Quantum entanglement is an essential resource to achieve quantum advantages in various nonclassical tasks, including quantum teleportation [1] and communication [2]. The fields of many-body physics have recognised entanglement as a useful quantity to characterise quantum ground states [3] and to witness quantum phase-transitions [4, 5]. For a bipartite pure state |Ψ⟩_{AB}, the most widely-studied measure to quantify entanglement is the entanglement entropy given by $E_{S}(\Psi) = S(\rho_B) = -\text{Tr}[\rho_B \log \rho_B]$, where $\rho_B = \text{Tr}_A |\Psi⟩_{AB}⟨\Psi|$ is the local quantum state. More generally, the Rényi entanglement entropy (REE) [6],

$$E_{\alpha}(\Psi) = S_{\alpha}(\rho_B) = \frac{1}{1-\alpha} \log \text{Tr}[\rho_B^{\alpha}],$$

has been studied as an extended class of entanglement quantifiers. The entanglement entropy $E_S$ can be retrieved in the limit $\alpha \to 1$. The REEs of low and high $\alpha$s behave differently given changes of the spectrum of $\rho_B$, which has made them useful to classify quantum phases [6, 7]. We note that $E_{\alpha}$ for an integer $\alpha > 1$ can be estimated without quantum-state tomography [10, 11] and $E_2$ has recently been measured in quantum many-body systems [12, 13].

One of the most important properties of entanglement is that we cannot increase it deterministically by any local operation and classical communication (LOCC) protocols. Nevertheless, it is possible to distill the maximally entangled state from a partially entangled state by allowing the success probability to be less than unity [16, 17]. The monotonicity condition for entanglement measures ensures that the degree of average entanglement for the total system does not increase by any stochastic LOCC (SLOCC) protocols. The REEs of order $0 \leq \alpha \leq 1$ satisfy such the condition [3], and they serve as reliable measures of entanglement. However, despite the advantages to be experimentally measurable, the REEs of order $\alpha > 1$ have a limitation as they do not satisfy the monotonicity condition [18]. Here, recalling that the monotonicity condition is based on the average entanglement, we ask the question whether the REEs of any order can be of use to characterise entanglement under SLOCC by taking into account their higher-order moments. There have been attempts to find refined conditions on the statistical properties of entanglement beyond its mean value [16, 17]. These have, however, been limited to the study of the success probability of nondeterministic transformations when the exact form, i.e., the Schmidt decompositions of outcome states, is given.

In this paper, we explore a general condition on entanglement transformation through SLOCC by focusing only on the REE of the outcome states, without characterising their Schmidt decompositions. We introduce a generalised entanglement entropy as an entanglement monotone, and based on this, establish a condition on the distribution of the REE under any SLOCC protocols. From this condition, we demonstrate that the success probability of raising entanglement exponentially decreases as the entanglement required to distill increases. This provides a strong limitation on entanglement manipulation under SLOCC protocols, for instance in distilling not only the maximally entangled states [16, 17] but also any increased entangled states. Our results can also be applied to the entanglement estimation of quantum many-body systems, as a lower bound on the REE is obtained from the higher-order moments of $E_2$ after applying an SLOCC protocol. Finally, we discuss how our results can be extended for mixed states.

**Condition on the REE distribution under SLOCC.**—Let us suppose that the initial bipartite pure state $|\Psi⟩_{AB}$ transforms through an SLOCC protocol as

$$|\Psi⟩_{AB} \xrightarrow{\mathcal{E}_{\text{SLOCC}}} \{p_m, |\Psi_m⟩_{AB}\}. \tag{1}$$

For simplicity, we shall denote bipartite states without the subscript $AB$, unless further specification is needed. Entanglement of the outcome state can increase depending on the SLOCC protocol, with probabilities $0 \leq p_m \leq 1$ satisfying $\sum_m p_m = 1$. The
monotonicity of the REE of order $0 \leq \alpha \leq 1$ guarantees that entanglement does not increase on average through any SLOCC protocol, i.e., $\langle \Delta E_\alpha \rangle \leq 0$, where $\langle O \rangle := \sum_m p_m O(\Psi_m)$ denotes an average over all possible outcomes, and $\Delta E_\alpha(\Psi_m) = E_\alpha(\Psi_m) - E_\alpha(\Psi)$ is the change in REE. However, the REE of order $\alpha > 1$ does not obey a monotonicity condition [18]. Nevertheless, we find, in this paper, the following statistical property that the REE of order $\alpha \in (0, \infty)$ should satisfy the following condition,

$$\langle e^{s(1-\alpha)\Delta E_\alpha} \rangle \begin{cases} 
\leq 1 & (0 < \alpha < 1 \text{ and } s \leq \frac{1}{\alpha}) \\
\geq 1 & (\alpha > 1 \text{ and } s \geq \frac{1}{\alpha})
\end{cases},$$

(2)

under any SLOCC protocols. We note that the condition is stronger than the monotonicity condition of the REEs of order $0 < \alpha < 1$ due to the convexity of the exponential function. It also provides more information about the higher-order moments $\langle \Delta E_\alpha \rangle^k$ beyond the mean value for any $\alpha \neq 1$. This result can be compared to entanglement fluctuation theorems [19, 20] showing the equality relations regarding the higher-order moments of outcome entanglement statistics. While previous works focus on a specific LOCC protocol [13] and the fluctuation of the Schmidt coefficients [20], our results do not depend on the form of LOCC protocol while dealing with the spectrum of entanglement measures given by the REE.

In order to prove Eq. (2), we introduce a family of entanglement monotones based on the generalised quantum entropy [21], given by $S_{(\alpha,s)}(\rho) := \frac{1}{s(1-\alpha)} \left( \exp(s(1-\alpha)S_\alpha(\rho)) - 1 \right)$. As $s \to 0$, $S_{(\alpha,s)}$ becomes the Rényi entropy $S_\alpha$, whereas $s = 1$ gives the Tsallis entropy [22]. Various properties of the generalised entropy, including the concavity on positive semi-definite matrices for $(\alpha, s) \in \Omega_{\text{conca}} = \{(\alpha, s) | 0 < \alpha < 1 \text{ and } s \leq \frac{1}{\alpha} \} \cup \{(\alpha, s) | \alpha > 1 \text{ and } s \geq \frac{1}{\alpha}\}$ have been studied in Refs. [21, 23, 24]. Vidal's work [26] then allows us to define the following entanglement monotone:

**Proposition 1.** The generalised entanglement entropy (GEE), defined as $$E_{(\alpha,s)}(\Psi) := \frac{1}{s(1-\alpha)} \left[ e^{s(1-\alpha)E_\alpha(\Psi)} - 1 \right],$$
is an entanglement monotone for $(\alpha, s) \in \Omega_{\text{conca}}$, satisfying the following conditions: (i) $E_{(\alpha,s)}(\Psi) = 0$ if and only if $|\Psi\rangle$ is separable, and (ii) $\sum_m p_m E_{(\alpha,s)}(\Psi_m) \leq E_{(\alpha,s)}(\Psi)$ via any SLOCC transform given by Eq. (1).

Detailed derivations of Propositions throughout this paper can be found in Appendix. As shown in Appendix, Eq. (2) naturally follows from Proposition 1 as $(\Delta E_{(\alpha,s)}) \leq 0$ holds for $(\alpha, s) \in \Omega_{\text{conca}}$.

We focus on the case $s = 1/\alpha$, where $E_{(\alpha,s)}$ becomes an entanglement monotone for all $\alpha \in (0, 1) \cup (1, \infty)$. When $\alpha$ approaches 1, the measure converges to the entanglement entropy, i.e., $\lim_{\alpha \to 1} E_{(\alpha,1/\alpha)} = E_S$, and Eq. (2) becomes $(\Delta E_S) \leq 0$. We also note that $E_{(\alpha,1/\alpha)}$ for $\alpha = 1/2$ has a direct connection to the entanglement negativity for pure bipartite states as $E_{(1/2,2)}(\Psi) = \| (|\Psi\rangle_{AB}\langle\Psi|)^{TP} \|_1 - 1$, where $T_B$ denotes the partial transpose on $B$ [27]. We will show that $s = 1/\alpha$ provides the tightest bounds on entanglement distillation and estimation tasks among all possible values of $s$.

**Probability bound for entanglement distillation via SLOCC.**—We first demonstrate that our condition on the probability distribution of the REE leads to a strong restriction on entanglement distillation via SLOCC. Let us clarify this problem by defining the accumulated success probability $P(E_\alpha \geq E_{\text{target}}) = \sum_{m \in \Omega_{\text{success}}} p_m$, where $\Omega_{\text{success}} = \{ m | E_\alpha(\Psi_m) \geq E_{\text{target}} \}$ is a set where the outcome states $|\Psi_m\rangle$ have entanglement $E_\alpha(\Psi_m)$ higher than the desired value $E_{\text{target}}$.

Finding the SLOCC protocol optimising $P(E_\alpha \geq E_{\text{target}})$ can be reformulated by using the necessary and sufficient condition for a pure state transition under SLOCC [16]. Furthermore, for any entanglement monotone $E$, we show that the optimal success probability is given by

$$\sup_{E_{\text{SLOCC}}} P(E \geq E_{\text{target}}) = \max_{\Psi} \left[ \min_{l \in \{1,2, \ldots, d\}} \left( \frac{\sum_{i=1}^{d} \lambda_i^l(\Psi)}{\sum_{i=1}^{d} \lambda_i^l(\Psi')} \right) \right] E(\Psi') = E_{\text{target}},$$

where $\lambda_i^l(\Psi)$ and $d$ are the Schmidt coefficients and the Schmidt rank of $|\Psi\rangle_{AB}$, respectively (see Appendix for details). Nevertheless, this optimisation problem is computationally challenging, even if the initial state's Schmidt coefficients are known, as it is not a convex problem. Thus, evaluating the highest success probability requires searching over all possible outcome states satisfying the constraints $E(\Psi') = E_{\text{target}}$, where the non-linearity of entanglement monotones, including the REE, makes the problem even more complicated especially for large $d$. Furthermore, no such optimisation can be applied to $E_\alpha$ for $\alpha > 1$, since it is not an entanglement monotone.

Instead of finding the exact solution to this problem, we investigate an upper bound on the success probability for entanglement distillation. We find that the condition on the REE given by Eq. (2) directly leads to the following bound:

**Proposition 2.** The accumulated success probability of achieving the outcome $E_{\alpha}$ larger than $E_{\text{target}}$ is upper bounded by

$$P(E_\alpha \geq E_{\text{target}}) \leq \frac{e^{s(1-\alpha)E_\alpha(\Psi)} - 1}{e^{s(1-\alpha)E_{\text{target}} - 1}},$$

(3)

for all $(\alpha, s) \in \Omega_{\text{conca}}$. Furthermore, the right-hand-side of the inequality is minimised when $s = 1/\alpha$. 
Let us focus on the case $s = 1/\alpha$, where the bound is minimised. For $0 < \alpha < 1$, this bound is tighter than the following probability bound derived from the monotonicity of the REE,

$$\langle \Delta E_\alpha \rangle \leq 0 \implies P(E_\alpha \geq E_{\text{target}}) \leq \frac{E_\alpha(\Psi)}{E_{\text{target}}}. \quad (4)$$

We further extend the bound for $\alpha = 0$ and $\alpha = 1$. For $E_\alpha(\Psi) = \log d < E_{\text{target}}$, the right-hand-side of Eq. (3) reaches zero when $\alpha$ approaches zero, implying that no SLOCC can increase the Schmidt rank. For the distillation of the maximally entangled state $|\Phi_d\rangle_{AB} \propto \sum_{i=1}^{d} |ii\rangle_{AB}$, i.e., $E_{\text{target}} = \log d$, the bound becomes $e^{\alpha [\log(d\rho_{AB})]_d}$, which coincides with the bound given by the G-concurrence monotone $\sum_{i=1}^{d} |ii\rangle_{AB}$. Although this bound does not reach the optimal rate found in Ref. [10], it can be obtained with less information about the initial state as one does not need to know all the Schmidt coefficients. When $\alpha$ approaches $1$, the bound reduces to $P(E_{S} \geq E_{\text{target}}) \leq E_{S}(\Psi)/E_{\text{target}}$.

By combining these results with the ordering between the REEs, $E_\alpha \leq E_\beta$ for $\alpha \geq \beta$, we get the probability bound for $\alpha \in [0, \infty)$ as

$$P(E_\alpha \geq E_{\text{target}}) \leq \min_{0 \leq \beta \leq \alpha} \left[ \frac{e^{\alpha [\log(d\rho_{AB})]_d} - 1}{e^{\alpha [\log(d\rho_{AB})]_d} - 1} \right]. \quad (5)$$

This shows a strong limitation for entanglement distillation via SLOCC as the upper bound on success probabilities exponentially decreases as $P(\Delta E_\alpha \geq x) = P(E_\alpha \geq E_\alpha(\Psi) + x) \leq e^{-\alpha [\log(d\rho_{AB})]_d}$ for $\alpha \in (0, 1)$, so that for the entanglement entropy we can always find $K_\alpha > 0$ and $k_\alpha > 0$ such that

$$P(\Delta E_{S} \geq x) \leq K_\alpha e^{-k_\alpha x}.$$ 

It is worth noting that the bounds in Eqs. (3) and (5) also hold for the REE of order $\alpha > 1$, even though the monotonicity is not guaranteed in this regime.

As an illustrative example, we choose the initial state $|\chi(r)\rangle_{AB} \propto \sum_{i=1}^{d} \sqrt{r} |ii\rangle_{AB}$ with $r = 0.86$ and $d = 500$ to show a dramatic improvement of the probability bound on raising the entanglement entropy. A significant gap between the bounds given in Eqs. (4) and (5) can be observed when the raised amount of entanglement $\Delta E_{S}$ becomes large (see Fig. 1). Even though evaluation of the optimal probability is computationally challenging, we can consider some SLOCC protocols to raise entanglement. We note that the distillation of the $k$-level maximally entangled state $|\Phi_k\rangle_{AB} \propto \sum_{i=1}^{k} |ii\rangle_{AB}$ is not an efficient protocol for raising the entanglement entropy as changing only the maximum and minimum Schmidt coefficients can provide a higher success probability than the $|\Phi_k\rangle$-distillation protocol (for details of the protocol we make use of, see Appendix. Figure 1 also shows that the GEE bound always gives a tighter bound than the monotonicity condition for randomly generated initial states and target entanglement entropies.

The probability bound on raising entanglement can also be derived when SLOCC is performed to multiple copies of the initial state, i.e., $|\Psi_m\rangle_{AB} \rightarrow |\Psi_m\rangle_{A'B'}$, where $A'$ and $B'$ are subsystems of $A \otimes B$ and $B \otimes A$, respectively. In this case, the raised amount of the REE per copy can be defined as $\Delta \epsilon_\alpha(\Psi_m) = (E_\alpha(\Psi_m) - E_\alpha(\Psi_{m})) / n = E_\alpha(\Psi_m) / n - E_\alpha(\Psi)$. We then obtain $P(\Delta \epsilon_\alpha \geq x) \leq e^{\alpha [\log(d\rho_{AB})]_d}$, which implies that the probability bound exponentially decreases as the number of copies increases.

**Estimating entanglement from the REE distribution via SLOCC.**— Until now, we have seen that the lower order of the REE restricts the success probabilities of raising the REE of the higher order via SLOCC. Conversely, we show that the distribution of the higher order REE can provide a new method to estimate the REE of lower orders after applying an SLOCC protocol. To do this, we rearrange the condition given by Eq. (2) to obtain the following inequality:

**Proposition 3.** For $(\alpha, s) \in \Omega_{\text{concave}}$ and $\alpha \leq \beta$, the REE under any SLOCC transformation satisfies

$$E_\alpha(\Psi) \geq \frac{1}{s(1 - \alpha)} \log \langle e^{s(1 - \alpha)E_{\beta}} \rangle, \quad (6)$$

where the right-hand-side of the inequality is maximised when $s = 1/\alpha$.

We apply our bounds on the REE to estimate entanglement from the distribution of experimentally observable quantities after applying SLOCC. In particular, $E_2$ can be experimentally detected from a single copy of a quantum state by using cross-correlations between randomised local measurements $\mathbf{12}$. We note that
the non-projective positive-operator valued measure. This can be done by allowing non-projective POVMs \( \{ \Pi_i(\epsilon) \} \) optimised over \( 0 \leq \epsilon \leq 1 \) on the spin at \( k = 4 \) (solid lines).

As a physical example, we consider the Heisenberg model in an \( N \)-spin system with the Hamiltonian

\[
H = -J \sum_{j=1}^{N} \vec{\sigma}(j) \cdot \vec{\sigma}(j+1)
\]

with periodic boundary condition. Here, \( J \) is the interaction strength and \( \vec{\sigma}(j) = (\sigma^x(j), \sigma^y(j), \sigma^z(j)) \) is the vector of Pauli matrices acting on the \( j \)th spin. We suppose that the system has \( N = 8 \) and initially prepared in the Neel state \( |\downarrow \uparrow \downarrow \cdots \uparrow \rangle \), where \( |\uparrow \rangle \) and \( |\downarrow \rangle \) are the eigenstates of \( \sigma_z \) with the eigenvalues \( +1 \) and \( -1 \), respectively. After the state evolves to

\[
|\Psi\rangle = e^{-iH\tau/\hbar} |\downarrow \uparrow \downarrow \cdots \uparrow \rangle
\]

we investigate entanglement between two parties, \( N_A = 6 \) and \( N_B = 2 \) (see Fig. 2). By performing the projection measurements onto \( |\uparrow\rangle_k \langle \uparrow| \) and \( |\downarrow\rangle_k \langle \downarrow| \) for the \( k \)th spin, we note that \( E_{0}(\Psi) \geq \tilde{E}_{0}(\{p_m, E_2(\Psi_m)\}) \geq E_2(\Psi) \) for \( 0 \leq \alpha \leq 0.55 \). Thus, acting with the local measurements, i.e., SLOCC, can provide better estimation of \( E_{0}(\Psi) \) than the direct evaluation of \( E_2(\Psi) \) without it. This bound can be improved by allowing non-projective positive-operator valued measurements (POVMs) \( \{ \Pi_i(\epsilon) \} = (1-\epsilon)|\uparrow\rangle \langle \uparrow| + \epsilon |\downarrow\rangle \langle \downarrow| \) and \( \Pi_i(\epsilon) = \epsilon |\uparrow\rangle \langle \uparrow| + (1-\epsilon) |\downarrow\rangle \langle \downarrow| \) as \( E_{0}(\Psi) \geq \tilde{E}_{0}(\{p_m, E_2(\Psi_m)\}) \geq E_2(\Psi) \) for all \( 0 \leq \alpha \leq 2 \). Further discussions for various values of \( \tau \) and other physical examples concerning the ground state of the transverse Ising model can be found in Appendix.

We also note that the right-hand-side of Eq. (6) for \( \beta = \alpha \) has been recently studied as a measure for accessible entanglement of indistinguishable particles \( [29] \) by considering a projection onto the Hilbert space with a fixed particle number in the subsystem.

**Generalisation for mixed states.**—We consider generalisation of our results to a mixed state. The generalised entanglement measure for a mixed state \( \rho \) can be constructed based on that for pure states as

\[
\text{co}(E_{\alpha,s}(\rho)) := \min_{\{q_{\alpha,s}\}} \sum_{\mu} q_{\mu} E_{\alpha,s}(\psi_{\mu}),
\]

where \( \text{co}(f) := \min_{\{q_{\alpha,s}\}} \sum_{\mu} q_{\mu} E_{\alpha,s}(\psi_{\mu}) \) is the convex roof construction of \( f \), obtained by optimising over all possible pure state decomposition \( \rho = \sum_{\mu} q_{\mu} |\psi_{\mu}\rangle \langle \psi_{\mu}| \). When \( s \to 0 \), \( \text{co}(E_{\alpha,s}) \) becomes the convex roof of the REE, i.e., \( \text{co}(E_{\alpha}) \), where its evaluation has been studied \( [30,31] \) for some classes of mixed states. We also note that for the case of \( \alpha \to 1 \), \( \text{co}(E_{(\alpha,1/\alpha)}) \) becomes the entanglement of formation \( [32] \).

Following on, we can consider a general SLOCC protocol, in which a mixed bipartite quantum state \( \rho \) is transformed into another bipartite state \( \rho_M \) with the outcome probability \( p_M \). This situation can be described by a coarse-grained LOCC instrument \( [33] \), where a coarse-grained outcome \( M \) consists of fine-grained outcomes \( m \in M \). In this case, the outcome state can be expressed as \( \mathcal{E}^{(M)}(\rho) = \sum_{m \in M} \mathcal{E}^{(m)}(\rho) = p_M \rho_M \), where \( p_M = \text{Tr}[\mathcal{E}^{(M)}(\rho)] \). We show the following inequality holds for mixed states:

**Proposition 4.** Suppose that an initial state \( \rho \) transforms by SLOCC. Then, the following inequality holds

\[
\text{co}(E_{\alpha,s}(\rho)) \geq \frac{1}{s(1-\alpha)} \left[ \langle \psi_{\alpha,s}\rangle_{\text{co}(E_{\beta,s})} - 1 \right]
\]

for \( 0 < \alpha < 1 \), \( \alpha \leq \beta \), and \( s \leq 1/\alpha \). Furthermore, the right-hand-side of the inequality is maximised when \( s = 1/\alpha \). Subsequently, the success probability of raising \( \text{co}(E_{\alpha}) \) for \( \alpha > 0 \) is upper bounded as

\[
P(\text{co}(E_{\alpha}) \geq E_{\text{target}}) \leq \min_{0 \leq \beta \leq \alpha^*} \left[ \left( \frac{1-\beta}{\beta} \right) \text{co}(E_{(\beta,1/\beta)}(\rho)) \right] - 1,
\]

where \( \alpha^* = \min\{\alpha, 1\} \).

Although finding an explicit expression for the convex roof measure \( \text{co}(E_{(\alpha,1/\alpha)}) \) for a general mixed state and \( \alpha \) is an open question, we point out that \( \text{co}(E_{(1/2,2)}) \) of the Werner states and isotropic states has been discovered \( [34] \) to have a less complicated form than the entanglement of formation \( [32] \).
Remarks.— We have shown that under any SLOCC process, there exist refined conditions on the outcome distribution of the REE, beyond its mean value. To this end, we have introduced a new family of entanglement measures based on the generalised entropy, the monotonicity of which involves the contribution of the higher-order moments of the outcome REE distribution after performing SLOCC. Our work provides a fundamental limitation for stochastic entanglement distillation, namely that its success probability exponentially decreases as the distilled amount of entanglement increases. The refined condition can also be utilised to obtain a lower bound on the initial state’s $E_\alpha$ from the distribution of the outcome $E_\beta$ for $\beta \geq \alpha$, for instance $E_2$ which can more readily be measured in experiments.

An interesting direction for future research is applying our results to other entanglement quantifiers related to the REE, such as the conditional Rényi entropy \[35\] and $\alpha$-logarithmic negativity \[36\]. Generalisation of our work to nondeterministic manipulation of multi-partite entanglement \[37\] could also be an intriguing topic as there exist distinct classes of entangled states that are not interconvertible by SLOCC \[38\].

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Appendix

Proof of Proposition 1

We show that the generalised entanglement entropy (GEE)

\[ E_{(\alpha,s)}(\Psi) := S_{(\alpha,s)}(\rho_B) = \frac{1}{s(1-\alpha)} \left[ e^{s(1-\alpha)E_\alpha(\Psi)} - 1 \right] \]

is an entanglement monotone for \((\alpha, s) \in \Omega_{\text{concave}} = \{(\alpha, s)|0 < \alpha < 1 \text{ and } s \leq 1/\alpha\} \cup \{(\alpha, s)|\alpha > 1 \text{ and } s \geq 1/\alpha\} \).

**Proof.** As every entanglement monotone for a pure state \(|\Psi\rangle_{AB}\) is defined by a concave function on the local density matrix \(\rho_B = \text{Tr}_A|\Psi\rangle_{AB}\langle\Psi|\) [21], it sufficient to show that

\[ S_{(\alpha,s)}(\rho_B) = \frac{1}{s(1-\alpha)} \left[ e^{s(1-\alpha)S_\alpha(\rho_B)} - 1 \right] , \]

is an operator concave function for positive semidefinite matrices. This has been proven in Ref. [21] based on Minkowski’s inequality, and we show a simplified version of the proof. We first show the following lemma, regarding

Lemma 1 (Concavity (convexity) of \(\|X\|_\alpha\) [21, 24]). For positive semidefinite matrices \(X\) and \(Y\) and \(0 \leq \mu \leq 1\),

\[ \|\mu X + (1 - \mu)Y\|_\alpha \left\{ \begin{array}{ll}
\geq \mu \|X\|_\alpha + (1 - \mu)\|Y\|_\alpha & (0 < \alpha \leq 1) \\
\leq \mu \|X\|_\alpha + (1 - \mu)\|Y\|_\alpha & (\alpha \geq 1)
\end{array} \right. . \]

**Proof.** Let us define \(\tilde{X} = X/\|X\|_p\) and \(\tilde{Y} = Y/\|Y\|_p\). Then we have

\[ \frac{\|\mu \tilde{X} + (1 - \mu)\tilde{Y}\|_\alpha}{\mu\|\tilde{X}\|_\alpha + (1 - \mu)\|\tilde{Y}\|_\alpha} = \left[ \text{Tr}(\mu \tilde{X} + (1 - \mu)\tilde{Y})^\alpha \right]^{1/\alpha}, \]

where \(\mu = \|X\|_\alpha/\|\mu X\|_\alpha + (1 - \mu)\|Y\|_\alpha\). By noting that \(f(t) = t^p\) is concave for \(0 < p < 1\) and convex for \(p > 1\) and by using \(\text{Tr}[X^\alpha] = 1 = \text{Tr}[Y^\alpha]\), we verify that the right-hand-side of the equation is greater or equal to 1 for \(0 < \alpha < 1\) while less or equal to 1 for \(\alpha > 1\).

Now we show the main proof by noting that \(S_{(\alpha,s)}(\rho_B) = \frac{1}{s(1-\alpha)} \|\rho_B\|_\alpha^{a\alpha} - 1\). For \(0 < \alpha < 1\) and \(s \leq 1/\alpha\), \(\|\rho_B\|_\alpha^{s\alpha}\) is concave since \(\|\rho_B\|_\alpha\) is a monotone increasing function and concave for \(0 < \alpha < 1\) and \(f(t) = t^{\alpha s}\) is a concave function for \(s \leq 1\). Conversely, for \(\alpha > 1\) and \(s \geq 1/\alpha\), \(\|\rho_B\|_\alpha^{s\alpha}\) is convex since \(\|\rho_B\|_\alpha\) is convex for \(\alpha > 1\) and \(f(t) = t^{\alpha s}\) is a convex function for \(s \geq 1\). By taking into account the term \((1 - \alpha)\), which is positive (negative) for \(0 < \alpha < 1\) (\(\alpha > 1\)), we conclude that \(S_{(\alpha,s)}(\rho_B)\) is concave on positive semidefinite matrices for \((\alpha, s) \in \Omega_{\text{concave}}\). In particular, for the case \(\alpha = 1\), \(E_{(\alpha,s)}\) becomes the entanglement entropy \(E_S\) for any \(s \neq 0\) as

\[ \lim_{\alpha \to 1} \frac{1}{s(1-\alpha)} \left[ e^{s(1-\alpha)S_\alpha(\rho_B)} - 1 \right] = S(\rho_B) = E_S(\Psi). \]

**Condition on the probability distribution of the REE under LOCC**

We start with the following inequality given by the monotonicity of \(E_{(\alpha,s)},\)

\[ \frac{1}{s(1-\alpha)} \left[ \sum_m p_m e^{s(1-\alpha)E_\alpha(\Psi_m)} - 1 \right] \leq \frac{1}{s(1-\alpha)} \left[ e^{s(1-\alpha)E_\alpha(\Psi)} - 1 \right] , \]

which can be rearranged to

\[ \frac{1}{s(1-\alpha)} \langle e^{s(1-\alpha)\Delta E_\alpha} \rangle \leq \frac{1}{s(1-\alpha)}, \]

We then have the desired inequality, depending on the sign of \((1 - \alpha)\),

\[ \langle e^{s(1-\alpha)\Delta E_\alpha} \rangle \left\{ \begin{array}{ll}
\leq 1 & (0 < \alpha < 1 \text{ and } s \leq \frac{1}{\alpha}) \\
\geq 1 & (\alpha > 1 \text{ and } s \geq \frac{1}{\alpha})
\end{array} \right. . \]
Optimal success probability of raising entanglement via SLOCC

We show that for an entanglement monotone \( E \), the optimal success probability of raising entanglement more than a desired value \( E_{\text{target}} \) via SLOCC is given by

\[
\sup_{\mathcal{E}_{\text{SLOCC}}} P \left( E \geq E_{\text{target}} \right) = \max_{\Psi} \left[ \min_{l \in \{1, 2, \ldots, d\}} \left( \frac{\sum_{i=l} d \lambda_i^2(\Psi)}{\sum_{i=l} d \lambda_i^2(\Psi')} \right) E(\Psi') = E_{\text{target}} \right],
\]

where \( \lambda_i^2(\Psi) \) are the Schmidt coefficients of \(|\Psi\rangle_{AB}\) in descending order.

**Proof.** We first show that a single state transformation is enough to achieve the optimal probability. In order to prove this, we use that the necessary and sufficient condition \[16\] for a pure state transition under an SLOCC protocol:

\[
|\Psi\rangle_{AB} \xrightarrow{\mathcal{E}_{\text{SLOCC}}} \{p_m, |\Psi_m\rangle_{AB}\} \iff \sum_{i=l} d \lambda_i^2(\Psi) \geq \sum_{m} p_m \left( \sum_{i=l} d \lambda_i^2(\Psi_m) \right) \quad \forall l = 1, 2, \ldots, d. \tag{8}
\]

Let us suppose that there exists an SLOCC protocol that transforms the initial bipartite state \(|\Psi\rangle_{AB} = \sum_{i=1} d \sqrt{\lambda_i^2(\Psi)} |ii\rangle_{AB}\) into \(|p_m, |\Psi_m\rangle_{AB}\rangle\), where \(|\Psi_1\rangle_{AB} = \sum_{i=1} ^d \sqrt{\lambda_i^2(\Psi_1)} |ii\rangle_{AB}\) and \(|\Psi_2\rangle_{AB} = \sum_{i=1} ^d \sqrt{\lambda_i^2(\Psi_2)} |ii\rangle_{AB}\) satisfy \(E(\Psi_1) \geq E_{\text{target}}\) and \(E(\Psi_2) \geq E_{\text{target}}\). Without loss of generality, we assume that all states have the same Schmidt basis since this could be achieved via local unitary operations. We then define the following bipartite state from \(|\Psi_1\rangle_{AB}\) and \(|\Psi_2\rangle_{AB}\):

\[
|\Psi'\rangle_{AB} = \sum_{i=l} d \sqrt{\left( \frac{p_1}{p_1 + p_2} \right) \lambda_i^2(\Psi_1) + \left( \frac{p_2}{p_1 + p_2} \right) \lambda_i^2(\Psi_2)} |ii\rangle_{AB} = \sum_{i=l} d \sqrt{\lambda_i^2(\Psi')} |ii\rangle_{AB},
\]

where \(\lambda_i^2(\Psi')\) is also given in descending order. We note that any entanglement monotone \(E\) of a pure bipartite state \(|\Psi\rangle_{AB}\) can be expressed by using a concave function \(V\) on its (positive-semidefinite) local density matrix \(\rho_B\) as \(E(\Psi) = V(\rho_B)\) \[9\]. We then have

\[
E(\Psi') = V \left( \left( \frac{p_1}{p_1 + p_2} \right) \rho_{B1} + \left( \frac{p_2}{p_1 + p_2} \right) \rho_{B2} \right) \\
\geq \left( \frac{p_1}{p_1 + p_2} \right) V(\rho_{B1}) + \left( \frac{p_2}{p_1 + p_2} \right) V(\rho_{B2}) \\
= \left( \frac{p_1}{p_1 + p_2} \right) E(\Psi_1) + \left( \frac{p_2}{p_1 + p_2} \right) E(\Psi_2) \\
\geq E_{\text{target}},
\]

where the first inequality comes from the concavity of \(V\) on a set of positive-semidefinite matrices. Here, \(\rho_{B1}\) and \(\rho_{B2}\) are reduced density matrices of \(|\Psi_1\rangle_{AB}\) and \(|\Psi_2\rangle_{AB}\), respectively. Also, from the necessary and sufficient condition for an SLOCC transformation given by Eq.\[8\], we have

\[
\sum_{i=l} d \lambda_i^2(\Psi) \geq p_1 \left( \sum_{i=l} d \lambda_i^2(\Psi_1) \right) + p_2 \left( \sum_{i=l} d \lambda_i^2(\Psi_2) \right) + \sum_{m \neq 1, 2} p_m \left( \sum_{i=l} d \lambda_i^2(\Psi_m) \right) \\
= (p_1 + p_2) \left[ \sum_{i=l} d \left( \frac{p_1}{p_1 + p_2} \right) \lambda_i^2(\Psi_1) + \left( \frac{p_2}{p_1 + p_2} \right) \lambda_i^2(\Psi_2) \right] + \sum_{m \neq 1, 2} p_m \left( \sum_{i=l} d \lambda_i^2(\Psi_m) \right) \\
= (p_1 + p_2) \left( \sum_{i=l} d \lambda_i^2(\Psi') \right) + \sum_{m \neq 1, 2} p_m \left( \sum_{i=l} d \lambda_i^2(\Psi_m) \right)
\]

for all \(l = 1, 2, \ldots, d\). Therefore, there exists an SLOCC protocol that transforms \(|\Psi\rangle_{AB}\) into \(|\Psi'\rangle_{AB}\) with success probability \(p_1 + p_2\).

Next, by noting that \(E(\Psi') \geq E_{\text{target}}\), we can always find a deterministic LOCC protocol \(\mathcal{E}_{\text{LOCC}}\) such that \(E(\mathcal{E}_{\text{LOCC}}(\Psi')) = E_{\text{target}}\). By combining these two protocols, we observe that for any SLOCC protocol that transforms
$|\Psi\rangle_{AB}$ into two outcome states $|\Psi_1\rangle_{AB}$ and $|\Psi_2\rangle_{AB}$ satisfying $E(|\Psi_{1(2)}\rangle) \geq E_{\text{target}}$ with probabilities $p_1$ and $p_2$, we can always find an SLOCC protocol that transforms $|\Psi\rangle_{AB}$ into $|\Psi'\rangle_{AB}$ satisfying $E(|\Psi'\rangle) = E_{\text{target}}$ with probabilities $p_1 + p_2$, i.e. with the same accumulated probability. This can be generalised for multiple outcome states of $|\Psi_m\rangle_{AB}$ satisfying $E(|\Psi_m\rangle) \geq E_{\text{target}}$ by applying multiple rounds of the pairwise merging process described above. Therefore, optimisation of the accumulated probability over all possible outcome states can be reduced to optimisation over a transition probability to a single state $|\Psi'\rangle_{AB}$ such that $E(|\Psi'\rangle) = E_{\text{target}}$.

\[ \square \]

**Entanglement manipulation protocol by varying the maximum and minimum Schmidt coefficients**

We introduce a simple protocol that can give a higher success probability than the distillation of the maximally entangled state. Suppose that the Schmidt decomposition of the initial bipartite state is given by $|\Psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_i^1(\Psi)} |ii\rangle_{AB}$. For the first round of the protocol, we vary the values of the maximum and minimum Schmidt coefficients $\lambda_1^1(\Psi)$ and $\lambda_d^1(\Psi)$ to $\lambda_1^1(\Psi) - \epsilon$ and $\lambda_d^1(\Psi) + \epsilon$, while keeping the others unchanged. In order to make all the Schmidt coefficients non-negative, $0 \leq \epsilon \leq \min\{\lambda_1^1, 1 - \lambda_d^1\}$. Then we have two possible situations:

(i) If there exists $\epsilon$ such that the outcome entanglement reaches $E_{\text{target}}$, we update the state to $|\Psi'\rangle$ having the same Schmidt coefficients as the initial state $|\Psi_{AB}\rangle$, except the two elements $\lambda_1^1(\Psi) - \epsilon$ and $\lambda_d^1(\Psi) + \epsilon$. Then entanglement of the target state becomes $E(|\Psi'\rangle) = E_{\text{target}}$, and we finish the protocol.

(ii) If there is no $\epsilon$ that can reach $E_{\text{target}}$ from varying $\lambda_1^1(\Psi) \rightarrow \lambda_1^1(\Psi) - \epsilon$ and $\lambda_d^1(\Psi) \rightarrow \lambda_d^1(\Psi) + \epsilon$, we update both coefficients to $(\lambda_1^1(\Psi) + \lambda_d^1(\Psi))/2$.

We repeat the protocol using the updated state $|\Psi'\rangle = \sum_{i=1}^{d} \sqrt{\lambda_i^1(\Psi')} |ii\rangle_{AB}$ recursively until the entanglement of the final state reaches $E_{\text{target}}$. We note that for the extreme case, $E_{\text{target}} = \log d$, this protocol ends up with the maximally entangled state after running sufficiently many rounds.

**Proof of Proposition 2**

**Proof.** We first show the following inequality

\[
P(E_{\alpha} \geq E_{\text{target}}) \leq \frac{e^{s(1-\alpha)E_{\alpha}(\Psi)} - 1}{e^{s(1-\alpha)E_{\text{target}} - 1}}
\]

for $(\alpha, s) \in \Omega_{\text{concave}}$. We note that

\[
\langle e^{s(1-\alpha)\Delta E_{\alpha}} \rangle = \sum_{m} p_{m} e^{s(1-\alpha)(E_{\alpha}(\Psi_{m}) - E_{\alpha}(\Psi))} = e^{-s(1-\alpha)E_{\alpha}(\Psi)} \left[\sum_{E_{\alpha}(\Psi_{m}) \geq E_{\text{target}}} p_{m} e^{s(1-\alpha)E_{\alpha}(\Psi_{m})} + \sum_{E_{\alpha}(\Psi_{m}) < E_{\text{target}}} p_{m} e^{s(1-\alpha)E_{\alpha}(\Psi_{m})}\right]
\]

\[
\geq e^{-s(1-\alpha)E_{\alpha}(\Psi)} \left[P(E_{\alpha} \geq E_{\text{target}}) e^{s(1-\alpha)E_{\alpha}(\Psi)} + P(E_{\alpha} < E_{\text{target}})\right] \quad (0 < \alpha < 1)
\]

\[
\leq e^{-s(1-\alpha)E_{\alpha}(\Psi)} \left[P(E_{\alpha} \geq E_{\text{target}}) e^{s(1-\alpha)E_{\alpha}(\Psi)} + P(E_{\alpha} < E_{\text{target}})\right] \quad (\alpha > 1)
\]

where the inequality comes from the fact that $E_{\alpha}(\Psi_{m}) \geq E_{\text{target}}$ for the first term and $E_{\alpha}(\Psi_{m}) \geq 0$ for the second term depending on the sign of $(1-\alpha)$. Then using $P(E_{\alpha} < E_{\text{target}}) = 1 - P(E_{\alpha} \geq E_{\text{target}})$ and combining with the condition given in Eq. (4), we have

\[
\begin{cases}
\left\{ e^{-s(1-\alpha)E_{\alpha}(\Psi)} P(E_{\alpha} \geq E_{\text{target}}) \right\} \left( e^{s(1-\alpha)E_{\alpha}(\Psi)} - 1 + 1 \right) \leq \langle e^{s(1-\alpha)\Delta E_{\alpha}} \rangle \leq 1 \quad (0 < \alpha < 1 \text{ and } s \leq \frac{1}{\alpha})
\end{cases}
\]

\[
\begin{cases}
\left\{ e^{-s(1-\alpha)E_{\alpha}(\Psi)} P(E_{\alpha} \geq E_{\text{target}}) \right\} \left( e^{s(1-\alpha)E_{\alpha}(\Psi)} - 1 + 1 \right) \geq \langle e^{s(1-\alpha)\Delta E_{\alpha}} \rangle \geq 1 \quad (\alpha > 1 \text{ and } s \geq \frac{1}{\alpha})
\end{cases}
\]

Finally be rearranging the inequalities and noting that $e^{s(1-\alpha)E_{\alpha}(\Psi)} - 1 \geq 0$ for $0 < \alpha < 1$ and $e^{s(1-\alpha)E_{\alpha}(\Psi)} - 1 \leq 0$ for $\alpha < 1$, we obtained the desired inequality.

Next, we show that $s = 1/\alpha$ gives the minimum bound for any $\alpha \in (0, \infty)$. It is enough to consider the case $E_{\text{target}} \geq E_{\alpha}(\Psi)$, otherwise $P(E_{\alpha} \geq E_{\text{target}}) \geq 1$ only gives a trivial bound. Let us define a function

\[
f(s) := \frac{e^{sx} - 1}{e^{sy} - 1}
\]
for given values of $x$ and $y$. We then note that for all $s > 0$,

$$
\frac{df(s)}{ds} = \left(\frac{e^{sx} - 1}{e^{sy} - 1}\right) \left[ \frac{x e^{sx}}{e^{sx} - 1} - \frac{y e^{sy}}{e^{sy} - 1} \right] \begin{cases} \geq 0 & (0 \geq x \geq y) \\ \leq 0 & (0 \leq x \leq y) \end{cases},
$$

since $\frac{e^{sx}}{e^{sy} - 1}$ is a monotonically increasing function on $x \in (-\infty, \infty)$ for $s > 0$. Then by taking $x = (1 - \alpha)E_\alpha(\Psi)$ and $y = (1 - \alpha)E_{\text{target}}$, we can observe that the bound is monotonically decreasing on $0 < s \leq 1/\alpha$ when $0 < \alpha < 1$, thus the minimum value is achieved for $s = 1/\alpha$. Conversely, for $\alpha > 1$, the bound is monotonically increasing on $s \geq 1/\alpha$, so the minimum value is again given by $s = 1/\alpha$.

\[ \Box \]

**Probability bounds for $\alpha = 0$ and $\alpha = 1$**

We show the limiting cases of the probability bound

$$P_{\text{bound}}(\alpha) := \frac{e^{\left(\frac{1-\alpha}{\alpha}\right)E_\alpha(\Psi)} - 1}{e^{\left(\frac{1-\alpha}{\alpha}\right)E_{\text{target}}} - 1}$$

when $\alpha$ approaches 0 and 1. We first consider the case $\alpha \to 0$. We can rewrite the bound as

$$\lim_{\alpha \to 0} P_{\text{bound}}(\alpha) = \lim_{\alpha \to 0} \left[ \frac{e^{\left(\frac{1-\alpha}{\alpha}\right)(E_\alpha(\Psi) - E_{\text{target}})} - e^{-\left(\frac{1-\alpha}{\alpha}\right)E_{\text{target}}}}{1 - e^{-\left(\frac{1-\alpha}{\alpha}\right)E_{\text{target}}}} \right],$$

then it is straightforward to see that

$$\lim_{\alpha \to 0} P_{\text{bound}}(\alpha) = \begin{cases} 0 & (E_0(\Psi) = \log d < E_{\text{target}}) \\ \infty & (E_0(\Psi) = \log d > E_{\text{target}}) \end{cases}.$$

For $E_0(\Psi) = \log d = E_{\text{target}}$, we note that

$$\lim_{\alpha \to 0} \frac{1 - \alpha}{\alpha} |E_\alpha(\Psi) - E_0(\Psi)| = \lim_{\alpha \to 0} \left[ \frac{\partial(E_\alpha(\Psi) - E_0(\Psi))}{\partial \alpha} \right] = \lim_{\alpha \to 0} \left[ -S(\tilde{\rho}_{B,\alpha}\|\rho_B) \right],$$

where $S(\sigma\|\rho) = \text{Tr}(\sigma \log \sigma - \log \rho)$ is the relative entropy and $\tilde{\rho}_{B,\alpha} = \rho_B^\beta / (\text{Tr} \rho_B^\beta)$. For $\alpha \to 0$, $\tilde{\rho}_{B,\alpha}$ becomes $\mathbb{1}_B/d$, then $\lim_{\alpha \to 0} \left(\frac{1 - \alpha}{\alpha}\right) [E_\alpha(\Psi) - E_0(\Psi)] = -S(\mathbb{1}_B/d\|\rho) = \log d - \text{Tr}(\log \rho_B)/d$. Hence, the probability bound becomes

$$\lim_{\alpha \to 0} P_{\text{bound}}(\alpha) = \begin{cases} 0 & (E_0(\Psi) = \log d < E_{\text{target}}) \\ d e^{-\text{Tr}(\log \rho_B)/d} & (E_0(\Psi) = \log d = E_{\text{target}}) \\ \infty & (E_0(\Psi) = \log d > E_{\text{target}}) \end{cases}.$$

When $\alpha$ approaches 1, we can rewrite the bound as

$$\lim_{\alpha \to 1} P_{\text{bound}}(\alpha) = \lim_{\alpha \to 1} \left(\frac{\alpha}{1 - \alpha}\right) \left[ e^{\left(\frac{\alpha}{1 - \alpha}\right)E_\alpha(\Psi)} - 1 \right] = E_{\text{S}(\Psi)} E_{\text{target}},$$

since $\lim_{\alpha \to 1} \left(\frac{\alpha}{1 - \alpha}\right) \left[ e^{\left(\frac{\alpha}{1 - \alpha}\right)E_\alpha(\Psi)} - 1 \right] = E_{\text{S}(\Psi)}$ and $\lim_{\alpha \to 1} \left(\frac{\alpha}{1 - \alpha}\right) \left[ e^{\left(\frac{\alpha}{1 - \alpha}\right)E_{\text{target}} - 1} \right] = E_{\text{target}}$.

**Proof of Proposition 3**

*Proof.* It is straightforward to obtain the inequality:

$$E_\alpha(\Psi) \geq \frac{1}{s(1 - \alpha)} \log \langle e^{s(1 - \alpha)E_\alpha} \rangle.$$
for \((\alpha, s) \in \Omega_{\text{concave}}\), by noting that
\[
\langle e^{s(1-\alpha)\Delta E_{\alpha}} \rangle = e^{-s(1-\alpha)E_{\alpha}(\Psi)} \langle e^{s(1-\alpha)E_{\alpha}} \rangle \begin{cases} 
\leq 1 & (0 < \alpha < 1 \text{ and } s \leq \frac{1}{\alpha}) \\
\geq 1 & (\alpha > 1 \text{ and } s \geq \frac{1}{\alpha}) 
\end{cases},
\]
where the inequality comes from Eq. (7). By rearranging the inequality and taking into account the sign of \((1-\alpha)\) we obtain the desired inequality. We note that the right-hand-side of the inequality is a monotone on \(E_{\alpha}(\Psi_m)\), thus we obtain
\[
E_{\alpha}(\Psi) \geq \frac{1}{s(1-\alpha)} \log(e^{s(1-\alpha)E_{\alpha}}) \geq \frac{1}{s(1-\alpha)} \log(e^{s(1-\alpha)E_{\beta}})
\]
as \(E_{\beta} \leq E_{\alpha}\) for \(0 < \alpha \leq \beta\).

We now show that \(s = 1/\alpha\) gives the maximum values of the bound,
\[
g(s) := \frac{1}{s} \log \left( \sum_m p_m e^{sx_m} \right)
\]
for a probability distribution \(\{p_m\}\) with outcome entities \(x_m\). We then note that
\[
\frac{dg(s)}{ds} = \frac{1}{s^2} \left[ \sum_m \left( \frac{p_m e^{sx_m}}{\sum_{m'} p_{m'} e^{sx_{m'}}} \right) \log \left( \frac{e^{sx_m}}{\sum_{m''} p_{m''} e^{sx_{m''}}} \right) \right] = \frac{H(\tilde{p}||p)}{s^2} \geq 0,
\]
where \(H(\tilde{p}||p) = \sum_m \tilde{p}_m \log(\tilde{p}_m/p_m)\) is the (classical) relative entropy between two probability distributions \(\{\tilde{p}_m = p_m e^{sx_m}/(\sum_{m'} p_{m'} e^{sx_{m'}})\}\) and \(\{p_m\}\). From this result, we note that \(\frac{1}{s(1-\alpha)} \log(e^{s(1-\alpha)E_{\beta}})\) is a monotonically increasing (decreasing) function of \(s\) when \(0 < \alpha < 1\) \((\alpha > 1)\). Hence, \(s = 1/\alpha\) gives the maximum bound for all \(\alpha \in (0, \infty)\) and any given distribution \(\{p_m, E_{\beta}(\Psi_m)\}\) after applying SLOCC.

Estimating entanglement of quantum many-body systems

As physical examples, we consider two different models in a 1-D spin system. First, we consider a Heisenberg model whose hamiltonian is given by
\[
H = -J \sum_j \vec{\sigma}(j) \cdot \vec{\sigma}(j+1)
\]
with periodic boundary condition \(\sigma^{(N+1)} = \sigma^{(1)}\). Let us suppose that the system is initially prepared in the Neel state \(|\uparrow \uparrow \downarrow \downarrow \cdots \uparrow\rangle\), which does not have entanglement. As the system undergoes the time evolution, the state \(|\Psi\rangle = e^{-iH \tau / \hbar} |\downarrow \uparrow \downarrow \downarrow \cdots \uparrow\rangle\) becomes entangled after some \(\tau\).

We investigate entanglement between subsystems of an \(N = 8\) spin system, divided into \(N_A = 6\) and \(N_B = 2\) after time evolution \(0 \leq J\tau \leq 10\) in units of \(\hbar = 1\). Figure 3 shows that the distribution of \(E_2(\Psi_m)\) after applying POVM

![Figure 3](image-url)

**FIG. 3:** \(E_{\alpha}(\Psi)\) (dot-dashed lines) and lower bounds given by \(E_{\alpha}(\{p_m, E_2(\Psi_m)\})\) (solid lines) for (a) \(\alpha = 0.01\), (b) \(\alpha = 0.1\), (c) \(\alpha = 0.2\), respectively. The lower bounds obtained by applying an optimal POVM on \(k = 4\) (green solid lines) and maximising over \(k = 1, 2, \ldots, N\) (blue solid lines) provide better estimation of \(E_{\alpha}(\Psi)\) than \(E_2(\Psi)\) (dotted lines).
Then, it is possible to express the outcome state as

\[ \hat{E}(\alpha, \beta) \text{ given by} \hat{E}(\alpha, \beta) \text{ for } 0 \leq \alpha \leq 1 \text{ at the critical point } J/h = 1. \] 

The bounds are given by the projection measurement \{ |\uparrow\rangle_k \rangle, \downarrow\rangle_k \rangle \} for \( k = 6 \) (green lines) and the POVM \{ \Pi_1(\epsilon), \Pi_1(\epsilon) \} optimised over \( 0 \leq \epsilon \leq 1 \) and \( k = 1, 2, \ldots, N \) (blue lines).

of \( \Pi_1(\epsilon) = (1 - \epsilon)|\uparrow\rangle_k \rangle + \epsilon|\downarrow\rangle_k \rangle \) and \( \Pi_k(\epsilon) = |\uparrow\rangle_k \rangle + (1 - \epsilon)|\downarrow\rangle_k \rangle \) on the \( k \)th spin provides an improved lower bound of \( E_\alpha(\Psi) \) than the direct estimation of \( E_2(\Psi) \), i.e., \( E_\alpha(\Psi) \geq E_\alpha(\{ p_m, E_2(\Psi_m) \}) \geq E_2(\Psi) \). However the gap between \( E_\alpha(\{ p_m, E_2(\Psi_m) \}) \) and \( E_2(\Psi) \) becomes smaller when \( \alpha \) increases.

Next, we consider a transverse Ising model with the following Hamiltonian

\[ H_{\text{Ising}} = -h \sum_{j=1}^{N} \sigma_x^{(j)} - J \sum_{j=1}^{N-1} \sigma_x^{(j)} \sigma_x^{(j+1)}, \]

where \( J \) is the interaction strength and \( h \) is the external magnetic field strength. Phase transition of this system occurs at \( J/h = 1 \). The ground state has no entanglement when \( J/h < 1 \) below the critical point, while it becomes entangled when \( J/h > 1 \). We investigate entanglement for the ground state of \( H_{\text{Ising}} \) with \( N_A = 6 \) and \( N_B = N - N_A = 2 \) by increasing the interaction strength \( J/h \).

\( E_\alpha(\Psi) \) shows behaviour different from \( E_2(\Psi) \) when the system undergoes the phase transition when \( \alpha \) is small. We plot in Fig. 4 that \( E_\alpha(\{ p_m, E_2(\Psi_m) \}) \) captures the tendency of the REE of lower order, which increases faster before the system reaches the critical point and changes more gradually than the higher order REE near the critical point. We also note that the projection measurement \{ \{ |\uparrow\rangle_k \rangle, |\downarrow\rangle_k \rangle \} \} on the single spin site is enough to observe this, while optimisation over POVMs only gives small improvement of the bound.

Proof of Proposition 4

**Proof.** We first show the inequality condition

\[
\co E_{(\alpha, s)}(\rho) \geq \frac{1}{s(1-\alpha)} \left[ \langle e^{s(1-\alpha)} \rangle_{\text{co}} E_{\alpha} \rho \right] - 1,
\]

for \( 0 < \alpha < 1 \), \( \alpha \leq \beta \), and \( s \leq 1/\alpha \). Let us suppose that \( \{ q_\mu, \langle \Psi_\mu | \rangle \} \), a pure state decomposition of \( \rho \), minimising the left-hand-side of the inequality, i.e.,

\[
\co E_{(\alpha, s)}(\rho) = \sum_\mu q_\mu^* E_{(\alpha, s)}(\Psi_\mu) = \sum_\mu q_\mu^* \left[ \frac{1}{s(1-\alpha)} \langle e^{s(1-\alpha)} \rangle_{\text{co}} E_{\alpha} \rho \right] - 1.
\]

Then, it is possible to express the outcome state as \( \rho_M = \frac{1}{\rho_M} \sum_{m \in M} |m\rangle \langle m| \) with \( \mathcal{E}(m)(|\Psi_\mu^s \rangle \langle \Psi_\mu^s|) = r_{\mu m} |\Psi_\mu^s \rangle \langle \Psi_\mu^s| \) satisfying \( \sum_m r_{\mu m} = 1 \). From the definition of the convex roof construction, we note
that \( \text{co}E_{\alpha}(\rho) \leq \sum_{m \in M} \sum_{\mu} \left( \frac{q_{\mu}^* r_{\mu m}}{p_M} \right) E_{\alpha}(\Psi^m_{\mu}) \). We then complete the proof as follows:

\[
\frac{1}{s(1 - \alpha)} \left[ e^{s(1 - \alpha)\text{co}E_{\alpha}} - 1 \right] = \frac{1}{s(1 - \alpha)} \left[ \sum_{M} p_M e^{s(1 - \alpha)\text{co}E_{\alpha}(\rho)} - 1 \right] \\
\leq \frac{1}{s(1 - \alpha)} \left[ \sum_{\mu,m} q_{\mu}^* r_{\mu m} e^{s(1 - \alpha)E_{\alpha}(\Psi^m_{\mu})} - 1 \right] \\
\leq \frac{1}{s(1 - \alpha)} \left[ \sum_{\mu} q_{\mu}^* e^{s(1 - \alpha)E_{\alpha}(\Psi^m_{\mu})} - 1 \right] \\
= \text{co}E_{(\alpha,s)}(\rho).
\]

The first inequality is obtained from the convexity of the exponential function and the second inequality comes from the monotonicity of \( \text{co}E_{(\alpha,s)}(\rho) \).

Next, we show that the right-hand-side of the inequality is maximised when \( s = 1/\alpha \). We note that for any \( x > 0 \),

\[
h(s) := \frac{1}{s} \left( e^{sx} - 1 \right)
\]

is monotonically increasing on \( s \). Therefore, the right-hand-side of Eq. (9) is maximised for the largest value of \( s \) within the valid regimes, which is \( s = 1/\alpha \). \( \square \)