Poisson Poincaré groups

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Abstract
We present almost complete list of normal forms of classical r-matrices on the Poincaré group.

Introduction
Classification of Poisson-Lie structures on a Lie group is closely related to the classification of quantum deformations of the group, cf. [1, 2]. Studying Poisson structures has here some advantages:

1. the notion of a Poisson-Lie group [3] (see also references in [2]) is simple, clear and general, whereas the notion of a quantum deformation is rather difficult, varies from author to author, and can be different for different types of groups (for instance, a separate definition for semidirect products),
2. calculations of Poisson structures are technically much easier (classical Yang-Baxter equation is quadratic whereas the quantum one is cubic) and have often a direct Lie-algebraic meaning,
3. in some cases it is easy to pass from a Poisson-Lie group to the corresponding quantum group [4]: the classical r-matrix can be used to
   (a) construct all remaining objects,
   (b) denote the deformation (convenient when communicating with other people),
4. it is easier to check whether the Poisson-Lie group is non-complete than to check if the corresponding quantum deformation (on the Hopf *-algebra level) can be formulated on the C*-algebra level [5].

We point out also, that models of quantum physical systems based on quantum symmetry correspond usually to (simpler) models of classical physical systems based on Poisson symmetry and some ideas of non-commutative geometry can be tested already on the semi-classical level [3, 5, 8].

The aim of this short report is to present main results of our study [9] of Poisson structures on the Poincaré group. Our classification agrees with the classification of quantum deformations of the Poincaré group obtained recently in [10]. In most cases, the quantum R-matrix (in the sense of [11]) turns out to be just the exponential of our corresponding classical r-matrix.

For some basic notation or notions we refer to [12, 2, 13, 4].
1 Inhomogeneous $o(p, q)$ algebras

We consider a $(p+q)$-dimensional real vector space $V \cong \mathbb{R}^{p+q}$, equipped with a scalar product $\eta$ of signature $(p, q)$. Let $\mathfrak{h} := o(p, q)$ denote the Lie algebra of the group $H \cong O(p, q)$ of endomorphisms of $V$ preserving $\eta$, and let $\mathfrak{g} := V \rtimes \mathfrak{h}$ be the corresponding ‘inhomogeneous’ Lie algebra.

**Theorem 1.1** (cf.[9]) For $\dim V > 2$ any cocycle $\delta: \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$ is a coboundary:

$$\delta(X) = \text{ad}_X r \quad \text{for } X \in \mathfrak{g}.$$  

Additionally, for $\dim V > 3$, $r \mapsto \text{ad} r$ is injective.

In view of this theorem, the classification of Poisson structures on $G = V \rtimes H$ consists in a description of equivalence classes (modulo $\text{Aut} \mathfrak{g}$) of $r \in \bigwedge^2 \mathfrak{g}$ such that $[r, r] \in (\bigwedge^3 \mathfrak{g})_{\text{inv}}$.

We have a decomposition

$$r = a + b + c,$$

corresponding to the decomposition

$$\bigwedge^2 \mathfrak{g} = \bigwedge^2 V \oplus (\bigwedge(V \wedge \mathfrak{h})) \oplus \bigwedge \mathfrak{h}.$$  

We have also the following decomposition of the Schouten bracket

$$[r, r] = 2[a,b] + (2[a,c] + [b,b]) + 2[b,c] + [c,c],$$

corresponding to the decomposition

$$\bigwedge^3 \mathfrak{g} = \bigwedge^3 V \oplus (\bigwedge^2 V \wedge \mathfrak{h}) \oplus (V \wedge \bigwedge^2 \mathfrak{h}) \oplus \bigwedge^3 \mathfrak{h}.$$  

We recall that $2 \bigwedge V$ is naturally isomorphic to $\mathfrak{h}$ as a $\mathfrak{h}$-module. The isomorphism is given by

$$\bigwedge^2 V \ni x \wedge y \mapsto \Omega_{x,y} := x \otimes \eta(y) - y \otimes \eta(x) \in \text{End} V$$

(here $\eta$ is interpreted as a map from $V$ to $V^*$). This isomorphism defines a canonical element in $(\bigwedge^2 V)^* \otimes \mathfrak{h}$, and, using the identification of $V$ and $V^*$, a canonical element $\Omega \in \bigwedge^2 V \otimes \mathfrak{h}$. If $e_1, \ldots, e_{p+q}$ denotes a basis of $V$, the canonical element $\Omega$ is given by
\[ \Omega = \eta^{km} \eta^{ln} e_k \wedge e_l \otimes \Omega_{m,n} \]

(summation convention), where \( \Omega_{m,n} := \Omega_{e_m, e_n} \) and \( \eta^{km} \) is the contravariant metric.

From the above theorem it follows that Poisson structures on \( G = V \times H \) are in one-to-one correspondence with \( r = a + b + c \in \mathfrak{g} \) such that

\[
\begin{align*}
\{c, c\} &= 0 \quad \text{(1)} \\
\{b, c\} &= 0 \quad \text{(2)} \\
2c + \{b, b\} &= t\Omega \quad (t \in \mathbb{R}) \quad \text{(3)} \\
\{a, b\} &= 0. \quad \text{(4)}
\end{align*}
\]

Equation (1) means that \( c \) is a triangular \( r \)-matrix on \( \mathfrak{h} \) (this is the semi-classical counterpart of a known theorem [14] excluding the case when the homogeneous part \( H \) is \( q \)-deformed).

Equation (2) tells that \( b \), as a map from \( \mathfrak{h}^* \) to \( V \), is a cocycle, the Lie bracket on \( \mathfrak{h}^* \) being defined by the triangular \( c \in \mathfrak{h} \) and the action of \( \mathfrak{h}^* \) on \( V \) is defined using the homomorphism from \( \mathfrak{h}^* \) to \( \mathfrak{h} \) given by \( b \).

Let us list some particular cases.

1. \( b = 0, c = 0, a \in \Lambda V \) arbitrary. This type of solutions we call ‘soft deformations’ [13].

2. \( a = 0, c = 0, [b, b] = t\Omega \). There is a family of solutions of the latter equation, parametrized by vectors in \( V \). Namely, for each \( x \in V \),

\[
b_x := \eta^{kl} e_k \otimes \Omega_{e_l, x}
\]

(\( e_k \) is any basis in \( V \)) satisfies this equation with \( t = -\eta(x, x) \).

3. \( a = 0, b = 0, c \in \mathfrak{h} \) triangular.

### 2 The case of the Poincaré group

Using the list of classical \( r \)-matrices for the Lorentz group from [2] (only triangular are needed), we have solved equations (1)–(4) in the case of the Poincaré group, assuming \( c \neq 0 \) or \( t = 0 \), and we have found several solutions in the case \( t \neq 0 \). The results are shown in the table below. Examples with \( t \neq 0 \) are provided by formula (5). Let \( e_0, e_1, e_2, e_3 \) be a Lorentz basis in \( V \). Let us introduce the standard generators of \( \mathfrak{h} \):

\[
M_i = \varepsilon_{ijk} e_k \otimes e^j, \quad L_i = e_0 \otimes e^i + e_i \otimes e^0
\]

\((i, j, k = 1, 2, 3)\). If we set \( x := e_0 \) in (5), we obtain

\[
b_{e_0} = \sum_{k=1}^{3} L_k \wedge e_k,
\]

which is the known [4] classical \( r \)-matrix corresponding to so called \( \kappa \)-deformation. Taking
\( x = e_3 \), we obtain another solution

\[
 b_{e_3} = M_1 \wedge e_2 - M_2 \wedge e_1 + L_3 \wedge e_0
\]

(this one is \( L_1, L_2, M_3 \)-invariant). As shown in the table below, both \( b_{e_0} \) and \( b_{e_3} \) are particular cases of more general families (thus, we have a ‘deformation’ of the \( \kappa \)-deformation).

The following table lists 23 cases labelled by the number \( N \) in the last column. The question mark in the table (case 9) reminds that the case \( c = 0 \), \([b, b] \neq 0\) is not yet completely solved (including the question mark, the list is complete).

| \( c \) | \( b \) | \( a \) | \# \( N \) |
|-----|-----|-----|-----|
| \( \gamma J H \wedge H \) | 0 | \( a e_+ \wedge e_- + \bar{a} e_1 \wedge e_2 \) | 2 1 |
| \( J X_+ \wedge X_+ \) | \( \beta_1 (e_1 \wedge X_+ - e_2 \wedge J X_+ + e_+ \wedge H) + \beta_2 e_+ \wedge J H \) | 0 | 1 2 |
| | \( \beta (e_1 \wedge X_+ - e_2 \wedge J X_+ + e_+ \wedge H) \) | \( a e_+ \wedge e_1 \) | 1 3 |
| | \( \beta (e_1 \wedge X_+ + e_2 \wedge J X_+) \) | \( e_+ \wedge (a_1 e_1 + a_2 e_2) - \beta^2 e_1 \wedge e_2 \) | 2 4 |
| \( H \wedge X_+ \) | 0 | 0 | 1 5 |
| \( J H \wedge J X_+ \) | \( \beta e_2 \wedge X_+ \) | 0 | 0 6 |
| \( \gamma J X_+ \wedge X_+ \) | 0 | \( e_1 \wedge L_1 + e_2 \wedge L_2 + e_3 \wedge L_3 + \beta e_0 \wedge M_3 \) | 0 | 1 7 |
| | \( e_2 \wedge M_1 - e_1 \wedge M_2 + e_0 \wedge L_3 + \beta e_3 \wedge M_3 \) | 0 | 1 8 |
| \( J H \wedge X_+ \) | \( e_1 \wedge J H \wedge X_+ + e_+ \wedge (H + \beta J H) \) | 0 | 1 9 |
| | \( e_1 \wedge (X_+ + \beta_1 J X_+) + e_+ \wedge (H + \beta_2 X_+) \) | \( a e_+ \wedge e_2 \) | 2 10 |
| | \( \beta_2 = 0 \) or \( \beta_1 = \beta_1 \) or \( \beta_2 = \pm 1 \) | | |
| | \( e_1 \wedge J X_+ + e_+ \wedge X_+ \) | \( e_- \wedge (a_1 e_1 + a_2 e_2) - a_2 e_+ \wedge e_- + a e_+ \wedge e_2 \) | 3 12 |
| | \( e_0 \wedge J H \) | \( a_1 e_0 \wedge e_3 + a_2 e_1 \wedge e_3 + a_3 e_3 \wedge e_2 \) | 3 13 |
| | \( e_1 \wedge H \) | \( a_1 e_0 \wedge e_3 + a_2 e_1 \wedge e_3 + a_3 e_3 \wedge e_2 + a e_+ \wedge e_2 \) | 3 14 |
| | \( e_2 \wedge X_+ \) | \( \bar{e}_+ \wedge e_- + e_+ \wedge e_1 + e_- \wedge (a_1 e_1 + a_2 e_2) \) | 2 15 |
| | \( e_3 \wedge J H \) | \( a_1 e_0 \wedge e_3 + a_2 e_0 \wedge e_1 + a_3 e_1 \wedge e_2 \) | 3 16 |
| | \( e_+ \wedge X_+ \) | \( e_- \wedge (a_1 e_1 + a_2 e_2) + a_3 e_3 \wedge e_2 \) | 2 17 |
| | \( e_+ \wedge (\beta_1 H + \beta_2 J H) \) | \( a e_1 \wedge e_2 + a_1 e_- \wedge e_1 - a e_+ \wedge e_- \) | 2 18 |
| | \( e_+ \wedge J H \) | \( a e_1 \wedge e_2 + a_1 e_- \wedge e_1 + a e_+ \wedge e_- \) | 2 19 |
| | \( e_+ \wedge H \) | \( a e_1 \wedge e_2 + a_1 e_- \wedge e_1 + a_3 e_3 \wedge e_2 \) | 2 20 |
| | \( e_+ \wedge \) | \( a e_1 \wedge e_2 + a_1 e_- \wedge e_1 + a e_+ \wedge e_- \) | 2 21 |
| | \( 0 \) | \( a_1 e_0 \wedge e_3 + a_2 e_1 \wedge e_2 \) | 2 22 |
| | \( e_1 \wedge e_+ \wedge e_+ \) | | 2 23 |

Now we explain the notation in the table. We have introduced the standard generators of \( \mathfrak{h} = sl(2, \mathbb{C}) \):

\[
 H = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad X_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\]

(recall that the action of \( X \in sl(2, \mathbb{C}) \) on a vector \( v \in V \) is given by \( X(v) := Xv + vX^+ \), the space \( V \) being identified with the set of hermitian \( 2 \times 2 \) matrices, where \( X^+ \) is the hermitian conjugate of \( X \)). We denote by \( J \) the multiplication by the imaginary unit in \( \mathfrak{h} \). It is also convenient to introduce the light-cone vectors \( e_\pm := e_0 \pm e_3 \).

In the forth column (labelled by \#) we indicate the number of essential parameters (more precisely – the maximal number of such parameters) involved in the deformation.
This number is in many cases less than the number of parameters actually occurring in the table. The reduction of the number of parameters can be achieved using two following one-parameter groups of automorphisms of \( \mathfrak{g} \):

- the group of dilations: \((v, X) \mapsto (\lambda v, X)\) \((in cases 1,2,3,4,6,17,18,19,20,21,22)\),
- the group of internal automorphisms generated by \( H \) \((in cases 15,17,18,19,20,21)\).

In order to make some comments on the relations between our classical \( r \)-matrices and the quantum Poincaré groups, let us set \( R := \exp(ir) \in \text{End}(C^5 \otimes C^5) \). We have two following remarks.

1. If \( c = 0 \) (cases 7–23), then \( r^2 = 0 \), hence \( R = 1 + ir \) and the corresponding to this \( R \)-matrix commutation relations for the elements of the \( 5 \times 5 \) matrix

\[
T = (T^a_b)_{a,b=0,...,4} = \begin{pmatrix} \Lambda & v \\ 0 & 1 \end{pmatrix}, \quad \Lambda = (\Lambda^\mu_{\nu})_{\mu,\nu=0,...,3}, \quad v = (v^\mu)_{\mu=0,...,3}
\]

arise simply by replacing the Poisson brackets (defined by \( r \)) by commutators (divided by \( \sqrt{-1} \)). As in [1], no ordering ambiguities arise in this case. Moreover, the right-hand-sides of the expressions for commutators satisfy (automatically) the Jacobi identity (the computation is the same as in the Poisson case in which it is true because we started with a Poisson structure). This is sufficient to show that the resulting algebra has a ‘correct size’.

Furthermore, from \( r_{12}r_{13}r_{23} = 0 = r_{23}r_{13}r_{12} \) it follows that

\[
R_{12}R_{13}R_{23} = 1 + r_{12} + r_{13} + r_{23} + r_{12}r_{13} + r_{12}r_{23} + r_{13}r_{23},
\]

\[
R_{23}R_{13}R_{12} = 1 + r_{23} + r_{13} + r_{12} + r_{23}r_{13} + r_{23}r_{12} + r_{13}r_{12},
\]

and

\[
R_{12}R_{13}R_{23} - R_{23}R_{13}R_{12} = [r, r].
\]

Hence \( R \) satisfies the Yang-Baxter equation if and only if \( [r, r] = 0 \) (cases 10–23).

2. If \( c \neq 0 \), then we can compare our results with [10] (in [10], only quantum Poincaré groups with non-trivial ‘Lorentz part’ were classified). Using [2] we have a clear correspondence between four types of triangular classical \( r \)-matrices \( c \) on the Lorentz Lie algebra and four types of triangular \( (q = 1) \) quantum Lorentz groups [1]. For a fixed quantum Lorentz group (corresponding to our \( c \)), the classification of quantum Poincaré groups given in [10] involves quantities similar to our \( b \) and \( a \) (they are denoted by \( H \) and \( T \)) and one can observe a clear correspondence between particular solutions presented in [10] and the cases 1–6 of our table. Whether and how the quantum \( R \)-matrix can be constructed from \( r \) is not completely clear yet. One can check that in the first case, \( R := \exp(ir) \) coincides with the \( R \)-matrix obtained in [10]. The same seems to be true for the second type of \( c \). In two remaining cases of \( c \) the relation may be more complicated, because already the \( R \)-matrix for the Lorentz part differs from \( \exp(ic) \) (however it is built of components \( \exp(ic_-), \exp(ic_+), \) see [2]).
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