Nonequilibrium inflaton dynamics and reheating. II. Fermion production, noise, and stochasticity

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(Submitted to Phys. Rev. D on August 13, 1997)

Abstract

We present a detailed and systematic analysis of the coarse-grained, nonequilibrium dynamics of a scalar inflaton field coupled to a fermion field in the latter stages of the reheating period of inflationary cosmology. We derive coupled nonperturbative equations for the inflaton mean field and variance at two loops in a general curved spacetime, and show that the equations of motion are real and causal, and that the gap equation for the two-point function is dissipative due to fermion particle production. We then specialize to the case of Minkowski space and small-amplitude inflaton oscillations, and derive the perturbative one-loop dissipation and noise kernels to fourth order in the Yukawa coupling constant; the normal-threshold dissipation and noise kernels are shown to satisfy a zero-temperature fluctuation-dissipation relation. We derive a Langevin equation for the dynamics of the inflaton zero mode, and show that its variance is non-negligible during reheating. Stochastic evolution of the inflaton may have important astrophysical consequences.

PACS number(s): 98.80.Cq, 04.62.+v, 05.70.Ln

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I. INTRODUCTION

In an earlier paper [1] we studied the effect of parametric particle creation on the inflaton dynamics in the post-inflation, “preheating” stage. It was shown that analysis of the dominant physical processes during the early stages of reheating necessitates consideration of self-consistent back reaction of the inflaton field variance on the inflaton mean field and quantum modes. This is a nonperturbative effect, and its description requires a consistent truncation of the Schwinger-Dyson equations [2,3,1]. Assuming initial conditions conducive to efficient parametric particle creation, the end state of the regime of parametric particle creation consists of a large inflaton variance (i.e., on the order of the tree level terms in the inflaton’s effective mass). Thus, both the inflaton mean field and variance should be treated on an equal footing. This requires a two-particle-irreducible (2PI) formulation of the effective action which is a subclass of the master effective action [2].

During the later stages of reheating, the dynamics of the inflaton field is thought, in the case of unbroken symmetry, to be dominated by damping from fermionic particle creation [4]. This stage of inflaton dynamics is the subject of this paper. We consider a model consisting of a scalar inflaton field $\phi$ (with $\lambda\phi^4$ self-coupling) coupled to a spinor field $\psi$ via a Yukawa interaction and we attempt to present as complete and rigorous a treatment as mandated by the self-consistency of the formalism and the actual solvability of the equations. Thus, we adopt a closed-time-path (CTP), two-particle-irreducible (2PI), coarse-grained effective action (CGEA) to derive the dynamics of the inflaton field. We have explained the significance of CTP and 2PI in our earlier papers [5–9,2,10] and their relevance to the study of inflaton dynamics in Ref. [1]. Here, an added feature of an open system is introduced: we wish to include the averaged effect of an environment on the system, and a useful method is via the coarse-grained effective action [11–15]. Let us explain the rationale for this.

Coarse-Grained Effective Action: In inflationary cosmology at the onset of the reheating period, the inflaton field’s zero mode typically has a large expectation value, whereas all other fields coupled to the inflaton, as well as inflaton modes with momenta greater than the Hubble constant, are to a good approximation in a vacuum state [20]. This suggests imposing a physical coarse-graining in which one regards the inflaton field as the system, and the various quantum fields coupled to it as the environment. From the closed-time-path, coarse-grained effective action (CTP-CGEA) [11–15] derived in Sec. [1] below, one obtains effective dynamical equations for the inflaton field, taking into account its effect on the environment, and back reaction therefrom. For the present problem, the system consists of the inflaton mean field and variance, and the environment consists of the spinor field(s) coupled to the system via a Yukawa interaction.

We wish to emphasize here a subtle yet important distinction between the system-environment division in nonequilibrium statistical mechanics and the system-bath division assumed in thermal field theory. In the latter, one assumes that the propagators for the bath degrees of freedom are fixed, finite-temperature equilibrium Green functions, whereas in the case of the CTP-CGEA, the environmental propagators are slaved (in the sense of [2]) to the dynamics of the system degrees of freedom, and are not fixed a priori to be equilibrium Green functions for all time. This distinction is important for discussions of fermion particle production during reheating, because it is only when the inflaton mean field amplitude is small enough for the use of perturbation theory, that the system-bath split implicit in
thermal field theory can be used. Otherwise, one must take into account the effect of the inflaton mean field on the bath (spinor) propagators.

**Earlier work:** The first studies of particle production during reheating in inflationary cosmology were [16,18], where reasonable estimates of particle production were made, but back reaction effects were not addressed. The earliest studies of fermionic particle creation during reheating used time-dependent perturbation theory to compute the imaginary part of the self-energy for the zero mode of the inflaton field, which was related to the damping parameter in a friction-type phenomenological term in the equation of motion for the inflaton zero mode [16,17,19,22,23]. In these studies, it was assumed that the effect of fermionic particle production on the dynamics of the inflaton zero mode is that of a $\Gamma \dot{\phi}$ friction term. However, it has been shown [25,7,26,27,24,28,21,62] that this assumption is not tenable for a wide variety of field-theoretic interactions and initial conditions. Rather, the effect of back reaction from particle production must be accounted for by deriving the effective evolution equation for the quantum expectation value of the inflaton zero mode, where the dynamics of the degrees of freedom of the produced particles are either coarse-grained (as in Sec. III of this paper), or accounted for through self-consistent coupled equations (as in Sec. I).

In general, particle creation leads to a nonlocal dissipation term in the inflaton mean field equation, and it is only under rather idealized conditions and specialized cases that one can expect the dissipation kernel to approach a delta function (i.e., a friction term) [29–32]. Therefore, [16,22,23] missed the time-nonlocal nature of fermion particle production and its effect on the dynamics of the inflaton field. In addition, these studies computed the self-energy in flat space, thereby neglecting the effect of curved spacetime on fermion production, and did not examine the stochastic noise arising from the coarse-graining of the fermion field.

In addition, most early studies of fermion production during reheating, in using time-dependent perturbation theory to compute the vacuum particle production rate [16,22,23], did not include the effect of back reaction of the produced fermion particles on the particle production process itself. In [21], the effect of a thermal initial fermion distribution on the particle production process was investigated (and a Pauli blocking effect was shown), but their analytic derivation of the Pauli blocking effect involves the same perturbative expansion (i.e., system-bath split) described above, and therefore does not incorporate the effect of the produced fermion quanta on the particle production process. In order to take this effect into account, it is necessary to include the effect of the time-varying inflaton mean field in the equation of motion for the spinor propagator, which amounts to a coarse-graining of the fermion field, in the system-environment sense, as described above. The perturbative amplitude expansion of the effective inflaton dynamics, in contrast, amounts to a system-bath coarse-graining which does not include this back reaction effect.

More recent studies of fermion production during reheating [33,34,21] obtained dynamical equations for the inflaton mean field using a one-loop factorization of the Lagrangian, and solved them numerically. However, these studies were carried out in flat space, and because they studied only the dynamics of the inflaton mean field (and at one loop), their analysis did not take into account the back reaction of the inflaton variance on the fermion quantum modes, nor the back reaction of particle production on the dynamics of the inflaton two-point function. The importance of the curved spacetime effect has been addressed in a previous paper [1], and we will discuss below the importance of treating the inflaton quantum variance on equal footing with the mean field.
Present work: In this study, we wish to describe the late stages of the reheating period, in which the damping of the inflaton mean field is dominated by fermionic particle production; our starting point is the end of the preheating period (in which the inflaton dynamics was dominated by back reaction from parametric particle creation, as discussed in our earlier paper [1]). Because the inflaton variance $\langle \varphi^2 \rangle$ can (for sufficiently strong self-coupling) be on the order of the square of the amplitude of mean field oscillations at the end of preheating (in the case of unbroken symmetry) [1], it is necessary to treat the inflaton mean field and variance on an equal footing in a study of the subsequent effective dynamics of the inflaton field. This requires a two-loop, two-particle-irreducible formulation of the coarse-grained effective action. At two loops, both the inflaton mean field and the inflaton variance couple to the spinor degrees of freedom, and are damped by back reaction from fermion particle production; all the previous studies mentioned above, in using the 1PI effective action, missed this possibly important effect.

In addition to having a large variance, the inflaton amplitude at the end of the preheating period may be large enough that the usual perturbative expansion in powers of the Yukawa coupling constant is not valid [see Eq. (3.1) below], in which case a nonperturbative derivation of the inflaton dynamics is required. In the construction of the CTP-2PI, coarse-grained effective action below, the spinor propagators obey one-loop dynamical equations in which the inflaton mean field acts as an external source. Studies of fermion particle production during reheating which relied on the use of perturbation theory in the Yukawa coupling constant [16,18,17,19,22–24] therefore do not apply to the case of fermion particle production at the end of preheating with unbroken symmetry, when the Yukawa coupling is sufficiently large. The dynamical equations derived in Sec. [4] below for the inflaton mean field and variance are applicable even when, as may be the case, the inflaton mean field amplitude is large enough that a perturbative expansion in powers of the Yukawa coupling is not justified.

Although, as discussed above, a proper treatment of the early stage of fermion production during reheating, starting at the end of preheating with unbroken symmetry, should in principle employ the CTP-2PI-CGEA to obtain coupled equations of motion for the inflaton mean field and variance, at very late times the inflaton mean field and variance will have been damped sufficiently (due to the dissipative mechanisms derived below in Sec. [4]) that the perturbative 1PI effective action will yield a qualitatively correct description of the inflaton mean-field dynamics. While curved spacetime effects should in principle be incorporated self-consistently for a quantitative calculation of the reheating temperature in a particular inflationary model (as discussed in [1]), for a general discussion of dissipative effective dynamics of the inflaton mean field in the case of weak cosmic expansion (where the Hubble constant $H$ is much smaller than the frequency of inflaton oscillations), it is reasonable to neglect curved spacetime effects in computing the spinor propagators. Therefore in Sec. [5], we derive the perturbative, flat-space CTP-1PI-CGEA to fourth order in the Yukawa coupling constant, and obtain an evolution equation for the inflaton mean field with nonlocal dissipation.

Another new feature of our work obtainable only from the stochastic approach adopted here is the derivation, in Sec. [6], of a Langevin equation for the inflaton mean field, with clear identification of the dissipation and noise kernels from the CGEA. We have calculated the energy dissipated and the fluctuations in the energy. From the latter we obtain the range
of parameters where the conventional “mean-field” approach breaks down. We believe the methodology presented here provides a better theoretical framework for the investigation of phase transitions in the early universe, as exemplified by our treatment of reheating in inflationary cosmology.

Organization and Notation: Our work is organized as follows. In Sec. II, we derive the coupled equations of motion for the inflaton mean field and variance, in a general curved spacetime, including diagrams in the coarse-grained CTP-2PI effective action of up to two-loop order. In Sec. III, we specialize to Minkowski space and small mean-field amplitude, and obtain a perturbative mean field equation including dissipative effects up to fourth order in the Yukawa coupling constant. In Sec. IV, we examine the dissipation and noise kernels obtained in Sec. III and show that they obey a fluctuation-dissipation relation. We then derive a Langevin equation which self-consistently includes the effect of noise on the dynamics of the inflaton zero mode. We summarize our results in Sec. V.

Throughout this paper we use units in which $c = 1$. Planck’s constant $\hbar$ is shown explicitly (i.e., not set equal to 1) except in those sections where noted. In these units, Newton’s constant is $G = \hbar M_P^{-2}$, where $M_P$ is the Planck mass. We assume a four-dimensional, globally hyperbolic, orientable spacetime manifold, and follow the sign conventions$^1$ of Birrell and Davies$^{36}$ for the metric tensor $g_{\mu\nu}$, the Riemann curvature tensor $R_{\mu\nu\rho\sigma}$, and the Einstein tensor $G_{\mu\nu}$. We use greek letters to denote spacetime indices. The beginning latin letters $a, b, c, d, e, f$ indicate the time branch (see Ref. $^{10}$, Sec. II). The Einstein summation convention over repeated indices is employed.

II. COARSE-GRAINED INFLATON DYNAMICS IN CURVED SPACETIME

In this section, we present a first-principles derivation of the nonequilibrium, nonperturbative, effective dynamics of a scalar inflaton field $\phi$ coupled to a spinor field $\psi$ via a Yukawa interaction, in a general, curved classical background spacetime. The use of the Schwinger-Keldysh closed-time-path (CTP) formalism allows formulation of the nonequilibrium dynamics of the inflaton from an appropriately defined initial quantum state. The evolution equations for the inflaton mean field and variance are derived from the two-loop, closed-time-path (CTP), two-particle-irreducible (2PI), coarse-grained effective action (CGEA). As the name suggests, there are two approximations of a statistical mechanical nature. One is the coarse graining of the environment—here the inflaton field is the system and the fermion field is the environment$^{1}$.

The other refers to a truncation of the correlation hierarchy for the inflaton field$^3$ —the two-particle-irreducible effective action. This formulation retains the inflaton mean field and variance as coupled dynamical degrees of freedom. Back Reaction of the scalar and spinor field dynamics on the spacetime is incorporated using the semiclassical Einstein equation, which follows from functional differentiation of the effective action with respect to the metric. It is shown that these dynamical equations are both real

\footnote{In the classification scheme of Misner, Thorne and Wheeler$^{17}$, the sign convention of Birrell and Davies$^{36}$ is classified as $(+, +, +)$.}
A. The model

We study a model consisting of a scalar field $\phi$ (the inflaton field) which is Yukawa-coupled to a spinor field $\psi$, in a curved, dynamical, classical background spacetime. The total action

$$S[\phi, \bar{\psi}, \psi, g^{\mu \nu}] = S^G[g^{\mu \nu}] + S^F[\phi, \bar{\psi}, \psi, g^{\mu \nu}], \quad (2.1)$$

consists of a part depicting classical gravity, $S^G[g^{\mu \nu}]$, and a part for the matter fields,

$$S^F[\phi, \bar{\psi}, \psi, g^{\mu \nu}] = S^\phi[\phi, g^{\mu \nu}] + S^\psi[\bar{\psi}, \psi, g^{\mu \nu}] + S^Y[\phi, \bar{\psi}, \psi, g^{\mu \nu}], \quad (2.2)$$

whose scalar (inflaton), spinor (fermion), and Yukawa-interaction parts are given by

$$S^\phi[\phi, g^{\mu \nu}] = -\frac{1}{2} \int d^4x \sqrt{-g} \left[ \phi (\Box + m^2 + \xi R) \phi + \frac{\lambda}{12} \phi^4 \right], \quad (2.3a)$$

$$S^\psi[\bar{\psi}, \psi, g^{\mu \nu}] = \int d^4x \sqrt{-g} \left[ i \frac{\bar{\psi}}{2} \left( \bar{\psi} \gamma^\mu \nabla_\mu \psi - (\nabla_\mu \bar{\psi}) \gamma^\mu \psi \right) - \mu \bar{\psi} \psi \right], \quad (2.3b)$$

$$S^Y[\phi, \bar{\psi}, \psi, g^{\mu \nu}] = -f \int d^4x \sqrt{-g} \bar{\phi} \psi \psi. \quad (2.3c)$$

For this theory to be renormalizable in semiclassical gravity, the bare gravity action $S^G[g^{\mu \nu}]$ of Eq. (2.1) should have the form \[37,36\]

$$S^G[g^{\mu \nu}] = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ R - 2\Lambda_c + cR^2 + bR\alpha^\alpha R\alpha_\beta + aR^\alpha\beta^\gamma\delta R\alpha_\beta\gamma\delta \right]. \quad (2.4)$$

In Eqs. (2.3a)–(2.3c), $m$ is the scalar field “mass” (with dimensions of inverse length); $\xi$ is the dimensionless coupling to gravity; $\mu$ is the spinor field “mass,” with dimensions of inverse length; $\Box$ is the Laplace-Beltrami operator in the curved background spacetime with metric tensor $g_{\mu\nu}$; $\nabla_\mu$ is the covariant derivative compatible with the metric; $\sqrt{-g}$ is the square root of the absolute value of the determinant of the metric; $\lambda$ is the self-coupling of the inflaton field, with dimensions of $1/\sqrt{\hbar}$; and $f$ is the Yukawa coupling constant, which has dimensions of $1/\sqrt{\hbar}$. In Eq. (2.4), $G$ is Newton’s constant (with dimensions of length divided by mass); $R$ is the scalar curvature; $R_{\mu\nu}$ is the Ricci tensor; $R_{\alpha\beta\gamma\delta}$ is the Riemann tensor; $a$, $b$, and $c$ are constants with dimensions of length squared; and $\Lambda_c$ is the cosmological constant, which has dimensions of inverse length-squared. The curved spacetime Dirac matrices $\gamma^\mu$ satisfy the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\}_+ = 2g^{\mu\nu}1_{1_{sp}}, \quad (2.5)$$

in terms of the contravariant metric tensor $g^{\mu\nu}$. The symbol $1_{1_{sp}}$ denotes the identity element in the Dirac algebra.
Due to the generalized Gauss-Bonnet theorem [38], the constants $a$, $b$, and $c$ are not all independent in four spacetime dimensions; let us therefore set $a = 0$. It is assumed that there is a definite separation of time scales between the stage of “preheating” discussed in our earlier work [1], and fermionic particle production, which is our primary focus in this work. However, this does not imply that perturbation theory in the Yukawa coupling constant $f$ is necessarily valid, which would require that condition (3.1) (defined in Sec. III below) be satisfied. In addition, the fermion field mass $\mu$ is assumed to be much lighter than the inflaton field mass $m$, i.e., the renormalized parameters $m$ and $\mu$ satisfy $m \gg \mu$.

**B. Closed-time-path, coarse-grained effective action**

We denote the quantum Heisenberg field operators of the scalar field $\phi$ and the spinor field $\psi$ by $\Phi_H$ and $\Psi_H$, respectively, and the quantum state $| \Omega \rangle$ by $| \Omega \rangle$. For consistency with the truncation of the correlation hierarchy at second order, we assume $\Phi_H$ to have a Gaussian moment expansion in the position basis [39], in which case the relevant observables are the scalar mean field

$$\hat{\phi}(x) \equiv \langle \Omega | \Phi_H(x) | \Omega \rangle,$$

and the mean-squared fluctuations, or variance, of the scalar field

$$\langle \Omega | \Phi_H^2(x) | \Omega \rangle - \langle \Omega | \Phi_H(x) | \Omega \rangle^2 \equiv \langle \Omega | \varphi_H^2(x) | \Omega \rangle,$$

where the last equality follows from the definition of the scalar fluctuation field

$$\varphi_H(x) \equiv \Phi_H(x) - \hat{\phi}(x).$$

As discussed above, at the end of the preheating period, the inflaton variance can be as large as the square of the amplitude of mean-field oscillations. On the basis of our assumption of separation of time scales in Sec. II A, and the conditions which prevail at the onset of reheating, the initial quantum state $| \Omega \rangle$ is assumed to be an appropriately defined vacuum state for the spinor field.

The construction of the CTP-2PI-CGEA for the $\phi\bar{\psi}\psi$ theory in a general, curved, background spacetime closely parallels the construction of the CTP-2PI effective action for the $O(N)$ model in curved spacetime discussed in an earlier paper [10]. Within the spacetime manifold (whose dynamics must be determined self-consistently through the semiclassical gravitational field equation), let $M$ be defined as the past domain of dependence of a Cauchy hypersurface $\Sigma_*$, i.e., $M \equiv D_-(\Sigma_*)$, where $\Sigma_*$ has been chosen to be far to the future of any dynamics we wish to study. We now define a “CTP” manifold $\mathcal{M}$ as the quotient space

$$\mathcal{M} \equiv (M \times \{+,-\})/\sim,$$

2 Although in this case the particular initial conditions constitute a pure state, this formalism can encompass general mixed-state initial conditions [3].
in terms of the discrete set \{+,−\} which labels the time branch. The volume form \(\epsilon_M\) on \(\mathcal{M}\) is defined to be \(\epsilon_M\) on the plus (\(+\)) branch and \(-\epsilon_M\) on the minus (\(-\)) branch. The restrictions of a function \(\phi\), defined on \(\mathcal{M}\), to the + and − time branches are denoted by \(\phi_+\) and \(\phi_-\), respectively. We can then define a matter field action on \(\mathcal{M}\),

\[
S^p[\phi_-, \bar{\psi}_-, \psi_-, g_-^{\mu \nu}; \phi_+, \bar{\psi}_+, \psi_+, g_+^{\mu \nu}] \equiv S^p[\phi_+, \bar{\psi}_+, \psi_+, g_+^{\mu \nu}] - S^p[\phi_-, \bar{\psi}_-, \psi_-, g_-^{\mu \nu}],
\]

(2.10)

where the spacetime integrations in \(S^p\) are now over \(\mathcal{M}\) only. We use the symbol \(S^p\) to distinguish it from the action \(S^p\) on \(\mathcal{M}\). Let us also simplify notation by suppressing time branch indices in the argument of functionals on \(\mathcal{M}\), i.e.,

\[
S^p[\phi, \bar{\psi}, \psi, g^{\mu \nu}] \equiv S^p[\phi_+, \bar{\psi}_+, \psi_+, g_+^{\mu \nu}; \phi_-, \bar{\psi}_-, \psi_-, g_-^{\mu \nu}].
\]

(2.11)

Let us also define the functional \(S^\gamma\) on \(\mathcal{M}\) by

\[
S^\gamma[\phi, \bar{\psi}, \psi, g^{\mu \nu}] \equiv S^\gamma[\phi_+, \bar{\psi}_+, \psi_+, g_+^{\mu \nu}] - S^\gamma[\phi_-, \bar{\psi}_-, \psi_-, g_-^{\mu \nu}],
\]

(2.12)

in analogy with Eq. (2.10). For a function \(\phi\) on \(\mathcal{M}\), the restrictions of \(\phi\) to the + and − time branches are subject to the boundary condition

\[
(\phi_+)|_{\Sigma_*} = (\phi_-)|_{\Sigma_*}
\]

(2.13)

at the hypersurface \(\Sigma_*\). The gravity action \(S^G\), promoted to a functional on \(\mathcal{M}\), takes the form

\[
S^G[g_+^{\mu \nu}, g_-^{\mu \nu}] = S^G[g_+^{\mu \nu}] - S^G[g_-^{\mu \nu}],
\]

(2.14)

where the range of spacetime integration in \(S^G\) on the right-hand side of Eq. (2.14) is understood to be over \(\mathcal{M}\).

To formulate the CTP-2PI-CGEA, our first step is to define a generating functional for \(n\)-point functions of the scalar field, in terms of the initial quantum state \(|\Omega\rangle\) which evolves under the influence of a local source \(J\), and a non-local source \(K\) coupled to the scalar field (in the interaction picture with the external sources being treated as the “interaction”). This generating functional depends on both \(J\) and \(K\), as well as the classical background metric \(g^{\mu \nu}\). In the path integral representation, the generating functional \(Z[J, K, g^{\mu \nu}]\) takes the form of a sum over scalar field configurations \(\phi\) and complex Grassmann-valued configurations \(\psi\) on the manifold \(\mathcal{M}\),

\[
Z[J, K, g^{\mu \nu}] \equiv \int_{\text{exp}} D\phi_- D\bar{\psi}_- D\psi_- D\phi_+ D\bar{\psi}_+ D\psi_+ \exp \left[ \frac{i}{\hbar} \left( S^p[\phi, \bar{\psi}, \psi, g^{\mu \nu}] ight. \right.
\]

\[
+ \int_M d^4x \sqrt{-g} c^{ab}_J \partial_a \phi_b \right.
\]

\[
+ \frac{1}{2} \int_M d^4x \sqrt{-g} \int_M d^4x' \sqrt{-g'} c^{ab} c^{cd} K_{ac}(x, x') \phi_b(x) \phi_d(x') \right) \right],
\]

(2.15)

where \(J_a(x)\) is a local c-number source and \(K_{ab}(x, x')\) is a nonlocal c-number source. The subscript CTP on the functional integral denotes a summation over field configurations \(\phi_\pm\),
The two-index symbol \( \psi_{\pm} \) and \( \psi_{\mp} \) which satisfy the boundary condition (2,13). The latin indices \( a, b, c, \ldots \), have the discrete index set \( \{+, -\} \), and denote the time branch \( \tilde{\mathcal{M}} \). The boundary conditions on the functional integral of Eq. (2.13) at the initial data surface determine the quantum state \( |\Omega\rangle \). The CTP indices have been dropped from \( g^{\mu\nu} \) for ease of notation; it will be clear how to reinstate them \( [10] \) in the two-loop CTP–2PI effective action shown below in Sec. [11]. The two-index symbol \( c^{ab} \) is defined by

\[
c^{ab} = \begin{cases} 
1 & \text{if } a = b = +, \\
-1 & \text{if } a = b = -, \\
0 & \text{otherwise.}
\end{cases}
\]  

(2.16)

The generating functional for normalized \( n \)-point functions is

\[
W[J, K, g^{\mu\nu}] = -i\hbar \ln Z[J, K, g^{\mu\nu}],
\]  

(2.17)
in terms of which we can define the classical scalar field on \( M \)

\[
\hat{\phi}_a(x)_{JK} = c_{ab} \frac{1}{\sqrt{-g}} \frac{\delta W[J, K, g^{\mu\nu}]}{\delta J_b(x)}
\]  

(2.18)

and the scalar two-point function on \( M \) in the presence of the sources \( J_a \) and \( K_{ab} \),

\[
hG_{ab}(x, x')_{JK} = 2c_{ac}c_{bd} \frac{1}{\sqrt{-g}} \frac{1}{\sqrt{-g'}} \frac{\delta W[J, K g^{\mu\nu}]}{\delta K_{cd}(x, x')} - \hat{\phi}_a(x)\hat{\phi}_b(x'),
\]  

(2.19)

where the \( JK \) subscripts indicate that \( \hat{\phi} \) and \( G \) are functionals of the \( J_a \) and \( K_{ab} \) sources. In the limit \( J_a = K_{ab} = 0 \), the classical field is the same on the two time branches,

\[
\hat{\phi}_+ \bigg|_{J=K=0} = \hat{\phi}_- \bigg|_{J=K=0} = \langle \Omega | \Phi_n | \Omega \rangle \equiv \hat{\phi}
\]  

(2.20)

and it is equivalent to the mean field \( \hat{\phi} \). In the same limit, \( G_{ab} \) becomes the CTP propagator for the fluctuation field defined in Eq. (2.8),

\[
\begin{align*}
hG_{++}(x, x')_{J=K=0} &= \langle \Omega | T(\varphi_h(x)\varphi_h(x')) | \Omega \rangle, \\
hG_{--}(x, x')_{J=K=0} &= \langle \Omega | \tilde{T}(\varphi_h(x)\varphi_h(x')) | \Omega \rangle, \\
hG_{+-}(x, x')_{J=K=0} &= \langle \Omega | \varphi_h(x)\varphi_h(x') | \Omega \rangle, \\
hG_{-+}(x, x')_{J=K=0} &= \langle \Omega | \varphi_h(x')\varphi_h(x) | \Omega \rangle,
\end{align*}
\]  

(2.21)

where \( T \) denotes temporal ordering and \( \tilde{T} \) denotes antitemporal ordering. In the coincidence limit\( [11] \), all four components (2.21a)–(2.21d) are equivalent to the variance \( \langle \varphi_h^2 \rangle \) defined in Eq. (2.7). Provided we can invert Eqs. (2.18) and (2.19) to obtain \( J_a \) and \( K_{ab} \) in terms of \( \hat{\phi}_a \) and \( G_{ab} \), the CTP–2PI effective action can be defined as the double Legendre transform (in both \( J_a \) and \( K_{ab} \)) of \( W[J, K, g^{\mu\nu}] \).

\[3\] The variance \( \langle \varphi(x)^2 \rangle \) is divergent in four spacetime dimensions, and should be regularized using a covariant procedure \( [36,40] \).
\[
\Gamma[\phi, G, g^{\mu\nu}] = W[J, K, g^{\mu\nu}] - \int_J d^4x \sqrt{-g} c^{ab} J_a(x) \hat{\phi}_b(x) \\
- \frac{1}{2} \int_M d^4x \sqrt{-g} \int d^4x' \sqrt{-g'} c^{ab} c^{cd} K_{ac}(x, x') \left[ h G_{bd}(x, x') + \hat{\phi}_b(x) \hat{\phi}_d(x') \right].
\]

(2.22)

The inverses of Eqs. (2.18) and (2.19) can be obtained by functional differentiation of Eq. (2.22) with respect to \(\hat{\phi}_a\),

\[
1 \frac{\delta \Gamma[\phi, G, g^{\mu\nu}]}{\delta \hat{\phi}_a(x)} = -c^{ab} J_b(x) \hat{\phi}_a(x) - \frac{1}{2} c^{ab} c^{cd} \int_M d^4x' \sqrt{-g'} [K_{bd}(x, x') \hat{\phi}_c(x') + K_{db}(x', x) \hat{\phi}_c(x)] \hat{\phi}_e(x'),
\]

(2.23)

and with respect to \(G_{ab}\),

\[
1 \frac{\delta \Gamma[\phi, G, g^{\mu\nu}]}{\delta G_{ab}(x, x')} = -\frac{\hbar}{2} c^{ac} c^{bd} K_{cd}(x, x') \hat{\phi}_a(x),
\]

(2.24)

where the \(\delta G\) subscript indicates that \(K_{ab}\) and \(J_a\) are functionals of \(\hat{\phi}_a\) and \(G_{ab}\). Inserting Eqs. (2.23) and (2.24) into Eq. (2.22) yields a functional integrodifferential equation for the CTP–2PI effective action in terms of \(\hat{\phi}\) and \(G\) only, so the \(JK\) subscripts can be dropped. It is useful to change the variable of functional integration to be the fluctuation field about \(\hat{\phi}_a\), defined by

\[
\varphi_a \equiv \phi_a - \hat{\phi}_a.
\]

(2.25)

Performing the change-of-variables \(D\phi \rightarrow D\varphi\), the equation for \(\Gamma\) is

\[
\Gamma[\phi, G, g^{\mu\nu}] = \int_M d^4x \int_M d^4x' \frac{\delta \Gamma[\phi, G]}{\delta G_{ab}(x', x)} G_{ab}(x, x') \\
- i\hbar \ln \left\{ \int_{\exp} e^{i\phi} D\varphi D\psi D\phi D\psi D\varphi D\psi \exp \left[ \frac{i}{\hbar} (S^\phi [\varphi + \phi, \bar{\psi}, \psi, g^{\mu\nu}] \\
- \int_M d^4x \frac{\delta \Gamma[\phi, G, g^{\mu\nu}]}{\delta \hat{\phi}_a} \varphi_a - \frac{1}{\hbar} \int_M d^4x \int_M d^4x' \frac{\delta \Gamma[\phi, G, g^{\mu\nu}]}{\delta G_{ba}(x', x)} \varphi_a(x) \varphi_b(x') \right) \right\},
\]

(2.26)

which has the formal solution

\[
\Gamma[\phi, G, g^{\mu\nu}] = S^\phi[\hat{\phi}] - \frac{i\hbar}{2} \ln \det G_{ab} - i\hbar \ln \det F_{ab} + \Gamma_2[\hat{\phi}, G] \\
+ \frac{i\hbar}{2} \int_M d^4x \sqrt{-g} \int_M d^4x' \sqrt{-g'} A^{ab}(x', x) G_{ab}(x, x'),
\]

(2.27)

where \(A^{ab}\) is the second functional derivative of the scalar part of the classical action \(S^\phi\), evaluated at \(\hat{\phi}\),

\[
i A^{ab}(x, x') = \frac{1}{\sqrt{-g}} \left( \frac{\delta^2 S^\phi}{\delta \phi_a(x) \delta \phi_b(x')} [\hat{\phi}] \right) \frac{1}{\sqrt{-g'}} \\
= - \left[ c^{ab}(\Box_x + m^2 + \xi R(x)) + c^{abcd} \frac{\lambda}{2} \hat{\phi}_c(x) \hat{\phi}_d(x) \right] \delta(x - x') \frac{1}{\sqrt{-g'}}.
\]

(2.28)
The symbol $F_{ab}$ denotes the one-loop CTP spinor propagator, which is defined by

$$F_{ab}(x, x') \equiv B_{ab}^{-1}(x, x'),$$

(2.29)

where we are suppressing spinor indices, and the inverse spinor propagator $B^{ab}$ is defined by

$$iB^{ab}(x, x') = \frac{1}{\sqrt{-g}} \frac{1}{\sqrt{-g}} \left[ \frac{\delta^2(\mathcal{S}^{\phi}[\bar{\psi}, \psi] + \mathcal{S}^{\phi}[\bar{\psi}, \psi; \phi])}{\delta \psi_a(x) \delta \bar{\psi}_b(x')} \right] \frac{1}{\sqrt{-g}}$$

(2.30)

$$= \left( c^{ab} (i\gamma^\mu \nabla_\mu - \mu) - c^{abc} f_{\phi_c}(x') \right) \delta(x' - x) \frac{1}{\sqrt{-g}} 1_{sp}. \quad (2.31)$$

It is clear from Eq. (2.31) that the use of the one-loop spinor propagators in the construction of the CTP-2PI-CGEA represents a nonperturbative resummation in the Yukawa coupling constant, which (as discussed above) goes beyond the standard time-dependent perturbation theory. The boundary conditions which define the inverses of Eqs. (2.28) and (2.31) are the boundary conditions at the initial data surface in the functional integral in Eq. (2.15), which in turn, define the initial quantum state $|\Omega\rangle$. The one-loop spinor propagator $F_{ab}$ is related to the expectation values of the spinor Heisenberg field operators for a spinor field in the presence of the c-number background field $\hat{\phi}$,

$$h F_{++}(x, x')|_{\hat{\phi}_+ = \hat{\phi}_-} = \langle \Omega | T(\Psi_h(x)\bar{\Psi}_h(x')) | \Omega \rangle, \quad (2.32a)$$

$$h F_{--}(x, x')|_{\hat{\phi}_+ = \hat{\phi}_-} = \langle \Omega | T(\Psi_h(x)\bar{\Psi}_h(x')) | \Omega \rangle, \quad (2.32b)$$

$$h F_{+-}(x, x')|_{\hat{\phi}_+ = \hat{\phi}_-} = -\langle \Omega | \bar{\Psi}_h(x')\Psi_h(x) | \Omega \rangle, \quad (2.32c)$$

$$h F_{-+}(x, x')|_{\hat{\phi}_+ = \hat{\phi}_-} = \langle \Omega | \Psi_h(x)\bar{\Psi}_h(x') | \Omega \rangle, \quad (2.32d)$$

where the spinor Heisenberg field operators obey the equations

$$(i\gamma^\mu \nabla_\mu - \mu - f\hat{\phi})\Psi = 0, \quad (2.33a)$$

$$(-i\gamma^\mu \nabla_\mu - \mu - f\hat{\phi})\bar{\Psi} = 0. \quad (2.33b)$$

The CTP spinor propagator components satisfy the relations (valid only when $\hat{\phi}_+ = \hat{\phi}_- = \hat{\phi}$)

$$F_{++}(x, x')^\dagger = F_{--}(x', x), \quad (2.34a)$$

$$F_{--}(x, x')^\dagger = F_{++}(x', x), \quad (2.34b)$$

$$F_{+-}(x, x')^\dagger = F_{-+}(x', x), \quad (2.34c)$$

$$F_{-+}(x, x')^\dagger = F_{+-}(x', x). \quad (2.34d)$$

The functional $\Gamma_2[\hat{\phi}, G]$ is defined as $-ih$ times the sum of all vacuum diagrams drawn according to the following rules:

1. Vertices carry spacetime ($x \in M$) and time branch ($a \in \{+, -\}$) labels.

2. Scalar field lines denote $hG_{ab}(x, x')$.

3. Spinor lines denote the one-loop CTP spinor propagator $hF_{ab}(x, x')$ (spinor indices are suppressed), defined in Eq. (2.29).
4. There are three interaction vertices, given by $iS^I/\hbar$, which is defined by

\begin{align}
S^I[\hat\phi, \varphi, \bar\psi, \psi] &= S^I[\hat\phi_+, \varphi_+, \bar\psi_+, \psi_+] - S^I[\hat\phi_-, \varphi_-, \bar\psi_-, \psi_-], \\
S^I[\hat\phi, \varphi, \bar\psi, \psi] &= -\int d^4x \sqrt{-g} \left[ f \bar\psi \psi + \frac{\lambda}{24} \varphi^4 + \frac{\lambda}{6} \hat\phi \varphi^3 \right],
\end{align}

where we have followed the notation of Eq. (2.12).

5. Only diagrams which are two-particle-irreducible with respect to cuts of scalar lines contribute to $\Gamma_2$.

The distinction between the CTP-2PI, coarse-grained effective action which arises here, and the fully two-particle-irreducible effective action (2PI with respect to scalar and spinor cuts), is due to the fact that we only Legendre-transformed sources coupled to $\hat\phi$; i.e., the spinor field is treated as the environment. In Eq. (2.25), the curved-spacetime Dirac $\delta$ function is defined as in [30]. Comparison of Eq. (2.27) above with Eq. (4.13) of Ref. [10] [which was computed for the $O(N)$ model] shows that the $\text{tr} \ln F_{ab}$ in Eq. (2.27) differs from the usual one-loop term by a factor of 2, owing to the difference (in the exponent) between the Gaussian integral formulas for real and complex fields [10].

The functional $\Gamma_2[\hat\phi, G, g^{\mu\nu}]$ can be evaluated in a loop expansion, which corresponds to an expansion in powers of $\hbar$,

$$
\Gamma_2[\hat\phi, G, g^{\mu\nu}] = \sum_{l=2}^{\infty} \hbar^l \Gamma^{(l)}[\hat\phi, G, g^{\mu\nu}],
$$

starting with the two-loop term, $\Gamma^{(2)}$, which has a diagrammatic expansion shown in Fig. 1. The $\lambda\varphi^4$ self-interaction leads to two terms in the two-loop part of the effective action, the second and third graphs of Fig. 1. They are the “setting sun” diagram, which is $O(\lambda^2)$, and the “double bubble,” which is $O(\lambda)$, respectively. The Yukawa interaction leads to only one diagram in $\Gamma^{(2)}$, the first diagram of Fig. 1.

$$
\frac{i f^2}{2} \epsilon^{\alpha a \alpha''} \epsilon^{\beta b \beta''} \int d^4x \sqrt{-g} \int d^4x' \sqrt{-g'} G_{ab}(x, x') \text{tr}_{sp} \left[ F_{a'b'}(x, x') F_{b'a''}(x', x) \right],
$$

where the trace is understood to be over the spinor indices which are not shown, and the three-index symbol $\epsilon^{abc}$ is defined by

$$
\epsilon^{abc} = \begin{cases} 
1 & \text{if } a = b = c = +, \\
-1 & \text{if } a = b = c = -, \\
0 & \text{otherwise}.
\end{cases}
$$

Here, we treat the $\lambda$ self-interaction using the time-dependent Hartree-Fock approximation [11], which is equivalent to retaining the $O(\lambda)$ (double-bubble) graph and dropping the $O(\lambda^2)$ (setting sun) graph. We assume for the present study that the coupling $\lambda$ is sufficiently small that the $O(\lambda^2)$ diagram is unimportant on the time scales of interest in the fermion production regime of the inflaton dynamics. The mean-field and gap equations including both the $O(\lambda)$ and the $O(\lambda^2)$ diagrams have been derived in a general curved spacetime in our previous paper [11].
C. Evolution equations for \( \hat{\phi} \) and \( G \) in curved spacetime

The (bare) semiclassical field equations for the two-point function, mean field, and metric can be obtained from the CTP-2PI-CGEA by functional differentiation with respect \( G_{ab} \), \( \hat{\phi}_a \), and \( g^{\mu\nu} \), followed by identifications of \( \hat{\phi} \) and \( g^{\mu\nu} \) on the two time branches [1],

\[
\frac{\delta S^C[g^{\mu\nu}]}{\delta g^{\mu\nu}_{ab}} + \frac{\Gamma[\hat{\phi}, G, g^{\mu\nu}]}{\delta g^{\mu\nu}_{+}} \bigg|_{\hat{\phi}_{+} = \hat{\phi}_{-} = \hat{\phi}; \quad g^{\mu\nu}_{+} = g^{\mu\nu}_{-} = g^{\mu\nu}} = 0, \quad (2.39a)
\]

\[
\frac{\delta \Gamma[\hat{\phi}, G, g^{\mu\nu}]}{\delta \phi_a} \bigg|_{\hat{\phi}_{+} = \hat{\phi}_{-} = \hat{\phi}; \quad g^{\mu\nu}_{+} = g^{\mu\nu}_{-} = g^{\mu\nu}} = 0, \quad (2.39b)
\]

\[
\frac{\delta \Gamma[\hat{\phi}, G, g^{\mu\nu}]}{\delta G_{ab}} \bigg|_{\hat{\phi}_{+} = \hat{\phi}_{-} = \hat{\phi}; \quad g^{\mu\nu}_{+} = g^{\mu\nu}_{-} = g^{\mu\nu}} = 0. \quad (2.39c)
\]

Equations (2.39a)–(2.39c) constitute the semiclassical approximation to the full quantum dynamics for the system described by the classical action (2.1). Equation (2.39a) should be understood as following after time branch indices have been reinstated on the metric tensor in the CTP-2PI-CGEA [10]. The field equation of semiclassical gravity (with bare parameters) is obtained directly from Eq. (2.39c),

\[
G_{\mu\nu} + \Lambda c g_{\mu\nu} + c^{(1)} H_{\mu\nu} + b^{(2)} H_{\mu\nu} = -8\pi G \langle T_{\mu\nu} \rangle, \quad (2.40)
\]

in terms of the (unrenormalized) quantum energy-momentum tensor defined by

\[
\langle T_{\mu\nu} \rangle = \frac{2}{\sqrt{-g}} \left. \frac{\delta \Gamma[\hat{\phi}, G, g^{\mu\nu}]}{\delta g^{\mu\nu}_{+}} \right|_{\hat{\phi}_{+} = \hat{\phi}_{-} = \hat{\phi}; \quad g^{\mu\nu}_{+} = g^{\mu\nu}_{-} = g^{\mu\nu}}. \quad (2.41)
\]

The energy-momentum tensor \( \langle T_{\mu\nu} \rangle \) is divergent in four spacetime dimensions, and must be regularized via a covariant procedure [33, 1].

Making the two-loop approximation to the CTP-2PI-CGEA, where we take \( \Gamma_2 \simeq \hbar^2 \Gamma^{(2)} \), and dropping the \( O(\lambda^2) \) diagram from \( \Gamma_2 \), the mean-field equation (2.41) becomes

\[
\left( \Box + m^2 + \xi R(x) + \frac{\lambda}{6} \hat{\phi}^2(x) + \frac{\lambda \hbar}{2} G(x, x) \right) \hat{\phi} + \hbar f \text{tr}_{sp}[F_{ab}(x, x)] - \hbar^2 g^3 \Sigma(x) = 0, \quad (2.42)
\]

where \( G(x, x) \) is the coincidence limit of \( G_{ab}(x, x') \), and in terms of a function \( \Sigma(y) \) defined by

\[
\Sigma(y) \equiv \int d^4 x \sqrt{-g} \int d^4 x' \sqrt{-g} \left\{ G_{++}(x, x') \text{tr}_{sp} [F_{++}(x, y) F_{++}(y, x') F_{++}(x', x)] ight. \\
- G_{+-}(x, x') \text{tr}_{sp} [F_{+-}(x, y) F_{++}(y, x') F_{+-}(x', x)] \\
- G_{-+}(x, x') \text{tr}_{sp} [F_{++}(x, y) F_{+-}(y, x') F_{-+}(x', x)] \\
+ G_{--}(x, x') \text{tr}_{sp} [F_{--}(x, y) F_{+-}(y, x') F_{--}(x', x)] \right\}. \quad (2.43)
\]
Making use of the curved spacetime definitions of the scalar and spinor field Hadamard kernels \[36\]
\[ G^{(1)}(x, x') = \langle \Omega | \{ \varphi_H(x), \varphi_H(x') \} | \Omega \rangle, \]  
(2.44a)
\[ F^{(1)}(x, x') = \langle \Omega | [\Psi_H(x), \bar{\Psi}_H(x')] | \Omega \rangle, \]  
(2.44b)
and retarded propagators
\[ G_R(x, x') = i\theta(x, x')\langle \Omega | [\varphi_H(x), \varphi_H(x')] | \Omega \rangle, \]  
(2.45a)
\[ F_R(x, x') = i\theta(x, x')\langle \Omega | [\Psi_H(x), \bar{\Psi}_H(x')] | \Omega \rangle, \]  
(2.45b)
the function \( \Sigma(y) \) can be recast in a manifestly real and causal form,
\[ \Sigma(y) = -2 \int d^4x \sqrt{-g} \int d^4x' \sqrt{-g'} \text{Re} \text{tr}_{sp} \left[ \left( \theta(x', x) G^{(1)}(x', x) F^{(1)}(x, x') \right. \right. \]
\[ \left. \left. -G_R(x, x')^* F_R(x, x') \right) F_R(y, x')^* F_R(y, x) \right] , \]
(2.46)
from which it is clear that the integrand vanishes whenever \( x \) or \( x' \) is to the future of \( y \). The “gap” equation for \( G_{ab} \) is obtained from Eq. (2.39c),
\[ (G^{-1})^{ba}(x, x') = A^{ba}(x, x') + \frac{i\lambda \hbar}{2} e^{ba} G(x, x) \delta(x - x') \frac{1}{\sqrt{-g}} + \hbar f^2 e^{ba} e^{cb} e^{bd} \text{tr}_{sp} [F_{a'b'}(x, x') F_{b'a'}(x', x)] . \]
(2.47)
Multiplying Eq. (2.47) through by \( G_{ab} \), performing a spacetime integration, and taking the \( ++ \) component, we obtain
\[ \left( \Box + m^2 + \xi R + \frac{\lambda}{2} \partial^2 + \frac{\lambda \hbar}{2} G(x, x) \right) G_{++}(x, x') \]
\[ + \hbar f^2 \int dx'' \sqrt{-g''} K(x, x'', x') = -i \delta(x - x') \frac{1}{\sqrt{-g}} , \]
(2.48)
in terms of a kernel \( K(x, x'') \) defined by
\[ K(x, x') = -i \text{tr}_{sp} [F_{++}(x, x') F_{++}(x', x) - F_{+-}(x, x') F_{-+}(x', x)] . \]
(2.49)
Making use of Eqs. (2.44b) and (2.45b), this kernel takes the form
\[ K(x, x') = \text{Re} \text{tr}_{sp} [F_R(x, x') F^{(1)}(x', x)] , \]
(2.50)
which shows that the gap equation (2.48) is manifestly real and causal. As will be shown below in Sec. (II) (in a perturbative limit), the kernel \( K(x, x') \) is dissipative, and it reflects back reaction from fermionic particle production induced by the time-dependence of the inflaton variance. The gap equation (2.48) is therefore damped for modes above threshold\[4\].

\[ ^4 \text{See } [\text{III}] \text{ for a similar discussion in the context of spinodal decomposition in quantum field theory.} \]
and this damping is not accounted for in the 1PI treatments of inflaton dynamics (where only the inflaton mean field is dynamical). In contrast to previous studies [21,33,28] which assumed a local equation of motion for the inflaton propagator, the two-loop gap equation obtained from the CTP-2PI-CGEA includes a nonlocal kernel, which is a generic feature of back reaction from particle production. As stressed above, the dissipative dynamics of the inflaton two-point function can be important when the inflaton variance is on the order of the square of the inflaton mean-field amplitude; such conditions may exist at the end of preheating.

The set of evolution equations (2.42) for $\hat{\phi}$ and (2.48) for $G$, is formally complete to two loops. Dissipation arises due to the coarse-graining of the spinor degrees of freedom. These dynamical equations are useful for general purposes, and are valid in a general background spacetime. However, in order to get explicit results, one needs to introduce approximations, as we now do.

III. DYNAMICS OF SMALL-AMPLITUDE INFLATON OSCILLATIONS

The effective evolution equations for the inflaton mean field $\hat{\phi}$ and variance $\langle \varphi^2 \rangle$ derived in the previous section are useful for studying fermion production when $\hat{\phi}_0$, the amplitude of the spatially homogeneous inflaton mean-field oscillations, is large, and the inflaton variance is of the same order-of-magnitude as $(\hat{\phi}_0)^2$. As discussed in Sec. II above, such conditions can prevail at the end of the preheating period in chaotic inflation with unbroken symmetry [3,1]. Because of the dissipative kernel $K(x,x')$ in the gap equation (2.48), which damps the evolution of $G$, and the back reaction terms in the mean-field equation, which damp the oscillations of $\hat{\phi}$, eventually the condition,

$$f\hat{\phi}_0 \ll m$$

will hold, at which point it is justifiable to follow the mean-field dynamics using the perturbative, 1PI, coarse-grained effective action [23]. Although in principle one should study this process in a general curved spacetime, for simplicity we assume spatial homogeneity, and that the inflaton mass is much greater than the Hubble constant, $m \gg H$. While this condition alone is in general not sufficient to ensure that curved spacetime effects are negligible during reheating (see, for example, [1], where cosmic expansion does affect preheating dynamics even though $m \gg H$), with the additional assumption of condition (3.1) it is reasonable to neglect the effect of cosmic expansion in the spinor propagators [34,21]. In this and the following section, we also neglect the self-coupling $\lambda$, because for the case of unbroken symmetry, the lowest-order $\lambda$-dependent contribution to the perturbative inflaton self-energy is $O(h^2)$ [28], and we are only concerned with one-loop dynamics in this section.

Let us therefore specialize to Minkowski space, and implement a perturbative expansion of the CTP effective action in powers of the mean field $\hat{\phi}$. This formally entails a solution of the gap equation (2.48) for $G$, a back-substitution of the solution into the CTP-2PI coarse-grained effective action, and a subsequent expansion of this expression (now a functional of $\hat{\phi}$ only) in powers of $\hat{\phi}$. The resulting perturbative expansion for the effective action contains only free-field propagators. For consistency, one should use an initial density matrix for the spinor degrees of freedom which corresponds to the end-state particle occupation numbers.
of the nonperturbative dynamics of Sec. I. For simplicity, however, we assume the initial quantum state for the spinor field is the vacuum state. Hereafter, $F_{ab}$ denotes the free-field, Minkowski-space, vacuum spinor CTP propagator, whose components are given by 12, 27, 13

$$F_{++}(x, x') = \int \frac{d^4 p}{(2\pi)^4} e^{-i p(x - x')} \frac{i(\not{p} + m)}{p^2 - \mu^2 + i\epsilon}, \quad (3.2a)$$

$$F_{--}(x, x') = -\int \frac{d^4 p}{(2\pi)^4} e^{-i p(x - x')} \frac{i(\not{p} + m)}{p^2 - \mu^2 - i\epsilon}, \quad (3.2b)$$

$$F_{-+}(x, x') = \int \frac{d^4 p}{(2\pi)^4} e^{-i p(x - x')} 2\pi(\not{p} + m)\delta(p^2 - \mu^2)\theta(p^0), \quad (3.2c)$$

$$F_{+-}(x, x') = \int \frac{d^4 p}{(2\pi)^4} e^{-i p(x - x')} 2\pi(\not{p} + m)\delta(p^2 - \mu^2)\theta(-p^0). \quad (3.2d)$$

The $++$ and $--$ propagators admit a representation in terms of a time-ordering function $\theta(x, y) = \theta(x^0 - y^0)$,

$$F_{++}(x, x') = \theta(x, x') F_{++}(x, x') + \theta(x', x) F_{+-}(x, x'), \quad (3.3a)$$

$$F_{--}(x, x') = \theta(x, x') F_{--}(x, x') + \theta(x', x) F_{-+}(x, x'). \quad (3.3b)$$

The CTP effective action can be expanded in powers of $f^2$, and we find

$$\Gamma[\hat{\phi}] = S^0[\hat{\phi}] - \frac{i\hbar}{2} \ln \det(\mathcal{A}^{ab})^{-1} - i\hbar \ln \det F_{ab} + \Gamma_1[\hat{\phi}], \quad (3.4)$$

where the kernel $\mathcal{A}$ is defined by

$$i\mathcal{A}^{ab}(x, x') = -e^{ab}(\Box x + m^2)\delta(x - x'), \quad (3.5)$$

and $\Gamma_1$ is defined as $-i\hbar$ times the sum of all one-particle-irreducible diagrams constructed with lines given by $\hbar \mathcal{A}^{-1}$ and $\hbar F_{ab}$, and vertices given by $S^Y[\hat{\phi}, \bar{\psi}, \psi]/\hbar$ and $S^Y[\varphi, \bar{\psi}, \psi]/\hbar$. Because the free-field propagators $\mathcal{A}^{-1}$ and $F_{ab}$ do not depend on $\phi$, the log(det) do not contribute to the variation of $\Gamma[\hat{\phi}]$ with respect to $\hat{\phi}$, and therefore, they can be dropped. The functional $\Gamma_1[\hat{\phi}]$ can be expanded in powers of $\hbar$,

$$\Gamma_1[\hat{\phi}] = \sum_{l=1}^{\infty} \hbar^l \Gamma^{(l)}[\hat{\phi}], \quad (3.6)$$

where the term $\Gamma^{(l)}[\hat{\phi}]$ is the sum of all 1PI $l$-loop graphs. Order by order in the loop expansion and the coupling constant, the CTP 1PI effective action must satisfy the unitarity condition

$$\Gamma_1|_{\hat{\phi} = \bar{\phi}}; \bar{\phi} = \bar{\phi} = 0, \quad (3.7)$$

which has been verified to two-loop order in the case of scalar $\lambda \phi^4$ field theory 14. The one-loop term in the loop expansion of the CTP effective action, $\Gamma^{(1)}[\hat{\phi}]$, can be further expanded in powers of $f^2$,
\[ \Gamma^{(1)}[\hat{\phi}] = \sum_{n=1}^{\infty} f^{2n} \Gamma^{(1)}_{2n}[\hat{\phi}], \]  

which corresponds to the usual amplitude expansion of the CTP effective action \[21\]. Figure 2 shows the diagrammatic expansion of \( \Gamma^{(1)} \). The terms \( \Gamma^{(1)}_{2n}[\hat{\phi}] \) are generally divergent, but since the theory is renormalizable in the standard “in-out” formulation, it is renormalizable in the closed-time-path, “in-in” formulation \[14,3\].

**A. One-loop perturbative effective action at \( O(f^2) \)**

The \( O(f^2) \) term in the expansion of the one-loop CTP effective action, which is the first term in Fig. 2, takes the form

\[ \Gamma^{(1)}_{2}[\hat{\phi}] = -\frac{i}{2} c^{abc} c^{d'c'} \int d^4x d^4x' \hat{\phi}_a(x) \hat{\phi}_{a'}(x') F_{bb'}(x,x') F_{cc'}(x',x). \]  

(3.9)

Making use of sum and difference variables

\[ \Sigma(x) = \frac{1}{2} [\hat{\phi}_+(x) + \hat{\phi}_-(x')], \]  

(3.10a)

\[ \Delta(x) = \hat{\phi}_+(x) - \hat{\phi}_-(x'), \]  

(3.10b)

the functional \( \Gamma^{(1)}_{2}[\hat{\phi}] \) can be recast in the form

\[ \Gamma^{(1)}_{2}[\hat{\phi}] = \int d^4x d^4x' \left[ \Sigma(x) \Delta(x') \mathcal{D}_2(x,x') + \frac{i}{2} \Delta(x') \Delta(x') \mathcal{N}_2(x,x') \right], \]  

(3.11)

in terms of manifestly real kernels \( \mathcal{D}_2(x,x') \) and \( \mathcal{N}_2(x,x') \) defined by

\[ \mathcal{D}_2(x,x') = \text{Im} \text{tr}_{sp} [F_{++}(x,x') F_{++}(x',x) + F_{--}(x,x') F_{--}(x',x)], \]  

(3.12a)

\[ \mathcal{N}_2(x,x') = -\text{Re} \text{tr}_{sp} [F_{++}(x,x') F_{++}(x',x)], \]  

(3.12b)

Only the kernel \( \mathcal{D}_2(x,x') \) contributes to the mean-field equation of motion. The kernel \( \mathcal{N}_2(x,x') \) constitutes a correlator for noise, and will be discussed in Sec. IV. The unitarity condition \( \text{(3.7)} \) requires that the sum of diagrams proportional to \( \Sigma(x) \Sigma(x') \) vanish identically. With the definitions of the retarded spinor propagator, Eq. \( \text{(2.44b)} \), and the spinor Hadamard kernel, Eq. \( \text{(2.45b)} \), which in Minkowski space take the form

\[ F_R(x,x') = i\theta(x,x') [F_{++}(x,x') - F_{--}(x,x')], \]  

(3.13a)

\[ F^{(1)}(x,x') = F_{++}(x,x') + F_{--}(x,x'), \]  

(3.13b)

the kernel \( \mathcal{D}_2(x,x') \) can be written in a manifestly causal form,

\[ ]

\[ ^5 \text{Note that there are no nonzero graphs with an odd number of vertices in this model.} \]
\[ D_2(x, x') = \frac{1}{2} \text{Re} \text{tr}_{sp}[F_R(x, x')F^{(1)}(x', x)]. \] (3.14)

Using Eq. (3.14), one can compute \( D_2(x, x') \) in an arbitrary curved background spacetime. It should be noted that \( D_2(x, x') \) is just the lowest-order term in the series expansion of \( \mathcal{K}(x, x') \) [defined in Eq. (2.50)] in powers of the coupling constant \( f \). The appearance of the retarded propagator in Eq. (3.14) guarantees that the contribution of \( \Gamma_2^{(1)} \) to the mean-field equation of motion is causal.

Let us now evaluate \( D_2(x, x') \) using dimensional regularization and the modified minimal subtraction (\( \text{MS} \)) renormalization prescription [45]. Dimensional regularization requires changing the coupling constant so that the interaction \( S \) has the correct dimensions in \( n \) spacetime dimensions,

\[ f \rightarrow f \Lambda^{(4-n)/2}, \] (3.15)

where we have introduced a parameter \( \Lambda \), the renormalization scale, which has dimensions of mass. By Lorentz invariance and causality, the product of Feynman propagators can be written in terms of an amplitude \( A_2 \) [45],

\[ \text{tr}_{sp}[F_{++}(x, x')F_{++}(x', x)] = i \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-x')} A_2(k^2 + i\epsilon), \] (3.16)

and with this choice of renormalization prescription, the amplitude \( A_2(k^2) \) takes the form

\[ A_2(k^2) = -\frac{3}{4\pi^2} \int_0^1 d\alpha E(\alpha; k^2) \log \left( \frac{E(\alpha; k^2)}{\Lambda^2} \right), \] (3.17)

where \( E(\alpha; k^2) \) is defined by

\[ E(\alpha; k^2) = \mu^2 - \alpha(1 - \alpha)k^2. \] (3.18)

Note that in Eq. (3.17), the \( \alpha \) integration shows up via the Feynman identity

\[ \frac{1}{C_1 \cdots C_N} = (N-1)! \int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_N \delta(\alpha_1 + \cdots + \alpha_{N-1})(\alpha_1 C_1 + \cdots + \alpha_N C_N)^{-N}. \] (3.19)

The \( i\epsilon \) appearing in Eq. (3.19) ensures that the amplitude \( A_2 \) is evaluated on the physical sheet [43,45]. The logarithm in Eq. (3.17) has a negative real argument when the two conditions \( k^2 > 4\mu^2 \) and \( |2\alpha - 1| < \sqrt{1 - 4\mu^2/k^2} \) are both satisfied. When \( |2\alpha - 1| < \sqrt{1 - 4\mu^2/k^2} \), the amplitude \( A_2(k^2) \) has a branch cut (considered as an analytically continued function of \( k^0 \)) for \( (k^0)^2 > \vec{k}^2 + 4\mu^2 \). The discontinuity across the branch cut is related to the “cut” version of the diagram [the second term in Eq. (3.12a)] via the Cutkosky rules [47,46,48–51],

\[ ^6 \text{The notation } E(\alpha; k^2) \text{ used here should not be confused with } E(k), \text{ the complete elliptic integral of second kind.} \]
\[ \text{tr}_{sp}[F_{+-}(x, x')F_{--}(x', x)] = -i \int \frac{d^4 k}{(2\pi)^4} e^{-ik(x-x')} \text{Disc}[A_2(k^2)] \theta(k^0). \] (3.20)

From Eqs. (3.16), (3.17), and (3.20), it is straightforward to obtain an expression for the dissipation kernel,

\[ \mathcal{D}_2(x, x') = \frac{3}{4\pi^2} \int \frac{d^4 k}{(2\pi)^4} e^{-ik(x-x')} \int_0^1 d\alpha E(\alpha; k^2) \left[ \log \left( \frac{|E(\alpha; k^2)|}{\Lambda^2} \right) - i\pi \theta[-E(\alpha; k^2)] \text{sgn}(k^0) \right], \] (3.21)

where we have now taken the limit \( \epsilon \to 0_+ \). One can verify by inspection that this kernel is real. However, the second term in Eq. (3.21) breaks time-reversal invariance and leads to dissipative mean field dynamics. The one-loop Fourier-transformed mean-field equation is (dropping the caret from \( \hat{d} \))

\[ \mathcal{D}_2(x, x') = \frac{3\hbar f^2}{4\pi^2} \int_0^1 d\alpha E(\alpha; k^2) \left[ \log \left( \frac{|E(\alpha; k^2)|}{\Lambda^2} \right) - i\pi \theta[-E(\alpha; k^2)] \text{sgn}(k^0) \right]. \] (3.22)

where \( \tilde{\gamma}_2(k) \) is the dissipation function, defined as \(-i\hbar f^2/k^0 \) times the Fourier transform of the second term in Eq. (3.21),

\[ \tilde{\gamma}_2(k) = \frac{\hbar f^2}{k^0} \text{Im} [\bar{\mathcal{D}}_2(k)] = \frac{\hbar f^2}{8\pi} \frac{k^2}{|k^0|} \left( 1 - \frac{4\mu^2}{k^2} \right)^{3/2} \theta(k^2 - 4\mu^2). \] (3.23)

The one-loop \( O(f^2) \) dissipation kernel agrees with previous calculations of the probability to produce a fermion particle pair of momentum \( k \) \([16, 22, 49, 52]\). In Eq. (3.22), \( \tilde{J}(k) \) is an external \( \epsilon \)-number source. The imaginary term \( i\hbar f^2/k^0 \) in Eq. (3.22) breaks time-reversal invariance and acts as a \( k \)-dependent dissipative force in the mean field equation. The \( \theta \) function enforces the energy threshold for the virtual fermion pair in the one-loop \( O(f^2) \) diagram to go on-shell. The dissipative mean-field equation (3.22) is essentially the linear-response approximation to the effective inflaton dynamics. It should be noted that the dissipation kernel \( \mathcal{D}_2(x, x') \) is nonlocal, in contrast with the local friction-type dissipation assumed in earlier studies of post-inflation reheating \([53]\). However, in the limit \( \mu^2 \to 0 \), the dissipation kernel does become time-local, as there is no longer a length scale in the expression for \( \mathcal{D}_2(x, x') \) which could define a time scale for nonlocal dissipation \([53]\).

### B. One-loop perturbative effective action at \( O(f^4) \)

The \( O(f^4) \) term in the one-loop CTP effective action consists of the “square” diagram, which is the second term in Fig. 2.

\[ \Gamma_4^{(1)}[\phi] = \frac{i}{4} e^{abc} e^{d' e' f'} e^{e f} \int d^4 x d^4 x' d^4 y d^4 y' \left[ \text{tr}_{sp}[F_{b\nu}(x, x') F_{c\epsilon}(x', y') F_{e\epsilon}(y', y) F_{f\epsilon}(y, x)] \right] \times \hat{\phi}_a(x) \hat{\phi}_{a'}(x') \hat{\phi}_d(y) \hat{\phi}_{d'}(y'). \] (3.24)
Expanding out the contracted CTP indices, we obtain

\[ \Gamma^{(1)}_4 [\phi] = \frac{i}{4} \int d^4 x d^4 x' d^4 y d^4 y' \left[ \hat{\phi}^+_4(x) \hat{\phi}^+_4(x') \hat{\phi}^-_4(y) \hat{\phi}^-_4(y') \text{tr} \{ F_{++}(x, x') F_{++}(x', y') F_{++}(y', y) F_{++}(y, x) \} 
+ \hat{\phi}^-_4(x) \hat{\phi}^-_4(x') \hat{\phi}^+_4(y) \hat{\phi}^+_4(y') \text{tr} \{ F_{--}(x, x') F_{--}(x', y') F_{--}(y', y) F_{--}(y, x) \} 
- 4 \hat{\phi}^+_4(x) \hat{\phi}^+_4(x') \hat{\phi}^-_4(y) \hat{\phi}^-_4(y') \text{tr} \{ F_{--}(x, x') F_{--}(x', y') F_{++}(y', y) F_{--}(y, x) \} 
- 4 \hat{\phi}^-_4(x) \hat{\phi}^-_4(x') \hat{\phi}^+_4(y) \hat{\phi}^+_4(y') \text{tr} \{ F_{++}(x, x') F_{++}(x', y') F_{--}(y', y) F_{++}(y, x) \} 
+ 2 \hat{\phi}^+_4(x) \hat{\phi}^+_4(x') \hat{\phi}^-_4(y) \hat{\phi}^-_4(y') \text{tr} \{ F_{--}(x, x') F_{--}(x', y') F_{--}(y', y) F_{--}(y, x) \} \right]. \tag{3.25} \]

When \( \Gamma^{(1)}_4 \) is expressed in terms of \( \Delta \) and \( \Sigma \) [defined in Eqs. (3.10a) and (3.10b)], only those terms with one factor of \( \Delta \) and three factors of \( \Sigma \) contribute to the mean field equation of motion. As a consequence of the unitarity condition (3.7), the sum of terms proportional to four factors of \( \Sigma \) must vanish. Keeping only those terms in the effective action which contribute to the mean field equation or are quadratic in \( \Delta \), we find

\[ \Gamma^{(1)}_4 [\phi] = \int d^4 x d^4 x' d^4 y d^4 y' \left[ \Delta(x) \Sigma(x') \Sigma(y') \Sigma(y) \mathcal{D}_4(x, x', y, y) \right. \quad \left. + \frac{i}{2} \Delta(x) \Delta(x') \Sigma(y') \Sigma(y) \mathcal{N}_4(x, x', y, y) \right], \tag{3.26} \]

in terms of a kernel \( \mathcal{D}_4(x, x', y, y) \) defined by

\[ \mathcal{D}_4(x, x', y, y) = - \text{Im} \text{tr}_p \left[ F_{++}(x, x') F_{++}(x', y') F_{++}(y', y) F_{++}(y, x) 
+ F_{++}(x, x') F_{--}(x', y') F_{--}(y', y) F_{++}(y, x) 
+ F_{--}(x, x') F_{--}(x', y') F_{++}(y', y) F_{++}(y, x) 
- F_{--}(x, x') F_{--}(x', y') F_{--}(y', y) F_{--}(y, x) 
- F_{++}(x, x') F_{++}(x', y') F_{--}(y', y) F_{--}(y, x) 
- F_{++}(x, x') F_{++}(x', y') F_{++}(y', y) F_{--}(y, x) \right], \tag{3.27} \]

and a “noise” kernel \( \mathcal{N}_4(x, x', y, y) \) defined by

\[ \mathcal{N}_4(x, x', y, y) = \text{Re} \text{tr}_p \left[ F_{++}(x, x') F_{++}(x', y') F_{++}(y', y) F_{++}(y, x) 
+ F_{++}(x, x') F_{--}(x', y') F_{--}(y', y) F_{--}(y, x) 
- F_{--}(x, x') F_{--}(x', y') F_{++}(y', y) F_{++}(y, x) 
- F_{--}(x, x') F_{--}(x', y') F_{--}(y', y) F_{--}(y, x) \right. \quad \left. + F_{++}(x, x') F_{++}(x', y') F_{++}(y', y) F_{++}(y, x) 
- F_{++}(x, x') F_{++}(x', y') F_{--}(y', y) F_{--}(y, x) \right]. \]
The term is logarithmically divergent, and as in Sec. III A, we use dimensional continuation over to a stochastic equation for $\hat{\phi}$, which will be studied in an upcoming paper [54].

The noise kernel $N_{\Delta}$ does not contribute to the mean field equation of motion. There are, of course, terms in $\Gamma^{(1)}$ which are higher order in $\Delta$, for example, $O(\Delta^4)$, but in passing over to a stochastic equation for $\hat{\phi}$ in Sec. [V], we will be assuming that $\Delta$ is small, so that higher-order terms in powers of $\Delta$ can be ignored. Such terms will in general contribute to non-Gaussian noise, which will be studied in an upcoming paper [54].

Let us evaluate the first term of Eq. (3.27), which consists of only Feynman propagators. The term is logarithmically divergent, and as in Sec. [III A], we use dimensional continuation and the modified minimal subtraction (MS) renormalization scheme. Because we are only interested in deriving the dissipative terms in the mean-field equation coming from this diagram, and because we are assuming $m \gg \mu$, we include only the one-loop logarithm. We find

$$\text{tr}_{sp} \left[ F_{++}(x, x') F_{++}(x', y') F_{++}(y', y) F_{++}(y, x) \right]_{\log} = i \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{d^4 k_3}{(2\pi)^4} e^{-i(k_1+k_2+k_3)3-\kappa_1.x + \kappa_2.x' + \kappa_3.y} A_4(k_1, k_2, k_3),$$

where the amplitude $A_4(k_1, k_2, k_3)$ is defined by

$$A_4(k_1, k_2, k_3) = -\frac{3}{2\pi^2} \int d\alpha_1 d\alpha_2 d\alpha_3 \log \left[ \frac{E_4(\alpha_1, \alpha_2, \alpha_3; k_1, k_2, k_3)}{\Lambda^2} \right],$$

in terms of a function $E_4$ defined by

$$E_4(\alpha_1, \alpha_2, \alpha_3; k_1, k_2, k_3) = \left[ (1-\alpha_1)k_1 + (1-\alpha_1-\alpha_2)k_2 + (1-\alpha_1-\alpha_2-\alpha_3)k_3 \right]^2$$

$$- (1-\alpha_1)k_1^2 - (1-\alpha_1-\alpha_2)(k_2^2 + 2k_1 \cdot k_2)$$

$$- (1-\alpha_1-\alpha_2-\alpha_3)(2k_1 \cdot k_3 + 2k_2 \cdot k_3 + k_3^2) + \mu^2.$$
in terms of a function $\mathcal{D}_4(k_1, k_2, k_3)$ defined by

$$
\mathcal{D}_4(x, x', y, y) = \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} e^{-i[(k_1 + k_2 + k_3) \cdot x + (k_1 + k_2 + k_3) \cdot y]} \mathcal{D}_4(k_1, k_2, k_3),
$$

(3.32)

and the functions $h(s)$ and $H(k_1, k_2, k_3)$ are defined by

$$
h(s) = \sqrt{1 - \frac{4\mu^2}{s} \theta(s - 4\mu^2)},
$$

(3.34a)

$$
H(k_1, k_2, k_3) = \int_{\alpha_1, \alpha_2, \alpha_3 > 0} \{ \theta[-E_4(\alpha_1, \alpha_2, \alpha_3; k_1, k_2, k_3)]
\times [\theta(k_1^0)\theta(-k_2^0)\theta(k_3^0) - \theta(-k_1^0)\theta(k_2^0)\theta(-k_3^0)] \}.
$$

(3.34b)

Equation (3.33) leads to the following mean-field equation at $O(f^4)$,

$$
\begin{align*}
& \left[ k^2 - m^2 + ik^0 \tilde{\gamma}_2(k) - \frac{3\hbar f^2}{4\pi^2} \int_0^1 d\alpha E(\alpha; k^2) \log \left( \frac{|E(\alpha; k^2)|}{\Lambda^2} \right) \right] \tilde{\phi}(k) \\
- & \frac{3\hbar f^4}{2\pi} \int \frac{d^4q}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \tilde{\phi}(k - q - l) \tilde{\phi}(q) \tilde{\phi}(l) \left[ \frac{i}{\pi} \int d\alpha_1 d\alpha_2 d\alpha_3 \log \left( \frac{|E_4(l + q - k, -q, -l)|}{\Lambda^2} \right) \right]
\times \sgn(q^0 + l^0)h[(q + l)^2] + \sgn(k^0)h(q^2) \\
+ & \sgn(k^0 - l^0)h[(k - l)^2] + \sgn(q^0)h(k^2) \\
+ & \sgn(k^0 - q^0 - l^0)h(l^2) \\
+ & \sgn(l^0)h[(k - q - l)^2] \\
+ & H(l + q - k, -q, -l)
\right] = -\tilde{J}(k).
\end{align*}
$$

(3.35)

The presence of terms of the form $i\sgn(p^0)h(p^2)$ in Eq. (3.33) clearly signifies dissipative dynamics. The $\theta$-function in Eq. (3.34a) enforces the energy threshold for the virtual fermion quanta created at a particular vertex to go on-shell. Comparing Eq. (3.33) and Eq. (3.22), and assuming spatial homogeneity, we see that the $O(f^4)$ dissipation kernel must be taken into account whenever the condition (3.1) fails to hold for the solution $\phi(t)$ to Eq. (3.22).

At the end of the regime of parametric resonance in chaotic inflaton, i.e. the “preheating” regime, the inflaton mean field may oscillate with an amplitude as large as $\sim m/g_{\phi\chi}$, where $g_{\phi\chi}$ is the coupling to another scalar field $\chi$, typically on the order of $10^{-4}$ [4]. Condition (3.1) would then be violated if $f > g_{\phi\chi}$. In this case it would be necessary, at a minimum, to take into account higher order terms (such as $\mathcal{D}_4$) in the mean-field equation, until such time as the amplitude $\phi_0(t)$ has decreased to the point where Eq. (3.1) is satisfied.
IV. NOISE KERNEL AND STOCHASTIC INFATON DYNAMICS

Although the kernels $N_2(x,x')$ and $N_4(x,x',y,y)$ do not contribute to the mean field equation, i.e., the equation of motion for $\hat{\phi}$, they contain information about stochasticity in a quasi-classical description of the effective dynamics of the inflaton field [29,31,30,32,58,13]. In this section, we study the effect of stochasticity on the dynamics of the inflaton mean field, within the perturbative framework established above.

A. Langevin equation and fluctuation-dissipation relation at $O(f^2)$

In this section we show how to obtain a classical stochastic equation for the inflaton field from the $O(f^2)$ perturbative CTP effective action. From Eq. (3.11), it follows that the $O(f^2)$ one-loop perturbative CTP effective action has the form

$$\Gamma[\hat{\phi}] = S[\hat{\phi}] + \int d^4xd^4x' \left[ \Sigma(x)\Delta(x')\mu_2(x,x') + \frac{i}{2}\Delta(x)\Delta(x')\nu_2(x,x') \right],$$  \hspace{1cm} (4.1)

where for simplicity we have defined

$$\nu_2(x,x') = \hbar f^2 N_2(x,x'),$$ \hspace{1cm} (4.2a)
$$\mu_2(x,x') = \hbar f^2 D_2(x,x').$$ \hspace{1cm} (4.2b)

In order to extract the stochastic noise arising from the kernel $N_2(x,x')$, we use the Gaussian identity [59]

$$\exp\left[ -\frac{1}{\hbar} \int d^4xd^4x'\Delta(x)\Delta(x')\nu_2(x,x') \right] = N \int D\xi_2 \exp\left[ -\frac{1}{2\hbar} \int d^4xd^4x'\xi_2(x)\nu_2^{-1}(x,x')\xi_2(x') \right. \\
\hspace{4cm} \left. + \frac{i}{\hbar} \int d^4x\xi_2(x)\Delta(x) \right],$$  \hspace{1cm} (4.3)

where $N$ is a normalization factor which does not depend on $\Delta$, and $\xi_2$ is a $c$-number functional integration variable. Following [73], we now define a functional

$$P[\xi_2] = N \exp\left[ -\frac{1}{\hbar} \int d^4xd^4x'\xi_2(x)\nu_2^{-1}(x,x')\xi_2(x') \right],$$  \hspace{1cm} (4.4)

and it follows from Eq. (4.3) that $P[\xi_2]$ is normalized in the sense of

$$\int D\xi_2 P[\xi_2] = 1.$$  \hspace{1cm} (4.5)

Using Eq. (4.3), we can rewrite the $O(f^2)$ one-loop CTP effective action, Eq. (4.1), as

$$\Gamma[\hat{\phi}] = -i\hbar \log \int D\xi_2 P[\xi_2] \exp\left[ \frac{i}{\hbar} \left( S[\hat{\phi}] + \int d^4xd^4x'\Sigma(x)\Delta(x')\mu_2(x,x') + \int d^4x\xi_2(x)\Delta(x) \right) \right].$$  \hspace{1cm} (4.6)
This suggests defining a new effective action which depends on both $\xi_2$ and $\hat{\phi}_x$ (dropping the carat from $\hat{\phi}$),

$$\Gamma[\phi, \xi_2] = S^0[\phi] + \int d^4x d^4x' \Sigma(x) \Delta(x') \mu_2(x, x') + \int d^4x \xi_2(x) \Delta(x). \quad (4.7)$$

Let us define a type of ensemble average

$$\langle\langle A \rangle\rangle = \int D\xi_2 P[\xi_2] A(\xi_2), \quad (4.8)$$

and note that Eqs. (4.4) and (4.8) imply that

$$\langle\langle \xi_2(x) \rangle\rangle = 0, \quad (4.9a)$$

$$\langle\langle \xi_2(x) \xi_2(x') \rangle\rangle = \hbar \nu_2(x, x'). \quad (4.9b)$$

Clearly then,

$$\left. \left( \frac{\delta}{\delta \phi_+} \langle\langle \Gamma[\phi, \xi_2] \rangle\rangle \right) \right|_{\phi_+ = \phi_- = \phi} = \left. \left( \frac{\delta}{\delta \phi_+} \Gamma[\phi] \right) \right|_{\phi_+ = \phi_- = \phi}. \quad (4.10)$$

Taking the variation of $\Gamma[\phi, \xi_2]$ with respect to $\phi_+$ and setting $\phi_+ = \phi_- = \phi$, we obtain (after a Fourier transform)

$$\left[ k^2 - m^2 + i k^0 \tilde{\gamma}_2(k) - \frac{3hf^2}{4\pi^2} \int_0^1 d\alpha E(\alpha; k^2) \log \left( \left| E(\alpha; k^2) \right| \right) \right] \phi(k) + \tilde{\xi}_2(k) = -\tilde{J}(k), \quad (4.11)$$

where $\tilde{\xi}_2(k)$ is defined by

$$\tilde{\xi}_2(k) = \int d^4x e^{ikx} \xi_2(x). \quad (4.12)$$

We now interpret Eq. (4.11) as a Langevin equation with stochastic force $\xi_2$. The inflaton Fourier mode $\hat{\phi}$ appearing in Eq. (4.11) should be viewed as a $c$-number stochastic variable, and the presence of the stochastic force $\xi_2$ indicates spontaneous breaking of spatial translation invariance by a Gaussian (but not white) noise source $\xi_2$. Moreover, this stochastic equation obeys a zero-temperature fluctuation-dissipation relation, as we now show. First, let us calculate the one-loop $O(f^2)$ noise kernel, $N_2(x, x')$ [defined in Eq. (3.12b) above], using dimensional regularization and modified minimal subtraction,

$$\nu_2(x, x') = \frac{hf^2}{8\pi} \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-x')} \tilde{\nu}_2(k), \quad (4.13)$$

in terms of the Fourier-transformed noise kernel

$$\tilde{\nu}_2(k) = \frac{hf^2}{8\pi} k^2 \left( 1 - \frac{4\mu^2}{k^2} \right)^{3/2} \theta(k^2 - 4\mu^2). \quad (4.14)$$

The noise kernel $\nu_2(x, x')$ is colored; colored noise has been observed in other interacting field theories [31,32,55]. By inspection of Eqs. (3.23) and (4.14), it follows that
\[ |k^0|\tilde{\gamma}(k) = \tilde{\nu}_2(k), \]  

which leads to the zero-temperature fluctuation-dissipation relation \[55\],

\[ \nu_2(t, \vec{k}) = \int_{-\infty}^{\infty} dt' K(t - t')\gamma_2(t', \vec{k}), \]  

in terms of the distribution-valued kernel \( K(t) \) defined by

\[ K(t) = \int_0^\infty \frac{d\omega}{\pi} \omega \cos(\omega t), \]  

and the spatially Fourier-transformed dissipation function and noise kernel,

\[ \nu_2(t, \vec{k}) = \int_{-\infty}^{\infty} \frac{dk_0}{2\pi e^{-ik_0t}\tilde{\nu}_2(k)}, \] \( 4.18a \)

\[ \gamma_2(t, \vec{k}) = \int_{-\infty}^{\infty} \frac{dk_0}{2\pi e^{-ik_0t}\tilde{\gamma}_2(k)}. \] \( 4.18b \)

This shows the physical significance of the noise kernel \( \nu_2(x, x') \) in an effective description of the dynamics of the scalar mean field.

**B. Langevin equation and fluctuation-dissipation relation at \( O(f^4) \)**

In this section we consider the \( O(f^4) \) one-loop noise kernel, \( N_4 \). The non-normal-threshold singularities in \( A_4 \) lead to a noise kernel which depends on \( \Sigma \), which is known to lead to ambiguities in the resulting Langevin equation \[50,51\]. The meaning and interpretation of the non-normal-threshold parts of \( N_4 \) and \( D_4 \) will be the subject of a future study \[53\]. Here, we focus on the effect of the normal-threshold singularities of \( A_4 \), which for the noise kernel, \( N_4 \), contribute a term

\[ \frac{i}{2} \int d^4xd^4x' \Delta(x)\Delta(x')\Sigma(x)\Sigma(x')\nu_4(x, x') \]  

(4.19)

to the CTP effective action, where the kernel \( \nu_4(x, x') \) is defined by

\[ \nu_4(x, x') = -\frac{3hf^4}{\pi} \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-x')} h(q^2), \]  

(4.20)

and the function \( h(s) \) was defined in Eq. (3.34a) above. The normal-threshold singularities of the dissipation kernel, \( D_4 \), [the second and third terms of Eq. (3.33)], lead to the following contribution to the CTP effective action,

\[ \int d^4xd^4x' \Delta(x)\Sigma(x) [\Sigma(x')]^2 \mu_4(x, x'), \]  

(4.21)

where the kernel \( \mu_4(x, x') \) is defined by
\[ \mu_4(x, x') = -\frac{3i\hbar f^4}{\pi} \int \frac{d^4q}{(2\pi)^4} e^{-iq(x'-x)} \text{sgn}(q^0)h(q^2). \] (4.22)

With the definitions
\[ \mu_4(x, x') = i \int \frac{d^4q}{(2\pi)^4} e^{-iq(x'-x)} q^0 \tilde{\gamma}_4(q), \] (4.23a)
\[ \nu_4(x, x') = \int \frac{d^4q}{(2\pi)^4} e^{-iq(x'-x)} \tilde{\nu}_4(q), \] (4.23b)

it follows immediately that the normal-threshold parts of \( D_4 \) and \( N_4 \) obey a fluctuation-dissipation relation identical in form to Eq. (1.13),
\[ |q^0| \tilde{\gamma}_4(q) = \tilde{\nu}_4(q). \] (4.24)

Making use of Eq. (1.3), the \( O(f^4) \) effective action (including only normal-threshold contributions) can be written in the form
\[ \Gamma[\phi, \xi_2, \xi_4] = S^\phi[\phi] + \int d^4xd^4x' \Delta(x) \Sigma(x') \mu_2(x, x') + \int d^4x \xi_2(x) \Delta(x) + \int d^4x \xi_4(x) \Delta(x) \Sigma(x), \] (4.25)

where the stochastic noise source \( \xi_4 \) satisfies the conditions
\[ \langle \langle \xi_4(x) \rangle \rangle = 0 \] (4.26a)
\[ \langle \langle \xi_4(x) \xi_4(x') \rangle \rangle = h \nu_4(x, x'). \] (4.26b)

Taking the functional derivative of Eq. (4.25) and making the usual identification, we obtain a Langevin equation with an additive noise \( \xi_2 \) and a multiplicative noise \( \xi_4 \),
\begin{align*}
\left[ q^2 - m^2 + i q^0 \gamma_2(q) - \frac{3h f^2}{4\pi^2} \int_0^1 d\alpha E(\alpha; q^2) \log \left( \frac{|E(\alpha; q^2)|}{\Lambda^2} \right) \right] \tilde{\phi}(q) \\
+ \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \tilde{\phi}(q-l) \tilde{\phi}(l-k) \left[ il^0 \gamma_4(l) - \frac{3h f^4}{2\pi^2} \int_0^1 d^4\alpha \log \left( \frac{|E_4(\alpha_1, \alpha_2, \alpha_3; l-q, -k, l-k)|}{\Lambda^2} \right) \right]
\end{align*}
\[ = -\tilde{\xi}_2(q) - \tilde{J}(q) - \int \frac{d^4k}{(2\pi)^4} \tilde{\xi}_4(q-k) \tilde{\phi}(k), \] (4.27)

where \( d^4\alpha = d\alpha_1 d\alpha_2 d\alpha_3 \). The stochastic force \( \xi_4 \) is clearly seen to contribute multiplicatively to the Langevin equation for \( \tilde{\phi} \).

**C. Homogeneous mean field dynamics at \( O(f^2) \)**

To make connection with post-inflationary reheating, it is customary to assume that the mean field \( \phi \) is spatially homogeneous \[62,1123\]. In this case, the Langevin equation (4.11) takes the form
\[ [\omega^2 - m^2 + i\omega\beta(\omega) + \eta(\omega)] \tilde{\phi}(\omega) + \tilde{\xi}_2(\omega) = -\tilde{J}(\omega), \]  
(4.28)

where we have defined
\[ \beta(\omega) = \frac{\hbar f^2}{8\pi} \omega^2 \left( 1 - \frac{4\mu^2}{\omega^2} \right)^{3/2} \theta(\omega^2 - 4\mu^2), \]  
(4.29a)
\[ \eta(\omega) = -\frac{3\hbar f^2}{4\pi^2} \int_0^1 d\alpha E(\alpha; \omega^2) \log \left( \frac{|E(\alpha; \omega^2)|}{\Lambda^2} \right). \]  
(4.29b)

The total energy dissipated to the fermion field over the history of the dynamical evolution of the mean field is given [at \( O(f^2) \)] by
\[ E = -\int_{-\infty}^\infty dt F_v(t) \frac{d\phi(t)}{dt}, \]  
(4.30)

where the friction force \( F_v(t) \) is the Fourier transform of \( i\beta(\omega)\omega \tilde{\phi}(\omega) \). After a bit of Fourier algebra, we obtain an expression for the ensemble-averaged, total dissipated energy,
\[ \langle \langle E \rangle \rangle = \hbar f^2 \frac{1}{8\pi^2} \int_{2\mu}^{\infty} d\omega \omega^3 \left( 1 - \frac{4\mu^2}{\omega^2} \right)^{3/2} \frac{|\tilde{J}(\omega)|^2}{[\omega^2 - m^2 + \eta(\omega)]^2 + \omega^2 \beta(\omega)^2}. \]  
(4.31)

It is straightforward to compute the variance in the total dissipated energy. Making use of Eq. (4.9b), we find
\[ ||\langle \langle E^2 \rangle \rangle - \langle \langle E \rangle \rangle^2|| = \frac{\hbar^4 f^6}{256\pi^6} \int_{2\mu}^{\infty} d\omega \omega^6 I(\omega) \left( 1 - \frac{4\mu^2}{\omega^2} \right)^3 \frac{|\tilde{J}(\omega)|^2}{[\omega^2 - m^2 + \eta(\omega)]^2 + \omega^2 \beta(\omega)^2}. \]  
(4.32)

where the function \( I(\omega) \) is defined by
\[ I(\omega) = \int_0^\sqrt{\omega^2 - 4\mu^2} dk k^2 (\omega^2 - k^2) \left( 1 - \frac{4\mu^2}{\omega^2 - k^2} \right)^{3/2}. \]  
(4.33)

Following [10], we assume that the inflaton field is held fixed via an external, constant c-number source \( J \) for \( t < 0 \), and that the source is removed for \( t \geq 0 \),
\[ J(t) = J \theta(-t). \]  
(4.34)

Setting \( \hbar = 1 \), assuming that \( m \gg \mu \), and expanding to lowest order in \( f \), we obtain for the ensemble averaged dissipated energy,
\[ \langle \langle E \rangle \rangle = \frac{f^2 J^2}{16\pi^2 m^2}. \]  
(4.35)

Let us now compute the variance in the total dissipated energy. Performing a regularization via dimensional continuation, we obtain
where $\delta$ is a constant of order unity defined by $
abla(x) = |119/60 - \gamma_{EM} - \log(4\pi m^2/\Lambda^2)|$, and $\gamma_{EM}$ is the Euler-Mascheroni constant, $\approx 0.5772$. Taking the ratio of the square root of Eq. (4.36) and Eq. (4.35), we obtain the relative strength of the rms fluctuations in the total dissipated energy density, $E_{\text{rms}}$,

$$
\frac{E_{\text{rms}}}{\langle\langle E\rangle\rangle} \equiv \frac{\sqrt{|\langle\langle E^2\rangle\rangle - \langle\langle E\rangle\rangle^2|}}{\langle\langle E\rangle\rangle} = \frac{2f m^3 \delta}{\sqrt{15\pi} J}.
$$

The parameter $J$ is related to the initial inflaton amplitude $\hat{\phi}_0(t_0)$ by $J = \hat{\phi}_0(t_0)m^2/2$, which leads to

$$
\frac{E_{\text{rms}}}{\langle\langle E\rangle\rangle} = \frac{4f m \delta}{\sqrt{15\pi} \hat{\phi}_0(t_0)} \approx 0.390 \frac{mf}{\hat{\phi}_0(t_0)}.
$$

The fundamental assumption which justified the perturbative expansion in $f$, Eq. (3.1), is seen to be independent of Eq. (4.38). Therefore, the ratio $E_{\text{rms}}/E$ is not required to be small by consistency with perturbation theory. As the initial inflaton amplitude $\hat{\phi}_0$ is made larger, the relative strength of the rms fluctuations of $E$ is seen to decrease, in accordance with the correspondence principle. It has been shown that the fluctuations in the total dissipated energy density are related to the fluctuations in the occupation numbers of modes [57].

Let us now examine whether the rms fluctuations in the total dissipated energy, as given by Eq. (4.38), is significant, given a reasonable value for the inflaton amplitude at the end of the preheating regime (the period of parametric resonance-induced particle production). In chaotic inflation with a scalar field $\chi$ coupled to the inflaton field via a coupling constant $g_{\phi\chi}$, the typical inflaton amplitude at the end of the preheating regime is on the order of $m/g_{\phi\chi}$ [4]. In this case, we would find $E_{\text{rms}}/\langle\langle E\rangle\rangle \approx f g_{\phi\chi}$, from which it is clear that fluctuations in the total dissipated energy are not significant relative to the ensemble-averaged total dissipated energy, and therefore should not appreciably affect the reheating temperature. However, in new inflation scenarios where the inflaton amplitude $\hat{\phi}_0$ can be on the order of $m$ at the onset of reheating, the ratio $m/\hat{\phi}_0$ can be of order unity [53]. In this case, the ratio $E_{\text{rms}}/\langle\langle E\rangle\rangle \approx f$, which may not be a negligible effect.

Although as shown above, stochasticity does not dramatically affect the total energy dissipated via fermion production in chaotic inflation, we may inquire whether the noise term in the Langevin equation for the inflaton zero-mode, Eq. (4.28), may nonetheless be non-negligible during the reheating period. Let us compute the rms fluctuations in the inflaton zero-mode, $\hat{\phi}_{\text{rms}}$. Following methods described above, we find that the rms fluctuations of the inflaton zero mode are given, to $O(f^2)$, by

$$
\hat{\phi}_{\text{rms}} = \frac{f}{\pi} \frac{m}{\sqrt{60\pi} \sigma},
$$

where $\sigma^2 = |61/30 - \gamma_{EM} - \log(4\pi m^2/\Lambda^2)|$. Equation (4.39) is seen to be independent of the inflaton zero-mode amplitude $\hat{\phi}_0$.  

28
In order to determine the relative importance of fluctuations in the inflaton zero-mode amplitude \( \hat{\phi}_0 \) during and at the end of the reheating period, we must introduce curved spacetime arguments. This is because the end of the reheating period is determined by the time \( t_{\text{end}} \) at which the Hubble constant becomes of the order of \( 3\beta(m) \) for the case of \( \lambda = 0 \) being discussed in this section \(^{23}\). Starting with the semiclassical Einstein equation \((2.40)\) for spatially flat Friedmann-Robertson-Walker (FRW) cosmology, setting \( b = c = \Lambda_c = 0 \) (following arguments similar to those of Sec. III D of Ref. \(^1\)), retaining only the inflaton zero mode as the dynamical degree of freedom (for consistency with FRW), and retaining both the \( \psi \) field energy density \( \rho_\psi \) and the classical, stochastic energy density of the inflaton zero mode, we have

\[
H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi}{3M_p^2} \left( \rho(\hat{\phi}^2) + \rho_\psi \right),
\]

where \( a \) is the scale factor, the dot denotes a derivative with respect to cosmic time, and \( \rho(\hat{\phi}^2) \) is the energy density as a function of the time-average (over one period of oscillation) of \( \hat{\phi}^2 \), which is given by the virial theorem,

\[
\rho(\hat{\phi}) \simeq m^2 \hat{\phi}^2 = \frac{1}{2}m^2(\hat{\phi}_0)^2.
\]

(4.41)

Making use of Eqs. (4.28), (4.34), and (4.40), we obtain an approximate expression for the (ensemble-averaged) energy density of the inflaton zero-mode at the end of the reheating period \(^7\) (to lowest order in \( f \)),

\[
\rho(t_{\text{end}}) \simeq \frac{3f^4M_p^2m^2}{(8\pi)^3e},
\]

(4.42)

where \( e \) is the base of the natural logarithm. Note that this expression is independent of the initial inflaton amplitude \(^4\). Equation (4.42) allows us to solve for the value of \( \hat{\phi}_0 \) at the end of reheating. We find

\[
\hat{\phi}_0(t_{\text{end}}) \simeq \frac{\sqrt{6/e}M_p f^2}{(8\pi)^{3/2}}.
\]

(4.43)

The rms fluctuations in the inflaton zero mode, \( \hat{\phi}_{\text{rms}} \), can only play a role in the inflaton zero-mode dynamics during reheating if the ratio \( \hat{\phi}_{\text{rms}}/\hat{\phi}_0 \) is not small relative to higher-order \([e.g., O(f^4)]\) processes which we are neglecting. In light of the minimum inflaton zero-mode amplitude attained during reheating, Eq. (4.43), we find that the ratio of fluctuations in the inflaton zero-mode to the zero-mode amplitude is given by

\[\text{We wish to emphasize, however, that this expression does not take into account the regime of nonperturbative dynamics discussed in Sec. \(^{11}\) and therefore should not be expected to yield a correct reheating temperature in a realistic inflationary scenario. However, it suffices for the present discussion of rms fluctuations of the inflaton amplitude, where we assume an idealized case similar to Eq. (106) of Ref. \(^{23}\).}\]

29
\[
\frac{\hat{\phi}_{\text{rms}}}{\hat{\phi}_0(t_{\text{end}})} \simeq \frac{8\sigma m}{f M_p} \sqrt{\frac{e}{45}} \simeq 2.37 \frac{m}{f M_p}. \tag{4.44}
\]

We estimate the ratio of the mean-squared inflaton amplitude fluctuations to the shift in the inflaton mass to be
\[
\left| \frac{\hat{\phi}_{\text{rms}}^2}{\eta(m)} \right| \simeq 0.01. \tag{4.45}
\]

If, prior to the end of reheating at \(t_{\text{end}}\), \(\hat{\phi}_{\text{rms}}/\hat{\phi}_0(t)\) becomes larger than higher-order terms which are neglected in our perturbative expansion, then fluctuations in the inflaton zero mode are a non-negligible effect. This will happen when \(\hat{\phi}_{\text{rms}}/\hat{\phi}_0(t_{\text{end}})\), given by Eq. (4.44), is not \(\ll 1\). This shows that stochasticity must be taken into account in the dynamics of the inflaton zero mode, during the late stages of the reheating period.

V. CONCLUSIONS

In this paper, we present the results of a study of (unbroken-symmetry) inflaton dynamics during the late stages of reheating, which is dominated by fermion particle production to a light spinor field coupled to the inflaton field via a Yukawa coupling. We derived coupled nonperturbative equations for the inflaton mean field and two-point function, in a general curved spacetime, and showed that, in addition to the dissipative mean-field equation, the gap equation for the two-point function is also dissipative, due to fermion particle production. Simultaneous evolution of the inflaton mean-field and two-point function is necessary for correctly following the inflaton dynamics after the end of the preheating period, because the large value of the variance invalidates use of the ordinary perturbative, 1PI effective action.

We also derived the dissipation and noise kernels for the small-amplitude dynamics of the inflaton field, valid in the late stages of reheating when the inflaton mean-field amplitude is very small. The \(O(f^2)\) noise and dissipation kernels, as well as the normal-threshold parts of the \(O(f^4)\) noise and dissipation kernels, are shown to obey a zero-temperature fluctuation-dissipation relation. With the noise and dissipation kernels, a Langevin equation for the inflaton zero mode is derived, and it is shown that the noise leads to a variance for the inflaton amplitude which is non-negligible before the end of reheating.

VI. ACKNOWLEDGMENTS

This work is supported in part by NSF Grant No. PHY94-21849. Part of this work was carried out at the Institute for Advanced Study, Princeton, where B.L.H. was a Dyson Visiting Professor. We enjoyed the hospitality of Los Alamos National Laboratory during the Santa Fe workshop “Nonequilibrium Phase Transitions” sponsored by the Center for Nonlinear Studies, and organized by Dr. E. Mottola. We thank Dr. E. Calzetta for discussions. S.A.R. wishes to thank the University of Buenos Aires for hospitality, and G. Stephens for helpful discussions.
APPENDIX A: DISCONTINUITIES OF THE SQUARE DIAGRAM

In this appendix, the seven terms of Eq. (3.27) involving cut propagators are explicitly evaluated using the Cutkosky rules. The second and third terms of Eq. (3.27) correspond to normal-threshold singularities in the $t$ and $s$ channels, and are given by

\[
\int \frac{d^4q}{(2\pi)^4} \text{tr}_{sp} \left[ F_{++}(q) F_{+-}(q + k_1) F_{--}(q + k_1 + k_2) F_{+-}(q + k_1 + k_2 + k_3) \right] = -i \text{Disc}[A_4(k_1, k_2, k_3)|_{\alpha_1 = \alpha_3 = 0}] \theta(k_2^0 + k_3^0) \tag{A1}
\]

and

\[
\int \frac{d^4q}{(2\pi)^4} \text{tr}_{sp} \left[ F_{+-}(q) F_{--}(q + k_1) F_{++}(q + k_1 + k_2) F_{++}(q + k_1 + k_2 + k_3) \right] = -i \text{Disc}[A_4(k_1, k_2, k_3)|_{\alpha_2 = 0; \alpha_1 + \alpha_3 = 1}] \theta(k_1^0 + k_2^0), \tag{A2}
\]

respectively. The third term in Eq. (3.27) corresponds to the leading-order singularity of the square diagram \[\text{!}\] (i.e., the solution of the Landau equations in which $\alpha_1, \alpha_2, \alpha_3$ are all nonzero),

\[
\int \frac{d^4q}{(2\pi)^4} \text{tr}_{sp} \left[ F_{++}(q) F_{+-}(q + k_1) F_{--}(q + k_1 + k_2) F_{--}(q + k_1 + k_2 + k_3) \right] = i \text{Disc}[A_4(k_1, k_2, k_3)|_{\alpha_1, \alpha_2, \alpha_3 > 0}] \theta(k_1^0) \theta(-k_2^0) \theta(k_3^0). \tag{A3}
\]

The last four terms in Eq. (3.27) correspond to the four remaining twice-contracted singularities, and are given by

\[
\int \frac{d^4q}{(2\pi)^4} \text{tr}_{sp} \left[ F_{++}(q) F_{+-}(q + k_1) F_{++}(q + k_1 + k_2) F_{++}(q + k_1 + k_2 + k_3) \right] = i \text{Disc}[A_4(k_1, k_2, k_3)|_{\alpha_2 = \alpha_3 = 0}] \theta(k_2^0), \tag{A4}
\]

\[
\int \frac{d^4q}{(2\pi)^4} \text{tr}_{sp} \left[ F_{++}(q) F_{++}(q + k_1) F_{+-}(q + k_1 + k_2) F_{--}(q + k_1 + k_2 + k_3) \right] = i \text{Disc}[A_4(k_1, k_2, k_3)|_{\alpha_3 = 0; \alpha_1 + \alpha_2 = 1}] \theta(k_3^0), \tag{A5}
\]

\[
\int \frac{d^4q}{(2\pi)^4} \text{tr}_{sp} \left[ F_{+-}(q) F_{++}(q + k_1) F_{++}(q + k_1 + k_2) F_{++}(q + k_1 + k_2 + k_3) \right] = i \text{Disc}[A_4(k_1, k_2, k_3)|_{\alpha_1 = \alpha_2 = 0}] \theta(k_1^0), \tag{A6}
\]

\[
\int \frac{d^4q}{(2\pi)^4} \text{tr}_{sp} \left[ F_{+-}(q) F_{+-}(q + k_1) F_{--}(q + k_1 + k_2) F_{+-}(q + k_1 + k_2 + k_3) \right] = i \text{Disc}[A_4(k_1, k_2, k_3)|_{\alpha_1 = 0; \alpha_2 + \alpha_3 = 1}] \theta(k_1^0 + k_2^0 + k_3^0). \tag{A7}
\]
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\[ \Gamma^{(2)} = \begin{array}{c}
\text{\includegraphics{figure1}}
\end{array} \]

FIG. 1. Diagrammatic expansion for \( \Gamma^{(2)} \), the two-loop part of the CTP-2PI-CGEA. Solid lines represent the spinor propagator \( F \) (as defined in Sec. II), and dotted lines represent the scalar propagator \( G \). The vertices terminating three \( \phi \) lines are proportional to the scalar mean field \( \hat{\phi} \). Each vertex carries spacetime \( (x) \) and CTP \((+,−)\) labels.

\[ \Gamma^{(1)} = \begin{array}{c}
\text{\includegraphics{figure2}}
\end{array} \]

FIG. 2. Diagrammatic expansion for \( \Gamma^{(1)} \), the one-loop part of the CTP-1PI-CGEA. Solid lines represent the spinor propagator \( F \) (as defined in Sec. II), and dotted lines represent multiplication by the scalar mean field \( \hat{\phi} \). Each vertex carries spacetime \( (x) \) and CTP \((+,−)\) labels.