2013).
[16] A. Mook, J. Henk, and I. Mertig, Phys. Rev. B 90, 024412 (2014).
[17] V. A. Zvyuzin and A. A. Kovalev, Phys. Rev. Lett. 117, 217203 (2016).
[18] S. A. Owerre, J. Phys.: Condens. Matter 28, 386001 (2016).
[19] K. Nakata, S. K. Kim, J. Klinovaja, and D. Loss, Phys. Rev. B 96, 224414 (2017).
[20] B. Li and A. A. Kovalev, Phys. Rev. B 97, 174413 (2018).
[21] R. Seshadri and D. Sen, Phys. Rev. B 97, 134411 (2018).
[22] P. A. McClarty, X.-Y. Dong, M. Gohlke, J. G. Rau, F. Pollmann, R. Moessner, and K. Penc, Phys. Rev. B 98, 060404 (2018).
[23] A. Fert, N. Reyren, and V. Cros, Nat. Rev. Mater. 2 (2017).
[24] Y. Zhou and M. Ezawa, Nat. Commun. 5 (2014).
[25] A. V. Chumak, V. I. Vasyuchka, A. A. Serga, and B. Hillebrands, Nature Phys. 11, 453 (2015).
[26] B. Göbel, A. F. Schäffer, J. Berakdar, I. Mertig, and S. S. P. Parkin, Sci. Rep. 9 (2019).
[27] V. Baltz, A. Manchon, M. Tsoi, T. Moriyama, T. Ono, and Y. Tserkovnyak, Rev. Mod. Phys. 90, 015005 (2018).
[28] K. Olejnık, T. Seifert, Z. Kašpar, V. Novák, P. Wadley, R. P. Campion, M. Baumgartner, P. Gambardella, P. Němec, J. Wunderlich, et al., Sci. Adv. 4, eaar3566 (2018).
[29] A. Manchon, J. Železný, I. M. Miron, T. Jungwirth, J. Sinova, A. Thiaville, K. Garello, and P. Gambardella, Rev. Mod. Phys. 91, 035004 (2019).
[30] J. Železný, H. Gao, K. Výborný, J. Zemen, J. Mašek, A. Manchon, J. Wunderlich, J. Sinova, and T. Jungwirth, Phys. Rev. Lett. 113, 157201 (2014).
[31] P. Wadley, B. Howells, J. elezny, C. Andrews, V. Hills, R. P. Campion, V. Novak, K. Olejník, F. Maccherozzi, S. S. Dhesi, et al., Science 351, 587 (2016).
[32] B. Göbel, A. Mook, J. Henk, and I. Mertig, Phys. Rev. B 96, 060406 (2017).
[33] R. Zarzuela, S. K. Kim, and Y. Tserkovnyak, Phys. Rev. B 100, 100408 (2019).
[34] S. K. Kim, O. Tchernyshyov, and Y. Tserkovnyak, Phys. Rev. B 92, 020402 (2015).
[35] J. Barker and O. A. Tretiakov, Phys. Rev. Lett. 116, 147203 (2016).
[36] H. Velkov, O. Gomonay, M. Beems, G. Schiwietz, A. Brataas, J. Sinova, and R. A. Duine, New J. Phys. 18, 075016 (2016).
[37] C. Jin, C. Song, J. Wang, and Q. Liu, Appl. Phys. Lett. 109, 182404 (2016).
[38] X. Zhang, Y. Zhou, and M. Ezawa, Sci. Rep. 6, 24795 (2016).
[39] C. A. Akosa, O. A. Tretiakov, G. Tatara, and A. Manchon, Phys. Rev. Lett. 121, 097204 (2018).
[40] P. M. Buhl, F. Freimuth, S. Blügel, and Y. Mokrousov, Phys. Status Solidi RRL 11, 1700007 (2017).
[41] M. W. Daniels, W. Yu, R. Cheng, J. Xiao, and D. Xiao, Phys. Rev. B 99, 244433 (2019).
[42] K. A. van Hoogdalem, Y. Tserkovnyak, and D. Loss, Phys. Rev. B 87, 024402 (2013).
[43] S. K. Kim, K. Nakata, D. Loss, and Y. Tserkovnyak, Phys. Rev. Lett. 122, 057204 (2019).
[44] A. A. Kovalev and S. Sandhoefer, Frontiers in Physics 6, 00098 (2018).
[45] U. Güngörđü, R. Nepal, O. A. Tretiakov, K. Belashchenko, and A. A. Kovalev, Phys. Rev. B 93, 064428 (2016).
[46] A. Vansteenkiste, J. Lelaiter, M. Dvornik, M. Helsen, F. Garcia-Sanchez, and B. V. Waeyenberge, AIP Advances 4, 107133 (2014).
[47] S. D. Yi, S. Onoda, N. Nagaosa, and J. H. Han, Phys. Rev. B 80, 054416 (2009).
[48] S.-Z. Lin, A. Saxena, and C. D. Batista, Phys. Rev. B 91, 224407 (2015).
[49] R. Ozawa, S. Hayami, K. Barros, G.-W. Chern, Y. Motome, and C. D. Batista, Journal of the Physical Society of Japan 85, 103703 (2016).
[50] M. Vousden, M. Albert, M. Beg, M.-A. Bisotti, R. Carey, D. Chernyshenko, D. Cortés-Ortuño, W. Wang, O. Hoverka, C. H. Marrows, et al., Appl. Phys. Lett. 108,
[51] X. Z. Yu, W. Koshiae, Y. Tokunaga, K. Shibata, Y. Taguchi, N. Nagaosa, and Y. Tokura, Nature 564, 95 (2018).
[52] A. A. Kovalev and Y. Tserkovnyak, EPL (Europhysics Letters) 97, 67002 (2012).
[53] M. W. Daniels, R. Cheng, W. Yu, J. Xiao, and D. Xiao, Phys. Rev. B 98, 134450 (2018).
[54] A. Auerbach, Interacting Electrons and Quantum Magnetism, Graduate Texts in Contemporary Physics (Springer New York, 1998).
[55] L. Lam, Journal of Mathematical Physics 12, 299 (1971).
[56] L. D. Landau, Quantum mechanics: non-relativistic theory (Butterworth Heinemann, Amsterdam, 1977).
[57] J. Colpa, Physica (Amsterdam) 93A, 327 (1978).
[58] D. R. Hofstadter, Phys. Rev. B 14, 2239 (1976).
[59] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. Den Nijs, Phys. Rev. Lett. 49, 405 (1982).
[60] B. Li, S. Sandhoefner, and A. A. Kovalev, Phys. Rev. Research 2, 013079 (2020).
[61] A. A. Kovalev and V. Zyuzin, Phys. Rev. B 93, 161106 (2016).
[62] B. Göbel, A. Mook, J. Henk, and I. Mertig, Phys. Rev. B 95, 094413 (2017).
[63] Y. Hatsugai, T. Fukui, and H. Aoki, Phys. Rev. B 74, 205414 (2006).
[64] D. N. Sheng, L. Sheng, and Z. Y. Weng, Phys. Rev. B 73, 233406 (2006).
[65] I. M. Lifshits, M. Y. Azbel', and M. I. Kaganov, Sov. Phys. JETP 4, 41 (1957).
[66] M. Arai and Y. Hatsugai, Phys. Rev. B 79, 075429 (2009).
[67] S. A. Diaz, J. Klinovaja, and D. Loss, Phys. Rev. Lett. 122, 187203 (2019).
[68] X. He, Y. Wang, N. Wu, A. N. Caruso, E. Vescovo, K. D. Belashchenko, P. A. Dowben, and C. Binek, Nat. Mater. 9, 579 (2010).
[69] Q. Shao, Y. Liu, G. Yu, S. K. Kim, X. Che, C. Tang, Q. L. He, Y. Tserkovnyak, J. Shi, and K. L. Wang, Nat. Electron. 2, 182 (2019).