Modified gravity in the interior of population II stars

Shaswata Chowdhury and Tapobrata Sarkar

Department of Physics, Indian Institute of Technology, Kanpur 208016, India

E-mail: shaswata@iitk.ac.in, tapo@iitk.ac.in

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Abstract. We study the effects of a beyond-Horndeski theory of modified gravity in the interior of a population II star. We consider a simple phenomenological model of a 1.1$M_{\odot}$ star that has left the main sequence, has a thin Hydrogen burning shell with a partially degenerate isothermal core, surrounded by a radiative envelope having two regions of distinct opacities. Using suitable matching conditions at the two internal boundaries, a numerical analysis of the resulting stellar equations in modified gravity is carried out. While overall, gravity may be weakened, resulting in a decrease of the luminosity and an increase of the radius of the star, some of these effects are reversed near the core. It is suggested how the model, within its limitations, can yield a bound on the modified gravity parameter.

Keywords: gravity, modified gravity

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Introduction

General relativity (GR), formulated by Einstein more than a century ago, is the most successful theory of gravity, and has been validated by several precision tests. More recently, issues relating to the observed cosmic acceleration and the cosmological constant seem to indicate the necessity of modifications to GR, where one adds extra degrees of freedom to the Einstein-Hilbert action of GR. Such theories of “modified gravity” (see, e.g. the excellent reviews of [1–4]), are becoming increasingly popular in the literature. Some of the best studied avatars of modified gravity are the scalar-tensor theories (STTs). In these scenarios, the compatibility of modifications to GR effects in solar system tests necessitate invoking some kind of screening mechanism, the most efficient being the Vainshtein mechanism [5] (see, e.g. [6, 7] for reviews). Here, GR is recovered in the solar system scale, via a non-linear screening of modified gravity.

The most general physical (i.e., ghost-free) theories of a scalar field coupled to gravity constitute the so called Horndeski theories [8]. Recall that, according to Lovelock’s theorem, Einstein’s equations constitute the unique set of second order equations that is obtained from a Lagrangian that depends on the metric, and its first and second order derivatives. A minimal additional degree of freedom alluded to above, involves a scalar field, called a Galileon. This was constructed in the work of [9]. Its covariant version and a further generalization was derived in [10, 11], and the latter was shown in [12] to be equivalent to Horndeski theory. Closely related to these are the Gleyzes-Langlois-Piazza-Vernizzi (GLPV) theories [13, 14] which extend the generalized Galileon theory to the “beyond Horndeski” class, which are physical, i.e., free from ghost modes. The degenerate higher order scalar tensor theories beyond Horndeski, in which the GLPV models arise as special cases, arose in the works of [15–17]. This was followed by the important work of Kobayashi, Watanabe and Yamauchi [18], who demonstrated that in such beyond-Horndeski theories of gravity, the Vainshtein mechanism might be partially effective, i.e., inside a stellar object, the effect of modified gravity is not fully screened. This was further studied in [19, 20], in the wake of the recent discovery of gravitational waves (GW170817) and stringent restrictions were put on the allowed theories (see also [21–24]).
A remarkable consequence of the partial breaking of the Vainshtein mechanism inside stellar objects is that in the low energy (Newtonian) limit, the pressure balance equation inside astrophysical objects is modified. For the beyond-Horndeski theories that we consider here, the modification of the pressure balance equation occurs via an additive term involving a dimensionless parameter \( \Upsilon \) (defined subsequently in subsection 2.2) which represents the effect of the extra degrees of freedom, and renormalizes the Newton’s constant. The pressure balance equation is a crucial ingredient in the derivation of analytical formulae corresponding to astrophysical observables that can be experimentally verified. Hence, one is immediately led to the conclusion that this class of modified gravity theories can be constrained by experimental data. Indeed, a number of works in this direction have been reported in the last few years, starting from the pioneering work of Koyama and Sakstein [25] (see, e.g., [26]–[35]). For a recent review of the effects of modification of gravity in stellar objects, see [36, 37].

We note here that some early works on the screening mechanism in modified gravity theories that contain an extra scalar degree of freedom appear in the works of Chang and Hui [38] and Davis et al. [39]. These were in the context of the chameleon mechanism discussed by Khoury and Weltman [40, 41], and stellar evolution from the main sequence to the red giant phase was considered. More recently in [27], Sakstein constrained the parameter of beyond Horndeski theories by considering the minimum mass of Hydrogen burning in brown dwarf stars. A similar constraint appears in the work of Crisostomi et al. [42], where the authors also considered possible constraints from the Chandrasekhar mass limit of white dwarfs, and the mass radius relation of such stars. Saltas et al. also obtained a strong constraint on beyond Horndeski theories of gravity, using helioseismology in [34]. Olmo et al. [43] considered brown dwarfs in an \( f(R) \) theory of gravity, in which one takes a modified action, with curvature dependent terms added to the Einstein-Hilbert one. From the astrophysical properties of brown dwarfs, they obtained a constraint on these theories. In [44], Wojnar studied lithium abundance in the stellar model of a low mass star in an \( f(R) \) gravity scenario. In [45], Wojnar also studied evolutionary tracks of low mass stellar objects in these theories.\(^1\) For more details, we refer the reader to the recent review by Olmo et al. [37] and references therein.

The results obtained from the works mentioned above confirm that beyond Horndeski theories broadly result in a weakening of gravity inside stellar objects if \( \Upsilon > 0 \). Similarly, gravity is strengthened if \( \Upsilon < 0 \) (see, e.g., [25, 27, 28, 32]). This is true at least far from the stellar core, as follows from the modified pressure balance equation, when the density of the stellar object falls away from the center. It is precisely due to this reason that main sequence stars in modified gravity, for example, are “dimmer and cooler” [25] than what one would normally expect from a GR analysis of the same star.

Most of the astrophysical studies of beyond Horndeski theories that have been carried out till date consider situations in which the entire stellar object is either in radiative or convective equilibrium. These can then be approximated as polytropes. Then, one usually assumes a suitable polytropic equation of state in studying the effects of the modification of the pressure balance equation. The situation however becomes more interesting and involved, in the presence of a distinct core-envelope structure, with a thin intermediate Hydrogen burning shell. It is well known that such a scenario might be difficult to deal with analytically — there are a large number of equations that one has to handle, even in a simplified model. In this paper, we carry out a detailed analysis of such a case, with a phenomenological model.

\(^1\)Literature on constraining theories of gravity via dark matter physics appears in [46], see also [47] and references therein.
proposed long back by Hoyle and Schwarzschild [48] (HS). Our results indicate that while overall the “dimmer and cooler” picture of [25] is indeed true, there is rich physics in the stellar interior in the presence of modified gravity. In particular, we find that the core radius of the star decreases, and its shell density increases with $\Upsilon$, while the core temperature decreases — results which might seem counter-intuitive given that increasing $\Upsilon$ weakens gravity, but we will explain why this is true. We also show how possible bounds on $\Upsilon$ can be given in such models.

We remind the reader at the outset that we are not constructing evolutionary tracks of stars here, a topic already studied earlier (there are several publicly available state of the art codes that accomplishes this). Neither are we attempting to construct an exact model of a stellar interior in modified gravity. Our study will be purely phenomenological, and based on the approximate model that we consider, and our aim is to obtain an analytic handle on the stellar structure equations in modified gravity scenarios.

This paper is organized as follows. In the next section 2, we will formulate the problem, by first elaborating on our model, reviewing the necessary equations in the Newtonian limit of GR, and then setting up the modified equations in beyond Horndeski theories. Section 3 contains a detailed exposition to our numerical analysis and results. The paper ends with discussions and conclusions in section 4.

2 The model and the setup

We now present our model and the modified gravity setup. This section has four parts. We first elaborate upon the model, and the various approximations used, in section 2.1. Then, we review the modified gravity setup in 2.2. in section 2.3, we write the stellar equations in the Newtonian formalism. Using these, in section 2.4, we derive the ones that will be used for our numerical analysis in the modified gravity scenario.

2.1 The model

Following HS, we model the interior of a $1.1M_\odot$ star that has left the main sequence, and study the new features that modified gravity predicts. The original model of HS sets the metallicity of the star to zero, in order to simplify the analytical equations, and this is what we will also assume here. We will thus look for effects of modified gravity in metal-poor population II (pop-II) stars, in globular clusters, which have left the main sequence. Population III stars which are even poorer in metals will not be of much relevance in the absence of data as of now.

As we will see in subsection 2.3, the metallicity $Z$ enters via the Kramer’s formula for the opacity $\kappa$ for free-free transitions, namely,

$$\kappa = 3.68 \times 10^{22} (1 - Z) (1 + X) \frac{\rho}{T^{3.5}} \text{ cm}^2 \text{ gm}^{-1},$$

with $X$ the hydrogen fraction, $\rho$ the density (in gm cm$^{-3}$) and $T$ the temperature in Kelvins.

The approximation that we use is $1 - Z \simeq 1$. This is a toy model, as we are ignoring the effects of metallicity. Indeed, elaborate fitting formulae for stellar evolution with small metallicity are well known (see, e.g. [56, 57]). However, as we have mentioned, our main interest here is to analytically capture the effects of modified gravity inside a star, rather
than a detailed study of its evolution. The assumption of zero metallicity, and the other simplifications on the internal stellar structure that we use (to be elaborated shortly) do not affect the gravitational part of the stellar equations. Hence, our toy model will still capture the essential physics in this scenario.

One has to make a large number of simplifying assumptions to set up the analytical equations governing the stellar structure. These are standard in the literature (see, e.g. [58]). Following HS, we will assume that in our model, apart from the metallicity, the radiation pressure and degeneracy arising out of relativistic effects are also negligible. Further, it is assumed that the (partially degenerate) core has a constant composition of Helium. The envelope, which is in radiative equilibrium, also has a constant composition, predominantly of Hydrogen. Further, the core mass fraction is taken to be 0.10 times the total mass (we will comment on this shortly). The thickness of the Hydrogen burning shell between the core and the envelope is assumed to be negligible. Further, the envelope is assumed to consist of two parts. The opacity of the inner part of the envelope arises due to free electron scattering and that in the outer part is determined by the free transitions of Hydrogen and Helium. Finally, the temperature and pressure of the star is taken to be zero at the surface. This last assumption needs to be carefully explained, as was done by HS. We will postpone a discussion on this to subsection 3.1.

In our model, we incorporate a further assumption that simplifies the numerical computations. Namely, we will assume that the dominant source of nuclear energy in the shell is through the carbon cycle (for an elaborate treatment of nuclear energy generation in stellar objects, see, e.g., [59]). This will be justified in sequel.

Our model, with the simplifying assumptions above, is a very good one for a $1.1M_\odot$ star near the turnoff point in the Hertzsprung-Russell diagram. This was recognized by HS, who also noted that in this particular setup, a higher mass (say $M = 1.2M_\odot$) will model stars later in their evolutionary tracks, closer to the giant phase. The basic features of our model is schematically depicted in figure 1, where, for future use, we indicate the position of the Hydrogen burning shell by $x_1$, and $x_s$ is the position where the formula for the opacity switches inside the envelope, as these result from free-free transitions and electron scattering. Even this simple model is rendered difficult to analyse in the presence of modified gravity.

Figure 1. Schematic diagram for the stellar model used in this paper.
To begin with, there are a large number of algebraic and coupled differential equations (a total of 45 such equations are listed in HS). Also, the presence of the two boundaries where suitable junction conditions need to be imposed, makes our task cumbersome. Nevertheless, we carry out a detailed numerical analysis here and are able to obtain the basic physics of the effects of modified gravity inside the class of stellar objects that we consider.

Broadly, our procedure can be illustrated in the following way (details will be provided in subsection 3.1). For a given $\Upsilon$, we start off with a suitably assumed value of the core temperature ($T_1$) and the relative position ($x_s$) in the envelope, where the mechanism of opacity changes. Then, we have a single parameter family of solutions for both the core and the envelope. Let us call the parameters as $A$ for the core, and $B$ for the envelope.\(^3\) Now, with our choice of $\Upsilon$, for different values of the envelope parameter $B$, we integrate the equations of the envelope, to obtain a family of envelope solutions. Similarly for different values of the core parameter $A$, we integrate the equations of the core to obtain a family of core solutions. The intersection of these two solutions on the plane of homology invariants (defined in subsection 2.3) give a valid tuple of the parameters $(A, B)$. Then, using suitable junction conditions at $x_1$ and $x_s$, we can compute “better values” of $T_1$ and $x_s$, as will be elaborated in subsection 3.1.

If these newly obtained values do not fall within 1% of the assumed values that we started with, we iterate the procedure. We continue the iterative process until the solution converges. The converged solution for $T_1$ and $x_s$ completely specifies the stellar configuration, from which observables can be computed for the given $\Upsilon$. Carrying out this procedure for different values of $\Upsilon$, we find the stellar observables as a function of $\Upsilon$.

In particular, having obtained the luminosity and the stellar radius in terms of $\Upsilon$ in our model, we will need an independent estimate of these, in order to put possible bounds on $\Upsilon$. For simplicity, if we assume a conservative 3% error margin on these observables, then such an estimate can be obtained. We will see that this leads to a reasonably tight bound on $\Upsilon$.

### 2.2 The modified gravity setup

To keep the discussion general at this stage, we will start by taking the Newtonian limit of the stress tensor inside a stellar object. In GR, this is given by $T_{\mu\nu} = \text{diag}(-\rho c^2, P_{\text{rad}}, P_{\perp}, P_{\perp})$ where $c$ is the speed of light and we have allowed for the fact that in general an anisotropy might be present, with $P_{\text{rad}}$ being the radial pressure and $P_{\perp}$ being the tangential one. Spherical symmetry dictates that such tangential pressures along the non-radial directions should be equal. Now we consider a generic Friedman-Robertson-Walker metric in the flat space-time limit, given by

$$ds^2 = -(1 + 2\Phi(r)) dt^2 + a(t)^2 (1 - 2\Psi(r)) \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] , \quad (2.2)$$

where we will set the scale factor $a(t)$ to unity, since we are interested only in a static situation. Here, $\Phi(r)$ is the Newtonian potential, and we have $\Phi(r), \Psi(r) \ll 1$. That the energy momentum tensor is covariantly conserved, i.e., $D_{\mu} T^{\mu\nu} = 0$ (with $D_{\mu}$ being the covariant derivative), then gives in the Newtonian limit,

$$\frac{dP_{\text{rad}}}{dr} = -\rho c^2 \frac{d\Phi}{dr} + \frac{2}{r} (P_{\perp} - P_{\text{rad}}) \left( 1 - r \frac{d\Psi}{dr} \right) . \quad (2.3)$$

---

\(^3\) $A$ is the quantity $\psi_c$ and $B$ is $Ckr$ in the next section.
We then see that terms involving $\Psi$ affects eq. (2.3) only for theories with an anisotropy. In beyond-Horndeski theories, the differential equations for the potential $\Phi(r)$ and $\Psi(r)$ were written down in [25] and read

$$\frac{d\Phi}{dr} = \frac{GM_r}{c^2 r^2} + \frac{\Upsilon}{4} \frac{G d^2 M_r}{c^2 dr^2} , \quad \frac{d\Psi}{dr} = \frac{GM_r}{c^2 r^2} - \frac{5\Upsilon}{4} \frac{G dM_r}{c^2 r^2} .$$

Here, $\Upsilon$, as mentioned before, is the parameter arising in the STT, with $G$ being the Newton’s constant and $M_r$ denoting the mass up to radius $r$. Now, substituting eq. (2.4) in eq. (2.3), and assuming an isotropic situation $P_{rad} = P_\perp = P$, we obtain the final form of the pressure balance equation

$$\frac{dP}{dr} = -\frac{GM_r \rho}{r^2} - \frac{\Upsilon}{4} \rho \frac{d^2 M_r}{dr^2} , \quad \frac{dM_r}{dr} = 4\pi r^2 \rho ,$$

with the second relation giving the mass in terms of the density. Note that as pointed out in [32], eq. (2.5) can be recast into standard form, but with an effective Newton’s constant $G_{\text{eff}}$, given as

$$\frac{G_{\text{eff}}}{G} = 1 + \frac{\Upsilon}{4} \frac{r^2}{M_r} \frac{d^2 M_r}{dr^2} .$$

The second term in eq. (2.6) determines the nature of modified gravity inside the star. Namely, from

$$\frac{d^2 M_r}{dr^2} = 8\pi r \rho + 4\pi r^2 \frac{d\rho}{dr} ,$$

we can glean that since $d\rho/dr < 0$, the second term in eq. (2.7) dominates over the first for large $r$. Hence, from eq. (2.6), we see that overall gravity weakens, at least for homogeneous low mass stars. However, close to $r = 0$, if the fall off of the density is not too steep, the first term dominates, and gravity gets stronger (which was noted by [26]). These are however just “rules of the thumb” statements as pointed out in [36], and for the stellar model that we consider here, a more detailed analysis is required.

### 2.3 The stellar equations

We will first write down the basic equations for the stellar interior that we consider, in the Newtonian limit of GR discussed above. We follow the notations of HS, and mention that the contents of this subsection can be found in textbooks, and we refer the reader to some of the excellent resources [58, 60–63]. In what follows, the subscripts have specific meanings. The subscript 1 on a variable stands for its value at the core envelope junction, $i$ stands for its value on the internal (core) side, $e$ stands for its value on the envelope side, and $s$ stands for its value at the junction at which the opacity mechanism changes (see figure 1).

Let us begin with the equation of state. In degenerate material (present in the core), we define the Fermi function as

$$F_\nu(\psi) = \int_0^{\infty} \frac{u^\nu}{e^u - \psi} du ,$$

where $\psi$ is the degeneracy function [48] (see also [64]). Then the pressure $P$ and the density $\rho$ are given by

$$P = \frac{8\pi}{3h^3} (2mkT)^{3/2} kTF_{3/2}(\psi) , \quad \rho = \frac{4\pi}{h^3} (2mkT)^{3/2} \mu_i HF_{1/2}(\psi) ,$$
where \( k \) is the Boltzmann constant, \( \mu_i \) is the mean molecular weight in the core and we will use \( \mu_e \) to denote that in the envelope, in sequel. Also, \( h \) is Planck’s constant, \( m \) denotes the electron mass, and \( H \) is the mass of the Hydrogen atom. Note that HS ignores the ion pressure in the core. However, a later computation due to Hayashi [65] shows that including this makes little difference in the radius and luminosity of stars that have just left the main sequence, a scenario that we are interested in here. Hence we will, following HS, ignore the ion pressure in the core.

In non-degenerate material, we assume the ideal gas relation

\[
P = \frac{k}{\mu_e H} \rho T .
\]

(2.10)

The hydrostatic equilibrium condition will play a central part in our analysis. In the Newtonian limit of GR, this is obtained by setting \( \Upsilon = 0 \) in eq. (2.5), i.e.,

\[
\frac{dP}{dr} = - \frac{GM_r \rho}{r^2} .
\]

(2.11)

First, we focus on the core. In terms of the non-dimensional variable

\[
\xi = \frac{4\pi \mu_i H}{h^{3/2}} \left( \frac{2m k T_1}{1/4} \right)^{1/2} \frac{r}{r^2} ,
\]

(2.12)

the hydrostatic equilibrium equation in the core reads

\[
\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\psi}{d\xi} \right) = - F_{1/2}(\psi)
\]

(2.13)

with boundary conditions

\[
\psi = \psi_c , \quad \frac{d\psi}{d\xi} = 0 \text{ at } \xi = 0 .
\]

(2.14)

In the core, the mass up to radius \( r \) is

\[
M_r = \frac{h^{3/2}}{4\pi \mu_i^2 H^2} (2m k T_1)^{3/4} (2m G)^{-3/2} \bar{\phi} , \quad \bar{\phi} = \int_0^\xi F_{1/2}(\psi) \xi^2 d\xi ,
\]

(2.15)

where, to make the notation more compact, we have defined the quantity \( \bar{\phi} \). Next, we come to the homology invariants (for a recent discussion on the use of the homology invariants in stellar physics, see e.g [63]), defined from

\[
U = \frac{d \ln M_r}{d \ln r} , \quad V = - \frac{d \ln P}{d \ln r} , \quad n + 1 = \frac{d \ln P}{d \ln T} .
\]

(2.16)

In the core, we have

\[
U = \frac{\xi^3 F_{1/2}(\psi)}{\phi} , \quad V = \frac{3}{2} \frac{F_{1/2}(\psi)}{\bar{\phi}} \frac{\bar{\phi}}{\xi} , \quad n + 1 = \infty .
\]

(2.17)

Finally, the core being isothermal in our model, we have \( d T_1 / d r = 0 \).

Next, we come to the equilibrium equations for the envelope. For radiative equilibrium in the envelope, we have

\[
\frac{dT}{dr} = - \frac{3 \kappa \rho L_r}{4ac T^3} ,
\]

(2.18)
where $L_r$ is the local radiative luminosity at distance $r$. Also, $a$ is the radiation constant, defined as $4\sigma/c$, with $\sigma$ being the Stefan-Boltzmann constant. In eq. (2.18), the opacities are defined from

$$\kappa = 3.68 \times 10^{22} (1 + X) \frac{\rho}{T^{3.5}} \text{ cm}^2 \text{ gm}^{-1}, \quad \text{and} \quad \kappa = 0.19 (1 + X) \text{ cm}^2 \text{ gm}^{-1}, \quad (2.19)$$

for free-free scattering taking place in the outer part of the envelope, and for electron scattering taking place in the inner part, respectively. Here, as discussed after eq. (2.1), we have, in Kramer’s formula, set $(1 - Z) \sim 1$.

Now, we define non-dimensional variables $p$, $q$, $t$, and $x$ through

$$P = \frac{pGM^2}{4\pi R^4}, \quad T = \frac{t\mu_e HGM}{k}, \quad M_r = qM, \quad r = xR, \quad (2.20)$$

with $M$ being the mass and $R$ being the radius of the star. Then, the equilibrium equations in the envelope in terms of the non-dimensional variables are

$$\frac{dp}{dx} = -\frac{q}{x^2} p, \quad \frac{dq}{dx} = \frac{px^2}{t}, \quad \frac{dt}{dx} = \begin{cases} -C_{kr} \frac{p^2}{x^2}, & x > x_s \\ -C_{el} \frac{p}{x^2}, & x < x_s \end{cases} \quad (2.21)$$

where $C_{kr}$ and $C_{el}$ are defined by

$$C_{kr} = \frac{3}{4ac} \frac{3.68 \times 10^{22} (1 + X)}{(4\pi)^3} \left( \frac{k}{\mu_e HG} \right)^{7.5} \frac{LR^{0.5}}{M^{5.5}}, \quad C_{el} = \frac{3}{4ac} \frac{0.19 (1 + X)}{(4\pi)^2} \left( \frac{k}{\mu_e HG} \right)^4 \frac{L}{M^2}. \quad (2.22)$$

These satisfy the relation

$$C_{el} = C_{kr} \frac{ps}{L^{1.5}}, \quad (2.23)$$

which arises due to the continuity of $\frac{dt}{dx}$ at $x_s$. The homology invariants in the envelope in terms of the non-dimensional variables are

$$U = \frac{px^3}{qt}, \quad V = \frac{q}{xt}, \quad n + 1 = \begin{cases} \frac{q}{C_{kr} \frac{L^{1.5}}{x}}, & \text{for } x > x_s \\ \frac{q}{C_{el} \frac{L^{1.5}}{x}}, & \text{for } x < x_s \end{cases} \quad (2.24)$$

Next we come to nuclear-energy production. The stellar luminosity is given by

$$L = 4\pi \int_0^R \epsilon \rho r^2 dr. \quad (2.25)$$

Here, $\epsilon = \epsilon_{CN} + \epsilon_{pp}$, where $\epsilon$ denotes the amount of energy released per unit mass per unit time and the subscripts denote the values for the Carbon-Nitrogen (C-N) cycle and the $p$-$p$ chain reactions, respectively. Now, we will take

$$\epsilon_{CN} = 600\rho X_{CN} \frac{T^{15}}{0.01 (20 \times 10^6)^{15}} \text{ erg gm}^{-1} \text{ sec}^{-1}, \quad \epsilon_{pp} = 0.5\rho X^2 \frac{T^4}{15 \times 10^6} \text{ erg gm}^{-1} \text{ sec}^{-1}. \quad (2.26)$$

While the second equation is standard, the first requires some careful explanation. We note that the exponent 15 appearing in eq. (2.26) was assumed by Hoyle and Schwarzschild and differs from the more conventional exponent close to 20 for the C-N cycle. This exponent of 15 is more appropriate for giant stars, for which the temperature at the thin hydrogen
burning shell is more than 20 million Kelvins. In our case, although the star is in the sub-giant phase, the choice of a negligible shell thickness (as discussed in subsection 2.1) makes it more appropriate to consider the lower exponent in eq. (2.26). That this gives reliable estimates of the temperature, luminosity etc. for a 1.1 \( M_\odot \) star at its turnoff point was noted by HS.

Now, we note that in eq. (2.26), along with the values given in eq. (2.29) below, the C-N cycle starts to dominate the energy generation process at shell temperatures more than 15.2 million Kelvins. This statement is independent of \( \Upsilon \), as in equating \( \epsilon_{pp} \) and \( \epsilon_{CN} \), \( \rho \) cancels, and we determine \( T \) as an appropriate power of a constant. As we will see later, in the presence of \( \Upsilon \), the shell temperature that we will be interested in here will range from 17.5 to 18.1 million Kelvins, for which this will indeed be true. In fact, for a shell temperature of 17.5 \( \times 10^6 \) K, in the model considered here, the energy generation by the C-N cycle is roughly 5 times that by the \( p-p \) chain, and increases to 7 times that of the \( p-p \) chain for a shell temperature of 18.1 \( \times 10^6 \) K. Therefore, as a convenient mathematical simplification, we will ignore the contribution of \( \epsilon_{pp} \) here.

With this in mind, we record the expressions for the energy production in the thin shell at \( x_1 \). Using the homology invariants of eq. (2.16), we obtain in the thin shell approximation,

\[
P \simeq P_1 \left( \frac{r}{r_1} \right)^{-V_{1e}} , \quad T \simeq T_1 \left( \frac{r}{r_1} \right)^{-V_{1e}/(n+1)_{1e}} . \tag{2.27}
\]

Substituting these results in the first relation of eq. (2.26), and using eq. (2.10) and eq. (2.25), we get

\[
L = L_{CN} , \quad L_{CN} \simeq 600 X_{CN} \rho_{1e}^2 \left( \frac{T_1}{20 \times 10^6} \right)^{15} \frac{4 \pi r_1^3}{V_{1e}[2 + 13/(n + 1)_{1e}]} - 3 . \tag{2.28}
\]

Finally, we record the constants assumed. Following HS, we take the Hydrogen fraction as \( X = 0.9 \) and \( X_{CN} = 5 \times 10^{-4} \). With \( Z = 0 \), the mean molecular weights are calculated by the formula for a fully ionized gas, and given by \( 1/\mu = 2X + 3/4(1 - X) \) \[62\]. Since the core is assumed to comprise primarily of Helium, we take \( \mu_i = 4/3 \). In the envelope, we then have \( \mu_e = 0.533 \). In summary, we use

\[
X = 0.9 \quad X_{CN} = 0.0005 \quad \mu_i = 1.333 \quad \mu_e = 0.533 . \tag{2.29}
\]

These are the most important equations that govern the physics of the star that we consider, in the Newtonian limit of GR. We now move over to the changes in these, in the presence of modified gravity.

### 2.4 The transformed stellar equations in modified gravity

The change in the hydrostatic equilibrium condition as given in eq. (2.5) modifies the entire analysis non-trivially in the context of the beyond Horndeski theory. In this subsection, we will write down the modified equations that follow. First, the hydrostatic equilibrium equation in the core, eq. (2.13), is modified to

\[
\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\psi}{d\xi} \right) = - \left[ F_{1/2}(\psi) \left( 1 + \frac{3\Upsilon}{2} \right) + \frac{3\Upsilon}{2} \xi \frac{d}{d\xi} F_{1/2}(\psi) + \frac{\Upsilon}{4} \xi^2 \frac{d^2}{d\xi^2} F_{1/2}(\psi) \right] , \tag{2.30}
\]

with the same boundary conditions as in eq. (2.14). Note that the quantity \( \psi(\xi) \) now depends on \( \Upsilon \) and hence affects all the core variables listed in the previous section, either explicitly or implicitly.
The temperature and the homology invariants in the core are
\[ T = T_1, \quad U = \xi^3 F_{1/2}(\psi) \frac{\phi}{\bar{\phi}}, \quad V = \frac{3}{2} F_{1/2}(\psi) \frac{d^2 \phi}{\xi} + \frac{3}{8} \xi F_{3/2}(\psi) \frac{d^2 \bar{\phi}}{\delta^2}, \]  
where the last relation indicates an explicit modification of \( V \) in the modified gravity theory, compared to the corresponding relation in eq. (2.17). Note that \( U \) is implicitly modified, as the solution to \( \psi \) is now \( \bar{\phi} \)-dependent, via eq. (2.30).

Next, we come to the modified equilibrium equations for the envelope. With the same non-dimensional variables defined in eq. (2.20), we find that these are now given by
\[ \frac{dp}{dx} = -q \frac{p}{x^2} - \frac{\Upsilon}{4} \frac{d^2 q}{dx^2}, \quad \frac{dq}{dx} = \frac{p x}{t}, \quad \frac{dt}{dx} = \begin{cases} -C_{kr} \frac{p^2}{x^2}, & x > x_s \\ -C_{el} \frac{p^2}{x^2}, & x < x_s \end{cases} \]  
where again the first relation indicates an explicit modification from that of eq. (2.21). Since the above equations are coupled, the other two relations are implicitly modified by \( \Upsilon \). Here, we will use the boundary conditions \( q = 1, \ t = 0, \ p = 0 \) at \( x = 1 \), and \( C_{kr} \) and \( C_{el} \) are defined in eq. (2.22). The homology invariants are
\[ U = \frac{p x^3}{q t}, \quad V = \frac{q x}{xt} + \frac{\Upsilon}{4} \frac{d^2 q}{dx^2}, \quad n + 1 = \begin{cases} \frac{q}{C_{kr}} \frac{t^{8.5}}{p}, & x > x_s \\ \frac{q}{C_{el}} \frac{t^{4}}{p}, & x < x_s \end{cases} \]  
where again these variables are modified, either explicitly or implicitly, compared to eq. (2.24).

We will now specify the matching conditions between the core and the envelope. There are four of these, which we list below as (a)–(d).

(a) Fitting of \( M_r \) at \( x_1 \) requires that \( M_{r1i} = q_1 M \)

(b) As is clear from eq. (2.16) along with eq. (2.29), matching of homology invariants at \( x_1 \) requires
\[ \frac{U_{1i}}{U_{1e}} = \frac{\mu_{1i}}{\mu_{1e}} = \frac{V_{1i}}{V_{1e}} = 2.5, \quad (n + 1)_{1e} \text{ finite} \]

(c) Fitting of \( r \) at \( x_1 \) requires that \( r_{1i} = x_{1e} R \)

(d) Fitting of opacity at \( x_s \) requires that \( L \) as computed from \( C_{kr} \) must equal \( L \) as computed from \( C_{el} \).

Before ending this section, we again emphasize that in the modified gravity scenario that we consider here, all stellar equations listed in subsection 2.3 are modified by \( \Upsilon \), either explicitly or implicitly. Our numerical analysis will consist of solving these coupled equations self consistently.

### 3 Numerical analysis and results

We now present the detailed numerical analysis of the modified equations, discussed in the last subsection.\(^4\) The reader would by now have realized the relevance of our qualitative discussion in the introduction. There, we schematically illustrated the procedure by two quantities that we called \( A \) and \( B \). Clearly, identifying \( A \) as \( \psi_c \) and \( B \) as \( C_{kr} \) brings us to the relevant discussion in the context of the model described in the last section as we now elaborate.

\(^4\)All numerical analysis was carried out at the High Performance Computing (HPC) facility at IIT, Kanpur, India and we have used C++ codes, along with appropriate Mathematica subroutines.
3.1 Numerical analysis

First, we fix the mass of the star $M$ and the mass fraction inside the core, $q_1$. In this paper, we will be mostly interested in $M = 1.1M_\odot$ and $q_1 = 0.1$. Now, we specify the initial values of the core temperature $T_1$, and the relative position of the envelope $x_s$, where the opacity formula switches. According to HS, these initial values of $T_1$ and $x_s$ are to be chosen shrewdly. This is crucial in our analysis and we have adhered to their suggestion. As should be clear from the discussion of the last section, once $M$ and $q_1$ are fixed, the solution to the core equations will depend only on the initial value of $\psi_c$. Similarly, the solution to the envelope equations will be depending solely upon the choice of the value of $C_{kr}$. Thus, $\psi_c$ gives us a single parameter family of core solutions, and $C_{kr}$ gives us a single parameter family of envelope solutions. These were respectively called $A$ and $B$ in subsection 2.1.

To begin with, for a given value of $C_{kr}$, we numerically solve the coupled equations in eq. (2.32) for $x > x_s$. Integrating these equations, we obtain the non-dimensional variables $(p_s, q_s, t_s)$ at our fixed $x = x_s$. Then from eq. (2.23), we obtain $C_{d1}$ and use this to further integrate eq. (2.32) for $x < x_s$. The position of the junction $x_{1e}$ from the envelope side is obtained when $q(x)$ reaches the given value of $q_1$. Once we have the value of $x_{1e}$, then $U_{1e}$ and $V_{1e}$ are computed via eq. (2.33).

A similar analysis is done for the solution in the core. Namely, we choose a $\psi_c$, and integrate eq. (2.30) upto $\xi_1$, where the mass inside the core reaches $q_1M$ (condition (a)). Once we obtain $\xi_1$, we can compute the homology invariants $U_{1i}$ and $V_{1i}$, via eq. (2.31) upon using eq. (2.30). Now in order to get the valid solution for the entire stellar object, the core solution is matched with the envelope solution at the core-envelope junction, via fitting condition (b). Let us elaborate briefly on this.

For our initially chosen $T_1$, we get the values of tuple $(U_{1i}, V_{1i})$ corresponding to different values of the free parameter $\psi_c$. We plot the corresponding curve (calling it C-curve, where $C$ stands for core) in a $U$-$V$ plane. Also, for our initially chosen $x_s$, we get the values of the tuple $(U_{1e}, V_{1e})$ corresponding to different values of the free parameter $C_{kr}$. After making an appropriate jump by a factor of 2.5 according to the fitting condition (b), we plot the corresponding curve (calling it E-curve, where $E$ stands for envelope) in the same $U$-$V$ plane. From the $U$-$V$ plane, we then get the values of $C_{kr}$ and $\psi_c$ corresponding to the intersecting point of the C-curve and the E-curve. This yields the fitted solution, where the fitting is done within a tolerance limit of $\pm 0.001$.

Now, we obtain $r_{1i}$ from the defining relation of eq. (2.12). Using this and the obtained value of $x_{1e}$, we compute the stellar radius $R$, from condition (c). Further, condition (d), via eq. (2.22) dictates that the ratio

$$\frac{p_s}{t_s^{\frac{4.5}{5}}} = K\frac{M^{2.5}}{R^{0.5}},$$  \hspace{1cm} (3.1)

with $K$ being an overall constant. With the right hand side of eq. (3.1) now determined, we evaluate a “better value” of $x_s$ numerically as the position in the envelope where $p(x)/t(x)^{\frac{4.5}{5}}$ satisfies the condition in eq. (3.1).

Next, having obtained $R$, and using the fitted value of $C_{kr}$, we get the stellar luminosity from eq. (2.22). This is then used in eq. (2.28) to obtain a better value of $T_1$. Thus having the newly obtained $T_1$ and $x_s$, if these are within 1% of their initially assumed values, then we stop the iteration, else we continue this process until we reach the desired convergence and get the final values of these quantities. Note that at every iteration, we start with the
obtained values of $T_1$ and $x_s$ from the previous one. The final converged value of $T_1$ and $x_s$ gives us the stellar configuration for the chosen value of $\Upsilon$.

The procedure described above is repeated with different values of the parameter $\Upsilon$. Thus, at the end, we obtain all stellar parameters as a function of $\Upsilon$.

There are a few computational subtleties that are best mentioned at this point. When we merge the expressions appearing in eq. (2.32), we obtain
\[
\frac{dp}{dx} = -\begin{cases} \frac{x^2 pt + \frac{\Upsilon}{2} x p^2 + \frac{C_{el}}{4} \frac{X}{r} p^3}{t^2 + \frac{\Upsilon}{2} x^2 p}, & \text{for } x > x_s \\ \frac{x^2 pt + \frac{\Upsilon}{2} x p^2 + \frac{C_{el}}{4} \frac{X}{r} p^3}{t^2 + \frac{\Upsilon}{4} x^2 p}, & \text{for } x < x_s \end{cases}
\]
\tag{3.2}
\]
Now, we had the boundary condition $q = 1, t = 0, p = 0$ at $x = 1$. In our numerical procedure, we have taken $t = 10^{-7}, p = 10^{-7}$ at $x = 1.0$, to start with. However, the third term in the numerator of this equation blows up, when we take such small numerical values. Note that initially near the surface $x = 1.0$, the third term turns out to be the most dominant term, in presence of $\Upsilon$. Thus
\[
\frac{dp}{dx} \simeq -\frac{C_{kr} p^3}{(t^2 + \frac{\Upsilon}{4} x^2 p)^2} \simeq -\frac{C_{kr} p^3}{x^2 t^{5.5}}
\]
\tag{3.3}
Without $\Upsilon$, only the first term remains, which is thus the dominating term for GR, i.e
\[
\frac{dp}{dx} = -\frac{qp}{x^2 t}
\]
\tag{3.4}
Hence the initial evolution of the equations from $x = 1.0$ to $x = (1.0 - 10^{-6})$ say, is distinctly different for the two cases, i.e., one with non zero $\Upsilon$ and the other with $\Upsilon = 0$. The values of $p, q, t$ are different at $x = (1 - 10^{-6})$ for the two cases, compared to their values at $x = 1.0$. Thus, once we evolve along the STT path, we cannot retrace back the GR results by simply putting $\Upsilon = 0$. Hence we have chosen to evolve our equations initially via the GR path (from $x = 1.0$ to $x = (1.0 - 10^{-6})$) and then evolved them using eq. (3.2). After the first step, the parameters take finite values, which no longer poses the blowing-up issue. This way, we have a handle of getting back GR results, by putting $\Upsilon = 0$ in all the featured equations in this paper.

### 3.2 Results and discussions

We are now ready to present our main results. First, we present the results on our numerical analysis based on the equations of section 2.4 and the methods of section 3.1. Then, we will discuss a possible bound on $\Upsilon$.

For $M = 1.1M_\odot$ and $q_1 = 0.10$ we perform the above numerical computation for different values of $\Upsilon$ and obtain a table 1 of different physical as well as non-dimensional variables. These results are summarised as follows.

- The radius $R$ of the star, the density $\rho_{1e}$, and the temperature on the envelope side $T_{1e}$ of the nuclear burning shell increase with increase in $\Upsilon$.

- The size of the core $r_1$, the core temperature $T_1$, the overall luminosity $L$ of the star and the effective temperature $T_{eff}$ at the surface decrease with increase in $\Upsilon$. 


| Parameter                  | $\Upsilon$ |
|----------------------------|------------|
| $T_1 \times (10^{-6})$    | 18.08      |
| $C_{ke} \times (10^8)$    | 2.90       |
| $\psi_c$                  | -0.65      |
| $\log x_s$                | -0.51      |
| $\log q_s$                | 1.23       |
| $\log g_s$                | -0.26      |
| $\log t_s$                | -0.27      |
| $\log x_1$                | -0.99      |
| $\log p_1$                | 1.90       |
| $\log t_1$                | 0.03       |
| $V_{1e}$                  | 0.84       |
| $U_{1e}$                  | 0.76       |
| $(n + 1)_{1e}$             | 1.95       |
| $r_1 \times (10^{-10})$   | 0.63       |
| $10 \times \log \left( \frac{R}{R_{\odot}} \right)$ | -0.54 |
| $10 \times \log \left( \frac{L}{L_{\odot}} \right)$ | 5.31 |
| $\log (\rho_{1e})$       | 1.74       |
| $\log (T_{1e})$           | 7.22       |
| $\log (T_{\text{eff}})$   | 3.92       |

Table 1. Dependence on $\Upsilon$ of the various physical parameters listed. ($M = 1.1 M_\odot$, $q_1 = 0.10$)

As a result of the above, with increasing $\Upsilon$, the span of the envelope increases, and the radial position ($x_s$), where the switch of opacity occurs, comes closer to the core. Clearly, these point to very non-trivial physics in the stellar interior, as we have mentioned before. The decrease in the luminosity $L$ and the effective temperature $T_{\text{eff}}$ is consistent with the fact that overall gravity weakens inside the stellar object (its radius increases), and it becomes dimmer and cooler. However, we note that near the core, the opposite happens. The core radius decreases appreciably as one increases $\Upsilon$, indicating that gravity is stronger near the core than what would happen in the $\Upsilon = 0$ case. In addition, the core temperature also decreases, although by a small amount. To balance the strong gravity, the density at the core increases by a large amount as one increases $\Upsilon$. We have checked that this is not an artefact of our choice of $M = 1.1 M_\odot$. The same trend is followed by models with $M = 1.15$ and $1.2 M_\odot$, although, as mentioned before, these masses are more appropriate for modeling stars near the giant phase.

At this point, it is convenient to discuss the simplifying boundary condition of zero temperature at the stellar surface (subsection 2.1). As pointed out by HS, this mathematical simplification is justified as long as the temperature very close to the stellar surface is of
the order of the effective temperature (computed via the luminosity). We find that this is indeed the case here. The non-dimensional quantity $t$ attains the value $3.3 \times 10^{-3}$ at $x = 1.0 - 10^{-6}$, compared to the assumed value of $t = 10^{-7}$ at $x = 1$. This value of $t$ is true irrespective of $\Upsilon$, since, as we have mentioned, for all $\Upsilon$, we evolved $t(x)$ from $x = 1.0$ to $x = 1.0 - 10^{-6}$ via the GR path. From the second relation of eq. (2.20), we see that the physical temperature depends on $\Upsilon$, since $R$ does. With this in mind, we computed the temperature at $x = 1.0 - 10^{-6}$, for the range of $\Upsilon$ given in table 1. We find that this temperature varies from $4.6 \times 10^4$ to $3.9 \times 10^4$ as $\Upsilon$ is varied from $-0.34$ to $0.24$. Clearly then, the temperature very close to the surface is of the order of $T_{\text{eff}}$ given in table 1, and our boundary condition for the temperature is justified.

Now, from table 1, we obtain a relationship between $L$ and $R$ as a function of $\Upsilon$. We see a monotonic decrease in $\log (R/R_\odot)$ and a monotonic increase in $\log (L/L_\odot)$ with decrease in $\Upsilon$. In figures 2 and 3, we plot the obtained value of the radius and the luminosity as a function of $\Upsilon$. These can be fitted as

$$\log \left(\frac{R}{R_\odot}\right) = 0.003 + 0.103 \Upsilon - 0.194 \Upsilon^2, \quad \log \left(\frac{L}{L_\odot}\right) = 0.440 - 0.269 \Upsilon,$$  

(3.5)

and we have plotted the actual results given by the dots, along with the fitted curve. One would now ideally like to use a mass-radius or a mass-luminosity relation for pop-II stars to get an estimate for $\Upsilon$ from eq. (3.5). Unfortunately, while such relations are available in the literature (see, e.g., [66] for a recent work, which also gives a nice historical account of the development of the subject in its introductory section) for solar neighborhood pop-I stars, these are not very well formulated for pop-II stars as of now.

We can however envisage a bound on $\Upsilon$ from possible error estimates in the measurement of the luminosity. For example, if we assume a very conservative $\sim 3\%$ error in $L$ (for $\Upsilon = 0$), it is checked from the second relation of eq. (3.5) that

$$-0.05 < \Upsilon < 0.04.$$  

(3.6)

Now, it has to be checked that such a range is consistent with the errors in the numerical analysis that we have carried out, namely that the numerical error in computing the luminosity is less than $\sim 3\%$. We have checked that this is indeed the case. One might ask if the conclusions above change if the model here was modified to include a small metallicity. This is an important source of degeneracy as pointed out by Koyama and Sakstein [25]. Note that in this model, the only places where an explicit dependence on metallicity will show up are
Table 2. Various bounds on Υ.

| Reference | Lower Bound | Upper Bound |
|-----------|-------------|-------------|
| [26]      | −0.67       | −           |
| [27]      | −           | 1.6         |
| [29]      | −0.22 (2σ)  | 0.27 (2σ)   |
| [30]      | −0.44       | −           |
| [32]      | −           | 0.14        |
| [34]      | −1.8 × 10^{-3} (2σ) | 1.2 × 10^{-3} (2σ) |
| [35]      | 0           | 0.47        |
| Present work | −0.05     | 0.04        |

in the mean molecular weights of the core and the envelope, and in Kramer’s opacity formula (through a multiplicative factor of (1 − Z)). Including, say, Z = 0.001 in our analysis results in negligible changes in the luminosity and the effective temperature at Υ = 0 (the change is ∼ 0.5% for L, while for T_{eff} it is even lesser). Hence, keeping the theory as GR and changing the metallicity in this model should not alter the conclusion of eq. (3.6).

4 Conclusions

It is well known that modified gravity theories beyond Horndeski, which are ghost free, typically exhibit a partial breaking of the Vainshtein mechanism inside stellar objects. This leads to a novel modification of gravity inside such objects, and in the Newtonian limit leads to an alteration of the pressure balance equation therein. This feature provides an exciting mechanism to test a class of modified gravity theories, using astrophysical signatures. Whereas many of the works in this direction concentrated on stellar objects that satisfy a polytropic equation of state (for example in dwarf stars), here we have taken the first step towards analytically studying the effects of modifications of gravity inside a stellar object that has a core-envelope structure. In such a situation, the effects of modified gravity have to be matched at appropriate junctions, which involves a detailed and involved numerical analysis. The main results of this paper are contained in section 3.2, in table 1 (and the discussions thereafter) and eq. (3.6).

We remind the reader that we have considered a toy model here, with zero metallicity and a number of simplifying assumptions, discussed in subsection 2.1. Indeed, more elaborate stellar models that relax some or all of these assumptions are well known in the literature. However, our purpose here was to obtain an analytic handle on the large number of coupled equations which result even in a simplified scenario. As we have mentioned, these simplifications refer only to the stellar matter and do not affect the gravity part of the stellar equations and hence bring out the essential physics in a modified gravity scenario. Also, as discussed in subsection 3.2, introducing a small metallicity while maintaining all other simplifying assumptions does not affect our results appreciably. It would however be interesting to consider a more realistic model, and we leave this for a future study.

One of the main results in this paper is a bound on the parameter Υ of the beyond Horndeski theory. For completeness, in table 2, we provide the currently available bounds on Υ (σ denotes standard deviation here, wherever available). Most of the bounds above
have been obtained in the context of dwarf stars, excepting the one in [34], where a precision constraint using helioseismology was obtained. We have, in this paper, constructed the bound of eq. (3.6) from an analysis of the internal structure of pop-II stars in globular clusters. One can see that even within the limitations of the model considered here, our bound is a good improvement from most of the ones available till now, from stellar structure constraints.

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