Remarks on the Spectral Action Principle

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Abstract
The presence of chiral fermions in the physical Hilbert space implies consistency conditions on the spectral action. These conditions are equivalent to the absence of gauge and gravitational anomalies. Suggestions for the fermionic part of the spectral action are made based on the supersymmetrisation of the bosonic part.
1 Introduction

In noncommutative geometry the interplay between geometry and physics is possible because a noncommutative space is defined by a spectral triple \((\mathcal{A}, \mathcal{H}, D)\) where \(\mathcal{A}\) is an algebra of operators, \(\mathcal{H}\) a Hilbert space of states, \(D\) is a compact operator acting on \(\mathcal{H}\). When the space is supplied with a real structure \(J\), an automorphism on elements of the algebra is equivalent to changing some initial metric \(D_0\) to

\[
D = D_0 + A + AJ^{-1}
\]

where \(A = \sum a_i [D, b_i]\) is a one form. It is conjectured in that the dynamics of the geometrical fields encoded in the metric operator \(D\) is given by

\[
Tr \left( F \left(D^2/m^2\right) \right) + (\psi, D\psi)
\]

where \(F\) is an arbitrary function of \(D\), the trace “\(Tr\)” is taken over the Hilbert space of states, \(\psi\) is a collection of states in the Hilbert space, and \((..,.)\) defines an inner product in the Hilbert space. The mass scale \(m\) is usually related to a cut-off scale. When this principle is used for the noncommutative space defined by the fermionic spectrum of the standard model, the spectral action gives, in the low energy limit, the standard model with all of its detailed structure.

In the considerations of two important points were not dealt with. The first has to do with the fact that states in the Hilbert space are chiral fermions. As the eigenvalue problem is only well defined for Dirac fermions one must define the trace over the chiral states only. We shall show that such a restriction makes the trace function non-invariant under chiral rotation, unless certain conditions on the fermionic representations of states in the Hilbert space are satisfied. These conditions coincide with the cancellation of gauge and gravitational anomalies. The second point not studied, is the obvious asymmetry between the bosonic and fermionic parts of the spectral action. The fermionic part is linear in \(D\) while the bosonic part involves a general function \(F\) of \(D\). We shall investigate this question, to find out whether more general fermionic actions are possible, by studying a supersymmetrization of some of the higher order bosonic terms obtained in the heat kernel expansion. The plan of this paper is as follows. In section two, an expression for the anomaly of the spectral action is derived. In section three higher order terms in the bosonic part of the spectral action of a simple Yang-Mills system, are obtained. In section four we supersymmetrize these higher order terms and postulate a general form for the fermionic action. Section five contains the conclusions.

2 Anomalies in the spectral action

Fermions that appear in the physical Hilbert space in any realistic model must be chiral. The grading operator \(\gamma\) is such that

\[
\gamma\psi_\pm = \pm\psi_\pm,
\]

\[
\gamma D\psi_\pm = \mp D\gamma\psi_\pm
\]

which implies that \(D\psi_\pm\) have opposite chirality to \(\psi_\pm\). Therefore, for chiral fermions with non-zero eigenvalues, one cannot set the eigenvalue problem for
\[ D^2 \psi_{n\pm} = \lambda_n \psi_{n\pm}, \quad \lambda_n = \lambda_n^* \]  

(4)

where

\[ \lambda_n = (\psi_n - |D^2| \psi_n^+). \]  

(5)

Assuming that all fermions in the spectrum have positive chirality (negative chirality fermions could always be written as the conjugate of positive chirality fermions). The bosonic spectral action in this case could be written as

\[ I_b = Tr \left( F(D^2) \right) = \sum_n (\psi_n - |F(D_- D_+) \psi_n^+ \right) \]  

(6)

where \( D^2 \) is replaced by \( D_- D_+ \) when acting on \( \psi_n \).

Let us now consider the behavior of this action under chiral transformations. The fermionic action is invariant under chiral rotations

\[ |\psi_+ \rangle \rightarrow e^{i \theta \gamma_5} |\psi_+ \rangle \]  

(7)

which implies that

\[ (\psi_+ | D | \psi_+ \rangle \rightarrow (\psi_+ | e^{i \theta \gamma_5} D e^{i \theta \gamma_5} | \psi_+ \rangle. \]  

(8)

It is a simple matter to see that

\[ e^{i \theta \gamma_5} D e^{i \theta \gamma_5} = D \]

implying the invariance of the fermionic action.

Under chiral rotations the bosonic action \( \{1\} \) transforms as

\[ I_b \rightarrow Tr \left\{ \sum_n \left( e^{-i \theta^a T^a \gamma_5} \psi_n - |F(D_- D_+) e^{i \theta^a T^a \gamma_5} \psi_n^+ \right) \right\} \]

\[ \rightarrow Tr \left\{ e^{i \theta^a T^a \gamma_5} \left( \psi_n - |F(D_- D_+) \psi_n^+ \right) \right\} \]  

(9)

where \( T^a \) are matrix representations for the fermions. We now use the heat kernel expansion for the function \( F \) \( \{11\} \)

\[ F(P) = \sum_{n \geq 0} f_n e_n(P) \]  

(10)

where \( e_n(P) \) are geometric invariants whose trace gives the Seeley-de Witt coefficients for the operator \( P = D_- D_+ \). Under an infinitesimal transformation equation \( \{1\} \) simplifies to

\[ I_b \rightarrow I_b + \sum_{n \geq 0} f_n Tr \left( 2i \theta^a T^a \gamma_5 e_n(P) \right). \]  

(11)

Because of the presence of \( \gamma_5 \) in the trace, the first non-vanishing gauge term comes from \( e_4(P) \) and is of the form

\[ 2i \theta^a T^a \left( \gamma_5 \gamma^{\mu \rho \sigma} T^b T^c \right) G_{\mu \nu}^b G_{\rho \sigma}^c \]  

\[ = i e^{\mu \rho \sigma} \theta^a C_{\mu \nu}^b G_{\rho \sigma}^c T \left( T^b, T^c \right). \]  

(12)
Therefore the bosonic action is non-invariant except when the condition on the fermionic group representations

\[ Tr \left( T^a \{ T^b, T^c \} \right) = 0 \] (13)

is satisfied. We can relate the anomaly generating term to non-conservation of currents. Let \( \theta = \theta^a(x) T^a \), then

\[ (\psi_+ | D | \psi_+) \rightarrow (\psi_+ | D | \psi_+) - i \theta^a \partial_\mu (\psi_+ | \gamma^\mu T^a | \psi_+) \] (14)

where we have used the identity

\[ e^{i \theta \gamma_5} D e^{i \theta \gamma_5} = D + i \theta^\mu \gamma_5 \partial_\mu \theta \]

and integrated by parts. Defining the fermionic current

\[ j_\mu^a = (\psi_+ | \gamma^\mu T^a | \psi_+) \] (15)

we then have

\[ \partial_\mu j_\mu^a = - \epsilon^{\mu \nu \rho \sigma} G_{\mu \nu} G_{\rho \sigma} Tr \left( T^a \{ T^b T^c \} \right). \] (16)

There is also one gravitational term that comes from \( e_4 (P) \)

\[ \epsilon^{\mu \nu \rho \sigma} \theta^a Tr \left( T^a \right) R_{\mu \nu} R_{\rho \sigma}. \] (17)

Therefore gravitational anomalies are absent if one further imposes the condition

\[ Tr \left( T^a \right) = 0. \] (18)

One can make similar analysis for anomalies on higher dimensional manifolds. It can be immediately seen that in ten dimensions the first non-vanishing trace in the heat-kernel expansion would result from a term of the form

\[ Tr \left( \gamma_{11} \gamma_{\mu_1 \mu_2} \ldots \gamma_{\mu_9 \mu_{10}} T^a_1 \ldots T^a_{10} \right) F_{\mu_1 \mu_2}^{a_1} \ldots F_{\mu_9 \mu_{10}}^{a_{10}} \] (19)

which is obviously related to the well known hexagon diagram \[ \text{[12]} \].

We conclude this section by observing that chiral gauge anomalies arise in the path integral formulation because the fermionic measure is not invariant under chiral rotations \[ \text{[8, 9, 10, 12]} \], while in the spectral action they arise because of the non-invariance of the trace operator. The anomaly cancellation conditions are, however, identical in both cases.

### 3 Fermionic part of the spectral action

Looking at the spectral action \[ \text{[2]} \] one cannot fail to notice the asymmetry between the bosonic and fermionic parts. The fermionic part has the simple form \( (\psi | D | \psi) \) while the bosonic part involves some (non-linear) function of the Dirac operator. If one hopes for some symmetry between the two parts of the action, there must exist a relation between them. As an example, in supersymmetric theories the bosonic and fermionic parts of the action are completely related to each other. This tends to indicate that the fermionic action must be more complicated than the simple form in \[ \text{[2]} \]. A first guess is to take

\[ I_f = (\psi | G(D) | \psi) \] (20)
where \( G(D) \) is some (odd) function of \( D \) (as follows from chirality). Although this is not the most general form because one can also write terms which are not quadratic but quartic or higher in the fermionic fields, it is the simplest form that involves an arbitrary function. To get a concrete idea on the possible structure of the fermionic terms we shall consider the supersymmetrization of the bosonic part of the spectral action associated with a simple noncommutative space. The spectral triple we are interested in is given by

\[
\mathcal{A} = C^\infty(M) \otimes M^N(C) \\
\mathcal{H} = L^2(M, S) \otimes M_N(C) \\
D_0 = \gamma^\mu \partial_\mu \otimes 1_N
\] (21)

which defines the product of a continuous four dimensional manifold \( M \) times the algebra of \( N \times N \) matrices. We add a real structure \( J \) (the charge conjugation operator) and supplement the fermions with their conjugate elements. The space of metrics has a natural foliations into equivalence classes. The internal fluctuations of a given metric are given by

\[
D = D_0 + A + JAJ^{-1}
\] (22)

where \( A = \sum_i a_i [D, b_i] \), \( a_i, b_i \in \mathcal{A} \). Automorphisms on elements of \( \mathcal{A} \) is equivalent to replacing \( D_0 \) by \( D \). For simplicity we shall write

\[
D = \gamma^\mu (\partial_\mu + A_\mu)
\] (23)

where \( A_\mu \) is traceless and \( D \) acts only on the Hilbert space of fermions, but not their conjugates. The bosonic action for this example was worked out, up to terms with two derivatives, in [7]. It is given by the expansion

\[
Tr F(D^2) = \sum_{n \geq 0} f_n a_n (D^2)
\] (24)

where

\[
f_0 = \int_0^\infty u F(u) du,
\]

\[
f_2 = \int_0^\infty F(u) du,
\]

\[
f_{2(n+2)} = (-1)^n F^{(n)}(0), \quad n \geq 0
\] (25)

To obtain terms with derivatives of order higher than two one must consider the Seeley-de Witt coefficient \( a_6 (D^2) \). To find out these terms we first write

\[
P = D^2 = -(\partial^\mu \partial_\mu + L^\mu \partial_\mu + B)
\] (26)

where for simplicity we assumed a flat manifold \( M \) with \( g_{\mu\nu} = \delta_{\mu\nu} \). For the space in [24] we have

\[
L^\mu = 2A^\mu \\
B = \partial^\mu A_\mu + A^\mu A_\mu + \frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu}.
\] (27)
The result of Gilkey [11] for $a_6 (P)$ is:

$$a_6 = \frac{1}{360} \text{Tr} \left( 8F^{\mu\nu\rho}F_{\mu\nu\rho} + 2F_{\mu\nu}^{:\rho}F^{\mu\rho} + 12F_{\mu\nu}F^{\mu\nu\rho} \rho 
- 12F_{\mu\nu}F^{\mu\nu\rho} \rho + 6E_{\mu\nu}^{\mu\nu} + 60EE_{\mu}^{\mu} \right)$$

$$+ 30E_{\mu}^{\mu}E_{\mu} + 60E^3 + 30E\Omega_{\mu\nu} \Omega^{\mu\nu} \right) \right) \right)$$

(28)

where

$$\omega_{\mu} = \frac{1}{2} \gamma_{\mu},$$

$$\Omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu],$$

$$E = B - (\partial^\mu A_\mu + A^\mu A_\mu).$$

(29)

In the simple example under consideration we have

$$\omega_{\mu} = A_\mu$$

$$\Omega_{\mu\nu} = F_{\mu\nu}$$

$$E = - \frac{1}{2} \gamma^{\mu\nu}F_{\mu\nu}. \right)$$

(30)

It is straightforward to work out the final form of $\int a_6 (P) d^4x \sqrt{g}$. After integrating by parts and ignoring boundary terms, we obtain

$$\int_M a_6 = \frac{1}{360} \int_M \text{Tr} \left( 11F^{\mu\nu\rho}F_{\mu\nu\rho} + 2F_{\mu\nu}^{:\rho}F^{\mu\rho} + 48F_{\mu\nu}F^{\mu\rho}F_{\rho}^{\mu} \right) \sqrt{g} d^4x. \right)$$

(31)

The last term vanishes in the abelian case ($N = 1$), as this is proportional to

$$f^{abc}F_{\mu\nu}^{a}F^{\nu b}F_{\rho}^{\mu c}$$

(32)

where $f^{abc}$ are the structure constants of $SU(N)$.

4 Supersymmetrisation of the bosonic terms

To find the supersymmetric extension of the higher order bosonic term in the heat-kernel expansion of the spectral action, the easiest way is to use the method of superfields and write an action in superspace. We shall use the notation of Bagger and Wess [13].

Let $V (x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ be a real vector superfield. Introduce the covariant supersymmetry and gauge operators

$$\nabla_\alpha = e^{-V} D_\alpha e^V$$

$$\nabla_{\dot{\alpha}} = e^{-V} D_{\dot{\alpha}} e^V$$

$$\nabla_\mu = e^{-V} \partial_\mu e^V$$

(33)

where $D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i(\sigma^\mu \bar{\theta})_\alpha \partial_\mu$ and $D_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i(\bar{\theta} \sigma^\mu)_{\dot{\alpha}} \partial_\mu$ are the supersymmetry covariant derivatives which satisfy

$$\{D_\alpha, D_\beta\} = 0 = \{\nabla_{\dot{\alpha}}, \nabla_{\dot{\beta}}\}$$

$$\{D_\alpha, \nabla_{\dot{\beta}}\} = -2i(\sigma^\mu)_{\alpha \dot{\beta}} \partial_\mu.$$
Notice that $\nabla_\alpha$ could be expressed in the alternative form

$$\nabla_\alpha = D_\alpha + (e^{-V} D_\alpha e^V)$$  \hspace{1cm} (35)$$

where the expression within brackets is not an operator. The operators $\nabla_\alpha$ and $\nabla_\dot{\alpha}$ satisfy

$$\{\nabla_\alpha, \nabla_\beta\} = 0 = \{\nabla_\dot{\alpha}, \nabla_\dot{\beta}\}$$

$$\{\nabla_\alpha, \nabla_\dot{\alpha}\} = -2i(\sigma^\mu)_{\alpha\dot{\alpha}} \nabla_\mu. \hspace{1cm} (36)$$

The field $W_\alpha = \mathcal{D}^\alpha (e^{-V} D_\alpha e^V)$ transforms covariantly under the transformation $e^V \rightarrow e^{-i\Lambda} e^V e^{i\Lambda}$ where $\Lambda$ and $\overline{\Lambda}$ are chiral and antichiral fields ($\mathcal{D}_\alpha \Lambda = 0$):

$$W_\alpha \rightarrow e^{-i\Lambda} W_\alpha e^{i\Lambda} \hspace{1cm} (37)$$

and therefore could be used in constructing a supersymmetric and gauge invariant action. In the Wess-Zumino gauge, the vector field $V$ takes the simple form

$$V = -\theta \sigma^\mu \overline{A}_\mu (x) + i \theta \theta \overline{X} + \frac{1}{2} \theta \theta \theta \theta \overline{X}. \hspace{1cm} (38)$$

One can verify that $W_\alpha$ contains the gauge field strength [13].

$$W_\alpha = -i \lambda_\alpha (y) + \theta_\beta \left( \delta^\beta \nabla_\alpha (y) - i (\sigma^{\mu\nu})_\beta \nabla^{\mu\nu}(y) \right) + \theta \theta \left( \sigma^\mu D_\mu \overline{X} (y) \right) \hspace{1cm} (39)$$

where $y^\mu = x^\mu + i \theta \sigma^\mu \overline{\theta}$.

A supersymmetric action that does not contain terms higher than two derivatives is

$$\frac{1}{16} \int d^2 \theta d^2 \overline{\theta} \text{Tr} (W^\alpha W_\alpha) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i \overline{\Lambda} \gamma^\mu D_\mu \Lambda + \frac{1}{2} X^a X^a \hspace{1cm} (40)$$

with an identical expression for the conjugate term.

To include terms with higher derivative, especially those found in the expression of $a_\alpha (P)$ we consider the symmetric and antisymmetric pieces of $\{\nabla_\alpha, W_\beta\}$ as these transform covariantly

$$\{\nabla_\alpha, W_\beta\} \rightarrow e^{-i\Lambda} \{\nabla_\alpha, W_\beta\} e^{i\Lambda}. \hspace{1cm} (41)$$

The invariant actions are

$$\text{Tr} \int d^2 \theta d^2 \overline{\theta} \left\{ \{\nabla_\alpha, W_\alpha\} \{\nabla_\beta, W_\beta\} \right\} + h.c \hspace{1cm} (42)$$

$$\text{Tr} \int d^2 \theta d^2 \overline{\theta} \left( \{\nabla_\alpha, W_\beta\} + \{\nabla_\beta, W_\alpha\} \right) \left\{ \{\nabla_\alpha, W_\beta\} \right\} + h.c. \hspace{1cm} (43)$$

The list of terms coming from the first action (42) are

$$\text{Tr} \left( F_{\mu\nu;\rho} F^{\mu\nu;\rho} \right), \quad \text{Tr} \left( D_\mu X D^\mu X \right)$$

$$\text{Tr} \left( \overline{X} \gamma^\mu D_\mu D^\mu \lambda \right), \quad \text{Tr} \left( \overline{X} \gamma^\mu \lambda \overline{X} \gamma^\mu \lambda \right)$$

$$\text{Tr} \left( D_\mu \overline{X} \gamma^\mu \lambda F_{\mu\nu} \right), \quad \text{Tr} \left( \overline{X} \gamma^\mu \lambda F_{\mu\nu;\rho} \right)$$

$$\text{Tr} \left( \overline{X} \gamma^\mu \lambda D_\mu X \right) \hspace{1cm} (44)$$
We notice the appearance of one term quartic in the fermions $Tr (\overline{\chi}^\nu \lambda \overline{\chi}_\mu \lambda)$, which is of the current-current interaction type. Similarly, from the second action (43) we have

\[ Tr (F_{\mu\nu} F^{\nu\rho} F^\rho) \quad Tr (F_{\mu\nu} \gamma^\rho F^{\mu\nu\rho}) \quad Tr (\overline{\chi} \gamma^{\mu\nu} \lambda D_\mu \lambda F_{\nu\rho}) \quad Tr (\overline{\chi} \gamma^{\mu\nu} \lambda \overline{\chi}_\mu \lambda). \] (45)

Again the quartic interaction term appears. It is possible to fix the coefficient between the two actions (42) and (43) to cancel the quartic fermionic interaction. Without evaluating the higher order terms, it is not clear whether this phenomena of obtaining non-trivial fermionic interactions would persist and whether one would be able to eliminate them.

We now compare these results with those obtained from the fermionic action

\[ I_f = (\Psi \mid G(D) \mid \Psi) \] (46)

where $G(D)$ is an odd function of $D$

\[ G(D) = \sum_{n=0}^{\infty} g_{2n+1} D^{2n+1}. \] (47)

This implies that

\[ I_f = \sum_{n=0}^{\infty} g_{2n+1} (\Psi \mid D^{2n+1} \mid \Psi). \] (48)

Note that there is no loss of generality by assuming that $g(D)$ is odd as $(\Psi \mid D^{2n} \mid \Psi)$ is zero by chirality. Evaluating $(\Psi \mid D^3 \mid \Psi)$ we obtain

\[ (\Psi \mid D^3 \mid \Psi) = (\Psi \mid -D^{\mu} \gamma^\rho D_\rho \mid \Psi) + \frac{1}{2} (\Psi \mid \gamma^{\mu\nu} F_{\mu\nu} \gamma^\rho D_\rho \mid \Psi). \] (49)

The field $X$ appearing in the supersymmetric theory could be obtained by replacing $D$ by $D + X$. Therefore, one sees that the simple expression for the fermionic action can recover all terms appearing in the supersymmetric theory, to that order in derivatives, except for the quartic fermionic terms which for certain combinations can be made to vanish. It is therefore natural, from the spectral action point of view where an action is defined in terms of the Dirac operator, to postulate that the fermionic action is given by $I_f$. It would be preferable to have some symmetry principle that relates the functions $F$ and $G$ appearing in the bosonic and fermionic terms to each other. In the spectral action of the superstring [14] the same expression generates both the bosonic and the fermionic actions. Unfortunately, in two-dimensions one has to work with world-sheet fermions and not space-time fermions which are connected in a non-transparent way, and this makes it difficult to obtain such a relation.

5 Conclusions

In this work we have studied two questions. The first is the issue of chiral fermions in the spectral action, and whether these could be introduced consistently, as the eigenvalue problem is not well defined. The trace has to be
restricted to the set of eigenfunctions for the operator $D_- D_+$. This makes the trace non-invariant under chiral rotations, resulting in gauge and gravitational anomalies. We obtained a set of conditions for cancellation of anomalies which coincided with the known conditions. The reasons for appearance of anomalies are, however, different. In the path integral formulation, chiral anomalies arise because of the lack of invariance of the chiral fermions measure under chiral rotations. The second question we addressed is the form of the fermionic spectral action, and whether one can find a simple form for it. We studied supersymmetrisation of terms containing derivatives of orders higher than two, appearing in the bosonic effective spectral action. We found that there are, in general, quartic fermionic terms which could not be written in a way that depends only on the operator $D$ as they are of the current-current type. These terms could be arranged to cancel. It is not possible to write the quartic fermionic term in a local way as a function of the operator $D$. One can simply adopt the point of view, that the fermionic action must be given by a simple form, quadratic in fermions, and dependent on some odd function of $D$. We therefore conjecture that the spectral action is given by

$$I = \text{Tr} (F \left( D^2/m^2 \right)) + (\Psi \mid G(D) \mid \Psi) \quad (50)$$

It is very likely that a relation exists between the functions $F$ and $G$, but without a guiding symmetry principle, such a relation is not easy to find.

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