Approximate Q-learning and SARSA(0) under the \(\epsilon\)-greedy Policy: a Differential Inclusion Analysis

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**Abstract**

Q-learning and SARSA(0) with linear function approximation, under \(\epsilon\)-greedy exploration, are leading methods to estimate the optimal policy in Reinforcement Learning (RL). It has been empirically known that the discontinuous nature of the greedy policies causes these algorithms to exhibit complex phenomena such as i.) instability, ii.) policy oscillation and chattering, iii.) multiple attractors, and iv.) worst policy convergence. However, the literature lacks a formal recipe to explain these behaviors and this has been a long-standing open problem (Sutton, 1999). Our work addresses this by building the necessary mathematical framework using stochastic recursive inclusions and Differential Inclusions (DIs). From this novel viewpoint, our main result states that these approximate algorithms asymptotically converge to suitable invariant sets of DIs instead of differential equations, as is common elsewhere in RL. Furthermore, the nature of these deterministic DIs completely governs the limiting behaviors of these algorithms.

1 Introduction

Algorithms in Reinforcement Learning (RL) control with function approximation and discontinuous action selection strategies are empirically known to exhibit complex phenomena such as i.) instability, ii.) policy oscillation and chattering, iii.) multiple attractors, and iv.) worst policy convergence (Bertsekas and Tsitsiklis, 1996; Bertsekas, 2011; De Farias and Van Roy, 2000; Young and Sutton, 2020). This holds true even in the simplest of cases such as Q-learning and SARSA(0) with linear function approximation and \(\epsilon\)-greedy exploration (Gordon, 1996, 2000). Accordingly, a mathematical framework to explain such behaviors has been a long standing open question (Sutton, 1999, Problem 1). In the related but simpler prediction problem, analyses have relied on the Stochastic Approximation (SA) and Ordinary Differential Equation (ODE) viewpoint. The same approach in control settings, however, has had limited success due to its heavy dependence on the continuity of the underlying vector field (Borkar, 2009, Section 2.1), (Benaïm, 1999, Section 4.1), (Benveniste et al., 2012, Section 1.5). Specifically, all results that apply to these settings require stringent smoothness properties on policy improvement (Perkins and Precup, 2002; Melo et al., 2008; Chen et al., 2019; Zou et al., 2019). In this work, we propose a novel framework based on Stochastic Recursive Inclusions (SRIs) and Differential Inclusions (DIs) to directly work with discontinuous driving functions.

To elaborate on the need for a novel framework, note that RL schemes have been viewed as SA algorithms so far, i.e., as update rules in \(\mathbb{R}^d\) having the form

\[
\theta_{n+1} = \theta_n + \alpha_n [h(\theta_n) + M_{n+1}], \quad n \geq 0, \tag{1}
\]

where \(h\) is a suitable driving function, \(\alpha_n\) is the stepsize, and \(M_{n+1}\) is noise. Further, when \(h\) is Lipschitz continuous, the ODE method has been used to show that the limiting dynamics of (1) is completely governed by the differential equation \(\dot{\theta}(t) = h(\theta(t))\) (Benaïm, 1999; Borkar, 2009).

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Approximate RL algorithms in general can also be cast as (1). However, if the resulting $h$ functions turn out to be discontinuous, as is the case with approximate Q-learning and SARSA(0) under \( \epsilon \)-greedy exploration, one is unable to apply the ODE method. Our proposed framework circumvents this by, in a sense, jointly considering all possible local update directions as a set and then viewing the resultant dynamics from a DI perspective. We remark that this has been a standard approach to handle a discontinuous driving function in classical control theory.

Formally, a DI (Aubin and Cellina [2012], Smirnov [2022]) is a relation of the form

\[
\dot{\theta}(t) \in h(\theta(t)),
\]

where $h$ is a set-valued map which associates with any point $\theta \in \mathbb{R}^d$ a set $h(\theta) \subseteq \mathbb{R}^d$. An ODE is a special case of a DI where $h(\theta)$ is a singleton. If $h$ is Marchaud, i.e., smooth in a set-valued sense (see Appendix A for the precise definition), the DI in (2) is guaranteed to have at least one continuous solution for every initial point (Borkar [2009], Lemma 5.3). However, unlike in ODEs, solutions of a DI need not be unique. An SRI (Benaim et al. [2005], Borkar [2009]), on the other hand, is an update rule in $\mathbb{R}^d$ of the form

\[
\theta_{n+1} = \theta_n + \alpha_n [y_n + M_{n+1}], \quad n \geq 0,
\]

where $y_n \in h(\theta_n) \subseteq \mathbb{R}^d$ for some set-valued function $h$, while $\alpha_n$ and $M_{n+1}$ are as before. Clearly, any SA scheme is also an SRI. If $h$ is Marchaud and satisfies some stability conditions and the sequences $(\alpha_n)$ and $(M_{n+1})$ satisfy the standard Robbins-Monro and martingale-difference conditions, respectively, then Borkar [2009] and Ramaswamy and Bhatnagar [2017] (see Theorem A.1 for the details) showed that the sequence $(\theta_n)$ is stable, i.e., almost surely bounded, and converges to the suitable invariant sets of the DI in (2).

Our proposed approach, a key contribution of this work, is to view approximate RL algorithms as SRIs and then appeal to the above result for understanding their limiting dynamics. Applying this perspective, our main result states that the asymptotic behaviors of linear Q-learning and SARSA(0) under \( \epsilon \)-greedy exploration are completely governed by their respective limiting DIs. The limiting deterministic DI thus provides the first pathway to systematically identify and explain the whole range of limiting phenomena that a general approximate method can exhibit, answering the question posed by Sutton (1999). As illustrations of this approach, we show in Section 3 how aspects of an algorithm’s limiting DI govern whether it possesses unique or multiple attractors, and whether it is susceptible to chattering in policy or parameter space. We are also able to identify a hitherto unseen ‘sliding mode’ phenomenon. Separately, we demonstrate that distinct algorithms for the same RL problem can have qualitatively different limiting DIs and, accordingly, different limiting behaviors.

Related Work: We provide here a brief survey of the literature on approximate RL algorithms.

Several works identify and report a variety of complex behaviors for approximate algorithms. These include the classic example of instability for Q-learning with linear function approximation (Baird, 1995) and the phenomenon of chattering in SARSA(0) (Gordon, 1996) and its approach to a bounded region (Gordon, 2000). De Farias and Van Roy (2000) show how approximate value iteration may not possess any fixed points, while Bertsekas (2011) argues that approximate policy iteration schemes may generally be prone to policy oscillations, chattering, and convergence to poor solutions. More recently, Young and Sutton (2020) show experimentally how a range of approximate RL algorithms involving value estimation and greedification exhibit pathological behaviors such as policy oscillation, multiple fixed points, and consistent convergence to the ‘worst’ policy. Our framework, along with illustrations in Section 3 helps explain many of these limiting phenomena in a rigorous fashion.

On the theoretical front, while global convergence results exist for Q-learning and SARSA for the tabular (no approximation) setting (see e.g., Bertsekas and Tsitsiklis, 1996), analysis with function approximation has been much more challenging. The difficulty in RL control primarily stems from the changing nature of policies (i.e., sampling distributions) which renders the iterations nonlinear (even with linear function approximation). One prominent stream of work focuses on extensions of linear SA theory to analyze nonlinear SA updates such as the ones in Q-learning with linear function approximation (Melo et al., 2008; Carvalho et al., 2020; Chen et al., 2019) and nonlinear (neural) function approximation (Fan et al., 2020; Xu and Gu, 2020). However, all these works hold the behavior policy fixed to mitigate the effect of changing sampling distributions. Another notable work is that of Lee and He (2020), which uses ideas from switched system theory to analyze nonlinear ODEs. This is perhaps similar in spirit to our endeavor as switched systems are also useful for analyzing discontinuous dynamics.
Another set of analyses apply SA techniques to study SARSA(0) with incrementally changing policies (Perkins and Preucil, 2002; Perkins and Pendrith, 2002; Melo et al., 2008; Zhang et al., 2022; Zou et al., 2019). However, a restrictive assumption used here is that the policy improvement operator is Lipschitz-continuous, e.g., softmax, which ensures the limiting ODE is ‘very smooth.’ Note that \( \epsilon \)-greedy policies do not fall in this class, which we are able to handle with the DI framework.

2 Setup and Main Result

We begin this section by recalling the RL problem of finding the optimal policy with linear function approximation. Thereafter, we discuss a generic learning scheme for solving this problem that includes as special cases the linear Q-learning and SARSA(0) algorithms with \( \epsilon \)-greedy exploration. Finally, we state our main result that describes the asymptotic behavior of this generic algorithm and, hence, of these approximate methods.

Let \( \Delta(U) \) denote the set of probability measures on a set \( U \). At the heart of RL, we have a Markov Decision Process (MDP) represented by the tuple \((S, A, \gamma, P, r)\), where \( S \) denotes a finite state space, \( A \) represents a finite action space equipped with a total order, \( \gamma \) is the discount factor, and \( P : S \times A \rightarrow \Delta(S) \) and \( r : S \times A \times S \rightarrow \mathbb{R} \) are deterministic functions such that \( P(s, a)(s') = P(s'|s, a) \) specifies the probability of moving from state \( s \) to \( s' \) under action \( a \), while \( r(s, a, s') \) is the one-step reward obtained in this transition. One of the main goals in RL control is to find the optimal \( Q \)-value function, associated with this MDP.

In practice, it is often the case that the state and action spaces are large. In such situations, to reduce the search space dimension, it is common to try and find an approximation of our generic algorithm at time \( n \), instead of \( Q_{\epsilon} \) itself. In this work, we focus on linear function approximation. That is, we presume we are given a feature matrix \( \Phi \in \mathbb{R}^{|S||A| \times d} \) and the goal is to find a \( \theta_{\epsilon} \in \mathbb{R}^d \) such that \( Q_{\epsilon} \approx \Phi \theta_{\epsilon} \).

Two algorithms to find such a \( \theta_{\epsilon} \) are linear Q-learning and SARSA(0) with \( \epsilon \)-greedy exploration. While these are widely used in practice, a rigorous understanding of their limiting behavior is missing in the literature and this is what our main result addresses. To enable a joint investigation, we first propose a generic learning scheme that includes the above two methods as special cases.

We need a few notations for stating this generic scheme. Let \( \phi^T(s, a) \), where \( T \) implies transpose, denote the \((s, a)\)-th row of \( \Phi \). Further, assume that \( \Phi \) and \( r \) satisfy the following condition.

\[ B_1 \] \( \Phi \) has full column rank. Further, there exist constants \( K_r, K_\phi \geq 0 \) such that \( \|\phi(s, a)\| \leq K_\phi \) and \( |r(s, a, s')| \leq K_r \) for all \( s, s' \in S \) and \( a \in A \).

Next, let \( \epsilon \in (0, 1] \) and \( \pi^\epsilon_n : S \rightarrow \Delta(A) \) the \( \epsilon \)-greedy policy at time \( n \geq 0 \). That is, if \( \theta_n \) is the estimate of our generic algorithm at time \( n \), then let

\[ \pi^\epsilon_n(a'|s) = \begin{cases} 1 - \epsilon(1 - 1/|A|) & \text{if } a' = \arg \max_a \phi^T(s, a)\theta_n, \\ \epsilon/|A| & \text{otherwise}. \end{cases} \]  

We presume that \( \arg \max_a \phi^T(s, a)\theta_n \) breaks ties between actions that have the same value of \( \phi^T(s, a)\theta_n \) using the total order on \( A \).

Since the state and action spaces are finite, the number of \( \epsilon \)-greedy policies is also finite. Suppose these policies satisfy the following condition.

\[ B_2 \] The Markov chains induced by each of the finitely many \( \epsilon \)-greedy policies are ergodic and, hence, have their own unique stationary distributions.

Finally, let \( \epsilon' \in [0, 1] \). Then, given some initial estimate \( \theta_0 \in \mathbb{R}^d \), the exact update rule of our generic algorithm at time \( n \geq 0 \) is

\[ \begin{align*}
\theta_{n+1} &= \theta_n + \alpha_n \delta_n \phi(s_n, a_n) \\
\delta_n &= r(s_n, a_n, s'_n) + \gamma \phi^T(s'_n, a'_n)\theta_n - \phi^T(s_n, a_n)\theta_n,
\end{align*} \]  

\[ (5) \]

\[ 1 \] Henceforth, we will use linear Q-learning (resp. linear SARSA(0)) to refer to Q-learning (resp. SARSA(0)) with linear function approximation.
where \( \alpha_n \) is the stepsize, \( s_n \) is the current state at time \( n \) which we presume is independently sampled from the stationary distribution corresponding to the Markov chain \( \pi_n \) induced by \( \pi_{n,t} \), \( \alpha_n \sim \pi_n(\cdot|s_n) \), while \( s'_n \sim \mathbb{P}(\cdot|s_n, a_n) \) and \( a'_n \sim \pi_n(\cdot|s'_n) \). Clearly, \( (9) \) with \( \epsilon' = 0 \) and \( \epsilon' = \epsilon \) corresponds to the linear Q-learning and SARSA(0) algorithms with \( \epsilon \)-greedy exploration, respectively.

With regards to the stepsize sequence \( (\alpha_n)_{n \geq 0} \), we presume that it satisfies the standard Robbins-Monro condition, i.e.,

\[
B_3: \quad \sum_{n \geq 0} \alpha_n = \infty \quad \text{and} \quad \sum_{n \geq 0} \alpha_n^2 < \infty.
\]

We need a few more notations before we can state our main result. For \( a \equiv (a(s))_s \in \mathcal{A}^S \), let the associated greedy region \( \mathcal{P}_a := \{ \theta \in \mathbb{R}^d : \forall s \in \mathcal{S}, a(s) = \arg \max_{a} \phi^{T}(s, a) \theta \} \), where \( \arg \max \) breaks ties using the total order. In other words, at each \( \theta \in \mathcal{P}_a \), \( a \) is the greedy policy corresponding to the Q-value function \( \Phi \theta \). Indeed, for some \( a \), it is possible that \( \mathcal{P}_a = \emptyset \). Nonetheless, \( \{ \mathcal{P}_a : a \in \mathcal{A}^S \} \) partitions \( \mathbb{R}^d \), i.e., for any \( \theta \in \mathbb{R}^d \), there exists a unique \( a \) such that \( \theta \in \mathcal{P}_a \).

Separately, let \( \pi_n^* \) (resp. \( \pi_n' \)) be the \( \epsilon \)-greedy (resp. \( \epsilon' \)-greedy) policy associated with the policy \( a \). It is easy to see that \( \pi_n^* = \pi_n' \) and \( \pi_n' = \pi_n'' \) whenever \( \theta_n \in \mathcal{P}_a \).

Next, let \( D_n \) denote the stationary distribution associated with the Markov chain induced by \( \pi_n \). Further, let

\[
b_n = \mathbb{E}[\phi(s_n, a_n)r(s_n, a_n, s'_n)] = \Phi^TD_n r
\]

and

\[
A_n = \mathbb{E}[\phi(s_n, a_n)\phi^T(s_n, a_n) - \gamma\phi(s_n, a_n)\phi^T(s'_n, a'_n)] = \Phi^TD_n'(\mathbb{I} - \gamma P_n')\Phi,
\]

where \( D_n \) is the diagonal matrix of size \( |\mathcal{S}||\mathcal{A}| \times |\mathcal{S}||\mathcal{A}| \) whose \((s, a)\)-th diagonal entry is \( \delta_n(s, a)\pi_n(a|s) \), \( r \) is the \( |\mathcal{S}||\mathcal{A}| \)-dimensional vector whose \((s, a)\)-th coordinate is \( r(s, a) = \sum_{s' \in \mathcal{S}} \mathbb{P}(s'|s, a)r(s, a, s') \), while \( P_n' \) is the matrix of size \( |\mathcal{S}||\mathcal{A}| \times |\mathcal{S}||\mathcal{A}| \) such that \( P_n'(s, a) = \Phi(s'|s, a)\pi_n(a'|s') \). Assume that the matrices \( A_n, a \in \mathcal{A}^S \), satisfy the following condition.

\[
B_4: \quad \text{For all } a \in \mathcal{A}^S, \text{ } A_n \text{ is positive definite, i.e., } \theta^T A_n \theta > 0 \forall \theta \in \mathbb{R}^d.
\]

Finally, for \( \theta \in \mathbb{R}^d \), let \( f : \mathbb{R}^d \rightarrow \mathbb{R}^d \) and the set-valued \( h : \mathbb{R}^d \rightarrow 2^{\mathbb{R}^d} \) be given by

\[
f(\theta) := \sum_{a \in \mathcal{A}^S} (b_n - A_n \theta) \mathbb{I}[\theta \in \mathcal{P}_a] \quad \text{and} \quad h(\theta) = \bigcap_{\delta > 0} \mathbb{C}(f(B(\theta, \delta)))
\]

where \( \mathbb{I} \) denotes the indicator function, \( \mathbb{C} \) is the convex closure, and \( B(\theta, \delta) \) is the open ball of radius \( \delta \) centered at \( \theta \). As we show later in the proof outline for Theorem 2.1, \( f(\theta) \) denotes our generic algorithm’s expected update direction at time \( n \) if \( \theta_n = \theta \), while \( h(\theta) \) is the collection of all possible local update directions in the vicinity of \( \theta \). Now, our main result basically states that the limiting behavior of \( (8) \) is completely governed by the DI

\[
\hat{\theta}(t) \in h(\theta(t));
\]

the convex closure in the definition of \( h \) ensures the above DI admits continuous solutions.

In relation to \( (9) \), we will say a set \( \Gamma \subset \mathbb{R}^d \) is invariant if, for every \( x \in \Gamma \), there is some solution trajectory \( (\theta(t))_{t \in (-\infty, \infty)} \) of \( (9) \) with \( \theta(0) = x \) that lies entirely in \( \Gamma \). An invariant set \( \Gamma \) is additionally said to be internally chain transitive if it is compact and, for \( x, y \in \Gamma \), \( \epsilon > 0 \), and \( T > 0 \), there exist \( m \geq 1 \) and points \( x_0 = x, x_1, \ldots, x_{m-1}, x_m = y \) in \( \Gamma \) such that one of the solution trajectories of \( (9) \) initiated at \( x_i \) meets the \( \epsilon \)-neighborhood of \( x_{i+1} \) for \( 0 \leq i < m \) after a time that is equal or larger than \( T \). Such characterizations are useful to restrict the possible choices for limiting sets of an SRI.

**Theorem 2.1 (Main Theorem: SRI-DI connection).** Suppose \( B_1 \ldots B_4 \) hold. Then, almost surely, the iterates of \( (5) \) are stable, i.e., \( \|\pi_n\| < \infty \), and converge to a (possibly sample path dependent) closed connected internally chain transitive invariant set of the differential inclusion in \( (9) \).
Proof Outline. The statement follows by verifying all the technical assumptions of Theorem 2.1 originally proved in (Borkar, 2009) and (Ramaswamy and Bhatnagar, 2017), that discusses the stability and convergence of SRIs and their connections to DIs. The full details are given in the Appendix A.

In the discussion below, we only show how the generic algorithm in (5) can be viewed as a SRI whose limiting DI is (9).

For \( n \geq 0 \), let \( F_n := \sigma(\theta_0, s_0, a_0, s'_0, \ldots, s_{n-1}, a_{n-1}, s'_{n-1}, a'_{n-1}) \), \( y_n = \mathbb{E}[\delta_n \phi(s_n, a_n) | F_n] \), and \( M_{n+1} = \delta_n \langle \phi(s_n, a_n) - y_n \rangle \). Separately, let \( M_0 = 0 \). Since \( \theta_n \in F_n \), i.e., \( \theta_n \) is measurable with respect to \( F_n \), we have

\[
y_n = f(\theta_n) = \sum_{a \in A^s} (b_a - A_a \theta_n) \mathbb{I} [\theta_n \in \mathcal{P}_a] \in h(\theta_n). \tag{10}
\]

Accordingly, the update rule in (5) can now be rewritten as \( \theta_{n+1} = \theta_n + \alpha_n [y_n + M_{n+1}] \), \( n \geq 0 \), which is of the desired form.

Corollary 2.2 (Isolated Equilibria). If the only internally chain transitive invariant sets for the DI in (9) are isolated equilibrium points, then \( (\theta_n) \) almost surely converges to a (possibly sample path dependent) equilibrium point.

Remark 2.3. The almost sure convergence shows that the limiting DI in (9) captures all possible asymptotic behaviors of our generic scheme given in (5).

Remark 2.4. If the DI in (9) satisfies the condition in Corollary 2.2 then our generic scheme given in (5) will not exhibit parameter chattering.

Remark 2.5. Our formal proof for stability builds upon the result in (Ramaswamy and Bhatnagar, 2017). As in (Gordon, 2000), the positive definiteness of the \( A_a \) matrices (Assumption B3) is what enables this result. For SARSA(0), this assumption can be verified as in (Gordon, 2000).

3 Illustration via Numerical Examples

We illustrate in this section how our main result (Theorem 2.1) can be used to explain the limiting dynamics of the Q-learning and SARSA(0) algorithms with linear function approximation and \( \epsilon \)-greedy exploration. We study these algorithms applied to a range of 2-state 2-action MDPs, whose rewards, probability transition matrices and state-action features have been generated via random search (each MDP’s details, such as probability transition function, reward function, discount factor, and the algorithm’s exploration parameter \( \epsilon \), are provided in Appendix B). We shall see that even with 2-dimensional linear function approximation, the algorithms’ trajectories exhibit complex and non-trivial phenomena, and show how our results explain their causes via the associated DIs.

3.1 Case study: The range of dynamics for Q-learning with only 2 greedy policies

This section explores some representative MDPs which, under linear approximation of their action-value functions in \( \mathbb{R}^2 \), exhibit only 2 greedy policies but behave very differently under Q-learning with function approximation. Note that we take up only the Q-learning update rule in the interest of 2-dimensional linear function approximation, depending on its appropriate DI.

Two self-consistent greedy regions – Multiple attractors. The MDP and features \( \Phi \) for this section induce 2 greedy regions, i.e., \( \mathbb{R}^2 \) is partitioned into 2 greedy regions (see definition of \( \mathcal{P}_a \) in Section 2) \( \mathcal{P}_{a_1} \) and \( \mathcal{P}_{a_2} \) via greedy policies \( a_1 \) and \( a_2 \), respectively.

Let us call the point \( A_a^{-1} b_a \), a landmark for the partition \( \mathcal{P}_a \), \( i = 1, 2 \), where the \( A_a \) and \( b_a \) are as in (6), (7) (all matrices our examples will be positive-definite, hence invertible). The landmark \( A_a^{-1} b_a \) is the (unique) point to which the ODE \( \dot{\theta}(t) = b_a - A_a \theta \) would converge. Put another way, it is the point to which the Q-learning algorithm would converge were it to always sample actions \( a_n \) in (5) according to the \( \epsilon \)-greedy policy based on \( a_\epsilon \).

The greedy regions in this example are both self-consistent. By this, we mean that each contains its own landmark: \( A_a^{-1} b_a \in \mathcal{P}_a \), for \( i = 1, 2 \). The landmarks, along with their respective color-shaded greedy regions, are shown as diamonds in Fig. 1a. Note that since each greedy region must be a cone and there are only 2 regions, each must be a halfspace with the origin 0 on its boundary.
Fig. 1b shows two sample paths of the iterates $\theta_n$ of Q-learning (update (5) and $\epsilon' = 0$), with $\epsilon = 0.1$ greedy exploration, $\gamma = 0.75$ discount factor, decaying stepsizes $\alpha_n = \Theta(1/n)$ and an iteration count of 20,000, against the backdrop of the partition diagram. The starting positions, marked with black dots, were chosen arbitrarily. Note that segments of a trajectory are often parallel to each other, because the number of directions to move from an iterate is finite in a finite MDP.

Though the trajectories may traverse both greedy regions, they eventually approach one of the two landmarks (this observation remains true after several repeated trials). How can this be explained via our theoretical framework?

To answer this, let us look at the DI $\dot{\theta}(t) = h(\theta(t))$ induced by Q-learning for this MDP, obtained via (8). Specifically, $h(\theta)$ equals the singleton set $\{b_{a_1} - A_{a_1}\theta\}$ for $\theta$ lying in the interior of the blue region (say $\mathcal{P}_{a_1}$), the singleton set $\{b_{a_2} - A_{a_2}\theta\}$ in the interior of the red region (say $\mathcal{P}_{a_2}$), and the connected segment $c((b_{a_1} - A_{a_1}\theta, b_{a_2} - A_{a_2}\theta))$ whenever $\theta$ is on the (1-dimensional) boundary of the two regions. In this DI, each point in the interior of a greedy region suffers a drift ‘towards’ the region’s landmark. Points on the boundary, however, can be subjected to any drift in the convex hull of the drifts induced by each partition’s affine dynamics at that point.

Theorem 2.1 says that with probability 1, the iterates must converge to an invariant and internally chain transitive set of the DI. Recall that an invariant set $I$ means that $\forall \theta \in I$ there is some trajectory $\theta(t), -\infty < t < \infty$ lying entirely in $I$ and passing through $\theta$ at time 0. Roughly, for $I$ to be internally chain transitive, for any pair of points $\theta_{\text{start}}, \theta_{\text{end}} \in I$, a closeness tolerance $\eta$ and a travel time $T$, we must be able to find a solution of the DI and intermediate waypoints in $I$ such that the solution starts close to $\theta_{\text{start}}$, ends close to $\theta_{\text{end}}$, and reaches close to each successive waypoint after time $T$.

Examining this DI leads to the conclusion that it possesses only 3 invariant, internally chain transitive singleton sets – the 2 limiting dynamics points, plus a point $\theta^c$ on the boundary whose convex hull contains the 0 vector[5] – for the following reasons. These 3 singletons are certainly invariant and internally chain transitive sets, because for each there is a solution of the DI that always stays at that point for all $-\infty < t < \infty$ by virtue of zero drift. It also follows that there can be no other invariant set for the DI, because (a) any solution of the DI that, at any time, is in the interior of a region continues within that region towards its respective landmark, and (b) any solution that, at any time, is on the boundary but not at $\theta^c$ must suffer a drift that eventually pushes it into some greedy region’s interior or along the boundary to $\theta^c$. We do not give a formal proof that these are the only invariant, internally chain transitive sets of the DI; the reader is directed to standard references on DIs (Aubin and Cellina [2012], Smirnov [2022]) and discontinuous dynamical systems (Cortes [2008]) for rigorous arguments.

Thus, according to our result, the stochastic iterates of Q-learning must eventually approach one of these 3 isolated points. We additionally remark that the point $\theta^c$ on the boundary is an unstable equilibrium point, because any neighborhood around it contains points (in the interiors of $\mathcal{P}_{a_1}$ and $\mathcal{P}_{a_2}$) from where the DI’s solutions will escape towards the landmarks. If the martingale difference noise $(M_n)$ is sufficiently ‘rich’, i.e., it does not put all its probability mass in a subspace of $\mathbb{R}^2$, then random chance will cause the iterates to eventually escape this unstable equilibrium point. This condition will hold if all state-actions’ features span $\mathbb{R}^2$, which they indeed do.

Our analysis also brings out the fact that asymptotically (as $n \to \infty$), there are no policy oscillations (iterates jumping regions infinitely often) nor parameter oscillations (iterates that ‘bounce between’ two regions at a positive distance) that can occur, because the iterates must eventually stabilize around one of the two landmarks in the interior of the greedy regions.

**No self-consistent greedy regions – Sliding mode attractor.** We consider a different MDP and feature set that induce 2 greedy regions, each of which has its landmark in the other region. Two trajectories of Q-learning, along with the partition structure, are shown in Fig. 2a with the same parameters and stepsizes as before, and appear to move eventually towards the boundary between the two regions.

Reasoning about the DI and its solutions for this case paints a qualitatively different picture than that of the previous example. As before, for each point $\theta$ in the interior of any greedy region, there is a

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[5] The coordinates of this (2-dimensional) point can be analytically obtained by solving 2 equations expressing that: (1) $\theta^c$ lies on the boundary of the greedy regions, (2) the vectors $b_{a_1} - A_{a_1}\theta^c$ and $b_{a_2} - A_{a_2}\theta^c$ point in opposite directions.
We show in this section that Q-learning and SARSA(0), both with the same linear Q-function approximation architecture and ϵ-exploration rate, can exhibit qualitatively different limiting behavior. The cause for this differing behavior will be again shown to lie in the associated differential inclusions (DIs). Figures 3a and 3b each show two stochastic trajectories of Q-learning and SARSA(0) with linear function approximation, overlaid on the (identical) greedy region partition diagram for the underlying 2-state 2-action MDP, along with the (non-identical) landmark points for the greedy regions induced by each algorithm. Recall that as per [3], the only way these algorithms differ is in the fact that the second action $a^*_n$ is chosen at time $n$ from an $\epsilon'$-greedy policy, with $\epsilon' = 0$ for Q-learning and $\epsilon' = \epsilon$ for SARSA(0). We have also ensured that both algorithms induce stable matrices $A_n$, thus their iterates cannot diverge.

3.2 Case study: Disparate behavior for Q-learning and SARSA(0) with identical feature sets

To our knowledge, the existence of sliding modes and associated equilibria has not been previously demonstrated for approximate RL algorithms. Also, our main result (Theorem 2.1) guarantees that asymptotically, the stochastic iterates $\theta_n$ will not oscillate or chatter, although policy oscillations are possible (the iterates may cross the boundary infinitely often on their way to $\theta^*$). This does not contradict previous claims about (parameter) chattering being observed in Q-learning or SARSA(0) ([Young and Sutton, 2020] [Gordon, 1996]), because we work with decaying Robbins-Monro type stepsizes, whereas previous work considers constant $(\Omega(1))$ stepsize schedules. We also remark that although our main result tells us only about the asymptotic nature of the iterates, more can perhaps be teased out about the nature of the trajectories via the connection between SREs and DIs; in fact, we conjecture that the existence of a sliding mode solution in the DI must imply that policy chattering is bound to take place almost surely.

**Only one self-consistent greedy region – Unique attractor.** We consider an MDP that, with linear Q-function approximation in $d = 2$, induces two greedy regions with exactly one of them being self-consistent. The DI for this setting has the property that points in the red region will experience a (vector) drift, towards the boundary, which is bounded away from 0 in norm. As a result, any trajectory that starts in the red region will, in finite time, hit the boundary. The driving (set-valued) function $h(\theta)$ for each boundary point can be shown to have all its constituent drift vectors ‘pointing into’ the blue region, so any solution from the boundary must instantaneously leave it. Continuing from here, the trajectory moves towards the blue landmark, which can be shown to be the sole invariant, internally chain transitive set of the DI. It is thus clear, from Theorem 2.1 that all the original stochastic iterates of the algorithm must eventually approach this point. This is a situation in which no policy or parameter chattering occurs. Also, as observed by [Young and Sutton, 2020], it is possible that this (single) landmark point to which every trajectory converges may represent a very poor quality approximation for the optimal value function.

We discuss other examples such as the chattering phenomenon observed in [Gordon, 1996]; [Young and Sutton, 2020] in the Appendix.
We see that the configuration of landmark points for each partition’s dynamics in the DI is different in each case. This makes the green and red greedy regions self-consistent (i.e., they contain their own limiting dynamics) for Q-learning, whereas they are both non-self-consistent for the SARSA(0) DI. This disparity in structure explains why trajectories for Q-learning are seen to approach the (stable) red and green points, and those for SARSA(0) approach a ‘sliding mode’ attractor on the boundary of the red and green greedy regions.

An observation in a similar spirit as ours has been made previously, albeit at a high level, about approximate policy iteration by Bertsekas (2011), that “... the choice of the iterative policy evaluation method ... does not seem crucial for the quality of the final policy obtained (as long as the methods converge)”, leading to “... an open question whether (this notable failure of approximate policy
iteration) is due to the inherent difficulty of the tetris problem, or whether it can be attributed to inherent pathologies of approximate policy iteration, such as oscillations/chattering between relatively poor policies, or weaknesses of the cost function approximation philosophy. "Q-learning and SARSA(0) differ only in their approach to policy evaluation via the sampling of the action $a'_n$. However, our DI framework leads us to believe that the complex dynamical possibilities of these algorithms may primarily be a result of the rich vector fields or DI structures that they induce rather than the MDP (e.g., tetris) itself being difficult.

![Figure 3: Example of an MDP for which Q-learning and SARSA(0) algorithms induce qualitatively different limiting structure. (Left) Two sample trajectories of the iterates of Q-learning, with a $\Theta(1/n)$ stepsize schedule, are overlaid on the partition structure and landmark point structure induced by Q-learning. In the landmark point structure induced by Q-learning, there is no ‘sliding mode’ point on the boundary of the green and red partitions, thus the trajectories move towards the red landmark in this case. (Right) Two sample trajectories of the iterates of SARSA(0), with a $\Theta(1/n)$ stepsize schedule and the same initial positions as for Q-learning, are overlaid on the partition diagram of the MDP. In the landmark structure induced by SARSA(0), there is a ‘sliding mode’ equilibrium point on the boundary of the green and red regions towards which both trajectories move (see inset).]

4 Conclusion and Future Directions

We have introduced the novel toolset of SRIs and DIs to aid the understanding of algorithmic dynamics of approximate RL algorithms. The recipe is conceptually simple – for a stochastic iteration, write down its update in expectation, view it as a DI, and use insights from the DI to infer the limiting behavior of the original iteration. We believe that the primary value of this work is in this bridge that has been uncovered to this hitherto unexplored toolset in the context of RL algorithms. This toolset is already capable of providing principled explanations of a range of algorithmic dynamics that have been well observed and documented.

We may have merely scratched the surface with regard to wielding the DI approach to its fullest potential. Our framework paves the way for a more comprehensive understanding of existing approximate-RL algorithms as well as the design of new schemes with ‘well-behaved’ DIs, hopefully resulting in value beyond just analysis to the realm of synthesis.

On a somber note, the insights we have uncovered about algorithms using arguably the simplest possible (linear) function approximation casts doubt on their utility in more complicated, nonlinear approximation architectures. It is plausible that there are many more failure modes in these settings than the range observed in the linear case, as also seen by [Young and Sutton (2020)] in neural approximation. Even with linear function approximation, we have shown in this paper that (a) one could converge to multiple attractors with disparate qualities, (b) one could oscillate between multiple policies’ greedy regions, making it difficult to devise simple stopping criteria (c) even if one converges always to a unique point within a greedy partition, the induced greedy policy could be arbitrarily bad – these spell doom for the practitioner unless carefully addressed and investigated. Separately, we acknowledge that our theory addresses only linear function approximation and more work is needed to extend it to nonlinear approximation architectures.
Yet another promising direction for future research, which is not taken up in this paper, is that of the quality of the limiting iterates (assuming there is some reasonable form of convergence). Again, Young and Sutton (2020, ‘Worst-case example’) and Bertsekas (2011) point out that approximate policy iteration schemes could very well converge to the ‘worst possible outcome’, which, perhaps, with suitable design considerations based on the DI connection, can be avoided in new algorithms to come. For instance, a whole new class of algorithms can arise by assigning arbitrary values of $\epsilon, \epsilon'$ in the generic update [5] with favourable properties. On a slightly different but related note, our work also reinforces the fact that merely ensuring stability of an incremental RL algorithm’s iterates is by no means sufficient to guarantee good performance – discontinuous policy updates often swamp out gains from stability by inducing complex and varied convergence modes.

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A Proof of Main Result

Our proof primarily builds upon the results from [Borkar (2009)] and [Ramaswamy and Bhatnagar (2017)] that study the asymptotic behavior of an SRI using a DI lens. Our proof also needs a result from [Molchanov and Pyatnitskiy (1989)] that discusses conditions for the origin to be a Globally Exponentially Stable (GES) equilibrium point of a DI. We first state these results along with the necessary definitions and, thereafter, provide the complete proof of Theorem 2.1.

For \( \theta \in \mathbb{R}^d \) and a set-valued function \( h : \mathbb{R}^d \to 2^{\mathbb{R}^d} \), let

\[
   h_c(\theta) = \frac{1}{c} h(c\theta) := \{ y : cy \in h(\theta) \}, \quad c \geq 1,
\]

and \( h_\infty(\theta) := \text{cl}\{ y : \lim_{c \to \infty} d(y, h_c(\theta)) = 0 \} \), where \( d(y, Z) := \inf\{ \| y - z \| : z \in Z \} \) and \( \text{cl} \) denotes closure. Separately, in relation to a DI, call an invariant set \( \Gamma \) an attractor if it is compact and has a fundamental neighborhood \( U \). The latter condition means that, for any \( \epsilon > 0 \), there exists \( T(\epsilon) \geq 0 \) such that all solution trajectories \( (\theta(t))_{t \geq 0} \) of this DI with \( \theta(0) \in U \) satisfy \( \theta(t) \in N^\epsilon(\Gamma) \) \( \forall t \geq T(\epsilon) \), where \( N^\epsilon(\Gamma) \) is the (open) \( \epsilon \)-neighborhood of \( \Gamma \).

The following result provides sufficient conditions for the stability and convergence of an SRI.

**Theorem A.1** (Corollary 4, Chapter 5 in [Borkar (2009)] and Theorem 1 in [Ramaswamy and Bhatnagar (2017)].) Consider a generic SRI of the form given in (3) and suppose the following conditions hold.

**C1.** \( h \) is Marchaud, i.e.,

(a) \( h(\theta) \) is convex and compact for all \( \theta \in \mathbb{R}^d \);
(b) \( \exists K_h > 0 \) such that, for all \( \theta \in \mathbb{R}^d \), \( \sup_{y \in h(\theta)} \| y \| \leq K_h (1 + \| \theta \|) \); and
(c) \( h \) is upper semicontinuous or, equivalently, \( \{ (\theta, y) \in \mathbb{R}^d \times \mathbb{R}^d : y \in h(\theta) \} \) is closed.

**C2.** \( \sum_{n=0}^\infty \alpha_n = \infty \), but \( \sum_{n=0}^\infty \alpha_n^2 < \infty \).

**C3.** \( (M_n) \) is a square-integrable martingale difference sequence adapted to an increasing family of \( \sigma \)-fields \( (\mathcal{F}_n) \). Further, \( \exists K_m \geq 0 \) such that

\[
   \mathbb{E}[\| M_{n+1} \|^2 | \mathcal{F}_n ] \leq K_m [1 + \| \theta_n \|^2] \quad \text{a.s.,} \quad n \geq 0.
\]

**C4.** \( h_\infty(\theta) \) is non-empty for all \( \theta \in \mathbb{R}^d \). Further, the differential inclusion \( \dot{\theta}(t) \in h_\infty(\theta(t)) \) has an attractor \( \Gamma \subseteq B(0, 1) \) such that \( \text{cl}(B(0, 1)) \) is a subset of one of its fundamental neighborhoods. Lastly, if the sequences \( (c_n), (z_n), \) and \( (x_n) \) are such that \( c_n \uparrow \infty \), \( z_n \in h_{c_n}(x_n) \forall n \geq 0 \), \( x_n \to \theta \), and \( z_n \to y \), then \( y \in h_\infty(\theta) \).

Then, almost surely, the iterates \( (\theta_n) \) of the SRI are stable, i.e., \( \sup_{n \geq 0} \| \theta_n \| < \infty \), and converge to a (possibly sample path dependent) closed connected internally chain transitive invariant set of the differential inclusion \( \dot{\theta}(t) \in h(\theta(t)) \).

For verifying Condition \( C_4 \) in the above result, we need an additional stability result from [Molchanov and Pyatnitskiy (1989)].

**Theorem A.2** (Theorem 1 and (2) in [Molchanov and Pyatnitskiy (1989)].) Consider the generic DI

\[
   \dot{\theta}(t) = g(\theta), \quad g(\theta) = \text{co}\{ y : y = A\theta, A \in \mathcal{X} \}
\]

where \( \mathcal{X} \) is a compact set of \( d \times d \) matrices. Suppose \( \exists V : \mathbb{R}^d \to \mathbb{R} \) such that

**D1.** \( V \) is strictly convex.

**D2.** \( V \) is of quasiquadratic form, i.e., for all \( \theta \in \mathbb{R}^d \), \( V(\theta) = \theta^T L(\theta) \theta \) for some symmetric \( d \times d \) matrix \( L(\theta) \in \mathbb{R}^{d \times d} \). Further, \( V \) is homogeneous of second order, i.e., \( V(\lambda \theta) = \lambda^2 V(\theta) \forall \theta \in \mathbb{R}^d \) and \( \lambda \in \mathbb{R} \). (These imply \( L(\lambda \theta) = \lambda L(\theta) \).)

**D3.** There exists \( \beta > 0 \) such that \( \nabla V(\theta)^T y \leq -\beta \| \theta \|^2 \) for all \( \theta \in \mathbb{R}^d \) and \( y \in g(\theta) \).
Then, the origin is the Globally Exponentially Stable (GES) equilibrium point of (11). That is, any solution \( (\theta(t))_{t \geq t_0} \) of (11) satisfies
\[
\|\theta(t)\| \leq K_1 \|\theta(t_0)\| e^{-K_2(t-t_0)}, \quad t \geq t_0,
\]
for some constants \( K_1 \geq 1 \) and \( K_2 > 0 \) that are independent of \( t, t_0, \) and \( \theta(t_0) \).

**Proof of Theorem 2.7.** The desired result follows from Theorem 2.1 and, in our discussion below, we only verify the sufficient conditions stated there, beginning with \( C_1 \).

For each \( \theta \in \mathbb{R}^d \), \( h(\theta) \) defined in (8) is convex since it is an intersection of convex sets. Similarly, it is closed and bounded, from which we have that it is also compact. These statements jointly establish \((C_1.a)\). Next, from the definition of \( h \) and by (arbitrarily) choosing \( \delta = 1 \), it follows that
\[
\sup_{y \in h(\theta)} ||y|| \leq \sup_{y \in \mathcal{C}(f(B(\theta, 1)))} ||y||.
\]
Since the argument set on the RHS above is closed and convex, we also have
\[
\sup_{y \in \mathcal{C}(f(B(\theta, 1)))} ||y|| \leq \sup_{y \in f(B(\theta, 1))} ||y||
\]
which, in turn, implies
\[
\sup_{y \in \mathcal{C}(f(B(\theta, 1)))} ||y|| \leq \sup_{\theta \in B(\theta, 1)} \sup_{a \in \mathcal{A}^d} ||b_a - A_n \theta'|| \leq \sup_{\theta \in B(\theta, 1)} \sup_{a \in \mathcal{A}^d} (||b_a|| + ||A_n||)(1 + ||\theta||).
\]
From (9), (7) and (10) note that \( ||b_a|| \leq K_0 K_r \) and \( ||A_n|| \leq (\gamma + 1) K_0^2 \). Hence, \((C_1.b)\) is satisfied for \( K_h := K_0 (K_r + (\gamma + 1) K_0) \). It remains to establish the upper semicontinuity of \( h \). That is, for any sequences \( (x_n) \) and \( (z_n) \) such that \( x_n \to \theta \), \( z_n \to y \), and \( z_n \in h(x_n) \forall n \geq 0 \), we need to show that \( y \in h(\theta) \). To see this, let \( \delta > 0 \) be arbitrary. Then, \( \exists N_\delta \geq 0 \) such that \( x_n \in B(\theta, \delta) \) for all \( n \geq N_\delta \).

Further, for each such \( n \), since \( B(\theta, \delta) \) is open, there is also a small ball around \( x_n \) that is contained in \( B(\theta, \delta) \) which, in turn, implies
\[
z_n \in h(x_n) \subseteq \mathcal{C}(f(B(\theta, \delta))).
\]
Because the set on the extreme right is closed and \( y \) is the limit of \( (z_n) \), it then follows that \( y \in \mathcal{C}(f(B(\theta, \delta))) \). The choice of \( \delta \) above being arbitrary finally shows that \( y \in h(\theta) \), as desired.

Condition \( C_2 \) holds trivially due to our stepsize assumption in (5).

Next, consider \( C_3 \). The fact that \( (M_n) \) is a martingale-difference sequence adapted to \( (\mathcal{F}_n) \) is a direct consequence of their respective definitions. Further, for any \( n \geq 0 \), it follows from (3) and (10) that
\[
||\delta_n \phi(s_n, a_n)|| \leq ||\delta_n|| \leq K_c (K_r + (\gamma + 1) K_0 ||\theta_n||)
\]
and, hence,
\[
||M_{n+1}|| \leq 2 K_c (K_r + (\gamma + 1) K_0 ||\theta_n||).
\]
Therefore, \( E[||M_{n+1}||^2 | \mathcal{F}_n] \leq K_m [1 + ||\theta_n||^2] \) for \( K_m = 8 K_c^2 \max\{K_r^2, (\gamma + 1) K_0^2\} \). It remains to show that \( (M_n) \) is a square integrable sequence, i.e., \( E[||M_n||^2] < \infty \) for all \( n \geq 0 \). The result is trivially true for \( n = 0 \). Separately, since \( ||\theta_0||^2 < \infty \), it follows from (3) and using (14) and (15) for \( n = 0 \) that \( E[||M_1||^2 < \infty \) and \( E[||\theta'||^2 < \infty \). The desired result now follows by induction.

Finally, we verify \( C_4 \). We begin by showing that \( h_\infty(\theta) \) is non-empty. For \( \theta \in \mathbb{R}^d \), let \( H_\theta := \{a \in \mathcal{A}^d : \theta \in \text{cl}(P_a)\} \). Since \( \{P_a : a \in \mathcal{A}^d\} \) partitions \( \mathbb{R}^d \), it follows that \( |H_\theta| \geq 1 \). Additionally, note that each \( P_a \) is a cone, i.e., \( \theta \in P_a \) implies \( c \theta \in P_a \) for all \( c > 0 \). Therefore, \( H_{c \theta} = H_\theta \) for any \( c > 0 \). From the definition of \( h \) in (8), it is now easy to see that \( h(c \theta) = \text{co}\{b_a - A_n \theta' : a \in H_\theta\} \) for \( c > 0 \); we don’t need to explicitly work with the closure of this set since a convex hull of finitely many vectors is closed by definition. This implies that \( h_\infty(\theta) = \text{co}\{-A_n \theta' : a \in H_\theta\} \), which is clearly non-empty.

Next, we claim that \( \Gamma := \{0\} \) is the desired attractor of the DI
\[
\dot{\theta}(t) \in h_\infty(\theta(t))
\]
that is contained in \( B(0, 1) \) and has \( \text{cl}(B(0, 1)) \) as a subset of one of its fundamental neighborhoods. To see this, first note that \( \Gamma \) is a subset of \( B(0, 1) \), is compact, and is invariant under (11). The
latter holds since the solution $\theta(t) \equiv 0$ passes through the origin and stays there entirely. Separately, since $B_δ$ holds, $A_δ + A_δ^T$ is a symmetric positive definite matrix for each $a$. Hence, with respect to Theorem A.2, if we let $\gamma := \gamma_0 = \min_{a \in A} \{ \lambda_{\min}(A_δ + A_δ^T) \}$, then all conditions of Theorem A.2 hold and we get that the origin is the GES equilibrium point of (16). The relation in (12) now implies that $B(0, δ)$ is a fundamental neighborhood of $\Gamma$ for any $δ > 0$. Picking $δ > 1$, we have $\text{cl}(B(0, 1)) \subset B(0, δ)$, as desired.

Lastly, let $(c_n, (x_n), (z_n), \theta)$, and $y$ be as in $\mathbb{C}_1$ and let $δ > 0$ be arbitrary. Then, $\exists N_δ \geq 0$ such that $x_n \in B(\theta, δ)$ and $H_x \subseteq H_0$ for all $n \geq N_δ$. Hence, for $n \geq N_δ$, using $z_n \in h_{c_n}(x_n) = \frac{1}{c_n} \lambda\{b_a - A_δ c_n x_n : a \in H_{x_n}\}$ and $h_{c_n}(\theta) = \frac{1}{c_n} \lambda\{b_a - A_δ c_n \theta : a \in H_0\}$, it follows that

$$d(z_n, h_{c_n}(\theta)) \leq \delta \sup_{a \in A^δ} \| A_a \|.$$  

A simple triangle inequality then shows that

$$\limsup_{n \to \infty} d(y, h_{c_n}(\theta)) \leq \delta \sup_{a \in A^δ} \| A_a \|.$$  

Since $δ > 0$ is arbitrary, we get

$$\lim_{n \to \infty} d(y, h_{c_n}(\theta)) = 0.$$  

It can be shown that the above conclusion holds for all sequences $(c_n')$ such that $c_n' \uparrow \infty$. From this, it follows that $y \in h_∞(\theta)$ as desired.

## B Details of MDPs used in Section 5

All MDPs used in the experiments have 2 states and 2 actions. In the following, MDP transition probabilities $\{P(s'|s, a)\}_{s, s', a}$ are displayed as separate $S \times S$ matrices for each action $a$, with each row indexing $s$ and each column indexing $s'$. Feature matrices $\Phi$ are displayed as $|S| \times |A|$ matrices, where rows are indexed as $(s_1, a_1), (s_1, a_2), \ldots, (s_2, a_1), (s_2, a_2), \ldots$, and reward vectors are displayed as $|S| \times 1$ vectors with the same row-indexing.

- **MDP used for Fig. 1** (Two self-consistent greedy regions):
  
  $\{P(s'|s, a)\}_{s, s', a} = \begin{bmatrix} 0.200 & 0.200 \\ 0.800 & 0.200 \end{bmatrix}$,  
  $\{P(s'|s, a_2)\}_{s, s', a} = \begin{bmatrix} 0.500 & 0.500 \\ 0.300 & 0.700 \end{bmatrix}$.

  $\Phi = \begin{bmatrix} 0.444 & 0.556 \\ 0.400 & 0.300 & 0.200 & 0.300 \\ -0.100 & -1.000 \\ -0.100 & -1.000 \end{bmatrix}$,  
  $r = \begin{bmatrix} 0.400 \\ 0.300 \\ 0.200 \\ -0.100 \end{bmatrix}$,  
  $\gamma = 0.75, \epsilon = 0.1$.

- **MDP used for Fig. 2a** (No self-consistent greedy regions):
  
  $\{P(s'|s, a_1)\}_{s, s', a} = \begin{bmatrix} 0.333 & 0.667 \\ 0.588 & 0.412 \end{bmatrix}$,  
  $\{P(s'|s, a_2)\}_{s, s', a} = \begin{bmatrix} 0.429 & 0.571 \\ 0.400 & 0.600 \end{bmatrix}$.

  $\Phi = \begin{bmatrix} -1.300 & -1.200 \\ -1.100 & -1.000 \\ 1.900 & -0.700 \\ 1.100 & -1.500 \end{bmatrix}$,  
  $r = \begin{bmatrix} 0.300 \\ 0.900 \\ -0.800 \end{bmatrix}$,  
  $\gamma = 0.75, \epsilon = 0.1$.

- **MDP used for Fig. 2b** (Only one self-consistent greedy region):
  
  $\{P(s'|s, a_1)\}_{s, s', a} = \begin{bmatrix} 0.286 & 0.714 \\ 0.250 & 0.750 \end{bmatrix}$,  
  $\{P(s'|s, a_2)\}_{s, s', a} = \begin{bmatrix} 0.500 & 0.500 \\ 0.200 & 0.800 \end{bmatrix}$.

  $\Phi = \begin{bmatrix} 0.800 & 0.400 \\ 1.000 & 0.700 \\ -1.600 & -0.700 \\ -0.200 & 1.400 \end{bmatrix}$,  
  $r = \begin{bmatrix} -0.800 \\ -2.000 \\ 0.500 \\ -2.100 \end{bmatrix}$,  
  $\gamma = 0.75, \epsilon = 0.1$.

- **MDP used for Fig. 3** (Disparate behavior for Q-Learning and SARSA):
We discuss in this section some examples of MDP settings provided in the literature that have been set-valued with a start, absorbing and two intermediate states, and the only choice among (two) actions being at \((2000)\). At this point, the conditions of A.1 can be verified to hold; thus the iterates \(\theta_n\) again use \(Q_\theta\) as the driving function. Taking the conditional expectation of the right-hand side of (17) yields the following stochastic approximation iteration in \(\mathbb{R}^3\):

\[
\theta_{n+1} = \theta_n + \alpha_n \delta_n, \quad \text{with } \delta_n = U_n \begin{bmatrix}
\theta^3_n - \theta^1_n \\
0 \\
2 - \theta^3_n
\end{bmatrix} + (1 - U_n) \begin{bmatrix}
0 \\
\theta^3_n - \theta^2_n \\
1 - \theta^3_n
\end{bmatrix}, \quad (17)
\]

Here, \(\theta^i_n\) denotes the \(i\)-th coordinate of \(\theta_n\), \(U_n\) denotes a Bernoulli random variable whose success probability conditioned on \(\theta_n\) is \(\epsilon\) if \(\theta^2_n > \theta^1_n\), and \((1 - \epsilon)\) if \(\theta^2_n \leq \theta^1_n\). Note that with respect to the original notation used in Gordon (1996), the variables \(\theta^1_n, \theta^2_n\) and \(\theta^3_n\) correspond to the variables \(Q_U, Q_L\) and \(Q_A\), respectively, at the \(n\)-th trajectory update. We will assume that \(\epsilon \in (0, 1/2)\).

Taking the conditional expectation of the right-hand side of (17) yields the \textit{discontinuous} ODE

\[
\dot{\theta} = b_\theta - A_\theta \theta,
\]

where \(b_\theta, A_\theta\) depend on which side of the half plane \(H = \{\theta \in \mathbb{R}^3 : \theta^1 = \theta^2\}\) the point \(\theta\) lies. (We again use \(\theta^i\) to mean the \(i\)-th coordinate of \(\theta\).) Specifically,

\[
b_\theta = \begin{cases} 
0, 0, 2 - \epsilon \end{cases}^{\top} & \text{if } \theta^1 \leq \theta^2 \\
0, 0, 1 + \epsilon \end{cases}^{\top} & \text{if } \theta^1 > \theta^2,
\]

and

\[
A_\theta = \begin{cases} 
\begin{bmatrix} 1 - \epsilon & 0 & \epsilon - 1 \\
0 & \epsilon & -\epsilon \\
0 & 0 & 1 \end{bmatrix}, & \text{if } \theta^1 \leq \theta^2 \\
\begin{bmatrix} \epsilon & 0 & -\epsilon \\
0 & 1 - \epsilon & \epsilon - 1 \\
0 & 0 & 1 \end{bmatrix}, & \text{if } \theta^1 > \theta^2
\end{cases}.
\]

(Note that the ODE’s driving function is arbitrary in case of a tie: \(\theta^1 = \theta^2\).)

This can be lifted to the DI

\[
\dot{\theta} \in h(\theta), \quad (18)
\]

with the \textit{set-valued} driving function \(h(\theta)\) equaling the singleton \(\{b_\theta - A_\theta \theta\}\) as defined above when \(\theta^1 \neq \theta^2\), and \(co(b_U - A_U \theta, b_L - A_L \theta)\) when \(\theta^1 = \theta^2\). Note that both the (triangular) matrices \(A_U\) and \(A_L\) are stable in the sense that their eigenvalues’ real parts are positive, which can be used to show that the iterates \((\theta_n)\) of (17) will be bounded almost-surely (this is also established by Gordon (2000)). At this point, the conditions of A.1 can be verified to hold; thus the iterates \((\theta_n)\) must converge to a closed connected internally chain transitive invariant set of the DI (18).

C The Chattering Phenomenon

We discuss in this section some examples of MDP settings provided in the literature that have been shown to exhibit pathological behavior such as parameter chattering. The case studies of Section 3 do not by any means cover all possible behavioral phenomena that the iterates can exhibit and the kinds of limiting DI structures that could be obtained. For example, there could be cases where the landmark point for the local dynamics of a greedy region could be located right on its boundary.

In this regard, a notable example is that of Gordon (1996). This paper constructs an absorbing MDP with a start, absorbing and two intermediate states, and the only choice among (two) actions being at the start state. Under state aggregation resulting in a 3-dimensional linear function approximation to the Q-value function, SARSA(0) with \(\epsilon\)-greedy exploration is shown to exhibit chattering both in the policy and parameter or Q-value estimates. Although this setting of an absorbing MDP with discount factor \(\gamma = 1\) is not the same as our discounted-cost ergodic MDP setting, we can still write down the SARSA(0) update \textit{after each absorbing trajectory} as the following stochastic approximation iteration in \(\mathbb{R}^3\):

\[
\theta_{n+1} = \theta_n + \alpha_n \delta_n, \quad \text{with } \delta_n = U_n \begin{bmatrix}
\theta^3_n - \theta^1_n \\
0 \\
2 - \theta^3_n
\end{bmatrix} + (1 - U_n) \begin{bmatrix}
0 \\
\theta^3_n - \theta^2_n \\
1 - \theta^3_n
\end{bmatrix}.
\]

\[
\begin{bmatrix}
0.716 & 0.284 \\
0.423 & 0.577 \\
-0.001 & 0.999
\end{bmatrix}, \begin{bmatrix}
0.109 & 0.891 \\
0.260 & 0.740 \\
-0.001 & 0.999
\end{bmatrix}, \Phi =
\]

\[
\begin{bmatrix}
0.411 & -1.051 \\
-0.768 & 0.240 \\
0.153 & 0.242 \\
-0.610 & -0.546
\end{bmatrix}, \begin{bmatrix}
-2.983 \\
-9.780 \\
-6.432 \\
9.052
\end{bmatrix}, \gamma = 0.95, \epsilon = 0.3.
\]
Further calculation shows that the landmarks for both sides’ dynamics are \( x_U^\ast := A_U^{-1}b_U = [2 - \epsilon, 2 - \epsilon, 2 - \epsilon]^T \) and \( x_L^\ast := A_L^{-1}b_L = [1 + \epsilon, 1 + \epsilon, 1 + \epsilon]^T \), both on the boundary hyperplane \( \mathcal{H} \).

The following observations (not necessarily exhaustive) can be made about the solution trajectories \((\theta(t))_{t \in \mathbb{R}}\) of the DI (18):

1. If \( \exists t_0 \in \mathbb{R} \) such that \( \theta(t) \notin \mathcal{H} \) for all \( t > t_0 \), then, by linear dynamical systems theory, \( \theta(t) \) must converge asymptotically to either \( x_U^\ast \) or \( x_L^\ast \). This is because after time \( t_0 \), the solution is confined to exactly one of the (open) half spaces \( \{ \theta^1 < \theta^2 \} \) or \( \{ \theta^1 > \theta^2 \} \).

2. If \( \exists t_0 \in \mathbb{R} \) such that \( \theta(t) \in \mathcal{H} \) for all \( t > t_0 \), then the solution is a ‘sliding motion’ confined to the half plane \( \mathcal{H} \) after time \( t_0 \). Let \( \theta \) denote \( \theta(t) \) for some \( t > t_0 \). Analyzing the DI dynamics with the additional restriction that \( \dot{\theta} \in \mathcal{H} \) shows that (a) if \( \theta^1 \neq \theta^3 \) (equivalent to \( \theta^2 \neq \theta^3 \)), then \( \theta(t) \) must converge asymptotically to the point \( [3/2, 3/2, 3/2]^T \) from within \( \mathcal{H} \); (b) if \( \theta^1 = \theta^3 = \theta^2 \), then (i) to retain equality between all 3 coordinates, the solution must stay put at \( \theta \) for all times after \( t \), which can happen if and only if \( \theta^1 = \theta^2 = \theta^3 \in [1 + \epsilon, 2 - \epsilon] \), else (ii) the equality between \( \theta^1 \) (or \( \theta^2 \)) and \( \theta^3 \) stops holding, and we get into the situation of (a) above.

The observations above, together with results from differential inclusion theory (Benaïm et al., 2005), lead us to conjecture that the only invariant, internally chain transitive sets of the DI (18) are all the singleton sets \( \{ [\eta, \eta, \eta]^T \} \) for \( \eta \in [1 + \epsilon, 2 - \epsilon] \). This would imply that no parameter chattering can take place for Gordon’s SARSA(0) MDP learning example with a Robbins-Monro decaying stepsize schedule \((\alpha_n)\), since the original iterates of (17) will almost-surely converge to a path-dependent singleton. Note, however, that policy-space oscillations are still possible.

Another related example of an absorbing MDP over which Q-learning is shown to exhibit (parameter) chattering, under a fixed stepsize, is given by Young and Sutton (2020, ‘Oscillating Example’). Although no formal analysis is carried out about the cause of the chattering, we surmise that if the associated DI does indeed have only isolated points as its invariant, internally chain transitive sets, then with decaying Robbins-Monro stepsizes, SARSA(0) should be free from such parameter chattering.