A Note on Objects Built From Bricks without Corners

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Abstract

We report a small advance on a question raised by Robertson, Schweitzer, and Wagon in [RSW02]. They constructed a genus-13 polyhedron built from bricks without corners, and asked whether every genus-0 such polyhedron must have a corner. A brick is a parallelopiped, and a corner is a brick of degree three or less in the brick graph. We describe a genus-3 polyhedron built from bricks with no corner, narrowing the genus gap.

1 Introduction

Sibley and Wagon [SW00] proved that any collection of parallelograms glued whole-edge to whole-edge must have at least one “elbow”: a parallelogram with at most two neighbors. This enabled them to prove that such tilings are 3-colorable. The analogous question in 3D is [Wag02, DO03]: Must every object built from parallelopipeds (henceforth, bricks) have at least one corner, a brick with at most three neighbors? The bricks must be properly joined: each pair is either disjoint, or intersects either in a single point, a single whole edge of each, or a single whole face of each. Two bricks in a collection are adjacent if they share a single whole face. Define the brick graph of a collection of bricks to have a node for each brick, and an arc for each pair of adjacent bricks. The question is whether there must exist a node of degree ≤ 3 in the brick graph. If so, 4-colorability could be established.

The answer is no: Robertson, Schweitzer, and Wagon found a polyhedron with no corner. This settled one question but raised another: Might this be true for a topological ball, i.e., an object of genus 0 (their Question 1)? Their example has a high genus; we show below its genus is 13. The main purpose of this note is to describe another example that has genus 3.

2 The Buttressed Octahedron

Fig. 1 shows the 52-brick example from [RSW02] in two views. As shown, it has corners, but when each brick is partitioned into eight sub-bricks from its center, it is an object built from bricks with no corners.

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The authors did not report its genus. We compute the genus from the Euler characteristic $V - E + F = \chi$, which is equal to $2 - 2g$, where $g$ is the genus. This computation is performed on the unrefined object in the figure; because $\chi$ is a topological invariant, it is unaltered by refinement/splitting. The calculations are shown in Table 1.

| Pieces          | $V$   | $E$   | $F$   |
|-----------------|-------|-------|-------|
| ring (4 quarters) | $4 \times 20 = 80$ | $4 \times 40 = 160$ | $4 \times 16 = 64$ |
| arch (2)        | $2 \times 30 = 60$   | $2 \times 66 = 132$  | $2 \times 32 = 64$  |
| buttress (8)    | 0     | $8 \times 4 = 32$   | $8 \times 4 = 32$   |
| sum             | 140   | 324   | 160   |

Table 1: Euler characteristic calculations $V - E + F = -24$.

\[
V - E + F = 140 - 324 + 160 = 2 - 2g \\
g = 13
\]

3 The ZZ-Object

The overall design of our new example is shown in Fig. 2. It consists of two Z-shaped paths connecting four cubes. Each of the long connectors has no corner when split lengthwise into four bricks. Similarly, the four cubes have no corners when split into eight cubes. Prior to splitting, the object consists of
only 10 bricks. However, as is evident from the figure, it is self-intersecting. In the figure, the centers of the four cubes are staggered at these coordinates:

\[(30, 40, 50)\]
\[(60, 10, 40)\]
\[(10, 20, 30)\]
\[(55, 30, 10)\]

The design of the object relies on this observation: If it can be arranged that every brick has two opposite faces covered by other bricks, then splitting will render it cornerless, with every sub-brick with \(\leq 2\) exposed faces, and so of degree \(\geq 4\).

The self-intersection can be removed by zig-zagging one of the \(Z\)s. The resulting object of 14 bricks is shown in Fig. 3 in two views.

We again compute the Euler characteristic \(\chi\). Because of its topological invariance, we compute it for the simpler object in Fig. 2 which has the same genus as the object in Fig. 3. We count all vertices as part of the cubes. All 12 edges of every cube are exposed, but only 3 faces of each. Each of the 6 connecting bricks contributes 4 edges and 4 faces. The calculations are shown in Table 2.

\[
V - E + F = 32 - 72 + 36 = 2 - 2g
\]

\[
g = 3
\]
Figure 3: A non-self-intersecting object of genus 3 with no corners (after refinement).
| Pieces          | $V$    | $E$     | $F$  |
|----------------|--------|---------|------|
| cubes (4)      | $4 \times 8 = 32$ | $4 \times 12 = 48$ | $4 \times 3 = 12$ |
| connectors (6) | $0$    | $6 \times 4 = 24$ | $6 \times 4 = 24$ |
| sum            | $32$   | $72$    | $36$ |

Table 2: Euler characteristic calculations $V - E + F = -4$.

4 Conclusion

We have constructed a 14-brick, genus-3 object, which when refined by splitting, has no corners. The question remains whether there exists an object without corners of smaller genus: 2, 1, or 0. We conjecture that all brick objects of genus 0 must have a corner. This gains some support from the work in [GO03] which shows that one class of topological balls always has at least four corners.

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