ON THE EQUIVALENCE OF TWO QUANTIFIER ELIMINATION TESTS

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Abstract. We prove that, for countable languages, two model-theoretic quantifier elimination tests, one proposed by J. R. Shoenfield and the other by L. van den Dries, are equivalent.

§1. Introduction. To facilitate the discussion we first introduce the following terminological and notational conventions.

Definition 1.1. Let $M$ be a model and $A \subseteq |M|$. Let $N$ be the model $\langle M, a \rangle_{a \in A}$.
1. The theory $\text{Th}(N)$, denoted by $\text{CD}(A, M)$, is called the complete diagram of $A$ in $M$. If $A = |M|$ we simply write $\text{CD}(M)$.
2. The set of all quantifier-free sentences in $\text{Th}(N)$, denoted by $\text{ED}(A, M)$, is called the elementary diagram of $A$ in $M$. Again if $A = |M|$ we simply write $\text{ED}(M)$.

Obviously if $N \preceq M$ then $\text{CD}(N, M) = \text{CD}(N)$ and if $N \subseteq M$ then $\text{ED}(N, M) = \text{ED}(N)$.

We say that a theory $T$ is model complete if and only if, for every pair of models $N, M \models T$, $N \subseteq M$ implies $N \preceq M$. Abraham Robinson showed that under certain conditions a model complete theory admits quantifier elimination (QE for short). This was one of the results that inaugurated the use of model-theoretic methods in the study of QE. Model-completeness has many equivalent formulations:

Fact 1.2. Let $T$ be any theory. The following are equivalent:
1. $T$ is model complete.
2. For any two models $N, M \models T$ with $N \subseteq M$ there is an $N^* \models T$ such that $N \preceq N^*$ and $M$ can be embedded into $N^*$ over $N$.
3. For any $M \models T$ the theory $T \cup \text{ED}(M)$ is complete.
4. For any two models $N, M \models T$ with $N \subseteq M$, every existential formula $\varphi(x)$, and every $\bar{b} \in |N|$, we have $M \models \varphi(\bar{b})$ if and only if $N \models \varphi(\bar{b})$.
5. For every existential formula $\varphi(x)$ there is a universal formula $\varphi^*(x)$ such that $T \vdash \varphi(x) \iff \varphi^*(x)$.

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6. For every formula $\varphi(\bar{x})$ there is a universal formula $\varphi^*(\bar{x})$ such that $T \vdash \varphi(\bar{x}) \leftrightarrow \varphi^*(\bar{x})$.

7. For every formula $\varphi(\bar{x})$ there is a universal formula $\varphi_1(\bar{x})$ and an existential formula $\varphi_2(\bar{x})$ such that $T \vdash \varphi_1(\bar{x}) \leftrightarrow \varphi(\bar{x}) \leftrightarrow \varphi_2(\bar{x})$.

For a proof of this fact see [1] and [3].

However, there are theories which are model complete but do not admit QE. For example, the complete theory of real closed fields in the language of rings is model complete, but the formula $\exists x \ x \times x = y$ is not equivalent to any quantifier-free formula in this theory. See [1] for details.

Over the years many model-theoretic properties have been proposed to strengthen model-completeness so that QE is implied without any additional assumptions on the theory in question. Some of these properties are logically equivalent to QE; others are strictly stronger than QE. Below we shall prove that two of the stronger ones, one proposed by J. R. Shoenfield and the other by L. van den Dries, are equivalent for countable languages.

§2. Some QE tests. Let $T$ be any theory. Here are some model-theoretic QE tests that are stronger than model-completeness:

**Definition 2.1.** $T$ is submodel complete if and only if for any model $M \models T$ and any $N \subseteq M$ the theory $T \cup \text{ED}(N)$ is complete.

This is a direct strengthening of [1,2,3].

**Definition 2.2.** $T$ has the submodel amalgamation property (SA-property for short) if and only if for any $M_1, M_2 \models T$ and any $N \subseteq M_1, M_2$ there is an $M^* \models T$ such that $M_1 \preceq M^*$ and $M_2$ can be embedded into $M^*$ over $N$ via a monomorphism $f$; that is, the following diagram

$$
\begin{array}{c}
M_1 \preceq M^* \\
\uparrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 0

2. For every two models $N, M \models T$ with $N \subseteq M$, every existential formula $\varphi(\vec{x})$, and every $b \in |N|$, we have $M \models \varphi(b)$ if and only if $N \models \varphi(b)$. In other words, $T$ is model complete.

When there is no danger of confusion we abuse $L(T)$ to denote both the language of $T$ and the set of all well-formed formulas in the language of $T$.

For two structures $N$ and $M$ in $L(T)$ we say that $M$ is a $T$-extension of $N$ if $|N| \subseteq |M|$ and $M \models T$.

**Definition 2.5.** $T$ has the van den Dries property (D-property for short) if and only if

1. For any model $N$, if there exists a model $M \models T$ such that $N \subseteq M$, then there is a $T$-closure $N^*$ of $N$, that is, a model $N^* \models T$ such that $N \subseteq N^*$ and $N^*$ can be embedded over $N$ into any $T$-extension of $N$;
2. If $N, M \models T$ and $N \subsetneq M$, then there is an $a \in |M| \setminus |N|$ such that $N + a$ can be embedded into an elementary extension of $N$ over $N$, where $N + a$ is the smallest submodel of $M$ that contains $|N| \cup \{a\}$.

The SS-property first appeared in Shoenfield’s textbook [4]. He subsequently modified it into the S-property and proved its equivalence to QE in [5]. The D-property was given by van den Dries in [6] and [7], which is a straightforward strengthening of the SS-property. However, the main result Theorem 2.7 below shows that, for countable languages, its main advantage over the SS-property is its conceptual concreteness rather than its logical strength.

**Theorem 2.6.** Let $T$ be a theory in a language with at least one constant symbol. For the following statements,

1. $T$ is submodel complete,
2. $T$ has the SA-property,
3. $T$ has the S-property,
4. $T$ has the SS-property,
5. $T$ has the D-property,
6. $T$ admits QE,

these logical implications hold:

![Diagram](image_url)

**Proof.** That $[1] \Rightarrow [2]$ and $[3]$ are equivalent to QE is well-known. See, for example, [3] and [5]. Here we give proofs to the remaining two implications. We also show directly how the first condition of the SS-property achieves QE on top of model-completeness. This proof is a modification of the standard proof of $[1] \Rightarrow [2][3]$ in the literature, which establishes a crucial connection between model-theoretic properties and syntactical properties.

$[4] \Rightarrow [6]$ Let $\varphi(\vec{x})$ be a formula in $L(T)$. Since $T$ is model complete, by $[1][2]$ $\varphi(\vec{x})$ is equivalent to both a universal formula and an existential formula. Hence we may assume that $\varphi(\vec{x})$ is a universal formula. Let $\varphi^*(\vec{x})$ be an existential
formula such that $T \models \varphi(x) \leftrightarrow \varphi^*(x)$. Let $\bar{c}$ be new constants. Let $\Gamma$ be a set that contains exactly the following formulas:

- $T \cup \{ \varphi(\bar{c}) \}$, and
- every quantifier-free formula $\neg \psi(\bar{c})$ such that $T \models \forall \bar{x} \ (\psi(\bar{x}) \rightarrow \varphi(\bar{x}))$.

Suppose for contradiction that $\Gamma$ is consistent. Take any model $M \models \Gamma$. Let $N \subseteq M$ be the minimal submodel generated by $\bar{c}$. Note that every element in $N$ can be written as a term that only involves $\bar{c}$, the constants of $L(T)$, and the functions of $L(T)$. Now, if $T \cup ED(N)$ does not prove $\varphi(\bar{c})$, then fix a model $M^* \models T \cup ED(N) \cup \{ \neg \varphi(\bar{c}) \}$. By the first condition of the SS-property we can find an $N_1 \models T \cup ED(N)$ in $M$ and an $N_2 \models T \cup ED(N)$ in $M^*$ such that they are isomorphic over $N$. Since $\varphi(\bar{x})$ is a universal formula and $M \models \varphi(\bar{c})$, we have $N_1 \models \varphi(\bar{c})$. So $N_2 \models \varphi(\bar{c})$, so $N_2 \models \varphi^*(\bar{c})$, so $M^* \models \varphi^*(\bar{c})$, so $M^* \models \varphi(\bar{c})$, contradiction. So $T \cup ED(N) \not\models \varphi(\bar{c})$. So there is a quantifier-free formula $\psi(\bar{c}) \in ED(N)$ such that $T \cup \{ \psi(\bar{c}) \} \not\models \psi(\bar{c})$, so $T \not\models \psi(\bar{c}) \rightarrow \varphi(\bar{c})$. But $\bar{c}$ are new constants, so $T \not\models \forall \bar{x} \ (\psi(\bar{x}) \rightarrow \varphi(\bar{x}))$. So $\neg \psi(\bar{c}) \in \Gamma$, contradiction again.

So $\Gamma$ is not consistent. This means that there are finitely many quantifier-free formulas $\psi_i(\bar{x})$ such that $T \models \forall \bar{x} \ (\psi_i(\bar{x}) \rightarrow \varphi(\bar{x}))$ for every $i$ and $T \models \forall \bar{x} \ (\varphi(\bar{x}) \rightarrow \bigvee_i \psi_i(\bar{x}))$. So $T \models \forall \bar{x} \ (\varphi(\bar{x}) \leftrightarrow \bigvee_i \psi_i(\bar{x}))$, as desired.

\[ \Rightarrow \] Let $M_1, M_2 \models T$, $N \subseteq M_1, M_2$, and let $M_2$ be $\|M_1\|^+$-saturated. By the first condition of the SS-property we can find two $T$-extensions $N_1, N_2$ of $N$ in $M_1, M_2$ respectively that are isomorphic over $N$. Let the isomorphism be $f$. Pick an $a \in |M_1| \setminus |N_1|$ and consider any quantifier-free formula $\varphi(x; \bar{b})$ with $\bar{b} \in |N_1|$ such that $M_1 \models \varphi(a; \bar{b})$. Since $M_1 \models \exists x \ \varphi(x; \bar{b})$, by the second condition of the SS-property we have $N_1 \models \exists x \ \varphi(x; \bar{b})$, so $N_2 \models \exists x \ \varphi(x; f(\bar{b}))$, so $M_2 \models \exists x \ \varphi(x; f(\bar{b}))$. Hence the quantifier-free type $f(p)$ is realized in $M_2$, say, by $d$, where $p$ is the set of all quantifier-free formulas in $tp(a/|N_1|, M_1)$. If we set $a \rightarrow d$ then we get an induced isomorphism between $N_1 + a$ and $N_2 + d$. Iterating this procedure to exhaust all elements in $M_1$ we see that $M_1$ can be embedded into $M_2$ over $N$.

\[ \Rightarrow \] Trivially the closure property, that is, the first condition of the D-property, implies the first condition of the SS-property. For the second condition of the SS-property, let $N, M \models T$ with $N \subseteq M$. Consider an existential formula $\exists \bar{x} \ \varphi(x; \bar{b})$ that is satisfied in $M$, where $\bar{b} \in |N|$ and $\varphi(x; \bar{b})$ is quantifier-free. So let $\bar{c}$ be such that $M \models \varphi(\bar{c}; \bar{b})$. We construct the following diagram:

$$
\begin{array}{cccc}
N_0 & \subset & N_0 + a_0 & \subset & N_1 & \subset & N_1 + a_1 & \subset & N_2 & \subset & \cdots & \subset & M \\
\ \subseteq & \ \subseteq & \ \subseteq & \ \subseteq & \ \subseteq & \ \subseteq & \ \subseteq & \ \subseteq & \ \subseteq & \ \subseteq & \ \subseteq & \ \subseteq \\
N_0^* & \uparrow & f_0 & \uparrow & f_1 & \uparrow & \cdots & \\
N_1^* & \subseteq & N_1 & \subseteq & N_2 & \subseteq & M \\
\end{array}
$$

where $N_0 = N$, each $N_{i+1}$ is the $T$-closure of $N_i + a_i$ promised by the closure property, each $a_i$ and $N_i^*$ are as described in the second condition of the D-property, all arrows are monomorphisms, and at the limit stage we simply take the union of all previous $N_i$’s.

Now, let $i$ be the least index such that $\bar{c} \in N_i$. Note that $i$ cannot be a limit ordinal. So $N_i \models \exists \bar{x} \ \varphi(\bar{x}; \bar{b})$, so $N_{i-1}^* \models \exists \bar{x} \ \varphi(\bar{x}; \bar{b})$, so $N_{i-1} \models \exists \bar{x} \ \varphi(\bar{x}; \bar{b})$, etc.
If \( \gamma \) is a limit ordinal and \( N_\gamma \models \exists x \varphi(x; \bar{b}) \), then there is a \( \bar{d} \in |N_\gamma| \) such that \( N_\gamma \models \varphi(\bar{d}; \bar{b}) \), so by the construction there is a \( j < \gamma \) such that \( \bar{d} \in |N_j| \), so \( N_j \models \varphi(\bar{d}; \bar{b}) \), so \( N_j \models \exists x \varphi(x; \bar{b}) \). As we trace back in the diagram we see that \( N = N_0 \models \exists x \varphi(x; \bar{b}) \).

The reason that we have assumed that the language of \( T \) has at least one constant symbol is to avoid certain pathology. That is, in the proof of \( \text{[4] \Rightarrow [6]} \) above, if \( \varphi \) is a sentence and \( L(T) \) has no constant symbol, then \( \bar{c} \) is the empty sequence and cannot generate any submodel as we do not allow an empty model. The reader should observe that in this case the proof will not go through if we simply use an arbitrary submodel. In the sequel we shall always assume that \( T \) has a constant symbol whenever we are in a similar situation.

There are still more model-theoretic tests that are equivalent to QE. They are all more or less variations of the three equivalent tests in the above theorem. See [2] for more details about this. On the other hand, it is tempting to ask if in the above theorem all of the statements are indeed equivalent.

Jeremy Avigad has an example which shows that QE is strictly weaker than the SS-property. Consider the set \( 2^\omega \) of all binary sequences of length \( \omega \). For each \( n \in \omega \) let \( Z_n \) be a unary predicate such that if \( n = 0 \) then \( Z_n(\eta) \) for any \( \eta \in 2^\omega \), otherwise \( Z_n(\eta) \) if and only if \( (\eta)_n = 0 \). Let \( T = \text{Th}(\langle 2^\omega, Z_n \rangle_{n \in \omega}) \).

Since except equality all predicates in the language are unary, every existential formula \( \exists x \varphi(x; \bar{y}) \) is equivalent to a formula of the form \( \bigvee_i (\theta_i(\bar{y}) \land \exists x \phi_i(x; \bar{y})) \), where \( \phi_i(x; \bar{y}) \) is a conjunction of literals each of which contains \( x \). If the unary predicates in the formula \( \exists x \phi_i(x; \bar{y}) \) describe a “consistent” finite sequence, then it can be translated into an equivalent quantifier-free formula that only involves \( \bar{y} \). So \( T \) proves that every existential formula is equivalent to a quantifier-free formula, which means that \( T \) admits QE. Now, it is not hard to see that any dense subset of \( 2^\omega \) is a model of \( T \). Let \( S_0 \subseteq 2^\omega \) be the set of those sequences that have only finitely many 0’s. Let \( S_1 \subseteq 2^\omega \) be the set of those sequences that have only finitely many 1’s and the constant sequence \( \bar{1} \). So both \( S_0 \) and \( S_1 \) are models of \( T \). Notice that \( \{1\} \) is a submodel of both models as there is no function symbol in the language. Clearly there cannot be isomorphic \( T \)-extensions of \( \{1\} \) in \( S_0 \) and \( S_1 \).

What about the SS-property and the D-property? First of all it is trivial that if a theory \( T \) admits QE then the second condition of the D-property holds, because, by \( \text{[12]} \) if \( N, M \models T \) and \( N \subseteq M \) then \( M \) itself is an elementary extension of \( N \). The closure property, however, is much harder to achieve. The rest of this paper is devoted to proving

**Theorem 2.7.** For countable languages the SS-property and the D-property are equivalent.

The argument is by a transfinite induction.

### §3. The base case of the induction

We need more concepts and Henkin’s Omitting Type Theorem.

**Definition 3.1.** Let \( \bar{x} \) be a sequence of variables and \( p \) a \( T \)-type in \( \bar{x} \). If there exists a formula \( \varphi(\bar{x}) \) such that \( T \cup \{ \varphi(\bar{x}) \} \) is consistent and \( \varphi(\bar{x}) \vdash p \), then we
say that $p$ is isolated by $\varphi(\bar{x})$ via $T$. If in context it is clear that which theory is being discussed then we omit $T$.

Note that if $p$ is a complete $T$-type then $p$ is isolated via $T$ if and only if there exists a $\varphi \in p$ such that $\varphi \vdash p$.

**Definition 3.2.** Let $M \models T$ and $A \subseteq |M|$. We say that $M$ is almost $T$-primary over $A$ if there exists an ordinal $\alpha$ and a sequence $\langle (N_i, b_i) : i < \alpha \rangle$ such that

1. $N_0$ is the minimal submodel of $M$ that contains $A$,
2. $b_1 \in |M| \setminus |N_0|$ and $N_{i+1} = N_i + b_i$ for each $i < \alpha$ (if $\alpha = \beta + 1$ then $b_\beta$ is not defined),
3. $N_\beta = \bigcup_{i < \beta} N_i$ if $\beta$ is a limit ordinal and $\bigcup_{i < \alpha} N_i = M$,
4. the type $tp(b_j/|N_j|, M)$ is isolated via $T_j$ for every $j < \alpha$, where $T_j = T \cup CD(N_j, M)$.

The sequence $\langle (N_i, b_i) : i < \alpha \rangle$ is called an almost isolating sequence for $M$ over $A$. The ordinal $\alpha$ is the length of the sequence.

For convenience, if $T = Th(M)$ then we omit $T$. Also, sometimes we allow an almost isolating sequence to have repeated consecutive $b_i$’s. Of course in this case we no longer require $b_i \notin |N_i|$ for the repeated occurrences. Note that this definition is a variation of the notion of a primary model, which plays an important role in the proof of Morley’s Theorem.

**Definition 3.3.** Let $M \models T$ and $A \subseteq |M|$. We say that $M$ is $T$-primary over $A$ if there exists an ordinal $\alpha$ and an enumeration $\langle b_i : i < \alpha \rangle$ of $|M| \setminus A$ such that the type

$$tp(b_j/A \cup \{b_i : i < j\}, M)$$

is isolated via $T_j$ for every $j < \alpha$, where $T_j = T \cup CD(A \cup \{b_i : i < j\}, M)$. The sequence $\langle b_i : i < \alpha \rangle$ is called an isolating sequence for $M$ over $A$. The ordinal $\alpha$ is the length of the sequence.

It is not hard to see that if $T$ is submodel complete and $N \subseteq M \models T$ then $M$ is almost $T$-primary over $N$ if and only if $M$ is $T$-primary over $N$. We prefer the concept of an almost primary model below because it is more explicit about what property is being exploited, namely submodel completeness.

**Theorem 3.4** (Henkin’s Omitting Type Theorem). If $L(T)$ is countable and $\Gamma$ is a countable collection of $T$-types such that $p$ is not isolated for every $p \in \Gamma$, then there exists a countable model $M \models T$ that omits all the types in $\Gamma$.

We proceed to develop a couple of technical lemmas. We have the following basic fact about an almost primary model satisfying a submodel complete theory:

**Lemma 3.5.** Suppose $T$ is submodel complete. Let $N \subseteq M \models T$. Then: if $M$ is almost $T$-primary over $N$, then for every model $M^* \models T \cup ED(N)$ there is an elementary embedding from $M$ into $M^*$ over $N$.

**Proof.** Since $T$ is submodel complete, the theory $T \cup ED(N)$ is complete. This means that for any formula $\varphi(\bar{x})$ and any $\bar{a} \in |N|$ we have

$$M \models \varphi(\bar{a}) \iff M^* \models \varphi(\bar{a}).$$
Let \( (N_i, b_i) : i < \alpha \) be an almost isolating sequence for \( M \) over \( N \). So by definition \( N_0 = N \). In order to prove the lemma it is enough to construct a continuous sequence of monomorphisms \( g_i : N_i \rightarrow M^* \) for \( i < \alpha \) such that

1. \( g_0 = \text{id}_N \),
2. \( N_i \models \varphi(\bar{a}) \iff M^* \models \varphi(g_i(\bar{a})) \) for each formula \( \varphi(x) \) and each \( \bar{a} \in N_i \),
3. if \( i < j < \alpha \) then \( g_i \subseteq g_j \), and
4. if \( \beta \) is a limit then \( g_\beta = \bigcup_{i<\beta} g_i \).

The embedding \( g = \bigcup_{i<\alpha} g_i \) is as desired. That \( g \) is elementary is because submodel completeness implies model completeness (see \[1.2\] and \[2.6\]).

Now we proceed to construct the sequence. Due to the clause 4 all we have to do is to make the successor case work. So suppose we have successfully constructed the sequence up to the ordinal \( i < \alpha \). Since the complete type \( p_i = \text{tp}(b_i/|N_i|, M) \) is isolated via \( T_i \) where \( T_i = T \cup \text{CD}(N_i, M) \), there exists a formula \( \varphi(x; \bar{a}) \in p_i \) isolating it. By the clause 2 we have

\[
\varphi(x; \bar{a}) \vdash p_i \Rightarrow \varphi(x; g_i(\bar{a})) \vdash g_i(p_i).
\]

Since \( M \models \varphi(b_i; \bar{a}) \), we have \( M \models \exists x \varphi(x; \bar{a}) \), so \( M^* \models \exists x \varphi(x; g_i(\bar{a})) \). Let \( c_i \in |M^*| \) such that \( M^* \models \varphi(c_i; g_i(\bar{a})) \). So by \[1.3\] \( c_i \) realizes the type \( g_i(p_i) \). Now define a function \( g_{i+1} \) by setting \( \tau(b_i) \mapsto \tau(c_i) \) for each term \( \tau(x) \) of \( L(T_i) \). It is easy to see that this is a well-defined monomorphism from \( N_{i+1} \) into \( M^* \) which extends \( g_i \) and takes \( b_i \) to \( c_i \). That the clause 2 is satisfied is, again, because \( T \) is submodel complete.

In order to build almost primary models we need the next crucial lemma.

**Lemma 3.6.** Suppose that \( L(T) \) is countable and \( T \) has the SS-property. Then for

1. every model \( M \models T \),
2. every countable submodel \( N \subseteq M \),
3. every formula \( \varphi(x; \bar{y}) \) and every \( \bar{a} \in |N| \) such that \( \exists x \varphi(x; \bar{a}) \in T \cup \text{ED}(N) \) but \( M \models \neg \varphi(b; \bar{a}) \) for every \( b \in |N| \),

there is an element \( c \in |M| \setminus |N| \) such that the type \( \text{tp}(c/|N|, M) \) is isolated and \( M \models \varphi(c; \bar{a}) \).

**Proof.** Fix an \( M \), an \( N \), an \( \bar{a} \), and a \( \varphi(x; \bar{y}) \) as above. Without loss of generality we may assume \( M \) is countable as well. Since \( T \) has the SS-property, by \[2.6\] the theory \( T \cup \text{ED}(N) \) is complete. So \( M \models \exists x \varphi(x; \bar{a}) \). So \( \varphi(M; \bar{a}) \neq \emptyset \) and, by the third condition, \( \varphi(M; \bar{a}) \subseteq |M| \setminus |N| \), where \( \varphi(M; \bar{a}) \) is the set \{ \( c \in |M| : M \models \varphi(c; \bar{a}) \)\}. Also note that \( T \) is model complete.

Suppose for contradiction we cannot find an element \( c \) in \( M \) as required. Define a collection \( \Gamma \) of \( T \cup \text{ED}(N) \)-types:

\[
\Gamma = \{ \text{tp}(c/|N|, M) : c \in |M| \setminus |N| \text{ and } M \models \varphi(c; \bar{a}) \}.
\]

Since \( \Gamma \) is countable, by Henkin’s Omitting Type Theorem there is a model \( O \models T \cup \text{ED}(N) \) that omits every type in \( \Gamma \). But \( T \) has the SS-property, so we can find two models \( M^* \subseteq M, O^* \subseteq O \) of \( T \) such that there is an isomorphism \( h : M^* \cong O^* \) whose restriction to \( N \) is \( \text{id}_N \). Since \( \exists x \varphi(x; \bar{a}) \in T \cup \text{ED}(N) \),
there must be some $c \in |M^*| \setminus |N|$ such that $M^* \models \varphi(c; \bar{a})$. Since $T$ is model complete, we deduce

$$\varphi(x; \bar{a}) \in \text{tp}(c/|N|, M^*) = \text{tp}(c/|N|, M).$$

This means that $h(c)$ realizes the $T \cup \text{ED}(N)$-type $\text{tp}(c/|N|, M)$ in $O$, contradicting the choice of $O$.

Note that in the above lemma, if $N$ is not a model of $T$, then there must exist a formula $\exists x \varphi(x; \bar{a}) \in T \cup \text{ED}(N)$ with $\bar{a} \in |N|$ such that $M \models \neg \varphi(b; \bar{a})$ for every $b \in |N|$, because otherwise $N$ would be a model of $T$ by the Tarski-Vaught Test as $T \cup \text{ED}(N)$ is complete. This property is important for our argument. We shall give it a name:

**Definition 3.7.** Let $M \models T$, $N \subseteq M$, and $\bar{a} \in |N|$. We say that $\varphi(x; \bar{a})$ is critical for $N$ if $\exists x \varphi(x; \bar{a}) \in T \cup \text{ED}(N)$ and $\varphi(M; \bar{a}) \subseteq |M| \setminus |N|$.

Now the SS-property enables us to construct almost primary models over countable submodels.

**Theorem 3.8.** If $L(T)$ is countable and $T$ has the SS-property then, for any model $M \models T$ and any countable submodel $N \subseteq M$, $N$ has a $T$-closure.

**Proof.** Fix $N \subseteq M \models T$ such that $N$ is countable. Again we may assume that $M$ is countable as well. So by Lemma 3.5 all we need to do is to build an almost $T$-primary model $N^*$ over $N$ inside $M$. For this it is enough to build an almost isolating sequence for some model of $T$ over $N$. The idea here is of course to find a suitable Skolem hull of $N$ inside $M$ such that the type of each “key” new element we find is isolated over all the previous elements.

To be precise, we want to build an almost isolating sequence $\langle (N_i, b_i) : i < \omega \cdot \omega \rangle$ over $N$ such that for

- each $n < \omega$,
- each $\bar{a} \in N_{\omega \cdot n}$, and
- each formula $\varphi(x; \bar{y})$ such that $M \models \exists x \varphi(x; \bar{a})$,

there is an $m < \omega$ such that $M \models \varphi(\tau(b_{\omega \cdot n + m}); \bar{a})$ for some term $\tau(x)$ in the language $L(T \cup \text{ED}(N_{\omega \cdot n + m}))$. It should be clear that $\bigcup_{i < \omega \cdot \omega} N_i = N^*$ is an elementary submodel of $M$, and hence is almost $T$-primary over $N$.

Now we carry out the construction. Start with $N_0 = N$ of course. Suppose $\langle (N_i, b_i) : i < \omega \cdot n \rangle$ is defined. Let $\langle \varphi_k(x; \bar{a}_k) : k < \omega \rangle$ be an enumeration of all the formulas in $T \cup \text{ED}(N_{\omega \cdot n})$ such that for every $k < \omega$ we have $M \models \exists x \varphi_k(x; \bar{a}_k)$ but $M \models \neg \varphi_k(d; \bar{a}_k)$ for every $d \in N_{\omega \cdot n}$. Now suppose we have extended the sequence all the way up to $(N_{\omega \cdot n + k}, b_{\omega \cdot n + k})$ for some $k < \omega$. Let $N_{\omega \cdot n + k + 1} = N_{\omega \cdot n + k} + b_{\omega \cdot n + k}$. If there is a $d \in N_{\omega \cdot n + k + 1}$ such that $M \models \varphi_{k+1}(d; \bar{a}_{k+1})$ then let $b_{\omega \cdot n + k + 1} = b_{\omega \cdot n + k}$. Otherwise by Lemma 3.3 we can pick a $b_{\omega \cdot n + k + 1} \in |M| \setminus |N_{\omega \cdot n + k + 1}|$ such that $M \models \varphi_{k+1}(b_{\omega \cdot n + k + 1}; \bar{a}_{k+1})$ and the type $\text{tp}(b_{\omega \cdot n + k + 1}/|N_{\omega \cdot n + k + 1}|, M)$ is isolated.

§4. The inductive step. The reader may ask: What is preventing us here from simply extending the above theorem to arbitrary theories and arbitrary submodels? One difficulty is this: We do not know how to extend Henkin’s
Omitting Type Theorem to uncountable languages and hence are unable to develop an analog of Lemma 3.6 for uncountable languages. In fact if we simply drop the countability requirement in Henkin’s Omitting Type Theorem then it is false. See [1] for discussions. However, in this last section we will show how to circumvent this difficulty if the language in question is countable. For this we need some basic concepts and facts in infinitary combinatorics, in particular stationary sets and Fodor’s Lemma.

Throughout the rest of this section $T$ is a theory in a countable language and has the SS-property. Our strategy is to establish an analog of Lemma 3.6 for any submodel. Let $M \models T$ and $N \subseteq M$ such that $N$ is uncountable and is not a model of $T$. We have two cases to consider, namely $\|N\|$ is regular and $\|N\|$ is singular.

**Definition 4.1.** Let $\alpha$ be an ordinal. A sequence $\langle N_i : i < \alpha \rangle$ is an $\alpha$-resolution of $N$ if

1. $N_i$ is a submodel of $N$ for all $i < \alpha$,
2. if $i < j < \alpha$ then $N_i \subseteq N_j$,
3. $\bigcup_{i<\alpha} N_i = N$.

If, in addition, $\bigcup_{i<\delta} N_i = N_\delta$ for every limit ordinal $\delta < \alpha$, then the sequence is a continuous $\alpha$-resolution of $N$.

**Lemma 4.2.** Suppose $\|N\| = \kappa$ is regular and $\varphi(x; \bar{a})$ is critical for $N$. Then there is an element $c \in \varphi(M; \bar{a})$ such that the type $\text{tp}(c/\|N\|, M)$ is isolated.

**Proof.** Without loss of generality we may assume $\|M\| = \kappa$. Fix a club $C = \langle \alpha_i : i < \kappa \rangle \subseteq \kappa$ and a continuous $\kappa$-resolution $\langle N_i : i < \kappa \rangle$ of $N$ such that

1. for all $\alpha_i, \alpha_j \in C$ and $i < j$ we have $|\alpha_i| \leq |\alpha_j \setminus \alpha_i|$,
2. $\|N_i\| = |\alpha_i|$,
3. $\bar{a} \in N_0$.

By the inductive hypothesis we construct a sequence $\langle b_i \in \varphi(M; \bar{a}) : i < \kappa \rangle$ such that each type $\text{tp}(b_i/\|N_i\|, M)$ is isolated. Fix an enumeration $\langle \phi_i : i < \kappa \rangle$ of all the formulas in the language of $T \cup \text{ED}(N)$ such that for each $\alpha_i \in C$ we have

$$\{i : \phi_i \text{ is a formula in the language of } T \cup \text{ED}(N_i)\} \subseteq \alpha_i.$$

Now define a function $f : C \to \kappa$ by letting $f(\alpha_i)$ be the least ordinal such that $\phi_f(\alpha_i)$ isolates the type $\text{tp}(b_i/\|N_i\|, M)$. Since $f$ is a pressing-down function on a stationary subset of $\kappa$ and $\kappa$ is regular, by Fodor’s Lemma, there is a $\gamma < \kappa$ such that $f^{-1}(\gamma) \subseteq C$ is stationary. Clearly for any $\alpha_i, \alpha_j \in f^{-1}(\gamma)$, if $\alpha_i < \alpha_j$ then $\text{tp}(b_i/\|N_j\|, M) = \text{tp}(b_j/\|N_j\|, M)$ as they are both isolated by $\phi_\gamma$. So $\text{tp}(b_i/\|N\|, M) = \text{tp}(b_j/\|N\|, M)$ for any $\alpha_i, \alpha_j \in f^{-1}(\gamma)$. And this type is isolated by $\phi_\gamma$ as desired.

For the case that $\|N\|$ is singular we need to work harder. First we formulate the following concept:

**Definition 4.3.** Let $\langle N_i : i < \alpha \rangle$ be an $\alpha$-resolution of $N$. Let $\bar{a} \in N_0$. Let $\varphi(x; \bar{a})$ be critical for $N$. We say that $F = \langle \varphi_i(x) : i < \alpha \rangle$ is a spinal sequence of $\varphi(x; \bar{a})$ for $\langle N_i : i < \alpha \rangle$ if:

1. each $\varphi_i(x)$ is a formula in the language of $T \cup \text{ED}(N_i)$,
2. \( \varphi_i(M) \neq \emptyset \) and \( \varphi_i(M) \subseteq \varphi(M; \bar{a}) \) for each \( i < \alpha \),
3. if \( b \in \varphi_i(M) \) then the type \( \text{tp}(b/|N_i|, M) \) is isolated by \( \varphi_i(x) \).

We write \( \text{dom}(\mathcal{F}) \) for the set
\[ \{ a \in |N| : a \text{ occurs as a parameter in some } \varphi_i(x) \in \mathcal{F} \} . \]

**Lemma 4.4.** Suppose \( ||N|| = \kappa \) is singular and \( \varphi(x; \bar{a}) \) is critical for \( N \). Then there is an element \( c \in \varphi(M; \bar{a}) \) such that the type \( \text{tp}(c/|N|, M) \) is isolated.

**Proof.** As above we may assume \( ||N|| = \kappa \). Let \( \lambda = \text{cf}(\kappa) < \kappa \). Let \( \langle \mu_i : i < \lambda \rangle \subseteq \kappa \) be a strictly increasing sequence of cardinals such that it is unbounded in \( \kappa \). Let \( \langle N_i : i < \lambda \rangle \) be a \( \lambda \)-resolution of \( N \) such that \( \bar{a} \in N_0 \) and \( ||N_i|| = \mu_i \).

Let \( \mathcal{F}_0 \) be a spinal sequence of \( \varphi(x; \bar{a}) \) for \( \langle N_i : i < \lambda \rangle \). Note that the existence of such a sequence is guaranteed by the inductive hypothesis. We have \( \text{dom}(\mathcal{F}_0) \leq \lambda \). Now let \( K_0 \subseteq N \) be the submodel generated by \( \text{dom}(\mathcal{F}_0) \cup \{ \bar{a} \} \). Note that \( \varphi(x; \bar{a}) \) is critical for \( K_0 \). Since \( ||K_0|| \leq \lambda < \kappa \), by the inductive hypothesis there is an element \( c_0 \in \varphi(M; \bar{a}) \) such that \( \text{tp}(c_0/|K_0|, M) \) is isolated by some formula \( \sigma_0(x) \) in \( L(T \cup \text{ED}(K_0)) \). Notice that if \( \mathcal{F}_0 \subseteq \text{tp}(c_0/|K_0|, M) \) then we are done: in this case \( \sigma_0(x) \) isolates the entire \( \mathcal{F}_0 \) and each \( \varphi_i(x) \in \mathcal{F}_0 \) isolates the type \( \text{tp}(c_0/|N_i|, M) \), so the type \( \text{tp}(c_0/|N|, M) \) is isolated by \( \sigma_0(x) \).

Next, since \( \varphi(x; \bar{a}) \wedge \sigma_0(x) \) is critical for \( N \) (because it contains \( \varphi(x; \bar{a}) \) as a conjunct), we can find a spinal sequence \( \mathcal{F}_1 \) of \( \varphi(x; \bar{a}) \wedge \sigma_0(x) \) for \( \langle N_i : i < \lambda \rangle \). Clearly \( \mathcal{F}_1 \) is also a spinal sequence of \( \varphi(x; \bar{a}) \) for \( \langle N_i : i < \lambda \rangle \). Let \( K_1 \subseteq N \) be the submodel generated by \( |K_0| \cup \text{dom}(\mathcal{F}_1) \). Then, similarly, we can find an element \( c_1 \in \varphi(M; \bar{a}) \) and a formula \( \sigma_1(x) \) in \( L(T \cup \text{ED}(K_1)) \) that isolates the type \( \text{tp}(c_1/|K_1|, M) \).

Continuing in this fashion we can construct a sequence \( \langle (\mathcal{F}_i, c_i, \sigma_i(x)) : i < \lambda^+ \rangle \) such that
1. \( c_i \in \varphi(M; \bar{a}) \),
2. \( \mathcal{F}_{i+1} \) is a spinal sequence of \( \varphi(x; \bar{a}) \wedge \sigma_i(x) \) for \( \langle N_i : i < \lambda \rangle \),
3. \( \sigma_i(x) \) is a formula in \( L(T \cup \text{ED}(K_i)) \) which isolates the type \( \text{tp}(c_i/|K_i|, M) \),
4. if \( i \) is a limit ordinal then \( \mathcal{F}_i \) is not defined.

Let \( K = \bigcup_{j<\lambda^+} K_j \). Let
\[ S^{\lambda^+}_\lambda = \{ \alpha < \lambda^+ : \text{cf}(\alpha) = \lambda \} , \]
which is a stationary subset of \( \lambda^+ \). Fix an enumeration of all the formulas in \( L(T \cup \text{ED}(K)) \) such that for each \( \alpha \in S^{\lambda^+}_\lambda \) we have
\[ \{ i : \phi_i \text{ is a formula in the language of } T \cup \text{ED}(K_\alpha) \} \subseteq \alpha . \]
So again by Fodor's Lemma there is a \( \sigma_j(x) \) and a stationary subset \( S \subseteq S^{\lambda^+}_\lambda \) such that for all \( \alpha \in S \) the type \( \text{tp}(c_\alpha/|K_\alpha|, M) \) is isolated by \( \sigma_j(x) \).

For any \( \alpha, \beta \in S \) with \( \alpha < \beta \), consider \( \mathcal{F}_{\alpha+1} \). Since \( \sigma_\alpha(x) \) is \( \sigma_j(x) \), \( \mathcal{F}_{\alpha+1} \) is a spinal sequence of \( \varphi(x; \bar{a}) \wedge \sigma_j(x) \) for \( \langle N_i : i < \lambda \rangle \). So
\[ M \models \exists x \ ( \varphi(x; \bar{a}) \wedge \sigma_j(x) \wedge \varphi_i(x) ) \]
for all \( \varphi_i(x) \in \mathcal{F}_{\alpha+1} \) (this is by the second condition in the definition of a spinal sequence above). Since \( \sigma_j(x) \) also isolates the complete type \( \text{tp}(c_\beta/|K_\beta|, M) \) and
We see that $\sigma_j(x)$ isolates the type $\text{tp}(c_{\beta}/ |N|, M)$. With these two lemmas we can now simply proceed to build an almost isolating sequence for some model of $T$ over $N$ much in the same way as in Theorem 2.7, only now the length of the almost isolating sequence can go up to $\|N\| \cdot \omega$. This proves Theorem 2.7.

We end this paper with a question:

**Question 4.5.** Is there an analog of Theorem 2.7 for uncountable languages?

Notice that, if $T$ is a theory in an uncountable language and the SS-property and the D-property are not equivalent for $T$, then there is an $M \models T$ and an $N \subseteq M$ such that the complete theory $T \cup \text{ED}(N)$ is not totally transcendental. This is because primary models always exist for totally transcendental theories.

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