PIECEWISE OBSERVERS OF RECTANGULAR DISCRETE FUZZY DESCRIPTOR SYSTEMS WITH MULTIPLE TIME-VARYING DELAYS

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Abstract. This paper investigates piecewise observer design for rectangular discrete fuzzy descriptor systems with multiple time-varying delays. Via a series of simple transformations, the considered rectangular descriptor plants are converted into standard ones with multiple time-varying delays. Then, two sufficient delay-dependent conditions for existence of piecewise fuzzy observers are derived based on piecewise Lyapunov functions. Finally, two numerical examples are presented to show the effectiveness of the theoretical results.

1. Introduction. In real word, many problems such as medication dosage and population growth, can be modeled by discrete-time dynamical systems. A great number of references have been reported on discrete-time systems. For example, the stability and robust stability for discrete-time linear systems has been investigated in paper [22, 14, 15]. Moreover, a discrete-time queueing model has been developed in paper [32], and performance of the model has been studied under realistic conditions. On the other hand, time-delay is a nature phenomenon of modern industry, which can not be avoided in many practical systems including communication systems, chemical systems and biological systems. Especially, time-delay makes a communication network difficult to maintain some good system properties. Recently, some researchers keep being interested in time-delayed systems. By defining minimum controllability realization index (MinCRI) [29], controllability was discussed for discrete-time linear time-delayed systems. For the same kind of systems, regulator problem has been dealt with in reference [13], where both delay-dependent and delay-independent conditions have been obtained. Moreover, in a

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general nonlinear time-delay system, the method of identifying unknown time-delays and model parameters was given [4].

In engineering applications, compared with continuous-time systems, discrete-time ones with time-varying delays can model more kinds of plants, which is very common in networked control systems (NCSs) [34]. Different from constant time-delay systems, time-varying delay can not be handled easily via traditional state augmentation approach. Much attention has been paid to discrete-time systems with time-varying delays. For asymptotic stability of discrete-time systems with time-varying delays, several sufficient conditions have been given in paper [9] via a new Lyapunov function, which are much better than some existing results. Reference [10] has focused on output-feedback stabilizing problem of discrete-time systems with time-varying delays. A stability condition related with bounds of delays has been derived firstly, by which stabilizing problem can be solved based on static and dynamic output-feedback controllers. However, the obtained results were not in the form of strict linear matrix inequality (LMI), leading that the process of cone complementary linearisation was used to solve the problem of nonconvex feasibility. Although many physical systems, such as chemical control process [18], biological systems [36], and electrical circuits [28], can be modeled by descriptor plants, the number of variables is always not required to be the same with that of equations.

On the other hand, a larger number of complex systems are difficult to describe and analyze, because of their strong nonlinearity, which has been overcome via Takagi-Sugeno (T-S) fuzzy model [30]. For different state regions of a T-S fuzzy model, local dynamics are described by linear systems, and the overall dynamic can be obtained using center-average defuzzifier. All subsystems can be connected in order to achieve some goals, although they are independent. Moreover, a nonlinear system could be well modeled by a T-S fuzzy system. Therefore, it becomes very meaningful to investigate relevant issues of T-S fuzzy models. For recent several years, unceasing efforts have been made to design controllers and investigate stability for T-S fuzzy systems [5, 3, 37]. In paper [23], an output feedback controller has been designed for a fuzzy system with disturbance such that the closed-loop system satisfied $H_{\infty}$ performance, and the premise variable space of the designed controller could be different from that of the original T-S fuzzy plant. Taking use of T-S model, adaptive control of a class of nonlinear plants has been researched in article [7]. Via relaxing the previous stability conditions, reference [20] has derived new stability results firstly. Then, $H_{\infty}$ controllers based on observers have been proposed for T-S fuzzy models. Combining with norm-bounded parameter uncertainties and time delays, a class of fuzzy discrete-time systems have been investigated [35], and a full order fuzzy output feedback controller has been obtained to guarantee $H_{\infty}$ performance and stability of the corresponding closed-loop system. For fuzzy discrete-time descriptor systems with multiple delays, article [19] has proposed D-stability criteria and designed a non-fragile controller. To our knowledge, the state estimation is often necessary for control problem when system states are not available. Consequently, how to design observers for fuzzy systems has been attracting much attention. Some results can be found in [24, 11, 31] and references therein. It is noted that most of above achievements are determined by common Lyapunov functions (CLFs).

However, there may be no appropriate CLFs for some fuzzy systems, which are still asymptotically stable [16]. For the purpose of reducing conservatism, instead of using CLF, a new approach based on piecewise Lyapunov function (PLF) has been
developed. Many results with piecewise idea, which can be applied to determining system stability, have been presented in papers [2, 12]. Via PLFs, $H_{\infty}$ control problem of continuous-time fuzzy affine systems has been researched in article [25], and piecewise affine static output feedback controllers have been designed such that the closed-loop systems were asymptotically stable with $H_{\infty}$ performance. Also, a method of designing observer-based output feedback controllers for fuzzy affine systems can be found in paper [26]. Using common or piecewise quadratic Lyapunov function, the existence of controllers has been analyzed in paper [26], and some new results in terms of LMIs have been given. For continuous-time case, it usually demands that the PLF must be continuous across region boundaries of state space. In contrast, such restriction is not required for discrete-time case [8], for the reason that considered discrete system may not take value on region boundaries. But few papers devote to observer design under PLFs, especially for fuzzy discrete-time descriptor systems. In this paper, we will discuss how to design piecewise fuzzy observers for rectangular fuzzy discrete-time descriptor systems with multiple time-varying delays. Firstly, the considered plants are converted to equivalent normal ones, for which existence of fuzzy observers are investigated via PLFs. Then, two sufficient conditions are presented in form of LMIs. With unknown inputs in dynamic equation and measurement one, results on rectangular descriptor systems are limited [6, 17]. Considering that, we also discuss observers of rectangular fuzzy descriptor systems with unknown inputs in this paper.

The rest of this paper is organized as follows. In Section 2, the considered rectangular fuzzy descriptor system is transformed into a standard plant under a simple assumption. Via PLFs, piecewise fuzzy observer is investigated in Section 3, and two sufficient delay-dependent conditions are derived. Then, observers of rectangular fuzzy descriptor systems with unknown inputs are researched in Section 4. For illustrating the effectiveness of the proposed results, two numerical examples are shown in Section 5. Section 6 is a short conclusion.

2. System transformation and problem formulation. In this section, we first make some equivalent transformations to the considered system, then propose the design problem of piecewise fuzzy observer. A rectangular fuzzy discrete-time descriptor system can be described as follows.

**Plant form:**

**Rule l:** IF $\theta_1(k)$ is $M_{l1}$, $\theta_2(k)$ is $M_{l2}$, ..., and $\theta_s(k)$ is $M_{ls}$, THEN

\[
\begin{cases}
E^* x(k + 1) = A_1^* x(k) + \sum_{r=1}^{R} A_1^* x(k - d_r(k)) + B_1^* u(k), \\
y_1^*(k) = C_1^* x(k) + \sum_{r=1}^{R} C_1^* x(k - d_r(k)), \\
x(k) = \phi(k), \quad k = -d, -d + 1, \ldots, 0,
\end{cases}
\]

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^p$ and $y_1^*(k) \in \mathbb{R}^q$ denote system state, input and measurable output, respectively. Matrix $E^* \in \mathbb{R}^{m \times n}$ is assumed to be singular, i.e., rank$E^* = r < \min\{m, n\}$, and $\phi(k)$ is initial condition. Furthermore, the numbers of inference rules and time-varying delays are $L$ and $R$, respectively. Suppose that $1 \leq l \leq L$. Time-varying delays $d_r(k) (1 \leq r \leq R)$ are known satisfying $d_{m_r} \leq d_r(k) \leq d_{M_r} (1 \leq r \leq R)$. Define $\bar{d} := \max_r \{d_{M_r}\}$ and $\underline{d} := \min_r \{d_{m_r}\}$. $M_{ls}(1 \leq s \leq p)$ are fuzzy sets, and $\theta(k) = [\theta_1(k), \theta_2(k), \ldots, \theta_s(k)]$ is premise variable vector. The $l$-th local model of fuzzy system (1) is denoted by $(A_1^*, A_{l1}, \ldots, A_{lR}, B_1^*, C_1^*, C_{l1}, \ldots, C_{lR})$. 
Since \( \text{rank}E^* = r \), there exists a nonsingular matrix \( P \) such that for each \( 1 \leq l \leq L, \ 1 \leq r \leq R \),
\[
PE^* = \begin{bmatrix} E \\ 0 \end{bmatrix}, \ PA^*_l = \begin{bmatrix} A_l \\ A_{ll} \end{bmatrix}, \ PA^*_{lr} = \begin{bmatrix} A_{lr} \\ A_{llr} \end{bmatrix}, \ PB^*_l = \begin{bmatrix} B_l \\ B_{ll} \end{bmatrix}.
\]

Note that the matrix \( E \) is full row rank. Moreover, denote \( v = m - r + q \), then we define
\[
y_l(k) = \begin{bmatrix} -B_{ll} u(k) \\ y^*_l(k) \end{bmatrix} \in \mathbb{R}^v, \ C_l = \begin{bmatrix} A_l \\ C^*_l \end{bmatrix} \in \mathbb{R}^{v \times n}, \ C_{lr} = \begin{bmatrix} A_{lr} \\ C^*_{llr} \end{bmatrix} \in \mathbb{R}^{v \times n},
\]
where \( 1 \leq r \leq R \). Immediately, system (1) can be rewritten as
\[
\begin{cases}
Ex(k + 1) = A_l x(k) + \sum_{r=1}^R A_{lr} x(k - d_r(k)) + B_l u(k), \\
y_l(k) = C_l x(k) + \sum_{r=1}^R C_{lr} x(k - d_r(k)), \\
x(k) = \phi(k), \ k = -d, -d + 1, \ldots, 0,
\end{cases}
\]
(2)
where \( E \in \mathbb{R}^{r \times n}, \ A_l \in \mathbb{R}^{r \times n}, \ A_{lr} \in \mathbb{R}^{r \times n} \ (1 \leq r \leq R) \) and \( B_l \in \mathbb{R}^{r \times p} \). Let
\[
\bar{E} = \begin{bmatrix} E \\ 0 \end{bmatrix} \in \mathbb{R}^{n \times n}, \ \bar{A}_l = \begin{bmatrix} A_l \\ 0 \end{bmatrix} \in \mathbb{R}^{n \times n},
\]
\[
\bar{B}_l = \begin{bmatrix} B_l \\ 0 \end{bmatrix} \in \mathbb{R}^{n \times p}, \ \bar{A}_{lr} = \begin{bmatrix} A_{lr} \\ 0 \end{bmatrix} \in \mathbb{R}^{n \times n} \ (1 \leq r \leq R).
\]

Then, the state equation of system (2) can be converted into
\[
\bar{E}x(k + 1) = \bar{A}_l x(k) + \sum_{r=1}^R \bar{A}_{lr} (x - d_r(k)) + \bar{B}_l u(k). \quad (3)
\]

Similar to reference [21], we make the following assumption for system (1).

**Assumption 1.**
\[
\text{rank} \begin{bmatrix} 0 & E^* \\ E^* & A^*_l \\ 0 & C_l^* \end{bmatrix} = n + \text{rank}E^*, \ 1 \leq l \leq L.
\]

**Remark 1.** If \( m = n \) in system (1), it is obvious that Assumption 1 is the condition of impulse observability of a traditional descriptor system. Thus Assumption 1 is generalization of common one and it is necessary for the existence of observer.

Under Assumption 1 on system (1), it is easy to derive that
\[
\text{rank} \begin{bmatrix} \bar{E} \\ C_l \end{bmatrix} = n, \ 1 \leq l \leq L. \quad (4)
\]
Thus, with system output \( y_l(k) \), equation (3) could be transformed into
\[
x(k + 1) = \bar{A}_l x(k) + \sum_{r=1}^R \bar{A}_{lr} x(k - d_r(k)) + \sum_{r=1}^R \bar{C}_{lr} x(k + 1 - d_r(k + 1)) + \bar{B}_l u(k) + N_l y_l(k + 1),
\]
where \( [\bar{A}_l, \ \bar{A}_{lr}, \ \bar{C}_{lr}, \ \bar{B}_l] = [T_l \bar{A}_l, \ T_l \bar{A}_{lr}, -N_l C_{lr}, \ T_l \bar{B}_l] \ (1 \leq r \leq R), \) and \( T_l, \ N_l \) are nonsingular matrices, needing to be designed such that
\[
T_l \bar{E} + N_l C_l = I_n, \ 1 \leq l \leq L. \quad (5)
\]
As for general solutions to equation (5), readers can refer to paper [33].
Now, under Assumption 1, by using center-average defuzzifier, an equivalent form of fuzzy system (1) can be obtained as

\[
\begin{align*}
    x(k + 1) &= \hat{A}(\theta)x(k) + \sum_{r=1}^{R} \hat{A}_{dr}(\theta)x(k - d_r(k)) + \sum_{r=1}^{R} \tilde{C}_{dr}(\theta)x(k + 1 - d_r(k + 1)) + \tilde{B}(\theta)u(k) + \sum_{r=1}^{L} h_l(\theta(k))N_l y_l(k + 1), \\
    y(k) &= C(\theta)x(k) + \sum_{r=1}^{R} C_{dr}(\theta)x(k - d_r(k)), \\
    x(k) &= \phi(k), \quad k = -d_r - d + 1, \ldots, 0,
\end{align*}
\]

(6)

where

\[
\begin{bmatrix}
    \hat{A}(\theta) & \hat{A}_{dr}(\theta) & \tilde{C}_{dr}(\theta) & \tilde{B}(\theta) & C(\theta) & C_{dr}(\theta)
\end{bmatrix}
\]

\[\triangleq \sum_{l=1}^{L} h_l(\theta(k)) \begin{bmatrix}
    \hat{A}_l & \hat{A}_{lr} & \tilde{C}_{lr} & \tilde{B}_l & C_l & C_{lr}
\end{bmatrix}, \quad 1 \leq r \leq R,
\]

and

\[h_l(\theta(k)) = \frac{\prod_{i=1}^{L} M_{is}(\theta_l(k))}{\sum_{i=1}^{L} M_{is}(\theta_l(k))}, \quad 1 \leq l \leq L,
\]

with denoting the grade of membership of \( \theta_l(k) \) in \( M_{is}(\theta_l(k)) \). Obviously, \( h_l(\theta(k)) \geq 0 \) (\( 1 \leq l \leq L \)) and \( \sum_{i=1}^{L} h_l(\theta(k)) = 1 \).

In this paper, a fuzzy observer will be piecewise designed for fuzzy system (6) via PLF. Taking use of the piecewise idea in paper [27], premise variable space can be partitioned as \( \bigcup_{i \in \varphi} \bar{S}_i \) by boundaries \( \{ \theta(k) \mid h_l(\theta(k)) = 1, \quad 0 \leq h_l(\theta(k) + \delta) < 1, \forall 0 < |\delta| \ll 1, \quad 1 \leq l \leq L \} \), where \( \varphi \) is the set of region indexes. For each region \( \bar{S}_i \) (\( i \in \varphi \)), we define

\[\mathcal{K}(i) \triangleq \{ m \mid h_m(\theta(k)) > 0, \quad \theta(k) \in \bar{S}_i, \quad 1 \leq m \leq L \}.
\]

Then, fuzzy system (6) can be expressed as

\[
\begin{align*}
    x(k + 1) &= \mathcal{A}_i(\theta)x(k) + \sum_{r=1}^{R} \mathcal{A}_{idr}(\theta)x(k - d_r(k)) + \sum_{r=1}^{R} \tilde{C}_{idr}(\theta)x(k + 1 - d_r(k + 1)) + \mathcal{B}_i(\theta)u(k) + \sum_{i \in \mathcal{K}(i)} h_l(\theta(k))N_l y_l(k + 1), \\
    y(k) &= \mathcal{C}_i(\theta)x(k) + \sum_{r=1}^{R} \mathcal{C}_{idr}(\theta)x(k - d_r(k)), \quad \theta(k) \in \bar{S}_i, \quad i \in \varphi,
\end{align*}
\]

(7)

where

\[
\begin{bmatrix}
    \mathcal{A}_i(\theta) & \mathcal{A}_{idr}(\theta) & \tilde{C}_{idr}(\theta) & \mathcal{B}_i(\theta) & \mathcal{C}_i(\theta) & \mathcal{C}_{idr}(\theta)
\end{bmatrix}
\]

\[= \sum_{i \in \mathcal{K}(i)} h_l(\theta(k)) \begin{bmatrix}
    \hat{A}_l & \hat{A}_{lr} & \tilde{C}_{lr} & \tilde{B}_l & C_l & C_{lr}
\end{bmatrix}
\]

with \( 0 < h_l(\theta(k)) \leq 1 \) and \( \sum_{i \in \mathcal{K}(i)} h_l(\theta(k)) = 1 \). For convenience, to represent all possible transitions among regions, a set \( \Phi \) is defined as

\[\Phi \triangleq \{(i, j) \mid \theta(k) \in \bar{S}_i, \theta(k + 1) \in \bar{S}_j, \quad i, j \in \varphi\}.
\]

It is noted that \( \theta(k) \) stays in the same region \( \bar{S}_i \) when \( i = j \), and jumps from \( \bar{S}_i \) to region \( \bar{S}_j \) when \( j \neq i \).

Now, given fuzzy system (7), we consider the following piecewise fuzzy observer. **Region Rule:** IF \( \theta(k) \in \bar{S}_i \), THEN

...
Observer Rule: IF $\theta_1(k)$ is $M_{11}$, $\theta_2(k)$ is $M_{12}$, ..., and $\theta_s(k)$ is $M_{1s}$, THEN
\[
\begin{align*}
\dot{x}(k+1) &= \bar{A}_i \dot{x}(k) + \sum_{r=1}^{R} \bar{A}_{ir} \dot{x}(k-d_r(k)) + \sum_{r=1}^{R} \bar{C}_{ir} \dot{x}(k+1-d_r(k)), \\
\dot{y}(k) &= C_i \dot{x}(k) + \sum_{r=1}^{R} C_{ir} \dot{x}(k-d_r(k)), \quad i \in \varphi, \quad l \in K(i),
\end{align*}
\]
where $\dot{x}(k) \in \mathbb{R}^n$ is the estimate of state $x(k)$, $y(k)$ and $\dot{y}(k)$ are final output of fuzzy system and fuzzy observer, respectively. Matrices $G_{it}$ ($l \in K(i)$, $i \in \varphi$) are observer gains to be designed. Using the same weight $h_i(\theta(k))$ and notations in fuzzy system (7), the global fuzzy observer can be written as
\[
\begin{align*}
\dot{x}(k+1) &= A_i(\theta) \dot{x}(k) + \sum_{r=1}^{R} A_{id_r}(\theta) \dot{x}(k-d_r(k)) + \sum_{r=1}^{R} C_{id_r}(\theta) \dot{x}(k+1-d_r(k)) \\
&+ \sum_{i \in K(i)} h_i(\theta(k)) N_i y_l(k+1), \\
\dot{y}(k) &= C_i(\theta) \dot{x}(k) + \sum_{r=1}^{R} C_{id_r}(\theta) \dot{x}(k-d_r(k)), \quad \theta(k) \in \bar{S}_i, \quad i \in \varphi,
\end{align*}
\]
where $G_i(\theta) \triangleq \sum_{i \in K(i)} h_i(\theta(k)) G_{it}$, $i \in \varphi$.

In summary, this paper dedicates to obtain matrices $G_{it}$ ($i \in \varphi$, $t \in K(i)$) such that $\dot{x}(k)$ in observer (9) asymptotically converges to $x(k)$ in system (7).

Finally, we introduce some notations, which will be used in this paper.

- Positive definite matrix $P$ is presented by $P > 0$;
- $0_{m \times n}$ means $m \times n$ matrix, whose entries are all zero;
- $I_n$ means $n \times n$ identity matrix;
- $A_{sd} \triangleq [ A_{s1} \ A_{s2} \ldots \ A_{sR} ]$; $A_{id}(\theta) \triangleq [ A_{id_1}(\theta) \ A_{id_2}(\theta) \ldots \ A_{id_R}(\theta) ]$, $s \in K(i)$, $i \in \varphi$;
- $\bar{C}_{sd} \triangleq [ \bar{C}_{s1} \ \bar{C}_{s2} \ldots \ \bar{C}_{sR} ]; \bar{C}_{id}(\theta) \triangleq [ \bar{C}_{id_1}(\theta) \ \bar{C}_{id_2}(\theta) \ldots \ \bar{C}_{id_R}(\theta) ]$, $s \in K(i)$, $i \in \varphi$;
- $C_{sd} \triangleq [ C_{s1} \ C_{s2} \ldots \ C_{sR} ]; C_{id}(\theta) \triangleq [ C_{id_1}(\theta) \ C_{id_2}(\theta) \ldots \ C_{id_R}(\theta) ]$, $t \in K(i)$, $i \in \varphi$;
- $e(k-d(k)) \triangleq [ e^T(k-d_1(k)) \ e^T(k-d_2(k)) \ldots \ e^T(k-d_R(k)) ]^T$;
- $\eta(k) \triangleq [ e^T(k) \ e^T(k-d(k)) \ldots \ e^T(k+1-d(k+1)) ]^T$;
- $U = \text{diag}\{U_1, \ U_2, \ldots, \ U_R\}$;
- $\bar{U} = (\bar{d} - \bar{d} + 1) \sum_{r=1}^{R} U_r$.

3. Existence of piecewise fuzzy observers. In this section, for rectangular discrete-time fuzzy descriptor systems with multiple time-varying delays, several results on designing piecewise fuzzy observers will be derived based on PLFs. First of all, we give Schur complement Lemma, which will be used in the sequel.

Lemma 3.1. [1] The LMI
\[
\begin{bmatrix}
M & S \\
S^T & R
\end{bmatrix} > 0,
\]
where $M = M^T$, $R = R^T$ is equivalent to
\[
R > 0, \quad M - SR^{-1}S^T > 0.
\]

Next, a sufficient condition for existence of fuzzy observer (8) is presented by the following theorem.
**Theorem 3.2.** Model (8) is a state observer of fuzzy system (1), if there exist matrices $U_r > 0$ ($1 \leq r \leq R$), $P_i > 0$ and $W_{is}$ ($i \in \varphi$, $s \in K(i)$) such that

$$
\Lambda_{ij}^0 + \Lambda_{ij}^0 < 0, \ s, t \in K(i), \ s \leq t,
$$

(10)

where $(i, j) \in \Phi$,

$$
\Lambda_{ij}^0 = \begin{bmatrix}
\bar{U} - P_i & 0_{n \times Rn} & 0_{n \times Rn} \\
- U & \bar{A}_P^TP_i + C_i^TW_{is} \\
* & - U & \bar{A}_P^TP_i + C_i^TW_{is} \\
* & * & - C_i^TNP_i \\
* & * & * \\
\end{bmatrix},
$$

and $*$ denotes matrix entries implied by the symmetry of a matrix through the paper.

Furthermore, observer gain matrices can be obtained as:

$$
G_{is} = P_i^{-1}W_{is}^T, \ s \in K(i), \ i \in \varphi.
$$

**Proof.** Denote estimation error by $e(k)$, which is defined as

$$
e(k) = x(k) - \hat{x}(k).
$$

Then, the following dynamic of error can be easily derived from system (7) and observer (9),

$$
e(k + 1) = [A_i(\theta) + G_i(\theta)C_i(\theta)]e(k) + \sum_{r=1}^{R} [A_{id_r}(\theta) + G_{id_r}(\theta)C_{id_r}(\theta)]e(k - d_r(k))
\begin{aligned}
+ \sum_{r=1}^{R} \bar{C}_{id_r}(\theta)e(k + 1 - d_r(k + 1)) \\
= [A_i(\theta) + G_i(\theta)C_i(\theta)]e(k) + [A_{id}(\theta) + G_{id}(\theta)C_{id}(\theta)]e(k - d(k))
\end{aligned}
\begin{aligned}
+ \bar{C}_{id}(\theta)e(k + 1 - d(k + 1)).
\end{aligned}
$$

(11)

In order to guarantee that model (8) is an observer of fuzzy system (1), we only need prove that error system (11) is asymptotically stable. Consider a PLF as

$$
V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k + 1) + V_5(k + 1),
$$

where

$$
V_1(k) = e^T(k)P_k e(k), \ V_2(k) = \sum_{r=1}^{R} \sum_{s=k-d_r(k)}^{k-1} e^T(s)U_r e(s),
$$

$$
V_3(k) = \sum_{r=1}^{R} \sum_{s=1-l}^{-d} \sum_{s=k+l}^{k-1} e^T(s)U_r e(s), \ P_i > 0 (i \in \varphi), \ U_r > 0 (1 \leq r \leq R),
$$

then

$$
\Delta V_1(k) = V_1(k + 1) - V_1(k)
\begin{aligned}
= e^T(k + 1)P_k e(k + 1) - e^T(k)P_k e(k), \ (i, j) \in \Phi,
\end{aligned}
$$

$$
\Delta V_2(k) = V_2(k + 1) - V_2(k)
\begin{aligned}
= \sum_{r=1}^{R} \sum_{s=k+1-d_r(k+1)}^{k} e^T(s)U_r e(s) - \sum_{s=k-d_r(k)}^{k-1} e^T(s)U_r e(s)
\end{aligned}
$$
\[
\begin{align*}
\Delta V_3(k) &= V_3(k+1) - V_3(k) \\
&= \sum_{s=k+1-d(k)}^{k} e^T(s)U_r e(s) - \sum_{s=k+1-d}^{k} e^T(s)U_r e(s) \\
&= \sum_{s=k+1-d}^{k} e^T(s)U_r e(s) - \sum_{s=k+1-d}^{k} e^T(s)U_r e(s) \\
&= (\tilde{d} - d) \sum_{r=1}^{R} e^T(k)U_r e(k) - \sum_{s=k+1-d}^{k} e^T(s)U_r e(s).
\end{align*}
\]

Thus,
\[
\Delta V(k) = \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) + \Delta V_2(k+1) + \Delta V_3(k+1)
\leq e^T(k+1) \left( P_2 + (\tilde{d} - d + 1) \sum_{r=1}^{R} U_r \right) e(k+1) - e^T(k)P_2 e(k) \\
+ e^T(k) \left( \sum_{r=1}^{R} (\tilde{d} - d + 1)U_r \right) e(k) - e^T(k - d(k))U e(k - d(k)) \\
- e^T(k+1 - d(k+1))U e(k+1 - d(k+1)) \\
\overset{(11)}{=} \eta^T(k) \Lambda_{ij} \eta(k), \ (i,j) \in \Phi,
\]

\[
\Lambda_{ij} = \begin{bmatrix}
\lambda_{11}^{ij}(	heta) & \lambda_{12}^{ij}(	heta) & \lambda_{13}^{ij}(	heta) \\
* & \lambda_{22}^{ij}(	heta) & \lambda_{23}^{ij}(	heta) \\
* & * & \lambda_{33}^{ij}(	heta)
\end{bmatrix}
\]
with
\[
\begin{align*}
\lambda_{11}^{ij}(\theta) &= [A_i(\theta) + G_i(\theta)C_i(\theta)]^T [P_j + \bar{U}] [A_i(\theta) + G_i(\theta)C_i(\theta)] + \bar{U} - P_i, \\
\lambda_{12}^{ij}(\theta) &= [A_i(\theta) + G_i(\theta)C_i(\theta)]^T [P_j + \bar{U}] [A_{id}(\theta) + G_i(\theta)C_{id}(\theta)], \\
\lambda_{13}^{ij}(\theta) &= [A_i(\theta) + G_i(\theta)C_i(\theta)]^T [P_j + \bar{U}] \bar{C}_id(\theta), \\
\lambda_{22}^{ij}(\theta) &= [A_{id}(\theta) + G_i(\theta)C_{id}(\theta)]^T [P_j + \bar{U}] [A_{id}(\theta) + G_i(\theta)C_{id}(\theta)] - U, \\
\lambda_{23}^{ij}(\theta) &= [A_{id}(\theta) + G_i(\theta)C_{id}(\theta)]^T [P_j + \bar{U}] \bar{C}_{id}(\theta), \\
\lambda_{33}^{ij}(\theta) &= \bar{C}_{id}(\theta) [P_j + \bar{U}] \bar{C}_{id}(\theta) - U.
\end{align*}
\]

The inequality \( \Delta V(k) < 0 \) holds for all \( \eta(k) \neq 0 \) if and only if
\[
\Lambda_{ij} < 0, \quad (i, j) \in \Phi. \quad (12)
\]

Obviously, condition (12) implies that error system (11) is asymptotically stable. From Lemma 3.1, inequality (12) is equivalent to
\[
\begin{bmatrix}
\bar{U} - P_i & 0_{n \times R_n} & 0_{n \times R_n} \\
* & -U & 0_{R_n \times R_n} \\
* & * & -U
\end{bmatrix}
\begin{bmatrix}
[A_i(\theta) + G_i(\theta)C_i(\theta)]^T \\
[A_{id}(\theta) + G_i(\theta)C_{id}(\theta)]^T \\
\bar{C}_{id}(\theta)
\end{bmatrix}
\begin{bmatrix}
P_i \\
P_i \\
\bar{P}_i
\end{bmatrix}
< 0,
\]

where \((i, j) \in \Phi\). Since \(-P_i(P_j + \bar{U})^{-1}P_i \leq (P_j + \bar{U}) - 2P_i\), pre- and post-multiplying inequality (13) by matrix \( \text{diag}\{I_n, I_{R_n}, I_{R_n}, P_i\} \), we can obtain that inequality (13) is derived by the following inequality
\[
\begin{bmatrix}
\bar{U} - P_i & 0_{n \times R_n} & 0_{n \times R_n} \\
* & -U & 0_{R_n \times R_n} \\
* & * & -U
\end{bmatrix}
\begin{bmatrix}
[A_i(\theta) + G_i(\theta)C_i(\theta)]^T P_i \\
[A_{id}(\theta) + G_i(\theta)C_{id}(\theta)]^T P_i \\
\bar{C}_{id}(\theta) P_i
\end{bmatrix}
\begin{bmatrix}
P_i \\
P_i \\
P_j + \bar{U} - 2P_i
\end{bmatrix}
< 0,
\]

where \((i, j) \in \Phi\). According to the denotation of coefficient matrices in system (7) and observer (9), we can, from inequality (14), obtain that
\[
\sum_s \sum_t h_s(\theta(k))h_t(\theta(k)) \begin{bmatrix}
\bar{U} - P_i & 0_{n \times R_n} & 0_{n \times R_n} \\
* & -U & 0_{R_n \times R_n} \\
* & * & -U
\end{bmatrix}
\begin{bmatrix}
(\bar{A}_s + G_isC_i)P_i \\
(\bar{A}_{id} + G_isC_{id})^T P_i \\
\bar{C}_{id} P_i
\end{bmatrix}
\begin{bmatrix}
P_i \\
P_i \\
P_j + \bar{U} - 2P_i
\end{bmatrix}
< 0,
\]

where \(s \in \mathcal{K}(i), \ t \in \mathcal{K}(i)\) and \((i, j) \in \Phi\). Let \( W_{is} = \mathcal{G}_{is}^T P_i \ (s \in \mathcal{K}(i), \ i \in \varphi)\), then inequality (15) becomes
\[
\sum_{s \in \mathcal{K}(i)} \sum_{t \in \mathcal{K}(i)} h_s(\theta(k))h_t(\theta(k))\Lambda_{ij}^{ij} < 0, \quad (i, j) \in \Phi.
\]

That is,
\[
\sum_{s \in \mathcal{K}(i)} h_s^2(\theta(k))\Lambda_{ij}^{ij} + \sum_{s \in \mathcal{K}(i)} \sum_{t \in \mathcal{K}(i)} h_s(\theta(k))h_t(\theta(k))(\Lambda_{ij}^{ij} + \Lambda_{ij}^{ij}) < 0, \quad (i, j) \in \Phi. \quad (16)
\]

It follows that inequality (16) can be implied by condition (10). Then, inequality (12) holds and error system (11) is asymptotically stable. □

In fact, condition (10) in Theorem 3.2 is strict LMI, so it can be easily solved by LMI Toolbox of Matlab. Next, another result is given as follows, which is stronger than Theorem 3.2.
Theorem 3.3. For fuzzy system (1), there is a fuzzy state observer in form of (8) if there exist matrices $U_r > 0$ $(1 \leq r \leq R)$, $Z_i > 0$, $P_i > 0$ and $W_{is}$ $(i \in \varphi$, $s \in K(i))$ such that
\[
\Psi_{st}^{ij} + \Psi_{ts}^{ij} < 0, \ s, t \in K(i), \ s \leq t, \quad (17)
\]
where $(i, j) \in \Phi$, and
\[
\Psi_{st}^{ij} = \begin{bmatrix}
\hat{U} - P_i & 0_{n \times R_n} & 0_{n \times R_n} \\
0_{R_n \times R_n} & \hat{A}_d^T P_i + C_d^T W_{is} & A_d^T P_i + C_d^T W_{is} - P_i \\
0_{R_n \times R_n} & -U & A_d^T P_i + C_d^T W_{is} \\
* & * & 0_{n \times R_n} \\
* & * & * \\
* & * & *
\end{bmatrix},
\]
Moreover, the observer gain matrices can be calculated as:
\[
G_{is} = P_i^{-1} W_{is}^T, \ s \in K(i), \ i \in \varphi. \quad (18)
\]
Proof. From proof of Theorem 3.2, we know that error system is depicted by equality (11). Take a PLF as
\[
V(k) = V_1(k) + V_2(k) + V_3(k) + V_2(k+1) + V_3(k+1) + V_4(k),
\]
where $V_i$ $(i = 1, 2, 3)$ have been defined in the proof of Theorem 3.2, and
\[
V_4(k) = \sum_{t = -d+1}^{0} \sum_{s = k + t - 1}^{k-1} [e(s + 1) - e(s)]^T Z[e(s + 1) - e(s)],
\]
where $Z > 0$, and
\[
\Delta V_4(k) = \sum_{t = -d+1}^{0} \left\{ \sum_{s = k + l}^{k} [e(s + 1) - e(s)]^T Z[e(s + 1) - e(s)] - \sum_{s = k + l - 1}^{k-1} [e(s + 1) - e(s)]^T Z[e(s + 1) - e(s)] \right\}
\]
\[
= \sum_{t = -d+1}^{0} \left\{ [e(k+1) - e(k)]^T Z[e(k+1) - e(k)] - [e(k + l) - e(k + l - 1)]^T Z[e(k + l) - e(k + l - 1)] \right\}
\]
\[
= d[e(k+1) - e(k)]^T Z[e(k+1) - e(k)]
\]
\[
\leq d[e(k+1) - e(k)]^T Z[e(k + l) - e(k + l - 1)]
\]
\[
\leq d[e(k+1) - e(k)]^T Z[e(k + l) - e(k)].
\]
Thus, we have
\[
\Delta V(k) = \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) + \Delta V_2(k+1) + \Delta V_3(k+1) + \Delta V_4(k)
\]
\[
\leq e^T(k + 1) (P_j + \hat{U}) e(k + 1) - e^T(k) P e(k) + e^T(k) U e(k)
\]
\[
- e^T(k - d(k)) U e(k - d(k)) - e^T(k + 1 - d(k + 1)) U e(k + 1 - d(k + 1))
\]
\[
+ d[e(k+1) - e(k)]^T Z[e(k+1) - e(k)]
\]
\[
\equiv \eta^T(k) \Psi_{ij} \eta(k), \ (i, j) \in \Phi,
\]
where

\[
\Psi_{ij} = \begin{bmatrix}
\omega_{11}^i(\theta) & \omega_{12}^i(\theta) & \omega_{13}^i(\theta) \\
\ast & \omega_{22}^i(\theta) & \omega_{23}^i(\theta) \\
\ast & \ast & \omega_{33}^i(\theta)
\end{bmatrix}
\]

with

\[
\omega_{11}^i(\theta) = \lambda_{11}^i(\theta) + [A_i(\theta) + G_i(\theta)C_i(\theta) - I]^T(dZ)[A_i(\theta) + G_i(\theta)C_i(\theta) - I],
\]

\[
\omega_{12}^i(\theta) = \lambda_{12}^i(\theta) + [A_i(\theta) + G_i(\theta)C_i(\theta) - I]^T(dZ)[A_{id}(\theta) + G_i(\theta)C_{id}(\theta)],
\]

\[
\omega_{13}^i(\theta) = \lambda_{13}^i(\theta) + [A_i(\theta) + G_i(\theta)C_i(\theta) - I]^T(dZ)\bar{C}_{id}(\theta),
\]

\[
\omega_{22}^i(\theta) = \lambda_{22}^i(\theta) + [A_{id}(\theta) + G_i(\theta)C_{id}(\theta)]^T(dZ)[A_{id}(\theta) + G_i(\theta)C_{id}(\theta)],
\]

\[
\omega_{23}^i(\theta) = \lambda_{23}^i(\theta) + [A_{id}(\theta) + G_i(\theta)C_{id}(\theta)]^T(dZ)\bar{C}_{id}(\theta),
\]

\[
\omega_{33}^i(\theta) = \lambda_{33}^i(\theta) + C_{id}^T(\theta)(dZ)\bar{C}_{id}(\theta).
\]

It is easy to see that

\[
\Psi_{ij} < 0, \ (i, j) \in \Phi,
\]

implying \(\Delta V(k) < 0\), which determines that error system (11) is asymptotically stable.

According to Lemma 3.1, inequality (19) is equivalent to

\[
\begin{bmatrix}
\bar{U} - P_i & 0_{n \times Rn} & 0_{n \times Rn} & D_i(\theta)^T & [D_i(\theta) - I]^T \\
\ast & -U & 0_{Rn \times Rn} & D_{id}(\theta)^T & D_{id}(\theta)^T \\
\ast & \ast & -U & \bar{C}_{id}(\theta)^T & \bar{C}_{id}(\theta)^T \\
\ast & \ast & \ast & -(P_j + \bar{U})^{-1} & 0_{n \times n} \\
\ast & \ast & \ast & P_j + \bar{U} - 2P_i & -P_i(dZ)^{-1}P_i
\end{bmatrix} < 0,
\]

where \(D_i(\theta) = A_i(\theta) + G_i(\theta)C_i(\theta)\), \(D_{id}(\theta) = A_{id}(\theta) + G_i(\theta)C_{id}(\theta)\) and \((i, j) \in \Phi\).

Define \(J_i = \text{diag}\{I_n, I_{Rn}, I_{Rn}, P_i, P_i\}\) \((i \in \varphi)\). Pre- and post-multiplying inequality (20) by matrix \(J_i\), we have that inequality (20) can be derived by condition

\[
- P_i [P_j + \bar{U}]^{-1}P_i \leq P_j + \bar{U} - 2P_i,
\]

and

\[
\begin{bmatrix}
\bar{U} - P_i & 0_{n \times Rn} & 0_{n \times Rn} & D_i(\theta)^T P_i & [D_i(\theta) - I]^T P_i \\
\ast & -U & 0_{Rn \times Rn} & D_{id}(\theta)^T P_i & D_{id}(\theta)^T P_i \\
\ast & \ast & -U & \bar{C}_{id}(\theta)^T P_i & \bar{C}_{id}(\theta)^T P_i \\
\ast & \ast & \ast & P_j + \bar{U} - 2P_i & 0_{n \times n} \\
\ast & \ast & \ast & \ast & -P_i(dZ)^{-1}P_i
\end{bmatrix} < 0.
\]

Take \(\bar{Z}_i = P_i Z^{-1} P_i\) and \(W_{is} = G_{id}^T P_i\) \((i \in \varphi, \ s \in \mathcal{K}(i))\). Consequently, from inequality (21), we have

\[
\sum_{s \in \mathcal{K}(i)} \sum_{t \in \mathcal{K}(i)} h_{s}(\theta(k))h_{t}(\theta(k))\Psi_{st} < 0, \ (i, j) \in \Phi,
\]

which can be rewritten as

\[
\sum_{s \in \mathcal{K}(i)} h_{s}^2(\theta(k))\Psi_{st} + \sum_{s \in \mathcal{K}(i)} \sum_{t \in \mathcal{K}(i)} h_{s}(\theta(k))h_{t}(\theta(k))\Psi_{st} < 0, \ (i, j) \in \Phi.
\]

Combining with inequality (17), we can know that inequality (22) holds. Then condition (19) can be satisfied, which proves that error system (11) is asymptotically stable. \(\square\)
Remark 2. Obviously, Theorem 3.2 can be implied by Theorem 3.3. In fact, with 

\[ \phi_{st}^{ij} = \begin{bmatrix} \Lambda_{st}^{ij} & \phi_{st}^{ij} \\ -\tilde{d}^{-1} \tilde{Z}_i \end{bmatrix}, \]

where matrices \( \Psi_{st}^{ij} \) and \( \Lambda_{st}^{ij} \) are introduced in Theorems 3.3 and 3.2, respectively. Then condition (10) can be easily obtained from inequality (17) according to Lemma 3.1. Therefore, the condition in Theorem 3.3 is stronger than that in Theorem 3.2. However, Theorem 3.2 is also useful when there are no compatible solutions to LMI (17).

Let \( P_i \equiv P \ (i \in \varphi) \), then corresponding CLF results can be derived from Theorems 3.2 and 3.3, which are omitted for brevity.

Remark 3. If \( E = I_n \), then fuzzy descriptor system (1) becomes a nominal fuzzy system. For this case, a piecewise fuzzy observer in form of model (8) with \([A_l, A_{ld}, \tilde{B}_l, N_l] = [A_l, A_{ld}, B_l, 0_{n \times q}] \ (1 \leq l \leq L, \ 1 \leq r \leq R)\), can be designed by solving LMIs (10) or (17) via replacing \( \hat{A}_s \) and \( \hat{A}_{sd} \) with \( A_s \) and \( A_{sd} = [A_{sd1}, A_{sd2}, \ldots, A_{sdn}] \) \((s \in K(i), \ i \in \varphi)\), respectively.

Remark 4. Specifically, different from continuous-time case, Lyapunov function in this paper does not need to be continuous across region boundaries of premise variables, because states of a discrete-time system may be meaningless on boundaries. In addition, when time-delays in fuzzy system (1) are constant, relevant results on existence of observer for fuzzy system (1) can be easily obtained from Theorems 3.2 and 3.3 with \( \tilde{d} = d \). In general, results based on PLF are less conservative than those via CLF [25, 26, 8].

4. Extension to rectangular fuzzy descriptor systems with unknown inputs. In this section, we will consider observers of rectangular fuzzy descriptor systems with unknown inputs. The model is described as follows:

**Plant form:**

**Rule j:** IF \( \theta_1(k) = M_{j1}, \ \theta_2(k) = M_{j2}, \ldots, \ \text{and} \ \theta_s(k) = M_{js}, \ \text{THEN} \)

\[
\begin{align*}
E^* x(k+1) &= A_j^* x(k) + \sum_{r=1}^{R} A_{jr}^* x(k-d_r(k)) + B_j^* u(k) + F_j^* w(k), \\
y_j^*(k) &= C_j^* x(k) + \sum_{r=1}^{R} C_{jr}^* x(k-d_r(k)) + G_j^* w(k), \\
x(k) &= \phi(k), \ k = -\tilde{d}, -\tilde{d}+1, \ldots, 0,
\end{align*}
\]

(23)

where \( 1 \leq j \leq L, \ w(k) \in \mathbb{R}^h \) means unknown input, \( F_j^* \in \mathbb{R}^{m \times h}, \ G_j^* \in \mathbb{R}^{q \times h} \) and remaining symbols are the same with those of system (1). For system (23), we make the following assumptions.

**Assumption 2.**

\[
\text{rank} \begin{bmatrix} F_j^* \\ G_j^* \end{bmatrix} = h_1 \leq h, \ \text{rank} \begin{bmatrix} E^* & F_j^* \\ 0 & G_j^* \end{bmatrix} = \text{rank}E^* + \text{rank} \begin{bmatrix} F_j^* \\ G_j^* \end{bmatrix}, \ 1 \leq j \leq L.
\]

**Assumption 3.**

\[
\text{rank} \begin{bmatrix} E^* & A_j^* & F_j^* \\ 0 & C_j^* & G_j^* \\ 0 & E^* & 0 \end{bmatrix} = n + \text{rank} \begin{bmatrix} E^* & F_j^* \\ 0 & G_j^* \end{bmatrix}, \ 1 \leq j \leq L.
\]
Remark 5. Existence of unknown inputs makes it hard to design observer for system (23). Thus, we need Assumption 2 to deal with unknown inputs. Moreover, Assumption 3 can be recognized as generalization of Assumption 1 (i.e., $F_j^* = G_j^* = 0$).

Since matrix $E^*$ is singular, an invertible matrix $P$ can be found such that for each $1 \leq j \leq L$, $1 \leq r \leq R$,

$$PE^* = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \quad PA_j^* = \begin{bmatrix} A_j & A_{jr} \\ A_{1j} & A_{1jr} \end{bmatrix}, \quad PA_j^* = \begin{bmatrix} A_j \\ A_{1j} \end{bmatrix}, \quad PB_j^* = \begin{bmatrix} B_j & B_{1j} \\ B_{1j} & B_{1jr} \end{bmatrix}, \quad PF_j^* = \begin{bmatrix} F_j \\ F_{1j} \end{bmatrix},$$

where $E \in \mathbb{R}^{r \times n}$ is full row rank. Similar to Section 2, system (23) can be converted to

$$\begin{cases} \dot{x}(k+1) = A_j x(k) + \sum_{r=1}^{R} A_{jr} x(k - d_r(k)) + B_j u(k) + F_j w(k), \\ y_j(k) = C_j x(k) + \sum_{r=1}^{R} C_{jr} x(k - d_r(k)) + G_j w(k), \\ x(k) = \phi(k), \ k = -d, -d + 1, \ldots, 0, \end{cases} \tag{24}$$

where $1 \leq j \leq L$. From Assumption 2, we know

$$\text{rank} G_j = \text{rank} \begin{bmatrix} F_j \\ G_j \end{bmatrix} = h_1, \ 1 \leq j \leq L.$$ 

Thus, for each $1 \leq j \leq L$, two nonsingular matrices $U_j$ and $V_j$ can be found such that

$$U_j G_j V_j = \begin{bmatrix} I_{h_1} & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$\begin{bmatrix} I_r & 0 \\ 0 & U_j \end{bmatrix} \begin{bmatrix} F_j \\ G_j \end{bmatrix} V_j = \begin{bmatrix} F_{1j} \\ I_{h_1} \\ 0 \end{bmatrix}.$$

In addition, for each $1 \leq r \leq R$, let

$$U_j y_j(k) = \begin{bmatrix} y_{j1}(k) \\ y_{j2}(k) \end{bmatrix}, \quad U_j C_j = \begin{bmatrix} C_{1j} \\ C_{2j} \end{bmatrix}, \quad U_j C_{jr} = \begin{bmatrix} C_{1jr} \\ C_{2jr} \end{bmatrix}, \quad V_j^{-1} w(k) = \begin{bmatrix} v_{1j}(k) \\ v_{2j}(k) \end{bmatrix}.$$

Thus, system (24) is equivalent to

$$\begin{cases} \dot{x}(k+1) = \Phi_j x(k) + \sum_{r=1}^{R} \Phi_{jr} x(k - d_r(k)) + B_j u(k) + F_{1j} y_{1j}(k), \\ y_{j1}(k) = C_{1j} x(k) + \sum_{r=1}^{R} C_{1jr} x(k - d_r(k)) + v_{1j}(k), \\ y_{j2}(k) = C_{2j} x(k) + \sum_{r=1}^{R} C_{2jr} x(k - d_r(k)), \\ x(k) = \phi(k), \ k = -d, -d + 1, \ldots, 0, \end{cases} \tag{25}$$

where $\Phi_j = A_j - F_{1j} C_{1j}$ and $\Phi_{jr} = A_{jr} - F_{1j} C_{1jr} (1 \leq r \leq R)$. Furthermore, the following square system can be easily obtained from system (25):

$$\begin{cases} \dot{x}(k+1) = \Phi_j x(k) + \sum_{r=1}^{R} \Phi_{jr} x(k - d_r(k)) + B_j u(k) + F_{1j} y_{1j}(k), \\ y_{j1}(k) = C_{1j} x(k) + \sum_{r=1}^{R} C_{1jr} x(k - d_r(k)) + v_{1j}(k), \\ y_{j2}(k) = C_{2j} x(k) + \sum_{r=1}^{R} C_{2jr} x(k - d_r(k)), \\ x(k) = \phi(k), \ k = -d, -d + 1, \ldots, 0, \end{cases} \tag{26}$$
where

\[
\begin{bmatrix}
E \\
0
\end{bmatrix} \in \mathbb{R}^{n \times n}, \quad \Phi_j = \begin{bmatrix}
\Phi_{1j} \\
\Phi_{2j}
\end{bmatrix} \in \mathbb{R}^{n \times n}, \quad \Phi_{jr} = \begin{bmatrix}
\Phi_{rj} \\
0
\end{bmatrix} \in \mathbb{R}^{n \times n},
\]

\[
\bar{B}_j = \begin{bmatrix}
B_{1j} \\
0
\end{bmatrix} \in \mathbb{R}^{n \times p}, \quad \bar{F}_{ij} = \begin{bmatrix}
F_{ij} \\
0
\end{bmatrix} \in \mathbb{R}^{n \times h_1}.
\]

Under Assumption 3 on system (23), it is easy to discover that

\[
\text{rank} \left[ \begin{array}{c}
\bar{E} \\
C_{2j}
\end{array} \right] = n, \quad 1 \leq j \leq L,
\]

so a matrix pair \([T_j, N_j]\) can be found such that

\[
T_j \bar{E} + N_j C_{2j} = I_n, \quad 1 \leq j \leq L, \quad (27)
\]

due to the invertible \(T_j\). Then, with output \(y_{2j}(k)\), the state equation of system (26) can be transformed into

\[
x(k + 1) = \Phi_j x(k) + \sum_{r=1}^{R} \Phi_{jr} x(k - d_r(k)) + \sum_{r=1}^{R} \bar{C}_{jr} x(k + 1 - d_r(k) + 1) + \bar{B}_j u(k)
\]

\[
+ \bar{F}_{ij} y_{ij}(k) + N_j y_{2j}(k + 1),
\]

where \([\Phi_j, \Phi_{jr}, \bar{B}_j, \bar{F}_{ij}] = T_j[\bar{F}_j, \bar{F}_{ir}, \bar{F}_{ij}]\) and \(\bar{C}_{jr} = - N_j C_{2jr}, \quad 1 \leq j \leq L\).

Next, using center-average defuzzifier, the final state and output of system (26) can be derived. According to the piecewise idea [27], similar to system (7) we can obtain a system as

\[
\begin{cases}
\dot{x}(k) = A_i(\theta) x(k) + \sum_{r=1}^{R} A_{idr}(\theta) x(k - d_r(k)) + \sum_{r=1}^{R} C_{ird}(\theta) x(k + 1 - d_r(k) + 1) \\
\dot{y}_1(k) = C_{i1}(\theta) x(k) + \sum_{r=1}^{R} C_{i1dr}(\theta) x(k - d_r(k)) + \sum_{t \in K(i)} h_t(\theta(k)) v_{1t}(k), \\
\dot{y}_2(k) = C_{i2}(\theta) x(k) + \sum_{r=1}^{R} C_{i2dr}(\theta) x(k - d_r(k)), \\
x(k) = \phi(k), \quad k = -d, -d + 1, \ldots, 0,
\end{cases}
\]

where

\[
A_i(\theta) = \begin{bmatrix}
A_{idr}(\theta) & C_{idr}(\theta) & B_i(\theta) & C_{i1d}(\theta) & C_{i2d}(\theta) & C_{i2d}(\theta)
\end{bmatrix},
\]

\[\triangleq \sum_{t \in K(i)} h_t(\theta(k)) \begin{bmatrix}
\hat{\Phi}_t & \hat{\Phi}_{tr} & \hat{\Phi}_{tr} & \hat{C}_t & \hat{C}_{1t} & \hat{C}_{2t} & \hat{C}_{2t}
\end{bmatrix}.
\]

The aim is to design an observer as follows:

**Region rule:** IF \(\theta(k) \in S_i\), THEN

**Observer rule:** IF \(\theta_1(k) = M_{j1}, \theta_2(k) = M_{j2}, \ldots, \theta_s(k) = M_{js}\), THEN

\[
\begin{cases}
\dot{x}(k) = \hat{\Phi}_j \dot{x}(k) + \sum_{r=1}^{R} \hat{\Phi}_{jr} \dot{x}(k - d_r(k)) + \sum_{r=1}^{R} \bar{C}_{jr} \dot{x}(k + 1 - d_r(k) + 1) \\
\dot{\gamma}_1(k) = C_{j1} \dot{x}(k) + \sum_{r=1}^{R} C_{j1dr} \dot{x}(k - d_r(k)) \\
\dot{\gamma}_2(k) = C_{j2} \dot{x}(k) + \sum_{r=1}^{R} C_{j2dr} \dot{x}(k - d_r(k)) \\
\dot{x}(k) = \phi(k), \quad k = -d, -d + 1, \ldots, 0
\end{cases}
\]

where \(\dot{x}(k) \in \mathbb{R}^n\) is the estimate of state \(x(k)\), and matrices \(D_{ij}\) \((i \in \varphi, j \in K(i))\) are observation error matrices. Denote \(\hat{\gamma}(k)\) the final output of fuzzy observer. With
the same weight $h_j(\theta(k))$ in system (28), the final estimated state and output of fuzzy observer (29) can be described as

$$
\begin{align*}
\dot{x}(k+1) &= \mathcal{A}_i(\theta)(k) + \sum_{r=1}^R \mathcal{A}_{id_r}(\theta)(k) \dot{x}(k-d_r(k)) + \sum_{r=1}^R \mathcal{C}_{id_r}(\theta) \dot{x}(k+1-d_r(k+1)) + B_i(\theta)u(k) + \sum_{t \in \mathcal{K}(i)} h_t(\theta(k)) P_{it} y_{it}(k) \\
&+ \sum_{t \in \mathcal{K}(i)} h_t(\theta(k)) N_t y_{2t}(k+1) + D_i(\theta)(\hat{y}(k) - y_2(k)), \\
\hat{y}(k) &= C_{2i}(\theta) \dot{x}(k) + \sum_{r=1}^R C_{2id_r}(\theta) \dot{x}(k-d_r(k)), \\
\hat{x}(k) &= \phi(k), \ k = -d, -d+1, \ldots, 0, \ i \in \varphi, \ j \in \mathcal{K}(i),
\end{align*}
$$

where $\mathcal{D}_i(\theta) \triangleq \sum_{t \in \mathcal{K}(i)} h_t(\theta(k)) D_{it}$.

The existence of observer (30) for system (28) can be depicted by following theorems.

**Theorem 4.1.** Model (29) is a state observer of fuzzy system (23), if there exist matrices $U_r > 0$ (1 $\leq r \leq R$), $P_i > 0$ and $W_{is}$ ($i \in \varphi, \ s \in \mathcal{K}(i)$) such that

$$
\mathcal{Y}_{st}^i + \mathcal{Y}_{ts}^i < 0, \ s, t \in \mathcal{K}(i), \ s \leq t,
$$

where $(i, j) \in \Phi$,

$$
\mathcal{Y}_{st}^i = \begin{bmatrix}
\bar{U} - P_i & \mathbf{0}_{n \times Rn} & 0_{n \times Rn} & \hat{\Phi}_s^T P_i + C_{2i}^T W_{is} \\
* & -U & 0_{n \times Rn} & \hat{\Phi}_{sd}^T P_i + C_{2id}^T W_{is} \\
* & * & -U & C_{2sd}^T N_i^T P_i \\
* & * & * & P_j + U - 2P_i
\end{bmatrix},
$$

and $\hat{\Phi}_{sd} = [\hat{\Phi}_{s1} \hat{\Phi}_{s2} \cdots \hat{\Phi}_{sR}]$, $C_{2id} = [C_{2i1} C_{2i2} \cdots C_{2iR}]$. Furthermore, observer gain matrices can be obtained as:

$$
D_{is} = P_i^{-1} W_{is}, \ s \in \mathcal{K}(i), \ i \in \varphi.
$$

Next, compared with above theorem, another stronger one is given.

**Theorem 4.2.** For fuzzy system (23), there is a fuzzy state observer in form of (29) if there exist matrices $U_r > 0$ (1 $\leq r \leq R$), $\mathcal{Z}_i > 0$, $P_i > 0$ and $W_{is}$ ($i \in \varphi, \ s \in \mathcal{K}(i)$) such that

$$
\Gamma_{st}^i + \Gamma_{ts}^i < 0, \ s, t \in \mathcal{K}(i), \ s \leq t,
$$

where $(i, j) \in \Phi$, and

$$
\Gamma_{st}^i = \begin{bmatrix}
\bar{U} - P_i & \mathbf{0}_{n \times Rn} & 0_{n \times Rn} & \hat{\Phi}_s^T P_i + C_{2i}^T W_{is} \\
* & -U & 0_{n \times Rn} & \hat{\Phi}_{sd}^T P_i + C_{2id}^T W_{is} \\
* & * & -U & C_{2sd}^T N_i^T P_i \\
* & * & * & P_j + U - 2P_i
\end{bmatrix}.
$$

Moreover, the observer gain matrices can be calculated as:

$$
D_{is} = P_i^{-1} W_{is}, \ s \in \mathcal{K}(i), \ i \in \varphi.
$$

**Remark 6.** Results of Theorems 4.1 and 4.2 can be easily derived according to Theorems 3.2 and 3.3, respectively. The proofs are omitted here. Additionally, if $P_i \equiv P$ in Theorems 4.1 and 4.2, then corresponding CLF results are obtained immediately.
5. **Numerical examples.** In this section, two numerical examples are introduced. The first one can be solved by either CLF approach or PLF one, and the other one can only be treated using PLF method.

**Example 1.** Consider a discrete-time fuzzy descriptor system with time-varying delays as follows:

**Plant Rule 1:** IF \( x_1(k) \) is \( M_l(x_1(k)) \), THEN

\[
\begin{align*}
E^* x(k+1) &= A^*_l x(k) + A^*_l x(k-d_1(k)) + A^*_l x(k-d_2(k)) + B^*_l u(k), \\
y^*_l(k) &= C^*_l x(k) + C^*_l x(k-d_1(k)) + C^*_l x(k-d_2(k)), \\
x(k) &= \phi(k), \ k = -d, -d+1, \ldots, 0, \ l \in \{1, 2, 3\},
\end{align*}
\]

where

\[
E^*_l = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A^*_l = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad A^*_1 = \begin{bmatrix} 0.15 & 0 \\ -0.15 & -0.1 \end{bmatrix},
\]

\[
A^*_2 = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.15 \end{bmatrix}, \quad A^*_3 = \begin{bmatrix} 0.15 & 0 \\ 0 & -0.25 \end{bmatrix}, \quad A^*_{21} = \begin{bmatrix} 0 & 0.1 \\ 0 & -0.25 \end{bmatrix},
\]

\[
A^*_{31} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.15 \end{bmatrix}, \quad A^*_{12} = \begin{bmatrix} 0.03 & 0.2 \\ -0.1 & 0 \end{bmatrix}, \quad A^*_{22} = \begin{bmatrix} 0.15 & 0 \\ -0.15 & -0.2 \end{bmatrix},
\]

\[
A^*_{32} = \begin{bmatrix} 0 & 0.2 \\ 0 & 1 \end{bmatrix}, \quad B^*_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B^*_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad B^*_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},
\]

\[
C^*_1 = \begin{bmatrix} 0.05 & 0 \end{bmatrix}, \quad C^*_2 = \begin{bmatrix} -0.05 & 0 \end{bmatrix}, \quad C^*_{31} = \begin{bmatrix} 0 & 0.1 \end{bmatrix},
\]

\[
C^*_{12} = \begin{bmatrix} 0 & 0.1 \end{bmatrix}, \quad C^*_{22} = \begin{bmatrix} -0.1 & 0 \end{bmatrix}, \quad C^*_{32} = \begin{bmatrix} 0.01 & 0 \end{bmatrix},
\]

\[
C^*_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad C^*_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad C^*_3 = \begin{bmatrix} -1 & 1 \end{bmatrix},
\]

and the membership functions are described in Fig. 1. It is assumed that \( d_1(k) = 2.5 + 0.5 \sin(k\pi + 1.5\pi) \), \( d_2(k) = 2.5 + 0.5 \cos(k\pi) \) and \( x_1(k) \in [-5, 4] \), so \( d = 2, \overline{d} = 3 \). Then according to the partition approach of this paper, which is given in Section

![Figure 1. Membership functions](image)

2, premise variable space can be covered by three regions:

\[
\bar{S}_1 = \{x_1 | -5 \leq x_1 \leq -3\}, \quad \bar{S}_2 = \{x_1 | -3 \leq x_1 \leq 3\}, \quad \bar{S}_3 = \{x_1 | 3 \leq x_1 \leq 4\}.
\]
As explained in Section 2, we have

\[
\dot{E} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} 0.15 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix},
\]

\[
\bar{A}_{11} = \begin{bmatrix} 0.15 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_{21} = \begin{bmatrix} 0 & 0.1 \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_{31} = \begin{bmatrix} 0.1 & 0.15 \\ 0 & 0 \end{bmatrix},
\]

\[
\bar{A}_{12} = \begin{bmatrix} 0.03 & 0.2 \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_{22} = \begin{bmatrix} 0.15 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_{32} = \begin{bmatrix} 0 & 0.2 \\ 0 & 0 \end{bmatrix},
\]

\[
C_{11} = \begin{bmatrix} 0 & -0.25 \\ 0.05 & 0 \end{bmatrix}, \quad C_{21} = \begin{bmatrix} 0 & -0.25 \\ -0.05 & 0 \end{bmatrix}, \quad C_{31} = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix},
\]

\[
C_{12} = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad C_{22} = \begin{bmatrix} -0.15 & -0.2 \\ -0.1 & 0 \end{bmatrix}, \quad C_{32} = \begin{bmatrix} 0 & 1 \\ 0.01 & 0 \end{bmatrix},
\]

\[
C_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -0.15 & -0.1 \\ 1 & 1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0.1 & 0.15 \\ -1 & 1 \end{bmatrix},
\]

\[
B_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T, \quad B_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T, \quad B_3 = \begin{bmatrix} -1 & 0 \end{bmatrix}^T.
\]

A group of solutions to equation (5) are derived as

\[
T_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 0 & 0 \\ 0.0001 & 1 \end{bmatrix},
\]

Using Theorem 3.2, it is obtained that

\[
U_1 = \begin{bmatrix} 0.4733 & 0.0099 \\ 0.0099 & 1.0814 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0.4363 & -0.1191 \\ -0.1191 & 0.8074 \end{bmatrix},
\]

\[
P_1 = \begin{bmatrix} 4.1976 & -0.1952 \\ -0.1952 & 6.8499 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 4.1860 & -0.2218 \\ -0.2218 & 7.3226 \end{bmatrix},
\]

\[
P_3 = \begin{bmatrix} 4.1129 & -0.2280 \\ -0.2280 & 7.4637 \end{bmatrix}, \quad G_{11} = \begin{bmatrix} 0.7851 & -0.0365 \\ -0.1704 & 0.0080 \end{bmatrix},
\]

\[
G_{21} = \begin{bmatrix} 0.7854 & -0.0879 \\ -0.1567 & 0.0735 \end{bmatrix}, \quad G_{22} = \begin{bmatrix} 0.9170 & -0.6688 \\ 0.0270 & -0.0159 \end{bmatrix},
\]

\[
G_{32} = \begin{bmatrix} 0.0619 & -0.9541 \\ -0.0033 & -0.0164 \end{bmatrix}, \quad G_{33} = \begin{bmatrix} 0.0441 & -0.5306 \\ 0.0084 & 0.0339 \end{bmatrix},
\]

Simulation results are presented in following figures, among which Fig. 2 and Fig. 3 show states of fuzzy system (27) and corresponding fuzzy observer, respectively, and error system is depicted in Fig. 4. Obviously, error system is asymptotically stable according to Fig. 4.

Remark 7. It is noted that above numerical example can also be solved by CLF methods. In order to illustrate reduction of conservatism via PLF, another example is given in the following, which does not exist appropriate observer based on CLF.

Example 2. A discrete-time fuzzy descriptor system with unknown inputs is given as:
Figure 2. State of the fuzzy descriptor system

Figure 3. State of the fuzzy observer

Figure 4. Error system

Figure 5. State of the fuzzy descriptor system
Plant Rule 1: IF \( x_1(k) \) is \( M_l(x_1(k)) \), THEN

\[
\begin{align*}
E^* x(k+1) &= A_l^* x(k) + A_{l1}^* x(k-d_l(k)) + A_{l2}^* x(k-d_l(k)) + B_l^* u(k) \\
&\quad + F_l^* w(k), \\
y_l^*(k) &= C_l^* x(k) + C_{l1}^* x(k-d_l(k)) + C_{l2}^* x(k-d_l(k)) + G_l^* w(k), \\
x(k) &= \phi(k), \quad k = -d, -d+1, \ldots, 0, \quad l \in \{1, 2, 3\},
\end{align*}
\]

where

\[
\begin{align*}
E^* &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1^* = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad A_2^* = \begin{bmatrix} 0.15 & 0 \\ -0.15 & -0.1 \end{bmatrix}, \quad A_3^* = \begin{bmatrix} 0.1 & 0 \\ 0 & -1 \end{bmatrix}, \\
A_{l1}^* &= \begin{bmatrix} 0.15 & 0 \\ 0 & -0.25 \end{bmatrix}, \quad A_{21}^* = \begin{bmatrix} 0 & 0.1 \\ 0 & -0.25 \end{bmatrix}, \quad A_{31}^* = \begin{bmatrix} 0.1 & 0.15 \\ 0 & 0 \end{bmatrix}, \\
A_{l2}^* &= \begin{bmatrix} 0.03 & 0.2 \\ -0.1 & 0 \end{bmatrix}, \quad A_{22}^* = \begin{bmatrix} 0.15 & 0 \\ -0.15 & -0.2 \end{bmatrix}, \quad A_{32}^* = \begin{bmatrix} 0 & 0.2 \\ 0 & 0.187 \end{bmatrix},
\end{align*}
\]

\[
\begin{align*}
C_{11}^* &= \begin{bmatrix} 0.05 \\ 0 \end{bmatrix}^T, \quad C_{21}^* = \begin{bmatrix} -0.05 \\ 0 \end{bmatrix}^T, \quad C_{31}^* = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}^T, \\
C_{12}^* &= \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}^T, \quad C_{22}^* = \begin{bmatrix} 0.09 \\ 0.18 \end{bmatrix}^T, \quad C_{32}^* = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}^T, \\
F_1^* &= \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}, \quad F_2^* = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad F_3^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \\
B_1^* &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_2^* = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad B_3^* = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \\
C_1^* &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T, \quad C_2^* = \begin{bmatrix} 0.15 \\ 1.1 \end{bmatrix}^T, \quad C_3^* = \begin{bmatrix} -1 \\ 1 \end{bmatrix}^T, \\
G_1^* &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T, \quad G_2^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T, \quad G_3^* = \begin{bmatrix} -1 \\ 0 \end{bmatrix}^T, \\
w(k) &= \begin{bmatrix} 0.02 \sin(k) \\ 0.02 \cos(k) \end{bmatrix} \quad \text{and} \quad u(k) = 0.02 \sin(k).
\end{align*}
\]

Take the same assumption and membership functions as those in Example 1. In addition, for region partition of premise variable space, this example is not different from Example 1. And a group of solutions to equation (26) can be derived as

\[
\begin{align*}
T_1 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\
N_1 &= \begin{bmatrix} 0 & 0.5 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.
\end{align*}
\]

With Matlab LMI Toolbox, it is easy to obtain a set of suitable solutions to LMIs in Theorem 4.1. Corresponding matrices are

\[
\begin{align*}
U_1 &= \begin{bmatrix} 154.9467 & 36.4743 \\ 36.4743 & 693.9168 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 133.8214 & -20.2563 \\ -20.2563 & 274.3549 \end{bmatrix}.
\end{align*}
\]
Remark 8. For Example 2, there are no compatible CLFs. However, a suitable fuzzy observer can be designed with PLF. It demonstrates that PLF approaches are less conservative than CLF ones. Furthermore, though Theorem 4.2 is stronger than Theorem 4.1, the latter is also very useful sometimes. In fact, Examples 1 and 2 can not be managed by Theorems 3.3 and 4.2, respectively.

6. Conclusion. For rectangular discrete-time fuzzy descriptor systems with multiple time-varying delays, approaches of piecewise designing fuzzy observers have been discussed in this paper. Similar to most methods of observer design, with LMI approach two sufficient delay-dependent conditions on existence of fuzzy observer have been obtained. The proposed PLF approaches are less conservative than corresponding CLF ones. Finally, two numerical examples have been given to demonstrate the effectiveness of presented methods.
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