Quantum information processing with geometric features of quantum states may provide promising noise-resilient schemes for quantum metrology. In this Letter, we theoretically explore geometric Sagnac interferometers with trapped atomic clocks for rotation sensing. We first present the Sagnac phase-matching condition (SPMC) which is critical and instrumental in experiments to extract the rotation frequency from the interference signal through the well-known Sagnac phase. Otherwise, if certain temporal profiles of the sweeping angular velocity and the total interrogation time are improperly used, the sensitivity could be detrimentally small. Furthermore, base on the SPMC, an scheme for unconventional geometric Sagnac interferometer is proposed, which could be intrinsically robust to certain decoherence noises and be more accessible in experiments compared with its pure geometric counterpart, due to a zero initial sweeping angular velocity. Such geometric Sagnac interferometers are capable of saturating the ultimate precision limit given by the quantum Cramér-Rao bound.

Introduction.—Coherent manipulation of atomic clock states can be used to sense rotation of a reference frame [1]. By enclosing a finite area with the two distinct internal states in real space, a Sagnac phase gate is constructed, which encodes the rotation frequency into the qubit phase as a matter-wave Sagnac phase. With quantum resources like coherence and entanglement, such quantum Sagnac interferometers are expected to achieve higher precision and sensitivity[1]. However, open system effects, e.g., decoherence caused by inevitable noises, may reduce the fidelity of the Sagnac phase gate and therefore the expected sensing precision could not be reached [2]. On the other hand, geometric quantum gates have been studied theoretically [3–11] and demonstrated in experiments [12–18] for quantum computation. Compared to the dynamical phase, geometric phase only depends on global geometrical features (e.g., area, volume, genus, etc.) of the state manipulation in the phase space. Consequently, it is intrinsically immune to local noise perturbations which preserves these geometrical features [19], and provides promising paradigm to construct various high-fidelity quantum phase gates. Therefore, it would be inspiring if such geometric properties could be harnessed for high-precision quantum sensing.

Stevenson et al. proposed a pioneering scheme for quantum rotation sensing with trapped atomic clock states in Ref. [20], and similar schemes were later considered in Ref. [21] with Fisher information analyses and in Ref. [22] with spin-orbital coupling. Whereas, the nature of the Sagnac phase $\phi_S$, i.e., whether the phase shift is dynamic, geometric or both, has not been clarified. And also, the fidelity and robustness of such Sagnac phase encoding protocols under decoherence were not investigated either. Besides, one of the claims in Ref. [20] that the accumulated phase difference of the interferometer remains independent of the temporal profile of the sweeping angular velocity $\omega_P(t)$, is actually not true.

In this Letter, we first explore geometric quantum rotation sensing with trapped atomic clocks, which could be potentially noise-resilient and achieve high sensitivity. Starting from a basic model in Ref. [20], we first present the Sagnac phase-matching condition (SPMC) for this system, under which the phase of the interferometer is exactly identical with $\phi_S$, which is significant for experiments to observe interference fringes induced by the rotation through the well-known Sagnac phase. Furthermore, based on the SPMC, we propose an experimentally more accessible scheme for unconventional geometric Sagnac interferometers, where the geometric Sagnac phase also involves a dynamic component [11]. Our results should be instrumental in experimentally implementing a potentially noise-resilient geometric Sagnac interferometer for rotation sensing with trapped atomic clocks.

Model and Sagnac phase-matching condition.—Within the basic scheme in Ref. [20], the interferometer protocol consists of two Ramsey $\pi/2$ pulses and two identical harmonic traps which counter-transport the clock states $|\uparrow\rangle$ and $|\downarrow\rangle$ along circular paths of radius $r$ in the $xy$ plane, with respective sweeping angular velocity $\pm\omega_P(t)\Omega (\omega_P(t) \geq 0)$ in the rotating frame $\mathcal{R}$. And $\mathcal{R}$ rotates in an angular velocity $\Omega = \Omega z$ with respect to an inertial frame $\mathcal{K}$. See Fig. 1 for a schematic illustration. The interrogation time $T$, when the two components are recombined for readout, is given by $\int_0^T \omega_P(t) dt = \pi$. Under the SPMC, the unitary time-evolution operators $U_1(T)$ and $U_2(T)$ for the two respective paths form a Sagnac phase-encoding gate $U_S (\phi_S)$ at $t = T$, which imprints the Sagnac phase into the qubit phase. Formally, the interferometer protocol can be written as [23]

$$V(T) = Y \left( \frac{\pi}{2} \right) U(T) Y \left( - \frac{\pi}{2} \right),$$

(1)

where $Y (\phi) = \exp \left( -i \phi \sigma_y / 2 \right)$ with $\sigma_y$ being the Pauli matrix, and $U(T) = T \exp \left( -i \int_0^T H(t) dt / \hbar \right)$ with $T$ being the time ordering operator and $H(t) = H_0(t) \Pi_0 + H_1(t) \Pi_1$ being the Hamiltonian, where we have used the notation $|0\rangle_s (|1\rangle_s) = |\uparrow\rangle (|\downarrow\rangle)$ and $\Pi_0 (\Pi_1) = |0\rangle_s \langle 0| (|1\rangle_s \langle 1|)$, with the subscript $s$ denoting the (pseudo)spin subspace. And it can
be shown directly that $U(T) = U_0(T)\Pi_0 + U_1(T)\Pi_1$, where $U_n(T) = T\exp\left(-i\int_0^T H_n(t)dt/\hbar\right)$ for $n = 0, 1$ and $H_n(t)$ is the time dependent single component Hamiltonian [24].

For the Sensing period in the interferometer scheme shown in Fig. 1, if the degrees of freedom in the radial and $z$ directions for atoms in the harmonic trap are tightly confined, then the time evolution can be described by the one dimensional Hamiltonian for atoms in the harmonic trap are in the ground state (vacuum) $|G\rangle_s$, respectively, and $|G\rangle_h$ is the ground state of atoms in the harmonic trap, with the subscript $h$ (s) denoting the harmonic oscillator (spin) subspace. In the Sensing period, the atoms in two traps are coherently split at $t = 0$ and are counter-transported along a circular path of radius $r$ with respective angular velocity $\Omega \pm \omega_P(t)$ in the inertial frame $\mathcal{K}$, and are recombined (RC) at time $T$. By properly selecting the $\omega_P(t)$ profile and the interrogation time $T$, a Sagnac phase gate $U_S(\phi_S)$ can be obtained, where $\phi_S = 2\pi r^2\Omega/\hbar$ is the Sagnac phase. The rotation frequency $\Omega$ can be read out from the population information after applying another $\pi/2$ pulse in the Readout stage.

The sweeping angular velocity $\omega_P(t)$ can be further extended to a function $W_P(t)$ defined on the whole real time axis $t \in [-\infty, +\infty]$, with $W_P(t) = \omega_P(t)$ for $t \in [0, T]$ and $W_P(t) = 0$ for the other, and the frequency spectrum of $W_P(t)$ can be obtained from its Fourier transform $W_P(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} W_P(t) \exp(-i\omega t) dt$.

The Hamiltonian in Eq. (2) describes a driven harmonic oscillator and the corresponding time evolution operator at time $t$ is given by [24]

$$U_\eta(t) = D[\alpha_\eta(t)] e^{-i\omega_0 t a^\dagger a + \frac{i}{2} \lambda_\eta(t)(a - a^\dagger)},$$

where $\omega_0$ is the trap frequency and $a(a^\dagger)$ is the annihilation (creation) operator for the trap mode. The second term in Eq. (2) represents the drive acting on the harmonic oscillator induced by the rotation of the frame, where $\lambda_\eta(t) = \sqrt{\pi h m \omega_0/2} \Gamma(1 - 2\eta) \omega_P(t)$, with $m$ being the particle mass and $\Omega$ being the rotation frequency to be measured. $\omega_P(t) \geq 0$ for $t \in [0, T]$ is experimentally designed sweeping angular velocity whose temporal profile determines the SPMC and the signal contrast, which will be shown below. The sweeping angular velocity $\omega_P(t)$ can be further extended to a function $W_P(t)$ defined on the whole real time axis $t \in [-\infty, +\infty]$, with $W_P(t) = \omega_P(t)$ for $t \in [0, T]$ and $W_P(t) = 0$ for the other, and the frequency spectrum of $W_P(t)$ can be obtained from its Fourier transform $W_P(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} W_P(t) \exp(-i\omega t) dt$.

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where $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ is the displacement operator for the harmonic oscillator, with

$$\alpha_\eta(t) = -\int_0^t \lambda_\eta(\tau) \exp[i\omega_0(\tau - t)] d\tau / \hbar, \quad \phi_\eta(t) = \int_0^t \int_0^{T_1} \lambda_\eta(\tau_1) \lambda_\eta(\tau_2) \sin[\omega_0(\tau_1 - \tau_2)] d\tau_2 d\tau_1 / h^2.$$

If the initial states for both $|0\rangle_s$ and $|1\rangle_s$ components in the harmonic trap are in the ground state (vacuum) $|G\rangle_h$ which is defined by $a |G\rangle_h = 0$, where the subscript $h$ denotes the harmonic trap subspace, then the state at time $t$ is given by $|\psi_\eta(t)\rangle_h = U_\eta(t) |G\rangle_h = |\alpha_\eta(t)\rangle_h \exp[i(\phi_\eta(t) - \omega t/2)]$, where $|\alpha_\eta(t)\rangle_h$ is the coherent state which is eigenstate of $a$ with eigenvalue $\alpha_\eta(t)$. See the Sensing period in Fig. 1.

For the initial state $\rho(0) = |G\rangle_h \langle G| \otimes |1\rangle_s \langle 1|$, the readout state reads $\rho(T) = V(T)\rho(0)V^\dagger(T)$ And the reduced density matrix for the spin subspace is given by $\rho_s(T) = T_{th}\rho(T)$ with $T_{th}$ being the trace operation, and reads $\rho_s(T) = |C_2 - \text{Re}(C_{1,0})\sigma_z - \text{Im}(C_{1,0})\sigma_y|/2$, where $C_{j,k}$ is the two-dimensional identity matrix, $C_{j,k} = \delta_{j,k} |C_1,0\rangle \langle C_1,0|$ and $\sigma_z = |0\rangle \langle 1| + |1\rangle \langle 0|$ are the Pauli operators. Therefore, the measurement signal, e.g., the population difference, is given by

$$s(\sigma_z) = \langle C_{1,0}\rangle \cos(\phi_I),$$

where the modulus $|C_{1,0}| = \exp(-|\Delta\alpha|^2/2)$ gives the signal contrast, with $\Delta\alpha = \alpha_0(T) - \alpha_1(T) \propto \bar{W}_P(\omega_0) [24]$, and $\phi_I = \text{arg}(C_{1,0})$ is the interferometer phase, with $\arg$ denoting the argument. From straightforward calculation one obtains the interferometer phase $\phi_I$, which is given by [24]

$$\phi_I = \phi_S \left[ 1 - \sqrt{\frac{2}{\pi}} \text{Re} \left[ \bar{W}_P(\omega_0) \right] \right],$$

where $\phi_S = 2\pi r^2\Omega/\hbar$ is the Sagnac phase and it can be shown that $0 \leq \phi_I \leq 2\phi_S [24]$. In contrast to Ref. [20], the phase of the interferometer in Eq. (4) is indeed dependent on the Fourier components of $\bar{W}_P(\omega)$ at the trap frequency $\omega_0$ and therefore depends on the temporal profile of $\omega_P(t)$. This

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**Figure 1.** (Color online) Schematic protocol of Sagnac interferometer with trapped atomic clock states for rotation sensing [20]. The protocol consists of Initialization, Sensing and Readout, where $Y(\pm \pi/2)$ denote the $\pi/2$ pulses and the clock states are $|0\rangle_s = |\uparrow\rangle$ and $|1\rangle_s = |\downarrow\rangle$, respectively, and $|G\rangle_h$ is the ground state of atoms in the harmonic trap, with the subscript $h$ (s) denoting the harmonic oscillator (spin) subspace. In the Sensing period, the atoms in two traps are coherently split at $t = 0$ and are counter-transported along a circular path of radius $r$ with respective angular velocity $\Omega \pm \omega_P(t)$ in the inertial frame $\mathcal{K}$, and are recombined (RC) at time $T$. By properly selecting the $\omega_P(t)$ profile and the interrogation time $T$, a Sagnac phase gate $U_S(\phi_S)$ can be obtained, where $\phi_S = 2\pi r^2\Omega/\hbar$ is the Sagnac phase. The rotation frequency $\Omega$ can be read out from the population information after applying another $\pi/2$ pulse in the Readout stage.
result is experimentally relevant because the form of $\omega_P(t)$ profile will strongly affect the interference fringes. And now we arrive at the SPMC, which is the main result of this section and is given by

$$\text{Re} \left[ \tilde{W}_P (\omega_0) \right] = 0, \quad (6)$$

such that $\phi_I = \phi_S$, i.e., the interferometer phase is exactly the Sagnac phase.

Note that not any $\omega_P(t)$ profile can satisfy the SPMC in Eq. (6). If certain temporal profile for $\omega_P(t)$ and interrogation time $T$ are improperly used in experiments, it may cause very small sensitivity, even with $\partial_1 \phi_I \approx 0$. For example, for a constant $\omega_P(t) = \pi/T$ and rapid interrogation with $T \ll \omega_0^{-1}$, $\text{Re} \left[ \tilde{W}_P (\omega_0) \right] = \sqrt{\pi/2} \sin (\omega_0 T) / (\omega_0 T) \approx \sqrt{\pi/2}$. Consequently, $\partial_1 \phi_I$ approaches 0 and the sensitivity for the rotation frequency $\Omega$ is negligibly weak. Therefore, the SPMC is a strong constraint which is significant in experiments, in particular when the contrast is not maximized, to observe the interferometer fringes induced by the Sagnac phase $\phi_S$ with high sensitivity. For an intuitive understanding of the origin of $\text{Re} \left[ \tilde{W}_P (\omega_0) \right]$ in $\phi_I$ in Eq. (5), see Ref. [24].

Unconventional geometric Sagnac interferometer.—Up to now we have obtained the SPMC, but the origins of the Sagnac phase have not been investigated yet. Next we analyze different components in the Sagnac phase $\phi_S$ and explore unconventional geometric [11, 14] Sagnac interferometers, which could be potentially resilient to noises and are promising for reaching high-precision quantum rotation sensing.

Quantum geometric phase is the phase change during a holonomy motion in the quantum state space [25–27], which was studied by Berry [28] for adiabatic cyclic motion and by Aharonov and Anandan [27] for any cyclic evolution. Here for the spin state $|\eta\rangle_\gamma (\eta = 0, 1)$, the total phase change for quantum evolution in each harmonic trap can be divided into dynamic and geometric components, which are given by [11, 19]

$$\gamma^d_\eta(T) = - \int_0^T \langle \psi_\eta(t)|H_\eta(t)|\psi_\eta(t) \rangle \, dt, \quad (7)$$

and

$$\gamma^g_\eta(T) = \frac{i}{2} \int_{T_\eta} \left( \alpha^*_\eta \delta \alpha_\eta - \alpha_\eta \delta \alpha^*_\eta \right) - \arg \langle \alpha_\eta(T)|G \rangle, \quad (8)$$

respectively. Note that here in Eqs. (7) (8) and hereafter we will drop the subspace subscript $\eta$ for convenience. From Eq. (8), one sees that the geometric phase consists of two parts, where the first is $-2$ times the area [29] subtended by the path $T_\eta = \{ \alpha_\eta(t) | t \in [0, T]\}$ of motion in the phase space and the second is the argument of the overlap between the initial and final states. Define $\Delta \gamma^d_\eta = \gamma^d_\eta(T) - \gamma^d_\eta(0)$ as the dynamic (geometric) phase difference of the interferometer, which satisfies $\phi_I = \Delta \gamma^d + \Delta \gamma^g + \arg \langle \alpha_\eta(T)|\alpha_\eta(0) \rangle$. The third component in $\phi_I$ is given by $\arg \langle \alpha_\eta(T)|\alpha_\eta(0) \rangle = |\alpha_\eta(T)| |\alpha_\eta(0) \rangle | \sin \left( \arg \langle \alpha_\eta(0) | - \arg \langle \alpha_\eta(T) \rangle \right|$, which only depends on the respective final positions of the $|0 \rangle$ and $|1 \rangle$ state atoms in the trap when they are recombined, and also has a clear geometric meaning. Therefore, it can be absorbed into the geometric phase difference. Consequently, the total phase difference of the Sagnac interferometer in Eq. (5) can be decomposed into

$$\phi_I = \Delta \gamma^d + \Delta \gamma^g, \quad (9)$$

where $\Delta \gamma^g = \gamma^g_\eta(T) - \gamma^g_\eta(0) + \arg \langle \alpha_\eta(T)|\alpha_\eta(0) \rangle$ is the purely geometric contribution related to the area and angle differences in the phase spaces, respectively.

Next, with a Theorem, we show that by properly selecting the interrogation time $T$ and the temporal profile of the sweeping angular velocity $\omega_P(t)$, the Sagnac phase can be an unconventional geometric phase. And then we propose an experimentally accessible scheme for unconventional geometric Sagnac interferometer.

Theorem.—For certain proper interrogation time $T$ and temporal profiles of $\omega_P(t) \geq 0$ with $t \in [0, T]$ which satisfy the SPMC, there exist system-parameter independent $\kappa \in \mathbb{R}$ such that

$$\Delta \gamma^d = (\kappa - 1) \Delta \gamma^g, \quad (10)$$

where for $\kappa = 1$, the Sagnac phase $\phi_S$ is purely geometric, and for $\kappa \neq 1$, $\phi_S$ is an unconventional geometric phase [30].

Proof and examples.—With the frequency spectrum $\tilde{W}_P (\omega)$ and straightforward calculations we obtain $\Delta \gamma^g = \sqrt{2/\pi} \Re \left[ \tilde{W}_P (\omega_0) \right]$ and $\Delta \gamma^d = \phi_S \left\{ 1 - \sqrt{2/\pi} \Re \left[ \tilde{W}_P (\omega_0) \right] \right\} - \Delta \gamma^g$, where $\xi (\omega_0, T) = \omega_0 \partial_\omega \Re \left[ \tilde{W}_P (\omega) \right]_{\omega=\omega_0} - \omega_0 T \Im \left[ \tilde{W}_P (\omega_0) \right]$ [24]. Recalling the SPMC, which is $\text{Re} \left[ \tilde{W}_P (\omega_0) \right] = 0$, $\kappa$ in Eq. (10) can be expressed as $\kappa = \sqrt{\pi/2 / \xi (\omega_0, T)}$. Therefore, for proper $\omega_P(t)$ and $T$, if there exists nonzero $\xi (\omega_0, T)$, then the above Theorem holds automatically. The unconventional geometric class with $\xi (\omega_0, T) \neq \sqrt{\pi/2}$ is more generic and could be more accessible. Below we give one example of this class with a sinusoidal temporal profile for $\omega_P(t)$, which could be a scheme for the unconventional geometric Sagnac interferometer. For comparison, a pure geometric scheme with flat $\omega_P(t)$ profile will be also presented.

(i) Unconventional geometric Sagnac phase. A sinusoidal angular velocity $\omega_P(t) = \pi^2 \sin (2\pi T/3) / (2T)$ for $t \in [0, T]$ with $T = 2\pi/\omega_0$, matches the Sagnac phase and maximizes the contrast at the same time, i.e., $\tilde{W}_P (\omega_0) = 0$ for this situation. This sinusoidal profile results in a nontrivial solution for $\kappa$ in Eq. (10), $\kappa = 8/\pi^2$, and the Sagnac phase $\phi_S = 8 \Delta \gamma^g / \pi^2$ is an unconventional geometric phase [24], by which we mean that the geometric $\phi_S$ also involves a dynamic component [11].

(ii) Pure geometric Sagnac phase. A constant angular velocity $\omega_P(t) = \pi/T$ for $t \in [0, T]$ with $T = 2K\pi/\omega_0$ ($K = 1, 2, 3, \cdots$) gives $\tilde{W}_P (\omega_0) = 0$, and therefore the Sagnac phase is automatically matched and the contrast is
maximized simultaneously. The solution for $\kappa$ in Eq. (10) is $\kappa = 1$ [24]. Furthermore, in this example $\gamma_0^S(T) = -K\pi$ for both branches with $\eta = 0$ and 1, which comes from the zero-energy contribution. So the Sagnac phase in this case is purely geometric.

The physical scenarios for above two examples are: When the two branches are recombined at $t = T$, the atoms in each trap accomplish 1 (for (i)) and $K$ (for (ii)) cyclic evolutions, respectively, and return to the initial vacuum state, during which the Sagnac phase is given by the area difference subtended by two trajectories. Plotted in Fig. 2 (a)-(c) are for the unconventional geometric Sagnac phase in example (i) and in Fig. 2 (d)-(f) are for the pure geometric counterpart in example (ii). In Fig. 2 (b) and (e), to satisfy the SPMC and to maximize the contrast at the same time, the trap frequency $\omega_0$ is given by the positive simultaneous zeros of real (Re) and imaginary (Im) parts of $\tilde{W}_P(\omega)$, which is $\omega_0 T = 2(2L + 1)\pi$ ($L = 0, 1, 2, \ldots$) for Fig. 2 (b) and $\omega_0 T = 2K\pi$ ($K$ is a positive integer) for Fig. 2 (e) [24]. Shown in Fig. 2 (c) and (f) are the phase space paths $\Gamma_\eta$ for $|\eta\rangle_s$ state atoms during the interrogation, with $\eta = 0$ and 1, respectively. Fig. 2 (c) is plotted with the sinusoidal $\omega_P(t)$ profile and $L = 0$, and Fig. 2 (f) is with the flat profile and $K = 1$. Note that in the example (i), only the $L = 0$ case can give a nontrivial solution for $\kappa$ in Eq. (10) [24]. The dashed blue line denotes $-\Gamma_1 = \{-\alpha_1(t) | t \in [0, T]\}$ which encloses the same area as $\Gamma_1$. Therefore, the area of the unfilled region inside $\Gamma_0$, which is equal to $\Delta\gamma^S/2$, is identical with $\phi_S/(2\kappa)$ with $\kappa = 8/\pi^2$ in Fig. 2 (c) while it equals $\phi_S/2$ in Fig. 2 (f), which are signatures of unconventional and pure geometric Sagnac phases, respectively.

So we have proved the above Theorem, and proposed a scheme for realizing unconventional geometric Sagnac rotation sensors. Compared to the purely geometric scheme in (ii), the unconventional geometric case in (i) could be more accessible in experiments due to a zero initial $\omega_P(t)$.

**Sensitivity and precision.**—The sensitivity of the Sagnac interferometer is limited by the uncertainty of the unbiased estimated value of $\Omega$, which is given by $\delta\Omega = \delta P(\phi_1)/\partial_1 P(\phi_1)$, where $P(\phi_1) = \langle \phi_2 \rangle$ is the signal given in Eq. (4). And straightforward calculation leads to

$$\frac{1}{(\delta\Omega)^2} = \frac{\langle \phi_\Omega \rangle^2}{|C_{1,0}|^2} \sin^2 \phi_1 \leq \mathcal{F},$$

where we have $|C_{1,0}| \leq 1$ and $\mathcal{F}$ is the quantum Fisher information (QFI) which determines the ultimate precision limit for quantum sensing via the quantum Cramér-Rao bound (QCRB) [31, 32]. And for the interferometer protocol $V(T)$ and the initial state $\rho(0)$ in this Letter, if the interrogation time is integer times the trap period, i.e., $T = 2K\pi/\omega_0$ ($K = 1, 2, 3, \ldots$), then the QFI in Eq. (11) is given by $\mathcal{F} = (\partial_1 \phi_1)^2$ [33]. Therefore, the conditions for attaining the equality in Eq. (11) and saturating the QCRB are $|C_{1,0}| = 1$ (or $\tilde{W}_P(\omega_0) = 0$) and $\omega_0 T = 2K\pi$. So both the two schemes in Fig. 2 with $P(\phi_1)$ measurements satisfy these conditions and thus saturate the QCRB.

**Conclusion.**—In summary, we have proposed a scheme for unconventional geometric Sagnac interferometer with trapped guided atomic clocks, where the geometric Sagnac phase also involves a dynamic component and could be more accessible in experiments, due to a zero initial sweeping angular velocity. Such unconventional geometric Sagnac phases could be potentially noise-resilient and be promising for high-sensitivity rotation sensors. And a scheme for pure geometric Sagnac interferometer is also provided for comparison. Besides, the Sagnac phase-matching condition for this system is presented, which is critical in experiments, in particular when the contrast is not maximized, to observe the interference fringes induced by the rotation through the well-known Sagnac phase with high sensitivity. Our work could stimulate further interests and studies on geometric rotation sensors with guided matter waves.

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[28] The analyses on the phase components here are made in the original Schrödinger picture, which is more reasonable for the Sagnac interferometer. If transformed into the Dirac or driven
SUPPLEMENTAL MATERIAL

In this supplemental material, we provide detailed derivations and calculations for the main results in the Letter. Concretely, in Sec. I we provide the interferometer protocol, derivation of the interferometer phase and the Sagnac phase-matching condition. In Sec. II we decompose the phase into dynamic and geometric components and give the corresponding analytic expressions. Finally in Sec. III, we present details of the examples for unconventional and pure geometric Sagnac phases in the main text.

I. INTERFEROMETER PROTOCOL AND SAGNAC PHASE-MATCHING CONDITION

The interferometer protocol is given by

\[ V(T) = Y \left( \frac{\pi}{2} \right) U(T) Y \left( -\frac{\pi}{2} \right), \]

where \( Y(\phi) = \exp(-i\phi\sigma_y/2) \) with \( \sigma_y \) being the Pauli matrix, and \( U(T) = \exp \left( -i \int_0^T H(t)dt/\hbar \right) \) with \( T \) being the time ordering operator and \( H(t) = H_0(t)\Pi_0 + H_1(t)\Pi_1 \) being the Hamiltonian, where we have used the notation \( |0\rangle_s (|1\rangle_s) = |\uparrow\rangle (|\downarrow\rangle) \) and \( \Pi_0 (\Pi_1) = |0\rangle_s\langle 0| (|1\rangle_s\langle 1|) \), with the subscript \( s \) (\( h \)) denoting the (pseudo)spin (harmonic trap) subspace.

With the properties of projection operators, \( \Pi_i \Pi_j = \delta_{ij} \Pi_s \) for \( i, j \in \{0, 1\} \), one can obtain

\[
U(T) = \mathcal{T} \exp \left( -i \int_0^T H(t)dt/\hbar \right)
\]

\[
= \mathcal{I}_h \otimes \mathcal{I}_s + (-i/\hbar) \int_0^T H(t)dt + \sum_{k=2}^{\infty} \frac{(-i/\hbar)^k}{k!} \int_0^T dt_0 \int_0^{t_0} dt_1 \cdots \int_0^{t_{k-2}} dt_{k-1} H(t_0) H(t_1) \cdots H(t_{k-1})
\]

\[
= \mathcal{I}_h \otimes \mathcal{I}_s + \left[ \mathcal{T} \exp \left( -i \int_0^T H_0(t)dt/\hbar \right) - \mathcal{I}_h \right] \Pi_0 + \left[ \mathcal{T} \exp \left( -i \int_0^T H_1(t)dt/\hbar \right) - \mathcal{I}_h \right] \Pi_1
\]

\[
= U_0(T)\Pi_0 + U_1(T)\Pi_1,
\]

where \( \mathcal{I} \) is the identity operator and \( U_\eta(T) = \mathcal{T} \exp \left( -i \int_0^T H_\eta(t)dt/\hbar \right) \) for \( \eta \in \{0, 1\} \), with \( H_\eta(t) \) being the time dependent single component Hamiltonian for the harmonic oscillator, and we have used the relation \( \Pi_0 + \Pi_1 = \mathcal{I}_s \).

For the one-dimensional model in Ref. [1], the Hamiltonian for the \( |\eta\rangle \) state atoms in the stationary reference frames relative to the corresponding transporting harmonic trap is given by

\[ H_\eta(t) = \hbar \omega_0 \left( a^\dagger a + \frac{1}{2} \right) + i\lambda_\eta(t) \left( a - a^\dagger \right), \]

where \( \omega_0 \) is the trap frequency and \( a (a^\dagger) \) is the annihilation (creation) operator for the trap mode. And

\[ \lambda_\eta(t) = \sqrt{\frac{m\omega_0^2}{2}} \left[ \Omega + (1 - 2\eta)\omega_P(t) \right], \]

with \( m \) being the particle mass and \( \Omega \) being the rotation frequency to be measured. \( \omega_P(t) \) for \( t \in [0, T] \) is experimentally designed sweeping angular velocity whose temporal profile determines the SPMC. The sweeping angular velocity \( \omega_P(t) \) can be further extended to a function \( W_P(t) \) defined on the whole real time axis \( t \in [ -\infty, +\infty ] \), with \( W_P(t) = \omega_P(t) \) for \( t \in [0, T] \) and \( W_P(t) = 0 \) for the other, and the frequency spectrum of \( W_P(t) \) can be obtained from its Fourier transform

\[ \tilde{W}_P(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} W_P(t) \exp(-i\omega t) \, dt. \]

The Hamiltonian in Eq. (S3) describes a forced harmonic oscillator and the corresponding time evolution operator at time \( t \) can be written as

\[ U_\eta(t) = U_{(0)}(t)U_\eta(t), \]

where \( U_{(0)}(t) = \exp \left[ -i\omega_0 \left( a^\dagger a + \frac{1}{2} \right) t \right] \) and \( U_\eta(t) \) satisfies

\[ i\hbar \frac{\partial}{\partial t} U_\eta(t) = i\lambda_\eta(t) \left[ \dot{a}(t) - \dot{a}^\dagger(t) \right] U_\eta(t), \]

(S6)
where $\hat{a}(t) = a \exp(-i \omega_0 t)$. Eq. (S6) can be solved from the Magnus expansion [2] and is given by

$$U_\eta(t) = D[\beta_\eta(t)] \exp \left[ i \phi_\eta(t) \right],$$

where $\beta_\eta(t) = -\int_0^t \lambda_\eta(\tau) \exp(i \omega_0 \tau) d\tau / \hbar,$

$$\phi_\eta(t) = \int_0^t \int_0^{\tau_1} \lambda_\eta(\tau_1) \lambda_\eta(\tau_2) \sin(\omega_0 (\tau_1 - \tau_2)) d\tau_2 d\tau_1 / h^2,$$

and $D(\beta) = \exp(\beta a^\dagger - \beta^* a)$ is the displacement operator for the oscillator. Therefore, the time evolution operator in Eq. (S5) reads

$$U_\eta(t) = U_{\eta(0)}(t) D[\beta_\eta(t)] U_{\eta(0)}^\dagger(t) \exp \left[ -i \omega_0 a^\dagger at \right] \exp \left[ i \left( \phi_\eta(t) - \omega_0 t / 2 \right) \right]$$

$$= D[\alpha_\eta(t)] \exp \left[ -i \omega_0 a^\dagger at \right] \exp \left[ i \left( \phi_\eta(t) - \omega_0 t / 2 \right) \right],$$

which is Eq. (3) in the main text, with

$$\alpha_\eta(t) = \beta_\eta(t) \exp \left[ -i \omega_0 t \right] = -\int_0^t \lambda_\eta(\tau) \exp \left[ i \omega_0 (\tau - t) \right] d\tau / \hbar.$$
respectively, where \( \phi_S = 2m\pi^2\Omega/\hbar \) is the Sagnac phase and we have used

\[
\int_0^T \int_0^{\tau_1} \omega_P(\tau_2) \sin [\omega_0 (\tau_1 - \tau_2)] d\tau_2 d\tau_1 = \int_0^T \int_0^T \omega_P(\tau_2) \sin [\omega_0 (\tau_1 - \tau_2)] d\tau_1 d\tau_2 \\
= \int_0^T \int_0^T \omega_P(\tau_2) \sin [\omega_0 (\tau_2 - \tau_1)] d\tau_2 d\tau_1. \tag{S15}
\]

The population difference, \( s\langle \sigma_z \rangle_s \), is given by

\[
s\langle \sigma_z \rangle_s = \text{Tr}_s [\rho_s(T)\sigma_z] \\
= -\text{Re} (C_{1,0}) \\
= -|C_{1,0}| \cos (\phi_I), \tag{S16}
\]

where the modulus \( |C_{1,0}| = \exp (-|\Delta \alpha|^2/2) \) gives the signal contrast, and the interferometer phase is given by Eq. (S12) and reads

\[
\phi_I = \text{arg} (C_{1,0}) \\
= \phi_0(T) - \phi_1(T) + \text{Im} (\alpha_1^*(T)\alpha_0(T)) \\
= \phi_S \left\{ 1 - \sqrt{\frac{2}{\pi}} \text{Re} \left[ \tilde{W}_P(\omega_0) \right] \right\}. \tag{S17}
\]

With the properties \( \omega_P(t) \geq 0 \) for \( t \in [0, T] \), \( |\cos \omega_0t| \leq 1 \) and \( \int_0^T \omega_P(t)dt = \pi \), one can obtain \( |\text{Re} \left[ \tilde{W}_P(\omega_0) \right] | \leq \sqrt{\pi}/2 \). Therefore, we have \( 0 \leq \phi_I \leq 2\phi_S \). In contrast to Ref. [1], this interferometer phase is indeed dependent of the Fourier components of \( \tilde{W}_P(\omega) \) at the trap frequency \( \omega_0 \) and therefore depends on the temporal profile of \( \omega_P(t) \). With Eq. (S17), we arrive at the Sagnac phase-matching condition (SPMC) which sets the interferometer phase to be exactly the Sagnac phase and is given by

\[
\text{Re} \left[ \tilde{W}_P(\omega_0) \right] = 0, \tag{S18}
\]

which is Eq. (6) in the main text. The origin of the real part of \( \tilde{W}_P(\omega_0) \) in \( \phi_I \) in Eq. (S17) may be understood intuitively from the following picture. The Hamiltonian in Eq. (S3) describes a parallel interaction between the qubit and the time dependent components of \( \tilde{W}_0 \) which is Eq. (6) in the main text. The origin of the real part of \( \tilde{W}_P(\omega_0) \) at the trap frequency \( \omega_0 \) and therefore depends on the temporal profile of \( \omega_P(t) \). With Eq. (S17), we arrive at the Sagnac phase-matching condition (SPMC) which sets the interferometer phase to be exactly the Sagnac phase and is given by

\[
\text{Re} \left[ \tilde{W}_P(\omega_0) \right] = 0,
\]

which is Eq. (6) in the main text. The origin of the real part of \( \tilde{W}_P(\omega_0) \) in \( \phi_I \) in Eq. (S17) may be understood intuitively from the following picture. The Hamiltonian in Eq. (S3) describes a parallel interaction between the qubit and the time dependent Hamiltonian \( H_I = i\gamma_0(\alpha(t)a - a^\dagger(t)\alpha^*) \), which leads to an energy shift \( \Delta E \) between the two clock states during the interrogation and results in an additional phase shift for the interferometer. Note that in the coherent state \( |\alpha\rangle \) basis, \( \langle \alpha|a(a^\dagger)|\alpha\rangle \) is always a real number. Therefore, in the frequency domain, this energy shift mainly comes from the real part of the Fourier transform of \( \tilde{W}_P(t) \) at the typical energy scale of the system, i.e. the trap frequency \( \omega_0 \), so \( \Delta E \propto \text{Re} \left[ \tilde{W}_P(\omega_0) \right] \). Consequently, the total phase difference of the interferometer is also dependent on \( \text{Re} \left[ \tilde{W}_P(\omega_0) \right] \).

II. DECOMPOSITION OF THE INTERFEROMETER PHASE

The total interferometer phase \( \phi_I \) in Eq. (S17) can be decomposed into dynamic and geometric components, respectively, which will be analyzed below. For the spin state \( |\eta\rangle_s \) (\( \eta = 0, 1 \)), the total phase change for quantum evolution in each harmonic trap can be divided into dynamic and geometric components \( \gamma^d_\eta(T) \) and \( \gamma^g_\eta(T) \), which are given by

\[
\gamma^d_\eta(T) = -\int_0^T \langle \psi_\eta(t) | H_\eta(t) | \psi_\eta(t) \rangle dt \\
= -\int_0^T \left[ \omega_0 \left( |\alpha_\eta(t)|^2 + \frac{1}{2} \right) - \frac{2\lambda_\eta(t)}{\hbar} \text{Im} \alpha_\eta(t) \right] dt \\
= 2\phi_\eta(T) - \omega_0 \int_0^T |\alpha_\eta(t)|^2 dt - \frac{1}{2} \omega_0 T, \tag{S19}
\]

\[
\gamma^g_\eta(T) = \phi_\eta(T) - \phi_0(T) + \text{Im} \left( \alpha_\eta(T) a - a^\dagger(T) \alpha^*_\eta(t) \right) dt.
\]
and
\[ \gamma^\beta_8(T) = \frac{i}{2} \int_{0}^{T} \alpha^*_\eta d\alpha_\eta - \alpha_\eta d\alpha^*_\eta - \arg \left[ \langle \alpha_\eta(T) | G \rangle \right] \]
\[ = - \int_{0}^{T} \text{Im} \left[ \alpha^*_\eta(t) \partial_t \alpha_\eta(t) \right] dt \]
\[ = -\phi_\eta(T) + \omega_0 \int_{0}^{T} |\alpha_\eta(t)|^2 dt, \]  
(S20)
respectively, where \( \phi_\eta(T) \) is given by Eq. (S8) and satisfies \( \phi_\eta(T) - \omega_0 T / 2 = \gamma^\beta_0(T) + \gamma^\beta_8(T) \), and \( \text{arg} \left[ \langle \alpha_\eta(T) | G \rangle \right] = 0 \). Note that here and hereafter we will drop the subspace subscript \( h \) for convenience. Define \( \Delta^\gamma_g(T) = \gamma^\beta_g(T) - \gamma^\beta_1(T) \) as the dynamic (geometric) phase difference of the interferometer, therefore the phase difference of the interferometer reads \( \phi_I = \Delta^\gamma_d + \Delta^\gamma_g + \text{arg} [\langle \alpha_1(T) | \alpha_0(T) \rangle] \), where the third component is given by
\[ \text{arg} [\langle \alpha_1(T) | \alpha_0(T) \rangle] = |\alpha^*_1(T)\alpha_0(T)| \sin \{ \text{arg} [\alpha_0(T)] - \text{arg} [\alpha_1(T)] \} \]
\[ = \text{Im} [\alpha^*_1(T)\alpha_0(T)], \]  
(S21)
which only depends on the respective final positions of the \( |0\rangle \) and \( |1\rangle \) state atoms in the trap when they are recombined, and also has a clear geometric meaning. Therefore, it can be absorbed into the geometric phase difference. Consequently, the total phase difference of the Sagnac interferometer in Eq. (S17) can be decomposed into
\[ \phi_I = \Delta^\gamma_d + \Delta^\gamma_g, \]  
(S22)
where \( \Delta^\gamma_g = \gamma^\beta_0(T) - \gamma^\beta_1(T) + \text{arg} [\langle \alpha_1(T) | \alpha_0(T) \rangle] \) is the purely geometric contribution related to the area and angle differences in the phase spaces, respectively.

In general, the calculations of explicit expressions for dynamic and geometric phases in each trap are difficult for an arbitrary \( \lambda_\eta(t) \). Whereas, the dynamic and geometric phase differences \( \Delta^\gamma_d \) and \( \Delta^\gamma_g \) in Eq. (S22) can be expressed in terms of \( \mathcal{W}_P(\omega) \) and its derivative at the trap frequency \( \omega_0 \), which will be shown below.

With
\[ \omega_0 \int_{0}^{T} |\alpha_\eta(t)|^2 dt = (\omega_0/h^2) \int_{0}^{T} dt \int_{0}^{t} d\tau_1 \int_{0}^{t} d\tau_2 \lambda_\eta(\tau_1)\lambda_\eta(\tau_2) \cos \omega_0 (\tau_1 - \tau_2) \]  
(S23)
and by defining \( \int_{0}^{T} \Delta |\alpha(t)|^2 dt = \int_{0}^{T} (|\alpha_0(t)|^2 - |\alpha_1(t)|^2)^2 dt \), one can easily obtain
\[ \omega_0 \int_{0}^{T} \Delta |\alpha(t)|^2 dt = (\omega_0/h^2) \int_{0}^{T} dt \int_{0}^{t} d\tau_1 \int_{0}^{t} d\tau_2 [\lambda_\eta(\tau_1)\lambda_\eta(\tau_2) - \lambda_\eta(\tau_1)\lambda_\eta(\tau_2)] \cos \omega_0 (\tau_1 - \tau_2) \]
\[ = (m\omega^2_0\Omega^2 / h) \int_{0}^{T} dt \int_{0}^{t} d\tau_1 \int_{0}^{t} d\tau_2 [\omega_P(\tau_1) + \omega_P(\tau_2)] \cos \omega_0 (\tau_1 - \tau_2) \]
\[ = \phi_S \left\{ 1 - \sqrt{\frac{2}{\pi}} \text{Re} \left[ e^{i\omega_0 T} \mathcal{W}_P(\omega_0) \right] - \sqrt{\frac{2}{\pi}} \omega_0 T \text{Im} \left[ \mathcal{W}_P(\omega_0) \right] - \frac{\omega_0}{\pi} \int_{0}^{T} \tau \omega_P(\tau) \sin \omega_0 \tau d\tau \right\} \]
\[ = \phi_S \left\{ 1 - \sqrt{\frac{2}{\pi}} \text{Re} \left[ e^{i\omega_0 T} \mathcal{W}_P(\omega_0) \right] - \sqrt{\frac{2}{\pi}} \omega_0 T \text{Im} \left[ \mathcal{W}_P(\omega_0) \right] + \sqrt{\frac{2}{\pi}} \omega_0 \partial_\omega \text{Re} \left[ \mathcal{W}_P(\omega) \right] \right\}, \]  
(S24)
where we have used the same integration method as in Eq. (S15) to obtain the third equation and we also have used the relation \( \partial_\omega \text{Re} \left[ \mathcal{W}_P(\omega) \right] = -\int_{0}^{T} \tau \omega_P(\tau) \sin \omega_0 \tau d\tau / \sqrt{2\pi} \). Together with Eqs. (S13), (S14), (S19), and (S20), we obtain
\[ \Delta^\gamma_g = \sqrt{\frac{2}{\pi}} \phi_S \omega_0 \left\{ \partial_\omega \text{Re} \left[ \mathcal{W}_P(\omega) \right] \right\}_{\omega=\omega_0} - T \text{Im} \left[ \mathcal{W}_P(\omega_0) \right], \]  
(S25)
and
\[ \Delta^\gamma_d = \phi_S \left\{ 1 - \sqrt{\frac{2}{\pi}} \text{Re} \left[ \mathcal{W}_P(\omega_0) \right] \right\} - \Delta^\gamma_g, \]  
(S26)
respectively, which are the expressions for \( \Delta^\gamma_g \) and \( \Delta^\gamma_d \) in the Proof of the main text.
III. GEOMETRIC SAGNAC PHASES

In the main text, we showed the existence of pure and unconventional geometric Sagnac interferometers and demonstrated corresponding examples. Here we give detailed calculations.

Theorem.—For certain proper interrogation time $T$ and temporal profiles of $\omega_P(t) \geq 0$ with $t \in [0, T]$ which satisfy the SPMC, there exist system-parameter independent $\kappa \in \mathbb{R}$ such that

$$\Delta \xi^d = (\kappa - 1) \Delta \xi,$$  \hspace{1cm} (S27)

where for $\kappa = 1$, the Sagnac phase $\phi_S$ is purely geometric, and for $\kappa \neq 1$, $\phi_S$ is an unconventional geometric phase. The expression for $\kappa$ in Eq. (S27) can be directly calculated from Eqs. (S25) and (S26), which is

$$\kappa = \sqrt{\frac{\pi}{2}} \text{Re} \left[ \frac{\mathcal{W}_P(\omega)}{\xi(\omega_0, T)} \right],$$ \hspace{1cm} (S28)

where $\xi(\omega_0, T) = \omega_0 \partial_\omega \text{Re} \left[ \mathcal{W}_P(\omega) \right]_{\omega = \omega_0} - \omega_0 \text{Im} \left[ \mathcal{W}_P(\omega_0) \right]$ and the condition under which Eq. (S28) holds is $\xi(\omega_0, T) \neq 0$ or $\partial_\omega \text{Re} \left[ \mathcal{W}_P(\omega) \right]_{\omega = \omega_0} \neq T \text{Im} \left[ \mathcal{W}_P(\omega_0) \right]$. Note that under the SPMC, $\kappa = \sqrt{\pi/2}/\xi(\omega_0, T)$. If the contrast is maximized at the same time, then an geometric Sagnac phase requires that the locations of $\omega_0$, i.e., the simultaneous zeros of $\text{Re} \left[ \mathcal{W}_P(\omega) \right]$ and $\text{Im} \left[ \mathcal{W}_P(\omega) \right]$, are not maxima or minima of the real part, and if $\omega_0 \partial_\omega \text{Re} \left[ \mathcal{W}_P(\omega) \right]_{\omega = \omega_0} \neq \sqrt{\pi/2}$, this geometric phase is unconventional.

Example (i): Unconventional geometric Sagnac phase with a sinusoidal temporal profile for $\omega_P(t)$. A sinusoidal angular velocity $\omega_P(t) = \pi^2 |\sin (2\pi t/T)| / (2T)$ with $t \in [0, T]$ gives the Fourier transform

$$\text{Re} \left[ \mathcal{W}_P(\omega) \right] = \frac{\sqrt{\pi/2} \cos^2 \left( \frac{\omega T}{4} \right) \cos \left( \frac{\omega T}{2} \right)}{1 - \left( \frac{\omega T}{2\pi} \right)^2}, \quad \text{Im} \left[ \mathcal{W}_P(\omega) \right] = -\frac{\sqrt{2\pi} \cos^3 \left( \frac{\omega T}{4} \right) \sin \left( \frac{\omega T}{4} \right)}{1 - \left( \frac{\omega T}{2\pi} \right)^2}. \hspace{1cm} (S29)

So the SPMC requires that $\omega_0 T = (2L + 1)\pi$ or $2(2L + 1)\pi$ with $L = 0, 1, 2, \cdots$ and $\mathcal{W}_P^*(\omega_0) = 0$ requires that $\omega_0 T = 2(2L + 1)\pi$ ($L = 0, 1, 2, \cdots$). The intersection is $\omega_0 T = (2L + 1)\pi$ ($L = 0, 1, 2, \cdots$). Further calculations show that only the $L = 0$ case with $T = 2\pi/\omega_0$ can give a solution of $\kappa$ in Eq. (S28), which is $\kappa = 8/\pi^2$. And therefore, the Sagnac phase $\phi_S = 8\Delta \xi^d/\pi^2$ is an unconventional geometric phase, by which we mean that the geometric $\phi_S$ also involves a dynamic component [3]. For the other cases with $L \neq 0$, $\partial_\omega \text{Re} \left[ \mathcal{W}_P(\omega) \right]_{\omega = \omega_0} \equiv 0$, and $\phi_S$ is completely dynamic.

Example (ii): Pure geometric Sagnac phase with a flat temporal profile for $\omega_P(t)$. A constant angular velocity $\omega_P(t) = \pi/T$ with $t \in [0, T]$ gives the Fourier transform

$$\text{Re} \left[ \mathcal{W}_P(\omega) \right] = \sqrt{\frac{\pi}{2} \sin \omega T}{\omega T}, \quad \text{Im} \left[ \mathcal{W}_P(\omega) \right] = \sqrt{\frac{\pi}{2} \cos \omega T - 1}{\omega T}. \hspace{1cm} (S30)

Therefore, the SPMC requires that $\omega_0 T = K\pi$ and the maximization of contrast, i.e., $\mathcal{W}_P^*(\omega_0) = 0$, requires that $\omega_0 T = 2K\pi$, with $K$ being a positive integer. If the interrogation time is selected to be $T = 2K\pi/\omega_0$ ($K = 1, 2, 3, \cdots$), then the Sagnac phase is automatically matched and the contrast is maximized simultaneously. For this case, the solution for $\kappa$ in Eq. (S28) is $\kappa = 1$. Furthermore, in this example $\gamma_\eta^d(T) = -K\pi$ for both branches with $\eta = 0$ and 1, which comes from the zero-energy contribution. So the Sagnac phase in this case only has a purely geometric component.

Next we provide one extra example besides (i) and (ii), which gives nontrivial solutions for $\kappa \neq 1$ in Eq. (S27), but only satisfy the SPMC while keeps a finite imaginary part of $\mathcal{W}_P^*(\omega_0)$.

Example (iii): Unconventional geometric Sagnac phase with small contrast. $\omega_P(t) = \pi^2 \sin (\pi t/T) / (2T)$ with $t \in [0, T]$, and the Fourier transform reads

$$\text{Re} \left[ \mathcal{W}_P(\omega) \right] = \frac{\sqrt{\pi/2} \cos^2 \left( \frac{\omega T}{4} \right)}{1 - \left( \frac{\omega T}{2\pi} \right)^2}, \quad \text{Im} \left[ \mathcal{W}_P(\omega) \right] = -\frac{\sqrt{\pi/8} \sin \omega T}{1 - \left( \frac{\omega T}{2\pi} \right)^2}. \hspace{1cm} (S31)

Therefore, the SPMC requires that $\omega_0 T = (2L + 1)\pi$ with $L = 0, 1, 2, \cdots$, while $\mathcal{W}_P^*(\omega_0) = 0$ requires that $\omega_0 T = (2K + 1)\pi$ ($K = 1, 2, 3, \cdots$), where for the latter $\partial_\omega \text{Re} \left[ \mathcal{W}_P(\omega) \right]_{\omega = \omega_0} = T \text{Im} \left[ \mathcal{W}_P(\omega_0) \right] \equiv 0$ and the Sagnac phase $\phi_S$ is completely dynamic. So we can take $L = 0$ and $T = \pi/\omega_0$, which satisfies the SPMC and gives a solution of $\kappa$ in Eq. (S28) with $\kappa = 8/\pi^2$, but leaves a finite imaginary part of $\mathcal{W}_P^*(\omega_0)$ and a small signal contrast.
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