Existence of ground states for a modified nonlinear Schrödinger equation

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Abstract
In this paper we prove the existence of ground state solutions of the modified nonlinear Schrödinger equation:

$$-\Delta u + V(x)u - \frac{1}{2} \Delta u^2 = |u|^{p-1}u, \quad x \in \mathbb{R}^N, \quad N \geq 3,$$

under some hypotheses on \( V(x) \). This model has been proposed in the theory of superfluid films in plasma physics. As a main novelty with respect to some previous results, we are able to deal with exponents \( p \in (1, 3) \). The proof is accomplished by minimization under a convenient constraint.

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1. Introduction

Let us consider the following modified version of the nonlinear Schrödinger equation:

$$i \phi_t - \Delta \phi + W(x)\phi - \frac{1}{2} \Delta |\phi|^2 = f(x, \phi), \quad x \in \mathbb{R}^N,$$

where \( N \geq 3, \phi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C} \) and \( f(x, \phi) \) is a nonlinear term. This quasilinear version of the nonlinear Schrödinger equation arises in several models of different physical phenomena, such as in the study of superfluid films in plasma physics, in condensed matter theory, etc (see [9, 10, 14, 17, 19, 20, 23, 30, 38, 40]).

In this paper we restrict ourselves to the case \( g(\phi) = \phi \) and a power nonlinearity \( f(x, \phi) \):

$$i \phi_t - \Delta \phi + W(x)\phi - \frac{1}{2} \phi \Delta |\phi|^2 = |\phi|^{p-1} \phi, \quad x \in \mathbb{R}^N. \tag{1}$$

This equation has been introduced in [7, 8, 16] to study a model of self-trapped electrons in quadratic or hexagonal lattices (see also [5, 6]). In those references numerical and analytical results have been given.

From a mathematical point of view, the local existence for the Cauchy problem has been first considered in [22, 35], being later improved in [13]. See also [18] for a result concerning very general quasilinear Schrödinger equations. In [13] the orbital stability of stationary studies is included, including blow-up, an issue also considered in [15].
Here we are interested in the existence of standing waves. By taking \( \phi = e^{-iu}u(x) \) with \( u : \mathbb{R}^N \to \mathbb{R} \), we are led to the equation:

\[
- \Delta u + V(x)u - \frac{1}{2}u \Delta u^2 = |u|^{p-1} u, \tag{2}
\]

where \( V(x) = W(x) + \omega \).

We define \( X = \{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^2|\nabla u|^2 < +\infty \} = \{ u \in H^1(\mathbb{R}^N) : u^2 \in H^1(\mathbb{R}^N) \} \).

Then, \( u \in X \) is a weak solution of (2) if

\[
\int_{\mathbb{R}^N} (1 + u^2) \nabla u \cdot \nabla \psi + u|\nabla u|^2 \psi + V(x)u\psi - |u|^{p-1} u \psi = 0 \quad \forall \psi \in C_0^\infty(\mathbb{R}^N). \tag{3}
\]

Observe that this is a weak formulation of a quasilinear problem with the principal part in divergence form. In a certain sense, weak solutions are critical points of the functional \( I : X \to \mathbb{R} \) defined by

\[
I(u) = \frac{1}{2} \left( \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 + u^2|\nabla u|^2 \right) - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}. \tag{4}
\]

By using Sobolev embedding and interpolation we conclude that \( X \subset L^q(\mathbb{R}^N) \) for any \( q \in [2, \frac{4N}{N-2}] \). Indeed, the exponent \( p = \frac{4N}{N-2} - 1 = \frac{4N+2}{N-2} \) is critical with respect to the existence of solutions, see [27], for instance.

It seems quite clear that \( X \) is the right space for studying the functional (4). However, the term \( \int_{\mathbb{R}^N} u^2|\nabla u|^2 \) is not convex and \( X \) is not even a vector space. So, the usual min–max techniques cannot be directly applied to \( I \).

In the literature several papers have considered this problem. Maybe the first one is [36], where both the one dimensional and the radial cases are studied. In [28] the higher dimension case is considered. In both papers the proofs are based on a minimization technique on the set:

\[
\left\{ u \in X : \int_{\mathbb{R}^N} |u|^{p+1} = \mu \right\}.
\]

Therefore, a Lagrange multiplier appears in the equation. Precisely, the existence of solutions is provided for a sequence of different multipliers.

In [12, 26], through a convenient change of variables, (2) is transformed into a related semilinear problem. That semilinear problem is more tractable since one can apply the well-known arguments of [3]. By so doing, solutions are found if either \( p \geq 3 \) or \( V \) is constant.

Finally, in [27] the authors use a minimization on a Nehari-type constraint to get existence results. Their argument does not depend on any change of variables, so it can be applied to more general problems. Moreover, they also prove the existence of sign-changing solutions, which seems to be a delicate issue in this kind of problem. But again \( p \geq 3 \) is assumed. See also [1, 25, 29, 31–34, 41, 42] for related results.

So, there are only a few partial results in the case \( p \in (1, 3) \). As previously mentioned, in [28, 36] a Lagrange multiplier appears in the equation. In [12] an existence result is given but only for constant potentials \( V \) (so one can consider the radially symmetric case). Moreover, a specific change of variables is used.

In this paper we give an existence result for nonconstant \( V \) and \( p \in (1, \frac{3N+2}{N-2}) \). Moreover, we avoid the use of any change of variables, so our techniques could be of use in more general situations. The main result of the paper is the following theorem.

**Theorem 1.1.** Assume \( p \in (1, \frac{3N+2}{N-2}) \) and \( V \in C^1(\mathbb{R}^N) \) satisfying the following:

\begin{itemize}
  \item [(V1)] \( 0 < V_0 \leq V(x) \leq \lim_{|x| \to +\infty} V(x) = V_\infty < +\infty \),
  \item [(V2)] the function \( x \mapsto x \cdot \nabla V(x) \) belongs to \( L^\infty(\mathbb{R}^N) \),
  \item [(V3)] the map \( s \mapsto s \frac{\sqrt{\pi}}{\Gamma(\frac{N}{2})} V(s \frac{\sqrt{\pi}}{\Gamma(\frac{N}{2})} x) \) is concave for any \( x \in \mathbb{R}^N \).
\end{itemize}
Then, there exists \( u \in X \) a positive solution of (2). Moreover, \( u \) is a ground state, that is, its energy \( I(u) \) is minimal among the set of nontrivial solutions of (2).

Let us comment briefly on the conditions assumed on \( V(x) \). Hypothesis (V1) is quite usual in this kind of problems; in such a case \( V(x) \) is called a trapping potential. This condition will allow us, by using the concentration-compactness principle of Lions [24], to deal with the lack of compactness due to the effect of translations in \( \mathbb{R}^N \). Other typical assumptions are periodicity or \( \lim_{|x| \to +\infty} V(x) = +\infty \).

Condition (V2) is technical but not very restrictive since, whenever \( \lim_{|x| \to +\infty} x \cdot \nabla V(x) \) exists, it must be zero (by (V1)). Here we can also handle bounded oscillations at infinity of the function \( x \mapsto x \cdot \nabla V(x) \).

More restrictive is the concavity hypothesis (V3). It is used at a unique technical but essential point, in the proof of lemma 3.1. Obviously, constant functions satisfy (V3). We now give a nontrivial example where (V3) is verified. Take \( W \in C^2(\mathbb{R}^N) \) such that

(W1) \( W(x) \leq \lim_{|x| \to +\infty} W(x) = W_\infty < +\infty \),

(W2) both functions \( x \mapsto x \cdot \nabla W(x) \), \( x \mapsto D^2 W(x)[x, x] \) belong to \( L^\infty(\mathbb{R}^N) \).

Again, (W1) means that \( W(x) \) is a trapping potential and, as above, condition (W2) is not very restrictive. If (W1) holds and the limits \( \lim_{|x| \to +\infty} x \cdot \nabla W(x) \), \( \lim_{|x| \to +\infty} D^2 W(x)[x, x] \) exist, they must be zero and then (W2) is satisfied.

Then, one can easily check that there exists \( \omega_0 \in \mathbb{R} \) such that for any \( \omega > \omega_0 \), \( V(x) = W(x) + \omega \) satisfies (V1), (V2) and (V3). Moreover \( \omega_0 \) can be made explicit upon the \( L^\infty \) norms of \( W \) and the functions given in assumption (W2). Therefore we prove existence of standing waves for (1) under conditions (W1) and (W2) (for phases \( \omega > \omega_0 \)).

The proof is based, again, on a constrained minimization procedure. But here the constraint is not of Nehari-type; instead, we use a Pohozaev identity. As far as we know, this kind of argument appears for the first time in [43] in the study of stationary solutions of the nonlinear Klein–Gordon problem (see also [2, 39] for different applications).

Once a minimizer has been found, we need to show that it is indeed a solution. In order to prove that we follow the ideas of lemma 2.5 of [27] (in turn inspired by [11]).

The paper is organized as follows. In section 2 we establish some preliminary results and state theorems 2.1 and 2.2, from which theorem 1.1 follows. Section 3 is devoted to the proofs of theorems 2.1 and 2.2.

2. Preliminaries and statement of the results

In this section we begin the study of (2). After some preliminaries, we show the minimization process that will be used in the proofs. Finally, we state theorems 2.1 and 2.2.

First of all, some comments are in order. In the appendix of [27] it is proved that weak solutions are bounded in \( L^\infty(\mathbb{R}^N) \). We point out that their arguments (based on Moser and De Giorgi iterations) work also for \( p \in (1, 3) \).

As mentioned in the introduction, problem (2) is a quasilinear elliptic equation with the principal part in divergence form. A density argument shows that the weak formulation (3) holds also for test functions in \( H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \). By [21] (see theorems 5.2 and 6.2 in chapter 4) it follows that \( u \in C^{1,\alpha} \). From Schauder theory we conclude that \( u \in C^{2,\alpha} \) is a classical solution of (2).

Moreover, if \( u \in X \) is a solution, \( u, Du, D^2 u \) have an exponential decay as \( |x| \to +\infty \) (see again the appendix in [27] and take into account the previous regularity discussion).
Finally, in [27] a Pohozaev identity for equation (2) is mentioned. Let us make it explicit. Just by applying proposition 2 of [37], assume that \( u \in X \) is a \( C^2 \) solution of (2) such that

\[
|\nabla u|^2 + V(x)u^2 + u^2|\nabla u|^2 + |u|^{p+1}, \quad \frac{|u|}{1+|x|}(|\nabla u| + u^2|\nabla u|) \in L^1(\mathbb{R}^N).
\]

Then, for any \( a \in \mathbb{R} \), \( u \) satisfies the identity:

\[
\left( \frac{2 - N}{2} + a \right) \int_{\mathbb{R}^N} |\nabla u|^2 + \left( a - \frac{N}{2} \right) \int_{\mathbb{R}^N} V(x)u^2 - \frac{1}{2} \int_{\mathbb{R}^N} \nabla V(x) \cdot x u^2
\]

\[
+ \left( 2a + \frac{2 - N}{2} \right) \int_{\mathbb{R}^N} |\nabla u|^2 u^2 + \left( \frac{N}{p+1} - a \right) \int_{\mathbb{R}^N} |u|^{p+1} = 0.
\]

We briefly check conditions (5). The first condition being obvious for any \( u \in X \), we consider the second one. By using Hölder and Hardy inequalities:

\[
\int_{\mathbb{R}^N} \frac{|u|}{1+|x|} |\nabla u| \leq \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|} \right)^{1/2} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \right)^{1/2} \leq C \int_{\mathbb{R}^N} |\nabla u|^2 < +\infty
\]

and

\[
\int_{\mathbb{R}^N} \frac{u^2}{1+|x|} |\nabla u| \leq \left( \int_{\mathbb{R}^N} \frac{u^4}{|x|^2} \right)^{1/2} \left( \int_{\mathbb{R}^N} u^2 |\nabla u|^2 \right)^{1/2} \leq C \int_{\mathbb{R}^N} |\nabla u|^2 < +\infty.
\]

It is worth saying a few words on the space \( X \):

\[X = \{ u \in H^1(\mathbb{R}^N) : u^2 \in H^1(\mathbb{R}^N) \}\]

where \( H^1(\mathbb{R}^N) \) is the usual Sobolev space. As mentioned in the introduction, \( X \) is not a vector space (it is not closed under the sum), nevertheless it is a complete metric space with distance:

\[d_X(u, v) = ||u - v||_{H^1} + ||\nabla u^2 - \nabla v^2||_{L^2}.
\]

It is easy to check that \( I \) is continuous on \( X \). Moreover, for any \( \psi \in C_0^\infty(\mathbb{R}^N) \) and \( u \in X \), \( u + \psi \in X \), and we can compute the Gateaux derivative:

\[
(I'(u), \psi) = \int_{\mathbb{R}^N} (1 + u^2)\nabla u \cdot \nabla \psi + u|\nabla u|^2 \psi + V(x)u\psi - |u|^{p-1}u\psi.
\]

Therefore, \( u \in X \) is a solution of (2) if and only if the Gateaux derivative of \( I \) along any direction in \( C_0^\infty(\mathbb{R}^N) \) vanishes.

For any \( u \in X \) we hereafter denote by \( u_t \) the map:

\[\mathbb{R}^+ \ni t \mapsto u_t \in X, \quad u_t(x) = tu(t^{-1}x).
\]

It is an exercise to check that \( t \mapsto u_t \) is indeed a continuous curve in \( X \) (for example, use Brezis–Lieb lemma [4]). Computing the functional on that curve we find

\[
f_u(t) := I(u_t) = \frac{t^N}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{t^{N+2}}{2} \int_{\mathbb{R}^N} V(t x)u^2
\]

\[
+ \frac{t^{N+2}}{2} \int_{\mathbb{R}^N} |\nabla u|^2 u^2 - \frac{t^{N+p+1}}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}.
\]

Since \( p + 1 > 2 \), we get that

- \( f_u(t) > 0 \) for \( t > 0 \) sufficiently small,
- \( \lim_{t \to +\infty} f_u(t) = -\infty \).
This implies that \( f_u \) attains its maximum. Moreover, thanks to (V2), \( f_u : \mathbb{R}^+ \to \mathbb{R} \) is \( C^1 \) and
\[
f_u'(t) = \frac{N}{2} t^{N-1} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{N + 2}{2} t^{N+1} \int_{\mathbb{R}^N} V(tx) u^2 + \frac{N + 1}{p} t^{N+1} \int_{\mathbb{R}^N} |u|^{p+1}.
\]

This motivates the following definition:
\[
M = \{ u \in X \setminus \{ 0 \} : J(u) = 0 \}
\]
where \( J : X \to \mathbb{R} \) is defined as
\[
J(u) = \frac{N}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{N + 2}{2} \int_{\mathbb{R}^N} (V(x)u^2 + |\nabla u|^2 u^2)
\]
\[
+ \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x) \cdot x u^2) - \frac{N + p + 1}{p + 1} \int_{\mathbb{R}^N} |u|^{p+1}.
\]

In other words, \( M \) is the set of functions \( u \) such that \( f'_u(1) = 0 \). Moreover, note that for any \( t, s \in \mathbb{R}^+ \), \( u_t = (ut)_s \), and so \( f_u(s) = f_u(ts) \). By taking the derivative at \( s = 1 \) we deduce that \( f_u'(1) = t f_u'(t) \) for any \( t > 0 \). Of course this inequality can also be checked by making the computations. Observe also that \( M \) is nothing but the set of functions \( u \in X \) such that the Pohozaev identity (6) holds for \( a = -1 \). Then, all solutions belong to \( M \).

We now state theorems 2.1 and 2.2, which will be proved in the next section.

**Theorem 2.1.** Define \( m := \inf_M I \). Then \( m \) is positive and is achieved at some \( u \in M \).

**Theorem 2.2.** The minimizer \( u \) given by theorem 2.1 is a ground state solution of equation (2). Moreover, it is positive (up to a change of sign).

### 3. Proof of the main results

In order to prove theorems 2.1 and 2.2, we need several auxiliary results. The proofs of the theorems will be given in two final subsections.

**Lemma 3.1.** For any \( u \in X \setminus \{ 0 \} \), the map \( f_u \) defined in (7) attains its maximum at exactly one point \( t_u \). Moreover, \( f_u \) is positive and increasing for \( 0 < t < t_u \) and decreasing for \( t > t_u \). Finally,
\[
m = \inf_{u \in X, u \neq 0} \max_{t>0} I(ut).
\]

**Proof.** First of all, by making the change of variable \( s = t^{N+p+1} \), we get
\[
f_u(s) = \frac{1}{2} s^{\frac{N}{N+p+1}} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{s^{\frac{N}{N+p+1}}}{2} \int_{\mathbb{R}^N} (V(s^{\frac{1}{N+p+1}} x)u^2 + |\nabla u|^2 u^2) - \frac{s}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}.
\]
By assumption (V3) (and this is the only but essential point in which (V3) is used!) this is a concave function. We already know that it attains its maximum; let \( t^a \) be the unique point at which this maximum is achieved. Then \( t^a \) is the unique critical point of \( f_a \) and \( f_a \) is positive and increasing for \( 0 < t < t^a \) and decreasing for \( t > t^a \).

In particular, for any \( u \neq 0 \), \( t^u \in \mathbb{R} \) is the unique value such that \( u_{t^u} \) belongs to \( M \), and \( I(u_{t^u}) \) reaches a global maximum for \( t = t^u \). This completes the proof. \( \square \)

The first claim of theorem 2.1 is proved here; it is indeed a consequence of a suitable comparison argument.
Lemma 3.2. There holds $m > 0$.

This readily implies that $d_X(M, 0) > 0$.

Proof. We introduce the functional
\[
\tilde{I}(u) = \frac{1}{2} \int_{\mathbb{R}^n} (|\nabla u|^2 + V_0 u^2 + u^2 |\nabla u|^2) - \frac{1}{p + 1} \int_{\mathbb{R}^n} |u|^{p+1},
\]
where $V_0$ comes from $(V1)$. Obviously, $\tilde{I}(u) \leq I(u)$, and this implies that
\[
m := \inf_{u \in X, u \neq 0} \max_{t > 0} \tilde{I}(u_t) = \inf_{u \in X, u \neq 0} \max_{t > 0} I(u) = m.
\]

It suffices then to prove that $\tilde{m} > 0$. Let us define
\[
\tilde{M} = \{u \in X \setminus \{0\} : g_u(1) = 0\} \quad \text{where } g_u(t) = \tilde{I}(u_t).
\]

By lemma 3.1 applied to $V \equiv V_0$ (actually a more direct proof can be given in this case), we know that
\[
\tilde{m} = \inf_{u \in \tilde{M}} \tilde{I}(u).
\]

For any $u \in \tilde{M}$,
\[
\frac{N + 2}{2} V_0 \int_{\mathbb{R}^n} u^2 + \frac{N + 2}{2} \int_{\mathbb{R}^n} |\nabla u|^2 u^2 \leq \frac{N + 2}{2} V_0 \int_{\mathbb{R}^n} u^2 + C \int_{\mathbb{R}^n} |u|^{\frac{2p}{p-1}}
\]
for a suitable constant $C > 0$. So, by using the Sobolev inequality,
\[
\frac{N + 2}{2} \int_{\mathbb{R}^n} |\nabla u|^2 u^2 \leq C \int_{\mathbb{R}^n} |u|^{\frac{2p}{p-1}} \leq C' \left( \int_{\mathbb{R}^n} |\nabla u|^2 u^2 \right)^{\frac{p}{2-p}}
\]
and this shows that $\int_{\mathbb{R}^n} |\nabla u|^2 u^2$ is bounded away from zero on $\tilde{M}$.

We conclude since the functional $\tilde{I}$ restricted to $\tilde{M}$ has the expression
\[
\tilde{I}(u) = \frac{1}{2} \int_{\mathbb{R}^n} (u^2 + |\nabla u|^2 + V_0 u^2) - \frac{1}{p + 1} \int_{\mathbb{R}^n} |u|^{p+1}
\]
\[
\geq c \int_{\mathbb{R}^n} (u^2 + |\nabla u|^2 + u^2 |\nabla u|^2).
\]

Proposition 3.3. There exists $c > 0$ such that for any $u \in \tilde{M},$
\[
I(u) \geq c \int_{\mathbb{R}^n} (u^2 + |\nabla u|^2 + u^2 |\nabla u|^2).
\]

Proof. Take $u \in \tilde{M}$ and choose $t \in (0, 1)$. We compute
\[
I(u_t) - t^{N+p+1} I(u) = \left( \frac{t^N}{2} - \frac{t^{N+p+1}}{2} \right) \int_{\mathbb{R}^n} |\nabla u|^2 + \left( \frac{t^{N+2}}{2} - \frac{t^{N+p+1}}{2} \right) \int_{\mathbb{R}^n} |\nabla u|^2 u^2
\]
\[
+ \frac{1}{2} \int_{\mathbb{R}^n} t^{N+2} V(tx) - \frac{t^{N+p+1}}{2} V(x) u^2.
\]

Observe that $V(tx) \geq V_0 \geq \delta V_\infty \geq \delta V(x)$, for some positive constant $\delta \in (0, 1)$ depending only on $V_0$ and $V_\infty$. By choosing a smaller $t$, if necessary, we get that
\[
\frac{t^{N+2}}{2} V(tx) - \frac{t^{N+p+1}}{2} V(x) \geq \left( \frac{\delta t^{N+2}}{2} - \frac{t^{N+p+1}}{2} \right) V(x) \geq \gamma
\]
for a positive fixed constant $γ > 0$. Recall that, by lemma 3.1, $I(u_t) ≤ I(u)$; by taking a smaller $γ$, if necessary, 

\[
(1 - t^{N+p+1})I(u) ≥ I(u_t) - t^{N+p+1}I(u) ≥ γ \int_{\mathbb{R}^N} (u^2 + |\nabla u|^2 + u^2|\nabla u|^2).
\]

We conclude by defining $c = \frac{γ}{1 - t^{N+p+1}}$. □

### 3.1. Proof of theorem 2.1

Take a sequence $u_n \in M$ so that $I(u_n) → m$. By proposition 3.3, both $u_n$ and $u_n^2$ are bounded in $H^1(\mathbb{R}^N)$. Passing to a convenient subsequence, both $u_n$ and $u_n^2$ converge weakly in $H^1(\mathbb{R}^N)$ and also pointwise. Therefore,

$$u_n \rightharpoonup u \quad \text{and} \quad u_n^2 \rightharpoonup u^2 \quad \text{in} \quad H^1(\mathbb{R}^N).$$

This implies, in particular, that $\{u_n\}$ is bounded in $L^{p+1}(\mathbb{R}^N)$.

The proof proceeds in several steps.

**Step 1:** \(\int_{\mathbb{R}^N} |u_n|^{p+1} \rightharpoonup 0\).

Let us recall the expression of $I$:

\[
I(u_n) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2 + u_n^2|\nabla u_n|^2) - \frac{1}{p+1} \int_{\mathbb{R}^N} |u_n|^{p+1} \quad \text{for any} \quad t > 1,
\]

so that $m ≥ I((u_n)_t)$

\[
= \frac{t^N}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 + \frac{t^{N+2}}{2} \int_{\mathbb{R}^N} V(tx)u_n^2 + \frac{t^{N+2}}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 u_n^2 - \frac{t^{N+p+1}}{p+1} \int_{\mathbb{R}^N} |u_n|^{p+1}
\]

\[
≥ \frac{t^N}{2} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V_0u_n^2 + |\nabla u_n|^2 u_n^2) - \frac{t^{N+p+1}}{p+1} \int_{\mathbb{R}^N} |u_n|^{p+1}
\]

\[
≥ \frac{t^N}{2} \delta - \frac{t^{N+p+1}}{p+1} \int_{\mathbb{R}^N} |u_n|^{p+1},
\]

where $δ$ is a fixed constant. It suffices to take $t > 1$ so that $\frac{t^N}{t} > 2m$ to get a lower bound for $\int_{\mathbb{R}^N} |u_n|^{p+1}$, proving step 1.

So, we can assume (passing to a subsequence, if necessary) that

\[
\int_{\mathbb{R}^N} |u_n|^{p+1} \rightharpoonup A \in (0, ∞).
\]

**Step 2:** Splitting by concentration compactness.

In this step we use in an essential way the concentration-compactness principle. We recall here the following result due to Lions (lemma I.1 of [24], part 2):

**Lemma 3.4.** Let $1 < r ≤ ∞$, $1 ≤ q < ∞$ with $q ≠ \frac{Nr}{N-r}$ if $N < r$. Assume that $ψ_n$ is bounded in $L^q(\mathbb{R}^N)$, $−\nabla ψ_n$ is bounded in $L^r(\mathbb{R}^N)$ and

\[
\sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |ψ_n|^q \rightarrow 0 \quad \text{for some} \quad R > 0.
\]

Then $ψ_n \rightharpoonup 0$ in $L^q(\mathbb{R}^N)$ for any $α \in (q, \frac{N r}{N-r})$.
We apply the previous lemma to \( \psi_n = u_n^2 \), \( q = \frac{p+1}{2} \) and \( r = 2 \); by step 1, \( \psi_n \) does not converge to 0 in \( L^q(\mathbb{R}^N) \). By interpolation, this implies that \( \psi_n \) does not converge to 0 in \( L^s(\mathbb{R}^N) \) for any \( s \in (1, \frac{2N}{2N-2}) \).

Then, there exists \( \delta > 0 \) and \( \{x_n\} \subset \mathbb{R}^N \) such that
\[
\int_{B_{x_n}(l)} |u_n|^{p+1} > \delta > 0.
\]

Fix \( \varepsilon > 0 \) and take \( \rho > \max\{1, \varepsilon^{-1}\} \), \( \eta_R(t) \) a smooth function defined on \([0, +\infty)\) such that
\begin{enumerate}[(a)]  
  \item \( \eta_R(t) = 1 \) for \( 0 \leq t \leq \rho \),  
  \item \( \eta_R(t) = 0 \) for \( t \geq 2\rho \),  
  \item \( \eta_R'(t) \leq 2/\rho \).  
\end{enumerate}

Define
\[
v_n(x) = \eta_R(|x - x_n|)u_n(x) \quad \text{and} \quad w_n(x) = (1 - \eta_R(|x - x_n|))u_n(x).
\]

Clearly \( v_n \) and \( w_n \) belong to \( X \) and \( u_n = v_n + w_n \). Observe that in particular
\[
\liminf_{n \to +\infty} \int_{B_{x_n}(\rho)} |u_n|^{p+1} \geq \delta. 
\]

**Step 3:** There exist constants \( C > 0 \), \( \tau > 0 \) independent of \( \varepsilon \) and \( n_0 = n_0(\varepsilon) \) such that
\[
\|v_n\|_{H^1} + \|w_n\|_{H^1} \leq C\varepsilon \quad \text{for all} \quad n \geq n_0.
\]

In the rest of the proof we denote by \( C \) certain positive constants, that may change from one expression to another, but all of them are independent of \( \varepsilon \) and \( n \).

Define \( z_n = u_n(\cdot + x_n) \). Clearly, \( z_n \rightharpoonup z \) and \( z_n^2 \rightharpoonup z^2 \) (both weak convergences are understood in \( H^1(\mathbb{R}^N) \)). By taking a larger \( R \), if necessary, we can assume that
\[
\int_{A_0(R, 2R)} |z|^{p+1} < \varepsilon, 
\]
where \( A_0(R, 2R) \) denotes the annulus centred in 0 with radii \( R \) and \( 2R \). Then, for \( n \) large enough,
\[
\left| \int_{\mathbb{R}^N} |u_n|^{p+1} - \int_{\mathbb{R}^N} |v_n|^{p+1} - \int_{\mathbb{R}^N} |w_n|^{p+1} \right| \leq 3\varepsilon. \tag{10}
\]

Since \( |\nabla z_n|^2 \) is uniformly bounded in \( L^1(\mathbb{R}^N) \), up to a subsequence, \( |\nabla z_n|^2 \) converges (in the sense of measure) to a certain positive measure \( \mu \) with \( \mu(\mathbb{R}^N) < +\infty \). By enlarging \( R \), if necessary, we can assume that \( \mu(A_0(R, 2R)) \leq \varepsilon \). Then, for \( n \) large enough,
\[
\int_{\mathbb{R}^N} |\nabla u_n|^2 \eta_R(|x - x_n|)(1 - \eta_R(|x - x_n|)) \, dx < \varepsilon.
\]

Taking this into account, straightforward computations show that for \( n \) large enough,
\[
\left| \int_{\mathbb{R}^N} |\nabla u_n|^2 - \int_{\mathbb{R}^N} |\nabla v_n|^2 - \int_{\mathbb{R}^N} |\nabla w_n|^2 \right| = \left| 2 \int_{\mathbb{R}^N} \nabla w_n \cdot \nabla v_n \right| 
\leq \frac{C}{R} + 2\varepsilon \leq C\varepsilon. \tag{11}
\]

Arguing as before, possibly choosing a larger \( R \), we get also
\[
\left| \int_{\mathbb{R}^N} |\nabla u_n|^2 u_n^2 - \int_{\mathbb{R}^N} |\nabla v_n|^2 v_n^2 - \int_{\mathbb{R}^N} |\nabla w_n|^2 w_n^2 \right| \leq C\varepsilon \tag{12}
\]
and finally
\[
\left| \int_{\mathbb{R}^N} V(tx)u_n^2 - \int_{\mathbb{R}^N} V(tx)v_n^2 - \int_{\mathbb{R}^N} V(tx)w_n^2 \right| \leq C\varepsilon. \tag{13}
\]
Putting together (10), (11), (12) and (13) we get that for \( n \) sufficiently large and \( t > 0 \),

\[
| I((u_n)_t) - I((v_n)_t) - I((w_n)_t) | \leq C \varepsilon (t^N + t^{N+1}).
\]

(14)

Now let \( t^{v_n} \) and \( t^{w_n} \) be the positive values which maximize \( f_{v_n}(t) \) and \( f_{w_n}(t) \), respectively, namely,

\[
I((v_n)_{t^{v_n}}) = \max_{t > 0} I((v_n)_t) \quad \text{and} \quad I((w_n)_{t^{w_n}}) = \max_{t > 0} I((w_n)_t).
\]

First, let us assume that \( t^{v_n} \leq t^{w_n} \) (the other case will be treated later). Then,

\[
I((v_n)_{t^{v_n}}) \geq 0 \quad \text{for} \quad t \leq t^{v_n}.
\]

(15)

The next aim is to find suitable bounds for the sequence \( \{ t^{v_n} \} \).

**Claim:** There exist \( 0 < \tilde{t} < 1 < \bar{t} \) independent of \( \varepsilon \) such that \( t^{v_n} \in (\tilde{t}, \bar{t}) \).

Indeed, take \( \bar{t} = ((p+1)A^{-1}B)^{\frac{1}{p+1}} \), where \( A \) comes from (8) and \( B \) is large enough such that \( \bar{t} > 1 \) and moreover

\[
B \geq \int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} V_{\infty} u_n^2 + \int_{\mathbb{R}^N} |\nabla u_n|^2 u_n^2.
\]

(16)

Then

\[
I((u_n)_{\bar{t}}) \leq \frac{\bar{t}^{N+1}}{2} \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} V(\bar{t}x) u_n^2 + \int_{\mathbb{R}^N} |\nabla u_n|^2 u_n^2 - \frac{2}{p+1} \bar{t}^{p-1} \int_{\mathbb{R}^N} |u_n|^{p+1} \right)
\]

\[
\leq -B \frac{\bar{t}^{N+2}}{2} < 0.
\]

By (14)

\[
I((u_n)_{\bar{t}}) \geq I((v_n)_{\bar{t}}) + I((w_n)_{\bar{t}}) - C\varepsilon \quad \forall t \in (0, \bar{t}].
\]

(17)

Then, taking a smaller \( \varepsilon \) if necessary,

\[
I((v_n)_{\bar{t}}) + I((w_n)_{\bar{t}}) < 0.
\]

Then \( I((v_n)_{\bar{t}}) < 0 \) or \( I((w_n)_{\bar{t}}) < 0 \). In any case lemma 3.1 implies that \( t^{v_n} \leq \bar{t} \) (recall that we are assuming \( t^{v_n} \leq t^{w_n} \)).

For the lower bound, take \( \tilde{t} = (\frac{m}{B})^{1/N} \), where \( B \) is chosen as in (16). Note that \( \tilde{t} < 1 \). For any \( t \leq \tilde{t} \),

\[
I((u_n)_{\tilde{t}}) \leq \frac{\tilde{t}^N}{2} \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} V(\tilde{t}x) u_n^2 + \int_{\mathbb{R}^N} |\nabla u_n|^2 u_n^2 \right)
\]

\[
\leq \frac{m}{2B} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V_{\infty} u_n^2 + |\nabla u_n|^2 u_n^2) \leq \frac{m}{2}.
\]

By (15) and (17),

\[
I((u_n)_{t^{v_n}}) \geq I((v_n)_{t^{v_n}}) + I((w_n)_{t^{v_n}}) - C\varepsilon \geq m - C\varepsilon
\]

(18)

and the right-hand side can be made greater then \( m/2 \), by choosing a small \( \varepsilon \). We conclude that \( t^{v_n} \geq \bar{t} \) and the claim is proved.

Since \( u_n \in M \), \( f_a \) reaches its maximum at \( t = 1 \). Then,

\[
m \Leftarrow I(u_n) \geq I((u_n)_{t^{w_n}})
\]
and using (18) we deduce, for \( n \) large, \( I((u_n)_t) \leq 2C\varepsilon \) for all \( t \in (0, t^{v_n}) \). Moreover, for any \( t \in (0, \tilde{t}) \):

\[
2C\varepsilon \geq I((u_n)_t) \geq \frac{\varepsilon^{N+2}}{2} \left[ \int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} V(tx)u_n^2 + \int_{\mathbb{R}^N} |\nabla u_n|^2 w_n^2 \right] - \frac{\varepsilon^{N+p+1}}{p+1} \int_{\mathbb{R}^N} |w_n|^{p+1} \geq \frac{\varepsilon^{N+2}}{2} q_n - Dt^{N+p+1}
\]

where

\[
q_n = \int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} u_n^2 + \int_{\mathbb{R}^N} |\nabla u_n|^2 w_n^2
\]

is bounded (independently of \( \varepsilon \)) and \( D > A \). Observe that \( \frac{\varepsilon^{N+2}}{2} q_n - Dt^{N+p+1} = \frac{\varepsilon^{N+2}}{2} q_n \) for \( t = (\frac{q_n}{2D})^{\frac{1}{p+1}} \). By taking a larger \( D \) we can assume that \( (\frac{q_n}{2D})^{\frac{1}{p+1}} \leq \tilde{t} \). With this choice of \( t \), we obtain

\[
2C\varepsilon \geq I((u_n)_t) \geq \left( \frac{q_n}{4D} \right)^{\frac{p+1}{2}} \frac{q_n}{4D} \geq c q_n^{\frac{p+1}{2}}.
\]

In other words,

\[
\|u_n\|_{H^1} + \|u_n^2\|_{H^1} \leq C\varepsilon \frac{\varepsilon^{N+1}}{\varepsilon^{\frac{p+1}{2}}} \quad \text{for some } C > 0 \text{ independent of } \varepsilon. \quad (19)
\]

In the case \( t^{v_n} > t^{w_n} \), we can argue analogously to conclude that \( \|v_n\|_{H^1} + \|v_n^2\|_{H^1} \leq C\varepsilon \frac{\varepsilon^{N+1}}{\varepsilon^{\frac{p+1}{2}}} \) for some \( C > 0 \). But, choosing small \( \varepsilon \), this contradicts (9), so (19) holds. This completes the proof of step 3.

We are now in a position to conclude the proof of theorem 2.1.

**Step 4:** The infimum of \( I \mid_{\mathcal{M}} \) is achieved.

Recall, see step 2, that \( z_n = u_n + x_n \), \( z_n \rightharpoonup z \) and \( z_n^2 \rightharpoonup z^2 \) (both weak convergences are in \( H^1(\mathbb{R}^N) \)). Moreover, by compactness, we have that \( z_n \rightharpoonup z \) in \( L^2_{\text{loc}}(\mathbb{R}^N) \). Finally, observe that \( z \neq 0 \) since, by (9),

\[
\delta < \liminf_{n \to +\infty} \int_{\mathbb{R}^N} |v_n|^{p+1} \leq \liminf_{n \to +\infty} \int_{B(0,2R)} |z_n|^{p+1} = \int_{B(0,2R)} |z|^{p+1}.
\]

Recall also that \( u_n = v_n + w_n \), with \( \|u_n\|_{H^1} + \|u_n^2\|_{H^1} \leq C\varepsilon \frac{\varepsilon^{N+1}}{\varepsilon^{\frac{p+1}{2}}} \). In the following estimate we use Hölder inequality to get

\[
\int_{\mathbb{R}^N} |u_n^2 - v_n^2| \leq \int_{\mathbb{R}^N} |u_n| |(|u_n| + |v_n|)| \leq \left( \int_{\mathbb{R}^N} u_n^2 \right)^{1/2} \left( \int_{\mathbb{R}^N} (|u_n| + |v_n|)^2 \right)^{1/2} \leq C\varepsilon \frac{\varepsilon^{N+1}}{\varepsilon^{\frac{p+1}{2}}}.
\]

(20)

On the other hand,

\[
\int_{\mathbb{R}^N} v_n^2 \leq \int_{B(0,2R)} z_n^2 \to \int_{B(0,2R)} z^2 \leq \int_{\mathbb{R}^N} z^2.
\]

Combining this estimate with (20), we obtain that

\[
\liminf_{n \to +\infty} \int_{\mathbb{R}^N} z_n^2 = \liminf_{n \to +\infty} \int_{\mathbb{R}^N} u_n^2 \leq \int_{\mathbb{R}^N} z^2 + C\varepsilon \frac{\varepsilon^{N+1}}{\varepsilon^{\frac{p+1}{2}}}.
\]

Since \( \varepsilon \) is arbitrary, we get that \( z_n \rightharpoonup z \) in \( L^2(\mathbb{R}^N) \) and, by interpolation, \( z_n \rightharpoonup z \) in \( L^q(\mathbb{R}^N) \) for all \( q \in [2, \frac{4N}{N-2}) \).
We discuss two cases:

**Case 1:** \( \{ x_n \} \) is bounded. Assume, passing to a subsequence, that \( x_n \to x_0 \). In this case \( u_n \to u, u_n^2 \to u^2 \) (both weak convergences are in \( H^1 \)), \( u_n \to u \) strongly in \( L^q(\mathbb{R}^N) \) for any \( q \in [2, \frac{4N}{N-2}) \), where \( u = z(\cdot - x_0) \).

In the following we just need to recall the expression of \( I((u_n)_t) \), see (7):

\[
m = \lim_{n \to +\infty} I(u_n) \geq \liminf_{n \to +\infty} I((u_n)_t) \geq I(u_t) \quad \forall t > 0.
\]

So, max, \( I(u_t) = m \) and \( u_n \to u, u_n^2 \to u^2 \) (both convergences are now strong in \( H^1(\mathbb{R}^N) \)). In particular, \( u \in M \) is a minimizer of \( I|_M \).

**Case 2:** \( \{ x_n \} \) is unbounded. In this case, by Lebesgue convergence theorem and condition (V1):

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^n} V(t(x)u_n^2(x)) \, dx = \lim_{n \to +\infty} \int_{\mathbb{R}^n} V(t(x + x_0))z_n^2(x) \, dx
\]

\[
= V_\infty \int_{\mathbb{R}^n} z^2 \geq \int_{\mathbb{R}^n} V(t(x))z^2(x) \, dx = \lim_{n \to +\infty} \int_{\mathbb{R}^n} V(t(x))z_n^2(x) \, dx
\]

for any \( t > 0 \) fixed. Therefore,

\[
m = \lim_{n \to +\infty} I(u_n) \geq \liminf_{n \to +\infty} I((u_n)_t) \geq I(z_t) \quad \forall t > 0.
\]

So, taking \( t^* \) so that \( f(t^*) = I(z_t) \) reaches its maximum, we get that \( z_{t^*} \in M \) and is a minimizer of \( I|_M \).

Having a minimum of \( I|_M \), the fact that it is indeed a solution of our equation, is based on a general idea used in [27].

### 3.2. Proof of theorem 2.2

Let \( u \in M \) be a minimizer of the functional \( I|_M \). Recall that, by lemma 3.1, \( I(u) = \inf_{v \in X, \sigma \neq 0} \max_{t > 0} \int_{\mathbb{R}^n} I(v_t) = m \).

We argue by contradiction by assuming that \( u \) is not a weak solution of (2). In such a case, we can choose \( \phi \in C_c^\infty(\mathbb{R}^N) \) such that

\[
\langle I'(u), \phi \rangle = \int_{\mathbb{R}^n} \nabla u \nabla \phi + \int_{\mathbb{R}^n} V(x)u\phi + \int_{\mathbb{R}^n} \nabla(u^2) \nabla(u\phi) - \int_{\mathbb{R}^n} |u|^{p-1} u\phi < -1.
\]

Then we fix \( \varepsilon > 0 \) sufficiently small such that

\[
\langle I'(u, + \sigma \phi), \phi \rangle \leq -1, \quad \forall |t - 1|, |\sigma| \leq \varepsilon
\]

and introduce a cut-off function \( 0 \leq \eta \leq 1 \) such that \( \eta(t) = 1 \) for \( |t - 1| \leq \varepsilon/2 \) and \( \eta(t) = 0 \) for \( |t - 1| \geq \varepsilon \).

We perturb the original curve \( u_t \) by defining, for \( t \geq 0 \)

\[
\gamma(t) = \begin{cases} u_t, & \text{if } |t - 1| \geq \varepsilon, \\
u_t + \varepsilon \eta(t) \phi, & \text{if } |t - 1| < \varepsilon. 
\end{cases}
\]

Note that \( \gamma(t) \) is a continuous curve in the metric space \((X, d)\) and, eventually choosing a smaller \( \varepsilon \), we obtain that \( d_X(\gamma(t), 0) > 0 \) for \( |t - 1| < \varepsilon \).

**Claim:** \( \sup_{t \geq 0} I(\gamma(t)) < m \).

Indeed, if \( |t - 1| \geq \varepsilon \), then \( I(\gamma(t)) = I(u_t) < I(u) = m \). If \( |t - 1| < \varepsilon \), by using the mean value theorem to the \( C^1 \) map:

\[
[0, \varepsilon] \ni \sigma \mapsto I(u_t + \sigma \eta(t) \phi) \in \mathbb{R},
\]
we find, for a suitable $\bar{\sigma} \in (0, \varepsilon),$
\[
I(u_t + \varepsilon \eta(t)\phi) = I(u_t) + \left( I(u_t + \bar{\sigma} \eta(t)\phi), \eta(t)\phi \right) \leq I(u_t) - \frac{1}{2} \eta(t) < m
\]
where in the first inequality we have used (21).

To conclude observe that $J(\gamma(1 - \varepsilon)) > 0$ and $J(\gamma(1 + \varepsilon)) < 0.$ By the continuity of the map $t \mapsto J(\gamma(t))$ there exists $t_0 \in (1 - \varepsilon, 1 + \varepsilon)$ such that $J(\gamma(t_0)) = 0.$ Namely, $\gamma(t_0) = u_{t_0} + \varepsilon \eta(t_0)\phi \in M$ and $I(\gamma(t_0)) < m;$ this gives the desired contradiction.

So far we have proved that the minimizer of $I \mid_M$ is a solution. Since any solution of (2) belongs to $M$ (see section 2), the minimizer is a ground state.

Moreover, consider $u \in M$ a minimizer of $I \mid_M.$ Then, the absolute value $|u| \in M$ is also a minimizer, and hence a solution. By the classical maximum principle (recall that solutions are $C^2), |u| > 0.$

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