Cohomology of $\mathfrak{osp}(1|2)$ acting on the space of bilinear differential operators on the superspace $\mathbb{R}^{1|1}$

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Abstract

We compute the first cohomology of the ortosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$ on the $(1,1)$-dimensional real superspace with coefficients in the superspace $\mathcal{D}_{\lambda,\nu,\mu}$ of bilinear differential operators acting on weighted densities. This work is the simplest superization of a result by Bouarroudj [Cohomology of the vector fields Lie algebras on $\mathbb{RP}^1$ acting on bilinear differential operators, International Journal of Geometric Methods in Modern Physics (2005), 2; N 1, 23-40].

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1 Introduction

The space of weighted densities of weight $\lambda$ on $\mathbb{R}$ (or $\lambda$-densities for short), denoted by:

$$\mathcal{F}_{\lambda} = \left\{ f dx^\lambda, f \in C^\infty(\mathbb{R}) \right\}, \quad \lambda \in \mathbb{R},$$

is the space of sections of the line bundle $(T^*\mathbb{R})^\lambda$. The Lie algebra $\text{Vect}(\mathbb{R})$ of vector fields $X_h = h \frac{\partial}{\partial x}$, where $h \in C^\infty(\mathbb{R})$, acts by the Lie derivative. Alternatively, this action can be written as follows:

$$X_h \cdot (f dx^\lambda) = L^\lambda_{X_h} f dx^\lambda \quad \text{with} \quad L^\lambda_{X_h} f = hf' + \lambda h f, \quad (1.1)$$

where $f', h'$ are $\frac{df}{dx}, \frac{dh}{dx}$. Each bilinear differential operator $A$ on $\mathbb{R}$ gives thus rise to a morphism from $\mathcal{F}_\lambda \otimes \mathcal{F}_\nu$ to $\mathcal{F}_\mu$, for any $\lambda, \nu, \mu \in \mathbb{R}$, by $f dx^\lambda \otimes g dx^\nu \mapsto A(f \otimes g) dx^\mu$. The Lie algebra $\text{Vect}(\mathbb{R})$ acts on the space $\mathcal{D}_{\lambda,\nu,\mu}$ of these differential operators by:

$$X_h \cdot A = L^\mu_{X_h} A - A \circ L^{(\lambda,\nu)}_{X_h} \quad (1.2)$$

where $L^{(\lambda,\nu)}_{X_h}$ is the Lie derivative on $\mathcal{F}_\lambda \otimes \mathcal{F}_\nu$ defined by the Leibnitz rule:

$$L^{(\lambda,\nu)}_{X_h} (f \otimes g) = L^\lambda_{X_h} f \otimes g + f \otimes L^\nu_{X_h} g.$$

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If we restrict ourselves to the Lie algebra \( \mathfrak{sl}(2) \) which is isomorphic to the Lie subalgebra of \( \text{Vect}(\mathbb{R}) \) spanned by
\[
\{X_1, X_x, X_{x^2}\},
\]
we have a family of infinite dimensional \( \mathfrak{sl}(2) \)-modules still denoted by \( \mathcal{D}_{\lambda,\nu,\mu} \). Bouarroudj, in \[2\], computes the cohomology space \( H^1_{\text{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\lambda,\nu,\mu}) \) where \( H^1_{\text{diff}} \) denotes the differential cohomology; that is, only cochains given by differential operators are considered.

In this paper we are interested in the study of the analogue super structures. More precisely, we consider here the superspace \( \mathbb{R}^{1|1} \) equipped with the standard contact structure given by the 1-form \( \alpha = dx + \theta d\theta \), we replace \( \mathfrak{sl}(2) \) by its analogue in the super setting, i.e the orthosymplectic Lie superalgebra \( \mathfrak{osp}(1|2) \) which can be realized as a subalgebra of the superalgebra \( \mathcal{K}(1) \) of contact vector fields. We introduce the superspace of \( \lambda \)-densities on the superspace \( \mathbb{R}^{1|1} \) (with respect to \( \alpha \)) denoted by \( \mathfrak{h}_\lambda \) and the superspace \( \mathcal{D}_{\lambda,\nu,\mu} \) of differential bilinear operators viewed as homomorphisms from \( \mathfrak{h}_\lambda \otimes \mathfrak{h}_\nu \) to \( \mathfrak{h}_\mu \). The superalgebra \( \mathfrak{osp}(1|2) \) acts naturally on \( \mathfrak{h}_\lambda \) and \( \mathcal{D}_{\lambda,\nu,\mu} \). We compute here the first cohomology spaces \( H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathfrak{sl}(2), \mathcal{D}_{\lambda,\nu,\mu}) \) and \( H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathcal{D}_{\lambda,\nu,\mu}) \), \( \lambda, \nu, \mu \in \mathbb{R} \), getting a result very close to the classical one \( H^1_{\text{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\lambda,\nu,\mu}) \). Moreover, we give explicit formulae for non trivial 1-cocycles which generate these spaces.

These spaces appear naturally in the problem of describing the deformations of the \( \mathfrak{osp}(1|2) \)-modules \( \mathcal{D}_{\lambda,\nu,\mu} \). More precisely, the first cohomology space \( H^1(\mathfrak{osp}(1|2), V) \) classifies the infinitesimal deformations of an \( \mathfrak{osp}(1|2) \) module \( V \) and the obstructions to integrability of a given infinitesimal deformation of \( V \) are elements of \( H^2(\mathfrak{osp}(1|2), V) \).

## 2 Definitions and Notation

### 2.1 The Lie superalgebra of contact vector fields on \( \mathbb{R}^{1|1} \)

We define the superspace \( \mathbb{R}^{1|1} \) in terms of its superalgebra of functions, denoted by \( C^\infty(\mathbb{R}^{1|1}) \) and consisting of elements of the form:
\[
F(x, \theta) = f_0(x) + f_1(x)\theta,
\]
where \( x \) is the even variable, \( \theta \) is the odd variable (\( \theta^2 = 0 \)) and \( f_0(x), f_1(x) \in C^\infty(\mathbb{R}) \). Even elements in \( C^\infty(\mathbb{R}^{1|1}) \) are the functions \( F(x, \theta) = f_0(x) \), the functions \( F(x, \theta) = \theta f_1(x) \) are odd elements. The parity of homogenous elements \( F \) will be denoted \( |F| \). We consider the contact bracket on \( C^\infty(\mathbb{R}^{1|1}) \) defined on \( C^\infty(\mathbb{R}^{1|1}) \) by:
\[
\{F, G\} = FG' - F'G + \frac{1}{2}\eta(F)\overline{\eta}(G),
\]
where \( \eta = \frac{\partial}{\partial x} + \theta \frac{\partial}{\partial \theta} \) and \( \overline{\eta} = \frac{\partial}{\partial x} - \theta \frac{\partial}{\partial \theta} \). The superspace \( \mathbb{R}^{1|1} \) is equipped with the standard contact structure given by the following 1-form:
\[
\alpha = dx + \theta d\theta.
\]

Let \( \text{Vect}(\mathbb{R}^{1|1}) \) be the superspace of vector fields on \( \mathbb{R}^{1|1} \):
\[
\text{Vect}(\mathbb{R}^{1|1}) = \left\{ F_0 \partial_x + F_1 \partial_\theta \mid F_i \in C^\infty(\mathbb{R}^{1|1}) \right\},
\]
where \( \partial_\theta \) stands for \( \frac{\partial}{\partial \theta} \) and \( \partial_x \) stands for \( \frac{\partial}{\partial x} \), and consider the superspace \( \mathcal{K}(1) \) of contact vector fields on \( \mathbb{R}^{1|1} \). That is, \( \mathcal{K}(1) \) is the Lie superalgebra of conformal vector fields on \( \mathbb{R}^{1|1} \) with respect to the 1-form \( \alpha \):

\[
\mathcal{K}(1) = \{ X \in \text{Vect}(\mathbb{R}^{1|1}) \mid \text{there exists } H \in C^\infty(\mathbb{R}^{1|1}) \text{ such that } \mathcal{L}_X(\alpha) = H \alpha \},
\]

where \( \mathcal{L}_X \) is the Lie derivative along the vector field \( X \). Any contact vector field on \( \mathbb{R}^{1|1} \) has the following explicit form:

\[
X_H = H \partial_x + \frac{1}{2} \eta(H) \eta, \quad \text{where } H \in C^\infty(\mathbb{R}^{1|1}).
\]

The bracket on \( \mathcal{K}(1) \) is given by

\[
[X_F, X_G] = X_{\{F,G\}}.
\]

### 2.2 The subalgebra \( \mathfrak{osp}(1|2) \)

The Lie algebra \( \mathfrak{sl}(2) \) is realized as subalgebra of the Lie algebra \( \text{Vect}(\mathbb{R}) \):

\[
\mathfrak{sl}(2) = \text{Span}(X_1, X_x, X_{x^2}).
\]

Similarly, we now consider the orthosymplectic Lie superalgebra as a subalgebra of \( \mathcal{K}(1) \):

\[
\mathfrak{osp}(1|2) = \text{Span}(X_1, X_x, X_{x^2}, X_{x\theta}, X_\theta).
\]

The space of even elements is isomorphic to \( \mathfrak{sl}(2) \), while the space of odd elements is two dimensional:

\[
(\mathfrak{osp}(1|2))_1 = \text{Span}(X_{x\theta}, X_\theta).
\]

The new commutation relations are:

\[
\begin{align*}
[X_{x^2}, X_\theta] &= -X_{x\theta}, \\
[X_{x^2}, X_{x\theta}] &= 0, \\
[X_{x\theta}, X_\theta] &= \frac{1}{2} X_x, \\
[X_x, X_{x\theta}] &= \frac{1}{2} X_1, \\
[X_1, X_{x\theta}] &= X_\theta,
\end{align*}
\]

### 2.3 The space of weighted densities on \( \mathbb{R}^{1|1} \)

We have analogous definition of weighted densities in super setting (see [1]) with \( dx \) replaced by \( \alpha \). The elements of these spaces are indeed (weighted) densities since all spaces of generalized tensor fields have just one parameter relative \( \mathcal{K}(1) \) — the value of \( X_x \) on the lowest weight vector (the one annihilated by \( X_\theta \)). From this point of view the volume element (roughly speaking, “\( dx \frac{\partial}{\partial \theta} \)”) is indistinguishable from \( \alpha \frac{\partial}{\partial \theta} \). We denote by \( \mathfrak{F}_\lambda \) the space of all weighted densities on \( \mathbb{R}^{1|1} \) of weight \( \lambda \):

\[
\mathfrak{F}_\lambda = \left\{ F(x, \theta) \alpha^\lambda \mid F(x, \theta) \in C^\infty(\mathbb{R}^{1|1}) \right\}.
\]

As a vector space, \( \mathfrak{F}_\lambda \) is isomorphic to \( C^\infty(\mathbb{R}^{1|1}) \), but the Lie derivative of the density \( F \alpha^\lambda \) along the vector field \( X_H \) in \( \mathcal{K}(1) \) is now:

\[
\mathcal{L}_{X_H}(F \alpha^\lambda) = \mathcal{L}_{X_H}^\lambda(F) \alpha^\lambda, \quad \text{with } \mathcal{L}_{X_H}^\lambda(F) = \mathcal{L}_{X_H}(F) + \lambda H' F.
\]
We will compute the first cohomology space of

\[ H^1_{\text{diff}}(\mathfrak{os}p(1|2), \mathfrak{D}_{\lambda,\nu;\mu}) \]

3.1 Cohomology

We will compute the first cohomology space of \( \mathfrak{os}p(1|2) \) with coefficients in \( \mathfrak{D}_{\lambda,\nu;\mu} \). Let us first recall some fundamental concepts from cohomology theory (see, e.g., \[3\]). Let \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) be a Lie superalgebra acting on a superspace \( V = V_0 \oplus V_1 \) and let \( \mathfrak{h} \) be a subalgebra of \( \mathfrak{g} \). (If \( \mathfrak{h} \)}
We say that $(\lambda, \nu, \mu)$ is resonant if $\mu - \lambda - \nu - 1 = k$ with $k \in \mathbb{N}$, and
\[(\lambda, \nu) = (-\frac{s}{2}, -\frac{t}{2}), \quad \text{where} \quad s, t \in \{0, \ldots, k\} \quad \text{and} \quad s + t \geq k. \quad (3.4)\]
We say that $(\lambda, \nu, \mu)$ is weakly resonant if $\mu - \lambda - \nu \in \mathbb{N}$ but $(\lambda, \nu, \mu)$ is not resonant.

2) We say that $(\lambda, \nu, \mu)$ is super resonant if $\mu - \lambda - \nu - 1 = k$ with $k \in \frac{1}{2}\mathbb{N}$, and
\[(\lambda, \nu) = (-\frac{s}{2}, -\frac{t}{2}), \quad \text{where} \quad s, t \in \{1, \ldots, [k]\} \quad \text{and} \quad s + t \geq [k + \frac{1}{2}] + 1. \quad (3.5)\]
We say that $(\lambda, \nu, \mu)$ is weakly super resonant if $\mu - \lambda - \nu = k + 1 \in \frac{1}{2}\mathbb{N}$, and
\[(\lambda, \nu) = (-\frac{s}{2}, -\frac{t}{2}), \quad s, t \in \{0, \ldots, [k] + 1\} \Rightarrow s + t < [k + \frac{1}{2}]. \quad (3.6)\]

**Remark 3.2.** The super resonance (respectively, weakly super resonance) of $(\lambda, \nu, \mu)$ express the resonance (respectively, weakly resonance) of:

- $(\lambda, \nu, \mu), (\lambda + \frac{1}{2}, \nu + \frac{1}{2}, \mu), (\lambda + \frac{1}{2}, \nu, \mu + \frac{1}{2})$ and $(\lambda, \nu + \frac{1}{2}, \mu + \frac{1}{2})$ if $\mu - \lambda - \nu$ is integer.
- $(\lambda, \nu, \mu + \frac{1}{2}), (\lambda + \frac{1}{2}, \nu + \frac{1}{2}, \mu + \frac{1}{2}), (\lambda + \frac{1}{2}, \nu, \mu)$ and $(\lambda, \nu + \frac{1}{2}, \mu)$ if $\mu - \lambda - \nu$ est semi-integer.

The main result in this paper is the following:
Theorem 3.1.

\[ H^1_{\text{diff}}(\mathfrak{osp}(1|2), D_{\lambda,\nu,\mu}) \simeq \begin{cases} \mathbb{R}^6 & \text{if } (\lambda,\nu,\mu) \text{ is super resonant}, \\ \mathbb{R} & \text{if } (\lambda,\nu,\mu) \text{ is weakly super resonant}. \end{cases} \]

The proof of Theorem 3.1 will be the subject of Section 4. Moreover, explicit formulae for non trivial 1-cocycles generating the corresponding cohomology spaces will be given. We will show that the spaces \( H^1_{\text{diff}}(\mathfrak{osp}(1|2), D_{\lambda,\nu,\mu}) \) and \( H^1_{\text{diff}}(\mathfrak{sl}(2), D_{\lambda,\nu,\mu}) \) are closely related. Therefore, for comparison and to build upon, we need to recall the description of \( H^1_{\text{diff}}(\mathfrak{sl}(2), D_{\lambda,\nu,\mu}) \).

Of course, there are some cases of \( (\lambda,\nu,\mu) \) which are neither super resonant nor weakly super resonant, these cases will be studied in Section 5.

3.3 The space \( H^1_{\text{diff}}(\mathfrak{sl}(2), D_{\lambda,\nu,\mu}) \)

For the sake of simplicity, the elements \( f dx^\lambda \) of \( \mathcal{F}_\lambda \) will be denoted \( f \). Any 1-cochain \( c \in Z^1_{\text{diff}}(\mathfrak{sl}(2), D_{\lambda,\nu,\mu}) \) should retain the following general form:

\[ c(X_h, f, g) = \sum_{i,j} \alpha_{i,j} h f^{(i)} g^{(j)} + \sum_{i,j} \beta_{i,j} h' f^{(i)} g^{(j)} + \sum_{i,j} \gamma_{i,j} h'' f^{(i)} g^{(j)}. \]

So, for any integer \( k \geq 0 \), we define the \( k \)-homogeneous component of \( c \) by

\[ c_k(X_h, f, g) = \sum_{i+j=k} \alpha_{i,j} h f^{(i)} g^{(j)} + \sum_{i+j=k-1} \beta_{i,j} h' f^{(i)} g^{(j)} + \sum_{i+j=k-2} \gamma_{i,j} h'' f^{(i)} g^{(j)}. \]

Of course, we suppose that \( \gamma_{i,j} = 0 \) if \( k \in \{0, 1\} \) and \( \beta_{i,j} = 0 \) if \( k = 0 \). The coboundary map \( \delta \) is homogeneous, therefore, we easily deduce the following lemma:

**Lemma 3.2.** Any 1-cochain \( c \in C^1_{\text{diff}}(\mathfrak{sl}(2), D_{\lambda,\nu,\mu}) \) is a 1-cocycle if and only if each of its homogeneous components is a 1-cocycle.

The following lemma gives the general form of any homogeneous 1-cocycle.

**Lemma 3.3.** Up to a coboundary, any \((k+2)\)-homogeneous 1-cocycle \( c \in Z^1_{\text{diff}}(\mathfrak{sl}(2), D_{\lambda,\nu,\mu}) \) can be expressed as follows. For all \( f \in \mathcal{F}_\lambda \), \( g \in \mathcal{F}_\nu \) and for all \( X_h \in \mathfrak{sl}(2) \):

\[ c(X_h, f, g) = \sum_{i=0}^{k+1} \beta_i h' f^{(i)} g^{(k+1-i)} + \sum_{i=0}^{k} \gamma_i h'' f^{(i)} g^{(k-i)}, \quad (3.7) \]

where \( \beta_i \) and \( \gamma_i \) are constants satisfying:

\[ 2(\mu - \lambda - \nu - k - 1) \gamma_i + (i+1)(i+2\lambda) \beta_{i+1} + (k+1-i)(k+i+2\nu) \beta_i = 0. \quad (3.8) \]

Proof. Any \((k+2)\)-homogeneous 1-cocycle on \( \mathfrak{sl}(2) \) should retain the following general form:

\[ c(X_h, f, g) = \sum_{i=0}^{k+2} \alpha_i h f^{(i)} g^{(k+2-i)} + \sum_{i=0}^{k+1} \beta_i h' f^{(i)} g^{(k+1-i)} + \sum_{i=0}^{k} \gamma_i h'' f^{(i)} g^{(k-i)}, \]
where $\alpha_i$, $\beta_i$ and $\gamma_i$ are, a priori, functions. First, we prove that the terms in $h$ can be annihilated by adding a coboundary. Let $b : \mathcal{F}_\lambda \times \mathcal{F}_\nu \to \mathcal{F}_\mu$ be a bilinear differential operator defined by

$$ b(f, g) = \sum_{i=0}^{k+2} b_i f^{(i)} g^{(k+2-i)}, $$

where $f \in \mathcal{F}_\lambda$, $g \in \mathcal{F}_\nu$ and the coefficients $b_i$ are functions satisfying $\frac{d}{dx}(b_i) = \alpha_i$.

Then, for all $X_h \in \mathfrak{sl}(2)$, we have

$$ \delta b(X_h, f, g) = \sum_{i=0}^{k+2} \alpha_i h f^{(i)} g^{(k+2-i)} + \sum_{i=0}^{k+1} (\mu - \lambda - \nu - k - 1)b_i h f^{(i)} g^{(k+2-i)} $$

$$ - \frac{1}{2} \sum_{i=0}^{k+1} ((i+1)(i+2\lambda)b_{i+1} + (k+2-i)(k+1-i+2\nu)b_i) h f^{(i)} g^{(k+1-i)}. $$

We replace $c$ by $\tilde{c} = c - \delta b$ and then we see that the 1-cocycle $\tilde{c}$ does not contain terms in $h$. So, up to a coboundary, any $(k+2)$-homogeneous 1-cocycle on $\mathfrak{sl}(2)$ can be expressed as follows:

$$ c(X_h, f, g) = \sum_{i=0}^{k+1} \beta_i h f^{(i)} g^{(k+1-i)} + \sum_{i=0}^{k} \gamma_i h f^{(i)} g^{(k-i)}. $$

Now, consider the 1-cocycle condition:

$$ c([X_{h_1}, X_{h_2}], f, g) - X_{h_1} \cdot c(X_{h_2}, f, g) + X_{h_2} \cdot c(X_{h_1}, f, g) = 0 $$

where $f \in \mathcal{F}_\lambda$, $g \in \mathcal{F}_\nu$ and $X_{h_1}, X_{h_2} \in \mathfrak{sl}(2)$. A direct computation proves that we have

$$ \frac{d}{dx}(\beta_i) = \frac{d}{dx}(\gamma_m) = 0 $$

and

$$ 2(\mu - \lambda - \nu - k - 1)\gamma_i + (i+1)(i+2\lambda)\beta_{i+1} + (k+1-i)(k-i+2\nu)\beta_i = 0. $$

**Corollary 3.3.** If $\mu - \lambda - \nu \neq k+1$, where $k+1 \in \mathbb{N}$, then any $(k+2)$-homogeneous 1-cocycle $c \in Z^1_{\text{diff}}(\mathfrak{sl}(2), D_{\lambda, \nu; \mu})$ is a coboundary. Especially, if $\mu - \lambda - \nu = k + 1$ then any 1-cocycle $c \in Z^1_{\text{diff}}(\mathfrak{sl}(2), D_{\lambda, \nu; \mu})$ is, up to a coboundary, $(k+2)$-homogeneous and if $\mu - \lambda - \nu \notin \mathbb{N}$ then $H^1_{\text{diff}}(\mathfrak{sl}(2), D_{\lambda, \nu; \mu}) = 0$.

Proof. If $\mu - \lambda - \nu \neq k+1$ we can easily show that the 1-cocycle $c$ defined by (3.7) is nothing but the operator $\delta b$ where

$$ b(f, g) = \frac{1}{\mu - \lambda - \nu - k - 1} \sum_{i=0}^{k+1} \beta_i f^{(i)} g^{(k+1-i)}. $$

\qed
Theorem 3.4. \cite{2}

\[ H^1_{\text{diff}}(\mathfrak{sl}(2), D_{\lambda, \nu, \mu}) \simeq \begin{cases} \mathbb{R}^3 & \text{if } (\lambda, \nu, \mu) \text{ is resonant}, \\ \mathbb{R} & \text{if } (\lambda, \nu, \mu) \text{ is weakly resonant}, \\ 0 & \text{otherwise.} \end{cases} \]

Proof. Let \( \mu - \lambda - \nu = k + 1 \); where \( k + 1 \in \mathbb{N} \), then, according to Corollary 3.3, any \( n \)-homogeneous 1-cocycle \( c \in Z^1_{\text{diff}}(\mathfrak{sl}(2), D_{\lambda, \nu, \mu}) \), where \( n \neq k + 2 \), is a coboundary. Thus, we consider only the \((k + 2)\)-homogeneous 1-cocycles given by Lemma 3.3. In this case, the relation \((3.8)\) becomes:

\[(i + 1)(i + 2\lambda)\beta_{i+1} + (k + 1 - i)(k - i + 2\nu)\beta_i = 0. \tag{3.9}\]

Let

\[ b(f, g) = \sum_{i=0}^{k+1} b_i f^{(i)} g^{(k+1-i)}. \]

By a direct computation we have

\[ \delta b(X_h, f, g) = -\frac{1}{2} \sum_{i=0}^{k} ((i + 1)(i + 2\lambda)b_{i+1} + (k + 1 - i)(k - i + 2\nu)b_i) h'' f^{(i)} g^{(k-i)}. \]

So, we are in position to complete the proof as Bouarroudj did in \cite{2}. We recall here the (slightly modified) explicit expressions of the 1-cocycles given in \cite{2}. Hereafter, \( \binom{x}{i} \) is the standard binomial coefficient: \( \binom{x}{i} = \frac{x(x-1)...(x-i+1)}{i!} \) that makes sense for arbitrary \( x \in \mathbb{R} \).

**Case 1:** (\( \lambda, \nu, \mu \) is weakly resonant). In this case, the corresponding cohomology space is one-dimensional, generated by the 1-cocycle \( a \) defined as follows:

(i) If \( \lambda \neq -\frac{s}{2} \), where \( s \in \{0, \ldots, k\} \), then

\[ a(X_h, f, g) = \sum_{i=0}^{k+1} \binom{k+1}{i} \binom{2\nu + k}{i} \binom{-2\lambda}{i}^{-1} h' f^{(i)} g^{(k+1-i)}. \tag{3.10} \]

(ii) If \( \nu \neq -\frac{t}{2} \), where \( t \in \{0, \ldots, k\} \), then

\[ a(X_h, f, g) = \sum_{i=0}^{k+1} \binom{k+1}{i} \binom{2\lambda + k}{k+1-i} \binom{-2\nu}{k+1-i}^{-1} h' f^{(i)} g^{(k+1-i)}. \tag{3.11} \]

(iii) If \( \lambda = -\frac{s}{2} \) and \( \nu = -\frac{t}{2} \), where \( s, t \in \{0, \ldots, k\} \) but \( s + t < k \), then

\[ a(X_h, f, g) = \sum_{i=s+1}^{k-t} (-1)^i \binom{k+1}{i} \binom{k-t-s-1}{i-s-1} h' f^{(i)} g^{(k+1-i)}. \tag{3.12} \]

Observe that if \( \mu - \lambda - \nu = 0 \) then \( (\lambda, \nu, \mu) \) is weakly resonant since \( \mu - \lambda - \nu \in \mathbb{N} \) but \( \mu - \lambda - \nu - 1 \notin \mathbb{N} \). In this case, the set \( \{0, \ldots, k\} \) is empty, so we are in the situations (i) and (ii) and the 1-cocycle \( a \) is then defined by \( a(X_h, f, g) = h' fg \).
Case 2: \((\lambda, \nu, \mu)\) is resonant. That is, \(\lambda = -\frac{s}{2}\) and \(\nu = -\frac{t}{2}\), where \(s, t \in \{0, \ldots, k\}\) with \(s + t \geq k\). In this case, the corresponding cohomology space is three-dimensional, generated by the 1-cocycles \(b, c\) and \(d\) defined as follows:

\[
\begin{align*}
    b(X_h, f, g) &= h'' f^{(k-t)} g^{(t)}, \\
    c(X_h, f, g) &= \sum_{i=0}^{k} \binom{k+1}{i} \binom{k-t}{i} \binom{s}{i}^{-1} h' f^{(i)} g^{(k+1-i)}, \\
    d(X_h, f, g) &= \sum_{i=s+1}^{k+1} \binom{k+1}{i} \binom{k-s}{i} \binom{t}{k+1-i} \binom{t}{k+1-i}^{-1} h' f^{(i)} g^{(k+1-i)}.
\end{align*}
\]

Observe that if \((\lambda, \nu) = (-\frac{s}{2}, -\frac{t}{2})\), where \(s \in \{0, \ldots, k\}\), then \((c+d)(X_h, f, g) = h'(fg)^{(k+1)}\). \(\square\)

3.4 Relationship between \(H^1_{\text{diff}}(\text{osp}(1|2), D_{\lambda,\nu,\mu})\) and \(H^1_{\text{diff}}(\text{sl}(2), D_{\lambda,\nu,\mu})\)

We need to present here some results illustrating the analogy between the cohomology spaces in super and classical settings.

**Proposition 3.4.**

1) As a \(\text{sl}(2)\)-module, we have

\[
\mathfrak{F}_\lambda \simeq \mathcal{F}_\lambda \oplus \Pi(\mathcal{F}_{\lambda + \frac{1}{2}}) \quad \text{and} \quad \text{osp}(1|2) \simeq \text{sl}(2) \oplus \Pi(\mathcal{H}),
\]

where \(\mathcal{H}\) is the subspace of \(\mathcal{F}_{\frac{1}{2}}\) spanned by \(\{dx^{-\frac{1}{2}}, xdx^{-\frac{1}{2}}\}\) and \(\Pi\) is the change of parity.

2) As a \(\text{sl}(2)\)-module, we have for the homogeneous relative parity components:

\[
\begin{align*}
    (D_{\lambda,\nu,\mu})_0 &\simeq D_{\lambda,\nu,\mu} \oplus D_{\lambda,\nu,\mu + \frac{1}{2}} \oplus D_{\lambda,\nu,\mu + \frac{1}{2},\nu + \frac{1}{2}} \oplus D_{\lambda,\nu,\mu + \frac{1}{2},\nu + \frac{1}{2}}, \\
    (D_{\lambda,\nu,\mu})_1 &\simeq \Pi \left( D_{\lambda,\nu,\mu + \frac{1}{2}} \oplus D_{\lambda,\nu,\mu + \frac{1}{2},\nu + \frac{1}{2}} \oplus D_{\lambda,\nu,\mu + \frac{1}{2},\nu + \frac{1}{2}} \oplus D_{\lambda,\nu,\mu + \frac{1}{2},\nu + \frac{1}{2}} \right).
\end{align*}
\]

**Proof.**

1) The first statement is immediately deduced from [2,3].

2) It is well known that if \(M = M_0 \oplus M_1\) and \(N = N_0 \oplus N_1\) are two \(g\)-modules, where \(g\) is a (super)algebra, then \(\text{Hom}(M, N)\) is a \(g\)-module, where the homogenous components are

\[
\begin{align*}
    \text{Hom}(M, N)_0 &= \text{Hom}(M_0, M_0) \oplus \text{Hom}(M_1, N_1) \quad \text{and} \quad \text{Hom}(M, N)_1 &= \text{Hom}(M_0, N_1) \oplus \text{Hom}(M_1, N_0)
\end{align*}
\]

and the \(g\)-action on \(\text{Hom}(M, N)\) is given by

\[
(X.A)(x) = X.(A(x)) - (-1)^{|A||X|} A(X.x).
\]

Moreover, if \(\varphi_1 : M \to M'\) and \(\varphi_2 : N \to N'\) are two \(g\)-isomorphisms, then the map \(\Psi : \text{Hom}(M, N) \to \text{Hom}(M', N')\) defined by

\[
\Psi(A) = \varphi_2 \circ A \circ \varphi_1^{-1}
\]

is a \(g\)-isomorphism. In our situation, as a \(\text{sl}(2)\)-module, we have for the homogeneous relative parity components:

\[
\begin{align*}
    (\mathfrak{F}_\lambda \otimes \mathfrak{F}_\nu)_0 &\simeq \mathcal{F}_\lambda \otimes \mathcal{F}_\nu \oplus \Pi(\mathcal{F}_{\lambda + \frac{1}{2}}) \otimes \Pi(\mathcal{F}_{\nu + \frac{1}{2}}), \\
    (\mathfrak{F}_\lambda \otimes \mathfrak{F}_\nu)_1 &\simeq \Pi(\mathcal{F}_{\lambda + \frac{1}{2}}) \otimes \mathcal{F}_\nu \oplus \mathcal{F}_\lambda \otimes \Pi(\mathcal{F}_{\nu + \frac{1}{2}}).
\end{align*}
\]
So, we deduce the two homogenous relative parity components of $\mathcal{D}_{\lambda,\nu;\mu}$ as a $\mathfrak{sl}(2)$-module. In fact, we have the following isomorphisms:

\[
\begin{align*}
\text{Hom}_{\text{diff}} \left( \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) \otimes \Pi(\mathcal{F}_{\nu+\frac{1}{2}}), \mathcal{F}_{\mu} \right) &\rightarrow D_{\lambda+\frac{1}{2},\nu+\frac{1}{2},\mu}, \quad A \mapsto A \circ (\Pi \otimes \Pi), \\
\text{Hom}_{\text{diff}} \left( \mathcal{F}_{\lambda} \otimes \Pi(\mathcal{F}_{\nu+\frac{1}{2}}), \Pi(\mathcal{F}_{\mu+\frac{1}{2}}) \right) &\rightarrow D_{\lambda,\nu+\frac{1}{2},\mu+\frac{1}{2}}, \quad A \mapsto \Pi \circ A \circ (\text{Id} \otimes \Pi), \\
\text{Hom}_{\text{diff}} \left( \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) \otimes \mathcal{F}_{\nu}, \Pi(\mathcal{F}_{\mu+\frac{1}{2}}) \right) &\rightarrow D_{\lambda+\frac{1}{2},\nu,\mu+\frac{1}{2}}, \quad A \mapsto \Pi \circ A \circ (\Pi \otimes \text{Id}).
\end{align*}
\]

Now, in order to compute $H^1(\mathfrak{osp}(1|2), \mathcal{D}_{\lambda,\nu;\mu})$, we need first to describe the $\mathfrak{sl}(2)$-relative cohomology space $H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathfrak{sl}(2); \mathcal{D}_{\lambda,\nu;\mu})$. So, we shall need the following description of some $\mathfrak{sl}(2)$-invariant mappings.

**Lemma 3.5.** Let

\[ A : \mathcal{H} \otimes \mathcal{F}_{\lambda} \otimes \mathcal{F}_{\nu} \rightarrow \mathcal{F}_{\mu}, \quad (h(dx)^{-\lambda}, f(dx)^{\lambda}, g(dx)^{\nu}) \mapsto A(h, f, g)(dx)^{\mu} \]

be a trilinear differential operator. If $A$ is a nontrivial $\mathfrak{sl}(2)$-invariant operator then

\[ \mu = \lambda + \nu + k - \frac{1}{2}, \text{ where } k \in \mathbb{N}. \]

For $k \geq 2$, the corresponding operator $A_k$ is of the form:

\[
A_k(h, f, g) = \sum_{i=0}^{k} c_i h f^{(i)} g^{(k-i)} + \sum_{i=0}^{k-1} [((i+1)(i+2\lambda)c_{i+1} + (k-i)(k-i-1+2\nu)c_i)] h' f^{(i)} g^{(k-i-1)},
\]

where the $c_i$ are constant characterized by the following recurrence formula:

\[
(i+1)(i+2)(i+2\lambda)(i+2\lambda+1)c_{i+2} + 2(i+1)(k-i-1)(k+i+2\lambda)(k-i-2+2\nu)c_{i+1} + (k-i-1)(k-i-2+2\nu)(k-i-1+2\nu)c_i = 0.
\]

For $k = 0, 1$, we have

\[ A_0(h, f, g) = c_0 h f g \quad \text{and} \quad A_1(h, f, g) = c_0 h f g + c_1 h f' g + (2\lambda c_1 + 2\nu c_0) h' f g. \]

Proof. Obviously, the operator $A$ is $\mathfrak{sl}(2)$-invariant if and only if each of its homogenous components is $\mathfrak{sl}(2)$-invariant. Moreover, the invariance with respect the vector field $X_1 = \partial_x$ yields that $A$ must be expressed with constant coefficients. Thus, let $k \in \mathbb{N}$ and consider

\[ A_k(h, f, g) = \sum_{i=0}^{k} c_i h f^{(i)} g^{(k-i)} + \sum_{i=0}^{k-1} d_i h' f^{(i)} g^{(k-i-1)}, \]
where the $c_i$ and $d_i$ are constants. The invariance property of $A$ with respect any vector fields $X_F$ reads:

\[
F(A(h, f, g))' + \mu F'A(h, f, g) = A(Fh' - \frac{1}{2} F' h, f, g) + A(h, F f' + \lambda F f, g) + A(h, f, F g' + \nu F' g). \tag{3.19}
\]

Consider any non vanishing coefficient $c_i$ and consider terms in $F'h f(i) g(k-i)$ in (3.19), we get

\[
\mu = \lambda + \nu + k - \frac{1}{2}.
\]

Considering respectively terms in $F'' h f(i) g(k-i-1)$ and (for $k \geq 2$) $F'' h f(i) g(k-i-2)$ yield

\[
d_i = (i + 1)(i + 2\lambda)c_{i+1} + (k - i)(k - i - 1 + 2\nu)c_i \tag{3.20}
\]

\[
0 = (i + 1)(i + 2\lambda)d_{i+1} + (k - i)(k - i - 1 + 2\nu)d_i. \tag{3.21}
\]

Combining (3.20) and (3.21) we have (3.18). Under these conditions we check that the operator $A_k$ is $\mathfrak{sl}(2)$-invariant. \hfill $\square$

**Proposition 3.6.** The $\mathfrak{sl}(2)$-relative cohomology spaces $H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathfrak{sl}(2); \mathfrak{d}_{\lambda, \nu; \mu})$ are all trivial. That is, any 1-cocycle $\Upsilon$ is a coboundary over $\mathfrak{osp}(1|2)$ if and only if its restriction to $\mathfrak{sl}(2)$ is a coboundary over $\mathfrak{sl}(2)$.

**Proof.** First, it is easy to see that any 1-cocycle $\Upsilon \in Z^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathfrak{d}_{\lambda, \nu; \mu})$ vanishing on $\mathfrak{sl}(2)$ is $\mathfrak{sl}(2)$-invariant. Indeed, the 1-cocycle relation of $\Upsilon$ reads:

\[
(-1)^{|F|} |\Upsilon| X_F \cdot \Upsilon(X_G) - (-1)^{|G|(|F|+|\Upsilon|)} X_G \cdot \Upsilon(X_F) - \Upsilon([X_F, X_G]) = 0,
\]

where $X_F, X_G \in \mathfrak{osp}(1|2)$. If $\Upsilon(X_F) = 0$ for all $X_F \in \mathfrak{sl}(2)$ then the previous equation becomes

\[
X_F \cdot \Upsilon(X_G) - \Upsilon([X_F, X_G]) = 0
\]

expressing the $\mathfrak{sl}(2)$-invariance of $\Upsilon$. Thus, the space $H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathfrak{sl}(2); \mathfrak{d}_{\lambda, \nu; \mu})$ is nothing but the space of cohomology classes of 1-cocycles vanishing on $\mathfrak{sl}(2)$.

Let $\Upsilon$ be a 1-cocycle vanishing on $\mathfrak{sl}(2)$, then, by the 1-cocycle condition, we have:

\[
\begin{align*}
X_f \cdot \Upsilon(X_{h\theta}) - \Upsilon([X_f, X_{h\theta}]) &= 0, \tag{3.22} \\
X_{h_1 \theta} \cdot \Upsilon(X_{h_2 \theta}) + X_{h_2 \theta} \cdot \Upsilon(X_{h_1 \theta}) &= 0, \tag{3.23}
\end{align*}
\]

where $f \in \mathbb{R}_2[x]$ and $h, h_1, h_2 \in \mathbb{R}_1[x]$. Here, $\mathbb{R}_n[x]$ is the space of polynomial functions in the variable $x$, with degree at most $n$.

1) If $\Upsilon$ is an even 1-cocycle then $\Upsilon$ is decomposed into four trilinear maps:

\[
\begin{align*}
\Pi(\mathcal{H}) \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) \otimes \mathcal{F}_\nu & \rightarrow \mathcal{F}_\mu, \\
\Pi(\mathcal{H}) \otimes \mathcal{F}_{\lambda} \otimes \Pi(\mathcal{F}_{\nu+\frac{1}{2}}) & \rightarrow \mathcal{F}_\mu, \\
\Pi(\mathcal{H}) \otimes \mathcal{F}_{\lambda} \otimes \mathcal{F}_\nu & \rightarrow \Pi(\mathcal{F}_{\mu+\frac{1}{2}}), \\
\Pi(\mathcal{H}) \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) \otimes \Pi(\mathcal{F}_{\nu+\frac{1}{2}}) & \rightarrow \Pi(\mathcal{F}_{\mu+\frac{1}{2}}).
\end{align*}
\]

The equation (3.22) is nothing but the $\mathfrak{sl}(2)$-invariance property of these maps. Therefore, the expressions of these maps are given by Lemma $\ref{Lemma 3.3}$ in fact, the change of parity functor
II commutes with the \( sl(2) \)-action. So, we must have \( \mu = \lambda + \nu + k \), where \( k + 1 \in \mathbb{N} \), otherwise, the operator \( \Upsilon \) is identically the zero map. More precisely:

If \( \mu = \lambda + \nu + k \) where \( k \in \mathbb{N}^* \), we have

\[
\Upsilon_k(X_{\theta h})(\theta f, g) = \sum_{i=0}^{k} c_i^1 h f^{(i)} g^{(k-i)} + \sum_{i=0}^{k-1} d_i^1 h' f^{(i)} g^{(k-i-1)},
\]

(3.24)

\[
\Upsilon_k(X_{\theta h})(f, \theta g) = \sum_{i=0}^{k} c_i^2 h f^{(i)} g^{(k-i)} + \sum_{i=0}^{k-1} d_i^2 h' f^{(i)} g^{(k-i-1)},
\]

(3.25)

\[
\Upsilon_k(X_{\theta h})(f, g) = \theta \sum_{i=0}^{k+1} c_i^3 h f^{(i)} g^{(k-i+1)} + \theta \sum_{i=0}^{k} d_i^3 h' f^{(i)} g^{(k-i)},
\]

(3.26)

\[
\Upsilon_k(X_{\theta h})(\theta f, g) = \theta \sum_{i=0}^{k} c_i^4 h f^{(i)} g^{(k-i)} + \theta \sum_{i=0}^{k-1} d_i^4 h' f^{(i)} g^{(k-i-1)},
\]

(3.27)

where

\[
d_i^1 = (i+1)(i+2\lambda+1)c_{i+1}^1 + (k-i)(k-i-1+2\nu)c_i^1,
\]

\[
d_i^2 = (i+1)(i+2\lambda)c_{i+1}^2 + (k-i)(k-i-1+2\nu+1)c_i^2,
\]

\[
d_i^3 = (i+1)(i+2\lambda)c_{i+1}^3 + (k-i)(k-i-1+2\nu)c_i^3,
\]

\[
d_i^4 = (i+1)(i+2\lambda+1)c_{i+1}^4 + (k-i)(k-i-1+2\nu+1)c_i^4
\]

and where the coefficients \( c_i^* \) are satisfying the recurrence formulae (3.18).

If \( \mu = \lambda + \nu - 1 \), we have

\[
\Upsilon_{-1}(X_{\theta h})(\theta f, g) = \Upsilon_{-1}(X_{\theta h})(f, \theta g) = \Upsilon_{-1}(X_{\theta h})(\theta f, \theta g) = 0,
\]

\[
\Upsilon_{-1}(X_{\theta h})(f, g) = c_0^3 \theta h f g.
\]

If \( \mu = \lambda + \nu \), we have

\[
\Upsilon_0(X_{\theta h})(\theta f, g) = c_0 h f g, \quad \Upsilon_0(X_{\theta h})(f, \theta g) = c_0^2 h f g, \quad \Upsilon_0(X_{\theta h})(\theta f, \theta g) = \theta c_0^3 h f g,
\]

\[
\Upsilon_0(X_{\theta h})(f, g) = \theta [c_0^3 h f g' + c_0^3 h' f' g + (2\lambda c_0^3 + 2\nu c_0^3)h' f g].
\]

The maps \( \Upsilon_k \) must satisfy the equation (3.29). More precisely, the maps \( \Upsilon_k \) satisfy the following four equations

\[
\frac{1}{2}\theta h_1(\Upsilon_k(X_{h^2\theta})(f, g))' + \mu \theta h_1'(\Upsilon_k(X_{h^2\theta})(f, g)) + \Upsilon_k(X_{h^2\theta})(\frac{1}{2}h_1 f, g)
\]

\[-\Upsilon_k(X_{h^2\theta})(f, \theta(\frac{1}{2}h_1 g' + \nu h_1' g)) + (h_1 \leftrightarrow h_2) = 0,
\]

\[
\frac{1}{2}\theta h_1(\Upsilon_k(X_{h^2\theta})(f, g\theta))' + \mu \theta h_1'(\Upsilon_k(X_{h^2\theta})(f, g\theta)) + \Upsilon_k(X_{h^2\theta})(\theta(\frac{1}{2}h_1 f' + \lambda h_1' f), g\theta)
\]

\[+\Upsilon_k(X_{h^2\theta})(f, \frac{1}{2}h_1 g) + (h_1 \leftrightarrow h_2) = 0,
\]

\[
\frac{1}{2}h_1 (\Upsilon_k(X_{h^2\theta})(f, g)) + \Upsilon_k(X_{h^2\theta})(\frac{1}{2}h_1 f' + \lambda h_1' f), g)
\]

\[+\Upsilon_k(X_{h^2\theta})(f, \theta(\frac{1}{2}h_1 g' + \nu h_1' g)) + (h_1 \leftrightarrow h_2) = 0,
\]

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Let Proposition 4.1.

The following statements hold:

First, according to Proposition 3.4, Proposition 3.6 and Theorem 3.4, we easily check that

\[ \frac{1}{2} h_1 \partial_0 (\Upsilon_k(X_{h_0})(f \theta, g \theta)) + \Upsilon_k(X_{h_0})(\frac{1}{2} h_1 f, g \theta) - \Upsilon_k(X_{h_0})(f \theta, \frac{1}{2} h_1 g)) + (h_1 \leftrightarrow h_2) = 0. \]

By a direct, but very hard, computation we show that \( \Upsilon_k \) is a coboundary. For instance, if \( \nu, \lambda \not\in \{0, -\frac{1}{2}, -1, \ldots, -\frac{k}{2} \} \), we check that \( \Upsilon_k = \delta B_k \) where

\[
B_k(f_0 + f_1 \theta, g_0 + g_1 \theta) = \theta \sum_i (-1)^i \left( \begin{array}{c} k - 1 \\ i \end{array} \right) \left( \begin{array}{c} 2

\nu + k - 1 \\ i \end{array} \right) \left( \begin{array}{c} 2 \nu + i \\ i \end{array} \right)^{-1} f_1^i (k - i), \]

where \( \delta \) is a coboundary. For instance, \( \Upsilon \) is an odd 1-cocycle then \( \Upsilon \) is decomposed into four components:

\[
\begin{align*}
\Pi(\mathcal{H}) \otimes \mathcal{F}_\lambda \otimes \mathcal{F}_\nu & \rightarrow \mathcal{F}_\mu, \\
\Pi(\mathcal{H}) \otimes \Pi(\mathcal{F}_{\lambda + \frac{1}{2}}) \otimes \Pi(\mathcal{F}_{\nu + \frac{1}{2}}) & \rightarrow \mathcal{F}_\mu, \\
\Pi(\mathcal{H}) \otimes \Pi(\mathcal{F}_{\lambda + \frac{1}{2}}) \otimes \mathcal{F}_\nu & \rightarrow \Pi(\mathcal{F}_{\mu + \frac{1}{2}}), \\
\Pi(\mathcal{H}) \otimes \mathcal{F}_\lambda \otimes \Pi(\mathcal{F}_{\nu + \frac{1}{2}}) & \rightarrow \Pi(\mathcal{F}_{\mu + \frac{1}{2}}).
\end{align*}
\]

The equation (3.22) is nothing but the \( \mathfrak{s}(2) \)-invariance property of these bilinear maps. Therefore, the expressions of these maps are given by Lemma 3.5. So, we must have \( \mu = \lambda + \nu + k - \frac{1}{2} \), where \( k \in \mathbb{N} \), otherwise, the operator \( \Upsilon \) is identically the zero map. If \( \mu = \lambda + \nu + k - \frac{1}{2} \), we check that \( \Upsilon \) is a coboundary.

2) Similarly, if \( \Upsilon \) is an odd 1-cocycle then \( \Upsilon \) is decomposed into four components:

\[
\begin{align*}
\Pi(\mathcal{H}) \otimes \mathcal{F}_\lambda \otimes \mathcal{F}_\nu & \rightarrow \mathcal{F}_\mu, \\
\Pi(\mathcal{H}) \otimes \Pi(\mathcal{F}_{\lambda + \frac{1}{2}}) \otimes \Pi(\mathcal{F}_{\nu + \frac{1}{2}}) & \rightarrow \mathcal{F}_\mu, \\
\Pi(\mathcal{H}) \otimes \Pi(\mathcal{F}_{\lambda + \frac{1}{2}}) \otimes \mathcal{F}_\nu & \rightarrow \Pi(\mathcal{F}_{\mu + \frac{1}{2}}), \\
\Pi(\mathcal{H}) \otimes \mathcal{F}_\lambda \otimes \Pi(\mathcal{F}_{\nu + \frac{1}{2}}) & \rightarrow \Pi(\mathcal{F}_{\mu + \frac{1}{2}}).
\end{align*}
\]

Lemma 3.5. Up to a coboundary, any 1-cocycle \( \Upsilon \in Z^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathcal{D}_{\lambda,\nu;\mu}) \) is invariant with respect the vector field \( X_1 = \partial_x \). That is, the map \( \Upsilon \) can be expressed with constant coefficients.

Proof. The 1-cocycle condition reads:

\[
X_1 \cdot \Upsilon(X_F) - (-1)^{||\Upsilon||} X_F \cdot \Upsilon(X_1) - \Upsilon([X_1, X_F]) = 0. \tag{3.28}
\]

But, from Theorem 3.1, up to a coboundary, we have \( \Upsilon(X_1) = 0 \), and therefore the equation (3.28) becomes

\[
X_1 \cdot (\Upsilon(X_F)) - \Upsilon([X_1, X_F]) = 0
\]

which is nothing but the invariance property of \( \Upsilon \) with respect the vector field \( X_1 = \partial_x \). \( \square \)

4 Proof of Theorem 3.1

First, according to Proposition 3.4, Proposition 3.6 and Theorem 3.4, we easily check that the following statements hold:

i) The space \( H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathcal{D}_{\lambda,\nu;\mu}) \) is trivial if \( 2(\mu - \lambda - \nu) + 1 \not\in \mathbb{N} \).

ii) The space \( H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathcal{D}_{\lambda,\nu;\mu}) \) is even if \( \mu - \lambda - \nu \) is integer and it is odd if \( \mu - \lambda - \nu \) is semi-integer.

Proposition 4.1. Let \( \Upsilon \in Z^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathcal{D}_{\lambda,\nu;\mu}) \), \( k \in \mathbb{N} \), \( h \in \mathbb{R}_{[x]} \) and \( f, g \in C^\infty(\mathbb{R}) \).

a) If \( \mu - \lambda - \nu = k \) then, up to a coboundary, \( \Upsilon(X_{\theta h})(\theta f, \theta g), \Upsilon(X_{\theta h})(f, \theta g) \) and \( \Upsilon(X_{\theta h})(\theta f, g) \) are \( k \)-homogeneous and \( \Upsilon(X_{\theta h})(f, g) \) is \( (k+1) \)-homogeneous.

b) If \( \mu - \lambda - \nu = k - \frac{1}{2} \) then, up to a coboundary, \( \Upsilon(X_{\theta h})(f, g), \Upsilon(X_{\theta h})(f, \theta g) \) and \( \Upsilon(X_{\theta h})(\theta f, g) \) are \( k \)-homogeneous and \( \Upsilon(X_{\theta h})(\theta f, \theta g) \) is \( (k-1) \)-homogeneous.
Proof. Let $\mu - \lambda - \nu = k$. Up to a coboundary, the operator $\Upsilon(X_{\theta})$ is an odd map. Therefore, the elements $\Upsilon(X_{\theta})((\theta f, \theta g), \Upsilon(X_{\theta})((f, \theta g), \ldots$ are all homogeneous (even or odd). Thus, the actions of $X_f$ and $X_{\lambda 0}$ on these elements are also homogeneous, see (2.3).

Now, for $h = x$ and $f = x^2$, the equation (3.22) becomes

$$X_{x^2} \cdot \Upsilon(X_{x^2}) = X_{x^2} \cdot \Upsilon(X_{x^2}).$$

So, using Lemma 3.3 and formulas (2.3), we obtain the statement a) for $h = x$. Besides, we use again the equation (3.22) but for $h = 1$ and $f = x$. The statement b) can be proved similarly.

Now, we explain our strategy to prove Theorem 3.1. Consider $\Upsilon \in Z^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathcal{D}_{\lambda, \nu, \mu})$ where $2(\mu - \lambda - \nu) + 1 \in \mathbb{N}$. That is,

$$\mu - \lambda - \nu = k \quad \text{or} \quad \mu - \lambda - \nu = k - \frac{1}{2} \quad \text{where} \quad k \in \mathbb{N}.$$

For instance, in the first case, the cohomology space is even, therefore, the restriction of $\Upsilon$ on $\mathfrak{sl}(2)$ is with values in $(\mathcal{D}_{\lambda, \nu, \mu})_0$ which is isomorphic, as $\mathfrak{sl}(2)$-module, to

$$D_{\lambda, \nu, \mu} \oplus D_{\lambda + \frac{1}{2}, \nu + \frac{1}{2}, \mu} \oplus D_{\lambda, \nu + \frac{1}{2}, \mu + \frac{1}{2}} \oplus D_{\lambda, \nu, \mu + \frac{1}{2}} \oplus D_{\lambda + \frac{1}{2}, \nu + \frac{1}{2}, \mu} \oplus D_{\lambda + \frac{1}{2}, \nu, \mu + \frac{1}{2}}$$

while the restriction of $\Upsilon$ on $\Pi(H)$ is with values in $(\mathcal{D}_{\lambda, \nu, \mu})_1$ which is isomorphic, as $\mathfrak{sl}(2)$-module, to

$$\Pi(D_{\lambda, \nu, \mu} \oplus D_{\lambda + \frac{1}{2}, \nu + \frac{1}{2}, \mu} \oplus D_{\lambda, \nu + \frac{1}{2}, \mu + \frac{1}{2}} \oplus D_{\lambda, \nu, \mu + \frac{1}{2}} \oplus D_{\lambda + \frac{1}{2}, \nu + \frac{1}{2}, \mu} \oplus D_{\lambda + \frac{1}{2}, \nu, \mu + \frac{1}{2}}).$$

Now, according to the decompositions (3.16) and (3.17), we have

$$H^1(\mathfrak{sl}(2), \mathcal{D}_{\lambda, \nu, \mu}) = H^1(\mathfrak{sl}(2), D_{\lambda, \nu, \mu} \oplus D_{\lambda + \frac{1}{2}, \nu + \frac{1}{2}, \mu} \oplus D_{\lambda, \nu + \frac{1}{2}, \mu + \frac{1}{2}} \oplus D_{\lambda, \nu, \mu + \frac{1}{2}} \oplus D_{\lambda + \frac{1}{2}, \nu + \frac{1}{2}, \mu} \oplus D_{\lambda + \frac{1}{2}, \nu, \mu + \frac{1}{2}})$$

$$\oplus H^1(\mathfrak{sl}(2), \Pi(D_{\lambda, \nu, \mu} \oplus D_{\lambda + \frac{1}{2}, \nu + \frac{1}{2}, \mu} \oplus D_{\lambda, \nu + \frac{1}{2}, \mu + \frac{1}{2}} \oplus D_{\lambda, \nu, \mu + \frac{1}{2}} \oplus D_{\lambda + \frac{1}{2}, \nu + \frac{1}{2}, \mu} \oplus D_{\lambda + \frac{1}{2}, \nu, \mu + \frac{1}{2}})).$$

Thus, the restriction of $\Upsilon$ to $\mathfrak{sl}(2)$ is, a priori, described by Theorem 3.1 while the general form of the restriction of $\Upsilon$ on $\Pi(H)$ is given by Proposition 4.1. Finally, the operator $\Upsilon$ will be completely given by the 1-cocycle conditions.

Hereafter, $F = f_0 + f_1 \theta$ and $G = g_0 + g_1 \theta$ where $f_0, g_0, f_1, g_1 \in C^\infty(\mathbb{R})$.

**Case 1:** $(\lambda, \nu, \mu)$ is weakly super resonant with $\mu - \lambda - \nu = k + 1 \in \mathbb{N}$. In this case, we describe the restriction of $\Upsilon$ to $\mathfrak{sl}(2)$ by using the 1-cocycles (3.10), (3.11) and (3.12).

a) Let $(\lambda, \nu) \neq (-\frac{s}{2}, -\frac{t}{2})$ where $s, t \in \{0, 1, \ldots, k + 1\}$. In this case, the 1-cocycle $\Upsilon$ is even and (if, for instance, $\lambda \neq -\frac{s}{2}$) its restriction to $\mathfrak{sl}(2)$ is given by

$$\Upsilon(X_h, F, G) = \alpha_1 a_1(X_h, f_0, g_0) + \alpha_2 a_2(X_h, f_1, g_1) + \theta \alpha_3 a_3(X_h, f_1, g_0) + \theta \alpha_4 a_4(X_h, f_0, g_1)$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ and the maps $a_1, a_2, a_3, a_4$ are as in (3.10). For instance, the expression of $a_2$ can be deduced from (3.10) by substituting respectively $\lambda + \frac{1}{2}, \nu + \frac{1}{2}$ and $k - 1$ to $\lambda, \nu$ and $k$, see (3.16). From the relation $\delta \Upsilon(X_{h_0}, X_{h_1}\theta)(F, G) = 0$ we deduce that

$$\alpha_4 = \alpha_1, \quad 2\lambda \alpha_3 = (2\lambda + k + 1)\alpha_1 \quad \text{and} \quad 2\lambda \alpha_2 = -(k + 1)\alpha_1.$$
Thus, according to Proposition 3.6, we have dim\(H^1(\mathfrak{osp}(1|2), (\mathfrak{D}_{\lambda,\nu,\mu})) \leq 1\). Now, using Lemma 3.3, Proposition 4.1, the isomorphism (3.10), and the 1-cocycle relations, we extend \(\Upsilon\) to \(\Pi(H)\). More precisely, we prove that we can choose
\[
\Upsilon(X_{h\theta}, F, G) = \alpha_1 \theta h' \left( \sum_{i=0}^{k+1} \binom{k+1}{i} \binom{2\nu + k}{i} \binom{-2\lambda}{i} f_0^{(i)}(k+1-i) \right)
- \frac{(k+1)}{2\lambda} f_1 g_{1}^{(k)} - \sum_{i=1}^{k} \binom{k}{i} \binom{2\nu + k - 1}{i} \binom{-2\lambda}{i} f_1^{(i)} g_{1}^{(k-i)}.
\]

Thus, in this case, dim\(H^1(\mathfrak{osp}(1|2), (\mathfrak{D}_{\lambda,\nu,\mu})) = 1\).

b) Let \((\lambda, \nu) = (-\frac{s}{2}, -\frac{t}{2})\) with \(s, t \in \{0, 1, \ldots, k + 1\}\) but \(s + t < k\). As in the previous case, the restriction of \(\Upsilon\) to \(\mathfrak{sl}(2)\) is given by
\[
\Upsilon(X_h, F, G) = \alpha_1 a_1(X_h, f_0, g_0) + \alpha_2 a_2(X_h, f_1, g_1) + \theta \alpha_3 a_3(X_h, f_1, g_0) + \theta \alpha_4 a_4(X_h, f_0, g_1)
\]
where \(\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}\), but here the maps \(a_1, a_2, a_3, a_4\) are as in (3.12). By using again the 1-cocycle relations we prove that
\[
\alpha_2 = -\frac{k+1}{k-s+1} \alpha_3, \quad \alpha_4 = -\frac{k-t+1}{k-s+1} \alpha_3, \quad \alpha_1 = -\frac{k-t-s}{k-s+1} \alpha_3.
\]
We prove that \(\Upsilon\) can be extended to \(\Pi(H)\). For instance, we can choose
\[
\Upsilon(X_{h\theta}, F, G) = -\frac{\alpha_1}{k-s+1} \theta h' \left( (k-t+1) \sum_{i=s+1}^{k-t} (-1)^i \binom{k+1}{i} \binom{k-t-s-1}{i-s-1} f_0^{(i)}(k-i) \right)
+ (k+1) \sum_{i=s}^{k-t} (-1)^i \binom{k+1}{i} \binom{k-t-s-1}{i-s-1} f_1^{(i)} g_{1}^{(k-i)}.
\]
Thus, in this case, dim\(H^1(\mathfrak{osp}(1|2), (\mathfrak{D}_{\lambda,\nu,\mu})) = 1\).

The case \(\mu - \lambda - \nu = k + \frac{3}{2}\) where \((\lambda, \nu) \neq (-\frac{s}{2}, -\frac{t}{2})\) or \((\lambda, \nu) = (-\frac{s}{2}, -\frac{t}{2})\) but \(s + t < k + 1\) with \(s, t \in \{0, 1, \ldots, k\}\) can be treated similarly. For instance, let \((\lambda, \nu) \neq (-\frac{s}{2}, -\frac{t}{2})\) where \(s, t \in \{0, 1, \ldots, k + 1\}\). The 1-cocycle \(\Upsilon\) is odd and (if, for instance, \(\lambda \neq -\frac{s}{2}\)) its restriction to \(\mathfrak{sl}(2)\) is given by
\[
\Upsilon(X_h, F, G) = \theta \alpha_1 a_1(X_h, f_0, g_0) + \theta \alpha_2 a_2(X_h, f_1, g_1) + \alpha_3 a_3(X_h, f_1, g_0) + \alpha_4 a_4(X_h, f_0, g_1)
\]
where \(\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}\) and the maps \(a_1, a_2, a_3, a_4\) are as in (3.10). For instance, the expression of \(a_1\) can be deduced from (3.10) by substituting \(k + 1\) to \(k\) while the expression of \(a_2\) can be deduced from (3.10) by substituting respectively \(\lambda + \frac{1}{2}\) and \(\nu + \frac{1}{2}\) to \(\lambda\) and \(\nu\), see (3.16). From the relation \(\delta \Upsilon(X_{h\theta}, X_{h\theta})(F, G) = 0\) we deduce that
\[
\alpha_2 = -\frac{2\lambda}{2\nu+k+1} \alpha_1, \quad \alpha_3 = \frac{2\nu+2\lambda+k+1}{2\nu+k+1} \alpha_1 \quad \text{and} \quad \alpha_4 = \frac{2\lambda}{2\nu+k+1} \alpha_1
\]
and we prove also that \(\Upsilon\) can be extended to \(\Pi(H)\).
Case 2: \((\lambda, \nu, \mu)\) is super resonant: \(\mu - \lambda - \nu = k + 1\) where \((\lambda, \nu) = (-\frac{s}{2}, -\frac{t}{2})\) with \(s, t \in \{1, \ldots, k\}\) and \(s + t \geq k + 1\). In this case the map \(\Upsilon|_{\mathfrak{sl}(2)}\) can be decomposed as follows: 

\[
\Upsilon|_{\mathfrak{sl}(2)} = B + C + D
\]

where the \(\lambda, \nu, \mu\) are neither super resonant nor weakly super resonant. We know that the non-vanishing spaces \(H^1(\mathfrak{osp}(1|2), (\mathbb{D}_{\lambda, \nu, \mu}))\) only can appear if \(2(\mu - \lambda - \nu) + 1 \in \mathbb{N}\), thus, we consider the following two situations:

A. Let \(\mu - \lambda - \nu = k + 1\) where \(k \in \mathbb{N}\) and \((\lambda, \nu) = (-\frac{s}{2}, -\frac{t}{2})\) with \(s, t \in \{0, \ldots, k + 1\}\). In this case the cohomology space is even and then we have to consider:

\[
(\lambda, \nu, \mu), \quad (\lambda + \frac{1}{2}, \nu + \frac{1}{2}, \mu), \quad (\lambda + \frac{1}{2}, \nu, \mu + \frac{1}{2}) \quad \text{and} \quad (\lambda, \nu + \frac{1}{2}, \mu + \frac{1}{2}).
\]

The cases for which \((\lambda, \nu, \mu)\) is neither super resonant nor weakly super resonant are:

(i) \(s + t = k\), in this case only \((\lambda, \nu, \mu)\) is resonant.
(ii) \(s \in \{0, \ldots, k\}\) and \(t = k + 1\), in this case only \((\lambda, \nu + \frac{1}{2}, \mu + \frac{1}{2})\) is resonant.
(iii) \(s = k + 1\) and \(t \in \{0, \ldots, k\}\), in this case only \((\lambda, \nu + \frac{1}{2}, \mu + \frac{1}{2})\) is resonant.

B. Let \(\mu - \lambda - \nu = k + \frac{3}{2}\) where \(k + 1 \in \mathbb{N}\) and \((\lambda, \nu) = (-\frac{s}{2}, -\frac{t}{2})\) with \(s, t \in \{0, \ldots, k + 1\}\). In this case the cohomology space is odd and then we have to consider:

\[
(\lambda + \frac{1}{2}, \nu, \mu), \quad (\lambda, \nu + \frac{1}{2}, \mu), \quad (\lambda, \nu, \mu + \frac{1}{2}) \quad \text{and} \quad (\lambda + \frac{1}{2}, \nu + \frac{1}{2}, \mu + \frac{1}{2}).
\]

We have to distinguish the following cases:

(i) \((s, t) = (k + 1, 0)\), in this case only \((\lambda + \frac{1}{2}, \nu, \mu)\) and \((\lambda, \nu + \frac{1}{2}, \mu)\) are resonant.
(ii) \((s, t) = (0, k + 1)\), in this case only \((\lambda, \nu + \frac{1}{2}, \mu)\) and \((\lambda, \nu, \mu + \frac{1}{2})\) are resonant.
(iii) \(s + t = k + 1\) with \(st \neq 0\), in this case only \((\lambda + \frac{1}{2}, \nu + \frac{1}{2}, \mu + \frac{1}{2})\) is non resonant.

5 Singular cases

Finally, we complete the study of the spaces \(H^1_{\text{diff}}(\mathfrak{osp}(1|2), (\mathbb{D}_{\lambda, \nu, \mu}))\) by considering the cases \((\lambda, \nu, \mu)\) which are neither super resonant nor weakly super resonant. We know that the non-vanishing spaces \(H^1_{\text{diff}}(\mathfrak{osp}(1|2), (\mathbb{D}_{\lambda, \nu, \mu}))\) only can appear if \(2(\mu - \lambda - \nu) + 1 \in \mathbb{N}\), thus, we consider the following two situations:

A. Let \(\mu - \lambda - \nu = k + 1\) where \(k \in \mathbb{N}\) and \((\lambda, \nu) = (-\frac{s}{2}, -\frac{t}{2})\) with \(s, t \in \{1, \ldots, k\}\) and \(s + t \geq k + 2\) (super resonance case with \(\mu - \lambda - \nu\) semi integer) can be treated similarly and we get the same result.

B. Let \(\mu - \lambda - \nu = k + \frac{3}{2}\) where \(k + 1 \in \mathbb{N}\) and \((\lambda, \nu) = (-\frac{s}{2}, -\frac{t}{2})\) with \(s, t \in \{1, \ldots, k\}\). In this case the cohomology space is even and then we have to consider:

\[
(\lambda, \nu, \mu), \quad (\lambda + \frac{1}{2}, \nu + \frac{1}{2}, \mu), \quad (\lambda + \frac{1}{2}, \nu, \mu + \frac{1}{2}) \quad \text{and} \quad (\lambda, \nu + \frac{1}{2}, \mu + \frac{1}{2}).
\]
(iv) $s = t = k + 1$, in this case only $(\lambda, \nu, \mu + \frac{1}{2})$ and $(\lambda + \frac{1}{2}, \nu + \frac{1}{2}, \mu + \frac{1}{2})$ are resonant.
(v) $s = k + 1$ and $t \in \{1, \ldots, k\}$, in this case only $(\lambda, \nu + \frac{1}{2}, \mu)$ is non resonant.
(vi) $t = k + 1$ and $s \in \{1, \ldots, k\}$, in this case only $(\lambda + \frac{1}{2}, \nu, \mu)$ is non resonant.

**Theorem 5.1.** Let $(\lambda, \nu, \mu)$ be neither super resonant nor weakly super resonant.

(a) If $\mu - \lambda - \nu = k + 1 \in \mathbb{N}^*$ and $(\lambda, \nu) = (-\frac{s}{2}, -\frac{t}{2})$ with $s, t \in \{0, \ldots, k + 1\}$ then

$$H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \nu, \mu}) \simeq \begin{cases} \mathbb{R} & \text{if } s + t = k, \\ \mathbb{R}^2 & \text{if } s = k + 1 \text{ or } t = k + 1 \text{ with } k + 2 \leq s + t \leq 2k + 1. \end{cases}$$

(b) If $\mu - \lambda - \nu - \frac{1}{2} = k + 1 \in \mathbb{N}$ and $(\lambda, \nu) = (-\frac{s}{2}, -\frac{t}{2})$ with $s, t \in \{0, \ldots, k + 1\}$ then

$$H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \nu, \mu}) \simeq \begin{cases} \mathbb{R}^5 & \text{if } s = k + 1 \text{ or } t = k + 1 \text{ with } k + 2 \leq s + t \leq 2k + 1, \\ \mathbb{R}^3 & \text{if } (s, t) = (0, k + 1), (k + 1, 0) \text{ with } k \neq -1, \\ \mathbb{R}^2 & \text{if } s = t = k + 1 \text{ or } s + t = k + 1 \text{ but } st \neq 0, \\ \mathbb{R} & \text{if } k = -1. \end{cases}$$

(c) Otherwise $H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \nu, \mu}) = 0$.

Proof. Recall that if $2(\mu - \lambda - \nu) + 1 \notin \mathbb{N}$ then $H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \nu, \mu}) = 0$. Thus, assume that $2(\mu - \lambda - \nu) + 1 \in \mathbb{N}$.

1) **Even cases:** $\mu - \lambda - \nu = k + 1$ where $k \in \mathbb{N}$.

i) $(\lambda, \nu) = (-\frac{s}{2}, -\frac{t}{2})$ with $s \in \{0, \ldots, k\}$. The restriction of $\Upsilon$ to $\mathfrak{sl}(2)$ is given by

$$\Upsilon(X_h)(f, g) = \alpha_1 h'' f^{(s)} g^{(k-s)} + a \sum_{0 \leq i \leq s} \binom{k + 1}{i} h'^{(k+1-i)} g^{(k+1-i)} + b \sum_{i=s+1}^{k+1} \binom{k + 1}{i} h'^{(k+1-i)} g^{(k-s)} ,$$

$$\Upsilon(X_h)(\theta f, \theta g) = \alpha_2 h'^{(s)} g^{(k-s)} , \quad \Upsilon(X_h)(\theta f, g) = \theta \alpha_3 h'^{(s)} g^{(k-s+1)} , \quad \Upsilon(X_h)(f, \theta g) = \theta \alpha_4 h'' f^{(s+1)} g^{(k-s)} .$$

As before, the 1-cocycle condition gives:

$$\Upsilon(X_h)(F, G) = \alpha_2 \left( h'^{(s)} g_1^{(k-s)} + \theta h'' \left( f_0^{(s+1)} g_1^{(k-s)} - f_1^{(s)} g_0^{(k-s+1)} \right) \right) ,$$

that is $a = b = \alpha_1 = 0$ and $\alpha_3 = -\alpha_4 = -\alpha_2$. We check that $\Upsilon$ can be extended to $\Pi(\mathcal{H})$ and we deduce that $\dim H^1(\mathfrak{osp}(1|2), (\mathfrak{D}_{\lambda, \nu, \mu})) = 1$.

ii) $(\lambda, \nu) = (-\frac{s}{2}, -\frac{k+1}{2})$ with $s \in \{1, \ldots, k\}$. The restriction of $\Upsilon$ to $\mathfrak{sl}(2)$ is, a priori, given
Thus, we have

\[ \Upsilon(X_h)(f, g) = \alpha_1 \sum_{i=s+1}^{k+1} \binom{k-s}{i-s-1} h'f(i)g^{(k+1-i)}, \]

\[ \Upsilon(X_h)(\theta f, \theta g) = \alpha_2 \sum_{i=s}^{k} \binom{k-s}{i-s} h'f(i)g^{(k-i)}, \]

\[ \Upsilon(X_h)(\theta f, g) = \alpha_3 \theta \sum_{i=s}^{k+1} \binom{k-s+1}{i-s} h'f(i)g^{(k+1-i)}, \]

\[ \Upsilon(X_h)(f, \theta g) = \theta \left( \alpha_4 \sum_{i=s+1}^{k+1} \binom{k+1-s}{i-s} h'f(i)g^{(k+1-i)} + \alpha_5 h''f^{(k)} + \alpha_6 h'f^{(k+1)} \right). \]

The 1-cocycle condition: \( \delta(\Upsilon)(X_x, X_\theta) = 0 \) gives:

\[ \alpha_6 = \alpha_4 = 0 \quad \text{and} \quad \alpha_1 = -\alpha_2 = \alpha_3. \]

We easily check that \( \Upsilon \) can be extended to \( \Pi(H) \), therefore \( \dim H^1(osp(1|2), D_{\lambda,\nu,\mu}) = 2 \). Of course, we have the same result if \( (\lambda, \nu) = (-\frac{1+k}{2}, -\frac{s}{2}) \) where \( s \in \{1, \ldots, k\} \).

2) **Odd cases**: \( \mu - \lambda - \nu + \frac{1}{2} = k+2 \in \mathbb{N} \) and \( (\lambda, \nu) = (-\frac{s}{2}, -\frac{k}{2}) \) with \( s, t \in \{0, \ldots, k+1\} \).

i) Let \( k = -2 \). Here we are in the situation (c) of Theorem 5.1. Obviously, in this case, \( (\lambda, \nu, \mu) \) is neither super resonant nor weakly super resonant. The restriction of \( \Upsilon \) to \( sl(2) \) is, a priori, given by:

\[ \Upsilon(X_h)(F, G) = \alpha h'f_0g_0, \]

where \( \alpha \in \mathbb{R} \) and \( h \in \mathbb{R}_2[x] \). The 1-cocycle condition: \( \delta(\Upsilon)(X_x, X_\theta) = 0 \) gives the following equation:

\[ x(\Upsilon(X_\theta)(f_0, g_0))' - \Upsilon(X_\theta)(xf_0', g_0) - \Upsilon(X_\theta)(f_0, xg_0') + \frac{1}{2} \alpha f_0g_0 = 0. \]

Thus, we have \( \alpha = 0 \) since \( \frac{1}{2} \alpha f_0g_0 \) is the unique term in \( f_0g_0 \) in the previous equation. By Proposition 3.6 we deduce that, in this case, \( H^1_{\text{diff}}(osp(1|2), D_{\lambda,\nu,\mu}) = 0 \).

ii) Let \( k = -1 \). In this case, \( (\lambda, \nu, \mu) \) is neither super resonant nor weakly super resonant if and only if \( (\lambda, \nu) = (0,0) \). So, let \( (\lambda, \nu) = (0,0) \) and consider the restriction of \( \Upsilon \) to \( sl(2) \) which is, a priori, given by:

\[ \Upsilon(X_h)(F, G) = h'f_1g_0 + \alpha_2 f_0g_1 - f_0g_1 + \alpha_3 \theta f_0g_0 + \alpha_4 \theta f_0g_0' + \alpha_5 \theta f_1g_1 + \alpha_6 \theta h''f_0g_0. \]

The 1-cocycle condition: \( \delta(\Upsilon)(X_x, X_\theta) = 0 \), gives

\[ \alpha_1 = \alpha_2 = -\alpha_3 = \alpha_4 = \alpha_6, \quad \alpha_5 = 0. \]

The restriction of \( \Upsilon \) to \( \Pi(H) \) can be given by

\[ \Upsilon(X_{h,\theta})(F, G) = \theta \alpha_4 h'_1FG. \]
Thus, \( \dim H_1^{\text{diff}}(\mathfrak{osp}(1|2), \mathcal{D}_{\lambda, \nu; \mu}) = 1 \). This proves the situation (b) when \( k = -1 \).

ii) Let \((\lambda, \nu) = \left(-\frac{k+1}{2}, -\frac{k+1}{2}\right)\) where \( k \in \mathbb{N}^* \). The restriction of \( \Upsilon \) to \( \mathfrak{sl}(2) \) is given by

\[
\Upsilon(X_h)(F, G) = \theta(\alpha_1 h'' f_0 g_0^{(k+1)} + \alpha_2 h' f_0 g_0^{(k+2)} + \alpha_3 h' f_0^{(k+2)} g_0 + \beta_1 h'' f_1 g_1^{(k+1)} + \beta_2 h' f_1 g_1^{(k+1)} + \beta_3 h' f_1 g_1^{(k+1)} g_1 + \gamma h' f_1^{(k+1)} g_0 + \delta h' f_0 g_0^{(k+1)}).
\]

where \( \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma, \delta \in \mathbb{R} \). From the relation \( \delta \Upsilon(X_{h_0}, X_{h_1 \theta})(F, G) = 0 \) we deduce that:

\[
\Upsilon(X_h)(F, G) = \theta(\alpha_1 h'' f_0 g_0^{(k+1)} + \beta_1 h'' f_1 g_1^{(k+1)}).
\]

We check that \( \Upsilon \) can be extended to \( \Pi(\mathcal{H}) \), therefore \( \dim H_1^{\text{diff}}(\mathfrak{osp}(1|2), \mathcal{D}_{\lambda, \nu; \mu}) = 2 \).

iii) Let \((\lambda, \nu) = \left(-\frac{k+1}{2}, -\frac{k+1}{2}\right)\) where \( k \in \mathbb{N}^* \) and \( t \in \{1, \ldots, k\} \). In this case the map \( \Upsilon|_{\mathfrak{sl}(2)} \) can be decomposed as follows: \( \Upsilon|_{\mathfrak{sl}(2)} = B + C + D \) where

\[
B(X_h, F, G) = h''(\theta_1 f_0^{(k-t+1)} g_0^{(t)}) + \theta_2 f_1^{(k-t+1)} g_1^{(t-1)} + \beta_3 f_1^{(k-t)} g_0^{(t)}),
\]

\[
C(X_h, F, G) = \theta_1 c_1(X_h, f_0, g_0) + \theta_2 c_2(X_h, f_1, g_1) + \gamma_3 c_3(X_h, f_1, g_1),
\]

\[
D(X_h, F, G) = \delta_1 \theta_1(X_h, f_0, g_0) + \delta_2 \theta_2(X_h, f_1, g_1) + \delta_3 \theta_3(X_h, f_1, g_1) + \alpha_1 a_1(X_h, f_0, g_1)
\]

where the \( c_i \), the \( \theta_i \) are as those defined in (3.14) and (3.15) and \( a_1 \) is as in (3.10). By the 1-cocycle relation: \( \delta \Upsilon(X_{h_0}, X_{h_1 \theta})(F, G) = 0 \) we prove

\[
\delta_1 = -\delta_2 = -\delta_3, \quad \gamma_3 = \frac{(k+1-t)}{k+1} \alpha_1, \quad \gamma_2 = \frac{-t}{k+1} \alpha_1, \quad \gamma_1 = -\alpha_1.
\]

We prove also that \( \Upsilon \) can be extended to \( \Pi(\mathcal{H}) \). Thus, \( \dim H_1^{\text{diff}}(\mathfrak{osp}(1|2), \mathcal{D}_{\lambda, \nu; \mu}) = 5 \). Similarly, we study the case \((\lambda, \nu) = \left(-\frac{t}{2}, -\frac{k+1-s}{2}\right)\) where \( k \in \mathbb{N}^* \) and \( s \in \{1, \ldots, k\} \).

iv) Let \((\lambda, \nu) = \left(-\frac{t}{2}, -\frac{k+1-s}{2}\right)\) where \( k \in \mathbb{N}^* \) and \( s \in \{1, \ldots, k\} \). In this case the map \( \Upsilon|_{\mathfrak{sl}(2)} \) can be decomposed as follows: \( \Upsilon|_{\mathfrak{sl}(2)} = B + C + D \) where

\[
B(X_h, F, G) = h''(\beta_1 f_1^{(s-1)} g_0^{(k-s+1)} + \beta_2 f_0^{(k-s)} g_1^{(k-s)} + \theta_3 f_0^{(k-t)} g_0^{(t)}),
\]

\[
C(X_h, F, G) = \theta_1 c_1(X_h, f_0, g_0) + \alpha_3 c_3(X_h, f_1, g_1) + \alpha_4 c_4(X_h, f_0, g_1),
\]

\[
D(X_h, F, G) = \alpha_1 \theta_1(X_h, f_0, g_0) + \alpha_2 \theta_3(X_h, f_1, g_1) + \alpha_3 \theta_2(X_h, f_1, g_1)
\]

where the \( c_i \), the \( \theta_i \) are as those defined in (3.14) and (3.15) and \( a_3 \) is as in (3.10). By the 1-cocycle relation: \( \delta \Upsilon(X_{h_0}, X_{h_1 \theta})(F, G) = 0 \) we prove:

\[
\begin{cases}
(k+2)\beta_2 = -(k-s+1)\beta_4 \\
(k+3)\beta_1 = -s\beta_4
\end{cases}
\]

We prove also that \( \Upsilon \) can be extended to \( \Pi(\mathcal{H}) \). Thus, \( \dim H_1^{\text{diff}}(\mathfrak{osp}(1|2), \mathcal{D}_{\lambda, \nu; \mu}) = 2 \).
References

[1] Agrebaoui B and Ben Fraj N, On the cohomology of the Lie superalgebra of contact vector fields on $S^{1|1}$, *Bell. Soc. Roy. Sci. Liège* **72**, 6, 2004, 365–375.

[2] S. Bouarroudj, *Cohomology of the vector fields Lie algebras on $\mathbb{RP}^1$ acting on bilinear differential operators*, International Journal of Geometric Methods in Modern Physics (2005), 2; N 1, 23-40.

[3] Fuchs D B, *Cohomology of infinite-dimensional Lie algebras*, Plenum Publ. New York, 1986.

[4] H. Gargoubi, N. Mellouli and V. Ovsienko *Differential Operators on Supercircle: Conformally Equivariant Quantization and Symbol Calculus*, Letters in Mathematical Physics (2007) **79**:5165.