HIERARCHIES OF NEW INVARIANTS AND CONSERVED INTEGRALS IN INVISCID FLUID FLOW

STEPHEN C. ANCO¹ AND GARY M. WEBB²

¹DEPARTMENT OF MATHEMATICS AND STATISTICS
BROCK UNIVERSITY
ST. CATHARINES, ON L2S3A1, CANADA

²CENTER FOR SPACE PLASMA AND AERONOMIC RESEARCH
THE UNIVERSITY OF ALABAMA IN HUNTSVILLE
HUNTSVILLE AL 35805, USA

Abstract. A vector calculus approach for the determination of advected invariants is presented for inviscid fluid flow. This approach describes invariants by means of Lie dragging of scalars, vectors, and skew-tensors with respect to the fluid velocity, which has the physical meaning of characterizing tensorial quantities that are frozen into the flow. Several new main results are obtained. First, simple algebraic and differential operators that can be applied recursively to derive a complete set of invariants starting from the basic known local and nonlocal invariants are constructed. Second, these operators are used to derive infinite hierarchies of local and nonlocal invariants for both adiabatic fluids and isentropic fluids that are either incompressible, or compressible with barotropic and non-barotropic equations of state. Each hierarchy is complete in the sense that no further invariants can be generated from the basic local and nonlocal invariants. All of the resulting new invariants are related to Ertel’s invariant, the Ertel-Rossby invariant, and Hollmann’s invariant. In particular, for incompressible fluid flow in which the density is non-constant across different fluid streamlines, a new variant of Ertel’s invariant and several new variants of Hollmann’s invariant are derived, where the entropy gradient is replaced by the density gradient. Third, the physical meaning of these new invariants and the resulting conserved integrals are discussed, and their relationship to conserved helicities and cross-helicities is described.

1. Introduction

Vorticity invariants and conserved helicity integrals have long been recognized to be important in the study of inviscid fluid flow in three dimensions, especially for understanding topological aspects of vortex flows and for studying existence, uniqueness, and stability of initial-value flows.

For a general hydrodynamical system on any spatial domain, an invariant is a material quantity constructed from the fluid variables (and possibly their spatial derivatives) such that it is advected by the flow. Physically, this means that the quantity is frozen into the

2000 Mathematics Subject Classification. Primary: 76N99, 37K05, 70S10; Secondary: 76M60.
Key words and phrases. fluid flow, conserved integral, constant of motion, vorticity, helicity, cross-helicity, circulation.

Email: sanco@brocku.ca, gmw0002@uah.edu
S.C.A. is supported by an NSERC research grant. G.M.W. is supported in part by NASA grant NNX15A165G.
flow, analogously to attaching it to fluid particles transported by the flow. Geometrically, an invariant has the property that its advective Lie derivative vanishes, where this derivative is defined by \( \mathcal{D}_t = \partial_t + \mathcal{L}_u \) in terms of the Lie derivative \( \mathcal{L}_u \) with respect to the fluid velocity \( \vec{u} \). The advective Lie derivative can be expressed alternatively as the usual material derivative \( \frac{d}{dt} = \partial_t + \vec{u} \cdot \nabla \) plus a rotation-dilation term that takes into account the tensorial nature of the quantity on which it acts.

A vorticity invariant in three dimensions refers to an invariant that has an essential dependence on the vorticity vector of the fluid flow, \( \vec{\omega} = \nabla \times \vec{u} \), given by the curl of the fluid velocity \( \vec{u} \). An invariant is local if its value at each point in the spatial domain is determined entirely by the values of the fluid variables and their spatial derivatives (up to some finite differential order) at that point, and otherwise an invariant is called nonlocal.

Inviscid isothermal fluid flow has only one local vorticity invariant, which is the densitized vorticity vector \((1/\rho)\vec{\omega}\), where \( \rho \) is the fluid density. For inviscid adiabatic fluid flow, the densitized vorticity vector \((1/\rho)\vec{\omega}\) is no longer an invariant, but there is a local vorticity invariant \( K_e = (1/\rho)\vec{\omega} \cdot \nabla S \) known as Ertel’s invariant \([8]\), where \( S \) is the fluid entropy. In the case of incompressible fluids, in which \( \rho \) is constant, the factor \( 1/\rho \) in these two invariants can be dropped. The vorticity vector invariant is well-known to be closely connected to the helicity conservation law \( \frac{d}{dt} \int_V (\vec{u} \cdot \vec{\omega}) \, dV = \oint_{\partial V} (\frac{1}{2} |\vec{u}|^2 - p/\rho) \vec{\omega} \cdot d\vec{A} \) which holds on moving volumes \( V(t) \) when the fluid pressure satisfies a barotropic equation of state, \( p = P(\rho) \). Likewise, Ertel’s invariant is related to the entropy circulation-flux conservation law \( \frac{d}{dt} \int_S (\vec{u} \times \nabla S) \cdot d\vec{A} = \oint_{\partial S} \left( \frac{1}{2} |\vec{u}|^2 - E - p/\rho \right) \nabla S \cdot d\vec{A} \), where \( E \) is the internal energy of the fluid as determined by the standard thermodynamic relation \( TdS = dE + pd(1/\rho) \) with \( T \) being the fluid temperature. Note that the helicity integral is conserved when the vorticity filaments are tangential to the moving boundary surface \( \partial V(t) \), and that the entropy circulation-flux is conserved when the entropy gradient is tangential to the moving boundary curve \( \partial S(t) \).

Apart from these well-known local vorticity invariants, there exist nonlocal vorticity invariants, which arise from Clebsch variables related Weber transformations. The oldest examples are the Ertel-Rossby invariant \([9]\) \( K_i = (1/\rho)(\vec{u} - \nabla \phi) \cdot \vec{\omega} \) for barotropic fluids, and Hollmann’s invariant \([10]\) \( K_h = (1/\rho)(\vec{u} - \nabla \phi) \cdot (\nabla S \times \nabla K_e) \) for adiabatic fluids, where \( \phi \) is a Clebsch variable defined by the transport equation \( \frac{d}{dt} \phi = \frac{1}{2} |\vec{u}|^2 - E - p/\rho \). The Ertel-Rossby invariant yields a conserved helicity integral \( \frac{d}{dt} \int_V (\vec{u} - \nabla \phi) \cdot \vec{\omega} \, dV = 0 \) which measures the self-linking of the vorticity filaments defined by \( \vec{\omega} \). In contrast to the familiar helicity integral, no boundary conditions are needed on \( V(t) \) for the integral to be conserved. Hollmann’s invariant yields a conserved cross-helicity integral \( \frac{d}{dt} \int_V (\vec{u} - \nabla \phi) \cdot (\nabla S \times \nabla K_e) \, dV = 0 \) which turns out to measure the mutual linking of filaments defined by \( \vec{\omega} \) and \( \nabla S \times \nabla K_e \).

An additional nonlocal invariant arises from the introduction of a Clebsch variable \( \psi \) defined by the transport equation \( \frac{d}{dt} \psi = T \) given in terms of the fluid temperature. This leads to a nonlocal vorticity vector invariant \( (1/\rho)\nabla \times (\vec{u} - \psi \nabla S) \), which generalizes the local vorticity vector invariant \((1/\rho)\vec{\omega}\) to inviscid adiabatic (non-barotropic) fluids. As an important consequence, there is a generalization of the Ertel-Rossby conserved helicity integral given by \( \frac{d}{dt} \int_V (\vec{u} - \nabla \phi - \psi \nabla S) \cdot \nabla \times (\vec{u} - \psi \nabla S) \, dV = 0 \) holding for adiabatic non-barotropic fluid flow, without no boundary conditions needed on \( V(t) \).
Helicity measures the knotting of vortex tubes in barotropic fluid flow. The generalized helicity in adiabatic non-barotropic fluid flow has interesting physical applications in the formation of tornadoes (see, e.g. [15]) and in solar magnetohydrodynamics (see, e.g. [16]). Both of these helicities are related to the generalized Aharonov-Bohm effect for fluids and MHD obtained in Ref.[17, 18, 19] (see also Ref.[20, 21]). More generally, vorticity invariants and associated conserved integrals are important in atmospheric and oceanic Rossby wave dynamics, where there is an interplay between the planetary vorticity and the local fluid vorticity (see, e.g. [15]).

Most strikingly, as indicated in Ref.[5] and further developed in Ref.[6], a method based on differential forms can be used to construct, in principle, an infinite set of vorticity invariants involving higher-order derivatives of the fluid variables. Those results give rise to several interesting open questions:

- How can the infinite set of local and nonlocal vorticity invariants for fluid dynamics be formulated in an explicit form?
- Can the set be produced directly from a minimal generating set of material operators and in what sense is the set complete?
- What is the physical meaning of these invariants and how are they connected to conserved helicity integrals and conserved cross-helicity integrals?

The purpose of the present work is to answer all of these questions and, as an important by-product, derive new invariants and corresponding new conserved integrals (which were not presented in Ref.[5, 6]). In particular, Ertel’s invariant and Hollmann’s invariant will be seen to be part of respective hierarchies of local and nonlocal invariants in inviscid adiabatic compressible fluid flow, while likewise the vorticity vector invariant and the related Ertel-Rossby invariant will be seen to be part of respective hierarchies of local and nonlocal invariants in inviscid isentropic compressible fluid flow. For each of these hierarchies there are corresponding material conserved integrals on moving volumes, surfaces, and curves. Specializations of both hierarchies to constant-density incompressible fluid flow also will be presented.

Furthermore, new counterparts of the hierarchies containing Ertel’s invariant and Hollmann’s invariant will be derived for incompressible fluid flow in which the density is non-constant across different fluid streamlines. In such flows the density is frozen-in and has no effect on the dynamics of the fluid velocity. These hierarchies contain a new variant of Ertel’s invariant, and several new variants of Hollmann’s invariant, where the entropy gradient is replaced by the density gradient.

Moreover, the hierarchies of invariants, covering adiabatic and isentropic inviscid compressible fluid flow, as well as constant and non-constant density inviscid incompressible fluid flow, are complete in the sense that no further invariants can be generated from the basic local and nonlocal invariants that are known for these types of fluid flow.

To make the presentation and main results accessible to the widest audience, all results will be stated using familiar vector calculus operations (and their extension to tensor calculus).

In Section 2, we discuss the different types and properties of advected invariants. We also review the definition and properties of the advective Lie derivative $\mathcal{D}_t = \partial_t + \mathcal{L}_{\vec{u}}$ as well as its relation to the material derivative, and we briefly explain the correspondence between invariants and conserved integrals on moving domains given by transported volumes, surfaces, and curves.
In Section 3, we formulate the basic algebraic and differential operations that can be applied to a set of scalar, vector, tensor invariants to yield further invariants. These material operations are the vector-calculus counterparts of the differential form methods discussed in Ref.[5, 6]. We emphasize that the formulation of material operations is a non-trivial problem because the vector dot product and cross product are not advected in fluids; in particular, the dot/cross product of two vector invariants is not an invariant.

In Sections 4 and 5, starting from the few known local and nonlocal invariants for inviscid fluid flow, we apply the algebraic and differential operations on invariants to derive complete (infinite) hierarchies of local and nonlocal invariants for both adiabatic fluid flow and isentropic fluid flow. These invariants physically describe frozen-in quantities given by advected scalars, vectors, and skew-tensors. We identify all of the invariants that are of vorticity type, and we discuss their physical meaning. New examples of vorticity invariants and their corresponding conserved integrals, including helicity and cross-helicity integrals, are shown.

In Section 6, we conclude with some further remarks.

Finally, in Appendix A, we review the basic algebraic and differential operations available in Euclidean space: dot, cross, exterior products; grad, curl, div, Lie derivative. In Appendix B, we provide a transcription between vector-calculus and differential-form notation. Additional mathematical background on differential forms and its application to fluid dynamics can be found in Ref.[4, 7].

Previous work on invariants in fluid flow can be found in Ref.[22, 23, 24, 25, 26, 27, 28, 29, 30, 31].

2. Types and properties of invariants

Throughout, we will work with scalars $K$, vectors $\vec{K}$, and skew tensors (bi-vectors) $\mathbf{K} = -\mathbf{K}^\dagger$, since these are the most familiar types of tensorial quantities in mechanics and in fluid dynamics. Simple examples of scalars are density $\rho$ and local entropy $S$; vector examples are velocity $\vec{u}$ and vorticity $\vec{\omega} = \vec{\nabla} \times \vec{u}$; examples of skew tensors are the antisymmetric derivative of the velocity $\vec{\nabla} \times \vec{u} = \vec{\nabla} \vec{u} - (\vec{\nabla} \vec{u})^\dagger$ and the entropy-circulation tensor $\vec{u} \times \vec{\nabla} S$.

Moreover, invariants of scalar type, vector type, and skew-tensor type give rise to conserved integrals defined respectively on moving volumes, surfaces, and curves in a fluid. In particular, helicity integral invariants and more general cross-helicity integral invariants are related to scalar invariants of a certain form. Flux integral invariants and circulation integral invariants are related respectively to vector invariants and skew-tensor invariants.

There is a simple way to transcribe everything into the setting of differential forms, which is explained in Appendix B.

2.1. Lie derivative advection and invariants. In fluid dynamics, a scalar, vector, or tensor quantity, $\mathbf{K}$, constructed from the fluid variables and their spatial derivatives, is an invariant (material quantity) iff it satisfies $\mathcal{D}_t \mathbf{K} = 0$ where

$$\mathcal{D}_t = \partial_t + \mathcal{L}_{\vec{u}}$$

(2.1)
is the advective Lie derivative with respect to the fluid velocity \( \vec{u} \). The Lie derivative \( \mathcal{L}_u \) is explicitly given by

\[
\mathcal{L}_u K = \vec{u} \cdot \nabla K,
\]

(2.2)

\[
\mathcal{L}_u \vec{K} = \vec{u} \cdot \nabla \vec{K} - \vec{K} \cdot \nabla \vec{u} = [\vec{u}, \vec{K}],
\]

(2.3)

\[
\mathcal{L}_u \vec{K} = \vec{u} \cdot \nabla \vec{K} - \vec{K} \cdot \nabla \vec{u} - (\nabla \vec{u})^t \cdot \vec{K},
\]

(2.4)

which gives the infinitesimal change of scalars, vectors, skew tensors under transport along fluid streamlines. In terms of the material derivative

\[
\frac{d}{dt} = \partial_t + \vec{u} \cdot \nabla,
\]

(2.5)

the advective Lie derivative can be expressed as

\[
\mathcal{D}_t K = \frac{d}{dt} K,
\]

(2.6)

\[
\mathcal{D}_t \vec{K} = \frac{d}{dt} \vec{K} - \vec{K} \cdot \nabla \vec{u},
\]

(2.7)

\[
\mathcal{D}_t \vec{K} = \frac{d}{dt} \vec{K} - \vec{K} \cdot \nabla \vec{u} - (\nabla \vec{u})^t \cdot \vec{K}.
\]

(2.8)

Physically, the (advective) Lie derivative differs from the (material) directional derivative by including rotation-dilation terms that take into account the tensorial nature of the quantity that it acts on. To explain these additional terms, consider the decomposition of the derivative of the fluid velocity into symmetric and antisymmetric tensors

\[
\nabla \vec{u} = \frac{1}{2} (s + \omega)
\]

(2.9)

where

\[
\omega = \nabla \wedge \vec{u} = \nabla \vec{u} - (\nabla \vec{u})^t
\]

(2.10)

is antisymmetric derivative of \( \vec{u} \) which measures rotation of streamlines, and where

\[
s = \nabla \circ \vec{u} = \nabla \vec{u} + (\nabla \vec{u})^t
\]

(2.11)

is the symmetric derivative of \( \vec{u} \) which measures stretching of streamlines. More specifically, the trace of \( \frac{1}{2}s \) given by \( \vec{\nabla} \cdot \vec{u} \) describes expansion/contraction, while the trace-free part of \( \frac{1}{2}s \) describes shear. Then, for vectors and skew tensors,

\[
\frac{d}{dt} \vec{K} - \mathcal{D}_t \vec{K} = \frac{1}{2} \vec{K} \cdot s + \frac{1}{2} \vec{K} \cdot \omega,
\]

(2.12)

\[
\frac{d}{dt} \vec{K} - \mathcal{D}_t \vec{K} = \frac{1}{2} (\vec{K} \cdot s + s \cdot \vec{K}) + \frac{1}{2} (\vec{K} \cdot \omega - \omega \cdot \vec{K}),
\]

(2.13)

while for scalars,

\[
\frac{d}{dt} K - \mathcal{D}_t K = 0.
\]

(2.14)

Basic examples of invariants in fluid dynamics are shown in Table 1. These invariants will be discussed further in Section 4. For understanding the physical meaning of the vector and skew tensor invariants, it is useful to recall how a vector \( \vec{K} \) can be decomposed into a magnitude \( |\vec{K}| = \sqrt{\vec{K} \cdot \vec{K}} \) and a direction represented by a unit vector \( \vec{K} = (1/|\vec{K}|)\vec{K} \);
similarly, a skew tensor \( K \) can be decomposed into a magnitude \( |K| = \sqrt{K : K} \) and a plane represented by a unit bi-vector \( \hat{K} = (1/|K|)K \).

**Table 1. Basic invariants**

| Invariant | Type       | Physical Meaning                                   | Fluid system                  |
|-----------|------------|---------------------------------------------------|-------------------------------|
| \( S \)  | scalar     | local entropy                                     | adiabatic                     |
| \( \rho \) | scalar     | density                                           | incompressible                |
| \( \vec{\omega} = \vec{\nabla} \times \vec{u} \) | vector     | vorticity                                         | constant density              |
| \( \vec{\nabla} \rho \times \vec{\nabla} S \) | vector     | transversality of isentropic and isodensity surfaces | adiabatic incompressible      |
| \( \epsilon \cdot \vec{\nabla} S \) | skew tensor | entropy gradient                                  | adiabatic                     |
| \( \epsilon \cdot \vec{\nabla} \rho \) | skew tensor | density gradient                                  | incompressible                |

2.2. **Integral invariants.** In fluid dynamics, the most physically useful type of conserved integrals (material conservation laws) are defined on moving domains given by volumes \( \mathcal{V}(t) \), surfaces \( \mathcal{S}(t) \), and curves \( \mathcal{C}(t) \) that are transported by the fluid flow. The points \( \vec{x}(t) \) comprising a transported domain obey

\[
\frac{d\vec{x}(t)}{dt} = \vec{u}(t, \vec{x}(t)).
\]  

(2.15)

It is physically natural to consider domains that are connected and (piece-wise) smooth. Integrals on a moving domain involve a density which is a function of the fluid variables and possibly their spatial derivatives as well as possibly \( t \) and \( \vec{x} \).

A moving volume integral has the form

\[
\int_{\mathcal{V}(t)} T dV
\]  

(2.16)

on a transported volume \( \mathcal{V}(t) \), where \( T \) is a scalar density and \( dV \) is the volume element. This integral (2.16) is conserved if its time derivative vanishes

\[
\frac{d}{dt} \int_{\mathcal{V}(t)} T dV = 0
\]  

(2.17)

when it is evaluated for all solutions of a given fluid system. Then \( \int_{\mathcal{V}(t)} T dV \) is an integral invariant (namely, a constant of motion).

It is well-known that a scalar density \( T \) yields a conserved volume integral if, and only if, it satisfies the advection equation

\[
\mathcal{D}_t T + (\vec{\nabla} \cdot \vec{u}) T = \rho \mathcal{D}_t ((1/\rho)T) = 0
\]  

(2.18)

holding for all solutions of a given fluid system. Here \( \mathcal{D}_t \) is the advective Lie derivative (2.1); since \( T \) is a scalar, this derivative coincides with the material derivative \( \frac{d}{dt} \). The scalar density advection equation (2.18) is often called the Reynolds transport theorem in the fluid mechanics literature [32], and it directly relates invariant moving volume integrals to scalar invariants.
There is a similar direct relation between invariant moving surface integrals and vector invariants, and also between invariant moving curve integrals and skew-tensor invariants, which are considered next.

A moving surface integral has the form

\[ \int_{S(t)} \vec{T} \cdot \hat{\nu} dA = \int_{S(t)} \vec{T} \cdot \hat{\nu} dA \]  

(2.19)
on a transported surface \( S(t) \), where \( \vec{T} \) is a vector density and \( d\vec{A} = \hat{\nu} dA \) is given by the surface element \( dA \) and the unit normal vector \( \hat{\nu} \) for \( S(t) \). When the surface \( S(t) \) is a closed (namely, it has no boundary), the normal vector \( \hat{\nu} \) is usually chosen to be outward directed; when \( S(t) \) is an open surface, the normal vector \( \hat{\nu} \) can be chosen in an arbitrary but continuous fashion at each point on the interior of the surface. An integral (2.19) is conserved if its time derivative vanishes

\[ \frac{d}{dt} \int_{S(t)} \vec{T} \cdot \hat{\nu} dA = 0 \]  

(2.20)
when it is evaluated for all solutions of a given fluid system. Then \( \int_{S(t)} \vec{T} \cdot \hat{\nu} dA \) is an integral invariant which describes the net flux of \( \vec{T} \) through the surface \( S(t) \) in the direction \( \hat{\nu} \).

It can be shown that a vector density \( \vec{T} \) yields a conserved flux integral if, and only if, it satisfies the advection equation

\[ \mathcal{D}_t \vec{T} + (\vec{\nabla} \cdot \vec{u}) \vec{T} = \rho \mathcal{D}_t ((1/\rho) \vec{T}) = 0 \]  

(2.21)
holding for all solutions of a given fluid system. Note that here the advective Lie derivative \( \mathcal{D}_t \) consists of the material derivative plus a rotation-dilation term that takes into account the vectorial nature of \( \vec{T} \):

\[ \mathcal{D}_t \vec{T} = \frac{d}{dt} \vec{T} - \frac{1}{2} \vec{T} \cdot s - \frac{1}{2} \vec{T} \cdot \omega, \]  

(2.22)
where \( s \) and \( \omega \) are the symmetric and antisymmetric parts (2.11)–(2.10) of the derivative of \( \vec{u} \).

Finally, a moving curve integral has the form

\[ \int_{C(t)} T : \hat{\nu} d\ell = \int_{C(t)} (\epsilon : T) \cdot d\vec{s} \]  

(2.23)
on a transported curve \( C(t) \), where \( T \) is a skew-tensor density and \( d\vec{s} = \epsilon : \hat{\nu} d\ell \) is given by the line element \( d\ell \) and the unit normal bi-vector \( \hat{\nu} \) for \( C(t) \), where \( \epsilon \) denotes the volume tensor. (Namely, \( \hat{\nu} \) belongs to the normal plane at each point on the curve, and hence \( \epsilon : \hat{\nu} = \hat{s} \) is a unit tangent vector along the curve.) This integral (2.23) is conserved if its time derivative vanishes

\[ \frac{d}{dt} \int_{C(t)} (\epsilon : T) \cdot d\vec{s} = 0 \]  

(2.24)
when it is evaluated for all solutions of a given fluid system. Then \( \int_{C(t)} (\epsilon : T) \cdot d\vec{s} \) is an integral invariant which describes the net circulation of \( \epsilon : T \) along the curve \( C(t) \) in the direction \( \hat{s} \).

The reason for formulating moving curve integrals in terms of skew-tensor densities, rather than the more common approach of using scalar or vector densities, is shown by the simplicity of the following condition relating invariant circulation integrals to invariant skew-tensors.
A skew-tensor density $T$ yields a conserved circulation integral if, and only if, it satisfies the advection equation
\[ \mathcal{D}_t T + (\vec{\nabla} \cdot \vec{u}) T = \rho \mathcal{D}_t ((1/\rho) T) = 0 \] (2.25)
holding for all solutions of a given fluid system. Here the advective Lie derivative $\mathcal{D}_t$ consists of the material derivative plus a rotation-dilation term that takes into account the tensorial nature of $T$:
\[ \mathcal{D}_t T = \frac{d}{dt} T - \frac{1}{2}(T \cdot s + s \cdot T) - \frac{1}{2}(T \cdot \omega - \omega \cdot T). \] (2.26)

Further discussion of material conservation laws and related developments from a modern viewpoint appears in Ref.[33, 34, 35].

2.2.1. Helicity and cross-helicity. Any scalar invariant of the form $K = (1/\rho) \vec{\xi} \cdot (\vec{\nabla} \times \vec{\xi})$ yields a conserved helicity integral \[ \frac{d}{dt} \int_{V(t)} \vec{\xi} \cdot (\vec{\nabla} \times \vec{\xi}) \ dV = 0 \] (2.27)
for the vorticity filaments defined by the integral curves of $\vec{\nabla} \times \vec{\xi}$ for any non-gradient vector field $\vec{\xi}$ in a fluid. Helicity integrals measure the topological self-linking (knottedness) of the vorticity filaments. When $(1/\rho) \vec{\xi} \cdot (\vec{\nabla} \times \vec{\xi})$ is advected in the fluid, helicity is conserved without the need for boundary conditions on the curl vector field $\vec{\nabla} \times \vec{\xi}$.

The notion of helicity of a curl vector field is known to have a generalization to the mutual linking of a pair of curl vector fields: \( \int_{V(t)} \vec{\xi} \cdot (\vec{\nabla} \times \vec{\xi}) \ dV = \int_{V(t)} \vec{\zeta} \cdot (\vec{\nabla} \times \vec{\zeta}) \ dV \) when $\vec{\xi} \times \vec{\zeta}$ is tangent to the moving boundary surface $\partial V(t)$. This type of moving volume integral is called a cross-helicity integral \[ \mathcal{C} \] \[ \] . It is conserved if either $K = (1/\rho) \vec{\xi} \cdot (\vec{\nabla} \times \vec{\xi})$ or $K = (1/\rho) \vec{\zeta} \cdot (\vec{\nabla} \times \vec{\zeta})$ is a scalar invariant, with their densitized difference being a total divergence $\vec{\nabla} \cdot (\vec{\xi} \times \vec{\zeta})$.

3. Operations on invariants

In fluid dynamics, the dot product and cross-product, as well as the gradient, divergence, and curl, have non-trivial transport properties that are important to understand when these operations act on invariants, $K$.

We begin with the dot product and cross-product. For any two invariant vectors $\vec{K}_1$ and $\vec{K}_2$, their dot product and cross-product satisfy the advection identities
\[ \mathcal{D}_t (\vec{K}_1 \cdot \vec{K}_2) = s : (\vec{K}_1 \otimes \vec{K}_2), \] (3.1)
\[ \mathcal{D}_t (\vec{K}_1 \times \vec{K}_2) = - (\vec{\nabla} \cdot \vec{u}) \vec{K}_1 \times \vec{K}_2 + \vec{K}_1 \times (\vec{K}_2 \cdot s) - \vec{K}_2 \times (\vec{K}_1 \cdot s), \] (3.2)
as shown in Appendix A. Thus, these two operations are not advected in a fluid, unless there is no shear and no expansion/contraction so that $s = 0$ (and hence $\vec{\nabla} \cdot \vec{u} = 0$). The triple product of three invariant vectors satisfies
\[ \mathcal{D}_t (\vec{K}_3 \cdot (\vec{K}_1 \times \vec{K}_2)) = - (\vec{\nabla} \cdot \vec{u}) \vec{K}_3 \cdot (\vec{K}_1 \times \vec{K}_2), \] (3.3)
which is advected in incompressible fluids and hence is an invariant.
Next we consider the gradient, divergence, and curl. As shown in Appendix A, the gradient of any scalar invariant $K$ satisfies the advection identity

$$\mathcal{D}_t(\vec{\nabla} K) = -s \cdot \vec{\nabla} K,$$  \hspace{1cm} (3.4)

and the divergence and curl of any vector invariant $\vec{K}$ satisfy the advection identities

$$\mathcal{D}_t(\vec{\nabla} \cdot \vec{K}) = \vec{K} \cdot \vec{\nabla} (\vec{\nabla} \cdot \vec{u}),$$  \hspace{1cm} (3.5)

$$\mathcal{D}_t(\vec{\nabla} \times \vec{K}) = \vec{\nabla} \times (\vec{K} \cdot \vec{s}) - (\vec{\nabla} \cdot \vec{u})\vec{\nabla} \times \vec{K}.$$  \hspace{1cm} (3.6)

Thus, none of these operations are advected in a fluid, due to the presence of shear and expansion/contraction.

All of the preceding advection properties arise from how the underlying metric, $g$, and volume tensor, $\epsilon$, in Euclidean space $\mathbb{R}^3$ behave under transport along fluid streamlines. The metric is a covariant symmetric tensor that directly defines the dot product, while the volume tensor is a totally antisymmetric contravariant tensor that defines the cross-product in combination with the metric, as explained in Appendix A. Under transport along streamlines, they obey

$$\mathcal{L}_{\vec{u}}g^{-1} = -s$$  \hspace{1cm} (3.7)

where $g^{-1}$ denotes the contravariant metric tensor, and

$$\mathcal{L}_{\vec{u}}\epsilon = -(\vec{\nabla} \cdot \vec{u})\epsilon, \quad \mathcal{L}_{\vec{u}}\epsilon = (\vec{\nabla} \cdot \vec{u})\epsilon$$  \hspace{1cm} (3.8)

where $\epsilon$ is a 3-form that is the covariant counterpart of $\epsilon$ and represents the volume element $dV$. These properties (3.7)–(3.8) are a geometrical version of the statements that $s$ physically measures stretching (shear and expansion/contraction) of fluid elements, and that $\vec{\nabla} \cdot \vec{u}$ physically measures expansion/contraction of fluid elements.

### 3.1. Material operations

We will now formulate a complete set of algebraic and differential material operations that take invariants into invariants for inviscid fluid flow in three dimensions. Specifically, any local operator acting on invariants can be expressed as a composition of the operations in this set. All of these operations will be presented in both geometrical and component forms.

To begin, we observe that the volume tensor $\epsilon$ has the advection property

$$\mathcal{D}_t\epsilon = -(\vec{\nabla} \cdot \vec{u})\epsilon.$$  \hspace{1cm} (3.9)

Thus, $\epsilon$ is advected in incompressible fluid flow. In compressible flows, the expansion/contraction of $\epsilon$ can be compensated by noting that the fluid density $\rho$ has the same advection property, since a fluid volume element physically expands/contracts by the factor $\vec{\nabla} \cdot \vec{u}$ in a compressible flow. Hence, the densitized volume tensor $(1/\rho)\epsilon$ obeys

$$\mathcal{D}_t((1/\rho)\epsilon) = 0,$$  \hspace{1cm} (3.10)

which holds in compressible as well as incompressible fluid flow.

This skew-tensor $(1/\rho)\epsilon$ along with the exterior product $\wedge$ and the vector derivative operator $\vec{\nabla}$ will be main ingredients in the sequel.
3.2. Material algebraic operations. We first consider scalar invariants, $K$. It is easy to see that any sum or product of scalar invariants is a scalar invariant. More generally, we have the following straightforward result.

**Proposition 3.1.** If $K_1$ and $K_2$ are scalar invariants, then so is $f(K_1, K_2)$ for any differentiable function $f$. Moreover, if $K$ is an invariant, then so is $f(K_1, K_2)K$.

The proof of the first part follows directly from the chain rule:

$$\frac{D}{dt}f = \frac{\partial f}{\partial K_1} \frac{D}{dt}K_1 + \frac{\partial f}{\partial K_2} \frac{D}{dt}K_2 = 0.$$

The second part follows from the product rule for $\frac{D}{dt}$.

Next we consider vector invariants, $\vec{K}$. Although the dot product and cross product of vector invariants are not invariant themselves, the exterior product of vector invariants is an invariant.

**Proposition 3.2.** If $\vec{K}_1$, $\vec{K}_2$ are vector invariants, then $\vec{K}_1 \wedge \vec{K}_2$ is an invariant skew-tensor.

The proof is simply the product rule for $\frac{D}{dt}$.

The triple product of vector invariants is also not an invariant (unless the fluid is incompressible), but it yields an invariant when it is multiplied by the fluid density $\rho$.

**Proposition 3.3.** If $\vec{K}_1$, $\vec{K}_2$, $\vec{K}_3$ are vector invariants, then $\rho(\vec{K}_1 \times \vec{K}_2) \cdot \vec{K}_3$ is an invariant scalar.

The proof amounts to combining the advection identity (3.3) and the mass continuity equation for $\rho$.

An analogous result holds for the product of a vector invariant with a skew-tensor invariant, as well as for the product of two skew-tensors. These products are related to the vector triple product in the following way:

$$K \times \vec{K} = \vec{K} \times K = 2\vec{K} \cdot (\vec{K}_1 \times \vec{K}_2) \quad \text{when} \quad K = \vec{K}_1 \wedge \vec{K}_2$$

(3.11)

and

$$K_1 \times K_2 = -K_2 \times K_1 = (\vec{K}_2 \times \vec{K}_2)\vec{K}_1 - (\vec{K}_1 \times \vec{K}_2)\vec{K}_2 \quad \text{when} \quad K_1 = \vec{K}_1 \wedge \vec{K}_2$$

(3.12)

Their definitions in component form for general skew tensors are stated in Appendix A.

**Proposition 3.4.** If $\vec{K}_1$ is a vector invariant and $K_2$ is a skew-tensor invariant, then $\vec{K}_1 \wedge K_2$ is an invariant totally antisymmetric tensor and $\rho \vec{K}_1 \times K_2$ is an invariant scalar.

The proof of the first part is simply the product rule for $\frac{D}{dt}$, while the second part then follows from the identity $\vec{K}_1 \wedge K_2 = (\vec{K}_1 \times K_2)e$ combined with the advection property (3.9).

**Proposition 3.5.** If $K_1$ and $K_2$ are skew-tensor invariants, then $\rho K_1 \times K_2$ is an invariant vector.

The proof is similar to that of the vector triple product.

Note that we do not need to consider totally antisymmetric tensors like the volume tensor because any such tensor can be converted into a corresponding scalar by contraction with the volume form.

Propositions 3.1 to 3.5 and their compositions, encompass all possible material algebraic operations. This can be demonstrated more easily by using differential forms, as shown in Appendix B.
Examples of the use of these operations will be presented in the application to fluid dynamics in Section 4.

3.3. Material differential operations. A primary differential operator that takes invariants into invariants is the Lie derivative with respect to an invariant vector, \( \vec{K} \). This result is a direct mathematical consequence of the commutator identity

\[
[\mathcal{D}_t, \mathcal{L}_{\vec{K}}] = \mathcal{L}_{\mathcal{D}_t \vec{K}} = 0. \tag{3.13}
\]

From a physical viewpoint, for any invariant vector \( \vec{K} \), its corresponding streamlines can be regarded as representing an invariant flow, and consequently the infinitesimal change of scalars, vectors, skew tensors under transport first along these streamlines and next along the fluid streamlines is the same as their infinitesimal change under transport first along the fluid streamlines and next along the streamlines of the invariant flow.

Hence we have the following main result.

**Proposition 3.6.** If \( K \) is an invariant, then its Lie derivative \( \mathcal{L}_{\vec{K}}K \) with respect to any vector invariant \( \vec{K} \) is an invariant of the same type as \( K \).

The expression for a Lie derivative on scalars, vectors, and tensors is closely related to the gradient and the divergence operators. A material version of these two operators is given by the following result.

**Proposition 3.7.** If \( K, \vec{K}, \mathbf{K} \) are invariants, then \( (1/\rho)\vec{\nabla} \cdot (\rho \vec{K}) \) is an invariant vector, \( (1/\rho)\vec{\nabla} \cdot (\rho \vec{K}) \) is an invariant skew-tensor, and \( (1/\rho)\vec{\nabla} \cdot (\rho \mathbf{K}) \) is an invariant vector.

The proof is a direct computation which uses the Lie derivative formulas (2.2)–(2.4). In particular, for the first part, \( \mathcal{D}_t((1/\rho)\vec{\nabla} \cdot (\rho \vec{K})) = (1/\rho)\vec{\nabla} \cdot \mathcal{D}_t(\rho \vec{K}) + (1/\rho)\vec{\nabla} \cdot (\rho \vec{K}) = (1/\rho)\vec{\nabla} \cdot \mathcal{D}_t \vec{K} = 0 \), after use of the advection identities (3.1) and (3.4). For the second part, \( \mathcal{D}_t((1/\rho)\vec{\nabla} \cdot (\rho \mathbf{K})) = \mathcal{D}_t(\vec{\nabla} \cdot \vec{K}) + \mathcal{D}_t(\vec{\nabla} \ln \rho \cdot \vec{K}) = \vec{\nabla} \cdot (\mathcal{D}_t \ln \rho + \vec{\nabla} \cdot \vec{u}) = 0 \) using the advection identity (3.5) combined with the mass continuity equation satisfied by \( \rho \). The third part is similar.

All material differential operators of first order are encompassed by Propositions 3.6 and 3.7 as can be demonstrated by using differential forms, which is shown in Appendix B.

Moreover, the Lie derivative operator in Proposition 3.6 can be constructed from the first-order differential operators in Proposition 3.7 combined with the algebraic operations in Propositions 3.2, 3.4, 3.5 as follows:

\[
\mathcal{L}_{\vec{K}} K_2 = (1/\rho)\vec{\nabla} \cdot (\rho K_2 \vec{K}_1) - (1/\rho)\vec{\nabla} \cdot (\rho \vec{K}_1) K_2, \tag{3.14}
\]

\[
\mathcal{L}_{\vec{K}} \vec{K}_2 = (1/\rho)\vec{\nabla} \cdot (\rho \vec{K}_1 \wedge \vec{K}_2) + (1/\rho)\vec{\nabla} \cdot (\rho \vec{K}_2) \vec{K}_1 - (1/\rho)\vec{\nabla} \cdot (\rho \vec{K}_1) \vec{K}_2, \tag{3.15}
\]

\[
\mathcal{L}_{\vec{K}} K_2 = (1/\rho)\vec{\nabla} \cdot (\rho K_1 \wedge \mathbf{K}_2) + (1/\rho)\vec{\nabla} \cdot (\rho \mathbf{K}_2) \wedge \vec{K}_1 - (1/\rho)\vec{\nabla} \cdot (\rho \mathbf{K}_1) \mathbf{K}_2 \tag{3.16}
\]

From these expressions, we obtain the following first-order material differential operators involving a vector invariant.

**Corollary 3.1.** If \( K, \vec{K}, \mathbf{K} \) are invariants, then for any vector invariant \( \vec{K}_1 \), \( (1/\rho)\vec{\nabla} \cdot (\rho K \vec{K}_1) \) is an invariant scalar, \( (1/\rho)\vec{\nabla} \cdot (\rho \vec{K} \wedge \vec{K}_1) \) is an invariant vector, and \( (1/\rho)\vec{\nabla} \cdot (\rho \mathbf{K} \wedge \vec{K}_1) \) is an invariant skew-tensor.
Each of these first-order material differential operators can be composed with itself to yield material differential operators of arbitrary order. In contrast, compositions of the first-order differential operators in Proposition 3.7 gives

\[
\frac{1}{\rho} \vec{\nabla} \cdot (\vec{\epsilon} \cdot \vec{\nabla} K) = -\frac{1}{\rho} \vec{\nabla} \times \vec{\nabla} K = 0,
\]

(3.17)

\[
\frac{1}{\rho} \vec{\nabla} \cdot (\vec{\nabla} \cdot (\rho K)) \equiv (1/\rho)(\frac{1}{2} \vec{\nabla} \wedge \vec{\nabla}) : (\rho K) = 0,
\]

(3.18)

\[
(1/\rho) \vec{\epsilon} \cdot \vec{\nabla} ((1/\rho) \vec{\nabla} \cdot (\rho \vec{K})) \neq 0.
\]

(3.19)

All further compositions vanish. Hence we have the following result.

**Corollary 3.2.** \((1/\rho) \vec{\epsilon} \cdot \vec{\nabla} ((1/\rho) \vec{\nabla} \cdot (\rho \vec{K}))\) is an invariant skew-tensor of order two constructed from an invariant vector \(\vec{K}\). There are no invariants of higher order constructed from a single invariant vector, and there are no invariants of second or higher order constructed from a single invariant scalar or skew-tensor.

### 3.4. A generating set of material operations.

To conclude these constructions, we summarize a generating set of all linearly independent material algebraic operations and material differential operations of first order, which arise from the preceding results.

**Theorem 3.1.**

(i) A generating set of material operations on scalar invariants, \(K\), consists of

\[
f, \quad K \cdot \text{grad},
\]

where \(f(K)\) is any differentiable function; and \(K \cdot \text{grad}(K) = K \cdot \vec{\nabla} K\) is a gradient with \(K = \vec{K}, K\), \((1/\rho) \vec{\epsilon}\) being any tensorial invariant.

(ii) A generating set of material operations on vector invariants and skew-tensor invariants, \(K = \vec{K}, K\), is given by

\[
f(K), \quad \wedge, \quad \rho K \times, \quad (1/\rho) \text{div} \rho,
\]

where \(f(K)K_1\) is multiplication by a differentiable function with \(K\) being any scalar invariant; \(K_1 \wedge K_2\) is the exterior product; \(\rho K \times (K_1) = \rho K \times K_1\) is a densitized cross-product; and \((1/\rho) \text{div} \rho K_1 = (1/\rho) \vec{\nabla} \cdot (\rho K_1)\) is a densitized divergence.

As an illustration of these material operations and how they can be composed to generate further material operators, in Table 2 we list, firstly, all functionally independent scalar invariants of at most order one that are constructed from one or two invariants, and secondly, all linearly independent vector and skew-tensor invariants of at most order one that are constructed from one or two invariants, up to multiplication by invariant scalar functions. Each invariant is shown in both geometrical notation and component notation (in Cartesian coordinates \(x^i\)).

The most important aspect of Theorem 3.1 is that it provides a simple explicit way to generate a hierarchy of invariants starting from one or more known invariants. Moreover, the resulting hierarchy will be complete in the sense that no additional invariants can be generated starting from the same known invariants.

### 3.5. Functional independence of invariants.

It will be useful to have a general notion of functional/linear independence for invariants.
Table 2. Construction of invariants

| Type      | Geometrical Form | Component Form |
|-----------|------------------|----------------|
| Scalar    | $\rho K_2 \times \bar{K}_1$ | $\rho \epsilon_{ijk} K_2^i K_2^j K_1^k$ |
|           | $(1/\rho) \nabla \cdot (\rho \bar{K}_1)$ | $(1/\rho) \nabla_i (\rho K_1^i)$ |
|           | $K_2 \times (\nabla \cdot (\rho K_1))$ | $\epsilon_{ijk} K_2^i \nabla_i (\rho K_1^i)$ |
|           | $\bar{K}_1 \cdot \nabla K_2$ | $K_1^i \nabla_i K_2$ |
| Vector    | $\rho K_1 \times K_2$, | $\rho K_1^i \epsilon_{ijkl} K_2^{jl}$ |
|           | $(1/\rho) \nabla \cdot (\rho K_1)$, | $(1/\rho) \nabla_j (\rho K_1^i)$ |
|           | $K_2 \cdot \nabla K_1$, | $K_2^j \nabla_j K_1$ |
|           | $(1/\rho) \nabla \cdot (\rho \bar{K}_1 \wedge \bar{K}_2)$ | $(1/\rho) \nabla_j (\rho (K_2^j K_2^k - K_1^j K_1^k))$ |
| Skew-Tensor | $K_1 \wedge K_2$, | $K_1^i K_2^j - K_1^j K_1^i$ |
|           | $(1/\rho) \epsilon \cdot \nabla K_1$ | $(1/\rho) \epsilon^{ijkl} \nabla_k K_1$ |
|           | $(1/\rho) \bar{K}_1 \wedge (\nabla \cdot (\rho K_1))$ | $(1/\rho) (K_2^i \nabla_k (\rho K_1^j) - K_2^j \nabla_k (\rho K_1^i))$ |
|           | $(1/\rho) \nabla \cdot (\rho \bar{K}_1 \wedge \bar{K}_2)$ | $(1/\rho) \nabla_k (\rho (K_1^i K_2^j + K_1^j K_2^i))$ |

Two scalar invariants $K_1$ and $K_2$ are said to be functionally dependent if $f(K_1, K_2) = 0$ holds identically for some non-constant function $f$, and otherwise the invariants $K_1$ and $K_2$ are said to be functionally independent.

Two vector or skew-tensor invariants $\mathcal{K}_1$ and $\mathcal{K}_2$ are said to be linearly dependent if $\mathcal{K}_1^i K_2^j + \mathcal{K}_2^i K_1^j = 0$ holds for some scalar invariants $\mathcal{K}_1^i \neq 0$ and $\mathcal{K}_2^j \neq 0$, and otherwise if $\mathcal{K}_1^i K_2^j + \mathcal{K}_2^i K_1^j = 0$ only holds with $\mathcal{K}_1^i = \mathcal{K}_2^j = 0$, then the invariants $\mathcal{K}_1$ and $\mathcal{K}_2$ are said to be linearly independent.

4. Hierarchies of local invariants

The equations governing inviscid adiabatic fluid flow consist of Euler’s equation

$$\bar{u}_t + \bar{u} \cdot \nabla \bar{u} = -(1/\rho) \nabla p$$

(4.1)

for the fluid velocity $\bar{u}$, together with the mass continuity equation

$$\rho_t + \nabla \cdot (\rho \bar{u}) = 0,$$

(4.2)

for the fluid density $\rho$, and the adiabatic transport equation

$$S_t + \bar{u} \cdot \nabla S = 0$$

(4.3)

for the fluid entropy $S$. The thermodynamic relation

$$dE = TdS - pd(1/\rho)$$

(4.4)

determines the internal fluid energy $E$, where $T$ is the fluid temperature and $p$ is the fluid pressure.

When the fluid is compressible, the fluid pressure is specified by an equation of state, $p = P(\rho, S)$. Then the thermodynamic relation (4.4) yields

$$E - E_0(S) = \int (1/\rho^2) P(\rho, S)d\rho = e(\rho, S),$$

(4.5)
and
\[ T = \int \left( \frac{1}{\rho^2} \right) P_S(\rho, S) d\rho + T_0(S) \]  
(4.6)
in terms of \( \rho \) and \( S \).

If the pressure depends only on the density, \( p = P(\rho) \), then the fluid flow is *barotropic*. In this case, the internal energy
\[ E - E_0(S) = \int \left( \frac{1}{\rho^2} \right) P(\rho) d\rho = e(\rho) \]  
(4.7)
also depends only on \( \rho \), while the temperature
\[ T = T_0(S) = E'_0(S) \]  
(4.8)
depends only on \( S \).

When the fluid is incompressible
\[ \vec{\nabla} \cdot \vec{u} = 0, \]  
(4.9)
the density satisfies the transport equation
\[ \rho_t + \vec{u} \cdot \vec{\nabla} \rho = 0, \]  
(4.10)
while the fluid pressure \( p \) satisfies a Laplacian equation \( \vec{\nabla} \cdot ((1/\rho) \vec{\nabla} p) = -(\vec{\nabla} \vec{u}) : (\vec{\nabla} \vec{u})^t = \frac{1}{2}(|\vec{u}|^2 - |\vec{s}|^2) \) arising from the divergence of the Euler equation (4.1).

In the physically important case when the density \( \rho \) is constant, the thermodynamic relation (4.3) then shows \( E = E_0(S) \) and \( e = 0 \), which can be viewed as a special case of the energy expression (4.7) due to \( d\rho = 0 \), while similarly \( T = T_0(S) \). In this case, both \( T \) and \( E \) satisfy the same transport equation as \( S \).

A fluid is *isentropic* (or homentropic) if the entropy \( S \) is constant through the fluid, and otherwise the fluid is *adiabatic*.

In both cases of incompressible and compressible fluids, the vorticity vector
\[ \vec{\omega} = \vec{\nabla} \times \vec{u} \]  
(4.11)
satisfies the dynamical equation
\[ \vec{\omega}_t + \vec{u} \cdot \vec{\nabla} \vec{\omega} - \vec{\omega} \cdot \vec{\nabla} \vec{u} = -(\vec{\nabla} \cdot \vec{u}) \vec{\omega} - \vec{\nabla} (1/\rho) \times \vec{\nabla} p \]  
(4.12)
obtained from the curl of the Euler equation. At each point in the fluid, the vorticity vector physically describes the local circulation of the fluid around an infinitesimal loop in the plane orthogonal to this vector. Vortex filaments are the integral curves of the vorticity vector. Because the vorticity vector is divergence free
\[ \vec{\nabla} \cdot \vec{\omega} = 0, \]  
(4.13)
a vortex filament never terminates in the fluid (except at a physical boundary if a fluid is confined to a finite domain).

There are three very useful alternative forms for both the fluid velocity equation and the vorticity equation.

First, the vector calculus identity
\[ \vec{u} \cdot \vec{\nabla} \vec{u} - \frac{1}{2} \vec{\nabla}(|\vec{u}|^2) = \vec{u} \cdot \vec{\omega} = \vec{\omega} \times \vec{u} \]  
(4.14)
leads directly to
\[ \vec{u}_t + \vec{\omega} \times \vec{u} = -\frac{1}{2} \vec{\nabla}(|\vec{u}|^2) - (1/\rho) \vec{\nabla} p \]  
(4.15)
which yields
\[ \vec{\omega}_t + \vec{\nabla} \times (\vec{\omega} \times \vec{u} + (1/\rho)\vec{\nabla}p) = 0. \tag{4.16} \]

This form of the fluid velocity equation is known as Crocco’s theorem [38].

Next,
\[ (1/\rho)\vec{\nabla}p = \vec{\nabla}h - T\vec{\nabla}S \tag{4.17} \]
is essentially a rearrangement of the thermodynamic relation (4.4), where
\[ h = E + p/\rho \tag{4.18} \]
is the enthalpy. This yields
\[ \vec{u}_t + \vec{u} \cdot \vec{\nabla} \vec{u} = -\vec{\nabla}h + T\vec{\nabla}S \tag{4.19} \]
and, hence,
\[ \vec{\omega}_t + \vec{\nabla} \times (\vec{\omega} \times \vec{u}) = \vec{\nabla}T \times \vec{\nabla}S. \tag{4.20} \]

Note that, apart from the source term \(\vec{\nabla}T \times \vec{\nabla}S\), this form of the vorticity equation is analogous to Faraday’s equation in barotropic MHD.

Last, through the vector calculus identity \(\vec{\nabla} \times (\vec{\omega} \times \vec{u}) = \vec{u} \cdot \vec{\nabla} \vec{\omega} - \vec{\omega} \cdot \vec{\nabla} \vec{u} + (\vec{\nabla} \cdot \vec{u})\vec{\omega}\), the vorticity equation (4.16) can be expressed as
\[ \vec{\omega}_t + \vec{u} \cdot \vec{\nabla} \vec{\omega} - \vec{\omega} \cdot \vec{\nabla} \vec{u} = -(\vec{\nabla} \cdot \vec{u})\vec{\omega} - \vec{\nabla}(1/\rho) \times \vec{\nabla}p, \tag{4.21} \]
or alternatively in the thermodynamic form
\[ \vec{\omega}_t + \vec{u} \cdot \vec{\nabla} \vec{\omega} - \vec{\omega} \cdot \vec{\nabla} \vec{u} = -(\vec{\nabla} \cdot \vec{u})\vec{\omega} + \vec{\nabla}T \times \vec{\nabla}S \tag{4.22} \]
for compressible fluids with a general equation of state.

4.1. Basic invariants. Inviscid fluid dynamics has several different basic local invariants, depending on whether the fluid flow is adiabatic or isentropic; compressible with a barotropic or non-barotropic equation of state; incompressible with constant or non-constant density.

To begin, we will consider each of the basic local invariants and explain the most general conditions under which each invariant holds for inviscid fluids. The conserved integrals arising from these invariants also will be discussed.

4.1.1. Vorticity invariant. The basic vorticity invariant is the densitized vorticity vector
\[ \vec{K}_\omega = (1/\rho)\vec{\omega} = (1/\rho)\vec{\nabla} \times \vec{u}. \tag{4.23} \]

To determine the conditions under which it is an invariant, consider the advective Lie derivative of the vorticity vector
\[ \mathcal{D}_t \vec{\omega} = -(\vec{\nabla} \cdot \vec{u})\vec{\omega} - \vec{\nabla}(1/\rho) \times \vec{\nabla}p \tag{4.24} \]
which is obtained from the vorticity equation (4.12). For constant density flows, this transport equation (4.24) shows that the vorticity vector is advected, since \(\vec{\nabla} \cdot \vec{u} = 0\) and \(\vec{\nabla} \rho = 0\). For non-constant density flows, the vorticity vector itself is no longer advected, due to the dilational term \(-(\vec{\nabla} \cdot \vec{u})\vec{\omega}\) and the pressure term \(-\vec{\nabla}(1/\rho) \times \vec{\nabla}p\). However, the dilational term can be compensated by expressing the mass continuity equation (1.1) in the transport form
\[ \mathcal{D}_t \rho = -(\vec{\nabla} \cdot \vec{u})\rho \tag{4.25} \]
and combining it with the vorticity transport equation (4.24), yielding $\mathbf{D}_t((1/\rho)\mathbf{\omega}) = -\nabla(1/\rho) \times \nabla(p/\rho)$. The pressure term then vanishes if (and only if) the flow has a barotropic equation of state, because $\nabla(1/\rho) \times \nabla(p/\rho) = -\rho P'(\rho) \nabla(1/\rho) \times \nabla(1/\rho) = 0$, whereby

$$\mathbf{D}_t((1/\rho)\mathbf{\omega}) = 0.$$  (4.26)

For non-constant density flows that are incompressible, note that the dilational term vanishes so that $\mathbf{D}_t\mathbf{\omega} = -\nabla(1/\rho) \times \nabla p$, but the pressure term does not vanish (as a consequence of the Laplacian equation holding on $p$).

The vorticity invariant (4.23) yields a conserved integral $\frac{d}{dt} \int_S (t) \mathbf{\omega} \cdot d\mathbf{A} = 0$ on moving surfaces $S(t)$. Since $\mathbf{\omega} = \nabla \times \mathbf{u}$ is a curl, this integral vanishes by Stokes’ theorem if $S(t)$ is closed. But if $S(t)$ has a boundary, then the integral instead reduces to a moving curve integral $\frac{d}{dt} \oint_C (t) \mathbf{u} \cdot d\mathbf{s} = 0$ on the closed moving boundary curve $C(t) = \partial S(t)$. This yields Kelvin’s circulation theorem for closed moving curves.

4.1.2. Entropy and Ertel’s invariant. In adiabatic fluid flow, there are three basic local invariants: the entropy

$$K_S = S,$$  (4.27)

the entropy gradient

$$K_S = (1/\rho)\mathbf{\epsilon} \cdot \nabla S,$$  (4.28)

and Ertel’s invariant

$$K_e = (1/\rho)\mathbf{\omega} \cdot \nabla S = (1/\rho)\mathbf{\nabla} \cdot (S\mathbf{\omega}) = (1/\rho)\mathbf{\nabla} \cdot (\mathbf{u} \times \nabla S).$$  (4.29)

The entropy (4.27) clearly is an invariant because it satisfies

$$\mathbf{D}_t S = 0$$  (4.30)

without any conditions on the fluid flow. This invariant is a local scalar of zeroth order.

The entropy gradient (4.28) arises from applying the material differential operator in part (i) of Theorem 3.1 to the entropy, which again does not require any conditions on the fluid flow. This invariant skew tensor represents a plane that is tangent to the surface of constant entropy at each point in the fluid, since $\nabla S \cdot K_S = 0$. Its magnitude $\sqrt{K_S} = (\sqrt{2}/\rho)|\nabla S|$ is inversely proportional to the distance between neighboring surfaces of constant entropy.

Verifying Ertel’s invariant (4.29) is much more involved and amounts to proving the potential vorticity theorem [8]:

$$\rho \mathbf{D}_t((1/\rho)\mathbf{\omega} \cdot \nabla F(\rho, S)) = \mathbf{\omega} \cdot \nabla \mathbf{D}_t F(\rho, S) = (\nabla(1/\rho) \times \nabla p) \cdot \nabla F(\rho, S)$$  (4.31)

which holds for any differentiable function $F(\rho, S)$. Substitution of $F = S$ into equation (4.31), followed by use of $\nabla p = P_\rho(\rho, S)\nabla \rho + P_S(\rho, S)\nabla S$ holding for the general equation of state $p = P(\rho, S)$, we see that

$$\mathbf{D}_t((1/\rho)\mathbf{\omega} \cdot \nabla S) = 0.$$  (4.32)

The physical meaning of Ertel’s invariant is that, at each point in a fluid, it measures the amount of penetration of a vortex filament into surfaces of constant entropy since $(\mathbf{\omega} \cdot \nabla S)/(|\mathbf{\omega}| |\nabla S|)$ is the alignment between the axis of the vortex filament and the normal direction to surfaces of constant entropy, while $|\mathbf{\omega}|$ is the vortex strength and $|\nabla S|$ is inversely
proportional to the distance between neighboring constant-entropy surfaces. In particular, wherever $K_e = 0$, vortex filaments lie in surfaces of constant entropy.

The conserved integral corresponding to Ertel’s invariant \((4.29)\) is \(\frac{d}{dt} \int_V (\vec{\omega} \cdot (\vec{u} \times \vec{\nabla} S)) dV = 0\) on moving volumes \(V(t)\). By Gauss’s theorem, this integral reduces to a conserved moving surface integral \(\frac{d}{dt} \int_{S(t)} (\vec{u} \times \vec{\nabla} S) \cdot d\vec{A} = 0\) on the closed moving boundary surface \(S(t) = \partial V(t)\). This yields a conserved entropy-circulation flux for closed moving surfaces.

Both Ertel’s invariant and the entropy-gradient invariant are local first-order invariants. They can be combined into an invariant function \(f(K_S, K_e)\), while the skew-tensor invariant \((4.28)\) can be generalized by multiplication with this function (cf. Proposition \(3.1\)). The resulting invariants

\[
f(S, K_e), \quad f(S, K_e)K_S
\]

comprise all local invariants of at most order one for inviscid adiabatic non-barotropic compressible fluid flow.

4.1.3. Density invariants. For incompressible non-constant density fluid flow, the density \(\rho\) is a scalar invariant

\[
K_\rho = \rho
\]

since the mass continuity equation \((4.25)\) reduces to \(\nabla \cdot \vec{u} = 0\) when \(\nabla \cdot \vec{u} = 0\). The gradient of the density then yields an invariant skew tensor \((1/\rho) \vec{\epsilon} \cdot \vec{\nabla} \rho = \vec{\epsilon} \cdot \vec{\nabla} \ln \rho\) through the material differential operator in part (i) of Theorem \(3.1\). This invariant can be combined with the density invariant to obtain a simpler skew-tensor invariant

\[
K_\rho = \vec{\epsilon} \cdot \vec{\nabla} \rho.
\]

At each point in an incompressible fluid, the skew-tensor invariant \((4.35)\) represents a plane that is tangent to the surface of constant density, since \(\vec{\nabla} \rho \cdot K_\rho = 0\). Its magnitude \(\sqrt{K_\rho} \cdot K_\rho = \sqrt{2} |\vec{\nabla} \rho|\) is inversely proportional to the distance between neighboring surfaces of constant density.

An additional new invariant arises from Ertel’s potential vorticity theorem \((4.31)\) by putting \(F = \rho\), which yields \(\rho \nabla \cdot (\vec{\omega} \cdot \vec{\nabla} \rho) = \vec{\omega} \cdot \vec{\nabla} \nabla \rho - (\vec{\nabla} (1/\rho) \times \vec{\nabla} p) \cdot \vec{\nabla} \rho = 0\) due to the density transport equation \((4.25)\). Note this does not rely on the vorticity being an invariant. Hence we see \((1/\rho) \vec{\omega} \cdot \vec{\nabla} \rho\) is an invariant scalar. Since the density itself is an invariant, this shows that

\[
K_\nu = \vec{\omega} \cdot \vec{\nabla} \rho = \vec{\nabla} \cdot (\rho \vec{\omega}) = \vec{\nabla} \cdot (\vec{u} \times \vec{\nabla} \rho)
\]

is an Ertel-type local scalar invariant. Its physical meaning measures the amount of penetration of a vortex filament into surfaces of constant density at each point in a fluid, since \((\vec{\omega} \cdot \vec{\nabla} \rho)/(|\vec{\omega}| |\vec{\nabla} \rho|)\) is the alignment between the axis of the vortex filament and the normal direction to the surfaces, while \(|\vec{\omega}|\) is the vortex strength and \(|\vec{\nabla} \rho|\) is inversely proportional to the distance between neighboring constant-density surfaces. In particular, wherever \(K_\nu = 0\), vortex filaments lie in constant-density surfaces.

Similarly to Ertel’s invariant \((4.29)\), the new density invariant \((4.36)\) yields a moving surface integral \(\frac{d}{dt} \int_{S(t)} (\vec{u} \times \vec{\nabla} \rho) \cdot d\vec{A} = 0\) describing a conserved density-circulation flux on closed moving surfaces \(S(t)\).
Clearly, any function of the two scalar invariants \( K_\rho \) and \( K_{e'} \) is also a scalar invariant, and the product of this function with \( K_\rho \) is a skew tensor invariant. This yields all local invariants of at most order one

\[ f(\rho, K_{e'}) \], \[ f(\rho, K_{e'}) K_\rho \]  

(4.37)

for incompressible isentropic fluid flow.

4.2. Higher-order invariants. From the basic invariants \( K_\rho, K_{e'}, K_S, K_e, \vec{K}_e, K_\rho, K_S \), we can obtain higher-order local invariants by applying Theorem 3.1 and taking into account Corollary 3.2, along with the fluid flow conditions under which each invariant holds. Many of these higher-order invariants are of vorticity type, which we will discuss.

We will start by considering adiabatic compressible fluid flow with a non-barotropic equation of state. Next we will specialize to a barotropic equation of state. Last we will consider incompressible flow, first with constant density and then with non-constant density.

4.2.1. Adiabatic compressible non-barotropic fluid flow. The basic local invariants holding for these flows consist of \( K_S, K_e, K_S \). Applying the material differential operators in part (i) of Theorem 3.1 yields an invariant skew-tensor

\[ \vec{K}_e^{(2)} = (1/\rho)e \cdot \vec{\nabla} K_e \]  

(4.38)

and an invariant vector

\[ \vec{K}_e^{(2)} = K_S \cdot \vec{\nabla} K_e = -K_e \cdot \vec{\nabla} K_S = (1/\rho)\vec{\nabla} K_e \times \vec{\nabla} S. \]  

(4.39)

These are local second-order invariants. All other material operations given by Theorem 3.1 yield trivial invariants. In particular, firstly, the material algebraic operations give \( \rho \vec{K}_S \times \vec{K}_e^{(2)} = 2\vec{K}_e^{(2)}, \rho \vec{K}_e^{(2)} \times K_S = \vec{\nabla} S \cdot (\vec{\nabla} K_e \times \vec{\nabla} S) = 0; \) and \( \rho \vec{K}_e^{(2)} \times \vec{K}_e^{(2)} = \vec{\nabla} K_e \cdot (\vec{\nabla} K_e \times \vec{\nabla} S) = 0. \) Secondly, the material differential operators give \( (1/\rho)\vec{\nabla} \cdot (\rho \vec{K}_S) = (1/\rho)\vec{\nabla} \cdot (e \cdot \vec{\nabla} S) = (1/\rho)\vec{\nabla} \times \vec{\nabla} S = 0, \vec{K}_e^{(2)} \cdot \vec{\nabla} K_S = (1/\rho)(\vec{\nabla} K_e \times \vec{\nabla} S) \cdot \vec{\nabla} S = 0, \) and likewise \( (1/\rho)\vec{\nabla} \cdot (\rho \vec{K}_e^{(2)}) = 0, \vec{K}_e^{(2)} \cdot \vec{\nabla} K_e = 0. \)

Since the preceding material operations are exhaustive, this establishes the following result.

**Theorem 4.1.** All independent local invariants for inviscid adiabatic compressible fluid flow with a non-barotropic equation of state are given by

\[ S, K_e, \vec{K}_e^{(2)}, K_S, K_e^{(2)}. \]  

(4.40)

Both \( \vec{K}_e^{(2)} \) and \( K_e^{(2)} \) are of vorticity type.

Physically, at each point in the fluid, the vorticity-type invariants have the following meaning: \( \vec{K}_e^{(2)} \) represents a plane that is tangent to the surface on which \( K_e \) is constant, while its magnitude \( \sqrt{\vec{K}_e^{(2)} \cdot \vec{K}_e^{(2)}} = (\sqrt{2}/\rho)||\vec{\nabla} K_e|| \) is inversely proportional to the distance between these neighboring surfaces; \( \vec{K}_e^{(2)} \) lies in the intersection of the respective surfaces on which \( S \) and \( K_e \) are constant, and its magnitude \( (1/\rho)||\vec{\nabla} K_e|| \sqrt{1 - (\vec{\nabla} K_e \cdot \vec{\nabla} S)^2/((||\vec{\nabla} K_e|| ||\vec{\nabla} S||)^2)} \) is proportional to the alignment between the respective tangent planes of the surfaces and to the inverse distance between neighboring surfaces.

The vector invariant \( \vec{K}_e^{(2)} \) gives rise to a conserved integral \( \frac{d}{dt} \int_S(t)(\vec{\nabla} K_e \times \vec{\nabla} S) \cdot d\vec{A} = 0 \) on moving surfaces \( S(t) \). By Stokes’ theorem, this integral vanishes if \( S(t) \) is closed, since
\[ \nabla K_e \times \nabla S \] is a curl. But if \( S(t) \) has a boundary, then the integral reduces to a moving curve integral on the closed moving boundary curve \( C(t) = \partial S(t) \). This yields a conserved circulation integral

\[
\oint_{C(t)} K_e \nabla S \cdot d\vec{s} = - \oint_{C(t)} S \nabla K_e \cdot d\vec{s}
\]

for closed moving curves.

### 4.2.2. Compressible barotropic fluid flow

We will now restrict attention to compressible barotropic fluid flow, but with \( S \) and \( T(S) \) being non-constant across different fluid streamlines. This describes barotropic fluids in which the fluid temperature is frozen-in and has no effect on the dynamics of the fluid velocity.

All of the invariants (4.40) for adiabatic non-barotropic fluid flow are invariants in barotropic fluid flow. More interestingly, the densitized vorticity vector invariant (4.23) holding in barotropic fluid flow gives rise to further local invariants as follows.

By applying part (i) of Theorem 3.1 to \( \vec{K}_\omega \) in combination with \( K_S \) and \( K_e \), we reproduce Ertel’s invariant

\[
\vec{K}_\omega \cdot \nabla K_S = \frac{1}{\rho} \vec{\omega} \cdot \nabla S = K_e
\]

and obtain another invariant scalar

\[
K_{\omega,e}^{(2)} = \vec{K}_\omega \cdot \nabla K_e = \left( \frac{1}{\rho} \vec{\omega} \cdot \nabla \right)^2 S.
\]

This local second-order invariant (4.43) measures the amount of penetration of vortex filaments into the surfaces on which \( K_e \) is constant.

Next, using the algebraic operations in part (ii) of Theorem 3.1 we find that no new invariants arise from \( \vec{K}_\omega \) in combination with \( K_{\omega, e}^{(2)} \), \( K_S \), and \( K_{\omega, e}^{(2)} \). In particular:

\[
\vec{K}_\omega \wedge K_{\omega, e}^{(2)} = (1/\rho^2) \vec{\omega} \wedge (\nabla K_e \times \nabla S) = K_e \vec{K}_e^{(2)} - K_{\omega, e}^{(2)} K_S; \quad \rho \vec{K}_\omega \times K_S = (1/\rho) \vec{\omega} \times (\vec{e} \cdot \nabla S) = 2K_e; \quad \rho \vec{K}_\omega \times K_{\omega, e}^{(2)} = (1/\rho) \vec{\omega} \times (\vec{e} \cdot \nabla K_e) = 2K_{\omega, e}^{(2)} \]

(after use of some algebraic identities in Appendix A). Further, applying the differential operator in part (ii) of Theorem 3.1 to \( \vec{K}_\omega \), we see that \((1/\rho) \nabla \cdot (\rho \vec{K}_\omega) = (1/\rho) \nabla \cdot \vec{\omega} = 0\) yields a trivial invariant.

Hence, all local invariants up to order two for barotropic fluid flow are given by

\[
f(S, K_e, K_{\omega, e}); \quad f(S, K_e, K_{\omega, e}) \vec{K}_\omega, \quad f(S, K_e, K_{\omega, e}) \vec{K}_{\omega, e}^{(2)}; \quad f(S, K_e, K_{\omega, e}) K_S, \quad f(S, K_e, K_{\omega, e}) K_{\omega, e}^{(2)}.
\]

Apart from \( S \) and \( K_S \), these are vorticity-type invariants.

A main difference compared to the non-barotropic case, however, is that the additional vorticity invariant \( \vec{K}_\omega \) enables the construction of higher-order invariants for barotropic fluid flow. It is straightforward to show that all new independent invariants at each successive order are produced by applying part (i) of Theorem 3.1 to the all of the lower-order invariants. In particular, similarly to what happens at second order, no new invariants arise from using the algebraic operations in part (ii) of Theorem 3.1.
Omitting the details, we find that all independent local third-order invariants consist of:

**two scalar invariants**

\[ \vec{K}_\omega \cdot \vec{\nabla}^2 K_{\omega,e}^{(2)} = ((1/\rho)\vec{\omega} \cdot \vec{\nabla})^3 S, \]

\[ \vec{K}_{e,S}^{(2)} \cdot \vec{\nabla}^2 K_{\omega,e}^{(2)} = (1/\rho)\vec{\nabla}(((1/\rho)\vec{\omega} \cdot \vec{\nabla})^2 S) \cdot (\vec{\nabla}((1/\rho)\vec{\omega} \cdot \vec{\nabla}) \times \vec{\nabla}) S); \]

**two vector invariants**

\[ (1/\rho)\vec{\nabla} K_S \times \vec{\nabla} K_{\omega,e}^{(2)} = (1/\rho)\vec{\nabla} S \times \vec{\nabla}(((1/\rho)\vec{\omega} \cdot \vec{\nabla})^2 S), \]

\[ (1/\rho)\vec{\nabla} K_e \times \vec{\nabla} K_{\omega,e}^{(2)} = (1/\rho)(\vec{\nabla}((1/\rho)\vec{\omega} \cdot \vec{\nabla}) \times \vec{\nabla}(((1/\rho)\vec{\omega} \cdot \vec{\nabla})^2 S); \]

**and a skew-tensor invariant**

\[ (1/\rho)\vec{\epsilon} \cdot \vec{\nabla} K_{\omega,e}^{(2)} = (1/\rho)\vec{\epsilon} \cdot \vec{\nabla}(((1/\rho)\vec{\omega} \cdot \vec{\nabla})^2 S); \]

Going to higher orders, we have the following main result.

**Theorem 4.2.** (i) For inviscid adiabatic compressible fluid flow with a barotropic equation of state, all independent local invariants of order \( n \geq 1 \) are recursively generated by

\[ K^{(n)} = \vec{K}^{(m)} \cdot \vec{\nabla} K^{(n-1)}, \]

\[ \vec{K}^{(n)} = K^{(m)} \cdot \vec{\nabla} K^{(n-1)}, \]

\[ K^{(n)} = (1/\rho)\vec{\epsilon} \cdot \vec{\nabla} K^{(n-1)}, \]

with \( m = 1, \ldots, n - 1 \), starting from \( K^{(1)} = K_e \) which is Ertel’s invariant \( (4.29) \), \( \vec{K}^{(1)} = \vec{K}_\omega \) which is the vorticity invariant \( (4.23) \), and \( K^{(1)} = K_S \) which is the entropy-gradient invariant \( (4.28) \). In this hierarchy, each invariant other than \( K_S \) is of vorticity type. (ii) For inviscid isentropic compressible fluid flow with a barotropic equation of state, the only local invariant is the vorticity \( (4.23) \).

**4.2.3. Constant-density fluid flow.** In these fluid flows, the densitized vorticity vector \( (4.23) \) which is an invariant for barotropic fluid flow is still an invariant vector.

Consequently, for constant-density flows in which the entropy \( S \) and temperature \( T(S) \) are frozen-in, whereby they do not affect the dynamics of the fluid velocity but are non-constant across different fluid streamlines, all of the invariants for adiabatic barotropic fluid flow shown in Theorem 4.2 continue to hold. Moreover, the density factor \( 1/\rho \) (being constant) can be dropped.

In contrast, for isentropic constant-density flows, where \( S \) and \( T \) are constant through the fluid, the basic adiabatic invariants \( K_S, K_e, K_S \) are obviously trivial. As a result, the only remaining local invariant is the vorticity vector \( (4.23) \).

**Theorem 4.3.** (i) For inviscid adiabatic constant-density fluid flow, all independent local invariants of order \( n \geq 1 \) are generated by the recursions \( (4.50), (4.51), (4.52) \) (with \( m = 1, \ldots, n - 1 \)), starting from Ertel’s invariant \( K^{(1)} = \vec{\omega} \cdot \vec{\nabla} S \), the vorticity invariant \( \vec{K}^{(1)} = \vec{\omega} \), and the entropy-gradient invariant \( K^{(1)} = \vec{\epsilon} \cdot \vec{\nabla} S \). (ii) For inviscid isentropic constant-density fluid flow, the only local invariant is the vorticity vector \( \vec{K}^{(1)} = \vec{\omega} \).
4.2.4. Incompressible non-constant density fluid flow. For these flows, the densitized vorticity vector \([4.23]\) is no longer an invariant. The basic invariants consist of \(\vec{K}_e\), \(K'_e\), \(\vec{K}_p\).

If the flow is also isentropic, then there are no other local invariants. But if the flow is adiabatic, then all of the local invariants given by Theorem \([4.1]\) for adiabatic non-barotropic fluid flow continue to hold. Moreover, additional local invariants arise by using material operations in Theorem \([3.1]\) to combine these adiabatic invariants \(S, K_e, \vec{K}_{e,S}, K_{S}, K_{e}'\) with the density invariants \(K_p, K_e', K_p\).

A useful observation here is that any invariant can be multiplied by \(K_p = \rho\). Consequently, we can work with the simpler adiabatic invariants

\[
\vec{K}_{e} = K_p K_e = \vec{\omega} \cdot \vec{\nabla} S = \vec{\nabla} \cdot (S \vec{\omega}) = \vec{\nabla} \cdot (\vec{u} \times \vec{\nabla} S),
\]

(4.53)

\[
\vec{K}_S = K_p K_S = \epsilon \cdot \vec{\nabla} S,
\]

(4.54)

\[
\vec{K}_{e}'(2) = K_p^2 K_e'(2) + K_e K_p = \epsilon \cdot \vec{\nabla} K_{e},
\]

(4.55)

\[
\vec{K}_{e,S}(2) = \vec{K}_S \cdot \vec{\nabla} K_{e} = -\vec{K}_{e} \cdot \vec{\nabla} S = \vec{\nabla} K_{e} \times \vec{\nabla} S.
\]

(4.56)

Firstly, the material differential operators give a first-order invariant vector

\[
\vec{K}_S^{(1)}_{S,p} = K_p \cdot \vec{\nabla} K_S = -\vec{K}_S \cdot \vec{\nabla} K_p = \vec{\nabla} S \times \vec{\nabla} \rho;
\]

four second-order invariant vectors

\[
\vec{K}_{e,S}(2) = \vec{K}_S \cdot \vec{\nabla} K_{e'} = \vec{\nabla} K_{e'} \times \vec{\nabla} S;
\]

(4.58)

\[
\vec{K}_{e,p}(2) = K_p \cdot \vec{\nabla} K_{e'} = \vec{\nabla} K_{e'} \times \vec{\nabla} \rho,
\]

(4.59)

\[
\vec{K}_{e}^{(2)} = K_p \cdot \vec{\nabla} K_{e} = -\vec{K}_{e} \cdot \vec{\nabla} \rho = \vec{\nabla} K_{e} \times \vec{\nabla} \rho,
\]

(4.60)

\[
\vec{K}_{e'}^{(2)} = \vec{K}_{e'}^{(2)} \cdot \vec{\nabla} K_{e'} = (1/\rho)\vec{\nabla} K_{e'} \times \vec{\nabla} K_{e};
\]

(4.61)

four second-order invariant scalars

\[
K_{e,S,p}^{(2)} = \vec{K}_{e,S}^{(2)} \cdot \vec{\nabla} \rho = -\vec{K}_{e,S}^{(2)} \cdot \vec{\nabla} S = \vec{K}_{S,p}^{(2)} \cdot \vec{\nabla} K_{e} = \vec{\nabla} K_{e} \cdot (\vec{\nabla} S \times \vec{\nabla} \rho),
\]

(4.62)

\[
K_{e,S,p}^{(2)} = \vec{K}_{e,S}^{(2)} \cdot \vec{\nabla} K_{e'} = -\vec{K}_{e,S}^{(2)} \cdot \vec{\nabla} S = \vec{K}_{e,S}^{(2)} \cdot \vec{\nabla} \rho = \vec{\nabla} K_{e'} \cdot (\vec{\nabla} S \times \vec{\nabla} \rho),
\]

(4.63)

\[
K_{e',S,p}^{(2)} = \vec{K}_{e',S}^{(2)} \cdot \vec{\nabla} K_{e} = -\vec{K}_{e',S}^{(2)} \cdot \vec{\nabla} S = \vec{K}_{e',S}^{(2)} \cdot \vec{\nabla} \rho = \vec{\nabla} K_{e} \cdot (\vec{\nabla} S \times \vec{\nabla} \rho),
\]

(4.64)

\[
K_{e',S,p}^{(2)} = \vec{K}_{e',S}^{(2)} \cdot \vec{\nabla} K_{e'} = -\vec{K}_{e',S}^{(2)} \cdot \vec{\nabla} S = \vec{K}_{e',S}^{(2)} \cdot \vec{\nabla} \rho = \vec{\nabla} K_{e'} \cdot (\vec{\nabla} S \times \vec{\nabla} \rho);
\]

(4.65)

and a second-order invariant skew-tensor

\[
\vec{K}_{e',p}^{(2)} = \epsilon \cdot \vec{\nabla} K_{e'}.
\]

(4.66)

The physical meaning of the new vector invariant \(\vec{K}_{S,p}^{(1)}\) is that it lies in the intersection of the respective surfaces on which \(S\) and \(\rho\) are constant, and its magnitude \(|\vec{\nabla} \rho| |\vec{\nabla} S| \sqrt{1 - (\vec{\nabla} S \cdot \vec{\nabla} \rho)^2 / (|\vec{\nabla} \rho| |\vec{\nabla} S|)^2}\) is measures the transversality of the respective tangent planes of the surfaces and to the inverse distance between neighboring surfaces. In particular, \(\vec{K}_{S,p}^{(1)}\) vanishes when the two tangent planes are aligned.

This vector invariant \(\vec{K}_{S,p}^{(2)}\) gives rise to a conserved integral \(\frac{d}{dt} \int_{S(t)} (\vec{\nabla} S \times \vec{\nabla} \rho) \cdot d\vec{A} = 0\) on moving surfaces \(S(t)\). By Stokes’ theorem, this integral vanishes if \(S(t)\) is closed. When
\( S(t) \) has a boundary, the integral instead reduces to a moving curve integral on the closed moving boundary curve \( C(t) = \partial S(t) \). This yields a conserved circulation integral

\[
\oint_{C(t)} S \nabla \rho \cdot ds = - \oint_{C(t)} \rho \nabla S \cdot ds
\]

(4.67)

for closed moving curves.

Similar physical properties hold for each of the four vector invariants (4.58)–(4.61).

The four scalar invariants (4.62)–(4.65) measure the triple alignment among the tangent planes of the respective surfaces on which \( S, \rho, K_\epsilon, K_\epsilon' \) are constant. In particular, if any two of the three surfaces are aligned at a point in the fluid, such that their normal vectors are parallel, then the corresponding scalar invariant vanishes.

Each of these scalar invariants yields a conserved moving volume integral. Since the invariants have the form of a divergence, these integrals reduce to moving surface integrals by Gauss’ theorem. This yields four conserved flux integrals

\[
\frac{d}{dt} \oint_{S(t)} K_1 (\nabla K_2 \times \nabla K_3) \cdot dA = 0,
\]

where \( K_1, K_2, K_3 \) are any three of the four scalar invariants.

The skew-tensor invariant \( \tilde{K}_e^{(2)} \) physically represents a plane that is tangent to the surface on which \( K_e' \) is constant, while its magnitude \( \sqrt{2|\nabla K_e'|} \) is inversely proportional to the distance between these neighboring surfaces.

Finally, the process of generating invariants can be continued to higher orders. Similarly to what happens at second order, all new independent invariants that arise at each successive order are produced by applying part (i) of Theorem 3.1 to the all of the lower-order invariants, as well as using multiplication by \( K_{\rho} \).

This establishes the following result.

**Theorem 4.4.** (i) For inviscid adiabatic incompressible fluid flow with non-constant density, all independent local invariants of order \( n \geq 1 \) are generated by the recursions

\[
K^{(n)} = \tilde{K}^{(m)} \cdot \nabla K^{(n-1)},
\]

(4.68)

\[
\tilde{K}^{(n)} = \tilde{K}^{(m)} \cdot \nabla K^{(n-1)},
\]

(4.69)

\[
\tilde{K}^{(n)} = \epsilon \cdot \nabla K^{(n-1)},
\]

(4.70)

with \( m = 1, \ldots, n - 1 \), starting from the Ertel-type invariants \( K^{(1)} = \tilde{\omega} \cdot \nabla S, \tilde{\omega} \cdot \nabla \rho \), the density-entropy surface transversality invariant \( \tilde{K}^{(1)} = \nabla \rho \times \nabla S \), and the entropy-gradient and density-gradient invariants \( \tilde{K}^{(1)} = \epsilon \cdot \nabla S, \epsilon \cdot \nabla \rho \). (ii) For inviscid isentropic incompressible fluid flow with non-constant density, the only local invariants are \( K = \rho, \tilde{\omega} \cdot \nabla \rho \), and \( \tilde{K} = \epsilon \cdot \nabla \rho \).

### 5. Hierarchies of nonlocal invariants

The starting point for deriving nonlocal invariants is a version of Weber’s formulation [39] of the fluid velocity equation in inviscid adiabatic compressible fluid flow. Weber’s original formulation involves the use of Lagrangian coordinates, but a simpler formulation can be obtained by the use of differential forms [40] [6] [7]. Here we will employ an alternative version that uses only tensorial quantities:

\[
\ddot{u} + s \cdot \ddot{u} = \nabla (\frac{1}{2} |\ddot{u}|^2 - E - p/\rho) + T \nabla S
\]

(5.1)
where $s$ is the symmetric derivative (2.11) of $\bar{u}$.

This form of the fluid velocity equation arises in a similar way to the derivation of Crocco’s theorem (4.15), with the important change that the vector calculus identity

$$\bar{u} \cdot \vec{\nabla} \bar{u} + \frac{1}{2} \vec{\nabla}(|\bar{u}|^2) = s \cdot \bar{u}$$  \hspace{1cm} (5.2)

is used in place of the vorticity relation (4.14).

A connection to invariants comes from taking the dot product of $\bar{u}_t + s \cdot \bar{u}$ with the densitized volume tensor $(1/\rho)\epsilon$, which yields

$$((1/\rho)\epsilon \cdot (\bar{u}_t + s \cdot \bar{u}) = \mathcal{D}_t((1/\rho)\epsilon \cdot \bar{u})$$  \hspace{1cm} (5.3)

by expressing the dot product in terms of the Euclidean metric $g$ and using the advection properties (3.7) for $g$ and (3.10) for $(1/\rho)\epsilon$. Then the fluid velocity equation (5.1) shows that $(1/\rho)\epsilon \cdot \bar{u}$ satisfies the transport equation

$$\mathcal{D}_t((1/\rho)\epsilon \cdot \bar{u}) = \frac{T}{\rho} \epsilon \cdot \vec{\nabla} S.$$  \hspace{1cm} (5.4)

We remark that this is the skew-tensor version of the differential-form equation

$$\mathcal{D}_t u = d \left(\frac{1}{2} |\bar{u}|^2 - E - p/\rho\right) + T dS$$  \hspace{1cm} (4.11, 4.2, 4.3)

where $u = \bar{u} \cdot d\bar{x}$ is a differential form corresponding to the fluid velocity, as explained in Appendix B.

After these preliminaries, the main idea is that now we will introduce all possible potentials that come naturally from the gradient term and the temperature term in the velocity skew-tensor equation (5.4). These potentials first arose in the work of Weber [39], Ertel [8], Rossby [9], and Hollmann [10], and they can be viewed alternatively as Clebsch variables which appear as Lagrange multipliers when an action principle is formulated for the fluid equations (4.1), (4.2), (4.3) (see, e.g. Ref.[6]).

To begin, consider a potential $\phi$ defined by the transport equation

$$\frac{d}{dt} \phi = \frac{1}{2} |\bar{u}|^2 - H(\rho, S)$$  \hspace{1cm} (5.5)

where

$$H(\rho, S) = E(\rho, S) + P(\rho, S)/\rho$$  \hspace{1cm} (5.6)

is the enthalpy (4.18) given by the sum of the internal energy density $E(\rho, S)$ and the flow energy density $P(\rho, S)/\rho$, with the fluid having a general equation of state $p = P(\rho, S)$. This transport equation (5.5) can be integrated to obtain $\phi$ along trajectories $\frac{d}{dt}\bar{x}(t) = \bar{u}(t, \bar{x}(t))$ of infinitesimal fluid elements. In particular,

$$\phi(t, \bar{x}(t)) = \phi(0, \bar{x}(0)) + \int_0^t \left(\frac{1}{2} |\bar{u}(t', \bar{x}(t'))|^2 - H(\rho(t', \bar{x}(t')), S(0, \bar{x}(0)))\right) dt'$$  \hspace{1cm} (5.7)

expresses $\phi$ as a nonlocal variable in terms of $\bar{u}$, $\rho$, and $S$. Note that $S(t, \bar{x}(t)) = S(0, \bar{x}(0))$ since $S$ is advected by the flow.

If we introduce the Clebsch velocity

$$\bar{v} = \bar{u} - \vec{\nabla} \phi$$  \hspace{1cm} (5.8)

whose curl is the vorticity

$$\vec{\nabla} \times \bar{v} = \vec{\nabla} \times \bar{u} = \bar{\omega},$$  \hspace{1cm} (5.9)

then we can combine equations (5.5) and (5.4) to get the transport equation

$$\mathcal{D}_t((1/\rho)\epsilon \cdot \bar{v}) = \frac{T}{\rho} \epsilon \cdot \vec{\nabla} S.$$  \hspace{1cm} (5.10)
The derivation uses the fact that the advective Lie derivative $\mathcal{D}_t$ coincides with the material derivative $\frac{d}{dt}$ on scalars, together with the commutator identity
\[
[\mathcal{D}_t, (1/\rho) \boldsymbol{\epsilon} \cdot \nabla]f = 0. \tag{5.11}
\]
The transport equations for $(1/\rho) \boldsymbol{\epsilon} \cdot \boldsymbol{v}$ and $\boldsymbol{\omega}$ give rise to interesting nonlocal invariants, including the Ertel-Rossby invariant and the Hollmann invariant. Most importantly, the material operations shown in Theorem 3.1 can be applied to generate several hierarchies of additional nonlocal invariants. Some of the higher-order scalar invariants in these hierarchies describe new nonlocal cross-helicities, which we will discuss.

We will begin by considering isentropic compressible fluid flow with a barotropic equation of state, and afterwards we will generalize the considerations to adiabatic compressible fluid flow, with both barotropic and non-barotropic equations of state. Last we will consider incompressible fluid flow.

Some final preliminary remarks are worth stating. Firstly, it is straightforward to show that equation (5.10) for $(1/\rho) \boldsymbol{\epsilon} \cdot \boldsymbol{v}$ is equivalent to a transport equation for $\boldsymbol{v}$:
\[
\mathcal{D}_t \boldsymbol{v} = -s \cdot \nabla + T \nabla S. \tag{5.12}
\]
This transport equation, together with the relation $\dot{\boldsymbol{u}} = \dot{\boldsymbol{v}} + \nabla \phi$ and the transport equation (5.5) for $\phi$, provides an equivalent dynamical description of the fluid velocity in inviscid adiabatic compressible fluid flow. Secondly, the vorticity has the transport equation
\[
\mathcal{D}_t ((1/\rho) \boldsymbol{\omega}) = \nabla \times \nabla S \tag{5.13}
\]
arising from the thermodynamic form of vorticity equation (4.22) expressed in terms of the advective Lie derivative (2.1). Alternatively, equation (5.13) can be obtained directly from the curl of equation (5.12) combined with the density transport equation (4.25) and the commutator identities
\[
[\mathcal{D}_t, \nabla]f = -s \cdot \nabla f, \quad [\mathcal{D}_t, \nabla \wedge] \vec{f} = -s \cdot \nabla \wedge \vec{f} \tag{5.14}
\]
which hold by a straightforward computation employing the advection property (3.7) for the Euclidean metric.

Finally, $\phi$ and $\boldsymbol{v}$ have the following physical meaning related to the total energy density $\frac{1}{2} |\vec{u}|^2 + H(\rho, S)$ of the fluid flow: We see from the transport equation (5.5) that $\frac{d}{dt} \phi$ is the deviation from equipartition of the kinetic energy density $\frac{1}{2} |\vec{u}|^2$ and the enthalpy energy density $H(\rho, S)$ in the total energy density. Hence, the Clebsch velocity (5.8) physically represents the part of the fluid velocity $\dot{\boldsymbol{u}}$ that is dynamically driven by enthalpy and heat transfer apart from a contribution $\nabla \phi$ due to any deviation from equipartition of the total energy.

5.1. Isentropic invariants. In isentropic compressible fluid flow with a barotropic equation of state $p = P(\rho)$, the fluid temperature $T$ and the entropy $S$ are constant throughout the fluid. Consequently, the two main transport equations (5.10) and (5.13) simplify to the respective forms
\[
\mathcal{D}_t ((1/\rho) \boldsymbol{\epsilon} \cdot \boldsymbol{v}) = 0, \tag{5.15}
\]
\[
\mathcal{D}_t ((1/\rho) \boldsymbol{\omega}) = 0. \tag{5.16}
\]
Hence, we see that
\[ K_v = (1/\rho) \epsilon \cdot \vec{v} \]  
(5.17)
is a nonlocal skew-tensor invariant. It represents a plane that is orthogonal to the direction of \( \vec{v} \) in the fluid, while its magnitude \( \sqrt{K_v : K_v} = (\sqrt{2}/\rho)|\vec{v}| \) is proportional to the magnitude of \( \vec{v} \).

We can now apply the material algebraic operation in part (ii) of Theorem 3.1 to the invariants \( K_v \) and \( K_\omega \), yielding \( K_v \times K_\omega = (2/\rho)\vec{\omega} \cdot \vec{v} \). Hence we obtain a nonlocal scalar invariant
\[ K_r = (1/\rho)\vec{\omega} \cdot \vec{v} \]  
(5.18)
which is the Ertel-Rossby invariant \( \mathcal{F} \). It measures the alignment between the streamlines of \( \vec{v} \) and the corresponding vorticity filaments defined by \( \nabla \times \vec{v} = \vec{\omega} \), and its corresponding conserved integral \( \frac{d}{dt} \int_{\gamma(t)} \vec{\omega} \cdot \vec{v} dV = 0 \) is the helicity of these filaments.

Next we can obtain additional nonlocal invariants from \( K_v \) and \( K_r \) by applying the material operations in part (i) of Theorem 3.1. This yields an invariant skew-tensor
\[ K_r^{(2)} = (1/\rho) \epsilon \cdot \nabla K_r, \]  
(5.19)
and an invariant vector
\[ K_{v,r}^{(2)} = K_v \cdot \nabla K_r = (1/\rho)\nabla K_r \times \vec{v}, \]  
(5.20)
as well as an invariant scalar
\[ K_{\omega,r}^{(2)} = K_\omega \cdot \nabla K_r = (1/\rho)\vec{\omega} \cdot \nabla K_r. \]  
(5.21)
These are second-order invariants of vorticity type. The material operations given by part (ii) of Theorem 3.1 yield no further non-trivial invariants. In particular, firstly, the material algebraic operations give \( \rho K_v \times K_r^{(2)} = 2K_{v,r}^{(2)}; \rho K_\omega \times K_r = 2K_{\omega,r}^{(2)}; \rho K_{v,t}^{(2)} \times K_r = \vec{v} \cdot (\nabla K_r \times \vec{v}) = 0; \rho K_{v,r}^{(2)} \times K_\omega^{(2)} = \nabla K_r \cdot (\nabla K_r \times \vec{v}) = 0; \) and \( K_\omega \times K_{v,t}^{(2)} = (1/\rho^2)\vec{\omega} \cdot (\nabla K_r \times \vec{v}) = K_r K_{v,t}^{(2)} - K_{\omega,r}^{(2)} K_v \).

Secondly, the material differential operators give \( (1/\rho)\nabla \cdot (\rho K_{v,t}^{(2)}) = 0; \) and \( (1/\rho)\nabla \cdot (\rho K_{\omega,r}^{(2)}) = K_{v,t}^{(2)}. \)

The three nonlocal vorticity-type invariants \( (5.19)-(5.21) \) have the following physical meaning, which is connected to the surfaces on which the Ertel-Rossby helicity \( K_r \) is constant in the fluid. \( K^{(2)} \) represents a plane that is tangent to each helicity surface, with \( \sqrt{K_r^{(2)} : K_r^{(2)}} = (1/\rho)|\nabla K_r| \) being inversely proportional to the distance between those neighboring surfaces; \( K_{\omega,r}^{(2)} \) measures the amount of penetration of vortex filaments into each helicity surface; and \( K_{v,r}^{(2)} \) lies in the intersection of each helicity surface and the plane orthogonal to the streamlines of \( \vec{v} \), while \( |K_{v,r}^{(2)}| = (1/\rho)|\nabla K_r||\vec{v}| \sqrt{1 - (\nabla K_r \cdot \vec{v})^2/(|\nabla K_r||\vec{v}|)^2} \) is proportional to the alignment between the streamline plane and the helicity surface as well as to the inverse distance between neighboring helicity surfaces.

The vector invariant \( K_{v,t}^{(2)} \) yields a conserved flux integral \( \frac{d}{dt} \int_{S(t)} (\nabla K_r \times \vec{v}) \cdot dA = 0 \) on moving surfaces \( S(t) \). For closed moving surfaces, this conserved flux integral can be expressed as a moving volume integral by Gauss’ theorem, which arises directly from the scalar invariant \( K_{\omega,r}^{(2)} \).

Together with the two basic invariants \( (5.17) \) and \( (5.18) \), the preceding vorticity-type invariants \( (5.19)-(5.21) \) comprise all independent nonlocal invariants of at most order two for inviscid isentropic fluid flow with a barotropic equation of state.
The process used to construct these invariants can be iterated to obtain a hierarchy of higher-order nonlocal invariants. This leads to the following main result.

**Theorem 5.1.** All independent nonlocal invariants of order \( n \geq 1 \) for inviscid isentropic compressible fluid flow with a barotropic equation of state are generated by the recursions (4.50)–(4.52) starting from \( K^{(0)} = K_{v} \) which is the velocity invariant (5.17), \( K^{(1)} = K_{r} \) which is the Ertel-Rossby helicity invariant (5.18), and \( \tilde{K}^{(1)} = \tilde{K}_{\omega} \) which is the local vorticity invariant (4.23). Other than \( K_{v} \), each invariant in this hierarchy is of vorticity type.

Some of the scalar invariants in this hierarchy describe new cross-helicities. The lowest-order example is given by \( K^{(3)} = (1/\rho)\vec{v} \cdot (\vec{\nabla}K_{\omega,t}^{(2)} \times \vec{\nabla}K_{r}) \), which yields the conserved integral

\[
\int_{V(t)} \vec{v} \cdot (\vec{\nabla}K_{\omega,t}^{(2)} \times \vec{\nabla}K_{r}) \, dV = \int_{V(t)} (K_{\omega,t}^{(2)} \vec{\nabla}K_{r}) \cdot \vec{\omega} \, dV = \int_{V(t)} (K_{r} \vec{\nabla}K_{\omega,t}^{(2)} \times \vec{\nabla}K_{r}) \cdot \vec{\omega} \, dV \tag{5.22}
\]

on moving volumes \( V(t) \) with \( \vec{v} \times \vec{\nabla}K_{r} \) being tangent to the moving boundary surface \( \partial V(t) \). This conserved integral is the cross-helicity of the pair of curl vector fields \( \vec{\omega} \) and \( \vec{\nabla}K_{\omega,t}^{(2)} \times \vec{\nabla}K_{r} \).

Physically, the cross-helicity describes the mutual linking of the vorticity filaments given by this pair of curl vector fields in the fluid.

5.2. Adiabatic non-barotropic invariants. To generalize the previous results to adiabatic (non-isentropic) compressible fluid flow, we follow the idea in Ref. [10, 11, 6] by introducing another potential (Clebsch variable) \( \psi \) defined by the transport equation

\[
\mathcal{D}_t \psi = T(\rho, S) \tag{5.23}
\]

using the fluid temperature (4.16). This transport equation can be integrated in the same way as the transport equation for \( \phi \), yielding

\[
\psi(t, \vec{x}(t)) = \psi(0, \vec{x}(0)) + \int_0^t T(\rho(t', \vec{x}(t')), S(0, \vec{x}(0))) \, dt' \tag{5.24}
\]

along trajectories \( \frac{d}{dt} \vec{x}(t) = \vec{u}(t, \vec{x}(t)) \) of infinitesimal fluid elements. Consequently, \( \psi \) represents a nonlocal variable in terms of \( \vec{u}, \rho, \) and \( S \).

Using both \( \psi \) and \( \phi \) to define an associated Clebsch velocity

\[
\vec{w} = \vec{u} - \vec{\nabla} \phi - \psi \vec{\nabla} S = \vec{v} - \psi \vec{\nabla} S, \tag{5.25}
\]

we can obtain the following transport formulation of the fluid velocity equation:

\[
\mathcal{D}_t \vec{w} = -s \cdot \vec{w}. \tag{5.26}
\]

The derivation is similar to the \( \vec{v} \) equation (5.12) in case of isentropic fluid flow. Specifically, the advective Lie derivative of \( \vec{w} \) is given by

\[
\mathcal{D}_t \vec{w} = \vec{u}_t - \vec{\nabla} \mathcal{D}_t \phi - \mathcal{D}_t \psi \vec{\nabla} S + s \cdot (\vec{\nabla} \phi + \psi \vec{\nabla} S) \tag{5.27}
\]

using the transport properties \( \mathcal{D}_t \vec{u} = \vec{u}_t \) (due to \( \mathcal{L}_u \vec{u} = 0 \)) and \( \mathcal{D}_t S = 0 \), combined with the commutator identity (5.14). Next, using the respective transport equations (5.23) and (5.5) for \( \psi \) and \( \phi \), we see that equation (5.27) becomes

\[
\mathcal{D}_t \vec{w} = \vec{u}_t + s \cdot (\vec{u} - \vec{w}) - \vec{\nabla}(\frac{1}{2}|\vec{u}|^2 - E - p/\rho) - T\vec{\nabla} S. \tag{5.28}
\]

This equation simplifies through the Weber-type equation (5.1) for \( \vec{u} \), directly yielding the velocity transport equation (5.26).
Here we are considering adiabatic compressible fluids with a general non-barotropic equation of state. The Clebsch formulation (5.26) and the subsequent results can be simplified in the case of a barotropic equation of state, which we will consider later.

Physically, \( \psi \nabla S \) represents the dynamical contribution to \( \vec{u} \) by heat transfer, and hence the Clebsch velocity \( \vec{w} \) describes the part of the fluid velocity \( \vec{u} \) that is dynamically driven by enthalpy apart from a contribution \( \nabla \phi \) due to any deviation from equipartition of the total energy into kinetic and enthalpy contributions.

The curl of \( \vec{w} \) is related to the vorticity of \( \vec{u} \) by
\[
\nabla \times \vec{w} = \nabla \times (\vec{u} - \psi \nabla S) = \vec{\omega} - \nabla \psi \times \nabla S,
\]
which describes the part of the fluid vorticity that has no contribution from heat transfer. Taking advective Lie derivative of \( (1/\rho) \nabla \times \vec{w} \) and using the advection properties (3.7) and (3.10), we get
\[
\mathcal{D}_t ((1/\rho) \nabla \times \vec{w}) = \mathcal{D}_t ((1/\rho) \vec{\omega}) - (1/\rho) \nabla \mathcal{D}_t \psi \times \nabla S = 0
\]
which follows from the transport equation (5.23) for \( \psi \). This shows that \( (1/\rho) \nabla \times \vec{w} \) is advected in adiabatic compressible fluid flow. Hence,
\[
K_w = (1/\rho) \nabla \times \vec{w} = (1/\rho)(\vec{\omega} - \nabla \psi \times \nabla S)
\]
is a nonlocal invariant vector.

Moreover, similarly to the derivation of the skew-tensor equation (5.4) satisfied by \( \vec{u} \), here the transport equation (5.26) for \( \vec{w} \) shows that
\[
\mathcal{K}_w = (1/\rho) \varepsilon \cdot \vec{w}
\]
is a nonlocal invariant skew-tensor. It physically represents a plane that is orthogonal to \( \vec{w} \) at each point in the fluid. We remark that this invariant skew-tensor has an alternative formulation as a differential form \( \omega = \vec{w} \cdot d\vec{x} \) that is advected, \( \mathcal{D}_t \omega = 0 \), as shown in Appendix B.

Each nonlocal invariant (5.31) and (5.32) gives rise to a conserved integral which describes a generalization of Kelvin’s circulation theorem. In particular, the conserved integral given by \( K_w \) is a moving curve integral
\[
\oint_{C(t)} \vec{w} \cdot d\vec{s} = \oint_{C(t)} (\vec{u} - \psi \nabla S) \cdot d\vec{s}
\]
which is the circulation of \( \vec{u} - \psi \nabla S \) on closed transported curves \( C(t) \). This conserved circulation integral can be expressed as a moving surface integral by Stokes’ theorem, giving
\[
\int_{S(t)} (\nabla \times \vec{w}) \cdot d\vec{A}
\]
where \( S(t) \) is any moving surface spanning the closed curve \( C(t) \) in the fluid, namely \( \partial S(t) = C(t) \). Clearly, this moving surface integral arises directly from the vector invariant \( K_w \).

We can combine these two nonlocal invariants (5.31) and (5.32) to get a scalar invariant by applying the material algebraic operation in part (ii) of Theorem 3.1. This yields
\[
\rho K_w \times \mathcal{K}_w = (2/\rho) \vec{w} \cdot (\nabla \times \vec{w})
\]
and thus
\[
K_w = (1/\rho) \vec{w} \cdot (\nabla \times \vec{w})
\]
is a generalization of the Ertel-Rossby invariant (5.18) to adiabatic compressible fluid flow. It measures the alignment between the streamlines of \( \vec{w} \) and the corresponding vorticity filaments defined by \( \nabla \times \vec{w} \). The helicity of these filaments is given by the corresponding conserved integral

\[
\frac{d}{dt} \int_{V(t)} (\nabla \times \vec{w}) \cdot \vec{w} \, dV = 0.
\] (5.36)

By applying part (i) of Theorem 3.1 to the entropy invariant \( S \) together with \( K_w \) and \( \tilde{K}_w \), we further obtain a nonlocal vector invariant

\[
K_w \cdot \tilde{\nabla} S = (1/\rho) \tilde{\nabla} S \times \vec{w} = (1/\rho) \tilde{\nabla} S \times \vec{v} = \tilde{K}_{v,S},
\] (5.37)

and a scalar invariant

\[
\tilde{K}_w \cdot \tilde{\nabla} S = (1/\rho) \tilde{\nabla} S \cdot (\tilde{\nabla} \times \vec{w}) = (1/\rho) \tilde{\nabla} S \cdot (\tilde{\nabla} \times \vec{v}) = \phi
\] (5.38)

which reproduces Ertel’s invariant (4.29). Here we have used the relation (5.25) between the Clebsch velocities \( \vec{w} \) and \( \vec{v} \), which gives

\[
(\vec{w} - \vec{v}) \times \tilde{\nabla} S = 0, \quad \tilde{\nabla} S \cdot (\vec{w} - \vec{v}) = 0.
\] (5.39)

The nonlocal vector invariant \( \tilde{K}_{v,S} \) has the following physical meaning: at each point in the fluid, it lies in the intersection of the plane orthogonal to the streamline of \( \vec{v} \) and the surface of constant entropy, while its magnitude \( |\tilde{K}_{v,S}| = (1/\rho)|\tilde{\nabla} S||\vec{v}|\sqrt{1-(\tilde{\nabla} S \cdot \vec{v})^2/(|\tilde{\nabla} S|\vec{v})^2} \) is proportional to the alignment between the streamline plane and the entropy surface as well as to the inverse distance between neighboring entropy surfaces. The conserved integral arising from \( \tilde{K}_{v,S} \) is a moving surface integral \( \int_{S(t)} (\tilde{\nabla} S \times \vec{v}) \cdot d\vec{A} = \int_{S(t)} (\tilde{\nabla} S \times (\vec{u} - \vec{v} \phi)) \cdot d\vec{A} \) describing a conserved flux. For closed moving surfaces, this integral reduces by Stokes’ theorem to \( \int_{S(t)} (\vec{u} \times \tilde{\nabla} S) \cdot d\vec{A} \), which is the conserved entropy-circulation flux arising from Ertel’s invariant (4.29).

Three more nonlocal invariants now arise by applying part (i) of Theorem 3.1 to Ertel’s invariant \( K_e \) together with \( K_w \), \( \tilde{K}_w \), and \( \tilde{K}_{v,S} \). This yields an invariant vector

\[
\tilde{K}_{w,e}^{(2)} = K_w \cdot \tilde{\nabla} K_e = (1/\rho) \tilde{\nabla} K_e \times \vec{w} = (1/\rho) \tilde{\nabla} ((1/\rho)\vec{w} \cdot \tilde{\nabla} S) \times \vec{w};
\] (5.40)

and two invariant scalars

\[
K_{w,e}^{(2)} = \tilde{K}_w \cdot \tilde{\nabla} K_e = (1/\rho) \tilde{\nabla} ((1/\rho)\vec{w} \cdot \tilde{\nabla} S) \cdot (\tilde{\nabla} \times \vec{w}),
\] (5.41)

\[
K_h = -\tilde{K}_{v,S} \cdot \tilde{\nabla} K_e = -(1/\rho) \vec{v} \cdot (\tilde{\nabla} K_e \times \tilde{\nabla} S).
\] (5.42)

These are second-order invariants of vorticity type.

The scalar invariant (5.42) is Hollmann’s invariant \([10]\). It is related to the triple alignment among the streamlines of \( \vec{v} \) and the surfaces on which \( S \) and \( K_e \) are respectively constant. The corresponding conserved integral is a moving volume integral \( \frac{d}{dt} \int_{V(t)} \vec{v} \cdot (\tilde{\nabla} K_e \times \tilde{\nabla} S) \, dV = 0 \). For moving volumes \( V(t) \) with \( \vec{v} \times \tilde{\nabla} S \) being tangent to the moving boundary surface \( \partial V(t) \), this conserved integral describes the cross-helicity of the pair of curl vector fields \( \tilde{\nabla} \times \vec{v} \) and \( \tilde{\nabla} S \times \tilde{\nabla} K_e \):

\[
\int_{V(t)} \vec{v} \cdot (\tilde{\nabla} K_e \times \tilde{\nabla} S) \, dV = \int_{V(t)} (K_e \tilde{\nabla} S) \cdot (\tilde{\nabla} \times \vec{v}) \, dV = \int_{V(t)} (S \tilde{\nabla} K_e) \cdot (\tilde{\nabla} \times \vec{v}) \, dV.
\] (5.43)
Physically, the cross-helicity describes the mutual linking of the vorticity filaments given by \( \vec{\nabla} \times \vec{v} \) and \( \vec{\nabla} K_e \times \vec{\nabla} S \) in the fluid.

Both additional invariants (5.10) and (5.11) are also related to the surfaces on which \( K_e \) is constant. The scalar invariant (5.11) measures the amount of penetration of the vorticity filaments of \( \vec{w} \) into those surfaces. The vector invariant (5.10) lies in the intersection of each surface and the plane orthogonal to the streamline of \( \vec{w} \) at each point in the fluid and is inversely proportional to the distance between neighboring surfaces. It yields a conserved flux integral \( \frac{d}{dt} \int_{S(t)} \left( \vec{\nabla} \left( (1/\rho) \vec{w} \cdot \vec{\nabla} S \right) \times \vec{w} \right) \cdot d\vec{A} = 0 \) on moving surfaces \( S(t) \). For closed moving surfaces, this moving surface integral can be expressed as a moving volume integral by Gauss' theorem, which arises directly from the scalar invariant (5.11).

More second-order nonlocal invariants of vorticity type arise in a similar way from the helicity invariant \( K_w \). This yields an invariant skew-tensor

\[
K_w^{(2)} = \left(1/\rho\right) e \cdot \vec{\nabla} K_w; \tag{5.44}
\]

three invariant vectors

\[
\vec{K}_w^{(2)} = K_w \cdot \vec{\nabla} K_w = (1/\rho) \vec{\nabla} K_w \times \vec{w}, \tag{5.45}
\]

\[
\vec{K}_w^{(2)} = K_e \cdot \vec{\nabla} K_w = -K_w \cdot \vec{\nabla} K_e = (1/\rho) \vec{\nabla} K_w \times \vec{w}, \tag{5.46}
\]

\[
\vec{K}_w^{(2)} = K_S \cdot \vec{\nabla} K_w = -K_w^{(2)} \cdot \vec{\nabla} S = (1/\rho) \vec{\nabla} K_w \times \vec{w}; \tag{5.47}
\]

and four invariant scalars

\[
K_w^{(2)} = \vec{K}_w^{(2)} \cdot \vec{\nabla} K_w = (1/\rho) (\vec{\nabla} \times \vec{w}) \cdot \vec{\nabla} K_w, \tag{5.48}
\]

\[
K_w^{(2)} = \vec{K}_w^{(2)} \cdot \vec{\nabla} K_w = -\vec{K}_w^{(2)} \cdot \vec{\nabla} S = (1/\rho) \vec{w} \cdot (\vec{\nabla} S \times \vec{\nabla} K_w), \tag{5.49}
\]

\[
K_e^{(2)} = \vec{K}_w^{(2)} \cdot \vec{\nabla} K_e = -\vec{K}_w^{(2)} \cdot \vec{\nabla} K_e = (1/\rho) \vec{w} \cdot (\vec{\nabla} K_e \times \vec{\nabla} K_w), \tag{5.50}
\]

\[
K_e^{(2)} = \vec{K}_w^{(2)} \cdot \vec{\nabla} K_e = -\vec{K}_w^{(2)} \cdot \vec{\nabla} K_e = (1/\rho) \vec{w} \cdot (\vec{\nabla} K_e \times \vec{\nabla} K_w), \tag{5.51}
\]

where \( \vec{K}_w^{(2)} \) is the adiabatic vector invariant (4.39). (Note here we have used the relation (5.39).)

The physical interpretation of these nonlocal invariants is similar to the previous ones. In particular, the two scalar invariants (5.49) and (5.50) give rise to conserved moving volume integrals

\[
\int_{\mathcal{V}(t)} \vec{v} \cdot (\vec{\nabla} \times (K_w \vec{\nabla} S)) \, dV = \int_{\mathcal{V}(t)} (K_w \vec{\nabla} S) \cdot (\vec{\nabla} \times \vec{v}) \, dV \tag{5.52}
\]

and

\[
\int_{\mathcal{V}(t)} \vec{w} \cdot (\vec{\nabla} \times (K_w \vec{\nabla} K_e)) \, dV = \int_{\mathcal{V}(t)} (K_w \vec{\nabla} K_e) \cdot (\vec{\nabla} \times \vec{w}) \, dV; \tag{5.53}
\]

which are cross-helicities of vorticity filaments given by \( \vec{\nabla} \times \vec{v} \) and \( \vec{\nabla} \times (K_w \vec{\nabla} S) \), as well as by \( \vec{\nabla} \times \vec{w} \) and \( \vec{\nabla} \times (K_w \vec{\nabla} K_e) \).

Higher-order nonlocal invariants can be constructed from the preceding invariants by use of the material operations in part (i) of Theorem 5.1. In a similar way to the isentropic case considered previously, this leads to the following main result which generalizes Theorem 4.1.

**Theorem 5.2.** All independent invariants (local and nonlocal) of order \( n \geq 1 \) for inviscid adiabatic compressible fluid with a non-barotropic equation of state are given by the recursions
starting from \( K^{(0)} = K_w \) which is the Clebsch velocity invariant (5.32), \( K^{(1)} = K_S \) which is the entropy-gradient invariant (4.28), \( K^{(1)} = \tilde{K}_w, \tilde{K}_{v,S} \) which are the adiabatic vorticity invariant (5.31) and the velocity-entropy alignment invariant (5.37), and \( K_w^{(1)} = K_w, K_e \) which are the adiabatic helicity invariant (5.35) and Ertel’s invariant (4.29). In this hierarchy, each invariant other than \( K_S \) and \( K_w \) is of vorticity type.

5.3. Adiabatic barotropic invariants. We will now restrict attention to a barotropic equation of state \( p = P(\rho) \), where \( S \) and \( T(S) \) are non-constant across different fluid streamlines.

All of the invariants from Theorem 5.2 for non-barotropic fluid flow are invariants in barotropic fluid flow, but some of the nonlocal invariants reduce to the local invariants shown in Theorem 4.2.

To explain how this works, it is simplest to return to the Weber-type equation (5.1) for \( \vec{u} \), and use the thermodynamic relation \( T = E_S \) with \( E = e(\rho) + E_0(S) \) being the internal fluid energy (4.7). This yields

\[
T(S)\nabla S = E_0(S)\nabla S = \nabla E_0(S). \tag{5.54}
\]

Then we have

\[
\tilde{u}_t \equiv s \cdot \tilde{u} = \tilde{\nabla}(\frac{1}{2}|\tilde{u}|^2 - E - p/\rho) + \tilde{\nabla}E_0 = \tilde{\nabla}(\frac{1}{2}|\tilde{u}|^2 - e(\rho) - p(\rho)/\rho). \tag{5.55}
\]

We can now introduce a Clebsch variable through the transport equation

\[
\frac{d}{dt} \theta = \frac{1}{2}(|\tilde{u}|^2 - e(\rho) - p(\rho)/\rho) \tag{5.56}
\]

where \( \theta \) is related to the previous Clebsch variables (5.5) and (5.23) by a line integral expression

\[
\theta - \phi = \int \psi \tilde{\nabla} S \cdot d\vec{x}. \tag{5.57}
\]

The fluid velocity equation (5.55) can again be expressed as a transport equation (5.26) but with

\[
\vec{w} = \tilde{u} - \tilde{\nabla} \theta \tag{5.58}
\]

being the Clebsch velocity.

The main consequence is that the curl of \( \vec{w} \) now yields the local vorticity

\[
\tilde{\nabla} \times \vec{w} = \tilde{\nabla} \times \tilde{u} = \vec{\omega}. \tag{5.59}
\]

Hence, in Theorem 5.2, any adiabatic non-barotropic invariant that is nonlocal only through a dependence on \( \tilde{\nabla} \times \vec{w} \) becomes a local invariant in the barotropic case.

5.4. Invariants in incompressible flows. In incompressible fluid flow, the Weber-type equation for \( \tilde{u} \) is given by

\[
\tilde{u}_t + s \cdot \tilde{u} = \tilde{\nabla}(\frac{1}{2}|\tilde{u}|^2 - p/\rho) + p\tilde{\nabla}(1/\rho), \tag{5.60}
\]

with the density satisfying the transport equation

\[
\rho_t + \tilde{u} \cdot \nabla \rho = 0, \tag{5.61}
\]

and the pressure satisfying the Laplacian equation \( \tilde{\nabla} \cdot ((1/\rho) \tilde{\nabla} p) = \frac{1}{4}(|\vec{\omega}|^2 - |s|^2) \). Although the fluid velocity equation (5.60) resembles the Weber-type formulation (5.1) for adiabatic
compressible fluid flow, the previous potentials $\phi$ and $\psi$ no longer exist. Nevertheless, we are able to introduce analogous potentials through the gradient term and the pressure in equation (5.60).

Consider the transport equation

$$\frac{d}{dt} \sigma = \frac{1}{2} |\vec{u}|^2 - \frac{p}{\rho}$$

(5.62)

defining a potential $\sigma$. Along trajectories $\frac{d}{dt} \vec{x}(t) = \vec{u}(t, \vec{x}(t))$ of infinitesimal fluid elements, this potential is a nonlocal variable

$$\sigma(t, \vec{x}(t)) = \sigma(0, \vec{x}(0)) + \int_0^t \left( \frac{1}{2} |\vec{u}(t', \vec{x}(t'))|^2 - \frac{p(t', \vec{x}(t'))}{\rho(t', \vec{x}(t'))} \right) dt'$$

(5.63)
given in terms of $\vec{u}$, $p$ and $\rho$. Now we introduce the Clebsch-type velocity

$$\vec{v} = \vec{u} - \vec{\nabla} \sigma.$$  

(5.64)

Similarly to the analogous velocity (5.8) for adiabatic compressible fluids, $\vec{v}$ obeys the transport equation

$$\mathcal{D}_t \vec{v} = -\vec{s} \cdot \vec{v} + p\vec{\nabla} \frac{1}{\rho}.$$  

(5.65)

The physical meaning of $\sigma$ and $\vec{v}$ is related to the total energy density $\frac{1}{2} |\vec{u}|^2 + \frac{p}{\rho}$ of the fluid flow. Specifically, the transport equation (5.62) shows that $\frac{d}{dt} \sigma$ is the deviation from equipartition of the kinetic energy density $\frac{1}{2} |\vec{u}|^2$ and the pressure-flow energy density $p/\rho$ in the total energy density. Consequently, the Clebsch-type velocity (5.64) has the physical meaning of the part of the fluid velocity $\vec{u}$ that is dynamically driven by the pressure apart from a contribution $\vec{\nabla} \sigma$ due to any deviation from equipartition of the total energy.

We note that the curl of $\vec{v}$ yields the vorticity

$$\vec{\nabla} \times \vec{v} = \vec{\nabla} \times \vec{u} = \vec{\omega}$$

(5.66)

which has the transport equation

$$\mathcal{D}_t \vec{\omega} = \vec{\nabla} p \times \vec{\nabla} \frac{1}{\rho}.$$  

(5.67)

To proceed, we will first look at constant-density fluids and afterwards consider non-constant density incompressible fluids.

5.4.1. Constant-density invariants. For constant-density fluid flow, the transport equations (5.65) for the Clebsch velocity and (5.67) for the vorticity reduce to the respective forms

$$\mathcal{D}_t \vec{v} = -\vec{s} \cdot \vec{v},$$  

(5.68)

$$\mathcal{D}_t \vec{\omega} = 0.$$  

(5.69)

From the transport equation (5.68) for $\vec{v}$, similarly to the derivation of the skew-tensor equation (5.4) satisfied by $\vec{u}$, we see that

$$K_v = \epsilon \cdot \vec{v}$$

(5.70)

is a nonlocal invariant skew-tensor. It physically represents a plane that is orthogonal to $\vec{v}$ at each point in the fluid. Both of the invariants $K_v$ and $\vec{\omega}$ give rise to a conserved integral

$$\frac{d}{dt} \oint_{C(t)} \vec{v} \cdot d\vec{s} = 0$$

which describes a generalization of Kelvin’s circulation theorem, holding for closed transported curves $C(t)$ in constant-density fluids.
We can now obtain more nonlocal invariants by applying the material operations Theorem 3.1. Note that, since the density $\rho$ is constant, it can be dropped in these operations.

Combining the two invariants $K_v$ and $\vec{\omega}$ by use of the material algebraic operation in part (ii) of Theorem 3.1, we get

$$K_v \times \vec{\omega} = 2\vec{v} \cdot (\vec{\nabla} \times \vec{v}). \tag{5.71}$$

This gives a scalar invariant

$$K_{v'} = \vec{v} \cdot (\vec{\nabla} \times \vec{v}) = \vec{\omega} \cdot \vec{v} \tag{5.72}$$

which is a counterpart of the Ertel-Rossby invariant (5.18). It measures the alignment between the streamlines of $\vec{v}$ and the corresponding vorticity filaments defined by $\vec{\nabla} \times \vec{v} = \vec{\omega}$.

The resulting conserved moving volume integral yields the helicity of $\vec{v}$.

For flows that are also isentropic, additional nonlocal invariants arise from the material differential operations in part (i) of Theorem 3.1 applied to the invariants $K_v, \vec{\omega}, K_v$. This yields the following result, analogous to Theorem 5.1 holding for isentropic compressible flows.

**Theorem 5.3.** All independent nonlocal invariants of order $n \geq 1$ for inviscid isentropic constant-density fluid flow are generated by the recursions (4.50)–(4.52) starting from $K^{(0)} = K_v, K^{(1)} = K_{v'},$ and $\vec{K}^{(1)} = \vec{\omega}$. Each invariant in this hierarchy is of vorticity type, other than $K_v$.

In contrast, for constant-density flows that are adiabatic, there are further nonlocal invariants which arise by using the material operations in Theorem 3.1 to combine $K_v, \vec{\omega}, K_v$ with the local adiabatic invariants from Theorem 4.3. In particular, the basic adiabatic invariants consist of the entropy-gradient invariant $\epsilon \cdot \vec{\nabla} S$ and Ertel’s invariant

$$K_{\tilde{e}} = \vec{\omega} \cdot \vec{\nabla} S = \vec{\nabla} \cdot (S\vec{\omega}) = \vec{\nabla} \cdot (\vec{u} \times \vec{\nabla} S) \tag{5.73}$$

specialized to constant-density flows. This leads to the following result.

**Theorem 5.4.** All independent nonlocal invariants of order $n \geq 1$ for inviscid adiabatic constant-density fluid flow are generated by the recursions (4.50)–(4.52) starting from $K^{(0)} = K_v, K^{(1)} = \epsilon \cdot \vec{\nabla} S, K^{(1)} = \vec{\omega} \cdot \vec{v}, \vec{\omega} \cdot \vec{\nabla} S$, and $\vec{K}^{(1)} = \vec{\omega}$. Each invariant in this hierarchy is of vorticity type, other than $K_v$.

One of the second-order scalar invariants in the hierarchy is a constant-density version of Hollmann’s invariant (5.42), given by

$$K_{\tilde{h}} = \vec{v} \cdot (\vec{\nabla} S \times \vec{\nabla} K_{\tilde{e}}). \tag{5.74}$$

Its corresponding conserved moving volume integral yields the cross-helicity of the pair of curl vector fields $\vec{\nabla} \times \vec{v}$ and $\vec{\nabla} \times (S \vec{\nabla} K_{\tilde{e}})$.

**5.4.2. Density invariants.** For incompressible fluid flow in which the density is non-constant, neither the vorticity $\vec{\omega}$ nor the Clebsch velocity skew-tensor $K_v$ are invariants, due to the density gradient terms occurring in the two respective transport equations (5.65) and (5.67). However, the density itself is now an invariant, since $\mathcal{D}_t \rho = 0$.

We can compensate the density gradient term in the transport equation (5.65) for the Clebsch velocity $\vec{v}$ by introducing another Clebsch variable $\vec{v}$ defined by

$$\frac{d}{dt} \vec{v} = p. \tag{5.75}$$
This transport equation can be integrated in the same way as the transport equation for $\sigma$, yielding

$$\vartheta(t, \vec{x}(t)) = \vartheta(0, \vec{x}(0)) + \int_0^t p(t', \vec{x}(t')) \, dt'$$  \hspace{1cm} (5.76)

along trajectories $\frac{d}{dt} \vec{x}(t) = \vec{u}(t, \vec{x}(t))$ of infinitesimal fluid elements. We use this Clebsch variable to define an associated Clebsch velocity

$$\vec{w} = \vec{u} - \vec{\nabla}\sigma - \vartheta \vec{\nabla}(1/\rho) = \vec{v} - \vartheta \vec{\nabla}(1/\rho)$$  \hspace{1cm} (5.77)

Then the advective Lie derivative of $\vec{w}$ is given by

$$\mathcal{D}_t \vec{w} = \mathcal{D}_t \vec{v} - \mathcal{D}_t \vartheta \vec{\nabla}(1/\rho) - \vartheta \mathcal{D}_t \vec{\nabla}(1/\rho) = \mathcal{D}_t \vec{v} - p \vec{\nabla}(1/\rho) + \vartheta \mathcal{D}_t \vec{\nabla}(1/\rho)$$  \hspace{1cm} (5.78)

using the transport equation (5.75) and the commutator identity (5.14). Finally, using the transport equation (5.64) for $\vec{v}$, we get

$$\mathcal{D}_t \vec{w} = -\vec{s} \cdot \vec{w}.$$  \hspace{1cm} (5.79)

This transport equation (5.79) for the Clebsch velocity $\vec{w}$, together with the relation $\vec{u} = \vec{w} + \vec{\nabla}\sigma + \vartheta \vec{\nabla}(1/\rho)$ and the transport equation (5.76) for $\vartheta$, provides an equivalent dynamical description of the fluid velocity in inviscid incompressible fluid flow. Physically, $\vec{w}$ represents the part of the fluid velocity $\vec{u}$ that is not dynamically driven by the respective contributions $\vartheta \vec{\nabla}(1/\rho)$ and $\vec{\nabla}\sigma$ due to density gradients and deviations from equipartition of the total energy density $\frac{1}{2} |\vec{u}|^2 + \frac{p}{\rho}$ of the fluid flow.

The curl of $\vec{w}$ is related to the vorticity of $\vec{u}$ by

$$\vec{\nabla} \times \vec{w} = \vec{\nabla} \times (\vec{v} - \vartheta \vec{\nabla}(1/\rho)) = \vec{\omega} - \vec{\nabla} \vartheta \times \vec{\nabla}(1/\rho).$$  \hspace{1cm} (5.80)

Similarly to the situation for adiabatic compressible fluid flow, the vorticity relation (5.80) leads to the transport equation

$$\mathcal{D}_t (\vec{\nabla} \times \vec{w}) = 0,$$  \hspace{1cm} (5.81)

showing that $\vec{\nabla} \times \vec{w}$ is advected in incompressible fluid flow.

Hence,

$$\vec{K}_w = \vec{\nabla} \times \vec{w}$$  \hspace{1cm} (5.82)

is a nonlocal invariant vector. Physically, this invariant describes the part of the fluid vorticity that has no contribution from density gradients.

Moreover, similarly to the constant-density case, here the transport equation (5.79) for $\vec{w}$ shows that

$$\vec{K}_w = \epsilon \cdot \vec{w}$$  \hspace{1cm} (5.83)

is a nonlocal invariant skew-tensor. It physically represents a plane that is orthogonal to $\vec{w}$ at each point in the fluid.

Both nonlocal invariants (5.82) and (5.83) gives rise to a conserved integral $\frac{d}{dt} \oint_{C(t)} \vec{w} \cdot d\vec{s} = 0$ which describes a generalization of Kelvin’s circulation theorem, holding for closed transported curves $C(t)$ in incompressible fluids with non-constant density.

Next we can combine these two nonlocal invariants (5.31) and (5.32) to get a scalar invariant by applying the material algebraic operation in part (ii) of Theorem 3.1. This yields

$$\rho \vec{K}_w \times \vec{K}_w = \rho \vec{w} \cdot (\vec{\nabla} \times \vec{w}),$$  \hspace{1cm} (5.84)
A useful observation now is that any invariant can be multiplied or divided by $\rho$. Thus we obtain

$$K_w = \vec{w} \cdot (\vec{\nabla} \times \vec{w}) \quad (5.85)$$

which is a generalization of the constant-density invariant (5.72) to incompressible fluid flows with non-constant density. The scalar invariant (5.85) measures the alignment between the streamlines of $\vec{w}$ and the corresponding vorticity filaments defined by $\vec{\nabla} \times \vec{w}$. Its corresponding conserved integral is the helicity of these filaments.

In the same way as for constant-density flows, there are additional nonlocal invariants for incompressible flows with non-constant density. We have the following generalization of Theorems 5.3 and 5.4.

**Theorem 5.5.** For inviscid incompressible fluid flow with non-constant density, all independent nonlocal invariants of order $n \geq 1$ are generated by the recursions (4.50)–(4.52) starting from $K^{(0)} = \vec{\epsilon} \cdot \vec{w}$, $K^{(1)} = \vec{w} \cdot (\vec{\nabla} \times \vec{w})$, $K^{(1)} = \vec{\nabla} \times \vec{w}$, when the flow is isentropic, as well as $K^{(1)} = \vec{\epsilon} \cdot \vec{\nabla} S, \vec{\omega} \cdot \vec{\nabla} \rho, K^{(1)} = \vec{\omega} \cdot \vec{\nabla} S$ when the flow is adiabatic. Each invariant in this hierarchy is of vorticity type, other than $\vec{\epsilon} \cdot \vec{w}$ and $\vec{\epsilon} \cdot \vec{\nabla} S$.

This hierarchy contains Ertel’s invariant (5.73) and its density-type variant (4.36), and Hollmann’s invariant (5.74), all of which hold for general incompressible fluid flows. In addition, the hierarchy contains several new density-type variants of Hollmann’s invariant:

$$K^{(1)} = \vec{w} \cdot (\vec{\nabla} \rho \times \vec{\nabla} K_e), \quad (5.86)$$

$$K^{(1)} = \vec{w} \cdot (\vec{\nabla} \rho \times \vec{\nabla} K_{e'}) \quad (5.87)$$

$$K^{(1)} = \vec{w} \cdot (\vec{\nabla} S \times \vec{\nabla} K_{e'}) \quad (5.88)$$

The first two of these nonlocal scalar invariants are related to the triple alignment among the streamlines of $\vec{w}$ and the respective surfaces on which $\rho$ and either $K_e$ or $K_{e'}$ are constant. The third nonlocal scalar invariant is similarly related to the triple alignment among the streamlines of $\vec{w}$ and the surfaces on which $S$ and $K_{e'}$ are constant. These invariants give rise to conserved moving volume integrals which yield the cross-helicity of the vorticity filaments $\vec{\nabla} \times \vec{w}$ with each of the curl vector fields $\vec{\nabla} \times (\rho \vec{\nabla} K_e), \vec{\nabla} \times (\rho \vec{\nabla} K_{e'}), \vec{\nabla} \times (S \vec{\nabla} K_{e'})$.

6. **Concluding remarks**

In this paper, we have developed a vector calculus approach to the determination of vorticity invariants in inviscid fluid flow. The main aim was to provide answers to several interesting open questions on advected invariants which arose in recent work [6, 5] using a formulation of the fluid equations based on differential forms.

Our approach uses the more familiar and common vector-calculus formulation of the fluid equations. Advected invariants in this formulation are naturally described by Lie dragging of scalars, vectors, and skew-tensors with respect to the fluid velocity and have the physical meaning of quantities that are frozen into the flow.

We have constructed algebraic and differential operations that can be applied recursively to derive a complete set of invariants starting from the basic known local and nonlocal invariants in inviscid fluid flow. Also we have explained how the invariants give rise to associated conserved integrals which are advected by the fluid flow. The basic types of invariants and conserved integrals consist of (i) advected scalars and corresponding conserved integrals on
moving domains; (ii) advected vectors and corresponding conserved flux integrals on moving surfaces (open or closed); (iii) advected skew-tensors and corresponding circulation integrals on moving curves (open or closed).

As main results, complete infinite hierarchies of local and nonlocal invariants are obtained for both adiabatic fluid flow and isentropic fluid flow that are either incompressible, or compressible with barotropic and non-barotropic equations of state. All of the new invariants are related to Ertel’s invariant and Hollmann’s invariant. In particular, for incompressible fluid flow in which the density is non-constant across different fluid streamlines, a new variant of Ertel’s invariant and several new variants of Hollmann’s invariant are derived, where the entropy gradient is replaced by the density gradient. The physical meaning of these new invariants and their corresponding conserved integrals has been discussed.

The ideas and methods developed in the present paper for vorticity-related invariants in inviscid fluid mechanics can clearly be extended to other fluid systems such as multi-phase fluids, magnetohydrodynamics, and two-fluid plasma models. We intend to explore this direction of work in further investigations.

In future work, we plan to develop a corresponding approach to find new helicity and cross-helicity conservation laws in both inviscid and viscous fluid flow.

Finally, an important question is what are the implications of the existence of an infinite number of conserved integral invariants in inviscid fluid flow? The invariants contain increasingly higher order derivatives of the basic fluid variables and the nonlocal Clebsch variables, and thus are closely related to regularity properties of solutions of the fluid equations. Moreover, the higher-order helicities and other higher-order invariants of vorticity type are plausibly connected to finer scale features in the solutions. Indeed, it is also possible to speculate that the higher-order vorticity invariants could be relevant to detecting development of turbulence. Investigation of such possibilities will require a deeper understanding of both the physical and mathematical meaning of these invariants.

**Appendix A. Geometric Tensors and Operations in Euclidean Space**

Euclidean space $\mathbb{R}^3$ has two fundamental geometrical structures: the metric tensor $g$ and the volume 3-form $\epsilon$. The metric tensor has the geometrical meaning that, for any two non-collinear vectors $\vec{a}$ and $\vec{b}$, the inner product $g(\vec{a}, \vec{b}) = \vec{a} \cdot \vec{b}$ is equal to the product of their lengths and the cosine of the angle between them. Similarly, the volume 3-form has the geometrical meaning that, for any three non-collinear vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, the triple product $\epsilon(\vec{a}, \vec{b}, \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ is equal to the volume of the parallelepiped spanned by these vectors. There is an associated volume tensor $\epsilon$ that is totally antisymmetric and is the dual of the volume 3-form such that $\vec{a} \wedge \vec{b} \wedge \vec{c} = \epsilon(\vec{a}, \vec{b}, \vec{c})\epsilon$.

Let $\hat{e}(i)$, $i = 1, 2, 3$, denote a Cartesian basis of orthonormal vectors in $\mathbb{R}^3$. Then the components of $g$ and $\epsilon$ are given by $g(\hat{e}(i), \hat{e}(j)) = \delta_{ij}$ (Kronecker symbol) and $\epsilon(\hat{e}(i), \hat{e}(j), \hat{e}(k)) = \delta_{ij} \cdot (\hat{e}(j) \times \hat{e}(k)) = \epsilon_{ijk}$ (Levi-Civita symbol). The components of $\epsilon$ are $\epsilon^{ijk}$, where indices are raised/lowered by use of the metric.

The components of a vector $\vec{a}$ are given by $a^i = g(\vec{a}, \hat{e}(i)) = \vec{a} \cdot \hat{e}(i)$, $i = 1, 2, 3$. Similarly, the components of a skew-tensor $\vec{a}$ are given by $a^{ij} = -a^{ji} = g(\vec{a}, \hat{e}(i)) \cdot \hat{e}(j)$, $i = 1, 2, 3$.

Next we recall the basic dot and cross product operations on vectors and skew-tensors, along with the exterior product operation. After that we review the different types of differential operations on scalars, vectors, and skew-tensors.
A.1. Algebraic operations: dot, cross, exterior products. Given two vectors \( \vec{a} \) and \( \vec{b} \), their dot product is a scalar \( c = \vec{a} \cdot \vec{b} \) and their cross product is a vector \( \vec{c} = \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \). In Cartesian components: \( \vec{a} \cdot \vec{b} = \delta_{ij} a^i b^j \) and \( (\vec{a} \times \vec{b})^i = \epsilon_{ijk} a^j b^k \).

Two more operations are the exterior product yielding a skew-tensor \( C = \vec{a} \wedge \vec{b} = \vec{a} \otimes \vec{b} - \vec{b} \otimes \vec{a} \), and the symmetric product yielding a symmetric tensor \( \vec{a} \otimes \vec{b} = \vec{a} \otimes \vec{b} + \vec{b} \otimes \vec{a} \). Their components are given by \( (\vec{a} \wedge \vec{b})^i = a^i b^j - b^i a^j \), and \( (\vec{a} \otimes \vec{b})^i = a^i b^j + b^i a^j \).

These products have a natural extension in which one vector is replaced by a skew-tensor. Given a vector \( \vec{a} \) and a skew-tensor \( \vec{b} \), their dot product is a vector \( \vec{c} = \vec{a} \cdot \vec{b} = -\vec{b} \cdot \vec{a} \), which has components \( \vec{c} = \epsilon_{ijk} a^i b^j \). Their cross product is a scalar \( c = \vec{a} \times \vec{b} = \vec{b} \times \vec{a} \) defined in components as \( \epsilon_{ijk} a^i b^j \). The exterior product \( \vec{a} \wedge \vec{b} \) is a totally antisymmetric tensor whose components are given by \( (\vec{a} \wedge \vec{b})^i = \epsilon_{ijk} a^j b^k \). This tensor is a scalar multiple of the volume tensor \( \vec{e} \), in particular \( \vec{a} \wedge \vec{b} = \frac{1}{2} (\vec{a} \times \vec{b}) \vec{e} \).

There is a further natural extension to products of two skew-tensors. The dot product of \( \vec{a} \) and \( \vec{b} \) is a tensor without symmetry. Antisymmetrizing this tensor yields a skew-tensor \( c = \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} \), while symmetrizing instead yields a symmetric tensor \( \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} \), where in components \( (\vec{a} \cdot \vec{b})^i = \delta_{ij} a^i b^j \). The cross product of \( \vec{a} \) and \( \vec{b} \) is a vector \( \vec{c} = \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \) whose components are given by \( (\vec{a} \times \vec{b})^i = \epsilon_{ijk} a^j b^k = -\epsilon_{jik} b^j a^k \). The double-dot product of \( \vec{a} \) and \( \vec{b} \) is a scalar \( a : b = \delta_{ij} \delta_{kl} a^i b^j b^k \).

Using the volume tensor, a vector \( \vec{a} \) can be converted into a skew-tensor by \( a_\epsilon = \vec{e} \cdot \vec{a} \). In components, \( a_\epsilon^{ij} = \epsilon^{ijk} a^k \). Conversely, a skew-tensor \( \vec{a} \) can be converted into a vector by \( a_\epsilon \vec{e} = \frac{1}{2} \vec{e} : \vec{a} \). The factor of \( \frac{1}{2} \) appears so that these two linear operations are inverses of each other due to the identity \( \delta_{ij} = \frac{1}{2} \epsilon^{ijk} \epsilon_{kl} \). Specifically, converting a vector \( \vec{a} \) to skew-tensor \( a_\epsilon \) and back to a vector \( \vec{a}_\epsilon = \frac{1}{2} \vec{e} : a_\epsilon = \vec{a} \) yields the original vector, and likewise converting a skew-tensor \( \vec{a} \) to a vector \( \vec{a}_\epsilon \) and back to a skew-tensor \( a_\epsilon = \vec{e} \cdot \vec{a}_\epsilon = \vec{a} \) yields the original skew-tensor. These operations are the three-dimension version of the Hodge dual operator (see, e.g. [4] [7]).

In a similar way, any totally antisymmetric tensor \( a^{ijk} \) can be converted into a scalar given by \( a_\epsilon = \frac{1}{6} \epsilon_{ijk} a^{ijk} \), and conversely \( a^{ijk} = a_\epsilon \epsilon^{ijk} \) is an identity.

The following identities are useful:

\[
(\vec{e} \cdot \vec{a}) : (\vec{e} \cdot \vec{b}) = 2 \vec{a} \cdot \vec{b},
\]

\[
(\vec{e} : \vec{a}) \cdot (\vec{e} : \vec{b}) = 2 \vec{a} : \vec{b},
\]

and

\[
\vec{a} \wedge \vec{b} = (\vec{a} \times \vec{b}) \cdot \vec{e},
\]

\[
\vec{a} \times \vec{b} = \frac{1}{2} \vec{e} : (\vec{a} \wedge \vec{b}) = (\vec{e} \cdot \vec{b}) \cdot \vec{a} = - (\vec{e} \cdot \vec{a}) \cdot \vec{b},
\]

\[
\vec{a} \wedge \vec{b} = \frac{1}{2} (\vec{a} \times \vec{b}) \vec{e},
\]

\[
\vec{a} \times \vec{b} = \vec{a} \cdot (\vec{e} : \vec{b}) = (\vec{e} \cdot \vec{a}) : \vec{b},
\]

\[
\vec{a} \wedge (\vec{b} \wedge \vec{c}) = ((\vec{a} \times \vec{b}) \cdot \vec{c}) \vec{e},
\]

\[
\vec{a} \wedge (\vec{b} \wedge \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c},
\]

\[
(\vec{a} \times \vec{b}) \cdot \vec{c} = \frac{1}{2} (\vec{e} : (\vec{a} \wedge \vec{b})) \cdot \vec{c}.
\]
A.2. Differential operations: grad, curl, div, Lie derivative. All differential operations involve the derivative $\hat{\nabla} = \hat{e}_i \partial_i + \hat{e}_j \partial_j + \hat{e}_k \partial_k$ as given in Cartesian coordinates $x^i$. The components of this vector operator will be denoted $\nabla^i$.

For a scalar function $a(x)$, the basic differential operator is the gradient $\nabla a = \text{grad } a$, which is a vector function with components $\nabla^i a$.

For a vector function $\vec{a}(x)$, the basic differential operators are the curl $\hat{\nabla} \times \vec{a} = \text{curl } \vec{a}$ and the divergence $\hat{\nabla} \cdot \vec{a} = \text{div } \vec{a}$, which are given by the components $(\hat{\nabla} \times \vec{a})^i = \epsilon^{ijk} \nabla_j a^k$ and $\hat{\nabla} \cdot \vec{a} = \delta_{ij} \nabla^i a^j$.

For a skew-tensor function $\vec{a}(x)$, the corresponding differential operators consist of $\hat{\nabla} \times \vec{a} = \text{curl } \vec{a}$ which is a scalar function, given in components by $\epsilon_{ijk} \nabla^i a^j$; and $\hat{\nabla} \cdot \vec{a} = \text{div } \vec{a}$, which is a vector function, with components $(\hat{\nabla} \cdot \vec{a})^i = \delta_{jk} \nabla^j a^{ki}$.

The Lie derivative with respect to a vector function $\vec{a}(x)$ is defined as follows on scalar functions $b(x)$, vector functions $\vec{b}(x)$, and skew-tensor functions $\vec{b}(x)$:

\[
\mathcal{L}_\vec{a}b = \vec{a} \cdot \hat{\nabla} b,
\]

\[
\mathcal{L}_\vec{a}\vec{b} = \vec{a} \cdot \hat{\nabla} \vec{b} - \vec{b} \cdot \hat{\nabla} \vec{a} = [\vec{a}, \vec{b}],
\]

\[
\mathcal{L}_\vec{a}\vec{b} = \vec{a} \cdot \hat{\nabla} \vec{b} - (\vec{b} \cdot \hat{\nabla})\vec{a} + ((\vec{b} \cdot \hat{\nabla})\vec{a})^t.
\]

Their Cartesian components are respectively given by $a^i \nabla_i b$, $a^i \nabla_j b^j - b^j \nabla_j a^i$, $a^k \nabla_k b^{ij} - b^k \nabla_k a^i - b^j \nabla_k a^i$. An important property is that the Lie derivative of the volume 3-form and volume tensor are given by

\[
\mathcal{L}_\vec{a} \epsilon = (\hat{\nabla} \cdot \vec{a}) \epsilon, \quad \mathcal{L}_\vec{a} \epsilon = -(\hat{\nabla} \cdot \vec{a}) \epsilon.
\]

The Lie derivative of the inverse metric tensor $g^{-1}$ is given by

\[
\mathcal{L}_\vec{a} g^{-1} = -\hat{\nabla} \o \vec{a}
\]

which is a symmetric derivative. In Cartesian components, $\mathcal{L}_\vec{a} \delta^{ij} = -(\nabla^i a^j + \nabla^j a^i)$.

A.3. Lie derivative identities. From the Lie derivatives (A.13) and (A.14), the following useful identities are straightforward to prove:

\[
\mathcal{L}_\vec{c}(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (\mathcal{L}_\vec{c}\vec{b}) + \vec{b} \cdot (\mathcal{L}_\vec{c}\vec{a}) + s : (\vec{a} \otimes \vec{b}),
\]

\[
\mathcal{L}_\vec{c}(\vec{a} \times \vec{b}) = \vec{a} \times (\mathcal{L}_\vec{c}\vec{b}) + \vec{b} \times (\mathcal{L}_\vec{c}\vec{a}) - (\hat{\nabla} \cdot \vec{c})\vec{a} \times \vec{b},
\]

\[
\mathcal{L}_\vec{c}(\hat{\nabla} f_1 \cdot \hat{\nabla} f_2) = \hat{\nabla} f_1 \cdot (\mathcal{L}_\vec{c}\hat{\nabla} f_2) + \hat{\nabla} f_2 \cdot (\mathcal{L}_\vec{c}\hat{\nabla} f_1) - s : (\hat{\nabla} f_1 \otimes \hat{\nabla} f_2),
\]

\[
\mathcal{L}_\vec{c}(\hat{\nabla} f_1 \times \hat{\nabla} f_2) = \hat{\nabla} f_1 \times (\mathcal{L}_\vec{c}\hat{\nabla} f_2) - \hat{\nabla} f_2 \times (\mathcal{L}_\vec{c}\hat{\nabla} f_1) - (\hat{\nabla} \cdot \vec{c})\hat{\nabla} f_1 \times \hat{\nabla} f_2.
\]

Appendix B. Transcription between vector/tensor calculus and differential forms

In Euclidean space $\mathbb{R}^3$, differential forms are a counterpart of vectors $\vec{a}$, skew-tensors $\vec{a}$, and totally antisymmetric tensors $\alpha \epsilon$. Specifically, a 1-form is a linear map from vectors into scalars; a 2-form is a linear map from skew-tensors into scalars; and a 3-form is a linear map from totally antisymmetric tensors into scalars.

With respect to a Cartesian basis of orthonormal vectors $\hat{e}_{(i)}$, $i = 1, 2, 3$, in $\mathbb{R}^3$, the components of a 1-form $\alpha$ are given by $\alpha_i = \alpha(\hat{e}_{(i)})$, and similarly the components of a
2-form $\beta$ are given by $\beta_{ij} = -\beta_{ji} = \beta(e_i \wedge e_j)$. The components of a 3-form $\gamma$ are a scalar multiple $\tilde{\gamma}$ of the components of the volume form $\epsilon$, namely $\gamma_{ijk} = \tilde{\gamma}\epsilon_{ijk} = \gamma(e_i \wedge e_j \wedge e_k)$; in particular, $\gamma = \tilde{\gamma}\epsilon$.

There is a one-to-one correspondence between 1-forms $\alpha$ and vectors $\tilde{a}$, which arises from the Euclidean metric. In components:

$$\alpha_i = \delta_{ij}a^j, \quad a^i = \delta^{ij}\alpha_j. \quad \text{(B.1)}$$

This correspondence extends to 2-forms $\beta$ and skew-tensors $\mathbf{a}$:

$$\beta_{ij} = \delta_{ik}\delta_{jl}a^{kl}, \quad a^{ij} = \delta^{ik}\delta^{jl}\beta_{kl}. \quad \text{(B.2)}$$

Another one-to-one correspondence arises from the volume tensor $\epsilon$ and the volume form $\epsilon$. Specifically, a 1-form $\alpha$ can be converted into a skew-tensor $\mathbf{a}$ given by

$$a^{ij} = \epsilon^{ijk}\alpha_k. \quad \text{(B.3a)}$$

Correspondingly, a skew-tensor $\mathbf{a}$ can be converted into a 1-form $\alpha$ given by

$$\alpha_i = \frac{1}{2}\epsilon_{ijk}a^{jk}. \quad \text{(B.3b)}$$

The factor of $\frac{1}{2}$ appears here so that these two linear operations are inverses of each other through the identity $\delta^i_j = \frac{1}{2}\epsilon_{ijk}\epsilon^{jl}$. In a similar way, through the previous correspondence between skew-tensors and 2-forms, a 2-form $\beta$ can be converted into a vector $\tilde{a}$ given by

$$a^i = \frac{1}{2}\epsilon^{ijk}\beta_{jk}, \quad \text{(B.4a)}$$

and conversely

$$\beta_{ij} = \epsilon_{ijk}a^k \quad \text{(B.4b)}$$

converts a vector $\tilde{a}$ back into a 2-form $\beta$.

It is natural to extend these correspondences by first defining a 0-form as a scalar $a$, and then using the correspondence between scalars and totally antisymmetric tensors to convert a 0-form $a$ into a totally antisymmetric tensor $\mathbf{a}$ having components $a^{ijk} = a\epsilon^{ijk}$. Conversely, a totally antisymmetric tensor is converted back into a 0-form given by $a = \frac{1}{6}\epsilon_{ijk}a^{ijk}$.

Next we recall the basic wedge product and exterior derivative operations on differential forms, along with contraction between differential forms and vectors and tensors. We show how these operations mirror the basic algebraic and differential operations on vectors and tensors.

### B.1. Operations on differential forms

The wedge product of 1-forms corresponds to the exterior product of vectors, and the wedge product of 1-form and 2-form corresponds to the exterior product of a vector and a skew-tensor. In particular, given two 1-forms $\alpha$ and $\mu$, their wedge product $\alpha \wedge \mu = \beta$ is a 2-form with components $\beta_{ij} = \alpha_i\mu_j - \mu_i\alpha_j$. Likewise, given a 1-form $\alpha$ and a 2-form $\beta$, their wedge product $\alpha \wedge \beta = \beta \wedge \alpha = \gamma$ is a 3-form with components $\gamma_{ijk} = \alpha_i\beta_{jk} + \alpha_j\beta_{ki} + \alpha_k\beta_{ij}$.

The wedge product of two 2-forms vanishes because there cannot exist a totally antisymmetric tensor of rank 4 in three dimensions.

Contraction of a vector $\tilde{a}$ with a 1-form $\alpha$ is defined by $\tilde{a} \cdot \alpha = a^i\alpha_i$ in components. Similarly, contraction of a vector $\tilde{a}$ with, respectively, a 2-form $\beta$ and a 3-form $\gamma = a\epsilon$ is defined by $\tilde{a} \cdot \beta = a^i\beta_{ij}$ and $\tilde{a} \cdot \gamma = a^i\gamma_{ijk}$. A skew-tensor $\mathbf{a}$ can be contracted with, respectively, a 2-form $\beta$ and a 3-form $\gamma$ by $\mathbf{a} \cdot \beta = a^i\beta_{ij}$ and $\mathbf{a} \cdot \gamma = a^i\gamma_{ijk}$.
The basic derivative operator on differential forms is the exterior derivative \( \mathbf{d} \) which corresponds to a curl on vectors and skew-tensors, and a gradient on scalars. In components, \( \mathbf{d} \) is defined on 0-forms \( a \), 1-forms \( \alpha \), 2-forms \( \beta \) by:

\[
(\mathbf{d}a)_i = \nabla_i a, \tag{B.5}
\]

\[
(\mathbf{d}\alpha)_{ij} = \nabla_i \alpha_j - \nabla_j \alpha_i, \tag{B.6}
\]

\[
(\mathbf{d}\beta)_{ijk} = \nabla_i \beta_{jk} + \nabla_j \beta_{ki} + \nabla_k \beta_{ij}, \tag{B.7}
\]

where \( \nabla_i = \partial_{x^i} \) in Cartesian coordinates \( x^i \). These differential forms are called exact. The exterior derivative of any exact differential form vanishes, due to the property \( \mathbf{d}^2 = 0 \), corresponding to the vector calculus properties \( \text{div } \text{curl} = 0 \) and \( \text{curl } \text{grad} = 0 \).

Through the two correspondences \( \text{(B.1)} \) and \( \text{(B.3)} \) between 1-forms and vectors, an exact 1-form \( \mathbf{d} \) represents a gradient vector \( \nabla a \) and a divergence of a totally antisymmetric tensor \( \nabla_k (e_{ijk} a) = e_{ijk} \nabla_k a \). Likewise, an exact 2-form \( \mathbf{d} \) represents an antisymmetric derivative of a vector \( \nabla^i a^j - \nabla^j a^i \) and a curl of a vector \( 2e_{jk} \nabla^j a^k \), by the correspondences \( \text{(B.2)} \) and \( \text{(B.4)} \). This curl can also be expressed as a divergence of a skew-tensor \( \nabla_j (e_{ijk} a^k) \).

In a similar way, an exact 3-form \( \mathbf{d} \) represents an antisymmetric derivative of a skew-tensor \( \nabla^i a^{jk} + \nabla^j a^{ki} + \nabla^k a^{ij} \) and a divergence of a vector \( 3 \nabla_i (e_{ijk} a^k) \).

Another derivative operator on differential forms is the Lie derivative with respect to any vector field \( \vec{a} \):

\[
\mathcal{L}_{\vec{a}} a = [\vec{a}, a] = \vec{a} \cdot \nabla a, \tag{B.8}
\]

\[
\mathcal{L}_{\vec{a}} \alpha = \vec{a} \cdot (\mathbf{d}\alpha) + \mathbf{d}(\vec{a} \cdot \alpha), \tag{B.9}
\]

\[
\mathcal{L}_{\vec{a}} \beta = \vec{a} \cdot (\mathbf{d}\beta) + \mathbf{d}(\vec{a} \cdot \beta), \tag{B.10}
\]

\[
\mathcal{L}_{\vec{a}} \gamma = \mathbf{d}(\vec{a} \cdot \gamma). \tag{B.11}
\]

The Lie derivative has the property that it commutes with the exterior derivative \( \mathbf{d} \):

\[
[\mathcal{L}_{\vec{a}}, \mathbf{d}] = 0. \tag{B.12}
\]

When the exterior derivative is applied to Cartesian coordinates \( x^i, i = 1, 2, 3 \), this yields a basis of three 1-forms \( (d x^1, d x^2, d x^3) \), denoted \( \mathbf{d} \vec{x} \). The wedge product of the these 1-forms then yields a basis of three 2-forms: \( (d x^1 \wedge d x^2, d x^2 \wedge d x^3, d x^3 \wedge d x^1) \), denoted \( \mathbf{d} \vec{x} \wedge \mathbf{d} \vec{x} \).

Every 1-form \( \alpha \) and 2-form \( \beta \) can be expanded in this basis in terms of Cartesian components

\[
\alpha = \alpha_i d x^i, \quad \beta = \beta_{ij} d x^i \wedge d x^j. \tag{B.13}
\]

### B.2. Euler’s equations using differential forms.

Euler’s equations \( \text{(4.1)} \) for inviscid fluid flow can be converted into differential forms by using the Cartesian components of the fluid velocity \( \vec{u} = u^i \vec{e}_i \) where \( \vec{e}_i, i = 1, 2, 3 \), is a Cartesian basis of orthonormal vectors in \( \mathbb{R}^3 \).

Write

\[
\mathbf{u} = \delta_{ij} u^i d x^j = \vec{u} \cdot \mathbf{d} \vec{x} \tag{B.14}
\]

for the 1-form corresponding to the fluid velocity. The curl of the fluid velocity corresponds to the exterior derivative of \( \mathbf{u} \):

\[
\boldsymbol{\omega} = \mathbf{d} \mathbf{u} = \epsilon_{ijk} (\vec{\nabla} \times \vec{u})^k d x^i \wedge d x^j = \epsilon_{ijk} \omega^k d x^i \wedge d x^j \tag{B.15}
\]
where the vorticity vector \( \vec{\omega} = \vec{\nabla} \times \vec{u} \) is identified with the vorticity 2-form \( \omega \) through the correspondence (B.4). The divergence free property of \( \vec{\omega} \) then corresponds to \( d\omega = 0 \).

To express Euler’s equations (4.1) in terms of \( \mathbf{u} \) and \( \omega \), we convert the vector calculus identity (4.4) into a corresponding 1-form identity

\[
\vec{u} \cdot \vec{\nabla} \mathbf{u} = \frac{1}{2} d(|\vec{u}|^2) - \vec{u} |\omega|.
\]  

This yields

\[
\mathbf{u}_t = \vec{u} |\omega| + \frac{1}{2} d(|\vec{u}|^2) - (1/\rho) dp.
\]  

Hence we have

\[
\omega_t = d(\vec{u} |\omega|) + (1/\rho^2) dp \land dp,
\]  

which is the 2-form version of the vorticity equation (4.16). Likewise, the entropy equation (4.3) becomes

\[
S_t + \vec{u} |dS| = 0.
\]  

These equations (B.17)–(B.19) have an elegant formulation using the advective Lie derivative (2.1):

\[
\mathcal{D}_t \mathbf{u} = d((1/2) |\vec{u}|^2 - E - p/\rho) + T dS,
\]

\[
\mathcal{D}_t \omega = dT \land dS, \quad \omega = d\mathbf{u},
\]

\[
\mathcal{D}_t S = 0,
\]  

where \( T \) is the fluid temperature and \( E \) is the internal energy of the fluid, given by the thermodynamic relation (4.4). This fluid velocity equation for \( \mathbf{u} \) is the differential-form version of Weber’s transformation [39, 40, 6, 7].

B.3. Invariants. Invariant vectors \( \vec{K} \) and invariant skew-tensors \( \mathbf{K} \) respectively correspond to invariant 2-forms \( \kappa = (1/2\rho)\vec{K} |\epsilon| \) and invariant 1-forms \( \lambda = (1/\rho)\mathbf{K} |\epsilon| \).

\[
\mathcal{D}_t \kappa = \partial_t \kappa + L_{\vec{a}} \kappa = 0, \quad \mathcal{D}_t \lambda = \partial_t \lambda + L_{\vec{a}} \lambda = 0.
\]  

The converse is \( \vec{K} = (1/\rho)\epsilon |\kappa \) and \( \mathbf{K} = (1/\rho)\epsilon |\lambda \). This correspondence arises from the property that the densitized volume form \( \rho \epsilon \) and the densitized volume tensor \( (1/\rho)\epsilon \) are invariants, \( \mathcal{D}_t (\rho \epsilon) = 0 \) and \( \mathcal{D}_t ((1/\rho)\epsilon) = 0 \), whereas the metric tensor \( g \) is not an invariant as show by its advection property (3.7).

In particular, in Cartesian components, \( \vec{K} = K^i \hat{e}^{(i)} \) is an invariant vector iff \( \kappa = \kappa_{ij} dx^j \land dx^k \) is an invariant 2-form, where \( \kappa_{ij} = 1/2 \rho \epsilon_{ijk} K^k \). The component form of the invariance property is given by \( (\mathcal{D}_t \vec{K})^i = \partial_t K^i + u^j \nabla_j K^i - K^j \nabla_j u^i = 0 \) and \( (\mathcal{D}_t \kappa)_{ij} = \partial_t \kappa_{ij} + u^k \nabla_k \kappa_{ij} + \kappa_{ik} \nabla_j u^k - \kappa_{kj} \nabla_i u^k = 0 \). Similarly, \( \mathbf{K} = K^{ij} \hat{e}^{(i)} \otimes \hat{e}^{(j)} \) is an invariant skew-tensor iff \( \lambda = \lambda_i dx^i \) is an invariant 1-form, where \( \lambda_i = 1/2 \rho \epsilon_{ijk} K^{jk} \). The component form of the invariance property is given by \( (\mathcal{D}_t \mathbf{K})^{ij} = \partial_t K^{ij} + u^k \nabla_k K^{ij} - K^{kj} \nabla_k u^i + K^{ki} \nabla_i u^j = 0 \) and \( (\mathcal{D}_t \lambda)_i = \partial_t \lambda_i + u^k \nabla_k \lambda_i + \lambda_j \nabla_i u^j = 0 \). In the terminology and notation of Ref.[5], scalar and vector invariants are respectively called a Lagrangian-type invariant \( I = K \) and a field-type invariant \( J = \vec{K} \), while skew-tensor invariants \( \mathbf{K} \) appear only in the form of a vector \( \vec{S} = \rho \epsilon : \mathbf{K} \). Such vectors are a kind of dual of skew-tensor invariants but are not themselves invariant since \( \mathcal{D}_t \vec{S} = -\vec{S} \cdot s \neq 0 \) follows from \( S^i = \delta^i_l \rho \epsilon_{ijk} K^{jk} \) by using the transport properties (3.7) and (3.8) of \( \delta^{ij} \) and \( \epsilon_{ijk} \). The proper way to view \( \vec{S} \) is that its associated 1-form \( S_i = \rho \epsilon_{ijk} K^{jk} \) is an actual invariant.
B.4. A generating set of material operators for advected differential forms. The material operators shown in Theorem 3.1 have a simple formulation in terms of differential forms.

Material differential operators:
- If \( \mu \) is an advected 0-form, 1-form, or 2-form, then \( d\mu \) is an advected 1-form, 2-form, or 3-form.
- If \( \mu \) is an advected 0-form, 1-form, 2-form, or 3-form, and \( \vec{K} \) is an invariant vector, then so is \( \mathcal{L}_{\vec{K}}\mu = \vec{K} \cdot d\mu + d(\vec{K} \cdot \mu) \).

Material algebraic operators:
- If \( \mu \) is an advected 1-form, 2-form, or 3-form, and \( \vec{K} \) is an invariant vector, then \( \vec{K} \cdot \mu \) is an advected 0-form, 1-form, or 2-form.
- If \( \mu \) and \( \nu \) are advected 1-forms, then \( \mu \wedge \nu \) is an advected 2-form. If \( \mu \) is an advected 1-form and \( \nu \) is an advected 2-form, then \( \mu \wedge \nu \) is an advected 3-form.
- If \( \mu \) is an advected 2-form, then \( (1/\rho)\epsilon \cdot \mu \) is an invariant vector. Conversely, if \( \vec{K} \) is an invariant vector, then \( \rho \epsilon \cdot \vec{K} \) is an advected 2-form.
- If \( \mu \) is an advected 3-form, then \( (1/\rho)\epsilon \cdot \mu \) is an advected 0-form, namely, an invariant scalar. Conversely, if \( K \) is an invariant scalar, then \( \rho K \epsilon \) is an advected 3-form.

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