BIQUADRATIC ADDITION LAWS ON ELLIPTIC CURVES IN $\mathbb{P}^3$ AND
THE CANONICAL MAP OF THE $(1, 2, 2)$-THETA DIVISOR

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ABSTRACT. We recall that a smooth ample surface $S$ in a general $(1, 2, 2)$-polarized abelian threefold, which is the pullback of the Theta divisor of a smooth plane quartic curve $D$, is a surface isogenous to the product $C \times C$, where $C$ is a genus 9 curve embedded in $\mathbb{P}^3$ as complete intersection of a smooth quadric and a smooth quartic. We show that the space of global holomorphic sections of the canonical bundle of this surface is generated by certain determinantal bihomogeneous polynomials of bidegree $(2, 2)$ on $\mathbb{P}^3$, which can be used to define biquadratic addition laws on the Jacobi model of elliptic curves, embedded in $\mathbb{P}^3$ as complete intersection of two quadrics. Finally, we use this interesting relationship with the biquadratic addition laws to describe the behavior of the canonical map of $S$.

1. Introduction

Let $(A, \mathcal{L})$ be a general $(1, 2, 2)$-polarized abelian threefold. We can consider an isogeny $p$ onto a principally polarized abelian threefold $(\mathcal{J}, \Theta)$, and we denote its kernel by $\mathcal{G}$, which is a group isomorphic to $\mathbb{Z}^2$ acting by translations on $A$. By our generality assumption on $A$, we can assume that $\mathcal{J}$ is the Jacobian variety of a non-hyperelliptic quartic plane $D$. Once identified $D$ with its embedded image in $\mathcal{J}$ through the Abel-Jacobi map, we can consider the pullback of $D$ through $p$, which we denote by $C$. The curve $C$ is a smooth projective curve of genus 9 with an unramified bidouble cover $\pi := p|_C$ onto $D$. It is well-known (see for instance [1] p. 226), that the Theta divisor $\Theta$ of $D$ is a translated with the vector of Riemann constants theta-characteristic of the subvariety $W_2(D) = \{ \mathcal{L} \in \text{Pic}^2(D) : h^0(D, \mathcal{L}) \geq 1 \}$.

According to Riemann’s Singularity Theorem (cf. [1] p.226), $W_2(D)$ is singular precisely when $D$ is hyperelliptic. Hence, by our generality assumption on $A$ and on $D$, we can identify $W_2(D)$ with the second symmetric product $D^{(2)}$, the latter defined as the quotient of the product $D \times D$ by the natural involution on the two factors. In particular, $D^{(2)}$ is a smooth surface which we regard as the set of effective divisors of degree 2 on $D$.

In this paper, we are interested in the problem of a purely geometrical description of the canonical map of the surface $S$ obtained by pulling back the Theta divisor $\Theta$ to $A$ through $p$. From the definition of $S$, it follows that $S$ is a bidouble unramified cover, which we can geometrically describe as a quotient

\[ S = C \times C / \Delta_\mathcal{G} \times \mathbb{Z}_2, \]

Key words and phrases. Surfaces of general type, Abelian varieties, Elliptic curves.

The present work was supported by the ERC Advanced grant n. 340258, TADMICANT.
where $\Delta_G$ denotes the diagonal subgroup of $G \times G$, acting naturally on the factors of the product $\mathcal{C} \times \mathcal{C}$, and where $\mathbb{Z}_2$ acts naturally on the two factors.

In the previous work [5], we studied the canonical map of $S$ by using projection methods: the isogeny $p$ factors through each of the three projections onto the three $(1,1,2)$-polarized abelian threefold, each of them obtained as a quotient of $A$ by a non-trivial element of $G$. Therefore, it is possible to investigate the behavior of canonical map of $S$ by looking closely at the canonical map of the corresponding quotients, which are surfaces of type $(1,1,2)$. The canonical map is of degree 2 and factors through a regular surface with 32 nodes, with $p_g = 4$ and $K^2 = 6$ (see [3]).

In this paper, we aim to study the canonical map of $S$ by using only the presentation 1.1 of $S$, which is, in particular, a surface isogenous to the product $\mathcal{C}^2$. We recall that surfaces isogenous to a product are the quotient of the form $\mathcal{C}_1 \times \mathcal{C}_2 / G$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are smooth projective curves and $G$ is a finite group acting freely of $\mathcal{C}_1 \times \mathcal{C}_2$. However, a satisfactory geometrical description of the canonical map of such surfaces has turned out to be in general a very challenging task (see [4]), which we leave aside in the hope to be able to address some aspects of this question in future. In our concrete case, it turned out that there exists a relationship between the canonical map of the previously defined surface $S$ and some bihomogeneous polynomials of bidegree $(2,2)$ in four variables which define addiction laws on certain elliptic curves in $\mathbb{P}_3$.

To introduce and describe more precisely this relationship, we start with the following lemma, which provides to us a very useful representation of the curve $\mathcal{C}$ and the unramified bidouble cover $p: \mathcal{C} \longrightarrow \mathcal{D}$ (cf. [5], lemma 2.6).

**Lemma 1.0.1.** Let $(A, \mathcal{L})$ be a general $(1,2,2)$-polarized Abelian 3-fold, let $p: A \longrightarrow \mathcal{J}$ be an isogeny onto the Jacobian of a general algebraic curve $\mathcal{D}$ of genus 3. Let us moreover consider the algebraic curve $\mathcal{C}$ obtained by pulling back to $A$ the curve $\mathcal{D}$, according to the following diagram:

$$(1.2) \quad \begin{array}{ccc} C & \longrightarrow & A \\ \downarrow & & \downarrow \\ D & \longrightarrow & \mathcal{J} \end{array}$$

Then, the following hold true:

- The genus 9 curve $\mathcal{C}$ admits $\mathcal{E}$ and $\mathcal{F}$ two distinct $G$-invariant $g_1$’s, with $\mathcal{E}^2 \neq \mathcal{F}^2$ and
  
  $$h^0(\mathcal{C}, \mathcal{E}) = h^0(\mathcal{C}, \mathcal{F}) = 2 \ .$$

- The line bundle $\mathcal{M} := \mathcal{E} \otimes \mathcal{F}$ is a very ample theta characteristic of type $g_3^3$.
- The image of $\mathcal{C}$ in $\mathbb{P}^3 = \mathbb{P}(\mathcal{M})$ is a complete intersection of the following type:

  $$(1.3) \quad \mathcal{C} : \begin{cases} X^2 + Y^2 + Z^2 + T^2 = 0 \\ q(X^2, Y^2, Z^2, T^2) = XYZT \ . \end{cases}$$
where \( q \) is a quadric, and there exist coordinates \([X, Y, Z, T]\) on \( \mathbb{P}^3 \) and two generators \( a, b \) of \( \mathcal{G} \) such that the projective representation of \( \mathcal{G} \) on \( \mathbb{P}^3 \) is represented by

\[
\begin{align*}
    a \cdot [X, Y, Z, T] &= [X, Y, -Z, -T] \\
    b \cdot [X, Y, Z, T] &= [X, -Y, Z, -T].
\end{align*}
\]

(1.4)

- The unramified covering \( p : \mathcal{C} \longrightarrow \mathcal{D} \) can be expressed as the map obtained by restricting to \( \mathcal{C} \) the rational map \( \psi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3 \) defined by

\[
\psi : [X, Y, Z, T] \mapsto [x, y, z, t] := [X^2, Y^2, Z^2, T^2].
\]

(1.5)

and the equations of \( \mathcal{D} \) in \( \mathbb{P}^3 = \mathbb{P}[x, y, z, t] \) are, according with (1.6), in the following form:

\[
\mathcal{D} : \begin{cases} 
    x + y + z + t = 0 \\
    q(x, y, z, t)^2 = xyzt.
\end{cases}
\]

(1.6)

We represent the points of \( \mathcal{S} \) by equivalence classes \([(P, Q)]\) of points of \( \mathcal{C} \times \mathcal{C} \) in \( \mathbb{P}^3 \times \mathbb{P}^3 \) with coordinates \([X_0 \cdots T_0]\) and \([X_1 \cdots T_1]\) on the two factors. Since the natural action of \( \mathcal{G} \) by translations in \( A \) is equivalent to the action of \( \mathcal{G} \) on the cosets of the diagonal subgroup in \( \mathcal{G}^2 \), the action of \( \mathcal{G} \) on \( \mathcal{S} \) can be naturally represented as the action of \( \mathcal{G} \) on the second component of \( \mathcal{C}^2 \); if \( g \) denotes an element of \( \mathcal{G} \) and \([(P, Q)]\) a point on \( \mathcal{S} \), we have

\[
g \cdot [P, Q] = [P, gQ].
\]

We can now easily exhibit a basis for \( H^0(\mathcal{S}, \omega_\mathcal{S}) \). By lemma 1.0.1, \( \omega_\mathcal{C} \) is the restriction of \( \mathcal{O}_{\mathbb{P}^2}(2) \) on \( \mathcal{C} \) and \( H^0(\mathcal{C}, \omega_\mathcal{C}) \) splits into a direct sum of \( \mathcal{G} \) invariant subspaces

\[
H^0(\mathcal{C}, \omega_\mathcal{C}) = W_{++} \oplus W_{+-} \oplus W_{-+} \oplus W_{--},
\]

according to the signs of the action of two fixed generators \( a \) and \( b \) of \( \mathcal{G} \) on the coordinates of \( \mathbb{P}^3 \) (see 1.4).

\[
\begin{align*}
    W_{++} &= \langle X^2|c, Y^2|c, Z^2|c \rangle \\
    W_{+-} &= \langle XY|c, ZT|c \rangle \\
    W_{-+} &= \langle XZ|c, YT|c \rangle \\
    W_{--} &= \langle XT|c, YZ|c \rangle
\end{align*}
\]

Recall that \( \mathcal{S} \) is defined as the quotient of \( \mathcal{C} \times \mathcal{C} \) by the action of \( \Delta_G \times \mathbb{Z}_2 \). Hence,

\[
H^0(\mathcal{S}, \omega_\mathcal{S}) = H^0(\mathcal{C} \times \mathcal{C}, \omega_\mathcal{C} \boxtimes \omega_\mathcal{C})^{\Delta_G \times \mathbb{Z}_2}
\]

where the latter vector space denotes the \( \Delta_G \times \mathbb{Z}_2 \)-invariant global holomorphic sections of \( \omega_\mathcal{C} \boxtimes \omega_\mathcal{C} \).
In conclusion, the following is a basis for $H^0(S, \omega_S)$:

$$\eta_{12} := \begin{vmatrix} X_0^2 & X_1^2 \\ Y_0^2 & Y_1^2 \end{vmatrix}$$

$$\eta_{13} := \begin{vmatrix} X_0^2 & X_1^2 \\ Z_0^2 & Z_1^2 \end{vmatrix}$$

$$\eta_{23} := \begin{vmatrix} Y_0^2 & Y_1^2 \\ Z_0^2 & Z_1^2 \end{vmatrix}$$

These determinantal polynomials of bidegree $(2, 2)$ turn out to be strictly related to some biquadratic addition laws on the elliptic curves in $\mathbb{P}^3$ defined as follows: we fix two non-zero complex numbers $u$ and $v$ such that $u + v$ is also non-zero. Then the following complete intersection is a smooth elliptic curve in $\mathbb{P}^3$:

$$\omega_{45} := \begin{vmatrix} X_0Y_0 & X_1Y_1 \\ Z_0T_0 & Z_1T_1 \end{vmatrix}$$

$$\omega_{67} := \begin{vmatrix} X_0Z_0 & X_1Z_1 \\ Y_0T_0 & Y_1T_1 \end{vmatrix}$$

$$\omega_{89} := \begin{vmatrix} X_0T_0 & X_1T_1 \\ Y_0Z_0 & Y_1Z_1 \end{vmatrix}$$

One can show that (cf. [2], p.22) that the rational map $\oplus : \mathbb{P}^3 \times \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ defined by the four biquadratic polynomials $[X_0^2Y_1^2 - Y_0^2X_1^2, X_0Y_0Z_1T_1 - Z_0T_0X_1Y_1, X_0Z_0Y_1T_1 - Y_0T_0X_1Z_1, X_0T_0Y_1Z_1 - Y_0Z_0X_1T_1]$ coincides, whenever defined, with the group law on $J_{u,v}$. If we compare these polynomials with $\eta_{01}, \omega_{45}, \omega_{67}, \omega_{89}$ in (1.7) we clearly notice that $\oplus|_{C \times C} = [\eta_{01}, \omega_{45}, \omega_{67}, \omega_{89}]$.

The paper is organized as follows. We recall in section 2 the general definition of addition law on a fixed, embedded abelian variety, and we specialize it to the case of biquadratic addition laws on embedded elliptic curves in $\mathbb{P}^3$. In the last section, we show how the previously mentioned relationship with the biquadratic addition laws on embedded elliptic curves of the form $J_{u,v}$ can be used to study the canonical map of $S$. We recall that, by the projection formula, it holds a decomposition in 1-dimensional vector spaces

$$H^0(A, \mathcal{L}) = \bigoplus_{\chi \in \mathcal{G}} H^0(J, O_J(\Theta) \otimes L_\chi)$$

where $L_\chi$ is a 2-torsion line bundle on $J$. Clearly, for every non-trivial element $g$ there exists a unique non-trivial character $\chi$ of $G$ whose kernel is generated by $g$, and we can define $S_g$ as the zero locus of a non-zero section of $H^0(J, O_J(\Theta) \otimes L_\chi)$. We conclude the third section with a proof of the following result:

**Theorem 1.0.2.** Let $U, V$ be points on $S$ such that $\phi_S(U) = \phi_S(V)$. Then one of the following cases occurs:

- $V = U$
• $V = -gU$ for some non-trivial element $g$ of $G$. This case arises precisely when $U$ and $V$ belong to the canonical curve $S \cap S_g$.
• $V = gU$ for some non-trivial element $g$ of $G$. This case arises precisely when $U$ and $V$ belong to the translate $S_h$, for every $h \in G - \{g\}$.
• $U$ and $V$ are two base points of $|S|$ which belong to the same $G$-orbit.

2. Addition laws of bidegree $(2, 2)$ on elliptic curves in $\mathbb{P}^3$

Throughout this section, we denote by $(A, \mathcal{L})$ a polarized abelian variety, with $\mathcal{L}$ a very ample line bundle. We denote by $\phi$ the holomorphic embedding $\phi|_{\mathcal{L}} : A \rightarrow \mathbb{P}^N$ defined by the linear system $|\mathcal{L}|$ on $A$. Moreover, we denote by $\mu, \delta : A \times A \rightarrow A$ the morphisms respectively defined by the sum and the difference in $A$, and by $\pi_1$ and $\pi_2$ the projections of $A \times A$ onto the respective factors. To prevent misunderstandings, we refer to $\mu$ as the group law on $A$, and we distinguish it from the notion of addition law which we are going to introduce in this section. An addition law $\oplus$ of bidegree $(m, n)$ is a rational map $\mathbb{P}^N \times \mathbb{P}^N \rightarrow \mathbb{P}^N$ defined by an ordered set of bihomogeneous polynomials $(f_0, \cdots , f_N)$, each of bidegree $(m, n)$, and such that there exists a non-empty Zariski open set $U$ of $A \times A$ on which $\oplus$ and $\mu$ coincide (see also [4]).

An addition law on $A$ can be viewed then as a rational map $\oplus : \mathbb{P}^N \times \mathbb{P}^N \rightarrow \mathbb{P}^N$ such that the following diagram commutes:

\[
\begin{array}{ccc}
A \times A & \xrightarrow{\phi \times \phi} & \mathbb{P}^N \times \mathbb{P}^N \\
\mu & \downarrow & \oplus \\
A & \xrightarrow{\phi} & \mathbb{P}^N
\end{array}
\]

Assigned an addition law $\oplus$ on $A$ defined by bihomogeneous polynomials $(f_0, \cdots , f_N)$, we denote by $W(\oplus)$ the sublinear system of $|O_{\mathbb{P}^N}(m) \boxtimes O_{\mathbb{P}^N}(n)|$ generated by $f_0, \cdots , f_N$.

In particular, the rational map $A \times A \rightarrow \mathbb{P}^N$, which in diagram (2.1) is defined as the composition $\phi \times \phi$ with $\oplus$, is defined by $N + 1$ linearly independent global sections of

\[
(\phi \times \phi)^* (O_{\mathbb{P}^N}(m) \boxtimes O_{\mathbb{P}^N}(n)) = \pi_1^* \mathcal{L}^m \boxtimes \pi_2^* \mathcal{L}^n .
\]

The morphism $\phi \circ \mu$ is defined, on the other side, by the complete linear system $|\mu^* \mathcal{L}|$ on $A \times A$. By applying the projection formula and by the fact that $\mu$ is a morphism with connected fibers, we have that

\[
H^0(A \times A, \mu^* \mathcal{L}) \cong H^0(A, \mathcal{L}) .
\]

Hence, a rational map $\oplus : \mathbb{P}^N \times \mathbb{P}^N \rightarrow \mathbb{P}^N$ of bidegree $(m, n)$ such that the previous diagram commutes can be expressed as a global section of

\[
(2.2) \quad \mathcal{M}_{m,n} := \mu^* \mathcal{L}^{-1} \boxtimes \pi_1^* \mathcal{L}^m \boxtimes \pi_2^* \mathcal{L}^n .
\]

Thus, our discussion justifies the following definition:

**Definition 2.0.1.** (Addition law, [6]) Let $(A, \mathcal{L})$ be a polarized abelian variety, where $\mathcal{L}$ is assumed to be very ample. Let $m, n$ two non-zero natural numbers. An addition law of bidegree $(m, n)$ on $A$ is a global section of $\mathcal{M}_{m,n}$, the latter defined as in (2.2).
Let \( s \in H^0(A \times A, \mathcal{M}_{m,n}) \) be a non-zero addition law of bidegree \((m, n)\). If we consider the rational map \( \oplus : \mathbb{P}^N \times \mathbb{P}^N \to \mathbb{P}^N \) defined by \( s \), and \( I_A \) the homogeneous defining ideal of \( A \) in \( \mathbb{P}^N \), then the restriction of \( \oplus \) to \( A \times A \) is given by some bihomogeneous polynomials \( f_0 \cdots f_N \) of bidegree \((m, n)\) in \( k[P_N]/I_A \) which define the group law \( \mu \) on \( A \) away from the base locus \( Z \) of \( W(\oplus) \). The locus \( Z \), which is the indeterminacy locus of the rational map \( \oplus \), will be called **exceptional locus of** \( s \). By looking at the map \( \mu \circ \phi \) in diagram 2.1 it can be seen now that this exceptional locus coincides with \( \text{div}(s) \), and it is, in particular, a divisor in \( A \times A \).

**Definition 2.0.2.** A set of addition laws \( s_1 \cdots s_k \) of bidegree \((m, n)\) is said to be a **complete set of addition laws** if:

\[
\text{div}(s_1) \cap \cdots \cap \text{div}(s_k) = \emptyset.
\]

In particular, there exists a complete set of addition laws of bidegree \((m, n)\) if and only if \( |\mathcal{M}_{m,n}| \) is base point free.

The problem of determining, whether for a given bidegree \((m, n)\) with \( m, n \geq 2 \) there exists an addition law (resp. a complete set of addition laws), has been solved by Lange and Ruppert (see [6] p. 610). Their main result is:

**Theorem 2.0.3.** Let \( A \) be an abelian variety embedded in \( \mathbb{P}^N \), and \( \mathcal{L} = \mathcal{M}^m \), with \( m \geq 3 \), a very ample line bundle defining the embedding of \( A \) in \( \mathbb{P}^N \). Then:

- There are complete systems of addition laws on \( A \subseteq \mathbb{P}^N \) of bidegree \((2, 3)\) and \((3, 2)\).
- There exists a system of addition laws on \( A \subseteq \mathbb{P}^N \) of bidegree \((2, 2)\) if and only if \( \mathcal{L} \) is symmetric. Furthermore, in this case, there exists a complete system of addition laws.

We focus now our attention on the case of biquadratic addition laws. When the line bundle \( \mathcal{L} \) is symmetric, by applying the projection formula (note moreover that \( \delta \) is a proper morphism with connected fibers) we have that

\[
H^0(A \times A, \mathcal{M}_{(2,2)}) = H^0(A \times A, \delta^* \mathcal{L}) \cong H^0(A, \mathcal{L})
\]

We see first a model of a smooth elliptic curve in \( \mathbb{P}^3 \) not contained in any hyperplane:

**Definition 2.0.4.** (Jacobi’s model, see also [2] p.21) Let \( u, v, w \) be three non-zero complex numbers such that \( u + v + w = 0 \). We denote by \( J_{u,v} \) the elliptic curve in \( \mathbb{P}^3 \) with coordinates \( X, \cdots, T \) defined as the complete intersection of two of the following three quadrics:

\[
J_{u,v} : \begin{cases}
u X^2 + Y^2 = Z^2 \\
u X^2 + Z^2 = T^2 \\
u X^2 + T^2 = Y^2.
\end{cases}
\]

On \( \mathbb{P}^3 \times \mathbb{P}^3 \), we denote by \([X_0 \cdots T_0]\) the coordinates on the first factor and by \([X_1 \cdots T_1]\) the coordinates for the second one. An explicit basis of the space of the biquadratic addition laws has been determined [2]:
Theorem 2.0.5. The vector space $H^0(J_{u,v} \times J_{u,v}, \mathcal{M}_{(2,2)})$ of the addition laws of bidegree $(2, 2)$ for the elliptic curve $J_{u,v}$ in $\mathbb{P}^3$ defined by the Jacobi quadratic equation is generated by:

$$\begin{align*}
\oplus_X := & [X_0^2Y_1^2 - Y_0^2X_1^2, X_0Y_0Z_1T_1 - Z_0T_0X_1Y_1, \\
& X_0Z_0Y_1T_1 - Y_0T_0X_1Z_1, X_0T_0Y_1Z_1 - Y_0Z_0X_1T_1] \\
\oplus_Y := & [X_0Z_0Y_1T_1 + Y_0T_0X_1Z_1, -uX_0T_0X_1T_1 + Y_0Z_0Y_1Z_1, \\
& wX_0^2X_1^2 + Z_0^2Z_1^2, vX_0Y_0X_1Y_1 + Z_0T_0Z_1T_1] \\
\oplus_Z := & [X_0Y_0Z_1T_1 + Z_0T_0X_1Y_1, uwX_0^2X_1^2 + Y_0^2Y_1^2, \\
& uX_0T_0X_1T_1 + Y_0Z_0Y_1Z_1, -wX_0Z_0X_1Z_1 + Y_0T_0Y_1T_1] \\
\oplus_T := & [u(X_0T_0Y_1Z_1 + Y_0Z_0X_1T_1), u(wX_0Z_0X_1Z_1 + Y_0T_0Y_1T_1), \\
& u(-vX_0Y_0X_1Y_1 + Z_0T_0Z_1T_1), -vY_0^2Y_1^2 - wZ_0^2Z_1^2 ].
\end{align*}$$

Moreover, for every $H \in \{X, Y, Z, T\}$, the exceptional divisor of $\oplus_H$ is $\delta^*(H)$, where $H$ denotes the corresponding hyperplane divisor $\mathbb{P}^3$.

Proof. See [2], p.22 \qed

Remark 2.0.6. Note that, by theorem 2.0.5, the exceptional divisor of $\oplus_X$ is $\delta^*((X = 0))$ and the divisor $(X = 0)$ on the elliptic curve $J_{u,v}$ is exactly $T_{id} + T_a + T_b + T_{ab}$, where $a$ and $b$ are generators of $G$ acting on the coordinates of $\mathbb{P}^3$ as in lemma 1.0.1 and

$$\begin{align*}
T_{id} := & \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} & T_a := & \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} & T_b := & \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} & T_{ab} := & \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}.
\end{align*}$$

As the notation suggests, for every element $g$ of $G$ the natural action of the point $T_g$ on $J_{u,v} - \Delta_2$ via $\oplus_X$ coincides with the action of $g$. Hence $\Delta_2 := \{T_{id}, T_a, T_b, T_{ab}\}$ is the group of 2-torsion points on $J_{u,v}$.

It is moreover possible to verify that, according to theorem 2.0.5, the addition law $\oplus_X$ is not defined precisely on the union of the four copies of $J_{u,v}$ in $J_{u,v} \times J_{u,v}$ which correspond to the 2-torsion points of $J_{u,v}$:

$$Z := \text{div}(\oplus_X) = \bigcup_{g \in G} \{ (P, g.P) \mid P \in J_{u,v} \} \subseteq \mathbb{P}^3 \times \mathbb{P}^3.$$
Definition 2.0.7. To simplify the notations we will denote the addition law $\oplus_X$ on $J_{u,v}$ simply by $\oplus$, and we denote the defining biquadratic polynomials by

$$
\eta_{12} := \begin{vmatrix}
X_0^2 & X_1^2 \\
Y_0^2 & Y_1^2
\end{vmatrix}
$$

$$
\omega_{45} := \begin{vmatrix}
X_0 Y_0 & X_1 Y_1 \\
Z_0 T_0 & Z_1 T_1
\end{vmatrix}
$$

$$
\omega_{67} := \begin{vmatrix}
X_0 Z_0 & X_1 Z_1 \\
Y_0 T_0 & Y_1 T_1
\end{vmatrix}
$$

$$
\omega_{89} := \begin{vmatrix}
X_0 T_0 & X_1 T_1 \\
Y_0 Z_0 & Y_1 Z_1
\end{vmatrix}
$$

(2.5)

Definition 2.0.8. (A more general model in $\mathbb{P}^3$) For our applications we need a slightly different model of smooth elliptic curve in $\mathbb{P}^3$. Under the hypothesis that $a,b,c$ and $d$ are all distinct complex numbers, the curve $\mathcal{E}$ in $\mathbb{P}^3$ defined by the following couple of quadrics is a smooth elliptic curve:

$$
\mathcal{E} := \begin{cases}
ax^2 + by^2 + cz^2 + dt^2 = 0 \\
x^2 + y^2 + z^2 + t^2 = 0
\end{cases}
$$

We can see now that, up to a choice of signs which represents the action of a 2-torsion point on $\mathcal{E}$, we can define an addition law which plays the role of the addition law $\oplus_X$ defined on the Jacobi model in definition and Theorem 2.0.5. The first step is to work out the equations (2.6) to obtain a Jacobi model isomorphic to $\mathcal{E}$ (see equations (2.3)). We have

$$
\mathcal{E} := \begin{cases}
\frac{a-d}{b-d} x^2 + \frac{c-d}{b-d} z^2 \\
-\frac{a-c}{b-c} x^2 + \frac{c-d}{b-c} t^2 = y^2
\end{cases}
$$

(2.7)

We consider now $\alpha$ and $\beta$ square roots of $\frac{c-d}{b-d}$ and $\frac{a-c}{b-c}$ respectively. By rescaling the coordinates $Z$ and $T$ with $\alpha$ and $\beta$ we see that $\mathcal{E} \cong J_{\frac{a-d}{b-d}, \frac{a-c}{b-c}}$ and we obtain on $\mathbb{P}^3 \times \mathbb{P}^3$ a rational map corresponding to $\oplus_X$, which represent an addition law of $\mathcal{E}$, up to the choice of the sign of $\alpha$ and $\beta$:

$$
\oplus_X : (P,Q) \mapsto \begin{bmatrix}
\eta_{12}(P,Q) \\
\alpha \beta \omega_{45}(P,Q) \\
\beta \omega_{67}(P,Q) \\
\alpha \omega_{89}(P,Q)
\end{bmatrix}
$$

(2.8)

Indeed, the rational map defined in (2.8) is an addition law up to the action of a 2-torsion point, according to remark 2.0.6. This means that this rational map $\oplus_X$ represents an operation on $\mathcal{E}$ of the following form:

$$
\tilde{\mu}(P,Q) = \mu(T, \mu(P,Q)) = T + P + Q,
$$

where $T$ is a 2-torsion point on $\mathcal{E}$.
3. The canonical map of the \((1,2,2)\) theta-divisor and its geometry

The sublinear system of \(|\omega_S|\) generated by the \(G\)-invariant sections \(\eta_{12}, \eta_{13}\) and \(\eta_{23}\) defines the Gauss map \(G : S \rightarrow \mathbb{P}^2\). This map factors through the isogeny \(p\) and the Gauss map of \(\Theta\), which can be seen as the map which associates to every divisor \(p + q\) on \(D\) the unique line \(l\) in \(\mathbb{P}^2 = \mathbb{P}(H^0(D, \omega_D))\) which cuts on \(D\) a canonical divisor greater than \(u + v\).

We aim now to describe the behavior of the component of the canonical map of \(S\) which is defined by the other three holomorphic sections of the canonical bundle of \(S\), which are \(\omega_{45}, \omega_{67}\) and \(\omega_{89}\). First, we have that the image of the restriction map \(H^0(A, \mathcal{O}_A(S)) \rightarrow H^0(S, \omega_S)\) is the subspace generated by \(\omega_{45}, \omega_{67}\) and \(\omega_{89}\).

**Definition 3.0.1.** In the decomposition in 1-dimensional vector spaces

\[
H^0(A, \mathcal{O}_A(S)) = \bigoplus_{\chi \in G} H^0(J, \mathcal{O}_J(D^{(2)}) \otimes L_\chi),
\]

where \(L_\chi\) are 2-torsion line bundle on \(J\), we have that

\[
H^0(J, \mathcal{O}_J(D^{(2)}) \otimes L_{\chi_a}) = \langle \omega_{45} \rangle, \\
H^0(J, \mathcal{O}_J(D^{(2)}) \otimes L_{\chi_b}) = \langle \omega_{67} \rangle, \\
H^0(J, \mathcal{O}_J(D^{(2)}) \otimes L_{\chi_{ab}}) = \langle \omega_{89} \rangle,
\]

where \(\chi_g\) denotes the unique non-trivial character of \(G\) such that \(\chi_g(g) = 1\). Clearly, for every non-trivial element \(g\) there exists a unique non-trivial character \(\chi\) of \(G\) whose kernel is generated by \(g\), and we can define \(S_g\) as the zero locus of the generator of \(H^0(J, \mathcal{O}_J(\Theta) \otimes L_{\chi_g})\).

The multiplication by \(-1\) in the Jacobian \(J\) corresponds to the Serre involution in \(\mathcal{D}^{(2)}\), which sends a divisor \(p + q\) to the unique divisor \(r + s\) such that \(p + q + r + s\) is a canonical divisor on \(D\). Hence, all global sections of \(H^0(A, S)\) are odd, being \(\mathcal{D}^{(2)}\) a translated of \(\Theta\) with an odd theta characteristic, and being \(\Theta\) the zero locus of the Riemann Theta function, which is an even function. Moreover, one can easily see that the base locus of \(|\mathcal{O}_A(S)|\) is a set of 16 points (\(A\) is supposed to be general), which on \(S\) is defined as the set where \(\omega_{45}, \omega_{67}\) and \(\omega_{89}\) vanish. In remark 3.0.4 we will characterize this locus in terms of the equation of the curve \(\mathcal{D}\).

**Definition 3.0.2.** Let \(U := \{(P, Q)\}\) be a point on \(S\), and \(r := G(U)\) the line \(\{ax + by + cz = 0\}\), where

\[
(a, b, c) = [\eta_{12}(U), -\eta_{02}(U), \eta_{01}(U)] \in \mathbb{P}^2.
\]

The pullback of this line through the rational map \(\psi : \mathbb{P}^3 \rightarrow \mathbb{P}^3\) which squares the coordinates (see 1.5) is the quadric

\[
\mathcal{R}_U : ax^2 + by^2 + cz^2 = 0.
\]

Finally, we denote by \(\mathcal{E}_U\) the locus defined by the intersection of \(\mathcal{R}_U\) with the \(G\)-invariant quadric of \(\mathbb{P}^3\) containing \(C\) (see equation 1.6):

\[
\mathcal{E}_U := \left\{ \begin{array}{ll}
ax^2 + by^2 + cz^2 = 0 \\
x^2 + y^2 + z^2 + t^2 = 0.
\end{array} \right.
\]
The curve $E_U$ is a smooth curve of genus 1 if and only if $a$, $b$ and $c$ are non zero and all distinct. In this case, (c.f. definition 2.0.8) there exist two constants $\alpha_U$ and $\beta_U$, which depend only on $a$, $b$, $c$, and a biquadratic addition law $\oplus^U$ on $E_U$, which is defined as follows:

$$
\oplus^U : (X, Y) \mapsto \begin{bmatrix}
\eta_0(X, Y) \\
\alpha_U \beta_U \omega_{45}(X, Y) \\
\beta_U \omega_{67}(X, Y) \\
\alpha_U \omega_{89}(X, Y)
\end{bmatrix}.
$$

By definition it follows that, if for two points $U = [P, Q]$ and $V = [R, S]$ we have that $\phi_{K_S}(U) = \phi_{K_S}(V)$, then $U$ and $V$ define the same locus $E_U$. We prove now that a closer relationship between the group law $E_U$ and the canonical group of $S$ holds.

**Lemma 3.0.3.** Let be $U = [P, Q]$ and $V = [R, S]$ two points of $S$ such that $E_U$ and $E_V$ are smooth. If $\phi_{K_S}(U) = \phi_{K_S}(V)$, then $E_U = E_V$ and $\mu_U(P, Q) = \mu_V(R, S)$ holds, where $\mu_U$ is the group law in $E_U$.

**Proof.** Let us consider the addition law $\oplus^U$ defined on $E_U$. For every point $W = [A, B]$ in a suitable neighborhood $U$ of $U$ in $S$, the locus $E_W$ is still a smooth elliptic curve, and we can then denote by $\tau_W$ a corresponding element in the Siegel upper half plane $H_1$ such that $E_W = \mathbb{C}/Z \oplus \tau_WZ$. Moreover, for every $W$ in such a neighborhood it is well-defined $\mu_W(W)$, where $\mu_W$ denotes the group law in $E_W$ and

$$
\mu_W(W) := \mu_W(A, B).
$$

Indeed, it can be easily seen that the definition does not depend on the choice of the representative of $W$.

We denote now by $\theta_0(z, \tau_W), \theta_1(z, \tau_W), \theta_2(z, \tau_W), \theta_3(z, \tau_W)$ the four theta functions defining the embedding of $E_W$ in $\mathbb{P}^3$, and by $\Psi$ the holomorphic map $\Psi : U \longrightarrow \mathbb{P}^3$ defined as follows:

$$
\begin{pmatrix}
1 \\
\alpha \\
\beta \\
\alpha \beta
\end{pmatrix} \circ \pi \circ \phi_S,
$$

where $\pi$ is the following projection $\mathbb{P}^5 \longrightarrow \mathbb{P}^3$:

$$
[\eta_0, \eta_1, \eta_2, \omega_{45}, \omega_{67}, \omega_{89}] \longrightarrow [\eta_0, \omega_{45}, \omega_{67}, \omega_{89}],
$$

and $\alpha$ and $\beta$ determinations of square roots of $-\frac{m_2}{m_1}$ and $-\frac{m_2}{\eta_3 \eta_2}$ respectively, which are defined according to definitions 3.1 and 2.0.8. The map $\Psi$ is defined everywhere on $U$ because, on every point of $U$, we have that $\eta_1 \neq 0$ and $\eta_2 \neq -\eta_3$ by definition of $U$, and in particular $\alpha$ and $\beta$ can be considered simply as holomorphic functions defined on $U$ as well and with values in $\mathbb{C}^*$. We remark, furthermore, that the choice of the branch of the square root used to define $\alpha$ and $\beta$ is not important because another choice leads to a sign-change of the coordinates to the function $\Psi$ accordingly to the action of the group $G$ on the coordinates of $\mathbb{P}^3$ (cf. definition 2.0.8). The map $\Psi$ is then:

$$
\Psi(W) = [\eta_0(W), \alpha \beta \omega_{45}(W), \beta \omega_{67}(W), \alpha \omega_{89}(W)] = \oplus^W(W) = [	heta_0 \circ \mu_W(W), \theta_1 \circ \mu_W(W), \theta_2 \circ \mu_W(W), \theta_3 \circ \mu_W(W)].
$$
Hence, if $\phi_S(U) = \phi_S(V)$ and $E_U = E_V$ are smooth elliptic curves, then $\Psi(U) = \Psi(V)$ and in particular there exists a non-zero $\zeta \in \mathbb{C}^*$ such that for every $j = 0, \ldots, 3$ we have

$$\theta_j \circ \mu_U(U) = \zeta \cdot \theta_j \circ \mu_U(V).$$

On the other hand, the sections $\theta_j$ on $E_U$, with $j = 0, \ldots, 3$, embed $E_U$ in $\mathbb{P}^3$, so we can conclude that $\mu_U(U) = \mu_U(V)$. \hfill \Box

**Remark 3.0.4.** In the notation of lemma 3.0.1 we consider the quartic curve $D$ in $\mathbb{P}^3$ defined by

$$D : \begin{cases} x + y + z + t = 0 \\ g(x, y, z, t)^2 = xyzt. \end{cases}$$

We see that the lines $x$, $y$, $z$ and $t$ in the plane $H : x + y + z + t = 0$ are bitangents. For every such a line $l$ we denote by $l_1 + l_2$ the effective divisor on $D$ such that

$$l \cdot D = 2(l_1 + l_2).$$

We select two points $L_1$ and $L_2$ in the respective preimages in $C$ with respect to $p$. Then, by remark 3.0.1 we see that $G([L_1, L_2])$ is a $G$-orbit of base points for $L$ in $A$, since $\omega_{45}$, $\omega_{67}$ and $\omega_{89}$ vanish on $[[L_1, L_2]]$. Since the set of base points of a $(1, 2, 2)$-polarization on a generic abelian variety $A$ is a finite set of 2-torsion points on $A$ of order 16, we have determined all base points.

**Theorem 3.0.5.** Let $U$, $V$ be points on $S$ such that $\phi_S(U) = \phi_S(V)$. Then one of the following cases occurs:

- $V = U$
- $V = -g \cdot U$ for some non-trivial element $g$ of $G$. This case arises precisely when $U$ and $V$ belong to the canonical curve $S \cap S_y$.
- $V = g \cdot U$ for some non-trivial element $g$ of $G$. This case arises precisely when $U$ and $V$ belong to the translate $S_h$, for every $h \in G - \{g\}$.
- $U$ and $V$ are two base points of $|S|$ which belong to the same $G$-orbit.

**Proof.** Let us consider $U = [P, Q]$ and $V = [R, S]$ two points on $S$, and let us assume that $\phi_S(U) = \phi_S(V)$. Let $p$, $q$, $r$ and $s$ denote, moreover, the corresponding points on $D$, and $[a, b, c] = [\eta_{23}, -\eta_{13}, \eta_{12}]$ the coefficients of the line $l := G(U) = G(V) \in \mathbb{P}^{2, V}$ according to 3.1.

Depending on the coefficients, the locus $E := E_U$ will be smooth or not. However, up to exchange $a$, $b$, and $c$ we can assume that we are in one of the following cases:

- i) $a$, $b$ and $c$ are all distinct and non-zero. In this case, $E$ is a smooth elliptic curve.
- ii) $c = 0$, but $b \neq 0 \neq a$ and $a \neq b$. In this case, the locus $E$ is the union of two irreducible conics in $\mathbb{P}^3$ which meet in a point not on $C$.
- iii) $c = 0$ and $b = 0$. In this case, $l$ is the bitangent $x$, and $E$ is a double conic contained in the hyperplane $\{X = 0\}$ in $\mathbb{P}^3$. This case occurs precisely when $U$ and $V$ are base points. (cf. definition 3.0.4)
- iv) $c = 0$ and $a = b \neq 0$. In this case, the locus $E$ is the union of four lines, each couple of them lying on a plane and intersecting in a point not belonging to $C$. 


We begin with the first case and we assume that $E$ is a smooth elliptic curve. Then by lemma 3.0.3 we have that:

\[(3.2) \quad \mu(P,Q) = \mu(R,S) ,\]

where $\mu$ is the group law in $E$, and we assume that $U \neq V$. Up to exchange $R$ and $S$ we can suppose that $R \neq P$ and $S \neq Q$ by the previous identity 3.2.

If $S$ belongs to the $G$-orbit of $P$, we can assume that $S = P$, because we can act on the representatives of $U$ and $V$ with the diagonal subgroup $\Delta_G$, and by (3.2) it follows that $R = Q$, and finally that $U = V$. Thus, we shall assume that $S$ does not belong to the $G$-orbit of $P$, and that the $G$-orbits of $R$ and $S$ are disjoint from the $G$-orbits of $P$ and $Q$. Thus, the points of the canonical divisor $p + q + r + s$ are such that $p \neq r$, $p \neq s$, $q \neq r$ and $q \neq s$, and the divisor $R + S$ on $C$ is the preimage of the Serre dual of the divisor $p + q$ on $D$. Hence, it must exist an element $g \in G$ such that:

\[V = -g.U\]

The element $g$ is not the identity because otherwise $U$ and $V$ were both base points (see definition 3.0.1), and in such a case we would reach a contradiction by remark 3.0.4 since $E$ cannot be smooth in this case. Hence, the theorem is proved in this case.

In the remaining cases is not possible to apply lemma 3.0.3 since $E$ is no longer smooth. Nevertheless, we can assume without loss of generality that $U = [P,Q]$ and $V = [P,R]$, where $P,Q,R$ are three points on $C$.

Suppose we are in the second case. Then $E$ is defined by the equations:

\[E := \begin{cases} aX^2 + bY^2 = 0 \\ X^2 + Y^2 + Z^2 + T^2 = 0 \end{cases} ,\]

where:

\[E = Q^+ \cup Q^- \]

\[Q^e = \begin{cases} Y = \epsilon i \sqrt{\frac{b}{a}} X \\ X^2 + Y^2 + Z^2 + T^2 = 0 \end{cases} \]

and $\epsilon$ denotes a sign. We choose the following parametrization $f^e : \mathbb{P}^1 \rightarrow Q^e \subseteq \mathbb{P}^3$:

\[(3.3) \quad f^e([u,v]) := \left[ \frac{uv}{\sqrt{1 - \frac{b}{a}}}, \epsilon i \sqrt{\frac{b}{a}} \frac{uv}{\sqrt{1 - \frac{b}{a}}}, \frac{i}{2} (u^2 + v^2), \frac{i}{2} (u^2 - v^2) \right] \]

\[Q^+ \cap Q^- = f^e([1,0]) = f^e([0,1]) \notin C .\]

The choice of the square roots in definition 3.3 is not important. Furthermore, we notice that the group $G$ acts in the following form:

\[a.f^e([u,v]) = f^e([u,-v]) \]

\[b.f^e([u,v]) = f^{-e}([v,u]) .\]
Hence, without loss of generality, we can assume that
\[ P := f^1([u, 1]) \]
\[ Q := f^1([v, 1]) \]
\[ R := f^s([w, 1]) \]
\[ \phi_S(U) = \phi_S(V) \ . \]

Moreover, without loss of generality we can assume that \( R \) does not belong to the \( G \)-orbit of \( P \). In this setting, we have to prove that \( v = w \) and that \( \epsilon = 1 \). First of all, we have that:
\[ \eta_02(V) = \eta_02([f^1([u, 1]), f^1([w, 1])]) = -\frac{1}{4(1 - \frac{a}{b})}(u^2 - w^2)(1 - u^2w^2) \]
\[ \eta_12(V) = \eta_12([f^1([u, 1]), f^1([w, 1])]) = -\frac{a}{4(1 - \frac{a}{b})}(u^2 - w^2)(1 - u^2w^2) \ . \]

In the same way, the following expressions of the sections \( \omega_{45}, \omega_{67} \) and \( \omega_{89} \) hold, up to a constant independent from \( u, v \) and \( \epsilon \):
\[ \omega_{45}(V) = \begin{vmatrix} u^2 & \epsilon w^2 \\ u^4 - 1 & w^4 - 1 \end{vmatrix} = \begin{cases} -(u^2 - w^2)(u^2w^2 + 1) & \text{if } \epsilon = 1 \\
(u^2 + w^2)(u^2w^2 - 1) & \text{if } \epsilon = -1 \end{cases} \]
(3.4)
\[ \omega_{67}(V) = \begin{vmatrix} u(u^2 + 1) & \epsilon w(u^2 + 1) \\ u(u^2 - 1) & w(u^2 - 1) \end{vmatrix} = \begin{cases} -2uw(u^2 - w^2) & \text{if } \epsilon = 1 \\
-2uw(u^2w^2 - 1) & \text{if } \epsilon = -1 \end{cases} \]
\[ \omega_{89}(V) = \begin{vmatrix} u(u^2 - 1) & \epsilon w(u^2 - 1) \\ u(u^2 + 1) & w(u^2 + 1) \end{vmatrix} = \begin{cases} 2uw(u^2 - w^2) & \text{if } \epsilon = 1 \\
-2uw(u^2w^2 - 1) & \text{if } \epsilon = -1 \end{cases} \]

Finally, by applying the previous expressions (3.4) to \( U \), we obtain:
\[ \phi_S(U) = \begin{bmatrix} \eta_01(U) \\ \eta_02(U) \\ \eta_12(U) \\ \omega_{45}(U) \\ \omega_{67}(U) \\ \omega_{89}(U) \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{4(1 - \frac{a}{b})}(u^2 - v^2)(1 - u^2v^2) \\ -\frac{a}{4(1 - \frac{a}{b})}(u^2 - v^2)(1 - u^2v^2) \\ -(u^2 - v^2)(u^2v^2 + 1) \\ -2uv(u^2 - v^2) \\ 2uv(u^2 - v^2) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{4(1 - \frac{a}{b})}(1 + u^2v^2) \\ \frac{a}{4(1 - \frac{a}{b})}(1 - u^2v^2) \\ 1 \\ u^2v^2 + 1 \\ 2uv \\ -2uv \end{bmatrix} . \]

If we had that \( \epsilon = -1 \), then we would have:
\[ \phi_S(V) = \begin{bmatrix} \eta_01(V) \\ \eta_02(V) \\ \eta_12(V) \\ \omega_{45}(V) \\ \omega_{67}(V) \\ \omega_{89}(V) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{4(1 - \frac{a}{b})}(u^2 - w^2)(1 - u^2w^2) \\ \frac{a}{4(1 - \frac{a}{b})}(u^2 - w^2)(1 - u^2w^2) \\ (u^2 + w^2)(u^2w^2 - 1) \\ -2uw(u^2w^2 - 1) \\ -2uw(u^2w^2 - 1) \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{4(1 - \frac{a}{b})}(u^2 - w^2) \\ -\frac{a}{4(1 - \frac{a}{b})}(u^2 - w^2) \\ u^2 + w^2 \\ -2uw \\ -2uw \end{bmatrix} , \]
which would imply that \( \phi_S(U) \neq \phi_S(V) \) since neither \( u \) nor \( w \) can vanish. Hence, we can conclude that \( \epsilon = \epsilon' = 1 \). In this case, we have, as points on \( \mathbb{P}^5 \):

\[
\phi_S(U) = \begin{bmatrix}
0 \\
\frac{1}{4(1 - \frac{u}{w})}(1 - u^2v^2) \\
\frac{1}{4(1 - \frac{u}{w})}(1 - u^2v^2) \\
2uv \\
-2uv
\end{bmatrix} = \phi_S(V) ,
\]

and it can be easily seen that \( v = w \) holds.

It only remains to consider the fourth case. The locus \( E \) is reducible and it is the union of four lines,

\[
E = r^{1,1} \cup r^{-1,1} \cup r^{1,-1} \cup r^{-1,-1}
\]

(3.5)

\[
r^{\gamma, \delta} = \begin{cases} 
Y = \gamma iX \\
T = \delta iZ 
\end{cases} \quad \gamma, \delta \in \{+1, -1\} .
\]

We can now easily parametrize these lines with parametrizations \( g^{\gamma, \delta} \), where \( g^{\gamma, \delta}([u, v]) := [u, \gamma iv, \nu, \delta iv] \). Denoted by \( \infty \) the point \([1, 0]\) on the projective line, it can be easily seen that \( g^{\gamma, \delta}(0) \) and \( g^{\gamma, \delta}(\infty) \) does not belong to \( C \), and that the group \( G \) acts on these lines as follows:

\[
a.g^{\gamma, \delta}([u, v]) = g^{\gamma, \delta}([-u, v]) \\
b.g^{\gamma, \delta}([u, v]) = g^{-\gamma, -\delta}([u, v]) .
\]

Let us consider now \( U := [g^{\gamma, \delta}(u), g^{\gamma, \delta}(u')] \) and \( V := [g^{\gamma, \delta}(u), g^{\gamma, \delta}(u'')] \). We assume that their image with respect to the canonical map is the same. By [1.7], the evaluation at \( U \) of the canonical map \( \phi_S \) can be expressed as follows:

\[
\phi_S(U) = \begin{bmatrix}
\eta_0(U) \\
\eta_1(U) \\
\eta_2(U) \\
\omega_4(U) \\
\omega_6(U) \\
\omega_89(U)
\end{bmatrix} = \begin{bmatrix}
0 \\
-\gamma u^2 \\
\frac{1}{2}(-\gamma' u^2) \\
\frac{1}{2}(u - u') \\
\frac{1}{2}(-\gamma' u^2) \\
\frac{1}{2}(\delta' u' - \gamma u)
\end{bmatrix} = \begin{bmatrix}
0 \\
-\gamma u^2 \\
\frac{1}{2}(u - u') \\
\frac{1}{2}(u' - u) \\
\frac{1}{2}(\delta - \gamma) \\
\frac{1}{2}(\delta' - \gamma)
\end{bmatrix}
\]

By the hypothesis that \( \phi_S(U) = \phi_S(V) \), it follows that there exists \( \lambda \in \mathbb{C}^* \) such that:

\[
\begin{align*}
\begin{cases}
u^2 - u''^2 = \lambda(u^2 - u'^2) \\
\gamma u^2 - \gamma' u'^2 = \lambda \left(\gamma u^2 - \gamma' u'^2\right) \\
\gamma'' u'' \Delta'' = \lambda \gamma' u' \Delta' \\
u'' \Delta'' = \lambda u' \Delta'
\end{cases}
\end{align*}
\]

(3.6)
where $\Delta' := \begin{vmatrix} \gamma & \gamma' \\ \delta & \delta' \end{vmatrix}$ and $\Delta'' := \begin{vmatrix} \gamma & \gamma'' \\ \delta & \delta'' \end{vmatrix}$. In consequence of the last two identities in 3.6 we can easily infer that $\gamma' = \gamma''$. In particular, we see that $\delta' = \delta''$ because $\Delta'$ vanishes if and only if $\Delta''$ does. Thus, $\Delta' = \Delta''$ and the equations 3.6 can be rewritten in the following form:

$$\begin{cases}
\gamma u^2 - \gamma' u''^2 &= \lambda(u^2 - u')^2 \\
\delta u^2 - \delta' u''^2 &= \lambda(u^2 - u')^2 \\
\gamma u^2 - \gamma' u''^2 &= \lambda(u^2 - u')^2 \\

\end{cases}$$

We finally obtain the following linear system in the variables $u^2, u'$:

$$\begin{align*}
\gamma\delta'(1 - \lambda)u^2 + \lambda\gamma'(1 - \lambda)u^2 &= 0 \\
(1 - \lambda)u^2 + (1 - \lambda)\lambda u^2 &= 0 .
\end{align*}$$

The determinant of this linear system must vanish because $u$ and $u'$ are supposed to be non-zero. Hence, we have that $\delta'\lambda(1 - \lambda)^2 \Delta = 0$, which leads to two possible cases: if $\lambda = 1$ we can conclude that $U = V$. Otherwise, $\Delta = 0$ and we have $\omega_{67} = \omega_{89} = 0$. Hence

$$\begin{cases}
u'' &= \lambda u' \\
u^2 &= -\lambda u'^2 \\
u^2 - u'^2 &= \lambda(u^2 - u')^2 ,
\end{cases}$$

and finally

$$(-\lambda - 1)u^2 = \lambda(-\lambda u^2 - u'^2) = -\lambda(\lambda + 1)u^2 .$$

In conclusion, $\lambda = -1$ and $(\gamma'', \delta'') = \pm(\gamma', \delta')$, and there exists then a nontrivial element $g$ of $G$ such that $g.U = V$. This completes the proof of the theorem. □

In [5] we proved that $\phi_S$ has actually injective differential. It is therefore an interesting question, whether the same result could be proved by using the approach used to prove theorem 3.0.5.
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