An Elementary Proof of the Meromorphy of $\int f^z$ for real-analytic $f$

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Suppose $U$ is a domain in $\mathbb{R}^n$, $f$ is a real-analytic function on $U$ and $M$ is an open semianalytic subset of $U$ with $\text{cl}(M) \subset U$. For $\phi \in C_c(U)$, we define

$$F(z) = \int_M (f(x))^z \phi(x) \, dx \quad (1.1)$$

To be clear what (1.1) means when $f$ takes on negative values, we fix some branch of the logarithm on the negative real axis and use it to define $f(x)^z$ when $f(x) < 0$. If $\text{Re}(z) > 0$, the integrand in (1.1) is integrable and standard methods show that $F$ is analytic in a neighborhood of $z$ with

$$F'(z) = \int_M \log(f(x))(f(x))^z \phi(x) \, dx \quad (1.2)$$

Here $\log(f(x))$ denotes the branch of the logarithm chosen above. It is natural to try to extend $F(z)$ to a meromorphic function of $z$ on all of $\mathbb{C}$ by doing appropriate integrations by parts, integrating $f(x)^z$ in some way. However, the zero set of an arbitrary real-analytic function can be quite complicated, so carrying this out may be rather involved. After Hironaka proved his monumental [H1] [H2], Gelfand suggested that by using these results one might be able to do the requisite analysis by reducing to the case where $f(x)$ is a monomial. This was done by Bernstein and Gelfand [BGe] when $f(x) = |P(x)|$ for a polynomial $P$ and when $M = U = \mathbb{R}^n$. In addition, Atiyah [A] did it for general semianalytic $M$ and general nonnegative real-analytic $f(x)$. Later, Bernstein [B] found an algebraic proof of the results in [BGe] not using resolution of singularities, by virtue of Bernstein-Sato polynomials, which allow one to integrate by parts directly in the original integral.

The purpose of this paper is to show that the elementary resolution of singularities algorithm of [G1] suffices to prove such results. In fact, we prove a very slightly more general result than that of [A] by removing the restriction that $f$ is nonnegative (the version of resolution of singularities used in [A] actually does use that $f \geq 0$ on all of $U$, so one can’t simply resolve $f^2$ or add a condition like $f < 0$ to the definition of $M$). The main result of this paper is:

**Theorem 1:** The function $F(z)$ of (1.1) extends to a meromorphic function on $\mathbb{C}$. If $K$ denotes any compact set containing $\text{supp}(\phi)$, then poles of the resulting function must be
at a point of the form $-\frac{1}{x}$, where $N$ is a fixed positive integer depending on $f$, $M$, and $K$, where $r$ is a positive integer. The order of any pole is at most the dimension $n$.

To prove Theorem 1, we use the following consequence of the Main Theorem of [G]:

**Resolution of Singularities Theorem:** Suppose $f$ is real analytic on a neighborhood of the origin. Then there is an neighborhood $V$ of the origin such that if $\phi(x) \in C_c(V)$ is nonnegative with $\phi(0) > 0$, then $\phi(x)$ can be written as $\phi(x) = \sum_{i=1}^p \phi_i(x)$, each $\phi_i(x)$ nonnegative, such that the following hold. Let $D_i = \{x : \phi_i(x) > 0\}$. There is a real-analytic diffeomorphism $\Psi_i$ from an open bounded $D'_i$ to $D_i$ such that on a neighborhood of $cl(D'_i)$, $f \circ \Psi_i(x) = d_i(x)m_i(x)$, $m_i(x)$ a monomial and $d_i(x)$ nonvanishing.

One can resolve several functions simultaneously in this fashion by resolving their product.

The Main Theorem of [G] also stipulates that each $\phi_i \circ \Psi_i(x)$ is a ”quasibump function”:

**Definition:** Let $E = \{x : x_i > 0 \text{ for all } i\}$. If $h(x)$ is a bounded, nonnegative, compactly supported function on $E$, we say $h(x)$ is a quasibump function if $h(x)$ is of the following form:

$$h(x) = a(x) \prod_{k=1}^l b_k(c_k(x) \frac{p_k(x)}{q_k(x)})$$

(1.2)

Here $p_k(x), q_k(x)$ are monomials, $a(x) \in C^\infty(cl(E))$, the $c_k(x)$ are nonvanishing real-analytic functions defined on a neighborhood of $\text{supp}(h)$, and $b_k(x)$ are functions in $C^\infty(\mathbb{R})$ such that there are $c_1 > c_0 > 0$ with each $b_k(x) = 1$ for $x < c_0$ and $b_k(x) = 0$ for $x > c_1$.

The reason this explicit form for $h(x)$ is useful for our purposes is that we will be doing some integrations by parts in integrals with $\phi_i \circ \Psi_i(x)$ appearing in the integrands, and derivatives will be landing on such $\phi_i \circ \Psi_i(x)$. Since we know the $\phi_i \circ \Psi_i(x)$ are all quasibump functions, we can get explicit estimates on the size of the derivatives. In addition, since the $b_k(x)$ are constant on $x < c_0$ and on $x > c_1$, the support of a derivative of $\phi_i \circ \Psi_i(x)$ will be substantially smaller than that of $\phi_i \circ \Psi_i$. After an appropriate coordinate change, this will effectively allow us to reduce the dimension of the problem and induct on the dimension $n$.

**Proof of Theorem 1:** We can assume that $M$ is of the form $\{x : g_k(x) > 0$ for $k = 1,...,p\}$, each $g_k$ real-analytic, since up to a set of measure zero every semianalytic set can be written as the finite union of sets of this form. At each point $x$ in $\text{supp}(\phi) \cap cl(M)$ one can find a neighborhood $N_x$ of $x$ such that the resolution of singularities theorem above applies to $f, g_1, ..., g_p$ simultaneously on $N_x$. Using a partition of unity, on $\text{supp}(\phi) \cap cl(M)$ one can write $\phi = \sum_{j} \alpha_j(x)$, where each $\alpha_j$ is in $C_c(N_x)$ for some $x$. Hence it suffices to prove Theorem 1 for an arbitrary $\alpha_j$ in place of $\phi$; adding the results will give Theorem 1 for $\phi$. Hence without loss of generality, we assume $\phi$ is one of these $\alpha_j$. Thus we may apply the resolution of singularities theorem to $f, g_1, ..., g_p$, and we write $\phi = \sum \phi_i$ accordingly.
Define
\[ F_i(z) = \int_M (f(x))^z \phi_i(x) \, dx \]  
(1.3)

It suffices to show each \( F_i(z) \) satisfies the conclusions of Theorem 1. We change coordinates in (1.3) to the blown up coordinates, obtaining
\[ F_i(z) = \int_E (f \circ \Psi_i(x))^z (\chi_M \circ \Psi_i(x))(\phi_i \circ \Psi_i(x)) \det \Psi_i(x) \, dx \]  
(1.4)

In the new coordinates, we have \( f \circ \Psi_i(x) = d_i(x)m_i(x) \), where \( d_i(x) \) is nonvanishing on a neighborhood of \( \text{supp}(\phi_i \circ \Psi_i) \) and \( m_i(x) \) is some monomial \( \prod_{j=1}^n x_j^{a_{ij}} \). Hence we can rewrite (1.4) as
\[ F_i(z) = \int_E \prod_{j=1}^n x_j^{a_{ij}} d_i(x)^z (\chi_M \circ \Psi_i(x))(\phi_i \circ \Psi_i(x)) \det \Psi_i(x) \, dx \]  
(1.5)

In the case that \( d_i(x) \) assumes negative values, one defines \( d_i(x)^z \) using the branch of the logarithm one used to define \( f(x)^z \) in the original coordinates. Since each \( g_k \circ \Psi_i(x) \) is also of the form \( d(x)m(x) \), either each \( g_k \) is positive throughout the domain of integration of (1.5), or there is at least one \( k \) for which \( g_k \) is negative throughout the domain. In the first case, \( \chi_M \circ \Psi_i(x) \) is always 1, in the second case it is always zero. In the latter case \( F_i(z) = 0 \) and there is nothing to prove, so we assume we are in the first case and write
\[ F_i(z) = \int_E \prod_{j=1}^n x_j^{a_{ij}} d_i(x)^z (\phi_i \circ \Psi_i(x)) \det \Psi_i(x) \, dx \]

We use the explicit form (1.2) of the quasibump function \( \phi_i \circ \Psi_i(x) \) and this becomes
\[ F_i(z) = \int_E \prod_{j=1}^n x_j^{a_{ij}} d_i(x)^z a(x) \prod_{k=1}^l b_k(c_k(x) \frac{p_k(x)}{q_k(x)}) \det \Psi_i(x) \, dx \]

We combine the smooth \( a(x) \) and \( \det \Psi_i(x) \) factors by letting \( A_i(x) = a(x) \det \Psi_i(x) \), and the above becomes
\[ F_i(z) = \int_E \prod_{j=1}^n x_j^{a_{ij}} d_i(x)^z A_i(x) \prod_{k=1}^l b_k(c_k(x) \frac{p_k(x)}{q_k(x)}) \, dx \]  
(1.5')

The idea behind the analysis of (1.5') is quite simple. One wishes to repeatedly integrate by parts, integrating first \( x_1^{a_{i1}z} \), then \( x_2^{a_{i2}z} \), going up to \( x_n^{a_{in}z} \). Then one integrates \( x_1^{a_{i1}z+1} \), cycles through the \( x_j \) again, and repeats ad nauseum. (One may skip any \( x_j \) for which \( a_{ij} = 0 \). Since the exponents of the \( x_j \) will increase each time, the integral will be analytic over a larger and larger \( z \)-domain as the integrations by parts proceed. One obtains poles
since one gets factors of \( \frac{1}{a_{ij} z + k} \) showing up with each integration by parts. Each pole should have order at most \( n \), since a given factor appears at most once per variable.

To make these heuristics work, one has to ensure that the derivatives landing on the \( b_k(c_k(x) \frac{p_k(x)}{q_k(x)}) \) factors don’t mess things up. In [A] or [BGe] such issues don’t arise since they use the stronger result of Hironaka which doesn’t require one to subdivide a neighborhood of the origin into different parts each having a different set of coordinate changes; instead there is one sequence \( \Psi \) of blow ups and the resulting \( \phi \circ \Psi \) is smooth.

Theorem 1 will follow from the following lemma:

**Lemma:** \( F_i(z) \) extends to a meromorphic function on \( \mathbb{C} \). There is a positive integer \( N \), depending on the \( a_{ij} \) and the various monomials \( p_k(x) \) and \( q_k(x) \), such that each pole of \( F_i(z) \) is at \( -\frac{k}{\eta} \) for some nonnegative integer \( r \). Let \( \eta > 0 \) such that the integrand of (1.5') is supported on \( (0, \eta)^n \). Then for any \( l \) and each compact subset \( K \) of \( \{ z : z > -\frac{l+1}{\eta} \} \), \( \sup_K \| \prod_{i=1}^{l} (z + \frac{x}{N}) F_i(z) \| \) can be bounded in terms of \( K, \eta \), the \( a_{ij} \), the monomials \( p_k(x) \) and \( q_k(x) \), and the \( C^m \) norms of the \( d_i(x) \), \( A_i(x) \), \( b_k(x) \), and \( c_k(x) \). Here \( m \) is some sufficiently large natural number.

**Proof:** We proceed by induction on \( n \). We do the \( n = 1 \) case at the same time as the \( n > 1 \). So we assume that either \( n = 1 \) or that \( n > 1 \) and we know the result for \( n-1 \). We perform an integration by parts in \( x_1 \) in (1.5'), turning the \( x_1^{-a_1 z} \) into an \( x_1^{-a_1 z - 1} \). If the derivative lands on \( d_i(x)^z \), we obtain another smooth function which does not interfere with future integrations by parts with respect to \( x_2, x_3, \) etc as described in the heuristics above. A similar situation occurs if the derivative lands on \( A_i(x) \). Things are more complicated when the derivative lands on one of the \( b_k(c_k(x) \frac{p_k(x)}{q_k(x)}) \). Let \( x_1^n \) denote the power of \( x_1 \) appearing in \( \frac{p_k(x)}{q_k(x)} \) (hence \( m \) could be positive, negative, or zero). Then we have

\[
\partial_{x_1} [b_k(c_k(x) \frac{p_k(x)}{q_k(x)})] = \frac{1}{x_1} (x_1 \partial_{x_1} [b_k(c_k(x) \frac{p_k(x)}{q_k(x)})] + m c_k(x)) \frac{p_k(x)}{q_k(x)}
\]

Then we write \( B_k(x) = x b_k'(x) \), we have

\[
\partial_{x_1} [b_k(c_k(x) \frac{p_k(x)}{q_k(x)})] = \frac{1}{x_1} (x_1 \partial_{x_1} [b_k(c_k(x) \frac{p_k(x)}{q_k(x)})] + m c_k(x)) \frac{p_k(x)}{q_k(x)}
\]

We consider \( \frac{x_1 \partial_{x_1} (\frac{p_k(x)}{q_k(x)}) + m c_k(x)}{c_k(x)} \) a smooth factor \( s_k(x) \), and thus we have

\[
\partial_{x_1} [b_k(c_k(x) \frac{p_k(x)}{q_k(x)})] = \frac{1}{x_1} s_k(x) B_k(c_k(x) \frac{p_k(x)}{q_k(x)})
\]

If we use the notation \( A_{ik}(x) = A_i(x)s_k(x) \), the integral corresponding to this term is given by

\[
\frac{1}{a_{i1} z + 1} \int_E \prod_j x_j^{a_{ij} z} d_i(x)^z [A_{ik}(x) B_k(c_k(x) \frac{p_k(x)}{q_k(x)}) \prod_{k \neq k} b_K(c_K(x) \frac{p_K(x)}{q_K(x)})] \, dx
\]
At first glance, (1.9) might appear to be little improved over (1.5'), since the exponents of the $a_{ij}z$ appearing are unchanged. However, there is a key difference. Namely, instead of having the quasibump function $\phi_i \circ \Psi_i(x)$ in the integrand, we have the bracketed expression in (1.9). Because $b_k(x)$ is constant for $0 < x < x_0$ and $x > x_1$, $B_k(x) = x b'_k(x)$ is supported on $[x_0, x_1]$. This means that the factor $B_k(c_k(x) p_k(x) / q_k(x))$ in the integrand is supported on the wedge $C_1 q_k(x) \leq p_k(x) \leq C_2 q_k(x)$ for some constants $C_1$ and $C_2$. After doing appropriate coordinate changes, we will be able to exploit this fact to reduce the problem to the $n - 1$ dimensional case. We break into three cases of increasing order of difficulty. We can assume $p_k(x)$ and $q_k(x)$ have no common factors.

Case 1): Either $p_k(x)$ or $q_k(x)$ is constant, and each $x_j$ appears to a positive power in whichever of $p_k(x)$ or $q_k(x)$ is nonconstant. Note that whenever $n = 1$ we are in case 1. Replacing $B_k(x)$ by $B_k(1/x)$ if necessary, we may assume $p_k(x)$ is constant. Because $B_k(x)$ is zero for $x > x_0 > 0$ for some $x_0$ and the integrand of (1.9) is compactly supported, there is some constant $C$ such that each $x_j > C$ when the integrand of (1.9) is nonzero. As a result, one can integrate by parts in (1.9) as many times as one likes; the integral is in fact an entire function.

Case 2): Either $p_k(x)$ or $q_k(x)$ is constant, but there is some $x_j$ not appearing in the nonconstant function of $p_k(x)$ or $q_k(x)$. Like before we may replace $B_k(x)$ by $B_k(1/x)$ if necessary and assume $p_k(x)$ is constant. Let $J$ be the set of $j$ for which $x_j$ appears in $q_k(x)$. Then in the integrand of (1.9), $x_j > C$ for all $j \in J$. Thus if in the integrand of (1.9) we freeze each $x_j$ at a constant for $j \in J$, then the integrand becomes that of an expression (1.5') corresponding to the $n - |J|$ dimensional case; we treat $A_{ik}(x) B_k(c_k(x) p_k(x) / q_k(x))$ like a single smooth factor $A_i(x)$. Hence by induction hypothesis, the integral in these $n - |J|$ variables is a meromorphic function satisfying the conclusions of the lemma in dimension $n - |J|$. By the uniform bounds given by the lemma, we conclude that (1.9) satisfies the conclusions of the lemma as well, and we are done with case 2.

Case 3): Both $p_k(x)$ and $q_k(x)$ are nonconstant. In this case we will have to break up (1.9) into several pieces. Some coordinate changes are done on each piece to reduce it to Cases 1 or 2. We first do a coordinate change $(x_1, ..., x_m) \rightarrow (x_1^{M_1}, ..., x_n^{M_n})$ so that each $x_j$ appearing in either $p_k(x)$ or $q_k(x)$ appears to the same power. We still get an expression of the form (1.9), after incorporating the determinant of this coordinate change into the $A_{ik}(x)$ factor. Suppose $x_l$ appears in $p_k(x)$ and $x_m$ appears in $q_k(x)$. Let $\alpha(x) \in C^\infty(0, \infty)$ be nonnegative such that $\alpha(y) + \alpha(1/y) = 1$ for all $y$, and such that $\alpha(y)$ is supported on $y < C$ for some $C$. In particular, in the integrand of (1.9) we have $\alpha(x_l / x_m) + \alpha(x_m / x_l) = 1$, and we correspondingly write the integral as $I_1(z) + I_2(z)$, where

\begin{align*}
I_1(z) &= \int_E \prod_j x_j^{a_{ij}z} d_i(x)^z [A_{ik}(x) B_k(c_k(x) p_k(x) / q_k(x))] \prod_{K \neq k} b_K(c_K(x) p_K(x) / q_K(x)) \alpha(x_l / x_m) dx \\
I_2(z) &= \int_E \prod_j x_j^{a_{ij}z} d_i(x)^z [A_{ik}(x) B_k(c_k(x) p_k(x) / q_k(x))] \prod_{K \neq k} b_K(c_K(x) p_K(x) / q_K(x)) \alpha(x_m / x_l) dx
\end{align*}
In (1.10) we do the variable change turning what was \( x_l \) into \( x_l x_m \), and in (1.11) we do the variable change turning what was \( x_m \) into \( x_m x_l \). The resulting integrals are still of the form (1.9). In addition, in the factor \( \frac{p_k(x)}{q_k(x)} \) of (1.10) the factor \( x_m \) dissapears, while in (1.11) the factor \( x_l \) dissappears.

If we iterate the above in (1.10) and (1.11), splitting into more and more terms, then eventually enough \( x_j \)'s will have dissappeared in \( \frac{p_k(x)}{q_k(x)} \) that we are in either case 1 or case 2. Thus (1.9) is the sum of finitely many terms that fall under case 1 or case 2, and thus we have the lemma in case 3 as well. This completes the proof of the lemma. We are also done with the proof of Theorem 1; as in the earlier heuristics we integrate by parts with respect to \( x_2, x_3 \), etc ad infinitum; one deals with these integrations by parts the way we dealt with the \( x_1 \) integration by parts above.

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