ON THE ADDITION OF QUANTUM MATRICES

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ABSTRACT We introduce an addition law for the usual quantum matrices $A(R)$ by means of a coaddition $\Delta t = t \otimes 1 + 1 \otimes t$. It supplements the usual comultiplication $\Delta t = t \otimes t$ and together they obey a codistributivity condition. The coaddition does not form a usual Hopf algebra but a braided one. The same remarks apply for rectangular $m \times n$ quantum matrices. As an application, we construct left-invariant vector fields on $A(R)$ and other quantum spaces. They close in the form of a braided Lie algebra. As another application, the wave-functions in the lattice approximation of Kac-Moody algebras and other lattice fields can be added and functionally differentiated.

1 Introduction

In recent years there has been a great deal of interest in quantum matrices. These algebras are modelled on the co-ordinate functions $t_{ij}$ on the ring of matrices $M_n$ say, whose value at matrix $M$ is its component $M_{ij}$. As is well-known by now, we keep their abstract algebraic properties but allow the generators $t_{ij}$ to be non-commutative. The non-commutativity can fruitfully be taken in a quadratic form controlled by a matrix $R$ and the resulting algebras $A(R)$ have nice properties particularly when $R$ obeys the quantum Yang-Baxter equations (QYBE). Moreover, by adding suitable further relations, one obtains for example $q$-deformations of the algebras of functions on the standard compact matrix groups.[1]

We further recall that the structure of $A(R)$ is that of a bialgebra with a coproduct $\Delta : A(R) \to A(R) \otimes A(R)$, $\Delta t = t \otimes t$ encoding the formal properties of matrix multiplication. This coproduct has numerous uses, allowing us to tensor product representations and other applications.

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In this paper we develop a further ‘coaddition’ structure $\Delta : A(R) \rightarrow A(R) \otimes A(R)$ corresponding this time to the addition of quantum matrices. Moreover, it is compatible with the existing coproduct $\Delta$ in a way that corresponds to the usual distributivity axiom. Our construction so far works for $R$ of Hecke type, but this is the most common and applies for example to the standard $GL_n$ $R$-matrices. In this way we complete the structure of $A(R)$ to fully model the ring structure – both addition and multiplication – of usual matrices. Such a structure can be called a quantum-braided ring for obvious reasons.

In order to formulate this structure we need the notion of a braided group or braided-Hopf algebra as introduced by the author in [2] and already applied in numerous contexts. For physics, the most important one is perhaps in [3][4]. We will see that the coaddition $\Delta$ does not form a usual quantum group or Hopf algebra, but a braided one. Nevertheless, it has group-like or vector space-like properties and one can construct for example braided-differential calculi on $A(R)$ using the technique of [4]. This we do in Section 3. As an application, we construct left-invariant vector fields on $A(R)$ and see that they close into some form of braided-Lie algebra.

Let us note in passing that some kind of braided ring of $2 \times 2$ matrices (with both braided comultiplication and braided coaddition) has already been found in the work of U. Meyer in his approach to $q$-deformations of Minkowski space[5]. This is different from our result but can be considered as one of our indirect motivations.

We also note that in [6] has been introduced a notion of rectangular quantum matrices $A(R_1 : R_2)$ associated to any pair of $R$-matrices. We will see in Section 4 that these too can be added via a coaddition, at least when $R_i$ are of Hecke type. Likewise, one may differentiate with respect to them. As an application, we construct rotational vector fields on quantum planes. We also propose a systematic approach to lattice wave-functions and their functional differentiation.

Finally, in the Section 5 we mention one context in the physics literature where these results have an immediate consequence. This is to the lattice approximation of Kac-Moody algebras in [7]. In this work there are quantum-matrix valued ‘wave-functions’ on the discretised line. We will see that the relations between them introduced in [7] have just the structure of a rectangular quantum matrix and hence can be added pointwise via our braided coaddition.
2 Braided-coaddition on quantum matrices

Here we prove the addition law on \( A(R) \) for \( R \)-matrices of Hecke type. Recall that a matrix solution of the QYBE \( R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \) is Hecke if

\[
(PR - q)(PR + q^{-1}) = 0
\]

for suitable \( q \). Here \( P \) denotes the usual permutation matrix. Thus they have two eigenvalues in their minimal polynomial. We use the standard notations where the suffices on \( R \) etc refer to the position in a matrix tensor product.

We recall also that a braided group (or braided-Hopf algebra) \([8][2]\) is \((B, \Delta, \varepsilon, S, \Psi)\) where the first four are like the coproduct, counit, antipode (when it exists) of a usual Hopf algebra but \( \Delta : B \rightarrow B \otimes B \) is an algebra homomorphism not to the usual tensor product algebra but to the braided tensor product one. This is defined by a linear map \( \Psi : B \otimes B \rightarrow B \otimes B \) which obeys the braid relations and which is compatible with the other maps in the sense that the braiding commutes with them (one says that the braiding is functorial) applied to either input in an obvious way. The braided tensor product is then \([8]\)

\[
(a \otimes b)(c \otimes d) = a\Psi(b \otimes c)d, \quad \forall a, b, c, d
\]

where we apply \( \Psi \) and then multiply the result from the left by \( a \) and from the right by \( d \) as shown. Thus the notion of a braided group or braided-Hopf algebra is a generalisation of the notion of super-group or super-Hopf algebra but with bose-fermi statistics replaced by braid ones. An introduction for physicists is in \([9]\). The formal mathematical picture is in \([10]\).

**Theorem 2.1** Let \( R \) be a solution of the QYBE of Hecke type and \( A(R) \) the usual bialgebra as in \([1][cf[11]]\) with a matrix of generators \( t = (t_i^j) \) and relations \( Rt_1t_2 = t_2t_1R \). This forms a braided-Hopf algebra with

\[
\Psi(t_1 \otimes t_2) = R_{21}t_2 \otimes t_1R, \quad \Delta t = t \otimes 1 + 1 \otimes t, \quad \varepsilon t = 0, \quad S t = -t
\]

**Proof** We give a direct proof. By definition, \( \Delta \) extends to products as an algebra homomorphism to the braided tensor product algebra. This is consistent because

\[
\Delta Rt_1t_2 = R(t_1 \otimes 1 + 1 \otimes t_1)(t_2 \otimes 1 + 1 \otimes t_2)
\]
\[
\begin{aligned}
\Delta t_2 t_1 R &= (t_2 \otimes 1 + 1 \otimes t_2)(t_1 \otimes 1 + 1 \otimes t_1) R \\
&= t_2 t_1 R \otimes 1 + 1 \otimes t_2 t_1 R + R t_1 \otimes t_2 R t_2 R + t_2 \otimes t_1 R \\
\end{aligned}
\]

which are equal because \( R_{21} R = 1 + (q^{-1} - q) P R \) and \( RR_{21} = 1 + (q^{-1} - q) R P \) from (1).

For a full picture here, we also have to check that \( \Psi \) shown here on the generators extends consistently to products of the generators in such a way as to be functorial with respect to products. The strategy is like that in [2,3] and reduces to the QYBE for \( R \). Explicitly, one has

\[
\Psi(t_1 t_2 \cdots t_M \otimes t_{M+1} \cdots t_{M+N}) = R_{M+1} R \cdots R_{M+11} R_{1M+1} \cdots R_{1M+N} \\
\vdots \quad \vdots \quad t_{M+1} \cdots t_{M+N} \otimes t_1 \cdots t_M \quad \vdots \quad \vdots \\
R_{M+N M} R_{M+N 1} R_{MM+M} R_{MM+1}
\]

where the blocks are to be multiplied in the order shown. We will give an alternative way to deduce this formula later in the section.

Another way to deduce the result without verifying directly is to reduce it to the problem to the construction of braided vector space Hopf algebras in [3], of which this is an example. To do this we introduce the notation \( t_I = t^{i_0}_{i_1} \) where \( I = (i_0, i_1) \), \( J = (j_0, j_1) \) etc are multindices. Then

\[
\begin{aligned}
\Psi(t_I \otimes t_J) &= t_B \otimes t_A R^A_{I} B_{J} \\
R^A_{I} B_{J} &= R^{i_0}_{b_0} i_0 a_0 R^{a_1}_{i_1} b_1 j_1. \\
\end{aligned}
\]

(3)

\[
\begin{aligned}
t_{I} t_{J} &= t_B t_A R'^A_{I} B_{J} \\
R'^A_{I} B_{J} &= R^{-1 i_0}_{a_0} j_0 b_0 R^{a_1}_{i_1} b_1 j_1. \\
\end{aligned}
\]

(4)

This is now exactly the framework of a covector braided group [3] and one just has to verify that these matrices \( R, R' \) obey the conditions there. These are \((PR + 1)(PR' - 1) = 0\) which reduces to \( R \) Hecke, and mixed QYBE-like relations of the form \( R_{12} R_{13} R'_{23} = R'_{23} R'_{13} R_{12} \) and \( R'_{12} R_{13} R_{23} = R_{23} R_{13} R'_{12} \). One has to write these out and see that they reduce to the QYBE for \( R \) many times. Finally, these is a matrix condition \( R'_{21} R = R_{21} R' \) which holds automatically and provides for a braided-antipode.

It is a natural question to ask how this (braided) coaddition structure, which clearly corresponds to the linear addition of underlying matrices in the classical case, relates to the usual comultiplication.
Definition 2.2 A quantum ring is a bialgebra \((B, \Delta, \epsilon)\) and a second Hopf algebra structure \((B, \bar{\Delta}, \bar{\epsilon})\) for the same algebra \(B\), which obeys the codistributivity axioms

\[
(id \otimes \cdot) \circ \Delta_{B \otimes B} \circ \Delta = (\Delta \otimes id) \circ \Delta
\]

\[
(\cdot \otimes id) \circ \Delta_{B \otimes B} \circ \bar{\Delta} = (id \otimes \bar{\Delta}) \circ \Delta
\]

where \(\Delta_{B \otimes B} = (id \otimes \tau \otimes id)(\Delta \otimes \Delta)\) is the usual tensor product coalgebra. Here \(\tau\) denotes the usual transposition of the middle two factors. We call \(\Delta\) the comultiplication and \(\bar{\Delta}\) the coaddition. If \(\bar{\Delta}\) forms a braided Hopf algebra rather than a usual one, we say that we have a quantum-braided ring.

Proposition 2.3 The braided-Hopf algebra structure on \(B = A(R)\) in Theorem 2.1 together with the usual comultiplication \(\Delta t = t \otimes t\) forms a quantum-braided ring.

Proof We have to prove the codistributivity conditions in Definition 2.2. On the generators they hold trivially. On products \(t_1 t_2\) of generators, we have for the first condition

\[
(id \otimes \cdot)\Delta_{B \otimes B}(t_1 t_2 \otimes 1 + 1 \otimes t_1 t_2 + t_1 \otimes t_2 + R_{21} t_2 \otimes t_1 R) = (id \otimes \cdot)\tau_{23}(t_1 t_2 \otimes 1 + 1 \otimes t_1 t_2 + t_1 \otimes t_2 \otimes t_2 + t_1 \otimes t_2 \otimes t_1 \otimes t_1 R) = t_1 t_2 \otimes 1 \otimes t_1 t_2 + 1 \otimes t_1 t_2 \otimes t_1 t_2 + t_1 \otimes t_2 \otimes t_1 t_2 + R_{21} t_2 \otimes t_1 \otimes t_2 t_1 R = (t_1 t_2 \otimes 1 + 1 \otimes t_1 t_2 + t_1 \otimes t_2 + R_{21} t_2 \otimes t_1 R) \otimes t_1 t_2 = (\bar{\Delta} \otimes id)(t_1 t_2 \otimes t_1 t_2)
\]

because of the relations in \(A(R)\). \(\tau_{23}\) denotes transposition in the middle two factors. One proves the general case in a similar way by induction. Similarly for codistributivity from the other side. \(\Box\)

For the canonical example, we let

\[
R = \begin{pmatrix}
q & 0 & 0 & 0 \\
0 & 1 & q - q^{-1} & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & q
\end{pmatrix}
\] (5)

the standard R-matrix for the \(2 \times 2\) quantum matrices \(M_q(2)\). This has generators \(t = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) and the usual relations

\[
qab = ba, \quad ca = acq, \quad qcd = dc, \quad db = dbq, \quad bc = cb, \quad ad - da = (q^{-1} - q)cb.
\]
This forms a quantum-braided ring with the usual comultiplication and the linear braided coaddition on the generators, with the braiding

\[ \Psi(a \otimes a) = q^2 a \otimes a \quad \Psi(a \otimes b) = q b \otimes a \]
\[ \Psi(a \otimes c) = q c \otimes a \quad \Psi(a \otimes d) = d \otimes a \]
\[ \Psi(b \otimes a) = qa \otimes b + (q^2 - 1)b \otimes a \quad \Psi(b \otimes b) = q^2 b \otimes b \]
\[ \Psi(b \otimes c) = c \otimes b + (q - q^{-1})d \otimes a \quad \Psi(b \otimes d) = q d \otimes b \]
\[ \Psi(c \otimes a) = qa \otimes c + (q^2 - 1)c \otimes a \quad \Psi(c \otimes c) = q^2 c \otimes c \]
\[ \Psi(c \otimes b) = b \otimes c + (q - q^{-1})d \otimes a \quad \Psi(c \otimes d) = q d \otimes c \]
\[ \Psi(d \otimes b) = q b \otimes d + (q^2 - 1)d \otimes b \quad \Psi(d \otimes c) = q c \otimes d + (q^2 - 1)d \otimes c \]
\[ \Psi(d \otimes a) = a \otimes d + (q - q^{-1})(c \otimes b + b \otimes c) + (q - q^{-1})^2 d \otimes a \quad \Psi(d \otimes d) = q^2 d \otimes d \]

Equivalently, one can work directly with the braided tensor product algebra $B \otimes B$. Denoting the second copy of $A(R)$ with a prime, the algebra $A(R) \otimes A(R)$ is generated by $t, t'$ with their usual relations and cross-relations from $\Psi$, namely the \textit{braid statistics}

\[ t'_1 t_2 = R_{21} t_2 t'_1 R. \quad (6) \]

The homomorphism property of $\Delta$ in Theorem 2.1 in this notation is that $t'' = t + t'$ then also obeys the FRT relations of $A(R)$.

In these terms then, let \( \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \) denote a second copy of $M_q(2)$ with braid statistics

\[ a' a = q^2 a a', \quad b' b = q^2 b b', \quad c' c = q^2 c c', \quad d' d = q^2 d d' \]
\[ a' b = q b a', \quad a' c = q c a', \quad a' d = d a', \quad b' d = q d b', \quad c' d = q d c' \]
\[ b' a = q a b' + (q^2 - 1)b a', \quad b' c = c b' + (q - q^{-1}) d a' \]
\[ c' a = q a c' + (q^2 - 1)c a', \quad c' b = b c' + (q - q^{-1}) d a' \]
\[ d' b = q b d' + (q^2 - 1)d b', \quad d' c = c q d' + (q^2 - 1)d c' \]
\[ d' a = a d' + (q - q^{-1})(c b' + b c') + (q - q^{-1})^2 d a' \]

Then

\[ \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \]
also obeys the relations of $M_q(2)$.

On the other hand, our construction works for the coaddition of any quantum matrix bialgebra associated to a Hecke-type R-matrix. This includes multiparameter and non-standard variants of the $GL_q(n)$ R-matrices and many others.

Finally, we explain the categorical setting which is that of bicomodules. Let us recall that in the abstract braided group theory, all structures are fully covariant under some quantum group. This quantum group has a (dual) quasitriangular structure which then induces the braidings, such as the braidings above. This point of view is explained in detail in [12].

**Lemma 2.4** $A(R)$ is a right $A(R)^{\text{cop}} \otimes A(R)$ comodule algebra. I.e. there is a coaction $A(R) \rightarrow A(R) \otimes A(R)^{\text{cop}} \otimes A(R)$ which is an algebra homomorphism. Explicitly, it is given by

$$t^i_j \rightarrow t^a_b \otimes \tau^i_a \otimes \sigma^b_j, \quad \text{i.e.,} \quad t \rightarrow \tau t \sigma$$

in a compact notation. Here $A(R)^{\text{cop}}$ denotes $A(R)$ with reversed coproduct and with matrix generator $\tau$. The matrix generator of the other coacting $A(R)$ is $\sigma$.

**Proof** We first note that if $A$ is a bialgebra or Hopf algebra then so is $A^{\text{cop}}$. This is the same algebra but with the reversed coproduct. Now $A$ coacts on itself from both the left and the right via its coproduct $\Delta : A \rightarrow A \otimes A$ (the left regular and right regular coactions), and this coproduct map is an algebra homomorphism. Now any left $A$-comodule algebra is the same thing as a right $A^{\text{cop}}$-comodule algebra. The coaction is naturally from the left but we write the same linear map now from the right and compensate by reversing the coproduct of $A$. This right coaction $A \rightarrow A \otimes A^{\text{cop}}$ is still an algebra homomorphism. In particular then, both $A$ and $A^{\text{cop}}$ coact from the right on $A$ via the coproduct homomorphism. Therefore $A^{\text{cop}} \otimes A$ also coacts from the right. This is the general situation. Explicitly for $A = A(R)$ the coaction is as stated.

In the compact notation $\tau t \sigma$ we write $\tau$ on the left for the purpose of matrix multiplication but it lives in the middle of $A(R) \otimes A(R)^{\text{cop}} \otimes A(R)$. One can also verify explicitly that the coaction shown extends consistently as an algebra homomorphism to products,

$$t^{i_1}_{j_1} \cdots t^{i_M}_{j_M} \rightarrow t^{a_1}_{b_1} \cdots t^{a_M}_{b_M} \otimes \tau^{i_1}_{a_1} \cdots \tau^{i_M}_{a_M} \otimes \sigma^{b_1}_{j_1} \cdots \sigma^{b_M}_{j_M}$$

as it must from the general reasons given. It is easy to see that this is consistent with the FRT relations $R t_1 t_2 = t_2 t_1 R$ by applying to both sides and using these and the corresponding relations $R \tau_1 \tau_2 = \tau_2 \tau_1 R$ and $R \sigma_1 \sigma_2 = \sigma_2 \sigma_1 R$. $\Box$
Equivalently, $A(R)$ is an $A(R)$-bicomodule algebra. For any bialgebra, a $A$-bicomodule is by definition something on which $A$ coacts from the left and from the right by maps $\beta_L$ and $\beta_R$ say, and these two coactions commute in the sense

$$(\beta_L \otimes \text{id})\beta_R = (\text{id} \otimes \beta_R)\beta_L.$$ 

This is just the same thing as a right $A^\text{cop} \otimes A$-comodule since the right $A(R)^\text{cop}$-comodule can be viewed just as well as $\beta_L$ and the right $A$-comodule part is $\beta_R$. This is exactly a dualisation of the notion of bimodule for algebras. For any dual-quasitriangular bialgebra $A$, the categories of left $A$-comodules, right $A$-comodules and hence of $A$-bicomodules are all braided [10].

**Proposition 2.5** $A(R)$ with its coaddition in Theorem 2.1 is a braided-bialgebra living in the braided category of $A(R)$-bicomodules.

**Proof** Suppose that $A$ is a dual-quasitriangular bialgebra [10] in the sense of a map $\mathcal{R} : A \otimes A \to \mathbb{C}$ obeying axioms dual to those of Drinfeld [11] for a universal $R$-matrix. Then $A^\text{cop}$ is also dual quasitriangular with $\mathcal{R}^\text{cop}(a \otimes b) = \mathcal{R}(b \otimes a)$. Hence also $A^\text{cop} \otimes A$ is dual-quasitriangular with the corresponding tensor product dual quasitriangular structure,

$$\mathcal{R}((a \otimes b) \otimes (c \otimes d)) = \mathcal{R}(c \otimes a)\mathcal{R}(b \otimes d), \quad \forall (a \otimes b), (c \otimes d) \in A^\text{cop} \otimes A.$$ 

So $A^\text{cop} \otimes A$ is a dual quantum group in the strict sense. In particular, from [12, Sec. 3] we know that $A(R)$ has dual quasitriangular structure $\mathcal{R} : A(R) \otimes A(R) \to \mathbb{C}$ such that $\mathcal{R}(t_1 \otimes t_2) = R_{12}$ and extended to products as a skew bicharacter. This is the reason that dual-quasitriangular structures were introduced by the author (and subsequently by other authors also). Hence we conclude that $A(R)^\text{cop} \otimes A$ has a dual-quasitriangular structure and that therefore its category of comodule is braided. From the general scheme explained in the present matrix context in [12], we compute the corresponding braiding as

$$\Psi(t^{i_1}_{j_1} \cdots t^{i_M}_{j_M} \otimes t^{k_1}_{l_1} \cdots t^{k_N}_{l_N})$$

$$= t^{c_1}_{d_1} \cdots t^{c_N}_{d_N} \otimes t^{a_1}_{b_1} \cdots t^{a_M}_{b_M} \mathcal{R}\left((\tau^{i_1}_{a_1} \cdots \tau^{i_M}_{a_M} \otimes \sigma^{b_1}_{j_1} \cdots \sigma^{b_M}_{j_M})(\tau^{k_1}_{c_1} \cdots \tau^{k_M}_{c_M} \otimes \sigma^{d_1}_{l_1} \cdots \sigma^{d_N}_{l_N})\right)$$

$$= t^{c_1}_{d_1} \cdots t^{c_N}_{d_N} \otimes t^{a_1}_{b_1} \cdots t^{a_M}_{b_M} Z_R\left(\frac{D}{b \overline{\otimes} l}\right) Z_R\left(\frac{I}{c \overline{\otimes} k}\right) A$$
Here
\[
\mathcal{R}(t_{i_1j_1} t_{i_2j_2} \cdots t_{i_Mj_M} \otimes t_{k_Nj_N} t_{k_{N-1}j_{N-1}} \cdots t_{k_1j_1})
= R_{m_{i_1}}^{k_1} n_{i_1} R_{m_{i_2}}^{k_2} n_{i_2} \cdots R_{m_{i_M}}^{k_M} n_{i_M}
R_{m_{j_1}}^{n_{j_1}} n_{j_1} R_{m_{j_2}}^{n_{j_2}} n_{j_2} \cdots
\vdots
\vdots
R_{m_{j_M}}^{n_{j_M}} n_{j_M} n_{j_M-1} \cdots n_{j_1} n_{j_1-1} \cdots n_{j_1} n_{j_1-1} \cdots n_{j_1} n_{j_1-1} \cdots n_{j_1} n_{j_1-1} \cdots n_{j_1} n_{j_1-1} \cdots n_{j_1} n_{j_1-1} \cdots n_{j_1} n_{j_1-1}
\]

is the description as a partition function[13, Sec. 5] and \(I = (i_1, \ldots, i_M)\) and \(K = (k_1, \ldots k_N)\) while \(\bar{I} = (i_M, \ldots, i_1)\) has the reversed order. This is the same result as \(\Psi\) in the proof of Theorem 2.1 above. \(\square\)

Another notation is to write \(t^I_J = t_{i_1j_1} \cdots t_{i_Mj_M}\) and \(\mathcal{R}(t^I_J \otimes t^K_L) = R^I_J K_L Z_R(\bar{K}^L_J)\).

Then
\[
\Psi(t^I_J \otimes t^K_L) = R^K_C t^C_D \otimes t^A_B R^B_J D_L
\]
which is of the same form as on the generators in Theorem 2.1. Note that the multi-index notation here is the one for partition functions and row-to-row transfer matrices in statistical mechanics and not the one above that reduced \(A(R)\) to a braided covector algebra. Thus we have two ways of thinking about \(A(R)\) as living in a braided category. The two are related by the bialgebra homomorphism
\[
A(R) \rightarrow A(R)^{\text{cop}} \otimes A(R), \quad t^{(i_0,i_1)}(j_0,j_1) \mapsto \tau^{j_0i_0} \otimes \sigma^{i_1j_1}.
\]

By this map an \(A(R)\)-comodule algebra in our braided-covector point of view[3] explained directly after Theorem 2.1, pushes out to our second \(A(R)^{\text{cop}} \otimes A(R)\)-comodule point of view.

3 Vector fields on quantum matrices

One thing that one can do with our new braided addition law on quantum matrices is to make an infinitesimal addition. According to[3] this defines braided differential operators or vector fields on the underlying braided space. The co-ordinate functions are \(t = (t^{i_0i_1}) = (t_I)\) so the corresponding differentials

\[
\partial^I \equiv \partial^{i_0i_1} \equiv \frac{\partial}{\partial t^{i_0i_1}} : A(R) \rightarrow A(R)
\]
defined by\,[4] \[
\partial^I f(t) = a_{-0}|(a_I^{-1}(f(a + t) - f(t))) = \text{coeff of } a_I \text{ in } f(a + t)
\]
where \(a = (a_I) = (a_{i_0 i_1})\) denotes the first copy of \(A(R)\) in \(A(R) \otimes A(R)\) (the second copy being denoted still by \(t\)). The two copies have braid statistics \(t_1 a_2 = R_{21} a_2 t_1 R\) with respect to each other from (\(\text{[}3\text{]}\)). We assume throughout that \(R\) is of Hecke type so that the results of Section 2 apply.

These differential operators \(\partial^I : A(R) \rightarrow A(R)\) are constructed explicitly in [4] in terms of braided-integers \([m, R]\) in the form \(\partial^I t_1 t_2 \cdots t_m = e^I_1 t_2 \cdots t_m[m : R]\) where \(R\) from Section 2 is the relevant braiding matrix for linear addition. Because they correspond to linear addition, they are the analogues of the usual cartesian partial derivatives \(\frac{\partial}{\partial v^j}\).

On the other hand, on a group or matrix space, there are also the left-invariant vector fields \(\tilde{\partial}^I\) say given by right-multiplication in the underlying ring of matrices. These are constructed in terms of the usual coproduct \(\Delta\) corresponding to the multiplication. They are related to the cartesian \(\partial^I\) via the codistributivity property in Definition 2.2. Our goal in this section is to compute this relationship explicitly. We will also see that they form some kind of braided-Lie algebra.

We begin by computing the algebra generated by \(\partial^I\) and its action on \(A(R)\).

**Proposition 3.1** The braided-differential operators \(\partial^I = \partial^{i_0 i_1}\) obey the relations \(\partial_1 \partial_2 R = R \partial_2 \partial_1\) of \(A(R_{21})\).

**Proof** In the general scheme of [4] the braided derivatives obey the braided vector algebra \(V(R')\) with relations and braiding

\[
\Psi(\partial^I \otimes \partial^J) = R^I_A J_B \partial^B \otimes \partial^A, \quad \partial^I \partial^J = R'^I_A J_B \partial^B \partial^A
\]

for \(R\) and \(R'\) as for the covector algebra (\(\text{[}3\text{]}\))–(\(\text{[}4\text{]}\)). This too forms a braided group with linear addition law, and is in some sense the dual of the covector braided group. Putting in the form of \(R'\) we have

\[
\partial^{i_0 i_1} \partial^{j_0 j_1} = R^{-1}_{a_0 b_0} b_0 R^{i_1}_{a_1 b_1} b_1 R^{i_1}_{a_1 b_1} b_1 \partial^{a_1 a_0}
\]

which gives the relations shown. \(\square\)

10
Both the vectors and the covectors live in a braided category hence there are braidings among them too. These are given in [14] in the general case, and we will need them explicitly

\[
\Psi(t_I \otimes \partial^J) = \tilde{R}^{A}_{I} \partial^B \otimes t_A, \quad \tilde{R}^{A}_{I} \partial^B = \tilde{R}^{b_0}_{i_0} a_0 \tilde{R}^{a_1}_{i_1} j_1 b_1 \tag{10}
\]

\[
\Psi(\partial^I \otimes t_J) = R^{-1}_{I} t_A \partial^B \otimes \partial^A, \quad R^{-1}_{I} t_A \partial^B = R^{-1}_{j_0} a_0 \tilde{R}^{-1}_{i_1} a_1 b_1 j_1 \tag{11}
\]

where \(\tilde{R}\) denotes the second inverse \(((R^t)^{-1})^t\) and \(t_2\) is transposition in the second matrix factor. In particular, the \(\partial^I\) obey a braided-Leibniz rule for computing \(\partial^I(f(t)g(t))\) whereby \(\partial^I\) is taken past \(f(t)\) with a \(\Psi^{-1}\). This is the generalisation of the philosophy of super-differentiation.

Next we compute the braided Heisenberg-Weyl algebra, which is the algebra of \(R\)-quantum mechanics generated by the vectors \(\partial^I\) acting on the co-ordinate functions \(t_J\). The general scheme in the braided setting is in [4] and gives the usual relations

\[
\partial^I t_J - t_A R^A_{J} \partial^B = \delta^I_J
\]

which in our case becomes

\[
\partial^{i_1} \epsilon^{j_0}_{j_1} - R^{a_1}_{j_0} b_1 c R^{c}_{i_1} \partial^d_a = \delta^{i_1}_{j_1} \epsilon^{j_0}_{i_0}. \tag{12}
\]

The structure here is that of a braided-semidirect product\([4]\) which we see is \(A(R) \triangleright A(R_{21})\). It is built on \(A(R) \otimes A(R_{21})\) with the cross relations \([4,2]\).

In this Heisenberg-Weyl algebra are the vector fields on \(A(R)\) as the subspace of the form \(f_I(t)\partial^I\). One can usually define on this set some form a braided Lie bracket corresponding to the Lie derivative of vector fields, i.e. some form of a diffeomorphism Lie algebra. This includes the case of the \(q\)-Virasoro Lie algebra but, as for this, the coalgebra and the general picture are both unknown. On the other hand, certain vector fields such as the left-invariant vector fields on a group or matrix space should close under the Lie bracket into some form of sub-braided Lie algebra similar to the corresponding quantum enveloping algebra in the standard cases.

We recall first the classical formula for the left-invariant vector fields on a matrix group \(G\). Let \(\xi\) be in the Lie algebra of \(G\). The corresponding left-invariant vector field is defined on \(f \in C(G)\) by

\[
(\xi \triangleright f)(g) = \frac{d}{d\epsilon} \bigg|_0 f(g(1 + \epsilon \xi)) = \frac{d}{d\epsilon} \bigg|_0 \left( f(g) + \epsilon (g \xi)^i_j \frac{\partial}{\partial g^i} f(g) \right) = g^i a \epsilon^a_j \frac{\partial}{\partial g^i} f(g).
\]
If we define $L_h^*$ by left translation in the group as $L_h^*(f)(g) = f(hg)$ for $h,g \in G$ then

$$(L_h^*(\tilde{\xi}f))(g) = (\tilde{\xi}f)(hg) = \frac{d}{d\epsilon}|_0 f(hg(1+\epsilon\xi)) = \frac{d}{d\epsilon}|_0 (L_h^*f)(g(1+\epsilon\xi)) = (\tilde{\xi}(L_h^*f))(g)$$

which expresses left-invariance of the $\tilde{\xi}$.

Abstractly in terms of the co-ordinate functions $t$ we analogously define the left-invariant vector fields

$$\tilde{\xi}f(t) = t^i a^j \xi_j \frac{\partial}{\partial t^i} f(t), \text{ i.e. } \tilde{\xi} = \xi^i_{i_0} \tilde{\partial}^i_{i_0}, \quad \tilde{\partial}^I \equiv \tilde{\partial}^{i_1}_{i_0} = t^a_{i_0} \tilde{\partial}^i_{a}$$ (13)

**Proposition 3.2** The vector fields $\tilde{\partial}^I : A(R) \rightarrow A(R)$ defined in (13) are left-invariant in the sense

$$(\text{id} \otimes \tilde{\partial}^I)\Delta = \Delta \tilde{\partial}^I.$$

**Proof** This follows from the codistributivity properties proven in Proposition 2.3. We first introduce the functionals

$$\epsilon^I = \epsilon^{i_1}_{i_0}, \quad \epsilon^I(t_J) = \delta^I_J = \delta^{i_1}_{j_1} \delta^{i_0}_{i_0}$$

with $\epsilon^I = 0$ on all other powers of $t$. With respect to the usual multiplicative coproduct $\Delta$ it obeys

$$(\text{id} \otimes \epsilon^I)\Delta = t^a_{i_0} \epsilon^{i_1}_{a}$$

since both sides are zero except when acting on anything of the form $t^a_{i_1}$. For each $a$, the coproduct $\Delta t^a_{i_1}$ contains the term $t^a_{i_0} \otimes t^0_{i_1}$ which is the only contribution to the left hand side. Then both sides give the same result. In terms of this linear functional $\epsilon^I$ we have the definition of $\tilde{\partial}^I$ as

$$\tilde{\partial}^I = (\epsilon^I \otimes \text{id})\Delta.$$

Now we can compute

$$\begin{align*}
(\text{id} \otimes \tilde{\partial}^I)\Delta &= (\text{id} \otimes \epsilon^I \otimes \text{id})(\text{id} \otimes \Delta)\Delta \\
&= (\text{id} \otimes \epsilon^I \otimes \text{id})(\cdot \otimes \text{id} \otimes \text{id})\tau_{23}(\Delta \otimes \Delta)\Delta \\
&= (\cdot \otimes \text{id})(\text{id} \otimes \text{id} \otimes \epsilon^I \otimes \text{id})\tau_{23}(\Delta \otimes \Delta)\Delta \\
&= (\cdot \otimes \text{id})(\text{id} \otimes \epsilon^I)(\Delta \otimes \Delta)\Delta
\end{align*}$$
\[
\begin{align*}
&= (\cdot \otimes \text{id})(\text{id} \otimes \Delta)(t^a_{i_0} \varepsilon_{i_1 a} \otimes \text{id}) \Delta \\
&= (\cdot \otimes \text{id})(\text{id} \otimes \Delta)t^a_{i_0} \otimes \partial^{i_1 a} \\
&= (t^a_{i_0} \otimes 1) \Delta \circ \partial^{i_1 a}
\end{align*}
\]

where the first equality is our new form for \( \partial^I \), the second equality is codistributivity, the third and fourth are rearrangements, the fifth is our property for \( \varepsilon^I \) above in relation to \( \Delta \). We then rearrange to write in terms of \( \partial \) again.

This result for \( \partial^I \) then implies

\[
(id \otimes t^b_{i_0} \partial^{i_1 b}) \Delta = (t^a_{i_0} \otimes t^b_{i_0}) \Delta \partial^{i_1 a} = \Delta \circ t^a_{i_0} \partial^{i_1 a}
\]

which is the left-invariance result for \( \tilde{\partial} \).

On any bialgebra or Hopf algebra \( A \) one has the left regular representation of \( A^* \) defined by

\[
h \triangleleft a = \sum a^{(1)} < a^{(2)}, h > \text{ where } \Delta a = \sum a^{(1)} \otimes a^{(2)} \text{ and } < , > \text{ is the pairing. These operators } A \to A \text{ are also left-invariant in the sense } \Delta(h \triangleleft a) = (\text{id} \otimes h \triangleleft) \Delta a \text{ as above. Conversely, every left-invariant operator is of this form for some } h \in A^*. \text{ Given a left-invariant operator } D, \text{ the required linear functional is } h = \varepsilon \circ D \text{ where } \varepsilon \text{ is the counit.}
\]

**Proposition 3.3**  Our left-invariant differential operators \( \tilde{\partial}^I \) correspond to linear functionals \( L^I \in A(A(R)^* \text{ given by } L^I_j(1) = 0 \) and

\[
L^I_j(t^{i_1}_{I_1} \cdots t^{i_m}_{I_m}) = \delta^i_{j_1} \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \cdots \delta^{i_m}_{j_m} + R^{i_2}_{i_1} R^{i_3}_{j_1 j_2} R^{i_4}_{j_3} \cdots \delta^{i_m}_{j_m} + \sum_{r=2}^m R^{r}_{a_{r-2} a_{r-1}} R^{a_{r-2}}_{a_{r-1}} \cdots R^{a_{r-1}}_{a_{r-2}} \Delta^{i_1}_{I_1} \Delta^{i_2}_{I_2} \cdots \Delta^{i_m}_{I_m}
\]

**Proof**  The action of \( \partial^I \) on monomials is given in \([3]\) in terms of \( R \). We write this in terms of \( R \) as in \([3]\), convert to \( \tilde{\partial}^I \) by another factor \( t \) and then evaluate against the counit. Thus we compute

\[
L^I_j(t_{I_1} \cdots t_{I_m}) = \epsilon(t^a_j \delta^{(a)}_{J_1} t_{J_1} \cdots t_{J_m} [m, R]^{J_1 \cdots J_m} )
\]

where \( J = (j_0, j_1) \) etc as before. The braided integers are defined in \([3]\) by \([m : R] = 1 + PR_{12} + \cdots + PR_{12} PR_{23} \cdots PR_{m-1, m} \). Putting the form of \( R \) into this and evaluating with 

\[
\epsilon(t_I) = \delta_I = \delta^{i_0 i_1}
\]

gives the result shown after conversion to another notation for the numbering of the indices. \( \Box \)
This is as a pair of single-row transfer matrices. It seems likely that at least for the standard $R$-matrices these linear functionals restrict to the group function algebras and can be written in terms of the FRT generators $l^\pm$ for the quantum enveloping algebra. We have not found any general formula of this form, however.

Now we study the algebra generated by these $\tilde{\partial}^I$ inside the braided Heisenberg-Weyl algebra. By definition any closed commutation relations inside here are to be thought of as restrictions of some braided-Lie derivative. In general we expect the $\tilde{\partial}^I$ to close and form a braided-Lie algebra version of $A^*$ via the correspondence in the last proposition.

Before describing this, we recall the braided-matrix algebras $B(R)$ introduced in [2]. This works for any bi-invertible $R$-matrix (not necessarily Hecke) and has a matrix of generators $u = (u^i_ia_i)$ with braiding and relations as follows. To be compatible with the left-handed conventions above it turns out we need $B(R_{21})$ rather than $B(R)$. Then

$$\Psi(u_I \otimes u_J) = u_B \otimes u_A \Psi^B_{J A} \Psi^K_{K L} = R^{c}_{i_0} d R^{-1i_1}_{a} a_{j_0} R^{a}_{k_1} j_1 b \tilde{R}^{k_0}_{c} b \; l_1,$$

$$u_I u_J = u_B u_A \Psi^B_{J A} \Psi^K_{K L} = R^{-1i_0}_{d} R_{a} d^{i_1}_{j_0} a_{k_1} j_1 b \tilde{R}^{k_0}_{c} b \; l_1.$$

These matrices are taken from [2] with some minor rearrangement of conventions for our present purposes. The relations and braiding can also be written in the compact form $R u_1 R_{21} u_2 = u_2 R u_1 R_{21}$ and $R u'_1 R^{-1} u_2 = u_2 R u'_1 R^{-1}$. The first of these is also known in other contexts, while the second arises only in the theory of braided groups. Our point of view in [2] was as defining the braided-commutative algebra of functions on a braided version of $M_n$. Thus $\Psi$ obeys the QYBE and $\Psi'$ is a variant of it so that the relations are like the super-commutativity for the functions on a super-space.

**Proposition 3.4** The braiding between $\tilde{\partial}^I$ and their relations inside the braided Heisenberg-Weyl algebra are

$$\Psi(\tilde{\partial}^I \otimes \tilde{\partial}^J) = \Psi^I_{A' B} \tilde{\partial}^B \otimes \tilde{\partial}^A, \quad \tilde{\partial}^I \tilde{\partial}^J - \Psi^I_{A' B} \tilde{\partial}^B \tilde{\partial}^A = \tilde{\partial}^{i_1}_{a_0} \delta^{i_1}_{j_0} - \Psi^I_{A' B} \tilde{\partial}^{a_1}_{b_0} \delta^{b_1}_{a_0}.$$ 

**Proof** The braidings are computed from (3), (9) and (10)-(11) for the braidings between $\partial, t$. To braid $t^a_{i_0} \otimes \partial^i_1 a$ past $t^b_{j_0} \otimes \partial^j_1 b$ first braid $\partial^i_1 a$ past $t^b_{j_0}$, then the $t$ resulting from this past $\partial^j_1 b$. Then braid $t^a_{i_0}$ to the right in the same way. The result is the braiding shown.
The computation for the relations is similar, using this time the relations of $A(R)$ for $\partial$ and the Heisenberg-Weyl relations (12) between $t$ and $\partial$. Between the $t$ and $\tilde{\partial}$ the relations come out as

$$\tilde{\partial}^{i_1}{}_{i_0} t^{j_0}{}_{j_1} = \delta^{i_1}{}_{j_1} t^{j_0}{}_{i_0} + t^{j_0}{}_{a} \tilde{\partial}^{b} c e^{R}_{a}{}^{d}{}_{i_0} d R^{i_1}{}_{j_1}{}^{b}{}_{b}$$

$$t^{i_0}{}_{i_1} \tilde{\partial}^{j_1}{}_{j_0} = \tilde{R}^{a}{}_{b}{}^{j_1}{}_{j_0} c e d^{i_0}{}_{a} R^{-1}{}^{d}{}_{j_0}{}^{b}{}_{i_1} - t^{i_0}{}_{a} \tilde{R}^{b}{}_{c}{}^{j_1}{}_{j_0} R^{-1}{}^{a}{}^{b}{}_{i_1}$$

The second of these is found by applying $\tilde{R}$ and $R^{-1}$ to both sides of the first. Using these relations one finds easily the result stated. □

The classical limit of these constructions is with $R = \text{id}$. Then the relations between $\tilde{\partial}$ become

$$[\tilde{\partial}^{i_1}{}_{i_0}, \tilde{\partial}^{j_1}{}_{j_0}] = \delta^{i_1}{}_{j_1} \tilde{\partial}^{j_1}{}_{i_0} - \delta^{i_1}{}_{i_0} \tilde{\partial}^{j_1}{}_{j_0}$$

which is how the Lie algebra $gl_n$ is realised as left-invariant vector fields on the functions on $M_n$.

Our constructions are exactly a braided version of this. On the other hand, we have not found a suitable braided coalgebra structure such that the relations in the proposition form a braided Lie algebra in the strict sense of [14]. Nevertheless, the two approaches are closely related and perhaps equivalent.

For our canonical example, we take the $GL_2$ R-matrix (5) as in Section 2 and write

$$[\tilde{\partial}^I, \tilde{\partial}^{J}]_{\psi'} = \tilde{\partial}^I \tilde{\partial}^J - \psi' A^I{}^J B \tilde{\partial}^B \tilde{\partial}^A,$$

$$(\tilde{\partial}^{i}_j) \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Then the $gl_2$ braided-Lie algebra-like structure from Proposition 3.4 comes out explicitly as

$$[\alpha, \beta]_{\psi'} = -\beta = -[\beta, \alpha]_{\psi'}, \quad [\alpha, \gamma]_{\psi'} = q^{-2} \gamma = -[\gamma, \alpha]_{\psi'}, \quad [\beta, \gamma]_{\psi'} = \delta - q^{-2} \alpha = -[\gamma, \beta]_{\psi'}$$

$$[\alpha, \delta]_{\psi'} = (q^{-2} - 1)(\delta - q^{-2} \alpha) = -[\delta, \alpha]_{\psi'}, \quad [\delta, \alpha]_{\psi'} = 0$$

$$[\beta, \delta]_{\psi'} = -q^{-4} \beta = -q^{-4}[\delta, \beta]_{\psi'}, \quad [\gamma, \delta]_{\psi'} = \gamma(1 + q^{-2} - q^{-4}), \quad [\delta, \gamma]_{\psi'} = -q^{-2} \gamma$$

with the remaining three zero. These can be compared with the braided-Lie algebra $gl_2$ in [14].

Meanwhile, the relations themselves from Proposition 3.4 compute as

$$\alpha \beta - q^{-2} \beta \alpha = -q^{-2} \beta, \quad \alpha \gamma - q^{-2} \gamma \alpha = \gamma$$

$$\beta \gamma - \gamma \beta = (1 + \alpha(q^2 - 1))(\delta - q^{-2} \alpha), \quad \alpha + \delta \quad \text{central.}$$
The braiding is computed from $\Psi$ and is similar to that for the $gl_2$ generators. We do not need it directly in the algebra. The algebra-relations themselves are an extension by linear terms of a variant of the quadratic braided-matrices algebra $BM_q(2) = U(gl_2)$. To see this note that one may rescale the $\tilde{\partial}$ so that a new parameter $\hbar$ appears in front of all the linear terms above. Then the limit $\hbar \to 0$ recovers a variant of these algebras, namely an algebra isomorphic to $BM_{q-1}(2)$ after a change and rescaling of the generators. The braided-matrices algebra was studied extensively in [2] where we explained the sense in which its relations were those of ‘braided-commutativity’. The equations above now add a linear right hand side to the corresponding braided-commutator.

4 Coaddition on rectangular quantum matrices

In this section we extend the results of the last two sections to coaddition and differentiation for rectangular quantum matrices. For $m \times 1$ or $1 \times n$ we recover the addition law and differential calculus for quantum planes in [3][3]. For square $n \times n$ matrices we recover of course the results above. Most importantly however, all of these are now unified into a single linear-differential calculus. Thus, as an application we compute the vector fields (orbital angular momentum operators) for the coaction of a quantum matrix group on a quantum space. From this point of view, the $1 \times n$ row vector is the $q$-deformed position co-ordinate vector $x = (x_i)$.

The general $m \times n$ quantum matrices also have a physical interpretation as follows. We can think of each row as a particle position co-ordinate and the entire matrix as a lattice model of a trajectory $\{x_i(t)\}$ say. Then the ability to add rectangular quantum matrices is crucial and in turn defines the notion $\frac{\delta}{\delta x_i(t)}$ of functional differentiation. Likewise, one can view a wave-function on space as a row vector $\psi(x)$ or a general vector-valued field as a rectangular quantum matrix $\{\psi^\alpha(x)\}$. If one is serious about $q$-deforming physical constructions, one needs to be able to add such fields pointwise, as well as to functionally differentiate with respect to them. Hence the results in this section are the first and most basic steps towards a systematic $q$-deformed or braided classical field theory and a theory of $q$-deformed or braided path-integration. One may also expect plenty of other applications of our basic notions of addition and differentiation.

We recall first the definition of rectangular quantum matrices as introduced previously in [3] in the theory of block decomposition of quantum matrices into quantum blocks. The idea
behind the definition is to use two independent solutions $R_1, R_2$ say of the QYBE, one for the rows and one for the columns. They can be of different dimensions. The associated quantum matrices may be (co)-multiplied whenever the columns of one matches the rows of the other, not only in dimension (as usual for matrices) but in flavour of QYBE solution also. The $R_i$ need not in fact obey the QYBE but that is the case of most immediate interest in the present paper.

Given $R_1 \in M_m \otimes M_m$ and $R_2 \in M_n \otimes M_n$, the associated quantum matrix algebra $A(R_1 : R_2)$ has a matrix of generators $x = (x^i_\mu)$ where greek indices run $\mu = 1, \cdots, m$ and latin ones $i = 1, \cdots, n$, and relations

$$R_1 x_1 x_2 = x_2 x_1 R_2,$$

i.e. $R_1^{\mu \nu} x^\alpha_i x^\beta_j = x^\nu_b x^\mu_a R_2^{a \beta}_{b \mu}$. \hfill (16)

The multiplication between compatible rectangular matrices is expressed now as a family of algebra homomorphisms:

$$\Delta_{R_1,R_2,R_3} : A(R_1 : R_3) \rightarrow A(R_1 : R_2) \otimes A(R_2 : R_3)$$ \hfill (17)

for any three matrices $R_i$. The map is given by the matrix coproduct of the relevant generators, $\Delta x^s = x^s_a \otimes x^a_s$ where $s$ runs in the range appropriate for $R_3$. Moreover, the family of maps taken together are coassociative in the sense:

$$(\text{id} \otimes \Delta_{R_2,R_3,R_4}) \circ \Delta_{R_1,R_2,R_3} = (\Delta_{R_1,R_2,R_3} \otimes \text{id}) \circ \Delta_{R_1,R_2,R_3,R_4}$$ \hfill (18)

as a map $A(R_1 : R_4) \rightarrow A(R_1 : R_2) \otimes A(R_2 : R_3) \otimes A(R_3 : R_4)$. This includes all possible coassociativity conditions arising from associativity of rectangular matrix multiplication. In summary, rectangular quantum matrices are not individually bialgebras but they all fit together into a weaker ‘co-groupoid’ structure on the entire family.

These maps include both coproducts for rectangular matrices, when (18) reduces to usual coassociativity, and matrix coactions

$$\beta_L : A(R_1 : R_2) \rightarrow A(R_1) \otimes A(R_1 : R_2), \quad \beta_R : A(R_1 : R_2) \rightarrow A(R_1 : R_2) \otimes A(R_2)$$

corresponding to matrix multiplication of a rectangular matrix by a square quantum matrix from the left or the right. Then (18) reduces to the comodule property for these maps.

Since we can add elements of quantum planes via a braided-coaddition, it is natural to do this too for general rectangular or square quantum matrices. We have covered the most
important square case in Section 2 and now we summarise the extension to the rectangular case. It was announced in [6]. From now on, we assume that all matrices $R_i$ are solutions of the QYBE and of Hecke type.

**Proposition 4.1** Let $R_i$ be Hecke solutions of the QYBE. Then the rectangular quantum matrices $A(R_1 : R_2)$ form a braided-Hopf algebra

$$\Psi(x_1 \otimes x_2) = (R_1)_{21} x_2 \otimes x_1 R_2, \quad \Delta_{R_1,R_2} x = x \otimes 1 + 1 \otimes x, \quad \epsilon x = 0.$$  

Moreover, the coaddition is codistributive with respect to the product of quantum matrices in the sense

$$(id \otimes \cdot) \circ (id \otimes \tau \otimes id) \circ (\Delta_{R_1,R_2,R_3} \otimes \Delta_{R_1,R_2,R_3}) \circ \Delta_{R_1,R_2} = (\Delta_{R_1,R_2} \otimes id) \circ \Delta_{R_1,R_2,R_3}$$

$$(\cdot \otimes id) \circ (id \otimes \tau \otimes id) \circ (\Delta_{R_1,R_2,R_3} \otimes \Delta_{R_1,R_2,R_3}) \circ \Delta_{R_1,R_2} = (id \otimes \Delta_{R_1,R_2}) \circ \Delta_{R_1,R_2,R_3}$$

where $\tau$ denotes usual transposition.

**Proof** The detailed proof follows the same strategy as in the proofs of Theorem 2.1 and Proposition 2.3 in Section 2. The refinement is to allow the indices in the matrix products to run over their relevant ranges and to use the corresponding $R$-matrices. ⊓⊔

We can also regard $A(R_1 : R_2)$ as a single quantum covector space as in [8] but now

$$R^A_{i} B_j = R_{i_0 a_0}^{j_0} a_0^{i_1} b_1^{j_1}, \quad R^t_{i} B_j = R_{i_0}^{j_0} a_0^{i_1} b_1^{j_1}. \quad (19)$$

The conventions are with $x_I = x_i^{0, i_1}$ so all 0-subscripted indices run over the range for $R_1$ and all 1-subscripted indices over the range for $R_2$. Finally, it is easy to see that $A(R_1 : R_2)$ lives as a braided-Hopf algebra in the braided category of $A(R_1) - A(R_2)$–bicomodules via left, right coactions $\beta_L, \beta_R$. This category is also that of right comodules of $A(R_1)^{\text{cop}} \otimes A(R_2)$ which is a dual-quasitriangular bialgebra. The two points of view are connected along the lines of [8]. This summarises the rectangular version of Section 2, which results are recovered now as special cases. But we also recover just as well the coaddition on the quantum planes and compatibility (‘colinearity’) of the coaction of the corresponding quantum matrix groups on them as a new special case. This will be our main application in the present section. Note that in our unified approach the one-dimensional $R$-matrices $(q)$ and $(-q^{-1})$ are perfectly good
solutions of the QYBE of Hecke type. Then \( A(R : q) \) and \( A(R : -q^{-1}) \) are two quantum column vector algebras, equipped with natural left-coactions of \( A(R) \). Likewise \( A(\frac{q}{R} : R) \) and \( A(-q^{-1} : \frac{R}{q}) \) are two quantum row vector algebras equipped with natural right-coactions of \( A(R) \). This includes the known facts about quantum planes associated to Hecke \( R \)-matrices into a single framework.

The results of Section 3 likewise generalise just as easily. Regarding \( \{x_I\} \) as quantum \( mn \)-dimensional covector space with braiding and relations from \( [13] \) we apply \( [4] \) and have at once rectangular \( n \times m \)-matrices \( \partial_i^a = \frac{\partial}{\partial x^a_i} \) or in the multi-index notation

\[
\partial^I = \partial^i_{i_0} : A(R_1 : R_2) \to A(R_1 : R_2)
\]
defined in the same way as an infinitesimal left translation. It sends \( f(x) \) to the coefficient of \( a^{\alpha_{i_1}} \) in \( f(a + x) \) where \( a \) is another rectangular quantum matrix with statistics \( x_1 a_2 = (R_1)_{21} x_2 a_1 R_2 \). This time a computation along the lines of Proposition 3.1 gives that \( \partial^I \) obey the relations \( \partial_1 \partial_2 R_1 = R_2 \partial_2 \partial_1 \) of \( A((R_2)_{21} : (R_1)_{21}) = A(R_2 : R_1)^{op} \). The general rule is to use the same formulae as in Section 3 but with \( R_1 \) or \( R_2 \) chosen according to the flavour of the indices. The same pattern applies to the braidings between \( \partial^I \) and \( x_J \) in \( (11)-(10) \). Finally, the braided Heisenberg-Weyl algebra \( (2) \) generalises in the same way as

\[
\partial^I x^{\nu j} - R_{1\mu}^{\alpha \nu} A_{\alpha \beta}^{\mu} \beta_\beta^\gamma a R_{2j}^{\alpha} i \beta_\alpha^\gamma = \delta_j^\nu \delta^\nu_{\mu}.
\]

(20)
The elements \( f_I(x) \partial^I \) are the quantum vector fields on \( A(R_1 : R_2) \).

Consider now such a quantum matrix algebra \( A(R_1 : R) \) as a right \( A(R) \)-comodule algebra by \( \beta_R \) corresponding to multiplication from the right of an \( m \times n \) quantum matrix \( x \) by an \( n \times n \) quantum matrix \( t \). This coaction induces quantum-vector fields on \( A(R_1 : R) \) in just the same fashion as the left-invariant vector fields in Section 3. The basis of these vector fields is provided now by

\[
\tilde{\partial}_j^I = x^{\alpha_j} \partial^\alpha_\alpha : A(R_1 : R) \to A(R_1 : R).
\]

These operators commute with the left \( A(R_1) \)-comodule algebra structure \( \beta_L \) corresponding to multiplication from the left by a \( m \times m \) quantum matrix. Thus they are left-invariant in the sense

\[
(id \otimes \tilde{\partial}_j^I) \beta_L = \beta_L \tilde{\partial}_j^I.
\]
Thus the construction of Proposition 3.2 generalises to the rectangular case. The proofs are strictly analogous.

Finally, the braiding and relations among the quantum vector fields $\tilde{\partial}^i_j$ can be computed just as in Section 3. They come out exactly as in Proposition 3.4 (independently of $R_1$). The reason is that while the intermediate computations for $\tilde{\partial}^i_j x^\mu_k$ etc involve both $R$ and $R_1$, the resulting formulae for the braiding and relations between $\tilde{\partial}$ involve only the latin indices associated to $R$. Thus we see that exactly the same braided-Lie algebra structure which acted in Proposition 3.4 as left-invariant vector fields on $A(R)$ now acts as $\beta_L$-invariant vector fields on $A(R_1 : R)$.

**Corollary 4.2** Let $R_1 = (q)$ and $R$ a Hecke solution of the QYBE. Then $A(q : R)$ is the covector algebra with generators $\mathbf{x} = (x_i)$ and relations $q x_1 x_2 = x_2 x_1 R$. The partial derivatives $\partial = (\partial^i)$ obey the relations $\partial_1 \partial_2 q = R \partial_2 \partial_1$ of the vector algebra $A(R_2, q)$. Inside the Heisenberg-Weyl algebra $\partial^i x_j - x_a R^a_{\beta j} \beta^\beta = \delta^i_j$ one has the vector fields for the coaction $\mathbf{x} \rightarrow \mathbf{x} t$ of $A(R)$. They are given by

$$\tilde{\partial}^i_j = x_j \partial^i$$

and obey the braided-Lie algebra relations and braiding in Proposition 3.4.

**Proof** The first part reduces, as it must, to the differential calculus for quantum planes in [4]. This part is not limited to the Hecke case. The second part concerning $\tilde{\partial}^i_j$ is our new result and follows at once from the theory above. One can also verify it directly. Note that if we introduce a $\hbar$ on the right hand side of the Heisenberg-Weyl algebra, it appears also on the right hand side of the Lie-algebra-like relations in Proposition 3.4. Moreover, sending $\hbar \rightarrow 0$ then recovers a variant of Proposition 3.5 of [12], of which this corollary is therefore a generalisation. □

For our canonical example with $R$-matrix (5) we have for $\mathbf{x} = (x, y)$ the usual quantum plane $yx = qxy$. Its partial derivatives $\partial = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ were recovered in our general scheme as the operators on polynomials in $x, y$ given in Example 2.4 of [4]. Using the formulae there, one may easily verify that

$$\alpha = \tilde{\partial}^1_1 = x \frac{\partial}{\partial x}, \quad \beta = \tilde{\partial}^1_2 = y \frac{\partial}{\partial x}, \quad \gamma = \tilde{\partial}^2_1 = x \frac{\partial}{\partial y}, \quad \delta = \tilde{\partial}^2_2 = y \frac{\partial}{\partial y}$$

obey the relations (14)–(15) as they must according to the above corollary.
These quantum vector fields are like orbital angular momentum because they are induced by the coaction of $A(R)$ and are indeed the action of some braided version of $gl_n$ as explained at the end of Section 3. Such orbital angular momentum realisations of quantum enveloping algebras or their braided-Lie algebras can surely be constructed by hand in low dimensions. However, we have derived them here in an entirely systematic way and one that works for all dimensions and general (Hecke) R-matrices. We note that some physically relevant cases such as rotations in $1+3$ dimensions can be expected to follow along broadly similar lines but require us to leave the Hecke setting. This is a topic for further work.

Finally, we recall a different kind of possible application. This is to view infinite-dimensional row or column matrices as fields or trajectories, as mentioned above. This time we could consider for example that $x^i_\mu = x_i(\mu)$ where $\mu$ is a discretised time variable. Matrix multiplication from the left would now be convolution of a time-dependent function against a matrix kernel. Moreover, our derivatives $\partial^i_\mu = \frac{\delta}{\delta x_i(\mu)}$ become functional derivatives. The point is that the superposition principle ensures that many constructions in field theory are nothing more than infinite-dimensional linear algebra. When discretised one has in the q-deformed setting many copies (one at each lattice site) of some non-commutative algebra. One also has in general the need for non-commutation relations between the sites. In this case one needs a systematic formulation of such algebras and one that expresses linearity and other familiar properties. The general theory of rectangular quantum matrices achieves this at least in the $GL_n$ or Hecke setting. For example, a classical but q-deformed trajectory in one dimension could be formulated as living in $A(q : R_\infty)$ where $R_\infty$ is the $GL_\infty$ R-matrix. Discretised wave-functions too can in principle be formulated this way. Some detailed applications of this point of view will be developed elsewhere, perhaps in relation to $\square$. For the present we limit ourselves to a remark about quantum loop groups arising from this general point of view. This is the topic of the next section.

5 Remark on lattice model of Kac-Moody Lie algebras

We conclude by explaining how the notion of rectangular quantum matrix studied in the last section leads naturally to the algebra of the discretised wave-functions on in the lattice approximation to Kac-Moody algebras in $\square$. We learn that we can add such fields, as well as differentiate with respect to them. There remarks then can be viewed as a small step towards
a braided-geometrical picture of a Kac-Moody algebra as a q-deformed or braided loop-group algebra.

Thus, let $G$ be a group and $LG$ the group of maps or trajectories $S^1 \to G$ with pointwise product. Now $C(LG)$ means functions on the space of such maps, i.e. in a discrete approximation it means $C(G \times G \times \cdots \times G)$. The q-deformed version of this is therefore $A(R) \otimes A(R) \otimes \cdots \otimes A(R)$ or a quotient if it by determinants etc. Here $\otimes$ has to be specified but it is natural to take here the braided-tensor product algebra as introduced in the theory of braided groups. We have explained in [16] that if one takes the braiding $\Psi(t_1 \otimes t_2) = t_2 \otimes t_1 R$ corresponding to $A(R)$ as a right $A(R)$-comodule by right-comultiplication, then one has from the definition of braided tensor product algebras as in (2) that the algebra $A(R)^{\otimes \infty}$ has generators $t(i)$ and relations

$$t_1(i) t_2(j) = \begin{cases} t_2(j) t_1(i) R_{21}^{-1} & \text{if } i < j \\ R_{12}^{-1} t_2(j) t_1(i) R_{12} & \text{if } i = j \\ t_2(j) t_1(i) R_{12} & \text{if } i > j \end{cases}$$

This is essentially the wave-function algebra algebra in [6] as we have remarked in [16].

One would still like the point of view whereby such braided tensor products are derived automatically rather than put in by hand. Such a point of view is the following, at least for the Hecke case such as corresponding to $G = GL_n$. We think of $LG$ directly as a wave-function or field on $S^1$. From this point of view it is natural to model it as a $n \times \infty$ (or $n \times nN$) rectangular quantum matrix. This consists of consecutive $n \times n$ blocks, each corresponding to one site on $S^1$. For the rows then we naturally take the $GL_n R$-matrix. For the columns we take the $GL_{\infty} R$-matrix $R_{\infty}$, or more precisely a $GL_{nN} R$-matrix to be finite.

**Proposition 5.1** The rectangular quantum matrix algebra $A(R : R_{\infty})$ can be identified by cutting into $n \times n$ blocks with the algebra $A(R)^{\otimes \infty}$ in (21).

**Proof** This is an application of the general theory of glueing of quantum block matrices introduced by the author and M. Markl in [3]. There we show that the $GL_{nN} R$-matrix can be built up as $R \oplus_q \cdots \oplus_q R$ where $\oplus_q$ is a certain associative glueing operation among Hecke R-matrices (which we also introduce). On the other hand, we show in [3, Sec. 4] that $A(R_1 : R_2 \oplus_q R_3) = A(R_1 : R_2) \otimes_{R_1} A(R_1 : R_3)$ where the braided tensor product is with braiding given by $R_1$. Iterating this formalism in our present example gives the result. □

Note that the discussion supposes that a limit $N \to \infty$ can also be taken in some way. The present remarks are purely algebraic and we do not address this point here. On the other
hand (at the $M_q(n)$ level) one can now add the wave-functions pointwise with braid statistics. One can also differentiate, both with respect to the field and in the group indices alone by using the techniques above. These are some of the ingredients used when formulating classical $\sigma$-models. In 2D-quantum gravity one also has exchange algebras which can be described as braided tensor product algebras\cite{16} and hence as rectangular quantum matrices in the same way. Thus the general scheme of using rectangular quantum matrices to systematically make $q$-deformed lattice approximations in field theory, as explained at the end of Section 4, seems to apply fairly generally. Moreover, the results in \cite{8} apply generally to provide a local description as quantum blocks with braided tensor product statistics between them. This indicates an interesting direction for further work.

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