Corrections to scaling in systems with thermodynamic constraints

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Using thermodynamic arguments treatment it is shown that, independently on whether Fisher renormalization changes the critical exponents near a phase transition in a constrained system or not, new corrections to scaling with correction exponents proportional to the specific heat index $\alpha$ appear. Because of the smallness $\alpha$ for the Ising, the XY, and the Heisenberg universality classes these corrections are dominant and can cause strong crossover effects. It is proven that the appearance of Fisher corrections to scaling is a quite general feature of the systems with constraints.

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The theoretical study of phase transitions in classical spin models has played an important role in the understanding of the key features of critical behaviour in real systems. It is well known that each model like the classical Ising, the XY and the Heisenberg model represents a certain universality class with a specific set of critical exponents. Apart from the spatial dimensionality and the range of the interactions a universality class is characterized by the number of components of the order parameter. In contrast to the case of ideal spin lattice models, real systems are exposed to various kinds of imperfections (lattice defects, impurities, etc.) which can have a significant influence on the true critical behaviour of the system considered. In many practical applications imperfections in the real systems result in a certain thermodynamic constraint. In the recent years, of notable interest were continuum spin fluid models \cite{19,20} which, in particular, are considered as a first step towards the modelling of ferrofluids \cite{21} and adsorption surface phenomena \cite{22}. Several important results, which have both theoretical and experimental interest, were obtained for such models. For example, it was found that, because of the interplay between spin and translational degrees of freedom, the phase diagrams for spin fluids are much more complex \cite{19,20} compared with pure liquids and could lead to the magnetic ordered state both in the gaseous and liquid phases. By applying an external magnetic field one could also shift significantly the locus of a gas-liquid transition \cite{19,20} and change the dynamic properties \cite{19,20}; both static and dynamic properties in the spin fluid models show the differences from pure fluids and/or spin lattice models. Note that in the case of spin liquid models we deal, in fact, with a system with thermodynamic constraint, namely, fixing the density of particles. Therefore, the general question arises how a thermodynamic constraint affects the critical singularities of the system considered. The answer to this question has been partly given a long time ago by M.Fisher \cite{23}. In particular, it was shown that, due to the constraint, the main singularities in the constrained system are renormalized if the critical exponent of the specific heat $\alpha$ is positive or remain the same as in the ‘ideal system’ (without constraint) when $\alpha$ is negative. This effect is known in the literature as ‘Fisher renormalization’, and for many examples of model and real systems with constraints this renormalization of critical exponents has been proven.

In Ref. \cite{22} it was found by Monte Carlo simulations that the critical exponents of a three-dimensional Heisenberg fluid model are in disagreement with the results known previously for the lattice model. Similar findings were later obtained for three- and two-dimensional (3d and 2d) Ising spin fluids \cite{23,24}. We note that the main difference between the cases of 3d Ising and 3d Heisenberg fluids, as systems with an annealed disorder, is that according to the Fisher renormalization \cite{22} the critical exponents have to be renormalized in the first case and remain the same as in the pure system in the second one. However, in both cases systematic deviations from the critical exponents predicted by theory were found in Monte Carlo calculations. We note also that in Refs. \cite{22,23} a weak dependence of universal quantities on a density of particles $n = N/V$ was observed. From these facts we conclude that the critical behaviour in such fluids with internal degrees of freedom are strongly influenced by nonlinear crossover effects which hide the true asymptotic critical behaviour leading to effective critical exponents in power law fits over the restricted temperature regions considered.

In this paper we will focus on the study of continuous spin fluid models and the main problem to be discussed is whether the magnetic transition in a fluid spin model lies in the same universality class as the corresponding transition in the lattice model and, if so, what is the reason for such a strong crossover, observed for these systems in Monte Carlo simulations \cite{22,24}. In this paper we will
follow the scheme of thermodynamic argumentation developed in the original paper of Fisher \[23\] more than thirty years ago. Due to the general arguments used, we believe that our results are of interest also to many other systems with thermodynamic constraints.

### I. GENERAL BACKGROUND

Let us consider the grand canonical ensemble with the thermodynamic potential \( \Omega = \Omega[\mu, h, T] \) of a continuum spin liquid at fixed volume \( V \). The pressure of the system is simply connected with \( \Omega[\mu, h, T] \), namely, \( P = -\frac{\partial \Omega}{\partial V} \), and from the thermodynamics we have the well-known expression

\[
dP = n d\mu + m dh + s dT, \tag{1}
\]

where \( n, m, \) and \( s \) are the particles’ density, the magnetization and entropy per unity of the volume, respectively. The conjugated thermodynamic fields in \( (1) \) are the chemical potential \( \mu \), the external magnetic field \( h \) and the temperature \( T \), respectively. In this study we are interested in the ferromagnetic phase transition of second order with \( m \) being the order parameter.

The first (and the most important) assumption, which has been made by Fisher in Ref. \[23\] and will be also used in our treatment, is that the nature of the phase transition remains ideal if observed at fixed ‘hidden’ thermodynamic force which is directly related in our case to chemical potential \( \mu \). In other words this means that in the ensemble with fixed \( \mu \) the magnetic critical exponents of the system considered belong to the same universality class as the corresponding lattice model (with the same spatial dimensionality and number of components of the order parameter and the same properties of the pair interactions). Of course, the critical temperature \( T_c \) of the ferromagnetic transition (as well as other nonuniversal quantities) will depend on the value of \( \mu \), so that in the ensemble with fixed \( \mu \) one has \( T_c(\mu) \).

Taking into account this assumption, the thermodynamic potential near the phase transition point can be written as the sum of two terms

\[
P = \bar{P}[T^*(T, h, \mu), h^*(T, h, \mu)] + \Delta P[T, h, \mu] \tag{2}
\]

where \( \bar{P}[T^*, h^*] \) describes the critical properties of the ‘ideal’ system with temperature \( T^* \) and magnetic field \( h^* \). \( P_{\text{reg}}[T^*, h^*] \) and \( \Delta P[T, h, \mu] \) are nonsingular functions of their arguments with continuous and smooth derivatives. The scaled field \( h^*(T, h, \mu) \) and the scaled field \( h^*(T, h, \mu) \) are also smooth functions, so that all the singular features of the system in the vicinity of the magnetic phase transition follows from the properties of \( P_{\text{sing}}[T^*(T, h, \mu), h^*(T, h, \mu)] \). Expression \( (2) \) can be considered as the mathematical formulation of the assumption made before. Note that using the ideas of scaling theory the form of \( P_{\text{sing}} \) can be further specified. In particular, if the so-called Wegner corrections \[29\] to asymptotic scaling behaviour are neglected, one has (see, e.g., \[27,28\]) the following expression

\[
P_{\text{sing}}[T^*, h^*] = |\tau^*|^{2-\alpha} f(z^*), \tag{4}
\]

where \( \tau^* = (T^* - T^*_c)/T^*_c, \) \( z^* = h*/|\tau^*|^{\beta+\gamma} \), and \( f(z) \) is a universal scaling function.

For the scaled field \( h^*(T, h, \mu) \), taking into account the symmetry properties of the magnetic field \( h \) we may use the following representation

\[
h^*(T, h, \mu) = h \cdot j(T, h, \mu) \tag{5}
\]

with \( j(T, h, \mu) = j(T, -h, \mu) > 0 \). This means that the real transition still occurs at \( h = 0 \).

For the ‘ideal’ part \( \bar{P}[T^*, h^*] \) we can write down the well known results for the main thermodynamic quantities. Namely, in the vicinity of critical point with \( h^* = 0 \) and \( T^* = T^*_c \) one has

\[
m^* = \left( \frac{\partial \bar{P}}{\partial h^*} \right)_{h^*=0} = A_m |\tau^*|^\beta, \tag{6}
\]

\[
\chi^* = \frac{\partial m^*}{\partial h^*} = A_\chi |\tau^*|^{-\gamma}, \tag{7}
\]

\[
s^* = \left( \frac{\partial \bar{P}}{\partial T} \right)_{h^*=0} = s_0 + s_1 \tau^* + A_s |\tau^*|^{1-\alpha}, \tag{8}
\]

\[
c_v^* = \frac{\partial s^*}{\partial T^*} = c_0 + c_1 \tau^* + A_c |\tau^*|^{-\alpha}, \tag{9}
\]

for the magnetization \( m^* \), the magnetic susceptibility \( \chi^* \), the entropy \( s^* \), and the specific heat \( c_v^* \) of the ‘ideal’ system, respectively. The coefficients \( A_m, A_\chi, s_0, s_1, A_s, c_0, c_1, \) and \( A_c \) in \( (6)-(9) \) do not depend on \( \tau^* \). The values of \( \beta, \gamma, \) and \( \alpha \) are the corresponding critical exponents of the ‘ideal’ system, respectively. It can easily be proved that the same critical singularities are observed at \( h = h^* = 0 \) for the unconstrained system \[2\] with the fixed chemical potential, when the function \( \bar{T}^*(T, h, \mu) \) has continuous and smooth derivatives. In this case the line of critical temperatures \( T_c(\mu) \) can be found from the equation \( \bar{T}^*(T_c(\mu), 0, \mu) = T^*_c \), and using the properties of the function \( \bar{T}^*(T, h, \mu) \) it can be shown that the critical exponents in the grand canonical ensemble under the assumption made above are the same as for the ‘ideal’ system.

Let us now consider the critical properties of the constrained system with the fixed density of particles \( n = \bar{n} =\text{const} \). In this case, if \( h = h^* = 0 \), one gets from \[3\]:

\[
n = \left. \left( \frac{\partial \bar{P}}{\partial \mu} \right) \right|_{T,h=0} = s^* \left( \frac{\partial \bar{T}^*}{\partial \mu} \right)_{T,h=0} \tag{9}
\]

\[
\left( \frac{\partial [P_{\text{reg}} + \Delta P[T, h, \mu]]}{\partial \mu} \right)_{T,h=0}.
\]
The critical temperature $T_c^0$ in the constrained system can be found from the condition $T_c^0 = T_c(\mu_c) = T_c(\bar{n})$, where the chemical potential $\mu_c$ satisfies the equation $n(\mu_c, T_c(\mu_c)) = \bar{n}$ with the function $T_c(\mu)$ defined above. In particular, this gives the equality $T^*(T_c^0, 0, \mu_c) = T_c^*$, so that in the vicinity of the critical point $[T_c^0, 0, \mu_c, h = 0]$, being of our interest, taking into account the properties of $T^*(T, h, \mu)$, one finds

$$T^*(T, 0, \mu) \simeq T_c^* + t_{\mu} \Delta \mu + t_T \Delta T,$$

(10)

with the coefficients $t_{\mu}$ and $t_T$, where $\Delta \mu = \mu - \mu_c$ and $\Delta T = T - T_c^0$ are assumed to be small enough for restricting of our consideration by the linear terms in (10). This expression can be rewritten as follows

$$\tau^* \simeq \tilde{t}_{\mu} \Delta \mu + \tilde{t}_T \tau^*$$

(11)

with $\tilde{t}_{\mu} = t_{\mu}/T_c^*$ and $\tilde{t}_T = t_T/T_c^*$. In a similar way, taking into account (6), one obtains from Eq. (6) the following expression for $\Delta n$:

$$\Delta n = n - \bar{n} = n_1 \tau^* + A_n |\tau^*|^{1-\alpha} + n_2 \tau + n_\mu \Delta \mu,$$

(12)

where $\tau = (T - T_c^0)/T_c^0$. The coefficients $n_1$ and $A_n$ can be expressed in terms of the coefficients $s_1$, $A_c$, and $t_{\mu}$, introduced in (8) and (10). For $\Delta n = 0$, combining Eqs. (10) and (12), we get the relation between the reduced temperature scales $\tau^*$ and $\tau$ in the ‘ideal’ and constrained systems, respectively:

$$\tau = b_0 \tau^* + b_1 \tau^* |\tau^*|^{-\alpha}$$

(13)

with

$$b_0 = \left\{ n_1 + n_\mu/\tilde{t}_{\mu} \left( \tilde{t}_T n_{\mu}/\tilde{t}_{\mu} - n_2 \right) \right\}^{-1}$$

and

$$b_1 = A_n \left( \tilde{t}_T n_{\mu}/\tilde{t}_{\mu} - n_2 \right)^{-1}.$$

Similar solutions can be found from (11) and (12) for another cases, being of interest in experimental situations. One may be the case when in the canonical ensemble the temperature $T$ is fixed at the critical value $T_c^0$, $\tau = 0$, but the density $\Delta n$ does change. In this case we obtain

$$\Delta n = c_0 \tau^* + c_1 |\tau^*|^{1-\alpha}$$

(14)

with $c_0 = n_1 + n_\mu/\tilde{t}_{\mu}$ and $c_1 = A_n$. The expressions (13) and (14) contain already the central result of Fisher analysis [24]: it is seen in (13) and (14) that, depending on the sign of the exponent $\alpha$, either the first ($\alpha < 0$) or second ($\alpha > 0$) term on the right hand side of (13) and/or (14) is dominant. This directly leads to Fisher renormalization of the critical exponents $\beta$ and $\gamma$ when $\alpha$ is positive. In this case the renormalized exponents $\beta'(1 - \alpha)$ and $\gamma'(1 - \alpha)$ describe the critical singularities of magnetization $m$ and magnetic susceptibility $\chi$ in the constrained system. The critical properties of the specific heat follow immediately from (7) when we use Eq. (13) for the scaled reduced temperature $\tau^*$. Taking the derivative in (7) with respect to $T$, one gets

$$c_v \sim \frac{\partial \tau^*}{\partial T} = \left\{ s_1 + A_\mu |\tau^*|^{-\alpha} \right\} \frac{1}{T_c^0} \frac{\partial \tau^*}{\partial \tau}.$$

On the other side, if $\alpha > 0$, from the expression (13), one has

$$\frac{\partial \tau^*}{\partial \tau} \sim \frac{1}{b_1 |\tau^*|^\alpha}.$$

Combining these two expressions, it is easily to show that the specific heat $c_v$ remains finite at the critical point in the constrained system, and a cusp-like singularity with the exponent $-\alpha/(1-\alpha)$ can be found when $\alpha > 0$. Hence, all the main results of Fisher consideration [24] are already reproduced for our specific case. Note also that the correction exponents $\Delta_i$, describing the Wegner corrections to scaling in the ‘ideal’ system, should be renormalized to $\Delta_i/(1-\alpha)$ if $\alpha$ is positive. In the opposite case, when $\alpha$ is negative, the critical exponents as well as the correction exponents $\Delta_i$ in the constrained system remain the same as in the ‘ideal’ model. These results are known since 1968, when the Fisher’s paper [24] was published.

**II. CORRECTIONS TO SCALING IN THE CONSTRAINED SYSTEM**

Let us consider some other consequences which follows from the thermodynamic analysis given in the previous section. Equations (13) and (14) may be solved with respect to $\tau^*$ by iterations, so that we obtain in first order

$$\tau^* = \text{sign}(\tau) B |\tau|^x \left\{ 1 + b |\tau|^\Delta x + \mathcal{O}(|\tau|^{2\Delta x}) \right\}$$

(15)

and

$$\tau^* = \text{sign}(\Delta n) C |\Delta n|^x \left\{ 1 + c |\Delta n|^\Delta x + \mathcal{O}(|\Delta n|^{2\Delta x}) \right\},$$

(16)

where the exponents $x_\alpha$ and $\Delta x_\alpha$ depend on the sign of the critical exponent of specific heat $\alpha$, namely,

$$x_\alpha = \left\{ \begin{array}{ll} \frac{1}{1-\alpha}, & \alpha > 0, \\ \frac{1}{\alpha}, & \alpha < 0, \end{array} \right.$$

(17)

and

$$\Delta x_\alpha = \left\{ \begin{array}{ll} \frac{\alpha}{1-\alpha}, & \alpha > 0, \\ \frac{-\alpha}{\alpha}, & \alpha < 0, \end{array} \right.$$

(18)
respectively. The coefficients $B$, $b$ and $C$, $c$ can be expressed by the initial parameters $b_0$, $b_1$ and $c_0$, $c_1$ in Eqs. (13) and (14), respectively. For example, one gets the following equations for the coefficients $B$ and $b$:

$$B = \begin{cases} b_1^{1/(1-\alpha)}, & \alpha > 0, \\ b_0^{-1}, & \alpha < 0, \end{cases}$$ (19)

and

$$b = \begin{cases} -(1-\alpha)^{-1}b_0b_1^{-1/(1-\alpha)}, & \alpha > 0, \\ -b_1b_0^{1+\alpha}, & \alpha < 0. \end{cases}$$ (20)

It is worth mentioning that the solutions (13) and (16) could be used only under some special demands which follows from thermodynamic stability conditions (see, e.g., [29,30]). In particular, it could be shown [29] that for $\alpha > 0$ a second order phase transition in the constrained system is observed only if $b_1$ in Eq. (13) is positive. For $b_1 < 0$ in the vicinity of phase transition (small $\tau^*$) the constrained system becomes unstable and the second order phase transition transforms to a first order transition. Note that the condition $b_1 = 0$ gives the position of a special tricritical point with the critical exponents of the ‘ideal’ system. Similar consideration may be applied for the opposite case with $\alpha < 0$. Therefore, let us assume that the parameters $b_0$ and $b_1$ ($c_0$ and $c_1$) are such that a second order phase transition still exists in the constrained system, and the main question then arises what are the singular properties of this transition in the nonasymptotic region.

The first important conclusion can be drawn from Eqs. (13) and (17). It is seen in Eq. (15) that, independently on either the Fisher renormalization changes the critical exponents in the constrained system or not, the new corrections to scaling with the exponent $\Delta_\alpha$ appear due to the constraint. The exponent of these corrections are proportional to $|\alpha|$ and, because this value is smaller for the Ising, the XY and the Heisenberg universality classes compared to the Wegner corrections $\Delta_\omega$, one can expect significant contributions from the new corrections to scaling just in the pre-asymptotic region.

For example, the magnetization $m$ in the constrained system when $\alpha < 0$ can be written as follows:

$$M = A_m|\tau|^\beta \left\{ 1 + a_1|\tau|^{\Delta_1} + a_2|\tau|^{\alpha} + O(|\tau|^{2\alpha}) \right\},$$ (21)

where $\Delta_1 = \omega \nu$ is well-known Wegner correction (see, e.g., [24]). Because of usually $|\alpha| < \Delta_1$, one can conclude, therefore, that Fisher correction with the exponent $|\alpha|$ is dominant in (21). In a similar way the case $\alpha > 0$ may be considered. It is evident for both cases that the width of asymptotic region in temperature scale, where we can restrict the description of critical singularities by considering the main critical exponents only, is reduced significantly.

The new corrections to scaling (see (15) and (17)) with the exponents (18) have appeared in our treatment as the result of the temperature rescaling due to the constraint. Similar idea was recently used by Krech [49] for the study of spin lattice models with constraints within the finite size scaling technique. However, we note that the temperature rescaling is not the only source for corrections of that type. For proving of this statement let us consider in more details the thermodynamic transformations from the ensemble with fixed $\mu$ to the ensemble with fixed $n$.

Before proceeding further we derive some additional relations to be useful for the subsequent calculations.

From Eq. (14) one derives

$$\left( \frac{\partial n}{\partial \mu} \right)_{T,h=0} = \left[ n_1 + n_c(1-\alpha)|\tau^*|^{-\alpha} \right] \left( \frac{\partial \tau^*}{\partial \mu} \right)_{T,h=0} + n_3.$$ (22)

The derivative $\partial \tau^*/\partial \mu$ can be easily found from Eq. (14), and inserting we obtain

$$\left( \frac{\partial m}{\partial \mu} \right)_{T,h=0} = \left[ c_0 + c_1(1-\alpha)|\tau^*|^{-\alpha} \right] \tilde{t}_\mu,$$ (23)

where the coefficients $c_0$ and $c_1$ are the same as in Eq. (14). Recalling now that the derivative (22) is directly related to the compressibility $\kappa_T$ of the system considered ($\partial n/\partial \mu)_{T,h=0} = n^2\kappa_T$), we can conclude that for $\alpha > 0$ a weak divergence appears in the isothermal compressibility in the both ensembles. The critical exponent describing such singular behaviour in the constrained system is equal to the renormalized index of specific heat $\alpha(1-\alpha)$, and

$$\kappa_T = \kappa_T^0 + A_\kappa|\tau|^{\alpha/(1-\alpha)}$$

with the parameters $\kappa_T^0$ and $A_\kappa = \tilde{t}_\mu n^2 c_1(1-\alpha)B$. This shows the main difference to the case of the specific heat, discussed above. In the opposite case, when $\alpha$ is negative, the compressibility in the both ensembles displays the same cusp-like singularity with the exponent $\alpha$ as is observed in the specific heat behaviour. Hence, our second conclusion is: for $\alpha > 0$ in the constrained system there are some thermodynamic quantities (e.g., the compressibility in our case) which are asymptotic weak divergent with the critical exponent $\alpha(1-\alpha)$. In general, these quantities can be expressed by the second order derivatives with respect to a ‘hidden’ thermodynamic force.

The result obtained above is important also for the study of the corrections to scaling in thermodynamic derivatives. In order to illustrate this statement let us use the well-known thermodynamic relation

$$\chi_{T,n} = \left( \frac{\partial M}{\partial h} \right)_{T,n} = \left( \frac{\partial M}{\partial h} \right)_{T,\mu} - \left( \frac{\partial M}{\partial \mu} \right)^2 \left( \frac{\partial n}{\partial \mu} \right)^{-1},$$

(24)
which gives the connection between the magnetic susceptibility in the canonical and the grand canonical ensembles. For the first term in the right-hand side of (24) the rescaling formula (13) for temperature dependence can be used. In the vicinity of the critical point, the contribution from the second term in (24) can be easily estimated by using Eqs. (3), (11), and (23). For the case of positive $\alpha$ this term produces the contributions proportional to $|\tau|^{|\gamma|+2+\alpha}$, $|\tau|^{|\gamma|+2}$, etc. Applying the hyperscaling relation $2\beta + \gamma = d\nu$, one obtains for these terms the following estimations $|\tau|^{|\gamma|+\alpha}$, $|\tau|^{|\gamma|+\alpha}$, etc., and, taking into account the expression (15), one can conclude that an additional correction term with the exponent $\alpha/(1-\alpha)$ appears in the canonical magnetic susceptibility. Similar consideration could be used for the case of negative $\alpha$. Hence, the third conclusion is that in addition to the rescaling mechanism, given by (15), there exists another source for the appearance of the same type of corrections to scaling (proportional to $\alpha$), due to the Legendre transformation. In particular, this means that the new corrections to scaling can not be included in the standard finite size scaling technique just by simple rescaling of the reduce temperature (13) as it was proposed in (12).

The results obtained can be easily generalized for: (i) the case of more complicate constraint, formulated in the form $F[n, T, \mu] = \theta =\text{const}$ (see, e.g., [21,30]); and (ii) the case, when several ‘hidden’ thermodynamic forces exist in a system. In the case (i), as it was shown in [24,80], the generalized constraint will modify the coefficients $b_0$ and $b_1$ in (13) and change the thermodynamic stability conditions, so that the phase transition of the second order may transform to a first-order transition. In fact, the constraint function $F[n, T, \mu] = \theta =\text{const}$ allows to reformulate all the results obtained in some ‘mixed’ ensemble with $\theta$ considered as a thermodynamic parameter, but the conclusions made above will be still valid. Some exceptions may be noted for special choices of the constraint function $F[n, T, \mu]$ (e.g., in the trivial case $F[n, T, \mu] = f(T, \mu)$).

In case (ii), when several ‘hidden’ thermodynamic forces $\{\mu_i\}$ with $i = 1, 2, \ldots, l$ exist in a system and some of them are constrained by external conditions, we can applied the scheme described above by performing consecutively Legendre transformations, starting from any constraint condition. It is evident that, for $\alpha > 0$ in the ‘ideal’ unconstrained system, after the first transformation we will find renormalized Fisher exponents and the new corrections to scaling, that is to say the critical exponent of specific heat will change sign ($\alpha \rightarrow -\alpha/(1-\alpha)$). This means that the critical exponents obtained are not changed any in further transformations, because of negative value of the specific heat exponent $-\alpha/(1-\alpha)$. Any next Legendre transformation may change only the amplitudes of the singular terms. Note also that the corrections to scaling, which will appear after the second (third and so on) Legendre transformation, will have the same exponents (multiple to $\alpha/(1-\alpha)$) as found after the first step. The case with $\alpha < 0$ in the ‘ideal’ unconstrained system is even more trivial, and after the first transformation we will find the same (‘ideal’) critical exponents with the new corrections to scaling proportional to $|\theta|$. This picture will not change in further transformations. Hence, we conclude that if in the constrained system with arbitrary constraints a second order phase transition is observed, one has to expect that the critical exponents, describing this transition in the asymptotic region, are defined by the rules, established by Fisher [34], and the leading corrections to scaling are given by (18).

### III. DISCUSSION AND CONCLUDING REMARKS

We end with a few concluding remarks:

(i) The results presented are quite general and important for many applications. It is worth to note that constrained equilibrium systems are widely studied in the literature, but only a few examples could be mentioned when the Fisher corrections to scaling were included into consideration (see, e.g., [21,27]). In general, such corrections may affect significantly the critical behaviour of compressible magnets [31,33], systems described by Hubbard model [32], binary and multicomponent mixtures [30,31], magnetic spin liquids [22–24], and 4He–3He mixtures [38,39].

(ii) It seems interesting to study more carefully the critical properties of magnets with quenched disorder. The motivation is the following. One of the approaches to the quenched systems is based on the idea of Morita [40,42]. In this approach a quenched system is considered as a quasi-equilibrium system with additional forces of constraints, keeping the quenched system in its equilibrium state and fixing the moments of random distribution that defines the disordered state. Therefore, the quenched system can be studied by means of the equilibrium theory, using some results presented in this paper. In this connection we point out some known results, which support this view: (a) the so-called Harris criterion [44], proposed for the quenched systems, is formally similar to the Fisher criterion [23], concerning the divergence of the specific heat in the constrained system; (b) in Ref. [31] within the $\epsilon$-expansion scheme it was proved analytically (using the replica trick) that the leading correction to scaling in a randomly diluted inhomogeneous $O(n)$ Heisenberg model is a term with the correction exponent $-\alpha$, as it is expected for the constrained Heisenberg model; (c) in Ref. [43] it was shown that if $\alpha > 0$ in the pure system, the traditional renormalization group flows, describing the disorder-induced universal critical behaviour are unstable with respect to replica-symmetry breaking potentials, found in spin glasses.

(iii) There were several studies of constrained systems, performed within the renormalization group technique [30,31,33,45,47]. In general the results obtained support the conclusions made within the thermodynamic
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