Research Article

Trapezoidal Type Fejér Inequalities Related to Harmonically Convex Functions and Application

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Some authors introduced the concepts of the harmonically arithmetic convex functions and establish some integral inequalities of Hermite Hadamard Fejér type related to the harmonically arithmetic convex functions. In this paper, a mapping $M(t)$ is considered to get some preliminary results and a new trapezoidal form of Fejér inequality related to the harmonically arithmetic convex functions. By using a mapping $M(t)$, the new theorems and corollaries are obtained. Taking advantage of these, applications were given for some real number averages.

1. Introduction

In 1906, the Hungarian mathematician L. Fejér proved the following integral inequalities known in the literature as Fejér inequality [1, 2]:

$$f \left( \frac{a+b}{2} \right) \int_a^b g(x) \, dx \leq \int_a^b f(x) g(x) \, dx \leq \frac{f(a) + f(b)}{2} \int_a^b g(x) \, dx$$

(1)

where $f : [a,b] \to \mathbb{R}$ is convex and $g : [a,b] \to \mathbb{R}^+$ is integrable and symmetric to $x = (a+b)/(2)$ ($g(x) = g(1/(1/a + 1/b - 1/x)), \forall x \in [a,b]$). For some other inequalities in connection with Fejér inequalities see [3–8] and the references therein.

In [9], Hwang found out the Fejér trapezoidal inequality related to convex functions as follows:

Theorem 1. Let $f : I \to \mathbb{R}$ be a differentiable mapping, where $a, b \in I$ with $a < b$, and let $g : [a,b] \to [0,\infty)$ be a continuous positive mapping symmetric to $(a+b)/2$. If the mapping $|f'|$ is convex on $[a,b]$, then the following inequality holds:

$$\frac{f(a) + f(b)}{2} \int_a^b g(x) \, dx - \int_a^b f(x) g(x) \, dx \leq \frac{b-a}{4} [||f'(a)|| + ||f'(b)||]$$

$$\cdot \int_{(1-t)/2a+(1+t)/2b}^{(1-t)/2a+(1+t)/2b} g(x) \, dx \, dt.$$  

(2)

In [8], Sarıkaya revealed new lemma and the difference between the right and middle part of (1) using Hölder’s inequality for convex function as follows:

Lemma 2 (see [8]). Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^a, a, b \in I^a$ with $a < b$, and let $w : [a,b] \to [0,\infty)$ be a differentiable mapping. If $f' \in L[a,b]$, then the following equality holds:

$$\frac{1}{b-a} \int_a^b f(x) w(x) \, dx - \frac{1}{b-a} f \left( \frac{a+b}{2} \right) \int_a^b w(x) \, dx$$

$$= (b-a) \int_0^1 k(t) f'(ta + (1-t)b) \, dt$$

(3)
for each \( t \in [0, 1] \), where

\[
k(t) = \begin{cases} 
\int_0^t w(as + (1 - s)b) \, ds, & 0 \leq t < \frac{1}{2} \\
- \int_t^{\frac{1}{2}} w(as + (1 - s)b) \, ds, & \frac{1}{2} \leq t < 1
\end{cases}
\] (4)

**Theorem 3** (see [8]). Let \( f : I^0 \subset \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping, \( a, b \in I^0 \) with \( a < b \), and let \( w : [a, b] \rightarrow \mathbb{R}^+ \) be a differentiable mapping symmetric to \((a+b)/2\). If \( f^{[q]} \) is convex on \([a, b], q > 1\), then the following inequality holds:

\[
\left| \frac{1}{b-a} \int_a^b f(a) - f(y) \, dx \right| \leq \frac{2}{(b-a)^2} \int_0^{(a+b)/2} w(x)(b-x) \, dx,
\] (5)

where \( 1/p + 1/q = 1 \) and

\[
g(x) = \int_0^{(b-(b-a)t)} w'(x) \, dx.
\] (6)

**Theorem 4** ([1] Theorem 2.1). Let \( f : I \rightarrow \mathbb{R} \) be a mapping that is differentiable on \( I^0 \), let \( a, b \in I^0 \) be points with \( a < b \), and let \( w : [a, b] \rightarrow \mathbb{R} \) be a nonnegative integrable mapping that is differentiable on \((a, b)\). If \( f \) is symmetric to \((a+b)/2\) and if \( f^{[q]} \) is convex on \([a, b], q > 1\), then

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{1}{2} \left[ \int_0^1 (g(x))^{1/p} \, dt \right]^{1/p} \cdot \left( \left| f'(a)^{[q]} \right| + \left| f'(b)^{[q]} \right| \right)^{1/q},
\] (7)

where \( g(x) = \int_0^{b-(b-a)t} w(x) \, dx \).

In [6], İscan identified the harmonically convex function and proved Hermite Hadamard type inequality connected with harmonically convex function as follows:

**Definition 5** (see [6]). Let \( I \subset \mathbb{R} \setminus \{0\} \) be a real interval. A function \( f : I \rightarrow \mathbb{R} \) is said to be harmonically convex if

\[
f \left( \frac{xy}{tx + (1-t)y} \right) \leq tf(y) + (1-t)f(x)
\] (8)

for \( \forall x, y \in I \) and \( t \in [0, 1] \). If the inequalities in (1) are reversed then \( f \) is said to be harmonically concave.

**Theorem 6** (see [6]). Let \( f : I \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) be a harmonically convex function and \( a, b \in I \) with \( a < b \). If \( f \in L[a, b] \) then the following inequalities hold

\[
f \left( \frac{2ab}{a+b} \right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \leq \frac{f(a) + f(b)}{2}
\] (9)

**Theorem 7** (see [6]). Let \( f : I \subset (0, \infty) \rightarrow \mathbb{R} \) be differentiable function on \( I^0 \), \( a, b \in I \), with \( a < b \) and \( f' \in L[a, b] \). If \( |f'|^{[q]} \) is harmonically convex on \([a, b], q \geq 1\), then

\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{ab(b-a)}{2} \lambda^{1-1/q} \left[ \lambda_2 |f'(a)|^q + \lambda_3 |f'(b)|^q \right]^{1/q}.
\] (10)

where

\[
\lambda_1 = \frac{1}{ab} - \frac{2}{(b-a)^2} \ln \left( \frac{(a+b)^2}{4ab} \right)
\]

\[
\lambda_2 = \frac{-1}{b-a} + \frac{3a+b}{(b-a)^2} \ln \left( \frac{(a+b)^2}{4ab} \right)
\]

\[
\lambda_3 = \frac{1}{a(b-a)} + \frac{3b+a}{(b-a)^2} \ln \left( \frac{(a+b)^2}{4ab} \right) = \lambda_1 - \lambda_2
\]

**Definition 8** (see [10]). A function \( f : [a, b] \rightarrow \mathbb{R} \) is said to satisfy Lipschitz condition on \([a, b]\) if there is a constant \( K \) so that for any two points \( x, y \in [a, b] \),

\[
|f(x) - f(y)| \leq K|x - y|.
\] (12)

In this paper, I obtain a new trapezoidal form of Fejér inequality via the absolute value of the derivative of the considered function is the harmonically convex function. In addition, I get features of the new theorems and new corollaries. Furthermore, some applications in connection with special means are given.

### 2. Main Results

In the section, we have obtained the new theorem and corollary about Hermite Hadamard Fejér type inequality for the both harmonically convex functions.

We use this lemma for harmonically convex function and motivated by above works and results we consider a mapping \( M(t) \) and obtain some introductory properties related to it. Also a new trapezoidal form of Fejér inequality is proved in the case that the absolute value of considered function is harmonically convex.

Related to a function \( g : [a, b] \rightarrow \mathbb{R} \) consider the mapping \( M : [0, 1] \rightarrow \mathbb{R} \) as the following:

\[
M(t) = \int_0^t g \left( \frac{ab}{sb + (1-s)a} \right) \, ds
\]

\[
- \int_0^t g \left( \frac{ab}{sb + (1-s)a} \right) \, ds.
\] (13)

There exist some properties for the mapping \( M(t) \), compiled in the following lemma which are used to obtain our main results.

**Lemma 9.** Suppose that \( I \subseteq \mathbb{R} \setminus \{0\} \) is an interval, \( a, b \in I^0 \) with \( a < b \) and \( g : [a, b] \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) is an integrable function on \([a, b]\).
(1) If \( g \) is symmetric to \( 2ab/(a+b) \), then
\[
M(t) = \begin{cases}
2^t \left(2^{1/2} \int_{t}^{1/2} g \left( \frac{ab}{sb + (1-s)a} \right) ds, & 0 < t < \frac{1}{2} \\
-2^t \left(-2^{1/2} \int_{1/2}^{t} g \left( \frac{ab}{sb + (1-s)a} \right) ds, & \frac{1}{2} < t < 1
\end{cases}
\]  
(14)

(2) For any \( t \in [0, 1] \),
\[
M(t) + M(1-t) = 0. 
\]  
(15)

(3) If \( g \) is a nonnegative function, then
\[
M(t) \geq 0, \quad 0 < t < \frac{1}{2} \\
M(t) \leq 0, \quad \frac{1}{2} < t < 1.
\]  
(16)

(4) The following inequalities hold.
\[
\int_{0}^{1} |M(t)| \, dt \leq \frac{1}{2} \|g\|_{\infty},
\]  
(17)

and
\[
\int_{0}^{1} |M(t)| \, dt \leq 2 \left\| g \right\|_{q} \int_{0}^{1} \left| t - \frac{1}{2} \right|^{1/p} \, dt
\]  
(18)

(5) Let \( I \subseteq \mathbb{R} \setminus \{0\} \) be an interval. \( f : I^\circ \rightarrow \mathbb{R} \) is a differentiable mapping on \( I^\circ \), \( I^\circ \) is an interior of \( I \), and \( g \) is a differentiable nonnegative mapping. If \( f' \in L[a, b] \), then we get the following equality:
\[
\begin{align*}
\frac{ab}{b-a} \left( \frac{f(a) + f(b)}{2} \right) &= \int_{a}^{b} \frac{g(x)}{x^2} \, dx \\
- \int_{a}^{b} f'(x) \frac{g(x)}{x^2} \, dx &= \frac{ab}{2} \left( b-a \right)
\end{align*}
\]  
(19)

Proof. (1) By taking the change of variable \( x = ab/(sb + (1-s)a) \) to (13), for \( 0 \leq t \leq 1/2 \), we obtain
\[
M(t) = \frac{ab}{b-a} \left[ \int_{a}^{ab/(sb + (1-s)a)} \frac{g(x)}{x^2} \, dx \\
- \int_{ab/(sb + (1-s)a)}^{b} \frac{g(x)}{x^2} \, dx \right],
\]  
(20)

where \( 2ab/(a+b) \leq ab/(tb+(1-t)a) \leq b \). Because of \( g \) being symmetric to \( 2ab/(a+b) \),
\[
\int_{2ab/(a+b)}^{b} \frac{g(x)}{x^2} \, dx = \int_{a}^{2ab/(a+b)} \frac{g(x)}{x^2} \, dx,
\]  
(21)

and so
\[
\int_{a}^{ab/(tb+(1-t)a)} \frac{g(x)}{x^2} \, dx = \int_{a}^{2ab/(a+b)} \frac{g(x)}{x^2} \, dx + \int_{2ab/(a+b)}^{ab/(tb+(1-t)a)} \frac{g(x)}{x^2} \, dx
\]  
(22)

However
\[
\int_{2ab/(a+b)}^{b} \frac{g(x)}{x^2} \, dx = \int_{2ab/(a+b)}^{ab/(tb+(1-t)a)} \frac{g(x)}{x^2} \, dx
\]  
(23)

If we use (22) and (23) to (20), we get
\[
M(t) = 2 \int_{t}^{1/2} \int_{a}^{ab/(tb+(1-t)a)} \frac{g(x)}{x^2} \, dx \, ds = \frac{ab}{sb + (1-s)a} \int_{a}^{1} \frac{g(x)}{x^2} \, dx,
\]  
(24)

where \( 0 \leq t \leq 1/2 \).

By using the same argument as above, we can prove that
\[
M(t) = -2 \int_{1/2}^{t} \int_{a}^{ab/(tb+(1-t)a)} \frac{g(x)}{x^2} \, dx \, ds
\]  
(25)

where \( 1/2 \leq t \leq 1 \).

(2) It is easy consequence of assertion of (1)
(3) By the assertion (3), we can get the following relations:
\[
\int_{0}^{1} |M(t)| \, dt = \int_{0}^{1/2} M(t) - \int_{1/2}^{1} M(t) \, dt
\]  
(26)

For the second part of (4), we conceive the following assertion which is not hard to prove:
\[
\int_{0}^{1} |M(t)| \, dt = 2 \int_{0}^{1} \left[ \int_{t}^{1/2} \frac{g(x)}{x^2} \, dx \right] \, dt.
\]  
(27)

By using Hölder inequality to the last inequality we get
\[
\left| \int_{1}^{1/2} g\left( \frac{ab}{sb + (1-s)a} \right) ds \right| \\
\leq \left| \int_{1}^{1/2} ds \right|^{1/p} \left| \int_{1}^{1/2} \left( g\left( \frac{ab}{sb + (1-s)a} \right) \right)^q ds \right|^{1/q}
\leq \|g\|_q \left| t - \frac{1}{2} \right|^{1/p}.
\]

Now using (29) in (27) inequality, we get
\[
\int_{0}^{1} |M(t)| dt \leq 2 \|g\|_{q} \int_{0}^{1} |t - \frac{1}{2}|^{1/p} dt.
\]

(4) Firstly we calculate the equality as follows:
\[
\int_{0}^{t} M(t) d\left( f\left( \frac{ab}{sb + (1-t)a} \right) \right)
= \frac{ab}{b-a} \int_{0}^{t} \left[ \int_{0}^{1/2} \left( \frac{g(x)}{x^2} \right) dx \right] \frac{g(x)}{x^2} dx \\
= \frac{ab}{b-a} \left( \int_{0}^{1/2} \left( \frac{g(x)}{x^2} \right) dx \right) \frac{g(x)}{x^2} \left[ \frac{ab}{b-a} \right]_{0}^{t}
+ \frac{b-a}{ab} \int_{0}^{1} f\left( \frac{ab}{sb + (1-t)a} \right) \left( \frac{g(x)}{x^2} \right) dx \\
+ \frac{g\left( \frac{ab}{tb + (1-t)a} \right)}{tb + (1-t)a} \left[ \int_{0}^{1/2} \left( \frac{g(x)}{x^2} \right) dx \right] \frac{f\left( \frac{ab}{sb + (1-t)a} \right)}{tb + (1-t)a} \\
+ \frac{g\left( \frac{ab}{tb + (1-t)a} \right)}{tb + (1-t)a} \left[ \int_{0}^{1/2} \left( \frac{g(x)}{x^2} \right) dx \right] \frac{f\left( \frac{ab}{sb + (1-t)a} \right)}{tb + (1-t)a} \\
- \frac{ab}{b-a} \int_{0}^{t} M(t) dt
\]
\[
\frac{f'(ab/(tb + (1-t)a)) dt}{(tb + (1-t)a)^2}
- \frac{ab}{b-a} \left[ \int_{0}^{1/2} \left( \frac{g(x)}{x^2} \right) dx \right] \frac{f\left( \frac{ab}{sb + (1-t)a} \right)}{tb + (1-t)a} \\
+ \frac{b-a}{ab} \int_{0}^{1} f\left( \frac{ab}{sb + (1-t)a} \right) \left( \frac{g(x)}{x^2} \right) dx \\
- \frac{ab}{b-a} \left[ \int_{0}^{1/2} \left( \frac{g(x)}{x^2} \right) dx \right] \frac{f\left( \frac{ab}{sb + (1-t)a} \right)}{tb + (1-t)a} \\
+ \frac{g\left( \frac{ab}{tb + (1-t)a} \right)}{tb + (1-t)a} \left[ \int_{0}^{1/2} \left( \frac{g(x)}{x^2} \right) dx \right] \frac{f\left( \frac{ab}{sb + (1-t)a} \right)}{tb + (1-t)a} \\
- \frac{ab}{b-a} \left[ \int_{0}^{1/2} \left( \frac{g(x)}{x^2} \right) dx \right] \frac{f\left( \frac{ab}{sb + (1-t)a} \right)}{tb + (1-t)a} \\
+ \frac{b-a}{ab} \int_{0}^{1} f\left( \frac{ab}{sb + (1-t)a} \right) \left( \frac{g(x)}{x^2} \right) dx \\
- \frac{ab}{b-a} \left[ \int_{0}^{1/2} \left( \frac{g(x)}{x^2} \right) dx \right] \frac{f\left( \frac{ab}{sb + (1-t)a} \right)}{tb + (1-t)a} \\
+ \frac{g\left( \frac{ab}{tb + (1-t)a} \right)}{tb + (1-t)a} \left[ \int_{0}^{1/2} \left( \frac{g(x)}{x^2} \right) dx \right] \Rightarrow \left| \int_{0}^{1} M(t) dt \right| \leq \frac{(b-a)(M-m)}{4ab} \int_{0}^{1} |M(t)| dt.
\]

**Proof.** From (5) of Lemma 9, we get
\[
\frac{ab}{b-a} \left( \frac{f(a) + f(b)}{2} \right) \int_{a}^{b} \frac{g(x)}{x^2} dx
- \int_{a}^{b} f(x) \frac{g(x)}{x^2} dx = \frac{ab}{2} \int_{0}^{1} M(t) dt
- \frac{f'(ab/(tb + (1-t)a))}{(tb + (1-t)a)^2} \left[ \frac{m + M}{2a^2b^2} \right] dt
+ \frac{ab}{2} \int_{0}^{1} M(t) dt
\]
\[
\frac{\left[ \frac{m + M}{2a^2b^2} \right] dt}{(tb + (1-t)a)^2}
\]
\[
\left| \frac{f'(ab/ (tb + (1-t)a))}{(tb + (1-t)a)^2} - \frac{m + M}{2a^2b^2} \right| \leq \frac{M - m}{2a^2b^2}
\]
\[
\left| f'(ab/ (tb + (1-t)a)) \right| \leq \frac{M - m}{2a^2b^2}
\]

**Theorem 10.** Suppose that \( f : I \subseteq \mathbb{R} \setminus \{0\} \) is a differentiable mapping on \( I^0 \), \( a, b \in I^0 \) with \( a < b \) and \( g : [a, b] \rightarrow \mathbb{R} \) is a differentiable mapping. Assume that \( f' \) is an integrable on \([a, b]\) and there exist constants \( m < M < \infty \) such that
\[
-\infty < m \leq x^2 f'(x) \leq M < \infty
\]
for all \( x \in [1/b, 1/a] \). Then
\[
\left| \frac{ab}{b-a} \left( \frac{f(a) + f(b)}{2} \right) \int_{a}^{b} \frac{g(x)}{x^2} dx \right|
- \int_{a}^{b} f(x) \frac{g(x)}{x^2} dx \right|
- \frac{m + M}{4ab}
\]
\[
- \frac{m + M}{4ab} \int_{0}^{1} |M(t)| dt.
\]
Remark 11. If we take $g$ is symmetric to $2ab/(a+b)$, then from Lemma 9, we get

$$\left| \frac{ab}{b-a} \left( \frac{f(a) + f(b)}{2} \int_a^b \frac{g(x)}{x^2} \, dx \right) \right| \leq \left( \frac{b-a}{2} \right) \left( \frac{M}{M-m} \right) \left\| g \right\|_{\infty},$$

(38)

and

$$\left| \frac{ab}{b-a} \left( \frac{f(a) + f(b)}{2} \int_a^b \frac{g(x)}{x^2} \, dx \right) \right| \leq \left( \frac{b-a}{2} \right) \left( \frac{M}{M-m} \right) \left\| g \right\|_{\infty},$$

(39)

Proof. From the definition of $M(t)$, claim of Lemma 9, and $|f'|$ being a harmonically convex functions, we have

$$\left| \frac{f(a) + f(b)}{2} \int_a^b \frac{g(x)}{x^2} \, dx - \int_a^b \frac{f(x)g(x)}{x^2} \, dx \right| \leq \left( \frac{b-a}{2} \right) \left( \frac{M(t)}{a} \right) \left\| f' \right\|_{L(1/\alpha, 1/\beta)} \left( \frac{ab}{(b+1-t)(a)} \right) \frac{dt}{(b+1-t)(a)^2},$$

(40)

where

$$A_1(x) = \int_0^{(1/\alpha-1/\beta)(1/\alpha-1/\beta)} \frac{t}{(tb + (1-t)(a)^2) dt}$$

(41)

and

$$A_2(x) = \int_0^{(1/\alpha-1/\beta)(1/\alpha-1/\beta)} \frac{1-t}{((1-t)(b+ta)^2) dt}$$

(42)

Proof. From the definition of $M(t)$, claim of Lemma 9, and $|f'|$ being a harmonically convex functions, we have

$$\left| \frac{f(a) + f(b)}{2} \int_a^b \frac{g(x)}{x^2} \, dx - \int_a^b \frac{f(x)g(x)}{x^2} \, dx \right| \leq \left( \frac{b-a}{2} \right) \left( \frac{M(t)}{a} \right) \left\| f' \right\|_{L(1/\alpha, 1/\beta)} \left( \frac{ab}{(b+1-t)(a)} \right) \frac{dt}{(b+1-t)(a)^2},$$

(43)

If we change the order of integration, we get

$$\left| \frac{f(a) + f(b)}{2} \int_a^b \frac{g(x)}{x^2} \, dx - \int_a^b \frac{f(x)g(x)}{x^2} \, dx \right| \leq \left( \frac{b-a}{2} \right) \left( \frac{M(t)}{a} \right) \left\| f' \right\|_{L(1/\alpha, 1/\beta)} \left( \frac{ab}{(b+1-t)(a)} \right) \frac{dt}{(b+1-t)(a)^2}$$

(44)
By using (45), (46), and (47) to (44), then we complete this proof. □

**Theorem 13.** Let \( f : \mathbb{R} \ni \{0\} \rightarrow \mathbb{R} \) be a mapping differentiable on \( I^* \), let \( a, b \in I^* \) be points with \( a < b \), and let \( g : [a, b] \rightarrow \mathbb{R} \) be a nonnegative integrable mapping that is differentiable on \([a, b]\). Assume that \( f' \) is integrable on \([a, b]\) and satisfies a Lipschitz condition for some \( K > 0 \). Then

\[
\int_{a}^{b} \frac{g(x)}{x^2} \left( f'(a) + (1 - t) f'(b) \right) dt dx = \int_{a}^{b} \frac{g(x)}{x^2} \left( f'(a) + (1 - t) f'(b) \right) dt dx.
\]

Also it is not hard to see that

\[
\int_{(1/a-1/b)/(1/a-1/b)}^{1} \frac{(1 - t) dt}{(1 - t) b + ta} = \int_{(1/a-1/x)/(1/a-1/b)}^{1} \frac{t dt}{(1 - t) b + ta}.
\]

By using (45), (46), and (47) to (44), then we complete this proof. □

**Theorem 13.** Let \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) be a mapping differentiable on \( I \), let \( a, b \in I \) be points with \( a < b \), and let \( g : [a, b] \rightarrow \mathbb{R} \) be a nonnegative integrable mapping that is differentiable on \([a, b]\). Assume that \( f' \) is integrable on \([a, b]\) and satisfies a Lipschitz condition for some \( K > 0 \). Then

\[
\int_{a}^{b} \frac{g(x)}{x^2} \left( f'(a) + (1 - t) f'(b) \right) dt dx = \int_{a}^{b} \frac{g(x)}{x^2} \left( f'(a) + (1 - t) f'(b) \right) dt dx.
\]

From the last equality, we get

\[
\int_{a}^{b} \frac{f(a) + f(b)}{2} \int_{a}^{b} \frac{g(x)}{x^2} dx
\]

\[
- \int_{a}^{b} \frac{f(x) g(x)}{x^2} dx - \frac{ab (b - a)}{2}
\]

\[
\int_{0}^{1} M(t) \frac{f'(2ab/(a + b))}{(tb + (1 - t)a)^2} dt \leq \frac{Ka^2b^2(b - a)^2}{2(a + b)}
\]

\[
\int_{0}^{1} |M(t)| \frac{1 - 2t}{(tb + (1 - t)a)^2} dt.
\]

**Proof.** By using (5) of Lemma 9, we have

\[
\int_{a}^{b} \frac{g(x)}{x^2} dx
\]

\[
- \int_{a}^{b} \frac{f(x) g(x)}{x^2} dx = \frac{ab (b - a)}{2}
\]

\[
\int_{0}^{1} M(t) \left[ f' \left( \frac{ab}{tb + (1 - t)a} \right) + f' \left( \frac{2ab}{a + b} \right) \right] dt = \frac{ab (b - a)}{2}
\]

\[
- f' \left( \frac{2ab}{a + b} \right) + f' \left( \frac{2ab}{a + b} \right) dt = \frac{ab (b - a)}{2}
\]

If \( f' \) satisfies a Lipschitz condition as (12) for some \( K > 0 \), then

\[
\left| f' \left( \frac{ab}{tb + (1 - t)a} \right) - f' \left( \frac{2ab}{a + b} \right) \right| \leq \frac{Ka^2b^2(b - a)}{2(a + b)}
\]

Because of this inequality, the proof is completed. □

**Remark 14.** In Theorem 13, suppose that \( g \) is symmetric to \( 2ab/(a + b) \). If we use (1) of Lemma 9, we obtain

\[
\int_{a}^{b} \frac{g(x)}{x^2} dx - \frac{ab (b - a)}{2}
\]

\[
\int_{0}^{1} M(t) \frac{1 - 2t}{(tb + (1 - t)a)^2} dt \leq \frac{Ka^2b^2(b - a)^2}{2(a + b)}
\]
Corollary 15. In Theorem 13, if we take 
logarithmic mean to positive to real numbers.

Recall the following means which could be considered extensions of arithmetic, geometric, harmonic, and generalized logarithmic means:

1. The arithmetic mean:
   \[ A = A(a, b) = \frac{a + b}{2}; \quad a, b \in \mathbb{R} \] (55)

2. The geometric mean:
   \[ G = G(a, b) = \sqrt{ab}; \quad a, b \in [0, \infty) \] (56)

3. The harmonic mean:
   \[ H = H(a, b) = \frac{2ab}{a + b}; \quad a, b \in \mathbb{R} \] (57)

4. The generalized logarithmic mean:
   \[ L_n(a, b) = \left( \frac{b^{n+1} - a^{n+1}}{(b - a)(n+1)} \right)^{1/n}; \quad n \in \mathbb{Z} \setminus \{-1, 0\}, \; a, b \in \mathbb{R}, \; a \neq b. \] (58)

Now, we use these means in Corollary 15, Theorem 12 and Remark 11.

**Proposition 16.** Let \( a, b \in (0, \infty) \), \( a < b \), and \( r > 2 \). Then
\[
\left| A(a', b') - G^2(a, b) L_{r-2}^{-2}(a, b) \right| \leq \frac{b(b-a)}{2a} \left[ Kba^{-1}(a, b) + \frac{r}{2} H^{-1}(a, b) \right] (59)
\]

**Proof.** The claim follows from Corollary 15 for \( f(x) = x^r \), \( x \in (0, \infty), \; r > 2 \). Since \( f \) is a convex and nondecreasing function, \( f \) is a harmonically convex function.

**Proposition 17.** Let \( a, b \in (0, \infty) \), \( a < b \), and \( r > 2 \). Then
\[
\left| A(a', b') - G^2(a, b) L_{r-2}^{-2}(a, b) \right| \leq \frac{b(b-a)}{2a} \left[ \frac{Kb}{(r-2)} \right] \]

where
\[
A_1(x) = \int_0^{(1/x-1)/(1/x-1)} \frac{t}{(tb + (1 - t)a)^2} dt + \int_0^{(1/x-1)/(1/x-1)} \frac{1 - t}{((1 - t)b + ta)^2} dt (61)
\]
and
\[
A_2(x) = \int_0^{(1/x-1)/(1/x-1)} \frac{1 - t}{((1 - t)b + ta)^2} dt + \int_0^{(1/x-1)/(1/x-1)} \frac{t}{(tb + (1 - t)a)^2} dt (62)
\]

**Proof.** The claim follows from Theorem 12 for \( f(x) = x^r \) and \( g(x) = 1, \; x \in (0, \infty), \; r > 2 \). Since \( f \) is a convex and nondecreasing function, \( f \) is a harmonically convex function.

**Proposition 18.** Let \( a, b \in (0, \infty) \), \( a < b \), and \( r > 2 \). Then
\[
\left| A(a', b') - G^2(a, b) L_{r-2}^{-2}(a, b) \right| \leq \frac{(b-a)(M-m)}{8ab} (63)
\]

**Proof.** The claim follows from Remark 11 for \( f(x) = x^r \) and \( g(x) = 1, \; x \in (0, \infty), \; r > 2 \). Since \( f \) is a convex and nondecreasing function, \( f \) is a harmonically convex function.

**Data Availability**
No data were used to support this study.

**Conflicts of Interest**
The author declares that they have no conflicts of interest.
References

[1] M. R. Delavar, S. S. Dragomir, and M. De La Sen, “Estimation type results related to Fejér inequality with applications,” Journal of Inequalities and Applications, vol. 2018, article 85, 2018.

[2] L. Fejér, “Über die fourierreihen, II,” Math. Naturwiss. Anz Ungar. Akad. Wiss, vol. 24, pp. 369–390, 1906.

[3] M. R. Delavar and S. S. Dragomir, “Trapezoidal type inequalities related to h-convex functions with applications,” Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, pp. 1–12, 2018.

[4] S. S. Dragomir, J. Pečarić, and L. E. Persson, “Some inequalities of Hadamard type,” Soochow Journal of Mathematics, vol. 21, pp. 335–341, 1995.

[5] I. Iscan, S. Turhan, and S. Maden, “Some Hermite Hadamard Fejer inequalities for harmonically convex functions via fractional integral,” Trends in Mathematical Sciences, vol. 4, no. 2, pp. 1–10, 2016.

[6] İ. İscan, “Hermite-Hadamard type inequalities for harmonically convex functions,” Hacettepe Journal of Mathematics and Statistics, vol. 43, no. 6, pp. 935–942, 2014.

[7] M. A. Latif, S. S. Dragomir, and E. Momanian, “Some Fejer type integral inequalities for geometrically-arithmetically-convex functions with applications,” Research Group in Mathematical Inequalities and Applications, vol. 18, article 25, 18 pages, 2015.

[8] M. Z. Sarikaya, “On new Hermite Hadamard Fejer type integral inequalities,” Studia Universitatis Babes-Bolyai Series Mathematica, vol. 57, no. 3, pp. 377–386, 2012.

[9] D.-Y. Hwang, “Some inequalities for differentiable convex mapping with application to weighted trapezoidal formula and higher moments of random variables,” Applied Mathematics and Computation, vol. 217, no. 23, pp. 9598–9605, 2011.

[10] A. W. Robert and D. E. Varberg, Convex Functions, Academic Press, San Diego, Calif, USA, 1973.
