Analysis of “SIR”
(“Signal”-to-“Interference”-Ratio) in Discrete-Time Autonomous Linear Networks with Symmetric Weight Matrices

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Abstract

It’s well-known that in a traditional discrete-time autonomous linear systems, the eigenvalues of the weight (system) matrix solely determine the stability of the system. If the spectral radius of the system matrix is larger than 1, then the system is unstable. In this paper, we examine the linear systems with symmetric weight matrix whose spectral radius is larger than 1.

The author introduced a dynamic-system-version of ”Signal-to-Interference Ratio (SIR)” in non-linear networks in [7] and [8] and in continuous-time linear networks in [9]. Using the same ”SIR” concept, we, in this paper, analyse the ”SIR” of the states in the following two N-dimensional discrete-time autonomous linear systems: 1) The system $x(k+1) = (I + \alpha(-rI + W))x(k)$ which is obtained by discretizing the autonomous continuous-time linear system in [9] using Euler method; where $I$ is the identity matrix, $r$ is a positive real number, and $\alpha > 0$ is the step size. 2) A more general autonomous linear system described by $x(k+1) = -\rho I + Wx(k)$, where $W$ is any real symmetric matrix whose diagonal elements are zero, and $I$ denotes the identity matrix and $\rho$ is a positive real number. Our analysis shows that: 1) The ”SIR” of any state converges to a constant value, called ”Ultimate SIR”, in a finite time in the above-mentioned discrete-time linear systems. 2) The ”Ultimate SIR” in the first system above is equal to $\frac{\alpha}{\lambda_{max}}$ where $\lambda_{max}$ is the maximum (positive) eigenvalue of the matrix $W$. These results are in line with those of [9] where corresponding continuous-time linear system is
examined. 3) The "Ultimate SIR" in the second system above is equal to $\frac{\rho}{\lambda_m}$ where $\lambda_m$ is the eigenvalue of $W$ which satisfy $|\lambda_m - \rho| = \max\{|\lambda_i - \rho|\}_{i=1}^N$ if $\rho$ is accordingly determined from the interval $0 < \rho < 1$.

In the later part of the paper, we use the introduced "Ultimate SIR" to stabilize the (originally unstable) networks. It's shown that the proposed Discrete-Time "Stabilized"-Autonomous-Linear-Networks-with-Ultimate-SIR" exhibit features which are generally attributed to Discrete-Time Hopfield Networks. Taking the sign of the converged states, the proposed networks are applied to binary associative memory design. Computer simulations show the effectiveness of the proposed networks as compared to traditional discrete Hopfield Networks.

Index Terms

Autonomous Discrete-Time Linear Systems, discrete Hopfield Networks, associative memory systems, Signal to Interference Ratio (SIR).

I. INTRODUCTION

Signal-to-Interference Ratio (SIR) is an important entity in communications engineering which indicates the quality of a link between a transmitter and a receiver in a multi transmitter-receiver environment (see e.g. [4], among many others). For example, let $N$ represent the number of transmitters and receivers using the same channel. Then the received SIR at receiver $i$ is given by (see e.g. [4])

$$SIR_i(k) = \frac{\gamma_i(k) = \frac{g_{ii}p_i(k)}{\nu_i + \sum_{j=1,j\neq i}^N g_{ij}p_j(k)}}{i = 1, \ldots, N}$$ (1)

where $p_i(k)$ is the transmission power of transmitter $i$ at time step $k$, $g_{ij}$ is the link gain from transmitter $j$ to receiver $i$ (e.g. in case of wireless communications, $g_{ij}$ involves path loss, shadowing, etc) and $\nu_i$ represents the receiver noise at receiver $i$. Typically, in wireless communication systems like cellular radio systems, every transmitter tries to optimize its power $p_i(k)$ such that the received SIR(k) (i.e., $\gamma_i(k)$) in eq.(1) is kept at a target SIR value, $\gamma_i^{tgt}$. In an interference dominant scenario, the receiver background noise $\nu_i$ can be ignored and then

$$\gamma_i(k) = \frac{g_{ii}p_i(k)}{\sum_{j=1,j\neq i}^N g_{ij}p_j(k)}, \quad i = 1, \ldots, N$$ (2)
The author defines the following dynamic-system-version of “Signal-to-Interference-Ratio (SIR)”, denoted by $\theta_i(k)$, by rewriting the eq.(1) with neural network terminology in [7] and [8]:

$$\theta_i(k) = \frac{a_{ii} x_i(k)}{\nu_i + \sum_{j=1, j\neq i}^{N} w_{ij} x_j(k)}, \quad i = 1, \ldots, N \tag{3}$$

where $\theta_i(k)$ is the defined ficticious “SIR” at time step $k$, $x_i(k)$ is the state of the $i$’th neuron, $a_{ii}$ is the feedback coefficient from its state to its input layer, $w_{ij}$ is the weight from the output of the $j$’th neuron to the input of the $j$’th neuron. For the sake of brevity, in this paper, we assume the ”interference dominant” case, i.e. $\nu_i$ is negligible.

A traditional discrete-time autonomous linear network is given by

$$x(k+1) = Mx(k), \quad x(k) \in \mathbb{R}^{N \times 1}, \quad M \in \mathbb{R}^{N \times N}, \tag{4}$$

where $x(k)$ shows the state vector at time $t$ and square matrix $M$ is called system matrix or weight matrix.

It’s well-known that in the system of eq.(4), the eigenvalues of the weight (system) matrix solely determine the stability of system. If the spectral radius of the matrix is larger than 1, then the system is unstable. In this paper, we examine the linear systems with system matrices whose spectral radius is larger than 1.

Using the same ”SIR” concept in eq.(3), we, in this paper, analyse the ”SIR” of the states in the following two discrete-time autonomous linear systems:

1) The following linear system which is obtained by discretizing the continuous-time linear system $\dot{x} = ( -rI + W)x$ in [9] using Euler method:

$$x(k+1) = (I + \alpha(-rI + W))x(k) \tag{5}$$

where $I$ denotes the identity matrix, $(-rI + W)$ is the real symmetric system matrix with zero-diagonal $W$, and $\alpha$ is the step size.

2) A more general autonomous linear system descibed by

$$x(k+1) = (-rI + W)x(k) \tag{6}$$
where $W$ is any real symmetric matrix whose diagonal elements are zero, and $I$ denotes the identity matrix and $r$ is a positive real number.

Our analysis shows that: 1) The “SIR” of any state converges to a constant value, called “Ultimate SIR”, in a finite time in the above-mentioned discrete-time linear systems. 2) The “Ultimate SIR” in the first system above is equal to $\frac{\rho}{\lambda_{max}}$ where $\lambda_{max}$ is the maximum (positive) eigenvalue of the matrix $W$. These results are in line with those of [9] where corresponding continuous-time linear system is examined. 3) The “Ultimate SIR” in the second system above is equal to $\frac{\rho}{\lambda_m}$ where $\lambda_m$ is the eigenvalue of $W$ which satisfy $|\lambda_m - \rho| = \max\{|\lambda_i - \rho|\}_{i=1}^N$ if $\rho$ is accordingly determined from the interval $0 < \rho < 1$.

In the later part of the paper, we use the introduced “Ultimate SIR” to stabilize the (originally unstable) network. It’s shown that the proposed Discrete-Time "Stabilized"-Autonomous-Linear-Networks-with-Ultimate-SIR” exhibit features which are generally attributed to Discrete-Time Hopfield Networks. Taking the sign of the converged states, the proposed networks are applied to binary associative memory design.

The paper is organized as follows: The ultimate ”SIR” is analysed for the autonomous linear discrete-time systems with symmetric weight matrices in section II. Section III presents the stabilized networks by their Ultimate "SIR” to be used as a binary associative memory system. Simulation results are presented in section IV, which is followed by the conclusions in Section V.

II. ANALYSIS OF “SIR” IN DISCRETE-TIME AUTONOMOUS LINEAR NETWORKS WITH SYMMETRIC WEIGHT MATRICES

In this section, we analyse the “SIR” of the states in the following two discrete-time autonomous linear systems: 1) The discrete-time autonomous system which is obtained by discretizing the continuous-time linear system in [9] using Euler method; and 2) A more general autonomous linear system described by $x(k+1) = (-\rho I + W)x(k)$, where $W$ is any real symmetric matrix whose diagonal elements are zero, and $I$ denotes the identity matrix and $r$ is a positive real number.

A. Discretized Autonomous Linear Systems with symmetric matrix case

The author examines the “SIR” in the following continuous-time linear system in [9]:

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\[
\dot{x} = (-rI + W)x \tag{7}
\]

where \(\dot{x}\) shows the derivative of \(x\) with respect to time, i.e., \(\dot{x} = \frac{dx}{dt}\). In this subsection, we analyse the following discrete-time autonomous linear system which is obtained by discretizing the continuous-time system of eq.(7) by using well-known Euler method:

\[
x(k + 1) = (I + \alpha(-rI + W))x(k) \tag{8}
\]

where \(I\) is the identity matrix, \(r\) is a positive real number, \((-rI + W)\) is the system matrix, \(x(k)\) shows the state vector at step \(k\), and \(\alpha > 0\) is the step size and

\[
rI = \begin{bmatrix}
    r & 0 & \ldots & 0 \\
    0 & r & \ldots & 0 \\
    \vdots & \ddots & \ddots & 0 \\
    0 & 0 & \ldots & r
\end{bmatrix}_{N \times N}, \quad W = \begin{bmatrix}
    0 & w_{12} & \ldots & w_{1N} \\
    w_{21} & 0 & \ldots & w_{2N} \\
    \vdots & \ddots & \ddots & \vdots \\
    w_{N1} & w_{N2} & \ldots & 0
\end{bmatrix}_{N \times N} \tag{9}
\]

In this paper, we examine only the linear systems with symmetric weight matrices, i.e., \(w_{ij} = w_{ji}, \ i, j = 1, 2, \ldots, N\). It’s well known that designing the weight matrix \(W\) as a symmetric one yields that all eigenvalues are real (see e.g. [6]), which we assume throughout the paper due to the simplicity and brevity of its analysis.

The reason of the notation in (9) is because we prefer to have the same notation as in [7].

**Proposition 1:**

In the autonomous discrete-time linear network of eq.(8), let’s assume that the spectral radius of the system matrix \((I + \alpha(-rI + W))\) is larger than 1. (This assumption is equal to the assumption that \(W\) has positive eigenvalue(s) and \(r > 0\) is chosen such that \(\lambda_{\text{max}} > r\), where \(\lambda_{\text{max}}\) is the maximum (positive) eigenvalue of \(W\).) If \(\alpha\) is chosen such that \(0 < \alpha r < 1\), then the defined "SIR" \(\theta_i(k)\) in eq.(3) for any state \(i\) converges to the following constant within a finite step number for any initial vector \(x(0)\) which is not completely perpendicular to the eigenvector corresponding to the largest eigenvalue of \(W\). \(^1\)

\(^1\) It’s easy to check in advance if the initial vector \(x(0)\) is completely perpendicular to the eigenvector of the maximum (positive) eigenvalue of \(W\) or not. If this is the case, then this can easily be overcome by introducing a small random variable to \(x(0)\) so that it’s not completely perpendicular to the mentioned eigenvector.
\[ \theta_i(k \geq k_T) = \frac{r}{\lambda_{\text{max}}}, \quad i = 1, 2, \ldots, N \]  

(10)

where \( \lambda_{\text{max}} \) is the maximum (positive) eigenvalue of the weight matrix \( W \) and \( k_T \) shows a finite time constant.

**Proof:**

From eq. (8), it’s obtained

\[ x(k) = (I + \alpha(-rI + W))^k x(0) \]  

(11)

where \( x(0) \) shows the initial state vector at step zero. Let us first examine the powers of the matrix \( (I + \alpha(-rI + W)) \) in (11) in terms of matrix \( rI \) and the eigenvectors of matrix \( W \):

It’s well known that any symmetric real square matrix can be decomposed into

\[ W = \sum_{i=1}^{N} \lambda_i v_i v_i^T = \sum_{i=1}^{N} \lambda_i V_i \]  

(12)

where \( \{\lambda_i\}_{i=1}^{N} \) and \( \{v_i\}_{i=1}^{N} \) show the (real) eigenvalues and the corresponding eigenvectors and the eigenvectors \( \{v_i\}_{i=1}^{N} \) are orthonormal (see e.g. [6]), i.e.,

\[ v_i v_j = \begin{cases} 
1 & \text{if } i = j, \\
0 & \text{if } i \neq j, 
\end{cases} \quad \text{where } i, j = 1, 2, \ldots, N \]  

(13)

Let’s define the outer-product matrices of the eigenvectors \( \{\lambda_i\}_{i=1}^{N} \) as \( V_j = v_i v_i^T, \quad i = 1, 2, \ldots, N \), which, from eq.(13), is equal to

\[ V_j = \begin{cases} 
I & \text{if } i = j, \\
0 & \text{if } i \neq j, 
\end{cases} \quad \text{where } i, j = 1, 2, \ldots, N \]  

(14)

where \( I \) is the identity matrix. Defining matrix \( M \),

\[ M = I + \alpha(-rI + W) \]  

(15)

which is obtained as

\[ M = (1 - \alpha r)I + \sum_{i=1}^{N} \beta_i(1)V_i \]  

(16)
where \( r > 0, \alpha > 0, \) and where \( \beta_i(1) \) is equal to

\[
\beta_i(1) = \alpha \lambda_i
\]  
(17)

The matrix \( M^2 \) can be written as

\[
M^2 = (1 - \alpha r)^2 I + \sum_{i=1}^{N} \beta_i(2) V_i
\]  
(18)

where \( \beta_i(2) \) is equal to

\[
\beta_i(2) = \alpha(1 - \alpha r) \lambda_i + (1 - \alpha r + \alpha \lambda_i) \beta_i(1)
\]  
(19)

Similarly, the matrix \( M^3 \) can be written as

\[
M^3 = (1 - \alpha r)^3 I + \sum_{i=1}^{N} \beta_i(3) V_i
\]  
(20)

where \( \beta_i(3) \) is equal to

\[
\beta_i(3) = \alpha(1 - \alpha r)^2 \lambda_i + (1 - \alpha r + \alpha \lambda_i) \beta_i(2)
\]  
(21)

So, \( M^4 \) can be written as

\[
M^4 = (1 - \alpha r)^4 I + \sum_{i=1}^{N} \beta_i(4) V_i
\]  
(22)

where \( \beta_i(4) \) is equal to

\[
\beta_i(4) = \alpha(1 - \alpha r)^3 \lambda_i + (1 - \alpha r + \alpha \lambda_i) \beta_i(3)
\]  
(23)

So, at step \( k \), the matrix \( (M)^k \) is obtained as

\[
M^k = (1 - \alpha r)^k I + \sum_{i=1}^{N} \beta_i(k) V_i
\]  
(24)

where \( \beta_i(k) \) is equal to

\[
\beta_i(k) = \alpha(1 - \alpha r)^{k-1} \lambda_i + (1 + \alpha(\lambda_i - r)) \beta_i(k - 1)
\]  
(25)
Using eq.(17) and (25), the $\beta_i(k)$ is obtained as

\begin{align*}
\beta_i(1) &= \alpha \lambda_i 
\beta_i(2) &= \alpha \lambda_i \left( (1 - \alpha r) + (1 + \alpha(\lambda_i - r)) \right) 
\beta_i(3) &= \alpha \lambda_i \left( (1 - \alpha r)^2 + (1 - \alpha r)(1 + \alpha(\lambda_i - r)) + (1 + \alpha(\lambda_i - r))^2 \right) 
\vdots 
\beta_i(k) &= \alpha \lambda_i \sum_{m=1}^{k} (1 - \alpha r)^{k-m} (1 + \alpha(\lambda_i - r))^{m-1}
\end{align*}

Defining $\lambda_i = \zeta_i(1 - \alpha r)$, we obtain

\begin{align*}
(1 - \alpha r)^{k-m} (1 + \alpha(\lambda_i - r))^{m-1} &= (1 - \alpha r)^{k-1} (1 + \alpha \zeta_i)^{m-1}
\end{align*}

Writing eq.(31) in eq.(30) gives

\begin{align*}
\beta_i(k) &= \alpha \zeta_i (1 - \alpha r)^k S(k)
\end{align*}

where $S(k)$ is

\begin{align*}
S(k) &= \sum_{m=1}^{k} (1 + \alpha \zeta_i)^{m-1}
\end{align*}

Summing $-S(k)$ with $(1 + \alpha \zeta_i)S(k)$ yields

\begin{align*}
S(k) &= \frac{(1 + \alpha \zeta_i)^k - 1}{\alpha \zeta_i}
\end{align*}

From eq.(32), (33) and (34), we obtain

\begin{align*}
\beta_i(k) &= (1 - \alpha r)^k (1 + \alpha \zeta_i)^k - (1 - \alpha r)^k
\end{align*}

Using the definition $\zeta_i = \lambda_i/(1 - \alpha r)$ in eq.(35) gives

\begin{align*}
\beta_i(k) &= (1 + \alpha(\lambda_i - r))^k - (1 - \alpha r)^k
\end{align*}

From eq.(24) and eq.(36),
\[ M^k = (1 - \alpha r)^k I + \sum_{i=1}^{N} (1 + \alpha (\lambda_i - r))^k V_i - \sum_{i=1}^{N} (1 - \alpha r)^k V_i \] (37)

Let’s put the \( N \) eigenvalues of matrix \( W \) into two groups as follows: Let those eigenvalues which are smaller than \( r \), belong to set \( T = \{ \lambda_{j_t} \}_{j_t=1}^{N_t} \) where \( N_t \) is the length of the set; and let those eigenvalues which are larger than \( r \) belong to set \( S = \{ \lambda_{j_s} \}_{j_s=1}^{N_s} \) where \( N_s \) is the length of the set. We write the matrix \( M^k \) in eq.(37) using this eigenvalue grouping

\[ M^k = M_{tp}(k) + M_{sp}(k) \] (38)

where

\[ M_{tp}(k) = (1 - \alpha r)^k I - \sum_{i=1}^{N} (1 - \alpha r)^k V_i + \sum_{j_t \in T} (1 + \alpha (\lambda_{j_t} - r))^k V_{j_t} \] (39)

and

\[ M_{sp}(k) = \sum_{j_s \in S} (1 + \alpha (\lambda_{j_s} - r))^k V_{j_s} \] (40)

We call the matrices \( M_{tp}(k) \) and \( M_{sp}(k) \) in (39) and (40) as transitory phase part and steady phase part, respectively, of the matrix \( M^k \).

It’s observed from eq.(39) that the \( M_{tp}(k) \) converges to zero in a finite step number \( k_T \) because relatively small step number \( \alpha > 0 \) is chosen such that \( (1 - \alpha r) < 1 \) and \( 1 + \alpha (\lambda_{j_t} - r) < 1 \). So,

\[ M_{tp}(k) \approx 0, \quad k \geq k_T \] (41)

Thus, what shapes the steady state behavior of the system in eq.(11) and (15) is merely the \( M_{sp}(k) \) in eq.(40). So, the steady phase solution is obtained from eq.(11), (15) and (40) using the above observations as follows

\[ x_{sp}(k) = M_{sp}(k)x(0) \] (42)

\[ = \sum_{j_s \in S} (1 + \alpha (\lambda_{j_s} - r))^k V_{j_s} x(0), \quad k \geq k_T \] (43)

Let’s define the interference vector, \( J_{sp}(k) \) as
\[ J_{sp}(k) = W x_{sp}(k) \] (44)

Using eq.(12) in (44) and the orthonormal features in (14) yields

\[ J_{sp}(k) = \sum_{j_s \in S} \lambda_{j_s} (1 + \alpha (\lambda_{j_s} - r))^k V_{j_s} x(0) \] (45)

First defining \( V_j x(0) = u_j \), and \( \xi = \frac{\alpha}{1-\alpha r} \), then dividing vector \( x_{sp}(k) \) of eq.(43) to \( J_{sp}(k) \) of eq.(45) elementwise and comparing the outcome with the “SIR” definition in eq.(3) results in

\[ \frac{x_{sp,i}(k)}{J_{sp,i}(k)} = \frac{1}{r} \theta_i(k), \quad i = 1, \ldots, N \] (46)

\[ = \frac{\sum_{j_s \in S} (1 + \xi \lambda_{j_s})^k u_{j_s,i}}{\sum_{j_s \in S} \lambda_{j_s} (1 + \xi \lambda_{j_s})^k u_{j_s,i}} \] (47)

In eq.(47), we assume that the \( u_j = V_j x(0) \) which corresponds to the eigenvector of the largest positive eigenvalue is different than zero vector. This means that we assume in the analysis here that \( x(0) \) is not completely perpendicular to the mentioned eigenvector. This is something easy to check in advance. If it is the case, then this can easily be overcome by introducing a small random number to \( x(0) \) so that it’s not completely perpendicular to the mentioned eigenvector.

From the analysis above, we observe that

1) If all the (positive) eigenvalues greater than \( r \) are the same, which is denoted as \( \lambda_b \), then it’s seen from (47) that

\[ \theta_i(k) = \frac{r}{\lambda_b}, \quad i = 1, \ldots, N, \quad k \geq k_T \] (48)

2) Similarly, if there is only one positive eigenvalue which is larger than \( r \), shown as \( \lambda_b \), then eq.(48) holds.

3) If there are more than two different (positive) eigenvalues and the largest positive eigenvalue is single (not multiple), then we see from (46) that the term related to the largest (positive) eigenvalue dominates the sum of the nominator. Same observation is valid for the sum of the denominator. This is because a relatively small increase in \( \lambda_j \) causes exponential increase as time step evolves, which is shown in the following: Let’s show the two largest (positive) eigenvalues as \( \lambda_{max} \) and \( \lambda_j \) respectively and the difference between them as \( \Delta \lambda \).
So, \(\lambda_{\text{max}} = \lambda_j + \Delta \lambda\). Let’s define the following ratio between the terms related to the two highest eigenvalues in the nominator

\[
K_n(k) = \frac{(1 + \xi \lambda_j)^k}{(1 + \xi (\lambda_j + \Delta \lambda))^k}
\] (49)

where

\[
\xi = \frac{\alpha}{1 - ar}
\] (50)

Similarly, let’s define the ratio between the terms related to the two highest eigenvalues in the denominator as

\[
K_d(k) = \frac{\lambda_j(1 + \xi \lambda_j)^k}{(\lambda_j + \Delta \lambda)(1 + \xi (\lambda_j + \Delta \lambda))^k}
\] (51)

From eq.(49) and (51), since \(\frac{\lambda_j}{\lambda_j + \Delta} < 1\) due to the above assumptions,

\[
K_d(k) < K_n(k)
\] (52)

We plot the ratio \(K_n(k)\) in Fig. 1 for some different \(\Delta \lambda\) values and for a typical \(\xi\) value. The Figure 1 and eq.(52) implies that the terms related to the \(\lambda_{\text{max}}\) dominate the sum of the nominator and that of the denominator respectively. So, from eq.(47) and (50),

\[
\frac{x_{sp,i}(k)}{J_{sp,i}(k)} = \frac{\sum_{j_s \in S} \lambda_j(1 + \xi \lambda_j) u_{j_s,i}}{\sum_{j_s \in S} \lambda_j(1 + \xi \lambda_j)^2 u_{j_s,i}} \rightarrow \frac{(1 + \xi \lambda_{\text{max}})^k}{\lambda_{\text{max}}(1 + \xi \lambda_{\text{max}})^k} = \frac{1}{\lambda_{\text{max}}}, \quad k \geq k_T
\] (53)

4) If the largest positive eigenvalue is a multiple eigenvale, then, similarly, the corresponding terms in the sum of the nominator and that of the demoninator become dominant, which implies from eq.(47), (49) and (51) that \(\frac{x_{sp,i}(k)}{J_{sp,i}(k)}\) converges to \(\frac{1}{\lambda_{\text{max}}}\) as step number increases.

Using the observations 1 to 4, eq.(41), the “SIR” definition in eq.(3), eq.(46) and (47), we conclude that

\[
\theta_i(k) = \frac{r x_{sp,i}(k)}{\sum_{j=1,j \neq i}^N w_{ij} x_{sp,j}(k)} = \frac{r}{\lambda_{\text{max}}}, \quad k \geq k_T \quad i = 1, \ldots, N
\] (54)

where \(\lambda_{\text{max}}\) is the largest (positive) eigenvalue of the matrix \(W\), and \(k_T\) shows the finite time constant (during which the matrix \(M_{tp}(k)\) in eq.(39) vanishes), which completes the proof.
**Definition:** Ultimate SIR value: In proposition 1, we showed that the SIR in (3) for every state in the autonomous discrete-time linear networks in eq.(8) converges to a constant value as step number goes to infinity. We call this converged constant value as ”ultimate SIR” and denote as $\theta_{ult}$.

**B. A more general autonomous linear discrete-time systems with symmetric matrix case**

In this subsection, we analyse the following discrete-time autonomous linear system

$$x(k+1) = (-\rho I + W)x(k)$$  \hspace{1cm} (55)

where $I$ is the identity matrix, $\rho$ is a positive real number, and $(-\rho I + W)$ is the symmetric system matrix. The real symmetric matrix $W$ is shown in eq.(9).

**Proposition 2:**

In the the discrete-time linear system of eq.(55), let’s assume that the spectral radius of symmetric matrix $W$ in (9) is larger than 1, i.e., the maximum of the norms of the eigenvalues is larger than 1.

If $\rho$ is chosen such that

1) \hspace{1cm} $0 < \rho < 1, \hspace{1cm} (56)$

and

2) Define the eigenvalue(s) $\lambda_m$ as

$$|\lambda_m - \rho| = \max\{|\lambda_i - \rho|\}^{N}_{i=1} > 1 \hspace{1cm} (57)$$

the eigenvalue $\lambda_m$ is unique. (In other words, $\rho$ is chosen in such a way that the equation eq.(57) does not hold for two eigenvalues with opposite signs. It would hold for a multiple eigenvalue as well, i.e., same sign.)
then the defined "SIR" \((\theta_i(k))\) in eq.(3) for any state \(i\) converges to the following ultimate SIR as step number \(k\) evolves for any initial vector \(x(0)\) which is not completely perpendicular to the eigenvector corresponding to the eigenvalue \(\lambda_m\) in (57) of \(W\).

\[
\theta_i(k \geq k_T) = \frac{\rho}{\lambda_m}, \quad i = 1, 2, \ldots, N
\]  

(58)

where \(\lambda_m\) is the eigenvalue of the weight matrix \(W\) which satisfy eq.(57) and \(k_T\) shows a finite time constant.

**Proof:**

From eq.(55),

\[
x(k) = (-\rho I + W)^k x(0)
\]

(59)

where \(x(0)\) shows the initial state vector at step zero. Let’s examine the powers of \((-\rho I + W)\) in (59) in terms of the eigenvectors of \(W\) using eqs.(12)-(14):

\[
(-\rho I + W) = -\rho I + \sum_{i=1}^{N} \eta_i(1)V_i
\]

(60)

where \(\eta_i(1)\) is equal to

\[
\eta_i(1) = \lambda_i
\]

(61)

The matrix \((-\rho I + W)^2\) can be written as

\[
(-\rho I + W)^2 = \rho^2 I + \sum_{i=1}^{N} \eta_i(2)V_i
\]

(62)

where \(\eta_i(2)\) is equal to

\[
\eta_i(2) = -\rho \lambda_i + (\lambda_i - \rho) \eta_i(1)
\]

(63)

Similarly, the matrix \((-\rho I + W)^3\) can be written as

It’s easy to check in advance if the initial vector \(x(0)\) is completely perpendicular to the eigenvector of the eigenvalue \(\lambda_m\) in (57) of \(W\) or not. If this is the case, then this can easily be overcome by introducing a small random number to \(x(0)\) so that it’s not completely perpendicular to the mentioned eigenvector.

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\[-\rho I + \mathbf{W}^3 = -\rho^3 I + \sum_{i=1}^{N} \eta_i(3) \mathbf{V}_i \tag{64}\]

where $\eta_i(3)$ is equal to

\[\eta_i(3) = \rho^2 \lambda_i + (\lambda_i - \rho) \eta_i(2) \tag{65}\]

So, at step $k$, the matrix $(-\rho I + \mathbf{W})^k$ is obtained as

\[-\rho I + \mathbf{W}^k = (-\rho)^k I + \sum_{i=1}^{N} \eta_i(k) \mathbf{V}_i \tag{66}\]

where $\eta_i(k)$ is

\[\eta_i(k) = (-\rho)^{k-1} \lambda_i + (\lambda_i - \rho) \eta_i(k-1) \tag{67}\]

So, from (61)-(67)

\[
\begin{align*}
\eta_i(1) &= \lambda_i \\
\eta_i(2) &= \lambda_i(-\rho + (\lambda_i - \rho)) \\
\eta_i(3) &= \lambda_i(\rho^2 - \rho(\lambda_i - \rho) + (\lambda_i - \rho)^2) \\
\vdots \\
\eta_i(k) &= \lambda_i \sum_{m=1}^{k} (-1)^{k-m} \rho^{k-m}(\lambda_i - \rho)^{m-1} \tag{72}
\end{align*}
\]

Defining $\lambda_i = \mu_i \rho$, we obtain

\[\rho^{k-m}(\lambda_i - \rho)^{m-1} = \rho^{k-1}(\mu_i - 1)^{m-1} \tag{73}\]

Writing eq.(73) in eq.(72) gives

\[\eta_i(k) = \lambda_i \rho^{k-1} S(k) \tag{74}\]

where $S(k)$ is

\[S(k) = \sum_{m=1}^{k} (-1)^{k-1}(\mu_i - 1)^{m-1} \tag{75}\]
Summing $S(k)$ with $(\mu_i - 1)S(k)$ yields

$$S(k) = \frac{(-1)^{k-1} + (\mu_i - 1)^k}{\mu_i} \quad (76)$$

From eq.(74), (75) and (76), we obtain

$$\eta_i(k) = \lambda_i \rho^{k-1} \frac{(-1)^{k-1} + (\mu_i - 1)^k}{\mu_i} \quad (77)$$

Using the definition $\mu_i = \lambda_i / \rho$ in eq.(77) gives

$$\eta_i(k) = (-1)^{k-1} \rho^k + (\lambda_i - \rho)^k \quad (78)$$

From eq.(66) and eq.(78),

$$(\rho I + W)^k = (-\rho I + W)^k + \sum_{i=1}^{N} (-1)^{k-1} \rho^k V_i + (\lambda_i - \rho)^k V_i \quad (79)$$

Let’s put the $N$ eigenvalues of matrix $W$ into two groups as follows: Let those eigenvalues which satisfy $|\lambda_j - \rho| < 1$ belong to set $T = \{\lambda_j\}_{j=1}^{N_t}$ where $N_t$ is the length of the set; and let all other eigenvalues (i.e. those which satisfy $|\lambda_j - \rho| > 1$) belong to set $S = \{\lambda_j\}_{j=1}^{N_s}$ where $N_s$ is the length of the set. Here, $\rho$ is chosen such that no eigenvalue satisfy $|\lambda_j - \rho| = 1$. We write the matrix $(-\rho I + W)^k$ in eq.(79) using this eigenvalue grouping as follows

$$(\rho I + W)^k = N_{tp}(k) + N_{sp}(k) \quad (80)$$

where

$$N_{tp}(k) = (-\rho I + W)^k = \sum_{i=1}^{N} (-1)^{k-1} \rho^k V_i + \sum_{j_t \in T} (\lambda_{jt} - \rho)^k V_{jt} \quad (81)$$

and

$$N_{sp}(k) = \sum_{j_s \in S} (\lambda_{js} - \rho)^k V_{js} \quad (82)$$

In (81), $|\lambda_{jt} - \rho| < 1$ from the grouping mentioned above and $\rho$ is chosen such that $0 < \rho < 1$, which means that the $N_{tp}(k)$ converges to zero in a finite step number $k_T$, i.e.,
Thus, what shapes the steady state behavior of the system in eq.(59) is merely the \(N_{sp}(k)\) in eq.(82). We call the matrices \(N_{tp}(k)\) and \(N_{sp}(k)\) in (81) and (82) as transitory phase part and steady phase part, respectively, of the matrix \(N^k\).

So, the steady phase solution is obtained from eq.(59), (80)(81) and (82) as follows

\[
x_{sp}(k) = N_{sp}(k)x(0)
= \sum_{j_s \in S} (\lambda_{j_s} - \rho)^k V_{j_s} x(0), \quad k \geq k_T
\]

Let’s define the interference vector, \(J_{sp}(k)\) as

\[
J_{sp}(k) = Wx_{sp}(k)
\]

Using eq.(12) in (86) and the orthonormal features in (14) yields

\[
J_{sp}(k) = \sum_{j_s \in S} \lambda_{j_s} (\lambda_{j_s} - \rho)^k V_{j_s} x(0)
\]

Defining \(V_j x(0) = u_j\), and then dividing vector \(x_{sp}(k)\) of eq.(85) to \(J_{sp}(k)\) of eq.(87) elementwise and comparing the outcome with the “SIR” definition in eq.(3) results in

\[
\frac{x_{sp,i}(k)}{J_{sp,i}(k)} = \frac{1}{r} \theta_i(k), \quad i = 1, \ldots, N
\]

In eq.(89), we assume that the \(u_j = V_j x(0)\) corresponding to the eigenvector \(\lambda_m\) in eq.(57) is different than zero vector. This means that we assume in the analysis here that \(x(0)\) is not completely perpendicular to the mentioned eigenvector. This is something easy to check in advance. If it is the case, then this can easily be overcome by introducing a small random number to \(x(0)\) so that it’s not completely perpendicular to the mentioned eigenvector.

Here it’s assumed that the eigenvalue \(\lambda_m\) satisfying the equation eq.(57) is unique. In other words, \(\rho\) is chosen in such a way that the equation eq.(57) does not hold for two eigenvalues with
opposite signs. (It holds for a multiple eigenvalue, i.e., with same sign). Using this assumption (which can easily be met by choosing $\rho$ accordingly) in eq.(89) yields the following: The term related to eigenvalue $\lambda_m$ in eq.(57) dominates the sum of the nominator. This is because a relatively small decrease in $\lambda_j$ causes exponential decrease as time step evolves, which is shown in the following: Let’s define the following ratio

$$\kappa_n(k) = \frac{(\lambda_j - \rho)^k}{(\lambda_j + \Delta \lambda - \rho)^k}$$  \hspace{1cm} (90)

where $\Delta \lambda$ represents the decrease. Similarly, for denominator

$$\kappa_d(k) = \frac{\lambda_j(\lambda_j - \rho)^k}{(\lambda_j + \Delta \lambda)(\lambda_j + \Delta \lambda - \rho)^k}$$  \hspace{1cm} (91)

From eq.(90) and (91), since $\frac{\lambda_j}{\lambda_j + \Delta} < 1$,

$$\kappa_d(k) < \kappa_n(k)$$  \hspace{1cm} (92)

We plot some typical examples of the ratio $\kappa_n(k)$ in Fig. 2 for some different $\Delta \lambda$ values. The Figure 2 and eq.(92) implies that the terms related to the $\lambda_m$ dominate the sum of the nominator and that of the denominator respectively. So, from eq.(89)

$$\frac{x_{sp,i}(k)}{J_{sp,i}(k)} = \frac{\sum_{j \in S}(\lambda_j - \rho)^k u_{j,i}}{\sum_{j \in S} \lambda_j(\lambda_j - \rho)^k u_{j,i}} \rightarrow \frac{(\lambda_m - \rho)^k u_{jm,i}}{\lambda_m(\lambda_m - \rho)^k u_{jm,i}} = \frac{1}{\lambda_{max}}, \hspace{1cm} k \geq k_T$$  \hspace{1cm} (93)

where $\lambda_m$ is defined by eq.(57). If the largest positive eigenvalue is a multiple eigenvalue, then, similarly, the corresponding terms in the sum of the nominator and that of the denominator become dominant, which implies from eq.(89), (90)-(93) that $\frac{x_{sp,i}(k)}{J_{sp,i}(k)}$ converges to $\frac{1}{\lambda_{max}}$ as step number evolves.

From eq.(83), and the "SIR" definition in eq.(3), we conclude from eq.(88)-(93) that

$$\theta_i(k) = \frac{r x_{sp,i}(k)}{\sum_{j=1,j \neq i}^N w_{ij} x_{sp,j}(k)} = \frac{r}{\lambda_m}, \hspace{1cm} k \geq k_T \hspace{0.5cm} i = 1, \ldots, N,$$  \hspace{1cm} (94)

where $\lambda_m$ is defined by eq.(57), and $k_T$ shows the finite time constant (during which the matrix $N_{tp}(k)$ in eq.(81) vanishes), which completes the proof.
III. Stabilized Discrete-Time Autonomous Linear Networks with Ultimate “SIR”

The proposed autonomous networks networks are

1)  \[
    x(k+1) = (I + \alpha(-rI + W))x(k)\delta(\theta(k) - \theta^{ult}) \quad (95)
    
    y(k) = sign(x(k)) \quad (96)
    
\]

where $W$ is defined in (9) $\alpha$ is step size, $I$ is identity matrix and $r > 0$ as in eq.(8), $\theta(k) = [\theta_1(k)\theta_2(k)\ldots\theta_N(k)]^T$, and $\theta^{ult} = [\theta_1^{ult}\theta_2^{ult}\ldots\theta_N^{ult}]^T$, and $y(k)$ is the output of the network. In eq.(95)

\[
    \delta(\theta - \theta^{ult}) = \begin{cases} 
    0 & \text{if and only if } \theta(k) = \theta^{ult}, \\
    1 & \text{otherwise}
    \end{cases} \quad (97)
\]

We call the network in ref.(95) as Discrete Stabilized Autonomous Linear Networks by Ultimate “SIR”1 (DSAL-U”SIR”1).

2)  \[
    x(k+1) = (-\rho I + W)x(k)\delta(\theta(k) - \theta^{ult}) \quad (98)
    
    y(k) = sign(x(k)) \quad (99)
    
\]

where $I$ is the identity matrix, $1 > \rho > 0$ and $W$ is defined in eq.(9), and $y(k)$ is the output of the network. We call the network in ref.(98) as DSAL-U”SIR”2.

**Proposition 3:**

The proposed discrete-time networks of DSAL-U”SIR”1 in eq.(95) is stable for any initial vector $x(0)$ which is not completely perpendicular to the eigenvector corresponding to the largest eigenvalue of $W$.  \(^3\)

**Proof:** The proof of proposition 1 above shows that in the linear networks of eq.(8), the defined SIR in eq.(3) for state $i$ converges to the constant ultimate SIR value in eq.(10) for any initial condition $x(0)$ within a finite step number $k_T$. It’s seen that the DSAL-U”SIR”1 in eq.(8)

\(^3\) See the comments of proposition 1.
is nothing but the underlying network of the proposed networks SAL-U"SIR"1 without the \( \delta \) function. Since the “SIR” in eq.(3) exponentially approaches to the constant Ultimate “SIR” in eq.(10), the delta function eq.(97) will stop the exponential increase once \( \theta(k) = \theta^{ult} \), at which the system outputs reach their steady state responses. So, the DSAL-U"SIR"1 is stable.

**Proposition 4:**

The proposed discrete-time network DSAL-U”SIR”2 in (98) is stable for any initial vector \( x(0) \) which is not completely perpendicular to the eigenvector corresponding to the eigenvalue described in eq.(57). \(^4\)

**Proof:** The proof of proposition 2 above shows that in the linear network of (55), the defined SIR in eq.(3) for state \( i \) converges to the constant ultimate SIR value in eq. (58), for any initial condition \( x(0) \) within a finite step number \( k_T \). It’s seen that the linear networks of eq.(55) is nothing but the underlying network of the proposed network DSAL-U”SIR”2 without the \( \delta \) function. Since the “SIR” in eq.(3) exponentially approaches to the constant Ultimate “SIR” in eq.(58), the delta function eq.(97) will stop the exponential increase once \( \theta(k) = \theta^{ult} \), at which the system output reach its steady state response. So, the DSAL-U”SIR”2 is stable.

So, from the analysis above for symmetric \( W \) and \( 0 < r < \lambda_{max} \) for the SAL-U”SIR”1 in eq.(95) and \( 0 < \rho < 1 \) for the DSAL-U”SIR”2 in eq.(98), we conclude that

1) The DSALU-”SIR”1 and DSALU-”SIR”2 does not show oscillatory behaviour because it’s assured by the design parameter \( r \) that \( \rho \), respectively, that there is no eigenvalue on the unit circle.

2) The transition phase of the ”unstable” linear network is shaped by the initial state vector and the phase space characteristics formed by the eigenvectors of \( W \). The network is stabilized by a switch function once the network has passed the transition phase. The output of the network then is formed taking the sign of the converged states. If the state converges to a plus or minus value is dictated by the phase space of the underlying linear network from the initial state vector at time 0.

---

\(^4\) See the comments of proposition 2.
Choosing the $r$ and $\rho$ in the DSALU-"SIR"1 and DSALU-"SIR"2 respectively such that the overall system matrix has positive eigenvalues makes the proposed networks exhibit similar features as Hopfield Network does as shown in the simulation results in section IV.

As far as the design of weight matrix $r\mathbf{I}$ ($\rho\mathbf{I}$) and $\mathbf{W}$ is concerned, well-known Hebb-learning rule ([5]) is one of the commonly used methods (see e.g. [2]). We proposed a method in [7] which is based on the Hebb-learning rule [5]. We summarize the design method here as well for the sake of completeness.

A. Outer products based network design

Let’s assume that $L$ desired prototype vectors, $\{\mathbf{d}_s\}_{s=1}^L$, are given from $(-1,+1)^N$. The proposed method is based on well-known Hebb-learning [5] as follows:

Step 1: Calculate the sum of outer products of the prototype vectors (Hebb Rule, [5])

$$\mathbf{Q} = \sum_{s=1}^L \mathbf{d}_s\mathbf{d}_s^T$$

(100)

Step 2: Determine the diagonal matrix $r\mathbf{I}$ and $\mathbf{W}$ as follows:

$$r = q_{ii} + \vartheta$$

(101)

where $\vartheta$ is a real number and

$$w_{ij} = \begin{cases} 0 & \text{if } i = j, \\ q_{ij} & \text{if } i \neq j \\ \end{cases}, \quad i, j = 1, \ldots, N$$

(102)

where $q_{ij}$ shows the entries of matrix $\mathbf{Q}$, $N$ is the dimension of the vector $\mathbf{x}$ and $L$ is the number of the prototype vectors ($N > L > 0$). From eq.(100), $q_{ii} = L$ since $\{\mathbf{d}_s\}$ is from $(-1,+1)^N$.

We assume that the desired prototype vectors are orthogonal and we use the following design procedure for matrices $\mathbf{A}$, $\mathbf{W}$ and $\mathbf{b}$, which is based on Hebb learning ([5]).

Proposition 5:

For the proposed network DSALU-"SIR"1 in eq.(8) whose weight matrix is designed by the proposed outer-products (Hebbian-learning)-based method above, if the prototype vectors are...
orthogonal, then the defined SIR in eq.(3) for any state converges to the following constant "ultimate SIR" awithin a finite step number

\[ \theta_i(k > k_T) = \frac{r}{N - L} \]  

(103)

where \( N \) is the dimension of the network and \( L \) is the prototype vectors, \( t_T \) shows a finite step number for any initial condition \( x(0) \) which is not completely orthogonal to any of the raws of matrix \( Q \) in eq.(100).

Proof:
The proof is presented in Appendix I.

Corollary 1:
For the proposed DSALU-"SIR"1 to be used as an associate memory system, whose weight matrix is designed by the proposed outer-products (Hebbian-learning)-based method in section III-A for \( L \) orthogonal binary vectors of dimension \( N \),

\[ \lambda_{max} = N - L \]  

(104)

where \( \lambda_{max} \) is the maximum (positive) eigenvalue of the weight matrix \( W \).

Proof:
From the proposition 1 and 4 above, the result in proposition 1 is valid for any real symmetric matrix \( W \) whose maximum eigenvalue is positive, while the result of proposition 4 is for only the symmetric matrix designed by the method in section III-A. So, comparing the results of the proposition 1 and 4, we conclude that for the network in proposition 4, the maximum (positive) eigenvalue of the weight matrix \( W \) is equal to \( N - L \).

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\[ ^5 \text{It's easy to check in advance if the initial vector } x(0) \text{ is completely orthogonal to any of the raws of matrix } Q \text{ in eq.(100) or not. If so, then this can easily be overcome by introducing a small random number to } x(0) \text{ so that it's not completely orthogonal to any of the raws of matrix } Q. \]
IV. SIMULATION RESULTS

We take similar examples as in [7], [8] and [9] for the sake of brevity and easy reproduction of the simulation results. We apply the same Hebb-based (outer-products-based) design procedure ([5]) in [7] and [9], which is presented in section III-A. So, the weight matrix $W$ in all the simulated networks (the proposed networks and Discrete-Time Hopfield Network) are the same.

In this section, we present two examples, one with 8 neurons and one with 16 neurons. As in [8], traditional Hopfield network is used a reference network. The discrete Hopfield Network [1] is

$$x^{k+1} = \text{sign} (W x^k)$$  \hspace{1cm} (105)

where $W$ is the weight matrix and $x^k$ is the state at time $k$, and at most one state is updated.

**Example 1:**

The desired prototype vectors are

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \end{bmatrix}$$  \hspace{1cm} (106)

The weight matrices $rI$ and $W$, and the threshold vector $b$ are obtained as follows by using the outer-products-based design mentioned above and $\theta$ is chosen as -1 and for the DSALU-U"SIR"2 network, $\rho = 0.5$.

$$A = 2I, \quad W = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 & -2 & -2 \\ 2 & 0 & 0 & 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 2 & -2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -2 & -2 & 0 & 0 \\ 0 & 0 & -2 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 2 & 0 & 0 & 0 & 0 \\ -2 & -2 & 0 & 0 & 0 & 0 & 0 & 2 \\ -2 & -2 & 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix}, \quad \nu = 0$$  \hspace{1cm} (107)

where $I$ shows the identity matrix of dimension $N$ by $N$. 

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The Figure 3 shows the percentages of correctly recovered desired patterns for all possible initial conditions $x(k = 0) \in (-1, +1)^N$, in the proposed DSALU-"SIR"1 and 2 as compared to traditional Hopfield network.

Let $m_d$ show the number of prototype vectors and $C(N, K)$ represents the combination $N, K$ (such that $N \geq K \geq 0$), which is equal to $C(N, K) = \frac{N!}{(N-K)!K!}$, where ! shows factorial. In our simulation, the prototype vectors are from $(-1, 1)^N$ as seen above. For initial conditions, we alter the sign of $K$ states where $K=0, 1, 2, 3$ and $4$, which means the initial condition is within $K$-Hamming distance from the corresponding prototype vector. So, the total number of different possible combinations for the initial conditions for this example is 24, 84 and 168 for 1, 2 and 3-Hamming distance cases respectively, which could be calculated by $m_d \times C(8, K)$, where $m_d = 3$ and $K = 1, 2$ and $3$.

As seen from Figure 3, the performance of the proposed networks DSALU-"SIR"1 and 2 are the same as that of the discrete-time Hopfield Network for 1-Hamming distance case (%100 for both networks) and are comparable results for 2 and 3-Hamming distance cases respectively.

**Example 2:**

The desired prototype vectors are

\[
D = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1
\end{bmatrix}
\] (108)

The weight matrices $rI$ and $W$ and threshold vector $b$ is obtained as follows by using the outer products based design as explained above. For matrix $rI$, $\vartheta$ is chosen as -2. The other network parameters are chosen as in example 1.

\[
rI = 3I,
\]
As seen from Figure 4 the performance of the proposed networks DSALU-"SIR"1 and 2 as compared to the traditional Hopfield network.

The total number of different possible combinations for the initial conditions for this example is 64, 480 and 2240 and 7280 for 1, 2, 3 and 4-Hamming distance cases respectively, which could be calculated by \( m_d \times C(16, K) \), where \( m_d = 4 \) and \( K = 1, 2, 3 \) and 4.

As seen from Figure 4 the performance of the proposed networks DSALU-"SIR"1 and 2 are the same as that of Hopfield Network for 1 and 2-Hamming distance cases (\( \%100 \) for both networks), and are comparable for 3,4 and 5-Hamming distance cases respectively.

V. Conclusions

Using the same “SIR” concept as in [7], and [8], we, in this paper, analyse the “SIR” of the states in the following two \( N \)-dimensional discrete-time autonomous linear systems:
1) The system \( x(k+1) = (I + \alpha(-rI + W))x(k) \) which is obtained by discretizing the autonomous continuous-time linear system in [9] using Euler method; where \( I \) is the identity matrix, \( r \) is a positive real number, and \( \alpha > 0 \) is the step size.

2) A more general autonomous linear system described by \( x(k+1) = -\rho I + Wx(k) \), where \( W \) is any real symmetric matrix whose diagonal elements are zero, and \( I \) denotes the identity matrix and \( \rho \) is a positive real number.

Our analysis shows that:

1) The “SIR” of any state converges to a constant value, called “Ultimate SIR”, in a finite time in the above-mentioned discrete-time linear systems.

2) The “Ultimate SIR” in the first system above is equal to \( \frac{\rho}{\lambda_{max}} \) where \( \lambda_{max} \) is the maximum (positive) eigenvalue of the matrix \( W \). These results are in line with those of [9] where corresponding continuous-time linear system is examined.

3) The “Ultimate SIR” in the second system above is equal to \( \frac{\rho}{\lambda_m} \) where \( \lambda_m \) is the eigenvalue of \( W \) which satisfy \( |\lambda_m - \rho| = \max\{|\lambda_i - \rho|\}_{i=1}^N \) if \( \rho \) is accordingly determined from the interval \( 0 < \rho < 1 \) as described in (57).

In the later part of the paper, we use the introduced “Ultimate SIR” to stabilize the (originally unstable) networks. It’s shown that the proposed Discrete-Time “Stabilized”-Autonomous-Linear-Networks-with-Ultimate-SIR” exhibit features which are generally attributed to Discrete-Time Hopfield Networks. Taking the sign of the converged states, the proposed networks are applied to binary associative memory design. Computer simulations show the effectiveness of the proposed networks as compared to traditional discrete Hopfield Networks.

As far as design of the design of the weight matrices are concerned, we also present an outer-products (Hebbian-learning)-based method, and show that if the prototype vectors are orthogonal in the proposed DSAL-U”SIR”1 network, then the ultimate SIR \( \vartheta^{ult} \) is equal to \( \frac{r}{N-L} \) where \( N \) is the dimension of the network and \( L \) is the prototype vectors.

**APPENDIX I**

*Proof of Proposition 5:*

The solution of the proposed network DSALU-”SIR”1 in eq.(8) is
\[ x(k) = \left( I + \alpha(-rI + W) \right)^k x(0) \]  

(110)

Let’s denote the system matrix as

\[ M = I + \alpha(-rI + W) \]  

(111)

From eq.(100) and (102),

\[ W = Q - LI \]  

(112)

where \( L \) is the number of orthogonal prototype vector. Using eq.(112) in (111) gives

\[ M = \left( 1 - \alpha(r + L) \right)I + \alpha Q \]  

(113)

and since \( d_s \in (-1, +1)^N \),

\[ Q^2 = NQ \]  

(114)

where \( N \) is the dimension of the system, i.e., the number of states, and \( Q \) is given in (100).

Next, we examine the powers of matrix \( M \) since the solution of the system is \( x(k) = M^k x(0) \):

First let’s define \( b \) and \( c \) as follows

\[ b = 1 - \alpha(r + L) \]  

(115)

\[ c = 1 - \alpha(r + L - N) \]  

(116)

From eq.(113) and (115),

\[ M = bI + \sigma(1)Q \]  

(117)

where

\[ \sigma(1) = \alpha \]  

(118)
The matrix $M^2$ is

$$M^2 = b^2 I + \sigma(2)Q$$

where $\sigma(2)$ is equal to

$$\sigma(2) = \alpha(b+c)$$

Similarly, the matrix $M^3$ is obtained as

$$M^3 = b^3 I + \sigma(3)Q$$

where $\sigma(3)$ is

$$\sigma(3) = \alpha(b^2 + bc + c^2)$$

For $k = 4$,

$$M^4 = b^4 I + \sigma(4)Q$$

where $\sigma(4)$ is

$$\sigma(4) = \alpha(b^3 + b^2 c + bc^2 + c^3)$$

So, when we continue, we observe that the $k$’th power of the matrix $M$ is obtained as

$$M^k = b^k I + \sigma(k)Q$$

where $\sigma(k)$ is

$$\sigma(k) = \alpha \sum_{m=1}^{k} b^{k-m} c^{m-1}$$

where $b = 1 - \alpha(r+L)$ and $c = 1 - \alpha(r+L-N)$ as defined in eq.(115) and (116), respectively.

Let’s define the following constant $\varphi$

$$\varphi = \frac{b}{c} = \frac{1 - \alpha(r+L)}{1 - \alpha(r+L-N)}$$
Using (127) in (126) results in

\[ \sigma(k) = \alpha c^{k-1} \sum_{m=1}^{k} \varphi^{k-m} \]  

(128)

Summing \(-\sigma(k)\) with \(\varphi \sigma(k)\) yields

\[ \sigma(k) = \frac{\varphi^k - 1}{\varphi - 1} \alpha c^{k-1} \]  

(129)

From eq.(125) and (129), the matrix \(M^k\) is written as follows

\[ M^k = b^k I + \alpha c^{k-1} \frac{\varphi^k - 1}{\varphi - 1} Q \]  

(130)

Using the definition of \(b\) and \(c\) in eq.(115) and (116), respectively, in (130) gives

\[ M^k = M_{tp}(k) + M_{sp}(k) \]  

(131)

where

\[ M_{tp}(k) = \left(1 - \alpha(r + L)\right)^k I - \frac{1}{\alpha N} \left(1 - \alpha(r + L)\right)^k Q \]  

(132)

and

\[ M_{sp}(k) = \frac{\left(1 - \alpha(r + L - N)\right)^k}{\alpha N} Q \]  

(133)

In above equations, the number of network dimension (\(N\)) is much larger than the number of prototype vector (\(L\)), i.e. \(N >> L\). In Hopfield networks, theoretically, \(L\) is in the range of \%15 of \(N\) (e.g. [3]). So, \(N > r + L\) by choosing \(r\) accordingly. The learning factor positive \(\alpha\) is also typically a relatively small number less than 1. Therefore, \(\left(1 - \alpha(r + L)\right) < 1\) and \(\left(1 - \alpha(r + L - N)\right) > 1\). This means that 1) the \(M_{tp}(k)\) in eq.(132) vanishes (approaches to zero) within a finite step number \(k_T\); and 2) what shapes the steady state behavior of the system is merely \(M_{sp}\).

\[ M_{tp}(k) \approx 0, \quad k \geq k_T \]  

(134)
We call the matrices $M_{tp}(k)$ and $M_{sp}(k)$ in (132) and (133) as transitory phase part and steady phase part, respectively, of the matrix $M^k$.

So, the steady phase solution is obtained from eq.(11), (131) and (133)

$$x_{sp}(k) = M_{sp}(k)x(0)$$
$$= \left(1 - \alpha(r + L - N)\right)^k_{\alpha N}Qx(0), \quad k \geq k_T$$

Let’s define the interference vector, $J_{sp}(k)$ as

$$J_{sp}(k) = Wx_{sp}(k)$$

From eq.(112), (114) and (137)

$$J_{sp}(k) = (Q - LI)x_{sp}(k)$$
$$= \left(1 - \alpha(r + L - N)\right)^k_{\alpha N}(N - L)Qx(0)$$

So, dividing vector $x_{sp}(k)$ of eq.(136) to $J_{sp}(k)$ of eq.(138) elementwise and comparing the outcome with the “SIR” definition in eq.(3) results in

$$\theta_i(k) = \frac{r}{N - L}, \quad i = 1, \ldots, N$$

which completes the proof.

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