The competition-common enemy graphs of digraphs satisfying Conditions $C(p)$ and $C'(p)$

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Abstract

S. -R. Kim and F. S. Roberts (2002) introduced the following conditions $C(p)$ and $C'(p)$ for digraphs as generalizations of the condition for digraphs to be semiorders. The condition $C(p)$ (resp. $C'(p)$) is: For any set $S$ of $p$ vertices in $D$, there exists $x \in S$ such that $N^+_D(x) \subseteq N^+_D(y)$ (resp. $N^-_D(x) \subseteq N^-_D(y)$) for all $y \in S$, where $N^+_D(x)$ (resp. $N^-_D(x)$) is the set of out-neighbors (resp. in-neighbors) of $x$ in $D$. The competition graph of a digraph $D$ is the (simple undirected) graph which has the same vertex set as $D$ and has an edge between two distinct vertices $x$ and $y$ if $N^+_D(x) \cap N^+_D(y) \neq \emptyset$. Kim and Roberts characterized the competition graphs of digraphs which satisfy Condition $C(p)$.

The competition-common enemy graph of a digraph $D$ is the graph which has the same vertex set as $D$ and has an edge between two distinct vertices $x$ and $y$ if it holds that both $N^+_D(x) \cap N^+_D(y) \neq \emptyset$ and $N^-_D(x) \cap N^-_D(y) \neq \emptyset$. In this note, we characterize the competition-common enemy graphs of digraphs satisfying Conditions $C(p)$ and $C'(p)$.

Keywords: competition-common enemy graph; semiorder; interval order; Condition $C(p)$

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1. Introduction

J. E. Cohen \[2\] introduced the notion of a competition graph in 1968 in connection with a problem in ecology. The competition graph \(C(D)\) of a digraph \(D\) is the (simple undirected) graph \(G = (V, E)\) which has the same vertex set as \(D\) and has an edge between two distinct vertices \(x\) and \(y\) if and only if \(N^+_D(x) \cap N^+_D(y) \neq \emptyset\), where \(N^+_D(x) := \{v \in V(D) \mid (x, v) \in A(D)\}\) is the set of out-neighbors of \(x\) in \(D\). It has been one of important research problems in the study of competition graphs to characterize the competition graphs of digraphs satisfying some conditions.

Definition. A digraph \(D = (V, A)\) is called a semiorder if there exist a real-valued function \(f : V \rightarrow \mathbb{R}\) on the set \(V\) and a positive real number \(\delta \in \mathbb{R}\) such that \((x, y) \in A\) if and only if \(f(x) > f(y) + \delta\).

A digraph \(D = (V, A)\) is called an interval order if there exists an assignment \(J : V \rightarrow 2^\mathbb{R}\) of a closed real interval \(J(x) \subset \mathbb{R}\) to each vertex \(x \in V\) such that \((x, y) \in A\) if and only if \(\min J(x) > \max J(y)\).

Kim and Roberts characterized the competition graphs of semiorders and interval orders as follows:

**Theorem 1.1** \([3]\). Let \(G\) be a graph. Then the following are equivalent.

(a) \(G\) is the competition graph of a semiorder,

(b) \(G\) is the competition graph of an interval order,

(c) \(G = K_r \cup I_q\) where if \(r \geq 2\) then \(q \geq 1\).

Moreover, Kim and Roberts \([3]\) introduced some conditions, which are called Condition \(C(p)\) and Condition \(C'(p)\), for digraphs as generalizations of the condition for digraphs to be semiorders, and they gave a characterization of the competition graphs of digraphs satisfying Condition \(C(p)\) to show Theorem 1.1 as its corollary.

D. D. Scott \([5]\) introduced the competition-common enemy graph of a digraph in 1987 as a variant of competition graph. The competition-common enemy graph of a digraph \(D\) is the graph which has the same vertex set as \(D\) and has an edge between two distinct vertices \(x\) and \(y\) if it holds that both \(N^+_D(x) \cap N^+_D(y) \neq \emptyset\) and \(N^-_D(x) \cap N^-_D(y) \neq \emptyset\), where \(N^-_D(x) := \{v \in V(D) \mid (v, x) \in A(D)\}\) is the set of in-neighbors of \(x\) in \(D\).

In this note, we characterize the competition-common enemy graphs of semiorders and interval orders as follows:

**Theorem 1.2.** Let \(G\) be a graph. Then the following are equivalent.

(a) \(G\) is the competition-common enemy graph of a semiorder,
2. Main Results

2.1. Conditions $C(p)$ and $C'(p)$

**Definition.** Let $D$ be a digraph. For a set $S$ of vertices in $D$, we define the following:

- $F_D^+(S) := \{ x \in S \mid N_D^+(x) \subseteq N_D^+(y) \text{ for all } y \in S \}$,
- $F_D^-(S) := \{ x \in S \mid N_D^-(x) \subseteq N_D^-(y) \text{ for all } y \in S \}$,
- $H_D^+(S) := \{ x \in S \mid N_D^+(x) \supseteq N_D^+(y) \text{ for all } y \in S \}$,
- $H_D^-(S) := \{ x \in S \mid N_D^-(x) \supseteq N_D^-(y) \text{ for all } y \in S \}$.

(Note that, in [3], an element in $F_D^+(S)$ is called a *foot* of $S$ and an element in $H_D^+(S)$ is called a *head* of $S$.)
Figure 2: Elements $x$ in $\mathcal{F}^+_D(S)$, $\mathcal{F}^-_D(S)$, $\mathcal{H}^+_D(S)$, and $\mathcal{H}^-_D(S)$

Let $p$ be a positive integer with $p \geq 2$. We say that $D$ satisfies Condition $C(p)$ (resp. Condition $C''(p)$, Condition $C^*(p)$, Condition $C^{**}(p)$) if the set $\mathcal{F}^+_D(S)$ (resp. $\mathcal{F}^-_D(S)$, $\mathcal{H}^+_D(S)$, $\mathcal{H}^-_D(S)$) is not empty for any set $S$ of $p$ vertices in the digraph $D$.

**Proposition 2.1** ([3]). Let $2 \leq p < q$. If a digraph $D$ satisfies Condition $C(p)$, then the digraph $D$ also satisfies Condition $C(q)$.

**Lemma 2.2.** Let $D$ be a digraph and $T, U$ be sets of vertices in $D$. If $\mathcal{F}^-_D(T) \cap U \neq \emptyset$, then $\mathcal{F}^-_D(U) \subseteq \mathcal{F}^-_D(T \cup U)$.

**Proof.** Take $t \in \mathcal{F}^-_D(T) \cap U$. Then $N^-_D(t) \subseteq N^-_D(t')$ for any $t' \in T \setminus U$. If $\mathcal{F}^-_D(U)$ is empty, then the lemma trivially holds. So we assume that
\( F_D(U) \neq \emptyset \). Take any \( u \in F_D(U) \). Then \( N_D^{-}(u) \subseteq N_D^{-}(u') \) for any \( u' \in U \).

Since \( t \in U \), we have \( N_D^{-}(u) \subseteq N_D^{-}(t) \). Therefore, \( N_D^{-}(u) \subseteq N_D^{-}(t') \) for any \( t' \in T \setminus U \). Thus \( N_D^{-}(u) \subseteq N_D^{-}(s) \) for any \( s \in (T \setminus U) \cup U = T \cup U \). Hence the lemma holds. \( \square \)

**Proposition 2.3.** Let \( 2 \leq p < q \). If a digraph \( D \) satisfies Condition \( C'(p) \), then the digraph \( D \) also satisfies Condition \( C'(q) \).

**Proof.** It is enough to show that \( D \) satisfies Condition \( C'(p+1) \). Let \( S \) be any set of \( p+1 \) vertices of \( D \), and let \( T \) be a subset of \( S \) with \( |T| = p \). Then \( F_D(T) \neq \emptyset \) since \( D \) satisfies Condition \( C'(p) \). Take an element \( x \) in \( F_D(T) \). Let \( U \) be a subset of \( S \) such that \( |U| = p \) and \( x \in U \). Since \( p \geq 2 \), it holds that \( T \cup U = S \). By Lemma 2.2, we have \( F_D(U) \subseteq F_D(T \cup U) = F_D(S) \). Since \( D \) satisfies Condition \( C'(p) \), \( F_D(U) \neq \emptyset \). Thus \( F_D(S) \) is not empty. \( \square \)

For a graph \( G \), we denote the set of all isolated vertices in \( G \) by \( I_G \). Then the graph \( G - I_G \) is the union of the nontrivial connected components of \( G \).

**Proposition 2.4.** Let \( G \) be the competition-common enemy graph of a digraph \( D \) which satisfies Conditions \( C(p) \) and \( C'(p) \) for some \( p \geq 2 \). Suppose that \( G - I_G \) has at least \( p \) vertices. Then \( G - I_G \) is a clique of \( G \).

**Proof.** Take any two vertices \( a \) and \( b \) in \( G - I_G \). Then \( a \) and \( b \) are not isolated. Let \( S \) be a set of \( p \) vertices in \( G - I_G \) containing the vertices \( a \) and \( b \). Since \( D \) satisfies Conditions \( C(p) \) and \( C'(p) \), there exist \( x \in F_D^+(S) \) and \( y \in F_D^-(S) \). Note that \( x \) and \( y \) are not isolated vertices. Take \( u \in N_D^+(x) \) and \( v \in N_D^-(y) \). By Condition \( C(p) \), we have \( u \in N_D^+(a) \cap N_D^-(b) \). By Condition \( C'(p) \), we have \( v \in N_D^-(a) \cap N_D^+(b) \). Therefore \( a \) and \( b \) are adjacent in \( G - I_G \). Hence the proposition holds. \( \square \)

2.2. Classification

**Theorem 2.5.** Let \( G \) be a graph and \( p \geq 2 \). Suppose that \( G - I_G \) has at least \( p \) vertices. Then \( G \) is the competition-common enemy graph of a loopless digraph satisfying Conditions \( C(p) \) and \( C'(p) \) if and only if \( G = K_r \cup I_q \) with \( r \geq p \) and \( q \geq 2 \).

**Proof.** First, we show the “only if” part. Let \( G \) be the competition-common enemy graph of a loopless digraph \( D \) satisfying Conditions \( C(p) \) and \( C'(p) \). Proposition 2.4 shows that \( G = K_r \cup I_q \) with \( r \geq p \) and \( q \geq 0 \). Suppose that \( q = 0 \) or \( q = 1 \). Since \( r \geq p \), by Propositions 2.1 and 2.3 \( D \) satisfies Conditions \( C(r) \) and \( C'(r) \). Let \( x \in F_D^+(S) \) and \( y \in F_D^-(S) \) where \( S := \)
we have $N_D^+(x) \neq \emptyset$ and $N_D^-(y) \neq \emptyset$. Let $u \in N_D^+(x)$ and $v \in N_D^-(y)$. If $u = v$, then $(s, u) \in A(D)$ and $(u, s) \in A(D)$ for any $s \in S$. Let $S'$ be a set of $p$ vertices containing the vertex $u$. Note that $S' \setminus \{u\} \subseteq S$ since $q \leq 1$. By Condition $C(p)$, $F_D^+(S') \neq \emptyset$. If $u \in F_D^+(S')$, then we have $s \in N_D^+(u) \subseteq N_D^+(s)$ for $s \in S' \setminus \{u\}$, i.e., $(s, s) \in A(D)$, which contradicts that $D$ is loopless. If $s \in F_D^+(S')$ for some $s \in S' \setminus \{u\}$, then we have $u \in N_D^+(s) \subseteq N_D^-(u)$, i.e., $(u, u) \in A(D)$, which contradicts that $D$ is loopless. Therefore $u$ and $v$ must be distinct. Since $q \leq 1$, at least one of $u$ and $v$ is in $S = V(G - I_G)$. If $u \in S$, then we have $u \in N_D^+(x) \subseteq N_D^+(u)$, i.e., $(u, u) \in A(D)$. If $v \in S$, then we have $v \in N_D^-(y) \subseteq N_D^-(v)$, i.e., $(v, v) \in A(D)$. In any case, we reach a contradiction. Thus we have $q \geq 2$.

Second, we show the “if” part. Let $G = K_r \cup I_q$ with $r \geq p$ and $q \geq 2$. Let $a$ and $b$ be distinct vertices in $I_q$. We define a digraph $D$ by $V(D) := V(G)$ and $A(D) := \{(a, x) \mid x \in V(K_r)\} \cup \{(x, b) \mid x \in V(K_r)\} \cup \{(a, b)\}$. Then $D$ is loopless, $D$ satisfies Conditions $C(p)$ and $C'(p)$, and the competition-common enemy graph of $D$ is equal to $G$. □

The double competition number $dk(G)$ of a graph $G$ is the minimum number $k$ such that $G$ with $k$ new isolated vertices is the competition-common enemy graph of an acyclic digraph.

**Theorem 2.6.** Let $G$ be a graph and $p \geq 2$. If $G$ is the competition-common enemy graph of an acyclic digraph $D$ satisfying Conditions $C(p)$ and $C'(p)$, then $G$ is one of the following graphs:

(a) $I_q \ (q \geq 1)$,
(b) $K_r \cup I_q \ (r \geq p, \ q \geq 2)$,
(c) $H \cup I_q$ where $|V(H)| < p$, $I_H = \emptyset$, and $q \geq dk(H)$.

**Proof.** Let $G$ be the competition-common enemy graph of an acyclic digraph $D$ satisfying Conditions $C(p)$ and $C'(p)$. If there is no nontrivial connected component in $G$, then (a) holds. Let $H$ be the union of all nontrivial connected components of $G$. Then we have $G = H \cup I_q$ with $q \geq 0$ and $I_H = \emptyset$. If $H$ has at least $p$ vertices, then it follows from Theorem 2.5 that $G = K_r \cup I_q$ with $r \geq p$ and $q \geq 2$, i.e., (b) holds. Suppose that the number of the vertices of $H$ is less than $p$. Since $G$ is the competition-common enemy graph of an acyclic digraph $D$, the double competition number $dk(G)$ of $G$ is equal to 0. Therefore, there must be at least $dk(H)$ vertices in $I_q$. Hence (c) holds. □
2.3. Proof of Theorem 1.2

Proof of Theorem 1.2. (a) ⇒ (b): Since semiorders are a special case of interval orders where every interval has the same length, (a) implies (b).

(b) ⇒ (c): We can easily check that any interval order satisfies Conditions $C(2)$ and $C'(2)$. By Theorem 2.6 with $p = 2$, we can conclude that if (b) then (c).

(c) ⇒ (a): Suppose that $G = I_q$ ($q \geq 1$) or $G = K_r \cup I_q$ ($r \geq 2, q \geq 2$). When $G = I_q$, we let $f_1(x) := 0$ for any $x \in V(G)$ and let $\delta_1 := 1$. Then $G$ is the competition-common enemy graph of the semiorder defined by $f_1$ and $\delta_1$. When $G = K_r \cup I_q$, we take a vertex $a$ in $I_q$, let $f_2(x) := 0$ for any $x \in V(K_r)$, $f_2(a) := 2$, and $f_2(b) := -2$ for any $b \in V(I_q) \setminus \{a\}$, and let $\delta_2 := 1$. Then $G$ is the competition-common enemy graph of the semiorder defined by $f_2$ and $\delta_2$.

Hence Theorem 1.2 holds. \qed

3. Concluding Remarks

In this section, we present some problems for further study.

In Theorem 2.5, we gave a characterization of the competition common-enemy graphs $G$ of digraphs satisfying Conditions $C(p)$ and $C'(p)$ if the number of the vertices of $G - I_G$ is at least $p$.

**Problem 3.1.** Characterize the competition-common enemy graphs $G$ of digraphs satisfying Conditions $C(p)$ and $C'(p)$ when the number of the vertices of $G - I_G$ is less than $p$.

In this note, we didn’t consider Conditions $C^*(p)$ and $C^{*'}(p)$.

**Problem 3.2.** Characterize the competition-common enemy graphs of digraphs satisfying Conditions $C^*(p)$ and $C^{*'}(p)$.

Niche graphs are another variant of competition graphs and were introduced by C. Cable, K. F. Jones, J.R. Lundgren, and S. Seager [1]. The niche graph of a digraph $D$ is the graph which has the same vertex set as $D$ and has an edge between two distinct vertices $x$ and $y$ if $N^+_D(x) \cap N^-_D(y) \neq \emptyset$ or $N^-_D(x) \cap N^+_D(y) \neq \emptyset$.

**Problem 3.3.** What are the niche graphs of digraphs satisfying Conditions $C(p)$, $C'(p)$, $C^*(p)$, or $C^{*'}(p)$?
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