On ‘rolling moduli’ solutions in string cosmology

A.A. Tseytlin**

Theoretical Physics Group, Blackett Laboratory
Imperial College, London SW7 2BZ, U.K.

Abstract
Given a static string solution with some free constant parameters (‘moduli’) it may be possible to construct a time-dependent solution by just replacing the moduli by some functions of time. We present several examples when such ‘rolling moduli’ ansatz is consistent. In particular, the anisotropic $D = 4$ cosmological solution of Nappi and Witten can be reinterpreted as a time-dependent generalisation of the (analytic continuation of) $D = 3$ ‘charged black string’ background with the ‘charge’ changing with time. We find some new $D = 4$ cosmological solutions which are ‘two rolling moduli’ generalisations of the previously known ones. We also comment on interplay between duality transformations and the replacement of moduli by functions of time.

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* e-mail: tseytlin@ic.ac.uk

* On leave from Lebedev Physics Institute, Moscow, Russia.
1. Introduction

Given a static string solution with some free parameters $\lambda_i$ (moduli of the corresponding 2d conformal field theory) one may try to look for cosmological solutions in which the spatial part remains the same (i.e. the same CFT at each fixed $t$) and only the moduli $\lambda_i$ change with time. A simple example is provided by the solution of [4] (see also [2][3][4]) where the moduli of a toroidal background and dilaton evolve with time. However, in general one is not guaranteed to get a solution by just replacing $\lambda_i$ in an $N$-dimensional static solution by some functions of $t$: a deformation just along moduli directions need not give a solution of the full set of string equations in $N + 1$-dimensions (for related remarks see, e.g., [5][6]). Depending on a ‘spatial’ CFT the only solution that may result may be the trivial one only ($\lambda_i = \text{const}$ and the dilaton linear in time).

The question under which conditions a particular trajectory in a moduli space can be identified with a string cosmological solution is of interest in connection with a possibility of a topology change in a process of cosmological evolution. A topology change can take place as one moves along moduli space of some Calabi-Yau manifolds [7]. It is not known whether the corresponding motion can actually be realised as a cosmological solution. A possibility of a topology change in string cosmology was recently discussed on a simple model in [8].

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1 The model discussed in [8] is essentially equivalent to the cosmological solution of [4] (based on the coset $[SL(2, R) \times SU(2)]/R \times R$) which is an $O(2, 2)$ duality rotation [10] of the anisotropic ‘direct product’ cosmological model of [11]. The spatial part of it represented by the coset model $[SU(2) \times U(1)]/U(1)$ is equivalent to an analytic continuation of the ‘charged black string’ solution of [12] and was discussed in detail in connection with a possibility of topology change in [13]. The topology here changes at the boundary of the moduli space and thus may not be of direct interest (with regions of different topology being separated by an infinite distance in theory space [14]). A smooth topology change was claimed to happen inside the moduli space [13][15] for a $\sigma$-model deformation of $SU(2)$ WZW model induced by a particular $O(2, 2)$ duality transformation but in that case it is not explicitly known which CFT corresponds to the deformed $\sigma$-model and whether the deformation can occur as a result of cosmological evolution.
The aim of this paper is to make some simple observations (Section 2) about the general structure of the string cosmological equations for the dilaton and moduli (assuming that the ansatz $\lambda_i \rightarrow \lambda_i(t)$ is consistent) and to present a number of examples in which the replacement of moduli by functions of time does, in fact, lead to string cosmological solutions (Section 3). In addition to reproducing some previously known solutions (in particular, the $D = 4$ solution of [9] can be obtained from the analytic continuation of the $D = 3$ charged black string solution [12] by making the ‘charge’ time-dependent; an equivalent observation was made in [8]) we will find a number of new $D = 4$ anisotropic cosmological string solutions and discuss the action of duality transformations.

2. String cosmological equations for the dilaton and moduli

Our discussion will be limited to the leading-order effective equations for the metric, antisymmetric tensor and dilaton couplings of the bosonic string $\sigma$-model. Consider an $N$-dimensional string background $G_{ab}(x)$, $B_{ab}(x)$, $\phi(x)$ ($a, b = 1, ..., N$) and generalise it to a $D = N + 1$-dimensional one by making the fields to depend on $t$. This corresponds to a particular gauge choice in $N + 1$ dimensions

$$ds^2 = \hat{G}_{\mu\nu}d\pi^\mu d\pi^\nu = -dt^2 + G_{ab}(t, x)dx^a dx^b,$$

$$\hat{B}_{\mu\nu} = (B_{ab}(t, x), 0), \quad \hat{G}_{\mu\nu}, \hat{B}_{\mu\nu} = \phi(t, x).$$

The leading-order string effective equations which follow from

$$S = \int d^Dx\sqrt{-G} e^{-2\phi} \left\{ \frac{D - 26}{6} - \frac{\alpha'}{4} \left[ \hat{R} + 4(\partial_\mu \phi)^2 - \frac{1}{12} \hat{H}^2_{\mu\nu\lambda} \right] + ... \right\},$$

are given by

$$\bar{\beta}^{G(D)}_{00} = \ddot{\phi} - \frac{1}{4} G^{ac} G^{bd} (\dot{G}_{ab} \dot{G}_{cd} + \dot{B}_{ab} \dot{B}_{cd}) = 0,$$

$$\bar{\beta}^{G(D)}_{ab} = \bar{\beta}^{G(N)}_{ab} + \frac{1}{2} \left[ \dot{G}_{ab} - \dot{\phi} G_{ab} - G^{cd} (\dot{G}_{ac} \dot{G}_{bd} - \dot{B}_{ac} \dot{B}_{bd}) \right] = 0.$$
\[ \tilde{\beta}^{B(D)}_{ab} = \tilde{\beta}^{B(N)}_{ab} + \frac{1}{2}(\ddot{B}_{ab} - \dot{\phi} \dot{B}_{ab} - 2G^{cd}\dot{G}_{c[a}\dot{\hat{B}}_{b]d}) = 0, \quad (5) \]

\[ \tilde{\beta}^{\phi(D)} = \tilde{\beta}^{\phi(N)} - \frac{1}{4}\dot{G}^{\mu\nu}\dot{G}^{D(D)} = \frac{c - 25}{6} + \frac{3}{2}\alpha'(\phi - \phi^2) = 0, \quad (6) \]

\[ c - 25 \equiv 6\tilde{\beta}^{\phi(N)} + 1 = N - 25 - \frac{3}{2}\alpha'[R - \frac{1}{12}H^2 - 4(\partial\phi)^2 + 4\nabla^2\phi], \quad (7) \]

\[ \tilde{\beta}^{G(D)}_{0a} = \nabla_c(G^{db}\dot{G}_{ba}) - \frac{1}{2}\ddot{B}_{db}H^a_{db} + 2\partial_a\dot{\phi} - G^{db}\dot{G}_{ba}\partial_d\varphi = 0, \quad (8) \]

\[ \tilde{\beta}^{0(D)}_{0a} = -G_{ab}\partial_d(G^{bc}G^{de}\dot{B}_{ce}) + 2\dot{B}_{ab}\nabla^b\varphi = 0, \quad (9) \]

where dot denotes a time derivative and we introduced the shifted dilaton

\[ \varphi \equiv 2\phi - \ln\sqrt{G}. \quad (10) \]

Eqs. (3)–(6) can be derived from the action which follows from (2) \((G_{00} = -1\) after the variation)

\[ S = \int dtd^N x\sqrt{-G_{00}} e^{-\varphi} \{ C - G^{00}\varphi^2 + \frac{1}{4}G^{ac}G^{bd}(\dot{G}_{ab}\dot{G}_{cd} + \ddot{B}_{ab}\ddot{B}_{cd}) \}, \quad (11) \]

\[ C \equiv -\frac{2}{3\alpha'}(c - 25). \quad (12) \]

Suppose \(G_{ab}(x, \lambda), B_{ab}(x, \lambda), \phi(x, \lambda)\) represent an \(N\)-dimensional string solution (CFT) depending on some free constant parameters \(\lambda_i\) \((i = 1, \ldots, r)\) (for clarity we shall first treat the constant part of the dilaton separately from the rest of the moduli). In that case the \(N\)-dimensional Weyl anomaly coefficients \(\tilde{\beta}^{G(N)}_{ab}, \tilde{\beta}^{B(N)}_{ab}\) in (4),(5) vanish and \(\tilde{\beta}^{\phi(N)}\) in (7) is equal to a constant \((c\) becomes the central charge of the spatial CFT\) which does not depend on the moduli \(\lambda_i\). Replacing \(\lambda_i\) by functions of time one may try to look for solutions of (3)–(9) such that

\[ G_{ab}(x, t) = G_{ab}(x, \lambda(t)), \quad B_{ab}(x, t) = G_{ab}(x, \lambda(t)), \quad \phi(x, t) = \phi_0(t) + \phi(x, \lambda(t)). \quad (13) \]

In general, such solutions will not exist: the \(x\)- and \(t\)-dependence will not decouple, the constraints (8),(9) will not be satisfied, etc. Let us, however, proceed by assuming that
the ‘rolling moduli’ ansatz (13) is consistent for a particular choice of a spatial CFT. Then \( \lambda_i \) and \( \phi_0(t) \) will be solutions of the equations that follow from the action

\[
S = \int dt \sqrt{-G_{00}} \ e^{-\varphi} \left\{ C + G^{00} \left[ -\varphi^2 + \gamma_{ij}(\lambda) \dot{\lambda}^i \dot{\lambda}^j \right] + \ldots \right\}, \tag{14}
\]

which follows from (11) upon integration over \( x \) (the consistency of the ansatz implies that such integration can be performed). Here \( \varphi(t) \) is the corresponding shifted dilaton (containing also the constant factor of the \( x \)-space volume), \( \gamma_{ij} \) is the ‘Zamolodchikov’s metric’ of the ‘static’ conformal field theory and dots stand for higher order \( \alpha' \)-corrections.

For example, in the case of the toroidal background when \( (G_{ab}, B_{ab}, \phi) \) are \( x \)-independent the integral over \( x \) decouples and (11) takes the manifestly \( O(N, N) \) duality-invariant form

\[
S = \int dt \ e^{-\varphi} \left[ C - \dot{\varphi}^2 - \frac{1}{8} \Tr (\dot{M} \eta \dot{M} \eta) + \ldots \right], \tag{15}
\]

\[
M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{16}
\]

In general, the action (14) admits an interesting interpretation as an action of a particle propagating in curved \( r + 1 \) dimensional space \( \lambda^A = (\varphi, \lambda^i) \) which has \textit{Minkowski} signature. Written in the form invariant under reparametrisations of \( t \) this action is

\[
S = \int dt \ e \left[ C + e^{-2\varphi} \gamma_{AB}(\Lambda) \dot{\Lambda}^A \dot{\Lambda}^B + \ldots \right], \tag{17}
\]

where we have defined the einbein \( e \) and the metric \( \gamma_{AB} \) on the space \( \Lambda^A = (\varphi, \lambda^i) \) by

\[
e^2 \equiv -e^{-2\varphi} G_{00}, \quad \gamma_{AB} = e^{-2\varphi} \begin{pmatrix} -1 & 0 \\ 0 & \gamma_{ij} \end{pmatrix}. \tag{18}
\]

\( \sqrt{-C} \) can be identified with the mass of the particle, so that the geodesics are time-like, null or space-like for negative, zero or positive \( C = -\frac{2}{3\alpha'}(c - 25) \) where \( c \) is the central charge of the spatial CFT.\footnote{\textsuperscript{2} A related discussion of cosmology in the presence of the dilaton and scalar moduli fields, and, in particular, the interpretation of the cosmological equations as geodesic motion in the ‘augmented’ or ‘extended’ moduli space \( (\Lambda^A) \) appeared in \cite{5} and was also mentioned in \cite{6}. The connection between the value of a central charge (or dimension) and the time-like, null or space-like nature of geodesics in the moduli space was noted in \cite{4}.}

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The dilaton $\varphi$ plays the role of a ‘time-like’ coordinate of the extended moduli space. For flat $\gamma_{ij}$ the metric in (18) is that of the Milne universe with a flat spatial section. The time reparametrisation invariance can be fixed by setting $G_{00} = -1$, i.e. $e = e^\varphi$. An alternative is to choose the standard particle theory gauge: $e = 1$, i.e. $G_{00} = -e^{2\varphi}$. In the gauge $G_{00} = -1$ we find the following evolution equations (cf. [16][17])

\[ C + \dot{\varphi}^2 - \gamma_{jk} \dot{\lambda}^j \dot{\lambda}^k = 2U , \]  
\[ \ddot{\lambda}^i + \Gamma^i_{jk} (\gamma) \dot{\lambda}^j \dot{\lambda}^k - \dot{\varphi} \dot{\lambda}^i = -\frac{\partial U}{\partial \lambda^i} , \]  
\[ \ddot{\varphi} - \gamma_{jk} \dot{\lambda}^j \dot{\lambda}^k = \frac{\partial U}{\partial \varphi} , \]  
where for generality we included a potential by replacing the ‘mass term’ $C$ in (17) by $V(\lambda) = C - 2U(\varphi, \lambda)$. It is interesting to observe that the solution for $\varphi$ is universal, i.e. it does not depend on a particular $\gamma_{ij}$. One can combine (19) and (21) to get (in what follows $U = 0$)

\[ \ddot{y} + Cy = 0 , \quad y \equiv e^{-\varphi} , \]  
with the solution (for definiteness we assume that $C \leq 0$ and $y(0) = 0$)

\[ \varphi = \varphi_0 - \ln \sinh 2bt , \quad 4b^2 = -C . \]  
This suggest a close relation between dilaton and time. The equation (20) or

\[ \ddot{\lambda}^i + \Gamma^i_{jk} (\gamma) \dot{\lambda}^j \dot{\lambda}^k = \dot{\varphi} \dot{\lambda}^i \]  
has a geometrical interpretation as an equation for a geodesic in the $\lambda^i$-space parametrised by a non-affine parameter. Defining the affine parameter $s$ (and using (23))

\[ \ddot{s} = \dot{\varphi} \dot{s} , \quad s(t) = \int_0^t dt' e^{\varphi(t')} = s_0 + s_1 \ln \tanh bt , \]  
we find that (24) takes the canonical form

\[ \frac{d^2 \lambda^i}{ds^2} + \Gamma^i_{jk} (\gamma) \frac{d\lambda^j}{ds} \frac{d\lambda^k}{ds} = 0 , \]
implying
\[ \gamma_{jk} \frac{d\lambda^j}{ds} \frac{d\lambda^k}{ds} = \mu^2 = \text{const}. \quad (26) \]

The solution of (25) is subject to a ‘zero energy’ constraint which follows from (19):
\[ C + \varphi^2 - s^2 \mu^2 = C + 4b^2 \coth^2 2bt - \mu^2 \left( \frac{2s_1 b}{\sinh 2bt} \right)^2 = 0. \quad (27) \]

This condition is satisfied if
\[ C = -4b^2, \quad s_1^2 \mu^2 = 1. \]

The dilaton (23) expressed in terms of the ‘proper time’ \( s \) is
\[ \varphi = \varphi_0' + \ln \sinh \mu(s_0 - s). \quad (28) \]

In the case when \( \gamma_{ij} = \text{const} \) we get \( d^2 \lambda^i/ds^2 = 0 \), i.e. (cf. [1][3])
\[ \lambda_i = \lambda_{0i} + \mu p_i s = \lambda_{0i}' + p_i \ln \tanh bt, \quad \gamma_{ij} p^i p^j = 1. \quad (29) \]

We can also express \( \lambda_i \) directly in terms of the dilaton. In a reparametrisation-invariant theory time is an ambiguous notion; we can use the component \( G_{00} \) of the metric or the dilaton itself as a ‘dynamical’ time. For example, in string theory we may set \( x^0 = \tau \) as a world-sheet gauge. Here the situation is similar: the (redefined) dilaton plays the role of time in the ‘third-quantised’ string picture.

3. Examples of ‘rolling moduli’ solutions

Let us now consider less trivial than torus examples of conformal backgrounds for which the rolling moduli ansatz (13) produces a consistent \( D = 4 \) cosmological solution. We shall start with the following (leading-order) \( N = 3 \) string solution corresponding to the coset \( SU(2)/U(1) \times U(1) \) (i.e., the \( SU(2) \) analog of the neutral black string background [18][12])
\[ ds_N^2 = dx^2 + a_1^2 \tan^2 b' x d\theta^2 + a_2^2 d\tilde{\theta}^2, \quad (30) \]
\[ \phi = \phi_0 - \ln \cos b' x , \quad \bar{\beta}^\phi = \frac{N - 26}{6} - \alpha' b'^2 . \]

This is a conformal background for arbitrary values of the constants \( a_i, \phi_0 \). Replacing these constants by functions of time we find that the full set of equations for an \( N + 1 \)-dimensional solution (3)–(9) is satisfied if \( a_i(t), \phi_0(t) \) solve the equations that follow from the action (14) or (17) with \( \gamma_{ij} = \delta_{ij} \) \((i, j = 1, 2)\), i.e.

\[ S = \int dt \, e^{-\varphi} \left( C - \dot{\varphi}^2 + \dot{\lambda}_1^2 + \dot{\lambda}_2^2 \right) , \quad \text{(31)} \]

\[ a_i \equiv e^{\lambda_i} , \quad \varphi(t) \equiv 2\phi_0(t) - \lambda_1(t) - \lambda_2(t) , \quad C = -\frac{2(N - 25)}{3\alpha'} + 4b'^2 . \]

The equations are thus the same as in the case of the flat two-torus so the solution is given by (29),(28)

\[ \varphi(t) = \varphi_0 - \ln \sinh 2bt , \quad 4b'^2 = -C , \quad \text{(32)} \]

\[ \lambda_i = \lambda_{i0} + p_i \ln \tanh b't , \quad p_1^2 + p_2^2 = 1 , \quad \text{(33)} \]

so that

\[ ds_D^2 = -dt^2 + dx^2 + a_{10}^2 \tanh^{2p_1} bt \tan^2 b' x \, d\theta^2 + a_{20}^2 \tanh^{2p_2} bt \, d\bar{\theta}^2 , \quad \text{(34)} \]

\[ \varphi(x, t) = \varphi_0 - \ln \sinh 2bt - \ln \sin 2b' x . \]

We have formally assumed that \( C < 0 \) (otherwise \( b \) is to be rotated to imaginary values) and picked up a solution with decreasing dilaton.

We can now generate other \( D = 4 \) cosmological solutions by considering the \( O(2, 2|R) \) duality rotations (see \[19\] for a review) of (34). The simplest special case of the solution (34) with \( p_1 = 0 \) (i.e. the direct product the \( x \)-dependent and \( t \)-dependent \( D = 2 \) backgrounds) was discussed in \[11\] and its duality rotations – in \[10\] \[20\]. A special duality rotation of (34) with \( p_1 = 0 \) gives the cosmological solution of \[9\] corresponding to the \( SL(2, R) \times SU(2)/R \times R \) gauged WZW model.

\[ \textbf{3} \quad \text{To satisfy the central charge condition } \bar{\beta}^\phi = 0 \text{ without introducing extra degrees of freedom one needs to take } b' \text{ to be imaginary.} \]
It turns out that in some cases the duality transformations in $\theta, \tilde{\theta}$ directions ‘commute’ with the replacement (13) of the moduli by functions of time. In particular, we can obtain the same generalisations of (34) by first applying the duality rotation to the spatial solution (30) and then using the ‘rolling moduli’ ansatz (13) for all the moduli (including the ones corresponding to the parameters of the duality transformation).

For example, we can give the following re-interpretation of the Nappi-Witten (NW) solution [9]: it is the cosmological generalisation of the $SU(2)$ analog of the charged black string background [12] (i.e. $SU(2) \times U(1)/U(1)$ gauged WZW model) with the modulus (charge) $q$ (and the constant mode of dilaton) replaced by a function of $t$. This solution thus explicitly illustrates the situation when a cosmological evolution is equivalent to a motion in the moduli space of a ‘spatial’ conformal field theory (equivalent observations were made in [8]). The corresponding $N = 3$ spatial background is [12][13]

$$ds^2_N = dx^2 + (1 + q) \left( \frac{1 - X}{X + 1 + 2q} d\theta^2 + q \frac{1 + X}{X + 1 + 2q} d\tilde{\theta}^2 \right),$$

$$B_{\theta\tilde{\theta}} = \frac{q(X - 1)}{X + 1 + 2q}, \quad \phi = \phi_0 - \frac{1}{2} \ln(X + 1 + 2q), \quad X \equiv \cos 2b'x. \tag{36}$$

This background solves the leading-order Weyl invariance conditions for an arbitrary value of $q$ (for a discussion of all-order generalisations and their scheme dependence see [21]). It is dual to (30) with

$$a_1 = 1, \quad a_2 = \frac{q}{q + 1} = e^{2\lambda}, \quad \lambda_1 = 0, \quad \lambda_2 \equiv \lambda, \quad q = \frac{e^{2\lambda}}{1 - e^{2\lambda}}. \tag{37}$$

4 The element of $SU(2)$ is parametrised as follows:

$$g = \exp(\frac{i}{2} \theta_L) \exp(\frac{i}{2} x \sigma_1) \exp(\frac{i}{2} \theta_R), \quad \theta_L = \theta + \tilde{\theta}, \quad \theta_R = \tilde{\theta} - \theta.$$

$q$ determines the embedding of $U(1)$ into $SU(2) \times U(1)$. For a generic $q$ the metric (35) has a topology of $S^3$, degenerating at $q = 0, -1$ (or $a_2 = 0, \infty$) to a product of a disc and a point (in the $SL(2,R)$ case the corresponding limits are the $D = 2$ black hole and its dual). The $q = \infty$ limit of (35),(36) corresponds to the $SU(2)$ WZW model.
It appears that the $O(2, 2)$ duality transformation ‘commutes’ with the replacement of the modulus $\lambda$ or $q$ by a function of $t$: starting with (35),(36) and replacing $q \rightarrow q(t)$ gives a consistent $D = 4$ cosmological solution.

The action (11) computed directly on the background (35),(36) with $q \rightarrow q(t)$ is a special case of (14),(31) (here $C$ is the same as in (31))

$$S = \int dt \, e^{-\phi} \left[ C - \dot{\phi}^2 + \frac{\dot{q}^2}{4q(q+1)} \right] = \int dt \, e^{-\phi} \left( C - \dot{\phi}^2 + \dot{\lambda}^2 \right).$$

(38)

The resulting time-dependent solution is thus given by $ds_D^2 = -dt^2 + ds_N^2(q(t))$ with (we choose $C < 0$ and growing solution)

$$\lambda = \lambda_0 + \ln \tanh bt, \quad q(t) = \frac{s_0^2 \tanh^2 bt}{1 - s_0^2 \tanh^2 bt},$$

(39)

$$4b^2 = -C = -\frac{2(N - 25)}{3\alpha'} + 4b'^2, \quad s_0 = e^{\lambda_0}.$$ 

The solution of (38) is reproduced in the case when there are no extra fields that shift the central charge, so that for $N = 3$ we have $C > 0$ and $b$ should be imaginary (i.e. tanh in (39) should be replaced by tg so that the space has finite life-time $0 < t < \pi/2$).

In (39) we have $q(0) = 0$ and $q(\infty) = s_0^2/(1 - s_0^2)$ so depending on $s_0$ the large $t$ limit corresponds to either $S^3$ or ‘dual black hole’ spatial topology. The topology change (see [8] for a detailed discussion) thus takes place only at the limiting points in the moduli space which are separated by an infinite distance [14]. It would be more interesting to find if there exists a cosmological realisation of a smooth topology change taking place inside the moduli space.

Let us now consider generalisations of the solution (39). As in (30) we can formally rescale $\theta$ and $\tilde{\theta}$ in the spatial solution (35),(36) by two arbitrary coefficients, replacing the factors $1 + q$ and $q$ by $(1 + q)a_1^2$ and $qa_2^2$. We can then consider the cosmological ansatz

$$5 \text{ Note the symmetry } q \rightarrow -(q + 1) \text{ or } a_2 \rightarrow a_2^{-1} \text{ which is the analog of the torus duality. Its presence is obvious in view of the relation between (35),(36) and (30).}$$
(13) with the $G_{ab}, B_{ab}$ given by (35),(36) where the coefficients $a_i$ and $q$ are replaced by functions of $t$. Such ansatz turns out to be consistent only if

$$a_1 = a_2^{-1} = a \equiv e^{\lambda'}$$

so that the determinant of the 3-metric and $B_{\theta \tilde{\theta}}$ retain their form unchanged while the metric (35) leads to the following $D = 4$ background

$$ds^2_D = -dt^2 + dx^2$$

$$+ [1 + q(t)]a^2(t) \frac{1 - X}{X + 1 + 2q(t)}d\theta^2 + q(t)a^{-2}(t) \frac{1 + X}{X + 1 + 2q(t)}d\tilde{\theta}^2.$$  \hspace{1cm} (41)

The solution for $q(t)$ and $a(t)$ is the same as in the 2-torus case (see (32),(33)), with the moduli $\lambda_i$ being $\lambda$ and $\lambda'$ (and $\gamma_{ij} = \delta_{ij}$). Computing the action (11),(14) one finds that it is given by (31) where now

$$\lambda = \frac{1}{2} \ln \frac{q}{q + 1} = \lambda_1 + \lambda_2, \quad \lambda' = \ln a = \lambda_1.$$  \hspace{1cm} (42)

The explicit form of the solution is found by replacing $q$ and $a$ in (36),(41) by

$$q = \frac{s_0^2 \tanh^{2(p_1 + p_2)} bt}{1 - s_0^2 \tanh^{2(p_1 + p_2)} bt}, \quad a^2 = a_0^2 \tanh^{2p_1} bt, \quad p_1^2 + p_2^2 = 1.$$  \hspace{1cm} (43)

The NW solution [9] corresponds to the special case of $p_1 = 0, p_2^2 = 1$ and imaginary $b$.

The background (41),(43),(36) can be also obtained by the $O(2,2)$ duality rotation of the solution (34). Namely, if (34) is written as

$$ds^2 = -dt^2 + dx^2 + g_1(x,t)d\theta^2 + g_2(x,t)d\tilde{\theta}^2,$$  \hspace{1cm} (44)

its duality rotation [10] can be put into the form

$$ds^2 = -dt^2 + dx^2 + \frac{g_1}{g_1g_2 + q_0} d\theta^2 + \frac{g_2}{g_1g_2 + q_0} d\tilde{\theta}^2,$$  \hspace{1cm} (45)

$$B_{\theta \tilde{\theta}} = \frac{q_0}{g_1g_2 + q_0}, \quad q_0 = \frac{s_0^2}{1 - s_0^2},$$
equivalent to the solution (41),(36).

We can also take special limits in $q$ in (35),(36) and then use rolling moduli ansatz (13). For example, one can start directly with the $SU(2)$ WZW model background (or $q \to \infty$ limit of (35),(36))

$$ds^2 = dx^2 + \frac{1}{2}(1 - X)d\theta^2 + \frac{1}{2}(1 + X)d\tilde{\theta}^2 , \quad X \equiv \cos 2b'x ,$$

$$B_{\theta\tilde{\theta}} = \frac{1}{2}(X - 1) , \quad \phi = \phi_0 .$$

Rescaling $\theta$ and $\tilde{\theta}$ one can take their ‘radii’ and the dilaton $\phi_0$ as free parameters. We can also add a constant $B_0$ to $B_{\theta\tilde{\theta}}$. Making these constants time-dependent one finds that this ansatz is consistent only if $a_1 = a_2^{-1}$ and $\dot{B}_0 = 0$ so that $\dot{B}_{\theta\tilde{\theta}} = 0$ and at the end we have a one-modulus problem but with the metric in (14) $\gamma = 2$. As a result, the corresponding $D = 4$ cosmological metric is given by

$$ds^2 = -dt^2 + dx^2 + a_0^2(tanh bt)^{\sqrt{2}} \sin^2 b'x \ d\theta^2 + a_0^{-2}(coth bt)^{\sqrt{2}} \cos^2 b'x \ d\tilde{\theta}^2 .$$

This solution can be related to (43) by formally taking the limit $q \to \infty$, $\lambda \to 0$, $\lambda_1 \to \lambda_2$ and $p_1 \to p_2 \to \sqrt{2}$.

Let us now comment on the relation between the solutions (34) (solution ‘$S_1$’), (41),(43) (solution ‘$S_2$’) and (48) (solution ‘$S_3$’). Let $D$ denote a ‘static’ $O(2,2)$ duality transformation and $T$ – the operation of adding a time direction to a conformal background and replacing some moduli by functions of time in a consistent way. Then acting by $D$ and $T$ on the corresponding (gauged) WZW models we have:

$$D(SU(2)) = [SU(2)/U(1)] \times U(1) , \quad D(SU(2)) = [SU(2) \times U(1)]/U(1) ,$$

$$T([SU(2)/U(1)] \times U(1)) = S_1 , \quad T([SU(2) \times U(1)]/U(1)) = S_2 , \quad T(SU(2)) = S_3 .$$

We have seen that $D(S_1) = S_2$, i.e. $D$ and $T$, in fact, commute when applied to $SU(2)/U(1) \times U(1)$ model. Note, however, that one cannot generate $S_1$ or $S_2$ by applying
a duality transformation to $S_3$ (while the first two solutions have two rolling moduli, the third has only one). Thus the commutativity of the duality with the rolling moduli ansatz (13) is not universal.

The generalised time-dependent solutions we have presented above do not have an obvious CFT interpretation: it is only for the NW solution [9] (and its special limits and analytic continuations [11],[8]) that we know that it corresponds to a gauged WZW model. There is also a similar example of $D = 3$ cosmological solution, which, like NW solution is obtained from a coset model and, at the same time, admits a ‘rolling moduli’ interpretation. Consider the 2-dimensional part of the conformal background (30) and rescale $x$ to introduce an additional constant (the solution of course depends only on $b'/a_1$)

$$ds^2_N = a_1^2dx^2 + a_2^2\tan^2b'xd\theta^2,$$

$$\phi = \phi_0 - \ln \cos b'x, \quad \tilde{\beta}^\phi = \frac{N-26}{6} - \alpha' b'^2 a_1^{-2}.$$  

Relaxing the $D = 2$ zero central condition (that fixes $b'/a_1$) we can look for $D = 3$ cosmological solutions by replacing $a_1$ and $a_2$ by functions of $t$. Such a solution exists if $a_1 = a_2^{-1}$ and is given by (cf. (34))

$$ds^2_D = -dt^2 + a_0^2\tanh^2bt \ dx^2 + a_0^{-2}\coth^2bt \ \tan^2b'x \ d\theta^2,$$

$$\varphi(x,t) = \varphi_0 - \ln \sinh 2bt - \ln \sin 2b'x.$$  

This solution is equivalent to the one obtained from the $SO(2,2)/SO(2,1)$ gauged WZW model [22]. This example suggests that one may be able to construct more general $D = 4$ cosmological solutions (with (34) and (41),(43) being special cases) by starting again with (30) or (35) and introducing three time-dependent parameters. Some of these models may have a coset CFT interpretation.

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