On the Linearized Artin Braid Representation

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Abstract

We linearize the Artin representation of the braid group given by (right) automorphisms of a free group providing a linear faithful representation of the braid group. This result is generalized to obtain linear representations for the coloured braid groupoid and pure braid group too. Applications to some areas of two-dimensional physics are discussed.
1 Introduction

Let us consider a free group $F_n$ of rank $n$ with generators $x_1, x_2, ..., x_n$. Artin [1, 2] proved that the braid group $B_n$ with generators $\sigma_1, ..., \sigma_n$ and defining relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| \geq 2$$

(1)

has a faithful representation as a group of right automorphisms $\hat{\sigma}_i, i = 1, ..., n - 1$ of $F_n$ given by

$$\hat{\sigma}_i : \begin{cases} 
  x_i & \mapsto x_i \cdot x_{i+1} \cdot x_i^{-1} \\
  x_{i+1} & \mapsto x_i \\
  x_j & \mapsto x_j \text{ for } j \neq i, i + 1
\end{cases}$$

(2)

On the other hand Magnus [3] obtained a class of representations of free groups which can be used to get in some cases representations of subgroups of the automorphism group of $F_n$. In this paper we prove that the linearization of the Artin transformations (2) using the concept of the Magnus representation leads to a faithful representation of the braid group which was already introduced in some different context in [4].

Let $\mathcal{J}B_n$ be the ring over the braid group $B_n$ with integer coefficients. The representatives of the braid group generators are the matrices

$$(\hat{\sigma}_i)_{jk} = 1_{(i-1,i-1)} \cdot \sigma_i \oplus \begin{pmatrix} \alpha_i & \beta_i \\ \sigma_i & 0 \end{pmatrix} \oplus 1_{n-2-i,n-2-i} \cdot \sigma_i$$

(3)

with

$$\alpha_i = \sigma_i(1 - \theta_i \theta_{i+1} \theta_i^{-1})$$
$$\beta_i = \sigma_i \theta_i$$

(4)

and

$$\theta_i = \sigma_1^{-1} ... \sigma_{i-1}^{-1} \sigma_i^2 \sigma_{i-1} ... \sigma_1$$

(5)

The fact that (3) provides a representation of $B_n$ with values in the braid ring $\mathcal{J}B_n$ can be seen from the following equalities in the braid ring:

$$\alpha_i \sigma_{i+2} \alpha_i + \beta_i \alpha_{i+1} \sigma_{i+1} = \sigma_{i+2} \alpha_i \sigma_{i+2}$$
\[
\begin{align*}
\alpha_i \sigma_{i+2} \beta_i &= \sigma_{i+2} \beta_i \alpha_{i+1} \\
\beta_i \beta_{i+1} \sigma_{i+1} &= \sigma_{i+2} \beta_i \beta_{i+1} \\
\sigma_{i+1} \sigma_{i+2} \alpha_i &= \alpha_{i+1} \sigma_{i+1} \sigma_{i+2} \\
\sigma_{i+1} \sigma_{i+2} \beta_i &= \beta_{i+1} \sigma_{i+1} \sigma_{i+2}
\end{align*}
\] (6)

A full tower of linear representations of the braid group is produced by iteratively replacing the old generators with the new representatives. The trivial diagonal representation \( \sigma_i = q \in \mathbb{R} \) generates for instance the Burau representation which again, introduced in (3), produces a new representation and so on. The representation (3) is reducible as a consequence of the fact that the element \( x_1 \cdot x_2 \cdot \ldots \cdot x_n \in F_n \) is left invariant under Artin transformations. Using the braid relations \( \alpha_i \) can be reexpressed to \( \alpha_i = (1 - \theta_i) \sigma_i \) [5].

Another proof of the fact that (3) is a representation of the braid group using homology was indicated to the authors by R. Lawrence [5]; it is based on her very interesting thesis [6]. It seems that ideas similar to ours can also be found in the book [7] and go back to Artin.

Here we also extend the construction which leads to (3) to the case of the coloured braid groupoid and pure braid group.

The representation (3) and especially its iterations have applications which are listed here: determination of braid and monodromy properties of generalized hypergeometric integrals, characterization of braid and monodromy representations through bilinear forms [8, 9], contour methods in two dimensional conformal quantum field theory (especially perturbation theory where the vertex structure is partially spoiled [9]) and quantum groups [10, 11]. These applications will be briefly discussed in section 5 of this paper. For more details see [4, 10, 11].

Concerning the organization of the paper, we put in section 2 some remarks about the free differential calculus and the Magnus representation of free groups and braid group. In section 3 we prove that (3) provides a linear faithful braid valued representation of the braid group; in section 4 we extend the results to the coloured and pure braid group cases.

2 The Magnus representation

We follow [12] with some minor modifications.

Let \( \Phi \) be an arbitrary homomorphism action on the free group \( F_n \) and let
$F_n^\Phi$ denote the image of $F_n$ under $\Phi$. Let $\mathcal{J}F_n^\Phi$ denote the group ring of $F_n^\Phi$ with integer coefficients. Elements in $\mathcal{J}F_n^\Phi$ are formal linear combinations of the form $\sum a_g g$ ($g \in F_n^\Phi$, $a_g \in \mathcal{J}$).

The operations in $\mathcal{J}F_n^\Phi$ are defined as follows:

\[
\sum a_g g + \sum b_g g = \sum (a_g + b_g) g
\]

\[
(\sum a_g g) \cdot (\sum b_g g) = \sum (\sum a_{gh^{-1}} b_h) g
\]

(7)

There is a well defined mapping $\frac{\partial}{\partial x_j}$, $j = 1, \ldots, n$, called the free differential:

\[
\frac{\partial}{\partial x_j} : \mathcal{J}F_n \to \mathcal{J}F_n
\]

given by

\[
\frac{\partial}{\partial x_j} (x_{\mu_1}^{\epsilon_1} \ldots x_{\mu_r}^{\epsilon_r}) = \sum_{i=1}^{r} \epsilon_i \delta_{\mu_i,j} x_{\mu_1}^{\epsilon_1} \ldots x_{\mu_i}^{\epsilon_i} \frac{1}{2} (\epsilon_i - 1)
\]

\[
\frac{\partial}{\partial x_j} (\sum a_g g) = \sum a_g \frac{\partial g}{\partial x_j}
\]

(9)

where $g \in F_n$, $a_g \in \mathcal{J}$, $\epsilon_i = \pm 1$.

We have the rules

\[
i) \quad \frac{\partial x_i}{\partial x_j} = \delta_{i,j}
\]

\[
i i) \quad \frac{\partial x_i^{-1}}{\partial x_j} = -\delta_{i,j} x_i^{-1}
\]

\[
i iii) \quad \frac{\partial (uv)}{\partial x_j} = \frac{\partial u}{\partial x_j} v + u \frac{\partial v}{\partial x_j}
\]

\[
i iv) \quad \text{chain rule}
\]

(10)

In $iii)$ $v^t$ means the sum of the integer-coefficients in $v$ and $iv)$ can be formulated as follows: let $v_1, \ldots, v_n$ be another system of generators in $F_n$. They are words $v_i(x_1, \ldots, x_n)$, $i = 1, \ldots, n$ in $F_n$. We have

\[
\frac{\partial}{\partial x_j} w(v_1, \ldots, v_n) = \sum_{k=1}^{n} \frac{\partial w}{\partial v_k} \frac{\partial v_k}{\partial x_j}
\]

Remark the unusual form of $ii)$ and $iii)$.

In section 4 we will consider the monodromy subgroup of the pure braid
group, given by the generators $\theta_i$ of formula (3). Every free group is isomorphic to the group generated by $\theta_i$. Via this identification we are able to consider the multiplication operation of elements of a free group by elements of the braid group $B_n$.

We are going now to recall to the reader the Magnus representations of a free group and of the braid group. Let $S_n$ be a free abelian semigroup with basis $s_1, \ldots, s_n$ and let $A(\mathcal{R}, S_n)$ be the semigroup ring of $S_n$ with respect to the ring $\mathcal{R}$. Let $\Phi$ be a homomorphism acting on the free group $F_n$; for $w \in F_n$ let $w^\Phi$ be the image of $w$ under $\Phi$. Then the mapping $w \to (w)^\Phi$,

$$(w)^\Phi = \left( \begin{array}{c} w^\Phi \sum_{j=1}^{n} \left( \frac{\partial w}{\partial x_j} \right)^\Phi s_j \\ 0 \\ 1 \end{array} \right)$$  \hspace{2cm} (11)$$

with the entries of $(w)^\Phi$ in $A(\mathcal{J}F_n^\Phi, S_n)$, is a representation of $F_n$ called the Magnus $\Phi$-representation. It is faithful if $F_n$ is abelian; otherwise its kernel is the commutator subgroup of $\ker \Phi$ \cite{12}. Now let $\mathcal{A}$ be any group of right (automorphisms) of $F_n$ such that

$$x^\Phi = x\alpha^\Phi$$  \hspace{2cm} (12)$$

for each $x \in F_n, \alpha \in \mathcal{A}$. Then the matrix

$$||\alpha||^\Phi = \left( \frac{\partial (x_\alpha)}{\partial x_j} \right)^\Phi$$  \hspace{2cm} (13)$$

with entries in $\mathcal{J}F_n^\Phi$ defines a linear representation of $\mathcal{A}$ called again Magnus representation. If the automorphism group $\mathcal{A}$ is the Artin automorphism \cite{4}, then we get the Magnus representation of the braid group. If $\Phi$ is defined by $x_i^\Phi = q \in \mathbb{R}$, $1 \leq i \leq n$, we recover the Burau representation as a special case of the Magnus representation. The proof of the above assertions is based on the chain rule of the free differential calculus and uses heavily the condition (12). In fact it turns out that this condition is rather restrictive because in the braid case it implies that $x_i^\Phi$ must be independent of $i$.

In the next section we will drop out the condition (12) in the particular case when $\mathcal{A}$ coincides with the Artin’s automorphism group and prove what we call the braid valued generalization of the Magnus representation for the braid group.
3 Braid valued representation of the braid group

Let $\tilde{F}_n$ be now a copy of $F_n$ with generators $s_i$. Certainly there will be a (generalized) Magnus representation of $F_n$ given by $w \rightarrow (w)$, $w \in F_n$ and

$$ (w) = \left( \begin{array}{c} w \\ \sum_{j=1}^{n} \frac{\partial w}{\partial x_j} s_j \\ 1 \end{array} \right) $$

(14)

where the entries of $(w)$ are in the free ring of $\tilde{F}_n$ with respect to the ring $JF_n$. For the generators this representation reduces to

$$ (x_i) = \left( \begin{array}{c} x_i \\ s_i \\ 1 \end{array} \right) $$

(15)

At the first glance this representation seems not to be very interesting, however we will prove that it provides a representation of the braid group via Artin automorphism.

We identify the free group $F_n$ with the monodromy subgroup of the pure braid group via the identification of the generators $x_i$ and $\theta_i$. For the monodromy group the Artin transformations are equivalent to the adjoint action of the generators $\sigma_i$:

$$ \sigma_i^{-1} \theta_k \sigma_i = \hat{\sigma}_i(\theta_k) \quad \text{for any} \quad 1 \leq i, k \leq n $$

(16)

where

$$ \hat{\sigma}_i(\theta_i) = \theta_i \cdot \theta_{i+1} \cdot \theta_i^{-1} $$

$$ \hat{\sigma}_i(\theta_{i+1}) = \theta_i $$

$$ \hat{\sigma}_i(\theta_j) = \theta_j \quad \text{for} \quad j \neq i, i + 1 $$

For given $i$ the right hand side of (14) provides a new system of generators of the monodromy group.

We go now from the free group $F_n$ to the free group $F'_n$ whose generators $\theta'_j$ are the images under the Artin transformation $\hat{\sigma}_i$ of the $F_n$ generators:

$$ \theta'_j = \sigma_i^{-1} \theta_j \sigma_i \quad , \quad 1 \leq i, j \leq n $$

6
Let us introduce for a given $i$ the jacobian map $J$ of the free differential calculus

$$J : \mathcal{J}F_n \to \mathcal{J}F_n$$

$$J = \left( \frac{\partial \hat{\sigma}_i(\theta_j)}{\partial \theta_k} \right)$$

(17)

which is the linearization of the Artin transformation. We proceed in the same way as before going from the free group $F'_n$ to $F''_n$ with generators

$$\theta'_j = \sigma_l^{-1} \sigma_l \tau_l \sigma_l^{-1} \theta_j \sigma_l, \quad 1 \leq l, j \leq n$$

The linearization of the Artin relations (applied to $\theta'_j$) is given by the jacobian $J' : \mathcal{J}F'_n \to \mathcal{J}F'_n$ defined as (17) with the replacement $\theta_j \to \theta'_j$.

It is possible to transport the linear mapping $J'$ to a linear mapping in $\mathcal{J}F_n$ by using the commutativity of the following diagram

$$\begin{array}{ccc}
\mathcal{J}F_n & \xrightarrow{J} & \mathcal{J}F_n \\
\downarrow & & \downarrow \\
\mathcal{J}F'_n & \xrightarrow{J'} & \mathcal{J}F'_n
\end{array}$$

(18)

where the vertical arrow denotes the adjoint map $ad_{\sigma}$:

$$ad_{\sigma} : \beta \mapsto \sigma_i^{-1} \beta \sigma_i \in \mathcal{J}F'_n, \quad \beta \in \mathcal{J}F_n$$

(19)

The above diagram makes possible to write

$$J' = \sigma_i^{-1} J \sigma_i$$

(20)

We get, using the braid relations:

$$(\sigma^{-1}_j J_i \sigma_j) J_j = (\sigma^{-1}_i J_j \sigma_i) J_i, \quad |i - j| \geq 2$$

(21)

and

$$[\sigma_i^{-1} (\sigma_i^{-1} J_i \sigma_1) \sigma_i] (\sigma_i^{-1} J_i \sigma_i) J_i = [\sigma_i^{-1} (\sigma_i^{-1} J_i \sigma_i) \sigma_i] (\sigma_i^{-1} J_i \sigma_i) J_i$$

(22)
and \((21)\) give, by inserting factors \(\sigma^{-1}\sigma\),
\[
\sigma_j^{-1}\sigma_i^{-1}\sigma_i J_i \sigma_j J_j = \sigma_i^{-1}\sigma_j^{-1}\sigma_j J_j \sigma_i J_i , \quad |i - j| \geq 2
\] (23)

and
\[
\sigma_i \sigma_i \sigma_i \sigma_i^{-1} \sigma_i J_i \sigma_i^{-1} \sigma_i J_i = \\
\sigma_i \sigma_i \sigma_i \sigma_i^{-1} \sigma_i J_i \sigma_i^{-1} \sigma_i J_i \sigma_i J_i \sigma_i J_i
\] (24)

The relations \((23, 24)\) can be written now by using again the braid relations as follows
\[
(\sigma_i J_i)(\sigma_j J_j) = (\sigma_j J_j)(\sigma_i J_i) , \quad |i - j| \geq 2
\] (25)
\[
(\sigma_i J_i)(\sigma_i J_i J_i)(\sigma_i J_i) = (\sigma_i J_i J_i)(\sigma_i J_i)(\sigma_i J_i J_i)
\] (26)

This shows that \(\sigma_i \rightarrow \sigma_i J_i\) is a linear representation of the braid group. By computing the matrix elements \(\frac{\partial \hat{\sigma}_i}{\partial \theta_i}\) of \(J_i\) following the rule of the free differential calculus, we get \((3)\). As an example take
\[
\frac{\partial \hat{\sigma}_i(\theta_i)}{\partial \theta_i} = \frac{\partial}{\partial \theta_i}(\theta_i \theta_i \theta_i^{-1}) = \frac{\partial \theta_i}{\partial \theta_i} + \theta_i \frac{\partial}{\partial \theta_i}(\theta_i \theta_i^{-1}) = \\
= 1 + \theta_i \theta_i \frac{\partial}{\partial \theta_i}(\theta_i^{-1}) = 1 - \theta_i \theta_i \theta_i
\] (27)

Since the representation of the braid group through the Artin right automorphism is faithful and \(\sigma_i^{-1} \theta_j \sigma_i\) for given \(i\) is an isomorphism of \(F_n\) on \(F'_n\), it follows that the braid valued representation \((3)\) is a linear faithful representation.

4 The coloured case and the pure braid group

In this section we apply the previous construction to the coloured case (representations of the coloured braid groupoid) and to the pure braid group.

We introduce first the coloured generators \(\sigma_i(\lambda, \mu)\) (where \(\lambda, \mu, \ldots\) can be looked as colours attached to strings) of the coloured braid groupoid \(B_n^c\); they satisfy the relations
\[
\sigma_i(\lambda, \mu) \sigma_{i+1}(\lambda, \nu) \sigma_i(\mu, \nu) = \sigma_{i+1}(\mu, \nu) \sigma_i(\lambda, \nu) \sigma_{i+1}(\lambda, \mu)
\] (28)
and
\[ \sigma_i(\lambda, \mu) \sigma_j(\nu, \rho) = \sigma_j(\nu, \rho) \sigma_i(\lambda, \mu), \quad \text{for}\quad |i - j| \geq 2 \]  

(29)

The inverses are introduced through the relation
\[ \sigma_i(\lambda, \mu) \sigma_i^{-1}(\mu, \lambda) = 1 \]  

(30)

The monodromies \( \theta_i \) are straightforwardly generalized to the coloured case as \( \theta_i^\rho(\ldots, \lambda, \mu) \), where \( \lambda, \mu \) are the last enclosed colours and \( \rho \) is the colour associated to \( \theta_i \):
\[ \theta_i^\rho(\ldots, \lambda, \mu) = \ldots \sigma_{i-2}^{-1}(\rho, \lambda) \sigma_{i-1}(\rho, \mu) \sigma_{i-1}(\mu, \rho) \sigma_{i-2}(\lambda, \rho) \ldots \]  

(31)

The considerations of section 3 can be generalized to the coloured braid groupoid and provide a matrix representation of the generators generalizing (3). The matrix \( B_c^i(\lambda, \mu) \) is obtained from (3) by replacing \( \sigma_i \rightarrow \sigma_i(\lambda, \mu) \), \( \theta_i \rightarrow \theta_i^\rho(\ldots, \mu, \lambda) \):
\[ (B_c^i)_{jk}(\ldots, \lambda, \mu) \equiv (B_c^i)_{jk}(\lambda, \mu) = \]
\[ 1_{(i-1,i-1)} \cdot \sigma_i(\lambda, \mu) \oplus \left( \begin{array}{cc} \alpha_i(\lambda, \mu) & \beta_i(\lambda, \mu) \\ \sigma_i(\lambda, \mu) & 0 \end{array} \right) \oplus 1_{n-2-i,n-2-i} \cdot \sigma_i(\lambda, \mu) \]  

(32)

where \( \alpha_i(\lambda, \mu), \beta_i(\lambda, \mu) \) can be easily read from (4).

In the particular case in which the generators \( \sigma_i(\lambda, \mu) \) are represented by \( q_i \in \mathbb{R} \), we get from the matrices \( B_c^i(\lambda, \mu) \) a representation of the coloured braid groupoid; when specialized to the generators of the pure braid group this representation coincides with the one discovered by Gassner [12]. It follows that the Gassner matrices satisfy not only the pure braid relations but also
\[ \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \left( \begin{array}{ccc} 1-t_{i+2} & t_i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \]  

(33)

and iterated relations of this kind, (see also [13]).

Equation (33) remembers the Yang-Baxter relation, but for the fact that it
lives on a direct sum instead of a tensor product. The problem of promoting a braid group representation from acting on a direct sum to a braid group representation on a tensor product space is of particular interest in the theory of quantum groups. A particular nice example was provided by L. Kauffman and H. Saleur [13] in connection with the quantum supergroup $U_q gl(1,1)$. The tensor product space is constructed in this case with the help of the exterior algebra over the direct sum space.

We come now to the pure braid group $P_n$. It can be algebraically defined as the subgroup of $B_n$ having generators

$$\theta^{(i)}_k = \sigma_i^{-1}...\sigma_{k-2}^{-1}\sigma_{k-1}^2\sigma_{k-2}...\sigma_i$$  \hspace{1cm} (34)

for $1 \leq i < k \leq n$ and defined relations which appear for instance in [12], p. 29. Here we will consider the pure braid group $P_{n+1}$ with the extra generators

$$\theta^{(0)}_k = \sigma_0^{-1}...\sigma_{k-2}^{-1}\sigma_{k-1}^2\sigma_{k-2}...\sigma_0$$ , \hspace{1cm} 1 < k \leq n

where $\sigma_0$ is the extra generator of $B_{n+1}$ as compared to $B_n$. The same procedure as before allows us to construct a representation of $P_{n+1}$ with matrices having entries in $J P_{n+1}$. It is convenient to express the $n \times n$ representative matrices $\hat{\theta}^{(i)}_k$ of the generators in terms of their action on the $n$ generators $y_j$ of a (right-) module over the ring $J P_{n+1}$. It turns out that:

i) For $j < i$ or $j > k$ we get

$$\hat{\theta}^{(i)}_k : y_j \mapsto \theta^{(i)}_k y_j$$

this relation being trivial.

ii)

$$\hat{\theta}^{(i)}_k : y_k \mapsto \theta^{(i)}_k (1 - \theta^{(0)}_i \theta^{(0)}_k \theta^{(0)}_i)y_i + \theta^{(i)}_k \theta^{(0)}_i y_k$$

iii) for $i \leq j < k$

$$\hat{\theta}^{(i)}_k : y_j \mapsto A_{(i,j,k)} y_j + B_{(i,j,k)} y_k + C_{(i,j,k)} y_i$$

where

$$A_{(i,j,k)} = \overline{\theta}^{(0)}_i \theta^{(i)}_k \theta^{(0)}_i$$

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\[ B_{(i,j,k)} = [1 - \overline{\theta}_i^{(0)} - \theta_j^{(0)} + \theta_j^{(0)}\overline{\theta}_i^{(0)}] \cdot \theta_k^{(i)}\theta_i^{(0)} \]

\[ C_{(i,j,k)} = [1 - \theta_j^{(0)} + \theta_j^{(0)}\overline{\theta}_i^{(0)}] \theta_k^{(i)} + \theta_j^{(0)}\theta_k^{(i)}\theta_i^{(0)} - \theta_k^{(i)}\theta_i^{(0)}\theta_j^{(0)} - \theta_i^{(0)}\theta_k^{(i)}\theta_j^{(0)}\overline{\theta}_i^{(0)} \]

In ii) and iii) the inverse of \( \theta_i^{(k)} \) has been denoted by \( \overline{\theta}_i^{(k)} \) for typographical reasons.

5 Applications

A geometric visualization of the representation (3) was given in [4]. We start by describing the connection of (3) with the realization of the braid group on analytic functions with isolated branch points singularities (for details see [4]). Let

\[ M_n^\geq = \{(z_1, ..., z_n) : |z_i| > |z_j|, \quad if \quad i < j, \quad -i\pi < arg z_k \leq i\pi \} \quad (35) \]

be a simply connected subset of \( \mathbb{C}^n \). Let \( \{f_j, j \in J\} \) be a family of holomorphic functions in \( M_n^\geq \). Singularities can appear only if two variables approach each other, \( z_i \to z_j \). We continue this function on the universal covering of \( M_n^\geq \) which is

\[ M_n = \{(z_1, ..., z_n) : \quad z_i \neq z_j \quad if \quad i \neq j \} \quad (36) \]

Choosing a point \( P \in M_n^\geq \) and denoting by \( \gamma \) a path in \( M_n \) starting at \( \gamma(0) = P \) and arriving at \( \gamma(1) = z \in M_n \), the expression \( f_j(z\gamma) \) denotes the (unique) analytic continuation of \( f_j \) from \( P \in M_n^\geq \) to \( z \in M_n \) along \( \gamma \). We introduce an action \( \tilde{\sigma}_i \) of the braid group \( B_n \) on functions \( f_j \) by

\[ (\tilde{\sigma}_i f_j)(z) = f_j(z, \gamma_i(z)) \quad (37) \]

for a path \( \gamma_i(z) \) in \( M_n \) running from \( P \) first interchanging \( P_i \) and \( P_{i+1} \) in positive direction and then connecting the resulting point to \( (z_1, ..., z_{i-1}, z_{i+1}, z_i, z_{i+2}, ..., z_n) = t_i(z) \) on a path in \( t_i M_n^\geq \).
The interesting point is that the Artin relations (2) can be realized on $\tilde{x}_i$ where

$$(\tilde{x}_i f_j)(z) = \int_{\gamma_i} f_j(t, z_1, ..., z_{n-1}) dt$$ (38)

with $\gamma_i$ a loop around $z_i$. Now the claim is that the relations

$$\tilde{x}_j \tilde{x}_i = \sum_k (B(i))_{jk} \tilde{x}_k$$ (39)

for $i = 1, 2, ..., n$ are verified [4], where $B(i)$ are the braid matrices (3). This allows us to get a representation of the braid groups on integrals if the braid representation on the integrands is known. The relation (38) can be understood as commuting the generators of the braid group through the integrals. Certainly this procedure can be iterated making possible a computation of the braid and monodromy properties of generalized hypergeometric integrals (in particular no “charge condition” is necessary). The graded structure which results from this construction as well as details and specific computations are contained in [4]. Let us remark that in the two-dimensional conformal perturbation theory braid properties of such generalized hypergeometric integrals without charge condition are needed. These problems were studied in [3]. Finally there are applications of the results and methods of this paper in the theory of quantum groups. A step in this direction was taken in [10, 11] where the vertex construction of Gomez and Sierra [14] was generalized and put in the framework of an abstract braid module. For some related ideas in the supergroup case $U_q gl(1, 1)$ see [13].

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