A stable self-similar singularity of evaporating drops: ellipsoidal collapse to a point

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Abstract

We study the problem of evaporating drops contracting to a point. Going back to Maxwell and Langmuir, the existence of a spherical solution for which evaporating drops collapse to a point in a self-similar manner is well established in the physical literature. The diameter of the drop follows the so-called $D^2$ law: the second power of the drop-diameter decays linearly in time. In this study we provide a complete mathematical proof of this classical law. We prove that evaporating drops which are initially small perturbations of a sphere collapse to a point and the shape of the drop converges to a self-similar ellipsoid whose center, orientation, and semi-axes are determined by the initial shape.

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1 Introduction

This paper addresses the question on the leading-order mechanism of the dynamics of evaporating drops. This question has attracted a great deal of attention in the physics society. The two situations that have been studied most are the “coffee ring” problem \(^1\) and the evaporation of a completely wetting liquid on a perfectly flat surface \(^2\) and references therein). Our study is motivated by the case where a drop wets completely but evaporation is significant. Many experiments reveal that the radius of such a drop goes to zero in finite time, with a characteristic scaling exponent close to 1/2 (see \(^7\), \(^8\), \(^33\)). The question is thus on the origin of the exponent 1/2.

Maxwell \(^17\) and Langmuir \(^46\) studied evaporation based on the following simplifying hypotheses: the evaporation process is diffusion-controlled, quasi-stationary, and isothermal; the interface of the drop is at local equilibrium (see \(^17\) for an historical account). These assumptions and the spherical symmetry lead to the classical \(^5\) law: the square of the drop-diameter \(D\) decreases linearly in time. In fact, such hypotheses are essentially related to the 1/2 exponent because the \(D^2\) law is exactly the case of the scaling exponent 1/2. Moreover, in many situations the evaporation of drops is well described by these hypotheses (e.g. \(^13\), \(^44\), \(^16\), \(^34\)). The assumptions can therefore be regarded as a leading-order approximation of more complex behavior (see \(^53\), \(^5\), \(^16\), \(^9\), and references therein).

In this study we provide a complete mathematical proof of the \(D^2\) law built on the simplifying hypotheses. The spherical symmetry will not be assumed and it will be verified that evaporating drops initially close to a sphere collapse to a point in finite time. The way to collapse obeys the \(D^2\) law.

We first introduce a precise mathematical formulation of the problem. Consider a drop that occupies an open, bounded, and simply connected domain \(\Omega(t) \subset \mathbb{R}^3\), where \(t\) denotes the time-variable. In the outside of the drop, there is the vapor concentration \(\phi(x, t), x \in \mathbb{R}^3\). The characteristic time of the evaporation process is much larger than the time that characterizes transfer rates across the interface of the drop. As a result, the concentration satisfies Laplace’s equation at each time, i.e. \(\Delta \phi = 0\). Moreover, the concentration reaches local equilibrium at the drop interface, i.e. \(\phi = 1\) on the boundary of \(\Omega(t)\) without loss of generality. Then the concentration far from the drop should be lower than 1, and hence we assume that \(\phi \to 0\) as \(|x| \to \infty\). In the inside of the drop, the liquid satisfies the Stokes equations because the Reynolds number of the drop is sufficiently small. Since the process is isothermal, only the normal stress must be balanced at the interface. The two conservation laws hold for the drop: linear momentum and angular momentum. The liquid-vapor interface will be moved by the conservation of mass at the interface. Here, the evaporative flux is proportional to the concentration gradient, i.e. \(\frac{\partial \phi}{\partial n}\), where \(\frac{\partial}{\partial n}\) is just for the simplicity of calculations and \(n\) is the outward normal of the boundary of \(\Omega(t)\). Finally we are led to the following system:

\[
\begin{align*}
-\nabla p + \Delta u &= 0 \quad \text{in } \Omega(t), \\
\text{div } u &= 0 \quad \text{in } \Omega(t), \\
T[u, p]n &= -\sigma \kappa n \quad \text{on } \Gamma(t), \\
\int_{\Omega(t)} u dV &= 0, \\
\int_{\Omega(t)} (u \times x) dV &= 0, \\
\Delta \phi &= 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega(t)}, \\
\phi &= 1 \quad \text{on } \Gamma(t), \\
\phi &\to 0 \quad \text{as } |x| \to \infty,
\end{align*}
\]

\[
\begin{align*}
v_n &= u \cdot n + \frac{1}{2} \frac{\partial \phi}{\partial n} \quad \text{on } \Gamma(t), \\
\Gamma(0) &= \Gamma^0,
\end{align*}
\]

where \(\Gamma(t)\) denotes the boundary of \(\Omega(t)\), \(\Gamma^0\) is the initial shape of the free boundary, \(T[u, p] = -pI + \nabla u + (\nabla u)^T\) is the stress tensor, \(\sigma\) is a given positive constant, \(\kappa\) is the mean curvature of \(\Gamma(t)\) (\(\kappa\) is positive for a sphere), and \(v_n\) is the velocity of the free boundary in the direction of the outward normal.
From the mathematical point of view, the problem (1.1)-(1.3) belongs to a class of free-boundary problems. Similar mathematical problems arise in various contexts in physics and biology: displacement of oil by water in porous media [3], evolution of uncharged or charged droplets (37, 31, 30, 21), evolution of tumors (3, 20, 22, 23, 24, 25), melting of spherical crystals [30], loop dislocations in crystals [26], Stefan problems (10, 11, 18, 28, 50, 38, 51), and Ostwald ripening [1], just to name a few. However, there is another interesting mathematical aspect in evaporating drops: formation of a finite-time singularity. Evaporating drops lose their mass but they retain constant density (we have assumed that the process is isothermal). As a consequence, the drop vanishes in a finite time and the boundary of the drop might collapse to a point, a collection of points, or a more complicated geometric object.

The system (1.1)-(1.3) has a self-similar solution given by

\[
\begin{cases}
p = \sigma(1 - t)^{-1/2}, \\
u = 0, \\
\phi = |x|^{-1}(1 - t)^{1/2}, \\
\Gamma(t) = \{|x| = (1 - t)^{1/2}\}.
\end{cases}
\]

The solution (1.4) implies that the drop vanishes at the extinction time \( t = 1 \). One can easily check that the only self-similar solutions of (1.1)-(1.3) with a power-law structure are the ones given by (1.4). Moreover, \( |x|^2 = (1 - t) \) means that \( D^2 \) decays linearly in time, i.e. the \( D^2 \) law is exactly satisfied by (1.4). The main aim of this study is therefore to prove the stability of the self-similar solution (1.4).

We now state our main result. We will begin with some notations. In this paper we will formulate the interface of the drop in terms of spherical coordinate systems, for example,

\[\Gamma(t) = \{ r = f(\theta, \varphi, t) \} ,\]

where \((r, \theta, \varphi)\) is a spherical coordinate system with respect to the origin and \( f \) is a real function whose variables \((\theta, \varphi)\) are defined on \( S = [0, \pi] \times [0, 2\pi) \). Then \( H^s(S) \) with \( s \geq 0 \) denote the Sobolev spaces. The spherical harmonics \( Y_{l, m} \) are on \( S \), and \( f = \sum_{l,m} \hat{f}_{l,m} Y_{l,m} \) is the spherical harmonic expansion as defined in Appendix A. Finally the main result reads as follows: the self-similar solution (1.4) is stable under small perturbations in \( H^s(S) \), for some appropriate \( s \geq 0 \). Moreover, the drop shape near extinction is universally an ellipsoid.

**Theorem 1.1** (Stable ellipsoidal collapse to a point). Assume that the initial shape \( \Gamma^0 \) is given by \( \{ r = 1 + e g^0(\theta, \varphi) \} \), where \( g^0 \in H^6(S) \). Then, for a sufficiently small \( \epsilon > 0 \), there exists a unique solution to the free-boundary problem (1.1)-(1.3) such that

\[
\frac{\Gamma(t) - \epsilon \mathbf{x}_0}{R(t)} = \{ r = 1 + e \epsilon^e(\theta, \varphi, t) \}, \quad R(t) = \left[ 1 - \frac{t}{1 + \epsilon \sigma 0(2\sqrt{\pi})^{-1}} \right]^{1/2}
\]

on the time interval \( 0 \leq t < 1 + \epsilon \sigma 0(2\sqrt{\pi})^{-1} \) for some \( (\mathbf{x}_0, t_0) = (\mathbf{x}_0(\epsilon), t_0(\epsilon)) \in \mathbb{R}^3 \times \mathbb{R} \), and we have:

(a) The solution \( \{ r = 1 + e \epsilon^e(\theta, \varphi, t) \} \), as well as the corresponding \( (u, p, \phi) \), of the transformed problem is unique in the function space \( Z^e \) to be defined in Theorem A.1.

(b) The quantity \( (\mathbf{x}_0, t_0) \) satisfies

\[
|\mathbf{x}_0 - \tilde{\mathbf{x}}_0| = O(\epsilon), \quad |t_0 - \tilde{t}_0| = O(\epsilon),
\]

where \( (\tilde{\mathbf{x}}_0, \tilde{t}_0) \in \mathbb{R}^3 \times \mathbb{R} \) merely depends on the \( l \leq 2 \) terms in the spherical harmonic expansion of \( g^0 \).

(c) There exist constants \( C > 0 \) and \( 0 < \lambda_0 < 1 \) such that

\[
\sup_{(\theta, \varphi) \in S} |g^e(\theta, \varphi, t) - g^e(\theta, \varphi)| \leq CR^\lambda(\epsilon) \]

for all \( t \in [0, 1 + \epsilon \sigma 0(2\sqrt{\pi})^{-1}] \),

where \( \{ r = 1 + e \epsilon^e(\theta, \varphi) \} \) is an ellipsoid which satisfies

\[
\sup_{(\theta, \varphi) \in S} |g^e(\theta, \varphi) - g^0_{0, 0} Y_{0, 0} - \sum_{|m| \leq 2} e^{-\sigma |m|^{2\pi} g^0_{2, m} Y_{2, m}}| = O(\epsilon).\]
The proof of Theorem 1.1 will be given at the end of Section 6. In Section 5 we will state the corresponding result to Theorem 1.1 for the reformulated problem on the fixed domain. In Section 2 we will introduce this reformulation.

Theorem 1.1 confirms that evaporating drops with arbitrary shapes close to a sphere also obey the D² law and the asymptotic shape of the drop is generically an ellipsoid. To our knowledge, this is the first result reporting the robustness of the D² law and its ellipsoidal structure near extinction.

Evaporation of macroscopic drops is well described by the simplifying hypotheses for the main part of their lifetimes, as discussed in [9]. At the very end of their lifetimes, we should consider another physics, e.g. changes of the equilibrium concentration at the interface, thermal effects, etc. However, self-similar singularities are intermediate-asymptotic solutions [2]. Hence, it could conceivably be hypothesized that, when macroscopic drops evaporate, the ellipsoidal structure near extinction is a building block in the transitional period between the macroscopic scale and the much smaller scale. The D² law for micrometer-sized or nanometer-sized droplets is discussed in [22] and references therein.

Theorem 1.1 has only examined evaporating drops which are freely suspended. However, in many situations where an evaporating drop is on a solid substrate, a characteristic scaling exponent of the radius of the drop is close to 1/2 (e.g. [7], [8], [18], [19], [33], [5], [16]). It may be the case therefore that the simplifying hypotheses are major factors but intricate conditions effecting such deviation happen, for example, at the contact line. The result of this study, as a basis for more involved analyses, may help us to understand such a phenomenon. Future work is required to establish this.

1.1 Overview of the proof

The key part of proving Theorem 1.1 is to treat the ellipsoidal solution $g^E$ as an operator of certain parameters (i.e. $g^E$ is determined completely by the parameters), and to construct a stable self-similar singularity of (1.1)-(1.3), which approaches the ellipsoidal solution in the rescaled domain.

In Section 2 we introduce self-similar variables ([35], [36]) and the Hanzawa transformation [39], and hence the system (1.1)-(1.3) can be transformed into a fixed domain. Moreover, we decompose the transformed system into the ellipsoidal part $g^E$ and the evolution part $g^S$. By this decomposition, the rescaled solution $g^E$ of (1.1)-(1.3) can be regarded as a perturbation of the ellipsoidal solution.

In Section 3 we state the main theorems in the reformulated framework. Theorem 3.1 provides the existence and uniqueness of the ellipsoidal solution that depends on certain parameters. In fact, the starting point for the decomposition defined in Section 2 is at Theorem 3.1. Theorem 3.2 analyzes the auxiliary linear problem for the evolution part of the decomposition. By Theorem 3.2, we can deduce the parameters that determine the ellipsoidal solution. Moreover, Theorem 3.2 shows that the ellipsoidal solution is approached exponentially in the logarithmic variable $\tau = -\ln R(t)$. Finally, we prove Theorem 3.3 for the nonlinear evolution problem of the decomposition. By reversing the procedure in Section 2 it readily follows that Theorem 3.3 is equivalent to Theorem 1.1.

In Section 4 we solve the system of ordinary differential equations deduced from the reformulated (1.1), in terms of the spherical harmonics and the vector spherical harmonics. By using the results in this section, we can compute the eigenvalues of certain linearized systems on $\mathbb{S}$ for $g^E$ and $g^S$, respectively.

In Section 5 we prove Theorem 5.1. By using the rescaling procedure in Section 2, one can easily check that the evaporative flux $\frac{1}{2} \frac{\partial}{\partial t} n$ overwhelms $\mathbf{u} \cdot \mathbf{n}$ in (1.2) near extinction. Consequently, the eigenvalues deduced from $\frac{1}{2} \frac{\partial}{\partial t} n$ are dominant as $\tau \to \infty$. However, such eigenvalues contain a vanishing eigenvalue corresponding to $Y_{2,m}$. This means that we have to look at the nonlinear behavior to establish the dynamics in the $Y_{2,m}$-direction, which is far from being trivial. We remark that the existence of this vanishing eigenvalue is not connected to a symmetry-breaking bifurcation like those found in free-boundary problems studied recently in connection to drops, tumor growth, etc. (cf. [20], [29], [24] as well as the papers [21], [6], [24], [25] making use of the Crandall-Rabinowitz theorem [12]). Hence, for this problem we have to rely on new ideas and techniques. Specifically we use a property of null quadrature domains, i.e. the boundary of $\Omega$ is an ellipsoid if and only if the gravity induced by the uniform mass on the complement of $\Omega$ is equal to zero in $\Omega$ ([15], [16], [32]).

In Section 6 we prove Theorem 6.2. By the results in Section 4, we have the system of ordinary differential equations for $g^S$. The stability is controlled by the eigenvalues of the linear operator of the system. However, there are two positive eigenvalues corresponding to $Y_{0,0}$ and $Y_{1,m}$, respectively. These positive eigenvalues are related to a small shifting on the position of the singularity in space and time, but do not indicate instability, as noted in [19], [54]. Therefore, we determine the singularity position such that the artifact-instability does not happen. Moreover, the eigenvalue corresponding to $Y_{2,m}$ vanishes. The motivation of the decomposition in Section 2 and Theorem 3.1 is fundamentally based on this vanishing eigenvalue. Here, the ellipsoidal solution is determined by the parameters in
In Appendix A we introduce properties of the spherical harmonics and the vector spherical harmonics, which are necessary for our proofs.

Notation. We will use the following notations throughout the paper: \( H^s(\Omega) \) denotes the Sobolev space of functions in \( \Omega \); the weighted Sobolev space \( H^s(\Omega; w) \) equipped with the norm

\[
\|G\|_{H^s(\Omega; w)}^2 = \sum_{|\alpha| \leq s} \int_{\Omega} |D^\alpha G|^2 w dV,
\]

where

\[
w(x) = \begin{cases} 1/|x|^2 & \text{if } \alpha = 0, \\ 1 & \text{if } \alpha > 0 \end{cases}
\]

holds; \( \overline{a} \) is the complex conjugate of \( a \in \mathbb{C} \).

2 Reformulation of the problem

In this section we transform the free-boundary problem into the fixed domain by using self-similar variables \([35], [36]\) and the Hanzawa transformation \([39]\). Moreover, we introduce a decomposition of the solution into the two components that will be studied separately: the ellipsoidal part and the evolution part.

As shown in \([31]\), it will be convenient to replace \((1.1)-(1.3)\) by the following system:

\[
\begin{align*}
\Delta p &= 0 \quad \text{in } \Omega(t), \\
-\nabla p + \Delta u &= 0 \quad \text{in } \Omega(t), \\
\text{div } u &= 0 \quad \text{on } \Gamma(t), \\
\mathbf{T}[u, p]n &= -\sigma \kappa \mathbf{n} \quad \text{on } \Gamma(t), \\
\int_{\Omega(t)} udV &= 0, \\
\int_{\Omega(t)} (u \times x) dV &= 0, \\
\Delta \phi &= 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}(t), \\
\phi &= 1 \quad \text{on } \Gamma(t), \\
\phi &\to 0 \quad \text{as } |x| \to \infty,
\end{align*}
\]

\( v_n = \mathbf{u} \cdot \mathbf{n} + \frac{1}{2} \frac{\partial \phi}{\partial n} \quad \text{on } \Gamma(t), \quad (2.2) \)

\( \Gamma(0) = \Gamma^0. \quad (2.3) \)

2.1 Reduction to the fixed domain

In this subsection we will use three different notations of variables in terms of spherical coordinate systems. The initial shape \( \Gamma^0 \) is given by \( \{ r' = 1 + \epsilon g^0(\theta', \varphi') \} \), here \( (r', \theta', \varphi') \) is the spherical coordinate system for the shrinking domain. Applying self-similar variables to the shrinking domain, we deduce the spherical coordinate system \( (r^\xi, \theta^\xi, \varphi^\xi) \) for the rescaled domain which still has the free boundary. Finally, by using the Hanzawa transformation, we arrive at the spherical coordinate system \( (r, \theta, \varphi) \) for the fixed domain. We give a simple diagram of these changes of variables:
As a first step, we introduce self-similar variables which reflect the scaling-property of \([21, 23]\) as well as the fact that the position of the singularity in space and time will be perturbed when the perturbed sphere collapses:

\[
\begin{align*}
\rho(x, t) &= (\sigma + \psi P^\xi(\xi, \tau))/R(t), \\
u(x, t) &= \psi U^\xi(\xi, \tau), \\
\phi(x, t) &= R(t)/|x - cx_0| + \epsilon \Phi^\xi(\xi, \tau), \\
(\Gamma(t) - cx_0)/R(t) &= \left\{ r^\xi = 1 + \epsilon g^\xi(\theta^\xi, \varphi^\xi, \tau) \right\},
\end{align*}
\] (2.4)

with

\[
\tau = -\ln R(t) \quad \text{and} \quad \xi = (x - cx_0)/R(t),
\] (2.5)

where

\[
R(t) = \left|1 - t/(1 + ct_0y_{0,0})\right|^{1/2},
\]

As a first step, we introduce self-similar variables which reflect the scaling-property of \([21, 23]\) as well as the fact that the position of the singularity in space and time will be perturbed when the perturbed sphere collapses:

\[
\begin{align*}
\frac{\Delta P^\xi}{\partial x^2} &= 0 \quad \text{in } \Omega^\xi(\tau), \\
-\nabla P^\xi + \Delta U^\xi &= 0 \quad \text{in } \Omega^\xi(\tau), \\
\text{div } U^\xi &= 0 \quad \text{on } \Gamma^\xi(\tau), \\
\mathbf{T}[U^\xi, P^\xi]n^\xi &= -\sigma(\kappa^\xi - 1)\epsilon^{-1}n^\xi \quad \text{on } \Gamma^\xi(\tau), \\
\int_{\Omega^\xi(\tau)} U^\xi dV &= 0, \\
\int_{\Omega^\xi(\tau)} (U^\xi \times n) dV &= 0, \\
\Delta \Phi^\xi &= 0 \quad \text{in } \mathbb{R}^3 \setminus \Omega^\xi(\tau), \\
\Phi^\xi &= g^\xi/(1 + \epsilon g^\xi) \quad \text{on } \Gamma^\xi(\tau), \\
\Phi^\xi &\to 0 \quad \text{as } |\xi| \to \infty,
\end{align*}
\] (2.6)

where \(\Gamma^\xi(\tau) = \{ r^\xi = 1 + \epsilon g^\xi(\theta^\xi, \varphi^\xi, \tau) \}\) holds, \(\kappa^\xi\) is the mean curvature of \(\Gamma^\xi(\tau)\), and \(n^\xi\) is the outward normal of the free boundary \(\Gamma^\xi(\tau)\). The kinematic condition \((2.2)\) and the initial condition \((2.3)\) will be treated at the end of this subsection.

We now transform the free-boundary system \((2.6)\) into the fixed domain by virtue of the Hanzawa transformation. The transformation is

\[
r^\xi = r + \Psi(1 - r)\epsilon g^\xi(\theta^\xi, \varphi^\xi, \tau), \quad (\theta^\xi, \varphi^\xi) = (\theta, \varphi),
\] (2.7)

with a cut-off function \(\Psi(z) \in C_0^\infty\),

\[
\Psi(z) = \begin{cases} 
0, & \text{if } |z| \geq 3\delta_0/4 \\
1, & \text{if } |z| < \delta_0/4
\end{cases}, \quad |\partial^k \Psi| \leq \frac{C}{\delta_0},
\]

where \(\delta_0\) is positive and small. It maps the domain bounded by \(\Gamma^\xi(\tau)\) into the sphere \(B := \{ r < 1 \}\). In terms of the new spherical coordinate system \((r, \theta, \varphi)\), by setting \(u^\xi(r^\xi, \theta^\xi, \varphi^\xi, \tau) = u(r, \theta, \varphi, \tau)\) we are led to

\[
\frac{\partial u^\xi}{\partial r^\xi} \frac{\partial^2 u^\xi}{\partial r^\xi} = \left( \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 r}{\partial r^2} \frac{\partial u}{\partial r} \right)^2 \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial r^2} \frac{\partial u}{\partial r}.
\]

with

\[
\frac{\partial r}{\partial r^\xi} = \frac{1}{1 - \Psi z \epsilon g}, \quad \frac{\partial^2 r}{\partial (r^\xi)^2} = \frac{-\Psi z \epsilon g}{(1 - \Psi z \epsilon g)^3},
\]
\[
\frac{\partial r}{\partial \xi} = \frac{-\Psi \epsilon g_0}{1 - \Psi \epsilon g}, \quad \frac{\partial^2 r}{\partial (\theta \xi)} = \frac{-\Psi \epsilon g_0}{1 - \Psi \epsilon g}, \quad \frac{2\Psi \Sigma (\epsilon g_0)^2}{(1 - \Psi \epsilon g)^2}, \quad \frac{\Psi^2 \phi \epsilon g (\epsilon g_0)^2}{(1 - \Psi \epsilon g)^3},
\]

where \(\Psi = 1 - \Phi/2, \Psi = 2\Phi/3, g_0(\theta, \phi, \tau) = g(\theta, \phi, \tau)\), and we can deduce the formulas for \(\phi\) in a similar fashion. Let \(\mathcal{B}\) denote the set \(\{r > 1\}\). By applying the Hanzawa transformation (2.7) to (2.6), we finally conclude that (2.6) is equivalent to the following system:

\[
\begin{aligned}
\Delta P &= f^1[P, g] \\
-\nabla P + \Delta U &= f^2[U, P, g] \\
\text{div } U &= f^3[U, g] \\
[-P I + \nabla U + (\nabla U)^T]e_r - \sigma(g + \frac{1}{2} \Delta g)e_r &= f^4[U, P, g] \\
\int_\partial \Phi = \int^\Phi[U, g], \\
\int_\partial (U \times (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)) dV &= f^5[U, g], \\
\Delta \Phi &= f^7[\Phi, g], \\
\Phi - g &= f^8[g], \\
\Phi &\to 0
\end{aligned}
\]

in \(\mathcal{B}\), \(\tau > 0\),

in \(\mathcal{S}\), \(\tau > 0\),
on \(\mathcal{S}\), \(\tau > 0\),

(2.8)

where

\[
f^1[P, g] = \left[ 1 - \frac{1}{(1 - \Psi \epsilon g)^2} \right] \frac{\partial^2 P}{\partial r^2} - \frac{-\Psi \epsilon g_0}{(1 - \Psi \epsilon g)^3} \frac{\partial P}{\partial r} + \frac{2}{r} \left( 1 - \frac{r}{r + \Psi \epsilon g} \right) \frac{\partial P}{\partial r} \\
+ \frac{1}{r^2} \left[ 1 - \frac{r^2}{(r + \Psi \epsilon g)^2} \right] \Delta \omega - \frac{1}{(r + \Psi \epsilon g)^2} \left( \frac{\cos \theta}{\sin \theta} \right) \frac{\partial P}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 P}{\partial \phi^2},
\]

(2.9)

\[
A^\theta[P, g] = \frac{-\Psi \epsilon g_0}{1 - \Psi \epsilon g} \frac{2}{(r + \Psi \epsilon g)^2} \frac{\partial^2 P}{\partial r^2} + \left[ \frac{-\Psi \epsilon g_0}{1 - \Psi \epsilon g} + \frac{-2\Psi \Sigma (\epsilon g_0)^2}{(1 - \Psi \epsilon g)^3} + \frac{-\Psi^2 \phi \epsilon g (\epsilon g_0)^2}{(1 - \Psi \epsilon g)^3} \right] \frac{\partial P}{\partial r} \\
+ \left( \frac{-\Psi \epsilon g_0}{1 - \Psi \epsilon g} \right)^2 \frac{\partial^2 P}{\partial r^2},
\]

(2.10)

and we have \(A^\phi[P, g]\) from replacing \(\theta\) by \(\phi\) in (2.10). In addition, we deduce

\[
f^2[U, P, g] = f^1[U, g] + \left( 1 - \frac{1}{1 - \Psi \epsilon g} \right) \frac{\partial P}{\partial r} e_r \\
+ \left[ \frac{1}{r} \left( 1 - \frac{r}{r + \Psi \epsilon g} \right) \frac{\partial P}{\partial \theta} - \frac{1}{r + \Psi \epsilon g} \frac{\Psi \epsilon g_0}{1 - \Psi \epsilon g} \frac{\partial P}{\partial r} \right] e_\theta \\
+ \frac{1}{\sin \theta} \left[ \frac{1}{r} \left( 1 - \frac{r}{r + \Psi \epsilon g} \right) \frac{\partial P}{\partial \phi} - \frac{1}{r + \Psi \epsilon g} \frac{\Psi \epsilon g_0}{1 - \Psi \epsilon g} \frac{\partial P}{\partial r} \right] e_\phi,
\]

(2.11)

\[
f^3[U, g] = 2 \left( 1 - \frac{1}{1 + \epsilon g} \right) (U \cdot e_r) + \left( 1 - \frac{1}{1 - \Psi \epsilon g} \right) \frac{\partial (U \cdot e_r)}{\partial r} \\
+ \left[ \frac{1}{1 + \epsilon g} \left( \frac{\cos \theta}{\sin \theta} \right) (U \cdot e_\theta) + \frac{\partial (U \cdot e_\theta)}{\partial \theta} \right] - \frac{1}{1 + \epsilon g} \frac{\Psi \epsilon g_0}{1 - \Psi \epsilon g} \frac{\partial (U \cdot e_\theta)}{\partial r} \\
+ \left( 1 - \frac{1}{1 + \epsilon g} \right) \frac{1}{\sin \theta} \frac{\partial (U \cdot e_\phi)}{\partial \phi} - \frac{1}{1 + \epsilon g} \frac{1}{\sin \theta} \frac{\Psi \epsilon g_0}{1 - \Psi \epsilon g} \frac{\partial (U \cdot e_\phi)}{\partial r},
\]

(2.12)

\[
f^4[U, P, g] = (-PI + B_1 + B_1^T) (e_r - \mathbf{n}) + (B_2 + B_2^T) e_r + \sigma(1 - \kappa \epsilon) \mathbf{c}^{-1} (\mathbf{n} - e_r) - \sigma f^4[g] e_r,
\]

(2.13)
Therefore, by means of the Hanzawa transformation (2.7), we arrive from (2.13) at the following equation for the kinematic condition (2.2):

\[
B_1 = \frac{1}{1 - \Psi_{\varepsilon g}} \frac{\partial U}{\partial r} e_r + \frac{1}{1 + \varepsilon g} \left( \frac{\partial U}{\partial \theta} e_\theta + \frac{1}{\sin \theta} \frac{\partial U}{\partial \varphi} e_\varphi \right) + \frac{1}{1 + \varepsilon g} \frac{\partial U}{\partial r} \left( -\Psi_{\varepsilon g} \varepsilon_{\theta} + \frac{1}{\sin \theta} \frac{\partial U}{\partial r} \right),
\]

\[
B_2 = \left( 1 - \frac{1}{1 - \Psi_{\varepsilon g}} \right) \frac{\partial U}{\partial r} e_r + \left( 1 - \frac{1}{1 + \varepsilon g} \right) \left( \frac{\partial U}{\partial \theta} e_\theta + \frac{1}{\sin \theta} \frac{\partial U}{\partial \varphi} e_\varphi \right) - \frac{1}{1 + \varepsilon g} \frac{\partial U}{\partial r} \left( -\Psi_{\varepsilon g} \varepsilon_{\theta} + \frac{1}{\sin \theta} \frac{\partial U}{\partial r} \right),
\]

and from Theorem 8.1 in [23], we have

\[
\kappa^\xi = 1 - \varepsilon(g + \frac{1}{2} \Delta \omega g + f^{4;\kappa}[g]), \quad f^{4;\kappa}[g] = O(\varepsilon).
\]

We also arrive at

\[
\mathbf{f}^5[U, g] = \int_{S} U \left[ 1 - (1 + \Psi_{\varepsilon g}/r)^2 (1 - \Psi_{\varepsilon g}) \right] dV,
\]

\[
\mathbf{f}^6[U, g] = \int_{S} U \times (r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta) \left[ 1 - (1 + \Psi_{\varepsilon g}/r)^3 (1 - \Psi_{\varepsilon g}) \right] dV,
\]

\[
\mathbf{f}^7[\Phi, g] = f^1[\Phi, g],
\]

\[
\mathbf{f}^8[g] = \frac{-\varepsilon g^2}{1 + \varepsilon g}.
\]

We emphasize that the order of the right-hand sides of (2.8) is $O(\varepsilon)$, which will be used in our proofs.

We now consider the kinematic condition (2.2). Using the self-similar variables (2.4) and (2.5), we deduce the following equation in terms of the spherical coordinate system $(\xi^\xi, \theta^\xi, \varphi^\xi)$:

\[
v_n = \frac{d}{dt} \left[ R(t) (1 + \varepsilon g^\xi) e_{(\tau^\xi)} \right] \cdot n^\xi = \frac{1}{2(1 + \varepsilon g^\xi)} e_{(\tau^\xi)} \cdot \left( -1 + \frac{\partial \varepsilon g^\xi}{\partial \tau} - \varepsilon g^\xi \right) e_{(\tau^\xi)} \cdot n^\xi \quad \text{on} \Gamma^\xi(\tau). \tag{2.11}
\]

In addition, one can easily see that

\[
n^\xi = \frac{1 + \varepsilon g^\xi}{\varepsilon^\xi} e_{(\tau^\xi)} + \frac{-\varepsilon_{\theta(\xi)}^\xi}{\varepsilon^\xi} e_{(\theta^\xi)} + \frac{-\varepsilon_{\varphi(\xi)}^\xi}{\varepsilon^\xi \sin \theta^\xi} e_{(\varphi^\xi)} \quad \text{with} \quad \varepsilon^\xi = \left[ (1 + \varepsilon g^\xi)^2 + (\varepsilon_{\theta(\xi)}^\xi)^2 + (\varepsilon_{\varphi(\xi)}^\xi / \sin \theta^\xi)^2 \right]^{1/2} \quad \text{holds.}
\]

Substituting (2.12) into (2.11) we can rewrite (2.2) in the form

\[
\frac{\partial \varepsilon}{\partial \tau} - 3g + t_0 Y_{0,0} - 2e^{-\tau} (U \cdot e_r) - \frac{\partial \phi}{\partial r} = e^{-\tau} f^9[U, g, t_0] + f^{10}[\Phi, g, t_0] \quad \text{on} \ S, \quad \tau > 0. \tag{2.14}
\]
where
\[ f^9[U, g, t_0] = 2(1 + \epsilon t_0 Y_{0,0}) \left( -U \cdot e_\theta \frac{\epsilon g_\theta}{1 + \epsilon g} - U \cdot e_\varphi \frac{\epsilon g_\varphi}{1 + \epsilon g \sin \theta} \right) + 2\epsilon t_0 Y_{0,0} U \cdot e_r, \]
\[ f^{10}[\Phi, g, t_0] = (1 + \epsilon t_0 Y_{0,0}) \left[ \frac{3 \epsilon g^2 + 2x^2g^3}{(1 + \epsilon g)^2} - \frac{1}{1 + \epsilon g} \left( \frac{\partial \Phi}{\partial \theta} + \frac{\partial \Phi}{\partial \varphi} \right) + \epsilon g_\theta \frac{1}{1 + \epsilon g \sin \theta} \right] + \epsilon t_0 Y_{0,0} \frac{\partial \Phi}{\partial r} + 2\epsilon t_0 Y_{0,0} g. \]

The order of \( f^9[U, g, t_0] \) and \( f^{10}[\Phi, g, t_0] \) is \( O(\epsilon) \).

We finally consider the initial condition \( 2.3 \). Assume that the initial shape
\[ \Gamma^0 = (1 + \epsilon g^0(\theta', \varphi')) e_{(r')} \]
is a perturbation of the self-similar solution \( 1.4 \), where \((r', \theta', \varphi')\) is the spherical coordinate system for the three-dimensional \( x \)-space and \( g^0(\theta', \varphi') \) is given. Then, by the self-similar variables \( 2.4 \) and \( 2.5 \), we are led to the following relation:
\[ (1 + \epsilon g^0(\theta', \varphi')) e_{(r')} = (1 + \epsilon g^\xi(\theta', \varphi', 0)) e_{(r')} + \epsilon x_0. \]

Writing \( \epsilon x_0 = \epsilon(x_{01}, x_{02}, x_{03}) \), we deduce from \( 2.14 \) that
\[ \begin{aligned}
(1 + \epsilon g^0(\theta', \varphi')) e_{(r')} &= (1 + \epsilon g^0(\theta', \varphi')) e_{(r')} - \epsilon x_0, \\
(1 + \epsilon g^\xi(\theta', \varphi', 0)) e_{(r')} &= (1 + \epsilon g^0(\theta', \varphi')) e_{(r')} - \epsilon x_0, \\
(1 + \epsilon g^\xi(\theta', \varphi', 0)) e_{(r')} &= (1 + \epsilon g^0(\theta', \varphi')) e_{(r')} - \epsilon x_0, \\
(1 + \epsilon g^\xi(\theta', \varphi', 0)) e_{(r')} &= (1 + \epsilon g^0(\theta', \varphi')) e_{(r')} - \epsilon x_0.
\end{aligned} \]

so that one can write
\[ (1 + \epsilon g^\xi(\theta', \varphi', 0))^2 = f^{\text{angle}}(\theta', \varphi') \]
with
\[ f^{\text{angle}}(\theta', \varphi') = (1 + \epsilon g^0(\theta', \varphi'))^2 + \epsilon^2|x_0|^2 - 2\epsilon (1 + \epsilon g^0(\theta', \varphi')) (x_{01} \sin \theta' \cos \varphi' + x_{02} \sin \theta' \sin \varphi' + x_{03} \cos \theta'). \]

Accordingly, it follows from \( 2.16 \) that \( \theta^\xi \) and \( \varphi^\xi \) are functions of \((\theta', \varphi')\):
\[ \begin{aligned}
\theta^\xi &= \cos^{-1} \left[ \frac{(1 + \epsilon g^0(\theta', \varphi')) \cos \theta' - \epsilon x_{03}}{\sqrt{f^{\text{angle}}(\theta', \varphi')}} \right] = \theta' + O(\epsilon), \\
\varphi^\xi &= \cos^{-1} \left[ \frac{(1 + \epsilon g^0(\theta', \varphi')) \sin \theta' \cos \varphi' - \epsilon x_{01}}{\sin \theta' \sqrt{f^{\text{angle}}(\theta', \varphi')}} \right] = \varphi' + O(\epsilon).
\end{aligned} \]

As a result, since there is no change of angles in the Hanzawa transformation \( 2.7 \) and \( g^\xi(\theta^\xi, \varphi^\xi, 0) = g(\theta, \varphi, 0) \) holds, we rewrite \( g(\theta, \varphi, 0) \) in the form
\[ g(\theta, \varphi, 0) = g^0(\theta, \varphi) + (x_{01} \sin \theta \cos \varphi + x_{02} \sin \theta \sin \varphi + x_{03} \cos \theta) = f^{11}[x_0], \]
where the order of \( f^{11}[x_0] \) is \( O(\epsilon) \). Moreover, by using
\[ Y_{1,1} = \sqrt{\frac{3}{8\pi}} \sin \theta e^{-i\varphi}, \quad Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos \theta, \quad Y_{1,-1} = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\varphi}, \]
the initial condition \( 2.3 \) is reduced to
\[ g(\theta, \varphi, 0) - g^0(\theta, \varphi) + \sum_{|m| \leq 1} b_{m} Y_{1,m} = f^{11}[x_0], \quad (2.17) \]
where
\[ \begin{pmatrix}
    x_{01} \\
    x_{02} \\
    x_{03}
\end{pmatrix} = \sqrt{\frac{3}{8\pi}}\begin{pmatrix}
    1 & 0 & -1 \\
    -i & 0 & -i \\
    0 & \sqrt{2} & 0
\end{pmatrix} \begin{pmatrix}
    b_{1,-1} \\
    b_{1,0} \\
    b_{1,1}
\end{pmatrix}. \quad (2.18) \]
2.2 The decomposition of the solution

We decompose the fixed-domain problem (2.13), (2.14), (2.17) into the ellipsoidal part and the evolution part. We will use the super-indexes $E$ and $T$ to denote the unknowns for such ellipsoidal and evolution problems, respectively. Accordingly, the decomposition is

$$(P, U, \Phi, g) = (P^E, U^E, \Phi^E, g^E) + (P^T, U^T, \Phi^T, g^T). \tag{2.19}$$

We define (2.19) more precisely. The ellipsoidal part satisfies the following overdetermined problem:

$$\begin{align*}
\Delta P^E &= f^1[P^E, g^E] \quad \text{in } \mathbb{B}, \\
-\nabla P^E + \Delta U^E &= f^2[U^E, P^E, g^E] \quad \text{in } \mathbb{B}, \\
\text{div } U^E &= f^3[U^E, g^E] \quad \text{on } \mathbb{S}, \\
[-P^E I + \nabla U^E + (\nabla U^E)^T]e_r - \sigma(g^E + \frac{1}{2}\Delta_\theta g^E)e_r &= f^4[U^E, P^E, g^E] \quad \text{on } \mathbb{S}, \\
\int_\mathbb{B} U^E dV &= f^5[U^E, g^E], \\
\int_\mathbb{B} (U^E \times (r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta)) dV &= f^6[U^E, g^E], \\
\Delta \Phi^E &= f^7[\Phi^E, g^E] \quad \text{in } \text{ext } \mathbb{B}, \\
\Phi^E - g^E &= f^8[g^E] \quad \text{on } \mathbb{S}, \\
\Phi^E &\to 0 \quad \text{as } r \to \infty, \tag{2.20}
\end{align*}$$

where $t_0$ is a part of the solution to (2.20), (2.22). On the other hand, the evolution part satisfies the following evolution problem:

$$\begin{align*}
\Delta P^T &= f^1[P, g] - f^1[P^E, g^E] \quad \text{in } \mathbb{B}, \quad \tau > 0, \\
-\nabla P^T + \Delta U^T &= f^2[U, P, g] - f^2[U^E, P^E, g^E] \quad \text{in } \mathbb{B}, \quad \tau > 0, \\
\text{div } U^T &= f^3[U, g] - f^3[U^E, g^E] \quad \text{on } \mathbb{S}, \quad \tau > 0, \\
[-P^T I + \nabla U^T + (\nabla U^T)^T]e_r - \sigma(g^T + \frac{1}{2}\Delta_\theta g^T)e_r &= f^4[U, P, g] - f^4[U^E, P^E, g^E] \quad \text{on } \mathbb{S}, \quad \tau > 0, \\
\int_\mathbb{B} U^T dV &= f^5[U, g] - f^5[U^E, g^E], \\
\int_\mathbb{B} (U^T \times (r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta)) dV &= f^6[U, g^T], \\
\Delta \Phi^T &= f^7[\Phi, g] - f^7[\Phi^E, g^E] \quad \text{in } \text{ext } \mathbb{B}, \quad \tau > 0, \\
\Phi^T - g^T &= f^8[g] - f^8[g^E] \quad \text{on } \mathbb{S}, \quad \tau > 0, \\
\Phi^T &\to 0 \quad \text{as } r \to \infty, \tag{2.23}
\end{align*}$$

$$\frac{\partial g^T}{\partial \tau} - 3g^T - 2e^{-\tau}(U \cdot e_r) - \frac{\partial \Phi^T}{\partial \tau} = e^{-\tau} f^9[U, g, t_0] + f^{10}[\Phi, g, t_0] - f^{10}[\Phi^E, g^E, t_0] \quad \text{on } \mathbb{S}, \quad \tau > 0, \tag{2.24}$$

$$g^T(\theta, \varphi, 0) - g^0 + \sum_{|m| \leq 1} b_{1,m} Y_{1,m} + g^E = f^{11}[x_0]. \tag{2.25}$$

Here, the solution to (2.20), (2.22) enters (2.23) and (2.24) so that the right-hand sides of (2.23) and (2.24) can be regarded as a perturbation of the ellipsoidal part. Moreover, $U^E$ and $g^E$ appear linearly on the left-hand sides of (2.24) and (2.25). This special structure of the system (2.20)-(2.25) will allow us to compute approximations to the eigenvalues of a certain linearized system.

3 The main theorems for the reformulated problem

The aim of this section is to state the main theorems for the reformulated problem (2.20)-(2.25). Using these main theorems and reversing the procedure in Section 2, we can readily prove Theorem 1.1.

Our first main theorem is the following:
Theorem 3.1 (Existence of ellipsoids). The set \( \{ r = 1 + \epsilon g^e \} \) of the overdetermined problem (2.20)-(2.22) must be an ellipsoid. Moreover, assume that \( g_{0,0}^e \) and \( \{ g_{2,m}^e \}_{m \leq 2} \) of \( g^e = \sum_{l,m} g_{l,m}^e Y_{l,m} \) are given complex constants such that
\[
g_{0,0}^e, g_{2,0}^e \in \mathbb{R}, \quad g_{2,-1}^e = -\overline{g_{2,1}^e} \in \mathbb{C}, \quad g_{2,-2}^e = \overline{g_{2,-2}^e} \in \mathbb{C},
\]
so that \( g^e \) is real. Then for a sufficiently small \( \epsilon > 0 \), (2.20)-(2.22) has a unique solution \((P^e, U^e, \Phi^e, g^e, t_0)\) such that
\[
\{ r = 1 + \epsilon g^e \} = \{ r = 1 + \epsilon g_{0,0}^e Y_{0,0} + \epsilon \sum_{|m| \leq 2} g_{2,m}^e Y_{2,m} + O(\epsilon^2) \}
\]
is an ellipsoid whose semi-axes are of lengths \( a, b, \) and \( c \) and \( t_0 \) satisfies
\[
1 + ct_0 Y_{0,0} = \frac{abc}{2} \int_0^\infty \frac{ds}{\sqrt{(a^2 + s)(b^2 + s)(c^2 + s)}} \quad (3.1)
\]
The following holds for any integer \( n \geq 4 \):
\[
\| P^e \|_{H_n^{-2}(B)} + \| U^e \|_{H_n^{-1}(B)} + \| \Phi^e \|_{H_n^{(\text{ext } B; w)}} + \| g^e \|_{H_n^{-2}(S)} + |t_0| \leq C(\| g_{0,0}^e \| + \sum_{|m| \leq 2} |g_{2,m}^e|) \quad (3.2)
\]
where \( C \) is some constant.

Proof. We will prove Theorem 3.1 in Section 5.

By Theorem 3.1, it immediately follows that the leading-order terms of the solution to (2.20)-(2.22) consist of \( g_{0,0}^e \) and \( \{ g_{2,m}^e \}_{m \leq 2} \). As a result, we can rewrite (2.14) and (2.15) in the form
\[
\frac{\partial g^e}{\partial \tau} - 3g^e - \frac{\partial \Phi^e}{\partial r} - 2e^{-\tau}(U^e \cdot e_r - \sigma_{10} \sum_{|m| \leq 2} g_{2,m}^e Y_{2,m}) = e^{-\tau} f_0[U, g, t_0] + f_{10}[\Phi, g, t_0] - f_{10}[\Phi^e, g^e, t_0] \quad (3.3)
\]
\[
+ 2e^{-\tau}(U^e \cdot e_r + \sigma_{10} \sum_{|m| \leq 2} g_{2,m}^e Y_{2,m}) + O(\epsilon)
\]
\[
g^e(\theta, \varphi, 0) - g^0 + \sum_{|m| \leq 1} b_{l,m} Y_{l,m} + g_{0,0}^e Y_{0,0} + \sum_{|m| \leq 2} g_{2,m}^e Y_{2,m} = f_{11}[x_0] - g^e + g_{0,0}^e Y_{0,0} + \sum_{|m| \leq 2} g_{2,m}^e Y_{2,m} + O(\epsilon) \quad (3.4)
\]
We have introduced the term \( \sigma_{10} \sum_{|m| \leq 2} g_{2,m}^e Y_{2,m} \) in (3.3) because it corresponds to the linearization of \( U^e \cdot e_r \), which will be apparent in Section 4. Consequently, the order of the right-hand side of (3.3) is \( O(\epsilon) \). Likewise, the order of the right-hand side of (3.4) is \( O(\epsilon) \).

We introduce the function spaces \( \mathcal{K} \) and \( \mathcal{Y} \). Define the following norms (see (2.24)):
\[
\| G \|_{s, B} = \left[ \int_0^\infty e^{2\lambda_0 \tau} \left( \| G(\cdot, \tau) \|_{H_s(B)} + \| G(\cdot, \tau) \|_{H_{s-1}(B)} + \| G(\cdot, \tau) \|_{H_2(B)} \right) d\tau \right]^{1/2},
\]
\[
\| G \|_{s, B} = \left[ \int_0^\infty e^{2\lambda_0 \tau} \left( \| G(\cdot, \tau) \|_{H_s(B)} + \| G(\cdot, \tau) \|_{H_{s-1}(B)} + \| G(\cdot, \tau) \|_{H_2(B)} \right) d\tau \right]^{1/2},
\]
\[
\| g \|_{s, B} = \left[ \int_0^\infty e^{2\lambda_0 \tau} \left( \| g(\cdot, \tau) \|_{H_s(S)} + \| g(\cdot, \tau) \|_{H_{s-1}(S)} + \| g(\cdot, \tau) \|_{H_2(S)} \right) d\tau \right]^{1/2},
\]
for \( s \geq 2 \). The number \( \lambda_0 \) is positive and will be chosen appropriately. We can define \( \| G \|_{s, \text{ext } B; w} \) in a similar way, but introducing the weighted Sobolev space \( H_s(\text{ext } B; w) \). Finally, we denote by \( \mathcal{X} \) the Banach space as the set of vectors \( x \) defined by
\[
x = \left( P^e, U^e, \Phi^e, g^e, g_{0,0}^e \in \mathbb{R}, \{ b_{l,n} \in \mathbb{C} \}_{|l| \leq 1}, \{ g_{2,m}^e \in \mathbb{C} \}_{|m| \leq 2} \right)
\]

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such that
\[ \|x\|_X = \|P^\xi\|_5,3 + \|U^T\|_6,3 + \|\Phi^T\|_{7,\text{ext}3,w} + \|g^\xi\|_{5,3} + |g_{6,0}^\xi| + \sum_{|n| \leq 1} |b_{1,n}| + \sum_{|m| \leq 2} |b_{2,m}^\xi| < \infty. \]

Moreover, Let \( \mathcal{Y} \) be the Banach space of elements
\[ y = (F^1;\xi, F^2;\xi, F^3;\xi, F^4;\xi, F^5;\xi, F^6;\xi, F^7;\xi, F^8;\xi, F^{9,10};\xi, F^{11};\xi) \]
such that
\[
\|y\|_\mathcal{Y} = \|F^1;\xi\|_{3,3} + \|F^2;\xi\|_{4,3} + \|F^3;\xi\|_{5,3} + \|F^4;\xi\|_{5,3} + \|F^5;\xi\|_{5,3} + \|F^6;\xi\|_{5,3} + \|F^7;\xi\|_{5,3} + \|F^8;\xi\|_{5,3} + \|F^9;\xi\|_{5,3} + \|F^{10};\xi\|_{5,3} + \|F^{11};\xi\|_{5,3} < \infty.
\]

We now introduce the linear operator \( \mathcal{L} \) defined by the left-hand sides of the system (2.23), (3.3), (3.4) and the nonlinear operator \( \mathcal{Y} \) defined by the right-hand sides of (2.23), (3.3), (3.4). By Theorem 3.1 \( \mathcal{L} \) and \( \mathcal{Y} \) can be regarded as operators defined on the Banach space \( \mathcal{X} \). In addition, \( \mathcal{Y}[x] \) is a perturbation of the ellipsoidal solution that comes from Theorem 3.1. By using these facts, one can easily check that \( \mathcal{Y}: \mathcal{X} \to \mathcal{Y} \). Therefore the nonlinear problem (2.23), (3.3), (3.4) can be compactly written in the form
\[ \mathcal{L}[x] = \mathcal{Y}[x], \quad \mathcal{L}: \mathcal{X} \to \mathcal{Y}, \quad \mathcal{Y}: \mathcal{X} \to \mathcal{Y}. \]

Consider first the auxiliary linear evolution problem
\[ \mathcal{L}[x] = y, \quad y \in \mathcal{Y}. \tag{3.5} \]

Then the following theorem holds for (3.5):

**Theorem 3.2** (The linear evolution problem). Assume that \( y \in \mathcal{Y} \) and \( g^0 \in H^6(S) \). Then there exists a unique solution \( x \in \mathcal{X} \) to (3.5), which satisfies the following estimate:
\[ \|x\|_X \leq C(\|g^0\|_{H^6(S)} + \|y\|_\mathcal{Y}), \tag{3.6} \]

for some constant \( C \).

**Proof.** It will be given in Section 6.

By Theorem 3.2 the operator \( \mathcal{L} \) is invertible, so that we can define the operator \( \mathcal{A} \):
\[ \mathcal{A}[x] := \mathcal{L}^{-1}[\mathcal{Y}[x]], \quad \mathcal{A}: \mathcal{X} \to \mathcal{X}. \tag{3.7} \]

Our last main theorem verifies that the perturbed solution (2.23) in the \( \xi \)-space converges to the ellipsoidal solution that is deduced from Theorem 3.1.

**Theorem 3.3** (The nonlinear evolution problem). If \( g^0 \in H^6(S) \), then for a sufficiently small \( \epsilon > 0 \), the nonlinear problem
\[ x = \mathcal{A}[x], \tag{3.8} \]

where \( \mathcal{A} \) is defined by (3.7), has a unique solution \( x \in \mathcal{X} \).

**Proof.** The proof is based on the contraction mapping principle. One can easily check that
\[ \|\mathcal{Y}[x]\|_\mathcal{Y} \leq C_0 \epsilon \|x\|_X \tag{3.9} \]
for any \( x \in \mathcal{X} \) such that \( \|x\|_X \leq 2C\|g^0\|_{H^6(S)} \), where \( C \) is the constant in (3.6) and \( C_0 \) is some appropriate constant. Therefore, using (3.6), we deduce
\[ \|\mathcal{A}[x]\|_X \leq C(\|g^0\|_{H^6(S)} + C_0 \epsilon \|x\|_X) \]
In this section we solve the following auxiliary problem:

4 The system of ordinary differential equations

so that \( A \) maps the ball of radius \( R = 2C\|g\|_{H^1(\mathcal{S})} \) into itself for a sufficiently small \( \epsilon > 0 \). We now verify that \( A \) is a strict contraction. By Theorem 3.2, for any \( \tilde{x}_1 \) and \( \tilde{x}_2 \) in \( \mathcal{X} \), there exist unique solutions \( x_1 \) and \( x_2 \) in \( \mathcal{X} \) such that \( \mathcal{L}[x_1] = \mathcal{Y}[\tilde{x}_1] \) and \( \mathcal{L}[x_2] = \mathcal{Y}[\tilde{x}_2] \), respectively. As a result, for \( \tilde{x}_1 \) and \( \tilde{x}_2 \) in the ball of radius \( R \), we conclude

\[
\|x_1 - x_2\|_{\mathcal{X}} \leq C\|\mathcal{Y}[\tilde{x}_1] - \mathcal{Y}[\tilde{x}_2]\|_{\mathcal{Y}} \leq C_0\epsilon\|\tilde{x}_1 - \tilde{x}_2\|_{\mathcal{X}}
\]

and hence \( A \) is a strict contraction for a sufficiently small \( \epsilon > 0 \). Finally, by the contraction mapping principle, (3.8) has a unique solution in the Banach space \( \mathcal{X} \).

In fact, the proof of the inequalities (3.9) and (3.10) is quite lengthy. However, one can show the inequalities in exactly the same way as Section 4 of [4], whose proof requires the following multiplicative property of the norms of the spaces \( \mathcal{X} \) and \( \mathcal{Y} \):

\[
\|fg\| \leq C_1\|f\|\|g\|
\]

for some appropriate constant \( C_1 \) (see [28]).

4 The system of ordinary differential equations

In this section we solve the following auxiliary problem:

\[
\Delta P = F^1 \quad \text{in } \mathbb{B}, \quad \text{(4.1)}
\]
\[
- \nabla P + \Delta U = F^2 \quad \text{in } \mathbb{B}, \quad \text{(4.2)}
\]
\[
\text{div } U = F^3 \quad \text{on } \mathbb{S}, \quad \text{(4.3)}
\]
\[
[-PI + \nabla U + (\nabla U)^T]e_r - \sigma(g + \frac{1}{2}\Delta \omega g)e_r = F^4 \quad \text{on } \mathbb{S}, \quad \text{(4.4)}
\]
\[
\int_{\mathbb{B}} U dV = F^5, \quad \text{(4.5)}
\]
\[
\int_{\mathbb{S}} (U \times (r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta)) dV = F^6, \quad \text{(4.6)}
\]
\[
\Delta \Phi = F^7 \quad \text{in ext } \mathbb{B}, \quad \text{(4.7)}
\]
\[
\Phi - g = F^8 \quad \text{on } \mathbb{S}, \quad \text{(4.8)}
\]
\[
\Phi \to 0 \quad \text{as } r \to \infty. \quad \text{(4.9)}
\]

The spherical harmonics form a complete set in a suitably weighted \( L^2 \) space. Likewise, the vector spherical harmonics form an orthonormal basis of a suitably weighted \( L^2 \) space. Thus we write the solution to (4.1)-(4.9) as the following expansions in terms of the spherical harmonics and the vector spherical harmonics:

\[
\begin{align*}
U & = \sum_{l,m} \left( U_{l,m}^V \mathbf{V}_{l,m} + U_{l,m}^X \mathbf{X}_{l,m} + U_{l,m}^W \mathbf{W}_{l,m} \right), \\
P & = \sum_{l,m} P_{l,m} Y_{l,m}, \\
\Phi & = \sum_{l,m} \Phi_{l,m} Y_{l,m}, \\
g & = \sum_{l,m} g_{l,m} Y_{l,m}.
\end{align*}
\]

The exact formulas and fundamental properties of these eigenfunctions are given in Appendix A. Analogously, the non-homogeneous terms in (4.1)-(4.3) have the following expansions:

\[
\begin{align*}
F^1 & = \sum_{l,m} F_{l,m}^1 Y_{l,m}, \\
F^2 & = \sum_{l,m} \left( F_{l,m}^{2,V} \mathbf{V}_{l,m} + F_{l,m}^{2,X} \mathbf{X}_{l,m} + F_{l,m}^{2,W} \mathbf{W}_{l,m} \right), \\
F^3 & = \sum_{l,m} F_{l,m}^{3,v} Y_{l,m}, \\
F^4 & = \sum_{l,m} \left( F_{l,m}^{4,V} \mathbf{V}_{l,m} + F_{l,m}^{4,X} \mathbf{X}_{l,m} + F_{l,m}^{4,W} \mathbf{W}_{l,m} \right), \\
F^7 & = \sum_{l,m} F_{l,m}^7 Y_{l,m}, \\
F^8 & = \sum_{l,m} F_{l,m}^8 Y_{l,m}.
\end{align*}
\]
Moreover, we write
\[
\begin{align*}
\mathbf{F}^5 &= F_1^5 \mathbf{e}_1 + F_2^5 \mathbf{e}_2 + F_3^5 \mathbf{e}_3, \\
\mathbf{F}^6 &= F_1^6 \mathbf{e}_1 + F_2^6 \mathbf{e}_2 + F_3^6 \mathbf{e}_3,
\end{align*}
\]
where \( \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \) is the standard basis for \( \mathbb{R}^3 \). We construct the explicit forms \( U_{l,m}^V, U_{l,m}^X, U_{l,m}^W, P_{l,m}, \Phi_{l,m} \) in terms of \( g_{l,m} \) and the coefficients of the expansions in (4.11), (4.12). As a consequence, we deduce the following expansions on \( r = 1 \), which are in (2.22), (2.24), and (3.3):
\[
\mathbf{U} \cdot \mathbf{e}_r = \sum_{l,m} (\mathbf{U} \cdot \mathbf{e}_r)_{l,m} Y_{l,m},
\]
\[
\frac{\partial \Phi}{\partial r} = \sum_{l,m} \left( \frac{\partial \Phi}{\partial r} \right)_{l,m} Y_{l,m}.
\]

4.1 The derivation of \( (\mathbf{U} \cdot \mathbf{e}_r)_{l,m} \) on \( r = 1 \)

By (A.5), the equation (4.1) becomes
\[
L_l P_{l,m} = F_{l,m}^l, \quad 0 < r < 1. \tag{4.13}
\]
Furthermore, by (A.5)–(A.9), the equation (4.2) becomes
\[
\begin{align*}
L_{r+1} U_{l,m}^V - \left( \frac{r+1}{2l+1} \right)^{1/2} \left( - \frac{\partial}{\partial r} + \frac{1}{r} \right) P_{l,m} &= F_{l,m}^{2V}, \quad r < 1, \\
L_{r} U_{l,m}^X &= F_{l,m}^{2X}, \quad r < 1, \\
L_{r-1} U_{l,m}^W - \left( \frac{r-1}{2l+1} \right)^{1/2} \left( \frac{\partial}{\partial r} + \frac{1}{r-1} \right) P_{l,m} &= F_{l,m}^{2W}, \quad r < 1.
\end{align*}
\]

The general solution to (4.13) and (4.14) is
\[
\begin{align*}
P_{l,m}(r) &= P_{l,m}^1 r^l + H_{l,m}^P, \\
U_{l,m}^V(r) &= V_{l,m}^1 r^{l+1} + H_{l,m}^V, \\
U_{l,m}^X(r) &= X_{l,m}^1 r^l + H_{l,m}^X, \\
U_{l,m}^W(r) &= W_{l,m}^1 r^{l-1} + \frac{1}{2} \left( \frac{r}{2l+1} \right)^{1/2} P_{l,m}^1 r^{l+1} + H_{l,m}^W,
\end{align*}
\]
where the solution to the non-homogeneous part is as follows:
\[
\begin{align*}
H_{l,m}^P &= -r^l \int_r^1 \frac{s^{l+1}}{2l+1} F_{l,m}^1 ds - r^{l-1} \int_0^r \frac{s^{l+2}}{2l+1} F_{l,m}^2 ds, \\
H_{l,m}^V &= -r^{l+1} \int_r^1 \frac{s^{l}}{2l+3} F_{l,m}^2 ds - r^{l-2} \int_0^r \frac{s^{l+3}}{2l+3} F_{l,m}^2 ds \\
&\quad + \left( \frac{l+1}{2l+1} \right)^{1/2} \frac{1}{2l+3} \frac{1}{2l+1} \left( r^{-l} - r^{-l+1} \right) \int_0^r s^{l+2} F_{l,m}^1 ds \\
&\quad + \frac{1}{2l+1} r^{l+1} \int_r^1 \left( s^{l+1} - s^{l+2} \right) F_{l,m}^1 ds + r^{-l-2} \int_0^r s^{l+2} \frac{r^2 - s^2}{2} F_{l,m}^1 ds, \\
H_{l,m}^X &= -r^l \int_r^1 \frac{s^{l+1}}{2l+1} F_{l,m}^2 ds - r^{l-1} \int_0^r \frac{s^{l+2}}{2l+1} F_{l,m}^2 ds, \\
H_{l,m}^W &= -r^{l-1} \int_r^1 \frac{s^{l+2}}{2l-1} F_{l,m}^2 ds - r^{l-1} \int_0^r \frac{s^{l+1}}{2l-1} F_{l,m}^2 ds \\
&\quad + \left( \frac{l}{2l+1} \right)^{1/2} \frac{1}{2l-1} \left( r^{l-1} \int_r^1 s^{l+1} s^2 - r^2 \right) F_{l,m}^1 ds \\
&\quad + \frac{r^{-l}}{2l+1} \left( \int_0^r s^{l+2} F_{l,m}^1 ds + \int_0^r s^{-l+1} r^{2l+1} F_{l,m}^1 ds \right),
\end{align*}
\]

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We deduce (4.3)-(4.6) in terms of the spherical harmonics expansions. By (A.10)-(A.12), the boundary condition (4.3) reduces to

\[ -\left( \frac{l + 1}{2l + 1} \right)^{1/2} \left( \frac{\partial}{\partial r} + l + 2 \right) U_{l,m}^{V} + \left( \frac{l}{2l + 1} \right)^{1/2} \left( \frac{\partial}{\partial r} - l + 1 \right) U_{l,m}^{W} = F_{l,m}^{4V}, \quad r = 1. \]  

(4.16)

A direct computation from Lemma 4.1 in [30] shows that (4.4) becomes

\[
\begin{align*}
\left( \frac{3l + 2}{2l + 1} \frac{\partial}{\partial r} - \frac{l^2 + 2l}{2l + 1} \right) U_{l,m}^{V} & - \frac{l^{1/2}(l + 1)^{1/2}}{2l + 1} \left( \frac{\partial}{\partial r} - l + 1 \right) U_{l,m}^{W} + \left( \frac{l + 1}{2l + 1} \right)^{1/2} P_{l,m} \\
+ \sigma \left( \frac{l + 1}{2l + 1} \right)^{1/2} \left( 1 - \frac{l^2 + 1}{2} \right) g_{l,m} & = F_{l,m}^{4V}, \quad r = 1, \\
\left( \frac{\partial}{\partial r} - 1 \right) U_{l,m}^{X} & = F_{l,m}^{4V}, \quad r = 1, \\
\left( \frac{1^{1/2}(l + 1)^{1/2}}{2l + 1} \left( \frac{\partial}{\partial r} + l + 2 \right) U_{l,m}^{V} + \left( \frac{3l + 1}{2l + 1} \frac{\partial}{\partial r} + \frac{l^2 - 1}{2l + 1} \right) U_{l,m}^{W} - \left( \frac{l}{2l + 1} \right)^{1/2} P_{l,m} \\
- \sigma \left( \frac{l + 1}{2l + 1} \right)^{1/2} \left( 1 - \frac{l^2 + 1}{2} \right) g_{l,m} & = F_{l,m}^{4W}, \quad r = 1. \\
\end{align*}
\]

(4.17)

(4.18)

(4.19)

Finally, by using Lemma 8.1 in [30] with its proof, the constraints (4.5) and (4.6) reduce to

\[
\sum_{m,j} \frac{1}{\sqrt{3}} \left( U_{l,m}^{W}(1) - \int_{0}^{1} x_j \frac{\partial U_{l,m}^{W}}{\partial r} \right) \left( \int_{S} x_j Y_{l,m} dS \right) e_j = \sum_{j} F_{j}^{5} e_j, \\
\sum_{m,j} -i\sqrt{2} \left( \int_{0}^{1} r^{3} U_{l,m}^{X} \frac{\partial}{\partial r} \right) \left( \int_{S} x_j Y_{l,m} dS \right) e_j = \sum_{j} F_{j}^{6} e_j. 
\]

(4.20)

(4.21)

Substituting (4.15) into the boundary conditions (4.10)-(4.13), we are led to the following linear system of equations for $P_{l,m}^{1}, V_{l,m}^{1}, X_{l,m}^{1}, W_{l,m}^{1}$ and $g_{l,m}$:

\[
\begin{align*}
-\left( \frac{l + 1}{2l + 1} \right)^{1/2} (2l + 3) V_{l,m}^{1} + \left( \frac{l}{2l + 1} \right) P_{l,m}^{1} & = J_{l,m}^{div}, \\
\frac{2l^2 + 3l + 2}{2l + 1} V_{l,m}^{1} + \left( \frac{l + 1}{2l + 1} \right)^{1/2} P_{l,m}^{1} + \sigma \left( \frac{l + 1}{2l + 1} \right)^{1/2} \left( 1 - \frac{l^2 + 1}{2} \right) g_{l,m} & = J_{l,m}^{\text{nabla V}}, \\
\left( l - 1 \right) X_{l,m}^{1} = J_{l,m}^{\text{nabla X}}, \\
-\left( \frac{l^{1/2}(l + 1)^{1/2}}{2l + 1} (2l + 3) V_{l,m}^{1} + 2(l - 1) W_{l,m}^{1} + \frac{2l^2 - 1}{2l + 1} \left( \frac{l}{2l + 1} \right)^{1/2} P_{l,m}^{1} - \sigma \left( \frac{l}{2l + 1} \right)^{1/2} \left( 1 - \frac{l^2 + 1}{2} \right) g_{l,m} & = J_{l,m}^{\text{nabla W}}. \\
\end{align*}
\]

(4.22)

(4.23)

(4.24)

(4.25)

The right-hand sides of (4.22)-(4.25) consist of $H_{l,m}^{P}, H_{l,m}^{V}, H_{l,m}^{X}, H_{l,m}^{W}, F_{l,m}^{3V}, F_{l,m}^{4V}, F_{l,m}^{4X}$ and $F_{l,m}^{4W}$. In the case $l = 1$ we will use (4.20) and (4.21) instead of (4.24) and (4.25), because $X_{l,m}^{1}$ and $W_{l,m}^{1}$ do not appear in (4.18) and (4.19). In the case $l = 0$ the relation $X_{0,0} = W_{0,0} = 0$ holds so that $U_{0,0} = U_{0,0}^{W} = 0$. Consider first the case $l \geq 2$. Solving (4.22)-(4.25), we conclude

\[
\begin{align*}
P_{l,m}^{1} & = \sigma \left( \frac{2l^4 + 7l^3 + 4l^2 - 7l - 6}{4l^2 + 8l + 6} \right) g_{l,m} + \frac{2l^2 + 3l + 2}{2l^2 + 4l + 3} J_{l,m}^{div} + \left( \frac{l + 1}{2l + 1} \right)^{1/2} \frac{4l^2 + 8l + 3}{2l^2 + 4l + 3} J_{l,m}^{\text{nabla V}}, \\
V_{l,m}^{1} & = \sigma \left( \frac{l + 1}{2l + 1} \right)^{1/2} \frac{3l^3 + l^2 - 2l}{4l^2 + 8l + 6} g_{l,m} - \left( \frac{l + 1}{2l + 1} \right)^{1/2} \frac{l + 1}{2l^2 + 4l + 3} J_{l,m}^{\text{nabla W}}. 
\end{align*}
\]

(4.26)

(4.27)
\[ X_{l,m}^1 = \frac{1}{l-1} j_{lsb}^l X, \quad (4.28) \]

\[ W_{l,m}^1 = -\sigma \left( \frac{l}{2l+1} \right)^{1/2} \frac{2l^4 + 9l^3 + 12l^2 + 4l}{8l^2 + 16l + 12} g_{l,m} \]

\[- \left( \frac{l}{2l+1} \right)^{1/2} \frac{2l^4 + 6l^2 + 6l + 2}{8l^2 + 8l - 4l - 12} j_{div}^{l,m} - \frac{1/2(1+l)^{1/2}(2l + 3)}{4l^2 + 8l + 6} j_{lsb}^{l,m} + \frac{1}{2l - 2} j_{lsb}^{l,m} W. \quad (4.29)\]

In the case \( l = 1 \), from (4.22) and (4.23) we have

\[ P_{1,m}^1 = \frac{7}{9} j_{div}^{1,m} + \frac{5}{3} \sqrt{\frac{2}{3}} j_{lsb}^{1,m}, \quad (4.30) \]

\[ V_{1,m}^1 = -\frac{2}{9} \sqrt{\frac{2}{3}} j_{div}^{1,m} + \frac{1}{9} j_{lsb}^{1,m}. \quad (4.31) \]

Substituting \( U_{1,m}^X \) and \( U_{1,m}^W \) in (4.15) into the constraints (4.20) and (4.21) we deduce the following linear system of equations for \( X_{1,m}^1 \) and \( W_{1,m}^1 \):

\[ \sqrt{2 \pi} \left[ \frac{1}{\sqrt{3}} W_{1,-1}^1 + \frac{1}{10} P_{1,-1}^1 - \frac{1}{\sqrt{3}} W_{1,1}^1 - \frac{1}{10} P_{1,1}^1 \right] = J_1^{mmt}, \quad (4.32) \]

\[ -i \sqrt{2 \pi} \left[ \frac{1}{\sqrt{3}} W_{1,-1}^1 + \frac{1}{10} P_{1,-1}^1 + \frac{1}{\sqrt{3}} W_{1,1}^1 + \frac{1}{10} P_{1,1}^1 \right] = J_2^{mmt}, \quad (4.33) \]

\[ 2 \sqrt{\frac{\pi}{3}} \left[ \frac{1}{\sqrt{3}} W_{1,0}^1 + \frac{1}{10} P_{1,0}^1 \right] = J_3^{mmt}, \quad (4.34) \]

\[ -i \frac{2}{5} \sqrt{\frac{\pi}{3}} (X_{1,-1}^1 - X_{1,1}^1) = J_4^{ang.mmt}, \quad (4.35) \]

\[ -i \frac{2}{5} \sqrt{\frac{\pi}{3}} (X_{1,-1}^1 + X_{1,1}^1) = J_5^{ang.mmt}, \quad (4.36) \]

\[ -i \frac{2}{5} \sqrt{\frac{2 \pi}{3}} X_{1,0}^1 = J_6^{mmt}. \quad (4.37) \]

where \( \{P_{1,m}^1\}_{|m| \leq 1} \) are given by (4.30) and the right-hand sides of (4.32)-(4.37) consist of \( H_{1,m}^X, H_{1,m}^W, F_5, \) and \( F_6 \). From (4.32)-(4.37) we obtain

\[ X_{1,-1}^1 = \frac{5}{4} \sqrt{\frac{3}{\pi}} j_1^{ang.mmt} - \frac{5}{4} \sqrt{\frac{3}{\pi}} j_2^{ang.mmt}, \quad (4.38) \]

\[ X_{1,0}^1 = \frac{5}{2} \sqrt{\frac{3}{2 \pi}} j_3^{ang.mmt}, \quad (4.39) \]

\[ X_{1,1}^1 = -\frac{5}{4} \sqrt{\frac{3}{\pi}} j_1^{ang.mmt} - \frac{5}{4} \sqrt{\frac{3}{\pi}} j_2^{ang.mmt}, \quad (4.40) \]

\[ W_{1,-1}^1 = - \frac{7}{30 \sqrt{3}} j_{div}^{1,-1} - \frac{1}{3 \sqrt{2}} j_{lsb}^{1,-1} + \frac{3}{2 \sqrt{2 \pi}} j_{mmt}^{1,-1} + i \frac{3}{2 \sqrt{2 \pi}} j_{mmt}^{1,-1}, \quad (4.41) \]

\[ W_{1,0}^1 = - \frac{7}{30 \sqrt{3}} j_{div}^{1,0} - \frac{1}{3 \sqrt{2}} j_{lsb}^{1,0} + \frac{3}{2 \sqrt{2 \pi}} j_{mmt}^{1,0} + i \frac{3}{2 \sqrt{2 \pi}} j_{mmt}^{1,0}. \quad (4.42) \]

\[ W_{1,1}^1 = - \frac{7}{30 \sqrt{3}} j_{div}^{1,1} - \frac{1}{3 \sqrt{2}} j_{lsb}^{1,1} - \frac{3}{2 \sqrt{2 \pi}} j_{mmt}^{1,1} + i \frac{3}{2 \sqrt{2 \pi}} j_{mmt}^{1,1}. \quad (4.43) \]

In the case \( l = 0 \) we already know that

\[ X_{0,0}^1 = W_{0,0}^1 = 0, \quad (4.44) \]
because $U^X_{0,0} = U^W_{0,0} = 0$. Moreover, solving (4.22) and (4.23) we have
\begin{align}
P^1_{0,0} &= -\sigma g_{0,0} + \frac{2}{3} f^{\text{div}}_{0,0} + J^{\text{ns,bV}}_{0,0}, \tag{4.45} \\
V^1_{0,0} &= -\frac{1}{3} J^{\text{div}}_{0,0}. \tag{4.46}
\end{align}

By (A.3)-(A.4), it is easily seen that
\begin{align}
(U \cdot e_r)_{l,m} &= - \left( \frac{l+1}{2l+1} \right)^{1/2} \left( \frac{l}{2l+1} \right)^{1/2} \left[ W^1_{l,m} + \frac{1}{2} \left( \frac{l}{2l+1} \right)^{1/2} P^1_{l,m} \right] + J^U_{l,m} e_r, \tag{4.47}
\end{align}

Substituting $U^V_{l,m}$ and $U^W_{l,m}$ in (4.45) into (4.47) on $r = 1$ we arrive at
\begin{align}
(U \cdot e_r)_{l,m} &= - \frac{l+2}{4l^3 + 4l^2 - 2l - 6} J^{\text{div}}_{l,m} - \left( \frac{l+1}{2l+1} \right)^{1/2} \left( \frac{l}{2l+1} \right)^{1/2} \left[ W^1_{l,m} + \frac{1}{2} \left( \frac{l}{2l+1} \right)^{1/2} P^1_{l,m} \right] + J^U_{l,m} e_r, \tag{4.48}
\end{align}

where $J^U_{l,m}$ consist of $H^V_{l,m}$ and $H^W_{l,m}$. Finally, substituting (4.20), (4.27), (4.29)-(4.31), (4.41)-(4.46) into (4.48) we conclude the following lemma:

**Lemma 4.1.** The vector $U = \sum_{l,m} \left( U^V_{l,m} V_{l,m} + U^X_{l,m} X_{l,m} + U^W_{l,m} W_{l,m} \right)$ in the solution of (4.1)-(4.6), evaluated at $r = 1$, satisfies the following relations:

\begin{align}
(U \cdot e_r)_{l,m} &= -\sigma l \frac{2l^2 + 5l + 2}{8l^2 + 16l + 12} J^{\text{div}}_{l,m} - \frac{l+2}{4l^3 + 4l^2 - 2l - 6} J^{\text{div}}_{l,m} - \left( \frac{l+1}{2l+1} \right)^{1/2} \left( \frac{l}{2l+1} \right)^{1/2} \left[ W^1_{l,m} + \frac{1}{2} \left( \frac{l}{2l+1} \right)^{1/2} P^1_{l,m} \right] + J^U_{l,m} e_r, \tag{4.49}
\end{align}

4.2 The derivation of \( \frac{\partial \phi}{\partial r} \) \( l,m \) on \( r = 1 \)

By (A.5), the equation (4.7) becomes
\begin{align}
L_l \Phi_{l,m} &= F^7_{l,m}, \quad r > 1. \tag{4.49}
\end{align}

In addition, (4.8) and (4.9) reduce to
\begin{align}
\Phi_{l,m} - g_{l,m} &= F^8_{l,m}, \quad r = 1, \tag{4.50} \\
\Phi_{l,m} &\to 0, \quad \text{as } r \to \infty. \tag{4.51}
\end{align}

We compute directly the solution to (4.49)-(4.51), and hence
\begin{align}
\Phi_{l,m}(r) &= \frac{1}{r^{l+1}} g_{l,m} + \frac{1}{r^{l+1}} F^8_{l,m} + H^\phi_{l,m}, \tag{4.52}
\end{align}

where
\begin{align}
H^\phi_{l,m} &= -\frac{1}{r^{l+1}} \int_1^r \rho^{2l} \int_0^\infty \frac{1}{s^{l-1}} F^7_{l,m} ds d\rho. \tag{4.53}
\end{align}

In conclusion one can easily show the following lemma:
Lemma 4.2. The solution \( \Phi = \sum_{l,m} \Phi_{l,m} Y_{l,m} \) to (4.7)-(4.9) satisfies the following relation at \( r = 1 \):

\[
\left( \frac{\partial \Phi}{\partial r} \right)_{l,m} = -(l+1)g_{l,m} - (l+1)F_{l,m}^8 - \int_1^\infty \frac{1}{s^{l-1}} F_{l,m}^7 ds.
\] (4.53)

5 Existence of ellipsoids

In this section we prove Theorem 5.1. By Lemma 4.2 one can easily verify that (2.22) reduces to the following nonlinear system of equations on \( r = 1 \) for \( \Phi_{l,m}^e, g_{l,m}^e, \) and \( t_0^e \):

\[
(l-2)g_{l,m}^e = (f^{10}[\Phi^e, g^e, t_0])_{l,m} + K_{l,m}^e, \quad l \geq 1,
\] (5.1)

\[-2g_{0,0}^e + t_0^e = (f^{10}[\Phi^e, g^e, t_0])_{0,0} + K_{0,0}^e,
\]

where \( K_{l,m}^e \) are given in terms of \( (f^7[\Phi^e, g^e]_{l,m} \text{ and } (f^8[g^e])_{l,m}} \). Main emphasis is on the case \( l = 2 \) for (5.1):

\[
0 = (f^{10}[\Phi^e, g^e, t_0])_{2,m} + K_{2,m}^e.
\]

Indeed, it is by no means a trivial task to prove Theorem 5.1 because of the complexity coming from the nonlinearity of \( f^7[\Phi^e, g^e], f^8[g^e], \) and \( f^{10}[\Phi^e, g^e, t_0] \). In order to resolve this drawback, we use a new approach based on the particular structure of (2.20)-(2.22).

5.1 Null quadrature domains

We first introduce null quadrature domains and the theorem without its proof on a characterization of null quadrature domains.

Definition 5.1. An open set \( \Xi \subset \mathbb{R}^n \) is called a null quadrature domain if

\[
\int_\Xi udV = 0
\]

for all harmonic and integrable functions \( u \) in \( \Xi \).

Theorem 5.2. Assume that \( \Xi \) has a bounded complement. Then \( \Xi \) is a null quadrature domain if and only if the boundary of \( \Xi \) is an ellipsoid.

We are now ready to prove the following lemma:

Lemma 5.3. If the overdetermined problem (2.20)-(2.22) has a classical solution, then \( \{ r = 1 + \epsilon g^e \} \) is an ellipsoid.

Proof. From (2.22), \( g^e \) is determined only by the solution to (2.21). We can verify the well-posedness of (2.20) in a similar fashion as in [31] and [30] when \( g^e \) is given. Therefore it is sufficient to consider (2.21) and (2.22). In the same way as in Section 2 we easily deduce that (2.21)-(2.22) is equivalent to the following system:

\[
\begin{cases}
\Delta \phi = 0, & r > R^+(t)(1 + \epsilon g^e), \\
\phi = 0, & r = R^+(t)(1 + \epsilon g^e), \\
\phi \to -1 & \text{as } r \to \infty,
\end{cases}
\] (5.2)

\[
v_n = -\frac{1}{2} \frac{\partial \phi}{\partial n}, & r = R^+(t)(1 + \epsilon g^e),
\]

\[
\phi = R^+(t)/r - 1 + \epsilon \Phi^e,
\] (5.4)

where \( R^+(t) = [1 + t/(1 + ct_0 Y_{0,0})]^{1/2} \). Let \( u \) be any harmonic and integrable function in \( \Xi(t) := \{ r > R^+(t)(1 + \epsilon g^e) \} \). It is well known that if \( f \) is a harmonic function in a neighborhood of infinity in \( \mathbb{R}^3 \), bounded by \( O(1/r) \), then there exists a constant \( c \) such that

\[
f = c + O\left(\frac{1}{r^2}\right), \quad \nabla f = \nabla\left(\frac{c}{r}\right) + O\left(\frac{1}{r^2}\right) \quad \text{as} \quad r \to \infty
\]
(see [14]). By using this property and Green’s formula, if (5.2)-(5.4) has a classical solution, we have
\[
d\int_{\Sigma(t)} udV = \int_{r=R(t)(1+\epsilon g^E)} u\omega_3 dS = \frac{1}{2} \int_{r=R(t)(1+\epsilon g^E)} \frac{\partial g}{\partial n} dS
\]
\[
= \frac{1}{2} \int_{r=M} \left( u \frac{\partial \phi}{\partial n} - \phi \frac{\partial u}{\partial n} \right) dV \to 0 \quad \text{as} \quad M \to \infty
\]
(as in [52, 14, 32, 13]). It immediately follows that for \(0 < t < T\),
\[
\int_{\Sigma(t)} udV = \int_{\Sigma(T)} udV \to 0 \quad \text{as} \quad T \to \infty,
\]
and hence \(\Sigma(t)\) is a null quadrature domain. Therefore, by Theorem 5.2, \(\{r = 1 + \epsilon g^E, 0\} \) is an ellipsoid.

5.2 The bijective representation of ellipsoids near the unit sphere

We show the following lemma which will be used to prove Theorem 3.1:

**Lemma 5.4.** For a sufficiently small \(\epsilon > 0\), there exists a unique ellipsoid \(\{r = 1 + \epsilon g^E, 0\}\) which satisfies the assumption on \(g^E\) in Theorem 3.1.

**Proof.** An arbitrarily oriented ellipsoid near the unit sphere, centered at the origin, is defined by the equation
\[
Y = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1, \quad (5.5)
\]
where \(\alpha_{ij} \in \mathbb{R}\) and the 3-by-3 matrix is positive definite. In the spherical coordinate system, (5.5) becomes
\[
r(\theta, \varphi) = 1 - \epsilon F[\{\alpha_{ij}\}] + G[\{\alpha_{ij}\}],
\]
where
\[
F[\{\alpha_{ij}\}] = \alpha_{33} \cos^2 \theta + \alpha_{11} \cos^2 \varphi \sin^2 \theta + \alpha_{13} \cos \varphi \sin \theta \sin^2 \varphi + \alpha_{23} \sin^\theta \sin \varphi,
\]
and
\[
G[\{\alpha_{ij}\}] = 1/(1 + \epsilon F[\{\alpha_{ij}\}])^{1/2} - 1 - \epsilon F[\{\alpha_{ij}\}]/2.
\]
By using (A.1), we easily deduce that \(F[\{\alpha_{ij}\}]\) is the linear combination of \(Y_{0,0}\) and \(Y_{2,2}\), so that
\[
r(\theta, \varphi) = 1 + \epsilon \sqrt{\frac{\pi}{30}}(-10(\alpha_{11} + \alpha_{22} + \alpha_{33})Y_{0,0} + \sqrt{30}(-\alpha_{11} + \alpha_{22} - 2i\alpha_{12})Y_{2,-2} - 2\sqrt{30}(\alpha_{13} + i\alpha_{23})Y_{2,-1}
\]
\[
+ 2\sqrt{5}(\alpha_{11} + \alpha_{22} - 2\alpha_{33})Y_{2,0} + 2\sqrt{30}(\alpha_{13} - \alpha_{23})Y_{2,1} + \sqrt{30}(-\alpha_{11} + \alpha_{22} + 2i\alpha_{12})Y_{2,2}) + G[\{\alpha_{ij}\}].
\]
The coefficients of \(Y_{0,0}\) and \(Y_{2,2}\) in (5.6) correspond to \(1 + \epsilon g^E_{0,0}\) and \(\{\epsilon g^E_{2,2}\}_{m \leq 2}\), respectively. Hence, from (5.6) we define the following system:
\[
f(\{\alpha_{ij}\}, \epsilon) := A = \begin{pmatrix} G_{0,0} \epsilon^{-1} \\ (G_{2,-2} + G_{2,2}) \epsilon^{-1} \\ (G_{2,-1} - G_{2,1}) \epsilon^{-1} \\ (G_{2,-2} - G_{2,2}) \epsilon^{-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]
\[
= 0,
\]
where \(G[\{\alpha_{ij}\}]\epsilon^{-1} = \sum_{l,m} G_{l,m} \epsilon^{-1} Y_{l,m} = O(\epsilon)\) and
\[
A = \sqrt{\frac{\pi}{30}} \begin{pmatrix} -10 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]
\[
\begin{pmatrix} 0 & -2\sqrt{30} & 0 & 0 \\ 2\sqrt{5} & 0 & 4\sqrt{30} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

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is an invertible matrix. It is readily seen that the Jacobian matrix of $f$ with respect to \{\alpha_{ij}\} is $A$ at $\epsilon = 0$. Therefore, by the implicit function theorem, the system (5.7) has a unique solution \{\alpha_{ij}\} for a sufficiently small $\epsilon > 0$. Moreover,

$$
\begin{pmatrix}
\alpha_{11} \\
\alpha_{22} \\
\alpha_{33} \\
\alpha_{12} \\
\alpha_{13} \\
\alpha_{23}
\end{pmatrix} = A^{-1} \begin{pmatrix}
g_{0,0}^e \\
g_{0,-2}^e + g_{2,2}^e \\
g_{2,-1}^e - g_{2,1}^e \\
g_{0,2}^e \\
(i(g_{2,-1}^e + g_{2,1}^e)) \\
(i(g_{2,-2}^e - g_{2,2}^e))
\end{pmatrix} + O(\epsilon).
$$

$\square$

### 5.3 The proof of Theorem 3.1

The system (5.2)-(5.4) is closely related to the so-called conductor problem for the ellipsoid, whose solution can be found in [45].

**Proposition 5.5.** [45] Let $\partial \Omega$ be an ellipsoid centered at the origin, which has the semi-axes are of lengths $a$, $b$, and $c$:

$$
\partial \Omega = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}.
$$

Then there exists a unique solution $\phi$ to the following overdetermined problem:

$$
\begin{align*}
\Delta \phi &= 0 & \text{in the exterior of } \partial \Omega, \\
\phi &= 1 & \text{on } \partial \Omega, \\
\phi &\to 0 & \text{as } r \to \infty, \\
\frac{\partial \phi}{\partial n} &= -\frac{E}{abc} d_0(x, y, z) & \text{on } \partial \Omega,
\end{align*}
$$

(5.8)

where $d_0(x, y, z)$ is the distance from the center of the ellipsoid to the plane tangent to $\partial \Omega$ at $(x, y, z)$ such that

$$
d_0(x, y, z) = \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right)^{-\frac{1}{2}},
$$

and $E$ is the capacity of the ellipsoid, which satisfies

$$
1 = \frac{E}{2} \int_0^\infty \frac{ds}{\sqrt{(a^2 + s)(b^2 + s)(c^2 + s)}}.
$$

(5.9)

We now give the proof of Theorem 3.1.

**Proof of Theorem 3.1.** By Lemma 5.3 the solution to (2.20)-(2.22) should be such that \{\epsilon + 1 + \epsilon g^e\} is an ellipsoid. By Lemma 5.4 we can construct a unique ellipsoid \{\epsilon + 1 + \epsilon g^e\} for given $g_{0,0}^e$ and $\{g_{2,m}^e\}_{|m| \leq 2}$. Therefore the remaining part is to show the well-posedness of (2.20)-(2.22) with the given ellipsoid \{\epsilon + 1 + \epsilon g^e\}. Using the proof of Lemma 5.3 we conclude that (2.21)-(2.22) is equivalent to the following system:

$$
\begin{align*}
\Delta \phi &= 0, & r > R(t)(1 + \epsilon g^e), \\
\phi &= 1, & r = R(t)(1 + \epsilon g^e), \\
\phi &\to 0 & \text{as } r \to \infty, \\
v_n = \frac{1}{2} \frac{\partial \phi}{\partial n}, & r = R(t)(1 + \epsilon g^e),
\end{align*}
$$

(5.10)

$$
\phi = R(t)/r + \epsilon \Phi^e,
$$

(5.11)
where $R(t) = [1 - t/(1 + ctY_{0,0})]^{1/2}$. Moreover, by the scaling-property, (5.10) reduces to

$$\begin{align*}
&\Delta \phi = 0, \quad r > 1 + eg^e, \\
&\phi = 1, \quad r = 1 + eg^e, \\
&\phi \to 0 \quad \text{as} \quad r \to \infty. 
\end{align*}$$

(5.12)

We now deduce a detailed calculation for (5.11). By the symmetry of (5.12) under rotation and translation, it is sufficient to consider $\{r = 1 + eg^e\}$ as a basic ellipsoid $\{x^2 + y^2 + z^2 = 1\}$. Let $v = (x, y, z) \in \mathbb{R}^3$ be a vector field satisfying $x^2 + y^2 + z^2 - 1 = 0$. Then (5.11) becomes

$$v_n = \left( \frac{d}{dt} R(t)v \right) \cdot n = \frac{1}{2} \frac{\partial \phi}{\partial n}, \quad r = R(t)(1 + eg^e).$$

(5.13)

By the scaling-property of (5.13), we are led to the following relation holding on $\{r = 1 + eg^e\}$:

$$\frac{\partial \phi}{\partial n} - \frac{dR^2}{dt} v \cdot n = -\frac{1}{1 + \varepsilon_0 Y_{0,0}} v \cdot n = -\frac{1}{1 + \varepsilon_0 Y_{0,0}} \nabla \left( x^2 + y^2 + z^2 - 1 \right) = -\frac{\varepsilon_0(x, y, z)}{1 + \varepsilon_0 Y_{0,0}}$$

(5.14)

If we set

$$t_0 = \frac{1}{\varepsilon_0 Y_{0,0}} \left( \frac{abc}{E} - 1 \right),$$

(5.15)

(5.14) is equivalent to (5.3), and hence by Proposition 5.3, the problem (2.21)-(2.22) has a unique solution $(\Phi^e, t_0)$. Here, we can show the well-posedness of (2.20) in a similar way as in [31] and [30]. Therefore the over-determined problem (2.20)-(2.22) has a unique solution $(\Phi^e, \Phi^e, g^e, t_0)$. Moreover, by (4.26)-(4.31), (4.38)-(4.46), (4.52), (5.6), and (5.15), one can see that the leading-order terms of $(P^e, \Phi^e, g^e, t_0)$ consists of $g_0^e$ and $\{g_{2,m}^e\}_{|m| \leq 2}$. Using this fact crucially, following a similar approach as in [31], [30], and using standard elliptic estimates, we immediately deduce that (3.2) holds.

6 The linear evolution problem

In this section we prove Theorem 5.2. By Lemmas 5.4 and 5.5, we conclude that (5.3) and (5.4) for the auxiliary linear problem (5.5) reduce to the following system of ordinary differential equations on $r = 1$ for $g_{l,m}^e$:

$$\begin{align*}
(g_{l,m}^e)_\tau - (l - 2) g_{l,m}^e + 2e^{-r}\sigma B(l)g_{l,m}^e &= 2e^{-r}J_{l,m}^e + K_{l,m}^e + F^{10,10}_{l,m}, \quad l \neq 2, \\
(g_{2,m}^e)_\tau + 2e^{-r}\sigma B(2)g_{2,m}^e - 20 \frac{g_{2,m}^e}{19} &= 2e^{-r}J_{2,m}^e + K_{2,m}^e + F^{10,10}_{2,m}, \\
B(l) &= \begin{cases} 
2l^3 + 5l^2 + 2l / (8l^2 + 16l + 12) & \text{if} \quad l \geq 2, \\
0 & \text{if} \quad l = 0, 1.
\end{cases}
\end{align*}$$

(6.1)

(6.2)

with the initial condition

$$\begin{align*}
g_{l,m}^e(\tau = 0) &= g_{l,m}^0 = F_{l,m}^{11,11}, \quad l > 2, \\
g_{2,m}^e(\tau = 0) &= g_{2,m}^0 + g_{2,m}^e = F_{2,m}^{11,11}, \quad |m| \leq 2, \\
g_{1,m}^e(\tau = 0) &= g_{1,m}^0 + b_{1,m} = F_{1,m}^{11,11}, \quad |m| \leq 1, \\
g_{0,0}^e(\tau = 0) &= g_{0,0}^0 + g_{0,0}^e = F_{0,0}^{11,11},
\end{align*}$$

(6.3)

(6.4)

(6.5)

(6.6)

where $J_{l,m}^e$ are given in terms of $H_{l,m}^{11,11}, F_{l,m}^{11,11}, F_{l,m}^{11,11}, F_{l,m}^{11,11}, F_{l,m}^{11,11}, F_{l,m}^{11,11}, F_{l,m}^{11,11}, F_{l,m}^{11,11}, F_{j}^{5,5},$ and $F_{j}^{6,6}$ and $K_{l,m}^{11,11}$ consists of $F_{l,m}^{11,11}$ and $F_{l,m}^{8,8}$. We will determine the unknowns $g_{0,0}^e, \{g_{1,m}^e\}_{|m| \leq 1}$, and $\{g_{2,m}^e\}_{|m| \leq 2}$ uniquely such that $g_{0,0}^e, \{g_{1,m}^e\}_{|m| \leq 1}$, and $\{g_{2,m}^e\}_{|m| \leq 2}$ go to zero as $\tau \to \infty$, which is essential to verify $\|x\| \not\to 1$. 

In order to establish estimates for (6.1)-(6.6) we will use the following lemma and its proof, which are slightly different from Lemma 3.1 in [28].
Lemma 6.1. Consider the initial value problem
\[
\frac{d y(t)}{d \tau} + \lambda y(t) + e^{-\tau} \gamma y(t) = f(t), \quad \tau > 0,
\]
where \(\gamma > 0\) and \(f \in L^2(0,T)\) for any \(T > 0\). If \(0 < \lambda_0 < \lambda\), then the following inequalities hold for all \(\tau > 0\):
\[
\int_0^\tau e^{2\lambda_0 s} y^2(s) ds \leq \frac{2}{(\lambda - \lambda_0)^2} \int_0^\tau e^{2\lambda_0 s} f^2(s) ds + \frac{y_0^2}{\lambda - \lambda_0},
\]
where \(C\) is some constant.

Proof. We conclude
\[
y(\tau) = \int_0^\tau e^{\gamma(\tau-s)} e^{-\lambda(\tau-s)} f(s) ds + e^{\gamma(\tau-1)} y(0) e^{-\lambda \tau}.
\]
Hence
\[
y^2(\tau) e^{2\lambda_0 \tau} \leq 2 \left( \int_0^\tau e^{-\lambda(\tau-s)} ds \right) \int_0^\tau e^{-\lambda(\tau-s)} \left( e^{\lambda s} f(s) \right)^2 ds + 2y_0^2 e^{-2\lambda(\tau-1)}
\]
\[
\leq \frac{2}{\lambda - \lambda_0} \int_0^\tau e^{\lambda s} f(s)^2 ds + 2y_0^2 e^{-2\lambda(\tau-1)}
\]
and, by integration over time,
\[
\int_0^\tau y^2(\tau) e^{2\lambda_0 \tau} d\tau \leq \frac{2}{(\lambda - \lambda_0)^2} \int_0^\tau \int_0^\tau e^{\lambda s} f(s)^2 ds du + \frac{y_0^2}{\lambda - \lambda_0}
\]
which implies (6.8). The inequality (6.9) follows easily from (6.8) by using (6.7). \(\square\)

We now give the proof of Theorem 6.2.

Proof of Theorem 6.2. We first solve the system (2.23) for the auxiliary linear problem (3.5) in the same way as Section 4. If the interface functions \(g^e\) and \(g^g\) are known, then (2.23) forms an elliptic system and one can show the well-posedness by elliptic theory. Therefore it is sufficient to consider (5.1)-(6.0).

We hereafter introduce a key ingredient in the proof. In the case \(l = 0\), we are led to
\[
y_{0,0}^\tau = e^{2\tau} \left( g_{0,0}^0 - g_{0,0}^e + F_{0,0}^{11,\tau} + \int_0^\tau e^{-2s} \left( 2e^{-s} J_{0,0}^\tau + K_{0,0}^\tau + F_{0,0}^{9,10,\tau} \right) ds \right)
\]
\[
- e^{2\tau} \int_\tau^\infty e^{-2s} \left( 2e^{-s} J_{0,0}^\tau + K_{0,0}^\tau + F_{0,0}^{9,10,\tau} \right) ds.
\]
Setting
\[
g_{0,0}^e = g_{0,0}^0 + F_{0,0}^{11,\tau} + \int_0^\infty e^{-2s} \left( 2e^{-s} J_{0,0}^\tau + K_{0,0}^\tau + F_{0,0}^{9,10,\tau} \right) ds,
\]
we deduce from (6.10) that
\[
g_{0,0}^\tau = - e^{2\tau} \int_\tau^\infty e^{-2s} \left( 2e^{-s} J_{0,0}^\tau + K_{0,0}^\tau + F_{0,0}^{9,10,\tau} \right) ds.
\]
We will use the following inequality: let \( y(\tau) = -e^{2\tau} \int_0^\infty e^{-2s} f(s) ds \), then by using Minkowski’s inequality for integrals, one can check that

\[
\left( \int_0^\tau e^{2\lambda_0 s} y^2 ds \right)^{1/2} = \left( \int_0^\tau \left( \int_0^\infty e^{-(2+\lambda_0)u} e^{\lambda_0(s+u)} f(s+u) du \right)^2 ds \right)^{1/2}
\leq \int_0^\infty e^{-(2+\lambda_0)u} \left( \int_0^\tau (e^{\lambda_0 f(s)} )^2 du \right)^{1/2} \leq C \left( \int_0^\infty e^{2\lambda_0 f^2(s)} ds \right)^{1/2}.
\]

By applying the inequality (6.13) to (6.12), we conclude

\[
\int_0^\tau e^{2\lambda_0 |g|_{0,0}^2} ds \leq C \left( \int_0^\infty e^{2\lambda_0 \left| e^{-s} J_{0,0}^\tau \right|^2 + |K_{0,0}^\tau|^2 + |F_{0,0}^{0,10;\tau}|^2 ds \right).
\]

In the case \( l = 1 \), following the same argument as in the case \( l = 0 \), we have

\[
\begin{align*}
b_{1,m} &= g_{0,m}^0 + F_{1,m}^{11;\tau} + \int_0^\infty e^{-s} (2e^{-s} J_{1,m}^\tau + K_{1,m}^\tau + F_{1,m}^{0,10;\tau}) ds, \\
\end{align*}
\]

and hence

\[
\int_0^\tau e^{2\lambda_0 |g|_{1,m}^2} ds \leq C \left( \int_0^\infty e^{2\lambda_0 \left| e^{-s} J_{1,m}^\tau \right|^2 + |K_{1,m}^\tau|^2 + |F_{1,m}^{0,10;\tau}|^2 ds \right).
\]

In the case \( l = 2 \), we are led to

\[
g_{2,m}^\tau = e^{\frac{20}{19} (e^{-\tau} - 1)} (g_{2,m}^0 + F_{2,m}^{11;\tau}) - g_{2,m}^\tau + \int_0^\tau e^{\frac{20}{19} (e^{-\tau} - 1)} (e^{-s} J_{2,m}^\tau + K_{2,m}^\tau + F_{2,m}^{0,10;\tau}) ds.
\]

Setting

\[
g_{2,m}^\tau = e^{-\frac{20}{19}} (g_{2,m}^0 + F_{2,m}^{11;\tau}) + \int_0^\infty e^{-\frac{20}{19} e^{-s}} (e^{-s} J_{2,m}^\tau + K_{2,m}^\tau + F_{2,m}^{0,10;\tau}) ds,
\]

we conclude

\[
g_{2,m}^\tau \to 0 \quad \text{as} \quad \tau \to \infty.
\]

As a result, differentiating (6.17) in \( \tau \), we have

\[
\left( \frac{g_{2,m}^\tau}{g_{2,m}^\tau} \right) = -\frac{20}{19} e^{-\tau} e^{\frac{20}{19} (e^{-\tau} - 1)} (g_{2,m}^0 + F_{2,m}^{11;\tau})
\]

\[
- \int_0^\tau \frac{20}{19} e^{-\tau} e^{\frac{20}{19} (e^{-\tau} - 1)} (e^{-s} J_{2,m}^\tau + K_{2,m}^\tau + F_{2,m}^{0,10;\tau}) ds
\]

\[
+ e^{-\tau} J_{2,m}^\tau + K_{2,m}^\tau + F_{2,m}^{0,10;\tau}
\]

so that, by the same arguments as in Lemma 6.1

\[
\int_0^\tau e^{2\lambda_0 s} \left| \left( \frac{g_{2,m}^\tau}{g_{2,m}^\tau} \right) \right|^2 ds \leq C \left( |g_{2,m}^0|^2 + |F_{2,m}^{11;\tau}|^2 + \int_0^\tau e^{2\lambda_0 s} (|e^{-s} J_{2,m}^\tau|^2 + |K_{2,m}^\tau|^2 + |F_{2,m}^{0,10;\tau}|^2) ds \right).
\]

By using (6.19) and the inequality (4.46) in [28], we deduce from (6.20) that

\[
\int_0^\tau e^{2\lambda_0 s} |g_{2,m}^\tau|^2 ds \leq C \left( |g_{2,m}^0|^2 + |F_{2,m}^{11;\tau}|^2 + \int_0^\tau e^{2\lambda_0 s} (|e^{-s} J_{2,m}^\tau|^2 + |K_{2,m}^\tau|^2 + |F_{2,m}^{0,10;\tau}|^2) ds \right).
\]
In the case \( l > 2 \), it immediately follows from Lemma 6.1 that

\[
\int_0^\tau e^{2\lambda_0 s}\left| g_{l,m} \right|^2 ds \leq \frac{C}{(l-2-\lambda_0)^2} \int_0^\tau e^{2\lambda_0 s}\left( \left| e^{-sJ_{l,m}} \right|^2 + \left| K_{l,m} \right|^2 + \left| F_{l,m}^{0,10} \right|^2 \right) ds + C \frac{\left| g_{l,m} \right|^2 + \left| F_{l,m}^{11} \right|^2}{l-2-\lambda_0}. \quad (6.22)
\]

The derivatives of \( g_{l,m} \) in \( \tau \) can be estimated similarly for all \( l \geq 0 \). Therefore, by using the estimates (6.14), (6.16), (6.21), (6.22) and the equalities (6.11), (6.15), (6.18), we can deduce the following estimate in a similar way as in [28]:

\[
\left\| g_{l,m} \right\| H_\infty(S) + \left\| g_{0,0} \right\| + \sum_{|n| \leq 1} b_{1,n} + \sum_{|m| \leq 2} g_{2,m} \leq C \left( \left\| g_{0} \right\|_{H^6(S)} + \left\| y \right\|_{Y} \right). \quad (6.23)
\]

Moreover, from standard elliptic estimates for \((P^\tau, U^\tau, \Phi^\tau)\), we immediately conclude that (3.6) holds.

**Remark 6.2.** By the Sobolev embedding theorem, the estimate (6.23) implies that \( g_{l,m} \) is uniformly bounded in space and time as well as \((P^\tau, \Phi^\tau)\).

**Remark 6.3.** The number \( \lambda_0 \) in the definition of the Banach spaces \( X \) and \( Y \) should be restricted to \( 0 < \lambda_0 < 1 \) to satisfy the estimates (6.14), (6.16), (6.21), and (6.22).

We now give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** By Theorem 3.3, the nonlinear problem \( x = A[x] \) has a unique solution

\[
x = \left( P^\tau, U^\tau, \Phi^\tau, g^\tau, g_{0,0}, \{ b_{1,n} \}_{|n| \leq 1}, \{ g_{2,m} \}_{|m| \leq 2} \right)
\]

in the space \( \mathcal{X} \), so that the solution \( x \) to the evolution problem (2.23)-(2.25) exists uniquely. Indeed, by Theorem 3.1 the parameters \( g_{0,0} \) and \( \{ g_{2,m} \}_{|m| \leq 2} \) of the solution \( x \) construct a unique ellipsoid \( \{ r = 1 + \epsilon g^\epsilon \} \) as well as a unique solution \((P^\epsilon, U^\epsilon, \Phi^\epsilon, g^\epsilon, t_0)\) to the ellipsoidal problem (2.20)-(2.22). We give a simple diagram to explain the procedure that transforms the free-boundary problem (1.1)-(1.3) into \( x = A[x] \) of Theorem 3.3:

Apply the self-similar variables (2.4), (2.5) and the Hanzawa transformation (2.7)

Decompose

By Theorem 3.1, the parameters \( g_{0,0}^\epsilon \) and \( \{ g_{2,m}^\epsilon \}_{|m| \leq 2} \) of the solution \( x \) construct the ellipsoidal part uniquely

\[
x = A[x] \text{ of Theorem 3.3}
\]

Therefore, the fixed-domain problem (2.8), (2.14), (2.17) has a unique solution of the form

\[
(P, U, \Phi, g) = (P^\epsilon, U^\epsilon, \Phi^\epsilon, g^\epsilon) + (P^\tau, U^\tau, \Phi^\tau, g^\tau) \quad \text{and} \quad (x_0, t_0).
\]
We remark that \((x_0, t_0)\) is a part of the solution, where \(\{b_{1,n}\}_{|n| \leq 1}\) determine \(x_0\) uniquely by (2.18). Finally, by the norm of \(\mathcal{X}\), we have \(\|g - g^e\|_{\mathcal{X}(\mathcal{S})} < \infty\), as well as the corresponding \((P, U, \Phi)\), i.e.

\[
\left[ \int_0^{2\lambda_0} e^{2\lambda_0 \tau} \left\| \frac{\partial^k}{\partial \tau^k} (g - g^e) \right\|_{\mathcal{H}^{k+\epsilon}(\mathcal{S})}^2 \ d\tau \right]^{1/2} < \infty,
\]

where \(0 < \lambda_0 < 1\) (see Remark 6.3). From here, reversing the transformation (2.24), (2.25), (2.27) and using the Sobolev embedding theorem, we easily conclude that the solution to the problem (1.1)-(1.3) exists uniquely and (1.5) and (1.7) in Theorem 1.1 are valid.

We now show the estimates (1.6) and (1.8) in Theorem 1.1. By using Theorem 3.1 and the formulas (2.15)-(2.17), we construct an ellipsoid represented by

\[
\{ r = 1 + \epsilon_0^0 Y_{0,0} + \epsilon \sum_{|n| \leq 1} g_{1,n}^0 Y_{1,n} + \epsilon \sum_{|m| \leq 2} g_{2,m}^0 Y_{2,m} + O(\epsilon^2) \},
\]

where the \(O(\epsilon^2)\)-terms merely depend on \(\{g_{0,m}\}_{|m| \leq 2}\). Let \(\tilde{x}_0\) be the center of mass of volume enclosed by (6.24). Then we are led to the following relation:

\[
\left( \begin{array}{c} \tilde{x}_{01} \\
\tilde{x}_{02} \\
\tilde{x}_{03} \end{array} \right) = \frac{3}{8\pi} \left( \begin{array}{ccc} 1 & 0 & -1 \\
-i & 0 & -i \\
0 & \sqrt{2} & 0 \end{array} \right) \left( \begin{array}{c} g_{0,-1}^0 \\
0 \\
g_{0,1}^0 \end{array} \right),
\]

(6.25)

where \(\epsilon_0 = \epsilon(\tilde{x}_{01}, \tilde{x}_{02}, \tilde{x}_{03})\). The same relation (2.18) is also satisfied by \(x_0\) and \(\{b_{1,n}\}_{|n| \leq 1}\). Let \(t_0\) be the real constant deduced from the capacity formula of (6.22):

\[
1 + \epsilon t_0 Y_{0,0} = \frac{\hat{a} \hat{b} \hat{c}}{2} \int_0^\infty \frac{ds}{(\hat{a}^2 + s)(\hat{b}^2 + s)(\hat{c}^2 + s)},
\]

(6.26)

where the semi-axes of (6.24) are of lengths \(\hat{a}, \hat{b}, \hat{c}\). Likewise, the quantity \(t_0\) has the same capacity formula (3.1). Indeed, from the equalities (6.11), (6.15), and (6.18), we have

\[
\begin{aligned}
g_{0,0}^\xi - g_{0,0}^0 &= O(\epsilon), \\
b_{1,n}^\xi - g_{1,n}^0 &= O(\epsilon), \quad |n| \leq 1, \\
g_{2,m}^\xi - e^{-\sigma \frac{\pi}{2}} g_{2,m}^0 &= O(\epsilon), \quad |m| \leq 2.
\end{aligned}
\]

(6.27)

Therefore, by using the estimates (6.27) and the relations (6.20), (2.18), (2.20), (3.1), it readily follows that (1.6) is valid. Moreover, (1.8) also follows easily from (6.27).

**A** The spherical harmonics and the vector spherical harmonics

In this appendix we collect properties of the spherical harmonics and the vector spherical harmonics. The spherical harmonics \(Y_{l,m}(\theta, \varphi)\) are defined on \(S\) by

\[
Y_{l,m}(\theta, \varphi) = (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{2(l+m)!}} P_{l,m}(\cos \theta) e^{im\varphi} \sqrt{\frac{2\pi}{(2l+m)!}}, \quad l \geq 0, \quad |m| \leq l,
\]

(A.1)

where \(P_{l,m}(z) = \frac{1}{2\pi i} (1 - z^2)^{m/2} \frac{d^{l+m}}{dz^{l+m}} (z^2 - 1)^l\). The spherical harmonics \(\{Y_{l,m}\}_{l,m}\) form a complete set in \(L^2(S)\) and

\[
\Delta_{\omega} Y_{l,m} = -(l+1) Y_{l,m},
\]

where \(\Delta_{\omega}\) is the Laplace operator on the surface of the sphere (see (2.49)). The vector spherical harmonics are defined by

\[
V_{l,m} = \left[ -\frac{1}{2l+1} \right]^{1/2} Y_{l,m} e_r + \left[ \frac{1}{(l+1)^{1/2}(2l+1)^{1/2}} \frac{\partial Y_{l,m}}{\partial \theta} \right] e_\theta + \left[ \frac{im Y_{l,m}}{(l+1)^{1/2}(2l+1)^{1/2} \sin \theta} \right] e_\varphi,
\]

(A.2)
Then there exist positive constants $c_1, c_2$ independent of $F$, $f$ such that

$$c_1\|F\|_{H^s(\mathbb{R}^3)}^2 \leq \sum_{j=0}^{s} \sum_{l \geq 0} (1 + l^{2(s-j)}) \sum_{m} \int_0^1 |D_l^j F_{l,m}|^2 r^2 dr \leq c_2 \|F\|_{H^s(\mathbb{R}^3)}^2,$$

$$c_1\|f\|_{H^{s+1/2}(\mathbb{S})}^2 \leq \sum_{l \geq 0} (1 + l^{2s+1}) \sum_{m} |f_{l,m}|^2 \leq c_2 \|f\|_{H^{s+1/2}(\mathbb{S})}^2,$$

where $s$ is a nonnegative integer. Moreover, we can define $\|F\|_{H^s(\text{ext } \mathbb{B}, w)}$ in a similar way.
Lemma A.2 (Lemmas 11.1 and 11.2 in [30]). Let

\[ F(r, \theta, \varphi) = \sum_{l,m} (A_{l,m}(r)V_{l,m} + B_{l,m}(r)X_{l,m} + C_{l,m}(r)W_{l,m}), \]

\[ f(\theta, \varphi) = \sum_{l,m} (\alpha_{l,m}V_{l,m} + \beta_{l,m}X_{l,m} + \gamma_{l,m}W_{l,m}). \]

Then there exist positive constants \( c_1, c_2 \) independent of \( F, f \) such that

\[ c_1 \| F \|_{H^s(\mathbb{B})}^2 \leq \sum_{l \geq 0} \sum_{m} (1 + l^{2(s-j)}) \int_{0}^{1} (|D^l_j A_{l,m}|^2 + |D^l_j B_{l,m}|^2 + |D^l_j C_{l,m}|^2) r^2 dr \leq c_2 \| F \|_{H^s(\mathbb{B})}^2, \]

\[ c_1 \| f \|_{H^{s+1/2}(\mathbb{B})}^2 \leq \sum_{l \geq 0} (1 + l^{2s+1}) \sum_{m} (|\alpha_{l,m}|^2 + |\beta_{l,m}|^2 + |\gamma_{l,m}|^2) \leq c_2 \| f \|_{H^{s+1/2}(\mathbb{B})}^2, \]

where \( s \) is a nonnegative integer.

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