WEIGHTED PROJECTIVE SPACES AND ITERATED THOM SPACES

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Abstract. For any \((n+1)\)-dimensional weight vector \(\chi\) of positive integers, the weighted projective space \(\mathbb{P}(\chi)\) is a projective toric variety, and has orbifold singularities in every case other than \(\mathbb{C}P^n\). We study the algebraic topology of \(\mathbb{P}(\chi)\), paying particular attention to its localisation at individual primes \(p\). We identify certain \(p\)-primary weight vectors \(\pi\) for which \(\mathbb{P}(\pi)\) is homeomorphic to an iterated Thom space over \(S^2\), and discuss how any \(\mathbb{P}(\chi)\) may be reconstructed from its \(p\)-primary factors. We express Kawasaki’s computations of the integral cohomology ring \(H^*(\mathbb{P}(\chi);\mathbb{Z})\) in terms of iterated Thom isomorphisms, and recover Al Amrani’s extension to complex \(K\)-theory. Our methods generalise to arbitrary complex oriented cohomology algebras \(E^*(\mathbb{P}(\chi))\) and their dual homology coalgebras \(E_*(\mathbb{P}(\chi))\), as we demonstrate for complex cobordism theory (the universal example). In particular, we describe a fundamental class in \(\Omega_{2n}^U(\mathbb{P}(\chi))\), which may be interpreted as a resolution of singularities.

1. Introduction

Weighted projective spaces \(\mathbb{P}(\chi)\) are defined for every integral weight vector \(\chi\), and constitute a family of singular toric varieties on which many hypotheses may be tested. Our central interest is to study their algebraic topology, and identify them up to homeomorphism with spaces whose singularities are more familiar, namely iterated Thom complexes. In the process of working towards this goal, it transpires that the cohomology algebras \(E^*(\mathbb{P}(\chi))\), and homology coalgebras \(E_*(\mathbb{P}(\chi))\), may be described in terms of iterated Thom isomorphisms for any complex oriented cohomology and homology theories \(E^*(-)\) and \(E_*(-)\). This provides a fruitful new perspective on algebraic objects of considerable complexity, and we present several explicit examples below.

Until the last few years, literature on the algebraic topology of weighted projective spaces has been sparse, and restricted mainly to work of Kawasaki [14] and Al Amrani [3, 4]. Immediately after the latter, Nishimura–Yosimura [15] took up the challenge of computing the real \(K\)-theory groups \(KO^*(\mathbb{P}(\chi))\), whose difficulty is increased by the lack of complex orientation for \(KO^*(-)\). More recently, it has become apparent that the toric structure of \(\mathbb{P}(\chi)\) is particularly
important, and our own work \[5\], \[6\] has exploited this fact. Most of our current results are independent of the toric framework, but its beauty is sufficiently compelling that we have retained here it for background and motivation.

We present our results in the following order.

Section 2 establishes notation by recalling definitions of $P(\chi)$ and the associated weighted lens spaces $L(k; \chi)$ in the toric setting. A crucial cofibre sequence of Kawasaki is expressed in terms of normal neighbourhoods over the orbit simplex $\Delta^n$ of the canonical torus action, and his resulting computation of the integral cohomology rings $H^*(P(\chi))$ and $H^*(L(k; \chi))$ are summarised. The important maps $\phi: \mathbb{C}P^n \rightarrow P(\chi)$ and $\psi: P(\chi) \rightarrow \mathbb{C}P^n$ are described in terms of homogeneous coordinates, and the complex line bundle classified by $\psi$ is identified for future use.

Section 3 introduces the concept of divisive weight vectors and their normalisations, and Theorem 3.8 employs an observation of Al Amrani (relating $\psi$ and normal neighbourhoods) to express $P(\chi)$ as an iterated Thom complex whenever $\chi$ is divisive. Fundamental examples are provided by $p$-primary weight vectors $\pi$, whose coordinates are powers of a fixed prime $p$. Archetypal examples are the $p$-contents $p\chi$ of $\chi$, which lead to the $p$-primary decomposition of $\chi$. Maps $e(\chi/\omega): P(\omega) \rightarrow P(\chi)$ are introduced as common generalisations of $\psi$ and $\sigma$, and determine extraction and inclusion maps between $P(\chi)$ and its $p$-primary parts $P(p\chi)$. The maps $e$ also encode a description of any weighted projective space as the quotient of any other by the action of a certain finite abelian group.

Section 5 considers complex oriented cohomology theories $E^*(-)$, whose coefficient rings $E_*$ are even dimensional and free of additive torsion. In Theorem 5.8 the complex orientation is exploited to describe $E^*(P(\chi))$ via iterated Thom isomorphisms and the $E$-theoretic formal group law, whenever $\chi$ is divisive. Illustrative examples are given using integral cohomology and complex $K$-theory, and the former are shown to recover Kawasaki’s calculations in the divisive (and therefore the $p$-primary) case.

Section 4 addresses the question that immediately arises: is it possible to reassemble the $p$-primary parts $P(p\chi)$, and recover $P(\chi)$ up to homeomorphism? Theorem 4.9 answers affirmatively, and offers the surprising addendum that reassembly may be effected in two contrasting ways, which describe $P(\chi)$ as an iterated limit and an iterated colimit of the $P(p\chi)$s respectively. Both constructions arise over $\mathbb{C}P^n$, utilising the maps $e(1/p\chi)$ and $e(p\chi)$. Examples are discussed which show that different reassemblies of the same $p$-primary parts can produce non-homeomorphic results.

Section 6 introduces cohomological reassembly as a natural analogue of the previous geometry. Theorem 6.10 describes $E^*(P(\chi))$ as both a limit and a colimit of the $p$-primary parts $E^*(P(p\chi))$, made explicit in terms of direct sums and iterated tensor products over $E^*(\mathbb{C}P^n)$ respectively. Most memorable is the resulting identification of $E^*(P(\chi))$ with an intersection of $p$-primary subalgebras of $E^*(\mathbb{C}P^n)$. Examples are provided in integral cohomology and complex $K$-theory to show that detailed computation is possible; the former recover Kawasaki’s calculations for arbitrary $\chi$, and the latter those of \[3\].
To date, toric topology has rarely dealt with the homology coalgebra $E_\ast(X_\Sigma)$ of a toric variety $X_\Sigma$. Section 7 redresses this situation, at least for weighted projective space, by studying the universal example $\mathcal{O}_U^* (\mathbb{P}(\chi))$ for divisive $\chi$. The relationship between Bott towers and iterated Thom spaces is recalled, and applied in Theorem 7.14 to identify explicit $\mathcal{O}_U^*$-generators for the coalgebra $\mathcal{O}_U^* (\mathbb{P}(\chi))$. In particular, a rational fundamental class is constructed that may be interpreted in terms of toric desingularisation.

Finally, Section 8 introduces homological reassembly by dualising the results of Section 6. Theorem 8.2 describes the homology coalgebra $E_\ast (\mathbb{P}(\chi))$ as both a limit and a colimit of the $p$-primary parts $E_\ast (\mathbb{P}(\chi))$ in terms of direct sums and iterated tensor products over $E_\ast (\mathbb{CP}^n)$ respectively. In particular, $E_\ast (\mathbb{P}(\chi))$ is identified with an intersection of $p$-primary subcoalgebras of $E_\ast (\mathbb{CP}^n)$, and an illustrative examples is given in complex $K$-theory.

Throughout our work we write $S^1$ for the circle as a topological space, and $T < \mathbb{C}^1$ for its realisation as the group of unimodular complex numbers with respect to multiplication. For any integer $k > 0$ we write $\mathbb{Z}/k$ for the integers modulo $k$, and $C_k < T$ for its realisation as the subgroup generated by a primitive $k$th root of unity. We interpret the standard simplex $\Delta^n$ as the intersection of the positive orthant $\mathbb{R}_{+}^{n+1}$ with the hyperplane $x_0 + \cdots + x_n = 1$, and denote its boundary by $\partial \Delta^n$. As an abstract simplicial complex, $\partial \Delta^n$ has $\binom{n+1}{k+1}$ faces of dimension $k$, for $-1 < k < n$.

For any generalised cohomology theory, we follow the convention that all homology and cohomology groups $E_\ast (X)$ and $E^\ast (X)$ are reduced for every space $X$. The unreduced counterparts are given by adjoining a disjoint basepoint, and considering $E_\ast (X_\pm)$ and $E^\ast (X_\pm)$. The coefficient ring $E_\ast$ is given by $E_\ast (S^0) \cong E^{-s}(S^0)$; we identify the homological and cohomological versions without further comment, and interpret $E_\ast (X_\pm)$ and $E^\ast (X_\pm)$ as $E_\ast$-modules and $E_\ast$-algebras respectively. We make the important assumption that $E_\ast$ is even dimensional and free of additive torsion, as holds for integral cohomology $H^\ast (-)$, complex $K$-theory $K^\ast (-)$, and complex cobordism $\Omega_U^\ast (-)$.

These theories are also complex oriented [11] Part II §2), by means of a class $x^E$ in $E^2 (\mathbb{CP}^\infty)$ whose restriction to $E^2 (\mathbb{CP}^1)$ is a generator. It follows that there exists a canonical isomorphism

$$E^\ast (\mathbb{CP}^\infty)_+ \cong E_\ast [x^E]$$

of $E_\ast$-algebras, and that complex vector bundles have associated $E$-theoretic Chern classes. In particular, $x^E$ is the first Chern class $c_1^E (\zeta)$ of the dual Hopf line bundle $\zeta$ over $\mathbb{CP}^\infty$. A minor abuse of notation allows $x^E$ to be confused with its restriction to $\mathbb{CP}^n$, and produces an isomorphism

$$E^\ast (\mathbb{CP}^n)_+ \cong E_\ast [x^E]/((x^E)^{n+1}) .$$

It is convenient to denote $x^E$ by $u$ in the universal case $\Omega_U^* (\mathbb{CP}^\infty)$, and to write $x^H$ as $x$ in $H^2 (\mathbb{CP}^\infty)$.

If the Thom space of $\zeta$ is identified with $\mathbb{CP}^\infty$, then $x^E$ may also be interpreted as a Thom class $t^E (\zeta)$, and extended to a universal Thom class $t^E \in E^0 (MU)$; thus $t^U$ is represented by the identity map on $MU$. 

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2. Weighted projective space

In this section basic notation is established, and the definitions of weighted projective space, weighted lens space, and their associated constructions are recalled. Readers are referred to Al Amrani [3], [4], Kawasaki [14], and the authors’ own work [5] for further details.

The standard action of the \((n + 1)\)-dimensional torus \(T^{n+1}\) on \(\mathbb{C}^{n+1}\) is by coordinatewise multiplication, and restricts to the unit sphere \(S^{2n+1}\). The orbit space of the latter is homeomorphic to the standard simplex \(\Delta^n \subset \mathbb{R}^{n+1}_{\geq 0}\), and the quotient map \(r: S^{2n+1} \to \Delta^n\) is given by \(r(z) = (|z_0|^2, \ldots, |z_n|^2)\).

A weight vector \(\chi\) is a sequence \((\chi_0, \ldots, \chi_n)\) of positive integers, and \(T_{\langle \chi \rangle}\) denotes the subcircle of elements \((t^{\chi_0}, \ldots, t^{\chi_n})\), as \(t\) ranges over \(T\). It is convenient to abbreviate the greatest common divisor \(\gcd(\chi_0, \ldots, \chi_n)\) and least common multiple \(\lcm(\chi_0, \ldots, \chi_n)\) to \(g = g(\chi)\) and \(l = l(\chi)\) respectively.

Definition 2.1. The weighted projective space \(\mathbb{P}(\chi)\) is the orbit space of the action of \(T_{\langle \chi \rangle}\) on \(S^{2n+1}\); it admits a canonical action of the quotient \(n\)-torus \(T^{n+1}/T_{\langle \chi \rangle}\), with orbit space \(\Delta^n\).

The respective quotient maps are

\[
S^{2n+1} \xrightarrow{p(\chi)} \mathbb{P}(\chi) \xrightarrow{q(\chi)} \Delta^n,
\]

whose composition is \(r\). The action of \(T_{\langle \chi \rangle}\) is free when \(\chi = (d, \ldots, d)\) for any positive integer \(d\), in which case \(\mathbb{P}(\chi)\) reduces to \(\mathbb{C}P^n\); in general, \(\mathbb{P}(\chi)\) has orbifold singularities. Weighted projective spaces provide an important class of singular examples in algebraic and symplectic geometry, although the focus of this article is on their algebraic topology.

It is sometimes convenient to assume that \(g(\chi) = 1\), because \(T\langle d\chi \rangle\) and \(T(\chi)\) produce homeomorphic orbit spaces for all \(d\).

Definition 2.3. For any positive integer \(k\), the weighted lens space \(L(k; \chi)\) is the orbit space of the action of the weighted cyclic subgroup \(C_k(\chi) < T(\chi)\) on \(S^{2n+1}\); it admits canonical actions of the quotient circle \(T(\chi)/C_k(\chi)\) with orbit space \(\mathbb{P}(\chi)\), and of the \((n + 1)\)-torus \(T^{n+1}/C_k(\chi)\) with orbit space \(\Delta^n\).

If \(k\) is prime to \(\chi_i\) for \(0 \leq i \leq n\), then \(L(k; \chi)\) is a standard lens space, and is smooth; otherwise, \(C_k(\chi)\) fails to act freely, and \(L(k; \chi)\) may be singular.
Restricting (2.2) to the hyperplane $z_n = 0$ yields orbit maps

$S^{2n-1} \xrightarrow{p(x') \mathbin{	ext{and}} q(x')} \mathbb{P}(x') \xrightarrow{q(x')} \Delta^{n-1}$,

where $\chi'$ denotes $(\chi_0, \ldots, \chi_{n-1})$ and $\Delta^{n-1}$ is the subsimplex $x_n = 0$ of $\Delta^n$. On the other hand, restricting to the cylinder $|z_n|^2 = 1/2$ gives

$S^{2n-1} \times S^1 \xrightarrow{p_{1/2} \mathbin{	ext{and}} q_{1/2}} L(\chi_n; \chi') \xrightarrow{q_{1/2}} \Delta_{1/2}$,

where $S^{2n-1} \subset S^{2n+1}$ is the subsphere $|z_0|^2 + \cdots + |z_{n-1}|^2 = 1/2$.

Thus $p_{1/2}$ is the orbit map for $T(\chi)$, and factors through $L(\chi_n; \chi') \times S^1$ under the actions of $C_{\chi_n}(\chi')$ and $T(\chi)/C_{\chi_n}(\chi')$ respectively; the latter is homotopy equivalent to $T$. Similarly, $q_{1/2}$ is the orbit map for the $n$–torus $T^n/C_{\chi_n}(\chi')$, whose orbit space $\Delta_{1/2}$ is the $(n-1)$–simplex $x_n = 1/2$. Finally, restricting to the circle $|z_n| = 1$ projects every point $(0, \ldots, 0, z_n)$ in $S^{2n+1}$ onto $[0, \ldots, 0, 1]$ in $\mathbb{P}(\chi)$, and thence to the vertex $(0, \ldots, 0, 1)$ in $\Delta^n$.

Now consider the decomposition of $\Delta^n$ into the union of subspaces $N(1/2)$ and $N(1/2)$, specified by $x_n \leq 1/2$ and $x_n \geq 1/2$ respectively; they are homeomorphic to the product $\Delta^{n-1} \times [0,1/2]$ and the cone $C\Delta_{1/2}$. So $\mathbb{P}(\chi)$ may be expressed as the pushout of

$\mathbb{P}(\chi') \leftarrow i \mathbin{	ext{and}} j \xrightarrow{\mathbb{P}(\chi)} CL(\chi_n; \chi')$,

where $N\mathbb{P}(\chi')$ denotes the neighbourhood $q^{-1}(N(1/2))$ of $\mathbb{P}(\chi')$ in $\mathbb{P}(\chi)$, and $CL(\chi_n; \chi')$ denotes the cone $q^{-1}(C(1/2))$, with basepoint $[0, \ldots, 0, 1]$. Equivalently, (2.4) arises by decomposing $S^{2n+1}$ as the pushout of

$S^{2n-1} \times D^2 \leftarrow i \mathbin{	ext{and}} j \xrightarrow{\mathbb{P}(\chi)} S^{2n-1} \times S^1 \xrightarrow{j} D^{2n} \times S^1$,

and forming orbit spaces under the action of $T(\chi)$. Reparametrisng $D^{2n}$ shows that $[0, \ldots, 0, 1]$ admits a neighbourhood of the form $C^n/C_{\chi_n}$; repeating at each point $[0, \ldots, 0, 1, \ldots, 0]$ confirms that $\mathbb{P}(\chi)$ is a complex orbifold.

Diagram (2.6) is cofibrant, and therefore expresses $\mathbb{P}(\chi)$ as the homotopy colimit of the diagram

$\mathbb{P}(\chi') \leftarrow f \mathbin{	ext{and}} j \xrightarrow{\mathbb{P}(\chi)} S(\chi_n; \chi') \xrightarrow{*}$,

where $f$ denotes the orbit map for the circle $T(\chi')/C_{\chi_n}(\chi')$, and $*$ is the point $[0, \ldots, 0, 1]$. This reinterprets Kawasaki’s cofibre sequence [14, page 245]

$L(\chi_n; \chi') \xrightarrow{f} \mathbb{P}(\chi') \xrightarrow{g} \mathbb{P}(\chi)$.

**Remark 2.9.** The category underlying diagram (2.7) may also be construed as $\text{cat}(\partial \Delta^1)$, whose objects are the faces $\varnothing$, 0 and 1 of $\partial \Delta^1$ and morphisms their inclusions. Iteration on $\mathbb{P}(\chi')$ leads to a description of $\mathbb{P}(\chi)$ as a homotopy colimit over $\text{cat}(\partial \Delta^n)$, in which the relevant diagram assigns an orbit space $T^n/T^k(\sigma)$ to each $(k-1)$–dimensional face $\sigma$ of $\Delta^n$; this is precisely the homotopy colimit of [19] Proposition 5.3.
Following Kawasaki, Al Amrani [3, I.1(b)] defines maps
\begin{equation}
\mathbb{C}P^n \xrightarrow{\phi} \mathbb{P}(\chi) \xrightarrow{\psi} \mathbb{C}P^n
\end{equation}
by \( \phi[z_0, \ldots, z_n] = [z_0^{\chi_0}, \ldots, z_n^{\chi_n}] \) and \( \psi[z_0, \ldots, z_n] = [z_0^{l(\chi)/\chi_0}, \ldots, z_n^{l(\chi)/\chi_n}] \). In both cases, the formulae for the homogeneous coordinates of the target values are understood to be normalised. It is sometimes important to make the weights explicit, by writing \( \phi(\chi) \) and \( \psi(\chi) \) respectively.

Usually, \( \psi \) is interpreted as a complex line bundle over \( \mathbb{P}(\chi) \), but may equally well be specified by its first Chern class \( c_1(\psi) \) in \( H^2(\mathbb{P}(\chi); \mathbb{Z}) \). The composition \( \psi \circ \phi \) has degree \( l = l(\chi) \) on \( H^2(\mathbb{C}P^n; \mathbb{Z}) \), so \( \phi^*(\psi) \) is the \( l \)th tensor power \( \mathcal{O}(z)^l \) of the dual Hopf bundle, and \( c_1(\phi^*(\psi)) = lx \) in \( H^2(\mathbb{C}P^n) \), following (1.1).

The rôle of \( \psi \) is clarified by identifying the total space \( S(\psi) \) of its associated circle bundle.

**Proposition 2.11.** The space \( S(\psi) \) is a \((2n + 1)\)-dimensional weighted lens space \( L(l; \chi) \).

**Proof.** By definition, \( S(\psi) \) is the pullback of the diagram
\begin{equation}
\mathbb{P}(\chi) \xrightarrow{\psi} \mathbb{C}P^n \xleftarrow{} S^{2n+1},
\end{equation}
and is a subspace \( X \subset S^{2n+1} \times \mathbb{P}(\chi) \). It contains all pairs \((y, [z])\) that satisfy the equation
\begin{equation}
t(y_0, \ldots, y_n) = (z_0^{l(\chi)/\chi_0}, \ldots, z_n^{l(\chi)/\chi_n})
\end{equation}
in \( S^{2n+1} \), for some \( t \in T \). So there exists a map \( h : L(l(\chi); \chi) \to X \), defined by
\begin{equation}
h[w_0, \ldots, w_n] = ([w_0^{l(\chi)/\chi_0}, \ldots, w_n^{l(\chi)/\chi_n}], [w_0, \ldots, w_n]);
\end{equation}
moreover, an inverse to \( h \) is given by mapping \((y, [z])\) to the equivalence class of those \((n + 1)\)-tuples \((z_0, \ldots, z_n)\) for which \( t = 1 \) in (2.12). It follows that \( h \) is the required homeomorphism. \( \square \)

**Corollary 2.13.** The circle \( T(\chi)/\mathcal{C}_1(\chi) \) acts freely on \( L(l; \chi) \), and has orbit space \( \mathbb{P}(\chi) \).

Of course the associated sphere bundle \( S(\phi^*(\psi)) \) is homeomorphic to the standard lens space \( L(l; 1, \ldots, 1) \), and is therefore a smooth manifold.

The following natural numbers are associated to \( \chi \), and were essentially introduced by Kawasaki; alternative descriptions are recovered in Theorem 6.15.

**Definition 2.14.** For any subset \( J \subseteq [n] \), the integer \( \chi_J \) is the product \( \prod_{j \in J} \chi_j \), and \( h_J = h_J(\chi) \) is the quotient \( \chi_J / \gcd(\chi_J : j \in J) \); for any \( 1 \leq j \leq n \), the integer \( l_j = l_j(\chi) = \text{lcm}(h_J : |J| = j) \), and \( m_j = m_j(\chi) = l(\chi)^j/l_j \).

Thus \( l_1 = l \) and \( m_1 = 1 \), whereas \( l_n = \chi_0 \cdots \chi_n/g \) and \( m_n = g l^n/\chi_0 \cdots \chi_n \).

Kawasaki applies the cofibre sequence (2.8) to identify the integral cohomology ring of \( \mathbb{P}(\chi) \) by means of an isomorphism
\begin{equation}
H^*(\mathbb{P}(\chi); \mathbb{Z}) \cong \mathbb{Z}[v_1, \ldots, v_n]/I(\chi),
\end{equation}
where \( v_j \) has dimension \( 2j \) and \( I(\chi) \) is the ideal generated by the elements \( v_1^j - m_j v_j \) for \( 1 \leq j \leq n \). Moreover, \( v_1 \) equals \( c_1(\psi) \), so \( \phi^*(v_1) = lx \) holds in
The same calculations identify the integral cohomology ring of $L(\chi_n; \chi')$ in terms of additive isomorphisms

$$H^j(L(\chi_n; \chi'); \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & \text{if } j = 2n - 1 \\
\mathbb{Z}/s_k & \text{if } j = 2k \text{ for } 1 \leq k \leq n - 1 \\
0 & \text{otherwise,}
\end{cases}$$

where $s_k = s_k(\chi)$ is given by $l_k/l'_k$, and $l'_k := l_k(\chi')$.

**Remark 2.17.** These results reveal that the maps $\phi$ and $\psi$ of (2.10) are mutually inverse rational homotopy equivalences, as their definitions suggest. The rationalisation $P(\chi)_Q$ is therefore the $2n$–skeleton of an Eilenberg–Mac Lane space $K(\mathbb{Q}, 2)$.

Many of Kawasaki’s calculations are recovered in Theorem 6.15.

### 3. Iterated Thom spaces and $p$-primary parts

In order to follow homotopy theoretical convention and study $P(\chi)$ prime by prime, it is convenient to introduce certain restrictions on the weights.

If $g(\chi) = 1$, then there exists an isomorphism

$$P(d\chi_0, \ldots, d\chi_{j-1}, \chi_j, d\chi_{j+1}, \ldots, d\chi_n) \cong P(\chi)$$

of algebraic varieties for any natural number $d$ such that $\gcd(d, \chi_j) = 1$, and every $0 \leq j \leq n$ [9, 13]. So no generality is lost by insisting that $\chi$ is normalised, in the sense that

$$\gcd(\chi_0, \ldots, \hat{\chi}_j, \ldots, \chi_n) = 1$$

for every $0 \leq j \leq n$.

**Definitions 3.2.** The weight vector $\chi$ is

1. weakly divisive if $\chi_j$ divides $\chi_n$ for every $0 \leq j < n$
2. divisive if $\chi_{j-1}$ divides $\chi_j$ for every $1 \leq j \leq n$
3. $p$-primary if every $\chi_j$ is a power of a fixed prime $p$.

If $\chi$ is divisive, then $q_j = q_j(\chi)$ denotes the integer $\chi_j/\chi_{j-1}$, for $1 \leq j \leq n$.

So divisive implies weakly divisive, but not conversely. In fact $\chi$ is divisive precisely when the reverse sequence $\chi_n, \ldots, \chi_0$ is well ordered, in the sense of Nishimura–Yosimura [15]; then

$$* = P(\chi_n), P(\chi_{n-1}, \chi_n) \setminus P(\chi_n), \ldots, P(\chi_0, \ldots, \chi_n) \setminus P(\chi_1, \ldots, \chi_n)$$

is a cell decomposition of $P(\chi)$ with one cell in every even dimension (compare [6, Remark 3.2] and [15, Proposition 2.3]). This decomposition describes the canonical cells that arise from Corollary 3.8 below.

**Lemma 3.3.** Every normalised $p$-primary weight vector $\pi$ may be ordered so as to take the form

$$\pi = (1, 1, p^{k_2}, \ldots, p^{k_n})$$

for some sequence $k_2 \leq \cdots \leq k_n$ of exponents.
Proof. Since $g(\pi) = 1$, it follows that $\pi_j = 1$ for some $j$. Applying (3.1), and reordering if necessary, yields the required form. \hfill \square

Of course (3.4) is divisive, and identifies $k_0$ and $k_1$ as 0.

**Definition 3.5.** If $(X_i)$ is a sequence of topological spaces for $0 \leq i \leq n$, and $(\theta_i)$ a sequence of vector bundles over $X_i$ for $0 \leq i < n$, then $X_n$ is an $n$-fold iterated Thom space over $X_0$ whenever $X_i$ is homeomorphic to the Thom space $Th(\theta_{i-1})$ for every $1 \leq i \leq n$.

**Example 3.6.** If $\zeta_i$ denotes the dual Hopf line bundle over $\mathbb{C}P^i$ for every $i \geq 0$, then the standard homeomorphisms $\mathbb{C}P^i \cong Th(\zeta_{i-1})$ display $\mathbb{C}P^n$ as an $n$-fold iterated Thom space over the one-point space $\mathbb{C}P^0$; or, alternatively, as an $(n-1)$-fold iterated Thom space over the 2-sphere $\mathbb{C}P^1$.

In Example 3.6, $\zeta_i$ may equally well be replaced by the Hopf bundle itself.

**Theorem 3.7.** If $\chi$ is weakly divisive, then $\mathbb{P}(\chi)$ is homeomorphic to the Thom space of a complex line bundle over $\mathbb{P}(\chi')$.

**Proof.** Al Amrani’s proof of [3, I.1(c)] applies to the line bundle $\psi'$ over $\mathbb{P}(\chi')$, and shows that the unit disc bundle $D(\psi')$ is homeomorphic to the neighbourhood $N\mathbb{P}(\chi')$ of $\mathbb{P}(\chi') \subset \mathbb{P}(\chi)$, as defined in (2.6). The unit sphere bundle $S(\psi')$ is therefore homeomorphic to the weighted lens space $L(\chi_n; \chi')$, and the cofibre sequence (2.8) identifies $\mathbb{P}(\chi)$ with the Thom space $Th(\psi')$. \hfill \square

**Corollary 3.8.** If $\chi$ is divisive, then $\mathbb{P}(\chi)$ is homeomorphic to an $n$-fold iterated Thom space of complex line bundles over the one-point space $*$. By analogy with Remark 2.9, an $n$-fold iterated Thom space may also be expressed as an iterated pushout, and therefore as a homotopy colimit over the category $\text{cat}(\partial \Delta^n)$.

In order to apply Corollary 3.8 further, the maps $\phi$ and $\psi$ of (2.10) must be generalised. A suitable context is provided by interpreting weight vectors as elements of the multiplicative monoid $\mathbb{N}^{n+1}$, with identity element $1 = (1, \ldots, 1)$. Given any two such $\chi$ and $\omega$, there exists a smallest positive integer $s = s(\chi, \omega)$ such that $\omega$ divides $s\chi$. The resulting quotient has coordinates $s\chi / \omega_j$ for $0 \leq j \leq n$, and is conveniently denoted by $\chi / \omega$; so equations such as

$$(3.9) \quad s = (s, \ldots, s), \quad \omega(\chi / \omega) = s\chi, \quad (\omega\chi)/\omega = \chi,$$

hold amongst weight vectors.

Every $\chi$ may then be expressed as a product of indecomposables. For any $0 \leq j \leq n$ and any prime $p$, write $\chi_j$ as $p^{a_j}\alpha_j$, where $p^{a_j}$ denotes the $p$-content of $\chi_j$ and $\text{gcd}(p, \alpha_j) = 1$.

**Definition 3.10.** The $p$-content of $\chi$ is the $p$-primary weight vector

$$p^\chi := (p^{a(0)}, \ldots, p^{a(n)}),$$

which satisfies $\chi = p^\chi \alpha$ in $\mathbb{N}^{n+1}$; the primary decomposition of $\chi$ is the factorisation $\chi = p_1^{a_1}\cdots p_m^{a_m}\chi$, as $p_i$ ranges over the prime factors of $\chi_j$. 

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Remark 3.11. Recent results of [6] show that, amongst weighted projective spaces, the homotopy type of $\mathbb{P}(\chi)$ is determined by the unordered coordinates of its non-trivial $p$-contents $p\chi$, for normalised $\chi$. The $p\chi$ may therefore assumed to be non-decreasing, and remultiplied to give a weight vector $\chi^*$ for which there exists a homotopy equivalence $\mathbb{P}(\chi) \simeq \mathbb{P}(\chi^*)$. Moreover $\chi^*$ is divisive by construction, so Corollary 3.8 implies that every $\mathbb{P}(\chi)$ is homotopy equivalent to an iterated Thom space. If the weights are pairwise coprime then $\chi^*$ takes the form $(1, \ldots, 1, c)$ and $\mathbb{P}(\chi^*)$ reduces to $Th(\zeta_{n-1}^c)$ over $\mathbb{C}P^{n-1}$, where $c = \prod_i \chi_i$.

Definitions 3.12.

1. The map $e(\chi/\omega) : \mathbb{P}(\omega) \to \mathbb{P}(\chi)$ is given by

$$e(\chi/\omega)([z_0, \ldots, z_n]) = [s^{\chi_0/\omega_0} z_0, \ldots, s^{\chi_n/\omega_n} z_n],$$

where $s = s(\chi, \omega)$, and coordinates are normalised as necessary:

2. the group $C_{\chi/\omega}$ is the product

$$C_{s^{\chi_0/\omega_0}} \times \cdots \times C_{s^{\chi_n/\omega_n}}$$

of cyclic groups, considered as a subgroup of $T^{n+1}$.

Following (3.9), the cases $e(\phi/\chi)$ and $C_{\phi/\chi}$ reduce to $e(\phi)$ and $C_{\phi}$ respectively. By definition, $e(r) = e(r, \ldots, r)$ raises homogeneous coordinates in $\mathbb{P}(\chi)$ to the $r$th power, and is therefore known as the $r$th power map on $\mathbb{P}(\chi)$.

Proposition 3.13. The map $e(\chi/\omega)$ is the orbit map of the natural action of $C_{\chi/\omega}$ on $\mathbb{P}(\omega)$.

Proof. Note first that $e(\chi/\omega)([y_0, \ldots, y_n]) = e(\chi/\omega)([z_0, \ldots, z_n])$ holds in $\mathbb{P}(\chi)$ precisely when

$$[y_0, \ldots, y_n] = \{[\lambda_0 z_0, \ldots, \lambda_n z_n] : \lambda_0^{h_{\omega_0}/\chi_0} = \cdots = \lambda_n^{h_{\omega_n}/\chi_n} = 1\}$$

in $\mathbb{P}(\omega)$. Since $e(\chi/\omega)$ is clearly surjective, the result follows. \qed

Corollary 3.14. Any weighted projective space arises as the orbit space of any other of the same dimension, under the action of a finite abelian group.

Remark 3.15. In the language of Definition 3.12 and Proposition 3.13, Al Amrani’s maps $\phi(\chi)$ and $\psi(\chi)$ are given by $e(\chi)$ and $e(1/\chi)$ respectively. Kawasaki [14, page 243] notes that $\phi(\chi)$ is the orbit map of the action of $C_{\chi}$ on $\mathbb{C}P^n$.

Proposition 3.16. For any weight vectors $\omega$ and $\chi$, the composition

$$\mathbb{P}(\omega) \xrightarrow{e(\chi/\omega)} \mathbb{P}(\sigma) \xrightarrow{e(\chi/\sigma)} \mathbb{P}(\chi)$$

factorises as $e(s') \circ e(\chi/\omega) = e(\chi/\omega) \circ e(s')$, where $s'$ denotes the natural number $s(\omega, \sigma)s(\sigma, \chi)/s(\omega, \chi)$.

Proof. It suffices to note that the given composition acts on homogeneous coordinates in $\mathbb{P}(\chi)$ by $z_i \mapsto z_i^{s(\omega, \sigma)s(\sigma, \chi)/s(\omega, \chi)}$, for $0 \leq i \leq n$. \qed
Proposition 3.16 implies the factorisations
\[ e(\chi/\omega) \circ e(\omega/\chi) = e(s'') \quad \text{and} \quad e(\omega/\chi) \circ e(\chi/\omega) = e(s''), \]
where \( s'' := s(\omega, \chi)s(\chi, \omega) \). Similarly, \( e(\chi) \circ e(1/\omega) = e(l(\omega)/s(\omega, \chi)) \) for any weight vectors \( \chi \) and \( \omega \).

**Definition 3.17.** For any weight vector \( \chi \) and any prime \( p \), the \( p \)-primary part of \( \mathbb{P}(\chi) \) is the weighted projective space \( \mathbb{P}(p\chi) \); the canonical maps
\[
e(p\chi/\chi): \mathbb{P}(\chi) \longrightarrow \mathbb{P}(p\chi) \quad \text{and} \quad e(\chi/p\chi): \mathbb{P}(p\chi) \longrightarrow \mathbb{P}(\chi)
\]
are \( p \)-extraction and \( p \)-insertion respectively.

In the notation of Definition 3.10 extraction and insertion are given by
\[
e(p\chi/\chi)[z_0, \ldots, z_n] = [z_0^{l(\alpha)/\alpha_0}, \ldots, z_n^{l(\alpha)/\alpha_n}]
\]
and \( e(\chi/p\chi)[z_0, \ldots, z_n] = [z_0^{\alpha_0}, \ldots, z_n^{\alpha_n}] \),

in terms of homogeneous coordinates. By Proposition 3.16 the compositions \( e(\chi/p\chi) \circ e(p\chi/\chi) \) and \( e(p\chi/\chi) \circ e(\chi/p\chi) \) reduce to the appropriate power maps \( e(l(\alpha)) \).

**Remark 3.19.** It follows immediately from Lemma 3.3 and Corollary 3.8 that every \( p \)-primary part \( \mathbb{P}(p\chi) \) is an iterated Thom space over \( * \).

**Example 3.20.** The 2-, 3-, and 5-primary parts of \( \mathbb{P}(3,4,5) \) are \( \mathbb{P}(1,4,1) \), \( \mathbb{P}(3,1,1) \), and \( \mathbb{P}(1,1,5) \) respectively. They form the codomains of the 2-, 3-, and 5-extraction maps, whose values on \( [z_0, z_1, z_2] \) are given by
\[
[z_0^5, z_1^{15}, z_2^3], \quad [z_0^{20}, z_1^5, z_2^4], \quad \text{and} \quad [z_0^4, z_1^3, z_2^2]
\]
respectively. The 2-, 3-, and 5-primary parts also form the domains of the 2-, 3-, and 5-insertion maps, whose values on \( [z_0, z_1, z_2] \) are given by
\[
[z_0^3, z_1^4, z_2^5], \quad [z_0, z_1^4, z_2^5], \quad \text{and} \quad [z_0^3, z_1^4, z_2]
\]
respectively.

4. Geometric reassembly

The problem of reassembling \( \mathbb{P}(\chi) \) from its \( p \)-primary parts must now be addressed. The solution is best understood in terms of weight vectors \( \sigma \) and \( \sigma' \), and commutative squares of the form
\[
\begin{array}{ccc}
\mathbb{P}(\sigma) & \xrightarrow{e(\sigma)} & \mathbb{P}(\sigma') \\
\mathbb{C}P^n & \xrightarrow{e(\sigma')} & \mathbb{C}P^n \\
\end{array}
\]
\[
\begin{array}{ccc}
\mathbb{P}(\sigma') & \xrightarrow{e(1/\sigma)} & \mathbb{P}(\sigma) \\
\mathbb{C}P^n & \xrightarrow{e(1/\sigma')} & \mathbb{C}P^n \\
\end{array}
\]
these may be incorporated into a single generalised square, although restrictions must be imposed upon \( \sigma \) and \( \sigma' \).

For any weight vector \( \chi \), it is convenient to write \( Q(\chi) \subset \mathbb{Z} \) for the set of primes that occur non-trivially in its \( p \)-primary decomposition.
Definition 4.2. Two weight vectors \( \sigma \) and \( \sigma' \) are coprime if \( Q(\sigma) \cap Q(\sigma') = \emptyset \); in other words, if \( \gcd(\sigma_j, \sigma'_k) = 1 \) for every \( 0 \leq j, k \leq n \).

In the context of Definition 4.12, this condition is equivalent to the coprimality of \( |C_\sigma| \) and \( |C_{\sigma'}| \). It also implies the weaker condition, that

\[
(4.3) \quad h(g_0, \ldots, g_n) = ((g_0^{\alpha_0}, \ldots, g_n^{\alpha_n}), (g_0^{\alpha'_0}, \ldots, g_n^{\alpha'_n}))
\]
defines an isomorphism \( h : C_{\sigma \sigma'} \to C_{\sigma'} \times C_{\sigma} \); or equivalently, that \( C_{\sigma \sigma'} \) is generated by the subgroups \( C_\sigma \) and \( C_{\sigma'} \).

Remark 4.4. Since \( Q(\sigma) = Q(1/\sigma) \), it follows that \( \sigma \) and \( \sigma' \) are coprime if and only if 1/\( \sigma \) and 1/\( \sigma' \) are coprime.

Proposition 4.5. If \( \sigma \) and \( \sigma' \) are coprime, then the left-hand diagram \( \mathbb{P}(\sigma') \to \mathbb{C}P^n \from \mathbb{P}(\sigma) \) is a pullback square.

Proof. The pullback of the diagram \( \mathbb{P}(\sigma') \to \mathbb{C}P^n \from \mathbb{P}(\sigma) \) is the subspace

\[
P := \{(z', z) : e(1/\sigma')(z') = e(1/\sigma)(z)\} \subset \mathbb{P}(\sigma') \times \mathbb{P}(\sigma),
\]
and the canonical map \( f : \mathbb{P}(\sigma \sigma') \to P \) acts by \( f(z) = (e(1/\sigma)(z), e(1/\sigma')(z)) \).

Since \( P \) is Hausdorff, if \( f \) is surjective, by equivariance. remains to show that \( f \) is bijective.

The maps \( e(1/\sigma) \) and \( e(1/\sigma') \) are quotients by \( C_{1/\sigma} \) and \( C_{1/\sigma'} \) respectively, by Proposition 3.13. So \( C_{1/\sigma'} \times C_{1/\sigma} \) acts on \( P \), and \( f \) is equivariant with respect to the isomorphism \( h \) of (4.3). The projection \( p : P \to \mathbb{P}(\sigma) \) is the quotient by \( C_{1/\sigma} \), and \( f \) is equivariant with respect to the projection \( C_{1/\sigma} \times C_{1/\sigma} \to C_{1/\sigma} \); the corresponding statement for \( p' : P \to \mathbb{P}(\sigma') \) holds by analogy. Both \( e(1/\sigma) \) and \( p \) are surjective, and the former factorises as \( p \circ f : \mathbb{P}(\sigma \sigma') \to \mathbb{P}(\sigma) \); therefore \( f \) is surjective, by equivariance.

To confirm injectivity, choose \( x \in \mathbb{P}(\sigma \sigma') \) and let \( G < C_{1/\sigma} \) be the isotropy group of \( y = f(x) \). Then \( c := |f^{-1}(y)| = |Gx| \) divides \( |G| \), so \( c \) divides \( |C_{1/\sigma}| \).

Replacing \( \sigma \) by \( \sigma' \) and applying the corresponding reasoning shows that \( c \) also divides \( |C_{1/\sigma'}| \). But \( \sigma \) and \( \sigma' \) are coprime, so \( c = 1 \) as sought. \( \square \)

Remark 4.6. Since all the maps in the square are algebraic, Proposition 4.5 also holds in the category of complex algebraic varieties.

The following example shows that the square is not generally a pullback.

Example 4.7. If \( \sigma = (1, 2, 2) \) and \( \sigma' = (2, 1, 2) \), then \( \mathbb{P}(\sigma) \cong \mathbb{P}(\sigma') \cong \mathbb{C}P^2 \) and \( e(1/\sigma) \) and \( e(1/\sigma') \) are homeomorphisms. So the pullback is also homeomorphic to \( \mathbb{C}P^2 \), and cannot be \( \mathbb{P}(\sigma \sigma') \cong \mathbb{P}(1, 1, 2) \) because the latter contains a singular point. Proposition 3.13 confirms that the canonical map \( f : \mathbb{P}(\sigma \sigma') \to \mathbb{C}P^2 \) is the quotient by \( C_2 \times C_2 \).

An example with normalised weights is given by increasing the dimensions, with additional weights 1; thus \( \sigma = (1, 1, 2, 2) \) and \( \sigma' = (1, 2, 1, 2) \).

Remarkably, weighted projective spaces may also be expressed as pushouts.

Proposition 4.8. If \( \sigma \) and \( \sigma' \) are coprime, then the right-hand diagram \( \mathbb{P}(\sigma') \to \mathbb{C}P^n \from \mathbb{P}(\sigma) \) is a pullback square.
Proof. The pushout of the diagram $\mathbb{P}(\sigma') \leftarrow \mathbb{C}P^n \rightarrow \mathbb{P}(\sigma)$ is the space

$$R = (\mathbb{P}(\sigma') \sqcup \mathbb{P}(\sigma))/\sim,$$

where the equivalence relation is generated by $e(\sigma')(z) \sim e(\sigma)(z)$ for any $z$ in $\mathbb{C}P^n$; the canonical map $g: R \rightarrow \mathbb{P}(\sigma')$ is defined by

$$g([e(\sigma')(z)]) = g([e(\sigma)(z)]) = e(\sigma')(z).$$

Since $R$ is compact, it suffices to show that $g$ is bijective.

Both $e(\sigma)$ and $e(\sigma')$ are surjective, and the natural map $q: \mathbb{C}P^n \rightarrow R$ is given by $q(z) = [e(\sigma)(z)] = [e(\sigma')(z)]$; so $q$ is surjective. It is also invariant with respect to the action of $C_{\sigma\sigma'}$, since the latter is generated by the subgroups $C_{\sigma}$ and $C_{\sigma'}$ (via (4.3)), which act trivially on $\mathbb{P}(\sigma)$ and $\mathbb{P}(\sigma')$ respectively. Hence $q$ induces a surjective map $\overline{q}: \mathbb{P}(\sigma') \rightarrow R$.

The map $e(\sigma')\mathbb{C}P^n \rightarrow \mathbb{P}(\sigma')$ factors as $g \circ q$, so $g \circ \overline{q} = 1$ on $\mathbb{P}(\sigma')$, and $\overline{q}$ is injective. Thus $\overline{q}$ is bijective, and has inverse $g$, as required. \qed

In order to state the main reassembly theorem, it is convenient to write $[m]$ for the simplicial complex consisting of $m$ disjoint vertices.

**Theorem 4.9.** The weighted projective space $\mathbb{P}(\chi)$ is homeomorphic to the limit of the $\text{CAT}([m])$-diagram

$$
\begin{array}{ccc}
\mathbb{P}(p_1, \chi) & \rightarrow & \mathbb{P}(p_m, \chi) \\
\downarrow & & \\
\mathbb{C}P^n & \leftarrow & \mathbb{P}(p_m, \chi),
\end{array}
$$

and the associated universal maps $\mathbb{P}(\chi) \rightarrow \mathbb{P}(p_i, \chi)$ may be identified with $e(p_i, \chi/\chi)$ for every $0 \leq i \leq m$. Similarly, $\mathbb{P}(\chi)$ is also homeomorphic to the colimit of the $\text{CAT}([m])$-diagram

$$
\begin{array}{ccc}
\mathbb{P}(p_1, \chi) & \leftarrow & \mathbb{P}(p_m, \chi) \\
\uparrow & & \\
\mathbb{C}P^n & \rightarrow & \mathbb{P}(p_m, \chi),
\end{array}
$$

and the associated universal maps $\mathbb{P}(p_i, \chi) \rightarrow \mathbb{P}(\chi)$ may be identified with $e(\chi/p_i, \chi)$ for every $0 \leq i \leq m$.

Proof. Proceed by induction on $m$, noting that the results are trivial for $m = 1$.

Suppose that $Q(\chi) = \{p_1, \ldots, p_k, p\}$ and $\chi_i = p^{\alpha(i)}\alpha_i$, as in Definition 3.10, so $Q(\alpha) = \{p_1, \ldots, p_k\}$. By the inductive hypotheses, $\mathbb{P}(\alpha)$ is homeomorphic to the pullback of the $\mathbb{P}(p_i, \chi)$ along the maps $e(1/p_i, \chi)$, and to the pushout of the $\mathbb{P}(p_i, \chi)$ along the maps $e(p_i, \chi)$; also, the universal maps $\mathbb{P}(\alpha) \rightarrow \mathbb{P}(p_i, \chi)$ and $\mathbb{P}(p_i, \chi) \rightarrow \mathbb{P}(\alpha)$ are given by $e(p_i, \chi/\chi)$ and $e(\chi/p_i, \chi)$ respectively, for $1 \leq i \leq k$. It therefore remains to prove that $\mathbb{P}(\chi)$ is homeomorphic to the pullback of

$$
\begin{array}{ccc}
\mathbb{P}(p, \chi) & \xrightarrow{e(1/p, \chi)} & \mathbb{C}P^n \\
& & \xleftarrow{e(1/\alpha)} \mathbb{P}(\alpha)
\end{array}
$$

and the pushout of

$$
\begin{array}{ccc}
\mathbb{P}(p, \chi) & \xleftarrow{e(p, \chi)} & \mathbb{C}P^n \\
& & \xrightarrow{e(\alpha)} \mathbb{P}(\alpha)
\end{array}
$$
respectively, and to confirm the identity of the associated maps \( \mathbb{P}(\chi) \to \mathbb{P}(p\chi) \), \( \mathbb{P}(\chi) \to \mathbb{P}(\alpha) \), \( \mathbb{P}(p\chi) \to \mathbb{P}(\chi) \), and \( \mathbb{P}(\alpha) \to \mathbb{P}(\chi) \). These follow directly from Proposition \ref{prop:5.1} and Proposition \ref{prop:4.8} respectively, because \( p\chi \) and \( \alpha \) are coprime. The induction is then complete. \( \square \)

**Example 4.12.** Theorem \ref{thm:4.19} applies to Example \ref{ex:3.20} and expresses \( \mathbb{P}(3, 4, 5) \) as the limit and colimit of the \( \text{cat}[3] \)-diagrams

\[ \begin{array}{ccc}
\mathbb{P}_3 & \xrightarrow{e_3} & \mathbb{P}_3 \\
\downarrow & & \uparrow \\
\mathbb{P}_2 & \xleftarrow{e_2} & \mathbb{C}P^2 & \xleftarrow{e_5} & \mathbb{P}_5 \\
\end{array} \quad \text{and} \quad \begin{array}{ccc}
\mathbb{P}_2 & \xleftarrow{e_2} & \mathbb{C}P^2 & \xrightarrow{e_5} & \mathbb{P}_5 \\
\end{array} \]

respectively, where \( \mathbb{P}_2 := \mathbb{P}(1, 4, 1) \), \( \mathbb{P}_3 := \mathbb{P}(3, 1, 1) \), and \( \mathbb{P}_5 := \mathbb{P}(1, 1, 5) \).

Observe that \( \mathbb{P}(1, 5, 12) \), \( \mathbb{P}(1, 4, 15) \), \( \mathbb{P}(1, 3, 20) \), and \( \mathbb{P}(1, 1, 60) \) may also be obtained by recombining \( \mathbb{P}(1, 4, 1) \), \( \mathbb{P}(3, 1, 1) \), and \( \mathbb{P}(1, 1, 5) \) with permuted coordinates. No two of the four are homeomorphic, as their singularity structure shows; but the results of \cite{[6]} (as described in Remark \ref{rem:5.11} above) prove that all four are homotopy equivalent to \( \mathbb{P}(3, 4, 5) \). The fact that their cohomology rings are isomorphic is noted in \cite{[5]}, and reproven in Theorem \ref{thm:6.15} below; it is also, of course, implicit in \cite{[14]}.

## 5. Iterated Thom Isomorphisms

From this point onwards, \( E^*(-) \) denotes a complex oriented cohomology theory, with orientation class \( t^E \). As described in Section \ref{sec:4}, the crucial examples are: \( H^*(-) \), with the Thom orientation; \( K^*(-) \), with the Conner-Floyd orientation; and \( \Omega^e_t(-) \), with the universal orientation. In particular, \( H^*(X) \) denotes the reduced integral cohomology ring of any space \( X \).

The existence of a Thom class \( t^E \) leads to the *Thom isomorphism*, which features in the following standard result.

**Proposition 5.1.** For any \( k \)-dimensional complex vector bundle \( \theta \) over \( X \), the \( E_\ast \)-algebra \( E^*(Th(\theta)) \) is a free module over \( E^*(X_\ast) \) on the single generator \( t^E(\theta) \); its multiplicative structure is determined by the relation

\[
(t^E(\theta))^2 = c_k^E(\theta) \cdot t^E(\theta)
\]

in \( E^{4k}(Th(\theta)) \).

More explicitly, the Thom isomorphism

\[
\cdot t^E(\theta) : E^*(X_\ast) \xrightarrow{\cong} E^{*+2k}(Th(\theta))
\]

is given by forming the relative cup product with \( t^E(\theta) \), and is induced by the relative diagonal map \( \delta : Th(\theta) \to X_\ast \wedge Th(\theta) \).

Proposition \ref{prop:5.1} applies to Definition \ref{def:3.5} whenever the bundles \( \theta_i \) are complex. In this case, the cohomology algebra \( E^*(X_i) \) arises from \( E^*(X_0) \) by means of \( i \)-fold iterated Thom isomorphisms. For example, if \( X_0 = * \) and \( \dim \theta_0 = k \), then the first iteration identifies \( E^*(X_1) \) with \( E_\ast(t)/t^2 \), where \( t \) lies in \( E^{2k}(X_1) \); this, of course, is because \( X_1 \) is homeomorphic to \( S^{2k} \). Further iterations require the Chern classes of the \( \theta_i \).
In subsequent applications, the \( \theta_i \) are line bundles. Over \( X \), the isomorphism classes of such bundles form an abelian group with respect to tensor product, which is isomorphic to \( H^2(X) \) under the Chern class \( c_1(\cdot) \). So any \( w \in H^2(X) \) gives rise to a complex line bundle

\[
\lambda = \lambda(w) \quad \text{such that} \quad c_1(\lambda(w)) = w;
\]

it is unique up to isomorphism, and \( \lambda(d_1w) = \lambda(w)^{d_1} \) for any integer \( d_1 \). The Thom class \( t(\lambda(w)^{d_1}) \) lies in \( H^2(Th(\lambda(w)^{d_1})) \), and the second stage of the iteration begins with a cohomology class \( d_2 t(\lambda(w)^{d_1}) \), for some integer \( d_2 \). The corresponding line bundle is \( \lambda(w)^{d_1,d_2} := \lambda(t(\lambda(w)^{d_1}))^{d_2} \) over \( Th(\lambda(w)^{d_1}) \). In this language, the \( n \)th stage identifies the bundle

\[
\lambda(w)^{d_1,\ldots,d_n} := \lambda(t(\lambda(w)^{d_1,\ldots,d_{n-1}}))^{d_n} \quad \text{over} \quad Th(\lambda(w)^{d_1,\ldots,d_{n-1}}).
\]

In other words, \( X_n = Th(\lambda(w)^{d_1,\ldots,d_n}) \) is an \( n \)-fold iterated Thom space over \( X_0 = X \), for which \( \theta_0 = \lambda(w)^{d_1} \) and \( \theta_{n-1} = \lambda(w)^{d_1,\ldots,d_n} \).

It is perfectly acceptable to choose \( X_0 = * \), in which case \( w = 0 \) and \( \lambda(w) \) is the trivial line bundle \( \mathcal{C} \). In this context, Corollary 3.8 may be combined with Al Amrani’s proof of [31, I.1(c)] to provide a homeomorphism

\[
P(\chi) \cong Th(\lambda(0)^{q_1,\ldots,q_n})
\]

for any divisive weight vector \( \chi \), where \( q_j = \chi_j/\chi_{j-1} \) as in Definition 3.2.

In general, \( E^*(X_n) \) may be computed by iterating Proposition 5.1 and exploiting two consequences of the fact that \( \theta \) has dimension 1. Firstly, the \( E \)-theory Thom class satisfies

\[
t^E(\theta) = c^E_1(\lambda(t(\theta)))
\]

in \( E^2(Th(\theta)) \), which follows directly from the universal example \( \zeta \) over \( \mathbb{C}P^\infty \).

Secondly, for any integer \( r \), the equation

\[
c^E(r^r) = [r](c^E_1(\theta))
\]

holds in \( E^2(X) \), where \([r]\) denotes the \( r \)-series of the formal group law \( F_E \) associated to \( x^E \) [12]. Thus

\[
[r](u) \equiv ru + \frac{1}{2}r(r-1)a^Eu^2 \mod (u^3)
\]

in \( E^*[u] \), where \( a^E \in E_2 \) is the coefficient of \( u_1u_2 \) in \( F_E(u_1,u_2) \).

**Theorem 5.8.** For any divisive \( \chi \), the \( E_\ast \)-algebra \( E_\ast(P(\chi)) \) is isomorphic to

\[
E_\ast[w_n, w_{n-1}w_n, \ldots, w_1w_2 \ldots w_n]/J^E,
\]

where \( w_hw_{h+1} \cdots w_n \) lies in \( E^{2(n-h+1)}(P(\chi)) \) for any \( h \leq i \), and \( J^E \) denotes the ideal generated by elements of the form

\[
(w_i - [q_i](w_{i-1})) w_{i+1} \cdots w_n
\]

for \( 1 \leq i \leq n; \) also \( w_0 = 0 \).

**Remark 5.10.** The elements \( w_i \) do not themselves exist in \( E^2(P(\chi)) \) for any \( i \neq n \), but appear only in monomials divisible by \( w_hw_{h+1} \cdots w_n \) for some \( h \leq i \). Nevertheless, the description provided by (5.9) isationally convenient, and encodes the product structure by repeated application of the relations in \( J^E \).
Proof of 4.8. Combine Proposition 5.1 with (5.4), (5.5), (5.6), and (5.7). The first stage identifies \( E^*([P, 1, 1]) \) as \( E_*[v_1]/(v_1^2) \), where \( v_1 := t^E(\lambda(0)) \) and \( \lambda(0) = \mathbb{C} \) over \( * \). The \( n \)th stage identifies \( E^*(\mathbb{P}(\chi)) \) as a free \( E^*(\mathbb{P}(\chi'))_+ \)-module on the single generator

\[
w_n := t^E(\lambda(0)^{q_1 \cdots q_n}),
\]

with the relation \( w_n^2 = [q_n(w_{n-1})_n]_n \) of (5.2).

\[\square\]

Example 5.11. The formal group law associated to integral cohomology is additive, and its \( r \)-series is given by \([r](u) = ru \) in \( \mathbb{Z}[u] \). So for any \( p \)-primary weight vector \( \pi = (1, 1, p^{k_2}, \ldots, p^{k_n}) \), (5.9) identifies \( H^*(\mathbb{P}(\pi)) \) with

\[
\mathbb{Z}[w_n, w_{n-1}w_n, \ldots, w_1w_2 \cdots w_n] / J,
\]

where \( J \) is generated by elements of the form

\[
(w_i - p^{k_i-k_{i-1}}w_{i-1}) w_i \cdots w_n
\]

for \( 1 \leq i \leq n \), and \( w_0 = 0 \). In fact (5.12) is isomorphic to \( \mathbb{Z}[v_1, \ldots, v_n]/I(\pi) \) of (2.15), where Kawasaki’s ideal \( I(\pi) \) is generated by the relations

\[
v_j^j = m(\pi)_j v_j, \quad \text{where} \quad m(\pi)_j = p^{(j-1)k_j}/p^{k_{n-1}+\cdots+k_{n-j+1}},
\]

for \( 2 \leq j \leq n \) [14] page 243. The isomorphism arises from the bijection of generators \( w_{n-j+1} \cdots w_n \leftrightarrow v_j \), by repeated application of (5.13); it is multiplicative because \( w_j^j = \prod_{h=1}^{j-1} p^{k_{n-k_h}+\cdots+k_{n-j+1}} \cdot w_{n-j+1} \cdots w_n \) by induction on \( j \).

Remarks 5.15.

1. Rationally, Theorem 5.8 states that \( E\mathbb{Q}^*(\mathbb{P}(\chi))_+ \) is isomorphic to \( E\mathbb{Q}^*[w_n]/(w_n^{n+1}) \).

2. If \( \chi = 1 \), then \( \mathbb{P}(\chi) \) reduces to \( \mathbb{C}P^n \), and Theorem 5.8 identifies \( w_n \) with \( x^E \), and \( w_j^j \) with \( w_{n-j+1} \cdots w_n \) for every \( j \geq 1 \).

These observations illustrate the homotopy equivalences of Remark 2.17.

Example 5.16. The 2-primary part of \( \mathbb{P}(1, 2, 3, 4) \) is \( \mathbb{P}(1, 2, 1, 4) \), which is a 2-fold iterated Thom space over \( \mathbb{P}(1, 1) = \mathbb{C}P^1 \). So \( E^*(\mathbb{P}(1, 1)) \) is isomorphic to \( E_*[w_1]/(w_1^2) \), where \( w_1 \) generates \( E^2(\mathbb{P}(1, 1)) \). Furthermore, \( \lambda(w_1) \cong \zeta \), and \( \mathbb{P}(1, 1, 2) \) is homeomorphic to \( Th(\lambda(w_1^2)) \); hence \( E^*(\mathbb{P}(1, 1, 2))_+ \) is isomorphic to \( E_*[w_2, w_1w_2] / J_1^E \), where \( w_2 := c_1^E(\lambda(w_1)_{2, 1}) \) and \( J_1^E \) is the ideal generated by \( w_1^2w_2 \) and \( w_2^2 - 2w_1w_2 \).

Similarly, \( \mathbb{P}(1, 2, 1, 4) \) is homeomorphic to \( Th(\lambda(w_1)) \); so \( E^*(\mathbb{P}(1, 2, 1, 4))_+ \) is isomorphic to \( E_*[w_3, w_2w_3, w_1w_2w_3] / J_2^E \), where \( w_3 := c_2^E(\lambda(w_1)_{2, 2, 1}) \), and \( J_2^E \) is generated by \( w_1^2w_2w_3, \ (w_2^2 - 2w_1w_2)w_3, \) and \( w_3^2 - 2w_2w_3 - a_2^E w_2^2w_3 \).

The 3-primary part of \( \mathbb{P}(1, 2, 3, 4) \) is \( \mathbb{P}(1, 1, 3, 1) \), which is a Thom space over \( \mathbb{P}(1, 1, 1) = \mathbb{C}P^2 \). So \( E^*(\mathbb{P}(1, 1, 1))_+ \) is isomorphic to \( E_*[w_2]/(w_2^3) \), where \( w_2 \)
generates $E^2(\mathbb{P}(1,1,1))$ and $w_2^2 = w_1 w_2$. Moreover, $\lambda(w_2) \cong \zeta$, and $\mathbb{P}(1,1,1,3)$ is homeomorphic to $Th(\lambda(w_1^{1,3}))$; so $E^*(\mathbb{P}(1,1,1,3))$ is isomorphic to
\[ E^*[w_3, w_2 w_3, w_1 w_2 w_3] / J^E_3, \]
where $w_3 := cE^*(\lambda(w_1)^{1,3,1})$, and $J^E_3$ is generated by
\[ w_2^2 w_3, \quad (w_2^2 - w_1 w_2) w_3, \quad \text{and} \quad w_3^2 - 3 w_2 w_3 - 3 a E w_2^2 w_3. \]

The multiplicative formal group law is associated to complex $K$-theory and the Conner-Floyd orientation. The coefficient ring is $K_* \cong \mathbb{Z}[z, z^{-1}]$, and the element $z x K \in K^0(\mathbb{C}P^\infty)$ is represented by the virtual Hopf bundle $\zeta : \mathbb{C}$. The $r$-series is induced by the tensor power map $\zeta \mapsto \zeta^r$, and is therefore given by
\[ [r](u) = z^{-1}((1 + z u)^r - 1) \]
in $K_*[u]$, for any integer $r$. Al Amrani’s results of [2] may then be recovered.

**Example 5.18.** Theorem 5.8 and (5.17) combine to show that, for any integer $r$, the $K_*$-algebra $K_*(\mathbb{P}(1, \ldots, 1, r))$ is isomorphic to
\[ K_*[w_n, w_{n-1} w_n, \ldots, w_1 w_2 \cdots w_n] / J^K, \]
where $J^K$ denotes the ideal generated by elements of the form
\[
(w_i - w_{i-1}) w_i \cdots w_n \quad \text{for} \quad 1 \leq i \leq n - 1, \\
\text{and} \quad (w_n - z^{-1}((1 + z w_{n-1})^r - 1)) w_n. 
\]
The latter is equivalent to
\[ w_n^2 = \sum_{s=1}^{r} \binom{r}{s} z^{s-1} w_{n-s} \cdots w_n, \quad \text{where} \quad w_0 = 0. \]

6. COHOMOLOGICAL REASSEMBLY

It is now possible to follow the lead of Theorem 4.9 by reassembling the $E_*$-algebra $E^*(\mathbb{P}(\chi))$ from its constituent components $E^*(\mathbb{P}(p_1 \chi))$.

For any weight vector $\chi$, recall that $Q(\chi) = \{ p_1, \ldots, p_m \}$ denotes the primes occurring in $\chi$. The decomposition of Definition 3.10 may then be expressed as $\chi = p_i \chi \alpha(i)$ for each $1 \leq i \leq m$, where $Q(\alpha(i)) = Q(\chi) \setminus \{ p_i \}$. It is convenient to write $\mathbb{Z}_\chi$ for the subring $\mathbb{Z}[p_1^{-1}, \ldots, p_m^{-1}] < Q$.

**Proposition 6.1.** The $E_*$-algebra $E^*(\mathbb{P}(\chi)_+)$ is a free $E_*$-module, with one generator in each even dimension $\leq 2n$.

**Proof.** Consider the insertion map $e(\alpha(j)) : \mathbb{P}(p_i \chi) \to \mathbb{P}(\chi)$ of (5.11), for some $1 \leq j \leq m$. By Proposition 3.13, it is the orbit map for the action of the finite group $C_{\alpha(j)}$, whose order is divisible by every $p_i$ such that $i \neq j$. It therefore induces an isomorphism
\[ e(\alpha(j))^*: H^*(\mathbb{P}(p_i \chi)_+; \mathbb{Z}_{\alpha(j)}) \longrightarrow H^*(\mathbb{P}(p_i \chi)_+; \mathbb{Z}_{\alpha(j)}). \]

Example 5.11 shows that the graded abelian group $H^*(\mathbb{P}(p_i \chi)_+)$ is free, with one generator in each even dimension $\leq 2n$. So $H^*(\mathbb{P}(\chi)_+)$ contains at most $p_i$ torsion, for $1 \leq i \neq j \leq m$. Repeating the argument for every $1 \leq j \leq m$ in turn proves that $H^*(\mathbb{P}(\chi)_+)$ is torsion free, and therefore has one generator in each even dimension $\leq 2n$. 

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Since $E_*$ is also torsion free and even dimensional, the Atiyah-Hirzebruch spectral sequence for $E^*(\mathbb{P}(\chi)_+)$ collapses, and the conclusion follows. □

**Corollary 6.3.** For any weight vectors $\chi$ and $\sigma$, the induced homomorphism

$$e(\sigma)^* \otimes 1: E^*([\mathbb{P}(\sigma \chi)_+]) \otimes \mathbb{Z}_\sigma \rightarrow E^*([\mathbb{P}(\chi)_+]) \otimes \mathbb{Z}_\sigma$$

is an isomorphism of algebras over $E_* \otimes \mathbb{Z}_\sigma$.

**Proof.** Proposition 6.1 implies that $e(\sigma)^*$ induces an isomorphism of $E_2$-terms of Atiyah-Hirzebruch spectral sequences, which collapse. It therefore induces the required isomorphism on their limits. □

Proposition 6.1 may, of course, be deduced from Kawasaki's calculations; as proven above, it follows from the theory of Thom spaces. Isomorphism (6.2) confirms that $e(\alpha): \mathbb{P}_{(p \chi)} \rightarrow \mathbb{P}(\chi)$ and $e(1/\alpha): \mathbb{P}(\chi) \rightarrow \mathbb{P}_{(p \chi)}$ are mutually inverse $p$-local homotopy equivalences, for any $p$ in $Q(\chi)$.

The next step is to identify a cohomological version of Proposition 4.5, by applying $E^*(-)$ to the first diagram (4.1).

**Proposition 6.4.** If $\sigma$ and $\sigma'$ are coprime, then the diagram

$$E^*(\mathbb{P}(\sigma \sigma')_+) \xleftarrow{e(1/\sigma')^*} E^*(\mathbb{P}(\sigma)_+)$$

is a pushout square; in other words, the canonical homomorphism

$$h: E^*(\mathbb{P}(\sigma)_+) \otimes E^*(\mathbb{CP}^n_+) \rightarrow E^*(\mathbb{P}(\sigma \sigma')_+)$$

is an isomorphism of $E_*$-algebras.

**Proof.** Corollary 6.3 ensures that the horizontal and vertical homomorphisms of (6.5) induce isomorphisms over $E_* \otimes \mathbb{Z}_{\sigma'}$ and $E_* \otimes \mathbb{Z}_{\sigma}$ respectively. The same is therefore true of the corresponding pushout square. Hence $h$ induces an isomorphism of $E_*$-algebras over both $E_* \otimes \mathbb{Z}_{\sigma'}$ and $E_* \otimes \mathbb{Z}_{\sigma}$. But $\sigma$ and $\sigma'$ are coprime, so $h$ is an isomorphism. □

The cohomological version of Proposition 4.8 has a similar proof, with arrows reversed.

**Proposition 6.6.** If $\sigma$ and $\sigma'$ are coprime, then the diagram

$$E^*(\mathbb{P}(\sigma \sigma')_+) \xrightarrow{e(\sigma)^*} E^*(\mathbb{P}(\sigma')_+)$$

is a pullback square; in other words, the canonical homomorphism

$$h: E^*(\mathbb{P}(\sigma \sigma')_+) \rightarrow E^*(\mathbb{P}(\sigma)_+) \times_{E^*(\mathbb{CP}^n_+)} E^*(\mathbb{P}(\sigma')_+)$$

is an isomorphism of $E_*$-algebras.
Remark 6.8. Since \( e(\sigma)^* \) and \( e(\sigma')^* \) are monic, the limit in Proposition 6.6 may be interpreted as an intersection
\[
E^\ast(\nabla_{\sigma}^+ \cap E^\ast(\nabla_{\sigma'}^+) < E^\ast(\mathbb{C}P^n_+).
\]

For \( p \) and \( p' \)-primary weight vectors \( \pi \) and \( \pi' \), this provides an illuminating description of \( E^\ast(\nabla_{\pi\pi'}^+) \) as a subalgebra of \( E^\ast(\mathbb{C}P^n_+) \).

The cohomological version of Theorem 4.9 is now within reach, essentially by applying \( E^\ast(\cdot) \) to the geometrical proof.

Theorem 6.10. For any weight vector \( \chi \), the \( E^\ast \)-algebra \( E^\ast(\nabla_{\chi}^+) \) is isomorphic to the colimit of the \( \text{cat}[m] \)-diagram
\[
\cdots \rightarrow E^\ast(\nabla_{\chi}^+) \leftarrow E^\ast(\mathbb{C}P^n_+) \rightarrow E^\ast((\nabla_m^+) \rightarrow \cdots,
\]
and the associated universal homomorphisms \( E^\ast(\nabla_{\chi}^+) \rightarrow E^\ast((\nabla_m^+) \rightarrow E^\ast(\nabla_{\chi}^+) \) may be identified with \( e(\nabla_{\chi}^+) \) for every \( 0 \leq i \leq m \). Similarly, \( E^\ast((\nabla_{\chi}^+) \) is also isomorphic to the limit of the \( \text{cat}^{op}[m] \)-diagram
\[
\cdots \rightarrow E^\ast((\nabla_{\chi}^+) \leftarrow E^\ast(\mathbb{C}P^n_+) \rightarrow E^\ast(\nabla_{\chi}^+) \rightarrow \cdots,
\]
and the associated universal homomorphisms \( E^\ast((\nabla_{\chi}^+) \rightarrow E^\ast(\nabla_{\chi}^+) \) may be identified with \( e(\chi/\nabla_{\chi}^+) \) for every \( 0 \leq i \leq m \).

Proof. Proceed by induction on \( m \), as in the proof of Theorem 4.9. The inductive steps appeal to Propositions 6.4 and 6.6 respectively.

Remark 6.11. Theorem 6.10 shows that \( E^\ast(\cdot) \) converts the geometric limits and colimits of Theorem 4.9 into the corresponding algebraic colimits and limits. Although the geometric pullbacks are not of fibrations, the induced algebraic pushouts are those of a collapsed Eilenberg-Moore spectral sequence.

The pullback and pushout descriptions of Theorem 6.10 yield isomorphisms
\[
E^\ast((\nabla_{\chi}^+ \rightarrow E^\ast(\nabla_{\chi}^+) \otimes E^\ast(\mathbb{C}P^n_+) \cdots \otimes E^\ast(\mathbb{C}P^n_+) \rightarrow E^\ast(\nabla_{\chi}^+) \]
and
\[
E^\ast((\nabla_{\chi}^+) \times E^\ast(\mathbb{C}P^n_+) \cdots \times E^\ast(\mathbb{C}P^n_+) \rightarrow E^\ast(\nabla_{\chi}^+) \rightarrow E^\ast((\nabla_m^+) \]
respectively; by analogy with (6.9), the latter may be rewritten as
\[
E^\ast((\nabla_{\chi}^+) \cap \cdots \cap E^\ast(\nabla_{\chi}^+) < E^\ast(\mathbb{C}P^n_+).
\]
Each of these \( E^\ast \)-algebras has one \( E^\ast \)-generator in each even dimension \( \leq 2n \), and leads directly back to Kawasaki’s original calculations.

Theorem 6.15. In the case of integral cohomology, the isomorphisms (6.12), (6.13), and (6.14) identify \( H^\ast((\nabla_{\chi}^+) \) with \( \mathbb{Z}[v(1), \ldots, v(n)]/I(\chi) \), as in (2.15).
Proof. For each prime $p_i \in Q(\chi)$, let $v_i \in H^{2j}(P(\chi)_+)\) denote the image of Kawasaki’s generator $1 \otimes \cdots \otimes v_i(j) \otimes \cdots \otimes 1$ under the isomorphism (6.12), where $v_i(j) \in H^{2j}(P(p_i\chi))$. Example 5.11 shows that $m_i(j)v_i(j) = v_i(1)^j$ lies in the image of $H^{2j}(\mathbb{C}P^n)$ for every $i$, where
\[ m_i(j) = \frac{p_i^{(j-1)k_i(n)}}{p_i^{k_i(n-1)+\cdots+k_i(n-j+1)}} \]
and hence that $H^{2j}(P(\chi))$ is free abelian, on generator $v(j)$. By construction, $v(1)^j = \prod_i m_i(j)v_i(j)$ for every $1 \leq j \leq n$, as required by (2.13).

Alternatively, the isomorphism (6.14) identifies $v_i(j) \in H^{2j}(P(p_i\chi))$ with the element $p_i^{k_i(n)+\cdots+k_i(n-j+1)}x^j \in H^{2j}(\mathbb{C}P^n)$. The intersection of the cyclic groups so generated is therefore infinite cyclic on
\[ v(j) = \prod_i p_i^{k_i(n)+\cdots+k_i(n-j+1)}x^j, \]
and the relation $v(1)^j = \prod_i m_i(j)v_i(j)$ follows again. \[ \Box \]

**Example 6.16.** By (6.14), $K^*(\mathbb{P}(3, 4, 5)_+)$ may be identified with
\[ (6.17) \quad K^*(\mathbb{P}(1, 4, 1)_+) \cap K^*(\mathbb{P}(3, 1, 1)_+) \cap K^*(\mathbb{P}(1, 1, 5)_+) < K^*(\mathbb{C}P^2_+). \]

Example 5.18 with $n = 2$ shows that $K^*(\mathbb{P}(1, 1, r)_+)$ is isomorphic to
\[ (6.18) \quad K_*(w_2, w_1w_2) / (w_1^2w_2, w_2^2 - rw_1w_2), \]
and that (6.14) identifies $w_2$ with $[r](x^K)$ in $K^2(\mathbb{C}P^2)$ and $w_1w_2$ with $r(x^K)^2$ in $K^4(\mathbb{C}P^2)$. Substituting $r = 3, 4, 5$ in turn confirms that $K^*(\mathbb{P}(3, 4, 5))$ has $y_1 = 60x^K + 90z(x^K)^2$ and $y_2 = 60(x^K)^2$ as $K_*$-generators, in dimensions 2 and 4 respectively. So there is an isomorphism
\[ (6.19) \quad K^*(\mathbb{P}(3, 4, 5)_+) \cong K_*[y_1, y_2] / (y_1^2 - 60y_2). \]

Setting $z = 1$ in (6.19) provides an example of Al Amrani’s abstract isomorphism [41 Corollary 3.2] between $K^*(\mathbb{P}(\chi))$ and $H^*(\mathbb{P}(\chi))$ for certain $\chi$.

Theorem 6.15 expresses Kawasaki’s discovery (which he did not make explicit) that the ring $H^*(\mathbb{P}(\chi)_+)$ depends only on the unordered coordinates of the vectors $p\chi$, as $p$ ranges over $Q(\chi)$. The same holds for the $E_*$-algebra $E^*(\mathbb{P}(\chi)_+)$. \[ \Box \]

**Remark 6.20.** These facts also follow from [6]. As explained in Remark 5.11, \( E^*(\mathbb{P}(\chi)_+) \) may always be described in terms of iterated Thom isomorphisms as $E^*(\mathbb{P}(\chi))$. The advantage presented by Theorem 6.10 is that the $p$-primary parts $E^p(\mathbb{P}(\chi))$ are each computed using Theorem 5.8 since the computations involve the $E$-theory $p^k$-series (rather than the $r$-series for composite $r$), the technical machinery of Brown-Peterson cohomology theory [16] may then be brought to bear.
7. Homology and fundamental classes

Since Davis and Januszkiewicz’s original work \[8\], toric topology has tended to focus on cohomological calculations to the detriment of their homological counterparts. For weighted projective spaces, however, the complex bordism coalgebras \(\Omega_*^U(\mathbb{P}(\chi))\) are of particular interest, and this section is devoted to understanding \(E_*^U(\mathbb{P}(\chi))\) for any complex oriented homology theory \(E_*(-)\).

For \(\mathbb{C}P^n\), the complex orientation reveals itself as an isomorphism

\[
E_*(\mathbb{C}P^n) \xrightarrow{\cong} E_*\langle b_0, b_1, \ldots, b_n \rangle
\]

of free \(E_*\)-coalgebras, where \(b_j\) has dimension \(2j\) and supports the coproduct

\[
\delta(b_j) = \sum_{i=0}^{j} b_i \otimes b_{j-i}
\]
in \(E_*(\mathbb{C}P^n) \otimes E_*(\mathbb{C}P^n)\); the \(b_j\) form the dual \(E_*\)-basis to the powers \((x^E)^j\) for \(0 \leq j \leq n\), and \(b_0\) is the counit 1.

For notational clarity, two conventions are adopted throughout the remainder of this section. Firstly, \(b_j\) is expanded to \(b_j^E\) whenever the homology theory needs emphasising; and secondly, following Chapter 1, the universal complex orientation is usually denoted by \(u\) in \(\Omega^2(\mathbb{C}P^n)\).

It is important to clarify the relationship between the \(b_j\) and the Poincaré duality isomorphism

\[
\cap \sigma: E^i(\mathbb{C}P^n) \xrightarrow{\cong} E_{2n-i}(\mathbb{C}P^n),
\]

defined by cap product with a fundamental class \(\sigma \in E_{2n}(\mathbb{C}P^n)\). This is best done in the context of the universal example, and has an interesting history.

During the early days of the theory, it was usual to identify \(\Omega_*^U(\mathbb{C}P^n)\) with the free \(\Omega_*^U\)-module on generators \(cp_j\), represented by the inclusions \(\mathbb{C}P^j \to \mathbb{C}P^n\) for \(0 \leq j \leq n\). From this viewpoint, \(cp_n\) is the bordism class of the identity map \(I_{\mathbb{C}P^n}\), and the most natural choice of fundamental class \(\sigma\). In particular, iteration of the formula

\[
u \cap cp_n = cp_{n-1}
\]
in \(\Omega_*^U(\mathbb{C}P^n)\) shows that \(cp_j\) is the Poincaré dual of \(u^{n-j}\) for every \(0 \leq j \leq n\).

On the other hand, the \(cp_j\) are certainly not Hom dual to the \(u^j\), but may be expanded by

\[
cp_j = [\mathbb{C}P^j]1 + [\mathbb{C}P^{j-1}]b_1 + \cdots + [\mathbb{C}P^1]b_{j-1} + b_j
\]
in terms of the basis \((7.1)\).

Formula \((7.4)\) is originally due to Novikov, and is an immediate consequence of \((7.3)\). It emphasises the fact that \(b_j\) lies in the reduced group \(\Omega^U_{2j}(\mathbb{C}P^n)\) for every \(1 \leq j \leq n\), whereas \(cp_j\) has obvious non-trivial augmentation. Nevertheless, \(b_n\) may be deployed equally well as a fundamental class, and determines an alternative Poincaré duality isomorphism. Since

\[
u \cap b_n^U = b_{n-1}^U
\]
holds in \(\Omega_*^U(\mathbb{C}P^n)\) by definition, \(b_j\) is the alternative Poincaré dual of \(u^{n-j}\) by analogy with \((7.3)\). Under the Thom orientation \(\Omega^U_*(-) \to H_*(-)\), both
$b_n$ and $cp_n$ map to the canonical fundamental class $b_n^H$ in $H_{2n}(\mathbb{C}P^n)$. They therefore induce the same Poincaré duality isomorphism in integral homology and cohomology.

The problem arises of identifying geometrical representatives $B_j \rightarrow \mathbb{C}P^n$ for the $b_j$, bearing in mind that the stably complex manifolds $B_j$ must bound when $j \geq 1$, because the $b_j$ are reduced. This was solved in \[14\], where the $B_j$ are constructed as iterated sphere bundles whose stably complex structures extend over the associated disc bundles. Subsequently, the $B_j$ were identified as Bott towers \[11\], and therefore as non-singular toric varieties with canonical complex structures; the stabilisations of these structures do not bound, having non-trivial Chern numbers. The language of iterated sphere bundles is documented in \[7\, Sections 2 \& 3\], and used extensively below.

Each Bott tower is determined by a list $(r_1, \ldots, r_n)$ of integral $j$-vectors $r_j$. Given any divisive weight vector $\chi_j$, let $q_j = \chi_j/\chi_{j-1}$ as in Definition 3.2 and define the Bott tower $(B_j(\chi) : 0 \leq j \leq n)$ by choosing

$$r_j = (0, \ldots, 0, q_j).$$

Then $B_0(\chi) = *$, and $B_j(\chi)$ is a $2j$-dimensional stably complex manifold equipped with canonical complex line bundles $\gamma_i = \gamma_i(\chi)$, for $0 \leq i \leq j \leq n$. It is defined inductively as the total space $S(\delta_j(\chi))$ of the 2–sphere bundle of

$$(7.5) \quad \delta_j(\chi) := \mathbb{R} \oplus \gamma_{j-1}^q
$$

over $B_{j-1}(\chi)$, where $\mathbb{R}$ denotes the trivial real line bundle. The unit $1 \in \mathbb{R}$ determines a section $i_{j-1}$ for $\delta_j(\chi)$, which features in the cofibre sequence

$$(7.6) \quad B_{j-1}(\chi) \xrightarrow{i_j} B_j(\chi) \xrightarrow{l_j} Th(\gamma_{j-1}^q).$$

In terms of the complex orientation $x^E$, the corresponding Thom class $t^E(\gamma_{j-1}^q)$ generates $E^2(Th(\gamma_{j-1}^q))$ and pulls back to the generator $v_j = v_j^E$ of $E^2(B_j(\chi))$, as described in \[7\, Chapter 3\]. The inductive description is completed by appealing to (7.3), and letting $\gamma_j$ be the complex line bundle $\lambda(v_j^H)$.

The stably complex structure on $B_j(\chi)$ is induced from the defining $S^2$-bundle (7.5), and extends over the 3-disc bundle $D(\delta_j(\chi))$ by Szczarba \[18\]. It is specified by a canonical isomorphism

$$(7.7) \quad c: \tau(B_j(\chi)) \oplus \mathbb{R} \xrightarrow{\cong} \gamma_0^q \oplus \cdots \oplus \gamma_{j-1}^q \oplus \mathbb{R}$$

of $SO(2j + 1)$-bundles, where $\gamma_0 = \mathbb{C}$.

Recall that (7.5) expresses $\mathbb{P}(\chi)$ as an iterated Thom space over *$. The sequence $X_0, X_1, \ldots, X_n$ of Thom spaces may be written as

$$(7.8) \quad *, \quad Th(\lambda(0)^{q_1}), \quad Th(\lambda(0)^{q_1,q_2}), \quad \ldots, \quad Th(\lambda(0)^{q_1,\ldots,q_n}),$$

or equivalently as

$$(7.9) \quad \mathbb{P}(1), \quad \mathbb{P}(1, \chi_1), \quad \mathbb{P}(1, \chi_1, \chi_2), \quad \ldots, \quad \mathbb{P}(1, \chi_1, \chi_2, \ldots, \chi_n).$$

For any $1 \leq j \leq n$, the Thom class $t^E(\lambda(0)^{q_1,\ldots,q_j})$ that arises from (7.8) coincides with the generator $w_j$ in $E^2(\mathbb{P}(1, \chi_1, \ldots, \chi_j))$ that arises from (7.9); it is convenient to denote them both by $t_j$ in $E^2(X_j)$.
Lemma 7.10. For every $1 \leq j \leq n$, there exists a map $f_{j,n} : B_j(\chi) \to X_n$ such that $f_{j,n}^*(t_n) = v_j$ in $E^2(B_j(\chi))$.

Proof. Proceed by induction on $j$, with base case $j = 1$.

For any $n \geq 1$, the map $f_{1,n} : B_1(\chi) \to X_n$ is necessarily the inclusion of the fibre $S^2 \subset \text{Th}(\lambda(0)^{n-1})$, and coincides with the map $\mathbb{P}(X_{n-1}, X_n) \to \mathbb{P}(\chi)$ induced on the final two homogeneous coordinates.

Assume that $f_{j-1,n}$ exists with the required properties, and choose $2 \leq j \leq n$. Thus $1 \leq j - 1 \leq n - 1$, and it follows that $f_{j-1,n-1} : B_{j-1}(\chi) \to X_{n-1}$ satisfies $f_{j-1,n-1}^*(t_{n-1}) = v_{j-1}$ in $E^2(B_{j-1}(\chi))$. Now define $f_{j,n}$ as the composition

$$B_j(\pi) \xrightarrow{t_j} \text{Th}(\gamma_{j-1}^{\chi}) \xrightarrow{f_{j-1,n-1}^*} X_n$$

for $2 \leq j \leq n$, where $\gamma_{j-1} = \lambda(v_{j-1}^H)$ and $f_{j-1,n-1}$ denotes $\text{Th}(f_{j-1,n-1})$. Thus $(f_{j-1,n-1})^*(t_n) = t^E(\gamma_{j-1}^{\chi})$ in $E^2(\text{Th}(\gamma_{j-1}^{\chi}))$, and applying $t_j^f$ yields the required equation.

Every complex orientation induces natural transformations $\Omega^U_*(-) \to E_*(-)$ and $\Omega^U_*(-) \to E^*(-)$, both of which are written $x^E_*$. They reduce to the identity in the universal case.

Definition 7.12. For any $0 \leq j \leq n$, the bordism class $b_j(\chi)$ is represented by the map $f_{j,n}$ of Lemma 7.10 and lies in $\Omega^U_2(\mathbb{P}(\chi)_+)$; its image $x^E_*(b_j(\chi))$ is also denoted by $b_j(\chi)$ (or $b^E_j(\chi)$ to avoid ambiguity), and lies in $E_{2j}(\mathbb{P}(\chi)_+)$. The $B_j(\chi)$ bound as stably complex manifolds for every $j > 0$, so the corresponding $b_j(\chi)$ actually belong to the reduced groups $E_{2j}(\mathbb{P}(\chi)_+)$. According to Theorem 5.8, the elements

$$\{w_{i+1} \ldots w_n : 0 \leq i \leq n\}$$

form an $\Omega^U_*$-basis for $\Omega^U_*(\mathbb{P}(\chi)_+)$, where the case $i = n$ is interpreted as 1.

Theorem 7.14. For any divisive $\chi$, the elements $\{b_j(\chi) : 0 \leq j \leq n\}$ form a basis for the $\Omega^U_*-$coalgebra $\Omega^U_*(\mathbb{P}(\chi)_+)$; this basis is dual to 7.13.

Proof. Proceed by induction on $n$, noting that the result is trivial for $n = 0$ and $\chi = (1)$. The inductive assumption is that $\{b_j(\chi') : 0 \leq j \leq n-1\}$ and

$$\{w_{i+1} \ldots w_{n-1} : 0 \leq i \leq n - 1\}$$

are dual $\Omega^U_*$-bases for $\Omega^U_*(\mathbb{P}(\chi')_+)$ and $\Omega^U_*(\mathbb{P}(\chi')_+)$ respectively. In other words, the Kronecker product $\langle w_{i+1} \ldots w_{n-1}, b_j(\chi') \rangle$ evaluates to $\delta_{n-i-1,j}$ in $\Omega^U_{2(j-i)}$, for every $0 \leq i, j \leq n - 1$.

Now consider 7.11, and the Thom isomorphisms

$$\phi^* : \Omega^U_*((X_{n-1})_+) \to \Omega^U_*+2(X_n) \quad \text{and} \quad \phi_* : \Omega^U_*+2(X_n) \to \Omega^U_*((X_{n-1})_+)$$

given by the Thom class $w^U(n) \in \Omega^U_2(X_n)$. Then $\phi^*$ satisfies

$$\phi^*(w_{i+1} \ldots w_{n-1}) = w_{i+1} \ldots w_n$$

for $0 \leq i \leq n - 1$, and Lemma 7.10 shows that $\phi_*$ satisfies

$$\phi_*(b_k(\chi)) = b_{k-1}(\chi')$$
for $1 \leq k \leq n$. To check that the stably complex structures behave as required by (7.10), appeal must be made to (7.7).

By (7.15) and (7.16), the Kronecker product $\langle w_{i+1} \cdots w_n, b_k(\chi) \rangle$ may be rewritten as

$$\langle \Phi^*(w_{i+1} \cdots w_{n-1}), b_k(\chi) \rangle = \langle w_{i+1} \cdots w_{n-1}, \Phi_*(b_k(\chi)) \rangle = \langle w_{i+1} \cdots w_{n-1}, b_{k-1}(\chi') \rangle$$

for $0 \leq i \leq n - 1$ and $1 \leq k \leq n$. This evaluates to $\delta_{n-i-1,k-1} = \delta_{n-i,k}$ by induction, so it remains only to confirm the cases $i = n$ and $k = 0$. They involve $1 \in \Omega_U^0((X_n)_+)$ and $1 \in \Omega_U^1((X_n)_+)$, and follow immediately. □

**Corollary 7.17.** Given any complex oriented homology theory $E_*(\cdot)$, the elements $\{b_j(\chi) : 0 \leq j \leq n\}$ form an $E_*$-basis for $E_*^*(\mathbb{P}(\chi)_+)$; it is dual to the basis $\{w_{i+1} \cdots w_n : 0 \leq i \leq n\}$ for $E^*(\mathbb{P}(\chi)_+)$ given by Theorem 5.8.

It follows from Corollary 7.17 that the coalgebra structure on $E_*(\mathbb{P}(\chi)_+)$ is determined by expressions of the form

$$(7.18) \quad \delta(b_j(\chi)) = \sum_{0 \leq k+i \leq j} e_{j,k,i} b_k(\chi) \otimes b_i(\chi)$$

in $E_*(\mathbb{P}(\chi)_+) \otimes E_*(\mathbb{P}(\chi)_+)$, for $0 \leq j \leq n$. The coefficient $e_{j,k,i}$ lies in $E_{2(j-k-l)}$, and is given by the coefficient of $w_{n-j+1} \cdots w_n$ in the product

$$w_{n-k+1} \cdots w_n \cdot w_{n-l+1} \cdots w_n.$$ 

**Example 7.19.** If $\chi = (1,1,1,r)$, then Example 5.16 shows that $E_*(\mathbb{P}(\chi)_+)$ is freely generated over $E_*$ by elements $\{b_j := b_j(\chi) : 0 \leq k \leq 2\}$, where $b_0 = 1$. Applying Theorem 5.8 to (7.18) yields $\delta(1) = 1 \otimes 1$, together with

$$\delta(b_1) = b_1 \otimes 1 + 1 \otimes b_1, \quad \delta(b_2) = b_2 \otimes 1 + rb_1 \otimes b_1 + 1 \otimes b_2,$$

and

$$\delta(b_3) = b_3 \otimes 1 + rb_2 \otimes b_1 + \left(\frac{r}{2}\right) aE b_1 \otimes b_1 + rb_1 \otimes b_2 + 1 \otimes b_3$$

in $E_*(\mathbb{P}(\chi)_+) \otimes E_*(\mathbb{P}(\chi)_+)$. In the case of integral homology, the top dimensional group $H_{2n}(\mathbb{P}(\chi))$ is isomorphic to $\mathbb{Z}$, and cap product with the generator $b_n^H(\chi)$ defines Poincaré duality over $\mathbb{Q}$; this does not, of course, extend to $\mathbb{Z}$, symptomising the existence of singularities. Nevertheless, $b_n^H(\chi)$ may still be thought of as a fundamental class, and is the image of the universal $b_n(\chi)$ under $x_n^H$. In this sense, the representing map $j_{n,n} : B_n(\chi) \to \mathbb{P}(\chi)$ may be interpreted as a desingularisation of $\mathbb{P}(\chi)$; it is closely related to the associated toric desingularisation [10] §2.6, as we shall explain in a future note.

8. Homological reassembly

It remains to consider the assembly problem for $E_*(\mathbb{P}(\chi))$, by dualising the results of Section 6. This approach is not strictly necessary, as explained in Remark 6.20; for $E_*(\mathbb{P}(\chi)_+)$ is isomorphic as $E_*$-coalgebras to $E_*(\mathbb{P}(\star))$, and the latter may be described in terms of iterated Thom isomorphisms. Nevertheless, the homological advantages of proceeding prime by prime are as valid as for cohomology.
**Proposition 8.1.** For any weight vector $\chi$, the $E_*$-coalgebra $E_*(\mathbb{P}(\chi)_+)$ is a free $E_*$-module, with one generator in each even dimension $\leq 2n$.

**Proof.** Because $E_*$ is even dimensional and torsion free, the result follows directly from applying $\text{Hom}_{E_*}(-, E_*)$ to Proposition 6.11. □

A more explicit description is obtained by dualising the free $E_*$-modules that appear in Theorem 6.10.

**Theorem 8.2.** For any weight vector $\chi$, the $E_*$-coalgebra $E_*(\mathbb{P}(\chi)_+)$ is isomorphic to the limit of the $\text{CAT}^{op}[m]$-diagram

$$E_*(\mathbb{P}(\chi)_+) \rightarrow E_*(\mathbb{C}P^n_m) \rightarrow E_*(\mathbb{P}(\chi)_+);$$

the corresponding universal map $E_*(\mathbb{P}(\chi)_+)$ may be identified with $e(\pi/\chi)_+$ for every $0 \leq i \leq m$. Similarly, $E_*(\mathbb{P}(\chi)_+)$ is also isomorphic to the colimit of the $\text{CAT}[m]$-diagram

$$E_*(\mathbb{P}(\chi)_+) \leftarrow E_*(\mathbb{C}P^n_m) \leftarrow E_*(\mathbb{P}(\chi)_+);$$

the corresponding universal map $E_*(\mathbb{P}(\chi)_+)$ may be identified with $e(\chi/\chi)_+$ for every $0 \leq i \leq m$.

The limit described by Theorem 8.2 is actually the iterated pullback $E_*(\mathbb{P}(\chi)_+) \times_{E_*(\mathbb{C}P^n_m)} \cdots \times_{E_*(\mathbb{P}(\chi)_+)} E_*(\mathbb{P}(\chi)_+)$ of $E_*$-coalgebras, and the colimit is the iterated pushout $E_*(\mathbb{P}(\chi)_+) \otimes_{E_*(\mathbb{C}P^n_m)} \cdots \otimes_{E_*(\mathbb{P}(\chi)_+)} E_*(\mathbb{P}(\chi)_+)$. By analogy with (6.14), the former may be rewritten as

$$E_*(\mathbb{P}(\chi)_+) \cap \cdots \cap E_*(\mathbb{P}(\chi)_+) < E_*(\mathbb{C}P^n_m).$$

**Example 8.4.** Expression (8.3) identifies $K_*(\mathbb{P}(3, 4, 5)_+)$ with $K_*(\mathbb{P}(1, 4, 1)_+) \cap K_*(\mathbb{P}(3, 1, 1)_+) \cap K_*(\mathbb{P}(1, 1, 5)_+) < K_*(\mathbb{C}P^2)$. Applying Corollary 7.17 with $\pi = (1, 1, r)$ shows that $K_*(\mathbb{P}(\pi)_+)$ is isomorphic to

$$K_*(1, b_1(\pi), b_2(\pi)),$$

and also that (8.3) identifies $b_1(\pi)$ with $b_1$ in $K_2(\mathbb{C}P^2)$ and $b_2(\pi)$ with $rb_2$ in $K_4(\mathbb{C}P^2)$. These identifications are compatible with the diagonals

$$\delta(b_1(\pi)) = b_1(\pi) \otimes 1 + 1 \otimes b_1(\pi) \quad \text{and} \quad \delta(b_2(\pi)) = b_2(\pi) \otimes 1 + rb_1(\pi) \otimes b_1(\pi) + 1 \otimes b_2(\pi).$$
Substituting $r = 3, 4$, and 5 and applying \[8.3\] confirms that $K_*(\mathbb{P}(3, 4, 5)_+)$ is isomorphic to the subcoalgebra

\[K_*(1, b_1, 60b_2) < K_*(\mathbb{C}P_+)\] (8.7)

as predicted by Proposition \[8.1\]. The resulting coalgebra is $K_*$-dual to the $K_*$-algebra description of $K^*(\mathbb{P}(3, 4, 5)_+)$ given by Example \[6.16\].

Setting $z = 1$ in \[8.7\] yields an abstract isomorphism of coalgebras between $K_*(P(\chi))$ and $H_*(P(\chi))$, and dualises Al Amrani’s algebra isomorphism of \[4\].

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