Modelling and Methods of Structural Analysis

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About applicability of semi-Markov models of operation

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Abstract. The methodology is proposed for assessing the applicability of stationary semi-Markov models for the problems of efficient operation of control and measuring equipment. The methodology is based on the Euler method of construction a fundamental system of solutions for a system of ordinary differential equations with constant coefficients. The methodology is founded on the calculation of all eigenvalues (spectrum) and eigenvectors of the matrix of the system of differential equations. It is shown that for the classical model of operation of measuring equipment for a typical range of variation of the main parameters of the model (probabilities of failure, false failure and undetected failure), the characteristic equation of the system has two invariant eigenvalues and four eigenvalues that can be considered us functions of the model parameters. A qualitative analysis of the dependence of the spectrum of the matrix on the parameters of the model is carried out. The dependence of the non-invariant real eigenvalue, which plays a key role in the convergence of the solution of the dynamic model to the solution of the stationary semi-Markov model, on the parameters of the model is investigated in detail. The results of mathematical simulation are presented.

Keywords: semi-Markov model of operation, stationary solution, eigenvalues, convergence.

1. Introduction

Markov and semi-Markov models are widely used to solve modeling problems and increase the operational efficiency of control and measuring equipment (CME) used to equip machine tools in numerical control and robotic production systems [1-4], aircrafts [5,6], technical systems and instruments, applied in the field of construction and housing and communal services [7]. Semi-Markov models for evaluating the effectiveness of the use of metrological support tools for CME are described in [8-15]. Most of these works are based on stationary semi-Markov models of operation and are aimed at assessing the readiness of an object for intended use. In [16] the new approach to the construction of semi-Markov models of operation is described, a solution to the problem of evaluating the impact of metrological support on the achievement of many goals of metrological support for the operation of organizational and technical systems was developed. In [17] the results of applying the finite element method to the problems of assessing the quality of metrological support for the operation of CME are described. In [8-9] the problems are investigated within the framework of the stationarity hypothesis. However, in all these papers there is no justification for the correctness of using the stationary model.

In this paper, we propose a methodology for assessing the applicability of stationary semi-Markov models based on the application of the Euler method [18,19] for solving systems of linear differential equations with constant coefficients. The results of applying the methodology for studying the classical model of operation of the CME [8,9] are presented.
2. Mathematical model of the operation of CME
Let us denote \( \{E_i, i = 0, 2, ..., n\} \) - a finite set of states in which the sample of CME sample can be.
Possible conditions:
- \( E_0 \) - operational state,
- \( E_1 \) - failure state,
- \( E_2 \) - verification of a failed CME sample state,
- \( E_3 \) - restoration state,
- \( E_4 \) - verification of a functional CME sample state,
- \( E_5 \) - undetected failure state.
The state transition graph is shown in figure 1.

![Figure 1. State transition graph.](image)

The transition probability matrix \( P^* = \begin{pmatrix} P_{ij} \end{pmatrix} \) has the following form:

\[
P^* = \begin{pmatrix}
0 & F(T_k) & 0 & 0 & 1 - F(T_k) & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 - \beta & 0 & \beta \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 - \alpha & 0 & 0 & \alpha & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix},
\]

Where \( F(\tau) \) is the integral function of the distribution of the failure time, \( F(T_k) \) is the probability of failure for the time between two checks, \( T_k \) is the time interval between two successive checks of the technical condition, \( \alpha \) is the conditional probability of a false failure, \( \beta \) is the conditional probability of an undetected failure.

It is assumed that the duration of control (checking the technical condition) and the duration of restoration (repair) are deterministic values equal to \( t_k \) and \( t_b \) respectively.

Let us consider a dynamic semi-Markov model.

\[
\begin{align*}
d\pi_0 / dt &= \pi_3 + (1 - \alpha)\pi_4 - \pi_0 \\
d\pi_1 / dt &= F(T_k)\pi_0 - \pi_1 \\
d\pi_2 / dt &= \pi_1 + \pi_5 - \pi_2 \\
d\pi_3 / dt &= (1 - \beta)\pi_2 + \alpha\pi_4 - \pi_3 \\
d\pi_4 / dt &= [1 - F(T_k)]\pi_0 - \pi_4 \\
d\pi_5 / dt &= \beta\pi_2 - \pi_5
\end{align*}
\]

It is the system of ordinary differential linear equations with constant coefficients.
The stationary solution of the semi-Markov model of operation has the form [8,9]:

\[
\begin{align*}
\pi_0 &= \frac{1}{A} (1 - \beta) \\
\pi_1 &= \frac{1}{A} F(T_i) (1 - \beta) \\
\pi_2 &= \frac{1}{A} F(T_i) \\
\pi_3 &= \frac{1}{A} \left\{ F(T_i) + \alpha (1 - F(T_i)) (1 - \beta) \right\} \\
\pi_4 &= \frac{1}{A} \left[ 1 - F(T_i) \right] (1 - \beta) \\
\pi_5 &= \frac{1}{A} \beta F(T_i)
\end{align*}
\]

Subject to:

\[
A = 2 \left[ 1 - \beta + F(T_i) \right] + \alpha \left[ 1 - F(T_i) \right] (1 - \beta).
\]

In [13] based on the stationary solution of the semi-Markov model, an estimate is given of the system availability coefficient \( \Gamma_K \) for four theoretical distribution laws: exponential, Rayleigh, Weibull, and truncated normal, which most closely describe the statistical distribution function of ST failures.

The parameters of the operation model are \( \alpha, \beta, F(T_i) \). Upper we will show that the time of the reach of the solution of system (2) to some small neighborhood of the stationary solution (1) depends on the specific values of these parameters and describe the methodology for estimating the time of the reach of the solution to the specified neighborhood.

### 3. Method

Let us consider the linear homogeneous system of differential equations [18,19] with constant coefficients in matrix form:

\[
\mathbf{Y}' = \mathbf{A} \mathbf{Y}
\]

The methodology is based on the Euler method of constructing a fundamental system of solutions for a system of ordinary differential equations with constant coefficients based on the calculation of all eigenvalues (spectrum) and eigenvectors of the matrix of the system.

We find the eigenvalues of the matrix \( \mathbf{A} \) from the equation \( \det (\mathbf{A} - \lambda \mathbf{I}) = 0 \), and the corresponding eigenvectors \( \mathbf{D} \) from the equation: \( (\mathbf{A} - \lambda \mathbf{I}) \cdot \mathbf{D} = 0 \).

The matrix \( \mathbf{A} \) has \( n \) eigenvalues, among which there can be real and complex ones, including multiple ones. The case of multiple roots is not considered in this article, since it does not occur in practical problems [8–15].

Thus, the solution of (3) have to satisfy the initial conditions

\[
\mathbf{Y}(0) = \mathbf{Y}_0.
\]

The solution of (3)-(4) can be represented as the sum of exponential terms \( \mathbf{Y}_{exp} \), damped oscillations \( \mathbf{Y}_{dum} \), and a stationary solution \( \mathbf{Y}_{st} \):

\[
\mathbf{Y} = \mathbf{Y}_{exp} + \mathbf{Y}_{dum} + \mathbf{Y}_{st}.
\]

For a finite nontrivial stationary solution \( \mathbf{Y}_{st} = (y_0^{st}, y_1^{st}, ..., y_n^{st}) \) to exist, it is necessary that:

A) there was one zero eigenvalue of the matrix \( \mathbf{A} \), and all other real eigenvalues were negative, 
B) each pair of complex conjugate eigenvalues had a negative real part.

In this case, as the argument \( x \) increases, any particular solution corresponding to a negative eigenvalue or an eigenvalue with a negative real part will tend to a stationary solution generated by a zero eigenvalue. The rate of convergence to the stationary solution depends on the magnitude of the
negative real part of the eigenvalue. The smaller the eigenvalue, the higher the rate of convergence. The larger the eigenvalue, the lower the convergence rate.

The $\varepsilon$ - percentage neighborhood of a stationary solution $\{y_i^\varepsilon, i = 1,..., n\}$ is the set of values $y_i, i = 1,..., n$ such that $|y_i - y_i^\varepsilon| < \varepsilon \cdot y_i^\varepsilon / 100$, $\forall i = 1,..., n$. We will assume that at the initial moment of time the solution of the system is outside the $\varepsilon$ - percentage neighborhood of the stationary solution. The intersection point of the solution to system (3) of the boundary $\varepsilon$ — the percentage neighborhood of the stationary solution depends on the eigenvalues of the matrix and the initial conditions.

In case of the real root of the characteristic equation, each component of the solution of the semi-Markov model has the form: $y_{ji} = C_{ji} \cdot d_{ji} \cdot \exp(-\lambda_j x)$ . In the case of a pair of complex conjugate roots, each pair of components of the solution has the form $y_{ji} = C_{ji} \cdot d_{ji} \cdot \exp(\alpha_j x) \cdot \cos \beta_j x$ and $y_{ji+1} = C_{ji+1} \cdot d_{ji+1} \cdot \exp(\alpha_j x) \cdot \sin \beta_j x$.

Let us determine the value of the argument at which the solution of system (3) falls into the $\varepsilon$ - percentage neighborhood of the stationary solution. To do this, we first evaluate the value of the argument at which the stationary solution of each component of the solution falls into the $\varepsilon$ - percent neighborhood of $y_i$ in the case of the real roots of the characteristic equation.

For each function, we calculate the estimate of the argument at which this function reaches the corresponding $\varepsilon$ - percentage of the stationary solution $y_i^\varepsilon$.

To do this, we first calculate the estimate of the argument $\bar{x}_{ji}$ at which each component of the solution $y_{ji}$ reaches a $\varepsilon / n$ - percentage neighborhood of the stationary solution $y_i^\varepsilon$:

$$\left| C_{ji} \cdot d_{ji} \right| \cdot \exp(\lambda_j x_{ji}) \leq \frac{\varepsilon \cdot y_i^\varepsilon}{n}.$$ Solving the inequality, we get:

$$x_{ji} \geq \bar{x}_{ji} = \frac{1}{\lambda_j} \ln \left( \frac{\varepsilon \cdot y_i^\varepsilon}{n \cdot C_{ji} \cdot d_{ji}} \right).$$

Having chosen the maximum value of the argument at which each component of the solution reaches a $\varepsilon / n$ - percentage neighborhood

$$\bar{x}_i^* = \max_j \bar{x}_{ji},$$

we get the value of the argument $\tilde{x}_i^*$ at which $y_i(x)$ it reaches its $\varepsilon$ - percentage neighborhood of the stationary solution.

Having chosen $\tilde{x}_i^* = \max_j \bar{x}_i^*$, we find the value of the argument at which each of the functions $y_i(x)$ will obviously be inside its own $\varepsilon$ - percentage neighborhood of the stationary solution $y_i^\varepsilon$.

Note that for an exponential function with a negative exponent, the following property is fulfilled: the smaller (the greater the absolute value) the exponent, the faster the exponential function will tend to zero, and the greater the exponent, the slower will be zero. Therefore, to study (analyze) the convergence of the solution of a dynamic system to a stationary solution, the key role is played by nonzero maximum eigenvalues of the system matrix that are nonzero in magnitude (minimum in absolute value).

Note that for the case of a pair of complex conjugate eigenvalues, the functions $\sin x$ and $\cos x$ entering into the corresponding solution should be replaced by their boundary values equal to unity, and the output to the boundary, the $\varepsilon$ - percentage neighborhood, should be estimated as described above.
4. Results

The calculations were carried out with the following parameters: \( \alpha = 0.1, \beta = 0.1, \lambda = 0.0025, \) \( F(T_K) = 0.9. \) The spectrum (set of eigenvalues) of the matrix: are \( \lambda_0 = 0, \lambda_1 = -1, \lambda_2 = -1.9851, \lambda_3 = -0.978, \lambda_{4,5} = -1.0184 \pm 0.6448i. \) These numerical values correspond to a typical model of interaction between metrological support tools and special control and measuring equipment [8,9].

The characteristic polynomial is a polynomial of sixth degree. It has two invariant eigenvalues \( \lambda_0 = 0, \lambda_1 = -1 \) that are independent on model parameters, and four eigenvalues depending on model parameters. The presence of a zero eigenvalue indicates the existence of a stationary solution, which we denote \( \pi^{\alpha}=(\pi^0_0, \pi^0_1, \ldots, \pi^0_i). \) The presence of a pair of complex conjugate eigenvalues suggests that damping oscillations of variables \( \pi_i \) are possible in the process of transition to a stationary solution. The calculations showed that the real part of the pair of complex conjugate roots is less than -1 for any values of the model parameters. A pair of complex conjugate eigenvalues from figure 2 is also not visible. Given two invariant eigenvalues, the characteristic polynomial can be represented in the form

\[
P(\lambda) = \lambda(\lambda+1)P_4(\lambda),
\]

where

\[
P_4(\lambda) = \alpha_1 \lambda^4 + \alpha_2 \lambda^3 + \alpha_3 \lambda^2 + \alpha_4 \lambda + \alpha_5,
\]

\[
\alpha_1 = 10 - (\beta + (1-\alpha)(1-F)), \quad \alpha_2 = 10 \beta + (1-\alpha)(1-F)), \quad \alpha_3 = \alpha(1-F),
\]

\[
\alpha_4 = 5 - 3(\beta + (1-\alpha)(1-F)), \quad \alpha_5 = 2(1-F) + (1-\alpha)(1-F) \beta - F(1-\beta).
\]

A general view of the graph of the characteristic polynomial is presented in figure 2.

![Figure 2. General view of the characteristic polynomial \( P(\lambda) \).](image)

Note that the eigenvalue \( \lambda_3 = -0.978 \), which is the maximum other than zero, is practically not visible on the graph due to the small scale. The location and dependence of this eigenvalue on the model parameters will be illustrated in figure 3. The figures show the dependences of the characteristic polynomial values in the vicinity of the eigenvalue \( \lambda = 1 \) on the probabilities \( \alpha \) - false and \( \beta \) - of undetected failures, as well as on the probability of failure in time \( T_K \).
Figure 3. Dependence of the characteristic polynomial on $\alpha$ in the vicinity of the eigenvalue $\lambda = 1$.

It takes to notice that the eigenvalue value $\lambda_j$ substantially depends on the model parameters. The parameter $F(T_k)$ has the greatest influence on the value $\lambda_j$.

In accordance with the methodology presented in the article, the $\varepsilon$ - percentage neighborhood of the stationary solution $\pi_i(t), i = 0,1,...,5$, we call the set of values such that:

$$|\pi_i(t) - \pi_i^{\mu}| < \varepsilon / 100, \quad \forall \ i = 0,1,...,5.$$ 

The maximum value of the time at which each component of the solution reaches a $\varepsilon/6$ - percentage neighborhood:

$$\tilde{t}_i^* = \max_j \tilde{t}_j \geq \frac{1}{\lambda_j} \ln \left| \frac{\varepsilon \cdot \pi_i^{\mu}}{6 \cdot C_j \cdot d_j} \right|.$$ 

Having chosen $\tilde{t}^* = \max_i \{\tilde{t}_i^*\}$, we will find the value of the moment in time at which each of the functions $\pi_i(t)$ will obviously be inside its own $\varepsilon$ - percentage neighborhood of the stationary solution.

5. Discussion

The novelty of the work lies in the fact that at the first time qualitative analysis of the classical model of operating of CME was made. The developed methodology has sufficient universality. It can be used to construction of estimates for models described by arbitrary systems of linear differential equations with both constant and piecewise constant coefficients. The methodology presented in the article is applicable for constructing estimates of convergence of technical systems and estimates of convergence of mathematical models of arbitrary order, including models of operating of control and measuring equipment used in the field of construction and housing and communal services.

Radio engineering, electronics, robot-technics, intelligent systems [20-24] are just some of the areas where the developed methodology can be effectively used. The developed methodology makes it possible, depending on the model parameters (probability of failure and probability of false failure) and typical modes of model functioning to assess the errors that occur when using the stationary model instead of the dynamic one.

Based on the analysis of results of the assessment, a decision may be made about the possibility (expediency) of applying the stationarity hypothesis to solve a particular practical problem.
The presented results of calculations confirm the validity of using stationary semi-Markov models in [8,9], instead of dynamic models.

6. Conclusion
The main results of the work are as follows:
1. A qualitative analysis of the influence of the model parameters on the eigenvalues of the matrix of the system of equations is carried out. Explicit analytical expressions are constructed for calculating the coefficients of the characteristic polynomial. The influence of the model parameters on the spectrum of the matrix is studied.
2. The methodology for assessing the applicability of stationary semi-Markov models of operation has been developed. To estimate the proximity of the solution of a dynamic and stationary problem, the concept of a \(\varepsilon\) - neighborhood of a stationary solution is proposed. The methodology allows to estimate the time required for reaching \(\varepsilon\) - neighborhood of a stationary solution for the models with an arbitrary finite number of states.
3. By means of the developed methodology, the time to reach a stationary solution for the classical model of operation of CME was estimated. It is shown that for typical values of the initial data parameters, the time to reach a stationary solution is 12-18% of the time between CME failures.

References
[1] Arbuzov V I, Mrochek J A, Panov A N and Harton V L 2001 Fundamentals of the quality management system of a machine-building enterprise. Moscow. Knowledge publishing house.
[2] Vishnyakov B and Egorov A 2013 The construction of confidence areas for the trajectories of motion of objects in machine vision problems. J. Theory and control systems, Vol. 3. pp.124-132.
[3] Gao W, Haitjema H, Fang F, Leach R, Cheung C, Savio E and Linares J 2019 J. CIRP Annals. On-machine and in-process surface metrology for precision manufacturing. Vol. 68, Iss. 2.
[4] Kornev A S 2016 Issues of metrological support of robotic weapons and military equipment. Proceedings of the first military scientific conference "Robotization of the Armed Forces of the Russian Federation". pp.111-113.
[5] Daletsky S V, Derkach O Ya and Petrov A N 2002 Efficiency of the technical operation of civil aircraft. Moscow. Air transport Pub.
[6] Volkov L I 1981 Management of the operation of aircraft. Moscow. High School Pub 368p.
[7] Khayrullin R Z and Ivanov P S 2018 Step-by-step control of target efficiency indices of the control and measuring equipment stock applied in construction and housing and communal services. 2-d International Conference of Material Science and nanotechnology. MATEC Web of Conferences. Vol. 170, 01010 5p.
[8] Sychev E B, Khramenkov V N and Shkitin A D 1993 Fundamentals of metrology of military equipment. Moscow. Military publishing house. 400p.
[9] Korolik V S, Turbin A F 1976 Semi-Markov processes and their applications. Kiev. Naukova Dumka Pub. 236p.
[10] Lavrik E, Frankenfeld U, Mehta S, Panasenko I and Schmidt H 2019 High-precision contactless optical 3D-metrology of silicon sensors. J. Nuclear Instruments and Methods in Physics Research. Vol. 935, pp.167-172.
[11] Francisco S, Guzmán J, Rosa B, Rodriguez C, Doimeadlos M and Ángel R 2019 Analytical metrology for nanomaterials: Present achievements and future challenges. J. Analytica Chimica Acta, Vol. 1059, pp.1-15.
[12] Khayrullin R Z and Safonov A A 2017 Semi-Markov model of operation of radio measuring equipment with metrological support. J. Scientific Review, №. 19. pp.167-170.
[13] Ershov D S, Smagin VA and Sysoev D O 2018 The statement of the problem of substantiating the optimal nomenclature and rational amount of calibration equipment of the metrological units of military units. J. Modern science: Actual problems of theory and practice. Series “Natural and Technical Sciences”, № 4, pp.31-34.

[14] Ershov D S 2018 Methodology for substantiating the optimal nomenclature and quantity of calibration equipment of metrological units of operational-tactical military formations, taking into account the predicted losses of military measuring instruments. J. Natural and technical sciences, № 4, pp.214-216.

[15] Ershov D S, Babak A V and Sysoev D O 2018 A mathematical model of the process of verification of military measuring instruments by metrological units of operational-tactical military formations in the course of hostilities. J. Natural and technical sciences, № 4, pp.217-220.

[16] Khayrullin R Z and Popenkov A Ya 2018 Distribution of control volumes according to the goals of metrological support of complex organizational and technical systems using semi-Markov models. The XI - International Conference "Management of the development of large-scale systems". 5p.

[17] Popenkov A Ya and Khayrullin R Z 2018 Application of the finite element method in the tasks of assessing the quality of metrological support for the operation of weapons, military equipment and military facilities. The XII - Russian Scientific and Technical Conference "Metrological Support of Defense and Security in the Russian Federation". Moscow. Povedniki. 5p.

[18] Khayrullin R Z and Nikitina I A 2019 To the study of semi-Markov models of operation of special equipment. J. Dynamics of complex systems - XXI century. Vol. 13, № 1, pp.5-12.

[19] Stepanov V V 2004 Differential equations. Moscow. Scince Pub. 472p.

[20] Yochan K and Jinkyun P 2019 Incorporating prior knowledge with simulation data to estimate PSF multipliers using Bayesian logistic regression. J. Reliability Engineering & System Safety. Vol.189, pp.210-217.

[21] Vishnyakov B V and Egorov A I 2013 The construction of confidence areas for the trajectories of motion of objects in machine vision problems. J. Theory and control systems. Vol. 3, pp.124-132

[22] Geweke J and Durham G 2019 Sequentially adaptive Bayesian learning algorithms for inference and optimization. J. Econometrics. Vol. 210, Iss. 1, pp.4-25.

[23] Michael D, Thomas A and Adcock A 2018 Prediction of tidal currents using Bayesian machine learning. J. Ocean Engineering. Vol. 158, pp.221-231.

[24] Wang L, Zhao L, Yu L, Wang J and Bi G 2019 Structured Bayesian learning for recovery of clustered sparse signal. J. Signal Processing. Vol. 166, 107255.