Nonthermal radiation of rotating black holes

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Abstract

Nonthermal radiation of a Kerr black hole is considered as tunneling of created particles through an effective Dirac gap. In the leading semiclassical approximation this approach is applicable to bosons as well. Our semiclassical results for photons and gravitons do not contradict those obtained previously. For neutrinos the result of our accurate quantum mechanical calculation is about two times larger than the previous one.

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1 Introduction

The amplification of an electromagnetic wave at the reflection from a rotating black hole, so called superradiation, was predicted by Zel’dovich [1], and also by Misner [2]. Then it was studied in detail by Starobinsky and Churilov [3] for electromagnetic and gravitational waves (see also [4]). It looks rather obvious that if the amplification of a wave at the reflection is possible, then its generation by a rotating black hole is possible as well. Indeed, direct calculation by Page [5] has demonstrated that the discussed, nonthermal radiation does exist, and not only for bosons, photons and gravitons, but for neutrinos as well. The last result looks rather mysterious since for fermions there is no superradiation.

In the present work these processes are considered from another point of view: as tunneling of quanta being created through the Dirac gap. Certainly, this approach by itself can be valid for fermions only. It is clear however that in the leading semiclassical approximation the production of fermions and bosons is described by same, up to the statistical weight, relationships.

Let us note that in the recent work by Calogeracos and Volovik [6] (we found out about it after the present paper had been written) an analogous mechanism was considered for the description of the friction experienced by a body rotating in superfluid liquid at $T = 0$: the quantum tunneling of quasiparticles to the region where their energy in the rotating frame is negative.

Our semiclassical results for photons and gravitons are in a reasonable qualitative agreement with the calculation [5]. One cannot expect here a quantitative coincidence since for the most essential partial waves the action inside the barrier exceeds unity not so much if any. As to neutrinos, the imaginary part of the action for them, at the total angular momenta of real importance, is considerably smaller then unity. Therefore, for spin $s = 1/2$ we have performed a complete numerical quantum mechanical calculation of the effect. Here our result exceeds that presented in [5] by a factor close to two. Unfortunately, the lack of details in [5] does not allow us to elucidate the origin of the discrepancy.

2 Scalar field

We will start with a problem of a methodological rather than direct physical interest, with the radiation of scalar massless particles by a rotating black hole.

The semiclassical solution of the problem is started from the Hamilton-Jacobi equations for the motion of a massless particle in the Kerr field (see, for instance, [7]):

$$\left(\frac{\partial S_r(r)}{\partial r}\right)^2 = -\frac{\kappa^2}{\Delta} + \frac{[(r^2 + a^2)\varepsilon - a l_z]^2}{\Delta^2}, \quad (1)$$

$$\left(\frac{\partial S_\theta(\theta)}{\partial \theta}\right)^2 = \kappa^2 - \left(a \varepsilon \sin \theta - \frac{l_z}{\sin \theta}\right)^2. \quad (2)$$

Here $S_r(r)$ and $S_\theta(\theta)$ are the radial and angular actions, respectively;

$$\Delta = r^2 - r_g r + a^2; \quad r_g = 2kM;$$

$a = J/M$ is the angular momentum of the black hole in the units of its mass $M$; $\varepsilon$ is the particle energy; $l_z$ is the projection of the particle angular momentum onto $a$ (the velocity of light $c$ is put to unity everywhere).
As to the constant $\kappa^2$ of the separation of variables, in the spherically symmetric limit $a \to \infty$ it is equal to the particle angular momentum squared $l^2$, or to $l(l+1)$ in quantum mechanics (the Planck constant $\hbar$ is also put to unity everywhere). The influence of the black hole rotation, i.e. of the finite $a$, upon $\kappa^2$ is taken into account by means of the perturbation theory applied to equ. (2). The result is

$$\kappa^2 = l(l+1) - 2\omega \alpha l_z + \frac{2}{3} \omega^2 \alpha^2 \left[ 1 + \frac{3l_z^2 - l(l+1)}{(2l-1)(2l+3)} \right].$$

Here and below we use the dimensionless variables $\omega = \varepsilon kM$, $x = r/kM$, $\alpha = a/kM$. Let us recall that in the semiclassical approximation the substitution

$$l(l+1) \to (l+1/2)^2$$

should be made.

Besides, in the exact quantum mechanical problem, under the reduction of the radial wave equation to the canonical form

$$R'' + p^2(r) R = 0,$$

in the expression for $p^2(r)$ additional (as compared to the right-hand side of equ. (1)) non-classical terms arise, which should be, strictly speaking, included for $l \simeq 1$. However, for the simplicity sake we neglect here and below these nonclassical corrections to $p^2(r)$, which does not influence qualitatively the results obtained.

The dependence of the classically inaccessible region, where the radial momentum squared $p^2$ is negative, on the distance $x$ is presented for different angular momenta in Figs. 1, 2. At the horizon the gap vanishes \[8\]. For $r \to \infty$ the boundaries of the classically inaccessible region behave as $\pm(l+1/2)/r$. In other words, the centrifugal term for massless particles plays in a sense the role of the mass squared. Let us note that for $l > 1$ both branches of the equation $p^2(r) = 0$ fall down, but for $l = 1$ one branch near the horizon grows up, and the second one falls down. Thus, the radiation mechanism consists in tunneling, in going out of particles from the dashed region to the infinity.

One should note the analogy between the emission of charged particles by a charged black hole and the effect discussed. In the first case the radiation is due to the Coulomb repulsion, and in the case considered here it is due to the repulsive interaction between the angular momenta of the particle and black hole \[9\].

The action inside the barrier for the radial equ. (2) is:

$$|S_r| = \int dx \sqrt{\frac{k^2}{(x-1)^2} - \frac{\omega(x^2+1) - l_z^2}{(x-1)^4}};$$

the integral is taken between two turning points. For simplicity sake we confine for the time being to the case of an extremal black hole, $a = kM$. Let us note that due to a singular dependence of $p$ on $x$, the action inside the barrier does not vanish at $l > 1$ even for the maximum energy $\omega = l_z/2$. So much the more, it stays finite at $l = 1$ (compare Figs. 1 and 2).

The repulsive interaction is proportional to the projection $l_z$ of the particle angular momentum and enters the tunneling probability through the exponent, but the barrier depends on the orbital angular momentum $l$ itself. Therefore, clearly the main contribution to the effect will be given by particles with $l_z$, close to $l$. The numerical calculation demonstrates that the contribution of the states with $l_z \neq l$ can be neglected at all. Besides, since the action inside
Figure 1: The energy gap for $l = 1$.

Figure 2: The energy gap for $l = 2$. 
the barrier decreases with the growing energy, the main contribution to the effect is given by
the particles with energy close to maximum.

Unfortunately, even for an extremal black hole an analytical calculation of the action inside
the barrier cannot be carried out to the end. Therefore, to obtain a qualitative idea of the
effect, we will use a simplified expression for $\kappa^2$:

$$\kappa^2 = l^2 + l - 2\omega l + \omega^2.$$  \hspace{1cm} (5)

(The results of a more accurate numerical calculation with expression (3) will be presented
below.) In this approximation one can obtain a simple analytical formula for action inside the
barrier for all angular momenta but $l = 1$. Let us assume that $\omega = (1 - \delta) l/2$ with $\delta \ll 1$; just
this range of energies contributes most of all into the radiation. Then, the turning points of
interest to us, which are situated to the right of the horizon, are:

$$x_{1,2} = 1 + \frac{2\delta}{2 \pm \sqrt{1 + 4/l}}.$$  \hspace{1cm} (6)

Now one finds easily

$$|S_{an}| = \frac{\pi l}{2} \left( 2 - \sqrt{3 - \frac{4}{l}} \right).$$  \hspace{1cm} (7)

One can see from this equation that the term $l$ (following $l^2$) in formula (3) is quite essential even
for large angular momenta: it generates the terms $4/l$ in formulae (5) and (7), thus enhancing
$|S|$ for $l \gg 1$ by $\pi/\sqrt{3}$. Correspondingly, the transmission factor $D = \exp(-2|S|)$ gets about
40 times smaller. Let us note that even the transition in $\kappa^2$ from $l(l + 1)$ to $(l + 1/2)^2$
makes the effect considerably smaller for $l$ comparable to unity; but this suppression dies out for large
angular momenta.

It follows from formula (7) that the action inside the barrier is large, it increases mono-
tonically with the growth of $l$, starting with $|S| = \pi$ for $l = 2$. As to $l = 1$, one can see by
comparing Fig. 1 with Fig. 2 that here the barrier is wider than for $l = 2$, and therefore the
action should be larger. Indeed, the numerical calculation of the action inside the barrier $|S|$ with $\kappa^2$
given by formula (3) confirms these estimates. Its results are presented in Table 1
where for the comparison sake we present also the analytical estimates $|S_{an}|$ with formula (5).

| $l$ | 1 | 2 | 3 |
|-----|---|---|---|
| $|S| \ $ | 3.45 | 3.15 | 3.33 |
| $|S_{an}| \ $ | 3.14 | 3.34 |

Table 1: Action inside the barrier for scalar particles.

very well. The numbers presented in the table refer to an extremal black hole and maximum
energy of emitted particles. It is clear however that the transition to nonextremal black holes,
lower energies, and larger $l$ will result in the growth of the action inside the barrier. Since here it proves to be always considerably larger than unity, the use of the semiclassical approximation inside the barrier is quite reasonable.

Let us check now whether it applies to the left of the barrier. The corresponding condition is of the usual form:

$$\frac{d}{dx} \frac{1}{p(x)} \ll 1.$$  \hspace{1cm} (8)

In other words, the minimum size of the initial wave packet to the left of the barrier should not exceed the distance from the horizon to the turning point. Near the horizon one can neglect in the expression for the momentum $p(x)$ the term related to the centrifugal barrier, so that

$$\frac{d}{dx} \frac{1}{p(x)} \approx \frac{d}{dx} \frac{(x - 1)^2}{\omega (x^2 + \alpha^2) - \alpha l}.$$  

One can see easily that for not so large $l$, which are of importance in our case, this expression is comparable to unity and condition (8) does not hold. Nevertheless, despite of this circumstance and of the neglect of the nonclassical corrections to $p^2(r)$ mentioned above, the results of the semiclassical calculation, presented below, are correct qualitatively.

Let us come back to the calculation of the radiation intensity. The radial current density of free particles in the energy interval $d\varepsilon$ is at $r \to \infty$

$$j_r(\varepsilon, l) d\varepsilon = \sum \frac{d^3p}{(2\pi)^3} \frac{\partial \varepsilon}{\partial p_r} = \sum_{l, l_z} \frac{2\pi dp_r}{(2\pi)^3 r^2} \frac{\partial \varepsilon}{\partial p_r}.$$  \hspace{1cm} (9)

Indeed, the initial summation here reduces to the integration over the azimuth angle of the vector $l$, which gives $2\pi$, and to the account for the contributions of all projections $l_z$ of the orbital angular momentum. As pointed out already, in our case it is sufficient to take into account only one of them, $l_z = l$. By means of the identity

$$\frac{\partial \varepsilon}{\partial p_r} dp_r = d\varepsilon,$$

we obtain in the result that the total flux of free particles at $r \to \infty$ is

$$4\pi r^2 j_r(\varepsilon, l) = 4\pi r^2 \sum_{l_z} \frac{2\pi}{(2\pi)^3 r^2} \to \frac{1}{\pi}.$$  \hspace{1cm} (10)

One can easily see that in our problem the total flux of radiated particles differs from the last expression by the barrier penetration factor only. Thus, in our semiclassical approximation we obtain for the loss of mass by a black hole in the unit time the following expression:

$$\frac{dM}{dt} = \frac{1}{\pi} \sum_{l=1}^{\infty} \varepsilon_{max} \int_0^{\varepsilon_{max}} \varepsilon \exp(-2 |S(\varepsilon, l)|) d\varepsilon.$$  \hspace{1cm} (11)

Here the maximum energy of radiated quanta is

$$\varepsilon_{max} = \frac{al}{r_h^2 + a^2};$$  \hspace{1cm} (12)
\( r_h = km + \sqrt{k^2M^2 - a^2} \) is the radius of the horizon of a Kerr black hole. The analogous expression for the loss of the angular momentum is

\[
\frac{dJ}{dt} = \frac{1}{\pi} \sum_{l=1}^{\infty} \varepsilon_{\text{max}} \int_0^\infty e^{-2|S(\varepsilon, l)|} d\varepsilon.
\] (13)

The results of the numerical calculation with formulae (11) and (13) of the loss by a black hole of its mass and angular momentum for different values of the rotation parameter \( \alpha \) are presented in Table 2. We present here and below, for spinning particles, results of calculations only for sufficiently rapid rotation, \( \alpha \approx 1 \). The point is that with further decrease of \( \alpha \), not only the thermal radiation grows rapidly, but the effect discussed falls down even more rapidly. For smaller \( \alpha \) this effect becomes much smaller than the thermal one, and thus its consideration there does not make much sense.

| \( \alpha \) | \( |dM/dt| \) | \( |dJ/dt| \) |
|---|---|---|
| 0.999 | 2.6 | 6.4 |
| 0.9 | 0.19 | 0.77 |

Table 2: Loss of mass (in the units of \( 10^{-3} \pi M^2 \)) and angular momentum (in the units of \( 10^{-3} \pi M \)), due to the radiation of scalar particles.

As one can see from Table 2, the rate of loss of the angular momentum is higher, in the comparable units, than the rate of loss of the mass. In fact, it follows immediately from expression (12). Already from this expression one can see that even for the maximum possible energy the ratio of the corresponding numbers is 2:1. Real ratios are even larger. Hence an important conclusion follows: extremal black holes do not exist. Even if an extremal hole is formed somehow, in the process of radiation it loses the extremality immediately.

3 Radiation of photons and gravitons

The investigation of the radiation of real particles we start with the electromagnetic field. Photon has two modes of opposite parity: the so called electric mode, with \( l = j \pm 1 \), and magnetic one, with \( l = j \) [10]. It follows from the duality invariance that the radiation intensities for these two modes are equal. Thus one can confine to the solution of the problem for the magnetic mode, and then just double the result.

One can demonstrate that the situation with the gravitational waves is analogous. Again, there are two modes which, due to a special duality, contribute equally to the radiation, and for one of these modes \( l = j \).

Obviously, for a mode with \( l = j \) the radial equation in the semiclassical approximation is the same as for the scalar field, but with different value of \( \kappa^2 \). It can be demonstrated also starting from the so called Teukolsky equation [11] (neglecting again nonclassical corrections...
to \( p^2(r) \)). The corresponding eigenvalues of the angular equation for particles of spin \( s \), found again in the perturbation theory, are [3]:

\[
\kappa^2 = j(j + 1) + \frac{1}{4} - 2\alpha \omega j_z - \frac{2\alpha \omega j_z s}{j(j + 1)} + \alpha^2 \omega^2 \left\{ \frac{2}{3} \left[ 1 + \frac{3j_z^2 - j(j + 1)}{(2j - 1)(2j + 3)} \right] \right. \\
\left. - \frac{2s^2}{j(j + 1)(2j - 1)(2j + 3)} + 2s^2 \left[ \frac{(j^2 - s^2)(j^2 - j_z^2)}{j^2(2j - 1)(2j + 1)} - \frac{(j + 1)^2 - j_z^2)((j + 1)^2 - s^2)}{(j + 1)^2(2j + 1)(2j + 3))} \right\},
\]

(14)

We have included into this expression the term \( 1/4 \), necessary for correct semiclassical description. Let us note that, as follows from the consideration of the helicity of a massless particle, the restriction \( j \geq s \) holds. Correspondingly, for a photon \( j \geq 1 \), for a graviton \( j \geq 2 \). As well as in the scalar case, the main contribution into the radiation is given by the states with the maximum projection of the angular momentum, \( j_z = j \).

Let us discuss first of all whether the semiclassical approximation is applicable here. As to the situation to the left of the barrier, it does not differ qualitatively from the scalar case. The situation inside the barrier is different. As one can see from equation (14), the presence of spin makes \( \kappa^2 \) smaller, and correspondingly, makes smaller the centrifugal repulsion. In result, both the barrier and the action inside it decrease. This qualitative argument is confirmed by a numerical calculation of \(|S|\) for photons and gravitons with the maximum projection of the angular momentum \( j_z = j \) and maximum energy for the case of an extremal black hole (see Table 3). Therefore in the present case, one should expect that the accuracy of semiclassical results is lower than in the scalar case.

The semiclassical formulae for electromagnetic and gravitational radiation differ formally from the corresponding scalar ones (11) and (13) by extra factor 2 only, which reflects the existence of two modes. The results of this calculation are presented in Table 4. In it we indicate in brackets for comparison the results of the complete quantum mechanical calculation [5] which takes into account as well the thermal radiation. It is clear from Table 4 that even for \( \alpha = 0.999 \), when the thermal radiation is negligibly small, our semiclassical calculation agrees with the complete one qualitatively only. It is only natural if one recalls that the semiclassical action in the present problem exceeds unity not so much, if any. This explanation is supported by the fact that for photon, where \(|S|\) is considerably larger (see Table 3), the agreement of the semiclassical calculation with the complete one is considerably better.

|       | s=1 | s=2 |
|-------|-----|-----|
| \( j \) | 1 | 2 | 2 | 3 |
| \(|S|\) | 1.84 | 2.17 | 1.0 | 1.7 |

Table 3: Action inside the barrier for photons and gravitons.
4 Radiation of neutrino

Let us consider at last the radiation by a rotating black hole of neutrinos, massless particles of spin 1/2. The wave function of a 2-component neutrino is written as (see, for instance [5]):

\[
\psi = \exp(-i\epsilon t + i j_z \phi) \begin{pmatrix} R_1 S_1 \\ R_2 S_2 \end{pmatrix}.
\]  

(15)

It is essential that the wave equations for neutrino in the Kerr metrics allow for the separation of variables as well [11]. The radial equations in dimensionless variables are:

\[
\frac{dR_1}{dx} - i \left( \frac{\omega(x^2 + \alpha^2) - j_z \alpha}{\Delta} \right) R_1 = \frac{\kappa}{\sqrt{\Delta}} R_2,
\]

\[
\frac{dR_2}{dx} + i \left( \frac{\omega(x^2 + \alpha^2) - j_z \alpha}{\Delta} \right) R_2 = \frac{\kappa}{\sqrt{\Delta}} R_1.
\]

(16)

The angular equations are:

\[
\frac{dS_1}{d\theta} + \left( \omega \alpha \sin \theta - \frac{j_z}{\sin \theta} \right) S_1 = \kappa S_2,
\]

\[
\frac{dS_2}{d\theta} - \left( \omega \alpha \sin \theta - \frac{j_z}{\sin \theta} \right) S_2 = -\kappa S_1.
\]

(17)

For \(\kappa^2\) the same formula (14) takes place, but now of course with \(s = 1/2\). As well as for bosons, it is sufficient practically to consider states with \(j_z = j\).

It is essential that \(R_1\) corresponds at infinity, for \(x \to \infty\), and at the horizon, for \(x \to 1\), to the wave running to the right, and \(R_2\) corresponds for \(x \to \infty\) and for \(x \to 1\) to the wave running to the left. (For this classification, it is convenient to use the so called “tortoise” coordinate \(\xi(x)\); for \(x \to \infty\) \(\xi \approx x + \infty\), for \(x \to 1\) \(\xi \approx \ln(x - 1) \to -\infty\).) It is quite natural that here the radial current density is

\[
j_r = |R_1|^2 - |R_2|^2.
\]

We are interested in the probability of penetrating the barrier for the state which is an outgoing wave at infinity. For a neutrino or antineutrino such a state has a fixed helicity,

| s=1       | s=2       |
|-----------|-----------|
| \(\alpha\) | \(|dM/dt|\) | \(|dJ/dt|\) | \(|dM/dt|\) | \(|dJ/dt|\) |
| 0.999     | 16.5(9.6) | 39(24)   | 66(228)  | 148(549) |
| 0.9       | 0.72(2.26)| 2.8(8.2) | 0.58(12.9)| 2(48)    |

Table 4: Loss of mass (in the units of \(10^{-3} \pi M^2\)) and angular momentum (in the units of \(10^{-3} \pi M\)), due to the radiation of photons and gravitons.
but has no definite parity. Meanwhile, the potential barrier depends in our problem, roughly speaking, on the orbital angular momentum, and therefore is much more transparent for the states of $l = j - 1/2$, than for the states of $l = j + 1/2$. (These states of given $l$ have definite parity, and are superpositions of neutrino and antineutrino.) Moreover, at $l = j - 1/2$ for small $j$, which give the main contribution to the radiation, the action either has no imaginary part at all, or its imaginary part is small, so that our above approach is inapplicable. Therefore, we will solve numerically the exact problem of the neutrino radiation.

Technically, it is convenient to find the reflection coefficient $R$ in the problem of the neutrino scattering off a black hole, and then use the obvious relationship for the transmission coefficient $D$:

$$D = 1 - R.$$ 

Here the expressions for the loss of mass and angular momentum by a black hole are:

$$\frac{dM}{dt} = \frac{1}{\pi} \sum_{j=1/2}^{\infty} \int_{0}^{\varepsilon_{\text{max}}} \varepsilon D(\varepsilon, j) \, d\varepsilon; \quad (18)$$

$$\frac{dJ}{dt} = \frac{1}{\pi} \sum_{j=1/2}^{\infty} \int_{0}^{\varepsilon_{\text{max}}} jD(\varepsilon, j) \, d\varepsilon; \quad (19)$$

$$\varepsilon_{\text{max}} = \frac{aj}{r_{h}^{2} + a^{2}}. \quad (20)$$

The results, obtained by numerical solution of the system of radial equations (16), are presented in Table 5. In brackets we present the results of [5], which include the Hawking radiation contribution. For a black hole close to extremal one, at $\alpha = 0.99$, where the thermal radiation is practically absent, our results are about twice as large as previous ones.

| $\alpha$ | $|dM/dt|$ | $|dJ/dt|$ |
|----------|-----------|-----------|
| 0.99     | 4.4 (2.1) | 11 (5.65) |
| 0.9      | 0.7 (1)   | 2.7 (3.25)|

Table 5: Loss of mass (in the units of $10^{-3} \pi M^{2}$) and angular momentum (in the units of $10^{-3} \pi M$), due to the radiation of neutrinos.

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