New $D_{n+1}^{(2)}$ K-matrices with quantum group symmetry

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Abstract

We propose new families of solutions of the $D_{n+1}^{(2)}$ boundary Yang–Baxter equation. The open spin-chain transfer matrices constructed with these K-matrices have quantum group symmetry corresponding to removing one node from the $D_{n+1}^{(2)}$ Dynkin diagram, namely, $U_q(B_{n-p}) \otimes U_q(B_p)$, where $p = 0, \ldots, n$. These transfer matrices also have a $p \leftrightarrow n - p$ duality symmetry. These symmetries help to account for the degeneracies in the spectrum of the transfer matrix.

Keywords: boundary Yang–Baxter equation, quantum group, integrable quantum spin chains

1. Introduction

An important class of solutions of the Yang–Baxter equation is given by the so-called trigonometric (or hyperbolic) R-matrices [1, 2], which are associated with affine Lie algebras. These R-matrices, which are basic building blocks of integrable quantum spin chains, were derived using quantum groups. Ironically, the periodic local integrable quantum spin chains of finite length constructed with these R-matrices do not exhibit any quantum group symmetry, because the periodic boundary conditions are not compatible with such symmetry. However, quantum group symmetry can be realized in open local integrable quantum spin chains of finite length, provided that the boundary conditions are suitably chosen. Integrable boundary conditions are provided by solutions of the boundary Yang–Baxter equation [6], which are often called K-matrices or reflection matrices.

$^3$ R-matrices have recently also been derived from four-dimensional gauge theory [3, 4].

$^4$ Periodic spin chains can have quantum group symmetry if one allows non-local interactions [5].
For trigonometric R-matrices and corresponding K-matrices [7, 8] associated with several series of affine Lie algebras \( \hat{g} \) (namely, \( A_{n}^{(2)}, A_{2n-1}^{(2)}, B_{n}^{(1)}, C_{n}^{(1)} \) and \( D_{n}^{(1)} \)), the corresponding open spin-chain transfer matrices [9, 10] have recently been shown to have quantum group symmetry corresponding to removing one node from the \( \hat{g} \) Dynkin diagram [11]. (The \( A_{n-1}^{(1)} \) case was discussed in [12]). However, the \( D_{n+1}^{(2)} \) case was not considered in [11], since most of the needed K-matrices were not known\(^5\). The principal aim of this note is to find the necessary K-matrices, and then to extend the program in [11] to the \( D_{n+1}^{(2)} \) case \((n = 1, 2, \ldots) \). We show that the open spin-chain transfer matrices constructed with these K-matrices indeed have quantum group symmetry corresponding to removing the \( p \)th node from the \( D_{n+1}^{(2)} \) Dynkin diagram, namely, \( U_q(B_{n-p}) \otimes U_q(B_p) \), where \( p = 0, \ldots, n \). A special feature of the \( D_{n+1}^{(2)} \) case, which is present also for \( C_{n}^{(1)} \) and \( D_{n}^{(1)} \) cases [11], is the existence of an additional \( p \leftrightarrow n-p \) duality symmetry.

The outline of this paper is as follows. The new K-matrices are presented in section 2. The construction of the open-chain transfer matrix is briefly reviewed in section 3. The quantum group symmetry of the transfer matrix is demonstrated in section 4, and the duality symmetry of the transfer matrix is shown in section 5. These symmetries are used in section 6 to explain the degeneracies in the spectrum of the transfer matrix for generic values of the anisotropy parameter \( \eta \). Section 7 contains a brief conclusion. Since the \( D_{n+1}^{(2)} \) case shares many similarities with other cases analyzed in [11], we try to avoid repeating here previous results as much as possible, and instead emphasize the results that are different.

2. R and K matrices

Let \( R(u) \) denote the \( D_{n+1}^{(2)} \) R-matrix [1], following the conventions in [14]. This is a \((2n+2)^2 \times (2n+2)^2\) matrix, which is a function of the spectral parameter \( u \) and the anisotropy parameter \( \eta \), and satisfies the Yang–Baxter equation

\[
R_{12}(u-v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u-v). \tag{2.1}
\]

We also use the following additional properties of this R-matrix: PT symmetry

\[
R_{21}(u) \equiv \mathcal{P}_{12} R_{12}(u) \mathcal{P}_{12} = R_{12}^{\alpha \gamma}(u), \tag{2.2}
\]

unitarity

\[
R_{12}(u) R_{21}(-u) = \zeta(u) \mathbb{I} \otimes \mathbb{I}, \tag{2.3}
\]

where \( \zeta(u) \) is given by

\[
\zeta(u) = \xi(u) \xi(-u), \quad \xi(u) = 4 \sinh(u + 2\eta) \sinh(u + 2m\eta), \tag{2.4}
\]

and crossing symmetry

\[
R_{12}(u) = V_1 R_{12}^{\alpha \gamma}(-u - \rho) V_1 = V_2 R_{12}^{\alpha \gamma}(-u - \rho) V_2^\dagger, \quad \rho = -2m\eta. \tag{2.5}
\]

\(^5\)The \( D_{n+1}^{(2)} \) R-matrix is—by far—the most complicated of the R-matrices in [1, 2], which may explain why few of the corresponding K-matrices had heretofore been found. Due to the complexity of the R-matrix, we do not prove that the proposed K-matrices satisfy the boundary Yang–Baxter equation (2.6) and the duality property (5.4), but only check these results for small values of \( n \).

\(^6\)The K-matrices for the case \( p = n \) were already known [8, 13], and the \( U_q(B_n) \) symmetry for this case was noted in [14].
where the crossing matrix $V$ can be found in [14].

Let $K^R(u)$ denote the ‘right’ $D_{n+1}^{(2)}$ K-matrix, which is a $(2n + 2) \times (2n + 2)$ matrix that satisfies the boundary Yang–Baxter equation [6, 9, 15]

$$R_{12}(u - v) K^R(u) R_{21}(u + v) K^R(v) = K^R(v) R_{12}(u + v) K^R(u) R_{21}(u - v).$$

(2.6)

In view of the expected $U_q(B_{n-p}) \otimes U_q(B_p)$ symmetry, we look for solutions of the following block-diagonal form

$$K^R(u) = K^R(u, p) = 
\begin{pmatrix}
  k_\pm(u) & g(u) \frac{1}{\eta + \frac{1}{2} \varepsilon} & k_1(u) & k_2(u) \\
  g(u) \frac{1}{\eta + \frac{1}{2} \varepsilon} & k_2(u) & k_1(u) \\
  k_1(u) & k_2(u) & g(u) \frac{1}{\eta + \frac{1}{2} \varepsilon} \\
  k_2(u) & g(u) \frac{1}{\eta + \frac{1}{2} \varepsilon} & k_2(u) & k_1(u)
\end{pmatrix},$$

(2.7)

where $p = 0, \ldots, n$ and $n = 1, 2, \ldots$. We impose the regularity constraint $K^R(0, p) = I$, and proceed to solve for the unknown functions following the method in the appendix of [16], for small values of $n$. The following pattern emerges

$$k_\pm(u) = e^{u \pm 2n},$$

$$g(u) = \frac{\cosh(u - (n - 2p)\eta + \frac{i}{2} \varepsilon)}{\cosh(u + (n - 2p)\eta - \frac{i}{2} \varepsilon)},$$

$$k_1(u) = \frac{\cosh(u) \cosh((n - 2p)\eta + \frac{i}{2} \varepsilon)}{\cosh(u + (n - 2p)\eta + \frac{i}{2} \varepsilon)},$$

$$k_2(u) = \frac{\sinh(u) \sinh((n - 2p)\eta + \frac{i}{2} \varepsilon)}{\cosh(u + (n - 2p)\eta + \frac{i}{2} \varepsilon)},$$

(2.8)

where $\varepsilon$ can take either of two values: $\varepsilon = 0$ or $\varepsilon = 1$. Unless otherwise noted, all the results in this paper hold for both values of $\varepsilon$. We have explicitly verified all of the solutions (2.7) and (2.8) up to $n = 10$, and we conjecture that they are valid for all $n$.

Note that the solutions with $p = \frac{n}{2}$ (even) and $\varepsilon = 0$ are diagonal; otherwise, the solutions are non-diagonal. The solutions with $p = n$, as well as the diagonal solution with $n = 2, p = 1, \varepsilon = 0$, were previously known [8, 13]; to our knowledge, all other solutions are new.

In order to maximize the symmetry of the spin chain, we impose the ‘same’ boundary conditions on the two ends, which corresponds to taking the ‘left’ K-matrix $K^L(u)$ to be

$$K^L(u) = K^L(u, p) = K^L(-u - \rho, p) M,$$

(2.9)

where $M$ is a diagonal matrix defined in terms of the crossing matrix $V$ by $M = V^t V$, whose explicit expression can be found in [14].

### 3. Transfer matrix

Using the R-matrix and the K-matrices (2.7) and (2.9), we can construct the transfer matrix of an integrable open spin chain of length $N$ [9]

$$t(u, p) = Tr_e K^R_e(u, p) T_e(u) K^R_e(u, p) \tilde{T}_a(u),$$

(3.1)
where the single-row monodromy matrices are defined by
\[
T_a(u) = R_{aN}(u) \cdots R_{a1}(u),
\]
\[
\tilde{T}_a(u) = R_{1a}(u) \cdots R_{N-1a}(u) \ R_{Na}(u),
\]
and the trace in (3.1) is over the ‘auxiliary’ space, which is denoted by \(a\). The transfer matrix satisfies the fundamental commutativity property
\[
[t(u, p), t(v, p)] = 0 \quad \text{for all } u, v,
\]
and contains the Hamiltonian \(\mathcal{H}(p) \sim t(0, p)\) as well as higher local conserved quantities.

4. Quantum group symmetry

We now argue that the transfer matrix (3.1) has the quantum group symmetry \(U_q(B_{n,p}) \otimes U_q(B_p)\). The argument is very similar to the one in [11]. The key idea is to perform a \(p\)-dependent gauge transformation of the \(R\) and \(K\) matrices
\[
\tilde{R}_{12}(u, p) = B_1(u, p) R_{12}(u) B_1(-u, p) = B_2(-u, p) R_{12}(u) B_2(u, p),
\]
and
\[
\tilde{K}_R(u, p) = B(u, p) K_R(u, p) B(u, p),
\]
\[
\tilde{K}_L(u, p) = B(-u, p) K_L(u, p) B(-u, p),
\]
where \(B(u, p)\) is the diagonal matrix
\[
B(u, p) = \text{diag} (\underbrace{e^u, \ldots, e^u}_p, \underbrace{1, \ldots, 1}_{2n+2-2p}, \underbrace{e^{-u}, \ldots, e^{-u}}_p).
\]

Evidently, the gauge-transformed K-matrix \(\tilde{K}_R(u, p)\) is the same as (2.7), except that the functions \(k_{\pm}(u)\) are replaced by 1. The gauge-transformed (single-row) monodromy matrix
\[
\tilde{T}_a(u, p) = \tilde{R}_{aN}(u, p) \tilde{R}_{aN-1}(u, p) \cdots \tilde{R}_{a1}(u, p)
\]
has the asymptotic behavior
\[
\tilde{T}_a(u, p) \sim e^{\pm 2N_a} \tilde{T}_a^\pm(p) \quad \text{for } u \to \pm \infty,
\]
where
\[
\tilde{T}_a^\pm(p) = \tilde{R}_{aN}^\pm(p) \tilde{R}_{aN-1}^\pm(p) \cdots \tilde{R}_{a1}^\pm(p)
\]
and
\[
\tilde{R}^\pm(p) = \lim_{u \to \pm \infty} e^{\mp 2a} \tilde{R}(u, p).
\]
The important point is that the operators \(\tilde{T}^\pm_{ij}(p) = (\tilde{T}^\pm_{ij}(p))_q\) can be expressed in terms of (the quantum enveloping algebra of) the unbroken \(D_{n+1}^{(2)}\) generators, i.e. the generators of \(U_q(B_{n,p}) \otimes U_q(B_p)\), see appendix. Hence, in order to demonstrate the quantum group symmetry of the transfer matrix, it suffices to show that
\[
\left[ \tilde{T}^\pm_{ij}(p), t(u, p) \right] = 0 \quad i, j = 1, 2, \ldots, 2n + 2.
\]
The proof of (4.8) in [11] carries over readily to the \(D_{n+1}^{(2)}\) case, except for lemma 1.
\[
[R_{12}^\pm(p), \hat{R}_2^\pm(u, p)] = 0.
\]

(4.9)

We have verified this relation explicitly for small values of \(n\) (for both \(\varepsilon = 0\) and \(\varepsilon = 1\)), and we conjecture that it is true for all \(n\).

The \((2n + 2)\)-dimensional vector space at each site decomposes under \(U_q(B_{n-p}) \otimes U_q(B_p)\) simply as the direct sum of vector representations of each factor, i.e.

\[
(2(n - p) + 1, 1) \oplus (1, 2p + 1).
\]

(4.10)

5. Duality symmetry

The transfer matrix (3.1) also has the \(p \leftrightarrow n - p\) duality symmetry

\[
\mathcal{U} t(u, p) \mathcal{U}^{-1} = f(u, p) t(u, n - p),
\]

(5.1)

where \(\mathcal{U}\) is the quantum-space operator

\[
\mathcal{U} = U_1 \ldots U_N,
\]

(5.2)

\(U\) is the block matrix

\[
U = \begin{cases} 
\begin{pmatrix} 
1 & 0 \\
0 & -1 
\end{pmatrix} & \text{for } n \text{ even,} \\
\begin{pmatrix} 
-1_{n \times n} & 1 \\
0 & -1 
\end{pmatrix} & \text{for } n \text{ odd,}
\end{cases}
\]

(5.3)

which satisfies \(UU' = I\), and \(f(u, p)\) is a scalar function given below (5.8). The proof of (5.1) is similar to the one in [11]. It makes use of the following properties of the \(D^{(2)}_{n+1}\) R-matrix

\[
U_1 R_{12}(u) U'_1 = W_2(u) R_{12}(u) (W_2(u))^{-1},
\]

(5.4)

\[
U_2 R_{12}(u) U'_2 = (W_1(u))^{-1} R_{12}(u) W_1(u),
\]

where \(W(u)\) is the block matrix

\[
W(u) = \begin{cases} 
\begin{pmatrix} 
-1 & 0 \\
0 & 1 
\end{pmatrix} & \text{for } n \text{ even,} \\
\begin{pmatrix} 
e^{-uI_{n \times n}} & 0 \\
0 & ne^{-uI_{n \times n}} 
\end{pmatrix} & \text{for } n \text{ odd.}
\end{cases}
\]

(5.5)
We have verified the properties (5.4) for small values of \( n \), and we conjecture that they are true for all \( n \). Moreover, the K-matrices (2.7) and (2.9) satisfy
\[
W(u)K^R(u, p)W(u) = f^R(u, p)K^R(u, n - p),
\]
\[
(W(u))^{-1}K^L(u, p)(W(u))^{-1} = f^L(u, p)K^L(u, n - p),
\]
where \( f^R(u, p) \) and \( f^L(u, p) \) are scalar functions given by
\[
f^R(u, p) = \frac{\cosh(u - (n - 2p)\eta + \frac{i\pi}{2}\varepsilon)}{\cosh(u + (n - 2p)\eta - \frac{i\pi}{2}\varepsilon)},
\]
\[
f^L(u, p) = \frac{\cosh(u - (n + 2p)\eta + \frac{i\pi}{2}\varepsilon)}{\cosh(u - (3n - 2p)\eta - \frac{i\pi}{2}\varepsilon)}.
\]

Using the properties (5.4) and (5.6), it is now straightforward to show [11] that the transfer matrix has the duality symmetry (5.1), where \( f(u, p) \) is given by
\[
f(u, p) = f^L(u, p)f^R(u, p).
\]

Consequently, for each eigenvalue \( \Lambda(u, n - p) \) of \( t(u, n - p) \), there is a corresponding eigenvalue \( \Lambda(u, n - p) \) of \( t(u, n - p) \) such that
\[
\Lambda(u, p) = f(u, p)\Lambda(u, n - p).
\]

### 5.1. Action of duality on the quantum group

Under a duality transformation, the operators \( \tilde{T}^\pm_a(p) \) (4.6) transform as follows
\[
\mathcal{U}\tilde{T}^\pm_a(p)\mathcal{U}^{-1} = U_a^{-1}\tilde{T}^\pm_a(n - p)U_a.
\]
The proof is similar to the one in [11]. In particular, the generators (A.5) and (A.6) transform as
\[
UH^{(0)}(p)U^{-1} = H^{(0)}(n - p), \quad UE^{(0)}(p)U^{-1} = \nu(p)E^{(0)}(n - p), \quad i = 1, 2, \ldots, n - p.
\]
\[
UH^{(1)}(p)U^{-1} = H^{(1)}(n - p), \quad UE^{(1)}(p)U^{-1} = E^{(1)}(n - p), \quad i = 1, 2, \ldots, n - p.
\]

where
\[
\nu(p) = \begin{cases} -1 & \text{if } n = \text{ even and } i = n - p, \\ +1 & \text{otherwise} \end{cases}
\]

and similarly for the coproducts.

### 5.2. Self-duality

For \( p = \frac{n}{2} \) with \( n \) even, the duality relation (5.1) implies that the transfer matrix is self-dual
\[
\mathcal{U}, t(u, \frac{n}{2}) = 0,
\]

The relations (5.11) are consistent by virtue of the identities
\[
U^2H^{(0)}(p) = H^{(0)}(p)U^2, \quad U^2E^{(0)}(p) = \nu(p)E^{(0)}(p)U^2.
\]
\[
U^2H^{(1)}(p) = H^{(1)}(p)U^2, \quad U^2E^{(1)}(p) = \nu(n - p)E^{(1)}(p)U^2.
\]
since \( f(u, \frac{n}{2}) = 1 \). This symmetry maps the representations \((1, R)\) and \((R, 1)\) (i.e. with ‘left’ and ‘right’ singlets, respectively) into each other; and therefore these states are degenerate (i.e. have the same transfer-matrix eigenvalue).

5.2.1. **Bonus symmetry for \( \varepsilon = 1 \).** For \( p = \frac{n}{2} \) (\( n \) even) and \( \varepsilon = 1 \), there is an additional (‘bonus’) symmetry, which leads to even higher degeneracies for the transfer-matrix eigenvalues. A similar phenomenon occurs for \( C_n^{(1)} \) and \( D_n^{(1)} \) \cite{11}. Indeed, one can show in a similar way that the transfer matrix obeys

\[
[D, t(u, \frac{n}{2})] = 0, 
\]

(5.14)

where \( D \) is the quantum-space operator given by

\[
D = D_1 = D \otimes \mathbb{1}^{(N-1)},
\]

(5.15)

and \( D \) is the (\( u \)-independent) matrix given by the gauge-transformed K-matrix

\[
D = \tilde{K}^B(u, \frac{n}{2}),
\]

(5.16)

which here is not diagonal.

A state \(|\Lambda\rangle\) that is a simultaneous eigenstate of \( t(u, \frac{n}{2}) \) and \( \mathcal{U} \) (recall (5.13)) is not necessarily an eigenstate of \( D \), since \( \mathcal{U} \) and \( D \) do not commute. In such case, \(|\Lambda\rangle\) and \( D|\Lambda\rangle\) are linearly independent eigenstates with the same transfer-matrix eigenvalue (recall (5.14)). In fact, the degeneracy of this eigenvalue becomes doubled as a consequence of the bonus symmetry \cite{11}.

6. **Degeneracies of the transfer-matrix spectrum**

For generic values of the anisotropy parameter \( \eta \), the degeneracies in the spectrum of the transfer matrix (3.1) mostly match with the predictions from the \( U_q(B_{n-p}) \otimes U_q(B_p) \) symmetry. Exceptions include when \( n \) is even and \( p = \frac{n}{2} \) (in which case there is a self-duality symmetry (5.13); and, if \( \varepsilon = 1 \), there is also a bonus symmetry (5.14)) or when \( n \) is odd and \( p = \frac{n+1}{2} \). Moreover, the spectrum exhibits a \( p \to n-p \) duality symmetry. We now consider some simple examples.

6.1. **Example 1: even \( n \)**

We first consider the case \( n = 4 \) (i.e. \( D_2^{(2)} \) and \( N = 2 \) (two sites)). By direct diagonalization of the transfer matrix \( t(u, p) \) for generic numerical values of \( u \) and \( \eta \), we find that the degeneracies are as follows:

\[
\begin{align*}
p &= 0 : & \{1, 1, 9, 9, 36, 44\} \\
p &= 1 : & \{1, 1, 9, 9, 36, 44\} \\
p &= 2 : & \{1, 1, 9, 9, 36, 44\} \\
p &= 3 : & \{1, 1, 9, 9, 36, 44\} \\
p &= 4 : & \{1, 1, 9, 9, 36, 44\}.
\end{align*}
\]

(6.1)
In other words, for $p = 0$, one eigenvalue is repeated 44 times, another eigenvalue is repeated 36 times, etc; and similarly for other values of $p$. The fact that the degeneracies are the same for $p$ and $n - p$ is a consequence of the duality symmetry (5.1) and (5.9).

On the other hand, the quantum group symmetry when $n = 4$ is $U_q(B_{4-p}) \otimes U_q(B_p)$, and the 10-dimensional representation at each site (4.10) is $(9 - 2p, 1) \oplus (1, 2p + 1)$. For generic values of $\eta$, the quantum group representations are the same as for the corresponding classical groups. Performing the tensor-product decompositions using LieART [17], we obtain

\[
p = 0 : B_4 \quad (9 \oplus 1)^{\otimes 2} = 2(1) \oplus 2(9) \oplus 36 \oplus 44
\]

\[
p = 1 : B_3 \otimes B_1 \quad [(7, 1) \oplus (1, 3)]^{\otimes 2} = 2(1, 1) \oplus (1, 3) \oplus (1, 5) \oplus 2(7, 3) \oplus (21, 1) \oplus (27, 1)
\]

\[
p = 2 : B_2 \otimes B_2 \quad [(5, 1) \oplus (1, 5)]^{\otimes 2} = 2(1, 1) \oplus (10, 1) \oplus (1, 10) \oplus 2(5, 5) \oplus (14, 1) \oplus (1, 14).
\]  

(6.2)

There is no need to display the tensor-product decompositions for $p > 2$ due to the symmetry $p \rightarrow n - p$.

Comparing the degeneracies (6.1) with the corresponding tensor-product decompositions (6.2), we see that they match, except for $p = 2$. For the latter case, the degeneracies are larger, due to the self-duality symmetry (5.13) for even $n$ and $p = \frac{n}{2}$, which here maps $(1, 10)$ to $(10, 1)$ (resulting in a 20-fold degeneracy), and also maps $(1, 14)$ to $(14, 1)$ (resulting in a 28-fold degeneracy). If $\varepsilon = 1$, then the bonus symmetry (5.14) implies that the two $(5, 5)$ are degenerate (giving rise to a 50-fold degeneracy), as well as the two $(1, 1)$ (resulting in a 2-fold degeneracy).

### 6.2. Example 2: odd $n$

As a second example, we consider the case $n = 5$ (i.e. $D_6^{(2)}$) and $N = 2$ (two sites). By direct diagonalization of the transfer matrix $t(u, p)$ for generic numerical values of $u$ and $\eta$, we find that the degeneracies are as follows:

\[
p = 0 : \quad \{1, 1, 11, 11, 55, 65\}
\]

\[
p = 1 : \quad \{1, 1, 3, 5, 27, 27, 36, 44\}
\]

\[
p = 2 : \quad \{1, 1, 10, 27, 49, 56\}
\]

\[
p = 3 : \quad \{1, 1, 10, 27, 49, 56\}
\]

\[
p = 4 : \quad \{1, 1, 3, 5, 27, 27, 36, 44\}
\]

\[
p = 5 : \quad \{1, 1, 11, 11, 55, 65\}.
\]  

(6.3)

We see again that the degeneracies are the same for $p$ and $n - p$, as a consequence of the duality symmetry (5.1) and (5.9).

On the other hand, the symmetry when $n = 5$ is $U_q(B_{5-p}) \otimes U_q(B_p)$, and the twelve-dimensional representation at each site (4.10) is $(11 - 2p, 1) \oplus (1, 2p + 1)$. The tensor-product decompositions are as follows:

\[
p = 0 : B_3 \quad (11 \oplus 1)^{\otimes 2} = 2(1) \oplus 2(11) \oplus 55 \oplus 65
\]

\[
p = 1 : B_5 \otimes B_1 \quad [(9, 1) \oplus (1, 3)]^{\otimes 2} = 2(1, 1) \oplus (1, 3) \oplus (1, 5) \oplus 2(9, 3) \oplus (36, 1) \oplus (44, 1)
\]

\[
p = 2 : B_3 \otimes B_2 \quad [(7, 1) \oplus (1, 5)]^{\otimes 2} = 2(1, 1) \oplus (1, 10) \oplus 2(7, 5) \oplus (1, 14) \oplus (21, 1) \oplus (27, 1).
\]  

(6.4)

Again, there is no need to display the tensor-product decompositions for $p > 2$ due to the symmetry $p \rightarrow n - p$. 


Comparing the degeneracies (6.3) with the corresponding tensor-product decompositions (6.4), we see that they match, except for \( p = 2 \). For the latter case, the degeneracies are larger: the \( (1, 14) \) and one \( (7, 5) \) are degenerate (resulting in a 49-fold degeneracy); and the \( (21, 1) \) and the other \( (7, 5) \) are degenerate (resulting in a 56-fold degeneracy). Similar degeneracies for odd \( n \) and \( p = n \pm \frac{1}{2} \) also occur for \( C_n^{(1)} \) and \( D_n^{(1)} \) [11].

7. Conclusions

We have found new \( D^{(2)}_{n+1} \) K-matrices \( K^{(u,p)} \) (2.7) and (2.8), for which the corresponding transfer matrix (3.1) has \( U_q(B_{n-p}) \otimes U_q(B_p) \) symmetry (4.8), as well as the \( p \leftrightarrow n - p \) duality symmetry (5.1). For the special case \( p = \frac{n}{2} (n \text{ even}) \), the transfer matrix has a self-duality symmetry (5.13), and an additional ‘bonus’ symmetry (5.14) if \( \epsilon = 1 \). These symmetries account for most of the degeneracies of the spectrum of the transfer matrix, as illustrated in the examples of section 6. The exceptions include the unusual degeneracy that occurs for \( p = \frac{n \pm 1}{2} (n \text{ odd}) \), noted in section 6.2, which is due to ‘mixing’ of representations of unequal dimensions. We expect that such degeneracies, which occur also for \( C_n^{(1)} \) and \( D_n^{(1)} \) [11], can be attributed to some discrete symmetries, which remain to be understood.

The following picture emerges about the symmetries of an integrable spin chain of length \( N \) constructed with a trigonometric R-matrix associated with an affine Lie algebra \( \hat{g} \): in the limit \( N \to \infty \), the spin chain has the infinite-dimensional \( U_q(\hat{g}) \) symmetry, regardless of boundary conditions, which is exploited in the vertex operator formalism (see e.g. [18–21]). However, for finite \( N \), the symmetry algebra of the spin chain is necessarily a finite-dimensional subalgebra of \( U_q(\hat{g}) \). Maximal subalgebras of \( U_q(\hat{g}) \) can presumably be obtained by removing one node from its Dynkin diagram. The present and previous [11, 12] work describe the boundary conditions and the corresponding integrable open spin chains with such symmetries, for all non-exceptional \( \hat{g} \). Other boundary conditions presumably lead to smaller symmetries.

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Appendix. Quantum group generators

We present here explicit expressions for the \( U_q(B_{n-p}) \otimes U_q(B_p) \) generators, in terms of which the operators \( \tilde{T}_i^{(u,p)}(p) \) (4.6) can be expressed. Following [11], we denote the generators corresponding to the simple roots of the ‘left’ algebra \( g^{(l)} \equiv B_{n-p} \) and the ‘right’ algebra \( g^{(r)} \equiv B_p \) by

\[
H_i^{(l)}(p), \quad E_i^{(l)}(p), \quad i = 1, \ldots, n - p,
\]

and

\[
H_i^{(r)}(p), \quad E_i^{(r)}(p), \quad i = 1, \ldots, p.
\]
respectively. The ‘left’ generators satisfy the commutation relations

\[
\begin{align*}
[ H_i^{(l)}(p), H_j^{(l)}(p) ] &= 0, \\
[ H_i^{(l)}(p), E_j^{\pm(l)}(p) ] &= \pm \alpha_i^{(l)} E_j^{\pm(l)}(p), \\
[ E_i^{+(l)}(p), E_j^{-(l)}(p) ] &= \delta_{ij} \sum_{k=1}^{n-p} \alpha_k^{(l)} H_k^{(l)}(p),
\end{align*}
(A.1)
\]

and the ‘right’ generators similarly satisfy the commutation relations

\[
\begin{align*}
[ H_i^{(r)}(p), H_j^{(r)}(p) ] &= 0, \\
[ H_i^{(r)}(p), E_j^{\pm(r)}(p) ] &= \pm \alpha_i^{(r)} E_j^{\pm(r)}(p), \\
[ E_i^{+(r)}(p), E_j^{-(r)}(p) ] &= \delta_{ij} \sum_{k=1}^{p} \alpha_k^{(r)} H_k^{(r)}(p).
\end{align*}
(A.2)
\]

Moreover, the ‘left’ and ‘right’ generators commute with each other. The simple roots \( \{ \alpha^{(1)}, \ldots, \alpha^{(m)} \} \) of \( B_m \) (where \( m \) is either \( n - p \) or \( p \)) in the orthogonal basis are given by

\[
\begin{align*}
\alpha^{(j)} &= e_j - e_{j+1}, & j &= 1, \ldots, m - 1, \\
\alpha^{(m)} &= e_m,
\end{align*}
(A.3)
\]

where \( e_j \) are the elementary \( m \)-dimensional basis vectors \( (e_j)_i = \delta_{ij} \).

In terms of the \( D_{n+1}^{(2)} \) generators

\[
\begin{align*}
H_i &= e_{ii} - e_{2n+3-1;2n+3-i}, & i &= 1, \ldots, n, \\
E_i^+ &= e_{i,i+1} + e_{2n+2-1;2n+3-i}, & i &= 1, \ldots, n - 1, \\
E_n^+ &= \frac{1}{\sqrt{2}} ( e_{n,n+1} + e_{n,n+2} - e_{n+2,n+3} - e_{n+1,n+3} ), \\
E_0^+ &= \frac{1}{\sqrt{2}} \eta ( e_{n+1,1} - e_{n+2,1} + e_{2n+2,n+1} - e_{2n+2,n+2} ), \\
E_i^- &= (E_i^+)^\dagger, & i &= 0, 1, \ldots, n,
\end{align*}
(A.4)
\]

where \( e_{ij} \) are the elementary \( (2n+2) \times (2n+2) \) matrices, the ‘left’ and ‘right’ generators are given by

\[
\begin{align*}
H_i^{(l)}(p) &= H_{p+i}, & E_i^{\pm(l)}(p) &= E_i^{\pm}(p), & i &= 1, \ldots, n - p, \\
H_i^{(r)}(p) &= -H_{p+1-i}, & E_i^{\pm(r)}(p) &= E_{p-i}^{\pm}, & i &= 1, \ldots, p,
\end{align*}
(A.5)
\]

respectively. The crucial point is that the broken generators \( E_p^{\pm} \) of \( D_{n+1}^{(2)} \) do not belong to either the ‘left’ or ‘right’ subalgebras.

The coproducts for the ‘left’ generators are given by

\[
\begin{align*}
\Delta(H_i^{(l)}) &= H_i^{(l)} \otimes 1 + 1 \otimes H_i^{(l)}, & j &= 1, \ldots, n - p, \\
\Delta(E_j^{\pm(l)}) &= E_j^{\pm(l)} \otimes e^{(n+1)e_0^{(0)\pm \eta_0^{(0)}}} + e^{-(n+1)e_0^{(0)\pm \eta_0^{(0)}}} \otimes E_j^{\pm(l)}, & j &= 1, \ldots, n - p - 1, \\
\Delta(E_{n-p}^{\pm(l)}) &= E_{n-p}^{\pm(l)} \otimes e^{\eta_{n-p}} + e^{-\eta_{n-p}} \otimes E_{n-p}^{\pm(l)}.
\end{align*}
(A.7)
\]
These coproducts satisfy
\[
\left[ \Delta(H_i^{(j)}), \Delta(E_j^{\pm (l)}) \right] = \pm \alpha_i^{(j)} \Delta(E_j^{\pm (l)}),
\]
and
\[
\Omega_y^{(j)} \Delta(E_j^{+(l)}) \Delta(E_j^{-(l)}) - \Delta(E_j^{-(l)}) \Delta(E_j^{+(l)}) \Omega_y^{(j)} = \delta_{ij} \frac{\sinh \left[ 2\eta \sum_{k=1}^{n-p} \alpha_k^{(j)} \Delta(H_k^{(j)}) \right]}{\sinh(2\eta)},
\]
where \( \Omega_y^{(j)} \) is given by
\[
\Omega_y^{(j)} = \begin{cases} 
\mathbb{I} & |i-j| = 1 \text{ and } 1 \leq \min(i,j) \leq n - p - 2, \\
\mathbb{I} \otimes \mathbb{I} & \text{otherwise}
\end{cases}
\]

The coproducts for the ‘right’ generators are given by
\[
\Delta(H_i^{(j)}) = H_i^{(j)} \otimes \mathbb{I} + \mathbb{I} \otimes H_i^{(j)},
\]
\[
\Delta(E_j^{+(r)}) = E_j^{+(r)} \otimes e^{-\eta H_1^{(1)}} \sinh(\eta H_1^{(1)}) + e^{\eta H_1^{(1)}} \sinh(\eta H_1^{(1)}) \otimes E_j^{+(r)},
\]
\[
\Delta(E_j^{-(r)}) = E_j^{-(r)} \otimes e^{\eta H_1^{(1)}} + e^{-\eta H_1^{(1)}} \otimes E_j^{-(r)}.
\]

These coproducts satisfy
\[
\left[ \Delta(H_i^{(j)}), \Delta(E_j^{\pm (r)}) \right] = \pm \alpha_i^{(j)} \Delta(E_j^{\pm (r)}),
\]
and
\[
\Omega_y^{(j)} \Delta(E_j^{+(r)}) \Delta(E_j^{-(r)}) - \Delta(E_j^{-(r)}) \Delta(E_j^{+(r)}) \Omega_y^{(j)} = \delta_{ij} \frac{\sinh \left[ 2\eta \sum_{k=1}^{n-p} \alpha_k^{(j)} \Delta(H_k^{(j)}) \right]}{\sinh(2\eta)},
\]
where \( \Omega_y^{(j)} \) is given by
\[
\Omega_y^{(j)} = \begin{cases} 
\mathbb{I} \otimes \mathbb{I} & |i-j| = 1 \text{ and } 1 \leq \min(i,j) \leq p - 2, \\
\mathbb{I} & \text{otherwise}
\end{cases}
\]

The operator \( \tilde{T}^{\pm (p)}(\eta) \) (4.6) can be expressed in terms of \( N \)-fold coproducts of these ‘left’ and ‘right’ generators, similarly to the other cases considered in [11].

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