Polynomial time guarantees for sampling based posterior inference in high-dimensional generalised linear models

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Abstract

The problem of computing posterior functionals in general high-dimensional statistical models with possibly non-log-concave likelihood functions is considered. Based on the proof strategy of [56], but using only local likelihood conditions and without relying on M-estimation theory, non-asymptotic statistical and computational guarantees are provided for gradient based MCMC algorithms. Given a suitable initialiser, these guarantees scale polynomially in key algorithmic quantities. The abstract results are applied to several concrete statistical models, including density estimation, nonparametric regression with generalised linear models and a canonical statistical non-linear inverse problem from PDEs.

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1 Introduction

Posterior inference for high-dimensional statistical models is increasingly important in contemporary applications, particularly in the physical sciences and in engineering [60, 65, 24]. Computing relevant functionals such as the posterior mean, mode or quantiles often relies on iterative sampling algorithms. Without additional structural assumptions, however, the mixing times of these algorithms can scale exponentially in the model dimension $p$ or the sample size $n$ [20, 71]. In this case, valid inference on an underlying ground truth requiring $p \asymp n^\rho$, $\rho > 0$, is intractable. Overcoming such computational hardness barriers is crucial to allow for efficient sampling based Bayesian procedures.

A canonical sampling approach uses Markov chain Monte Carlo (MCMC) algorithms (see, e.g., [61]). They generate a specifically designed Markov chain $(\vartheta_k)_{k=1}^\infty$, whose

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laws $\mathcal{L}(\theta_k)$ approximate up to a target precision level the posterior distribution with probability density
\[
\pi(\theta|Z^{(n)}) \propto e^{\ell_n(\theta)} \pi(\theta), \quad \theta \in \mathbb{R}^p.
\] (1)

Here, the data $Z^{(n)} = (Z_i)_{i=1}^n$ are observations in a statistical model depending on a ground truth $\theta_0 \in \ell^2(N)$ with log-likelihood function $\ell_n$ and prior density $\pi$. Suppose that the computational complexity of some MCMC algorithm arises mainly from the number of iterations. In this case, existing guarantees for practically feasible mixing times, growing at most polynomially in $n$, $p$, are essentially limited to strongly concave $\ell_n$ with Lipschitz-gradients \([13, 19, 43, 75]\). When $\ell_n$ is non-linear, both properties are generally not satisfied, even for Gaussian priors. Relevant examples include mixture models, statistical non-linear inverse problems \([49, 48, 31, 1, 52]\) and generalised linear models (GLMs) \([47]\). For related discussions on MCMC in different statistical settings and closely related optimisation algorithms see \([4, 59, 66, 39, 44, 57, 6, 12]\).

In a recent contribution Nickl and Wang \([56]\) obtain polynomial time sampling guarantees in a specific non-linear example involving a partial differential equation (PDE). To go beyond the non-concave setting, the key-idea, which was later extended by \([8]\) to other PDEs, is to rely on the Fisher information for providing a natural statistical notion of curvature for the log-likelihood function near $\theta_0$. By combining empirical process techniques with tools from Bayesian nonparametrics \([28]\), there exists a high-dimensional region $B \subset \mathbb{R}^p$ of parameters near $\theta_0$, where the posterior measure concentrates most of its mass and where $\ell_n$ is locally strongly concave with high probability. Convexifying $-\ell_n$ yields a surrogate posterior measure, whose log-density $\log \tilde{\pi}(\cdot|Z^{(n)})$ is globally strongly concave with Lipschitz-gradients, and which is close to the true posterior measure in Wasserstein distance with high probability. Given a problem-specific initialiser $\theta_{\text{init}} \in \mathbb{R}^p$ to identify the region $B$ in a data-driven fashion, $\tilde{\pi}(\cdot|Z^{(n)})$ can be leveraged to generate approximate samples of the posterior by a gradient-based Langevin MCMC-scheme with cost depending polynomially on $p$ and $n$, based on recent results by Durmus and Mouline \([18]\).

In the present work we extend this proof strategy beyond the PDE setting to general high-dimensional statistical models. While \([56, 8]\) rely upon M-estimation theory for the maximum-a-posteriori (MAP) estimator to show the log-concave approximation of the posterior measure, our proof takes a novel route and is fully Bayesian. This turns out to be crucial in regression models with unbounded and possibly non-Lipschitz regression functions. Assuming only local likelihood conditions, we obtain non-asymptotic sampling guarantees for posterior functionals with polynomial dependence of key algorithm-specific parameters simultaneously on $n$, $p$ and the target precision level. The abstract hypotheses are verified for concrete statistical models in density estimation and non-parametric regression under Bernstein conditions, including Gaussian and Poisson measurement errors. To demonstrate our approach in a canonical example from the inverse problem literature, we discuss in detail sampling for an elliptic PDE, sometimes called
Darcy’s problem (see [53] and Section 3.7 of [65]), with a non-Lipschitz forward map.

We further prove that the Langevin Markov chain based on the surrogate density takes exponentially in \( n \) many steps to leave the region of local curvature, where it coincides with \( \pi(|Z^{(n)}) \). Our results therefore imply that upon initialising into a region of sufficient local curvature even a standard vanilla Langevin MCMC algorithm is able to compute posterior aspects at polynomial cost. This is consistent with related results for gradient based optimisation algorithms that local curvature near the global optimum can improve the rate of convergence [5]. Sampling algorithms, on the other hand, necessarily have to explore the full parameter space and therefore depend more heavily on global properties of the underlying target distribution. Note that, even if an initialiser near \( \theta_0 \) is available, it is not clear if the computation of posterior functionals, which depend on the whole posterior measure, is feasible. The existence of a suitable initialiser is postulated here, and finding one in polynomial time may be in itself a non-trivial task. We discuss this issue in some concrete examples. Since Gaussian priors are of particular interest in practice, we cannot restrict to compact parameter regions \( a\text{-priori} \), which introduces substantial technical challenges.

Bayesian inference in high-dimensional models has been intensely studied in the literature [34, 25, 27, 38, 68, 73, 26]. Guarantees for MCMC-based posterior sampling algorithms were obtained, e.g., by [32, 6], showing that sampling at polynomial cost is possible, in principle. Their assumptions are, however, rather restrictive and not explicit in their quantitative dependence on \( n \) and \( p \), see [56] for a discussion and additional references. Starting with [13] several works focus on obtaining non-asymptotic results for Langevin-type algorithms and strongly log-concave target measures [50, 17]. To break the ‘curse of dimensionality’, and often also the ‘curse of non-linearity’, other sampling approaches replace the complex posterior measure by a more simple object [58, 74, 63], yielding empirically efficient procedures, but with unclear relation to the true posterior measure.

Conceptually, the approximation by a log-concave measure is different from the more traditional Gaussian Laplace-type approximations [64, 33, 63, 6], where the posterior is replaced by a quadratic with constant covariance matrix relative to a well-chosen centring point (often the MAP estimator). In contrast, we leave the log-likelihood function locally unchanged. This added flexibility seems to be crucial for obtaining the fast convergence towards the posterior measure in our results. Related to this are Bernstein-von Mises theorems, which generally do not hold in high-dimensional settings [11, 23]. In particular, [51] have recently shown that no such theorem exists for Darcy’s problem, while we show that efficient MCMC-sampling is possible (see also [53, Remark 5.4.2]).

This paper is organised as follows: In Section 2 the main results are demonstrated for nonparametric generalised linear models. Section 3 develops the approximation by the surrogate posterior measure in a general context and presents convergence guarantees for the surrogate and vanilla Langevin sampler. In Section 4 applications to several statistical models are discussed in detail, including density estimation, nonparametric
regression and Darcy’s problem. Proofs are deferred to Section 5 and to the Appendix.

We write $a \lesssim b$ if $a \leq Cb$ for a universal constant $C$, and $a \asymp b$ if $a \lesssim b$ and $b \lesssim a$. For a measurable space $(\mathcal{O}, \mathcal{A})$, equipped with a measure $\nu_\mathcal{O}$, let $L^p(\mathcal{O})$, $1 \leq p \leq \infty$, be the spaces of $p$-integrable $\mathcal{A}$-measurable functions with respect to $\nu_\mathcal{O}$, normed by $\|\cdot\|_{L^p}$, and denote by $\ell^p(\mathbb{N})$ the usual spaces of $p$-summable sequences with norm $\|\cdot\|_{\ell^p}$. Set $\|\cdot\| = \|\cdot\|_{\ell^2}$. For a matrix $M \in \mathbb{R}^{p \times p}$ let $\|M\|_{op}$ be the operator norm. The minimal and maximal eigenvalues of a positive symmetric matrix $\Sigma$ are denoted by $\lambda_{\min}(\Sigma)$, $\lambda_{\max}(\Sigma)$. For two Borel probability measures $\mu_1$, $\mu_2$ on $\mathbb{R}^p$ with finite second moments the (squared) Wasserstein distance is defined as $W_2^2(\mu_1, \mu_2) = \inf \int_{\mathbb{R}^p \times \mathbb{R}^p} \|\theta - \theta'\|^2 d\mu(\theta, \theta')$, where the infimum is computed over all couplings $\mu$ of $\mu_1$, $\mu_2$. Denote by $C^k(\mathcal{O})$, $0 \leq k \leq \infty$, the spaces of $k$-times differentiable real-valued functions. For a real-valued function $f : \mathbb{R}^p \to \mathbb{R}$, its gradient and Hessian, if existing, are denoted by $\nabla f$, $\nabla^2 f$, respectively. We say that $f$ is Lipschitz if the norm $\|f\|_{\text{Lip}} = \sup_{x, y \in \mathbb{R}^p, x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|}$ is finite. Moreover, we say $f$ is globally $m_f$-strongly concave and has $\Lambda_f$-Lipschitz gradients for $\Lambda_f, m_f > 0$, if for all $\theta, \theta' \in \mathbb{R}^p$

$$\|\nabla f(\theta) - \nabla f(\theta')\| \leq \Lambda_f \|\theta - \theta'\|,$$

$$f(\theta') \leq f(\theta) + (\theta - \theta')^T \nabla f(\theta) - \frac{m_f}{2} \|\theta - \theta'\|^2.$$

2 Main results for generalised linear models

In this section we illustrate our main results in the concrete setting of the GLMs introduced by Nelder and Wedderburn in [51]. They comprise several important non-linear statistical regression models such as Gaussian, Poisson and logistic regression. Bayesian inference for GLMs is a classical topic [16], for posterior contraction in high-dimensional statistical regression models such as Gaussian, Poisson and logistic regression. Bayesian produced by Nelder and Wedderburn in [51]. They comprise several important non-linear

Let $\Theta \subset \ell^2(\mathbb{N})$ be a parameter space containing $\mathbb{R}^p$. Let $(\mathcal{X}, \mathcal{A})$ be a measurable space equipped with a measure $\nu_{\mathcal{X}}$ and let $\xi$ be a probability measure on $\mathbb{R}$. Set $\nu = \xi \otimes \nu_{\mathcal{X}}$. For an orthonormal basis $(e_k)_{k \geq 1}$ of $L^2(\mathcal{X})$ let $\Phi(\theta) = \sum_{k=1}^{\infty} \theta_k e_k$ and let $g : \mathcal{I} \to \mathbb{R}$ be an invertible and continuous link function on some interval $\mathcal{I} \subset \mathbb{R}$.

Suppose that $n$ independent observations $Z^{(n)} = (Y_i, X_i)_{i=1}^{n}$ with values in $(\mathbb{R} \times \mathcal{X})^n$ are drawn from a distribution $P_\theta^n = \otimes_{i=1}^{n} P_\theta$, $\theta \in \Theta$, such that the law of the response variables $Y_i$ follows conditional on $X_i = x$ a one-parameter exponential family with $g(\mathbb{E}_{\theta}[Y_i|X_i]) = \Phi(\theta)(X_i)$. If $p_{\mathcal{X}}$ denotes the $\nu_{\mathcal{X}}$-density of the covariates $X_i$, then this means that the coordinate $\nu$-densities $p_{\theta}$ are of the form

$$p_{\theta}(y, x) = \exp\left(yb(\theta)(x) - A(b(\theta)(x))\right) p_{\mathcal{X}}(x), \quad y \in \mathbb{R}, x \in \mathcal{X},$$

(2)
with $A(h) = \log \int e^{\nu h} d\xi (y)$ and where the natural parameter is given by the generally non-linear scalar-valued function
\[
b(\theta) = (A')^{-1} \circ g^{-1} \circ \Phi(\theta). \tag{3}\]

The observations $Z^{(n)}$ follow then the nonparametric regression model
\[
Y_i = g^{-1} \circ \Phi(\theta)(X_i) + \epsilon_i, \quad E_{\theta}[\epsilon_i|X_i] = 0.
\]

2.1 Prior and posterior

Introduce for $\alpha \in \mathbb{R}$ the $\ell^2(\mathbb{N})$-Sobolev spaces
\[
h^\alpha(\mathbb{N}) = \left\{ \theta \in \ell^2(\mathbb{N}) : \|\theta\|^2_\alpha = \sum_{k=1}^{\infty} k^{2\alpha} \theta_k^2 < \infty \right\}.
\]

Assuming that the data are generated according to $P_{\theta_0}$ for a parameter $\theta_0 \in h^\alpha(\mathbb{N})$ with regularity $\alpha \geq 0$, a popular ‘sieve’ prior distribution puts independent scalar Gaussian priors with increasing variances on the first $p$ coefficients $\theta \in \mathbb{R}^p$, that is,
\[
\theta \sim \Pi \equiv \Pi_n = N(0, n^{-1/(2\alpha + 1)}\Sigma_\alpha^{-1}), \quad \Sigma_\alpha = \text{diag}(1, 2^{2\alpha}, \ldots, p^{2\alpha}). \tag{4}\]

Let $\pi \equiv \pi_n$ denote the density of $\Pi$. The posterior measure $\Pi(\cdot|Z^{(n)})$ then arises from the observations $Z^{(n)}$ using Bayes’ formula with probability density
\[
\pi(\theta|Z^{(n)}) = \frac{\prod_{i=1}^{n} p_\theta(Y_i, X_i)\pi(\theta)}{\int_\Theta \prod_{i=1}^{n} p_\theta(Y_i, X_i)\pi(\theta)d\theta} \propto e^{\ell_n(\theta) - n^{1/(2\alpha + 1)}\|\theta\|_\alpha^2/2}, \quad \theta \in \mathbb{R}^p, \tag{5}\]

where the log-likelihood function of the data $Z^{(n)}$ equals up to additive constants (not depending on $\theta$)
\[
\ell_n(\theta) = \sum_{i=1}^{n} (Y_i b(\theta)(X_i) - A(b(\theta)(X_i))). \tag{6}\]

Note that this is independent of $p_X$. Together with the Gaussian prior the log-posterior density is strongly concave if $\ell_n$ is concave, and has Lipschitz gradients if $\ell_n$ does. Both properties generally fail for an arbitrary link function $g$ and hold even for the canonical link only in exceptional cases.

Example 1. Consider the canonical link function $g = (A')^{-1}$. Then $b(\theta) = \Phi(\theta)$ is linear, $A$ is convex and $\ell_n$ concave. For Gaussian and logistic regression with $A(x) = x$ and $A(x) = \log(1 + e^x)$, respectively, $A''$ is bounded and $\nabla \ell_n$ uniformly Lipschitz. See Remark 8 for a comment on sampling guarantees in this case.

Example 2. In Poisson regression with the canonical link we have $A(x) = e^x - 1$ and the Lipschitz constant of $\nabla \ell_n$ grows exponentially as $\|\theta\| \to \infty$. Since the Gaussian prior is supported on all of $\mathbb{R}^p$, the Lipschitz-property cannot be enforced by restricting to $\theta$ in a ball of fixed Euclidean radius.
2.2 Local curvature

Instead of sampling directly from $\Pi(\cdot|Z(n))$ let us first determine a high-dimensional and statistically informative set of parameters, where the curvature of $\ell_n$ can be quantified depending on $n$ and $p$. Our candidate for this is

$$B = \{\theta \in \mathbb{R}^p : \|\theta - \theta_{*,p}\| \leq \eta\},$$

where $\eta > 0$ and $\theta_{*,p} = (\theta_{0,1}, \ldots, \theta_{0,p})$ is the $\mathbb{R}^p$-projection of $\theta_0$, which we assume to be a sufficiently good approximation of $\theta_0$. To identify the region $B$ suppose also that we dispose of a ‘proxy’ $\theta_{\text{init}} \in B$, which will also serve as the initialiser of the MCMC scheme in the next section. We will establish that the eigenvalues of $-\nabla^2 \ell_n(\theta)$ are on $B$, up to an absolute factor, contained in the interval $[n, np^{1/2}]$ with high $\mathbb{P}_{\theta_0}$-probability as soon as the map $b$ in (3) is uniformly bounded on $B$. Sufficient conditions for this are as follows.

**Condition 3.** Suppose that $\theta_0 \in h^\alpha(N)$ for $\alpha > 1$ and let $p \leq C n^{1/(2\alpha+1)}$, $C > 0$. The radius of $B$ is $\eta = p^{-1/2}$, and $\theta_{*,p}, \theta_{\text{init}}$ are such that

$$\|\theta_0 - \theta_{*,p}\| \leq c_0 n^{-\alpha/(2\alpha+1)}, \quad \|\theta_{\text{init}} - \theta_{*,p}\| \leq \eta/8,$$

$c_0 > 0$. The design is bounded in the sense that $c_X^{-1} \leq p_X(x) \leq c_X$ for all $x \in X$ and some $c_X > 0$, and the basis functions satisfy $\sup_{k \geq 1} \sup_{x \in X} |e_k(x)| \leq c_X$.

A suitable proxy $\theta_{\text{init}}$ can be computed at polynomial cost from the data $Z(n)$, for instance, by the estimators in [22, 45, 72]. Note that the radius $\eta$ is much larger than the minimax rate $n^{-\alpha/(2\alpha+1)}$ for an $\alpha$-smooth ground truth in the underlying regression model, cf. [10]. While the region $B$ shrinks as $p \to \infty$, its radius $\eta \gtrsim n^{-\alpha/(2\theta+1)}$ is large relative to the typical size of a ball around $\theta_{*,p}$ on which most of the posterior mass is concentrated with high $\mathbb{P}_{\theta_0}$-probability (for a precise definition see (14) below). For an alternative initialisation with constant radius $\eta$ see Remark 7.

With this we define in (17) a surrogate log-likelihood function $\tilde{\ell}_n$, which coincides with $\ell_n$ on $B$ and which is $\tilde{m}$-strongly concave with $\tilde{\Lambda}$-Lipschitz gradients for $\tilde{m} \asymp n$, $\tilde{\Lambda} \asymp np^{1/2}$. This induces the surrogate posterior measure $\tilde{\Pi}(\cdot|Z(n))$ with density

$$\tilde{\pi}(\theta|Z(n)) \propto e^{\tilde{\ell}_n(\theta)-n^{1/(2\alpha+1)}\|\theta\|_\alpha^2}.$$  

It coincides with $\Pi(\cdot|Z(n))$ on measurable subsets of $B$ up to random normalising factors.

2.3 Sampling guarantees

A standard MCMC approach for sampling from the Gibbs-type measure with density (9) is the unadjusted Langevin algorithm [62]. It takes an initialiser $\tilde{\theta}_0$, a step size $\gamma > 0$ and
independent $p$-dimensional Gaussian innovations $\xi_k \sim N(0, I_{p \times p})$ as input, and produces a Markov chain with iterates $\theta_k \in \mathbb{R}^p$, where

$$
\tilde{\theta}_{k+1} = \tilde{\theta}_k + \gamma \nabla \log \tilde{\pi}(\tilde{\theta}_k|Z^{(n)}) + \sqrt{2\gamma} \xi_{k+1},
$$

(10)

$$
= \tilde{\theta}_k + \gamma \left( \nabla \ell_n(\tilde{\theta}_k) - n^{1/(2\alpha+1)} \Sigma_\alpha \tilde{\theta}_k \right) + \sqrt{2\gamma} \xi_{k+1}.
$$

(11)

We initialise at $\tilde{\theta}_0 \equiv \theta_{\text{init}} \in \mathcal{B}$. Since $\ell_n$ coincides with $\ell_n$ on a set where the posterior puts most of its mass, we expect that the invariant measure of the Markov chain is close to the true posterior measure, while the global concavity leads to fast mixing.

In our first main result we derive an exponential concentration inequality under the law of the Markov chain, denoted by $P$, for the approximation of posterior functionals by ergodic averages. It requires a sufficiently small $\gamma$, a burn-in time $J_{\text{in}} \geq 1$ and a precision level $\varepsilon > 0$, which is lower bounded according to the sample size and a discretisation 'bias' relative to the continuous time formulation of (11) as $\gamma \to 0$. Note that the theorem is explicit in the dependence of constants on $p, n$ and non-asymptotic in the sense that whenever the hypotheses hold for pairs $p, n$, then so do the conclusions, which are informative only as $n \to \infty$.

**Theorem 4.** Suppose the data arise in a GLM with coordinate densities (2) and $g \in C^3(T)$. Let Condition 3 be satisfied and let $(\tilde{\theta}_k)_{k \geq 1}$ be the Markov chain with iterates (11). For $c > 0$ suppose that $\gamma \leq cn^{-1}p^{-1/2}$,

$$
\varepsilon \geq c \max \left( e^{-n^{1/(2\alpha+1)/2}}, \gamma^{1/2} p, \gamma^2 p^{3/2} n^{1/2} \right), \quad J_{\text{in}} = \frac{\log(c \varepsilon^2)}{\log(1 - c n \gamma)}.
$$

Then there exist $c_1, c_2, c_3 > 0$ such that for all $J \geq 1$ with $\mathbb{P}_{\tilde{\theta}_0}^\varepsilon$-probability at least $1 - c_2 \exp(-c_1 n^{1/(2\alpha+1)})$ the following holds:

(i) $W_2 \left( \mathcal{L}(\tilde{\theta}_{J+J_{\text{in}}}), \Pi(\cdot|Z^{(n)}) \right) \leq \varepsilon$.

(ii) For any Lipschitz function $f : \mathbb{R}^p \to \mathbb{R}$ with $\|f\|_{\text{Lip}} = 1$

$$
P \left( \left| \frac{1}{J} \sum_{k=1+J_{\text{in}}}^{J+J_{\text{in}}} f(\tilde{\theta}_k) - \int f(\theta) d\Pi(\theta|Z^{(n)}) \right| > \varepsilon \right) \leq 2 \exp \left( -c_3 \frac{\varepsilon^2 n^2 J \gamma}{1 + 1/(n J \gamma)} \right).
$$

This result implies that if $\gamma^{-1}$ depends polynomially on $p, n$, then the number of iterations $J + J_{\text{in}} = O(n^{\rho_0} p^{\rho_1} e^{-\rho_2})$, $\rho, \rho_1, \rho_2 > 0$, necessary to approximate posterior functionals at precision $\varepsilon$, grows at most polynomially in $n, p$ and $\varepsilon^{-1}$ with high $\mathbb{P}_{\tilde{\theta}_0}^\varepsilon \times \mathbb{P}$-probability. Consider next the Markov chain with iterates depending on the true posterior density

$$
\theta_{k+1} = \theta_k + \gamma \nabla \log \pi(\theta_k|Z^{(n)}) + \sqrt{2\gamma} \xi_{k+1}, \quad \theta_0 \equiv \theta_{\text{init}}.
$$

(12)
Theorem 5. Under the assumption of Theorem 4 suppose that \( P \) again yields polynomial time sampling guarantees for the posterior using many steps of Gaussian innovations. We will prove, however, that it takes in average exponentially in \( J_{\text{out}} \geq J_{\text{in}} \) to do so for the first time. Combined with Theorem 4 this yields polynomial time sampling results for log-concave measures, e.g. [18, Theorem 5], which depend critically on the condition number \( \Lambda n / m \approx p^{2/3} \). Using our machinery this is replaced by the smaller quantity \( \Lambda / m \approx p^{1/2} \), leading to faster mixing, even in the ideal case of a concave log-likelihood function.

Since we start from \( \theta_{\text{init}} \in B \), the iterates coincide with \( \bar{\vartheta}_k \), as long as the latter has not exited from the region of local curvature \( B \). This will happen ‘eventually’ due to the Gaussian innovations. We will prove, however, that it takes in average exponentially in \( n \) many steps \( J_{\text{out}} \gg J_{\text{in}} \) to do so for the first time. Combined with Theorem 4 this yields polynomial time sampling guarantees for the posterior using \( (\vartheta_k)_{k \geq 0} \). We write again \( P \) for the law of this Markov chain.

**Theorem 5.** Under the assumption of Theorem 4 suppose that \( \gamma \leq c n^{-1} p^{-1} \), \( J_{\text{out}} = e^{c'' n^{1/(2\alpha+1)}} \) for \( c', c'' > 0 \) and \( p \leq C (\log n)^{-(2\alpha+1)/2} n^{2\alpha+1/2 + 1/\gamma} \). Then there exist \( c_1, c_2, c_3, c_4 > 0 \) such that with \( P_{\theta_0} \)-probability at least \( 1 - c_2 \exp(-c_1 n^{1/(2\alpha+1)}) \) for all \( J + J_{\text{in}} \leq J_{\text{out}} \)

\[
\mathbb{P} \left( \frac{1}{J} \sum_{k=1+J_{\text{in}}}^{J+J_{\text{in}}} f(\vartheta_k) - \int_{\Theta} f(\theta) d\Pi(\theta | Z^{(n)}) \right) > \varepsilon \right) 
\leq c_4 \exp \left( -c_3 \min \left( \frac{\varepsilon^2 m^2 J^{1/2} \gamma}{1 + 1/(m J \gamma)}, \frac{n^{1/(2\alpha+1)}}{1 - e^{-m(J+J_{\text{in}})}}, \frac{n^{1/(2\alpha+1)}}{\gamma \Lambda^2 / m} \right) - 1 \right).
\]

In the final result of this section we recover the ground truth \( \theta_0 \) with high \( P_{\theta_0} \times P \)-probability by approximating the posterior mean using \( f(\theta) = \theta \) in the last theorem. We clearly see the impact of statistical and computational errors on the approximation. Recall that \( n^{-\alpha/(2\alpha+1)} \) is the frequentist minimax rate of convergence for estimating an \( \alpha \)-smooth \( \theta_0 \).

**Theorem 6.** Under the assumptions of Theorem 5 there exist \( c_1, c_2, c_3 > 0 \) such that with \( P_{\theta_0} \times P \)-probability at least \( 1 - c_2 \exp(-c_1 n^{1/(2\alpha+1)}) \) for all \( J + J_{\text{in}} \leq J_{\text{out}} \)

\[
\| \frac{1}{J} \sum_{k=1+J_{\text{in}}}^{J+J_{\text{in}}} \vartheta_k - \theta_0 \| \leq c_3 n^{-\alpha/(2\alpha+1)} + \varepsilon.
\]

**Remark 7** (Initialisation in the GLM model). Deviating from the general setting below, it can be shown that the results in this setting remain true if the set \( B \) in the construction of \( \tilde{\ell}_n \) is replaced by \( B_{\log n} \), where \( B_\rho = \{ \theta \in \mathbb{R}^p : \| \theta - \theta_\alpha \|_\rho \leq c_1, \| \theta \|_\alpha \leq \rho \} \) for any large enough \( c_1 > 0 \) (it is actually sufficient to take \( B_{\rho_0} \) if an upper bound \( \| \theta_0 \|_\rho \leq c_0 \) is known), yielding a larger Lipschitz constant \( \tilde{\Lambda} \). This means it is enough to initialise into a compact parameter set, which can be done by grid search in logarithmic time. This observation seems less relevant for GLMs, because good initialisers exist, but may be useful for other statistical models.

**Remark 8** (Log-concave likelihood functions). In Example 1 we can directly apply known sampling results for log-concave measures, e.g. [18, Theorem 5], which depend critically on the condition number \( \Lambda n / m \approx p^{2/3} \). Using our machinery this is replaced by the smaller quantity \( \Lambda / m \approx p^{1/2} \), leading to faster mixing, even in the ideal case of a concave log-likelihood function.
3 General sampling guarantees

Let us now consider a more general statistical setup. As before, \( \Theta \subset \ell^2(\mathbb{N}) \) denotes a parameter space containing \( \nu \) with values in \( Z^n \), where \((Z,A)\) is a measurable space, drawn from a product measure \( P_0^n = \otimes_{i=1}^n P_\theta, \theta \in \Theta \). Let \( p_\theta \) denote the probability density of \( P_\theta \) with respect to a dominating measure \( \nu \). The log-likelihood function is

\[
\ell_n(\theta) \equiv \ell_n(\theta, Z^{(n)}) = \sum_{i=1}^n \log p_\theta(Z_i) = \sum_{i=1}^n \ell(\theta, Z_i).
\]

The Bayesian approach assumes \( \theta \sim \Pi \) for a prior probability measure \( \Pi \equiv \Pi_n \) on \( \Theta \), supported on \( \mathbb{R}^p \), and which may depend on \( n \). We suppose that it has a Lebesgue density \( \pi \equiv \pi_n \). The posterior distribution is then induced by the density \( 1 \) up to normalising factors.

3.1 Main assumptions

The convergence guarantees below will be formulated with respect to a fixed ground truth \( \theta_0 \in \Theta \) generating the data \( Z^{(n)} \). Let \( \theta_{*,p} \in \mathbb{R}^p \) be a high-dimensional approximation of \( \theta_0 \) (not necessarily its \( \mathbb{R}^p \)-projection as in Section 2). A fundamental assumption for all the following considerations is that the posterior measure contracts around a certain ’rate’ \( \delta_n \) with high probability and that a small ball condition holds for the ’normalising factors’. Both conditions can be verified by standard tools from Bayesian nonparametrics, following the seminal work \[25\].

**Assumption A.** The data \( Z^{(n)} \) arise from the law \( P^n_0 \) for a fixed \( \theta_0 \in \Theta \). There exist \( c_0 > 0 \) and \( \theta_{*,p} \in \mathbb{R}^p \) with \( \|\theta_{*,p}\| = c_0 \) as well as a sequence \( 0 < \delta_n \to 0 \) such that for some \( \beta \geq 1 \), any \( c > 0 \) and any sufficiently large \( L \) there are \( C_1, C_2, C_3, C_4 > 0 \) with

\[
\mathbb{P}^n_0 \left( \theta \in \mathbb{R}^p : \|\theta - \theta_{*,p}\|^\beta > L\delta_n \left| Z^{(n)} \right. \right) \geq e^{-cn\delta_n^2} \leq C_2 e^{-C_1 n\delta_n^2}, \tag{14}
\]

\[
\mathbb{P}^n_{\theta_0} \left( \int_{\|\theta - \theta_{*,p}\| \leq \delta_n} e^{\ell_n(\theta) - \ell_n(\theta_0)} \pi(\theta) d\theta \right) \leq e^{-C_3 n\delta_n^2}. \tag{15}
\]

Besides asking for posterior contraction, we also require the prior density to be strongly log-concave and have a Lipschitz gradient.

**Assumption B.** The prior log-density \( \log \pi \) is \( m_\pi \)-strongly concave and has \( \Lambda_\pi \)-Lipschitz gradients for some \( \Lambda_\pi, m_\pi > 0 \). The unique maximiser \( \theta_{\pi,\max} \in \mathbb{R}^p \) satisfies \( \|\theta_{\pi,\max}\| \leq c_0 \). Moreover, the fourth moments of the prior are uniformly (in \( n \) and \( p \)) bounded.
The Gaussian prior in (4) satisfies Assumption \(\mathbb{B}\) for \(\theta_0 \in \mathcal{C}^a(\mathbb{N})\) with \(\Lambda_\pi = n^{1/(2\alpha+1)}p^{2\alpha}\), \(m_\pi = n^{1/(2\alpha+1)}\), \(\theta_{\pi,\text{max}} = 0\). Other practically relevant priors for which the last two assumptions can be verified are suitable finite-dimensional approximations of 'p-exponential' priors \([3, 2]\).

In order to approximate the posterior by a log-concave surrogate measure, the curvature and growth of the log-likelihood function need to be quantified relative to the sample size and the model dimension on a local region close to \(\theta_{*,p}\) with high \(\mathbb{P}_{\theta_0}\)-probability.

**Assumption C.** There exist \(0 < \eta \leq 1\), an event \(\mathcal{E}\) with \(\mathbb{P}_{\theta_0}(\mathcal{E}) \geq 1 - c'e^{-cn_\delta^2}\) for \(c,c' > 0\), and a region \(\mathcal{B} = \{\theta \in \mathbb{R}^p : \|\theta - \theta_{*,p}\| \leq \eta\}\) such that \(\theta \mapsto \ell_n(\theta) \in C^2(\mathcal{B})\), \(\mathbb{P}_{\theta_0}\)-almost surely, and such that for some \(c_{\max} \geq c_{\min} > 0\), \(\kappa_1, \kappa_2, \kappa_3 \geq 0\) the following holds on \(\mathcal{E}\):

(i) (local boundedness) \(\|\nabla \ell_n(\theta_{*,p})\| \leq c_{\max}n\delta n^\kappa_1\) and \(\sup_{\theta \in \mathcal{B}} \|\nabla^2 \ell_n(\theta)\|_{\text{op}} \leq c_{\max}np^{\kappa_2}\).

(ii) (local curvature) \(\inf_{\theta \in \mathcal{B}} \lambda_{\min}(-\nabla^2 \ell_n(\theta)) \geq c_{\min}n p^{-\kappa_3}\).

Note that \(\ell_n\) is not restricted outside of \(\mathcal{B}\). The assumption \(\eta \leq 1\) is natural given the local nature of the assumption, and simplifies some proofs. We conclude with a condition on the magnitudes of \(p, \eta\) as well as a 'curvature' parameter \(K\), which appears in the Lipschitz constant \(\hat{\Lambda}\) in Theorem \(\mathbf{10}\). The upper and lower bounds on \(p\) and \(\eta\) are used, among others, in the proof of the Wasserstein approximation of the posterior.

**Assumption D.** Suppose that Assumptions \([4, 2, 4, \mathbb{C}]\) hold, and that there exists \(\theta_{\text{init}} \in \mathbb{R}^p\) with \(\|\theta_{\text{init}} - \theta_{*,p}\| \leq \eta/8\). In addition, with the function \(v : \mathbb{R}^p \to [0, 1]\) below and with curvature adjusted rate \(\tilde{\delta}_{n,p} = \max(\delta_{n}^{1/\beta}, \delta n^\kappa_1 + \kappa_3)\), the dimension \(p\), the radius \(\eta\) and a curvature parameter \(K\) satisfy for some \(C > 0\)

\[ p \leq Cn\delta_n^2, \quad \eta \geq (\log n)\tilde{\delta}_{n,p}, \quad K \geq 60c_{\max}\|v\|_{C^2}n(1 + p^{\kappa_2})\. \]

**Remark 9.** The lower bound \(\delta n^{\kappa_1 + \kappa_3}\) for \(\eta\) can be improved to \(\delta n^{\kappa_3/2}\) by an analysis of the MAP estimator, cf. \([8]\), Condition 3.5]. This may be difficult in concrete cases, e.g. in the example of Section 4.3. On the other hand, our lower bound is satisfied in the examples considered here and by \([5, 14, 8]\). Compared to the latter two references, our lower bound on \(K\) is typically much smaller and independent of \(\eta\).

### 3.2 The surrogate posterior

We construct a globally concave surrogate log-likelihood function \(\tilde{\ell}_n : \mathbb{R}^p \to \mathbb{R}\) such that \(\tilde{\ell}_n\) agrees with \(\ell_n\) on

\[ \tilde{\mathcal{B}} = \{\theta \in \mathbb{R}^p : \|\theta - \theta_{*,p}\| \leq 3\eta/8\} \subset \mathcal{B}. \]
Our construction is similar to Definition 3.5 of [56], but leads generally to a smaller Lipschitz constant $\tilde{\Lambda}$. Set
\[
\tilde{\ell}_n(\theta) = v \left( \frac{\|\theta - \theta_{\text{init}}\|}{\eta} \right) (\ell_n(\theta) - \ell_n(\theta_{\text{init}})) + \ell_n(\theta_{\text{init}}) - K v \eta \left( \|\theta - \theta_{\text{init}}\| \right) \tag{17}
\]
for $K > 0$ and two smooth auxiliary functions: a 'cut-off function' $v : \mathbb{R}^p \to [0, 1]$ and a globally convex function $v_\eta : \mathbb{R}^p \to [0, \infty)$, $v_\eta(t) = (\varphi_{n/8} \ast \gamma_\eta)(t)$, where $\ast$ is the convolution product, and where
\[
v(t) = \begin{cases} 1, & t \leq 3/4, \\ 0, & t > 7/8, \end{cases} \quad \gamma_\eta(t) = \begin{cases} 0, & t < 5\eta/8, \\ (t - 5\eta/8)^2, & t \geq 5\eta/8. \end{cases}
\]
The function $\varphi_t(x) = t^{-1}\varphi(x/t)$, $t > 0$, is a mollifier for some smooth function $\varphi : \mathbb{R} \to [0, \infty)$ with support in $[-1, 1]$, satisfying $\varphi(-x) = \varphi(x)$, $\int_{\mathbb{R}} \varphi(x)dx = 1$. We define now analogously to (1) the 'surrogate' posterior measure $\tilde{\Pi}(|Z^{(n)})$ with density
\[
\tilde{\pi}(\theta|Z^{(n)}) = \frac{e^{\tilde{\ell}_n(\theta)}\pi(\theta)}{\int_\Theta e^{\tilde{\ell}_n(\theta)}\pi(\theta)d\theta} \propto e^{\tilde{\ell}_n(\theta)}\pi(\theta). \tag{18}
\]
It has the following properties.

**Theorem 10.** Under Assumption $\Box$ the following holds on the event $\mathcal{E}$:

(i) $\ell_n(\theta) = \tilde{\ell}_n(\theta)$ for all $\theta \in \hat{\Theta}$.

(ii) $\tilde{\ell}_n$ is $\tilde{m}$-strongly concave and has $\tilde{\Lambda}$-Lipschitz gradients with $\tilde{\Lambda} = 7K$ and $\tilde{m} = c_{\min}np^{-\kappa_3}$.

In particular, under Assumptions $\mathbb{D}$ and $\mathbb{C}$ log $\tilde{\pi}(|Z^{(n)})$ is $m$-strongly concave and has $\Lambda$-Lipschitz gradients with $\Lambda = 7K + \Lambda_\pi$, $m = c_{\min}np^{-\kappa_3} + m_\pi$.

### 3.3 Log-concave approximation of the posterior

We show next that the surrogate posterior concentrates around $\theta_{\ast,p}$ with high probability at the same rate $\delta_n$ as the true posterior. The result and its proof are of independent interest, since it dispenses with the usual construction of Hellinger tests and relies only on the log-concavity of $\tilde{\Pi}(|Z^{(n)})$.

**Proposition 11.** Under Assumption $\mathbb{D}$ there exist for any $c > 0$ and any sufficiently large $L > 0$ constants $c_1, c_2 > 0$ with
\[
\mathbb{P}_0^n \left( \tilde{\Pi} \left( \theta \in \mathbb{R}^p : \|\theta - \theta_{\ast,p}\|^3 > L\delta_n \right) \bigg| Z^{(n)} \right) \geq e^{-cn\delta_n^2} \leq c_2 e^{-c_1n\delta_n^2}. \tag{19}
\]

We conclude by showing that the surrogate and true posteriors are exponentially close in Wasserstein distance. The proof generalises the specific argument of Theorem 4.14 in [56] and requires, in particular, no analysis of the MAP estimator.

**Theorem 12.** Under Assumption $\mathbb{D}$ there exist $c, c' > 0$ and an event $\tilde{\mathcal{E}}$ with $\mathbb{P}_0^n(\tilde{\mathcal{E}}) \geq 1 - c' e^{-cn\delta_n^2}$ on which $W_2^2 \left( \tilde{\Pi}(|Z^{(n)}), \Pi(|Z^{(n)}) \right) \leq e^{-n\delta_n}$.
3.4 Polynomial time sampling guarantees

Suppose that the gradient $\nabla \tilde{\ell}_n$ can be evaluated at polynomial cost and we are therefore left with quantifying the number of iterates $k$ in (10) to approximate the posterior up to a target precision level. Combining the Wasserstein approximation in Theorem 12 with standard non-asymptotic sampling bounds for strongly log-concave potentials with Lipschitz-gradients, the distance of the law $\mathcal{L}(\tilde{\vartheta}_k)$ to the true posterior measure can be quantified in terms of $k$ and a sufficiently small step size $\gamma$. Convergence as $k \to \infty$ is only achieved if $\gamma \to 0$ and $n \to \infty$.

**Theorem 13.** Set $B(\gamma) = 36\gamma pA^2/m^2 + 12\gamma^2 pA^4/m^3$ and let $\gamma \leq 2/(m + \Lambda)$. Let $(\tilde{\vartheta}_k)_{k \geq 1}$ be the Markov chain with iterates (10). Under Assumption 12 there exists a constant $c_W \equiv C(c_0, c_{\max}, c_{\min})$ such that for all $k \geq 1$ on the event $\mathcal{E}$

$$W_2^2 \left( \mathcal{L}(\tilde{\vartheta}_k), \Pi(\cdot|Z^{(n)}) \right) \leq 2e^{-n\delta_\gamma^2} + 4(1 - m\gamma/2)^k (c_W \max(\eta, \Lambda_\pi/m)^2 + p/m) + B(\gamma).$$

This yields the following result on the computation of posterior functionals by ergodic averages up to a target precision $\varepsilon$ after a burn-in period $J_m$.

**Theorem 14.** Let $\gamma \leq 2/(m + \Lambda)$, consider a precision level $\varepsilon \geq \sqrt{16e^{-n\delta_\gamma^2} + 8B(\gamma)}$ and suppose that the burn-in time satisfies

$$J_m \geq \frac{1}{\log(1 - m\gamma/2)} \log \frac{\varepsilon^2}{32(c_W \max(\eta, \Lambda_\pi/m)^2 + p/m)}.$$

Under Assumption 12 there exists a constant $c_F > 0$ such that for all Lipschitz functions $f : \mathbb{R}^p \to \mathbb{R}$ with $\|f\|_{\text{Lip}} = 1$ and all $J \geq 1$ on the event $\mathcal{E}$

$$P \left( \frac{1}{J} \sum_{k=1+J_m}^{1+J} f(\tilde{\vartheta}_k) - \int_{\mathcal{B}} f(\theta) d\Pi(\theta) | Z^{(n)} \right) > \varepsilon \right) \leq 2 \exp \left( -c_F \frac{\varepsilon^2 m^2 J \gamma}{1 + 1/(mJ\gamma)} \right).$$

Polynomial time convergence guarantees after $J + J_m$ iterations with high probability under $\mathbb{P}_{\theta_0} \times P$ are obtained from this when $\gamma^{-1}, \varepsilon^{-1}, m, \eta^{-1}$ and $\Lambda_\pi$ exhibit most polynomial growth in $p, n$. Next, let $\tau = \inf\{k \geq 1 : \tilde{\vartheta}_k \notin \mathcal{B} \}$ denote the first time the Markov chain $(\tilde{\vartheta}_k)_{k \geq 0}$ leaves from the region $\mathcal{B}$, where $\ell_n$ and $\tilde{\ell}_n$ coincide according to Theorem 10. We can quantify the probability (under $P$) to exit before a time $J$.

**Theorem 15.** Grant Assumption 12 and suppose that

$$\gamma \leq m/(\sqrt{54A^2}), \quad \|\nabla \log \pi(\theta, \pi)\| \leq \eta m/16. \quad (19)$$

Then there exist $c_1, c_2, c_3, c_4 > 0$ such that on an event $\mathcal{E}$ with $\mathbb{P}_{\theta_0}(\mathcal{E}) \geq 1 - c_1 e^{-c_2 n\delta_\gamma^2}$ for all $J \geq 1$

$$P (\tau \leq J) \leq c_3 P \exp \left( -c_4 \frac{\eta^2 m}{p(1 - e^{-mJ\gamma})} \right) + c_4 J P \exp \left( -c_2 \frac{\eta^2 m^2}{\gamma p A^2} \right).$$
If $\eta^2 m^2/(\gamma p \Lambda^2) \geq n^\rho$ for $\rho > 0$ (this holds for the applications in the next section), and given a computational budget of at most $J \times n^{\rho'}$, $\rho' > 0$, many steps, we conclude that the Markov chain $(\vartheta_k)_{0 \leq k \leq J}$, obtained by the vanilla Langevin-MCMC algorithm in (12), will stay within the region of local curvature with high $P_{\theta_0} \times P$-probability. Combining the last two theorems yields the following result.

**Theorem 16.** Grant Assumption (13) and let the conditions in (14) be satisfied. Let $J_{in}$, $\varepsilon$, $C_F$ be as in Theorem (14) and let $(\vartheta_k)_{k \geq 1}$ be the Markov chain with iterates (12). Then there exist $c_1, c_2, c_3, c_4 > 0$ such that on an event $E$ with $P_{\theta_0}(E) \geq 1 - c_1 e^{-c_2 n^{\delta_n^2}}$ for all Lipschitz functions $f : \mathbb{R}^p \to \mathbb{R}$ with $\|f\|_{Lip} = 1$ and all $J \geq 1$

\[
P\left(\left| \frac{1}{J} \sum_{k=1}^{J_{in}} f(\vartheta_k) - \int f(\vartheta) d\Pi(\vartheta[Z^{(n)}]) \right| > \varepsilon \right) \leq 2 \exp \left(-c_F \frac{\varepsilon^2 m^2 J \gamma}{1 + 1/(m J \gamma)} \right) + c_3 p \exp \left(-c_4 \frac{\eta^2 m}{p(1 - e^{-m(J_{in} + J)})} \right) + c_4 (J_{in} + J)p \exp \left(-c_4 \frac{\eta^2 m^2}{\gamma p \Lambda^2} \right).
\]

In particular, taking $f(\vartheta) = \vartheta$ and assuming that $\theta_{*, p}$ is sufficiently close to the data-generating truth $\theta_0$, we obtain the following guarantee on recovering $\theta_0$ by an ergodic average of $(\vartheta_k)_{k \geq 0}$. In view of the statistical error it is enough to restrict to a target precision level $\varepsilon \gtrsim \delta_n^{1/3}$.

**Corollary 17.** In the setting of Theorem (16) assume $\|\theta_0 - \theta_{*, p}\| \leq c_0 \delta_n^{1/3}$. Then there exist $c, c_1, c_2, c_3, c_4 > 0$ such that on an event $E$ with $P_{\theta_0}(E) \geq 1 - c_1 e^{-c_2 n^{\delta_n^2}}$ for all $J \geq 1$

\[
P\left(\left\| \frac{1}{J} \sum_{k=1}^{J_{in}} \vartheta_k - \theta_0 \right\| > c \delta_n^{1/3} + \varepsilon \right) \leq 2p \exp \left(-c_F \frac{\varepsilon^2 m^2 J \gamma / p^2}{1 + 1/(m J \gamma)} \right) + c_3 p^2 \exp \left(-c_4 \frac{\eta^2 m}{p(1 - e^{-m(J_{in} + J)})} \right) + c_4 (J_{in} + J)p^2 \exp \left(-c_4 \frac{\eta^2 m^2}{\gamma p \Lambda^2} \right).
\]

4 Applications

In this section we verify the assumptions from Section 3.1 for density estimation and for nonparametric regression models with error distributions in general exponential families. For regression, first a general setting is considered, followed by Darcy’s problem. Polynomial time sampling guarantees are obtained from the general results in Section 3.3. We focus on the Gaussian prior from (4). As in Section 2, $(e_k)_{k \geq 1}$ is an orthonormal basis of $L^2(\mathcal{X})$ with respect to a measure $\nu_X$ on $\mathcal{X}$, $\Phi(\vartheta) = \sum_{k=1}^{\infty} \theta_k e_k$ for $\vartheta \in \ell^2(\mathbb{N})$. 

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4.1 Density estimation

Suppose that we observe an i.i.d. sample \((X_i)_{i=1}^n\) from a density \(p_\theta\) relative to \(\nu_X\). Following [73], who study posterior contraction with different priors, assume

\[
p_\theta(x) = \frac{e^{\Phi(\theta)(x)}}{\int_X e^{\Phi(\theta)(x)} d\nu_X(x)} = e^{\Phi(\theta)(x) - A(\Phi(\theta))}, \quad \theta \in \Theta, x \in \mathcal{X},
\]

(20)

where \(A(\Phi(\theta)) = \log \int_X e^{\Phi(\theta)(x)} d\nu_X(x)\). The log-likelihood function

\[
\ell_n(\theta) = \sum_{i=1}^n (\Phi(\theta)(X_i) - A(\Phi(\theta)))
\]

is strongly concave only on bounded subsets in \(\Theta\). Implementations of the posterior distribution using MCMC have been discussed in various works [40, 41, 68, 46], but computational guarantees have not been addressed previously. Since constants are not identifiable, we may assume the basis functions \(e_k\) are centered with respect to \(\nu_X\). Suitable initialisers with \(\eta = p^{-1/2}\) can be obtained from [72, 70]. Remark [7] applies here as well.

**Theorem 18.** Suppose the data arise according to the coordinate densities (20). Let Condition [3] be satisfied, but without the design restriction, and assume \(\int_X e_k(x) d\nu_X(x) = 0, k \geq 1\). Then the results of Theorems [4-6] hold true under the same restrictions on \(\gamma\), \(\varepsilon\), \(J_{\text{in}}\), \(J_{\text{out}}\).

4.2 Nonparametric regression

Suppose that we observe independent random vectors \((Y_i, X_i)_{i=1}^n\) from a regression model with marginal densities [2] and with

\[
b(\theta) = (A')^{-1} \circ g^{-1} \circ G(\theta)
\]

(21)

for a known forward operator \(G : \Theta \rightarrow L^2(\mathcal{X})\). This includes the GLMs from Section [2] and non-linear operators \(G\), which appear in the context of non-linear inverse problems [48, 31, 1]. Posterior sampling guarantees have been obtained by [54, 8] for Gaussian measurement errors with the canonical link function for specific operators \(G\), which are globally bounded and Lipschitz. Here we allow for general exponential families and operators \(G\).

Let us first translate the assumptions from Section [3.1] into conditions on \(G\). Following ideas from the Bayesian inverse problem literature, posterior contraction follows from establishing posterior contraction around \(G(\theta_{\ast, \beta})\), combined with stability and local Lipschitz properties. The growth bounds in Assumption [C] correspond to \(L^\infty(\mathcal{X})\)- and \(L^2(\mathcal{X})\)-bounds for \(G\) on \(B\).
**Condition G1.** Suppose that \( \theta_0 \in h^\alpha(\mathbb{N}) \), \( \alpha > 1/2 \), and that there exist \( \theta_{*,p} \in \mathbb{R}^p \), \( \theta_{\text{init}} \in \mathbb{R}^p \), \( C, c_0 > 0 \) with \( p \leq C n^{1/(2\alpha+1)} \), \( \|\theta_0\|_\alpha, \|\theta_{*,p}\|_\alpha \leq c_0 \), \( \|\theta_{\text{init}} - \theta_{*,p}\| \leq \eta/8 \). The design is bounded and there exists \( \beta \geq 1 \) such that the following holds for all \( \theta, \theta' \in \ell^2(\mathbb{N}) \):

(i) \( \|G(\theta_0) - G(\theta_{*,p})\|_{L^2} \leq c_0 n^{-\alpha/(2\alpha+1)} \).

(ii) If \( r > 0 \) and \( \|\theta\|_\alpha, \|\theta'\|_\alpha \leq r \), then there exists \( c_r > 0 \) with \( \|G(\theta)\|_{L^\infty} \leq c_r \) and

\[
\frac{c_r^{-1}}{\|\theta - \theta'\|^\beta} \leq \|G(\theta) - G(\theta')\|_{L^2} \leq \frac{c_r}{\|\theta - \theta'\|}.
\]  

**Condition G2.** For all \( x \in \mathcal{X} \), \( \theta \mapsto G(\theta)(x) \in C^2(\mathcal{B}) \). There exist \( \bar{c}_{\text{max}} \geq \bar{c}_{\text{min}} > 0 \), \( k_i \geq 0 \), \( i = 1, \ldots, 5 \) such that the following holds for all \( v \in \mathbb{R}^p \) with \( \|v\| \leq 1 \) and all \( \theta \neq \theta' \in \mathcal{B} \):

(i) \( \|G(\theta_{*,p}) - G(\theta)\|_{L^2} \leq \bar{c}_{\text{max}} \eta \).

(ii) \( \|G(\theta)\|_{L^\infty} \leq \bar{c}_{\text{max}} \), \( \|v^\top \nabla G(\theta)\|_{L^\infty} \leq \bar{c}_{\text{max}} p^{k_1} \) and

\[
\|\nabla^2 G(\theta)\|_{L^\infty(\mathcal{X},\mathbb{R}^p \times \mathbb{R}^p)} + \frac{\|\nabla^2 G(\theta) - \nabla^2 G(\theta')\|_{L^\infty(\mathcal{X},\mathbb{R}^p \times \mathbb{R}^p)}}{\|\theta - \theta'\|} \leq \bar{c}_{\text{max}} p^{k_2}.
\]

(iii) \( \|v^\top \nabla G(\theta)\|_{L^2} \leq \bar{c}_{\text{min}} p^{k_3} \), \( \|v^\top \nabla^2 G(\theta)v\|_{L^2} \leq \bar{c}_{\text{max}} p^{k_4} \).

(iv) \( \|v^\top \nabla G(\theta)\|_{L^2}^2 \geq \bar{c}_{\text{min}} p^{-k_5} \).

With this we establish Assumption [D] It immediately yields the polynomial time sampling guarantees from Section 3.3.

**Proposition 19.** Suppose the data arise in a nonparametric regression model with coordinate densities [2], \( b \) as in [21] and \( g \in C^3(\mathcal{I}) \) for a forward operator \( G \) satisfying Conditions [G1] and [G2] and such that for all large enough \( n \)

\[
\eta \geq n^{-\alpha/(2\alpha+1)} \max(n^{(\alpha-\alpha/\beta)/(2\alpha+1)}, p^{k_3+k_5}) \log n,
\]

\[
p^{-k_5} \geq n^{-\alpha/(2\alpha+1)} \max(p^{\max(k_1,2k_3,k_4)}, \eta p^{\max(3k_1,k_2)}) \log n.
\]

Then Assumption [D] holds for \( \kappa_1 = k_3 \), \( \kappa_2 = \max(k_1,2k_3,k_4) \), \( \kappa_3 = k_5 \) and \( K \geq c n p^{\max(k_1,2k_3,k_4)}, c > 0 \).

**4.3 Darcy’s problem**

Suppose that \( \mathcal{X} \) is a bounded domain in \( \mathbb{R}^d \) with smooth boundary \( \partial \mathcal{X} \). Let \( \nabla u \) and \( \nabla \cdot u = \sum_{i=1}^d \partial_i u \) denote the gradient and divergence operators, respectively. For a conductivity \( f \in C^\gamma(\mathcal{X}) \), \( \gamma \in \mathbb{N} \), consider the divergence form operator

\[
L_f u = \nabla \cdot (f \nabla u).
\]
For source \( g_1 \in C^\infty(\mathcal{X}) \) and boundary values \( g_2 \in C^\infty(\partial \mathcal{X}) \) let \( u \equiv u_f \) be the solution to the boundary value problem

\[
\begin{aligned}
    & \mathcal{L}_f u = g_1 \quad \text{in } \mathcal{X}, \\
    & u = g_2 \quad \text{on } \partial \mathcal{X}.
\end{aligned}
\] (26)

For strictly positive \( f \) the operator \( \mathcal{L}_f \) is uniformly elliptic and classical solutions \( u_f \in C^2(\mathcal{X}) \) exist by standard elliptic PDE theory (e.g., Theorem 6.14 in [29]). Details on the analytical properties of the PDE (26) relevant to our analysis are collected in Section 6.1.3.

The function \( u_f \) typically represents the density of some quantity within the region \( \mathcal{X} \) and the PDE describes diffusion within \( \mathcal{X} \) at equilibrium [21]. Determining the unknown conductivity \( f \) from noisy observations of \( u_f \) is a popular example in the inverse problem literature, called Darcy’s problem, see [14, 9, 55] and the references therein. For a fixed \( f_{\min} > 0 \) let

\[
f_\theta = f_{\min} + \exp(\Phi(\theta)), \quad \theta \in \ell^2(\mathbb{N}),
\] (27)

and consider the measurement model in Section 4.2 with \( \mathcal{G}(\theta) = u_{f_\theta} \) for known \( g_1 \) and \( g_2 \). For Gaussian measurement errors, posterior contraction in this model for different Gaussian process priors is studied by [31], while [54] show that no Bernstein-von Mises theorem holds.

In the following, in order to use elliptic PDE-regularity theory in \( L^2 \), we choose for \( (e_k)_{k \geq 1} \) the eigenbasis of the negative Dirichlet Laplacian \( -\Delta = -\nabla \cdot \nabla \) with associated eigenvalues \( (\lambda_k)_{k \geq 1} \). This leads to a rather large regularity assumption on \( \alpha \) and can possibly be relaxed by using Schauder estimates instead. Moreover, we restrict to \( d \leq 3 \) to simplify the proofs using Sobolev embeddings.

**Condition 20.** Suppose that \( \theta_0 \in h^\alpha(\mathbb{N}), \alpha \geq 21/d, d \leq 3 \), and that there exist \( \theta_{s,p} \in \mathbb{R}^p, \theta_{\text{init}} \in \mathbb{R}^p, C, c_0 > 0 \) with \( p \leq Cn^{-1/(2\alpha+1)} \), \( \eta = p^{-8/d} \), \( \|\theta_0\|_\alpha, \|\theta_{s,p}\|_\alpha \leq c_0 \), \( \|\theta_{\text{init}} - \theta_{s,p}\| \leq \eta/8 \), \( \|\mathcal{G}(\theta_0) - \mathcal{G}(\theta_{s,p})\|_{L^2} \leq c_0 n^{-1/(2\alpha+1)} \). The design is bounded and the solutions \( u_{f_\theta} \) satisfy for all \( c > 0 \), some \( \mu, c' > 0 \), possibly depending on \( c \) and \( \alpha' > 1/d + 1/2 \)

\[
\inf_{x \in \mathcal{X}, \|\Phi(\theta)\|_{C^1} \leq c} \left( \frac{1}{2} \Delta u_{f_\theta}(x) + \mu \|\nabla u_{f_\theta}(x)\|_{\mathbb{R}^d}^2 \right) \geq c', \quad \theta \in h^{\alpha'}(\mathcal{X}).
\] (28)

Condition (28) ensures injectivity of the forward operator, which is necessary to show (22). It holds for a large class of models \( f, g_1, g_2 \) (see [53, Proposition 2.1.5]), for instance, as soon as \( g_1 > 0 \) on \( \mathcal{X} \).

**Theorem 21.** Suppose the data arise from the nonparametric regression model with coordinate densities (2), \( b \) as in (21) and \( g \in C^3(\mathcal{I}) \) for the forward operator \( \mathcal{G}(\theta) = \)
u_{fg}$. Assume that Condition [20] is satisfied. Then the results of Theorem [4] hold for 
\[ \gamma \leq c n^{-1} p^{-2/d}, \quad c > 0, \]
\[ \varepsilon \geq c \max(e^{-n^{1/(2a+1)}}/2, \gamma^{1/2} p^{1/2+8/d}, \gamma p^{1/2+13/d} n^{1/2}), \]
\[ J_{in} = \frac{\log(c\varepsilon^2 p^{-12/d})}{\log(1 - c\gamma np^{-6/d})}. \]
Moreover, the results of Theorems [5] and [6] hold for 
\[ p \leq c (\log n)^{-1} \frac{2^k}{2a+1} \]
after replacing (13) with
\[ \| \frac{1}{J} \sum_{k=1}^{J+J_{in}} \theta_k - \theta_0 \| \leq c_3 n^{-(\alpha-\delta)/2} + \varepsilon. \]

5 Proofs

5.1 Proofs for Section 2: GLMs

The specific results for GLMs follow from applying the general statements in Section 3 to
the regression model in Section 4.2 with \( Z(n) = (Y_i, X_i)_{i=1}^n, Z = \mathbb{R} \times \mathcal{X}, \nu = \xi \otimes \nu_X, \)
\( \delta_n = n^{-\alpha/(2a+1)} \) and \( G(\theta) = \Phi(\theta) = \sum_{k=1}^\infty \theta_k e_k. \) By modifying the final constant \( c_2 \)
in the statements of Theorems [4] [5] [6] it is enough to consider any sufficiently large \( n. \)
Observe first the following lemma.

**Lemma 22.** Suppose the data arise in a GLM with coordinate densities \( g \in C^3(I) \). Let Condition [3]
be satisfied. Then Conditions [G1] and [G2] are satisfied for \( \beta = 1 \)
and \( k_1 = 1/2, k_2 = 0, k_3 = 0, k_4 = 0, k_5 = 0. \)

**Proof.** It is enough to verify Conditions [G1(i,ii)] and [G2(i-iv)]. The operator \( \Phi : l^2(N) \to L^2(\mathcal{X}) \)
being an isometry, [6] yields immediately Conditions [G1(i), G2(i)]. Since the basis functions \( e_k \) are bounded, we have 
\( \| \Phi(\theta) \|_{L^\infty} \lesssim \| \theta \|_{l^1} \lesssim \| \theta \|_{\alpha}. \) Obtain from this Condition [G1(ii)], noting that the inequalities in [22] are equalities with \( c_\theta = 1 \)
and \( \beta = 1. \) The remaining statements in Condition [G2] follow with the claimed values for the \( k_i \) by observing that 
\( \nabla^2 \Phi(\theta) = 0 \) and that for \( v \in \mathbb{R}^p, \theta \in \mathcal{B} \subset \{ \theta \in \mathbb{R}^p : \| \theta \|_\alpha \leq c_1 \}, \)
c\( c_1 > 0, \) implying \( \| v^\top \nabla \Phi(\theta) \|_{L^2} = \| \sum_{k=1}^p v_k e_k \| \leq \| v \| \) and
\[ \| v^\top \nabla \Phi(\theta) \|_{L^\infty} = \| \sum_{k=1}^p v_k e_k \|_{L^\infty} \lesssim \| v \|_{l^1} \leq \| v \|_{l^1} \leq p^{1/2} \| v \|. \]

For the \( k_i \) from this lemma we can establish [23], [24], since \( \eta \geq \delta_n \log n \) for large enough \( n \) and \( \delta_n p \log n \leq 1. \) This allows us to apply Proposition [19] to verify Assumption [14] for \( k_1 = 0, k_2 = 1/2, k_3 = 0 \) and \( K \geq cnp^{1/2}, c > 0. \) Let us now prove the
three theorems in Section 2.3. Consider \( \gamma, \varepsilon \) and \( J_{in} \) as stated in Theorem [4] As
m_\pi = n^{1/(2\alpha + 1)}, \Lambda_\pi = n^{1/(2\alpha + 1)} p^{2\alpha}, the curvature and Lipschitz constants from Theorem 10 satisfy
\[ m \gtrsim n, \quad \Lambda \asymp n^{1/2}, \quad \max(\eta, \Lambda_{\pi}/m)^2 + p/m \lesssim 1. \]

This gives \( B(\gamma) \lesssim \gamma p^2 + \gamma^3 n \) and Theorem 4 follows from Theorems 13 and 14. Next, assume \( p \leq (\log n)^{-\alpha/(2\alpha + 1)} 2^{2\alpha + 1} \gamma^3 \lesssim n\delta_n^2 \) such that \( \Lambda_{\pi} \log n \lesssim \eta m \), and
\[
\| \nabla \log \pi(\theta, \rho) \| \leq \Lambda_{\pi} \| \theta \| \lesssim \eta m / 16, \quad \eta \sqrt{m/p} \gtrsim \sqrt{n} \gtrsim cn^{-1/(2\alpha + 1)}
\]
for any \( c > 0 \) and large enough \( n \), implying for any \( C > 0 \) and some \( C > 0 \) also \( (J + Jm)pe^{-C'[m/(\gamma p\Lambda^2)]} \lesssim e^{-C'n^{-1/(2\alpha + 1)}} \). Theorems 5 and 6 are then obtained from Theorem 10 and Corollary 17.

### 5.2 Proofs for Section 3.3: The surrogate posterior

**Proof of Theorem 10** Part (i) is true by the construction of \( \hat{\ell}_n \) in (17) and the condition on the initialiser in Assumption D. The supplement follows immediately from part (ii).

For the proof of (ii) let us restrict to the event \( \mathcal{E} \) and write \( \tilde{v} = v(\| \cdot - \theta_{\text{init}} \| / \eta) \), \( \tilde{v} = v_\eta(\| \cdot - \theta_{\text{init}} \|) \). We consider first the set \( V = \{ \theta \in \mathbb{R}^p : \| \theta - \theta_{\text{init}} \| \leq 3\eta/4 \} \subset B \). On \( V \), \( \tilde{v} \) vanishes and \( \tilde{v} = 1 \). Hence, by the local curvature bound from Assumption C(ii) we have
\[
\inf_{\theta \in V} \lambda_{\min} \left( -\nabla^2 \hat{\ell}_n(\theta) \right) \geq \inf_{\theta \in B} \lambda_{\min} \left( -\nabla^2 \ell_n(\theta) \right) \geq c_{\min} np^{-\kappa_3}. \tag{29}
\]

Next, Lemma B.5 and the proof of Lemma B.6 in [30] (with \( \lambda_{\max}(I) = 1 \)) imply that \( \| \nabla \tilde{v}(\theta) \| \leq \| v \|_{C_1 \eta^{-1}}, \| \nabla^2 \tilde{v}(\theta) \|_{\text{op}} \leq 4\| v \|_{C_2 \eta^{-2}} \) for all \( \theta \in B \), as well as \( \lambda_{\min}(\nabla^2 \tilde{v}(\theta)) \geq 1/3, \| \nabla^2 \tilde{v}(\theta) \|_{\text{op}} \leq 6 \) Assumption C(i) therefore gives
\[
\sup_{\theta \in B} |\ell_n(\theta) - \ell_n(\theta_{\text{init}})| \leq \| \nabla \ell_n(\theta_{\text{init}}) \| \eta + \sup_{\theta \in B} \| \nabla^2 \ell_n(\theta) \|_{\text{op}} \eta^2 / 2
\]
\[
\quad \leq c_{\max} n \left( \delta_n p^{\kappa_4} \eta + \eta^{\kappa_2} \eta^2 / 2 \right), \quad \sup_{\theta \in B} \| \nabla \ell_n(\theta) \| \leq \| \nabla \ell_n(\theta_{\text{init}}) \| + \sup_{\theta \in B} \| \nabla^2 \ell_n(\theta) \|_{\text{op}} \eta \leq c_{\max} n \left( \delta_n p^{\kappa_4} + \eta^{\kappa_2} \right).
\]

By the triangle inequality \( \sup_{\theta \in B} |\ell_n(\theta) - \ell_n(\theta_{\text{init}})| \leq 2c_{\max} n (\delta_n p^{\kappa_4} + \eta^{\kappa_2} \eta^2 / 2) \). Combining the last two displays and noting that \( \tilde{v} \) vanishes outside of \( B \) yields by the chain rule for \( \theta \in \mathbb{R}^p \)
\[
\| \nabla^2 (\tilde{v}(\ell_n - \ell_n(\theta_{\text{init}})))(\theta) \|_{\text{op}} \leq \sup_{\theta \in B} (\| \nabla^2 \tilde{v}(\theta) \|_{\text{op}} \| \ell_n(\theta) - \ell_n(\theta_{\text{init}}) \| + 2\| \nabla \tilde{v}(\theta) \| \| \nabla \ell_n(\theta) \| + \| \tilde{v}(\theta) \| \| \nabla^2 \ell_n(\theta) \|_{\text{op}})
\]
\[
\leq 10c_{\max} \| v \|_{C_2 n} (\eta^{-1} \delta_n p^{\kappa_4} + \eta^{\kappa_2}) \leq K/6,
\]
using $\eta \geq \delta_n p^{\kappa_1 + \kappa_3}$ and with $K$ from Assumption D. Consequently,

$$\inf_{\theta \in \mathbb{V}^c} \lambda_{\min} \left( -\nabla^2 \hat{\ell}_n(\theta) \right) \geq - \sup_{\theta \in \mathbb{R}^p} \| \nabla^2 (\bar{\nu} \ell_n)(\theta) \|_{op} + K/3 \geq K/6.$$ 

Together with (29) and $K/6 \geq c_{\min n}$ we thus obtain the wanted curvature bound of $\hat{\ell}_n$:

$$\inf_{\theta \in \mathbb{R}^p} \lambda_{\min} \left( -\nabla^2 \hat{\ell}_n(\theta) \right) \geq \min \left( c_{\min n} p^{-\kappa_3}, K/6 \right) = c_{\min n} p^{-\kappa_3}.$$ 

At last, the gradient-Lipschitz bound follows for $\theta \neq \theta' \in \mathbb{R}^p$ from

$$\frac{\| \nabla (\hat{\ell}_n(\theta) - \hat{\ell}_n(\theta')) \|}{\| \theta - \theta' \|} \leq \sup_{\theta \in \mathbb{R}^p} \| \nabla^2 \hat{\ell}_n(\theta) \|_{op}$$

$$\leq \sup_{\theta \in \mathbb{R}^p} \| \nabla^2 (\bar{\nu} \ell_n)(\theta) \|_{op} + K \sup_{\theta \in \mathbb{R}^p} \| \nabla^2 \bar{\nu}(\theta) \|_{op} \leq K/6 + 6K \leq 7K. \quad \Box$$

**Proof of Proposition 11** Let $c > 0$, $L \geq 1$. Define two balls with centres $\theta_{s,p}$

$$\mathcal{U} = \left\{ \theta \in \mathbb{R}^p : \| \theta - \theta_{s,p} \| \leq L^{1/2} \delta_n^{1/2} \right\},$$

$$\tilde{\mathcal{U}} = \left\{ \theta \in \mathbb{R}^p : \| \theta - \theta_{s,p} \| \leq L^{1/2} \tilde{\delta}_n \right\}.$$ 

Let $\mathcal{D}^c$ denote the event in (15) and consider also $\tilde{\mathcal{D}} = \{ \Pi(\mathcal{U}|Z^{(n)}) \geq 1 - e^{-c_n \delta^2_n/4} \}$. In view of Assumptions A and C, by taking $L$ large enough, we may restrict ourselves in the proof to the high probability event $\tilde{\mathcal{E}} = \mathcal{E} \cap \mathcal{D} \cap \tilde{\mathcal{D}}$.

We first study the surrogate posterior measure of the set $\tilde{\mathcal{U}}$. For large enough $n$ (and by increasing the final constant $c_2$ in the statement), Assumption D and $L \geq 1$ provide us with the relation

$$\max(\delta_n, L^{1/2} \delta_n^{1/2}) \leq L^{1/2} \tilde{\delta}_n \leq (\log n)^{-1} \eta \leq 3\eta/8. \quad (30)$$ 

On the event $\mathcal{E}$, this means by Theorem 10(i)

$$\ell_n(\theta) = \hat{\ell}_n(\theta) \quad \text{for any } \theta \text{ from the set } \tilde{\mathcal{U}}. \quad (31)$$

For $C_1 = L^{1/2} \max / (4c_{\max})$ and $L$ large enough to ensure $C_1 \geq 1$ let $\theta \in \tilde{\mathcal{U}}^c$ such that $\| \theta - \theta_{s,p} \| > (4C_1 \max / c_{\min}) \delta_n p^{\kappa_1 + \kappa_3}$. Lemma 23 below shows $\hat{\ell}_n(\theta) \leq -(4C_1^2 c_{\max} / c_{\min}) n \delta_n^2 + \ell_n(\theta_{s,p})$. Using (31) to lower bound the normalising factors in the posterior density, we thus get for $C_4 > 0$ on $\mathcal{E} \cap \mathcal{D}$

$$\tilde{\pi}(\theta|Z^{(n)}) = \frac{e^{\hat{\ell}_n(\theta) - \ell_n(\theta)} \pi(\theta)}{\int_{\Theta} e^{\hat{\ell}_n(\theta) - \ell_n(\theta)} \pi(\theta) d\theta} \leq \frac{e^{\hat{\ell}_n(\theta) - \ell_n(\theta)} \pi(\theta)}{\int_{\Theta} e^{\ell_n(\theta) - \ell_n(\theta)} \pi(\theta) d\theta}$$

$$\leq e^{C_4 n \delta_n^2} e^{\hat{\ell}_n(\theta) - \ell_n(\theta)} \pi(\theta) \leq e^{-(4C_1^2 c_{\max} / c_{\min}) n \delta_n^2 + \ell_n(\theta_{s,p}) - \ell_n(\theta_0)} \pi(\theta). \quad (32)$$

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Setting $L = L^{2/\beta}c_{\min}/4 - C_4 - c$, the Markov inequality gives

$$
P_{\theta_0}^n \left( \tilde{\Pi}(\tilde{U}^c|Z^{(n)}) > e^{-cn \delta_n^2}/2, \tilde{E} \right) \leq P_{\theta_0}^n \left( e^{-L \ln \delta_n^2} e^{\tilde{\ell}_n(\theta_{s.p}) - \ell_n(\theta_0)} \int_{\tilde{U}^c} \pi(\theta) d\theta > 1/2 \right) \leq 2e^{-L \ln \delta_n^2} \mathbb{P}_{\theta_0}^n \left[ e^{\tilde{\ell}_n(\theta_{s.p}) - \ell_n(\theta_0)} \right] \leq 2e^{-L \ln \delta_n^2}.
$$

Taking $L$ possibly larger $\bar{L} > 0$ and the last line is indeed exponentially small.

Next, (31) also implies $p_n \pi(\theta|Z^{(n)}) = \tilde{\pi}(\theta|Z^{(n)})$ for $\theta \in \mathcal{U}$ with random normalising factors $0 < p_n < \infty$. In particular, noting $\mathcal{U} \subset \tilde{\mathcal{U}}$ due to (30), we have on the event $\mathcal{E} \cap \mathcal{D}$

$$p_n^{-1} \geq p_n^{-1} \Pi(\tilde{U}|Z^{(n)}) = \Pi(\mathcal{U}|Z^{(n)}) \geq \Pi(\mathcal{U}|Z^{(n)}) \geq 1 - e^{-cn \delta_n^2}/4.
$$

This yields $p_n \leq (1 - e^{-cn \delta_n^2}/4)^{-1} \leq 2$ and

$$\Pi(\mathcal{U}^c \cap \tilde{U}|Z^{(n)}) = p_n \Pi(\mathcal{U}^c \cap \tilde{U}|Z^{(n)}) \leq 2\Pi(\mathcal{U}^c|Z^{(n)}).
$$

Splitting $\mathcal{U}^c$ into the sets $\mathcal{U}^c \cap \tilde{U}$ and $\mathcal{U}^c \cap \tilde{U}^c = \tilde{\mathcal{U}}^c$, conclude from (33)

$$\mathbb{P}_{\theta_0}^n \left( \tilde{\Pi}(\mathcal{U}^c|Z^{(n)}) > e^{-cn \delta_n^2}, \tilde{E} \right) \leq \mathbb{P}_{\theta_0}^n \left( \tilde{\Pi}(\mathcal{U}^c \cap \tilde{U}|Z^{(n)}) > e^{-cn \delta_n^2}/2, \tilde{D} \right) + 2e^{-L \ln \delta_n^2} \leq \mathbb{P}_{\theta_0}^n \left( \Pi(\mathcal{U}^c|Z^{(n)}) > e^{-cn \delta_n^2}/4, \tilde{D} \right) + 2e^{-L \ln \delta_n^2} = 2e^{-L \ln \delta_n^2}.
$$

**Lemma 23.** Grant Assumption $\Box$ If $C_1 \geq 1$, $\theta \in \mathbb{R}^p$ satisfies $||\theta - \theta_{s.p}|| > (4C_1 c_{\max}/c_{\min}) \delta_n p^\kappa_1 + \kappa_3$, then we have $\tilde{\ell}_n(\theta) - \ell_n(\theta_{s.p}) < -(4C_1^2 c_{\max}/c_{\min}) n \delta_n^2$ on $\mathcal{E}$.

**Proof.** Theorem 10 i) yields $\tilde{\ell}_n(\theta_{s.p}) = \ell_n(\theta_{s.p})$, $\nabla \tilde{\ell}_n(\theta_{s.p}) = \nabla \ell_n(\theta_{s.p})$. Hence, Assumption $\Box$ Theorem 10 ii) and the Cauchy-Schwarz inequality imply

$$\tilde{\ell}_n(\theta) - \ell_n(\theta_{s.p}) \leq c_{\max} np^{\kappa_1} \delta_n ||\theta - \theta_{s.p}|| - (c_{\min}/2) np^{\kappa_3} ||\theta - \theta_{s.p}||^2.
$$

If $||\theta - \theta_{s.p}|| > (4C_1 c_{\max}/c_{\min}) \delta_n p^{\kappa_1 + \kappa_3}$ and $C_1 \geq 1$, then

$$||\theta - \theta_{s.p}|| < \frac{p^{-\kappa_1 - \kappa_3}}{(4c_{\max}/c_{\min}) \delta_n} ||\theta - \theta_{s.p}||^2
$$

and therefore

$$\tilde{\ell}_n(\theta) - \ell_n(\theta_{s.p}) < (c_{\min}/4) np^{\kappa_3} ||\theta - \theta_{s.p}||^2 - (c_{\min}/2) np^{\kappa_3} ||\theta - \theta_{s.p}||^2 = -(c_{\min}/4) np^{\kappa_3} ||\theta - \theta_{s.p}||^2 < -(4C_1^2 c_{\max}/c_{\min}) n \delta_n^2.
$$

$\square$

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**Proof of Theorem 12** Recall the sets $U$, $\tilde{U}$ and the high-probability event $\tilde{E}$ from the proof of Proposition 11. Taking $n$ and $L$ large enough we have on $\tilde{E}$ for any $c > 0$ that $\Pi(\tilde{U}^c|Z^{(n)}) \leq \Pi(\tilde{U}|Z^{(n)}) \leq e^{-cn\delta_n^2}/2$, as well as for some $C_2 > 0$

$$\pi(\theta|Z^{(n)}) = \frac{e^{\ell_n(\theta)-\ell_n(\theta_0)}\pi(\theta)}{\int_{\Theta} e^{\ell_n(\theta)-\ell_n(\theta_0)}\pi(\theta)d\theta} \leq e^{C_2 n\delta_n^2 e^{\ell_n(\theta)-\ell_n(\theta_0)}\pi(\theta)}, \quad \theta \in \mathbb{R}^p. \quad (35)$$

We begin by applying Theorem 6.15 of Villani (2009) to upper bound the squared Wasserstein distance between the posterior and the surrogate posterior as

$$W_2^2(\Pi(\cdot|Z^{(n)}), \Pi(\cdot|Z^{(n)})) \leq 2\int_{\mathbb{R}^p} \|\theta - \theta_{*,p}\|^2|\tilde{\pi}(\theta|Z^{(n)}) - \pi(\theta|Z^{(n)})|d\theta.$$ 

Decompose the integral as $I_1 + I_2 + I_3$ with

$$I_1 = \int_{\tilde{U}} \|\theta - \theta_{*,p}\|^2|\tilde{\pi}(\theta|Z^{(n)}) - \pi(\theta|Z^{(n)})|d\theta,$$

$$I_2 = \int_{\tilde{U}^c} \|\theta - \theta_{*,p}\|^2\tilde{\pi}(\theta|Z^{(n)})d\theta,$$

$$I_3 = \int_{\tilde{U}^c} \|\theta - \theta_{*,p}\|^2\pi(\theta|Z^{(n)})d\theta.$$ 

It is enough to show that each of these terms exceeds $e^{-n\delta_n^2}/3$ on $\tilde{E}$ only with exponentially small $\mathbb{P}^0_{\theta_0}$-probability. Arguing as in (34) for the random normalising factors $p_n$, we have

$$p_n \geq p_n \Pi(\tilde{U}|Z^{(n)}) = \tilde{\Pi}(\tilde{U}|Z^{(n)}) \geq 1 - e^{-cn\delta_n^2}/2. \quad (36)$$

Together with (34) this means $1 - e^{-n\delta_n^2}/2 \leq p_n \leq (1 - e^{-cn\delta_n^2}/2)^{-1}$, or equivalently,

$$-\frac{e^{-cn\delta_n^2}/2}{1 - e^{-cn\delta_n^2}/2} \leq 1 - p_n \leq e^{-cn\delta_n^2}/2,$$

implying $|1 - p_n| \leq e^{-cn\delta_n^2}$. For large enough $n$ we know from (33) that $L^{1/\beta}\tilde{\delta}_{n,p} \leq 1/3$. Consequently, with $p_n \pi(\theta|Z^{(n)}) = \tilde{\pi}(\theta|Z^{(n)})$ for $\theta \in \tilde{U}$, we obtain on $\tilde{E}$

$$I_1 \leq L^{2/\beta}\tilde{\delta}_{n,p}^2 \int_{\tilde{U}} \left|\tilde{\pi}(\theta|Z^{(n)}) - \pi(\theta|Z^{(n)})\right|d\theta \leq \frac{1}{3} \frac{p_n}{\Pi(\tilde{U}|Z^{(n)})} \leq e^{-cn\delta_n^2}/3.$$ 

On the other hand, we find from the Cauchy-Schwarz inequality and (32) for $C_1 \geq 1$ with $4C_1^2c^2_m/c_{\min} - C_4 \geq 0$ that

$$I_2^2 \leq \tilde{\Pi}(\tilde{U}^c|Z^{(n)}) \int_{\tilde{U}^c} \|\theta - \theta_{*,p}\|^4\tilde{\pi}(\theta|Z^{(n)})d\theta$$

$$\leq e^{-cn\delta_n^2}e^{\ell_n(\theta_{*,p})-\ell_n(\theta_0)} \int_{\Theta} \|\theta - \theta_{*,p}\|^4\pi(\theta)d\theta. \quad (37)$$
Noting $\|\theta_{*p}\| \leq c_0$ by Assumption A and because the prior has uniformly bounded fourth moments according to Assumption B, we infer from the Markov inequality and Fubini’s theorem that

$$\mathbb{P}_{\hat{\theta}_0} \left( I_2 > \frac{e^{-n\delta n^2}}{3}, \tilde{\mathcal{E}} \right) = \mathbb{P}_{\hat{\theta}_0} \left( I_2^2 > \frac{e^{-2n\delta n^2}}{9}, \tilde{\mathcal{E}} \right) \lesssim e^{(2-c)n\delta n^2} \int_{\Theta} \|\theta - \theta_{*p}\|^4 \pi(\theta) d\theta \lesssim e^{(2-c)n\delta n^2}.$$  

The same upper bound holds for the probability with respect to $I_3$ because of $\Pi(\tilde{U} \mid Z(n)) \leq e^{-cn\delta n^2/2}$ and (35). The result follows by taking $c > 2$. □

5.3 Exit time of the surrogate Markov chain

In this section we prove Theorem 15. The main idea is to relate the discrete time Markov chain $(\tilde{\vartheta}_k)_{k \geq 0}$ to a continuous time Langevin diffusion process with gradient potential $\nabla \tilde{\pi}(\cdot \mid Z^{(n)})$. This reduces the problem of computing the exit time of $(\tilde{\vartheta}_k)_{k \geq 0}$ from $\tilde{\mathcal{B}}$ to the corresponding exit time of the diffusion process. This is achieved by comparing to a suitable Ornstein-Uhlenbeck process, whose exit time can be bounded analytically.

Proof of Theorem 15. For the proof we may restrict to the event $\mathcal{E}$. By increasing the final constant $c_1$ in the statement, it is enough to consider any sufficiently large $n$. By extending the probability space carrying the Markov chain we can further assume without loss of generality that it also supports a $p$-dimensional Brownian motion $(W_t)_{t \geq 0}$ with respect to a filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions (see [37, Section 5.2.A]). For fixed data $Z^{(n)}$ denote by $f(\theta) = \log \tilde{\pi}(\cdot \mid Z^{(n)})$ the log-density of the surrogate posterior measure. With this associate two $p$-dimensional stochastic differential equations

$$dL_t = \nabla f(L_t) dt + \sqrt{2} dW_t,$$

$$d\tilde{L}_t = \nabla f(\tilde{L}_{\lceil t/\gamma \rceil} \gamma) dt + \sqrt{2} dW_t,$$

for $t \geq 0$, both starting at $L_0 = \tilde{L}_0 = \theta_{\text{init}}$. Since $f$ is strongly $m$-concave and has $\Lambda$-Lipschitz gradients on $\mathcal{E}$, cf. the supplement in Theorem 10 classical results for stochastic differential equations (e.g., [37, Theorem 5.2.9]) verify that (38) has a unique strong solution $(L_t)_{t \geq 0}$ with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$. The process $(\tilde{L}_t)_{t \geq 0}$ is simply the continuous time interpolation of $(\tilde{\vartheta}_k)_{k \geq 0}$ in the sense that

$$\mathcal{L}(\tilde{L}_{\gamma_1}, \ldots, \tilde{L}_{J\gamma}) = \mathcal{L}(\vartheta_1, \ldots, \vartheta_J).$$

This means

$$P(\tau \leq J) = P \left( \sup_{k=1, \ldots, J} \|\tilde{L}_{k\gamma} - \theta_{*p}\| > 3\eta/8 \right),$$

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and the result follows from the triangle inequality

\[ \|L_k - \theta_{s,p}\| \leq \|L_k - \theta_{s,p}\| + \|L_k\| \]

together with Lemmas 24, 25 below, noting \( \eta \sqrt{m/p} \gtrsim \delta_n p^{\gamma_3} \sqrt{np^{-\gamma_3}/(n \delta_n)} \log n \gtrsim \log n. \)

**Lemma 24.** In the setting of Theorem 15 we have for some \( c, c' > 0 \), all large enough \( n \) and \( x \sqrt{m/p} \gtrsim \log n \)

\[
P \left( \sup_{0 \leq t \leq \gamma} \|L_t - \theta_{s,p}\| > x/8 \right) \leq c' \exp \left( -c \frac{x^2 m}{p(1 - e^{-m \gamma})} \right).
\]

**Proof.** Recall that \( p \)-dimensional Brownian motion for \( p \geq 2 \) does not hit points \( \mathbb{P} \)-almost surely [36, Theorem 18.6]. By Girsanov’s theorem this also holds for the diffusion process \( L \) on any finite time interval. We can therefore apply Itô’s formula to the function \( \theta \mapsto \|\theta - \theta_{s,p}\| \) (which is only non-smooth at the point \( \theta_{s,p} \)) such that

\[
\|L_t - \theta_{s,p}\| = \int_0^t \left( \frac{L_s - \theta_{s,p}}{\|L_s - \theta_{s,p}\|} \cdot \nabla f(L_s) + \frac{p - 1}{2} \frac{p - 1}{\|L_s - \theta_{s,p}\|} \right) ds + \sqrt{2} \tilde{W}_t,
\]

where \( \tilde{W}_t = \int_0^t (L_s - \theta_{s,p}) \|L_s - \theta_{s,p}\|^{-1} dW_s \) is a scalar Brownian motion by Lévy’s characterisation of Brownian motion. The strong \( m \)-concavity of \( f \) implies

\[
(\theta - \theta_{s,p}) \cdot \nabla f(\theta) \leq (\theta - \theta_{s,p}) \cdot \nabla f(\theta_{s,p}) - \frac{(m/2)(\theta - \theta_{s,p})^2}{\|L_s - \theta_{s,p}\|}
\]

\[
= (\theta - \theta_{s,p}) \cdot (-\frac{(m/2)(\theta - \tilde{\theta})}{\|L_s - \theta_{s,p}\|}) , \quad \tilde{\theta} = \theta_{s,p} - \frac{2}{m} \nabla f(\theta_{s,p}).
\]

Let \((V_t)_{t \geq 0}\) be a \( p \)-dimensional Ornstein-Uhlenbeck process satisfying

\[
dV_t = -(m/2)(V_t - \tilde{\theta}) dt + \sqrt{2} \tilde{dW}_t, \quad V_0 = \theta_{\text{init}}.
\]

By a comparison argument for scalar Itô processes [35] we get

\[
\|L_t - \theta_{s,p}\| \leq \|V_t - \theta_{s,p}\| \quad \mathbb{P} \text{-almost surely for all } t \geq 0.
\] (39)

The process \((V_t)_{t \geq 0}\) has the explicit solution

\[
V_t = \theta_{\text{init}} e^{-(m/2)t} + \bar{\theta}(1 - e^{-(m/2)t}) + \sqrt{2/m} \tilde{W}_1 e^{-mt}
\]

\[
= \theta_{s,p} + (\theta_{\text{init}} - \theta_{s,p}) e^{-(m/2)t} - \frac{2}{m} \nabla f(\theta_{s,p})(1 - e^{-(m/2)t}) + \sqrt{2/m} \tilde{W}_1 e^{-mt}.
\]

Assumption [C] the lower bound on \( \eta \) from Assumption [D] as well as \( m \geq c_{\min} n p^{-\gamma_3} \) imply

\[ \|\nabla \ell_n(\theta_{s,p})\| \leq (c_{\max}/c_{\min})(\log n)^{-1} \eta m. \] Together with (19) this means for large enough
n, \( \| \nabla f(\theta_{\star,p}) \| \leq \| \nabla \ell_n(\theta_{\star,p}) \| + \| \nabla \log \pi(\theta_{\star,p}) \| \leq \eta m/8 \). Since also \( \| \theta_{\text{init}} - \theta_{\star,p} \| \leq \eta/8 \) by Assumption D and using that \( ac + a(1 - c) = a \) for \( a, c \in \mathbb{R} \) we have

\[
\| \theta_{\text{init}} - \theta_{\star,p} \| e^{-(m/2)t} + (2/m) \| \nabla f(\theta_{\star,p}) \| (1 - e^{-(m/2)t}) \leq \eta/8. 
\]

With this conclude from (39)

\[
P \left( \sup_{0 \leq t \leq J_{\gamma}} \| L_t - \theta_{\star,p} \| > x + \eta/8 \right) \leq P \left( \sup_{0 \leq t \leq J_{\gamma}} \| \tilde{W}_{1,e^{-mt}} \| > x \sqrt{m/2} \right).
\]

Let \((\tilde{W}_{t})_{t \geq 0}\) denote the coordinate processes of \((\tilde{W}_{t})_{t \geq 0}\). Using \( \| x \| \leq p^{1/2} \max_{1 \leq i \leq p} |x_i| \) for \( x \in \mathbb{R}^p \) together with a union bound the last probability is upper bounded by

\[
pP \left( \sup_{0 \leq s \leq 1-e^{-mJ_{\gamma}}} |\tilde{W}_{1,s}| > x \sqrt{m/(2p)} \right).
\]

We can now apply a well-known result on the exit time of a scalar Brownian motion from an interval, cf. [37, Remark 2.8.3], which gives

\[
P( \sup_{0 \leq s \leq t} |\tilde{W}_{1,s}| \geq b) \leq (\sqrt{2t/(b\sqrt{\pi})})e^{-b^2/(2t)}, \quad b > 0, t > 0. \tag{40}
\]

Obtain the claim from \( x \sqrt{m/p} \geq \log n \). \( \square \)

**Lemma 25.** In the setting of Theorem 15 we have for some \( c, c' > 0 \) and large enough \( n \)

\[
P \left( \sup_{k=1,\ldots,J} \| L_{k\gamma} - \bar{L}_{k\gamma} \| > 3\eta/16 \right) \leq c' p \exp \left( -c \frac{\eta^2 m}{p(1 - e^{-mJ_{\gamma}})} \right) + c' Jp \exp \left( -c \frac{\eta^2 m^2}{\gamma p \Lambda^2} \right).
\]

**Proof.** We begin by applying Lemma 22 of [18] (their equation (51)) to the strongly convex function \( U = -f \) and \( \kappa = (2m\Lambda)/(m + \Lambda), \ \varepsilon = \kappa/4 \) such that for all \( k \geq 1 \)

\[
\| L_{k\gamma} - \bar{L}_{k\gamma} \|^2 \leq (1 - \gamma\kappa/2) \| L_{(k-1)\gamma} - \bar{L}_{(k-1)\gamma} \|^2 + (\gamma + 2/\kappa) \int_{(k-1)\gamma}^{k\gamma} \| \nabla f(L_s) - \nabla f(L_{(k-1)\gamma}) \|^2 ds.
\]

Since \( L_0 = \bar{L}_0 \), this yields inductively

\[
\| L_{k\gamma} - \bar{L}_{k\gamma} \|^2 \leq (\gamma + 2/\kappa) \sum_{i=1}^k (1 - \gamma\kappa/2)^{k-i} \int_{(i-1)\gamma}^{i\gamma} \| \nabla f(L_s) - \nabla f(L_{(i-1)\gamma}) \|^2 ds.
\]

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Now, $\nabla f$ is $\Lambda$-Lipschitz, and so using $\gamma \leq m/(\sqrt{\Lambda^4}) \leq \Lambda^{-1}$, $\kappa \geq m$ and recalling that $L$ solves (38), we have with $V = \sup_{0 \leq t \leq J\gamma} \|W_t - W_{[t/\gamma]}\|$,
\[
\|L_{k\gamma} - \tilde{L}_{k\gamma}\| \leq \sqrt{2(\gamma/\kappa + 2/\kappa^2)}\Lambda \sup_{0 \leq t \leq k\gamma} \|L_t - L_{[t/\gamma]}\|
\]
\[
\leq (\sqrt{6}\Lambda/m) \sup_{0 \leq t \leq k\gamma} \left( \int_{[t/\gamma]}^t \|\nabla f(L_s)\| ds + \frac{\sqrt{2}}{\gamma} W_t - W_{[t/\gamma]} \right)
\]
\[
\leq (\sqrt{6}\Lambda/m) \sup_{0 \leq t \leq k\gamma} \left( \int_{[t/\gamma]}^t \|\nabla f(L_s) - \nabla f(\theta_{s,p})\| ds + \gamma \|\nabla f(\theta_{s,p})\| \right) + (\sqrt{12}\Lambda/m)V
\]
\[
\leq (\sqrt{6}\Lambda^2/m) \sup_{0 \leq t \leq J\gamma} \|L_t - \theta_{s,p}\| + \sqrt{6}\Lambda\gamma\eta/8 + (\sqrt{12}\Lambda/m)V,
\]

because $\|\nabla f(\theta_{s,p})\| \leq \eta m/8$ for large enough $n$ as established in the previous lemma. Note $\sqrt{6}\Lambda\gamma\eta/8 < \eta/16$ and $\eta m/(16\sqrt{\Lambda^2}\gamma) - \eta/8 \geq \eta/16$. Applying the triangle inequality to the probability in question and the result of Lemma (34) to $x = \eta/16$ therefore shows for some $c, c' > 0$

\[
P \left( \sup_{k=1,\ldots,J} \|L_{k\gamma} - \tilde{L}_{k\gamma}\| > 3\eta/16 \right)
\]
\[
\leq P \left( \sup_{0 \leq t \leq J\gamma} \|L_t - \theta_{s,p}\| > \eta m/ \left( 16\sqrt{6}\Lambda^2 \gamma \right) \right) + P \left( \gamma^{-1/2} V > \eta m/ \left( 16\sqrt{12}\Lambda \gamma^{1/2} \right) \right)
\]
\[
\leq c' \exp \left( -c \frac{\eta^2 m}{p(1 - e^{-m\gamma})} \right) + JpP \left( \sup_{k-1 \leq i \leq k} |W_{i,t} - W_{i|[t]}| > \eta m/ \left( 16\sqrt{12}\Lambda p \right) \right),
\]

where we have used again the inequality $\|x\| \leq p^{1/2} \max_{1 \leq i \leq p} |x_i|$ for $x \in \mathbb{R}^p$ and a union bound together with

\[
V \overset{d}{=} \gamma^{1/2} \sup_{0 \leq t \leq J} \|W_t - W_{[t]}\| \leq \gamma^{1/2} p^{1/2} \max_{1 \leq i \leq p, 1 \leq k \leq J} \sup_{k-1 \leq i \leq k} |W_{i,t} - W_{i|[t]}|.
\]

To conclude, use $(W_{i,t} - W_{i|[t]})_{k-1 \leq i \leq k} \overset{d}{=} (\tilde{W}_{1,t})_{0 \leq t \leq 1}$ and apply (40), noting $\gamma \leq m/\Lambda^2$ and $\eta \sqrt{m/p} \geq \log n$.

5.4 Proofs for Section 3.4: Polynomial time sampling guarantees

We prove now the remaining results in Section 3.4. Observe first the following crude upper bound on the distance between the initialiser and the mode of the surrogate posterior.
Lemma 26. Grant Assumption [D] and let $\theta_{\text{max}} \in \mathbb{R}^p$ be the unique maximiser of the surrogate posterior density $\hat{\pi}(\cdot|Z^{(n)})$. Then there exists a constant $c_W \equiv C(c_0, c_{\max}, c_{\min})$ such that we have on the event $\mathcal{E}$

$$
\|\theta_{\text{init}} - \theta_{\text{max}}\| \leq c_W^{1/2} \max(\eta, \Lambda_\pi/m).
$$

Proof. Let $\hat{\theta}_{\max}$ and $\theta_{\pi,\max}$ be the unique maximisers of the strongly concave maps $\hat{\ell}_n$ and $\log \pi$. The triangle inequality and Assumption [D] give

$$
\|\theta_{\text{init}} - \theta_{\text{max}}\| \leq \eta/8 + \|\theta_{*,p} - \hat{\theta}_{\max}\| + \|\hat{\theta}_{\max} - \theta_{\text{max}}\|.
$$

Suppose first $\|\hat{\theta}_{\max} - \theta_{*,p}\| > (4c_{\max}/c_{\min})\delta_n p^{k_1+k_3}$ such that by Lemma 23

$$
\hat{\ell}_n(\hat{\theta}_{\max}) - \hat{\ell}_n(\theta_{*,p}) \leq -(4c_{\max}/c_{\min}) n\delta_n < 0.
$$

Since $\hat{\theta}_{\max}$ is the unique maximiser of $\hat{\ell}_n$, this means necessarily $\|\hat{\theta}_{\max} - \theta_{*,p}\| \leq (4c_{\max}/c_{\min})\delta_n p^{k_1+k_3} \leq \eta$ using Assumption [D]. This yields in (41) already the claim if $\hat{\theta}_{\max} = \theta_{\text{max}}$. Suppose now $\hat{\theta}_{\max} \neq \theta_{\text{max}}$. By the previous argument and Assumption [D] we have $\|\theta_{\text{max}}\| + \|\theta_{\pi,\max}\| \leq 1$. Since $\log \pi$ is concave and has $\Lambda_\pi$-Lipschitz gradients, we get

$$
\log \pi(\theta_{\text{max}}) - \log \pi(\hat{\theta}_{\max}) \leq \nabla \log \pi(\hat{\theta}_{\max})^T (\theta_{\text{max}} - \hat{\theta}_{\max})
= \left(\nabla \log \pi(\theta_{\text{max}}) - \nabla \log \pi(\theta_{\pi,\max})\right)^T (\theta_{\text{max}} - \hat{\theta}_{\max})
\leq \Lambda_\pi \|\theta_{\text{max}} - \hat{\theta}_{\max}\| \|\theta_{\text{max}} - \theta_{\pi,\max}\| \leq \Lambda_\pi \|\theta_{\text{max}} - \hat{\theta}_{\max}\|,
$$

and therefore

$$
0 \leq \hat{\ell}_n(\hat{\theta}_{\max}) - \hat{\ell}_n(\theta_{\text{max}})
= \log \hat{\pi}(\hat{\theta}_{\max}|Z^{(n)}) - \log \hat{\pi}(\theta_{\text{max}}|Z^{(n)}) + \log \pi(\theta_{\text{max}}) - \log \pi(\hat{\theta}_{\max})
\leq -\frac{m}{2}\|\hat{\theta}_{\max} - \theta_{\text{max}}\|^2 + \Lambda_\pi \|\theta_{\text{max}} - \hat{\theta}_{\max}\|.
$$

Hence, $\|\theta_{\text{max}} - \hat{\theta}_{\max}\| \leq \Lambda_\pi/m$, and we conclude again by (41). \hfill \Box

Proof of Theorem 13. Apply Theorem 5 of [18] (in the form stated as Proposition A.4 in [56]) to the strongly log-concave measure $\mu = \tilde{\Pi}(\cdot|Z^{(n)})$ from Theorem 11 with unique maximiser $\theta_{\text{max}}$ such that

$$
W_2^2(\mathcal{L}(\hat{\theta}_k), \tilde{\Pi}(\cdot|Z^{(n)}) \leq 2(1 - m\gamma/2)^k (\|\theta_{\text{init}} - \theta_{\text{max}}\|^2 + p/m) + B(\gamma)/2, \quad k \geq 0.
$$

The claim follows from the triangle inequality for the Wasserstein distance, Theorem 12 and Lemma 26. \hfill \Box
Proof of Theorem 14. For \( k \geq J_{\text{in}} \) we get \( 4(1 - m\gamma/2)^k(c_W \max(\eta, \Lambda_p/m)^2 + p/m) \leq \varepsilon^2/8 \), hence by Theorem 13, \( W_2^2(\Lambda(\bar{\theta}_k), \Pi(\cdot|Z^{(n)})) \leq \varepsilon^2/4 \). The claim follows now from the proof of [56, Theorem 3.8].

Proof of Theorem 16. If \( A \) is the event whose probability we want to upper bound, then \( P(A) \leq P(A, \tau > J + J_{\text{in}}) + P(\tau \leq J + J_{\text{in}}) \). Since \( \bar{\theta}_k = \bar{\theta}_k \) for all \( k \leq J + J_{\text{in}} < \tau \), the result follows immediately from Theorems 14 and 15.

We proceed the proof of the final result by the following Lemma on contraction of the posterior mean around the ground truth, which adapts arguments from [49] to our setting.

**Lemma 27.** Suppose that \( \|\theta_0 - \theta_{*,p}\| \leq c_0 \delta_n^{1/\beta} \). Under Assumptions [A] and [B] there exist \( c, c' > 0 \) such that for any large enough \( L \)

\[
P_{\theta_0}^n \left( \left\| \int_{\Theta} \theta d\Pi(\theta|Z^{(n)}) - \theta_0 \right\| > e^{-n\delta_n^2} + L^{1/\beta} \delta_n^{1/\beta} \right) \leq c' e^{-cn^2}.
\]

**Proof.** Recall the set \( U = \left\{ \theta \in \mathbb{R}^p : \|\theta - \theta_{*,p}\| \leq L^{1/\beta} \delta_n^{1/\beta} \right\} \) from the proof of Proposition 11 and the high probability event \( \tilde{E} \) defined there such that, on \( \tilde{E} \), \( \Pi(U^c|Z^{(n)}) \leq e^{-cn^2} \) for any \( c > 0 \) and sufficiently large \( L \). The Jensen inequality shows

\[
\left\| \int_{\Theta} \theta d\Pi(\cdot|Z^{(n)}) - \theta_0 \right\| \leq \int_{\Theta} \left\| \theta - \theta_{*,p} \right\| \|\pi(\theta|Z^{(n)})\| d\theta + \int_{U^c} \left\| \theta - \theta_{*,p} \right\| \|\pi(\theta|Z^{(n)})\| d\theta + \|\theta_{*,-p} - \theta_0\|.
\]

Arguing as in (37) we thus find that with high probability the first term in the last line is smaller than \( e^{-n\delta_n^2} \). Using the bias condition \( \|\theta_0 - \theta_{*,p}\| \leq c_0 \delta_n^{1/\beta} \), we obtain for the last display with high probability the upper bound \( e^{-n\delta_n^2} + (L^{1/\beta} + c_0) \delta_n^{1/\beta} \). Modifying the constant \( L \) yields the claim.

**Proof of Corollary 14.** For large enough \( c > 0 \) we have by Lemma 27 with sufficiently high \( P_{\theta_0}^n \)-probability \( \left\| \int_{\Theta} \theta d\Pi(\theta|Z^{(n)}) - \theta_0 \right\| \leq c_0 \delta_n^{1/\beta} \). Hence, with the coordinate functions \( f_i(x) = x_i \),

\[
P \left( \left\| \frac{1}{J} \sum_{k=1+J_{\text{in}}}^{J+J_{\text{in}}} \vartheta_k - \theta_0 \right\| > c_0 \delta_n^{1/\beta} + \varepsilon \right) \leq P \left( \left\| \frac{1}{J} \sum_{k=1+J_{\text{in}}}^{J+J_{\text{in}}} \vartheta_k - \int_{\Theta} \theta d\Pi(\theta|Z^{(n)}) \right\| > \varepsilon \right)
\]

\[
\leq p \max_{i=1,\ldots,p} P \left( \left\| \frac{1}{J} \sum_{k=1+J_{\text{in}}}^{J+J_{\text{in}}} f_i(\vartheta_k) - \int_{\Theta} f_i(\theta) d\Pi(\theta|Z^{(n)}) > \varepsilon/p \right\| \right),
\]

using in the last line the inequality \( \|x\| \leq \sqrt{p \max_{i=1,\ldots,p} |x_i|} \). Conclude now by Theorem 16 for the 1-Lipschitz maps \( f_i \).

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5.5 Sufficient moment conditions for constructing the surrogate likelihood

In this section we verify the local growth conditions on the log-likelihood function from Assumption [3] under moment conditions. We use the notations from Section 3.

Theorem 28. Let \(0 < \eta \leq 1, \kappa_1, \kappa_2, \kappa_3, \kappa_4 \geq 0\) and \(B = \{\theta \in \mathbb{R}^p : \|\theta - \theta^*\| \leq \eta\}\) and suppose that \(\theta \mapsto \ell_n(\theta) \in C^2(B)\), \(P_{\theta^0}\)-almost surely. Then Assumption [28] holds for some \(c_{\text{min}}, c_{\text{max}} > 0\), if there exist \(C > 0, C_1 \geq C_2 > 0\) such that for all \(v \in \mathbb{R}^p\) with \(\|v\| \leq 1\) and all \(i = 1, \ldots, n\) the following conditions hold:

(i) (growth conditions) \(p \leq Cn\eta^2, C \max(\delta_n p^{\kappa_2}, \eta\delta_n p^{\kappa_4}, \delta_n^2 p^{\kappa_4}) \log n \leq p^{-\kappa_3}\).

(ii) (local mean boundedness) \(|E_{\theta^0} \theta^T \nabla \ell(\theta_{*p}, Z_i)| \leq C_1 \delta_n p^{\kappa_1}\) and for all \(q \geq 2\)
\[
E_{\theta^0} \theta^T \nabla \ell(\theta_{*p}, Z_i)|^q \leq (q!/2)C_1^q p^{\kappa_1+q(q-2)\kappa_2},
\]
\[
\sup_{\theta \in B} E_{\theta^0} \theta^T \nabla^2 \ell(\theta, Z_i) \theta|^q \leq (q!/2)C_1^q p^{2\kappa_2+q(q-2)\kappa_4},
\]
\[
\sup_{\theta, \theta' \in B} E_{\theta^0} \theta^T \nabla^2(\ell(\theta, Z_i) - \ell(\theta', Z_i)) \theta'|^q \leq (q!/2) \left(C_1 p^{\kappa_4} \|\theta - \theta'\|^q\right).
\]

(iii) (local mean curvature) \(\inf_{\theta \in B} \lambda_{\min} \left(E_{\theta^0} \left[-\nabla^2 \ell(\theta, Z_i)\right]\right) \geq C_2 p^{-\kappa_3}\).

The proof of this theorem is based on the classical Bernstein inequality (see, e.g., Proposition 3.1.8 in [30]) and a chaining argument for empirical processes with mixed tails, cf. Theorem 3.5 of [15]. In the proof we denote for a metric space \(T\) and a metric \(d\) by \(N(T, d, \varepsilon)\) the minimal number of closed \(d\)-balls of radius \(\varepsilon\) necessary to cover \(T\).

Lemma 29 (Bernstein’s inequality). Let \(X_1, \ldots, X_n\) be real-valued centred and independent random variables such that \(E_{\theta^0} |X_i|^q \leq (q!/2)\sigma^2 e^{q^2-2}\) for some \(\sigma > 0, c > 0\) and all \(1 \leq i \leq n, q \geq 2\). Then
\[
P_{\theta^0} \left(\sum_{i=1}^n X_i \geq \sqrt{2n\sigma^2 t + ct}\right) \leq 2e^{-t}, \quad t \geq 0.
\]

Lemma 30. Let \(U\) be a measurable subset of \(\mathbb{R}^p\) with diameter \(\sup_{\theta, \theta' \in U} \|\theta - \theta'\| = D > 0\). Let \(h_{\theta} : Z \mapsto \mathbb{R}, \theta \in \mathcal{U}\), be a family of functions such that for some \(\sigma_p, c_p > 0, all q \geq 2\) and all \(i = 1, \ldots, n\)
\[
E_{\theta^0} |h_{\theta}(Z_i)|^q \leq (q!/2)e^{q^2}c_p^{-2} \sigma_p^2, \quad \theta \in \mathcal{U},
\]
\[
E_{\theta^0} |h_{\theta}(Z_i) - h_{\theta'}(Z_i)|^q \leq (q!/2)C_p^q \|\theta - \theta'\|^q, \quad \theta, \theta' \in \mathcal{U}.
\]
Consider the empirical process \((Z_n(\theta), \theta \in \mathcal{U})\) with \(Z_n(\theta) = \sum_{i=1}^n (h_{\theta}(Z_i) - E_{\theta^0} h_{\theta}(Z_i))\).
Then there exists a universal constant \(M \geq 1\) such that for all \(t \geq 1, t' \geq 0\)
\[
P_{\theta^0} \left(\sup_{\theta \in \mathcal{U}} |Z_n(\theta)| \geq Mc_p D \left(\sqrt{n}p + \sqrt{n}t + t\right) + 3 \left(\sqrt{2n\sigma_p^2 t' + c_p t'}\right)\right) \leq e^{-t} + 2e^{-t'}.
\]
Proof. Write $Z_n(\theta) = \sum_{i=1}^n h_{\theta,i}$ with independent and centred random variables $h_{\theta,i} = h_\theta(Z_i) - \mathbb{E}_{\theta_0} h_\theta(Z_i)$. The moment assumptions in (42) and (43) hold for the $h_{\theta,i}$ with constants $3\sigma_p$ and $3c_p$. Fix any $\theta, \theta' \in \mathcal{U}$. Then the Bernstein inequality in Lemma 29 gives for $t \geq 0$

$$\mathbb{P}_{\theta_0} \left( |Z_n(\theta')| \geq 3\sqrt{2n\sigma_p^2 t} + 3c_p t \right) \leq 2e^{-t}, \quad (44)$$

$$\mathbb{P}_{\theta_0} \left( |Z_n(\theta) - Z_n(\theta')| \geq 3c_p ||\theta - \theta'|| \sqrt{2nt} + 3c_p ||\theta - \theta'|| t \right) \leq 2e^{-t}. \quad (45)$$

The last line implies that $Z_n$ has a mixed tail with respect to the metrics $d_1(\theta, \theta') = 3c_p ||\theta - \theta'||$, $d_2(\theta, \theta') = \sqrt{2n}d_1(\theta, \theta')$ in the sense of [13, Equation (3.8)]. Since the set $\mathcal{U}$ has diameter $\sup_{\theta, \theta' \in \mathcal{U}} d_1(\theta, \theta') = 3c_p D$ with respect to $d_1$ and diameter $3c_p D\sqrt{2n}$ with respect to $d_2$, using Proposition 4.3.34 and equation (4.171) in [30] yields for the metric entropy integrals with respect to $d_1$ and $d_2$ the upper bounds

$$\gamma_{d_1}(\mathcal{U}) = \int_0^\infty \log N(\mathcal{U}, d_1, \varepsilon) d\varepsilon \leq \int_0^{3c_p D} \log N(\{ \theta \in \mathbb{R}^p : ||\theta|| \leq D \}, ||\cdot||, \varepsilon/(3c_p)) d\varepsilon$$

$$= \int_0^{3c_p D} \log N(\{ \theta \in \mathbb{R}^p : ||\theta|| \leq 1 \}, ||\cdot||, \varepsilon/(3c_p D)) d\varepsilon$$

$$\leq \int_0^{3c_p D} p \log(9c_p D/\varepsilon) d\varepsilon = 3c_p D p \int_0^1 \log(3/\varepsilon) d\varepsilon \lesssim c_p D p,$$

and in the same way

$$\gamma_{d_2}(\mathcal{U}) = \int_0^\infty \sqrt{\log N(\mathcal{U}, d_2, \varepsilon)} d\varepsilon \leq 3c_p D \sqrt{np} \int_0^1 \sqrt{\log(3/\varepsilon)} d\varepsilon \lesssim c_p D \sqrt{np}.$$

Together with the mixed tail property in (45) infer from Theorem 3.5 of [15] the existence of an absolute constant $M \geq 1$ such that for any $t \geq 1$

$$\mathbb{P}_{\theta_0} \left( \sup_{\theta \in \mathcal{U}} |Z_n(\theta) - Z_n(\theta')| \geq M c_p D (\sqrt{np} + p + \sqrt{nt} + t) \right) \leq e^{-t}.$$ 

The result follows from the triangle inequality and from applying (44) to $t = t'$.

With this let us prove the main result of this section.

Proof of Theorem 28 For constants $C_3, C_4 > 0$, to be determined lateron, set $\tau_1 = C_3n\delta_{\lambda^2}k_1$, $\tau_2 = C_4np^{-k_3}$ and define $b(\theta) = \nabla \ell_n(\theta) - \mathbb{E}_{\theta_0} \nabla \ell_n(\theta)$, $\Sigma(\theta) = \nabla^2 \ell_n(\theta) - \mathbb{E}_{\theta_0} \nabla^2 \ell_n(\theta)$. We will prove the claim for the event $\mathcal{E} = \mathcal{E}_1 \cap \mathcal{E}_2$, where

$$\mathcal{E}_1 = \{ \|b(\theta_n)\| \leq \tau_1 \}, \quad \mathcal{E}_2 = \left\{ \sup_{\theta \in \mathcal{B}} \|\Sigma(\theta)\|_{op} \leq \tau_2 \right\}.$$
Recall the min-max characterisation of the eigenvalues of a symmetric matrix $A \in \mathbb{R}^{p \times p}$ such that
\[
\|\nabla^2 A\|_{\text{op}} = \sup_{v \in \mathbb{R}^p : \|v\| \leq 1} v^T \nabla^2 A v.
\]
With this conclude using the mean boundedness assumptions in part (ii) of the statement of the theorem (with $q = 2$) for $\theta \in B$ that $\|\mathbb{E}_{\theta_0}^n \nabla^2 \ell_n(\theta)\|_{\text{op}} \leq C_1 n p^{\kappa_2}$, and therefore that we have on $\mathcal{E}$
\[
\|\nabla^2 \ell_n(\theta)\|_{\text{op}} \leq \tau_2 + C_1 n p^{\kappa_2} \leq (C_4 + C_1) n p^{\kappa_2}.
\]
In the same way, $\|\nabla \ell_n(\theta_{s,p})\| \leq (C_3 + C_1) n \delta_n p^{\kappa_1}$ on $\mathcal{E}$, proving Assumption $[\mathcal{C}^i]$ for $c_{\text{max}} = \max(C_3, C_4) + C_1$, while the mean curvature lower bound in part (iii) yields by Weyl’s inequality for $\theta \in B$
\[
\lambda_{\min} \left( - \nabla^2 \ell_n(\theta) \right) = \lambda_{\min} \left( \mathbb{E}_{\theta_0}^n \left[ - \nabla^2 \ell_n(\theta) \right] - \Sigma(\theta) \right) \\
\geq \sum_{i=1}^n \lambda_{\min} \left( \mathbb{E}_{\theta_0}^n \left[ - \nabla^2 \ell(\theta, Z_i) \right] \right) - \|\Sigma(\theta)\|_{\text{op}} \geq C_2 n p^{-\kappa_3} - C_4 n p^{-\kappa_3} = (C_2 - C_4) n p^{-\kappa_3}.
\]
From this obtain Assumption $[\mathcal{C}^ii]$ for $c_{\text{min}} = C_2 - C_4$, as long as $C_4 < C_2$.

We are therefore left with showing $\mathbb{P}_{\theta_0}^n(\mathcal{E}^c) \leq C' e^{-C n \delta^2}$ for suitable $C_3, C_4$. By adjusting $C'$ it suffices to prove this for $n$ large enough. We will use a contraction argument for quadratic forms, commonly used in random matrix theory. For $0 < \delta \leq 1$ and $N = N(\{v \in \mathbb{R}^p : \|v\| \leq 1, \|\cdot\|, \delta\})$ let $v_1, \ldots, v_N$ be the centres of a minimal open cover for the Euclidean unit ball with radius $\delta$. This implies for $v \in \mathbb{R}^p$ with $\|v\| \leq 1$ and $i = 1, \ldots, N$ with $\|v - v_i\| \leq \delta$ that
\[
v^T \Sigma(\theta)v = v_i^T \Sigma(\theta)v_i + (v - v_i)^T \Sigma(\theta)(v - v_i) + 2(v - v_i)^T \Sigma(\theta)v_i \\
\leq v_i^T \Sigma(\theta)v_i + \delta^2 \|\Sigma(\theta)\|_{\text{op}} + 2\delta \|\Sigma(\theta)\|_{\text{op}} \leq v_i^T \Sigma(\theta)v_i + 3\delta \|\Sigma(\theta)\|_{\text{op}}.
\]
For the same $v_i$ we also get $|v^T b(\theta_{s,p})| \leq |v_i^T b(\theta_{s,p})| + \delta \|b(\theta_{s,p})\|$. Taking $\delta = 1/4$ and maximising over $v$ in the unit ball and over $i$ then gives
\[
|b(\theta_{s,p})| \leq \frac{4}{3} \max_{i=1,\ldots,N} |v_i^T b(\theta_{s,p})|,
\]
\[
\|\Sigma(\theta)\|_{\text{op}} \leq 4 \max_{i=1,\ldots,N} \sup_{\theta \in B} |v_i^T \Sigma(\theta)v_i|.
\]
By applying union bounds this means for $j = 1, 2$
\[
\mathbb{P}_{\theta_0}^n(\mathcal{E}^c) \leq \mathbb{P}_{\theta_0}^n(\mathcal{E}_1^c) + \mathbb{P}_{\theta_0}^n(\mathcal{E}_2^c) \\
\leq N \sup_{v \in \mathbb{R}^p : \|v\| \leq 1} \left( \mathbb{P}_{\theta_0}^n \left( |v^T b(\theta_{s,p})| > 3\tau_1/4 \right) + \mathbb{P}_{\theta_0}^n \left( \sup_{\theta \in B} |v^T \Sigma(\theta)v| > \tau_2/4 \right) \right).
\]
Proposition 4.3.34 of [30] and the growth conditions in part (i) yield $N \leq e^{\rho \log 12} \leq e^{3Cn\delta_n^2}$.

To prove the wanted high probability bounds we are left with establishing that the probabilities in the last display are each smaller than $C' e^{-4Cn\delta_n^2}$ for some $C' > 0$. For this we apply the two lemmas above to the empirical processes $b(\theta_{x,p})$ and $\Sigma(\theta)$, uniformly for $\theta \in \mathcal{B}$. First, consider $b(\theta_{x,p}) = \sum_{i=1}^n (h_\theta(Z_i) - E_\theta h_\theta(Z_i))$ with $h_\theta(Z_i) = v^\top \nabla \ell_n(\theta_{x,p}, Z_i)$. The mean boundedness conditions in (ii) show $E_\theta |h_\theta(Z_i)|^q \leq (q!/2) \sigma^2 e^{r^2}$ for $\sigma = C_1 p^{\kappa_1}$, $c = C_1 p^{\kappa_2}$ and all $q \geq 2$. We can therefore apply Lemma 29 to $t = 4Cn\delta_n^2$. The growth conditions in part (i) imply $\delta_n p^{\kappa_2} \leq p^{-\kappa_3} \leq p^{\kappa_1}$. This means, if we set $C_3 = 4(2\sqrt{2C_1} + 4CC_1)/3$, then

$$\sqrt{2n\sigma^2 t} + ct = 2\sqrt{2C_1 n\delta_n p^{\kappa_1} + 4CC_1 n\delta_n^2 p^{\kappa_2}} = 3\tau_1/4,$$

and therefore $\mathbb{P}_{\theta_0}^n (|v^\top b(\theta_{x,p})| > 3\tau_1/4) \leq 2e^{-4Cn\delta_n^2}$. Next, consider $\Sigma(\theta) = \sum_{i=1}^n (h_\theta(Z_i) - E_\theta h_\theta(Z_i))$ with $h_\theta(Z_i) = v^\top \nabla \ell_n(\theta, Z_i)v$. Using again the conditions in part (ii) verifies (42) and (43) with $\sigma_p = C_1 p^{\kappa_2}$, $c_p = C_1 p^{\kappa_1}$. The set $\mathcal{U} = \mathcal{B}$ has diameter $D = \sup_{\theta, \theta' \in \mathcal{B}} ||\theta - \theta'|| = 2\eta$. If $M$ is the constant from the statement of Lemma 30 and $t = t' = 4Cn\delta_n^2$, then

$$Mc_p D \left( \sqrt{np} + p + \sqrt{nt} + t \right) + 3 \left( \sqrt{2n\sigma_p^2 t'} + c_p t' \right) \leq MC_1 p^{\kappa_4} 2\eta \left( 4\sqrt{Cn\delta_n} + 8Cn\delta_n^2 \right) + 3 \left( 2\sqrt{2C_1 n\delta_n p^{\kappa_2}} + 4CC_1 n\delta_n^2 p^{\kappa_4} \right).$$

Taking $n$ large enough, the growth conditions in part (i) provide us for any $c > 0$ with the upper bound $\max(\delta_n p^{\kappa_2}, \eta\delta_n p^{\kappa_1}, \delta_n^2 p^{\kappa_2}) \leq cp^{-\kappa_3}$. This implies that the expression in the last display is upper bounded by $\tau_2/4 = (C_4/4) np^{-\kappa_3}$ for a suitable $C_4 < C_2$. Lemma 30 now implies the wanted upper bound $\mathbb{P}_{\theta_0}^n (\sup_{\theta \in \mathcal{B}} |v^\top \Sigma(\theta)v| > \tau_2/4) \leq 3e^{-4Cn\delta_n^2}$. This finishes the proof.

6 Appendix

6.1 Proofs for specific models in Section 4

6.1.1 Density estimation

The density estimation model fits into the setting in Section 3 with $Z(n) = (X_i)_{i=1}^n$, $Z = \mathcal{X}$, $v = \nu \chi$, $\delta_n = n^{-\alpha/(2\alpha+1)}$. By modifying the final constant $c_2$ in the statements of Theorems 4, 5, 6 it is enough to consider any sufficiently large $n$. We begin by checking the assumptions in Section 3.4 for the density estimation model.

**Proposition 31.** Consider the setting of Theorem 18. Then Assumption D holds for $\kappa_1 = \kappa_3 = 0$, $\kappa_2 = 1/2$, $K = cnp^{1/2}$, $c > 0$ large enough.

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Proof. Let us make a few preliminary observations, which will be used in the proof without further mention. Since the basis functions $e_k$ are bounded, we have $\|\Phi(\theta)\|_{L^\infty} \lesssim \|\theta\|_{\ell^1}$ for $\theta \in \ell^1(\mathbb{N})$. Fixing a radius $r > 0$ therefore yields the existence of a constant $c_r > 0$ with

$$\|\Phi(\theta)\|_{L^\infty} \leq c_r, \quad c_r^{-1} \leq p_\theta(x) \leq c_r, \quad x \in \mathcal{X}, \|\theta\|_{\ell^1} \leq r. \quad (46)$$

The operators $A : L^\infty(\mathcal{X}) \to \mathbb{R}$, $A(u) = \log \int_{\mathcal{X}} e^{u(x)}d\nu_X(x)$, are Fréchet differentiable with derivatives obtained according to the chain rule for $u, h \in L^\infty(\mathcal{X})$ by

$$DA(u)[h] = \frac{\int_{\mathcal{X}} h(x)e^{u(x)}d\nu_X(x)}{\int_{\mathcal{X}} e^{u(x)}d\nu_X(x)},$$

$$D^2A(u)[h, h] = \frac{\int_{\mathcal{X}} h(x)h'(x)e^{u(x)}d\nu_X(x)}{\int_{\mathcal{X}} e^{u(x)}d\nu_X(x)} - \left(\frac{\int_{\mathcal{X}} h(x)e^{u(x)}d\nu_X(x)}{\int_{\mathcal{X}} e^{u(x)}d\nu_X(x)}\right)^2.$$

In particular, if $\theta, \theta' \in \ell^1(\mathbb{N})$, then

$$DA(\Phi(\theta))[\Phi(\theta')] = \langle \Phi(\theta'), p_\theta \rangle_{L^2},$$

$$D^2A(\Phi(\theta))[\Phi(\theta'), \Phi(\theta')] = \int_{\mathcal{X}} (\Phi(\theta')(x) - \langle \Phi(\theta'), p_\theta \rangle_{L^2} - \langle \Phi(\theta'), p_\theta \rangle_{L^2})^2 p_\theta(x)d\nu_X(x). \quad (48)$$

Fix now $\theta, \theta'$ with $\|\theta\|_{\ell^1}, \|\theta'\|_{\ell^1} \leq r$. Using that $\int_{\mathcal{X}} \Phi(\theta')d\nu_X = 0$ by the centring of the $e_k$, we have by (46) and the Cauchy-Schwarz inequality

$$\int_{\mathcal{X}} (\Phi(\theta')(x) - \langle \Phi(\theta'), p_\theta \rangle_{L^2})^2 d\nu_X(x) = \|\Phi(\theta')\|_{L^2}^2 + \|\Phi(\theta'), p_\theta\|_{L^2} \leq \|\theta'\|^2 + c_r\|\theta'\|^2,$

and therefore

$$c_r^{-1}\|\theta'\|^2 \leq D^2A(\Phi(\theta))[\Phi(\theta'), \Phi(\theta')] \leq c_r(1 + \sqrt{c_r})\|\theta'\|^2. \quad (49)$$

On the other hand, it follows from (46) and (47) that $A(\Phi(\cdot))$ is uniformly Lipschitz on the set $\{\theta : \|\theta\|_{\ell^1} \leq r\}$, because

$$|A(\Phi(\theta)) - A(\Phi(\theta'))| = \left|\int_0^1 DA(\Phi(\theta' + t(\theta - \theta')))D\Phi(\theta - \theta')dt\right| \leq \sup_{0 \leq t \leq 1} \|D^2A(\Phi(\theta'))\|_{L^\infty}\|\Phi(\theta - \theta')\|_{L^2} \leq c_r\|\theta - \theta'\|. \quad (50)$$

We write $\Phi^x_\theta = \Phi(\theta)(x)$ and denote by $X$ a generic copy of $Z_i = X_i$. Observe for $v \in \mathbb{R}^p$ the identities

$$\ell(\theta) = \Phi^X_\theta - A(\Phi(\theta)), \quad E_{\theta_0}\ell(\theta) = \langle \Phi(\theta), p_{\theta_0} \rangle_{L^2} - A(\Phi(\theta)),$$

$$v^T \nabla \ell(\theta) = \Phi^X_v - \langle \Phi(v), p_{\theta_0} \rangle_{L^2}, \quad E_{\theta_0}v^T \nabla \ell(\theta) = \langle \Phi(v), p_{\theta_0} - p_{\theta_0} \rangle_{L^2},$$

$$v^T \nabla^2 \ell(\theta)v = -D^2A(\Phi(\theta))[\Phi(v), \Phi(v)]. \quad (51)$$
Note that $v^\top \nabla\ell(\theta)v$ is not random. After these preparations let us verify Assumptions [A, C] and [D]. For this it suffices to check the conditions of Theorems 40 and 28 for suitable $\kappa_1, \ldots, \kappa_4$.

**Theorem 40(i): Approximation.** Consider $r > 0$ and $\theta \in B_{n,r}$. Since $\alpha > 1/2$, there exists $r' \equiv r'(r)$ with

$$U \subset \{ \theta \in \ell^2(\mathbb{N}) : \|\theta\|_\alpha \leq r \} \subset \{ \theta \in \ell^1(\mathbb{N}) : \|\theta\|_{\ell^1} \leq r' \}.$$  

(52)

It follows for $\theta$ and $\theta_0 \in h^\alpha(\mathbb{N})$

$$\mathbb{E}_{\theta_0} (\ell(\theta_0) - \ell(\theta)) = (\Phi(\theta_0 - \theta), p_{\theta_0})_{L^2} - (A(\Phi(\theta_0)) - A(\Phi(\theta)))$$

$$= DA(\Phi(\theta_0))[\Phi(\theta_0 - \theta)] - (A(\Phi(\theta_0)) - A(\Phi(\theta)))$$

$$= \int_0^1 (1 - t)D^2A(\Phi(\theta' + t(\theta - \theta')))[\Phi(\theta - \theta'), \Phi(\theta - \theta')]dt.$$  

The first part of inequality (51) follows from (49), (8) such that $\theta$ holds for $\theta, \theta_0 \in h^\alpha(\mathbb{N})$

$$\|\mathbb{E}_{\theta_0} (\ell(\theta_0) - \ell(\theta))\| \leq \|\theta_0 - \theta\|^2 \leq \delta_n^2 + \|\theta - \theta_{*,p}\|^2 \leq \delta_n^2.$$  

For the second part note that the log-likelihood function $\theta \to \ell(\theta) = \log p_\theta$ is uniformly bounded on $B_{n,r}$ such that for $q \geq 2$ by (50)

$$\mathbb{E}_{\theta_0} |\ell(\theta) - \ell(\theta_0)|^q \leq C^{q-2} \mathbb{E}_{\theta_0} |\ell(\theta) - \ell(\theta_0)|^2$$

$$\lesssim C^{q-2} \left( \mathbb{E}_{\theta_0} |\Phi_{\theta} X - \Phi_{\theta_0} X|^2 + |A(\Phi(\theta)) - A(\Phi(\theta_0))|^2 \right)$$

$$\lesssim C^{q-2} (\|\Phi(\theta - \theta_0)\|_{L^2}^2 + \|\theta - \theta_0\|^2) \lesssim C^{q-2} \|\theta - \theta_0\|^2 \lesssim C^{q-2}\delta_n^2.$$  

**Theorem 40(ii): Hellinger distance.** Let $r > 0$, $\theta, \theta' \in \ell^2(\mathbb{N})$ with $\|\theta\|_\alpha, \|\theta'\|_\alpha \leq r$ and set $u = (\Phi_{\theta} X + \Phi_{\theta'} X)/2$. Arguing as in the proof of the corresponding statement in Proposition 19 we get $h^2(\theta, \theta') = 2 - 2e^{-x}$ where

$$x = \frac{1}{4} \int_0^1 t \int_0^1 D^2A \left( \frac{1 - t}{2} + tt' \right) \Phi(\theta - \theta') \Phi(\theta' - \theta') dt' dt.$$  

Upper and lower bounding this non-random quantity gives by (19)

$$c_r^{-1} \|\theta - \theta'\|^2 \leq x \leq c_r (1 + \sqrt{c_r}) \|\theta - \theta'\|^2 \leq c_r (1 + \sqrt{c_r}) (2r)^2.$$  

The result follows from (18).

**Theorem 28(ii): Local mean boundedness.** A key step is to note that there exists $r' > 0$ with $B \subset \{ \theta \in \ell^1(\mathbb{N}) : \|\theta\|_{\ell^1} \leq r' \}$, because for $\theta \in B$

$$\|\theta\|_{\ell^1} \leq \|\theta - \theta_{*,p}\|_{\ell^1} + \|\theta_{*,p}\|_{\ell^1} \lesssim p^{1/2} \|\theta - \theta_{*,p}\| + \|\theta_{*,p}\|_\alpha \lesssim 1.$$  

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Fix now $\theta \in \mathcal{B}$, $v \in \mathbb{R}^p$ with $\|v\| \leq 1$. It follows from the identities (51) and (49)

$$\|p_{\theta_0} - p_{\theta_{*p}}\|^2 \lesssim \mathbb{E}_{\theta_0} \left[ e^{\ell(\theta_0)} - e^{\ell(\theta_{*p})} \right]^2 \lesssim \mathbb{E}_{\theta_0} |\ell(\theta_0) - \ell(\theta_{*p})|^2 \lesssim \delta_n^2. \quad (53)$$

Together with the last display this means for $q \geq 2$, $\theta \in \mathcal{B}$ and some $C > 0$

$$\mathbb{E}_{\theta_0} |v^T \nabla \ell(\theta_{*p})|^q \leq C^{q-2} \|v^T \nabla \ell(\theta_{*p})\|^2 \mathbb{E}_{\theta_0} \|v^T \nabla \ell(\theta_{*p})\|^2 \lesssim C^{q-2} p^{(q-2)/2} (2\|\Phi(v)\|^2 + 2\langle \Phi(v), p_{\theta_{*p}} \rangle^2) \lesssim C^{q-2} p^{(q-2)/2},$$

$$\mathbb{E}_{\theta_0} |v^T \nabla^2 \ell(\theta)v|^q \leq C^q.$$

At last, we have for $\theta' \in \mathcal{B}$,

$$\begin{align*}
|v^T (\nabla^2 \ell(\theta) - \nabla^2 \ell(\theta'))v| & \lesssim \int_X \left( \Phi_v^x - \langle \Phi(v), p_\theta \rangle_{L^2} \right)^2 |p_\theta - p_{\theta'}| (x) d\nu_X(x) \\
+ \int_X \left( \Phi_v^x - \langle \Phi(v), p_\theta \rangle_{L^2} \right)^2 - (\Phi_v^x - \langle \Phi(v), p_{\theta'} \rangle_{L^2})^2 d\nu_X(x) \\
& \lesssim \|\Phi_v\|_{L^\infty} \|\Phi(v)\|_{L^2} \|p_\theta - p_{\theta'}\|_{L^2} \\
+ \langle \Phi(v), p_{\theta'} - p_\theta \rangle_{L^2} \int_X (2\Phi_v^x - \langle \Phi(v), p_\theta + p_{\theta'} \rangle_{L^2}) p_{\theta'}(x) d\nu_X(x) \\
& \lesssim p^{1/2} \|\theta - \theta'\| + \langle \Phi(v), p_{\theta'} - p_\theta \rangle_{L^2} \lesssim p^{1/2} \|\theta - \theta'\|.
\end{align*}$$

In all, we verify the assumptions in Theorem \[\text{28}(\text{ii})\] with $\kappa_1 = 0$, $\kappa_2 = 1/2$, $\kappa_4 = 1/2$.

**Theorem \[\text{28}(\text{iii})\]: Local curvature.** Use the identities (51) and (49) to obtain the result with $\kappa_3 = 0$ from

$$\inf_{\theta \in \mathcal{B}} \mathbb{E}_{\theta_0} \left[ -v^T \nabla^2 \ell(\theta)v \right] = \inf_{\theta \in \mathcal{B}} D^2 A(\Phi(\theta))[\Phi(v), \Phi(v)] \gtrsim \|v\|^2 = 1.$$

**Theorem \[\text{28}(\text{i})\]: Growth conditions.** These follow immediately from the conditions in the proposition for the obtained $\kappa_i$, noting $\delta_n p^{\kappa_2 + \kappa_3} = \delta_n p^{1/2} \lesssim (\log n)^{-1},$ $(\eta \delta_n + \delta_n^2) p^{\kappa_3 + \kappa_4} \lesssim \delta_n p^{1/2}$ for $\alpha > 1$.

**Assumption \[\text{12}\].** Clearly, for large enough $n$, $\eta = p^{-1/2} \gtrsim (\log n) \delta_n$ and $K \gtrsim n p^{1/2} \gtrsim n (p^{1/2} \delta_n + p^{1/2})$.

By the help of this proposition we can now obtain the three theorems in Section \[\text{2.3}\] from exactly the same proof as in Section \[\text{5.1}\] with the same choices for $\gamma, \epsilon, J_{\text{in}}$ and $J_{\text{out}}$. 

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6.1.2 Nonparametric regression

Proof of Proposition [19]. Let us make a few preliminary observations used in the proof without further mention. Recall $b(\theta) = (A')^{-1} \circ g^{-1} \circ G(\theta)$. By Assumptions [G1](ii) and [G2](ii), the range of the map $(\theta, x) \mapsto G(\theta)(x)$, evaluated for $\theta$ in the convex hull $R = \text{conv}(B \cup \{\theta_0\})$ and $x \in X$, is a bounded subset of $\mathbb{R}$. Since $A$ is smooth and convex, and $g \in C^2(I)$ is invertible, this means there exists a constant $M > 0$ such that $\sup_{\theta \in R} ||b(\theta)||_{L^\infty} \leq M$, $\sup_{\theta \in R} ||f(b(\theta))||_{L^\infty} \leq M$ for $f \in \{A, A', A'', A'''\}$ and $\inf_{\theta \in R} ||A''(b(\theta))||_{L^\infty} \geq M^{-1}$.

We write $b_\theta^x = b(\theta)(x)$ and denote by $(Y, X)$ a generic copy of $(Y_i, X_i)$. Frequently, we will use that the bounded design implies for $\theta, \theta' \in \ell^2(\mathbb{N})$

$$||b(\theta) - b(\theta')||_{L^2}^2 \lesssim \mathbb{E}_{\theta_0} (b_\theta^X - b_{\theta'}^X)^2 \lesssim ||b(\theta) - b(\theta')||_{L^2}^2. \quad (54)$$

The assumption that $\theta \mapsto G(\theta)(x) \in C^2(B)$ at every $x \in X$ implies $\theta \mapsto b_\theta^x \in C^2(B)$. The properties of $G$ in Assumption [G1] all transfer to the map $b$ immediately, as do the statements on $G$ and its first derivative in Assumption [G2]. Regarding second derivatives we find for $v \in \mathbb{R}^p$, $||v|| = 1$,

$$\|\nabla^2 b(\theta)\|_{L^\infty(X, \mathbb{R}^{p \times p})} \lesssim \max(p^{2k_1}, p^{k_2}),$$

$$\|\nabla^2 b(\theta) - \nabla^2 b(\theta')\|_{L^\infty(X, \mathbb{R}^{p \times p})} \lesssim \max(p^{3k_1}, p^{k_1 + k_2}) \|\theta - \theta'\|,$$

$$\|v^\top \nabla^2 b(\theta)v\|_{L^2} \lesssim \max(p^{2k_1}, p^{k_1}). \quad (55)$$

At last, observe for $v \in \mathbb{R}^p$ the identities

$$\ell(\theta) = Y b_\theta^X - A(b_\theta^X), \quad \mathbb{E}_{\theta_0} Y = \mathbb{E}_{\theta_0} A'(b_{\theta_0}^X),$$

$$v^\top \nabla \ell(\theta) = (Y - A'(b_\theta^X))v^\top \nabla b_\theta^X,$$

$$v^\top \nabla^2 \ell(\theta)v = (Y - A'(b_\theta^X))v^\top \nabla^2 b_\theta^X v - A''(b_\theta^X)(v^\top \nabla b_\theta^X)^2. \quad (56)$$

With these preparations let us to verify Assumptions [1] for some $\kappa_1, \ldots, \kappa_4$. For this it suffices to check the conditions of Theorems [10] and [28].

Theorem [10(i)]: Approximation. Consider $r > 0$ and $\theta \in B_{n,r}$. The first part of inequality [14] follows from a Taylor expansion of $A$ at $b_{\theta_0}^X$ such that by (52) and Assumption [G1](i,ii) (with $b$ instead of $G$)

$$||\mathbb{E}_{\theta_0} (\ell(\theta_0) - \ell(\theta))|| = ||\mathbb{E}_{\theta_0} (A'(b_{\theta_0}^X)(b_{\theta_0}^X - b_{\theta_0}^X) + A(b_{\theta_0}^X) - A(b_{\theta_0}^X))||$$

$$\leq \frac{M}{2} \mathbb{E}_{\theta_0} (b_{\theta_0}^X - b_{\theta_0}^X)^2 \lesssim \|b(\theta) - b(\theta')\|_{L^2}^2 \lesssim \|b(\theta_0) - b(\theta_{*,p})\|_{L^2}^2 + \|\theta_{*,p} - \theta_{*,p}\|_{L^2}^2 \lesssim \delta_n^2.$$

For the part inequality let $\lambda \in \mathbb{R}$ be sufficiently small such that for some $c > 0$

$$\mathbb{E}_{\theta_0} [\exp(\lambda Y) X] = \exp (A(\lambda + b_{\theta_0}^X) - A(b_{\theta_0}^X)) \leq c,$$

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and $E_{\theta_0}(\exp(\lambda|Y|)|X) \leq 2c$. In particular, for some $c_\lambda > 0$ and all $q \geq 2$, 
\[ E_{\theta_0} [|Y|^q | X] \leq c_\lambda^q, \]  
(57)
and by conditioning on $X$
\[ E_{\theta_0} \left| b(b_\theta^X - b_0^X) \right|^q \leq c_\lambda^q E_{\theta_0} \left| bX - b_0 \right|^q \leq c_\lambda^q \|b(\theta) - b(\theta_0)\|_{L^2}^2 \leq c_\lambda^q \delta_n^2. \]
This gives $E_{\theta_0} |\ell(\theta) - \ell(\theta_0)|^q \lesssim c_\lambda^q \delta_n^2$.

**Theorem 13(ii): Hellinger distance.** Let $r > 0, \theta, \theta' \in \ell^2(N)$ with $\|\theta\|_\alpha, \|\theta'\|_\alpha \leq r$. By convexity of $A$, the squared Hellinger distance between $P_\theta$ and $P_{\theta'}$, cf. (73), equals
\[ h^2(\theta, \theta') = 2 - \int_z 2\sqrt{p_\theta(z)p_{\theta'}(z)}d\nu(z) = 2(1 - E_{\theta_0}e^{-x}) \]
with $x = (A(b_\theta^X) + A(b_{\theta'}^X))/2 - A(u) \geq 0$ and $u = (b_\theta^X + b_{\theta'}^X)/2$. Rewrite $x$ as
\[ x = \frac{(b_\theta^X - b_{\theta'}^X)^2}{4} \int_0^1 t \int_0^1 A'' \left( \left( \frac{1 - t}{2} + tt' \right) (b_\theta^X - b_{\theta'}^X) \right) \, dt \, dt'. \]
We obtain $(b_\theta^X - b_{\theta'}^X)^2 \lesssim ||x - (b_\theta^X - b_{\theta'}^X)^2$ and thus by (22) (with $b$ instead of $G$)
\[ ||\theta - \theta'||^2 \lesssim E_{\theta_0} x \lesssim ||\theta - \theta'||^2. \]
Arguing now as in (31, Proposition 1], with $0 \leq x \leq c'$ for some $c' \equiv c'(c, r)$ and convexity
\[ e^{-x} \leq \frac{x}{c'} e^{-c'} + \left( 1 - \frac{x}{c'} \right) = \frac{e^{-c'} - 1}{c'} - x + 1, \]  
(58)
the result follows from $h^2(\theta, \theta') \leq 2E_{\theta_0} x$ and $h^2(\theta, \theta') \geq 2\frac{1 - e^{-x}}{x} E_{\theta_0} x$.

**Theorem 28(ii): Local mean boundedness.** In the following $c, c_\lambda > 0$ are constants changing from line to line. Recalling the identities (59), it follows by the Cauchy-Schwarz inequality and Assumption G1(i)
\[ \left| E_{\theta_0} v^\top \nabla \ell(\theta, s, p) \right| = \left| E_{\theta_0} [(A'(b_\theta^X) - A'(b_{\theta'}^X))v^\top v b_\theta^X] \right| \leq \|b(\theta) - b(\theta, s, p)\|_{L^2} \|v^\top \nabla b(\theta, s, p)\|_{L^2} \leq \delta_n b^k. \]
We find for $q \geq 2$ and all $\theta \in \mathcal{B}$
\[ E_{\theta_0} \left| v^\top \nabla^2 b_\theta^X v \right|^q + E_{\theta_0} \left( v^\top \nabla b_\theta^X \right)^2q \]
\[ \leq c_\lambda^q \left( \max (p(2(q-2)k_1, p(q-2)k_2), \|v^\top \nabla^2 b_\theta^X v \|^2 + p(2q-2)k_1, \|v^\top \nabla b_\theta^X \|^2) \right) \]
\[ \leq c_\lambda^q p^2 \max (2k_3, k_4) + (q-2) \max (2k_1, k_2). \]
Hence, (57) and (55) give
\[
\mathbb{E}_{\theta_0} \left| v^T \nabla \ell(\theta^*; p)^q \right|^q \leq c_\lambda^q \mathbb{E}_{\theta_0} \left| v^T \nabla_\theta X \theta^* \right|^q \lesssim c_\lambda^q p^{2k_3 + (q-2)k_1},
\]
\[
\mathbb{E}_{\theta_0} \left| v^T \nabla^2 \ell(\theta) v \right|^q \leq c_\lambda^q \left( \mathbb{E}_{\theta_0} \left| v^T \nabla^2 b_0^X v \right|^q + \mathbb{E}_{\theta_0} (v^T \nabla b_0^X)^2 q \right)
\lesssim c_\lambda^q p^{2 \max(2k_3,k_4) + (q-2) \max(2k_1,k_2)}.
\]

Next, decompose \( v^T (\nabla^2 \ell(\theta) - \nabla^2 \ell(\theta')) v \) as
\[
(A'(b_0^X) - A'(b_0^X))v^T \nabla^2 b_0^X v - (Y - A'(b_0^X))v^T (\nabla^2 b_0^X - \nabla^2 b_0^X)v
+ (A''(b_0^X) - A''(b_0^X))v^T (\nabla^2 b_0^X)^2 - A''(b_0^X)((v^T \nabla b_0^X)^2 - (v^T \nabla b_0^X)^2).
\]

From this we find
\[
\mathbb{E}_{\theta_0} \left| v^T \nabla^2 \ell(\theta) v \right|^q
\lesssim c_\lambda^q p^{qk_1} \left( \mathbb{E}_{\theta_0} \left| v^T \nabla^2 b_0^X v \right|^q + \mathbb{E}_{\theta_0} (v^T \nabla b_0^X)^2 q \right) \left\| \theta - \theta' \right\|^q
+ c_\lambda^q \mathbb{E}_{\theta_0} \left| v^T (\nabla^2 b_0^X - \nabla^2 b_0^X)v \right|^q \lesssim c_\lambda^q p^{q \max(3k_1,k_2)} \left\| \theta - \theta' \right\|^q.
\]

This verifies the assumptions in Theorem 28(ii) with \( \kappa_1 = k_3, \kappa_2 = \max(k_1,2k_3,k_4), \kappa_4 = \max(3k_1,k_2) \).

**Theorem 28(iii): Local curvature.** It follows from (56) and \( \mathbb{E}_{\theta_0} Y = \mathbb{E}_{\theta_0} A'(b_0^X) \) that
\[
-\mathbb{E}_{\theta_0} (v^T \nabla^2 \ell(\theta) v) = \mathbb{E}_{\theta_0} [(A'(b_0^X) - A'(b_0^X))v^T \nabla^2 b_0^X v] + \mathbb{E}_{\theta_0} [A''(b_0^X)(v^T \nabla b_0^X)^2].
\]

By (52) (55), the Cauchy-Schwarz inequality and
\[
\|b(\theta_0) - b(\theta)\|_{L^2} \lesssim \|b(\theta_0) - b(\theta^*; p)\|_{L^2} + \|b(\theta^*; p) - b(\theta)\|_{L^2} \lesssim \eta,
\]
the last line is up to a constant lower bounded by
\[
-\|b(\theta_0) - b(\theta)\|_{L^2} \|v^T \nabla^2 b(\theta) v\|_{L^2} + \|v^T \nabla b(\theta)\|_{L^2}^2 \gtrsim -\eta p^{\max(2k_3,k_4)} + p^{-k_5}.
\]

Obtain the claim with \( \kappa_3 = k_5 \) by the assumption that \( \eta p^{\max(2k_3,k_4)} \log n \leq p^{-k_5} \).

**Theorem 28(i): Growth conditions.** These follow immediately from the conditions in the proposition for the obtained \( \kappa_i \), noting that \( \eta \geq \delta_n, k_4 \leq k_2, k_3 \leq k_1 \) such that
\[
C \max(\delta_n p^{k_2}, \eta \delta_n p^{k_4}, \delta_n^2 p^{k_4}) \log n
= \delta_n \max(p^{k_1},p^{2k_3},p^{k_4},\eta p^{3k_1}, \eta p^{k_2}, \delta_n p^{3k_1}, \delta_n p^{k_2}) \log n
= \delta_n \max(p^{\max(k_1,2k_3,k_4)}, \eta p^{\max(3k_1,k_2)}) \log n \leq p^{-k_5}.
\]

**Assumption 4** The form of \( K \) follows by plug-in of \( \kappa_2 \). □
6.1.3 Darcy’s problem

We will recall first some relevant analytical properties of the PDE (26), give a stability estimate for the forward operator \( G \) based on the condition (28), analyse further analytical properties of \( G \) and conclude with the proof of Theorem 21.

### Some PDE facts

Let us state a few well-known facts from the theory of elliptic PDEs. Details can be found in [67, Section 5A]. For multi-indices \( i = (i_1, \ldots, i_d) \) let \( D^i \) be the weak partial derivative operators. Denote the classical \( L^2(X) \)-Sobolev spaces of integer order \( s \geq 0 \) by

\[
H^s(X) = \left\{ w \in L^2(X) : \|w\|_{H^s}^2 = \sum_{|i| \leq s} \|D^i w\|_{L^2}^2 < \infty \right\}.
\]

They satisfy a Sobolev embedding [67, Proposition 4.3],

\[
H^s(X) \subset C^k(X), \quad s > k + d/2,
\]

(59)

Let \( H^s_0(X) \) be the subspace of functions in \( H^s(X) \) that vanish on the boundary of \( X \) in the trace sense. Their topological dual spaces are denoted by \( (H^s_0(X))^* \). For \( f \in C^1(X) \) the divergence form operator \( L_f \) takes functions in \( H^2_0(X) \) to \( L^2(X) \). If \( f \) is strictly positive on \( X \), then it has (e.g., by [21, Theorem 6.3.4]) a linear, continuous inverse operator \( L_f^{-1} : L^2(X) \rightarrow H^2_0(X) \). In particular, we have

\[
g_2 = 0 \Rightarrow u_f = G(f) = L_f^{-1} g_1.
\]

(60)

Another scale of Sobolev spaces \( \tilde{H}^s(X) \) is induced by the eigensystem \( (\lambda_k, e_k)_{k=1}^n \) of the negative Dirichlet Laplacian, where

\[
\tilde{H}^s(X) = \left\{ f \in L^2(X) : \|f\|_{\tilde{H}^s}^2 = \sum_{k=1}^\infty \lambda_k^s \langle f, e_k \rangle^2_{L^2} < \infty \right\},
\]

which is equipped with the inner product

\[
\langle f, g \rangle_{\tilde{H}^s} = \sum_{k=1}^\infty \lambda_k^s \langle f, e_k \rangle_{L^2} \langle g, e_k \rangle_{L^2}.
\]

Due to the presence of a boundary they generally differ from the Sobolev spaces \( H^s(X) \), but it can be shown that

\[
\tilde{H}^s(X) = H^s_0(X), \quad s = 1, 2, \quad \tilde{H}^s(X) \subset H^s_0(X), \quad s \in \mathbb{N},
\]

(61)
and the $\|\cdot\|_{H^s}$- and $\|\cdot\|_{\tilde{H}^s}$-norms are equivalent on $H^s(\mathcal{X})$. By Weyl’s law the eigenvalues satisfy for constants $0 < c_1 < c_2 < \infty$

$$c_1 k^{2/d} \leq |\lambda_k| \leq c_2 k^{2/d}, \quad k \geq 1,$$

and hence the map $\Phi : h^{s/d}(N) \to \tilde{H}^s(\mathcal{X})$, $\Phi(\theta) = \sum_{k=1}^{\infty} \theta_k e_k$ is an isomorphism with

$$c_1 \|\theta\|_{s/d}^2 \leq \|\Phi(\theta)\|_{H^s}^2 \leq c_2 \|\theta\|_{s/d}^2.$$

It follows from the last three displays and the Sobolev embedding (59) that for $\gamma > k + d/2$, $k \in \mathbb{N} \cup \{0\}$,

$$\|\Phi(\theta)\|_{C^k} \lesssim \|\Phi(\theta)\|_{H^\gamma} \lesssim \|\theta\|_{\gamma/d} \lesssim p^{\gamma/d} \|\theta\|, \quad \theta \in h^\gamma(N). \tag{62}$$

In particular, if $\|\theta\|_{\gamma/d} \leq r$, then we have for a constant $C_r > 0$

$$\|f \theta\|_{C^k} \leq f_{\min} + \|e^{\Phi(\theta)}\|_{C^k} \leq C_r. \tag{63}$$

We require the following quantitative elliptic regularity estimates with explicit constants depending on the conductivity.

**Lemma 32.** We have for $f \in C^{\gamma+1}(\mathcal{X})$, $\gamma \geq 0$, and $w \in H^{\gamma+2}(\mathcal{X})$

$$\|L_f w\|_{H^\gamma} \leq 2 \|f\|_{C^{\gamma+1}} \|w\|_{H^{\gamma+2}}.$$

**Proof.** It suffices to note that

$$\|L_f w\|_{H^\gamma} = \|f \Delta w + \nabla f \cdot \nabla w\|_{H^\gamma} \leq 2 \|f\|_{C^{\gamma+1}} \|w\|_{H^{\gamma+2}}. \quad \square$$

**Lemma 33.** For $c > 0$ consider $f \in C^1(\mathcal{X})$ with $f \geq f_{\min}$, $\|f\|_{C^1} \leq c$. Then there exists a constant $C \equiv C(f_{\min}, c)$ such that the following statements hold:

(i) $w \in L^2(\mathcal{X})$: $\|L_f^{-1} w\|_{H^2} \leq C \|w\|_{L^2},$

(ii) $w \in (H^2_0)^*(\mathcal{X})$: $\|L_f^{-1} w\|_{L^2} \leq C \|w\|_{(H^2_0)^*},$

(iii) $w \in H^1(\mathcal{X}; \mathbb{R}^d)$: $\|L_f^{-1}(\nabla \cdot w)\|_{H^1} \leq C \|w\|_{L^2}$.

**Proof.** Parts (i) and (ii) follow from [55, Lemmas 21 and 23]. For (iii) use duality to find for $z \in H^1_0(\mathcal{X})$

$$\|L_f z\|_{(H^2_0)^*} = \sup_{\varphi \in H^2_0, \|\varphi\|_{H^2} \leq 1} |\langle L_f z, \varphi \rangle_{L^2}| \geq \|L_f z, z\|_{L^2} \|z\|_{H^1}^{-1}$$

$$= \langle f \nabla z, \nabla z \rangle_{L^2} \|z\|_{H^1}^{-1} \|\nabla z\|_{L^2} \|z\|_{H^1}^{-1} \geq \|z\|_{H^1}.$$
concluding by the Poincaré inequality. Applying this to \( z = L_f^{-1}(\nabla \cdot w) \) for \( w \in H^1(\Omega; \mathbb{R}^d) \) yields
\[
\|L_f^{-1}(\nabla \cdot w)\|_{H^1} \lesssim \|\nabla \cdot w\|_{(H_0^1)^*}.
\]
Since the partial derivative operators are bounded operators from \( H^1(\Omega) \) to \( L^2(\Omega) \), the result follows by duality, the divergence theorem and the Cauchy-Schwarz inequality such that
\[
\|\nabla \cdot w\|_{(H_0^1)^*} = \sup_{\varphi \in H^1_0(\Omega), \|\varphi\|_{H^1} \leq 1} \left| \int_{\Omega} (-\nabla \cdot \varphi) w \, d\nu \right| \
\leq \sup_{\varphi \in L^2(\Omega), \|\varphi\|_{L^2} \leq 1} \left| \int_{\Omega} \varphi w \, d\nu \right| = \|w\|_{L^2}.
\]

**Lemma 34.** Let \( \gamma > k + d/2 \) for \( k \in \mathbb{N} \cup \{0\} \) and let \( f \in H^\gamma(\Omega) \), \( f \geq f_{\min} \). Then for all \( c > 0 \) there exists a constant \( C \equiv C(\gamma, f_{\min}, \Omega, g_1, g_2, c) \) such that
\[
\sup_{\|f\|_{H^{\gamma+1}} \leq c} \|u_f\|_{H^{\gamma+1}} \leq C, \quad \sup_{\|f\|_{H^\gamma} \leq c} \|u_f\|_{C^{k+1}} \leq C.
\]

**Proof.** If \( u_f = g_2 = 0 \) on \( \partial \Omega \), then \([55, \text{Lemma 23}]\) shows the first inequality for a constant \( C \equiv C(\gamma, f_{\min}, \Omega, g_1) \) and the second one follows from the Sobolev embedding \( H^\gamma(\Omega) \subset C^k(\Omega) \). For general \( g_2 \in C^\infty(\partial \Omega) \) we can assume without loss of generality that it extends to a function in \( C^\infty(\Omega) \) (e.g., by taking \( g_2 \) as the solution of the PDE \([24]\) for the standard Laplacian with \( f \equiv 1, g_1 = 0 \), which is smooth, cf. \([24, \text{Theorem 8.14}]\)) and note that \( \tilde{u}_f = u_f - g_2 \) solves the PDE \([26]\) with right hand side \( g_1 = f - L_fg_2 \) and \( \tilde{u}_f = 0 \) on \( \partial \Omega \). Then what has been shown so far applies to \( \tilde{u}_f \) and we obtain the second inequality (and thus also the first) with
\[
\sup_{\|f\|_{H^\gamma} \leq c} \|u_f\|_{C^{k+1}} \leq \sup_{\|f\|_{H^\gamma} \leq c} \|\tilde{u}_f\|_{C^{k+1}} + \|g_2\|_{C^{k+1}} \leq C + \|g_2\|_{C^{k+1}}.
\]

**A stability estimate**

**Lemma 35.** Let \( f, f' \in C^1(\Omega) \) with \( f = f' \) on \( \partial \Omega \) and \( \|f\|_{C^1}, \|f'\|_{C^1} \leq c \) for some \( c > 0 \) and suppose for \( \mu, c' > 0 \) that
\[
\inf_{x \in \Omega} \left( \frac{1}{2} \Delta u_f(x) + \mu \|\nabla u_f(x)\|_{\mathbb{R}^d}^2 \right) \geq c'.
\]
Then there exists a constant \( C \equiv C(\gamma, f_{\min}, \Omega, g_1, g_2, c) > 0 \) such that
\[
(i) \ h \in H^1_0(\Omega): \|L_hu_f\|_{L^2} \geq C\|h\|_{L^2},
\]
\[
(ii) \ \|f - f'\|_{L^2} \leq C\|u_f - u_{f'}\|_{H^2}.
\]
Proof. For $h \in C_c^\infty(\mathcal{X})$ the claim in (i) follows from Lemma 1 (it is easy to check that the requirement $f \in C_c^\infty(\mathcal{X})$ in the proof can be reduced to $f \in C^1(\mathcal{X})$), and extends by taking limits to $h \in H_0^1(\mathcal{X})$. For (ii) the proof of Proposition 3 applies: Take $h = f - f' \in H_0^1(\mathcal{X})$ such that $L_h u_f = L_F (u_f - u_f)$ (cf. (65) below) and hence by (i) and Lemma 32

$$\|f - f'\|_{L^2} = \|h\|_{L^2} \lesssim \|L_h u_f\|_{L^2} \lesssim \|u_f - u_f\|_{H^2}.$$  

\[\square\]

**Lemma 36.** Let $\theta, \theta' \in h^\alpha(\mathbb{N})$, $d \alpha > 1 + d/2$, with $\|\theta\|_\alpha, \|\theta'\|_\alpha \leq c$ for some $c > 0$, and suppose that (64) holds for $f = f_\theta$. Then there exists a constant $C \equiv C(\alpha, f_{\min}, \mathcal{X}, g_1, g_2, c) > 0$ such that

$$\|\theta - \theta'\|_\beta \leq C \|G(\theta) - G(\theta')\|_{L^2}, \quad \beta = \frac{\alpha + d}{\alpha - d}.$$  

**Proof.** Let $\gamma = d \alpha$ such that $\beta = (\gamma + 1)/(\gamma - 1)$. Use first $x \leq e^x - 1$ for $x \geq 0$ and (62) to the extent that

$$\|\theta - \theta'\| = \|\Phi(\theta) - \Phi(\theta')\|_{L^2} \leq \|e^{\Phi(\theta')}\|_{L^\infty} \|e^{\Phi(\theta)} - e^{\Phi(\theta')}\|_{L^2} \lesssim \|f_\theta - f_\theta\|_{L^2}.$$  

Apply Lemma 35(ii) to the last term. To conclude observe for $w = u_{f_\theta} - u_{f_\theta'} = G(\theta) - G(\theta')$ that $\|w\|_{H^{\gamma+1}} \leq C$ for a constant depending on $\alpha, f_{\min}, \mathcal{X}, g_1, g_2, c$ by Lemma 34 and (62), and that by an interpolation inequality for Sobolev spaces \[76\] Theorems 1.15 and 1.35

$$\|w\|_{H^2} \lesssim \|w\|_{L^2}^{(\gamma-1)/(\gamma+1)} \|w\|_{H^{\gamma+1}}^{2/(\gamma+1)} \lesssim \|w\|_{L^2}^{1/\beta}.$$  

\[\square\]

**Analytical properties of the forward map**

**Lemma 37.** Let $\theta, \theta' \in h^\alpha(\mathbb{N})$, $d \alpha > 1 + d/2$, with $\|\theta\|_\alpha, \|\theta'\|_\alpha \leq c$ for some $c > 0$. Then there exists a constant $C \equiv C(\alpha, c)$ such that

$$\|G(\theta) - G(\theta')\|_{L^2} \leq C \|\theta - \theta'\|.$$  

**Proof.** We have $(G(\theta) - G(\theta'))(x) = g_2(x) - g_2(x) = 0$ for $x \in \partial X$ such that $G(\theta) - G(\theta') \in H_0^1(\mathcal{X})$ by Lemma 34 and

$$\mathcal{L}_{f_{\theta'}} (G(\theta) - G(\theta')) = (\mathcal{L}_{f_{\theta'}} - \mathcal{L}_{f_\theta}) G(\theta) + \mathcal{L}_{f_\theta} G(\theta) - \mathcal{L}_{f_{\theta'}} G(\theta') = \mathcal{L}_{f_{\theta'}} f_\theta G(\theta) + g_1 - g_1 = \mathcal{L}_{f_{\theta'}} - f_\theta G(\theta).$$  

This allows for applying $\mathcal{L}_{f_{\theta'}}^{-1}$ and we get

$$G(\theta) - G(\theta') = \mathcal{L}_{f_{\theta'}}^{-1} \mathcal{L}_{f_{\theta'}} - f_\theta G(\theta).$$  

(65)
Lemma 33 combined with (63) yields
\[ \|G(\theta) - G(\theta')\|_{L^2} \leq C_{v} \|\mathcal{L}_{f_{\theta} - f_{\theta}} G(\theta)\|_{(H^0_\alpha)^*}. \]

By duality, the divergence theorem and writing \( G(\theta) = u_{f_{\theta}} \) the last term equals
\[
\sup_{\varphi \in H^2_\alpha, \|\varphi\|_{H^2_\alpha} \leq 1} \left| \int_{\mathcal{X}} \varphi \nabla \cdot (f_{\theta'} - f_{\theta}) \nabla u_{f_{\theta}} \right| = \sup_{\varphi \in H^2_\alpha, \|\varphi\|_{H^2_\alpha} \leq 1} \left| \int_{\mathcal{X}} (f_{\theta'} - f_{\theta}) \nabla \varphi \cdot \nabla u_{f_{\theta}} \right|
\leq \|f_{\theta'} - f_{\theta}\|_{L^2} \sup_{\|\varphi\|_{H^2_\alpha} \leq 1} \|\nabla \varphi \cdot \nabla u_{f_{\theta}}\|_{L^2}
\leq \|f_{\theta'} - f_{\theta}\|_{L^2} \|u_{f_{\theta}}\|_{C^1}.
\]

The result follows then from Lemma 33 and (62) such that
\[ \|f_{\theta'} - f_{\theta}\|_{L^2} \lesssim \|\Phi(\theta' - \theta)\|_{L^2} = \|\theta' - \theta\|. \]

**Proposition 38.** Let \( \theta \in \mathbb{R}^p \), \( v \in \mathbb{R}^p \) and set \( f_{\theta,v} = e^{\Phi(\theta)}\Phi(v), f_{\theta,v,2} = e^{\Phi(\theta)}\Phi(v)^2 \). Then we have for \( x \in \mathcal{X} \) the formulas
\[
v^\top \nabla \mathcal{G}(\theta)(x) = -\left( \mathcal{L}_{f_{\theta}}^{-1} \mathcal{L}_{f_{\theta,0}} u_{f_{\theta}} \right)(x),
v^\top \nabla^2 \mathcal{G}(\theta)(x)v = 2 \left( \mathcal{L}_{f_{\theta}}^{-1} \mathcal{L}_{f_{\theta,0}} L_{f_{\theta,0}} u_{f_{\theta}} \right)(x) - \left( \mathcal{L}_{f_{\theta}}^{-1} \mathcal{L}_{f_{\theta,0,2}} u_{f_{\theta}} \right)(x).
\]

**Proof.** Let us write \( G(f) = u_f \) such that \( G(\theta) = G(f_{\theta}) \). We will establish for
\[
G : H^\alpha(\mathcal{X}) \cap \{ f : f(x) > 0, x \in \bar{\mathcal{X}} \} \to C(\mathcal{X})
\]
and \( h, h' \in H^\alpha(\mathcal{X}) \) as \( \|h\|_{H^\alpha} \to 0 \) and \( \|h'\|_{H^\alpha} \to 0 \), respectively, that
\[
\|G(f + h) - G(f) - A_1(f)[h]\|_{L^\infty} = O \left( \|h\|^2_{H^\alpha} \right), \quad (66)
\|A_1(f + h'[h] - A_2(f)[h, h']\|_{L^\infty} = O \left( \|h'\|^2_{H^\alpha} \right), \quad (67)
\]
with continuous linear operators \( A_1(f) : H^\alpha(\mathcal{X}) \to C(\mathcal{X}), A_2(f) : H^\alpha(\mathcal{X}) \times H^\alpha(\mathcal{X}) \to C(\mathcal{X}) \) given by
\[
A_1(f)[h] = -\mathcal{L}_{f_{\theta}}^{-1} \mathcal{L}_{h} G(f), \quad (68)
A_2(f)[h, h'] = \mathcal{L}_{f_{\theta}}^{-1} \mathcal{L}_{h'} \mathcal{L}_{f_{\theta}}^{-1} \mathcal{L}_{h} G(f) + \mathcal{L}_{f_{\theta}}^{-1} \mathcal{L}_{h} \mathcal{L}_{f_{\theta}}^{-1} \mathcal{L}_{h'} G(f). \quad (69)
\]

This implies that \( G \) is two-times continuously Fréchet differentiable with derivatives \( D G(f) = A_1(f), D^2 G(f) = A_2(f) \). Since the map \( \theta \mapsto f_{\theta} = f_{\min} + e^{\Phi(\theta)} \) in (27) satisfies on \( \mathbb{R}^p \)
\[
v^\top \nabla f_{\theta} = e^{\Phi(\theta)}\Phi(v) = f_{\theta,v}, \quad v^\top \nabla^2 f_{\theta,v} = e^{\Phi(\theta)}\Phi(v)^2 = f_{\theta,v,2},
\]
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the claim follows from the chain rule and from replacing $G(f_0) = u_{f_0}$.

Consider now $h, h' \in H^\alpha(\mathcal{X})$ with sufficiently small $\|\cdot\|_{H^\alpha}$-norms such that $f + h$, $f + h'$ are strictly positive on $\mathcal{X}$ and $G(f + h)$, $G(f + h')$, $G(f)$ are well-defined with values in $H^2(\mathcal{X})$. Using $\text{(65)}$ twice we get

$$G(f + h) - G(f) - A_1(f)[h] = -\mathcal{L}_f^{-1}\mathcal{L}_h(G(f + h) - G(f))$$

$$= \mathcal{L}_f^{-1}\mathcal{L}_h\mathcal{L}_f^{-1}\mathcal{L}_hG(f + h).$$

(70)

Since $G(f + h) = \mathcal{L}_f^{-1}g_1$, applying several times Lemmas $\text{(33)(i)}$ and $\text{(2)}$ and suppressing constants depending on $\|f\|_{C^1}$, we get

$$\|\mathcal{L}_f^{-1}\mathcal{L}_h\mathcal{L}_f^{-1}\mathcal{L}_hG(f + h)\|_{H^2} \lesssim (1 + \|h\|_{C^1})\|h\|_{C^1}^2,$$

implying (66) by (70) and the Sobolev embedding $H^2(\mathcal{X}) \subset L^\infty(\mathcal{X})$ for $d \leq 3$. Next, we have

$$\mathcal{L}_f(A_1(f + h')[h] - A_1(f)[h])$$

$$= -\mathcal{L}_f h \mathcal{L}_f^{-1} \mathcal{L}_f^{-1} \mathcal{L}_h G(f + h') + \mathcal{L}_f^{-1} \mathcal{L}_h G(f)$$

$$= \mathcal{L}_h \mathcal{L}_f^{-1} \mathcal{L}_f^{-1} \mathcal{L}_h (G(f + h') - G(f))$$

$$= \mathcal{L}_h \mathcal{L}_f^{-1} \mathcal{L}_f^{-1} \mathcal{L}_h G(f + h').$$

With this write

$$A_1(f + h')[h] - A_1(f)[h] - A_2(f)[h, h']$$

$$= \mathcal{L}_f^{-1} \mathcal{L}_h (\mathcal{L}_f^{-1} \mathcal{L}_f^{-1} \mathcal{L}_h G(f + h') + \mathcal{L}_f^{-1} \mathcal{L}_h (G(f + h') - G(f))$$

$$+ \mathcal{L}_f^{-1} \mathcal{L}_h \mathcal{L}_f^{-1} \mathcal{L}_h (G(f + h') - G(f)) =: R_1 + R_2 + R_3.$$  

Arguing as after (70) gives

$$\|R_2\|_{L^\infty} = \|\mathcal{L}_f^{-1} \mathcal{L}_h \mathcal{L}_f^{-1} \mathcal{L}_h \mathcal{L}_f^{-1} \mathcal{L}_h (G(f + h'))\|_{L^\infty}$$

$$\lesssim (1 + \|h\|_{C^1})\|h\|_{C^1}\|h\|_{C^1}^2,$$

and the same upper bound applies to $\|R_3\|_{L^\infty}$. At last, for $w \in L^2(\mathcal{X})$ observe that

$$(\mathcal{L}_f^{-1} - \mathcal{L}_f^{-1})w = \mathcal{L}_f^{-1} (\mathcal{L}_f - \mathcal{L}_f^{-1})w = -\mathcal{L}_f^{-1} \mathcal{L}_h \mathcal{L}_f^{-1} w,$$  

(71)

and so (67) follows from arguing as in the last display, such that

$$\|R_1\|_{L^\infty} \lesssim (1 + \|h\|_{C^1}^2)\|h\|_{C^1}\|h\|_{C^1}^2.$$  

\[\square\]
Proof of Theorem 21. We first verify the conditions of Proposition 19 in the following lemma.

Lemma 39. Consider the setting of Theorem 21 with \(d \leq 3\), \(||\theta_0||_\alpha \leq c_0\) for \(\alpha \geq 7/d\). Then Assumptions G1 and G2 are satisfied for \(\beta = (\alpha + d)/(\alpha - d)\), \(\eta = p^{-8/d}\) and \(k_1 = 1/d, k_2 = 7/d, k_3 = 0, k_4 = 2/d, k_5 = 6/d\).

Proof. Since \(d \alpha \geq 7 > 1 + d/2\), Assumption G1 ii) can be verified from the stability and Lipschitz estimates in Lemmas 36 and 37 for the stated \(\beta\). The Lipschitz estimate and \(\text{g}\) also yield Assumption G1 i). We are therefore left with checking Assumption G2.

Let \(\theta \in \mathcal{B}\) and \(v \in \mathbb{R}^p, ||v|| \leq 1\). Then \(||\theta_{*,p}||_\alpha \leq ||\theta_0||_\alpha \leq c_0\) for \(\alpha \geq 7/d\) gives

\[
||\theta||_{7/d} \leq ||\theta - \theta_{*,p}||_{7/d} + ||\theta_{*,p}||_{7/d} \leq p^{7/d}||\theta - \theta_{*,p}|| + c_0 \lesssim 1,
\]

which in view of (62), (63) and \(7 > 5 + d/2\) implies for a constant \(C \equiv C(f_{\min}, c_0)\) that \(||\Phi(\theta)||_C^5 \leq C, ||\Phi(v)||_{L^1} \leq C p^{1/d}\), \(||\Phi(v)||_{L^\infty} \leq C p^{3/d}\), \(||\Phi(v)||_{C^1} \leq C p^{3/d}\) and

\[
\sup_{\theta \in \mathcal{B}} ||f_\theta||_{H^5} \lesssim \sup_{\theta \in \mathcal{B}} ||f_\theta||_{C^5} \leq f_{\min} + \sup_{\theta \in \mathcal{B}} ||e^{\Phi(\theta)}||_{C^5} \leq C.
\]

In particular, by Lemma 34 for \(C' > 0\)

\[
\sup_{\theta \in \mathcal{B}} ||u_{f_\theta}||_{C^4} \leq C'.
\]

These properties will be used tacitly in the following proof. As before we will suppress constants not depending on \(n, p\). The gradient and Hessian of \(G\) were computed in Proposition 38.

Assumption G2 i): Use (72) and 37.

Assumption G2 ii): The required differentiability follows from Proposition 38. For the sup-norm bounds observe first by Lemma 33 i)

\[
||L_{f_\theta}^{-1} L_{f_{\theta,v}} u_{f_\theta}||_{H^2} \lesssim ||\nabla \cdot \left(e^{\Phi(\theta)} \Phi(v) \nabla u_{f_\theta}\right)||_{L^2} \lesssim ||e^{\Phi(\theta)}||_{C^1} ||\Phi(v)||_{H^1} ||u_{f_\theta}||_{C^2} \lesssim p^{1/d};
\]

and similarly

\[
||L_{f_\theta}^{-1} L_{f_{\theta,v}} L_{f_{\theta,v}} u_{f_\theta}||_{H^2} \lesssim ||\nabla \cdot \left(e^{\Phi(\theta)} \Phi(v) \nabla L_{f_\theta}^{-1} L_{f_{\theta,v}} u_{f_\theta}\right)||_{L^2} \lesssim ||e^{\Phi(\theta)}||_{C^1} ||\Phi(v)||_{C^1} ||L_{f_\theta}^{-1} L_{f_{\theta,v}} u_{f_\theta}||_{H^2} \lesssim p^{4/d},
\]

\[
||L_{f_\theta}^{-1} L_{f_{\theta,v}} u_{f_\theta}||_{H^2} \lesssim ||\nabla \cdot \left(e^{\Phi(\theta)} \Phi(v)^2 \nabla u_{f_\theta}\right)||_{L^2} \lesssim ||e^{\Phi(\theta)}||_{C^1} ||\Phi(v)||_{H^1} ||u_{f_\theta}||_{C^2} \lesssim p^{4/d}.
\]
The Sobolev embedding $H^2(\mathcal{X}) \subset C(\mathcal{X})$ in $d \leq 3$ therefore shows
\[
\|v^T \nabla G(\theta)\|_{L^\infty} \lesssim p^{1/d}, \quad \|v^T \nabla^2 G(\theta)v\|_{L^\infty} \lesssim p^{4/d}.
\]
Next, for $\theta' \in B, h, h' \in C^1(\mathcal{X})$ write
\[
\mathcal{L}^{-1}_{f_0} - \mathcal{L}^{-1}_{f_{0'}} = \mathcal{L}^{-1}_{f_0} \mathcal{L}_{f_0 - f_{0'}} \mathcal{L}^{-1}_{f_{0'}}, \quad \mathcal{L}_h - \mathcal{L}_{h'} = \nabla \cdot (h - h') \nabla,
\]
such that Lemma 33(i) implies for $w \in L^2(\mathcal{X}), w' \in H^2(\mathcal{X})$
\[
\| (\mathcal{L}^{-1}_{f_0} - \mathcal{L}^{-1}_{f_{0'}}) w \|_{H^2} \lesssim \| e^{\Phi(\theta)} - e^{\Phi(\theta')}) \|_{C^1} \| \mathcal{L}^{-1}_{f_{0'}} w \|_{H^2},
\]
\[
\| (\mathcal{L}_{f_0,v} - \mathcal{L}_{f_{0'},v}) w' \|_{L^2} \lesssim \| e^{\Phi(\theta)} - e^{\Phi(\theta')}) \|_{C^1} \| \Phi(v) \|_{C^1} \| w' \|_{H^2},
\]
\[
\| (\mathcal{L}_{f_0,v} - \mathcal{L}_{f_{0'},v}) w \|_{L^2} \lesssim \| e^{\Phi(\theta)} - e^{\Phi(\theta')}) \|_{C^1} \| \Phi(v) \|_{C^1} \| u_{f_0} \|_{C^2},
\]
\[
\| (\mathcal{L}_{f_0,v} - \mathcal{L}_{f_{0'},v}) u_{f_0} \|_{L^2} \lesssim \| e^{\Phi(\theta)} - e^{\Phi(\theta')}) \|_{C^1} \| \Phi(v) \|_{C^1} \| u_{f_0} \|_{C^2}.
\]
Then, $\| e^{\Phi(\theta)} - e^{\Phi(\theta')}) \|_{C^1} \lesssim \| \Phi(\theta - \theta') \|_{C^1}$, and so the terms in the last display are upper bounded up to constants by $p^{3/d} \| \theta - \theta' \|_{\| w \|_{L^2}, p^{6/d} \| \theta - \theta' \|_{\| w \|_{H^2}, p^{4/d} \| \theta - \theta' \|}$ and $p^{4/d} \| \theta - \theta' \|$, respectively. Combining these estimates with Lemmas 32(ii), 33(i) and with the Lipschitz bound from Lemma 37 we obtain
\[
\| \nabla^2 G(\theta) - \nabla^2 G(\theta') \|_{L^\infty(\mathcal{X})_p} \lesssim p^{7/d} \| \theta - \theta' \|.
\]
In all, Assumption C2(i) holds with $k_1 = 1/d, k_2 = 7/d$.

Assumption C2(iii): Use Lemma 33(iii) to the extent that
\[
\| \mathcal{L}^{-1}_{f_0} \mathcal{L}_{f_0,v} u_{f_0} \|_{H^1} \lesssim \| e^{\Phi(\theta)} \Phi(v) \nabla u_{f_0} \|_{L^2} \lesssim \| e^{\Phi(\theta)} \|_{L^\infty} \| \Phi(v) \|_{L^2} \| u_{f_0} \|_{C^1} \lesssim 1,
\]
as well as
\[
\| \mathcal{L}^{-1}_{f_0} \mathcal{L}_{f_0,v} u_{f_0} \|_{L^2} \lesssim \| e^{\Phi(\theta)} \Phi(v) \nabla (\mathcal{L}^{-1}_{f_0} \mathcal{L}_{f_0,v} u_{f_0}) \|_{L^2} \lesssim \| e^{\Phi(\theta)} \|_{L^\infty} \| \Phi(v) \|_{L^\infty} \| \mathcal{L}^{-1}_{f_0} \left( \nabla \cdot (e^{\Phi(\theta)} \Phi(v) \nabla u_{f_0}) \right) \|_{H^1} \lesssim p^{2/d} \| e^{\Phi(\theta)} \Phi(v) \nabla u_{f_0} \|_{L^2} \lesssim p^{2/d} \| \Phi(v) \|_{L^2} \| u_{f_0} \|_{C^1} \lesssim p^{2/d},
\]
\[
\| \mathcal{L}^{-1}_{f_0} \mathcal{L}_{f_0,v} u_{f_0} \|_{H^1} \lesssim \| e^{\Phi(\theta)} \Phi(v) \nabla u_{f_0} \|_{L^2} \lesssim \| e^{\Phi(\theta)} \|_{L^\infty} \| \Phi(v) \|_{L^\infty} \| \Phi(v) \|_{L^2} \| u_{f_0} \|_{C^1} \lesssim p^{2/d}.
\]
From this and $\| \cdot \|_{L^2} \leq \| \cdot \|_{H^1}$ obtain the result with $k_3 = 0, k_4 = 2/d$.

Assumption C2(iv): An interpolation inequality for Sobolev spaces (see e.g., 76, Theorems 1.15 and 3.35) yields
\[
\| w \|_{H^2} \lesssim \| w \|_{H^1}^{1/2} \| w \|_{H^1}^{1/2}, \quad 0 \neq w \in H^4(\mathcal{X}).
\]
Applying this to \( w = \mathcal{L}_{f_0}^{-1} \mathcal{L}_{f_0,v} u_{f_0} \) and observing the inequalities \( \| \mathcal{L}_f w \|_{L^2} \lesssim \| w \|_{H^2} \), \( \| w \|_{H^2} \lesssim \| \mathcal{L}_f w \|_{H^2} \) shows

\[
\| u^T \nabla G(\theta) \|_{L^2} = \| w \|_{L^2} \gtrsim \frac{\| \mathcal{L}_{f_0,v} u_{f_0} \|_{L^2}^2}{\| \mathcal{L}_{f_0,v} u_{f_0} \|_{H^2}^2}.
\]

Recall the stability estimate from Lemma \ref{lem:stability}(ii), which yields for \( h = e^{\Phi(\theta)} \Phi(v) \in H^1_0(\mathcal{X}) \) (here we use that \( \Phi(\theta) \in H^1_0(\mathcal{X}) \))

\[
\| \mathcal{L}_{f_0,v} u_{f_0} \|_{L^2}^2 \gtrsim \| e^{\Phi(\theta)} \Phi(v) \|_{L^2}^2 \gtrsim e^{-2\| \Phi(\theta) \|_{L^\infty}} \| \Phi(v) \|_{L^2}^2 \gtrsim 1,
\]

while Lemma \ref{lem:stability} and \((62)\) provide us with the upper bound

\[
\| \mathcal{L}_{f_0,v} u_{f_0} \|_{H^2} \lesssim \| e^{\Phi(\theta)} \Phi(v) \|_{H^2} \| u_{f_0} \|_{C^4} \lesssim \| e^{\Phi(\theta)} \|_{C^4} \| \Phi(v) \|_{H^2} \lesssim \beta^{3/d}.
\]

The last three displays yield the wanted lower bound with \( k_5 = 6/d \).

By Proposition \ref{prop:assumptions} we establish Assumptions \ref{assumption:A}, \ref{assumption:C} and \ref{assumption:D} for \( \kappa_1 = 0, \kappa_2 = 2/d, \kappa_3 = 6/d \) and \( K \geq c n p^{2/d}, c > 0 \), noting that the required conditions in \ref{assumption:D} hold because

\[
p \lesssim n \delta_n^2 = \delta_n^{-1/\alpha} \text{ and } p^{-8/d} \lesssim \delta_n^{8/(ad)} \text{ yield for } \alpha \geq 21/d \text{ and } d \leq 3 \text{ that } (\alpha - d)/(\alpha + d) > 8/(ad), \delta_n p^{6/d} \lesssim \delta_n 1/(6/ad) = \delta_n (\alpha - 6/(ad)) \lesssim \delta_n 8/(ad) \text{ and thus } \delta_n (\alpha - d)/(\alpha + d) \lesssim \delta_n 8/(ad),
\]

such that for large enough \( n \)

\[
\max(\delta_n^{1/\beta}, \delta_n p^{3+k_3+k_5}) \log n \leq \max(\delta_n^{(\alpha - d)/(\alpha + d)}, n^{-1/(2\alpha + 1)} p^{6/d}) \log n \leq p^{-8/d} = \eta,
\]

\[
\delta_n \max(p^{\max(k_1,2k_3,k_4)}, \eta p^{\max(k_1,2k_2)}) \log n = \delta_n \max(p^{2/d}, p^{7/d-8/d}) \log n \leq \delta_n p^{2/d} \log n \lesssim \delta_n^{-2/\alpha}.
\]

Let us now prove the three theorems in Section \ref{sec:results}. Consider \( \gamma, \varepsilon \) and \( J_{in} \) as stated in Theorem \ref{thm:main} and note that, using \( m_\pi = n^{1/(2\alpha + 1)}, \Lambda_\pi = n^{1/(2\alpha + 1)} p^{2\alpha} \), the curvature and Lipschitz constants from Theorem \ref{thm:connections} satisfy

\[
m \gtrsim \eta \log n \lesssim \gamma p^{12/d}, \quad \Lambda \gtrsim \eta \log n \lesssim \gamma p^{12/d}.
\]

This gives \( B(\gamma) \lesssim \gamma p^{16/d} + \gamma^2 p^{26/d} n \) and the results in Theorem \ref{thm:main} follow in the present model from Theorems \ref{thm:sufficient} and \ref{thm:necessary}.

Next, assume \( p \leq (\log n)^{-2(2\alpha + 14)/d} n^{\frac{26}{2\alpha + 14}} \). This gives \( \max(\eta, \Lambda_\pi/m)^2 + p/m \lesssim \eta^{12/d} \).

For any \( c > 0 \) and large enough \( n \), we have \( \| \mathcal{L}_{f_0,v} u_{f_0} \|_{L^2} \lesssim c \delta_n^2 \), such that \( \Lambda_\pi \log n \leq \eta m \) and

\[
\| \nabla \log \pi(\theta_{s,p}) \| \leq \Lambda_\pi \| \theta_{s,p} \| \leq \eta m / 16, \quad \eta \sqrt{m/p} \gtrsim p^{-11/d-1/2} \gtrsim \eta \gtrsim \eta m / 16.
\]

This gives \( \| \mathcal{L}_{f_0,v} u_{f_0} \|_{L^2} \lesssim c \delta_n^2 \). We find for large enough \( n \) and suitable \( C, C' > 0 \) that \( J p^2 e^{-C_0^2 m/p} \lesssim e^{-C'' n^{1/(2\alpha + 1)}} \) and thus Corollary \ref{cor:connections} implies Theorem \ref{thm:main}.
6.2 Posterior contraction with rescaled Gaussian priors

In this section we formulate a general contraction result for the posterior \( \Pi(\cdot | Z^{(n)}) \) from Section 3 around a sufficiently regular ground truth \( \theta_0 \in h^\alpha(N) \) for the Gaussian prior in \([4]\), achieving the high probability bounds in Assumption A. The proof follows closely \([30, \text{Theorem 7.3.1}] \) and \([31, \text{Theorem 13}] \), using Bernstein-type moment conditions and a stability condition for the Hellinger distance. We include a proof for the convenience of the reader. In the following, we use the notation of Section 3 and define for \( \theta, \theta' \in \Theta \) the squared Hellinger distance as

\[
h^2(\theta, \theta') = \int_Z \left( \sqrt{p_\theta(z)} - \sqrt{p_{\theta'}(z)} \right)^2 d\nu(z). \tag{73}
\]

For a definition of the metric entropy see Section 5.5.

**Theorem 40.** Let \( \theta_0 \in h^\alpha(N) \) with \( \|\theta_0\|_\alpha \leq c_0 \) for \( \alpha > 1/2 \), \( c_0 > 0 \) and suppose that the data \( Z^{(n)} \) arise from the law \( P^*_{\theta_0} \). Let \( \Pi(\cdot | Z^{(n)}) \) be the posterior distribution with the rescaled Gaussian prior \( \Pi \) from \([4]\). For \( \delta_n = n^{-\alpha/(2\alpha+1)} \) and \( C > 0 \) suppose \( p \leq Cn\delta^2_n \) and set for \( \theta, p \in \mathbb{R}^p \) with \( \|\theta, p\|_\alpha \leq c_0 \)

\[
B_{n,r} = \{ \theta \in \mathbb{R}^p : \|\theta - \theta, p\| \leq \delta_n, \|\theta\|_\alpha \leq r \}, \quad r > 0.
\]

Then the posterior distribution concentrates around \( \theta, p \) at the rate \( \delta_n \) and satisfies Assumption A if there exists \( \beta \geq 1 \) such that for any \( r > 0 \) and some \( c_r > 0 \) the following holds:

(i) For all \( \theta \in B_{n,r} \) and all \( q \geq 2 \)

\[
\mathbb{E}_{\theta_0}(\ell(\theta_0) - \ell(\theta)) \leq c_r \delta^2_n, \quad \mathbb{E}_{\theta_0}|\ell(\theta_0) - \ell(\theta)|^q \leq (q!/2)\delta^2_n c_r^q. \tag{74}
\]

(ii) For all \( \theta, \theta' \in \ell^2(N) \) with \( \|\theta\|_\alpha, \|\theta'\|_\alpha \leq r \),

\[
c_r^{-1}||\theta - \theta'||^\beta \leq h(\theta, \theta') \leq c_r||\theta - \theta'||. \tag{75}
\]

**Proof.** Let \( \mathcal{D} \) denote the high probability event considered in Lemma \([12]\). Since that lemma already shows \([15]\), we only have to prove the posterior contraction in \([14]\). Consider for \( L, L' > 0 \) the sets

\[
\mathcal{A} = \{ \theta \in \mathbb{R}^p : \|\theta\|_\alpha \leq L' \}, \quad \mathcal{U} = \{ \theta \in \mathcal{A} : h(\theta, \theta_0) \leq L\delta_n \}. \tag{76}
\]

Invoking the stability bound in \([16]\) yields

\[
\mathcal{U} \subset \{ \theta \in \mathbb{R}^p : \|\theta - \theta, p\| \leq c_{L'} L\delta_n \}. \tag{77}
\]
It is therefore enough to show that the posterior contracts in Hellinger distance on the event $\mathcal{D}$, that is, for any $C_1 > 0$ and large enough $L, L'$ there exist $C_2, C_3 > 0$ with
\[
\mathbb{P}_{\theta_0}^n \left( \Pi(U^n|Z^{(n)}) > e^{-C_1 n \delta_n^2}, \mathcal{D} \right) \leq C_3 e^{-C_2 n \delta_n^2}. \tag{77}
\]

First, the entropy bound in Lemma 43 below implies by [30, Theorem 7.1.4] the existence of tests $\Psi$ with values in $[0, 1]$ such that for all $n$, any large enough $L, L'$ and some $C' > 0$
\[
\mathbb{E}_{\theta_0}^n \Psi \leq e^{-C' n \delta_n^2}, \quad \sup_{\theta \in U^c \cap A} \mathbb{E}_{\theta_0}^n (1 - \Psi) \leq e^{-C' n \delta_n^2}. \tag{78}
\]

It is therefore enough to prove (77) for the probability in question restricted to $\{\Psi = 0\}$.

On $\mathcal{D}$, we can lower bound the normalising factors in the posterior density such that for all $\theta \in \mathbb{R}^p$ and some $c' > 0$
\[
\pi(\theta|Z^{(n)}) = \frac{e^{\ell_n(\theta) - \ell_n(\theta_0)} \pi(\theta)}{\int_{\Theta} e^{\ell_n(\theta) - \ell_n(\theta_0)} \pi(\theta) d\theta} \leq e^{c' n \delta_n^2} e^{\ell_n(\theta) - \ell_n(\theta_0)} \pi(\theta). \tag{79}
\]

The Markov inequality and Fubini’s theorem now yield
\[
\mathbb{P}_{\theta_0}^n \left( \Pi(U^n|Z^{(n)}) > e^{-C_1 n \delta_n^2}, \{\Psi = 0\} \right) \leq \mathbb{P}_{\theta_0}^n \left( (1 - \Psi) \int_{U^c} e^{\ell_n(\theta) - \ell_n(\theta_0)} \pi(\theta) d\theta > e^{-(C_1 + c') n \delta_n^2} \right)
\]
\[
\leq e^{(C_1 + c') n \delta_n^2} \int_{U^c} \mathbb{E}_{\theta_0}^n (1 - \Psi) e^{\ell_n(\theta) - \ell_n(\theta_0)} \pi(\theta) d\theta
\]
\[
\leq e^{(C_1 + c') n \delta_n^2} \int_{U^c} \mathbb{E}_{\theta_0}^n (1 - \Psi) \pi(\theta) d\theta.
\]

Integrating separately over the sets $U^c \cap A$ and $U^c \cap A^c$, the second bound on the tests in (78) and the excess mass condition from Lemma 43, together with $\Psi \leq 1$, give for large enough $L'$ and $c > 0$
\[
\int_{U^c \cap A} \mathbb{E}_{\theta_0}^n (1 - \Psi) \pi(\theta) d\theta \leq e^{-C' n \delta_n^2},
\]
\[
\int_{U^c \cap A^c} \mathbb{E}_{\theta_0}^n (1 - \Psi) \pi(\theta) d\theta \leq \Pi(A^c) \leq e^{-c n \delta_n^2}.
\]

This shows (77) and finishes the proof. $\square$

Let us now state and prove the auxiliary results used in the proof above.

**Lemma 41.** In the setting of Theorem 40 there exists for any large enough $r \geq \max(4, 2c_0)$ a constant $c \equiv c(C, \alpha, c_0, r) > 0$ with $\Pi(B_{n,r}) \geq e^{-cn \delta_n^2}$. 

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Proof. We can write \( \Pi \sim (n\delta_{\alpha}^2)^{-1/2} \Pi \) for the unscaled probability measure \( \Pi \sim N(0, \Lambda_{\alpha}^{-1}) \). The reproducing kernel Hilbert space of \( \Pi \) is equipped with the norm \( \| \cdot \|_{\alpha} \) on \( \mathbb{R}^p \). Since \( \| \theta, p \|_{\alpha} \leq c_{\alpha} \), we have for \( r \geq 2c_{\alpha} \)

\[
\{ \theta \in \mathbb{R}^p : \| \theta - \theta, p \| \leq \delta_{\alpha}, \| \theta - \theta, p \|_{\alpha} \leq r/2 \} \subset B_{n,r}.
\]

The small-ball calculus from [30, Corollary 2.6.18] allows then for lower bounding the wanted probability as

\[
\Pi(B_{n,r}) \geq e^{-n\delta_{\alpha}^2\|\theta, p\|_{\alpha}^2/2} \Pi \left( \theta \in \mathbb{R}^p : \| \theta \| \leq n^{1/2}\delta_{\alpha}^2, \| \theta \|_{\alpha} \leq (r/2)(n\delta_{\alpha}^2)^{1/2} \right)
\]

\[
\geq e^{-n\delta_{\alpha}^2c_{\alpha}/2} \left( \Pi \left( \theta \in \mathbb{R}^p : \| \theta \| \leq n^{1/2}\delta_{\alpha}^2 \right) - \Pi \left( \theta \in \mathbb{R}^p : \| \theta \|_{\alpha} > (r/2)(n\delta_{\alpha}^2)^{1/2} \right) \right).
\] (80)

Observe for any \( \varepsilon > 0 \) the metric entropy bound

\[
\log N \left( \{ \theta \in \mathbb{R}^p : \| \theta \|_{\alpha} \leq 1 \}, \| \cdot \|_{\alpha}, \varepsilon \right) \leq \log N \left( H^\varepsilon((0, 1)), \| \cdot \|_{L^2((0,1))}, \varepsilon \right) \lesssim \varepsilon^{-1/\alpha},
\] (81)

where \( H^\varepsilon((0, 1)) \), \( s \in \mathbb{R} \), is the \( L^2((0, 1)) \)-Sobolev space of fractional order \( s \), concluding by [69, Theorem 4.10.13]. It follows from [42, Theorem 1.2] with \( J \equiv 1 \) and \( \varepsilon = n^{1/2}\delta_{\alpha}^2 \) for a universal constant \( c' > 0 \) that

\[
\Pi \left( \theta \in \mathbb{R}^p : \| \theta \|_{\alpha} \leq (r/2)(n\delta_{\alpha}^2)^{1/2} \right) \geq e^{-c'(n^{1/2}\delta_{\alpha}^2)^{-2/(2\alpha - 1)}} = e^{-c'n\delta_{\alpha}^2}.
\]

On the other hand, let \( V \) be a \( p \)-dimensional standard Gaussian random vector, defined on probability space with probability measure \( \mathbb{P} \) and expectation operator \( \mathbb{E} \). Noting \( \| \Lambda_{\alpha}^{-1/2}V \|_{\alpha} = \| V \| \), we have

\[
\Pi \left( \theta \in \mathbb{R}^p : \| \theta \|_{\alpha} > (r/2)(n\delta_{\alpha}^2)^{1/2} \right) = \mathbb{P} \left( \| V \| > (r/2)(n\delta_{\alpha}^2)^{1/2} \right).
\]

For \( r \geq 4 \) we get from \( p \leq Cn\delta_{\alpha}^2 \) that \( \mathbb{E}\| V \| \leq p^{1/2} \leq (r/4)(Cn\delta_{\alpha}^2)^{1/2} \). By a standard concentration inequality for Lipschitz-functionals of Gaussian random vectors (Theorem 2.5.7 of [30] with \( F = \| \cdot \| \)) this means that the last display is upper bounded by

\[
\mathbb{P} \left( \| V \| - \mathbb{E}\| V \| > (r/4)(n\delta_{\alpha}^2)^{1/2} \right) \leq e^{-r^2/16Cn\delta_{\alpha}^2}.
\] (82)

Conclude now with (80).

\( \square \)

**Lemma 42.** In the setting of Theorem 41 there exists \( c' > 0 \) such that

\[
\mathbb{P}_{\theta, r} \left( \| \theta - \theta, p \|_{\alpha} \leq \delta_{\alpha}, e^{f_{\alpha}(\theta)} - e^{f_{\alpha}(\theta_0)} \pi(\theta) d\theta > e^{-c'n\delta_{\alpha}^2} \right) \leq 2e^{-n\delta_{\alpha}^2}.
\]

**Proof.** Let \( \mathcal{D}^c \) denote the event in question. For large enough \( r \) and \( c > 0 \) Lemma 41 shows \( \Pi(B_{n,r}) \geq e^{-cn\delta_{\alpha}^2} \). With \( c_r \) the constant from (74) choose \( c' = 2c + 7c_r \) and consider
the probability measure \( \nu_n = \Pi(\cdot \cap B_{n,r})/\Pi(B_{n,r}) \), supported on \( B_{n,r} \). Introducing the functions \( h(x) = \int_{B_{n,r}} (\ell(\theta_0, x) - \ell(\theta, x))d\nu_n(\theta) \), the Jensen inequality implies

\[
\mathbb{P}_{\theta_0}^n(D^c) = \mathbb{P}_{\theta_0}^n \left( \Pi(B_{n,r}) \int_{\Theta} e^{\ell_n(\theta) - \ell_n(\theta_0)}d\nu_n(\theta) \leq e^{-c' n \delta_n^2} \right)
\leq \mathbb{P}_{\theta_0}^n \left( \sum_{i=1}^n h(Z_i) \geq 7c_r n \delta_n^2 \right).
\]

We find from Fubini’s theorem and (74)

\[
|\mathbb{E}_{\theta_0} h(Z_i)| = |\int_{B_{n,r}} \mathbb{E}_{\theta_0}(\ell(\theta_0) - \ell(\theta))d\nu_n(\theta)| \leq c_r \delta_n^2,
\]

while we get for \( q \geq 2 \) from (74)

\[
\mathbb{E}_{\theta_0} |h(Z_i) - \mathbb{E}_{\theta_0} h(Z_i)|^q \leq 2^{q+1} \mathbb{E}_{\theta_0} |h(Z_i)|^q \\
\leq 2^{q+1} \int_{B_{n,r}} \mathbb{E}_{\theta_0} |\ell(\theta_0) - \ell(\theta)|^q d\nu_n(\theta) \leq (q!)/2 (c_r \delta_n^2)^2 (2c_r)^{q-2}.
\]

The claim follows from Lemma [29] applied to \( \sigma^2 = 8c_r^2 \delta_n^2 \), \( c = 2c_r \) and \( t = n \delta_n^2 \) such that

\[
\mathbb{P}_{\theta_0}^n \left( \sum_{i=1}^n h(Z_i) \geq 7c_r n \delta_n^2 \right) \leq \mathbb{P}_{\theta_0}^n \left( \sum_{i=1}^n (h(Z_i) - \mathbb{E}_{\theta_0} h(Z_i)) \geq 6c_r n \delta_n^2 \right) \leq 2e^{-n \delta_n^2}.
\]

\[\square\]

**Lemma 43.** Consider the setting of Theorem [40] and let \( \mathcal{A} = \{\theta \in \mathbb{R}^p : \|\theta\|_\alpha \leq L'\} \) for \( L' > 0 \). If \( L' \) is large enough, then there exists \( c > 0 \) such that

\[
\log N(\mathcal{A}, h, \delta_n) \leq cn \delta_n^2, \quad \Pi(\mathcal{A}) \geq 1 - e^{-cn \delta_n^2}.
\]

**Proof.** Apply first the upper bound on the Hellinger distance in (73) and then (81) with \( \varepsilon = (c_{L'/L'}) \delta_n \) to the extent that, noting \( \delta_n^{-1/\alpha} = n \delta_n^2 \),

\[
\log N(\mathcal{A}, h, \delta_n) \leq \log N(\mathcal{A}, \|\cdot\|, c_{L'/\delta_n}) \\
= \log N \left( \{\theta \in \mathbb{R}^p : \|\theta\|_\alpha \leq 1, \|\cdot\|, (c_{L'/L'}) \delta_n \} \right) \lesssim (L'/c_{L'})^{-1/\alpha} n \delta_n^2.
\]

This proves the wanted metric entropy bound. Next, if \( \theta \in \mathbb{R}^p \) has norm \( \|\theta\| \leq C^{-\alpha}(L'/2) \delta_n \), then \( p \leq C n \delta_n^2 = C \delta_n^{-1/\alpha} \) yields \( \|\theta\|_\alpha \leq p^\alpha \|\theta\| \leq L'/2 \), and thus

\[
\{\theta = \theta_1 + \theta_2 \in \mathbb{R}^p : \|\theta_1\| \leq C^{-\alpha}(L'/2) \delta_n, \|\theta_2\|_\alpha \leq L'/2 \} \subset \mathcal{A}.
\]

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Denoting by $\Phi$ the standard Gaussian distribution function and recalling $\Pi \sim (n\delta_n^2)^{-1/2} \bar{\Pi}$ from the proof of Lemma 41, Borell’s inequality \cite{28, Theorem 11.17} provides the lower bound

$$\Pi(A) \geq \Phi\left(\frac{\theta_2}{\bar{\Pi} - \frac{1}{2}}\right) \Sigma_{\left\{ \theta \in \mathbb{R}^p : \|\theta\| \leq C^{-\alpha}(L'/2)n^{1/2}\delta_n^2 \right\}} (L'/2)n^{1/2}\delta_n^2).$$

Applying now \cite[Theorem 1.2]{42} to $J \equiv 1$ and $\varepsilon = C^{-\alpha}(L'/2)n^{1/2}\delta_n^2$, and using the inequality $y \geq -2\Phi^{-1}(e^{-y^2/4})$ for $y = (L'/2)n^{1/2}\delta_n^2$ and large enough $L'$, which holds for $y \geq 2\sqrt{2\pi}$ by standard computations for $\Phi$, we find for $c' > 0$ that

$$\Pi(A) \geq \Phi\left(\Phi^{-1}\left(e^{-c'\varepsilon}\delta_n^2\right) - 2\Phi^{-1}\left(e^{-((L')^2)/16}\delta_n^2\right)\right).$$

Possibly increasing $L'$ even further, we ensure that $c' \leq (L')^2/16$. This implies at last

$$\Pi(A) \geq \Phi(-\Phi^{-1}(e^{-((L')^2)/16}\delta_n^2)) \geq 1 - e^{-((L')^2)/16}\delta_n^2.$$ 

From this obtain the claim. \hfill \Box

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