COHERENT TANNAKA DUALITY AND ALGEBRAICITY OF HOM-STACKS

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Abstract. We establish Tannaka duality for noetherian algebraic stacks with affine stabilizer groups. Our main application is the existence of $\mathrm{Hom}$-stacks in great generality.

1. Introduction

Classically, Tannaka duality reconstructs a group from its category of finite dimensional representations. Various incarnations of this have been studied for decades. In [Lur04], a Tannaka duality for algebraic stacks was introduced. This reformulation centered on the properties of the functor

$$\omega_X(T) : \mathrm{Hom}(T, X) \to \mathrm{Hom}_{c\otimes}(\mathrm{Coh}(X), \mathrm{Coh}(T)),$$

$$(f : T \to X) \mapsto (f^* : \mathrm{Coh}(X) \to \mathrm{Coh}(T)),$$

where $T$ and $X$ are algebraic stacks and the subscript $c\otimes$ denotes cocontinuous symmetric monoidal functors. In [Lur04] it was established that if $X$ is quasi-compact with affine diagonal (e.g., the quotient of a variety by an affine algebraic group), then $\omega_X(T)$ is fully faithful with image consisting of tame functors for every algebraic stack $T$. Even though tameness of a functor is a difficult condition to verify, Lurie was able to establish some striking applications of his results.

Lurie’s result is well-suited to the types of algebraic stacks that appear in algebraic topology and geometric representation theory—such stacks tend to have affine diagonal. In moduli theory, however, there are natural stacks that do not have quasi-affine diagonal (e.g., stacks of singular curves [AKL14, §4.1]) nor even separated diagonal (e.g., log stacks). The main result of this article is the following theorem, which besides removing Lurie’s hypothesis of affine diagonal, obviates tameness.

**Theorem 1.1.** Let $X$ be a noetherian algebraic stack with affine stabilizers. For every locally excellent algebraic stack $T$, the natural functors

$$\mathrm{Hom}(T, X) \to \mathrm{Hom}_{c\otimes, \simeq}(\mathrm{Coh}(X), \mathrm{Coh}(T)) \to \mathrm{Hom}_{c\otimes, \simeq}(\mathrm{QCoh}(X), \mathrm{QCoh}(T))$$

are equivalences of groupoids.
That $X$ has \textit{affine stabilizers} means that $\text{Aut}(x)$ is affine for every field $k$ and point $x$: $\text{Spec}(k) \to X$; equivalently, the diagonal of $X$ has affine fibers. We also wish to emphasize that we do not assume that the diagonal of $X$ is separated in Theorem 1.1. The subscript $r \otimes \simeq$ (resp. $c \otimes \simeq$) denotes right exact (resp. cocontinuous) symmetric monoid functors and natural isomorphisms. The restriction to stacks with affine stabilizers is a necessary condition for the equivalence in Theorem 1.1 (see Theorem 10.1). An algebraic stack is \textit{locally excellent} if there exists a smooth presentation by an excellent scheme (see Remark 7.2); this includes every algebraic stack that is locally of finite type over a field, over $\mathbb{Z}$, or over a complete local noetherian ring.

\textbf{Applications.} In future work with J. Alper [AHR14], Theorem 1.1 is applied to resolve Alper’s conjecture on the local quotient structure of algebraic stacks [Alp10]. A more immediate application of Theorem 1.1 is the following algebraicity result for Hom-stacks, generalizing all previously known results.

\textbf{Theorem 1.2.} Let $Z \to S$ be a proper and flat morphism of algebraic stacks that is of finite presentation. If $X \to S$ is a morphism of algebraic stacks that is locally of finite presentation with quasi-compact and quasi-separated diagonal and affine stabilizers, then so is $\text{Hom}_S(Z, X) \to S$. If $X \to S$ has affine (resp. quasi-affine, resp. separated) diagonal, then so has $\text{Hom}_S(Z, X)$.

Theorem 1.2 was motivated by [AOV11, Question 1.4]. Theorem 1.2 has already seen applications to log geometry [Wis14], an area which provides a continual source of stacks that are neither Deligne–Mumford nor have separated diagonal. In general, the condition that $X$ has affine stabilizers is necessary (see Theorem 10.4). However, if $Z \to S$ is finite and flat of finite presentation, then $\text{Hom}_S(Z, X)$ is algebraic for every algebraic stack $X$ [Ryd11c, Thm. 3.12]. Establishing the separation properties of $\text{Hom}_S(Z, X)$ in complete generality requires some extensions of the results of Olsson [Ols06b], which we isolate in Appendix C.

There are analogous algebraicity results for Weil restrictions (equivalently, restrictions of scalars).

\textbf{Theorem 1.3.} Let $f: Z \to S$ be a proper and flat morphism of algebraic stacks that is of finite presentation. If $X \to Z \to S$ is a morphism of algebraic stacks that is locally of finite presentation with quasi-compact and quasi-separated diagonal and affine stabilizers, then so is the Weil restriction $f_\ast X = R_{Z/S}(X) \to S$. If $X \to Z$ has affine (resp. quasi-affine, resp. separated) diagonal, then so has $f_\ast X$.

In the case where $Z$ has finite diagonal and $X$ has quasi-finite and separated diagonal, Theorems 1.2 and 1.3 were proved in [HR14c, Thms. 3 & 4]. They were subsequently generalized in [HR14b, Thm. 2.3 & Cor. 2.4], where the finite presentation assumptions on $X \to S$ were excised. Using a similar argument, we also excise the finite presentation assumptions from $X \to S$ (Corollary 9.2). Besides yielding much stronger results, we feel that the techniques developed in this paper to prove Theorems 1.2 and 1.3 are conceptually simpler and more robust than the approach of [HR14c].
Refinement of Tannaka duality. For stacks with quasi-affine diagonal, Theorem 1.1 can be strengthened: no excellency assumptions and we can include all natural transformations (not just the natural isomorphisms as in Theorem 1.1).

**Theorem 1.4.** Let $X$ be a quasi-compact algebraic stack with quasi-affine diagonal and let $T$ be an algebraic stack. Then the natural functor

$$\omega_X(T) : \text{Hom}(T, X) \to \text{Hom}_{\mathcal{O}}(\mathbb{Coh}(X), \mathbb{Coh}(T))$$

is fully faithful. If $T$ is locally noetherian, then $\omega_X(T)$ is an equivalence.

Theorems 1.1 and 1.4 are consequences of a more precise Tannaka duality result for non-noetherian stacks with affine stabilizers (see Theorem 8.4).

**Further applications.** Note that $\text{Hom}(\mathbb{Coh}(X), \mathbb{Coh}(\bullet))$ is a stack for the fpqc topology, whereas $X$ is only known to be a stack for the fppf topology—unless it has quasi-affine diagonal [LMB, Cor. 10.7]. This indicates that the restrictions on $T$ in Theorem 1.1 could be necessary when $\Delta_X$ is not quasi-affine. In particular, we do not know if noetherian algebraic stacks with affine stabilizers are stacks in the fpqc topology. Nonetheless, we are able to establish the following result.

**Corollary 1.5.** Let $X$ be a quasi-separated algebraic stack with affine stabilizers. Let $\pi : T' \to T$ be an fpqc covering such that $T$ is a locally excellent stack and $T'$ is locally noetherian. Then $X$ satisfies effective descent for $\pi$.

Another application concerns completions.

**Corollary 1.6.** Let $A$ be a noetherian ring and let $I \subseteq A$ be an ideal. Assume that $A$ is complete with respect to the $I$-adic topology. Let $X$ be a noetherian algebraic stack and consider the natural morphism

$$X(A) \to \lim_{\leftarrow} X(A/I^n)$$

of groupoids. This morphism is an equivalence if either

(i) $X$ has affine stabilizers and $A$ is excellent (or merely a G-ring); or
(ii) $X$ has quasi-affine diagonal.

The special case of Corollary 1.6(ii) where $X$ has quasi-finite and separated diagonal, was already settled by previous work of the authors: it is a special case of [HR14c, Thm. 4.3] via [Ryd15, Thm. D]. Another special case of Corollary 1.6(ii) is when $X$ is a quasi-separated algebraic space. In this case the equivalence $X(A) \to \lim_{\leftarrow} X(A/I^n)$ has recently been extended to non-noetherian complete rings $A$ by Bhatt [Bha14, Thm. 1.1]. That $X$ has affine stabilizers in Corollary 1.6 is necessary (see Theorem 10.5).

**On the proof.** The equivalence of $\omega_X(T)$ when $X$ has the resolution property has appeared in various forms in the work of others (cf. Schäppi [Sch12, Thm. 1.3.2], Savin [Sav06] and Brandenburg [Bra14, Cor. 5.7.12]). Since this case is an essential stepping stone in the proof of our main theorem (Theorem 8.4), we have included a full proof in Theorem 4.9. A key point, which has been observed by many, is that vector bundles on $X$ are dualizable objects in $\mathbb{Coh}(X)$ and that any tensor functor preserves dualizable objects.
In general, there are not enough vector bundles. Indeed, if $X$ has enough vector bundles, then $X$ has affine diagonal \cite[Prop. 1.3]{Tot04}. Our proof uses the following three ideas to overcome this problem:

(i) If $U \subseteq X$ is a quasi-compact open immersion and $\text{QCoh}(X) \to \text{QCoh}(T)$ is a tensor functor, then there is an induced tensor functor $\text{QCoh}(U) \to \text{QCoh}(V)$ where $V \subseteq T$ is the “inverse image of $U$”. The proof of this is based on ideas of Brandenburg and Chirvasitu \cite{BCL14}. (Section 5)

(ii) If $X$ is an infinitesimal neighborhood of a stack with the resolution property, then $\omega_X(T)$ is an equivalence for all $T$. (Section 6)

(iii) There is a constructible stratification of $X$ into stacks with affine diagonal and the resolution property (Proposition 8.2). We deduce the main theorem by induction on the number of strata using formal gluings \cite{MB96}. This step uses special cases of Corollaries 1.5 and 1.6. (Sections 7 and 8)

In Appendix B we extend the results of \cite{MB96} to non-separated algebraic spaces. We are thus able to prove the main theorems without assuming that the diagonal of $X$ is separated.

Open questions. Concerning (ii), it should be noted that we do not know the answer to the following two questions.

Question 1.7. If $X_0$ has the resolution property and $X_0 \xrightarrow{i} X$ is a nilpotent closed immersion, then does $X$ have the resolution property?

The question has an affirmative answer if $X_0$ is cohomologically affine, e.g., $X_0 = B_k G$ where $G$ is a linearly reductive group scheme over $k$. The question is open if $X_0 = B_k G$ where $G$ is not linearly reductive, even if $X = B_k[e] G_e$ where $G_e$ is a deformation of $G$ over the dual numbers \cite{Con10}.

Question 1.8. If $X_0 \xrightarrow{i} X$ is a nilpotent closed immersion and $\omega_{X_0}(T)$ is an equivalence, is then $\omega_X(T)$ an equivalence?

Step (ii) answers neither of these questions but uses a special case of the first question (Lemma 6.2) and the conclusion (Main Lemma 6.1) is a special case of the second question.

The following technical question also arose in this research.

Question 1.9. Let $X$ be an algebraic stack with quasi-compact and quasi-separated diagonal and affine stabilizers. Let $k$ be a field. Is every morphism $\text{Spec } k \to X$ affine?

If $X$ étale-locally has quasi-affine diagonal, then Question 1.9 has an affirmative answer (Lemma 6.5). This makes finding counterexamples extraordinarily difficult and thus very interesting. This question arose because if $\text{Spec } k \to X$ is non-affine, then $\omega_X(\text{Spec } k)$ is not fully faithful (Theorem 10.2). This explains our restriction to natural isomorphisms in Theorem 10.1. Note that every morphism $\text{Spec } k \to X$ as in Question 1.9 is at least quasi-affine \cite[Thm. B.2]{Ryd11b}. We do not know the answer to the question even if $X$ has separated diagonal and is of finite type over a field.
On the applications. Let $T$ be a noetherian and locally excellent algebraic stack and let $Z$ be a closed substack defined by a coherent ideal $J \subseteq \mathcal{O}_T$. Let $Z^{[n]}$ be the closed immersion defined by $J^{n+1}$. Assume that the natural functor $\text{Coh}(T) \to \lim_{\leftarrow n} \text{Coh}(Z^{[n]})$ is an equivalence of categories. Then an immediate consequence of Tannaka duality (Theorem 1.1) is that $\text{Hom}(T, X) \to \lim_{\leftarrow n} \text{Hom}(Z^{[n]}, X)$ is an equivalence of categories for every noetherian algebraic stack $X$ with affine stabilizers. This applies in particular if $A$ is excellent and $I$-adically complete and $T = \text{Spec}(A) = \text{Spec}(A/I)$ Spec(A/I) (Grothendieck’s existence theorem). Theorem 1.2 follows quite easily from this latter case. There are also non-proper stacks $T$ satisfying $\text{Coh}(T) \to \lim_{\leftarrow n} \text{Coh}(Z^{[n]})$, such as global quotient stacks with proper good moduli spaces (see [GZ14, AB05] for some special cases). Such statements, and their derived versions, were recently considered by Halpern-Leistner–Preygel [HP14]. There, they considered variants of our Theorem 1.2. For their algebraicity results, their assumption was similar to assuming that $\text{Coh}(T) \to \lim_{\leftarrow n} \text{Coh}(Z^{[n]})$ was an equivalence (though they also considered other derived versions), and that $X \to S$ was locally of finite presentation with affine diagonal.

Relation to other work. As mentioned in the beginning of the Introduction, Lurie identifies the image of $\omega_X(T)$ with the tame functors when $X$ is quasi-compact with affine diagonal [Lur04]. Tameness means that faithful flatness of objects is preserved. This is a very strong assumption that makes it possible to directly pull back a smooth presentation of $X$ to a smooth covering of $T$ and deduce the result by descent. Lurie’s methods also work for non-noetherian $T$.

Brandenburg and Chirvasitu have shown that $\omega_X(T)$ is an equivalence for every quasi-compact and quasi-separated scheme $X$ [BC14], also for non-noetherian $T$. The key idea of their proof is the tensor localization that we have adapted in Section 5. Using this technique, we give a slightly simplified proof of their theorem in Theorem 5.9.

A variant of a special case of Theorem 1.4 was considered by Fukuyama–Iwanari [FI13, Thm. 5.10]. Indeed, it was shown that if $X$ is a separated Deligne–Mumford stack of finite type over a field $k$ with quasi-projective coarse moduli space over $k$ such that $X = [U/G]$, where $U$ is quasi-projective over $k$ and the linear algebraic group $G$ acts $k$-linearly on $U$, then any cocontinuous symmetric monoidal functor $F: D(X) \to D(T)$, where $D(S)$ denotes the stable $\infty$-category of quasi-coherent sheaves on the stack $S$, arises from a $k$-morphism of algebraic stacks $f: T \to X$. In a recent preprint, Bhatt partially extended the results of Fukuyama–Iwanari to quasi-compact and quasi-separated algebraic spaces [Bha14, Thm. 1.5] and gave several interesting applications. In general, it is not obvious that a cocontinuous symmetric monoidal functor $F: D(X) \to D(T)$ induces a cocontinuous symmetric monoidal functor
$H^0(F) : \text{QCoh}(X) \to \text{QCoh}(T)$ (equivalently, $F$ preserves connective complexes). Indeed, for algebraic spaces, this is \cite[Lem. 2.6]{Bha}, which requires scallop decompositions and compact generation. These techniques are not available for general stacks with infinite stabilizers \cite{HNR}. 

In fact, there are symmetric monoidal functors $F : D(BG_m) \to D(k)$ that do not preserve connective complexes \cite{BZ}. Thus, the derived results of Bhatt and Fukuyama–Iwanari do not extend to geometric stacks with infinite stabilizers without further assumptions. Bhatt has recently communicated to us that he and Halpern-Leistner can show that if $X$ is noetherian with quasi-affine diagonal and $T$ is arbitrary, then every symmetric monoidal functor $F : D(X) \to D(T)$ which preserves pseudo-coherent and connective complexes is representable by a morphism of stacks $f : T \to X$. Note, however, that since $F$ is assumed to preserve connective complexes, it induces a symmetric monoidal functor $H^0(F) : \text{QCoh}(X) \to \text{QCoh}(T)$. In particular, if $T$ is locally noetherian, then Bhatt–Halpern-Leistner’s result follows from our Theorem 1.4.

However, it is not at all obvious that a cocontinuous symmetric monoidal functor $f^* : \text{QCoh}(X) \to \text{QCoh}(T)$ is derivable to a cocontinuous symmetric monoidal functor $Lf^* : D(X) \to D(T)$. Indeed, this essentially requires that there are sufficiently many acyclics for $f^*$ in $D(X)$. Since $\text{QCoh}(X)$ rarely has enough projectives, this acyclicity condition is non-trivial. In particular, if $T$ is locally noetherian, then our Theorems 1.1 and 1.4 are stronger than any derived result for stacks with infinite stabilizers.

Bhatt’s main application \cite[Thm. 1.1]{Bha} is a variant of our Corollary 1.6: $X$ is required to be an algebraic space but $A$ is permitted to be non-noetherian.

In contrast to \cite{FI, Bha}, our methods are completely underived. The key idea in \cite{Bha} is to replace vector bundles with perfect complexes which are the dualizable objects in the derived category. Whereas $\text{QCoh}(X)$ is often not (or not known) to be generated by vector bundles, the derived category $D(X)$ is often generated by perfect complexes. In particular, this is true when $X$ is an algebraic space, has quasi-finite diagonal, or is a $\mathbb{Q}$-stack of s-global type \cite{HRd}.

While we are currently unable to establish Theorem 1.4 for all non-noetherian $T$, in \cite{HRd} we make some progress towards this goal.

We wish to point out that we do not address the Tannaka recognition problem, i.e., which symmetric monoidal categories arise as the category of quasi-coherent sheaves on an algebraic stack. This has been done in characteristic 0 for gerbes \cite{De}, Schäppi \cite[Thm. 1.4]{Sch} also characterised those symmetric monoidal categories that are generated by dualizable, which was recently recovered using different methods by Tonini \cite{Ton}. Similar results from the derived perspective have been considered by Wallbridge \cite{Wal} and Iwanari \cite{Iwa}.

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2. Symmetric monoidal categories

A symmetric monoidal category is the data of a category $\mathcal{C}$, a tensor product $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and a unit $\mathcal{O}_\mathcal{C}$ that together satisfy various naturality, commutativity, and associativity properties [ML98, VII.7].

A symmetric monoidal category $\mathcal{C}$ is closed if for any $B \in \mathcal{C}$ the functor $- \otimes B: \mathcal{C} \to \mathcal{C}$ admits a right adjoint, which we denote as $\text{Hom}_\mathcal{C}(B, -)$.

A functor $F: \mathcal{C} \to \mathcal{D}$ between symmetric monoidal categories is lax symmetric monoidal if for each $C$ and $C'$ of $\mathcal{C}$ there are natural maps $F(C) \otimes_D F(C') \to F(C \otimes_C C')$ and $\mathcal{O}_D \to F(\mathcal{O}_\mathcal{C})$ that are compatible with the symmetric monoidal structure. If these maps are both isomorphisms, then $F$ is symmetric monoidal. Note that if $F: \mathcal{C} \to \mathcal{D}$ is a symmetric monoidal functor, then a right adjoint $G: \mathcal{D} \to \mathcal{C}$ to $F$ is always lax symmetric monoidal.

If $\mathcal{C}$ is a symmetric monoidal category, then a commutative $\mathcal{C}$-algebra consists of an object $A$ of $\mathcal{C}$ together with a multiplication $m: A \otimes_C A \to A$ and a unit $e_A: \mathcal{O}_\mathcal{C} \to A$ with the expected properties [ML98, VII.3]. Let $\text{CAlg}(\mathcal{C})$ denote the category of commutative $\mathcal{C}$-algebras. The category $\text{CAlg}(\mathcal{C})$ is naturally endowed with a symmetric monoidal structure that makes the forgetful functor $\text{CAlg}(\mathcal{C}) \to \mathcal{C}$ symmetric monoidal.

The following observation will be used frequently: if $F: \mathcal{C} \to \mathcal{D}$ is a lax symmetric monoidal functor and $A$ is a commutative $\mathcal{C}$-algebra, then $F(A)$ is a commutative $\mathcal{D}$-algebra.

3. Abelian tensor categories

An abelian tensor category is an abelian symmetric monoidal category. A Grothendieck abelian tensor category is an abelian tensor category such that the underlying abelian category is Grothendieck abelian and the tensor product is cocontinuous in each variable. By the Special Adjoint Functor Theorem [KS06, Prop. 8.3.27(iii)], if $\mathcal{C}$ is a Grothendieck abelian tensor category, then it is also closed.

A tensor functor $F: \mathcal{C} \to \mathcal{D}$ is an additive symmetric monoidal functor between abelian tensor categories. Let $\text{GTC}$ be the 2-category of Grothendieck abelian tensor categories and cocontinuous tensor functors. By the Special Adjoint Functor Theorem, if $F: \mathcal{C} \to \mathcal{D}$ is a cocontinuous tensor functor, then $F$ admits a (lax symmetric monoidal) right adjoint.

Example 3.1. Let $T$ be a ringed site. The category of $\mathcal{O}_T$-modules $\text{Mod}(T)$ is a Grothendieck abelian tensor category with unit $\mathcal{O}_T$ and the internal Hom is the functor $\text{Hom}_{\mathcal{O}_T}(M, -)$.

Example 3.2. Let $X$ be an algebraic stack. The category of quasi-coherent sheaves $\text{QCoh}(X)$ is a Grothendieck abelian tensor category with unit $\mathcal{O}_X$ [Stacks, Tag 0781]. The internal Hom is $\text{QC}(\text{Hom}_{\mathcal{O}_X}(M, -))$, where QC denotes the quasi-coherator (the right adjoint to the inclusion of the category of quasi-coherent sheaves in the category of lisse-étale $\mathcal{O}_X$-modules). If $X$ is...
an algebraic stack, then $\mathcal{C} \text{Alg}(\mathcal{Q}\text{Coh}(X))$ is the symmetric monoidal category of quasi-coherent $\mathcal{O}_X$-algebras.

If $f : X \to Y$ is a morphism of algebraic stacks, then the resulting functor $f^* : \mathcal{Q}\text{Coh}(Y) \to \mathcal{Q}\text{Coh}(X)$ is a cocontinuous tensor functor. If $f$ is flat, then $f^*$ is exact. We always denote the right adjoint of $f^*$ by $f_* : \mathcal{Q}\text{Coh}(X) \to \mathcal{Q}\text{Coh}(Y)$. If $f$ is quasi-compact and quasi-separated, then $f_*$ coincides with the pushforward of lisse-étale $\mathcal{O}_X$-modules [Ols07b, Lem. 6.5(i)]. In particular, if $f$ is quasi-compact and quasi-separated, then $f_* : \mathcal{Q}\text{Coh}(X) \to \mathcal{Q}\text{Coh}(Y)$ preserves directed colimits (work smooth-locally on $Y$ and then apply [Stacks] Tag 0738).

**Definition 3.3.** Given abelian tensor categories $\mathcal{C}$ and $\mathcal{D}$, we let $\text{Hom}_{\mathcal{C}\otimes}(\mathcal{C}, \mathcal{D})$ (resp. $\text{Hom}_{\mathcal{C}\otimes}(\mathcal{C}, \mathcal{D})$) denote the category of cocontinuous (resp. right exact) tensor functors and natural transformations (natural with respect to both homomorphisms and the symmetric monoidal structure). We let $\text{Hom}_{\mathcal{C}\otimes,\otimes}(\mathcal{C}, \mathcal{D})$ (resp. $\text{Hom}_{\mathcal{C}\otimes,\otimes}(\mathcal{C}, \mathcal{D})$) denote its core, that is, the groupoid of cocontinuous (resp. right exact) tensor functors and natural isomorphisms.

We conclude this section with some useful facts for the paper. We first consider modules over algebras, which are addressed, for example, in Brandenbourg’s thesis [Bra14, §5.3] in even greater generality.

Let $\mathcal{C}$ be a Grothendieck abelian tensor category and let $A$ be a commutative $\mathcal{C}$-algebra. Define $\text{Mod}_\mathcal{C}(A)$ to be the category of $A$-modules. That is, it has objects pairs $(M, a)$, where $M \in \mathcal{C}$ and $a : A \otimes_M M \to M$ is an action of $A$ on $M$. Morphisms $\phi : (M, a) \to (M', a')$ in $\text{Mod}_\mathcal{C}(A)$ are those $\mathcal{C}$-morphisms $\phi : M \to M'$ that preserve the respective actions. We identify $A$ with $(A, m) \in \text{Mod}_\mathcal{C}(A)$ where $m : A \otimes_M A \to A$ is the multiplication. It is straightforward to show that $\text{Mod}_\mathcal{C}(A)$ is a Grothendieck abelian tensor category, with tensor product $\otimes_A$ and unit $A$, and the natural forgetful functor $\text{Mod}_\mathcal{C}(A) \to \mathcal{C}$ preserves all limits and colimits [KS06, §4.3].

If $s : A \to B$ is a $\mathcal{C}$-algebra homomorphism, then there is a natural cocontinuous tensor functor:

$$s^* : \text{Mod}_\mathcal{C}(A) \to \text{Mod}_\mathcal{C}(B), \quad (M, a) \mapsto (B \otimes_A M, B \otimes_A a).$$

Also, let $f^* : \mathcal{C} \to \mathcal{D}$ be a cocontinuous tensor functor with right adjoint $f_* : \mathcal{D} \to \mathcal{C}$. If $A$ is a commutative $\mathcal{C}$-algebra, then there is a naturally induced cocontinuous tensor functor:

$$f_\mathcal{C}^* : \text{Mod}_\mathcal{C}(A) \to \text{Mod}_\mathcal{D}(f^* A), \quad (M, a) \mapsto (f^* M, f^* a).$$

Noting that $\eta : f^* f_* \mathcal{O}_\mathcal{D} \to \mathcal{O}_\mathcal{D}$ is a $\mathcal{D}$-algebra homomorphism, there is a naturally induced cocontinuous tensor functor:

$$f^* : \text{Mod}_\mathcal{C}(f_* \mathcal{O}_\mathcal{D}) \xrightarrow{f_\mathcal{C}^* \mathcal{O}_\mathcal{D}} \text{Mod}_\mathcal{D}(f^* f_* \mathcal{O}_\mathcal{D}) \xrightarrow{\eta^*} \text{Mod}_\mathcal{D}(\mathcal{O}_\mathcal{D}) = \mathcal{D}.$$

Moreover, if we let $\epsilon : \mathcal{O}_\mathcal{C} \to f_* f^* \mathcal{O}_\mathcal{C} = f_* \mathcal{O}_\mathcal{D}$ denote the unit, then $f^* = f^* \epsilon^*$. We have the following striking characterization of module categories [Bra14, Prop. 5.3.1].

**Proposition 3.4.** Let $\mathcal{C}$ be a Grothendieck abelian tensor category and let $A$ be a commutative algebra in $\mathcal{C}$. Then for every Grothendieck abelian tensor
category $\text{D}$, there is an equivalence of categories

\[
\text{Hom}_{\text{Coh}}(\text{Mod}_C(A), \text{D}) \simeq \{(F, h) : F \in \text{Hom}_{\text{Coh}}(C, \text{D}), h \in \text{Hom}_{\text{Alg}}(F(A), \mathcal{O}_D)\},
\]

where a morphism $(F, h) \rightarrow (F', h')$ is a natural transformation $\alpha : F \rightarrow F'$ such that $h = h' \circ \alpha(A)$.

The following corollary is immediate (see [Bra14, Cor. 5.3.7]).

**Corollary 3.5.** Let $p : Y' \rightarrow Y$ be an affine morphism of algebraic stacks. Let $X$ be an algebraic stack and let $g^* : \text{QCoh}(Y) \rightarrow \text{QCoh}(X)$ be a cocontinuous tensor functor. If $X'$ is the affine $X$-scheme $\text{Spec}_X(g^*p_*\mathcal{O}_{Y'})$ with structure morphism $p' : X' \rightarrow X$, then there is a 2-cocartesian diagram in GTC:

\[
\begin{array}{ccc}
\text{QCoh}(X') & \xrightarrow{g^*} & \text{QCoh}(Y') \\
p'^* & & \downarrow p^* \\
\text{QCoh}(X) & \xrightarrow{g^*} & \text{QCoh}(Y).
\end{array}
\]

Moreover, the natural transformation $g^*p_* \Rightarrow p'_*g'^*$ is an isomorphism.

Note that if $g^*$ is represented by a morphism $g : X \rightarrow Y$, then $X' \cong X \times_Y Y'$.

Let $X$ be a noetherian algebraic stack and let $I$ be a coherent ideal of $\mathcal{O}_X$. For each integer $n \geq 0$, let $Z_n = V(I^{n+1})$. Let $\text{Coh}(X, I)$ denote the category $\text{Lim}_n \text{Coh}(Z_n)$. The arguments of [Stacks Tag 087X] easily extend to establish the following:

(i) $\text{Coh}(X, I)$ is an abelian tensor category;

(ii) if $p : U \rightarrow X$ is smooth and quasi-compact, then the restriction $\text{Coh}(X, I) \rightarrow \text{Coh}(U, p^*I)$ is an exact tensor functor; and

(iii) exactness in $\text{Coh}(X, I)$ may be checked on a smooth covering of $X$.

If $\{f_n\}_{n \geq 0}, \{M_n\}_{n \geq 0} \rightarrow \{N_n\}_{n \geq 0}$ is a morphism in $\text{Coh}(X, I)$, then it is easily determined that $\ker(\{f_n\}_{n \geq 0}) \cong \{\ker(f_n)\}_{n \geq 0}$. Computing $\text{coker}(\{f_n\}_{n \geq 0})$ is more involved. We will need the following lemma.

**Lemma 3.6.** $\text{Coh}(X, I)$ is the limit of the inverse system of categories $\{\text{Coh}(Z_n)\}_{n \geq 0}$ in the core of the 2-category of abelian tensor categories and right exact tensor functors.

**Proof.** It remains to verify that if $F_n : C \rightarrow \text{Coh}(Z_n)$ is a compatible sequence of right exact abelian tensor functors, then the induced abelian tensor functor $\text{Lim}_n F_n : C \rightarrow \text{Coh}(X, I)$ is right exact. The explicit description of cokernels in $\text{Coh}(X, I)$ shows that this is the case. \qed

4. Tensorial Algebraic Stacks

Let $T$ and $X$ be algebraic stacks. The induced functor

\[
\omega_X(T) : \text{Hom}(T, X) \rightarrow \text{Hom}_{\text{Coh}}(\text{QCoh}(X), \text{QCoh}(T))
\]

takes a morphism $f$ to $f^*$, factors canonically through a functor

\[
\omega^0_X(T) : \text{Hom}(T, X) \rightarrow \text{Hom}_{\text{Coh}}(\text{QCoh}(X), \text{QCoh}(T)).
\]
As noted in the introduction, since $\text{QCoh}(-)$ is a stack in the fpqc topology, the right hand side of both functors are stacks in the fpqc topology when varying $T$—for an elaborate proof of this, see [LT12, Thm. 1.1]. Also, the left hand side is a stack for the fppf topology—even the fpqc topology when $X$ has quasi-affine diagonal [LMB, Cor. 10.7].

**Definition 4.1.** Let $T$ and $X$ be algebraic stacks. Then a tensor functor $f^*: \text{QCoh}(X) \to \text{QCoh}(T)$ is representable if it arises from a morphism of algebraic stacks $f: T \to X$. If $f, g: T \to X$ are morphisms, then a natural transformation $\tau: f^* \Rightarrow g^*$ of tensor functors is realizable if it is induced by a 2-morphism $f \Rightarrow g$. We say that $X$ is tensorial if $\omega_X(T)$ is an equivalence for every algebraic stack $T$, or equivalently, for every affine scheme $T$ [Bra14, Def. 3.4.4].

If $X$ is a quasi-compact algebraic stack with affine diagonal, then, as remarked in the introduction, $\omega_X(T)$ has essential image the cocontinuous tensor functors that are tame [Lur04]. We wish to point out that the notion of a tame tensor functor is quite subtle, particularly for stacks with non-affine diagonal [Lur04, Rmk. 5.12].

We begin by recalling two basic lemmas on tensorial stacks. The first is the combination of [Bra14, Cor. 5.3.4 & 5.6.4].

**Lemma 4.2.** Let $q: X' \to X$ be a quasi-affine morphism of algebraic stacks. If $T$ is an algebraic stack and $\omega_X(T)$ is faithful, fully faithful or an equivalence; then so is $\omega_{X'}(T)$. In particular, if $X$ is tensorial, then so is $X'$.

*Proof.* Since $q$ is the composition of a quasi-compact open immersion followed by an affine morphism, it suffices to treat these two cases separately. When $q$ is affine the result is an easy consequence of Proposition 3.3. If $q$ is a quasi-compact open immersion, then the counit $q^*q_* \to \text{id}_{\text{QCoh}(X')}$ is an isomorphism; the result now follows from [BC14, Prop. 2.3.6]. □

The second lemma is well-known (e.g., it is a very special case of [BCT14, Thm. 3.4.2]).

**Lemma 4.3.** Every quasi-affine scheme is tensorial.

*Proof.* By Lemma 4.2, it is sufficient to prove that $X = \text{Spec} \mathbb{Z}$ is tensorial, which is well-known. We refer the interested reader to [BCT14, Cor. 2.2.4] or [Bra14, Cor. 5.2.3]. □

Lacking an answer to Question 1.9 in general, we are forced to make the following definition to treat natural transformations that are not isomorphisms.

**Definition 4.4.** An algebraic stack $X$ is affine-pointed if every morphism $\text{Spec} k \to X$, where $k$ is a field, is affine.

Note that if $X$ is affine-pointed, then it has affine stabilizers. The following lemma shows that many algebraic stacks with affine stabilizers that are encountered in practice are affine-pointed.

**Lemma 4.5.** Let $X$ be an algebraic stack.

(i) If $X$ has quasi-affine diagonal, then $X$ is affine-pointed.
Proof. Throughout, we fix a field $k$ and a morphism $x: \text{Spec} k \to X$.

For (i) since $k$ is a field, every extension in $\mathcal{QCoh}(\text{Spec} k)$ is split; thus $x_*$ is cohomologically affine [Alp13 Def. 3.1]. Since $X$ has quasi-affine diagonal, this property is preserved after pulling back $x$ along a smooth morphism $p: U \to X$, where $U$ is an affine scheme [Alp13 Prop. 3.9(vii)]. By Serre’s Criterion [EGA] II.5.2.1, IV.1.7.17–18], the morphism $\text{Spec} k \times_X U \to U$ is affine; and this case follows.

For (ii) the pullback of $g$ along $x$ gives a quasi-finite and faithfully flat morphism $g_0: V_0 \to \text{Spec} k$. Since $V_0$ is discrete with finite stabilizers, there exists a finite surjective morphism $W_0 \to V_0$ where $W_0$ is a finite disjoint union of spectra of fields. By assumption $W_0 \to V_0 \to V$ is affine; hence so is $V_0 \to V$ (by Chevalley’s Theorem [Ryd15 Thm. 8.1] applied smooth-locally on $V$). By descent, Spec $k \to X$ is affine and the result follows. □

The following lemma highlights the benefits of affine-pointed stacks.

**Lemma 4.6.** Let $f_1$, $f_2: T \to X$ be morphisms of algebraic stacks and let $\gamma: f_1^* \Rightarrow f_2^*$ be a natural transformation of cocontinuous tensor functors. If $X$ is affine-pointed, then the induced maps of topological spaces $[f_1]$, $[f_2]: |T| \to |X|$ coincide.

**Proof.** It suffices to prove that if $T = \text{Spec} k$, where $k$ is a field, then $\gamma$ is realizable. Since $X$ is affine-pointed, the morphisms $f_1$ and $f_2$ are affine. Also, the natural transformation $\gamma$ induces, by adjunction, a morphism of quasi-coherent $O_X$-algebras $\gamma^*(O_T): (f_2)_*O_T \to (f_1)_*O_T$. In particular, $\gamma^*(O_T)$ induces a morphism $\nu: T \to T$ over $X$. We are now free to replace $X$ by $T$, $f_2$ by $\text{id}_T$, and $f_1$ by $\nu$. Since $T$ is affine, the result now follows from Lemma 4.5. □

We can now prove the following proposition (generalizing Lurie’s corresponding result for an algebraic stack with affine diagonal).

**Proposition 4.7.** Let $X$ be an algebraic stack.

(i) Let $T$ be a quasi-affine scheme and let $f_1$, $f_2: T \to X$ be quasi-affine morphisms.

(a) If $\alpha$, $\beta: f_1 \Rightarrow f_2$ are 2-morphisms and $\alpha^* = \beta^*$ as natural transformations $f_1^* \Rightarrow f_2^*$, then $\alpha = \beta$.

(b) Let $\gamma: f_1^* \Rightarrow f_2^*$ be a natural transformation of cocontinuous tensor functors. If $\gamma$ is an isomorphism or $X$ is affine-pointed, then $\gamma$ is realizable.

(ii) If $T$ is an algebraic stack and $X$ has quasi-affine diagonal, then the functor $\omega_X(T)$ is fully faithful.

**Proof.** For (ii) we may assume that $T$ is an affine scheme. Then every morphism $T \to X$ is quasi-affine and the result follows by (i) and Lemma 4.6(i).

For (i) there are quasi-compact open immersions $i_k: T \hookrightarrow V_k$ over $X$, where $V_k := \text{Spec} X((f_k)_*O_T)$ and $k = 1, 2$. Let $\nu_k: V_k \to X$ be the induced 1-morphism.
We first treat $(i)(a)$ The hypotheses imply that $\alpha_s = \beta_s$ as natural isomorphisms of functors from $(f_2)_s$ to $(f_1)_s$. In particular, $\alpha_s$ and $\beta_s$ induce the same 1-morphism from $V_1$ to $V_2$ over $X$. Since $i_1$ and $i_2$ are open immersions, they are monomorphisms; hence $\alpha = \beta$.

We now treat $(i)(b)$ The natural transformation $\gamma: f_1^* \Rightarrow f_2^*$ uniquely induces a natural transformation of lax symmetric monoidal functors $\gamma^\vee: (f_2)_s \Rightarrow (f_1)_s$. In particular, there is an induced morphism of quasi-coherent $O_X$-algebras $\gamma^\vee(O_T): (f_2)_s O_T \rightarrow (f_1)_s O_T$; hence a morphism of algebraic stacks $g: V_1 \rightarrow V_2$ over $X$. Note that $\gamma^\vee$ uniquely induces a natural transformation of lax symmetric monoidal functors $(i_2)_s \Rightarrow g_*(i_1)_s$, and by adjunction we have a uniquely induced natural transformation of tensor functors $\gamma': (g \circ i_1)^* \Rightarrow i_2^*$.

Replacing $X$ by $V_2$, $f_1$ by $g \circ i_1$, $f_2$ by $i_2$, and $\gamma$ by $\gamma'$, we may assume that $f_2$ is a quasi-compact open immersion such that $O_X \rightarrow (f_2)_s O_T$ is an isomorphism.

If $\gamma$ is an isomorphism, then $f_1$ is also a quasi-compact open immersion. Let $Z_1$ and $Z_2$ denote closed substacks of $X$ whose complements are $f_1(T)$ and $f_2(T)$, respectively. Then $f_1^* O_{Z_2} \cong f_2^* O_{Z_2} \cong 0$, so $f_1(T) \subseteq f_2(T)$. Arguing similarly, we obtain the reverse inclusion and we see that $f_1(T) = f_2(T)$. Since $f_1$ and $f_2$ are open immersions, we obtain the result when $\gamma$ is assumed to be an isomorphism.

Otherwise, Lemma 4.6 implies that $f_1$ factors through $f_2(T) \subseteq X$. We may now replace $X$ by $T$ and $\gamma$ with $(f_2)_s(-)$. [BC14, Prop. 2.3.6]. Then $X$ is quasi-affine and the result follows from Lemma 4.9

From Proposition $\square$ of Lemma 4.6, we obtain an analogue of Lemma 4.6 for natural isomorphisms of functors.

Corollary 4.8. Let $f_1, f_2: T \rightarrow X$ be morphisms of algebraic stacks and let $\gamma: f_1^* \simeq f_2^*$ be a natural isomorphism of cocontinuous tensor functors. If $X$ has affine stabilizers and quasi-compact diagonal, then the induced maps of topological spaces $[f_1], [f_2]: |T| \rightarrow |X|$ coincide.

Proof. It suffices to prove the result when $T = \text{Spec } k$, where $k$ is a field. Since $X$ has affine stabilizers and quasi-compact diagonal the morphisms $f_1$ and $f_2$ are quasi-affine [Ryd11b, Thm. B.2]. The result now follows from Proposition $(i)(b)$

The following result, in a slightly different context, was proved by Schäppi [Scha12, Thm. 1.3.2]. Using the Gross–Totaro theorem, we can simplify Schäppi’s arguments in the algebro-geometric setting.

Theorem 4.9. Let $X$ be a quasi-compact and quasi-separated algebraic stack with affine stabilizers. If $X$ has the resolution property, then it is tensorial.

Proof. Let $T$ be an algebraic stack. By Gross–Totaro [Gro13], there is a quasi-affine morphism $g: X \rightarrow \text{BGL}_{N, \mathbb{Z}}$. By Lemma 4.12 it is enough to consider $X = \text{BGL}_{N, \mathbb{Z}}$. Note that $X$ is quasi-compact with affine diagonal, so the functor $\omega_X(T)$ is fully faithful (Proposition 4.7). It remains to prove that every cocontinuous tensor functor $f^*: \text{QCoh}(X) \rightarrow \text{QCoh}(T)$ is representable.
Let $p: \text{Spec}(\mathbb{Z}) \to BGL_{N,\mathbb{Z}}$ be the universal $GL_N$-bundle and let $\mathcal{A} = p_* \mathbb{Z}$ be the regular representation. There is an exact sequence

$$0 \to \mathcal{O}_{BGL_{N,\mathbb{Z}}} \to \mathcal{A} \to \mathcal{Q} \to 0$$

of flat quasi-coherent sheaves. Write $\mathcal{A}$ as the directed colimit of its finitely generated subsheaves $\mathcal{A}_\lambda$ containing the unit and let $\mathcal{Q}_\lambda = \mathcal{A}_\lambda / \mathcal{O}_{BGL_{N,\mathbb{Z}}} \subseteq \mathcal{Q}$. Then $\mathcal{A}_\lambda$ and $\mathcal{Q}_\lambda$ are vector bundles.

It is well-known that (1) any tensor functor $f^*: \text{QCoh}(X) \to \text{QCoh}(T)$ preserves dualizable objects and exact sequences of dualizable objects (for example, see [Bra14, Def. 4.7.1 & Lem. 4.7.10]) and (2) the dualizable objects in $\text{QCoh}(Y)$ are the vector bundles for any algebraic stack $Y$ [Bra14, Prop. 4.7.5]. Since $f^*$ is cocontinuous, we thus have exact sequences

$$0 \to \mathcal{O}_T \to f^* \mathcal{A}_\lambda \to f^* \mathcal{Q}_\lambda \to 0$$

$$0 \to \mathcal{O}_T \to f^* \mathcal{A} \to f^* \mathcal{Q} \to 0$$

of vector bundles and flat quasi-coherent sheaves. In particular, $f^* \mathcal{A}$ is a faithfully flat algebra.

Let $V = \text{Spec}_T(f^* \mathcal{A})$; then $r^*: V \to T$ is faithfully flat. By Corollary 3.5, we have a cocartesian diagram

$$\begin{array}{ccc}
\text{QCoh}(V) & \xrightarrow{f'^*} & \text{QCoh}(\text{Spec } \mathbb{Z}) \\
\downarrow{r^*} & & \downarrow{p^*} \\
\text{QCoh}(T) & \xleftarrow{f^*} & \text{QCoh}(X).
\end{array}$$

Since Spec $\mathbb{Z}$ is tensorial (Lemma 4.3), the functor $f'^*$ is representable. Thus $f'^* p^*$ is representable and by fpqc descent, $f^*$ is representable. □

5. Tensor localizations

Let $\mathcal{C}$ be a Grothendieck abelian category. A Serre subcategory is a full non-empty subcategory $\mathcal{K} \subseteq \mathcal{C}$ closed under taking subquotients and extensions. Serre subcategories are abelian and the inclusion functor is exact. By [Gab62, Ch. III], there is a quotient $\mathcal{Q}$ of $\mathcal{C}$ by $\mathcal{K}$ and an exact cocontinuous tensor functor $q^*: \mathcal{C} \to \mathcal{Q}$ that vanishes on $\mathcal{K}$. The quotient $q^*: \mathcal{C} \to \mathcal{Q}$ is a localization if $q^*$ admits a right adjoint $q_*: \mathcal{Q} \to \mathcal{C}$. It follows that $q_*$ is fully faithful and that $q^* q_* \simeq \text{id}_{\mathcal{Q}}$.

Let $\mathcal{C}$ be a Grothendieck abelian tensor category. A subcategory $\mathcal{K} \subseteq \mathcal{C}$ is a tensor ideal if $\mathcal{C} \to \mathcal{C}/\mathcal{K}$ is a localization and $\mathcal{K}$ is closed under tensor products with objects in $\mathcal{C}$. If $q^*: \mathcal{C} \to \mathcal{Q}$ is an exact cocontinuous tensor functor, then $\ker(q^*)$ is a tensor ideal. Conversely, if $\mathcal{K} \subseteq \mathcal{C}$ is a tensor ideal, then the quotient $\mathcal{Q} = \mathcal{C}/\mathcal{K}$ is a Grothendieck abelian tensor category, the functor $q^*: \mathcal{C} \to \mathcal{Q}$ is an exact cocontinuous tensor functor and $\ker(q^*) = \mathcal{K}$. We say that $q^*$ is a tensor localization.

Example 5.1. Let $f: X \to Y$ be a morphism of algebraic stacks. If $f$ is flat, then $f^*$ is exact. If $f$ is a quasi-compact flat monomorphism (e.g., a quasi-compact open immersion), then $\text{QCoh}(X)$ is the quotient of $\text{QCoh}(Y)$ by $\ker(f^*)$. This follows from the fact that the counit $f^* f_* \to \text{id}$ is an isomorphism so that $f_*$ is a section of $f^*$ [Gab62, Prop. III.2.5].
Definition 5.2. Let $C$ be a Grothendieck abelian tensor category. For $M \in C$ let $\varphi_M : O_C \to \mathcal{H}om_C(M, M)$ denote the adjoint to the canonical isomorphism $O_C \otimes_C M \to M$. Let the annihilator $\text{Ann}_C(M)$ of $M$ be the kernel of $\varphi_M$, which we consider as an ideal of $O_C$.

Example 5.3. Let $X$ be an algebraic stack and let $F \in \text{QCoh}(X)$. Then $\text{Ann}_{\text{QCoh}(X)}(F) = QC(\text{Ann}_{\text{Mod}(X)}(F))$. In particular, if $F$ is of finite type, then $\text{Ann}_{\text{QCoh}(X)}(F) = \text{Ann}_{\text{Mod}(X)}(F)$.

Recall that an object $c \in C$ is finitely generated if the natural map:
$$\lim_{\lambda} \text{Hom}_C(c, d_\lambda) \to \text{Hom}_C(c, \lim_{\lambda} d_\lambda)$$
is bijective for every direct system $\{d_\lambda\}_\lambda$ in $C$ with monomorphic bonding maps. A category $C$ is locally finitely generated if it is cocomplete and has a set $A$ of finitely generated objects such that every object $c$ of $C$ is a directed colimit of objects from $A$.

Example 5.4. Let $X$ be a quasi-compact and quasi-separated algebraic stack. The finitely generated objects in $\text{QCoh}(X)$ are the quasi-coherent sheaves of finite type. Thus $\text{QCoh}(X)$ is locally finitely generated [Ryd14].

We also require the following definition.

Definition 5.5. Let $q^* : C \to Q$ be a tensor localization. Then it is supported if $q^*(O_C/\text{Ann}(K)) \cong 0$ for every finitely generated object $K$ of $C$ such that $q^*(K) \cong 0$.

The notion of a supported tensor localization is very natural.

Example 5.6. If $f : X \to Y$ is a quasi-compact and flat monomorphism of algebraic stacks, then the tensor localization $f^* : \text{QCoh}(Y) \to \text{QCoh}(X)$ of Example 5.1 is supported. Indeed, if $M$ is a quasi-coherent $O_Y$-module of finite type in the kernel of $f^*$, then $f^* \text{Ann}_{\text{QCoh}(Y)}(M) = \text{Ann}_{\text{QCoh}(X)}(f^*M) = O_X$.

We have the following crucial theorem generalizing [BC14, Lem. 3.3.6].

Theorem 5.7. Let $C$ be a locally finitely generated Grothendieck abelian tensor category and let $q^* : C \to Q$ be a supported tensor localization. If $f^* : C \to D$ is a cocontinuous tensor functor such that $f^*(K) \cong 0$ for every finitely generated object $K$ of $C$ such that $q^*(K) \cong 0$, then $f^*$ factors essentially uniquely through $Q$.

Note that Theorem 5.7 is trivial if $f^*$ is exact. The challenge is to use the symmetric monoidal structure to deduce this also when $f^*$ is merely right-exact. The proof we give is a straightforward generalization of [BC14, Lem. 3.3.6]. To prove Theorem 5.7 we require the following lemma.

Lemma 5.8 ([BC14, Lem. 3.3.2]). Let $f^* : C \to D$ be a cocontinuous tensor functor. If $I \subseteq O_C$ is an $O_C$-ideal such that $f^*(O_C/I) \cong 0$, then $f^*(I) \to f^*(O_C)$ is an isomorphism.

Proof. Since $f^*$ is right-exact and $f^*(O_C/I) = 0$, it follows that $f^*(I) \to f^*(O_C) = O_D$ is surjective. Let $J = f^*(I)$ and $K = \ker(J \to O_D)$. 


The surjection \( J \to \mathcal{O}_D \) gives an injective homomorphism
\[
J \cong \text{Hom}_D(\mathcal{O}_D, J) \hookrightarrow \text{Hom}_D(J, J).
\]
This homomorphism is the adjoint of the composition \( g: J \otimes J \to J \otimes \mathcal{O}_D \cong J \). Since the multiplication homomorphism \( m: I \otimes I \to I \) factors as \( I \otimes I \to I \otimes \mathcal{O}_C \to I \), it follows that \( g = f^*(m) \). The multiplication map also factors as \( I \otimes I \to \mathcal{O}_C \otimes I \to I \) which shows that the composition of \( K \otimes J \to J \otimes J \) and \( g \) is zero. It follows that its adjoint
\[
K \leftarrow J \cong \text{Hom}_D(\mathcal{O}_D, J) \hookrightarrow \text{Hom}_D(J, J)
\]
is zero so \( K = 0 \).

**Proof of Theorem 5.7.** If \( K \in \mathcal{C} \), since \( \mathcal{C} \) is locally finitely generated, it may be written as a directed colimit \( K = \varinjlim K_\lambda \), where \( K_\lambda \subseteq K \) and \( K_\lambda \) is finitely generated. If \( K \in \ker(q^*) \), then \( q^*K_\lambda \subseteq q^*K = 0 \). In particular, \( K := \ker(q^*) \subseteq \ker(f^*) \). Let \( 0 \to K \to M \to N \to Q \to 0 \) be an exact sequence in \( \mathcal{C} \) with \( K, Q \in \mathcal{K} \). We have to prove that \( f^*(M \to N) \) is an isomorphism in \( \mathcal{D} \). Let \( N_0 \) be the image of \( M \) in \( N \). By right-exactness, we have an exact sequence \( f^*(K) \to f^*(M) \to f^*(N_0) \to 0 \). Since \( f^*(K) = 0 \), we have that \( f^*(M) = f^*(N_0) \). We may thus replace \( M \) with \( N_0 \) and assume that \( K = 0 \) and \( M \to N \) is injective.

Write \( N \) as the directed colimit of finitely generated subobjects \( N_\lambda \subseteq N \). Let \( N_\lambda = M + N_\lambda \subseteq N \) and \( I_\lambda = \text{Ann}(N_\lambda/M) \). By definition, we have that \( I_\lambda \otimes N_\lambda/M \to N_\lambda/M \) is zero; hence \( I_\lambda \otimes N_\lambda \to N_\lambda \) factors through \( M \).

Note that \( N_\lambda/M = N_\lambda/(N_\lambda \cap M) \) is a quotient of a finitely generated object and a subobject of \( Q \), so \( \mathcal{O}_C/I_\lambda \in \mathcal{K} \) since \( q^* \) is supported. We conclude that \( f^*(I_\lambda) \to f^*(\mathcal{O}_C) \) is an isomorphism using Lemma 5.8. Now consider the commutative diagrams:
\[
\begin{array}{ccc}
I_\lambda \otimes M & \longrightarrow & M \\
\downarrow & & \downarrow \\
I_\lambda \otimes N_\lambda & \longrightarrow & N_\lambda
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
f^*(M) & \overset{\simeq}{\longrightarrow} & f^*(M) \\
\downarrow & & \downarrow \\
f^*(N_\lambda) & \overset{\simeq}{\longrightarrow} & f^*(N_\lambda),
\end{array}
\]
where the right diagram is obtained by applying \( f^* \) to the left diagram. It follows that \( f^*(M) \to f^*(N_\lambda) \) is an isomorphism. Since \( f^* \) is cocontinuous, it follows that \( f^*(M) \to f^*(N) = \varprojlim f^*(N_\lambda) \) is an isomorphism.

To show how powerful tensor localization is, we can quickly prove that tensoriality is local for the Zariski topology—even for stacks.

**Theorem 5.9.** Let \( X \) be a quasi-compact and quasi-separated algebraic stack. Let \( X = \bigcup_{k=1}^n X_k \) be an open covering by quasi-compact open substacks. If every \( X_k \) is tensorial, then so is \( X \).

**Proof.** Let \( \iota_k: X_k \to X \) denote the open immersion and let \( I_k \) be an ideal of finite type defining a closed substack complementary to \( X_k \) [Ryd14, Prop. 7.2].

Let \( T \) be a scheme. First we will show that \( \omega_X(T) \) is fully faithful. Thus, let \( f, g: T \to X \) be two morphisms and suppose that we are given a natural transformation of cocontinuous tensor functors \( \gamma: f^* \Rightarrow g^* \). Then \( f^*(\mathcal{O}_X/I_k) \to g^*(\mathcal{O}_X/I_k) \) so there is an inclusion \( f^{-1}(X_k) \subseteq g^{-1}(X_k) \) for
every $k$. Let $T_k = f^{-1}(X_k)$, let $j_k: T_k \to T$ denote the corresponding open immersion and let $f_k, g_k: T_k \to X_k$ denote the restrictions of $f$ and $g$. Since $(f_k)^* = j_k^* f^*(i_k)_*$ and $(g_k)^* = j_k^* g^*(i_k)_*$, we obtain a natural transformation $\gamma_k: f_k^* \Rightarrow g_k^*$, hence a unique 2-isomorphism $f_k \Rightarrow g_k$.

Since $T = \bigcup_{k=1}^{N} T_k$, it follows by fppf-descent that $\omega_X(T)$ is fully faithful.

For essential surjectivity, let $f^*: \text{QCoh}(X) \to \text{QCoh}(T)$ be a cocontinuous tensor functor. The surjection $O_T \twoheadrightarrow f^*(O_X/I_k)$ defines a closed subscheme and we let $j_k: T_k \to T$ denote its open complement. If $K \in \text{QCoh}(X)$ is a finitely generated object such that $i_k^* K \cong 0$, then $I_k^m K = 0$ for sufficiently large $m$. Thus $j_k^* f^* K = 0$. Since $i_k^*: \text{QCoh}(X) \to \text{QCoh}(X_k)$ is a supported localization (Example 5.6), we may apply Theorem 5.7 and deduce that $j_k^* f^*$ factors via $i_k^*$ and a tensor functor $f_k^*: \text{QCoh}(X_k) \to \text{QCoh}(T_k)$. The latter is representable by assumption, hence so is $j_k^* f^* = f_k^* i_k^*$.

Finally, since $O_X/I_1 \otimes \cdots \otimes O_X/I_n = 0$, it follows that $f^*(O_X/I_1) \otimes \cdots \otimes f^*(O_X/I_n) = 0$ so $T = \bigcup_{k=1}^{n} T_k$ is an open covering. We conclude that $f$ is representable by fppf descent.

Combining Theorem 5.9 with Lemma 4.3 we obtain a short proof of the main result of [BCT14].

**Corollary 5.10** (Brandenburg–Chirvasitu). Every quasi-compact and quasi-separated scheme is tensorial.

6. The Main Lemma

The main result of this section is the following technical lemma, which proves that the tensorial property extends over nilpotent thickenings of quasi-compact algebraic stacks with affine stabilizers having the resolution property.

**Lemma 6.1** (Main Lemma). Let $i: X_0 \to X$ be a closed immersion of algebraic stacks defined by a quasi-coherent ideal $I$ such that $I^n = 0$ for some integer $n > 0$. Suppose that $X_0$ is quasi-compact and quasi-separated with affine stabilizers. If $X_0$ has the resolution property, then $X$ is tensorial.

We have another lemma that will be crucial for proving Lemma 6.1.

**Lemma 6.2.** Consider a 2-cocartesian diagram of algebraic stacks:

$$
\begin{array}{ccc}
U_0 & \xrightarrow{i} & U \\
\downarrow{p_0} & & \downarrow{p} \\
X_0 & \xrightarrow{j} & X
\end{array}
$$

such that the following conditions are satisfied.

(i) $i$ is a nilpotent closed immersion;
(ii) $U_0$ is an affine scheme; and
(iii) $X_0$ is quasi-compact and quasi-separated with affine stabilizers.

If $X_0$ has the resolution property, then so has $X$.

**Proof.** Note that $X_0$ has affine diagonal by the Gross–Totaro theorem; hence $p_0$ is affine. By [Hal14b Prop. A.2], the square is a geometric pushout. In particular, $j$ is a nilpotent closed immersion, $p$ is affine, and the natural
map $\mathcal{O}_X \to p_*\mathcal{O}_U \times_{p_*\mathcal{O}_U} j_*\mathcal{O}_X$ is an isomorphism. By the Gross–Totaro Theorem \cite[Cor. 5.9]{Gro13}, there exists a vector bundle $V_0$ on $X_0$ such that the total space of the frame bundle of $V_0$ is quasi-affine. Let $E_0 = p_0^*V_0$; then, since $U_0$ is affine, there exists a vector bundle $E$ on $U$ equipped with an isomorphism $\alpha : i^*E \to E_0$. Let $V$ be the quasi-coherent $\mathcal{O}_X$-module $p_*E \times_{p_*j} V_0$. By \cite[Thm. 2.2(iv)]{Fer03}, $V$ is a vector bundle on $X$ and there is an isomorphism $j^*V \cong V_0$. By \cite[Prop. 5.7]{Gro13}, it follows that $X$ has the resolution property.

Proof of Lemma \ref{lem:resolution}. We observe at the outset that the Gross–Totaro Theorem \cite[Cor. 5.9]{Gro13} implies that $X_0$ has affine diagonal; thus, $X$ has affine diagonal. Thus, by Proposition \ref{prop:resolution} it suffices to prove that if $T$ is an affine scheme and $f^*: \text{QCoh}(X) \to \text{QCoh}(T)$ is a cocontinuous tensor functor, then there exists an étale and surjective morphism $c: T' \to T$ such that $c^*f^*$ is representable.

By Corollary \ref{cor:resolution} there is a 2-cartesian diagram in $\text{GTC}$

$$
\begin{array}{ccc}
\text{QCoh}(T_0) & \xrightarrow{f_0^*} & \text{QCoh}(X_0) \\
\downarrow{j^*} & & \downarrow{i^*} \\
\text{QCoh}(T) & \xleftarrow{f^*} & \text{QCoh}(X),
\end{array}
$$

where $j: T_0 \to T$ is the closed immersion defined by the image $J$ of $f^*I$ in $\mathcal{O}_T$. In particular, $j$ is a nilpotent closed immersion. Since $X_0$ has the resolution property, $f_0^*$ is representable by a morphism of algebraic stacks $f_0: T_0 \to X_0$ (Theorem \ref{thm:resolution}).

Let $p: U \to X$ be a smooth and surjective morphism, where $U$ is an affine scheme; then, $p$ is affine. The pullback of $p$ along the morphism $i \circ f_0: T_0 \to X$ results in a smooth and affine surjective morphism of schemes $q_0: V_0 \to T_0$. By \cite[IV.17.16.3(ii)]{EGA}, there exists an affine étale and surjective morphism $q_0: T'_0 \to T_0$ such that the pullback $q_0^*: V_0' \to T'_0$ of $q_0$ to $T'_0$ admits a section. By \cite[IV.18.1.2]{EGA}, there exists a unique affine étale morphism $c: T' \to T$ lifting $q_0: T'_0 \to T_0$. After replacing $T$ with $T'$ and $f^*$ with $c^*f^*$, we may thus assume that $q_0$ admits a section.

Let $X' = \text{Spec}_X(f_*\mathcal{O}_T)$. Let $I' = I(f_*\mathcal{O}_T)$ be the $\mathcal{O}_{X'}$-ideal generated by $I$ and let $X'_0 = V(I')$. Then $X'$ is a quasi-compact stack with affine diagonal, $X'_0 \to X'$ is a closed immersion defined by an ideal whose $n$th power vanishes and $X'_0$ has the resolution property. Let $f'^* = f'^*: \text{QCoh}(X') \to \text{QCoh}(T)$ be the resulting tensor functor. Since $f'$ is right-exact, it follows that $J = \text{im}(f'^*I' \to \mathcal{O}_T)$. Also, $I' \subseteq f'_*J \subseteq \mathcal{O}_{X'}$. Thus $V(f'_*J) \subseteq X'_0$, so has the resolution property. Note that $f'^*I' \to f'^*f'_*J \to J$ is surjective. Since $f'_*$ is lax symmetric monoidal, for each integer $k \geq 1$ the morphism $(f'_*J)^{\otimes k} \to \mathcal{O}_{X'}$ factors through $f'_*(J^{\otimes k}) \to \mathcal{O}_{X'}$. In particular, $(f'_*J)^{\otimes 2} \subseteq f'_*(J^{k+1})$ and $(f'_*J)^n = 0$.

We may thus replace $X$ by $X'$, $X_0$ by $V(f'_*J)$, $f^*$ by $f'^*$, $I$ by $f'_*J$ and assume henceforth that

\begin{itemize}
  \item[(i)] $\mathcal{O}_X \to f_*\mathcal{O}_T$ is an isomorphism,
  \item[(ii)] $I = f_*J$ for some $\mathcal{O}_T$-ideal $J$ with $J^n = 0$,
  \item[(iii)] $f_*(J^{\otimes l}) \subseteq f_*(J^{l+1})$ for each integer $l \geq 1$, and
\end{itemize}
(iv) $q_0 : V_0 \to T_0$ admits a section.

For each integer $l \geq 0$ let $i_l : X_l \to X$ be the closed immersion defined by $f_*(J^{l+1})$ and let $j_l : T_l \to T$ be the closed immersion defined by $J^{l+1}$. Since $f^* f_* (J^{l+1}) \to f^* \mathcal{O}_X = \mathcal{O}_T$ factors through $J^{l+1}$, it follows that the tensor functor $j_l^* f^* : \text{QCoh}(X) \to \text{QCoh}(T_l)$ factors through a tensor functor $f_l^* : \text{QCoh}(X_l) \to \text{QCoh}(T_l)$ such that $f_l^* i_l^* \cong j_l^* f^*$.

We will now prove by induction on $l \geq 0$ that $X_l$ has the resolution property. Since $X_{n-1} = X$, the result will then follow from Theorem 4.9.

By hypothesis, $X_0$ has the resolution property. Fix an integer $l \geq 0$ and assume that $X_l$ has the resolution property; hence, the tensor functor $f_l^*$ is representable by an affine morphism $f_l : T_l \to X_l$. It remains to prove that $X_{l+1}$ has the resolution property.

**Claim 1.** Let $M \in \text{QCoh}(T)$. If $J^{l+1} M = 0$, then the natural morphism $f^* f_* M \to M$ is surjective.

**Proof of Claim 1.** By hypothesis, the adjunction morphism $M \to (j_l)_*( j_l^* M)$ is an isomorphism. Since $f_l$ is an affine morphism, $f_l^* (j_l)_*( j_l^* M) \to j_l^* M$ is surjective. Thus,

$$f^* f_* M \cong f^* f_* (j_l)_*( j_l^* M) \cong f^* (j_l)_*( f_l)_*( j_l^* M)
\cong (j_l)_*( f_l)_*( j_l^* M) \to (j_l)_*( j_l^* M) \cong M,$$

and the claim follows. △

**Claim 2.** Let $M \in \text{QCoh}(T)$ and let $Q$ be a quotient of $f_* M$. If $J^{l+1} M = 0$, then the natural morphism $Q \to p_* p^* Q$ is split injective.

**Proof of Claim 2.** Form the 2-cartesian diagram of algebraic stacks

\[
\begin{array}{ccc}
V_0 & \overset{q_0}{\longrightarrow} & V_l \overset{q_l}{\longrightarrow} U_l \overset{u_l}{\longrightarrow} U \\
T_0 & \overset{q_l}{\longrightarrow} & T_l \overset{j_l}{\longrightarrow} X_l \overset{i_l}{\longrightarrow} X.
\end{array}
\]

By construction, $q_0$ admits a section. Since $q_l$ is smooth and $T_l$ is affine, the section that $q_0$ admits lifts to a section of $q_l$. It follows that the morphism $\mathcal{O}_{T_l} \to (q_l)_* \mathcal{O}_{V_l}$ is split injective. Pushing this morphism forward along $i_l \circ f_l$ we obtain a split injective morphism $(i_l)_*( f_l)_*( q_l)_* \mathcal{O}_{V_l}.$

Now $(i_l)_*( f_l)_*( j_l^* M) \cong f_* M$ and so $Q$ is a $(i_l)_*( f_l)_* \mathcal{O}_{T_l}$-module. Applying the functor $Q \otimes (i_l)_*( f_l)_* \mathcal{O}_{T_l}$ to the split injective morphism above, we obtain a split injective morphism:

$$v : Q \to Q \otimes (i_l)_*( f_l)_* \mathcal{O}_{T_l} (i_l)_*( f_l)_*( q_l)_* \mathcal{O}_{V_l}.$$  

By functoriality, the morphism $\mathcal{O}_U \to (u_l)_* (q_l)_* \mathcal{O}_{V_l}$ induces a natural $\mathcal{O}_X$-module homomorphism $p_* \mathcal{O}_U \to (i_l)_*( f_l)_*( q_l)_* \mathcal{O}_{V_l}$. There is also a natural morphism $\mathcal{O}_X \to (i_l)_*( f_l)_* \mathcal{O}_{T_l}$ and thus there is an induced homomorphism

$$p_* p^* Q \cong Q \otimes \mathcal{O}_X p_* \mathcal{O}_U \to Q \otimes (i_l)_*( f_l)_* \mathcal{O}_{T_l} (i_l)_*( f_l)_*( q_l)_* \mathcal{O}_{V_l}$$

such that the pre-composition with $Q \to p_* p^* Q$ coincides with $v$. Since $v$ is split, it follows that $Q \to p_* p^* Q$ is split. △
Let $Q_l = f_*(J^{l+1})/f_*(J^{l+2})$. By (iii), $X_{l+1}$ is a square zero extension of $X_l$ by $Q_l$. Retaining the notation of Claim 2, there is a 2-commutative diagram of algebraic stacks:

\[
\begin{array}{ccc}
U_l & \rightarrow & U_{l+1} \\
\downarrow & & \downarrow \\
X_l & \rightarrow & X_{l+1},
\end{array}
\]

where both the inner and outer squares are 2-cartesian and the inner square is 2-cocartesian. The morphism $X_l \rightarrow X_{l+1}$ is a square zero extension of $X_l$ by $(p_l)_*p_l^*Q_l$ and the morphism $X_{l+1} \rightarrow X_{l+1}$ is the morphism of $X_l$-extensions given by the natural map $Q_l \rightarrow (p_l)_*p_l^*Q_l$. By Claim 2, the morphism $Q_l \rightarrow (p_l)_*p_l^*Q_l$ is split injective and so there is an induced splitting $X_{l+1} \rightarrow X_{l+1}$ which is affine. By [Gro13, Prop. 4.3(i)], it remains to prove that $X_{l+1}$ has the resolution property, which follows from Lemma 6.2.

\[\square\]

7. Formal gluings

Let $S$ be an algebraic space, let $Z \hookrightarrow S$ be a finitely presented closed immersion and let $U = S \setminus Z$ denote its complement. A flat Mayer–Vietoris square is a cartesian square of algebraic spaces

\[
\begin{array}{ccc}
U' & \rightarrow & S' \\
\downarrow & & \downarrow f \\
U & \rightarrow & S
\end{array}
\]

such that $f$ is flat and $f|_Z$ is an isomorphism [HR14d, App.]. If $F : \text{Sch}^{\text{op}} \rightarrow \text{Cat}$ is a fibered category, then there is a natural functor:

\[
\Phi_F : F(S) \rightarrow F(S') \times_{F(U')} F(U).
\]

The following theorem is a reformulation of the main results of [MB96].

**Theorem 7.1.** Consider a flat Mayer–Vietoris square and assume that $f : S' \rightarrow S$ is affine. Let $X$ be an algebraic stack and consider the fibered category $X (\cdot) = \text{Hom}_\otimes(\text{QCoh}(X), \text{QCoh}(\cdot))$ on the category of schemes.

(i) $\Phi_X (\cdot)$ is an equivalence of categories;

(ii) $\Phi_X$ is fully faithful;

(iii) $\Phi_X$ is an equivalence if $\Delta_X$ is quasi-affine; and

(iv) $\Phi_X$ is an equivalence if $S$ is locally excellent, $S'$ is noetherian and $\Delta_X$ is quasi-separated.

**Proof.** By [MB96, 0.3] (or [FR70, App.]), there is an equivalence

\[
\text{QCoh}(S) \rightarrow \text{QCoh}(S') \times_{\text{QCoh}(U')} \text{QCoh}(U).
\]

Thus we have (i). The claims (ii), (iii), and (iv), follow immediately from [MB96, 6.5.1] and Corollary 6.2. \[\square\]
Remark 7.2. Recall that a noetherian ring $A$ is excellent [Mat80, p. 260], [Mat80 Ch. 13] or [EGA IV.7.8.2], if

(i) $A$ is a G-ring, that is, $A_p \to \hat{A}_p$ has geometrically regular fibers;
(ii) the regular locus $\text{Reg} B \subseteq \text{Spec} B$ is open for every finitely generated $A$-algebra $B$; and
(iii) $A$ is universally catenary.

If (i) and (ii) hold, then we say that $A$ is quasi-excellent. All excellency assumptions originate from Corollary 8.2 via Theorem 7.1. It is used to guarantee that the formal fibers are geometrically regular so that Néron–Popescu desingularization applies. We can thus replace “locally excellent” with “locally the spectrum of a G-ring”. Note that whereas being a G-ring and being quasi-excellent are local for the smooth topology [Mat89, 32.2], excellency does not descend even for finite étale coverings [EGA IV.18.7.7].

Corollary 7.3. Let $X$ be an algebraic stack. Let $A$ be a ring and let $I \subset A$ be a finitely generated ideal. Let $S = \text{Spec}(A)$, $Z_n = V(I^{n+1})$ and $U = S \setminus Z_0$. Let $j_n: Z_n \to S$ and $i: U \to S$ be the resulting immersions.

(i) Let $f_1, f_2: S \to X$ be morphisms of algebraic stacks.
   (a) Assume that $\ker(O_S \to i_*O_U) \cap \bigcap_{n=0}^{\infty} f^n = 0$. Let $\alpha, \beta: f_1 \Rightarrow f_2$ be 2-morphisms. If $\alpha_U = \beta_U$ and $\alpha_{Z_n} = \beta_{Z_n}$ for all $n$, then $\alpha = \beta$.
   (b) Assume that $S$ is noetherian and that $\omega_X(T)$ is faithful for all noetherian $T$. Let $t: f_1^* \Rightarrow f_2^*$ be a natural transformation of cocontinuous tensor functors. If $i^*(t)$ and $j_n^*(t)$ are realizable for all $n$, then $t$ is realizable.

(ii) Assume either that $S$ is excellent and $X$ has quasi-separated diagonal or that $S$ is noetherian and $X$ has quasi-affine diagonal. Further, assume that $\omega_X(T)$ is fully faithful for all noetherian $T$. If $f^*: \text{QCoh}(X) \to \text{QCoh}(S)$ is a cocontinuous tensor functor such that $i^*f^*$ and $j_n^*f^*$ are representable for all $n$, then $f^*$ is representable.

Proof. First, we show (ii). By assumption, the induced functor $j_n^*f^*$ comes from a morphism $f_n: Z_n \to X$. Pick an étale cover $q_0: Z'_n \to Z_0$ such that $f_0 \circ q_0: Z'_0 \to X$ has a lift $g_0: Z'_0 \to W$, where $p: W \to X$ is a smooth morphism and $W$ is affine. After replacing $S$ with an étale cover $S' \to S$, we may assume that $Z'_0 = Z_0$ and that $f_0$ has a lift $g_0: Z_0 \to W$.

Since $p$ is smooth, we may choose compatible lifts $g_n: Z_n \to W$ for all $n$. Now since $W$ is affine, we have an induced morphism $\hat{g}: \hat{S} \to W$, where $\hat{S} = \text{Spec}(\hat{A})$ and $\hat{A}$ denotes the completion of $A$ at the ideal $I$. Let $\hat{f} = p \circ \hat{g}$. Then $(j_n)^*\hat{f}^* = (j_n)^*f^*$ for all $n$. Since $\text{Coh}(\hat{S}) = \varprojlim_n \text{Coh}(Z_n)$ (Lemma 3.6), it follows that $\hat{f}^* \simeq \pi^*f^*$ where $\pi: \hat{S} \to S$ is the completion morphism. Indeed, this last equivalence may be verified after restricting both sides to the quasi-coherent $O_{\hat{S}}$-modules of finite type (Example 5.1) and both sides send quasi-coherent $O_X$-modules of finite type to $\text{Coh}(\hat{S})$.

That $f^*$ is representable now follows from Theorem 7.1.

For (i)(b), we proceed similarly. Consider the representable morphism $E \to S$ given by the equalizer of $f_1$ and $f_2$. Then 2-isomorphisms between
$f_1$ and $f_2$ correspond to $S$-sections of $E$. By assumption, we have compatible sections $\tau_U \in E(U)$ and $\tau_n \in E(Z_n)$ for all $n$. After replacing $S$ with an étale cover, we may assume that $\tau_n$ lifts to $E'$ where $E' \to E$ is étale and $E'$ is an affine scheme. In particular, there are compatible lifts of all the $\tau_n$ to $E'$. Since $E'$ is affine, we get an induced morphism $\hat{S} \to E'$; thus, a morphism $\hat{S} \to E$. Equivalently, we get a 2-isomorphism between $f_1 \circ \pi$ and $f_2 \circ \pi$. The induced 2-isomorphism between $\pi^*f_1^*$ and $\pi^*f_2^*$ equals $\pi^*t$ since it coincides on the truncations. We may now apply Theorem 7.1 to deduce that $t$ is realized by a 2-morphism $\tau : f_1 \Rightarrow f_2$.

For (i)(a), we consider the representable morphism $r : R \to S$ given by the equalizer of $\alpha$ and $\beta$. It suffices to prove that $r$ is an isomorphism. Note that $r$ is automatically a monomorphism and locally of finite presentation. By assumption, there are compatible sections of $r$ over $U$ and $Z_n$ for all $n$, thus $r_U$ and $r_{Z_n}$ are isomorphisms for all $n$. By Proposition A.3 $r$ is an isomorphism.

8. Tannaka duality

In this section, we prove our general Tannaka duality result (Theorem 8.4) and as a consequence also establish Theorems 1.1 and 1.4. To accomplish this, we consider the following refinement of [HR14a, Def. 2.5].

**Definition 8.1.** Let $X$ be a quasi-compact algebraic stack. A **finitely presented filtration of $X$** is a sequence of finitely presented closed immersions $\emptyset = X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \ldots \hookrightarrow X_r \hookrightarrow X$ such that $|X_r| = |X|$. The **strata** of the filtration are the locally closed finitely presented substacks $Y_k := X_k \setminus X_{k-1}$. The $n$th **infinitesimal neighborhood of $X_k$** is the finitely presented closed immersion $X_k^{[n]} \hookrightarrow X$ which is given by the ideal $I_{k+1}$ where $X_k \hookrightarrow X$ is given by $I_k$. The $n$th infinitesimal neighborhood of $Y_k$ is the locally closed finitely presented substack $Y_k^{[n]} := X_k^{[n]} \setminus X_{k-1}$.

Stacks that have affine stabilizers can be stratified into stacks with the resolution property.

**Proposition 8.2.** Let $X$ be an algebraic stack. The following are equivalent:

(i) $X$ is quasi-compact and quasi-separated with affine stabilizers;
(ii) $X$ has a finitely presented filtration $(X_k)$ with strata of the form $Y_k = [U_k/GL_{N_k}]$ where $U_k$ is quasi-affine.
(iii) $X$ has a finitely presented filtration $(X_k)$ with strata $Y_k$ that are quasi-compact with affine diagonal and the resolution property.

**Proof.** That (i) $\implies$ (ii) is [HR14a, Prop. 2.6(i)]. That (ii) $\iff$ (iii) is the Gross–Totaro theorem [Gro13]. That (iii) $\implies$ (i) is straightforward. □

When in addition $X$ is noetherian or has finitely presented inertia, this result is due to Kresch [Kre99, Prop. 3.5.9] and Drinfeld–Gaitsgory [DG13, Prop. 2.3.4]. They construct stratifications by quotient stacks of the form $[V_k/GL_{N_k}]$, where each $V_k$ is quasi-projective and the action is linear. This implies that the strata have the resolution property. This case is slightly simpler since $X$ can then be stratified into gerbes [Ryd14, Cor. 7.4], something which is not possible in general.
**Remark 8.3.** In [DG13 Def. 1.1.7], Drinfeld and Gaitsgory introduces the notion of a QCA stack. These are (derived) algebraic stacks that are quasi-compact and quasi-separated with affine stabilizers and finitely presented inertia. The condition on the inertia is presumably only used for [DG13 Prop. 2.3.4] and could be excused using Proposition 8.2.

We now state and prove the main result of the paper.

**Theorem 8.4.** Let $T$ and $X$ be algebraic stacks and consider the functors

\[
\omega_X(T) : \text{Hom}(T, X) \to \text{Hom}_{\text{QCoh}}(\text{QCoh}(X), \text{QCoh}(T)),
\]

\[
\omega_X^\omega(T) : \text{Hom}(T, X) \to \text{Hom}_{\text{QCoh}}(\text{QCoh}(X), \text{QCoh}(T)).
\]

Assume that $X$ is quasi-compact and quasi-separated.

(i) If $X$ has quasi-affine diagonal, then

(a) $\omega_X(T)$ is fully faithful; and

(b) $\omega_X(T)$ is essentially surjective if $T$ is locally noetherian.

(ii) If $X$ has affine stabilizers, then

(a) $\omega_X(T)$ is faithful if $T$ is locally noetherian or has no embedded components;

(b) $\omega_X^\omega(T)$ is full if $T$ is locally noetherian;

(c) $\omega_X(T)$ is full if $X$ is affine-pointed and $T$ is locally noetherian.

(d) $\omega_X(T)$ is essentially surjective if $T$ is locally excellent.

In particular, $\omega_X^\omega(T)$ is an equivalence if $X$ has affine stabilizers and $T$ is locally excellent, and $\omega_X(T)$ is an equivalence if $X$ has quasi-affine diagonal and $T$ is locally noetherian.

**Proof.** When $X$ has quasi-affine diagonal, we have already seen that $\omega_X(T)$ is fully faithful for all $T$ (Proposition 4.7). This is (i)(a).

Choose a filtration $(X_\ell)$ with strata $(Y_\ell)$ as in Proposition 8.2. We will prove the theorem by induction on the number of strata $r$. If $r = 0$, then $X = \emptyset$ and there is nothing to prove. If $r \geq 1$, then $V := X \setminus X_1$ has a filtration of length $r - 1$; thus by induction the theorem holds for $V$. The theorem also holds for $X^{[n]}_1 = Y^{[n]}_1$ and all $n$, since $\omega^{[n]}_X(T)$ is an equivalence of categories by the Main Lemma 6.1. Note that if $r = 1$, then $V = \emptyset$ and $X = X^{[n]}_1 = Y^{[n]}_1$ for sufficiently large $n$.

We can assume that $T = \text{Spec}(A)$ is affine. Let $I \subseteq O_X$ be the ideal defining $X_1$. Let $i_n : X^{[n]}_1 \hookrightarrow X$ be the closed substack defined by $I^{n+1}$ and let $g : V \to X$ be its complement.

For (ii)(a), pick two maps $f_1, f_2 : T \to X$ and 2-isomorphisms $\tau_1, \tau_2 : f_1 \Rightarrow f_2$ and assume that $\omega_X(T)(\tau_1) = \omega_X(T)(\tau_2)$. We need to prove that $\tau_1 = \tau_2$.

For (ii)(b) (resp. (ii)(c)), pick two maps $f_1, f_2 : T \to X$ and a natural isomorphism (resp. transformation) $\gamma : f_1^* \Rightarrow f_2^*$ of cocontinuous tensor functors. We need to prove that $\gamma$ comes from a 2-morphism $g : f_1 \Rightarrow f_2$.

For (i)(b) and (ii)(d) pick a cocontinuous tensor functor $f^* : \text{QCoh}(X) \to \text{QCoh}(T)$. We need to prove that $f^*$ is representable.

When we prove (ii)(d) (resp. (ii)(b) and (ii)(c)), we assume that (ii)(b) (resp. (ii)(a)) already has been established. When we prove (i)(b), we note that $\omega_X(T)$ is fully faithful for all $T$. 
In cases [ii](a), [ii](b) and [ii](c) let $J = \text{Im}(f_2^* I \to f_2^* O_X = O_T)$, which is a finitely generated ideal because $f_2$ is a morphism. In cases [i](b) and [ii](d) let $J = \text{Im}(f^* I \to f^* O_X = O_T)$, which is a finitely generated ideal because $T$ is noetherian. Let $j_n: Z^{[n]} \to T$ be the finitely presented closed immersion defined by $J^{n+1}$ and let $r: U \to T$ be its complement, a quasi-compact open immersion.

In cases [ii](a), [ii](b) and [ii](c) we have that $f_1^{-1}(V) = f_2^{-1}(V) = U$ and $f_1^{-1}(X^{[1]}_1) = f_2^{-1}(X^{[1]}_1) = Z^{[n]}$; in the first case this is obvious and for the other two cases this follows from Corollary 4.8 and Lemma 4.6, respectively. Thus, after restricting to either $Z^{[n]}$ or $U$ we have that $\tau_1 = \tau_2$ in case [ii](a) and that $\gamma$ is realizable in cases [ii](b) and [ii](c).

In cases [i](b) and [ii](d) we have that $(j_n)^* f^* = (j_n)^* f^*(i_n)_* i_n^*$ and $(j_n)^* f^*(i_n)_*: \text{QCoh}(X^{[1]}_1) \to \text{QCoh}(Z^{[n]})$ is a tensor functor. Hence $(j_n)^* f^*$ is representable by the inductive assumption. Recall that $q^*: \text{QCoh}(X) \to \text{QCoh}(U)$ is a supported localization (Example 5.6). If $F \in \text{QCoh}(X)$ is a quasi-coherent sheaf of finite type such that $q^* F = 0$, then $I^m F = 0$ for some integer $m$ and hence $J^m q^* F = 0$ and thus $r^* f^* F = 0$. By Theorem 5.7 we obtain a factorization of $r^* f^*$ via $q^*$ and a tensor functor $\text{QCoh}(V) \to \text{QCoh}(U)$. The latter is representable by the inductive assumption. Thus after restricting to either $Z^{[n]}$ or $U$, the tensor functor is representable.

If $T$ is noetherian or has no embedded associated points, then the stratification $\emptyset \subset Z \subset T$ is separating by Lemma A.2. The result now follows from Corollary 7.3.

**Remark 8.5.** Let $X$ be a quasi-compact and quasi-separated algebraic stack with affine stabilizers. Let $T$ be a locally noetherian stack and let $\pi: T' \to T$ be a flat morphism. Assume that we have morphisms $f_1, f_2: T \to X$. Then $\text{Hom}(f_1 \circ \pi, f_2 \circ \pi) \to \text{Hom}(\pi^* f_1^*, \pi^* f_2^*)$ is injective even if $T'$ is not noetherian. Indeed, the stratification on $T'$ constructed in the proof of Theorem 8.4(ii)(a) is the pull-back along $\pi$ of a stratification on $T$, hence separating by Lemma A.2.

We conclude with the proof of Theorem 1.4.

**Proof of Theorem 1.4.** We note that

$$\text{Hom}_{r_{\otimes, \simeq}}(\text{Coh}(X), \text{Coh}(T)) \to \text{Hom}_{c_{\otimes, \simeq}}(\text{QCoh}(X), \text{QCoh}(T))$$

is fully faithful. It is thus enough to prove that $\omega_X^\otimes(T): \text{Hom}(T, X) \to \text{Hom}_{c_{\otimes, \simeq}}(\text{QCoh}(X), \text{QCoh}(T))$ is an equivalence of groupoids. This follows from Theorem 8.4(ii).

**Theorem 1.4** is Theorem 8.4(ii).

### 9. Applications

In this section, we address the applications outlined in the introduction.

**Proof of Corollary 1.5.** Let $T' \to T$ be an fpqc covering with $T$ locally excellent and $T'$ locally noetherian. Since $X$ is an fpfp-stack, we may assume that $T$ and $T'$ are affine and that $T' \to T$ is faithfully flat. Let $T'' = T' \times_T T'$. Since $X$ has affine stabilizers, the functor $\omega_X^\otimes(T)$ is an equivalence, the functor $\omega_X^\otimes(T')$ is fully faithful and the functor $\omega_X(T'')$ is
faithful for morphisms $T'' \to T' \to X$ (Theorem 8.3 and Remark 8.3). Since $\text{Hom}_{\mathcal{C}, \mathcal{C}}(\text{QCoh}(X), \text{QCoh}(-))$ is an fpqc stack, it follows that $T' \to T$ is a morphism of effective descent for $X$.

Proof of Corollary 8.4. It is readily verified that we can assume that $X$ is quasi-compact. As $A$ is noetherian, $\text{Coh}(A) = \varprojlim A/I^n$. Thus,

$$X(A) \cong \text{Hom}_{\mathcal{C}, \mathcal{C}}(\text{Coh}(X), \text{Coh}(A))$$

$$\cong \text{Hom}_{\mathcal{C}, \mathcal{C}}(\text{Coh}(X), \varprojlim A/I^n)$$

$$\cong \varprojlim \text{Hom}_{\mathcal{C}, \mathcal{C}}(\text{Coh}(X), \text{Coh}(A/I^n))$$

$$\cong \varprojlim X(A/I^n).$$

Proof of Theorem 1.2. We begin with the following standard reductions: we can assume that (1) $S$ is affine, (2) $X \to S$ is quasi-compact, hence of finite presentation, and (3) $S$ is of finite type over $\text{Spec}(\mathbb{Z})$, hence excellent. We will prove the algebraicity of $\text{Hom}_S(Z, X)$ using a variant of Artin’s criterion for algebraicity due to the first author [Hal14b Thm. A]. By descent, $\text{Hom}_S(Z, X)$ is stack for the étale topology and by standard limit methods it is limit preserving. In the notation of [Hal14b §9], we observe that associating a graph to a morphism produces a $1$-morphism of $S$-groupoids

$$\text{Hom}_S(Z, X) \to \text{Mor}_{\mathcal{Z}_{X/S}}(Z, X/S),$$

which is formally étale (because $Z \to S$ is flat). By [Hal14b Lemmas 1.5(4,9) and 9.3], we see that $\text{Hom}_S(Z, X)$ is homogeneous. By [Hal13b Lemmas 6.11 and 6.12 and §9] and [Hal14a Thm. C], the conditions on automorphisms, deformations, and obstructions are satisfied. What remains to be shown is the effectiveness of the $S$-groupoid $\text{Hom}_S(Z, X)$. Thus, let $T = \text{Spec}(B) \to S$, where $(B, m)$ is a complete local noetherian ring. Let $T_n = \text{Spec}(B/m^{n+1})$. Since $Z \to S$ is proper, for every noetherian algebraic stack $W$ with affine stabilizers there are equivalences

$$\text{Hom}(Z \times S T, W) \cong \text{Hom}_{\mathcal{C}, \mathcal{C}}(\text{Coh}(W), \text{Coh}(Z \times S T))$$

$$\cong \text{Hom}_{\mathcal{C}, \mathcal{C}}(\text{Coh}(W), \varprojlim \text{Coh}(Z \times S T_n))$$

$$\cong \varprojlim \text{Hom}_{\mathcal{C}, \mathcal{C}}(\text{Coh}(W), \text{Coh}(Z \times S T_n))$$

$$\cong \varprojlim \text{Hom}(Z \times S T_n, W).$$

Since $X$ and $S$ have affine stabilizers, it follows that

$$\text{Hom}_S(Z \times S T, X) \cong \varprojlim \text{Hom}_S(Z \times S T_n, X);$$

that is, the stack $\text{Hom}_S(Z, X)$ is effective.

Let $P$ be one of the properties: affine, quasi-affine, separated, quasi-compact and quasi-separated with affine fibers; and assume that $\Delta_X$ has $P$. Let $T$ be an affine scheme. Let $Z_T$ and $X_T$ denote $Z \times_S T$ and $X \times_S T$, respectively. Suppose we are given two $T$-morphisms $f_1, f_2 : Z_T \to X_T$ and consider $Q := \text{Isom}_{Z_T}(f_1, f_2) = X \times_{X \times_S X} Z_T$. Then $Q \to Z_T$ is representable, of finite presentation and $P$. If $\pi : Z_T \to T$ denotes the structure morphism, then $\pi_* Q$ is an algebraic space which is locally of finite presentation, being the pull-back of the diagonal of $\text{Hom}_S(Z, X)$ along the morphism $T \to \text{Hom}_S(Z, X) \times_S \text{Hom}_S(Z, X)$ corresponding to $(f_1, f_2)$. 


It remains to prove that the induced morphism $\pi_\ast Q \to T$ is locally of finite presentation and has $P$. For the properties affine, quasi-affine and separated, this is [HR14b, Thm. 2.3 (i), (ii) & (iv)]: for the first two properties applied to $Q \to Z$ and for the third applied to the closed immersion $\overline{Q} \to \overline{Q} \times \overline{T}$ and the Weil restriction $\pi_\ast Q \to \pi_\ast Q \times T \pi_\ast Q = \pi_\ast (\overline{Q} \times \overline{T})$. For the property quasi-compact and quasi-separated with affine stabilizers, it follows from Theorem [C.1] that $\pi_\ast Q \to T$ is quasi-compact and quasi-separated; we claim that this morphism has affine fibers. To see this, we may assume that $T$ is the spectrum of an algebraically closed field. In this situation, either $\pi_\ast Q$ is empty or $f_1 \simeq f_2$; it suffices to treat the latter case. Now $\overline{Q} \to \overline{T}$ is a group with affine fibers and the result follows from Theorem [C.1]. □

Lemma 9.1. Let $f: Z \to S$ be a proper and flat morphism of finite presentation between algebraic stacks. For any morphism $X \to Z$ of algebraic stacks, the forgetful morphism $f_\ast X \to \text{Hom}_S(Z, X)$ is an open immersion.

Proof. It is sufficient to prove that if $T$ is an affine $S$-scheme and $h: Z \times_S T \to X \times_S T$ is a $T$-morphism, then the locus of points where $f_T \circ h: Z \times_S T \to Z \times S T$ is an isomorphism is open on $T$.

First, consider the diagonal of $f_T \circ h$. This morphism is proper and representable and the locus on $T$ where this map is a closed immersion is open [Ryd11c, Lem. 1.8 (iii)]. We may thus assume that $f_T \circ h$ is representable. Repeating the argument on $f_T \circ h$, we may assume that $f_T \circ h$ is a closed immersion. That the locus in $T$ where $f_T \circ h$ is an isomorphism is open now follows easily by studying the étale locus of $f_T \circ h$, cf. [Ols06b, Lem. 5.2]. The result follows. □

Proof of Theorem 1.3. That $f_\ast X \to S$ is algebraic, locally of finite presentation, with quasi-compact and quasi-separated diagonal and affine stabilizers follows from Theorem [1.2] and Lemma [9.1]. The additional separation properties of $f_\ast X$ follows from [HR14b, Thm. 2.3 (i), (ii) & (iv)] applied to the diagonal and double diagonal of $X \to Z$. □

As claimed in the introduction, we now extend [HR14b, Thm. 2.3 & Cor. 2.4]. The statement of the following corollary uses the notion of a morphism of algebraic stacks that is locally of approximation type [HR14b, §1]. A trivial example of a morphism locally of approximation type is a quasi-separated morphism that is locally of finite presentation. It is hoped that every quasi-separated morphism of algebraic stacks is locally of approximation type, but this is currently unknown. It is known, however, that morphisms of algebraic stacks that have quasi-finite and locally separated diagonal are locally of approximation type [Ryd15]. In particular, all quasi-separated morphisms of algebraic stacks that are relatively Deligne–Mumford are locally of approximation type.

Corollary 9.2. Let $f: Z \to S$ be a proper and flat morphism of finite presentation between algebraic stacks.

(i) Let $h: X \to S$ be a morphism of algebraic stacks with affine stabilizers that is locally of approximation type. Then $\text{Hom}_S(Z, X)$ is algebraic and locally of approximation type with affine stabilizers. If $h$ is locally of finite presentation, then so is $\text{Hom}_S(Z, X) \to S$. If
the diagonal of \( h \) is affine (resp. quasi-affine, resp. separated), then so is the diagonal of \( \text{Hom}_S(Z, X) \to S \).

(ii) Let \( g: X \to Z \) be a morphism of algebraic stacks such that \( f \circ g: X \to S \) has affine stabilizers and is locally of approximation type. Then the \( S \)-stack \( f_*X \) is algebraic and locally of approximation type with affine stabilizers. If \( g \) is locally of finite presentation, then so is \( f_*X \to S \). If the diagonal of \( g \) is affine (resp. quasi-affine, resp. separated), then so is the diagonal of \( f_*X \to S \).

Proof. For (i), we may immediately reduce to the situation where \( S \) is an affine scheme. Since \( f \) is quasi-compact, we may further assume that \( h \) is quasi-compact. By [HR14b, Lem. 1.1], there is a fpqc covering \( \{ S_i \to S \} \) such that each \( S_i \) is affine and \( X \times_S S_i \to S_i \) factors as \( X_0 \to S_i \) of finite presentation and \( X \times_S S_i \to X_0 \) is affine.

Combining the results of [HR14a, Thm. 2.8] with [Ryd15, Thms. D & 7.10], we can arrange so that each \( X_0 \to S \) has affine stabilizers (or has one of the other desired separation properties).

Thus, we may now replace \( S \) by \( S_i \) and may assume that \( X \to S \) factors as \( X \to X_0 \to S \), where \( q \) is affine and \( X_0 \to S \) is of finite presentation with the appropriate separation condition. By Theorem 1.2, the stack \( \text{Hom}_S(Z, X_0) \) is algebraic and locally of finite presentation with the appropriate separation condition. By [HR14b, Thm. 2.3(i)], the morphism \( \text{Hom}_S(Z, X) \to \text{Hom}_S(Z, X_0) \) is representable by affine morphisms; the result follows.

For (ii) we argue exactly as in the proof of Theorem 1.3.

10. COUNTEREXAMPLES

In this section we give four counter-examples (Theorems 10.1, 10.2, 10.4, and 10.5):

- in Theorems 1.1, 1.2 and 8.3(ii)(a) it is necessary that \( X \) has affine stabilizer groups;
- in Theorem 8.3(ii)(c) it is necessary that \( X \) is affine-pointed;
- in Theorem 1.2 it is necessary that \( X \) has affine stabilizer groups; and
- in Corollary 1.6 it is necessary that \( X \) has affine stabilizer groups.

Theorem 10.1. Let \( X \) be a quasi-separated algebraic stack. If \( k \) is an algebraically closed field and \( x: \text{Spec} \ k \to X \) is a point with non-affine stabilizer, then \( \text{Aut}(x) \to \text{Aut}_\otimes(x^*) \) is not injective. In particular, \( \omega_X(\text{Spec} \ k) \) is not faithful and \( X \) is not tensorial.

Proof. By assumption, the stabilizer group scheme \( G_x \) of \( x \) is affine. Let \( H = (G_x)_{\text{ant}} \) be the largest anti-affine subgroup of \( G_x \); then \( H \) is a non-trivial anti-affine group scheme over \( k \) and the quotient group scheme \( H / H \) is affine \([DG70, \S III.3.8]\). The induced morphism \( B_k H \to B_k G_x \to X \) is thus quasi-affine by \([Ryd11b, \text{Thm. B.2}]\).

By [Bri09, Lem. 1.1], the morphism \( p: \text{Spec} \ k \to B_k H \) induces an equivalence of abelian tensor categories \( p^*: \text{QCoh}(B_k H) \to \text{QCoh}(\text{Spec} \ k) \). Since \( \text{Aut}(p) = H(k) \neq \{ \text{id}_p \} = \text{Aut}_\otimes(p^*) \), the functor \( \omega_{B_k H}(\text{Spec} \ k) \) is not faithful. Hence \( \omega_X(\text{Spec} \ k) \) is not faithful by Lemma 1.2. □
Theorem 10.2. Let $X$ be a quasi-compact and quasi-separated algebraic stack with affine stabilizers. If $k$ is a field and $x_0: \text{Spec } k \to X$ is a non-affine morphism, then there exists a field extension $K/k$ and a point $y: \text{Spec } K \to X$ such that $\text{Isom}(y, x) \to \text{Hom}_o(y^*, x^*)$ is not surjective, where $x$ denotes the $K$-point corresponding to $x_0$. In particular, $\omega_X(\text{Spec } K)$ is not full.

Proof. To simplify notation, we let $x = x_0$. Since $X$ has quasi-compact diagonal, $x$ is quasi-affine [Ryd11 Thm. B.2]. By Lemma 4.2, we may replace $X$ by $\text{Spec } X(x, k)$ and consequently assume that $x$ is a quasi-compact open immersion and $\mathcal{O}_X \to \mathcal{I}$ is an isomorphism. In particular, $x$ is a section to a morphism $f: X \to \text{Spec } k$. Since $x$ is not affine, it follows that there exists a closed point $y$ disjoint from the image of $x$. In particular, there is a field extension $K/k$ and a $k$-morphism $y: \text{Spec } K \to X$ whose image is a closed point disjoint from $x$.

We now base change the entire situation by $\text{Spec } K \to \text{Spec } k$. This results in two morphisms $x_K, y_K: \text{Spec } K \to X \otimes_k K$, where $x_K$ is a quasi-compact open immersion such that $\mathcal{O}_{X \otimes_k K} \cong (x_K)_K$ and $y_K$ has image a closed point disjoint from the image of $x_K$. We replace $X, k, x,$ and $y$ by $X \otimes_k K, K, x_K,$ and $y_K$ respectively.

Let $\mathcal{G}_y \subseteq X$ be the residual gerbe associated to $y$, which is a closed immersion. We define a natural transformation $\gamma^\vee: x_s \Rightarrow y_s$ at $k$ to be the composition $x_s, k \cong \mathcal{O}_X \to \mathcal{O}_{\mathcal{G}_y} \to y_s, k$ and extend to all of $\mathcal{QCoh}(\text{Spec } k)$ by taking colimits. By adjunction, there is an induced natural transformation $\gamma: y^* \Rightarrow x^*$. A simple calculation shows that $\gamma$ is a natural transformation of cocontinuous tensor functors. Since its adjoint $\gamma^\vee$ is not an isomorphism, $\gamma$ is not an isomorphism; thus $\gamma$ is not realizable. The result follows. □

The following lemma is a variant of [Bha14 Ex. 4.12], which B. Bhatt communicated to the authors.

Lemma 10.3. Let $k$ be an algebraically closed field and let $G/k$ be an anti-affine group scheme of finite type. Let $Z/k$ be a regular scheme with a closed subscheme $C$ that is a nodal curve over $k$. Then there is a compatible system of $G$-torsors $E_n \to C[n]$ such that there does not exist a $G$-torsor $E \to Z$ that restricts to the $E_n$'s.

Proof. Recall that $G$ is smooth, connected and commutative [DG70 §III.3.8]. Furthermore, by Chevalley’s theorem, there is an extension $0 \to H \to G \to A \to 0$, where $A$ is an abelian variety (of positive dimension) and $H$ is affine. Let $x_A \in A(k)$ be an element of infinite order and let $x \in G(k)$ be any lift of $x_A$.

Let $\tilde{C}$ be the normalization of $C$. Let $F_0 \to C$ be the $G$-torsor obtained by gluing the trivial $G$-torsor on $\tilde{C}$ along the node by translation by $x$. Note that the induced $A$-torsor $F_0/H \to C$ is not torsion as it is obtained by gluing along the non-torsion element $x_A$.

We may now lift $F_0 \to C$ to $G$-torsors $F_n \to C[n]$. Indeed, the obstruction to lifting $F_{n-1}$ to $F_n$ lies in $\text{Ext}^1_{O_{\tilde{C}}}(L^\bullet_{BG/k}, I^n/I^{n+1})$, where $g_0: C \to BG$ is the morphism corresponding to $F_0 \to C$ and $I$ is the ideal defining $C$ in $Z$. Since $G$ is smooth, the cotangent complex $L^\bullet_{BG/k}$ is concentrated in degree
1 and since $C$ is a curve, it has cohomological dimension 1. It follows that the obstruction group is zero.

Now given a $G$-torsor $F \to Z$, there is an induced $A$-torsor $F/H \to Z$. Since $Z$ is regular, the torsor $F/H \to Z$ is torsion in $H^1(Z,A)$ [Ray70b, XIII 2.4 & 2.6]. Thus, $F/H \to Z$ cannot restrict to $F_0/H \to C$ and the result follows. □

We now have the following theorem, which is a counterexample to [Aok06a, Thm. 1.1] and [Aok06b, Case I].

**Theorem 10.4.** Let $X \to S$ be a quasi-separated morphism of algebraic stacks. If $k$ is an algebraically closed field and $x: \text{Spec } k \to X$ is a point with non-affine stabilizer, then there exists a morphism $\mathbb{A}^1_k \to S$ and a proper and flat family of curves $Z \to \mathbb{A}^1_k$, where $Z$ is regular, such that $\text{Hom}_{\mathbb{A}^1_k}(Z, X \times_S \mathbb{A}^1_k)$ is not algebraic.

**Proof.** Let $Q$ be the stabilizer group scheme of $x$ and let $G$ be the largest anti-affine subgroup scheme of $Q$; thus, $G$ is a non-trivial anti-affine group scheme over $k$ and the quotient group scheme $Q/G$ is affine [DG70, §III.3.8].

Let $Z$ be a proper family of curves over $T = \mathbb{A}^1_k = \text{Spec } k[t]$ with regular total space and a nodal curve $C$ as the fiber over the origin; for example, take $Z = \text{Proj } T(k[t][x,y,z]/(y^2z-x^2z-x^3-z^3))$ over $T$. Let $T_n = V(t^n+1)$, $\hat{T} = \text{Spec } \hat{O}_{T,0}$, $Z_n = Z \times_T T_n$, and $\hat{Z} = Z \times_T \hat{T}$. We now apply Lemma [10.3] to $C$ in $\hat{Z}$ and $G$. Since $Z_n = C[t^n]$, this produces an element in

$$\lim_{\leftarrow n} \text{Hom}_T(Z, BG_T)(T_n) = \lim_{\leftarrow n} \text{Hom}(Z_n, BG)$$

that does not lift to $\text{Hom}_{\hat{T}}(Z, BG_T)(\hat{T}) = \text{Hom}(\hat{Z}, BG)$.

This shows that $\text{Hom}_{\hat{T}}(Z, BG_T)$ is not algebraic.

By [Ryd11b, Thm. B.2], the morphism $x$ factors as $\text{Spec } k \to BQ \to Q \to X$, where $Q$ is the residual gerbe, $Q \to X$ is quasi-affine and $BQ \to Q$ is affine. Since $Q/G$ is affine, it follows that the induced morphism $BG \to BQ \to X$ is quasi-affine. By [HR14b, Thm. 2.3(ii)], the induced morphism $\text{Hom}_{\hat{T}}(Z, BG_T) \to \text{Hom}_{\hat{T}}(Z, X \times_S T)$ is quasi-affine. In particular, if $\text{Hom}_{\hat{T}}(Z, X \times_S T)$ is algebraic, then $\text{Hom}_{\hat{T}}(Z, BG_T)$ is algebraic, which is a contradiction. The result follows. □

The following theorem extends [Bha14, Ex. 4.12].

**Theorem 10.5.** Let $X$ be an algebraic stack with quasi-compact diagonal. If $X$ does not have affine stabilizers, then there exists a noetherian two-dimensional regular ring $A$, complete with respect to an ideal $I$, such that $X(A) \to \lim X(A/I^n)$ is not an equivalence of categories.

**Proof.** Let $x \in |X|$ be a point with non-affine stabilizer group. Arguing as in the proof of Theorem [10.4] there exists an algebraically closed field $k$, an anti-affine group scheme $G/k$ of finite type and a quasi-affine morphism $BG \to X$. An easy calculation shows that it is enough to prove the theorem for $X = BG$. 

Let $A_0 = k[x, y]$ and let $A$ be the completion of $A_0$ along the ideal $I = (y^2 - x^3 - x^2)$. Then $Z = \text{Spec}(A)$ and $C = \text{Spec}(A/I)$ satisfies the conditions of Lemma 10.3 and we obtain an element in $\varprojlim_n X(A/I^n)$ that does not lift to $X(A)$.

\section*{Appendix A. Monomorphisms and stratifications}

In this appendix, we introduce some notions and results needed for the faithfulness part of Theorem \[\text{[8.4]}\] when $T$ is not noetherian. This is essential for the proof of Corollary \[\text{[1.6]}\].

\begin{definition}{A.1} We say that a finitely presented filtration $(X_k)$ of $X$ is separating if the family $\{j_k^n: Y_k^{[n]} \to X\}_{k,n}$ is separating \[\text{[EGA IV.11.9.1]}\], that is, if the intersection $\bigcap_{k,n} \ker(\mathcal{O}_X \to (j_k^n)_* \mathcal{O}_{Y_k^{[n]}})$ is zero as a lisse-étale sheaf.

\begin{lemma}{A.2} Every finitely presented filtration $(X_k)$ on $X$ is separating if either

(i) $X$ is noetherian; or

(ii) $X$ has no embedded (weakly) associated point.

If $X$ is noetherian with a filtration $(X_k)$ and $X' \to X$ is flat, then $(X_k \times_X X')$ is a separating filtration on $X'$.

\end{lemma}

\begin{proof} As the question is smooth-local, we can assume that $X$ and $X'$ are affine schemes. If $X$ is noetherian, then by primary decomposition there exists a separating family $\coprod_{i=1}^m \text{Spec}(A_i) \to X$ where the $A_i$ are artinian. As every $\text{Spec}(A_i)$ factors through some $Y_k^{[n]}$, it follows that $(X_k)$ is separating. In general, $(\text{Spec}(\mathcal{O}_{X,x}) \to X)_{x \in \text{Ass}(X)}$ is separating \[\text{[Laz64 1.2, 1.5, 1.6]}\]. If $x$ is a non-embedded associated point, then $\text{Spec}(\mathcal{O}_{X,x})$ is a one-point scheme and factors through some $Y_k^{[n]}$ and the first claim follows.

For the last claim, we note that a finite number of the infinitesimal neighborhoods of the strata suffices in the noetherian case and that flat morphisms preserve kernels and finite intersections.

\end{proof}

\begin{proposition}{A.3} Let $X$ be an algebraic stack with a finitely presented filtration $(X_k)$. Let $f: Z \to X$ be a morphism locally of finite type. If $f|_{Y_k^{[n]}}$ is an isomorphism for every $k$ and $n$, then $f$ is a surjective closed immersion. If in addition $(X_k)$ is separating, then $f$ is an isomorphism.

\end{proposition}

\begin{proof} Note that $f$ is a surjective and quasi-compact monomorphism. We will prove that $f$ is a closed immersion by induction on the number of strata $r$. If $r = 0$, then $X = \emptyset$ and there is nothing to prove. If $r = 1$, then $X = X_1^{[n]} = Y_1^{[n]}$ for sufficiently large $n$ and the result follows. If $r \geq 2$, then let $U = X \setminus X_1$. By the induction hypothesis, $f|_U$ is a surjective closed immersion.

It is enough to show that $f$ is a closed immersion in a neighborhood of every $x \in |X_1|$. This can be checked locally in the étale topology and we may thus assume that $Z = Z_0 \amalg Z_1$ where $Z_0 \to X$ is a closed immersion and $Z_1 \cap f^{-1}(x) = \emptyset$. Note that $f(Z_0) \cap U$ and $f(Z_1) \cap U$ are disjoint closed and open subsets.
It is further enough to show that \( f \) is a closed immersion after replacing \( X \) with either \( X_1, f(Z_0) \cap U \) or \( f(Z_1) \cap U \). In the first and second case, \( f \) is certainly a closed immersion. In the third case, \( f(Z) \) is set-theoretically contained in \( X_1 \). Let \( W = f(Z_0) \cap X_1 \); this is an open and closed substack of \( X_1 \). Thus, \( f|_{Z_0} : Z_0 \to X \) factors through \( W[n] \) for sufficiently large \( n \). By hypothesis, this means that \( Z_0 \sim W[n] \sim W[N] \) for all sufficiently large \( n \) and all \( N \geq n \). This implies that \( W[n] \hookrightarrow X \) is an open immersion and we have proved that \( f \) is a closed immersion in a neighborhood of \( x \). The result follows.

The last claim is obvious. □

The following example illustrates that a closed immersion \( f : Z \to X \) as in Proposition A.3 need not be an isomorphism even if \( f \) is of finite presentation.

**Example A.4.** Let \( A = k[x, z_1, z_2, \ldots]/(xz_1, \{z_k - \sum_{i,j} z_i z_j \} \mid k \geq 1), \{z_i z_j \} \mid i,j \geq 1) \) and \( B = A/(z_1) \). Then \( A/(x^n) = k[x]/(x^n) = B/(x^n) \) and \( A_x = k[x]/x = B_x \) but the surjection \( A \to B \) is not an isomorphism.

**Appendix B. Mayer–Vietoris squares and étale morphisms**

In this appendix, we remove the separatedness assumption that is present in many results of [MB96]. We do this by establishing the gluing of étale and unramified morphisms along certain Mayer–Vietoris squares (Theorem B.7).

Let \( S \) be an algebraic space, let \( Z \hookrightarrow S \) be a finitely presented closed immersion and let \( U = S \setminus Z \) denote its complement. A Mayer–Vietoris square [HR14] App. is a cartesian square of algebraic spaces

\[
\begin{array}{ccc}
U' & \longrightarrow & S' \\
\downarrow f' & & \downarrow f \\
U & \longrightarrow & S
\end{array}
\]

such that

(i) \( f \) is quasi-compact, quasi-separated and representable;

(ii) \( f|_Z \) is an isomorphism; and

(iii) \( Lf^*O_Z \to f^*O_Z \) is an isomorphism, that is, \( Z \hookrightarrow S \) and \( f \) are Tor-independent.

A Mayer–Vietoris square such that \( f \) is affine is the same as a triple \( (S, Z, S') \) satisfying the (TI) condition in the terminology of [MB96] 0.2, 0.6. Note that (iii) is always satisfied if \( f \) is flat.

The main example of a Mayer–Vietoris square is obtained by completions: \( S \) is affine and noetherian and \( S' \) is the completion of \( S \) along \( Z \). Our main theorem in this context is the gluing of algebraic spaces that are locally of finite presentation and quasi-separated.

**Theorem B.1.** Consider a Mayer–Vietoris square as in (B.1) with \( f : S' \to S \) flat and affine, \( S \) locally excellent and \( S' \) locally noetherian. Then the natural functor

\[
\Phi : \text{AlgSp}_{lfp,qp}(S) \to \text{AlgSp}_{lfp,qp}(S') \times \text{AlgSp}_{lfp,qp}(U') \times \text{AlgSp}_{lfp,qp}(U)
\]

is an equivalence of categories.
For separated algebraic spaces (SLFP), this is [MB96, Cor. 5.6]. We then obtain the following generalization of [MB96, Cor. 6.5.1] (which assumes that $X$ has separated diagonal).

**Corollary B.2.** Consider a Mayer–Vietoris square as in (B.1) with $f : S' \to S$ flat and affine, $S$ locally excellent and $S'$ locally noetherian. Let $X$ be an algebraic stack such that $\Delta_X$ is quasi-separated (that is, $\Delta \Delta_X$ is quasi-compact). Then the natural functor

$$X(S) \to X(S') \times_{X(U')} X(U)$$

is an equivalence of groupoids. Equivalently, $S$ is the pushout of $S'$ and $U$ in the 2-category of algebraic stacks with quasi-separated diagonal.

Just as in [MB96] we prove Theorem B.1 by constructing a descent datum along $S' \to S$ via the Mayer–Vietoris square [MB96, Prop. 2.5.2]:

\begin{equation}
\begin{CD}
U' @>>> S' \\
@V{\Delta_{U'}} VV @VV{\Delta_f} V \\
U' \times_U U' @>>> S' \times_S S'.
\end{CD}
\end{equation}

In contrast to the original square, the morphism $\Delta_f$ is typically not flat, nor $S' \times_S S'$ noetherian. A notable exception is when $S' \to S$ is étale; this case is treated in detail in [Ryd11b].

When $f$ is not flat, we have to impose some flatness on the objects instead. As in [MB96, §2] we say that $X \to S$ is $f$-flat if $\text{Tor}_i^S(\mathcal{O}_X, \mathcal{O}_{S'}) = 0$ for all $i > 0$. Recall that

- $X \to S$ is $f$-flat if and only if $X|_{U} \to U$ is $f_U$-flat [MB96, Cor. 3.4.3];
- if $X \to S$ is $f$-flat, then the pull-back of the Mayer–Vietoris square along $X \to S$ is a Mayer–Vietoris square [MB96 Prop. 2.4]; and
- if $X \to S$ is $f$-flat, then $Z \to X$ is $X \times_S S'$-flat if and only if the composition $Z \to X \to S$ is $f$-flat [MB96 Lem. 2.1].

The fundamental result on Mayer–Vietoris squares is that we have an equivalence of categories

$$\text{QCoh}_{f-\text{fl}}(S) \to \text{QCoh}(S') \times_{\text{QCoh}(U')} \text{QCoh}_{f_U-\text{fl}}(U),$$

cf. [HR14d, Thm. A.4] for the general case, [MB96, Thm. 3.1] when $f$ is affine and [FR70, App.] when $f$ is affine and flat.

We will now show that $\pi : \tilde{S} = S' \sqcup_U S \to S$ is universally submersive and that we have an equivalence of categories

$$\text{Et}(S) \to \text{Et}(S') \times_{\text{Et}(U')} \text{Et}(U),$$

at least when $f$ is a closed immersion. When $\pi$ is universally subtrusive, this is an easy consequence of the main result of [Ryd10]. However, $\pi$ is typically not universally subtrusive which makes the situation more delicate.

The following reduction on Mayer–Vietoris squares will be important.

**Remark B.3 (Zariskification).** If $S' = \text{Spec}(A')$ is affine, then we may localize $S'$ in $Z = \text{Spec}(A'/I')$ by taking $S'' = \text{Spec}(A'')$ where $A'' = (1+I')^{-1}A'$. Since $S'' \to S'$ is flat and an isomorphism over $Z$, the resulting triple $(S, Z, S'')$ is a Mayer–Vietoris triple.
Lemma B.4. Consider a Mayer–Vietoris square as in \([B.1]\) and assume that every non-empty closed subset of \(|S|\) meets \(Z\). Then every non-empty closed subset of \(|U|\) meets \(f_U(U')\).

Proof. Let \(W_U \hookrightarrow U\) be a closed subscheme. If \(W_U \cap f_U(U') = \emptyset\), then \(W_U\) is trivially \(f_U\)-flat and we can glue the closed immersions \(W_U \hookrightarrow U\), \(\emptyset \hookrightarrow S'\) to a closed immersion \(W \hookrightarrow S\) \([MB96, Cor. 3.7.2]\). Then \(W \cap Z = \emptyset\), hence \(W = \emptyset\) by the assumption on \(S\). \(\square\)

Proposition B.5. Suppose we are given a Mayer–Vietoris square as in \([B.1]\) with \(f : S' \to S\) affine. Then the morphism \(\pi : \tilde{S} = S' \cap U \to S\) is universally submersive.

Proof. The question is fppf-local on \(S\), so we can assume that \(S = \text{Spec}(A)\) is affine.

As the general case is somewhat technical, let us first prove that \(\pi\) is submersive. Note that \(\pi\) is surjective and quasi-compact, since \(f|_Z\) is an isomorphism. To show that \(\pi\) is submersive, it is enough to prove that a subset \(W \subset |S|\) is closed if \(W|_U\) and \(f^{-1}(W)\) are closed. Since \(\pi\) is quasi-compact and surjective, it follows that \(W\) is pro-constructible. It is thus enough to prove that \(W\) is stable under specializations \([EGA, IV.1.10.1]\). To verify this, we may assume that \(S\) is the spectrum of a local ring with closed point \(s\) and it is enough to prove that if \(W \neq \emptyset\), then \(s \in W\). Since \(S' = \text{Spec}(A')\) is affine, we may also replace \(S'\) with the Zariskification of \(S'\) along \(Z\). We can thus assume that every closed subset of \(S'\) intersects \(Z\).

By assumption \(W \cap U\) is closed. If \(W \cap U = \emptyset\), then \(W = W \cap Z = f^{-1}(W) \cap Z\) is closed. If \(W \cap U \neq \emptyset\), then \(f^{-1}(W) \neq \emptyset\) by Lemma B.4. By assumption \(f^{-1}(W)\) is closed; hence \(s \in W\).

For universal submersiveness, there is a valuative criterion: for every valuation ring \(V\) and morphism \(\text{Spec}(V) \to S\), the base change \(\tilde{S} \times_S \text{Spec}(V) \to \text{Spec}(V)\) is submersive \([Ryd10, Thm. 2.8]\).

To verify this, we may assume that \(S = \text{Spec}(A)\) is local with closed point \(s\) and that \(A \to V\) is local. The open subscheme \(\text{Spec}(V) \times_S U \subset \text{Spec}(V)\) is quasi-compact and hence has a unique closed point \(v_1\). Let \(P \subseteq V\) be the prime ideal corresponding to \(v_1\) and let \(V_1 = V/P\). Then \(V\) is composed of the valuation rings \(V_1\) and \(V_P\) in the sense that \(V = V_1 \times_{\kappa(v_1)} V_P\). Let \(s_1\) be the image of \(v_1\) in \(S\). Let \(K = \kappa(s_1)\) and consider the intersection \(W_1 := V_1 \cap K\) inside \(\kappa(v_1)\). Then \(W_1\) is a valuation ring with fraction field \(K\). Finally let \(W = W_1 \times_K \mathcal{O}_{S,s_1}\). The local ring \(W\) is a semi-valuation ring in the terminology of Temkin and a local ring of the relative Riemann–Zariski space \(\text{RZ}_U(S)\) \([Tem11]\). Note that \(A \to V\) factors via \(W\). Moreover, \(\text{Spec}(W) \to \text{Spec}(S)\) restricts to \(\text{Spec}(\mathcal{O}_{S,s_1}) \to U\) over \(U\) and equals \(\text{Spec}(W_1) \to S_1\) over \(S_1 = \{s_1\}\). In particular, \(\text{Spec}(W) \to \text{Spec}(S)\) is \(f\)-flat so we may replace \(A\) with \(W\).

Now \(U\) is the spectrum of a local ring with closed point \(s_1\) and \(S_1 = \{s_1\}\) is the spectrum of a valuation ring \(W_1\). As before, we may also replace \(S'\) by its Zariskification along \(Z\). By Lemma B.4, there is a point \(s'_1 \in S'\) above \(s_1\). Let \(S'_1 = \{s'_1\}\). Then \(s \in S'_1\) so \(S'_1 \to S_1\) is faithfully flat.
Thus, we have shown that \( f \) is universally submersive over \( S_1 \). Since \( \text{Spec}(V) \) is totally ordered it follows that \( \tilde{S} \times \text{Spec}(V) \to \text{Spec}(V) \) is submersive.

**Lemma B.6.** Consider a Mayer–Vietoris square as in \((\mathcal{B.1})\). Assume that \( S \) and \( U \) are affine and that \( f_U(U') \subseteq |U| \) is closed. If \((S, Z)\) is a henselian pair, then \((U, f_U(U'))\) is a henselian pair.

**Proof.** Let \( E_U \to U \) be an affine étale morphism which is an isomorphism along \( f_U(U')_{\text{red}} \). Then \( E_U \to U \) is an isomorphism over the schematic image of \( f_U \). We may thus glue \( E_U \to U \), \( E_U = U' \), \( E' = S' \) to an affine étale morphism \( E \to S \) \([\text{MB96}]\) Prop. 4.1 (iii)]. Since \( E|_Z \to Z \) is an isomorphism, it follows that \( E \to S \) has a section. Hence, so has \( E_U \to U \) and we have shown that \((U, f_U(U'))\) is a henselian pair \([\text{Ray70a}]\) XI §2. \( \square \)

Let \( \mathcal{E}t(S) \) denote the category of representable étale morphisms \( X \to S \). Similarly, let \( \mathcal{U}nr(S) \) denote the category of representable unramified morphisms. We follow the modern terminology where unramified means locally of finite type and formally unramified. We do not impose any further conditions on the morphisms such as quasi-compactness or quasi-separatedness. We will identify \( \mathcal{E}t(S) \) with the category of sheaves of sets on the small étale site of \( S \).

**Theorem B.7.** Consider a Mayer–Vietoris square as in \((\mathcal{B.1})\). Assume that \( f: S' \to S \) is a closed immersion. Then the natural functors

\[
\Phi_{\mathcal{E}t}: \mathcal{E}t(S) \to \mathcal{E}t(S') \times_{\mathcal{E}t(U')} \mathcal{E}t(U), \quad \text{and} \quad \Phi_{\mathcal{U}nr}: \mathcal{U}nr_{f-\text{flat}}(S) \to \mathcal{U}nr(S') \times_{\mathcal{U}nr(U')} \mathcal{U}nr_{f_U-\text{flat}}(U)
\]

are equivalences of categories.

**Proof.** That both functors are fully faithful follows from Proposition \([\text{B.5}]\) since the diagonal of an unramified morphism is an open immersion. We will now prove that \( \Phi_{\mathcal{E}t} \) is essentially surjective. By fppf-descent, the question is local and we may assume that \( S \) is affine.

Consider the blow-up \( \text{Bl}_Z(S) \to S \). This is a proper f-flat morphism since \( \text{Bl}_Z(S) \times_S U \to U \) is an isomorphism and hence \( f_U \)-flat. The open immersion \( U \to \text{Bl}_Z(S) \) is affine since it is the complement of the exceptional divisor \( E \). It may happen that \( \text{Bl}_Z(S) \to S \) is not surjective, but \( \text{Bl}_Z(S) \bowtie Z \to S \) is surjective and \( f \)-flat. Proper surjective morphisms are morphisms of effective descent for \( \mathcal{E}t \) \([\text{SGA4_2}]\) Exp. VIII, Thm. 9.4], so we may replace \( S \) and \( Z \) with \( \text{Bl}_Z(S) \bowtie Z \) and \( E \bowtie Z \). We may thus assume that both \( S \) and \( U \) are affine.

We will treat \( \mathcal{E}t(S') \times_{\mathcal{E}t(U')} \mathcal{E}t(U) \) as the category of triples \((F', F_U, F_U) \in \mathcal{E}t(S') \times \mathcal{E}t(U') \times \mathcal{E}t(U)\) together with isomorphisms \( i^* F' \cong F_U \cong f_U^* F_U \). Let \( k = i \circ f_U = f \circ i': U' \to S \). The functor \( \Phi_{\mathcal{E}t} \) has a right-adjoint \( \Psi \) that takes a triple \((F', F_U', F_U)\) to the sheaf \( f_U^* F' \times_{k* F_U'} i_* F_U \). Since \( \Phi_{\mathcal{E}t} \) is fully faithful, it is enough to verify that the counit \( \epsilon: \Phi_{\mathcal{E}t} \Psi \to \text{id} \) of the adjunction is an isomorphism.

It is obvious that \( \epsilon_\Psi(F', F_U', F_U) \to F_U \) is an isomorphism since the pull-back of the Mayer–Vietoris triple \((S, Z, S')\) along \( U \to S \) results
in the Mayer–Vietoris triple \((U, \emptyset, U')\). It is thus enough to verify that 
\(\epsilon_f: f^*\Psi(F', F_{U'}, F_U) \to F'\) is an isomorphism.

Let \(s': \text{Spec}(\bar{k}) \to S'\) be a geometric point and let \(s = f \circ s'\). It is 
 enough to verify that \((\epsilon_f)_{s'}\) is an isomorphism. To verify this, we may 
replace \(S = \text{Spec}(A)\) with the strict henselization at \(s\). Then \((S', s')\) is 
also henselian as is \((U, U')\) by Lemma B.3 and \((\epsilon_f)_{s'} = (\epsilon_f)(S')\). We note 
that \((f^*\Psi(F', F_{U'}, F_U))(S') = \Psi(F', F_{U'}, F_U)(S) = F'(S') \times_{F'(U')} F_U(U) = 
F'(S)\) since \(S, Z\), \((S', Z)\) and \((U, U')\) are henselian pairs. This proves that 
\((\epsilon_f)_{s'}\) is an isomorphism; hence we have established that the counit is an 
isomorphism and that \(\Phi_{\text{Et}}\) is an equivalence.

To see that \(\Phi_{\text{Unr}}\) is an equivalence, we recall that every unramified 
morphism has a canonical factorization as a closed immersion followed by an 
étale morphism \[\text{[Ryd11a]}. \] The result then follows from the gluings of \(f\)-flat 
closed immersions \[\text{[MB96 Cor. 3.7.2]} \] and the étale case. \(\square\)

In particular, Theorem B.7 allows us to glue monomorphisms that are 
locally of finite type. We also need the following special case of Theorem 
B.1 (for which no noetherian hypotheses are needed) that allow us to 
glomer monomorphisms when \(f\) is flat. We let \(\text{QAff}(S)\) denote the category 
of quasi-affine morphisms \(X \to S\) and let \(\text{LQF}_{\text{sep}}(S)\) denote the category 
of locally quasi-finite and separated morphisms \(X \to S\).

**Lemma B.8.** Consider a Mayer–Vietoris square as in \(\text{[B.1]}\). Assume that 
\(f: S' \to S\) is flat and affine. Then the natural functors 
\[
\Phi_{\text{QAff}} \colon \text{QAff}_{f-\text{fl}}(S) \to \text{QAff}(S') \times_{\text{QAff}(U')} \text{QAff}_{f_{U'}-\text{fl}}(U)
\]
\[
\Phi_{\text{LQF}_{\text{sep}}} \colon \text{LQF}_{\text{sep}, f-\text{fl}}(S) \to \text{LQF}_{\text{sep}}(S') \times_{\text{LQF}_{\text{sep}}(U')} \text{LQF}_{\text{sep}, f_{U'}-\text{fl}}(U)
\]
are equivalences of categories.

**Proof.** That \(\Phi_{\text{QAff}}\) is an equivalence is \[\text{[MB96 Thm. 1.1]}\]. Fully faithfulness 
of \(\Phi_{\text{LQF}_{\text{sep}}}\) follows from \[\text{[MB96 Cor. 3.7.2 and Prop. 4.1 (v)]}\]. To show that 
\(\Phi_{\text{LQF}_{\text{sep}}}\) is essentially surjective, it is enough, by \[\text{[MB96 Prop. 5.7.1]}\], to 
prove that it is essentially surjective on quasi-compact objects. Since a quasi-
finite separated morphism is quasi-affine \[\text{[LM93 A.2]}\], the result follows. \(\square\)

We may now remove the separatedness assumption in \[\text{[MB96 Thm. 4.5]}\] 
at the expense of some additional hypotheses.

**Corollary B.9.** Consider a Mayer–Vietoris square as in \(\text{[B.1]}\). Assume that 
\(f: S' \to S\) is either a closed immersion or flat and affine. Then the 
natural functor 
\[
\Phi_{\text{AlgSp}} \colon \text{AlgSp}_{f-\text{fl}}(S) \to \text{AlgSp}(S') \times_{\text{AlgSp}(U')} \text{AlgSp}_{f_{U'}-\text{fl}}(U)
\]
is fully faithful.

**Proof.** Let \(X\) and \(Y\) be \(f\)-flat algebraic spaces over \(S\). We may assume that 
\(X = S\). Any section \(S \to Y\) is then an \(f \times_S Y\)-flat monomorphism, 
locally of finite type. The result thus follows from Theorem B.7 (when \(f\) is a closed 
immersion) and Lemma B.8 (when \(f\) is flat and affine) together with \[\text{[MB96 Prop. 4.1 (v)]}\]. \(\square\)
Proof of Theorem B.1. That the functor is fully faithful is a special case of Corollary B.9. By [MB96, Prop. 5.7.1], it is enough to prove that the functor is essentially surjective on the subcategory of finitely presented morphisms. Thus, let $X' \to S'$ and $X_U \to U$ be representable morphisms of finite presentation.

The diagonal of $f$ fits in the Mayer–Vietoris square (B.2). Since $\Delta f$ is a closed immersion, we may apply Corollary B.9 to obtain a gluing datum of $X' \to S'$ along $S' \to S$. Effectivity of the descent datum then follows as in the proof of [MB96, Prop. 5.6 (iii)] by reduction to fppf descent using Néron–Popescu desingularization.

Proof of Corollary B.2. Reason as in [MB96, Cor. 6.5.1] using Theorem B.1 to avoid the hypothesis that $X$ has separated diagonal.

Remark B.10. We proved Corollary B.9 in the non-flat case using a descent result for unramified morphisms. As monomorphisms of finite type are quasi-finite and separated, another approach is to prove a descent result for quasi-affine morphisms as in the flat case. A problem with this approach is that the affine hull does not commute with non-flat morphisms and hence does not glue. A potential solution is to instead work with the derived affine hull which does commute with pull-back (in the Tor-independent setting that we always assume). Derived objects can be glued in Mayer–Vietoris squares, cf. [HR14d, Lem. 5.6] or [Bha14, Thm. 1.4 (2)]. The derived affine hull is, however, not connective so one would probably need to generalize Proposition B.5 to Mayer–Vietoris squares of non-connective derived schemes.

Remark B.11. Another recent proof of Corollary B.9, valid without assumptions on $f$, is given in [Bha14, Thm. 1.4 (1)]. It is based on derived Tannaka duality for algebraic spaces [Bha14, Thm. 1.5].

Appendix C. A boundedness result for Weil restrictions

In this section we prove the following boundedness result for Weil restrictions (or equivalently, for Hom stacks).

Theorem C.1. Let $f : Z \to S$ be a proper, flat and finitely presented morphism of algebraic stacks. Let $X \to Z$ be a representable morphism of finite presentation with affine fibers. Then the Weil restriction $f_* X = R_{Z/S}(X) \to S$ is of finite presentation. If in addition $X \to Z$ is a group, then $f_* X \to S$ is a group with affine fibers.

Theorem C.1 is used in Theorem 1.2 (resp. 1.3) to establish the quasi-compactness of the diagonal of Hom-stacks (resp. Weil restrictions). Without using Theorem C.1 the proofs give the algebraicity of Hom-stacks (resp. Weil restrictions) and that the diagonal is (quasi-)affine when the diagonal of $X \to S$ (resp. $X \to Z$) is (quasi-)affine. In the setting of Theorem C.1 $X \to Z$ is representable and hence has quasi-affine diagonal. Thus, we can conclude that $f_* X \to S$ is representable, quasi-separated and locally of finite presentation. It remains to prove that $f_* X \to S$ is quasi-compact.

As the question is local on $S$, we may assume that $S$ is an affine scheme. By standard approximation results, we may assume that $S$ is of finite type.
over Spec \( Z \). For the remainder of the appendix, all stacks will be of finite presentation over \( S \) and hence excellent with finite normalization etc.

We fix once and for all a proper flat morphism \( f: Z \to S \) and a representable morphism \( X \to Z \) of finite presentation. We abbreviate \( R_{Z/S} = R_{Z/S}(X) \). We will give three reduction results of increasing generality for which we do not assume that \( X \to Z \) has affine fibers.

By noetherian induction on \( S \), to prove that \( R_{Z/S}(X) \) is quasi-compact, we may assume that \( S \) is integral and replace \( S \) with a suitable dense open subscheme. Moreover, we may also replace \( Z \to S \) with the pull-back along a dominant map \( S' \to S \). Recall that there exists a field extension \( K' \to K(S) \) such that \((Z_K')_{\text{red}} \) (resp. \((Z_K')_{\text{norm}} \)) is geometrically reduced (resp. geometrically normal) over \( K' \). After replacing \( S \) with a dense open subset of the normalization in \( K' \), we may thus assume that

\begin{enumerate}[(i)]  
  \item \( Z_{\text{red}} \to S \) is flat with geometrically reduced fibers; and  
  \item \( Z_{\text{norm}} \to S \) is flat with geometrically normal fibers;
\end{enumerate}

since these properties are constructible [EGA IV.9.7.7 (iii) and 9.9.4 (iii)].

Our first reduction result is similar to [Ols06b, Lem. 5.11].

**Proposition C.2.** Let \( Z_0 \to Z \) be a square-zero nil-immersion of algebraic stacks that are flat and proper over \( S \). If \( R_{Z_0/S} \) is quasi-compact, then so is \( R_{Z/S} \).

**Proof.** There is a natural morphism \( \pi: R_{Z/S} \to R_{Z_0/S} \) and it is enough to prove that this morphism is quasi-compact. Let \( T \) be an affine scheme and let \( T \to R_{Z_0/S} \) be a morphism. This corresponds to a \( Z \)-morphism \( s_0: Z_0 \times_S T \to X \). The pull-back of \( \pi \) to \( T = \text{Spec } A \) is the functor that takes a morphism \( T' \to T \) of affine schemes to lifts \( s \) in the following diagram

\[
\begin{array}{ccc}
Z_0 \times_S T' & \xrightarrow{s_0} & X \\
\downarrow & & \downarrow \\
Z \times_S T' & \xrightarrow{s} & Z
\end{array}
\]

By Illusie, the obstruction to the existence of a lift \( s \) lies in \( O(T') = E^1(T') := \text{Ext}^1_{O_{Z_0 \times_S T'}}(L(s_0)^*L_{X/Z}, \mathcal{I}_T) \) and if the obstruction vanishes, the set of lifts is a torsor under \( D(T') = E^0(T') := \text{Ext}^0_{O_{Z_0 \times_S T'}}(L(s_0)^*L_{X/Z}, \mathcal{I}_T) \) (apply [Ols06a] Thm. 1.5) with \( J = K = 0 \). Here \( \mathcal{I}_T \) denotes the ideal of \( Z_0 \times_S T' \hookrightarrow Z \times_S T' \), which by flatness of \( Z_0 \to S \) equals the pull-back of the ideal \( I \) defining \( Z_0 \to \emptyset \).

To show that \( \pi \) is quasi-compact, we may assume that \( T \) is integral and it is enough to prove that \( \pi \) is quasi-compact after replacing \( T \) with a dense open subscheme. We may thus assume that \( E^j(T) \) are vector bundles for \( j = 0, 1 \) and that \( E^j(T) \otimes_A A' \cong E^j(T') \), for every morphism \( T' = \text{Spec } A' \to T = \text{Spec } A \) (by the Semi-continuity Theorem [Hal14a] Thm. A) applied to the generic point of \( T \). Let \( o \in O(T) \) be the obstruction. If \( o \neq 0 \), then the situation is obstructed for all \( T' \to T \) and \( R_{Z/S} \times_{R_{Z_0/S}} T = \emptyset \). If \( o = 0 \), then the situation is unobstructed and the restriction of \( \pi \) to \( T \)
is (non-canonically) isomorphic to the vector bundle Spec(Sym$(E^0(T)^\vee)$), hence quasi-compact.

**Corollary C.3.** If $R_{Z'/S'}$ is quasi-compact for every scheme $S'$, morphism $S' \to S$ and nil-immersion $Z' \to Z \times_S S'$ such that $Z' \to S'$ is flat with geometrically reduced fibers, then $R_{Z/S}$ is quasi-compact.

**Proof.** Assume that the condition holds. To prove that $R_{Z/S}$ is quasi-compact, we may assume that $S$ is integral. We may also assume that $Z_{\text{red}} \to S$ has geometrically reduced fibers. Pick a sequence of square-zero nil-immersions $Z_{\text{red}} = Z_0 \hookrightarrow Z_1 \hookrightarrow \ldots \hookrightarrow Z_n = Z$. After replacing $S$ with a dense open subset, we may assume that all the $Z_i \to S$ are flat. The corollary then follows from Proposition C.2.

The second reduction is for a (partial) normalization:

**Proposition C.4.** Let $W \to Z$ be a finite birational morphism between reduced algebraic stacks over $S$. Let $Z_0 \hookrightarrow Z$ and $W_0 \hookrightarrow W$ be the closed substacks defined by the conductor ideal of $W \to Z$. Assume that $W$, $Z$, $W_0$ and $Z_0$ are flat over $S$. Then $R_{Z/S}$ is quasi-compact if $R_{W/S}$ and $R_{Z_0/S}$ are quasi-compact.

**Proof.** The square

\[
\begin{array}{ccc}
W_0 & \to & W \\
\downarrow & & \downarrow \\
Z_0 & \to & Z
\end{array}
\]

is a bicartesian square and remains so after arbitrary base change over $S$ since $W_0 \to S$ is flat. This gives a cartesian square

\[
\begin{array}{ccc}
R_{W_0/S} & \leftarrow & R_{W/S} \\
\uparrow & & \uparrow \\
R_{Z_0/S} & \leftarrow & R_{Z/S}
\end{array}
\]

Since $R_{Z_0/S}$ and $R_{W/S}$ are quasi-compact, so is $R_{Z/S}$.

**Corollary C.5.** If $R_{Z'/S'}$ is quasi-compact for every scheme $S'$, morphism $S' \to S$ and finite morphism $Z' \to Z \times_S S'$ such that $Z' \to S'$ is flat with geometrically normal fibers, then $R_{Z/S}$ is quasi-compact.

**Proof.** By Corollary C.3, we may assume that $Z \to S$ is flat with geometrically reduced fibers. We will use induction on the maximal fiber dimension $d$ of $Z \to S$. After modifying $S$, we may assume that $W := Z_{\text{norm}} \to S$ is flat with geometrically normal fibers. Let $Z_0 \hookrightarrow Z$ and $W_0 \hookrightarrow W$ be the closed substacks given by the conductor ideal of $W \to Z$.

After replacing $S$ with a dense open subset, we may assume that $Z_0 \to S$ and $W_0 \to S$ are flat and that $W \to Z$ is an isomorphism over an open subset $U \subseteq Z$ that is dense in every fiber. In particular, since $Z_0 \cap U = \emptyset$, the dimensions of the fibers of $Z_0 \to S$ are strictly smaller than $d$. Thus, by induction we may assume that $R_{Z_0/S}$ is quasi-compact. It follows that $R_{Z/S}$ is quasi-compact by Proposition C.4.
Lemma C.6. Let $S$ be an algebraic stack, let $T$ be an algebraic $S$-stack and let $g: T' \to T$ be a universally subtrusive (e.g., proper and surjective) morphism of finite presentation such that $g$ is flat over an open substack $U \subseteq T$. If $T$ is weakly normal in $U$ (e.g., $T$ normal and $U$ open dense), then for every representable morphism $X \to S$, the following sequence is exact:

$$
\begin{CD}
X(T) @>>> X(T') @>>> X(T' \times_T T)
\end{CD}
$$

where $X(T) = \text{Hom}_S(T, X)$ etc.

Proof. It is enough to prove that given a morphism $f: T' \to X$ such that $f \circ \pi_1 = f \circ \pi_2: T' \times_T T' \to X$, there exists a unique morphism $h: T \to X$ such that $f = h \circ g$. By fpqc-descent over $U$, there is a unique $h|_U: U \to X$ such that $f|_{g^{-1}(U)} = h|_U \circ g|_{g^{-1}(U)}$. Consider the morphism $\tilde{g}: \tilde{T}' = T' \amalg_U T \to X$. The morphism $\tilde{f} = (f, h|_U): \tilde{T}' \to X$ satisfies $\tilde{f} \circ \tilde{\pi}_1 = \tilde{f} \circ \tilde{\pi}_2$ where $\tilde{\pi}_i$ denotes the projections of $\tilde{T}' \times_X \tilde{T}' \to \tilde{T}'$. By assumption, $\tilde{g}$ is universally subtrusive and weakly normal. Thus, by $h$-descent ([Ryd10], Thm. 7.4), we have an exact sequence

$$
\begin{CD}
X(T) @>>> X(\tilde{T}') @>>> X((\tilde{T}' \times_T \tilde{T}')_{\text{red}}).
\end{CD}
$$

Indeed, by smooth descent we can assume that $S$, $T$ and $\tilde{T}'$ are schemes so that ([Ryd10], Thm. 7.4) applies. We conclude that $\tilde{f}$ comes from a unique morphism $h: T \to X$.

Our last general reduction result generalizes ([Ols06b], Lem. 5.13) (which requires $X \to Z$ to be finite).

Proposition C.7. Let $W \to Z$ be a proper surjective morphism over $S$. Assume that $Z \to S$ is flat with geometrically normal fibers. If $R_{W/S}$ is quasi-compact, then so is $R_{Z/S}$.

Proof. We may assume that $S$ is an integral scheme. After replacing $S$ with an open subscheme, we may also assume that $W \to Z$ is flat over an open subset $U \subseteq Z$ that is dense in every fiber over $S$. Consider the sequence

$$
\begin{CD}
R_{Z/S} @>>> R_{W/S} @>>> R_{W \times_Z W/S}
\end{CD}
$$

There is a canonical morphism $\varphi: R_{Z/S} \to E$ where $E$ denotes the equalizer of the parallel arrows. Since $R_{W/S}$ is quasi-compact (and $R_{W \times_Z W/S}$ is quasi-separated), the equalizer $E$ is quasi-compact. It is thus enough to show that $\varphi$ is quasi-compact. Thus, pick a scheme $T$ and a morphism $T \to E$ and let us show that $R_{Z/S} \times_E T$ is quasi-compact.

By noetherian induction on $T$, we may assume that $T$ is normal. The morphism $T \to E$ gives an element of $\text{Hom}_Z(W \times_S T, X)$ such that the two images in $\text{Hom}_Z(W \times_Z W \times_S T, X)$ coincide. Noting that $Z \times_S T$ is normal, the lemma applies and gives a unique element in $\text{Hom}_Z(Z \times_S T, X) = \text{Hom}_S(T, R_{Z/S})$. Thus, $R_{Z/S} \times_E T \to T$ has a section. Repeating the argument with $T = \text{Spec}\ k(t)$ for every point $t \in T$, we see that $R_{Z/S} \times_E T \to T$ is injective so the section is surjective. It follows that $R_{Z/S} \times_E T$ is quasi-compact.  

\[\square\]
Proof of Theorem [C.7]. As usual, we may assume that \( S \) is an affine integral scheme. By Corollary C.5 we may in addition assume that \( Z \to S \) has geometrically normal fibers. Let \( W \to Z \) be a proper surjective morphism with \( W \) a projective \( S \)-scheme [Ols05]. By replacing \( S \) with a dense open, we may assume that \( W \to S \) is flat. By Proposition C.7, we may replace \( Z \) with \( W \) and assume that \( Z \) is a (projective) scheme. Repeating the first reduction, we may still assume that \( Z \to S \) has geometrically normal fibers.

Let \( Y = \text{Spec}_{Z}(g_{*}O_{X}) \) where \( g : X \to Z \) and let \( X \to Y \to Z \) be the induced factorization. Since \( X \to Z \) has affine fibers \( X \to Y \) is an isomorphism over an open dense subset \( U \subseteq Z \). After replacing \( S \) with a dense open subscheme, we may assume that \( U \) is dense in every fiber over \( S \). Since \( R_{Z/S}(Y) \to S \) is affine, it is enough to prove that \( R_{Z/S}(X) \to R_{Z/S}(Y) \) is quasi-compact. We may thus replace \( X, Z, U \) and \( S \) with \( X \times_{Y} (Z \times_{S} R_{Z/S}(Y)), Z \times_{S} R_{Z/S}(Y), U \times_{S} R_{Z/S}(Y) \) and \( R_{Z/S}(Y) \). We may thus assume that \( X \to Z \) is an isomorphism over \( U \).

Since \( X \) is an algebraic space, there exists a finite surjective morphism \( X' \to X \) such that \( X' \) is a scheme. In particular, there is a finite field extension \( L/K(U) \) such that the normalization of \( X' \) in \( L \) is a scheme. Take a splitting field \( L'/L \) and let \( Z' \) be the normalization of \( Z \) in \( L' \). Then \( X' := (X \times Z Z')_{\text{norm}} = X_{\text{norm}/L'} \) is a scheme. By replacing \( S \) with a normalization in an extension of \( K(S) \) and shrinking, we may assume that \( Z' \to S \) and \( X' \to S \) are flat with geometrically normal fibers. By Proposition C.7, it is enough to prove that \( R_{Z'/S}(X \times_{Z} Z') \) is quasi-compact.

There is a natural morphism

\[
R_{Z'/S}(X') \to R_{Z'/S}(X \times_{Z} Z').
\]

We claim that this morphism is surjective. To see this, we may assume that \( S \) is a field. Then \( Z' \) and \( X' \) are normal and any section \( Z' \to X \times_{Z} Z' \) lifts uniquely to a section \( Z' \to X' \). Indeed, \( Z' \times_{X \times_{Z} Z'} X' \to Z' \) is finite and an isomorphism over \( U \), hence has a canonical section. We can thus replace \( X \) and \( Z \) with \( X' \) and \( Z' \) and assume that \( X \) is a scheme.

Since \( X \) is a scheme, and hence locally separated, by Raynaud–Gruson there is a blow-up \( Z' \to Z \) such that the strict transform \( X' \to Z' \) of \( X \to Z \) is étale [RG71, Thm. 5.7.11]. After shrinking \( S \), we may assume that \( Z' \to S \) is flat. Then since \( U \subseteq Z' \) remains dense after arbitrary pull-back over \( S \), we have that \( R_{Z'/S}(X \times_{Z} Z') = R_{Z'/S}(X') \). Using Proposition C.7 and replacing \( X \to Z \) with \( X' \to Z' \), we may thus assume that \( X \to Z \) in addition is étale.

Finally, we note that the étale morphism \( X \to Z \) corresponds to a constructible sheaf and that \( R_{Z/S}(X) \) is nothing but the étale sheaf \( f_{*}X \), which is constructible by a special case of the proper base change theorem [SGA4\(_{3}\), XIV.1.1]. That is, the étale morphism \( R_{Z/S}(X) \to S \) is of finite presentation.

For the second part of the theorem on groups, we may assume that \( S \) is the spectrum of a field \( k \). By the first part \( R_{Z/S}(X) \) is then a group scheme \( G \) of finite type over \( k \). Let \( H = G_{\text{ant}} \) be the largest anti-affine subgroup of \( G \). It is normal, connected and smooth and the quotient \( G/H \) is affine [DG70, §III.3.8].
The counit $G \times_k Z \to X$ of the adjunction $(f^*, f_*)$ is a group homomorphism and induces a group homomorphism $H \times_k Z \to X$. It is enough to show that this factors through the unit section of $X \to Z$, because this forces $H = 0$ by adjunction.

Note that for every stack $W \to \text{Spec} k$, the pull-back $H \times_k W \to W$ is an anti-affine group in the sense that the push-forward of $\mathcal{O}_{H \times_k W}$ is $\mathcal{O}_W$ by flat base change. Since $X \to Z$ has affine fibers, there is a finitely presented filtration $(Z_i)$ of $Z$ with strata $Y_i^{[n]}$ over which $X \times_Z Y_i^{[n]} \to Y_i^{[n]}$ is affine. Since $H \times_k Y_i^{[n]}$ is anti-affine, it follows that the restricted counit $H \times_k Y_i^{[n]} \to X \times_Z Y_i^{[n]}$ factors through the unit section $Y_i^{[n]} \to X \times_Z Y_i^{[n]}$.

Let $E$ be the equalizer of $H \times_k Z \to X$ and the constant map $H \times_k Z \to Z \to X$. The above discussion shows that the monomorphism $E \to H \times_k Z$ is an isomorphism over every strata $Y_i^{[n]}$, hence an isomorphism (Proposition A.3) using that the filtration is separating since $Z$ is noetherian.

Let us conclude with a brief discussion on how the methods of this appendix can be used to extend [Ols07a]. The following theorem generalizes both Theorem C.1 and [Ols07a] Thm. 1.1. Indeed, the latter is the special case where $X \to Z$ is quasi-finite and proper and $Z \to S$ is Deligne–Mumford.

**Theorem C.8.** Let $f: Z \to S$ be a proper, flat and finitely presented morphism of algebraic stacks. Let $X \to Z$ be a morphism of finite presentation that has unramified and finite inertia. Moreover, assume that the relative coarse moduli space $X_{\text{cms}} \to Z$ has affine fibers. Then the Weil restriction $f_*X = R_{Z/S}(X) \to S$ is of finite presentation.

The results of C.2–C.5 hold as stated even if $X \to Z$ is not representable. Lemma C.6 holds for $X \to S$ relatively Deligne–Mumford with the modification that the conclusion is that $T' \to T$ is a morphism of effective descent for the fibered category $X$ (the proof of [Ryd10] Thm. 7.4) is via descent of étale morphisms and goes through verbatim). Proposition C.7 holds for $X \to Z$ relatively Deligne–Mumford: in the proof one replaces the equalizer $E$ with the stack of proper descent data $\text{Des}(W/Z)$ as in [Ols07a] 4.2 and it follows that $R_{Z/S} \to \text{Des}(W/Z)$ is an isomorphism over normal schemes by the lemma.

The proof of Theorem C.8 now proceeds as the proof of Theorem C.1 with the following modifications. We can only arrange so that $X|_U \to U$ is a coarse moduli space and not an isomorphism. We can replace $X$ with its normalization and assume that $X$ is normal. Then we have a rigidification $X \to Y \to Z$ where $X \to Y$ is a gerbe and $Y \to Z$ is an orbifold. If we first treat the case when $X = Y$ is an orbifold, then we may use the same argument as when $X$ was an algebraic space to reduce to $X$ a scheme and then to $X \to Z$ étale. We can thus assume that $X \to Z$ is a proper étale gerbe. By replacing $Z$ with a normalization in an extension of $K(Z)$, we may assume that $X \to Z$ admits a section.

We complete the argument when $X \to Z$ is a proper étale gerbe admitting a section as in [Ols07a] Prop. 3.1 but correcting one small error. We apply proper base change instead of smooth base change (the latter only applies if the gerbe is tame). The sheaves $R^1f_*G$ etc are then only constructible and
not locally constant but this is not a problem: after passing to a dense open subset of $S$, all constructible sheaves in question becomes locally constant. With this modification of the proof, we do not need to assume that $Z \to S$ is smooth and we do not need to use alteration of singularities (as in [Ols07a, 6.4]).

We expect that the assumption in Theorem C.8 on unramified inertia can be replaced with tameness as in [AOV11, Thm. C.2 (ii)].

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