The Sign Problem, $\mathcal{PT}$ Symmetry and Abelian Lattice Duality

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Lattice field theories with complex actions are not easily studied using conventional analytic or simulation methods. However, a large class of these models are invariant under $CT$, where $C$ is charge conjugation and $T$ is time reversal, including models with non-zero chemical potential. For Abelian models in this class, lattice duality maps models with complex actions into dual models with real actions. For extended regions of parameter space, calculable for each model, duality resolves the sign problem for both analytic methods and computer simulations. Explicit duality relations are given for models for spin and gauge models based on $Z(N)$ and $U(1)$ symmetry groups. The dual forms are generalizations of the $Z(N)$ chiral clock model and the lattice Frenkel-Kontorova model, respectively. From these equivalences, rich sets of spatially-modulated phases are found in the strong-coupling region of the original models.

The sign problem is a fundamental issue in Euclidean lattice field theories at non-zero chemical potential, manifesting as complex weights in the path integral \[ \frac{1}{Z} = \int D\phi \exp \left( \sum_n x_n^2 + \text{interaction terms} \right) \] In this letter, we provide a technique that maps a large class of Abelian lattice models with complex weights to dual models with real weights. The dual form of these models can then be studied using familiar analytical and computational methods. The methods in this class possess a generalized $\mathcal{PT}$ symmetry. In recent years, substantial progress has been made in the study of models with this symmetry \[ \frac{1}{Z} = \int D\phi \exp \left( \sum_n x_n^2 + \text{interaction terms} \right) \] The methods developed here are applicable to models with a non-zero chemical potential or a Minkowski-space electric field, which also has a sign problem \[ \frac{1}{Z} = \int D\phi \exp \left( \sum_n x_n^2 + \text{interaction terms} \right) \] The utility of lattice duality for the sign problem was shown some time ago \[ \frac{1}{Z} = \int D\phi \exp \left( \sum_n x_n^2 + \text{interaction terms} \right) \] and has recently been systematically studied \[ \frac{1}{Z} = \int D\phi \exp \left( \sum_n x_n^2 + \text{interaction terms} \right) \] in an intermediate form, particularly in connection with the worm algorithm \[ \frac{1}{Z} = \int D\phi \exp \left( \sum_n x_n^2 + \text{interaction terms} \right) \]. The explicit duality relations we derive here based on generalized $\mathcal{PT}$ symmetry represent a solution to the sign problem for Abelian lattice models over a wide range of parameter space. The dual forms generalize the well-known chiral $Z(N)$ and Frenkel-Kontorova models and typically have a rich phase structure with spatially-modulated phases \[ \frac{1}{Z} = \int D\phi \exp \left( \sum_n x_n^2 + \text{interaction terms} \right) \]. Such phases are also known to occur in $(1 + 1)$-dimensional fermionic models \[ \frac{1}{Z} = \int D\phi \exp \left( \sum_n x_n^2 + \text{interaction terms} \right) \] and would also appear naturally in a quarkyonic phase \[ \frac{1}{Z} = \int D\phi \exp \left( \sum_n x_n^2 + \text{interaction terms} \right) \]. However, spatially-modulated phases are not special to fermions at finite density, as shown by a continuum model of $(1 + 1)$-dimensional QCD with heavy particles \[ \frac{1}{Z} = \int D\phi \exp \left( \sum_n x_n^2 + \text{interaction terms} \right) \]. The appearance of spatially-modulated phases is natural in $\mathcal{PT}$-symmetric models \[ \frac{1}{Z} = \int D\phi \exp \left( \sum_n x_n^2 + \text{interaction terms} \right) \].

In the models discussed here, the fundamental fields are elements $z = \exp(i\theta)$ of $Z(N)$ or $U(1)$. The lattice actions are complex, but invariant under the simultaneous application of the operators $C$ and $T$, where $C$ is a linear charge conjugation operator that takes $\theta$ to $-\theta$, and hence $z$ to $z^*$, and $T$ is time reversal implemented as complex conjugation. Thus these models have $CT$ symmetry as a generalized $\mathcal{PT}$ symmetry. In a lattice model, this symmetry ensures that the eigenvalues of the transfer matrix are either real or occur in complex conjugate pairs. The presence of complex eigenvalues gives rise to a rich phase structure not possible with Hermitian transfer matrices. Broadly speaking, there are three possibilities: all eigenvalues are real (region I); the dominant eigenvalue of the transfer matrix is real, but other eigenvalues form complex pairs (region II); the dominant eigenvalues form a complex conjugate pair (region III). It is known that models in region I are equivalent to a Hermitian theory \[ \frac{1}{Z} = \int D\phi \exp \left( \sum_n x_n^2 + \text{interaction terms} \right) \]. In regions II and III, the occurrence of complex conjugate pairs gives rise to spatially-modulated behavior \[ \frac{1}{Z} = \int D\phi \exp \left( \sum_n x_n^2 + \text{interaction terms} \right) \]. Consider the transfer matrix $T$ of a lattice model with $Z(N)$ or $U(1)$ variables such that $CTC = T^*$. The Fourier transform operator $\mathcal{F}$ is a unitary matrix satisfying $\mathcal{F}^T = \mathcal{F}$ and $\mathcal{F}^2 = \mathcal{F}^* = 1$. $\mathcal{F}$ acts on $T$ to give a real transfer matrix $\tilde{T} = \mathcal{F}T\mathcal{F}^*$ satisfying $\tilde{T}^* = \tilde{T}$ \[ \frac{1}{Z} = \int D\phi \exp \left( \sum_n x_n^2 + \text{interaction terms} \right) \]. The use of the Fourier transform in the construction of a real transfer matrix is closely related to the use of lattice duality transformations in finding real actions for models invariant under $CT$.

We begin with duality for $d = 2$ $Z(N)$ models with a chemical potential using the methods of \[ \frac{1}{Z} = \int D\phi \exp \left( \sum_n x_n^2 + \text{interaction terms} \right) \] for the Villain, or heat kernel, action. Defining the site-based spin variables as $\exp(2\pi im(x)/N)$, with $m(x)$ an integer between 0 and $N$, the partition function is given by

$$Z[J, \mu \delta, \nu] = \sum_m \sum_{\nu} \exp \left[ -\frac{J}{2} \sum_{x,\nu} \left( \frac{2\pi}{N} \partial_x m(x) - i\mu \delta - 2\pi \nu(x) \right)^2 \right]$$ \hspace{1cm} (1)$$

where $\partial_x m(x) \equiv m(x + \nu) - m(x)$ and the sum over link variables $\nu(x) \in \mathbb{Z}$ ensures periodicity. Using the properties of the Villain action, we can write
where $V$ is the number of sites on the lattice such that $dV$ is the number of links. Summation over the $m(x)$’s give a set of delta function constraints:

$$Z[J, \mu \delta_{\nu,2}] = (2\pi J)^{-dV/2} \sum_{p, \mu} \exp \left[ -\frac{1}{2J} \sum_{x, \mu} p_{\mu}^2 (x) + \sum_{x, \mu} p_{\mu} (x) \left( \frac{2\pi}{N} \partial_{\nu} m(x) - i\mu \delta_{\nu,2} \right) \right]$$  

(2)
generalizing the well-known self-duality of the $d = 3$ Abelian Higgs system. The $d = 3$ $Z(N)$ gauge field is dual to the $d = 3$ chiral $Z(N)$ spin model, which has been extensively studied \[21, 22\]. In the strong-coupling limit where $K$ is small and thus $J$ large, the response of the system to an external real (Minkowski-space) electric field reveals an infinite number of commensurate inhomogeneous phases separating the disordered, confining phase of the gauge theory from a phase with a constant induced field.

The duality between $\mathcal{CT}$-symmetric interactions and chiral interactions is not restricted to the Villain action, but holds more generally. Consider a $Z(N)$ model on a $d$-dimensional lattice with $N \geq 3$ and local interactions on sites, links, plaquettes and et cetera. A typical term in the lattice action is a function $V(z)$ where $z \in Z(N)$. It can be expanded as

$$V(z) = \sum_{j=0}^{N-1} v_j z^j.$$  

We define $V_j = V(\omega^j)$ where $\omega = \exp(2\pi i/N)$. The $\mathcal{CT}$ operator takes $V(z)$ into $V^*(z^*)$. For a $\mathcal{CT}$-symmetric interaction, this implies the $v_j$'s are real and $V_{N-j} = V_j^*$. A duality transform on $V$ is implemented as a Fourier transform

$$\exp(-\tilde{V}_j) = \sum_{k=0}^{N-1} \omega^{jk} \exp(-V_k).$$  

$\mathcal{CT}$ symmetry thus implies that the dual weights $\exp(-\tilde{V}_j)$ are real. If the dual weights are all positive, we can expand the dual interaction as

$$\tilde{V}(w) = \sum_{j=0}^{N-1} \tilde{v}_j w^j$$  

and we must have $\tilde{v}_j = \tilde{v}_{N-j}$, but the $\tilde{v}_j$'s need not be real. This in general induces a chirality, a handedness to the interactions, generalizing the chiral $Z(N)$ models to a larger class of lattice models. There are large regions of parameter space for which all the dual weights are positive, in which case we say the model is in the positive dual weight (PDW) class. Such models may be simulated by standard computational methods such as the Metropolis algorithm, and familiar theoretical tools such as mean field theory may be applied. This represents a solution of the sign problem for Abelian lattice models in the PDW class. Bochner’s theorem states that the strict positivity of $\exp(-\tilde{V})$ is equivalent to requiring that $\exp(-V)$ be positive-definite. This in turn leads to the requirement $V_0 < (V_j + V_{N-j})/2$ for $1 \leq j \leq N - 1$, immediately excluding antiferromagnetic interactions from the PDW class.

Within the PDW region, the Perron-Frobenius theorem applies to the dual representation of the transfer matrix, so there is a single dominant eigenvalue. Thus the PDW region is disjoint from region III. Figure 1 shows the PDW region for a $Z(3)$ interaction in term of the variables $v_2 = v_1 + v_2$ and $v_1 = v_1 - v_2$. The behavior shown is periodic in $v_i$ with a period of $2\pi/3\sqrt{3} \approx 1.2092$. In several models, the PDW region completely includes the region associated with a non-zero chemical potential.

Many powerful techniques can be applied within the PDW region, including low- and high-temperature expansions, variational methods such as mean field theory, and renormalization group analysis. One particularly powerful analytic method for spatially modulated phases combines mean field theory in $d - 1$ directions with the transfer matrix in the spatially modulated direction \[39\]. Correlation functions for local operators in the original complex representation of the model can be constructed in the dual theory in a well-known way \[37, 38\].

Further insight can be obtained from models based on $U(1)$. Here we apply the duality techniques pioneered by Jose et al. \[38\]. The partition function of the two-dimensional XY model with an imaginary chemical potential term has the form

$$Z[K, \mu \delta_{\nu,2}] = \int_{S^1} [d\theta] \sum_{\nu} \exp \left[ -\frac{K}{2} \sum_{x,\nu} (\partial_\nu \theta(x) - i \mu \delta_{\nu,2} - 2 \pi n_\nu(x))^2 \right].$$  

Again using the properties of the Villain action, we have

$$Z[K, \mu \delta_{\nu,2}] = \int_{S^1} [d\theta] \prod_{x,\nu} \sum_{\nu} \frac{1}{\sqrt{2\pi K}} e^{-\nu^2(x)/2K} e^{i \nu \theta(x)} e^{(\nu \theta(x) - i \delta_{\nu,2} \nu)}.$$
Integration over the $\theta$ variables leads to the constraint $\nabla_v p_\nu (x) = 0$. This in turns allows us to write $p_\rho (x) = \epsilon_{\rho \nu} \nabla_\nu m(X)$ where $m(X)$ is an integer-valued field on the dual lattice site $X$ which is displaced from $x$ by half a lattice spacing in each direction. The partition function is now

$$Z = \sum_{\{m(X)\} \in \mathbb{Z}} \frac{1}{\sqrt{2\pi K}} e^{-\sum_X [\nabla_\nu m(X)]^2 / 2K + \mu \nabla_1 m(X)}.$$  

(18)

The final step is to introduce a new field $\phi(x) \in R$ using a periodic $\delta$-function, effectively performing a Poisson resummation:

$$Z = \int_R [d\phi(X)] e^{-\sum_X [\nabla_\nu \phi(X)]^2 / 2K + \mu \nabla_1 \phi(X)} \sum_{\{m(X)\} \in \mathbb{Z}} e^{2\pi i m(X) \phi(X)}.$$  

(19)

If we keep only the $m = 1$ contributions, we have a lattice sine-Gordon model

$$Z = \int_R [d\phi(X)] \exp \left[ -\sum_{X,\mu} \frac{1}{2K} (\nabla_\mu \phi(X))^2 - \sum_X \mu \nabla_1 \phi(X) + \sum_X 2y \cos (2\pi \phi(X)) \right]$$  

(20)

with $y = 1$. This will be recognized as a two-dimensional lattice version of the Frenkel-Kontorova model, a sine-Gordon model with an additional term proportional to $\mu$. For each fixed value of $X_2$, the term $\sum_X \nabla_1 \phi(X)$ counts the number of kinks on that slice: The particles in the original representation manifest as lattice kinks in the dual representation. This generalizes to other lattice models based on $U(1)$, and can also be applied to $Z(N)$ models realized by explicit breaking of $U(1)$ down to $Z(N)$. From a continuum point of view, this model can be further mapped to a massive Thirring model with $\mu$ coupling to the conserved fermion current.

All of the Abelian lattice models in the $CT$-symmetric class studied here have real dual representations. These models typically exhibit a rich phase structure in regions of parameter space where the dual weights are positive. The properties of these models can be studied in the dual representation with both computational and analytical tools. Spatially-modulated phases can be detected in simulations using appropriate two-point functions; analytical studies combined with known results from condensed matter physics can provide valuable guidance. Patel has recently suggested that an oscillatory signal might appear in baryon number correlators in heavy ion collisions at RHIC and the LHC [39, 40]. We believe that the complex phase structure seen in Abelian systems is likely to appear in non-Abelian systems. As an intermediate step, application of duality to an effective Abelian model associated with the reduction of $SU(N)$ to $U(1)^{N-1}$ [41,44] appears possible with the results developed here.

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