CHARACTERIZATION FOR THE EXISTENCE OF BOUNDED SOLUTIONS TO ELLIPTIC EQUATIONS

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(Communicated by Isabeau Birindelli)

Abstract. We give necessary and sufficient conditions for which the elliptic equation
\[ \Delta u = \rho(x) \Phi(u) \] in \( \mathbb{R}^d \) \((d \geq 3)\)
has nontrivial bounded solutions.

1. Introduction. We are interested in nonnegative solutions of the semilinear elliptic equation
\[ \Delta u = \rho(x) \Phi(u) \] \( \text{(1)} \)
which are defined and bounded on all of \( \mathbb{R}^d \) \((d \geq 3)\), where \( \rho \) is nonnegative locally bounded function on \( \mathbb{R}^d \) and \( \Phi \) is nonnegative continuous function on \([0, \infty]\) such that \( \Phi(0) = 0 \). Such a solution is understood in the distributional sense and is called an entire bounded solution. An entire solution is said to be nontrivial, if it is not identically zero. Throughout this paper, we will look only for nonnegative solutions, and many times we will omit the term “nonnegative”.

The interest of the study of semilinear elliptic equations becomes from its deep relations to some problems of physics, differential geometry, branching processes, diffusions and other fields; see, for instance, [8, 11, 13, 14, 16, 20].

The existence of nontrivial entire bounded solutions of Eq.(1) has been dealt with by many authors. Various hypothesis on \( \rho \) and many examples of \( \Phi \) have been considered; see, for instance, [1, 7, 9, 10, 12, 15, 16, 21] and the references therein. By way of illustration, we give a brief account of the results obtained. In [16], Ni proved the existence of nontrivial entire bounded solutions of
\[ \Delta u = \rho(x) u^{\frac{2\alpha}{2\alpha + 2}} \]
provided \( \rho \) is bounded and locally Hölder continuous on \( \mathbb{R}^d \) such that \( \rho(x) \leq C/|x|^l \) for some \( l > 2 \). Two years later, Kawano [10] and Naito [15] extended the existence result of Ni to some classes of nonlinearity \( \Phi \) by considering the weaker condition:

2010 Mathematics Subject Classification. Primary: 31B05, 35B08; Secondary: 35J08, 35J91.
Key words and phrases. Elliptic equation, bounded solution, thinness at \( \infty \), Green function.
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there exists a locally Hölder continuous function $a^*$ on $[0, \infty[$ such that $\rho(x) \leq a^*(|x|)$ for all $x \in \mathbb{R}^d$ and

$$\int_0^\infty r a^*(r) \, dr < \infty.$$  

Similar conditions on $\rho$ was given by Lair and Wood [12] for the sublinear case $\Phi(u) = u^\gamma, \ 0 < \gamma \leq 1$. The authors proved the existence of a nontrivial entire bounded solution if

$$\int_0^\infty r\rho^*(r) \, dr < \infty,$$

while there is no solution if

$$\int_0^\infty r\rho_*(r) \, dr = \infty,$$

where $\rho_*(r) = \inf_{|x|=r} \rho(x)$ and $\rho^*(r) = \sup_{|x|=r} \rho(x)$ for all $r \geq 0$. Ye and Zhou [21] investigated Eq.(1) for a locally bounded $\rho$ and a continuous nonlinearity $\Phi$ satisfying

$$\lim_{u \to 0^+} \frac{\Phi(u)}{u} = 0 \quad \text{or} \quad \lim_{u \to \infty} \frac{\Phi(u)}{u} = 0.$$

They proved the existence of nontrivial entire bounded solutions provided the equation

$$-\Delta u = \rho(x)$$

has a bounded solution in $\mathbb{R}^d$, or equivalently, for every $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \frac{\rho(y)}{|x-y|^{d-2}} \, dx < \infty.$$

This condition on $\rho$ was also considered by Dupaigne et al. [7] for superlinear $\Phi$ in the sense that

$$\int_0^\infty \frac{dt}{\sqrt{\int_t^\infty \Phi(s) \, ds}} < \infty.$$

It is worth noting that all conditions, on the weight function $\rho$, given in the aforementioned works do not characterize the existence of entire bounded solutions, they are only sufficient conditions. Necessary and sufficient condition was first given by El Mabrouk [9] for the sublinear case where $\Phi(u) = u^\gamma, \ 0 < \gamma < 1$. He proved that the sublinear equation

$$\Delta u = \rho(x)u^\gamma$$

admits a nontrivial entire bounded solution, if and only if, there exist a set $A$ thin at $\infty$ and $x \in \mathbb{R}^d$ such that

$$\int_{A^c} \frac{\rho(y)}{|x-y|^{d-2}} \, dy < \infty. \quad (2)$$

This result is in fact true for any $\Phi$ which is concave and nondecreasing on $[0, \infty[$, since the method of [9] demands only these two properties which have already been satisfied by the sublinear function $u^\gamma$.

Our purpose here consists in extending this characterization result to a large class of nonlinearity $\Phi$ which includes functions which are increasing, concave, convex, superlinear, sublinear, etc. More precisely, assume that there exists a constant $\eta > 0$ such that

$$\Phi(t) > 0 \quad \text{for all} \quad 0 < t < \eta.$$
Then Eq. (1) admits a nontrivial entire bounded solution if and only if condition (2) holds. In this case we show, in addition, the existence of infinitely many nontrivial entire bounded solutions with values in $[0, \eta]$. It should be noted that if $\Phi(a) = 0$ for some $a > 0$, then the constant function $a$ is a nontrivial entire bounded solution of Eq. (1). This holds without any condition on $\rho$. Therefore, it will eventually appear that the class of nonlinearity $\Phi$ that we consider here is much more satisfactory than one might expect.

Similar to that of [9], our approach make substantial use of classical potential theory. Our methods have the advantage of being intrinsic, they will essentially still work to give similar results for some differential operators such as second order elliptic operators, parabolic differential operators, fractional Laplace operators, etc. In this context, we note that an analogous characterization result for nonlocal operators was given in [3] where $\Delta$ is substituted by its fractional powers $\Delta^{\alpha/2}$, $0 < \alpha < 2$.

2. Preliminaries. For an open set $U$ of $\mathbb{R}^d$, let $B(U)$ be the set of all Borel measurable functions on $U$, $B_b(U)$ be the set of all bounded Borel measurable functions on $U$, $C(U)$ be the set of all continuous functions on $U$ and $C_0(U)$ be the set of all continuous functions on $U$ such that $u = 0$ on $\partial U$. We denote by $C_c^\infty(U)$ the set of all infinitely differentiable functions on $U$ with compact support. The uniform norm will be denoted by $\|\cdot\|$.

Let $\Omega$ is the set of all continuous functions from $[0, \infty]$ to $\mathbb{R}^d$. Set $B_t(\omega) = \omega(t)$ for $t \geq 0$ and $\omega \in \Omega$. Let $\mathcal{F} = \sigma(B_t; t \geq 0)$ denotes the smallest $\sigma$-field on $\Omega$ such that $B_t$ is $\mathcal{F}$-measurable for all $t \geq 0$. For every $x \in \mathbb{R}^d$, let $P^x$ be the unique probability distribution on $\Omega$ to brownian motion starting at $x$ and $E^x$ the corresponding expectation. $(\Omega, \mathcal{F}, B_t, P^x)$ is called the canonical Brownian motion on the Euclidean space $\mathbb{R}^d$. For $x \in \mathbb{R}^d$ and $t > 0$, the normal density $p(t, x, \cdot)$ of $(B_t)$ is given by

$$p(t, x, y) = \frac{1}{(4\pi t)^{d/2}} \exp \left(-\frac{|x-y|^2}{4t}\right).$$

In other word, for $A \in B(\mathbb{R}^d)$,

$$P^x(B_t \in A) = \frac{1}{(4\pi t)^{d/2}} \int_A \exp \left(-\frac{|x-y|^2}{4t}\right) dy.$$

It is well known that the classical Laplacian $\Delta$ is the infinitesimal generator of $(B_t)$.

We collect some results from analytic and probabilistic classical potential theory, which will be used later. These results are well known and they can be found, for example, in [2, 4, 17]. Let $D$ be a bounded domain in $\mathbb{R}^d$ and let $\tau_D$ be the first exit time from $D$ by $(B_t)$, i.e.,

$$\tau_D = \inf\{t > 0; B_t \not\in D\}.$$ 

A point $x \in \partial D$ is called regular for the set $D$ if $P^x(\tau_D = 0) = 1$. The bounded domain $D$ is called regular if all $x \in \partial D$ are regular for $D$. Let us note that bounded open sets which are Lipschitz or satisfying the “exterior cone condition” are regular.
For every $x \in \mathbb{R}^d$, let $H_D(x, \cdot)$ be the harmonic measure relative to $x$ and $D$, i.e., for $A \in \mathcal{B}(\mathbb{R}^d)$,

$$H_D(x, A) = P^x(B_{\tau_D} \in A).$$

Since Brownian motion has continuous paths, it is easy to see that $H_D(x, \cdot) = \delta_x$ the Dirac measure at $x \in \mathbb{R}^d \setminus \overline{D}$ and $H_D(x, \cdot) = \delta_x$ for $x \in \partial D$ if we assume in addition that $D$ is regular. For $x \in D$, the measure $H_D(x, \cdot)$ is concentrated on $\partial D$ and is absolutely continuous with respect to the Haussdorff surface measure on $\partial D$. Let $K_D(x, \cdot)$ denotes its corresponding density function. For $f \in \mathcal{B}(\mathbb{R}^d)$, it will be convenient to denote again

$$H_D f(x) = \begin{cases} \int_{\partial D} f(z) K_D(x, z) \, dz & \text{if } x \in D \\ f(x) & \text{if } x \in \mathbb{R}^d \setminus D. \end{cases}$$

A lower semi-continuous function $s : U \to ]-\infty, +\infty]$ is said to be superharmonic if $H_D s \leq s$ for every regular bounded open set $D$ such that $\overline{D} \subset U$. Every function $v$ such that $-v$ is superharmonic is called subharmonic. A function $h \in \mathcal{B}(U)$ is harmonic in $U$ if and only if one of the following three equivalent assertions holds:

- $h \in \mathcal{C}^2(U)$ and $\Delta h(x) = 0$ for every $x \in U$.
- $h \in \mathcal{C}(U)$ and, for every regular bounded open set such that $\overline{D} \subset U$.
- $h \in \mathcal{C}(U)$ and, for every $\varphi \in \mathcal{C}_c^\infty(U)$,

$$\int_U h(x) \Delta \varphi(x) \, dx = 0.$$

If $D$ is regular then, for each $f \in \mathcal{C}(\partial D)$, $H_D f$ is the unique continuous extension $h$ of $f$ on $\overline{D}$ such that $\Delta h = 0$ on $D$, i.e., $H_D f$ is the unique solution of the following Dirichlet problem

$$\begin{cases} \Delta h = 0 & \text{in } D \\ h = f & \text{on } \partial D. \end{cases}$$

The Green kernel $G_{\mathbb{R}^d}$ of $\mathbb{R}^d$, $d \geq 3$, is defined for $x, y \in \mathbb{R}^d$ by

$$G_{\mathbb{R}^d}(x, y) = \int_0^\infty p(t, x, y) \, dt = \begin{cases} c_d |x - y|^{2-d} & \text{if } x \neq y \\ \infty & \text{if } x = y, \end{cases}$$

where $c_d$ is a positive constant depending only on $d$. The Green kernel $G_D$ of a regular bounded open set $D$ is defined by $G_D(x, y) = 0$ if $x$ or $y$ belongs to $D^c$ and

$$G_D(x, y) = \int_0^\infty p^D(t, x, y) \, dt; x, y \in D,$$

where $p^D(t, x, y)$ is the transition density of the Brownian motion $(B^D_t)$ killed upon exiting $D$, which is given by

$$p^D(t, x, y) = p(t, x, y) - E^x[p(t - \tau_D, B_{\tau_D}, y), \tau_D < t].$$

This identity leads, using the strong Markov property, to

$$G_D(x, y) = G_{\mathbb{R}^d}(x, y) - \int_{\partial D} G_{\mathbb{R}^d}(z, y) K_D(x, z) \, dz; x, y \in D.$$
Let \( \rho \in C(\partial D) \) be a regular bounded open set such that \( \partial D \subset U \). The second one is a nontrivial extension of [9, Lemma 3] from concave nonlinearity \( u^\gamma \), \( 0 < \gamma < 1 \), to the more general case \( \Phi(u) \).

3. **Necessary condition.** We recall that \( \rho : \mathbb{R}^d \to [0, \infty[ \) is a locally bounded Borel measurable function and \( \Phi : [0, \infty[ \to [0, \infty[ \) is a continuous function such that \( \Phi(0) = 0 \). By a solution of Eq. (1) in an open subset \( U \) of \( \mathbb{R}^d \), we mean every nonnegative function \( u \in C(U) \) such that

\[
\int_U u(x) \Delta \varphi(x) \, dx = \int_U \rho(x) \Phi(u(x)) \varphi(x) \, dx
\]

holds for every nonnegative function \( \varphi \in C_c^\infty(U) \). Supersolutions and subsolutions of Eq. (1) have to be understood in the same way replacing “=” in (4) by “≤” and “≥” respectively.

The main result of this section is to show that if Eq. (1) has a nontrivial bounded solution then the weight function \( \rho \) should satisfies condition (2). We start with the following two preparatory lemmas. The first one is similar to [9, Lemma 1]. The second one is a nontrivial extension of [9, Lemma 3] from concave nonlinearity \( u^\gamma \), \( 0 < \gamma < 1 \), to the more general case \( \Phi(u) \).

**Lemma 3.1.** Let \( U \) be an open set and \( u \in C(U) \). Then \( u \) is a solution of Eq. (1) in \( U \) if and only if

\[
u(x) + G_D(\rho \Phi(u))(x) = H_D u(x), \quad x \in U,
\]

for every regular bounded open set \( D \) such that \( \overline{D} \subset U \).

**Proof.** Let \( D \) be a regular bounded open set such that \( \overline{D} \subset U \) and define

\[ h(x) := u(x) + G_D(\rho \Phi(u))(x), \quad x \in U. \]

The function \( \rho \Phi(u) \) is bounded on \( D \) and then \( G_D(\rho \Phi(u)) \in C_0(D) \). This implies, in particular, that \( h \in C(D) \) and \( h = u \) on \( \partial D \). On the other hand, by (3), for every \( \varphi \in C_c^\infty(D) \),

\[
\int_D h(x) \Delta \varphi(x) \, dx = \int_D u(x) \Delta \varphi(x) \, dx + \int_D G_D(\rho \Phi(u))(x) \Delta \varphi(x) \, dx
\]

\[
= \int_D u(x) \Delta \varphi(x) \, dx - \int_D \rho(x) \Phi(u(x)) \varphi(x) \, dx.
\]

Therefore, \( u \) is a solution of Eq. (1) in \( D \) if and only if \( h \) is harmonic in \( D \). But \( \Delta h = 0 \) in \( D \) is equivalent to \( h = H_D h \) which means that \( h = H_D u \) since \( h = u \) on \( \partial D \).
Lemma 3.2. Let \( u \) be a bounded solution of Eq. (1). Then, for every \( x \in \mathbb{R}^d \),
\[
    u(x) + G_{\mathbb{R}^d}(\rho \Phi(u))(x) = \|u\|. 
\]
(5)

Proof. Let \( B_n \) be the ball of radius \( n \geq 1 \) centred at the origin of \( \mathbb{R}^d \). By Lemma 3.1, we have
\[
    u + G_{B_n}(\rho \Phi(u)) = H_{B_n}u \quad \text{on} \quad B_n. 
\]
For every \( n \geq 1 \), \( G_{B_n} \leq G_{B_{n+1}} \) and hence \( H_{B_n}u \leq H_{B_{n+1}}u \). Let \( h \) be the function defined for \( x \in \mathbb{R}^d \) by
\[
    h(x) := \sup_{n \geq 1} H_{B_n}u(x). 
\]
Clearly, \( h \) is nonnegative, bounded by above by \( \|u\| \) and harmonic in \( \mathbb{R}^d \). Thus, by Liouville’s theorem,
\[
    h \equiv c \in \mathbb{R}_+. 
\]
Now, seeing that \( \lim_{n \to \infty} G_{B_n} = G_{\mathbb{R}^d} \), we immediately deduce that
\[
    u + G_{\mathbb{R}^d}(\rho \Phi(u)) = c \quad \text{on} \quad \mathbb{R}^d. 
\]
Since \( G_{\mathbb{R}^d}(\rho \Phi(u)) \) is a potential on \( \mathbb{R}^d \),
\[
    \inf_{x \in \mathbb{R}^d} G_{\mathbb{R}^d}(\rho \Phi(u))(x) = 0, 
\]
and hence
\[
    \sup_{x \in \mathbb{R}^d} u(x) = c 
\]
as desired. \( \square \)

Before stating our first main result, we first clarify some terminology.
Let \( A \in \mathcal{B}(\mathbb{R}^d) \) and let \( T_A \) be the first hitting time of \( A \)
\[
    T_A := \inf\{t > 0 ; B_t \in A\}. 
\]
The set \( A \) is said to be \textit{recurrent} if \( P^x(T_A < \infty) = 1 \) for every \( x \in \mathbb{R}^d \) and \textit{transient} otherwise, see for instance [17, p. 24] or [5, p. 121]. The set \( A \) is called thin at \( \infty \) if there exists \( s \in \mathcal{S}^+ \) the set of all nonnegative superharmonic functions on \( \mathbb{R}^d \) such that \( s \geq 1 \) on \( A \) but not on the whole space \( \mathbb{R}^d \), see for example [2, p. 215]. The \textit{regularized reduced function} (or balayage) of the constant function \( 1 \) relative to \( A \) in \( \mathbb{R}^d \) is given by
\[
    \hat{R}_1^A(x) = \liminf_{y \to x} R_1^A(y) \quad ; \quad x \in \mathbb{R}^d, 
\]
where
\[
    R_1^A(x) = \inf\{v(x) ; v \in \mathcal{S}^+ \text{ and } v \geq 1 \text{ on } A\} 
\]
\[
    = \inf\{v(x) ; v \in \mathcal{S}^+, \text{ } v = 1 \text{ on } A \text{ and } v \leq 1 \text{ on } \mathbb{R}^d\}. 
\]
It is well known from [4, p. 263] or [19, p. 231] that
\[
    \hat{R}_1^A(x) = P^x(T_A < \infty) \quad ; \quad x \in \mathbb{R}^d. 
\]
Hence, the following assertions are obviously equivalent.

\( (a) \): \( A \) is transient.

\( (b) \): \( A \) is thin at \( \infty \).
For $d > 3$, the thorn

$$A = \{(x_1, ..., x_d) \in \mathbb{R}^d; \ x_1 \geq e \text{ and } x_2^2 + \cdots + x_d^2 \leq x_1^2 / (\ln x_1)^{2\beta}\}, \ \beta > \frac{1}{d-3},$$

(6)

is thin at $\infty$ since it is transient as shown in [17, Proposition 3.3.6].

**Theorem 3.3.** Assume that $\Phi(t) > 0$ for $t > 0$. Let $u$ be a nontrivial bounded solution of Eq.(1). Then condition (2) holds.

**Proof.** Let $A = \{2u < \|u\|\}$ and $s$ be the nonnegative function defined on $\mathbb{R}^d$ by

$$s(x) = 2 \frac{\|u\| - u(x)}{\|u\|}.$$

Clearly, $s$ is superharmonic on $\mathbb{R}^d$ and $s \geq 1$ on $A$ but not on the whole of $\mathbb{R}^d$. Therefore, $A$ is thin at $\infty$. Let

$$m := \inf \left\{ \Phi(t); \ \frac{1}{2} \|u\| \leq t \leq \|u\| \right\}.$$

It is easy to see that $m > 0$ and $\Phi(u) \geq m$ on $A^c$. Thus, for every $x \in \mathbb{R}^d$,

$$m \int_{A^c} G_{\mathbb{R}^d}(x, y) \rho(y) \, dy \leq \int_{A^c} G_{\mathbb{R}^d}(x, y) \rho(y) \Phi(u(y)) \, dy$$

$$\leq \int_{\mathbb{R}^d} G_{\mathbb{R}^d}(x, y) \rho(y) \Phi(u(y)) \, dy$$

$$\leq \|u\|$$

by (5), which completes the proof. □

4. **Sufficient conditions.** Throughout this section, $\rho : \mathbb{R}^d \to [0, \infty]$ will denote a locally bounded Borel measurable function and $\Phi : [0, \infty] \to [0, \infty]$ a continuous function such that $\Phi(0) = 0$. The purpose of this section is to investigate, under various conditions on $\rho$, the existence of nontrivial bounded solutions to Eq.(1). For the sake of clarity, we first consider the particular case where the nonlinearity $\Phi$ is nondecreasing. Next, we turn to the general case.

4.1. **The case where $\Phi$ is nondecreasing.** Throughout this subsection we assume that $\Phi$ is nondecreasing on $[0, \infty[$. We first state the following useful results which have been proved in [9].

**Lemma 4.1.** If $u$ (resp. $v$) is a supersolution (resp. subsolution) of Eq.(1) in a bounded open set $D$ and $\liminf_{z \to x} (u - v)(x) \geq 0$ for all $z \in \partial D$, then $u \geq v$ on $D$.

**Lemma 4.2.** For every regular bounded open set $D$ and every $f \in C^+(D^c)$, there exists one and only one $u \in C^+(\mathbb{R}^d)$ such that

$$\begin{cases} \Delta u = \rho(x) \Phi(u) & \text{in } D \\ u = f & \text{on } D^c. \end{cases}$$

Furthermore,

$$u(x) + G_D(\rho \Phi(u))(x) = H_Df(x); \quad x \in \mathbb{R}^d.$$

(7)

For every $\lambda > 0$ and every integer $n \geq 1$, let $u^n_\lambda$ be the solution of the problem

$$\begin{cases} \Delta u = \rho(x) \Phi(u) & \text{in } B_n \\ u = \lambda & \text{on } B^n_\lambda. \end{cases}$$

(8)
where $B_n$ denotes the ball of radius $n$ and centred at the origin of $\mathbb{R}^d$. It is easy to check using Lemma 4.1 that

$$0 \leq u_\lambda^{n+1} \leq u_\lambda^n \leq \lambda.$$  

Moreover, by (7), we have

$$u_\lambda^n + G_{B_n}(\rho \Phi(u_\lambda^n)) = \lambda \quad \text{on} \quad \mathbb{R}^d. \quad (9)$$

Let $u_\lambda$ be the function defined on $\mathbb{R}^d$ by

$$u_\lambda(x) := \inf_n u_\lambda^n(x).$$

**Lemma 4.3.** (a) The nonnegative bounded function $u_\lambda$ is a solution of Eq.(1) in $\mathbb{R}^d$.

(b) If $\rho$ is radially symmetric on $\mathbb{R}^d$ then so does $u_\lambda$.

**Proof.** (a) Let $D \subset \mathbb{R}^d$ be a regular bounded open set and let $n_0 \geq 1$ such that $\overline{D} \subset B_{n_0}$. Since $u_\lambda^n$ is a solution of Eq.(1) in $B_n$, it follows from Lemma 3.1 that, for every $n \geq n_0$,

$$u_\lambda^n(x) + G_D(\rho \Phi(u_\lambda^n))(x) = H_D u_\lambda^n(x); \quad x \in \mathbb{R}^d.$$  

For every $n \geq n_0$ and every $x, y \in \mathbb{R}^d$, we have

$$G_D(x,y)\rho(y)\Phi(u_\lambda^n(y)) \leq \Phi(\lambda) G_D(x,y)\rho(y).$$

The right quantity is integrable in $y$ since $\rho$ is locally bounded on $\mathbb{R}^d$. Thus, by letting $n$ tend to $\infty$, we deduce using the dominated convergence theorem that

$$u_\lambda(x) + G_D(\rho \Phi(u_\lambda))(x) = H_D u_\lambda(x).$$

This implies that $u_\lambda \in C(D)$ since $G_D(\rho \Phi(u_\lambda))$ and $H_D u_\lambda$ are both continuous on $D$, and hence $u_\lambda$ is a solution of Eq.(1) in $\mathbb{R}^d$ by Lemma 3.1.

(b) Assume that $\rho$ is radially symmetric on $\mathbb{R}^d$. Let $\tau$ be an orthogonal transformation in $\mathbb{R}^d$. It is easy to check that $u_\lambda^n \circ \tau$ is a solution of problem (8). Thus, $u_\lambda^n \circ \tau = u_\lambda^n$ since $u_\lambda^n$ is the unique solution of this problem. This means that $u_\lambda^n$ is radially symmetric, and hence so does $u_\lambda$.

We do not know whether or not the solution $u_\lambda$ is nontrivial (not identically zero). In the following, our focus will be on sufficient conditions on $\rho$ under which $u_\lambda$ is nontrivial.

**Proposition 1.** Let $\lambda > 0$. If the set $\{ \rho > 0 \}$ is thin at infinity then $u_\lambda$ is nontrivial.

**Proof.** Since the set $\{ \rho > 0 \}$ is thin at infinity, there exist a nonnegative superharmonic function $s$ on $\mathbb{R}^d$ and $x_0 \in \mathbb{R}^d$ such that $s \geq \lambda$ on $\{ \rho > 0 \}$ and $s(x_0) < \lambda$. By (9), we have

$$G_{B_n}(\rho \Phi(u_\lambda^n)) \leq \lambda.$$  

Seeing that $s \geq \lambda$ on $\{ \rho > 0 \}$, there holds

$$G_{B_n}(\rho \Phi(u_\lambda^n)) \leq s \quad \text{on} \quad \{ \rho > 0 \}.$$  

Maria-Frostman domination principle [2, Theorem 5.1.11] yields $G_{B_n}(\rho \Phi(u_\lambda^n)) \leq s$ on the whole $\mathbb{R}^d$. Hence,

$$u_\lambda^n + s \geq u_\lambda^n + G_{B_n}(\rho \Phi(u_\lambda^n)) = \lambda \quad \text{on} \quad \mathbb{R}^d.$$
This implies that $\lambda - s(x_0) \leq u_\lambda^n(x_0)$ from which we conclude, by letting $n$ tend to $\infty$, that $u_\lambda(x_0) > 0$ as desired. \hfill \Box

The following result is probably well known, but we do not have a reference.

Lemma 4.4. Let $p : \mathbb{R}^d \to [0, \infty]$ be defined by

$$p(x) := \int_{\mathbb{R}^d} \frac{\rho(y)}{|x - y|^{d-2}} dy.$$ 

Then $p \in C(\mathbb{R}^d)$, and hence is finite every where on $\mathbb{R}^d$, provided $p(x_0) < \infty$ for some $x_0 \in \mathbb{R}^d$.

Proof. Let $x_0 \in \mathbb{R}^d$ such that $p(x_0) < \infty$. Let $D$ be a regular bounded open set such that $x_0 \in D$. For every $x \in D$ and every $y \in \mathbb{R}^d$, we have

$$G_{\mathbb{R}^d}(x, y) = G_D(x, y) + \int_{\partial D} G_{\mathbb{R}^d}(z, y) K_D(x, z) dz.$$ 

Multiplying both sides by $\rho(y)$ and integrating with respect to $y$, there holds

$$p(x) = G_D(\rho)(x) + H_{DP}(x).$$ 

The function $H_{DP}$ is not identically $\infty$ on $D$ since $H_{DP}(x_0) < p(x_0) < \infty$. This yields that $H_{DP}$ is harmonic and hence continuous on $D$. On the other hand, $G_D(\rho) \in C(D)$ since $\rho$ is bounded on $D$. Hence $p$ is continuous on $D$ and consequently on $\mathbb{R}^d$ since $D$ is arbitrary in $\mathbb{R}^d$. \hfill \Box

Proposition 2. Let $\lambda > 0$. Assume that there exists $x \in \mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} \frac{\rho(y)}{|x - y|^{d-2}} dy < \infty. \quad (10)$$

Then $u_\lambda$ is nontrivial. Furthermore,

$$\|u_\lambda\| = \lambda.$$ 

Proof. By the above Lemma, (10) holds for every $x \in \mathbb{R}^d$. Let $x \in \mathbb{R}^d$. By (9), we have

$$u_\lambda^n(x) + G_{B_n}(\rho \Phi(u_\lambda^n))(x) = \lambda. \quad (11)$$

Recalling that $G_{B_n} \uparrow G_{\mathbb{R}^d}$, there holds

$$G_{B_n}(x, y) \rho(y) \Phi(u_\lambda^n(y)) \leq \Phi(\lambda) G_{\mathbb{R}^d}(x, y) \rho(y).$$

The right quantity is integrable in $y$ by the hypothesis (10). Thus, by letting $n$ tend to $\infty$ in (11), we deduce using the dominated convergence theorem that

$$u_\lambda(x) + G_{\mathbb{R}^d}(\rho \Phi(u_\lambda))(x) = \lambda.$$ 

This identity immediately implies that $u_\lambda$ is not identically zero. Otherwise, it leads to the contradiction that $0 = \lambda$. Moreover,

$$\sup_{x \in \mathbb{R}^d} u_\lambda(x) = \lambda - \inf_{x \in \mathbb{R}^d} G_{\mathbb{R}^d}(\rho \Phi(u_\lambda))(x) = \lambda$$

since $G_{\mathbb{R}^d}(\rho \Phi(u_\lambda))$ is a potential on $\mathbb{R}^d$. \hfill \Box

Proposition 3. Assume that $\rho$ is radially symmetric on $\mathbb{R}^d$. Let $\lambda > 0$. Then $u_\lambda$ is nontrivial if and only if

$$\int_0^{\infty} r \rho(r) dr < \infty. \quad (12)$$
Proof. Assume that \( u_\lambda \) is nontrivial. The function \( u_\lambda \) is radially symmetric by the statement (b) of Lemma 4.3. Let \( x_0 \in \mathbb{R}^d \) such that \( u_\lambda(x_0) \neq 0 \). A simple application of the maximum principle to the radial subharmonic function \( u_\lambda \) shows that

\[
u \lambda(x) \geq u_\lambda(x_0) \quad \text{whenever} \quad |x| \geq |x_0|.
\]

Using the fact that \( \Phi \) is nondecreasing and spherical coordinates, there holds

\[
G_{\mathbb{R}^d}(\rho \Phi(u_\lambda))(0) = c_d \int_{\mathbb{R}^d} \frac{\rho(y)\Phi(u_\lambda(y))}{|y|^{d-2}} \, dy
\geq c_d \int_{\{|y| \geq |x_0|\}} \frac{\rho(y)\Phi(u_\lambda(y))}{|y|^{d-2}} \, dy
\geq c_d \Phi(u_\lambda(x_0)) \int_{\{|y| \geq |x_0|\}} \frac{\rho(y)}{|y|^{d-2}} \, dy
= c_d \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \Phi(u_\lambda(x_0)) \int_{|x_0|}^{\infty} \rho(r) \, dr.
\]

This implies that \( \int_{|x_0|}^{\infty} \rho(r) \, dr < \infty \) since \( \Phi(u_\lambda(x_0)) \neq 0 \) and, by (5), \( G_{\mathbb{R}^d}(\rho \Phi(u_\lambda))(0) < \infty \). The fact that \( \rho \) is locally bounded yields \( \int_{|x_0|}^{\infty} \rho(r) \, dr < \infty \), and hence (12) holds. The converse follows from Proposition 2 by seeing that

\[
G_{\mathbb{R}^d}(\rho)(0) = \kappa_d \int_{0}^{\infty} \rho(r) \, dr,
\]

where \( \kappa_d \) is a positive constant depending only on \( d \).

We have shown, in the previous proposition, that the sufficient condition (10) is also necessary for the existence of nontrivial entire bounded solution of Eq. (1) when \( \rho \) is radial. However, as we shall see now, condition (10) is not necessary when \( \rho \) is not radially symmetric. This is very similar to previous results in [9].

Remark 1. Consider the thorn \( A \) of \( \mathbb{R}^d, \ d > 3 \), given by (6) and take \( \rho = 1_A \). Let \( x \in \mathbb{R}^d \) and choose \( R > e \) and \( c > 0 \) such that \( |x - y|^{2-d} \geq c |y|^{2-d} \) for all \( y \in \mathbb{R}^d \) satisfying \( |y| \geq R \). Using spherical coordinates in \( \mathbb{R}^{d-1} \), there holds

\[
\int_{\mathbb{R}^d} \frac{\rho(y)}{|x - y|^{d-2}} \, dy = \int_{A \setminus \{|y| \geq R\}} \frac{dy}{|y|^{d-2}}
\geq c \int_{A \cap \{|y| \geq R\}} \frac{dy}{|y|^{d-2}}
= c' \int_{R}^{\infty} \int_{0}^{h(r)} \frac{t^{d-2}}{(r^2 + t^2)^{\frac{d-2}{2}}} \, dt \, dr
\geq c' \int_{R}^{\infty} \left( \int_{0}^{h(r)} t^{d-2} \, dt \right) \frac{dr}{(r^2 + h^2(r))^{\frac{d-2}{2}}}
= c' \int_{R}^{\infty} \frac{(h(r))^{d-1}}{(r^2 + h^2(r))^{\frac{d-2}{2}}} \, dr
= \infty
\]

where \( h(r) = r/(\ln r)^d \). Hence (10) fails, while Eq. (1) admits a nonnegative nontrivial entire bounded solution by Proposition 1 since the set \( \{\rho > 0\} = A \) is thin at \( \infty \) by hypothesis.
The next result consists in proving that the necessary condition (2) is also sufficient for the existence of nontrivial entire bounded solution of Eq. (1).

**Theorem 4.5.** Let $\lambda > 0$. If there exist a set $A$ thin at $\infty$ and $x \in \mathbb{R}^d$ such that

$$\int_{A^c} \frac{\rho(y)}{|x - y|^{d-2}} \, dy < \infty,$$

then $u_\lambda$ is nontrivial.

**Proof.** Let $\rho_1$ and $\rho_2$ be the functions defined by $\rho_1 = 1_A \rho$ and $\rho_2 = 1_{A^c} \rho$. For every integer $n \geq 1$, let $v^n_\lambda$, $w^n_\lambda$ and $u^n_\lambda$ be the solution of problem (8) for $\rho = \rho_1$, $\rho = \rho_2$ and $\rho = \rho_1 + \rho_2$ respectively. Using the fact that the function $\Phi$ is nondecreasing, it follows from Lemma 4.1 that the three sequences of functions $(v^n_\lambda)_n$, $(v^n_\lambda)_n$ and $(w^n_\lambda)_n$ are decreasing and

$$u^n_\lambda \leq \inf(v^n_\lambda, w^n_\lambda).$$

This inequality implies that $\Phi(u^n_\lambda) \leq \inf(\Phi(v^n_\lambda), \Phi(w^n_\lambda))$ which yields

$$\Delta (\lambda + u^n_\lambda - v^n_\lambda - w^n_\lambda) = (\rho_1 + \rho_2) \Phi(u^n_\lambda) - \rho_1 \Phi(v^n_\lambda) - \rho_2 \Phi(w^n_\lambda) \leq 0 \quad \text{in} \quad B_n.$$  

Seeing that $\lambda + u^n_\lambda - v^n_\lambda - w^n_\lambda = 0$ on $\partial B_n$, it follows from the minimum principle for superharmonic functions that

$$\lambda + u^n_\lambda - v^n_\lambda - w^n_\lambda \geq 0 \quad \text{on} \quad B_n. \quad (13)$$

Put $u_\lambda = \inf_n u^n_\lambda$, $v_\lambda = \inf_n v^n_\lambda$ and $w_\lambda = \inf_n w^n_\lambda$. By letting $n$ tend to infinity in (13), there holds

$$\lambda + u_\lambda - v_\lambda - w_\lambda \geq 0 \quad \text{on} \quad \mathbb{R}^d.$$ 

This immediately implies that

$$\left(\|v_\lambda\| - v_\lambda\right) + \left(\lambda - w_\lambda\right) + u_\lambda \geq \|v_\lambda\| \quad \text{on} \quad \mathbb{R}^d. \quad (14)$$

Since the set $A$ is thin at $\infty$, it follows from Proposition 1 (resp. Proposition 2) that $\|v_\lambda\| \not\equiv 0$ (resp. $\|w_\lambda\| = \lambda$). Now, plugging the identity (5) applied to the both functions $v_\lambda$ and $w_\lambda$ into (14), we obtain

$$G_{\mathbb{R}^d}(\rho_1 \Phi(v_\lambda)) + G_{\mathbb{R}^d}(\rho_2 \Phi(w_\lambda)) + u_\lambda \geq \|v_\lambda\| \quad \text{on} \quad \mathbb{R}^d.$$ 

This inequality immediately implies that $u_\lambda$ is nontrivial. Otherwise, we arrive to the contradiction

$$2\lambda \geq p := G_{\mathbb{R}^d}(\rho_1 \Phi(v_\lambda)) + G_{\mathbb{R}^d}(\rho_2 \Phi(w_\lambda)) \geq \|v_\lambda\| > 0,$$

since $p$ is a potential on $\mathbb{R}^d$. This completes the proof of the theorem. \(\square\)

As an immediate corollary of Theorems 3.3 and 4.5, we find:

**Corollary 1.** Assume that $\Phi : [0, \infty] \to [0, \infty]$ is continuous and increasing such that $\Phi(0) = 0$. Then, Eq. (1) has a nontrivial entire bounded solution if and only if condition (2) holds.

**Remark 2.** Assume that $\rho$ satisfies condition (2).

(a) For every $\eta > 0$ there exists infinitely many entire solutions of Eq. (1) which are bounded above by $\eta$. Indeed, let $(\lambda_n)_n$ be the sequence of real numbers defined by

$$\lambda_0 = \eta \quad \text{and} \quad \lambda_{n+1} = \frac{1}{2} \|u_{\lambda_n}\|, \quad n \geq 0.$$
For every $n \geq 0$, $\lambda_n \neq 0$ since $u_{\lambda_n}$ is nontrivial by the above theorem. Thus, using the fact that $\|u_{\lambda}\| \leq \lambda$, it follows that

$$\|u_{\lambda_{n+1}}\| < 2\lambda_{n+1} = \|u_{\lambda_n}\|.$$  

Hence, $u_{\lambda_{n+1}} \neq u_{\lambda_n}$ and $\|u_{\lambda_n}\| < \|u_{\lambda_0}\| \leq \eta$ as desired.

(b) For the existence of entire solutions of Eq.(1), we do not need monotonicity of $\Phi$ on the whole interval $[0, \infty]$. In fact, it will be sufficient to assume that $\Phi$ is nondecreasing on $[0, \eta]$ for some constant $\eta > 0$. Indeed, let $\widetilde{\Phi}$ be a continuous nondecreasing extension of $\Phi$ on $[0, \infty]$. By the statement (a) of this remark, the equation

$$\Delta u = \rho(x) \widetilde{\Phi}(u)$$

has many entire solutions which are bounded above by $\eta$. The fact that $\widetilde{\Phi} = \Phi$ on $[0, \eta]$ completes the proof.

4.2. The general case. Our aim here is to show that condition (2) is still sufficient for the existence of nontrivial entire bounded solutions of Eq.(1) under the weaker hypothesis that $\Phi > 0$, rather than just increasing, on an open interval $(0, \eta)$ for some constant $\eta > 0$. As usual, we keep continuity of $\Phi$ on $[0, \eta]$ and $\Phi(0) = 0$. One can, for example, check that $\Phi(t) = t(\sin 1/t + 2)$ satisfies $\Phi(t) > 0$ for $t > 0$, while $\Phi$ is not increasing on any neighbourhood of 0. Before we go on, we need the following key lemma.

**Lemma 4.6.** Let $\eta > 0$ and assume that $\Phi$ is continuous and nonnegative on $[0, \eta]$. If Eq.(1) admits a nontrivial nonnegative entire bounded subsolution $v \in C(\mathbb{R}^d)$ such that $\|v\| \leq \eta$, then there exists a solution $u$ such that $v \leq u \leq \|v\|$.

**Proof.** Let $v \in C(\mathbb{R}^d)$ be a nontrivial entire bounded subsolution of Eq.(1). For every $n \geq 1$, let $B_n$ denotes the ball of radius $n$ and centred at the origin of $\mathbb{R}^d$. We consider the problem

$$\begin{cases}
\Delta u = \rho(x) \Phi(u) & \text{in } B_n \\
u = v & \text{on } B_n^c.
\end{cases}$$

Seeing that the constant function $\|v\|$ is a supersolution of Eq.(1), it follows from [6] (see also [18]) that this problem possesses at least one solution $v_n$ such that $v \leq v_n \leq \|v\|$ on $B_n$.

For every $k \geq 1$, the sequence of functions $(v_n)_n$ is bounded in $X_k := (C(\overline{B_k}), \| \cdot \|)$ the space of continuous functions on $\overline{B_k}$ the closure of $B_k$ and equipped with the uniform norm. Let $(v_{1,n})_n$ be a convergent subsequence of $(v_n)_n$ in $X_1$ and denote by $u_1$ its limit. Such convergent subsequence exists since $X_1$ is compact. By induction on $k \geq 2$, let $(v_{k,n})_n$ be a convergent subsequence of $(v_{k-1,n})_n$ in $X_k$ and denotes by $u_k$ its limit. It is routine to verify that, for every $k \geq 1$, $u_k$ is a solution of Eq.(1) in $B_k$ and $u_{k+1} = u_k$ on $B_k$.

Thus, the function $u$ defined on $\mathbb{R}^d$ by

$$u(x) := u_k(x) \quad \text{for } |x| \leq k$$

is an entire solution of Eq.(1) satisfying $v \leq u \leq \|v\|$, which completes the proof. \hfill $\Box$
We now come to the theorem which was alluded to in the introduction of the present subsection.

**Theorem 4.7.** Let \( \eta > 0 \) and assume that \( \Phi \) is continuous on \([0, \eta]\) such that \( \Phi(0) = 0 \) and \( \Phi(t) > 0 \) for \( 0 < t < \eta \). Then Eq.(1) admits a nontrivial entire bounded solution provided \( \rho \) satisfies condition (2).

**Proof.** From Lemma 4.6, it suffices to prove, under condition (2), the existence of a nontrivial nonnegative entire subsolution \( v \in C(\mathbb{R}^d) \) of Eq.(1) such that \( \|v\| \leq \eta \).

Let \( \tilde{\Phi} \) be the function defined on \([0, \infty]\) by

\[
\tilde{\Phi}(t) := \begin{cases} 
\sup_{s \in [0,t]} \Phi(s) & \text{if } 0 \leq t \leq \eta \\
\sup_{s \in [0,\eta]} \Phi(s) & \text{if } t \geq \eta. 
\end{cases}
\]

It is easy to check that \( \tilde{\Phi} \) is continuous and nondecreasing on \([0, \infty]\) such that \( \tilde{\Phi}(0) = 0 \). Now, assume that condition (2) holds. Then, by Theorem 4.5, the equation

\[
\Delta u = \rho(x) \tilde{\Phi}(u)
\]

has a nontrivial nonnegative entire bounded solution \( v \in C(\mathbb{R}^d) \). By the statement (a) of Remarks 2, we can suppose that \( \|v\| \leq \eta \). Thus, seeing that

\[
\tilde{\Phi}(t) \geq \Phi(t), \quad 0 \leq t \leq \eta,
\]

we immediately deduce that \( v \) is a subsolution of Eq.(1) as desired. \( \square \)

As a consequence of Theorems 3.3 and 4.7, we have:

**Corollary 2.** Let \( \eta > 0 \) and assume that \( \Phi \) is continuous on \([0, \eta]\) such that \( \Phi(0) = 0 \) and \( \Phi(t) > 0 \) for \( 0 < t < \eta \). Then Eq.(1) admits a nontrivial entire bounded solution with values in \([0, \eta]\) if and only if condition (2) holds.

It is worth noting that the condition “\( \Phi(0) = 0 \)” is not necessary here, it can be simply replaced by “\( \Phi(a) = 0 \)” for some \( a \geq 0 \). In this case, nontrivial solution is understood to mean the nonconstant function \( a \).

**Corollary 3.** Assume that \( \Phi \) is continuous on an interval \([a, b]\) such that \( \Phi(a) = 0 \) and \( \Phi(t) > 0 \) for \( 0 < a < t < b \). Then Eq.(1) admits a nonconstant entire bounded solution if and only if condition (2) holds.

**Proof.** Denote \( \eta := b - a \) and define, for \( t \in [0, \eta] \),

\[
\Psi(t) := \Phi(a + t).
\]

It is clear that \( u \in C^+(\mathbb{R}^d) \) is a solution of \( \Delta u = \rho(x) \Psi(u) \), if and only if, \( v = u + a \) is a solution of Eq.(1). By the above corollary, a non identically zero solution \( u \) exists if and only if condition (2) holds. This completes the proof. \( \square \)

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Received May 2018; revised August 2018.

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