Extremes of the stochastic heat equation with additive Lévy noise

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Abstract

We analyze the spatial asymptotic properties of the solution to the stochastic heat equation driven by an additive Lévy space-time white noise. For fixed time $t > 0$ and space $x \in \mathbb{R}^d$, we determine the exact tail behavior of the solution both for light-tailed and for heavy-tailed Lévy jump measures. Based on these asymptotics we determine for any fixed time $t > 0$ the almost-sure growth rate of the solution as $|x| \to \infty$.

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1 Introduction

We consider the stochastic heat equation (SHE) on $\mathbb{R}^d$ driven by an additive Lévy space-time white noise $\dot{\Lambda}$, with zero initial condition, given by

$$
\begin{align*}
\partial_t Y(t, x) &= \frac{\kappa}{2} \Delta Y(t, x) + \dot{\Lambda}(t, x), \\
Y(0, \cdot) &= 0,
\end{align*}
$$

(1.1)

where $\Delta$ stands for the Laplacian, $\kappa > 0$ is the diffusion constant, and the measure $\Lambda$ is given by

$$
\Lambda(dt, dx) = m dt\, dx + \int_{(0,\infty)} z \mu(dt, dx, dz) + \int_{(0,1]} z (\mu - \nu)(dt, dx, dz),
$$

(1.2)

Here, $m \in \mathbb{R}$ and $\mu$ is a Poisson random measure on $(0,\infty) \times \mathbb{R}^d \times (0,\infty)$ whose intensity measure $\nu$ takes the form $\nu(dt, dx, dz) = dt\, dx\, \lambda(dz)$, with a Lévy measure satisfying $\int_{(0,\infty)}(1 \wedge z^2) \lambda(dz) < \infty$.

To exclude trivialities, we always assume that $\lambda$ is not identically zero.

In this case the mild solution to (1.1) can be written explicitly in the form

$$
Y(t, x) = \int_0^t \int_{\mathbb{R}^d} g(t - s, x - y) \Lambda(ds, dy),
$$

(1.3)

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where
\[
g(t, x) = \frac{1}{(2\pi\kappa t)^{d/2}} e^{-\frac{|x|^2}{4\kappa t}}, \quad t > 0, \ x \in \mathbb{R}^d,
\] (1.4)
is the heat kernel. In our earlier paper \[8\] we obtained a complete description of the almost-sure growth behavior of \(Y(t, x)\) at a fixed spatial point \(x \in \mathbb{R}^d\) as \(t \to \infty\). In particular, \(t \mapsto Y(t, x)\) satisfies a weak law of large numbers but surprisingly violates the strong law of large numbers. In the present paper we continue these investigations and analyze the almost-sure behavior for fixed time \(t > 0\), as \(|x| \to \infty\).

To this end, we determine in Section 2 the exact tail asymptotics for \(Y(t, x)\) both for light-tailed and for heavy-tailed Lévy measures. Note that since the heat kernel is singular at the origin, the results of \[13, 14, 24\] for moving-average processes with bounded kernels do not apply. In \[7\] we proved that for any jump measure \(\lambda\), the \((1 + \frac{2}{d})\)-moment of \(Y(t, x)\) is infinite, which suggests a power-law tail behavior. In Theorem 2 we show that this is indeed the case, regardless of whether the noise itself is light- or heavy-tailed. Section 3 contains the tail asymptotics for \(\sup_{x \in A} Y(t, x)\), where \(A\) is a bounded Borel set. Based on these results, we determine in Section 4 the almost-sure growth behavior of \(Y(t, x)\) as \(|x| \to \infty\). The behavior is very different from the behavior of the Gaussian case, in which
\[
\lim \sup_{|x| \to \infty} \frac{Y(t, x)}{(\log |x|)^{1/2}} = \left(\frac{4t}{\pi\kappa}\right)^{\frac{1}{4}}
\] (1.5)
almost surely; see \[18, Eq. (6.3)\]. All the proofs are gathered together in Section 5. In our companion paper \[9\] we address the same questions for the SHE with multiplicative Lévy noise.

Let us end this introductory section by stating necessary and sufficient conditions in terms of the jump measure \(\lambda\) for the existence of the integral (1.3). To our best knowledge, this result is new. While many works have studied sufficient conditions for existence \[11, 14, 5, 26\], necessary and sufficient conditions have only been derived for multiplicative noise \[2\] or for specific types of noises such as \(\alpha\)-stable noise \[11\]. Introduce the measure \(\eta\) as
\[
\eta(B) = \nu\left(\{(s, y, z) : s \leq t, g(s, y)z \in B\}\right),
\] (1.6)
where \(B \subseteq (0, \infty)\) is a Borel set.

**Theorem 1.** Suppose that \(\Lambda\) is of the form (1.2).

(i) The integral defining \(Y(t, x)\) in (1.3) exists if and only if (iff)
\[
\int_{(1,\infty)} (\log z)^{d/2} \lambda(dz) < \infty \quad \text{and} \quad \begin{cases} 
\int_{(0,1]} z^2 \lambda(dz) < \infty & \text{if } d = 1, \\
\int_{(0,1]} z^2 |\log z| \lambda(dz) < \infty & \text{if } d = 2, \\
\int_{(0,1]} z^{1+2/d} \lambda(dz) < \infty & \text{if } d \geq 3.
\end{cases}
\] (1.7)

In this case, \(\eta\) is a Lévy measure and \(Y(t, x)\) is infinitely divisible with characteristic function
\[
\mathbb{E}[e^{i\theta Y(t,x)}] = \exp \left\{ i\theta A + \int_{(0,\infty)} \left( e^{i\theta u} - 1 - i\theta u 1(u \leq 1) \right) \eta(du) \right\},
\] (1.8)
where 1 stands for the indicator function and \(A \in \mathbb{R}\) is some explicit constant.
\( (ii) \) The integral
\[
\int_0^t \int_{\mathbb{R}^d} \int_{(0,\infty)} g(t-s, x-y) z \mu(ds, dy, dz)
\]
exists iff
\[
\int_{(1,\infty)} (\log z)^{d/2} \lambda(dz) < \infty \quad \text{and} \quad \int_{(0,1]} z \lambda(dz) < \infty.
\]

Remark 1. Note that (1.7) is identical to the necessary and sufficient condition found in [2] for the existence of solutions to the SHE with multiplicative noise in dimensions \( d = 1, 2 \) but is weaker than the necessary condition found in [2, Prop. 2.2] for \( d \geq 3 \). In other words, if \( d \geq 3 \), there are Lévy noises for which the SHE with additive noise has a solution but the SHE with multiplicative noise does not.

Whenever \( \int_{(0,1]} z \lambda(dz) < \infty \), there is no need for compensation, so we assume without loss of generality that \( m = \int_{(0,1]} z \lambda(dz) \). In this case,
\[
Y(t, x) = \int_0^t \int_{\mathbb{R}^d} \int_{(0,\infty)} g(t-s, x-y) z \mu(ds, dy, dz) = \sum_{\tau_i \leq t} g(t-\tau_i, x-\eta_i) \zeta_i,
\]
where \( (\tau_i, \eta_i, \zeta_i) \) are the points of the Poisson random measure \( \mu \). In what follows we always assume that (1.7) holds.

2 Tail asymptotics

Since \( Y(t, x) \) is infinitely divisible, its tail behavior is the same as the tail behavior of its Lévy measure \( \eta \), whenever the tail is subexponential. This result was proved by [12] for nonnegative infinitely divisible random variables and by [21, 22] in the general case. Therefore, we need to determine the tail of the Lévy measure \( \eta \) in (1.6).

For \( \gamma > 0 \) introduce the moments and truncated moments of \( \lambda \) as
\[
m_\gamma(\lambda) = \int_{(0,\infty)} z^\gamma \lambda(dz) \quad \text{and} \quad M_\gamma(x) = \int_{(0,x]} z^\gamma \lambda(dz).
\]

Lemma 1. Let \( D = (2\pi \kappa t)^{d/2} \). For any \( r > 0 \),
\[
\overline{\eta}(r) = \eta((r, \infty)) = r^{-(1+2/d)} \frac{d^{d/2}}{\pi \kappa (d+2)^{d/2+1} \Gamma(\frac{d}{2} + 1)} \int_0^\infty e^{-u} u^{d/2} M_{1+2/d}(Dre^{ud/(d+2)}) du.
\]

From the representation above we immediately see that as soon as \( m_{1+2/d}(\lambda) < \infty \), then \( \overline{\eta}(r) \sim c r^{-1-2/d} \), for some \( c > 0 \). We can determine the tail even if this moment condition does not hold, provided that \( \overline{\lambda} \) is regularly varying.

In the following, the class of regularly varying functions with index \( \rho \in \mathbb{R} \) is denoted by \( \mathcal{RV}_\rho \). For general theory on regular variation we refer to [3]. Write \( \overline{\lambda}(r) = \lambda((r, \infty)) \). By Karamata’s theorem, for \( \alpha > 0 \), \( \overline{\lambda} \in \mathcal{RV}_{-\alpha} \) iff the truncated moment \( M_{1+2/d} \) in (2.1) is also regularly varying. However, for \( \alpha = 0 \), the latter holds iff \( \overline{\lambda} \) belongs to the de Haan class (see e.g. [3, Thm. 3.7.1]). Therefore, it is more difficult to determine the asymptotics of \( \overline{\eta} \) for \( \alpha = 0 \), and in fact the result itself is surprising.
Lemma 2. Let $\lambda$ satisfy (1.7).

(i) Assume that $m_{1+2/d}(\lambda) < \infty$. Then
\[
\overline{\eta}(r) \sim r^{1-2/d} \frac{d^{d/2}}{\pi \kappa (d+2)^{d/2+1}} m_{1+2/d}(\lambda), \quad r \to \infty.
\]

(ii) Assume that $\overline{\lambda}(r) = \ell(r)r^{-\alpha}$ for $\alpha \in (0, 1 + \frac{2}{d}]$, where $\ell$ is slowly varying, and if $\alpha = 1 + \frac{2}{d}$, further assume that $\int_1^\infty \ell(u)u^{-1} \, du = \infty$. Then as $r \to \infty$,
\[
\overline{\eta}(r) \sim \begin{cases} 
\ell(r)r^{-\alpha} \frac{D^{1+2/d-\alpha}}{d \pi \kappa \alpha^{d/2}(1 + \frac{2}{d} - \alpha)} & \text{if } \alpha < 1 + \frac{2}{d}, \\
L(r)r^{-1-2/d}(d \pi \kappa (1 + \frac{2}{d})^{d/2})^{-1} & \text{if } \alpha = 1 + \frac{2}{d},
\end{cases}
\]
where
\[
L(r) = \int_1^r \ell(u)u^{-1} \, du.
\]

(iii) Assume that $\alpha = 0$ and $\overline{\lambda}(x) = \ell(x)$ is slowly varying. Then as $r \to \infty$,
\[
\overline{\eta}(r) \sim L_0(r)\frac{D^{1+2/d}}{2 \pi \kappa \Gamma\left(\frac{d}{2} + 1\right)(1 + \frac{2}{d})},
\]
where
\[
L_0(r) := \int_1^\infty \ell(r y) y^{-1} (\log y)^{d/2-1} \, dy
\]
is slowly varying and $L_0(r)/\ell(r) \to \infty$ as $r \to \infty$.

Example 1. Assume that $\overline{\lambda}(r) = (\log r)^{-\beta}$ for $r > e$. Then (1.7) holds iff $\beta > \frac{d}{2}$. By substituting $u = (1 + \log y/\log r)^{-1}$, we obtain
\[
L_0(r) = (\log r)^{d/2-\beta} \beta\left(\frac{d}{2}, \beta - \frac{d}{d} \right), \quad r \to \infty,
\]
where $\beta$ is the usual beta function.

To determine the tail of the spatial supremum, we need the tail of the largest contribution to $Y(t, x)$ by a single atom. Without loss of generality, consider $x = 0$ and define
\[
\overline{Y}(t) = \sup_{\tau_i \leq t} g(t - \tau_i, \eta_i) \zeta_i.
\]
For $r > 0$ large, let
\[
S_r = \{(s, y, z) : s \in [0, t], g(s, y)z > r\}.
\]
Clearly, $\overline{Y}(t) \leq r$ iff $\mu(S_r) = 0$, which shows that
\[
P(\overline{Y}(t) \leq r) = e^{-\nu(S_r)} = e^{-\overline{\eta}(r)}.
\]
As a result we obtain the following.
Theorem 2. Let $Y(t, x)$ be given in (1.3) and assume (1.7).

(i) The tail function $\eta$ has extended regular variation at infinity $[\text{[3, p. 65]}]$, that is, there are $\theta_1, \theta_2 \in \mathbb{R}$ such that for any $c > 1$,

$$c^{\theta_1} \leq \liminf_{x \to \infty} \frac{\eta(cx)}{\eta(x)} \leq \limsup_{x \to \infty} \frac{\eta(cx)}{\eta(x)} \leq c^{\theta_2}. \quad (2.6)$$

(ii) As $r \to \infty$,

$$P(Y(t, x) > r) \sim P(\tilde{Y}(t) > r) \sim \eta(r). \quad (2.7)$$

(iii) For $\alpha \in [0, 1 + \frac{2}{d})$, $\eta \in \mathcal{RV}_\alpha$ iff $\overline{\eta} \in \mathcal{RV}_{-\alpha}$. For $\alpha = 1 + \frac{2}{d}$, we have $\eta \in \mathcal{RV}_{-1-2/d}$ if $r \mapsto \int_0^r u^{2/d} \int_1^\infty \overline{\eta}(uv)(\log v)^{d/2-1}v^{-1}dvdu$ is slowly varying. In particular, the latter holds if $m_{1+2/d}(\lambda) < \infty$.

3 Spatial supremum

Let $A \in \mathcal{B}(\mathbb{R}^d)$ be a Borel subset of $\mathbb{R}^d$ with finite and positive Lebesgue measure and define

$$X_A(t) = \begin{cases} \sum_{\eta_i \in A, \tau_i \leq t} (2\pi \kappa (t - \tau_i))^{-d/2} \zeta_i \mathbb{1}_{\{(2\pi \kappa (t - \tau_i))^{-d/2} \zeta_i > 1\}} & \text{if } d = 1, \\ \sum_{\eta_i \in A, \tau_i \leq t} (2\pi \kappa (t - \tau_i))^{-d/2} \zeta_i & \text{if } d \geq 2, \end{cases} \quad (3.1)$$

where $\overline{A}$ is the closure of $A$. Since $X_A(t)$ is a functional of a Poisson random measure, one easily obtains necessary and sufficient conditions for the existence.

Define the measure $\tau$ as

$$\tau(B) = (\text{Leb} \times \lambda)\left(\{(s, z) : (2\pi \kappa s)^{-d/2} z \in B \cap (\mathbb{1}_{(d=1)}, \infty), s \leq t\}\right), \quad B \in \mathcal{B}(\mathbb{R}^d), \quad (3.2)$$

where Leb is the Lebesgue measure. For a Borel set $A$ let $|A|$ be its Lebesgue measure.

Theorem 3. Suppose that $|A| \in (0, \infty)$. The sum $X_A(t)$ is finite a.s. iff

$$\int_{(0,1)} z^{2/d} \log z \mathbb{1}_{(d=2)} \lambda(dz) < \infty. \quad (3.3)$$

Furthermore, if (3.3) holds then

$$E[e^{i\theta X_A(t)}] = \exp \left\{ |A| \int_{(0,\infty)} (1 - e^{-i\theta u}) \tau(du) \right\}. \quad (3.4)$$

Note that (3.3) holds for any Lévy measure if $d = 1$. From (3.2) we obtain that for $r > 1$

$$\tau(r) = \tau((r, \infty)) = \int_{(0,\infty)} (2\pi \kappa)^{-1}(z/r)^{2/d} \land t \lambda(dz) \quad (3.5)$$

$$= r^{-2/d}(2\pi \kappa)^{-1} M_{2/d}(rD) + t \overline{\lambda}(rD)$$

In specific cases, we can determine the asymptotic behavior of $\tau$ explicitly.
Lemma 3. Assume (3.3).
(i) If \( m_{2/d}(\lambda) < \infty \), then \( \tau(r) \sim (2\pi\kappa)^{-1} m_{2/d}(\lambda) r^{-2/d} \) as \( r \to \infty \).
(ii) Assume that \( \lambda(r) = \ell(r) r^{-\alpha} \) for \( \alpha \in [0, \frac{2}{d}] \), where \( \ell \) is slowly varying, and further assume \( \int_1^\infty \ell(u) u^{-1} \, du = \infty \) if \( \alpha = \frac{2}{d} \). Recalling the definition of \( L \) from (2.3), we have as \( r \to \infty \) that
\[
\tau(r) \sim \begin{cases}
\frac{2 ld^{-\alpha}}{2 - d \alpha} \ell(r) r^{-\alpha} & \text{if } \alpha < \frac{2}{d}, \\
\frac{2}{d} (2\pi\kappa)^{-1} L(r) r^{-2/d} & \text{if } \alpha = \frac{2}{d}.
\end{cases}
\]

Introduce the notation
\[
\lambda_A(t) = \sup \{ (2\pi\kappa(t - \tau_i))^{-d/2} \zeta_i : \tau_i \leq t, \eta_i \in A \}. \tag{3.6}
\]
To determine the tail of \( \lambda_A(t) \), let \( T_r = \{(s, z) : s \leq t, (2\pi\kappa s)^{-d/2} z > r \} \). Then \( \lambda_A(t) \leq r \) iff \( \mu(A \times T_r) = 0 \), thus
\[
P(\lambda_A(t) > r) = 1 - e^{-|A| \tau(r)}. \tag{3.7}
\]

Theorem 4. Assume (3.3).
(i) The tail function \( \tau \) has extended regular variation at infinity.
(ii) For every bounded Borel set \( A \),
\[
P(X_A(t) > r) \sim P(\lambda_A(t) > r) \sim |A| \tau(r), \quad r \to \infty. \tag{3.8}
\]
(iii) For \( \alpha \in [0, \frac{2}{d}] \), \( \tau \in \mathcal{RV}_- \) iff \( \lambda \in \mathcal{RV}_- \). For \( \alpha = \frac{2}{d} \), we have \( \tau \in \mathcal{RV}_{-2/d} \) iff \( r \mapsto \int_0^r u^{2/d-1} \lambda(u) \, du \) is slowly varying. In particular, the latter holds if \( m_{2/d}(\lambda) < \infty \) or if \( \lambda \in \mathcal{RV}_{-2/d} \).

In order to determine the tail asymptotics of the local supremum of the solution, let us introduce for each \( A \in \mathcal{B}(\mathbb{R}^d) \) the measure
\[
\eta_A(B) = \nu \left( \{(s, y, z) : s \leq t, (2\pi s)^{-d/2} e^{-\frac{\text{dist}(y, A)^2}{2s}} z \in B \cap (\mathbf{1}_{\{d=1\}}, \infty) \} \right), \quad B \in \mathcal{B}(\mathbb{R}^d). \tag{3.9}
\]
If \( m_{2/d}(\lambda) < \infty \) or if \( \lambda \) is regularly varying with index \( -\alpha \) for some \( \alpha \in (0, \frac{2}{d}] \), one can express \( \overline{\eta}_A \) in terms of \( \tau \) or \( \lambda \).

Lemma 4. Let \( A \) be a bounded Borel set. Assume (3.3) and \( \int_{(1, \infty)} \log z^{d/2} \lambda(dz) < \infty \).
(i) If \( m_{2/d}(\lambda) < \infty \) or \( \lambda(r) = \ell(r) r^{-2/d} \) and \( \ell \) is slowly varying with \( \int_1^\infty \ell(u) u^{-1} \, du = \infty \), then
\[
\overline{\eta}_A(r) \sim |A| \tau(r), \quad r \to \infty
\]
(ii) If \( \lambda(r) = \ell(r) r^{-\alpha} \) for \( \alpha \in (0, \frac{2}{d}) \), where \( \ell \) is slowly varying, then
\[
\overline{\eta}_A(r) \sim \lambda(r) \int_0^r \int_{\mathbb{R}^d} (2\pi\kappa s)^{-\alpha d/2} e^{-\frac{\text{dist}(y, A)^2}{2ks}} \, ds \, dy, \quad r \to \infty.
\]
**Theorem 5.** Let $A$ be a bounded Borel set. Assume (3.3) and $\int_{(1, \infty)} (\log z)^{d/2} \lambda(dz) < \infty$. If $d = 1$, further assume that
\[ \exists q \in (0, 2) : M_q(1) < \infty. \quad (3.10) \]
Then under the assumptions of Lemma 4 (i) or (ii) we have that
\[ P \left( \sup_{x \in A} Y(t, x) > r \right) \sim \eta_A(r), \quad r \to \infty. \]

**Remark 2.** As the proof shows, even without the assumptions of Lemma 4 (i) and (ii), the statement of Theorem 5 continues to hold provided $\eta_A$ is subexponential. We were not able to prove or disprove this in general.

## 4 Growth rate

In what follows we assume (3.3). For $r > 0$ and $0 \leq r_1 < r_2$, we write
\[ V(r) = \{ (s, z) : \frac{z}{(2\pi \kappa s)^{d/2}} > r, s \leq t \}, \quad V(r_1, r_2) = \{ (s, z) : \frac{z}{(2\pi \kappa s)^{d/2}} \in (r_1, r_2], s \leq t \}. \quad (4.1) \]
Recalling (3.2) we have for $r > 1$
\[ (\text{Leb} \times \lambda)(V(r)) = \overline{\nu}(r). \quad (4.2) \]
Note that $\overline{\nu}(r)$ is a continuously decreasing function, with $\overline{\nu}(\infty) = 0$ and $\overline{\nu}(0+) = \infty$ whenever $\lambda((0, 1)) = \infty$. If $m_{2/d}(\lambda) < \infty$, then by (3.5)
\[ \overline{\nu}(r) \leq r^{-2/d} (2\pi \kappa)^{-1} m_{2/d}(\lambda). \quad (4.3) \]
From (3.5) we further see that whenever $\int_{(0, 1]} z^{2/d} \lambda(dz) = \infty$ we have for any $r > 0$
\[ \sup_{y \mid x - y \leq r} \sup_{y \mid x - y \leq r} Y(t, y) = \infty. \quad (4.4) \]
Therefore, our standing assumption (3.3) is optimal for $d \geq 3$ and almost optimal for $d = 2$. For a more general result in this direction, see [6, Thm. 3.3]. Furthermore, by [6, Thm. 3.1], if $\int_{(0, 1]} z^p \lambda(dz) < \infty$ for some $p < \frac{2}{d}$, then for any fixed $t$ the function $x \mapsto Y(t, x)$ is a.s. continuous.

If $\int_{0}^{\infty} z^{2/d} \lambda(dz) = m_{2/d}(\lambda) < \infty$, the non-Gaussian analogue of (1.5) (see also [10, Thm. 1.2]) reads as follows.

**Theorem 6.** Let $f : (0, \infty) \to (0, \infty)$ be nondecreasing and assume that (3.3) holds. If $d = 1$, further assume (3.10). Then almost surely
\[ \limsup_{x \to \infty} \sup_{y \mid x - y \leq r} \frac{Y(t, y)}{f(x)} = \infty \quad \text{or} \quad \limsup_{x \to \infty} \sup_{y \mid x - y \leq r} \frac{Y(t, y)}{f(x)} = 0, \]
according to whether the following integral diverges or converges:
\[ \int_{1}^{\infty} r^{d-1} \overline{\nu}(f(r)) \, dr. \quad (4.5) \]
The result says that there is no proper normalization. If $m_{2/d}(\lambda) < \infty$, then almost surely there are infinitely many peaks in $B(x) = \{ y : |y| \leq x \}$ that are larger than $x^{d/2}(\log x)^{d/2}$ but only finitely many that are larger than $x^{d/2}(\log x)^{d/2+\varepsilon}$.

**Remark 3.** If the Lévy measure is small in the sense that $m_{2/d}(\lambda) < \infty$, then the large peaks of $Y(t, x)$ are caused by points very close to the time $t$. (If we remove jumps close to $(t, x)$, this is equivalent to removing the singularity of $g$ in (1.3). The local spatial supremum of the resulting process would have a finite moment of order $\frac{2}{d}$. In particular, its tail probability would be $o(r^{-2/d})$, which by the arguments of the proof of the theorem implies that the peaks will be of smaller order.) However, if $X \in R\mathcal{V}_-\alpha$ with $\alpha < \frac{2}{d}$, then the peaks are caused by large jumps, which are not necessarily close to $t$. Indeed, assume that the integral in (4.5) diverges. For some $\delta \in (0, t)$ and large $K > 0$ define

$$\tilde{A}_n = \{ \mu(\{(s, y, z) : s \leq t - \delta, |y| \in [n, n+1], z > Kf(n)\}) \geq 1 \}.$$ 

Then as $n \to \infty$,

$$P(\tilde{A}_n) \sim Cn^{-d-1}\lambda(f(n)) \sim Cn^{-d-1}\pi(f(n)),$$

showing that $\sum_{n=1}^{\infty} P(\tilde{A}_n) = \infty$. By the second Borel–Cantelli lemma $\tilde{A}_n$ occurs infinitely often.

In line with the previous remark we show in our next and final result that the largest peaks of $x \mapsto Y(t, x)$ are typically not attained at integer locations if $m_{2/d}(\lambda) < \infty$. To this end, introduce the process

$$Y_0(t, x) = \begin{cases} \int_0^t \int_{(0,\infty)} g(t-s, x-y) \mathbb{1}_{\{|x-y| \leq \frac{1}{2} \}} \mu(ds, dy, dz), & \text{if } d = 1, \\ \int_0^t \int_{(0,\infty)} g(t-s, x-y) \mathbb{1}_{\{|x-y| \leq \frac{1}{2} \}} \mu(ds, dy, dz), & \text{if } d \geq 2, \end{cases}$$

which is infinitely divisible with Lévy measure

$$\eta_0(B) = \nu(\{(s, y, z) : s \leq t, |y| \leq \frac{1}{2}, g(s, y)z \in B \cap (\mathbb{1}_{\{d=1\}}, \infty)\}).$$

**Theorem 7.** Let $f : (0, \infty) \to (0, \infty)$ be nondecreasing and assume that (1.7) holds. If $d = 1$, further assume (4.10). Then

$$\int_1^{\infty} r^{d-1}\eta(f(r)) \, dr < \infty \implies \lim_{x \to \infty} \sup_{y \in \mathbb{Z}^d, |y| \leq x} \frac{\max_{y \in \mathbb{Z}^d, |y| \leq x} Y(t, y)}{f(x)} = 0,$$

$$\int_1^{\infty} r^{d-1}\eta_0(f(r)) \, dr = \infty \implies \lim_{x \to \infty} \sup_{y \in \mathbb{Z}^d, |y| \leq x} \frac{\max_{y \in \mathbb{Z}^d, |y| \leq x} Y(t, y)}{f(x)} = \infty.$$

The result above is optimal if $\pi(r) \asymp \eta_0(r)$ (i.e., $0 < \liminf_{r \to \infty} \frac{\pi(r)}{\eta(r)} \leq \limsup_{r \to \infty} \frac{\pi(r)}{\eta(r)} < \infty$). We end with a sufficient condition for the asymptotic equivalence of $\overline{\eta}$ and $\overline{\eta}_0$ and an example where they are not.

**Lemma 5.** (i) If $m_{1+2/d}(\lambda) < \infty$, or if there exist $\delta > 0$ and $C > 0$ such that for $r, y > 1$ large enough

$$\frac{\overline{X}(ry)}{\overline{X}(r)} \leq Cy^{-\delta},$$

then $\overline{\eta}_0(r) \asymp \overline{\eta}(r)$ as $r \to \infty$.

(ii) Under the assumptions of Lemma $\mathbb{Z}$ (iii) we have $\overline{\eta}_0(r) = o(\overline{\eta}(r))$ as $r \to \infty$. 

8
5 Proofs

5.1 Proofs for Section I

Proof of Theorem I. We start with the part (ii). By standard results on Poisson integrals (see e.g. [19, Thm. 2.7]), the integral in (1.9) exists a.s. iff
\[
\int\int\int (1 \land g(s, y)z) \, ds \, dy \, \lambda(dz) < \infty,
\]
where \( \int\int\int = \int_0^t \int_{\mathbb{R}^d} \int_{(0, \infty)} \). For any \( u > 0 \)
\[
g(s, y) \leq u \iff s \geq (2\pi\kappa u^{2/d})^{-1} =: H_1(u) \quad \text{or} \quad \left\{ s \in (0, H_1(u)) \quad \text{and} \quad |y| \geq \sqrt{-\kappa ds \log(2\pi\kappa s u^{2/d})} =: H_2(s, u) \right\}. \tag{5.1}
\]
Note that if \( z \leq (2\pi\kappa t)^{d/2} =: D \), then \( H_1(1/z) \leq t \). Let
\[
A_1 = \{(s, y, z) : z \leq D, s \leq H_1(1/z), |y| \leq H_2(s, 1/z)\},
\]
\[
A_2 = \{(s, y, z) : z > D, s \leq t, |y| \leq H_2(s, 1/z)\}
\]
and
\[
B_1 = \{(s, y, z) : s \geq H_1(1/z)\},
\]
\[
B_{21} = \{(s, y, z) : z \leq D, s \leq H_1(1/z), |y| > H_2(s, 1/z)\},
\]
\[
B_{22} = \{(s, y, z) : z > D, s \leq t, |y| > H_2(s, 1/z)\}.
\]

Then \( A_1, A_2, B_1, B_{21}, B_{22} \) form a partition of \((0, t] \times \mathbb{R}^d \times (0, \infty)\). Moreover, by (5.1), \( 1 \leq g(s, y)z \) iff \( (s, y, z) \in A_1 \cup A_2 \).

Consider the upper incomplete gamma function \( \Gamma(s, x) = \int_x^\infty u^{s-1}e^{-u} \, du \). For \( r \leq H_1(1/z) \), by a change of variables \( v = \log(H_1(1/z))/s \),
\[
\int_0^r H_2(s, 1/z)^d \, ds = \int_0^r \left( \kappa ds \log \frac{H_1(1/z)}{s} \right)^{d/2} \, ds
\]
\[
= (kd)^{d/2} H_1(1/z)^{d/2+1} \int_0^\infty \frac{e^{-v(d/2+1)}v^{d/2}}{\log H_1(1/z)^{d/2}} \, dv
\]
\[
= (kd)^{d/2} \left( \frac{2}{d+2} \right)^{d/2+1} H_1(1/z)^{d/2+1} \Gamma(d/2+1, (d/2+1 \log(H_1(1/z)/r)). \tag{5.2}
\]

Therefore, on \( A_1 \), after simplifying the constant,
\[
\int\int\int_{A_1} (1 \land g(s, y)z) \, ds \, dy \, \lambda(dz) = \int_{(0, D]} \int_0^{H_1(1/z)} v_d H_2(s, 1/z)^d \, ds \lambda(dz)
\]
\[
= \frac{d^{d/2}}{\pi \kappa (d+2)^{d/2+1}} \int_{(0, D]} z^{1+2/d} \, \lambda(dz),
\]
where \( v_d = \pi^{d/2}/\Gamma\left(\frac{d}{2} + 1\right) \) is the volume of the unit ball \( B(1) \). We see that this integral is finite iff \( \int_{(0,1]} z^{1+2/d} \lambda(dz) < \infty \). On \( A_2 \),

\[
\iiint_{A_2} (1 \land g(s,y)z) \, ds \, dy \, \lambda(dz) = \frac{d^{d/2}}{\Gamma\left(\frac{d}{2} + 1\right)\pi \kappa (d + 2)^{d/2 + 1}} \int_{(D,\infty)} z^{1+2/d} \Gamma\left(\frac{d}{2} + 1, \left(\frac{d}{2} + 1\right) \log \frac{z^{2/d}}{2\pi \kappa t}\right) \lambda(dz).
\]

Since \( \Gamma\left(\frac{d}{2} + 1, u\right) \sim e^{-u} u^{d/2} \) as \( u \to \infty \),

\[
z^{1+2/d} \Gamma\left(\frac{d}{2} + 1, \left(\frac{d}{2} + 1\right) \log \frac{z^{2/d}}{2\pi \kappa t}\right) \sim \left(2\pi \kappa t\right)^{d/2 + 1} \left(1 + \frac{d}{2}\right)^{d/2} \left(\log z\right)^{d/2},
\]

as \( z \to \infty \), which implies that

\[
\iiint_{A_2} (1 \land g(s,y)z) \, ds \, dy \, \lambda(dz) < \infty \iff \int_{(1,\infty)} \left(\log z\right)^{d/2} \lambda(dz) < \infty.
\]

On \( B_1 \),

\[
\iiint_{B_1} g(s,y)z \, ds \, dy \, \lambda(dz) = \int_{(0,D]} z(t - H_1(1/z)) \lambda(dz),
\]

which is finite iff \( \int_{(0,1]} z \lambda(dz) < \infty \). For any \( h > 0 \),

\[
\int_{|y|>h} g(s,y) \, dy = dv_d(2\pi \kappa s)^{-d/2} \int_{h}^{\infty} e^{-\frac{z^2}{2\pi \kappa s} r^{d-1}} \, dr = \frac{\Gamma\left(\frac{d}{2}, \frac{h^2}{2\pi \kappa s}\right)}{\Gamma\left(\frac{d}{2}\right)}.
\]

(5.3)

Furthermore, for any \( a > 0 \),

\[
\int_{0}^{a} \Gamma\left(\frac{d}{2}, \frac{h^2}{2\pi \kappa s} \log \frac{a}{\xi}\right) \, ds = a \Gamma\left(\frac{d}{2}\right) \left(1 - (1 + \frac{d}{2})^{-d/2}\right).
\]

(5.4)

Therefore, by (5.3) and (5.4),

\[
\iiint_{B_{21}} g(s,y)z \, ds \, dy \, \lambda(dz) = (2\pi \kappa)^{-1} (1 - (1 + \frac{d}{2})^{-d/2}) \int_{(0,D]} z^{1+2/d} \lambda(dz).
\]

Finally, on \( B_{22} \), we use (5.3) and the asymptotics \( \Gamma\left(\frac{d}{2}, u\right) \sim e^{-u} u^{d/2-1} \) to obtain that

\[
\iiint_{B_{22}} g(s,y)z \, ds \, dy \, \lambda(dz) < \infty \iff \int_{(1,\infty)} \left(\log z\right)^{d/2-1} \lambda(dz) < \infty.
\]

By \[19\] Thm. 2.7 (ii)] the characteristic function of the integral in (1.9) is

\[
\exp \left\{ - \iiint (1 - e^{ig(s,y)z}) \, ds \, dy \, \lambda(dz) \right\} = \exp \left\{ - \int_{(0,\infty)} (1 - e^{i\theta u}) \eta(du) \right\}.
\]

(5.5)
To prove the existence of $Y(t, x)$ defined as a compensated integral, we use the stochastic integration theory of [23]. By writing

$$Y(t, x) = mt + \int_0^t \int_{\mathbb{R}} \int_{(0,1]} g(t - s, x - y) z (\mu - \nu)(ds, dy, dz)
\quad + \int_0^t \int_{\mathbb{R}} \int_{(1,\infty)} g(t - s, x - y) z \mu(ds, dy, dz)
\quad =: mt + Y_s(t, x) + Y_b(t, x)$$

(5.6)

and the previously proved existence result for $Y_b(t, x)$, it is enough to deal with $Y_s(t, x)$, that is, we may assume that there are only small jumps. Spelling out [23, Thm. 2.7] to our setting, we obtain that $Y(t, x)$ exists iff

$$\int_0^t \int_{\mathbb{R}} \int_{(0,1]} g(s, y) z \mathbb{1}(g(s, y) z > 1) ds dy \lambda(dz) < \infty,$$

(5.7)

and

$$\int_0^t \int_{\mathbb{R}} \int_{(0,1]} (1 \wedge g(s, y)^2 z^2) ds dy \lambda(dz) < \infty.$$  

(5.8)

To check (5.7), as in (5.3) write

$$\int_{|y| \leq h} g(s, y) dy = \frac{1}{\Gamma(\frac{d}{2})} \int_0^{\frac{h^2}{2\pi s}} e^{-u/d-1} du.$$

Thus, as in (5.4),

$$\int_0^{H_1(1/z)} \int_0^{H_2(s,1/z)/(2\kappa s)} e^{-u/d-1} du ds = H_1(1/z)(1 + \frac{2}{d})^{-d/2} \Gamma(\frac{d}{2}),$$

which gives that

$$\int_0^t \int_{\mathbb{R}} \int_{(0,1]} g(s, y) z \mathbb{1}(g(s, y) z > 1) ds dy \lambda(dz) = (1 + \frac{2}{d})^{-d/2} \int_{(0,1]} z H_1(1/z) \lambda(dz).$$

The latter integral exists iff $\int_{(0,1]} z^{1+2/d} \lambda(dz) < \infty$.

For (5.8), by the previous calculations, we only have to deal with the integral on $B_1 \cup B_{21}$. As

$$\int_{\mathbb{R}^d} g(s, y)^2 dy = 2^{-d}(\pi\kappa)^{-d/2} s^{-d/2},$$

we obtain that

$$\iint_{B_3} g(s, y)^2 z^2 ds dy \lambda(dz) < \infty$$

iff the second part of (1.7) holds. Finally, for $h > 0,$

$$\int_{|y| > h} g(s, y)^2 dy = \frac{\Gamma(\frac{d}{2}, \frac{h^2}{2\pi s})}{2d(\pi\kappa s)^{d/2} \Gamma(\frac{d}{2})},$$
and for \(a > 0\),
\[
\int_0^a s^{-d/2} \frac{d}{2} \log \frac{d}{2} \, ds = a^{1-d/2} \Gamma\left(\frac{d}{2}\right) \frac{2\left[\frac{1}{2} + \frac{1}{d}\right]}{d - 2},
\]
where the last fraction is \(\frac{1}{2}\) if \(d = 2\). Thus,
\[
\int\int\int_{B_{21}} g(s, y)^2 z^2 \, ds \, dy \, \lambda(dz) = 2^{-d}\left(\frac{\pi \kappa}{2}\right)^{\frac{d}{2}} \frac{\Gamma\left(\frac{d}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{d}{2} + 1\right)} \frac{a^{1-d/2} \Gamma\left(\frac{d}{2}\right)}{d - 2} - 2 - \frac{2\pi}{2} - 2 - 2\left(\frac{1}{2} + \frac{1}{d}\right) - \frac{2\pi}{2},
\]
which is finite iff \(\int_{[0,1]} z^{1+2/d} \lambda(dz) < \infty\). In summary, (5.7) and (5.8) hold iff (1.7) holds.

By [23, Thm. 2.7 (iv)], the characteristic function of \(Y_s(t, x)\) is
\[
\mathbb{E}\left[e^{i\theta Y_s(t, x)}\right] = \exp\left\{-i\theta \int_0^t \int_{\mathbb{R}^d} \int_{(0,1]} 1(g(s, y) z > 1) g(s, y) z \, ds \, dy \, \lambda(dz)\right\} + \int_0^\infty \left(e^{i\theta u} - 1 - i\theta(u \wedge 1)\right) \eta(du)\right\}.
\]
Combining with (5.5), we obtain (1.8).

### 5.2 Proofs for Section 2

**Proof of Lemma 1.** By (5.1) and (5.2) and Fubini’s theorem, we have
\[
\eta(r) = \int_{(0,\infty)} \int_0^{H_1(r/z)^{\wedge t}} v_d H_2(s, r/z)^d \, ds \, \lambda(dz) = \int_{(0,\infty)} \frac{d^{d/2}}{\pi \kappa (d+2)^{d/2+1}} \left(\frac{z}{r}\right)^{1+2/d} \Gamma\left(\frac{d}{2} + 1, \frac{d}{2} + 1\right) \log \left(\frac{H_1(r/z)}{H_1(r/z)^{\wedge t}}\right) \lambda(dz) \tag{5.9}
\]
\[
= r^{-(1+2/d)} \frac{d^{d/2}}{\pi \kappa (d+2)^{d/2+1}} \Gamma\left(\frac{d}{2} + 1\right) \int_0^\infty e^{-u u^{d/2}} M_{1+2/d}(D e^{u d/(d+2)}) \, du,
\]
proving the exact formula for \(\eta(r)\).

**Proof of Lemma 2.** (i) If \(m_{1+2/d}(\lambda) < \infty\), the asymptotic result follows immediately from (2.2).

(ii) Integration by parts gives for any \(\gamma > 0\) that
\[
M_\gamma(r) = \int_{[0,r]} z^{\gamma} \lambda(dz) = \int_{0}^{r} \gamma u^{\gamma-1} \lambda(u) \, du - r^{\gamma} \lambda(r). \tag{5.10}
\]
Thus, as \(r \to \infty\), we have by Karamata’s theorem (see [3, Prop. 1.5.8 and 1.5.9a]) that for \(\gamma > \alpha\),
\[
M_\gamma(r) \sim r^{\gamma} \lambda(r) \frac{\alpha}{\gamma - \alpha} = r^{\gamma-\alpha} \ell(r) \frac{\alpha}{\gamma - \alpha}, \tag{5.11}
\]
while for \(\gamma = \alpha\),
\[
M_\alpha(r) \sim \alpha \int_1^r \ell(y) y^{-1} \, dy = \alpha L(r), \tag{5.12}
\]
where \(L\) is slowly varying and \(L(r)/\ell(r) \to \infty\) as \(r \to \infty\).
By \((5.11)\) with \(\gamma = 1 + \frac{2}{d}\) (or \((5.12)\) for \(\alpha = 1 + \frac{2}{d}\)) and properties of slowly varying functions,
\[
M_{1+2/d}(r) \int_0^\infty e^{-u} u^{d/2} \frac{M_{1+2/d}(D r e^{u/d} (d+2))}{M_{1+2/d}(r)} du \sim M_{1+2/d}(r) \int_0^\infty e^{-u} u^{d/2} (D r e^{u/d} (d+2))^{1+2/d-\alpha} du
\]
\[
= M_{1+2/d}(r) D^{1+2/d-\alpha} \left( \frac{d+2}{\alpha d} \right)^{d/2+1} \Gamma \left( \frac{d}{2} + 1 \right),
\]
where the use of Lebesgue’s dominated convergence theorem is justified by Potter’s bounds.

(iii) For \(\alpha = 0\) the truncated moment \(M_{1+2/d}\) is not necessarily regularly varying, therefore more care is needed. First we analyze \(L_0\), which is finite by \((1.7)\) and satisfies, for any large \(K\),
\[
L_0(r) \geq \int_1^K \ell(ry) y^{-1} (\log y)^{d/2-1} dy \sim \ell(r) \int_1^K y^{-1} (\log y)^{d/2-1} dy.
\]
Since the latter integral goes to infinity as \(K \to \infty\), we obtain that \(L_0(r) / \ell(r) \to \infty\) as \(r \to \infty\). Next, for \(a > 1\),
\[
L_0(ar) = \int_a^\infty \ell(ry) y^{-1} (\log y/a)^{d/2-1} dy,
\]
thus
\[
L_0(r) - L_0(ar) = \int_1^a \ell(ry) y^{-1} (\log y)^{d/2-1} dy + \int_a^\infty \ell(ry) y^{-1} ((\log y)^{d/2-1} - (\log y/a)^{d/2-1}) dy,
\]
which implies
\[
\lim_{r \to \infty} \frac{L_0(r) - L_0(ar)}{L_0(r)} = 0,
\]
that is, \(L_0(r)\) is indeed slowly varying. Furthermore, for any \(a > 1\),
\[
L_0(r) \sim \int_a^\infty \ell(ry) y^{-1} (\log y)^{d/2-1} dy, \quad r \to \infty. \tag{5.13}
\]

Next we turn to \(\eta(r)\). Changing variables \(y = D r e^{u/d} (d+2)\) in \((2.2)\), we obtain
\[
\eta(r) = \frac{D^{1+2/d}}{d \pi \kappa \Gamma \left( \frac{d}{2} + 1 \right)} \int_D^{\infty} y^{-2-2/d} \left( \log \frac{y}{D r} \right)^{d/2} M_{1+2/d}(y) dy.
\]
By Fubini’s theorem,
\[
\int_0^\infty y^{-2-2/d} (\log y/r)^{d/2} M_{1+2/d}(y) dy = \int_{(0,r]} z^{1+2/d} \lambda(dz) r^{-1-2/d} \int_1^\infty u^{-2-2/d} (\log u)^{d/2} du
\]
\[
+ \int_{(r,\infty)} z^{1+2/d} r^{-1-2/d} \int_{z/r}^\infty u^{-2-2/d} (\log u)^{d/2} du \lambda(dz)
\]
\[
= r^{-1-2/d} \int_{(0,\infty)} z^{1+2/d} f(1 \vee z/r) \lambda(dz),
\]
13
where \( a \land b = \max\{a, b\} \) and \( f(y) = \int_y^\infty u^{-2-2/d}(\log u)^{d/2} \, du \). Using the fundamental theorem of calculus to write \( z^{1+2/d} f(1 \land z/r) \) as an integral, exchanging the two resulting integrals by Fubini’s theorem, and changing variables \( y = z/r \), we obtain

\[
\begin{align*}
  r^{-1-2/d} \int_{(0, \infty)} z^{1+2/d} f(1 \land z/r) \lambda(dz) \\
  = \int_0^\infty \int_0^1 \lambda(ry) \left[ (1 + \frac{2}{d}) y^{2/d} f(1 \land y) - \mathbbm{1}(y > 1)(\log y)^{d/2} y^{-1} \right] \, dy \\
  = \int_0^1 \int_0^\infty \lambda(ry)(1 + \frac{2}{d}) y^{2/d} f(1) \, dy \\
  + \int_1^\infty \int_y^\infty \lambda(ry)(1 + \frac{2}{d}) y^{2/d} f(1) \, dy \\
  \quad \times \int_{(0, \infty)} u^{-2-2/d} \left[ (\log u)^{d/2} - (\log y)^{d/2} \right] \, du \, dy.
\end{align*}
\]

Using that

\[
(1 + \frac{2}{d}) y^{2/d} \int_y^\infty u^{-2-2/d} \left[ (\log u)^{d/2} - (\log y)^{d/2} \right] \, du = \frac{4}{d} y^{2/d} \int_y^\infty v^{-2-2/d} (\log v)^{d/2-1} \, dv,
\]

we end up with

\[
\eta(r/D) = \frac{D^{1+2/d}}{d \pi \kappa \Gamma(\frac{d}{2} + 1)} \left[ \int_0^1 \lambda(ry)(1 + \frac{2}{d}) y^{2/d} f(1) \, dy \\
  + \int_1^\infty \lambda(ry) \frac{4}{d} y^{2/d} \int_y^\infty v^{-2-2/d} (\log v)^{d/2-1} \, dv \, dy \right].
\]  

(5.14)

As \( y \to \infty \),

\[
y^{2/d} \int_y^\infty v^{-2-2/d} (\log v)^{d/2-1} \, dv \sim (1 + \frac{2}{d})^{-1} y^{-1}(\log y)^{d/2-1},
\]

so for \( K \) large enough,

\[
\int_K^\infty \lambda(ry)y^{2/d} \int_y^\infty v^{-2-2/d} (\log v)^{d/2-1} \, dv \, dy \sim (1 + \frac{2}{d})^{-1} \int_K^\infty \lambda(ry)y^{-1}(\log y)^{d/2-1} \, dy
\]

\[
\sim (1 + \frac{2}{d})^{-1} L_0(r),
\]

where the last asymptotic follows from (5.13). Since \( \ell(r)/L_0(r) \to 0 \) as \( r \to \infty \),

\[
\eta(r/D) \sim \frac{D^{1+2/d}}{2 \pi \kappa \Gamma(\frac{d}{2} + 1)(1 + \frac{2}{d})} L_0(r),
\]

as claimed.

Proof of Theorem 2. (i) Starting from the first line of (5.9), we can also write \( \eta(r) \) as

\[
\begin{align*}
  \eta(r) &= v_d(\kappa d)^{d/2} \int_0^t s^{d/2} \int_{(2 \pi \kappa s)^{d/2}(r, \infty)} \left( \log \frac{z^{2/d}}{2 \pi \kappa s r^{2/d}} \right)^{d/2} \lambda(dz) \, ds \\
  &= \frac{(2t)^{1+d/2} \kappa^{d/2} v_d}{d} r^{-1-2/d} \int_0^r v^{2/d} \int_{(Dv, \infty)} \left( \log \frac{z}{Dv} \right)^{d/2} \lambda(dz) \, dv,
\end{align*}
\]  

(5.15)
where we changed variables \( v = (s/t)^{d/2} r \) to go from the first to the second line. By the fundamental theorem of calculus we have \( \log \overline{\eta}(r) = C + \int_1^r \xi(v)v^{-1} \, dv \) with

\[
C = \log \left( \frac{2t}{1 + d/2} \right)^{d/2} v^{d/2} + \log \int_0^1 v^{2/d} \int_{(Dv, \infty)} \left( \log \frac{z}{Dv} \right)^{d/2} \lambda(\,d z) \, dv,
\]

\[
\xi(v) = \frac{v^{1 + 2/d} \int_{(Du, \infty)} (\log \frac{z}{Du})^{d/2} \lambda(\,d z)}{\int_0^v u^{2/d} \int_{(Du, \infty)} (\log \frac{z}{Du})^{d/2} \lambda(\,d z) \, du} - (1 + \frac{2}{d}).
\]

Since \( u \mapsto \int_{(Du, \infty)} (\log \frac{z}{Du})^{d/2} \lambda(\,d z) \) is decreasing in \( u \), we have \(-(1 + \frac{2}{d}) \leq \xi(v) \leq 0\). The claim now follows from \[3\] Thm. 2.2.6.

(ii) By (i) and \[3\] Thm. 2.0.7, \( \overline{\eta} \) has dominated variation \[3\] p. 54 and \( \overline{\eta}(r+s)/\overline{\eta}(r) \to 1 \) as \( r \to \infty \) for any \( s > 0 \). Hence \( \overline{\eta} \) is subexponential \[15\] Thm. 1 and \[21\] Thm. 3.1 (see also \[22\] Thm. 5.1).

(iii) By the representation theorem of regularly varying functions \( \overline{\eta} \in \mathcal{RV}_{-\alpha} \) iff \( \lim_{r \to \infty} \xi(r) = 1 + \frac{2}{d} - \alpha \). By Karamata’s theorem (\[3\] Thm. 1.6.1) this holds for \( \alpha < 1 + \frac{2}{d} \) iff \( f \in \mathcal{RV}_{-\alpha} \), where

\[
f(r) = \int_{(r, \infty)} \left( \log \frac{z}{r} \right)^{d/2} \lambda(\,d z) = \frac{d}{2} \int_r^\infty \overline{\lambda}(z) \left( \log \frac{z}{r} \right)^{d/2} \, dz.
\]

Consider the kernel \( k(u) = (\log u^{-1})^{d/2-1} \mathbb{1}_{(0,1)}(u) \). Define the Mellin convolution of \( f_1 \) and \( f_2 \) by

\[
f_1 \star^M f_2(r) = \int_0^\infty f_1(r/t)f_2(t)t^{-1} \, dt,
\]

see e.g. \[3\] Sect. 4. With this notation \( f(r) = k \star^M \overline{\lambda}(r) \). The Mellin transform of \( k \), that is,

\[
\tilde{k}(z) = \int_0^1 t^{-z-1}(\log t)^{d/2-1} \, dt = i^d \sqrt{\pi}^{-d} \Gamma(d/2),
\]

is defined and nonzero whenever \( \Re z < 0 \) and \( \sqrt{\pi} \) is chosen such that \( \arg(\sqrt{\pi}) \in (\frac{1}{2} \pi, \frac{3}{2} \pi) \). Therefore, we can apply \[3\] Thm. 4.8.3. (It is easy to check that the Tauberian condition is satisfied since \( \overline{\lambda} \) is decreasing; see also \[3\] Exercise 1.11.14.) Therefore, \( k \star^M \overline{\lambda} \in \mathcal{RV}_{-\alpha} \) implies that \( \overline{\lambda} \in \mathcal{RV}_{-\alpha} \), as claimed. The other direction was proved in Lemma \[2\].

If \( \alpha = 1 + \frac{2}{d} \), then \( \lim_{r \to \infty} \xi(r) = 0 \) iff

\[
\frac{r^{1+2/d}f(r)}{\int_0^r u^{2/d}f(u) \, du} \to 0,
\]

which holds iff \( \int_0^r u^{2/d}f(u) \, du \) is slowly varying, see \[3\] Thm. 8.3.1 or \[16\] Thm. 1.1. Using the first identity in \[5\] we can easily verify that the latter holds if \( m_{1+2/d}(\lambda) < \infty \).

### 5.3 Proofs for Section 3

**Proof of Theorem 3** Without loss of generality, assume \( \kappa = \frac{1}{2\pi} \). If \( d = 1 \), then by \[19\] Thm. 2.7 \( X_A(t) \) exists if

\[
\iiint 1(s^{-d/2}z > 1, y \in \overline{A}) \, ds \, dy \lambda(\,d z) = |\overline{A}| \int_{(0, \infty)} (z^2 \wedge t) \lambda(\,d z) < \infty,
\]

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which is true for any Lévy measure. If \( d \geq 2 \), \( X_A(t) \) exists iff

\[
\int \int \int (1 \wedge s^{-d/2} z \mathbb{1}(y \in A)) \, ds \, dy \, \lambda(dz) = |A| \int_0^t \int_{(0,\infty)} (1 \wedge s^{-d/2} z) \, ds \, \lambda(dz) < \infty.
\]

For \( z \leq t^{d/2} \), we have \( \int_0^t (1 \wedge s^{-d/2} z) \, ds = z^{2/d} + z \int_{s^{d/2}}^{t^d} s^{-d/2} \, ds \), while for \( z > t^{d/2} \), we have \( \int_0^t (1 \wedge s^{-d/2} z) \, ds = t \). Thus,

\[
\int_0^t \int_{(0,\infty)} (1 \wedge s^{-d/2} z) \, ds \, \lambda(dz) = \int_{[0,t^{d/2}]} z^{2/d} \lambda(dz) + \int_{[t^{d/2},t]} z \int_{s^{d/2}}^{t^d} s^{-d/2} \, ds \, \lambda(dz) + t \bar{\lambda}(t^{d/2}),
\]

which is finite iff \( \text{(3.3)} \) holds. The identity \( \text{(3.4)} \) follows from [19] Thm. 2.7 (ii). \( \square \)

**Proof of Lemma 4.** (i) is an immediate consequence of \( \text{(3.5)} \). (ii) follows from \( \text{(3.5)} \) combined with Karamata’s theorem. \( \square \)

**Proof of Theorem 4.** Recall \( \text{(3.7)} \). Then, as in Theorem 2 claims (i) and (ii) follow by writing

\[
\log \pi(r) = C + \int_0^1 \xi(u) u^{-1} \, du
\]

where \( \xi \) satisfies \( \frac{-2}{3} < \xi(u) \leq 0 \). For (iii), we have as in Theorem 2 that \( \pi \in \mathcal{RV}_\alpha \) iff \( \lim_{r \to \infty} \xi(r) = 2/d - \alpha \), which for \( \alpha < \frac{2}{3} \) holds iff \( \bar{\lambda} \in \mathcal{RV}_\alpha \), as claimed. If \( \alpha = \frac{2}{3} \) then using [3] Thm. 8.3.1 \( \lim_{r \to \infty} \xi(r) = 0 \) iff \( \int_0^1 u^{2/d-1} \lambda(u) \, du \) is slowly varying. This holds if \( m_{2/d}(\lambda) < \infty \) or \( \bar{\lambda} \in \mathcal{RV}_{2/d-\frac{2}{3}} \).

**Proof of Lemma 4.** (i) If \( m_{2/d}(\lambda) < \infty \), choose \( \varepsilon > 0 \) and observe that for \( r > 1 \),

\[
\pi_A(r) \leq \int \int \mathbb{1}((2\pi \kappa s)^{-d/2} z > r) \mathbb{1}_{\{y \in A^\varepsilon\}} \, ds \, dy \, \lambda(dz) + \int \int \mathbb{1}((2\pi \kappa s)^{-d/2} e^{-\frac{\text{dist}(y,A)^2}{2\varepsilon s}} z > r) \mathbb{1}_{\{y \notin A^\varepsilon\}} \, ds \, dy \, \lambda(dz)
\]

\[
\leq |A^\varepsilon| \pi(r) + \int \int \mathbb{1}((2\pi \kappa s)^{-d/2} e^{-\frac{\text{dist}(y,A)^2}{2\varepsilon s}} z > r) \mathbb{1}_{\{y \notin A^\varepsilon\}} \, ds \, dy \, \lambda(dz).
\]

where \( A^\varepsilon = \{ x \in \mathbb{R}^d : \text{dist}(x,A) < \varepsilon \} \). Since \( \int \int ((2\pi \kappa s)^{-1} e^{-\frac{\text{dist}(y,A)^2}{2\varepsilon s}} \mathbb{1}_{\{y \notin A^\varepsilon\}})^{2/d} \, ds \, dy \, \lambda(dz) < \infty \), the last term in the previous display is \( o(r^{-2/d}) \), which together with Lemma 3 (i) shows that \( \limsup_{r \to \infty} \pi_A(r)/\pi(r) \leq |A^\varepsilon| \), which converges to \( |A| \) by letting \( \varepsilon \to 0 \). The opposite relation follows from the fact that

\[
\pi_A(r) \geq \int \int \mathbb{1}((2\pi \kappa s)^{-d/2} z > r) \mathbb{1}_{\{y \in A\}} \, ds \, dy \, \lambda(dz) = |A| \pi(r).
\]

If \( \bar{\lambda}(r) = r^{-2/d} \ell(r) \), one can use Potter’s bounds, dominated convergence and Lemma 3 (ii) to show that the last integral in \( \text{(5.17)} \) is

\[
\sim r^{-2/d} \ell(r) \int \int (2\pi \kappa s)^{-1} e^{-\frac{\text{dist}(y,A)^2}{2\varepsilon s}} \mathbb{1}_{\{y \notin A^\varepsilon\}} \, ds \, dy = o(\pi(r)).
\]
The remaining proof is the same as in the case \( m_{2/d}(\lambda) < \infty \).

(ii) If \( \lambda(r) = r^{-\alpha} \ell(r) \) for some \( \alpha \in (0, \frac{2}{d}) \), a direct calculation shows that for \( r > 1 \),

\[
\eta_A(r) = \iint (2\pi s)^{-d/2} e^{-\frac{d\text{dist}(y, A)^2}{2s}} \, ds \, dy \lambda(dz)
\sim r^{-\alpha} \ell(r) \int_0^r \iint (2\pi s)^{-d/2} e^{-\frac{d\text{dist}(y, A)^2}{2s}} \, ds \, dy.
\]

\( \square \)

**Proof of Theorem**

Note that for \( d \geq 2 \) condition (3.3) implies summable jumps, in which case we assume that \( Y(t, x) \) has the form (1.11). For \( d = 1 \), note that \( Y(t, x) = Y_d(t, x) + Y_s(t, x) + Y'_s(t, x) \), where

\[
Y_d(t, x) = mt + \iint g(t - s, x - y)z\{ (2\pi s)^{-1/2} z < 1 \} - \{ z > 1 \} \, ds \, dy \lambda(dz),
\]

\[
Y_s(t, x) = \iint g(t - s, x - y)z\{ (2\pi s)^{-1/2} z < 1 \} (\mu - \nu) (ds, dy, dz),
\]

\[
Y'_s(t, x) = \iint g(t - s, x - y)z\{ (2\pi s)^{-1/2} z < 1 \} \mu (ds, dy, dz).
\]

A straightforward computation shows that \( Y_d(t, x) < \infty \) for all Lévy measures \( \lambda \). Furthermore, by (3.11) and the proof of [20, Thm. 3.8] one can show that

\[
\mathbb{P}\left( \sup_{x \in A} |Y'_s(t, x)| < \infty \right) = 1.
\]

(5.19)

For completeness, we sketch the proof. We use [20, Thm. 1] (with \( \alpha = p = 2 \)) and Minkowski’s integral inequality to obtain

\[
\mathbb{E}[|Y'_s(t, x) - Y'_s(t, x')|^2] \leq C \iint |g(t - s, x - y) - g(t - s, x' - y)|^2 z^2 \{ (2\pi s)^{-1/2} z < 1 \} \nu (ds, dy, dz)
\]

for all \( x, x' \in \mathbb{R} \). We have on the set \( (2\pi s)^{-1/2} z < 1 \)

\[
|g(t - s, x - y) - g(t - s, x' - y)|^2 z^2 = C(t - s)^{-1} z^2 e^{-\frac{|x - y|^2}{2(t - s)}} e^{-\frac{|x' - y|^2}{2(t - s)}} \leq C|g((t - s), x - y) - g((t - s), x' - y)|^q z^2,
\]

where \( q \) is the exponent from (3.10) (which satisfies \( q < 2 \)). With this estimate and again [26, Lemme A2], we conclude that

\[
\mathbb{E}[|Y'_s(t, x) - Y'_s(t, x')|^2] \leq C|x - x'|^{3-q}.
\]

Since \( 3 - q > 1 \), it follows from [17, Thm. 4.3] that

\[
\mathbb{E}\left[ \sup_{x \in A} Y'_s(t, x)^2 \right] \leq \mathbb{E}[Y'_s(t, 0)^2] + \mathbb{E}\left[ \sup_{x, x' \in A} |Y'_s(t, x) - Y'_s(t, x')|^2 \right] < \infty,
\]

which shows (5.19).
Next, choose \( r > 0 \) such that \( A \subseteq B(r) = \{ x \in \mathbb{R}^d : |x| \leq r \} \). Then
\[
\sup_{x \in A} Y'_d(t, x) \leq \int \int \int (2\pi \kappa(t - s))^{-d/2} z \mathbb{1}_{\{ y \in B(r) \}} \mathbb{1}_{\{ d \geq 2 \text{ or } (2\pi \kappa(t - s))^{-1/2} z > 1 \}} \mu(ds, dy, dz) \\
+ \int \int \int (2\pi \kappa(t - s))^{-d/2} e^{-\frac{\text{dist}(y, B(r))^2}{2(2\pi \kappa(t - s))}} z \mathbb{1}_{\{ y \in B(r) \}} \mathbb{1}_{\{ d \geq 2 \text{ or } (2\pi \kappa(t - s))^{-1/2} z > 1 \}} \mu(ds, dy, dz).
\]

The first term on the right-hand side is simply \( X_{B(r)}(t) \), which is finite a.s. by Theorem 3.3. The second term has the same distribution as \( Y'_d(t, 0) \), which is finite a.s. as well. Therefore, \( \sup_{x \in A} Y(t, x) < \infty \) a.s. for all \( d \). The assertion of the theorem now follows from Lemma 4 (which implies that \( \pi_A \) is subexponential under the stated assumptions) and [25, Thm. 3.1].

### 5.4 Proofs for Section 4

For \( 0 < r < r' \) let \( B(r, r') = \{ x \in \mathbb{R}^d : r < |x| \leq r' \} \).

**Proof of Theorem 4.** First assume that (4.5) converges and let \( K > 0 \). We start with \( d \geq 2 \). Since \( B(n, n + 1) \) can be covered with \( O(n^{d-1}) \) many unit cubes and \( Y \) is stationary in space, Theorem 4 shows that
\[
P \left( \sup_{y \in B(n, n + 1)} Y(t, y) > \frac{f(n)}{K} \right) \leq C n^{d-1} P \left( \sup_{y \in [0,1]^d} Y(t, y) > \frac{f(n)}{K} \right) \\
\leq C n^{d-1} P \left( X_{[0,1]^d}(t) > \frac{f(n)}{K} \right) \\
\leq 2 C n^{d-1} \tau(f(n)/K) \leq C' n^{d-1} \tau(f(n)),
\]
which is summable by hypothesis. In the last inequality we also used that \( \tau \) is extended regularly varying. So by the first Borel–Cantelli lemma,
\[
\sup_{y \in B(n, n + 1)} Y(t, y) > \frac{f(n)}{K}
\]
only happens finitely often and hence
\[
\limsup_{x \to \infty} \frac{\sup_{|y| \leq x} Y(t, y)}{f(x)} \leq K^{-1}
\]
almost surely, proving the claim since \( K \) was arbitrary. If \( d = 1 \), recall the decomposition (5.18). We can apply (5.20) to \( Y'_d(t, x) \), while \( |Y'_d(t, x) + X'_d(t, x)| \) has a smaller tail by (5.19) (in \( d = 1 \), the tail of \( Y'_d(t, x) \) is no smaller than \( C r^{-2} \) by Lemma 3 (i)). Therefore, the final bound in (5.20) remains true.

For the converse statement, assume that the integral in (4.5) diverges. If \( d = 1 \), we consider again the decomposition (5.18). As before, we let \( Y'_d(t, x) = Y(t, x) \) if \( d \geq 2 \). For \( K > 0 \) large consider the events
\[
A_n = \{ \mu(\{(s, y, z) : (s, z) \in V(K f(n + 1)), y \in B(n, n + 1)\}) \geq 1 \}, \quad n \geq 1.
\]

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By \((4.2)\) and Theorem 4 (i),
\[
P(A_n) \sim v_d((n+1)^d - n^d)\tau(Kf(n+1)) \geq Cn^{d-1}\tau(f(n+1)).
\]
Since the integral in \((4.3)\) diverges, we have that \(\sum_{n=1}^{\infty} P(A_n) = \infty\). Noting that the \(A_n\)'s are independent, the second Borel–Cantelli lemma implies that \(A_n\) occurs infinitely often. On \(A_n\),
\[
\sup_{y \in B(n,n+1)} Y'_b(t,y) \geq Kf(n+1).
\]
Thus, almost surely,
\[
\limsup_{x \to \infty} \frac{\sup_{|y| \leq x} Y'_b(t,y)}{f(x)} \geq K,
\]
which proves the claim for \(d \geq 2\) since \(K > 0\) is arbitrarily large.

If \(d = 1\), note that the proof above shows that \(Y'_b(t,x)\) develops infinitely many peaks larger than \(x^{1/2}\) on \(B(x)\) (because \(\tau(r)\) decreases no faster than shown in Lemma 3 \(i)\)). So if we show that \(|Y'_b(t,x)| + |Y''_b(t,x)|\) from \((6.18)\) can only have finitely many peaks of that size, then the proof in \(d = 1\) will be complete. For \(|Y'_b(t,x)|\), this is trivial. For \(|Y''_b(t,x)|\), this is a simple consequence of \((5.19)\) and the arguments in the first part of the proof.

**Proof of Theorem 2.** The upper bound proof is essentially the same as for Theorem 6 except that \((5.20)\) should be replaced by

\[
P \left( \max_{y \in \mathbb{Z}^d, y \in B(0,n+1)} Y(t,y) > \frac{f(n)}{K} \right) \leq Cn^{d-1}P \left( Y(t,0) > \frac{f(n)}{K} \right)
\]
\[
\leq Cn^{d-1}\tau(f(n)/K) \leq C'n^{d-1}\tau(f(n)).
\]

For the lower bound proof, if \(d = 1\), we consider the decomposition \(Y(t,x) = At + Y''_b(t,x) + Y''_b(t,x)\), where \(A\) is the same constant as in Theorem 1 and

\[
Y''_b(t,x) = \iiint g(t-s,x-y)zI_{\{g(t-s,x-y)z \leq 1\}}(\mu - \nu)(ds,dy,dz),
\]
\[
Y''_b(t,x) = \iiint g(t-s,x-y)zI_{\{g(t-s,x-y)z > 1\}}(\mu)(ds,dy,dz).
\]

If \(d \geq 2\), we let \(Y''_b(t,x) = Y(t,x)\). Clearly, \(Y''_b(t,x) \geq Y_0(t,x)\) from \((4.6)\) and \(P(Y_0(t,x) > r) \sim \tau_0(r)\) similarly to Theorem 2. Because the \((Y_0(t,x))_{x \in \mathbb{Z}^d}\) are independent and

\[
\sum_{n=1}^{\infty} \sum_{y \in \mathbb{Z}^d \cap B(0,n+1)} P(Y_0(t,y) > Kf(n+1)) \geq C \sum_{n=1}^{\infty} n^{d-1}\tau_0(f(n+1)K) = \infty,
\]
the second Borel–Cantelli lemma shows that \(Y''_b(t,x)/f(x) \geq Y_0(t,x)/f(x) \geq K\) for infinitely many \(x \in \mathbb{Z}^d\). If \(d = 1\), then as in the proof of Theorem 4 one can show that the peaks of \(|Y''_b(t,x)|\) are of lower order.

\[\square\]
Proof of Lemma 5. Recall $H_1$ and $H_2$ from (5.1). For $r > 1$

$$\overline{\eta}_0(r) = \eta_0((r, \infty)) = \int_{(0, \infty)} H_1(r/z)^{s} \lambda(dz).$$

For fixed $u > 0$ the map $s \mapsto H_2(s, u)$ is increasing on $[0, (2\pi keu^{2/d})^{-1}]$, and decreasing on $[(2\pi keu^{2/d})^{-1}, H_1(u)]$, with global maximum $H_2((2\pi keu^{2/d})^{-1}, u) = \sqrt{d/(2\pi e)}u^{-1/d}$. In particular, $H_2(s, u) \leq \frac{1}{2}$ whenever $u \geq (2d/\pi e)^{d/2}$. Therefore, as in the proof of Lemma 1

$$\int_{0,(\pi e/(2d))^{d/2}}^{H_1(r/z)^{s}} \lambda(dz) \leq \frac{d^{d/2}}{\pi \Gamma(d + 1)} \int_{0,(2\pi e/d)^{d/2}}^{(z)^{d/2}} \Gamma\left(d/2 + 1, \log \frac{H_1(r/z)}{H_1(r/z)} \right) \lambda(dz) \geq c_1 r^{-1/2} M_{1+2/d}(c_2 r).$$

At the same time, if $u > 0$ is small enough, then

$$\int_{0}^{H_1(u)^{s}} \left(\frac{1}{2} \wedge H_2(s, u)^{d}\right) \lambda(dz) \geq t/3.$$

Thus there exists $c_3$ such that

$$\int_{(c_3 r, \infty)}^{H_1(r/z)^{s}} \lambda(dz) \geq c_4 \overline{\lambda}(c_3 r).$$

It follows that there are finite constants $c_1, c_2, C_1, C_2 > 0$ depending only on $d$ and $t$ such that

$$c_1 r^{-1/2} M_{1+2/d}(c_2 r) + c_1 \overline{\lambda}(c_2 r) \leq \overline{\eta}_0(r) \leq C_1 r^{-1/2} M_{1+2/d}(C_2 r) + C_1 \overline{\lambda}(C_2 r). \quad (5.22)$$

(The second inequality is an easy consequence of the first two displays in this proof.)

From (5.22) and Lemma 2 (i) we see that $\overline{\eta}_0(r) \asymp \overline{\eta}(r)$ whenever $m_{1+2/d}(\lambda) < \infty$. If (4.7) holds, using $\Gamma(d/2 + 1, r) \sim e^{-r} r^{d/2}$ as $r \to \infty$, we have

$$\int_{(r, \infty)} \left(\frac{z}{r}\right)^{d/2} \Gamma\left(d/2 + 1, (d/2 + 1) \log \frac{H_1(r/z)}{H_1(r/z)} \right) \lambda(dz) \leq C \int_{(r, \infty)} \log(z/r)^{d/2} \lambda(dz) = C \int_{(r, \infty)} \log(z/r)^{d/2-1} \lambda(z) z^{-1} dz \leq C \overline{\lambda}(r) \int_{(r, \infty)} \log(y)^{d/2-1} y^{-1-\delta} dy = C \overline{\lambda}(r),$$

which implies $\overline{\eta}_0(r) \asymp \overline{\eta}(r)$.

On the other hand, (ii) follows from Lemma 2 (iii) and (5.22).

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