An Integer Linear Programming Model for Solving Radio Mean Labeling Problem

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This work was supported by the Deanship of Scientific Research, Majmaah University, under Project RGP-2019-29.

ABSTRACT A Radio mean labeling of a connected graph \( G \) is an injective function \( h \) from the vertex set, \( V(G) \), to the set of natural numbers \( N \) such that for any two distinct vertices \( x \) and \( y \) of \( G \),

\[
\left\lceil \frac{h(x)+h(y)}{2} \right\rceil \geq \text{diam} + 1 - d(x, y).
\]

where \( \text{diam} \) is the diameter of \( G \) and \( d(x, y) \) denotes the distance between the two vertices \( x \) and \( y \). The number of radio mean of \( h \), \( rmn(h) \), is the maximum number assigned to any vertex of \( G \). The radio mean number of \( G \), \( rmn(G) \), is the minimum value of \( rmn(h) \), taken over all radio mean labeling \( h \) of \( G \). This work has three contributions. The first one is proving two theorems which find the radio mean number for cycles and paths. The second contribution is proposing an approximate algorithm which finds an upper bound for radio mean number of a given graph. The third contribution is that we introduce a novel integer linear programming formulation for the radio mean problem. Finally, the experimental results analysis and statistical test proved that the Integer Linear Programming Model overcame the proposed approximate algorithm according to CPU time only. On the other hand, both the Integer Linear Programming Model and the proposed approximate algorithm had the same upper bound of the radio mean number of \( G \).

INDEX TERMS Channel assignment problem, radio mean number, upper bound, path and cycle.

I. INTRODUCTION

Let \( V(G) \) and \( E(G) \) denote the set of vertices and the set of edges for the graph \( G \) respectively. Hale [1] proposed the channel assignment problem. The radio labeling of graphs (multilevel distance labeling) is proposed by Chartrand et al. [2] in 2001 due to the regulations for channel assignments of FM radio stations. Zhang [3] determined the upper bounds of the radio numbers of cycles. Liu and Zhu [4] introduced the exact formula for the radio numbers for paths and cycles. Badr and Moussa [5] introduced the algorithm that determines the upper bound of radio \( k \)-chromatic number for a graph. This algorithm overcame the algorithm that was due to Saha and Panigrahi [6]. Saha and Panigrahi [7] proposed two radio \( k \)-coloring methods for a given graph which will find radio \( k \)-colorings.

Ponraja et al. [8] and Ponraja and Narayanan [9] proposed the radio mean labeling of graphs as follows: let \( h \) be an injective function from the vertex set, \( V(G) \), to the set of natural numbers \( N \) where where for any two distinct vertices \( x \) and \( y \) of \( G \),

\[
\left\lceil \frac{h(x)+h(y)}{2} \right\rceil \geq \text{diam} + 1 - d(x, y).
\]

where \( \text{diam} \) is the diameter of \( G \) and \( d(x, y) \) denotes the distance between the two vertices \( x \) and \( y \). The number of radio mean of \( h \), \( rmn(h) \), is the maximum number assigned to any vertex of \( G \). The number of radio mean of \( G \), \( rmn(G) \), is the minimum value of \( rmn(h) \), taken over all radio mean labeling \( h \) of \( G \). Ponraja et al. [8] found the number of radio mean for networks with diameter 3, lotus with a circle, Sunflower networks and Helms. Ponraja and Narayanan [9] determined the number of radio mean for some networks that are related to cycles and complete graph. In [10] they found the number of radio mean for triangular ladder network, \( P_n \odot \bar{K}_2 \) (It consists of a path \( P_n \) in which every vertex \( x_i \) joined to two vertices \( y_1 \) and \( z_1 \) of \( \bar{K}_2 \)), \( K_6 \odot \bar{K}_2 \) (It consists of a complete graph \( K_6 \) in which every vertex \( x_i \) joined to two vertices \( y_1 \) and \( z_1 \) of \( \bar{K}_2 \)) and \( W_n \odot \bar{K}_2 \) (It consists of a wheel \( W_n \) in which every vertex \( x_i \) joined to two vertices \( y_1 \) and \( z_1 \) of \( \bar{K}_2 \)). Since the radio mean labeling problem is derived from the radio
k-coloring application, so, the radio mean labeling application is NP-hard problem for a graph. In [11] the authors introduced an application for radio frequency identification (RFID). On the other hand, the metaheuristic approaches for the linear wireless sensor networks were proposed in [12]

This work has three contributions. The first contribution is proving two theorems which find the radio mean number for cycles and paths. The second contribution is proposing an approximate algorithm which finds an upper bound for a radio mean number of a given graph. The third contribution is that we introduce a novel integer linear programing formulation for finding a radio mean number of a given graph. The third contribution is that we introduce a novel integer linear programing formulation for finding the radio mean number of a cycle and path are introduced in Section 2. An approximate algorithm which finds the upper bound of radio mean number of a graph is proposed with an example in Section 3. A novel integer linear programing formulation for finding a radio mean number of a graph is introduced with an example in Section 4. In Section 5 the numerical results analysis and statistical test proved that the Integer Linear Programming Model overcame the proposed approximate algorithm according to CPU time only. On the other hand, both the Integer Linear Programming Model and the proposed approximate algorithm had the same upper bound of the radio mean number of G.

The rest of this work is organized as the following: the radio mean number of cycle and path are introduced in Section 2. An approximate algorithm which finds the upper bound of radio mean number of a graph is proposed with an example in Section 3. A novel integer linear programing formulation for finding a radio mean number of a graph is introduced with an example in Section 4. In Section 5 the numerical results analysis and statistical test proved that the Integer Linear Programming Model overcame the proposed approximate algorithm are provided. Finally, conclusions are drawn in Section 6.

II. MAIN RESULTS

In this section, we introduce some basic definitions before proving the theorems that determine the radio mean number of cycle and path.

Definition 1: The distance from a vertex u to a vertex v is the number of edges in a shortest u – v path in G and it is denoted by d(u, v).

Definition 2: Let G be a connected graph, the eccentricity e(v) of a vertex v is the distance between v and a vertex farthest from v in G.

Definition 3: The diameter diam(G) of G is the greatest eccentricity among the vertices of G.

Theorem 1: The radio mean number of the cycles Cₙ is given by:

\[ \text{rmn}(Cₙ) = \begin{cases} \left\lceil \frac{3n}{2} - 4 \right\rceil & \text{if } 3 \leq n \leq 7 \\ \frac{n}{2} & \text{if } n \geq 8 \end{cases} \]

Proof: Let \( x₁, x₂, \ldots, xₙ \) be cycle of length n so \( \text{diam}(Cₙ) = \left\lceil \frac{n}{2} \right\rceil \). Define a function \( h:V(Cₙ)\rightarrow \mathbb{N} \) by the following cases:

Case a: For \( 3 \leq n \leq 7 \), we have three subcases as the following:

Case a.1: At \( 3 \leq n \leq 5 \), in this subcase the vertices are labeled by the following function:

\[ h(x_i) = i; \quad 1 \leq i \leq n \]

Case a.2: At \( n= 6 \), in this subcase the vertices are labeled by the following function:

\[ \begin{align*} h(x_{i+1}) &= n - 1 - 2i; \quad 0 \leq i < \frac{n}{2} - 1 \\ h(x_{n-j}) &= 2j + 2; \quad 0 \leq j < \frac{n}{2} - 1 \\ h(x_\frac{n}{2}) &= n, \quad h(x_\frac{n}{2} + 1) = 1 \end{align*} \]

Case a.3: At \( n = 7 \), we may label the vertices of C₇ as the following

\[ \begin{align*} h(x_{i+1}) &= n - 1 - 2i; \quad 0 \leq i < \frac{n+1}{2} \\ h(x_{n-j}) &= 2j + 2; \quad 0 \leq j < \frac{n-1}{2} \end{align*} \]

Therefore for any pair \((x_i, x_j), i \neq j, 1 \leq i \leq n, 1 \leq j \leq n\), we have:

\[ d(x_i, x_j) + \left\lceil \frac{h(x_i) + h(x_j)}{2} \right\rceil \geq 1 + \left\lfloor \frac{n}{2} \right\rfloor = 1 + \text{dim}(Cₙ) \]

Case b: For \( n \equiv 0(\text{mod} 4) \) i.e. \( n = 4k + 8, k \geq 0 \), we may label the vertices of Cₙ as follows:

\[ \begin{align*} h(x_\frac{n}{2}) &= 1, \quad h(x_1) = n + 2k - 1 \\ h(x_{i+2}) &= \frac{n}{2} - 1 + 2i, \quad 0 \leq i < \frac{n}{2} - 2 \\ h(x_{n-j}) &= \frac{n}{2} - 2 + 2j, \quad 0 \leq j < \frac{n}{2} \end{align*} \]

Therefore for any pair \((x_i, x_j), i \neq j, 1 \leq i \leq n, 1 \leq j \leq n\), we have:

\[ d(x_i, x_j) + \left\lceil \frac{h(x_i) + h(x_j)}{2} \right\rceil \geq 1 + \left\lfloor \frac{n}{2} \right\rfloor = 1 + \text{dim}(Cₙ) \]

Case c: \( n \equiv 1(\text{mod} 4) \) i.e. \( n = 4k + 9, k \geq 0 \), in this case the vertices are labeled by the following function:

\[ \begin{align*} h(x_\frac{n+1}{2}) &= 1, \quad h(x_1) = n + 2k, \quad h(x_\frac{n}{2}) = \left\lfloor \frac{n}{2} \right\rfloor - 2 \\ h(x_{n-1}) &= \left\lceil \frac{n}{2} \right\rceil + 2, \quad h(x_{n-2}) = \left\lceil \frac{n}{2} \right\rceil \\ h(x_{i+2}) &= \left\lceil \frac{n}{2} \right\rceil - 1 + 2i, \quad 0 \leq i < \left\lfloor \frac{n}{2} \right\rfloor - 1 \\ h(x_{n-3-j}) &= \left\lceil \frac{n}{2} \right\rceil + 4 + 2j, \quad 0 \leq j < \left\lfloor \frac{n}{2} \right\rfloor - 3 \end{align*} \]

Therefore for any pair \((x_i, x_j), i \neq j, 1 \leq i \leq n, 1 \leq j \leq n\), we have:

\[ d(x_i, x_j) + \left\lceil \frac{h(x_i) + h(x_j)}{2} \right\rceil \geq 1 + \left\lfloor \frac{n}{2} \right\rfloor = 1 + \text{dim}(Cₙ) \]

Case d: \( n \equiv 2(\text{mod} 4) \) i.e. \( n = 4k + 10, k \geq 0 \), in this case the vertices are labeled by the following function:

\[ \begin{align*} h(x_\frac{n+1}{2}) &= 1, \quad h(x_1) = n + 2k + 1 \\ h(x_{i+2}) &= \frac{n}{2} - 1 + 2i, \quad 0 \leq i < \frac{n}{2} - 2 \\ h(x_{n-j}) &= \frac{n}{2} - 2 + 2j, \quad 0 \leq j < \frac{n}{2} - 1 \end{align*} \]
Therefore for any pair \((x_i, x_j)\), \(i \neq j, 1 \leq i \leq n, 1 \leq j \leq n\), we have:

\[
d \left( x_i, x_j \right) + \left[ \frac{h(x_i) + h(x_j)}{2} \right] \geq 1 + \left\lceil \frac{n}{2} \right\rceil = 1 + \dim(C_n)
\]

**Case e:** \(n \equiv 3 \pmod{4}\) i.e. \(n = 4k + 7, k \geq 1\), in this case the vertices are labeled by the following function:

\[
h \left( \frac{x_{n+1}}{2} \right) = 1, \ h(x_1) = n + 2k - 1, \ h(x_n) = \left\lceil \frac{n}{2} \right\rceil - 2
\]

\[
h \left( x_{n-1} \right) = \left\lceil \frac{n}{2} \right\rceil + 2, \ h(x_{n-2}) = \left\lceil \frac{n}{2} \right\rceil
\]

\[
h \left( x_{i+2} \right) = \left\lceil \frac{n}{2} \right\rceil - 1 + 2i, \ 0 \leq i < \left\lceil \frac{n}{2} \right\rceil - 1
\]

\[
h \left( x_{n-3-j} \right) = \left\lceil \frac{n}{2} \right\rceil + 4 + 2j, \ 0 \leq j < \left\lceil \frac{n}{2} \right\rceil - 3
\]

Therefore for any pair \((x_i, x_j)\), \(i \neq j, 1 \leq i \leq n, 1 \leq j \leq n\), we have:

\[
d \left( x_i, x_j \right) + \left[ \frac{h(x_i) + h(x_j)}{2} \right] \geq 1 + \left\lceil \frac{n}{2} \right\rceil = 1 + \dim(C_n)
\]

Thus, the radio mean condition is satisfied for all pairs of vertices. Now, we have the upper bound of the radio mean labeling of \(C_n\) as the following inequality:

\[
\text{rmn}(C_n) \leq \text{rmn}(h) = \begin{cases} \left\lceil \frac{3n}{2} \right\rceil - 4 & \text{if } 3 \leq n \leq 7 \\ \left\lceil \frac{3n}{2} \right\rceil & \text{if } n \geq 8 \end{cases}
\]

(1)

Since \(h\) is an injective mapping (i.e. we can’t label two or more vertices in \(V(C_n)\) with the same natural number in \(N\)) then the lower bound of the radio mean labeling of \(C_n\) is determined by the following inequality:

\[
\text{rmn}(C_n) \geq \begin{cases} n & \text{if } 3 \leq n \leq 7 \\ \left\lceil \frac{3n}{2} \right\rceil - 4 & \text{if } n \geq 8 \end{cases}
\]

(2)

From Inequalities 1 and 2, we have:

\[
\text{rmn}(C_n) = \begin{cases} n & \text{if } 3 \leq n \leq 7 \\ \left\lceil \frac{3n}{2} \right\rceil - 4 & \text{if } n \geq 8 \end{cases}
\]

Therefore, the labeling \(h:V(C_n) \rightarrow N\) defined by the above cases satisfies the radio mean condition.

**Theorem 2:** The number of radio mean for the path graph \(P_n\) is given by:

\[
\text{rmn}(P_n) = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ 2n - 3 & \text{if } n > 2 \end{cases}
\]

**Proof:** Let \(x_1, x_2, \ldots, x_n\) be path of length \(n - 1\), i.e. \(\text{diam}(P_n) = n - 1\). Define a function \(h:V(P_n) \rightarrow N\), as follows:

\[
h(x_1) = 1,
\]

\[
h(x_{n-i}) = n-1+i, 0 \leq i, n - 1
\]

Therefore for any pair \((x_i, x_j)\), \(i \neq j, 0 \leq i, j \leq n\), we have:

\[
d \left( x_i, x_j \right) + \left[ \frac{h(x_i) + h(x_j)}{2} \right] \geq 1+n-1 = 1 + \dim(P_n)
\]

Algorithm 1 The Upper Bound of Bound of the Radio Mean Number of a Graph \(G\)

**Input:** The adjacency matrix of the graph \(G\) and the diameter of \(G\) (\(\text{dim}\)).

**Output:** The upper bound of radio mean number of \(G\).

**Begin**

1: Choose a vertex \(u\) and \(\text{lab}(u) = \text{dim}\).
2: \(S = \{u\}\).
3: For all \(v \in V(G) - S\), compute,

\[
\text{temp}(v) = \max_{i \in s} \left( \text{lab}(i) + \text{ceil} \left( \frac{\text{dim} - d(u, v) + 1}{\text{dim}} \right) \right)
\]

4: Let \(\text{min} = \min_{v \in V(G) - S} \{\text{temp}(v)\}\).
5: Choose a vertex \(v \in V(G) - S\), where \(\text{temp}(v) = \text{min}\).

6: Assign, \(\text{lab}(v) = \text{min}\).
7: \(S = S \cup \{v\}\).
8: Repeat Step 3 to Step 6 until all vertices are labeled.
9: Repeat Step 1 to Step 7 for every vertex \(x \in V(G)\).

**End**

for any pair \((x_i, x_j)\), \(i \neq j, 0 \leq i \leq n, 0 \leq j \leq n\). Thus, the radio mean condition is satisfied with all pairs of vertices. Now, we have the upper bound of the radio mean labeling of \(P_n\) as the following inequality:

\[
\text{rmn}(P_n) \leq \text{rmn}(h) = 2n - 3
\]

(3)

Since \(h\) is an injective mapping (i.e. we can’t label two or more vertices in \(V(C_n)\) with the same natural number in \(N\)) then the lower bound of the radio mean labeling of \(P_n\) is determined by the following inequality:

\[
\text{rmn}(P_n) \geq 2n - 3
\]

(4)

for all radio mean labeling \(h\). From Inequalities 3 and 4, we have:

\[
\text{rmn}(P_n) = 2n - 3, n \geq 3.
\]

Therefore, the labeling \(h:V(P_n) \rightarrow N\) defined by the above satisfies the radio mean condition.

**III. A NEW GRAPH RADIO MEAN ALGORITHM**

In this section, we propose an algorithm that determines an upper bound of the radio mean for an arbitrary graph \(G\). The main idea of the proposed algorithm is that the algorithm changes the initial vertex to improve the upper bound. The diameter of \(G\) is assigned to some vertex. The next vertex is labeled by the minimum possible integer. After all vertices are labeled, the algorithm changes the initial vertex over all vertices.

**Complexity of Algorithms 1:** It is clear that step 1 and step 2 both have one operation. On the other hand, step 3 has a nested loop which has \(O(n^2)\) time complexity. Three steps (step 4, step 5 and step 6) have \(O(n)\) time complexity. Step 7 has one operation but the last two steps (step 8, step 9)
have \(O(n^3)\) and \(O(n^4)\) respectively. The proposed algorithm has the following time complexity:

\[
2O(1) + 3O(n) + O(1) + O(n^3) + O(n^4) = O(n^4)
\]

**Example 1:** This example shows how to compute the upper bound of the number of radio mean for the path \(P_5\). We suppose that \(x_i\) are the labels of the vertices \(v_i\) such that \(1 \leq i \leq 5\). So, Algorithm 1 determines the upper bound of the number of radio mean as the following:

It is known that \(diam(P_5) = 4\). We choose a vertex \(x_1\) and \(\text{lab}(x_1) = 4\). Let \(S = \{x_1\}\) and for all \(v \in V(G) - S\), compute,

\[
\begin{align*}
temp(x_2) &= \max_{x_1} \left\{ 4 + \left\lceil \frac{\text{MAX}(\{(4 + 1 - 1, 1\})}{10} \right\rceil \right\} = 5 \\
temp(x_3) &= \max_{x_1} \left\{ 4 + \left\lceil \frac{\text{MAX}(\{(4 + 1 - 2, 1\})}{10} \right\rceil \right\} = 5 \\
temp(x_4) &= \max_{x_1} \left\{ 4 + \left\lceil \frac{\text{MAX}(\{(4 + 1 - 3, 1\})}{10} \right\rceil \right\} = 5 \\
temp(x_5) &= \max_{x_1} \left\{ 4 + \left\lceil \frac{\text{MAX}(\{(4 + 1 - 4, 1\})}{10} \right\rceil \right\} = 5.
\end{align*}
\]

Let \(\min_{v \in V(G) - S} \{\text{temp}(v)\} = 5\) we choose a vertex \(x_2 \in V(G) - S\). Such that \(temp(x_2) = 5\). Give \(\text{lab}(x_2) = 5\) and \(S = \{x_1, x_2\}\)

\[
\begin{align*}
temp(x_3) &= \max_{x_1, x_2} \left\{ 4 + \left\lceil \frac{\text{MAX}(\{(4 + 1 - 1, 1\})}{10} \right\rceil \right\} = 6 \\
temp(x_4) &= \max_{x_1, x_2} \left\{ 4 + \left\lceil \frac{\text{MAX}(\{(4 + 1 - 2, 1\})}{10} \right\rceil \right\} = 6 \\
temp(x_5) &= \max_{x_1, x_2} \left\{ 4 + \left\lceil \frac{\text{MAX}(\{(4 + 1 - 3, 1\})}{10} \right\rceil \right\} = 6.
\end{align*}
\]

Let \(\min_{v \in V(G) - S} \{\text{temp}(v)\} = 6\) we choose a vertex \(x_3 \in V(G) - S\), where \(temp(x_3) = 6\). Give \(\text{lab}(x_3) = 6\) and \(S = \{x_1, x_2, x_3\}\)

\[
\begin{align*}
temp(x_4) &= \max_{x_1, x_2, x_3} \left\{ 4 + \left\lceil \frac{\text{MAX}(\{(4 + 1 - 1, 1\})}{10} \right\rceil \right\} = 7 \\
temp(x_5) &= \max_{x_1, x_2, x_3} \left\{ 4 + \left\lceil \frac{\text{MAX}(\{(4 + 1 - 3, 1\})}{10} \right\rceil \right\} = 7.
\end{align*}
\]

Let \(\min_{v \in V(G) - S} \{\text{temp}(v)\} = 7\), we choose a vertex \(x_4 \in V(G) - S\), where \(temp(x_4) = 7\). Give \(\text{col}(x_4) = 7\) and \(S = \{x_1, x_2, x_3, x_4\}\)

\[
\begin{align*}
temp(x_5) &= \max_{x_1, x_2, x_3} \left\{ 4 + \left\lceil \frac{\text{MAX}(\{(4 + 1 - 1, 1\})}{10} \right\rceil \right\} = 8
\end{align*}
\]

Let \(\min_{v \in V - S} \{\text{temp}(v)\} = 8\), we choose a vertex \(x_5 \in V(G) - S\), such that \(temp(x_5) = 8\). Give \(\text{col}(x_5) = 8\) and \(S = \{x_1, x_2, x_3, x_4, x_5\}\). It is clear that all vertices are labeled and \(\text{rmm}(P_5) = 8\).

**IV. CASTING AS AN INTEGER LINEAR PROGRAMMING MODEL**

In this section, we introduce a new mathematical model [13]–[21] for the radio mean labeling application.

Let \(V(G) = \{v_1, v_2, \ldots, v_n\}\) are the vertices of the connected graph \(G\) with order \(n\), and let \(D = \{d_{ij}\}\) be the distance matrix of \(G\), that is, \(d_{ij} = d(v_i, v_j)\) for \(1 \leq i, j \leq n\). Let \(x_i\) be the labels of the vertices \(v_i\) where, \(1 \leq i \leq n\). We define the function \(F\) by \(F = x_1 + x_2 + \ldots + x_n\).

Minimizing \(F\) subject to the \(n\) constraints

\[
2n \left[ \frac{x_i + x_j}{2} \right] \geq \text{diam} + 1 - d(v_i, v_j) \quad \text{for} \quad 1 \leq i \leq n - 1, \quad 2 \leq j \leq n \text{ and } i < j \quad \text{where} \quad x_1, x_2, \ldots, x_n \in \{0, 1\}.
\]
Example 2: This example shows how to formulate the radio mean problem as the mathematical model for cycle $C_3$. Let $x_i$ labels the vertices $v_i$ where $1 \leq i \leq 3$. Now, this mathematical model (integer programming model) is the following:

$$\min f = x_1 + x_2 + x_3$$

subject to:

$$6|x_1 - x_2| \geq \text{diam} + 1 - d(v_1, v_2);$$

$$6|x_1 - x_3| \geq \text{diam} + 1 - d(v_1, v_3);$$

$$6|x_2 - x_3| \geq \text{diam} + 1 - d(v_2, v_3)$$

where $x_1, x_2, x_3 \geq 0$

Since $\text{diam} = \lfloor n/2 \rfloor$ (the diameter of $C_n$) then diam = 1 for $C_3$ and the distance matrix of the cyclic graph $C_3$ is

$$D = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

So, the final form of the above mathematical model is:

$$\min f = x_1 + x_2 + x_3$$

subject to:

$$6|x_1 - x_2| \geq 1; 6|x_1 - x_3| \geq 1;$$

$$6|x_2 - x_3| \geq 1; \quad x_1, x_2, x_3 \geq 0$$

MATLAB solver has 3 for the solution of the above example.

### Table 3. Comparison between Standard Radio mean number, Algorithm and Integer Linear Programming for the upper bound of radio mean number for the cycle graph.

| n | Standard RM | Proposed Algorithm | Integer Linear Programming |
|---|-------------|--------------------|---------------------------|
|   | $\text{min}(C_n)$ | CPU Time | $\text{min}(C_n)$ |
| 1 | - | - | - |
| 2 | - | - | - |
| 3 | 3 | 0.00388 | 3 | 0.192434 |
| 4 | 4 | 0.007196 | 5 | 0.197589 |
| 5 | 5 | 0.014144 | 6 | 0.204881 |
| 6 | 6 | 0.209584 | 7 | 0.208477 |
| 7 | 7 | 0.34647 | 8 | 0.210394 |
| 8 | 8 | 0.38613 | 9 | 0.210055 |
| 9 | 9 | 0.37399 | 10 | 0.210209 |
| 10 | 10 | 0.37588 | 11 | 0.217347 |
| 11 | 11 | 0.131079 | 12 | 0.219025 |
| 12 | 12 | 0.133592 | 13 | 0.219235 |
| 13 | 13 | 0.219854 | 14 | 0.220924 |
| 14 | 14 | 0.225044 | 15 | 0.222914 |
| 15 | 15 | 0.313199 | 16 | 0.225655 |
| 16 | 16 | 0.375647 | 17 | 0.225115 |
| 17 | 17 | 0.474422 | 18 | 0.229262 |
| 18 | 18 | 0.574174 | 19 | 0.23136 |
| 19 | 19 | 0.685679 | 20 | 0.237609 |
| 20 | 20 | 0.827998 | 21 | 0.243028 |
| 21 | 21 | 1.171796 | 22 | 0.245212 |
| 22 | 22 | 3.180393 | 23 | 0.248772 |
| 23 | 23 | 3.548829 | 24 | 0.250364 |
| 24 | 24 | 4.139955 | 25 | 0.253585 |
| 25 | 25 | 5.27022 | 26 | 0.254351 |
| 26 | 26 | 2.541999 | 27 | 0.256721 |
| 27 | 27 | 2.979835 | 28 | 0.26002 |
| 28 | 28 | 3.442036 | 29 | 0.2711 |
| 29 | 29 | 4.360193 | 30 | 0.302073 |
| 30 | 30 | 3.975796 | 31 | 0.302639 |
| 31 | 31 | 31.025694 | 32 | 0.459412 |

V. COMPUTATIONAL STUDY

In this section, we describe our numerical experiments and present computational results, which prove that the Integer Linear Programming Model overcomes the proposed approximate algorithm according to CPU time only. We test the proposed approaches on path graphs and cycle graphs. Table 1 describes the computing environment. MATLAB solver was used to solve the mathematical model.

In Tables 2 and 3 the abbreviations Standard RM, $\text{min}(P_n)$, and CPU Time are used to denote the exact radio mean number for path graphs and cycle graphs respectively. Table 2 shows that the proposed Integer Linear Programming Model and the proposed Algorithm 1 determine the same upper bound for the number of radio mean which closes to the exact radio mean number of the path $P_n$. On the other hand, Integer Linear Programming Model overcomes Algorithm 1 according to CPU time as shown in Figure 1.
Table 3 shows that the proposed Integer Linear Programming Model and the proposed Algorithm 1 determine the same upper bound for the number of radio mean which is close to the exact radio mean number of the cycle $C_n$. On the other hand, Table 2 and Table 3 show that the Integer Linear Programming Model overcomes the proposed algorithm Algorithm 1 according to CPU time only as shown in Figure 2.

VI. CONCLUSION

In this work, we propose three contributions. The first contribution is that we proved two theorems which determine the radio mean number for cycle graphs and path graphs. The second contribution is proposing a new approximate algorithm which finds the upper bound for the number of radio mean for a given graph. The third contribution is that we proposed a new mathematical model for finding the upper bound for the number of radio mean of a graph. Finally, the experimental results analysis and statistical test prove that the Integer Linear Programming Model overcame the proposed approximate algorithm according to CPU time only. In future work, we will adopt new approaches for determining the radio mean number of large graphs. These approaches are parallel processing and metaheuristic algorithms.

REFERENCES

[1] W. K. Hale, “Frequency assignment: Theory and applications,” Proc. IEEE, vol. 68, no. 12, pp. 1497–1514, Dec. 1980.
[2] G. Chartrand, D. Erwin, P. Zhang, and F. Harary, “Radio labelings of graphs,” Bull. Inst. Ars Combinatoria Appl., vol. 33, pp. 77–85, Sep. 2001.
[3] P. Zhang, “Radio labelings of cycles,” Ars Combinatoria, vol. 65, pp. 21–32, Oct. 2002.
[4] D. D.-F. Liu and X. Zhu, “Multilevel distance labelings for paths and cycles,” SIAM J. Discrete Math., vol. 19, no. 3, pp. 610–621, Jan. 2005.
[5] E. M. Badr and M. I. Moussa, “An upper bound of radio $k$-coloring problem and its integer linear programming model,” Wireless Netw., vol. 26, no. 7, pp. 4955–4964, Oct. 2020, doi: 10.1007/s11276-019-01979-8.
[6] L. Saha and P. Panigrahi, “A graph radio $k$-coloring algorithm,” in Combinatorial Algorithms (Lecture Notes in Computer Science) Vol. 7643, S. Arumugan and W. F. Smyth Eds, Berlin, Germany: Springer, 2012.
[7] L. Saha and P. Panigrahi, “A new graph radio $k$-coloring algorithm,” Discrete Math., Algorithms Appl., vol. 11, no. 1, 2019, Art. no. 1950005.
[8] R. Ponraj, S. S. Narayanan, and R. Kala, “Radio mean labeling of a graph,” AKCE Int. J. Graphs Combinatorics, vol. 12, nos. 2–3, pp. 224–228, Nov. 2015.
[9] R. Ponraj and S. S. Narayanan, “On radio mean number of some graphs,” Int. J. Math. Combinatorics, vol. 3, pp. 41–48, 2014.
[10] K. Suthitha, C. D. Raj, and A. Subramanian, “Radio mean labeling of Path and Cycle related graphs,” Global J. Math. Sci., Theory Practical, vol. 9, no. 3, pp. 337–345, 2017.
[11] M. N. Shafique, M. M. Kharshid, H. Rahman, A. Khanna, and D. Gupta, “The role of big data predictive analytics and radio frequency identification in the pharmaceutical industry,” IEEE Access, vol. 7, pp. 9013–9021, 2019.
[12] S. Varshney, C. Kumar, A. Swaroop, A. Khanna, D. Gupta, J. Rodrigues, P. Pinheiro, and V. de Albuquerque, “Energy efficient management of pipelines in buildings using linear wireless sensor networks,” Sensors, vol. 18, no. 8, p. 2618, Aug. 2018, doi: 10.3390/18082618.
[13] R. Balakrishnan and K. Renganathan, A Textbook of Graph Theory. New York, NY, USA: Springer, 2012.
[14] E. M. Badr, “On graceful labeling of the generalization of cyclic snakes,” J. Discrete Math. Sci. Cryptogr, vol. 18, no. 6, pp. 773–783, Nov. 2015.
[15] M. S. Bazaraa, J. J. Jarvis, and H. D. Sherali, Linear Programming and Network Flows, 3rd ed. New York, NY, USA: Wiley, 2004.
[16] G. Chartrand, L. Eroh, M. A. Johnson, and O. R. Oellermann, “Resolvability in graphs and the metric dimension of a graph,” Discrete Appl. Math., vol. 105, nos. 1–3, pp. 99–113, Oct. 2000.
[17] E. S. Badr, K. Paparrizos, N. Samaras, and A. Sifaleras, “On the basis inverse of the exterior point simplex algorithm,” in Proc. 17th Nat. Conf. Hellenic Oper. Res. Soc. (HELORS), Rio, Greece, Jun. 2005, pp. 677–687.
[18] E. S. Badr, K. Paparrizos, and B. G. T. Varkas, “Some computational results on the efficiency of an exterior point algorithm,” in Proc. 18th Nat. Conf. Hellenic Oper. Res. Soc. (HELORS), Rio, Greece, Jun. 2006, pp. 1103–1115.
[19] M. I. Moussa, E. M. Badr, and S. Almotairi, “A data hiding algorithm based on DNA and elliptic curve cryptosystems,” J. Inf. Hiding Multimedia Signal Process., vol. 10, no. 3, pp. 458–469, 2019.
[20] E. M. Badr and H. Eldengy, “A Hybrid water cycle-particle swarm optimization for solving the fuzzy underground water confined steady flow,” Indonesian J. Elect. Eng. Comput. Sci., vol. 19, no. 1, pp. 492–504, 2019.
[21] E. Badr and S. Almotairi, “On a dual direct cosine simplex type algorithm and its computational behavior,” Math. Problems Eng., vol. 2020, pp. 1–8, May 2020.

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