Strong Edge-Coloring of Cubic Bipartite Graphs: A Counterexample

Daniel W. Cranston*

May 27, 2022

Abstract

A strong edge-coloring $\varphi$ of a graph $G$ assigns colors to edges of $G$ such that $\varphi(e_1) \neq \varphi(e_2)$ whenever $e_1$ and $e_2$ are at distance no more than 1. It is equivalent to a proper vertex coloring of the square of the line graph of $G$. In 1990 Faudree, Schelp, Gyárfás, and Tuza conjectured that if $G$ is a bipartite graph with maximum degree 3 and sufficiently large girth, then $G$ has a strong edge-coloring with at most 5 colors. In 2021 this conjecture was disproved by Lužar, Mačajová, Škoviera, and Soták. Here we give an alternative construction to disprove the conjecture.

1 Introduction

A strong edge-coloring $\varphi$ of a graph $G$ assigns colors to edges of $G$ such that $\varphi(e_1) \neq \varphi(e_2)$ whenever $e_1$ and $e_2$ are at distance no more than 1. (This is equivalent to a proper vertex coloring of the square of the line graph.) The strong chromatic index of $G$, denoted $\chi'_s(G)$ is the smallest number of colors that admits a strong edge-coloring. This notion was introduced in 1983 by Fouquet and Jolivet [6, 7]. In 1985 Erdős and Nešetřil conjectured, for every graph $G$ with maximum degree $\Delta$, that $\chi'_s(G) \leq \frac{5}{2} \Delta^2$ and that the lower order terms can be improved slightly when $\Delta$ is odd. This problem has spurred much work in the area, and Deng, Yu, and Zhou [3] survey results through 2019. In this note we focus on a conjecture from 1990 of Faudree, Schelp, Gyárfás, and Tuza [5].

Conjecture 1 [5]. Let $G$ be a graph with $\Delta(G) = 3$.

1. Now $\chi'_s(G) \leq 10$.
2. If $G$ is bipartite, then $\chi'_s(G) \leq 9$.
3. If $G$ is planar, then $\chi'_s(G) \leq 9$.
4. If $G$ is bipartite and for each edge $xy \in E(G)$ we have $d(x) + d(y) \leq 5$, then $\chi'_s(G) \leq 6$.
5. If $G$ is bipartite and has no 4-cycle, then $\chi'_s(G) \leq 7$.
6. If $G$ is bipartite and its girth is large, then $\chi'_s(G) \leq 5$.

Four parts of this conjecture have been confirmed. In the early 1990s Andersen [1] and Horák, Qing, and Trotter [8] proved (1). In 1993 Steger and Yu [12] proved (2). In 2016 Kostochka, Li, Ruksasakchai, Santana, Wang, and Yu [9] proved (3). And in 2008 Wu and Lin [13] proved (4). As far as we know, (5) remains open. In 2021 (6) was disproved by Lužar, Mačajová, Škoviera, and Soták [10]. Here we give an alternate (and, arguably, simpler) construction to disprove (6).

*Department of Computer Science, Virginia Commonwealth University, Richmond, VA, USA; dcranston@vcu.edu
2 Main Result

Our Main Theorem is motivated by the special case of $k$-regular graphs where $k = 3$, which is all that is needed to disprove Conjecture [3][6]. However, with only a bit more work we prove the result for all $k \geq 2$.

Main Theorem. For every positive integer $g$ and every integer $k \geq 2$, there exists a $k$-regular bipartite graph $G$ such that $G$ has girth at least $g$ and $\chi'_s(G) \geq 2k$.

We first prove the Main Theorem assuming two lemmas. We prove the lemmas below.

Proof. Fix positive integers $g$ and $k \geq 2$. By Lemma[2] if $n$ is sufficiently large then there exists a bipartite $k$-regular graph on $2n$ vertices with girth at least $g$. We choose such $n$ that is not divisible by $2k - 1$. Since $G$ is $k$-regular, $|E(G)| = \frac{3}{2}|V(G)| = kn$. Since $(2k - 1) \mid n$, and $k$ is relatively prime to $2k - 1$, also $(2k - 1) \mid |E(G)|$. Thus, Lemma[1] implies that $\chi'_s(G) \geq 2k$. □

We consider an arbitrary edge $e$ in a $k$-regular graph and the $2k - 2$ edges that share one endpoint with $e$; in the square of the line graph, the corresponding vertices form a clique. So each color in a strong edge-coloring of $G$ is used on at most one of these $2k - 1$ edges. By repeating this argument for every edge $e$, and averaging, we deduce that every color in a strong edge-coloring is used on at most $1/(2k - 1)$ of all edges. We formalize this idea below.

Lemma 1. If $G$ is $k$-regular and simple, for some $k \geq 2$, then in every strong edge-coloring $\varphi$ of $G$ every color class of $\varphi$ has size at most $|E(G)|/(2k - 1)$. In particular, if $(2k - 1) \not\mid |E(G)|$, then $\chi'_s(G) \geq 2k$.

Proof. Fix a simple $k$-regular graph $G$ and a strong edge-coloring $\varphi$ of $G$. Let $C$ be a set of edges receiving the same color under $\varphi$. For each $e \in E(G)$, let $N(e)$ denote the set of edges sharing at least one endpoint with $e$. Note that $e \in N(e)$ and $|N(e)| = 2k - 1$ for every $e \in E(G)$, since $G$ is $k$-regular. Furthermore, $e \in N(e')$ for exactly $2k - 1$ edges $e'$ (one of which is $e$), for each $e \in E(G)$. Since $\varphi$ is a strong edge-coloring, we get $|N(e) \cap C| \leq 1$ for every $e \in E(G)$. Thus,

$$\sum_{e \in E(G)} |C \cap N(e)| \leq \sum_{e \in E(G)} 1 = |E(G)|.$$

So $(2k - 1)|C| \leq |E(G)|$, giving $|C| \leq |E(G)|/(2k - 1)$. If also we have $(2k - 1) \not\mid |E(G)|$, then $|C| < |E(G)|/(2k - 1)$. Since $C$ is arbitrary, we get $\chi'_s(G) > |E(G)|/(|E(G)|/(2k - 1)) = 2k - 1$. That is, $\chi'_s(G) \geq 2k$. □

Lemma 2. Fix integers $k \geq 2$ and $g \geq 3$ and $n \geq g$. If also $n \geq 3 \cdot (k - 1)^{g-1}/(k - 2)$ when $k \geq 3$, then there exists a simple $k$-regular bipartite graph on $2n$ vertices with girth at least $g$.

Erdős and Sachs [4][11] each proved the existence of regular graphs with arbitrary degree and arbitrary girth. We follow the outline of [4] (see [2] Theorem III.1.4'), but we must adapt the proof to ensure that $G$ is also bipartite.

Proof. Fix $k$, $g$, and $n$ as in the lemma. Our proof is by induction on $k$. The base case, $k = 2$, holds by letting $G$ be a Hamiltonian cycle on $2n$ vertices. For the induction step, let $G$ be a $(k - 1)$-regular bipartite graph on $2n$ vertices with girth at least $g$. For each $A \subseteq E(G)$, we write $G + A$ to denote the graph formed from $G$ by adding each edge in $A$. We iteratively build an edge set $A$ such that $G + A$ is $k$-regular, bipartite, and has girth at least $g$. Since $G$ is bipartite, denote its parts by $X$ and $Y$. Given $A$, let $X_{low} := \{x \in X \mid d_{G+A}(x) = k - 1\}$ and $X_{high} := \{x \in X \mid d_{G+A}(x) = k\}$. Define $Y_{low}$ and $Y_{high}$ analogously. Note, for each $A$, that $X_{low}, X_{high}$ partition $X$ and $Y_{low}, Y_{high}$ partition $Y$. Since $G + A$ is bipartite, also $|X_{low}| = |Y_{low}|$.
and \(|X_{\text{high}}| = |Y_{\text{high}}|\). For all \(v, w \in V(G)\), denote by \(\text{dist}(v, w)\) the distance in \(G+A\) from \(v\) to \(w\).

For \(W_1 \subseteq V(G)\) and \(W_2 \subseteq V(G)\), let \(\text{dist}(W_1, W_2) := \min_{w_1 \in W_1, w_2 \in W_2} \text{dist}(w_1, w_2)\).

Initially, let \(A = \emptyset\). If \(|A| < n\), then we will show how to enlarge \(A\), either by adding a single edge, or by removing one edge and adding two.

If \(X_{\text{low}} = \emptyset\), then we are done. So assume both \(X_{\text{low}}\) and \(Y_{\text{low}}\) are nonempty. If there exist \(x_\ell \in X_{\text{low}}\) and \(y_\ell \in Y_{\text{low}}\) such that \(\text{dist}(x_\ell, y_\ell) \geq g - 1\), then we add edge \(x_\ell y_\ell\) (and are done). So assume no such \(x_\ell, y_\ell\) exist. The set of vertices at distance no more than \(g - 2\) from any \(x_\ell \in X_{\text{low}}\) has size at most \(1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{(g-2)} < (k-1)^{g-1}/(k-2)\). This set contains all of \(Y_{\text{low}}\), so we assume \(|X_{\text{low}}| = |Y_{\text{low}}| < (k-1)^{g-1}/(k-2)\). Note that \(|X_{\text{low}}| < |X|\), so \(|A| > 0\). Fix arbitrary \(x_\ell \in X_{\text{low}}\) and \(y_\ell \in Y_{\text{low}}\). We show there exists an edge \(x_h y_h \in A\) such that \(\text{dist}(x_\ell, y_\ell), \{x_h, y_h\}) \geq g - 1\); see Figure 1. Let \(A_{\text{bad}}\) denote the set of edges in \(A\) that fail this criteria; note that \(|A_{\text{bad}}| < 2(k-1)^{g-1}/(k-2)\). Since \(|X| \geq \lceil 3*(k-1)^{g-1}/(k-2) \rceil\) and \(|X_{\text{low}}| < (k-1)^{g-1}/(k-2)\), we have \(|A| - |A_{\text{bad}}| = |X_{\text{high}}| - |A_{\text{bad}}|) > 3*(k-1)^{g-1}/(k-2) - (k-1)^{g-1}/(k-2) = 0. Thus, the desired edge \(x_h y_h \in A\).

Form \(A'\) from \(A\) by removing \(x_h y_h\) and adding edges \(x_\ell y_h\) and \(y_\ell x_h\). Evidently, \(|A'| = |A| + 1\) and \(G + A'\) is bipartite with maximum degree \(k\). Thus, it suffices to check that \(G + A'\) has girth at least \(g\). By construction, each of \(x_\ell, y_\ell\) is distance at least \(g - 1\) from each of \(x_h, y_h\), so any cycle \(C\) of length less than \(g\) in \(G + A'\) must use both of edges \(x_\ell y_h\) and \(y_\ell x_h\). Since \(x_h y_h \in A\) and \(G + A\) has girth at least \(g\), every \(x_h, y_h\)-path in \(G + A - x_h y_h\) has length at least \(g - 1\). Thus, \(C\) contains vertices \(x_\ell, x_h, y_\ell, y_h\) in that cyclic order. But this contradicts that \(C\) has length less than \(g\), since (by construction) every \(x_\ell, x_h\)-path in \(G + A\) has length at least \(g - 1\).

Conjecture 2 below slightly weakens Conjecture 1(6), and generalizes it to graphs with \(\Delta = k\).

Conjecture 2. For each integer \(k \geq 3\), there exists a girth \(g_k\) such that if \(G\) is bipartite with girth at least \(g_k\), with \(\Delta(G) = k\), and with \(m\) edges, then \(\chi'_c(G) \leq 2k\) and \(G\) has a strong edge-coloring with colors \(1, \ldots, 2k\) that uses color \(2k\) on at most \(m - (2k-1)|m/(2k-1)|\) edges.

Acknowledgments

Thanks to two anonymous referees for helpful comments which improved the presentation.

References

[1] L. D. Andersen. The strong chromatic index of a cubic graph is at most 10. Discrete Math., 108(1-3):231–252, 1992. Topological, algebraical and combinatorial structures. Frolík’s memorial volume.
[2] B. Bollobás. *Extremal graph theory*, volume 11 of *London Mathematical Society Monographs*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1978.

[3] K. Deng, G. Yu, and X. Zhou. Recent progress on strong edge-coloring of graphs. *Discrete Math. Algorithms Appl.*, 11(5):1950062, 21, 2019.

[4] P. Erdős and H. Sachs. Reguläre Graphen gegebener Taillenweite mit minimaler Knotenzahl. *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe*, 12:251–257, 1963.

[5] R. J. Faudree, R. H. Schelp, A. Gyárfás, and Z. Tuza. The strong chromatic index of graphs. *Ars Combin.*, 29(B):205–211, 1990. Twelfth British Combinatorial Conference (Norwich, 1989).

[6] J.-L. Fouquet and J.-L. Jolivet. Strong edge-colorings of graphs and applications to multi-k-gons. *Ars Combin.*, 16(A):141–150, 1983.

[7] J.-L. Fouquet and J.-L. Jolivet. Strong edge-coloring of cubic planar graphs. In *Progress in graph theory (Waterloo, Ont., 1982)*, pages 247–264. Academic Press, Toronto, ON, 1984.

[8] P. Horák, H. Qing, and W. T. Trotter. Induced matchings in cubic graphs. *J. Graph Theory*, 17(2):151–160, 1993.

[9] A. V. Kostochka, X. Li, W. Ruksasakchai, M. Santana, T. Wang, and G. Yu. Strong chromatic index of subcubic planar multigraphs. *European J. Combin.*, 51:380–397, 2016, arXiv:1507.08959.

[10] B. Lužar, E. Mačajová, M. Škoviera, and R. Soták. Strong edge colorings of graphs and the covers of kneser graphs. *J Graph Theory*, pages 1–12, 2022, arXiv:2101.04768.

[11] H. Sachs. Regular graphs with given girth and restricted circuits. *J. London Math. Soc.*, 38:423–429, 1963. doi:10.1112/jlms/s1-38.1.423.

[12] A. Steger and M.-L. Yu. On induced matchings. *Discrete Math.*, 120(1-3):291–295, 1993.

[13] J. Wu and W. Lin. The strong chromatic index of a class of graphs. *Discrete Math.*, 308(24):6254–6261, 2008.