Varying-$\alpha$ Theories and Solutions to the Cosmological Problems

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Abstract

If the fine structure constant $\alpha = e^2/(\hbar c)$ were to change, then a number of interpretations would be possible, attributing this change either to variations in the electron charge, the dielectric constant of the vacuum, the speed of light, or Planck’s constant. All these variations should be operationally equivalent and can be related by changes of standard units. We show how the varying speed of light cosmology recently proposed can be rephrased as a dielectric vacuum theory, similar to the one proposed by Bekenstein. The cosmological problems will therefore also be solved in such a theory.

1 Theories with varying constants

Numerous experiments have attempted to establish whether or not the traditional fundamental constants of physics are indeed constants. In such efforts it is important to recognise that one should consider only dimensionless constants \cite{1,2}. Measurements of dimensional quantities represent ratios with respect to standard units. In reference frames moving relative to each other, or at different points in spacetime, one cannot be sure of the equivalence of these units. Statements of constancy of dimensional constants must therefore be circular, because they postulate the correspondence of the standard units used \cite{3}. Conversely, any experimental evidence for varying dimensional constants could always be absorbed into a redefinition of units. Thus attention must be focussed upon dimensionless ratios of dimensional constants.

Suppose that evidence is found for varying dimensionless constants. Any theory explaining the phenomenon would necessarily have to make use of dimensional quantities. It would be a matter of choice as to which dimensional quan-
tities were taken to vary. Any theory based on a choice to vary one dimensional constant could always be reformulated as a theory based on another choice of varying constant. Thus, evidence for time-variation on the fine structure constant could be accommodated within a theory assuming constant electron charge, \(e\), and varying speed of light, \(c\), or one postulating varying \(e\) and constant \(c\). The two formulations should be equivalent as far as dimensionless observations are concerned. However, the simplest theories based on different choices would not necessarily be the same.

Evidence has recently been found [4] that is consistent with a time-varying fine structure constant \(\alpha = e^2/(\hbar c)\). The observations make use of high-quality Keck Telescope data and a new theoretical technique to compare quasar spectral lines of in different multiplets, simultaneously analysing the MgII 2796/2803 doublet and up to five FeII transitions (FeII 2344, 2374, 2383, 2587, 2600 Å), from three different multiplets. This technique improves our observational sensitivity to changes in \(\alpha\) by an order of magnitude. New upper limits on possible time-variation in \(\alpha\) are found at low redshifts \((z < 1)\) with \(\Delta \alpha/\alpha = -1.9 \pm 0.5 \times 10^{-5}\), consistent with strongest known limits from the Oklo natural reactor [5], but evidence is found for a variation is detected at high redshift \((z > 1)\) with \(\Delta \alpha/\alpha = -1.1 \pm 0.4 \times 10^{-5}\). As yet, this cannot be taken as positive evidence for time variation in the fine structure constant at high redshift because it cannot be excluded that the observed spectral variation arises from subtle line blendings. However, these observations and the new technique they introduce provide a new level of precision in testing the constancy of constants which is significantly better than direct laboratory measurement. Since string theories permit the variation of \(\alpha\), we would also like to be able to place cosmological limits on the variation of \(\alpha\) by examining its consequences in the very early universe. In order to do this rigorously we need a self-consistent theory which incorporates varying \(\alpha\) into the cosmological evolution equations.

Changes in \(\alpha\) can be interpreted in different ways. A theory of varying electric charge was first proposed by Bekenstein [6]. In this theory the vacuum may be seen as a dielectric medium effectively screening the electric charge. A varying speed of light theory (with \(\hbar \propto c\)) was proposed by Albrecht and Magueijo [7]. These two theories correspond to different representations of a varying \(\alpha\) in terms of varying dimensional constants [10]. There should exist a set of duality transformations between these two representations. The purpose of this paper is to provide these transformations and explore the interconnection between these two classes of theory.

The minimal varying-\(e\) theory is of interest because it offers a means of solving the so-called cosmological problems: the horizon, flatness, cosmological constant, entropy, and homogeneity problems. We would like to identify the varying-\(e\) reformulation of this theory. We will find that this theory is far
from the minimal varying-$e$ theory. It is similar to a Brans-Dicke theory in which the dielectric field $\epsilon = e/e_0$ plays the role of the inverse of the Gravitational constant $\phi = 1/G$. Such a theory is formally Lorentz invariant, a major improvement over the varying-$c$ formulation.

The plan of the paper is as follows. In Sec. 2 we review the varying speed of light proposal. In Sec. 3 we derive the transformation of units that maps varying $c$ theories into varying-$e$ theories. In Sec. 4 we write down the varying-$e$ dual to the minimal varying speed of light theory. Then in Sec. 5 we derive the cosmological equations in this theory. The cosmological problems pose dimensionless questions. Hence they should also be solved in this theory. We spell out the solution to the flatness problem in Sec. 6. We conclude with some remarks on the proposed theory.

For obvious reasons, we do not use units in which $G = c = \hbar = 1$.

### 2 Varying speed of light theories

In varying speed of light (VSL) theories a varying $\alpha$ is interpreted as $c \propto \hbar \propto \alpha^{-1/2}$. The electromagnetic coupling $e$ is constant, Lorentz invariance is broken, and so by construction there is a preferred frame for the formulation of the physical laws. In the minimally coupled theory one then simply replaces $c$ by a field in this preferred frame. Hence, the action is still

$$S = \int dx^4 \left( \sqrt{-g} \left( \frac{\psi(R + 2\Lambda)}{16\pi G} + \mathcal{L}_M \right) + \mathcal{L}_\psi \right)$$

(1)

with $\psi(x^\mu) = c^4$. The dynamical variables are the metric $g_{\mu\nu}$, any matter field variables contained in $\mathcal{L}_M$, and the scalar field $\psi$ itself. The Riemann tensor (and the Ricci scalar) is to be computed from $g_{\mu\nu}$ at constant $\psi$ in the usual way. This can only be true in one frame: additional terms in $\partial_\mu \psi$ must be present in other frames.

Varying the action with respect to the metric and ignoring surface terms leads to

$$G_{\mu\nu} - g_{\mu\nu}\Lambda = \frac{8\pi G}{\psi} T_{\mu\nu}.$$  

(2)

Therefore, Einstein’s equations do not acquire new terms in the preferred frame. Minimal coupling at the level of Einstein’s equations is at the heart of the model’s ability to solve the cosmological problems. It requires of any
action-principle formulation that $\mathcal{L}_\psi$ must not contain the metric explicitly, and so does not contribute to the energy-momentum tensor.

In a cosmological context, the Friedmann metric can still be written as

$$ds^2 = -c^2 dt^2 + a^2 \left[ \frac{dr^2}{1 + Kr^2} + r^2 d\Omega \right],$$

and the Einstein’s equations are still,

$$(\dot{a}/a)^2 = \frac{8\pi G}{3} \rho - \frac{Kc^2}{a^2}$$

$$\dot{a}/a = \frac{4\pi G}{3} \left( \rho + 3\frac{p}{c^2} \right)$$

However, the conservation equation that follows from (4)-(5) is now

$$\dot{\rho} + \frac{3}{a} \dot{a} \left( \rho + \frac{p}{c^2} \right) = \frac{3Kc^2}{4\pi Ga^2} \dot{c}$$

The definition of $\mathcal{L}_\psi$ controls the dynamics of $\psi$. It is only required that $\psi$ does not couple to the metric. In [7] $c$ changes in an abrupt phase transition, but one could also imagine $c \propto a^n$ as in [8]. The latter scenario would result from a Lagrangian of the Brans-Dicke type, with

$$\mathcal{L}_\psi = -\frac{\omega}{16\pi G\psi} \dot{\psi}^2$$

(where $\omega$ is a dimensionless coupling constant) and is being investigated [9]. The addition of an appropriate temperature-dependent potential $V(\psi)$ could induce a phase transition, as required in the scenario developed in [7]. In this respect the abrupt scenario of Albrecht and Magueijo [7] and the smooth scenario of Barrow [8] are the analogues of inflationary cosmological evolution with and without a phase transition, respectively.

3 Mapping varying-$c$ theories into varying-$e$ theories

Given a variable $\alpha$, and a VSL theory, it must always be possible to redefine units so that $c$ and $\hbar$ are constant, and $e$ varies. The two descriptions should be equivalent with respect of dimensionless quantities. To find the transformation let us assume that measurements of intervals of length $dx$, time $dt$, and energy $dE$, are made in the VSL system of units. In this system $c \propto h \propto 1/\sqrt{\alpha}$ and
\( e = e_0 \). Now define a new system of units such that the same measurements in the new system of units lead to results \( d\hat{x}, d\hat{t}, \) and \( d\hat{E} \) such that

\[
\begin{align*}
    c_0 d\hat{x} &= c \, dx \\
    c_0^2 d\hat{t} &= c^2 \, dt \\
    \frac{d\hat{E}}{c_0^3} &= \frac{dE}{c^3}
\end{align*}
\]

(8) (9) (10)

where \( c_0 \) is a constant, to be identified with the fixed speed of light. These relations fully specify the new system of units. One may then construct dimensionless ratios in order to identify the constants in the new system:

\[
\begin{align*}
    \frac{\hat{c} dt}{dx} &= \frac{c \, dt}{dx} \\
    \frac{\hat{h} d\hat{E} dt}{d\hat{E} dx} &= \frac{\hbar}{dE dt} \\
    \frac{\hat{G} d\hat{E}}{dx c^4} &= \frac{G \, dE}{dx c^4} \\
    \frac{\hat{e}^2}{d\hat{E} d\hat{x}} &= \frac{e_0}{dE dx}.
\end{align*}
\]

(11) (12) (13) (14)

From these we find that in the new system of units

\[
\begin{align*}
    \hat{c} &= c_0 \\
    \hat{h} &= \hbar \frac{c_0}{c} = \hbar_0 \\
    \hat{G} &= G \\
    \hat{e} &= e_0 \frac{c_0}{c} \propto \sqrt{\alpha}.
\end{align*}
\]

(15) (16) (17) (18)

Hence, in the new system, \( c \) and \( \hbar \) are constants and \( \varepsilon \) varies as \( \sqrt{\alpha} \). The transformation (11) can also be defined by

\[
\begin{align*}
    d\hat{x} &= dx/\epsilon \\
    d\hat{t} &= dt/\epsilon^2 \\
    d\hat{E} &= dE \epsilon^3
\end{align*}
\]

(19) (20) (21)

where \( \epsilon = \hat{c}/e_0 \) is the dielectric constant of the vacuum in the varying-\( \varepsilon \) system.

The transformation (19) between units of energy, length, and time, determines the relations between any measurements in the two systems of units. For instance, of mass \( \hat{M} = M/\epsilon \), of mass density \( \hat{\rho} = \rho/\epsilon^4 \), and of pressure \( \hat{p} = p/\epsilon^4 \). The
dual to minimal VSL therefore does not predict $M \propto \epsilon^2$ as in Bekenstein’s theory [6], a first indication that the minimal theories in the two formulations are not equivalent. Notice that $dt/dt > 0$, so that the arrow of time is not reversed by this transformation. Also notice that although the Compton wavelengths of all particles are adiabatic invariants in VSL theories, they decay as $\epsilon$ increases in the changing-$\epsilon$ dual of VSL (as indeed in Bekenstein’s theory).

4 The dual of minimal VSL

Action (1) is defined in a preferred frame where one postulates that it includes no terms in the derivatives of $\psi$. In a cosmological setting this frame is defined by the proper time and the conformal space coordinates which ensure that $K = 0, \pm 1$. In general, this frame would be defined by a 4 vector $u^\mu$, and the metric could be written as

$$g_{\mu\nu} = u_\mu u_\nu + h_{\mu\nu}$$

(22)

with $h_{\mu\nu}u^\mu = 0$ and $u^\mu_\mu = 1$. In the new units one may choose $u^\mu$ to be the same, and the new metric to be

$$\hat{g}_{\mu\nu} = u_\mu u_\nu + h_{\mu\nu}/\epsilon^2.$$  

(23)

This is a statement of the preferred VSL frame.

Under the transformation of units (19) the action (1) becomes

$$S = \int d\hat{x}^4 \left( \sqrt{-\hat{g}} \epsilon^{-2} \left( \frac{c_0^4 (\hat{R} + 2\hat{\Lambda})}{16\pi G} + \hat{\mathcal{L}}_M \right) + \mathcal{L}_\epsilon \right),$$

(24)

where the curvature is to be computed from the old metric in the usual way. However, the variation is to be performed with respect to the new metric. Therefore, the new action, apart from the $\epsilon^{-2}$ factor, is just the standard Einstein-Hilbert action to which a “spatial” conformal transformation has been applied. This is reminiscent of Brans-Dicke theory, which is Einstein’s gravity subject to a conformal transformation dependent on a field $\phi$ which represents the inverse of the gravitational ‘constant’.

The ADM formalism may be used to write this Lagrangian more explicitly as

$$S = \int d\hat{x}^4 \sqrt{-\hat{g}} \left[ \frac{c_0^4}{16\pi G} \left( (3)\hat{R} (\hat{h}_{\mu\nu} \epsilon^2) + \kappa_{ij} \kappa^{ij} - \kappa^2 + 2\hat{\Lambda} \right) + \hat{\mathcal{L}}_M \right] + \mathcal{L}_\epsilon,$$

(25)
where $h_{\mu\nu}$ and $\kappa_{\mu\nu} = h^\alpha_{\mu} h^\beta_{\nu} \nabla_\alpha u_\beta$ are the first and second fundamental forms, and $^{(3)}R$ is the Ricci scalar derived from the spatial metric $h_{\mu\nu}$.

The Einstein equations derived from this action are more simply written using the Hamiltonian formalism, with a 3+1 split induced by vector $u^\mu$ (see for instance [12]). With a unit lapse and zero shift (temporal gauge), the Hamiltonian density in Einstein’s theory takes the form

$$\mathcal{H} = h^{1/2} \left[ -^{(3)}R + h^{-1} \left( \Pi^{\mu\nu} \Pi_{\mu\nu} - \frac{1}{2} \Pi^2 \right) \right]$$

The second fundamental form is given by

$$\kappa_{\mu\nu} = \frac{\dot{h}_{\mu\nu}}{2}$$

and the momenta conjugate to the $h_{\mu\nu}$ are given by

$$\Pi_{\mu\nu} = \frac{\partial L}{\partial \dot{h}_{\mu\nu}} = h^{1/2} (\kappa_{\mu\nu} - \kappa h_{\mu\nu}).$$

The Hamiltonian constraint is $\mathcal{H} = 0$ and the momentum constraint is

$$\nabla_\mu (h^{-1/2} \Pi^{\mu\nu}) = 0$$

The dynamical equations are

$$\dot{h}_{\mu\nu} = \frac{\delta \mathcal{H}}{\delta \Pi_{\mu\nu}} = 2h^{-1/2} \left( \Pi_{\mu\nu} - \frac{1}{2} h_{\mu\nu} \Pi \right)$$

and

$$\dot{\Pi}_{\mu\nu} = -\frac{\delta \mathcal{H}}{\delta \dot{h}_{\mu\nu}} = -h^{1/2}$$

\begin{align*}
&\quad \left( ^{(3)}R_{\mu\nu} - \frac{1}{2} h^{(3)}_{\mu\nu} R \right) \\
&\quad + \frac{1}{2} h^{-1/2} h_{\mu\nu} \left( \Pi_{\alpha\beta} \Pi^{\alpha\beta} - \frac{1}{2} \Pi^2 \right) \\
&\quad - 2h^{-1/2} \left( \Pi_{\mu}^{\alpha} \Pi_{\alpha\nu} - \frac{1}{2} \Pi \Pi_{\mu\nu} \right)
\end{align*}

These are Einstein’s equations in vacuum. Addition of matter is straightforward. If one performs the transformation

$$h_{\mu\nu} \rightarrow h_{\mu\nu} \epsilon^2$$
upon these equations one obtains Einstein’s equations, in Hamiltonian form, in the dual of VSL theory.

The varying-e dual of VSL is therefore not the minimal varying-e theory (encoded in Postulate P8 of Bekenstein’s theory [6] which assumes that the Einstein field equations are left unchanged by the introduction of varying e). Instead, it resembles a Brans-Dicke theory of $\epsilon$, in which a conformal transformation dependent on $\epsilon$ is applied to standard gravity. However, this transformation is only applied to the spatial sections defined by a vector field $u^\mu$. The Lagrangian does not break Lorentz invariance explicitly, but of course the presence of vector $u^\mu$ does. In this respect this theory resembles the Lagrangian written by Coleman and Glashow [11].

5 Cosmology with a non-minimal varying-e theory

When applied to cosmology, the transformation (19) changes the line element (3) into

$$d\hat{s}^2 = ds^2/\epsilon^2 = -c_0^2 dt^2 + \hat{a}^2 \left[ \frac{dr^2}{1 + \hat{K} \hat{r}^2} + \hat{r}^2 d\hat{\Omega} \right]$$  (34)

with $\hat{a} = a/\epsilon$, and $\hat{K} = K$. Note that after the transformation of units one must perform a spatial coordinate transformation so that $\hat{K} = K = \{0, \pm 1\}$. The Friedmann equations in the new units are therefore

$$\left( \frac{\dot{a}'}{a} + \frac{\epsilon'}{\epsilon} \right)^2 = \frac{8\pi G}{3} \hat{\rho} - \frac{K c_0^2}{a^2},$$  (35)

$$\frac{\ddot{a}''}{a} + \frac{\epsilon''}{\epsilon} - 2 \left( \frac{\epsilon'}{\epsilon} \right)^2 = -\frac{4\pi G}{3} \left( \dot{\rho} + 3\dot{\hat{\rho}}/c_0^2 \right),$$  (36)

with a prime denoting $d/d\hat{t}$. One can derive these equations by applying the transformation (19) to the VSL Friedmann equations, or by writing the Einstein’s equations associated with the action (25) for line element (34).

The conservation equation in this theory becomes

$$\dot{\rho} + 3 \frac{\dot{a}'}{a} \left( \dot{\rho} + \frac{\dot{\hat{\rho}}}{c_0^2} \right) = \frac{\epsilon'}{\epsilon} \left( \dot{\rho} - 3\frac{\dot{\hat{\rho}}}{c_0^2} \right) - \frac{3K c_0^2}{4\pi G a^2} \frac{\epsilon'}{\epsilon}$$  (37)

The first source term on the right-hand side is zero in the radiation-dominated era. The second term on the right-hand side couples to spatial curvature in
the same way as in the VSL theory (see eq. (6) above). In this non-minimal varying-\(e\) theory the dielectric properties of the vacuum induce violations of energy conservation in curved space times. The coupling is such that if \(\epsilon'/\epsilon > 0\), only the \(K = 0\) universe is stable. Energy is produced for subcritical densities, and taken away for supercritical densities. One should therefore expect a solution to the flatness problem.

One can derive simple radiation-dominated (\(\dot{\rho} = \dot{\rho}c_s^2/3\)) solutions to these equations, by noting that \(a \propto t^{1/2}\), and so

\[
\dot{a} = \frac{1}{\epsilon} \left( \int_0^{\dot{t}} d\dot{t}' \epsilon^2 \right)^{1/2}
\]  

(38)

For a sudden phase transition at time \(\dot{t} = \dot{t}_c\) in which \(\epsilon\) jumps from \(\epsilon_-\) to \(\epsilon_+\) one has

\[
\dot{a} = \dot{t}^{1/2} \quad \text{for} \quad \dot{t} < \dot{t}_c
\]

\[
= \sqrt{\left(\frac{\epsilon_-}{\epsilon_+}\right)
\dot{t}_c + \dot{t} - \dot{t}_c} \quad \text{for} \quad \dot{t} > \dot{t}_c
\]  

(39)

If \(\epsilon_- \ll \epsilon_+\) then effectively

\[
\dot{a} = \dot{t}^{1/2} \quad \text{for} \quad \dot{t} < \dot{t}_c
\]

\[
= \sqrt{\dot{t} - \dot{t}_c} \quad \text{for} \quad \dot{t} > \dot{t}_c
\]  

(40)

and there is a second big bang at \(t_c\). The expansion factor \(\dot{a}\) drops to zero at \(\dot{t}_c\), and evolution restarts as if \(t_c\) had been the big bang.

Scenarios in which \(c \propto a^m\) were considered in [8]. If \(-1 < m < 0\) (a scenario in which the quasi-flatness problem is solved [9]), then \(\dot{a} = \dot{t}^{1/2}\), and we have a normal radiation-dominated universe expansion factor. However, if the flatness problem is to be solved in a radiation-dominated universe then we require \(m < -1\). Since \(\dot{a} \propto a^{m+1}\), again we have that the expansion factor decreases while \(\epsilon\) varies. This type of variation should therefore only occur during a short period in the very early universe. In both cases we see that radiation-dominated universes in which the flatness problem is solved display a decrease in the expansion factor \(\dot{a}\), and we have deflation in these coordinates. Deflation is the natural cosmological setting for a decreasing \(\alpha\) theory. In such scenarios the Compton wavelengths of all particles decay in time. The unusual coupling to gravity that exists in this theory ensures that the universe deflates, so that the ratio of Compton wavelengths to the Hubble length does not decrease faster than in the standard theory.
6 The flatness problem

We now spell out how the flatness problem can be solved in these cosmologies. In standard VSL cosmology if we assume that

\[ c = c_0 a^m \] (41)
\[ p = (\gamma - 1) \rho c^2 \] (42)

then, as we found in [8], the flatness problem is solved as \( a \to \infty \) if

\[ 2m \leq 2 - 3\gamma \] (43)

which reduces to the standard inflationary condition if \( m = 0 \). This is just the condition that the ratio between the energy and the curvature contributions to expansion evolves as

\[ F_{VSLT} \equiv \frac{\rho}{Kc_0^2/a^2} \propto a^{2-3\gamma-2m} \to \infty \]

for \( a \to \infty \). In deriving this expression we used the solution for \( \rho \) that comes from integrating the conservation equation (6) after substituting (41)-(42).

Now, in the dual theory with constant \( c \) and varying \( e \), let us assume

\[ \epsilon = \epsilon_0 \hat{a}^n \]
\[ \hat{p} = (\gamma - 1) \hat{\rho} \hat{c}^2 \]

and let us study a similar ratio:

\[ \hat{F}_{dual} \equiv \frac{\hat{\rho}}{K\hat{c}_0^2/\hat{a}^2} \]

Again, we can integrate when \( K = 0 \) to find \( \hat{\rho} = B\hat{a}^{4n-3\gamma(n+1)} \). This is what must be approached asymptotically when \( K \neq 0 \) for the flatness problem to be solved. We then note that \( \dot{a} = \alpha \dot{c}^{-1} \) so that \( a \propto \hat{a}^{n+1} \). Therefore \( a \to \infty \) corresponds to \( \dot{a} \to \infty \) iff \( n > -1 \). Also, since \( \epsilon \propto \dot{c}^{-1} \) in the dual theory, we have the relations

\[ \epsilon \propto \dot{a}^n \propto c^{-1} \propto a^{-m} \propto \hat{a}^{-m(n+1)} \]
and we see that the constants \( n \) and \( m \) that we have introduced in the two frames are related by

\[
    n + 1 = \frac{1}{m + 1} \tag{44}
\]

and therefore

\[
    F_{\text{dual}} \propto \hat{\rho} \hat{a}^2 \propto \hat{a}^{4n+2-3\gamma(n+1)}
\]

Hence in the dual theory the flatness problem is solved as \( \hat{a} \to \infty \) (which corresponds to \( a \to \infty \) when \( n + 1 > 0 \)) if

\[
    4n + 2 - 3\gamma(n + 1) > 0. \tag{45}
\]

If we have \( n + 1 < 0 \) then \( a \to \infty \) corresponds to \( \hat{a} \to 0 \) and the condition for solving the flatness problem in this limit is

\[
    4n + 2 - 3\gamma(n + 1) < 0. \tag{46}
\]

We can show that in both cases the conditions on \( n \), (45)-(46) for the resolution of the flatness problem transform, using (44) into the condition (43) for its resolution in the VSL theory.

More generally we note that under transformation (19)

\[
    \delta_\Omega = \frac{\rho - \rho_c}{\rho_c} = \frac{\hat{\rho} - \hat{\rho}_c}{\hat{\rho}_c} = \delta_\Omega \tag{47}
\]

Deviations from critical density are therefore the same regardless of the system of units, as expected, since the flatness problem is a dimensionless question. The varying-\( \epsilon \) duals of VSL scenarios which solve the flatness problem must therefore solve this problem also.

\[\text{From Eqns.5 and 37 one can derive}\]

\[
    \delta_\Omega' = (1 + \delta_\Omega)\delta_\Omega \left( \frac{\dot{a}'}{a'} + \frac{\dot{\epsilon}'}{\epsilon} \right)(3\gamma - 2) - 2\frac{\dot{\epsilon}'}{\epsilon} \delta_\Omega \tag{48}
\]

To first order in \( \delta_\Omega \) we therefore have

\[
    \delta_\Omega' = \delta_\Omega(3\gamma - 2)\sqrt{\frac{8\pi G\rho}{3}} - 2\frac{\epsilon'}{\epsilon} \delta_\Omega \tag{49}
\]
For a sufficiently fast increase in $\epsilon$ one will therefore always have $\dot{\delta}_\Omega/\dot{\delta}_\Omega \ll -1$, and the flatness problem is solved.

7 Conclusions

Theories of varying $\alpha$ have been proposed in the past [6], attributing this change to a change in $\epsilon$. These theories couple minimally to gravity and therefore do not solve the cosmological problems. The VSL proposal is a varying-$\alpha$ theory which attributes this change to $\hbar$ and $c$ instead. VSL theories that are minimally coupled to gravity can solve the cosmological problems. In this paper we stressed the existence of a duality between varying-$\epsilon$ theories and VSL theories. We derived the standard unit transformation linking these two types of theory and derived the dual of minimal VSL. The resulting varying-$\epsilon$ theory is of Brans-Dicke type where the vacuum dielectric field $\epsilon = e/e_0$ behaves like the inverse of a gravitational coupling $\phi = 1/G$. Standard Brans-Dicke theory may be thought of as a $\phi$-related conformal transformation applied to Einstein’s gravity. The VSL dual derives from an $\epsilon$-related conformal transformation that only acts on spatial sections of the metric, as defined by a given vector $u^\mu$. The presence of this vector in the Lagrangian breaks Lorentz invariance, as in the theory of Coleman and Glashow [11]. The resulting theory is comparable to an ether theory in the sense that the dielectric medium is not just another cosmic ingredient (as is the case for Bekenstein’s theory [6]) but also participates in the formulation of the laws of physics, and defines a preferred frame. We have showed how in this theory, in scenarios in which the flatness problem is solved, one has “deflation”. The increasing $\epsilon$ field induces a period of contraction of the universe, in accord with the decreasing Compton wavelengths of all particles.

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