A remark on the minimal dilation of the semigroup generated by a normal UCP-map

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Abstract

There are known three ways to construct the minimal dilation of the discrete semigroup generated by a normal unital completely positive (UCP-) map on a von Neumann algebra, which are given by Arveson, Bhat-Skeide and Muhly-Solel. In this paper, we clarify the relation of the constructions by Bhat-Skeide and Muhly-Solel, and show that they are essentially the same.

1 Introduction

A dynamical transformation in a quantum physical system is described by a completely positive (CP-) map on an operator algebra in a broad sense. We consider a von Neumann algebra $M$ acting on a Hilbert space $H$ and a normal unital completely positive (UCP-) map $T$ on $M$. Stinespring’s dilation theorem ensures the existence of a normal representation $(\pi, \mathcal{K})$ of $M$ and an isometry $v$ on $\mathcal{K}$ such that $T(x) = v^* \pi(x)v$ for all $x \in M$. When we consider a time evolution, the $n$-times transformation $T^n$ is important, but it is difficult to deal with representations $\{\pi_n\}_{n=1}^{\infty}$ associated with $\{T^n\}_{n=1}^{\infty}$.

Now we consider the minimal dilation of the semigroup $\{T^n\}$ that is a large von Neumann algebra $N \supset M$ and a $*$-endomorphism $\alpha$ on $N$ such that $T^n$ is represented by $\alpha^n$ for each $n \in \mathbb{N}$, and it is desirable that $(N, \alpha)$ is minimal. To be accurate, the notion of minimal dilations is introduced in [5] as the following.

Definition 1.1. Let $M$ be a von Neumann algebra and $T$ be a normal UCP-map on $M$. A triplet $(N, \alpha, p)$ of a von Neumann algebra $N \supset M$, a $*$-endomorphism $\alpha$ on $N$ and a projection $p \in N$ is called a dilation of $T$ if
$M = pNp$ and $T^n(x) = p\alpha^n(x)p$ for all $x \in M$ and $n \in \mathbb{Z}_{\geq 0}$. Moreover, a dilation $(N, \alpha, p)$ of $T$ is called minimal if $N$ is generated by $\bigcup_{n=0}^{\infty} \alpha^n(M)$ and the central projection $c(p)$ of $p$ coincides $1_N$.

Dilations for a $C^*$-algebra $A$ and those for a continuous semigroup $\{T_t\}_{t \geq 0}$ consisting of CP-maps on $A$ are also defined in a similar way. It is known that a minimal dilation is unique if it exists. Then the question of the existence of the minimal dilation arises. Bhat[8] proved the existence of the minimal dilation in the case when $A = \mathcal{B}(\mathcal{H})$ which consists of all bounded operators on a Hilbert space $\mathcal{H}$, and each $T_t$ is unital. In [9], he generalized a way of the construction in stages and constructed a minimal dilation on a $C^*$-algebra $A$ under the assumption that $A$ is unital and $\|T_t(1_A)\| \leq 1$ holds for all $t \geq 0$. These are called the minimal dilation theory for $C^*$-algebras.

After that, Bhat-Skeide[10] constructed the minimal dilation on a von Neumann algebra $N \supset A$ in the case when $A$ is a von Neumann algebra and a semigroup $\{T_t\}_{t \geq 0}$ of normal CP-maps on $A$ has a continuity with respect to $t \geq 0$, by using inductive limits of the tensor products of Hilbert bimodules. On the other hand, Arveson[1],[2] introduced the product systems and gave a one-to-one correspondence between product systems and semigroups $\{\alpha_t\}_{t \geq 0}$ of $\ast$-endomorphisms called the $E_0$-semigroups. Consequently, he classified product systems. But after that, it is understood that Arveson’s theory contains the dilation theory substantially, and his idea affected the constructions of dilations. Muhly-Solel[13] proved the result in [10] for normal UCP-maps $\{T_t\}_{t \geq 0}$ by the similar way as [10]. But the constructions are different in its appearance and no direct relation was known.

In this paper, we overview the constructions in [10] and [13], of the minimal dilation in the sense of Definition 1.1 in the case when given semigroup is a discrete semigroup $\{T^n\}_{n=0}^{\infty}$ generated by a normal UCP-map. We shall make their direct relationship clear and reveal that these constructions are essentially the same. The dilation of a discrete semigroup is applicable to the theory of non-commutative Poisson boundaries in [12].

In what follows, we assume that all Hilbert spaces are separable, and $\mathcal{B}(\mathcal{H}, \mathcal{K})$ means the set of all bounded operators from $\mathcal{H}$ to $\mathcal{K}$. If $\mathcal{K} = \mathcal{H}$, we denote $\mathcal{B}(\mathcal{H}, \mathcal{K})$ by $\mathcal{B}(\mathcal{H})$. For a set $X$, the identity map on $X$ is denoted by $\text{id}_X$ and $F^0 = \text{id}_X$ for every map $F : X \to X$. The unit of a unital algebra $A$ is denoted by $1_A$.

The author is deeply grateful to Prof. Shigeru Yamagami for insightful
2 Preliminaries

We recall the notion of $W^*$-modules and the related notation about them.

**Definition 2.1.** (1) For von Neumann algebras $N$ and $M$, a Hilbert space $\mathcal{H}$ with normal $\ast$-representations of $N$ and the opposite von Neumann algebra $M^\circ$ of $M$ is a $W^*-N-M$-bimodule if their representations commute. When $N = \mathbb{C}$ or $M = \mathbb{C}$, we call $\mathcal{H}$ a right $W^*-M$-module or a left $W^*-N$-module, respectively. We write a $W^*-N-M$-bimodule, a right $W^*-M$-module and a left $W^*-N$-module by $N\mathcal{H}M$, $\mathcal{H}M$ and $N\mathcal{H}$, respectively.

(2) Let $N$ be a von Neumann algebra, $X_N$ and $Y_N$ be right $W^*-N$-modules, and $NZ$ and $NW$ be left $W^*-N$-modules. $\text{Hom}(X_N, Y_N)$ and $\text{Hom}(NZ, NW)$ are the sets of all right and left $N$-linear bounded maps, respectively. If $X = Y$ and $Z = W$, they are denoted by $\text{End}(X_N)$ and $\text{End}(NZ)$, respectively.

(3) We denote the standard representation space of a von Neumann algebra $M$ in [11] by $L^2(M)$.

We recall Hilbert modules which are tools to construct the minimal dilation in the ways by Bhat-Skeide and Muhly-Solel. It is a module over a von Neumann algebra $M$ with an $M$-valued inner product.

**Definition 2.2.** Let $M$ be a von Neumann algebra and $E$ be a right $M$-module. If a map $(\cdot, \cdot) : E \times E \to M$ is defined and satisfies the following properties, then $E$ is called a Hilbert $M$-module.

1. $(x, \alpha y + \beta z) = \alpha(x, y) + \beta(x, z)$ ($x, y, z \in E$, $\alpha, \beta \in \mathbb{C}$).
2. $(x, ya) = (x, y)a$ ($x, y \in E$, $a \in M$).
3. $(x, y)^\ast = (y, x)$ ($x, y \in E$).
4. $(x, x) \geq 0$ ($x \in E$).
5. For every $x \in E$, $x = 0$ if and only if $(x, x) = 0$. 

comments and suggestions.
(6) $E$ is complete with respect to the norm defined by $\|x\| = \|(x, x)\|^\frac{1}{2}$.

Suppose $E$ and $F$ are Hilbert $M$-modules. For a right module homomorphism $b : E \to F$, a right module homomorphism $b^* : F \to E$ is called the adjoint of $b$ when $(y, bx) = (b^* y, x)$ holds for every $x \in E$ and $a \in M$. We denote the set of all right module homomorphism with the adjoint by $\mathcal{B}^a(E, F)$. Automatically, $b \in \mathcal{B}^a(E, F)$ is bounded and $\mathcal{B}^a(E, E)$ is a C*-algebra.

If a surjection $u \in \mathcal{B}^a(E, F)$ satisfies that $(ux, uy) = (x, y)$ for every $x, y \in E$, it is called an isomorphism. Then $E$ and $F$ are said to be isomorphic and we write $E \cong F$.

**Definition 2.3.** Let $M$ and $N$ be von Neumann algebras and $E$ be a Hilbert $N$-module. We call $E$ a Hilbert $M$-$N$-bimodule when it is an $M$-$N$-bimodule satisfying

$$(x, ay) = (a^* x, y)$$

for every $x, y \in E$ and $a \in M$.

**Definition 2.4.** Let $M, N$ and $P$ be von Neumann algebras, $E$ be a Hilbert $N$-$M$-bimodule and $F$ be a Hilbert $M$-$P$-bimodule. Left and right actions of $a \in M$ and $c \in P$ on the algebraic tensor product $E \otimes_{\text{alg}} F$ are defined by $a(x \otimes y)c = (ax) \otimes (yc)$ for each $x \in E$ and $y \in F$. We define that

$$(x \otimes y, x' \otimes y') = (y, (x, x')y')$$

for each $x, x' \in E$ and $y, y' \in F$, and put $\mathcal{N} = \{z \in E \otimes_{\text{alg}} F \mid (z, z) = 0\}$. The tensor product $E \otimes_M F$ of $E$ and $F$ is defined by the completion of $(E \otimes_{\text{alg}} F)/\mathcal{N}$ with respect to the norm induced from the above inner product. The left and right actions can be extended on $E \otimes_M F$, thus $E \otimes_M F$ becomes as Hilbert $N$-$P$-bimodules.

The tensor product is associative, and for a Hilbert $M$-$M$-bimodule $E$, we can identify that $\mathcal{B}^a(E) \subset \mathcal{B}^a(E) \otimes_M 1_E \subset \mathcal{B}^a(E \otimes_M E)$.

We introduce the GNS-construction with respect to a normal UCP-map, see [16] for example.

**Definition 2.5.** Suppose $M$ is a von Neumann algebra and $T : M \to M$ is a normal UCP-map. We define a Hilbert $M$-$M$-bimodule $E(M, T)$ by the
completion of \((M \otimes_{\text{alg}} M)/\mathcal{N}\) with respect to a norm induced from an inner product
\[
(a \otimes b, a' \otimes b')_T = b^* T(a^* a') b' \quad (a, a', b, b' \in M),
\]
where \(\mathcal{N} = \{z \in M \otimes_{\text{alg}} M \mid (z, z)_T = 0\}\). If we put \(\xi = 1_M \otimes 1_M + \mathcal{N}\), then span\((M\xi M)\) is dense in \(E(M, T)\) and \(T(a) = (\xi, a\xi)\) holds for all \(a \in M\).

We call the couple \((E(M, T), \xi)\) the GNS-representation with respect to \(T\).

There is an important identification in Bhat-Skeide’s construction as the following.

**Definition 2.6.** Let \(M\) be a von Neumann algebra acting on a Hilbert space \(H\) and \(E\) be a Hilbert \(M\)-module. Then \(H\) and \(E\) are a Hilbert \(M\)-\(C\)-bimodule and a Hilbert \(C\)-\(M\)-bimodule, respectively, and hence we can define the tensor product \(E \otimes M H\) as Hilbert bimodules. For \(\xi \in E\), we define \(L_\xi: H \ni h \mapsto \xi \otimes h \in E \otimes_M H\). Then we can identify \(E\) as a right \(M\)-submodule of \(B(H, E \otimes_M H)\) by \(B_a(E) \subset B(H, E \otimes_M H)\) is a von Neumann subalgebra; see [16].

A tensor product defined as follows is used in Muhly-Solele’s construction.

**Definition 2.7.** Let \(M\) be a von Neumann algebra acting on a Hilbert space \(H\) and \(T\) be a normal UCP-map on \(M\). We define a sesquilinear form on the algebraic tensor product \(M \otimes_{\text{alg}} H\) by
\[
(x \otimes \xi, y \otimes \eta) = (\xi, T(x^* y) \eta) \quad (x, y \in M, \ \xi, \eta \in H).
\]
We define the Hilbert space \(M \otimes_T H = (M \otimes_{\text{alg}} H)/\mathcal{N}\), where \(\mathcal{N} = \{z \in M \otimes_{\text{alg}} H \mid (z, z) = 0\}\).

A representation \(\pi_T\) of \(M\) on \(M \otimes_T H\) is defined by
\[
\pi_T(y)(x \otimes \xi) = yx \otimes \xi \quad (x \in M, \ \xi \in H).
\]
3 Some isomorphisms between $W^*$-bimodules

In this section, some new results on isomorphisms between $W^*$-bimodules are stated as Proposition 3.3–Corollary 3.6. In Subsection 4.4, they will be used to see a relation between two constructions of the minimal dilation, which are given by Bhat-Skeide and Muhly-Solel.

First, we introduce notations with respect to $W^*$-modules and the relative tensor products in [15], and recall the facts about them (cf. [17] and [6]).

Fact 3.1. (1) Let $M$ be a von Neumann algebra and $\mathcal{H}_M$ be a $W^*$-$M$-module. For each positive normal functional $\phi$ on $M$, let $(\pi_\phi, \mathcal{H}_\phi, \xi_\phi)$ be the GNS-representation of $M$ with respect to $\phi$. We denote $\pi_\phi(x)\xi_\phi$ for each $x \in M$. Since $\mathcal{H}$ is decomposable into cyclic representations, there exists a family of vectors $\{\xi_i\}_{i \in I}$ in $\mathcal{H}$ such that $\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_{\omega_i}$ where $\omega_i(x) = (\xi_i, x\xi_i)$. Moreover, if we denote the support of $\omega_i$ by $q_i$, we have

$$\mathcal{H} \cong \bigoplus_{i \in I} (L^2(M)q_i) \cong \bigoplus_{i \in I} L^2(M)q_i$$

as $W^*$-$M$-module where $q$ is the diagonal matrix whose diagonal entries are $\{q_i\}_{i \in I}$.

(2) For a $W^*$-$M$-$N$-bimodule $\mathcal{M}\mathcal{H}_N$, we denote the dual Hilbert space of $\mathcal{H}$ by $\mathcal{H}^*$. For every $\xi^* \in \mathcal{H}^*$, the right action of $x \in M$ and the left action of $y \in N$ to $\xi^*$ are defined by

$$y\xi^*x = (x^*\xi^*y^*)^* \in \mathcal{H}^*.$$ 

Then $\mathcal{H}^*$ becomes an $N$-$M$-bimodule.

(3) For each right $W^*$-$M$-module $\mathcal{H}_M$ and left $W^*$-$M$ module $\mathcal{M}\mathcal{K}$, we denote the relative tensor product of $\mathcal{H}$ and $\mathcal{K}$ with respect to $M$ by $\mathcal{H} \otimes^M \mathcal{K}$. The relative tensor product is associative. For a faithful semi-finite normal weight $\phi$, the subspace of sums of the form $\xi\phi^{-\frac{1}{2}}\eta$’s is dense in $\mathcal{H} \otimes^M \mathcal{K}$. The relative tensor products have the following property.

$$\mathcal{H} \otimes^M L^2(M) \cong \mathcal{H}, \ L^2(M) \otimes^M \mathcal{K} \cong \mathcal{K},$$

$$\mathcal{K} \otimes^{(M)^*} \mathcal{K}^* \cong L^2(M), \ K^* \otimes^M \mathcal{K} \cong L^2(M')$$

where these isomorphisms mean as $W^*$-modules.
We fix a von Neumann algebra $M$. Let $X_M$ be a Hilbert $M$-module and $H_M$ be a right $W^*$-$M$-module. We can define the right $W^*$-$M$-module $H(X)_M$ and the Hilbert $M$-module $X(H)_M$ as the following.

$$H(X)_M = (X \otimes_M L^2(M))_M;$$

$$(x \otimes \xi, y \otimes \eta)_{H(X)} = (\xi, (x, y)\eta) \quad (x \otimes \xi, y \otimes \eta \in H(X)),$$

$$X(H) = \text{Hom}(L^2(M)_M, H_M)_M,$$

$$(x, y)_{X(H)} = x^*y \in \text{End}(L^2(M)_M = M \quad (x, y \in X(H)).$$

This gives a one-to-one correspondence between Hilbert $M$-modules and right $W^*$-$M$-modules.

From now on, we fix a von Neumann algebra $M$ acting on a Hilbert space $\mathcal{H}$ and a normal UCP-map $T$ on $M$. We see relations between the relative tensor product $\otimes^M$ and the tensor product $\otimes_T$ defined in Section 1.

**Definition 3.2.** Since $M$ acts on the standard space $L^2(M)$ of $M$, we can define a left $W^*$-$M$-module $\mathcal{H}(M, T) = M \otimes_T L^2(M)$ (Definition [2.7]). We define a right action of $M$ on $\mathcal{H}(M, T)$ by $(x \otimes \xi)y = x \otimes \xi y$ for each $x, y \in M$ and $\xi \in L^2(M)$. Then $\mathcal{H}(M, T)$ is a $W^*$-$M$-$M$-bimodule.

**Proposition 3.3.** An isomorphism $\mathcal{H}(M, T) \otimes^M \mathcal{H}(M, T) \cong M \otimes_T (M \otimes_T L^2(M))$ holds as $W^*$-bimodules.

*Proof.* Let $\phi$ be a faithful semi-finite normal weight on $M$. We define a correspondence from an each vector

$$((x \otimes_T y) \phi^{\frac{1}{2}})(z \otimes_T \phi^{\frac{1}{2}}w) \in (M \otimes_T L^2(M)) \otimes^\phi (M \otimes_T L^2(M)) \cong (M \otimes_T L^2(M)) \otimes^M (M \otimes_T L^2(M)) = \mathcal{H}(M, T) \otimes^M \mathcal{H}(M, T);$$

and to the vector

$$x \otimes_T ((yz) \otimes_T (\phi^{\frac{1}{2}} \phi^{-\frac{1}{2}} \phi^{\frac{1}{2}} w)) = x \otimes_T ((yz) \otimes_T (\phi^{\frac{1}{2}} w)) \in M \otimes_T (M \otimes_T L^2(M)).$$

Then this correspondence gives a $W^*$-bimodule isomorphism. \qed

**Proposition 3.4.** An isomorphism $\mathcal{H}(M, T) \otimes^M \mathcal{H} \cong M \otimes_T \mathcal{H}$ holds as $W^*$-bimodules.
Proof. Let $\phi$ be a faithful semi-finite normal weight on $M$. By Fact 3.1 (1) with respect to the decomposition of $\mathcal{H}$, each vector $\xi \in \mathcal{H}$ can be represented as $\bigoplus_{i \in I} \xi_i$ for some $\xi_i \in p_i L^2(M)$ and the projection $p_i$. We define a correspondence which maps
\[
(x \otimes_T y) \phi^{-\frac{1}{2}} \bigoplus_{i \in I} \xi_i \in (M \otimes_T L^2(M)) \otimes^M \mathcal{H}
\]
to $x \otimes_T \left( \bigoplus_{i \in I} \xi_i \right) \in M \otimes_T \mathcal{H}$. This correspondence is a unitary.

Now, we have
\[
\mathcal{H}(M, T) \otimes^M \mathcal{H}(M, T) \otimes^M \mathcal{H}(M, T) = (M \otimes_T L^2(M)) \otimes^M (M \otimes_T L^2(M)) \otimes^M (M \otimes_T L^2(M))
\]
\[
\cong (M \otimes_T L^2(M)) \otimes^M (M \otimes_T (M \otimes_T L^2(M)))
\]
\[
\cong (M \otimes_T (M \otimes_T (M \otimes_T L^2(M))))
\]

Indeed the first isomorphism is implied from Proposition 3.3 and the third isomorphism is given by a unitary defined by
\[
(x_1 \otimes_T x_2 \phi^{\frac{1}{2}}) \phi^{-\frac{1}{2}} (x_3 \otimes_T (x_4 \otimes_T \phi^{\frac{1}{2}} x_5)) \mapsto x_1 \otimes_T ((x_2 x_3) \otimes_T (x_4 \otimes_T \phi^{\frac{1}{2}} x_5))
\]
for each $x_1, x_2, x_3, x_4, x_5 \in M$ similarly to the proof of Proposition 3.4. In the same way, we have
\[
\underbrace{\mathcal{H}(M, T) \otimes^M \cdots \otimes^M \mathcal{H}(M, T)}_{n \text{ times}} \cong \underbrace{M \otimes_T (M \otimes_T L^2(M)) \cdots}_{n \text{ times}}.
\]

We define a $W^*-M-M$-bimodule
\[
_M \mathcal{H}_n(M, T)_M = \underbrace{M \otimes^M \cdots \otimes^M \mathcal{H}(M, T)}_{n \text{ times}}
\]
and a $W^*-M'-M'$-bimodule
\[
(M')^\phi \mathcal{H}'_n(M, T)(M')^\phi = (M')^\phi \underbrace{\mathcal{H}^* \otimes^M \mathcal{H}_n(M, T) \otimes^M \mathcal{H}(M')^\phi}_{\text{for each } n \in \mathbb{N}}.
\]
Proposition 3.5. We have an isomorphism
\[ \mathcal{H}'_n(M, T) \cong \mathcal{H}'_1(M, T) \otimes (M')^n \mathcal{H}'_1(M, T) \]
as \(W^*\)-bimodules for all \(n \in \mathbb{N}\).

Proof. By Fact 3.1 (3), we have isomorphisms
\[
\mathcal{H}'_1(M, T) \otimes (M')^n \mathcal{H}'_1(M, T)_{M'} \\
= \mathcal{H}^* \otimes M \mathcal{H}(M, T) \otimes M (M')^n \mathcal{H}^* \otimes M \mathcal{H}(M, T) \otimes M \mathcal{H} \\
\cong \mathcal{H}^* \otimes M \mathcal{H}(M, T) \otimes M L^2(M) \otimes M \mathcal{H}(M, T) \otimes M \mathcal{H} \\
\cong \mathcal{H}^* \otimes M \mathcal{H}(M, T) \otimes M \mathcal{H}(M, T) \otimes M \mathcal{H} \\
= \mathcal{H}'_2(M, T)
\]
as \(W^*\)-\((M')^n\)-\((M')^n\)-bimodules. \(\square\)

Corollary 3.6. We have an isomorphism
\[ \mathcal{H}_n(M, T) \otimes M \mathcal{H} \cong M \otimes_T (M \otimes_T \cdots (M \otimes_T (M \otimes_T \mathcal{H})) \cdots) \]
as \(W^*\)-bimodules for all \(n \in \mathbb{N}\).

A map defined by
\[
\text{Hom}_{(M, M)}(M \otimes_T L^2(M)) \otimes M \mathcal{H} \\
\ni X \mapsto X' \in \text{Hom}_{(M', M')}(M' \otimes M \mathcal{H}, M' \otimes M \mathcal{H}) \\
X' : \mathcal{H}^* \otimes M \mathcal{H} \ni \eta \gamma \xi \mapsto \eta \gamma \xi X' \in \mathcal{H}^* \otimes M \mathcal{H}
\]
induces isomorphisms
\[
\text{Hom}_{(M', M)}(M \otimes_T L^2(M)) \otimes M \mathcal{H} \\
\cong \text{Hom}_{(M', M)}(M' \otimes M \mathcal{H}, M' \otimes M \mathcal{H}) \\
\cong \text{Hom}_{(M', M)}(L^2(M'), M' \otimes M \mathcal{H}) \quad (\because \text{Fact 3.1 (3)})
\]
Then \(\text{Hom}_{(M', M)}(L^2(M'), M' \otimes M \mathcal{H})\) corresponds to \(\mathcal{H}^* \otimes M \mathcal{H} = \mathcal{H}'_1(M, T)\) by Fact 3.1 (4).
4 Two constructions of the minimal dilation

In this section, we describe two constructions of the minimal dilation by Bhat-Skeide\cite{10} and Muhly-Solel\cite{13}, and see a relation between these constructions. We fix a von Neumann algebra $M$ acting on a Hilbert space $\mathcal{H}$ and a normal UCP-map $T$ on $M$.

4.1 Bhat-Skeide’s construction

Let $(E(M, T), \xi)$ be the GNS-representation with respect to $T$. We put

$E_n = E(M, T) \otimes_M \cdots \otimes_M E(M, T),$

$\xi_n = \xi \otimes \cdots \otimes \xi$

$n$ times

Then $(E_n, \xi_n)$ is the GNS-representation with respect to $T^n$ for each $n \in \mathbb{N}$ by the uniqueness of the GNS-representation. Let $E$ be an the inductive limit of the inductive system $(\{E_n\}_{n=0}^{\infty}, \{\xi_{n-m} \otimes \text{id}_{E_n}\}_{n, m=0}^{\infty})$. We define $\mathcal{K}_n = E_n \otimes_M \mathcal{H}$ for each $n \in \mathbb{N}$ and $\mathcal{K} = E \otimes_M \mathcal{H}$. By the identification in Definition 2.6 and \cite{16}, each $E^n \subset \mathcal{B}(\mathcal{H}, \mathcal{K}_n)$ is a von Neumann $M$-$M$-bimodule and $E^n \subset \mathcal{B}(\mathcal{H}, \mathcal{K})$ is so, where $\overline{-}$ means the strong closure.

We define an endomorphism $\theta$ on $\mathcal{B}^a(\overline{E^n})$ by

$\theta(b) = b \otimes \text{id}_{\overline{E^n}} \in \mathcal{B}^a(\overline{E^n} \otimes \overline{E^n}) \cong \mathcal{B}^a(\overline{E^n})$ \quad ($b \in \mathcal{B}^a(\overline{E^n})$).

For each $a \in M$, we define $j_0(a) \in \mathcal{B}^a(\overline{E^n})$ by

$j_0(a)(\eta) = \xi a(\xi, \eta)$ \quad ($\eta \in E$)

and $j_n = \theta^n \circ j_0 \in \mathcal{B}^a(\overline{E^n})$ for each $n \in \mathbb{N}$. Then we have

$j_m(1_M)j_n(a)j_m(1_M) = j_m(T^{n-m}(a))$

for all $n \geq m$ and $a \in M$. We can identify that $M = j_0(M)$. Let $N$ be a von Neumann algebra generated by $j_{\mathbb{Z}_{\geq 0}}(M)$, $p$ be $j_0(1_M)$ and $\alpha$ be a restriction of $\theta$ to $N$. Then the conditions in Definition 1.1 are satisfied.
4.2 Muhly-Solel’s construction

Put \( E(0) = M' \). For each \( n \in \mathbb{N} \), we define \( \mathcal{H}_n = (M \otimes_T (\cdots \otimes_T (M \otimes_T \mathcal{H}) \cdots)) \)
and \( E(n) = \text{Hom}(M \mathcal{H}, M \mathcal{H}_n) \). Each \( E(n) \) admits an \( M' \)-valued inner product defined by
\[
(X, Y) = X^*Y \in M' \quad (X, Y \in E(n)),
\]
and we can define left and right actions of \( M' \) on \( E(n) \) by
\[
(xX)\xi = (1_M \otimes \cdots \otimes 1_M \otimes x)X\xi \quad (x \in M', X \in E(n), \xi \in \mathcal{H}),
\]
\[
(Xx)\xi = X(x\xi) \quad (x \in M', X \in E(n), \xi \in \mathcal{H}).
\]
Then \( E(n) \) becomes a \( W^* \)-correspondence over \( M' \) in the sense of [13], and we identify \( E(n) \otimes_{M'} E(m) \) with \( E(n+m) \) by a map
\[
U_{n,m} : E(n) \otimes_{M'} E(m) \ni X_n \otimes X_m \mapsto (1_M \otimes \cdots \otimes 1_M \otimes X_n)X_m \in E(n+m)
\]
for each \( n, m \in \mathbb{Z}_{\geq 0} \).

Now, we put \( \bar{P}_0 = \text{id}_{E(0)} \) and \( \mathcal{L}_0 = \mathcal{H} \), and for each \( n \in \mathbb{N} \) define a map \( P_n : E(n) \to \mathcal{B}(\mathcal{H}) \) by \( P_n(X) = i^* \circ X \) for each \( X \in E(n) \). Let \( \mathcal{L}_n \) be a Hilbert space which is given by the completion of \( E(n) \otimes_{\text{alg}} \mathcal{H} \) with respect to an inner product defined by
\[
(X \otimes \xi, Y \otimes \eta) = (X\xi, Y\eta) \quad (X, Y \in E(n), \xi, \eta \in \mathcal{H}).
\]
For each \( 0 < m < n \), we define isometric operators \( u_{n,m} \) by
\[
u_{n,m} = (U_{n,m-n} \otimes 1_{\mathcal{B}(\mathcal{H})})(\text{id}_{E(m)} \otimes \bar{P}_{n-m}^*) : \mathcal{L}_m \to \mathcal{L}_n,
\]
\[
u_{n,0} = \bar{P}_n^* : \mathcal{L}_0 \to \mathcal{L}_n,
\]
\[
u_{n,n} = 1_{\mathcal{B}(\mathcal{L}_n)} : \mathcal{L}_n \to \mathcal{L}_n,
\]
where for all \( Q : E(n) \to \mathcal{B}(\mathcal{H}) \), a map \( \bar{Q} : \mathcal{L}_n \to \mathcal{H}_n \) is defined by \( \bar{Q}(X \otimes \xi) = Q(X)\xi \) for each \( X \in E(n) \) and \( \xi \in \mathcal{H}_n \). Let \( \mathcal{L} \) be the inductive limit of \( (\{\mathcal{L}_n\}_{n=0}^\infty, \{u_{nm}\}_{n,m=0}^\infty) \) and \( \iota_n : \mathcal{L}_n \to \mathcal{L} \) be the canonical embedding for each \( n \in \mathbb{Z}_{\geq 0} \). For each \( m \in \mathbb{Z}_{\geq 0} \) and \( X_n \in E(n) \), we define \( V_n(X_n) \in \mathcal{B}(\mathcal{L}) \) by
\[
V_n(X_n)(\iota_m(X_m \otimes \xi)) = \iota_{m+n}(U_{n,m}(X_n \otimes X_m) \otimes \xi) \quad (X_m \in E(m), \xi \in \mathcal{H}).
\]
We put \( N = V_0(M')' \) and define \( \alpha(x) = \tilde{V}_1(\id_{E(1)} \otimes x)\tilde{V}_1^* \) for each \( x \in N \). Then \( \alpha \) is a normal unital \( \ast \)-endomorphism on \( N \) such that

\[
\begin{align*}
\iota_0^* N \iota_0 &= M, \\
T^n(\iota_0^* x \iota_0) &= \iota_0^* \alpha^n(x) \iota_0 \quad (n \in \mathbb{Z}_{\geq 0}, \ x \in N), \\
T^n(y) &= \iota_0^* \alpha^n(\iota_0 y \iota_0^*) \iota_0 \quad (n \in \mathbb{Z}_{\geq 0}, \ y \in M).
\end{align*}
\]

We identify \( M \) with \( \iota_0^* M \iota_0^* \) and define a projection \( p = \iota_0^* \iota_0^* \in N \). Then we have

\[
M \cong \iota_0^* M \iota_0^* = \iota_0^* \iota_0^* N \iota_0 \iota_0^* = p N p \subset N.
\]

Thus the semigroup \( \{\alpha^n\}_{n=0}^\infty \) is the minimal dilation of the semigroup \( \{T^n\}_{n=0}^\infty \) in the sense of [3] and [4]. We have constructed the minimal dilation in the sense of Definition [1].

### 4.3 The minimal dilation on the standard space

We see Muhly-Solel’s construction of the minimal dilation when \( \mathcal{H} = L^2(M) \). When we use the notation in Subsection 4.2, \( E(0) = M' \) and for each \( n \in \mathbb{N} \),

\[
\begin{align*}
\mathcal{H}_n &= (M \otimes_T (\cdots \otimes_T (M \otimes_T L^2(M)) \cdots))_{n \text{ times}}, \\
E(n) &= \Hom(M L^2(M), M \mathcal{H}_n), \\
\mathcal{L}_n &= E(n) \otimes L^2(M).
\end{align*}
\]

Then for \( n \in \mathbb{Z}_{\geq 0} \), a map \( U_n : E(n) \otimes_M L^2(M) \ni X \otimes \xi \mapsto X \xi \in \mathcal{H}_n \) gives an isomorphism \( \mathcal{L}_n \cong \mathcal{H}_n \) as Hilbert spaces. Now, for \( n \geq m \), we define an isometry

\[
v_{n,m} = U_n u_{n,m} U_m^* : \mathcal{H}_m \to \mathcal{H}_n
\]

where \( u_{n,m} : \mathcal{L}_m \to \mathcal{L}_n \) is the isometry defined in Subsection 4.2. Then \( (\{\mathcal{H}_n\}_{n=0}^\infty, \{v_{nm}\}_{n,m=0}^\infty) \) is an inductive system, and let \( \mathcal{H}' \) be the inductive limit of it. Similarly as Subsection 4.2, for each \( n \in \mathbb{Z}_{\geq 0} \), let \( \kappa_n : \mathcal{H}_n \to \mathcal{H}' \) be the canonical embedding and we define \( V'_n(X_n) \in \mathcal{B}(\mathcal{H}') \) for each \( X_n \in E(n) \) by

\[
V'_n(X_n)(\kappa_m(x_1 \otimes \cdots \otimes x_m \otimes \xi)) = \kappa_{n+m}(x_1 \otimes \cdots \otimes x_m \otimes X_n \xi)
\]

\[
(m \in \mathbb{Z}_{\geq 0}, \ x_1 \otimes \cdots \otimes x_m \otimes \xi \in \mathcal{H}_m).
\]
Then we can prove an analogue of the result in Subsection 4.2 by looking the proof of the original theorem \([13]\) i.e., if we define

\[
R = V_0'(M')',
\]

\[
\beta(x) = V_1' (\text{id}_{E(1)} \otimes x) \tilde{V}_1' (x \in R),
\]

then \(\beta\) is a normal unital \(*\)-endomorphism on \(R\) such that

\[
\begin{align*}
\kappa_0^* R \kappa_0 &= M, \\
T^n (\kappa_0^* x \kappa_0) &= \kappa_0^* \alpha^n (x) \kappa_0 \quad (n \in \mathbb{Z}_{\geq 0}, x \in N), \\
T^n (y) &= \kappa_0^* \alpha^n (\kappa_0 y \kappa_0^*) \kappa_0 \quad (n \in \mathbb{Z}_{\geq 0}, y \in M).
\end{align*}
\]

### 4.4 The relation between the two constructions

In this subsection, we use the notations in Section 3, Subsection 4.1 and 4.2.

By Proposition 3.4,

\[
E(1) = \text{Hom}(M \mathcal{H}, MM \otimes_T \mathcal{H}) \cong \text{Hom}(M \mathcal{H}, M (M \otimes_T L^2(M)) \otimes^M \mathcal{H})
\]

holds, and hence \(E(1)\) corresponds to \(\mathcal{H}^* \otimes^M (M \otimes_T L^2(M)) \otimes^M \mathcal{H}\). Hence we get a one-to-one correspondence

\[
E(n) \cong E(1) \otimes_{M'} \cdots \otimes_{M'} E(1) \longleftrightarrow \mathcal{H}^* \otimes^M \mathcal{H}_n(M, T) \otimes^M \mathcal{H}.
\]

for each \(n \in \mathbb{N}\).

On the other hand, for each \(n \in \mathbb{N}\), we can define the tensor product \(E_n \otimes_T L^2(M)\) similarly as Definition 2.7 where \(E_n\) is in Subsection 4.1. Then we have \(\mathcal{H}_2 = M \otimes_T (M \otimes_T L^2(M)) \cong E_1 \otimes_T L^2(M)\) as left \(W^*\)-module. Indeed for all \(x_1, x_2, y_1, y_2 \in M\) and \(\xi_1, \xi_2 \in L^2(M)\),

\[
(x_1 \otimes y_1 \otimes \xi_1, x_2 \otimes y_2 \otimes \xi_2)_{\mathcal{H}_2} = (y_1 \otimes \xi_1, T(x_1^* x_2^* y_2 \otimes \xi_2)_{\mathcal{H}_1},
\]

\[
= \xi, (y_1^* T(x_1^* x_2^* y_2 \xi_2))_{L^2(M)},
\]

\[
= ((x_1 \otimes y_1, x_2 \otimes y_2)_{\mathcal{H}} \xi_2)_{L^2(M)},
\]

holds. By induction, we have

\[
\mathcal{H}_{2n}(M, T) \cong E_n \otimes_T L^2(M)
\]

for each \(n \in \mathbb{N}\).

This concludes that the constructions of the dilation by Bhat-Skeide and Muhly-Solel are essentially the same.
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