Notes on MV-modules over integral domains

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Abstract

An MV-module is an MV-algebra endowed with a scalar multiplication with scalars in a PMV-algebra (i.e. an MV-algebra endowed with a binary “ring-like” product). We investigate the class of semisimple MV-modules over a semisimple and totally ordered integral domain, and prove an adjunction with a special class of linear spaces.

Keywords: MV-algebra, MV-module, integral domain, linear space, tensor product.

1 Introduction

MV-algebras were defined in 1958 as the algebraic counterpart Łukasiewicz infinite-valued logic. They are structures $(A, \oplus, \ast, 0)$ such that $(A, \oplus, 0)$ is an abelian monoid, $x^{**} = x$, and the equations $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$ and $x \oplus 0^* = 0^*$ are satisfied for any $x, y \in A$. The literature on the subject is very wide and we suggest [4, 5, 15] for further details.

The standard model for an MV-algebra is the unit interval $[0, 1]$, with $x \oplus y = \min(x + y, 1)$ and $x^* = 1 - x$ and it generates the variety of MV-algebra. One of the most important achievement in the field is the categorical equivalence between MV-algebras and lattice-ordered groups with strong unit ($\ell u$-groups). We suggest [1, 16] for further details on $\ell u$-groups and related structures.

In more details, given an $\ell u$-group $(G, u)$, the interval $[0, u]_G = \{x \in G \mid 0 \leq x \leq u\}$ is an MV-algebra if we define $x \oplus y = (x + y) \wedge u$ and $x^* = u - x$. This gives us a functor, denoted by $\Gamma$, from auG (the category whose objects are Abelian lattice-ordered groups with strong unit and whose morphisms are maps that are at the same time groups homomorphisms and lattices homomorphisms) to MV (the category whose objects are MV-algebras and whose morphisms are homomorphism of MV-algebras).

Product MV-algebras (PMV-algebra for short) are obtained when we endow an MV-algebras with a binary and internal “ring-like” product [12, 6]. When the product is a scalar one, with scalars chosen in a PMV-algebra, we obtain the notion of MV-module. A Riesz MV-algebra is an MV-module over $[0, 1]$. We remark that $[0, 1]$ can be seen as a PMV-algebra or a MV-module over itself, when the product (either scalar or internal) coincide with the usual product between real numbers.

The functor $\Gamma$ naturally extends to PMV-algebras and MV-modules. We obtain
a categorical equivalence between PMV-algebras and a proper subclass of lattice-ordered rings with strong unit (ℓu-rings) [6]; we will denote this functor by \( \Gamma \). In the same way, in [7] it is proved the categorical equivalence between MV-modules over a fixed PMV-algebra \( P \) and lattice-ordered modules over the ℓu-rings that corresponds to \( P \) via \( \Gamma \); we will denote this functor by \( \Gamma_R \), where \( \Gamma(R, u) = P \).

In this short paper we study MV-modules over a special class of PMV-algebras, that is PMV-algebras without zero-divisors. The main result is an adjunction between such MV-modules and linear spaces over a totally ordered and Archimedean field. In order to apply some fundamental results from literature, we need to restrict our work to totally ordered and semisimple PMV-algebras.

2 Preliminaries

The main tool in our development is the tensor product of MV-algebras, defined by Mundici in [14] and further investigated in [5, 10, 11].

Given two MV-algebras \( A \) and \( B \), the tensor product is the MV-algebra \( A \otimes_{mv} B \) uniquely defined by the universal bimorphism \( \beta: A \times B \to A \otimes_{mv} B \) such that \( \beta(a, b) = a \otimes_{mv} b \). We recall that a bimorphism is a bilinear map that commutes with \( \lor \) and \( \land \) on both argument.

An important subclass of MV-algebras is the one of semisimple algebras. An MV-algebra \( A \) is semisimple if the intersection of all maximal ideal (called Radical of \( A \), and denoted by \( \text{Rad}(A) \)) is zero. We have that, via \( \Gamma \), semisimple MV-algebras correspond to Archimedean ℓu-groups, where an ℓ-group \( G \) is said to be Archimedean if \( nx \leq y \) for any \( n \in \mathbb{N} \) and \( x \geq 0 \), implies \( x = 0 \).

Since the class of semisimple MV-algebras is not closed under tensor product, in [14] the semisimple tensor product \( \otimes_{ss} \) is defined as the quotient

\[
A \otimes_{ss} B = A \otimes_{mv} B / \text{Rad}(A \otimes_{mv} B),
\]

for any \( A \) and \( B \) semisimple MV-algebras. It satisfies the following universal property, with respect to semisimple MV-algebras:

for any semisimple MV-algebra \( C \) and for any bimorphism \( \beta: A \times B \to C \), there is a unique homomorphism of MV-algebras \( \omega: A \otimes_{ss} B \to [0, \beta(1, 1)] \leq_i C \) such that \( \omega \circ \beta_{A, B} = \beta \), where \( \beta_{A, B}: A \times B \to A \otimes_{ss} B \) is defined by \( \beta_{A, B}(a, b) = a \otimes_{ss} b \).

The notation \( [0, a] \leq_i A \) means that \( [0, a] \) is an interval MV-algebra of \( A \). See [14, 8] for further details.

In [10] the following is proved.

**Theorem 2.1.** Let \( A \) be a unital and semisimple PMV-algebra, and \( B \) be a semisimple MV-algebra. Then \( A \otimes_{ss} B \) is an \( A \)-MV-module.

Moreover, denoted by \( \mathcal{U}_A(M) \) the MV-reduct of \( M \), an MV-module with scalars in \( A \), the following universal property holds.

**Theorem 2.2.** [10] Let \( A \) be a unital, semisimple and totally ordered PMV-algebra, let \( B \) be a semisimple MV-algebra. Then for any unital and semisimple \( A \)-MV-module \( M \) and for any homomorphism of MV-algebras \( f: B \to \mathcal{U}_A(M) \) there is a unique homomorphism of \( A \)-MV-modules \( f: A \otimes_{ss} B \to M \) such that \( f \circ \iota_{B, A} = f \), where \( \iota_{B, A}: B \to A \otimes_{ss} B \) is the embedding in the tensor product.
In [3] the lattice-ordered counterpart of $\otimes_{ss}$ is introduced: the authors define the tensor product of Archimedean lattice-ordered groups with strong unit. Given $(G, u_G)$ and $(H, u_H)$ $\ell u$-groups, $(G \otimes_a H, u_G \otimes_a u_H)$ is an $\ell u$-group uniquely defined, up to isomorphism, by a universal property with respect to Archimedean structures.

In [10] the following is proved.

Theorem 2.3. If $(G_A, u_A), (G_B, u_B)$ are Archimedean $\ell u$-groups and $A, B$ are semisimple MV-algebras such that $A \cong \Gamma(G_A, u_A)$ and $B \cong \Gamma(G_B, u_B)$ then $A \otimes_{ss} B \cong \Gamma(G_A \otimes_a G_B, u_A \otimes_a u_B)$.

Remark 2.1. In [6] PMV-algebras are defined in the most general case, while in [12] any PMV-algebra is unital and commutative. In the sequel we will use the definition from [12].

3 MV-domains

We start this section with the definition of an MV-domain.

Definition 3.1. A PMV-algebra $P$ is called MV-domain if $x \cdot y = 0$ implies $x = 0$ or $y = 0$.

Remark 3.2. In [13], Montagna defines the quasi variety of $PMV^+$-algebras, as PMV-algebras that satisfies the quasi-identity $x^2 = 0$ implies $x = 0$. $PMV^+$-algebras are therefore algebras without nilpotent elements, and by definition any MV-domain is a $PMV^+$-algebras. The converse is not true in general. We recall that for $PMV^+$-algebras several important results holds, like the subdirect representation theorem.

Proposition 3.1. Let $P$ be a totally ordered PMV-algebra. $P$ is an MV-domain if and only if the corresponding $\ell u$-ring is a integral domain.

Proof. One direction is obvious. For the other direction, let $P$ be a MV-domain such that $P = \Gamma(\cdot) (R, u)$, with $(R, u)$ $\ell u$-ring. Let $x, y$ be elements of $R^+$ such that $x \cdot y = 0$. There exist $x_1, \ldots x_n, y_1, \ldots y_m$ in $P$ such that $x = \sum_{i=1}^n x_i$, $y = \sum_{i=1}^m y_j$. Therefore

$$x \cdot y = \sum_{i,j} x_i \cdot y_j = 0.$$ 

Hence for any $i = 1, \ldots, n$ and $j = 1, \ldots, m$ $x_i \cdot y_j = 0$. By hypothesis we have:

(i) there exists one $i$ such that $a_i \neq 0$, then we have all $b_j = 0$, and then $b = 0$;

(ii) for any $i$ we have $a_i = 0$, then $a = 0$.

The result follows from the total order on $R$. 

In the follow, we will denote by $MV\text{ArDom}P$ the category whose objects are semisimple MV-modules over a semisimple and totally ordered MV-domain $P$, and whose morphisms are homomorphisms of MV-modules.
Remark 3.3. (i) A $P$-ideal $I$ for a $P$-MV-module $A$ is an ideal that satisfies the condition $\alpha x \in I$ for any $\alpha \in P$ and any $x \in A$. This condition is always satisfied when $P$ is a unital PMV-algebra.

(ii) By [7] Proposition 3.16 any object in $\text{MVVarDomP}$ is a subdirect product of totally ordered $P$-MV-modules.

(iii) By [1] Chapter XIV Section 6 Lemma 2, in a totally ordered and Archimedean $\ell$-group any positive element is a strong unit. In particular the product-unit is a strong unit.

4 The categorical adjunction

Let $\text{LinSpArK}$ be the category whose objects are Archimedean and lattice-ordered linear spaces with strong unit over $K$, Archimedean and totally ordered field with strong unit, and whose morphisms are homogeneous homomorphisms of $\ell$-groups.

Proposition 4.1. Let $(V, u)$ be an object in $\text{LinSpArK}$, and $h: V_1 \to V_2$ a morphism between objects $(V_1, u_1)$ and $(V_2, u_2)$ of $\text{LinSpArK}$. Denoted by $P$ the PMV-algebra $\Gamma_{(\ell)}(K, e)$, where $e$ is the unit in $K$, $\Gamma_{(K, e)}(V, u)$ is an element of $\text{MVVarDomP}$, the category of Archimedean MV-modules over $P$. Moreover, $h_{|\Gamma_{(K, e)}(V_1, u_1)}$ is an homomorphism of MV-modules $\Gamma_{(K, e)}(V_1, u_1)$ and $\Gamma_{(K, e)}(V_2, u_2)$.

Proof. It follows directly from Remark 3.3 and [7] Proposition 4.1.

Lemma 4.2. If $R$ is an Archimedean and totally ordered integral domain, its quotient field $F$ is Archimedean and totally ordered.

Proof. $F$ is totally ordered by [2] Theorem 10.4. Let $a, b \in F^+$ such that $na \leq b$ for any $n \in \mathbb{N}$. By definition, this comes to $n\frac{a}{b} \leq \frac{b}{b}$, with $x_1, x_2 \in R^+$ and $y_1, y_2 \in R^+ \setminus \{0\}$ such that $a = \frac{x_1}{y_1}, b = \frac{x_2}{y_2}$. The latter is equivalent to $\frac{x_2 y_1 - x_1 y_2}{y_1 y_2} \leq 0 \Rightarrow \frac{a}{b} \leq b/a$. Therefore, $x_2 y_1 - x_1 y_2 \leq 0$ and $nx_1 y_2 \leq x_2 y_1$. Since $R$ is an Archimedean integral domain, we get $x_1 y_2 = 0$ and $a = 0$. Trivially, the unit in $R$ is unit in $F$.

Theorem 4.3. Let $M$ be an object in the category $\text{MVVarDomP}$. There exists an Archimedean and lattice-ordered linear space with strong unit $(V, u)$ over a totally ordered and Archimedean field $(K, e)$ uniquely associated to $M$.

Proof. By [7] Corollary 4.8, there exists an Archimedean $\ell u$-group $(G, u)$ and a totally ordered and Archimedean $\ell u$-ring $(R, e)$ such that $P \simeq \Gamma_{(\ell)}(R, e)$ and $M \simeq \Gamma_{(R, e)}(G, u)$ by [6] Theorem 3.3, $e$ is unit in $R$ and by Proposition 3.3 it is an integral domain. By Lemma 4.2 the quotient field $K = \{\frac{a}{b} | a, b \in R \setminus b \neq 0\}$ is Archimedean, totally ordered and unital.

By Theorem 2.3 $\Gamma(K \otimes_a G, e \otimes_a u) \simeq \Gamma(K, e) \otimes_{ss} \Gamma(G, u)$ and by Theorem 2.1 $\Gamma(K, e) \otimes_{ss} \Gamma(G, u)$ is a MV-module over $\Gamma(K, e)$, then by [7] Corollary 4.8 $K \otimes_a G$ is $\ell$-module over $K$ and since $K$ is a field, $K \otimes_a G \in \text{LinSpArK}$. The uniqueness of $K \otimes_a G$ follows by construction.

Proposition 4.4. Let $M$ be an object in $\text{MVVarDomP}$, with $P$ semisimple and totally ordered MV-domain. Let $(G, v)$ be the $\ell u$-group such that $M \simeq \Gamma(G, v)$,
Lemma 4.6. Let \((R, e)\) be the integral domain such that \(P = \Gamma_{(K)}(R, e)\) and let \(K\) be the quotient field of \(R\). For any object \((V, u)\) in \(\text{LinSpArK}\) and any \(f : M \to \Gamma_{(K,e)}(V, u)\) homomorphism of \(P\)-MV-modules there exists unique \(f^\sharp : K \otimes_a G \to V\) homomorphism in \(\text{LinSpArK}\) such that \(\Gamma_{(K,e)}(f^\sharp) \circ \iota_M = f\).

Proof. By definition, \(\Gamma_{(K,e)}(V, u)\) is a \(\Gamma_{(0)}(K, e)\)-MV-module and since \(P \subseteq \Gamma(K, e)\), \(f\) is well defined as homomorphisms of \(P\)-MV-modules. By Theorem 2.2 there exists \(f^* : \Gamma(K, e) \otimes_{ss} M \to \Gamma(V, u)\), homomorphism of \(\Gamma(K, e)\)-MV-modules. By Theorem 2.3, \(\Gamma(K, e) \otimes_{ss} M \simeq \Gamma(K \otimes_a G, v \otimes_a v)\). Therefore by [7, Corollary 4.8], \(f^*\) extends in a unique way to \(f^\sharp : K \otimes_a G \to V\), morphism in \(\text{LinSpArK}\). We remark that by Theorem 2.2 \(f^* \circ \iota_M = f\), where \(\iota_M\) is the standard embedding of \(M\) in \(\Gamma(K, e) \otimes_{ss} M\).

Proposition 4.5. Let \(h\) be a morphism between the two objects \(M\) and \(N\) in the category \(\text{MVArDomP}\), with \(P \simeq \Gamma_{(\cdot)}(R, e)\), \(M \simeq \Gamma(G, v_G)\), \(N \simeq \Gamma(H, v_H)\) and let \(K\) be the quotient field of \(R\). Then there exists a unique morphism \(h^\sharp : K \otimes_a G \to K \otimes H\) in \(\text{LinSpArK}\) such that \(\Gamma_{(K, e)}(h^\sharp) \circ \iota_M = \iota_N \circ h\).

Proof. Let \(\iota_M\) and \(\iota_N\) be the standard embeddings in the tensor products [10]. By Theorem 2.3, \(\Gamma(K, e) \otimes_{ss} M \simeq \Gamma(K \otimes_a G, v \otimes v_G)\) and \(\Gamma(K, e) \otimes_{ss} N \simeq \Gamma(K \otimes_a H, \epsilon \otimes v_H)\). With abuse of notation, we will denote by \(\iota_M\) and \(\iota_N\) the composite maps from \(M\) and \(N\) in \(\Gamma(K \otimes_a G, e \otimes v_G)\) and \(\Gamma(K \otimes_a H, e \otimes v_H)\) respectively. By Proposition 4.4 applied on \(\iota_M \circ h\) and \(\iota_N\) there exists a unique \(h^* : \Gamma(K, e) \otimes M \to \Gamma(K \otimes_a H, e \otimes v_H)\), such that \(h^* \circ \iota_M = \iota_N \circ h\).

\[
\begin{array}{ccc}
M & \xrightarrow{h} & N \\
\iota_M \downarrow & & \iota_N \downarrow \\
\Gamma(K, e) \otimes M & \xrightarrow{h^*} & \Gamma(K, e) \otimes N
\end{array}
\]

Figure 1

Again by Theorem 2.3 and [7, Corollary 4.8] there exists a map \(h^\sharp : K \otimes_a G \to K \otimes H\). The uniqueness of \(h^*\) gives us the desired conclusion.

Let \(P\) be a semisimple and totally ordered MV-domain, let \((R, e)\) be the lattice such that \(P = \Gamma_{(\cdot)}(R, e)\), and let \(K\) be the quotient field \(R\). We have two functors:

- \(\Gamma_{(K, e)} : \text{LinSpArK} \to \text{MVArModP}\), which is the functor from [7];
- \(\mathcal{L} : \text{MVArModP} \to \text{LinSpArK}\) such that for any \(P\)-MV-module \(M\), \(\mathcal{L}(M)\) is the linear space \(K \otimes_a G\) defined in Theorem 4.3 for any morphism \(h\), \(\mathcal{L}(h)\) is the map \(h^\sharp\) defined in Proposition 4.5.

Lemma 4.6. \(\mathcal{L}\) is a functor.
Proof. Let $h: A \to B$ and $g: B \to C$ be homomorphisms of $P$-MV-modules, with $A = \Gamma(G, u_G)$, $B = \Gamma(H, u_H)$, and $C = \Gamma(L, u_L)$. As in Proposition 4.5, there exists $h^*: \Gamma(K, e) \to \Gamma(K \otimes_a H, e \otimes u_H)$, such that $h^* \circ \iota_A = \iota_B \circ h$, and there exists $g^*: \Gamma(K, e) \otimes B \to \Gamma(K \otimes_a L, e \otimes u_L)$, such that $g^* \circ \iota_B = \iota_C \circ g$. Then

$$(g^* \circ h^*) \circ \iota_A = g^* \circ (h^* \circ \iota_A) = g^* \circ (\iota_B \circ h) = (g^* \circ \iota_B) \circ h = \iota_C \circ (g \circ h).$$

Thus, $(g \circ h)^* = g^* \circ h^*$. Since $\mathcal{L}(g \circ h)$ is the extension of $(g^* \circ h^*)$ by the inverse functor of $\Gamma(K, e)$, $(g \circ h)^2 = g^2 \circ h^2$.

**Proposition 4.7.** Let $M$ be an element in $\text{MVArDomP}$. Then $(\iota_M)_{M \in \text{MVArModP}}$, with $\iota_M: M \to \Gamma(K, e) \otimes M$, are a natural transformation between the identity functor on $\text{MVArDomP}$ and the composite functor $\Gamma_{(K, e)} \mathcal{L}$.

**Proof.** Let $N, L \in \text{MVArModP}$ and let $h: N \to L$ an homomorphism of $P$-MV-modules. We have to prove that $\Gamma_{(K, e)} \mathcal{L}(h) \circ \iota_N = \iota_L \circ h$. This is straightforward, since by definition $\mathcal{L}(h)$ is the extension on linear spaces of $h^*$, then $\Gamma_{(K, e)} \mathcal{L}(h) = h^*$ and the conclusion follows by Proposition 4.5.

**Theorem 4.8.** The pair $(\Gamma_{(K, e)}, \mathcal{L})$ is an adjoint pair.

**Proof.** $\mathcal{L}$ is a left adjoint of $\Gamma_{(K, e)}$ if, for any element $M \in \text{MVArModP}$, any $(V, u) \in \text{LinSpArK}$, and any homomorphism of $P$-MV-module $h: M \to \Gamma(V, u)$ there exists a morphism in $\text{LinSpArK}$ $h^2: K \otimes_a G \to V$, where $M \simeq \Gamma(G, u)$, such that $\Gamma_{(K, e)}(h^2) \circ \iota_M = h$. This is proved in Proposition 4.4.

**Remark 4.1.** We remark that we cannot have an equivalence between the categories $\text{MVArDomP}$ and $\text{LinSpArK}$. Indeed if $(R, u) = (2, 1)$, $P = \{0, 1\}$ and $M = L_3 \in \text{MVArModP}$ then $K = \mathbb{Q}$ and $\Gamma(M) = \{0, 1 \cap \mathbb{Q}\} \not\simeq \mathbb{L}_3$.

**Lemma 4.9.** Let $P$ be a totally ordered and semisimple MV-domain such that $P = \Gamma_{(K, e)}$, with $K$ totally ordered and Archimedean field. Let $M$ be an semisimple MV-module over $P$. If $\alpha x = 0$, then $\alpha = 0$ or $x = 0$.

**Proof.** By [7] Corollary 4.8, there exists a semisimple $\ell$-module with strong unit $(V, u)\text{ over }K$ such that $M = \Gamma_{(K, e)}(V, u)$. Since $K$ is a field, $(V, u)$ is actually a linear space. The result follows by the remark that the property holds in any linear space.

**Proposition 4.10.** Let $P$ be a totally ordered and semisimple MV-domain such that $P = \Gamma_{(K, e)}$, with $K$ totally ordered and Archimedean field. Let $M$ be an Archimedean MV-module over $P$. Then the map

$$\iota: P \to M, \quad \iota(a) = a1$$

is an embedding of MV-algebras.

**Proof.** By [7] Lemma 3.11(a)], $\iota(0) = 0$; by [7] Definition 3.1] if $a + b$ is defined, then $\iota(a + b) = (a + b)1 = a1 + b1 = \iota(a) + \iota(b)$ and $\iota$ is linear; by [7] Lemma 3.11(f)], $(a1)^* = a^*1$, then $\iota(a^*) = \iota(a)^*$. Moreover, $a \oplus b = (a \wedge b^*) + b$. Since $P$ is totally ordered, and any linear map is isotone by [9] Proposition 3.9], it follows that $\iota(a \oplus b) = \iota(a) \oplus \iota(b)$. Finally, by Lemma 4.9 $\iota$ is injective.
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