Compact spacelike surfaces in the 3-dimensional de Sitter space.

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In this paper we establish several sufficient conditions for a compact spacelike surface in the 3-dimensional de Sitter space to be totally geodesic or spherical.

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Let \( E_1^4 \) be a 4-dimensional Lorentz-Minkowski space, that is, the space endowed with the Lorentzian metric tensor \( \langle \cdot, \cdot \rangle \) given by

\[
\langle \cdot, \cdot \rangle = (dx_1)^2 + (dx_2)^2 + (dx_3)^2 - (dx_0)^2,
\]

where \( (x_1, x_2, x_3, x_0) \) are the canonical coordinates of \( E_1^4 \). The 3-dimensional unitary de Sitter space is defined as the following hyperquadric of \( E_1^4 \).

\[
S_3^1 = \{ x \in \mathbb{R}^4 : \langle x, x \rangle = 1 \}
\]

As it is well known, \( S_3^1 \) inherits from \( E_1^4 \) a time-orientable Lorentzian metric which makes it the standard model of a Lorentzian space of constant sectional curvature one. A smooth immersion \( \psi : F \rightarrow S_3^1 \subset E_1^4 \) of a 2-dimensional connected manifold \( M \) is said to be a spacelike surface if the induced metric via \( \psi \) is a Riemannian metric on \( M \), which, as usual, is also denoted by \( \langle \cdot, \cdot \rangle \). The time-orientation of \( S_3^1 \) allows us to define a (global) unique timelike unit normal field \( n \) on \( F \), tangent to \( S_3^1 \), and hence we may assume that \( F \) oriented by \( n \). We will refer to \( n \) as the Gauss map of \( F \).

We note that Lobachevsky space \( L^3 \) is the set of points

\[
L^3 = \{ x \in E_1^4 : \langle x, x \rangle = -1, x_0 > 0 \}.
\]

It is well known that a compact spacelike surface in the 3-dimensional de Sitter space \( S_3^1 \) is diffeomorphic to a sphere \( S^2 \). Thus, it is interesting to look for additional assumptions for such a surface to be totally geodesic or totally umbilical round sphere.

There are two possible kinds of geometric assumptions: extrinsic, that is relative to the second fundamental form, and intrinsic, namely, concerning to the Gaussian curvature of the induced metric. As regards to the extrinsic approach, Ramanathan [10] proved that every compact spacelike surface in \( S_3^1 \) of constant mean curvature is totally umbilical. This result was generalized to hypersurface of any dimension by Montiel [9]. J.Aledo and A.Romero characterize the compact spacelike surfaces in \( S_3^1 \) whose second fundamental form defines a Riemannian metric. They studied the case of constant Gaussian curvature \( K_{II} \) of the second fundamental form, proving that the totally umbilical round spheres are the only compact spacelike surfaces in \( S_3^1 \) with \( K < 1 \) and constant \( K_{II} \) [2]. With respect to the intrinsic approach Li [8] obtained that compact spacelike surface of constant Gaussian curvature is totally umbilical. And he proved there is no complete spacelike surface in \( S_3^1 \) with constant Gaussian curvature \( K > 1 \). J.Aledo and A.Romero proved the same result without condition that Gaussian curvature is constant [2]. But it is true more general result.

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Theorem. 1. Let \( F \) be a \( C^2 \)-regular complete spacelike surface in de Sitter space \( S^3_1 \). If Gaussian curvature \( K \geq 1 \) then the surface \( F \) is totally geodesic great sphere with Gaussian curvature \( K = 1 \).

S. N. Bershtein proved that an explicitly given saddle surface over a whole plane in the Euclidean space \( E^3 \) with slower than linear growth at infinity must be a cylinder. He proved this theorem for surfaces of class \( C^2 \) [4], and it was generalized to the non-regular case in [1].

A surface \( F^2 \) of smoothness class \( C^1 \) in \( S^3 \) may be projected univalently into a great sphere \( S_0^2 \) if the great spheres tangent to \( F^2 \) do not pass through points \( Q_1, Q_2 \) polar to \( S_0^2 \).

The surface \( F^2 \) in \( S^3 \) is called a saddle surface if any closed rectifiable contour \( L \), that is, in the intersection of \( F^2 \) with an arbitrary great sphere \( S^2 \) in \( S^3 \), lies in an open hemisphere, and is deformable to a point in the surface can be spanned by a two-dimensional simply connected surface \( Q \) contained in \( F^2 \cap S^2 \). In other words, from the surface it is impossible to cut off a crust by a great sphere \( S^2 \), that is, on \( F^2 \) there do not exist domains with boundary that lie in an open great hemisphere of \( S^2 \) and are wholly in one of the great hemispheres of \( S^3 \) into which it is divided by the great sphere \( S^2 \). In this case when \( F \) is a regular surface of class \( C^2 \), the saddle condition is equivalent to the condition that the Gaussian curvature of \( F^2 \) does not exceed one. We have the following result.

Theorem. 2. [5]/[6]/[7]. Let \( F \) be an explicitly given compact saddle surface of smoothness class \( C^1 \) in the spherical space \( S^3 \). Then \( F \) is a totally geodesic great sphere.

This theorem is a generalization of a theorem of Bernshtein to a spherical space. For regular space we obtain the following corollary.

Theorem. 3. [5]/[6] Let \( F \) be an explicitly given compact surface that is regular of class \( C^2 \) in the spherical space \( S^3 \). If the Gaussian curvature \( K \) of \( F \) satisfies \( K \leq 1 \) then \( F \) is a totally geodesic great sphere.

This theorem was stated in [6]. Really theorems 2,3 had been proved in [7] but were formulated there for a centrally symmetric surfaces. The final version was in [5].

It seems to us that the following conjecture must hold under a restriction on the Gaussian curvature of the surface. Suppose that \( F \) is an embedded compact surface, regular of class \( C^2 \), in the spherical space \( S^3 \). If the Gaussian curvature \( K \) of \( F \) satisfies \( 0 < K \leq 1 \), then \( F \) is a totally geodesic great sphere.

A.D. Aleksandrov [3] had proved that an analytical surface in Euclidean space \( E^3 \) homeomorphic to a sphere is a standard sphere if principal curvatures satisfy the inequality

\[
(k_1 + c)(k_2 + c) \leq 0
\]  

(1)

This result had been generalized for analytic surfaces in spherical space \( S^3 \) and Lobachevsky space \( L^3 \) [7]:

a) in \( S^3 \) with additional hypothesis of positive Gaussian curvature;

b) in \( L^3 \) under additional assumptions that principal curvatures \( k_1, k_2 \) satisfy \( |k_1|, |k_2| > c_0 > 1 \).

But in Lobachevsky space the result is true under weaker analytic restriction.

Theorem. 4. Let \( F \) be a \( C^3 \) regular surface homeomorphic to the sphere in the Lobachevsky space \( L^3 \). If \( |k_1|, |k_2| > c_0 > 1 \) and principle curvatures \( k_1 \) and \( k_2 \) satisfy (1), then the surface is an umbilical round sphere in \( L^3 \).

Analogical result it is true for surfaces in the de Sitter space \( S^3_1 \).

Theorem. 5. Let \( F \) be a \( C^3 \) regular compact spacelike surface in the de Sitter space \( S^3_1 \). If \( |k_1|, |k_2| < 1 \) and principal curvatures satisfy (1), then the surface is an umbilical round sphere in \( S^3_1 \).
Let $S^3_1$ be a simply-connected pseudo-Riemannian space of curvature 1 and signature $(+,+,+,-)$. It can be isometrically embedded in the pseudo-Euclidean space $E^4_1$ of signature $(+,+,+,-)$ as the hypersurface given by the equation $x_1^2 + x_2^2 + x_3^2 - x_0^2 = 1$. Together with $E^4_1$ we consider the superimposed Euclidean space $E$ with unit sphere $S^3$ given by the equation $x_1^2 + x_2^2 + x_3^2 + x_0^2 = 1$. We specify a mapping of $S^3_1$ into $S^3$. To the point $P$ of $S^3_1$ with position vector $r$ we assign the point $\tilde{P}$ with position vector $\tilde{r} = r/\sqrt{1 + 2x_0^2}$. Under the mapping, to a surface $F \subset S^3_1$ corresponds a surface $\tilde{F} \subset S^3$. Let $b_{ij}$ and $\tilde{b}_{ij}$ be the coefficients of the second quadratic forms of $F$ and $\tilde{F}$, and $n = (n_1, n_2, n_3, n_0)$ be a normal vector field on $F$.

**Lemma 1.** [7] $\tilde{b}_{ij} = b_{ij}/\sqrt{1 + 2x_0^2}\sqrt{1 + 2n_0^2}$.

**Proof of theorem 1.** From the condition $K \geq 1$ it follows that $F$ is a compact spacelike surface in the de Sitter space $S^3_1$. Locally a spacelike surface is explicitly given over totally geodesic great sphere $S^2_0 \subset S^3_1$ and the orthogonal projection $p: F \to S^2_0$ in $S^3_1$ is covering. Indeed, $p$ is a local diffeomorphism. The compactness of $F$ and the simply connectedness of $S^2_0$ imply that $p$ is a global diffeomorphism $F$ on $S^2_0$ and the surface $F$ is globally explicitly given over $S^2_0$.

We map a surface $F$ in $S^3_1$ into a surface $\tilde{F}$ in $S^3$. If $F$ has a definite metric and Gaussian curvature $K \geq 1$, then $\tilde{F}$ has Gaussian curvature not greater than 1. This follows immediately from Lemma 1, Gauss’s formula and the fact that $\langle n, n \rangle = -1$ for normals to $F$. In a pseudo-Euclidean space, the analogous correspondence between surfaces and their curvatures was used by Sokolov [11].

The surface $\tilde{F}$ satisfies the conditions of theorem 3. It follows that $\tilde{F}$ is a totally geodesic great sphere. By lemma 1 the ranks of the second quadratic forms of $\tilde{F}$ and $F$ coincide and we obtain that the surface $F$ is a totally geodesic surface in $S^3_1$.

**Proof of Theorem 4 and 5.** The normal $n(u_1, u_2)$ to $F$ is chosen so that the principal curvatures satisfy (1). In a neighborhood of an arbitrary nonumbilical point $P$ we choose coordinate curves consisting of the lines of curvature, and an arbitrary orthogonal net in the case of umbilical point. At $P$ the coefficients of the first quadratic form are $e = g = 1, f = 0$. Let $F_1$ be the surface with radius vector $\rho = (r - cn)/\sqrt{|c^2 - 1|}$.

In both cases the surface $F_1$ lies in $S^3_1$. Moreover

$$\rho_{u_1} = \frac{(1 + ck_1)}{\sqrt{|c^2 - 1|}} r_{u_1}, \quad \rho_{u_2} = \frac{(1 + ck_2)}{\sqrt{|c^2 - 1|}} r_{u_2}.$$ 

The unit normal $n_1 = \frac{c r - n}{\sqrt{|c^2 - 1|}}$. From the conditions on the principal curvatures of $F$ in theorems 4, 5 it follows that

$$\langle \rho_{u_1}, \rho_{u_1} \rangle > 0, \quad \langle \rho_{u_2}, \rho_{u_2} \rangle > 0$$

and $F_1$ is a spacelike surface in $S^3_1$. The coefficients of the second quadratic form of the surface $F_1$ are

$$L_1 = \frac{(1 + ck_1)(k_1 + c)}{\sqrt{c^2 - 1}}, \quad N_1 = \frac{(1 + ck_2)(k_2 + c)}{\sqrt{c^2 - 1}}.$$ 

The Gaussian curvature of $F_1$ at the point $P_1$ is equal to

$$K = 1 - \frac{(k_1 + c)(k_2 + c)|c^2 - 1|}{(1 + k_1 c)^2(1 + k_2 c)^2} \geq 1$$

The same is true in umbilical points too. The surface $F_1$ satisfies the conditions of theorem 1. It follows that the surface $F_1$ is a totally geodesic great sphere in $S^3_1$ and $F$ is an umbilical surface in $L^3$ or $S^3_1$. 



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