Rotating Dirac fermions in a magnetic field in 1+2,3 dimensions

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We consider the effects of an external magnetic field on rotating fermions in 1+2,3 dimensions. The dual effect of a rotation parallel to the magnetic field causes a net increase in the fermionic density by centrifugation, which follows from the sinking of the particle lowest Landau level in the Dirac sea for free Dirac fermions. In 1+d = 2n dimensions, this effect is related to the chiral magnetic effect in 2n-2 dimensions. This phenomenon is discussed specifically for both weak and strong inter-fermion interactions in 1+2 dimensions. For QCD in 1+3 dimensions with Dirac quarks, we show that in the strongly coupled phase with spontaneously broken chiral symmetry, this mechanism reveals itself in the form of an induced pion condensation by centrifugation. We use this observation to show that this effect causes a shift in the chiral condensate in leading order in the pion interaction, and to discuss the possibility for the formation of a novel pion super-fluid phase in off-central heavy ion collisions at collider energies.

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I. INTRODUCTION

The combined effects of rotations and magnetic fields on Dirac fermions are realized in a wide range of physical settings ranging from macroscopic spinning neutron stars and black holes [1], all the way to microscopic anomalous transport in Weyl metals [2]. In any dimensions, strong magnetic fields reorganize the fermionic spectra into Landau levels, each with a huge planar degeneracy that is lifted when a paralell rotation is applied. The past decade has seen a large interest in the chiral and vortical effects and their relationship with anomalies [3] (and references therein).

Perhaps, a less well known effect stems from the dual combination of a rotation and magnetic field on free or interacting Dirac fermions. Recently, it was noted that this dual combination could lead to novel effects for composite fermions at half filling in 1+2 dimensions under the assumption that they are Dirac fermions [4], and more explicitly for free and interacting Dirac fermions in 1+3 dimensions [5–7]. Indeed, when a rotation is applied along a magnetic field, the charge density was observed to increase in the absence of a chemical potential. A possible relationship of this phenomenon to the Chern-Simons term in odd dimensions, and the chiral anomaly in even dimensions was suggested.

The purpose of this paper is to revisit these issues in a more explicit way in 1+2,3 dimensions. The case of 1+2 dimensions is of interest to planar materials in the context of solid state physics, while the case of 1+3 dimensions is of more general interest with relation to QCD. Recently, there have been few studies along these lines using effective models of the NJL type in 1+3 dimensions, where the phenomenon of charge density enhancement was also confirmed with new observations [6, 7]. Also, recent analyses using pion effective descriptions have suggested the possibility of Bose condensation in strong magnetic fields [8] and dense matter with magnetism or rotations [9].

This paper consists of a number of new results: 1/ a full analysis of the combined effects of a rotation and magnetic field on free and interacting Dirac fermions in 1+2 dimensions, both at weak and strong coupling; 2/ a correspondence with anomalies in arbitrary dimensions; 3/ a deformation of the current densities by centrifugation in the presence of a magnetic field; 4/ a depletion of the QCD chiral condensate in leading order in the pion interaction; 5/ a charge pion condensation induced by centrifugation in a magnetic field.

The outline of the paper is as follows: In section II we detail the Landau level problem for free Dirac fermions in 1+2 dimensions in the presence of an arbitrary rotation described using a local metric. In section III we explore the effects of the interaction on the free results through a 4-Fermi interaction both in the weak and strong coupling regime. In section IV and V we extend our chief observations to 1+3 dimensions to the free and interacting fermionic cases with particular interest to the shift in the chiral condensate in QCD. In section VI, we discuss the possibility for the formation of a pion BEC phase in off-central heavy ion collisions. Our conclusions are in section VII. We record in the Appendices useful details regarding some of the calculations.

II. DIRAC FERMIONS IN 1+2

In this section we will outline how to implement a global rotation through a pertinent metric. We will then use it to derive explicit results for massless Dirac fermions with a global $U(2)$ symmetry in the presence of a parallel magnetic field in 1+2 dimensions. The basic mechanism of the shift caused by the rotation on the LLL will be
clearly elucidated, and both the scalar and vector densities evaluated.

A. Metric for a rotating frame

To address the effects of a finite rotation $\Omega$ in $1 + 2$ dimensions we define the rotating metric

$$ds^2 = (1 - \Omega^2 \rho^2)dt^2 + 2y\Omega dx dt - 2x\Omega dy dt$$  \hspace{1cm} (1)

The frame fields or vielbeins are defined as $\eta^{\mu\nu} = \delta^{\mu\nu} \eta_{ab}$ with signature $\sqrt{-g} = 1$, in terms of which the co-moving frame is $\theta^a = \eta^a_{\mu} dx^\mu$ and $e_a = \eta^c_{\mu} \partial_{\mu}$ are explicitly given by

$$(\theta^0, \theta^1, \theta^2) = (dt, dx - y\Omega dt, dy + x\Omega dt)$$
$$e_0, e_1, e_2 = (\partial_t + y\Omega \partial_x - x\Omega \partial_y, \partial_1, \partial_2)$$  \hspace{1cm} (2)

with the spin connections

$$\omega^1_0 = \omega^0_1 = +\Omega(dy - \Omega dx dt)$$
$$\omega^2_0 = \omega^0_2 = -\Omega(dx + \Omega dy dt)$$  \hspace{1cm} (3)

In a fixed area of size $S = \pi R^2$, the time-like nature of the metric [1] and therefore causality are maintained for $\Omega R \leq 1$. The importance of a finite size for rotating fermions was emphasized in [7]. This will be understood throughout.

B. Rotation plus magnetic field

The Lagrangian that describes free rotating Dirac fermions in a fixed magnetic field in $1 + 2$ dimensions, reads

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu (D_\mu + \Gamma_\mu) - M)\psi$$
$$= \bar{\psi}(i\gamma^0(\partial_t - \Omega(x \partial_y - y \partial_x + iS^z)) + i\gamma^1 D_1 - M)\psi$$  \hspace{1cm} (4)

with the long derivative $D = \partial - i eA$, and the following choice of gamma matrices, $\gamma^a$ as $\gamma^0 = \text{diag}(\sigma_3, -\sigma_3), \gamma^1 = \text{diag}(i\sigma_1, -i\sigma_1), \gamma^2 = \text{diag}(i\sigma_2, -i\sigma_2)$, to accomodate for both particles and anti-particles.

A thorough analysis of [4] for an external vector potential in a rotationally non-symmetric gauge was given in [13]. Here we insist on preserving rotational symmetry by choosing $A_\mu = (0, By/2, -Bx/2, 0)$. As a result, the LL spectrum is characterized explicitly by both energy and angular momentum conservation which are described in terms of the anti-commutative harmonic oscillator $a, b$ operators

$$a = \frac{i}{\sqrt{2eB}}(D_x + iD_y) = -\frac{i}{\sqrt{2eB}} \left(2\partial + \frac{eBz}{2}\right)$$
$$b = \frac{1}{\sqrt{2eB}} \left(2\partial + \frac{eBz}{2}\right)$$  \hspace{1cm} (5)

Throughout, we will assume $eB > 0$ unless specified otherwise. The rotating Landau levels are labelled by $m, n$ as

$$E^\pm + \Omega(m - n + \frac{1}{2}) = \pm \sqrt{M^2 + 2eBn} = \pm \tilde{E}$$  \hspace{1cm} (6)

for particles and anti-particles. The corresponding normalized scalar wave functions for the $n$-th Landau level with good angular momentum $l_z = xp_y - yp_x = b^0 b - a^0 a$ with eigenvalue $m - n$, are

$$f_{nm} = \frac{(a^0)^n (b^0)^m}{\sqrt{n!m!}} f_00$$  \hspace{1cm} (7)

with the lowest Landau level (LLL) $f_{00} \propto e^{-\frac{1}{2}eB(x^2 + y^2)}$. Note that for $n=0$, we have only one positive energy state with spin up, and one negative energy state with spin down, each with degeneracy $N = eBS/2\pi$. For $\Omega = 0$ and $n > 0$ all Landau level (LL) have degeneracy $2N = eBS/\pi$. The degeneracy is lifted by centrifugation for $\Omega \neq 0$.

In terms of (7) the quantized Dirac fields follow in the form

$$\psi(t, \vec{x}) = \sum_{nm} (u^i_{nm}(\vec{x})e^{-iE^+ t} a^i_{nm} + v^i_{nm}(\vec{x})e^{-iE^- t} b^i_{nm})$$  \hspace{1cm} (8)

where $a^i_{nm}$ annihilates a particle with positive energy $E^+$ and spin $i = \pm\frac{1}{2}$, and $b^i_{nm}$ creates a hole with negative energy $E^-$ and spin $i = \mp\frac{1}{2}$. Their corresponding wavefunctions are

$$u_{0m} = (f_{0m}, 0, 0, 0)$$
$$v_{0m} = (0, 0, f_{0m}, 0)$$
$$u_{nm} = \sqrt{\frac{E + M}{2E}} \left( f_{nm}, i\sqrt{2eB} \frac{E + M}{E - M} f_{n-1, m}, 0, 0 \right)$$
$$u_{nm} = \sqrt{\frac{E - M}{2E}} \left( 0, 0, f_{nm}, \frac{i\sqrt{2eB}}{E - M} f_{n-1, m} \right)$$
$$v_{nm} = \sqrt{\frac{E + M}{2E}} \left( 0, 0, f_{nm}, i\sqrt{2eB} \frac{E + M}{E - M} f_{n-1, m} \right)$$
$$v_{nm} = \sqrt{\frac{E - M}{2E}} \left( f_{nm}, \frac{i\sqrt{2eB}}{E - M} f_{n-1, m}, 0, 0 \right)$$  \hspace{1cm} (9)
needs to be filled. The rotating vacuum now includes the particle \( LLL \) which for \( \Omega = 0 \) are shown in (a) each with degeneracy \( N \). For \( \Omega \neq 0 \) the degeneracy is lifted. In (b) we illustrate how the centrifugation lifts the degeneracy on the states with angular momentum \( N \) by shifting them down by \( \pm M - \Omega(N + \frac{1}{2}) \). The rotating vacuum now includes the particle LLL which needs to be filled.

\[ \psi = \begin{pmatrix} \psi_+ \cr \psi_- \end{pmatrix}, \quad U \psi = p \psi. \]

\[ \langle \bar{\psi} \psi \rangle (r) = \frac{eB}{2\pi} \sum_m \frac{e^{-\frac{mB^2r^2}{2}}}{m!} \left( \frac{eB^2r^2}{2} \right)^m \times (n_F(-\beta\Omega(m+1/2)) + n_F(\beta\Omega(m+1/2)) - 1) = 0 \]

which is identically zero even for zero temperature \( \beta = \infty \). So any finite rotation, however infinitesimal will cause the scalar density to vanish for free rotating fermions at finite \( B \) in \( 1 + 2 \) dimensions.

### D. Vector density

The local density of Dirac fermions in the rotating frame in \( 1 + 2 \) dimensions is readily found using (8) in the current density

\[ \langle j^0(x) \rangle = \langle : \bar{\psi} \gamma^0 \psi : \rangle = \sum_{n=0}^\infty j^0_n(x) \]

The normal ordering is carried with respect to the true vacuum at finite \( \Omega \). Each LL in (11) including the LLL contribute through a tower of rotational states \( -n < m < N - n \) for both particles and anti-particles. This finite range in the angular momentum is further detailed in Appendix I. Specifically, and for finite temperature \( 1/\beta \), the contributions of the LL and the LLL are respectively

\[ j^0_{n>0}(x) = \sum_m |f_{nm}|^2 + |f_{n-1,m}|^2 \times (n_F(E_{nm}^+) - n_F(E_{nm}^-)) \]

\[ j^0_{n=0}(x) = \frac{eB}{2\pi} \sum_m \frac{e^{-\frac{mB^2r^2}{2}}}{m!} \left( \frac{eB^2r^2}{2} \right)^m \times \sinh(\beta\Omega(m+\frac{1}{2})/2)/\cosh(\beta\Omega(m+\frac{1}{2})/2) \]

with the definition

\[ E_{nm}^\pm = E_n \mp \left( m - n + \frac{1}{2} \right) \Omega \]

\[ = \sqrt{\varepsilon Bn} \mp \left( m - n + \frac{1}{2} \right) \Omega \]

We first note that the particle density is inhomogeneous in the plane and peaks at the edge of the disc \( S = \pi R^2 \) under the effects of centrifugation. For small \( \beta\Omega \ll 1 \), i.e. small rotations or high temperature, the inhomogeneous particle density carried by the LLL is

\[ j^0_{n>0}(x) = \sum_m |f_{nm}|^2 + |f_{n-1,m}|^2 \times (n_F(E_{nm}^+) - n_F(E_{nm}^-)) \]

\[ j^0_{n=0}(x) = \frac{eB}{2\pi} \sum_m \frac{e^{-\frac{mB^2r^2}{2}}}{m!} \left( \frac{eB^2r^2}{2} \right)^m \times \sinh(\beta\Omega(m+\frac{1}{2})/2)/\cosh(\beta\Omega(m+\frac{1}{2})/2) \]

\[ E_{nm}^\pm = E_n \mp \left( m - n + \frac{1}{2} \right) \Omega \]

\[ = \sqrt{\varepsilon Bn} \mp \left( m - n + \frac{1}{2} \right) \Omega \]
\[ j_0^0 \phi(r) = \frac{\beta \Omega e^B}{4\pi} \sum_m e^{-\frac{eb^2}{m!}} \left( \frac{eB r^2}{2} \right)^m \left( m + \frac{1}{2} \right) \]

Under the combined effect of the rotation and the magnetic field, the particle density undergoes a centrifugal effect with a maximum at the edge of the rotational plane. This effect will persist even in the presence of interactions as we will discuss below (see Fig. 6).

The total number of particles follow from (11-13) by integration over \( S = \pi R^2 \). The results for the LL and LLL are respectively

\[
\begin{align*}
n_n &= 2 \sum_m (n_F(E^+_{nm}) - n_F(E^-_{nm})) \\
n_0 &= \sum_m \sinh(\beta \Omega (m + \frac{1}{2})/2) / \cosh(\beta \Omega (m + \frac{1}{2})/2)
\end{align*}
\]

For small \( \beta \Omega \), which is similar to small \( \Omega \) or large temperature, the results in (15) simplify

\[
\begin{align*}
n_n|\Omega &= 4\beta \Omega \sum_m \left( m - n + \frac{1}{2} \right) e^{\beta E_n} / (1 + e^{2\beta E_n})^2 \\
&= 4\beta \Omega \left( \frac{N^2 + 2N}{2} - n \right) e^{\beta E_n} / (1 + e^{2\beta E_n})^2 \\
n_0|\Omega &= \frac{1}{2} 2 \Omega \sum_m \left( m + \frac{1}{2} \right) = \beta \Omega (N^2 + 2N) / 4
\end{align*}
\]

We note that in \( 1 + 2 \) dimensions, the LLL generates a net density at \( \beta \Omega \ll 1 \). For strictly zero temperature \( \beta \Omega \rightarrow 0 \), the result

\[ n_0|_{\beta\rightarrow\infty} = \text{sgn}(\Omega)N \]

which can be understood from Fig. 1, for \( M \rightarrow 0 \). Since the normal ordered density operator : \( \psi^\dagger \psi \because (a^\dagger a - b^\dagger b)u^\dagger u \sim (1 - 0)u^\dagger u \) which precisely gives \( N \). Note that for a rotation opposite to the magnetic field, the LLL shift up and above the zero energy mark. Therefore, we have instead : \( \psi^\dagger \psi \because (a^\dagger a - b^\dagger b)u^\dagger u \sim (0 - 1)u^\dagger u \) which precisely gives \( -N \), as expected from (17).

These observations are not restricted to only finite temperature. Indeed, at zero temperature but finite chemical potential, the rotation induces changes in the population of the LLL. This can be seen through the substitution (7)

\[ \beta \Omega \left( m + \frac{1}{2} \right) \rightarrow \beta \left( \mu + \Omega \left( m + \frac{1}{2} \right) \right) \]

in (18), with the result

\[ n_0(\mu) = N, \quad \mu \geq -\frac{\Omega}{2} \]

\[ n_0(\mu) \approx N + 1 + \frac{2\mu}{\Omega}, \quad -\left( N + \frac{1}{2} \right) \Omega \leq \mu \leq -\frac{\Omega}{2} \]

\[ n_0(\mu) = -N, \quad \mu \leq -\left( N + \frac{1}{2} \right) \]

III. INTERACTING FERMIONS IN 1 + 2

Consider now fermions in \( 1 + 2 \) dimensions interacting via 4-Fermi interactions, as a way to model QCD\( _{1+2} \) in strong and rotating magnetic fields. The advantage of this reduction is that it will allow for closed form results with physical lessons for QCD\( _{1+3} \), which even when modeled with 4-Fermi interactions is only tractable numerically. Following [15] [14], we now consider \( N_c \) copies of the preceding Dirac fermions, interacting via local 4-Fermi \( U(2) \) symmetric interactions

\[ \mathcal{L}_{\text{int}} = \frac{G}{2} (|\bar{\psi}\psi|^2 + |\bar{\psi}i\gamma^5\psi|^2 + |\bar{\psi}\gamma^3\phi|^2) \]

Standard bosonization gives

\[ \mathcal{L}_{\text{int}} \rightarrow -\frac{1}{G} (\bar{\psi} (\sigma + \gamma^3 \tau + i\gamma^5 \pi) \psi - \frac{1}{2G} (\sigma^2 + \pi^2 + \tau^2) \]

with the scalar fields

\[ \frac{1}{G} (\sigma, \tau, \pi) = (\bar{\psi}\psi, \bar{\psi}\gamma^3\psi, i\bar{\psi}\gamma^5\psi) \]

For large \( N_c \), (108) can be analyzed in the leading \( 1/N_c \) approximation using the loop expansion for the effective action. Explicit \( U(2) \) symmetry makes the effective action only a function of \( \sigma^2 + \pi^2 + \tau^2 \), so it is sufficient to search for saddle points with \( \tau = \pi = 0 \), as others follow by symmetry.

The effective potential stemming from (108) can be organized in three parts

\[ \mathcal{V} = \mathcal{V}_0 + \mathcal{V}_T = \frac{\sigma^2}{2G} + \mathcal{V}_\Lambda + \mathcal{V}_T \]

The zero temperature (vacuum) contribution from the fermion loop is

\[ \mathcal{V}_\Lambda = -\frac{N_c}{4\pi^2} \frac{1}{s^2} \int_0^\infty \frac{ds}{s^2} e^{-s\sigma^2} eB \coth(eBs) \]

which is cut off in the UV by \( 1/\Lambda^2 \), while the thermal contribution is
\[ V_T = -\frac{N_c T}{S} \sum_{j=1,-1} \sum_{n=0}^{N} \sum_{l=-n}^{N-n} \ln(1 + e^{-\beta(E_n - j\Omega(l + \frac{1}{2}))}) \]  

(25)

with \( E_n = \sqrt{\sigma^2 + 2eBn} \) and \( N/S = eB/2\pi \). A complementary but numerically useful approximation to (25) is given in Appendix II using the proper time formalism.

### A. Weak coupling regime

At zero temperature and in the absence of \( B, \Omega \), the effective potential (110) for the interacting Dirac fermions in 1 + 2 dimensions simplifies

\[ V \to \frac{\sigma^2}{2G} - \frac{N_c}{4\pi^2} \int_{\frac{1}{\Lambda}}^{\infty} \frac{ds}{s^2} e^{-s\sigma^2} \]  

(26)

If we set \( g = \frac{G\Lambda}{\pi} \), then (26) exhibits a minimum at \( \sigma = \Lambda/g_r \) with \( 1/g_r = 1/g - 1/g_c \), only for sufficiently strong coupling \( g > g_c = \sqrt{\pi} \). The minimum breaks spontaneously \( U(2) \to U(1) \times U(1) \) with a finite \( \langle \bar{\psi}\psi \rangle = -N_c\sigma/G \). The putative chargeless Goldstone mode signals a BKT phase at any finite \( N_c \).

At zero temperature and zero rotation \( \Omega = 0 \) but with \( B \neq 0 \), the effective potential (110) can be made more explicit by rescaling and expanding in \( \frac{1}{\Lambda} \). For small \( \sigma \) and large \( \Lambda \) the dominant contributions are

\[ V_\Lambda = + \frac{N_c \sigma^3}{4\pi^2} \int_{\frac{1}{\Lambda}}^{\infty} \frac{dx}{s^2} \frac{eBx}{\Lambda} \coth \left( \frac{eBx}{\Lambda} \right) \]
\[ - \frac{N_c \sigma^2}{2\pi^2} + \frac{N_c \sigma^3}{3\pi} \]
\[ + \frac{N_c}{4\pi^2} \int_{\frac{1}{\Lambda}}^{\infty} \frac{ds}{s^2} (e^{-s\sigma^2} - 1)(eBs\coth(eBs) - 1) \]
\[ + O \left( \frac{1}{\Lambda} \right) \]  

(27)

The first contribution is independent of \( \sigma \), so we will ignore it. Therefore, the vacuum contribution to the effective potential combines the first term in (110) and the second and third contributions in (27)

\[ \frac{V_0}{N_c} \approx \frac{\Lambda \sigma^2}{2\pi g_r} - \frac{eB\sigma}{2\pi} + \frac{\sigma^3}{3\pi} \]  

(28)

In the weak coupling regime

\[ 0 \leq \left( \frac{1}{g_r} \equiv 1 - \frac{1}{g} - \frac{1}{g_c} \right)^{-1} \leq \frac{\Lambda}{eB} \]  

(29)

we can ignore the cubic contribution in (28). A minimum of (28) always exists for arbitrarily weak coupling, with a mass gap \( \sigma = \pi g_r N/S \Lambda \) and a finite chiral condensate \( \langle \bar{\psi}\psi \rangle = -N_c N/S (1 - g/g_c) \approx -N_c N/S \). The latter is in agreement with the result for free Dirac fermions. This is the phenomenon of magnetic catalysis [13].
1. Vacuum with $\Omega \neq 0$

At zero temperature, the effective potential for rotating Dirac particles in a strong magnetic field is given by the first two contributions in (110), plus the contribution from the rotating anti-particles in the LL, 

$$\frac{\mathcal{V}}{N_c} = \frac{N \sigma^2}{2 \pi g_r} - eB \frac{\sigma}{2\pi} - eB \sum_{l=0}^{N} \left( (l + \frac{1}{2}) \Omega - \sigma \right) \theta \left( (l + \frac{1}{2}) \Omega - \sigma \right)$$

(30)

For small rotation the summation can be approximated by a continuous integration with the result

$$\frac{\mathcal{V}}{N_c} \approx \frac{N \sigma^2}{2 \pi g_r} - eB \frac{\sigma}{2\pi} - \frac{1}{2\Omega} \left( E_{\Omega} - \sigma \right) \left( E_{\Omega} - \sigma \right)^2$$

(31)

with $E_{\Omega} = (N + \frac{1}{2}) \Omega$. For $\sigma > E_{\Omega}$, the effective potential is independent of $\Omega$, and develops a minimum for

$$\sigma_2 = \frac{\pi g_r eB}{\Lambda}$$

$$\frac{\mathcal{V}_2}{N_c} = - \frac{\pi g_r}{2\Lambda} \left( \frac{eB}{2\pi} \right)^2$$

(32)

In contrast, for $\sigma < E_{\Omega}$, (31) depends on $\Omega$ through

$$\frac{\mathcal{V}}{N_c} \approx \left( \frac{\Lambda}{2 \pi g_r} - \frac{eB}{4\pi N \Omega} \right) \sigma^2 + \frac{eB}{4\pi N} \sigma - \frac{eB \Omega}{4\pi} \left( N + \frac{1}{2} \right)$$

(33)

and prefers always

$$\sigma_1 = 0$$

$$\frac{\mathcal{V}_1}{N_c} = - \frac{E_{\Omega} eB}{2 \pi^2}$$

(34)

For $E_{\Omega} < \frac{\pi g_r eB}{2\pi}$ the 2-minimum (34) is dominant. The rotating vacuum develops a scalar condensate $\langle \bar{\psi} \psi \rangle \neq 0$ with finite $\sigma_2$ but zero fermion density $\langle \bar{\psi} \gamma^0 \psi \rangle = 0$. In the opposite, with $E_{\Omega} > \frac{\pi g_r eB}{2\pi}$, the 1-minimum (33) takes over. The rotating vacuum prefers a gapless solution with $\sigma_1 = 0$ and zero scalar condensate $\langle \bar{\psi} \psi \rangle = 0$, but a finite fermion density $\langle \bar{\psi} \gamma^0 \psi \rangle \neq 0$. In large $N$, the critical value for which this occurs is

$$\Omega_c = \frac{g_r eB}{2N + 1} \frac{1}{\Lambda}$$

(35)

This is the phenomenon of rotational inhibition of the magnetic catalysis noted in 1 + 3 dimensions in [7]. At finite but large $N$ and without the use of the continuum approximation and keeping the $\sigma^3$ term, the results remain quantitatively almost the same, with one exception that the local minimum $\sigma_1 = 0$ can overtake the finite local minimum $\sigma_2$ slightly before the $\Omega_c$. For $\Lambda = 10\sqrt{eB}$ and $N = 100$, (35) yields $\Omega_c = 0.0000497\sqrt{eB}$. We note that in the free case with $\Lambda \rightarrow \infty$, (35) yields $\Omega_c \rightarrow 0$ in agreement with the observation in (10). Any finite rotation destroys the free scalar condensate.

In Fig. 2 we show the behavior of the effective potential for finite but small $\Omega$ with the two local minima (32) and (33). We have used $\Lambda/\sqrt{eB} = 10$, $N = 100$ and $g_r = 1$. A transition sets in numerically at $\Omega_c = 0.000488\sqrt{eB}$ in agreement with (35). In Fig. 3 we display the effective mass as a function of $\sqrt{eB}$ and $\Omega$ in units of $\Lambda$, for $g_r = 1$ (weak coupling regime) and $T = 0$. While the mass gap is seen to increase slightly faster than linearly with $\sqrt{eB}$ at $\Omega = 0$, the effects of the rotation is to cause it to disappear at the critical value (35) through a first order transition at weak coupling.

2. Thermal state with $\Omega \neq 0$

First we note that the existence of a mass gap for any finite temperature does not contradict the Mermin-Wagner-Coleman (MWC) theorem, since the thermal state is in a BKT phase rather than a spontaneously broken or Goldstone phase. Having said that, at finite temperature and weak coupling, we note that since $\sigma_2 \ll \sqrt{eB}$, the temperatures of interest for the vanishing of the mass gap, are in the low range with $T \ll \sqrt{eB}$. Therefore, only the $j = \pm 1$ LLL contribute in (25). For $T \approx T_c \approx \sigma_2$, the potential flattens out and the centrifugation near $\sigma = 0$ becomes visible leading to a small value for the critical $\Omega_c$.

In Fig. 4 we show the behaviour of the effective potential for $\Lambda/\sqrt{eB} = 10$, $N = 100$ and $g_r = 1$ (weak coupling) for $\beta = 80/\sqrt{eB}$ and $\beta = 43/\sqrt{eB}$. For $\beta \geq 80/\sqrt{eB}$ the transition occurs at $\Omega_c \approx 0.0005\sqrt{eB}$, and for $\beta = 43/\sqrt{eB}$, the transition is around $\Omega_c = 0.0001\sqrt{eB}$. The critical temperature is numerically in the range $\beta_c \approx (40 - 43)/\sqrt{eB}$. The behavior of the effective mass is shown in in Fig. 5 for the same value of $g_r = 1$ (weak coupling) and $\Lambda = 10\sqrt{eB}$, as a function of $\beta$ and $\Omega$ for the ranges $50 < \beta < 80$ and $0.0003 \leq \Omega \leq 0.0006$ in units of $\sqrt{eB}$.

In Fig. 6 we show the analogue of the profile density (14) in units of $\sqrt{eB}$, in the weak coupling regime with $g_r = 1$ and for $1/\beta \ll \Omega$ as a function of $x = eB r^2/2$. The first figure from the top is for $\Omega = 0.00005\sqrt{eB}$ for $1/\beta = 0$. It is roughly constant and drops sharply at the edge of the causality disc fixed by $\Omega R = 1$. However, for $\Omega \ll 1/\beta \ll \sqrt{eB}$ a linear behavior sets in in the middle of the disc, to drop only sharply at the edge. The second and third figures from the top are for $\beta = 100/\sqrt{eB}$ and $\Omega = 0.0001\sqrt{eB}$ and $\Omega = 0.0005\sqrt{eB}$ respectively. The
fourth figure is for $\beta = 40/\sqrt{eB}$ at $\Omega = 0.0001\sqrt{eB}$. As we indicated in section IID for the free case, this centrifugation effect holds for the interacting case as well and carries to higher dimensions as we show below. We will suggest a possible physical application in 1+3 dimensions. Finally, the occurrence of surface or edge modes was noted recently in [11]. We show in Appendix IX that they do not alter our current discussion for large $N$.

3. Dense state with $\Omega \neq 0$

For completeness, we now explore the effects of a finite chemical potential $\mu$ on the mass gap for $\bar{\psi}\psi$ pairing. Just as a caution, we note that a more complete treatment would require the inclusion of the competing $\psi\psi$ channel as well. However, we note that in leading order in $1/N_c$ the $\psi\psi$ channel is $1/N_c$ suppressed in comparison to the $\bar{\psi}\psi$ channel and can be ignored. With this in mind, the effect of a finite chemical potential follows from (98) through the substitution $\Omega(l + \frac{1}{2}) \rightarrow \mu + \Omega(l + \frac{1}{2})$, which we now briefly address.

In Fig. 7 we show the behavior of the effective potential $V$ for $\beta = 80/\sqrt{eB}$ and $\mu = 0.007/\sqrt{eB}$ as a function of $\sigma$ in units of $\sqrt{eB}$. The top figure is for $\Omega = 0$ and the bottom figure is for $\Omega = 0.0003\sqrt{eB}$. The increase in the rotation causes the loss of the gapped solution. In particular, for $g_r = 1$ (weak coupling), $\beta = 80/\sqrt{eB}$ and $\Omega = 0$, the critical value is $\mu_c = 0.02\sqrt{eB}$, while for
FIG. 7: Finite temperature effective potential $V(\sigma)$ at $\beta = 80/\sqrt{eB}$ and $\mu = 0.007/\sqrt{eB}$ as a function of $\sigma$ in units of $\sqrt{eB}$: $\Omega = 0$ (top) and $\Omega = 0.0003\sqrt{eB}$ (bottom).

$\Omega = 0.0003\sqrt{eB}$, the critical value is $\mu_c = 0.007\sqrt{eB}$.

Finally and for completeness, we discuss in Appendix III the dense state with negative $\mu$. Since the model under consideration can be viewed as an effective description of planar condensed matter systems [14], a negative chemical potential is experimentally accessible.

**B. Strong coupling regime**

In the opposite regime of strong coupling with $g > g_c$, a mass gap also forms. In the regime where the ratio $\Lambda/\sqrt{eB}$ is large and $g > g_c$ or $g_r < 0$, the minimum of the effective potential is now controlled by the first and third contributions in (28) namely

$$V_0 \approx -\frac{\Lambda \sigma^2}{2\pi |g_r|} + \frac{\sigma^3}{3\pi}$$

with a mass gap $\bar{\sigma} = \Lambda/|g_r|$. For $\sqrt{eB}/\Lambda < 1$, the leading contribution shifts the mass and the scalar condensate quadratically,

$$\frac{\langle \bar{\psi}\bar{\psi} \rangle}{\langle \psi\psi \rangle_0} - 1 \approx \frac{(eB)^2}{12(\Lambda/|g_r|)^4}$$

We note that the ratio of the mass gap to the LL gap $\bar{\sigma}/\sqrt{eB}$ can be very large. Therefore, the critical $\Omega_c$ for which the mass gap can be depleted is much larger in strong coupling than in weak coupling. For fixed $\Omega$, the mass $\bar{\sigma}$ decreases as the ratio $\Lambda/\sqrt{eB}$ decreases. For instance, for $g_r = -4$ and $\Lambda/\sqrt{eB} = 5$, $\Omega_c \approx 0.008\Lambda$,

but for $\Lambda/\sqrt{eB}|g_r| = 3$, $\Omega_c \approx 0.009\Lambda$. In Fig. 11 we show the behavior of the mass gap for strong coupling with $g_r = -4$ versus $\sigma$ in units of $\Lambda$ as a function of $\Omega$ expressed in units of $\Lambda/10^4$. The top figure is for $\Lambda/\sqrt{eB} = 5$ and the bottom figure is for $\Lambda/\sqrt{eB} = 3$.

**IV. FREE DIRAC FERMIONS IN 1+3**

The extension of the previous analysis to $1+3$ dimensions for free Dirac fermions is straightforward. In Appendix IV we detail the rotating wavefunctions in the presence of a magnetic field, for the free case. The interacting case is more challenging for say the case of QCD which is strongly coupled and gapped in the vacuum. Below, we will focus on the combined effects of a rotation and magnetic field on the QCD chiral condensate in the spontaneously broken phase using mesoscopic arguments, and leading order chiral perturbation.

**A. Free left currents**

We now extend the analysis for the left or $L$-currents to show the generic nature of the observations made in $1+2$ dimensions above. From Appendix IV, the $L$-wavefunctions in $1+3$ dimensions take the simplifying form
While the current density at the origin reproduces the expected result, the distribution of the current density in the radial direction is not homogeneous. Indeed, the centrifugation causes it to peak at the edge as in 1 + 2 dimensions. This is readily seen from the contribution of the LLL which can be worked out explicitly with the result

$$J_{L=0}^3 = \frac{eB}{4\pi^2} \sum_{m=0} \left( \frac{eB r^2}{2} \right)^m \frac{(m+1/2)\Omega + \mu_L}{m!}$$

The sum can be performed exactly with the result

$$J_{L=0}^3(r) = \frac{eB}{4\pi^2} \left( \mu_L + \Omega \left( \frac{1}{2} + \pi N r^2 \right) \right)$$

The centrifugal effect causes the current density to peak at the edge of the rotational plane in 1 + 3 dimensions. A possible application of this phenomenon maybe in current heavy ion collisions at collider energies such as RHIC and LHC. Indeed, for semi-central collisions both the rotational (orbital) and electric magnetic fields are sizable with $\Omega \sim eB \sim m_\pi$ which may induce partonic densities of the type [45] that are largely deformed in the transverse plane. While the rotation and magnetic fields tend to separate the partonic charges in concert along the rotational axis, the centrifugation causes this separation to peak in the orthogonal direction where the observed particle flow is more important. If true, this effect should be seen as an enhancement of $v_4$ in the charged particle flow.

### B. Number of free left particles

As we noted in 1 + 2 dimensions, the number of free left particles increases in 1 + 3 dimensions due to the sinking of the particle LLL in the Dirac sea. More explicitly, we have

$$n_L = \int dxdy \langle \bar{\psi}_L \gamma^0 \psi_L \rangle$$

$$= \sum_m \int_{-\infty}^0 \frac{dp}{2\pi} \left( n_F(-p - \mu_m) - n_F(-p + \mu_m) \right)$$

$$+ \sum_{n=1,m} \int_{-\infty}^\infty \frac{dp}{2\pi} \left( n_F(E_n - \mu_m) - n_F(E_n + \mu_m) \right)$$

with $\mu_m = \frac{B}{3} + \mu_L$ and $\mu_{10} = -\frac{B}{3} + \mu_L$. For small $B$ and zero $\mu_L$, the summation in [45] gives

$$\sum \frac{eB}{2\pi} f(\sqrt{p^2 + 2gBn}) \rightarrow \int \frac{kdk}{2\pi} f(\sqrt{p^2 + k^2})$$

This reproduces the known result at $B = 0$ [1]

$$-\frac{T\Omega}{12\pi^2} \left( \Omega + 2\mu_L \right)^3 + \frac{(\Omega - 2\mu_L)^3}{96\pi^2}$$

Here $\mu_{nm} = (m - n + \frac{1}{2})\Omega + \mu_L$ and $E_n = \sqrt{p^2 + 2eBn}$. The flowing left current along the rotational-magnetic axis is
The first contribution in (47) was noted in \([5, 7]\). (46-47) generalize to arbitrary \(1 + d\) dimensions. In particular, for \(\mu L = 0\)

\[
\begin{align*}
n_{L0} &= 2 \frac{d-3}{d^2-2} \frac{\pi^2}{2} \frac{V_{d-2}}{2} \text{sgn}(\Omega) |\Omega|^{d-2} \sum_{m=1}^{N} \left( m + \frac{1}{2} \right)^{d-2} \\
\text{with the volume } V_{d-2} &= \pi^{\frac{d-1}{2}} / \Gamma\left(\frac{d}{2}\right).
\end{align*}
\]

\section*{C. Relation to anomalies}

These observations can be used to generalize (49) to arbitrary \(1 + d = 2n\) dimensions. Consider the case with non-vanishing and non-parallel magnetic fields \(B_{2k,2k+1} \neq 0\) with \(1 \leq k \leq n - 3\). The general anomaly induced chiral magnetic effect for the left current is \([12]\)

\[
J_{n,L}^{2n-1} = -\frac{\mu L}{2\pi} \left( \frac{e}{2\pi} \right)^{n-1} B_{12}B_{34}...B_{2n-4,2n-3} \quad (49)
\]

We now observe from (45) that the role of the rotation is to tag to \(\mu L\) in \(2n = 4\) dimensions as

\[
e\frac{B}{2\pi} \left( \mu L + \Omega \left( \frac{1}{2} + \pi N r^2 \right) \right) \equiv \mu L \frac{eB}{2\pi} + \Omega J(r) \quad (50)
\]

The anomalous result (49) relates to the rotationally induced current by a similar substitution in \(2n\) dimensions, namely

\[
J_{n,L}^{2n-1}(r) = -\frac{1}{2\pi} \left( \frac{e}{2\pi} \right)^{n-2} B_{12}B_{34}...B_{2n-6,2n-5}(\Omega, J(r)) \quad (51)
\]

where \(J(r)\) refers to the current spin density in the radial direction within the \(2n = 4, 2n - 3\) plane

\[
J_{2n-4,2n-3}(r) = \frac{eB_{2n-4,2n-3}}{2\pi} \left( \frac{1}{2} + B_{2n-4,2n-3} r^2 / 2 \right) \quad (52)
\]

The rotational contribution to the current density (51) in \(2n\) dimensions is related to the chiral magnetic effect (49) in \(2n - 2\) dimensions.

\section*{D. Charge neutral volume}

Most of the analyses for the fermions presented above hold for the absolute ground state with overall charge conservation not enforced (open volume \(V\)). If we require total charge neutrality of the system (closed volume \(V\)) then we expect an induced charge chemical potential \(\mu_{in}\) such that \((\Omega \cdot \vec{B} > 0)\)

\[
\begin{align*}
\sum_{n,m=0}^{N} \int \frac{dp}{2\pi} n_F \left( E_n - \mu_{in} - \Omega \left( \frac{1}{2} + m - n \right) \right) &= 0 \\
\sum_{n,m=0}^{N} \int \frac{dp}{2\pi} n_F \left( E_n + \mu_{in} + \Omega \left( \frac{1}{2} + m - n \right) \right) &= 0
\end{align*}
\]

where the number of \(\pi^+\) (first contribution) balances the number of \(\pi^-\) (second contribution). For large \(eB\) or small temperature \(T\), only the \(n = 0\) term survives as before. In this case, the solution for \(\mu_{in}\) follows by inspection

\[
\mu_{in} = -\frac{\Omega}{2} - \frac{N \Omega}{2} \quad (54)
\]

The ground state consists of negative charge filling the LLL with \(m = 0\) to \(m = N / 2\), and positive charge filling the LLL with \(m = N / 2\) to \(N\). The corresponding charge density for massless fermions is

\[
\langle J_{L,n=0}^{0}(x) \rangle = \frac{eB}{4\pi^2} \sum_{m=0}^{[\frac{N}{2}]} e^{-\frac{\omega m^2}{2}} \frac{eBr^2 m}{m!} \quad (55)
\]

The first line is the contribution from all negative charge contributions, and the second line from all positive charge contributions. After integration, the total negative charge density is

\[
\left\langle \int d^2x J_{L,n=0}^{0}(x) \right\rangle_{\text{negative}} = \frac{1}{2\pi} \sum_{m=0}^{[\frac{N}{2}]} \left( m - \frac{N}{2} \right) \Omega \quad (56)
\]

and similarly for the positive charge density. In Fig. 3 we display the charge density in the LLL in a closed volume \(V = SL\) with total charge neutrality as given by (55). We expect the same distribution of charge around a fluid vortex when overall charge neutrality is enforced, which is to be contrasted with a vortex with only positive (negative) charge accumulation when the charge neutrality constrain is not enforced [5].
The vacuum diffusion constant is \( D \) with \( \pi \) Goldstone modes and the correlator in (58) is dominated by the lightest absence of magnetism, the vacuum is isospin symmetric QCD vacuum in Euclidean 4-dimensional space. In the \( u,d \) for 2 light flavors. The averaging in (58) is over the QCD vacuum in Euclidean 4-dimensional space. In the absence of magnetism, the vacuum is isospin symmetric and the correlator in (58) is dominated by the lightest Goldstone modes \( \pi^{0,\pm} \)

\[
P(0, \tau) = 2 (P_0(0, \tau) + P_{\pm}(0, \tau)) \approx \sum_{Q} e^{-\tau D(0,0)Q^2} \quad (59)
\]

The sum is over the pions or diffusons with momenta \( Q_p = n_p 2\pi/L \) in a periodic \( V_4 = L^4 \) Euclidean box. The vacuum diffusion constant is \( D(0,0) \).

\[
I = \int_{0}^{\infty} (P(0, \tau) - P(0,0)) d\tau \quad (62)
\]

In the chiral limit, replacing the sums over free momenta by integrals allows to get rid of the explicit \( \Omega \) dependence in (61) by shiting \( p_4 \). So the dependence on \( \Omega, B \) in \( P_0 \) is only through \( D(0, B) \). Clearly, in the absence of \( B \) a rotation \( \Omega \) alone cannot change the return probability, and therefore the chiral or scalar condensate as the vacuum is rotationally symmetric. This is not the case in the presence of an externally fixed magnetic field \( B \) as rotational symmetry is broken. Indeed, the LL dependence in \( P_{\pm} \) does not drop but can be resummed exactly with the result

\[
I = \frac{e BV_4}{16\pi^2 D} \int_{0}^{\infty} \left( \frac{1}{z \sinh z} - \frac{1}{z^2} \right) dz = - \frac{\ln2}{16\pi^2} e BV_4 \quad (63)
\]

Using the value of the diffusion constant we arrive at

\[
B. \text{ Diffusion with } B, \Omega \neq 0
\]

Under rotations all \( \pi^{0,\pm} \) are affected by centrifugation, while only the \( \pi^{\pm} \) are affected by magnetism. As a result, the squared and Euclideanized pion spectra are

\[
Q_0^2 = p_r^2 + p_3^2 + (p_4 + i\Omega l)^2 + m_\pi^2
\]

\[
Q_{j=\pm}^2 = e B(2n + 1) + p_3^2 + (p_4 + i\Omega j l)^2 + m_\pi^2 \quad (60)
\]

Each chargeless mode carries \( l = 0, \pm 1, \ldots \), while each charged mode is in a LL \( n \) where \( -n \leq l \leq N - n \) with degeneracy \( N \). Note that the rotational energy shift in Euclidean space is purely imaginary. The change in each of the return probabilities in (59) following from (60) is

\[
P_0(\Omega, \tau) = \sum_{n_3,n_4} \sum_{n=\pm \infty} \sum_{m} \sum_{p} e^{-\tau D(\Omega, B)(p_3^2 + p_4^2 + (p_4 + i\Omega j l)^2 + m_\pi^2)}
\]

\[
P_{j=\pm}(B, \Omega, \tau) = \sum_{n_3,n_4} \sum_{n=\pm \infty} \sum_{m} \sum_{p} e^{-\tau D(\Omega, B)(e B(2n + 1) + p_3^2 + (p_4 + i\Omega j l)^2 + m_\pi^2)}
\]

with \( p_{3,4} = n_{3,4}(2\pi/L) \) in an Euclidean box of 4-volume chosen cylindrical with \( V_4 \rightarrow \pi R^2 L^2 \) and the causal constraint \( \Omega R < 1 \). In general, in the rotating vacuum with magnetism the diffusion constant \( D(\Omega, B) \) is \( \Omega, B \) dependent.

The change in the quark return probability is the change in the charged diffusion modes and is captured by the difference

\[
I = \int_{0}^{\infty} (P(\Omega, B, \tau) - P(0,0,\tau)) d\tau \quad (62)
\]

\[
I = \frac{e BV_4}{16\pi^2 D} \int_{0}^{\infty} \left( \frac{1}{z \sinh z} - \frac{1}{z^2} \right) dz = - \frac{\ln2}{16\pi^2} e BV_4 \quad (63)
\]
\[
\frac{\langle \bar{\psi} \psi \rangle_{\Omega,B}}{\langle \bar{\psi} \psi \rangle_{0,0}} - 1 = \frac{\ln 2 \, eB \, D(0,0)}{16\pi^2 \, F_+^2 \, D(\Omega, B)}
\]  

(64)

For \( \Omega = 0 \) and \( B \neq 0 \), \( \langle \bar{\psi} \psi \rangle_{0,0} \) is in agreement with chiral perturbation theory in leading order \( \mathcal{O}(\alpha) \). This linear magnetic field is supported by lattice simulations \( \mathcal{O}(\alpha) \).

(64) is the analogue of \( \langle \bar{\psi} \psi \rangle \) in 1 + 2 dimensions at strong coupling, with the difference that it grows linearly not quadratically. The quadratic growth follows from the absence of charged Goldstone modes. As indicated earlier, in 1 + 2 dimensions the gapped phase is a BKT phase not a Goldstone phase. We now give an independent determination that fixes \( D(\Omega, B) \) in \( \langle \bar{\psi} \psi \rangle \).

C. Energy densities of a BEC of chiral pions

To assess the dual action of \( \vec{\Omega} \cdot \vec{B} > 0 \) in the QCD vacuum energy, requires vacuum loops in the presence of \( \Omega, B \). When the magnetic field is sufficiently weak, i.e. \( |eB| \ll (4\pi F_+)^2 \) with \( F_+ \) the pion decay constant constant, the loop momenta are small and QCD is well described by an effective theory of chiral pions. In leading order, the pion interactions which are soft can be ignored. The \( \Omega, B \) dependent parts in the QCD vacuum energy follow from a one-pion loop with arbitrary \( \Omega, B \) insertions in leading order, with the rotation acting as an effective chemical potential. In the presence of a fixed magnetic field in the +z direction \( \vec{B} = B\hat{z} \), the charged \( \pi^\pm \) pion spectrum is characterized by highly degenerate LL with energies

\[
E_{np} = (|eB|(2n + 1) + p^2 + m_\pi^2)^{1/2}
\]

(65)

Each LL \( n \) for fixed pion 3-momentum \( p \) carries a degeneracy \( N \), labeled by the \( z \)-component of the angular momentum \( L_z = l \) with \( -n \leq l \leq N - n \) as detailed in Appendix XIV. When a rotation \( \Omega \) is applied, the spectrum \( \langle \bar{\psi} \psi \rangle \) shifts so that in the rotating frame we have \( \langle \bar{\psi} \psi \rangle(\Omega \cdot \vec{B}, \Omega) \) (66)

Here \( j = +1 \) for positively charged pions (particles) and \( j = -1 \) for negatively charged pions (anti-particles). As a result, the degeneracy of each LL is lifted. The mechanism of \( \pi^\pm \) splitting by a rotation can cause \( \pi^\pm \) pion condensation \( \mathcal{O}(\alpha) \). We now explore this condensation in the vacuum and also matter for different overall charge constraints.

1. Open volume

We first consider the open volume \( V = SL \) case, where charge is free to move in and out of \( V \). In leading order in the pion interaction, the QCD vacuum energy per unit volume in \( V \) is the sum of a purely \( B \) dependent contribution \( \mathcal{E}_\pi B \) and a mixed \( B, \Omega \) dependent contribution \( \mathcal{E}_\pi \Omega \)

\[
\mathcal{E}_\pi(\Omega, B) = \mathcal{E}_\pi B + \mathcal{E}_\pi \Omega
\]

(67)

If we denote by \( n \) the number of condensed \( \pi^+ \) per unit length \( L \) along the rotational axis, then

\[
\mathcal{E}_\pi B = \frac{2N}{S} \int_{-\infty}^{+\infty} dp \sum_{n=0}^{\infty} \frac{1}{2} \epsilon_n(p)
\]

\[
\mathcal{E}_\pi \Omega = -\frac{n}{S} (N\Omega - m_0) + \frac{n^2}{2}
\]

(68)

with \( \epsilon_n(p) = p^2 + m_\pi^2 \) and \( m_\pi^2 = (2n + 1)eB + m_0^2 \). The first contribution stems from the pion loop with charged \( \pi^\pm \) pions, while the second contribution stems from the Bose condensation of the \( \pi^+ \) in the LLL when the rotationally induced chemical potential \( \mu_N = N\Omega \) exceeds the effective pion mass \( m_0 \). In the open volume case, the accumulation of the charge at the edge of \( V \) is compensated by a deficit outside of \( V \) to maintain overall charge conservation. The last contribution in \( \mathcal{E}_\pi \Omega \) is the Coulomb repulsion in the condensed droplet of \( \pi^+ \) by centrifugation.

To assess the Coulomb contribution, we note that the 2-dimensional charge distribution in this state is given by \( \rho_N(\vec{x}) = e^2 |f_{0N}(x, y)|^2 \) where \( f_{0N}(x, y) \) is the N-LL

\[
f_{0N}(x, y) \approx \left( \frac{1}{\sqrt{2\pi B}} \left( \frac{2}{\partial \partial_z + \frac{eBz}{2}} \right)^N \right) e^{-\frac{1}{2}eBz^2}
\]

(69)

with \( z = x + iy \) and valued in \( S = \pi R^2 \). The condensate lies at the edge of the rotational plane with a Coulomb factor

\[
c_N = \frac{e^2}{2L} \int_{L \times S} d^3x \, d^3x' \rho_N(\vec{x}) \frac{1}{|x - x'|} \rho_N(\vec{x}')
\]

(70)

In the large degeneracy \( N \) limit, we can approximate this distribution by a uniform radial distribution within the area \( N - \sqrt{N} \leq \frac{eBz^2}{2} \leq N + \sqrt{N} \) with total charge \( e \). It follows that the Coulomb factor is \( c_N \approx e^2/12\pi \sqrt{N} \).

The condensate density \( n \) is fixed by minimizing the energy density \( \mathcal{E}_\pi \Omega \) in \( (68) \), with the result

\[
n = \frac{\theta(N\Omega - m_0) \, N\Omega - m_0}{2c_N}
\]

(71)

for which the energy density in \( (68) \) is

\[
\mathcal{E}_\pi \Omega \rightarrow -\frac{3\pi \sqrt{N}}{e^2 S} (N\Omega - m_0)^2 \theta(N\Omega - m_0)
\]

(72)
For \( eB = 0.1 \, m^2 \), and \( N = 1000 \), the threshold for developing non-zero \( n \) is \( \Omega_{\text{min}} = 0.001 \, m \). For \( \Omega = 0.0015 \, m \), we have \( n = 268 \, m \), and for \( \Omega = 0.002 \, m \), we have \( n = 566 \, m \).

The condensation of charged pions by rotation in a magnetic field is for bosons, what the accumulation of vector charge in a vortex threaded by a magnetic field is for fermions [5], and in general in any rotating frame with a magnetic field [5 23]. For Dirac fermions this phenomenon is related to spectral flow and therefore to anomalies [5 23], of which the charged pionic condensate is its low energy manifestation in the QCD vacuum. In both cases, the charge accumulation in the finite volume \( V = LS \) is compensated by a deficiency of opposite charge in the outside of the volume \( V \). Overall charge conservation is maintained by allowing the charge to move in or out of \( V \) as also suggested in [5] for fermions.

2. Closed volume

If the volume \( V = SL \) is closed with no charge allowed to flow in or out, then charge conservation is to be enforced strictly in \( V \) [23]. Let \( \mu \) be the charged chemical potential in the co-moving frame. Charge neutrality at finite \( T, \mu \) requires

\[
\sum_{l=0}^{N} \int \frac{dp}{2\pi \, e^\pm (E_{\pi} - \mu)} = \frac{1}{2} \sum_{l=0}^{N} \int \frac{dp}{2\pi \, e^\pm (E_{\pi} + \mu)} = 1
\]

(73)

with the pion spectrum [5 23]. is solved for \( \mu = - \frac{N \Omega}{2} \) at any temperature \( T \). Therefore, \( l = N - m \) and \( l = m \) state for \( \pi^+ \) and \( \pi^- \) will have the same occupation number. For \( N \Omega > 2m_0 \) simultaneous condensation occurs for \( m = 0 \), i.e. \( \pi^+ \) with \( l = N \) and \( \pi^- \) with \( l = 0 \). For \( (N-2)\Omega > 2m_0 \) the condensation involves both \( m = 0,1 \). As we increase \( \Omega \) all \( m \leq \frac{N}{2} \) will condense, i.e. \( \pi^\pm \) with \( \frac{N}{2} \leq l \leq N \) and \( \pi^- \) with \( 0 \leq l \leq \frac{N}{2} \).

An alternative way to see this without solving for \( \mu \) is to note that for all terms in (73) to be meaningful, the inequalities

\[
\ldots \leq -m_0 \leq \mu \leq m_0 - N \Omega \leq \ldots
\]

must hold. Thus, as long as \( m_0 - N \Omega < -m_0 \) or \( N \Omega > 2m_0 \), the occupation number of the \( l = N \) state for \( \pi^+ \) and the \( l = 0 \) state for \( \pi^- \) are no longer meaningful, and condensation may follow. For increasing \( \Omega \) such that \( m_0 - N \Omega + \Omega < -m_0 - \Omega \) or \( (N-2)\Omega > 2m_0 \), the condensation for the \( l = N - 1 \) state of \( \pi^+ \) and the \( l = 1 \) state for \( \pi^- \) will also follow, which is consistent with the above argument based on the solution for \( \mu \). We note that in the charge-conserving case, the critical \( \Omega \) is twice the critical \( \Omega \) in the non-conserving case.

Now consider the rotating ground state with \( T = 0 \) and \( N \Omega > 2m_0 \) but \( (N-2)\Omega < 2m_0 \), so that only the \( l = N \) state for \( \pi^+ \) and \( l = 0 \) state for \( \pi^- \) condense. The analogue of (68) is then

\[
\mathcal{E}_{\pi\Omega} = -n (N \Omega - 2m_0) + d_N n^2
\]

(75)

with the new Coulomb factor

\[
d_N \approx \frac{e^2}{2} \int \frac{R}{2\pi r} \left( \frac{1}{2\pi r} \right)^2 = \frac{e^2}{4\pi} \ln \frac{R}{a} \approx \frac{e^2}{8\pi} \ln N
\]

(76)

d_N is the electric field energy stored between two charged rings with radius \( l_M \sim 1/\sqrt{eB} \) and charge \(-1 \) \( (\pi^-) \), and radius \( R \gg l_M \) and charge \(+1 \) \( (\pi^+) \). The Coulomb self-energy is now subleading as \( c_N/d_N \) at large \( N \) and omitted. The pion condensate density that minimizes (75) is the same as (71) with the substitution \( m_0 \to 2m_0 \) and \( c_N \to d_N \).

3. Magnetic back-reaction

To order \( \alpha = e^2/4\pi \), the charged pion condensate at the edge of the volume \( V \) induces a magnetic field that adds to the applied external magnetic field, for both the open and closed case. To assess it, consider the QED part of the charged pion Lagrangian in leading order

\[
\mathcal{L} = \frac{-f_{\text{QED}}^2}{4} + |(d + ie(A + a)) \Pi|^2
\]

(77)
in form notations with \( f = da \). Here \( A \) is the external vector potential for the background magnetic field, and \( a \) is a fluctuation which is 0 in leading order. At next to leading order \( a = a[J^\mu] = a[n \mu] \), with \( J^\mu = \langle n | J^\mu | n \rangle \) the current induced by the pion condensation with

\[
| n \rangle_a = (a_{p=0,n=0,l=0}^L | n_{p=0,n=0,l=0}^L | 0) \\
| n \rangle_b = (a_{p=0,n=0,l=0}^L | n_{p=0,n=0,l=0}^L | 0)
\]

(78)

More details regarding the quantization of free pions at finite \( \Omega, B \) can be found in Appendix XIV. The sub-label \( a \) refers to the closed volume case with charge conservation, while \( b \) refers to the open volume case. For both cases, the induced current is azimuthal

\[
J^\theta | n \rangle = \langle n | j^\theta | n \rangle = \frac{eN}{m_{\pi}} | f_{\text{QED}} |^2 \\
\approx \frac{eN}{2m_0 \pi R^2} \delta(r - R) = \frac{e^2 B n}{4\pi m_0} \delta(r - R)
\]

(79)

with \( f_{\text{QED}} \) the LLL. [79] sources a uniform magnetic field in \( V = SL \) in the z-direction,
which adds to the applied external magnetic field $B \to B + b_z[n]$. We can solve anew the LL problem in the modified magnetic field $B(1 + \frac{e^2 \mathbf{n}}{4\pi m_0})$, which amounts to the following substitutions

$$m_0^2 \to m_0^2[n] = m_0^2 + eB \left(1 + \frac{e^2 \mathbf{n}}{4\pi m_0}\right)$$

$$N \to N[n] = N \left(1 + \frac{e^2 \mathbf{n}}{4\pi m_0}\right)$$

In addition, $(79)$ induces a magnetic energy per unit length in $V$

$$\frac{\mathcal{E}}{2} = \frac{n^2 e^4 B^2 R^2}{32 \pi m_0^2[n]} = \frac{e^3 B N n^2}{16 \pi m_0^2[n]}$$

The Coulomb factors in the back-reacted case are now $c_N = \frac{\epsilon^2}{12\pi\sqrt{N(m)}}$ (open volume) and $d_N = \frac{\epsilon^2}{8\pi}$ (closed volume). With all in mind, the pion energies per unit volume for the closed (a) and open case (b) are respectively

$$\mathcal{E}_{\pi \Omega}[\Omega, n] =$$

$$-(N[n])\Omega - 2m_0[n] + n^2 e^2 \left(\frac{eBN}{16\pi m_0^2[n]} + \frac{\ln N[n]}{8\pi}\right)$$

$$\mathcal{E}_{\pi \Omega}[\Omega, n] =$$

$$-(N[n])\Omega - m_0[n] + n^2 e^2 \left(\frac{eBN}{16\pi m_0^2[n]} + \frac{1}{12\pi \sqrt{N[n]}}\right)$$

We have checked that the dependence of $m_0[n]$ and $N[n]$ on $n$ is rather weak, and the threshold for pion condensation remains the same in both cases.

### D. Shift in the chiral condensate

In leading order in $(eB)/(4\pi F_{\pi}^2)$, the chiral condensate can be extracted from $(67)$ as $\langle \bar{\psi} \psi \rangle_{\Omega, B} = \partial \mathcal{E}_{\pi \Omega} / \partial m$ modulo vacuum renormalization. Using the GOR relation $m_0^2 F_{\pi}^2 = -m \langle \bar{\psi} \psi \rangle_{0,0}$ in the absence of $\Omega, B$, we can trade the derivative with respect to the current mass $m$ with the derivative with respect to the pion mass $m_{\pi}$. For the $\Omega$ independent pion contribution in $(67)$, we explicitly have

$$\frac{\partial \mathcal{E}_{\pi B}}{\partial m} = \langle \bar{\psi} \psi \rangle_{0,0} \left(\frac{eB}{4\pi F_{\pi}^2}\right) \int ds \frac{e^B e^{-sm_0^2}}{s \sinh(EBs)}$$

The corresponding shift in the chiral condensate for $\Omega = 0$ but finite $B$ is

$$\frac{\langle \bar{\psi} \psi \rangle_{B, \Omega}}{\langle \bar{\psi} \psi \rangle_{0,0}} - 1 = \frac{\ln 2}{eB}$$

in agreement with chiral perturbation theory in leading order $[16]$. This linear magnetic catalysis is supported by lattice simulations $[17]$. The quadratic magnetic catalysis in NJL-type models at strong coupling, was initially proposed in $[13]$. A rerun of the same arguments for the $\Omega$ dependent contribution in $(68)$, yields the net shift of the chiral condensate for the open case (no back-reaction)

$$\frac{\langle \bar{\psi} \psi \rangle_{\Omega, B}}{\langle \bar{\psi} \psi \rangle_{0,0}} - 1 =$$

$$\frac{1}{2} \left(\frac{eB}{F_{\pi}} \ln 2 - \frac{3}{e^2 \sqrt{N}} \left(\frac{\Omega \mathcal{E}}{m_0^2 - 1}\right) \theta(N\Omega - m_0)\right)$$

and for the closed case (no back-reaction)

$$\frac{\langle \bar{\psi} \psi \rangle_{\Omega, B}}{\langle \bar{\psi} \psi \rangle_{0,0}} - 1 =$$

$$\frac{eB}{2F_{\pi}} \left(\frac{\ln 2}{8\pi^2} - \frac{4}{e^2 N \ln N} \frac{\Omega \mathcal{E}}{m_0^2 - 1}\right) \theta(N\Omega - 2m_0)$$

in leading order in the pion interaction.

Finally, the back-reacted energy densities $(83)$ can be used to correct $(86-87)$. A rerun of the preceding arguments yield in the closed case with back-reaction

$$\frac{\langle \bar{\psi} \psi \rangle_{B, \Omega}}{\langle \bar{\psi} \psi \rangle_{0,0}} - 1 =$$

$$\frac{eB \ln 2}{16\pi^2 F_{\pi}^2} + \frac{\theta(N\Omega - m_0) B}{NF_{\pi}^2 m_0 e}$$

$$\times \left(\frac{8m_0 - 4N\Omega}{2\ln N + \frac{eBN}{m_0^2}} + \frac{2eBN(2m_0 - N\Omega)^2}{m_0^3(2\ln N + \frac{eBN}{m_0^2})^2}\right)$$

and in the open case with back-reaction

$$\frac{\langle \bar{\psi} \psi \rangle_{B, \Omega}}{\langle \bar{\psi} \psi \rangle_{0,0}} - 1 =$$

$$\frac{eB \ln 2}{16\pi^2 F_{\pi}^2} + \frac{\theta(N\Omega - m_0) B}{NF_{\pi}^2 m_0 e}$$

$$\times \left(\frac{4m_0 - 4N\Omega}{3\sqrt{N} + \frac{eBN}{m_0^2}} + \frac{2eBN(2m_0 - N\Omega)^2}{m_0^3(3\sqrt{N} + \frac{eBN}{m_0^2})^2}\right)$$

The change of the chiral condensate under the combined effects of a magnetic field and a rotation was initially suggested using arguments from random matrix
theory and anomalies \[4, 18\]. It was clarified and detailed in the context of the NJL model in \[6, 7\]. The effect of the rotation is to inhibit the so-called magnetic catalysis as emphasized in \[7\]. Note that all the shifts are of order \( N_c^{-1} \) and would be missed in an effective calculation with constituent quarks such as in the NJL model in the leading loop or \( N_c^0 \) approximation. A critical rotation rotation can compensate the increase induced by the magnetic field. The shifted condensates \[66, 67\] (open volume), \[87\] (closed volume) and \[88, 89\] (back-reaction) when compared to the diffusive result \[64\] fix the ratio of the diffusion constants for the different charge conservation cases, with or without magnetic back-reaction.

VI. PION SUPERFLUID IN HEAVY-ION COLLISIONS

In a heavy ion collision at collider energies, very large angular momenta \( l \sim 10^3 - 10^6 \hbar \) \[21\] and large magnetic fields \( B \sim m^2_\pi \) \[24\] are expected in off central collisions, in the early parts of the collision. Assuming that they persist in the freeze-out part where the constituents are hadrons, i.e. \( R \sim 10 \mathrm{fm} \) with still \( eB \sim m^2_\pi \), this would translates to a LL degeneracy \( N = eBR^2/2 \sim (m_\pi \times 10 \mathrm{fm})^2 \sim 10^7/4 \) and a rotational chemical potential \( \mu_N = N \Omega \sim 1.25 m_\pi \). The pion chemical potentials at freeze-out are \( \mu_f \sim 0.5 m_\pi \) at RHIC, and \( \mu_f \sim 0.70 m_\pi \) at the LHC \[25\]. When combined with the rotationally induced chemical potential, we have \( \mu_x = \mu_N + \mu_f \sim 1.75 m_\pi \) and \( 1.96 m_\pi \) respectively. These chemical potentials may induce charged pion condensation, in the form of a rotating BEC of pions ad the edge of the fire ball. The specifics of this BEC depends on whether the volume \( V \) is open or closed as we now detail.

In the open volume case without magnetic back-reaction, the mean number of condensed \( \pi^+ \) is

\[
N_{\pi^+} = \sum_{n=0}^{\infty} \frac{n e^{n(N + \mu_f - m_\pi)} - n^2}{12 \pi \sqrt{\pi} T L}.
\] (90)

For \( L \sim 10 \mathrm{fm} \), \( eB \sim m^2_\pi \) and \( N \approx 25 \), we show in Fig. \[11\] the average number of condensed \( \pi^+ \) for temperatures in the range \( 0.5 m_\pi \leq T \leq 1.5 m_\pi \) and rotations in the range \( 0.02 m_\pi \leq \Omega \leq 0.5 m_\pi \). As \( \Omega \) exceeds the critical \( \Omega_{\text{min}} \), the number of \( \pi^+ \) increases.

For the closed volume case without magnetic back-reaction, the mean number of condensed \( \pi^\pm \) pions are

\[
N_{\pi^\pm} = \sum_{n=0}^{\infty} \frac{n e^{n(N + \mu_f - 2m_\pi)} - n^2}{12 \pi \sqrt{\pi} T L}.
\] (91)

For \( eB = m^2_\pi \), \( \Omega_c = 2 \sqrt{2} / N \sqrt{eB} \) and \( R \sqrt{eB} = \sqrt{2N} \), so that \( \Omega_c R = \sqrt{N} \). In this case, we must have \( N \geq 16 \)

\[
\text{FIG. 10: The mean number of superfluid pions } N_{\pi^\pm} \text{ in the range } 0.5 m_\pi \leq T \leq 1.5 m_\pi, \mu_f = 0.5 m_\pi \text{ and } 0.03 m_\pi \leq \Omega \leq 0.04 m_\pi.
\]

\[
\text{FIG. 11: The mean number of superfluid pions } N_{\pi^\pm} \text{ in the range } 0.5 m_\pi \leq T \leq 1.5 m_\pi \text{ and } 0.03 m_\pi \leq \Omega \leq 0.04 m_\pi, \text{ for } \mu_f = 0.8 m_\pi.
\]

When the magnetic back-reaction is taken into account for both the closed (a) and open (b) volume case, the mean number of condensed pions is

\[
N_{\pi^\pm, a, b} = \sum_{n=0}^{\infty} \frac{n e^{n(N + \mu_f - 2m_\pi)} - n^2}{12 \pi \sqrt{\pi} T L} - \sum_{n=0}^{\infty} \frac{n e^{n(N + \mu_f - 2m_\pi)} - n^2}{12 \pi \sqrt{\pi} T L}.
\]

with \( \kappa_a = 2 \) (closed volume) and \( \kappa_b = 1 \) (open volume). Below we plot the number of the condensation for \( N = 50, L = 10 \mathrm{fm} \), in the range of \( 0.03 m_\pi \leq \Omega \leq 0.09 m_\pi \).

VII. CONCLUSIONS

We analyzed the combined effects of a rotation and a magnetic field on free and interacting Dirac fermions in 1+2 dimensions. Our results show that the rotation causes massless positive states in the LLL to sink into the Dirac sea, followed by an increase in the density of particles. The scalar density of particles does not change
FIG. 12: The mean number of superfluid pions $N_{\pi}^{\pm}$ in the range $0.5m_{\pi} \leq T \leq 1.5m_{\pi}$ and $0.03m_{\pi} \leq \Omega \leq 0.05m_{\pi}$, for case a (upper) and case b (lower).

in the free case, but is modified in the interacting case. These results strengthen our earlier observation that an increase in the density of composite fermions in the quantum Hall effect at half filling under rotation would signal their Dirac nature [4]. They may also be of relevance to planar condensate matter systems when subject to a parallel rotation plus a magnetic field.

We showed that the mechanism behind the sinking of the LLL for free Dirac fermions, holds in any dimension, leading to a finite increase in the density of particles that is related to anomalies. For QCD in the spontaneously broken phase with Dirac fermions, this mechanism manifests itself in a novel way through the condensation of charged pions. We used this observation to derive the shift in the chiral condensate in leading order in the pion interaction.

On a more speculative way in QCD, the charged separation caused by the dual combination of a rotation parallel to a magnetic field, may impact on the flow of charged particles in semi-central collisions of heavy ions at present collider energies, provided that the magnetic field is still strong in the freeze-out region. While both the rotation and the magnetic field separate charges along the rotational axis as known through the standard chiral vortical and magnetic effect, the combined effect causes them to centrifuge. The resulting charge separation is quadrupolar as opposed to polar with some consequences for the charged particle flow. Also, the possibility of an induced and coherent charge accumulation by rotation in a magnetic field, whether in the form of partons or pions, may affect the fluctuations in the charge and pion number, the transport coefficients such as the viscous coefficients, and potentially the electromagnetic emissivities in the prompt and intermediate part of the collision, especially their distribution and flow in the low mass region. These issues are worth further investigations.

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IX. APPENDIX I: RANGE OF $l$

To better understand the nature of the range in the orbital angular momentum $l$ for each LL, we recall that for $l \geq 0$ the wavefunction is typically of the form

$$z^l e^{-\frac{eBr^2}{4} L_l^l} (93)$$

The requirement that (93) stays within the area $S = \pi R^2$ implies that $l + n < N$, meaning that both $l, n < N$. Conversely, for $l < 0$ the wavefunctions are of the form

$$z^{|l|} e^{-\frac{eBr^2}{4} L_{n-|l|}^{|l|}} (eBr^2/2) (94)$$

which requires $n \leq N$. But for this case, we always have $n \geq -l$. These observations imply that the orbital angular momentum is bracketed with $-n \leq l \leq N - n$. This range of $l$ helps keep the angular shift $\Omega n$ smaller than the magnetic shift $\sqrt{eBr}$ for large $n$. Indeed, this requirement together with the causality bound $\Omega R < 1$, implies that

$$\sqrt{2eBr n} - \Omega |l| \geq \frac{1}{R} (\sqrt{4N^2} - N) \approx \sqrt{NeB} (95)$$

X. APPENDIX II: ALTERNATIVE $\nu_T$

The one-loop finite temperature contribution to the effective potential relates to the scalar condensate through

$$\frac{\partial V_T}{\partial \sigma} = - \int d^2 x \langle \bar{\psi} \psi \rangle |_\beta. (96)$$

Using the quantized fields (8) and the proper time construction, we have
For positive \( l \), the constraint is \( l \leq N - n \), thus the upper bound for \( n \) is \( N - l \). For negative \( l \), we also have \(|l| \leq N \) and \(|l| \leq n \leq N \). Thus, the summation over \( n \) gives for positive \( l \)

\[
\frac{\partial V_T}{\partial \sigma} = -4\sigma \int \frac{d\omega}{2\pi} \sum_{l} f_F(\omega, l)
\]

\[
\times \text{Im} \int_0^\infty idse^{-is(\omega^2-\sigma^2-ir)} \left( \sum_{n,m} (2 - \delta_{n,0}) e^{i2eBs} \right)
\]

(97)

For positive \( l \), the constraint is \( l \leq N - n \), thus the upper bound for \( n \) is \( N - l \). For negative \( l \), we also have \(|l| \leq N \) and \(|l| \leq n \leq N \). Thus, the summation over \( n \) gives for positive \( l \)

\[
\frac{\partial V_T}{\partial \sigma} = -4\sigma \int \frac{d\omega}{2\pi} \sum_{l} f_F(\omega, l)
\]

\[
\times \text{Im} \int_0^\infty idse^{-is(\omega^2-\sigma^2-ir)} \left( \sum_{n,m} (2 - \delta_{n,0}) e^{i2eBs} \right)
\]

(97)

Since we have

\[
f_F(\omega, l) = \frac{\theta(\omega)}{e^{i(\omega-\Omega(l+1/2)-\mu)}} + \frac{-\theta(\omega)}{e^{i(\omega+\Omega(l+1/2)+\mu)}}
\]

(99)

it is clear that \(|f_F| \leq 2 \). Thus the summation of the second term in (98) is of order

\[
1 + e^{2ieBs(N-l)} - 2\frac{e^{2ieBs(N-l)}}{1 - e^{2ieBs}}
\]

(100)

After analytical continuation to the imaginary axis, this contribution vanishes in the thermodynamical limit. The only contribution is to the residue which is \( l \)-independent. For negative \( l \) we have

\[
e^{2iN + Bs} - e^{2iN + Bs}|l|/1 - e^{2ieBs}
\]

(101)

After analytic continuation, neither the residue nor the integrand part survive. With all in mind, the result is now

\[
\frac{\partial V_T}{\partial \sigma} = -4\sigma \int \frac{d\omega}{2\pi} \sum_{l} f_F(\omega, l)
\]

\[
\times \text{Im} \int_0^\infty idse^{-is(\omega^2-\sigma^2-ir)} \left( 1 + e^{2ieBs(N-l)} - 2\frac{e^{2ieBs(N-l)}}{1 - e^{2ieBs}} \right)
\]

(102)

For \( \omega^2 - \sigma^2 \leq 0 \), the analytical continuation of the integrand to the positive imaginary axis yields zero imaginary part. For \( \omega^2 - \sigma^2 \geq 0 \) the analytical continuation of the first and second contributions to the negative and positive real axis respectively, yield adding residues with a net imaginary part. The result is

\[
\frac{\partial V_T}{\partial \sigma} = -4\sigma \int \frac{d\omega}{2\pi} \theta(\omega^2 - \sigma^2) \sum_{l=0}^N f_F(\omega, l)
\]

\[
\times \left( \frac{1}{2} + \sum_{n=1}^\infty \cos \left( \frac{\pi n}{eB} (\sigma^2 - \omega^2) \right) \right)
\]

(103)

which integrates to

\[
V_T = \int d\omega \sum_{l=0}^N f_F(l, \omega) \theta(\omega^2 - \sigma^2)
\]

\[
\times \left( \frac{\omega^2 - \sigma^2}{eB} + \frac{2}{\pi} \sum_{n=1}^\infty \sin \left( \frac{\pi n}{eB} (\omega^2 - \sigma^2) \right) \right)
\]

(104)

Through a change of variable, we can recast each \( l \)-contribution in (104) in the form

\[
\int f_F(l, \omega) \theta(\omega^2 - \sigma^2) \frac{\omega^2 - \sigma^2}{eB} - 2\frac{n + 1}{\pi}
\]

(105)

By partial integration we found that the first term cancels the last term, with only boundary terms left. The final result for the thermal contribution to the effective potential takes the canonical form

\[
V_T = \frac{1}{\beta} \sum_{l=0}^\infty \sum_{n=1}^\infty \sum_{j=1}^{1} \ln(1 + e^{-\beta(E_n - j(\mu + \Omega(l+1/2)))})
\]

(106)

This result is equivalent to [25] in the thermodynamical limit.

**XI. EDGE MODES IN 1+2**

Recently, it was noted in [11] that for a negative fermion mass and when the boundary condition at the luminal radius \( R \) was enforced (for example through an MIT bag boundary condition, see also [22]), there is one imaginary solution to the radial wave number \( k_{\perp} = \sqrt{E^2 - M^2} \) for each angular momentum \( m \) (in the infinite area case \( k_{\perp} = 2eBN \)). These solutions were referred to as edge modes as they peak near the edge in the absence of a magnetic field. For a finite magnetic field, the corresponding wave function reads

\[
e^{-i\frac{\omega^2}{2} \times e^{im\phi}} F_1 \left( -\frac{k_{\perp}^2}{2eB}, m + 1, \frac{eB^2}{2} \right)
\]

(107)
The increasing hypergeometric function $1_{F_1}$ may overcome the pre-factor $e^{-\sqrt{eB}r}$, and become dominant at large $r$. However, for large degeneracies with $N \gg 1$ this does not take place. Indeed, in the parameter range discussed here with $N = 100$ and $M = -\sqrt{eB}$, the edge solution for $m = 0$ reads $k_1^2 \approx -10^{-42} eB$, and for $k_\perp$ this small, the hypergeometric function remains almost constant for all $r$. This edge mode is simply the deeply confined LLL mode $e^{-\sqrt{eB}r}$. For $m = 80$, the edge solution is about $k_1^2 \approx -0.4 eB$. The $1_{F_1}$ function for this value at the edge is about 2 times the value at the origin or $r = 0$, which should be viewed as a moderate enhancement of the LLL wave function with $1_{F_1}$ set to 1. Specifically, $m^m e^{-\frac{\sqrt{eB}}{2}}$ for $m \approx N$ already peaks near the boundary, the edge enhancement by $1_{F_1}$ changes nothing qualitatively. For large $N$, the LLL wave function remains a good approximation for the low lying modes and needs no further amendment. The only effect is that the energy of these edge states become slightly lighter (for the case considered it is 0.8:1), which could turn to a moderate statistical enhancement.

XII. NEGATIVE $\mu$ IN 1+2

The use of a negative potential $\mu$ maybe more than academic in 1+2 dimensions, since effective descriptions of planar condense matter systems are described by the model we presented in the main text using Dirac fermions [14]. In Fig. 13 we show the behavior of the effective potential $V$ as a function of $\sigma$ for $T = 0$ and $\Omega = 0$, but large negative $\mu = -0.031 \sqrt{eB}$, where the gap solution is lost (top). The critical value for which this happens is $\mu_c = -0.025 \sqrt{eB}$. Amusingly, with increasing $\Omega$, the mass gap is recovered at $\Omega_{c_1}$, then lost at $\Omega_{c_2}$. For instance, at $T = 0$ and $\mu = -0.03 \sqrt{eB}$, we have $\Omega_{c_1} = 0.00011 \sqrt{eB}$ and $\Omega_{c_2} = 0.00096 \sqrt{eB}$ as illustrated in Fig. 13 middle and bottom respectively. In Fig. 14 we show the effective mass as a function of $\Omega$ for also $T = 0$ and $\mu = -0.03 \sqrt{eB}$.

XIII. FREE DIRAC FERMION IN 1+3

In 1+3 dimensions, the rotating metric [1] is minimally changed to $ds^2 \rightarrow ds^2 - dz^2$, with the pertinent changes to the co-moving coordinates. In the chiral Dirac basis for the gamma matrices, the rotating LL levels [6] are now changed to

$$(E^\pm + \Omega(m - n + \frac{1}{2})) = \pm \sqrt{p^2 + M^2 + 2eBn} = \pm \tilde{E}$$

(108)

with the corresponding wavefunctions for particles

$$u^T_{nm1} = e^{-iE^+ t + ipz} \frac{1}{\sqrt{2\tilde{E}(\tilde{E} + p)}} \times (M_{nm}, 0, (\tilde{E} + p) f_{nm}, -\sqrt{2eBn} f_{n-1,m})$$

$$u^T_{nm2} = e^{-iE^+ t + ipz} \frac{1}{\sqrt{2\tilde{E}(\tilde{E} + p)}} \times (\sqrt{2eBn} f_{nm}, (\tilde{E} + p) f_{n-1,m}, 0, M f_{n-1,m})$$

(109)
and anti-particles

\[ v_{nm1}^T = e^{-iE^t - ipz} \frac{1}{\sqrt{2\dot{E}(\dot{E} + p)}} \times (M f_{nm, 0} - (\dot{E} + p)f_{nm, 0} - \sqrt{2}Be f_{n-1,m}) \]

\[ v_{nm2}^T = e^{-iE^t - ipz} \frac{1}{\sqrt{2\dot{E}(\dot{E} + p)}} \times (\sqrt{2}Be f_{nm, 0} - (\dot{E} + p)f_{n-1,m}, 0, M f_{n-1,m}) \]  

(110)

The quantized fields are now

\[ \psi(t, \vec{x}) = \int \sum_{nm} \frac{dp}{2\pi} \left( e^{-iE^t + ipz} a_{nm\ell}(x)_a \tilde{a}_{nm\ell}(p) \right. - \left. e^{-iE^t - ipz} v_{nm\ell}(x)_b \tilde{b}_{nm\ell}(p) \right) \]  

(111) 

with the anti-commutation rules

\[ [a_{nm\ell}(p), a_{pq\ell}(p')] = \delta_{np} \delta_{mq} \delta_{ij} 2\pi \delta(p - p') \]  

(112)

**XIV. FREE PION IN 1 + 3**

We now present and explicit derivation of the pion spectrum in a rotating frame for infinite volume. The rotating metric is the same as for the Dirac fermions in 1+3 dimensions. The co-moving Dirac frame is defined similarly with \( e_a = e^\mu_a \partial_\mu \) and \( (e_0, \vec{e}) = (\partial_0 + y \Omega \partial_y - x \Omega \partial_x, \nabla) \). In the rest frame, the circular vector potential reads \( A_R = -\frac{Br^2}{2} \partial_\theta \) in form notation. Using the coordinate transform to the rotating frame \( t_M = t, \theta_M = \theta + \Omega t \) yields

\[ A = -\frac{Br^2}{2} \partial \theta - \frac{\Omega Br^2}{2} dt \]  

(113)

In the rotating frame there is in addition to the magnetic field \( B\hat{z} \), an induced electric field \( \vec{E} = \Omega \vec{B} \). This is expected from a Lorentz transformation from the fixed frame with \( B\hat{z} \) to the co-moving frame \( B\hat{z} \) and \( \vec{E} = \Omega \vec{B} \).

In the rotating frame, a charged scalar is described by the Lagrangian

\[ \mathcal{L} = |(D_t + y\Omega D_x - x\Omega D_y)\Pi|^2 - |D_i\Pi|^2 - m^2\Pi^2 \]  

(114) 

with the long derivative \( D = \partial + ieA \). The electric field drops out in \[113\], thanks to the identity

\[ D_t + y\Omega D_x - x\Omega D_y = \partial_t + y\Omega \partial_x - x\Omega \partial_y \]  

(115)

The co-moving frame corresponds only to a frame change with no new force expected. In the rotating frame, the charged field satisfies

\[ -(\partial_t + y\Omega \partial_x - x\Omega \partial_y)^2 \Pi - D^t D_x \Pi + m^2 \Pi = 0 \]  

(116)

In the infinite volume case, we solve \[116\] using the ladder operators

\[ a = \frac{i}{\sqrt{2\pi}B} (D_x + iD_y) \]

\[ a^\dagger = \frac{i}{\sqrt{2\pi}B} (D_x - iD_y) \]

\[ b = \frac{1}{\sqrt{2\pi}B} (2\partial + \frac{eB}{2} \partial^z) \]

\[ b^\dagger = \frac{1}{\sqrt{2\pi}B} (-2\partial + \frac{eB}{2} \partial^z) \]  

(117)

Hence, the identities

\[ D^t_x D_x + D^t_y D_y = eB(2a^\dagger a + 1) \]

\[ L_z = i(-x\partial_y + y\partial_x) = b^\dagger b - a^\dagger a \]  

(118)

The general stationary solution to \[116\] is of the form \( \Pi = e^{ip_z - iEt} f \) with \( f \) solving

\[ (E + \Omega L_z)^2 f = (m^2 + p^2) f + eB(2a^\dagger a + 1) f \]  

(119)

The normalizable solutions form a tower of LL of the form

\[ f_{mn} = \frac{1}{\sqrt{m!n!}} (a^\dagger)^n (b^\dagger)^m f_{00} \]

\[ (E_{mn} + \Omega(m - n))^2 = eB(2n + 1) + m_n^2 \]  

(120)

with \( f_{00} \sim e^{-\frac{eB}{2}(x^2 + y^2)} \) as the LLL. Therefore, the quantized charged field \( \Pi \) in the rotating frame takes the form

\[ \Pi = \int \frac{dp}{2\pi} \sum_{nm} f_{mn} \sqrt{2E_n} (a_{nm\ell} e^{-iE^t + ipz} + b_{nm\ell}^\dagger e^{iE^t - ipz}) \]  

(121)

with the bosonic canonical rules

\[ [b_{nm\ell}, b^\dagger_{nm',m',\ell'}] = [a_{nm\ell}, a^\dagger_{nm',m',\ell'}] = 2\pi \delta_{nn'} \delta_{mm'} \delta(p - p') \]  

(122)

\( a_{nm\ell}^\dagger \) creates a \( \pi^+ \) with energy \( E_t^+ = E_n - \Omega(m - n) \), charge +e and \( l = m - n \). \( b_{nm\ell}^\dagger \) creates a \( \pi^- \) with energy \( E_t^+ = E_n + \Omega(m - n) \), charge -e and \( l = -m + n \).
Hence, the relation between the rotating frame and the rest frame energies are $E^\text{rotating} = E^\text{rest} - \Omega L_z$ with $L_z = j l$, $l = m - n$. In particular, $j = +1$ for $\pi^+$ (particle) and $j = -1$ for $\pi^-$ (anti-particle) as in \([66]\). For completeness, the solutions to the Klein-Gordon equation can be found in \([23]\).