SYMPLECTIC MAPS AND HYPERKÄHLER MOMENT MAP GEOMETRY

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Abstract. We obtain a correspondence between the group of symplectic diffeomorphisms of a 4-dimensional real torus and the vanishing locus of a certain hyperKähler moment map. This observation gives rise to a new flow, called the modified moment map flow. The construction can be adapted to the polyhedral setting, for which we prove a Duistermaat type theorem. This paper lays out the groundwork for some effective polyhedral symplectic geometry and for a potential Morse-Bott theory, with applications to the topology of the space of symplectic maps of the 4-torus.

1. Introduction

1.1. Motivations. Symplectic geometry is the natural mathematical framework for Hamiltonian mechanics. The group of symplectic diffeomorphisms is a cornerstone of the theory, but its topology remains unknown, to a large extent [20, §10.4]. The interest for the topology of these groups has stirred dazzling developments in the field of symplectic topology. In particular, the Arnold conjecture, about the number of fixed points of a symplectic diffeomorphism, can be understood as a natural extension of the Poincaré-Birkhoff fixed point theorem. Beautiful techniques were introduced in the field of symplectic topology for solving the Arnold conjecture, intimately related to the existence of pseudoholomorphic curves and Floer homology. More generally, Gromov showed that certain symplectic properties are flexible, thanks some h-principle and convex integration methods [11], whereas others are rigid, due to the existence of pseudoholomorphic curves [12]. The Hofer geometry of the symplectic diffeomorphism group was subsequently developed and gave new insights about such rigidity properties [16]. Although the local structure of the group of symplectic diffeomorphism is completely described as an infinite dimensional Lie group [3], its large scale topology remains deeply mysterious, in spite of all the efforts of...
the community, except for some special cases (cf. [1], for instance). In particular, nothing more than the local structure is known about the topology of the symplectic diffeomorphism group of a 4-dimensional torus.

One of the famous result about the symplectic diffeomorphism groups is due to Eliashberg and Gromov [12, 8], who proved a striking theorem: although it is possible to approximate any smooth diffeomorphism by volume preserving diffeomorphisms, in the $C^0$-sense, it is not possible to do so with symplectic diffeomorphisms. More precisely, the Eliashberg-Gromov theorem states that every smooth diffeomorphism, which is a $C^0$-limit of a sequence of symplectic diffeomorphism, must be symplectic as well. This rigidity result is a strong incentive to relax the regularity of symplectic maps and gave rise to the field of $C^0$-symplectic geometry which is, in some sense, an exploration of the closure of the group of symplectic diffeomorphisms.

Our paper deals with the case of a 4-dimensional real torus $M$, endowed with a canonical symplectic form $\omega_M$ and a conjugate hyperKähler structure. We introduce a new flow, called the modified moment map flow, associated to a hyperKähler moment map geometry on a moduli space of differential forms on $M$ and a triply Hamiltonian gauge group action on the moduli space. The fixed point locus of the modified moment map flow is closely related to the group $\text{Symp}(M, \omega_M)$ of symplectic diffeomorphisms of $M$. The flow seems to be an interesting tool to investigate the topology of $\text{Symp}(M, \omega_M)$, in the spirit of Morse-Bott theory. Some essential properties of the modified moment map flow are proved in this paper. They are, in some sense, the first steps toward an infinite dimensional Morse-Bott theory, that would lead to interesting topological invariants of the group of symplectic diffeomorphisms.

We show that the hyperKähler moment map construction can be readily adapted to polyhedral geometry. We consider the space of polyhedral maps of the torus $M$, with respect to a prescribed triangulation $\mathcal{T}$. The space of polyhedral symplectic maps $\text{Symp}(M, \omega_M, \mathcal{T})$ can be regarded as a finite dimensional approximation of the space $\text{Symp}^{PL}(M, \omega_M)$ of piecewise linear symplectic maps of the torus (cf. §7.6.3 for more details). We show that all our constructions, in the smooth setting, have analogues in the polyhedral setting and we define a polyhedral modified moment map flow. This flow is an ordinary differential equation and much stronger results can be proved. In the spirit of the Duistermaat theorem, we construct a continuous retraction thanks to the polyhedral modified moment map flow at Theorem G. This result is a strong incentive to carry out numerical computer experiments and produce effective examples of polyhedral symplectic maps of the 4-torus [13].

Very little is known about piecewise linear symplectic geometry, although some seminal works have been carried out by Gratza [10],
Bertelson-Distexe [6, 4], Jauberteau-Rollin-Tapie [14, 22] and Etourneau [9]. In each case, the authors try to extend some elementary properties of smooth symplectic geometry to the piecewise linear symplectic category. The task turns out to be unexpectedly hard, which could be a manifestation of symplectic rigidity. For example [22] shows that every smoothly immersed torus in $\mathbb{R}^4$ can be approximated, in the $C^0$ sense, by piecewise linear immersed Lagrangian tori. This result can be understood as an extension of the Gromov-Lees Lagrangian flexibility theorem [12, 17].

A tentative definition for piecewise linear symplectic manifold can be found in [4], but most of the basic properies of smooth symplectic geometry are lost in the piecewise linear case. Indeed, the classical local Darboux theorem is a conjecture for piecewise linear symplectic manifolds. The fact that the group smooth symplectic diffeomorphism is locally arc connected is classical [3, 20]. However the same statement is a conjecture for the space of piecewise linear symplectic maps. There is even worse: the only constructions of polyhedral symplectic maps of the 4-dimensional quotient torus $M$ are the trivial examples of affine maps of $M$. It is also possible to consider some product constructions induced by area preserving polyhedral maps of the 2-torus, but no general construction or deformation theory is available, like in the smooth setting.

A natural continuation of this work is to develop a Morse-Bott theory following [15], associated to the polyhedral modified moment map flow and designed to obtain topological invariants for the space of polyhedral symplectic maps of the 4-torus $\text{Symp}(M, \omega_M, \mathcal{T})$ and, ultimately, for the space $\text{Symp}^{PL}(M, \omega_M)$ of piecewise linear symplectic maps.

1.2. Notations. A hyperKähler torus $M$ of real dimension 4 is a quotient $V/\Gamma$ where

- $V$ is an affine space of real dimension 4.
- $\Gamma$ is a lattice of the underlying vector space $\overrightarrow{V}$.
- The vector space $\overrightarrow{V}$ is identified via a given linear isomorphism with the space of quaternions $\mathbb{H}$.

The quaternionic multiplication by complex numbers on the right endows $\mathbb{H}$, hence $\overrightarrow{V}$, with a structure of complex vector space. In addition, the vector space $\overrightarrow{V}$ carries a Euclidean inner product $g_V$, deduced from the canonical inner product on $\mathbb{H}$ and the linear isomorphism. Hence the affine space $V$ is endowed with a flat Riemannian metric $g_V$ and a compatible integrable almost complex structure denoted $i$. In conclusion, we have a flat Kähler structure on the affine space $(V, g_V, i, \omega_V)$, where $\omega_V$ is the Kähler form. The Kähler structure on $V$ descends to the quotient $M$ as a flat Kähler structure

$$(M, g_M, i, \omega_M)$$
called the canonical Kähler structure. We are now interested in the group of symplectic diffeomorphisms of the torus denoted

\[ \text{Symp}(M, \omega_M). \]

There are many other \( g_M \)-compatible complex structures on \( M \): the quaternionic multiplication by \( i, j \) and \( k \), on the left induce three \( g_M \)-compatible integrable almost complex structures \( I, J \) and \( K \) on \( M \), which define a conjugate hyperKähler structure with Kähler forms \( \hat{\omega}_I, \hat{\omega}_J \) and \( \hat{\omega}_K \). By definition the hyperKähler forms \( \hat{\omega}_\bullet \) are compatible with the reverse orientation of \( \omega_M \).

1.3. Statement of results in the smooth setting. The gauge group

\[ T = C^\infty(M, S^1) \]

acts by complex multiplication on the moduli space

\[ F = \Omega^1(M, \overrightarrow{V}) \]

of \( \overrightarrow{V} \)-valued differential 1-forms on \( M \). We show that the moduli space \( F \) carries a natural formal hyperKähler structure, together with a triply Hamiltonian action of the gauge group \( T \).

**Theorem A.** We consider a hyperKähler quotient torus \( M = V/\Gamma \) of real dimension 4, endowed with its canonical symplectic form \( \omega_M \) and conjugate hyperKähler structure, given by the almost complex structures \( I, J \) and \( K \), with associated Kähler forms \( \hat{\omega}_\bullet \) for \( \bullet = I, J, K \). The moduli space \( F = \Omega^1(M, \overrightarrow{V}) \) of \( \overrightarrow{V} \)-valued differential 1-forms on \( M \), carries a canonical hyperKähler structure

\[ (F, \mathcal{G}, I, J, K), \]

where \( \mathcal{G} \) is a Euclidean inner product compatible with the almost complex structures \( I, J \) and \( K \) on \( F \). The associated Kähler forms are denoted respectively \( \Omega_I, \Omega_J \) and \( \Omega_K \).

The moduli space \( F \) is endowed with a canonical action of the gauge group \( T = C^\infty(M, S^1) \), by complex multiplication. The hyperKähler structure of \( F \) is invariant under the \( T \)-action. Furthermore, the gauge group action is Hamiltonian with respect to \( \Omega_I, \Omega_J \) and \( \Omega_K \) and the corresponding moment maps

\[ \mu_I, \mu_J, \mu_K : F \to t \simeq C^\infty(M, \mathbb{R}), \]

are given by the formulas

\[ \mu_\bullet(F) = -\frac{(F^*\omega_V) \wedge \hat{\omega}_\bullet}{\text{vol}_M}, \]

for \( \bullet = I, J, K \), where \( \text{vol}_M = \frac{\sqrt{2}}{2} \) and

\[ (F^*\omega_V)(\eta_1, \eta_2) = \omega_V(F(\eta_1), F(\eta_2)). \]
It is convenient to gather the various moment maps $\mu_\bullet$ into a single map

$$\mu : \mathcal{F} \to \mathfrak{t}^3$$ \hspace{1cm} (1.1)

given by $\mu = (\mu_1, \mu_J, \mu_K)$, called the hyperKähler moment map.

A smooth map $f : M \to M$ defines a tangent map $f_* : TM \to TM$. Using the triviality of the tangent bundle of a quotient torus, the tangent map $f_*$ can be understood as a $\mathcal{V}$-valued differential form, denoted $Df \in \mathcal{F}$ and closely related to the differential operator $d$ in local coordinates (cf. §2.3 for a more detailed definition). Thus $D$ defines a linear differential operator between the moduli spaces

$$D : \mathcal{M} \to \mathcal{F},$$

where $\mathcal{M} = \{f : M \to M, f \in C^\infty\}$ is the space of smooth maps of $M$. Our next theorem provides an interpretations of symplectic diffeomorphisms of $M$ as zeroes of the moment map $\mu$:

**Theorem B.** Let $M$ be a hyperKähler quotient torus of real dimension 4, with canonical symplectic form $\omega_M$. Let $f : M \to M$ be a smooth map of the torus satisfying the cohomological condition $f^*[\omega_M] = [\omega_M]$. Then the following properties are equivalent:

1. $f^*[\omega_M] = [\omega_M].$
2. $\mu(Df) = 0$, where $\mu$ is the hyperKähler moment map.

Considering the downward gradient flow of the functional

$$\phi : \mathcal{F} \to \mathbb{R}$$

$$F \mapsto \frac{1}{2} \|\mu(F)\|_{L^2}^2$$

seems most natural when contemplating Theorem A and Theorem B. The functional $\phi$ is to be treated as some type of Morse-Bott function on the moduli space $\mathcal{F}$ and the corresponding downward gradient flow is designed to obtain some topological information about the critical locus of $\phi$.

The original groundbreakimg idea, that the norm squared $\phi$ of a moment map should behave like a Morse-Bott function, is due to Atiyah-Bott. They applied an equivariant Morse theory to the Yang-Mills functional and calculated the Betti numbers of the moduli space of semistable bundles over a Riemann surface [2]. Then, Donaldson relied on the Yang-Mills flow to give a purely gauge theoretic proof of the Narasimhan-Seshadri theorem [7].

In the finite dimensional Kähler setting, Kirwan showed that the function $\phi$ can be treated as a Morse-Bott function [15] and induces corresponding Morse-Bott inequalities [21, §6]. The crucial property, that the moment map flow provides retractions, is attributed to an
observation of Duistermaat in [21, footnote, p.167], but a nice self-contained proof can be found in [18].

However, the standard moment map flow on $\mathcal{F}$ is not interesting in our case, as far as we are concerned with the topology of $\text{Symp}(\mathcal{M}, \omega_\mathcal{M})$. Indeed, the action of the gauge group $\mathbb{T}$ does not preserve the subspace $\text{Im} \mathcal{D}$ and neither does the gradient flow of $\phi$. Thus, such a flow does not capture the topology of $\mathcal{M}$. We get around this issue by introducing a variant of the moment map flow: given a cohomology class $\alpha \in H^1(M, V)$, the subspace of closed forms with cohomology class contained in the line $\mathbb{R}\alpha$ is denoted

$$\mathcal{F}_\alpha = \{ F \in \mathcal{F}, dF = 0 \text{ and } [F] \in \mathbb{R}\alpha \}.$$ 

We consider the downward gradient flow of the restricted functional $\phi : \mathcal{F}_\alpha \to \mathbb{R}$, called the modified moment map flow and we prove the following basic properties:

**Theorem C.** The fixed point locus of the modified moment map flow on $\mathcal{F}_\alpha$, in other words the critical locus of $\phi : \mathcal{F}_\alpha \to \mathbb{R}$, agrees with the vanishing locus of $\phi : \mathcal{F}_\alpha \to \mathbb{R}$. Furthermore,

1. the modified moment map flow has the short time existence property and

2. the $L^2$-norm is non increasing along the flow.

**Remark 1.3.1.** The short time existence property hides some technical aspects of the problem. The exact statement involves a completed version the moduli space $\mathcal{F}$ with respect to some Hölder norm. The technical version of this result is stated at Theorem 4.4.1.

**Remark 1.3.2.** The restriction of the functional $\phi$ to the vector space $\mathcal{F}_\alpha$ may seem somewhat arbitrary. At first glance, it makes sense to restrict $\phi$ to other subspaces. For example, the restriction of $\phi$ to the cohomology class $\alpha$, understood as an affine subspace of $\mathcal{F}$, could be considered as well. It turns out that the $L^2$-decay property of Theorem C does not hold in this case. In fact, the decay property extends to the polyhedral setting introduced at §7. This feature of the polyhedral moment map flow is a crucial ingredient for the construction of a retraction proved at Theorem G, which motivates our point of view.

We say that a cohomology class $\alpha \in H^1(M, V)$ is symplectic, if there exists a smooth map $f : M \to M$ such that $f^* [\omega_\mathcal{M}] = [\omega_\mathcal{M}]$ and $\alpha = [\mathcal{D}f]$. The component $\mathcal{M}_\alpha$ of $\mathcal{M}$ is defined as the subspace of maps $f$ with cohomology class $[\mathcal{D}f] = \alpha$. Similarly, we denote $\text{Symp}_\alpha(M, \omega_\mathcal{M})$ the subspace of $\text{Symp}(M, \omega_\mathcal{M})$ contained in $\mathcal{M}_\alpha$. By Proposition 2.3.2, the map

$$\mathcal{D} : \text{Symp}_\alpha(M, \omega_\mathcal{M}) \to \mathcal{F}_\alpha$$

is injective up to the action of $V$ by translations. Furthermore, the cone on its image agrees with the vanishing locus of $\phi$ in $\mathcal{F}_\alpha \setminus 0$ by
Corollary 5.3.5. In an ideal situation, we would expect some Duistermaat type theorem, where the modified moment map flow produces a continuous retraction by deformation of $\mathcal{F}_\alpha$ onto the vanishing locus of $\phi$. This would be an important step to investigate the topology of $\text{Symp}_\alpha(M, \omega_M)$. However, we are facing several issues, raised in the next questions:

**Questions 1.3.3.**

1. Does the modified moment map flow enjoy a long time existence property?
2. Are the flow lines convergent?
3. Are there some non trivial flow lines with limit $0 \in \mathcal{F}_\alpha$?

In fact we have little prospect to answer questions (1) and (2), since the modified moment map flow does not seem to have any nice regularizing properties, like parabolic flows for instance. Nevertheless, the answer to these questions is settled in the context of polyhedral geometry, where we obtain the best possible result at Theorem G.

Question (3) is more conceptual. Indeed $F = 0$ is the differential of a constant map of $M$, which is definitely not symplectic. This singular phenomenon, if it ever occurs, is an obstruction for pushing homotopies of $\mathcal{M}_\alpha$ into $\text{Symp}_\alpha(M, \omega_M)$ via the modified moment map, as explained in details at §5.1. This motivates the introduction at §5 of some classical tools of blowup analysis, devoted to study the flow lines converging toward $F = 0$. In particular we define the real blowup of the moduli space, the soliton equation and the renormalized flow along the unit sphere $S_\alpha \subset \mathcal{F}_\alpha$. We show that the renormalized flow, which is the downward gradient flow of the restricted functional $\phi : S_\alpha \to \mathbb{R}$, can be interpreted as a blow up of the modified moment map flow at $F = 0$.

The space of solitons of $\mathcal{F}_\alpha$ is a cone, which contains the vanishing set of $\phi$, called the space of non proper solitons. The remaining solitons are called the proper solitons and it turns out that each proper soliton defines a flowline of the modified moment map flow converging toward $F = 0$, as proved by Lemma 5.4.1. The space of solitons of $\mathcal{F}_\alpha$ admits a partition

$$
\{0\} \sqcup \mathcal{I}_{\alpha,p} \sqcup \mathcal{I}_{\alpha,np}
$$

where $\mathcal{I}_{\alpha,p}$ is the space of proper solitons and $\mathcal{I}_{\alpha,np}$ is the space of non proper solitons in $\mathcal{F}_\alpha \setminus 0$. By Corollary 5.3.5, $\mathcal{I}_{\alpha,np}$ is spanned by the image of $\text{Symp}_\alpha(M, \omega_M)$ by $\mathcal{D}$ and the action of $\mathbb{R}^*$. From these formal considerations, it seems most natural to regard $\phi : S_\alpha \to \mathbb{R}$ as a Morse-Bott function. An adapted Morse-Bott theory would provide a dynamical interaction between the critical components of $\phi$, in particular between the space of proper and non proper solitons of $S_\alpha$. It seems sensible to expect that topological invariants could be associated to the group of symplectic diffeomorphisms via a Morse-Bott cohomological construction for $\phi$. In this paper, we show that the renormalized flow has the following basic properties:
**Theorem D.** The renormalized flow along $S_\alpha$ has the short time existence property. The fixed points of the renormalized flow are the solitons contained in $S_\alpha$ and the minimum of $\phi : S_\alpha \to \mathbb{R}$ is the critical locus that consists of non proper solitons in the sphere.

However, many essential questions are opened at this stage:

**Questions 1.3.4.** Is the space of proper solitons of $\mathcal{F}_\alpha$ empty? Are there some proper solitons with vanishing cohomology class? Do the components of the space of solitons have a structure of manifold, in some sense?

The first question is directly related to the existence of solutions of the modified moment map flow converging toward zero. We show at Theorem H that the answer to this question is always positive in the case of polyhedral geometry.

We also answer partially the last question in the smooth setting, by proving at Theorem 6.1.1 that $\mathcal{I}_{\alpha,p}$ and $\mathcal{I}_{\alpha,np}$ are topologically separated, with respect to a suitable Hölder topology. In this introduction, we state a non technical version of this result, as some type of rigidity theorem: trying to deform a non proper soliton into a proper soliton may seem like a natural idea, but we prove that it is not possible.

**Theorem E.** Let $\alpha$ be a symplectic cohomology class. For every map $\psi : X \to \mathcal{F}_\alpha$ from a topological connected space $X$, continuous with respect to the Whitney topology, such that $\psi(X) \subset \mathcal{I}_{\alpha,p} \cup \mathcal{I}_{\alpha,np}$ and $\psi(X) \cap \mathcal{I}_{\alpha,np} \neq \emptyset$, we have $\psi(X) \subset \mathcal{I}_{\alpha,np}$.

### 1.4. Statement of results in the polyhedral setting.

All the constructions and results presented in the smooth setting at §1.3 were initially motivated by the polyhedral symplectic framework, with the piecewise linear symplectic geometry in mind. Working in the smooth setting helped us unveil the relevant objects in the discrete and finite dimensional world of polyhedral geometry. We quickly introduce the polyhedral concepts necessary for stating our results in the following paragraphs, but the reader may refer to §7 for more details.

A *Euclidean triangulation* of the quotient torus $M$ is a triangulation $\mathcal{T} = (M, \mathcal{K}, \ell)$, in the usual sense, with the additional condition that the restriction of the homeomorphism $\ell : |\mathcal{K}| \to M$ is affine along each simplex of the simplicial complex $\mathcal{K}$. The *space of polyhedral maps* with respect to $\mathcal{T}$, denoted $\mathcal{M}(\mathcal{T})$, is the space of continuous maps, affine along each simplex to the triangulation.

The restriction of a polyhedral map $f|_\sigma$ to a simplex of the triangulation is differentiable, although the map $f$ is generally not differentiable everywhere on $M$. In particular the pull back of the differential
form $f^*\omega_M$ is well defined along each simplex of the triangulation. Accordingly, a polyhedral map such that the pullback of $f^*\omega_M$ agrees with the pullback of $\omega_M$ along each simplex is called a polyhedral symplectic map and the space of polyhedral symplectic maps is denoted $\text{Symp}(M, \omega_M, \mathcal{T})$.

By analogy with the smooth setting, we introduce the space $F_T$ as the space of families $F = (F_\sigma)_{\sigma \in \mathcal{K}_4}$, where $\sigma$ belongs to the set of 4-simplices $\mathcal{K}_4$ and $F_\sigma$ is a constant $\nabla$-valued differential 1-form along $\sigma$. Similarly to the smooth case, there is a well-defined linear operator associated to the differentiation of polyhedral maps $D : \mathcal{M}(T) \to F(T)$. In the case where all the pullbacks of the family $F = (F_\sigma) \in F_T$ agree along common faces, we say that $F$ is a Whitney form. The subspace of Whitney forms in $F_T$ is denoted $F_c(T)$. In particular, the operator $D$ takes values in the space of Whitney forms. A Whitney form in $F_c(T)$ defines a Whitney cohomology class and we can introduce the subspace $F_\alpha(T)$ of Whitney forms with cohomology class in the line $\mathbb{R}\alpha$ for any $\alpha \in H^1(M, \nabla) \setminus 0$. The gauge group $T(T)$ is the space of functions $\lambda : \mathcal{K}_4 \to S^1$ and its Lie algebra $t(T)$ is the space of functions $\zeta : \mathcal{K}_4 \to \mathbb{R}$. The group $T$ is a real torus with dimension the number of facets of $\mathcal{K}$ and the group $T(T)$ acts on $F_T$ by complex multiplication, as in the smooth case. Finally, all the constructions of hyperKähler structures and moment maps in the smooth setting hold formally for $F_T$ acted on by $T(T)$ and we obtain the following result:

**Theorem F.** The statements of Theorem A and Theorem B hold in the polyhedral setting.

Continuing the analogy with the smooth case, the energy functional

$$\phi : \mathcal{F}(T) \to \mathbb{R}$$

is given similarly by $\phi(F) = \|\mu(F)\|^2_{L^2}$, where $\mu : \mathcal{F}(T) \to t(T)^3$ is the polyhedral hyperKähler moment map. The polyhedral modified moment map flow is then defined as the downward gradient flow of the restricted functional $\phi : \mathcal{F}_\alpha(T) \to \mathbb{R}$, for $\alpha \in H^1(M, \nabla) \setminus 0$. We may also consider the downward gradient flow of $\phi$ along the unit sphere $S_\alpha(T)$ in $\mathcal{F}_\alpha(T)$ and obtain a polyhedral renormalized flow.

The polyhedral modified moment map flow and the polyhedral renormalized flow are both gradient flows on finite dimensional manifolds and strong properties follow from the ordinary differential equation theory. In the case of polyhedral modified moment map flow, we prove the following Duistermaat type theorem:

**Theorem G.** Let $\alpha$ be a cohomology class in $H^1(M, \nabla) \setminus 0$. For every $F \in \mathcal{F}_\alpha(T)$, there exists a unique solution of the polyhedral modified
moment map flow $F_t$ defined for $t \in [0, +\infty)$, such that $F_0 = F$. Furthermore, the flow line is convergent and its limit $F_\infty$ belongs to the vanishing locus of $\phi$. The extended flow

$$\Theta : [0, +\infty] \times F_\alpha(\mathcal{F}) \to F_\alpha(\mathcal{F})$$

given by $\Theta(t, F) = F_t$ and $\Theta(+\infty, F) = F_\infty$, defines a continuous retraction of $F_\alpha(\mathcal{F})$ onto the vanishing locus of $\phi$. Finally, the flow $\Theta$ has exponential convergence rate in a neighborhood of every regular point (cf. Definition 7.9.1) of the vanishing locus of $\phi$.

Given a symplectic class $\alpha \in H^1(M, \hat{\nabla})$, we consider the subspaces $\mathcal{M}_\alpha(\mathcal{F}) \subset \mathcal{M}(\mathcal{F})$ and $\text{Symp}_\alpha(M, \omega, \mathcal{F}) \subset \text{Symp}(M, \omega, \mathcal{F})$ of maps $f$ with cohomology class $[\mathcal{F}] = \alpha$, as in the smooth setting.

We discuss at §5.1 a potential application of the modified moment map flow to identify the homotopy type of the space of symplectic maps. Theorem G shows that hypothesis (H1) and (H2) of §5.1 always hold in the polyhedral setting. Under the additional hypothesis (H3) that $\Theta$ does not admit any non trivial flow line converging toward $F = 0$, the evaluation map $\text{ev} : \mathcal{M}_\alpha(\mathcal{F}) \to M$ at some point $x_0 \in M$ would be a homotopy equivalence. But hypothesis (H3) is too strong, in fact it never holds in the polyhedral case. Indeed, Theorem H states that non trivial flow lines converging toward $F = 0$ always exist for the flow $\Theta$. This motivates the introduction of blowup analysis, 

**polyhedral soliton equation and polyhedral renormalized flow** similar to the smooth setting. The prospect to obtain a Morse-Bott theory adapted to $\phi : S_\alpha(\mathcal{F}) \to \mathbb{R}$ seems much higher than in the smooth situation, thanks to the ODE theory and the finite dimensional context. At this stage we can show that the polyhedral renormalized flow satisfies the following elementary properties:

**Theorem H.** Let $\alpha \in H^1(M, \hat{\nabla})$ be a symplectic class. The polyhedral renormalized flow is a complete gradient flow on the sphere $S_\alpha(\mathcal{F})$. The solitons of the sphere are the fixed points of the flow and the limiting orbits of the renormalized flow are contained in the space of solitons. Non proper solitons of $S_\alpha(\mathcal{F})$ are in correspondence with elements of $\text{Symp}_\alpha(M, \omega, \mathcal{F})$, up to the action of $\hat{\nabla}$ and the the antipodal map of the sphere. The space of proper solitons in $S_\alpha(\mathcal{F})$ is not empty. Furthermore, each proper soliton defines non trivial solutions of the polyhedral modified moment map flow, converging toward $F = 0$.

Some more effort has to be undertaken to prove that $\phi : S_\alpha \to \mathbb{R}$ is Morse-Bott. Our attempts to prove this have been unsuccessful so far. We also failed to prove an analogue of Theorem E in the polyhedral setting and we conclude our introduction with the following conjecture:

**Conjecture I.** Given a symplectic cohomology class $\alpha$ and a Euclidean triangulation $\mathcal{R}_1$ of a 4-dimensional hyperKähler quotient torus $M$,
there exists a refinement $\mathcal{T}_2$ of $\mathcal{T}_1$ such that the energy functional $\phi : S_\alpha(\mathcal{T}_2) \to \mathbb{R}$ has a well defined Morse-Bott theory.

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2. K"{a}hler moment map

2.1. Smooth symplectic maps. We recall some basic facts in the general case of a connected symplectic manifold \((M, \omega_M)\) of real dimension \(2n\). A symplectic form \(\omega_M\) is a non degenerate closed differential 2-form, defined on the smooth manifold \(M\). The symplectic form determines a volume form

\[
\text{vol}_M = \frac{\omega_M^n}{n!},
\]

hence a symplectic orientation for \(M\). The moduli space of smooth maps of \(M\), denoted

\[
\mathcal{M} = \{f : M \rightarrow M, f \in C^\infty\}.
\]

is considered from a purely formal perspective in most of this paper. However, H"{o}lder versions of \(\mathcal{M}\) shall be introduced, when the analysis on Banach spaces is needed. The group of diffeomorphisms \(\text{Diff}(M)\) is contained in \(\mathcal{M}\) and the subgroup of symplectic maps (or symplectomorphisms), denoted \(\text{Symp}(M, \omega_M)\), consists of smooth map \(f \in \mathcal{M}\) such that

\[
f^* \omega_M = \omega_M. \quad (2.1)
\]
Formula 2.1 implies that $f$ is a diffeomorphism, under some sensible assumptions and we recall the classical result:

**Proposition 2.1.1.** Let $(M, \omega_M)$ be a closed connected symplectic manifold and let $f : M \to M$, be a symplectic map. Then $f$ is an orientation preserving diffeomorphism of $M$. Thus $\text{Symp}(M, \omega_M)$ is a subgroup of $\text{Diff}(M)$ and we have the inclusions

$$\text{Symp}(M, \omega_M) \subset \text{Diff}(M) \subset M.$$

**Proof.** The non-degeneracy of $\omega_M$ implies that the tangent map $f_* : TM \to TM$ is a fiberwise isomorphism of the tangent bundle. By the *implicit function theorem*, it follows that the map $f$ is a local diffeomorphism. A symplectic map satisfies the identity $f^* \text{vol}_M = \text{vol}_M$ so that $f$ preserves the symplectic orientation and preserves the cohomology class $[\text{vol}_M] \in H^{2n}(M, \mathbb{R})$; in particular, $f$ has degree 1. An orientation preserving local diffeomorphism of degree 1 is injective. The fact that $f$ is an injective local diffeomorphism of a closed manifold implies that $f$ is a global diffeomorphism. \qed

2.2. **Quotient Hermitian torus.** From this point, we assume that the manifold $M$ is a *quotient torus* of the form

$$M = V/\Gamma,$$

where $V$ is an *affine space* and $\Gamma$ is a *lattice* of the underlying vector space $\overrightarrow{V}$. We consider the case where $\overrightarrow{V}$ is a complex vector space, endowed with a *Hermitian metric* $h_V$. Recall that a Hermitian metric is a definite positive sesquilinear form on $\overrightarrow{V}$, anti-$\mathbb{C}$-linear in the first variable. Then, we can write

$$h_V(\eta_1, \eta_2) = g_V(\eta_1, \eta_2) + i\omega_V(\eta_1, \eta_2),$$

for every $\eta_1, \eta_2 \in \overrightarrow{V}$, where $g_V$ and $\omega_V$ are the real and imaginary parts of $h_V$. By definition, $g_V$ defines a *Euclidean metric* on $V$ and $\omega_V$, a symplectic form.

The structure of complex vector space on $\overrightarrow{V}$ can be regarded as an *almost complex structure*, deduced from the complex multiplication by $i$, understood a linear endomorphism:

$$i : \overrightarrow{V} \to \overrightarrow{V},$$

$$v \mapsto iv.$$

The action by translation of $\eta \in \overrightarrow{V}$ on $x \in V$, denoted $x \mapsto x + \eta$, descends as a canonical action of $\overrightarrow{V}$ on the quotient torus $M = V/\Gamma$. The Hermitian and complex structures on $V$ also descend to the quotient $M$ as well. The induced almost complex structure is still denoted $i : TM \to TM$. The induced Hermitian structure, symplectic structure
and Riemannian metric on the quotient $M$ are denoted respectively $h_M$, $\omega_M$ and $g_M$. In conclusion, we have a flat Kähler structure
\[ (M, g_M, i, \omega_M) \]
on the quotient torus $M$, preserved by the action of $\nabla$ on $M$.

2.3. **Differentials and differentiation.** The tangent bundle $\pi : TV \to V$ of the affine space $V$ is canonically isomorphic to the trivial bundle $\pi_1 : V \times \nabla \to V$, where $\pi_1$ is the first canonical projection. The action of $\nabla$ on $V$ induces an action on the tangent bundle $TV \to V$, which is trivial on the fibers. The tangent bundle $\pi : TM \to M$ is obtained as a quotient of $TV \to V$ by the lattice $\Gamma$. It follows that the tangent bundle $TM \to M$ is also canonically isomorphic to the trivial bundle $\pi_1 : M \times \nabla \to M$. Hence we obtain a smooth $\nabla$-invariant bundle map trivialization
\[ \rho : TM \to \nabla, \]given by $\pi_2 : M \times \nabla \to \nabla$, where $\pi_2$ is the canonical projection on the second factor. In addition, the map $\rho$ respects the Kähler structures, in the sense that
\[ \rho \circ i = i \circ \rho, \quad \rho^* \omega_V = \omega_M, \quad \rho^* g_V = g_M \quad \text{and} \quad \rho^* h_V = h_M, \]
where the pullbacks are understood in the linear sense:
\[ (\rho^* \omega_V)(\eta_1, \eta_2) = \omega_V(\rho(\eta_1), \rho(\eta_2)), \]etc...
The tangent map of $f \in \mathcal{M}$ sits in the commutative diagram
\[ \begin{array}{ccc}
TM & \xrightarrow{\mathcal{D} f} & \nabla \\
\downarrow \pi & & \downarrow \pi \\
M & \xrightarrow{f} & M
\end{array} \]where
\[ \mathcal{D} f : TM \to \nabla, \]is the $\nabla$-valued differential 1-form, called the **differential of $f$**, defined by
\[ \mathcal{D} f = \rho \circ f_*. \]
The operator $\mathcal{D}$ can be regarded as a linear differential operator of order 1
\[ \mathcal{D} : \mathcal{M} \to \mathcal{F} = \Omega^1(M, \nabla), \]
where $\mathcal{F}$ denotes the space of $\nabla$-valued differential 1-forms on $M$. The operator $\mathcal{D}$ is closely related to the usual differential $d$ according to Remark 2.3.1.
Remark 2.3.1. An alternate construction of $\mathcal{D}f$ proceeds as follows: let $\tilde{f} : V \to \tilde{V}$ be a lift of $f : M \to M$ to the universal cover $p : V \to M$. In other words, $\tilde{f}$ is a smooth map such that $p \circ \tilde{f} = f \circ p$. The differential $d\tilde{f}$ is a $\tilde{V}$-valued 1-form on $V$, which is to say an element of $\Omega^1(V, \tilde{V})$. By construction $\tilde{f}$ is $\Gamma$-invariant, so that $d\tilde{f}$ descends to the quotient as an element of $\Omega^1(M, V)\tilde{\Gamma}$, which agrees with $\mathcal{D}f$.

Since $\tilde{V}$ acts on $M$ by translations, the space of maps $\mathcal{M}$ admits a canonical action by the space of $\tilde{V}$-valued functions, $\Omega^0(M, \tilde{V})$: given $f \in \mathcal{M}$ and $\eta \in \Omega^0(M, \tilde{V})$, we define $f + \eta \in \mathcal{M}$ by

$$(f + \eta)(x) = f(x) + \eta(x)$$

for every $x \in M$. In particular, the space of constant $\tilde{V}$-valued functions, identified to $\tilde{V}$, acts on $\mathcal{M}$. By construction, we have the following proposition:

Proposition 2.3.2. Given two maps $f$ and $h \in \mathcal{M}$, we have

$$\mathcal{D}f = \mathcal{D}h$$

if, and only if, there exists a vector $\eta \in \tilde{V}$, such that $f = h + \eta$. In particular, the map $\mathcal{D}$ descends as a bijection

$$\mathcal{M}/\tilde{V} \xrightarrow{\mathcal{D}} \mathcal{D}(\mathcal{M}) \subset \mathcal{F},$$

where $\tilde{V}$ acts on $\mathcal{M}$ by translation.

Proof. The construction of $\mathcal{D}f$ and $\mathcal{D}h$ by Remark 2.3.1 is given by $d\tilde{f}$ and $d\tilde{h}$, where the maps $\tilde{f}, \tilde{h} : V \to V$ are lifts of $f$ and $g$. The equation $\mathcal{D}f = \mathcal{D}h$ is equivalent to $d\tilde{f} = d\tilde{h}$, where $d$ is the usual differential. Therefore $\tilde{f}$ and $\tilde{h}$ agree up to a translation by a vector $\eta \in \tilde{V}$, which proves the proposition. □

2.4. Cohomology. In this section, we recall some basic facts about the cohomology of a torus and homotopy classes of maps of a torus. The usual exterior differentials for real valued differential forms are denoted $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$, for $k \geq -1$, with the convention that $d_{-1} : \Omega^0(M) \to \Omega^0(M)$ is the 0-map. These operators admit a canonical extension to $\tilde{V}$-valued differential forms, denoted similarly $d_k : \Omega^k(M, \tilde{V}) \to \Omega^{k+1}(M, \tilde{V})$, and defined by the property that for every real valued differential $k$-form $\beta$ and vector $\eta \in \tilde{V}$, we have

$$d_k(\beta \otimes \eta) = (d_k \beta) \otimes \eta,$$  \hspace{1cm} (2.6)

where $d_k \beta$ is the usual exterior differential. Thus we have a complex

$$0 \xrightarrow{d_{-1}} \Omega^0(M, \tilde{V}) \xrightarrow{d_0} \Omega^1(M, \tilde{V}) \xrightarrow{d_1} \ldots \xrightarrow{d_k} \ldots$$

where $d_k$ is the usual exterior differential.
with associated De Rham cohomology spaces

\[ H^k(M, \overrightarrow{V}) = \frac{\ker d_k}{\text{Im } d_{k-1}} \]

defined for every \( k \geq 0 \) and canonical isomorphisms

\[ H^k(M, \mathbb{R}) \otimes \overrightarrow{V} \cong H^k(M, \overrightarrow{V}). \]

**Lemma 2.4.1.** The image of the map \( \mathcal{D} : \mathcal{M} \to \mathcal{F} \) consists of closed forms. In other words \( d_1 \circ \mathcal{D} = 0 \), which is to say

\[ \text{Im } \mathcal{D} \subset \ker d_1. \]

**Proof.** The result follows immediately from Remark 2.3.1, Formula (2.6) and the fact that \( d^2 = 0 \). \( \square \)

Every form \( \mathcal{D}f \in \mathcal{F} \) is a closed \( \overrightarrow{V} \)-valued 1-form by Lemma 2.4.1 and defines a cohomology class denoted \([\mathcal{D}f] \in H^1(M, \overrightarrow{V})\). Thus, we have a canonical map

\[ \alpha : \mathcal{M} \to H^1(M, \overrightarrow{V}) \]

\[ f \mapsto \alpha(f) = [\mathcal{D}f]. \]

The quotient torus \( M \) is equipped with a canonical isomorphism

\[ \psi : \overrightarrow{\mathcal{V}}^* \to H^1(M, \mathbb{R}). \]

Indeed, any element \( \beta \in \overrightarrow{\mathcal{V}}^* \) can be understood as a constant differential 1-form on \( V \). The constant form \( \beta \) descends to the quotient \( M \) as a closed 1-form denoted \( \beta \) as well. Finally, \( \beta \) defines a cohomology class \( \psi(\beta) = [\beta] \in H^1(M, \mathbb{R}) \). The fact that \( \psi \) is an isomorphism is a classical fact for tori. The dual lattice \( \Gamma^* \subset \overrightarrow{\mathcal{V}}^* \) is defined by

\[ \Gamma^* = \{ \beta \in \overrightarrow{\mathcal{V}}^*, \forall \gamma \in \Gamma, \beta(\gamma) \in \mathbb{Z} \}. \]

Using the isomorphism \( \psi \), the dual lattice \( \Gamma^* \subset \overrightarrow{\mathcal{V}}^* \) is understood as a lattice in \( H^1(M, \mathbb{R}) \) identified to the group of integral classes \( H^1(M, \mathbb{Z}) \subset H^1(M, \mathbb{R}) \) and we have a canonical isomorphism

\[ \psi : \Gamma^* \hookrightarrow H^1(M, \mathbb{Z}). \]

Every map \( f \in \mathcal{M} \), defines a canonical morphism

\[ f^* : H^1(M, \mathbb{Z}) \to H^1(M, \mathbb{Z}) \]

which depends only on the homotopy class of \( f \). Using the isomorphism \( \psi \), the map \( f^* \in \text{End}(H^1(M, \mathbb{Z})) \) is identified to a lattice endomorphism \( f^* \in \text{End}(\Gamma^*) \). The canonical isomorphisms \( \text{End}(\Gamma^*) \cong \text{End}(\Gamma) \cong \Gamma^* \otimes_{\mathbb{Z}} \Gamma \) show that

\[ \text{End}(\Gamma^*) \cong H^1(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \Gamma \subset H^1(M, \mathbb{R}) \otimes_{\mathbb{R}} \overrightarrow{V} = H^1(M, \overrightarrow{V}). \]
If a class \( \alpha \in H^1(M, \overrightarrow{V}) \) belongs to the image of \( \text{End}(\Gamma^*) \), we say that it is an \textit{integral cohomology class}. We recall the classical result of topology, well known in the case of a circle:

**Proposition 2.4.2.** For \( f \in \mathcal{M} \), the morphism \( f^* \in \text{End}(\Gamma^*) \), understood as an integral cohomology class in \( H^1(M, \overrightarrow{V}) \), agrees with \( \alpha(f) = [\mathcal{D}f] \).

Two smooth maps \( f \) and \( h : M \to M \) are homotopic if, and only if, \( \alpha(f) = \alpha(h) \). Furthermore, the map

\[
\mathcal{M} \to \text{End}(\Gamma^*)
\]

\[
f \mapsto f^*
\]

is surjective and induces bijection between \( \text{End}(\Gamma^*) \) and the set of homotopy classes of smooth maps \( f : M \to M \).

According to the above proposition, homotopy classes of maps are parametrized by integral cohomology classes \( \alpha \in H^1(M, \overrightarrow{V}) \). Thus, it makes sense to use the notation

\[
\mathcal{M}_\alpha = \{ f \in \mathcal{M}, [\mathcal{D}f] = \alpha \}
\]

for each homotopy component of the space of smooth maps of the torus. Similarly we use the notations

\[
\text{Symp}_\alpha(M, \omega_M) \quad \text{and} \quad \text{Diff}_\alpha(M),
\]

for the subspaces of maps with integral cohomology class \( \alpha \).

**Remark 2.4.3.** The surjectivity property of Proposition 2.4.2 can be realized using \textit{affine maps}: the canonical map \( \text{End}(\Gamma) \to \text{End}(\Gamma^*) \) is an isomorphism, so that an element of \( \alpha \in \text{End}(\Gamma^*) \) corresponds to a morphism \( \tilde{f} \) of the lattice \( \Gamma \). The endomorphism \( \tilde{f} \) extends canonically as a linear map \( \tilde{f} \in \text{End}(\overrightarrow{V}) \). Any affine map \( f : V \to V \) with linear part \( \tilde{f} \) descends to the quotient as a map \( f : M \to M \) such that \( \alpha(f) = \alpha \).

**Definition 2.4.4.** A cohomology class \( \alpha \in H^1(M, \overrightarrow{V}) \) is called \textit{symplectic}, if there exists \( f \in \mathcal{M} \), such that \( \alpha = [\mathcal{D}f] \) and \( f^*[\omega_M] = [\omega_M] \).

**Remark 2.4.5.** Let \( \alpha \in H^1(M, \overrightarrow{V}) \) be a symplectic cohomology class. According to Remark 2.4.3, there exists an affine map \( f : V \to V \) which descends to the quotient as \( f : M \to M \) with the property that \( \alpha = [\mathcal{D}f] \). We have \( f^*[\omega_M] = [\omega_M] \) since \( \alpha \) is symplectic. From the fact that \( f \) is affine we deduce that \( f^*\omega_M \) is a constant differential form, which forces \( f^*\omega_M = \omega_M \). Thus \( f \) is an affine symplectic map. We conclude that the set of symplectic classes is in 1 : 1-correspondence with the group of symplectic isomorphisms of \( \Gamma \)

\[
\text{Symp}(\Gamma, \omega_V) = \text{End}(\Gamma) \cap \text{Symp}(\overrightarrow{V}, \omega_V)
\] (2.8)
where $\text{Symp}(\overrightarrow{V}, \omega_V)$ is the group of linear symplectic endomorphisms of $\overrightarrow{V}$. In the above definition, every element $\tilde{f} \in \text{End}(\Gamma)$ induces a canonical linear endomorphism of $V$ and we may consider that $\text{End}(\Gamma) \subset \text{End}(\overrightarrow{V})$.

Remark 2.4.6. The surjectivity property of Proposition 2.4.2 can also be established directly, relying on integration: let $\alpha$ be an integral cohomology class in $H^1(M, \overrightarrow{V})$ and $F \in \mathcal{F}$, a representative of $\alpha$. Let $\tilde{F} = p^*F \in \Omega^1(V, \overrightarrow{V})$ be the lift of $F$ to the universal cover $p : V \to M$. Given $y_0, y_1$ and $y \in V$, let $\tilde{f} : V \to V$ be the function defined by

$$\tilde{f}(y) = y_1 + \int_{y_0}^{y} \tilde{F}, \quad (2.9)$$

where the integral is taken along any smooth path in $V$ from $y_0$ to $y$. The integral is independent of the choice of path between $y_0$ and $y$ since $\tilde{F}$ is closed and $V$ is simply connected. By construction, $\tilde{f}(y + \gamma) - \tilde{f}(y) \in \Gamma$ for every $\gamma \in \Gamma$, since $\alpha = [F]$ is an integral class. Hence $\tilde{f}$ descends as a smooth map $f : M \to M$. Using the notations $x_0 = p(y_0)$, $x_1 = p(y_1)$, we have $f(x_0) = x_1$, $\mathcal{D}f = F$ and $[\mathcal{D}f] = [F] = \alpha$, by definition. In conclusion we have an integral

$$\chi : \alpha \to \mathcal{M}_\alpha$$

$$F \mapsto f$$

such that $f(x_0) = x_1$ and $f$ is defined by Formula (2.9).

2.5. Bundle structures. In §2.2, we outlined how a Hermitian structure on $\overrightarrow{V}$ induces a flat Kähler structure $(M, g_M, i_M, \omega_M)$ on the quotient torus $M = V/\Gamma$. We consider a second almost complex structure $J_V : \overrightarrow{V} \to \overrightarrow{V}$ on $\overrightarrow{V}$, compatible with the Riemannian metric $g_V$, in the sense that

$$g_V(J_V \eta_1, J_V \eta_2) = g_V(\eta_1, \eta_2),$$

for every $\eta_1, \eta_2 \in \overrightarrow{V}$. The compatible almost complex structure $J_V$ provides an alternate symplectic form on $V$ defined by

$$\hat{\omega}_V = g_V(J_V \cdot, \cdot).$$

The almost complex structure $J_V$ and the symplectic form $\hat{\omega}_V$ descend to the quotient $M$ and provide an alternate Kähler structure $(M, g_M, J_M, \hat{\omega}_M)$, where the Kähler form $\hat{\omega}_M$ is induced by $\hat{\omega}_V$. In conclusion, we have two competing Kähler structures on the torus

$$(M, g_M, i_M, \omega_M) \text{ and } (M, g_M, J_M, \hat{\omega}_M).$$

The Riemannian metric induces a musical isomorphism between the cotangent bundle $T^*M \to M$ and the tangent bundle $TM \to M$. Thus
the cotangent bundle and all the bundles of $k$-forms $\Lambda^k M \to M$ are endowed with an induced fiberwise Euclidean inner product deduced from $g_M$ and denoted $g$. It follows that the complex vector bundles

$$\Lambda^k M \otimes_R \overrightarrow{V} \to M$$

have several induced structures:

1. The vector bundles $\Lambda^k M \otimes_R \overrightarrow{V} \to M$ carry fiberwise Euclidean inner products, denoted $g$ as well, induced by $g$ and $g_V$.

2. The vector bundles $\Lambda^k M \otimes_R \overrightarrow{V} \to M$ carry a canonical structure of complex vector bundle deduced from the structure of complex vector space on $\overrightarrow{V}$.

3. The vector bundle $T^* M \otimes_R \overrightarrow{V} \to M$ carries a second fiberwise almost complex structure $J$ deduced from $J_M$ and defined by

$$J \cdot F = -F \circ J_M$$

for every $F \in T^* M \otimes_R \overrightarrow{V}$. The almost complex structure $J$ commutes with the multiplicative action of $\mathbb{C}$ of complex vector bundle. Hence $J$ is complex linear in this sense. In particular $i$ and $J$ commute:

$$i(J \cdot F) = -iF \circ J_M = J \cdot (iF).$$

4. The metrics are compatible with the almost complex structures:

$$g(iF_1, iF_2) = g(F_1, F_2) \quad \text{and} \quad g(J \cdot F_1, J \cdot F_2) = g(F_1, F_2)$$

for every $F_1, F_2 \in T^*_x M \otimes \overrightarrow{V}$.

5. The bundle $T^* M \otimes_R \overrightarrow{V} \to M$ carries a fiberwise anti-symmetric 2-form, denoted $\hat{\omega}$, defined by

$$\hat{\omega}(F_1, F_2) = g(J \cdot F_1, F_2).$$

**Remark 2.5.1.** Once the Euclidean inner product $g$ is understood, we will often use the more compact notations

$$\langle F_1, F_2 \rangle = g(F_1, F_2) \quad \text{and} \quad |F|^2 = \sqrt{g(F, F)}.$$

2.6. **An involution.** We define a bundle endomorphism

$$T^* M \otimes_R \overrightarrow{V} \xrightarrow{R} T^* M \otimes_R \overrightarrow{V}$$

by the formula

$$RF = iJ \cdot F$$

for every $F \in T^* M \otimes_R \overrightarrow{V}$. We have $R^2 F = i^2 J^2 \cdot F = F$ since $i$ and $J$ commute, which shows that $R$ is an *involution*. This provides a splitting

$$T^* M \otimes_R \overrightarrow{V} = \Lambda^{1,+} \oplus \Lambda^{1,-}.$$
where the $\Lambda^{\pm}$ are the eigenspaces of $R$ associated to the eigenvalues $\pm 1$. By definition, the elements $F \in \Lambda^{\pm}$ (resp. $\Lambda^{-}$) are the $J_M$-complex (resp. anti-complex) morphisms $F : TM \to \mathbb{V}$.

The involution $R$ induces a canonical involution of the space of differential forms

$$R : \mathcal{F} \to \mathcal{F}$$

defined by $(RF)_x = R(F_x)$ for every $x \in M$. Similarly, the fiberwise almost complex structure $J$ of $T^*M \otimes \mathbb{V}$ defines a canonical almost complex structure

$$J : \mathcal{F} \to \mathcal{F}$$
given by $(JF)_x = J(F_x)$ for $F \in \mathcal{F}$ and every $x \in M$. Then $RF = iJF$ and we have a similar splitting

$$\mathcal{F} = \mathcal{F}^+ \oplus \mathcal{F}^-$$

into complex and anti-complex differential forms. It follows that every differential form $F \in \mathcal{F}$ admits a canonical decomposition

$$F = F^+ + F^-,$$

with components $F^\pm \in \mathcal{F}^\pm$.

2.7. **Gauge group action.** We recall that the complex vector bundle structure of $T^*M \otimes_{\mathbb{R}} \mathbb{V} \to M$ is deduced from the complex vector space structure of $\mathbb{V}$ (cf. 2.5, item (2)). The space of differential forms $\mathcal{F}$ inherits a module structure over the ring of complex valued functions $\lambda : M \to \mathbb{C}$, with multiplication given by

$$(\lambda F)_x = \lambda(x)F_x \quad \text{for every } x \in M.$$ 

We define an action of the *infinite dimensional complex torus*

$$\mathbb{T}^C = C^\infty(M, \mathbb{C}^*)$$
on $\mathcal{F}$ as follows: for every $\lambda \in \mathbb{T}^C$ and $F \in \mathcal{F}$, we put

$$\lambda \cdot F = \bar{\lambda}^{-1}F^+ + \lambda F^-,$$ 

(2.11)

where multiplication in the RHS comes from the module structure and $\bar{\lambda}$ denotes the complex conjugate. The gauge group $\mathbb{T}^C$ contains the *real torus*

$$\mathbb{T} = C^\infty(M, S^1) \subset \mathbb{T}^C,$$

where $S^1 \subset \mathbb{C}$ is identified to complex numbers of module 1. If $\lambda \in \mathbb{T}$, the action on $\mathcal{F}$ is given by the standard complex multiplication $\lambda \cdot F = \lambda F$ since $\lambda^{-1} = \bar{\lambda}$ in this case.

**Remark 2.7.1.** The action of $\mathbb{T}^C$ defined by (2.11) depends on the splitting $\mathcal{F}^\pm$, and ultimately on $J_M$. However, the action of $\mathbb{T}$, which is merely the standard multiplication by a complex function, does not depend on $J_M$. 
2.8. **Infinitesimal action.** Formally, $\mathbb{T}^C$ is understood as a Lie group acting on $\mathcal{F}$ with Lie algebra $\mathfrak{t}^C$ identified to $C^\infty(M, \mathbb{C})$. The exponential map

$$\exp : \mathfrak{t}^C \to \mathbb{T}^C$$

is defined by the formula

$$\exp(\zeta) = e^{i\zeta}, \quad \text{for } \zeta \in \mathfrak{t}^C = C^\infty(M, \mathbb{C}).$$

With these conventions, the Lie algebra $\mathfrak{t}$ of the subgroup $\mathbb{T}$ is identified to $C^\infty(M, \mathbb{R})$ via the exponential map. As usual, in the context of a Lie group action, the infinitesimal action of an element of the Lie algebra $\zeta \in \mathfrak{t}^C$ is given by the vector field $X_\zeta$ on $\mathcal{F}$, defined by the formula

$$X_\zeta(F) = \frac{\partial}{\partial t} \exp(t\zeta) \cdot F \bigg|_{t=0}.$$ 

**Lemma 2.8.1.** The vector field $X_\zeta$ associated to the infinitesimal action of $\zeta \in \mathfrak{t}^C$ on $\mathcal{F}$ satisfies the formula

$$X_\zeta(F) = i\bar{\zeta}F^+ + i\zeta F^-,$$

for every $F \in \mathcal{F}$. In the special case where $\zeta \in \mathfrak{t}$ we have

$$X_\zeta(F) = i\zeta F.$$  

(2.12)

If $\zeta \in i\mathfrak{t}$, then

$$X_\zeta(F) = -i\zeta \mathbb{R}F = \zeta \mathcal{J} \cdot F$$

and for every $\zeta \in \mathfrak{t}$, we have

$$\mathcal{J} \cdot X_\zeta(F) = i\zeta \mathcal{J} \cdot F = X_{i\zeta}(F).$$  

(2.13)

**Proof.** This follows immediately from formula (2.11) and the definition of the exponential map. \hfill $\square$

2.9. **Kähler structure on the moduli space.** The moduli space $\mathcal{F}$ of $\mathbb{V}$-valued 1-forms is endowed with a formal canonical Kähler structure described below:

1. For $F_1$ and $F_2$ in $\mathcal{F}$, we define a Euclidean metric $\mathcal{G}$ on $\mathcal{F}$ by

$$\mathcal{G}(F_1, F_2) = \int_M g(F_1, F_2)\text{vol}_M,$$

(2.14)

where $\text{vol}_M$ is the volume form associated to $\omega_M$. As $g$ is defined on every tensor bundle, it follows that the Euclidean metric $\mathcal{G}$ is defined for every tensor space as well, by the same formula. The short notation

$$\langle \langle \cdot, \cdot \rangle \rangle = \mathcal{G}(\cdot, \cdot)$$

and the corresponding $L^2$-norm notation

$$\|F\|_{L^2} = \sqrt{\langle \langle F, F \rangle \rangle}$$

will often be used instead of $\mathcal{G}$. 

(2) The space \( F \) is endowed with the almost complex structures \( J \) defined above: for \( F \in F \),
\[
(J \cdot F)_x = J \cdot F_x \tag{2.15}
\]
for every \( x \in M \). Then \( J \) is compatible with \( G \) in the sense that
\[
G(J \cdot J \cdot) = G.
\]
(3) We define a symplectic form \( \Omega \) on \( F \) by
\[
\Omega(F_1, F_2) = G(J \cdot F_1, F_2) = \int_M \hat{\omega}(F_1, F_2) \text{vol}_M. \tag{2.16}
\]
In particular \((F, G, J, \Omega)\) is a formal Kähler structure. We summarize our construction in the next proposition.

**Proposition 2.9.1.** Let \( M = V/\Gamma \) be a quotient torus, where \( V \) is an affine space with lattice \( \Gamma \) and \( \hat{V} \), a complex vector space endowed with a Hermitian structure \( h_V \), as an underlying vector space. For every almost complex structure \( J_V \) compatible with the Euclidean metric \( g_V \) deduced from \( h_V \), there exists a natural formal Kähler structure \((G, G, J, \Omega)\) on \( F \) defined by Formulas (2.14), (2.15) and (2.16), acted on by the gauge group \( T^C = C^\infty(M, \C^*) \), with an action given by Formula (2.11).

**Proposition 2.9.2.** The \( T^C \)-action on \( F \) preserves the almost complex structure \( J \). Furthermore, the action of \( T^C \) is the \( J \)-complexified action of \( T \).

**Proof.** It is sufficient to prove that the \( T^C \) action commutes with \( J \) and this proves the first statement. By definition \( R = iJ \), hence \(-iR = J \). Furthermore, \( R \) is a module morphism over the ring of complex valued functions. Then for every \( \lambda \in T^C \), we have \( J \cdot \lambda \cdot F = -iR\lambda \cdot F \) and, by definition of the torus action (2.11),
\[
-iR\lambda \cdot F = -iR(\lambda^{-1}F^+ + \lambda F^-)
\]
\[
= -(\lambda^{-1}RF^+ + \lambda RF^-)
\]
\[
= -\lambda^{-1}iF^+ + \lambda iF^-
\]
\[
= \lambda \cdot (-iRF)
\]
\[
= \lambda \cdot JRF.
\]
We conclude that \( \lambda \cdot JF = J \lambda \cdot F \).

The second part of the statement follows from Corollary 2.8.1. By construction we have the splitting of the Lie algebra with trivial Lie bracket \( t^C = t \oplus it \), where \( t \) is identified to real valued functions and \( t^C \) complex valued functions. By Formula (2.13), the infinitesimal action satisfies \( JX_\zeta = X_{i\zeta} \) for every \( \zeta \in t \), which proves the second statement of the lemma. \( \Box \)
Proposition 2.9.3. The Kähler structure \((\mathcal{F}, \mathcal{G}, \mathcal{J}, \Omega)\) is invariant under the action of \(\mathbb{T}\).

Proof. For \(\lambda \in \mathbb{T}\), the action is just the usual complex multiplication \(\lambda \cdot F = \lambda F\), so that \(g(\lambda F, \lambda F) = |\lambda|^2 g(F, F) = g(F, F)\). Therefore \(\mathcal{G}(\lambda \cdot F, \lambda \cdot F) = \mathcal{G}(F, F)\) and the \(\mathbb{T}\)-action preserves \(\mathcal{G}\). By Proposition 2.9.2, the action preserves the almost complex structure \(\mathcal{J}\), hence the Kähler form \(\Omega\) is preserved as well. \(\square\)

2.10. Hamiltonian gauge group action. The \(\mathbb{T}\)-action on the moduli space \(\mathcal{F}\) turns out to be Hamiltonian, as we are going to show in this section. Let \(\mu : \mathcal{F} \to \mathfrak{t} = C^\infty(M, \mathbb{R})\) be the map given by the formula

\[
\mu(F) = -\frac{1}{2} g(\mathcal{R} F, F). \tag{2.17}
\]

The map \(\mu\) is \(\mathbb{T}\)-invariant and simple computations give

\[
D\mu|_F \cdot \dot{F} = -\frac{1}{2} \left( g(\mathcal{R} F, \dot{F}) + g(\mathcal{R} \dot{F}, F) \right) \tag{2.18}
\]

\[
= -g(\mathcal{R} F, \dot{F}) \tag{2.19}
\]

\[
= -g(i\mathcal{J} F, \dot{F}) \tag{2.20}
\]

\[
= -\hat{\omega}(iF, \dot{F}) \tag{2.21}
\]

Using the \(L^2\)-inner product on \(\mathfrak{t}\), we compute

\[
\langle \langle D\mu|_F \cdot \dot{F}, \zeta \rangle \rangle = -\int_M \hat{\omega}(iF, \dot{F})\zeta \sigma_M
\]

\[
= -\int_M \hat{\omega}(i\mathcal{J} F, \dot{F}) \text{vol}_M
\]

\[
= -\Omega(i\mathcal{J} F, \dot{F})
\]

\[
= -\iota_{X_\zeta}(F) \Omega(\dot{F}),
\]

where \(X_\zeta\) is the infinitesimal action of \(\zeta \in \mathfrak{t}\) on \(\mathcal{F}\), given by Formula (2.12). The above identity shows that \(\mu\) is a moment map for the action of \(\mathbb{T}\). We summarize our construction as follows:

Theorem 2.10.1. Let \(M = V/\Gamma\) be a quotient torus, where \(V\) is an affine space with \(\overrightarrow{V}\) as an underlying vector space. We assume that \(\overrightarrow{V}\) is a complex Hermitian vector space, endowed with an an alternate almost complex structure \(\mathcal{J}_V\) as in Proposition 2.9.1. Then the gauge group action of \(\mathbb{T}\) on the moduli space \(\mathcal{F}\) is formally Hamiltonian with respect to \(\Omega\), with moment map \(\mu\) given by Formula (2.17). In other words, for every \(\zeta \in \mathfrak{t}\), we have

\[
D\langle \langle \mu, \zeta \rangle \rangle = -\iota_{X_\zeta} \Omega,
\]

where \(X_\zeta\) is the infinitesimal action of \(\zeta\) on \(\mathcal{F}\).
2.11. **Moment map and symplectic density.** The moment map defined by Formula (2.17) is some type of *symplectic density*, as we are going to see. For $F \in \mathcal{F}$, we have by definition

$$2\mu(F) = g(iF \circ J_M, F).$$

We choose a local oriented orthonormal frame $(e_1, f_1, \cdots, e_n, f_n)$ of $T_xM$ adapted to the Kähler structure $(g_M, J_M, \hat{\omega}_M)$. This means that $J_M e_j = f_j$, so that the symplectic form is given by the formula

$$\hat{\omega}_M = \sum_{k=1}^n e_k^* \wedge f_k^*$$
on $T_xM$. Then by definition of the metric $g$

$$2\mu(F) = \sum_{k=1}^n g_V(iF \circ J_M(e_k), F(e_k)) + g_V(iF \circ J_M(f_k), F(f_k))$$

$$= \sum_{k=1}^n ig_V(iF(f_k), F(e_k)) - g_V(iF(e_k), F(f_k))$$

$$= -2\sum_{k=1}^n g_V(iF(e_k), F(f_k))$$

$$= -2\sum_{k=1}^n \omega_V(F(e_k), F(f_k))$$

The pullback of the Euclidean symplectic form $\omega_V$ by $F$ is defined by

$$F^*\omega_V(\eta_1, \eta_2) = \omega_V(F(\eta_1), F(\eta_2)),$$
and the above computation shows that

$$\mu(F) = -\sum_{k=1}^n F^*\omega_V(e_k, f_k).$$

Assuming $n = 2$ for simplicity, we have

$$\hat{\omega}_M = e_1^* \wedge f_1^* + e_2^* \wedge f_2^*$$
so that

$$F^*\omega_V \wedge \hat{\omega}_M = -\mu(F)\text{vol}_M$$
and we deduce the following result:

**Proposition 2.11.1.** In the case where $M = V/\Gamma$ is a torus of real dimension 4, we have

$$\mu(F) = -\frac{(F^*\omega_V) \wedge \hat{\omega}_M}{\text{vol}_M}, \quad (2.22)$$
for every $F \in \mathcal{F}$. In particular, if $F = \mathcal{D}f$ for some $f \in \mathcal{M}$, then

$$\mu(\mathcal{D}f) = -\frac{(f^*\omega_M) \wedge \hat{\omega}_M}{\text{vol}_M}.$$
3. HyperKähler moment map

3.1. HyperKähler torus. We restrict our attention to the case of a hyperKähler torus of real dimension 4. More precisely, we assume that the vector space $\mathbf{V}$ is equipped with a linear isomorphism $\mathbf{V} \cong \mathbb{H}$ with the space of quaternions $\mathbb{H}$. Recall that the space of quaternions $\mathbb{H}$ can be constructed as a 4-dimensional real vector space: every quaternion $q \in \mathbb{H}$ can be written as a linear combination

$$q = a + bi + cj + dk,$$

where $a, b, c, d$ are real numbers and the quaternions $i, j$ and $k$ satisfy the quaternionic relations

$$i^2 = j^2 = k^2 = ijk = -1.$$

In particular, we have a canonical inclusion $\mathbb{C} \subset \mathbb{H}$, since every complex number of the form $\lambda = a + bi$ is by definition a quaternion. Among the various possible structures of complex vector space on $\mathbb{H}$, we define the action of $\lambda \in \mathbb{C}$, on $q \in \mathbb{H}$ by the **quaternionic multiplication on the right**:

$$\lambda \cdot q = q \lambda.$$

This multiplicative action of $\mathbb{C}$ endows $\mathbb{H}$ with a structure of complex vector space. In particular, this complex structure is such that the map

$$\mathbb{C}^2 \longrightarrow \mathbb{H}
\begin{pmatrix} z_1, z_2 \end{pmatrix} \longmapsto z_1 + jz_2.$$  \hspace{1cm} (3.1)

is a complex linear isomorphism.

We define 3 almost complex structures $I, J, K$ on $\mathbb{H}$, given by the quaternionic multiplication by $i, j$ and $k$ on the left:

$$I \cdot q = iq, \quad J \cdot q = jq \quad \text{and} \quad K \cdot q = kq,$$

and we also denote by $I, J$ and $K$ the almost complex structures deduced on $\mathbb{C}^2$ using the isomorphism (3.1).

From now on $V$ is an affine space with the underlying vector space $\mathbf{V}$ identified to $\mathbb{H}$. Then $\mathbf{V}$ is isomorphic to the complex vector space $\mathbb{C}^2$ via (3.1), equipped with its canonical Hermitian structure. Furthermore $\mathbf{V}$ is endowed with three additional almost complex structure $\bullet = I, J$ and $K$, deduced from the one on $\mathbb{H}$. The three almost complex structures are compatible with the metric $g_V$ and their corresponding symplectic forms $\hat{\omega}_I = \hat{\omega}_J$ and $\hat{\omega}_K$ are given by

$$\hat{\omega}_I = g_V(I \cdot \cdot), \quad \hat{\omega}_J = g_V(J \cdot \cdot) \quad \text{and} \quad \hat{\omega}_K = g_V(K \cdot \cdot).$$

3.2. Selfduality. The metric $g_V$ and the orientation induced by $\omega_V$ provide a **Hodge operator**, which is an involution $\ast : \Lambda^2 \mathbf{V}^* \rightarrow \Lambda^2 \mathbf{V}^*$ defined by

$$\gamma_1 \wedge \ast \gamma_2 = g_V(\gamma_1, \gamma_2) \text{vol}_V.$$
where $\text{vol}_V = \frac{\omega_V^2}{2}$. We have the well known splitting of 2-forms

$$\Lambda^2 V^* = \Lambda^{2+} V^* \oplus \Lambda^{2-} V^*$$

into selfdual and anti-selfdual 2-forms, corresponding to the $\pm 1$ eigenvalues of the Hodge $\star$ operator. The isomorphism between $V^*$ and $\mathbb{C}^2$ together with the canonical coordinates $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, give the formulas

$$\hat{\omega}_I = dx_1 \wedge dy_1 - dx_2 \wedge dy_2,$$

$$\hat{\omega}_J = dx_1 \wedge dx_2 + dy_1 \wedge dy_2,$$

$$\hat{\omega}_K = -dx_1 \wedge dy_2 - dx_2 \wedge dy_1.$$

Notice that these forms are quite different from $\omega_V = dx_1 \wedge dy_2 + dx_2 \wedge dy_2$. Using the fact that $\star(dx_1 \wedge dy_1) = dx_2 \wedge dy_2$, $\star(dx_1 \wedge dx_2) = -dy_1 \wedge dy_2$ and $\star(dx_1 \wedge dy_2) = dy_1 \wedge dx_2$, we observe that the forms $\hat{\omega}_\bullet$ span the space of anti-selfdual 2-forms $\Lambda^2- V^*$, whereas $\omega_V$ is a selfdual 2-form. In particular, we have the decomposition

$$\Lambda^2- V^* = \mathbb{R} \hat{\omega}_I \oplus \mathbb{R} \hat{\omega}_J \oplus \mathbb{R} \hat{\omega}_K$$

and the formulas

$$\hat{\omega}_I^2 = \hat{\omega}_J^2 = \hat{\omega}_K^2 = -\omega_V^2 = -2\text{vol}_V.$$

Given a lattice $\Gamma$ of $V$, we obtain a torus $M = V/\Gamma$ of real dimension 4. The metric $g_V$, the almost complex structures $i, I, J, K$ and the Kähler forms $\omega_V, \hat{\omega}_\bullet$ descend to the quotient $M$. We obtain a flat canonical Kähler structure $(M, g_M, i, \omega_M)$ and a conjugate hyperKähler structure $(M, g_M, I, J, K)$ with three Kähler forms denoted $\hat{\omega}_\bullet$ as well. By construction the canonical Kähler form $\omega_M$ is selfdual, whereas the $\hat{\omega}_\bullet$ are anti-selfdual.

### 3.3. Triple moment map.

According Proposition 2.9.1, the almost complex structures $I, J$ and $K$ on $M$ induce three almost complex structures $I, J$ and $K$ on the moduli space $\mathcal{F}$. Together with the metric $g$, they provide a hyperKähler structure, with three corresponding Kähler forms on $\mathcal{F}$, denoted $\Omega_I, \Omega_J$ and $\Omega_K$. We also denote by $\mathcal{R}_I, \mathcal{R}_J$ and $\mathcal{R}_K$ the three involutions of $\mathcal{F}$ induced by the almost complex structures.

By Proposition 2.9.3, the action of $T$ on $\mathcal{F}$ preserves the hyperKähler structure. By Theorem 2.10.1 the action of $T$ is Hamiltonian with respect to the three symplectic forms and the moment maps are given by

$$\mu_\bullet(F) = -\frac{1}{2}g(\mathcal{R}_F, F)$$

or, equivalently by Proposition (2.11.1)

$$\mu_\bullet(F) = -\frac{(F^* \omega_V) \wedge \hat{\omega}_\bullet}{\text{vol}_M}$$
for \( \bullet = I, J, K \). These identities can be gathered into a single one

\[(F^*\omega_V)^- = \frac{1}{2} \sum \mu_\bullet(F)\hat{\omega}_\bullet\]

where \( \beta^- \) denotes the anti-selfdual component of a differential 2-form. Indeed, the forms \( \hat{\omega}_\bullet \) provide an orthogonal frame for the space of anti-selfdual forms and they satisfy \( |\hat{\omega}_\bullet|^2 = 2 \). We check that

\[(F^*\omega_V)^- \wedge \hat{\omega}_\bullet = F^*\omega_V \wedge \hat{\omega}_\bullet = -\mu_\bullet(F)\text{vol}_\mathcal{M}\]

and introduce the hyperKähler moment map

\[
\mu : \mathcal{F} \to \mathbb{R}^3 \cong C^\infty(M, \mathbb{R})^3
\]

\[F \mapsto (\mu_I(F), \mu_J(F), \mu_K(F)).\]  

(3.4)

**Remark 3.3.1.** The forms \( \hat{\omega}_\bullet \) are anti-selfdual and provide an isomorphism

\[
\xi : \mathbb{R}^3 \to \Omega^{2-}(\mathcal{M})
\]

\[(\zeta_I, \zeta_J, \zeta_K) \mapsto \frac{1}{\sqrt{2}} \sum \zeta_\bullet \hat{\omega}_\bullet,\]

which respects the metrics. By definition

\[
\sqrt{2}(F^*\omega_V)^- = \xi(\mu_I(F), \mu_J(F), \mu_K(F))
\]

(3.6)

and it makes sense to interpret the hyperKähler moment map as a map

\[
\tilde{\mu} : \mathcal{F} \to \Omega^{2-}(\mathcal{M})
\]

\[F \mapsto \sqrt{2}(F^*\omega_V)^-\]

(3.7)

related to \( \mu \) by the identity

\[
\tilde{\mu} = \xi \circ \mu.
\]

**Proposition 3.3.2.** For every \( F \in \mathcal{F} \), the following properties are equivalent:

1. \( F^*\omega_V \) is selfdual;
2. \( \mu(F) = 0 \).

Proof. The equivalence is an immediate consequence of Formula (3.6). \( \square \)

In the case where \( F = \mathcal{D}f \), we have an interesting interpretation of the moment map which is the leitmotif to this paper:

**Proposition 3.3.3.** Let \( f \in \mathcal{M} \) be a smooth map. Then the following properties are equivalent:

1. \( \mu(\mathcal{D}f) = 0 \);
2. \( f^*\omega_M \) is a selfdual harmonic form.
In particular, if \( f^*[\omega_M] = [\omega_M] \), then \( f \) is a symplectomorphism if, and only if, \( \mu(\mathcal{D}f) = 0 \), which is to say:

\[
\text{Symp}(M, \omega_M) = \{ f \in \mathcal{M}, f^*[\omega_M] = [\omega_M] \text{ and } \mu \circ \mathcal{D}(f) = 0 \}.
\]

Proof. Assume that \( f \in \mathcal{M} \) and \( \mu(\mathcal{D}f) = 0 \). By Proposition 3.3.2, \( \mu(\mathcal{D}f) = 0 \) is equivalent to the fact that \( F^*\omega_V \) is a selfdual form, where \( F = \mathcal{D}f \). Since \( F^*\omega_V = f^*\omega_M \), we deduce that \( f^*\omega_M \) is selfdual and closed. In particular, it is harmonic and the first part of the proposition follows.

If \( f^*[\omega_M] = [\omega_M] \), then \( f^*\omega_M - \omega_M \) is an exact form. The Kähler form \( \omega_M \) is selfdual. If \( f^*\omega_M \) is selfdual as well, then \( f^*\omega_M - \omega_M \) is an exact selfdual form, which implies that it vanishes identically. We conclude that \( f \) is a symplectic map. \( \Box \)

4. HyperKähler flow

4.1. Prescribed cohomology classes and Hodge projector. The moduli space of \( \mathcal{V} \)-valued differential 1-forms \( \mathcal{F} \) contains the subspace of closed forms:

\[
\mathcal{F}_c = \{ F \in \mathcal{F}, dF = 0 \}.
\]

Let \( \alpha \in H^1(M, \mathcal{V}) \setminus 0 \) be a fixed cohomology class. We introduce the subspace of closed forms \( \mathcal{F}_\alpha \) with cohomology class contained in the line \( \mathbb{R}\alpha \):

\[
\mathcal{F}_\alpha = \{ F \in \mathcal{F}, dF = 0 \text{ and } [F] \in \mathbb{R}\alpha \} \subset \mathcal{F}_c \subset \mathcal{F}.
\]

It is a standard fact that Hodge theory extends to \( \mathcal{V} \)-valued forms. The formal adjoint of \( d \) with respect to the \( L^2 \)-metric \( \mathcal{G} \) is denoted \( d^* \) and the Laplacian operator is defined by \( \Delta = dd^* + d^*d \). Hodge theory provides an orthogonal projection with respect to the metric \( \mathcal{G} \):

\[
\Pi_c : \mathcal{F} \to \mathcal{F}_c.
\]

Indeed, any 1-form \( F \in \mathcal{F} \) admits an orthogonal decomposition

\[
F = F_h + d\beta + d^*b,
\]

where \( F_h \) is harmonic, \( \beta \in \Omega^0(M, \mathcal{V}) \) and \( b \in \Omega^2(M, \mathcal{V}) \). Then, the projection \( \Pi_c : \mathcal{F} \to \mathcal{F}_c \) onto the closed component is given by

\[
\Pi_c(F) = F_h + d\beta.
\]

Cohomology classes are represented by a unique harmonic form. This leads to an orthogonal decomposition into harmonic components

\[
F_h = F_\alpha + F_\alpha^\perp,
\]

where \([F_\alpha] \in \mathbb{R}\alpha\) and \( F_\alpha^\perp \) is orthogonal to the harmonic representative of \( \alpha \). The orthogonal projection

\[
\Pi_\alpha : \mathcal{F} \to \mathcal{F}_\alpha.
\]
is then defined by
\[ \Pi_\alpha(F) = F_\alpha + d\beta. \]

4.2. **Energy of the moment map.** The energy functional of the hyperKähler moment map
\[ \phi : \mathcal{F} \to \mathbb{R} \]
is defined by
\[ \phi(F) = \frac{1}{2} \| \mu(F) \|^2_{L^2} \]
\[ = \frac{1}{2} \| \mu_I(F) \|^2_{L^2} + \frac{1}{2} \| \mu_J(F) \|^2_{L^2} + \frac{1}{2} \| \mu_K(F) \|^2_{L^2}. \]
The functional \( \phi \) is non negative and its vanishing locus agrees with the zero set the hyperKähler moment map
\[ \mu^{-1}(0) = \phi^{-1}(0). \]
Furthermore, the vanishing set of \( \phi \) agrees with the set of critical values according to the following proposition:

**Proposition 4.2.1.** The critical points of \( \phi : \mathcal{F} \to \mathbb{R} \) are the zeroes of the hyperKähler moment map \( \mu \). Similarly, the critical points of the restricted functional \( \phi : \mathcal{F}_\alpha \to \mathbb{R} \) are the zeroes of \( \mu \) contained in \( \mathcal{F}_\alpha \).

The proof of Proposition 4.2.1 follows from the elementary computations carried out in the rest of this section. The differential of \( \phi \) is readily computed:
\[ D\phi|_F \cdot \dot{F} = \langle \langle D\mu_I|_F \cdot \dot{F}, \mu_I(F) \rangle \rangle \]
\[ + \langle \langle D\mu_J|_F \cdot \dot{F}, \mu_J(F) \rangle \rangle \]
\[ + \langle \langle D\mu_K|_F \cdot \dot{F}, \mu_K(F) \rangle \rangle \]
where \( \langle \langle \cdot, \cdot \rangle \rangle \) denotes the \( L^2 \) inner product on \( t \). By Formula (2.19),
\[ D\mu_I|_F \cdot \dot{F} = -g(\mathcal{R}_I F, \dot{F}), \]
hence
\[ \mu_I(F)D\mu_I|_F \cdot \dot{F} = -g(\mu_I(F)\mathcal{R}_I F, \dot{F}) \]
and more generally
\[ \mu_\bullet(F)D\mu_\bullet|_F \cdot \dot{F} = -g(\mu_\bullet(F)\mathcal{R}_\bullet F, \dot{F}). \]
Integrating the above formula, we have
\[ \langle \langle D\mu_\bullet|_F \cdot \dot{F}, \mu_\bullet(F) \rangle \rangle = -\mathcal{G}(\mu_\bullet(F)\mathcal{R}_\bullet F, \dot{F}) \]
\[ = -\Omega_\bullet(\mu_\bullet(F) i F, \dot{F}) \]
\[ = -\iota_{X_{\mu_\bullet(F)}}(F)\Omega_\bullet(\dot{F}). \]
where \( X_{\mu_\bullet(F)} \) is the vector field corresponding to the infinitesimal action of \( \mu_\bullet(F) \in t \) on \( \mathcal{F} \).
Lemma 4.2.2. For every $F, \dot{F} \in \mathcal{F}$, we have

$$D\phi|_F \cdot \dot{F} = \langle \sum \bullet W*(F), \dot{F} \rangle $$

(4.6)

where

$$W*(F) = -\mu*(F)R*F.$$ 

(4.7)

In particular

$$D\phi|_F \cdot F = 4\phi(F).$$ 

(4.8)

Proof. Formulas (4.6) and (4.7) are deduced from (4.2) and (4.3). The identity (4.8) is a simple consequence:

$$D\phi|_F \cdot F = -\sum \bullet G(\mu*(F)R*F, F) = -\sum \bullet \int g(\mu*(F)R*F, F) \text{vol}_M$$

Since $g(\mu*(F)R*F, F) = \mu*(F)g(R*F, F) = -2\mu*(F)^2$, we deduce that $D\phi|_F \cdot F = 2\sum \bullet \|\mu*(F)\|^2 = 4\phi(F)$.  

$\square$

Proof of Proposition 4.2.1. If $F \in \mathcal{F}$ satisfies $\mu(F) = 0$, then $\mu*(F) = 0$ hence $W*(F) = 0$ and $D\phi|_F = 0$. Conversely, if $F$ is a critical point of $\phi: \mathcal{F} \rightarrow \mathbb{R}$, then $4\phi(F) = D\phi|_F \cdot F = 0$, by Formula (4.8), which implies $\mu(F) = 0$.

If $F$ is a critical point of the restriction $\phi: \mathcal{F}_\alpha \rightarrow \mathbb{R}$, then $D\phi|_F \cdot F$ vanishes and the same proof shows that $\mu(F) = 0$.  

As a direct consequence of Formula (4.6) we obtain the following result:

Corollary 4.2.3. The gradient of $\phi: \mathcal{F} \rightarrow \mathbb{R}$ at $F$ is given by the formula

$$\nabla\phi(F) = \sum \bullet W*(F).$$

The gradient $\nabla^\alpha\phi$ of the restriction $\phi: \mathcal{F}_\alpha \rightarrow \mathbb{R}$ at $F \in \mathcal{F}_\alpha$ is given by the projection

$$\nabla^\alpha\phi(F) = \Pi_\alpha(\nabla\phi(F)).$$

For every $F \in \mathcal{F}$, we have

$$\langle \nabla\phi(F), F \rangle = 4\phi(F)$$

and, similarly, for every $F \in \mathcal{F}_\alpha$

$$\langle \nabla^\alpha\phi(F), F \rangle = \langle \Pi_\alpha \nabla\phi(F), F \rangle = 4\phi(F).$$
4.3. Modified moment map flow. The downward gradient flow of the moment map energy functional \( \phi : \mathcal{F} \to \mathbb{R} \) is the classical hyperKähler moment map flow used in many gauge theoretic settings. However the subspace \( \mathcal{F}_\alpha \subset \mathcal{F} \) is not invariant under the gauge group and the flow may not preserve this subspace. This is problematic, since we are mostly interested in closed differential, with Proposition 3.3.3 in mind. We get around this issue by considering the flow of the restricted functional \( \phi : \mathcal{F}_\alpha \to \mathbb{R} \), defined by

\[
\frac{\partial F}{\partial t} = -\Pi_\alpha \nabla \phi(F).
\] (4.9)

The above flow is called the modified hyperKähler moment map flow.

**Proposition 4.3.1.** The following conditions are equivalent for \( F \in \mathcal{F}_\alpha \):

1. \( F \) is a zero of \( \mu \)
2. \( F \) is a zero of \( \phi \)
3. \( F \) is a critical point of the functional \( \phi : \mathcal{F}_\alpha \to \mathbb{R} \)
4. \( F \) is a fixed point of the modified moment map flow.

**Proof.** The proof is obvious since fixed points are by definition the critical point of the restricted functional \( \phi : \mathcal{F}_\alpha \to \mathbb{R} \), which were identified at Proposition 4.2.1. \( \square \)

An essential feature of the modified moment map flow is the decay property of the \( L^2 \) norm:

**Proposition 4.3.2.** Let \( F_t \in \mathcal{F}_\alpha \) be a solution of the modified moment map flow, for \( t \) in some interval \( I \). Then

\[
\frac{\partial}{\partial t} \| F_t \|^2_{L^2} = -8\phi(F_t),
\]

for every \( t \in I \). In particular \( t \mapsto \| F_t \|^2_{L^2} \) is a non increasing function on \( I \).

**Proof.** We compute

\[
\frac{\partial}{\partial t} \| F_t \|^2_{L^2} = 2\langle \frac{\partial F_t}{\partial t}, F_t \rangle = -2\langle \Pi_\alpha \nabla \phi(F_t), F_t \rangle = -2D\phi|_{F_t} \cdot F_t.
\]

an the result follows by Equation (4.8). \( \square \)

**Remark 4.3.3.** In the case of an ordinary differential equation, Proposition 4.3.2 would guaranty the long time existence of the flow. An analogue of this proposition holds indeed in the polyhedral setting (cf. Proposition 7.7.1) and insures the long time existence of the flow (cf. Corollary 7.7.2).
4.4. Short time existence. The modified hyperKähler moment map flow has been considered so far from a purely formal perspective. We are going to show that the Cauchy-Lipschitz theorem applies, once suitable Hölder spaces have been introduced.

We consider the $C^{k,\nu}$-Hölder norms, denoted $\| \cdot \|_{k,\nu}$, defined via the metrics $g_M$ and $g$ on the spaces of functions and tensors on the 4-torus $M$, where $\nu \in (0,1)$ is the Hölder regularity exponent and $k$ is the number of derivatives controlled by the norm. The spaces of smooth functions and differential forms can be completed into Banach spaces with respect to these Hölder norms. For example, we denote by $\mathcal{F}^{k,\nu}$ the completion of the space of smooth differential forms $\mathcal{F}$ with respect to the $C^{k,\nu}$-Hölder norm.

If $k \geq 2$, the exterior derivative $d$, its adjoint $d^*$ and the Laplacian operator $\Delta = dd^* + d^*d$ are defined on $\mathcal{F}^{k,\nu}$. The Hodge decomposition theory holds and, in particular, we may consider the subspace of closed forms $\mathcal{F}_c^{k,\nu}$. We can also consider the subspace of closed forms with cohomology class in $\mathbb{R} \alpha$, denoted $\mathcal{F}_\alpha^{k,\nu}$ and the orthogonal projection $\Pi_\alpha : \mathcal{F}^{k,\nu} \to \mathcal{F}_\alpha^{k,\nu}$ (cf. Lemma 4.4.2), which extends the projection defined at §2.4 in the smooth settings.

Theorem 4.4.1. Let $k \geq 2$ and $\nu \in (0,1)$ be some Hölder exponents. For every $\alpha \in H^1(M, \overrightarrow{V}) \setminus 0$ and $F \in \mathcal{F}_\alpha^{k,\nu}$, there exists $\varepsilon > 0$ and a unique differentiable map

$$[-\varepsilon, \varepsilon] \to \mathcal{F}_\alpha^{k,\nu}$$

$$t \mapsto F_t$$

such that

- We have the initial condition $F_0 = F$ and
- the map is a solution of the modified hyperKähler moment map flow.

Theorem 4.4.1 follows immediately from the Cauchy-Lipschitz theorem. A key step to prove the Lipschitz condition is the following classical result of Hodge theory:

Lemma 4.4.2. For $k \geq 2$, the projection $\Pi_\alpha : \mathcal{F}^{k,\nu} \to \mathcal{F}_\alpha^{k,\nu}$ defines a continuous map with respect to the $C^{k,\nu}$-Hölder norm. In other words, there exists a constant $C > 0$, such that for every $F \in \mathcal{F}^{k,\nu}$

$$\| \Pi_\alpha F \|_{k,\nu} \leq C \| F \|_{k,\nu}.$$ 

Remark 4.4.3. More generally, Hodge projectors are all continuous provided suitable Hölder spaces are introduced. For example, the projection of differential $p$-forms onto their closed, co-closed or harmonic components are continuous as well. However we include a proof of the above theorem for clarity.
Proof of Lemma 4.4.2. For \( k \geq 2 \), the classical Hodge theory shows that every 1-form \( F \in \mathcal{F}^{k,\nu} \) admits a \( G \)-orthogonal decomposition

\[
F = F_h + \Delta G,
\]
where \( G \in \mathcal{F}^{k+2,\nu} \) is a 1-form orthogonal to harmonic forms and \( F_h \) is the harmonic component of \( F \). A \( C^{k,\nu} \)-estimate on \( F \) provides a \( C^{k-2,\nu} \)-estimate for \( \Delta F \). Taking the Laplacian of both sides of the above identity gives \( \Delta F = \Delta^2 G \), since \( F_h \) is harmonic. The \( C^{k,\nu} \)-norm of \( F \) controls the \( C^{k,\nu} \)-norm of \( \Delta^2 G \). The operator \( \Delta \) is self-adjoint, hence \( \Delta G \) is orthogonal to the kernel of \( \Delta \). The Schauder estimates for the elliptic operator \( \Delta \) provide a control on the \( C^{k,\nu} \)-norm of \( \Delta G \). Since \( G \) was chosen orthogonal to harmonic forms, we deduce a \( C^{k+2,\nu} \)-control on \( G \), by the Schauder estimates. In conclusion, there exists a constant \( c_1 > 0 \), independent of \( F \), such that

\[
\| G \|_{k+2,\nu} \leq c_1 \| F \|_{k,\nu}.
\]

In particular

\[
\| dd^* G \|_{k,\nu} \leq c_2 \| F \|_{k,\nu}, \tag{4.10}
\]

for some constant \( c_2 > 0 \), independent of \( F \).

It turns out that there exists a universal constant \( c_3 > 0 \), such that

\[
\| F_h \|_{L^2} \leq c_3 \| F \|_{k,\nu}. \tag{4.11}
\]

The proof goes by contradiction. If this is not true, we can find a sequence \( F_j \) of differential forms such that

\[
\lim_{j \to \infty} \| F_j \|_{k,\nu} = 0 \quad \text{and} \quad \| F_{j,h} \|_{L^2} = 1.
\]

Since the space of harmonic form is finite dimensional, we may assume, up to extracting a subsequence, that \( F_{j,h} \) converges to a harmonic form \( F_h \) with \( \| F_h \|_{L^2} = 1 \). In particular, \( F_h \) is non vanishing. Computing the \( L^2 \) inner product, we have

\[
\langle F_j, F_h \rangle = \langle F_{j,h}, F_h \rangle.
\]

In particular

\[
\lim \langle F_j, F_h \rangle = \| F_h \|_{L^2}^2 = 1.
\]

By the Cauchy-Schwartz inequality, \( \langle F_j, F_h \rangle \leq \| F_j \|_{L^2} \| F_h \|_{L^2} \), but \( \| F_j \|_{L^2} \leq \| F_j \|_{k,\nu} \text{vol}(M) \) hence \( \lim \| F_j \|_{L^2} = 0 \). We conclude that

\[
\lim \langle F_j, F_h \rangle = 0
\]

which is a contradiction and estimate (4.11) is proved.

The orthogonal decomposition into harmonic components \( F_h = F_\alpha + F_\perp_\alpha \) gives the estimate

\[
\| F_\alpha \|_{L^2} \leq \| F_h \|_{L^2}.
\]

Together with estimate (4.11), we deduce that

\[
\| F_\alpha \|_{L^2} \leq c_3 \| F \|_{k,\nu}. \tag{4.12}
\]
All norms are equivalent on a finite dimensional space, in particular on the space of harmonic forms. Therefore, there exists a universal constant $c_4 > 0$ such that for every harmonic form $\beta_h$

$$c_4 \| \beta_h \|_{k,\nu} \leq \| \beta_h \|_{L^2}. $$

Together with estimate (4.12), we deduce that

$$\| F_\alpha \|_{k,\nu} \leq c_5 \| F \|_{k,\nu}$$

where $c_5 = \frac{c_3}{c_4}$.

Finally $F = F_\alpha + F^\perp_\alpha + dd^*G + d^*dG$, and $\Pi_\alpha F = F_\alpha + dd^*G$ so that

$$\| \Pi_\alpha F \|_{k,\nu} \leq \| F_\alpha \|_{k,\nu} + \| dd^*G \|_{k,\nu}$$

and the lemma follows from estimates (4.13) and (4.10).

**Proof of Theorem 4.4.1.** The map of $F \mapsto \nabla\phi(F)$ is polynomial of order 3 with respect to the coefficients of $F$ by Corollary 4.2.3. Therefore, the map $F \mapsto \nabla\phi(F)$ is locally Lipschitzian with respect to the $C^{k,\nu}$-Hölder norm. The projector $\Pi_\alpha$ is continuous by Lemma 4.4.2, hence the composition $F \mapsto \Pi_\alpha \nabla\phi(F)$ is locally Lipschitzian as well. The standard proof of existence of solution of an evolution equation rests on the existence of fixed points of a functional

$$\Upsilon : B \to B$$

where $B$ is the Banach space of continuous maps

$$C^0([-\epsilon, \epsilon], \mathfrak{F}^{k,\nu}_\alpha),$$

endowed with the norm

$$\| F_t \|_{\infty} = \sup_{t \in [-\epsilon, \epsilon]} \| F_t \|_{k,\nu}.$$ 

More precisely, $\Upsilon$ is defined for each $F \in B$, by

$$\Upsilon(F)_t = F_0 + \int_0^t \Pi_\alpha \nabla\phi(F_s) ds$$

For $\epsilon > 0$ sufficiently small, the functional $\Upsilon$ is a locally $\frac{1}{2}$-contractant map of Banach space for $(B, \| \cdot \|_{\infty})$ and the Banach fixed point theorem applies. 

5. Renormalized flow and real blowup

5.1. **Homotopy of the symplectomorphism group.** We give a sketch of a potential application of the modified moment map flow, for investigating the homotopy type of $\text{Symp}_\alpha(M,\omega_M)$. Let $B^{n+1}$ be a closed Euclidean ball of dimension $n + 1$ and $S^n$ be its boundary sphere. Let

$$h : S^n \to \text{Symp}_\alpha(M,\omega_M)$$

be a continuous map, with respect to the $C^1$-topology on $\mathcal{M}_\alpha$. We denote $h(\tau)$ by $h_\tau$ as usual, for $\tau \in S^n$. We choose a marked point $x_0 \in M$ and assume, for simplicity, that $h_\tau(x_0) = x_0$ for every $\tau \in S^n$. 

The map $H : S^n \to \mathcal{F}_\alpha$, defined by $H_\tau = \mathcal{D}h_\tau$, is continuous with respect to the $C^0$-topology. The cohomology class $\alpha$ is can be understood as an affine subspace of $\mathcal{F}_\alpha$. In particular this space is contractible, hence $H$ extends as a continuous homotopy $H : B^{n+1} \to \alpha$. Since $\alpha$ is an integral cohomology class, there exists a unique family of maps $h : B^{n+1} \to \mathcal{M}_\alpha$ which extend the map $h : S^n \to \text{Symp}_\alpha(M,\omega_M)$, with the property that $h_\tau(x_0) = x_0$ and $\mathcal{D}h_\tau = H_\tau$ for every $\tau \in B^{n+1}$. This shows that the initial homotopy $h : S^n \to \text{Symp}_\alpha(M,\omega_M)$ is trivial in $\mathcal{M}_\alpha$. We would like to know whether it is also trivial in $\text{Symp}_\alpha(M,\omega_M)$.

The idea is to try to push back the homotopy in the symplectic group via the modified moment map flow. For every $\tau \in S^n$, the point $H_\tau$ is a fixed point of the modified moment map flow, by assumption. We now make some extremely strong hypothesis:

(H1) We assume that, for every $\tau \in B^{n+1}$, the modified moment map flow $\mathcal{F}_\tau$, with initial condition $\mathcal{F}_0 = H_\tau$, exists for every $t \geq 0$ and converges toward a limit $\tilde{H}_\tau$ as $t \to +\infty$.

(H2) We suppose that the limit $\tilde{H}_\tau$ belongs to the vanishing set of $\phi$ and depends continuously on the initial condition.

(H3) We assume that the limit $\tilde{H}_\tau$ is different from $\mathcal{F}_\tau = 0$, for every $\tau \in B^{n+1}$.

Properties (H1) and (H2) are conjectural in the smooth case. However they hold in the polyhedral context, as proved by Theorem G. Under these hypothesis, we have a deformed $H$ into continuous map $\tilde{H} : B^{n+1} \to \mathcal{F}_\tau$, which agrees with $H$ along $S^n$ and takes values in the vanishing set of $\phi$. If the hypothesis (3) holds as well, then $\tilde{H}_\tau$ belongs to $\mathcal{F}_\alpha \setminus 0$. By Lemma 5.3.4 we can deduce that the cohomology class $[\tilde{H}_\tau]$ is non zero and we can write

$$[\tilde{H}_\tau] = \kappa(\tau)\alpha$$

for some non vanishing continuous function $\kappa : B^{n+1} \to \mathbb{R}$. Furthermore, $\kappa = 1$ along the boundary $S^n$. We consider the rescaled family $\tilde{H}_\tau = \kappa^{-1}(\tau)\tilde{H}_\tau$. By definition $[\tilde{H}_\tau] = \alpha$ and the exists a unique family of maps $f_\tau \in \mathcal{M}_\alpha$ such that $\mathcal{D}f_\tau = \tilde{H}_\tau$ and $f_\tau(x_0) = x_0$. By constructions the maps $f_\tau$ belong to $\text{Symp}_\alpha(M,\omega_M)$ by Theorem B and $h_\tau = f_\tau$ for every $\tau \in S^n$. We conclude that the map $h : S^n \to \text{Symp}_\alpha(M,\omega_M)$ is homotopically trivial in $\text{Symp}_\alpha(M,\omega_M)$.

If hypothesis $(H1 - H2 - H3)$ hold for every map $h$ as above, it would readily follow that the evaluation map $ev : \text{Symp}_\alpha(M,\omega_M) \to M$ at $x_0$ is a homotopy equivalence. However this is too good to be true. In the polyhedral setting, we prove at Theorem H that there always exists non trivial solutions of the polyhedral modified moment map flow converging toward $F = 0$. Hence, hypothesis (H3) is generally false in the polyhedral context and we shall see that this is directly related to the existence of proper solitons. We are not able to prove a similar
result in the smooth setting, but it is likely that the same phenomenon
arise. The conclusion of this discussion is that proper solitons should
play a role in the topological description of the space of symplectic
maps of the torus and contribute, somehow, to its complexity.

5.2. **Blownup moduli space.** For any cohomology class \( \alpha \in H^1(M, \overline{\mathcal{V}}) \setminus 0 \), the sphere of radius \( r > 0 \) in \( \mathcal{F}_\alpha \) is defined by
\[
\mathbb{S}_\alpha(r) = \{ F \in \mathcal{F}_\alpha, \| F \|_{L^2} = r \}
\]
and the sphere is denoted \( \mathbb{S}_\alpha \) in the case \( r = 1 \). The real blowup of the moduli space
\[
p : \tilde{\mathcal{F}}_\alpha \to \mathcal{F}_\alpha,
\]
is defined by \( \tilde{\mathcal{F}}_\alpha = [0, +\infty) \times \mathbb{S}_\alpha \), where the blowdown map is given by \( p(r,F) = rF \). The real blowup contains an exceptional divisor
\[
\mathcal{E} = p^{-1}(0) = \{ 0 \} \times \mathbb{S}_\alpha \subset \tilde{\mathcal{F}}_\alpha,
\]
such that the restriction of blowdown map \( p : \tilde{\mathcal{F}}_\alpha \setminus \mathcal{E} \to \mathcal{F}_\alpha \setminus \{ 0 \} \) is bijective. One can think of the real blowup as a system of spherical coordinates for \( \mathcal{F}_\alpha \), with an extra boundary component, corresponding to \( r = 0 \).

5.3. **Solitons.** A critical points \( F \in \mathbb{S}_\alpha(r) \) of the functional \( \phi : \mathcal{F}_\alpha \to \mathbb{R} \) is also a critical point of the restricted functional \( \phi : \mathbb{S}_\alpha(r) \to \mathbb{R} \), but the converse is not necessarily true. A differential form \( F \in \mathbb{S}_\alpha(r) \) is a critical points of the restricted functional \( \phi : \mathbb{S}_\alpha(r) \to \mathbb{R} \) if, and only if, the gradient of \( \phi : \mathcal{F}_\alpha \to \mathbb{R} \) at \( F \) is orthogonal to the sphere \( \mathbb{S}_\alpha(r) \). This property is equivalent to have a radial gradient \( \Pi_\alpha \nabla \phi(F) = \kappa F \), for some constant \( \kappa \in \mathbb{R} \). However by Corollary 4.2.3, the identity \( \langle \Pi_\alpha \nabla \phi(F), F \rangle = 4\phi(F) \) implies \( \kappa r^2 = 4\phi(F) \). We deduce that \( F \) satisfies the following equation, called the **soliton equation**:

\[
\| F \|_{L^2}^2 \Pi_\alpha \nabla \phi(F) = 4\phi(F)F.
\]  

(5.1)

**Definition 5.3.1.** A solution \( F \in \mathcal{F}_\alpha \) of Equation (5.1) is called a soliton. Then either \( \phi(F) \neq 0 \), and we say that \( F \) is a proper soliton, of \( \phi(F) = 0 \) and \( F \) is called a non proper soliton..

**Remark 5.3.2.** The soliton equation (5.1) is **real homogeneous**. Therefore, the space of solitons is a conical subspace of \( \mathcal{F}_\alpha \).

By definiton, we have the following property:

**Lemma 5.3.3.** A differential form \( F \in \mathcal{F}_\alpha \) is a soliton if, and only if, either \( F = 0 \) or \( F \) is a critical point of the restricted functional \( \phi : \mathbb{S}_\alpha(r) \to \mathbb{R} \), where \( r = \| F \|_{L^2} \).

Following the ideas of Atiyah-Bott [2], Donaldson [7] and Kirwan [15, 21] in the finite dimensional case, the functional $\phi: S_\alpha \to \mathbb{R}$ should be regarded as a Morse-Bott function. By definition, the critical set of the restricted functional is the space of solitons in $S_\alpha$. Furthermore, Morse-Bott homology theory provides important topological information about the critical set. Finally, the subspace of non proper solitons in $S_\alpha$ is directly related to the topology of the the symplectic group as we are going to see.

**Lemma 5.3.4.** If $F \in \mathcal{F}_\alpha$ is a non proper soliton with vanishing cohomology class, then $F = 0$.

**Proof.** Let $F$ be a non proper soliton with vanishing cohomology class. By definition $\phi(F) = 0$, which is equivalent to $\tilde{\mu}(F) = (F^*\omega_V)^- = 0$, which implies that $F^*\omega_V$ is selfdual. Since $[F] = 0$, there exists a map $f: M \to M$, homotopic to a constant map, such that $F = Df$. Using the identity $F^*\omega_V = f^*\omega_M$, we conclude that $f^*\omega_M$ is selfdual. On the other hand $[f^*\omega_M] = 0$, since $f$ is homotopic to a constant map. A selfdual exact form must vanish, hence $f^*\omega_M = 0$, which implies $f_\ast = 0$ by the non degeneracy of $\omega_M$. Finally, $f$ is a constant map, so that $Df = F = 0$. 

**Corollary 5.3.5.** The cone spanned by the action of $\mathbb{R}^*$ and the image of $\text{Symp}_\alpha(M, \omega_M)$ by $\mathcal{D}$ agrees with the space of non proper solitons in $\mathcal{F}_\alpha \setminus 0$. Equivalently, the map

$$\varpi: \text{Symp}_\alpha(M, \omega_M) \to S_\alpha, \quad f \mapsto \frac{\mathcal{D}f}{\|f_\ast\|_{L^2}}$$

induces a bijection between $\text{Symp}_\alpha(M, \omega_M)/\overrightarrow{V}$ and the space of non proper solitons in $S_\alpha$, modulo the antipodal map of $S_\alpha$.

**Remark 5.3.6.** The Corollary is stated informally, but it is possible to show that $\varpi$ induces a diffeomorphism between moduli spaces, once suitable Hölder spaces have been introduced.

**Proof.** Given $f \in \text{Symp}_\alpha(M, \omega_M)$, we have $\mu(Df) = 0$ hence $\phi(Df) = 0$, hence $Df$ is a non proper soliton. The fact that $f$ is a diffeomorphism implies that $Df \neq 0$.

Conversely, let $G$ be a non proper soliton in $\mathcal{F}_\alpha \setminus 0$. By Lemma 5.3.4, the cohomology class of $G$ is non zero. Hence there exist $\kappa \in \mathbb{R} \setminus 0$, such that $\kappa[G] = \alpha$. In particular $F = \kappa G$ has integral cohomology class $\alpha$. It follows that there exists $f \in \mathcal{M}_\kappa$ such that $F = Df$. By construction $G$ belongs to the ray spanned by $Df$. The fact that $\phi(G) = 0$ implies $\phi(F) = 0$, which shows that $f \in \text{Symp}_\alpha(M, \omega_M)$. The second statement is an immediate consequence of the first. 

□
5.4. **Solitons and flow lines.** A soliton provides a natural solution of the modified moment map flow, according to the following lemma.

**Lemma 5.4.1.** Let $G \in \mathcal{F}_\alpha \setminus 0$ be a soliton and $r : I \to [0, +\infty)$ be a solution of the ordinary differential equation

$$
\|G\|_{L^2}^2 \frac{dr}{dt} = -4r^3 \phi(G)
$$

defined on some interval $I$. Explicitly, either

1. $G$ is not proper, then $r(t) = r_0$ with $I = \mathbb{R}$ for some constant $r_0 \geq 0$.
2. or $G$ is proper and

$$
r(t) = \frac{1}{\sqrt{8(t-t_0)\phi(G)}}
$$

for some constant $t_0 \in \mathbb{R}$ with $I = (t_0, +\infty)$.

Then

$$
F_t = p(r(t), G) = r(t)G \in \mathcal{F}_\alpha
$$

is a solution of the modified moment map flow with the following dichotomy:

1. If $G$ is not proper, then $F_t$ is a stationary solution of the flow.
2. If $G$ is proper, then $F_t$ is not stationary and $F_t$ converges toward $0 \in \mathcal{F}_\alpha$.

**Proof.** Put $F_t = r(t)G$, where $G$ and $r$ satisfy the assumptions of the lemma. By definition

$$
\frac{\partial}{\partial t} F_t = \frac{dr}{dt} G = -4\|G\|_{L^2}^{-2}r^3 \phi(G) G.
$$

Since $G \in S_\alpha$ is a soliton, then $\|G\|_{L^2}^{-2} \Pi_\alpha \nabla \phi(G) = 4\phi(G)G$, hence

$$
\frac{\partial}{\partial t} F_t = -r^3 \Pi_\alpha \nabla \phi(G) = -\Pi_\alpha \nabla \phi(F_t),
$$

which shows that $F_t$ is a flow line of the modified moment map. □

5.5. **Blowup flow.** In this section, we show that the modified moment map flow admits a natural lift to the real blowup $\hat{\mathcal{F}}_\alpha$.

From now on, $F_t \in \mathcal{F}_\alpha$ is assumed to be a smooth solution of the modified moment map flow, defined for $t \in [t_0, t_1)$, with $t_1 \in (t_0, +\infty]$. The function

$$
r : [t_0, t_1) \to \mathbb{R}
$$

defined by

$$
r(t) = \|F_t\|_{L^2}
$$

satisfies the ODE

$$
\frac{dr^2}{dt} = -8\phi(F_t)
$$

according to Proposition 4.3.2. In particular $r(t)$ is a non increasing function of $t \in [t_0, t_1)$. If fact $r$ must be decreasing unless it is constant. Indeed, if the derivative of $r$ vanishes at some $t_2 \in [t_0, t_1)$, then
\( \phi(F_{t_2}) = 0 \), which implies that \( F_{t_2} \) is a fixed point of the modified moment map flow. By the local uniqueness property of the Cauchy-Lipschitz Theorem 4.4.1, the flow is static on the interval \([t_0, t_1]\). Similarly, \( r > 0 \) unless the flow is static. Indeed, it \( r(t_2) = 0 \) for some \( t_2 \in [t_0, t_1] \), then \( \frac{dr}{dt} \) vanishes at \( t_2 \) and the above argument shows that the flow is static on the interval. Thus we have proved the following lemma:

**Lemma 5.5.1.** The function \( r : [t_0, t_1) \to \mathbb{R} \) is positive and decreasing, unless \( F_t \) is a static solution of the modified moment map flow along the interval.

Given a smooth solution \( F_t \) of the modified moment map flow defined for \( t \in [t_0, t_1) \), we know that \( F_t \) does not vanish, unless \( F_t = 0 \) along the interval. Then, there exists a unique point \( (r(t), G_t) \in \hat{\mathcal{F}}_\alpha \), such that \( p(r(t), G_t) = F_t \), defined by

\[
r(t) = \| F_t \|_{L^2}, \quad G_t = \frac{F_t}{r(t)}.
\]

We are going to see that \( (r, G) \) is solution of an evolution equation defined on the real blowup. First \( 2r \frac{dr}{dt} = -8\phi(F) = -8\phi(rG) = -8r^4\phi(G) \), so that

\[
\frac{dr}{dt} = -4r^3\phi(G). \tag{5.2}
\]

By the chain rule, we have

\[
\frac{\partial F}{\partial t} = \frac{dr}{dt}G + r \frac{\partial G}{\partial t} = -4r^3\phi(G)G + r \frac{\partial G}{\partial t}. \tag{5.3}
\]

Since \( F \) is a flow line, we have

\[
\frac{\partial F}{\partial t} = -\Pi_\alpha \nabla \phi(F) = -r^3\Pi_\alpha \nabla \phi(G). \tag{5.4}
\]

By Equations (5.3) and (5.4), we deduce that

\[
\frac{-r}{4} \frac{\partial G}{\partial t} = 4\phi(G)G - \Pi_\alpha \nabla \phi(G). \tag{5.5}
\]

In conclusion, we have the following proposition

**Proposition 5.5.2.** Let \( (r(t), G_t) \in \hat{\mathcal{F}}_\alpha \) be a path defined for \( t \in [t_0, t_1) \), such that \( F_t = r(t)G_t \) is solution of the modified moment map flow. Then \( (r, G) \) is solution of the system of differential equations

\[
\frac{dr}{dt} = -4r^3\phi(G) \tag{5.6}
\]

\[
\frac{\partial G}{\partial t} = r^2 \left( 4\phi(G)G - \Pi_\alpha \nabla \phi(G) \right) \tag{5.7}
\]

for \( t \in [t_0, t_1) \).
5.6. **Renormalized flow.** Let \((r(t), G_t) \in \hat{F}_\alpha\) be a solution of the lifted flow as above. We introduce a function \(s : [t_0, t_1) \to \mathbb{R}\), solution of the ODE

\[
\frac{ds}{dt} = r^2.
\] (5.8)

If \(F_t\) does not vanish, then \(r > 0\) and \(s\) is strictly increasing. Thus, \(s\) can be used as a reparametrization of the evolution equation. In particular \(r^{-2} = \frac{dt}{ds}\) so that

\[
r^{-2} \frac{\partial G}{\partial t} = \frac{dt}{ds} \frac{\partial G}{\partial t} = \frac{\partial G}{\partial s}.
\]

We deduce the differential system of equations,

\[
\frac{\partial G}{\partial s} = 4\phi(G)G - \Pi_\alpha \nabla \phi(G)
\] (5.9)

and

\[
\frac{dr}{ds} = -4r\phi(G)
\] (5.10)

Equation (5.9), which is the downward gradient flow of the restricted functional \(\phi : S_\alpha \to \mathbb{R}\), is called the **renormalized flow**. The ODE given by Equation (5.10) and Equation (5.8) are used to pass from the renormalized flow to the modified moment map flow and vice versa: let \(G_s\) be a solution of the renormalized flow for \(s\) in some interval \(I\). Then we can solve the ODE (5.10) on the interval \(I\). The general solution has the form \(r(s) = r_0 e^{-4\tau(s)}\), where \(\tau\) is an integral of \(\phi(G_s)\) on \(I\) and \(r_0 \geq 0\), by convention. If \(r_0 = 0\), then \(r = 0\) and we obtain \(F = rG = 0\). If \(r_0 > 0\), we have \(r > 0\) and we can solve the ODE \(\frac{dt}{ds} = \frac{1}{r^2} = \frac{1}{r_0^2} e^{8\tau(s)}\) deduced from Equation (5.8). Hence \(t(s) = \frac{1}{r_0^2} \int e^{8\tau(s)} ds\) is defined on \(I\) and can be used as a reparametrization to obtain a solution \(F_t = rG\) of the modified moment map flow. From our discussion, we deduce the following proposition:

**Proposition 5.6.1.** The space of fixed points of the evolution equations given by the renormalized flow (5.9) and Equation (5.10) consists of pairs \((r, G) \in \hat{F}_\alpha\) such that \(G\) is a soliton of \(S_\alpha\) and \(r \geq 0\). If \(r > 0\) then \(G\) is a non proper soliton. In particular, the only fixed points \((r, G)\) such that \(G\) is a proper soliton belong to the exceptional divisor \(\mathcal{E}\) of the real blowup.

In the spirit of Morse-Bott theory, we show that the downward gradient flow of \(\phi : S_\alpha \to \mathbb{R}\), which is the renormalized flow, has short time existence:

**Theorem 5.6.2.** Let \(\alpha\) be a cohomology class in \(H^1(M, \mathbb{R})\setminus 0\) and \((k, \nu)\) some Hölder parameters with \(k \geq 2\). For every \(G \in \mathbb{F}_\alpha^{k,\nu}\), such that \(\|G\|_{L^2} = 1\), there exists \(\varepsilon > 0\) and a continuous map \(G_s \in \mathbb{F}_\alpha^{k,\nu}\) defined
for \( s \in [-\varepsilon, \varepsilon] \), which is a solution of the renormalized flow and such that \( G_0 = G \). Furthermore, this solution is unique on the interval.

Proof. We notice that the term \( 4\phi(G)G \) is polynomial in the coefficients of \( G \). The proof relies on the Cauchy-Lipschitz theorem applied to Equation (5.9) and follows the same argument as in Theorem 4.4.1. \( \square \)

5.7. Proofs of the main theorems for the smooth case. In this section, we collect our results to prove the main statement of the introduction.

Proof of Theorem A. The canonical Kähler structure and the conjugate hyperKähler structure on a quotient torus is described at §3.1. The resulting hyperKähler structure on the moduli space \( \mathcal{F} \) follows from Proposition 2.9.1 and the hyperKähler moment map is introduced at §3.3. The theorem then follows from Lemma 3.3.2 and Formula (3.3). \( \square \)

Proof of Theorem B. The theorem is a restatement of Proposition 3.3.3. \( \square \)

Proof of Theorem C. The fixed points of the modified moment map flow are the zeroes of \( \phi \) by Proposition 4.3.1. The \( L^2 \) decay properties is a consequence of Proposition 4.3.2 and Theorem 4.4.1 shows the short time existence of the flow. \( \square \)

Proof of Theorem D. This is a restatement of Theorem 5.6.2. \( \square \)

6. Rigidity and solitons

The goal of this section is to give a proof of Theorem E and, more specifically, the stronger result given at Theorem 6.1.1.

6.1. Main result. Given a symplectic cohomology class \( \alpha \), we denote by

\[
\mathcal{I}^{k,\nu}_\alpha \subset \mathcal{I}^{k,\nu}_\alpha \setminus 0
\]

the space of non zero solitons with Hölder regularity \((k, \nu)\), for some \( k \geq 2 \). We denote by define \( \mathcal{I}^{k,\nu}_{\alpha,p} \) the subspace of proper solitons and by \( \mathcal{I}^{k,\nu}_{\alpha,np} \) the subspace of non proper solitons of \( \mathcal{I}^{k,\nu}_\alpha \). Then we have the obvious partition

\[
\mathcal{I}^{k,\nu}_\alpha = \mathcal{I}^{k,\nu}_{\alpha,p} \sqcup \mathcal{I}^{k,\nu}_{\alpha,np}.
\]

It turns out that the components of the above decomposition are also topologically separated for \( k \geq 3 \), in the sense of the following theorem:

Theorem 6.1.1. Let \( \alpha \in H^1(M, \tilde{V}) \) be a symplectic cohomology class and \( k \geq 3 \). Then, the spaces of solitons \( \mathcal{I}^{k,\nu}_{\alpha,p} \) and \( \mathcal{I}^{k,\nu}_{\alpha,np} \) are open and closed subspaces of \( \mathcal{I}^{k,\nu}_\alpha \), endowed the the Hölder \( C^{k,\nu} \)-topology.

In particular, if \( \psi : X \to \mathcal{I}^{k,\nu}_\alpha \) is a continuous map from a connected topological space \( X \) such that \( \psi(X) \cap \mathcal{I}^{k,\nu}_{\alpha,np} \neq \emptyset \), then \( \psi(X) \subset \mathcal{I}^{k,\nu}_{\alpha,np} \).
The proof of Theorem 6.1.1 is postponed at the end of this section. We notice that the result implies one of our main theorems:

**Proof of Theorem E.** The Whitney topology is stronger than the Hölder topology and the result is an immediate consequence of Theorem 6.1.1. □

6.2. **Linearized soliton equation.** By Definition 5.3.1, a soliton \( F \in \mathcal{F}_\alpha \) is a solution of the equation \( E(F) = 0 \), where \( E : \mathcal{F}_\alpha \to \mathcal{F}_\alpha \) is the functional
\[
E(F) = 4\phi(F)F - \|F\|_{L^2}^2\Pi_\alpha \nabla \phi(F).
\]
Alternatively, we may consider the dual functional
\[
e : \mathcal{F}_\alpha \to L(\mathcal{F}_\alpha, \mathbb{R})
\]
with values in the dual of \( \mathcal{F}_\alpha \), given by
\[
e(F) \cdot \dot{F} = \langle \langle E(F), \dot{F} \rangle \rangle
\]
for every \( \dot{F} \in \mathcal{F}_\alpha \). By definition
\[
e(F) \cdot \dot{F} = 4\phi(F)\langle \langle F, \dot{F} \rangle \rangle - \|F\|_{L^2}^2 \mathcal{D}\phi \cdot \dot{F}.
\]
In the particular case where \( f : M \to M \) is a symplectomorphism of \((M, \omega_M)\) with \([\mathcal{D}f] = \alpha\), we have \( \phi(\mathcal{D}f) = 0 \) and \( F = \mathcal{D}f \) is a critical point of \( \phi : \mathcal{F}_\alpha \to \mathbb{R} \). Since \( \|f_*\|_{L^2} = \|\mathcal{D}f\|_{L^2} \), the differential of \( e \) is given by
\[
De|_{\mathcal{D}f} \cdot (\dot{F}_1, \dot{F}_2) = -\|f_*\|_{L^2}^2 \mathcal{D}^2\phi|_{\mathcal{D}f} \cdot (\dot{F}_1, \dot{F}_2)
\]
and \( De|_{\mathcal{D}f} \) agrees with the Hessian of \( \phi \) up to a constant factor.

**Lemma 6.2.1.** Let \( F \in \mathcal{F}_\alpha \) be a non proper soliton. Then, the Hessian of \( \phi \) at \( F \) is given by the formula:
\[
D^2\phi|_F \cdot (\dot{F}_1, \dot{F}_2) = \int_M \left( \sum \bullet g(\mathcal{R}\bullet F, \dot{F}_1)g(\mathcal{R}\bullet F, \dot{F}_2) \right) \text{vol}_M \quad (6.1)
\]
for every \( \dot{F}_1, \dot{F}_2 \in \mathcal{F}_\alpha \).

**Proof.** By Lemma 4.2.2 and Equation (4.6)
\[
D\phi|_F \cdot F_1 = \sum \bullet \langle \langle W_\bullet (F), \dot{F}_1 \rangle \rangle.
\]
where \( W_\bullet (F) = -\mu_\bullet (F)\mathcal{R}_\bullet F \) by Equation (4.7). Thus
\[
D W_\bullet |_F \cdot \dot{F}_2 = -(D\mu_\bullet |_F \cdot \dot{F}_2)\mathcal{R}_\bullet F,
\]
since \( \mu_\bullet (F) = 0 \). By Equation (2.19), we deduce that
\[
D W_\bullet |_F \cdot \dot{F}_2 = g(\mathcal{R}_\bullet F, \dot{F}_2)\mathcal{R}_\bullet F,
\]
which implies
\[ D^2 \phi|_F (\dot{F}_1, \dot{F}_2) = \sum \langle (R_* F_1 \cdot F_2) R_* F, \dot{F}_1 \rangle \]
\[ = \sum \langle (R_* F_1 \cdot F_2), g(R_* F, \dot{F}_1) \rangle \]
and the lemma follows. \(\square\)

The following corollaries are immediate:

**Corollary 6.2.2.** Let \( F \) be a non proper soliton in \( \mathcal{F}_\alpha \). The Hessian of \( \phi : \mathcal{F}_\alpha \to \mathbb{R} \) at \( F \) is a non negative symmetric bilinear form. Moreover its kernel consists of vectors \( \dot{F} \in \mathcal{F}_\alpha \) such that \( D\mu|_F \cdot \dot{F} = 0 \).

**Corollary 6.2.3.** Let \( f \) be a symplectomorphism of \((M,\omega_M)\) and \( F = Df \in \mathcal{F}_\alpha \), where \( \alpha = [F] \). Then, the differential of the soliton equation \( DE|_F \) is a non positive selfadjoint operator, whose kernel (and cokernel) is identified to \( \ker D\mu|_F \subset \mathcal{F}_\alpha \).

We continue our computation at \( F = Df \), where \( f \in \text{Symp}_\alpha(M,\omega_M) \): we can write Formula (6.1) as
\[ D^2 \phi|_F (\dot{F}_1, \dot{F}_2) = \int_M \sum \langle (D_\mu_\ast|_F \cdot \dot{F}_1)(D_\mu_\ast|_F \cdot \dot{F}_2) \rangle \text{vol}_M \]  
(6.2)
and, using the natural inner product on \( \mathbb{R}^3 \), the above formula can be expressed as
\[ D^2 \phi|_F (\dot{F}_1, \dot{F}_2) = \langle D\mu|_F \cdot \dot{F}_1, D\mu|_F \cdot \dot{F}_2 \rangle \].  
(6.3)

Let \( v \) be a tangent vector field along \( f : M \to M \). Then, there exists a smooth family of maps \( f_t : M \to M \) defined for \( t \) in a neighborhood of 0, such that \( f_0 = f \) and \( \frac{\partial f_t}{\partial t}|_{t=0} = v \). By definition \( Df_t = \rho \circ (f_t)_\ast \), hence
\[ \frac{\partial}{\partial t} Df_t \bigg|_{t=0} = \frac{\partial}{\partial t} (\rho \circ (f_t)_\ast) \bigg|_{t=0} = d\rho \circ v_\ast = d(\rho \circ v) = d\eta \]
where \( \eta = \rho \circ v \).

We will use the notations summarized in the commutative diagram below: if \( f \) is a diffeomorphism, we may write \( v = f_* u \), for some vector field \( u \) on \( M \). We put \( \eta = \rho \circ v \) and obtain the commutative diagram:
\[ \begin{array}{ccc}
TM & \xrightarrow{f_*} & TM \\
\downarrow \pi & & \downarrow \pi \\
M & \xrightarrow{f} & M
\end{array} \]
\[ \begin{array}{ccc}
V & \xrightarrow{\rho} & \tilde{V} \\
\downarrow \pi & & \downarrow \pi \\
\tilde{M} & \xrightarrow{\eta} & \tilde{M}
\end{array} \]  
(6.4)
Hence, if \( F = \mathcal{G} f \) and \( F_i = d\eta_i \), where \( \eta_i = \rho \circ v_i \) for some vector fields \( v_i \) along \( f \), then
\[
D^2 \phi_{\mathcal{G} f} \cdot (d\eta_1, d\eta_2) = \langle D\mu_{\mathcal{G} f} \cdot d\eta_1, D\mu_{\mathcal{G} f} \cdot d\eta_2 \rangle. \tag{6.5}
\]
According to Remark 3.3.1, the hyperKähler moment map \( \mu \) can be considered as a map \( \tilde{\mu} \) with values in anti-selfdual forms. This point of view makes computations evident, as in the next lemma :

**Lemma 6.2.4.** Let \( f \) be a diffeomorphism of \( M \) and \( v \) be a vector field along \( f \). Then for every vector field \( v \) along \( f \), we have
\[
D \tilde{\mu}_{\mathcal{G} f} \cdot d\eta = \sqrt{2} d^{-}(\iota_u f^* \omega_M)
\]
where \( u \) is the vector field on \( M \) defined by \( v = f_* u \) and \( \eta = \rho \circ v \).

**Proof.** Let \( f_t \) be a smooth family of maps defined for \( t \) in some neighborhood of \( 0 \in \mathbb{R} \), such that \( f_0 = f \) and \( \frac{df}{dt} \big|_{t=0} = v \). By definition
\[
\tilde{\mu}(\mathcal{G} f_t) = \sqrt{2} (f_t^* \omega_M)^{-}
\]
and its variation at \( t = 0 \) is given by
\[
D \tilde{\mu}_{\mathcal{G} f} \cdot d\eta = \sqrt{2} \frac{\partial}{\partial t} \bigg|_{t=0} (f_t^* \omega_M)^{-}
\]
If we write \( v = f_* u \), where \( u \) is a vector field on \( M \), then the RHS can be expressed as
\[
(\mathcal{L}_u f^* \omega_M)^{-} = d^{-}(\iota_u f^* \omega_M) + (\iota_u df^* \omega_M)^{-},
\]
where \( \mathcal{L}_u \) is the Lie derivative, and we obtain the lemma since \( f^* \omega_M \) is closed. \( \square \)

**Lemma 6.2.5.** Let \( v_1, v_2 \) be two tangent vector fields along a symplectomorphism \( f : M \to M \). Using the notation conventions of Diagram (6.4), we have
\[
D^2 \phi_{\mathcal{G} f}(d\eta_1, d\eta_2) = 2 \langle d^{-}(u_1^b), d^{-}u_2^b \rangle
\]
where \( b : TM \to T^* M \) is symplectic musical isomorphism, defined by \( u^b = \iota_u \omega_M \).

**Proof.** We know that \( \tilde{\mu} \) and \( \mu \) have the same \( L^2 \)-norms, by definition (cf. Remark 3.3.1), and the result follows from Lemma 6.2.4 and Equation (6.5). \( \square \)

If \( f \) is a diffeomorphism, there is a correspondence between vector fields \( u \) on \( M \) and vector field \( v \) along \( f \) via the equation \( f_* u = v \). We denote the inverse of this map
\[
P_f = (f^{-1})_* = (f_*)^{-1},
\]
so that \( u = P_f v \).
Corollary 6.2.6. Let \(v_1\) and \(v_2\) be two vector fields along a symplectomorphism \(f \in \mathcal{M}_\alpha\). Then, we have
\[
D\varepsilon|_{\mathcal{G}f \cdot (d\eta_1, d\eta_2)} = -2\|f_*\|^2_L \langle d^- (P_f v_1)^\flat, (P_f v_2)^\flat \rangle
\]
where \(\eta_i = \rho \circ v_i\).

Corollary 6.2.7. Let \(v\) be a vector field along a symplectomorphism \(f \in \text{Symp}_\alpha(M, \omega_M)\). Then, we have
\[
d^* D\varepsilon|_{\mathcal{G}f \cdot d\eta} = -2\|f_*\|^2_L \langle d^+ (P_f v_1)^\flat, (P_f v_2)^\flat \rangle
\]
where \(P_f = (f_*)^{-1}\) and \(\beta : TM \to T^*M\) and \(\sharp : T^*M \to TM\) denote the symplectic duality musical operators.

Proof. By definition of \(d^*\), we have
\[
\langle d^* D\varepsilon|_{\mathcal{G}f \cdot d\eta_1, d\eta_2} \rangle = \langle (D\varepsilon|_{\mathcal{G}f \cdot d\eta_1, d\eta_2}) \rangle = D\varepsilon|_{\mathcal{G}f (d\eta_1, d\eta_2)}.
\]
By Corollary 6.2.6, we deduce
\[
\langle d^* D\varepsilon|_{\mathcal{G}f \cdot d\eta_1, d\eta_2} \rangle = -2\|f_*\|^2_L \langle d^+ (P_f v_1)^\flat, (P_f v_2)^\flat \rangle
\]
Now \(\langle \beta, \nu^\flat \rangle = \langle \beta^\sharp, \nu \rangle\) and \(\langle v_1, v_2 \rangle = \langle \rho v_1, \rho v_2 \rangle\) therefore,
\[
\langle d^* D\varepsilon|_{\mathcal{G}f \cdot d\eta_1, d\eta_2} \rangle = -2\|f_*\|^2_L \langle (d^+ (P_f v_1)^\flat, (P_f v_2)^\flat) \rangle
= -2\|f_*\|^2_L \langle \rho \cdot P_f^* \cdot (d^+ (P_f v_1)^\flat, (P_f v_2)^\flat) \rangle
\]
and the corollary follows. \qed

6.3. Rigidity of non proper solitons in the case of maps. A symplectomorphisms \(f \in \text{Symp}_\alpha(M, \omega_M)\) provides a non proper soliton \(E \circ \mathcal{D} f \in \mathcal{F}_\alpha \setminus 0\). By definition, this is a solution of the equation \(E \circ \mathcal{D} f = 0\). The symplectic deformations of the map \(f\) provide an infinite dimensional family of solutions of the equation \(E \circ \mathcal{D} f = 0\).

We are going to prove that they are locally the only ones:

Theorem 6.3.1. Given some Hölder space parameters \((k, \nu)\) with \(k \geq 4\) and a symplectomorphism \(f \in \mathcal{M}_\alpha^{k, \nu}\), there exists \(\varepsilon > 0\) with the following property : for every \(h \in \mathcal{M}_\alpha^{k, \nu}\) such that \(\mathcal{D} h\) is a soliton and \(\|f - h\|_{k, \nu} \leq \varepsilon\), then \(h\) is a symplectomorphism.

Proof. The result is an immediate consequence of Lemma 6.3.2 and Theorem 6.3.3. \qed

For technical reasons, it is convenient to replace the soliton equation for maps \(E \circ \mathcal{D} f = 0\) with the weaker equation
\[
\Psi(f) = 0,
\]
where \(\Psi\) is the functional
\[
\Psi : \mathcal{M}_\alpha \to \Omega^0(M, \overrightarrow{V})
\]
\[f \mapsto d^* E(\mathcal{D} f).\]
Obviously, $E(\mathcal{D}f) = 0$ implies $\Psi(f) = 0$. Conversely, $\Psi(f) = 0$ implies that $E(\mathcal{D}f)$ is orthogonal to closed forms. Since $E(\mathcal{D}f)$ is closed, by definition of $E$, it follows that $E(\mathcal{D}f)$ is harmonic and we have the following result:

**Lemma 6.3.2.** The solutions $f \in \mathcal{M}_\alpha$ of the equation $\Psi(f) = 0$ are the maps such that $E(\mathcal{D}f)$ is a harmonic form.

In the rest of this section, we prove a rigidity result for $\Psi$. Loosely stated, near a symplectic map $f$, every zero of $\Psi$ is also a symplectic map:

**Theorem 6.3.3.** Given some Hölder space parameters $(k, \nu)$ with $k \geq 4$ and a symplectomorphism $f \in \mathcal{M}_\alpha^{k,\nu}$, there exists $\varepsilon > 0$ with the following property: for every $h \in \mathcal{M}_\alpha^{k,\nu}$ such that $\Psi(h) = 0$ and $\|f - h\|_{k,\nu} \leq \varepsilon$, then $h$ is a symplectomorphism.

*Proof.* The theorem is a restatement of Corollary 6.6.3. □

**Lemma 6.3.4.** The map $\Psi$ is given by

$$\Psi(h) = 4\|(h^*\omega_M)^-\|_{L^2}^2 \mathcal{D}h + \|h_*\|_{L^2}^2 \mathcal{D}^* \nabla(\mathcal{D}h)$$

*Proof.* The formula is obtained from the definition of $E(F)$, replacing $F$ with $\mathcal{D}h$ and using the fact that $\|\mathcal{D}h\|_{L^2} = \|h_*\|_{L^2}$. We also notice that $\mathcal{D}P_\alpha F = \mathcal{D}F$ for every $F \in \mathcal{F}$. Indeed, the orthogonal projection $P_\alpha$ operates by adding some harmonic and coclosed terms, which belong to the kernel of $\mathcal{D}^*$. Finally, we have $\phi(\mathcal{D}h) = \frac{1}{2}\|\hat{\mu}(\mathcal{D}h)\|_{L^2}^2 = \| (h^*\omega_M)^- \|_{L^2}^2$ which proves the formula. □

**Lemma 6.3.5.** The variation of $\Psi$ at a symplectomorphism $f$ is given by

$$D\Psi|_f \cdot v = -2\|\mathcal{D}f\|_{L^2}^2 \rho \cdot P_f^* \cdot (\mathcal{D}^* \mathcal{D}(P_f \cdot v))^1$$

where $P_f^*$ is the formal adjoint of $P_f = (f_*)^{-1}$ and $v$ is a vector field along $f$. The differential is formally selfadjoint in the sense that

$$\langle \langle D\Psi|_f \cdot v_1, \rho \cdot v_2 \rangle \rangle = \langle \langle \rho \cdot v_1, D\Psi|_f \cdot v_2 \rangle \rangle.$$ 

Furthermore, $D\Psi|_f$ is non positive, in the sense that

$$\langle \langle D\Psi|_f \cdot v, \rho \cdot v \rangle \rangle = -2\|\mathcal{D}f\|_{L^2}^2 \|\mathcal{D}(P_f v)^b\|_{L^2}^2,$$

and the kernel of $D\Psi|_f$ is identified to symplectic vector fields along $f$.

*Proof.* By definition $D\Psi|_f \cdot v$ is the variation of $\mathcal{D}(P_f \cdot d\eta)$, where $\eta = \rho \circ v$ and the formula follows from Corollary 6.2.7. Now,

$$\langle \langle \rho \cdot P_f^* \cdot (\mathcal{D}^* \mathcal{D}(P_f v)^b)^2, \rho \cdot v_2 \rangle \rangle = \langle \langle P_f^* \cdot (\mathcal{D}^* \mathcal{D}(P_f v)^b)^2, P_f v_2 \rangle \rangle$$

$$= \langle \langle (\mathcal{D}^* \mathcal{D}(P_f v_1)^b)^2, P_f v_2 \rangle \rangle$$

$$= \langle \langle \mathcal{D}^* \mathcal{D}(P_f v_1)^b, (P_f v_2)^b \rangle \rangle$$

$$= \langle \langle \mathcal{D}^* (P_f v_1)^b, d(P_f v_2)^b \rangle \rangle$$

$$= \langle \langle d^* (P_f v_1)^b, d^* (P_f v_2)^b \rangle \rangle$$
so that
\[ \langle D\Psi|_f \cdot v_1, \rho \cdot v_2 \rangle = \langle \rho \cdot v_1, D\Psi|_f \cdot v_2 \rangle, \]
and we conclude that \( D\Psi|_f \) is formally selfadjoint. In the case where \( v = v_1 = v_2 \), the above computation proves the last identity of the lemma. \( \Box \)

6.4. **Weinstein chart.** The Weinstein charts provide nice local coordinates on the moduli space \( \mathcal{M} \). They are particularly well suited for applying the implicit function theorem in the proof of Theorem 6.3.3. We recall the basic steps for the construction of a Weinstein chart (for more details see [3, 20]):

1. The product \( N = M \times M \) is endowed with the product symplectic form
   \[ \omega_N = p_1^*\omega_M - p_2^*\omega_M, \]
   where \( p_i : N \to M \) are the canonical projections on the \( i \)-th factor of \( N = M \times M \). It is well known that \( f \) is a symplectomorphism of \( (M, \omega_M) \) if, and only if, its graph
   \[ G_f = \{(x, f(x)), x \in M\} \subset N \]
is a Lagrangian submanifold of \( (N, \omega_N) \).

2. The total space \( T^*M \) of the contangent bundle is endowed with the canonical symplectic form
   \[ \omega_{T^*} = d\Lambda, \]
   where \( \Lambda \) is the Liouville form on \( T^*M \), defined by
   \[ \Lambda(w) = \tilde{\pi}(w)(\pi_*(w)) \]
   where \( \pi : T^*M \to M \) is the canonical projection, \( \pi_* : TT^*M \to TM \) is its tangent map and \( \tilde{\pi} : TT^*M \to T^*M \) is the canonical projection of the tangent bundle of \( T^*M \).

3. If \( f \) is a symplectomorphism, the Lagrangian tubular neighborhood theorem shows that there exists an open neighborhood \( U \subset N \) of \( G_f \) and an open neighborhood \( \mathcal{V} \subset T^*M \) of the 0-section \( \beta_0 : M \to T^*M \), together with a diffeomorphism
   \[ \theta : \mathcal{V} \to U, \]
such that
   \[ \theta^*\omega_N = \omega_{T^*} \quad \text{and} \quad \theta \circ \beta_0(x) = (x, f(x)) \forall x \in M. \]

4. In particular, any differential 1-form \( \beta \) on \( M \), such that \( \beta(M) \subset \mathcal{V} \), defines an embedding \( \theta \circ \beta : M \to N \), whose image, denoted \( G_{f,\beta} \), is a deformation of \( G_f \). Another classical result is that \( G_{f,\beta} \) is Lagrangian if, and only if, \( \beta \) is a closed differential 1-form.
(5) By construction $p_1 \circ \theta \circ \beta_0 = \text{id}_M$. It follows that $\varphi_\beta = p_1 \circ \theta \circ \beta$ is a diffeomorphism of $M$ as well, provided the differential 1-form $\beta$ has sufficiently small $C^1$-norm. Then we can define

$$f_\beta = p_2 \circ \theta \circ \beta \circ \varphi_\beta^{-1},$$

so that $G_{f,\beta}$ is the graph of $f_\beta : M \to M$. Notice that, by definition $f_{\beta_0} = f$.

(6) The Weinstein chart at $f$ is given by

$$W(\beta) = f_\beta,$$

and is defined on a sufficiently small $C^1$-neighborhood of the 0-section $\beta_0$. By construction, $f_\beta$ is homotopic to $f$, hence $W(\beta) \in \mathcal{M}_\alpha$, where $\alpha$ is the symplectic class $[\mathcal{D} f]$. In the sequel is more convenient to use the $(k, \nu)$-Hölder norms, for some $k \geq 2$ and we can summarize the construction in Proposition 6.4.1.

**Proposition 6.4.1.** Let $\alpha$ be a symplectic class and $(k, \nu)$ some Hölder parameters with $k \geq 2$. For every symplectic map $f \in \mathcal{M}_\alpha^{k,\nu}$, there exists an open neighborhood $\mathcal{U}$ of $\beta_0 = 0$ in the space $\Omega^1(M)_{k,\nu}$ of differential 1-forms with Hölder regularity $C^{k,\nu}$ and an open neighborhood $\mathcal{V}$ of $f$ in the space of Hölder maps $\mathcal{M}_\alpha^{k,\nu}$, such that the Weinstein map at $f$

$$W : \mathcal{U} \to \mathcal{V}$$

is a local diffeomorphims. Furthermore, the Weinstein chart restricts as a diffeomorphism

$$W : \mathcal{U}_c \to \mathcal{V}_\omega$$

where $\mathcal{U}_c \subset \mathcal{U}$ is the subspace of closed forms and $\mathcal{V}_\omega \subset \mathcal{V}$ is the subspace of symplectic maps with respect to $\omega_M$.

We can summarize Proposition 6.4.1 by a diagram,

$$\begin{array}{c}
\mathcal{U} \\
\downarrow \quad W \\
\mathcal{U}_c \\
\downarrow \\
\mathcal{V} \\
\downarrow \quad W \\
\mathcal{V}_\omega
\end{array} (6.6)$$

where vertical arrows are the canonical inclusion maps and horizontal maps are diffeomorphisms.

6.5. **Differential of the Weinstein map.** In this section we consider the Weinstein chart at a symplectic map $f$ and we compute its differential at $\beta_0 = 0$. Let $\beta$ be a smooth differential 1-form on $M$ and $\beta_t = t \beta$, with $\varphi_t = \varphi_{\beta_t} = p_1 \circ \theta \circ \beta_t$. Then $\varphi_t$ is a 1-parameter family of diffeomorphisms, for $t$ sufficiently small, such that $\varphi_0 = \text{id}_M$. We put $f_t = f_{\beta_t} = W(\beta_t)$ and we compute

$$\left. \frac{\partial \varphi_t}{\partial t} \right|_{t=0} = (p_1)_* \cdot \theta_* \cdot \beta$$
where \( \dot{\beta} \) is understood as a vertical tangent vector field along the zero section of \( T^*M \to M \). Hence

\[
\frac{\partial \varphi_i^{-1}}{\partial t} \bigg|_{t=0} = -(p_1)_* \cdot \theta_* \cdot \dot{\beta}
\]

The vector field \( \theta_* \cdot \dot{\beta} \) along \( G_f \) is neither vertical nor horizontal in \( N = M \times M \). However

\[
\frac{\partial f_i}{\partial t} \bigg|_{t=0} = \frac{\partial}{\partial t} \theta \circ \beta \circ \varphi_i^{-1} \bigg|_{t=0} = \theta_* \cdot \dot{\beta} - \theta_* \cdot (\beta_0)_* \cdot (p_1)_* \cdot \theta_* \cdot \dot{\beta}
\]

is parallel to the fibers of \( p_2 : N \to M \). Indeed \( (p_1)_* \cdot \theta_* \cdot (\beta_0)_* = \text{id} \), so that the first projection of the above vector vanishes. In conclusion

\[
\frac{\partial W(\beta_i)}{\partial t} \bigg|_{t=0} = (p_2)_* \cdot \theta_* \cdot \dot{\beta}
\]

which is to say

\[
DW|_{\beta_0} = (p_2)_* \circ \theta_*.
\]

We deduce the following lemma

**Lemma 6.5.1.** The 1-form \( \dot{\beta} \) is closed if, and only if, \( v = (p_2)_* \cdot \theta_* \cdot \dot{\beta} \) is a symplectic vector field along \( f \), in other words

\[
d\dot{\beta} = 0 \iff dv \cdot \omega_M = 0,
\]

where \( v \) is the vector field defined by \( f_* u = v \).

**Proof.** The Weinstein chart is a local diffeomorphism that maps closed differential 1-forms to symplectic maps and vice-versa, according to Proposition 6.4.1. The lemma follows from the computation of \( DW|_{\beta_0} \) given by Formula (6.7).

\[\square\]

### 6.6. Implicit function theorem.

The map \( \Psi \), defined in the smooth setting at §6.3, admits a canonical extension to Hölder spaces

\[
\Psi : \mathcal{V} \to \Omega^0(M, \vec{V})_{k-2,\nu},
\]

where \( \mathcal{V} \) is an open neighborhood of \( f \) in \( \mathcal{M}^{k,\nu}_{k} \), chosen according to Proposition 6.4.1, and \( \Omega^0(M, \vec{V})_{k-2,\nu} \) is the space of \( \vec{V} \)-valued functions on \( M \) with Hölder regularity \( (k-2, \nu) \), for some \( k \geq 2 \).

By definition, \( \Psi \) vanishes along the space of symplectomorphism \( \mathcal{Y}_\omega \). Our goal is to prove that there are no other solutions of the equation \( \Psi = 0 \) in \( \mathcal{V} \), provided it is a sufficiently small neighborhood of \( f \) in \( C^{k,\nu} \)-norm.

It is convenient to work on an open set \( \mathcal{U} \subset \Omega^1(M)_{k,\nu} \), using the Weinstein chart \( W : \mathcal{U} \to \mathcal{V} \), where \( \mathcal{U} \) is a neighborhood of \( \beta_0 = 0 \), as in Proposition 6.4.1. The we introduce the reparametrized map

\[
\Phi = \Psi \circ W : \mathcal{U} \to \Omega^0(M, \vec{V})_{k-2,\nu}
\]

and our problem is equivalent to ask whether the points of \( \mathcal{U} \) are the only zeroes of \( \Phi \) in \( \mathcal{U} \).
By Lemma 6.3.5 and Lemma 6.5.1, the kernel of $D\Phi|_{\beta_0}$ is the space of closed differential 1-forms $\mathcal{W}^{k,\nu}_c$ with regularity $C^{k,\nu}$. In particular, Hodge theory provides an orthogonal splitting

$$\Omega^1(M)_{k,\nu} = \mathcal{W}^{k,\nu}_c \oplus \mathcal{W}^{k,\nu}_d$$

where $\mathcal{W}^{k,\nu}_d$ is the space of coexact forms with regularity $C^{k,\nu}$.

We have a similar splitting

$$\Omega^1(M)_{k-2,\nu} = \mathcal{W}^{k-2,\nu}_c \oplus \mathcal{W}^{k-2,\nu}_d$$

for Hölder regularity $C^{k-2,\nu}$, under the assumption that $k \geq 4$. The symplectic duality $\sharp : \Omega^1(M) \to \Gamma(TM)$ composed with $\mathcal{D} f : \Gamma(TM) \to \Omega^0(M, \overrightarrow{V})$ provides an isomorphism

$$\mathcal{D} f \circ \sharp : \Omega^1(M)_{k-2,\nu} \to \Omega^0(M, \overrightarrow{V})_{k-2,\nu}.$$ 

By Lemma 6.3.5, the image of the factor $\hat{\mathcal{W}}^{k-2,\nu}_c = \mathcal{D} f \circ \sharp(\mathcal{W}^{k-2,\nu}_c)$ is the cokernel of $D\Phi|_{\beta_0}$, and we have a splitting

$$\Omega^0(M, \overrightarrow{V})_{k-2,\nu} = \hat{\mathcal{W}}^{k-2,\nu}_c \oplus \hat{\mathcal{W}}^{k-2,\nu}_d$$

where

$$\hat{\mathcal{W}}^{k-2,\nu}_d = \mathcal{D} f \circ \sharp(\mathcal{W}^{k-2,\nu}_d)$$

Finally Hodge orthogonal projection

$$\Pi_d : \Omega^1(M)_{k-2,\nu} \to \mathcal{W}^{k-2,\nu}_d$$

induces a projector

$$\Pi_d : \Omega^0(M, \overrightarrow{V})_{k-2,\nu} \to \mathcal{W}^{k-2,\nu}_d$$

via the isomorphism $\mathcal{D} f \circ \sharp$. and we can consider the projected map

$$\hat{\Phi} : \mathcal{W} \to \mathcal{W}^{k-2,\nu}_d$$

defined by

$$\hat{\Phi} = \Pi_d \circ \Phi.$$ 

By Remark 4.4.3, the projector $\Pi_d$ is continuous so that $\hat{\Phi}$ is a differentiable map. By construction $\mathcal{D} \hat{\Phi}|_{\beta_0}$ is surjective and its kernel is $\mathcal{W}^{k,\nu}_c$. Therefore the operator

$$\mathcal{D} \hat{\Phi}|_{\beta_0} : \mathcal{W}^{k,\nu}_d \to \mathcal{W}^{k-2,\nu}_d$$

is an isomorphism.

By the implicit function theorem, we deduce the following results

**Theorem 6.6.1.** Let $(k, \nu)$ be some Hölder parameters with $k \geq 4$. If $\mathcal{W}$ is a sufficiently small neighborhood of $\beta_0 = 0$ in $\Omega^1(M)_{k,\nu}$, then the restriction of $\hat{\Phi}$ to every affine subspace parallel to $\mathcal{W}^{k,\nu}_d$ is an embedding. More precisely for every closed form $\beta \in \mathcal{W}_c$,

$$\hat{\Phi} : \mathcal{W} \cap (\beta + \mathcal{W}^{k,\nu}_d) \to \mathcal{W}^{k-2,\nu}_d.$$
is an embedding. In particular $\beta$ is the only zero of $\hat{\Phi}$ in this affine subspace.

**Corollary 6.6.2.** With the assumptions of Theorem 6.6.1, the map $\Phi$ defined at (6.8) satisfies

$$\Phi^{-1}(0) = \mathcal{Z}_c.$$  

*Proof.* If $\beta$ is a zero of $\Phi$, it is also a zero of the projected map $\hat{\Phi}$ and the result follows from Theorem 6.6.1. $\square$

**Corollary 6.6.3.** With the assumption of Theorem 6.6.1, the map $\Psi$ defined on $\mathcal{V}$ satisfies

$$\Psi^{-1}(0) = \mathcal{V}_\omega.$$  

Furthermore, if $G \in \mathcal{F}_k^{k-1,\nu}$ is a soliton of the form $G = D h$, with $h \in \mathcal{V}$, then $h$ is a symplectomorphism.

*Proof of Theorem 6.1.1.* The space $\mathcal{S}_k,\nu,\alpha,\text{np}$ is the vanishing locus of the restricted energy functional $\phi : \mathcal{S}_k,\nu,\alpha,\text{np} \to \mathbb{R}$. This functional is continuous, hence $\mathcal{S}_k,\nu,\alpha,\text{np}$ is a closed subspace of $\mathcal{S}_k,\nu,\alpha$. We now show that the space $\mathcal{S}_k,\nu,\alpha,\text{np}$ is open in the space of non zero solitons. Let $F$ be a non proper soliton of $\mathcal{F}_k^{k,\nu} \setminus 0$. According to Lemma 5.3.4, the cohomology class $[F]$ is non vanishing and we may assume that $[F] = \alpha$, up to rescaling by a constant factor. Thus, there exists a symplectic map $f \in \mathcal{M}_k^{k+1,\nu}$ such that $F = Dh$. More precisely, we can define $f = \chi(F)$, where $\chi$ is the integral defined by (2.10). The map $\chi$ was defined in the smooth case, however the construction extends in an obvious way as a continuous map:

$$\chi : \{ G \in \mathcal{F}_k^{k,\nu}, [G] = \alpha \} \to \mathcal{M}_k^{k+1,\nu}.$$  

For $k \geq 3$, we choose an open neighborhood $\mathcal{V}$ of $f$ in $\mathcal{M}_k^{k+1,\nu}$, according to Corollary 6.6.3. By definition, if $h \in \mathcal{V}$ is such that $G = Dh$ is a soliton, then $h$ is symplectic. By continuity, $\chi^{-1}(\mathcal{V})$ is an open neighborhood of $F$ in the source space of $\chi$ and the rescaling map

$$q : \mathcal{F}_k^{k,\nu} \to \{ G \in \mathcal{F}_k^{k,\nu}, [G] = \alpha \}$$  

$$G \mapsto \frac{\alpha}{[G]} G$$  

is well defined and continuous in a neighborhood of $F$. We consider the neighborhood of $F$ given by $\mathcal{V}' = (\chi \circ q)^{-1}(\mathcal{V}) \subset \mathcal{F}_k^{k,\nu}$. If $F' \in \mathcal{V}'$ is a soliton, then $G = q(F')$ is also a soliton, and $h = \chi(G) \in \mathcal{V}$ is such that $G = Dh$. By Corollary 6.6.3, this implies that $h$ is a symplectic map, which shows that $G$ is not a proper soliton. It follows that $F'$ is not proper as well. In conclusion, every soliton of $\mathcal{V}'$ is not proper. This proves that $\mathcal{F}_k^{k,\nu}$ is open in $\mathcal{F}_k^{k,\nu}$.

Finally, $\mathcal{F}_k^{k,\nu}$ is the complement of $\mathcal{F}_k^{k,\nu}$, in the space $\mathcal{F}_k^{k,\nu}$. It follows that $\mathcal{F}_k^{k,\nu}$ is open and closed in $\mathcal{F}_k^{k,\nu}$. The last statement of the theorem is classical, by the connexity of $X$. $\square$
7. Polyhedral symplectic geometry

In this section, we introduce polyhedral analogues of all the geometrical objects introduced in the smooth setting, in the first part of this work. The constructions stem from a quotient torus $M = V/\Gamma$, as in §3, where the underlying vector space $\bar{V}$ is equipped with a linear isomorphism with the space of quaternions $\mathbb{H}$. In §3, we saw that the quotient construction various structures on $M$, including a flat Riemannian metric $g_M$ and several integrable compatible almost complex structures, $i, I, J$ and $K$ on $M$, with Kähler forms $\omega_M, \hat{\omega}_I, \hat{\omega}_J$ and $\hat{\omega}_K$. We will continue to use the same background geometry on the quotient torus $M$, as in the smooth setting.

7.1. Triangulations and polyhedral maps. In the next paragraphs, we recall some standard definitions and introduce some basic concepts related to polyhedral geometry.

A $k$-simplex, contained in some ambient affine space, is the convex hull of $k+1$ affinely independent points of the affine space. A simplicial complex $\mathcal{K}$ is a collection of simplices contained in some fixed affine ambient space. The collection must be stable by intersection and by passing to subfaces.

Given a simplex $\sigma$, we say that $f : \sigma \to M$ is an affine map if it admits an affine lift $\tilde{f}$ to the universal cover $\tilde{\sigma} \to \tilde{M}$.

Similarly a map $v : \sigma \to TM$ is affine if it admits an affine lift to the tangent space of the universal cover $\tilde{v} : \tilde{\sigma} \to \tilde{TV}$.

A triangulation $\mathcal{T} = (M, \mathcal{K}, \ell)$ of $M$ is given by a triple where:

- $\mathcal{K}$ is a geometric simplicial complex contained in some ambient affine space,
- $\ell : |\mathcal{K}| \to M$ is a homeomorphism, where $|\mathcal{K}|$ is the topological space associated to $\mathcal{K}$.

A Euclidean triangulation is a triangulation such that the restriction $\ell : \sigma \to M$ to every simplex $\sigma \in \mathcal{K}$ is an affine map. We say that a triangulation is finite, if the underlying simplicial complex is finite.

In this paper, we only consider finite Euclidean triangulations of the quotient torus $M$. For simplicity, a triangulation of $M$ always refers to finite Euclidean triangulation, unless stated otherwise.

We have an alternate definition by passing to the universal cover: a finite Euclidean triangulation of $M$ is given equivalently by a locally finite $\Gamma$-invariant simplicial decomposition of $V$. 
Definition 7.1.1. A map \( f : M \to \mathbb{R} \) is called polyhedral with respect to a Euclidean triangulation \( \mathcal{T} \) of \( M \) if the following conditions are satisfied:

1. The map \( f \) is continuous.
2. The restriction of the map \( f \circ \ell : \sigma \to \mathbb{R} \) is affine for every \( \sigma \in \mathcal{K} \).

We define similarly polyhedral maps on \( M \) with values in a vector space, an affine space, in \( M \) or in \( TM \).

The moduli space of polyhedral maps \( f : M \to M \), with respect to some fixed Euclidean triangulation \( \mathcal{T} \) of \( M \), is denoted \( \mathcal{M}(\mathcal{T}) = \{ f : M \to M, \, f \text{ is polyhedral with respect to } \mathcal{T} \} \).

The moduli space is the obvious analogue of the space of smooth maps \( \mathcal{M} \). One of the main features of \( \mathcal{M}(\mathcal{T}) \) is that it is a finite dimensional manifold.

Once a triangulation \( \mathcal{T} \) is prescribed, we will often identify a simplex \( \sigma \in \mathcal{K} \) with its image \( \ell(\sigma) \subset M \). Bearing this in mind, it makes sense to talk about the restriction of a map defined on \( M \) to any simplex \( \sigma \) of the triangulation understood as a domain in \( M \). Furthermore, a vertex \( \sigma_0 \in \mathcal{K}_0 \) is assimilated to a point \( \sigma_0 \in M \) via \( \ell \).

A vector field \( v \), along a polyhedral map \( f \in \mathcal{M}(\mathcal{T}) \), is a map defined on the set of 0-simplexes of the triangulation, such that the following diagram is commutative

\[
\begin{array}{ccc}
TM & \xrightarrow{v} & \mathcal{K}_0 \\
\downarrow{\pi} & & \downarrow{f} \\
M & \to & M
\end{array}
\]

where \( \pi : TM \to M \) is the tangent bundle and the set of vertices \( \mathcal{K}_0 \) is understood as a finite subset of \( M \). With such a definition, a vector field can be extended uniquely as a map \( v : M \to TM \), polyhedral with respect to \( \mathcal{T} \), such that the following diagram commutes

\[
\begin{array}{ccc}
TM & \xrightarrow{v} & TM \\
\downarrow{\pi} & & \downarrow{\pi} \\
M & \xrightarrow{f} & M
\end{array}
\]

The tangent space of \( \mathcal{M}(\mathcal{T}) \) at \( f \), denoted \( T_f \mathcal{M}(\mathcal{T}) \), is the vector space of polyhedral vector fields along \( f \). The tangent space provides an exponential local chart at \( f \) for the moduli space. The exponential map if defined by

\[
T_f \mathcal{M}(\mathcal{T}) \to \mathcal{M}(\mathcal{T}) \quad \quad v \mapsto f + v
\]
where \( f + v \) is given by
\[
(f + v)(x) = f(x) + v(x),
\]
for every \( x \in M \) and \( v(x) \in T_{f(x)}M \) acts by translation on the torus \( M \). By definition \( \frac{\partial}{\partial t}[(f + tv)]_{t=0} = v \), which shows that the space of polyhedral vector fields along \( f \) is indeed the tangent space.

7.2. Differential for polyhedral maps. A polyhedral map \( f \in M \) is generally not differentiable. However its restriction \( f|_{\sigma} : \sigma \to M \) is affine, hence differentiable, for every \( \sigma \in \mathcal{K} \). Thus, we have a collection of differentials
\[
Df|_{\sigma} = \rho \circ (f|_{\sigma})_*
\]
for every \( \sigma \in \mathcal{K} \). Since \( f|_{\sigma} \) is affine, it follows that \( Df|_{\sigma} \) is constant on \( \sigma \) and in particular \( df|_{\sigma} = 0 \). If \( \sigma_2 \) is a face of \( \sigma_1 \in \mathcal{K} \), then \( j^* Df|_{\sigma_1} = Df|_{\sigma_2} \), where \( j : \sigma_2 \to \sigma_1 \) is the canonical inclusion. In particular, all the informations of the family \( (Df|_{\sigma})_{\sigma \in \mathcal{K}} \) can be deduced only from the facets \( (Df|_{\sigma})_{\sigma \in \mathcal{K}_4} \). Depending on the context, \( (Df|_{\sigma}) \) can be understood as a family where \( \sigma \in \mathcal{K} \) or \( \sigma \in \mathcal{K}_4 \) without any loss of generality.

We define the space of families of constant differential forms on facets
\[
\mathcal{F}(\mathcal{F}) = \{(F_\sigma)_{\sigma \in \mathcal{K}_4}, \quad F_\sigma \in \Omega^1(\sigma, \vec{V})
\]
and \( F_\sigma \) is constant on \( \sigma \}
By definition, the differential \( D \) induces a linear map
\[
D : M(\mathcal{F}) \to \mathcal{F}(\mathcal{F}).
\]

We denote by \( \Omega^k(M, \mathcal{F}) \) the space of families of differential forms \( \beta = (\beta_\sigma)_{\sigma \in \mathcal{K}} \) where \( \beta_\sigma \) is a smooth differential \( k \)-form on \( \sigma \), understood as a domain in \( M \). An element \( \beta = (\beta_\sigma) \in \Omega^k(M, \mathcal{F}) \) satisfies the Whitney condition, if for every \( \sigma_1, \sigma_2 \in \mathcal{K} \), with \( \sigma_2 \) a face of \( \sigma_1 \), we have \( j^* \gamma_\sigma_1 = \gamma_\sigma_2 \). The subspace of elements in \( \Omega^k(M, \mathcal{F}) \) that satisfy the Whitney condition is called the space of Whitney forms and is denoted \( \Omega^k_w(M, \mathcal{F}) \). We define a differential operator \( d \) on the space of Whitney forms: for \( \beta = (\beta_\sigma) \), we put \( d\beta = (d \beta_\sigma) \). It is easy to check that the operator \( d \) preserves the Whitney condition and \( d^2 = 0 \). Thus we have recalled the definition of the Whitney complex
\[
d_k : \Omega^k_w(M, \mathcal{F}) \to \Omega^{k+1}_w(M, \mathcal{F}).
\]
We can extends the construction to the case of \( \vec{V} \)-valued differential forms and we obtain the Whitney cohomology spaces
\[
H^k_w(M, \vec{V}, \mathcal{F}).
\]
In particular, we have the following lemma:

**Lemma 7.2.1.** The family of \( \vec{V} \)-valued differential 1-forms, \( F = (Df|_{\sigma}) \), where \( \sigma \in \mathcal{K} \), is a closed Whitney form.
Smooth forms always satisfy the Whitney condition. Hence we have a canonical map \( H^k(M, \bar{V}) \to H^*_w(M, \bar{V}, \mathcal{T}) \) from the DeRham cohomology. It turns out that this map is an isomorphism by the Whitney theorem. In particular, for every polyhedral map \( f \in \mathcal{M}(\mathcal{T}) \), the family \( F = (\mathcal{D}f|_\sigma) \) defines a cohomology class in \( H^1_w(M, \bar{V}, \mathcal{T}) \), hence in \( H^1(M, \bar{V}) \). For simplicity of notations, the family of differential forms \((\mathcal{D}f|_\sigma)\) is often referred to as \( \mathcal{D}f \). Eventually, we have defined a map

\[
\alpha : \mathcal{M}(\mathcal{T}) \to H^1(M, \bar{V})
\]
given by \( \alpha(f) = [\mathcal{D}f] \)

For an integral class \( \alpha \in H^1(M, \bar{V}) \setminus 0 \), we define the subspace \( \mathcal{M}_\alpha(\mathcal{T}) \) of polyhedral maps \( f \) such that \( \alpha(f) = \alpha \). We also introduce the subspace of Whitney forms of \( \mathcal{F}(\mathcal{T}) \) with cohomology class in \( \mathbb{R} \alpha \), denoted \( \mathcal{F}_\alpha(\mathcal{T}) \). By construction, \( \mathcal{D} \) defines a linear map

\[
\mathcal{D} : \mathcal{M}_\alpha(\mathcal{T}) \to \mathcal{F}_\alpha(\mathcal{T}),
\]

and its image is identified to \( \mathcal{M}_\alpha(\mathcal{T}) / \bar{V} \).

Given \( f \in \mathcal{M}(\mathcal{T}) \), the map tangent map \( f_* \) is only defined along simplexes of \( \mathcal{K} \). The pullback of a smooth form \( \beta \) by \( f \), defined by \( f^*\beta = (f^*\beta|_\sigma) \), satisfies the Whitney condition as well. If \( \beta \) is closed, so is \( f^*\beta \), in the sense of Whitney. As an application, \( f^*\omega_M = (f^*\omega_M|_\sigma) \) is well a defined closed Whitney 2-form. Furthermore, the fact that \( f \) is affine along every simplex implies that \( f_* \) is constant along every simplex and that the pullback \( f^*\omega_M \) is constant along every simplex of the triangulation.

**Definition 7.2.2.** A polyhedral map \( f \in \mathcal{M}(\mathcal{T}) \) such that the pullback \( f^*\omega_M \) vanishes along every simplex of the triangulation is called a polyhedral symplectic map. The space of polyhedral symplectic maps with respect to \( \mathcal{T} \) is denoted \( \text{Symp}(M, \omega_M, \mathcal{T}) \).

**Remark 7.2.3.** The space \( \text{Symp}(M, \omega_M, \mathcal{T}) \) does not have a group structure, since polyhedral maps cannot be composed without passing to some direct limit over all possible triangulations of \( M \). Proposition 2.1.1 does no apply out of the box, and it is not even clear whether a symplectic polyhedral map is a homeomorphism of \( M \).

Similarly to the smooth setting, we introduce the notation

\[
\text{Symp}_\alpha(M, \omega_M, \mathcal{T})
\]

for the space of polyhedral symplectic maps such that \( f \in \mathcal{M}_\alpha(\mathcal{T}) \).

### 7.3. HyperKähler structure in the polyhedral case.

The space of families \( F = (F_x)_{x \in \mathcal{K}} \) of constant \( \bar{V} \)-valued constant differential 1-forms \( \mathcal{F}(\mathcal{T}) \), is endowed with a HyperKähler structure similar to the smooth setting. The metric \( g \) defined on the bundle \( T^*M \otimes \bar{V} \to M \) induces a
$L^2$-metric $\mathcal{G}$ on $\mathcal{F}(\mathcal{T})$, similar to the smooth case (cf. Formula (2.14)) and given by

$$\mathcal{G}(F, G) = \sum_{\sigma \in \mathcal{K}_4} \int_{\sigma} g(F_{\sigma}, G_{\sigma}) \text{vol}_M$$

for every $F = (F_{\sigma})$ and $G = (G_{\sigma}) \in \mathcal{F}(\mathcal{T})$. Following the construction in the smooth case (cf. §3.3), the almost complex structures $I, J$ and $K$ on $M$ induce three almost complex structures $\mathcal{I}, \mathcal{J}$ and $\mathcal{K}$ on $\mathcal{F}(\mathcal{T})$, defined by

$$\begin{align*}
(\mathcal{I}F)_{\sigma} & = -F_{\sigma} \circ I, \\
(\mathcal{J}F)_{\sigma} & = -F_{\sigma} \circ J, \\
(\mathcal{K}F)_{\sigma} & = -F_{\sigma} \circ K,
\end{align*}$$

compatible with the metric $\mathcal{G}$, with corresponding Kähler forms

$$\begin{align*}
\Omega_I & = \mathcal{G}(\mathcal{I}, \cdot), \\
\Omega_J & = \mathcal{G}(\mathcal{J}, \cdot) \quad \text{and} \quad \Omega_K = \mathcal{G}(\mathcal{K}, \cdot). 
\end{align*}$$

In conclusion, the moduli space $\mathcal{F}(\mathcal{T})$ is endowed with a natural hyperKähler structure $(\mathcal{F}(\mathcal{T}), \mathcal{G}, \mathcal{I}, \mathcal{J}, \mathcal{K})$.

7.4. **Torus action in the polyhedral case.** The group $\mathbb{T}(\mathcal{T})$ is defined as the space of functions $\lambda : \mathcal{K}_4 \to S^1$ on the set of facets of the triangulation. Equivalently, $\lambda$ can be understood as a collection $(\lambda_{\sigma})_{\sigma} \in \mathcal{K}_4$ of constant functions $\lambda_{\sigma} : \sigma \to S^1$ on each facet of the triangulation. In fact $\mathbb{T}(\mathcal{T})$ is isomorphic as a group to the real torus $(S^1)^m$, where the dimension $m$ is the number of facets of the triangulation.

The Lie algebra $\mathfrak{t}(\mathcal{T})$ of the group $\mathbb{T}(\mathcal{T})$, is the vector space of real valued functions $\zeta : \mathcal{K}_4 \to \mathbb{R}$ defined on the set of facets of the triangulation. As in the case of the group, such a function $\zeta$ can be thought of as collection $(\zeta_{\sigma})_{\sigma} \in \mathcal{K}_4$ of constant real valued functions $\zeta_{\sigma} : \sigma \to \mathbb{R}$ on each facet of the triangulation. The Lie algebra is endowed with a $L^2$ inner product, defined by

$$\langle \zeta, \zeta' \rangle = \sum_{\sigma \in \mathcal{K}_4} \int_{\sigma} \zeta_{\sigma} \overline{\zeta'_{\sigma}} \text{vol}_M$$

and the exponential map $\exp : \mathfrak{t}(\mathcal{T}) \to \mathbb{T}(\mathcal{T})$ is defined by $\exp(\zeta) = \lambda$, with $\lambda(\sigma) = e^{i\zeta(\sigma)}$ for each $\sigma \in \mathcal{K}_4$.

The fact that $\mathcal{V}$ is a complex vector space induces a multiplicative action by complex valued functions on the space of $\mathcal{V}$-valued differential forms. In particular, the group $\mathbb{T}(\mathcal{T})$ acts by complex multiplication on the moduli space $\mathcal{F}(\mathcal{T})$. More precisely, given $\lambda = (\lambda_{\sigma}) \in \mathbb{T}(\mathcal{T})$ and $F = (F_{\sigma}) \in \mathcal{F}(\mathcal{T})$, we define $\lambda \cdot F \in \mathcal{F}(\mathcal{T})$ by

$$(\lambda \cdot F)_{\sigma} = \lambda_{\sigma} F_{\sigma}$$

for every $\sigma \in \mathcal{K}_4$, where the product in the RHS is the multiplication by the complex valued function $\lambda_{\sigma}$.
Proposition 7.4.1. The hyperKähler structure \((\mathcal{F} (\mathcal{T}), \mathcal{G}, \mathcal{I}, \mathcal{J}, \mathcal{K})\) is invariant under the action of the torus \(T (\mathcal{T})\).

Proof. The proof is formally the same as in the smooth setting (cf. Proposition 2.9.3). \(\square\)

As in the smooth case, we have three involutions \(\mathcal{R}_\bullet\) of \(\mathcal{F} (\mathcal{T})\), given by

\[
\mathcal{R}_I F = iF, \quad \mathcal{R}_J F = iJ F, \quad \mathcal{R}_K F = iK F
\]

with three maps \(\mu_\bullet : \mathcal{F} (\mathcal{T}) \to t (\mathcal{T})\) defined by

\[
\mu_\bullet (F) (\sigma) = -\frac{1}{2} \langle \mathcal{R}_\bullet F_\sigma, F_\sigma \rangle
\]

Following the analogy (cf. §2.10), the action of \(T (\mathcal{T})\) on \(\mathcal{F} (\mathcal{T})\) is Hamiltonian with respect to each Kähler forms \(\Omega_\bullet\):

Theorem 7.4.2. The group action of \(T (\mathcal{T})\) on \(\mathcal{F} (\mathcal{T})\) is Hamiltonian with respect to each Kähler forms \(\Omega_\bullet\). The moment maps are given by \(\mu_\bullet\) for \(\bullet = I, J, K\). More precisely, \(\mu_\bullet\) is \(T (\mathcal{T})\)-invariant and for every \(\zeta \in t (\mathcal{T})\)

\[
D \langle \langle \mu_\bullet, \zeta \rangle \rangle = -\iota_{X_\zeta} \Omega_\bullet
\]

where \(X_\zeta\) is the vector field on \(\mathcal{F} (\mathcal{T})\) associated to the infinitesimal action of \(\zeta\). In this sense, \(\mu_\bullet\) is a moment map for the group action with respect to the Kähler form \(\Omega_\bullet\).

Proof. The proof of Theorem 2.10.1, in the smooth setting, can be repeated formally in the polyhedral context. \(\square\)

7.5. Symplectic density in the polyhedral case. Formula 2.22 was used in the smooth setting to express the moment map in terms of symplectic density. However the formula is a consequence of algebraic pointwise computation. It follows that a similar formula holds in the polyhedral setting: for \(F = (F_\sigma) \in \mathcal{F} (\mathcal{T})\), we deduce that \(\mu_\bullet (F) \in t (\mathcal{T})\) is given by the family

\[
\mu_\bullet (F)(\sigma) = -\frac{F_\sigma^* \omega_V \wedge \hat{\omega}_\bullet}{\text{vol}_M}
\]

The three moment maps can be gathered in a hyperKähler moment map

\[
\mu = (\mu_I, \mu_J, \mu_K)
\]

and using the isomorphism \(\xi\) (cf. Formula 3.5), we have

\[
\xi \circ \mu (F)(\sigma) = \sqrt{2} (F_\sigma^* \omega_V)^-.
\]

Hence the moment can be interpreted as a map \(\tilde{\mu}\) on \(\mathcal{F}\), with values in the space of families of constant anti-selfdual forms on each facet of the triangulation and given by \(\tilde{\mu} = \xi \circ \mu\).
7.6. Polyhedral Hodge theory. Stokes theorem extends to Whitney form: in the case of our torus $M$ endowed with a Euclidean triangulation $\mathcal{T}$, let $\beta = (\beta_\sigma) \in \Omega_w^p(M, \mathcal{T})$ be a Whitney form. For every 4-simplex $\sigma \in \mathcal{K}_4$, we have
\[ \int_{\partial \sigma} j^* \beta_\sigma = \int_\sigma d\beta_\sigma \]
where $j : \partial \sigma \to \sigma$ is the canonical inclusion, by the usual Stokes theorem on a simplex. By the Whitney condition, the boundary contributions cancel out and we have
\[ \int_M d\beta := \sum_{\sigma \in \mathcal{K}_4} \int_\sigma d\beta_\sigma = 0. \]
We can define the exterior product for families $\beta = (\beta_\sigma)$ and $\gamma = (\gamma_\sigma)$ by $\beta \wedge \gamma = (\beta_\sigma \wedge \gamma_\sigma)$. By the usual Leibnitz formula
\[ d(\beta \wedge \gamma) = d\beta \wedge \gamma + (-1)^p \beta \wedge d\gamma \]
if $\beta$ is a $p$-form. If $\beta$ is a Whitney $p$-form and $\gamma$ is a Whitney $q$-form with $p + q = 3$, we deduce from the Stokes theorem
\[ (-1)^p \int_M d\beta \wedge \gamma + \int_M \beta \wedge d\gamma = 0. \]
We define a Hodge $\star$-operator on families on $\Omega^p(M, \mathcal{T})$ by $\star \beta = (\star \beta_\sigma)$, where $\star$ is the usual Hodge operator. If $\beta \in \Omega^2(M, \mathcal{T})$ satisfies $\star \beta = \beta$, we say that $\beta$ is selfdual 2-form, as in the smooth setting. Then
\[ \int_M \beta \wedge \beta = \sum_{\sigma \in \mathcal{K}_4} \int_\sigma \beta_\sigma \wedge \beta_\sigma \]
\[ = \sum_{\sigma \in \mathcal{K}_4} \int_\sigma (\beta_\sigma, \star \beta_\sigma) \text{vol}_M \]
\[ = \sum_{\sigma \in \mathcal{K}_4} \int_\sigma (\beta_\sigma, \beta_\sigma) \text{vol}_M = \| \beta \|^2_{L^2} \]
so that $\int_M \beta \wedge \beta > 0$ or $\beta = 0$. We deduce the following result, analogue to the smooth case:

**Lemma 7.6.1.** The only selfdual exact Whitney 2-form is the zero form.

**Proof.** Is $\beta$ is an exact Whitney 2-form, there exists a Whitney 1-form $\gamma$ such that $\beta = d\gamma$. By Stokes theorem, we deduce that $\int_M \beta \wedge \beta = \int_M d(\gamma \wedge \beta) = 0$, hence $\beta = 0$. \[\Box\]

**Corollary 7.6.2.** Let $f \in \mathcal{M}(\mathcal{T})$ be a polyhedral map of the torus such that $f^* [\omega_M] = [\omega_M]$. Then the following conditions are equivalent
\[ (1) \quad \mu \circ \mathcal{D} f = 0. \]
\[ (2) \quad f \text{ is symplectic, in the polyhedral sense.} \]
Proof. The proof is similar to the one in the smooth setting (cf. Proposition 3.3.3). Assuming that $\mu \circ Df = 0$, we have the equivalent equation $\hat{\mu}(Df) = 0$ which means that the family $f^*\omega_M|_\sigma$ is self-dual on every 4-simplex $\sigma \in \mathcal{K}_4$. Hence $f^*\omega_M$ is a self-dual Whitney 2-form. By assumption $f^*\omega_M - \omega_M$ is an exact Whitney 2-form and by Lemma 7.6.1, we conclude that $f^*\omega_M = \omega_M$.

Conversely, if $f$ is symplectic in the polyhedral sense, then $f^*\omega_M = \omega_M$ along every 4-simplex, which shows that $f^*\omega_M$ is selfdual, hence $\hat{\mu}(Df) = 0$. □

We conclude this section with a proof of one of our main theorems:

Proof of Theorem $F$. The analogue of Theorem $A$ is given by Theorem 7.4.2 for the polyhedral case. The analogue of Theorem $B$ is given by Corollary 7.6.2. □

Remark 7.6.3. Let $\mathcal{T}_i = (M, \mathcal{K}_i, \phi_i)$ be two Euclidean triangulations of the torus for $i = 1, 2$, such that $\mathcal{T}_2$ is a refinement of $\mathcal{T}_1$. By definition of polyhedral maps, there is a canonical injection $\mathcal{M}(\mathcal{T}_1) \hookrightarrow \mathcal{M}(\mathcal{T}_2)$. Hence the family of moduli spaces $\mathcal{M}(\mathcal{T})$ parametrized by triangulations and ordered by refinement is an inductive family. The direct limit

$$\mathcal{M}^{PL} = \varinjlim \mathcal{M}(\mathcal{T})$$

is a topological space endowed with the final topology. As a set $\mathcal{M}^{PL}$ is the space of piecewise linear maps of the torus, by definition. We can consider the subspace of piecewise linear symplectic maps, obtained as the direct limit of the spaces of polyhedral symplectic maps

$$\text{Symp}^{PL}(M, \omega_M) = \varinjlim \text{Symp}(M, \omega_M, \mathcal{T}).$$

The moduli spaces $\mathcal{F}(\mathcal{T})$ and the groups $\mathcal{T}(\mathcal{T})$ also have direct limits $\mathcal{F}^{PL}$ and $\mathcal{T}^{PL}$. Formally, we have an interesting framework where our moment map constructions extend to the piecewise linear setting without any reference to the choice of a particular triangulation.

7.7. Polyhedral modified moment map flow. Given $\alpha \in H^1(M, \overrightarrow{V}) \setminus 0$, we defined at §7.2 the subspace $\mathcal{F}_\alpha(\mathcal{T})$ as the subspace of Whitney forms of $\mathcal{F}(\mathcal{T})$ with cohomology class in $\mathbb{R}\alpha$. The Euclidean metric $\mathcal{G}$ induces an orthogonal projector, which is the analogue of the Hodge projector in the smooth setting, denoted

$$\Pi_\alpha : \mathcal{F}(\mathcal{T}) \to \mathcal{F}_\alpha(\mathcal{T}).$$

We consider the energy of the moment map also have a polyhedral analogue defined by the functional

$$\phi : \mathcal{F}(\mathcal{T}) \to \mathbb{R},$$
given by

\[ \phi(F) = \frac{1}{2} \| \mu(F) \|_{L^2}^2. \]

The gradient of \( \phi \) on \( \mathcal{F}(\mathcal{T}) \) is denoted \( \nabla \phi \), whereas the gradient of the restriction \( \phi : \mathcal{F}_\alpha(\mathcal{T}) \to \mathbb{R} \), denoted \( \nabla^\alpha \phi \), is related to \( \nabla \phi \) via the orthogonal projection

\[ \nabla^\alpha \phi(F) = \Pi_\alpha \nabla \phi(F) \]

for every \( F \in \mathcal{F}_\alpha(\mathcal{T}) \). We consider the downward gradient flow of \( \phi \) along \( \mathcal{F}_\alpha(\mathcal{T}) \):

\[ \frac{\partial F}{\partial t} = -\Pi_\alpha \nabla \phi(F). \]  \hfill (7.1)

The evolution equation (7.1) is called the \textit{polyhedral modified moment map flow}. This ordinary differential equation is an analogue of the modified moment map flow defined in the smooth setting. However, the Cauchy-Lipschitz theorem for ODE insures the local existence and uniqueness of solutions of equation (7.1).

All the identities computed in the smooth setting have formal analogues in the polyhedral case. The most useful properties are summarized in Proposition 7.7.1. Before stating the results, we introduce the vector fields \( W_\bullet \), defined on \( \mathcal{F}(\mathcal{T}) \) by the formula

\[ W_\bullet(F) = -\mu_\bullet(F) \mathcal{R}_\bullet F \]

for \( \bullet = I, J, K \). The proof of Lemma 4.2.2 is immediately adapted to the polyhedral setting and gives the following result:

**Proposition 7.7.1.** The gradient of \( \phi \) on \( \mathcal{F}(\mathcal{T}) \) is given by the formula

\[ \nabla \phi(F) = \sum_\bullet W_\bullet(F). \]

In addition

\[ \langle \langle \nabla \phi(F), F \rangle \rangle = 4 \phi(F). \]

For every solution \( F_t \) of the modified moment map flow, for \( t \) in some interval, we have

\[ \frac{d}{dt} \| F_t \|_{L^2}^2 = -8 \phi(F_t). \]

In particular, the \( L^2 \)-norm is non-increasing along the flow.

**Corollary 7.7.2.** The critical points of \( \phi : \mathcal{F}_\alpha(\mathcal{T}) \to \mathbb{R} \) are the fixed points of the polyhedral modified moment map flow and agree with the vanishing locus of \( \phi \) in \( \mathcal{F}_\alpha(\mathcal{T}) \).

Furthermore, a flow line \( F_t \) defined for \( t \) in some interval \([t_0, t_1]\) admits a unique extension for \( t \in [t_0, +\infty) \), as a solution of the modified moment map flow. Finally, the limiting orbits of the flow are non-empty compact sets.
Proof. If $F \in \mathcal{T}_\alpha$, then

$$\langle \nabla \phi(F), F \rangle = \langle \Pi_\alpha \nabla \phi(F), F \rangle = \langle \nabla^\alpha \phi(F), F \rangle.$$  

By Proposition 7.7.1, we deduce that $\langle \nabla^\alpha \phi(F), F \rangle = 4\phi(F)$. This shows that a critical point of the restriction $\phi: \mathcal{T}_\alpha \to \mathbb{R}$ is a zero of $\phi$. Conversely $\phi(F) = 0$ implies $\mu_*(F) = 0$ hence $W_*(F) = 0$, by definition. Proposition 7.7.1 then shows that $F$ is a critical point of $\phi$.

The monotonicity of the $L^2$ norm along flow lines stated in Proposition 7.7.1 insures that the flow cannot blowup: if $F_t$ is a solution of the flow defined for $t \in [t_0, t_1)$, then $\|F_t\|_{L^2} \leq \|F_{t_0}\|_{L^2}$. The existence of the flow for $t \in [t_0, +\infty)$ follows from a classical result of ODE theory and the flow line is bounded by $\|F_{t_0}\|_{L^2}$. The limiting orbit of $F_t$ is closed, by definition, and the compactness property follows from the boundedness. The non emptyness of the limiting orbit of $F_t$ is clear, as we can always extract a converging subsequence of $F_t$, by the Bolzano-Weirstrass theorem. □

7.8. A Duistermaat theorem. In this section, we will often omit the reference to the triangulation $\mathcal{T}$ and denote $\mathcal{T}(\mathcal{T})$ by $\mathcal{T}$, for simplicity. The moment map and the modified moment map flow are always in reference to the polyhedral framework, for some fixed triangulation $\mathcal{T}$. By continuity of $\phi$, the sets

$$V_\varepsilon = \{F \in \mathcal{T}_\alpha, \phi(F) < \varepsilon\} \quad (7.2)$$

are open neighborhood of the vanishing locus of $\phi$ in $\mathcal{T}_\alpha$, for every $\varepsilon > 0$.

The function $\phi$ is non increasing along the modified moment map flow, by definition of a downward gradient flow. This implies that the set $V_\varepsilon$ is invariant under the flow:

**Lemma 7.8.1.** Let $F_t$ be a solution of the modified moment map flow, defined for $t$ in some interval $I$. If $F_t \in V_\varepsilon$ for $t \in I$, then $F_{t'} \in V_\varepsilon$ for every $t' \in I$ with $t' \geq t$.

We consider the closed ball or radius $R > 0$ centered at the origin in $\mathcal{T}_\alpha$

$$B_R = \{F \in \mathcal{T}_\alpha, \|F\| \leq R\}$$

and $C_\varepsilon$ be the compact subset of $B_R$ defined by

$$C_\varepsilon = B_R \setminus V_\varepsilon.$$

We introduce the

$$\delta = \inf_{F \in C_\varepsilon} N(F) > 0$$

where $N: \mathcal{T}_\alpha \to \mathbb{R}$ is the continuous function defined by

$$N(F) = \|\nabla^\alpha \phi(F)\|^2. \quad (7.3)$$

By continuity of $N$ and compactness of $C_\varepsilon$, the infimum $\delta$ is a minimum, with $\delta = N(F_{\text{min}})$ for some $F_{\text{min}} \in C_\varepsilon$. By Corollary 7.7.2, $\phi$ and $N$
have the same vanishing locus on $\mathcal{F}_\alpha$. Hence $N$ does not vanish on $C_\varepsilon$ and it follows that $\delta = N(F_{\min}) > 0$. Notice that $\delta = \delta(\varepsilon, R)$ depends on the choice of $\varepsilon$ and $R$.

**Lemma 7.8.2.** Let $F_t$ be a solution of the modified moment map flow, defined for $t \in [0, +\infty)$. If $F_0 \in \bar{B}_R$, then $F_t \in \bar{B}_R \cap V_\varepsilon$ for every $t \geq \frac{\phi(F_0)}{\delta}$.

**Proof.** We consider a solution $F_t$ of the modified moment map flow, with the assumptions of the lemma. By Proposition 7.7.1 the $L^2$-norm is non increasing along the flow, hence $F_t \in \bar{B}_R$ for every $t \geq 0$. By Lemma 7.8.1, $V_\varepsilon$ is stable under the flow. In conclusion

$$\tilde{V}_\varepsilon = \bar{B}_R \cap V_\varepsilon$$

is also stable under the flow. Notice that we have a partition of the ball $\bar{B}_R = \tilde{V}_\varepsilon \cup C_\varepsilon$. As a solution of a downward gradient flow, the flow line satisfies the ODE

$$\frac{d\phi(F_t)}{dt} = -\|\nabla^\alpha\phi(F_t)\|_{L^2}^2 = -N(F_t).$$

In particular, if $F_t \in C_\varepsilon$, we have

$$\frac{d\phi(F_t)}{dt} \leq -\delta. \quad (7.4)$$

If $F_t \in C_\varepsilon$ for every $t \in [0, T]$, we obtain by integration of the differential inequality (7.4)

$$\phi(F_T) \leq \phi(F_0) - T\delta. \quad (7.5)$$

Using the fact that $\phi(F_T) \geq 0$, we deduce that $T \leq T_0 = \frac{\phi(F_0)}{\delta}$. Now there are two possibilities:

- $F_t \in \tilde{V}_\varepsilon$ for some time $t_0 \leq T_0$. We argued above that $\tilde{V}_\varepsilon$ is stable under the flow. Hence $F_t \in \tilde{V}_\varepsilon$ for every $t \geq t_0$, in particular, for every $t \geq T_0$.

- If, on the contrary, $F_t \in C_\varepsilon$ for every $t \in [0, T_0]$, we deduce that $\phi(F_{T_0}) \leq 0$ from Inequality (7.5). Since $\phi$ is non negative, we have $\phi(F_{T_0}) = 0$. This is a contradiction since $F_{T_0} \in C_\varepsilon$, hence $\phi(F_{T_0}) > 0$, by assumption.

We conclude that $F_t$ hits $\tilde{V}_\varepsilon$ for some time $t_0 \leq T_0$. Since $\tilde{V}_\varepsilon$ is stable under the flow, we have $F_t \in \tilde{V}_\varepsilon$ for every $t \geq t_0$, in particular, for every $t \geq T_0$, which proves the lemma. $\square$

**Corollary 7.8.3.** For every solution of the modified moment map flow $F_t$, defined for $t \in [0, +\infty)$, we have

$$\lim_{t \to +\infty} \phi(F_t) = 0.$$

In particular every limiting orbit of the modified moment map flow is contained in the zero locus of $\phi$.  

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Proof. We choose $R = \|F_0\|$ in the above constructions. Then for every $\varepsilon > 0$, there exists $\delta$ such that for every $t \geq \frac{\phi(F_0)}{\delta}$, we have $\phi(F_t) < \varepsilon$, by Lemma 7.8.2. This shows that the limit of $\phi(F_t)$ is zero. The statement about limiting orbits is immediate, by continuity of $\phi$. □

Eventually, we prove our main result for the modified moment map flow:

**Theorem 7.8.4.** Every solution $F_t \in \mathcal{F}_\alpha(\mathcal{F})$ of the modified moment map flow, defined for $t$ in some interval $[0, +\infty)$, admits a limit

$$F_\infty = \lim_{t \to +\infty} F_t \in \mathcal{F}_\alpha(\mathcal{F})$$

such that $\phi(F_\infty) = 0$. In particular, we have an extended flow

$$\Theta : [0, +\infty] \times \mathcal{F}_\alpha(\mathcal{F}) \to \mathcal{F}_\alpha(\mathcal{F})$$

defined as follows: for $F \in \mathcal{F}_\alpha(\mathcal{F})$ and $t \in [0, +\infty)$, we put

- $\Theta(t, F) = F_t$, where $F_t$ is the solution of the modified moment map flow with initial condition $F_0 = F$.
- If $t = +\infty$, we put $\Theta(+\infty, F) = F_\infty = \lim_{t \to +\infty} F_t$.

The extended flow $\Theta$ is a continuous map and defines a continuous retraction of $\mathcal{F}_{\alpha}(\mathcal{F})$ onto the zero locus of $\phi$ in $\mathcal{F}_{\alpha}(\mathcal{F})$.

The proof of Theorem 7.8.4 rests on the /Lojasiewicz estimate (cf. [18, Lemma 2.2] or Proposition 1 p. 67 of [19], or Proposition 6.8 in [5]):

**Lemma 7.8.5** (Lojasiewicz gradient inequality). Let $\phi$ be a real analytic function on an open set $W \subset \mathbb{R}^k$ and $x \in W$, a critical point of $\phi$, such that $\phi(x) = 0$. Then, there exists a neighborhood $U$ of $x$ and some constants $c > 0$ and $\nu \in (0, 1)$, such that

$$\|\nabla \phi(y)\| \geq c|\phi(y)|^\nu$$

for every $y \in W$, where $\|\cdot\|$ is an Euclidean norm on $\mathbb{R}^k$ and $\nabla$ is the gradient with respect to the Euclidean inner product.

Proof of Theorem 7.8.4. The function $\phi : \mathcal{F}_\alpha(\mathcal{F}) \to \mathbb{R}$ is polynomial function of degree 4. In particular, $\phi$ is analytic and Lemma 7.8.5 applies. As before, we consider the closed ball $\bar{B}_R \subset \mathcal{F}_{\alpha}$ or radius $R > 0$ centered at the origin. The vanishing locus $K = \phi^{-1}(0) \cap \bar{B}_0$ is closed in $\bar{B}_0$, hence compact. For each point $F \in K$, we choose an open neighborhood $U_F$ of $F$ in $\mathcal{F}_{\alpha}$ and constants $c_F > 0$ and $\nu_F \in (0, 1)$ such that the Lojasiewicz estimate holds.

The family of sets $U_F$ for $F \in K$ is an open cover $K$. By compactness, we can extract a finite cover $U_1, \cdots, U_k$ and we define

$$U = \bigcup U_i, \quad \nu = \max \nu_i \quad \text{and} \quad c = \min c_i.$$
In particular, for every $F \in U$, we have
\[ \|\nabla^\alpha \phi(F)\| \geq c \phi(F)^\nu \]  
provided $\phi(F) \leq 1$. In other words, Inequality 7.6 holds on $U \cap V_1$, where $V_1$ is defined by (7.2).

If $\bar{B}_R \subset V_1 \cap U$, we have proved that (7.6) holds on the ball. Suppose this is not the case, in other words that $\bar{B}_R \setminus (U \cap V_1)$ is not empty. Then, the function $\phi$ admits a minimum $\epsilon_0 > 0$ on the compact set $\bar{B}_R \setminus (U \cap V_1)$. In particular $V_{\epsilon_0}$ does not intersect this compact set, hence
\[ V_{\epsilon_0} \cap \bar{B}_R \subset U \cap V_1 \cap \bar{B}_R. \]

In conclusion, an open neighborhood $\tilde{V}_{\epsilon_0} = V_{\epsilon_0} \cap \bar{B}_0$ of $K$, with respect to the induced topology on the ball, such that the Lojasiewicz inequality (7.6) holds for every $F \in \tilde{V}_{\epsilon_0}$.

Notice that $\tilde{V}_{\epsilon_0}$ cannot cover the ball $\bar{B}_R$. Indeed, $\tilde{V}_{\epsilon_0}$ is a subset of $U \cap V_1$ which does not cover the ball. Hence $C_{\epsilon_0} = \bar{B}_R \setminus V_{\epsilon_0}$ is a non empty compact subset of $\bar{B}_R$. We consider the infimum $\delta_0$ of the function $N$ (cf. Formula (7.3)) on $C_{\epsilon_0}$ as before and we have $\delta_0 > 0$.

For $F \in \mathcal{F}_\alpha$ and we define $R = \|F\| + 1$. In particular $\bar{B}_R$ is a neighborhood of $F$ in $\mathcal{F}_\alpha$. For every $G \in \bar{B}_R$, we denote by $G_t$, the corresponding flow line, defined for $t \in [0, +\infty)$, with $G_0 = G$. By Lemma 7.8.2
\[ G_t \in \tilde{V}_{\epsilon_0} \quad \forall t \geq \frac{M}{\delta_0}, \]
where $M$ is the maximum of $\phi$ on the ball $\bar{B}_R$. From the flow equation, we deduce
\[ \frac{d}{dt} \phi(G_t) = -\|\nabla^\alpha \phi(G_t)\|^2, \]
hence
\[ -\frac{d}{dt} \phi(G_t)^{1-\nu} = (1 - \nu)\phi(G_t)^{-\nu}\|\nabla^\alpha \phi(G_t)\|^2. \]

For every $t \geq \frac{M}{\delta_0}$, we have by (7.6)
\[ -\frac{d}{dt} \phi(G_t)^{1-\nu} \geq c(1 - \nu)\|\nabla^\alpha \phi(G_t)\|. \]

Hence, for every $t_1 > t_0 \geq \frac{M}{\delta_0}$, we obtain by integrating the above inequality
\[ \phi(G_{t_0})^{1-\nu} - \phi(G_{t_1})^{1-\nu} = -\int_{t_0}^{t_1} \frac{d}{dt} \phi(G_t)^{1-\nu} dt \]
\[ \phi(G_{t_0})^{1-\nu} - \phi(G_{t_1})^{1-\nu} \geq c(1 - \nu) \int_{t_0}^{t_1} \|\nabla \phi(G_t)\| dt. \]
By the Jensen inequality, we have
\[ d(G_{t_0}, G_{t_1}) \leq \int_{t_0}^{t_1} \| \frac{d}{dt} G_t \| \, dt \]
\[
\leq \int_{t_0}^{t_1} \| \nabla \phi(G_t) \| \, dt
\tag{7.8}
\]
where \( d(F,G) = \| F - G \| \) is the distance induced by the Euclidean norm on \( \mathcal{F}_\alpha \). By Inequalities (7.7) and (7.8), we have
\[ d(G_{t_0}, G_{t_1}) \leq c_2 \left( \phi(G_{t_0})^{1-\nu} - \phi(G_{t_1})^{1-\nu} \right), \tag{7.9} \]
where \( c_2 = \frac{1}{c(1-\nu)} \).

By Corollary 7.8.3, \( \lim_{t \to +\infty} \phi(G_t) = 0 \), hence \( \lim_{t \to +\infty} \phi(G_t)^{1-\nu} = 0 \). This implies that the function \( t \mapsto \phi(G_t)^{1-\nu} \) satisfies the Cauchy criterion. By Inequality (7.9), we deduce that for every \( \epsilon > 0 \), there exists \( t_0 \geq \frac{M}{\nu} \), such that for every \( t_1 > t_0 \), we have \( d(G_{t_0}, G_{t_1}) \leq \epsilon \). We see that \( t \mapsto G_t \) satisfies the Cauchy criterions as well. By completness of \( \mathcal{F}_\alpha \), the flow line \( G_t \) converges as \( t \) goes to infinity. In particular \( F_\infty = \lim_{t \to +\infty} \Theta(t, F) \) exists for every \( F \in \mathcal{F}_\alpha \) and we have proved that the extended flow is well defined on the entire space \([0, +\infty) \times \mathcal{F}_\alpha\).

By the classical ODE theory, the flow \( \Theta \) is continuous on \([0, +\infty) \times \mathcal{F}_\alpha\). We are going to show that \( \Theta \) is also continuous at every point of \([+\infty) \times \mathcal{F}_\alpha\), hence everywhere. First observe that passing to the limit in Inequality (7.9) provides a control between \( G_t \) and its limit \( G_\infty \). More precisely, let \( G_t \) be any solution of the modified moment map flow defined for \( t \in [0, +\infty) \), with \( G_0 \in \bar{B}_R \) as before. Using the fact that \( \lim_{t_t \rightarrow \infty} \phi(G_{t_t}) = 0 \), we deduce that
\[ d(G_t, G_\infty) \leq c_2 \phi(G_t)^{1-\nu}, \tag{7.10} \]
for every \( t \geq \frac{M}{\nu} \). This implies
\[ d(G_\infty, F_\infty) \leq d(G_\infty, G_t) + d(G_t, F_t) + d(F_t, F_\infty) \]
\[ \leq d(G_t, F_t) + c_2 \phi(F_t)^{1-\nu} + c_2 \phi(G_t)^{1-\nu}. \tag{7.11} \]
for every \( t \geq \frac{M}{\nu} \).

Given \( \epsilon > 0 \), we denote by \( \delta > 0 \) the constant provided by Lemma 7.8.2 and we put \( T = M \max\left( \frac{1}{\delta}, \frac{1}{\nu} \right) \). Then for every \( t \geq T \), we have \( \phi(G_t) \leq \epsilon \) for every flow line \( G_t \) with \( G_0 = G \in \bar{B}_R \). By Inequality 7.11, we have
\[ d(G_\infty, F_\infty) \leq d(G_T, F_T) + 2c_2 \epsilon^{1-\nu}. \]
Using the continuity of \( \Theta \) at \((T, F) \in [0, +\infty) \times \mathcal{F}_\alpha\), we deduce that there exists a neighborhood \( W \) of \( F \) in \( \bar{B}_R \), such that for every \( G \in W \), we have \( d(\Theta(T, F), \Theta(T, G)) \leq \epsilon \). In conclusion
\[ d(\Theta(+\infty, G), \Theta(+\infty, F)) \leq \epsilon + 2c_2 \epsilon^{1-\nu}. \]
for every $F \in W$, which shows that $G \mapsto \Theta(+\infty, G)$ is continuous at $F$. Furthermore, for every $t \geq T$ and $G \in W$,
\[
d(G_t, F_\infty) \leq d(G_t, G_\infty) + d(G_\infty, F_\infty) \\
\leq c_2(\phi(G_t)^{1-\nu}) + d(G_\infty, F_\infty) \\
\leq \varepsilon + 3c_2\varepsilon^{1-\nu},
\]
which shows that $\Theta$ is continuous at $(+\infty, F)$. \hfill \Box

7.9. Flow and regular locus. The vanishing locus of restriction of the hyperKähler moment map $\mu : \mathcal{F}_\alpha(T) \to t^3(\mathcal{T})$ agrees with the vanishing locus of $\phi$ in $\mathcal{F}_\alpha(T)$. This set is real algebraic, but little is understood about the regularity of the vanishing locus. More precisely, we introduce the following definition:

Definition 7.9.1. A point $F$ of the vanishing locus of $\phi : \mathcal{F}_\alpha(T) \to \mathbb{R}$ is called regular, if the differential of the hyperKähler moment map $\mu : \mathcal{F}_\alpha(T) \to t(\mathcal{T})^3$ has constant rank in neighborhood of $F$.

Remarks 7.9.2. (1) The vanishing set of $\phi : \mathcal{F}_\alpha \to t(\mathcal{T})^3$ has the structure of a smooth manifold in a neighborhood of a regular point.

(2) We do not have a good understanding of the regularity condition. In fact it is not clear whether there exists any regular point. It seems sensible to expect that regular points should be generic, perhaps up to passing to a suitable refinement of the triangulation $\mathcal{T}$.

If the regularity condition is met at some point $F$, the polyhedral version of the modified moment map flow is well behaved in a neighborhood of $F$, in the sense that the flow has exponential convergence. This property is a strong incentive to explore numerical versions of the flow, that should provide effective examples of polyhedral symplectic maps of the torus $M$ [13].

Theorem 7.9.3. Let $F \in \mathcal{F}_\alpha$ be a regular point of the vanishing locus of $\phi$. Then, there exists an open neighborhood $V$ of $F$ in $\mathcal{F}_\alpha(T)$, such that

1. The vanishing locus of $\phi$ is $V$ is a manifold.
2. The set $V$ is invariant under the flow $G \mapsto \Theta(t, G)$ for $t \geq 0$.
3. For every $G \in V$, the flow line $t \mapsto \Theta(t, G)$ converges exponentially fast as $t \to +\infty$ toward a zero of $\phi$ in $V$.

Proof. We choose first an open neighborhood $V$ of $F$ in $\mathcal{F}_\alpha(T)$, such that the rank of the differential of $\mu$ is constant on $V$. Then the vanishing locus of $\phi$ in $V$ is a manifold. By Proposition 7.7.1, $\nabla \phi(F) = \sum W_*(F)$. Following the formal computations, as in the smooth case,
we deduce the analogue of Formula (6.3) for the Hessian of \( \phi \) at \( G \in V \) with \( \phi(G) = 0 \), given by

\[
D^2 \phi|_G(\dot{F}, \dot{F}) = \|D\mu|_G \cdot \dot{F}\|_{L^2}
\]

Therefore, the kernel of the Hessian of \( \phi \) at \( G \) is the tangent space to the vanishing locus of \( \phi \) in \( \mathcal{F}_\alpha(\mathcal{T}) \) whereas the Hessian is positive in transverse directions. Therefore, the restriction \( \phi : V \to \mathbb{R} \) is a Morse-Bott function and its critical set agrees with its vanishing set in \( V \). Furthermore, the vanishing set is stable. The theorem then follows from classical result of ODE in the context of Morse-Bott theory. □

**Proof of Theorem G.** The first part of the theorem is given by Theorem 7.8.4 and the statement about exponential convergence is given by Theorem 7.9.3. □

**7.10. Polyhedral renormalized flow.** In this section \( \alpha \) is a symplectic cohomology class in \( H^1(M, \mathcal{V}) \). The image of \( \text{Symp}_\alpha(M, \mathcal{T}, \omega_M) \) by \( \mathcal{D} \) is denoted \( S_\alpha(\mathcal{T}) \subset \mathcal{F}_\alpha(\mathcal{T}) \). Recall that \( \mathcal{D} \) induces a homeomorphism between \( \mathcal{M}_\alpha/\mathcal{V} \) and its image in \( \mathcal{F}_\alpha(\mathcal{T}) \). It follows that the topology of \( \text{Symp}_\alpha(M, \omega_M, \mathcal{T}) \) and \( S_\alpha(\mathcal{T}) \) are closely related. Furthermore, we have the following lemma, relating the topology of \( S_\alpha(\mathcal{T}) \) with the vanishing set of \( \phi \):

**Lemma 7.10.1.** The real cone

\[ \mathbb{R}^* \cdot S_\alpha(\mathcal{T}) \subset \mathcal{F}_\alpha(\mathcal{T}) \setminus 0, \]

spanned by \( S_\alpha(\mathcal{T}) \), agrees with the vanishing locus of \( \phi \) in \( \mathcal{F}_\alpha(\mathcal{T}) \setminus 0 \).

**Proof.** If \( f \in \text{Symp}_\alpha(M, \omega_M, \mathcal{T}) \), then \( \mu(\mathcal{D}f) = 0 \) hence \( \phi(\mathcal{D}f) = 0 \). Notice that \( [\mathcal{D}f] = \alpha \), hence \( \mathcal{D}f \neq 0 \).

Conversely, if \( F \in \mathcal{F}_\alpha \setminus 0 \) and \( \phi(F) = 0 \), then \( [F] \neq 0 \). This fact is the polyhedral analogue of Lemma 5.3.4. Indeed, \( \phi(F) = 0 \) is equivalent to the fact that \( F^*\omega_V|_{\sigma} \) is selfdual for every facet \( \sigma \) of the triangulation. If \( [F] = 0 \), there exists a map \( f \in \mathcal{M}_\alpha(\mathcal{T}) \) homotopic to the identity such that \( F = \mathcal{D}f \). In particular, \( F^*\omega_V = f^*\omega_M \) so that \( F^*\omega_V \) is is an exact Whitney form. By Lemma 7.6.1, we deduce that \( F \) vanishes, which is a contradiction.

In conclusion \( [F] \neq 0 \) and we can write \( [F] = \kappa \alpha \), for some \( \kappa \in \mathbb{R}^* \). By rescaling, we obtain \( G = \kappa^{-1}F \) with \( [G] = \alpha \). Since \( \alpha \) is integral, there exists \( h \in \mathcal{M}_\alpha(\mathcal{T}) \) such that \( G = \mathcal{D}h \). The condition \( \phi(F) = 0 \) implies \( \phi(G) = 0 \) and \( h \) is a symplectic polyhedral map. Finally \( F = \kappa \mathcal{D}h \) and \( F \) belongs to the cone spanned by \( S_\alpha \). □

**Corollary 7.10.2.** The vanishing locus of \( \phi \) in \( \mathcal{F}_\alpha(\mathcal{T}) \setminus 0 \) admits a continuous retraction onto \( S_\alpha \cup -S_\alpha \). The latter space is homeomorphic to the vanishing locus of \( \phi : S_\alpha(\mathcal{T}) \to \mathbb{R} \), where \( S_\alpha(\mathcal{T}) \) is the unit sphere in \( \mathcal{F}_\alpha(\mathcal{T}) \).
Proof. By Lemma 7.10.1, the vanishing locus of $\phi$ in $F_\alpha \setminus 0$ is the cone $\mathbb{R}^* \cdot S_\alpha$. There exists a continuous retraction of $\mathbb{R}^*$ onto $\{-1, 1\}$ which proves the first part of the corollary. The homeomorphism of the last statement is given by the restriction of the radial projection $F_\alpha(\mathcal{I}) \setminus 0 \to S_\alpha(\mathcal{I})$. □

Remark 7.10.3. Theorem 7.8.4, provides a continuous retraction of $F_\alpha(\mathcal{I})$ onto the vanishing locus of $\phi$. Together with Corollary 7.10.2, these facts contribute to the belief that topological insights about the space of symplectic polyhedral maps $\text{Symp}_\alpha(M, \omega_M, \mathcal{I})$ could be derived from the modified moment map flow. Unfortunately Lemma 7.10.1 and Corollary 7.10.2 deal with the space $F_\alpha \setminus 0$ whereas the full space $F_\alpha$ is involved in Theorem 7.8.4.

By Corollary 7.10.2, the vanishing locus of $\phi : S_\alpha(\mathcal{I}) \to \mathbb{R}$ contains significant topological information about $\text{Symp}_\alpha(\mathcal{I})$. We would like to treat this function as a Morse-Bott function to obtain topological invariants for $\text{Symp}_\alpha(\mathcal{I})$.

7.11. Polyhedral solitons. As introduced at §5.3 in the smooth setting, we can introduce the space of polyhedral solitons $F \in F_\alpha(\mathcal{I})$, which are the solitons of the polyhedral soliton equation
\[
\|F\|^2_{L^2} \nabla^\alpha \phi(F) = 4\phi(F)F. \tag{7.12}
\]
The non zero solitons are denoted $\mathcal{I}_\alpha(\mathcal{I})$ and we have a partition
\[
\mathcal{I}_\alpha(\mathcal{I}) = \mathcal{I}_{\alpha,np}(\mathcal{I}) \sqcup \mathcal{I}_{\alpha,p}(\mathcal{I})
\]
into proper solitons and non proper solitons. The latter coincide with the vanishing locus of $\phi$ in $F_\alpha(\mathcal{I}) \setminus 0$. Finally, the polyhedral renormalized flow is the downward gradient flow of the restriction $\phi : S_\alpha \to \mathbb{R}$ given by the formula
\[
\frac{\partial G}{\partial s} = 4\phi(G)G - \nabla^\alpha \phi(G). \tag{7.13}
\]
By definition, the fixed points of the renormalized flow are the solitons contained in $S_\alpha(\mathcal{I})$.

Theorem 7.11.1. The polyhedral renormalized flow on $S_\alpha(\mathcal{I})$ is complete. Furthermore, the limiting orbits of the flow are contained in the space of solitons in $S_\alpha(\mathcal{I})$.

Proof. This follows from classical result of ODE theory on compact manifolds, in the case of a gradient flow. □

Conjecture I is motivated by the fact that the regularity of the critical locus of $\phi$ is not well understood. This is a central question if we would like to regard $\phi$ as a Morse-Bott function. In contrast with the smooth setting, we have the following existence result for proper solitons:

Lemma 7.11.2. The set of proper solitons in $F_\alpha(\mathcal{I})$ is not empty.
Proof. There exists an affine symplectic map $f$ in $\mathcal{M}_\alpha$. We can deform $f$ by moving on of the values $f(\sigma_0)$ where $\sigma_0$ is a vertex of the triangulation $\mathcal{T}$. For generic choice of $f(\sigma_0)$, the deformations $h$ is not symplectic. It follows that $\phi(\mathcal{D}h) \neq 0$ and this shows that $\phi$ does not vanish identically on $S_\alpha$. In particular, the supremum of $\phi$ is positive. By compactness, there exists $G \in S_\alpha$ such that $\phi(G) = \sup_{F \in S_\alpha} \phi(F) > 0$. Hence, $G$ is a critical point of $\phi : S_\alpha \to \mathbb{R}$, in other words, a soliton, and it is proper.

Proof of Theorem H. The result is a restatement of Theorem 7.11.1 together with Lemma 7.11.2, which shows existence of proper solitons. The fact that every proper soliton defines a solution of the modified moment map flow converging toward $F = 0$ is an immediate consequence of Lemma 5.4.1, which applies formally to the polyhedral setting. □

References

[1] M. Abreu and D. McDuff. Topology of symplectomorphism groups of rational ruled surfaces. J. Am. Math. Soc., 13(4):971–1009, 2000.
[2] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. Philos. Trans. R. Soc. Lond., Ser. A, 308:523–615, 1983.
[3] A. Banyaga. The structure of classical diffeomorphism groups, volume 400 of Math. Appl., Dordr. Dordrecht: Kluwer Academic Publishers, 1997.
[4] M. Bertelson and J. Distexhe. PL approximations of symplectic manifolds. J. Symplectic Geom., 2024. to appear.
[5] E. Bierstone and P. D. Milman. Semianalytic and subanalytic sets. Publ. Math., Inst. Hautes Étud. Sci., 67:5–42, 1988.
[6] J. Distexhe. Triangulating symplectic manifolds. Université Libre de Bruxelles, PhD thesis, 2019.
[7] S. K. Donaldson. A new proof of a theorem of Narasimhan and Seshadri. J. Differ. Geom., 18:269–277, 1983.
[8] Y. M. Eliashberg. A theorem on the structure of wave fronts and its applications in symplectic topology. Funct. Anal. Appl., 21(1-3):227–232, 1987.
[9] S. Etourneau. Approximation $C^1$ d’immersions isotropes lisses par des immersions isotropes PL. Nantes University, PhD thesis, 2023.
[10] B. Gratza. Piecewise linear approximations in symplectic geometry. ETH Zurich, PhD thesis, 1998.
[11] M. Gromov. Pseudo holomorphic curves in symplectic manifolds. Invent. Math., 82:307–347, 1985.
[12] M. Gromov. Partial differential relations, volume 9 of Ergeb. Math. Grenzgeb., 3. Folge. Springer, Cham, 1986.
[13] F. Jauberteau and Y. Rollin. Numerical flow for polyhedral symplectic maps. In preparation, 2024.
[14] F. Jauberteau, Y. Rollin, and S. Tapie. Discrete geometry and isotropic surfaces. Mém. Soc. Math. Fr., Nouv. Sér., 161:1–99, 2019.
[15] F. C. Kirwan. Cohomology of quotients in symplectic and algebraic geometry, volume 31 of Math. Notes (Princeton). Princeton University Press, Princeton, NJ, 1984.
[16] F. Lalonde and D. McDuff. The geometry of symplectic energy. Ann. Math. (2), 141(2):349–371, 1995.
[17] J. A. Lees. On the classification of Lagrange immersions. *Duke Math. J.*, 43:217–224, 1976.

[18] E. Lerman. Gradient flow of the norm squared of a moment map. *Enseign. Math. (2)*, 51(1-2):117–127, 2005.

[19] S. Lojasiewicz. Ensembles semi-analytiques. *IHES preprint*, 1965.

[20] D. McDuff and D. Salamon. *Introduction to symplectic topology*, volume 27 of *Oxf. Grad. Texts Math.* Oxford: Oxford University Press, 3rd edition edition, 2016.

[21] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory.*, volume 34 of *Ergeb. Math. Grenzgeb.* Berlin: Springer-Verlag, 3rd enl. ed. edition, 1994.

[22] Y. Rollin. Polyhedral approximation by Lagrangian and isotropic tori. *J. Symplectic Geom.*, 20(6):1349–1383, 2022.

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