Experimental determination of the degree of quantum polarisation of continuous variable states

Christian Kothe, Lars Skovgaard Madsen, Ulrik Lund Andersen, and Gunnar Björk

1Department of Physics, Technical University of Denmark, Fysikvej, DK-2800 Kongens Lyngby, Denmark
2Department of Applied Physics, Royal Institute of Technology (KTH), AlbaNova University Center, SE-106 91 Stockholm, Sweden

(Dated: May 2, 2014)

We demonstrate excitation-manifold resolved polarisation characterisation of continuous-variable (CV) quantum states. In contrast to traditional characterisation of polarisation that is based on the Stokes parameters, we experimentally determine the Stokes vector of each excitation manifold separately. Only for states with a given photon number does the methods coincide. For states with an indeterminate photon number, for example Gaussian states, the employed method gives a richer and more accurate description. We apply the method both in theory and in experiment to some common states to demonstrate its advantages.

PACS numbers: 42.25.Ja, 03.65.Ca, 42.50.Dv

Polarisation is one of the key parameters of the electro-magnetic field as demonstrated by the plethora of different applications. To mention a few, the classical polarisation is used in thin-film ellipsometry [1], near-field microscopy [2], remote sensing [3] and light scattering [4]. In recent years, the concept of polarisation has found a footing in quantum optics and in quantum information science where the information is efficiently encoded in a time-efficient manner.

Despite the seemingly correct translation from classical to quantum polarisation, the definition in (3) yields an inconsistent quantification of the degree of polarisation [21]. This can be illustrated by some simple examples: According to the definition in (3), the state $|\Psi\rangle_H$ is fully polarised (that is $P^{cl}_1 = 1$) for any pure state $|\Psi\rangle_H \neq |0\rangle$. This implies the unpalatable

$$P^{cl}_1(|\Psi\rangle) = \sqrt{\langle \hat{S}_1 \rangle^2 + \langle \hat{S}_2 \rangle^2 + \langle \hat{S}_3 \rangle^2}.$$  

\[ (3) \]

where the polarisation of a classical, electro-magnetic field is uniquely described by the Stokes parameters which can be written as

$$S_0 = |a_1|^2 + |a_2|^2, S_1 = a_H^* a_V + a_H a_V^*,$$

$$S_2 = -i (a_H^* a_V - a_H a_V^*), S_3 = |a_H|^2 - |a_V|^2, \quad (1)$$

where $a_H$ and $a_V$ denote the amplitudes of the field in two linearly polarised orthogonal modes $H$ and $V$. From these, the classical degree of polarisation $P^{cl}$ is defined as

$$P^{cl} = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0}.$$  

\[ (2) \]

The degree of quantum polarisation has previously been defined as a direct translation of the classical degree [11–13]:

$$P^{cl}_1(\hat{\rho}) = \sqrt{\langle \hat{S}_1 \rangle^2 + \langle \hat{S}_2 \rangle^2 + \langle \hat{S}_3 \rangle^2}.$$  

\[ (3) \]
consequence that states arbitrarily close to the
two-mode vacuum are fully polarised and renders
the measure discontinuous as a function of the state excitation.
As another example of its failure, we consider
the state $|Ξ\rangle = \frac{1}{\sqrt{3}} \left( \sqrt{2} |1, 0\rangle_H V + e^{i\varphi} |0, 2\rangle_H V \right)$ where $
\varphi \in \{0; 2\pi\}$. Applying the semi-classical polarisation measure $\mathbb{P}_1^c(|Ξ\rangle) = 0$ and thus implies that $|Ξ\rangle$ is unpolari-
sed. This means that the state should be polarisation trans-
formation invariant [20]. However, $|Ξ\rangle$ is not polarisa-
tion transformation invariant: Under, a $\pi/2$ polarisation rotation of the state, the state is transformed into
$\frac{1}{\sqrt{3}} \left( \sqrt{2} |0, 1\rangle_H V - e^{i\varphi} |2, 0\rangle_H V \right)$, which is orthogonal to
$|Ξ\rangle$. Thus $|Ξ\rangle$ is not invariant under polarisation trans-
formation and it is therefore clear that the definition in
(5) falls short in quantifying the degree of quantum polar-
isation. We note that a common property of the afore-
mentioned examples is that the photon number $n$ is not a
fixed quantity.

From the above discussion, it is clear that the semi-
classical definition is unsuitable for determining the de-
gree of polarisation for many quantum states. As the
main result of this paper, we propose a new definition
of the degree of polarisation that circumvents the short-
comings of the previous definition:

$$\mathbb{P}_1(\hat{\rho}) = \sum_{N=1}^{\infty} p_N \sqrt{\left\langle \hat{S}_1, N \right\rangle^2 + \left\langle \hat{S}_2, N \right\rangle^2 + \left\langle \hat{S}_3, N \right\rangle^2} / \left\langle \hat{S}_0, N \right\rangle, (4)$$

where $p_N = \text{Tr}(\hat{1}_N \rho)$, $\hat{\rho}_N = (\hat{1}_N \hat{\rho} \hat{1}_N) / p_N$, $\left\langle \hat{S}_j, N \right\rangle = \text{Tr}(\hat{S}_j \hat{\rho}_N)$, and $\hat{1}_N = \sum_{m=0}^{N} |m, N-m\rangle \langle m, N-m|$, so that $\hat{\rho}_N$ is the normalised $N$-photon projection of the state’s density matrix. The polarisation degree is quanti-
fied by a weighted sum of the semi-classical degree of polarisation in each excitation manifold of the state (except for $N = 0$). In other words, every excitation manifold is treated separately. As is clear from the definition, $\mathbb{P}_1$ coincides with $\mathbb{P}_1^c$ when the number of photons is a fixed quantity. The two definitions also become approx-
imately equal for classical-like states such as coherent states with $\left\langle \hat{S}_0 \right\rangle \gg 1$. For many other states the two definitions give different results, and in the following we argue that $\mathbb{P}_1$ gives a better assessment of the polarisation properties of quantum states than $\mathbb{P}_1^c$. For example, we find the intuitively correct results $\mathbb{P}_1(|\psi\rangle) = 1$ and $\mathbb{P}_1(|\psi_{sqz}\rangle) \neq 1$. Below, we will examine, theoretically and experimentally, some properties of $\mathbb{P}_1$ for continuous variable Gaussian states, and we will in particular focus on moderately excited coherent and squeezed states [22].

First we consider a two-mode state in which one of
two orthogonal polarisation modes is vacuum whereas the other one is a coherent state: $|\psi\rangle = |\alpha\rangle_H \otimes |0\rangle_V$, where $\alpha = ae^{i\phi}$ is the complex amplitude of the coherent state ($a \in \mathbb{R}_+^\times$, $\phi \in [0; 2\pi]$). For this state, the semi-
classical degree of polarisation is unity, $\mathbb{P}_1^{sc}(|\psi\rangle) = 1$,

for all values of $\alpha$ except $\alpha = 0$. On the other hand, using the new measure we find

$$\mathbb{P}_1(|\psi\rangle) = 1 - e^{-|\alpha|^2}. (5)$$

which is continuous $\forall \alpha$; $\mathbb{P}_1(|\psi\rangle) \rightarrow 0$ when $\alpha \rightarrow 0$, and for large amplitudes, $\mathbb{P}_1(|\psi\rangle) \rightarrow 1$ when $|\alpha| = a \gg 1$. Therefore, the classical and quantum limits, re-
spectively corresponding to large and small amplitudes, are smoothly connected. $\mathbb{P}_1(|\psi\rangle)$ is illustrated in
Fig. 1 by the solid line, $\mathbb{P}_1^c(|\psi\rangle)$ is shown by the dotted line. Eq. (5) can be easily generalised to any two-
mode coherent states, $|\alpha\rangle_H \otimes |\beta\rangle_V$, which after an appro-
priate transformation can be written as a one mode coherent state, $|\alpha'\rangle = |0\rangle_V$ in some other polarisation basis with $|\alpha'|^2 = |\alpha|^2 + |\beta|^2$. The degree of quantum polarisation of any two-mode coherent state is therefore given by Eq. (5) with $|\alpha|^2 \rightarrow |\alpha|^2 + |\beta|^2$.

Next we consider the degree of quantum polarisation for single-mode squeezed states: $|\varphi(\xi, \alpha)\rangle = D(\alpha)S(\xi)|0\rangle_H \otimes |0\rangle_V$ where $D$ is the displacement operator and $S(\xi) = \exp \frac{[\xi a - \xi \dot{a}]/2}{2}$ is the squeezing operator with $\xi = re^{i\theta}$, $r \in \mathbb{R}_+^\times$, $\theta \in [0; 2\pi)$ being the squeezing parameter. For this state we find

$$\mathbb{P}_1^{sc}(|\varphi(\xi, \alpha)\rangle) = 1, (6)$$

which is illustrated in Fig. 1 as a function of the average number of photons for a squeezed vacuum state (that is, $\alpha = 0$ and $\langle n_{sqz}\rangle = \sinh(r)$). We clearly see that $\mathbb{P}_1$ differs significantly from $\mathbb{P}_1^c$ for basically all practical squeezing values. We also note that the degree of quantum polarisation (according to the new measure) is not
parameters whereas for different squeezing parameters, the degree of polarisation (both squeezed state and squeezed vacuum state) is illustrated for different squeezing degrees. Both measures exhibit zero polarisation degree for equal squeezing angles, $\theta$. Whereas the right plots show $P_D$, the degree of polarisation also depends on the squeezing angle, $\theta$.

Finally, we consider the generalised pure two-mode squeezed state

$$\hat{D}(\alpha_H)\hat{S}(\xi_H)|0\rangle_H \otimes \hat{D}(\alpha_V)\hat{S}(\xi_V)|0\rangle_V,$$

and plot the degree of polarisation (both $P^c_1$ (left column) and $P_1$ (right column)) in Fig. 2 for three different states. In Fig. 2 (a), a two-mode vacuum state ($\alpha = 0$) is illustrated for different squeezing degrees. Both measures exhibit zero polarisation degree for equal squeezing parameters whereas for different squeezing parameters, $P_1$ gives lower values than $P^c_1$. If we now set $\xi_H = 0$ and $\alpha_V = 0$ (corresponding to a coherent state in the $H$-mode and a squeezed vacuum state in the $V$-mode), the behaviour of the two polarisation measures is very different as illustrated in Fig. 2 (b). Finally, we plot the two-mode displaced squeezed state (with $\xi_H = 0.2$ and $\xi_V = 0.6$) in Fig. 2 (c). The plot for the semi-classical measure once again illustrates its inappropriateness to be a good measure of polarisation: According to $P^c_1$, by displacing a squeezed state further away from the vacuum, the state becomes more unpolarised. This incorrect behaviour is not seen by the new measure.

**Experimental realisation** - Since $\hat{S}_x$ commutes with all other Stokes operators, the Stokes vectors per excitation manifold and thus $P_1$ can be directly accessed by using a proper waveplate configuration, a polarising beam splitter and two photon number resolving detectors (PNRDs). Such detectors are currently capable of efficiently detecting more than 6 photons and due to the rapid progress in developing such detectors more advanced versions with increased optical power range might soon become available [23]. For very high excitations, standard intensity detectors can be used [10, 24, 25].

Since we wish to characterise the degree of polarisation in different regimes from low to high photon numbers, we have chosen to use a homodyne detector. Using such a detection device, a full tomographic reconstruction of the state [27, 28] from low to relatively high photon numbers is possible, and from this reconstruction we deduce the degree of polarisation.

We produce displaced two-mode squeezed states using the setup shown in Fig. 3. Two optical parametric oscillators (OPOs) based on nonlinear downconversion in periodically poled KTP crystals are used to generate vacuum squeezed states. The OPOs are injected with modulated coherent states to enable the production of displaced squeezed states [29]. To form the two-mode state, the outputs from the OPOs are combined on a polarising beam splitter.

In contrast to previous realisations on CV polarisation quantum states, we solely define our state to be residing in periodically poled KTP crystals and terminated with a polarising beam splitter.
FIG. 4. Degree of polarisation for experimental data of a squeezed state in mode $H$ and a displaced coherent state in mode $V$. We vary the displacement from $-6$ dB ($\alpha = 0.25$, 0.0625 photons) to 6 dB ($\alpha = 4$, 16 photons). The squeezed state has a squeezing of 3.2 dB and an antisqueezing of 7.4 dB, corresponding to 1.0 photons. Error bars correspond to 1% uncertainty in the shot noise. Simulations starting from the initial squeezed state are shown with the solid lines. The inset shows the polarisation contributions of the different manifolds for the 0 dB (1 photon) state. Only $\langle \hat{S}_y \rangle$ contributes to the polarisation of the states produced here (i.e., $\langle \hat{S}_y \rangle = \langle \hat{S}_z \rangle = 0$) and therefore one has $P^s_1 = |\sum$ blue bars| and $P_1 = \sum$ red bars.

at a sideband frequency of 4.9 MHz. Such a definition of the polarisation state enables us to investigate a large variety of different polarisation states from a low excitation to a relatively high excitation. We measure each mode, $H$ and $V$, by splitting the polarisation state on a polarising beam splitter and using two homodyne detectors. The measured currents of the homodyne detectors are sampled at 500 kHz with a frequency bandwidth of 90 kHz, and subsequently sent to a computer for analysis. Since the generated states have Gaussian wavefunctions, it suffices to estimate the covariance matrix of the state for full characterisation. From this we calculate the first 50 excitation manifolds of the two-mode density matrix and take the expectation values of the Stokes operators (per manifold) from which the degree of polarisation is estimated.

We start our experimental analysis with one-mode squeezed or one-mode coherent states as defined in Eq. (5) and Eq. (6) (when $\alpha = 0$), respectively. These states are produced by blocking OPO2 while operating either the EOM (for producing the coherent state) or the OPO1 (for producing the squeezed states). The excitation of the coherent state is controlled by the modulation depth of the EOM whereas the squeezing degree (or the average number of photons associated with the squeezing process) of the squeezed state is controlled by the pump power. Our results for $P^s_1$ and $P^c_1$ are plotted in Fig. 4 where the error bars indicate the 1% uncertainty in determining the shot-noise limit. The experimental values for the squeezed state deviates slightly from the theoretical prediction (Eq. (6) with $\alpha = 0$) which is a consequence of the small impurity of the generated state. As also predicted by theory, we see that both states become increasingly more polarised as the photon numbers from the coherent state or from the squeezed state increases.

Next, we investigate another particularly interesting state in which a coherent state is excited in the $H$-mode while the $V$-mode is a squeezed vacuum state corresponding to $\alpha_V = 0$ and $r_H = 0$. The squeezed state is squeezed by 3.2 dB below the shot noise limit and the coherent excitation of the $H$ mode is varied.

We present the experimental results for this state in Fig. 4. For a coherent amplitude of 0.25, $P^c_1$ yields a large degree of polarisation of 0.88 although this state is very close to the vacuum state. Furthermore, when increasing the coherent modulation, $P^c_1$ decreases to zero which occurs when the number of photons in each polarisation mode is unity. This result is erroneous as the state is not invariant to rotation (permutation of the $H$ and $V$ mode) for any value of the displacement. In contrast, the new measure is behaving as expected: The degree of polarisation is reasonably small for low excitations and increases near monotonically for larger excitations. These different behaviours can be understood by looking at the contributions of the different manifolds in definition (4). $\hat{S}_1$ is the only operator contributing to the polarisation and we plot the expectation value of this per manifold in inset of Fig. 4. Here, we see that it points in opposite directions for the different manifolds which then sum up to zero for the $P^c_1$, and thus the polarisation becomes hidden. However, for the $P_1$-measure, the polarisation is not hidden since in this case the absolute value of the $\langle \hat{S}_1 \rangle$-values from the different manifolds are added.

As a final experiment we operate both OPOs and modulators in order to produce the generalised state in Eq. (7). Also for these generalised states we measure a degree of polarisation which is monotonically increasing as a function of the coherent excitation.

Conclusion - We have proposed a measure of a states first order polarisation (using the first moments of the Stokes operators) that overcomes the fallacies of the conventional measure. Specifically, in contrast to the conventional measure, it detects first order hidden polarisation and it is continuous. Due to its suitability in quantifying polarisation and its extraordinary simplicity - can be directly measured - we believe that this measure of polarisation will have wide applicability in different sciences.

We acknowledges support from the Danish Research Council (Sapere Aude project no. 10-081599). GB acknowledges financial support from the Swedish Research Council (VR) through its Linnaeus Center of Excellence ADOPT and contract 319-2010-7332. We acknowledge discussions with Jonas Söderholm.
I. Ghiu, G. Björk, P. Marian, and T. A. Marian,
A. B. Klimov, L. L. Sánchez-Soto, E. C. Luis, Phys. Rev. A 66, 013806 (Jul 2002).

The subscript indicates that it only takes into account the
first moments of the Stokes vectors. That is, for this
paper we disregard all higher Stokes operator moments
\(S^r\), where \(r > 1\). Higher order of Stokes operators
have been investigated in [21] (second moments) and [32] (higher moments).

H. Prakash and N. Chandra, Phys. Rev. A 4, 796 (Aug 1971),
http://link.aps.org/doi/10.1103/PhysRevA.4.796
1021 (Feb 1974), http://link.aps.org/doi/10.1103/PhysRevA.9.1021

A. B. Klimov, G. Björk, J. Söderholm, L. S. Madsen, M. Lassen, U. L. Andersen, J. Heersink, R. Dong, C. Marquardt, G. Leuchs, and L. L. Sánchez-Soto, Phys. Rev. Lett. 105, 153602 (2010)

U. L. Andersen, G. Leuchs, and C. Silberhorn, Laser Photon. Rev. 4, 337 (2010)

D. Fukuda, G. Fujii, T. Numata, K. Amemiya, A. Yoshizawa, H. Tsuchida, H. Fujino, H. Ishii, T. Itatani, S. Inoue, and T. Zama, Opt. Express 19, 870 (Jan 2011),
http://www.opticsexpress.org/abstract.cfm?URI=oe-19-2-870

J. Heersink, V. Josse, G. Leuchs, and U. L. Andersen, Opt. Lett. 30, 1192 (May 2005),
http://ol.osa.org/abstract.cfm?url={$UL}$=ol-30-10-1192

T. Iskhakov, M. V. Chkhikvadze, and G. Leuchs, Phys. Rev. Lett. 102, 183602 (May 2009),
http://link.aps.org/doi/10.1103/PhysRevLett.102.183602

L. S. Madsen, A. Berni, M. Lassen, and U. L. Andersen, Phys. Rev. Lett. 109, 030402 (Jul 2012),
http://link.aps.org/doi/10.1103/PhysRevLett.109.030402

D. T. Smithey, M. Beck, M. G. Raymer, and D. F. Walls, J. Mod. Opt. 48, 171 (2001), ISSN 0950-0340,
http://link.aip.org/link/?JMO/48/171/1

K. Vogel and H. Risken, Phys. Rev. A 40, 2847 (Sep 1989),
http://link.aps.org/doi/10.1103/PhysRevA.40.2847

In practice we first displace and subsequently squeeze.
However, these two operations are interchangeable
when replacing \(\alpha_{H,V}\) by \(\alpha_{H,V} = \alpha_{H,V} \cosh(r_{H,V}) + \alpha_{H,V} \sinh(r_{H,V})\) \[33, 34\], and we choose the
notation of Eq. \[6\] to be consistent with the order in
Eq. \[6\].

A. B. Klimov, G. Björk, J. Söderholm, L. L. Sánchez-Soto, A. B. Klimov, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Rev. Mod. Phys. 84, 621 (2012),
http://link.aps.org/doi/10.1103/RevModPhys.84.621

G. Björk, J. Söderholm, Y.-S. Kim, Y.-S. Ra, H.-T. Lim, C. Kothe, Y.-H. Kim, L. L. Sánchez-Soto, and A. B. Klimov, Phys. Rev. A 85, 053835 (May 2012),
http://link.aps.org/doi/10.1103/PhysRevA.85.053835

K. B. Moller, T. G. Jørgensen, and J. P. Dahl, Phys. Rev. A 54, 5378 (Dec 1996),
http://link.aps.org/doi/10.1103/PhysRevA.54.5378

R. Loudon and P. L. Knight, J. Mod. Opt. 34, 709 (Jun 1987), ISSN 0950-3340