The conformal boundary states for SU(2) at level 1

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Abstract

For the case of the SU(2) WZW model at level one, the boundary states that only preserve the conformal symmetry are analysed. Under the assumption that the usual Cardy boundary states as well as their marginal deformations are consistent, the most general conformal boundary states are determined. They are found to be parametrised by group elements in SL(2, \mathbb{C}).
1. Introduction

Much has been understood in the last few years about conformal field theories on surfaces with a boundary. In particular, if the conformal field theory is rational with respect to some chiral algebra, the boundary states that preserve this symmetry algebra have been classified at least for a large class of cases [1,2]. However, much less is known about boundary conditions that only preserve a symmetry algebra with respect to which the original theory is not rational.

In many applications, including string theory, one is ultimately interested in boundary conditions that preserve the conformal symmetry but not necessarily any larger symmetry. The only conformal field theories which are rational with respect to the Virasoro algebra alone are the minimal models. All other conformal field theories are at best rational with respect to some larger symmetry algebra $W$ (but not with respect to $Vir \subset W$). The physically important boundary conditions may then break the symmetry to $Vir$. In this paper we construct such boundary states for a simple example, the SU(2) WZW model at level 1. This theory is rational with respect to the affine $\hat{su}(2)$ symmetry, but not with respect to the Virasoro algebra. We will arrive at results that are similar, but not identical, to claims made by Friedan in unpublished work [3]; see Section 5 for further comments.

In order to obtain constraints on the possible conformal boundary states of this model we make use of one of the sewing relations of [4], the so-called factorisation (or cluster) condition. As we explain in detail in Section 2, this constraint requires that certain bulk-boundary structure constants satisfy a set of non-linear equations (sometimes referred to as the classifying algebra in this context). The coefficients in these equations are in principle determined in terms of bulk operator product coefficients and fusing matrices. For rational theories these are sometimes available, but for the case at hand (where we take the chiral algebra to be the Virasoro algebra alone and therefore deal with a non-rational theory) they are not known explicitly. We therefore use an indirect argument to determine the coefficients appearing in the cluster condition. To this end, we observe that these coefficients can be deduced from the knowledge of a sufficiently large class of solutions. In our case, the latter arise from the boundary states that preserve the full affine symmetry; these have been constructed before (see for example [5,6]), and are believed to be consistent. In fact, they can be obtained from the Cardy solution [1] by marginal deformations.

Having determined the cluster condition explicitly we then classify all its solutions (i.e. all the one-dimensional representations of the classifying algebra), and therefore all possible (fundamental) D-branes of the theory. As it turns out, the most general D-branes are parametrised by group elements in $SL(2,\mathbb{C})$. On the level of our analysis we cannot
prove that all of these D-branes are actually consistent (i.e. satisfy all remaining sewing constraints as well), but our arguments imply that the theory does not have any other (fundamental) D-branes. This is quite a surprising result since one may have thought that the space of D-branes that only preserve the conformal symmetry would be much larger.

In Section 4 we then check that this larger class of boundary conditions satisfies Cardy’s condition. In particular, we find that the spectrum of open strings between two such D-branes organises itself into representations of the Virasoro algebra; however, in general, the conformal weights that appear are complex (rather than real). We conclude in Section 5 with some comments about the interpretation of these D-branes.

2. Constraints on boundary conditions for \( \hat{su}(2)_{k=1} \)

Let us begin by recalling some properties of the theory on the plane (the bulk theory) which in our case is the SU(2) WZW model at level \( k = 1 \). This theory has left- and right-moving currents \( J^a(z) \) and \( \bar{J}^a(\bar{z}) \) whose modes \( J^a_n \) and \( \bar{J}^a_n \) define two commuting copies of the affine algebra \( \hat{su}(2)_1 \). The space of states of the bulk theory \( \mathcal{H}_{\text{bulk}} \) can therefore be decomposed into highest weight representations of these two algebras.

There are two highest weight representations of the affine algebra \( \hat{su}(2) \) at level 1 that actually define representations of the conformal field theory (or the corresponding vertex operator algebra): the vacuum representation \( \mathcal{H}_0 \) that is generated from a state transforming in the singlet representations of the horizontal \( su(2) \) algebra (spanned by the zero modes \( J^a_0 \)), and the representation \( \mathcal{H}_{\frac{1}{2}} \) for which the highest weight states transform in the doublet representation of the horizontal \( su(2) \) algebra. For \( k = 1 \) there is only one modular invariant combination of these representations and thus a unique bulk theory, whose space of states is of the form

\[
\mathcal{H}_{\text{bulk}} = \left( \mathcal{H}_{0}^{su(2)} \otimes \mathcal{H}_{0}^{\bar{su}(2)} \right) \oplus \left( \mathcal{H}_{\frac{1}{2}}^{su(2)} \otimes \mathcal{H}_{\frac{1}{2}}^{\bar{su}(2)} \right). \tag{2.1}
\]

Because of the Sugawara construction, every highest weight representation of \( \hat{su}(2)_1 \) also forms a representation of the Virasoro algebra \( \text{Vir} \) with \( c = 1 \). The generators of the Virasoro algebra commute with the current zero modes, and thus the \( \hat{su}(2) \) representations can be decomposed into representations of \( su(2) \oplus \text{Vir} \). If we denote the \((2j + 1)\)-dimensional spin \( j \) representation of \( su(2) \) by \( V^j \) and the Virasoro algebra irreducible highest weight representation of weight \( h \) by \( \mathcal{H}^\text{Vir}_h \), then we have

\[
\mathcal{H}_{j}^{\hat{su}(2)} = \sum_{n=0}^{\infty} V^{n+j} \otimes \mathcal{H}^{\text{Vir}}_{(n+j)^2}. \tag{2.2}
\]
It thus follows that the bulk state space can be decomposed with respect to the algebra \(\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R \oplus \text{Vir}_L \oplus \text{Vir}_R\) as

\[
\mathcal{H}_{\text{bulk}} = \sum_{s, \bar{s} = 0}^{\infty} \left( \mathcal{V}^{s/2} \otimes \bar{\mathcal{V}}^{\bar{s}/2} \right) \otimes \left( \mathcal{H}_{s^{2}/4} \otimes \bar{\mathcal{H}}_{s^{2}/4}^{\text{Vir}} \right).
\]  

(2.3)

The representations of the Virasoro algebra that appear in this decomposition are all degenerate, i.e. the corresponding Verma modules contain non-trivial null vectors. Indeed, for \(c = 1\) and \(h = \frac{j^2}{2}\) with \(j \in \frac{1}{2} \mathbb{Z}\), there is a single independent null vector at level \((2j+1)\); if we introduce the notation

\[
\vartheta_{s}(q) = q^{\frac{j^2}{2}} \eta(q), \quad \eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} \left(1 - q^n\right)
\]

(2.4)

(where \(\eta(q)\) is the usual Dedekind \(\eta\)-function) then the character of the Virasoro highest weight representation with \(h = \frac{j^2}{2}\) is

\[
\chi^{\text{Vir}}_{h}(q) = \vartheta_{\sqrt{2}j}(q) - \vartheta_{\sqrt{2}(j+1)}(q).
\]

(2.5)

The decomposition (2.3) implies that there are \((2j+1)^2\) Virasoro primary fields of weight \(h = \bar{h} = j^2\) for each \(j = 0, 1/2, 1, \ldots\); they transform in the \(V^j \otimes \bar{V}^j\) representation of \(\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R\) and are denoted by \(\varphi^{j}_{m,n}(z, \bar{z})\) with labels \(m, n = -j, -j + 1, \ldots, j\). These fields can be thought of as being associated to the matrix elements of the SU(2) representation \(j\), and transform in the appropriate way under \(\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R\). (See the Appendix for our conventions for representations of \(\mathfrak{su}(2)\) and SU(2).)

In order to be able to compute the partition functions (overlaps) between two conformal boundary states we shall also need some details about the relation between fields and the corresponding in- and out-states. We shall first describe our conventions in some more generality and then concentrate on the case at hand. If we denote the different Virasoro primary fields by \(\varphi_{a}(z, \bar{z})\) (where in our case, \(a\) stands for \((j; m, n)\)), they satisfy the operator product expansion (OPE)

\[
\varphi_{a}(z, \bar{z}) \varphi_{b}(w, \bar{w}) = \frac{g_{ab}}{(z-w)^{2h_{a}}(\bar{z}-\bar{w})^{2h_{a}}} + \ldots,
\]

(2.6)

where we have only made the coupling to the identity field explicit. Here \(g_{ab}\) is some metric which we shall not assume to be equal to \(\delta_{ab}\). (Indeed, for our case where \(a = (j; m, n)\), it is not.) We introduce the corresponding in-states by the standard convention

\[
|a\rangle = \lim_{z \to 0} \varphi_{a}(z, \bar{z}) |0\rangle.
\]

(2.7)
The out-states can then be given in terms of the action on the out-vacuum as

\[ \langle a | = \sum_b \lim_{z \to \infty} z^{2h_a} \bar{z}^{2\bar{h}_a} \langle 0 | g^{ab} \varphi_b(z, \bar{z}), \]  

(2.8)

where \( g^{ab} \) is the matrix inverse of \( g_{ab} \). By definition, \( \langle 0 | \) satisfies \( \langle 0 | 0 \rangle = 1 \), and this implies then that \( \langle a | b \rangle = \delta_{ab} \) as usual.

If we label the fields by \((j; m, n)\), it follows from the \(su(2)_L \oplus su(2)_R\) symmetry that the metric is proportional to

\[ g_{(j; m, n), (j'; m', n')} = \delta_{j, j'} \delta_{m, -m'} \delta_{n, -n'} (-1)^{m-n}. \]  

(2.9)

Indeed, for either \(su(2)\) the singlet is contained in the tensor product of two representation with \(su(2)\) spins \(j\) and \(j'\) only if \(j = j'\); if this is the case, the singlet state is proportional (in our conventions) to

\[ \sum_{m=-j}^{j} (-1)^m |j, m\rangle \otimes |j, -m\rangle. \]  

(2.10)

Applying this argument to both \(su(2)_L\) and \(su(2)_R\), it follows that the metric agrees with (2.9) up to some overall (possibly \(j\)-dependent) normalisation. In order to fix this, we normalise our fields in accordance with the usual free-field construction of \(\hat{su}(2)_1\) in which the currents are given by

\[ J^+(z) = \exp(i\sqrt{2}\phi(z)), \quad J^3(z) = i \partial \phi(z)/\sqrt{2}, \quad J^+(\bar{z}) = \exp(-i\sqrt{2}\bar{\phi}(\bar{z})). \]  

(2.11)

\[ \bar{J}^+(z) = \exp(i\sqrt{2}\bar{\phi}(z)), \quad \bar{J}^3(z) = i \bar{\partial} \bar{\phi}(\bar{z})/\sqrt{2}, \quad \bar{J}^+(\bar{z}) = \exp(-i\sqrt{2}\bar{\phi}(\bar{z})). \]  

(2.12)

In these conventions, the special Virasoro primary fields labelled \((j; j, j)\) and \((j; -j, -j)\) are given by

\[ \varphi_{j,j}^j(z, \bar{z}) = \hat{c}_j \exp(i \sqrt{2} j(\phi(z) + \bar{\phi}(\bar{z}))), \quad \varphi_{-j,-j}^j(z, \bar{z}) = \hat{c}_{-j} \exp(-i \sqrt{2} j(\phi(z) + \bar{\phi}(\bar{z}))), \]  

(2.13)

where \(\hat{c}_j = (-1)^{2j(J_0^3 - J_0^3)}\) is a cocycle needed to ensure locality of all the fields with \(j\) half-integer. The other Virasoro primary fields can be obtained from these by taking suitable contour integrals of \(J^\pm\) and \(\bar{J}^\pm\) around these fields. It then follows from (2.13) that \(g_{(j; j, j), (j; -j, -j)} = 1\) for all \(j\), and therefore that the normalisation in (2.9) is correct. The signs in (2.9) will pop up later on whenever we relate in-coming and out-going boundary states.
2.1. Boundary conditions and structure constants

Every boundary condition of a conformal field theory has to satisfy two sets of constraints: the sewing constraints [4] that come from considering the consistency of a boundary conformal field theory on a genus zero world-sheet, and the Cardy condition [1] that arises from analysing a cylinder (i.e., an open string one-loop diagram) with two not necessarily equal boundary conditions. On the face of it, these constraints are independent, although there is circumstantial evidence suggesting that they may not be.

In this paper we shall analyse in detail one of the sewing constraints of [4] (that corresponds to the so-called classifying algebra of [7]). As we shall see, for $\mathfrak{su}(2)_1$ this constraint is already fairly restrictive and allows one to cut down the possible space of conformal D-brane states quite significantly, although the theory is not rational with respect to the Virasoro algebra. We shall also verify that all solutions of this constraint actually satisfy the Cardy condition.

Let us begin by introducing some notation. We take, without loss of generality, the genus zero world-sheet to be the upper half-plane with coordinates $z = x + iy$, bounded by the real line. The bulk structure constants arise in the operator product expansions of the primary bulk fields,

$$
\varphi_a(z, \bar{z}) \varphi_b(w, \bar{w}) = \sum_c C_{ab}^c (z - w)^{h_c - h_a - h_b} (\bar{z} - \bar{w})^{h_c - \bar{h}_a - \bar{h}_b} (\varphi_c(w, \bar{w}) + \ldots), \quad (2.14)
$$

where, as before, the label $a$ denotes the different Virasoro primary fields, and the ellipses correspond to higher order corrections in $(z - w)$ or $(\bar{z} - \bar{w})$. Similarly, the boundary structure constants arise in the operator product expansion of boundary fields

$$
\psi_i^{\alpha\beta}(x) \psi_j^{\beta\gamma}(y) = \sum_k c_{ij}^{(\alpha\beta\gamma)k} (x - y)^{h_k - h_i - h_j} (\psi_k^{\alpha\gamma}(y) + \ldots), \quad x > y. \quad (2.15)
$$

Finally, when a primary bulk field $\varphi_a$ approaches the real line with boundary condition $\alpha$, it can be expanded over the fields on the boundary, defining the bulk-boundary structure constants $(\alpha)B^i_a$,

$$
\varphi_a(z, \bar{z}) = \sum_i (\alpha)B^i_a (2y)^{h_i - h_a - \bar{h}_a} (\psi_i^{\alpha\alpha}(x) + O(y)) \quad (2.16)
$$

with $z = x + iy$. Taken together, these structure constants determine the operator algebra uniquely, and therefore all correlation functions, at least in principle. In practice, the correlation functions are found as sums of chiral blocks (which express the factorisation of the correlation functions on internal channels) multiplied by the appropriate structure
constants. The chiral blocks are uniquely determined by the representation theory of the symmetry algebra of the boundary CFT (the Virasoro algebra, in our case). Different possible ways to factorise on internal channels lead to different representations for the same correlation function, and the equality of these representations leads to constraints on the structure constants, known as the sewing constraints.

The constraint we are most interested in arises from considering a two-point function of bulk primary fields on the upper half-plane,

\[
\varphi_a(z, \bar{z}) \varphi_b(w, \bar{w}) = \langle \varphi_a(z, \bar{z}) \varphi_b(w, \bar{w}) \rangle. \tag{2.17}
\]

The gluing conditions for the energy-momentum tensor imply that (2.17) can be described in terms of four-point chiral blocks where we insert chiral vertex operators of weight \( h_a, \bar{h}_a, h_b \) and \( \bar{h}_b \) at \( z, \bar{z}, w \) and \( \bar{w} \), respectively.

This four-point function can be factorised in two ways, which lead to two different representations of the correlation function, as shown below. In the first picture, one applies the bulk-boundary OPE to both bulk fields first and then evaluates two-point functions of boundary fields. In the second picture, the bulk OPE is performed first, and the bulk-boundary OPE is then applied to the various terms in the OPE of the bulk fields.

In writing down these equations, we have dropped some superscripts labelling boundary conditions, and we have specialised to the case where \( h_a = \bar{h}_a \) and \( h_b = \bar{h}_b \). The \( f^i \) and \( f^c \)
denote the different chiral four-point blocks, and \( \eta \) is the cross-ratio \( \eta = |(z - w)/(z - \bar{w})|^2 \), which is real with \( 0 \leq \eta \leq 1 \). In both equations, the \( k \)-summation is over all boundary fields of zero conformal weight, and the fields \( \psi_i \) and \( \psi_j \) are necessarily of the same conformal weight; both of these statements follow from \( \text{SL}(2, \mathbb{R}) \)-covariance of the boundary vacuum correlation functions. Normally one assumes that there is a single bulk field of weight zero (the identity) but for boundary fields there are many situations in which one may want to have a multiplicity of weight zero fields, for example to consider superpositions of branes or to introduce Chan-Paton factors.

The two sets of chiral blocks are related by the so-called fusing matrices (explicit expressions for the fusing matrices of degenerate Virasoro representations at \( c = 1 \) relevant here are given in [8])

\[
f^c_{\begin{bmatrix} b & b \\ a & a \end{bmatrix}} (1 - \eta) = \sum_i F_{ci} \begin{bmatrix} b & b \\ a & a \end{bmatrix} f^i_{\begin{bmatrix} a & b \\ a & b \end{bmatrix}} (\eta),
\]

so that substituting (2.20) into (2.19) and comparing with (2.18), one finds the sewing relation

\[
\sum_{i,j}^{(\alpha)} B_{a}^{i} \cdot (\alpha) B_{b}^{j} c_{ij}^{k} \langle \psi_{k} \rangle = \sum_{c}^{c} C_{a b}^{c} F_{ci} \begin{bmatrix} b & b \\ a & a \end{bmatrix}^{(\alpha)} B_{c}^{k} \langle \psi_{k} \rangle.
\]

If the algebra of weight zero boundary fields is commutative (and satisfies suitable further conditions, for example semi-simplicity), one can find a basis of projectors onto ‘fundamental boundary conditions’ which support a single weight zero field (see e.g. [8] for some examples). We shall now restrict attention to such fundamental boundary conditions \( \alpha \) and denote the unique boundary field of weight 0 by \( \mathbb{1}_{\alpha} \). Taking the \( k = \mathbb{1} \) channel in (2.21), we then arrive at the relation

\[
^{(\alpha)} B_{a}^{\mathbb{1}} \cdot (\alpha) B_{b}^{\mathbb{1}} = \sum_{c}^{c} C_{a b}^{c} F_{c}^{\mathbb{1}} \begin{bmatrix} b & b \\ a & a \end{bmatrix}^{(\alpha)} B_{c}^{\mathbb{1}}.
\]

Let us add some remarks on the interpretation of this relation, and on the mathematical structure it defines. First of all, the condition can be recognised as the cluster condition on the correlation functions of the boundary OPE: the projection on \( h = 0 \) channels that occurred in (2.18) and (2.19) also arises if the separation of the two bulk fields parallel to the boundary is increased, so when checking (2.22) we are investigating the ‘long-range behaviour’ of the correlator (2.17), which should ‘cluster’ as usual; see also [6]. Ignoring issues of physical interpretation, it is tempting to regard (2.22) as the defining relations of an algebra, with ‘structure constants’ \( M_{a b}^{c} \equiv C_{ab}^{c} \cdot F_{c}^{\mathbb{1}} \begin{bmatrix} b & b \\ a & a \end{bmatrix} \),

\[
B_{a} B_{b} = \sum_{c} M_{a b}^{c} B_{c},
\]

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where, for the moment, the $B_a$ are abstract symbols. If these relations define a commutative and associative algebra, the constraint on the fundamental boundary conditions simply says that the bulk-boundary coefficients $\langle \alpha \rangle B_a^\Pi$ have to form a one-dimensional representation of (2.23); this is the reasoning behind calling it the ‘classifying algebra’ [7]. However, one should recall that it is not yet clear whether every boundary condition that satisfies this constraint will also satisfy the other sewing constraints, or Cardy’s condition (and therefore whether this algebra really provides a one-to-one classification of the boundary conditions).

In general, surprisingly little is known about the structure defined by (2.23). It is easy to show that the algebra is commutative since both $C^{c}_{ab}$ and $F^{c}_{b} [a_{\alpha} a_{\beta}]$ are symmetric under the exchange of $a$ and $b$. (Since we are only considering the Virasoro symmetry, every primary field is self-conjugate; the symmetry of the fusing matrix then follows from [9], eq. (4.9).) On the other hand, as far as we are aware, a general proof that the algebra is associative is still missing. In simple cases such as rational conformal field theories with charge-conjugate partition function and with standard gluing conditions imposed (which, in particular, preserve the full symmetry algebra), one can show that the classifying algebra is nothing but the fusion ring, $M_{ab}^c = N_{ab}^c$, see [10,7,2]. Under the same assumptions, solutions to all sewing relations were found in [11], expressed in terms of the representation category of the chiral algebra. The structure of (2.23) was further clarified and extended to certain non-diagonal modular invariants of SU(2) and Virasoro minimal models in [2]; for these cases, the $M_{ab}^c$ are structure constants of a Pasquier algebra, which in turn opens up interesting connections to quantum symmetries, see [12].

2.2. The boundary state formalism

The bulk-boundary constants $\langle \alpha \rangle B_a^\Pi$ are also an essential ingredient in the boundary state formalism that is particularly suited to discuss the Cardy condition. Here a boundary condition $\alpha$ is represented by a generalised coherent state $\| \alpha \|$ in the bulk theory that satisfies a number of conditions. In order to relate boundary states with bulk-boundary constants, let us consider the bulk theory as being defined in the region outside the unit disk; the bulk correlators in the presence of the boundary then provide a map from the space of states, with field insertions outside the unit disk, to the complex numbers, and thus can be represented by a boundary state as

$$\langle \psi | \mapsto \langle \psi | \alpha \rangle . \quad (2.24)$$

The condition that the boundary preserves the conformal symmetry becomes

$$\langle L_m - \bar{L}_{-m} \| \alpha \rangle = 0 . \quad (2.25)$$
By regarding this as an intertwiner of the left and right actions of the Virasoro algebra, one can prove that a basis of solutions is given by the Ishibashi states $|a\rangle$ associated to bulk Virasoro primary fields $\varphi_a$ with $h_a = \bar{h}_a$ [13]. Thus we can write

$$\|\alpha\rangle = \sum_{(j,m,n)} A_{(j;m,n)}^{\alpha} |j; m, n\rangle,$$  \hspace{1cm} (2.26)

where $A_{(j;m,n)}^{\alpha}$ are some constants (that characterise the boundary condition).

Alternatively, we can consider the bulk theory to be defined on the interior of the unit disk, and describe the boundary condition in terms of an ‘out’-boundary state $\langle\langle \alpha |$ which satisfies the obvious analogue of (2.25), and for which a basis can be written in terms of Ishibashi states $\langle\langle a |$; the general setup is represented diagrammatically as below

$$\varphi(z, \bar{z})$$

If we take the boundary condition $\alpha$ to be defined by (2.26), then the boundary state $\langle\langle \alpha |$ that corresponds to the same boundary condition $\alpha$, now viewed as ‘out-going’, is given by

$$\langle\langle \alpha | = \sum_{(j,m,n)} \sum_{(j',m',n')} g_{(j;m,n),(j',m',n')} A_{(j;m,n)}^{\alpha} \langle\langle j'; m', n' |,$$  \hspace{1cm} (2.28)

because with this convention both boundary states have the same leading behaviour in the limit in which an arbitrary bulk field approaches the boundary: for (2.26) the relevant limit is

$$\alpha B_{(j;m,n)}^{\|} = \lim_{|z| \to 1} (|z|^2 - 1)^{2h_j} \frac{\langle 0 | \varphi_{m,n}^j(z, \bar{z}) \| \alpha \rangle}{\langle 0 \| \alpha \rangle}$$

$$= \frac{1}{\langle 0 \| \alpha \rangle} \sum_{(j,m,n)} \sum_{(j',m',n')} g_{(j;m,n),(j',m',n')} \langle j'; m', n' \| \alpha \rangle$$

$$= (-1)^{m-n} A_{(j;-m,-n)}^{\alpha}/A_{(0;0,0)}^{\alpha},$$  \hspace{1cm} (2.29)
where we have used the conformal symmetry of the amplitudes as well as (2.8) and the fact that (in our conventions) \( g_{0,0} = 1 \). In the last line we have furthermore replaced \( g \) by the explicit expression given in (2.9). This then agrees with the relevant limit for (2.28)

\[
\alpha B_{(j;m,n)}^{a} = \lim_{|z| \rightarrow 1} (1 - |z|^2)^{2h_j} \frac{\langle \alpha | \varphi_{m,n}(z, \bar{z}) \rangle 0}{\langle \alpha | 0 \rangle}
\]

\[
= \frac{1}{\langle \alpha | 0 \rangle} \langle \alpha | j; m, n \rangle \tag{2.30}
\]

\[
= (-1)^{m-n} A_{(j; -m, -n)}^{\alpha} / A_{(0,0,0)}^{\alpha}.
\]

Incidentally, as is clear from the discussion of the previous subsection, the relevant numerical coefficients are the bulk-boundary structure constants that occur in (2.22).

As an aside, let us note that the same relation between incoming and outgoing boundary states (2.26) and (2.28), which we have derived from an analysis of the bulk-boundary OPE, arises if one defines the outgoing boundary state with the help of the CPT operator \( \Theta \). Using that the latter acts trivially on \( \tilde{\text{su}}(2)_1 \) Ishibashi states, we have

\[
\langle \alpha | = (\Theta | \alpha \rangle\rangle^* \tag{2.31}
\]

where the star denotes the ordinary conjugation in the bulk Hilbert space that maps \( |a\rangle \) to \( \langle a| \). The CPT operator was used in the computation of open string amplitudes in [14,15]; geometrically, it arises because the two boundary components of the cylinder diagram have opposite orientations.

Having introduced this machinery, including a careful treatment of the relative normalisation of Virasoro primaries, we can now discuss Cardy’s condition in more detail. Let us consider a cylindrical world-sheet with boundary conditions \( \alpha \) and \( \beta \), where the cylinder has length \( L \) and circumference \( R \). By a conformal transformation this can be mapped to an annulus of inner radius \( \exp(-2\pi L/R) \) and outer radius 1, or to a semi-annulus on the upper half-plane of inner radius 1 and outer radius \( \exp(\pi R/L) \). The corresponding partition function can therefore be expressed in two different ways,

\[
Z_{\alpha\beta}(L, R) = \langle \alpha | q^{\frac{1}{2}(L_0 + \bar{L}_0 - c/12)} | \beta \rangle = \text{Tr}_{H_{\alpha\beta}}(\tilde{q}^{L_0 - c/24}) \tag{2.32}
\]

where in the second expression \( H_{\alpha\beta} \) is the Hilbert space for the upper half-plane with boundary condition \( \alpha \) on the positive real axis and \( \beta \) on the negative real axis; furthermore, \( q := \exp(-4\pi L/R) \) and \( \tilde{q} := \exp(-\pi R/L) \). Since the last expression is a trace in a representation space of the Virasoro algebra, we can decompose \( Z_{\alpha\beta}(L, R) \) in terms of irreducible Virasoro representations,

\[
Z_{\alpha\beta}(L, R) = \sum_i n_{i\alpha}^{\beta} \chi_i(\tilde{q}) \tag{2.33}
\]
Cardy’s condition is then the requirement that the multiplicities $n_{i\alpha \beta}$ are non-negative integers. Strictly speaking, this condition only makes sense if the spectrum of Virasoro representations that occurs in (2.33) is discrete (as is necessarily the case for rational theories). The $\hat{su}(2)_1$ model we are considering here is only quasi-rational with respect to the Virasoro algebra (i.e. the model contains an infinite number of Virasoro representations, but each operator product of primary fields only contains a finite number of primary fields). However, as we shall see, the Virasoro spectrum that actually arises for the different overlaps (corresponding to pairs of boundary states that satisfy (2.22)) will always be discrete, and indeed, will always lead to non-negative integers $n_{i\alpha \beta}$. This is a strong consistency check on our construction.

3. The general solution to the cluster condition

Our next aim is to compute the structure constants of the algebra (2.22) for the case of $\hat{su}(2)_1$. The idea of the derivation is to deduce these structure constants from the knowledge of a sufficiently large class of solutions.

As we have seen above, every fundamental boundary condition (that only preserves the conformal symmetry) has to solve (2.22). While the general solution to this problem has not been given so far (we shall present the complete solution in the next section), a large class of boundary conditions that are believed to be fully consistent have been constructed before. These boundary conditions are characterised by the property that they preserve the full $\hat{su}(2)$ current symmetry. The corresponding boundary states satisfy the gluing conditions

$$ (J^a_m + \text{Ad}_{(g,\iota)}(\bar{J}^a_m)) \| g \rangle = 0, \quad m \in \mathbb{Z}, \quad \iota = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (J_1) $$

for some $g \in SU(2)$, see [5,6]. (We have included the matrix $\iota$ in this definition for later notational convenience; our conventions are such that the usual Cardy boundary state $\| 0 \rangle \rangle_{\text{Cardy}}$ associated with the $\hat{su}(2)_1$ vacuum representation is given by $\| 0 \rangle \rangle_{\text{Cardy}} = \| -\iota \rangle \rangle$. If one wants to avoid the $\iota$, one could work with ‘Neumann-like’ Virasoro Ishibashi states $\| j; m, -n \rangle \rangle$ instead of the ‘Dirichlet-like’ $\| j; m, n \rangle \rangle$ used in the following.) The boundary states in (3.1) are marginal deformations of $\hat{su}(2)_1$ Cardy states. In terms of our Virasoro Ishibashi states, they are explicitly given as

$$ \| g \rangle \rangle = \frac{1}{2^{1 \over 2}} \sum_{(j; m, n)} D^j_{m, n}(g) \| j; m, n \rangle \rangle, \quad (3.2) $$
where $D_{m,n}^j(g)$ is the matrix element of $g$ in the representation $j$.

Because of (2.29) the corresponding boundary structure constants for these boundary conditions are just matrix elements of SU(2),

$$gB^g_{(j;m,n)} = (-1)^{m-n} D_{-m,-n}^j(g) = D_{m,n}^j(i \cdot g \cdot i^{-1}). \quad (3.3)$$

An explicit formula for the matrix elements can be found in [16]

$$D_{m,n}^j(g) = \sum_{l=\max(0,n-m)}^{\min(j-m,j+n)} \frac{[(j+m)!(j-m)!(j+n)!(j-n)!]^{1/2}}{(j-m-l)!(j+n-l)!(m-n+l)!} \times a^{j-n-l} b^{j-m-l} c^{m-n+l}, \quad (3.4)$$

where we have written $g \in SU(2)$ as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (3.5)$$

with $c = -b^*$ and $d = a^*$. Some useful relations involving the matrix elements $D_{m,n}^j(g)$ have been collected in the appendix.

If we set $g = e$ in (3.2), the boundary state describes the standard Dirichlet brane (at $x = 0$). On the other hand, standard Neumann branes correspond to the choice $a = d = 0$ in (3.5) – see the remarks after (3.1).

If we assume that the boundary states $\|g\|$ define fundamental boundary conditions in the sense defined before† the different normalisation constants have to satisfy the sewing relation (2.22)

$$D_{m_1,n_1}^{j_1}(g) D_{m_2,n_2}^{j_2}(g) = \sum_{j;m,n} M_{(j_1;m_1,n_1),(j_2;m_2,n_2)}^{(j;m,n)} D_{m,n}^j(g), \quad (3.6)$$

where a priori, the coefficients $M$ on the right hand side are products of bulk structure constants and elements of the (Virasoro) fusing matrix.

The key observation is now that this family of equations (one for each group element $g \in SU(2)$) already determines $M$ uniquely. This can be seen as follows: because of

† While it is possible to normalise each boundary condition $\alpha$ so that $Z_{\alpha\alpha}$ contains the vacuum representation of the Virasoro algebra precisely once, there is a non-trivial check that the resulting expressions for the pairwise overlaps $Z_{\alpha\beta}$ are also given by a sum of Virasoro characters with non-negative integer coefficients. As we shall see below, the above boundary states have this property (see also [17,18,6]).
the Peter-Weyl Theorem (see for example [19]), the matrix elements $D_{m,n}^j(g)$ define an orthogonal basis of $L^2$ functions on the group manifold SU(2). Since the product of two matrix elements $D_{m,n}^j(g)$ is again an $L^2$ function on SU(2), it can be expanded uniquely in terms of these matrix elements; the expansion coefficients are precisely the structure constants $M$, that are thus uniquely determined by (3.6).

Next we observe that (3.6) is solved if $M$ equals (see for example [16] eq. (5-116))

$$M_{(j_1;m_1,n_1),(j_2;m_2,n_2)}^{(j;m,n)} = (j_1m_1,j_2m_2|jm)(jn|j_1n_1,j_2n_2), \quad (3.7)$$

where $(j_1m_1,j_2m_2|jm)$ and $(jn|j_1n_1,j_2n_2)$ are the Clebsch-Gordan coefficients that describe the decomposition of the tensor product $j_1 \otimes j_2$ in terms of the representation $j$. Indeed, the left hand side of (3.6) is the matrix element between the states labelled by $(m_1 \otimes m_2)$ and $(n_1 \otimes n_2)$ in the tensor product of the representations $j_1$ and $j_2$; the Clebsch-Gordan coefficients describe the decomposition of this tensor product into irreducible representations, and therefore these matrix elements must agree with the right hand side of (3.6). Since the structure constants $M$ are uniquely determined by (3.6), we may conclude that $M$ must be given by (3.7).

The derivation of (3.7) is somewhat indirect, and indeed assumes that the familiar set of boundary states $|g\rangle$ for $\widehat{su}(2)_1$ actually define consistent boundary conditions. As a consistency check on our analysis we have in a few cases verified by explicit computation that the constants $M$ so derived agree with the formula in terms of the bulk structure constants and the fusing matrices.

3.1. The general solution

As we have seen before, the normalisation constants of every fundamental D-brane define a one-dimensional representation of the algebra (2.22). Now that we have identified the structure constants of this algebra, we can classify all its one-dimensional representations, and thus obtain an 'upper bound' on the set of all fundamental branes of the theory, irrespective of whether these boundary conditions preserve $\widehat{su}(2)_1$ or just the Virasoro algebra.

Let us denote by $B_{m,n}^j = ^\alpha B_{m,n}^j$ the relevant bulk-boundary structure constants for such a general boundary state labelled $\alpha$. We are looking for the most general solution to the (non-linear) equation

$$B_{m_1,n_1}^{j_1} B_{m_2,n_2}^{j_2} = \sum_{j;m,n} M_{(j_1;m_1,n_1),(j_2;m_2,n_2)}^{(j;m,n)} B_{m,n}^j \cdot (3.8)$$
If the solution is non-trivial (i.e. if at least one $B^j_{m,n} \neq 0$), then we have to have $B^0_{0,0} = 1$, since the Clebsch-Gordan coefficients satisfy $(j'n'jn,00) = \delta_{jj'}\delta_{nn'}$. Next we observe that once we have chosen $B^j_{1,2} m,n = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)_{m,n}$, all other coefficients $B^j_{m,n}$ are uniquely determined (if they can be consistently found); this is simply a consequence of the fact that every representation of SU(2) is contained in multiple tensor products of the $j = \frac{1}{2}$ representation. It thus remains to analyse what consistency condition on (3.9) guarantees that a solution for all $B^j_{m,n}$ can be obtained.

One such consistency condition can be easily found: using the explicit form of the Clebsch-Gordan coefficients, see e.g. eq. (9-122) of [16], we have that

$$B^j_{\frac{1}{2},\frac{1}{2}} - B^j_{-\frac{1}{2},-\frac{1}{2}} = \frac{1}{2}B^1_{0,0} - \frac{1}{2}B^0_{0,0},$$

$$B^j_{\frac{1}{2},\frac{1}{2}} - B^j_{-\frac{1}{2},-\frac{1}{2}} = \frac{1}{2}B^1_{0,0} + \frac{1}{2}B^0_{0,0}.$$

This implies, in particular, that the matrix in (3.9) must have determinant one:

$$B^j_{\frac{1}{2},-\frac{1}{2}} - B^j_{\frac{1}{2},\frac{1}{2}} = ad - bc = B^0_{0,0} = 1.$$

We shall now argue that this is, in fact, the only consistency condition. It is clear from the structure of (3.8) that the consistency conditions are polynomial relations in $a, b, c, d$, and therefore, that the relations do not involve the complex conjugate of any of these coefficients. Furthermore, if (3.9) is a matrix in SU(2), then the consistency conditions are manifestly satisfied. However, the only polynomial relation that is actually satisfied by every SU(2) matrix is the condition that the determinant is equal to one. Thus (3.11) must be the only consistency condition. We therefore conclude that the (fundamental) conformal boundary conditions of $\hat{su}(2)$ at level $k = 1$ are (at most) parametrised by group elements in SL(2, $\mathbb{C}$).

4. Cardy condition and the boundary spectrum revisited

In this section we want to check whether the family of boundary states that are associated to group elements in SL(2, $\mathbb{C}$) actually satisfy Cardy’s condition. To this end we need to calculate the various overlaps between the in- and out-boundary states we have defined before.
The annulus partition function \( \mathcal{A} \equiv Z_{g_1, g_2} \) for an annulus with boundary condition \( g_1 \) on the outer circumference and \( g_2 \) on the inner one is given in terms of boundary states as

\[
\mathcal{A} = \langle g_1 \mid q^{L_0 + \bar{L}_0} \mid g_2 \rangle = \frac{1}{\sqrt{2}} \sum_{j \in \mathbb{Z}_+} \sum_{m,n} (-1)^{m-n} D^j_{m-n}(g_1) D^j_{m,n}(g_2) \chi_{j^2}(q). \tag{4.1}
\]

Because of (A.6) the sum over \( m \) and \( n \) simplifies to

\[
\sum_{m,n} D^j_{n,m}(g_1^{-1}) D^j_{m,n}(g_2) = \sum_n D^j_{n,n}(g_1^{-1} g_2), \tag{4.2}
\]

where we have used the property that the matrix elements \( D^j_{m,n} \) also define a representation for \( \text{SL}(2, \mathbb{C}) \). (Again, this follows from the fact that the representation property is a polynomial relation in \( a, b, c, d \), and that the only polynomial relation that is satisfied by \( \text{SU}(2) \) is \( ad - bc = 1 \).) Next we observe that (4.2) is simply the trace in the \( j^\text{th} \) representation of \( g = g_1^{-1} g_2 \). By conjugation by an element of \( \text{SL}(2, \mathbb{C}) \) (that does not modify the value of the trace), we can bring \( g \) into ‘Jordan normal form’ \( \hat{g} \) (i.e. we can choose \( \hat{c} = 0 \)); the matrix elements \( D^j_{n,n}(\hat{g}) \) then simplify to

\[
D^j_{n,n}(\hat{g}) = \hat{a}^{j+n} \hat{d}^{-n} = \hat{a}^{2n}, \tag{4.3}
\]

where we have used that \( \hat{a} \hat{d} = 1 \) since \( \hat{g} \) has determinant one. The sum over \( n \) can now be easily performed, and we find

\[
\sum_n D^j_{n,n}(g_1^{-1} g_2) = \frac{\sinh((2j + 1)\alpha)}{\sinh(\alpha)}, \tag{4.4}
\]

where \( \hat{a} = e^\alpha \).

Next, we rewrite the Virasoro character \( \chi_{j^2} \) in terms of the functions \( \vartheta \) as in (2.5), and thus obtain

\[
\sqrt{2} \mathcal{A} = \sum_{j \in \mathbb{Z}_+} \frac{\sinh((2j + 1)\alpha)}{\sinh(\alpha)} \left( \vartheta \sqrt{2} j(q) - \vartheta \sqrt{2} (j+1)(q) \right)
\]

\[
= \vartheta_0(q) + \sum_{j=1}^{\infty} \vartheta \sqrt{2} j(q) \left( \frac{\sinh((2j + 1)\alpha)}{\sinh(\alpha)} - \frac{\sinh((2j - 1)\alpha)}{\sinh(\alpha)} \right)
\]

\[
+ 2 \cosh(\alpha) \vartheta \sqrt{2} \frac{\vartheta_0(q)}{\sqrt{2}} + \sum_{j=\frac{3}{2}}^{\infty} \vartheta \sqrt{2} j(q) \left( \frac{\sinh((2j + 1)\alpha)}{\sinh(\alpha)} - \frac{\sinh((2j - 1)\alpha)}{\sinh(\alpha)} \right) \tag{4.5}
\]

\[
= \sum_{j \in \mathbb{Z}} \cosh(2j\alpha) \vartheta \sqrt{2} j(q),
\]

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where the sum in the second (third) line is over integers (half-odd integers) only, and where we have used the identity
\[
\frac{\sinh((2j+1)\alpha)}{\sinh(\alpha)} - \frac{\sinh((2j-1)\alpha)}{\sinh(\alpha)} = 2 \cosh(2j\alpha)
\] 
(4.6)
as well as the fact that \(\vartheta_s(q)\) only depends on \(|s|\). Under a modular transformation, the \(\vartheta\)-functions transform as
\[
\vartheta_s(q) = \int_{-\infty}^{\infty} dt \, e^{2\pi it s} \vartheta_t(\tilde{q}),
\] 
(4.7)
where \(\tilde{q}\) is the ‘open string’ parameter. (More precisely, if we write \(q = e^{2\pi i \tau}\) with \(\text{Im} \tau > 0\), then \(\tilde{q} = e^{-2\pi i \tau}\).) Putting these results together we then find that in the open string description the overlap becomes
\[
A = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} dt \vartheta_t(\tilde{q}) \sum_{j \in \frac{1}{2} \mathbb{Z}} e^{2\pi i \sqrt{2}jt \cosh(2j\alpha)}
\]
\[
= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} dt \vartheta_t(\tilde{q}) \sum_{l \in \mathbb{Z}} e^{il(\sqrt{2}\pi t - i\alpha)}
\]
\[
= \frac{2\pi}{\sqrt{2}} \int_{-\infty}^{\infty} dt \vartheta_t(\tilde{q}) \sum_{n \in \mathbb{Z}} \delta(\sqrt{2}\pi t - i\alpha + 2\pi n)
\]
\[
= \sum_{n \in \mathbb{Z}} \vartheta_{\omega_{\frac{1}{\sqrt{2}}n}}(\tilde{q}).
\] 
(4.8)
For all values of \(\alpha\) this defines a positive integer linear combination of Virasoro characters.

Since \(\alpha\) only depends on \(\text{Tr}(g_1^{-1} g_2)\), the open string spectrum between any of the above D-branes (labelled by \(g_1 \in \text{SL}(2, \mathbb{C})\)) and itself is the same, irrespective of \(g_1\). (For the case when \(g_1 \in \text{SU}(2)\), this was observed before in [17,18,6].) In fact, in this case \(\alpha = 0\), and we therefore obtain only Virasoro representations for which \(h\) is non-negative. Furthermore, the vacuum representation (\(i.e.\) the character with \(h = 0\)) occurs precisely once, as was claimed previously for the case with \(g_1 \in \text{SU}(2)\). Indeed, the fact that this is true for all \(g_1 \in \text{SL}(2, \mathbb{C})\) is necessary for the consistency of our approach: the constraint that a boundary state has to provide a representation of the algebra (2.22) only arises for ‘fundamental’ D-branes, \(i.e.\) for those boundary states for which the vacuum representation appears precisely once in the open string spectrum.

For the case where \(g = g_1^{-1} g_2 \in \text{SU}(2)\), \(\alpha\) is purely imaginary, and thus all Virasoro highest weights that occur in the open string between two such D-branes are positive. However, in general also negative or even complex values for the conformal weights occur in the open string partition function. In particular, if \(g_1\) is an arbitrary element in \(\text{SL}(2, \mathbb{C}) \backslash \text{SU}(2)\), we can always find \(g_2 \in \text{SU}(2)\) so that the overlap between \(g_1\) and \(g_2\) leads to imaginary conformal weights in the open string.
5. Discussion

Let us recapitulate what we have shown in this paper. Under the assumption that the well-known boundary states that are marginal deformations of $\hat{\mathfrak{su}}(2)_1$ Cardy boundary states and are parametrised by $g \in \text{SU}(2)$ define consistent boundary conditions, we have derived the structure constants of the ‘classifying algebra’ (2.22). We have then shown that the most general one-dimensional representation of (2.22) can be described in terms of group elements in $\text{SL}(2, \mathbb{C})$. Since every fundamental D-brane has to satisfy this condition, this implies that the complete set of fundamental conformal boundary conditions for $\hat{\mathfrak{su}}(2)_1$ is contained in this family of boundary states. Naively, one might have expected that the space of conformal boundary conditions (which preserve the Virasoro algebra only and render the boundary theory non-rational) is much larger.

One can ask whether the branes parametrised by $\text{SL}(2, \mathbb{C}) \backslash \text{SU}(2)$ are indeed ‘new’ in the sense that they cannot be written as superpositions of the boundary states associated to elements in $\text{SU}(2)$. Since we are dealing with a continuum of boundary states here, this is not a purely algebraic question, but some analytic considerations come into play. It is natural to believe that the relevant superpositions are of the form

$$\|B\rangle = \int_{\text{SU}(2)} d\mu(g) F_B(g) \|g\rangle,$$

where $F_B$ are tempered distributions on the manifold $\text{SU}(2)$ with Haar measure $d\mu$. (For example, $F_B$ is a delta-function if $\|B\rangle$ is taken from our $\text{SU}(2)$ family of boundary states.) The boundary states associated to $\text{SL}(2, \mathbb{C}) \backslash \text{SU}(2)$ cannot be written in this form, and are therefore likely to be genuinely new.

As we have stressed before, it is a priori not clear whether all of these boundary states actually define consistent boundary conditions. In particular, while all of these more general boundary states satisfy Cardy’s condition, we have not checked whether they satisfy all other sewing constraints.

In general, we have little to say about this problem. However, there exist various subclasses of boundary states which, very plausibly define consistent boundary conditions. In particular, we can consider the cosets of $\text{SU}(2)$ in $\text{SL}(2, \mathbb{C})$. Since the overlap between two boundary states depends only on $\text{Tr}(g_1^{-1}g_2)$, it follows that the relative overlaps of this subset of D-branes are precisely the same as those of the original $\text{SU}(2)$ family. Thus the boundary field content is the same and therefore these branes presumably define consistent boundary conditions. In fact, if we write $g = h \cdot u$ where $h$ is a fixed element $h \in \text{SL}(2, \mathbb{C})$
that characterises the coset, and $u \in SU(2)$ is arbitrary, we can relate the corresponding boundary states as
\[
| h \cdot u \rangle = \exp(\theta_a J_a^0) | u \rangle ,
\]
using a representation $h = \exp(\theta_a t^a)$ in terms of Lie algebra generators. It then follows that the boundary conditions associated to different cosets define completely equivalent field theories since they can be related by conjugation of the bulk fields by $h$.

While (at least) these cosets of boundary states seem to define consistent boundary conditions from the point of view of conformal field theory, the corresponding D-branes are presumably not of interest in string theory (except, of course, for the original D-branes that are associated to group elements in $SU(2)$). Indeed, the simplest example of a group element in $SL(2, \mathbb{C}) \setminus SU(2)$ is
\[
g = \left( \begin{array}{cc} \exp a & 0 \\ 0 & \exp(-a) \end{array} \right),
\]
where $a \neq 0$ is real. In the free-field construction, the corresponding D-brane can be thought of as a Dirichlet brane whose position takes a purely \textit{imaginary} value. In general, one can similarly show that group elements in $SL(2, \mathbb{C}) \setminus SU(2)$ lead to D-branes that have imaginary couplings to some of the bulk fields. If we discard these ‘unphysical’ boundary conditions then the results of our paper imply that all ‘physical’ D-branes of the $\mathfrak{su}(2)_1$ theory preserve actually the full $SU(2)$ symmetry!

The problem of classifying all conformal boundary conditions of a single free boson at the self-dual radius was discussed by Friedan in unpublished work [3]. By the Frenkel-Kac-Segal construction this theory is equivalent to the $SU(2)$ WZW model at $k = 1$ that we have been considering in this paper. Friedan’s paper contains a bare outline of some of the steps in the calculation, and then claims that the space of conformal boundary conditions is $SU(2)$. Since he only provides very few details about the properties he requires of a conformal boundary condition, it is not possible to say quite why he finds this space, and not $SL(2, \mathbb{C})$ as we do. One possible interpretation is that he has implicitly assumed that the in- and out- boundary states are related as in (2.31), without the CPT operator $\Theta$. Alternatively, he may have discarded these additional boundary states because of their unphysical properties from a string theory point of view. However, since his text does not contain any statement about these matters, it is impossible to make a thorough comparison with our results.

Our analysis is quite specific to the $SU(2)$ WZW at level $k = 1$ and does not directly generalise to $k > 1$. Indeed, as was shown recently in [20], the $\mathfrak{su}(2)_k$ theory for $k > 1$ possesses (physical) D-branes that do not preserve the affine symmetry. Extensions to
other examples of truly symmetry breaking boundary conditions (rendering the boundary
theory non-rational) require a relatively detailed understanding of how representations
of the chiral algebra of the bulk theory decompose into those of the smaller symmetry
algebra. On the other hand, the results of this paper should generalise fairly directly to
the case of $\widehat{\text{su}}(n)$ at $k = 1$ where, for $n > 2$, the Virasoro algebra is replaced by $W_n$, the
Casimir algebra of $\widehat{\text{su}}(n)_1$. (The $W$-algebra $W_n$ is the commutant of $\text{su}(n)$ in $\widehat{\text{su}}(n)$.) We
then expect that D-branes that respect the $W_n$ symmetry are parametrised by elements
in $\text{SL}(n, \mathbb{C})$, rather than just by elements in $\text{SU}(n)$. This would provide another class of
examples where there are far fewer symmetry breaking boundary conditions than one may
have naively thought.

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Appendix A. Some properties of matrix elements

In this appendix we shall collect some useful identities involving the matrix elements
(3.4) of $SU(2)$ and fix our conventions for the action of $\text{su}(2)$ generators. First of all, since
the coefficients of the sum in (3.4) are real, we have

$$\left\{ D^j_{m,n} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}^* = D^j_{m,n} \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}, \quad (A.1)$$

Next we observe that

$$D^j_{-n,-m} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = D^j_{m,n} \begin{pmatrix} d & b \\ c & a \end{pmatrix}, \quad (A.2)$$

as follows directly from the definition of (3.4). Similarly we find that

$$(-1)^{m-n} D^j_{m,n} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = D^j_{m,n} \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}. \quad (A.3)$$
Furthermore, by writing out the left hand side and changing the summation variable from \( l \) to \( \tilde{l} = n - m + l \) we derive that
\[
D_{n,m}^j \left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] = D_{m,n}^j \left[ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \right].
\] (A.4)

Taking (A.2), (A.3) and (A.4) together we then obtain
\[
(-1)^{m-n} D_{-m,-n}^j \left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] = D_{m,n}^j \left[ \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \right].
\] (A.5)

This implies that, for all \( g \in \text{SL}(2, \mathbb{C}) \), we have
\[
(-1)^{m-n} D_{-m,-n}^j (g) = D_{n,m}^j (g^{-1}) .
\] (A.6)

For the case where \( g \in \text{SU}(2) \), we can use \( c = -b^* \) and \( d = a^* \) to show that
\[
[D_{m,n}^j (g)]^* = (-1)^{m-n} D_{-m,-n}^j (g) .
\] (A.7)

Finally, the action of \( \text{su}(2) \) in the representation \( V^j \) with basis \( |j,m\rangle \) is determined by (3.4) to be
\[
J^3 |j,m\rangle = m |j,m\rangle ,
\]
\[
J^+ |j,m\rangle = \sqrt{(j-m)(j+m+1)} |j,m+1\rangle ,
\] (A.8)
\[
J^- |j,m\rangle = \sqrt{(j+m)(j-m+1)} |j,m-1\rangle .
\]

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