Higher-order topological phases in a spring-mass model on a breathing kagome lattice

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We propose a realization of higher-order topological phases in a spring-mass model with a breathing kagome structure. To demonstrate the existence of the higher-order topological phases, we characterize the topological properties with the $Z_3$ Berry phase, and show that the corner states appear under the fixed boundary condition. Moreover, we suggest that the corner states can be detected experimentally through a forced vibration.

I. INTRODUCTION

Topological insulators (TIs)\cite{1,2} are distinctive class of insulators where topologically-nontrivial structures of Bloch wave functions give rise to characteristic boundary states. Various unique phenomena are caused by these boundary states, such as quantization of Hall conductivity\cite{3}, and spin-Hall conductivities\cite{4}, and electromagnetic responses\cite{5,6}. In TIs, bulk topological invariants which characterize the nontrivial topology of Bloch wave functions are known to be related to the $(d-1)$-dimensional boundary states (with $d$ being spacial dimension of the bulk Hamiltonian); this relation is nowadays established as bulk-boundary correspondence\cite{7,8}.

Recently, a novel class of TIs, called higher-order topological insulators (HOTIs), was introduced\cite{9-15}. In HOTIs, $d-2$ or less dimensional boundary states appear in $d$-dimensional models, which is predicted by topological invariants in the bulk. Examples of such topological invariants include the multipole moment\cite{10,16}, the nested Wilson loops\cite{11,15}, the quantized Wannier centers\cite{12}, the entanglement polarization\cite{17}, and the $Z_Q$ Berry phase\cite{18-20}. In that sense, a novel kind of bulk-boundary correspondence emerges in HOTIs.

In parallel with the theoretical developments, realization of HOTIs in solids has actively been pursued\cite{21-27}. In addition, higher-order topological phases in classical systems have also been studied intensively. These systems are advantageous compared with solids from the viewpoints of simplicity of experimental set-up and high tunability of parameters, which enable us to implement desirable structures to realize higher-order topological phases. Indeed, the higher-order topological phases were realized in mechanical systems\cite{28,29}, photonic crystals\cite{30}, phononic crystals\cite{31,32}, and electrical circuits\cite{33,34}.

In this paper, we propose a realization of higher-order topological phases in a spring-mass model. Spring-mass models, composed of a periodic alignment of springs and mass points, serve as a simple platform to realize topological phenomena governed by Newton’s equation of motion\cite{35-40}. Indeed, topological phases accompanied by characteristic boundary states, such as Chern insulators\cite{36,37}, nodal-line semimetals\cite{38}, and Weyl semimetals\cite{39}, were proposed. These results motivated us to study the higher-order topological phase in a spring-mass model.

As a concrete example, we study the spring-mass model on a breathing kagome lattice. To demonstrate the realization of the higher-order topological phase, we present the characterization of topological phases by the $Z_3$ Berry phase, and show the existence of a corner states. We further propose how to observe the corner states experimentally. To this aim, we study the dynamics under the external force and show the characteristic behavior of corner states distinct form the bulk states.

The rest of this paper is organized as follows. In Sec. II, we introduce the spring-mass model on a breathing kagome lattice and explain how to describe the motion of mass points in this model. In Sec. III, we show the band structures under the periodic boundary condition. We also show that the $Z_3$ Berry phase characterizes the topological phases. In Sec. IV, we elucidate the existence of the corner states in this model under the fixed boundary conditions with a triangle arrangement. In Sec. V, we demonstrate that the corner states can be observed by the forced vibration. In Sec. VI, we present a summary of this paper. In Appendix A, we explain the details of the $Z_3$ Berry phases defined in a momentum space. In Appendix B, we show the band structure on the cylinder geometry. In Appendix C, we see the $Z_3$ Berry phase for lower five bands which accounts for the corner states under the weak tension. In Appendix D, we show the existence of the corner state under the fixed boundary conditions with a parallelogram arrangement.

II. A SPRING-MASS MODEL ON A BREATHING KAGOME LATTICE

We consider a system consists of the mass points aligned on a breathing kagome lattice and springs connecting the masses (Fig. 1). The spring constants of red springs on upward triangles are $t_a$, and those of blue springs on downward triangles are $t_b$. We label a red spring (a blue spring) as $\alpha = (a, b)$, respectively. Henceforth, we set the mass as unity for simplicity.

We set unit vectors, $\hat{a}_1$ and $\hat{a}_2$, as

$$\hat{a}_1 = (R_a + R_b) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (1a)$$
\[ \dot{\mathbf{a}}_2 = (R_a + R_b) \left( -\frac{1}{2} \right), \tag{1b} \]

where \( R_a \) and \( R_b \) denote lengths of red springs and blue springs in equilibrium, respectively. To make the model in equilibrium, we have to take into account a balance of forces, which is satisfied for

\[ t_a(R_a - l_a) = t_b(R_b - l_b), \tag{2} \]

where \( l_a \) is a natural length of a spring. In the following, we set \( l_a = l_b = 1 \).

Let us see how to describe the motion of mass points. Dynamical variables are \( \vec{x}_{R,p} \), which are displacements of the mass points from the position in equilibrium. Here, a pair of indices \( R \) and \( p \) specifies the lattice points; \( R \) denotes the position of the unit cell, and \( p = 1, 2, 3 \) denotes a sublattice. For later use, we introduce \( \vec{r}_p \), which denotes the position of sublattice \( p \) within the unit cell. The Lagrangian describing the motion of masses is written as

\[ \mathcal{L} = \frac{1}{2} \sum_{R} \sum_{p} \left( \dot{x}_R^\mu_{R,p} \right)^2 \]

\[-\frac{1}{2} \sum_{\langle R, p; R', q \rangle} (x^\mu_{R,p} - x^\mu_{R',q}) \gamma^\mu\nu_{R+p-R'-q} (x^\nu_{R,p} - x^\nu_{R',q}), \tag{3} \]

where \( \langle R, p; R', q \rangle \) means nearest-neighbor pairs of the mass points, \( \mu, \nu = x, y \) are directions in a two-dimensional space, and the explicit form of \( \gamma_{R+p-R'-q}^\mu\nu \) is

\[ \gamma_{R+p-R'-q}^\mu\nu = t_\alpha \left\{ 1 - \eta_\alpha \delta_{\mu\nu} + \eta_\alpha \hat{X}^\mu \hat{X}^\nu \right\}. \]

Here, \( \hat{X} = R + \vec{r}_p - \vec{R}' - \vec{r}_q \) and \( \hat{X}^\mu = X^\mu / |\hat{X}| \). The parameter \( \eta_\alpha \) is defined as

\[ \eta_\alpha = \frac{l_\alpha}{R_\alpha}, \tag{4} \]

which denotes the strength of tensions of springs. Notice that the index \( \alpha \) in \( \gamma_{R+p-R'-q}^\mu\nu \) is naturally determined once we specify the nearest-neighbor pair \( \langle R, p; R', q \rangle \).

The first term of Eq. (3) is the kinetic energy, while the second term is the potential energy of the springs.

Then, the Euler-Lagrange equation for \( x^\mu_{R,p} \)

\[ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}_R^\mu_{R,p}} \right) - \frac{\partial \mathcal{L}}{\partial x^\mu_{R,p}} = 0, \tag{5} \]

can be written as a coupled differential equation,

\[ \ddot{\vec{x}} + D \vec{x} = 0, \tag{6} \]

where \( \vec{x} \) is the column vector obtained by aligned \( x^\mu_{R,p} \).

The matrix \( D \),

\[ (D)_{R,p; R', q, \mu, \nu} = \left( \sum_{\langle \vec{R}, p; \vec{R}', q \rangle} \gamma_{\vec{R}+\vec{r}_p-\vec{R}'-\vec{r}_q}^\mu\nu \delta_{\vec{R}, \vec{R}'} \delta_{p, q} - \gamma_{\vec{R}+\vec{r}_p-\vec{R}'-\vec{r}_q}^\mu\nu \right), \tag{7} \]

is a real-space dynamical matrix. Assuming the mass points oscillate with a frequency \( \omega \), we can write \( x^\mu_{R,p} = e^{i\omega t} \xi^\mu_{R,p} \). Substituting this into Eq. (6), we obtain

\[ -\omega^2 \xi + D \xi = 0. \tag{8} \]

Equation (8) is an eigenvalue equation of the matrix \( D \) whose basis is \( \xi^\mu_{R,p} \) and eigenvalue is \( \omega^2 \). This equation describes the motion of masses in a spring-mass model in a real-space.

Under the periodic boundary condition, the translational invariance of the system results in the eigenvalue equation in the momentum space. First, we apply the Fourier transformation

\[ x^\mu_{R,p} = \frac{1}{N} \sum_k e^{i\vec{k} \cdot \vec{R}} u^\mu_{\vec{k},p}. \tag{9} \]

Substituting Eq. (9) into Eq. (3), the Lagrangian is written as

\[ \mathcal{L} = \frac{1}{N} \sum_k \left\{ \frac{1}{2} \sum_p \dot{u}^\mu_{\vec{k},p} \dot{u}^\mu_{\vec{k},p} - \frac{1}{2} \sum_{pq} \Gamma_{\vec{k},\vec{k}',\mu,\nu}^{\mu,\nu} (\vec{k}' - \vec{k}, \mu, \nu) \right\}, \tag{10a} \]

with

\[ \Gamma_{\vec{k},\vec{k}',\mu,\nu}^{\mu,\nu} = \sum_{\vec{R}} (D)_{\vec{R},\vec{R}'; \mu, \nu} e^{i\vec{k} \cdot \vec{R}}. \tag{10b} \]

The matrix \( \Gamma \) is called a momentum-space dynamical matrix. The dimension of the momentum-space dynamical matrix in this model is six since there are three
sublattices and two spatial coordinates. Under the basis \(\left(u^x_{k,1}, u^x_{k,2}, u^x_{k,3}, u^y_{k,1}, u^y_{k,2}, u^y_{k,3}\right)^T\), the explicit form of \(\Gamma(\hat{k})\) is

\[
\Gamma = \begin{pmatrix}
D_1 & -\gamma_{12}(\hat{k}) & -\gamma_{13}(\hat{k}) \\
-\gamma_{12}(\hat{k}) & D_2 & -\gamma_{23}(\hat{k}) \\
-\gamma_{13}(\hat{k}) & -\gamma_{23}(\hat{k}) & D_3
\end{pmatrix},
\]

(11a)

with

\[
\begin{align*}
\gamma_{12}(\hat{k}) &= \gamma_{12a} + e^{-i(\alpha_1 + \alpha_2)\cdot\hat{k}}\gamma_{12b}, \\
\gamma_{13}(\hat{k}) &= \gamma_{13a} + e^{-i\alpha_2\cdot\hat{k}}\gamma_{13b}, \\
\gamma_{23}(\hat{k}) &= \gamma_{23a} + e^{-i\alpha_2\cdot\hat{k}}\gamma_{23b},
\end{align*}
\]

(11b)-(11d)

and

\[
\begin{align*}
D_1 &= \gamma_{12a} + \gamma_{13a} + \gamma_{12b} + \gamma_{13b}, \\
D_2 &= \gamma_{12a} + \gamma_{23a} + \gamma_{12b} + \gamma_{23b}, \\
D_3 &= \gamma_{23a} + \gamma_{13a} + \gamma_{23b} + \gamma_{13b}.
\end{align*}
\]

(11e)-(11g)

Here, \(\gamma_{12a}, \gamma_{13a},\) and \(\gamma_{23a}\) are defined as

\[
\begin{align*}
\gamma_{12a} &= \eta_a \left\{ (1 - \eta_a) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \eta_a \left( \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \frac{1}{2} \\ 1 \end{pmatrix} \right) \right\}, \\
\gamma_{13a} &= \eta_a \left\{ (1 - \eta_a) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \eta_a \left( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \right\}, \\
\gamma_{23a} &= \eta_a \left\{ (1 - \eta_a) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \eta_a \left( \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \frac{1}{2} \\ 1 \end{pmatrix} \right) \right\}.
\end{align*}
\]

(11h)-(11j)

The Euler-Lagrange equation for \(u^\mu_{k,p}\) is written as

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial u^\mu_{k,p}} \right) - \frac{\partial \mathcal{L}}{\partial u^\mu_{k,p}} = 0.
\]

(12)

Writing the time-independence of \(u^\mu_{k,p}\) as

\[
u^\mu_{k,p} = e^{-i\omega t} \phi^\mu_k(\hat{k}),
\]

we obtain the Euler-Lagrange equation reduced to the eigenvalue equation

\[
-\omega^2 \phi^\mu_k(\hat{k}) + \sum_q \Gamma_{pq}^{\mu\nu}(\hat{k}) \phi^\nu_q(\hat{k}) = 0.
\]

(15)

By solving the eigenvalue equation (15), we obtain the dispersion relation, which we will discuss in the next section.

Before closing this section, we address the correspondence between the spring-mass model and the tight-binding model. In fact, the spring-mass model is reduced to two copies of the tight-binding model if we set \(\eta_a = 0\), \(\eta_b = 0\), i.e., the tension is infinitely strong, since the off-diagonal parts of the matrix \(\gamma_{ij\alpha}\) vanish [see Eqs. (11h)-(11j)].

### III. BULK PROPERTIES

In this section, we present the properties of this model under the periodic boundary condition. To be specific, we investigate the dispersion relation and the bulk topological invariant. First, we show the band structures obtained by diagonalizing the momentum-space dynamical matrix [Fig. 2(b)-(e)]. Here, the horizontal axis denotes the high-symmetry lines in the Brillouin zone with \(\Gamma, K,\) and \(M\) denoting the high-symmetry points [Fig. 2(a)].

There are six bands; for \(\eta_a = 0\), we see three bands, each of which is doubly degenerate. For \(\eta_a \neq 0\), two-fold degeneracy is lifted because of the coupling between transverse waves and longitudinal waves [Fig. 2(b) and (c)]. Moreover, there are no flat bands unless \(\eta_a = 0\). These are unique characters of the spring-mass model which are different from the tight-binding model.

Next, we see bulk topological properties by calculating the \(Z_3\) Berry phase. The \(Z_3\) Berry phase for the lowest
precisely, \( \Delta \) relates \( t \in \Delta \) \( \in \mathbb{Z} \). The lower second and third bands is closed. To map out the difficulties in the gauge choice. Note that we employ the method introduced in Ref. 42 to avoid the Berry phase with \( \nu \) details, see Appendix A.

\[ \nu \times \nu \text{ Berry connection matrix:} \]
\[ \vec{A}^\nu(\vec{k}) = i \Phi^\nu(\vec{k}) \frac{\partial}{\partial \vec{k}} \Phi^\nu(\vec{k}), \]
\[ \Phi^\nu(\vec{k}) = [\vec{\phi^\nu_1}(\vec{k}), \ldots, \vec{\phi^\nu_{\nu}}(\vec{k})]. \]

is the 6 \( \times \nu \) matrix composed of the eigenvectors of the momentum-space dynamical matrix \( \Gamma(\vec{k}) \), represented by \( \vec{\phi^\nu}(\vec{k}) \). Then, the Berry phase for the lowest \( \nu \) bands is expressed as
\[ \gamma^\nu = \int_{L_1} \text{Tr} [\vec{A}^\nu(\vec{k})] \cdot d\vec{k}. \]

where \( L_1 \) is a path in a momentum space, \( L_1 : \Gamma_1 \to \Gamma_2 \to \Gamma_3 \). Note that the Berry phase in Eq. (18) is the momentum-space analogue of the one defined with respect to the local twist parameters of the Hamiltonian 18.

The Berry phase is quantized in \( \mathbb{Z}_3 \),
\[ \gamma^\nu = \frac{2\pi k}{3} \quad (\text{mod } 2\pi), \]

with \( k = 0, 1, 2 \), because of the \( C_3 \) symmetry. For further details, see Appendix A.

In Fig. 3, we show the numerical results for the \( \mathbb{Z}_3 \) Berry phase with \( \nu = 2 \). For the numerical calculation, we employ the method introduced in Ref. 42 to avoid the difficulties in the gauge choice. Note that \( \mathbb{Z}_3 \) Berry phase cannot be defined for \( \eta_a = 1 \) because the gap between lower second and third band is closed. To map out the \( \mathbb{Z}_3 \) Berry phase in the parameter space, we introduce \( \Delta \in [-1, 1] \), describing the degrees of breathing. More precisely, \( \Delta \) relates \( t_a \) and \( t_b \) as
\[ t_a = 1 + \Delta, \]
and
\[ t_b = 1 - \Delta. \]

For \( \Delta = 1 \) \((-1) \), the system is reduced to a set of isolated triangles of mass points connected by red (blue) springs, respectively. We note that \( t_b = 0 \) \((t_a = 0) \) holds for \( \Delta = 1 \) \((-1) \), respectively. For \( \Delta = 0 \), the mass points form an isotropic kagome lattice because \( t_a = t_b \) holds.

We see in Fig. 3(b) that there are three phases where the \( \mathbb{Z}_3 \) Berry phase takes \( \gamma = 0, \frac{2\pi}{3}, \) and \( \frac{4\pi}{3} \), respectively. This is a sharp contrast to the tight-binding model where there are only two phases \( (\gamma = 0, \frac{2\pi}{3}, \) and originates from the coupling between longitudinal waves and transverse waves.

In this section, we demonstrate that system hosts corner states due to topological properties in the bulk. This bulk-corner correspondence serves as a direct evidence of the higher-order topological phase. Specifically, considering a triangle arrangement (Fig. 5), we numerically show that the corner states emerge for \( \gamma^2 = 4\pi/3 \) while they do not for \( \gamma^2 = 0 \).

Before going to the numerical results, we consider two limits, i.e., \( t_b = 0 \) \([\text{Fig. 5(b)}] \), and \( t_a = 0 \) \([\text{Fig. 5(b)}] \), to gain an insight of the boundary states. Note that, in these limits, the equilibrium condition of Eq. (2) is inevitably broken. Nevertheless, it is helpful to consider these limits, as we explain below. For \( t_a = 0 \), \( t_b \neq 0 \)

\[ \mathbb{Z}_3 \text{ Berry phase takes } \gamma = 0, \frac{2\pi}{3}, \text{ and } \frac{4\pi}{3}, \text{ respectively.} \]
FIG. 5. (Color online) Schematic figures for the spring-mass model in a triangle arrangement for (a) $t_a \neq 0, t_b \neq 0$, (b) $t_a = 0, t_b \neq 0$, and (c) $t_a \neq 0, t_b = 0$. Black lines represent walls. (d) Eigenfrequencies as a function of $t_a/t_b$ for $\eta_a = 0.1$ with a triangle arrangement. (e) Kinetic-energy distribution in the real space of the corner states for $t_a/t_b = 0.4$. We write the number of small triangles along the edge as $L$; the total number of masses is $3L(L + 1)/2$.

[Fig. 5(c)], there exist three isolated mass points connected only to the wall, at three corners of the triangle. This configuration supports the eigenmodes localized at the corners. In contrast to this, there are no isolated mass points for $t_a \neq 0$, $t_b = 0$ [Fig. 5(c)]. From these, we expect that three corner states exist for $t_a \ll t_b$, while they do not for $t_b \ll t_a$.

Keeping this observation in mind, let us move on to the numerical results. In Fig. 5(d), we plot the energy spectra as a function of $t_a/t_b$ for $\eta_a = 0.1$ and $L = 20$. We see the existence of the in-gap states for a certain region of $t_a/t_b < 1$, encircled by a red ellipse in Fig. 5(d). Note that these corner states have quasi-three-fold degeneracy for $\eta_a \neq 0$. We also note that, even for $t_a/t_b < 1$, the corner states may be energetically buried in the bulk or edge states, thus they cannot be seen in the energy spectra of Fig. 5(d); for the edge state of the present model, see Appendix B.

Figure 5(e) indicates that the in-gap states observed above corresponds to the corner states. This figure shows the kinetic energy distribution

$$\bar{E}_{R,p} = \frac{1}{4N_c} \sum_{\ell=1}^{N_c} \left( \omega_\ell I_{R,p}^\ell \right)^2,$$

(21)

for these quasi-degenerate in-gap states. Here, $I_{R,p}^\ell = \sqrt{\left(\xi_{R,p}^{\ell,x}\right)^2 + \left(\xi_{R,p}^{\ell,y}\right)^2}$ is the amplitude of the displacement of the mass point $(\vec{R}, p)$ of the mode $\ell$. The summation in Eq. (21) is taken over the $N_c$-fold (quasi-)degenerate states. In Fig. 5(e), we can clearly see that the kinetic energy distribution is localized at the corners, manifesting the existence of the higher-order topological phase in the present model.

Combining the above results and the fact that $\mathbb{Z}_3$ Berry phase $\gamma^2$ takes $4\pi/3$ for $t_a/t_b < 1$ (see Fig. 3), we can confirm that the bulk-corner correspondence holds for our spring-mass model. This is a direct consequence of the adiabatic connection argument we have presented in Sec. III.

In the above we have focused on the case where $\eta$ is small ($\eta = 0.1$). We can also observe the bulk-corner correspondence for $\eta_a = 0.9$ where the in-gap states are between the third and the fourth bands (see Appendix B). We also note that the corner states are also found in a parallelogram arrangement, which can also be understood in the same adiabatic connection argument (see Appendix C)

V. FORCED VIBRATION

FIG. 6. (Color online) (a) Schematic figure for the experimental setup for the forced vibration. The force $f(t)$ is added to the mass point of the top corner of the triangle (encircled by a green circle). (b) $\omega_0$ is near a corner mode, and (c) $\omega_0$ is away from a corner mode. In panels (b) and (c), the amplitudes $A_{R,p}$ at $t = 1000$ are plotted.

In this section, we point out that corner states emerging our spring-mass model can be experimentally observed by analyzing a forced vibration. The forced vibration is caused by an external force $f(t)$. The equation of motion with an external forces is

$$\ddot{x} + D\dot{x} = \bar{f}(t),$$

(22)
where $(f)_{\vec{r},p,\mu}(t) = f^\mu_{\vec{r},p}(t)$ is an external force which is added to the mass at $\vec{R} + \vec{r}$ along the $\mu$ direction.

Consider the case where the external force is given by $f^\mu_{\vec{r},p}(t) = F^\mu_{\vec{r},p} \cos(\omega_0 t)$, i.e., it oscillates in time with the frequency $\omega_0$. We further set $F^\mu_{\vec{r},p}$ such that it has amplitudes only at the top corner [Fig. 6(a)]. Then, as a particular solution of the equation of motion of Eq. (22), $\xi^\mu_i$, which governs the behaviors of the long time scale, we get

$$\xi^\mu_i = \begin{cases} \sum_i U_{ij} E_j \cos \omega_0 t & \text{for } \omega_i \neq \omega_0, \\ \frac{\sum_i U_{ij} E_j}{2\omega_0} t \sin \omega_0 t & \text{for } \omega_i = \omega_0, \end{cases} \quad (23)$$

where $U$ is a matrix obtained by aligning the eigenvectors of the equation of motion in the absence of $f(t)$ [i.e., Eq. (8)], $i$ specifies the eigenmode, $j$ is the abbreviation of the position of the mass point $\vec{R} + \vec{r}$, and the direction of the motion $\mu$, and $\omega_i$ is the eigenfrequency of the $i$th mode. From Eq. (23), we see the resonance occurs at $\omega_i = \omega_0$. This indicates that for $\omega_0$ being close to the eigenfrequency of the corner state, one obtains the large vibration amplitude only at the corner; in contrast, when it is close to the eigenfrequency of the bulk state, the vibration spreads over the bulk. Thus, by looking at the time evolution of the forced vibration upon changing $\omega_0$, one can find whether the corner states exist or not.

To demonstrate this, we numerically solve the equation of motion using the Euler method on a triangle arrangement. In the Euler method, the time evolution is described as

$$\begin{pmatrix} \dot{x}(t + \Delta t) \\ \dot{x}(t + \Delta t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -D & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix} \Delta t$$

where $\Delta t$ is a small time step. $D$ is given in Eq. (22). For the numerical simulations, we set $\eta_a = 0.9$, $t_a/t_b = 0.1$, and $\Delta t = 0.0001$. An initial state is set as $x^\mu_{\vec{R},p}(0) = 0$ and $\dot{x}^\mu_{\vec{R},p}(0) = 0$, i.e., the system is in equilibrium. After running a simulation to $t = t_{\text{max}} = 1000$, we observe the amplitudes of vibration

$$A_{\vec{R},p} = \sqrt{[x^x_{\vec{R},p}(t_{\text{max}})]^2 + [x^y_{\vec{R},p}(t_{\text{max}})]^2}. \quad (25)$$

The results are shown in Figs. 6(b) and 6(c). As expected, when the $\omega_0$ is near the frequency of the corner state$^{13}$, the large vibration is seen only near the corner, while the vibration propagates in the bulk when the $\omega_0$ is near the frequency of the bulk state.

The above results suggest that the corner state induced by the topological properties in the bulk can be experimentally observed; the resonance frequency corresponds to the frequency of corner states.

VI. SUMMARY

In summary, we have shown that the higher-order topological phase is realized in the spring-mass model on a breathing kagome lattice. We have found that the bulk topological invariant, i.e., the $\mathbb{Z}_3$ Berry phase, characterizes the topologically nontrivial phase. In contrast to the tight-binding model, we have found the topologically nontrivial phase with $\gamma = \frac{4\pi}{3}$, in addition to the phase with $\gamma = \frac{2\pi}{3}$ that corresponds to the two copies of the topological phase in the tight-binding model. This is due to the coupling between longitudinal and transverse modes, inherent in the spring-mass model. We have also found that the characteristic corner states appear under the fixed boundary condition, both in the triangle and in the parallelogram arrangements. Furthermore, we have proposed that the corner states can be detected experimentally through a forced vibration. By the numerical simulation, we have found that the corner-selective vibration is observed when the external frequency is close to that of the corner modes.

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Appendix A: $\mathbb{Z}_3$ Berry phase in a momentum space

![Diagram](image)

FIG. 7. (Color online) (a) The triangle area whose vertices are $\Gamma_1$, $\Gamma_2$, $\Gamma_3$. (b,c) Schematic figures for the operations which keeps Hamiltonian invariant. (b) The $120^\circ$ rotation of the triangle area in a momentum space. (c) The translation of the triangle area in a momentum space. Combining these two operations, the point $\Gamma_1$ is transformed to $\Gamma_{i+1}$ with $\Gamma_4 := \Gamma_1$. Consequently, the path $L_i$ is transformed to $L_{i+1}$ with $L_4 = L_1$.

In this Appendix, we introduce the $\mathbb{Z}_3$ Berry phase defined in a momentum space. The key idea originates from the quantized Berry phase with respect to the local twists of the Hamiltonian$^{18–20,44–53}$. Such a Berry phase has been used as a topological order parameter for various topological phases, especially in correlated systems such as spin systems$^{44–52}$. The Berry phase is quantized due to...
symmetries, e.g., time-reversal symmetry, inversion symmetry and discrete rotational symmetry. Recently, it is also used to characterize the HOTI phases\textsuperscript{19,53}.

Here we show that such a Berry phase can also be defined in the momentum-space representation, taking a breathing kagome lattice tight-binding model as an example. Extension of the $\mathbb{Z}_3$ Berry phase in a momentum space is of importance from the viewpoint of computational costs. Namely, to calculate the quantized Berry phase with respect to the local twists of the Hamiltonian, one has to calculate the many-body ground state wave function, while there is the phase factor from the local twist. On the other hand, the single-particle eigenfunctions are enough to calculate the $\mathbb{Z}_3$ Berry phase in a momentum space, thus we can save computational costs when dealing with non-interacting quantum systems and classical systems.

In the tight-binding model on a breathing kagome lattice, the Hamiltonian is invariant under the three-fold rotation in the momentum space (Fig. 7). We define this operator as $U$. The Hamiltonian has a symmetry which is expressed as,

$$UH(k_i)U^{-1} = H(C_3k_i), \quad (A1)$$

with $k_i \in L_i$ and $L_i$ being a path in a momentum space, $L_i : \Gamma_i \rightarrow G \rightarrow \Gamma_{i+1}$. Here, we have supposed that the momentum $k_i$ is mapped to $C_3k_i \in L_{i+1}$ by applying $C_3$ rotation. This relation indicates that the Berry phases $\gamma(L_i)$ computed along each path takes the same value,

$$\gamma(L_1) = \gamma(L_2) = \gamma(L_3). \quad (A2)$$

In addition, the integral along the path $L_1 + L_2 + L_3$ is equal to zero,

$$\sum_{i=1}^3 \gamma(L_i) = 0 \mod 2\pi. \quad (A3)$$

From (A2) and (A3),

$$\gamma \equiv \gamma(L_i) = \frac{2\pi k}{3} \mod 2\pi, \quad (A4)$$

where $k$ is 0, 1 or 2. The same argument can be applied to a spring-mass model on a breathing kagome lattice by replacing the Hamiltonian $H$ with the momentum-space dynamical matrix $\Gamma$, since $\Gamma$ preserves a three-fold rotational symmetry and a translational symmetry. Note that, in the spring-mass models, the rotation is applied not only to the momentum and sublattice degrees of freedom, but also to directions of the displacement ($\mu = x, y$).

In fact, the $\mathbb{Z}_3$ Berry phase in a momentum space is equivalent to the local-twist Berry phase discussed in Ref. 18. To see this, we consider the $\mathbb{Z}_3$ Berry phase in a momentum space, where the upward triangle is a unit cell. In this case, there is the phase factor from the Bloch wave vector for hoppings from a site in the upward triangle to a site in the downward triangle. By replacing the factors $e^{-i\vec{k} \cdot \vec{a}_1}$ and $e^{-i\vec{k} \cdot \vec{a}_2}$ with the twisting parameters $e^{i\theta_1}$ and $e^{i\theta_2}$ respectively, we find that the $\mathbb{Z}_3$ Berry phase in a momentum space is same to the local-twist Berry phase for the $1 \times 1$ unit cell of the downward triangle. Similar correspondence also holds for the Su-Schirrer-Heeger model\textsuperscript{54} and a breathing pyrochlore model.

Appendix B: The cylinder of the spring-mass model

In this Appendix, we show the dispersion relations on the cylinder geometry, focusing on the features of the edge states. Here, we set the number of red springs in the axial direction as 20, and we write the momentum in the azimuth direction $k|_l$.

The results are shown in Fig. 8. We see that there exists a mode between the bulk continuum, whose real-space distribution is localized at the edge. Importantly, the edge mode is not energetically connected to the bulk continuum, meaning that the lower-dimensional boundary states, i.e., the corner states, are allowed to exist between the edge modes and the bulk modes.

![Fig. 8](image)

FIG. 8. (Color online) The dispersion relations in the cylinder for (a) $\eta_a = 0$, $t_a/t_b = 0.4$, and (b) $\eta_a = 0.1$, $t_a/t_b = 0.4$. There exist edge modes between the bulk continuum (encircled by red ellipses).

Appendix C: $\mathbb{Z}_3$ Berry phase for the lowest five bands

In this Appendix, we show the result for $\gamma^5$. The reason for calculating $\gamma^5$ is that there exist the corner states between the third and the fourth band for $\eta_a \sim 1$, contrary to the case of $\eta_a \sim 0$. To be concrete, we show the bulk band structure and the energy spectra in a triangle arrangement at $\eta = 0.9$ in Figs. 10(a) and 10(b), respectively. We find that the band gap exists between the third and the fourth bands and between the fifth and the sixth bands, in contrast to the case of $\eta_a \sim 0$. Correspondingly, the corner states appear for a certain region for $t_a < t_b$, while there are no corner states for $t_a > t_b$.
which is again inferred from the adiabatic connection argument.

We plot $\gamma^5$ in Fig. 10(c), showing that $\mathbb{Z}_3$ is independent of $\eta_a$ and the phase translation occurs at $\Delta = 0$ i.e. $t_a = t_b$. Similarly to $\gamma^2$ discussed in Sec. III, $\gamma^5 = 0$ indicates that the system is adiabatically connected to the decoupled triangles with red springs [Fig. 4(a)], while $\gamma^5 = 4\pi/3$ indicates that the system is adiabatically connected to the decoupled triangles with red springs [Fig. 4(b)].

Appendix D: Parallelogram arrangement under the fixed boundary condition

In this Appendix, we show the results for a parallelogram arrangement (Fig. 9). In this arrangement, we write the number of small triangles along the edge as $L$. Then, the number of total masses is $3L^2 + 4L + 1$.

Considering the limiting cases of $t_a = 0$ and $t_b = 0$, there exists an isolated mass on the right-top and left-bottom, respectively [Figs. 9(b) and 9(c)]. This means that the system is expected to have corner states not only for $t_a < t_b$ but also $t_a > t_b$, in contrast to the triangle arrangement.

In Fig. 9(d), we plot the energy spectra as a function of $t_a/t_b$ for $L = 20$ and $\eta_a = 0.1$. As expected, we see the in-gap states for both $t_a < t_b$ and $t_a > t_b$. Looking at the spatial distribution of the kinetic energy of the in-gap states for $t_a < t_b$, we find that it is localized at the left-bottom corner [Fig. 9(e)], which is again inferred from the adiabatic connection argument.

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