POSITIVE LIOUVILLE THEOREM AND ASYMPTOTIC BEHAVIOUR FOR
(p, A)-LAPLACIAN TYPE ELLIPTIC EQUATIONS WITH FUCHSIAN
POTENTIALS IN MORREY SPACE

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Dedicated to Volodya Maz’ya on the occasion of his 80th birthday

Abstract. We study Liouville-type theorems and the asymptotic behaviour of positive solutions near an isolated singular point \( \zeta \in \partial \Omega \cup \{\infty\} \) of the quasilinear elliptic equations

\[- \text{div}(|\nabla u|^{p-2} A \nabla u) + V |u|^{p-2} u = 0 \text{ in } \Omega \setminus \{\zeta\},\]

where \( \Omega \) is a domain in \( \mathbb{R}^d \) (\( d \geq 2 \)), and \( A = (a_{ij}) \in L^\infty_{\text{loc}}(\Omega; \mathbb{R}^{d \times d}) \) is a symmetric and locally uniformly positive definite matrix. The potential \( V \) lies in a certain local Morrey space (depending on \( p \)) and has a Fuchsian-type isolated singularity at \( \zeta \).

1. Introduction

Let \( \Omega \) be a domain in \( \mathbb{R}^d \), \( d \geq 2 \), and consider the quasilinear elliptic partial differential equation

\[ Q(u) = Q_{p,A,V}(u) := -\Delta_{p,A}(u) + V |u|^{p-2} u = 0 \text{ in } \Omega. \tag{1.1} \]

and let \( \zeta \in \{0, \infty\} \) be a fixed isolated singular point of \( Q_{p,A,V} \) which belongs to the ideal boundary of \( \Omega \) (to be explained in the sequel).

Here \( 1 < p < \infty \), \( V \) is a real valued potential belonging to a certain local Morrey space, and

\[ \Delta_{p,A}(u) := \text{div}(|\nabla u|^{p-2} A \nabla u) \]

is the \((p, A)\)-Laplacian, where \( A = (a_{ij}) \in L^\infty_{\text{loc}}(\Omega; \mathbb{R}^{d \times d}) \) is a symmetric and locally uniformly positive definite matrix, and

\[ |\xi|^2_{A(x)} := A(x) \xi \cdot \xi = \sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \quad x \in \Omega \text{ and } \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d. \]

We note that (1.1) is the Euler-Lagrange equation associated to the energy functional

\[ Q(\varphi) = Q_{p,A,V}(\varphi) := \int_\Omega (|\nabla \varphi|^p + V |\varphi|^p) \, dx \quad \varphi \in C^\infty_c(\Omega). \tag{1.2} \]

The quasilinear equation (1.1) satisfies the homogeneity property of linear equations but not the additivity (therefore, such an equation is sometimes called half-linear or quasilinear elliptic equations with natural growth terms). Consequently, one expects that positive solutions of (1.1) would share some properties of positive solutions of linear elliptic equations. Indeed, criticality theory for (1.1), similar to the linear case, was established in [14–16].

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In [2], Frass and Pinchover studied Liouville theorems and removable singularity theorems for positive classical solutions of (1.1) under the assumptions that $A$ is the identity matrix, $V \in L^\infty_{\text{loc}}(\Omega)$, and $V$ has a pointwise Fuchsian-type singularity near $\zeta \in \{0, \infty\}$, namely,

$$|V(x)| \leq \frac{C}{|x|^p} \quad \text{near } \zeta. \tag{1.3}$$

Moreover, in the same paper and in [3], the asymptotic behavior of the quotient of two positive solutions near the singular point $\zeta$ has been obtained. The results in [2, 3] extend the results obtained in [13, and the references therein] for second-order linear elliptic operators (not necessarily symmetric) to the quasilinear case. We note that an affirmative answer to Problem 51 of Maz’ya’s recent paper [10] follows from [2, Theorem 1.1].

The aim of the present paper is to study Liouville-type theorems, Picard-type principles, and removable singularity theorems for positive weak solutions of (1.1), by relaxing significantly the condition on the potential $V \in L^\infty_{\text{loc}}(\Omega)$. More precisely, we enable a symmetric, locally bounded, and locally uniformly positive definite matrix $A$, and a potential $V$ that lies in a certain local Morrey space and has a generalized Fuchsian-type singularity at $\zeta$ (in term of a weighted Morrey norm of $V$). In fact, our local regularity assumptions on $A$ and $V$ are almost the weakest to keep the validity of the local Harnack inequality and the local Hölder continuity of weak solutions.

The outline of the present paper is as follows. In Section 2 we provide a short summary on the local theory of positive solutions of (1.1) with potentials in local Morrey spaces and prove Harnack convergence principle under minimal assumptions on the sort of convergence of the coefficients of the sequence of operators. In Section 3 we introduce the notion of a (generalized) Fuchsian singularity for the operator $Q$ at a point $\zeta$, and prove a uniform Harnack inequality near such a singular point which is a key result for proving (under further assumptions) that the quotient of two positive solutions near $\zeta$ admits a limit in the wide sense. Section 4 is devoted to the asymptotic behaviour of positive $(p, A)$-harmonic functions near an isolated singular point for the case where $A \in \mathbb{R}^{d \times d}$ is a symmetric and positive definite matrix. In Section 5 we assume that $Q$ has a weak Fuchsian singularity at $\zeta$ and prove that it is a sufficient condition for the validity of a positive Liouville-type theorem. Finally, Section 6 is devoted to the study of Liouville-type theorem in the elliptically symmetric case.

2. Preliminaries

We begin with notation, some definitions and assumptions. Throughout the paper, $\Omega$ is a domain (i.e., a nonempty open connected set) in $\mathbb{R}^d$, $d \geq 2$. By $B_r(x_0)$ and $S_r(x_0) = \partial B_r(x_0)$, we denote the open ball and the sphere of radius $r > 0$ centered at $x_0$, respectively, and we set $B_r := B_r(0), S_r := S_r(0)$. Denote $B^*_r := \mathbb{R}^d \setminus \overline{B_r}$ and $(\mathbb{R}^d)^* := \mathbb{R}^d \setminus \{0\}$, the corresponding exterior domains. For $R > 0$ we denote by $\mathcal{A}_R$ the annuls $\mathcal{A}_R := \{x \in \mathbb{R}^d \mid R/2 \leq |x| < 3R/2\}$, and for a domain $\Omega \subset \mathbb{R}^d$ and $R > 0$, we define the dilated domain $\Omega/R := \{x \in \mathbb{R}^d \mid x = R^{-1}y, \text{where } y \in \Omega\}$. Let $f, g \in C(\Omega)$ be two positive functions. The notation $f \asymp g$ in $\Omega$ means that there exists positive constant $C$ such that

$$C^{-1}g(x) \leq f(x) \leq Cg(x) \quad \text{for all } x \in \Omega.$$

We write $\Omega_1 \Subset \Omega_2$ if $\Omega_2$ is open and $\overline{\Omega_1}$ is compact (proper) subset of $\Omega_2$. By a compact exhaustion of a domain $\Omega$, we mean a sequence of smooth relatively compact domains in $\Omega$ such...
that $\Omega_1 \neq \emptyset$, $\Omega_i \subseteq \Omega_{i+1}$, and $\cup_{i=1}^{\infty} \Omega_i = \Omega$. Finally, throughout the paper $C$ refers to a positive constant which may vary from line to line.

We first introduce a certain class of Morrey spaces, in which the potential $V$ of the operator $Q_{p,A,V}$ belongs to.

**Definition 2.1 (Morrey spaces).** A function $f \in L^1_{\text{loc}}(\Omega; \mathbb{R})$ is said to belong to the local Morrey space $M^q_{\text{loc}}(\Omega; \mathbb{R})$, $q \in [1, \infty]$ if for any $\omega \subseteq \Omega$

$$\|f\|_{M^q(\omega)} := \sup_{y \in \omega \atop 0 < r < \text{diam}(\omega)} \frac{1}{r^{d/q'}} \int_{\omega \cap B_r(y)} |f| \, dx < \infty,$$

where $q' = q/(q - 1)$ is the Hölder conjugate exponent of $q$. By applying Hölder inequality, it can be seen that $L^q_{\text{loc}}(\Omega) \subsetneq M^q_{\text{loc}}(\Omega) \subsetneq L^1_{\text{loc}}(\Omega)$ for any $q \in (1, \infty)$. For $q = 1$ we have $M^1_{\text{loc}}(\Omega) = L^1_{\text{loc}}(\Omega)$, and for $q = \infty$ we have $M^\infty_{\text{loc}}(\Omega) = L^\infty_{\text{loc}}(\Omega)$ (as vector spaces).

Next we define a special local Morrey space $M^q_{\text{loc}}(p; \Omega)$ which depends on the underlying exponent $1 < p < \infty$.

**Definition 2.2 (Special Morrey spaces).** For $p \neq d$, we define

$$M^q_{\text{loc}}(p; \Omega) := \begin{cases} M^q_{\text{loc}}(\Omega) \text{ with } q > d/p & \text{if } p < d, \\ L^1_{\text{loc}}(\Omega) & \text{if } p > d, \end{cases}$$

while for $p = d$, the Morrey space $M^q_{\text{loc}}(d; \Omega)$ consists of all those $f$ such that for some $q > d$ and any $\omega \subseteq \Omega$

$$\|f\|_{M^q(d, \omega)} := \sup_{y \in \omega \atop 0 < r < \text{diam}(\omega)} \varphi_q(r) \int_{\omega \cap B_r(y)} |f| \, dx < \infty,$$

where $\varphi_q(r) := \log^{q/d'} \left( \frac{\text{diam}(\omega)}{r} \right)$ (see [8, Theorem 1.94], and references therein).

Throughout the article we consider the following $(p,A)$-Laplace type equation

$$Q(u) = Q_{p,A,V}(u) := -\Delta_{p,A}(u) + V |u|^{p-2}u = 0 \quad \text{in } \Omega, \tag{2.1}$$

under the following assumptions on $A$ and $V$.

**Assumptions 2.3.**

- $A = (a_{ij})_{i,j=1}^d \in L^\infty_{\text{loc}}(\Omega; \mathbb{R}^{d \times d})$ is a symmetric matrix.
- $A$ is locally uniformly elliptic in $\Omega$, that is, for any compact $K \subseteq \Omega$ there exists $\Theta_K > 0$ such that

$$\Theta_K^{-1} \sum_{i=1}^d \xi_i^2 \leq \sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \leq \Theta_K \sum_{i=1}^d \xi_i^2 \quad \forall \xi \in \mathbb{R}^d \text{ and } \forall x \in K.$$  

- $V \in M^q_{\text{loc}}(p; \Omega)$ is a real valued function.

Recall that a function $v$ is said to be a (weak) solution of the equation $Q_{p,A,V}(u) = 0$ in $\Omega$ if $v \in W^{1,p}_{\text{loc}}(\Omega)$ and $v$ satisfies

$$\int_{\Omega} (|v|^{p-2}A \nabla v \cdot \nabla \varphi + V |v|^{p-2}v \varphi) \, dx = 0 \quad \forall \varphi \in C_0^\infty(\Omega). \tag{2.2}$$
Further, we say the $v \in W^{1,p}_{\text{loc}}(\Omega)$ is a supersolution of (2.1) if the integral in (2.2) is nonnegative for every nonnegative test function $\varphi \in C^{\infty}_c(\Omega)$. A function $v$ is a subsolution of (2.1) if $-v$ is supersolution of (2.1). It should be noted that the above definitions make sense because of the following Morrey-Adams Theorem (see for example, [14, Theorem 2.4] and references therein).

**Theorem 2.4** (Morrey-Adams theorem). Let $\omega \subset \mathbb{R}^d$ and $V \in M^q(p; \omega)$.

(i) There exists a constant $C = C(d, p, q) > 0$ such that for any $\delta > 0$ and all $u \in W^{1,p}_{\text{loc}}(\omega)$

$$\int_{\omega} |V||u|^p \, dx \leq \delta \|\nabla u\|_{L^p(\omega; \mathbb{R}^d)}^p + \frac{C}{\delta^{d/(pq-d)}} \|V\|_{M^q(p; \omega)}^{pq/(pq-d)} \|u\|_{L^p(\omega)}^p. \tag{2.3}$$

(ii) For any $\omega' \subset \omega$ with Lipschitz boundary, there exist $0 < C = C(d, p, q, \omega', \omega, \delta, \|V\|_{M^q(p; \omega)})$ and $\delta_0$ such that for $0 < \delta \leq \delta_0$ and all $u \in W^{1,p}(\omega')$

$$\int_{\omega'} |V||u|^p \, dx \leq \delta \|\nabla u\|_{L^p(\omega'; \mathbb{R}^d)}^p + C \|u\|_{L^p(\omega')}^p.$$

We recall the Allegretto-Piepenbrink-type theorem (see, [14, Theorem 4.3]). This theorem states that $Q_{p, A, V}(\varphi) \geq 0$ for all $\varphi \in C^{\infty}_c(\Omega)$ (in short, $Q_{p, A, V} \geq 0$ in $\Omega$) if and only if the equation $Q_{p, A, V}(u) = 0$ possesses a positive (super)solution in $\Omega$.

**Throughout the paper, we assume that** $Q_{p, A, V}(\varphi) \geq 0$ for all $\varphi \in C^{\infty}_c(\Omega)$.

The above assumption implies the solvability of the Dirichlet problem in bounded subdomains (see [14] Theorem 3.10 and Proposition 5.2):

**Lemma 2.5.** Assume that $Q_{p, A, V} \geq 0$ in $\Omega$. Then for any Lipschitz subdomain $\omega \subset \Omega$, $0 \leq g \in C(\omega)$ and $0 \leq f \in C(\partial \omega)$, there exists a nonnegative solution $u \in W^{1,p}(\omega)$ of the problem

$$Q_{p, A, V}(v) = g \text{ in } \omega, \text{ and } v = f \text{ on } \partial \omega.$$  

Moreover, the solution $u$ is unique if either $f = 0$ or $f > 0$ on $\partial \omega$.

We recall the local Harnack inequality of nonnegative solutions of (2.1), see for example, [8, Theorem 3.14] for the case $p \leq d$ and [17, Theorem 7.4.1] for the case $p > d$.

**Theorem 2.6** (Local Harnack inequality). Let $A, V$ satisfy Assumptions 2.3, and let $\omega' \subset \omega \subset \Omega$. Then for any nonnegative solution $v$ of (2.1) in $\Omega$ we have

$$\sup_{\omega'} v \leq C \inf_{\omega'} v, \tag{2.4}$$

where $C$ is a positive constant depending only on $d, p, \text{dist}(\omega', \partial \omega)$, $\|A\|_{L^{\infty}(\omega'; \mathbb{R}^{d \times d})}$, the ellipticity constant of $A$ in $\omega$, and $\|V\|_{M^q(p; \omega)}$ but not on $v$.

The next result concerns the Harnack convergence principle for a sequence of normalized positive solutions of equations of the form (2.1) (cf. [14, Proposition 2.11] where $A \in L^{\infty}_{\text{loc}}(\Omega; \mathbb{R}^{d \times d})$ is fixed, and $\{V_i\}_{i=1}^\infty \subset M^q_{\text{loc}}(p; \Omega_i)$ converges strongly in $M^q_{\text{loc}}(p; \Omega)$ to $V \in M^q_{\text{loc}}(p; \Omega)$).

**Proposition 2.7** (Harnack convergence principle). Let $\{\Omega_i\}$ be a compact exhaustion of $\Omega$. Assume that $\{A_i\}_{i=1}^\infty$ is a sequence of symmetric and locally uniformly positive definite matrices such that the local ellipticity constant does not depend on $i$, and $\{A_i\}_{i=1}^\infty \subset L^{\infty}_{\text{loc}}(\Omega_i; \mathbb{R}^{d \times d})$ converges weakly in $L^{\infty}_{\text{loc}}(\Omega_i; \mathbb{R}^{d \times d})$ to a matrix $A \in L^{\infty}_{\text{loc}}(\Omega; \mathbb{R}^{d \times d})$. Assume also that $\{V_i\}_{i=1}^\infty \subset M^q_{\text{loc}}(p; \Omega_i)$ converges weakly in $M^q_{\text{loc}}(p; \Omega)$ to $V \in M^q_{\text{loc}}(p; \Omega)$.

For each $i \geq 1$, let $v_i$ be a positive weak solution of the equation $Q_{p, A_i, V_i}(u) = 0$ in $\Omega_i$ such that $v_i(x_0) = 1$, where $x_0$ is a fixed reference point in $\Omega_i$. 

**Remark 2.8.** The local Harnack inequality (2.4) is a consequence of Theorem 2.4 and Proposition 2.7.
Then there exists $0 < \beta < 1$ such that, up to a subsequence, $\{v_i\}$ converges weakly in $W^{1,p}_\text{loc}(\Omega)$ and in $C^\beta_{\text{loc}}(\Omega)$ to a positive weak solution $v$ of the equation $Q_{\beta , A, \gamma}(v) = 0$ in $\Omega$.

Proof. Since the sequence $\{A_i\}$ is locally uniformly elliptic and converges weakly in $L^\infty(\Omega; \mathbb{R}^{d \times d})$, it follows that that $\|A_i\|_{L^\infty(\Omega'; \mathbb{R}^{d \times d})} \leq C$ for every $\Omega' \Subset \Omega$, and hence, $A_i$ are uniformly bounded in every $\Omega' \Subset \Omega$ expect for a set of measure zero. By the definition of $v_i$ being a positive weak solution to $Q_{A_i, \gamma}(v_i) = 0$ in $\Omega_i$, we have

$$
\int_{\Omega_i} |\nabla v_i|_{A_i}^{p-2} A_i \nabla v_i \cdot \nabla u \, dx + \int_{\Omega_i} V_i u_i^{p-1} u \, dx = 0 \quad \text{for all } u \in W^{1,p}_\text{loc}(\Omega_i).
$$

(2.5)

Also, by elliptic regularity, $v_i$ are Hölder continuous for all $i \geq 1$. Fix $k \in \mathbb{N}$. Thus, for $u \in C^\infty_0(\Omega_k)$, by plugging $v_i|u|^p \in W^{1,p}_\text{loc}(\Omega_k)$, $i \geq k$, as a test function in (2.5) we get

$$
\|\nabla v_i|_{A_i} \|_{L^p(\Omega_k)}^p \leq p \int_{\Omega_k} |\nabla v_i|_{A_i}^{p-1} |u|^{p-1} v_i |\nabla u|_{A_i} \, dx + \int_{\Omega_k} \|V_i| e_i^p \| u^p \, dx.
$$

For the first term of the right-hand side of the above equation, we apply Young’s inequality: $pab \leq \varepsilon a^{p'} + \frac{(p-1)}{\varepsilon} \|\nabla v_i|_{A_i}^{p-1} |u|^{p-1}$, for a $b = v_i|\nabla u|_{A_i}$. On the second term, we use the Morrey-Adams theorem (Theorem 2.4). Then we arrive at

$$(1 - \varepsilon) \|\nabla v_i|_{A_i} \|^p_{L^p(\Omega_k)} \leq \left(\frac{p-1}{\varepsilon}\right)^{p-1} \|v_i| |\nabla u|_{A_i} \|_{L^p(\Omega_k)}^p + \delta \|\nabla(v_i u)\|_{L^p(\Omega_k; \mathbb{R}^d)} + C \|v_i u\|_{L^p(\Omega_k)}^p.
$$

(2.6)

Since the sequence $\{A_i\}$ is locally uniformly elliptic and bounded a.e., and by the inequality

$$
\|\nabla(v_i u)\|_{L^p(\Omega_k; \mathbb{R}^d)} \leq 2^{p-1} \left(\|v_i| u| \|_{L^p(\Omega_k; \mathbb{R}^d)} + \|u| v_i \|_{L^p(\Omega_k; \mathbb{R}^d)}\right),
$$

we obtain the following estimates valid for $i \geq k$ and for any $u \in C^\infty_c(\Omega_k)$

$$
\left((1 - \varepsilon)C^p_{\Omega_k} - 2^{p-1} \delta C^{-p}_{\Omega_k}\right) \|\nabla v_i| |u| \|_{L^p(\Omega_k)} \leq \left(\frac{p-1}{\varepsilon}\right)^{p-1} C^p_{\Omega_k} - 2^{p-1} \delta \|v_i| |\nabla u|_{A_i} \|_{L^p(\Omega_k)} + C(d, p, q, \delta, \|\nabla|_{M^\gamma_p(\Omega_k)}) \|v_i u\|_{L^p(\Omega_k)}^p.
$$

(2.7)

We now take an arbitrary $\omega \Subset \Omega$ and without loss of generality we assume that $x_0 \in \omega$. Picking a subdomain $\omega' \Subset \Omega$ such that $\omega \Subset \omega'$, we can find $k \geq 1$ such that $\omega' \Subset \Omega_k$. Then we choose $\delta < (1 - \varepsilon)2^{1-p} C^2_{\Omega_k}$ and specialize $u \in C^\infty_c(\Omega_k)$ such that

$$
\text{supp}\{u\} \subset \omega', \quad 0 \leq u \leq 1, \quad u = 1 \text{ in } \omega \text{ and } |\nabla u| \leq \frac{1}{\text{dist}(\omega, \partial \omega')} \text{ in } \omega'.
$$

(2.7)

Due to the local Harnack inequality (2.6), the sequence $\{v_i\}_{i=1}^\infty$ of solutions is bounded in $L^\infty(\omega)$. In fact, by elliptic regularity $\{v_i\}_{i=1}^\infty$ is bounded in $C^0(\omega)$, where $0 < \alpha \leq 1$. Moreover, by plugging $u$ as in (2.7) to the inequality (2.6), we get

$$
\|\nabla v_i\|_{L^p(\omega; \mathbb{R}^d)} + \|v_i\|_{L^p(\omega)} \leq C(d, p, q, \varepsilon, \delta, \text{dist}(\omega, \partial \omega'), C_{\Omega_k}, \|\nabla|_{M^\gamma_p(\Omega_k)}) \|v_i u\|_{L^p(\Omega_k)}^p.
$$

(2.7)

for all $i \geq k$. This implies that the sequence $\{v_i\}_{i=1}^\infty$ is bounded in $W^{1,p}(\omega)$. Hence up to a subsequence, still denoted by $\{v_i\}_{i=1}^\infty$, we obtain that $\{v_i\}_{i=1}^\infty$ converges uniformly in $\omega$, and weakly to a nonnegative function $\nu \in W^{1,p}(\omega) \cap C^0(\omega)$ with $\nu(x_0) = 1$. So, we have

$$
v_i \rightarrow \nu \quad \text{uniformly in } \omega, \quad \text{and } \nabla v_i \rightharpoonup \nabla \nu \text{ in } L^p(\omega; \mathbb{R}^d).$$
We now show that \(v\) is a weak solution of \(Q_{p,A,V}(u) = 0\) in \(\bar{\omega} \Subset \omega\) such that \(x_0 \in \bar{\omega}\). Using the uniform convergence in \(\omega\) of \(v_i \to v\), we obtain
\[
\left| \int_{\omega} (V_i v_i^{p_i-1} \varphi - v v^{p-1} \varphi) \, dx \right| \leq C \|v_i - v\|_{L_\infty(\omega)} \int_{\omega} |V_i| \, dx + \int_{\omega} (V_i - V) v^{p-1} \varphi \, dx.
\]

Since the sequence \(\{V_i\}\) converges weakly to \(V\) in \(M^q_{\text{loc}}(\rho; \Omega)\), it is bounded in \(L^1_{\text{loc}}(\Omega)\), therefore, the first term tends to zero, while the second term tends to zero by the weak convergence of \(\{V_i\}\) to \(V\). Hence,
\[
\int_{\omega} V_i v_i^{p_i-1} \varphi \, dx \to \int_{\omega} V v^{p-1} \varphi \, dx \quad \text{for all } \varphi \in C_0^\infty(\omega).
\] (2.8)

It remains to show that
\[
\xi_i := |\nabla v_i|^{p_i-2} A_i \nabla v_i \to_{i \to \infty} |\nabla v|^{p-2} \hat{A} \nabla v := \xi \quad \text{in } L^{p'}(\bar{\omega}; \mathbb{R}^d).
\] (2.9)

To prove this claim, it is enough to prove that \(\xi_i \to \xi\) a.e., and that \(\{\xi_i\}\) is bounded in \(L^{p'}(\bar{\omega}; \mathbb{R}^d)\) (see, [9] and [5, Lemma 3.73]). The boundedness of \(\{\xi_i\}\) in \(L^{p'}(\bar{\omega}; \mathbb{R}^d)\) clearly follows from the boundedness of \(\{\nabla v_i\}\) in \(L^p(\bar{\omega}; \mathbb{R}^d)\). So, we need to prove the a.e. convergence of \(\{\xi_i\}\) to \(\xi\).

By our assumption, \(\{A_i\}_{i=1}^\infty \subset L^p_{\text{loc}}(\Omega; \mathbb{R}^{d \times d})\) converges weakly to \(\hat{A} \in L^p_{\text{loc}}(\Omega; \mathbb{R}^{d \times d})\) to a matrix \(\hat{A} \in L^\infty(\Omega; \mathbb{R}^{d \times d})\). Then let us consider \(u\) as in (2.7) but with \(\omega\) and \(\omega'\) replaced by \(\bar{\omega}\) and \(\omega\), respectively. So, by plugging \(u(v_i - v)\) as a test function in (2.5), we obtain
\[
\int_{\omega} u \xi_i \cdot \nabla (v_i - v) \, dx = -\int_{\omega} (v_i - v) \xi_i \cdot \nabla u \, dx - \int_{\omega} V_i v_i^{p_i-1} u(v_i - v) \, dx.
\] (2.10)

For the first integral on the right-hand side of (2.10), apply Hölder’s inequality to get
\[
\left| -\int_{\omega} (v_i - v) \xi_i \cdot \nabla u \, dx \right| \leq C_{\omega}^{p/p'} \|v_i - v\|_{L_p(\omega; \mathbb{R}^d)} \|\nabla v_i\|_{L_{p'}(\omega; \mathbb{R}^d)}^{p/p'} \leq C(p, C_{\omega}, \text{dist}(\bar{\omega}, \partial \omega)) \|(v_i - v)\|_{L_p(\omega)} \|\nabla v_i\|_{L_{p'}(\omega; \mathbb{R}^d)}^{p/p'} \to_{i \to \infty} 0,
\]
since \(\|\nabla v_i\|_{L_{p'}(\omega; \mathbb{R}^d)}\) are uniformly bounded and \(v_i \to v\) in \(L^p(\omega)\). A similar argument leading to (2.8) implies that the second integral on the right-hand side of (2.10) also converges to 0. Thus,
\[
\int_{\omega} u \xi_i \cdot \nabla (v_i - v) \, dx \to_{i \to \infty} 0.
\] (2.11)

Notice that
\[
(\xi_i - \xi) \cdot (\nabla v_i - \nabla v) = (|\nabla v_i|^{p_i-2} A_i \nabla v_i - |\nabla v|^{p-2} A \nabla v) \cdot (\nabla v_i - \nabla v)
+ (|\nabla v|^{p-2} A \nabla v - |\nabla v|^{p_i-2} A_i \nabla v) \cdot (\nabla v_i - \nabla v) \geq (|\nabla v|^{p-2} A \nabla v - |\nabla v|^{p_i-2} A_i \nabla v) \cdot (\nabla v_i - \nabla v).
\]

Since \(A_i\) converges weakly to \(A\) in \(L^\infty(\Omega, \mathbb{R}^{d \times d})\), it follows [11, Proposition 2.9] that \(A_i \to A\) a.e.. Therefore,
\[
|\nabla v(x)|_{A_i(x)}^{p_i-2} A_i(x) \nabla v(x) \to |\nabla v(x)|_{A(x)}^{p-2} A(x) \nabla v(x) \quad \text{for a.e. in } \omega,
\]
and also
\[
|\nabla v|_{A_i}^{p_i-2} A_i \nabla v - |\nabla v|_{A}^{p_i-2} A \nabla v|^{p'} \leq 2^{p-1}(||\nabla v|_{A_i}^{p} + |\nabla v|_{A}^{p}) \leq C|\nabla v|^{p}.
\]
since the sequence \(\{A_i\}\) is bounded a.e in \(\omega\). Thus, the dominated convergence theorem implies
\[
\lim_{i \to \infty} \int_{\omega} |\nabla v|_{A_i}^{p_i-2} A_i \nabla v - |\nabla v|_{A_i}^{p-2} A \nabla v|^{p'} \, dx = 0.
\]
Therefore, the Hölder inequality and the boundedness of $\nabla v_i$ in $L^p(\omega; \mathbb{R}^d)$ implies

\[
\left| \int_\omega u(|\nabla v|_{A_i}^{p-2} A_i \nabla v - |\nabla v|_h^{p-2} A \nabla v) \cdot (\nabla v_i - \nabla v) \, dx \right| \to_{i \to \infty} 0. \tag{2.12}
\]

Now by using above vectors inequality we get

\[
0 \leq \int_\omega \left[ (\xi_i - \xi) \cdot (\nabla v_i - \nabla v) - (|\nabla v|_{A_i}^{p-2} A_i \nabla v - |\nabla v|_h^{p-2} A h \nabla v) \cdot (\nabla v_i - \nabla v) \right] \, dx
\]

\[
\leq \int_\omega u \left[ (\xi_i - \xi) \cdot (\nabla v_i - \nabla v) - (|\nabla v|_{A_i}^{p-2} A_i \nabla v - |\nabla v|_h^{p-2} A h \nabla v) \cdot (\nabla v_i - \nabla v) \right] \, dx
\]

\[
= \int_\omega u(\xi_i - \xi) \cdot (\nabla v_i - \nabla v) \, dx - \int_\omega u(|\nabla v|_{A_i}^{p-2} A_i \nabla v - |\nabla v|_h^{p-2} A h \nabla v)(\nabla v_i - \nabla v) \, dx \to_{i \to \infty} 0,
\]

where we have used (2.11), (2.12) and $\nabla v_i \to \nabla v$ in $L^p(\omega; \mathbb{R}^d)$. It follows that

\[
\lim_{i \to \infty} \int_\omega \left( |\nabla v_i|_{A_i}^{p-2} A_i \nabla v_i - |\nabla v|_{A_i}^{p-2} A_i \nabla v \right) \cdot (\nabla v_i - \nabla v) \, dx = 0. \tag{2.13}
\]

To prove the claim (2.9), we proceed as in [5, Lemma 3.73]. Denote

\[
D_i = (|\nabla v_i|_{A_i}^{p-2} A_i \nabla v_i - |\nabla v|_{A_i}^{p-2} A_i \nabla v) \cdot (\nabla v_i - \nabla v).
\]

Since $D_i$ is a nonnegative function, (2.13) implies that $D_i \to 0$ in $L^1(\tilde{\omega})$. Extracting a subsequence we have $D_i \to 0$ a.e. in $\tilde{\omega}$. Therefore, there exists a subset $Z$ of $\tilde{\omega}$ of zero measure such that for $x \in \tilde{\omega} \setminus Z$ we have $D_i(x) \to 0$.

Fix $x \in \tilde{\omega} \setminus Z$. Without loss of generality, we may assume that $|\nabla v(x)| < \infty$. Since $A_i$ are locally uniformly elliptic and bounded, we have

\[
D_i(x) \geq |\nabla v_i(x)|_{A_i}^p - |\nabla v(x)|_{A_i}^p - (|\nabla v(x)|_{A_i} |\nabla v_i(x)|_{A_i}^{-1} + |\nabla v_i(x)|_{A_i} |\nabla v(x)|_{A_i}^{-1})
\]

\[
\geq C_0 |\nabla v_i(x)|_{A_i}^p - C^{-p}_0 (|\nabla v_i(x)|_{A_i}^p + |\nabla v_i(x)|_{A_i}^p)
\]

\[
\geq C_0 |\nabla v_i(x)|_{A_i}^p - C(|\nabla v_i(x)|_{A_i}^p + |\nabla v_i(x)|)
\]

where $C = \max(|\nabla v(x)|_{A_i}, |\nabla v(x)|_{A_i}^{-1})C_0^{-p}$. From the above inequality, it readily follows that $|\nabla v_i(x)|$ is uniformly bounded with respect to $i$, since $D_i(x) \to 0$.

Let $\eta$ be a limit point of $\nabla v_i(x)$. Then $|\eta| < \infty$ and

\[
0 = \lim_{i \to \infty} \left( |\nabla v_i(x)|_{A_i}^{p-2} A_i \nabla v_i(x) - |\nabla v(x)|_{A_i}^{p-2} A_i \nabla v(x) \right) \cdot (\nabla v_i(x) - \nabla v(x))
\]

\[
= (|\nabla v(x)|_{A_i}^{p-2} A_i \nabla v(x) - |\nabla v(x)|_{A_i}^{p-2} A_i \nabla v(x)) \cdot (\eta - \nabla v(x)).
\]

This implies that $\eta = \nabla v(x)$. Thus we get $\nabla v_i(x) \to \nabla v(x)$ for every $x \in \tilde{\omega} \setminus Z$, i.e., $\nabla v_i \to \nabla v$ a.e. in $\tilde{\omega}$ and $|\nabla v_i(x)|_{A_i}^{p-2} A_i \nabla v_i(x) \to |\nabla v(x)|_{A_i}^{p-2} A_i \nabla v(x)$ a.e. in $\tilde{\omega}$. Recall that the $L^{p'}$-norm of $|\nabla v_i|_{A_i}^{p-2} A_i \nabla v_i$ is bounded in $\tilde{\omega}$, therefore, (2.9) follows.

Finally, we formulate a weak comparison principle (WCP) for the case $A \in L^\infty_{loc}(\Omega; \mathbb{R}^{d \times d})$, and $V \in M^d_{loc}(p; \Omega)$. For the proof see Theorem 5.3 in [14].
point to be \(\zeta\). We allow the domain \(\Omega\) to be unbounded and with nonsmooth boundary, and the singular
neighborhood of infinity in \(\hat{\Omega}\) is either 0 or \(\infty\), and that \(\zeta\) is an isolated component of \(\partial \hat{\Omega}\).

With some abuse of notation, we write \(a \to \zeta\) if
\[
\begin{cases}
  a \to 0 & \text{in } \mathbb{R} \text{ and } \zeta = 0, \\
  a \to \infty & \text{in } \mathbb{R} \text{ and } \zeta = \infty.
\end{cases}
\]

We extend the definition of pointwise Fuchsian-type singularity (see (1.3)).

**Definition 3.1** (Fuchsian singularity). Let \(\Omega\) be a domain in \(\mathbb{R}^d\), and \(A\) and \(V\) satisfy Assumptions 2.3. Let \(\zeta \in \partial \Omega\) be an isolated point of \(\partial \hat{\Omega}\), where \(\zeta = 0\) or \(\zeta = \infty\). We say that the operator \(Q_{p,A,V}\) has a Fuchsian-type singularity at \(\zeta\) (in short, Fuchsian singularity at \(\zeta\)) if the following two conditions are satisfied:

\[
\begin{cases}
  (1) \text{ The matrix } A \text{ is bounded and uniformly elliptic in a punctured neighbourhood } \Omega' \subset \Omega \text{ of } \zeta, \text{ that is, there is a positive constant } C \text{ such that,} \\
  \quad \quad C^{-1} |\xi|^2 \leq |\xi_A|^2 \leq C^{-1} |\xi|^2 \quad \forall x \in \Omega' \text{ and } \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d. \\
  (2) \text{ There exists a positive constant } C \text{ and } R_0 > 0 \text{ such that} \\
  \quad \quad \|x|^{p-d/q} V\|_{M^q(p;A_R)} \leq C \quad \text{if } p \neq d, \\
  \quad \quad \|V\|_{M^q(d;A_R)} \leq C \quad \text{if } p = d, \\
  \quad \quad \text{for all } 0 < R < R_0 \text{ if } \zeta = 0, \text{ and } R > R_0 \text{ if } \zeta = \infty, \text{ where } A_R := \{x \mid R/2 \leq |x| < 3R/2\}.
\end{cases}
\]

**Definition 3.2.** A set \(A \subset \Omega\) is called an essential set with respect to the singular point \(\zeta\) if there exist real numbers \(0 < a < 1 < b < \infty\), and a sequence of positive numbers \(\{R_n\}\) with \(R_n \to \zeta\) such that \(A = \cup A_n\), where \(A_n = \{x \in \Omega \mid aR_n < |x| < bR_n\}\).

**Remark 3.3.** Similar to the linear case [13], it turns out that it is sufficient to assume that inequality (3.2) is satisfied only on some essential subset of a neighbourhood of \(\zeta\). More precisely,
instead of (3.2), we may assume that for some essential set $\mathcal{A} \subset \Omega$ with respect to $\zeta$
\[
\begin{cases}
\|x|^{-d/q} V\|_{M^q(p; A_n)} \leq C & \text{if } p \neq d,
\|V\|_{M^q(d; A_n)} \leq C & \text{if } p = d,
\end{cases}
\]
where $C$ is independent of $n$. \hspace{1cm} (3.3)

**Example 3.4.** Let $\Omega = \mathbb{R}^d \setminus \{0\}$ and fix $1 < p < \infty$. Consider the $p$-Laplace operator with the Hardy potential $V(x) = \lambda |x|^{-p}$
\[-\Delta_p u - \lambda \frac{|u|^{p-2} u}{|x|^p} = 0 \text{ in } \Omega,
\]
where $\lambda \leq C_H := |(p-d)/p|^p$ is the Hardy constant. A straightforward computation shows that (3.4) has Fuchsian singularity both at $\zeta = 0$ and $\zeta = \infty$ (in the sense of definition (3.1)).

The above example implies:

**Example 3.5.** Let $\Omega = \mathbb{R}^d \setminus \{0\}$ and fix $1 < p < \infty$. Consider the operator $Q_{p,A,V} = -\Delta_{p,A} + V|u|^{-2}u$, and assume that the matrix $A$ satisfies (3.1) in $\Omega$, and $|V(x)| \leq C|x|^{-p}$ in $\Omega$. Then $Q_{p,A,V}$ has Fuchsian singularity at the origin and at infinity.

We now present a dilation process which uses the quasi-invariance of Fuchsian equations of the form (2.1) under the scaling $x \mapsto Rx$ for $R > 0$. Let $A_R$ and $V_R$ be the scaled matrix and potential defined by
\[A_R(x) := A(Rx), \quad V_R(x) := R^d V(Rx) \quad x \in \Omega/R.
\]
Consider the annular set $\mathcal{A}_R = (B_{3R/2} \setminus \bar{B}_{R/2}) \cap \Omega$. By our assumption that $\zeta$ is an isolated singular point it follows that $\mathcal{A}_R/R$ is fixed annular set $\tilde{\mathcal{A}}$ for $R$ ‘near’ $\zeta$, and for such $R$ we have
\[
\|V_R\|_{M^q(p, \tilde{\mathcal{A}})} = \|V_R\|_{M^q(p, A_R/R)} = R^{p-d/q}\|V\|_{M^q(p, A)} \leq C,
\]
for $p \neq d$ and while for $p = d$,
\[
\|V_R\|_{M^q(d, \tilde{\mathcal{A}})} = \|V_R\|_{M^q(d, A_R/R)} = \|V\|_{M^q(d, A)} \leq C.
\]

Let $Y := \lim_{n \to \infty} \Omega/R_n$. Note that since by our assumptions, $\zeta$ is an isolated component of $\partial \tilde{\Omega}$, it follows that $Y = (\mathbb{R}^d)^* = \mathbb{R}^d \setminus \{0\}$.

The limiting dilation process is defined as follows. Let $\zeta = 0$ or $\zeta = \infty$, and assume that there is a sequence $\{R_n\}$ of positive numbers satisfying $R_n \to \zeta$ such that
\[
\begin{cases}
A_{R_n} \to A \quad \text{in the weak topology of } L^\infty_{\text{loc}}(Y; \mathbb{R}^{d \times d}), \quad \text{and}
V_{R_n} \to V \quad \text{in the weak topology of } M^q_{\text{loc}}(p; Y).
\end{cases}
\]
Motivated by the Harnack convergence principle (see, Proposition 2.7), we define the limiting dilated equation with respect to equation (2.1) and the sequence $\{R_n\}$ that satisfies (3.7) by
\[
D^{\{R_n\}}(Q)(w) := -\Delta_{p,A}(w) + |w|^{p-2} w = 0 \text{ on } Y.
\]
The following proposition establishes a key invariance property of the limiting dilation process.

**Proposition 3.6.** Let $A, V$, satisfy Assumptions 2.3. Suppose that the quasilinear equation (2.1) has a Fuchsian singularity at $\zeta \in \{0, \infty\} \subset \partial \tilde{\Omega}$, and let $D^{\{R_n\}}(Q)(w)$ be a limiting dilated operator corresponding to a sequence $R_n \to \zeta$. Then the equation $D^{\{R_n\}}(Q)(w) = 0$ in $Y$ has Fuchsian singularity at $\zeta$. 


Proof. It is trivial to verify that the proposition holds true when \( p = d \). Now for \( p \neq d \), by Remark 3.3, there exists \( C > 0 \) and an essential set \( \mathcal{A} = \cup \mathcal{A}_n \), where \( \mathcal{A}_n = \{ x \in \Omega \mid a R_n < |x| < b R_n \} \) such that

\[
\| x \|^p \leq C \forall n \in \mathbb{N}.
\]

We claim that

\[
\| x \|^p \leq \| x \|^{p/q} \| v \|^q \leq \| x \|^{p/q} \| v \|^q \| v \|^q \| M^q(\Omega) \| \leq C \forall n \in \mathbb{N}.
\]

Recall that for each \( n \), \( \mathcal{A}_n / R_n \) is a fixed annular set \( \tilde{A} = \{ x \mid a < |x| < b \} \). Assume that \( p < d \), so \( p - d/q > 0 \). Then we have

\[
\| x \|^p \leq b^{p/d} \| v \|^q \| M^q(\Omega) \| \leq b^{p/d} \lim_{n \to \infty} \| v \|^q \| M^q(\Omega) \| \leq C.
\]

For \( p > d \), \( M^q(\Omega) = L^1(\Omega) \), and similarly we get \( \| x \|^p \leq \| v \|^q \| M^q(\Omega) \| \leq C. \)

\[\Box\]

**Definition 3.7.** Let \( \mathcal{G}_\zeta \) be the germs of all positive solutions \( u \) of the equation \( Q_{p,A,V}(w) = 0 \) in relative punctured neighbourhoods of \( \zeta \). We say that \( \zeta \) is a regular point of the above equation if for any two positive solutions \( u, v \in \mathcal{G}_\zeta \)

\[
\lim_{x \to \zeta} u(x) \quad \text{exists in the wide sense.}
\]

Next, we define two types of positive solutions of minimal growth which was introduced by Agmon [1] for the linear case, and was later extended to \( p \)-Laplacian type equations in [2,15,16].

**Definition 3.8.** (1) Let \( K_0 \) be a compact subset of \( \Omega \). A positive solution \( u \) of the equation \( Q_{p,A,V}(u) = 0 \) in \( \Omega \setminus K_0 \) is said to be a positive solution of minimal growth in a neighbourhood of infinity in \( \Omega \) if for any smooth compact subset \( K \) of \( \Omega \) with \( K_0 \subseteq \text{int} K \) and any positive supersolution \( v \in C(\Omega \setminus \text{int} K) \) of the equation \( Q_{p,A,V}(u) = 0 \) in \( \Omega \setminus K \), we have

\[
u \leq v \quad \text{on} \quad \partial K \quad \Rightarrow \quad u \leq v \quad \text{in} \quad \Omega \setminus K.
\]

(2) A positive solution of the equation \( Q_{p,A,V}(u) = 0 \) in \( \Omega \) which has minimal growth in a neighbourhood of infinity in \( \Omega \) is called a ground state of \( Q_{p,A,V} \) in \( \Omega \).

(3) Let \( \zeta \in \partial \Omega \) and let \( u \) be a positive solution of the equation \( Q_{p,A,V}(u) = 0 \) in \( \Omega \). Then \( u \) is said to be a positive solution of minimal growth in a neighbourhood of \( \partial \Omega \setminus \{\zeta\} \) if for any relative neighbourhood \( K \subseteq \Omega \) of \( \zeta \) such that \( \Gamma := \partial K \cap \Omega \) is smooth, and any positive supersolution \( v \in C((\Omega \setminus K) \cup \Gamma) \) of the equation \( Q_{p,A,V}(u) = 0 \) in \( \Omega \setminus K \), we have

\[
u \leq v \quad \text{on} \quad \Gamma \quad \Rightarrow \quad u \leq v \quad \text{in} \quad \Omega \setminus K.
\]

**Proposition 3.9.** Suppose that \( Q_{p,A,V} \geq 0 \) in \( \Omega \), and \( Q_{p,A,V} \) has an isolated Fuchsian singularity at \( \zeta \in \partial \Omega \), where \( \zeta \in \{0, \infty\} \). Then equation (2.1) admits a positive solution in \( \Omega \) of minimal growth in a neighbourhood of \( \partial \Omega \setminus \{\zeta\} \).

**Proof.** Let \( \zeta = 0 \). By [14, Theorem 5.7], for any \( x_0 \in \Omega \), the equation \( Q_{p,A,V}(u) = 0 \) admits a positive solution \( u_{x_0} \) of the equation \( Q_{p,A,V}(u) = 0 \) in \( \Omega \setminus \{x_0\} \) of minimal growth in a neighbourhood of infinity in \( \Omega \). Note that the proof of [14, Theorem 5.7] applies also in case that \( x_0 \in \partial \Omega \) is an isolated singular point of \( \partial \Omega \). Hence, (2.1) admits a positive solution in \( \Omega \) of minimal growth in a neighbourhood of \( \partial \Omega \setminus \{0\} \).
Now consider the case when $\zeta = \infty$. Let $\{x_n\} \subset \Omega$ be a sequence such that $x_n \to \infty$. Fix a reference point $x_0 \in \Omega$, and a compact smooth exhaustion $\{\Omega_k\}$ of $\Omega$. For $n \in \mathbb{N}$, denote by $u_{x_n}$ a positive solution of the equation $Q_{p,A,V}(u) = 0$ in $\Omega \setminus \{x_n\}$ of minimal growth in a neighborhood of infinity in $\Omega$, and let $v_{x_n}(x) := u_{x_n}(x)/u_{x_n}(x_0)$. By the Harnack convergence principle, up to a subsequence, $v_{x_n}$ converges locally uniformly to $v$ which is a positive solution of the equation $Q_{p,A,V}(u) = 0$ in $\Omega$.

We claim that $v$ is a positive solution in $\Omega$ of minimal growth in a neighborhood of $\partial \Omega \setminus \{\infty\}$. Indeed, let $K \subset \hat{\Omega}$ be a punctured neighborhood of $\infty$ with smooth boundary, and let $w$ be a positive supersolution of the equation $Q_{p,A,V}(u) = 0$ in $\Omega' = \Omega \setminus K$, such that $v \leq w$ on $\partial K$. Then for any $\varepsilon > 0$ there exists $n_\varepsilon$ such that $v_{x_n} \leq (1 + \varepsilon)w$ on $\partial K$ for all $n \geq n_\varepsilon$.

Recall that by the construction of $v_{x_n}$, for a fixed $n$ we have, $v_{x_n} = \lim_{k \to \infty} v_{n,k}$, where $v_{n,k}$ restricted to $\Omega_k \cap \Omega'$ is a positive solution of the equation $Q_{p,A,V}(u) = 0$ which vanishes on $\partial \Omega_k \cap \Omega'$. Therefore, by the weak convergence principle, $v_{x_n} \leq (1 + \varepsilon)w$ in $\Omega_k \cap \Omega'$, for all $n \geq n_\varepsilon$ and therefore, $v \leq w + \varepsilon$ in $\Omega'$. Since $\varepsilon$ is arbitrarily small, we have $v \leq w$ in $\Omega'$. Thus, $v$ is a positive solution in $\Omega$ of minimal growth in a neighborhood of $\partial \Omega \setminus \{\zeta\}$.

We extend Conjecture 1.1 in [2] to the more general setting considered in the present paper. Our main goal is to prove it under some further relatively mild assumptions.

**Conjecture 3.10.** Assume that Equation (2.1) has a Fuchsian-type isolated singularity at $\zeta \in \partial \Omega$ and admits a (global) positive solution. Then

(i) $\zeta$ is a regular point of equation (2.1).

(ii) Equation (2.1) admits a unique (global) positive solution of minimal growth in a neighborhood of $\partial \Omega \setminus \{\zeta\}$.

Next, we recall the notions of subcriticality and criticality (for more details see [14]).

**Definition 3.11.** Assume that $Q_{p,A,V} \geq 0$ in $\Omega$. Then $Q_{p,A,V}$ is called *subcritical* in $\Omega$ if there exists a nonzero nonnegative function $W \in M^q_{\text{loc}}(p; \Omega)$ such that

\[
Q_{p,A,V}(\phi) \geq \int_{\Omega} W|\phi|^p \, dx \quad \text{for all } \phi \in C^\infty_c(\Omega). \tag{3.9}
\]

If this is not the case, then $Q_{p,A,V}$ is called *critical* in $\Omega$.

**Theorem 3.12** ([14]). Let $Q_{p,A,V} \geq 0$ in $\Omega$. Then the following assertions are equivalent:

(i) $Q_{p,A,V}$ is critical in $\Omega$.

(ii) The equation $Q_{p,A,V}(u) = 0$ in $\Omega$ admits a unique positive supersolution.

(iii) The equation $Q_{p,A,V}(u) = 0$ in $\Omega$ admits a ground state $\phi$.

The following uniform Harnack inequality near an isolated Fuchsian singular point $\zeta$ is a key ingredient for obtaining regularity results at $\zeta$.

**Theorem 3.13** (Uniform Harnack inequality). Let $A$ and $V$ satisfy Assumptions 2.3, and assume that $Q = Q_{p,A,V}$ has an isolated Fuchsian singularity at $\zeta \in \partial \Omega$, where $\zeta = 0$ or $\zeta = \infty$. Let $u, v \in G_\zeta$. Consider the annular set $A_r := (B_{3r/2} \setminus \bar{B}_{r/2}) \cap \Omega'$, where $\Omega'$ is a punctured neighborhood of $\zeta$. Denote

\[
a_r := \inf_{x \in A_r} \frac{u(x)}{v(x)}, \quad A_r := \sup_{x \in A_r} \frac{u(x)}{v(x)}. \]

Then there exists $C > 0$ independent of $r$, $u$ and $v$ such that

\[A_r \leq Ca_r \quad \forall r \text{ near } \zeta.\]
Proof. Fix positive solutions \( u \) and \( v \) in \( \Omega' \subset \Omega \), a fixed punctured neighbourhood of \( \zeta \). For \( r > 0 \), let us consider the annular set \( \tilde{A}_r := (B_{2r} \setminus \bar{B}_{r/4}) \cap \Omega' \). Since \( \zeta = 0 \) (respectively, \( \zeta = \infty \)) is an isolated singular point, then for \( r < r_0 \) (respectively, \( r > r_0 \)) \( A_r/r \) and \( \tilde{A}_r/r \) are fixed annulus \( A \) and \( \tilde{A} \) respectively and \( A \subseteq \tilde{A} \).

Now for such \( r \), we define \( u_r(x) := u(rx) \) for \( x \in \Omega'/r \). Then the function \( u_r \) is a positive solution of the equation

\[
Q_r[u_r] := -\Delta_{p,A_r}(u_r) + V_r(x)|u_r|^{p-2}u_r = 0 \quad \text{in} \quad \tilde{A},
\]

where \( A_r(x) := A(rx) \) and \( V_r = r^p V(rx) \). Similarly, \( v_r(x) := v(rx) \) for \( x \in \Omega'/r \) satisfies \( Q_r[v_r] = 0 \) in \( \tilde{A} \). In light of estimates (3.5) and (3.6), the norms \( \|V_r\|_{M^q(\rho,\tilde{A})} \) of the scaled potentials are uniformly bounded \( \tilde{A} \). Also, by (3.1), the matrices \( A_r(x) \) are uniformly bounded and uniformly elliptic in \( \tilde{A} \). Therefore, the local Harnack inequality (Theorem 2.6) in the annular domain \( \tilde{A} \) implies

\[
A_r = \sup_{x \in \tilde{A}} \frac{u(x)}{v(x)} = \sup_{x \in \tilde{A}} \frac{u_r(x)}{v_r(x)} \leq C \inf_{x \in \tilde{A}} \frac{u_r(x)}{v_r(x)} = C \inf_{x \in \tilde{A}} \frac{u(x)}{v(x)} = C a_r,
\]

where the constant \( C \) is independent of \( u \) and \( v \) for \( r \) near \( \zeta \). \( \square \)

The weak comparison principle (Theorem 2.8) implies the following monotonicity useful result:

**Lemma 3.14.** Let \( A \) and \( V \) satisfy Assumptions 2.3, and assume that \( u, v \in \mathcal{G}_\zeta \) are defined in a punctured neighbourhood \( \Omega' \) of \( \zeta \). For \( r > 0 \), denote

\[
m_r := \inf_{S_r \subset E \setminus \{\zeta\}} \frac{u(x)}{v(x)}, \quad M_r := \sup_{S_r \subset E \setminus \{\zeta\}} \frac{u(x)}{v(x)}.
\]

(i) The functions \( m_r \) and \( M_r \) are finally monotone as \( r \to \zeta \). Specifically, there are numbers \( 0 \leq m \leq M \leq \infty \) such that

\[
m := \lim_{r \to \zeta} m_r, \quad M := \lim_{r \to \zeta} M_r.
\]

(ii) Suppose further that \( u \) and \( v \) are both positive solutions of (2.1) in \( \Omega \) of minimal growth in \( \partial \Omega \setminus \{\zeta\} \), then \( 0 < m \leq M < \infty \) and \( m_r \searrow m \) and \( M_r \nearrow M \) when \( r \to \zeta \).

The proof of Lemma 3.14 follows the same steps as in [2, Lemma 4.2] (where \( A \) is the identity matrix and \( V \in L^\infty_{\text{loc}}(\Omega) \)), and therefore it is omitted.

**Remark 3.15.** Let \( A \in \mathbb{R}^{d \times d} \) be a symmetric, positive definite matrix. Then clearly, Lemma 3.14 also holds if the sphere \( S_r \) is replaced by the set \( \partial E_A(r) = \{ x \in \mathbb{R}^d \mid |x|_{A^{-1}} = r \} \).

The following result readily follows from the second part of Lemma 3.14.

**Corollary 3.16.** Suppose that (2.1) has a Fuchsian isolated singularity at \( \zeta \in \partial \Omega \). Let \( u, v \) be two positive solutions of (2.1) of minimal growth in a neighbourhood of \( \partial \Omega \setminus \{\zeta\} \). Then

\[
m v(x) \leq u(x) \leq M v(x) \quad x \in \Omega,
\]

where \( 0 < m \leq M < \infty \) are defined in (3.12).

As in [2], the regularity at \( \zeta \) implies a positive Liouville-type theorem.

**Proposition 3.17.** Suppose that \( Q_{p,A,V} \) has a regular and isolated Fuchsian singularity at \( \zeta \in \partial \Omega \). Then equation (2.1) admits a unique positive solution in \( \Omega \) of minimal growth in a neighbourhood of \( \partial \Omega \setminus \{\zeta\} \).
Thus, we may assume that $u$ and $v$ be two solutions of \((2.1)\) of minimal growth in a neighbourhood of $\partial \Omega \setminus \{\zeta\}$. Then by Corollary \ref{cor:exist}, we have
\[ mv(x) \leq u(x) \leq M v(x) \quad x \in \Omega, \]
where $0 < m \leq M < \infty$ are defined in \((3.11)\) and \((3.12)\). In addition, since $\zeta$ is a regular point, it follows that
\[ \lim_{x \to \zeta} \frac{u(x)}{v(x)} \quad \text{exists and is positive.} \]
Thus, we have $m = M$ and $u(x) = M v(x)$.

The following proposition asserts that the regularity of a Fuchsian singular point with respect to a limiting dilated equation implies the regularity of the corresponding singular point for the original equation \((2.1)\). The proposition extends Proposition 2.2 in [2], where $A$ is the identity matrix and $V \in L^\infty_{\text{loc}}(\Omega)$ satisfies \((1.3)\). As in [2], the proof of below relies upon the Harnack convergence principle, the WCP and the uniform Harnack inequality.

**Proposition 3.18.** Let $A, V$, satisfy Assumptions 2.3. Suppose that the operator $Q = Q_{p,A,V}$ has an isolated Fuchsian singularity at $\zeta \in \partial \hat{\Omega}$, and there is a sequence $R_n \to \zeta$, such that either $0$ or $\infty$ is a regular point of a limiting dilated equation $D^{(R_n)}(Q)(w) = 0$ in $\Omega$. Then $\zeta$ is a regular point of the equation $Q(u) = 0$ in $\Omega$.

**Proof.** Let $u, v \in \mathcal{G}_\zeta$ and set
\[ m_r := \inf_{S_r \cap \Omega'} \frac{u(x)}{v(x)}, \quad M_r := \sup_{S_r \cap \Omega'} \frac{u(x)}{v(x)}, \]
where $\Omega'$ is a punctured neighbourhood of $\zeta$. By Lemma \ref{lemma:exist}, $M := \lim_{r \to \zeta} M_r$ and $m := \lim_{r \to \zeta} m_r$ exist in the wide sense, and we need to prove that $M = m$.

Now if $M := \lim_{r \to \zeta} M_r = \infty$ (respectively, $m := \lim_{r \to \zeta} m_r = 0$), then by the uniform Harnack inequality, Lemma \ref{lemma:uniharn}, we have $m = \infty$ (respectively, $M = 0$), and hence the limit
\[ \lim_{x \to \zeta} \frac{u(x)}{v(x)} \quad \text{exists in the wide sense.} \]
Thus, we may assume that $u \asymp v$ in some neighbourhood $\Omega' \subset \Omega$ of $\zeta$. Fix $x_0 \in \mathbb{R}^d$ such that $R_n x_0 \in \Omega$ for all $n \in \mathbb{N}$ and define
\[ u_n(x) := \frac{u(R_n x)}{u(R_n x_0)}, \quad v_n(x) := \frac{v(R_n x)}{v(R_n x_0)}. \]
Then by the definition of the set $\mathcal{G}_\zeta$, $u_n$ and $v_n$ are positive solutions of the equation
\[ -\Delta_{p,A_n}(w) + V_n(x) |w|^{p-2} w = 0 \quad \text{in } \Omega'/R_n, \]
where $A_n(x) := A_{R_n}(x)$ and $V_n(x) := V_{R_n}(x)$, are the associated scaled matrix and potential, respectively. Since $u_n(x_0) = 1$ and $v_n(x_0) \asymp 1$, the Harnack convergence principle (Proposition \ref{prop:harn}) implies that $\{R_n\}$ admits a subsequence (still denoted by $\{R_n\}$) such that
\[ \lim_{n \to \infty} u_n(x) := u_\infty(x), \quad \lim_{n \to \infty} v_n(x) := v_\infty(x) \]
locally uniformly in $Y = \lim_{n \to \infty} \Omega'/R_n$, and $u_\infty$ and $v_\infty$ are positive solutions of the limiting dilated equation
\[ D^{(R_n)}(Q)(w) = -\Delta_{p,A}(w) + \nabla |w|^{p-2} w = 0 \quad \text{on } Y. \]
Thus, for any fixed \( R > 0 \) we have

\[
\sup_{x \in S_R} \frac{u_\infty(x)}{v_\infty(x)} = \sup_{x \in S_R} \lim_{n \to \infty} \frac{u_n(x)}{v_n(x)} = \lim_{n \to \infty} \sup_{x \in S_R} \frac{u_n(x)}{v_n(x)} = \lim_{n \to \infty} \sup_{x \in S_R} \frac{u(R_n x)}{v(R_n x)} = \lim_{n \to \infty} \sup_{x \in S_R} \frac{u(R_n x)}{v(R_n x)} = \lim_{n \to \infty} M_{RR_n} = M,
\]

where we have used the existence of \( \lim_{r \to \zeta} M_r = M \), and the local uniform convergence of the sequence \( \{u_n/v_n\} \) in \( Y \). Similarly, we have \( \inf_{x \in S_R} \frac{\mu_\infty(x)}{\mu_\infty(x)} = m \) Now by our assumption, either \( \zeta_1 = 0 \) or \( \zeta_1 = \infty \) is a regular point of the dilated equation (3.17), so the limit

\[
\lim_{x \to \zeta} \frac{u_\infty(x)}{v_\infty(x)} \quad \text{exists.}
\]

Hence, \( m = M \), which implies that

\[
\lim_{x \to \zeta} \frac{u(x)}{v(x)} \quad \text{exists.}
\]

Thus, \( \zeta \) is a regular point of the equation \( Q(u) = 0 \) in \( \Omega \).

\[
\Box
\]

4. Asymptotic behaviour of \((p, \mathbb{A})\)-harmonic functions

In this section, we study the regularity of positive \((p, \mathbb{A})\)-harmonic functions at \( \zeta \), when \( \mathbb{A} \in \mathbb{R}^{d \times d} \) is a fixed symmetric and positive definite matrix. We prove the following theorem.

**Theorem 4.1.** Assume that \( 1 < p < \infty \) and \( d \geq 2 \). Let \( \mathbb{A} \in \mathbb{R}^{d \times d} \) be a fixed symmetric and positive definite matrix. Then

(i) for \( p \leq d \), \( \zeta = 0 \) is a regular point of the equation \(-\Delta_{p, \mathbb{A}}(u) = 0 \) in \( \mathbb{R}^d \setminus \{0\} \).

(ii) for \( p \geq d \), \( \zeta = \infty \) is a regular point of the equation \(-\Delta_{p, \mathbb{A}}(u) = 0 \) in \( \mathbb{R}^d \).

The proof of Theorem 4.1 depends on the asymptotic behaviour of positive solutions near an isolated singularity. Before proving Theorem 4.1, we first establish the existence of a ‘fundamental solution’ (given by an explicit form) for the \((p, \mathbb{A})\)-Laplacian

\[
-\Delta_{p, \mathbb{A}}(u) = \text{div}\left(|\nabla u|^p \mathbb{A} \nabla u\right),
\]

where \( \mathbb{A} \in \mathbb{R}^{d \times d} \) is a fixed symmetric and positive definite matrix.

**Lemma 4.2** (Fundamental solution). Let \( \mathbb{A} \in \mathbb{R}^{d \times d} \) be a fixed symmetric, positive definite matrix, and let \( \mathbb{A}^{-1} \) be its inverse matrix. Fix \( y \in \mathbb{R}^d \). Let

\[
\mu(x - y) := C_{p, d, \mathbb{A}} \begin{cases} 
|x - y|_{\mathbb{A}^{-1}}^{(p-d)/(p-1)} & x \in \mathbb{R}^d, p \neq d, \\
-\log |x - y|_{\mathbb{A}^{-1}} & x \in \mathbb{R}^d, p = d,
\end{cases}
\]

where

\[
C_{p, d, \mathbb{A}} := \begin{cases} 
\frac{p-1}{d-p} (|\mathbb{A}|^{1/2} \omega_d)^{-1/(p-1)} & p \neq d, \\
(|\mathbb{A}|^{1/2} \omega_d)^{-1/(d-1)} & p = d,
\end{cases}
\]

\( |\mathbb{A}| \) is the determinant of \( \mathbb{A} \), and \( \omega_d \) is the hypersurface area of the unit sphere in \( \mathbb{R}^d \). Then

\[
-\Delta_{p, \mathbb{A}}(\mu(x - y)) = \delta_y(x) \quad \text{in } \mathbb{R}^d.
\]
Remarks 4.3. 1. Note that \( \mu \) is a positive function if and only if \( p < d \), which implies that 
\[- \Delta_{p, \mathbb{A}} \] (where \( \mathbb{A} \) is a constant matrix) is subcritical in \( \mathbb{R}^d \) if and only if \( p < d \) (see Theorem 4.6).

2. In the sequel we abuse the notation and write \( \mu(|x - y|) := \mu(x - y) \).

Proof. Denote \( C := C_{p,d,\mathbb{A}}, \) and assume first that \( p \neq d \). Without loss of generality, we may assume that \( y = 0 \). Recall that for a fixed symmetric matrix \( \mathbb{A} \in \mathbb{R}^{d \times d} \), the gradient of the associated quadratic form is given by
\[
\nabla(A.x \cdot x) = 2A.x. \tag{4.2}
\]

Therefore, the chain rule and (4.2) implies
\[
\nabla \mu(x) = C \frac{p-d}{p-1} |x|^{-(d)/(p-1)} \nabla \left( (A^{-1}x \cdot x)^{1/2} \right) = C \frac{p-d}{p-1} |x|^{-(d)/(p-1)} \frac{1}{2|x|A^{-1}} \nabla(A^{-1}x : x) = C \frac{p-d}{p-1} |x|^{-(d)/(p-1)} A^{-1} 1_x.
\]

So,
\[
|\nabla \mu(x)|_{A} = |C| \left| \frac{p-d}{p-1} \right| |x|^{-(d)/(p-1)} |A^{-1}x|_{A} = |C| \left| \frac{p-d}{p-1} \right| |x|^{-(d)/(p-1)}.
\]

Consequently,
\[
\eta(x) := |\nabla \mu(x)|_{A}^{p-2} A \nabla \mu(x) = C |C|^{p-2} c(p,d) |x|^{(1-2d)/(p-1)} |A^{-1}x|_{A} = C |C|^{p-2} c(p,d) |x|^{-d} A^{-1} x,
\]

where \( c(p,d) = \left| \frac{p-d}{p-1} \right| \). Denote \( \eta_{i}(x) := C |C|^{p-2} c(p,d) |x|^{-d} A_{i} x. \) Then by (4.2)
\[
\frac{\partial \eta_{i}(x)}{\partial x_{i}} = C |C|^{p-2} c(p,d) \left( |x|^{-d} A_{i} - d|x|^{-d-2} (A^{-1}x)_{i} x_{i} \right).
\]

Therefore, for all \( x \in \mathbb{R}^{d} \setminus \{0\} \) we have
\[
- \Delta_{p, \mathbb{A}}(\mu(x)) = - \text{div} \eta(x) = - C |C|^{p-2} c(p,d) \left( d|x|^{-d} - d|x|^{-d-2} \sum_{i=1}^{d} (A^{-1}x)_{i} x_{i} \right) = 0.
\]

Similarly, for \( p = d \), we obtain that \( C_{d,\mathbb{A}} \log(|x|_{A^{-1}}) \) satisfies \( - \Delta_{d, \mathbb{A}}(u) = 0 \) in \( \mathbb{R}^d \setminus \{0\} \).

We now find the constant \( C = C_{p,d,\mathbb{A}} \in \mathbb{R} \) such that \( \mu \) satisfies
\[
- \Delta_{p, \mathbb{A}}(\mu) = \delta_0 \quad \text{in} \quad \mathbb{R}^d, \tag{4.4}
\]
in the sense of distributions. Recall that the ellipsoid \( E_{\mathbb{A}}(r) = \{ x \in \mathbb{R}^d \mid |x|_{A^{-1}} < r \} \) with ‘center’ at the origin and ‘radius’ \( r > 0 \), is affinely equivalent to the ball \( B_r(0) \). Hence, \( E_{\mathbb{A}}(r) \) is a relatively compact, convex subset of \( \mathbb{R}^d \).

Let us first consider the case \( p \neq d \). Note that if \( p < d \), then \( \lim_{x \to 0} \mu(x) = \infty \), but nevertheless, \( \mu \) is integrable near the origin. Using the divergence theorem on the ellipsoid \( E_{\mathbb{A}}(r) \), it follows that the function \( \mu \) should satisfy
\[
-1 = \int_{E_{\mathbb{A}}(r)} \text{div} \left( |\nabla \mu|_{A}^{p-2} A \nabla \mu \right) dx = \int_{\partial E_{\mathbb{A}}(r)} |\nabla \mu|_{A}^{p-2} A \nabla \mu \cdot \mathbf{n} dS, \tag{4.5}
\]

where \( \mathbf{n} = A^{-1} x / |x|_{A^{-1}} \) is the unit outward normal to the boundary of the ellipsoid \( E_{\mathbb{A}}(r) \) and \( dS \) is the hypersurface element area. Recall that by (4.3) we have
\[
|\nabla \mu(x)|_{A}^{p-2} A \nabla \mu(x) = C |C|^{p-2} \left| \frac{p-d}{p-1} \right|^{p-2} \frac{p-d}{p-1} \frac{x}{|x|^{d}_{A^{-1}}}.
\]
and the hypersurface area of $\partial E_{d}(r)$ is given by $r^{d-1}|A|^{1/2} \omega_{d}$ (see for example, [12, p. 238]), it follows from (4.5) that

$$-1 = C|C|^{p-2} \left| \frac{p-d}{p-1} \right| \int_{\partial E_{d}(r)} \frac{1}{|x|^{d-1}} \ dS = C|C|^{p-2} \left| \frac{p-d}{p-1} \right| |A|^{1/2} \omega_{d}.$$ 

So, for $p \neq d$, we have

$$C = C_{p,d,A} = \frac{p-1}{d-p} (|A|^{1/2} \omega_{d})^{-1/(p-1)}.$$ 

Similarly, for $p = d$ one obtains $C(d, A) = (|A|^{1/2} \omega_{d})^{-1/(d-1)}$. \hfill \Box

**Theorem 4.4.** Let $1 < p \leq d$ and $A \in \mathbb{R}^{d \times d}$ be a fixed symmetric positive definite matrix. Suppose that $u$ is a positive solution of the equation $-\Delta_{p,A}(v) = 0$ in a punctured neighbourhood of 0 which has a non-removable singularity at 0, then

$$u(x) \sim \mu(x),$$ 

as $x \to 0$.

where $\mu$ is the fundamental solution of $-\Delta_{p,A}$ in $\mathbb{R}^{d}$ given by (4.1).

**Remark 4.5.** For the case when $A = I$, Theorem 4.4 has been proved in [6, Theorem 1.1 and 7], see also [4]. We give a slightly different proof of Theorem 4.4 by using Lemma 3.14.

**Proof of Theorem 4.4.** Assume that $1 < p < d$, the proof for the case when $p = d$ needs only minor modifications, and therefore, it is omitted.

It is known [18, 19] that any positive solution $v$ of the equation $-\Delta_{p,A}(u) = 0$ in a punctured ball $B_{r} \setminus \{0\}$ has either a removable singularity at 0, or else,

$$v(x) \asymp \mu(x) \quad \text{as } x \to 0.$$ 

Since $u$ has a nonremovable singularity at 0, it follows that there exists $C > 0$ such that $C^{-1} \mu(x) \lesssim v(x) \lesssim C \mu(x)$ for all $x$ in a small punctured neighbourhood of 0.

Let $\{x_{n}\}$ be a sequence converging to 0. Denote $r_{n} = |x_{n}|_{A}^{-1}$, and define

$$M_{n} := \sup_{\partial E_{d}(r_{n})} u(x)/\mu(x), \quad m_{n} := \inf_{\partial E_{d}(r_{n})} u(x)/\mu(x).$$

Then the sequence $\{M_{n}\}$ is bounded and bounded away from 0. Moreover, by Lemma 3.14 and Remark 3.15, the sequence $\{M_{n}\}$ is finally monotone. Let $M := \lim_{n \to \infty} M_{n}$. Then

$$\lim_{n \to \infty} \left[ \sup_{\partial E_{d}(r_{n})} (u/\mu - M) \right] = 0.$$ 

Fix $x_{0} \in \mathbb{R}^{d}$ such that $r_{n} x_{0} \in \Omega^{\ast} = \Omega \setminus \{0\}$ for all $n \in \mathbb{N}$ and consider the following functions

$$u_{n}(x) := \frac{u(r_{n}x)}{\mu(r_{n}x_{0})}, \quad \mu_{n}(x) := \frac{\mu(r_{n}x)}{\mu(r_{n}x_{0})}.$$ 

Then $u_{n}$ and $\mu_{n}$ are positive solution of the equation $-\Delta_{p,A}(w) = 0$ in $\Omega^{\ast} / r_{n}$. Note that $\mu_{n}(x) = |x_{0}|_{A}^{-d} / |x|_{A}^{-d}$ with $\mu_{n}(x_{0}) = 1$, hence, $\mu_{n}$ does not depend on $n$. On the other hand, $u_{n}(x_{0}) \propto 1$, hence, the Harnack convergence principle implies that, up to a subsequence,

$$\lim_{n \to \infty} u_{n}(x) = u_{\infty}(x)$$ 

locally uniformly in \( \mathbb{R}^d \setminus \{0\} \) and \( u_\infty \) is a positive solution of \( -\Delta_{p,\mathcal{A}}(w) = 0 \) in \( \mathbb{R}^d \setminus \{0\} \). Then for any fixed \( R > 0 \), as in Proposition 3.18, it follows that

\[
M = \sup_{x \in \partial E_h(R)} \frac{u_\infty(x)}{\mu(x)}.
\]

Hence, for any \( R > 0 \), we have \( u_\infty(x) \leq M \mu(x) \) for all \( x \in \partial E_h(R) \). Note that \( \nabla \mu \neq 0 \). Recall the strong comparison principle, [2, Theorem 3.2] which is proved for the case where the principal part of the operator \( Q \) is the \( p \)-Laplacian. Nevertheless, it is easy to check that the proof applies also to our setting, and in particular, for the \((p,\mathcal{A})\)-operator. Hence, the strong comparison principle implies that \( u_\infty(x) = M \mu(x) \).

Similarly, let \( m := \lim_{n \to \infty} m_n \), it follows that for any \( R > 0 \), we have \( m \mu(x) \leq u_\infty(x) \) for all \( x \in \partial E_h(R) \), and consequently, \( u_\infty(x) = m \mu(x) \). Therefore, \( M = m \), and this implies that

\[
\lim_{n \to \infty} \|u/\mu - M\|_{L^\infty(\partial E_h(|x|_{\mathcal{A}}-1))} = 0.
\]

In other words, \( u \) is almost equal to \( M \mu \) on a sequence of concentric ellipsoids converging to 0. Using the WCP in the annuli

\[
\mathcal{A}_n := \{|x_{n+1}|_{\mathcal{A}}-1 \leq |x|_{\mathcal{A}}-1 \leq |x_n|_{\mathcal{A}}-1\}, \quad n \geq 1,
\]

it follows that

\[
\lim_{r \to 0} \|u/\mu - M\|_{L^\infty(\partial E_h(r))} = 0.
\]

Finally we note that it can be easily verified that \( M \) is independent of the choice of the sequence \( \{x_n\} \). Thus, the theorem is proved.

Similar to the case of the \( p \)-Laplacian in \( \mathbb{R}^d \) we have:

**Theorem 4.6.** Assume that \( \mathcal{A} \in \mathbb{R}^{d \times d} \) is a symmetric, positive definite matrix. Then the operator \( -\Delta_{p,\mathcal{A}} \) is critical in \( \mathbb{R}^d \) if and only if \( p \geq d \).

**Proof.** If \( p < d \), then by the Hardy inequality for the \( p \)-Laplacian

\[
\int_{\mathbb{R}^d} |\nabla \varphi|^p \, dx \geq \left( \frac{d - p}{p} \right)^p \int_{\mathbb{R}^d} \frac{|\varphi|^p}{1 + |x|^p} \, dx \quad \forall \varphi \in C_c^\infty(\mathbb{R}^d),
\]

the operator \( -\Delta_{p,\mathcal{A}} \) is subcritical in \( \mathbb{R}^d \).

Suppose now that \( p \geq d \). By Theorem 3.12, the following Dirichlet problem admits a unique positive solution \( w_k \):

\[
\begin{cases}
\Delta_{p,\mathcal{A}}(w_k) = 0 & \text{in } B_k \setminus \overline{B}_1, \\
w_k(x) = 1 & \text{on } S_1, \\
w_k(x) = 0 & \text{on } S_k.
\end{cases}
\]

By the WCP, \( \{w_k\}_{k \in \mathbb{N}} \) is an increasing sequence satisfying \( 0 \leq w_k \leq 1 \), and therefore, converging to a positive solution \( w \) of \( \Delta_{p,\mathcal{A}}(v) = 0 \) in \( \mathbb{R}^d \setminus \overline{B}_1 \), that clearly has minimal growth at infinity in \( \mathbb{R}^d \). Thus, it is enough to show that \( w = 1 \) in \( \mathbb{R}^d \setminus \overline{B}_1 \). We obviously have \( w \leq 1 \). On the other hand, since \( |\mu(x)| \to \infty \) as \( x \to \infty \), it follows that for any \( \varepsilon > 0 \), there is \( k_\varepsilon \) such that \( 1 - \varepsilon |\mu(x)| \leq w_k \) obviously on \( S_1 \) and also on \( S_k \) for every \( k \geq k_\varepsilon \). Invoking again the WCP, it follows that \( 1 - \varepsilon |\mu| \leq w \) in \( B_k \setminus B_1 \) and it follows \( 1 - \varepsilon |\mu| \leq w \in \mathbb{R}^d \setminus B_1 \). By letting \( \varepsilon \to 0 \), we conclude that \( 1 \leq w \). Thus, \( w = 1 \) in \( \mathbb{R}^d \setminus B_1 \), and \( u_0 = 1 \) is a ground state. Hence, by Theorem 3.12, the operator \( -\Delta_{p,\mathcal{A}} \) is critical in \( \mathbb{R}^d \). \( \square \)
Corollary 4.7. Assume that $1 < p < \infty$, and $\Lambda \in \mathbb{R}^{d \times d}$ is a symmetric, positive definite matrix. Let $u$ be a positive solution of the equation $-\Delta_{p,\Lambda}(u) = 0$ in a neighbourhood of infinity. Then $\lim_{x \to \infty} u(x)$ exists in the wide sense.

Moreover, if $p \geq d$ (resp., $p < d$), then $\lim_{x \to \infty} u(x) \neq 0$ (resp., $\lim_{x \to \infty} u(x) \neq \infty$).

Proof. From Lemma 3.14 (with $v = 1$), it follows that the functions given by

$$m_r := \inf_{x \in S_r} u(x), \quad M_r := \sup_{x \in S_r} u(x)$$

are monotone for large enough $r$. If $\lim_{r \to \infty} m_r = \infty$, then clearly $\lim_{x \to \infty} u(x) = \infty$.

Assume now that $m = \lim_{r \to \infty} m_r < \infty$. Then for any $\varepsilon > 0$ the function $u - m + \varepsilon$ is a positive solution of $\Delta_{p,\Lambda}(w) = 0$ in some neighbourhood infinity. Then by the uniform Harnack inequality (3.13), we get

$$M_r - m + \varepsilon \leq C(M_r - m + \varepsilon),$$

for large enough $r$. By taking $r \to \infty$, we get $0 \leq M - m \leq (C - 1)\varepsilon$. This implies that $M = m < \infty$, and $\lim_{|x| \to \infty} u(x) = m = M < \infty$. Thus, $u$ has a finite limit as $x \to \infty$.

Let $p < d$, and suppose that there exists a positive $(p, \Lambda)$-harmonic function $u$ in a neighbourhood such that $\lim_{x \to \infty} u(x) = \infty$. By repeating the proof of Theorem 4.6, with $u$ replacing $\mu$, it would follow that $-\Delta_{p,\Lambda}$ is critical in $\mathbb{R}^d$, a contradiction to Theorem 4.6.

It remains to prove that $\lim_{x \to \infty} u(x) \neq 0$ if $p \geq d$. By Theorem 4.6, $\Delta_{p,\Lambda}$ is critical in $\mathbb{R}^d$ with a ground state $u_0 = 1$. Since a ground state is an entire positive of minimal growth at infinity, it follows that $\lim_{x \to \infty} u(x) \neq 0$. Hence, the lemma follows. \hfill $\square$

Next we discuss the asymptotic behaviour of positive $(p, \Lambda)$-harmonic functions at $\infty$ for $p \geq d$.

Theorem 4.8. Assume that $p \geq d$ and $\Lambda \in \mathbb{R}^{d \times d}$ is a symmetric, positive definite matrix. Let $u$ be a positive solution of the equation $-\Delta_{p,\Lambda}(w) = 0$ in a neighbourhood of infinity in $\mathbb{R}^d$. Then either $u$ has a (finite) positive limit as $x \to \infty$, or

$$u(x) \sim_{x \to \infty} -\mu(x),$$

where $\mu$ is the fundamental solution of $-\Delta_{p,\Lambda}$ in $\mathbb{R}^d$ given by (4.1).

To show this theorem, we use a Kelvin-type transform (see, Definition A.1 of [2] for $\Lambda = I$).

Definition 4.9. For $x \in \mathbb{R}^d$, we denote by $\bar{x} := x/|x|_{\Lambda^{-1}}^2$. Then $\bar{x}$ is the inverse point with respect to the ellipsoid $E_{\Lambda}(1)$. In particular, $|\bar{x}|_{\Lambda^{-1}} = 1/|x|_{\Lambda^{-1}}$. Let $u$ be a function either defined in the ellipsoid $E_{\Lambda}(1) \setminus \{0\}$, or on $\mathbb{R}^d \setminus E_{\Lambda}(1)$. The generalized Kelvin transform of $u$ is given by

$$K[u](x) := u(\bar{x}) = u(x/|x|_{\Lambda^{-1}}^2).$$

For $p = d$, the Dirichlet integral $\int_{\Omega} |\nabla u|^d_{\Lambda} \, dx$ is conformally invariant since $\lambda_{\min} |\nabla u|^d_{\Lambda} \leq |\nabla u|^d_{\Lambda} \leq \lambda_{\max} |\nabla u|^d_{\Lambda}$, where $\lambda_{\min}, \lambda_{\max}$ are the lowest and greatest eigenvalues of $\Lambda$. The $(d, \Lambda)$-harmonic equation $-\Delta_{d,\Lambda}(u) = 0$ is therefore, invariant under the generalized Kelvin transform. In particular, if $u$ is $(d, A)$-harmonic, then $K[u]$ is also $(d, A)$-harmonic (see also, Lemma 4.10). Hence, for $p = d$, Theorem 4.8 follows from Theorem 4.4.

Lemma 4.10. Assume that $p > d$, and let $\Lambda \in \mathbb{R}^{d \times d}$ be a symmetric, positive definite matrix. Set $\beta := 2(p - d)$. Suppose that $u$ is a solution of $-\Delta_{p,\Lambda}(u) = 0$ in a neighbourhood of infinity (respectively, in a punctured neighbourhood of origin).
Then \( v := K[u] \) is a solution of the equation
\[
- \text{div}(B(v)) := -\text{div}(|x|^{p-2}_A \nabla v) = 0,
\]
in a punctured neighbourhood of origin (respectively, in a neighbourhood of infinity).

Proof. Denote \( \tilde{x}_i := x_i/|x|_A^{2} \). By using the chain rule and (4.2), it follows that
\[
\nabla v(x) = |\tilde{x}|_A^{2} \nabla u(\tilde{x}) - 2(\nabla u(\tilde{x}) \cdot \tilde{x})\tilde{x},
\]
where \( \tilde{\nabla} \) denotes the gradient with respect to \( \tilde{x} \). Therefore,
\[
|\nabla v(x)|_A^{2} = [\tilde{x}| \nabla u(\tilde{x}) - 2(\nabla u(\tilde{x}) \cdot \tilde{x})\tilde{x}] \cdot [\tilde{x}| \nabla u(\tilde{x}) - 2(\nabla u(\tilde{x}) \cdot \tilde{x})\tilde{x}] = |\nabla u(\tilde{x})|_A^{4} - 4|\nabla u(\tilde{x})|_A^{2} + |\nabla u(\tilde{x})|_A^{4}.
\]
Thus, \( |\nabla v(x)|_A = |\nabla u(\tilde{x})|_A^{2} \).

Consider \( B(v) = |x|_A^{\beta-2} \nabla v \), where \( \beta = 2(p - d) \). Following the same steps of the computation in [2, Lemma A.1], we conclude that
\[
\text{div}(B(v)) = \text{div}(|x|_A^{\beta} \nabla v) = |x|_A^{2d} \Delta_{p,A}(u(\tilde{x})) = 0.
\]
\( \square \)

Remark 4.11. By Lemma 4.2, \( x|_A^{(p-d)/(p-1)} \) is a positive \((p,A)\)-harmonic function in the punctured space. Lemma 4.10 implies that \( \text{div}(B(|x|_A^{d-p/(p-1)})) = 0 \) in the punctured space.

The following two lemmas are the analogous results for the \( p \)-Laplacian proved in [2, Appendix A]. For the completeness, we provide the proof.

Lemma 4.12. Assume that \( p > d \), and \( A \in \mathbb{R}^{d \times d} \) is symmetric, positive definite matrix. Let \( u \) be a solution of the equation \( -\Delta_{p,A}(u) = 0 \) in a neighbourhood of infinity with \( \lim_{x \to \infty} u(x) = \infty \). Choose \( R > 0 \) and \( c > 0 \) such that \( v_c := K[u](x) - c \) is positive near the origin and negative on \( \partial E_{A}(R) \). Then there exists \( C > 0 \) such that for any \( \varphi \in C_0^1(E_{A}(R)) \) which equals 1 near the origin, we have
\[
\int_{E_{A}(R)} B[v_c] \cdot \nabla \varphi \, dx = C.
\]

Proof. The difference of any two such \( \varphi \) has a compact support in \( E_{A}(R) \setminus \bar{E}_{A}(0, \varepsilon) \) for some \( \varepsilon > 0 \). Since \( v_c \) satisfies \( -\text{div}(B(v_c)) = 0 \) in \( E_{A}(R) \setminus \bar{E}_{A}(0, \varepsilon) \), therefore it follows that
\[
\int_{E_{A}(R)} B[v_c] \cdot \nabla \varphi \, dx = \text{constant} = C.
\]

We show that the constant \( C \) is positive. For this, we choose the following test function:
\[
\varphi_{\nu}(x) := \begin{cases} 0 & v_c(x) \leq 0, \\ v_c(x) & 0 < v_c(x) < \nu, \\ \nu & v_c(x) \geq \nu. \end{cases}
\]

Therefore, we have
\[
C = \int_{E_{A}(R)} B[v_c] \cdot \nabla \varphi_{1} \, dx = \int_{\{x \in E_{A}(R) : 0 < v_c < 1\}} |x|_A^{\beta-2} \nabla v_c \cdot \nabla \varphi_{1} \, dx > 0. \quad \square
\]

Lemma 4.13. Assume that \( p > d \), and \( A \in \mathbb{R}^{d \times d} \) is a symmetric, positive definite matrix. Let \( v_c(x) \) be the solution as in Lemma 4.12. Then there exists \( \varepsilon > 0 \) such that
\[
v_c > |x|_A^{d-p/(p-1)} \quad \text{in} \quad E_{A}(\varepsilon) \setminus \{0\}. \quad (4.9)
\]
Proof. For $0 < r < R$, consider

$$m_r = \inf_{x \in E_r(R)} v_c(x) \quad \text{and} \quad M_r = \sup_{x \in \partial E_r(R)} v_c(x).$$

Since $\lim v_c(x) = \infty$, Remark 3.15 (with $v = 1$) implies that the functions $m_r, M_r$ are non-decreasing when $r \to 0$. We show that there exists constants $C_1$ and $C_2$ such that

$$m_r \leq C_1 r^{(d-p)/(p-1)} \leq C_2 M_r \quad \text{for all} \ 0 < r < r_0,$$

for some $r_0 > 0$. Then by applying the uniform Harnack inequality to the $(p, \mathbb{A})$-harmonic function $u$, the claim of the lemma will follow.

Let $\varphi$ as defined above, and note that $\varphi(x) = \nu$ near origin. Thus, by Lemma 4.12, we have

$$Cm_r = \int_{E_r(R)} B[v_c] \cdot \nabla \varphi_m \, dx = C_1 \int_{E_r(R)} |x|^\beta_A \nabla \varphi \, dx \geq C' \frac{\lambda_{\min}^p}{\lambda_{\max}^p} m_r^p \cap_{p, \beta}(B_r, R),$$

where $\lambda_{\min}$ and $\lambda_{\max}$ denote the lowest and greatest eigenvalue of the matrix $\mathbb{A}$ and $\cap_{p, \beta}(B_r, R)$ is the weighted $p$-capacity of the ball $B_r$ in $B_R$ with respect to the measure $|x|^\beta$. Then by [5, Example 2.2], it follows that

$$\cap_{p, \beta}(B_r, R) = C'(r^{(p-d-\beta)/(p-1)} - R^{(p-d-\beta)/(p-1)})^{1-p}.$$ 

Since $(p - d - \beta)/(p - 1) = (d - p)/(p - 1)$, we have

$$Cm_r^{1-p} \geq C'(r^{(d-p)/(p-1)} - R^{(d-p)/(p-1)})^{1-p},$$

which implies that

$$m_r \leq C'_1 \left(r^{(d-p)/(p-1)} - R^{(d-p)/(p-1)}\right) \leq C_1 r^{(d-p)/(p-1)}.$$

Next we show $r^{(d-p)/(p-1)} \leq C_2 M_r$, for some $C_2 > 0$. Denote $\alpha = (d-p)/(p-1)$. For $0 < r < R$, consider the following test function

$$\psi_r(x) := \begin{cases} 1 & |x|_{\mathbb{A}} - 1 < r, \\ |x|_{\mathbb{A}} - 1 \leq |x|_{\mathbb{A}} - 1 \leq R, \\ 0 & |x|_{\mathbb{A}} - 1 > R. \end{cases}$$

By Lemma 4.12 and using the Hölder inequality, we have

$$C = \int_{E_{\mathbb{A}}(R)} \nabla \psi_r \cdot B[v_c] \, dx \leq \left( \int_{E_{\mathbb{A}}(R) \setminus E_{\mathbb{A}}(R)} |x|_{\mathbb{A}}^\beta \nabla \psi_r |^p_A \, dx \right)^{1/p} \left( \int_{E_{\mathbb{A}}(R) \setminus E_{\mathbb{A}}(R)} |x|_{\mathbb{A}}^\beta \nabla v_c |^p_A \, dx \right)^{(p-1)/p}. \quad (4.10)$$

Now,

$$\int_{E_{\mathbb{A}}(R) \setminus E_{\mathbb{A}}(R)} |x|_{\mathbb{A}}^\beta \nabla \psi_r |^p_A \, dx = \frac{C'}{(r^\alpha - R^\alpha) \alpha} \left(R^{(\alpha+1)p+\beta+d} - R^{(\alpha-1)p+\beta+d}\right) = \frac{C'}{(r^\alpha - R^\alpha)^{p-1}};$$

where we used $(\alpha - 1)p + \beta + d = \alpha$. Thus, for small $r$, we get

$$\int_{E_{\mathbb{A}}(R) \setminus E_{\mathbb{A}}(R)} |x|_{\mathbb{A}}^\beta \nabla \psi_r |^p_A \, dx \leq C'r^{-\alpha(p-1)}. \quad (4.11)$$

For the second term of (4.10), we note that $v_c = \psi_M$ in $\{0 \leq v_c \leq M_r\}$ which is a subset of $E_{\mathbb{A}}(R) \setminus E_{\mathbb{A}}(R)$. Thus we have

$$\int_{E_{\mathbb{A}}(R) \setminus E_{\mathbb{A}}(R)} |x|_{\mathbb{A}}^\beta \nabla v_c |^p_A \, dx \leq \int_{\{0 \leq v_c \leq M_r\}} |x|_{\mathbb{A}}^\beta \nabla v_c |^p_A \, dx \leq \int_{E_{\mathbb{A}}(R)} B[v_c] \cdot \nabla v_M \, dx = CM_r. \quad (4.12)$$
Therefore, from (4.10), (4.11) and (4.12), we get
\[ C_2' \leq r^{\alpha(p-1)/p} M_r^{(p-1)/p}, \]
which shows that \( r^{(d-p)/(p-1)} \leq C_2 M_r. \]

**Proof of Theorem 4.8.** Let \( p > d \). In light of Corollary 4.7, we need only to consider the case \( u(x) \to \infty \) as \( x \to \infty \). Since Lemma 4.13 implies that \( v(x) := K[u](x) \propto |x|^{(d-p)/(p-1)} \) near the origin, we need to show that in fact, \( v(x) := K[u](x) \sim |x|^{(d-p)/(p-1)} \) as \( x \to 0 \). Then in light of Lemma 4.10, \( u(x) \sim |x|^{(p-d)/(p-1)} \) as \( x \to \infty \).

We follow the proof of [2, Theorem 2.3]. For \( 0 < \sigma < 1 \), define \( w_\sigma(x) := v(x)/\sigma^\alpha \) where \( \alpha = (d-p)/(p-1) \). Since \( v_c \propto |x|^{(\alpha)}_{A-1} \) in \( E_\delta(\epsilon) \setminus \{0\} \), it follows that \( w_\sigma(x) \propto |x|^{(\alpha)}_{A-1} \) in \( E_\delta(\epsilon/\sigma) \setminus \{0\} \) for some \( \epsilon > 0 \) and also the family \( \{w_\sigma\}_{0<\sigma<1} \) is locally bounded. By extracting a subsequence \( \sigma_n \to 0 \), we have that the sequence \( \{w_{\sigma_n}\} \) converges locally uniformly to \( w(x) \) in \( \mathbb{R}^d \setminus \{0\} \). Moreover, \( w \) is a positive solution of the equation
\[-\text{div} \ (B(u)) = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \{0\}.\]

Then by Remark 3.15, we have
\[ m := \lim_{r \to 0} m_r = \lim_{r \to 0} \inf_{x \in \partial E_\delta(r)} \frac{v(x)}{r^\alpha}, \quad M := \lim_{r \to 0} M_r = \lim_{r \to 0} \sup_{x \in \partial E_\delta(r)} \frac{v(x)}{r^\alpha}. \]
This implies that \( m|x|^{(\alpha)}_{A-1} \leq w(x) \leq M|x|^{(\alpha)}_{A-1} \). Indeed, for any \( R > 0 \) we have
\[ \inf_{x \in \partial E_\delta(R)} \frac{w(x)}{|x|^{(\alpha)}_{A-1}} = \inf_{x \in \partial E_\delta(R)} \frac{w_{\sigma_n}(x)}{|x|^{(\alpha)}_{A-1}} = \lim_{n \to \infty} \inf_{x \in \partial E_\delta(R)} \frac{w_{\sigma_n}(x)}{|x|^{(\alpha)}_{A-1}} = \lim_{n \to \infty} \frac{v_{\sigma_n}(x)}{|x|^{(\alpha)}_{A-1}} = \lim_{n \to \infty} \frac{v(x)}{|x|^{(\alpha)}_{A-1}} = \lim_{n \to \infty} \frac{v_{\sigma_n}(x)}{|x|^{(\alpha)}_{A-1}} = \lim_{n \to \infty} m_{\sigma_n R} = m, \]
where we used the local uniform convergence of \( \{w_{\sigma_n}(x)/|x|^{(\alpha)}_{A-1}\} \). Similarly, we have
\[ \sup_{x \in E_\delta(R)} \frac{w(x)}{|x|^{(\alpha)}_{A-1}} = M \quad \forall R > 0. \]
Hence,
\[ m|x|^{(\alpha)}_{A-1} \leq w(x) \leq M|x|^{(\alpha)}_{A-1} \quad \forall R > 0. \]
Note that \( |x|^{(\alpha)}_{A-1} \) is a positive solution of \(-\text{div} \ (B(u)) = 0 \) in \( \mathbb{R}^d \setminus \{0\} \) and the function \( |x|^{(\alpha)}_{A-1} \) does not have any critical point. Hence, by the strong comparison principle (see, [2, Theorem 3.2]) which is valid also for the \((p, A)\)-operator, we obtain \( m|x|^{(\alpha)}_{A-1} = w(x) = M|x|^{(\alpha)}_{A-1} \), and hence, \( m = M \).

**Proof of Theorem 4.1.** The proof follows directly from theorems 4.4 and 4.8.

**5. Weak Fuchsian singularity and positive Liouville theorems**

In this section we introduce the notion of weak Fuchsian singularity, and prove Conjecture 3.10 for \( Q \) which has weak Fuchsian singularity at \( \zeta \) (see, Theorem 5.4).

**Definition 5.1.** Let \( A \) and \( V \) satisfy Assumptions 2.3. Assume that \( Q \) has an isolated Fuchsian singularity \( \zeta \in \partial \Omega \), where \( \zeta = 0 \) or \( \zeta = \infty \). The operator \( Q = Q_{p,A,V} \) is said to have a weak
Fuchsian singularity at \( \zeta \) if there exist \( m \) sequences \( \{R_n^{(j)}\}_{n=1}^\infty \subset \mathbb{R}_+ \), \( 1 \leq j \leq m \), satisfying \( R_n^{(j)} \to \zeta^j \), where \( \zeta^{(1)} = \zeta \), and \( \zeta^{(j)} = 0 \) or \( \zeta^{(j)} = \infty \) for \( 2 \leq j \leq m \), such that
\[
\mathcal{D}^{(R_n^{(m)})} \circ \cdots \circ \mathcal{D}^{(R_n^{(1)})}(Q) = -\Delta_{p,A}(w) \quad \text{on } Y,
\]
where \( \Lambda \in \mathbb{R}^{d \times d} \) is a symmetric, positive definite matrix, and \( Y = \lim_{n \to \infty} \Omega/R_n^{(1)}. \)

**Remark 5.2.** Example 2.1 in [2] demonstrates that \( m \) in (5.1) might be greater than 1. Moreover, although in this example \( V \not\in M^q(p; B_1 \setminus \{0\}) \), the corresponding operator has a weak Fuchsian singularity at \( \zeta = 0 \).

The next example shows that if \( \zeta = 0 \) and \( V \in M^q(p; \Omega) \) for some punctured neighborhood \( \Omega \) of the origin, and \( A \) is continuous at 0, then \( Q \) has weak Fuchsian singularity at \( \zeta = 0 \).

**Example 5.3.** Assume that \( A \in L^\infty_{\text{loc}}(\Omega; \mathbb{R}^{d \times d}) \) is continuous at the isolated singular point \( \zeta = 0 \).

Let \( V \in M^q_{\text{loc}}(p; \Omega) \) has a Fuchsian singularity at \( 0 \in \partial \hat{\Omega} \). Further suppose that \( V \in M^q(p; B_1 \cap \Omega) \). Then for any smooth function \( \varphi \) having compact support in \( B_\varepsilon \setminus \{0\} \) we have
\[
\left| \int_{\Omega/R} R^pV(Rx)\varphi(x)dx \right| \leq R^{n-d} \int_{\Omega} |V(x)||\varphi(x/R)|dx \leq R^{p-d} \|\varphi\|_\infty \int_{\Omega \cap B_{Rr}} |V(x)|dx.
\]
Take \( R > 0 \) small enough such that \( \Omega \cap B_{Rr} \subset \Omega \cap B_1 \). Then for \( p < d \), (5.2) implies
\[
\left| \int_{\Omega/R} R^pV(Rx)\varphi(x)dx \right| \leq R^{p-d}(Rr)^{d/q'} \|\varphi\|_\infty \frac{1}{(Rr)^{d/q'}} \int_{\Omega \cap B_{Rr}} |V(x)|dx
\]
while for \( p > d \)
\[
\left| \int_{\Omega/R} R^pV(Rx)\varphi(x)dx \right| \leq \|\varphi\|_\infty \|V\|_{M^q(p;\Omega \cap B_1)} R^{p-d/q} \to 0.
\]
Similarly, for \( p = d \) it can be seen that
\[
\left| \int_{\Omega/R} R^pV(Rx)\varphi(x)dx \right| \leq \|\varphi\|_\infty \|V\|_{M^q(p;\Omega \cap B_1)} \frac{1}{\log^{d/q'}(1/Rr)} \to 0.
\]
Therefore, the operator \( Q_{p,A,V} \) has a weak Fuchsian singularity at 0.

**Theorem 5.4** (Liouville theorem). Let \( A \) and \( V \) satisfy Assumptions 2.3. Suppose that \( \zeta \in \partial \hat{\Omega} \) is an isolated singular point. Assume that the operator \( Q = Q_{p,A,V} \) has a weak Fuchsian singularity at \( \zeta \). Then \( \zeta \) is a regular point of Equation (2.1).

In other words, if \( u \) and \( v \) are two positive solutions of the equation \( Q_{p,A,V}(w) = 0 \) in a punctured neighborhood of \( \zeta \), then
\begin{enumerate}
  \item \( \lim_{x \to \zeta} u(x) \) exists in the wide sense.
  \item the equation \( Q_{p,A,V}(w) = 0 \) admits a unique positive solution in \( \Omega \) of minimal growth in a neighborhood of \( \partial \hat{\Omega} \setminus \{\zeta\} \).
\end{enumerate}

**Proof.** By Proposition 3.17, we have (i) \( \Rightarrow \) (ii). Thus, we only need to show that \( \lim_{x \to \zeta} u(x) \) exists in the wide sense. Since the operator \( Q \) has a weak Fuchsian singularity at \( \zeta \), we have
\[
\mathcal{D}^{(R_n^{(m)})} \circ \cdots \circ \mathcal{D}^{(R_n^{(1)})}(Q) = -\Delta_{p,A}(w) = 0 \quad \text{in } \mathbb{R}^d \setminus \{0\},
\]

(5.3)
where \( A \in \mathbb{R}^{d \times d} \) is a symmetric, positive definite matrix. Recall that by Theorem 4.1 either 0 or \( \infty \) is a regular point of \(-\Delta_{p,A}\). Therefore, Proposition 3.18 and a reverse induction argument implies that \( \zeta \) is a regular point of the equation \( Q_{p,A,V}(w) = 0 \).

\[ \square \]

6. Positive Liouville theorem in the elliptically symmetric case

This section is devoted to the proof of Conjecture 3.10 in the elliptically symmetric case.

**Definition 6.1.** Let \( A \in \mathbb{R}^{d \times d} \) be a symmetric, positive definite matrix. We say that \( f : \Omega \to \mathbb{R} \) is \emph{elliptically symmetric with respect to} \( A \) if \( f(x) = f(|x|_{A^{-1}}) \) for all \( x \in \Omega \), where \( \tilde{f} : \mathbb{R}_+ \to \mathbb{R} \).

In the sequel, with some abuse of notation, we omit the distinction between \( f \) and \( \tilde{f} \).

Throughout the present section we fix \( A \in \mathbb{R}^{d \times d} \) and assume that the potential \( V \in M^q_{\text{loc}}(\Omega) \) is elliptically symmetric with respect to \( A \) i.e., \( V(x) = V(|x|_{A^{-1}}) \).

Denote \( r = |x|_{A^{-1}} \), and let us calculate \( \Delta_{p,A}(f(r)) = \text{div}(|\nabla f(r)|_A^{p-2}A\nabla f(r)) \).

Using (4.2), we obtain

\[
\nabla f(r) = f'(r)\frac{A^{-1}x}{r} \quad \text{and} \quad |\nabla f(r)|_A = \frac{|f'(r)|}{r} |A^{-1}x|_A = |f'(r)|.
\]

Consequently, \( \eta := |\nabla f(r)|_A^{p-2}A\nabla f(r) = |f'(r)|^{p-2}f'(r)x/r \), and

\[
\partial \eta_i/\partial x_i = \frac{\frac{f'(r)^{p-2}f''(r)}{r} + \frac{x_i(A^{-1}x)_i}{r} \left[ - \frac{f'(r)^{p-2}f''(r)}{r^2} + (p-1) \frac{f'(r)^{p-2}f''(r)}{r} \right]}{r}.
\]

Therefore, we get

\[
\Delta_{p,A}(f(r)) = \sum_{i=1}^d \frac{\partial \eta_i}{\partial x_i} = |f'(r)|^{p-2} \left[ (p-1)f''(r) + d \frac{d-1}{r} f'(r) \right], \quad \text{where } r = |x|_{A^{-1}}.
\]

\[ (6.1) \]

**Lemma 6.2.** Let \( A \in \mathbb{R}^{d \times d} \) be a symmetric, positive definite matrix. Assume that the domain \( \Omega \) and the potential \( V \) are elliptically symmetric with respect to \( A \) and the equation \( Q_{p,A,V}(u) = 0 \) possess a positive solution. Further, suppose that the operator \( Q_{p,A,V} \) has a Fuchsian isolated singularity at \( \zeta \in \{0, \infty\} \). Then for any \( u \in G_\zeta \), there exists an elliptically symmetric (with respect to \( A \)) solution \( \tilde{u} \in G_\zeta \) such that \( u \asymp \tilde{u} \).

\[ \text{Proof.} \quad \text{We consider the case } \zeta = 0, \text{the case when } \zeta = \infty, \text{can be shown similarly. Fix } R > 0 \text{ such that } u \text{ is defined in the punctured ellipsoid } E_h(2R) \setminus \{0\}. \text{ Then for } 0 < \varrho < R, \text{ consider the following Dirichlet problem}
\]

\[
\begin{cases}
Q_{p,A,V}(w) = 0 & \text{in } E_h(R) \setminus \bar{E}_h(\varrho), \\
w(x) = m_R & x \in \partial E_h(R) \\
w(x) = m_\varrho & x \in \partial E_h(\varrho),
\end{cases}
\]

(6.2)

where \( m_r = \inf_{x \in \partial E_h(r)} u(x). \) By Lemma 2.5, there exists a unique solution \( u_{\varrho,R} \) to the Dirichlet problem (6.2). Moreover, from the unique solvability of the one-dimensional Dirichlet problem it follows that \( u_{\varrho,R} \) is elliptically symmetric with respect to \( A \). Moreover, by the uniform Harnack inequality (Theorem 3.13) and the WCP we have

\[
u_{\varrho,R} \leq u \leq Cu_{\varrho,R} \quad \text{in } E_h(R) \setminus \bar{E}_h(\varrho),
\]

where \( C > 0 \) is independent of \( \varrho \).
Applying the Harnack converging principle, it follows that there exists a sequence $q_n \to 0$ such that $u_{q_n} \to \bar{u}$ locally uniformly in $E_{\hat{A}}(R) \setminus \{0\}$, where $\bar{u}$ is an elliptically symmetric positive solution of the equation $Q_{p,\hat{A},V}(w) = 0$ in $E_{\hat{A}}(R) \setminus \{0\}$. \hfill \Box

**Theorem 6.3.** Let $\hat{A} \in \mathbb{R}^{d \times d}$ be a symmetric, positive definite matrix. Assume that the domain $\Omega$ and the potential $V$ are elliptically symmetric with respect to $\hat{A}$ and the corresponding equation (2.1) possess a positive solution. Further, suppose that the operator $Q_{p,\hat{A},V}$ has a Fuchsian isolated singularity at $\zeta \in \{0, \infty\}$. Then

(i) $\zeta$ is a regular point of (2.1).

(ii) the equation $Q_{p,\hat{A},V}(w) = 0$ possess a unique positive solution in $\Omega$ of minimal growth in a neighbourhood of $\partial \Omega \setminus \{\zeta\}$.

**Proof.** (i) Assume first that $u, v \in \mathcal{G}_\zeta$, where $u$ is elliptically symmetric with respect to $\hat{A}$. Since the operator $Q_{p,\hat{A},V}$ has a Fuchsian isolated singularity at $\zeta$, hence Lemma 3.14, Proposition 3.18, and the uniform Harnack inequality Theorem 3.13, imply that either

$$
\lim_{x \to \zeta \atop x \in \Omega} \frac{u(x)}{v(x)} \text{ exists, and equal either to 0 or } \infty,
$$

or else, $u \asymp v$ in some punctured neighbourhood $\Omega' \subset \Omega$ of $\zeta$. For a sequence $\{R_n\}$ which converges to $\zeta$, define $u_n(x)$ and $v_n(x)$ as in the proof of Proposition 3.18 (see, (3.14)). Then, $u_n$ and $v_n$ are positive solutions of (3.15). Following the arguments as in Proposition 3.18, it follows that up to a subsequence

$$
\lim_{n \to \infty} u_n(x) = u_\infty(x), \quad \lim_{n \to \infty} v_n(x) = v_\infty(x),
$$

locally uniformly in $\mathbb{R}^d \setminus \{0\}$, and $u_\infty, v_\infty$ are positive solutions of the limiting dilated equation

$$
-\Delta_{p,\hat{A}}(w) + V|w|^{p-2}w = 0 \text{ in } \mathbb{R}^d \setminus \{0\}.
$$

Note that the potential $V$ and the solution $u_\infty$ are elliptically symmetric with respect to $\hat{A}$. Moreover, as in Proposition 3.18, for any fixed $R > 0$, we have

$$
\sup_{x \in S_R} u_\infty(x) = M, \quad \inf_{x \in S_R} u_\infty(x) = m,
$$

where $M = \lim_{r \to \zeta} M_r$ and $m = \lim_{r \to \zeta} m_r$ and $M_r, M$ are defined as in Lemma 3.14. Assume that the potential $V$ is nonzero, otherwise it has a weak Fuchsian singularity at $\zeta$ and the theorem follows from Theorem 5.4.

Let $S_{u_\infty}$ be the set of critical points of $u_\infty$. Then it is closed and elliptically symmetric. Now if $\zeta$ is an interior point of $\hat{S}_{u_\infty}$ then $|\nabla u_\infty| = 0$ in some punctured neighbourhood $\Omega'$ of $\zeta$. This implies that $u_\infty$ is constant near $\zeta$ which contradicts our assumption that $\nabla \neq 0$ near $\zeta$. Therefore, there exists an annular set $\hat{A} = E_{\hat{A}}(R) \setminus E_{\hat{A}}(r)$ close to $\zeta$ such that $S_{u_\infty} \cap \hat{A} = \emptyset$. Hence by the strong comparison principle (see [2, Theorem 2]), which is also valid for $Q_{p,\hat{A},V}$, we obtain $mv_\infty = u_\infty = Mv_\infty$ in $\hat{A}$. So, $m = M$, and the theorem follows.

Assume now that $u, v \in \mathcal{G}_\zeta$. Then by Lemma 6.2, there exists a elliptically symmetric solution $\bar{u} \in \mathcal{G}_\zeta$ such that $u \asymp \bar{u}$. By the proof before if follows that $u \sim \bar{u}$ and the limit

$$
\lim_{x \to \zeta \atop x \in \Omega'} \frac{v(x)}{\bar{u}(x)} \text{ exists in the wide sense, and } \lim_{x \to \zeta \atop x \in \Omega'} \frac{u(x)}{\bar{u}(x)} = C > 0,
$$

which shows that
\[
\lim_{x \to \zeta} \frac{u(x)}{v(x)} \text{ exists in the wide sense.}
\]

(ii) Follows from Proposition 3.17. □

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