Rainbow vertex pair-pancyclicity of strongly edge-colored graphs

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An edge-colored graph is rainbow if no two edges of the graph have the same color. An edge-colored graph $G^c$ is called properly colored if every two adjacent edges of $G^c$ receive distinct colors in $G^c$. A strongly edge-colored graph is a proper edge-colored graph such that every path of length 3 is rainbow. We call an edge-colored graph $G^c$ rainbow vertex pair-pancyclic if any two vertices in $G^c$ are contained in a rainbow cycle of length $\ell$ for each $3 \leq \ell \leq n$. In this paper, we show that every strongly edge-colored graph $G^c$ of order $n$ with minimum degree $\delta \geq \frac{2n}{3} + 1$ is rainbow vertex pair-pancyclicity.

Keywords: edge-coloring; strongly edge-colored graph; rainbow cycle; rainbow vertex pair-pancyclicity.

1 Introduction

In this paper, we only consider finite, undirected and simple graphs. Let $G$ be a graph consisting of a vertex set $V(G)$ and an edge set $E = E(G)$. We use $d(v)$ to denote the number of edges incident with vertex $v$ in $G$. A strongly edge-colored graph is a proper edge-colored graph such that every path of length 3 is rainbow. We call an edge-colored graph $G^c$ rainbow vertex pair-pancyclic if any two vertices in $G^c$ are contained in a rainbow cycle of length $\ell$ for each $3 \leq \ell \leq n$. In this paper, we show that every strongly edge-colored graph $G^c$ of order $n$ with minimum degree $\delta \geq \frac{2n}{3} + 1$ is rainbow vertex pair-pancyclicity.

Keywords: edge-coloring; strongly edge-colored graph; rainbow cycle; rainbow vertex pair-pancyclicity.

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The classical Dirac’s theorem states that every graph $G$ is Hamiltonian if $\delta(G) \geq \frac{n}{2}$. Inspired by this famous theorem, Hendry (1990) show that every graph $G$ of order $n$ with minimum degree $\delta \geq \frac{n+1}{2}$ is vertex-pancyclic. During the past few decades, the existence of cycles in graphs have been extensively studied in the literatures. We recommend Abouelaoualim et al. (2010); Chen (2018); Chen and Li (2021, 2022); Chen et al. (2019); Czygrinow et al. (2021); Ehard and Mohr (2020); Fujita et al. (2019); Guo et al. (2022); Kano and Li (2008); Li et al. (2022) for more results.

For edge-colored graphs, Lo (2014) proved the following asymptotic theorem about properly colored cycles.

**Theorem 1.1 (Lo (2014))** For any $\varepsilon > 0$, there exists an integer $n_0$ such that every edge-colored graph $G^c$ with $n$ vertices and $\delta^c(G) \geq \left(\frac{2}{3} + \varepsilon\right)n$ and $n \geq n_0$ contains a properly edge-colored cycle of length $l$ for all $3 \leq l \leq n$, where $\delta^c(G)$ is the minimum number of distinct colors of edges incident with a vertex in $G^c$.

Cheng et al. (2019) considered the existence of rainbow Hamiltonian cycles in strongly edge-colored graph and proposed the following two conjectures.

**Conjecture 1.2 (Cheng et al. (2019))** Every strongly edge-colored graph $G^c$ with $n$ vertices and degree at least $\frac{n+1}{2}$ has a rainbow Hamiltonian cycle.

**Conjecture 1.3 (Cheng et al. (2019))** Every strongly edge-colored graph $G^c$ with $n$ vertices and degree at least $\frac{n}{2}$ has a rainbow Hamiltonian path.

To support the above two conjectures, they presented the following theorem.

**Theorem 1.4 (Cheng et al. (2019))** Let $G^c$ be a strongly edge-colored graph with minimum degree $\delta$, if $\delta \geq \frac{2|V(G)|}{3}$, then $G^c$ has a rainbow Hamiltonian cycle.

Wang and Qian (2021) showed that every strongly edge-colored graph $G^c$ on $n$ vertices is rainbow vertex-pancyclic if $\delta \geq \frac{2n+1}{3}$. Li and Li (2022) further considered the rainbow edge-pancyclicity of strongly edge-colored graphs and proposed the following theorem.

**Theorem 1.5 (Li and Li (2022))** Let $G^c$ be a strongly edge-colored graph on $n$ vertices. If $\delta(G^c) \geq \frac{2n+1}{3}$, then $G^c$ is rainbow edge-pancyclic. Furthermore, for every edge $e$ of $G^c$, one can find a rainbow $l$-cycle containing $e$ for each $3 \leq l \leq n$ in polynomial time.

In this paper, we consider the rainbow vertex pair-pancyclicity of strongly edge-colored graph. Our main result is as follows.

**Theorem 1.6** Let $G^c$ be a strongly edge-colored graph with $n$ vertices and minimum degree $\delta$. If $\delta \geq \frac{2n}{3} + 1$, then $G^c$ is rainbow vertex pair-pancyclicity.

## 2 Proof of Theorem 1.6

First, we introduce some useful notations. Given a rainbow cycle $C$ in graph $G^c$, a color $s$ is called a $C$-color (resp., $\overline{C}$-color) if $s \in c(C)$ (resp., $s \notin c(C)$). Correspondingly, we call an edge $e$ a $C$-color edge (resp., $\overline{C}$-color edge) if $c(e) \in c(C)$ (resp., $c(e) \notin c(C)$). Two adjacent vertices $u$ and $v$ are called $C$-adjacent (resp., $\overline{C}$-adjacent) if $c(uv) \in c(C)$ (resp., $c(uv) \notin c(C)$). For two disjoint adjacent subsets $V_1$ and $V_2$ of $V(G)$, let $E(V_1, V_2)$ denote the set of edges between $V_1$ and $V_2$. We denote the subsets
of $E(V_1, V_2)$ consisting of the $C$-color edges (resp., $\bar{C}$-color edges) by $E_C(V_1, V_2)$ (resp., $E_{\bar{C}}(V_1, V_2)$).

Similarly, for two subgraphs $H_1$ and $H_2$, we denote the set of $C$-color edges (resp., $\bar{C}$-color edges) between $V(H_1)$ and $V(H_2)$ by $E_C(H_1, H_2)$ (resp., $E_{\bar{C}}(H_1, H_2)$). For any two vertices $v_i$ and $v_j$ of cycle $C = v_1v_2 \ldots v_i v_1$, we identify the two subscripts $i$ and $j$ if $i \equiv j \pmod{l}$. Let $v_i C^+ v_j$ be the path $v_i v_{i+1} \ldots v_j v_j$ and $v_i C^- v_j$ the path $v_i v_{i-1} \ldots v_j v_j$, respectively. For any vertex $v \in V(G^c)$, let $CN(v)$ be the set of colors used by the edges incident with $v$.

From the definition of strongly edge-coloring, we can easily get the following observation.

**Observation 2.1** Each cycle of length at most 5 in a strongly edge-colored graph is rainbow.

**Proof of Theorem 1.6**: Recall that the colors on the edges incident with $v$ are pairwise distinct for each vertex $v$ of a strongly edge-colored graph. So we do not distinguish the colors of adjacent edges in the following. If $n \leq 8$, $G$ is complete since $\delta \geq \frac{2n}{3} + 1$, and so the result clearly holds. Thus we suppose that $n \geq 9$. Let $a$ and $b$ be two arbitrary vertices of $G$. If $a$ and $b$ are adjacent, then $a$ and $b$ are contained in a rainbow cycle of length $l$ for each $l$ with $3 \leq l \leq n$ from Theorem 1.5. So we consider that $a$ and $b$ are not adjacent. Since $\delta \geq \frac{2n}{3} + 1$, we have that $a$ and $b$ are contained in a 4-cycle which is rainbow from Observation 2.1. Suppose to the contrary that the result is not true. Then there is an integer $l$ with $4 \leq l \leq n - 1$ such that there is a rainbow $l$-cycle containing $a$ and $b$, but there is no rainbow $(l + 1)$-cycle containing both $a$ and $b$. Let $C := v_1 v_2 \ldots v_l v_1$ be a rainbow $l$-cycle containing $a$ and $b$.

Without loss of generality, we assume that $c(v_i v_{i+1}) = i$ for $1 \leq i \leq l$. For $1 \leq i \leq l$, let $N_i$ be the set of the vertices of $C$ which are adjacent to $v_i$, that is, $N_i = N(v_i) \cap V(C)$. We then proof the following claim.

**Claim 1** $l \geq \frac{4n + 12}{3}$. In particular, $l \geq 7$ when $n \geq 9$.

**Proof.** Since $G^c$ is strongly edge-colored, for any $v_i \in N_1$, the color $j$ does not occur in $CN(v_1)$. So the number of $C$-colors not contained in $CN(v_1)$ is at least $|N_1| - 1$, and therefore, the number of $C$-colors contained in $CN(v_1)$ is at most $l - (|N_1| - 1)$. Since $1$ and $l$ are $C$-colors in $CN(v_1)$, we have that the number of $C$-colors contained in $E(v_1, V(G) \setminus V(C))$ is at most $l - (|N_1| - 1) = l - |N_1| - 1$. Hence, we have $|E_C(v_1, V(G) \setminus V(C))| \leq l - |N_1| - 1$. Since $|E(v_1, V(G) \setminus V(C))| \geq \delta - |N_1|$, we have that

$$|E_C(v_1, V(G) \setminus V(C))| = |E(v_1, V(G) \setminus V(C))| - |E_C(v_1, V(G) \setminus V(C))|$$

$$\geq (\delta - |N_1|) - (l - |N_1| - 1)$$

$$= \delta - l + 1.$$

Similarly, we can also deduce that $|E_C(v_i, V(G) \setminus V(C))| \geq \delta - l + 1$ for all $1 \leq i \leq l$. For any two vertices $v_i$ and $v_{i+1}$ with $1 \leq i \leq l$, if there exists a vertex $w \in V(G) \setminus V(C)$ such that both $v_i w$ and $v_{i+1} w$ are $C$-color edges, then both $a$ and $b$ are contained in a rainbow $(l + 1)$-cycle $C' := v_i w v_{i+1} C^+ v_i$, a contradiction. Thus, for any common neighbor $w \in V(G) \setminus V(C)$ of $v_i$ and $v_{i+1}$, either $v_i w$ or $v_{i+1} w$ is not a $\bar{C}$-color edge. Then we have that $|E_C(v_i, w)| + |E_C(v_{i+1}, w)| \leq 1$. Therefore, we have

$$n \geq |E_C(v_i, V(G) \setminus V(C))| + |E_C(v_{i+1}, V(G) \setminus V(C))| + l \geq 2(\delta - l + 1) + l = 2\delta - l + 2.$$

Hence,

$$l \geq 2\delta - n + 2 \geq 2 \cdot \left(\frac{2n}{3} + 1\right) - n + 2 = \frac{n + 12}{3}.$$
Since

Let $H = K_k$ be the maximal rainbow complete graph in $G^c[V(G) \setminus V(C)]$ such that every edge in $H$ is $\tilde{C}$-colored, and let $R = G^c[V(G) - (V(C) \cup V(H))]$. It is clearly that for any $w \in V(H)$, if there is a vertex $v_i \in V(C)$ such that $v_iw$ is a $\tilde{C}$-color edge, then $c(v_iw) \notin c(H)$ since $G^c$ is a strongly edge-colored graph.

For two $\tilde{C}$-color edges $v_iw_1$ and $v_jw_2$ with $w_1, w_2 \in V(H)$ and $1 \leq i < j \leq l$, if $w_1 = w_2$ and $j - i = 1$, we say $v_iw_1$ and $v_jw_2$ are forbidden pair of type 1; if $w_1 \neq w_2$, both $a$ and $b$ are contained in $v_iC^2v_j$, and $2 \leq j - i \leq k$, we say $v_iw_1$ and $v_jw_2$ are forbidden pair of type 2. Clearly, if $E_{\tilde{C}}(C, H)$ has a forbidden pair of type 1, then there exists a rainbow $(l+1)$-cycle $C' := v_iw_1v_jC^2v_i$ containing both $a$ and $b$, and if $E_{\tilde{C}}(C, H)$ has a forbidden pair of type 2, then there exist a rainbow $(l+1)$-cycle $C' := v_iw_1Hw_2v_jC^2v_i$ containing both $a$ and $b$, where $w_1Hw_2$ is a path of length $|E(v_iC^2v_j)| - 1$ with endpoints $w_1$ and $w_2$ in $H$.

**Claim 2** $k \geq 3$.

**Proof.** For each $w \in V(H)$, let

\[
\tilde{s}_w = |E_{\tilde{C}}(w, C)|, s_w = |E_{C}(w, C)|, \\
\tilde{t}_w = |E_{\tilde{C}}(w, R)|, t_w = |E_{C}(w, R)|.
\]

We have

\[
s_w + t_w \leq l - (\tilde{s}_w + s_w),
\]

and so, we have

\[
\tilde{s}_w + 2s_w + t_w \leq l.
\]

Let $v_{i_1}, v_{i_2}, \ldots, v_{i_{\tilde{s}_w}}$ be the vertices on $C$ which are $\tilde{C}$-adjacent to $w$. Without loss of generality, we suppose that $1 \leq i_1 < i_2 < \ldots < i_{\tilde{s}_w} \leq l$. Then $i_{j+1} - i_j \geq 2$ for each $1 \leq j \leq \tilde{s}_w - 1$ and $i_{\tilde{s}_w} - i_1 \leq l - 2$. Let $I = \{i_1 - 1, i_1, i_2 - 1, i_2, \ldots, i_{\tilde{s}_w} - 1, i_{\tilde{s}_w}\}$. Clearly, we have $|I| = 2\tilde{s}_w$ and $I \cap CN(w) = \phi$. Thus, we can deduce that

\[
2\tilde{s}_w + s_w + t_w = |I| + s_w + t_w \leq l.
\]

Since $|V(R)| = n - l - k$, we have $t_w + \tilde{t}_w \leq n - l - k$. Together with inequalities (2) and (3), we have

\[
3\tilde{s}_w + 3s_w + 3t_w + \tilde{t}_w \leq l + l + n - l - k = n + l - k.
\]

Let

\[
\tilde{S} = \sum_{w \in V(H)} \tilde{s}_w, S = \sum_{w \in V(H)} s_w, T = \sum_{w \in V(H)} \tilde{t}_w, T = \sum_{w \in V(H)} t_w.
\]

Then

\[
3\tilde{S} + 3S + 3T + \tilde{T} \leq k(n + l - k).
\]
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Since $k$ is maximal, each vertex of $R$ has at most $k - 1$ number of $\tilde{C}$-color edges to $H$, which implies that
\[
\tilde{T} = \sum_{w \in V(H)} \tilde{t}_w \leq (k - 1)(n - l - k). \tag{5}
\]

Recall that $w \in V(H)$. By (1) and the arbitrariness of $w$, we have
\[
k\delta \leq \sum_{w \in V(H)} (\tilde{s}_w + s_w + \tilde{t}_w + t_w + (k - 1))
\[
= \tilde{S} + S + \tilde{T} + T + k(k - 1). \tag{6}
\]
Combining inequalities (4), (5) and (6), we can get the following inequality
\[
3k\delta \leq 3\tilde{S} + 3S + 3T + 3\tilde{T} + 3k(k - 1)
\]
\[
\leq k(n + l - k) + 2(k - 1)(n - l - k) + 3k(k - 1)
\]
\[
\leq n(3k - 2) + l(2 - k) - k.
\]
If $k = 1$, then $l > n$, a contradiction. If $k = 2$, then $\delta \leq \frac{2n-1}{3}$, again a contradiction. So we have $k \geq 3$. Claim 2 follows.

Since $H$ is a rainbow complete graph, we can deduce that
\[
S + T \leq l. \tag{7}
\]

Claim 3 $\tilde{S} \geq l + 1$.

**Proof.** Suppose, by way of contradiction, that $\tilde{S} \leq l$. Combining with inequality (6), we can get that
\[
k\delta \leq \tilde{S} + S + \tilde{T} + T + k(k - 1) \leq l + l + (k - 1)(n - l - k) + k(k - 1),
\]
which implies that $k(n - l - \delta) \geq n - 3l$. Since $\delta \geq \frac{2n}{3} + 1$ and $l \geq \frac{n+12}{3}$ from Claim 1, we have $n - l - \delta \leq 0$. Thus we have $3(n - l - \delta) \geq k(n - l - \delta) \geq n - 3l$ from Claim 2, and therefore $\delta \leq \frac{2n}{3}$, a contradiction. Claim 3 follows.

Without loss of generality, we suppose that $a = v_1$ and $b = v_m$, where $2 \leq m \leq l - 1$, and let $P^1 = aC + b$. Then we design an algorithm to generate a sequence of disjoint sub-paths $P^1_1, P^1_2, ..., P^1_{h_1}$ of $C$ respect to $P^1$ and $H$. 


Algorithm A1

**Input:** a strongly edge-colored graph $G^c$, a rainbow cycle $C = v_1v_2\ldots v_1$, a path $P^1 = v_1v_2\ldots v_m$ and a rainbow complete subgraph $H = K_k$ of $G^c - V(C)$.

**Output:** a sequence of disjoint paths $P^1_1, P^1_2, \ldots, P^1_{h_1}$ such that $P^1_i$ is a subgraph of $C$.

1: Set $i = 1$
2: While $V(P^1_i) \neq \phi$
   If $E_{\bar{C}}(P^1_i, H) = \phi$
     stop
   Else
     Set $d$ be the smallest subscript such that $E_{\bar{C}}(v_d, H) \neq \phi$
     If $d + k \geq m$ then
       Set $P^1_i = v_dv_{d+1}\ldots v_m$
       stop
     Else If $|E_{\bar{C}}(v_d, H)| \geq 2$ then
       Set $P^1_i = v_dv_{d+1}\ldots v_{d+k}$
       If $|E_{\bar{C}}(v_d, H)| = 1$ then
         Set $P^1_i = v_dv_{d+1}\ldots v_{d+k+1}$
       Set $P^1_i = P^1_i \setminus P^1_i$
       stop
   $i = i + 1$
3: return $P^1_1, P^1_2, \ldots, P^1_{h_1}$

Claim 4: $|E_{\bar{C}}(P^1_i, H)| \leq |V(P^1_i)| - 1$ for any $1 \leq i \leq h_1 - 1$, $|E_{\bar{C}}(P^1_{h_1}, H)| \leq k$ if $|V(P^1_{h_1})| \in \{1, 2\}$, and $|E_{\bar{C}}(P^1_{h_1}, H)| \leq k + 1$ if $3 \leq |V(P^1_{h_1})| \leq k + 1$.

**Proof.** For $1 \leq i \leq h_1 - 1$, we distinguish the following two cases.

**Case 1.** $|E_{\bar{C}}(v_d, H)| \geq 2$. Then we have $P^1_i = v_dv_{d+1}\ldots v_{d+k}$. Let $w_1$ and $w_2$ be two vertices in $H$ such that $v_dw_1, v_dw_2 \in E_{\bar{C}}(v_d, H)$. Since there exist no forbidden pairs of type 1 for any vertex $w \in V(H)$, then we have $|E_{\bar{C}}(v_d, H)| + |E_{\bar{C}}(v_{d+1}, H)| \leq k$. For any $j$ with $d + 2 \leq j \leq d + k$, if $w_1$ and $v_j$ are $\bar{C}$-adjacent, then $v_jw_1$ and $v_dw_2$ form a forbidden pair of type 2; if $w_2$ and $v_j$ are $\bar{C}$-adjacent, then $v_jw_2$ and $v_dw_1$ form a forbidden pair of type 2; if $v_j$ and $w$ are $\bar{C}$-adjacent for some $w$ with $w \neq w_1$ and $w \neq w_2$, then $v_jw$ and $v_dw$ form a forbidden pair of type 2. Therefore, we have $|E_{\bar{C}}(v_j, H)| = 0$. Thus,

$$|E_{\bar{C}}(P^1_i, H)| = \sum_{j=d}^{d+k} |E_{\bar{C}}(v_j, H)| = |E_{\bar{C}}(v_d, H)| + |E_{\bar{C}}(v_{d+1}, H)| \leq k = |V(P^1_i)| - 1.$$

**Case 2.** $|E_{\bar{C}}(v_d, H)| = 1$. Then we have $P^1_i = v_dv_{d+1}\ldots v_{d+k+1}$. Let $w_1$ be a vertex in $H$ such that $v_dw_1 \in E_{\bar{C}}(v_d, H)$. We further distinguish the following three cases.

**Case 2.1.** $|E_{\bar{C}}(v_{d+1}, H)| = 0$. For any $w \in V(H) \setminus \{w_1\}$, we have that $v_j$ and $w$ cannot be $\bar{C}$-adjacent for any $d + 2 \leq j \leq d + k + 1$ since otherwise $v_jw$ and $v_dw_1$ form a forbidden pair of type 2. Thus, we
have $|E_C(v_j, H)| \leq 1$ and $\sum_{j=d+2}^{d+k+1} |E_C(v_j, H)| \leq k - 1$. Therefore,

$$|E_C(P_1^1, H)| = \sum_{j=d}^{d+k+1} |E_C(v_j, H)|$$

$$= |E_C(v_d, H)| + |E_C(v_{d+1}, H)| + \sum_{j=d+2}^{d+k+1} |E_C(v_j, H)|$$

$$\leq 1 + 0 + (k - 1)$$

$$= k$$

$$\leq |V(P_1^1)| - 1.$$

**Case 2.2.** $|E_C(v_{d+1}, H)| = 1$. Let $w_2$ be a vertex in $H$ such that $v_{d+1} w_2 \in E_C(v_d, H)$. Clearly, $w_1 \neq w_2$. If $v_{d+2}$ and $w_2$ are $\bar{C}$-adjacent, we have that $v_{d+2} w_2$ and $v_d w_1$ form a forbidden pair of type 2, a contradiction. If $v_{d+2}$ and $w$ are $\bar{C}$-adjacent for some $w \in V(H)$ with $w \neq w_1$ and $w \neq w_2$, then $v_{d+2} w$ and $v_d w_1$ form a forbidden pair of type 2, again a contradiction. So, $|E_C(v_{d+2}, H)| \leq 1$.

For any $j$ with $d + 3 \leq j \leq d + k + 1$, if $w_1$ and $v_j$ are $\bar{C}$-adjacent, then $v_j w_1$ and $v_{d+1} w_2$ form a forbidden pair of type 2; if $w_2$ and $v_j$ are $\bar{C}$-adjacent, then $v_j w_2$ and $v_{d+1} w_1$ form a forbidden pair of type 2; if $v_j$ and $w$ are $\bar{C}$-adjacent for some $w \in V(H)$ with $w \neq w_1$ and $w \neq w_2$, then $v_j w$ and $v_{d+1} w_1$ form a forbidden pair of type 2. We obtain a contradiction in the above three cases, and therefore, we have $\sum_{j=d+3}^{d+k+1} |E_C(v_j, H)| = 0$. Therefore,

$$|E_C(P_1^1, H)| = \sum_{j=d}^{d+k+1} |E_C(v_j, H)|$$

$$= |E_C(v_d, H)| + |E_C(v_{d+1}, H)| + |E_C(v_{d+2}, H)| + \sum_{j=d+3}^{d+k+1} |E_C(v_j, H)|$$

$$\leq 1 + 1 + 1 + 0$$

$$\leq k$$

$$\leq |V(P_1^1)| - 1.$$

**Case 2.3.** $|E_C(v_{d+1}, H)| \geq 2$. Let $Q_1^1 = P_1^1 \setminus \{v_d\} = v_{d+1}v_{d+2}...v_{d+k+1}$. Similar to the discussion of Case 1, we have that $|E_C(Q_1^1, H)| \leq |V(Q_1^1)| - 1 = (k + 1) - 1 = k$. Thus, $|E_C(P_1^1, H)| = |E_C(v_d, H)| + |E_C(Q_1^1, H)| \leq 1 + k = |V(P_1^1)| - 1$.

Then we analyse the value of $|E_C(P_{h_1}^{1}, H)|$. If $|V(P_{h_1}^{1})| = 1$, the inequality $|E_C(P_{h_1}^{1}, H)| \leq k$ clearly holds. If $|V(P_{h_1}^{1})| = 2$, that is, $P_{h_1}^{1} = v_d v_{d+1}$, we have $|E_C(v_d, H)| + |E_C(v_{d+1}, H)| \leq k$ since $v_d$ and $v_{d+1}$ are adjacent. Therefore, $|E_C(P_{h_1}^{1}, H)| = |E_C(v_d, H)| + |E_C(v_{d+1}, H)| \leq k$. If $3 \leq |V(P_{h_1}^{1})| \leq k + 1$, we have $|E_C(P_{h_1}^{1}, H)| \leq k$ when $|E_C(v_d, H)| \geq 2$ by the similar analysis of the above Case 1 (taking $m$ as $d + k$), and $|E_C(P_{h_1}^{1}, H)| \leq k + 1$ when $|E_C(v_d, H)| = 1$ by the similar analysis of the above Case 2 (taking $m$ as $d + k + 1$). The proof is thus completed.
Let \( P^2 = aC^{-}b \). Then we design another algorithm to generate a sequence of disjoint sub-paths \( P^2_1, P^2_2, \ldots, P^2_{h_2} \) of \( C \) respect to \( P^2 \) and \( H \) in the following.

**Algorithm AII**

**Input:** a strongly edge-colored graph \( G \), a rainbow cycle \( C = v_1v_2 \ldots v_lv_1 \), \( P^2 = aC^{-}b = v_{l+1}v_{l+1-1} \ldots v_m \) and a rainbow complete subgraph \( H = K_k \) of \( G^2 - V(C) \).

**Output:** a sequence of disjoint paths \( P^2_1, P^2_2, \ldots, P^2_{h_2} \) such that \( P^2_{i} \) is a subgraph of \( C \).

1: \( \text{Set } i = 1 \)
2: \( \text{While } V(P^2_{i}) \neq \phi \) do  
   If \( E_{\tilde{C}}(P^2_{i}, H) = \phi \)  
      stop  
   Else Set \( d \) be the biggest subscript for which \( E_{\tilde{C}}(v_d, H) \neq \phi \)  
      If \( d - k \leq m \) then  
         Set \( P^2_{i} = v_d \ldots v_{d-k} \)  
         stop  
      Else If \( |E_{\tilde{C}}(v_d, H)| \geq 2 \) then  
         Set \( P^2_{i} = v_d \ldots v_{d-k} \)  
      End If  
      If \( |E_{\tilde{C}}(v_d, H)| = 1 \) then  
         Set \( P^2_{i} = v_d \ldots v_{d-k-1} \)  
      End If  
      Set \( P^2 = P^2_{i} \setminus P^2_{i} \)  
      Set \( i = i + 1 \)  
3: \( \text{return } P^2_{1}, P^2_{2}, \ldots, P^2_{h_2} \)

Similar to Claim 4, we can get the following Claim.

**Claim 5** \( |E_{\tilde{C}}(P^2_{i}, H)| \leq |V(P^2)| - 1 \) for all \( 1 \leq i \leq h_2 - 1 \), \( |E_{\tilde{C}}(P^2_{h_2}, H)| \leq k \) if \( |V(P^2_{h_2})| \in \{1, 2\} \) and \( |E_{\tilde{C}}(P^2_{h_2}, H)| \leq k + 1 \) if \( 3 \leq |V(P^2_{h_2})| \leq k + 1 \).

According to the above claims, we have

\[
|E_{\tilde{C}}(C, H)| = |E_{\tilde{C}}(aC^{-}b, H)| + |E_{\tilde{C}}(aC^{-}b, H)| - |E_{\tilde{C}}(a, H)| - |E_{\tilde{C}}(b, H)|
\leq \sum_{i=1}^{h_1-1} |V(P^1_i)| - (h_1 - 1) + |E_{\tilde{C}}(P^1_{h_1}, H)|
+ \sum_{i=1}^{h_2-1} |V(P^2_i)| - (h_2 - 1) + |E_{\tilde{C}}(P^2_{h_2}, H)|
- |E_{\tilde{C}}(a, H)| - |E_{\tilde{C}}(b, H)|
\leq |l - |V(P^1_{h_1})| - |V(P^2_{h_2})| + 1| - (h_1 + h_2) + 2
+ |E_{\tilde{C}}(P^1_{h_1}, H)| + |E_{\tilde{C}}(P^2_{h_2}, H)| - |E_{\tilde{C}}(a, H)| - |E_{\tilde{C}}(b, H)|
= l - (|V(P^1_{h_1})| + |V(P^2_{h_2})|) - (h_1 + h_2) + 3
+ |E_{\tilde{C}}(P^1_{h_1}, H)| + |E_{\tilde{C}}(P^2_{h_2}, H)| - |E_{\tilde{C}}(a, H)| - |E_{\tilde{C}}(b, H)|.
\] (8)

**Claim 6** \( \bar{s} \leq l + 2k - 4 \).
**Proof.** We show that \( \bar{S} \leq \max\{2k+2, l+k-1, l+2k-4\} \), which implies \( \bar{S} \leq l+2k-4 \) since \( l \geq 7 \) from Claim 1 and \( k \geq 3 \) from Claim 2.

Let \( h = h_1 + h_2 \). By symmetry, we suppose \( h_1 \geq h_2 \) and \( |V(P_{h_1}^1)| \geq |V(P_{h_2}^2)| \). From Claim 3, we have \( h \geq 1 \). Then we proceed our proof by distinguishing the following four cases.

**Case 1.** \( h_1 = 1 \) and \( h_2 = 0 \). From Algorithm AII, we have \( E_\mathcal{C}(uC \setminus v, H) = \phi \). Thus, \( E_\mathcal{C}(u, H) = \phi \) and \( E_\mathcal{C}(b, H) = \phi \). From Algorithm AI, we have \( |V(P_{h_1}^1)| \geq 2 \). If \( |V(P_{h_2}^2)| = 2 \), let \( u \) be the vertex distinct from \( b \) in \( C \) such that \( E_\mathcal{C}(u, H) \neq \phi \). Thus we have \( \bar{S} = |E_\mathcal{C}(u, H)| \leq k < 2k + 2 \). If \( |V(P_{h_1}^1)| \geq 3 \), from Claim 4, we have \( \bar{S} = E_\mathcal{C}(P_{h_1}^1, H) \leq k + 1 < 2k + 2 \). The claim follows.

**Case 2.** \( h_1 \geq 2 \) and \( h_2 = 0 \). From Algorithm AI and AII, we have \( E_\mathcal{C}(u, H) = \phi \), \( E_\mathcal{C}(b, H) = \phi \) and \( |V(P_{h_1}^1)| \geq 2 \). If \( |V(P_{h_1}^1)| = 2 \), since \( E_\mathcal{C}(b, H) = \phi \), we have \( |E_\mathcal{C}(P_{h_1}^1, H)| + |E_\mathcal{C}(P_{h_2}^2, H)| - |E_\mathcal{C}(b, H)| = |E_\mathcal{C}(P_{h_1}^1, H)| \leq k \). Applying inequality (8), we have \( \bar{S} \leq l - 2 - 3 + k + 0 = l + k - 1 \). If \( |V(P_{h_1}^1)| \geq 3 \), from Claim 4, we have \( \bar{S} \leq l - 2 - 3 + k + 1 = l + k - 1 \). The claim follows.

**Case 3.** \( h_1 = 1 \) and \( h_2 = 1 \). By Claim 4 and 5, if \( |V(P_{h_1}^1)| \in \{1, 2\} \) and \( |V(P_{h_2}^2)| \in \{1, 2\} \), we have \( \bar{S} \leq |E_\mathcal{C}(P_{h_1}^1, H)| + |E_\mathcal{C}(P_{h_2}^2, H)| \leq 2k + 2 \). If \( |V(P_{h_1}^1)| \geq 3 \) and \( |V(P_{h_2}^2)| \in \{1, 2\} \), we have \( \bar{S} \leq |E_\mathcal{C}(P_{h_1}^1, H)| + |E_\mathcal{C}(P_{h_2}^2, H)| \leq 2k + 1 < 2k + 2 \). If \( |V(P_{h_1}^1)| \geq 3 \) and \( |V(P_{h_2}^2)| \geq 3 \), we have \( \bar{S} \leq |E_\mathcal{C}(P_{h_1}^1, H)| + |E_\mathcal{C}(P_{h_2}^2, H)| \leq 2k + 2 \). The claim holds.

**Case 4.** \( h \geq 3 \) and \( h_2 \geq 1 \). We consider the following six cases.

**Case 4.1.** \( |V(P_{h_1}^1)| = 1 \) and \( |V(P_{h_2}^2)| = 1 \). It is clearly that

\[
V(P_{h_1}^1) = V(P_{h_2}^2) = \{b\}
\]

and

\[
|E_\mathcal{C}(P_{h_1}^1, H)| + |E_\mathcal{C}(P_{h_2}^2, H)| - |E_\mathcal{C}(b, H)| = |E_\mathcal{C}(b, H)| \leq k.
\]

By inequality (8), we have

\[
\bar{S} = |E_\mathcal{C}(C, H)| \leq l - 2 - 3 + k = l + k - 2 < l + k - 1.
\]

**Case 4.2.** \( |V(P_{h_1}^1)| = 2 \) and \( |V(P_{h_2}^2)| = 1 \). It is clearly that \( V(P_{h_2}^2) = \{b\} \). From Claim 4, we have

\[
|E_\mathcal{C}(P_{h_1}^1, H)| + |E_\mathcal{C}(P_{h_2}^2, H)| - |E_\mathcal{C}(b, H)| = |E_\mathcal{C}(P_{h_1}^1, H)| \leq k.
\]

By inequality (8) and \( h \geq 3 \), we have

\[
\bar{S} \leq l - 3 - 3 + k + 0 = l + k - 3 < l + k - 1.
\]

**Case 4.3.** \( |V(P_{h_1}^1)| \geq 3 \) and \( |V(P_{h_2}^2)| = 1 \). It is clearly that \( V(P_{h_2}^2) = \{b\} \). From Claim 4, we have

\[
|E_\mathcal{C}(P_{h_1}^1, H)| + |E_\mathcal{C}(P_{h_2}^2, H)| - |E_\mathcal{C}(b, H)| = |E_\mathcal{C}(P_{h_1}^1, H)| \leq k + 1.
\]

By inequality (8) and \( h \geq 3 \), we have

\[
\bar{S} = |E_\mathcal{C}(C, H)| \leq l - 4 - 3 + k = l + k - 3 < l + k - 1.
\]
Case 4.4. \(|V(P_{h_1}^1)| = 2\) and \(|V(P_{h_2}^2)| = 2\). From Claim 4 and 5, we have

\[|E_C(P_{h_1}^1, H)| + |E_C(P_{h_2}^2, H)| - |E_C(b, H)| \leq 2k.\]

By inequality (8) and \(h \geq 3\), we have

\[\tilde{S} = |E_C(C, H)| \leq l - 4 - 3 + 3 + 2k + 0 = l + 2k - 4 < l + k - 1.\]

Case 4.5. \(|V(P_{h_1}^1)| \geq 3\) and \(|V(P_{h_2}^2)| = 2\). It is clearly that

\[|E_C(P_{h_1}^1, H)| + |E_C(P_{h_2}^2, H)| - |E_C(b, H)| \leq k + k + 1 = 2k + 1.\]

By inequality (8) and \(h \geq 3\), we have

\[\tilde{S} = |E_C(C, H)| \leq l - 5 - 3 + 3 + 2k + 1 + 0 = l + 2k - 4.\]

Case 4.6. \(|V(P_{h_1}^1)| \geq 3\) and \(|V(P_{h_2}^2)| \geq 3\). From Claim 4 and 5, we have

\[|E_C(P_{h_1}^1, H)| + |E_C(P_{h_2}^2, H)| - |E_C(b, H)| \leq k + 1 + k + 1 = 2k + 2.\]

By inequality (8), we have

\[\tilde{S} = |E_C(C, H)| \leq l - 6 - 3 + 3 + 2k + 2 + 0 = l + 2k - 4.\]

The Claim follows.

From Claim 6, inequalities (5) (6) and (7), we can deduce that

\[k\delta \leq \tilde{S} + S + \tilde{T} + T + k(k - 1)\]
\[\leq l + 2k - 4 + l + (k - 1)(n - l - k) + k(k - 1)\]
\[= l + 2k - 4 + k(n - l) + 2l - n.\]

Therefore, we have \(k(n - l - \delta + 2) \geq n - 3l + 4\). Since \(l \geq \frac{n+12}{3}\) from Claim 1 and \(\delta \geq \frac{2n}{3} + 1\), we have \(n - l - \delta + 2 < 0\). Then from Claim 2, we have

\[3(n - l - \delta + 2) \geq k(n - l - \delta + 2) \geq n - 3l + 4,\]

which implies that \(\delta \leq \frac{2n+2}{3}\), a contradiction. We complete the proof of Theorem 1.6.

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