Can the viscosity in astrophysical black hole accretion disks be close to its string theory bound?

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String theory and gauge/gravity duality suggest the lower bound of shear viscosity ($\eta$) to entropy density ($s$) for any matter to be $\sim \mu \hbar / 4\pi k_B$, when $\hbar$ and $k_B$ are reduced Planck and Boltzmann constants respectively and $\mu \leq 1$. Motivated by this, we explore $\eta/s$ in black hole accretion flows, in order to understand if such exotic flows could be a natural site for the lowest $\eta/s$. Accretion flow plays an important role in black hole physics in identifying the existence of the underlying black hole. This is a rotating shear flow with insignificant molecular viscosity, which could however have a significant turbulent viscosity, generating transport, heat and hence entropy in the flow. However, in presence of strong magnetic field, magnetic stresses can help in transporting matter independent of viscosity, via celebrated Blandford-Payne mechanism. In such cases, energy and then entropy produces via Ohmic dissipation. In addition, certain optically thin, hot, accretion flows, of temperature $\gtrsim 10^9$K, may be favourable for nuclear burning which could generate/absorb huge energy, much higher than that in a star. We find that $\eta/s$ in accretion flows appears to be close to the lower bound suggested by theory, if they are embedded by strong magnetic field or producing nuclear energy, when the source of energy is not viscous effects. A lower bound on $\eta/s$ also leads to an upper bound on the Reynolds number of the flow.

**Keywords:** infall, accretion, and accretion disks; gauge/string duality; shear rate dependent viscosity; magnetic field; black holes; relativistic plasmas

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I. INTRODUCTION

In order to explain the Quark Gluon Plasma (QGP) observables at Relativistic Heavy Ion Collider (RHIC) having temperature \( T \gtrsim 10^{12} \text{K} \), it has become apparent that a non-zero shear viscosity, \( \eta \), is needed. One way of characterizing \( \eta \), which is a dimensionful number, is to take its ratio with the entropy density, \( s \). It turns out that RHIC plasma has

\[
\frac{\eta}{s} \sim 0.1 \frac{\hbar}{k_B}.
\]

(1)

Although \( \eta \) itself is large (in CGS units \( \sim 10^{12} \text{gm/cm/s} \)), its ratio with \( s \) is small. Note that this ratio for a known fluid, say water, is about two orders of magnitude greater than the above value. Hydrodynamic simulations at RHIC and Large Hadron Collider (LHC) (which provide the environments created around a few microseconds after the big bang) as well suggest a small value for this ratio in the hot QGP state. Such a small value points towards strongly interacting matter. In fact, string theory arguments and the famous gauge/gravity duality in the form of the AdS/CFT correspondence suggest a lower bound \([1]\) for \( \eta/s \) for any matter given by

\[
\frac{\eta}{s} \geq \frac{\mu}{4\pi} \frac{\hbar}{k_B} \sim 0.08\mu \frac{\hbar}{k_B},
\]

(2)

where \( \mu \ll 1 \) \([2]\) but non-zero \([3, 4]\) (see, e.g., \([5]\)). For the present purpose, we assume that there is a bound and that \( \mu \sim 1 \).

Is there any natural site revealing an \( \eta/s \) close to above lower bound? Note that even if \( \mu \ll 1 \), still the same question remains. Such a value may be possible to arise naturally, if the temperature and/or density of the systems are/is same as that in RHIC/LHC. In the search of such an exotic system, our story starts which could serve as a natural verification of gauge/gravity duality in the form of the AdS/CFT correspondence.

Observed data suggest that the accretion flows around certain black holes must be optically thin, geometrically thick and very hot. Such flows often exhibit outflows/jets in their hard spectral states. The ion temperature \( (T_i) \) in such flows/disks is as high as \( 7 \times 10^{11} \text{K} \). The supermassive black hole system at the centre of our galaxy, Sgr A* \([6]\), (most of the temporal classes of) the X-ray binary Cyg X-1 \([7]\) are some of the examples of this kind of hot accretion disks, which exhibit radiatively inefficient flows. They are particularly different from their radiatively efficient counter part, namely the Keplerian accretion disk, which is geometrically thin, optically thick, cooler, and of temperature \( T \sim 10^7 \text{K} \) \([8]\). The flow in
certain temporal classes (and soft spectral states) of micro-quasar GRS 1915+105 is an example of Keplerian disk. While the temperature of the latter cases could be same order in magnitude as that in the center of a star, due to much low density they practically do not exhibit any thermonuclear burning. Hence any source of energy therein must be due to magnetic and viscous effects. On the other hand, the former cases may be favourable for thermonuclear reactions due to their very high temperature, even if their density is low. Hence, the energy released/absorbed due to nuclear reactions in an optically thin, hot, accretion flow could be comparable to or even dominating over its viscous counterpart. Being very hot, such flows are also highly ionized and hence expected to be strongly magnetized (see e.g. [12]) rendering magnetic dissipation. In fact, the celebrated Blandford-Payne mechanism is based on such a magnetized accretion disk in the Keplerian regime, which argues for angular momentum transport due to outflows/jets through the outgoing magnetic field lines, in absence of shear (turbulent) viscosity. Now an obvious question arises: what is the value of $\eta/s$ in optically thin accretion flows (e.g. Advection Dominated Accretion Flow (ADAF), General Advective Accretion Flow (GAAF)) when the temperature is close to that of the QGP matter in RHIC? Primarily guess is that it is not too much different from its lower bound as in equation (1). Is this that simple to anticipate? Note that accretion flows with very small molecular viscosity (and then very large Reynolds number) exhibit turbulence and then turbulent viscosity. If the turbulent viscosity dominates, then the energy dissipated due to viscosity in disks is comparable/dominant to/over the magnetic/nuclear energy, as will be discussed in detail below. Note also that at low densities (zero chemical potential), lattice QCD calculations suggest a crossover from the hadronic state to the QGP state at around $1.52 \times 10^{12}$K. At such high temperatures, there is expected to be copious pion production in a strongly interacting system. This QGP state is thought to have existed a few microseconds after the big bang and is being extensively studied at RHIC and LHC. Of course the density in early universe is large as well, unlike that in accretion flows, a large temperature is well enough to reveal such a phase which has been mimicked in RHIC/LHC. However, at a very large density (several orders of magnitude larger than that in accretion flows) the lattice QCD results are not valid.

Recently, Sinha & Mukhopadhyay initiated to look at the above issue and argued that the Shakura-Sunyaev turbulent viscosity parameter $\alpha$ should not be constant throughout
the flow, and be decreasing with the increase of temperature and/or density of the flow. Based on the general relativistic model of a viscous ADAF, they showed that \( \eta/s \) for an astrophysical black hole, at finite \( \alpha \), is always several orders of magnitude higher than its value in a QGP fluid. Then they argued for the flow to become very weakly viscous (not exhibiting turbulence) close to a black hole, rendering a smaller \( \eta/s \). The results appear to be independent of the choice of equation of state (EoS); whether of ideal gas or of QCD which is appropriate for flows with \( T \gtrsim 10^{12} \text{K} \) and low density (close to zero chemical potential). However, they could not clarify the natural circumstances when \( \alpha \) is small. Note that small \( \alpha \) would reveal negligible dissipation of energy in the flow. Hence, question arises what is the source of energy and entropy in the accretion flows, which is also very important in order to address \( \eta/s \)? In addition, how is the accretion possible in a very small-\( \alpha \) flow? Moreover, as will be explained in the next section, above work did not model the entropy of the flow adequately, while generally interpreted the results correctly. Instead, that was the first step forward in the direction of evaluating \( \eta/s \) for any astrophysical flow, even though incomplete, which is a potential natural site for small \( \eta/s \). In the present work, we have removed the above uncertainties lying in the previous work. Then we plan to establish the underlying physics giving rise to the astrophysical values of \( \eta/s \). More precisely, we plan to address: (1) regime of accretion flows giving rise to \( \eta/s \) close to its theoretical lower bound, (2) observational implications of such regimes, (3) black hole sources presumably revealing lower bound of \( \eta/s \), (4) constraining physical parameters (e.g., Reynolds number) of accretion flows based on \( \eta/s \) bound.

In the next section, we discuss the salient features of hot advective accretion flows and the importance of strong magnetic field and nuclear energy produced therein in order to obtain small \( \eta/s \). Subsequently in §3 we analyse the solutions of the equations and address the flow parameters making \( \eta/s \) to be close to the lower limit. Finally §4 summarizes our findings with discussion and implication.

II. ADVECTIVE ACCRETION FLOWS AROUND BLACK HOLES

In order to understand the underlying physics, let us consider the equations describing optically thin, viscous, magnetized, advective accretion disk \[24\]. We describe the model in the pseudo-Newtonian framework with the Paczyński-Wiita potential \[25, 26\]. This is par-
particularly because, for the present purpose, the pure general relativistic results for a rotating black hole qualitatively do not reveal any new physics. Hence, the vertically averaged hydromagnetic equations of energy-momentum balance along with the equations of continuity, induction and divergence of magnetic field in the limit of very large conductivity are

\[
\dot{M} = 4\pi x \Sigma \vartheta, \\
\vartheta \frac{d\vartheta}{dx} + \frac{1}{\rho} \frac{dP}{dx} - \frac{\lambda^2}{x^3} + \frac{1}{2(x-1)^2} = 0, \\
\vartheta \frac{d\lambda}{dx} = \frac{1}{x \Sigma} \frac{d}{dx} (x^2 W_{x\phi}) + \frac{h x}{\Sigma} \left( B_z \frac{dB_c}{dx} + B_\phi \frac{dB_\phi}{dx} + \frac{B_z B_\phi}{x} \right), \\
\vartheta h T \frac{ds}{dx} = \frac{\vartheta h}{\Gamma_3 - 1} \left( \frac{dP}{dx} - \frac{\Gamma_1 P}{\rho} \frac{d\rho}{dx} \right) = Q^+ - Q^-, \\
\frac{d}{dx} (xB_x) = 0, \\
\frac{d}{dx} \left( \vartheta B_\phi - \frac{B_x \lambda}{x} \right) = 0, \\
\frac{d}{dx} (x \vartheta B_z) = 0,
\]

assuming that the variables do not vary significantly in the vertical direction such that \( d/dz \to 1/z \), which is indeed true for the disk flows. Here \( \dot{M} \) is the conserved mass accretion rate, \( \Sigma \) and \( \rho \) are the vertically integrated density and density of the flow respectively, \( \vartheta \) is the radial velocity, \( P \) the total pressure, \( \lambda \) the specific angular momentum, \( W_{x\phi} \) the vertically integrated shearing stress, \( h \) the half-thickness, \( s \) the entropy per unit volume, \( T \) the temperature of the flow, \( Q^+ \) and \( Q^- \) are the vertically integrated net energy released and absorbed rates in/from the flow respectively, \( \Gamma_1, \Gamma_3 \) indicate the polytropic indices depending on the gas and radiation content in the flow (see, e.g., [16] for exact expressions) and \( B_x, B_\phi \) and \( B_z \) are the components of magnetic field. Note that, the independent variable \( x \) is the radial coordinate of the flow expressed in the units of \( \frac{2GM}{c^2} \), where \( G \) is the gravitation constant, \( M \) the mass of the black hole and \( c \) the speed of light. Accordingly, all the above variables are expressed in dimensionless units. For any other details, see the existing literature [16, 27].

Let us now pay a special attention to the energy equation of the equation set (3). We can rewrite it as

\[
\vartheta T \frac{ds}{dx} = q_{vis} + q_{mag} + q_{Nex} - q_{rad} - q_{Nend},
\]

where \( q_{vis}, q_{mag} \) and \( q_{Nex} \) respectively define the energies released per unit volume per unit
time due to viscous dissipation, magnetic dissipation and thermonuclear reactions and \( q_{\text{rad}} \) and \( q_{\text{Nend}} \) respectively indicate the energy radiated away per unit volume per unit time by various cooling processes like Bremsstrahlung, synchrotron and inverse-Comptonization of soft photons, and the energy absorbed per unit volume per unit time due to thermonuclear reactions.

It was shown earlier that the nuclear energy released/absorbed in a hot accretion disk need not be negligible \([10, 11, 14, 28]\) and the thermonuclear reactions therein could produce over-abundant metals \([10, 29–31]\) which are observed in galaxies. However, the value of \( \alpha \) viscosity determines whether the energy due to nuclear reactions is comparable to that due to the viscous effects \([10, 28]\) and Ohmic dissipation or not. From equation (4) we can write down the entropy per unit volume, namely the entropy density, of the flow as

\[
s = \int \frac{q_{\text{vis}} + q_{\text{mag}} + q_{\text{Nex}} - q_{\text{rad}} - q_{\text{Nend}}}{\partial T} \, dx.
\]

(5)

Now the turbulent kinematic viscosity \( \nu = \partial_{l}l/3 \), when \( \partial_{t} \) and \( l_{t} \) are respectively the turbulent Eddy velocity and the size of Eddy, which are respectively to be scaled linearly with sound speed \( (c_{s}) \) of the disk and \( h \) and hence

\[
\nu = \frac{\alpha_{\phi}c_{s} \alpha_{l}h}{3} = \alpha c_{s}h,
\]

(6)

where \( \alpha_{\phi} \), \( \alpha_{l} \) and hence \( \alpha \) are constants. Therefore, the shear viscosity is given by

\[
\eta = \alpha \rho c_{s}h.
\]

(7)

In an optically thin flow, often \( q_{\text{rad}} \) is approximately chosen to be proportional to \( q_{\text{vis}} + q_{\text{mag}} \), in order to understand the hydrodynamic properties of the flow without much loss of generality, such that \( q_{\text{rad}} = (1 - f)(q_{\text{vis}} + q_{\text{mag}}) \) (see, e.g., \([10, 15, 27]\)), when \( f \) is a constant. For an ADAF \( f = 1 \) strictly (and \( \dot{M} \ll \dot{M}_{\text{Edd}} \), when \( \dot{M}_{\text{Edd}} \) is the Eddington accretion rate) and for a GAAF \( 0 < f \leq 1 \) (and \( \dot{M} \) is larger than that in an ADAF \([16]\)). Note that \( f = 0 \) for an optically thick Keplerian accretion disk. Hence, following previous work (e.g. \([10, 24, 27]\)), we can write

\[
q_{\text{vis}} + q_{\text{mag}} - q_{\text{rad}} = f (q_{\text{vis}} + q_{\text{mag}}) = f \left( \alpha (I_{n+1}c_{s}^{2} + I_{n} \partial^{2}) \rho \frac{d}{dx} \left( \frac{\lambda}{x} \right) + \frac{3|\vec{B}|^{2} \partial}{16 \pi x} \right),
\]

(8)

where \( I_{n} = (2^{n}n!)^{2}/(2n + 1)! \). Therefore, if \( q_{\text{mag}}, q_{\text{Nex}}/f, q_{\text{Nend}}/f \ll q_{\text{vis}} \), which we name as viscous regime, \( \eta/s \) does not explicitly depend on \( \alpha \) (and \( \eta \)) \([37]\), depends only on the
hydrodynamic quantities whose numerical values, in a fixed range of radii, do not vary significantly in the optically thin regime with the change of $\alpha$. In the viscous regime, $\eta/s$ simply scales with $M$. Note that viscosity depends on the size of the system, which in turn is determined by the mass of the black hole. Hence, only for a primordial black hole of $M \lesssim 10^{16}$ gm, $\eta/s$ is close to its theoretical lower bound, as was suggested earlier [21]. It was then also suggested that $\alpha$ has to be very small in an ADAF to obtain $\eta/s \sim \hbar/4\pi k_B$ for an astrophysical black hole. However, for a small $\alpha$, viscous dissipation (and then $s$) in an ADAF (or generally in a GAAF) should be small as well, as is apparent from equations (4) and (8). Hence, the ratio of $\eta$ to $s$ should not be invariably small, as described above, in the viscous regime. Therefore, while the earlier work [21] is very interesting and the first attempt to evaluate $\eta/s$ in an astrophysical flow, the flow (and hence any conclusion) should be amended with extra physics. In other words, the conclusion is incomplete.

Indeed $\eta/s$ could be small at a small $\alpha$ when $s$ does not decrease in that extent with the decrease of $\alpha$. However, this is possible only when $q_{\text{mag}} \gg f q_{\text{vis}}$ or $q_{\text{Nex}} \gg f q_{\text{vis}}$, which we name magnetic and nuclear regimes respectively. The magnetic regime essentially corresponds to the Blandford-Payne mechanism in the hard spectral state, when accretion takes place via magnetic stress, independent of the presence of shear/turbulent viscosity [13]. This plausibly corresponds to the case of accretion flows with outflows/jets, e.g. Sgr A*, hard states of GRS 1915+105. Hence, even at a negligibly small $\alpha$ accretion is possible with a negligible $\eta/s$. On the other hand, the nuclear reactions and corresponding energies depend on the density and temperature of the system. As shown earlier [16, 22, 23, 32, 34] that even in the inviscid limit, temperature and density of the optically thin flows remain as high as those of a viscous accretion flow for a fixed $M$, which successfully explain many observation [35, 36]. Indeed, it was shown [10, 11] that significant nuclear energy ($\sim 10^{15}$ ergs/cc/sec) throughout the disk may be released/absorbed in an inviscid flow which could lead to disk instability. Hence, from equation (5) it is naturally possible that $s$ is large even at a small $\alpha$. Therefore, $\eta/s \sim \hbar/4\pi k_B$ also plausibly in the nuclear regime of the accretion flow, when $|q_{\text{Nex}} - q_{\text{Nend}}| \sim |q_{\text{Nex}}|$ or $|q_{\text{Nend}}|$ which could be the case in the hot accretion flows [10, 11, 14].
III. SOLUTIONS AND $\eta/S$

Let us understand the above physics quantitatively. Figures 1a,b,c show hydrodynamic/hydromagnetic properties of a disk around a $10M_\odot$ nonrotating black hole in the magnetic regime, accreting mass at the rate $\sim \dot{M}_{\text{Edd}}$. In this case of weakly viscous hot flow with $\alpha \sim 5 \times 10^{-17}$ (but not zero exactly), $\eta/s$ is shown to be close to the theoretical lower bound $\hbar/4\pi k_B$. For comparison, $\eta/s$ is also shown for a viscous regime, which is quite large for $\alpha = 0.05$ and $f = 0.1$. We also show the variations of Mach number and magnetic field in the disk for the cases mentioned above for reference. Interestingly, increase of magnetic field in the magnetic regime increases rate of transport which further renders advancing the Keplerian disk towards the black hole, making the sub-Keplerian regime, which is shown here, smaller. Figures 1d,e,f depict the properties for $\dot{M} \sim 10^{-2}\dot{M}_{\text{Edd}}$ in the magnetic regime. A very small (but nonzero) $\alpha$ ($\sim 2 \times 10^{-19}$) reveals $\eta/s$ to exhibit the theoretical lower limit. The impact of change of the magnetic field strength to the disk is same as that in the high accreting case discussed above. Interestingly, independent of the accretion rate, magnetic field $\gtrsim kG$ is able to transport matter efficiently in the sub-Keplerian flow in absence of viscosity, as was already argued by Blandford & Payne [13] in the Keplerian, self-similar framework. A smaller field strength would also transport matter as well, except with the increase of size of the sub-Keplerian regime.

Figure 2 is devoted for hydrodynamic/hydromagnetic properties around a $10^7M_\odot$ nonrotating black hole in the magnetic regime. While Figs. 2a,b,c correspond to $\dot{M} \sim \dot{M}_{\text{Edd}}$, Figs. 2d,e,f correspond to $\dot{M} \sim 10^{-4}\dot{M}_{\text{Edd}}$. In either of the cases, very weakly viscous ($\alpha \sim 5 \times 10^{-23}$ for $\dot{M} \sim \dot{M}_{\text{Edd}}$ and $\alpha \sim 10^{-25}$ for $\dot{M} \sim 10^{-4}\dot{M}_{\text{Edd}}$) hot flows reveal small $\eta/s$, close to its theoretical lower bound $\hbar/4\pi k_B$. For comparison, $\eta/s$ is also shown for a viscous, high accreting regime which is quite large for $\alpha = 0.05$ and $f = 0.5$. Interestingly, magnetic field $\gtrsim mG$ is able to transport matter efficiently in the sub-Keplerian flow around a supermassive black hole in absence of viscosity, as was already argued by Blandford & Payne [13] in the Keplerian, self-similar framework. As is the case for a stellar mass black hole, a smaller field strength would also transport matter, except the fact that the Keplerian to sub-Keplerian transition would arise at a larger radius.

Let us also look into the nuclear regime. Figure 3 shows hydrodynamic properties of a disk around a $10M_\odot$ nonrotating black with $\dot{M} = 0.5\dot{M}_{\text{Edd}}$ in the nuclear regime. In the cases
FIG. 1: Variations of (a) Mach number, (b) magnetic field in G, (c) $\eta/s$ in CGS unit when the horizontal line (long-dashed) refers to $h/4\pi k_B$, as functions of flow radius for $M = 10M_\odot$ at $\dot{M} = \dot{M}_{\text{Edd}}$. Solid line corresponds to the flow in viscous regime and dotted and dashed lines correspond to the flows in magnetic regime. (d), (e), (f) describe same as in (a), (b), (c) respectively, except for $\dot{M} = 0.01\dot{M}_{\text{Edd}}$. See Table 1 for other details.

of weakly viscous hot flow, $\eta/s$ is shown to be inversely proportional to the nuclear energy released per unit volume. When the ratio of nuclear energy per unit volume to viscous energy per unit volume is chosen to be $Q_N/Q_{\text{vis}} \sim 10^{16}$ (which corresponds to $Q_{\text{max}} \sim 3.4 \times 10^{17}$ erg/sec, depicted by the solid line), $\eta/s$ appears to be close to the theoretical lower bound $h/4\pi k_B$. Note that the production of nuclear energy also depends on the residence time of matter in the disk which is determined by $\alpha$, polytropic constant $\gamma$, initial mass fraction of the inflowing isotopes etc. For comparison, $\eta/s$ is also shown for a viscous regime, which
FIG. 2: Variations of (a) Mach number, (b) magnetic field in G, (c) η/s in CGS unit when the horizontal line (long-dashed) refers to $\hbar/4\pi k_B$, as functions of flow radius for $M = 10^7 M_\odot$ at $\dot{M} = \dot{M}_{\text{Edd}}$. Solid line corresponds to the flow in viscous regime and dotted line corresponds to the flow in magnetic regime. (d), (e), (f) describe same as in (a), (b), (c) respectively, except for $\dot{M} = 10^{-4} \dot{M}_{\text{Edd}}$. See Table 2 for other details.

is quite large for $\alpha = 0.05$ and $f = 0.1$. We also show the variations of Mach number and temperature in the disk for the cases mentioned above for reference. Interestingly, the latter profiles overlap each other in the nuclear regime, independent of $Q_{\text{N}}/Q_{\text{vis}}$, and are very similar to that in the viscous regime. This also verifies that the Mach number and temperature do not depend much on $\alpha$, in the optically thin flows.

In Fig. 4 we show a couple of sets of solutions for highly sub-Eddington hotter flows and a solution set for a low angular momentum, high mass accreting, hot flow, in the nuclear
regimes. Interestingly, a very low angular momentum ($\lambda \sim 1$), gas dominated, hot accretion flow with $\dot{M} = 10^{-4}\dot{M}_{\text{Edd}}$ (solid line) violates the theoretical lower limit of $\eta/s$. This restricts $\lambda$ of the flow. If $\lambda$ increases from 1 to 2, the flow becomes relatively cooler due to decrease of radial velocity which restricts nuclear energy, rendering slightly larger $\eta/s > \hbar/4\pi k_B$ (dotted line). A flow with similar $\lambda$ ($\sim 1.6$) and $\alpha$, but $\dot{M} = 0.5\dot{M}_{\text{Edd}}$ and hence revealing a radiation dominated accretion (dashed line) flow, exhibits however a similar $\eta/s > \hbar/4\pi k_B$. Note that while the nuclear energy among the three cases considered here is highest in the last flow, due to presence of radiation the cooling efficiency therein is maximum, which renders its $\eta/s$ above the lower limit throughout.

### Table 1

| $\dot{M}$ in $\dot{M}_{\text{Edd}}$ units | $\gamma$ in inner disk | $\lambda$ in inner disk | Regime | $|\vec{B}|$ in CGS in inner disk | $f$ | Figure, line |
|-----------------------------------------|------------------------|------------------------|--------|---------------------------------|-----|----------------|
| 1                                       | 1.335                  | 3.2                    | Magnetic, $\alpha = 5 \times 10^{-17}$ | $1.85 \times 10^3 G$ | 0.5 | Figs. 1a,b,c, dotted |
| 1                                       | 1.335                  | 3.2                    | Magnetic, $\alpha = 5 \times 10^{-17}$ | $1.85 \times 10^5 G$ | 0.5 | Figs. 1a,b,c, dashed |
| 1                                       | 1.335                  | 3.2                    | Viscous, $\alpha = 0.05$              | $-$                | 0.1 | Figs. 1a,c, solid    |
| 0.01                                    | 1.5                    | 3.25                   | Magnetic, $\alpha = 2 \times 10^{-19}$ | $220 G$            | 0.5 | Figs. 2a,b,c, dotted |
| 0.01                                    | 1.5                    | 3.25                   | Magnetic, $\alpha = 2 \times 10^{-19}$ | $2.2 \times 10^4 G$ | 0.5 | Figs. 2a,b,c, dashed |

### Table 2

| $\dot{M}$ in $\dot{M}_{\text{Edd}}$ units | $\gamma$ in inner disk | $\lambda$ in inner disk | Regime | $|\vec{B}|$ in CGS in inner disk | $f$ | Figure, line |
|-----------------------------------------|------------------------|------------------------|--------|---------------------------------|-----|----------------|
| 1                                       | 1.335                  | 3.2                    | Magnetic, $\alpha = 2 \times 10^{-23}$ | $1.85 G$            | 0.5 | Figs. 2a,b,c, dotted |
| 1                                       | 1.335                  | 3.2                    | Viscous, $\alpha = 0.05$              | $-$                | 0.5 | Figs. 2a,c, solid    |
| $10^{-4}$                                | 1.53                   | 2.6                    | Magnetic, $\alpha = 10^{-25}$         | $0.0185 G$          | 0.5 | Figs. 2a,b,c, dotted |
FIG. 3: Variations of (a) Mach number, (b) temperature in K, (c) energy density in CGS unit, (d) $\eta/s$ in CGS unit, as functions of flow radius for $M = 10M_\odot$ at high mass accretion rates. Solid and dotted lines correspond to the flows of very high and high nuclear regimes respectively and dashed line corresponds to the flow of viscous regime. See Table 3 for other details.
Table 3
Parameters for the results shown in Figures 3 and 4 when $M = 10M_\odot$

| $\dot{M}$ in $\dot{M}_{\text{Edd}}$ units | $\gamma$ | $\lambda$ | Regime | $Q_{\text{max}}$ in CGS | $f$ | Figure, line |
|-----------------------------------------|---------|----------|--------|--------------------------|----|--------------|
| 0.5                                     | 1.34    | 3.2      | Nuclear | $3.4 \times 10^{17}$     | –  | Fig. 3 solid |
| 0.5                                     | 1.34    | 3.2      | Nuclear | $3.4 \times 10^{11}$     | –  | Fig. 3 dotted |
| 1                                       | 1.335   | 3.2 (inner) | Viscous, $\alpha = 0.05$ | $1.8 \times 10^{19}$ | 0.1 | Fig. 3 dashed |
| 0.0001                                  | 1.53    | 1        | Nuclear | $1.3 \times 10^{14}$     | –  | Fig. 4 solid |
| 0.0001                                  | 1.53    | 2        | Nuclear | $3 \times 10^{12}$       | –  | Fig. 4 dotted |
| 0.5                                     | 1.34    | 1.6      | Nuclear | $8.4 \times 10^{19}$     | –  | Fig. 4 dashed |

IV. SUMMARY AND DISCUSSION

We have shown the astrophysical existence of small $\eta/s$, close to its theoretical lower bound. As our physical inputs, governing the values of $\eta/s$, are in accordance with observed data, this could be argued as a natural evidence for $\eta/s$ close to its lower bound. This is possible in the cases of hot, optically thin, accretion flows. More precisely, in order to have a small $\eta/s$, flow should be magnetically dominated and/or producing huge energy/entropy other than that due to viscosity.

The celebrated Blandford-Payne mechanism \[13\] was already shown to produce accretion via magnetic stress (which helps in producing outflows/jets and removing angular momentum) in absence of viscosity in the framework of Keplerian, self-similar flow. We have employed this idea, but relaxing self-similar assumption, in order to obtain $\eta/s$ close to $\hbar/4\pi k_B$ in our accretion flows. We have found that in the presence of magnetic field of the order of mG to kG, the disk flows exhibit this value of $\eta/s$ for supermassive to stellar mass black holes. Observed hard X-rays in Cyg X-1, some temporal classes (e.g. $\chi$) of GRS 1915+105, Sgr A* are observed to be associated with jets which argues for the sources to exhibit strong magnetic fields. Hence, those sources could be considered to be natural sites for $\eta/s \sim \hbar/4\pi k_B$.

Additionally, the flow exhibiting nuclear energy could be a natural site for small $\eta/s$. Nuclear energy depends on the density and temperature of the flow. The temperature in an optically thin, sub-Keplerian accretion flow, producing observed hard X-rays, could be
FIG. 4: Variations of (a) Mach number, (b) temperature in K, (c) energy density in CGS unit, (d) $\eta/s$ in CGS unit, as functions of flow radius for $M = 10M_\odot$. Solid and dotted lines correspond to the flows of very high and high nuclear regimes respectively at low mass accretion rates and dashed line corresponds to the flow of a nuclear regime at a high mass accretion rate. See Table 3 for other details.

$\gtrsim 10^9$K. Therefore, eventhough the density therein is small, very high temperature would suffice for nuclear burning. We have recalled the previous work discussing possible nuclear burning in accretion flows and its observational evidences/consequences [10, 11, 28–31]. Based on them, we have modeled the hot, optically thin accretion flows for the present purpose and shown how the nuclear energy in the flow determines the value of $\eta/s$. Thus we argue that the flows producing hard X-rays in Cyg X-1, some temporal classes (e.g. $\chi$) of GRS 1915+105 could be hot enough to have $\eta/s \sim \hbar/4\pi k_B$, atleast close to the
black hole for certain nuclear regimes, when flows are necessarily sub-Keplerian due to the dominance of gravitational power of the black hole. It will be now interesting to explore $\eta/s$ for other astrophysical flows, having, e.g., strong magnetic field or/and high temperature and/or density.

Finally, we end by enlightening maximum Reynolds number ($R_e$) to the flows, restricted by the lower bound of $\eta/s$. $R_e$, which is the ratio of typical (characteristic) velocity times length scale to the kinematic viscosity of the flow, can be recast as $R_e = M_{ac} (x/h)/\alpha$. Now from Figs. 1 and 3, $M_{ac}$ close to the black hole is $\sim 5 - 10$. However, the magnetic and nuclear regimes correspond to very low but nonzero viscosity revealing $\alpha \gtrsim 10^{-16}$ for $\eta/s$ lower bound to be satisfied. As typically $h/x \sim 0.1 - 0.5$ for an optically thin, geometrically thick flow, for stellar mass black holes $R_e \lesssim 10^{17}$. If we look at Fig. 2 showing $M_{ac}$ to be of the similar order in magnitude around supermassive black holes, however for smaller values of $\alpha$ ($\gtrsim 10^{-22}$) in larger accretion disks (as size scales with the mass of the black hole), we obtain $R_e \lesssim 10^{23}$ for those flows.

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[37] In general, $\eta/s \sim T\eta/(W_{x\varphi}/\eta) = T/\frac{d}{dx} (\lambda/x)$, when $q_{Nex}, q_{Nend} \ll \eta_{vis}$.