Resolutions for Equivariant Sheaves over Toric Varieties

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July 2004

Abstract

In this work we construct global resolutions for general coherent equivariant sheaves over toric varieties. For this, we use the framework of sheaves over posets. We develop a notion of gluing of posets and of sheaves over posets, which we apply to construct global resolutions for equivariant sheaves. Our constructions give a natural correspondence between resolutions for reflexive equivariant sheaves and free resolutions of vector space arrangements.

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1 Introduction

An important part of the theory of vector bundles over homogeneous spaces $G/P$ is the study of homogeneous vector bundles. This class of vector bundles has first been investigated by Kostant and Bott in the 50’s, who clarified the relation between the representation theory of $G$ and homogeneous bundles. This relation in many cases allows to determine properties of homogeneous vector bundles very explicitly, and so homogeneous bundles have played a great role in the field of studying general vector bundles, notably over projective spaces.

In a more general situation, one considers a quasi-homogeneous space, i.e. a space $X$ together with the action of an algebraic group $G$ such that this action has a dense open orbit in $X$. In this context it is customary to speak about equivariant rather than homogeneous vector bundles; denote $\sigma, p_2 : G \times X \to X$ the group action and the projection onto the second factor, respectively, then a vector bundle (or a more general sheaf) $E$ on $X$ is equivariant if there exists an isomorphism

$$\Phi : \sigma^* E \xrightarrow{\cong} p_2^* E$$

such that

$$(\mu \times 1_X)^* \Phi = p_{23}^* \phi \circ (1_G \times \sigma)^* \Phi,$$

where $\mu$ is the group multiplication morphism and $p_{23}$ the projection onto the second and third factor of $G \times G \times X$ (see also [MFK94]). This situation in general is considerably more difficult than the case of homogeneous spaces, as (at least) the following two things can happen: in general, $X$ has a rather complicated orbit structure, such that there are lower-dimensional invariant loci which allow equivariant vector bundles to degenerate to more general equivariant sheaves, if considered in families in a suitable sense; moreover, the representation theory of $G$ contributes only marginal information. So the conclusion is that one has to study the complete category of equivariant sheaves over $X$, which in particular means:

(i) construct good invariants for equivariant sheaves over $X$,

(ii) study moduli spaces with respect to these invariants.

In this work, we attempt to carry out part of such a program for equivariant sheaves over toric varieties, which are probably the easiest examples of quasi-homogeneous spaces.

Reflexive Sheaves. Our approach is based on the framework of $\Delta$-families which we have developed in earlier work ([Per04a]), which in turn generalizes the characterization of Klyachko ([Kly90], [Kly91]) of equivariant reflexive sheaves. Klyachko’s observation
was that every such sheaf $\mathcal{E}$ is equivalent to a finite dimensional vector space $E$ together with a finite set of full filtrations

$$
\cdots \subset E^\rho(i) \subset E^\rho(i+1) \subset \cdots \subset E
$$

for $i \in \mathbb{Z}$ and every torus invariant divisor $\rho \in \Delta(1)$ (see section 3 for notation). Naively, one can separate two kinds of data from such a set of filtrations: first, the indices $i$, preferably those where the dimension of the filtration jumps $E^\rho(i) \subsetneq E^\rho(i+1)$, and second, the flags underlying the filtrations, when we forget about the indices. One could think of the indices as a discrete invariant for $E$, and the flags as moduli for the sheaf. However, it turns out that the indices essentially only determine the first equivariant Chern class of $E$, and the moduli of flags do not behave very well in sheaf theoretic sense. This has been investigated in detail in [Per04b] for case of equivariant vector bundles of rank two over toric surfaces.

One could proceed now and declare the equivariant Chern classes as invariants for equivariant sheaves and construct moduli with respect to these (this has been done in [Per04b]), but we are interested in a more direct approach and want to analyze the flags underlying the filtrations. These flags and their intersections determine a subvector space arrangement of $E$, and as there is no more data left to describe $E$, one intuitively assumes that all further properties of $E$ are somehow encoded in this arrangement.

Our approach is to construct a global resolution for any given equivariant sheaf $\mathcal{E}$ over $X$. From the point of view of homogeneous coordinate rings (see [Cox95]) it has been observed ([BV97]) that every such sheaf has a finite global resolution

$$
0 \longrightarrow \mathcal{F}_s \longrightarrow \cdots \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{E} \longrightarrow 0
$$

where $\mathcal{F}_i \cong \bigoplus_j \mathcal{O}(D_{ij})$ for every $i$. Here, the $D_{ij}$ are torus invariant Weil divisors, and the sheaves $\mathcal{O}(D_{ij})$ are equivariant reflexive sheaves of rank one; in the case where $X$ is smooth, these sheaves always are invertible. We will give an explicit construction for such resolutions, which for the case of reflexive sheaves will only depend on the underlying vector space arrangements. Our results generalize a result of Klyachko, who in [Kly90] constructed a canonical resolution in the case where $\mathcal{E}$ is locally free and $X$ is smooth and complete.

**Vector space arrangements.** An interesting aspect of our construction is the solution of the following problem; consider any subvector space arrangement in some vector space $E$, and its underlying poset $\mathcal{P}$ which is given by the set of subvector spaces in the arrangement together with the partial order which is given by inclusion. Then, does there exist a vector space $\mathcal{F}$ together with a coordinate vector space arrangement such that the underlying poset is isomorphic to $\mathcal{P}$? The answer is yes, and it is rather straightforward to see that one just needs to choose $\mathcal{F}$ large enough, such that the combinatorics of $\mathcal{P}$ can be modelled by coordinate spaces of $\mathcal{F}$. As a byproduct, we obtain
a surjection $F \to E$ such that for every element $V \in \mathcal{P}$ and its corresponding subvector space $F_V$ of $F$, we have a commutative exact diagram

$$
\begin{array}{c}
0 \longrightarrow K \longrightarrow F \longrightarrow E \longrightarrow 0 \\
0 \longrightarrow K_V \longrightarrow F_V \longrightarrow V \longrightarrow 0.
\end{array}
$$

The vector spaces $K_V$ again form a vector space arrangement in $K$ whose underlying poset is a subset of the original poset $\mathcal{P}$. We call the arrangement $K_V$ the \textit{first syzygy arrangement} of $\mathcal{P}$.

By iterating this procedure, we obtain an exact sequence of vector spaces

$$
0 \longrightarrow F_s \longrightarrow \cdots \longrightarrow F_0 \longrightarrow E \longrightarrow 0
$$

where every $F_i$ contains a coordinate vector space arrangement whose underlying poset coincides with the poset underlying the $i$-th syzygy arrangement. In the case where the arrangement in $E$ is closed under performing intersections, we have even a good notion of \textit{minimal} resolutions; we obtain a unique representation of such an arrangement in terms of the purely combinatorial information encoded in the successive coordinate space arrangements. In a sense, we can think of the resolution as providing a “K-theory”-class in a suitable category of vector space arrangements. We formulate the following

\textbf{Conjecture:} Let $E$ be a reflexive equivariant sheaf over a toric variety $X$, then every property of $E$ depends only on the indices of the filtrations $E^p(i)$ and the $K$-theory class of the underlying vector space arrangement.

One can read this conjecture also the way that the “K-theory”-class of a vector space arrangement is its finest possible invariant. The class of coordinate vector space arrangements is a well-studied subject (see [BP00]), and it would be interesting to see whether properties of general arrangements can be studied through free resolutions.

\textbf{Poset representations.} The construction of some global resolution for an arbitrary equivariant coherent sheaf over $X$ is not necessarily a difficult task, but in general the organization of all the needed data is rather elaborate. Any nuts and bolts approach, starting from scratch, would probably be rather cumbersome for the reader to follow; therefore we adopt in this paper a more formal approach, by developing a certain amount of framework in the context of poset representations. Such representations, as a subtopic of quiver representations [Gab72], have been studied since long (see [Naz80]). Any poset $\mathcal{P}$ with a partial order $\leq$ in a natural way is equivalent to some category. In this category the objects are the elements of $\mathcal{P}$, and the morphisms are the relations $x \leq y$, i.e. there exists at most one morphism between two objects $x, y \in \mathcal{P}$. A \textit{representation} of $\mathcal{P}$ is a
functor \( F : \mathcal{P} \rightarrow k\text{-}\text{Vect}, \ x \mapsto F_x \), the category of vector spaces over some field \( k \). The representations themselves form an abelian category whose morphisms are the natural transformations.

On a poset \( \mathcal{P} \) there exists a natural topology, which is generated by the basis \( U(x) = \{ x \leq y \in \mathcal{P} \} \). Using this topology, every representation \( F \) of \( \mathcal{P} \) induces a sheaf over \( \mathcal{P} \) by setting \( \mathcal{F}(U(x)) := F_x \), and conversely, any sheaf over \( \mathcal{P} \) with values in \( k\text{-}\text{Vect} \) induces a representation of \( \mathcal{P} \). In fact, the categories of representations of \( \mathcal{P} \) and of sheaves over \( \mathcal{P} \) with values in \( k\text{-}\text{Vect} \) are equivalent. However, it will be more comfortable for us to have both points of view in mind and to switch the picture freely. The additional bonus of sheaves over \( \mathcal{P} \) is that by the continuation to the whole topology of \( \mathcal{P} \), they automatically incorporate inverse limits via \( \mathcal{F}(U) = \lim_{\leftarrow} \mathcal{F}(U(x)) \), where the limit runs over all \( x \in U \). For us, this is a very natural way to encode all possible pullback diagrams over the poset \( \mathcal{P} \). Sheaves over posets have been in the literature before, the first reference we are aware of being [Bac75]. More recently, this kind of sheaves has been used in similar contexts like ours, for the study of certain modules over semigroup rings [Yan01], and for vector space arrangements [DGM00].

**Posets and graded modules.** Our general principle will be to start with local constructions and to globalize these by some gluing procedure, where 'local' and 'global' means over affine and general toric varieties, respectively. Recall that an affine toric variety \( U_\sigma \) over some algebraically closed field \( k \) on which the torus \( T \) acts, is equivalent to the spectrum of a normal semigroup ring \( k[\sigma_M] \). The semigroup \( \sigma_M \) is a subsemigroup of the character group \( M \cong \mathbb{Z}^r \) of \( T \), which is given by the intersection of a convex rational polyhedral cone \( \hat{\sigma} \) in \( M \otimes \mathbb{Z} \mathbb{R} \) with \( M \). Any equivariant sheaf \( \mathcal{E} \) over an affine toric variety \( U_\sigma \) is equivalent to an \( M \)-graded \( k[\sigma_M] \)-module \( E^\sigma = \Gamma(U_\sigma, \mathcal{E}) \), i.e.

\[
E^\sigma = \bigoplus_{m \in M} E^\sigma_m.
\]

A fundamental observation is that this grading is the reason that equivariant sheaves over toric varieties have still a semi-combinatorial nature, in contrast to the completely combinatorial description of the toric varieties themselves. To see this, note that \( \sigma_M \) endows \( M \) with the structure of a poset by setting \( m \leq_\sigma m' \) iff \( m' - m \in \sigma_m \) (we simplify here, as in fact this in general only defines a preorder). This way, \( E^\sigma \) is equivalent to a representation of \( M \) which maps every \( m \) to the vector space \( E^\sigma_m \), and every relation \( m \leq_\sigma m' \) is mapped to the vector space homomorphism \( E^\sigma_m \rightarrow E^\sigma_{m'} \), which is given by multiplication with the monomial \( \chi(m' - m) \). It turns out that the category of representations of the poset \( M \) is equivalent to the category of equivariant quasicoherent sheaves over \( U_\sigma \).

To be able to work truly with a finitely generated module, one needs an expedient finite representation for it. For this, we introduce the notion of a polyhedral decom-
position of \( M \). For any \( \rho \in \sigma(1) \) and any integer \( n_\rho \) we have the shifted halfspace \( \{ m \in M_\mathbb{R} \mid \langle m, n(\rho) \rangle \geq n_\rho \} \) in \( M_\mathbb{R} \), and for any tuple \( \underline{n} = (n_\rho \mid \rho \in \sigma(1)) \in \mathbb{Z}^{\sigma(1)} \) the intersection of half spaces \( P_{\underline{n}} = \{ m \mid \langle m, n(\rho) \rangle \geq n_\rho \text{ for all } \rho \in \sigma(1) \} \). We call such an unbounded domain \( P_{\underline{n}} \) a polyhedron. Note that the dual cone \( \hat{\sigma} \) itself is a polyhedron which has the zero face as its unique compact face. Figure 1 shows an example of a cone where \( \sigma(1) \) consists of four rays, and a polyhedron defined with respect to these four rays. The intersection of two polyhedra \( P_{\underline{n}_1}, P_{\underline{n}_2} \) is again a polyhedron, \( P_{\underline{n}} \), where

\[ \underline{n} = (\max\{n_{1,\rho}, n_{2,\rho}\} \mid \rho \in \sigma(1)). \]

This way, any collection of polyhedra \( P_{\underline{n}_1}, \ldots, P_{\underline{n}_s} \) gives rise to a partition of \( M \) as follows. Define the 'least common multiple' \( \underline{n} \) of any collection \( n_{i_1}, \ldots, n_{i_k} \), by the componentwise maximum of the \( n_{i_j} \). Then the equivalence classes \( T_{\underline{n}} \) contain all \( m \in M \) with \( \langle m, n(\rho) \rangle \geq n_{i,\rho} \) for all \( \rho \in \sigma(1) \), for which there is no bigger least common multiple \( \underline{n}' \) satisfying these inequalities. Figure 2 shows a partition of \( \mathbb{Z}^2 \) generated by three polyhedra. The set of lcm’s of the \( \underline{n}_1, \ldots, \underline{n}_s \) in a natural way becomes a poset, as a subposet of \( \mathbb{Z}^{\sigma(1)} \) with partial order induced by the componentwise order. The lcm’s are a special case of a polyhedral decomposition which is induced by an admissible poset. A finite poset \( P^\sigma \subset \mathbb{Z}^{\sigma(1)} \) is admissible if for any

\[
\begin{align*}
Figure 1: \text{Example of a cone with four maximal faces and a polyhedron} \\
\underline{n} = (\max\{n_{1,\rho}, n_{2,\rho}\} \mid \rho \in \sigma(1)). \end{align*}
\]

\[
\begin{align*}
Figure 2: \text{A polyhedral decomposition of } \mathbb{Z}^2 \text{ into seven regions} \\
\end{align*}
\]
There exists a unique maximal element $\mathbf{n} \in \mathcal{P}^\sigma$ such that $\langle m, n(\rho) \rangle \geq n(\rho)$ for all $\rho \in \sigma(1)$. $\mathcal{P}^\sigma$ is admissible with respect to $E^\sigma$ if moreover for every $\mathbf{n} \in \mathcal{P}^\sigma$ there exists a vector space $E_{\mathbf{n}}$ such that $E_{\mathbf{n}} \cong E^\sigma_m$ for all $m \in T_{\mathbf{n}}$. The vector spaces $E_{\mathbf{n}}$ together with appropriate morphisms $E_{\mathbf{n}} \rightarrow E_{\mathbf{n}'}$ (whose existence is part of our definition 4.11 for admissible posets), yield a representation of $\mathcal{P}^\sigma$, which encodes the complete structure of $E^\sigma$. We can think of it, euphemistically, as a compression of $E^\sigma$.

The most important feature of our constructions is that the compression of $E^\sigma$ is functorial, because we systematically exploit the formalism of sheaves on posets; we finally arrive at an equivalence of categories between sheaves over $\mathcal{P}^\sigma$ and $k[\sigma_M]$-modules with respect to which $\mathcal{P}^\sigma$ is admissible (Theorem 4.15). This in particular enables us to construct resolutions of $E^\sigma$ in terms of free resolutions of the sheaf $E_{\mathbf{n}}$ over $\mathcal{P}^\sigma$.

The resolutions obtained this way are not free resolutions, but rather resolutions by reflexive modules of rank one. Any $\mathbf{n} \in \mathbb{Z}^\sigma(1)$ gives rise to a $T$-invariant Weil divisor $D_{\mathbf{n}} = -\sum_{\rho \in \sigma(1)} n(\rho) D_{\rho}$ on $U_{\sigma}$, and thus to a reflexive sheaf of rank one $\mathcal{O}_{U_{\sigma}}(D_{\mathbf{n}})$. Write $S_{(\mathbf{n})}$ for the associated reflexive $k[\sigma_M]$-module, then its $M$-graded decomposition is given by

$$S_{(\mathbf{n})} \cong \bigoplus_{m \in \mathbb{P}^\sigma \cap M} k \cdot \chi(m).$$

Every equivalence class $T_{\mathbf{n}}$ has the shape of the forepart of the polyhedron $P_{\mathbf{n}}$, and thus provides a ’slot’ by which we can define a map $S_{(\mathbf{n})} \rightarrow E^\sigma_m$ without missing any $M$-degree in $T_{\mathbf{n}}$. This leads to a somewhat different philosophy of resolutions than the usual one — instead of a generating set of $E^\sigma$ as basic input for our resolutions, we use a polyhedral decomposition. This at least leads to finite resolutions and reduces in many cases the problem to understanding the modules $S_{(\mathbf{n})}$ (see Theorem 5.13 for such an application).

We want to remark that our notions of admissible posets and polyhedral decompositions are very close, though not entirely identical, to the sector partitions in [HM04].

### Gluing of posets and globalization.

A sheaf $\mathcal{E}$ is equivalent to a collection of $k[\sigma_M]$-modules $E^\sigma$, where $\sigma$ runs over the fan associated to $X$, which glues in an appropriate sense over the $U_{\sigma}$. On the other hand, $\mathcal{E}$ can be represented by a collection of sheaves over some admissible posets $\mathcal{P}^\sigma$, which we have to glue — in an appropriate sense.

The problem of gluing posets might be interesting in a somewhat broader mathematical context, so that we decided to define it slightly more general than necessary. We remark that the naive idea of gluing posets like topological spaces, which of course can be done, probably does not lead to anything interesting. For instance, one can easily show that a topological space which is covered by two open sets, each of which is homeomorphic to a poset, can globally be given the structure of a poset. By induction, one concludes that every set which has a finite cover by posets is a poset again. Our notion of gluing is different from this, and indeed it is a derived concept which comes very naturally from toric geometry, suitable for us to construct global resolutions.
Our idea is to realize gluing by passing from posets to preordered sets. In contrast to our statements above, a semigroup $\sigma_M$ in general induces only a preorder on $M$, rather than a partial order. For any two $m, m' \in M$ we have $m \leq_\sigma m'$ and $m' \leq_\sigma m$ iff $m - m' \in \sigma_M^\bot$, the maximal subgroup of $\sigma_M$. We can turn $\leq_\sigma$ into a proper partial order if we pass to the induced order on the quotient $M/\sigma_M^\bot$. For any $M$-graded module $E$ and any pair $m, m'$ with $m \leq_\sigma m'$ and $m' \leq_\sigma m$, the multiplication homomorphisms by $\chi(m' - m)$ and $\chi(m - m')$ necessarily are isomorphisms, and in fact, the categories of $M$-graded $k[\sigma_M]$-modules and of $M/\sigma_M^\bot$-graded $k[\sigma_M/\sigma_M^\bot]$-modules are equivalent.

Now for simplicity assume that $\leq_\sigma$ is a partial order and let $\tau < \sigma$ be a proper face, such that $\leq_\tau$ is a proper preorder. $\tau_M$ is of the form $\sigma_M + \mathbb{Z}_{\geq 0} \cdot (-m_\tau)$ for some $m_\tau \in \sigma_M$ such that $\tau^\bot \cap \sigma$ is a proper face of $\sigma$ and $\tau_M^\bot = (\sigma_M \cap \tau_M^\bot) + \mathbb{Z}_{\geq 0} \cdot (-m_\tau)$ is a nontrivial subgroup.

The set $\tau_M^\bot \cap \sigma_M$ is a subsemigroup of $\tau_M^\bot$, giving rise to a partial order on $\tau_M^\bot$. For any $m \in M$, we can think of the affine subset $m + \tau_M^\bot$ as a slice in $M$, and every such slice has its own partial order. With respect to such a slice, we can consider the directed system $E^\sigma_m$, with $m' \in m + \tau_M^\bot$, and the directed limit of this system:

$$E^\tau_m := \lim_{\rightarrow} E^\sigma_{m'}, \quad m' \in m + \tau_M^\bot.$$ 

It turns out that $\bigoplus_{m \in M} E^\sigma_m \cong E^\sigma_{\chi(m_\tau)}$, i.e. the localization of $E^\sigma$ by the character $\chi(m_\tau)$, which we now can interpret as some kind of limit figure of $E^\sigma$ along the direction $m_\tau$.

This example is our prototype for defining gluing of partially ordered sets and sheaves over them. Let $P$ be an abstract poset with some partial order $\leq$. Then a localization of $\leq$ is a preorder $\leq'$, such that $x \leq y$ implies $x \leq' y$, and $x \leq' y$ implies $x \leq w$ for some element $w$ with $w \leq' y$ and $y \leq' w$. This is the abstract analogue of the slicing above, where the preorder $\leq'$ groups together certain subsets of $P$. By this definition, if we pass to the quotient $P/\sim$, where $x \sim y$ iff $x \leq' y$ and $y \leq' x$, every representation $F$ of $P$ induces a representation $\tilde{F}$ of $P/\sim$, where

$$\tilde{F}[x] = \lim_{\rightarrow} F_y,$$

the limit is taken over all $y \in x$ with respect to the partial order $\leq$. This way, we obtain a quite canonical procedure for gluing sheaves $\tilde{F}_1, \tilde{F}_2$ over two posets $P_1, P_2$: we simply require that the posets have localizations $\leq_1, \leq_2$ such that there exists isomorphisms $l : P_1/\sim \to P_2/\sim$ and $\phi : l^* \tilde{F}_2 \to \tilde{F}_1$. We refer to subsections 2.3 and 2.4 for the precise definitions.

**Overview of the paper.** This paper tries to be self-contained in the sense that all required notation related to toric geometry are introduced. However, we refrain from giving any account on these subjects; we refer to [Per04a] for a more details.
The paper consists of four principal parts. In section 2, we present our principal technical framework from the theory of poset representations; in addition to well-known material, in this section gluing of posets and of sheaves over posets are introduced. In section 3, we recall general notions from toric geometry and the formalism of ∆-families as developed in [Per04a]. We present a partial reformulation of the material in view of the formalism of section 2. We show that the Krull-Schmidt theorem holds in the category of equivariant coherent sheaves over any toric variety. Section 4 contains the biggest part of the work; starting from polynomial rings (subsection 4.2), and then generalizing to normal semigroup rings (subsection 4.3), we construct resolutions for finitely generated modules over affine toric varieties. In subsections 4.4 and 4.5 we construct global resolutions, both from the point of view of gluing over posets, and homogeneous coordinate rings. In section 5 we analyze the special case of reflexive modules, and in particular we amplify their close relationship to vector space arrangements. As an application, in subsection 5.4 we show that our resolutions in the case of reflexive modules behave well in sense of homological algebra. In subsection 5.5 we discuss how resolutions of vector space arrangements can effectively be computed in terms of associated modules over polynomial rings.

This work extends results of my thesis [Per03]. Most of this paper has been written during my stay at the Abdus Salam ICTP, Trieste for whose hospitality I am deeply grateful.

2 Preliminaries on Preordered Sets

In this section let \( \mathcal{P} \) be a countable set on which a preorder \( \leq \) is defined. Recall that a preorder is defined by the same axioms as a partial order, except for the reflexivity axiom, i.e. there may exist elements \( x, y \in \mathcal{P} \) such that \( x \leq y \) and \( y \leq x \), but \( x \neq y \). For such pairs we write \( x \preceq y \); in the sequel we will frequently put indices on the symbol \( \leq \), such as \( \leq', \leq_\sigma \), etc.; then these indices also apply to \( \preceq \). If there is no ambiguity in the preorder chosen, we just write \( \mathcal{P} \), else we write \((\mathcal{P}, \leq)\).

2.1 Representations of preordered sets

Any preordered set \( \mathcal{P} \) in a natural way forms a category; its objects are given by the set underlying \( \mathcal{P} \) and the morphisms for \( x, y \in \text{Ob}(\mathcal{P}) \) are:

\[
\text{Mor}(x, y) = \begin{cases} 
\text{the pair } (x, y) & \text{if } x \leq y \\
\emptyset & \text{else.}
\end{cases}
\]

Here the pair \((x, x)\) represents the identity morphism for all \( x \in \mathcal{P} \).
Definition 2.1: A functor from \( P \) to \( k\text{-Vect} \), the category of vector spaces over the field \( k \), is called a \( k\text{-linear representation} \) of \( P \).

As a general notation, if \( E \) denotes a \( k\text{-linear representation} \) of a preordered set \( P \), an element \( x \in P \) is mapped to the vector space denoted \( E_x \), and the relation \( x \leq y \) is mapped to a vector space homomorphism \( E(x, y) \). The \( k\text{-linear representations} \) of \( P \) form an abelian category whose morphisms are natural transformations.

Representations of \( P \) are equivalent to \textit{sheaves} over \( P \). On \( P \) there is defined a topology which is generated by the basis \( U(x) := \{ y \geq x \} \) for all \( x \in P \). The continuous maps between posets then are precisely the order preserving maps. A sheaf \( E \) on \( P \) with respect to this topology with values in \( k\text{-Vect} \) automatically induces a representation of \( P \). On the other hand, for any representation \( E \), following [GD71] §0.3.2, we obtain a presheaf \( E \) on \( P \) by setting \( E(U(x)) := E_x \) for all \( x \in P \) and \( E(U) := \lim \leftarrow E(U(x)) \) for some open set \( U \), where the limit runs over all \( x \in U \). Note that the stalk \( E_x \) is isomorphic to \( E(U(x)) \). By observing that for some \( U(x) \) every open cover necessarily contains \( U(x) \), and applying the criterion of §0.3.2.2 in [GD71], it follows that every presheaf automatically is a sheaf.

A distinguished class of representations are the representations \( F_x \) associated to some element \( x \in P \), which are given by:

\[
    y \mapsto \begin{cases} 
        k & \text{if } x \leq y \\
        0 & \text{else,}
    \end{cases}
\]

and relations \( y \leq z \) mapped to identity if \( x \leq y \), and to the zero map else. In terms of sheaves over \( P \), one can alternatively define \( F^x \) as follows. Denote \( j_x \) the canonical inclusion \( U(x) \hookrightarrow P \), and let \( k \) be the constant sheaf with \( k(U(y)) = k \) for all \( y \in P \). Then \( F^x \) corresponds to \( j_x j_x^* k \). We say that a representation of \( P \) is \textit{free} if it is isomorphic to a direct sum of objects of the form \( F^x \). We have:

**Proposition 2.2:** The representations \( F^x \) are projective objects in the category of \( k\text{-linear representations} \) of \( P \).

Using the notion of free objects, we can introduce \textit{free resolutions}.

**Definition 2.3:** Let \( P \) be any preordered set and \( E \) a \( k\text{-linear representation} \). Then a \textit{free resolution} of \( E \) is an exact sequence

\[
    \ldots \rightarrow F_i \rightarrow \ldots \rightarrow F_0 \rightarrow E \rightarrow 0
\]

where for every \( i : F_i \cong \bigoplus_j F^{x_{ij}} \) for some \( x_{ij} \in P \).
Let $x \in P$, then we consider the subvector space of $E_x$ which is generated by the image of all $E_y$, $y < x$, by the morphisms $E(x, y)$, $E_{<x} := \sum_{y < x} E(y, x)E_y$, where we set $E_{<x} := 0$ if the set $\{y < x\}$ is empty. $\text{codim}_{E_x} E_{<x}$ is the free dimension of $E_x$.

**Proposition 2.4:** Let $P$ be a finite preordered set. Then for every $k$-linear representation of $P$ there exists a finite free resolution, that is, there exists a free resolution as above and some $n \geq 0$ such that $F_i = 0$ for all $i > n$.

**Proof.** Let $X \subset P$ be the set of elements such that $E_x$ has positive free dimension. For every $x \in X$ we consider the short exact sequence of vector spaces

$$0 \longrightarrow E_{<x} \longrightarrow E_x \longrightarrow E_x / E_{<x} \longrightarrow 0,$$

where we have chosen some section $\mu_x$. For every such $x$, we can consider the constant sheaf $E_x / E_{<x}$ on $P$ and its restriction $E^x := \sum_{y < x} F^x E_y$ for every pair $x \leq y$ and the zero map for all $x \not\leq y$. The sheaf $E^x$ is isomorphic to $(F^x)^f_x$, where $f_x$ is the free dimension of $E_x$. Thus we define $F_0 = \bigoplus_{x \in X} E^x \cong \bigoplus_{x \in X} (F^x)^f_x$ and a homomorphism $\phi_0 : F_0 \longrightarrow E$ by setting $\phi_0 := \sum_{x \in X} \phi_x$. By construction, $\phi_0$ is a surjective map, and we obtain thus a short exact sequence of representations of $P$:

$$0 \longrightarrow K_0 \longrightarrow F_0 \longrightarrow E \longrightarrow 0.$$

Now we can repeat this construction with $K_0$, and by iterating we obtain a free resolution of $E$ which is concatenated of short exact sequences $0 \longrightarrow K_{i+1} \longrightarrow F_{i+1} \longrightarrow K_i \longrightarrow 0$.

Now observe that for $K_{i+1,x} = 0$ whenever the free dimension of $K_{i,x}$ is equal to $\dim K_{i,x}$, and $K_{i,x} = 0$ implies that $K_{i+1,x} = 0$. The set of such $K_{i,x}$ whose free dimension is equal to $\dim K_{i,x}$ is always nonempty as long as $K_i$ is nontrivial, because the set contains at least the minimal elements $x \in P$ which have nontrivial $K_{i,x}$. So, as $P$ is finite, it follows that there exists some $r > 0$ for which $K_{i,x} = 0$ for all $i > r$. □

**Definition 2.5:** Let $0 \longrightarrow F_r \xrightarrow{\phi_r} \ldots \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} E \longrightarrow 0$ be a free resolution of a $k$-linear representation $E$, then we call the kernel of $\phi_i$ the $i$th syzygy representation of $E$.

### 2.2 Direct and inverse limits

Now we recall some basic facts about direct and inverse limits in the category of vector spaces. This is only intended as a reminder to the reader, as we will be using limits extensively during the rest of this paper.
As we have seen in the previous subsection, every preordered set \( \mathcal{P} \) in a natural way is a directed family. Thus, a representation \( E \) of \( \mathcal{P} \) becomes a directed family of vector spaces. Recall, that the inverse limit of \( E \) is a vector space

\[
\lim_{\leftarrow} E := E^i
\]

which has the following universal properties:

(i) for every element \( x \in \mathcal{P} \) there exists a unique homomorphism \( \phi_x : E^i \to E_x \) such that \( E(x, y) \circ \phi_x = \phi_y \) for every \( x \leq y \);

(ii) for every vector space \( F \) with homomorphisms \( \psi_x : F \to E_x \), where \( \psi_y = E(x, y) \circ \psi_x \) for every \( x \leq y \), there exists a unique homomorphism \( \delta : F \to E^i \) with \( \psi_x = \phi_x \circ \delta \) for all \( x \in \mathcal{P} \).

**Definition 2.6:** We denote the vector space homomorphism \( \delta : F \to E^i \) diagonal homomorphism from \( F \) to \( E^i \).

Explicitly, such a limit can be constructed as the subvector space of the direct product \( \prod_{x \in \mathcal{P}} E_x \) consisting of sequences \( (e_x \mid x \in \mathcal{P}) \) such that \( E(x, y)(e_x) = e_y \) for every pair \( x \leq y \). If \( \mathcal{P} \) has a unique minimal element \( x_{\text{min}} \), then \( \phi_{x_{\text{min}}} : E^i \to E_{x_{\text{min}}} \) becomes an isomorphism. This construction is a straightforward generalization of the pullback in the category of vector spaces; the pullback is the special case where the poset consists of three elements \( x, y, z \) with \( x < z \) and \( y < z \).

Dually, there exists the direct limit

\[
\lim_{\rightarrow} E := E^d
\]

which generalizes pushout. It can explicitly be constructed as the quotient of the vector space \( \prod_{x \in \mathcal{P}} E_x \) by the subvector space generated by vectors \( E(x, y)(e_x) - e_x \). For every \( x \in \mathcal{P} \) there exists a homomorphism \( \phi^x : E_x \to E^d \) such that universal properties analogously to the inverse limit are fulfilled. Note that in case that there exists a unique maximal element \( x_{\text{max}} \), the homomorphism \( \phi^{x_{\text{max}}} \) is an isomorphism.

Both limits behave covariantly; consider two preordered sets \( \mathcal{P}, \mathcal{Q} \), and any two representations \( E, F \) of \( \mathcal{P} \) and \( \mathcal{Q} \), respectively, and a order preserving map \( f : \mathcal{P} \to \mathcal{Q} \). Then any natural transformation \( r : E \to f^*F \) induces a homomorphism of limits:

\[
\lim_{\leftarrow} r : \lim_{\leftarrow} E \to \lim_{\leftarrow} f^*F \text{ resp. } \lim_{\rightarrow} r : \lim_{\rightarrow} E \to \lim_{\rightarrow} f^*F.
\]

In particular, if \( \mathcal{P} \) and \( \mathcal{Q} \) have unique minimal elements \( x_{\text{min}} \) and \( y_{\text{min}} \), respectively, and \( f(x_{\text{min}}) = y_{\text{min}} \), we obtain

\[
\lim_{\leftarrow} r : \lim_{\leftarrow} E \to \lim_{\leftarrow} F,
\]

and analogously for the direct limit with respect to maximal elements.
2.3 Gluing of preordered sets

**Definition 2.7:** Let \((\mathcal{P}, \leq)\) be a preordered set. Then we denote \(\mathcal{P}_{\leq}\) the quotient of \(\mathcal{P}\) by the equivalence relation which is given by \(x \sim y\) iff \(x \leq y\).

Clearly, \(\leq\) induces a partial order on the set \(\mathcal{P}_{\leq}\).

**Definition 2.8:** A localization of \(\mathcal{P}\) is a preorder \(\leq'\) on \(\mathcal{P}\) such that the following conditions are fulfilled:

(i) for all \(x, y \in \mathcal{P}\), \(x \leq y\) implies \(x \leq' y\),

(ii) for all \(x \leq' y\) there exists some \(w \leq' y\) such that \(x \leq w\).

Let \(\leq'\) be a localization of \((\mathcal{P}, \leq)\), then \(\leq\) induces a relation on \(\mathcal{P}_{\leq'}\) by setting \([x] \leq [y]\) iff there exist \(u \in [x], v \in [y]\) with \(u \leq v\).

**Proposition 2.9:** Let \(\leq'\) a localization of \((\mathcal{P}, \leq)\), then the relation on \(\mathcal{P}_{\leq'}\) induced by \(\leq\) coincides with the partial order induced by \(\leq'\).

**Proof.** We check the poset axioms for \(\leq\):

1) \([x] \leq [x]\) follows because \(u \leq' u\) implies that there exists \(v \leq' u\) such that \(u \leq v\).

2) Let \([x] \leq [y]\) and \([y] \leq [x]\); then there exist \(u, p \in [x], v, q \in [y]\) such that \(u \leq v\) and \(p \leq q\); then \(u \leq' v \leq' p \leq' u\), and thus \(v \leq' u\), hence \([x] = [y]\).

3) Let \([x] \leq [y]\) and \([y] \leq [z]\); then there exist \(u \in [x], v, p \in [y]\) and \(q \in [z]\) such that \(u \leq v\) and \(p \leq q\); thus \(u \leq' v \leq' p \leq' q\) and there exists \(w \leq' q\) such that \(u \leq w\), and thus \([x] \leq [z]\).

Now the equivalence of the partial orders \(\leq\) and \(\leq'\) on \(\mathcal{P}_{\leq'}\) is trivial. \(\Box\)

Let \((\mathcal{P}_\alpha, \leq_\alpha)\) be a finite family of preordered sets, together with a family of representations \(F_\alpha\), where \(\alpha\) runs over some index set \(A\). Our aim is to glue these representations when certain conditions on the \(\mathcal{P}_\alpha\) are fulfilled. For this, we need the following notion:

**Definition 2.10:** Let \(f : \mathcal{P} \rightarrow \mathcal{Q}\) be an order preserving map between preordered sets. Then \(f\) is a contraction if

(i) for every \(x \in \mathcal{P}\) there exists some \(y \in \mathcal{Q}\) such that \(f(U(x)) = U(y)\),

(ii) for every \(y \in \mathcal{Q}\) \(f^{-1}(U(y)) = U(x)\) for some \(x \in \mathcal{P}\).

These conditions imply that \(f\) is surjective and that for every \(x \in \mathcal{P}\) with \(f(U(x)) = U(y)\) there exists \(z \in \mathcal{P}\) with \(U(x) \subset U(z) = f^{-1}(U(y))\). By this we can define a map \(h : \mathcal{Q} \rightarrow \mathcal{P}\) by mapping \(y \mapsto z\). This map is an order preserving injection of \(\mathcal{Q}\) into \(\mathcal{P}\).

**Definition 2.11:** Let \(f : \mathcal{P} \rightarrow \mathcal{Q}\) be a contraction. Then the unique map \(h : \mathcal{Q} \rightarrow \mathcal{P}\) mapping \(y \in \mathcal{Q}\) to \(z \in \mathcal{P}\) such that \(f^{-1}(U(y)) = U(z)\) is called hooking of \(\mathcal{Q}\) into \(\mathcal{P}\).
Using this definition, one can think of our gluing of posets as a process of hooking different posets along common contractions.

Let \( P, Q \) be two finite preordered sets, \( f : P \rightarrow Q \) a contraction, and \( E \) a representation of \( Q \). Then the pullback \( f^*F^x \) for any free representation for some \( x \in Q \) then is isomorphic to the free representation \( F^{h(x)} \) of \( P \). For any free resolution \( 0 \rightarrow F_r \rightarrow \cdots \rightarrow F_0 \rightarrow E \rightarrow 0 \), one can consider the pullback sequence \( 0 \rightarrow f^*F_r \rightarrow \cdots \rightarrow f^*F_0 \rightarrow f^*E \rightarrow 0 \). We observe:

**Lemma 2.12:** The sequence \( 0 \rightarrow f^*F_r \rightarrow \cdots \rightarrow f^*F_0 \rightarrow f^*E \rightarrow 0 \) is isomorphic to the free resolution of \( f^*E \) in the sense of proposition 2.4.

**Proof.** It suffices to check the first step of the resolution \( 0 \rightarrow K_0 \rightarrow f^*F_0 \rightarrow f^*E \rightarrow 0 \) and to show that \( K_0 \) and \( f^*F_0 \) coincide with the representations obtained by the procedure of proposition 2.4. But this follows directly from the fact that for every \( y \in Q \), the homomorphisms \( (f^*E)_{h(x)} \rightarrow (f^*E)_y \) are isomorphisms for all \( h(x) \leq y \in f^{-1}(x) \).

**Definition 2.13:** Let \( (A, \preceq) \) be a finite poset and \( P_\alpha, \alpha \in A \) be a family of preordered sets. We say that the posets \( P_\alpha \) glue over \( A \) if

(i) for every \( \beta < \alpha \in A \) there exists a localization \( \leq_\beta \) of \( \leq_\alpha \) and a contraction \( l_{\alpha \beta} : (P_\alpha)_{\leq_\beta} \rightarrow (P_\beta)_{\leq_\beta} \);

(ii) for every triple \( \gamma \preceq \beta \preceq \alpha \in A \), the composition of maps \( P_\alpha \rightarrow (P_\alpha)_{\leq_\beta} \xrightarrow{l_{\alpha \beta}} (P_\beta)_{\leq_\beta} \rightarrow (P_\gamma)_{\leq_\gamma} \) coincides with \( P_\alpha \rightarrow (P_\alpha)_{\leq_\gamma} \xrightarrow{l_{\alpha \gamma}} (P_\gamma)_{\leq_\gamma} \).

Our principal example, where the maps \( l_{\alpha \beta} \) actually are isomorphisms, will be the preorderings associated to a fan \( \Delta \) in section 3.

### 2.4 Gluing of sheaves over preordered sets

Let \( (P, \leq) \) be some preordered set; if \( E \) is some representation of \( P \), then for any pair \( x \preceq y \), the map \( E(x, y) : E_x \rightarrow E_y \) is an isomorphism whose inverse is \( E(y, x) \). Thus \( E \) descends to a representation of \( P_{\leq} \) by setting \( E_{[x]} := \lim_y E_y \), where the direct limit is taken over all elements \( y \leq x \). For any \( y \leq x \) there is the canonical inclusion of directed systems \( \{E_z \mid z \leq y\} \hookrightarrow \{E_z \mid z \leq x\} \), which induces a functorial homomorphism \( E_{[y]} \rightarrow E_{[x]} \). On the other hand, every representation \( F \) of \( P_{\leq} \) lifts to a representation of \( P \) by setting \( E_x := E_{[x]} \) and \( E(x, y) := E([x], [y]) \). By descend and lift, we have:

**Lemma 2.14:** Let \( (P, \leq) \) be a preordered set. The category of representations of \( P \) is equivalent to the category of representations of \( P_{\leq} \).
Let \( \leq' \) be a localization of \( \leq \). For any \( x \in P \), denote \( P_x = \{ z \in P \mid z \leq' x \} \).

We construct a representation on \( P_{\leq'} \) by mapping \( [x]' \in (P)_{\leq'} \) to the vector space \( E_{[x]'} := \lim_{\to} E_z \), the direct limit taken over \( P_x \) with respect to the partial order \( \leq \). The inclusion \( P_x \hookrightarrow P_y \) induces an inclusion of directed sets with respect to \( \leq \), and thus we obtain a morphism \( E_{[x]'} \rightarrow E_{[y]'} \). By lemma 2.14, this representation lifts to a representation of \( (P, \leq') \).

**Definition 2.15:** Let \((P, \leq)\) be a preordered set, \( \leq' \) a localization of \( \leq \), and \( E \) a representation of \( (P, \leq) \). Consider the poset \( P_{\leq'} \). Then we call the induced sheaf on \( (P, \leq') \) a localization of \( F \).

Now we assume that we are given some partially ordered set \((A, \preceq)\), a collection of preordered sets \( P_\alpha \) which glues over \( A \), and a collection of sheaves \( E_\alpha \) over \( P_\alpha \) for every \( \alpha \in A \). We want glue this collection of sheaves to give some kind of global object over the glued preordered sets.

**Definition 2.16:** We say that the collection \( E_\alpha \) glues over the collection \( P_\alpha \), if

(i) for every \( \beta \preceq \alpha \), and morphism of posets \( l_{\alpha \beta} : (P_\alpha)_{\leq'\beta} \rightarrow (P_\beta)_{\leq\beta} \) there is an isomorphism of sheaves \( \phi_{\alpha\beta} : l_{\alpha \beta}^* E_{\beta} \cong E_{\alpha} \).

(ii) for every triple \( \gamma \preceq \beta \preceq \alpha \): \( \phi_{\alpha\gamma} = \phi_{\alpha\beta} \circ l_{\alpha \beta}^* \phi_{\beta\gamma} \).

We call a such a collection a sheaf over \( P_\alpha \).

Let \( E_\alpha, F_\alpha \) be sheaves over \( P_\alpha \), where we denote the gluing homomorphisms \( \phi_{\alpha\beta} \) and \( \psi_{\alpha\beta} \), respectively. A homomorphism from \( E_\alpha \) to \( F_\alpha \) is given by a collection of homomorphisms \( f_\alpha : E_\alpha \rightarrow F_\alpha \) such that \( f_\alpha \circ \phi_{\alpha\beta} = \psi_{\alpha\beta} \circ l_{\alpha \beta}^* f_\alpha \) for every pair \( \beta \preceq \alpha \).

One checks straightforwardly that this is compatible with the cocycle conditions on \( \phi_{\alpha\beta} \) and \( \psi_{\alpha\beta} \), and moreover that the corresponding families of kernels and cokernels of \( f_\alpha \) glues over \( A \):

**Proposition 2.17:** The category of sheaves over \( P_\alpha \) is abelian.

**Compression of sheaves over preordered sets.** Let \( A \) be a finite poset and denote \( P_A^f \) the category of collections of finite preordered sets \( \{ P_\alpha \mid \alpha \in A \} \) which glue over \( A \). Let \( \{ Q_\alpha \} \) be any collection of not necessarily finite preordered sets which glues over \( A \). Denote \( C \) any subcategory of the category of sheaves which glue over the collection \( Q_\alpha \).

A compression of \( C \) is any object \( \{ P_\alpha \} \) of \( P_A^f \) together with a pair of functors

\[
\text{zip} : C \rightarrow \text{Sheaves}(P_\alpha) \\
\text{unzip} : \text{Sheaves}(P_\alpha) \rightarrow C.
\]

which induce an equivalence of categories between \( C \) and \( \text{Sheaves}(P_\alpha) \).
3 Toric Varieties and $\Delta$-Families

In this section we briefly recall basic facts for toric varieties and our results from [Per04a] on equivariant sheaves over toric varieties. For general information about toric varieties we refer to [Oda88] and [Ful93]. In this work $X$ will always denote an $r$-dimensional toric variety over a fixed algebraically closed field $k$, and $T$ the open dense torus contained in $X$. Moreover, we use the following notation:

- $M \cong \mathbb{Z}^n$ is the character group of $T$, and $N$ the $\mathbb{Z}$-module dual to $M$;
- $M_R := M \otimes \mathbb{Z} \mathbb{R}, N_R := N \otimes \mathbb{Z} \mathbb{R}$;
- elements of $M$ are denoted $m, m'$ etc. if written additively and $\chi(m), \chi(m')$ etc. if written multiplicatively, i.e. $\chi(m + m') = \chi(m)\chi(m')$;
- $\Delta$ denotes the fan associated to $X$, and cones in $\Delta$ are denoted by small Greek letters $\rho, \sigma, \tau$, etc.; the natural order among cones is denoted by $\tau < \sigma$;
- $\Delta(i) := \{ \sigma \in \Delta | \dim \sigma = i \}$ the set of all cones of fixed dimension $i$, $\sigma(i) := \{ \tau \in \Delta(i) | \tau < \sigma \}$;
- $\hat{\sigma} := \{ m \in M_R | \langle m, n \rangle \geq 0 \text{ for all } n \in \sigma \}$ is the cone dual to $\sigma$,
- $\sigma^\perp = \{ m \in M_R | \langle m, n \rangle = 0 \text{ for all } n \in \sigma \}$,
- $\sigma_M := \hat{\sigma} \cap M$ is the subsemigroup of $M$ associated to $\sigma$,
- $\sigma_M^\perp := \sigma^\perp \cap M$ is the maximal subgroup of $\sigma_M$;
- the affine toric variety associate to a cone $\sigma$ is denoted $U_\sigma$,
- $U_\sigma \cong \text{spec}(k[\sigma_M])$, where $k[\sigma_M]$ is the semigroup ring over $\sigma_M$;
- elements of $\Delta(1)$ are called rays, and the torus invariant Weil divisor associated to some ray $\rho \in \Delta(1)$ is denoted $D_\rho$.

3.1 Equivariant sheaves and $\Delta$-families

Consider any rational polyhedral convex cone $\sigma$, then the subsemigroup $\sigma_M$ induces a directed preorder $\leq_\sigma$ on $M$ by setting $m \leq_\sigma m'$ iff $m' - m \in \sigma_M$. The following properties of $\leq_\sigma$ are easy to see:

(i) $m \leq_\sigma m'$ and $m' \leq_\sigma m$ iff $m' - m \in \sigma_M^\perp$.
(ii) If $\tau \leq \sigma$, then $m \leq_\sigma m'$ implies $m \leq_\tau m'$.
(iii) If $\sigma$ is of maximal dimension in $N_R$, then $\leq_\sigma$ is a partial order.

Let $E$ be an equivariant sheaf over $X$ and denote $E^\sigma := \Gamma(U_\sigma, E)$ for every affine open $T$-invariant subvariety $U_\sigma$ of $X$. The dual action of $T$ on $E^\sigma$ induces an isotypical decomposition

$$E^\sigma = \bigoplus_{m \in \Delta} E_{m}^\sigma$$

For any two $m \leq_\sigma m'$, there exists a distinguished $k$-linear map spaces $\chi_{m,m'}^\sigma : E_m \rightarrow E_{m'}$ which is given by multiplication by the monomial $\chi(m' - m) \in k[\sigma_M]$. These distinguished maps completely specify the module structure of $E^\sigma$ over $k[\sigma_M]$. Observing
that $\chi(m''-m')\chi(m'-m) = \chi(m''-m)$ and $\chi(m-m) = 1$, we even obtain a functorial description of $E^\sigma$. By mapping $m \mapsto E^\sigma_m$ and $(m, m') \mapsto \chi^\sigma_{m,m'}$ for $m \leq \sigma m'$, every $M$-graded $k[\sigma_M]$-module $E^\sigma$ defines a functor from the preordered set $(M, \leq \sigma)$ to the category $k$-$\text{Vec}$ of $k$-vector spaces.

**Proposition 3.1** ([Per04a], Proposition 5.5): Let $U_\sigma = \text{spec}(k[\sigma_M])$ be an affine toric variety. Then the following categories are equivalent:

(i) equivariant quasicoherent sheaves over $U_\sigma$,  
(ii) $M$-graded $k[\sigma_M]$-modules,  
(iii) $k$-linear representations of the preordered set $(M, \leq \sigma)$.

**Definition 3.2:** We call a representation of $(M, \leq \sigma)$ a $\sigma$-family.

In the sequel, we will use the notation $E^\sigma$ exchangeably for the $k[\sigma_M]$-module and for the $\sigma$-family.

Now for any pair $\tau < \sigma$, there exists some $m_\tau \in \sigma_1^M$ such that $\tau_M = \sigma_M + \mathbb{Z}_{\geq 0}(-m_\tau)$ and $\tau_M^\sigma = (\tau_M^\sigma \cap \sigma_M) + \mathbb{Z}_{\geq 0}(-m_\tau)$. In terms of preordered sets, this translates the way that we can consider $(M, \leq \tau)$ as a localization of $(M, \leq \sigma)$ in the sense of subsection 2.3. Moreover, the localization of $(M, \leq \sigma)$ by $\leq \tau$ coincides with $(M, \leq \tau)$, and thus the contractions $l_{\tau} : M_{\leq \sigma} \to M_{\leq \tau}$ are isomorphisms. We have:

**Proposition 3.3:** The family of preordered sets $(M, \leq \sigma)$, $\sigma \in \Delta$, glues over $\Delta$.

The restriction of $E_{|U_\sigma}$ to $U_\tau$ corresponds to the localization $E^\sigma_{\chi(m_\tau)}$. To understand this in terms of $\sigma$-families, we first observe that the canonical map $E^\sigma \to E^\sigma_{\chi(m_\tau)}$ at the same time is a homomorphism of directed systems.

**Proposition 3.4:** For every $m \in M$ there exists a natural isomorphism $E^\sigma_m \cong \lim_{\to} E^\sigma_{m'}$, where the limit is taken over the directed system of all $E^\sigma_{m'}$ with $m' \leq \tau m$ with respect to the preorder $\leq \sigma$.

**Proof.** By definition of localization, the vector space $E^\sigma_m$ is the set of equivalence classes $\{[\frac{e}{\chi^\sigma_{m'}}] | \deg_M e = m + m'\}$, where $\frac{e_{m_1}}{\chi^\sigma_{m_1}} \sim \frac{e_{m_2}}{\chi^\sigma_{m_2}}$ if and only if $\chi(m_1) \cdot e_2 = \chi(m_2) : e_1$ in $E^\sigma$, where without loss of generality, $m_1$ and $m_2$ can be chosen from $\sigma_M$. In other notation, this reads $\chi^\sigma_{m+m_1,m+m_1+m_2} : e_2 = \chi^\sigma_{m+m_2,m+m_1+m_2} : e_1$. So, in a natural way, we can identify $E^\tau_m$ with the direct limit $\lim_{\to} E^\sigma_{m'}$. \quad \square

By this proposition, we see that the localization of $E^\sigma$ by $\chi(m_\tau)$ translates into the localization of $E^\sigma$, considered as sheaf over $(M, \leq \sigma)$, to $(M, \leq \tau)$. We get:

**Definition 3.5** (see also [Per04a], Definition 5.8): A $\Delta$-family is a collection $\{E^\sigma | \sigma \in \Delta\}$ of $\sigma$-families which glues over $\Delta$. 
**Theorem 3.6 ([Per04a], Theorem 5.9):** The category of equivariant sheaves over $X$ is equivalent to the category of $\Delta$-families.

### 3.2 The Krull-Schmidt property

Let $\mathcal{C}$ be any category in which direct sums exist. We say that the Krull-Schmidt theorem holds in $\mathcal{C}$ if for every object $A$ in $\mathcal{C}$ and for every two decompositions into indecomposable objects

$$A \cong X_1 \oplus X_2 \oplus \cdots \oplus X_n \cong Y_1 \oplus Y_2 \oplus \cdots \oplus Y_m$$

we have $m = n$, and there exists a permutation $\pi$ of $\{1, \ldots, n\}$ such that $X_i \cong Y_{\pi(i)}$ for every $i$. It is well known that the Krull-Schmidt theorem holds in the category of coherent sheaves over a complete variety. For the category of equivariant coherent sheaves over a toric variety, we can drop the completeness condition:

**Theorem 3.7:** Let $X$ be any toric variety, then the Krull-Schmidt theorem holds for the category of equivariant coherent sheaves over $X$.

**Proof.** According to a classical result of Atiyah ([Ati56]), it suffices to show that for every two equivariant sheaves $\mathcal{E}$ and $\mathcal{F}$, the vector space $\text{Hom}(\mathcal{E}, \mathcal{F})^T$ of $T$-equivariant sheaf homomorphisms is finite-dimensional. As we are dealing only with finite fans, it is enough to consider the case where $X = U_\sigma$ is an affine toric variety such that $\mathcal{E}$ and $\mathcal{F}$ correspond to finitely generated $k[\sigma_M]$-modules $E^\sigma$ and $F^\sigma$. In this case the statement follows because every generator of $E^\sigma$ of degree $m$ must be mapped to some element $f \in F_m^\sigma$ and every vector space $F_m^\sigma$ is finite dimensional ([Per04a], Proposition 5.11). \qed

### 3.3 The quotient representation of a toric variety

Every toric variety can be represented as a good quotient of a quasi-affine toric variety (see [Cox95]). This representation starts with the exact sequence

$$0 \longrightarrow M_0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Delta(1)} \longrightarrow A \longrightarrow 0$$

where the map from $M$ to $\mathbb{Z}^{\Delta(1)}$ is given by $m \mapsto ((m, n(\rho)) \mid \rho \in \Delta(1))$. In the sequel we will assume that the fan $\Delta$ is not contained in a proper subvector space of $N_\mathbb{R}$. In this case $M_0$ is the zero module.

We consider the polynomial ring $S = k[x_\rho \mid \rho \in \Delta(1)]$; this ring is endowed with a natural $\mathbb{Z}^{\Delta(1)}$-grading by setting $\deg x_\rho = n$ for every monomial $x_\rho$. Via the surjection of $\mathbb{Z}^{\Delta(1)}$ onto $A$, the ring $S$ automatically acquires an $A$-grading,

$$S \cong \bigoplus_{\alpha \in A} S_\alpha.$$
We define the irrelevant ideal \( B = \langle x^\sigma \mid \sigma \in \Delta \rangle \), where \( x^\sigma = \prod_{\rho \in \Delta(1)} x^\rho \) for every \( \sigma \in \Delta \). The variety \( \mathbf{V}(B) \) defined by \( B \) is a finite union of linear subspaces of \( \text{spec}(S) \cong k^{\Delta(1)} \), which has codimension at least two. The complement of \( \mathbf{V}(B) \), which we denote \( \hat{X} \), is a quasi-affine toric variety, on which the torus \( \hat{T} \cong (k^*)^{\Delta(1)} \) acts. Denote \( e_\rho \) the standard basis vectors of \( \mathbb{R}^{\Delta(1)} \), then the fan of \( \hat{X} \) is generated by the cones \( \hat{\sigma} = \sum_{\rho \in \sigma(1)} \mathbb{R}_{\geq 0} e_\rho \), for every \( \sigma \in \Delta \). The affine open subsets \( U_\sigma \) form a cover of \( \hat{X} \), and we will call \( \hat{\Delta} = \{ \hat{\sigma} \mid \sigma \in \Delta \} \) the fan of \( \hat{X} \), although in general \( \hat{\Delta} \) is not a proper fan, unless \( X \) is a simplicial toric variety. There is a canonical morphism \( \pi : \hat{X} \to X \) which is described by the map of fans induced by the linear map given by \( e_\rho \mapsto n(\rho) \). By this morphism, \( X \) becomes a good quotient of \( \hat{X} \) by the diagonalizable group \( G = \text{Hom}(A, k^*) \). The coordinates \( x_\rho \), then serve as global coordinates for \( X \), and \( S \) is denoted the homogeneous coordinate ring of \( X \).

**A-graded \( S \)-modules.** Any \( A \)-graded \( S \)-module \( F \) defines a \( G \)-equivariant sheaf over \( k^{\Delta(1)} \) and thus over \( \hat{X} \), and it has been shown (see [Mus02]) that every quasicoherent sheaf over \( X \) can be represented as a descend of an \( A \)-graded \( S \)-module \( F \) of the form \( (\pi_*(\hat{F}|_X))^G \), where \( ^{-} \) denotes the usual sheafification functor over the affine space \( k^{\Delta(1)} \).

We abbreviate the descend of a module \( F \) over \( X \) gives rise to an \( A \)-graded \( S \)-module \( \Gamma(\hat{X}, \pi^*F) \). There is always an isomorphism \( \Gamma(\hat{X}, \pi^*F)^- \cong F \), but in general there is no isomorphism between any \( A \)-graded module \( F \) and \( \Gamma(\hat{X}, \pi^*F) \).

**Fine-graded \( S \)-modules.** For the study of equivariant sheaves, we have to consider fine graded modules, i.e. \( \mathbb{Z}^{\Delta(1)} \)-graded \( S \)-modules. Such a module \( F \) is equivalent to \( \hat{T} \)-equivariant sheaf over \( k^{\Delta(1)} \), and its descend \( \hat{F} \) then in a natural way is a \( T \)-equivariant sheaf over \( X \). On the other hand, the pullback \( \pi^*E \) of some \( T \)-equivariant sheaf over \( X \) has a natural \( \hat{T} \)-equivariant structure, and thus \( \hat{E} := \Gamma(\hat{X}, \pi^*E) \) is fine graded. The most important examples for us are the modules which are defined as the descend of free \( S \)-modules of rank one. These are the modules of the form \( S(\underline{m}) \), the degree shift of \( S \) by some element \( \underline{m} \in \mathbb{Z}^{\Delta(1)} \), where \( S(\underline{m})_{\underline{m}} = S_{\underline{m}+\underline{m}} \). The descend \( \hat{S}(\underline{m}) \) is isomorphic to \( \mathcal{O}_X(D_{\underline{m}}) \), the reflexive sheaf of rank one which is associated to the Weil-divisor \( D_{\underline{m}} := \sum_{\rho \in \Delta(1)} -n_\rho D_\rho \). As a general notation, we write \( S(\underline{m}) \) instead of \( S(\underline{m})_{\underline{m}} \); note that this shift is in the \( \mathbb{Z}^{\Delta(1)} \)-grading, not in the \( A \)-grading and therefore fixes a unique equivariant structure on \( \hat{S}(\underline{m}) \).

**Global and local quotient representations.** For any \( \sigma \in \Delta \) there is an exact sequence

\[
0 \to \sigma^\Delta_M \to M \to \mathbb{Z}^{\sigma(1)} \to A^\sigma \to 0,
\]
by which we have a splitting $M \cong \sigma_M^+ \oplus M/\sigma_M^+$, where we identify $M/\sigma_M^+ \cong M_{\leq \sigma}$ with the image of $M$ in $\mathbb{Z}^{\sigma(1)}$. This induces a splitting $U_\sigma \cong T_\sigma \times U_{\sigma'}$, where $T_\sigma \cong \text{spec}(k[\sigma_M])$ is the minimal orbit of $U_\sigma$, and $U_{\sigma'}$ is the affine toric variety associated to the subsemigroup $\sigma'_M = \sigma_M/\sigma_M^+$ of $M/\sigma_M^+$. Below, every construction with respect to $(M, \leq \sigma)$ will up to natural equivalence only depend on $M_{\leq \sigma}$, and so for clearer presentation we will always neglect the factor $\sigma_M^+$ and identify any $m \in M$ with its image in $\mathbb{Z}^{\sigma(1)}$.

The embedding of $M$ in $\mathbb{Z}^{\sigma(1)}$ is in a natural way compatible with the partial order $\leq$ on $\mathbb{Z}^{\sigma(1)}$ induced by the subsemigroup $\mathbb{N}^{\sigma(1)}$, i.e. $m \leq m'$ iff $m \leq m'$. We consider the order $\leq$ as an extension of $\leq_{\sigma}$ to $\mathbb{Z}^{\sigma(1)}$. For any $\tau < \sigma$, the localization of $\leq_{\sigma}$ by $\tau_{\sigma}$ extends to a localization of $\leq$ by the preorder $\leq'_{\tau}$ induced by the subsemigroup $\mathbb{N}^{\tau(1)} \oplus \mathbb{Z}^{\sigma(1)}/\tau(1)$, and we have a natural identification $(\mathbb{Z}^{\sigma(1)})_{\leq'_{\tau}} = \mathbb{Z}^{\tau(1)}$. This localization is naturally compatible with the localization of $M$ by $\leq_{\tau}$ and we have the following commutative exact diagram:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & \sigma_M^+ & \rightarrow & M & \rightarrow & \mathbb{Z}^{\sigma(1)} & \rightarrow & A^\sigma & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \pi & & \downarrow & & \\
0 & \rightarrow & \tau_M^+ & \rightarrow & M & \rightarrow & \mathbb{Z}^{\tau(1)} & \rightarrow & A^\tau & \rightarrow & 0,
\end{array}
$$

where $\pi$ is the canonical projection from $\mathbb{Z}^{\sigma(1)}$ onto $\mathbb{Z}^{\tau(1)}$. Having these natural compatibilities in mind, in the sequel we will use the notation $\leq_{\sigma}$ for both preorders on $M$ and on $\mathbb{Z}^{\sigma(1)}$, we will write $n \leq_{\sigma} m$ and the like for $n \in \mathbb{Z}^{\sigma(1)}$ and $m \in M$.

We now describe more precisely the relation between the $\Delta$-family of $E$ and the $\hat{\Delta}$-family of $\pi^*E$. For any $\sigma \in \Delta$ consider the quotient representation $\pi_\sigma : U_\hat{\sigma} \rightarrow U_{\sigma}$, where $U_{\hat{\sigma}} \cong k(\sigma)$, Here without loss of generality we assume for the moment that $\sigma$ has full dimension in $N_\mathbb{R}$. Denote $E^\sigma := \Gamma(U_{\sigma}, \mathcal{E})$ and $\hat{E}^\sigma := \Gamma(U_{\hat{\sigma}}, \pi_\sigma^*\mathcal{E})$. The homogeneous coordinate ring has a natural $A^\sigma$-grading $S^\sigma = \bigoplus_{\alpha \in A^\sigma} S^\sigma_\alpha$, with $S_0 \cong k[\sigma_M]$, so that we can write:

$$
\hat{E}^\sigma \cong E^\sigma \otimes S^\sigma_0, S^\sigma \cong E^\sigma \otimes S^\sigma_0 \left( \bigoplus_{\alpha \in A^\sigma} S^\sigma_\alpha \right) \cong \bigoplus_{\alpha \in A^\sigma} (E^\sigma \otimes S^\sigma_0 S^\sigma_\alpha).
$$

By $E^\sigma \otimes S^\sigma_0 S^\sigma_0 \cong E^\sigma$, we find that $E^\sigma$ is naturally embedded in $\hat{E}^\sigma$, and thus the $\sigma$-family of $E^\sigma$ is a subfamily of the $\hat{\sigma}$-family of $\hat{E}^\sigma$.

Denote $\leq_{\Delta}$ the preorder on $\mathbb{Z}^{\Delta(1)}$, then every $(\mathbb{Z}^{\sigma(1)}, \leq_{\sigma})$ is isomorphic to the localization of $(\mathbb{Z}^{\Delta(1)}, \leq_{\Delta})$ by the preorder $\leq'_{\sigma}$ where $n \leq'_{\sigma} n'$ iff $n - n' \in \mathbb{Z}^{\Delta(1)} \setminus \sigma(1)$. By the natural projection $\mathbb{Z}^{\Delta(1)} \rightarrow \mathbb{Z}^{\sigma(1)}$, every fine graded module over the localization $S_{\times, \sigma} \cong k[\mathbb{Z}^{\Delta(1)} \setminus \sigma(1)]$ is equivalent to a fine graded module over $S^\sigma \cong k[\mathbb{N}^{\sigma(1)}]$. The localization $\hat{E}_{x, \sigma}$ of $E$ then is equivalent to a representation of $(\mathbb{Z}^{\sigma(1)}, \leq_{\sigma})$, and by naturality, the $\Delta$-family $E^\Delta$ glues as a subfamily of the $\hat{\Delta}$-family $\hat{E}^\Delta$. 

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Resolutions of $\hat{E}$. As for every equivariant coherent sheaf $\mathcal{E}$ we can consider its associated fine graded module $\hat{E}$, in principle there is nothing which prevents us from doing this and to compute some finite free resolution of $\hat{E}$ over $S$, which then descends to a resolution of $\mathcal{E}$ of the desired type:

$$0 \rightarrow \hat{F}_s \rightarrow \cdots \rightarrow \hat{F}_0 \rightarrow \mathcal{E} \rightarrow 0,$$

where $\hat{F}_i \cong \bigoplus_{j=1}^k \mathcal{O}_X(D_{n_j})$ and the length $s$ by Hilbert’s syzygy theorem being bounded by the numbers of rays in $\Delta$. At this point we could stop with this paper and leave the problem as an application of traditional methods. However, there are some drawbacks of this point of view, which motivate our further investigations. One problem is that the pullback of a coherent sheaves along good quotients so far seems not to be understood very well, not even for the toric case — and as we will see in example 3.8 below, such pullbacks might show pathological behaviour, such as acquiring additional torsion. Another problem is that due to its global nature, the module $\hat{E}$ contains much more relations which might be irrelevant to consider for getting a resolution.

Example 3.8: Consider the subsemigroup $\sigma_M$ of $M \cong \mathbb{Z}^2$ which is generated by the elements $(1,0)$, $(1,1)$, $(1,2)$ and its associated semigroup ring $S_0 = k[\sigma_M]$. Its fan is spanned by the primitive vectors $\underline{u}_1 = (2,-1)$ and $\underline{u}_2 = (0,1)$ in $\mathbb{N}_R$, and the homogeneous coordinate ring $S = S^\sigma$ is $\mathbb{Z}_2$-graded. Denote $\underline{n} := (-1,0)$ and consider the reflexive $S_0$-module $S(\underline{n}) \cong S_1$. For the pullback we have:

$$S(\underline{n}) \otimes_{S_0} (S_0 \otimes S_1) \cong (S_1 \otimes_{S_0} S_0) \oplus (S_1 \otimes_{S_0} S_1) \cong S_1 \otimes (S_1 \otimes_{S_0} S_1).$$

To compute $(S_1 \otimes_{S_0} S_1)$, we directly evaluate it as an $M$-graded tensor product. The module $S(\underline{n})$ is a $M$-graded, where

$$S_{(\underline{n}),m} = \begin{cases} k & \text{if } \langle m, \underline{n}_i \rangle \geq 0 \text{ for } i = 1,2 \\ 0 & \text{else.} \end{cases}$$

In degree $m$, $S(\underline{n}) \otimes_{k[\sigma_M]} S(\underline{n})$ is generated by all elements $\chi(m_1) \otimes \chi(m_2)$ such that $m_1 + m_2 = m$ modulo the relation that $\chi(m_1) \otimes \chi(m_2)$ is equivalent to $\chi(m_1 - m') \otimes \chi(m_2)$ and $\chi(m_1) \otimes \chi(m_2 - m'')$, respectively, whenever there exist some $\chi(m')$ or $\chi(m'')$ such that $\chi(m_1 - m')$ and $\chi(m_2 - m'')$, respectively, are in $S(\underline{n})$. It turns out that these relations cancel most of the generators in every degree, so that for all nonzero degrees:

$$\dim(S(\underline{n}) \otimes_{S_0} S(\underline{n}))_m = \begin{cases} 2 & \text{if } m = (1,1) \\ 1 & \text{else.} \end{cases}$$

Note that the nonzero degrees are precisely those contained in the intersection of the half spaces $\langle m, \underline{n}_2 \rangle \geq 0$, $\langle m, 2\underline{n}_1 \rangle \geq 0$ and $\langle m, (1,0) \rangle \geq 0$. So, in degree $(1,1)$, our module
has dimension two, whereas in all other degrees it has at dimension one, which implies that it has torsion in degree $(1, 1)$, as for any character $(1, 1) \leq \sigma m$ the homomorphism $\chi_{(1,1),m}$ cannot be injective. This is indeed an example where pullback of a torsion free, and even reflexive, module along a geometric quotient acquires some new torsion.

Another phenomenon which we want to mention is that there are also other relevant effects which one has to consider if one tries to choose some alternative module instead of $\hat{E}$ whose descend coincides with that of $\hat{E}$. For instance, for any affine toric variety $U_{\sigma}$ for which $A_{\sigma}$ is nontrivial, there exist nonzero $S$-modules $F$ whose zero component vanishes; the most easiest example is the one-dimensional module $S_{\sigma}/\langle x_{\rho} \mid \rho \in \sigma(1) \rangle$ whose degree gets shifted by some nonzero $\alpha \in A_{\sigma}$.

4 Compression and Resolutions

4.1 $\text{lcm}$-lattices in $\mathbb{Z}^r$

The partial order on $\mathbb{Z}^r$ induced by $N^r$ coincides with the partial order given by componentwise ordering, i.e. if we write $\underline{n} = (n_1, \ldots, n_r)$, $\underline{n}' = (n_1', \ldots, n_r')$, then $\underline{n} \leq \underline{n}'$ iff $n_i \leq n_i'$ for every $1 \leq i \leq r$. We set $\mathbb{Z} := \{-\infty\} \cup \mathbb{Z}$ which is totally ordered by $-\infty < n$ for all $n \in \mathbb{Z}$. Like $\mathbb{Z}^r$, the set $\overline{\mathbb{Z}}^r$ is partially ordered by the componentwise total order, and the the canonical inclusion $\mathbb{Z}^r \hookrightarrow \overline{\mathbb{Z}}^r$ is order preserving. We call any element in $\overline{\mathbb{Z}}^r \setminus \mathbb{Z}^r$ infinitary.

For any element $\underline{n} \in \mathbb{Z}^r$ we can consider the subset $\underline{n} + N^r$, which is the intersection of the shifted cone $\underline{n} + R^r_{\geq 0}$ with $\mathbb{Z}^r$. It is easy to see that for any finite set of elements $\underline{n}_1, \ldots, \underline{n}_s$ in $\mathbb{Z}^r$, the intersection $\bigcap_{i=1}^s (\underline{n}_i + N^r)$ is again of the form $\underline{n} + N^r$. The element $\underline{n}$ is called the least common multiple of $\underline{n}_1, \ldots, \underline{n}_s$, denoted lcm$(\underline{n}_1, \ldots, \underline{n}_s)$, and it is given by componentwise maximum of the $n_i$. The lcm extends canonically to $\overline{\mathbb{Z}}^r$. In the geometric picture, for some infinitary element $\underline{n} = (n_1, \ldots, n_r)$ with $n_{i_j} = -\infty$ for some $\{i_1, \ldots, i_k\} \subset \{1, \ldots, r\}$, we write $\underline{n} + C$ for the cone, where $C = \{c \in \mathbb{R}^r \mid c_i \geq 0$ if $i \notin \{i_1, \ldots, i_k\}\}$. One can think of the cone $C$ of the standard orthant moved to minus infinity in the directions $i_1, \ldots, i_k$.

In our actual definition of the lcm-lattice, we will need infinitary elements to generate the lattice, but after generation, we throw away all these elements. Instead, we close every lcm-lattice from below by adding the unique minimal element $(-\infty, \ldots, -\infty) =: \hat{0}$.

**Definition 4.1:** Let $\mathcal{P} \subset \mathbb{Z}^r$ be some poset and lcm$(\mathcal{P})$ the lattice generated by the lcm’s of elements in $\mathcal{P}$. Then we denote the set $(\text{lcm}(\mathcal{P}) \cap \mathbb{Z}^r) \cup \hat{0}$ the lcm-lattice of $\mathcal{P}$.

Every lcm-lattice $\mathcal{L}$ gives rise to a partition of $\mathbb{Z}^r$, respectively to an equivalence relation, on $\mathbb{Z}^r$. Namely, for every element $\underline{n} \in \mathbb{Z}^r$, there exists a unique maximal element $\underline{n}' \in \mathcal{L}$ with $\underline{n}' \leq \underline{n}$.
Definition 4.2: Let $\underline{n} \in \mathbb{Z}^r$ and $\underline{n}' \in \mathcal{L}$ maximal such that $\underline{n}' \leq \underline{n}$. Then we call $\underline{n}'$ the anchor element $A(\underline{n})$ of $\underline{n}$ in $\mathcal{L}$. Any two elements $\underline{n}_1, \underline{n}_2 \in \mathbb{Z}^r$ are equivalent iff $A(\underline{n}_1) = A(\underline{n}_2)$. We denote $T_\underline{n}$ the equivalence class associated to $\underline{n} \in \mathcal{L}$.

4.2 Polynomial rings

In this subsection we consider the special case where $X$ is an affine toric variety isomorphic to the affine space $k^r$, so that we can assume without loss of generality that $\sigma$ and $\sigma_M$ coincide with the standard orthant $\mathbb{R}_{\geq 0}^r$ in $\mathbb{R}^r$, and the subsemigroup $\mathbb{N}^r$ of $\mathbb{Z}^r$, respectively. We denote $S \cong k[\mathbb{N}^r]$ the coordinate ring of $X$ and $E$ a nonzero finitely generated $S$-module. We formally extend the representation of $(\mathbb{Z}^r, \leq)$ by $E$ to a representation of $(\hat{\mathbb{Z}}^r, \leq)$ by setting $E_\hat{0} = 0$ for all infinitary $\underline{n}$. In order to construct a compression functor for $E$, we have to extract all nontrivial maps (i.e. the nonisomorphisms) of the corresponding $\sigma$-family, as well as all possible relations among them.

Definition 4.3: Let $\underline{n} \in \mathbb{Z}^r$, then we define the set $I_E(\underline{n})$ to contain those elements $\underline{n}'$ in $\hat{\mathbb{Z}}^r$ which are minimal with the property that for all $\underline{n}'' \in \hat{\mathbb{Z}}^r$ with $\underline{n}' \leq \underline{n}'' \leq \underline{n}$ the morphisms $\chi_{\underline{n}'',\underline{n}} : E_{\underline{n}'} \to E_{\underline{n}''}$ and $\chi_{\underline{n}'',\underline{n}} : E_{\underline{n}''} \to E_{\underline{n}}$ are isomorphisms. We denote $I_E := \bigcup_{\underline{n} \in \mathbb{Z}^r} I_E(\underline{n})$.

Note that the case where $I_E(\underline{n})$ contains an infinitary element can only (but not necessarily has to) occur when $E_\hat{0}$ is zero. Moreover, note that it follows immediately from the finitely generatedness of $E$ that $I_E$ and the $I_E(\underline{n})$ are finite sets.

Definition 4.4: We denote $\mathcal{L}_E$ the lcm-lattice generated by $I_E$. For any $\underline{n} \in \mathbb{Z}^r$, we denote the corresponding anchor element by $A_E(\underline{n})$.

We can depict the set of equivalence classes as a tiling of $\mathbb{R}^r$ by cubic, possibly non-compact blocks, where the anchor elements are precisely those elements sitting on the smallest vertex with respect to $\leq$. Observe that $\text{lcm}\{\underline{n}_1, \ldots, \underline{n}_s\} \in \mathbb{Z}^r$ as soon as at least one of the $\underline{n}_j$ is non-infinitary. Moreover, if $I_E(\underline{n})$ contains an infinitary element, this implies that $E_\hat{0} = 0$. In general, the set $I_E(\underline{n})$ will contain infinitary elements only if there exists no $\underline{n}' < \underline{n}$ such that $E_{\underline{n}'} \neq 0$. In that case, $I_E(\underline{n})$ will contain $\hat{0}$ as its only element. An exception are those modules $E$, which are of rank zero, and thus are torsion modules. The infinitary elements $I_E(\underline{n})$ for all $\underline{n} \in \mathbb{Z}^r$ in that case describe the support of $E$.

Example 4.5: Let $J \subset S$ be a monomial ideal, generated by monomials $x^{\underline{m}_1}, \ldots, x^{\underline{m}_s}$. Then we have

$$I_J(\underline{n}) = \begin{cases} \{\underline{n}' \leq \underline{n}\} & \text{if } x^{\underline{n}} \in J \\ \hat{0} & \text{else} \end{cases}$$
and the anchor element \( A_J(n) \) being \( \text{lcm}\{n, \leq n\} \). The lattice \( L_J \) then coincides with the lcm-lattice introduced in [GPW99].

**Example 4.6:** Consider the torsion module \( T = k[x, y]/(x^2, xy, y^2) \). We have

\[
I_T(n) = \begin{cases} 
\{(0, 0)\} & \text{for } n \in \{(0, 0), (1, 0), (0, 1)\} \\
\{(1, 1)\} & \text{for } n = (1, 1) \\
\{(2, -\infty)\} & \text{for } n = (k, 0), k > 1 \\
\{(-\infty, 2)\} & \text{for } n = (0, k), k > 1 \\
\{(1, 1), (2, -\infty)\} & \text{for } n = (k, 1), k > 1 \\
\{(1, 1), (-\infty, 2)\} & \text{for } n = (1, k), k > 1 \\
\{(1, 1), (2, -\infty), (-\infty, 2)\} & \text{for } (2, 2) \leq n \\
\{0\} & \text{else.}
\end{cases}
\]

The corresponding lcm-lattice then is the set \( \{0, (0, 0), (2, 0), (0, 2), (1, 1), (1, 2), (2, 1), (2, 2)\} \). Figure 3 shows the partitioning of \( \mathbb{Z}^2 \) by the lcm-lattice. The rectangular figure indicates the degrees \( (0, 0), (1, 0), (0, 1) \), where \( T \) is nonzero; the light grey triangles indicate all the initial elements \( I_T(n) \), and the darker grey triangles denote the additional elements of the lcm-lattice. The infinitary elements become merged to \( \hat{0} \) in \( L_T \).

Denote \( \mathcal{L}_E\text{-Rep} \) the category of finite-dimensional \( k \)-linear representations of \( \mathcal{L}_E \); denote \( \mathcal{M}_E \) the full subcategory of the category of fine-graded \( S \)-modules whose objects are those modules \( F \) whose associated lcm-lattice \( L_F \) is a sublattice of \( L_E \). Let \( \iota_E : \mathcal{L}_E \hookrightarrow \mathbb{Z}^r \) be the canonical inclusion. Then we define the functor \( \text{zip}^E \) from \( \mathcal{M}_E \) into \( \mathcal{L}_E\text{-Rep} \) by

\[
\text{zip}^E(F) := \iota_E^* F
\]
where \( i_E \) denotes the sheaf pullback.

To define the unzip functor, we have to do a little bit more. Let \( F \) be some representation of \( \mathcal{L}_E \), mapping \( \underline{n} \) to \( F(\underline{n}) \), and \( \underline{n} \leq \underline{n}' \) to \( F(\underline{n}, \underline{n}') \). Then we define a representation of \( \mathbb{Z}^r \) by setting \( F_{\underline{n}} := F(A_E(\underline{n})) \) and \( \chi_{\underline{n}, \underline{n}'} := F(A_E(\underline{n}), A_E(\underline{n}')) \) for every pair \( \underline{n}, \underline{n}' \in \mathbb{Z}^r \). This indeed establishes a well defined functor, where \( F(A_E(\underline{n}), A_E(\underline{n}')) = \text{id} \) whenever \( A_E(\underline{n}) = A_E(\underline{n}') \) and \( F(A_E(\underline{n}), A_E(\underline{n}'')) = F(A_E(\underline{n}'), A_E(\underline{n}'')) \) \( \circ F(A_E(\underline{n}), A_E(\underline{n}')) \) whenever \( \underline{n} \leq \underline{n}' \leq \underline{n}'' \).

**Theorem 4.7:** The pair of functors \( \text{zip and unzip} \) establishes an equivalence of categories between \( \mathcal{M}_E \) and \( \mathcal{L}_E - \text{Rep} \).

**Proof.** We show that \( \text{unzip} \circ \text{zip} \cong 1_{\mathcal{M}_E} \) and \( \text{zip} \circ \text{unzip} \cong 1_{\mathcal{L}_E - \text{Rep}} \). In the first case, let \( F \) be some representation of \( (\mathbb{Z}^r, \leq) \). Denote \( F':= \text{unzip}(F) \) and define \( h : F'_\underline{n} \rightarrow F_{\underline{n}} \) by setting \( h := \chi_{\underline{n}, \underline{n}'} \). Now \( h \) is an isomorphism for every \( \underline{n} \in \mathbb{Z}^r \), and moreover, for every pair \( \underline{n} \leq \underline{n}' \), we have \( \chi_{\underline{n}, \underline{n}'} \circ \chi_{\underline{n}', \underline{n}''} = \chi_{\underline{n}, \underline{n}''} \). So we obtain \( \text{unzip} \circ \text{zip} \cong 1_{\mathcal{M}_E} \).

The other direction is immediate, and we even obtain \( \text{zip} \circ \text{unzip} = 1_{\mathcal{L}_E - \text{Rep}} \).

**Corollary 4.8:** \( \mathcal{M}_E \) is an abelian category.

Let \( \underline{n} \) be any element in \( \mathcal{L}_E \), then we can consider the free representation \( F_\underline{n} \) of \( \mathbb{Z}^r \). Its unzipping has a particularly easy structure, namely \( \text{unzip}(F_\underline{n}) \cong S(-\underline{n}) \), i.e. the free fine-graded \( S \)-module with degree shifted by \( -\underline{n} \). \( \text{unzip}(F_\underline{n}) \) is the unique \( S \)-module which has the property that its \( \underline{n}' \)-th degree is one-dimensional if \( \underline{n} \leq \underline{n}' \) and zero else.

Now we can consider a free resolution of \( \text{zip}(E) \) in terms of free representations of \( \mathcal{L}_E \):

\[
0 \rightarrow F_s \rightarrow \cdots \rightarrow F_0 \rightarrow \text{zip}(E) \rightarrow 0
\]

where for every \( 1 \leq i \leq s \):

\[
F_i \cong \bigoplus_{\underline{n} \in \mathcal{L}_E} (F_{\underline{n}})^{f_{\underline{n}}^i}
\]

where \( f_{\underline{n}}^i \) is the free dimension of the vector space associated to \( \underline{n} \) in the \((i-1)\)-th syzygy representation. By unzipping, we obtain an exact sequence of free representations:

\[
0 \rightarrow \text{unzip}(F_s) \rightarrow \cdots \rightarrow \text{unzip}(F_0) \rightarrow E \rightarrow 0
\]

where for every \( 1 \leq i \leq s \):

\[
\text{unzip}(F_i) \cong \bigoplus_{\underline{n} \in \mathcal{L}_E} S(-\underline{n})^{f_{\underline{n}}^i}.
\]

In order to show, that this is a minimal free resolution of \( E \) over \( S \), we consider the first step of the resolution \( 0 \rightarrow K_0 \rightarrow \text{unzip}(F_0) \rightarrow E \rightarrow 0 \). We define a map \( \phi : \mathcal{L}_E \rightarrow \mathcal{L}_{K_0} \)
by mapping every \( \underline{n} \in \mathcal{L}_E \) to its anchor element in \( \mathcal{L}_{K_0} \):

\[
\phi(\underline{n}) := A_{K_0}(\underline{n}).
\]

We have the following:

**Proposition 4.9:** The map \( \phi \) is a contraction.

**Proof.** We first show that \( \phi(U_E(\underline{n})) = U_{K_0}(A_{K_0}(\underline{n})) \) for all \( \underline{n} \in \mathcal{L}_E \), where we write \( U_E \) and \( U_{K_0} \) for open subsets in \( \mathcal{L}_E \) and \( \mathcal{L}_{K_0} \), respectively. Clearly, \( \phi(U_E(\underline{n})) \subset U(A_{K_0}(n)) \); by construction of \( K_0 \), the lattice \( \mathcal{L}_{K_0} \) is a sublattice of \( \mathcal{L}_E \), so that for any \( \underline{n}' \in U_{K_0}(A_{K_0}) \) there is \( \underline{n}'' \in U_E(\underline{n}) \) with \( \phi(\underline{n}'') = \underline{n}' \). Now let \( \underline{n} \in \mathcal{L}_{K_0} \) and consider the set \( \phi^{-1}(U_{K_0}(\underline{n})) \), which consists of all \( \underline{n}' \in \mathcal{L}_E \) such that \( \underline{n} \leq A_{K_0}(\underline{n}') \). \( \underline{n} \leq \underline{n}' \) implies \( \underline{n} = A_{K_0}(\underline{n}) \leq A_{K_0}(\underline{n}') \), and thus \( U_E(\underline{n}) \subset \phi^{-1}(U_{K_0}(\underline{n})) \). Moreover, \( \phi^{-1}(U_{K_0}(\underline{n})) = \{ \underline{n}' \in \mathcal{L}_E \mid \underline{n} \leq A_{K_0}(\underline{n}') \} \), and thus \( \phi^{-1}(U_{K_0}(\underline{n})) \subset U_E(\underline{n}) \). Hence, \( \phi^{-1}(U_{K_0}(\underline{n})) = U_E(\underline{n}) \), and \( \phi \) is a contraction. \( \square \)

**Theorem 4.10:** Sequence (1) is a minimal free resolution of \( E \) over \( S \).

**Proof.** Observe that the number of \( k \)-linear independent generators of the module \( E \) degree \( \underline{n} \) is the codimension of the subvector space \( \sum_{n' < n} x^{\underline{n}-\underline{n}'} \cdot E_{\underline{n}'} \) of \( E_{\underline{n}} \), which coincides with the free dimension of \( E_{\underline{n}} \). Thus unzipping \( F_0 \) is the minimal free module which surjects onto \( E \). Using proposition 4.9 and lemma 2.12, we see that a resolution of \( K_0 \) over \( \mathcal{L}_E \) is a lift of some resolution of \( K_0 \) restricted to \( \mathcal{L}_{K_0} \). Hence, the theorem follows by induction. \( \square \)

### 4.3 Admissible posets and normal semigroup rings

To extend our considerations to the case of normal semigroup rings, consider the map \( M \to \mathbb{Z}^{a(1)} \), which without loss of generality we assume to be injective. This corresponds to a quotient representation \( \pi : k^{a(1)} \to U_\sigma \) together with an \( A \)-graded homogeneous coordinate ring \( S := k[x_\rho \mid \rho \in \sigma(1)] \). For any coherent sheaf \( \mathcal{E} \) over \( U_\sigma \), we can consider its pullback \( \pi^* \mathcal{E} \) over \( k^{a(1)} \).

Applying the machinery from subsection 4.2, we can obtain a reflexive resolution for \( \mathcal{E} \) by sheafification of the resolution of \( \mathring{E} \) with respect to the lcm-lattice \( \mathcal{L}_{\mathring{E}} \):

\[
0 \to \text{unzip}(F_1)^\vee \to \cdots \to \text{unzip}(F_0)^\vee \to \mathcal{E} \to 0,
\]

where \( \mathcal{E} \cong (\text{unzip} \, \pi^* \mathcal{E})^\vee \). For any anchor element \( \underline{n} \in \mathcal{L}_{\mathring{E}} \), the unzipping of the associated free representation of \( \mathcal{L}_{\mathring{E}} \) is isomorphic to \( S(-\underline{n}) \). Unlike the case of smooth toric varieties, in the general case such a resolution is not uniquely defined, and it can be possible to obtain shorter resolutions which are of this type.
**Definition 4.11:** Let $E$ be a $M$-graded $k[\sigma_M]$-module. A finite subposet $\mathcal{P} \subset \mathbb{Z}^r \cup \hat{0}$ is admissible with respect to $E$ if

(i) for all $m \in M$ there exists a unique $\underline{n} \in \mathcal{P}$ with $\underline{n} \leq m$, such that $\underline{n}' \leq m$ implies $\underline{n}' \leq \underline{n}$ for all $\underline{n}' \in \mathcal{P}$;

(ii) consider the open set $U_{\underline{n}} = \bigcup_{\underline{n} \leq m} U(m)$ in $M$ and the vector space $E(U_{\underline{n}}) = \lim_{\leftarrow} E_m$, there exists a vector space $E_{\underline{n}}$ and a diagonal homomorphism $E_{\underline{n}} \to E(U_{\underline{n}})$ such that every induced homomorphism $E_{\underline{n}} \to E_m$ is an isomorphism for all $m \in T_{\underline{n}}$.

We call $E_{\underline{n}}$ the anchor completion of $E$ at $\underline{n}$ and we denote $A_E(m)$ the unique maximal element $n \in \mathcal{P}$ with $n \leq m$.

Note that in the definition we have identified the elements $m \in M$ with their image in $\mathbb{Z}^{\sigma(1)}$. For any $\underline{n} \in \mathcal{P}$, the homomorphism $E_{\underline{n}} \to E(U_{\underline{n}})$ necessarily is injective, and for every $\underline{n} \leq \underline{n}'$, the composition

$$E_{\underline{n}} \to E(U_{\underline{n}}) \to E(U_{\underline{n}'})$$

is a diagonal morphism, which factors through the image of $E_{\underline{n}'}$, such that we obtain a morphism between the anchor completions $E_{\underline{n}} \to E_{\underline{n}'}$.

**Lemma 4.12:** Assume that $U_\sigma$ is smooth and thus $M \cong \mathbb{Z}^{\sigma(1)}$ and let $\mathcal{P}$ be some admissible poset with respect to $E$. Then for every subset $m_1, \ldots, m_s$ of $\mathcal{P}$, $\text{lcm}\{m_1, \ldots, m_s\}$ is also contained in $\mathcal{P}$. In particular, $\mathcal{P}$ contains the lcm-lattice $\mathcal{L}_E$.

**Proof.** Denote $m_t := \text{lcm}\{m_1, \ldots, m_s\}$. There exists a unique $m \in \mathcal{P}$ such that $m \geq_\sigma m_t$; but such an $m$ must coincide with $m_t$. \qed

From the observation that $\mathcal{L}_E$ is admissible, we conclude:

**Proposition 4.13:** Every finitely generated $k[\sigma_M]$-module $E$ has an admissible poset.

**Proof.** We take the poset of all $\underline{n} \in \mathcal{L}_E$ such that $\{m \in M \mid A_E(m) = \underline{n}\} \neq \emptyset$. \qed

Let $\mathcal{P} \subset \mathbb{Z}^{\sigma(1)}$ be an admissible poset, and denote $\mathcal{M}_\mathcal{P}$ the category of finitely generated, $M$-graded $k[\sigma_M]$-modules for which $\mathcal{P}$ is admissible. Then we define the functor $\text{zip}^\mathcal{P}$ from $\mathcal{M}_\mathcal{P}$ to the category of $k$-linear representations of $\mathcal{P}$ by:

$$\text{zip}^\mathcal{P}(E)_{\underline{n}} := E_{\underline{n}},$$

where $E_{\underline{n}}$ is the anchor completion at $\underline{n}$.

**Remark 4.14:** Our definition also allows to add anchor elements $\underline{n}$ such that the corresponding set $T_{\underline{n}}$ is empty. In that case we set $E_{\underline{n}} = \lim_{\leftarrow} E_{\underline{n}'}$ for all $\underline{n} < \underline{n}' \in \mathcal{P}$ such that $T_{\underline{n}'} \neq \emptyset$.

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In the opposite direction, from every representation $E$ of an admissible poset $\mathcal{P}$ one can construct a representation of $M$. We define $\text{unzip}^\mathcal{P}(E)$ by setting:

(i) $\text{unzip}^\mathcal{P}(E)_m := E_{A(m)}$,

(ii) $\chi_{m,m'} := E(A(m), A(m'))$.

**Theorem 4.15:** The pair $\text{zip}^\mathcal{P}$ and $\text{unzip}^\mathcal{P}$ is a compression of $M$, i.e. $\text{zip}^\mathcal{P}$ and $\text{unzip}^\mathcal{P}$ are functors which establish an equivalence of categories.

**Proof.** By construction, $\text{unzip}^\mathcal{P} \circ \text{zip}^\mathcal{P}(E) \cong E$ for every $k[\sigma_M]$-module for which $\mathcal{P}$ is admissible. To obtain functors, we show that any morphism $E \rightarrow F$ of objects in $M$ induces a morphism of the corresponding representations of $\mathcal{P}$ and vice versa. First, any homomorphism $E \rightarrow F$ is a homomorphism of sheaves over $(M, \leq)$, and thus there is an induced homomorphism $E_n \rightarrow E(U_n) \rightarrow F(U_n)$ for every $n \in \mathcal{P}$, which factors through the diagonal $F_n$, hence we obtain a homomorphism $E_n \rightarrow F_n$; the family of such morphisms for every $n \in \mathcal{P}$ in a natural way represents a homomorphism of representations of $\mathcal{P}$. In the other direction, a homomorphism $f : \text{zip}^\mathcal{P}(E) \rightarrow \text{zip}^\mathcal{P}(F)$ unzips componentwise as $f_m := f_{A(m)} : E_{A(m)} \rightarrow F_{A(m)}$. 

**Proposition 4.16:** Let $n \in \mathcal{P}$, then $\mathcal{P}$ is admissible with respect to the reflexive module $S(n)$, and moreover, $S_n \cong \text{unzip}^\mathcal{P} F_n$.

**Proof.** Let $m \in M$, then $n \leq A_E(m)$ iff $n \leq m$: the first implication is clear, because $m \leq A_E(m)$; for the second, observe that $A_E(m) \geq \text{lcm}\{A_E(m), n\}$, and thus $n \leq A_E(m)$. So $\mathcal{P}$ is admissible with respect to $S(n)$ and $\text{unzip}^\mathcal{P} F_n \cong S_{(n)}$.

As in the case for polynomial rings, we obtain a reflexive resolution for $E$:

$$0 \rightarrow \text{unzip}^\mathcal{P}(F_\ast) \rightarrow \cdots \rightarrow \text{unzip}^\mathcal{P}(F_0) \rightarrow E \rightarrow 0.$$ 

**Example 4.17:** Consider the semigroup $\sigma_M$ from 3.8 and the torsion sheaf $T$ which is given by:

$$T_m = \begin{cases} k & m = (p, 0), p \geq 0 \\ k & m = (0, 1) + p \cdot (1, 2), p \geq 0 \\ 0 & \text{else,} \end{cases}$$

and $\chi_{m,m'} = \text{id}$ whenever $T_m, T_{m'} \neq 0$. We can compare the following two admissible posets,

$$\mathcal{P}_1 = \{\hat{0}, (0, 0), (-1, 1), (0, 1)\}$$

and

$$\mathcal{P}_2 = \{\hat{0}, (-1, 0), (0, 1)\}.$$
We have \((\text{zip}^{\mathcal{P}_1} T)_{(0,0)} = k, (\text{zip}^{\mathcal{P}_1} T)_{(-1,1)} = k, (\text{zip}^{\mathcal{P}_1} T)_{(0,1)} = 0,\) and \(\text{zip}^{\mathcal{P}_2} T_{(-1,0)} = k, \text{zip}^{\mathcal{P}_2} T_{(0,1)} = 0.\) But in the latter case, we have that \(T(U_{(-1,0)})\) is is the fiber product \(k \times_0 k \cong k^2\), such that \(\text{zip}^{\mathcal{P}_2} T_{(-1,0)}\) corresponds to a proper diagonal homomorphism \(k \to k^2\). These compressions give rise to two somewhat different resolutions. Via resolving over \(\mathcal{P}_1\) and by unzip\(^{\mathcal{P}_1}\), we obtain:

\[
0 \longrightarrow S_{(0,-1)} \longrightarrow S_{(1,-1)} \oplus S_{(0,0)} \longrightarrow T \longrightarrow 0
\]

and for \(\mathcal{P}_2\):

\[
0 \longrightarrow S_{(0,-1)} \longrightarrow S_{(1,0)} \longrightarrow T \longrightarrow 0.
\]

In a sense, the module \(T\) is like the module \(k[x,y]/(xy)\) over the polynomial ring \(k[x,y]\), whose lcm-lattice is isomorphic to \(\mathcal{P}_2\). However, there exists no unique minimal element in \(m \in M\) with \(T_m \neq 0\), so that the consideration of the diagonal morphism indeed is necessary to obtain a resolution which is like the minimal resolution of \(k[x,y]/(xy)\).

The left part of figure 4 shows a part of the lattice \(\mathbb{Z}^2\); the light grey areas indicate the degrees, where \(T_m\) is nonzero. The right part of figure 4 shows the partitioning of \(\mathbb{Z}^2\) according to the two admissible posets \(\mathcal{P}_1\) and \(\mathcal{P}_2\).

![Figure 4](image)

**Figure 4:** The module from example 4.17 and the partitions of \(\mathbb{Z}^2\) with respect to \(\mathcal{P}_1\) and \(\mathcal{P}_2\).

### 4.4 Extension of a module to the homogeneous coordinate ring

Consider \(E\) any \(M\)-graded \(k[\sigma_M]\)-module and \(S\) the homogeneous coordinate ring for \(U_\sigma\). As we have seen in the previous subsection, one can in a natural way associate the \(S\)-module \(\hat{E} = E \otimes_{S_0} S\) to \(E\). In this subsection we want to discuss another way to associate a module, denoted \(EE\), to \(E\) which also has the property that \(EE \cong E\), but which behaves better, for instance it preserves the property of torsion freeness. For this, for every \(\underline{n} \in \mathbb{Z}^{\sigma(1)}\) we denote \(U_{\underline{n}} := \bigcup_{m \leq \underline{n}} U(m)\) an open subset of \((M, \leq_\sigma)\). To see
that the set \( \{n \leq m\} \) is always nonempty, just choose some \( m \in \sigma_M \) with \( \langle m, n(\rho) \rangle > 0 \) for every \( \rho \in \sigma(1) \), then for every \( n \in \mathbb{Z}^{\sigma(1)} \) we can choose an integer \( c > 0 \) such that \( n \leq c \cdot m \). Thus for every \( n \in \mathbb{Z}^{\sigma(1)} \) the vector space \( E(U_n) \) exists, and can be identified with \( \lim_{\leftarrow} E_m \). If \( E \) is finitely generated, then the \( E(U_n) \) are finite dimensional.

**Definition 4.18:** We define a representation of \( (\mathbb{Z}^{\sigma(1)}, \leq) \) by:

\[
EE_n := E(U_n).
\]

For every \( n \leq n' \), the set \( U_n' \) is contained in \( U_n \), and thus we have a functorial homomorphism \( EE_n \to EE_{n'} \), and indeed we obtain a well-defined representation of \( \mathbb{Z}^{\sigma(1)} \).

**Proposition 4.19:** \( EE \) has the following properties:

(i) \( EE = E \).

(ii) If \( E \) is finitely generated, then also \( EE \) is finitely generated.

(iii) If \( E \) is torsion free, then also \( EE \) is torsion free.

**Proof.** (i): By definition, if \( n = m \in M \), then \( EE_m = E(U(m)) = E_m \), thus \( EE_0 = E \).

(ii): We apply the criteria of [Per04a], §5.3. We have already stated that the \( EE_n \) are finite dimensional. For all infinite chains \( \cdots < n_i < n_{i+1} < \cdots \), we know that there exists an index \( i_0 \) such that the \( E_m \) vanish for \( m \leq n_{i_0} \), and thus the \( EE_n \) are zero. To see that there are only finitely many \( n \) such that \( \bigoplus_{n < n'} EE_{n'} \to EE_n \) is not surjective, we choose some finite poset in \( \mathbb{Z}^{\sigma(1)} \) which is admissible with respect to \( E \); using this, we find that there are only finitely many isomorphism classes of vector spaces \( EE_{n'} \). (iii): As the morphisms \( \chi_{m,m'} \) are injective for every \( m \leq m' \), the induced morphisms of the limits \( EE_n \to EE_{n'} \) are also injective for every \( n \leq n' \).

Note that in general \( EE \) is not just \( \hat{E} \) modulo torsion. For instance, the module \( \hat{E} \) of example 3.8 modulo torsion is not reflexive, whereas \( EE \) is reflexive (see subsection 5.2).

**Proposition 4.20:** The lcm-lattice of \( EE \) is admissible with respect to \( E \).

**Proof.** Denote \( L \) the lcm-lattice of \( E \). Let \( m \in M \) and \( n \in L \) its anchor element. By definition, the map \( EE_n \to E_m \) is an isomorphism, and thus \( L \) is admissible.

So we can use \( EE \) as alternative module by which we can construct resolutions of \( \mathcal{E} \). In subsection 5.2, we will do a more explicit analysis of \( EE \) for the case where \( \mathcal{E} \) is reflexive.

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4.5 Global resolutions for $\Delta$-families

Now let $\mathcal{E}$ be an equivariant coherent sheaf over $X$ and $E^\Delta$ its associated $\Delta$-family. To obtain global resolution of $\mathcal{E}$, we want to extend the techniques considered in the previous two subsections. Denoting $E^\sigma := \Gamma(U_\sigma, \mathcal{E})$, we assume that we have a family $\mathcal{P} = \{P^\sigma | \sigma \in \Delta\}$ of posets and compressions $\text{zip}^{\mathcal{P}^\sigma}$, $\text{unzip}^{\mathcal{P}^\sigma}$ with respect to these posets. For nicer notation, we write $\text{zip}^\sigma$ and $\text{unzip}^\sigma$ instead of $\text{zip}^{\mathcal{P}^\sigma}$ and $\text{unzip}^{\mathcal{P}^\sigma}$. For any $m \in M$ we write $A^\sigma_E(m)$ for the anchor element of $m$ in $\mathcal{P}^\sigma$. We denote $l_{\sigma \tau}$ and $k_{\sigma \tau}$ the gluing maps for the families $(\mathcal{M}, \leq_{\sigma})$ and $P^\sigma$.

Definition 4.21: The collection $\mathcal{P} = \{P^\sigma | \sigma \in \Delta\}$ is called admissible with respect to $E^\sigma$ if it glues over $\Delta$ and for every $\sigma \in \Delta$, the poset $P^\sigma$ is admissible with respect to $E^\sigma$.

Compressions of $\Delta$-families.

Proposition 4.22: Let $\mathcal{P} = \{P^\sigma | \sigma \in \Delta\}$ be a collection of posets which is admissible with respect to $\mathcal{E}$ and assume that we have a family of sheaves $F^\sigma$ which glues over the collection $\mathcal{P}^\sigma$, then the family $\text{unzip}^\sigma F^\sigma$ is a $\Delta$-family.

Proof. We show that $l_{\sigma \tau}^* \text{unzip}^\tau F^\tau \cong \text{unzip}^\sigma k_{\sigma \tau}^* F^\tau$ for every $\tau < \sigma$. This follows componentwise from $(l_{\sigma \tau}^* \text{unzip}^\tau F^\tau)_m = (\text{unzip}^\tau F^\tau)_{l_{\sigma \tau}(m)} = (\text{unzip}^\tau F^\tau)_m = F^\tau_{\sigma \tau}(m)$ and $(\text{unzip}^\sigma k_{\sigma \tau}^* F^\tau)_m = (k_{\sigma \tau}^* F^\tau)_{A^\sigma_m} = F^\tau_{k_{\sigma \tau}(A^\sigma_m)} = F^\tau_{A^\sigma_m}$, where the last isomorphism follows from the fact that $k_{\sigma \tau}$ is a contraction. Denote $\Psi^{\sigma \tau} : k_{\sigma \tau}^* F^\tau \cong F^\sigma$ the gluing maps over the family $\mathcal{P}^\sigma$, then we set $\Phi^{\sigma \tau} := \text{unzip}^\sigma \Psi^{\sigma \tau}$. By the isomorphisms $l_{\sigma \tau}^* \text{unzip}^\tau F^\tau \cong \text{unzip}^\sigma k_{\sigma \tau}^* F^\tau$ for all $\tau < \sigma$ and the functoriality of $l_{\sigma \tau}^*$, we have for any triple $\rho < \tau < \sigma$ the natural identification $\Phi^{\sigma \rho} = \Phi^{\sigma \tau} \circ l_{\sigma \tau}^* \Phi^{\tau \rho}$, and the proposition follows. \hfill $\Box$

Denote $\mathcal{S}^\mathcal{P}$ the category of coherent equivariant sheaves over $X$ with respect to which the collection $\mathcal{P}^\sigma$ is admissible. The operations $\text{zip}^\Delta$ and $\text{unzip}^\Delta$ are, up to natural isomorphism, mutually inverse functors from $\mathcal{S}^\mathcal{P}$ to the category sheaves over $\mathcal{P}^\sigma$. Thus, we have:

Theorem 4.23: $\text{zip}^\Delta$ and $\text{unzip}^\Delta$ are a compression of $\mathcal{S}^\mathcal{P}$.

In general, there is no canonical choice for admissible posets which automatically glues over $\Delta$. However, below we will give a gluing procedure starting from a family of admissible posets over $\Delta_{\text{max}}$, which yields a set of admissible posets together with a compression for any coherent $\Delta$-family. For smooth toric varieties, the lcm-lattices already will do the job:

Proposition 4.24: Assume that $X$ is a smooth toric variety, then the family $\text{zip}^\sigma E^\sigma$, with respect to the lcm-lattices of the modules $E^\sigma$, glues over $\Delta$.
where we write $D$ of the lcm-lattices can lead to more convenient resolutions.

Example 4.25: Consider the toric surface

$\begin{align*}
(0, 0, 1) \quad \text{other, and an indication of the associated lcm-lattices. The squares indicate the degrees}
\quad \text{\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ morally allows foraper of the choice of other admissible posets instead of the lcm-lattices can lead to more convenient resolutions.

Proof. Without loss of generality assume that $\sigma$ has full dimension in $N_\mathbb{R}$. Let $\tau < \sigma$ and for $m \in M$ denote $\bar{m}$ its class in $M/\tau_M$. For any $m' \leq_{\sigma} m \in M$, an isomorphism $\chi_{m',m}^\tau : E^\sigma_{m'} \to E^\sigma_m$ implies an isomorphism $\chi_{m',m}^\tau : E^\tau_{m'} \to E^\tau_m$, and for any $m \in M$, we have that $\bar{m}' \in I(\bar{m})$ implies that there exists some $m'' \in \bar{m}$ with $m'' \in I(\bar{m})$. Therefore, if we denote $P^\sigma$, $P^\tau$ the lcm-lattices of $E^\sigma$ and $E^\tau$, respectively, we have a canonical contraction $(P^\sigma)_{\leq_{\sigma}} \to (P^\tau)_{\leq_{\tau}}$, given by $P^\sigma \ni m \mapsto A_{E^\tau}(\bar{m})$. \qed

Refining compressions. For some arbitrary choice of $\Psi$, the category $S^\Psi$ in general contains not enough reflexive sheaves of rank one to construct global resolutions. This is true even for the collection of lcm-lattices of $E$ over a smooth toric variety. The reason for this is that relations of the module $\rho$ elements which come from the transition from one $\rho$-family into another some of which have to enter a local resolution. One possible resolution would be:

$$
0 \to \mathcal{O}(-D_1 - D_2 - 2D_3) \oplus \mathcal{O}(-3D_2 - 3D_3) \to \\
\mathcal{O}(-D_1 - 2D_3) \oplus \mathcal{O}(-D_2 - 2D_3) \oplus \mathcal{O}(-2D_2 - 3D_3) \oplus \mathcal{O}(-3D_2 - 2D_3) \to \\
\mathcal{O}(-2D_3) \oplus \mathcal{O}(-2D_2 - 2D_3) \to \mathcal{S} \to 0
$$

where we write $D_i$ instead of $D_{\rho_i}$. Note that the choice of other admissible posets instead of the lcm-lattices can lead to more convenient resolutions.
We consider any family of posets $\mathcal{P} = \{P_\sigma \mid \sigma \in \Delta\}$, which glues over $\Delta$ and which is admissible with respect to $\mathcal{E}$. We are going to construct a family of posets $\tilde{\mathcal{P}} = \{\tilde{P}_\sigma \mid \sigma \in \Delta\}$ which glues over $\Delta$, is admissible with respect to $\mathcal{E}$, and whose associated category $\mathbf{S}_{\tilde{\mathcal{P}}}$ has enough reflexive sheaves. We start bottom-up and we consider $P_\rho \subset Z_\rho(1) \sim Z$ for some $\rho \in \Delta(1)$. In fact, $P_\rho$ is a linear chain, i.e. a totally ordered subset of $Z$. For every $\sigma > \rho$, we consider $(P_\sigma)_{\rho \sigma}$ as a subset of $Z_\rho(1)$, such that the hooking $h_{\sigma \rho}$ becomes the natural inclusion $P_\rho \subset (P_\sigma)_{\rho \sigma}$ in $Z_\rho(1)$. We set

$$\tilde{P}_\rho := \bigcup_{\rho < \sigma} (P_\sigma)_{\rho \sigma}.$$

Now fix some $\sigma \in \Delta$ together with its admissible lattice $P_\sigma \subset Z^{\sigma(1)}$. For every $\tau < \sigma$, we consider the natural embedding $Z^{\tau(1)} \hookrightarrow Z^{\sigma(1)}$ which is induced by the inclusion $\tau(1) \subset \sigma(1)$. In particular, every element $i \in \tilde{P}_\rho$ becomes an element of $Z^{\sigma(1)}$ which is nonzero only at the $\rho$th position. For every $m \in M$ there exists a unique anchor element $A(m) \in P_\sigma$. We refine now by setting:

$$\tilde{A}(m) = \text{lcm}\{i \in \tilde{P}_\rho \mid i \leq \langle m, n(\rho)\rangle\}_{\rho \in \sigma(1)}$$

and

$$\tilde{P}^\sigma := \{\tilde{A}(m) \mid m \in M\} \cup P^\sigma,$$

where we observe that $A(m)$ is of the form

$$A(m) = \left(\max\{i \in \tilde{P}_\rho \mid i \leq A(m)_{\rho \rho}\} \mid \rho \in \sigma(1)\right)$$
and thus $P^\sigma \subset \tilde{P}^\sigma$. Clearly, $\tilde{P}^\sigma$ is admissible with respect to $E^\sigma$, and we can consider the compressions $\text{zip} \tilde{P}^\sigma$, $\text{unzip} \tilde{P}^\sigma$. Using the identification of $(P^\sigma)_{\leq \sigma}$ with its image in $Z^{(1)}$, we have:

**Proposition 4.26:** For any $\tau \in \Delta$, $\tilde{P}^\tau = \bigcup_{\tau < \sigma} (P^\sigma)_{\leq \sigma}$, where the union runs over all $\sigma \in \Delta_{\text{max}}$ with $\tau < \sigma$.

**Proof.** This follows because for any $\eta < \tau$, $P^\eta = (P^\eta)_{\leq \eta} \in Z^{(1)}$ is a subset of the image of $(P^\sigma)_{\leq \sigma}$ in $Z^{(1)}$. Thus $\tilde{P}^\eta = \bigcup_{\rho < \sigma} (P^\rho)_{\leq \rho}$ where the union runs over all maximal cones. Now the proposition follows from $P^\tau \subset (P^\sigma)_{\leq \tau}$ and by the generatedness of $\tilde{P}^\sigma$ by $P^\sigma$ and the $\tilde{P}^\rho$. 

By this proposition, we can conclude that the choice of any collection of admissible posets leads to a collection of admissible posets which glue over $\Delta$:

**Corollary 4.27:** The family $\tilde{P}^\tau$ is generated by the $P^\sigma$, where $\tau$ runs over $\Delta_{\text{max}}$.

**Corollary 4.28:** $(\tilde{P}^\sigma)_{\leq \tau} = \tilde{P}^\tau$ for all $\tau < \sigma \in \Delta$.

By combining these two corollaries, we obtain:

**Proposition 4.29:** The family of sheaves $\text{zip} \tilde{P}^\sigma E^\sigma$ glues over $\Delta$.

**Global resolutions.** Recall from section 4.4 that for every $\sigma \in \Delta$ we can construct the extension module $EE^\sigma$ of $E^\sigma$ over the ring $S^\sigma$. In the equivariant setting, the category of modules over $S^\sigma$ is equivalent to that of the ring $S_{\hat{\sigma}}$, and we can extend $EE^\sigma$ to a module over this ring. By naturality of the construction, the $EE^\sigma$ glue to a sheaf $E\hat{E}$ over $\hat{X}$, and we obtain the $S$-module $E\hat{E} := \Gamma(k^{\Delta(1)}, EE)$. We have the following properties for $E\hat{E}$, which immediately follow from the corresponding properties of proposition 4.19:

**Proposition 4.30:** $E\hat{E}$ has the following properties:

(i) $E\hat{E} \cong E$.

(ii) If $E$ is coherent, then $E\hat{E}$ is finitely generated.

(iii) if $E$ is torsion free, then $E\hat{E}$ is torsion free.

This way, a global resolution can be constructed as the descend of a resolution of the $S$-module $E\hat{E}$ with respect to its lcm-lattice. However, there are more possibilities to resolve $E$ which use $E\hat{E}$ but do not require the cost of computing the whole lcm-lattice of $E\hat{E}$. For this, we give a more precise picture of $E\hat{E}$.

For every $\sigma \in \Delta$ and every $\tau < \sigma$, denote $\pi_{\sigma} : Z^{(1)} \rightarrow Z^{\sigma(1)}$ and $\pi_{\tau} : Z^{\sigma(1)} \rightarrow Z^{\tau(1)}$ the canonical projections. For any $n \in Z^{\sigma(1)}$, $EE^\sigma_n$ is defined to be the inverse
limit $E^\sigma(U_m) := \lim_{\sigma} E^\sigma(U_m)$. For any $\tau < \sigma \in \Delta$, there is the canonical map induced by localization: $E^\sigma(U_m) \to E^\tau(U_{\pi^\sigma m})$, and for any $n \in \mathbb{Z}^{\Delta(1)}$, we obtain the directed system

$$
\begin{array}{c}
E\hat{E}_m \xrightarrow{E^\sigma} E_{\pi^\sigma m} \\
\downarrow \pi_\tau \\
E E_{\pi^\tau m}
\end{array}
$$

whose final object is the vector space $E E^0_{\pi^\sigma m}$. The component $E\hat{E}_m$ then has the universal property of the inverse limit of this system:

$$
E\hat{E}_m = \lim_{\sigma} E^\sigma_m = \lim_{\sigma} E^\sigma m
$$

where the latter limit runs over all $\sigma \in \Delta$ and the system of all $m \in M$ such that $n \leq \sigma m$.

**Definition 4.31:** A lift $\tilde{\mathcal{P}}^\lambda$ of the collection $\tilde{\mathcal{P}}^\sigma$ is a collection of injective, order-preserving maps $\lambda_\sigma : \tilde{\mathcal{P}}^\sigma \hookrightarrow \mathbb{Z}^{\Delta(1)} \cup \tilde{\mathcal{P}}^0$ such that

(i) $\pi_\sigma \left( \lambda_\sigma (n) \right) = n$ for all $n \in \tilde{\mathcal{P}}^\sigma$;

(ii) $\lambda_\sigma (n) = \text{lcm} \left\{ \lambda_\sigma \left( \left( \pi_\tau^{-1}(n) \cap \tilde{\mathcal{P}}^\sigma \right) \right) \right\}$ for every $n \in \tilde{\mathcal{P}}^\tau$;

(iii) for all $n \in \mathcal{P}^\sigma$ the composition $E E_{\lambda_\sigma (n)} \to (E E_{\lambda_\sigma (n)})_{\leq \sigma} \to E E^\sigma n$ is surjective.

We identify $\tilde{\mathcal{P}}^\lambda$ with the poset given by the image of the maps $\lambda_\sigma$.

There is, of course, no most natural choice for a lift $\lambda$, but a general choice which always works, is:

$$
\lambda_\sigma (n)_\rho = \begin{cases} 
 n_\rho & \text{if } \rho \in \sigma(1), \\
 \max\{i \in \tilde{\mathcal{P}}^\rho\} & \text{if } \tilde{\mathcal{P}}^\rho \neq \hat{0}, \\
 0 & \text{else}.
\end{cases}
$$

In the case where $\mathcal{E}$ is reflexive, it is possible to do a more efficient general choice, as we will see in subsection 5.2. With respect to a lift $\lambda$, we can define the submodule $E E\hat{\lambda} \subset E\hat{E}$ as follows. For every $\sigma \in \Delta$ and all $n \in \tilde{\mathcal{P}}^\sigma$ we choose a subvector space $E'_{\lambda_\sigma (n)} \subset E E_{\lambda_\sigma (n)}$ such that the induced morphism $E'_{\lambda_\sigma (n)} \to E E^\sigma n$ is surjective. Then we define $E E\hat{\lambda}$ to be the module generated by the $E'_{\lambda_\sigma (n)}$. Note that in spite we do not make explicit this choice in the notation, we always assume it implicitly.

**Proposition 4.32:** $(E E\hat{\lambda})^\sim \cong \mathcal{E}$.

**Proof.** As the homomorphism $E E_{\lambda_\sigma (n)} \to E E^\sigma n$ is surjective, the map $(E E_{\lambda_\sigma (n)})_{\leq \sigma} \to E E^\sigma n$ becomes an isomorphism. Moreover, $\pi_\sigma \left( \tilde{\mathcal{P}}^\sigma \right) = \tilde{\mathcal{P}}^\sigma$, so that the induced representation on $\tilde{\mathcal{P}}^\sigma$ is the same as for $E\hat{E}$. \qed
So, we can also use $E \hat{E}_\lambda$ to construct resolutions:

**Corollary 4.33:** Every lift $\tilde{P}^\lambda$ gives rise to a resolution of $\mathcal{E}$.

Instead of taking $E \hat{E}_\lambda$, we can also take directly the poset $\tilde{P}^\lambda$. Denote $i : \tilde{P}^\lambda \hookrightarrow \mathbb{Z}^{\Delta(1)}$ the canonical inclusion. We define:

$$\text{zip}^\lambda \mathcal{E} := i^* E \hat{E}_\lambda.$$ 

For any given representation $E$ of $\tilde{P}^\lambda$, by the canonical projection we obtain back the admissible poset $\tilde{P}^\sigma$ as the image of $\tilde{P}^\lambda$ in $\mathbb{Z}^{\sigma(1)}$, together with the localization of the representation $E$. These representations glue naturally over $\tilde{P}^\sigma$, and we can use them reconstruct the module $E \hat{E}$. By taking the submodule generated in the degrees given by $\tilde{P}^\lambda$, we obtain $E \hat{E}_\lambda$. Using either $E \hat{E}$ or $E \hat{E}_\lambda$, by sheafification we get back the sheaf $\mathcal{E}$. We denote this procedure $\text{unzip}^\lambda E$.

**Proposition 4.34:** The category of representations of $\tilde{P}^\lambda$ is a full subcategory of the category of sheaves which glue over the collection $\tilde{P}^\sigma$.

**Proof.** We only remark that these categories in general can not be equivalent, as the lift $\lambda^\sigma$ for every $\underline{n} \in \tilde{P}^\sigma$ fixes the choice of free representations of $\tilde{P}^\sigma$, $\sigma' \in \Delta$, which glue together with the free representation of $\underline{n}$ over $\tilde{P}^\sigma$. \qed

Using this correspondence, we obtain finally the finest class of global resolutions for $\mathcal{E}$. For the representation $E^\lambda$ of $\tilde{P}^\lambda$, for some lift $\lambda$, we construct the free resolution in the category of $\tilde{P}^\lambda$-representations, and by unzipping we obtain:

$$0 \rightarrow \text{unzip}^\lambda F_s \rightarrow \cdots \rightarrow \text{unzip}^\lambda F_0 \rightarrow \mathcal{E} \rightarrow 0.$$ 

However, nothing prevents us from taking a different lift $\lambda$ for every syzygy of $\mathcal{E}$, and as we will see below, this will be quite natural for doing so in the case of reflexive sheaves. So we can consider a sequence of lifts $\lambda_0, \ldots, \lambda_s$ and a corresponding resolution:

$$0 \rightarrow \text{unzip}^{\lambda_s} F_s \rightarrow \cdots \rightarrow \text{unzip}^{\lambda_0} F_0 \rightarrow \mathcal{E} \rightarrow 0.$$ 

Here, the finiteness of the sequence follows that in every step we eliminate minimal elements of the induced representations of the admissible posets $\tilde{P}^\sigma$, but the length $s$ finally may depend on the successive choice of the lifts.

**Example 4.35:** Consider the variety $\mathbb{P}^1 \times \mathbb{P}^1$ and skyscraper sheaf $\mathcal{S}$ similar to that of example 4.25, but this time with slightly different gradings:

$$\Gamma(U_{12}, \mathcal{S}) = k \cdot \chi(1,1), \quad \Gamma(U_{23}, \mathcal{S}) = k \cdot \chi(-1,1).$$
Figure 6 shows the corresponding posets. Explicitly, we have:

\[ \tilde{P}^{12} = \{ \hat{0}, (1, 1), (2, 1), (1, 2), (2, 2) \} \]
\[ \tilde{P}^{23} = \{ \hat{0}, (1, 1), (2, 1), (1, 2), (2, 2) \} \]
\[ \tilde{P}^2 = \{ \hat{0}, 1, 2 \} \]

(for brevity, we suppress the ray \( \rho_4 \)). We consider the lifts:

\[ \lambda_{12}(\tilde{P}^{12}) = \{ \hat{0}, (1, 1, 1), (2, 1, 2), (1, 2, 1), (2, 2, 2) \} \]
\[ \lambda_{12}(\tilde{P}^{23}) = \{ \hat{0}, (1, 1, 1), (1, 2, 1), (2, 1, 2), (2, 2, 2) \} \]
\[ \lambda_{12}(\tilde{P}^2) = \{ \hat{0}, (2, 1, 2), (2, 2, 2) \} \]

The sheaf \( \hat{E} \) is given by

\[ \hat{E}_{\mathbb{P}^1} \cong \begin{cases} k^2 & \text{if } \mathbf{n} = (1, 1, 1) \\ 0 & \text{else.} \end{cases} \]

Choosing the one-dimensional diagonal \( k \subset \hat{E}_{(1,1,1)} \), we obtain as resolution:

\[ 0 \longrightarrow \mathcal{O}(-2D_1 - 2D_2 - 2D_3) \longrightarrow \mathcal{O}(-2D_1 - D_2 - 2D_3) \oplus \mathcal{O}(-D_1 - 2D_2 - D_3) \]
\[ \longrightarrow \mathcal{O}(-D_1 - D_2 - D_3) \longrightarrow S \longrightarrow 0 \]

### 5 Reflexive Sheaves and Vector Space Arrangements

#### 5.1 Reflexive sheaves and their canonical admissible posets

For an equivariant reflexive sheaf over a toric variety, i.e. a sheaf \( \mathcal{E} \) which is isomorphic to its bidual, \( \mathcal{E} \cong \mathcal{E}^\vee \), the associated \( \Delta \)-family has a quite efficient representation. To every equivariant coherent sheaf \( \mathcal{E} \) over \( U_\sigma \), one can associate a limit vector space
\(E^\sigma := \lim_{\to} E^\sigma_m\), and by the gluing of the \(E^\sigma\) over the collection of posets \((M, \leq_\sigma)\), there is a functorial isomorphism \(E^\sigma \to E^0 =: E\), where 0 denotes the zero cone in \(\Delta\), and moreover, \(\dim E = \rk E\). As explained in detail in [Per04a], section 5 (see also [Kly90], [Kly91]), every equivariant reflexive sheaf \(E\) is determined by a set of filtrations

\[
\cdots \subset E^\rho(i) \subset E^\rho(i + 1) \subset \cdots \subset E
\]

for every ray \(\rho \in \Delta(1)\). These filtrations must be full, i.e. \(E^\rho(i) = 0\) for very small \(i\), and \(E^\rho(i) = E\) for \(i\) very large. The corresponding \(\sigma\)-families then can be constructed from these filtrations by setting

\[
E^\sigma_m = \bigcap_{\rho \in \sigma(1)} E^\rho(\langle m, n(\rho) \rangle).
\]

In fact, this construction establishes an equivalence of categories between equivariant reflexive sheaves and vector spaces with full filtrations. The morphisms in the latter category are vector space homomorphisms which are compatible with the filtrations in the \(\Delta\)-family sense ([Per04a], Theorem 5.29).

Consider a reflexive module \(E^\sigma\) over the ring \(k[\sigma_M]\), where without loss of generality we assume that \(\sigma\) has full dimension in \(N_\mathbb{R}\). To any such module there is associated the subvector space arrangement \(\{E^\sigma_m \mid m \in M\}\) in the limit vector space \(E\), where \(E^\sigma_m = \bigcup_{\rho \in \sigma(1)} E^\rho(\langle m, n(\rho) \rangle)\). This arrangement in a natural way is a poset, where the partial order is given by inclusion. We will show that we can embed this poset into \(\mathbb{Z}^{\sigma(1)}\) such that it becomes an admissible poset for \(E^\sigma\).

**Definition 5.1:** For every \(m \in M\), we define \(\kappa_\rho(m) = \min\{i \in \mathbb{Z} \mid E^\sigma_m \subset E^\rho(i)\}\) and the anchor of \(m\) by:

\[
A(m) = (\kappa_\rho(m) \mid \rho \in \sigma(1)) \in \mathbb{Z}^{\sigma(1)}.
\]

We denote \(\mathcal{P}_{E^\sigma}\) the subposet \(\{A(m) \mid m \in M\}\) of \(\mathbb{Z}^{\sigma(1)}\).

**Proposition 5.2:** \(\mathcal{P}_{E^\sigma}\) is admissible with respect to \(E^\sigma\).

**Proof.** First, clearly, \(A(m) \leq m\) for all \(m \in M\). Now assume that \(A(m') \leq m\) for some \(m' \in M\). \(A(m') \leq m\) implies that \(E^\sigma_{m'} \subset E^\sigma_m\), and thus \(A(m') \leq A(m)\). Now, by definition \(\bigcap_{\rho \in \sigma(1)} E^\rho(n_\rho) = E^\sigma_{m'}\) for all \(m \in T_m\) for some \(n_\rho \in \mathcal{P}_{E^\sigma}\).

**Definition 5.3:** We call \(\mathcal{P}_{E^\sigma}\) the canonical admissible poset of \(E^\sigma\).

An important fact for understanding the structure of reflexive modules is the following

**Lemma 5.4:** Let \(\mathcal{P}_{E^\sigma}\) be the canonical admissible poset of \(E^\sigma\). Then \(E^\sigma_m \subset E^\sigma_{m'}\) iff \(A(m) \leq A(m')\). Moreover, \(E^\sigma_m = E^\sigma_{m'}\) iff \(A(m) = A(m')\).
Proof. Assume first that $E^\sigma_m \subset E^\sigma_{n'}$. Then for every $\rho \in \sigma(1)$ it follows that $\min\{i \mid E^\sigma_m \subset E^\rho(i)\} \leq \min\{i \mid E^\sigma_{n'} \subset E^\rho(i)\}$, and thus $A(m) \leq_{\sigma} A(m')$. In the other direction, denote $n := A(m)$, $n' := A(m')$, then $n_\rho \leq n'_\rho$ for every $\rho \in \sigma(1)$ and $E^\rho(n_\rho) \subseteq E^\rho(n'_\rho)$, and thus $E^\sigma_m \subset E^\sigma_{n'}$. □

**Proposition 5.5:** If $U_\sigma$ is smooth, then, as a poset, the vector space arrangement associated to $E^\sigma$ is isomorphic to its lcm-lattice.

Proof. Because $U_\sigma$ is smooth, for every $m \in M$, the anchor element $A(m)$ is an element of $M$, and we conclude from the proof of proposition 5.2, that $A(m)$ is the unique member of $I(m)$. For any two $A(m) \neq A(m')$, the vector spaces $E^\sigma_m$ and $E^\sigma_{m'}$ do not coincide, and thus the vector space $E^\sigma_{m''}$, where $m'' = \text{lcm}\{m, m'\}$, contains at least the sum $E^\sigma_m + E^\sigma_{m'}$. Moreover, we have that $E^\sigma_{m''} = \bigcap_{\rho \in \sigma(1)} E^\rho(\langle m'', n(\rho) \rangle)$, where for every $\rho \in \sigma(1)$ $E^\rho(\langle m'', n(\rho) \rangle)$ contains $E^\rho_m$ and $E^\rho_{m'}$, and thus $\langle m'', n(\rho) \rangle \geq \max\{\langle m, n(\rho) \rangle, \langle m', n(\rho) \rangle\}$. So $m''$ is the minimal element of $M$ with respect to the partial order $\sigma_M$, such that $E^\sigma_{m''}$ contains both, $E^\sigma_m$ and $E^\sigma_{m'}$. □

**Example 5.6:** We give an example which shows that the choice of another admissible poset instead of the canonical one can improve the resolution. Consider the subsemigroup $\sigma_M$ of $\mathbb{Z}^2$ which is generated by $(1, 0)$, $(1, 1)$ and $(1, 2)$; the corresponding cone $\sigma$ has two rays $\rho_1, \rho_2$ with primitive elements $n(\rho_1) = (2, 1)$, $n(\rho_2) = (0, 1)$. Let $E \cong k^3$ and consider the filtrations

$$E^{\rho_1}(i) = \begin{cases} 0 & \text{for } i < 0 \\ E_1 & \text{for } i = 0 \\ E & \text{for } i > 0 \end{cases}$$

$$E^{\rho_2}(i) = \begin{cases} 0 & \text{for } i < 1 \\ E_2 & \text{for } i = 1 \\ E & \text{for } i > 1. \end{cases}$$

with $\dim E_i = 2$ and the $E_i$ in general position. The corresponding canonical admissible poset is $\mathcal{P} = \{\hat{0}, (0, 2), (1, 1), (1, 2)\}$ and it leads to the resolution

$$0 \longrightarrow S_{(2,2)} \longrightarrow S^2_{(1,1)} \oplus S^2_{(0,2)} \longrightarrow E \longrightarrow 0$$

If we choose instead the poset $\mathcal{P}' = \{\hat{0}, (0, 1), (0, 2), (1, 1), (1, 2)\}$, the associated representation of $\mathcal{P}'$ maps $(0, 1)$ to the the subvector space $E_1 \cap E_2$ of $E$. The corresponding vector space arrangements are shown as linear configurations in $\mathbb{P}E \cong \mathbb{P}^2$ in figure 7.

The grey dot in the right figure denotes the intersection $E_1 \cap E_2$. The corresponding resolution becomes:

$$0 \longrightarrow S_{(0,1)} \oplus S_{(0,2)} \oplus S_{(1,1)} \longrightarrow E \longrightarrow 0,$$

i.e. $E$ splits into a direct sum of reflexive sheaves of rank one.
5.2 Extensions to the homogeneous coordinate ring

We first investigate the structure of the module $E\hat{E}$ where $E$ is reflexive. For this, we first consider the module $EE^\sigma$ for any $\sigma \in \Delta$. Its determination is a straightforward computation:

**Proposition 5.7:** Let $E^\sigma$ be a reflexive $k[\sigma_M]$-module given by filtrations $E^\rho(i)$. Then its extension is given by:

$$EE^\sigma_n = \bigcap_{\rho \in \sigma(1)} E^\rho(n\rho).$$

**Proof.** We have $EE^\sigma_n = \lim\leftarrow E^\sigma_m$, where the limit runs over all $n \leq m$. As all morphisms $\chi_{m,m'}^\sigma$ are injective, this direct limit immediately translates into an intersection in $E^\sigma$:

$$\lim\leftarrow E^\sigma_m \subseteq \bigcap_{n \leq m} \bigcap_{\rho \in \sigma(1)} E^\rho(\langle m, n(\rho) \rangle).$$

It is always possible to find $m \in M$ for some $\tau \in \sigma(1)$ such that $\langle m, n(\tau) \rangle = n_\tau$ and $\langle m, n(\rho) \rangle >> 0$ for any $\tau \neq \rho$, such that $\bigcap_{\rho \in \sigma(1)} E^\rho(\langle m, n(\rho) \rangle) = E^\tau(n_\tau)$. Thus we obtain $\bigcap_{\rho \in \sigma(1)} E^\rho(n_\rho) \subseteq EE^\sigma_n \subseteq \bigcap_{\rho \in \sigma(1)} E^\rho(n_\rho)$ and the proposition follows. $\square$

So the module $E\hat{E}^\sigma$ can explicitly be described by the filtrations for $E$ and in fact, it is a reflexive module. To describe its filtrations more explicitly, we use the quotient representation $\pi : k^{\sigma(1)} \rightarrow U_\sigma$. For each $\rho \in \sigma(1)$, the restriction of $E$ to $U_\rho$ is a locally free sheaf and thus if we restrict $\pi$ to $U_\rho$, the pullback

$$\hat{E}^\hat{\rho} := (\pi|_{U_\rho})^*E|_{U_\rho}$$

is locally free over $U_\rho$. To determine the filtration associated to $\hat{E}^\hat{\rho}$, consider the injective map

$$\alpha_\rho : M/\rho_M^1 \rightarrow \mathbb{Z}^{\rho(1)}.$$
Then every element \( i \in \mathbb{Z}^{\Delta(1)}/\rho_{\Delta M} \) lies in a unique interval \( \alpha_{\rho}(j) \leq i < \alpha_{\rho}(j + 1) \) for some \( j \in M/\rho_{\Delta M} \cong \mathbb{Z} \). \( \mathcal{E}^\rho \) then can be described by a filtration of \( E \), which is given by

\[ E \mathcal{E}^\rho(i) = E^\rho(j) \text{ for } \alpha_{\rho}(j) \leq i < \alpha_{\rho}(j + 1). \]

The reflexive \( S \)-module defined by set of filtrations \( E \mathcal{E}^\rho(i) \) for every \( \rho \in \Delta(1) \) then can be identified with \( E \mathcal{E} \).

**Proposition 5.8:** Let \( \mathcal{E} \) be a reflexive sheaf, then there is an isomorphism \( \hat{E}^{-} \cong E \mathcal{E} \).

### 5.3 Resolutions for vector space arrangements and reflexive equivariant sheaves

**The affine case.** First we consider resolutions for a reflexive \( M \)-graded module \( E^\sigma \) over \( k[\sigma_M] \) with filtrations \( E^\rho(i) \) for \( \rho \in \sigma(1) \). Revisiting the resolution process of proposition 2.4 for the corresponding representation of the canonical admissible poset \( \mathcal{P}_{E^\sigma} \), we find by lemma 5.4 that for any \( \underline{n} \in \mathcal{P}_{E^\sigma} \), the vector space \( E^n_{<\underline{n}} \) is the subvector space of \( E^n_{\underline{n}} \) which is spanned by all its subvector spaces in the arrangement \( \mathcal{P}_{E^\sigma} \). We have the first step of its resolution

\[ 0 \to K_0 \to F_0 \to E^\sigma \to 0 \]

such that \( F_0 \) is a reflexive module \( F_0 \cong \bigoplus_{\underline{n} \in \mathcal{P}_{E^\sigma}} S^I_{\underline{n}} \) which is defined by filtrations \( F^\rho(i) \) in a limit vector space \( F \), defining a vector space arrangement \( Q := \{ F_m \mid m \in M \} \).

**Proposition 5.9:** The poset underlying the vector space arrangement \( Q \) is isomorphic to \( \mathcal{P}_{E^\sigma} \).

**Proof.** The dimension of the vector space \( F_{0,\underline{n}} \) is given by the number of \( \underline{n}' \leq \sigma \underline{n} \); by lemma 5.4 we have that \( E^\sigma_{m} \subseteq E^\sigma_{m'} \) iff \( A(m) < A(m') \), and thus the number of \( \underline{n}'' \in \mathcal{P} \) for which \( E^n_{\underline{n}''} \) has positive free dimension and which \( \underline{n}'' \leq A(m) \) is smaller than the number of such elements with \( \underline{n}'' \leq A(m') \). \( \square \)

The kernel \( K_0 \) is a reflexive module, given by filtrations \( K^\rho(i) = \ker(F^\rho(i) \to E^\rho(i)) \) of the kernel vector space \( K = \ker(F \to E) \). However, the canonical admissible poset of \( K_0 \) is no longer isomorphic to \( \mathcal{P}_{E^\sigma} \), but we have the following:

**Proposition 5.10:** The canonical admissible poset of \( K_0 \) is a contraction of \( \mathcal{P}_{E^\sigma} \).

**Proof.** We define the retraction morphism \( r : \mathcal{P}_{E^\sigma} \to \mathcal{P}_{K_0} \) by mapping \( A_{E^\sigma}(m) \) to \( A_{K_0}(m) \) for all \( m \in M \). For any \( E^\sigma_m \subset E^\sigma_{m'} \), we have \( K_{0,m} \subset K_{0,m'} \), and thus \( r(U(A_{E^\sigma}(m))) \subset U(A_{K_0}(m)) \). The other inclusion follows because \( \mathcal{P}_{K_0} \) is admissible for \( K_0 \). On the other hand, let \( \underline{n} \in \mathcal{P}_{K_0} \), then \( \underline{n} \leq \underline{n}' \) for every \( \underline{n}' \in r^{-1}(U(\underline{n})) \), and \( \underline{n} \in r^{-1}(U(\underline{n})) \), thus \( r^{-1}(U(\underline{n})) = U(\underline{n}) \) in \( \mathcal{P}_{E^\sigma} \). \( \square \)
By 2.12 this in particular implies that we can iterate and the resolution of the vector space arrangement $\mathcal{P}_{E^\rho}$ is equivalent to a resolution of $E^\rho$. We have:

$$0 \to F_s \to \cdots \to F_0 \to E^\rho \to 0$$

where $F_i \cong \bigoplus_{\underline{u} \in \mathcal{P}_{E^\rho}} S_{(\underline{u})}^{f_i}$.

The shape of the resolution can be changed by choosing another admissible poset for $E^\rho$. This in turn is equivalent to adding any set of intersections of vector spaces in $\mathcal{P}_{E^\rho}$.

To see this, we pass to the module $EE^\rho$. The arrangement of this module is complete with respect to intersections, and every anchor element of the canonical admissible poset of $E^\rho$ is by definition an anchor element of the lcm-lattice of $EE^\rho$. In particular, for every $\underline{n} \in \mathcal{L}_{E^\rho}$ with $\underline{n} \leq \underline{m}$, we have $\underline{n} \leq A_{E^\rho}(\underline{m})$ by lemma 5.4, so that condition (i) of definition 4.11 is fulfilled. Moreover, as $T_\underline{n}$ is empty if $\underline{n}$ is not from $\mathcal{P}_{E^\rho}$, condition (ii) is trivially fulfilled.

**The global case.** Now we assume that $\mathcal{E}$ is a reflexive sheaf over an arbitrary toric variety $X$, represented by filtrations $E^\rho(i)$ of some vector space $E$ for every $\rho \in \Delta(1)$.

We denote $\mathcal{P}$ the canonical admissible posets for every $E^\rho$. To make contact with the formalism of section 4.5, we first consider the refinements $\tilde{\mathcal{P}}^\sigma$.

**Lemma 5.11:** $\mathcal{P}^\sigma$ is a contraction of $\tilde{\mathcal{P}}^\sigma$ for every $\sigma \in \Delta$.

**Proof.** For every $\rho \in \Delta(1)$, the canonical admissible poset $\mathcal{P}^\rho$ is given by $\hat{0}$ and some sequence $i_1^\rho < \cdots < i_{k^\rho}^\rho$ in $\mathbb{Z}$, where $k^\rho < \text{rk} \mathcal{E}$, such that $E^\rho(i) = E^\rho(i + j)$ for $j \geq 0$ if and only if there exists no $i_p^\rho$ for some $\rho \in \{1, \ldots, k^\rho\}$ such that $i < i_p^\rho \leq i + j$. For every $\rho \in \Delta(1)$ and every $\rho < \sigma$, we have $(\mathcal{P}^\sigma)_{<\rho} = \mathcal{P}^\rho$, and thus $\mathcal{P}^\sigma = \tilde{\mathcal{P}}^\sigma$. Recall that $\tilde{A}^\sigma$ was defined as the least common multiple of the elements $\max\{i \in \tilde{\mathcal{P}}^\rho \mid i \leq \langle m, n(\rho) \rangle\}$, where $\tilde{\mathcal{P}}$ is considered as subset of $\mathbb{Z}^{\sigma(1)}$ via the canonical embedding $\mathbb{Z}^\rho \hookrightarrow \mathbb{Z}^{\sigma(1)}$.

Denote $r : \tilde{\mathcal{P}}^\sigma \to \mathcal{P}^\sigma$, mapping the anchor $\tilde{A}^\sigma(m)$ to $A^\sigma(m)$. Clearly, $r$ is surjective. Then for any $\tilde{A}^\sigma(m) \in \tilde{\mathcal{P}}^\sigma$, the image of $U(\tilde{A}^\sigma(m))$ is $U(A^\sigma(m))$. For any $\underline{n} \in \mathcal{P}^\sigma$, $r^{-1}(\underline{n}) = \underline{n}$, so $r^{-1}(U(\underline{n})) = U(\underline{n})$ (the latter as an open subset of $\tilde{\mathcal{P}}^\sigma$, and the lemma follows.

For resolving $\mathcal{E}$, we now must define a lift $\lambda$ of the collection $\tilde{\mathcal{P}}^\sigma$ to $\mathbb{Z}^{\Delta(1)}$. For every $\sigma \in \Delta$, we define $\lambda_\sigma : \tilde{\mathcal{P}}^\sigma \to \mathbb{Z}^{\Delta(1)}$ by

$$\lambda_\sigma(\underline{n})_\rho = \begin{cases} \min\{i \mid E^\sigma(i) \subset E^\rho(i)\} & \text{for } \rho \in \Delta(1) \setminus \sigma(1) \\ n_\rho & \text{for } \rho \in \sigma(1). \end{cases}$$

**Proposition 5.12:** The collection $\lambda_\sigma$ is a lift of $\tilde{\mathcal{P}}^\sigma$. 

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Proof. By definition, \((\pi_\sigma \circ \lambda_\sigma)(\underline{n}) = \underline{n}\) for every \(\underline{n} \in \tilde{P}_\sigma\). We show that \(\lambda_\tau(\underline{n}) = \operatorname{lcm}\left\{\lambda_\sigma((\pi_\sigma)^{-1}(\underline{n}) \cap \tilde{P}_\tau) \mid \tau < \sigma\right\}\) for every \(\tilde{P}_\tau\). For this, observe that \(E\hat{E}_{\lambda_\sigma(\underline{n})} = E^\sigma_{\underline{n}}\), because

\[
E^\sigma_{\underline{n}} = \bigcap_{\rho \in \sigma(1)} E^\rho(\underline{n}_\rho) \subset E\hat{E}_{\lambda_\sigma(\underline{n})} = \bigcap_{\rho \in \Delta(1)} E^\rho(\lambda_\sigma(\underline{n})_\rho)
\]

\[
= E^\sigma_{\underline{n}} \cap \left( \bigcap_{\rho \in \Delta(1) \setminus \sigma(1)} E^\rho(\lambda_\sigma(\underline{n})_\rho) \right) \subset E^\sigma_{\underline{n}}.
\]

Now, the lift \(\lambda\) gives rise to a subarrangement of the subvector space arrangement of the arrangement associated to \(E\hat{E}\), which is given by the union of arrangements in \(E\):\

\[
P^\Delta := \bigcup_{\sigma \in \Delta} P^\sigma = \left\{ \bigcap_{\rho \in \sigma(1)} E^\rho((m, n(\rho))) \mid \sigma \in \Delta, m \in M \right\} = \bigcup_{\sigma \in \Delta} \{\lambda_\sigma(\underline{n}) \mid \underline{n} \in P^\sigma\}
\]

The first step \(0 \to K_0 \to \mathcal{F}_0 \to \mathcal{E} \to 0\) of the global resolution of \(E\) then is given by the sheaf

\[
\mathcal{F}_0 \cong \bigoplus_{\underline{n} \in P^\Delta} \mathcal{O}(D_{\lambda_\sigma(\underline{n})})^{f^0_{\underline{n}}}_{\underline{n}}
\]

where \(f^0_{\underline{n}}\) is the free dimension of the vector space \(E_{\underline{n}}\). By iteration, we get a free resolution, which at the same time is a resolution of the vector space arrangement \(P^\Delta\). Note that this resolution coincides with the resolution of the module \(E\hat{E}_{\lambda}\) over \(S\). The global resolution of \(E\) constructed using \(E\hat{E}\) is given by the minimal resolution given by the vector space arrangement in \(E\) which is generated by all intersections of the vector spaces \(E^\rho(\iota)\).

### 5.4 Resolutions of Cohen-Macaulay modules

Let \(E\) be a (maximal) Cohen-Macaulay module over \(k[\sigma_M]\), where \(\sigma\) has full dimension in \(N^R\). We show that our resolutions behave well in the sense that the maximal length of regular sequences does not decrease. We follow [BH98] §1.5, and say that the graded module \(E\) is Cohen-Macaulay if \(\operatorname{grade}_m E = \dim k[\sigma_M]\), where \(m\) is the maximal homogeneous ideal of \(k[\sigma_M]\) which is generated by all non-unit monomials.

**Theorem 5.13:** Let \(E\) be an \(M\)-graded Cohen-Macaulay module over \(k[\sigma_M]\) and consider the resolution

\[
0 \to F_s \to \cdots \to F_0 \to E \to 0
\]

corresponding to the canonical admissible poset of \(E\). Then every \(F_i\) is a direct sum of Cohen-Macaulay modules of rank one.
Proof. We need only to consider the first step of the resolution $0 \to K_0 \to F_0 \to E \to 0$, as $K_0$ will be Cohen-Macaulay if $F_0$ and $E$ are Cohen-Macaulay; the result then follows by induction. If we restrict the surjection from $F_0$ to $E$ to a direct summand of rank one $R$ of $F_0$, we necessarily obtain an injection $0 \to R \to E$. We show that any $E$-regular sequence by construction also is a $R$-sequence of $R/\mathfrak{m}$. By induction. If we restrict the surjection from $F_0$ to $E$ to a direct summand of rank one $R$ of $F_0$, we necessarily obtain an injection $0 \to R \to E$. We show that any $E$-regular sequence by construction also is a $R$-sequence of $R/\mathfrak{m}$.

If $\alpha$ is injective, then also $\beta$ is injective, and the element $x_{i+1}$ is a nonzero divisor of $R/x_iR$, as it is a nonzero divisor of $E/x_iE$. To show that $\alpha$ is injective, we show that there exists no $e_1, \ldots, e_i \in E$ such that $y := \sum_{j=1}^i x_j e_j$ is in $R$ but not in $x_iR$. This sum decomposes into homogeneous summands $y = \sum_{m \in M} y_m$ where $y_m = \sum_{j=1}^i \sum_{m' \in M} x_j m' \cdot e_{j,m-m'}$. If we write $x_j m' = a_{j,m'}$, this sum can be written as $\sum_{j=1}^i \sum_{m' \in M} a_{j,m'} \cdot e_{j,m-m'}$. Now we split the set $\{m' \in M \mid e_{j,m-m'} \neq 0\} = U_j \coprod V_j$, where $U_j = \{m' \mid R_{m-m'} \neq 0\}$. By construction of the inclusion of $R$ in $E$, there does not exist any $m'' \in V_j$ such that $E_{m''}$ contains a one dimensional subvector space whose image in $E$ coincides with the image of $R$. For any $m' \in V_j$, the elements $\chi(m') \cdot e_{j,m-m'}$ must be contained in the subvector space $F_m$ spanned by all $E_{m''}$ with $m'' < m$, and writing the equations modulo $F_m$, we can replace every $e_j$ by some $f_j$ such that $f_{j,m-m'} = 0$ if $m' \in V_j$ and $\sum_j x_j f_j = y$. Thus we have for every $m$ the equation $x_m = \sum_{j=1}^i \sum_{m' \in U_j} a_{j,m'} \chi(m') \cdot f_{j,m-m'}$. For $m' \in U_j$, we can project every $f_{j,m-m'}$ to some appropriate $r_{j,m-m'} \in R_{m-m'}$, such that $x_m = \sum_{j=1}^i \sum_{m' \in U_j} a_{j,m'} \chi(m') \cdot r_{j,m-m'}$. Therefore, we have $x_m \in x_iR$, from which follows that $\alpha$ is injective.

Corollary 5.14 (from proof of theorem 5.13): Let $E$ be any reflexive $k[\sigma_M]$-module and $F_i$ as in the theorem, then $\text{grade}_m F_i \geq \text{grade}_m E$ for all $0 \leq i \leq s$. 

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5.5 Reflexive models for vector space arrangements

In this subsection we want to make a few remarks on how resolutions of vector space arrangements can efficiently be constructed by passing to appropriate reflexive modules over the polynomial ring. The point here is resolutions of such modules are a standard task for many computer algebra systems. However, to make use of such systems, one has to construct appropriate input data from the arrangement. Let \( \mathcal{V} \) be a subvector space arrangement of some vector space \( \mathbb{V} \). We make two assumptions on \( \mathcal{V} \); the first is that \( \mathcal{V} \) is complete with respect to intersections, that is, for any subset \( W_1, \ldots, W_r \in \mathcal{V} \), the intersection \( W_1 \cap \cdots \cap W_n \) is also in \( \mathcal{V} \). The second assumption is that the input data for \( \mathcal{V} \) is given by a set of vectors \( v_{W_1}, \ldots, v_{W_i} \) such that \( W \) is the span over \( k \) of all \( v_{V_i} \) where \( V \subset W \) and \( i = 1, \ldots, i_V \). Moreover, we assume that this set is irredundant, i.e. \( i_W = \text{codim}_W \sum_{V \subseteq W} V \). Using this input data, the first step of the resolution of \( \mathcal{V} \) is nearly tautological. Assume that we have chosen a basis for \( \mathcal{V} \), then \( F_0 \) is given by a basis \( e_{W_i} \), \( i = 1, \ldots, i_W \) in one-to-one correspondence to the vectors \( v_{W_i} \), and the matrix \( M \) then can simply be chosen as having the vectors \( v_{W_i} \) as its columns, i.e. \( M = (v_{W_i}^{W_j}) \). By associating to \( \mathcal{V} \) the structure of some appropriate fine-graded module, the matrix \( M \) becomes a monomial matrix for which syzygies can be computed.

**Definition 5.15:** (i) A reflexive model for \( \mathcal{V} \) is an inclusion \( \mathcal{V} \hookrightarrow (\mathbb{Z}^r, \leq) \) for some \( r > 0 \) such that that its image in \( \mathbb{Z}^r \) is an lcm-lattice.

(ii) A set of generating flags of \( \mathcal{V} \) is a set of tuples \( \{E_1^1 \subset \cdots \subset E_1^1, \ldots, \{E_r^1 \subset \cdots \subset E_r^1, \ldots \} \} \subset \mathcal{V} \) such that \( E_{n_i}^i = \mathcal{V} \) for every \( i \) and \( \mathcal{V} \) is the set of all intersections among the \( E_j^i \).

Let \( E_j^i \) be any set of generating flags, then we associate to each of the flags a tuple of integers \( \chi_j^i := (k_1^i < \cdots < k_{n_i}^i) \). This data defines a reflexive model, where we map every \( W \in \mathcal{V} \) to the tuple \( \chi_W := (\min\{k_j^i \mid W \subset E_j^i \} \mid i = 1, \ldots, r) \). As easily can be seen, this reflexive model gives rise to a reflexive fine-graded module \( E \) over the polynomial ring \( S = k[x_1, \ldots, x_r] \) which is given by filtrations

\[
E^i(j) = \begin{cases} 
0 & \text{if } j < k_1^i, \\
E_j^i & \text{if } k_l^i \leq j < k_{l+1}^i, l < n_i, \\
\mathcal{V} & \text{if } n_i \leq j.
\end{cases}
\]

\( E \) has an embedding into the free module \( S(-\chi_{\min})^{\dim \mathcal{V}} \), where \( \chi_{\min} = (k_1^i, \ldots, k_r^i) \), and the module \( F_0 \) is given by the direct sum \( \bigoplus_{W \in \mathcal{V}} S(-\chi_W)^{f_W} \), where \( f_W \) is the free
dimension of $V$. So, we have

$$0 \rightarrow K_0 \rightarrow \bigoplus_{W \in V} S(-k_W)^{f_V} \xrightarrow{\bar{M}} S(-k_{\min})^{\dim V},$$

where $\bar{M}$ is a monomial matrix whose entries are of the form $(v_{ij}^W x^k_{V} - k_{\min})$, where the $v_{ij}^W$ are the corresponding entries of the matrix $M$. The image of $M$ then is the module $E$. So the only effect seen by the choice of the reflexive model for $V$ are the number of variables in the ring $S$ and the degrees of the monomials in $\bar{M}$ and the subsequent matrices in the resolution, whereas the coefficients in $\bar{M}$ are precisely the entries of the matrix $M$.

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