Hilbert’s 6-th problem and principle of completeness in dynamics

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Abstract. The following offers a new axiomatic basis of mechanics and physics in their most important dynamics domain, i.e. a principle (axiom) of completeness intended to generalize Newton’s second law of motion for the case of a non-stationary variable-mass point (system) that varies with time. This generalization leads to hyperdynamic dependencies describing such motion from new accurate qualitative standpoints.

1. Introduction
This work can be considered a modified version of the theory of hyperreactive motion proposed by the author [1], [2], [3], [4]. The incentive was to contribute a mite to the solution of David Hilbert’s sixth problem on axiomatization. The new axiomatization of mechanics and physics in dynamics is suggested, namely, the principle (axiom) of completeness called to generalize Isaac Newton’s second law of dynamics in the case of a non-stationary variable-mass of point (system) changed with time.

Introduction of the new hyperreactive forces into analysis also required a qualitatively new structural approach to the basic dynamic principles of mechanics, their refinement and modernization. In this sense, the results of Meshchersky’s theory [5] on derivation of equations of reactive motion, as well as Tsiolkovsky’s calculation scheme [6] cannot be deemed entirely satisfactory.

2. Principle (axiom) of completeness
The hyperreactive model based on the new differential law of motion (principle or axiom of completeness) contains those summands in the equation of motion that depend not only on point mass $M(t)$ at a time moment $t$ and on the speed of its change $dM(t)/dt$, but also on the acceleration of the mass change $d^2M(t)/dt^2$, which is crucially important from the perspective of global description of material bodies’ motion process.

Assuming the mass to be one more generalized coordinate, the total momentum of the point (system) $Q(t)$ can thus be written as follows:

$$Q(t) = M(t) \frac{dR(t)}{dt} + \frac{dM(t)}{dt} R(t), \quad (1)$$
where vector $R(t) = r(t) - \rho(t)$, let us call it reactive vector, $d r(t)/dt = v(t)$ is the absolute velocity of motion of the point, $d \rho(t)/dt = u(t)$ is the absolute exhaust velocity of particles of the radiating center that represents a time vector function defined over $[t_0, t]$. Here $r(t), \rho(t)$ are current radius vectors of the point and the particle in the absolute frame of reference, correspondingly. The second term in the right-hand side of equation (1) represents the momentum that appears due to the change of mass over the space vector segment $R(t)$.

Let us call the product of the point mass $M(t)$ and the reactive vector $R(t)$ the motion composition vector $S(t) = M(t) R(t)$. Then, in accordance with expression (1), we obtain the new differential law of motion: principle (axiom) of completeness or the theorem of changing of the motion composition.

**Theorem 1.** Time derivative of the motion composition vector of a system equals to the vector of its total momentum:

$$\frac{dS(t)}{dt} = Q(t).$$

We use the momentum principle for the value $Q(t)$ (1):

$$\frac{dQ(t)}{dt} = F(t),$$

where $F(t)$ is the external active force acting on the point. We get

$$M \frac{d^2 R}{dt^2} + 2 \frac{dM}{dt} \frac{dR}{dt} + \frac{d^2 M}{dt^2} R = F$$

and, hence,

$$M \frac{dv}{dt} = F + 2\Phi_1 + \Phi_2 + \Phi_3,$$

where the following notations are introduced:

$$\Phi_1 = - \frac{dM}{dt} \frac{dR}{dt}, \quad \Phi_2 = \frac{dM}{dt} V, \quad \Phi_3 = - \frac{d^2 M}{dt^2} R.$$

Here, $\Phi_1$ is the standard reactive force with the relative exhaust velocity of particles $V = u - v$, $\Phi_2$ is the force arising due to the effect of non-stationarity of velocity $u$. The force $\Phi_3$ generated by the acceleration of the point mass change will be referred to as hyperreactive.

Thus, the general equation of hyperreactive motion (2) is written down as

$$\frac{d(M\dot{R} + \dot{M}R)}{dt} = F,$$

where the over-dot denotes the time derivative. One can clearly see due to which term and why the effect of hyperreactivity appears.

### 3. Energy of a variable-mass point

Let the motion of the point be not constrained by geometrical (holonomic or non-holonomic) constraints. Choose the coordinates characterizing the position of the point in space. We have the following chain of identical transformations (under the assumption that the point mass $M(t)$ is a continuously differentiable function of only time $t$):

$$F = \frac{d}{dt} (M\dot{R} + \dot{M}R) = \frac{d}{dt} (M\dot{R} + \dot{M}R) \frac{\partial R}{\partial R}$$
\[ = \frac{d}{dt} \left[ (M \dot{R} + \dot{M} R) \frac{\partial R}{\partial R} \right] - (M \dot{R} + \dot{M} R) \frac{d}{dt} \left( \frac{\partial R}{\partial R} \right) \]  

Thus, using relation (3), the efficient energy \( T_e \) of point motion in the field of the external force \( F \) can be represented in the form of the following sum:

\[ T_e = T_k + T_r, \]

where the following values are denoted in terms of the reactive vector \( R \):

\[ T_k = \frac{M \dot{R}^2}{2} \] is kinetic energy, \( T_r = \dot{M} R \dot{R} \) is the newly introduced reactive energy of the point.

Along with the theorem of changing of the total momentum of a point, based on relations (3) and (4), we will give the theorem of changing of the efficient energy of a variable-mass point.

**Theorem 2.** Time derivative of the gradient vector of the efficient energy scalar field \( T_e (4) \) of a point with respect to the elements of vector \( \dot{R} \) (or negative relative velocity \(-V \) of particle outflow) is equal to the vector of the external active force acting on the point:

\[ \frac{d}{dt} (\nabla_{\dot{R}} T_e) = F. \]

Here \( \nabla_{\dot{R}} = \partial / \partial \dot{R} \) denotes the vector of partial differentiation (gradient) with respect to the corresponding components of the vector \( \dot{R} \). This allows us to conclude that the point has the highest rate of change of the efficient energy in the direction of the velocity vector \( \dot{R} \).

One can assign to the coordinate \( q_j \) the generalized external force \( F_j (\partial R_i / \partial q_j) = G_j \). Then, in the field of potential acting forces: \( G = \text{grad} U, G_j = \partial U / \partial q_j \), where \( U(q_1, ..., q_s) \) is potential function, we can write the following for the total kinetic potential \( T_* \) of the system: \( T_* = T_e + U = L + T_r \), where \( L = T_k + U \) is the Lagrange function. Hence, we have

\[ \frac{d}{dt} \frac{\partial T_*}{\partial q_j} - \frac{\partial T_*}{\partial q_j} = P_j, \quad P_j = -\dot{M}_i \dot{R}_i \frac{\partial R_i}{\partial q_j}, \]

where \( P_j \) is a generalized hyperreactive force corresponding to coordinate \( q_j \).

4. **Hamilton’s variational principle**

Now let us choose the functional \( S_H \), which is called the Hamiltonian action, as the measure of mechanical motion. We derive Hamilton’s variational principle from the equation of hyperreactive motion of a variable-mass point and establish the extremal properties of the action \( S_H \) for real motions.

Thus, let a vector universal equation of hyperreactive motion be given, and

\[ \left( F - \frac{d}{dt} (M \dot{R} + \dot{M} R) \right) \delta R = 0, \]

where \( \delta R \) denotes the variation of the reactive vector. We suppose that the field of acting forces is potential, i.e.

\[ F \delta R = \text{grad} U \delta R = \delta U. \]

For other terms in equation (5), the following transformations can be performed:

\[ \frac{d}{dt} (M \dot{R} + \dot{M} R) \delta R \]
\[
\frac{d}{dt} \left[ (\dot{M} \dot{R} + \dot{M}R) \delta R \right] - \delta \left( \frac{\dot{M} \dot{R}^2}{2} + \dot{M}RR \right) + \dot{M}R \delta R, 
\]
(7)

where \( \dot{M} \dot{R}^2/2 + \dot{M}RR = T_e \). Here, \( M(t) \) is a known continuously differentiable function of time and \( M(t) \), \( \dot{M}(t) \) are not varied. Thus, expression (7) is written as

\[
\frac{dQ}{dt} \cdot \delta R = \frac{d}{dt} (Q \delta R) - \delta T_e + \dot{M} \dot{R} \delta R. 
\]
(8)

Taking into account relations (6) and (8), the universal equation of hyperreactive motion (5) can be represented in the following form:

\[
\delta T_e + \delta U = \frac{d}{dt} (Q \delta R) + \dot{M} \dot{R} \delta R, 
\]

T

\[
\delta T_e + U = T_e. 
\]

Let us integrate the latter expression with respect to time from \( t_0 \) to \( t_* \). We obtain

\[
\int_{t_0}^{t_*} \delta T_e \, dt = Q \delta R \bigg|_{t_0}^{t_*} + \int_{t_0}^{t_*} \dot{M} \dot{R} \delta R \, dt. 
\]

In case of synchronous variation, for fixed time moments \( t = t_0 \) and \( t = t_* \) we have \( \delta R = 0 \), i.e.

\[
\int_{t_0}^{t_*} (\delta T_e - \dot{M} \dot{R} \delta R) \, dt = 0. 
\]
(9)

Thus, Hamilton’s principle for hyperreactive motion has a mathematical statement in the form of (9).

The structure of the variational integral \( \int f(g) \delta g \) [1] allows for writing down Hamilton’s principle (9) in the standard form. In fact, we have

\[
\delta S_{H_e} = \delta \int_{t_0}^{t_*} L_e \, dt = \delta \int_{t_0}^{t_*} (T_e - \int M \dot{R} \delta R) \, dt = 0, 
\]
(10)

where

\[
S_{H_e} = \int_{t_0}^{t_*} L_e \, dt, \quad L_e = T_e - \int M \dot{R} \delta R, \quad T_e = \frac{\dot{M} \dot{R}^2}{2} + \dot{M}RR + U. 
\]

Here, \( S_{H_e} \) is the total Hamiltonian action, \( L_e \) is the generalized Lagrange function, and the following equality holds in relation (10): \( \delta \int M \dot{R} \delta R = M \dot{R} \delta R \).

5. Conclusions

The concept of a new type of reactive motion, namely, hyperreactive motion of a variable-mass point is substantiated. The new axiomatic principle of dynamics was called ”principle (axiom) of completeness”. The key point of the new approach is consideration of hyperreactive components.

References

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