ON THE ELLIPTIC CURVE $y^2 = x^3 - 2rDx$ AND FACTORING INTEGERS

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Abstract. Let $D = pq$ be the product of two distinct odd primes. Assume the parity conjecture, we construct infinitely many $r \geq 1$ such that $E_{2rD} : y^2 = x^3 - 2rDx$ has conjectural rank one and $v_p(x(k|Q)) \neq v_q(x(k|Q))$ for any odd integer $k$, where $Q$ is the generator of the free part of $E(Q)$. Furthermore, under the Generalized Riemann hypothesis, the minimal value of $r$ is in $O(\log^4 D)$. As a corollary, one can factor $D$ by computing the generator $Q$.

1. Introduction and Main Results

In this paper, we study the elliptic curve $E_{2rD}/\mathbb{Q}$ defined by

$$E_{2rD} : y^2 = x^3 - 2rDx, \quad r \in \mathbb{Z}_{\geq 1},$$

where $D = pq$ is a product of two distinct odd primes and $2rD$ is square-free. Burhanuddin and Huang [4] studied $E_D : y^2 = x^3 - Dx$ with $D = pq, p \equiv q \equiv 3 \pmod{16}$. And they proved that, under the parity conjecture, $E_D$ has conjectural rank one and $v_p(x(Q)) \neq v_q(x(Q))$, where $Q$ is the generator of $E_D(\mathbb{Q})/E_D(\mathbb{Q})_{\text{tors}}$, so one can recover $p$ and $q$ from $D$ and $x(Q)$. Furthermore, they conjectured that factoring $D$ is polynomial time equivalent to computing the generator $Q$.

In this paper, we generalize their results to all the $D = pq$ with odd primes $p \neq q$. Moreover, for a given $D$, we prove that there are infinitely many $r \geq 1$ such that $E_{2rD}$ has conjectural rank one and $v_p(x([k]Q)) \neq v_q(x([k]Q))$ for any odd integer $k$, where $Q$ is the generator of $E_{2rD}(\mathbb{Q})/E_{2rD}(\mathbb{Q})_{\text{tors}}$.

The main results are summarized as follows,

**Theorem 1.1.** Let $D = pq$, where $p$ and $q$ are distinct odd primes. Under the parity conjecture, there are infinitely many positive integer $r$ such that $E_{2rD} : y^2 = x^3 - 2rDx$ has conjectural rank one. Let $Q$ be the generator of $E_{2rD}(\mathbb{Q})/E_{2rD}(\mathbb{Q})_{\text{tors}}$, then $v_p(x([k]Q)) \neq v_q(x([k]Q))$ for any odd integer $k$. Furthermore, assume the Generalized Riemann hypothesis, the minimal value of $r$ is in $O(\log^4 D)$.

**Corollary 1.2.** Given $D$, a product of two distinct odd primes. Assume the parity conjecture and the Generalized Riemann hypothesis. Then there exists an odd integer $r \in O(\log^4 D)$ such that $D$ can be recovered from the data $x([2k+1]Q)$ and $D$, where $Q$ is the generator of $E_{2rD}(\mathbb{Q})/E_{2rD}(\mathbb{Q})_{\text{tors}}$ and $k$ is any rational integer.

**Remark 1.3.** Burhanuddin and Huang [4] have proved that, if the naive height of the generator of the free part of $E_D(\mathbb{Q})$ with $D = pq, p \equiv q \equiv 3(\text{mod}16)$, grows polynomially in $\log \Delta$, where $\Delta$ is the minimal discriminant of $E_D$, then factoring $D$ is polynomial time reducible to computing the generator of the group $E_D(\mathbb{Q})$. On the other hand, for general $D$, Lang [9] conjectured that the log height of the generator is (approximately) bounded above by $D^{1/12}$ and is widely believed that this upper bound is accurate for “most” elliptic curves. More precisely, it is a folklore conjecture that the probability of an elliptic curve $E_D(\mathbb{Q})$ has a generator with height $h(P) \leq D^{1/13}$ is less than $D^{-c}$ for some absolute constant $c$. So although the additional parameter $r$ leads to a family of curves which are suitable for factoring $D$, it’s unlikely to find one, whose height of generator grows polynomially in $\log \Delta$.

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The paper is organized as follows. In section 2 we use the 2-descent method to compute the Selmer groups of these elliptic curves \( E_{2rD} \) and their isogenous curves \( E'_{2rD} \), which will give an upper bound of rank of Mordell-Weil group \( E_{2rD}(\mathbb{Q}) \), and we construct infinitely many such curves \( E_{2rD} \) such that \( r_{E_{2rD}} \leq 1 \) for an appropriately chosen value of the parameter \( r \). In section 3 we calculate the global root number of these elliptic curves \( E_{2rD} \), which will give a lower bound of rank of Mordell-Weil group \( E_{2rD}(\mathbb{Q}) \) under the weak parity conjecture. In section 4 we study the property of non-torsion points of Mordell-Weil group \( E_{2rD}(\mathbb{Q}) \) such as arithmetic property and canonical height. In section 5 we prove our main results.

2. Computation of the Selmer groups

In this section, we determine the Selmer group \( S^{(\phi)}(E_{2rD}/\mathbb{Q}) \) of the elliptic curve \( E_{2rD} \) defined as in the introduction.

Let \( E_{2rD} : y^2 = x^3 + 8rDx \) be its 2-isogenous elliptic curve and 
\[
\phi : E_{2rD} \rightarrow E'_{2rD} \quad \text{and} \quad \phi' : E'_{2rD} \rightarrow E_{2rD}
\]
be the corresponding 2-isogeny defined as
\[
\phi((x, y)) = \left( \frac{y^2}{x^2}, -\frac{y(2rD + x^2)}{x^2} \right) \quad \text{and} \quad \phi'((x, y)) = \left( \frac{y^2}{4x^2}, \frac{y(8rD - x^2)}{8x^2} \right)
\]
respectively.

Let \( S = \{ \infty \} \cup \{ \text{primes dividing } 2rD \} \) and \( \mathbb{Q}(S, 2) = \langle S \rangle \) be the subgroup of \( \mathbb{Q}^*/\mathbb{Q}^{*2} \). For \( d \in \mathbb{Q}(S, 2) \), the corresponding homogeneous spaces are defined as follows,
\[
C_d : dW^2 = d^2 + 8rDZ^4 \quad \text{and} \quad C'_d : dW^2 = d^2 - 2rDZ^4.
\]

By Proposition 4.9 in [11], the Selmer groups have the following identifications:
\[
\{1, 2rD\} \subseteq S^{(\phi)}(E_{2rD}/\mathbb{Q}) \cong \{ d \in \mathbb{Q}(S, 2) : C_d(\mathbb{Q}_v) \neq \emptyset \text{ for all } v \in S \},
\]
\[
\{1, -2rD\} \subseteq S^{(\phi)}(E'_{2rD}/\mathbb{Q}) \cong \{ d \in \mathbb{Q}(S, 2) : C'_d(\mathbb{Q}_v) \neq \emptyset \text{ for all } v \in S \}.
\]

For \( d \in \mathbb{Q}(S, 2) \) and \( 2 \nmid d \), the following two lemmas give the sufficient and necessary conditions for \( d \in S^{(\phi)}(E_{2rD}/\mathbb{Q}) \) or \( d \in S^{(\phi)}(E'_{2rD}/\mathbb{Q}) \).

Lemma 2.1. Let \( C_d : dW^2 = d + \frac{8rD}{d}Z^4 \) with \( d \nmid D \), then

1. \( C_d(\mathbb{Q}_2) \neq \emptyset \) if and only if \( d \equiv 1(\text{mod } 8) \);
2. \( C_d(\mathbb{Q}_3) \neq \emptyset \) if and only if \( (\frac{d}{3}) = 1 \) for any prime \( t \mid D \); 
3. \( C_d(\mathbb{Q}_3) \neq \emptyset \) if and only if \( (\frac{2rD/d}{3}) = 1 \) for any prime \( |d| \).

Proof. For simplicity, let \( f(Z, W) := W^2 - \frac{8rD}{d}Z^4 - d \).

1. If \( C_d(\mathbb{Q}_2) \neq \emptyset \), take any point \((z, w) \in C_d(\mathbb{Q}_2)\), then \( v_2(z) \geq 0, v_2(w) = 0 \) and \( w^2 = d + \frac{8rD}{d}z^4 \). Taking the valuation \( v_2 \) at 2 of both sides, we have \( d \equiv 1(\text{mod } 8) \). On the other hand, if \( d \equiv 1(\text{mod } 8) \), then \( v_2(f(0, 1)) > 2v_2(\frac{\partial f}{\partial W}(0, 1)) \). By Hensel’s lemma in [11], \( f(Z, W) = 0 \) has a solution in \( \mathbb{Q}_2^2 \), that is, \( C_d(\mathbb{Q}_2) \neq \emptyset \).

2. For any prime \( t \mid D \), if \( C_d(\mathbb{Q}_3) \neq \emptyset \), take any point \((z, w) \in C_d(\mathbb{Q}_3)\), then \( v_t(z) \geq 0, v_t(w) = 0 \) and \( w^2 = d + \frac{8rD}{d}z^4 \). Taking the valuation \( v_t \) at \( t \) of both sides, we have \( (\frac{d}{t}) = 1 \). On the other hand, if \( (\frac{d}{t}) = 1 \), then there exists an integer \( a \) such that \( a^2 \equiv d(\text{mod } t) \), therefore \( v_t(f(0, a)) > 2v_t(\frac{\partial f}{\partial W}(0, a)) \). By Hensel’s lemma, \( f(Z, W) = 0 \) has a solution in \( \mathbb{Q}_3^2 \), that is, \( C_d(\mathbb{Q}_3) \neq \emptyset \).
(3) For any prime \( l \mid d \), let \( g(Z_1, W_1; l, i) := W_1^2 - \frac{8rD}{d} Z_1^4 - dl^{4i} \), for some non-negative integer \( i \). If \( C_d(\mathbb{Q}_l) \neq \emptyset \), take any point \((z, w) \in C_d(\mathbb{Q}_l)\), then \( z = l^{-i} z_1 \), \( w = w_1^{-2i} \), where \( v_l(z_1) = v_l(w_1) = 0 \), \( i \geq 0 \) and \((z_1, w_1)\) satisfying \( w_1^2 = dl^{4i} + \frac{8rD}{d} z_1^4 \). Taking the valuation \( v_l \) at \( l \) of both sides, we have \( \left( \frac{2rD/d}{d} \right) = 1 \). On the other hand, if \( \left( \frac{2rD/d}{d} \right) = 1 \), then there exists an integer \( b \) such that \( b^2 \equiv \frac{8rD}{d} (\text{mod } l) \), therefore we have \( v_l(f(0, b)) > 2v_l\left( \frac{\partial f}{\partial y}(0, b) \right) \). By Hensel’s lemma, \( g(Z_1, W_1; l, i) = 0 \) has a solution in \( \mathbb{Q}_l^2 \), that is, \( C_d(\mathbb{Q}_l) \neq \emptyset \).

Lemma 2.2. Let \( C'_d : W^2 = d - \frac{2rD}{d} Z^4 \) with \( d \mid rD \), then

1. \( C'_d(\mathbb{Q}_2) \neq \emptyset \) if and only if \( d - 2rD/d = 1 (\text{mod } 8) \) or \( d = 1 (\text{mod } 8) \);
2. \( C'_d(\mathbb{Q}_3) \neq \emptyset \) if and only if \( \left( \frac{4}{3} \right) = 1 \) for any prime \( l \mid D \);
3. \( C'_d(\mathbb{Q}_l) \neq \emptyset \) if and only if \( \left( \frac{-2rD/d}{d} \right) = 1 \) for any prime \( l \mid d \).

Proof. The proof is similar to lemma 2.1.

We now determine the Selmer groups of the elliptic curves appeared in our theorems in the introduction, by the lemmas above. For simplicity, the class of odd integer \( m \) in \( \mathbb{Z}/8\mathbb{Z} \) will be denoted by \( \overline{m} \), and define \( \alpha_7 = \overline{3} \) and \( \alpha_{11} = \overline{5} \). Define the set \( A_D \) of parameters \( r \) as follows,

\[
A_D = \begin{cases}
\{ l \mid l \in \overline{7}, (\frac{p}{l}) = -(\frac{1}{l}) = -1 \}, & \text{if } D \equiv 7 \pmod{8}, p \not\equiv 1 \pmod{8} \\
\{ l \mid l \in \overline{7}, (\frac{p}{l}) = -(\frac{1}{l}) = -1 \}, & \text{if } D \equiv 3 \pmod{5}, p \not\equiv 1 \pmod{8} \\
\{ l \mid l \in \alpha_7, (\frac{2}{l}) = -1 \}, & \text{if } D \equiv 1 \pmod{8}, p \not\equiv 1 \pmod{8} \\
\{ l \mid l \in \alpha_{11}, (\frac{2}{l}) = -1 \}, & \text{if } D \equiv 1 \pmod{8}, p \equiv 1 \pmod{8} \\
\{ l_1, l_2 \mid l_1 \in \overline{3}, l_2 \in \overline{7}, (\frac{p}{l_1}) = (\frac{p}{l_2}) = -1, (\frac{l_1 l_2}{p}) = -1 \}, & \text{if } D \equiv 1 \pmod{8}, p \equiv 1 \pmod{8}
\end{cases}
\]

where \( l, l_1, l_2 \) are primes. By density theorem, \( A_D \) is an infinite set. The Selmer group of \( E_{2rD} \) is determined by the following proposition.

Proposition 2.3. Let \( D = pq \) be the product of two distinct odd primes \( p, q \). Then for any \( r \in A_D \), we have

\[
S^{(\phi)}(E_{2rD}/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad S^{(\phi)}(E'_{2rD}/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2.
\]

Proof. By the lemmas above, we can determine the elements in the Selmer groups of \( E_{2rD} \) and \( E'_{2rD} \) easily, where \( r = 1, l, l_1 l_2 \). We only take \( r = l_1 l_2 \) for an example.

In this case, \( S = \{ \infty \} \cup \{ 2, p, q, l_1, l_2 \} \) and \( Q(S, 2) = \{ -1, 2, p, q, l_1, l_2 \} \), where \( p \equiv q \equiv 1, l_1 \equiv 3, l_2 \equiv 7 \pmod{8} \) and they satisfying \( (\frac{2}{p}) = (\frac{2}{q}) = -1 \) and \( (\frac{p}{l_1}) (\frac{p}{l_2}) = -1 \). Without loss of generality, we can assume \( (\frac{l_1}{p}) = 1, (\frac{l_2}{p}) = -1 \) and \( (\frac{l_1 l_2}{p}) = 1 \). By Lemma 2.1, \( S^{(\phi)}(E_{2rD}/\mathbb{Q}) \cong \{ 1, 2Dl_1 l_2 \} \cong \mathbb{Z}/2\mathbb{Z} \).

For \( S^{(\phi)}(E'_{2rD}/\mathbb{Q}) \), by Lemma 2.2, one can check

\[
S^{(\phi)}(E'_{2rD}/\mathbb{Q}) = \begin{cases}
\{ 1, -pl_2, ql_1, -2rD \}, & \text{if } (\frac{q}{p}) = 1 \\
\{ 1, -2pl_1, ql_2, -2rD \}, & \text{if } (\frac{q}{p}) = -1.
\end{cases}
\]

It follows that \( S^{(\phi)}(E'_{2rD}/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2 \).

Use the exact sequences listed in [11], we have

\[
\begin{array}{c}
0 \longrightarrow E_{2rD}(\mathbb{Q}) \longrightarrow S^{(\phi)}(E_{2rD}/\mathbb{Q}) \longrightarrow TS(E_{2rD}/\mathbb{Q})[\phi] \longrightarrow 0 \\
0 \longrightarrow E'_{2rD}(\mathbb{Q}) \longrightarrow S^{(\phi)}(E'_{2rD}/\mathbb{Q}) \longrightarrow TS(E'_{2rD}/\mathbb{Q})[\phi] \longrightarrow 0 \\
0 \longrightarrow E_{2rD}(\mathbb{Q}) \longrightarrow E'_{2rD}(\mathbb{Q}) \longrightarrow E_{2rD}(\mathbb{Q}) \longrightarrow E_{2rD}(\mathbb{Q}) \longrightarrow 0
\end{array}
\]
which follows that

\[ r_{E_{2r}} + \dim_2(\text{TS}(E_{2rD}/Q)[\phi]) + \dim_2(\text{TS}(E'_{2rD}/Q)[\hat{\phi}]) \]

\[ = \dim_2(S(\phi)(E_{2rD}/Q)) + \dim_2(S(\hat{\phi})(E'_{2rD}/Q)) - 2, \]

where \( \dim_2 \) denotes the dimension of an \( \mathbb{F}_2 \)-vector space. The equality above gives an upper bound of \( r_{E_{2r}} \) as follows,

\[ r_{E_{2r}} \leq \dim_2(S(\phi)(E_{2rD}/Q)) + \dim_2(S(\hat{\phi})(E'_{2rD}/Q)) - 2, \]

which implies that

\[ r_{E_{2r}} \leq 1, \quad \text{for any } r \in A_D. \] (2.1)

In the next section, we study the global root number of \( E_{2rD} \), which will give a lower bound of \( r_{E_{2r}} \).

3. Computation of the conjectural rank

Let \( E \) be an elliptic curve over \( \mathbb{Q} \) with conductor \( N_E \). By the Modularity Theorem \([6]\), the L-function admits an analytic continuation to an entire function satisfying the functional equation

\[ \Lambda_E(2-s) = W(E)\Lambda_E(s), \quad \text{where } \Lambda_E(s) = N_E^{s/2}(2\pi)^{-s}\Gamma(s)L_E(s), \]

and \( W(E) = \pm 1 \) is called the global root number.

Let \( r_{E_{2r}}^{an} \) and \( r_{E_{2r}} \) be the analytic rank and arithmetic rank of \( E \) respectively, where \( r_{E_{2r}}^{an} \) is the order of vanishing of \( L_E(s) \) at \( s = 1 \) and \( r_{E_{2r}} \) is the rank of the abelian group \( E(\mathbb{Q}) \). For the parity of \( r_{E_{2r}} \), there is a famous conjecture:

**Conjecture 3.1.** (Parity Conjecture) \((-1)^{r_E} = W(E)\).

**Remark 3.2.** In this paper, we only need the weak form of the parity conjecture:

\[ W(E) = -1 \implies r_E \geq 1. \]

For the detail of the parity conjecture, see the recent works by Tim Dokchiter \([5]\).

This weak parity conjecture gives a lower bound of \( r_E \), while \( W(E) = -1 \). Due to the special choice of our elliptic curve, the global root number is equal to \(-1\) by the following lemma.

**Lemma 3.3.** Let \( E_{2N} : y^2 = x^3 - 2Nx \) be an elliptic curve over \( \mathbb{Q} \) with \( 2N \) square-free, then \( W(E_{2N}) = -1 \).

**Proof.** By \([2]\), for any integer \( d \) such that \( d \not\equiv 0 \pmod{4} \), the global root number of the elliptic curve \( E_d : y^2 = x^3 - dx \), has the following formula,

\[ W(E_d) = \text{sgn}(-d) \cdot \epsilon(d) \cdot \prod_{p^2 \mid d, p \geq 3} \left( \frac{-1}{p} \right) \]

where

\[ \epsilon(d) = \begin{cases} 
-1 & \text{if } d \equiv 1, 3, 11, 13(\pmod{16}) \\
1 & \text{if } d \equiv 2, 5, 6, 7, 9, 10, 14, 15(\pmod{16}),
\end{cases} \]

which tells us \( W(E_{2N}) = -1 \). \( \square \)

**Corollary 3.4.** Assume the Parity conjecture, then for any \( r \in A_D \) we have,

\[ r_{E_{2r}} = 1 \quad \text{and} \quad \text{TS}((E'_{2rD}/Q)(\hat{\phi})) = 0. \]

**Proof.** Inequality 2.1, lemma 3.3 and the parity conjecture imply the conclusion. \( \square \)
Proof. \[ \text{Lemma 4.1.} \]

For any \( E \) residue field of \( E \) that is, canonical height of non-torsion points of \( E \), theorem tells us that \( v \). For any finite place \( v \), Proposition 6.1 in \( \text{Section 2, section 3 and the density theorem state that there are infinitely many integers} (2) \]

Case I. For \( v \) \( = \) \( 1 \), then for any nonzero integer \( k \), 

\[
\begin{align*}
0 \geq v_t(x([2kQ])) &\equiv 0(\text{mod } 2) \\
1 \leq v_t(x([2k+1]Q)) &\equiv 1(\text{mod } 2) \\
1 \leq v_t(x([2k]Q + T)) &\equiv 1(\text{mod } 2) \\
-2 \geq v_t(x([2k+1]Q + T)) &\equiv 0(\text{mod } 2).
\end{align*}
\]

(2) if \( v_t(x(Q)) \leq 0 \), then for any nonzero integer \( k \),

\[
\begin{align*}
0 \geq v_t(x([2k]Q)) &\equiv 0(\text{mod } 2) \\
0 \geq v_t(x([2k+1]Q)) &\equiv 0(\text{mod } 2) \\
1 \leq v_t(x([2k]Q + T)) &\equiv 1(\text{mod } 2) \\
1 \leq v_t(x([2k+1]Q + T)) &\equiv 1(\text{mod } 2).
\end{align*}
\]

Proof. Case I. For \( v_t(x(Q)) = a \geq 3 \) (note that \( a \) is odd) : Using Tate’s algorithm in [12], we know the Tamagawa number \( c_t \) is equal to 2, which means \( E_{2r,D}(Q_t)/E_{2r,D,0}(Q_t) \cong \mathbb{Z}/2\mathbb{Z} \). Since \( Q \notin E_{2r,D,0}(Q_t) \), we have 

\[ [2k]Q \in E_{2r,D,0}(Q_t), \ [2k+1]Q \in E_{2r,D}(Q_t) \setminus E_{2r,D,0}(Q_t). \]

By the duplication formula, 

\[ x([2]Q) = \frac{x(Q)^4 + 4rDx(Q)^2 + 4(rD)^2}{4x(Q)^2 - 8rDx(Q)}, \]

so \( v_t(x([2]Q)) = -(a - 1) < 0 \), hence \([2]Q \in E_{2r,D,1}(Q_t) \). The isomorphism \( E_{2r,D,1}(Q_t) \cong \tilde{E}_{2r,D}(\mathcal{M}), (x, y) \mapsto (z(x) = -\frac{x}{y}, w(z)) \) together with Proposition 2.3 in [11] implies that \( v_t(z([2k]Q)) = v_t(2k) + \frac{a-1}{2} \). So 

\[ v_t(x([2k]Q)) = -2v_t(2k) - (a - 1), \]

hence \( v_t(x([2k+1]Q)) = -2v_t(2k + 1) - (a - 1) \), again the duplication formula gives the equality \( v_t(x([2k+1]Q)) = 1 - v_t(x([2k+1]Q)) \), thus 

\[ v_t(x([2k+1]Q)) = 2v_t(2k + 1) + a. \]
On the other hand, by the chord and tangent principal, we have
\[ x(R) \cdot x(R + T) = -2rD, \quad R \in E_{2rD}(\mathbb{Q}) \setminus \{O, T\} \]
so
\[ v_t(x([2k]Q + T)) = 2v_t(2k) + a, \]
\[ v_t(x([2k + 1]Q + T)) = -2v_t(2k + 1) - (a - 1). \]
By the argument above, we have
\[
\begin{cases}
-2 \geq v_t(x([2k]Q)) \equiv 0 \pmod{2} \\
3 \leq v_t(x([2k + 1]Q)) \equiv 1 \pmod{2} \\
3 \leq v_t(x([2k]Q + T)) \equiv 1 \pmod{2} \\
-2 \geq v_t(x([2k + 1]Q + T)) \equiv 0 \pmod{2}.
\end{cases}
\]
Case II. For \( v_t(x(Q)) = 1 \) : the duplication formula gives \( v_t(x([2]Q)) = 0 \), so \( \bar{E}_{ns}(\mathbb{F}_t) = \langle [2]Q \rangle \), where \([2]Q\) is the reduction of \([2]Q\) at the place \( t \). Since \( |\bar{E}_{ns}(\mathbb{F}_t)| = t \), hence \([t]([2]Q) = O\). Let \( b := -v_t(x([2]Q)) \), then \( b \) is an even integer bigger than one. Similarly, we have \( v_t(x([k]Q) = 1 + b \). Since \( \bar{E}_{ns}(\mathbb{F}_t) = \langle [2]Q \rangle \), for any integer \( k \) not divided by \( t \), the valuation \( v_t(x([k]Q)) \) is equal to zero. While \( t|k \), similarly to the Case I, we have
\[ v_t(x([2k]Q) = -2v_t(2k) - (b - 2). \]
Thus for any non-zero integer \( k \), we have
\[ v_t(x([2k]Q) = \begin{cases} 0 & \text{if } t \not| k \\ -2v_t(2k) - (b - 2) & \text{if } t|k, \end{cases} \]
\[ v_t(x([2k + 1]Q)) = \begin{cases} 1 & \text{if } t \not| 2k + 1 \\ 2v_t(2k + 1) + (b - 1) & \text{if } t|2k + 1, \end{cases} \]
\[ v_t(x([2k]Q + T) = \begin{cases} 1 & \text{if } t \not| k \\ 2v_t(2k) + (b - 1) & \text{if } t|k, \end{cases} \]
and
\[ v_t(x([2k + 1]Q + T) = \begin{cases} 0 & \text{if } t \not| 2k + 1 \\ -2v_t(2k + 1) - (b - 2) & \text{if } t|2k + 1. \end{cases} \]
so
\[
\begin{cases}
0 \geq v_t(x([2k]Q)) \equiv 0 \pmod{2} \\
1 \leq v_t(x([2k + 1]Q)) \equiv 1 \pmod{2} \\
1 \leq v_t(x([2k]Q + T)) \equiv 1 \pmod{2} \\
-2 \geq v_t(x([2k + 1]Q + T)) \equiv 0 \pmod{2}.
\end{cases}
\]
From Case I and Case II, we know that if \( v_t(x(Q)) \geq 1 \), then
\[
\begin{cases}
0 \geq v_t(x([2k]Q)) \equiv 0 \pmod{2} \\
1 \leq v_t(x([2k + 1]Q)) \equiv 1 \pmod{2} \\
1 \leq v_t(x([2k]Q + T)) \equiv 1 \pmod{2} \\
-2 \geq v_t(x([2k + 1]Q + T)) \equiv 0 \pmod{2}.
\end{cases}
\]
(4.1)
Case III. For \( v_t(x(Q)) = 0 \) : The nonsingular part of the reduction curve is generated by one point i.e. \( \bar{E}_{2rD,ns}(\mathbb{F}_t) = \langle \bar{Q} \rangle \), where \( \bar{Q} \) is the reduction of \( Q \). Similarly to the discussion in the Case II, for any non-zero integer \( k \), we have
\[
\begin{cases}
0 \geq v_t(x([2k]Q)) \equiv 0 \pmod{2} \\
0 \geq v_t(x([2k + 1]Q)) \equiv 0 \pmod{2} \\
1 \leq v_t(x([2k]Q + T)) \equiv 1 \pmod{2} \\
1 \leq v_t(x([2k + 1]Q + T)) \equiv 1 \pmod{2}.
\end{cases}
\]
Case IV. For $v_t(x(Q)) \leq -2$: Similarly to Case I, we have
\[
\begin{aligned}
-2 &\geq v_t(x([2k]Q)) \equiv 0 \pmod{2} \\
-2 &\geq v_t(x([2k+1]Q)) \equiv 0 \pmod{2} \\
3 &\leq v_t(x([2k]Q+T)) \equiv 1 \pmod{2} \\
3 &\leq v_t(x([2k+1]Q+T)) \equiv 1 \pmod{2}.
\end{aligned}
\]

From Case III and Case IV, we know that if $v_t(x(Q)) \leq 0$, then
\[
\begin{aligned}
0 &\geq v_t(x([2k]Q)) \equiv 0 \pmod{2} \\
0 &\geq v_t(x([2k+1]Q)) \equiv 0 \pmod{2} \\
1 &\leq v_t(x([2k]Q+T)) \equiv 1 \pmod{2} \\
1 &\leq v_t(x([2k+1]Q+T)) \equiv 1 \pmod{2}.
\end{aligned}
\tag{4.2}
\]

Using lemma 4.1, one can obtain the following observation.

**Corollary 4.2.** Assume the Parity conjecture, then for any $r \in A_D$,
\[
v_p(x([k]Q)) \neq v_q(x([k]Q)),
\]
where $Q$ is the generator of $E_{2rD}(\mathbb{Q})/E_{2rD}(\mathbb{Q})_{\text{tors}}$.

**Proof.** We first claim that, there exists two points $R_1, R_2 \in E_{2rD}(\mathbb{Q})$ such that
\[
v_p(x(R_i)) + 1 \equiv v_q(x(R_i)) \pmod{2} \quad \text{and} \quad v_p(x(R_i)) \cdot v_q(x(R_i)) \leq 0, \quad \text{for } i = 1, 2.
\tag{4.3}
\]
We take the case $(p, q) \equiv (5, 5) \pmod{8}$ and $(\frac{q}{p}) = 1$ for example. The proof for other cases are similar.

In this case, the prime $l$ is in the class $\alpha_T = 3$ and $(\frac{D}{p}) = -1$. For the elliptic curve $E_{2lD}$, we have $TS(E_{2lD}(\mathbb{Q})[\hat{\alpha}_l]) = 0$ and $S(\hat{\alpha}_l)(E_{2lD}(\mathbb{Q}) = \{1, -q, 2pl, -2Dl\}$. Thus $C_{-q}(\mathbb{Q}) \neq \emptyset, C_{2pl}(\mathbb{Q}) \neq \emptyset$. Take $(z_1, w_1) \in C_{2pl}(\mathbb{Q})$ and $(z_2, w_2) \in C_{-q}(\mathbb{Q})$. One can see $v_p(z_1) \leq 0, v_q(z_1) \geq 0$ and $v_p(z_2) \geq 0, v_q(z_2) \leq 0$. Denote the images of $(z_1, w_1)$ and $(z_2, w_2)$ under the following isomorphism
\[
\begin{aligned}
C_d &\rightarrow E_{2rD}, \quad (z, w) \mapsto \left(\frac{d}{z^2}, \frac{dw}{z^3}\right),
\end{aligned}
\]
by $R_1$ and $R_2$ respectively. So $R_1 := (\frac{2plw_1}{z_1^2}, \frac{2plw_1}{z_1^2})$ and $R_2 := (\frac{-q}{z_2^2}, \frac{-qw_2}{z_2^2})$. Thus $R_1, R_2$ satisfy the relations (4.3).

Now, by Lemma 4.1, we have $v_p(x(Q)) \geq 1, v_q(x(Q)) \leq 0$ or $v_p(x(Q)) \leq 0, v_q(x(Q)) \geq 1$. Hence, for any $k \in \mathbb{Z}$, (4.1) and (4.2) give
\[
v_p(x([2k+1]Q)) \neq v_q(x([2k+1]Q)).
\]

**Remark 4.3.** Corollary 4.2 implies that, given the generator $Q$ of $E_{2rD}(\mathbb{Q})/E_{2rD}(\mathbb{Q})_{\text{tors}}$, one can recover the prime divisor $p$ from $x([k]Q)$ and $D$. The complexity of searching rational points of elliptic curves over $\mathbb{Q}$ depends on the height of the rational points. For the elliptic curve $E_{2rD}$, the canonical height of rational points is known as follows,

**Proposition 4.4.** For any $r \in A_D$, let $P \in E_{2rD}(\mathbb{Q})$ be a non-torsion rational point, and write the $x-$coordinate of $P$ as $x = \frac{a}{D}$. Then,
\[
\hat{h}(P) \geq \frac{1}{16} \log(4rD),
\tag{4.4}
\]
\[
\frac{1}{4} \log(\frac{a^2 + 2rDd^4}{2rD}) \leq \hat{h}(P) \leq \frac{1}{4} \log(a^2 + 2rDd^4) + \frac{1}{12} \log 2.
\tag{4.5}
\]
Proof. We first estimate the archimedean contribution $\hat{h}_\infty$ to the canonical height by using Tate’s series. Arguing as Proposition 2.1 in [3] we have

$$0 \leq \hat{h}_\infty(P) - \frac{1}{4} \log(x(P)^2 + 2rD) + \frac{1}{12} \log |\Delta| \leq \frac{1}{12} \log 2. \quad (4.6)$$

Let $P_1 = 2P$, from the Lemma 4.1, it follows that $P_1 \in E_{2rD,0}(\mathbb{Q}_t)$ for any rational prime $t$, the local height of $P_1$ at $t$ is given by the formula Theorem 4.1 in [12]

$$\hat{h}_t(P) = \frac{1}{2} \max \{0, -\text{ord}_t(x(P)) \log t\}$$

$$+ \frac{1}{12} \text{ord}_t(\Delta) \log t - \begin{cases} 
0, & P \in E_{2rD,0}(\mathbb{Q}_t), \\
\frac{1}{4} \text{ord}_t(2rD) \log t, & P \notin E_{2rD,0}(\mathbb{Q}_t). 
\end{cases} \quad (4.7a)$$

$$\hat{h}_t(P_1) = \frac{1}{2} \max \{0, -\text{ord}_t(x(P_1)) \log t\} + \frac{1}{12} \text{ord}_t(\Delta) \log t.$$ 

Writing $x(P_1) = \frac{\alpha}{\delta^2}$, and combining (4.6) we obtain:

$$\hat{h}(P_1) \geq \frac{1}{4} \log(\alpha^2 + 2rD\delta^4)$$

Since $P_1 \in E_{2rD,0}(\mathbb{R})$, then $\alpha/\delta^2 \geq \sqrt{2rD}$, so in fact $\alpha^2 + 2rD\delta^4 \geq 4rD\delta^4 \geq 4rD$. This gives the lower bound

$$\hat{h}(P_1) \geq \frac{1}{4} \log(4rD)$$

and then the formula $\hat{h}(P_1) = \hat{h}(2P) = \hat{h}(P)$ gives the desired inequality as Proposition 4.4.

In order to prove the upper and lower bounds, rewrite the inequalities (4.6), (4.7a) and (4.7b) respectively, as

$$0 \leq \hat{h}_\infty(P) - \frac{1}{4} \log(x(P)^2 + 2rD) + \frac{1}{12} \log |\Delta| \leq \frac{1}{12} \log 2$$

$$- \frac{1}{4} \text{ord}_t(2rD) \log t \leq \hat{h}_t(P) - \text{ord}_t(d) \log t - \frac{1}{12} \text{ord}_t(\Delta) \log t \leq 0 (t \geq 2)$$

Adding these estimates over all places yields

$$- \frac{1}{4} \log(2rD) \leq \hat{h}(P) - \frac{1}{4} \log(\alpha^2 + 2rDd^4) \leq \frac{1}{12} \log 2.$$ 

Therefore the required inequality follows from the inequality above.

Remark 4.5. The lower bound (4.4) is a special case of a conjecture of Serge Lang [9], which says that the canonical height of a non-torsion point on elliptic curve should satisfy

$$\hat{h}(P) \gg \log |\Delta|,$$

where $\Delta = 2^9(rD)^3$ is the discriminant of $E_{2rD}$.

Remark 4.6. Using inequality (4.5), one can obtain the difference between the canonical height and the Weil height such as

$$- \frac{1}{4} \log(4rD) \leq \hat{h}(P) - \frac{1}{2} h(P) \leq \frac{1}{4} \log(2rD + 1) + \frac{1}{12} \log 2,$$

where $h(P)$ is the Weil height of $P$ defined as $h(P) = \frac{1}{2} \log \max\{|a|, |d^2|\}$. Thus if one can give an efficient and practical algorithm for an upper bound estimate of the canonical height, then by Zagier’s Theorem 1.1 in [10] one can find the generator of $E_{2rD}$.
ON THE ELLIPTIC CURVE $y^2 = x^3 - 2rDx$ AND FACTORING INTEGERS 9

5. THE PROOF OF THE MAIN RESULTS

Now let’s come to prove our main results theorem 1.1 and Corollary 1.2.

**Proof of theorem 1.1.** The definition of $A_D$, Corollary 3.4, Corollary 4.2 and the density theorem state that there are infinitely many integers $r$ such that $E_{2rD}$ has conjectural rank one and

$$v_p(x([k]Q)) \neq v_q(x([k]Q)),$$  

for any odd integer $k$,

where $Q$ is the generator of $E_{2rD}(\mathbb{Q})/E_{2rD}(\mathbb{Q})_{\text{tors}}$.

The following lemma shows that the least additional parameter $r$ in $A_D$ has a small upper bound.

**Lemma 5.1.** Given positive integers $k, m$, there exits a constant $c$ dependent only on $m$ and $k$ such that for any sequences of pairwise different odd primes $p_1, \ldots, p_k$, signs $\epsilon_1, \ldots, \epsilon_k \in \{-1, 1\}$, and an integer $a$ coprime to $m$ such that for any sequences of pairwise different odd primes $p_1, \ldots, p_k$, signs $\epsilon_1, \ldots, \epsilon_k \in \{-1, 1\}$, and an integer $a$ coprime to $m$ satisfying

$$(\frac{p_i}{l}) = \epsilon_i, \quad 1 \leq i \leq k, \quad \text{and} \quad l \equiv a \mod m$$

is upper bounded by $c \log^2(\prod_{i=1}^k p_i)$.

**Proof.** Let

$$d_i = \begin{cases} 
p_i & \text{if } p_i \equiv 1 \mod 4, \\
4p_i & \text{otherwise},
\end{cases} \quad (1 \leq i \leq k)$$

and $K_i = \mathbb{Q}(\sqrt{d_i}), 1 \leq i \leq k$. Then $\text{Gal}(K_i/\mathbb{Q}) \cong \{1, -1\}$. Identifying the two groups, we have

$$(\frac{p_i}{l}) = (\frac{K_i/\mathbb{Q}}{l}).$$

The latter is the Artin symbol.

Let $K_0 = \mathbb{Q}(\zeta_m)$, where $\zeta_m$ is a $m$-th primary root of unity. We have $\text{Gal}(K_0/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^*$, and $l \equiv a \mod m$ if and only if $\left(\frac{K_0/Q}{l}\right) = a \mod m$.

Let $K = \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_k}, \zeta_m)$, which is an abelian extension of $\mathbb{Q}$. An odd prime $l$ is un-ramified in $K$ if and only if $l \not| m \prod_{i=1}^k d_i$. For such $l$, we have

$$\left(\frac{K/Q}{l}\right) \equiv \left(\frac{K_i/Q}{l}\right), \quad (0 \leq i \leq k).$$

Since $p_1, \ldots, p_k$ are pairwise different and $(p_1 \ldots p_k, m) = 1$, we have $K_i \cap K_j = \mathbb{Q}$ for $0 \leq i \neq j \leq k$, and so we have the isomorphism

$$\text{Gal}(K/Q) \cong \prod_{0 \leq i \leq k} \text{Gal}(K_i/Q) \quad \sigma \mapsto (\sigma|_{K_i})_{0 \leq i \leq k}.$$  

Thus there exists a unique $\sigma \in \text{Gal}(K/Q)$ such that $\sigma|_{K_i} = \epsilon_i$ for $1 \leq i \leq k$ and $\sigma|_{K_0} = a \mod m$.

By Theorem 3.1(3) in [1], there exists a prime $l \leq (1 + o(1)) \log^2(|\Delta_K|)$ such that $\left(\frac{K/Q}{l}\right) = \sigma$, where $\Delta_K$ is the discriminant of $K$. For fixed $k$ and $m$, one can get easily that there exists a constant $c' > 0$ satisfying $|\Delta_K| < c'(\prod_{i=1}^k p_i)^{m^2k - 1}$. Thus we can find a constant $c$, dependent on $m$ and $k$, such that there exists a prime $l < c \log^2(\prod_{i=1}^k p_i)$ satisfying $\left(\frac{p_i}{l}\right) = \epsilon_i$ for $1 \leq i \leq k$, and $l \equiv a \mod m$.

Lemma 5.1 shows that there exists an integer $r \in O(\log^4 D)$ such that the elliptic curve $E_{2rD}$ has conjectural rank one and

$$v_p(x([k]Q)) \neq v_q(x([k]Q)),$$  

for any odd integer $k$, where $Q$ is the generator of $E_{2rD}(\mathbb{Q})/E_{2rD}(\mathbb{Q})_{\text{tors}}$.

**Proof of Corollary 1.2.** It follows directly from the theorem 1.1. \qed
Remark 5.2. Theorem 1.1 tells us that any point of infinite order in $E_{2rD}(\mathbb{Q})$ can be used to factor $D$ (if the point is of even order, use the duplication formula to "halving" it). Since $\text{rank}(E_{2rD}(\mathbb{Q}))$ is equal to one, beside standard methods, there are two other methods of searching for a point of infinite order. One method (see [13]) is the canonical height search algorithm, the other method (see[7] and [8]) is the Heegner point algorithm, both methods are still quite efficiently for conductor of moderately large(say $10^5$ to $10^7$). But both seem quite impractical in general if $D$ is greater than $10^8$.

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