A unifying view on the irreversible investment exercise boundary in a stochastic, time-inhomogeneous capacity expansion problem

Maria B. Chiarolla *

Dipartimento di Scienze dell’Economia, Campus Ecotekne, University of Salento, 73047 Lecce, Italy

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Abstract. This paper devises a way to solve by the Bank and El Karoui Representation Theorem a quite complex stochastic, continuous time capacity expansion problem with irreversible investment on the finite time interval \([0, T]\), despite the presence of a state dependent scrap value associated with the production facility at the terminal time \(T\). Standard variational methods are not feasible but the functional to be maximized admits a supergradient, hence the optimal control satisfies some first order conditions which are solved by means of the Representation Theorem. The devise introduced is new in singular stochastic control and of interest in its own right. As far as we know the Representation Theorem has never been applied to this extent. In fact, contrary to the no scrap value case, a non integral term depending on the initial capacity \(y\) through the scrap value function arises in the supergradient making non trivial the application of the Representation Theorem and the existence of the so-called base capacity \(l^*_{y}(t)\), the unique solution of an integral equation. The base capacity is a positive level depending on \(y\) which the optimal investment process is shown to become active at.

In the special case of deterministic coefficients, discount factor, conversion factor, wage rate and interest rate, it is known from the literature that standard variational methods may be applied to show that the optimal investment process becomes active at a deterministic investment exercise boundary \(\hat{y}(t)\) arising from the optimal stopping problem classically associated to the original capacity problem. It is also known that, in the absence of scrap value at the terminal time \(T\), the base capacity \(l^*_{y}(t)\) given by the Representation Theorem is independent of \(y\) and equal to the boundary \(\hat{y}(t)\) obtained by variational methods. This paper shows that such result may be generalized to the scrap value case but under a further assumption that allows to obtain monotonicity and positiveness of \(\hat{y}(t)\) by means of probabilistic methods. As a byproduct of monotonicity, continuity of the optimal investment process is also obtained. Therefore, getting a unifying view on the curve at which is optimal to invest is possible even in the presence of scrap value but it requires adding some extra conditions. The advantage is that the integral equation of the base capacity may then be used to characterize \(\hat{y}(t)\).

Keywords: irreversible investment; singular stochastic control; multivariable production function; investment exercise boundary; the Bank and El Karoui Representation Theorem; boundary integral equation

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1 Introduction

This paper contributes to the literature on stochastic, continuous time capacity expansion with irreversible investment on a finite time interval \([0, T]\), introducing a devise to solve by the Bank and El Karoui Representation Theorem a quite complex model for which standard variational methods are not feasible.

The reader may find an extensive review of literature in [6], [8], among others, about singular stochastic control of expanding capacity, by optimally choosing the investment process. The classical approach is based on the study of the optimal stopping problem naturally associated to the control problem (cf. for example [4], [8]), the so-called optimal risk of not investing. The optimal investment process is then usually obtained in terms of the left-continuous inverse of the optimal stopping time, and that can be done even in general models under time-inhomogeneous diffusions, allowing random,
time dependent coefficients in the dynamics of the capacity process, and in some cases even under multivariable production functions as in \cite{6}, that might be awkward to handle (cf. \cite{7}). However, in order to obtain the free boundary $\hat{y}(t)$ at which it is optimal to invest, one must apply variational methods and hence coefficients and discount factor must be reasonably reconsidered, quite often taken constant. In that case, under some extra conditions on the production function (e.g. of Cobb-Douglas type), the investment exercise boundary $\hat{y}(t)$ may be characterized by an integral equation that often requires a priori continuity of the boundary itself. The investment boundary makes the strip $[0, T) \times (0, \infty)$ split into the Continuation Region $\Delta$ where it is not optimal to invest as the capital’s replacement cost is strictly greater than the shadow value of installed capital, and its complement, the Stopping Region $\Delta^c$ where it is optimal to invest instantaneously.

In a simplified version of the irreversible investment problem in \cite{6}, without scrap value and employment choice, \cite{5} was the first to study the investment exercise boundary by successfully exploiting some first order conditions together with the Bank and El Karoui Representation Theorem. In essence the Representation Theorem (cf. \cite{2}), applied to an optional process \( \{X(t), t \in [0, T]\} \) such that \( X(T) = 0 \), provides a representation of the form

\[
X(t) = E\left\{ \int_{(t,T]} f(s, \sup_{t \leq u' < s} \xi^*(u')) \mu(ds) \bigg| F_t \right\}
\]

in terms of a prescribed function \( f(t, \xi) \) strictly decreasing in \( \xi \), and a nonnegative optional random measure \( \mu \), with the progressively measurable process \( \xi^*(t) \) to be determined. Its connection with some stochastic optimization problems was shown in \cite{2}, \cite{3}. A good review of applications may be found in \cite{5}. Application to a different model of singular stochastic control problem of irreversible investment with inventory risk is more recently found in \cite{1}, after the earlier \cite{13}.

In \cite{5}, due to the lack of a scrap value at the terminal time \( T \), the first order conditions satisfied by the optimal investment process could be written in a form similar to (1.1) (see (3.7)); in fact, a non-zero scrap value would have added inside the conditional expectation a non integral term depending on the initial capacity. Then the Representation Theorem was used to obtain the optimal investment process in terms of the solution \( \xi^*(t) \) of (1.1), and hence in terms of the related base capacity \( l^*(t) \) (interpreted as the optimal capacity level for a firm starting at time \( t \) from zero capacity, cf. \cite{13}, Definition 3.1), which was independent of the initial capacity \( y \). Under the assumption of deterministic coefficients, \cite{5} managed to use some results in \cite{6} to show that the base capacity and the investment exercise boundary \( \hat{y}(t) \) were the same, and therefore \( l^*(t) \) had to be deterministic and its integral equation could be used to characterize \( \hat{y}(t) \) without any further assumption on it.

This paper devises a way to solve by the Bank and El Karoui Representation Theorem a quite complex stochastic, continuous time capacity expansion problem with irreversible investment entering additively a time-inhomogeneous diffusion on the finite time interval \([0, T]\), in the presence of a state dependent scrap value associated with the production facility at the terminal time \( T \). Standard variational methods are not feasible but the functional to be maximized admits a supergradient, hence the optimal control satisfies some first order conditions. To solve such conditions by means of the Representation Theorem a devise is introduced which is new in singular stochastic control and of interest in its own right. As far as we know the Representation Theorem has never been applied to this extent. In fact, contrary to the no scrap value case, the first order conditions do not have a form similar to (1.1) but an extra, non integral term, depending on the initial capacity \( y \) through the scrap value function, arises in the supergradient and makes non trivial the application of the Representation Theorem and the existence of the base capacity \( l_y^*(t) \), the unique solution of the corresponding integral equation.
In the present model the levels of production capacity \( C \), labor input \( L \) (at current random wage rate \( w(t) \)) and operating capital \( K \) (at current random interest rate \( r(t) \)) contribute to the firm's production and are optimally chosen in order to maximize the expected total discounted profits under the three-variable production function \( R(C, L, K) \). The setting is borrowed by the equilibrium model of a stochastic continuous time economy including also a financial market in \([10]\). So \( R(C, L, K) \) is handled by the corresponding reduced production function \( \tilde{R}(C, w, r) \), given in terms of conjugate functions (cf. \([14]\)). The optimal capacity expansion problem is then the maximization of the expected total discounted profit plus scrap value, net of investment,

\[
\mathcal{J}_y(\nu) = E\left\{ \int_0^T e^{-\int_0^t \mu F(u)du} \tilde{R}(C^y(t; \nu), w(t), r(t)) \, dt + e^{-\int_0^T \mu F(u)du} G(C^y(T; \nu)) - \int_{[0,T]} e^{-\int_0^t \mu F(u)du} \, d\nu(t) \right\}
\]

over the set of non-decreasing investment processes \( \nu(t) \) entering the time-inhomogeneous dynamics of the capacity process \( C^y \) starting at time zero from \( y > 0 \),

\[
dC^y(t; \nu) = C^y(t; \nu)[-\mu_C(t) \, dt + \sigma_C(t) \, dW(t)] + f_C(t) \, d\nu(t),
\]

where \( \mu_C \geq 0, \sigma_C, f_C \) are bounded, measurable, adapted processes. Here \( f_C \) is a conversion factor as one unit of investment is converted into \( f_C \) units of production capacity. It is evident that in such setting the classical variational approach is not easy to push through due also to the dependence on the stochastic processes \( w(t), r(t) \). Instead, by exploiting the existence of a supergradient of the strict concave functional \( \mathcal{J}_y(\nu) \), the optimal \( \nu \) may be characterized by first order conditions which we solve by means of the Bank and El Karoui Representation Theorem.

We handle the presence of the scrap vale at the terminal time \( T \) by suitably defining the optimal process \( X(t) \), the function \( f(t, \xi) \) and the optional random measure \( \mu(dt) \) on \([0, \infty)\) rather than on \([0, T]\). The method is quite involved while allowing quite general \( R, G, \) and random coefficients and discount factor. Notice that sometimes, in much simpler contexts, the time interval is fictitiously extended to \( \infty \) only in order to apply the Representation Theorem to a process \( X \) that satisfies \( X(\infty) = 0 \) but not \( X(T) = 0 \) (see point 1. below \([5,3]\)), as is the case with the discounted stock price \( e^{-rt}P_t, t \in [0, T] \), in \([3]\), Corollary 2.4. There, with \( f(t, x) = -x \) and \( \mu(dt) = re^{-rt} \, dt \), the process \( e^{-rs}P_{s \wedge T} \) is written as \( \int_{[s, \infty]} re^{-rt}P_{s \wedge T} \, dt \) for \( s \in [0, \infty) \), and it is 0 for \( s = \infty \).

For each initial capacity \( y \), we solve the optimal capacity expansion problem by obtaining the optimal investment process in terms of the base capacity \( l^*_y(t) \), a process linked to the unique solution \( \xi^*_y(t) \) of the integral equation given by the Representation Theorem. In fact, the base capacity is a positive level depending on \( y \) which the optimal investment process is shown to become active at.

In the special case of deterministic coefficients, discount factor, conversion factor, wage rate and interest rate, we prove that even in the scrap value case the base capacity \( l^*_y(t) \) may be independent of \( y \) and equal to the boundary \( \hat{y}(t) \) obtained by variational methods but, for that, we need a further assumption that allows to obtain monotonicity and strict positiveness of \( \hat{y}(t) \) by means of probabilistic methods. As a byproduct of monotonicity, continuity of the optimal investment process is also obtained, despite the lack of knowledge of continuity of \( \hat{y}(t) \). Such result is of interest in singular stochastic control with time-inhomogeneous dynamics. A similar result was obtained in \([6]\) but for the constant coefficients case. Obviously the optimal investment process is continuous if the boundary is so, and that is the case for most models in the literature which are restricted to geometric dynamics with constant coefficients or, more generally, limited to time-homogeneous diffusions.

In conclusion, getting a unifying view on the curve at which it is optimal to invest is possible even in the presence of scrap value but it requires adding some extra conditions. The advantage is that the
integral equation of the base capacity may then be used to characterize \( \hat{y}(t) \), whatever “cost” functions \( w(t) \) and \( r(t) \) one chooses, and without appealing to PDE methods that could be difficult to use in complex settings. Notice also that equality of \( \hat{y}(t) \) with the base capacity implies that the latter must be deterministic.

This paper is organized as follows. In Section 2 we set the model and recall the main properties of the reduced production function. In Section 3 we solve the first order conditions for optimality finding the solution via the Bank and El Karoui Representation Theorem obtaining the optimal investment process in terms of the base capacity \( I_0^X(t) \). In Section 4 the special case of deterministic coefficients is considered in the presence of scrap value. Under some extra conditions, monotonicity and strict positiveness of the investment exercise boundary \( \hat{y}(t) \) on \([0, T]\) are obtained, and then used in Section 5 to prove that the base capacity coincides with the investment exercise boundary. Section 4 also contains the proof of the continuity of the optimal investment process. An Appendix A completes the paper recalling the variational approach in [6] and generalizing it to the present setting.

2 The Model

To make the paper self-contained we recall the setting in [10] of a production facility (“the firm”) with finite horizon \( T \) on a complete probability space \((\Omega, \mathcal{F}, P)\) with filtration \( \{\mathcal{F}_t : t \in [0, T]\} \) given by the usual augmentation of the filtration generated by an exogenous \( n \)-dimensional Brownian motion \( \{W(t) : t \in [0, T]\} \). The firm produces output at rate \( R(C, L, K) \) when it has capacity \( C \), employs \( L \) units of labour and borrows \( K \) units of operating capital at current wages \( w \) and interest rate \( r \), respectively. The capital invested in technology on the time interval \([0, t]\) to increase capacity is denoted \( \nu(t) \), it is given by a process almost surely finite, left-continuous with right limits (“lcrl”), non-decreasing and adapted. The non-decreasing nature of \( \nu \) expresses the irreversibility of investment.

The three-variable production function \( R(C, L, K) \) is defined on all of \( \mathbb{R}^3 \) for convenience, but it may take the value \(-\infty\). It is finite on \( \text{dom}(R) := \{(C, L, K)^\top : R(C, L, K) > -\infty\} \). The \( C \)-section of \( \text{dom}(R) \) is defined by \( \text{dom}(R(C)) := \{(L, K)^\top : R(C, L, K) > -\infty\} \). We set \( \nabla_{L,K} R(C, L, K) := (R_L(C, L, K), R_K(C, L, K))^\top \) where \( R_X \) denotes the partial derivative with respect to \( X \). We denote \( \mathbb{R}^3_+ \) the positive orthant in \( \mathbb{R}^3 \), \( \mathbb{R}^3_+ \) its closure (i.e. the non-negative orthant), and \( \text{bdy}(A) \) the boundary of the set \( A \). Let \( \kappa_L \) and \( \kappa_K \) denote respectively the labour supply and the money supply, and let \( \kappa_w, \kappa_r \) be given constants. Then we make the following

**Assumption-[R]**

![Assumption-[R]](image)

**Assumption-[LK]**

![Assumption-[LK]](image)

The manager of the firm chooses labour \( L \) and operating capital \( K \) in order to maximize at each moment in time the production profits \( R(C, L, K) - wL - rK \), at current capacity \( C \), wages \( w \) and interest rate \( r \). Set \( A := [0, \kappa_L] \times [0, \kappa_K] \) and \( \bar{A} := [0, \infty) \times A \). Let \( R^A(C, \cdot, \cdot) \) denote \( R(C, \cdot, \cdot) \) modified.
as \(-\infty\) off \(A\). Define the “reduced production function”

\[
\hat{R}(C, w, r) := \max_{(L,K) \in A} \left[ R(C, L, K) - wL - rK \right],
\]

it is the maximal production profit rate. Notice that, for fixed \(C\), \(\hat{R}(C, w, r)\) is the negative of the concave conjugate of \(R^A(C, \cdot, \cdot)\) (cf. [4]), hence it is convex in \(w\) and in \(r\), and strictly concave in \(C\) (by a generalization of Proposition 5.1 in the Appendix of [6]). As in [6], Section 2 (with \(L\) replaced by \((L, K)\)) a unique solution exists and it is denoted by (with \(\top\) denoting “transpose”)

\[
(L^C(w, r), K^C(w, r)) := I^{R_A(C, \cdot, \cdot)}(w, r)
\]

where \(I^{R_A(C, \cdot, \cdot)}\) is an extension of the inverse of \(\nabla_{L,K} R^A(C, \cdot, \cdot)\) (cf. [7], Proposition 3.2).

**Remark 2.1** It holds the growth condition \(\sup_{C \geq 0} \max_{0, \kappa \in [0, \kappa]} \{ \tilde{R}(C, w, r) - \varepsilon C \} = \kappa_\varepsilon \) with \(\kappa_\varepsilon\) depending on \(\kappa_L\), \(\kappa_w\), \(\kappa_K\), \(\kappa_r\) and \(\varepsilon\). It follows by [7], Proposition 3.3, thanks to Assumption-[R] (see also [6]), and it is needed below in the proof of Proposition 2.2

The capacity process \(C^y(t; \nu)\) starting at time zero from level \(y > 0\) under the control \(\nu\) is given by

\[
\begin{align*}
\{ dC^y(t; \nu) &= C^y(t; \nu)[-\mu_C(t) dt + \sigma^\top_C(t) dW(t)] + f_C(t) d\nu(t), \quad t \in (0, T], \\
C^y(0; \nu) &= y > 0,
\end{align*}
\]

where \(f_C\) is a conversion factor as one unit of investment is converted into \(f_C\) units of production capacity, and \(\nu \in S\) with

\[
S := \{ \nu : [0, T] \to \mathbb{R} : \text{non-decreasing, lcrl, adapted process with } \nu(0) = 0 \text{ a.s.} \},
\]

a convex set. For the coefficients it holds the following

**Assumption-[C]**

\[
\begin{align*}
(\text{i}) \quad & \mu_C \geq 0, \sigma_C, f_C \text{ are bounded, measurable, adapted processes on } [0, T]; \\
(\text{ii}) \quad & f_C \text{ is continuous with } 0 < k_f \leq f_C \leq \kappa_f.
\end{align*}
\]

The firm has a finite planning horizon \(T\). At that time its value is \(G(C^y(T; \nu))\), the so-called *scrap value* of the firm. For \(G\) it holds the following

**Assumption-[G]**

\[
\begin{align*}
(\text{i}) \quad & G : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is concave, non-decreasing, continuously differentiable}; \\
(\text{ii}) \quad & \lim_{C \to \infty} G'(C) = 0, \quad G'(0)f_C(T) \leq 1 \text{ a.s.}.
\end{align*}
\]

An alternative to the limit condition in (ii) above is: \(G(C) \leq a_o + b_o C, b_0 \kappa_f < 1, a_o, b_o \geq 0\).

For every fixed \(s \in [0, T]\), define the exponential martingale

\[
\mathcal{M}_s(t) = e^{\int_s^t \sigma^\top_C(u) dW(u) - \frac{1}{2} \int_s^t \|\sigma_C(u)\|^2 du}, \quad t \in [s, T],
\]

then \(E\{[\mathcal{M}_s(t)]^p\} < \infty\) for any \(p\). The decay of a unit of initial capital in the absence of investment is denoted

\[
C^\alpha(t) := C^1(t; 0) = e^{-\int_0^t \mu_C(u) du} \mathcal{M}_0(t),
\]
then the solution of (2.4) is

\[ C^y(t; \nu) = C^o(t) [y + \int_{[0,t]} \frac{f_C(u)}{C_o(u)} d\nu(u)] = C^o(t)[y + \mathcal{V}(t)], \]

where \( \mathcal{V}(t) := \int_{[0,t]} \frac{f_C(u)}{C_o(u)} d\nu(u) \).

Over the finite planning horizon the firm chooses the investment process \( \nu \in \mathcal{S} \) in order to maximize the expected total discounted profit plus scrap value, net of investment,

\[ J_y(\nu) = E \left\{ \int_0^T e^{-\int_0^t \mu_F(u) du} \tilde{R}(C^y(t; \nu), w(t), r(t)) dt + e^{-\int_0^T \mu_F(u) du} G(C^y(T; \nu)) \right\}. \]

Here the firm discount rate \( \mu_F \) is assumed to be uniformly bounded in \( (t, \omega) \), nonnegative, measurable, adapted with \( \tilde{\mu} := \mu_C + \mu_F \geq \varepsilon_0 > 0 \) a.s. Note that \( (t, \omega) \mapsto \tilde{R}(C^y(t; \nu), w(t), r(t)) \) is measurable due to the continuity of \( \tilde{R}(C, w, r) \). The firm’s optimal capacity expansion problem is then

\[ V(y) := \max_{\nu \in \mathcal{S}} J_y(\nu). \]

Problem (2.11) is a slight modification of the “social planning problem” discussed by Baldursson and Karatzas [4]. Its solution may be found as in [6], Section 3 by means of the associated optimal stopping problem known as “the optimal risk of not investing”. [6] generalized the results in [8], which was based on some results in [4]. The unboundness of the reduced production function \( \tilde{R} \) is handled by the following estimates which allow to use some results in [4]. Its proof is as [6], Proposition 2.1.

**Proposition 2.2** There exists a constant \( K_J \) depending on \( T, \kappa_L, \kappa_w, \kappa_K, \kappa_r, \kappa_f, k_f \) only such that

(a) \( J_y(\nu) \leq K_J (1 + y) \) on \( \mathcal{S} \),

(b) \( E \left\{ \int_{[0,T]} e^{-\int_0^t \mu_F(u) du} d\nu(t) \right\} \leq 2K_J (1 + y) \) if \( J_y(\nu) \geq 0. \)

Moreover, \( \tilde{R} \) strictly concave in \( C, G \) concave, and \( C \) affine in \( \nu \) imply that \( J_y \) is concave on \( \mathcal{S} \); in fact it is strictly concave since \( 0 < k_f \leq f_C \).

The strict concavity of \( J_y(\nu) \) has a double implication. First, it implies that the solution of the optimal capacity expansion problem (2.11) is unique, if it exists. If we denote it by \( \hat{\nu} \), then the corresponding unique solution of (2.3) is given by

\[ (\hat{L}(t), \hat{K}(t)) := (L^{C^y(t; \hat{\nu})}(w(t), r(t)), K^{C^y(t; \hat{\nu})}(w(t), r(t))) = I^{R^A(C^y(t; \hat{\nu}); \cdot)}(w(t), r(t)). \]

Secondly, the functional \( J_y \) admits the supergradient (3.1) in the next section.

**Remark 2.3** Notice that there is a simpler way to calculate \( \tilde{R}_C(yC^o(u), w(u), r(u)) \). In fact, a generalization of Proposition 5.1 in the Appendix of [6] (based on the results in [7]) implies that

\[ \tilde{R}_C(C, w, r) = R_C(C, I^{R^A(C; \cdot)}(w, r)), \]

for \( w \) and \( r \) fixed, since \( \nabla_{L,K} R(C, I^{R^A(C; \cdot)}(w, r)) - (w, r)^\top \frac{\partial}{\partial C} I^{R^A(C; \cdot)}(w, r) = 0. \)
If $R$ is of the Cobb-Douglas type with zero shift, i.e. $R(C, L, K) = \frac{1}{\alpha \beta \gamma} C^\alpha L^\beta K^\gamma$ with $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma < 1$, then

$$\tilde{R}_C(C, w(t), r(t)) = \left[ \frac{1}{\beta \gamma} \left( \frac{\beta}{\alpha w(t)} \right)^\beta \left( \frac{\gamma}{\alpha r(t)} \right)^\gamma \right]^\frac{1}{1-\beta-\gamma}.$$ 

\[\square\]

3 Solving by the Bank-El Karoui Representation Theorem approach

To solve the optimal capacity expansion (2.11) is not an easy task. The underline dynamics has random time-depending coefficients and the production function is also a function of the stochastic processes $w(t), r(t)$. So variational methods are precluded and we must take a different approach.

We look for some first order conditions. The strict concavity of $J_y(\nu)$ implies the existence of the supergradient,

$$\nabla_{\nu} J_y(\nu)(\tau) = \frac{f_C(\tau)}{C_o(\tau)} E \left\{ \int_\tau^T e^{-\int_0^s \mu_F(u) du} C^o(t) \tilde{R}_C(C_y(t; \nu), w(t), r(t)) \, dt \right. $$

$$+ e^{-\int_0^\tau \mu_F(u) du} C^o(t) \left. \frac{C_y(T; \nu)}{1_{[0, T]}(\tau)} \right|_{\mathcal{F}_\tau} - e^{-\int_0^\tau \mu_F(u) du} 1_{[0, T]}(\tau)$$

for all $\{\mathcal{F}_t\}_{t \in [0, T]}$-stopping times $\tau : \Omega \rightarrow [0, T]$ and $\omega \in \Omega$ such that $\tau(\omega) < T$ (recall that $G'(\infty) = 0$).

For convenience we denote by $\Upsilon[0, T]$ the set of all $\{\mathcal{F}_t\}_{t \in [0, T]}$-stopping times taking values in $[0, T]$ and we characterize optimality through first-order conditions.

**Proposition 3.1** The control $\hat{\nu} \in \mathcal{S}$ is optimal for the capacity problem (2.11) if and only if it satisfies the first-order conditions

$$\left\{\begin{array}{l}
\nabla_{\nu} J_y(\hat{\nu})(\tau) \leq 0 \quad \text{a.s. for } \tau \in \Upsilon[0, T], \\
E \left\{ \int_{[0, T]} \nabla_{\nu} J_y(\hat{\nu})(t) \, d\hat{\nu}(t) \right\} = 0.
\end{array}\right.$$ 

In particular, (3.2) says that $\nabla_{\nu} J_y(\hat{\nu})(t)$ is zero at times $t$ of strict increase for $\hat{\nu}$. The arguments of the proof generalize those in [5], Theorem 3.2, to take into account that the conditional expectation in the supergradient of the present model contains also the non integral term involving $G'$ (and hence the initial capacity $y$) due to the scrap value at the terminal time $T$.

We aim to solve (3.2) by means of the Bank-El Karoui Representation Theorem (cf. [2], Theorem 3) which we briefly recall here. This celebrated Theorem, for $T$ not necessarily finite, guarantees the representation

$$X(\tau) = E \left\{ \int_{(\tau, T]} f(t, \sup_{\tau \leq u < t} \xi^*(u)) \mu(du) \bigg| \mathcal{F}_\tau \right\}, \quad \tau \in \Upsilon[0, T],$$

in terms of a unique optional process $\xi^*$ for given

1. $X(\omega, t) : \Omega \times [0, T] \rightarrow \mathbb{R}$ an optional process of class (D) (i.e. $\sup_{\tau \in \Upsilon[0, T]} E\{X(\tau)\} < +\infty$), lower-semicontinuous in expectation (i.e. for any sequence of stopping times $\tau_n \nearrow \tau$ a.s. for some $\tau \in \Upsilon[0, T]$, it holds $\liminf_n E\{X(\tau_n)\} \geq E\{X(\tau)\}$) and such that $X(T) = 0$,

2. $\mu(\omega, dt)$ a nonnegative optional random Borel measure,
3. \( f(\omega, t, x) : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R} \) a random field such that
   (i) for any \( x \in \mathbb{R} \), the stochastic process \((\omega, t) \to f(\omega, t, x)\) is progressively measurable and belonging to \( L^1(P(\omega) \otimes \mu(\omega, dt))\),
   (ii) for any \((\omega, t) \in \Omega \times [0, T]\), the mapping \( x \to f(\omega, t, x)\) is continuous and strictly decreasing from \(+\infty\) to \(-\infty\).

The unique optional process \( \xi^* \) is found by introducing, for \( \xi < 0 \) and \( t \in [0, T] \), the optimal stopping problem
\[
\Gamma^\xi(t) := \text{ess inf}_{\tau \leq t \leq T} \mathbb{E}\left\{ \int_t^\tau f(u, \xi) \mu(du) + X(\tau) \mid \mathcal{F}_t \right\}.
\]
In fact, an application of \[2\], Lemma 4.12, implies the continuity of the mapping \( \xi \to \Gamma^\xi(\omega, t) \) for any fixed \((\omega, t) \in \Omega \times [0, T]\), its non-increasing property from \( \Gamma^{-\infty}(t) := \lim_{\xi \to -\infty} \Gamma^\xi(t) = X(t) \), and the optimality of the stopping time
\[
\tau^\xi(t) := \inf\{ u \in [t, T) : \Gamma^\xi(u) = X(u) \} \wedge T, \quad t \in [0, T),
\]
for \( \Gamma^\xi(t) \), which may be taken right-continuous in \( t \). Then the Bank-El Karoui Representation Theorem (cf. \[2\], equation (23) and Lemma 4.13) shows that the solution \( \xi^* \) is the optional process
\[
\xi^*(t) := \sup\{ \xi < 0 : \Gamma^\xi(t) = X(t) \}, \quad t \in [s, T),
\]
which is also uniquely determined up to optional sections if shown to be upper right-continuous à la Bourbaki (cf. \[2\], Theorem 1), i.e. if \( \xi^*(s) = \lim_{\varepsilon \downarrow 0} \sup_{u \in [s, (s+\varepsilon) \wedge T]} \xi^*(u) \) (such limit is greater than or equal to what is commonly called limit superior, in fact it is its upper envelope).

The Bank-El Karoui Representation Theorem was successfully employed in \[5\] to solve the much simpler capacity expansion model with no scrap value, no labor and interest rate, therefore involving a single variable production function \( R \). There the conditional expectation in the supergradient contained only the integral term (no \( G' \) term and hence no starting capacity \( y \)) and a look at it inspired that the optional process to be represented would be \( X(\omega, t) := e^{-\int_0^t \mu_F(v) dv} \frac{C^\alpha(\omega, t)}{C^\beta(\nu, t)} \mathbb{I}_{[0, T]}(t) \). Its representation in terms of a unique optional process \( \xi^*(t) \) (cf. \[5\], Lemma 4.1) was
\[
X(\tau) = \mathbb{E}\left\{ \int_\tau^T e^{-\int_\tau^\tau \mu_F(u) du} C^\alpha(t) R_C\left( \frac{C^\alpha(\omega, t)}{\sup_{\tau \leq u < t} \xi^*(u)} \right) dt \bigg| \mathcal{F}_\tau \right\}.
\]
The Inada condition holding for the production function \( R \) allowed to satisfy point 3(iii) above as well as the non-increasing property of the corresponding \( \Gamma^\xi \) from \( \Gamma^{-\infty}(t) := \lim_{\xi \to -\infty} \Gamma^\xi(t) = X(t) \). As one can see, in the absence of scrap value, \( X, \Gamma, \xi^* \) were independent of the initial capacity \( y \). It cannot be so in our model with non-zero scrap value as the supergradient involves \( G' \), whose argument \( \frac{C^\beta(T, w)}{C^\beta(\nu, t)} \mathbb{I}_{[0, T]}(t) \) depends on \( y \).

As pointed out before, in order to apply the Bank-El Karoui Representation Theorem we need to assume some sort of Inada condition. Hence we make the further
\textbf{Assumption-[I]}
\[
\begin{align*}
\lim_{C \to -\infty} \tilde{R}_C(C, w, r) & = +\infty \quad \text{for any } (w, r) \in (0, \kappa_w) \times (0, \kappa_r), \\
G' : \mathbb{R}_+ & \to \mathbb{R}_+ \text{ strictly decreasing.}
\end{align*}
\]
To handle the scrap value at terminal time $T$, that is the non integral term in the conditional expectation appearing in the supergradient (3.1), we shall suitably define the processes $X, \mu, f$ on $[0, \infty)$ in order to satisfy all the requirements while trying to recover the non integral term by integrating over the extra time interval $[T, \infty)$. Such devise is new in singular stochastic control and of interest in its own right. As far as we know the Representation Theorem has never been applied to this extent.

By (3.9) it is clear that

\begin{equation}
X^y(\omega, t) := \begin{cases} 
  e^{-\int_0^t \mu_F(\omega, \omega, t)} C^\alpha(\omega, t) f_C(\omega) \, dt, & t \in [0, T), \\
  e^{\int_0^T \mu_F(\omega, \omega, t)} e^{\mu_F(T)(t-T)} C^\alpha(\omega, T) G'(y C^\alpha(\omega, T)), & t \in [T, \infty); 
\end{cases}
\end{equation}

(3.10)

Whereas for

\begin{equation}
\mu(\omega, dt) := \begin{cases} 
  e^{-\int_0^t \mu_F(\omega, \omega, t)} C^\alpha(\omega, t) dt, & \text{on } [0, T), \\
  e^{-\int_0^T \mu_F(\omega, \omega, t)} e^{\mu_F(T)(t-T)} C^\alpha(\omega, T) dt, & \text{on } [T, \infty); 
\end{cases}
\end{equation}

(3.11)

\begin{equation}
f(\omega, t, x) := \begin{cases} 
  \tilde{R}_C(\frac{C^\alpha(\omega, t)}{x}) w(t, r(t)), & x < 0, t \in [0, T), \\
  \mu_F(\omega, T) G'(\frac{C^\alpha(\omega, T)}{x}), & x < 0, t \in [T, \infty), \\
  -x, & x \geq 0. 
\end{cases}
\end{equation}

(3.12)

\textbf{Proposition 3.2} Under Assumption-[I], for $y > 0$ and $X, \mu, f$ given by (3.9), it holds the representation

\begin{equation}
X^y(\tau) = E\left\{ \int_0^\infty f(t, \sup_{\tau \leq u < \tau} \xi^y_u(u')) \mu(dt) \bigg| F_\tau \right\}, \quad \tau \in \mathcal{Y}[0, T],
\end{equation}

(3.13)

for a unique optional, right-continuous, strictly negative process $\xi^y(\tau)$.

\textbf{Proof:} Set (cf. (3.11))

\begin{equation}
\Gamma^y(\tau) := \operatorname{ess inf}_{\tau \leq \tau < \infty} E\left\{ \int_\tau^\infty f(u, \xi) \mu(du) + X^y(\tau) \bigg| F_\tau \right\} \quad \text{for } \xi < 0, t \in [0, \infty).
\end{equation}

(3.14)

By (3.9) it is clear that $X^y(\infty) = 0$, and all the requirements about the process $X$, the random measure $\mu(\omega, dt)$ and the random field $f(\omega, t, x)$ are satisfied. In fact for $t \geq T$ our $f(\omega, t, x)$ is strictly decreasing from $\mu_F(T) G'(0)$ to $-\infty$, rather than from $-\infty$ to $-\infty$, but this is all that is needed to obtain $\Gamma^y(\infty) = X^y(t)$. Then the solution $\xi^y(\tau)$ exists and is unique. Its upper right-continuity and strict negativity on $[0, T]$ follow from arguments as in [5], Lemma 4.1, based on the definition of $f(\omega, t, x)$ and the right-continuity of $\Gamma^y(\tau)$.

It follows that

\begin{equation}
\tau^y(\tau) := \inf\{u \in [t, \infty) : \Gamma^y_u(u) = X(u)\}
\end{equation}

(3.15)

is optimal for $\Gamma^y(\tau)$, for $t \geq 0$ (cf. (3.5)). The next Lemma provides the explicit calculation of $\Gamma^y(\tau)$.

\textbf{Lemma 3.3} Under Assumption-[I], for $y > 0$, $\xi < 0$, and $t \in [0, T)$,

\begin{equation}
\Gamma^y(\tau) = \operatorname{ess inf}_{\tau \leq \tau < T} E\left\{ \int_\tau^T e^{-\int_u^\tau \mu_F(r) dr} C^\alpha(r) \tilde{R}_C(\frac{C^\alpha(r)}{-\xi}) w(r), r(\tau) \right\} du
\end{equation}

(3.16)

\begin{equation}
\quad + e^{-\int_0^T \mu_F(t) dr} C^\alpha(T) \mathbb{1}_{\{\tau < T\}} + e^{-\int_0^T \mu_F(t) dr} C^\alpha(T) G'(y C^\alpha(T)) \mathbb{1}_{\{\tau = T\}} \bigg| F_\tau \bigg\}
\end{equation}

(3.17)

\textbf{Whereas for } $t \in [T, \infty)$,

\begin{equation}
\Gamma^y(\tau) = \operatorname{ess inf}_{\tau \leq \tau < \infty} E\left\{ e^{-\int_u^\tau \mu_F(r) dr} C^\alpha(T)
\end{equation}

(3.18)

\begin{equation}
\times \left[ (1 - e^{-\mu_F(t)(T-T)}) G'(\frac{C^\alpha(T)}{-\xi}) + e^{-\mu_F(t)(T-T)} G'(y C^\alpha(T)) \right] \bigg| F_\tau \right\}.
\end{equation}

(3.19)
Proof: For \( t \in [T, \infty) \) an explicit calculation of \( \Gamma^{y, \xi}(T) \) gives (3.14). For \( t \in [0, T) \) one has

\[
\Gamma^{y, \xi}(t) := \text{ess inf}_{t \leq \tau < \infty} E \left\{ \int_t^{\tau \wedge T} e^{-\int_t^u \mu_F(r) dr} C^0(u) \tilde{R}_C \left( \frac{C^0(u)}{-\xi}, w(u), r(u) \right) du + e^{-\int_t^T \mu_F(r) dr} C^0(T) \mathbf{1}_{\{\tau < T\}} \right. \\
+ e^{-\int_t^T \mu_F(r) dr} G'(yC^0(T)) \mathbf{1}_{\{\tau = T\}} \left| \mathcal{F}_t \right. \right\}
\]

due to \( \mu_F > 0 \) and \( G' > 0 \). On the other hand, by restricting the set of stopping times, one has

\[
\Gamma^{y, \xi}(t) \leq \text{ess inf}_{t \leq \tau \leq T} E \left\{ \int_t^T e^{-\int_t^u \mu_F(r) dr} C^0(u) \tilde{R}_C \left( \frac{C^0(u)}{-\xi}, w(u), r(u) \right) du + e^{-\int_t^T \mu_F(r) dr} C^0(T) \mathbf{1}_{\{\tau < T\}} + e^{-\int_t^T \mu_F(r) dr} G'(yC^0(T)) \mathbf{1}_{\{\tau = T\}} \left| \mathcal{F}_t \right. \right\}
\]

and the result follows. \( \square \)

Since the terminal time of our capacity problem is \( T \), we need to know what values \( \xi^*_y \) takes on \([T, \infty] \).

Lemma 3.4 Under Assumption-[I], for \( y > 0 \),

\[
(3.15) \quad \xi^*_y(t) = -\frac{1}{y} \quad \text{for all } t \geq T.
\]

Proof: Recall that \( \xi^*_y(t) := \sup\{ \xi < 0 : \Gamma^{y, \xi}(t) = X^y(t) \} \) (cf. (3.16)).

By (3.14) for \( t = T \), if \( -\frac{1}{y} < \xi < 0 \) then \( \Gamma^{y, \xi}(T) = e^{-\int_0^T \mu_F(r) dr} C^0(T) G'(\frac{C^0(T)}{-\xi}) < X^y(T) \) with the infimum attained at \( \tau^{y, \xi}(T) = \infty \). Whereas for \( \xi < -\frac{1}{y} \) the infimum is attained at \( \tau^{y, \xi}(T) = T \) and \( \Gamma^{y, \xi}(T) = e^{-\int_0^T \mu_F(r) dr} C^0(T) G'(yC^0(T)) \) is independent of \( \xi \) (so \( \Gamma^{y, \infty}(T) = X^y(T) \) is satisfied); that same value of \( \Gamma^{y, \xi}(T) \) is found for \( \xi = -\frac{1}{y} \) since \( \Gamma^{y, -\frac{1}{y}}(T) \) is the infimum of a constant argument, independent of \( \tau \). Therefore \( \xi^*_y(T) = -\frac{1}{y} \).

Similarly, for \( t > T \) (cf. (3.14))

\[
\Gamma^{y, \xi}(t) = \text{ess inf}_{t \leq \tau < \infty} E \left\{ e^{-\int_t^T \mu_F(r) dr} C^0(T) \right. \times \left[ G'(\frac{C^0(T)}{-\xi}) + e^{-\mu_F(T)(t-T)} \left( e^{-\mu_F(T)(t-T)} G'(yC^0(T)) - G'(\frac{C^0(T)}{-\xi}) \right) \right] \left| \mathcal{F}_t \right. \right\}
\]

Therefore, if \( e^{-\mu_F(T)(t-T)} G'(yC^0(T)) > G'(\frac{C^0(T)}{-\xi}) \), then the infimum is attained at \( \tau^{y, \xi}(t) = \infty \), \( \Gamma^{y, \xi}(t) = e^{-\int_0^T \mu_F(r) dr} C^0(T) G'(\frac{C^0(T)}{-\xi}) < X^y(t) \) and \( \xi > -\frac{1}{y} \), since \( G'(yC^0(T)) > G'(\frac{C^0(T)}{-\xi}) \). Instead, \( G'(yC^0(T)) - G'(\frac{C^0(T)}{-\xi}) \leq 0 \) for \( \xi \leq -\frac{1}{y} \), so that \( G'(yC^0(T)) - e^{-\mu_F(T)(t-T)} G'(\frac{C^0(T)}{-\xi}) < 0 \). Hence the
in infimum is attained at \( \tau^{y,\xi}(t) = t \) and \( \Gamma^{y,\xi}(t) = e^{-\int_0^T \mu_F(r)dr} C^0(T)e^{-\mu_F(T)(t-T)}G'(yC^0(T)) = X^y(t) \) is independent of \( \xi \) (again \( \Gamma^{y,-\infty}(t) = X^y(t) \), and (3.15) follows.

\[ \square \]

Notice that for \( t \in [0, T] \) and \( \xi \leq -\frac{1}{y} \) the optimal stopping time \( \tau^{y,\xi}(t) \) of \( \Gamma^{y,\xi}(t) \) (cf. (3.12)) reduces to

\[ \tau^{y,\xi}(t) = \inf \{ u \in [t, T) : \Gamma^{y,\xi}(u) = e^{-\int_0^u \mu_F(r)dr} \frac{C^0(u)}{f_C(u)} \} \wedge T. \]

Using Lemma 3.4 we obtain

**Proposition 3.5** Under Assumption-1, for \( y > 0 \) there exists a unique optional, upper right-continuous, negative process \( \xi^*_y(t) \) that, for all \( \tau \in \Upsilon[0, T] \), solves the representation problem

\[ e^{-\int_0^t \mu_F(r)dr} \frac{C^0(\tau)}{f_C(\tau)} \mathbb{1}_{[0, T]}(\tau) = E \left\{ \int_T^\infty e^{-\int_0^t \mu_F(r)dr} C^0(t) \tilde{R}_C \left( \sup_{\tau \leq u \leq t} \frac{C^0(u)}{\xi^*_y(u)} w(t), r(t) \right) dt \right. 
+ e^{-\int_0^t \mu_F(r)dr} C^0(T) G' \left( \sup_{\tau \leq u \leq T} \frac{C^0(u)}{\xi^*_y(u)} \right) \left. \frac{f^*_y(t)}{f_C(t)} \right\}. \]

**Proof:** It suffices to apply the Bank-El Karoui Representation Theorem to the optional process \( X^y(t) \), the nonnegative optional random Borel measure \( \mu(\omega, dt) \), and the random field \( f(\omega, t, x) \) defined in (3.9). From (3.10) with \( \tau = T \) we have

\[ X^y(T) = e^{-\int_0^T \mu_F(r)dr} C^0(T) E \left\{ \int_T^\infty e^{-\mu_F(T)(t-T)} C^0(t) \tilde{R}_C \left( \sup_{\tau \leq u \leq t} \frac{C^0(u)}{\xi^*_y(u)} w(t), r(t) \right) dt \right\}. \]

Uniqueness of \( \xi^*_y \) implies \( \sup_{T \leq u \leq t} \xi^*_y(u) \) is zero for all \( t > T \) and such constant is \( -\frac{1}{y} \) by (3.15). Now take \( \tau \in \Upsilon[0, T] \) in (3.10), then

\[ e^{-\int_0^t \mu_F(r)dr} \frac{C^0(\tau)}{f_C(\tau)} \mathbb{1}_{[0, T]}(\tau) = E \left\{ \int_T^\infty e^{-\int_0^t \mu_F(r)dr} C^0(t) \tilde{R}_C \left( \sup_{\tau \leq u \leq t} \frac{C^0(u)}{\xi^*_y(u)} w(t), r(t) \right) dt \right. 
+ e^{-\int_0^t \mu_F(r)dr} C^0(T) \int_T^\infty e^{-\mu_F(T)(t-T)} C^0(T) \tilde{R}_C \left( \sup_{\tau \leq u \leq t} \frac{C^0(u)}{\xi^*_y(u)} \right) \left. \frac{f^*_y(t)}{f_C(t)} \right\}. \]

(3.18)

and integrating the exponential in the last integral provides (3.17). Notice that in the last line \( G' \) is zero for all \( \omega \) such that \( \tau(\omega) \notin [0, T] \), since \( G'(+\infty) = 0 \), and the indicator is a reminder of the requirement \( \tau(\omega) < T \) for the validity of the equation.

\[ \square \]

Now we set

\[ l^*_y(t) := -\frac{C^0(t)}{\xi^*_y(t)}, \quad t \in [0, T]. \]
The strictly positive process $l^*_y(t)$ is the base capacity of the present model (see [13] for the original Definition 3.1). The reason for being so-called will be clear soon after the concluding Theorem below.

**Corollary 3.6** Under Assumption-[I], for $y > 0$ the optional process $l^*_y(t)$ is the unique upper right-continuous, positive solution of the representation problem

$$
(3.20) \quad E \left\{ \int_\tau^T e^{-\int_0^\tau \mu_F(r) \, dr} C^o(t) \tilde{R}_C \left( C^o(t) \sup_{\tau \leq \tau' < t} \frac{l^*_y(u')}{C^o(u')} , w(t), r(t) \right) \, dt \right. 
+ e^{-\int_0^\tau \mu_F(r) \, dr} C^o(T) G' \left( C^o(T) \sup_{\tau \leq \tau' < T} \frac{l^*_y(u')}{C^o(u')} \vee y \right) \bigg| \mathcal{F}_\tau \right\} = e^{-\int_0^\tau \mu_F(r) \, dr} \frac{C^o(\tau)}{f_C(\tau)} 1_{[0,T]}(\tau),
$$

for all $\tau \in \Upsilon[0,T]$.

**Proof:** The proof is straightforward after plugging (3.19) into (3.17) (see also [5], Lemma 4.1). In fact,

$$
\frac{1}{-\sup_{\tau \leq \tau' \leq \tau} l^*_y(\omega')} = \frac{1}{\inf_{\tau \leq \tau' \leq \tau} C^o(\omega')} = \sup_{\tau \leq \tau' \leq \tau} \frac{l^*_y(\omega')}{C^o(\omega')} \text{ for any } t \in T.
$$

At this point, a more careful look at (3.20) brings back to mind the supergradient (3.1) and we can finally solve the very general original capacity expansion problem, for each initial capacity $y$. In fact, the optimal control investment process may be obtained in terms of the base capacity $l^*_y(t)$ (cf. the simpler case of [5] with no scrap value).

**Theorem 3.7** Under Assumption-[I], the stochastic control process

$$
(3.21) \quad \nu^y_t(t) := \int_{[0,t]} \frac{C^o(u)}{f_C(u)} \, d\tilde{\tau}^y_t(u), \quad t \in [0,T],
$$

with

$$
(3.22) \quad \left\{ \begin{array}{l}
\tilde{\tau}^y_t(0) = 0,
\tilde{\tau}^y_t(t) := \sup_{0 \leq u < t} \left[ \frac{l^*_y(u')}{C^o(u')} \vee y \right] - y = \sup_{0 \leq u < t} \left[ \frac{l^*_y(u')}{C^o(u')} - y \right]^+,
\end{array} \right. \quad t \in (0,T),
$$

solves the first-order conditions (3.2). Hence $\nu^y_t(t)$ is the unique optimal investment process of problem (2.11).

**Proof:** Recall (2.9), then $C^y(t; \nu^y_t) = C^o(t)[y + \tilde{\tau}^y_t(t)] = C^o(t) \sup_{0 \leq u < t} \left[ \frac{l^*_y(u')}{C^o(u')} \vee y \right]$. Then, for all $\tau \in \Upsilon[0,T]$ and $\omega \in \Omega$ such that $\tau(\omega) < T$,

$$
E \left\{ \int_\tau^T e^{-\int_0^\tau \mu_F(r) \, dr} C^o(t) \tilde{R}_C \left( C^o(t; \nu^y_t), w(t), r(t) \right) \, dt + e^{-\int_0^\tau \mu_F(r) \, dr} C^o(T) G' \left( C^o(T; \nu^y_t) \right) \bigg| \mathcal{F}_\tau \right\} 
\leq E \left\{ \int_\tau^T e^{-\int_0^\tau \mu_F(r) \, dr} C^o(t) \tilde{R}_C \left( C^o(t) \sup_{\tau \leq \tau' < t} \frac{l^*_y(u')}{C^o(u')}, w(t), r(t) \right) \, dt \right. 
+ e^{-\int_0^\tau \mu_F(r) \, dr} C^o(T) G' \left( C^o(T) \sup_{\tau \leq \tau' < T} \frac{l^*_y(u')}{C^o(u')} \vee y \right) \bigg| \mathcal{F}_\tau \right\} = e^{-\int_0^\tau \mu_F(r) \, dr} \frac{C^o(\tau)}{f_C(\tau)} 1_{[0,T]}(\tau),
$$

by the decreasing property of $R_C$ and $G'$ in $C$ and by (3.20). Hence $\nabla_\nu J_y(\nu^y_t(\tau)) \leq 0$.

Moreover, at times $\tau$ of strict increase for $\tilde{\tau}^y_t$, the above holds with equality since $C^y(t; \nu^y_t) = C^o(t) \sup_{\tau \leq \tau' < t} \frac{l^*_y(u')}{C^o(u')}$ for $\tau < t \leq T$. Therefore $\nabla_\nu J_y(\nu^y_t)(\tau) = 0$ when $d\tilde{\tau}^y_t(\tau) > 0$ and (3.2) follows.

\[ \square \]
For a firm starting at time 0 from a capacity level \( y \) lower than \( l_y^0(0) \), it is easy to see from (3.22) that \( \tau_y^0((0+) \) is the jump size required to instantaneously reach the optimal capacity level \( l_y^*(0) \). Similarly, the base capacity \( l_y^*(t) \) represents the optimal capacity level at time \( t \) for a firm starting at time 0 from capacity \( y \). For that reason the optimal capacity process \( C^y(t; \nu^y) \) can be referred to as the capacity process that tracks \( l_y^* \).

There is no closed form solution for the solution \( l_y^*(t) \) of equation (3.20), but it might be found numerically by backward induction on discretized version of the equation, as suggested by Bank and Föllmer [8], Section 4, for their model.

4 The case of deterministic coefficients

When the following Assumption-[det]

\[
\mu_C, \sigma_C, f_C, \mu_F, w, r \text{ are deterministic functions}
\]

holds, the singular stochastic control problem of capacity expansion may be studied by variational methods through the optimal stopping problem associated to it, the so-called optimal risk of not investing until time \( t \), and the free boundary \( \hat{y}(t) \) arising from it. It is then natural to ask whether and how the base capacity \( l_y^*(t) \) (which depends on the initial capacity \( y \)) is linked to the free boundary \( \hat{y} \). In [5] it was shown that \( l^*(t) = \hat{y}(t) \) but there \( l^* \) did not depend on \( y \) as the model had no scrap value at the terminal time \( T \). In the present case with scrap value, as we shall see, some further conditions are needed in order to obtain the equality, and hence the independence of the base capacity from \( y \).

It can be shown that the capacity problem may be imbedded allowing the capacity process to start at any time \( s \in [0, T] \) from level \( y > 0 \) (by generalizing, among others, the model with scrap value, constant coefficients and additive production function depending on labor in [6], the model with constant coefficients but no scrap value in [8], the model with deterministic time-dependent coefficients but no scrap value, no labor, no interest rate in [5]) . The details are provided in Appendix A for completeness, but here we recall the main points. First of all, the value function of the firm’s optimal capacity problem \( V(s, y) \) (see (A.3) as compared to (2.11)) and the value function \( v(s, y) \) of the associated optimal stopping problem are linked; in fact, \( v(t, y) \) is the shadow value of installed capital, that is \( v(s, y) = \frac{\partial}{\partial y} V(s, y) \) (cf. (A.12)). The free boundary \( \hat{y}(t) \) makes the strip \( [0, T] \times (0, \infty) \) split into the Continuation Region

\[
\Delta = \left\{ (s, y) \in [0, T] \times (0, \infty) : v(s, y) < \frac{1}{f_C(s)} \right\}
\]

where it is not optimal to invest as the capital’s replacement cost is strictly greater than the shadow value of installed capital, and its complement, the Stopping Region \( \Delta^c \) where it is optimal to invest instantaneously (cf. (A.16), (A.17)).

Also, setting \( C^s(t) := C^s(t; \nu^s) = e^{-\int_0^t \mu_C(r) dr} M_s(t) \) (cf. (2.7)) and \( Y^{s,y}(t) := y C^s(t) \) for \( s \in [0, T] \), \( t \geq s, y > 0 \), and performing a change of probability measure allows to rewrite the “optimal risk of not investing” \( v(s, y) \) under the new discount factor \( \hat{\mu}(t) := \mu_C(t) + \mu_F(t) \geq \varepsilon_0 > 0 \) (cf. (A.15)), and then prove that \( \hat{y}(s) \) is the lower boundary of \( \Delta \) in the \( (s, y) \)-plane (cf. (A.19)),

\[
\hat{y}(s) = \sup \left\{ z \geq 0 : v(s, z) = \frac{1}{f_C(s)} \right\},
\]
and that the unique optimal investment process $\hat{v}^{s,y}$ of the capacity problem can be obtained in terms of $\nu^{s,y}$ (cf. (A.9)), the left-continuous inverse of the optimal stopping time for $v(s,y)$. It follows that

$$\nu^{s,y}(t) := \sup_{s \leq u < t} \left[ \frac{\hat{y}(u)}{C^s(u)} - y \right]^+ \quad \text{for } t > s,$$

with $\nu^{s,y}(s) := 0$ and $C^{s,y}(t; \hat{v}^{s,y}) \geq \hat{y}(t)$ a.s.

Apparently the function $\hat{y}(t)$ might be related to the stochastic process $l^*_y(t)$. In fact the controls $\nu^{s,y}$ and $\nu^s_y$ are similar and we know that the optimal investment process is unique, so by identifying the controls we would like to conclude that $\hat{y}(t)$ and $l^*_y(t)$ are the same but, again, the latter depends on the initial capacity $y$. To handle this $y$-dependence some properties of $\hat{y}(t)$ are needed.

We stress that all the effort in the rest of this Section is not needed in the case of no scrap value. We start by looking at the $s$-sections of $\Delta$ (cf. (A.16)).

**Proposition 4.1** Under Assumption-[det], for each $s \in [0,T]$ fixed,

(i) $v(s, \cdot)$ is non-increasing in $y$;

(ii) $v(s, \cdot)$ is strictly decreasing on $y > \hat{y}(s)$;

(iii) the set $\{y > 0 : v(s,y) < 1/f_C(s)\}$ is connected.

**Proof:** Recall that $\hat{R}(C,w,r)$ is strictly concave in $C$, hence $\hat{R}_C$ is strictly decreasing in $C$, and $G'$ is non-increasing. Thus, $Y^{s,y_1}(t) < Y^{s,y_2}(t)$ for $y_1 < y_2$ implies that $v(s,y)$ is non-increasing in $y$ and strictly decreasing for $y > \hat{y}(s)$. It follows that the set $\{y > 0 : v(s,y) < 1/f_C(s)\}$ is connected for each fixed $s$. \(\square\)

On the other hand, the study of the $y$-sections of $\Delta$ is complicated by the time dependence of $f_C$. The following result takes care of it. Some of the arguments in the proof are similar to those needed to prove Theorem A.3 (see the discussion above it) but this proof is much more tricky and requires the extra conditions below, so we provide it for completeness.

**Proposition 4.2** Under Assumption-[det] and the further conditions

$$\begin{align*}
\begin{cases}
\hat{R}_C(C,w,r) \text{ and } G'(C) \text{ are convex in } C \in (0,\infty), \\
\|\sigma_C\|^2 \leq \mu_C \quad \text{a.e. in } [0,T], \\
f_C \in C^1([0,T]) \quad \text{with } \bar{\mu} \leq -f'_C/f_C \quad \text{a.e. on } [0,T],
\end{cases}
\end{align*}$$

for $y > 0$ fixed,

(i) $v(s,y) - \frac{1}{f_C(s)}$ is non-increasing in $s \in [0,T]$;

(ii) the set $\{s \geq 0 : v(s,y) < 1/f_C(s)\}$ is connected.

**Proof:** If, for fixed $y > 0$, we show that $v(s,y) - \frac{1}{f_C(s)}$ is non-increasing in $s$, then point (ii) will follow from $v(s,y) - \frac{1}{f_C(s)} \leq 0$ (which is obtained by taking $\tau = s$ in (A.15)). So we show point (i). Fix $s_1 < s_2$, then (see (A.15))

$$v(s_1, y) = \inf_{\tau \in [s_1,s_2]} E^{Q,s_1} \left\{ \int_{s_1}^{s_2} e^{-\int_{s_1}^{u} \bar{\mu}(r)dr} \hat{R}_C(Y^{s_1,y}(u),w(u),r(u)) du + e^{-\int_{s_1}^{s_2} \bar{\mu}(r)dr} \frac{1}{f_C(\tau)} \mathbb{I}_{\{\tau < T\}} + e^{-\int_{s_1}^{s_2} \bar{\mu}(r)dr} G'(Y^{s_1,y}(T)) \mathbb{I}_{\{\tau = T\}} \right\}$$

$$= \inf_{\tau \in [s_1,s_2]} E^{Q,s_1} \left\{ \mathbb{I}_{\{\tau < s_2\}} \left[ \int_{s_1}^{\tau} e^{-\int_{s_1}^{u} \bar{\mu}(r)dr} \hat{R}_C(Y^{s_1,y}(u),w(u),r(u)) du + e^{-\int_{s_1}^{s_2} \bar{\mu}(r)dr} \frac{1}{f_C(\tau)} \right] \right\}$$
Then
\begin{align*}
&\left\{ \int_{s_1}^{\bar{s}} e^{-\int_{s_1}^{u} \bar{\mu}(r)dr} \tilde{R}_C(Y^{s_1,y}(u), w(u), r(u))du + e^{-\int_{s_1}^{\bar{s}} \bar{\mu}(r)dr} \right\} \\
&\left\{ \int_{s_2}^{\bar{s}} e^{-\int_{s_2}^{u} \bar{\mu}(r)dr} \tilde{R}_C(Y^{s_2,y}(u), w(u), r(u))du + e^{-\int_{s_2}^{\bar{s}} \bar{\mu}(r)dr} \right\} \\
&\left\{ \int_{s_2}^{\bar{s}} e^{-\int_{s_2}^{u} \bar{\mu}(r)dr} \tilde{R}_C(Y^{s_2,y}(u), w(u), r(u))du + e^{-\int_{s_2}^{\bar{s}} \bar{\mu}(r)dr} \right\} + e^{-\int_{s_2}^{\bar{s}} \bar{\mu}(r)dr} G'(Y^{s_1,y}(T))(1_{\{\tau = T\}})
\right]\}.
\end{align*}

Set \( \bar{\tau} := \tau \lor s_2 \) and notice that
\begin{align*}
&\left\{ \int_{s_1}^{\bar{\tau}} e^{-\int_{s_1}^{u} \bar{\mu}(r)dr} \tilde{R}_C(Y^{s_1,y}(u), w(u), r(u))du + e^{-\int_{s_1}^{\bar{\tau}} \bar{\mu}(r)dr} \right\} \\
&\left\{ \int_{s_2}^{\bar{\tau}} e^{-\int_{s_2}^{u} \bar{\mu}(r)dr} \tilde{R}_C(Y^{s_2,y}(u), w(u), r(u))du + e^{-\int_{s_2}^{\bar{\tau}} \bar{\mu}(r)dr} \right\} \\
&\left\{ \int_{s_2}^{\bar{\tau}} e^{-\int_{s_2}^{u} \bar{\mu}(r)dr} \tilde{R}_C(Y^{s_2,y}(u), w(u), r(u))du + e^{-\int_{s_2}^{\bar{\tau}} \bar{\mu}(r)dr} \right\} + e^{-\int_{s_2}^{\bar{\tau}} \bar{\mu}(r)dr} G'(Y^{s_1,y}(T))(1_{\{\tau = T\}})
\right]\}.
\end{align*}

Then
\begin{align*}
v(s_1, y) &= \inf_{\tau \in \mathcal{Y}_{s_1}[s_1, T]} \mathbb{E}Q_{s_1} \left\{ 1_{\{\tau < s_2\}} \int_{s_1}^{\tau} e^{-\int_{s_1}^{u} \bar{\mu}(r)dr} \tilde{R}_C(Y^{s_1,y}(u), w(u), r(u))du \\
&+ e^{-\int_{s_1}^{\tau} \bar{\mu}(r)dr} \right\} \\
&\left\{ 1_{\{s_2 \leq \tau \leq T\}} \int_{s_1}^{s_2} e^{-\int_{s_1}^{u} \bar{\mu}(r)dr} \tilde{R}_C(Y^{s_1,y}(u), w(u), r(u))du + e^{-\int_{s_1}^{s_2} \bar{\mu}(r)dr} \right\} \\
&\left\{ 1_{\{s_2 \leq \tau \leq T\}} \int_{s_2}^{s_1} e^{-\int_{s_2}^{u} \bar{\mu}(r)dr} \tilde{R}_C(Y^{s_2,y}(u), w(u), r(u))du + e^{-\int_{s_2}^{s_1} \bar{\mu}(r)dr} \right\} + e^{-\int_{s_2}^{s_1} \bar{\mu}(r)dr} G'(Y^{s_1,y}(T))(1_{\{\tau = T\}})
\right]\}.
\end{align*}

Assumption-[G] provides an upper bound on the second negative term and hence
\begin{align*}
v(s_1, y) &\geq \inf_{\tau \in \mathcal{Y}_{s_1}[s_1, T]} \mathbb{E}Q_{s_1} \left\{ 1_{\{\tau < s_2\}} \int_{s_1}^{\tau} e^{-\int_{s_1}^{u} \bar{\mu}(r)dr} \tilde{R}_C(Y^{s_1,y}(u), w(u), r(u))du \\
&+ e^{-\int_{s_1}^{\tau} \bar{\mu}(r)dr} \right\} \\
&\left\{ 1_{\{s_2 \leq \tau \leq T\}} \int_{s_1}^{s_2} e^{-\int_{s_1}^{u} \bar{\mu}(r)dr} \tilde{R}_C(Y^{s_1,y}(u), w(u), r(u))du + e^{-\int_{s_1}^{s_2} \bar{\mu}(r)dr} \right\} \\
&\left\{ 1_{\{s_2 \leq \tau \leq T\}} \int_{s_2}^{s_1} e^{-\int_{s_2}^{u} \bar{\mu}(r)dr} \tilde{R}_C(Y^{s_2,y}(u), w(u), r(u))du + e^{-\int_{s_2}^{s_1} \bar{\mu}(r)dr} \right\} + e^{-\int_{s_2}^{s_1} \bar{\mu}(r)dr} G'(Y^{s_1,y}(T))(1_{\{\tau = T\}})
\right]\}.
\end{align*}
For the latter expectation we proceed as in the proof of Theorem \[\text{A.3}\]. We consider the canonical probability space \((\bar{\Omega}, \bar{P})\) where now \(\bar{\Omega} = C_0[\bar{s}_1, T]\) (the space of continuous functions on \([\bar{s}_1, T]\) which are zero at \(\bar{s}_1\)) and \(\bar{P}\) is the Wiener measure on \(\bar{\Omega}\). We define \(W_{Q_1} \cdot \bar{\omega} = W_{Q_1}(t, \omega) = \bar{w}(t)\) the coordinate mapping on \(C_0[\bar{s}_1, T]\) with \(\bar{w} = (\bar{w}_1, \bar{w}_2)\), where \(\bar{w}_1 = \{W_{Q_1}(u) - W_{Q_1}(s_1) : s_1 \leq u \leq s_2\}\), \(\bar{w}_2 = \{W_{Q_1}(u) - W_{Q_1}(s_2) : s_2 \leq u \leq T\}\). Hence \(\bar{P}\) is a product measure on \(C_0[\bar{s}_1, T] = C_0[\bar{s}_1, s_2] \times C_0[\bar{s}_2, T]\), due to independence of the increments of \(W_{Q_1}\). Then, for each \(\bar{\omega}_1 \in \bar{\Omega}\) fixed, \(\bar{\tau}_{\bar{\omega}_1}(\cdot) := \tau(\bar{\omega}_1, \cdot) \vee s_2 \in \bar{Y}_{s_2}[s_2, T]\) with \(\tau(\bar{\omega}_1, \cdot)\) measurable w.r.t. \(\bar{F}_{s_2, T}\). If \(E_{\bar{\omega}_1}^Q\{\cdot\}\) denotes expectation over \(\bar{\omega}_1\) and, for each \(\bar{\omega}_1 \in \bar{\Omega}\), the expectation over \(\bar{\omega}_2\) is denoted by \(\Phi(t, Y_{s_1, y}(t); \bar{\tau}_{\bar{\omega}_1})\) (as in the proof of Theorem \[\text{A.3}\]), then the last expectation above is written as \(E_{\bar{\omega}_1}^P\{\Phi(t, Y_{s_1, y}(t); \bar{\tau}_{\bar{\omega}_1})\}\). Therefore for the last infimum above it holds \(\inf_{\tau' \in Y_{s_2}[s_2, T]} E_{\bar{\omega}_1}^P\{\Phi(t, Y_{s_1, y}(t); \bar{\tau}_{\bar{\omega}_1})\} = \inf_{\tau' \in Y_{s_2}[s_2, T]} E_{\bar{\omega}_1}^P\{\Phi(t, Y_{s_1, y}(t); \tau')\}\), and hence

\[v(s_1, y) \geq \inf_{\tau' \in Y_{s_1}[s_1, T]} E_{Q_1}^s \left\{ \int_{s_1}^{\tau'} e^{-\int_{s_1}^{u} \hat{\mu}(r) dr} \bar{R}_C(Y_{s_1, y}(u), w(u), r(u)) du + \mathbb{1}_{\{\tau' < s_2\}} \left[ e^{-\int_{s_2}^{\tau'} \hat{\mu}(r) dr} \frac{1}{f_C(\tau')} - e^{-\int_{s_1}^{s_2} \hat{\mu}(r) dr} \frac{1}{f_C(s_2)} \right] \right\}
\]

\[+ e^{-\int_{s_1}^{s_2} \hat{\mu}(r) dr} \inf_{\tau' \in Y_{s_2}[s_2, T]} E_{Q_1}^s \left\{ \int_{s_2}^{\tau'} e^{-\int_{s_2}^{u} \hat{\mu}(r) dr} \bar{R}_C(Y_{s_1, y}(u), w(u), r(u)) du + \mathbb{1}_{\{\tau' < s_2\}} \left[ e^{-\int_{s_2}^{\tau'} \hat{\mu}(r) dr} \frac{1}{f_C(\tau')} + e^{-\int_{s_1}^{s_2} \hat{\mu}(r) dr} G'(Y_{s_1, y}(T)) \mathbb{1}_{\{\tau' = T\}} \right] \right\} \]

(4.3)

Now consider the second infimum in (4.3). For the first term we have

\[E_{Q_1}^s \left\{ \int_{s_2}^{s_1} e^{-\int_{s_2}^{u} \hat{\mu}(r) dr} \bar{R}_C(Y_{s_1, y}(u), w(u), r(u)) du \right\} \]

\[= E_{Q_1}^s \left\{ \int_{s_2}^{\tau'} e^{-\int_{s_2}^{u} \hat{\mu}(r) dr} \bar{R}_C(Y_{s_1, y}(u), w(u), r(u)) \mathbb{1}_{\{\tau' \in \bar{F}_{s_2, T}\}} du \right\} \]

\[\geq E_{Q_1}^s \left\{ \int_{s_2}^{\tau'} e^{-\int_{s_2}^{u} \hat{\mu}(r) dr} \bar{R}_C(Y_{s_1, y}(u), w(u), r(u)) du \right\} \]

since \(\bar{R}_C\) is convex in \(C\) by (4.2) and \(Q_{s_1}|\bar{F}_{s_2, T}\) is a regular conditional probability distribution. Moreover \(C_{s_2}(\cdot)\) is \(\bar{F}_{s_2, T}\)-measurable and \(Y_{s_1, y}(s_2)\) is independent of \(\bar{F}_{s_2, T}\). Then, for \(u > s_2\), using \(\mu_C \geq \|\sigma_C\|^2\) a.e. (cf. (4.2)2) we get

\[E_{Q_1}^s \{Y_{s_1, y}(u)|\bar{F}_{s_2, T}\} = E_{Q_1}^s \{Y_{s_1, y}(s_2)|C_{s_2}(u)|\bar{F}_{s_2, T}\} = E_{Q_1}^s \{Y_{s_1, y}(s_2)|\bar{F}_{s_2, T}\} = E_{Q_1}^s \{Y_{s_1, y}(s_2)|\bar{F}_{s_2, T}\} \]

\[= E_{Q_1}^s \{Y_{s_1, y}(s_2)|\bar{F}_{s_2, T}\} \leq y C_{s_2}(u) \]

where \(M_{\tau_1}^{Q_1}(t) = e^{\int_0^t \frac{\sigma_C(r)^2}{2} dr} \int_0^t e^{-\int_{s_2}^{r} \hat{\mu}(r) dr} \bar{R}_C(Y_{s_1, y}(u), w(u), r(u)) du \)

and \(Q_{s_1} = Q_{s_1,s_2} \otimes Q_{s_2}\) gives

\[E_{Q_1}^s \left\{ \int_{s_2}^{\tau'} e^{-\int_{s_2}^{u} \hat{\mu}(r) dr} \bar{R}_C(Y_{s_1, y}(u), w(u), r(u)) du \right\} \]
\[
E^{Q_{s_2}} \left\{ \int_{s_2}^{\tau_s} e^{-\int_{s_2}^{u} \bar{\mu}(r) \, dr} \tilde{R}_C(y C^u s_2(u), w(u), r(u)) \, du \right\}
\]

being the last integral independent of \(F_{s_1, s_2}\).

Similar arguments apply to the term involving \(G^t\) in (4.3), using the fact that \(G^t\) is non-increasing and convex by (4.2). Therefore the last infimum in (4.3) is greater or equal \(v(s_2, y)\), and we have

\[
v(s_1, y) \geq \inf_{\tau \in \mathcal{T}_{s_1}[s_1, T]} E^{Q_{s_1}} \left\{ \mathbf{1}_{\{\tau < s_2\}} \int_{s_1}^{\tau} e^{-\int_{s_1}^{u} \bar{\mu}(r) \, dr} \tilde{R}_C(Y^{s_1, y}(u), w(u), r(u)) \, du \right\}
\]

\[
+ \mathbf{1}_{\{\tau < s_2\}} \left[ e^{-\int_{s_1}^{\tau} \bar{\mu}(r) \, dr} - e^{-\int_{s_1}^{s_2} \bar{\mu}(r) \, dr} \frac{1}{f_C(\tau)} \right] + e^{-\int_{s_1}^{s_2} \bar{\mu}(r) \, dr} v(s_2, y).
\]

Now subtracting \(\frac{1}{f_C(s_1)}\) from both sides, adding and subtracting \(e^{-\int_{s_1}^{s_2} \bar{\mu}(r) \, dr} \frac{1}{f_C(s_2)}\) on the right-hand side, and recalling that \(\bar{\mu} > 0\) give

\[
v(s_1, y) - \frac{1}{f_C(s_1)} \geq \inf_{\tau \in \mathcal{T}_{s_1}[s_1, T]} E^{Q_{s_1}} \left\{ \mathbf{1}_{\{\tau < s_2\}} \int_{s_1}^{\tau} e^{-\int_{s_1}^{u} \bar{\mu}(r) \, dr} \tilde{R}_C(Y^{s_1, y}(u), w(u), r(u)) \, du \right\}
\]

\[
+ \mathbf{1}_{\{\tau < s_2\}} \left[ e^{-\int_{s_1}^{\tau} \bar{\mu}(r) \, dr} - e^{-\int_{s_1}^{s_2} \bar{\mu}(r) \, dr} \frac{1}{f_C(\tau)} \right] - \frac{1}{f_C(s_1)}
\]

\[
+ e^{-\int_{s_1}^{s_2} \bar{\mu}(r) \, dr} \frac{1}{f_C(s_2)} + e^{-\int_{s_1}^{s_2} \bar{\mu}(r) \, dr} v(s_2, y) - \frac{1}{f_C(s_2)}.
\]

By condition (4.2)3 the last expectation above is non-negative, hence \(e^{-\int_{s_1}^{s_2} \bar{\mu}(r) \, dr} v(s, y) - \frac{1}{f_C(s)}\) is non-increasing in \(s\), but the exponential is decreasing and \(v(s, y) - \frac{1}{f_C(s)} \leq 0\) by (A.15), therefore \(v(s, y) - \frac{1}{f_C(s)}\) itself must be non-increasing in \(s\) and point (i) is proved. 

Now the above results enable us to prove the following Theorem that will be needed in Section 5.1 to identify the free boundary with the base capacity in the present time-inhomogeneous model with scrap value at the terminal time \(T\). It is a generalization of [6], Proposition 3.3 which was limited to the constant coefficients case.

**Theorem 4.3** Under Assumption-[det] and conditions (4.2),

(i) \(\hat{y}(s)\) is non-increasing on \([0, T]\);

(ii) \(\hat{y}(s)\) is strictly positive on \([0, T]\) if also Assumption-[I]1 of Section 3 holds.

**Proof:** Point (i) follows from (A.19) and point (i) of Proposition 4.2. In fact, \(0 \geq v(s_1, y) - \frac{1}{f_C(s_1)} \geq v(s_2, y) - \frac{1}{f_C(s_2)}\) for \(s_2 > s_1\) implies \(\hat{y}(s_2) \leq \hat{y}(s_1)\). The lower boundary of the Borel set \(\Delta\) is the continuation region of problem (A.15), is graph(\(\hat{y}\)), and \(\Delta\) lies above it. The first exit time of \((t, Y^{s,y}(t))\) from \(\Delta\) is \(\hat{\tau}(s, y)\) by (A.15). To prove point (ii), suppose to the contrary that Assumption-[I]1 holds and \(\hat{y}(t) = 0\) for some \(t < T\), then \(\hat{y}(s) \equiv 0\) for \(s \in [t, T]\) since \(\hat{y}\) is non-increasing. Then by the dynamics of \(Y^{s,y}\) it follows that \(\hat{\tau}(s, y) = T\) for all \((s, y) \in [t, T] \times (0, \infty)\) and hence

\[
v(s, y) = E^{Q_s} \left\{ \int_{s}^{T} e^{-\int_{s}^{u} \bar{\mu}(r) \, dr} \tilde{R}_C(Y^{s,y}(u), w(u), r(u)) \, du \right\} + e^{-\int_{s}^{T} \bar{\mu}(r) \, dr} G^t(Y^{s,y}(T)) < \frac{1}{f_C(s)},
\]
which is impossible since the expected value blows up as \( y \downarrow 0 \) by Assumption-[I]1. \( \square \)

As a byproduct we obtain continuity of the optimal investment process.

**Corollary 4.4** Under Assumption-[det] and conditions \((4.2)\), the optimal investment process \( \tilde{\nu}^{s,y}(t) \) is continuous except possibly for an initial jump, hence so is the optimal capacity process \( C^{s,y}(t;\tilde{\nu}^{s,y}) \).

**Proof:** Recall that \( \hat{\tau}(s,y) \) is non-decreasing in \( y \) a.s. as pointed out below \((A.9)\). Let \( z > y > \hat{y}(s) \), then the connectness properties proved in Proposition \((4.1)\) and Proposition \((4.2)\) imply \( (s,z),(s,y) \in \Delta \) and \( \hat{\tau}(s,y) > s \). Also \( (t,Y^{s,y}(t)) \) lies strictly below \( (t,Y^{s,z}(t)) \) since \( Y^{s,z}(t) - Y^{s,y}(t) = (z-y)C^s(t) > 0 \). Hence at \( u = \hat{\tau}(s,y) \) the process \( (u,Y^{s,y}(u)) \) lies in the boundary, but \( (u,Y^{s,z}(u)) \) still lies in the interior of \( \Delta \) since its boundary is non-increasing. It follows that \( \hat{\tau}(s,y) < \hat{\tau}(s,z) \). So \( \hat{\tau}(s,y) \) is strictly increasing a.s. on \( y > \hat{y}(s) \), therefore its left-continuous inverse (modulo a shift) \( \bar{\nu}^{s,y} \) is continuous except possibly for an initial jump, and so is \( C^{s,y}(t;\tilde{\nu}^{s,y}) \). \( \square \)

## 5 A unifying view on the optimal investment boundary

Under Assumption-[I] of Section 3, Assumption-[det] of Section 4 and conditions \((4.2)\) of Proposition \((4.2)\) in this Section we manage to show that the investment exercise boundary obtained by variational methods coincides with the base capacity provided by the Representation Theorem approach. Hence in the presence of scrap value at the terminal time \( T \) getting a unifying view on the curve at which is optimal to invest is possible but it requires adding several conditions.

Certainly uniqueness of the optimal control, Proposition \((A.4)\) and Theorem \((3.7)\) imply the identification of \( \hat{\nu}^{0,y}(t) \) with the optimal control \( \nu^y_0(t) \) obtained via the Bank and El Karoui Representation Theorem. Moreover \( \hat{y}(t) \) is strictly positive and its non-increasing property implies upper right-continuity à la Bourbaki, properties naturally enjoyed by the base capacity \( l^y_0(t) \).

Recalling \((A.15)\), writing \((3.13)\) under the new probability measure \( Q_0 \) and taking care of the conditioning by arguments as in the proof of Theorem \((A.3)\) (see also \([5]\), Proposition 5.2) provide

\[
(5.1) \quad \hat{\Gamma}^{y^c,t} \hat{\gamma}^{-\frac{1}{y^c}}(t) := e^{\int_0^t \mu_F(r)dr} \frac{1}{C^0(t)} \frac{1}{\hat{\gamma}^{-\frac{1}{y^c}}(t)} = v(t,yC^0(t)) \leq \frac{1}{f_C(t)}, \quad t \in (0,T), \text{ a.s.}
\]

but, contrary to the no-scrap value case of \([5]\), Proposition 5.3, \( \hat{\Gamma}^{y,\xi}(t) \) cannot be written in terms of \( v \) for a generic value of \( \xi \) due to the \( y \)-parameter dependence of \( \Gamma^{y,\xi} \) (carried over to \( \xi^\mu(t) \) and \( l^y_0(t) \)) which accounts for the term \( G'(yC^0(T)) \) due to the scrap value. Therefore (cf. \((3.19)\)), for \( t \in [0,T) \),

\[
(5.2) \quad l^y_0(t) := \frac{C^0(t)}{\xi^\mu(t)} = \frac{C^0(t)}{-\sup\left\{ \xi < 0 : \hat{\Gamma}^{y,\xi}(t) = \frac{1}{f_C(t)} \right\}} = \sup\left\{ -\frac{1}{\xi} C^0(t) > 0 : \hat{\Gamma}^{y^c,\xi}(t) = \frac{1}{f_C(t)} \right\}
\]

does not appear immediately linked to \( \hat{y}(t) \) (cf. \((A.19)\)). So some work is needed in order to compare the two approaches. Notice that, for \( t \in [0,T) \) and \( z > y > 0 \), \( l^y_0(t) \geq l^y_z(t) \) since \( \hat{\Gamma}^{y^c,\xi}(t) \geq \hat{\Gamma}^{y,\xi}(t) \) by the decreasing property of \( G'() \), which together with the decreasing property of \( \hat{R}_C(\cdot,w,r) \) also implies

\[
\frac{1}{f_C(t)} \geq \hat{\Gamma}^{y^c,t} \hat{\gamma}^{-\frac{1}{y^c}}(t) \geq \hat{\Gamma}^{y^z,t} \hat{\gamma}^{-\frac{1}{y^z}}(t) \geq \hat{\Gamma}^{y^z,t} \hat{\gamma}^{-\frac{1}{y^z}}(t) = \hat{\Gamma}^{z^c,t} \hat{\gamma}^{-\frac{1}{z^c}}(t).
\]

The main result of this Section is the following
Theorem 5.1 Under Assumption-[I], Assumption-[det] and conditions (4.2), for $t \in [0, T)$, the process $l_y^\ast(t)$ equals $\hat{y}(t)$ a.s., hence it may be assumed deterministic and independent of $y$.

Proof: If $\omega$ is such that $0 < yC^o(\omega, t) \leq \hat{y}(t)$ (cf. (A.19)), then $\tilde{\Gamma}^{y_i - \frac{1}{2}y}(\omega, t) = \frac{1}{f_{C}(t)}$ and $\tilde{\Gamma}^{\hat{y} - \frac{1}{2}y}(\omega, t) = \frac{1}{f_{C}(t)}$ for all $0 < z < y$ as well. Therefore

$$l_y^\ast(\omega, t) = C^o(\omega, t) \sup \left\{ z \geq y : \tilde{\Gamma}^{y_i - \frac{1}{2}y}(\omega, t) = \frac{1}{f_{C}(t)} \right\} = C^o(\omega, t) \sup \left\{ z \geq y : \tilde{\Gamma}^{\hat{y} - \frac{1}{2}y}(\omega, t) = \frac{1}{f_{C}(t)} \right\}.$$

Also

$$\left\{ zC^o(\omega, t) \geq yC^o(\omega, t) : v(t, zC^o(\omega, t)) = \frac{1}{f_{C}(t)} \right\} \sup \left\{ z' \geq yC^o(\omega, t) : v(t, z') = \frac{1}{f_{C}(t)} \right\} = \sup \left\{ z \geq y : v(t, zC^o(\omega, t)) = \frac{1}{f_{C}(t)} \right\}.$$

with $\hat{z}(\omega) := \frac{1}{f_{C}(\omega, t)}$. Clearly the last set is contained into the first one, hence all sets coincide and

$$l_y^\ast(\omega, t) \geq \sup \left\{ z' \geq yC^o(\omega, t) : v(t, z') = \frac{1}{f_{C}(t)} \right\} = \hat{y}(t).$$

We claim $l_y^\ast(\omega, t) = \hat{y}(t)$. In fact assume not, then there exists $z_0 > \hat{y}(t)$ such that $\tilde{\Gamma}^{y_i - \frac{1}{2}y}(\omega, t) = \frac{1}{C^o(\omega, t)}$. It follows that $\tilde{\Gamma}^{y_i - \frac{1}{2}y}(\omega, t) = \frac{1}{f_{C}(t)}$ for all $z < \frac{z_0}{C^o(\omega, t)}$ since $\tilde{R}_C$ and $G'$ are decreasing in $C$. As $\tilde{\Gamma}^{y_i - \frac{1}{2}y}(\omega, t)$ is obtained at $\tau^{y_i - \frac{1}{2}y}(\omega, t) = t$, it coincides with the infimum over $\tau \in [t, T)$ (which does not involve $G'$, thus not $y$); that is,

$$\frac{1}{f_{C}(t)} = \liminf_{t \to \tau < T} \tilde{\Gamma}^{y_i - \frac{1}{2}y}(\omega, t) = \liminf_{t \to \tau < T} \tilde{\Gamma}^{y_i - \frac{1}{2}y}(\tau) = \liminf_{t \to \tau < T} \tilde{\Gamma}^{y_i - \frac{1}{2}y}(\tau)$$

where, for convenience, we have denoted $\tilde{\Gamma}^{y_i z}(\tau)$ the argument of the ess inf of the corresponding $\tilde{\Gamma}$. Also, $\hat{y}(t) < zC^o(\omega, t)$ implies $\tilde{\Gamma}^{y_i - \frac{1}{2}y}(\omega, t) = v(t, zC^o(\omega, t)) = \frac{1}{f_{C}(t)}$, and hence $\tilde{\Gamma}^{y_i - \frac{1}{2}y}(\omega, t)$ is not obtained before $T$ for all $zC^o(\omega, t) \in (\hat{y}(t), z_0]$ (cf. (A.18)). In other words, the process $zC^o(\omega, t)G'(t)$, for $u > t$, never reaches the boundary $\hat{y}(u)$ before $T$, and that may only happen if $\hat{y}(u) \equiv 0$ for $u > t$, contradicting point (ii) of Theorem 4.3 or else if $\omega$ is in a null set.

On the other hand, if $\omega$ is such that $yC^o(\omega, t) = \hat{y}(t)$, then $\tilde{\Gamma}^{y_i - \frac{1}{2}y}(\omega, t) \leq \tilde{\Gamma}^{y_i - \frac{1}{2}y}(\omega, t) < \frac{1}{C^o(\omega, t)}$ for all $0 < z < y$; as well as $\tilde{\Gamma}^{y_i - \frac{1}{2}y}(\omega, t) \leq \tilde{\Gamma}^{y_i - \frac{1}{2}y}(\omega, t) \leq \tilde{\Gamma}^{y_i - \frac{1}{2}y}(\omega, t) < \frac{1}{f_{C}(t)} = \tilde{\Gamma}^{y_i - \frac{1}{2}y}(\omega, t) \cdot \frac{C^o(\omega, t)}{\hat{y}(t)}$ for all $\frac{\hat{y}(t)}{C^o(\omega, t)} < z < y$. Therefore to determine $l_y^\ast(\omega, t)$ it suffices to consider $z \leq \frac{\hat{y}(t)}{C^o(\omega, t)}$; in fact,

$$l_y^\ast(\omega, t) = \frac{C^o(\omega, t)}{- \sup \left\{ - \frac{1}{z} \in (0, \frac{1}{f_{C}(t)}) : \tilde{\Gamma}^{y_i - \frac{1}{2}y}(\omega, t) = \frac{1}{f_{C}(t)} \right\} = \sup \left\{ zC^o(\omega, t) \leq \frac{1}{f_{C}(t)} \right\} \leq \sup \left\{ zC^o(\omega, t) \leq \hat{y}(t) : v(t, zC^o(\omega, t)) = \frac{1}{f_{C}(t)} \right\}.$$
hence also
\begin{equation}
(5.4) \quad l^*_y(t) \leq \sup \left\{ z' \leq \hat{y}(t) : v(t, z') = \frac{1}{f_C(t)} \right\} = \hat{y}(t).
\end{equation}

To prove \( l^*_y(\omega, t) = \hat{y}(t) \) assume not, then for \( z_0 \in (l^*_y(\omega, t), \hat{y}(t)] \) it holds \( \hat{y}_y - \frac{C^0(u_0)}{z_0} (\omega, t) < \frac{1}{f_C(t)} \) and \( \hat{y}_y - \frac{C^0(u_0)}{z_0} (\omega, t) = v(t, z_0) = \frac{1}{f_C(t)} \) with \( \tau \frac{\hat{y}_y - C^0(u_0)}{z_0} (\omega, t) = t \). It follows that must necessarily be \( \tau \frac{\hat{y}_y - C^0(u_0)}{z_0} (\omega, t) = \inf \left\{ u \in [t, T) : \hat{y}_y - \frac{C^0(u_0)}{z_0} (\omega, u) = \frac{1}{f_C(t)} \right\} \wedge T = T \) (see (3.3) and (3.10)), since the two \( \hat{\Gamma} \) differ only for the scrap value, and this is a contradiction since
\[ \operatorname{ess inf}_{t \leq \tau < T} \hat{y}_y - \frac{C^0(u_0)}{z_0} (\tau) = \operatorname{ess inf}_{t \leq \tau < T} \hat{y}_y - \frac{C^0(u_0)}{z_0} (\tau) = \frac{1}{f_C(t)}. \]

The above Theorem has an interesting consequence. The integral equation in Corollary 3.6 whose unique upper right-continuous, positive solution is \( l^*(t) \), may be used to characterize uniquely the investment exercise boundary \( \hat{y}(t) \) (see also [5], Theorem 5.5 for the no scrap value model). In fact, by writing the equation for \( \tau = t \in [0, T) \) and \( y = \hat{y}_y (\omega, t) \), switching to the new probability measure \( Q := Q_t \), using the fact that \( \frac{C^0(u_0)}{C_t^0(t)} \) for \( u > t \) is independent of \( F_t \), we obtain the following

**Corollary 5.2** Under Assumption-[I], Assumption-[det] and conditions (4.2), the investment exercise boundary \( \hat{y}(t) \) is the unique upper right-continuous, strictly positive solution of the integral equation
\begin{equation}
(5.5) \quad E^{Q_t} \left\{ \int_t^T e^{-\int_t^r \mu(u) \, du} \hat{R}_C \left( \sup_{t \leq u' < T} \hat{y}(u')C^{u'}(u), w(u), r(u) \right) \right\} \right. \\
+ e^{-\int_t^T \mu(u) \, du} G \left( \sup_{t \leq u' < T} \hat{y}(u')C^{u'}(T) \right) \left. \right\} = \frac{1}{f_C(t)}, \quad \forall t \in [0, T).
\end{equation}

This is a useful result as it allows to find numerically the investment exercise boundary for any choice of “cost” functions \( w \) and \( r \), in a quite complex singular stochastic control problem of capacity expansion.

In conclusion, putting together the steps, the following algorithm for the continuous optimal investment process \( \hat{\nu}^{s,y} \) holds,
1. given the cost functions \( w(t), r(t) \), use equality (2.13) to calculate the partial \( C \)-derivative \( \hat{R}_C(C, w, r) \) of the reduced production function;
2. plug it into (5.5) and solve such integral equation (numerically after a discretization) to find its unique upper right-continuous, positive solution \( \hat{y}(t) \);
3. plug \( \hat{y}(t) \) into (A.21) to determine the continuous process \( \hat{T}^{s,y} \);
4. finally plug \( \hat{T}^{s,y} \) into (A.10) and obtain the optimal investment process \( \hat{\nu}^{s,y} \).

### A Time-inhomogeneous model with scrap value: variational approach

Under Assumption-[det] of Section 4 as in [8], [9], [5], the problem may be imbedded by considering the dynamics of the capacity process \( C^{s,y}(t; \nu) \) starting at time \( s \in [0, T] \) from \( y > 0 \) and controlled by \( \nu \),
\begin{equation}
(A.1) \quad \begin{cases}
\frac{dC^{s,y}(t; \nu)}{dt} = C^{s,y}(t; \nu) \left[ -\mu_C(t) dt + \sigma_C(t) dW(t) \right] + f_C(t) d\nu(t), \\
C^{s,y}(s; \nu) = y > 0,
\end{cases} \quad t \in (s, T],
\end{equation}

\( C^{s,y}(t; \nu) \) is a solution of the integral equation
\begin{equation}
(5.6) \quad C^{s,y}(t; \nu) = \left( \int_t^{T} e^{\int_t^r \mu(u) \, du} \hat{R}_C \left( \sup_{t \leq u' < T} \hat{y}(u')C^{u'}(u), w(u), r(u) \right) \right. \\
+ e^{\int_t^{T} \mu(u) \, du} G \left( \sup_{t \leq u' < T} \hat{y}(u')C^{u'}(T) \right) \left. \right\} \frac{1}{f_C(t)}, \quad \forall t \in [s, T).
\end{equation}
with \( \nu \in \mathcal{S}_s := \{ \nu : [s, T] \to \mathbb{R} : \text{non-decreasing, lcrl, adapted process s.t. } \nu(s) = 0 \text{ a.s.} \} \). Set (cf. (2.7))

\[
(A.2) \quad C^s(t) := \frac{C^u(t)}{C^0(s)} = e^{-\int_s^t \mu C(r)dr} \mathcal{M}_s(t),
\]

then the solution of (A.1) is \( C^{s,y}(t; \nu) = C^s(t)\left[y + \int_{[s,t]} \frac{f_C(u)}{C^0(u)} \, d\nu(u)\right] = C^s(t)[y + \Upsilon(t)] \) where \( \Upsilon(t) := \int_{[s,t]} \frac{f_C(u)}{C^0(u)} \, d\nu(u) \). The expected total discounted profit plus scrap value, net of investment, is

\[
(A.3) \quad J_{s,y}(\nu) = E\left\{ \int_s^T e^{-\int_s^t \mu F(u)du} \, \hat{R}(C^{s,y}(t; \nu), w(t), r(t)) \, dt + e^{-\int_s^T \mu F(u)du} G(C^{s,y}(T; \nu)) \right\},
\]

and the firm’s optimal capacity expansion problem is

\[
(A.4) \quad V(s, y) := \max_{\nu \in \mathcal{S}_s} J_{s,y}(\nu).
\]

Notice that, due to Assumption-[det], \( \nu(t) \) and \( C^s(t) \) are \( \mathcal{F}_{s,t} \)-measurable where

\[
\mathcal{F}_{s,t} := \sigma \{W(u) - W(s) : s \leq u \leq t\}.
\]

The variational approach to the singular stochastic control problem (A.4) is based on the study of the optimal stopping problem naturally associated to it,

\[
(A.5) \quad Z^{s,y}(t) := \text{ess inf}_{\tau \in \mathcal{Y}_s[t,T]} E\left\{ \zeta^{s,y}(\tau) \mid \mathcal{F}_{s,t} \right\}, \quad t \in [s, T],
\]

where \( \mathcal{Y}_s[t,T] \) denotes the set of all \( \{\mathcal{F}_{s,u}\}_{u \in [t,T]} \) stopping times taking values in \( [t, T] \) (i.e. such that \( \{\tau < u\} \in \mathcal{F}_{s,u} \), and

\[
(A.6) \quad \zeta^{s,y}(t) := \int_s^t e^{-\int_s^u \mu F(r)dr} C^s(u)\hat{R}_C(yC^s(u), w(u), r(u)) \, du + e^{-\int_s^t \mu F(r)dr} \frac{C^s(t)}{f_C(t)} 1_{\{t < T\}} + e^{-\int_s^t \mu F(r)dr} C^s(T)G'(yC^s(T)) 1_{\{t = T\}}
\]

is the opportunity cost of not investing until time \( t \) when the capacity is \( y \) at time \( s \). Then the optimal risk of not investing until time \( t \) is defined as

\[
(A.7) \quad \nu(s, y) := Z^{s,y}(s),
\]

where \( Z^{s,y}(\cdot) \) is a modification of \( Z^{s,y}(\cdot) \) having right-continuous paths with left limits (“rcll”). Hence, up to a null set, \( \nu(s, y) = Z^{s,y}(s) \).

Consider the optimal stopping time

\[
(A.8) \quad \hat{\tau}(s, y) := \inf\{t \in [s, T] : Z^{s,y}(t) = \zeta^{s,y}(t)\} \wedge T,
\]

and take its left-continuous inverse (modulo a shift)

\[
(A.9) \quad \begin{cases} \tau^{s,y}(t) := [\sup\{z \geq y : \hat{\tau}(s, z+) < t\} - y]^+ & \text{if } t > s, \\ \tau^{s,y}(s) := 0. \end{cases}
\]

Notice that \( \tau^{s,y}(s) \) is non-decreasing in \( y \) a.s. (cf. [4], Lemma 1). The following result follows by arguments as in the proof of [6], Theorem 3.1, based on the estimates in Proposition [2.2] and the strict concavity of \( J_{s,y} \) on \( \mathcal{S}_s \).
Proposition A.1 Under Assumption-[det], for fixed \( y > 0 \) and \( s \in [0, T) \) set

\[
\begin{align*}
\hat{\nu}^{s,y}(t) &:= \int_{[s,t]} \frac{C^s(u)}{f_C(u)} dP^s,y(u), \quad t \in (s, T), \\
\hat{\nu}^{s,y}(s) &:= 0,
\end{align*}
\]

then

(i) \( \hat{\nu}^{s,y} \) is the unique optimal solution of \( V(s, y) := \max_{\nu \in \mathcal{S}_s} J_{s,y}(\nu) \),

(ii) \( E\{\|\hat{\nu}^{s,y}\|_T\} \leq 2KJ(1 + y) \max_{t \in [s, T]} e\int_s^t \mu_F(r) dr \),

(iii) if \( C(T; \hat{\nu}^{s,y}) \equiv 0 \) a.s. then \( y = 0, \hat{\nu}^{s,y} \equiv 0 \); moreover

\[
E\left\{ \int_t^T e^{-\int_s^t \mu_F(r) dr} \tilde{R}_C(yC^s(u), w(u), r(u)) \frac{C^s(u)}{C^s(t)} du \\
+ e^{-\int_s^t \mu_F(r) dr} G'(yC^s(T)) \frac{C^s(T)}{C^s(t)} \mathbb{1}_{\mathcal{F}_{s,t}} \right\} \leq e - \int_s^t \mu_F(r) dr \frac{1}{f_C(t)} \quad \text{a.e., a.s.}
\]

Remark A.2 If \( R \) is of the Cobb-Douglas type (cf. Remark 2.3), then \( \tilde{R}_C(0, wt), r(t) = +\infty \) for any \( t \), so [A.11] fails and \( C^s(y; \hat{\nu}) > 0 \) for all \( t > 0 \) (see also [6], Remark 3.2). \( \square \)

A generalization of [4], Proposition 2 and Theorem 3, shows that the stopping time \( \hat{\tau}(s, y) \) (cf. [A.8]) is optimal for \( v(s, y) \) and the value function \( v(s, y) \) is the shadow value of installed capital, i.e.

\[
v(s, y) = \frac{\partial}{\partial y} V(s, y).
\]

Having defined our ingredients for all \( s \) and \( y \), observe that if \( \hat{\tau}(s, y) = s \) for \( s < T \), then \( \frac{1}{f_C(s)} = \zeta^{s,y}(s) = Z^{s,y}(s) \leq Z^{s,z}(s) \leq \zeta^{s,z}(s) = \frac{1}{f_C(z)} \) for all \( z < y \), since \( \tilde{R}_C \) and \( G' \) are non-increasing. Hence \( \hat{\tau}(s, z) = s \) for all \( z < y \) and the mapping

\[
\hat{y}(s) := \sup\{z > 0 : \hat{\tau}(s, z) = s\}
\]

is well defined. In essence, \( \hat{y}(s) \) is the maximal initial capacity at time \( s \) for which it is optimal to invest instantaneously.

Define the probability measure \( Q_s \sim P \) by

\[
dQ_s = \exp \left\{ \int_s^T \sigma_C(t) dW(t) - \frac{1}{2} \int_s^T \|\sigma_C(t)\|^2 du \right\} = C^s(T) e^{\int_s^T \mu_C(t) dt},
\]

then \( W^{Q_s}(t) := W(t) - W(s) - \int_s^t \sigma_C(u) du \) is a Wiener process. Under the new probability \( Q_s \) the process \( Y^{s,y}(t) := yC^s(t) \) evolves according to the dynamics

\[
\begin{align*}
dY^{s,y}(t) &:= Y^{s,y}(t) \left[ (\|\sigma_C(t)\|^2 - \mu_C(t)) dt + \sigma_C(t) dW^{Q_s}(t) \right], \quad t \in (s, T], \\
Y^{s,y}(s) &= y,
\end{align*}
\]

and the “optimal risk of not investing” becomes

\[
v(s, y) = \inf_{\tau \in \mathcal{Y}_{s,T}} E^{Q_s} \left\{ \int_s^T e^{-\int_s^t \bar{\mu}(r) dr} \tilde{R}_C(Y^{s,y}(u), w(u), r(u)) du \\
+ e^{-\int_s^t \bar{\mu}(r) dr} \frac{1}{f_C(\tau)} \mathbb{1}_{\{\tau < T\}} + e^{-\int_s^\tau \bar{\mu}(r) dr} G'(Y^{s,y}(\tau)) \mathbb{1}_{\{\tau = T\}} \right\}
\]
with \( \hat{\mu}(t) := \mu_C(t) + \mu_F(t) \geq \varepsilon_0 > 0 \) (as it is in Section 2). Taking \( \tau = s \) in (A.15) shows that \( v(s, y) \leq \frac{1}{f_C(s)} \) for all \( y > 0 \), \( s \in [0, T) \). Hence the strip \([0, T) \times (0, \infty)\) splits into two regions (see also [6]), the Continuation Region (or inaction region) where it is not optimal to invest as the capital’s replacement cost strictly exceeds the shadow value of installed capital,

\[
\Delta = \left\{ (s, y) \in [0, T) \times (0, \infty) : v(s, y) < \frac{1}{f_C(s)} \right\},
\]

and its complement, the Stopping Region (or action region),

\[
\Delta^c = \left\{ (s, y) \in [0, T) \times (0, \infty) : v(s, y) = \frac{1}{f_C(s)} \right\},
\]

where it is optimal to invest instantaneously.

Under our reduced production function \( \tilde{R} \), time-inhomogeneous capacity process and scrap value at terminal time \( T \), the following result holds and shows that the optimal stopping time \( \hat{\tau}(s, y) \) of (A.15) may be characterized as the first time \( (t, Y_{s,y}(t)) \) exits the Borel set \( \Delta \). The proof is omitted as it is a generalization of [8], Theorem 4.1, established for the model with constant coefficients, discount factor and conversion factor \( f_C \) but without scrap value, later on extented to the model with scrap value and additive production function in [6], Proposition 3.3. Similar arguments were then used in [5], Theorem A.1, in the case of time-dependent coefficients, discount factor and conversion factor \( f_C \), but without scrap value. In all these results the derivative of the production function w.r.t. capacity \( C \) was a function of \( C \) only.

**Theorem A.3** Under Assumption-[det], the optimal stopping time of problem (A.15) is characterized as

\[
\hat{\tau}(s, y) = \inf \left\{ t \in [s, T) : v(t, Y_{s,y}(t)) = \frac{1}{f_C(t)} \right\} \wedge T
\]

for each starting time \( s \in [0, T) \) and initial state \( y \in (0, \infty) \).

It follows that (A.13) is now written as

\[
\hat{y}(s) = \sup \left\{ z \geq 0 : v(s, z) = \frac{1}{f_C(s)} \right\},
\]

hence it is the lower boundary of the Borel set \( \Delta \) in the \((s, y)\)-plane; (A.18) is equivalently written as

\[
\hat{\tau}(s, y) = \inf \{ t \in [s, T) : [Y_{s,y}(t) - \hat{y}(t)]^+ = 0 \} \wedge T
\]

and so, recalling \( Y_{s,y}(t) = yC_s(t) \), its left-continuous inverse (A.9) may be written in the form

\[
\begin{cases}
\varphi^{s,y}(t) := \sup_{s \leq u < t} \left[ \frac{\hat{y}(u)}{C_s(u)} - y \right]^+ & \text{for } t > s, \\
\varphi^{s,y}(s) := 0.
\end{cases}
\]

Hence \( y + \varphi^{s,y}(t) \geq \frac{\hat{y}(t)}{C_s(t)} \) a.s. (i.e. \( C_s^{y}(t; \varphi^{s,y}) \geq \hat{y}(t) \) a.s.), and \( \varphi^{s,y}(t) = \frac{\hat{y}(t) - yC_s(t)}{C_s(t)} \) at times of strict increase.
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