A NOTE ON AN ORDER BETWEEN OBJECT-ORIENTED SOFT CONCEPTS IN A SOFT CONTEXT

Won Keun Min
Department of Mathematics, Kangwon National University, Chuncheon 24341, Korea.
ORCID: 0000-0002-3439-2255

Abstract:
The purpose of this work is to study the algebraic structure in the family of all the \(m\)-concepts, so we introduce the notion of an order in the set of all \(m\)-concepts and show that the ordered set is a complete lattice. And, we discover what is the condition for the isomorphic relation between two \(m\)-concept lattices in a given soft context.

1. INTRODUCTION

Formal concept analysis [10] was introduced by Wille, which is an important theory for the research of information structures induced by a binary relation between the set of attributes and objects attributes. The three basic notions of FCA are formal context, formal concept, and concept lattice. A formal context is a kind of information system, which is a tabular form of an object-attribute value relationship [2, 3, 9]. A formal concept is a pair of a set of objects as called the extent and a set of attributes as called the intent. The set of all formal concepts together with the order relation forms a complete lattice called the concept lattice [9,10]. In order to deal complicated problems, Molodtsov introduced the concept of soft set in [8]. The operations for the soft set theory was introduced by Maji et al. in [4]. In [1], Ali et al. proposed new operations modified some concepts introduced by Maji. We have formed a soft context by combining the concepts of the formal context and the soft set defined by the set-valued mapping in [6]. Additionally, we introduced and studied the new concepts named soft concepts and soft concepts lattices. In [11], Yao introduced a new concept called an object oriented formal concept in a formal context by using the notion of approximation operations.

And also, by using the two operation, we investigated the new concept of \(m\)-concepts related closely the object oriented concept in formal context in [7].

In this paper, we introduce the notion of an order in the set of all \(m\)-concepts and show that the ordered set is a complete lattice. And, we discover what is the condition for the isomorphic relation between two \(m\)-concept lattices in a given soft context.

2. PRELIMINARIES

A formal context is a triplet \((U, V, I)\), where \(U\) is a non-empty finite set of objects, \(V\) is a nonempty finite set of attributes, and \(I\) is a relation between \(U\) and \(V\). Let \((U, V, I)\) be a formal context. For a pair of elements \(x \in U\) and \(y \in V\), if \((x, y) \in I\), then it means that object \(x\) has attribute \(y\) and we write \(xIy\). The set of all attributes with a given object \(x \in U\) and the set of all objects with a a given attribute \(y \in V\) are denoted as the following [9,10]:

\[
x^* = \{y \in V | xIy\}; \quad y^* = \{x \in U | xIy\}.
\]

And, the operations for the subsets \(X \subseteq U\) and \(Y \subseteq V\) are defined as:

\[
X^* = \{y \in V | \text{for all } x \in X, xIy\}; \quad Y^* = \{x \in U | \text{for all } y \in Y, xIy\}.
\]

In a formal context \((U, V, I)\), a pair \((X, Y)\) of two sets \(X \subseteq U\) and \(Y \subseteq V\) is called a formal concept of \((U, V, I)\) if \(X = Y^*\) and \(B = Y^*\), where \(X\) and \(Y\) are called the extent and the intent of the formal concept, respectively.

Let \(U\) be a universe set and \(E\) be a collection of properties of objects in \(U\). We will call \(E\) the set of parameters with respect to \(U\).

A pair \((F, E)\) is called a soft set [8] over \(U\) if \(F\) is a set-valued mapping of \(E\) into the set \(P(U)\) of all subsets of the set \(U\), i.e.,

\[
F : E \rightarrow P(U).
\]

In other words, for \(a \in E\), \(F(a)\) may be considered as the set of all elements of the soft set \((F, E)\) at \(a\).

Let \(U = \{z_1, z_2, \ldots, z_m\}\) be a non-empty finite set of objects, \(E = \{e_1, e_2, \ldots, e_n\}\) a non-empty finite set of attributes, and \(F : E \rightarrow P(U)\) a soft set. Then the triple \((U, E, F)\) is called a soft context [6].

And, in a soft context \((U, E, F)\), we introduced the following mappings:

For each \(Z \subseteq P(U)\) and \(Y \subseteq P(E)\),

1. \(F^+ : P(E) \rightarrow P(U)\) is a mapping defined as \(F^+(Y) = \bigcap_{y \in Y} F(y)\);
2. \(F^- : P(U) \rightarrow P(E)\) is a mapping defined as \(F^-(Z) = \{a \in E | Z \subseteq F(a)\}\);
3. \(\Psi : P(U) \rightarrow P(U)\) is an operation defined as \(\Psi(Z) = F^+ F^-(Z)\).

Then \(Z\) is called a soft concept [6] in \((U, E, F)\) if \(\Psi(Z) = F^+ F^-(Z) = Z\). The set of all soft concepts is denoted by \(sC(U, E, F)\).

In [7], we introduced the notion of \(m\)-concepts which is independent of the notion of soft concepts to each other as the following: For each \(X \in P(U)\),

1. \(F : P(A) \rightarrow P(U)\) is a mapping defined as \(F(C) = \bigcup_{c \in C} F(c)\);
(2) \( F : P(U) \rightarrow P(A) \) is a mapping defined as \( F(X) = \{ c \in A : F(c) \subseteq X \} \).

**Theorem 2.1** ([7]). Let \((U, A, F)\) be a soft context, \(S, T \subseteq U\) and \(B, C \subseteq A\). Then we have:

1. If \( S \subseteq T \), then \( F(S) \subseteq F(T) \); if \( B \subseteq C \), then \( F(B) \subseteq F(C) \);
2. \( F \left( F(S) \right) \subseteq S \); \( F \left( F(B) \right) \subseteq B \);
3. \( F(S \cap T) = F(S) \cap F(T), F(B \cup C) = F(B) \cup F(C) \);
4. \( F(S) = F \circ F(S), F(B) = F \circ F(B) \).

Let \( \Phi : P(U) \rightarrow P(U) \) be an operation defined by \( \Phi(X) = \overline{F} F(X) \) for \( X \in P(U) \).

Then for \( X \in P(U) \), \( X \) is called an \( m \)-concept (or object oriented soft concept) [7] in \((U, A, F)\) if \( \Phi(X) = \overline{F} F(X) = X \).

The set of all \( m \)-concepts is denoted by \( m(U, A, F) \).

### 3. MAIN RESULTS

First, for a soft context \((U, A, F)\) and \( C \subseteq A \), we consider a set-valued mapping \( F_C : C \rightarrow P(U) \) defined by \( F_C(c) = F(c) \) for all \( c \in C \). Then the set-valued mapping \( F_C \) induces a soft set \((F_C, C)\) and a soft context \((U, C, F_C)\).

Then we consider the operations \( F_C, \overline{F}_C, \Phi_C \) as the following:

\[ F_C : P(C) \rightarrow P(U) \text{ is a mapping defined by } F_C(B) = \bigcup_{B \in F(C)} b \text{ for each } B \in P(C), \]
\[ \overline{F}_C : P(U) \rightarrow P(C) \text{ is a mapping defined by } \overline{F}_C(X) = \{ c \in C : F_C(c) \subseteq X \} \text{ for each } X \in P(U). \]

An associated operation \( \Phi_C : P(U) \rightarrow P(U) \) is also well defined by for each \( X \in P(U) \), \( \Phi_C(X) = \overline{F}_C \overline{F}_C(X) \).

**Lemma 3.1.** Let \((U, A, F)\) be a soft context, \( C \subseteq A \) and \( X \subseteq U \). Then:

1. \( F_C(X) \subseteq \overline{F}_C(X) \).
2. \( \overline{F}_C(X) = \overline{F}_C(X) \cap C \).

**Proof.** Obvious.

**Theorem 3.2.** Let \((U, A, F)\) be a soft context, \( X, Y \subseteq U \) and \( B, C, E \subseteq A \). Then we have the following things:

1. If \( X \subseteq Y \), then \( F_C(X) \subseteq F_C(Y) \); if \( B \subseteq E \), then \( F_C(B) \subseteq F_C(E) \);
2. \( F_C F_C(X) \subseteq X \); \( F_C F_C(B) \subseteq B \);
3. \( F_C(X \cap Y) = F_C(X) \cap F_C(Y), F_C(B \cup E) = F_C(B) \cup F_C(E) \);
4. \( \overline{F}_C(X) = \overline{F}_C \overline{F}_C(X), \overline{F}_C(B) = \overline{F}_C \overline{F}_C(B) \).

**Proof.** It is obvious from the notions of \( F_C, \overline{F}_C \) and \( \Phi_C \).

Let \((U, A, F)\) be a soft context, \( X \subseteq P(U) \) and \( C \subseteq A \). Then \( X \) is called \( m \)-concept in \((U, C, F_C)\) if \( \Phi_C(X) = \overline{F}_C \overline{F}_C(X) = X \). The set of all \( m \)-concepts will be denoted by \( m(U, C, F_C) \).

**Theorem 3.3** ([7]). Let \((U, A, F)\) be a soft context. Then we have:

1. \( \Phi(\emptyset) = \emptyset \).
2. \( \Phi(X) \) is an \( m \)-concept.
3. For \( B \subseteq A \), \( \overline{F}(B) \) is an \( m \)-concept.

(4) For \( a \in A \), \( F(a) \) is an \( m \)-concept.

(5) \( X \) is an \( m \)-concept if and only if there is some \( B \subseteq A \) such that \( X = \overline{F}(B) \).

By Theorem 3.3, the next theorem is obviously obtained:

**Theorem 3.4.** Let \((U, A, F)\) be a soft context, \( X \subseteq U \) and \( B, C, E \subseteq A \). Then:

1. \( \Phi_C(\emptyset) = \emptyset \).
2. \( \Phi_C(X) \) is an \( m \)-concept in \((U, C, F_C)\).
3. For each \( B \subseteq C \), \( F_C(B) \) is an \( m \)-concept in \((U, C, F_C)\).
4. For each \( c \in C \), \( F(c) \) is an \( m \)-concept in \((U, C, F_C)\).
5. \( X \) is an \( m \)-concept in \((U, C, F_C)\) if and only if \( X = \overline{F}(C) \) for some \( B \subseteq P(C) \).

**Theorem 3.5.** Let \((U, A, F)\) be a soft context and \( C \subseteq A \). Then:

1. If \( \Phi_C(B_1), \ldots, \Phi_C(B_n) \in \text{Im}(F_C) \), then \( \Phi_C(B_1) \cup \ldots \cup \Phi_C(B_n) \in \text{Im}(F_C) \).

**Proof.** By (1) of Theorem 3.4, it is easily obtained.

(2) For \( B_1, \ldots, B_n \in P(C) \), by (3) of Theorem 3.2, \( \Phi_C(B_1) \cup \ldots \cup \Phi_C(B_n) = \Phi_C(B_1 \cup \ldots \cup B_n) \). Since \( B_1 \cup \ldots \cup B_n \in P(C) \), by (3) of Theorem 3.4, the statement (2) is obtained.

**Theorem 3.6.** Let \((U, A, F)\) be a soft context and \( S_C = \{ F_C(c) \mid c \in C \subseteq A \} \) for the soft set \((F_C, C)\). Then:

1. \( S_C \subseteq m(U, C, F_C) \).
2. For each \( X \in m(U, C, F_C) \), there is \( S_1, S_2, \ldots, S_n \in S_C \) satisfying \( X = S_1 \cup S_2 \cup \ldots \cup S_n \).

**Proof.** By (1) of Theorem 3.4, it is obvious.

(2) By (4) of Theorem 3.4, there is a \( B \in P(C) \) satisfying \( X = \overline{F}(B) \). So, \( X = \overline{F}(B) = \bigcup_{B \subseteq F_C(B)} B \in S_C \).

**Theorem 3.7.** Let \((U, A, F)\) be a soft context. Then for \( C \subseteq A \), \( m(U, C, F_C) \subseteq m(U, A, F) \).

**Proof.** For \( X \in m(U, C, F_C) \), by Theorem 3.4, there is \( B \in P(C) \) satisfying \( X = \overline{F}(B) \). From Lemma 3.1, \( X = \overline{F}(B) = \overline{F}(B) \) for \( B \in P(C) \). From \( m(U, A, F) = \text{Im}(F) \) in [7], it implies \( X \in m(U, A, F) \). So, \( m(U, C, F_C) \subseteq m(U, A, F) \).

Now, we define an order between two \( m \)-soft concepts in \((U, A, F)\) as the following:

**Definition 3.8.** Let \((U, A, F)\) be a soft context and \( X, Y \in m(U, A, F) \).

\[ X \preceq Y \text{ if and only if } X \subseteq Y. \]

\( X \) is called a \( m \)-soft concept and \( Y, X \) is called a \( m \)-concept of \( X \). For the ordered set \((m(U, A, F), \preceq)\), the infimum and supremum are defined by:

\[ X \land Y = \Phi(X \cap Y) \quad X \lor Y = X \cup Y. \]

**Example 3.9.** For \( U = \{1, 2, 3, 4, 5\} \), \( A = \{a, b, c, d, e\} \). Let us consider a soft context \((U, A, F)\) as shown in Table 1.
Then, \((F, A)\) is a soft set as follows:
\[
\begin{align*}
F(a) &= \{1, 2\}; \quad F(b) = \{1, 3\}; \quad F(c) = \{2, 5\}; \\
F(d) &= \{1, 2, 3\}; \quad F(e) = \{1, 2, 5\}.
\end{align*}
\]

For the soft context \((U, A, F)\),
\[
mL(U, A, F) = \Im(F) = \{\Im(C) \mid C \subseteq P(A)\}
\]
\[
= \{\emptyset, \{1, 2\}, \{1, 3\}, \{2, 5\}, \{1, 2, 3\}, \{1, 2, 5\}, \{1, 2, 3, 5\}, U\}.
\]

Hence, \(mL(U, A, F)\) is obtained as shown in the below diagram:

\[
\begin{align*}
&\quad U \\
&\quad \quad \{1, 2, 3, 5\} \\
&\quad \quad \quad \{1, 2, 5\} \\
&\quad \quad \quad \quad \{1, 2\} \\
&\quad \quad \quad \quad \quad \{1, 3\} \\
&\quad \quad \quad \quad \quad \emptyset \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad mL(U, A, F), \text{ where } A = \{a, b, c, d, e\}
\end{align*}
\]

**Theorem 3.10.** Let \((U, A, F)\) be a soft context. Then \((mL(U, A, F), \leq, \wedge, \vee)\) is a complete lattice.

**Proof.** (1) Let \(X, Y \in mL(U, A, F)\). Then from Theorem 3.4, there exist \(B, C \in P(A)\) such that \(\Im(B) = X\) and \(\Im(C) = Y\). By Theorem 3.2, \(\Im(B) \cup \Im(C) = \Im(B \cup C)\) and \(X \cup Y = \Im(B \cup C)\). It implies \(X \cup Y \in mL(U, A, F)\), and so \(X \cup Y = X \cup Y \in mL(U, A, F)\).

(2) For \(X, Y \in mL(U, A, F)\), let \(Z \in mL(U, A, F)\) satisfying \(Z \subseteq X \cap Y\) and \(X \cap Y \subseteq Z\). Then from \(X \cap Y \subseteq Z\), \(\Phi(X \cap Y) \subseteq Z\). Since \(Z \subseteq X \cap Y\), from Theorem 3.4, \(\Phi(Z) \subseteq \Phi(X \cap Y)\). It implies \(Z = \Phi(Z) = \Phi(X \cap Y) = X \cap Y\), and so \(X \cap Y = Z \in mL(U, A, F)\).

The complete lattice \((mL(U, A, F), \leq, \wedge, \vee)\) is called \(m\)-concept lattice (or object oriented soft concept lattice) and simply will be denoted by \(mL(U, A, F)\).

**Definition 3.11.** Let \(mL(U, B, F)\) and \(mL(U, C, G)\) be two \(m\)-concept lattices. \(mL(U, B, F)\) is said to be finer than \(mL(U, C, G)\), which is denoted by \(mL(U, B, F) \leq mL(U, C, G)\) if and only if \(mL(U, C, G) \subseteq mL(U, B, F)\). If \(mL(U, B, F) \leq mL(U, C, G)\) and \(mL(U, C, G) \leq mL(U, B, F)\), then two \(m\)-concept lattices are said to be isomorphic to each other, and denoted by \(mL(U, B, F) \cong mL(U, C, G)\).

**Theorem 3.12.** Let \(mL(U, A, F)\) be an \(m\)-concept lattice. Then for \(C \subseteq A\), \(mL(U, A, F) \leq mL(U, C, F)\).

**Proof.** From Theorem 3.7, we know that \(mL(U, C, F_C) \subseteq mL(U, A, F)\). So, we have \(mL(U, A, F) \leq mL(U, C, F_C)\).

**Theorem 3.13.** Let \((U, A, F)\) be a soft context and \(C \subseteq A\). Then \(mL(U, A, F) \cong mL(U, C, F_C)\) if and only if \(\Im(F) = \Im(F_C)\).

**Proof.** By Theorem 3.5, \(\Im(F) = \Im(F_C)\) if and only if \(mL(U, A, F) = mL(U, C, F_C)\) if and only if \(mL(U, A, F) \cong mL(U, C, F_C)\). So, the theorem is obtained.

**Example 3.14.** As in Example 3.9, let us consider a soft context \((U, A, F)\). For a subset \(C = \{a, b, c\}\) of \(A\), \((U, C, F_C)\) is a soft context. Then we easily find that \(mL(U, C, F_C) = \{\emptyset, \{1, 2\}, \{1, 3\}, \{2, 5\}, \{1, 2, 3\}, \{1, 2, 5\}, \{1, 2, 3, 5\}, U\}\). So, \(mL(U, A, F) \cong mL(U, C, F_C)\). Consequently, \(mL(U, A, F) \cong mL(U, C, F_C)\). The following diagrams are induced by \(A\) and \(C\) respectively.

\[
\begin{align*}
A &= \{a, b, c, d, e\} & C &= \{a, b, c\} \\
&\uparrow & \uparrow \\
&\{1, 2, 3, 5\} & \{1, 2, 3, 5\} \\
&\quad \{1, 2, 5\} & \quad \{1, 2, 5\} \\
&\quad \quad \{1, 2\} & \quad \quad \{1, 2\} \\
&\quad \quad \quad \{1, 3\} & \quad \quad \quad \{1, 3\} \\
&\quad \quad \quad \emptyset & \quad \quad \quad \emptyset
\end{align*}
\]

\[
\begin{align*}
mL(U, A, F) &\cong mL(U, C, F_C)
\end{align*}
\]

**4. CONCLUSION**

We showed that the set of all \(m\)-concepts of a given \(m\)-context together with the order relation between two \(m\)-concepts is a complete lattice, and found what is the condition for the isomorphic relation between two \(m\)-concept lattices. In the next research, we will study the relationships between \(m\)-concept lattices and formal concept lattices.

**ACKNOWLEDGMENTS**

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (No. NRF-2017R1D1A1B03031399).

**REFERENCES**

[1] M. I. Ali, F. Feng, X. Y. Liu, W. K. Min, M. Shabir, On some new operations in soft set theory, Computers and Mathematics with Applications, 57, 2009, 1547–1553.

[2] B. Ganter, R. Wille, Formal Concept Analysis: Mathematical Foundations, Springer, Berlin, 1999.

[3] J. Jin, K. Qin, Z. Pei, Reduction-based approaches towards constructing Galois (concept) lattices, Lecture Notes in Artificial Intelligence, 4062, Springer, Berlin, 2006, 107—113.

[4] P. K. Maji, R. Biswas, A. R. Roy, On soft set theory, Comput. Math. Appl., 45, 2003, 555–562.
[5] Min W. K., Soft sets over a common topological universe, Journal of Intelligent and Fuzzy Systems, 26(5), 2014, 2099–2106.

[6] W. K. Min, Y. K. Kim, Soft concept lattice for formal concept analysis based on soft sets: Theoretical foundations and Applications, Soft Computing, 23(19), 2019, 9657–9668. https://doi.org/10.1007/s00500-018-3532-z

[7] W. K. Min, Y. K. Kim, On Object-oriented Concepts in a Soft Context Defined by a Soft Set, International Journal of Engineering Research and Technology, 12(11), 2019, 1914–1918.

[8] D. Molodtsov, Soft set theory first results, Computers and Mathematics with Applications, 37, 1999, 19–31.

[9] R. Wille, Concept lattices and conceptual knowledge systems, Computers Mathematics with Applications, 23(6–9), 1992, 493–515.

[10] R. Wille, Restructuring the lattice theory: an approach based on hierarchies of concepts, in: I. Rival (Ed.), Ordered Sets, Reidel, Dordrecht, Boston, 1982, 445–470.

[11] Y. Y. Yao, A comparative study of formal concept analysis and rough set theory in data analysis, RSCTC 2004: Rough Sets and Current Trends in Computing, 2004, 59–68.