Transverse linear and orbital angular momenta of beam waves and propagation in random media

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Abstract

For paraxial propagation of scalar waves, the classic electromagnetic theory definition of transverse linear (TLM) and orbital angular (OAM) momenta of the beam wave are represented in terms of the coherence function. We show in examples that neither is the presence of optical vortices necessary for the intrinsic OAM, nor does the presence of optical vortices warrant the non-zero intrinsic OAM. The OAM is analyzed for homogeneously coherent and twisted partially coherent beam waves. A twisted Gaussian beam has an intrinsic OAM with a per-unit power value that can be continuously changed by varying the twist parameters. Using the parabolic propagation equation for the coherence function, we show that both the total TLM and OAM are conserved for the free-space propagation, but not for propagation in an inhomogeneous medium. In the presence of the random inhomogeneous medium, the total TLM and OAM are conserved in average, but the OAM fluctuations grow with the propagation path. This growth is slower for beams with rotation-symmetric irradiance.

Keywords: wave propagation, atmospheric optics, atmospheric turbulence, singular optics, OAM, coherence

1. Introduction

Orbital angular momentum (OAM) has been one of the most popular topics of beam propagation studies in the last decade, as evidenced by several recent books [1–5] and review papers [6–12]. This interest is motivated by two major application areas: optical manipulation of small particles and free-space optical communication [13, 14]. In most cases discussed in the literature, the OAM is associated with the optical vortices of the beam wave front. For the purposes of this work, optical vortices are the isolated nulls of the irradiance, in the vicinity of which phase necessarily has helical form.

The common misconception that the intrinsic OAM is inevitably associated with the phase vortices is apparently based on misinterpretation of the seminal Allen et al paper [15]. This work properly stated that vortex Laguerre–Gaussian (LG) beams, carry intrinsic OAM, but it never claimed that the presence of vortices is necessary for the OAM existence. There are many examples in the OAM literature [6, 16, 17] that show that the presence of the optical vortices is not necessary for the existence of the OAM. Still, in the literature, beam waves carrying the intrinsic OAM are almost inevitably associated with helically phased beams. One of the intents of this paper is to reiterate that the intrinsic OAM and the optical vortices are not tightly connected. The second is to examine evolution of OAM on propagation in the free-space and inhomogeneous medium.

In section 2, we revisit the classic definitions of the transverse linear momentum (TLM) and OAM of a paraxial scalar wave and relate both to the wave coherence function. In section 3, we use the coherence function representation to show several examples of coherent and partially coherent beam waves that generalize the typical helical structure of the optical vortices, and demonstrate that the intrinsic OAM and optical vortices are only loosely related. In section 4, we discuss the TLM and OAM evolution during propagation in a free-space, inhomogeneous medium, and TLM and OAM...
statistics for propagation in a random inhomogeneous medium such as atmospheric turbulence.

2. TLM and OAM of scalar paraxial waves

Starting from the general definition of the linear and angular momentum densities of the electromagnetic field [6, 15, chapter 6 in 18],

\[ \mathcal{L} = \varepsilon_0 \mathbf{E} \times \mathbf{B}, \quad \mathcal{M} = \varepsilon_0 \mathbf{E} \times (\mathbf{E} \times \mathbf{B}). \] (1)

For the case of a linear polarized narrowband paraxial wave propagating along the z-direction, where the electric field \( \mathbf{E} \) is defined in terms of a scalar complex envelope by

\[ \mathcal{E} = \hat{x} \text{Re}[u(r, z) \exp(ikz - i\omega t)]. \] (2)

The period-averaged TLM density \( \mathbf{L}(r, z) \) and z-component of the OAM density \( M(r, z) \) were calculated in [15] as

\[ \mathbf{L}(r, z) = \text{Im}[u^\dagger(r, z) \nabla u(r, z)], \quad M(r, z) \hat{z} = \mathbf{r} \times \mathbf{L}(r, z). \] (3)

Here, \( \mathbf{L}(r, z) \) is the component of the linear momentum density \( \mathcal{L} \), which is orthogonal to the nominal propagation direction \( \hat{z} \), \( \mathbf{r} = (x, y) \) is the transverse coordinate, and we dropped the inconsequential factor \( \omega_0 \Omega \). An equivalent, but more intuitive definition for the OAM density

\[ M(r, \theta, z) = \text{Im} \left[ u^\dagger(r, \theta, z) \frac{\partial}{\partial \theta} u(r, \theta, z) \right] \] (4)

was used in [19] based on the polar coordinates in the transverse plane.

Both TLM and OAM densities are related to the coherence function of the optical wave

\[ \gamma(r, \rho, z) \equiv u^\dagger(r + \frac{\rho}{2}, z) u(r - \frac{\rho}{2}, z). \] (5)

The overbar in equation (5) indicates the averaging over the possible fluctuations of the radiation source. The coherence function models the ‘slow detector’ situation when the characteristic time of the source fluctuations is much smaller than the optical detector response time, as is the case in regular daylight imaging. In the case of coherent wave or ‘fast detector’ conditions, assumed in equations (3) and (4), the coherence function is just a product of field and conjugate in two points separated by vector \( \rho \). It is straightforward to show that for the coherent case

\[ \mathbf{L}(r, z) = -i \nabla_\rho \gamma(r, 0, z), \quad M(r, z) \hat{z} = -\mathbf{r} \times \nabla_\rho \gamma(r, 0, z). \] (6)

After presenting \( u(r, z) \) in the amplitude and phase form and introducing the wave irradiance \( I(r, z) \)

\[ u(r, z) = A(r, z) \exp[i\phi(r, z)], \quad A(r, z) \geq 0, \]

\[ I(r, z) = A^2(r, z) = \gamma(r, 0, z). \] (7)

the TLM and OAM densities are related to the irradiance-weighted phase gradient

\[ \mathbf{L}(r, z) = I(r, z) \nabla \phi(r, z), \quad M(r, z) \hat{z} = I(r, z) \mathbf{r} \times \nabla \phi(r, z). \] (8)

Using polar coordinates in equation (8), the TLM and OAM densities can be presented as

\[ \mathbf{L}(r, \theta, z) = I(r, \theta, z) \left[ \frac{\partial \phi(r, \theta, z)}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \phi(r, \theta, z)}{\partial \theta} \hat{\theta} \right], \]

\[ M(r, \theta, z) = I(r, \theta, z) \frac{\partial \phi(r, \theta, z)}{\partial \theta}. \] (9)

The total TLM and OAM of the beam wave are

\[ \mathbf{L}(z) = \int d^2 \mathbf{r} \mathbf{L}(r, z), \quad M(z) = \int d^2 r M(r, z). \] (10)

Clearly, the total OAM of the beam depends on the location of the coordinate origin. Namely, if the OAM is measured relative to point \( \mathbf{r}_c \), then

\[ M_{r_c}(z) \hat{z} = -i \int d^2 r (\mathbf{r} - \mathbf{r}_c) \times \nabla_\rho \gamma(r, 0, z) \quad = M(z) \hat{z} - \mathbf{r}_c \times \mathbf{L}(z). \] (11)

Only in the case when the total TLM is zero, the beam is said to have intrinsic OAM that is frame invariant [20]. An equivalent representation of the total TLM and OAM in terms of the first and second geometrical moments of the Wigner function was presented in [16, 17].

Equation (6) can be readily extended to the partially coherent waves’ case by recognizing that it is still valid for the instantaneous random TLM, but ‘slow’ detector measures the mean TLM over the source fluctuations. In other words, for the partially coherent waves equation (6), provides the mean TLM and OAM densities of a partially coherent beam wave. Note that equations (8) and (9) are valid for the coherent waves only. Formal Mercer theorem-based generalization of the OAM for partially coherent beams was discussed in [21] in terms of Wigner function formalism.

3. Examples

In this section, we present several simple examples of the TLM and OAM calculations. These examples illustrate the relation between the OAM and the optical vortices. The first three examples deal with coherent beams where the phase can be identified, and give examples of the waves with vortex OAM (VOAM) and asymmetry OAM (AOAM), as introduced in [17]. Partially coherent beam waves are used in the last two cases. All examples consider the field in a single plane, and we drop the z-dependence in our notations for this section.
3.1. Radial irradiance-angular phase (RI-AP) beams

Consider a special coherent beam case when irradiance is rotationally symmetric, phase depends only on the angular variable, and complex field envelope is
\[ u(r, \theta) = \sqrt{I(r)} e^{i \phi(\theta)}, \quad I(r) \geq 0. \] (12)

This type of beam wave includes the LG beams that are ubiquitous in the OAM literature. However, unlike the LG beams, the RI-AP beams do not, in general, preserve their functional form in propagation. TLM and OAM densities for this case are
\[ L(r, \theta) = \frac{1}{r} I(r) \frac{d \phi(\theta)}{d \theta} (-\hat{x} \sin \theta + \hat{y} \cos \theta), \]
\[ M(r, \theta) = I(r) \frac{d \phi(\theta)}{d \theta}. \] (13)

In order to have intrinsic OAM, it is necessary that the total TLM is zero, as was discussed in the previous section, equation (11)
\[ L = \int_0^\infty r dr \int_{-\pi}^{\pi} d \theta \ L(r, \theta) = \left[ \int_0^\infty dr I(r) \int_{-\pi}^{\pi} d \theta \frac{d \phi(\theta)}{d \theta} \right] \times (-\hat{x} \sin \theta + \hat{y} \cos \theta) = 0. \] (14)

This implies that the Fourier series for \( d \phi(\theta)/d \theta \) lack the 2\( \pi \)-periodic term. Namely,
\[ \frac{d \phi(\theta)}{d \theta} = a_0 + a_2 \cos 2\theta + b_2 \sin 2\theta + a_3 \cos 3\theta + b_3 \sin 3\theta + ... \] (15)

In order to have a continuous field, it is necessary that
\[ \exp[i \phi(\theta + 2\pi)] = \exp[i \phi(\theta)] \] (16)

and, this constraint excludes the ‘fractional vortex’ beams [22] from consideration. Accounting for equation (15), the phase is presented as
\[ \phi(\theta) = l \theta + \frac{a_2}{2} \sin 2\theta - \frac{b_2}{2} \cos 2\theta + \frac{a_3}{3} \sin 3\theta - b_3 \cos 3\theta + ... \quad l = 0, \pm 1, \pm 2, \pm 3, ... \] (17)

For \( l = 0 \) continuity of the complex field \( u(r) \) at \( r = 0 \) requires that \( I(0) = 0 \).

Figure 1 shows an example of a phase front given by equation (17) for \( l = 1 \). All coefficients \( a_i \) and \( b_i \) are zero, besides \( b_0 = 5 \).

With the phase given by equation (17) and OAM density by equation (13), the total OAM of the beam is
\[ M = \int_0^\infty r dr I(r) \int_0^{2\pi} d\theta \frac{d \phi(\theta)}{d \theta} = 2P, \] (18)

where \( P \) is the total beam power. This relationship between the OAM and total power is well known for the LG [15] Bessel–Gaussian [23, 24] and Airy beams [25], but here it is extended to the arbitrary irradiance distributions, and a phase that is not a simple helix.

![Figure 1](image-url)

**Figure 1.** Phase described by equation (17) with \( l = 1 \) and \( b_0 = 5 \). Spiral curve shows trace of conventional helical phase.

3.2. Vortex beam without intrinsic OAM

Consider a slightly modified version of the RI-AP beams, equation (12), for the field of a coherent where irradiance has harmonic angular dependence
\[ u(r, \theta) = \sqrt{I(r)(1 + \eta \cos \theta)} e^{i \phi(\theta)}, \quad I(r) \geq 0, \quad |\eta| < 1. \] (19)

Using equations (9) and (10), the total TLM and OAM of this beam are:
\[ L = \int_0^\infty r dr \int_{-\pi}^{\pi} d \theta \left[ -\hat{x} \int_0^{2\pi} d \theta (1 + \eta \cos \theta) \frac{d \phi(\theta)}{d \theta} \sin \theta \right. \]
\[ + \int_0^{2\pi} d \theta (1 + \eta \cos \theta) \frac{d \phi(\theta)}{d \theta} \cos \theta \left. \sin \theta \right] \]
\[ M = \frac{P}{2\pi} \int_0^{2\pi} \frac{d \theta}{\cos \theta} (1 + \eta \cos \theta) \frac{d \phi(\theta)}{d \theta}. \] (20)

For this example, we keep the first three terms of the Fourier series for the phase and require that the total TLM is zero, which leads to equations
\[ 2b_1 + \eta b_2 = 0, \quad 2\eta + 2a_1 + \eta b_2 = 0. \] (21)

The total OAM of this beam is
\[ M = P \left( l + \frac{2}{2} a_2 \right). \] (22)

and the coherent beam with the field
\[ u(r, \theta) = \sqrt{I(r)(1 + \eta \cos \theta)} \exp \left[ i \theta - \frac{2}{\eta} \frac{1}{\sin \theta} \sin \theta \right. \]
\[ + i \left( \frac{2}{\eta} - 1 \right) I \sin 2\theta \left. \right] \] (23)

has a phase vortex but zero OAM. This is an example of VOAM and AOAM [17] canceling each other. Figure 2 shows phase and irradiance of such a beam for \( l = 1 \), \( \eta = 0.75 \) and \( I(r) = r^2 \exp(-r^2) \).
3.3. Intrinsic OAM beams without phase vortices

Consider a coherent beam wave with irradiance and phase distributions:

\[ I(r, \theta) = A^2(r)[1 + \eta(r^2 - 2rd \cos \theta + d^2)], \]
\[ \text{Im}[A(r)] = 0, \quad \text{Re}[A(r)] > 0, \quad \eta > 0, \]
\[ \phi(r, \theta) = r(a_1 \sin \theta + b_1 \cos \theta + a_2 \sin 2\theta + b_2 \cos 2\theta). \]  

This choice of the irradiance and phase distributions is motivated by the integral formula for the OAM that follows from equations (9) and (10)

\[ M = \int_0^\infty r dr \int_0^{2\pi} d\theta I(r, \theta) \frac{\partial \phi(r, \theta)}{\partial \theta}. \]

Examination of the integral over the angular variable, \( \theta \), for a fixed radius \( r \) suggests that the presence of the angular dependence of the irradiance distribution can possibly result in the non-zero value of this integral even for the zero-average periodic phase, as given in equation (24).

The irradiance in equation (24) is a zero-centered rotationally symmetric distribution modified by a ‘bump’ displaced by distance \( d \) along the \( x \)-axis. The phase is continuous across the whole plane due to the absence of the linear in \( \theta \) term, compared to equation (17), and the presence of the \( r \) factor.

Direct calculation of the integrals for TLM and OAM, equation (10), leads to

\[ L = \left[ b_1(P_1 + \eta d^2P_1 + \eta P_3) - \frac{3}{2}b_2 \eta dP_2 \right] \hat{\xi} + \left[ a_1(P_1 + \eta d^2P_1 + \eta P_3) - \frac{3}{2}a_2 \eta dP_2 \right] \hat{\eta}, \]
\[ M = -a_1 \eta dP_3. \]

Here, we introduced shorthand notations for the geometrical moments of \( A^2(r) \)

\[ P_n = 2\pi \int_0^\infty r^n dr A^2(r). \]  

Total TLM can be set to zero by choosing

\[ a_2 = \frac{2}{3\eta dP_2} (P_1 + \eta d^2P_1 + \eta P_3) a_1, \quad b_2 = \frac{2}{3\eta dP_2} (P_1 + \eta d^2P_1 + \eta P_3) b_1, \]

and intrinsic OAM does not vanish if \( a_1 \neq 0 \). Figure 3 show an example of such a beam wave with Gaussian \( A(r) = \exp(-r^2/2w^2) \). The values of the parameters are given in the figure captions. We emphasize that the irradiance of this beam is positive, the unwrapped phase is continuous and has no vortices or any kind of singularities, but the beam has intrinsic total OAM \( M \approx -0.21P \).

Without performing the detailed calculations of the first and second moments of the Wigner function, one cannot be certain that this is an example of the pure AOAM beam. Indeed, example E of [17] is a vortex-less beam with both VOAM and OAM components of its intrinsic OAM. An example of vortex-less beams’ pure AOAM intrinsic OAM was constructed in [6] by passing an elliptical Gaussian beam through an astigmatic lens. Similar to our example, it was noted in [6] that the total OAM per unit power can be continuously varied by adjusting the beam and lens parameters, and can greatly exceed unity.

3.4. Homogeneously coherent partially coherent (HCPC) beams

We now turn our attention to the more general case of partially coherent beams. Equation (6) relates the average over the source fluctuations TLM and OAM densities to the wave coherence function. The simplest case of the partially coherent beam wave is the HCPC beam waves with the coherence...
function

$$\gamma(\mathbf{r}, \rho) = A(\mathbf{r} + \frac{\rho}{2})A(\mathbf{r} - \frac{\rho}{2})$$

$$\times \exp \left[ i\phi(\mathbf{r} + \frac{\rho}{2}) - i\phi(\mathbf{r} - \frac{\rho}{2}) \right] \chi(\rho).$$  (29)

Here,

$$A(\mathbf{r}) \geq 0, \quad \text{Im}[\phi(\mathbf{r})] = 0, \quad \chi(0) = 1,$$

$$\chi(\kappa) = \frac{1}{4\pi^2} \int d^2 \rho \chi(\rho) \exp(-i\kappa \cdot \rho) \geq 0, \quad (30)$$

and the first three constraints do not cause any loss of generality, while the last constraint on the coherence coefficient spectrum warrants the necessary semi-positive definiteness (SPD) property of the coherence function. Clearly, ubiquitous Gaussian-Schell model beams form a specific case of the HCPC class of beam waves.

As was discussed in [26], this type of wave is formed when radiation from a virtual incoherent source with spatial brightness distribution proportional to $$\tilde{\chi}(kr/L)$$ passes through an aperture with complex transmission coefficient $$A(\mathbf{r})\exp[i\phi(\mathbf{r})]$$.

The TLM corresponding to equation (29) is calculated with the help of equation (6) as

$$\mathbf{L}(\mathbf{r}) = A^2(\mathbf{r}) \nabla \varphi(\mathbf{r}) - i\mathbf{f}(\mathbf{r}) \nabla \chi(0).$$  (31)

The first term in the right-hand part is related to the coherent effects introduced by the aperture function, and is not related to the partial coherence. It is noteworthy that the coherence coefficient enters the TLM density only as its gradient at zero separation, and the overall shape of the coherence coefficient $$\chi(\mathbf{r})$$ is irrelevant here. It is easy to show using the propagation model described in the next section that $$\nabla \chi(0)$$ is proportional to the centroid position of the above-mentioned virtual incoherent source, and the size and brightness distribution of the source are of no importance. In fact, a coherent point source can be used with the same effect on the TLM density. The conclusion here is that the HCPC beam waves do not add much variety to the TLM and OAM in comparison to coherent waves.

### 3.5. Twisted Gaussian beams

Simon and Mukunda [27–29] introduced and investigated a partially coherent twisted beam model with coherence function that can be written as

$$\gamma(\mathbf{r}, \rho) = \frac{b_0^2}{b^2} \exp \left( -\frac{R^2}{2b^2} - \frac{\rho^2}{4b_c^2} + i\frac{k\sigma_x \sigma_y}{b^2} (\mathbf{r} \times \rho) \cdot \hat{z} \right).$$  (32)

It was shown in [26] that this partially coherent beam wave can be presented as a result of averaging of the randomly displaced and tilted collimated Gaussian beam

$$u(\mathbf{r}) = \exp \left( -\frac{(\mathbf{r} - \mathbf{d})^2}{2b_0^2} + ik\gamma \cdot (\mathbf{r} - \mathbf{d}) \right).$$  (33)

Here, $$\mathbf{d} = (\xi, \eta)$$ is the random beam displacement, and $$\gamma = (\alpha, \beta)$$ is the random phase front tilt. Assuming that $$\mathbf{d}$$ and $$\gamma$$ have zero-average normal distribution with a covariance matrix of special form

$$\begin{pmatrix} \xi \\ \eta \\ \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \xi & \eta & \alpha & \beta \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sigma_d^2 & 0 & 0 & i\sigma_d \sigma_\gamma \\ 0 & \sigma_\gamma^2 & -i\sigma_d \sigma_\gamma & 0 \\ 0 & -i\sigma_d \sigma_\gamma & \sigma_d^2 & 0 \\ i\sigma_d \sigma_\gamma & 0 & 0 & \sigma_\gamma^2 \end{pmatrix},$$  (34)

where $$|t| \leq 1$$ is the correlation coefficient of displacements and tilts, it is straightforward to check that the coherence

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**Figure 3.** Phase, left panel, and irradiance, right panel, of the vortex-less beam wave with non-vanishing intrinsic OAM described by equation (24) with $$a_i = 1.0, b_i = 0, w = 0.5, d = 0.5$$ and $$\eta = 10$$ Arrows show the TLM vector field.
function corresponding to this wave has the form of equation (32) if
\[ \rho_{\ell}^2 = \frac{b_0^2 b^2}{b_0^2 (1 + k^2 \sigma_x^2 b_0^2) + \sigma_{\rho}^2 (1 + k^2 \sigma_y^2 b_0^2 (1 - r^2))}. \] (35)
and the beam width is \( b^2 = b_0^2 + \sigma_{\rho}^2 \). Note that this ‘constructive’ approach [26] to the coherence function formation guarantees its SPD property.

Irradiance and TLM and OAM densities for coherence function, equation (32), can be calculated with the help of equation (8) as
\[ I(r) = \frac{b_0^2}{b^2} \exp \left( - \frac{r^2}{b^2} \right), \]
\[ L(r) = I(r) \frac{k \sigma_x \sigma_y}{b_0^2} (-\gamma \mathbf{\hat{x}} + \mathbf{\hat{y}}), \]
\[ M(r) = I(r) \frac{k \sigma_x \sigma_y}{b_0^2} r^2. \] (36)

Figure 4 shows an example of the TLM vector field superimposed on the contour plot of the beam irradiance, both given by equation (36). The total power and intrinsic OAM of the twisted beam, equation (32) are
\[ P = \pi b_0^2, \quad M = P k \sigma_x \sigma_y. \] (37)

It is essential to note that the individual samples of this beam, given by equation (33), do not carry intrinsic OAM. Namely, the total instantaneous TLM and OAM of the beam wave given by equation (33) are
\[ L = P k \gamma, \quad M = P k (\mathbf{d} \times \gamma). \] (38)

In the statistical averaging process, the total TLM vanishes, but the OAM is preserved to some extent by statistical correlations between the random displacement \( \mathbf{d} \) and tilt \( \gamma \), which allows the intrinsic OAM to develop.

3.6. Intrinsic OAM of nonstationary beam wave
Discussion of the intrinsic OAM emergence for the Simon and Mukunda twisted beam suggests a simple way of generating the intrinsic OAM beams by averaging nonstationary, but not stochastic beam waves. The structure of the covariance matrix, equation (34), indicates that the random tilts \( \gamma \) are always orthogonal to the random displacements \( \mathbf{d} \). We introduce a rotating tilted Gaussian beam:
\[ u(r, t) = \exp \left( \frac{(r - r_{c}(t))^2}{2 b_0^2} + ik \gamma(t) \cdot (r - r_{c}(t)) \right), \]
\[ r_{c}(t) = l (\mathbf{\hat{x}} \cos t + \mathbf{\hat{y}} \sin t), \]
\[ \gamma(t) = \delta (-\mathbf{\hat{x}} \sin t + \mathbf{\hat{y}} \cos t). \] (39)

The center of this beam traces a circle with radius \( l \), and the phase front tilt remains orthogonal to the instantaneous radius-vector of the beam center position. For brevity, we use a unit angular velocity here, but of course the actual angular rotation frequency \( \omega_{\ell} \) must be much smaller than the carrier frequency. The nonstationary coherence function of this beam is
\[ \gamma(r, \rho, t) = \exp \left( \frac{(x - l \cos t)^2 + (y - l \sin t)^2}{b_0^2} - \frac{r^2}{4 b_0^2} - ik \delta (\xi \sin t - \eta \cos t) \right), \]
\[ r = x \mathbf{\hat{x}} + y \mathbf{\hat{y}}, \quad \rho = \xi \mathbf{\hat{x}} + \eta \mathbf{\hat{y}}. \] (40)
The total power, TLM and OAM of this beam are readily calculated using equations (6) and (10) as

\[ P(t) = \pi b_0^2, \quad L(t) = P k \delta(-\lambda \sin t + \gamma \cos t), \]

\[ M(t) = P k \delta. \]  

(41)

Period-averaged values are

\[ P = \frac{1}{2\pi} \int_0^{2\pi} \text{d}t P(t) = \pi b_0^2, \]

\[ L = \frac{1}{2\pi} \int_0^{2\pi} \text{d}t L(t) = 0, \]

\[ M = \frac{1}{2\pi} \int_0^{2\pi} \text{d}t M(t) = P k \delta, \]  

(42)

and the beam has a non-zero average intrinsic OAM. For optical manipulation applications, in a case when the particle response time is larger than the period of rotation, this nonstationary beam would exert rotational momentum just as the stationary beam with intrinsic OAM would.

4. Beam propagation effects on TLM and OAM

In this section, we discuss the evolution of the TLM and OAM during paraxial beam wave propagation, first in the free space and then in the presence of an inhomogeneous medium.

4.1. Free space propagation

Propagation of the paraxial beam wave field is described by the parabolic equation

\[ 2i k \frac{\partial u(r, z)}{\partial z} + \Delta_r u(r, z) = 0. \]  

(43)

The corresponding equation for propagation of the coherence function is

\[ i k \frac{\partial \gamma(r, \rho, z)}{\partial z} + \nabla_r \cdot \nabla_\rho \gamma(r, \rho, z) = 0, \]  

(44)

and it extends the paraxial propagation formalism to propagation of partially coherent waves. Setting \( \rho = 0 \) in equation (44) leads to the equation

\[ \frac{\partial H(r, z)}{\partial z} = -\frac{1}{k} \nabla_r \cdot L(r, z), \]  

(45)

which, expectedly, relates the change of the power density (irradiance) to the divergence of the TLM. For a bounded beam-type wave, where the field vanishes quickly with distance from the beam axis, integration of equation (45) over the transverse plane leads to the energy conservation principle

\[ \frac{\partial}{\partial z} \iint d^2 r H(r, z) = \frac{\partial}{\partial z} P(z) = 0. \]  

(46)

Total TLM can be calculated directly from the solution of equation (44), e.g. \([30–32]\)

\[ \gamma(r, \rho, z) = \frac{k^2}{4\pi z^2} \iint d^2 r_0 \iint d^2 \rho_0 \gamma(r_0, \rho_0, 0) \times \exp \left[ -\frac{ik}{z} (r - r_0)(\rho - \rho_0) \right]. \]  

(47)

Using equations (8) and (10) the total TLM after propagation from the plane \( z = 0 \) to plane \( z \) is

\[ L(z) = \frac{k^2}{4\pi z^2} \iint d^2 r_0 \iint d^2 \rho_0 \gamma(r_0, \rho_0, 0) \times \iint d^2 r (r - r_0) \exp \left[ -\frac{ik}{z} (r_0 - r)(\rho_0 - \rho) \right]. \]  

(48)

The last integral in equation (48) is recognized as being proportional to the gradient of the Dirac delta-function \( \nabla \delta(\rho_0) \), leading to

\[ L(z) = -i \iint d^2 r_0 \nabla_\rho \gamma(r_0, 0, 0) = L(0). \]  

(49)

and the conclusion that the total TLM of beam waves is preserved in the free-space propagation. In particular, this implies that a beam wave with intrinsic OAM in the initial plane \( z = 0 \) will maintain some intrinsic OAM in the propagation process.

In order to calculate the value of this OAM, following equations (8) and (10), we apply \( \nabla_\rho \) operator to equation (44), set \( \rho = 0 \), cross-multiply the result by \( r \) and integrate over \( r \). The result is the equation for the total OAM change on free-space propagation

\[ \frac{d}{dz} M(z) = \frac{1}{k} \iint d^2 r \frac{d}{dz} z \cdot \{ r \times \nabla_\rho \nabla_\rho \gamma(r, \rho, z) \} |_{\rho=0}. \]  

(50)

The second integral identity, equation (A8), implies that the right-hand term is identically zero, and

\[ M(z) = M(0). \]  

(51)

Hence, both total TLM and OAM are conserved on the free-space propagation, and intrinsic OAM, if exists, does not change on propagation.

Conservation of OAM is widely cited in the literature, and is the foundation for application of the OAM carrying beams for free-space optical communication. However, OAM conservation is usually stated without any proof, with the exception of [17] where the Wigner function moments formalism was used to demonstrate the OAM conservation in free-space paraxial propagation. Here, derivation is based on the paraxial propagation model presented by parabolic equation (44). In particular, all intrinsic OAM values calculated for the coherent and partially coherent examples of vortex and vortex-less beams in the previous section are conserved if the corresponding fields or coherence functions are used as initial conditions for wave propagation in equations (43) or (44).
4.2. TLM and OAM in an inhomogeneous medium

Paraxial propagation through an inhomogeneous medium with large and smooth variations of refractive index \( n(\mathbf{r}, z) \) is described by parabolic equation \([30–32]\)
\[
2i k \frac{\partial u(r, z)}{\partial z} + \Delta_n(\mathbf{r}, z) + 2k^2 n(\mathbf{r}, z) u(\mathbf{r}, z) = 0. \quad (52)
\]

A corresponding equation for the coherence function propagation is \([31]\)
\[
i k \frac{\partial^2 \gamma(r, \rho, z)}{\partial z^2} + \nabla_r \cdot \nabla_r \gamma(r, \rho, z)
+ k^2 \left[ n \left( r + \frac{\rho}{2}, z \right) - n \left( r - \frac{\rho}{2}, z \right) \right] \gamma(r, \rho, z) = 0.
\]
\[
(53)
\]

In the case of a random inhomogeneous medium, e.g., atmospheric turbulence, this equation describes averaged over the source fluctuations, but instantaneous in terms of the random medium fluctuations coherence function. Thus, it implicitly assumes that the source fluctuations are much faster than the medium fluctuations.

Setting \( \rho = 0 \) in equation (53) leads to equation (45), and the integration transverse plane leads to the energy conservation principle, equation (46). This is a well-known result that refractive turbulence does not affect the total beam power \([31]\).

The equation for the TLM density can be derived by applying the \( \nabla_r \) operator to equation (53) and setting \( \rho = 0 \)
\[
\frac{\partial}{\partial z} L(r, z) - \frac{1}{k} \nabla_r [\nabla_r \cdot \nabla_r \gamma(r, \rho, z)] \bigg|_{\rho = 0} - k \nabla n(r, z) I(r, z) = 0.
\]
\[
(54)
\]

The equation for the total TLM can be presented as
\[
\frac{\partial}{\partial z} L(z) - \frac{1}{k} \iint d^2 r \nabla_r [\nabla_r \cdot \nabla_r \gamma(r, \rho, z)] \bigg|_{\rho = 0} = k \iint d^2 r \nabla n(r, z) I(r, z),
\]
\[
(55)
\]

and according to the first integral identity, equation (A4), the second term in the left-hand part of equation (55) is zero. Therefore, the total TLM evolution in an inhomogeneous medium is described by the equation \([33]\)
\[
\frac{\partial}{\partial z} L(z) = k \iint d^2 r \nabla n(r, z) I(r, z).
\]
\[
(56)
\]

The right-hand part of this equation is just the irradiance-weighted mean gradient of the refractive index. With reference to equation (8), one can speculate that equation (55) represents the changes of the mean phase tilt caused by the ‘wedge’ component of inhomogeneity. Total TLM changes indicate that, in general, beam waves lose their initial intrinsic OAM immediately upon entering an inhomogeneous medium.

The equation for the OAM density can be obtained by cross-multiplying equation (55) by \( r \), and projecting the result on the \( z \)-axis.
\[
\frac{\partial}{\partial z} M(r, z) - \frac{1}{k} i \left[ r \nabla_r (\nabla_r \cdot \nabla_r \gamma(r, \rho, z)) \right] \bigg|_{\rho = 0}
\]
\[
- k \frac{\partial}{\partial z} [r \nabla n(r, z)] I(r, z) = 0,
\]
\[
(57)
\]

and the equation for the total OAM can be obtained by integrating equation (57) over the transverse plane:
\[
\frac{\partial}{\partial z} M(z) - \frac{1}{k} \iint d^2 r [r \nabla_r (\nabla_r \cdot \nabla_r \gamma(r, \rho, z))] \bigg|_{\rho = 0}
\]
\[
= k \iint d^2 r [r \nabla n(r, z)] I(r, z).
\]
\[
(58)
\]

According to the second integral identity, equation (A8), the second term in the left-hand part of equation (57) is zero. Therefore, the total OAM evolution in an inhomogeneous medium is described by the equation
\[
\frac{\partial}{\partial z} M(z) = k \iint d^2 r \left[ \frac{\partial}{\partial y} \gamma(r, z) - \frac{\partial}{\partial x} \gamma(r, z) \right] I(r, z).
\]
\[
(59)
\]

Just like the total TLM, the total OAM is not conserved in the inhomogeneous medium. This equation was also used in \([33]\).

4.3. Mean TLM and OAM in a random medium

Results of the previous subsection are valid for instantaneous TLM and OAM values. For random inhomogeneous media such as atmospheric turbulence, it is sensible to consider statistics of the TLM and OAM densities and totals over an ensemble of possible realizations of the random field \( n(\mathbf{r}, z) \). The simplest statistics are the mean values \( \langle L(r, z) \rangle \), \( \langle M(r, z) \rangle \), \( \langle L(z) \rangle \), and \( \langle M(z) \rangle \). Here, angular brackets indicate the averaging over the realizations (or time scales) of the medium fluctuations. In the case of the partially coherent sources, we assume that source fluctuations have a very short correlation time, and are not resolved by detector. The mean TLM and OAM are still related to the coherence function by equation (6), if the mean coherence function \( \Gamma(r, \rho, z) = \langle \gamma(r, \rho, z) \rangle \) is used. The equation for the mean coherence function can be derived from equation (53) using the Markov approximation \([31, 34]\)
\[
\frac{\partial \Gamma(r, \rho, z)}{\partial z} - \frac{i}{k} \nabla_r \cdot \nabla_r \Gamma(r, \rho, z)
\]
\[
+ \frac{\pi k^2}{4} H(\rho) \Gamma(r, \rho, z) = 0.
\]
\[
(60)
\]

Here,
\[
H(\rho) = 8 \iint d^2 \rho \Phi_0(\mathbf{k}, 0) [1 - \cos(\mathbf{k} \cdot \rho)],
\]
\[
(61)
\]

and \( \Phi_0(\mathbf{k}, \rho) \) is a 3D spectrum of the refractive index fluctuations. The solution of equation (60) is well known,
e.g. [31, 32]:

\[
\Gamma (\mathbf{r}, \rho, z) = \frac{k^2}{4 \pi^2 z^2} \int d^2 r_0 \int d^2 \rho_0 \gamma (\mathbf{r}_0, \rho_0, 0) \\
\times \exp \left( \frac{i k}{z} (\mathbf{r} - \mathbf{r}_0)(\rho - \rho_0) - \frac{\pi k^2}{4} \right) \\
\times \int_0^z \text{d}H \left[ \rho_0 \left( 1 - \frac{\zeta}{z} \right) + \rho_0 \frac{\zeta}{z} \right]
\]  
(62)

and offers a straightforward path for calculation of the mean TLM and OAM densities. We will not display these results here, but instead analyze the total mean TLM and OAM densities. We will not display these results either, e.g. [J. Opt. 20 (2018) 205602, M. Charmotskii].

4.4. Fluctuations of the total TLM and OAM

As follows from equations (56), (59) and (63), (64), a random inhomogeneous medium affects the instantaneous total TLM and OAM, but not the mean TLM and OAM values. The simplest statistic that provides insight into the OAM fluctuations is the normalized variance of the total TLM and OAM

\[
\sigma_{\text{TLM}}^2 = \frac{\langle L^2 (z) \rangle - \langle L (z) \rangle^2}{\langle L (z) \rangle^2}, \quad \sigma_{\text{OAM}}^2 = \frac{\langle M^2 (z) \rangle - \langle M (z) \rangle^2}{\langle M (z) \rangle^2}.
\]  
(65)

Using equation (8), the second statistical moments of TLM and OAM can be presented as

\[
\langle L^2 (z) \rangle = -\int d^2 r_1 \int d^2 r_2 \nabla_\rho \cdot \nabla_\rho \\
\Gamma_4 (\mathbf{r}_1, \rho_1 = 0, \mathbf{r}_2, \rho_2 = 0, z),
\]

\[
\langle M^2 (z) \rangle = -\int d^2 r_1 \int d^2 r_2 (\mathbf{r}_1 \times \nabla_\rho) \cdot (\mathbf{r}_2 \times \nabla_\rho) \\
\Gamma_4 (\mathbf{r}_1, \rho_1 = 0, \mathbf{r}_2, \rho_2 = 0, z).
\]  
(66)

Here, the fourth-order mean coherence function is introduced as

\[
\Gamma_4 (\mathbf{r}_1, \rho_1, \mathbf{r}_2, \rho_2, z) = \langle \gamma (\mathbf{r}_1, \rho_1, z) \gamma (\mathbf{r}_2, \rho_2, z) \rangle
\]  
(67)

and propagation of the second-order instantaneous coherence function is described by equation (55). The propagation equation for \( \Gamma_4 \) is well known under the Markov approximation [31, 34]

\[
\frac{\partial}{\partial z} \Gamma_4 (\mathbf{r}_1, \rho_1, \mathbf{r}_2, \rho_2) = \frac{i}{k} \text{grad} (\text{grad} \Gamma_4) + \frac{\pi k^2}{4} \Psi (\mathbf{r}_1, \rho_1, \mathbf{r}_2, \rho_2) \Omega_4, \quad \Omega_4 = 0,
\]

\[
\Psi (\mathbf{r}_1, \rho_1, \mathbf{r}_2, \rho_2) = H (\rho_1) + H (\rho_2)
\]

\[
H (\mathbf{r}_1 - \mathbf{r}_2 + \frac{\rho_1 - \rho_2}{2}) + H (\mathbf{r}_1 - \mathbf{r}_2 - \frac{\rho_1 - \rho_2}{2})
\]

\[
- H (\mathbf{r}_1 - \mathbf{r}_2 + \frac{\rho_1 + \rho_2}{2}) + H (\mathbf{r}_1 - \mathbf{r}_2 - \frac{\rho_1 + \rho_2}{2})
\]  
(68)

After applying the \( \nabla_\rho \) and \( \nabla_\rho \) operators to equation (68), setting \( \rho_{1,2} = 0 \), and integrating over \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \), with the help of the first integral identity in equation (A2), the equation for the mean square of the total TLM is

\[
\frac{\partial}{\partial z} \langle L^2 (z) \rangle = \frac{\pi k^2}{4} \int d^2 r_1 \int d^2 r_2 \langle I (\mathbf{r}_1, z) I (\mathbf{r}_2, z) \rangle
\]

\[
\times \Delta H (\mathbf{r}_1 - \mathbf{r}_2).
\]  
(69)

Similarly, after additional cross-multiplications by \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \), the equation for the mean square of the total OAM is

\[
\frac{\partial}{\partial z} \langle M^2 (z) \rangle = \frac{\pi k^2}{4} \int d^2 r_1 \int d^2 r_2 \langle I (\mathbf{r}_1, z) I (\mathbf{r}_2, z) \rangle
\]

\[
\times (\mathbf{r}_1 \times \nabla) (\mathbf{r}_2 \times \nabla) \Delta H (\mathbf{r}_1 - \mathbf{r}_2).
\]  
(70)

The second moment of irradiance in equations (69) and (70) complicates further development, and here we consider only perturbation solutions, where the product of the free-space irradiances replaces the said second moment.

\[
\langle I (\mathbf{r}_1, z) I (\mathbf{r}_2, z) \rangle \approx \hat{I}_0 (\mathbf{r}_1, z) \hat{I}_0 (\mathbf{r}_2, z).
\]  
(71)

Using the spectral representation, equation (61) and the normalized irradiance spectrum

\[
\hat{I}_0 (\kappa, z) = \frac{1}{4 \pi^2} \int d^2 r \exp (-i \kappa \cdot \mathbf{r}),
\]  
(72)

we present normalized (per unit power) total TLM variance as

\[
\sigma_{\text{TLM}}^2 (z) = 2 \pi k^2 \int_0^z d \zeta \int d^2 \kappa \kappa^2 \psi_0 (\kappa, \zeta) \hat{I}_0 (\kappa, \zeta)^2.
\]  
(73)

This equation strongly resembles the well-known equation for the variance of the beam wander, e.g. [35]

\[
\sigma_{\text{H}}^2 (z) = 2 \pi \int_0^z d \zeta \left( 1 - \frac{\zeta}{z} \right)^2 \int d^2 \kappa \kappa^2 \psi_0 (\kappa, \zeta) \hat{I}_0 (\kappa, \zeta)^2,
\]  
(74)

and it can be shown that equation (73) is just the variance of the phase tilt or angle of arrival of the beam wave, which is not surprising, considering equations (8) and (9).
Similarly, normalized (per unit power) total OAM variance in the first-order of perturbation theory is

$$\sigma^2_M(z) = 2\pi k^2 \int_0^\infty d\zeta \int d^2k \Phi_b(\kappa, \zeta) |\nabla_\kappa \tilde{I}_0(\kappa, \zeta)| \kappa^2.$$

(75)

It is easy to see that for any rotationally symmetric irradiance distribution, the cross-product in the right-hand part of equation (75) is zero. It can be shown that even in this case, the second-order perturbation term does not vanish, and becomes the leading term of the perturbation series for \(\sigma^2_M(z)\). Note that this includes the ubiquitous case of LG beams. The details of these elaborate calculations are outside of the scope of this paper. Instead, we consider a simple example of a coherent collimated elliptical Gaussian beam with initial field distribution

$$u(r, 0) = \exp\left(-\frac{x^2}{2a^2(0)} - \frac{y^2}{2b^2(0)}\right).$$

(76)

Free-space irradiance of this beam is

$$I(r, z) = \exp\left(-\frac{x^2}{a^2(z)} - \frac{y^2}{b^2(z)}\right),$$

$$a^2(z) = a^2(0) + \frac{z^2}{k^2a^2(0)},$$

$$b^2(z) = b^2(0) + \frac{z^2}{k^2b^2(0)}.$$  

(77)

Also, the first-order \(\sigma^2_M(z)\) can be readily calculated for the standard Kolmogorov turbulence spectrum

$$\Phi_b(\kappa) = 0.033C_n^2 \kappa^{-\frac{11}{3}}.$$  

(78)

at the plane \(z = L\) from equation (75) as

$$\sigma^2_M(L) = \frac{0.033}{\sqrt{2}} \pi k^2 \int_0^L dz C_n^2 |a^2(z) - b^2(z)|^{\frac{5}{2}} \int_0^\infty dx x^{\frac{5}{2}} \times \exp\left(-\frac{x a^2(z) + b^2(z)}{|a^2(z) - b^2(z)|}\right) I_0(x).$$

(79)

Figure 5 shows an example of the OAM variance calculated by numerical integration of equation (79) for turbulence conditions with a coherence radius of approximately 4 cm. The range of Fresnel numbers \(N = k a^2(0)/\zeta\) corresponding to the \(a(0)\) values in figure 5 is \(0.1 < N < 100\), and for the left side of the chart beams are divergent with \(b(z) > a(z)\) for the largest part of the propagation path. For the right side of the chart, wide collimated beams propagate similarly to a plane wave, and maintain \(a(z) > b(z)\). OAM variances are larger for beams with larger ellipticity in both cases. In the middle part, \(N \approx 1\), and \(a(z) > b(z)\) at the start, changing to \(b(z) > a(z)\) closer to the end of the path and having \(b(z) \approx a(z)\) in the middle. This results in the smaller average ellipticity values along the path, and smaller OAM fluctuations. Note that in our normalization, the OAM of a standard first-order LG beam is unity. Hence, for the case considered, random turbulence-induced OAM fluctuation can be comparable to the initial beam OAM, related to the original optical vortex.

Note that both perturbation results, equations (73) and (75) can be extended by using the mean irradiance \(\langle \tilde{I}(\kappa, z) \rangle\) instead of the free-space irradiance \(\tilde{I}_0(\kappa, z)\), as was proposed for the beam wander calculations in [36], and for variance of the total OAM in [33].

Mean values and variances of total OAM were considered for partially coherent Gaussian beams in [37], for Bessel–Gaussian beams in [38], and for LG beams in [39] using the technique described in [33], which is very similar to the one used here.

4.5. TLM and Shack–Hartmann wave front sensor (WFS)

Consider a field with coherence function \(\gamma(r, \rho)\) incident on a Shack–Hartmann lenslet array. Let the \(m\)th subaperture complex amplitude transmission function be

$$A(r - \mathbf{R}_m)\exp\left(-\frac{i k (r - \mathbf{R}_m)^2}{2F}\right), \quad \text{Im}[A(r)] = 0.$$  

(80)

Using equation (47) for the free-space propagation from the aperture plane to the focal plane, one can calculate the irradiance distribution in the focal plane \(z = F\) as

$$I_f(r) = \frac{k^2}{4\pi^2F^2} \int d^2r_\alpha \int d^2\rho_\lambda A\left(r + \frac{\rho_\lambda}{2}\right) \times A\left(r - \frac{\rho_\lambda}{2}\right) \gamma(r + \mathbf{R}_\alpha, \rho_\lambda)$$

$$\times \exp\left[-\frac{ik}{F} \cdot \rho_\lambda\right].$$  

(81)

Figure 5. Dependence of the OAM variance, \(\sigma^2_M\), equation (79), on the initial beam size \(a(0)\) for \(L = 1\) km, \(C_n^2 = 10^{-14} \text{ m}^{-2/3}\), and wavelength, \(\lambda = 1.0\ \mu\text{m}\). Parameter is the ratio \(b(0)/a(0)\) of the initial beam widths.
The total power in the focal plane calculated from equation (81) as

$$P_n = \iint d^2 r I_n(r) = \iint d^2 r A^2(r) \gamma(r_A + R_n, 0), \quad (82)$$

and is equal to the power flux through the aperture, as expected. The first geometrical moment (FGM) of this irradiance distribution is

$$G_n = \iint d^2 r r I_n(r) = -\frac{i F}{k} \iint d^2 r A^2(r) \nabla_{r_A} \gamma(r_A + R_n, 0). \quad (83)$$

Comparing this to the first equation (6) reveals that the FGM is proportional to the TLM averaged over the $n$th aperture, specifically

$$\bar{L}_n = \frac{1}{S_{A_n}} \iint d^2 r A^2(r) L(r_A + R_n) = G_n k S_{A_n} F,$$

$$S_A = \iint d^2 r A^2(r_A). \quad (84)$$

Equation (84) shows that the FGM data obtained from the Shack–Hartmann sensor provide the information about the TLM density. Of course, as is the case with any realistic sensor, the TLM data are sub-aperture—averaged, discrete, and are limited to the wave front area intercepted by the aperture array. The OAM density and total OAM estimate can be calculated using the discrete version of the second equations (6) and (10):

$$M_n = \hat{z}(R_n \times L_n) = \frac{k}{S_{A_n}} \hat{z}(R_n \times G_n),$$

$$M = S_A \sum_n \hat{z} \cdot (R_n \times \bar{L}_n) = \frac{k}{F} \sum_n \hat{z} \cdot (R_n \times G_n). \quad (85)$$

Note that the described procedure is valid for the general case of partially coherent beam waves, when the focal plane detector registers the averaged over-the-fast source fluctuations irradiance.

In comparison, the typical WFS use of the Shack–Hartmann sensor assumes a coherent incident wave, when equation (8) holds and the $n$th sub-aperture FGM is

$$G_n = \frac{F}{k} \iint d^2 r A^2(r) I(r_A + R_n) \nabla \phi(r_A + R_n). \quad (86)$$

The WFS strives to recover the sub-aperture–averaged phase slopes

$$\gamma_n = \frac{1}{k S_{A_n}} \iint d^2 r A^2(r) \nabla \phi(r_A + R_n), \quad (87)$$

and for this purpose, irradiance and power-flux fluctuations are treated as noise factors. Therefore, the mean phase-tilt estimate uses normalization by the measured power flux through the sub-aperture, $P_n$, equation (82):

$$\bar{\gamma}_n = \frac{1}{P_n} \bar{G}_n. \quad (88)$$

Phase slopes are further used to reconstruct the phase of the incident wave. In particular, the ‘slope discrepancy’ [40] can be used to detect the presence of optical vortices. However, the OAM cannot be estimated from the slopes alone.

5. Conclusions

For the paraxial scalar waves case, TLM and OAM densities of the electromagnetic theory are simply related to the wave coherence function in a fixed transverse plane. This allows the straightforward extension of the TLM and OAM densities concepts to the partially coherent waves case, and makes the powerful parabolic equations apparatus available for investigation of the TLM and OAM evolution on propagation.

In contrast to common beliefs, there is no definite connection between the intrinsic OAM and phase vorticity. In particular, we gave examples of single-vortex beams with zero intrinsic OAM (section 3.2) and vortex-less beams with nonzero intrinsic OAM (section 3.3). In the last case, the OAM per unit power does not have discrete values. We also showed examples of the RI-AP vortex beams that have more complicated phase fronts than the simple helixes of the LG-type beams, but still have discrete values of the OAM per unit power similar to the LG beams.

We discussed the TLM and OAM of some partially coherent beam waves. For the simple case of the homogeneously coherent, or generalized Schell-type beam waves, introduction of the partial coherence does not provide any new options for creation and control of the intrinsic OAM. However, the twisted Gaussian beam waves provide an example of a partially coherent beam with intrinsic OAM. This OAM emerges as a result of statistical averaging of instantaneous beams that do not have an intrinsic OAM.

Based on the latter example, we proposed a technique for the OAM beams generation based on the rotation of a simple coherent vortex-less tilted beam. The per-unit power OAM of these beams can be varied continuously by adjusting the rotation radius and tilt.

The fundamental relation between the TLM density and coherence function, allows to use a paraxial parabolic equation to study evolution of the TLM and OAM during wave propagation including propagation in inhomogeneous media.

Both total TLM and OAM are conserved in the free-space propagation case. For the paraxial propagation in inhomogeneous media, in general, neither TLM nor OAM are conserved.

In the case of random inhomogeneous medium, such as atmospheric turbulence, when the Markov approximation is feasible, the mean total TLM and OAM are conserved but instantaneous TLM and OAM values fluctuate.

The TLM fluctuations are closely related to the well-known effect of the wave front tilt. However, the OAM fluctuations are more delicate, and do not appear in the first-order perturbation theory for the beams with rotationally symmetric irradiance. This implies that the OAM fluctuation
development will be slower for the RI-AP beams, including the LG beams, than it would be for the less symmetric beams.

We calculated the first-order perturbation term for the OAM variance of the elliptical Gaussian beams and found that for realistic atmospheric propagation conditions, the OAM fluctuations can be comparable to the intrinsic OAM of a lower-order LG beam with the same power.

Finally, we showed that the TLM density can be measured by a conventional Shack–Hartmann sensor parallel with the conventional phase slopes.

Appendix. Two integral identities

In coordinate representation \( \gamma(\mathbf{r}, \rho, z) = \gamma(x, y, \xi, \eta, z) \), we introduce shorthand notations

\[
\begin{align*}
 f(x, y, z) &= \frac{\partial^2 \gamma(\mathbf{r}, \rho, z)}{\partial \xi^2} \\
g(x, y, z) &= \frac{\partial^2 \gamma(\mathbf{r}, \rho, z)}{\partial \xi \partial \eta} \\
h(x, y, z) &= \frac{\partial^2 \gamma(\mathbf{r}, \rho, z)}{\partial \eta^2}
\end{align*}
\]

Consider vector

\[
\mathbf{A} = \iint d^2r \nabla(\nabla \cdot \nabla_{\rho} \gamma(\mathbf{r}, 0, z))|_{\rho=0}
\]

having Cartesian components

\[
A_x = \iint d^2r \left( \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right), \quad A_y = \iint d^2r \left( \frac{\partial g}{\partial x} + \frac{\partial h}{\partial y} \right)
\]

By Green’s theorem, these integrals are equal to the flux of the vector fields \( f \hat{\mathbf{x}} + g \hat{\mathbf{y}} \) and \( g \hat{\mathbf{x}} + h \hat{\mathbf{y}} \) through the closed contour at infinity. For beam waves with a field quickly vanishing away from the axis, this flux is zero. This proves that

\[
\mathbf{A} = \iint d^2r \nabla(\nabla \cdot \nabla_{\rho} \gamma(\mathbf{r}, 0, z))|_{\rho=0} = 0.
\]

Consider integral

\[
\mathbf{B} = \hat{z} \iint d^2r [\mathbf{r} \times \nabla(\nabla \cdot \nabla_{\rho} \gamma(\mathbf{r}, \rho, z))]|_{\rho=0}.
\]

Using Cartesian coordinates, equation (A5) can be written as:

\[
\mathbf{B} = \iint dx dy \left[ \left( \frac{\partial g}{\partial x} + \frac{\partial h}{\partial y} \right) - \frac{1}{2} \left( \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) \right].
\]

The integrand in the right-hand part of equation (A1) can be presented as a divergence of auxiliary vector field \( \mathbf{F}(x, y) \)

\[
\mathbf{F}(x, y) = (-y \hat{\mathbf{y}} + x \hat{\mathbf{x}}) \mathbf{F}(x, y),
\]

\[
 \mathbf{F}(x, y) = (xh - yf) \hat{\mathbf{y}}.
\]

By Green’s theorem, this integral is equal to the flux of \( \mathbf{F} \) through the closed contour at infinity. For beam waves with a field quickly vanishing away from the axis, this flux is zero. This proves that

\[
\mathbf{B} = \iint d^3r [\mathbf{r} \times \nabla(\nabla \cdot \nabla_{\rho} \gamma(\mathbf{r}, \rho, z))]|_{\rho=0} = 0.
\]

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