I. INTRODUCTION

Atomic physics and condensed matter physics now enjoy strong connections through the study of quantum transport in ultracold dilute gases [1]. Long-standing problems of condensed matter physics may be addressed by preparing controllable environments for the dynamics of cold atoms and by continuously tuning their interactions. This in turn allows for the study of fundamental features of high-temperature superconductivity using ultracold gases as controllable, analog computers of various model hamiltonians [2].

Degenerate Fermi gases were first produced in 1999 [3], and more recently Fermi superfluid behaviour has been conclusively evidenced through the generation of vortices [4] and the onset of critical velocities [5] in degenerate samples of 6Li. Weakly interacting Fermi gases are difficult to bring to quantum degeneracy mainly due to fundamental obstacles in adapting cooling techniques successfully used for bosonic species. In particular, the Pauli principle inhibits efficient evaporative cooling among identical fermions as they reach degeneracy. This issue has been circumvented by developing two cooling techniques, namely mutual evaporative cooling among identical fermions as they reach degeneracy, consolidating in a thermodynamic setting earlier arguments based on more restrictive assumptions. In particular, we confirm that the minimum temperature of the mixture is obtained at the crossover between boson and fermion heat capacities, and that below such a temperature sympathetic cooling vanishes. When the effect of partial overlap is taken into account, optimal sympathetic cooling of the Fermi species may be achieved by properly tuning the relative trapping strength of the two species in a time-dependent fashion. Alternatively, the dimensionality of the trap in the final stage of cooling can be changed by increasing the confinement strength, which also results in a crossover of the heat capacities at deeper Fermi degeneracies. This technique may be extended to Fermi-Bose degenerate mixtures in optical lattices.

To date, the smallest Fermi degeneracy achieved with both cooling techniques is in the $T/T_F \geq 5 \times 10^{-2}$ range [8, 9]. This limitation has not precluded the study of temperature-independent features of degenerate Fermi gases, such as quantum phase transitions related to unbalanced spin populations [10, 11, 12, 13] or the effect of Fermi impurities in the coherence properties of a Bose gas [14, 15]. However, the study of more conventional phase transitions in which the temperature is the key parameter is still uncharted territory and, as discussed for instance in [10, 17, 18], this requires the achievement
of degeneracy factors $T/T_F \simeq 10^{-3}$ or lower. Unconventional pairing mechanisms that are unstable at higher $T/T_F$ could then be observed, and the phase diagram of Fermi atoms in the degenerate regime could be mapped in a wider range of parameter space. Moreover, the study of ultracold Fermi-Bose mixtures is an interesting subject in itself, acting as the counterpart of the $^3$He-$^4$He liquid mixtures extensively investigated at much higher densities and temperatures.

Considering the novel physical insights that deeper Fermi degenerate gases and Fermi-Bose mixtures may provide, it is relevant to discuss the limitations to reaching the lowest $T/T_F$ in realistic settings available by means of sympathetic cooling, and ways to overcome them. Here, we discuss two different techniques to overcome the apparent $T/T_F \simeq 10^{-2}$ limit observed so far, based on optimized heat capacity matching with species-selective traps or with lower dimensionality traps. The paper is organized as follows. In Section II, after briefly reviewing previous results on sympathetic cooling in species-selective traps, we determine the time evolution of the temperature of the mixture in a particular class of species-selective traps through a thermodynamical analysis, and we subsequently include the effect of the spatial overlap between the thermal component of the Bose gas and the Fermi gas. The main novelty with respect to previous semi-quantitative analyses is that including both temporal and spatial dependence in the thermodynamics of sympathetic cooling leads to an optimization of the heat capacity provided that a time-dependent trapping frequency ratio is implemented. As an alternative to this optimization procedure, in Section III we discuss heat capacity matching resulting from lower effective dimensions for trapping, exploiting the strong dependence on dimensionality of the density of states of the Bose gas. A simpler protocol for optimizing heat capacity is available with nearly one-dimensional Fermi-Bose mixtures, which requires time modulation of the trapping strengths of the Fermi and Bose gases in the last cooling stage. Broader considerations on generic trapping settings are then discussed in the conclusions.

II. HEAT CAPACITY MATCHING THROUGH SPECIES-SELECTIVE TRAPPING

After discussing the evidence for a correlation between the degeneracy factor $T/T_F$ and the trapping frequency ratio between the Fermi and the Bose species, we review previous results on the use of species-selective traps and their limiting assumptions. We then relax these assumptions with a thermodynamical analysis also including the effect of the partial overlap between the Fermi gas and the thermal component of the Bose gas in the specific case of bichromatic traps.

A. Qualitative considerations on heat capacity matching

Evaporative cooling has been instrumental in reaching Bose degeneracy for dilute atomic gases. Extensive analysis have already addressed the dynamics of evaporative cooling of a Bose gas [19, 20, 21, 22], using Monte-Carlo [23, 24], mean-field analysis [25], and beyond [26], including also more detailed effects [27, 28, 29]. These studies have been also extended to the case of separate Bose and Fermi clouds [30] or Fermi-Bose mixtures using the quantum Boltzmann equation [31] or other semiclassical models [32]. Some generic features of the effectiveness in cooling fermions through a Bose gas can be addressed based on the insights first discussed in [33] and then analyzed in more detail in [34, 35]. The heat capacity of a degenerate Fermi gas depends linearly on its temperature, being for a harmonically trapped gas equal to $C_F \simeq \pi^2 k_B N_f T/T_F$, while a harmonically trapped degenerate Bose gas has a cubic dependence on temperature $C_b \simeq 10.8 k_B N_b (T/T_c)^3$. The degeneracy parameter can be written in terms of the ratio of heat capacities: $T/T_F \simeq 0.35 (\omega_f/\omega_b)^{3/2} (C_b/C_f)^{1/2}$ [35]. By assuming that sympathetic cooling loses efficiency when the heat capacity of the Bose gas matches exactly that of the Fermi gas ($C_b = C_f$) we obtain a conservative limit on the attainable $T/T_F$ vs. $\omega_f/\omega_b$ space, depicted in Fig. 1 by the upper line. In the hypothesis that some residual cooling occurs when $C_b < C_f$, for instance with cooling stopping when $C_b/C_f \lesssim 0.1$, we obtain the lower line in Fig. 1. Realistically, we do expect that sympathetic cooling will be quenched when $0.1 \leq C_b/C_f \lesssim 1$, i.e. in the region delimited by the two lines. In Fig. 1 we also plot the minimum $T/T_F$ as obtained by the various running experiments with Fermi-Bose mixtures. Although the number of explored Fermi-Bose mixtures is limited and the diverse technical solutions for trapping and cooling may provide alternative explanations [36], a correlation between the trapping frequency ratio $\omega_f/\omega_b$ and the minimum achieved $T/T_F$ seems corroborated by the actual results and invites more quantitative attention. Further analysis provided insight into the cooling limitations in different trapping conditions and for different species combinations [41]. It was found that significant gains in $T/T_F$ could be achieved for stronger relative Fermi-Bose confinements than the natural $\omega_f/\omega_b$ provided by the mass ratio between the two species. In [41] the focus was on an equilibrium situation at nearly zero temperature, and to develop a more comprehensive understanding of the cooling process a dynamical framework is needed in which crucial finite temperature effects for the Bose gas are taken into account. Here we specialize the analysis of the thermodynamics as $^6$Li is sympathetically cooled by evaporating $^{87}$Rb in an optical dipole trap. This mixture, expected to optimize cooling efficiency [41, 42], is currently used in various laboratories [43, 44, 45] both for studies of Fermi superfluidity and for the formation of ultracold molecules [46], taking
In all previous discussions of species-selective traps a static picture was assumed for the cooling of the Fermi-Bose mixture, relying on the analysis already available in the simple case of evaporative cooling of a single Bose species, allowing us to either maintain a constant \( \omega_f/\omega_b \) throughout cooling or adjust the relative trapping strengths during the process.

The exact way in which the temperature is determined from the trapping parameters depends upon both the model and the proposed cooling strategy. Eq. (1) merely identifies a limit to the cooling rate if thermodynamic equilibrium is to be maintained throughout the process, and assuming a constant ratio \( \eta \), as it was discussed in [34]. In practice, these assumptions can be relaxed by using a more dynamical model based on energy balance, as we discuss in the next section.

\begin{align}
\frac{\Delta U(t)}{\Delta U_i} &= \left( 1 + \frac{t}{\tau} \right)^{\eta_c} \tag{1}
\end{align}

where \( \Delta U_i \) is the initial potential depth, \( \eta_c = -2(\eta' - 3)/\eta' \), and \( \tau^{-1} = (2/3)\eta'(\eta - 4)\exp(-\eta)\gamma_i \), with \( \eta' = \eta + (\eta - 5)/(\eta - 4) \) and \( \gamma_i \) the initial elastic collision rate. With the time dependence of \( \Delta U_i \) determined in this way we can obtain the other relevant quantities in the process (number of particles, temperature, density, scattering rates) resulting in scaling laws similar to Eq. (1). A potential depth/temperature ratio of \( \eta = 5 \) to 10 is considered to yield optimal efficiency in evaporative cooling [49]. This approach can be applied to either the simple case of evaporative cooling of a single Bose species, or the sympathetic cooling in a Fermi-Bose mixture. In both cases, Eq. (1) describes the potential depth of the Bose species \( \Delta U_b \), with the implicit assumption that the presence of the fermions does not drastically affect the evaporation and cooling of the bosons. This is justified by all dual-species systems trap at least an order of magnitude more bosons than fermions, but this approximation may suffer towards the end of the cooling process when a majority of the bosons have been evaporated.

In the case of a bichromatic optical dipole trap, from \( \Delta U_b \) we determine the required power \( P_1 \) of the laser confining both species, and then for a targeted trapping strength ratio \( \omega_f/\omega_b \) we can calculate the required power \( P_2 \) for the Bose-deconfining laser, and finally determine the fermion potential depth \( \Delta U_f \). In this way we have independent control of the spatial size and potential depth for each of the two species, allowing us to either maintain a constant \( \omega_f/\omega_b \) throughout cooling or adjust the relative trapping strengths during the process.

B. Thermodynamical balance

To move beyond the limiting assumptions in [34], we start our analysis by considering a dual-species system at thermodynamic equilibrium. The trap parameters are then suddenly changed in order to force some bosons to evaporate; then one waits for a new thermodynamic equilibrium before applying another evaporation step, in analogy to the scheme discussed in [50]. The step-by-step temperature reached in this way is determined by energy conservation for the Fermi and Bose gases [51].

For concreteness, we consider a species-selective trapping scheme as described in [52]. A mixture of \( N_f \) fermions and \( N_b \) bosons is confined into a bichromatic optical dipole trap tailored by two lasers of wavelengths \( \lambda_1, \lambda_2 \) and powers \( P_1, P_2 \). In order to obtain quasi analytic results, we will approximate the trap potential by a truncated harmonic potential properly reproducing the bottom curvature and the depth of the well. This approximation becomes exact for energies small with respect to the trap depth, a condition satisfied in the cases discussed below. The presence of the second laser allows one to make the trap parameters of the fermion species, namely the characteristic frequencies \( \omega_{fx}, \omega_{fy}, \omega_{fz} \) and the depth \( \Delta U_f \), different from the corresponding boson parameters, \( \omega_{bx}, \omega_{by}, \omega_{bz} \) and \( \Delta U_b \). As a consequence, the ratio \( \omega_f/\omega_b \) for each species \( s = b, f \) we define \( \omega_s = (\omega_{sx}/\omega_{sy}/\omega_{sz})^{1/3} \), can be varied from its mass-determined value for \( P_2/P_1 = 0 \) to an ideally arbitrary large value for \( P_2/P_1 \) approaching a positive critical value [53]. The evaporation steps are carried out by decreasing the power \( P_1 \) of the reference laser while maintaining the...
ratio $P_2/P_1$ at a constant value. In this way, the trap
depths $\Delta U_b$, $\Delta U_f$, which are proportional to $P_1$, and the
frequencies $\omega_b$, $\omega_f$ (proportional to $\sqrt{P_1}$), decrease while
the ratio $\omega_f/\omega_b$ remains constant \[53\].

At the end of an evaporation step in which the power
of the reference laser is changed from $P_1$ to $P_1 + dP_1$
and once thermodynamic equilibrium is reestablished,
the temperature changes from $T$ to $T + dT$ according
to an energy balance equation of the form

$$ (\Delta U_b + \xi k_B T) \frac{dN_b^{ex}}{dT} = dE_b + dE_f, \quad (2) $$

where $dN_b^{ex}$ is the number of bosons in the excited states
at temperature $T$ that evaporate and $\Delta U_b + \xi k_B T$ (with
$0 \leq \xi \leq 1$) is the mean energy per evaporated boson \[29\].
The quantities $dE_b$ and $dE_f$ are the energy changes of the
trapped boson and fermion species, both at temperature $T$.

By observing that due to a change of $P_1$, all the quan-
tities $T$, $\omega_b$, $\Delta U_b$, $\omega_f$, $\Delta U_f$ change and using for $N_b^{ex}$, $E_b$
and $E_f$ the expressions provided by Eqs. \[A9\], \[A8\] and
\[A12\], we have

$$
\begin{align*}
\frac{dN_b^{ex}}{dT} &= \frac{\partial N_b^{ex}}{\partial T} dT + \frac{\partial N_b^{ex}}{\partial \omega_b} d\omega_b + \frac{\partial N_b^{ex}}{\partial \Delta U_b} d\Delta U_b, \quad (3) \\
dE_b &= \frac{\partial E_b}{\partial T} dT + \frac{\partial E_b}{\partial \omega_b} d\omega_b + \frac{\partial E_b}{\partial \Delta U_b} d\Delta U_b, \quad (4) \\
dE_f &= \frac{\partial E_f}{\partial T} dT + \frac{\partial E_f}{\partial \omega_b} d\omega_f. \quad (5)
\end{align*}
$$

It is worth to point out that, within the approximation
used, $E_f$ does not depend on $\Delta U_f$ and the number of
fermions $N_f$ remains constant. By evaluating the above
partial derivatives and inserting the result into Eq. \[2\],
we arrive at

$$
\frac{d \frac{T}{P_1}}{dT} = \frac{3}{2P_1} \left( \frac{\omega_b}{\omega_f} \right)^3 \left( \frac{T}{P_1} \right)^2 \frac{3}{\pi^2} \left( \frac{\omega_b}{\omega_f} \right)^3 \left( \frac{T}{P_1} \right)^2 \frac{\Delta U_b^{1/2}}{k_BT_F} \left( \frac{T}{P_1} \right)^{-1} P_1^{1/2} - 1, \quad (6)
$$

where we used $d(T/T_F) = dT/T_F - (T/T_F) dT/T_F$ and
$dT/T_F = dP_1/(2P_1)$, which stems from the proportion-
ality of $T_F$ to $\omega_f$, namely $k_BT_F = (6N_f)^{1/3} \omega_f$, see Eq.
\[A13\]. Like $\omega_f/\omega_b$, the value of $\Delta U_b P_1^{-1/2}/k_BT_F$
is a constant determined by the value of the ratio $P_2/P_1$, and
$p(x)$ and $q(x)$ are defined as

$$
\begin{align*}
p(x) &= \int_0^x \frac{t^3}{e^t - 1} dt + \frac{\xi^3}{e^{\xi} - 1}, \quad (7) \\
q(x) &= 3x \int_0^x \frac{t^2}{e^t - 1} dt - 4 \int_0^x \frac{t^3}{e^t - 1} dt \\
&\quad + 3 \xi^3 \int_0^x \frac{t^2}{e^t - 1} dt - \xi^3 \frac{e^x - 1}{e^x - 1}. \quad (8)
\end{align*}
$$

Note that $p(0) = q(0) = 0$ whereas $p(x) \simeq 6\xi(4)$ and
$q(x) \simeq 6\xi(3)x - 24\xi(4) + 6\xi(3)\xi$ for $x \gg 1$, $\xi$ being the
Riemann zeta function.

Equation \[8\] is a nonlinear ordinary differential equation
which allows for the determination of $T/T_F(P_1)$
during the evaporative cooling. Observing that both func-
tions $p(x)$ and $q(x)$ are non negative for $x \geq 0$ and
assuming for simplicity $\xi = 0$, the qualitative behavior of
$T/T_F(P_1)$ is as follows. The numerator of the last fraction
in Eq. \[8\] is always positive. If we start from initial
values of $P_1$ and $T/T_F$ respectively not too small and
not too large, the argument of the functions $p$ and
$q$ is large with respect to unity, which is equivalent to state
that $\Delta U_b \gg k_BT$, a fact that also justifies the
choice $\xi = 0$. The denominator of the last fraction in Eq. \[8\]
is thus also positive so that $T/T_F$ decreases by decreasing
$P_1$. The decrease may be faster or slower than
$P_1^{1/2}$ depending on the value of the constants $\omega_f/\omega_b$
and $\Delta U_b P_1^{-1/2}/k_BT_F$. Eventually, however, the last fraction
in Eq. \[8\] becomes larger than 1 so that a second regime
starts in which $T/T_F$ decreases faster and faster. As a
consequence, the argument of the functions $p$ and $q$
decreases and the denominator of the last fraction in Eq.
\[8\] approaches 0. A singular point is thus reached in
which $d(T/T_F)/dP_1 = \infty$ and $T/T_F > 0$. A numerical
study also shows that Eq. \[8\] has a discontinuity at the
singular point with unphysical negative values of
$T/T_F$ on the left. The value of $T/T_F$ at the right of the
singular point represents the minimum achievable $T/T_F$
during the cooling, provided that the initial number of
bosons is sufficiently large so that they are not completely
evaporated before the singular point is reached.

The behavior of $T/T_F$ as a function of $P_1$ is shown in
Fig. 2 for different initial conditions and different values
of $\omega_f/\omega_b$. We stress that whereas the minimum $T/T_F$
depends very little on the details of the cooling, the number
of atoms of both species and initial values of temperature
and reference laser power, we observe a substantial
decrease of the minimum achievable $T/T_F$, by increasing
the ratio $\omega_f/\omega_b$. This is in agreement with a previous pre-
diction based on a rough matching of boson and fermion
heat capacities \[34\]. In fact, the singular point of Eq. \[8\]
is defined by the condition

\[
(\Delta U_b + \xi k_B T) \frac{\partial N_b^e}{\partial T} - \frac{\partial E_b}{\partial T} - \frac{\partial E_f}{\partial T} = 0 .
\]

For \(\Delta U_b \gg k_B T\), a condition well satisfied at the singular point, we have

\[
\frac{\partial N_b^e}{\partial T} = \frac{1}{k_B T} \frac{\partial E_b}{\partial T} \frac{6\zeta(3)}{24\zeta(4)},
\]

therefore Eq. (10) is equivalent to

\[
\left( 0.28 \frac{\Delta U_b}{k_B T} - 1 \right) C_b \simeq C_f,
\]

where \(C_s = \partial E_s/\partial T\) with \(s = b, f\).

The sharp drop observed in \(T/T_F\) before the singular point deserves some comments. At temperatures sufficiently low with respect to \(T_F\) and \(T_c\), the energy of the mixture \(E_b + E_f\) is dominated by the zero-temperature Fermi energy \(E_f(N_f,0)\), see Appendix A. In this case, the right-hand side of the energy balance equation (2) can be approximated by \((\partial E_f(N_f,0)/\omega_f) d\omega_f\), and in the proximity of the singular point (i.e. at low temperatures), following Eq. (11) the left-hand side of the same balance can be written as \((\gamma C_b - C_f) dT\), where \(\gamma \simeq 0.28 \Delta U_b/k_B T - 1\). The approximated energy balance thus gives

\[
dT \simeq \frac{d\omega_f}{\omega_f} \frac{\frac{4}{3} N_f k_B T_F}{\gamma C_b - C_f}.
\]

From Eq. (12) we see that a small decrease of the fermion trapping frequency induces a temperature decrease, the size of which depends on the value of \(N_f k_B T_F/(\gamma C_b - C_f)\). Note that the denominator of this ratio contains a difference, not a sum, of the boson and fermion specific heats. The divergence of the derivative \(dT/d\omega_f\) predicted at the critical point is certainly unphysical: we expect that a long time is needed to re-equilibrate the system in a freezing step \(T \to T + dT\) with \(dT\) large. In this case, dissipative phenomena should be taken into account by a more complicated model in which the singular point will be substituted by a minimum. However, this does not change the meaning of the lowest reachable \(T/T_F\) which is the point where fermion and boson specific heats do match.

In Fig. 3 we plot the dependence of \(T/T_F\) upon the
number of bosons, normalized to the fermion number (assumed to be constant during the evaporation process). It is manifest that deeper Fermi degeneracy factors are obtained for higher trapping frequency ratios. This plot has to be compared to the one presented in Fig. 2 of [32] in which \( T/T_F \) was shown versus a similar quantity (in our notation \((N_b(0) - N_b)/N_f\) where \(N_b(0)\) is the number of initial bosons prior to evaporation). While our analysis confirms that the initial decrease in \( T/T_F \) is faster for lower \( \omega_f/\omega_b \) ratios, thus suggesting more efficient cooling - if measured by the drop in \( T/T_F \) per unit of boson removed in the evaporation process - we also notice that the evaporation process extends much further for larger \( \omega_f/\omega_b \) and allows to reach deeper Fermi degeneracy factors before stopping. Therefore, if the goal of the cooling process is the achievement of the lowest \( T/T_F \) degeneracy parameters rather than saving bosons during the evaporative process [54], the use of larger \( \omega_f/\omega_b \) is beneficial, at variance with [32].

The main issue in [32] is that no temporal dependence for the trapping frequencies was assumed (unlike in Eqs. (33)), which is unphysical for any realistic evaporative cooling strategy involving species-selective trapping strengths. The discussion in [32] also struggles with issues arising from more practical limitations, as commented in [41], in particular the fact that all the Fermi-Bose species available in practice will be affected by issues of spatial overlap in usual confining potentials, due to the smaller mass of the Fermi species (apart from the never considered \(^{40}\)K-\(^{23}\)Na mixture). Also, for \( \omega_b/\omega_f > 1 \) the superfluid critical velocity of the Bose gas will become larger than the Fermi velocity, inhibiting scattering between fermions and bosons and then sympathetic cooling [53, 54]. Finally, a weaker trapping frequency for the Fermi species corresponds to a lower potential energy depth with respect to that of the bosons, resulting in significant fermion losses during the forced evaporative process of the bosons. As we discuss in the following, a large \( \omega_f/\omega_b \) ratio is also beneficial in terms of improving the spatial overlap between fermions and bosons and therefore the cooling efficiency.

C. Spatial overlap between the fermion and the thermal Bose atoms

An even more realistic analysis of the cooling dynamics must also account for the intrinsic inhomogeneous character of the trapping potential as this will result in incomplete overlap between the Fermi and the Bose gases and consequently a decreased cooling efficiency. As introduced in [41], we express the spatial overlap \( \epsilon_{ij} \) between two clouds of densities \( \rho_i \) and \( \rho_j \), as

\[
\epsilon_{ij} = (N_i N_j)^{-1/2} \int \rho_i^{1/2}(r) \rho_j^{1/2}(r) d^3r ,
\]

where \( i, j \) refer to the fermion (F), Bose condensate (B) or thermal boson (T) density profiles. The fraction of atoms that share the same region of space is thus given by \( \epsilon_{ij}^2 \). The Bose atoms available for cooling are those having a non-zero specific heat, \( \text{i.e.} \) those in a thermal state. Finite overlap between the thermal bosons and the Fermi atoms will result in a decrease in the cooling rate with respect to the case of ideal overlap. An accurate evaluation of the cooling rate should take into account kinetic equations for the two interacting gases. In a pessimistic, conservative fashion, we can assume that the cooling rate \( \dot{q} \) is decreased by a factor equal to the fraction of atoms that can actually exchange energy without any mass transport involved, as \( q_{cool} \rightarrow q_{cool}^{2}/\varepsilon_{FB} \). The minimum attainable degeneracy parameter \( T/T_F \) correspondingly increases as \( T/T_F \rightarrow \varepsilon_{FB}^{-2} T/T_F \). This static estimate does not take into account the timescale over which fermions and bosons exchange energy through elastic collisions, and the particle relocation along the trap volume, but it can be considered as an upper limit to the effect of partial overlap. This analysis requires the density profiles of the condensate and the non-condensed thermal boson to be computed independently, following the discussion of a Fermi-Bose mixture at finite temperature reported in [57].

The dependence of the spatial overlap parameters \( \epsilon_{FB} \) and \( \epsilon_{FT} \) on temperature, boson number, and trapping frequency ratio is shown in Fig. 4. As the temperature drops below \( T_c \) a finite condensate fraction appears; the fermion-thermal boson overlap starts to decrease, and the fermion-condensate overlap increases. There is a limit to \( \epsilon_{FB} \) for a given \( \omega_f/\omega_b \), however, since the cloud of less massive fermions will have a much larger spatial radius and thus a strong relative confinement is required to improve the fermion-Bose condensate overlap. The overlap dependence on number ratio is rather straightforward, with a gradual decrease in fermion-condensate overlap as bosons are evaporated and an almost flat behaviour for \( \epsilon_{FT} \), since \( N_b^{cool} \) is roughly constant for a given temperature and boson losses manifest as a decrease in the condensate fraction and have minimal effect on the thermal cloud.

The effect of the spatial overlap on the minimum reachable Fermi degeneracy factor \( T/T_F \) is depicted in Fig. 5, with the choice of interspecies scattering length of the Li-Rb system corresponding to the pessimistic scenario of repulsive interaction (see [58] for the issues related to its measurement). It is evident that about one order of magnitude may be lost in the achievable minimum degeneracy factor when the overlap factor is taken into account, although we conjecture that with a full kinetic analysis the actual result will be located in between the two curves. The situation is analyzed in more detail in Fig. 6, which shows \( \epsilon_{FT} \) versus \( \omega_f/\omega_b \) and \( T/T_c \). During the cooling process, \( \text{i.e.} \) as \( T/T_c \) decreases, the optimal overlap is shifted to larger values of \( \omega_f/\omega_b \), until the minimum of the Fermi degeneracy is reached as shown in the contour plot of Fig. 7. This suggests the use of a time-variable trapping strategy, initial with lower values of \( \omega_f/\omega_b \), then increased in time by increasing the power ratio \( P_f/P_t \). Such a time-dependent relative con-
FIG. 4: (Color online) Overlap parameters $\epsilon_{FB}$ (dashed line) and $\epsilon_{FT}$ (solid line) versus (a) $T/T_c$, (b) $N_b/N_f$, and (c) $\omega_f/\omega_b$. Plots made with other two parameters held at given values of $N_b/N_f=15$, $T/T_c=0.3$, $\omega_f/\omega_b=3$, in the case of a scattering length for Rb of $a_{bb}=+5.8$ nm and an interspecies scattering length $a_{fb}=+0.5$ nm (with the same values also used for the plots in Figs. 5-7).

Plots made with other two parameters held at given values of $N_b/N_f=15$, $T/T_c=0.3$, $\omega_f/\omega_b=3$, in the case of a scattering length for Rb of $a_{bb}=+5.8$ nm and an interspecies scattering length $a_{fb}=+0.5$ nm (with the same values also used for the plots in Figs. 5-7).

FIG. 5: (Color online) Plots of the minimum $T/T_F$ reached during the evaporative cooling process in an optical dipole trap, without (line) and with (dots) the fermion-thermal spatial overlap taken into account. The curve and dots start at the lowest possible value of $\omega_f/\omega_b=2.443$, which is the 'natural' trapping frequency ratio for a $^6$Li-$^{87}$Rb mixture with the deconfining laser beam switched off ($P_2=0$).

FIG. 6: (Color online) Three-dimensional plot and two-dimensional contour plot of the $\epsilon_{FT}$ dependence on $\omega_f/\omega_b$, and $T/T_c$. Significant overlap values for trapping ratios in the range of $\omega_f/\omega_b \simeq 3-7$ are evident at relatively large temperature ratios $T/T_c$. At lower temperatures, the optimal overlap is achieved at higher trapping frequencies ratios. The optimal path maximizing the overlap is highlighted by the dashed line in both plots.

finement strategy is not the only way to optimize the Fermi degeneracy factor, however. In the following, we will discuss a similar procedure which exploits the advantages of reducing the dimensionality of the system when Fermi degeneracy is approached.

III. HEAT CAPACITY MATCHING THROUGH LOWER DIMENSIONALITY

As an alternative to the cooling strategy described above, we discuss here the possibility to match the heat capacity of Bose and Fermi gases at the lowest possible $T/T_F$ by exploiting lower dimensionality traps. Ultimately, the mismatching between the specific heats of Bose and Fermi degenerate gases depend on the scaling of the heat capacities with temperature, and this in turn depends upon the dimensionality of the Bose gas. As discussed in and demonstrated in , a dramatic increase in the trapping frequency in one (or two) trapping axes will result in an effective two- (or one-) dimensional system. This in turn allows for a better matching of the heat capacities since the Bose gas dependence on temperature will become milder than in the full 3D case. In order to gain quantitative insights on how to realize such a matching, we first consider noninteracting gases in a harmonic potential, with the number $N_{f,b}$ of particles fixed.
as (+ for fermions, − for bosons):
\[
N_{f,b}(\mu, T) = \sum_{j=0}^{\infty} \frac{g_j}{e^{(E_j - \mu)/k_B T} + 1},
\]
where \(g_j\) is the degeneracy of energy level \(E_j\), \(\mu\) the chemical potential, \(k_B\) Boltzmann constant, and \(T\) the temperature, with the number of particles \(N_{f,b}\) fixed. Solving this equation numerically for \(\mu = \mu(T)\), we can then calculate the total energy
\[
E_{f,b}(T) = \sum_{j=0}^{\infty} \frac{g_j E_j}{e^{(E_j - \mu(T))/k_B T} + 1},
\]
and from this we obtain the heat capacity as \(C(T) = \partial E/\partial T\). In the calculation below, we assign \(N_{f,b} = 10^4\), and assume the initial trap to be isotropic with the trapping frequency ratio \(\omega_f/\omega_b\). The numerical calculations for 2D and 3D are straightforward, while for 1D some approximations are necessary to reduce the simulation time to realistic values.

### A. 2D and 3D traps

For atoms trapped in a three dimensional harmonic potential \(V(r) = m\omega^2 r^2/2\), the energy eigenvalues \(E_j\) \((j=0,1,2,...)\) are given by \(E_j^{3D} = (j+3/2)\hbar \omega\). Since the trap is three-dimensional and isotropic, the degeneracy \(g_j\) of the energy levels is given by \(g_j = (j+1)(j+2)/2\). The number of particles for bosons (−) and fermions (+) is:
\[
N_{f,b}^{3D} = \frac{1}{2} \sum_{j=0}^{Q} \frac{(j+1)(j+2)}{e^{((j+3/2)\hbar \omega - \mu)/k_B T} + 1}.
\]

**FIG. 7:** (Color online) Contour plot of the Fermi degeneracy factor \(T/T_F\) versus the Bose degeneracy factor \(T/T_c\) and the trapping frequency ration \(\omega_f/\omega_b\). The lower Fermi degeneracy factor of \(T/T_F \approx 0.02\) is obtained for \(\omega_f/\omega_b \approx 8\).

The upper limit \(Q\) in the summation should be infinity in principle, but a value of \(Q=1500\) is sufficient for numerical convergence. For the given parameters \(N_{f,b}\) and \(\omega\) we solve the above two equations for \(\mu\) at different temperatures, and then calculate the heat capacities. As depicted in Fig. 8, the Bose and Fermi heat capacities intersect each other at \(T \approx 0.293T_F\), with the three-dimensional Fermi temperature \(T_F^{3D} = (6N_f)^{1/3}\hbar \omega/k_B\). The situation is very similar for atoms trapped in a 2D isotropic harmonic potential, except that now the energy eigenvalues are given by \(E_j^{2D} = (j+1)\hbar \omega\), with degeneracy \(g_j = j+1\), and \(Q\) needs to be increased to \(\sim 10^8\) to achieve adequate convergence. The two heat capacities curves intersect each other at \(T \approx 0.308 T_F\) where the 2D Fermi temperature \(T_F^2\) is given by \(T_F^2 = (2N_f)^{1/2}\hbar \omega/k_B\). From Fig. 8 we see that going from a full 3D to a 2D system actually slightly worsens the heat capacity matching, yielding a higher \(T/T_F\) at the point where \(C_b\) and \(C_f\) intersect each other. However, further reduction to a 1D system results in complete matching of the heat capacities, as we will see below.

**FIG. 8:** (Color online) Heat capacity curves of bosons and fermions in 3D (dashed curves) and 2D (solid curves) in an isotropic harmonic trap, with the bosons exhibiting nonmonotonic behaviour. The crossing point between the Bose and Fermi curves for the 2D case occurs at a slightly higher \(T/T_F\) value, ruling out its use for a more favourable cooling of fermions.

**B. 1D trap**

For atoms in a 1D trap, the energy eigenvalues are given by \(E_j^{1D} = (j+1/2)\hbar \omega\) with degeneracy \(g_j = 1\). If we follow the same steps as the 2D and 3D cases, the upper summation limit \(Q\) (see Eq. (16)) must be extremely large in order to reach convergence and approximations are required for the one-dimensional trap analysis. These approximations are outlined in Appendix B.
and with them we obtain the Fermi and Bose energies, from which the heat capacities are evaluated. To take full advantage of the gain in the heat capacity matching in 1D traps it is helpful to investigate how heat capacities of bosons and fermions change when we gradually reduce the dimension of the trap. We assume that the atoms are first trapped in a 2D trap with trapping frequencies along the two dimensions ω and kω, and then this relative confinement parameter k is gradually increased from 1 to infinity. The system will be effectively 1D when kB T ≪ ℏω. We evaluate the heat capacities of bosons and fermions at different values of k. The results are shown in Fig. 9 (a-d). As k increases, the shape of the heat capacity curve of bosons becomes more similar to that of fermions, in the sense that it slowly loses the peak structure and the curvature near zero temperature begins to resemble that of fermions. If k is further increased (case (c)) there will be a region where the two curves completely coincide with each other. This is consistent with the previous result we obtained for an ideal 1D trap. In Fig. 9 (d) the heat capacity curves of bosons and fermions are shown at an even higher aspect ratio, to emphasize the crossover from the 1 D case to the 2 D case at high temperatures. In the ideal 1D case, the heat capacity curves are identical, as simply explained in the canonical ensemble approach [63, 64]. Indeed, the total internal energy of fermions in a 1D harmonic trap only differs from that of the bosons by the Fermi zero-point energy E0 = Nf(Nf−1)ℏω/2, and therefore the two systems have identical heat capacities.

The existence of this crossover indicates that we can control the heat capacity matching of bosons and fermions by changing the ratio of the two trapping frequencies in a 2D trap. Thus one possible solution to improve the cooling efficiency is to first evaporate in a 3D trap and then, when the Fermi degeneracy starts to reach about T/TF ≃ 0.3, to increase the trapping frequencies achieving a quasi-one-dimensional system, then continuing the evaporation process. A possible limitation of this technique comes from the larger collisional loss rate as a result of the increased confinement. Also, as studied for achieving Bose condensation of hydrogen atoms, the nearly 1D character of the evaporative cooling may lead to non-ergodic evaporation limiting its efficiency [65], although this has not prevented achievement of Bose degeneracy [67].

![Heat Capacity Curves](image-url)

**FIG. 9:** (Color online) Heat capacity curves of bosons (black, dashed) and fermions (red, continuous). We consider for simplicity the case of equal number of bosons and fermions NB = NF = 10⁴ with equal mass mb = mf. Curves in plots (a), (b), (c) and (d) are in 2D traps with k = ωf/ωc equals to 1, 2.5 × 10⁻³, 2 × 10⁻³, and 5 × 10⁻⁴ respectively, with the case in (d) showing an initial plateau of the heat capacities at ≃ 1 before reaching the high temperature limit of a 2 D gas.

**IV. CONCLUSIONS**

We have examined the thermodynamics of evaporative and sympathetic cooling in a Fermi-Bose mixture, and identified possible ways to achieve a lower Fermi degeneracy factor T/TF. Thermodynamical considerations are based on general assumptions and measurable, phenomenological inputs, like heating rate and specific heat, and provide a solid framework to discuss cooling dynamics regardless of sophisticated microscopic models [68].

They also allow for comparison with experimental results, such as those discussed in [63, 71, 72, 73, 74], or for the inclusion of more realistic inputs such as the specific heat of an interacting gas [72]. Although we have focused the attention on the particular ⁴Li-³⁷Rb mixture, the extension to other Fermi-Bose combinations is straightforward, furthermore benefiting from more favourable interspecies thermalization properties with respect to this particular mixture, for which limitations in elastic scattering and use of Feshbach resonances have been experimentally evidenced [43, 76].

Among the main results we have obtained, we have discussed two different cooling strategies: constructing a species-selective trap with independently tunable Fermi and Bose trapping frequencies, and creating traps with reduced dimensionality in the latest stage of evaporation. We have shown that different trapping ratios lead to distinctly different cooling trajectories. However, incomplete spatial overlap will not only result in a longer cooling time needed to attain a given temperature, but will also increase the temperature at with the heating rate will balance the cooling rate. When the progressive depletion of the bosonic thermal cloud is taken into account, optimized cooling requires time-dependent trapping strengths. Additionally, we have discussed how to exploit the strong dependence of the bosonic specific heat upon dimensionality to create nearly one-dimensional traps in the ultimate stage of sympathetic cooling. This will be of particular relevance for various planned studies of Fermi gases in optical lattices [77] in which bichromatic optical traps are not viable.
APPENDIX A: BOSE AND FERMI GASES IN A 3D HARMONIC TRAP OF FINITE DEPTH

We report here the expressions for the energy and the number of particles of ideal degenerate Bose and Fermi gases confined in a three-dimensional harmonic trap of finite depth.

For the trapped bosons we start with the exact expressions of the energy and of the number of particles in a general 3D harmonic potential:

\[ E_b = \sum_{n_x=0}^{\infty} \sum_{n_y=0}^{\infty} \sum_{n_z=0}^{\infty} \frac{E_{n_x, n_y, n_z}}{(E_{n_x, n_y, n_z} - \mu)/k_B T - 1}, \quad (A1) \]

\[ N_b = \sum_{n_x=0}^{\infty} \sum_{n_y=0}^{\infty} \sum_{n_z=0}^{\infty} \frac{1}{(E_{n_x, n_y, n_z} - \mu)/k_B T - 1}, \quad (A2) \]

where \( E_{n_x, n_y, n_z} = \hbar \omega_x (n_x + 1/2) + \hbar \omega_y (n_y + 1/2) + \hbar \omega_z (n_z + 1/2) \). As usual, we write \( N_b = N_b^0 + N_b^{\text{exc}} \), where \( N_b^0 = (e^{E_b/\hbar \omega_x \omega_y \omega_z} - 1)^{-1} \) is the number of bosons in the ground state and \( N_b^{\text{exc}} \) the number of those thermally excited. In order that the number of particles remains positive, it is necessary for the chemical potential to satisfy \( \mu(T) \leq E_{b,0,0} \). At temperatures \( T \leq T_c \), where \( T_c \) is the Bose-Einstein condensation critical temperature, the chemical potential is frozen to its maximum value. In general, \( \mu(T) \) and all the other thermodynamic quantities can be evaluated explicitly as a power series expansion in the two parameters \( \mu(T) - E_{b,0,0} \) and \( \hbar \omega / k_B T \), where \( \omega_b = (\omega_x \omega_y \omega_z)^{1/3} \). In the case \( T \leq T_c \) and \( \hbar \omega_b \ll k_B T \), which is relevant to the experimental situations discussed here, we can restrict to the lowest order and write

\[ E_b = 3\zeta(4)(\hbar \omega_b)^3, \quad (A3) \]

\[ N_b^{\text{exc}} = \zeta(3) \left( \frac{k_B T}{\hbar \omega_b} \right)^3, \quad (A4) \]

where \( \zeta \) is the Riemann zeta function. Alternatively, the above two results can be obtained using the semiclassical density of states

\[ \varrho_b(E) = \frac{d}{dE} \frac{1}{\hbar \omega_x \hbar \omega_y \hbar \omega_z} = \frac{E^2}{2(\hbar \omega_b)^3} \quad (A5) \]

and the continuum approximation

\[ E_b = \int_0^{\Delta U_b} \frac{E}{e^{E/k_B T} - 1} \varrho_b(E) dE, \quad (A6) \]

\[ N_b^{\text{exc}} = \int_0^{\Delta U_b} \frac{1}{e^{E/k_B T} - 1} \varrho_b(E) dE. \quad (A7) \]

For a trap of finite depth schematized as a harmonic potential truncated at energy \( \Delta U_b \), we thus write

\[ E_b = \int_0^{\Delta U_b} \frac{E}{e^{E/k_B T} - 1} \varrho_b(E) dE \]

\[ = \frac{(k_B T)^3}{(\hbar \omega_b)^3} \int_0^{\Delta U_b} \frac{t^3}{e^t - 1} dt, \quad (A8) \]

\[ N_b^{\text{exc}} = \int_0^{\Delta U_b} \frac{1}{e^{E/k_B T} - 1} \varrho_b(E) dE \]

\[ = \frac{(k_B T)^3}{(\hbar \omega_b)^3} \int_0^{\Delta U_b} \frac{t^2}{e^t - 1} dt. \quad (A9) \]

Consider now a system of \( N_f \) fermions confined in a harmonic trap having characteristic frequencies \( \omega_{f_x}, \omega_{f_y}, \omega_{f_z} \). Under the condition \( \hbar \omega_{f_x}, \hbar \omega_{f_y}, \hbar \omega_{f_z} \ll k_B T \), a continuum approximation holds as in the boson case, so that the fermion counterparts of Eqs. (A1) and (A2) can be simplified to

\[ E_f = \int_0^\infty \frac{E}{e^{(E-\mu)/k_B T} - 1} \varrho_f(E) dE, \quad (A10) \]

\[ N_f = \int_0^\infty \frac{1}{e^{(E-\mu)/k_B T} - 1} \varrho_f(E) dE, \quad (A11) \]

where \( \varrho_f(E) = E^2/2(\hbar \omega_f)^3 \) and \( \omega_f = (\omega_{f_x} \omega_{f_y} \omega_{f_z})^{1/3} \). Note that in the above expressions we have neglected the zero-point energy which is justified since for fermions \( \mu \gg E_{F,0,0} \). The accuracy of the continuum approximation introduced above is discussed in detail in [78]. The chemical potential \( \mu(T) \) can be eliminated between Eqs. (A10) and (A11) by means of the standard Sommerfeld expansion in powers of temperature. By keeping terms only to second order in \( T \), we have the well known result

\[ E_f(N_f, T) \approx E_f(N_f, 0) + \frac{\pi^2}{6}(k_B T)^2 \varrho_f(E_F), \quad (A12) \]

where the Fermi energy \( E_F \) is related to \( N_f \) by

\[ N_f = \int_0^{E_F} \varrho_f(E) dE = \frac{1}{6} \left( \frac{E_F}{\hbar \omega_f} \right)^3 \quad (A13) \]

and the zero temperature term \( E_f(N_f, 0) \) is given by

\[ E_f(N_f, 0) = \int_0^{E_F} \varrho_f(E) dE = \frac{E_F^4}{8(\hbar \omega_f)^3}. \quad (A14) \]

Making use of the relationship between the Fermi energy and the number of fermions, we finally obtain

\[ E_f = \frac{3}{4} \left( \frac{1}{6} \frac{N_f^{1/3}}{\hbar \omega_f} \right)^2 \frac{\pi^2}{2} \left( \frac{k_B T}{\hbar \omega_f} \right)^2 \quad (A15) \]

Equations (A13) and (A15) are valid also for fermions trapped into a harmonic potential truncated at energy \( \Delta U_f \), provided that \( \Delta U_f \gg E_F, k_B T \).
APPENDIX B: INTERNAL ENERGY FOR A 1D TRAPPED IDEAL FERMI GAS

The total atom numbers in one dimension \( N_{1D}^{1} \) and \( N_{1D}^{2} \) are given by Eq. (14), with \( g_{j} = 1 \). However, the upper summation limit \( Q \) (see Eqs. (13)) needed for convergence becomes unreasonably large in one dimension \( \mathbb{R}^{1} \), and thus approximations are required.

For fermions, we can directly replace the summation with an integral in the case of \( k_{B}T \gg \hbar \omega \):

\[
N_{1D}^{1} = \sum_{j=0}^{+\infty} \frac{1}{e^{((j+1/2)\omega/\hbar - \mu)/k_{B}T} - 1} \approx \frac{1}{\hbar\omega} \int_{\hbar\omega/2}^{+\infty} \frac{dE}{e^{(E - \mu)/k_{B}T} + 1}.
\]

Similarly, the total energy is given by

\[
E_{f} \approx \frac{1}{\hbar\omega} \int_{\hbar\omega/2}^{+\infty} EdE e^{(E - \mu)/k_{B}T} + 1.
\]

The total number of bosons is given by

\[
N_{b}^{1D} = \sum_{j=0}^{+\infty} \frac{1}{e^{((j+1/2)\omega/\hbar - \mu)/k_{B}T} - 1} = \frac{1}{e^{(\hbar\omega/2 - \mu)/k_{B}T} - 1} + \sum_{j=1}^{+\infty} \frac{1}{e^{((j+1/2)\omega/\hbar - \mu)/k_{B}T} - 1}.
\]

The second summation term can be evaluated by integral using the Euler-Maclaurin formula:

\[
\sum_{x=a}^{b} F(x) = \int_{a}^{b} F(x)dx + \frac{F(a)}{2} + \frac{F(b)}{2} + \sum_{k=1}^{n} B_{2k} (2k)! \int_{a}^{b} \frac{F^{(2k-1)}(x) - F^{(2k-1)}(a)}{x^{2k}} dx + R
\]

where \( B_{2} = 1/6, B_{4} = -1/30, \ldots \) are the Bernoulli numbers, and \( R \) is the remainder term. In the case of \( k_{B}T \gg \hbar\omega \), the first term \( (k = 1) \) in the expansion is sufficient, giving:

\[
N_{b}^{1D} = \frac{1}{e^{(\hbar\omega/2 - \mu)/k_{B}T} - 1} + \int_{1}^{+\infty} \frac{dx}{e^{((x+1/2)\hbar\omega/\hbar - \mu)/k_{B}T} - 1} + \frac{1}{2} \frac{e^{(3\hbar\omega/2 - \mu)/k_{B}T} - 1}{2} - \frac{\hbar\omega}{12k_{B}T} \frac{e^{(3\hbar\omega/2 - \mu)/k_{B}T} - 1}{2}.
\]

Having fixed \( N \), we solve this equation numerically for \( \mu \) at different temperatures to obtain \( \mu = \mu(T) \), and then apply the Euler-Maclaurin formula again to obtain the total energy \( E_{b}^{1D} \).

The discussion above holds for a one dimensional Bose gas. In the case of a trapping frequency range for which there is a smooth crossover from two dimensions to one dimension, we recall that the number of particles in a 2D trap is given by:

\[
N_{2D}^{1D} = \sum_{i,j=0}^{+\infty} \frac{1}{e^{((i+1/2)\hbar\omega + (j+1/2)\hbar\omega - \mu)/k_{B}T} + 1}.
\]

Running two independent indexes is computationally very inefficient and slow. We therefore assume \( \omega_{1} = \omega \) and \( \omega_{2} = k\omega \) where \( k \) is a positive integer. The summation is run until \( (i + kj) = Q \) which, with \( Q \) large enough, yields a good approximation. Now \( N_{2D}^{1D} \) becomes:

\[
N_{2d} = \sum_{i,j=0}^{+\infty} \frac{1}{e^{((i+1/2)\hbar\omega + (1/2+kj)\hbar\omega - \mu)/k_{B}T} + 1} = \sum_{i=0}^{Q} \frac{\text{floor}(i/k) + 1}{e^{(\mu + (i+1)\hbar\omega)/k_{B}T} + 1}.
\]

where \( \text{floor}(j/k) \) is the nearest integer less than or equal to \( j/k \). In this way only one index is present, allowing to improve significantly the computational speed.

ACKNOWLEDGMENTS

MBH and QW acknowledge support from the Dartmouth Graduate Fellowship program, and MBH also acknowledges support from the Gordon Hull and NSF-GAANN fellowships. CP and RO acknowledge partial support through Cofinanziamento MIUR protocollo 2002027798_001, and RO also acknowledge partial support by the NSF through the Institute for Theoretical Atomic and Molecular Physics at Harvard University and the Smithsonian Astrophysical Observatory.
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