LOGARITHMIC TRACE AND ORBIFOLD PRODUCTS

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ABSTRACT. The purpose of this paper is to give a purely equivariant definition of orbifold Chow rings of quotient Deligne-Mumford stacks. This completes a program begun in [JKK] for quotients by finite groups. The key to our construction is the definition (Section 6.1), of a twisted pullback in equivariant K-theory, $K_G(X) \rightarrow K_G(I_G^2(X))$ taking non-negative elements to non-negative elements. (Here $I_G^2(X) = \{(g_1, g_2, x)|g_1 x = g_2 x = x\} \subset G \times G \times X$.) The twisted pullback is defined using data about fixed loci of elements of finite order in $G$, but depends only on the underlying quotient stack (Theorem 6.3). In our theory, the twisted pullback of the class $T \in K_G(X)$, corresponding to the tangent bundle to $[X/G]$, replaces the obstruction bundle of the corresponding moduli space of twisted stable maps. When $G$ is finite, the twisted pullback of the tangent bundle agrees with the class $R(\mathfrak{m})$ given in [JKK] Definition 1.5. However, unlike in [JKK] we need not compare our class to the class of the obstruction bundle of Fantechi and Göttsche [FG] in order to prove that it is a non-negative integral element of $K_G(I_G^2(X))$.

We also give an equivariant description of the product on the orbifold $K$-theory of $[X/G]$. Our orbifold Riemann-Roch theorem (Theorem 7.3) states that there is an orbifold Chern character homomorphism which induces an isomorphism of a canonical summand in the orbifold Grothendieck ring with the orbifold Chow ring. As an application we show (Theorem 8.7) that if $\mathcal{X} = [X/G]$, then there is an associative orbifold product structure on $K(\mathcal{X}) \otimes \mathbb{C}$ distinct from the usual tensor product.

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1. Introduction

1.1. Statement of results. In this paper we work in the algebraic category and consider quasi-free actions of arbitrary algebraic groups on arbitrary smooth varieties (or more generally algebraic spaces). In practice most smooth, separated Deligne-Mumford stacks have natural presentations as quotients of the form \([X/G]\) with \(X\) smooth and \(G\) an algebraic group acting properly on \(X\). (The paper [EHKV] gives criteria for a Deligne-Mumford stack to be a quotient stack. There are in fact no known examples of separated Deligne-Mumford stacks which are provably not quotient stacks.)

Our first main result is a purely equivariant description of the orbifold product on the Chow groups of the inertia stack \(\mathcal{I}_X = [I_G(X)/G]\), where

\[I_G(X) = \{(g, x)|gx = x\} \subseteq G \times X.\]

The product depends only on data about fixed loci of elements of finite order and makes no reference to moduli spaces and obstruction bundles. This completes a program begun in [JKK]. In particular we show, without use of the character formula of [JKK, Lemma 8.5], that when \(G\) is finite the class \(R(m)\) given in [JKK, Definition 1.5] is a non-negative integral element of \(K\)-theory. To do this
we define (Section 6.1), for arbitrary $G$ acting quasi-freely on a smooth variety $X$, an explicit twisted pullback map on integral equivariant $K$-theory $K_G(X) \to K_G(X/G)$, where $K_G^2(X) = I_G(X) \times_X I_G(X)$. The twisted pullback takes non-negative elements to non-negative elements and depends only on the underlying quotient stack $\mathcal{X} = [X/G]$—not on the specific choice of presentation $X$ and $G$. The twisted pullback we define for quotient Deligne-Mumford stacks is a special case of a more general construction given in Sections 4 and 5. For the reader’s convenience we outline the construction in Section 1.2 below.

When $G$ is finite, the class $R(m)$ of [JJK] is a component of the twisted pullback of the tangent bundle $TX$. More generally, for $G$ acting with finite stabilizer on $X$ we define a product on the equivariant Chow groups $A_G^*(I_G(X))$ by the formula

$$\alpha \ast_G \beta = \mu_* (e_i^* \alpha \cup e_2^* \beta \cup \varepsilon(T^\text{tw})).$$

Here $\mu: I_G^2(X) \to I_G(X)$ is induced from the group multiplication $G \times G \to G$, the map $e_i$ is the projection onto the $i$-th factor, $T^\text{tw}$ is the twisted pullback of the class $T \in K_G(X)$ corresponding to the tangent bundle of $[X/G]$, and $\varepsilon(T^\text{tw})$ denotes the Euler class. In Theorem 6.10 we prove that the $\ast_G$ product is commutative and associative by showing that it satisfies the sufficiency conditions (Propositions 3.7, 3.9, 3.12) for an inertial product to be commutative and associative. Example 6.15 gives an example of a different associative product on $A_G^*(I_G(X))$. An interesting question is to classify all possible associative products on the Chow groups $A_G^*(I_G(X))$.

The $\ast_G$ product can be explicitly calculated using the decomposition of $I_G(X)$ into a disjoint sum of components $\bigsqcup_{\Psi} I(\Psi)$, where the sum runs over all conjugacy classes $\Psi \subset G$ of elements of finite order. Here $I(\Psi) = \{(g, x)|gx = x, g \in \Psi\}$. Note that this disjoint sum is finite, as $I(\Psi) = \emptyset$ for all but finitely many conjugacy classes $\Psi$. The equivariant Chow groups $A_G^*(I(\Psi))$ may be identified with $A_G^*(X^\Psi)$, where $g \in \Psi$ is any element and $Z = Z_G(g)$ is the centralizer of $g$ in $G$. In this way the $\ast_G$ product can be computed purely in terms of the fixed point data of elements of finite order for the action of $X$ on $G$.

When $X$ is a smooth scheme an analogous definition can be made in equivariant $K$-theory, replacing the Euler class of $T^\text{tw}$ in the equivariant Chow group with the K-theoretic Euler class $\lambda_{-1}((T^\text{tw})^*)$. In this case we define an orbifold Chern character

$$c_G: K_G(I_G(X)) \otimes \mathbb{Q} \to A_G^*(I_G(X)) \otimes \mathbb{Q}$$

and prove (Theorem 7.3) that it preserves the corresponding orbifold products. The orbifold Chern character is not an isomorphism but it restricts to an isomorphism on a summand in $K_G(I_G(X)) \otimes \mathbb{Q}$ which depends only on the stack $[X/G]$. When $G$ is finite this summand equals the small orbifold $K$-theory defined in [JJK].

Finally in Section 8 we combine the orbifold Riemann-Roch theorem with the non-Abelian localization theorem of [EC3] to obtain a twisted, or orbifold, product
on $K_G(X) \otimes \mathbb{C}$. Again this product depends only on the underlying quotient stack $[X/G]$ and not on the particular presentation.

1.2. Twisted pullbacks. Let $G$ be an algebraic group acting on a space $X$ with arbitrary stabilizers. Let $\mathbf{m} = (m_1, \ldots, m_l)$ be an $l$-tuple of elements in $G$ (not necessarily of finite order) which lie in a compact subgroup $K \subset G$ and satisfy $\prod_{i=1}^l m_i = 1$. And let $Z = Z_G(\mathbf{m}) = \bigcap_{i=1}^l Z_G(m_i)$. In Section 5.2 we define a twisted pullback map $K_G(X) \to K_Z(X^\mathbf{m})$, called the logarithmic restriction, where $X^\mathbf{m}$ consists of the subset of $X$ consisting of points fixed by $m_i$ for all $i = 1, \ldots, l$. This map takes non-negative elements to non-negative elements and can be used to define (Section 2) a twisted pullback map $K_G(X) \to K_G(\mathbb{C}[\Phi(\mathbf{m})])$, where $\mathbb{C}[\Phi(\mathbf{m})]$ is the set of pairs $(\mathbf{g}, x) \in G^l \times X$ such that $\mathbf{g} = (g_1, \ldots, g_l)$ is conjugate to $\mathbf{m}$ under the diagonal conjugation action of $G$ on $G^l$ and $x$ is fixed by each $g_i$ for $i = 1, \ldots, l$. When $G$ acts quasi-freely, the twisted pullback $K_G(X) \to K_G(\mathbb{C}[\Phi(\mathbf{m})])$ is defined using the decomposition of $\mathbb{C}[\Phi(\mathbf{m})]$ into open and closed components indexed by diagonal conjugacy classes in $G \times G$.

In a subsequent paper, we plan to use the general twisted pullback construction to define "stack products" for Artin quotient stacks.

1.3. Connection to orbifold cohomology and other literature. Orbifold cohomology was originally defined by Chen and Ruan in their landmark paper [CR]. They showed that there is a $\mathbb{Q}$-graded, associative, super-commutative product structure on the cohomology groups of the inertia orbifold $\mathcal{X}$ associated to an orbifold $\mathcal{X}$. The orbifold cohomology ring is the degree-zero part of the quantum cohomology of the orbifold $\mathcal{X}$, and the product is defined via integration against the virtual fundamental class of the moduli stack of ghost maps.
Subsequently there has been a great deal of interest in the orbifold product. Simpler descriptions of orbifold cohomology have been given for global quotient stacks, that is stacks of the form $[X/G]$ with $X$ a manifold and $G$ a finite group, [FG, JKK]. The theory has also been extended to Chow groups [AGV, JKK] and $K$-theory [JKK]. Borisov, Chen and Smith calculated the orbifold Chow rings of toric stacks [BCS] and in symplectic geometry an orbifold cohomology for torus actions was computed by Goldin, Holm and Knutson in [GHK].

The orbifold Chow and $K$-theory rings we define here extend earlier definitions of [JKK] (and implicitly [FG]) given for actions of finite groups. In this paper, we do not work with equivariant cohomology, but the formalism we develop works equally well for actions of compact Lie groups on almost complex manifolds. The character formula of [JKK, Lemma 8.5] implies that our product on equivariant cohomology of the inertia group scheme agrees, after tensoring with $\mathbb{Q}$, with that defined by Fantechi and Göttsche in [FG]. However in both [FG] and [JKK], the orbifold product is defined on the $G$-invariant part of the stringy cohomology (resp. Chow groups) of $X$. These groups are rationally isomorphic to the equivariant cohomology of $I_G(X)$, but in general they are a coarser invariant than equivariant cohomology. For example, if $G = \mu_n$ and $X = \text{Spec} \mathbb{C}$, then the additive structure on Fantechi and Göttsche’s orbifold cohomology is the Abelian group $\mathbb{Z}^n$, while the $\mu_n$ equivariant cohomology of $I_{\mu_n}(X)$ is additively isomorphic to the Abelian group $(\mathbb{Z}/nt)^n$.

For symplectic orbifolds which are quotients of tori, the equivariant cohomology version of our product agrees with that defined by Goldin, Holm and Knutson in [GHK]. The results of this paper may be viewed as a method (using different techniques) of extending their work to non-Abelian group actions.

When $X$ is a point and $G$ is finite, $K_G(I_G(X))$ is additively isomorphic to $K_G(G)$. The orbifold product then endows an exotic product on $K_G(G)$. This product was previously studied by Lusztig [Lu] in the context of Hecke algebras. Furthermore, Lusztig’s ring $K_G(G)$ admits an interpretation [AtSc] as the Verlinde algebra of the finite group $G$ at level 0 (see also [KP]).

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2. Background

**Conventions:** In this paper all schemes and algebraic spaces are assumed to be of finite type over the complex numbers \( \mathbb{C} \). All algebraic groups are assumed to be linear, that is they are isomorphic to closed subgroups of \( \text{GL}_n(\mathbb{C}) \) for some \( n \). We will sometimes use the term **linear algebraic group** for emphasis.

However, most of the formalism we develop also works with equivariant cohomology replacing equivariant Chow groups, for Lie group actions on almost complex manifolds.

We introduce some notation associated to groups which we will need. If \( G \) is an algebraic group, we denote the Lie algebra of \( G \) by \( \text{Lie}(G) \). For all \( m \) in \( G \), let \( Z_G(m) \) denote the **centralizer of** \( m \) **in** \( G \). For all \( m = (m_1, \ldots, m_n) \) in \( G^n \), denote the **centralizer of** \( m \) **in** \( G \) by \( Z_G(m) \). It consists of all elements commuting with \( m_i \) for all \( i = 1, \ldots, n \). For all \( n \), the set \( G^n \) has a **diagonal conjugation action** of \( G \) defined by \( g \cdot (m_1, \ldots, m_n) := (gm_1g^{-1}, \ldots, gm_ng^{-1}) \) for all \( g \) and \( m_i \) in \( G \).

A \( G \)-orbit \( \Phi \) of \( G^n \) is called a **diagonal conjugacy class (of length** \( n \)), while \( \Phi(m) \) denotes the diagonal conjugacy class containing \( m \) in \( G^n \). The set of diagonal conjugacy classes of length \( n \) is denoted by \( G^n[n] \).

2.1. **Group actions and quotient stacks.** In this paper we will consider three related notions for the action of an algebraic group \( G \) on a scheme (or more generally algebraic space) \( X \).

**Definition 2.1.** Let \( G \) be an algebraic group acting on an algebraic space \( X \). The **inertia group scheme** \( I_G(X) \) is defined as

\[
I_G(X) := \{(g, x) \mid gx = x\} \subseteq G \times X.
\]

**Remark 2.2.** If \( X \) is an algebraic space then \( I_G(X) \) is also an algebraic space. However the map \( I_G(X) \to X \) is representable in the category of schemes; i.e \( I_G(X) \) is an \( X \)-scheme. For this reason we refer to \( I_G(X) \) as the inertia group scheme even when \( X \) is algebraic space.

**Definition 2.3.** Let \( G \) be an algebraic group acting on an algebraic space \( X \).

(i) We say that \( G \) acts **properly** on \( X \) if the map \( G \times X \to X \times X \), defined by \( (g, x) \mapsto (x, gx) \) is proper.

(ii) We say that \( G \) acts with **finite stabilizer** if the projection \( I_G(X) \to X \) is finite.

(iii) We say that \( G \) acts **quasi-freely** if the projection \( I_G(X) \to X \) is quasi-finite.

Since \( G \) is affine, the map \( G \times X \to X \times X \) is finite if it is proper. The projection \( I_G(X) \to X \) is obtained from the map \( G \times X \to X \times X \) by base change along the diagonal morphism \( X \to X \times X \). Hence (i) implies (ii). Moreover the geometric fibers of the map \( I_G(X) \to X \) are the stabilizer groups and condition (iii) is equivalent to the requirement that the stabilizer group of any geometric
point is finite. Since we work in characteristic 0, the quotient stack \([X/G]\) is a Deligne-Mumford stack (DM stack) if and only if \(G\) acts quasi-freely. If \(G\) is a finite group, then the action is automatically proper. In general, \(G\) acts properly if and only if \([X/G]\) is a separated DM stack.

In order to construct the orbifold product, we need to push-forward along the morphism \(I_G(X) \times_X I_G(X) \to I_G(X)\). As a result we require throughout most of the paper that \(G\) acts with finite stabilizer. We also remark that the condition that \(G\) act with finite stabilizer is the necessary separation hypothesis required for the existence of a coarse moduli space of the quotient stack \([X/G]\) [KM].

**Definition 2.4.** Following [EHKV], we say that a stack \(X\) is a quotient stack if \(X\) is equivalent to a stack of the form \([X/G]\), where \(X\) is an algebraic space and \(G\) is a linear algebraic group.

Most stacks that naturally arise in algebraic geometry are quotient stacks. The papers [EHKV] and [Tot] deal with criteria for determining when a stack is a quotient stack.

**Definition 2.5.** An algebraic orbifold is a smooth DM stack which is generically represented by a scheme; i.e., the automorphism group at a general point is trivial.

**Proposition 2.6.** [EHKV] Any algebraic orbifold is a quotient stack.

### 2.2. The inertia group scheme and inertia stack.

**Definition 2.7.** If \(G\) is an algebraic group acting on an algebraic space \(X\), then it induces a \(G\)-action on \(G \times X\) via

\[
g \cdot (m, x) := (gmg^{-1}, gx).
\]

This action preserves \(I_G(X)\), and the quotient

\[
\mathcal{J}_\mathcal{X} = [I_G(X)/G]
\]

is the inertia stack of \(\mathcal{X} = [X/G]\).

**Remark 2.8.** If \(G\) acts with finite stabilizer on \(X\), then the projection map \(\mathcal{J}_\mathcal{X} \to \mathcal{X}\) is finite.

**Definition 2.9.** Let \(\Psi\) be a conjugacy class in \(G\) and let

\[
I(\Psi) := \{(g, x)|gx = x, g \in \Psi\} \subseteq G \times X.
\]

If \(G\) acts quasi-freely (in particular if the action is proper), then \(I(\Psi) = \emptyset\) unless \(\Psi\) consists of elements of finite order. Since we work in characteristic 0, any element of finite order is semi-simple, so its conjugacy class is closed in \(G\) by [Bor, Theorem 9.2]. It follows that \(I(\Psi)\) is closed in \(I_G(X)\), since it is the inverse image of \(\Psi\) under the projection \(I_G(X) \to G\).

**Lemma 2.10.** If \(G\) acts quasi-freely, then all but finitely many of the \(I(\Psi)\) are empty.
Proof. Without loss of generality, we may assume that \( G \) acts transitively on the set of connected components of \( X \). Let \( X^0 \) be a connected component, and let \( U \subset X^0 \) be an open set over which the fibers of the map \( I_G(X) \to X \) are finite and flat (and hence étale). Let \( W = GU \). Over the \( G \)-invariant open set \( W \), all stabilizers are conjugate to a fixed finite subgroup \( H \subset G \). The complement, \( X \setminus W \) is a union of \( G \)-invariant subspaces of strictly smaller dimension. By Noetherian induction, it suffices to prove the proposition for the open set \( W \). Thus we are reduced to proving the proposition under the assumption that the stabilizer at every point of \( X \) is conjugate to a fixed subgroup \( H \subset G \).

Let \( \Psi \subset G \) be a conjugacy class. Under the assumptions on \( X \), we have \( I(\Psi) = \emptyset \) unless \( \Psi \cap H \neq \emptyset \). Since \( H \) is finite, there can be only finitely many such \( \Psi \). \( \square \)

**Proposition 2.11.** [EG3] If \( G \) acts quasi-freely on \( X \), then \( I_G(X) \) is the disjoint union of the finitely many non-empty \( I(\Psi) \). In particular, the \( I(\Psi) \) are disjoint sums of connected components of \( I_G(X) \).

Proof. Since distinct conjugacy classes are disjoint, the \( I(\Psi) \) are also disjoint. As noted above, the \( I(\Psi) \) are also closed. By definition, every closed point in \( I_G(X) \) lies on some non-empty \( I(\Psi) \). Since there are only finitely many such \( I(\Psi) \), it follows that the complement of the union of the \( I(\Psi) \) is a Zariski open set which contains no closed points. Since we work over an algebraically closed field, the complement must be empty. \( \square \)

### 2.3. Multiple inertia schemes.

**Definition 2.12.** Let \( \mathbb{I}_G^2(X) = I_G(X) \times_X I_G(X) \). As a set, we have
\[
\mathbb{I}_G^2(X) = \{(g_1, g_2, x) | g_1 x = g_2 x = x\}.
\]
We refer to \( \mathbb{I}_G^2(X) \) as the *double inertia scheme*. More generally, we define
\[
\mathbb{I}_G^l(X) := I_G(X) \times_X \cdots \times_X I_G(X) \subseteq G^l \times X.
\]
The multiple inertia \( \mathbb{I}_G^l(X) \) has \( G \)-action taking
\[
m \cdot (g_1, g_2, \ldots, g_l, x) := (mg_1 m^{-1}, mg_2 m^{-1}, \ldots, mg_l m^{-1}, mx).
\]
We call the quotient stack \([\mathbb{I}_G^2(X)/G] \) the *double inertia stack* \( \mathfrak{I}_X \). It is equivalent to the fiber product \( \mathfrak{I}_X \times_X \mathfrak{I}_X \).

**Definition 2.13.** Given an \( l \)-tuple of elements \( g := (g_1, g_2, \ldots, g_l) \in G^l \) define \( \Phi(g_1, g_2, \ldots, g_l) \in G^l \) to be their orbit under the diagonal action of \( G \) by conjugation on each factor. Define
\[
\mathbb{I}^l(\Phi) := \{(g_1, g_2, \ldots, g_l, x) | g \in \Phi\} \subseteq \mathbb{I}_G^l(X)
\]
Clearly, if \( G \) acts quasi-freely, then \( \mathbb{I}^l(\Phi(g)) \) is empty unless \( g_i \) has finite order for all \( i = 1, \ldots, l \). A key observation is that something stronger holds.
Lemma 2.14. If \((m_1, \ldots, m_l, x) \in \mathbb{P}^l_G(X)\), then \(H = \langle m_1, \ldots, m_l \rangle\) is a finite group.

Proof. If \(h \in H\), then \(hx = x\) (since \(m_ix = x\) for all \(m_i\)). Since \(G\) acts quasi-freely, we conclude that \(H\) must be a finite group. \(\square\)

Lemma 2.15. If \(G\) acts quasi-freely on \(X\), then \(\mathbb{I}^l(\Phi)\) is closed in \(\mathbb{I}^l_G(X)\) for all \(\Phi\) in \(G^{[l]}\).

Proof. If \(\Phi \in G^{[l]}\) is a diagonal conjugacy class, then \(\mathbb{I}^2(\Phi)\) is the inverse image of \(\Phi\) under the projection \(\mathbb{I}^l_G(X) \to G^{[l]}\). Thus to prove the proposition it suffices to show that conjugacy classes of \(l\)-tuples of semi-simple elements are closed in \(G^l\). Since an \(l\)-tuple \((g_1, \ldots, g_l) \in G^l\) normalizes the diagonal subgroup \(G \subset G^l\), we can again invoke [Bor, Theorem 9.2] to conclude that if \(g_1, \ldots, g_l\) are semi-simple, then \(\Phi(g_1, \ldots, g_l)\) is closed in \(G^l\). \(\square\)

Lemma 2.16. If \(G\) acts quasi-freely then \(\mathbb{I}^l(\Phi)\) is empty for all but finitely many \(\Phi\) in \(G^{[l]}\).

Proof. The proof is almost identical to the proof of Lemma 2.10. Again by Noetherian induction we may reduce to the case when the stabilizer at every closed point of \(X\) is conjugate to a fixed finite subgroup \(H \subset G\). If \(\Phi \subset G \times G\) is a diagonal conjugacy class, then for all \(\Phi\) in \(G^{[l]}\), \(\mathbb{I}^l(\Phi) = \emptyset\) unless \(\Phi \cap (H^l) \neq \emptyset\). Since \(H^l\) is finite, there can be only finitely many such \(\Phi\). \(\square\)

The same argument used in the proof of Proposition 2.11 yields a decomposition result for \(\mathbb{I}_G^l(X)\).

Proposition 2.17. If \(G\) acts quasi-freely on \(X\), then \(\mathbb{I}_G^l(X)\) is the disjoint union of the finitely many non-empty \(\mathbb{I}^l(\Phi)\). In particular, the \(\mathbb{I}^l(\Phi)\) are disjoint sums of connected components of \(\mathbb{I}_G^l(X)\).

2.4. Equivariant Chow groups. Equivariant Chow groups were defined in [EG1] for actions of linear algebraic groups on arbitrary algebraic spaces over a field. They are algebraic analogues of equivariant cohomology groups and the formalism of this paper also goes through for equivariant cohomology. If \(G\) is an algebraic group and \(X\) is a \(G\)-space, then in this paper we use the notation \(A^*_G(X)\) to denote the infinite direct sum \(\bigoplus_{i=0}^\infty A^i_G(X)\), where \(A^i_G(X)\) is the “codimension-\(i\)” equivariant Chow group. An element of \(A^0_G(X)\) is represented by a codimension-\(i\) cycle on a quotient \(X \times_G U\), where \(U\) is an open set in a representation on which \(G\) acts freely and such that the complement of \(U\) has codimension more than \(i\) in \(V\). The space \(X \times_G U\) may be viewed as an approximation of the Borel construction in equivariant cohomology.

Remark 2.18. Even if \(X\) is a scheme, the quotient \(X \times_G U\) may exist only in the category of algebraic spaces. For this reason, the natural category for equivariant intersection theory is that of algebraic spaces of finite type over a field.
By their definition, the basic properties of equivariant Chow groups follow from the corresponding properties of ordinary Chow groups of algebraic spaces. As discussed in \([GE1\text{, Section 6}]\), the definition of Chow groups of schemes given in \([Ful]\text{ Chapters 1-6}]\) can be carried over essentially unchanged.

Since representations may have arbitrarily large dimension, the groups \(A^i_G(X)\) can be non-zero in arbitrarily high degree. If \(G\) acts freely on \(X\), then \(A^i_G(X) = A^i(X/G)\), where \(X/G\) is the quotient in the category of algebraic spaces (which always exists). If \(G\) acts quasi-freely, then \(A^i_G(X) \otimes \mathbb{Q} = 0\) for \(i > \dim X\), and \(A^i_G(X) \otimes \mathbb{Q} = A^i(X/G) \otimes \mathbb{Q}\) when the quotient exists in the category of algebraic spaces \([EG1, \text{ Theorem 3}]\). More generally, \([EG1, \text{ Proposition 19}]\) states that \(A^*_G(X)\) may be identified with the Chow groups of the quotient stack \([X/G]\). In particular, they depend only on the underlying quotient stack and not on the group \(G\) and space \(X\).

**Remark 2.19.** In \([EG1]\), the notation \(A^i_G(X)\) was used for the “codimension-\(i\)” operational Chow group, rather than the codimension-\(i\) group of cycles. However, if \(X\) is smooth, then these two groups are identified. In this paper we work exclusively with smooth spaces so the notational difference is immaterial.

**Remark 2.20.** In this paper we will often consider equivariant Chow groups of smooth but disconnected spaces. If \(X = \coprod_{k=1}^m X_k\), then \(A^i_G(X) = \bigoplus_{k=1}^m A^i_G(X_k)\) so any “codimension-\(i\)” cycle is a sum of “codimension-\(i\)” cycles on the each connected component \(X_k\).

Equivariant Chow groups enjoy the same formal properties as ordinary Chow groups. In particular, for \(X\) smooth there is an intersection product which makes \(A^*_G(X)\) a graded, commutative ring. If \(f: Y \to X\) is a morphism of smooth varieties, then there is a pullback \(f^*: A^*_G(X) \to A^*_G(Y)\) which is a ring homomorphism. If \(f\) is proper, then there is a push-forward \(f_*: A^*_G(Y) \to A^*_G(X)\) which shifts degrees by the relative codimension of the morphism \(f\).

If \(G\) acts properly on \(X\), then there is a pushforward isomorphism \([EG1, \text{ Theorem 3}]\) \(p_*: A^*_G(X) \otimes \mathbb{Q} \to A^*(X/G) \otimes \mathbb{Q}\), where \(X/G\) is the geometric quotient (which always exists in the category of algebraic spaces by \([KM]\)). Let \(pr: X/G \to \text{Spec } \mathbb{C}\) be the projection to a point.

**Definition 2.21.** If \(G\) acts properly on \(X\) and the quotient \(X/G\) is complete, then we define the quotient degree map \(\int_{[X/G]}: A^*_G(X) \otimes \mathbb{Q} \to \mathbb{Q} = A^*(\text{Spec } \mathbb{C})\) by the formula

\[
\int_{[X/G]} \alpha := pr_* p_* \alpha.
\]

**Remark 2.22.** The stack \([X/G]\) is complete if and only if \(G\) acts properly and the quotient \(X/G\) is a complete algebraic space.
**Remark 2.23.** The pushforward $A^*_G(X) \otimes \mathbb{Q} \to A_*(X/G)$ commutes with equivariant pushforward for finite morphisms. This implies two facts about the quotient degree which we will use below:

(i) If $Y \to X$ is a finite $G$-equivariant morphism such that $[X/G]$ (and hence $[Y/G]$) is complete, and if $\alpha \in A^*_G(Y) \otimes \mathbb{Q}$, then

$$\int_{[Y/G]} \alpha = \int_{[X/G]} f_* \alpha.$$ 

(ii) If $\sigma : X \to X$ is an automorphism that commutes with the $G$-action, then

$$\int_{[X/G]} \sigma^* \alpha = \int_{[X/G]} \alpha$$

for all classes $\alpha \in A^*_G(X) \otimes \mathbb{Q}$.

Equivariant vector bundles have equivariant Chern classes with values in the equivariant Chow ring. Equivariant Chern classes have the same formal properties as ordinary Chern classes. The only difference is that, because $A^*_G(X)$ may be non-zero in arbitrarily high degree, the Chern character and Todd classes must, a priori, be viewed as elements in the formal completion $\prod_{i=0}^{\infty} A^i_G(X) \otimes \mathbb{Q}$. However, in this paper we only consider quasi-free actions so the formal completion is the same as $A^*_G(X) \otimes \mathbb{Q}$.

We will often consider $G$-equivariant vector bundles on non-connected algebraic spaces whose rank varies on the connected components.

**Definition 2.24.** If $V$ is such a bundle on a space $X$, then we use the notation $\varepsilon(V)$ for the Euler class of $V$, that is the class in $A^*_G(X)$ whose restriction to equivariant the Chow group of each connected component is the top Chern class of the restriction of $V$ to that component.

If $Z \subset G$ is a closed subgroup and $X$ is a $Z$-space, then we write $G \times_Z X$ for the quotient of the $(G \times Z)$-space $G \times X$ by the subgroup $1 \times Z$, where $G \times Z$ acts by the rule $(k, z) \cdot (g, x) = (kgz^{-1}, zx)$. The quotient has an action of $G$ and we may identify $A^*_Z(X)$ with $A^*_G(G \times_Z X)$ ([EG2, Proposition 3.2a]). We refer to this identification as Morita equivalence. If $X$ is also a $G$-space, then $G \times_Z X = G/Z \times X$ and flat pullback along the morphism $G/Z \times X \to X$ induces a restriction morphism $A^*_G(X) \to A^*_Z(X)$.

### 2.5. Equivariant K-theory

Let $G$ be an algebraic group acting on an algebraic space $X$. We use the notation $K_G(X)$ to denote the Grothendieck ring of $G$-equivariant vector bundles on $X$. An element $K_G(X)$ is positive if it is equivalent to a positive integral sum of classes of equivariant vector bundles. An element is non-negative if it is either 0 or positive. If $\alpha$ is a non-negative, then its Euler class is well defined, as is the corresponding $K$-theory class $\lambda_-(\alpha^*)$. 
Given a morphism \( f: Y \to X \) of \( G \)-spaces, there is a naturally defined pullback \( f^*: K_G(X) \to K_G(Y) \). In order to construct the twisted product on equivariant \( K \)-theory we need the existence of pushforwards for finite local complete intersection morphisms of smooth spaces. A sufficient condition for this to hold is if \( Y \) and \( X \) satisfy the equivariant resolution property—that is every \( G \)-coherent sheaf is the quotient of a \( G \)-equivariant locally free sheaf [Köc, Section 3]. By Thomason’s resolution theorem [Tho], the equivariant resolution property holds if \( X \) satisfies the non-equivariant resolution property. The resolution property is known to hold for smooth schemes, but no general result exists for algebraic spaces.

In order to prove associativity of the orbifold product, we need to use the equivariant self intersection formula for finite local complete intersection morphisms. This follows from the excess intersection formula for \( G \)-projective morphisms proved by Köck [Köc, Theorem 3.8] for schemes which satisfy the resolution property. Consequently, when we work in equivariant \( K \)-theory we will assume that we work with smooth schemes rather than smooth algebraic spaces.

Suppose that \( G \) acts properly on a scheme \( X \) with geometric quotient \( X/G \) (which need not be a scheme). Let \( \pi: X \to X/G \) be the quotient map. By [EG4, Lemma 6.2] the assignment \( \mathcal{E} \to (\pi_*\mathcal{E})^G \) is an exact functor from the category \( G \)-equivariant vector bundles on \( X \) to the category of coherent sheaves on \( X/G \).

**Definition 2.25.** If \( G \) acts properly on \( X \) and the quotient \( X/G \) is complete, then we define the quotient Euler characteristic map \( \chi_{[X/G]}: K_G(X) \to \mathbb{Z} \) by the formula

\[
\chi_{[X/G]}(\mathcal{E}) = \sum_i (-1)^i \dim H^i(X/G, (\pi_*\mathcal{E})^G).
\]

**Remark 2.26.** As is the case for the quotient degree, the quotient Euler characteristic commutes with finite equivariant morphisms and is invariant under automorphisms which commute with the action of \( G \).

If \( Z \subset G \) is a closed subgroup and \( X \) is a \( G \)-space, then we again have a Morita equivalence identification of \( K_Z(X) = K_G(G \times_Z X) \) as described in [EG2, Proposition 3.2(a)]. When \( X \) is also a \( G \)-space, then \( G \times_Z X = G/Z \times X \) and the pullback along the \( G \)-equivariant morphism \( G/Z \times X \to X \) corresponds to the restriction map \( K_G(X) \to K_Z(X) \).

As explained in [EG3, Section 3.2], the Morita equivalence identification of equivariant \( K \)-theory follows from an explicit equivalence between the category of \( Z \)-locally free sheaves on \( X \) and \( G \)-locally free sheaves on \( G \times_Z X \). If \( V \) is \( G \)-module on \( G \times_Z X \), then its pullback to \( G \times X \) is a \((G \times Z)\)-module on \( G \times X \). The subsheaf, \( \mathcal{Y} \) of \( G \times 1 \)-invariant sections is a \( Z \)-module on \( X \).

### 2.6. Fixed loci, conjugacy classes and Morita equivalence.

Let \( G \) be an algebraic group acting on an algebraic space \( X \). Consider a diagonal conjugacy class \( \Phi \) in \( G^{[l]} \). Given \( m = (m_1, \ldots, m_l) \in \Phi \), let \( X^m \) be the intersection of
the fixed loci $X^{m_1} \cap \ldots \cap X^{m_l}$. Define a map $G \times X^m \to \mathbb{I}^l(\Phi)$ by $(g, x) \mapsto (gm_i g^{-1}, \ldots, gm_l g^{-1}, gx)$.

**Lemma 2.27.** (cf. [EC83, Lemma 4.3]) The map $G \times X^m \to \mathbb{I}^l(\Phi)$ is a $Z_m$-torsor, where $Z_m$ acts by $z \cdot (g, x) = (gz^{-1}, zx)$. In particular, $\mathbb{I}^l(\Phi)$ is smooth if $X$ is smooth.

As a consequence we obtain the following decompositions of $A^*(\mathcal{I}_X)$ (resp. $K(\mathcal{I}_X)$) and $A^*(\mathcal{I}_X)$ (resp. $K(\mathcal{I}_X)$).

**Proposition 2.28.**

\[ A^*(\mathcal{I}_X) = \bigoplus_{\Psi} A^*_{Z_G(m)}(X^m) \]

\[ K(\mathcal{I}_X) = \bigoplus_{\Psi} K_{Z_G(m)}(X^m), \]

where the sum is over every conjugacy class $\Psi$ of $G$ such that $I(\Psi) \neq \emptyset$, and $m$ is a choice of representative for each $\Psi$.

Likewise

\[ A^*(\mathcal{I}_X) = \bigoplus_{\Phi} A^*_{Z_G(m_1, m_2)}(X^{m_1, m_2}) \]

\[ K(\mathcal{I}_X) = \bigoplus_{\Phi} K_{Z_G(m_1, m_2)}(X^{m_1, m_2}), \]

where the sum is over all diagonal conjugacy classes $\Phi$ in $G^{[2]}$ such that $\mathbb{I}^2(\Phi) \neq \emptyset$, and where $(m_1, m_2)$ is a choice of a representative for each $\Psi$.

**Definition 2.29.** We define multiplication maps of the form $\mu_i : G^l \to G^{l-1}$ defined by $(g_1, \ldots, g_l) \mapsto (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_l)$, where $i \in \{1, \ldots, l-1\}$.

We also define evaluation maps of the form $e_j : G^l \to G^{l-1}$ defined by $(g_1, \ldots, g_l) \mapsto (g_1 \cdot g_{j-1}, g_{j+1}, \ldots, g_l)$, where $j \in \{1, \ldots, l\}$.

Since these maps commute with the diagonal action of $G$ on $G^l$ and $G^{l-1}$, they induce maps, also denoted by the same symbols, $\mu_i : \mathbb{I}^l(\Phi) \to \mathbb{I}^{l-1}(\mu(\Phi))$ and $e_j : \mathbb{I}^l(\Phi) \to \mathbb{I}^{l-1}(e_j(\Phi))$.

Now let $V$ be coherent $G$-module on $\mathbb{I}^{l-1}(\mu_i(\Phi))$ (resp. $\mathbb{I}^{l-1}(e_j(\Phi))$). Then $V$ pulls back to a coherent $G$-module $W = \mu_i^* V$ (resp. $e_j^* V$) on $\mathbb{I}^l(\Phi)$. Given $m = (m_1, \ldots, m_l) \in \Phi$, let $\mathcal{V}$ be the Morita equivalent $Z_m$-module on $X^m$. Likewise let $\mathcal{V}$ be the $Z_{\mu_i(m)}$ (resp. $Z_{e_j(m)}$)-module on $X^{\mu_i(m)}$ (resp. $X^{e_j(m)}$) Morita equivalent to $V$. There is an inclusion of centralizers $Z_m \subset Z_{\mu_i(m)}$ (resp. $Z_m \subset Z_{e_j(m)}$) as subgroups of $G$. Let $\mathcal{V}|_{X^m}$ be the $Z_m$-module on $X^m$ obtained by first restricting the action to a $Z_m$-action and then pulling back via the natural inclusion $X^m \hookrightarrow X^{\mu_i(m)}$ (resp. $X^m \hookrightarrow X^{e_j(m)}$).

**Lemma 2.30.** With the notation as above $\mathcal{V} = \mathcal{V}|_{X^m}$.
Proof. We only prove the statement for the multiplication map $\mu_i$, as the proof for the evaluation map is essentially identical. The proof follows from the fact that the diagram of torsors for the groups $Z_m$ and $Z_{\mu_i(m)}$

$$
\begin{array}{ccc}
G \times X^m & \hookrightarrow & G \times X^{\mu_i(m)} \\
\downarrow & & \downarrow \\
I(\Phi) & \overset{\mu_i}{\rightarrow} & I(\mu(\Phi))
\end{array}
$$

commutes and the upper horizontal arrow is $G \times Z_m$-equivariant. $\square$

Lemma 2.27 also yields the following useful proposition.

**Proposition 2.31.** If $\Phi$ in $G^{[\ell+1]}$ is the diagonal conjugacy class of $(m_1, \ldots, m_{\ell+1})$, where $\prod_{i=1}^{\ell+1} m_i = 1$, then $e_{\ell+1} : I(\Phi) \rightarrow I(e_{\ell+1}(\Phi))$ is a $G$-equivariant isomorphism which induces a $G$-equivariant embedding $\mathbb{P}^\ell_G(X) \rightarrow \mathbb{P}^{\ell+1}_G(X)$.

### 3. Inertia group scheme products

**Background hypotheses.** Throughout this section, we assume that $X$ is a smooth algebraic space when working with equivariant Chow groups and a smooth scheme when working with equivariant $K$-theory. If a group $G$ acts on a scheme or space $X$, then we assume that the action has finite stabilizer.

If we view $I_G(X)$ as a group scheme over $X$, there are three maps

$$
\mathbb{P}_G^2(X) = I_G(X) \times_X I_G(X) \rightarrow I_G(X).
$$

Let $e_1, e_2$ be the projections onto the first and second factors respectively and let $\mu$ be the multiplication map. Under the assumption that $G$ acts on $X$ with finite stabilizer, all three of the above maps are finite.

**Definition 3.1.** Given a class $c \in A^*_G(I_G(X))$, we may define a binary operation $\star_c$ on $A^*_G(I_G(X))$ by the formula

$$
\alpha \star_c \beta = \mu_*(e_1^* \alpha \cdot e_2^* \beta \cdot c). \quad (1)
$$

If we let $\mathcal{O} = [X/G]$, then identifying $A^*_G(I_G(X)) = A^*(\mathcal{O})$, the $\star_c$ product may be viewed as a product on $A^*(\mathcal{O})$.

When $X$ has the resolution property for coherent sheaves (e.g., if $X$ is a scheme), then, given a class $\mathcal{E} \in K_G(\mathbb{P}_G^2(X))$ we may define a product $\star_{\mathcal{E}}$ on $K_G(I_G(X))$ by the formula

$$
\mathcal{F} \star_{\mathcal{E}} \mathcal{G} = \mu_*(e_1^* \mathcal{F} \otimes e_2^* \mathcal{G} \otimes \mathcal{E}) \quad (2)
$$

Again this product corresponds to a product on $K(\mathcal{O})$. 
Definition 3.2. Suppose that \([X/G]\) is complete (i.e., \(G\) acts properly on \(X\) and \(X/G\) is complete). Let \(\sigma : I_G(X) \to I_G(X)\) be the involution taking \((g, x)\) to \((g^{-1}, x)\). We define the pairing \(\eta : A^*_G(I_G(X)) \otimes \mathbb{Q} \otimes A^*_G(I_G(X)) \otimes \mathbb{Q} \to \mathbb{Q}\) as

\[
\eta(\alpha_1, \alpha_2) := \int_{I_G(X)/G} \alpha_1 \cdot \sigma^* \alpha_2
\]

for any classes \(\alpha_1, \alpha_2 \in A^*_G(I_G(X)) \otimes \mathbb{Q}\). Similarly we define the pairing \(\eta : K_G(X) \otimes K_G(X) \to \mathbb{Z}\) as

\[
\eta(\mathcal{F}, \mathcal{G}) = \chi_{[I_G(X)/G]}(\mathcal{F} \otimes \sigma^* \mathcal{G})
\]

Remark 3.3. Since \(\sigma\) is an involution, observe that \(\alpha_2 \cdot \sigma^* \alpha_1 = \sigma^*(\alpha_1 \cdot \sigma^* \alpha_2)\). Hence by Remark 2.23 \(\eta(\alpha_1, \alpha_2) = \eta(\alpha_2, \alpha_1)\).

Definition 3.4. If \([X/G]\) is complete, the \(\star_c\) product on \(A^*_G(I_G(X)) \otimes \mathbb{Q}\) is Frobenius if

\[
\eta(\alpha_1 \star_c \alpha_2, \alpha_3) = \eta(\alpha_1, \alpha_2 \star_c \alpha_3)
\]

for all classes \(\alpha_1, \alpha_2, \alpha_3 \in A^*_G(I_G(X)) \otimes \mathbb{Q}\). We define the Frobenius property of the \(\star_e\) product on \(K_G(X)\) analogously.

Proposition 3.5. A sufficient condition for the \(\star_c\) product to be Frobenius is that the class \(c \in A^*(\mathbb{L}^2_G(X))\) satisfy cyclic invariance; that is,

\[
\tau^*(c) = c,
\]

where \(\tau : \mathbb{L}^2_G(X) \to \mathbb{L}^2_G(X)\) is the map taking \((g_1, g_2, x)\) to \((g_2, (g_1g_2)^{-1}, x)\).

The analogous statement holds for the \(\star_e\) product.

Proof. This follows from the projection formula, the automorphism invariance of the quotient degree (Remark 2.23) and the following simple properties of \(\tau\):

\[
\begin{align*}
e_2 &= e_1 \circ \tau \quad &e_1 &= \sigma \circ \mu \circ \tau \\
\sigma \circ \mu &= e_2 \circ \tau.
\end{align*}
\]
We have
\[
\eta(\alpha_1 \star_c \alpha_2, \alpha_3) = \int \mu_*(e_1^* e_2 \alpha_2 \cdot c) \cdot \sigma^* \alpha_3 \\
= \int \mu_*(e_1^* e_2 \alpha_2 \cdot c \cdot \mu^* \sigma^* \alpha_3) \\
= \int (\tau^* \mu^* \sigma^* \alpha_1 \cdot \tau^* e_1^* e_2 \alpha_2 \cdot \tau^* c \cdot \tau^* e_3 \alpha_3) \\
= \int (e_1^* e_2 \alpha_2 \cdot e_3 \alpha_3 \cdot c \cdot \mu^* \sigma^* \alpha_1) \\
= \int \mu_*(e_1^* e_2 \alpha_2 \cdot e_3 \alpha_3 \cdot c) \cdot \sigma^* \alpha_1 \\
= \eta(\alpha_2 \star_c \alpha_3, \alpha_1) = \eta(\alpha_1, \alpha_2 \star_c \alpha_3)
\]

The decomposition of $I_G(X)$ into pieces $I(\Psi)$ and the decomposition of $\mathbb{P}_G^2(X)$ into pieces $\mathbb{P}_G^2(\Phi)$ gives a more refined description of the $\star_c$ and $\star_c^e$ products. We will use this description to give sufficient conditions for the products to be commutative and associative.

**Definition 3.6.** Given a conjugacy class $\Psi \in G^{[1]}$ and a class $\alpha \in A_G^*(I_X)$, let $\alpha_\Psi$ be the component in the summand $A_G^*(I(\Psi))$ of $A_G^*(I_X)$.

Likewise if $\Phi \in G^{[2]}$ is a diagonal conjugacy class, we let $c_\Phi \in A_G^*(I_G^2(X))$ denote the component of a class $c$ in the summand $A_G^*(\mathbb{P}(\Phi))$. We will use similar notation for elements of the equivariant Grothendieck group.

The $\star_c$ product can be expressed as a sum over diagonal conjugacy classes in $G \times G$ as follows. Let $\Phi \in G^{[2]}$ be a diagonal conjugacy class and let $\Psi_1 = e_1(\Phi)$ and $\Psi_2 = e_2(\Phi)$. Likewise, let $\Psi_3 = \mu(\Phi)$. Given classes $\alpha_1 \in A_G^*(I(\Psi_1))$ and $\alpha_2 \in A_G^*(I(\Psi_2))$ the product $\alpha_1 \star_c \alpha_2$ will have a contribution in $A_G^*(I(\Psi_3))$ given by
\[
\mu_*(e_1^* \alpha_1 \cdot e_2^* \alpha_2 \cdot c_\Phi)
\]
(3)
The product $\alpha_1 \star_c \alpha_2$ is obtained by summing over terms of the form (3) for all diagonal conjugacy classes $\Phi$ such that $e_1(\Phi) = \Psi_1$ and $e_2(\Phi) = \Psi_2$.

Not all choices of a class $c \in A_G^*(\mathbb{P}_G^2(X))$ produce interesting products. We begin with a necessary and sufficient condition for $\star_c$ to have an identity.

**Proposition 3.7.** Let $1 \in A_G^*(I_G(X))$ be the fundamental class of $I(\{1\}) = X \subset I_G(X)$. Then 1 is the identity for the $\star_c$ product if and only if $c_\Phi = \mathbb{P}_G^2(\Phi)$ for all diagonal conjugacy classes such that $e_1(\Phi) = \{1_G\}$ or $e_2(\Phi) = \{1_G\}$.

**Proof.** If $\beta \in A_G^*(I(\Psi))$, then the only diagonal conjugacy class $\Phi \in G^{[2]}$ satisfying $e_1(\Phi) = \{1\}$ and $e_2(\Phi) = \Psi$ is the class $\Phi = (1, \Psi) \subset G \times G$. Thus the only
contribution to $1 \star_e \beta$ is in the $\Psi$ component. On $\mathbb{P}^2(\Phi)$ $\mu = e_2$ and $e_1^*1 = [\mathbb{P}^2(\Phi)]$. Thus the projection formula gives

$$1 \star_e \alpha = \alpha \cdot \mu_\ast c_\Phi.$$ 

Since $\mu = e_2$ is an isomorphism $\mathbb{P}^2(\Phi) \rightarrow I(\Psi)$, it follows that $1 \star_e \alpha = \alpha$ if and only if $c_\Phi = [\mathbb{P}^2(\Phi)]$. Exchanging the roles of $1$ and $\alpha$ yields the condition that $c_\Phi = [\mathbb{P}^2(\Phi)]$ for $\Phi = (\Psi, 1)$.

An essentially identical argument yields the following criterion for the $\star_e$ product.

**Proposition 3.8.** The class $[\mathcal{O}_X] \in K_G(I_G(X))$ is the identity for the $\star_e$ product if and only if $\mathcal{E}_\Phi = [\mathcal{O}_{\mathcal{E}(\Phi)}]$ for all diagonal conjugacy classes $\Phi$ in $G^{[2]}$ such that $e_1(\Phi) = \{1_G\}$ or $e_2(\Phi) = \{1_G\}$.

Next we give a condition that ensures the $\star_e$ product is commutative. Let $i : \mathbb{P}^2_G(X) \rightarrow \mathbb{P}^2_G(X)$ be the involution induced by the involution on $G \times G$ that exchanges the factors.

**Proposition 3.9.** A sufficient condition for the $\star_e$ product to be commutative is that for each diagonal conjugacy class $\Phi$ we have

$$i^*c_\Phi = c_{i(\Phi)}.$$ 

Similarly, the $\star_e$ product is commutative provided that $i^*\mathcal{E}_\Phi = \mathcal{E}_{i(\Phi)}$.

**Proof.** The proof is a straightforward application of [3] and the fact that $\mu(\Phi) = \mu(i(\Phi))$, because if $m_1, m_2 \in G$, then $m_1m_2$ and $m_2m_1$ are conjugate. □

3.1. **A criterion for associativity of the $\star_e$ and $\star_\epsilon$ products.** In this section, we give a sufficient condition for the $\star_e$ product to be associative. An analogous condition, which we do not write down, also holds for the $\star_\epsilon$ product. We closely follow Section 5 of [JJK]. Given $m_1, m_2, m_3 \in G$, with conjugacy classes $\Psi_1, \Psi_2, \Psi_3$, respectively, let $\Psi_{12}$ and $\Psi_{23}$ be the conjugacy class of the products $m_1m_2$ and $m_2m_3$ and let $\Psi_{13}$ be the conjugacy class of the product $m_1m_2m_3$. Let $\Phi_{1,2}$ and $\Phi_{2,3}$ be diagonal conjugacy classes of the pairs $(m_1, m_2)$ and $(m_2, m_3)$ respectively. Let $\Phi_{12,3}$ be the diagonal conjugacy class of the pair $(m_1m_2, m_3)$ and $\Phi_{1,23}$ the diagonal conjugacy class of the pair $(m_1, m_2m_3)$.

**Lemma 3.10.** A sufficient condition for associativity to be satisfied is if

$$\mu_\ast\left(e_1^*\mu_\ast(e_1^*\alpha_1 \cdot e_2^*\alpha_2 \cdot c_{\Phi_{1,2}}) \cdot e_2^*\alpha_3 \cdot c_{\Phi_{12,3}}\right) = \mu_\ast\left(e_1^*\alpha_1 \cdot e_2^*\mu_\ast(e_1^*\alpha_2 \cdot c_{\Phi_{2,3}}) \cdot c_{\Phi_{1,23}}\right)$$

for all diagonal conjugacy classes $\Phi_{1,2}$, $\Phi_{12,3}$, $\Phi_{2,3}$, $\Phi_{1,23}$ in $G^{[2]}$ determined by a triple of elements $(m_1, m_2, m_3) \in G^3$ and all classes $\alpha_1, \alpha_2, \alpha_3 \in A_G^\epsilon(I(\Psi_i))$. The analogous statement holds for the $\star_\epsilon$ product.
Proof. We only give the proof for the \( \ast_c \) product, as the proof for the \( \ast_e \) product is essentially identical. Let \( \Psi_1, \Psi_2, \Psi_3, \Psi_{123} \) be conjugacy classes in \( G \). Given classes \( \alpha_1 \in A^*_G(I(\Psi_1)), \alpha_2 \in A^*_G(I(\Psi_2)), \alpha_3 \in A^*_G(I(\Psi_3)) \), the component of the product

\[
(\alpha_1 \ast_c \alpha_2) \ast_c \alpha_3
\]

in \( A^*_G(I(\Psi_{123})) \) is the sum

\[
\sum_{\Phi_{1,2}} \sum_{\Phi_{123}} \mu_\ast(e^1_1 \mu_\ast(e^1_2 \alpha_1 \cdot e^2_2 \alpha_2 \cdot c_{\Phi_{1,2}}) \cdot e^3_2 \alpha_3 \cdot c_{\Phi_{123}}), \tag{5}
\]

where the sum is over all pairs of diagonal conjugacy classes \( \Phi_{1,2}, \Phi_{123} \) in \( G^2 \) satisfying

\[
e_1(\Phi_{1,2}) = \Psi_1, \quad e_2(\Phi_{1,2}) = \Psi_2, \quad \mu(\Phi_{1,2}) = e_1(\Phi_{123}),
\]

\[
e_2(\Phi_{123}) = \Psi_3, \quad \mu(\Phi_{123}) = \Psi_{123}. \tag{6}
\]

Similarly, the \( \Psi_{123} \) component of the product \( \alpha_1 \ast_c (\alpha_2 \ast_c \alpha_3) \) is calculated by the following sum.

\[
\sum_{\Phi_{1,23}} \sum_{\Phi_{2,3}} \mu_\ast(e^1_2 \alpha_1 \cdot e^2_2 \mu_\ast(e^1_3 \alpha_2 \cdot e^2_3 \alpha_3 \cdot c_{\Phi_{2,3}}) \cdot c_{\Phi_{1,23}}), \tag{7}
\]

where the sum is over all pairs of diagonal conjugacy classes \( \Phi_{1,23}, \Phi_{2,3} \) in \( G^2 \) satisfying

\[
e_1(\Phi_{2,3}) = \Psi_2, \quad e_2(\Phi_{2,3}) = \Psi_3, \quad e_1(\Phi_{1,23}) = \Psi_1.
\]

\[
e_2(\Phi_{1,23}) = \mu(\Phi_{2,3}), \quad \mu(\Phi_{1,23}) = \Psi_{123}. \tag{8}
\]

A sufficient condition for the \( \ast_c \) products \( (\alpha_1 \ast_c \alpha_2) \ast_c \alpha_3 \) and \( (\alpha_1 \ast_c \alpha_2) \ast_c \alpha_3 \) to be equal is that the terms in sums (7) and (5) be identified. The assignments \( (m_1, m_2, m_3) \mapsto \Phi_{1,2}, \Phi_{12,3} \) and \( (m_1, m_2, m_3) \mapsto \Phi_{2,3}, \Phi_{1,23} \) gives a bijection between the set of of pairs of conjugacy classes satisfying (6) and those satisfying (8). \( \square \)

Following [JKK] the condition of Lemma 3.10 can be expressed in terms of excess normal bundles. Given \( m_1, m_2, m_3 \in G \), let \( \Phi_{1,2,3}, \Phi_{12,3}, \Phi_{2,3}, \Phi_{1,23} \) in \( G^2 \) denote the diagonal conjugacy class of the tuple \( (m_1, m_2, m_3) \), \( \Phi_{12,3} \) in \( G^2 \) the diagonal conjugacy class of \( (m_1 m_2, m_3) \), \( \Phi_{1,23} \) in \( G^2 \) the diagonal conjugacy class of \( (m_1, m_2 m_3) \), \( \Phi_{ij} \) in \( G^2 \) the diagonal conjugacy class of \( (m_i, m_j) \) for \( 1 \leq i < j \leq 3 \) and \( \Psi_{123} \) the conjugacy class of \( m_1 m_2 m_3 \). For \( 1 \leq i < j \leq 3 \) there are evaluation maps \( e_{ij} : \Sigma^3(\Phi_{1,2,3}) \to \Sigma^2(\Phi_{i,j}) \). Also, for \( 1 \leq i \leq 3 \) there are evaluation maps \( e_i : \Sigma^3(\Phi_{1,2,3}) \to \Sigma^2(\Phi_{i,3}) \). Likewise there are product maps \( \mu_{12,3} : \Sigma^3(\Phi_{1,2,3}) \to \Sigma^2(\Phi_{12,3}) \) and \( \mu_{1,23} : \Sigma^3(\Phi_{1,2,3}) \to \Sigma^2(\Phi_{1,23}) \) and \( \mu_{123} : \Sigma^3(\Phi_{1,2,3}) \to \Sigma^2(\Phi_{123}) \).

Lemma 3.11. The diagrams

\[
\begin{array}{ccc}
\Sigma^3(\Phi_{1,2,3}) & \xrightarrow{e_{1,2}} & \Sigma^2(\Phi_{1,2}) \\
\mu_{12,3} \downarrow & & \mu \downarrow \\
\Sigma^2(\Phi_{12,3}) & \xrightarrow{e_1} & \Sigma^2(\Phi_{12,3})
\end{array}
\]
and
\[
\begin{align*}
\mathbb{P}^3(\Phi_{1,2,3}) & \xrightarrow{\varepsilon_{2,3}} \mathbb{P}^2(\Phi_{2,3}) \\
\mu_{1,23} & \downarrow \\
\mathbb{P}^2(\Phi_{1,2,3}) & \xrightarrow{\varepsilon_2} I(\Psi_{23})
\end{align*}
\]
(10)
are Cartesian and the horizontal arrows are finite local complete intersection morphisms.

**Proof.** Since \(X\) is smooth, the maps are l.c.i. because the \(\mathbb{P}(\Phi)\) and \(I(\Psi)\) are all smooth. An easy calculation with fixed loci shows that the diagrams are Cartesian. The map \(e_1\) is the composition
\[
\mathbb{P}^2(\Phi_{12,3}) \hookrightarrow I(\Psi_{12}) \times_X I(\Psi_{12}) \to I(\Psi_{12}).
\]
By Proposition 2.17, the first map is an open and closed embedding. We assume that the action has finite stabilizer so the second map is finite, since it is obtained by base change from the projection \(I_G(X) \to X\). Hence by base change the map \(e_{1,2}\) is finite. An identical argument shows that \(e_2\) and \(e_{2,3}\) are also finite. \(\Box\)

Let \(E_{1,2}\) be the excess normal bundle for diagram (9) and let \(E_{2,3}\) be the excess normal bundle for diagram (10).

**Proposition 3.12.** A sufficient condition for the \(*_{c}\) product to be associative is that the following identity holds in \(A^*_c(\mathbb{P}^3(\Phi_{1,2,3}))\):
\[
e_{1,2}c_{\Phi_{1,2}} \cdot \mu_{1,23}^*c_{\Phi_{12,3}} \cdot \varepsilon(E_{1,2}) = e_{2,3}^*c_{\Phi_{2,3}} \cdot \mu_{1,23}^*c_{\Phi_{1,23}} \cdot \varepsilon(E_{2,3})
\]
(11)
for all triples of elements \((m_1, m_2, m_3) \in G^3\), and where \(\varepsilon\) denotes the Euler class, as in Definition 2.24. A sufficient condition for the \(*_{d}\) product to be associative is that
\[
e_{1,2}^*\sigma_{\Phi_{1,2}} \otimes \mu_{1,23}^*\sigma_{\Phi_{12,3}} \otimes \lambda_{-1}(E_{1,2}^*) = e_{2,3}^*\sigma_{\Phi_{2,3}} \otimes \sigma_{\Phi_{1,23}}^* c_{\Phi_{1,23}} \otimes \lambda_{-1}(E_{2,3}^*)
\]
(12)

**Proof.** Again we only give the proof in equivariant Chow theory as the proof in equivariant \(K\)-theory is essentially identical. We wish to compare the two sides of (1). As in [JKK] we will use the excess intersection formula. However the morphisms we consider are finite local complete intersection morphisms. In equivariant intersection theory, the excess intersection formula follows from the definition of equivariant Chow groups and the corresponding non-equivariant excess intersection formula for algebraic spaces [Ful Theorem 6.5] (see Remark 2.18 above). In equivariant \(K\)-theory, the excess intersection formula for finite l.c.i. morphisms of schemes satisfying the resolution property follows from [Koc Theorem 3.8].

By the equivariant excess intersection formula for l.c.i. morphisms, if \(x \in A^*_c(\mathbb{P}^2(\Phi_{1,2}))\), then \(e_1^*\mu_*x = \mu_{12,3}^*(\varepsilon(E_{1,2} \cdot e_{1,2}^*x))\). Thus the left-hand side of (11) can be rewritten as
\[
\mu_* \left[ \mu_{12,3} \cdot (e_{1,2}^*e_{1,2}^*c_{\Phi_{1,2}} \cdot e_{1,2}^*c_{\Phi_{1,2}} \cdot \varepsilon(E_{1,2})) \cdot e_{2}^*c_{\Phi_{1,2}} \right]
\]
(13)
Since $\mu \circ \mu_{12,3} = \mu_{12,3}$ and $e_1 \circ e_{1,2} = e_1$, $e_2 \circ e_{1,2} = e_2$, we may apply the projection formula for the map $\mu_{12,3}^*$ to rewrite (13) as

$$
\mu_{1,2,3}^* \left( \epsilon_1^* \alpha_1 \cdot \epsilon_2^* \alpha_2 \cdot \epsilon_{1,2}^* c_{\Phi_{1,2}} \cdot \varepsilon(E_{1,2}) \right) \cdot \mu_{12,3}^* \epsilon_2^* \alpha_3 \cdot \mu_{12,3}^* c_{\Phi_{12,3}}.
$$

(14)

Finally, we note that $\mu_{12,3} \circ e_2 = \varepsilon_3$, so (14) can be rewritten as

$$
\mu_{1,2,3}^* \left( \epsilon_1^* \alpha_1 \cdot \epsilon_2^* \alpha_2 \cdot \epsilon_{1,2}^* c_{\Phi_{1,2}} \cdot \varepsilon(E_{1,2}) \right).
$$

(15)

A similar calculation shows that the right-hand side of (14) can be rewritten as

$$
\mu_{1,2,3}^* \left( \epsilon_1^* \alpha_1 \cdot \epsilon_2^* \alpha_2 \cdot \epsilon_{1,2}^* c_{\Phi_{1,2}} \cdot \varepsilon(E_{1,2}) \right) \cdot \mu_{12,3}^* \epsilon_3^* \alpha_3 \cdot \mu_{12,3}^* c_{\Phi_{12,3}} \cdot \varepsilon(E_{2,3}).
$$

(16)

Clearly (11) implies that (15) and (16) are equal. □

4. Logarithmic traces

Let $X$ be an algebraic space with the action of algebraic group $Z$. Suppose that we are given a rank-$n$ $Z$-equivariant vector bundle $V$ on $X$ and matrices $g_1, \ldots, g_l \in U(n)$ satisfying $\prod_{i=1}^l g_i = 1$ which act on the fibers of $V \to X$ and whose action commutes with the action of $Z$. In this section we define the logarithmic trace $L(g)(V) \in K_Z(X)$ and show that

$$
\sum_{i=1}^l L(g_i)(V) - V + \bigcap_{i=1}^l V^{g_i}
$$

is represented by a non-negative integral element of $K_Z(X)$.

4.1. The logarithmic trace on a complex vector space.

**Definition 4.1.** Let $V$ be an $n$-dimensional complex vector space and let $g \in \text{GL}(V)$ be an element which lies in some compact subgroup $K \subset \text{GL}(V)$. This is equivalent to assuming that $g$ is conjugate to a unitary matrix. Write the eigenvalues of $V$ as

$$
\exp(2\pi \sqrt{-1} \alpha_1), \ldots, \exp(2\pi \sqrt{-1} \alpha_n)
$$

with $0 \leq \alpha_i < 1$ for all $i = 1, \ldots, n$. Define the logarithmic trace of $g$ by the formula

$$
L(g) = \sum_{i=1}^n \alpha_i
$$

(17)

The key fact we need about the logarithmic trace is the following proposition:

**Proposition 4.2.** [FW] Let $V$ be an $n$-dimensional complex vector space and let $g_1, \ldots, g_l \in \text{GL}(V)$ be elements which lie in a common compact subgroup $K \subset
GL(V) and satisfy $g_1 \ldots g_l = 1$. Then $\sum_{i=1}^l L(g_i)$ is a non-negative integer satisfying the inequality

$$\sum_{i=1}^l L(g_i) \geq n - n_0$$

(18)

where $n_0 = \dim \bigcap_{i=1}^l V^{g_i}$ is the dimension of the invariant subspace.

**Proof.** The fact that $\sum_{i=1}^l L(g_i) \in \mathbb{Z}$ follows from the fact that $\exp(2\pi \sqrt{-1} L(g_i)) = \det g_i$ and $\prod_{i=1}^l \det g_i = 1$. Since $K$ lies in a subgroup of GL(V) isomorphic to $U(n)$, the inequality (18) follows from Theorem 2.2 of [FW], which states that for any $l$ unitary matrices $A_1, \ldots, A_l \in U(n)$ with product equal to the identity matrix $1$, the sum of the logarithmic traces $\sum_{i=1}^l L(A_i)$ is greater than or equal to $n - n_0$. □

4.2. The logarithmic trace on vector bundles. Let $X$ be an algebraic space with the action of an algebraic group $Z$. The definition of the logarithmic trace (17) and the inequality (18) of Proposition 4.2 can be easily generalized to $Z$-equivariant vector bundles. The $K$-theory version of the logarithmic trace will be used to define the twisted pullback bundles used in the construction of the inertial group scheme product.

**Definition 4.3.** Let $V$ be a rank-$n$ vector bundle on $X$ and let $g$ be a a unitary automorphism of the fibers of $V \to X$. If we assume that the action of $g$ commutes with action of $Z$ on $V$, the eigenbundles for the action of $g$ are all $Z$-subbundles, and we define the logarithmic trace of $V$ by the formula

$$L(g)(V) = \sum_{k=1}^m \alpha_k V_k \in K_Z(X) \otimes \mathbb{R}$$

(19)

on each connected component of $X$. Here $\exp(2\pi \sqrt{-1} \alpha_1), \ldots, \exp(2\pi \sqrt{-1} \alpha_m)$ are the distinct eigenvalues of $g$ acting on $V$, with $0 \leq \alpha_k < 1$ for all $k \in \{1, \ldots, m\}$, and $V_1, \ldots, V_m$ are the corresponding eigenbundles.

**Definition 4.4.** Let $V$ be a rank-$n$ vector bundle on $X$ and let $g$ be a unitary automorphism of the fibers of $V \to X$ which commutes with $Z$. The age $a_V(g)$ of $g$ on $V$ is the rank of $L(g)(V)$, which is a locally constant function on $X$

$$a_V(g) := \sum_{k=1}^m \alpha_k \dim(V_k)$$

**Remark 4.5.** If $g$ has finite order and acts on a variety $X$, then $a_V(g)$ is a locally constant function on $X^g$ which takes values in $\mathbb{Q}$.

The following proposition is an easy generalization of Proposition 4.2.
Proposition 4.6. Let \( g = (g_1, \ldots g_l) \) be an \( l \)-tuple of elements which lie in a compact subgroup of a reductive group \( H \) and satisfy \( g_1 \cdots g_l = 1 \). Let \( V \) be a \( Z \times H \)-equivariant vector bundle on \( H \), where \( H \) is assumed to act trivially on \( X \). Then
\[
\sum_{i=1}^{l} L(g_i)(V) - V + V^g
\]
is a non-negative element of \( K_Z(X) \), where \( V^g \) is subbundle of \( V \) invariant under the action of the \( g_i \) for all \( i \in \{1, \ldots, l\} \). In particular, the map \( V \mapsto \sum_{i=1}^{l} L(g_i)(V) - V + V^g \) is an additive homomorphism \( K_Z \times H(X) \rightarrow K_Z(X) \) which takes non-negative elements to non-negative elements.

To prove the proposition, we first need the following lemma.

Lemma 4.7. (cf. \[Seg\], Prop 2.2) Let \( Z \) be an algebraic group acting on an algebraic space \( X \) and let \( H \) be a linearly reductive group acting trivially on \( X \). Then any \( Z \times H \)-equivariant vector bundle \( V \rightarrow X \) has a canonical decomposition as a direct sum \( \bigoplus_E V_E \otimes E \), where \( E \) runs over the set of irreducible \( H \)-modules and \( V_E \) is a \( Z \)-bundle. In particular, \( K_{Z \times H}(X) \) is canonically isomorphic to \( K_Z(X) \otimes \text{Rep}(H) \).

Proof of Proposition 4.7. The proof is more or less identical to the proof given in \[Seg\]. If \( E \) is an \( H \)-module, then we can consider the \( Z \times H \)-module \( \mathcal{O}_X \otimes E \). If \( V \) is a \( Z \times H \)-module, then \( \mathcal{H}om(\mathcal{O}_X \otimes E, V) \) has a natural structure as \( Z \times H \)-module. Let \( V_E \) be the invariant subbundle
\[
\mathcal{H}om(\mathcal{O}_X \otimes E, V)^{1 \times H}.
\]
There is a natural map of \( Z \times H \)-vector bundles \( V_E \otimes E \rightarrow V \). As noted in the proof of Proposition 2.2 of \[Seg\] the map \( \bigoplus_E E \otimes V_E \rightarrow V \) is an isomorphism, where the sum runs over all irreducible \( H \)-modules \( E \). \( \square \)

Proof of Proposition 4.6. By Lemma 4.7 we may assume that \( V = V_E \otimes E \), where \( V_E \) is a \( Z \)-module and \( E \) is an irreducible representation of \( H \). Thus \( L(g)(V) = (L(g)(E))(V_E) \). By Proposition 4.2, \( \sum_{i=1}^{l} L(g_i)(E) \geq \dim E - \dim E^g \), and the proposition follows. \( \square \)

Remark 4.8. A priori the class \( \sum_{i=1}^{l} L(g_i)(V) \) is only well defined up to torsion. However, the proof of Proposition 4.6 shows that there is a canonical choice of integral representative. More precisely, if \( V = \sum E \otimes E \), then \( \sum_{i=1}^{l} L(g_i)(V) \) is represented by the integral class \( \sum E(\sum_{i=1}^{l} L(g_i)(E))[V_E] \).
on the action of $G$ on $X$. We let $\mathbf{m} := (m_1, \ldots, m_l)$ be an $l$-tuple of elements (not necessarily of finite order) which lie in a compact subgroup of $K \subset G$ and satisfy $m_1 \ldots m_l = 1$. Let $X^\mathbf{m} = \bigcap_{i=1}^l X^{m_i}$ be the fixed locus of the $m_i$. We let $\Phi(\mathbf{m}) \in G^{[l]}$ be the diagonal conjugacy class of the tuple $\mathbf{m}$.

In this section, we use the logarithmic trace to construct a twisted pullback map $K_G(X) \to K_G(\mathbb{I}^l(\Phi(\mathbf{m})))$. The twisted pullback map takes non-negative elements to non-negative elements and depends only on the conjugacy class $\Phi(\mathbf{m})$.

In Section 6 we apply this construction when $G$ acts with finite stabilizer to define a twisted pullback map $K_G(X) \to K(\mathbb{I}^2_G(X))$. The Euler class of the twisted pullback of the tangent bundle $T$, corresponding to the tangent bundle of $[X/G]$, produces a class $c \in A^*_G(\mathbb{I}^2_G(X))$ which satisfies the conditions necessary for the $\star_c$ product to be a commutative and associative.

5.1. The logarithmic restriction map in equivariant $K$-theory. There is an action of $Z = Z_G(\mathbf{m})$ on $X^\mathbf{m}$, and if $V$ is $G$-bundle on $X$, then $V|_{X^\mathbf{m}}$ has a canonical $Z$-action which necessarily commutes with the action of the $m_i$ on the fibers of $V|_{X^\mathbf{m}}$.

**Lemma 5.1.** Let $H \subset G$ be the Zariski closure of the group generated by the $m_i$. Then $H$ also acts trivially on $X^\mathbf{m}$.

**Proof.** For every point $x \in X^\mathbf{m}$, the isotropy group $G_x$ contains each $m_i$. Since $G_x$ is an algebraic subgroup of $G$, it must also contain $H$. Therefore, $H$ acts trivially on $X^\mathbf{m}$. $\square$

**Lemma 5.2.** The subgroups $Z$ and $H$ commute.

**Proof.** The commutator map $Z \times H \to G$, $(z, h) \mapsto [z, h]$ is constant on the dense subset $Z \times \langle m_1, \ldots, m_l \rangle$, so the image of the commutator map is constant. Hence $Z$ and $H$ are commuting subgroups of $G$. $\square$

As a consequence of Lemmas 5.1 and 5.2 the restriction $V|_{X^\mathbf{m}}$ has a natural $Z \times H$-module structure.

**Definition 5.3.** Given a $G$-equivariant vector bundle $V$ define the logarithmic restriction of $V$ to be the class $K_Z(X^\mathbf{m})$ defined by the formula

$$V(\mathbf{m}) = \sum_{i=1}^l L(m_i)(V|_{X^\mathbf{m}}) + V^\mathbf{m} - V|_{X^\mathbf{m}}.$$ (20)

By Proposition 4.6 and Remark 4.8, $V(\mathbf{m})$ is a non-negative integral element in $K_Z(X^\mathbf{m})$. The assignment $V \mapsto V(\mathbf{m})$ extends linearly to give a map $K_G(X) \to K_Z(X^\mathbf{m})$ taking non-negative elements to non-negative elements. We refer to this map as the logarithmic restriction map.
5.2. The twisted pullback in equivariant $K$-theory. The map $G \times X^m \to \mathbb{P}^l(\Phi)$ given by $(h, y) \mapsto (hy, hmyh^{-1}, \ldots, hmyh^{-1})$ gives an identification $\mathbb{P}^l(\Phi) = G \times_Z X^m$. Hence by Morita equivalence, we may identify $K_Z(X^m)$ with $K_G(\mathbb{P}^l(\Phi))$. Let $V(\Phi) \in K_G(\mathbb{P}^l(\Phi))$ be the class identified with $V(m)$.

Lemma 5.4. The class $V(\Phi) \in K_G(\mathbb{P}^l(\Phi))$ is independent of the choice of representative $(m_1, \ldots, m_l) \in \Phi$.

Proof. Let

$$g = (g_1, \ldots, g_l) = (h^{-1}m_1h, h^{-1}m_2h, \ldots, h^{-1}m_lh)$$

be another $l$-tuple in $\Phi$. Then $hX^g = X^m$ and $hZgh^{-1} = Z_m$. Let $p: \mathbb{P}^l(\Phi) \to X$ be the map induced by the projection $G^l \times X \to X$. By Lemma 2.30, the bundle $p^*V$ on $\mathbb{P}^l(\Phi)$ is equivalent to the bundle $V|_{X^m}$ (resp. $V|_{X^g}$) under the Morita equivalences between the category of $G$-modules on $\mathbb{P}^l(\Phi)$ and the category of $Z_m$-modules (resp. $Z_g$-modules) on $X^m$ (resp. $X^g$).

Since $g_i$ is conjugate to $m_i$, the eigenvalues for the action of $g_i$ on $V|_{X^g}$ are the same as the eigenvalues for the action of $m_i$ on $V|_{X^m}$. If $\alpha$ is an eigenvalue, let $V_{\alpha,m_i}$ be the $\alpha$-eigenbundle of $V|_{X^m}$ and $V_{\alpha,g_i}$ the $\alpha$-eigenbundle of $V|_{X^g}$. To complete the proof it suffices to show that $V_{\alpha,m_i}$ and $V_{\alpha,g_i}$ identify with the same $G$-bundle on $\mathbb{P}^l(\Phi)$.

This can be seen as follows. Consider the commutative triangle

$$
\begin{array}{ccc}
G \times X^g & \xrightarrow{k_g} & G \times X^m \\
\downarrow & & \downarrow \\
\mathbb{P}^l(\Phi) & & 
\end{array}
$$

where the horizontal map $k_g$ is the isomorphism given by $(h, x) \mapsto (ghg^{-1}, gx)$. If $W$ is a $G \times Z_m$-module on $G \times X^m$ then the $1 \times Z_m$-invariant subsheaf of $W$ is the same as the $1 \times Z_g$-invariant subsheaf of $k^*W$. Since $V_{\alpha,g_i} = k_g^*V_{\alpha,m_i}$, the lemma follows. \qed

As an immediate consequence of Lemma 4.12 and the definition we obtain the following proposition.

Proposition 5.5. The map $K_G(X) \to K_G(\mathbb{P}^l(\Phi))$, $V \mapsto V(\Phi)$ takes non-negative elements to non-negative elements.

Definition 5.6. We refer to the map $K_G(X) \to K_G(\mathbb{P}^l(\Phi))$ as the twisted pullback map.

The twisted pullback $K_G(X) \to K_G(\mathbb{P}^l(\Phi))$ is the composition of the logarithmic restriction $K_G(X) \to K_Z(X^m)$ with the Morita equivalence identification $K_Z(X^m) = K_G(\mathbb{P}^l(\Phi))$.

Remark 5.7. As noted in Remark 4.8, if $V$ is a $G$-equivariant vector bundle on $X$, then there is a canonical choice of a $Z$-bundle on $X^m$ whose class represents...
\[ \sum_{i=1}^l L(m_i)(V|_X^m). \] As a result there is also a canonical choice of representative for the class \( V(m) \) (and hence \( V(\Phi) \)).

More precisely, if \( V|_X^m \) decomposes as \( (V^m \otimes 1) \oplus \bigoplus_E V_E \otimes E \) where \( 1 \) is the trivial representation of the group \( H = \langle m \rangle \) and the direct sum is over all non-trivial irreducible representations, then \( V(m) \) is represented by the bundle \( \bigoplus_E V_E^{r_E} \) where \( r_E \) is the non-negative integer \( \sum_{i=1}^l (L(m_i)(E) - \dim E) \) and the sum is over all non-trivial irreducible representations of \( H \). We will also use the notation \( V(m) \) (resp. \( V(\Phi) \)) to refer to the corresponding bundles on \( X^m \) (resp. \( \mathbb{I}(\Phi) \)).

### 5.3. Identities for logarithmic restrictions

The following identities will be helpful in proving the associativity of the products we are most interested in, namely, the \( \star_{cv} \) and \( \star_{cv'} \) products.

Let \( m_1, m_2, m_3 \) be elements of \( G \) which lie in a common compact subgroup \( K \) and let \( m_4 = (m_1 m_2 m_3)^{-1} \). Consider the tuples \( m_{1,2} = (m_1, m_2, (m_1 m_2)^{-1}), m_{2,3} = (m_2, m_3, (m_2 m_3)^{-1}), m_{1,2,3} = (m_1 m_2 m_3, m_4) \), and \( m_{1,23} = (m_1, m_2 m_3, m_4) \). Each of these tuples lies in \( K \), so the logarithmic restriction is defined for each tuple. Let \( m = (m_1, m_2, m_3, m_4) \). There is a natural inclusion of \( X^m \) into the fixed locus of each of the tuples defined above. This inclusion induces restriction maps in equivariant \( K \)-theory \( K_{Z_{i,j}}(X^{m_{i,j}}) \to K_Z(X^m) \), etc., where \( Z_{i,j} = Z_G(m_i, m_j) \). We will abuse notation and indicate the restriction of a class in \( K_{Z_{i,j}}(X^{m_{i,j}}) \) to \( K_Z(X^m) \) by the same name.

**Lemma 5.8.** Let \( V \) be a \( G \)-bundle on \( X \). Then following identities hold in \( K_Z(X^m) \).

\[
V(m_{1,2}) + V(m_{12,3}) = \sum_{i=1}^4 L(m_i)(V) + V^{m_{1,2}} + V^{m_{12,3}} - V^{m_1 m_2} - V \quad (21)
\]

\[
V(m_{2,3}) + V(m_{1,23}) = \sum_{i=1}^4 L(m_i)(V) + V^{m_{2,3}} + V^{m_{1,23}} - V^{m_2 m_3} - V \quad (22)
\]

Similarly, for all \( r \geq 4 \), \( m := (m_1, \ldots, m_r) \) in \( K^r \) such that \( \prod_{i=1}^r m_i = 1 \), then for all \( j = 2, \ldots, r - 2 \), set \( m := \prod_{i=1}^j m_i \). We have the equality

\[
V(m) = V(m_1, \ldots, m_j, m^{-1}) + V(m, m_{j+1}, \ldots, m_r) + \mathcal{E}_j(V)(m) \quad (23)
\]

in \( K_{Z_G(m)}(X^m) \), where

\[
\mathcal{E}_j(V)(m) := V^m - V^{(m_1, \ldots, m_j, m^{-1})} - V^{(m, m_{j+1}, \ldots, m_r)} + V^m. \quad (24)
\]

**Proof.** The proof follows from the definition of the restricted pullback in terms of the logarithmic trace and the identity \( L(g)(V) + L(g^{-1})(V) = V - V^g \). \( \square \)
6. THE TWISTED PRODUCT

Background hypotheses. In Section 6.1 we assume that all group actions are quasi-free. In Section 6.2 we assume that all group actions have finite stabilizer. When working with equivariant Chow groups we assume that quasi-free. In Section 6.2 we assume that all group actions have finite stabilizer.

In this section we define, for each positive element \( V \in K_G(X) \), twisted products \(*_{c,v}\) and \(*_{d,v}\) on \( A^*_G(I_G(X)) \) and \( K_G(I_G(X)) \), respectively. In general the product will be commutative but not associative. When \( V = \mathbb{T} \) is the element of \( K_G(X) \) corresponding to the tangent bundle of the quotient stack \( [X/G] \), then the product is associative.

6.1. The twisted pullback \( K_G(X) \to K_G([L^2_G(X)]) \). Let \( G \) be an algebraic group acting quasi-freely on an algebraic space \( X \). The construction of Section 5 allows us to define twisted pullbacks \( K_G(X) \to K_G([L^2_G(X)]) \) and, more generally, \( K_G(X) \to K_G([L^2_G(X)]) \).

**Definition 6.1.** If \( V \) is an equivariant vector bundle on \( X \) and \( \Phi = \Phi(m_1, \ldots, m_l) \in G^{[l]} \) define \( V^{tw}(\Phi) \in K_G([L^1_G(X)]) \) to be the class identified with \( V(\Phi(m_1, \ldots, m_l, (m_1 \ldots m_l)^{-1})) \) under the isomorphism

\[
I^{l+1}(\Phi(m_1, \ldots, m_l, (m_1 \ldots m_l)^{-1})) \cong I^l(\Phi(m_1, \ldots, m_l))
\]

of Proposition 2.31.

**Definition 6.2.** Define a twisted pullback \( K_G(X) \to K_G([L^2_G(X)]) \) (and, more generally, \( K_G(X) \to K_G([L^2_G(X)]) \)) taking \( V \mapsto V^{tw} \) by setting \( V^{tw}|_{I_2(\Phi)} = V^{tw}(\Phi) \) for each diagonal conjugacy class \( \Phi = \Phi(m_1, m_2) \in G^{[2]} \) (and, more generally, \( \Phi(m_1, \ldots, m_l) \in G^{[l]} \)) such that \( I^2(\Phi) \neq \emptyset \) (and, more generally, \( I^l(\Phi) \neq \emptyset \)).

(Note that a necessary condition for \( I^2(\Phi) \) to be non-empty is that \( m_1 \) and \( m_2 \) generate a finite—hence unitarizable—subgroup \( H \subset G \).) The crucial fact about the twisted pullback is that it does not depend on the presentation for \([X/G]\) as a quotient stack.

**Theorem 6.3.** If \( V \) is a \( G \)-equivariant vector bundle on \( X \), then the bundle \( V^{tw} \) defined above is independent of the choice presentation of \( \mathcal{X} = [X/G] \) as a quotient stack.

**Proof.** Suppose that \( \mathcal{X} = [X/G] \) is equivalent to the quotient \([Y/H]\) for some group \( H \). Let \( W = X \times_Y Y \). The two identifications of \( \mathcal{X} \) as a quotient stack imply that there are commuting free actions of \( G \) and \( H \) on \( W \) such that \( W/H = X \) and \( W/G = Y \) and \([W/(G \times H)] = \mathcal{X} \). To prove the theorem it suffices to show that \( V^{tw} \) as computed on \( I^2_G(X) \) may be identified with \( V^{tw} \) as computed on \( I^2_{G \times H}(W) \) under the common identification with \( I^2_{\mathcal{X}} \).
The projection map \( p: W \to X \) induces a morphism (which we also call \( p \)) \( \mathbb{P}^2_G(W) \to \mathbb{P}^2_G(X) \). Let \( V^{tw,G} \) be the bundle on \( \mathbb{P}^2_G(X) \) obtain by the logarithmic twist for the \( G \) action on \( X \) and let \( V^{tw,G \times H} \) be the \( G \times H \)-equivariant bundle on \( \mathbb{P}^2_G(W) \). To identify the two twists, we must show that \( p^* (V^{tw,G}) = V^{tw,G \times H} \).

Let \( \Phi \in G^{[2]} \) be a diagonal conjugacy class, then \( p^{-1}(\mathbb{P}^2(\Phi)) = \bigsqcup_{\Phi'} \mathbb{P}^2(\Phi') \), where the sum is over the finitely many diagonal conjugacy classes \( \Phi' \subset (G \times H \times G \times H) \) whose image in \( G \times G \) is \( \Phi \) and for which \( \mathbb{P}^2(\Phi') \neq \emptyset \). Let \( \Phi' \) be a conjugacy class in \( G \times H \times G \times H \) whose image in \( G \times G \) is \( \Phi \). To prove the theorem, we must show that \( p^*(V^{tw}(\Phi))(\Phi') = V^{tw}(\Phi') \) for all such \( \Phi \) and \( \Phi' \).

Given a representative \((g_1 \times h_1, g_2 \times h_2) \in \Phi'\), the map \( p: W \to X \) restricts to a map \( p': W^{g_1 \times h_1, g_2 \times h_2} \to X^{g_1 \times g_2} \). There is a free action of \( 1 \times Z_H(h_1, h_2) \) on \( W^{g_1 \times h_1, g_2 \times h_2} \), and the map \( p' \) is a torsor for this group. Let \( K' = \langle g_1 \times h_1, g_2 \times h_2 \rangle \).

If \( V \) is \( G \)-equivariant vector bundle on \( X \) then there is a natural isomorphism of \( Z_{G \times H}(g_1 \times h_1, g_2 \times h_2) \times K' \)-equivariant vector bundles,

\[
p'^* (V|_{X^{g_1 \times g_2}}) \to (p^* V)|_{W^{g_1 \times h_1, g_2 \times h_2}}
\]

**Lemma 6.4.** If \( p(w) = x \), then the projection \( G \times H \to G \) induces an isomorphism \( \text{Stab}_{G \times H}(w) \to \text{Stab}_G(x) \).

**Proof of Lemma 6.4.** If \( (g, h)w = w \), then, since \( p \) is \( 1 \times H \)-equivariant map, \( gp(w) = p(w) \). Thus the projection \( G \times H \to G \) restricts to a map \( \text{Stab}_{G \times H}(w) \to \text{Stab}_G(x) \). Since the normal subgroup \( 1 \times H \subset G \times H \) acts freely on \( W \) with quotient \( X = W/(1 \times H) \), the fibers of \( p \) are \( (1 \times H) \)-orbits. Hence \( gx = x \) if and only if \( (g \times 1)w = (1 \times h)^{-1}w \) for a unique element \( 1 \times h \in 1 \times H \). Thus the map \( \text{Stab}_{G \times H}(w) \to \text{Stab}_G(x) \) is bijective.

By Lemma 6.4, we see that if \( g_1 \times h_1, g_2 \times h_2 \in \text{Stab}_{G \times H}(w) \), then the projection induces an isomorphism of groups

\[
K' = \langle g_1 \times h_1, g_2 \times h_2 \rangle \to K = \langle g_1, g_2 \rangle.
\]

Now if \( V|_{X^{g_1 \times g_2}} \) decomposes as a sum \( \bigoplus_E V_E \otimes E \), with \( V_E \) a \( Z_{G}(g_1, g_2) \)-equivariant bundle and with the sum running over irreducible \( K \)-modules \( E \), then \( p'^* V|_{X^{g_1 \times g_2}} = \bigoplus_E p'^* V_E \otimes E \). Thus

\[
p'^* V(g_1, g_2) = \bigoplus p'^* (V_E^{g_1 a(E)}) = p^* V(g_1 \times h_1, g_2 \times h_2),
\]

where \( a(E) = L(g_1)(E) + L(g_2)(E) + L((g_1 g_2)^{-1})(E) \). This proves our theorem.

6.2. The twisted product. Given a positive element \( V \) in \( K_G(X) \), we may formally define twisted products \( *_{cV} \) and \( *_{\varepsilon V} \), where \( c_V = \varepsilon(V^{tw}) \) and \( \varepsilon_V = \lambda_{-1}((V^{tw})^*) \) on \( A_G^r(I_G(X)) = A^r(\mathcal{M}_X) \) and \( K_G(I_G(X)) = K(\mathcal{M}_X) \), respectively.

**Proposition 6.5.** For any positive element \( V \) in \( K_G(X) \), the \( *_{cV} \) product on \( A^r_2(I_G(X)) \) is commutative with identity \( I(\{1\}) \) and the pairing \( \eta \) on \( A^r_2(I_G(X)) \otimes \mathbb{Q} \) satisfies the Frobenius property when \( [X/G] \) is complete. Similarly, the \( *_{\varepsilon_V} \)
products on \(K_G(I_G(X))\) is commutative with identity \([O_{\tilde{I}(1)}]\) and the pairing \(\eta\) on \(K_G(I_G(X))\) satisfies the Frobenius property when \([X/G]\) is complete.

**Proof.** From the definition we see that if \(\Phi\) is the conjugacy class of the pair \((1, m)\), then \(V((1, m, m^{-1})) = 0\), so \(e(V(\Phi)) = [I_2(\Phi)]\). Similarly, \(\lambda_1((V(\Phi))^*) = [\tilde{O}_2(\Phi)]\).

Hence by Propositions 3.7 and 3.8, the well known fact that if \(P\) is independent of the presentation of \(G\), then the \(*_{cv}\) and \(*_{\delta V}\) products have an identity.

Also, given \(m_1, m_2 \in G\), \(i^*V(\Phi(m_1, m_2)) = V(\Phi(m_2, m_1))\), where \(i\) is the involution on \(\tilde{I}_G(X)\) that switches the factors. Hence by Proposition 3.9 the \(*_{cv}\) and \(*_{c^g}\) products are commutative. It is immediate from the definition that \(\tau^*_c = c_v\) and \(\tau^*_\delta V = \delta V\), so by Proposition 3.5 the \(*_{cv}\) and \(*_{c^g}\) products are Frobenius when \([X/G]\) is complete.

For general \(V\) the \(*_{cv}\) product (resp. \(*_{c^g}\) product) will not be associative.

**Lemma 6.6.** Let \(X = [X/G]\) be a DM quotient stack, and let \(p: X \to [X/G]\) be the universal \(G\)-torsor. Then \(p^*T_{\mathcal{X}} = T_X - g\) in \(K_G(X)\), where \(g\) is the Lie algebra of \(G\).

**Proof.** The map \(X \to [X/G]\) is a representable \(G\)-torsor, so the result follows from the well known fact that if \(P \to Y\) is \(G\)-torsor, then \(T_P = g\). For a proof see [EG3, Lemma A.1].

**Definition 6.7.** We define \(T := T_X - g\) in \(K_G(X)\).

By Lemma 6.6 \(T\) is a positive element of \(K_G(X)\) and its image in \(K(\mathcal{X})\) is independent of the presentation of \(\mathcal{X}\) as a quotient stack.

**Definition 6.8.** The rational grading on \(A^*_G(I_G(X))\) assigns the rational number \(|v| := k + \alpha_T(g)(v)\) to each homogeneous element \(v\) in \(A^*_{2G(g)}(X^g) \subseteq A^*_G(I_G(X))\) where \(\alpha_T(g)(v)\) denotes the age of \(g\) on \(\mathbb{T}\), as given in Definition 4.1, evaluated on the support of \(v\), and \(\mathbb{T}\) is understood to be restricted to \(X^g\).

The next proposition shows that when \([X/G]\) is complete the rational grading is natural with respect to the pairing \(\eta\) on \(A^*_G(I_G(X)) \otimes \mathbb{Q}\).

**Proposition 6.9.** If \([X/G]\) is complete and if \(\Phi_1, \Phi_2\) are conjugacy classes in \(G\) and \(v_i \in A^*_G(I_G(\Phi_i)) \otimes \mathbb{Q} \subseteq A^*_G(I_G(X)) \otimes \mathbb{Q}\) (resp. \(v_i \in K_G(I_G(\Phi_i)) \subseteq K_G(I_G(X))\)) then \(\eta(v_1, v_2) = 0\) unless \(\Phi_2 = (\Phi_1)^{-1}\) and \(\eta(v_1, v_2) = 0\) for all homogeneous \(v_i\) in \(A^*_G(I_G(\Phi_i)) \otimes \mathbb{Q}\) unless \(|v_1| + |v_2| = \dim[X/G]|\).

**Proof.** The vanishing conditions on the pairing \(\eta\) follow immediately from the definition and dimensional considerations.

We now come to our main theorem.

**Theorem 6.10.** The \(*_{cv}\) and \(*_{c^g}\) products on \(A^*_G(I_G(X))\) and \(K_G(I_G(X))\) are associative, and the orbifold product \(*_{cv}\) on \(A^*_G(I_G(X))\) preserves the rational grading.
(i) \( f : X \to Y \) is an étale \( G \)-equivariant morphism then the induced pullback \( f^* : A_G^*(I_G(Y)) \to A_G^*(I_G(X)) \) (resp. \( f^* : K_G(I_G(Y)) \to K_G(I_G(X)) \)) respects the \( \ast_{ct} \) product (resp. \( \ast_{ct} \) product).

(ii) For all \( \ell \geq 2 \), we have the identity
\[
v_1 \ast_{ct} \cdots \ast_{ct} v_\ell = \mu_{s}(e_1^* v_1 \cdot e_2^* v_2 \cdots e_\ell^* v_\ell \cdot \varepsilon(\mathbb{T}^{tw}))
\] in \( A_G^*(I_G(X)) \), for all \( v_i \) in \( A_G^*(I_G(X)) \), where \( e_j : \mathbb{I}^\ell(X) \to I_G(X) \) is the \( j \)-th evaluation map taking \((m_1, \ldots, m_\ell, x) \mapsto (m_j, x) \) and \( \mu : \mathbb{I}^\ell(X) \to I_G(X) \) takes \((m_1, \ldots, m_\ell, x) \mapsto (m_1 m_2 \cdots m_\ell, x) \). The analogous identity also holds in \( K_G(I_G(X)) \) for \( \ast_{ct} \).

Proof of Theorem 6.11. Given \( m_1, m_2, m_3 \) we use the same notation as in Section 3.1 and consider the conjugacy classes of pairs \( \Phi_{1,2}, \Phi_{2,3}, \Phi_{12,3}, \Phi_{12,3} \in G \times G \) as well as the conjugacy class \( \Phi_{1,2,3} \in G^3 \).

By Proposition 3.12 it suffices to prove that equations (11) and (12) hold for the \( \ast_{ct} \) and \( \ast_{ct} \) products for all triples \((m_1, m_2, m_3) \in G^3 \) such that \( \mathbb{I}^3(\Phi(m_1, m_2, m_3)) \neq \emptyset \). To do this we will show that the following equation holds in \( K_G(\mathbb{I}^3(\Phi(m_1, m_2, m_3))) \)
\[
e_{1,2} \mathbb{T}^{\mathbb{I}^3(\Phi_{1,2})} + \mu_{2,3}^* \mathbb{T}^{\mathbb{I}^3(\Phi_{2,3})} + E_{1,2} = \epsilon_{2,3}^* \mathbb{T}^{\mathbb{I}^3(\Phi_{2,3})} + \mu_{1,2}^* \mathbb{T}^{\mathbb{I}^3(\Phi_{1,2})} + E_{2,3}
\] (26)

Let \( m = (m_1, m_2, m_3, (m_1 m_2 m_3)^{-1}) \) and let \( Z = Z_G(m_1, m_2, m_3) \). As usual the map \( G \times X^m \to \mathbb{I}^3(\Phi_{1,2,3}) \), \((g, x) \mapsto (gx, gm_1 g^{-1}, gm_2 g^{-1}, gm_3 g^{-1})\) identifies \( \mathbb{I}^3(\Phi_{1,2,3}) \) with the quotient \( G \times Z X^m \). Let \( Z_{i,j} = Z_G(m_{i,j}) \) and let \( \mathfrak{z}_{i,j} = \text{Lie}(Z_{i,j}) \), the Lie algebra of \( Z_{i,j} \). Finally, let \( \mathfrak{z}_k \) be the Lie algebra of \( Z_G(m_k) \) for \( k = 12, 23 \).

Lemma 6.11. Under the Morita equivalence identification of \( K_G(\mathbb{I}^3(\Phi_{1,2,3})) \) \( = K_Z(X^m) \) the class \( E_{1,2} \) is identified with
\[
TX^{m_1 m_2}X^m - TX^{m_2 m_3}X^m + TX^{m_1}X^m + \mathfrak{z}_{1,2} + \mathfrak{z}_{12,3} - \mathfrak{z}_{12} + \mathfrak{z}
\] (27)
and the class \( E_{2,3} \) is identified with
\[
TX^{m_2 m_3}X^m - TX^{m_1 m_3}X^m + TX^{m_2}X^m + \mathfrak{z}_{2,3} + \mathfrak{z}_{12,3} - \mathfrak{z}_{23} + \mathfrak{z}
\] (28)

Proof of Lemma 6.11. We only prove the identity (27), as the proof of (28) is identical. By definition
\[
E_{1,2} = \mu_{12,3}^* N_{e_1} - N_{e_{1,2}} = \mu_{12,3}^* T^1(\Phi_{1,2}) - \mu_{12,3}^* T^2(\Phi_{1,2,3}) - e_{1,2}^* T^2(\Phi_{1,2}) + T^3(\Phi_{1,2,3})
\] (29)

If \( \Phi \) is the conjugacy class of an \( l \)-tuple \( g = (g_1, g_2, \ldots, g_l) \), then \( \mathbb{I}^l(\Phi) \simeq G \times \mathbb{Z}_l X^g \). Hence, under the Morita equivalence identification of \( K_G(\mathbb{I}^l(\Phi)) \) with \( K_{Z_G(g)}(X^g) \) we have that
\[
T^l(\Phi) = TX^g - \text{Lie}(Z_G(g))
\] (30)
Substituting (30) into (29) yields the identity of (27), provided that the various composition of pullbacks correspond to the restriction along the inclusions of the various fixed loci. This follows from Lemma 2.30 □

Now, for all $g = (g_1, \ldots, g_l)$ in $G^l$, we have $(TX|_{X^g})^g = TX^g$. Likewise, $g^g = \mathfrak{g}$, where $\mathfrak{g} = \text{Lie}(Z_G(g))$.

Combining the formulas of Lemma 6.11 with equations (21) and (22) with $V = T = TX - \mathfrak{g}$ yields,

$$T(m_1, m_2, m_3, m_4) = T(m_{1,2}) + T(m_{1,2,3}) + E_{1,2}$$

$$= T(m_{2,3}) + T(m_{1,2,3}) + E_{2,3}.$$  (32)

Therefore the $\star_{ct}$ and $\star_{tT}$ products are associative.

The fact that $\star_{ct}$ preserves the rational grading follows immediately from Equation (20), where $V = T$.

If $f : X \rightarrow Y$ is an étale $G$-equivariant morphism then the induced pullback $f^* : A^*_G(I_G(Y)) \rightarrow A^*_G(I_G(X))$ respects the orbifold products since $f^*(TY - \mathfrak{g}) = TX - \mathfrak{g}$. The argument is identical for K-theory.

To prove Equation (25), we first observe that the $\ell = 3$ case follows from either Equations (31) or (32). The general case follows by induction. Suppose that Equation (25) holds for $\ell$ then write

$$v_1 \star_{ct} \cdots \star_{ct} v_\ell \star_{ct} v_{\ell+1} = (v_1 \star_{ct} \cdots \star_{ct} v_\ell) \star_{ct} v_{\ell+1}$$

and apply the induction hypothesis to the expression in the parenthesis, using excess intersection theory and Equation (23) for $r = \ell + 2$ and $j = \ell$ to obtain the desired result. □

**Remark 6.12.** When $G$ is finite, then the character formula of [JKK, Lemma 8.5] implies that $T^{tw}$ equals the class of the obstruction bundle defined in the paper of Fantechi-Göttsche [FG]. Hence, the $\star_{ct}$ product on $A^*_G(X) \otimes \mathbb{Q} = A^*(X)^G$ equals their product.

**Remark 6.13.** The analogue of Equation (25) in the context of torus actions on symplectic manifolds was proven in [GHK].

**Example 6.14.** Let $G$ be a finite group acting upon a point $X$. In this case, $I_G(X) = G$ where $G$ acts upon itself by conjugation. Therefore, the orbifold $K$-theory $K_G(I_G(X))$ is additively equal to $K_G(G)$, the Grothendieck group of $G$-equivariant vector bundles on $G$, but the orbifold product is not the usual product on equivariant $K$-theory. Since the tangent bundle to $G$ has rank zero, $T^{tw}$ vanishes and the orbifold product of two $G$-equivariant vector bundles $V$ and $W$ over $G$ is given by

$$V \star W = \mu_* (e_1^* V \otimes e_2^* W)$$

where $e_i : G \times G \rightarrow G$ are the projections onto the $i$-th factor and $\mu : G \times G \rightarrow G$ is the multiplication. The $\star$ product on $K_G(G)$ coincides with a product introduced
by Lusztig [Lu]. Kaufmann and Pham [KP, Theorem 3.13] show that this product also coincides with a product on the representation ring of the Drinfeld double of $G$ which appears in the physics literature (cf. [DPR]).

**Example 6.15.** If we set $c_{\Phi(g, 1)} = [\mathbb{P}^2(\Phi(g, 1))]$ and $c_{\Phi(1, g)} = [\mathbb{P}^2(\Phi(g, 1))]$ and $c(\Phi) = 0$ for all other conjugacy classes in $G^{[2]}$, then the $\star_c$ product is commutative and associative. This is the restriction of the $\star_{ct}$ product obtained by setting all products $\alpha_{g_1} \star \alpha_{g_2}$ to be 0 unless $\Psi_1$ or $\Psi_2$ is the conjugacy class $\{1\}$. If $g \in \Psi$ is a representative element, and we identify $A^*_G(I(\Psi)) = A^*_G(Z_G(g))(X^g)$ then we have a simple formula for the $\star_c$ product:

$$
\alpha_{\{1\}} \star_c \alpha_\Psi = j^*\alpha_{\{1\}} \cdot \alpha_g,
$$

(33)

where $j^*: A^*_G(X) = A^*_G(I(\{1\})) \to A^*_G(Z_G(g))(X^g)$ is the composition of the restriction of groups map $A^*_G(X) \to A^*_G(Z_G(g))(X)$ with pullback in $Z_G(g)$-equivariant Chow groups for the regular embedding $X^g \hookrightarrow X$, and $\alpha_g \in A^*_G(Z_G(g))(X^g)$ is the class which is Morita equivalent to $\alpha_\Psi \in A^*_G(I(\Psi))$.

7. **Orbifold Riemann-Roch**

**Background hypotheses.** In this section, all spaces are required to be schemes and all group actions have finite stabilizer.

In this section, we extend the orbifold Riemann-Roch theorem of [JKK] to quotient stacks $\mathcal{X} = [X/G]$ with $G$ an arbitrary linear algebraic group acting with finite stabilizer on a smooth scheme $X$. In particular, we show that a generalization of the stringy Chern character defines a ring homomorphism

$$
\mathfrak{ch}: K_G(I_G(X)) \to A^*_G(I_G(X)) \otimes \mathbb{Q}
$$

of orbifold $\star_{\mathfrak{c}_I}$ and $\star_{\mathfrak{c}_T}$ products. Note, however, that (when $X$ is complete) $\mathfrak{ch}$ does not preserve the pairing. Applying the equivariant Riemann-Roch theorem of [EC2], the map $\mathfrak{ch}$ factors through an isomorphism $K_G(I_G(X))_1 \to A^*_G(I_G(X)) \otimes \mathbb{Q}$, where $K_G(I_G(X))_1$ is a distinguished summand in $K_G(I_G(X)) \otimes \mathbb{Q}$ which generalizes the small orbifold $K$-theory of [JKK] for quotients by a finite group.

After tensoring with $\mathbb{C}$, the summand $K_G(I_G(X))_1$ may be identified via the Riemann-Roch isomorphism of [EC2] with $K_G(X) \otimes \mathbb{C} = K([X/G]) \otimes \mathbb{C}$. As a corollary we obtain an orbifold product on $K(\mathcal{X}) \otimes \mathbb{C}$. We give an explicit description of this product in Section 8.

7.1. **Background on equivariant Riemann-Roch.** We recall some basic facts that were proved in [EC2] and [EC3] about the equivariant Riemann-Roch theorem. Suppose that $Y$ is a smooth algebraic space on which an algebraic group $G$ acts. The Chern character defines a ring homomorphism $\mathfrak{ch}: K_G(Y) \to \prod_{i=0}^{\infty} A^*_G(Y) \otimes \mathbb{Q}$. When the group acts quasi-freely, then by [EG1,EG2], $A^*_G(Y) \otimes \mathbb{Q} = 0$ for $i$
sufficiently large, so the infinite direct product may be identified with the usual rational equivariant Chow ring. If the quotient stack \([Y/G]\) satisfies the resolution property (in particular if \(Y\) is a smooth scheme), then we may identify the Grothendieck group of equivariant vector bundles \(K_G(Y)\) with the Grothendieck group of \(G\)-linearized coherent sheaves.

In this case, the equivariant Riemann-Roch theorem of [EG2] implies that the map \(\text{ch}: K_G(Y) \to \prod_{i=0}^{\infty} A_G^i(Y) \otimes \mathbb{Q}\) factors through an isomorphism \(\widehat{K}_G(Y) \to \prod_{i=0}^{\infty} A_G^i(Y)\), where \(\widehat{K}_G(Y)\) is the completion of \(K_G(X) \otimes \mathbb{Q}\) at the augmentation ideal of \(\text{Rep}(G) \otimes \mathbb{Q}\). When \(G\) acts quasi-freely then \(K_G(Y)\) is supported at a finite number of points of the representation ring \(\text{Rep}(G)\), and we may identify the augmentation completion \(\widehat{K}_G(Y)\) with the localization at the same ideal. This localization, which we denote by \(K_G(Y)_1\), is a summand in \(K_G(Y) \otimes \mathbb{Q}\). If we let \(\mathcal{Y} = [Y/G]\) be the quotient stack, then \(K_G(Y)_1\) may be identified with the completion of \(K(\mathcal{Y}) \otimes \mathbb{Q}\) at its augmentation ideal. In particular, it is independent of the choice of presentation for \(\mathcal{Y}\) as a quotient stack.

**Theorem 7.1.** (cf. [EG2]) The equivariant Chern character homomorphism \(\text{ch}: K_G(Y) \to A_G^*(Y) \otimes \mathbb{Q}\) factors through an isomorphism \(K_G(Y)_1 \to A_G^*(Y) \otimes \mathbb{Q}\).

**Proof.** Since \(Y\) is smooth, [EG2, Theorem 3.3] states that the map

\[
\tau^G_X: K_G(Y)_1 \to A_G^*(Y), \quad \text{given by } V \mapsto \text{ch}(V) \text{Td}(\mathbb{T})
\]

is an isomorphism. However, the class \(\text{Td}(\mathbb{T})\) is invertible in \(A_G^*(Y) \otimes \mathbb{Q}\), so the Chern character homomorphism is also an isomorphism, after completing at the augmentation ideal. \(\square\)

### 7.2. The orbifold Chern character.

As observed in [JKK], the Todd class map \(K(X) \to A^*(X) \otimes \mathbb{Q}\) can be formally extended to a map \(K(X) \otimes \mathbb{Q} \to A^*(X) \otimes \mathbb{Q}\) by defining \(\text{Td}(\frac{1}{m!}V)\) to be the unique element of the form \(t = 1 + t_1 + \ldots + t_m\) with \(t_i \in A^i(X)\) that satisfies the equation \(t^m = \text{Td}(V) \in A^*(X)\). The same argument works for equivariant Chow groups and we can define \(\text{Td}(\frac{1}{m!}V) \in \prod_{i=0}^{\infty} A_G^i(X)\) for any equivariant vector bundle \(V\).

If \(\Psi\) is a conjugacy class in \(G\) and \(V\) is a \(G\)-equivariant vector bundle on \(X\), let \(L(\Psi)(V) \in K_G(I_{\Psi}) \otimes \mathbb{Q}\) be the class which is Morita equivalent to \(L(m)(V|_{X^m})\) for any \(m \in \Psi\). The argument used to prove Lemma 5.4 shows that \(L(\Psi)(V)\) is independent of the choice of \(m\).

**Definition 7.2.** Define the orbifold Chern character

\[
\text{ch}: K_G(I_G(X)) \to A_G^*(I_G(X)) \otimes \mathbb{Q}
\]

by the formula

\[
\text{ch}(\mathcal{F}_\Psi) = \text{ch}(\mathcal{F}_\Psi) \text{Td}(-L(\Psi)(\mathbb{T}))
\]

for a class \(\mathcal{F}_\Psi \in K_G(I(\Psi))\).
We now obtain the following generalization of [JKK, Theorem 6.1] to groups of positive dimension.

**Theorem 7.3.** The map $\text{ch}: K_G(I_G(X)) \to A_G^*(I_G(X)) \otimes \mathbb{Q}$ is a homomorphism when $K_G(I_G(X))$ has the $\ast_{\varepsilon_1}$ product and $A_G^*(I_G(X)) \otimes \mathbb{Q}$ has the $\ast_{c_1}$ product. Moreover, the map $\text{ch}$ factors through an isomorphism

$$\text{ch}: K_G(I_G(X))_1 \to A_G^*(I_G(X)) \otimes \mathbb{Q}.$$  

**Proof.** The proof is analogous to the proof of Theorem 6.1 in [JKK]. Given conjugacy classes $\Psi_1$ and $\Psi_2$ and $\alpha_1 \in K_G(I(\Psi_1))$, $\alpha_2 \in K_G(I(\Psi_2))$, then by definition

$$\text{ch}(\alpha_1 \ast_{\varepsilon_1} \alpha_2) = \text{ch}\left(\sum_{\Phi_{1,2}} \mu_*(e_1^*\alpha_1 \otimes e_2^*\alpha_2 \otimes \lambda_{-1}(T(\Phi_{1,2}))^*)\right),$$  

where the sum on the right-hand side of (35) is over all $\Phi_{1,2} \in G^{[2]}$ satisfying $e_1(\Phi_{1,2}) = \Psi_1$, $e_2(\Phi_{1,2}) = \Psi_2$.

Similarly,

$$\text{ch}(\alpha_1) \ast_{c_1} \text{ch}(\alpha_2) = \sum_{\Phi_{1,2}} \mu_*(\text{ch}(\alpha_1)\text{ch}(\alpha_2)\varepsilon(T(\Phi_{1,2})))$$  

To prove that $\text{ch}$ commutes with the twisted product, it suffices to show that the right-hand sides of (35) and (36) are termwise equal. Consider $\Phi_{1,2} \in G^{[2]}$ satisfying $e_1(\Phi_{1,2}) = \Psi_1$, $e_2(\Phi_{1,2}) = \Psi_2$. Let $\Psi_{12} = \mu(\Phi_{1,2})$. By definition

$$\text{ch}(\mu_*(e_1^*\alpha_1 \otimes e_2^*\alpha_2 \otimes \lambda_{-1}(T(\Phi_{1,2}))^*)) = \text{ch}(\mu_*(e_1^*\alpha_1 \otimes e_2^*\alpha_2 \otimes \lambda_{-1}(T(\Phi_{1,2}))^*)) Td(-L(\Psi_{12})(\mathbb{T}))$$  

By the equivariant Grothendieck-Riemann-Roch theorem, the right-hand side of (37) equals

$$\mu_*[\text{ch}(e_1^*\alpha_1 \otimes e_2^*\alpha_2 \otimes \lambda_{-1}(T(\Phi_{1,2}))^*) Td(T_\mu)] Td(-L(\Psi_{12})(\mathbb{T}))$$  

Now if $V$ is a positive element in equivariant $K$-theory, then $\text{ch}(\lambda_{-1}(V^*)) Td(V) = \varepsilon(V)$. Thus, using the multiplicity of the ordinary equivariant Chern character, we may rewrite the right hand side of (38) as

$$\mu_*[\text{ch}(e_1^*\alpha_1) Td(T_\mu)] Td(-T(\Phi_{1,2})) Td(-L(\Psi_{12})(\mathbb{T}))$$  

Applying the projection formula, (39) can be rewritten as

$$\mu_*[e_1^*\text{ch}(\alpha_1) Td(T_\mu)] Td(-T(\Phi_{1,2})) Td(-\mu^*L(\Psi_{12})(\mathbb{T}))$$  

On the other hand,

$$\text{ch}(\alpha_1) \ast_{c_1} \text{ch}(\alpha_2)$$  

$$= \mu_*[e_1^*(\text{ch}(\alpha_1) Td(-L(\Psi_1)(\mathbb{T}))) e_2^*(\text{ch}(\alpha_2) Td(-L(\Psi_2)(\mathbb{T}))) \varepsilon(T(\Phi_{1,2}))]$$  

$$= \mu_*[e_1^* \text{ch}(\alpha_1) e_2^* \text{ch}(\alpha_2) \varepsilon(T(\Phi_{1,2})) Td(-\mu^*L(\Psi_1)(\mathbb{T})) Td(-e_2^*L(\Psi_2)(\mathbb{T}))]$$
Comparing (40) and (41), it suffices to show that the equation
\[ T_\mu - \mathcal{T}(\Phi_{1,2}) - \mu^*L(\Psi_1)(\mathcal{T}) = -\epsilon_1^*L(\Psi_1)(\mathcal{T}) - \epsilon_2^*L(\Psi_2)(\mathcal{T}) \]  
holds in \( K_G(I(\Psi_{1,2}) \rightleftharpoons K_G(I(\Psi_1,\Psi_2)) \). If \((m_1, m_2) \in \Psi\), then by Morita equivalence it suffices to prove the corresponding identity in \( K_{\mathbb{Z}_m}(X^m) \), where \( m = (m_1, m_2, (m_1m_2)^{-1}) \).

By definition,
\[ T_\mu = \mu^*TI(\Psi_1) - T\mathcal{I}^2(\Phi_{1,2}) \]  
As noted above, if \( \Phi \) is the conjugacy class of an \( l \)-tuple \( \mathbf{g} = (g_1, \ldots, g_l) \), then \( T\mathcal{I}^l(\Phi) = \mathcal{T}\mathbf{g} \). Hence by Lemma 2.30, the right-hand side of (43) is Morita equivalent to
\[ T^{m_1m_2}|_{X^m} - \mathcal{T}^m. \]  
Substituting (44) and the definition of the logarithmic trace we see that the left-hand side of (42) simplifies to
\[ T^{m_1m_2}|_{X^m} - \mathcal{T}(\mathbf{m}) - L(m_1m_2)(\mathcal{T})|_{X^m}. \]  
Applying the identity \( L(g)(V) + L(g^{-1})(V) = V - V^g \) with \( g = m_1m_2 \) and \( V = \mathcal{T} \), we can rewrite (45) as
\[ -\mathcal{T}(\mathbf{m}) + \mathcal{T}^m - T^{m_1m_2}|_{X^m} + L(m_1m_2)(\mathcal{T})|_{X^m}. \]  
By definition
\[ \mathcal{T}(\mathbf{m}) = L(m_1)(\mathcal{T})|_{X^m} + L(m_2)(\mathcal{T})|_{X^m} + L((m_1m_2)^{-1})(\mathcal{T})|_{X^m} + \mathcal{T}^m|_{X^m} - \mathcal{T}^m. \]  
Substituting (47) into (46), we see that the left-hand side of (42) simplifies to
\[ -L(m_1)(\mathcal{T})|_{X^m} - L(m_2)(\mathcal{T})|_{X^{m_2}}, \]  
but the right hand side of (42) is Morita equivalent to the same class, since \( L(\Psi_i)(\mathcal{T}) \) is Morita equivalent to \( L(m_i)(\mathcal{T}) \). Therefore \( \phi \) defines a ring homomorphism.

Moreover, since the class \( \text{Td}(L(\Psi)) \) is invertible in \( A^*_G(I(\Psi)) \otimes \mathbb{Q} \) and the \( *_{e_G} \) commutes with the natural \( \text{Rep}(G) \) action on \( K_G(X) \) we see that the map \( \phi \) factors through a ring isomorphism of twisted products \( K_G(X_1) \rightarrow A^*_G(I_G(X)) \otimes \mathbb{Q} \).

**Remark 7.4.** The localization \( K_G(I_G(X))_1 \) is the analogue, for \( \dim G > 0 \), of the small orbifold \( K \)-theory of [JKK]. In particular, when \( G \) is a finite group then \( K_G(X)_1 \) is isomorphic to \( K_{\text{orb}}([X/G]) \), the small orbifold \( K \)-theory of \([X/G]\). This follows from the fact that both are isomorphic to the orbifold Chow ring.

### 8. A Twisted Product on \( K_G(X) \otimes \mathbb{C} \)

Given a commutative, twisted product \( * \) on \( K_G(I_G(X)) \) which commutes with the \( \text{Rep}(G) \)-algebra structure on \( K_G(I_G(X)) \), the results of [EG2] and [EG3] allow one to define a corresponding product on \( K_G(X) \otimes \mathbb{C} \). In particular, the twisted product \( *_{e_G} \) induces an orbifold product on \( K_G(X) \otimes \mathbb{C} \). In this case there is an orbifold Chern character isomorphism of \( K_G(X) \otimes \mathbb{C} \) with \( A^*_G(I_G(X)) \otimes \mathbb{C} \).
Remark 8.1. The isomorphism of vector spaces \( K_G(X) \otimes \mathbb{C} \rightarrow A^*_G(I_G(X)) \otimes \mathbb{C} \) may be viewed as an algebraic analogue to Adem and Ruan’s isomorphism \([AR]\) between equivariant topological \( K \)-theory and equivariant cohomology of \( I_G(X) \). However, the products that appear in the Chern character described in \([AR]\) are not the same as ours. In particular, the product they use on equivariant topological \( K \)-theory is the tensor product, and the product they use on the equivariant cohomology of \( I_G(X) \) is not the Chen-Ruan orbifold product.

8.1. A decomposition of \( K_G(X) \otimes \mathbb{C} \) and the non-Abelian localization theorem. When \( G \) acts quasi-freely then by \([EG2, EG3, VV]\) the \( \text{Rep}(G) \otimes \mathbb{C} \)-module \( K_G(X) \otimes \mathbb{C} \) is supported at a finite number of maximal ideals \( m_\Psi \subset \text{Rep}(G) \otimes \mathbb{C} \), corresponding to conjugacy classes \( \Psi \) such that \( I(\Psi) \neq \emptyset \). As in \([EG3]\), we use the notation \( m_\Psi \) to refer to the maximal ideal of virtual representations whose character vanishes on \( \Psi \). As a result, equivariant \( K \)-theory decomposes into a direct sum of its localizations.

\[
K_G(X) \otimes \mathbb{C} = \bigoplus_\Psi K_G(X)_{m_\Psi}, \tag{49}
\]

where the sum is over the finite number of conjugacy classes \( \Psi \) such that \( I(\Psi) \neq \emptyset \).

Remark 8.2. If we view \( \text{Rep}(G) \otimes \mathbb{C} \) as the ring of polynomial class functions on \( G \), then the component \( \alpha_\Psi \) of a class \( \alpha \) in the summand \( K_G(X)_{m_\Psi} \) equals \( 1_\Psi \alpha \), where \( 1_\Psi \in \text{Rep}(G) \otimes \mathbb{C} \) is any polynomial satisfying \( 1_\Psi(\Psi) = 1 \) and \( 1_\Psi(\Psi') = 0 \) for all other \( \Psi' \in \text{Supp}(K_G(X) \otimes \mathbb{C}) \). Since the support of \( K_G(X) \otimes \mathbb{C} \) is finite, such a function always exists. If \( G \) is infinite, there is no canonical choice for the function \( 1_\Psi \) because we impose no conditions on the value away from the support of \( K_G(X) \otimes \mathbb{C} \). However, different choices for \( 1_\Psi \) yield the same product \( 1_\Psi \alpha \in K_G(X) \otimes \mathbb{C} \).

Let \( f: I_G(X) \rightarrow X \) be the projection. Since \( G \) is assumed to act with finite stabilizer, the map \( f \) is a finite l.c.i. morphism. Fix a conjugacy class \( \Psi \in G \) such that \( I(\Psi) \neq \emptyset \). By Proposition \(2.11\), \( I(\Psi) \) is open and closed in \( I_G(X) \), so the restriction of \( f \) to \( I(\Psi) \) is also finite and l.c.i. Choose \( h \in \Psi \) and identify \( I(\Psi) = G \times_Z X^h \), where \( Z = Z_G(h) \) is the centralizer of \( h \) in \( G \). Let \( m_\Psi \subset \text{Rep}(Z) \otimes \mathbb{C} \) be the ideal of virtual representations whose character vanishes at the central conjugacy class \( \{h\} \subset Z \). Since \( Z \) acts with finite stabilizer on \( X^h \), we have that \( K_Z(X^h)_{m_\Psi} \) is a summand in \( K_Z(X^h) \otimes \mathbb{C} \). Following \([EG3]\), we denote by \( K_G(I(\Psi))_{\text{cent}_\Psi} \) the summand in \( K_G(I(\Psi)) \otimes \mathbb{C} \) which is Morita equivalent to \( K_Z(X^h)_{m_\Psi} \). By \([EG3, Lemma 4.6]\), \( K_G(I(\Psi))_{\text{cent}_\Psi} \) is independent of the choice of representative \( h \in \Psi \). Having established the necessary notation, the non-Abelian localization theorem is as follows.
Theorem 8.3. [EG3, Theorem 5.1] The pushforward $f_* : K_G(I(\Psi))_{\text{cent}} \to K_G(X)_{m}$ is an isomorphism. If $\alpha \in K_G(X)_{m}$, then

$$\alpha = f_* \left( \frac{f^* \alpha}{\lambda-1(N_f^*)} \right)$$

(50)

8.2. Multiplicative twisting in equivariant $K$-theory. Let $Y$ be an algebraic space with the action of an algebraic group $Z$ and let $V$ be a $Z$-bundle on $Y$. Let $h$ be an automorphism of the fibers of $V/Y$ which commutes with the action of $Z$ on $V$. We can consider an exponential analogue of the logarithmic trace and define the multiplicative twist $V^{\text{mult}}(h)$ of $V$ in $K_Z(Y) \otimes \mathbb{C}$ by the formula

$$V^{\text{mult}}(h) = \sum_{\chi} \chi V_{\chi},$$

(51)

where the sum is over all eigenvalues $\chi$ for the action of $h$ on the fibers of $V/X$ and $V_{\chi}$ denotes the $\chi$-eigenspace. The formula (51) extends to an automorphism $t_g : K_Z(X) \otimes \mathbb{C} \to K_Z(X) \otimes \mathbb{C}$ with inverse $t_g^{-1}$.

As in [EG3, EG4], we will consider the twist in the following special case. If $h \in \Psi$, then $h$ is central in $Z = Z_G(h)$ and acts trivially on $X^h$. Hence there is an action of $h$ on the fibers of any $Z$-bundle on $X^h$. In this case, the automorphism $t_h$ induces an isomorphism of the summands $K_Z(X^h)_1$ and $K_Z(X^h)_{m_h}$. Via the usual Morita equivalence there is an induced isomorphism $t : K_G(I_G(X))_1 \to K_G(I_G(X))_{\text{cent}}$ which is independent of the choice of representative $h \in \Psi$. To simplify notation, let

$$t : K_G(I_G(X))_1 \to \bigoplus_{\Psi} K_G(I_G(X))_{\text{cent}},$$

be the map whose restriction to the summand $K_G(I(\Psi))_1$ is $t_{\Psi}$.

In order to define the twisted product on $K_G(X) \otimes \mathbb{C}$, we will need to combine the non-Abelian localization map with the multiplicative twist.

Definition 8.4. Let $f^1 : K_G(X) \otimes \mathbb{C} \to K_G(I_G(X))_1$ given by the formula

$$f^1 \alpha_{\Psi} = t^{-1} \left( \frac{f^* \alpha_{\Psi}}{\lambda-1(N_f^*)} \right)$$

(52)

for $\alpha_{\Psi} \in K_G(X)_{m}$. 

Proposition 8.5. The map $f^1$ is an isomorphism $K_G(X) \to K_G(I_G(X))_1$ and $(f^1)^{-1} = f_* \circ t$
Proof. The restriction of $f^!$ to $K_G(X)_{m\Psi}$ is the composition of isomorphisms $f^* : K_G(X)_{m\Psi} \to K_G(I(\Psi))_{\text{cent}\Psi}$ and $t^{-1}_\Psi : K_G(I(\Psi))_{\text{cent}\Psi} \to K_G(I(\Psi))_1$, so it is an isomorphism. Also, given $\alpha_\Psi \in K_G(X)_{m\Psi}$, we have

$$(f_* \circ t)(f^! \alpha_\Psi) = f_* \circ f^*( \frac{\alpha_\Psi}{\lambda_1(N_\Psi^*)})$$

where the second equality follows from (50). Conversely, if $\beta_\Psi \in K_G(I(\Psi))_1$, then

$$f^!(f_*(t(\beta_\Psi))) = t^{-1}_\Psi f^*( f_*(\frac{t(\beta_\Psi)}{\lambda_1(N_\Psi^*)}))$$

$$= t^{-1}(\lambda_1(N_\Psi^*) \frac{t(\beta_\Psi)}{\lambda_1(N_\Psi^*)})$$

$$= \beta_\Psi,$$

where the second equality follows from the self-intersection formula for the map $f$. □

8.3. The twisted product on $K_G(X) \otimes \mathbb{C}$. We now have the necessary terminology to define the twisted product on $K_G(X) \otimes \mathbb{C}$.

Definition 8.6. Given classes $\alpha_1, \alpha_2 \in K_G(X) \otimes \mathbb{C}$ set

$$\alpha_1 \star_T \alpha_2 = f_* t(f^!(\alpha_1) \star_{\text{et}} f^!(\alpha_2)).$$

(53)

The following is an immediate consequence Theorem 6.10.

Theorem 8.7. The $\star_T$ product on $K_G(X) \otimes \mathbb{C}$ is commutative and associative with identity element equal to the the projection of $[\mathcal{O}_X]$ in $K_G(X)_1$.

We also obtain as corollary of the Riemann-Roch isomorphism.

Corollary 8.8. The map $\text{ch} \circ f^! : K_G(X) \otimes \mathbb{C} \to A^*_G(I_G(X)) \otimes \mathbb{C}$ is a ring isomorphism for the $\star_T$ product on $K_G(X) \otimes \mathbb{C}$ and the $\star_{\text{et}}$ product on $A^*_G(I_G(X))$.

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