TORSION THEORY OF COHERENT FUNCTORS

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ABSTRACT. Let $\mathcal{C}$ be an additive category with cokernels and let $\text{Mod}(\mathcal{C})$ be the category of additive functors from $\mathcal{C}^{\text{op}}$ to the category $\text{Ab}$ of abelian groups. Let $\text{mod}(\mathcal{C})$ be the full subcategory of $\text{Mod}(\mathcal{C})$ consisting of coherent functors. In this paper, we first study some basic properties of pseudo-kernels of morphisms in $\mathcal{C}$. When $\mathcal{C}$ has pseudo-kernels, $\text{mod}(\mathcal{C})$ is abelian and then, in this case, we study radical functors, half exact functors, left exact functors and injective objects in $\text{mod}(\mathcal{C})$. At last, we extend the results for $\text{Mod}(\mathcal{C})$.

1. Introduction

The notion of coherent first time in 1940 was introduced by Cartan [C] and Oka [O] on several complex variables. Serre in 1955 in his famous paper [S], showed that the same notions could be carried over to algebraic geometry and since then coherent sheaves and their cohomology have been ubiquitous in algebraic geometry. In 1962, Auslander [A], developed the notion coherent in functors category. Hartshorne [H], explained the general theory of coherent functors on the category of finitely generated modules over a noetherian ring, in order to study coherent sheaves on projective space. The various characterization of functors category was given by Krause [K1]. He investigated certain subcategories which are defined in terms of coherent functors.

Throughout this paper we assume that $\mathcal{C}$ is an additive category with cokernels. We denote by $\text{Mod}(\mathcal{C})$ the category of additive functors $F : \mathcal{C}^{\text{op}} \to \text{Ab}$ which is an abelian category. A functor $F : \mathcal{C}^{\text{op}} \to \text{Ab}$ in $\text{Mod}(\mathcal{C})$ is called coherent if there exists an exact sequence $\text{Hom}_\mathcal{C}(-, A) \to \text{Hom}_\mathcal{C}(-, B) \to F \to 0$ of functors such that $A$ and $B$ are objects in $\mathcal{C}$. We denote by $\text{mod}(\mathcal{C})$, the full category of $\text{Mod}(\mathcal{C})$ which consists coherent functors. By Yoneda lemma for a functor $F$ with the presentation $\text{Hom}_\mathcal{C}(-, A) \to \text{Hom}_\mathcal{C}(-, B) \to F \to 0$, there is a correspondence morphism $A \to B$ and if we set $v(F) = \text{Coker}(A \to B)$, then $v : \text{mod}(\mathcal{C}) \to C$ is an additive functor and according to [K2, Lemma 2.8], if $\mathcal{C}$ has cokernels, then $v$ has a right adjoint functor $h_\mathcal{C}$ by the assignment $h_\mathcal{C}(C) = \text{Hom}_\mathcal{C}(-, C)$. We denote by $\text{mod}(\mathcal{C})$ a full subcategory of $\text{mod}(\mathcal{C})$ consisting of all functors $F$ such that $v(F) = 0$.

In section 2, we find a characterization of functors in $\text{mod}(\mathcal{C})$ and we show that $(\text{mod}(\mathcal{C}), \mathcal{D})$ is a torsion theory in which $\mathcal{D}$ is a full subcategory of $\text{mod}(\mathcal{C})$ consisting of all functors which send epimorphisms to monomorphisms (cf. Proposition 2.1 and Proposition 2.2). We recall that a morphism $B \to A$ is a pseudo-kernel of a morphism $A \to C$ in $\mathcal{C}$ provided that the induced sequence of functors $\text{Hom}_\mathcal{C}(-, B) \to \text{Hom}_\mathcal{C}(-, A) \to \text{Hom}_C(-, C)$ is exact. We study some basic properties of pseudo-kernels which have a key role in the other sections. We show that $v$ preserves pseudo-kernels if and only if $\mathcal{C}$ is abelian (cf. Theorem 2.3).

In section 3, we assume that $\mathcal{C}$ has pseudo-kernels and so by [F, Theorem 1.4], $\text{mod}(\mathcal{C})$ is an abelian category. A full subcategory $\mathcal{X}$ of $\text{mod}(\mathcal{C})$ is called Serre if it is closed under subobject, quotients and extension. In this section we assume that $\text{mod}(\mathcal{C})$ is a Serre subcategory of $\text{mod}(\mathcal{C})$. We show that a functor $G \in \text{mod}(\mathcal{C})$ is left exact if and only if $\text{Ext}_{\text{mod}(\mathcal{C})}^i(F, G) = 0$ for $i = 0, 1$ and all $F \in \text{mod}(\mathcal{C})$ (cf. Proposition 3.2). We prove that for any functor $F \in \text{mod}(\mathcal{C})$, there exist a unique left exact functor $L(F)$ and a morphism $\eta_F : F \to L(F)$ such that $\text{Ker} \eta_F$ and $\text{Coker} \eta_F$ are in $\text{mod}(\mathcal{C})$ and so we define a radical functor on $\text{mod}(\mathcal{C})$ with $r(F) = \text{Ker} \eta_F$ for any $F \in \text{mod}(\mathcal{C})$ (cf. Proposition 3.3). We show that $r : \text{mod}(\mathcal{C}) \to \text{mod}(\mathcal{C})$ is left exact and $r : \text{mod}(\mathcal{C}) \to \text{mod}(\mathcal{C})$

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is a right adjoint functor of the inclusion functor. We denote by $\mathcal{F}_r$ a subclass of $\text{mod}(C)$ consisting of all functors $F$ such that $r(F) = 0$ and we show that $D = \mathcal{F}_r$ and if $C$ has kernels, every $F \in \mathcal{F}_r$ has projective dimension $\leq 1$ (cf. Theorem 3.6). Finally, in Theorem 3.7 we prove that $r(F) = 0$ if and only if $F$ has a presentation $\text{Hom}_C(\cdot, A) \xrightarrow{\partial} \text{Hom}_C(\cdot, B) \xrightarrow{\pi} F \to 0$ such that $v(\partial)$ is pseudo-kernel of $v(\pi)$ for any functor $F \in \text{mod}(C)$.

In section 4, we study half exact functors in $\text{mod}(C)$. A functor $F$ in $\text{mod}(C)$ is said to be half exact if for any exact sequence $A \to B \to C \to 0$ of objects of $\mathcal{C}$ with $A \to B$ a pseudo-kernel of $B \to C$, the sequence $F(C) \to F(B) \to F(A)$ is exact. We show that a functor $G$ is half exact if and only if $\text{Ext}^1_{\text{mod}(C)}(F, G) = 1$ for all $F \in \text{mod}(C)$. In particular if $\text{mod}(C)$ is Serre and $G$ is half exact, then so is $r(G)$. For any functors $F$ and $G$ in $\text{mod}(C)$, there is a homomorphism $\theta_{F,G} : F(v(G)) \to \text{Hom}_{\text{mod}(C)}(F, G)$ which is natural in $F$ and $G$ and we show that $F \in D$ if and only if $\theta_{F,G}$ is monic for all $G \in \text{mod}(C)$ (cf. Theorem 4.4 and Corollary 4.5). We also show that $F$ is half exact if and only if $\theta_{F,G}$ is epic for all $G \in \text{mod}(C)$; in particular, $\theta_{F,G}$ is isomorphism for all $G \in \text{mod}(C)$ if and only if $F$ is left exact (cf. Corollary 4.6). In the rest of this section we study injective objects in $\text{mod}(C)$. In Proposition 5.3, we prove that the subcategory $\text{mod}(C)$ is abelian and $\text{mod}(C)$ is exact. We show that a functor $F$ in $\text{mod}(C)$ is Serre and if there exist objects $F \in \text{mod}(C)$ and $L$-preenvelopes.

2. CATEGORIES WITH COKERNELS

Let $\mathcal{U}$ be a universe in the sense of Grothendieck [G]. We say that a category $\mathcal{C}$ is a $\mathcal{U}$-category if the objects and the morphisms of $\mathcal{C}$ are both sets which are elements of $\mathcal{U}$ (for details see [A] and [G]). Throughout this paper $\mathcal{C}$ is an additive $\mathcal{U}$-category and we denote by $\text{Mod}(C)$ the category of additive functors from $C^{op}$ to the category $\text{Ab}$ of abelian groups. For any objects $A, B$ in $\mathcal{C}$, the set of all morphisms from $A$ to $B$ is denoted $\text{Hom}_C(A, B)$ which is an abelian group. It follows from Yoneda lemma that $\text{Hom}_C(\cdot, B)$ is a projective object of $\text{Mod}(C)$ for any object $B$ in $\mathcal{C}$.

Let $\mathcal{C}$ be an additive category. A functor $F$ in $\text{Mod}(C)$ is called coherent if there exist objects $A$ and $B$ in $\mathcal{C}$ such that and there is an exact sequence of functors $\text{Hom}_C(A, \cdot) \to \text{Hom}_C(B, \cdot) \to F \to 0$. We denote by $\text{mod}(C)$ a full subcategory of $\text{Mod}(C)$ consisting of coherent functors.

Using Yoneda’s lemma, there exists a full and faithful functor $h_C : \text{mod}(C) \to C, A \mapsto \text{Hom}_C(\cdot, A)$ and according to [K2, Lemma 2.8], if $C$ has cokernels, then $h_C$ has a left adjoint functor $v : \text{mod}(C) \to C$. It follows from [K2, Universal Property 2.1] that $v = 1_C$ is right exact and $v h_C = 1_C$. To be more precise, if $F \in \text{mod}(C)$, then there is a presentation $\text{Hom}(\cdot, X) \to \text{Hom}(\cdot, Y) \to F \to 0$ and we define $v(F) = \text{Coker}(X \to Y)$. We also define a full subcategory $\text{mod}(C)$ of $\text{mod}(C)$ which is $\text{mod}(C) = \{F \in \text{mod}(C) | v(F) = 0\}$. It is straightforward to show that $\text{mod}(C)$ is closed under quotients and extensions. Throughout this section $\mathcal{C}$ has cokernels. We start this section by a results about $\text{mod}(C)$.

Proposition 2.1. Let $F \in \text{mod}(C)$. Then the following are equivalent.

(i) $F \in \text{mod}(C)$. 

(ii) there exists an epimorphism $A_2 \to A_1$ such that $(-, A_2) \to (-, A_1) \to F \to 0$ is exact in $\text{mod}(\mathcal{C})$.

(iii) $\text{Hom}_{\text{mod}(\mathcal{C})}(F, G) = 0$ for those $G \in \text{mod}(\mathcal{C})$ which send epimorphisms to monomorphisms.

Proof. (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are clear. (iii) $\Rightarrow$ (i) Since $v$ is a left adjoint of $h$ and by the assumption we have $\text{Hom}_\mathcal{C}(v(F), v(F)) \cong \text{Hom}_{\text{mod}(\mathcal{C})}(F, h_\mathcal{C}v(F)) = 0$ which implies that $v(F) = 0$.

Let $\mathcal{D}$ be a subcategory of $\text{mod}(\mathcal{C})$ consisting of all functors which send epimorphisms to monomorphisms. Then we have the following proposition.

Proposition 2.2. $(\text{mod}(\mathcal{C}), \mathcal{D})$ is a torsion theory.

Proof. Assume that $G \in \text{mod}(\mathcal{C})$ such that $\text{Hom}_{\text{mod}(\mathcal{C})}(F, G) = 0$ for all functors $F \in \text{mod}(\mathcal{C})$. For any epimorphism $A \to B$ in $\mathcal{C}$, there exists $F \in \text{mod}(\mathcal{C})$ such that the following sequence is exact

$\text{Hom}_\mathcal{C}(-, A) \to \text{Hom}_\mathcal{C}(-, B) \to F \to 0$.

According to Yoneda lemma, $G(B) \to G(A)$ is monomorphism and so the proof completes by using Proposition 2.4. □

We recall that a morphism $B \to A$ is a pseudo-kernel of a morphism $A \to C$ in $\mathcal{C}$ provided that the induced sequence of functors $\text{Hom}_\mathcal{C}(-, B) \to \text{Hom}_\mathcal{C}(-, A) \to \text{Hom}_\mathcal{C}(-, C)$ is exact. This concept introduced by Freyd and he proved that $\mathcal{C}$ has pseudo-kernels iff $\text{mod}(\mathcal{C})$ is abelian (cf. [F, Theorem 1.4]). The pseudo-cokernel is defined dually. In the rest of this section we assume that $\mathcal{C}$ has pseudo-kernels. We study some basic properties for this category.

Proposition 2.3. Cokernel of pseudo-kernels of any morphism of $\mathcal{C}$ is unique up to isomorphisms.

Proof. Assume that $\omega_1 : \Omega_1 \to A$ and $\omega_2 : \Omega_2 \to A$ are pseudo-kernels of $\gamma : A \to B$. By the definition there exist morphisms $\theta_1 : \Omega_1 \to \Omega_2$ and $\theta_2 : \Omega_2 \to \Omega_1$ such that $\omega_1 \theta_2 = \omega_2$ and $\omega_2 \theta_1 = \omega_1$. Assume that $\beta_1 : A \to C_1$ are cokernels of $\omega_i$ for $i = 1, 2$. Since $\beta_1 \omega_2 = \beta_2 \omega_1 \theta_2 = 0$ and $\beta_2 \omega_1 = \beta_2 \omega_2 \theta_1 = 0$, there exist the morphisms $\alpha_1 : C_1 \to C_2$ and $\alpha_2 : C_2 \to C_1$ such that $\alpha_1 \alpha_2 = 1_{C_2}$ and $\alpha_2 \alpha_1 = 1_{C_1}$. □

Lemma 2.4. If $\omega : \Omega \to A$ is a pseudo-kernel of a morphism, then $\omega$ is a pseudo-kernel of $\text{Coker} \omega$.

Proof. Assume that $\omega$ is pseudo-kernel of $\gamma : A \to B$ and $\alpha : A \to C$ is the cokernel of $\omega$. Then there exists a morphism $\theta : C \to B$ such that $\theta \alpha = \gamma$. Assume that $\delta : X \to A$ is any morphism such that $\alpha \delta = 0$. Then $\gamma \delta = 0$; and hence there is a morphism $\eta : W \to X$ such that $\delta \eta = \omega$. □

Lemma 2.5. If any morphism of $\mathcal{C}$ factors through an epimorphism and a monomorphism and $\mathcal{C}$ has pseudo-kernels, then it has kernels.

Proof. Assume that $\varphi : Y \to Z$ is a morphism and $\psi : X \to Y$ is a pseudo-kernel of $\varphi$. Then there is a factorization of $\psi$ as $X \xrightarrow{i} I \xrightarrow{j} Y$ where $i$ is monomorphism. It is straightforward to show that $i$ is a pseudo-kernel of $\varphi$ and since $i$ is monomorphism, it is kernel of $\varphi$. □

The following theorem shows that when a category having cokernels and pseudo-kernels is abelian.

Theorem 2.6. Let $\mathcal{C}$ have pseudo-kernels. Then the following conditions are equivalent:

(i) The functor $v$ preserves pseudo-kernels.

(ii) The functor $v : \text{mod}(\mathcal{C}) \to \mathcal{C}$ is left exact.

(iii) $\mathcal{C}$ is abelian.
Proof. (i) ⇐ (ii) follows from [K2, Lemma 2.6]. (ii)⇒ (iii). Assume that $A \rightarrow B$ is a morphism in $C$ and assume that $F$ is the kernel of $\theta : \text{Hom}_C(-, A) \rightarrow \text{Hom}_C(-, B)$. Since $v$ is left exact, it preserves monomorphisms and pseudo-kernels; hence $v(F)$ is the kernel of $v(\theta) = A \rightarrow B$. On the other hand, since $\text{mod}(C)$ is abelian, we have $\text{Coker Ker} \theta \cong \text{Ker Coker} \theta$ and since $v$ is exact, we have $\text{Coker Ker} v(\theta) \cong \text{Ker Coker} v(\theta)$. Further, since $\text{mod}(C)$ is abelian, it has finite products, and since $v$ is exact and an additive functors, it is easy to prove that $C$ has finite product. (iii)⇐(ii) follows from [A].

Let $f : C \rightarrow D$ be a functor between additive categories. We denote by $f^* : \text{mod}(C) \rightarrow \text{mod}(D)$ the unique right exact functor extending $f$. By [K2, Universal Property 2.1] such a functor always exists.

Proposition 2.7. Let $C$ and $D$ be categories with cokernels and $f : C \rightarrow D$ be an additive functor. Then $f$ preserves epimorphisms iff $f^*(\text{Ker}(v_D)) \subseteq \text{Ker}(v_D)$. Moreover, if $C$ has kernels, $D$ has pseudo-kernels and $f$ preserves pseudo-kernels, then $f$ preserves epimorphisms iff $f$ is exact.

Proof. Assume that $f$ preserves epimorphisms and $F \in \text{Ker}(v_D)$. Then there exists an exact sequence $\text{Hom}_C(-, B) \rightarrow \text{Hom}_C(-, A) \rightarrow F \rightarrow 0$ such that $B \rightarrow A$ is an epimorphism in $C$. In view of the assumption $f(B) \rightarrow f(A)$ is epimorphism in $D$ so that $f^*(F) = \text{Coker}(\text{Hom}_D(-, f(B))) \rightarrow \text{Hom}_D(-, f(A)) \in \text{Ker}(v_D)$. The proof of the converse is similar. To prove the second claim, using [K2, Lemma 2.6], $f$ is left exact and so it suffices to show that it is right exact. Given an exact sequence $0 \rightarrow B \rightarrow A \rightarrow C \rightarrow 0$, by the assumption $f(B) \rightarrow f(A)$ is pseudo-kernel of $f(A) \rightarrow f(C)$ and hence $\text{Hom}_D(-, f(A)) \rightarrow \text{Hom}_D(-, f(B)) \rightarrow \text{Hom}_D(-, f(C))$ is exact. On the other hand, $F = \text{Coker}(\text{Hom}_C(-, A) \rightarrow \text{Hom}_C(-, C)) \in \text{Ker}(v_D)$ and so using the first assertion, $f(B) \rightarrow f(A) \rightarrow f(C) \rightarrow 0$ is exact. The converse is clear.

3. Radical functors

A full subcategory $\mathcal{X}$ of $\text{mod}(C)$ is said to be Serre if it is closed under quotients, subobjects and extensions. Throughout this section we assume that $C$ has pseudo-kernels and cokernels and $\text{mod}(C)$ is a Serre subcategory of $\text{mod}(C)$

Lemma 3.1. Let $B \rightarrow A$ be an epimorph of $C$. If $C \rightarrow B$ is a pseudo-kernel of $B \rightarrow A$, then $C \rightarrow B \rightarrow A \rightarrow 0$ is exact (i.e $B \rightarrow A = \text{Coker}(C \rightarrow B)$).

Proof. According to Proposition 2.1 there exists an exact sequence $\text{Hom}_C(-, C) \rightarrow \text{Hom}_C(-, B) \rightarrow \text{Hom}_C(-, A) \rightarrow F \rightarrow 0$ of functors such that $F \in \text{mod}(C)$. If $B \rightarrow D$ is the cokernel of $C \rightarrow B$, then $B \rightarrow A$ factors through an epimorphism $D \rightarrow A$. It follows from Lemma 2.3 that $C \rightarrow B$ pseudo-kernel of $B \rightarrow D$ and so there exists $F_0 \in \text{mod}(C)$ such that the following diagram with exact rows commutes

$$
\begin{array}{cccccccc}
\text{Hom}_C(-, C) & \rightarrow & \text{Hom}_C(-, B) & \rightarrow & \text{Hom}_C(-, D) & \rightarrow & F_0 & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\text{Hom}_C(-, C) & \rightarrow & \text{Hom}_C(-, B) & \rightarrow & \text{Hom}_C(-, A) & \rightarrow & F & \rightarrow 0.
\end{array}
$$

Hence there is an isomorphism $\text{Ker} \gamma \cong \text{Ker} \theta$ and since $\text{mod}(C)$ is Serre, $\text{Ker} \theta \in \text{mod}(C)$. Hence Proposition 2.2 implies that $\text{Ker} \gamma = 0$ so that the epimorphism $D \rightarrow A$ is isomorphism.

We now find a characterization of left exact functors.

Proposition 3.2. Let $G \in \text{mod}(C)$. Then $\text{Ext}^i_{\text{mod}(C)}(F, G) = 0$ for $i = 0, 1$ and all $F \in \text{mod}(C)$ if and only if $G$ is left exact.
Proof. Assume that $\operatorname{Ext}^i_{\mod (C)}(F,G) = 0$ for $i = 0,1$ and all $F \in \mod (C)$. There exists a morphism $A_1 \to A_0$ and an exact sequence of functors $\operatorname{Hom}_C(-,A_1) \to \operatorname{Hom}_C(-,A_0) \to G \to 0$. Assume that $A_2 \to A_1$ is a pseudo-kernel of $A_1 \to A_0$ and also assume that $B = \operatorname{Coker}(A_2 \to A_1)$. Then $v(G) = \operatorname{Coker}(B \to A_0)$ and by virtue of Lemma 2.1, the morphism $A_2 \to A_1$ is a pseudo-kernel of $A_1 \to B$. Therefore, we have the following commutative diagram with the exact rows

$$
\begin{array}{cccccccccc}
\operatorname{Hom}_C(-,A_2) & \operatorname{Hom}_C(-,A_1) & \operatorname{Hom}_C(-,B) & G_0 & 0 \\
\operatorname{Hom}_C(-,A_2) & \operatorname{Hom}_C(-,A_1) & \operatorname{Hom}_C(-,A_0) & G & 0 \\
& & \operatorname{Hom}_C(-,v(G)) & & & & \\
\end{array}
$$

where by using Proposition 2.1 we have $G_0 \in \mod (C)$. Since $\mod (C)$ is Serre, we have $\operatorname{Ker} \theta \in \mod (C)$ and an isomorphism $\operatorname{Ker} \gamma \cong \operatorname{Ker} \theta$. Then it follows from Proposition 2.1 that $\operatorname{Ker} \theta = \operatorname{Ker} \gamma = 0$. We observe that $\operatorname{Hom}_C(-,B) \to \operatorname{Hom}_C(-,A_0)$ is monic so that the induced morphism $B \to A_0$ is monic. If we put $F = \operatorname{Coker}(\operatorname{Hom}_C(-,B) \to \operatorname{Hom}_C(-,A_0))$, then $v(G) = v(F) = \operatorname{Coker}(B \to A_0)$ so that $0 \to \operatorname{Hom}_C(-,B) \to \operatorname{Hom}_C(-,A_2) \to \operatorname{Hom}_C(-,v(F))$ is exact. Then using the above diagram, we have an exact sequence $0 \to G_0 \to G \to \operatorname{Hom}_C(-,v(G)) \to G_1 \to 0$ where $G_1 = \operatorname{Coker}(\operatorname{Hom}_C(-,A_0) \to \operatorname{Hom}_C(-,v(G)))$ and so using Proposition 2.1 it belongs to $\mod (C)$. Since $G_0 \in \mod (C)$, the assumption implies that $G_0 = 0$ and $0 \to G \to \operatorname{Hom}_C(-,v(G)) \to G_1 \to 0$ splits so that $G_1 = 0$. Conversely, assume that $G$ is left exact and $F \in \mod (C)$. Then there exist objects $B$ and $C$ and an epimorphism $\alpha : B \to C$ such that $\operatorname{Hom}_C(-,B) \to \operatorname{Hom}_C(-,C) \to F \to 0$ is exact. If $K$ is the pseudo-kernel of $\alpha$, it follows from Lemma 3.1 that $K \to B \to C \to 0$ is exact and so there is an exact sequence of functors $\operatorname{Hom}_C(-,K) \to \operatorname{Hom}_C(-,B) \to \operatorname{Hom}_C(-,C) \to F \to 0$. Now, application of the functor $\operatorname{Hom}_C(-,G)$ and using the assumption and Yoneda lemma, the result follows.

Proposition 3.3. For any $F \in \mod (C)$, there exists a left exact functor $L(F)$ and a morphism $\eta_F : F \to L(F)$ such that $\operatorname{Ker} \eta_F$ and $\operatorname{Coker} \eta_F$ are in $\mod (C)$. Furthermore, $L(F)$ and $\eta_F$ with the mentioned property is unique up to isomorphism.

Proof. By a similar proof of Proposition 3.2 there exists an exact sequence of functors $E(F) : 0 \to F_0 \to F \xrightarrow{\eta \phi} \operatorname{Hom}_C(-,v(F)) \to F_1 \to 0$ such that $F_0$ and $F_1$ are in $\mod (C)$. Then we consider $L(F) = h_C(v(F)) = \operatorname{Hom}_C(-,v(F))$. A similar proof of [A, Proposition 3.4], shows that the above exact sequence is unique up to isomorphism.

A preradical functor $r$ of $\mod (C)$ is a subfunctor of $1_{\mod (C)}$. The preradical $r$ is called idempotent if $r^2 = r$ and it is called radical if for any functor $F$ in $\mod (C)$ we have $r(F/r(F)) = 0$. If $\mod (C)$ is Serre, then we define a preradical of $\mod (C)$ as $r(F) := \operatorname{Ker} \eta_F$ for any $F$ in $\mod (C)$. We observe that $\eta(-) : (-) \to L(-)$ is a natural transform. To be more precise, if $f : F \to G$ is a morphism of functors in $\mod (C)$, then applying the functor $\operatorname{Hom}_C(-,L(G))$ to $E(F)$, there exist unique morphisms $L(f) : L(F) \to L(G)$ and $r(F) \to r(G)$ such that the following diagram is commutative

$$
\begin{array}{cccccccccc}
0 & \to & r(F) & \to & F & \xrightarrow{\eta_F} & L(F) & \to & \operatorname{Coker} \eta_F & \to & 0 \\
0 & \to & r(G) & \to & G & \xrightarrow{\eta_G} & L(G) & \to & \operatorname{Coker} \eta_G & \to & 0 \\
\end{array}
$$

Clearly, $r$ is an idempotent preradical functor. We also define the torsion-free class $T_r$ a subclass of $\mod (C)$ consisting of those $F \in \mod (C)$ which $\eta_F$ is monic; or equivalently $r(F) = 0$.

Lemma 3.4. The functor $r : \mod (C) \to \mod (C)$ is a right adjoint of the inclusion functor $i : \mod (C) \to \mod (C)$.
Proof. For any $G \in \text{mod}(\mathcal{C})$ and $F \in \text{mod}(\mathcal{C})$, applying $\text{Hom}_{\text{mod}(\mathcal{C})}(G, -)$ to the exact sequence

$$0 \rightarrow r(F) \rightarrow F \xrightarrow{\eta_F} h_{\mathcal{C}}v(F) \rightarrow \text{Coker} \eta_F \rightarrow 0$$

we deduce that $\text{Hom}_{\text{mod}(\mathcal{C})}(G, r(F)) = \text{Hom}_{\text{mod}(\mathcal{C})}(G, r(F)) \cong \text{Hom}_{\text{mod}(\mathcal{C})}(G, F)$ so that $r : \text{mod}(\mathcal{C}) \rightarrow \text{mod}(\mathcal{C})$ is the right adjoint functor of the inclusion functor $i : \text{mod}(\mathcal{C}) \rightarrow \text{mod}(\mathcal{C})$. \hfill $\square$

Since $r$ is a right adjoint of $i$, the functor $r : \text{mod}(\mathcal{C}) \rightarrow \text{mod}(\mathcal{C})$ is left exact and since $h_{\mathcal{C}}$ is a right adjoint of $v$, $r(F)$ is the largest subfunctor of $F$ contained in $\text{mod}(\mathcal{C})$.

Lemma 3.5. The functor $r : \text{mod}(\mathcal{C}) \rightarrow \text{mod}(\mathcal{C})$ is left exact. In particular, $r$ is radical.

Proof. Let $0 \rightarrow F_1 \xrightarrow{\phi} F_2 \xrightarrow{\theta} F_3 \rightarrow 0$ is an exact sequence in $\text{mod}(\mathcal{C})$ and so there is an sequence of functors $0 \rightarrow r(F_1) \xrightarrow{r(\phi)} r(F_2) \xrightarrow{r(\theta)} r(F_3)$. We notice that $r(\phi)$ is monic and if $K$ is the kernel of $r(\theta)$, since $\text{mod}(\mathcal{C})$ is Serre, we have $K \in \text{mod}(\mathcal{C})$ and there is a monomorphism $\alpha : r(F_1) \rightarrow K$; also there is a monomorphism $h : K \rightarrow F_1$ such that $h \circ \alpha = i_{F_1} : r(F_1) \rightarrow F_1$ is the inclusion functor. Since $r(F_1)$ is the largest subfunctor of $F_1$ contained in $\text{mod}(\mathcal{C})$, $\alpha$ is an isomorphism. In order to prove the second assertion, applying the functor $r$ to the exact sequence $0 \rightarrow F/r(F) \rightarrow h_{\mathcal{C}}v(F)$ yields that $r(F/r(F)) = 0$. \hfill $\square$

The following theorem shows that the subclass $\mathcal{F}_r$ coincides with the class $\mathcal{D}$ mentioned in Section 2.

Theorem 3.6. Let $F$ be a functor in $\mathcal{C}$ and consider the following conditions.
(i) $F \in \mathcal{F}_r$.
(ii) $F \in \mathcal{D}$.
(iii) $\text{pd} F \leq 1$. Then (i) and (ii) are equivalent. The implication (iii) $\Rightarrow$ (i) holds and if $\mathcal{C}$ has kernels, then the implication (iii) $\Rightarrow$ (i) holds too.

Proof. (i)$\Rightarrow$ (ii). There exists an exact sequence of functors $0 \rightarrow F \xrightarrow{\eta_F} (-, v(F))$ in $\text{mod}(\mathcal{C})$. For any exact sequence of objects $A \rightarrow B \rightarrow 0$ in $\mathcal{C}$, we have $\text{Hom}(f, v(F))\eta_F(B) = \eta_F(A)F(f)$ which implies that $F(f)$ is monic. (ii)$\Rightarrow$ (i). According to Proposition 2.2 we have $\text{Hom}_{\mathcal{C}}(r(F), F) = 0$ and so the exact sequence $0 \rightarrow r(F) \rightarrow F \rightarrow \text{Hom}_{\mathcal{C}}(-, v(F))$ implies that $r(F) = 0$. (iii)$\Rightarrow$ (i). If $\text{pd} F = 0$, then $F$ is a direct summand of $\text{Hom}_{\mathcal{C}}(-, A)$ where $A$ is an object of $\mathcal{C}$. It follows from Proposition 2.1 that $(r(\text{Hom}_{\mathcal{C}}(-, A)), \text{Hom}_{\mathcal{C}}(-, A)) = 0$ which implies that $r(\text{Hom}_{\mathcal{C}}(-, A)) = 0$; and hence $r(F) = 0$. If $\text{pd} F = 1$, then there exists an exact sequence of functors $0 \rightarrow \text{Hom}_{\mathcal{C}}(-, A) \rightarrow \text{Hom}_{\mathcal{C}}(-, B) \rightarrow F \rightarrow 0$ which yields an exact sequence $0 \rightarrow A \rightarrow B \rightarrow v(F) \rightarrow 0$ of object of $\mathcal{C}$. Application of $\text{Hom}_{\mathcal{C}}(-, -)$ induces the following exact sequence of functors

$$0 \rightarrow \text{Hom}_{\mathcal{C}}(-, A) \rightarrow \text{Hom}_{\mathcal{C}}(-, B) \rightarrow \text{Hom}_{\mathcal{C}}(-, v(F))$$

which induces an exact sequence $0 \rightarrow F \rightarrow \text{Hom}_{\mathcal{C}}(-, v(F))$ of functors in $\mathcal{C}$. Application of the functor $\text{Hom}_{\mathcal{C}}(r(F), -)$ and using Proposition 2.1 yield $r(F) = 0$. (i)$\Rightarrow$ (iii). Since $r(F) = 0$, there is the exact sequence $0 \rightarrow F \xrightarrow{\eta_F} \text{Hom}_{\mathcal{C}}(-, v(F))$ of functors and since $F \in \text{mod}(\mathcal{C})$, there is an exact sequence of functors $\text{Hom}_{\mathcal{C}}(-, A) \xrightarrow{\alpha} \text{Hom}_{\mathcal{C}}(-, B) \xrightarrow{\beta} F \rightarrow 0$. Assume that $f : C \rightarrow B$ is the kernel the morphism $B \rightarrow v(F)$. Then there exists a unique morphism $g : A \rightarrow C$ such that $f \circ g = v(\alpha)$. Further, there is an exact sequence $0 \rightarrow \text{Hom}_{\mathcal{C}}(-, C) \xrightarrow{\gamma} \text{Hom}_{\mathcal{C}}(-, B) \rightarrow \text{Hom}_{\mathcal{C}}(-, v(F))$ in $\text{mod}(\mathcal{C})$ where $v(\gamma) = f$. We have the following commutative diagram with exact rows where $v(\beta) = g$

$$
\begin{array}{ccc}
\text{Hom}_{\mathcal{C}}(-, A) & \xrightarrow{\alpha} & \text{Hom}_{\mathcal{C}}(-, B) \\
\downarrow{\beta} & & \downarrow{1_{\text{Hom}_{\mathcal{C}}(-, B)}} \\
0 & \xrightarrow{\gamma} & \text{Hom}_{\mathcal{C}}(-, B) \\
\downarrow{0} & & \downarrow{\eta_F} \\
0 & \xrightarrow{0} & \text{Hom}_{\mathcal{C}}(-, v(F)) \\
\end{array}
$$
Since, $\eta_F$ is monic, $\beta$ is epic; and hence; we deduce that $F = \text{Coker} \alpha = \text{Coker} \gamma$ which forces $\text{pd} F \leq 1$.

We show that the class $\mathcal{F}_r$ can be specified in terms of pseudo-kernels. We find a characterization of functors in $\mathcal{F}_r$.

**Theorem 3.7.** Let $F \in \text{mod}(C)$. Then $r(F) = 0$ if and only if there exists a presentation $\text{Hom}_C(-, A) \xrightarrow{\theta} \text{Hom}_C(-, B) \to F \to 0$ of $F$ such that $v(\theta)$ is the pseudo-kernel of $\text{Coker} v(\theta)$.

**Proof.** Assume that $r(F) = 0$. Then there exists a presentation $\text{Hom}_C(-, A) \xrightarrow{\theta} \text{Hom}_C(-, B) \xrightarrow{\phi} F \to 0$ of $F$ and an exact sequence of functors $\text{Hom}_C(-, A) \xrightarrow{\theta} \text{Hom}_C(-, B) \xrightarrow{\phi} \text{Hom}_C(-, v(F))$ such that $f = \eta_F \circ \pi$ and $\eta_F$ is monic. Therefore $v(\theta) : A \to B$ is the pseudo-kernel of $v(f) : B \to v(F)$. It now follows from Lemma 2.4 that $v(\theta)$ is pseudo-kernel of $v(\pi)$. Conversely assume that $\text{Hom}_C(-, A) \xrightarrow{\theta} \text{Hom}_C(-, B) \xrightarrow{\phi} F \to 0$ is a presentation of $F$ such that $v(\theta)$ is a pseudo-kernel of $v(\pi)$. Then we have an exact sequence $\text{Hom}_C(-, A) \xrightarrow{\theta} \text{Hom}_C(-, B) \xrightarrow{\phi} \text{Hom}_C(-, v(F))$ of functors and we set $I = \text{Im} \theta$. Now, there exists the following commutative diagram with the exact rows

$$
\begin{array}{ccc}
\text{Hom}_C(-, A) & \xrightarrow{\theta} & \text{Hom}_C(-, B) \\
p & & \downarrow 1_{\text{Hom}_C(-, B)} \\
0 & \longrightarrow & I & \longrightarrow & \text{Hom}_C(-, B) & \longrightarrow & \text{Hom}_C(-, v(F)) \\
\downarrow & & \downarrow \alpha & & \downarrow \\
0 & \longrightarrow & F & \longrightarrow & 0
\end{array}
$$

which implies that $\alpha$ is monic. Since $r$ is left exact, we have $r(F) = 0$. \hfill $\square$

### 4. Half Exact Functors

Throughout this section, we assume that $C$ has pseudo-kernels and cokernels.

A functor $F$ in $\text{mod}(C)$ is called half exact if for any exact sequence $A \to B \to C \to 0$ of objects of $C$ with $A \to B$ a pseudo-kernel of $B \to C$, the sequence $F(C) \to F(B) \to F(A)$ is exact.

In the following proposition, we show how half exact functors coincides with those which have been mentioned in the abelian categories.

**Proposition 4.1.** Assume that $C$ has kernels and $F$ is a functor in $\text{mod}(C)$. Then $F$ is half exact if and only if for any exact sequence $0 \to A \to B \to C \to 0$, $F(C) \to F(B) \to F(A)$ is exact.

**Proof.** Assume that $A \to B \to C \to 0$ is an exact sequence such that $A \to B$ is a pseudo-kernel of $B \to C$. If $K \to B$ is the kernel of $B \to C$, it follows from Proposition 2.3 that $0 \to K \to B \to C \to 0$ is an exact sequence and $K$ is a direct summand of $A$. Using the assumption, one can easily show that $F(C) \to F(B) \to F(A)$ is exact. \hfill $\square$

We now give a characterization of half exact functors.

**Lemma 4.2.** Let $G \in \text{mod}(C)$. Then $G$ is half exact if and only if $\text{Ext}_1^{\text{mod}(C)}(F, G) = 0$ for all $F \in \text{mod}(C)$. In particular, if $\text{mod}(C)$ is Serre and $G$ is half exact, then so is $r(G)$.

**Proof.** For any $F \in \text{mod}(C)$ there exists an epimorphism $B \to C$ such that the sequence $\text{Hom}_C(-, B) \to \text{Hom}_C(-, C) \to F \to 0$ is exact. Then there exists an object $A$ such that $A \to B$ is a pseudo-kernel of $B \to C$. It follows from Lemma 3.1 that $A \to B \to C \to 0$ is exact so that $\text{Hom}_C(-, A) \to \text{Hom}_C(-, B) \to \text{Hom}_C(-, C) \to F \to 0$. Now, the result follows by applying $\text{Hom}_{\text{mod}(C)}(-, G)$ and using the Yoneda lemma. To prove the second assertion, for any $F \in \text{mod}(C)$, applying the functor $\text{Hom}_{\text{mod}(C)}(F, -)$ to the exact sequence $0 \to r(G) \to G \to G/r(G) \to 0$ and using Theorem 3.9 Proposition 3.2 the assumption and the first assertion, the result follows. \hfill $\square$
Proposition 4.3. Let \( \text{mod} (C) \) be a Serre subcategory of \( \text{mod} (C) \) and \( G \in \text{mod} (C) \). Then the following conditions hold.

(i) \( G \) preserves epimorphisms to monomorphisms if and only if \( \eta_G \) is monic.
(ii) \( G \) is left exact if and only if \( \eta_G \) is isomorphism.

Proof. (i) is clear by Proposition 2.2 and Theorem 3.6(ii) If \( G \) is left exact, then according to Proposition 2.1 we have \( \text{Hom}_{\text{mod}(C)} (r (G), G) = 0 \) and so \( r (G) = 0 \). Then using Proposition 3.2 the exact sequence \( 0 \to G \xrightarrow{\eta} h_C v (G) \to F \to 0 \) splits so that \( F = 0 \). The converse of the implication is clear.

In the following theorem we give a generalization of the Yoneda lemma.

Theorem 4.4. Let \( F \) and \( G \) be functors in \( \text{mod} (C) \). Then there is a homomorphism \( \theta_{F,G} : F (v (G)) \to \text{Hom}_{\text{mod}} (C, F) \) of abelian groups which is natural in \( F \) and \( G \). Moreover, if \( F \) is in \( \mathcal{D} \), then \( \theta_{F,G} \) is monic.

Proof. Since \( G \in \text{mod} (C) \), there exists an exact sequence of functors \( \text{Hom}_{\text{mod}} (C, -) \to \text{Hom}_{\text{mod}} (C, A) \to G \to 0 \). Application of the functor \( v (\cdot) \) yields the exact sequence \( B \xrightarrow{\beta} A \xrightarrow{v (\cdot)} v (G) \to 0 \) of objects of \( C \). If we put \( H = \text{Im} (\text{Hom}_{\text{mod}} (C, -) \to \text{Hom}_{\text{mod}} (C, A)) \) and \( C = v (H) \), then we have a natural epimorphism \( \beta : B \to C \) and a morphism \( \gamma : C \to A \) such that \( \alpha = \gamma \beta \) and \( C \xrightarrow{\gamma} A \xrightarrow{v (\cdot)} v (G) \to 0 \) is an exact sequence of objects in \( C \). Then, there is a sequence \( F (v (G)) \xrightarrow{F (\pi)} F (A) \xrightarrow{F (\gamma)} F (C) \) of abelian groups and further, by Yoneda lemma, there exist an exact sequence \( 0 \to \text{Hom}_{\text{mod}} (C, F) \to F (A) \xrightarrow{F (\alpha)} F (B) \) of abelian groups and a unique homomorphism \( \theta_{F,G} : F (v (G)) \to (G, F) \) such that the following diagram is commutative

\[
\begin{array}{ccc}
F (v (G)) & \xrightarrow{F (\pi)} & F (A) \\
\downarrow{\theta_{F,G}} & & \downarrow{1_{F(A)}} \\
\text{Hom}_{\text{mod}} (C, (G, F)) & \xrightarrow{\lambda} & F (A) \\
\end{array}

\begin{array}{ccc}
 & & \downarrow{F (\alpha)} \\
 & & F (B) \\
\end{array}

\begin{array}{ccc}
0 & \xrightarrow{0} & F (A) \\
\downarrow{\theta_{F,G}} & & \downarrow{F (\beta)} \\
\text{Hom}_{\text{mod}} (C, (G_2, F)) & \xrightarrow{\lambda} & F (A) \\
\end{array}
\]

If \( F \) is in \( \mathcal{D} \), then \( F (\pi) \) is monic; and hence \( \theta_{F,G} \) is monic. In order to show that \( \theta \) is natural, assume that \( \eta : G_1 \to G_2 \) is a morphism in \( \text{mod} (C) \). Then for \( i = 1, 2 \), there is exact sequences \( \text{Hom}_{\text{mod}} (C, -, B_i) \) such that \( A_i \) and \( B_i \) are objects in \( C \). Using Yoneda lemma there is the following commutative diagram of abelian groups with exact rows

\[
\begin{array}{ccc}
0 & \xrightarrow{0} & \text{Hom}_{\text{mod}} (C, (G_2, F)) \\
\downarrow{\theta_{F,G_1}} & & \downarrow{F (v (G_2))} \\
\text{Hom}_{\text{mod}} (C, (G_1, F)) & \xrightarrow{\theta_{F,G_2}} & F (A_1) \\
\end{array}
\]

The naturality \( \theta_{F,G} \) in \( F \) is similar.

Corollary 4.5. Let \( F \) and \( G \) be in \( \text{mod} (C) \). Then \( F \) is in \( \mathcal{D} \) if and only if \( \theta_{F,G} \) is monic for all functors \( G \in \text{mod} (C) \).

Proof. Assume that \( \theta_{F,G} \) is monic for all \( G \in \text{mod} (C) \) and that \( B \to C \to 0 \) is an exact sequence of objects of \( C \). Then there exists an exact sequence of functors \( \text{Hom}_{\text{mod}} (C, -), B \to \text{Hom}_{\text{mod}} (C, -) \to G \to 0 \) such that \( G \in \text{mod} (C) \). Using Yoneda lemma, there is an exact sequence of abelian groups \( 0 \to \text{Hom}_{\text{mod}} (C, F) \to F (C) \to F (B) \to F (A) \). Since \( \theta_{F,G} \) is isomorphism, \( \text{Hom}_{\text{mod}} (C, F) \simeq F (v (G)) = 0 \) and so \( F \) is in \( \mathcal{D} \). The converse follows from the previous theorem.
Corollary 4.6. Let \( F \in \text{mod}(\mathcal{C}) \). Then \( F \) is half exact if and only if \( \theta_{F,G} \) is epic for any \( G \in \text{mod}(\mathcal{C}) \). In particular, if \( \text{mod}(\mathcal{C}) \) is Serre, then \( F \) is left exact if and only if \( \theta_{F,G} \) is isomorphism for all \( G \in \text{mod}(\mathcal{C}) \).

Proof. Given an exact sequence \( A \to B \to C \to 0 \) such that \( A \to B \) is a pseudo-kernel of \( B \to C \), we have an exact sequence of functors \( \text{Hom}_\mathcal{C}(-,A) \to \text{Hom}_\mathcal{C}(-,B) \to \text{Hom}_\mathcal{C}(-,C) \). Putting \( H = \text{Coker}(\text{Hom}_\mathcal{C}(-,A) \to \text{Hom}_\mathcal{C}(-,B)) \), since the functor \( v \) preserves cokernels we have \( v(H) \cong C \) and so \( F(C) \to F(B) \to F(A) \) is an exact sequence of abilian groups if and only if \( F(v(H)) \to F(B) \to F(A) \) is exact. Now, we have the following commutative diagram with the bottom row exact

\[
\begin{array}{ccc}
F(v(H)) & \to & F(B) \\
\downarrow \theta_{F,H} & \cong & \downarrow \theta_{F,\text{Hom}_\mathcal{C}(-,B)} \\
0 & \to & \text{Hom}_\mathcal{C}(H,F) \\
\end{array}
\]

It is straightforward to prove that \( \theta_{F,H} \) is epic if and only if the top row is exact. The second assertion follows by the first and the previous corollary, Lemma 4.2 and Proposition 3.2. \( \square \)

In the rest of this section we assume that \( \text{mod}(\mathcal{C}) \) is a Serre subcategory of \( \text{mod}(\mathcal{C}) \).

Corollary 4.7. Let \( F,G \) be in \( \text{mod}(\mathcal{C}) \). Then there is an isomorphism \( \text{Ker} \theta_{F,G} \cong \text{Ker} \theta_{r(F),G} \).

Proof. There is an exact sequence of functors \( 0 \to r(F) \to F \xrightarrow{\eta} h_\mathcal{C}(v(F)) \). Then for any functor \( G \) we have the following commutative diagram with exact rows

\[
\begin{array}{ccc}
0 & \to & r(F)(v(G)) \\
\downarrow \theta_{r(F),G} & & \downarrow \theta_{F,G} \\
0 & \to & \text{Hom}_\mathcal{C}(G,r(F)) \\
\end{array}
\]

\[
\begin{array}{ccc}
F(v(G)) & \to & h_\mathcal{C}(v(F))(v(G)) \\
\downarrow \varepsilon & & \downarrow \varepsilon \\
\text{Hom}_\mathcal{C}(G,F) & \to & \text{Hom}_\mathcal{C}(G,h_\mathcal{C}(v(F)))
\end{array}
\]

The previous corollary implies that \( \varepsilon \) is isomorphism and so the result follows. \( \square \)

Corollary 4.8. Let \( F,G \) be in \( \text{mod}(\mathcal{C}) \). Then there is an isomorphism \( \text{Ker} \theta_{F,G} \cong \text{Ker} \theta_{F,G/r(G)} \) and \( \text{Coker} \theta_{F,G/r(G)} \) is isomorphic to a subobject of \( \text{Coker} \theta_{F,G} \). In particular, if \( r(F) = 0 \), then \( \text{Coker} \theta_{F,G/r(G)} \cong \text{Coker} \theta_{F,G} \).

Proof. There are the following commutative diagram

\[
\begin{array}{ccc}
F(v(G/r(G))) & \xrightarrow{\cong} & F(v(G)) \\
\downarrow \theta_{F,G/r(G)} & \downarrow \theta_{F,G} & \\
0 & \xrightarrow{(\pi,F)} & \text{Hom}_\mathcal{C}(G,F) \\
\end{array}
\]

It is clear that \( \text{Ker} \theta_{F,G} \cong \theta_{F,G/r(G)} \). This also implies that \( \text{Im} \theta_{F,G/r(G)} \cong \text{Im} \theta_{F,G} \). Then we have the following commutative diagram with exact rows

\[
\begin{array}{ccc}
0 & \to & \text{Hom}_\mathcal{C}(G/r(G),F) \\
\downarrow & & \downarrow \text{Coker} \theta_{F,G/r(G)} \\
0 & \to & \text{Hom}_\mathcal{C}(G,F) \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \to & \text{Coker} \theta_{F,G} \\
\downarrow & & \downarrow \text{Coker} \theta_{F,G} \\
0 & \to & \text{Coker} \theta_{F,G} \\
\end{array}
\]

which implies that \( \text{Ker} (\pi,F) = 0 \). If \( r(F) = 0 \), then \( \text{Hom}_{\text{mod}(\mathcal{C})}(r(G),F) = 0 \) and then \( (\pi,F) \) is isomorphism so that \( \text{Coker} (\pi,F) = 0 \). \( \square \)
In the rest of this paper we study the injective objects in \( \text{mod}(\mathcal{C}) \).

**Lemma 4.9.** If \( G \) is a functor in \( \text{mod}(\mathcal{C}) \) such that \( \text{Ext}^i_{\text{mod}(\mathcal{C})}(F,G) = 0 \) for \( i = 1,2 \) and all \( F \in \text{mod}(\mathcal{C}) \), then \( G \) is injective in \( \text{mod}(\mathcal{C}) \).

**Proof.** For any functor \( F \) in \( \text{mod}(\mathcal{C}) \) consider the exact sequence of functors

\[
0 \to r(F) \to F \overset{\eta_F}{\to} hCv(F) \to \text{Coker}(\eta_F) \to 0
\]

in which \( \text{Ker}(\eta_F) \) and \( \text{Coker}(\eta_F) \) are in \( \text{mod}(\mathcal{C}) \). We observe that \( hCv(F) = \text{Hom}_\mathcal{C}(\cdot,v(F)) \) is projective in \( \text{mod}(\mathcal{C}) \) and hence \( \text{Ext}^1_{\text{mod}(\mathcal{C})}(hCv(F),G) = 0 \). Then applying the functor \( \text{Hom}_{\text{mod}(\mathcal{C})}(\cdot,G) \) to the above exact sequence and using the assumption, we deduce that \( \text{Ext}^1_{\text{mod}(\mathcal{C})}(F,G) = 0 \). \( \square \)

**Proposition 4.10.** If \( G \in \text{mod}(\mathcal{C}) \) is injective, then so is \( r(G) \).

**Proof.** By Lemma 3.3, since \( r : \text{mod}(\mathcal{C}) \to \text{mod}(\mathcal{C}) \) is a right adjoint functor of the inclusion functor \( i : \text{mod}(\mathcal{C}) \to \text{mod}(\mathcal{C}) \), \( r(G) \) is injective in \( \text{mod}(\mathcal{C}) \); and hence \( \text{Ext}^i_{\text{mod}(\mathcal{C})}(F,r(G)) = 0 \) for all \( i > 0 \) and all \( F \in \text{mod}(\mathcal{C}) \). Now, Proposition 4.11 implies that \( r(G) \) is injective. \( \square \)

**Proposition 4.11.** Let \( \mathcal{C} \) be abelian. If \( A \) is an object of \( \mathcal{C} \) such that \( \text{id}A = n \), then \( \text{Ext}^n_C(\cdot,-) \) is injective in \( \text{mod}(\mathcal{C}) \). Moreover, if \( n \geq 1 \), then \( \text{Ext}^n_C(\cdot,-) \) is in \( \text{mod}(\mathcal{C}) \).

**Proof.** The case \( n = 0 \), since by Theorem 2.6, \( v \) is left exact, it is straightforward to show that \( hC(F) = \text{Hom}_\mathcal{C}(\cdot,v(F)) \) is injective in \( \text{mod}(\mathcal{C}) \). Assume that \( n = 1 \) and consider an injective resolution \( 0 \to A \to E_0 \to E_1 \to 0 \) for \( A \). By virtue of the case \( n = 0 \), there exists an exact sequence of functors

\[
0 \to \text{Hom}_\mathcal{C}(\cdot,-) \to \text{Hom}_\mathcal{C}(\cdot,E_0) \to \text{Hom}_\mathcal{C}(\cdot,E_1) \to \text{Ext}^1_C(\cdot,-) \to 0
\]

which implies that \( \text{Ext}^1_C(\cdot,-) \in \text{mod}(\mathcal{C}) \). Assume that \( G = \text{Coker}(\text{Hom}_\mathcal{C}(\cdot,-) \to \text{Hom}_\mathcal{C}(\cdot,E_0)) \). For any functor \( F \in \mathcal{C} \), using the adjointness \( \text{Hom}_\mathcal{C}(\cdot,E_i) \) are injective for \( i = 0,1 \), we conclude that \( \text{Ext}^i_{\text{mod}(\mathcal{C})}(F,\text{Ext}^i_C(\cdot,-)) = \text{Ext}^i_{\text{mod}(\mathcal{C})}(F,(-),A) = 0 \) because \( \text{pd}F \leq 2 \). Thus \( \text{Ext}^1_C(-,A) \) is injective. Assume that \( n > 1 \) and hence there exists an exact sequence of objects \( 0 \to A \to E \to B \to 0 \) in \( \mathcal{C} \) such that \( E \) is injective and \( \text{id}B = n - 1 \). An easy induction and the isomorphism \( \text{Ext}^n_C(\cdot,-) \cong \text{Ext}^{n-1}_C(B,\cdot) \) imply that \( \text{Ext}^n_C(-,A) \) is in \( \text{mod}(\mathcal{C}) \) and it is injective in \( \mathcal{C} \). \( \square \)

**Remark 4.12.** We notice that if \( \mathcal{A} \) is an abelian category with enough projective, then it follows from [AB, Proposition 1.8] that \( hCv(F) \cong R^0F \) and so \( \text{Ker}(F \overset{\eta_F}{\to} hCv(F)) = r(F) \) is projective stable. We also observe that by using [CE, Chap V, Theorem 5.3] that for any \( n \geq 0 \), we have an equivalence of functors \( R^nF(-) \cong \text{Ext}^n(-,v(F)) \) where \( R^n \) is the \( n \)-th right derived functor of \( F \). We observe that \( \text{mod}(\mathcal{C}) \) consists of all projective stable functors. It is clear that \( F \in \text{mod}(\mathcal{C}) \) if and only if \( R^nF = 0 \) and hence in this case \( R^nF = 0 \) for all \( n \geq 0 \).

If \( 0 \to F_1 \to F_2 \to F_3 \to 0 \) is an exact sequence in \( \text{mod}(\mathcal{C}) \), then there is a long exact sequence of right derived functors in \( \text{Mod}(\mathcal{C}) \). To be more precise, we have an exact sequence

\[
0 \to R^0(F_1) \to R^0(F_2) \to R^0(F_3) \to R^1(F_1) \to \ldots
\]

We observe that if \( \mathcal{C} \) has enough injective objects, then \( R^nF \in \text{mod}(\mathcal{C}) \) for any \( F \in \text{mod}(\mathcal{C}) \). Because, there is an exact sequence \( 0 \to v(F) \to E \to C \to 0 \) such that \( E \) is injective. Then there is an exact sequence \( \text{Hom}_\mathcal{C}(\cdot,E) \to \text{Hom}_\mathcal{C}(\cdot,C) \to R^1F \to 0 \) so that \( R^1F \in \text{mod}(\mathcal{C}) \). An induction on \( n \) implies that \( R^nF \in \text{mod}(\mathcal{C}) \) for all \( n \geq 1 \).
5. LIMITS IN COHERENT FUNCTORS

Throughout this section $\text{mod}(\mathcal{C})$ is a Serre subcategory of $\text{mod}(\mathcal{C})$.

A non-empty category $\mathcal{I}$ is said to be filtered provided that for each pair of objects $\lambda_1, \lambda_2$ of $\mathcal{I}$, there are morphisms $\varphi_1 : \lambda_1 \to \mu$ for some $\mu \in \mathcal{I}$, and for each pair of morphisms $\varphi_1, \varphi_2 : \lambda \to \mu$, there is a morphism $\psi : \mu \to \nu$ with $\psi \varphi_1 = \psi \varphi_2$. We call the colimit $\lim_{\rightarrow} X_\lambda$ of a functor $X : \mathcal{I} \to \mathcal{C}, \lambda \mapsto X_\lambda$ a direct limit if $\mathcal{I}$ is a skeletally small filtered category. We denote by $\overline{\text{mod}(\mathcal{C})}$ the full subcategory of $\text{Mod}(\mathcal{C})$ which consists of direct limit $\lim_{\rightarrow} F_i$ with each $F_i \in \text{mod}(\mathcal{C})$.

Proposition 5.1. There are the following equalities:

$$\{X \in \text{Mod}(\mathcal{C}) | \text{Hom}_{\text{mod}(\mathcal{C})}(\overline{\text{mod}(\mathcal{C})}, X) = 0\}$$

$$= \{X \in \text{Mod}(\mathcal{C}) | \text{Hom}_{\text{mod}(\mathcal{C})}(\overline{\text{mod}(\mathcal{C})}, X) = 0\} = \overline{\mathcal{D}}.$$

Proof. The first equality is clear. Clearly $\overline{\mathcal{D}} \subseteq \{X \in \text{Mod}(\mathcal{C}) | \text{Hom}_{\text{mod}(\mathcal{C})}(\overline{\text{mod}(\mathcal{C})}, X) = 0\}$. Conversely, assume that $X \in \{X \in \text{Mod}(\mathcal{C}) | \text{Hom}_{\text{mod}(\mathcal{C})}(\overline{\text{mod}(\mathcal{C})}, X) = 0\}$. By [Cr, Lemma 4.1], it suffices to show that for any $Y \in \text{mod}(\mathcal{C})$, any morphism $Y \to X$ factors through some $D \in \mathcal{D}$. Consider the exact sequence $0 \to r(Y) \to Y \to Y/r(Y) \to 0$ where $r(Y) \in \text{mod}(\mathcal{C})$ and $Y/r(Y) \in \mathcal{D}$. Since $\text{Hom}_{\text{mod}(\mathcal{C})}(r(Y), X) = 0$, the morphism $Y \to X$ factors through $Y \to Y/r(Y)$.

Let $\mathcal{A}$ be an abelian category. A torsion theory $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$ is said to be of finite type if the corresponding right adjoint $r : \mathcal{A} \to \mathcal{T}$ of the inclusion functor $i : \mathcal{T} \to \mathcal{A}$ commutes with direct limits. Let $\mathcal{S}$ be a Serre subcategory of $\mathcal{A}$ and let $\mathcal{A}/\mathcal{S}$ be the corresponding quotient category with the canonical functor $q : \mathcal{A} \to \mathcal{A}/\mathcal{S}$ (for details we refer the readers to [G, P]). The subcategory $\mathcal{S}$ of $\mathcal{A}$ is called localizing provided that $q$ admits a right adjoint functor $s : \mathcal{A}/\mathcal{S} \to \mathcal{A}$ which is called section functor. For each $X \in \mathcal{A}$, the adjointness induces a natural morphism $\psi_X : X \to s \circ q(X)$ with Ker $\psi_X$, Coker $\psi_X \in \mathcal{S}$. An object $X \in \mathcal{A}$ is said to be $\mathcal{S}$-closed provided $\psi_X$ is an isomorphism.

Proposition 5.2. $\overline{\text{mod}(\mathcal{C})}$ is a localizing subcategory of finite type of $\text{Mod}(\mathcal{C})$.

Proof. As $\text{Mod}(\mathcal{C})$ is locally coherent, the result follows by [K3, Theorem 2.8].

Given a subclass $\mathcal{S}$ of an abelian category $\mathcal{A}$, we recall that the right perpendicular category $\mathcal{S}^\perp$ of $\mathcal{S}$ is the full subcategory of $\mathcal{A}$ consisting of all objects $M$ satisfying $\text{Hom}_\mathcal{A}(X, M) = \text{Ext}^1_{\mathcal{A}}(X, M) = 0$ for all $X \in \mathcal{S}$. If $\mathcal{S}$ is a localizing subcategory of $\mathcal{A}$, by virtue of [G, III.2, Lemme 1], the right perpendicular category $\mathcal{S}^\perp$ coincides with the full subcategory of $\mathcal{S}$-closed objects.

Proposition 5.3. There are the equalities $\overline{\text{mod}(\mathcal{C})}^\perp = \text{mod}(\mathcal{C})^\perp = \mathcal{L}$ where $\mathcal{L}$ is a subclass of $\text{mod}(\mathcal{C})$ consisting of left exact functors.

Proof. The first equality follows from [K3, Corollary 2.11]. According to Proposition 5.2, $\overline{\text{mod}(\mathcal{C})}$ is a localizing subcategory of $\text{Mod}(\mathcal{C})$ and so by [G, III.2, Lemma 1], it suffices to show that $X$ is $\overline{\text{mod}(\mathcal{C})}$-closed. Assume that $q : \text{Mod}(\mathcal{C}) \to \text{Mod}(\mathcal{C})/\overline{\text{mod}(\mathcal{C})}$ is the quotient functor with the section functor $s : \text{Mod}(\mathcal{C})/\overline{\text{mod}(\mathcal{C})} \to \text{Mod}(\mathcal{C})$. Since, by Proposition 5.2, $\overline{\text{mod}(\mathcal{C})}$ is of finite type, it follows from [K3, Lemma 2.4] that $s$ preserves direct limits. If $X \in \mathcal{L}$, then $X = \lim_{\rightarrow} X_i$ where $X_i \in \mathcal{L}$ for each $i$. Using Proposition 3.2, we have $\text{Ext}^1_{\text{mod}(\mathcal{C})}(\text{mod}(\mathcal{C}), X_i) = 0$ for $i = 0, 1$. For each $i$, we have an exact sequence $0 \to \text{Ker} \psi_{X_i} \to X_i \overset{\psi_{X_i}}{\to} s \circ q(X_i) \to \text{Coker} \psi_{X_i} \to 0$ where Ker $\psi_{X_i}$ and Coker $\psi_{X_i}$ are in $\overline{\text{mod}(\mathcal{C})}$. Now the previous argument implies that Ker $\psi_{X_i} = 0$. If Coker $\psi_{X_i} = \lim_{\rightarrow} C_j$ with $C_j \in \text{mod}(\mathcal{C})$, for each $j$, applying the functor $\text{Hom}_{\text{mod}(\mathcal{C})}(C_j, -)$ to the above exact sequence and the fact that $\text{Ext}^1(C_j, X_i) = 0$ we conclude that each the canonical morphism $\varphi_j : C_j \to \text{Coker} \psi_{X_i}$ factors through $s \circ q(X_i)$ so that $\varphi_j = 0$. Thus Coker $\psi_{X_i} = 0$
and so $\psi_X$ is isomorphism for each $i$. Now since $s$ and $q$ preserves direct limits, we have $X \cong s \circ q(X)$. Conversely assume that $X \in \mathcal{mod}(C)^{\perp}$ and $Y \in \mathcal{mod}(C)$. Then applying the functor $\text{Hom}_{\mathcal{Mod}(C)}(-, X)$ to the exact sequence $0 \to r(Y) \to Y \xrightarrow{\eta_V} h_Cv(Y) \to \text{Coker} \eta_V \to 0$, we deduce that any morphism $Y \to X$ factors through $\eta_V : Y \to h_Cv(Y)$. Since $h_Cv(Y) \in \mathcal{L}$, the result follows from [Cr, Lemma 4.1].

Let $\mathcal{M}$ be a subcategory of $\mathcal{Mod}(C)$ and let $F$ be a functor in $\mathcal{Mod}(C)$. A morphism $f : F \to M$ with $M \in \mathcal{M}$ is an $\mathcal{M}$-preenvelope (or a left $\mathcal{M}$-approximation) of $F$ provided that the induced abelian group homomorphism $\text{Hom}_{\mathcal{Mod}(C)}(f, X) : \text{Hom}_{\mathcal{Mod}(C)}(M, X) \to \text{Hom}_{\mathcal{Mod}(C)}(F, X)$ is surjective for each $X \in \mathcal{M}$.

**Corollary 5.4.** If $\mathcal{C}$ has direct limits, then any functor in $\mathcal{L}$ has an $\mathcal{L}$-preenvelope.

**Proof.** Assume that $F \in \mathcal{L}$. Then $F = \lim_i F_i$, where $F_i \in \mathcal{L}$. Using Proposition 5.3, there exists some object $A_i$ such that $F_i = \text{Hom}_C(-, A_i)$ for each $i$. Assume that $\sigma_i : F_i \to F$ is the canonical morphism for each $i$. If $F : \mathcal{I} \to \text{Mod}(\mathcal{C})$ defined as $F(i) = F_i$, is the corresponding functor where $\mathcal{I}$ is an skeletally small filtered category, then $v(F) : \mathcal{I} \to \mathcal{C}$ defined as $v(F)(i) = A_i$ is a functor and since $\mathcal{C}$ has direct limits, $\lim_i v(F) = \lim_i A_i$ exists with the canonical morphism $\varphi_i : A_i \to \lim_i A_i$ for each $i$. We observe that there exists a unique morphism $\delta : F \to \text{Hom}_C(-, \lim_i A_i)$ such that $\delta \circ \sigma_i = h_C(\varphi_i)$. We assert that $\delta$ is an $\mathcal{L}$-preenvelope. Suppose that $G = \text{Hom}_C(-, C)$ is in $\mathcal{L}$ and $f : F \to G$ is a morphism. Then there exists a unique morphism $\varepsilon : \lim_i A_i \to C$ such that $\varepsilon \circ \varphi_i = v(f \circ \sigma_i)$. Thus for each $i$, we have $h_C(\varepsilon) \circ \delta \circ \sigma_i = h_C(\varepsilon) \circ h_C(\varphi_i) = h_C(\varepsilon f \circ \sigma_i) = f \circ \sigma_i$. The unicity implies that $h_C(\varepsilon) \circ \delta = f$.

The following results shows that $\mathcal{L}$ is the class of right exact functors in $\mathcal{Mod}(\mathcal{C})$.

**Proposition 5.5.** Let $\mathcal{C}$ has pseudo-kernels and $F \in \mathcal{Mod}(\mathcal{C})$. Then $F \in \mathcal{L}$ if and only if $F$ is left exact.

**Proof.** If $F \in \mathcal{L}$, then $F = \lim_i F_i$, where $F_i \in \mathcal{L}$. Then $F$ is left exact as the direct limits are exact in the category of abelian groups. Conversely assume that $F$ is left exact. For any $G \in \mathcal{mod}(\mathcal{C})$, using Proposition 4.3 there exists an epimorphism $B \to A$ in $\mathcal{C}$ such that $\text{Hom}_C(-, B) \to \text{Hom}_C(-, A) \to G \to 0$ is exact. Assume that $C \to B$ is a pseudo-kernel of $B \to A$. It follows from Lemma 3.7 that $C \to B \to A \to 0$ is exact and since $F$ is left exact, $0 \to F(B) \to F(A) \to F(C)$ is an exact sequence of abelian groups. On the other hand, there is an exact sequence $\text{Hom}_C(-, C) \to \text{Hom}_C(-, B) \to \text{Hom}_C(-, A) \to G \to 0$. Applying $\text{Hom}_{\mathcal{Mod}(\mathcal{C})}(-, F)$ to this exact sequence and using Yoneda lemma, we deduce that $\text{Hom}_{\mathcal{Mod}(\mathcal{C})}(G, F) = \text{Ext}^1_{\mathcal{Mod}(\mathcal{C})}(G, F) = 0$ so that $F \in \mathcal{mod}(\mathcal{C})^{\perp}$. Now the result follows by using Proposition 5.3.

**Proposition 5.6.** For any functor $F \in \mathcal{Mod}(\mathcal{C})$, there exists a morphism $\eta_F : F \to L(F)$ such that $L(F) \in \mathcal{L}$ and $\text{Ker} \eta_F$ and $\text{Coker} \eta_F$ are in $\mathcal{mod}(\mathcal{C})$.

**Proof.** Since $\mathcal{Mod}(\mathcal{C})$ is locally coherent, we have $F = \lim_i F_i$ where $F_i \in \mathcal{mod}(\mathcal{C})$ for each $i$. Then for each $i$, there is an exact sequence

$$0 \to r(F_i) \to F_i \xrightarrow{\eta_{F_i}} \text{Hom}_C(-, v(F_i)) \to c(F_i) \to 0$$

where $r(F_i), c(F_i) \in \mathcal{mod}(\mathcal{C})$ for all $i$. If $\mathcal{I}$ is an skeletally small filtered category such that $F : \mathcal{I} \to \mathcal{Mod}(\mathcal{C})$ is a functor with $F(i) = F_i$, then $r(F) : \mathcal{I} \to \mathcal{Mod}(\mathcal{C})$ defined as $r(F)(i) = r(F_i)$ is a functor which has direct limit $\lim_i r(F_i)$. Then we have an exact sequence of functors

$$0 \to \lim_i r(F_i) \to F \xrightarrow{\eta_F} \lim_i \text{Hom}_C(-, v(F_i)) \to \text{lim}_i c(F_i) \to 0$$

in which $L(F) = \lim_i \text{Hom}_C(-, v(F_i)) \in \mathcal{L}$.
We consider a functor \( r : \text{Mod}(\mathcal{C}) \to \text{mod}(\mathcal{C}) \) defined as \( r(F) = \text{Ker} \eta_F \) for any functor \( F \in \text{Mod}(\mathcal{C}) \). It follows from Proposition \ref{prop:coherent-functors} that \( r(F) \) is the largest subfunctor of \( F \) contained in \( \text{mod}(\mathcal{C}) \). Since \( \text{mod}(\mathcal{C}) \) is a localizing subcategory of \( \text{Mod}(\mathcal{C}) \), the functor \( r : \text{Mod}(\mathcal{C}) \to \text{mod}(\mathcal{C}) \) is left exact. Using again Proposition \ref{prop:coherent-functors} the functor \( r : \text{Mod}(\mathcal{C}) \to \text{mod}(\mathcal{C}) \) is a right adjoint functor of the inclusion functor.

**Corollary 5.7.** \((\text{mod}(\mathcal{C}), D)\) is a hereditary torsion theory of \( \text{Mod}(\mathcal{C}) \).

**Proof.** Assume that \( \text{Hom}_{\text{Mod}(\mathcal{C})}(X, Y) = 0 \) for all \( Y \in D \). It is clear that \( X/r(X) \in D \) and so \( X \in \text{mod}(\mathcal{C}) \). The other part follows by Proposition \ref{prop:coherent-functors} \( \square \)

**Proposition 5.8.** There exists a right exact functor \( v^* : \text{Mod}(\mathcal{C}) \to \mathcal{C} \) such that \( v^*|_{\text{mod}(\mathcal{C})} = v \) and it is a left adjoint functor of \( h_C : \mathcal{C} \to \text{Mod}(\mathcal{C}) \).

**Proof.** According to [K2, Universal Property 5.6] the functor \( v^* \) exists such that commutes with direct limits so that it is right exact. For the second assertion, for any \( F \in \text{Mod}(\mathcal{C}) \) and \( A \in \mathcal{C} \), since \( \text{Mod}(\mathcal{C}) \) is locally coherent, we have \( F = \lim_{\leftarrow} F_i \), where \( F_i \in \text{mod}(\mathcal{C}) \); and hence there are the following isomorphisms

\[
\text{Hom}_\mathcal{C}(v^*(F), A) = \text{Hom}_\mathcal{C}(\lim_{\leftarrow} v(F_i), A) \cong \lim_{\leftarrow} \text{Hom}_\mathcal{C}(v(F_i), A)
\]

\[
\cong \lim_{\leftarrow} \text{Hom}_{\text{mod}(\mathcal{C})}(F_i, h_C(A)) \cong \text{Hom}_{\text{mod}(\mathcal{C})}(F, h_C(A)).
\]

\( \square \)

**Proposition 5.9.** If \( \mathcal{C} \) has direct limits, then every \( F \in \text{Mod}(\mathcal{C}) \) has \( \mathcal{L} \)-preenvelopes.

**Proof.** Since \( \text{Mod}(\mathcal{C}) \) is locally coherent, we have \( F = \lim_{\leftarrow} F_i \), where \( F_i \in \text{mod}(\mathcal{C}) \) for each \( i \). By the proof of Proposition \ref{prop:coherent-functors} \( i \) we have \( L(F) = \lim_{\leftarrow} \text{Hom}_\mathcal{C}(-, v(F_i)) \) and using Corollary \ref{cor:hereditary-torsion-theory} \( f \) the morphism \( \delta : \lim_{\leftarrow} \text{Hom}_\mathcal{C}(-, v(F_i)) \to h_C v^*(F) \) is an \( \mathcal{L} \)-preenvelope. We claim that \( \delta \circ \eta_F : F \to h_C v^*(F) \) is an \( \mathcal{L} \)-preenvelope. For any \( G \in \mathcal{L} \), applying the functor \( \text{Hom}_{\text{Mod}(\mathcal{C})}(-, G) \) to the exact sequence

\[
0 \to \lim_{\leftarrow} \text{Ker}(F_i) \to F \xrightarrow{\eta_F} L(F) \to \lim_{\leftarrow} \text{Coker}(F_i) \to 0
\]

and using Proposition \ref{prop:coherent-functors} \( i \) we deduce that for any morphism \( \theta : F \to G \), there exists a morphism \( \varepsilon : L(F) \to G \) such that \( \varepsilon \circ \eta_F = \theta \) and using Corollary \ref{cor:hereditary-torsion-theory} there exists a morphism \( \alpha : h_C v^*(F) \to G \) such that \( \alpha \circ \delta = \varepsilon \). Thus \( \alpha \circ \delta \circ \eta_F = \theta \). \( \square \)

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