The Quark Propagator in Momentum Space

Jon Ivar Skullerud and Anthony G. Williams

CSSM, University of Adelaide, Adelaide 5000, Australia

The quark propagator is calculated in the Landau gauge at $\beta = 6.0$. A method for removing the dominant, tree-level lattice artefacts is presented, enabling a calculation of the momentum-dependent dynamical quark mass.

1. Introduction

The quark propagator is one of the fundamental quantities in QCD. By studying the mass function, which is the scalar part of the quark propagator, we can gain insight into the mechanisms of chiral symmetry breaking. The momentum dependence of the quark propagator is also used extensively as input in Dyson–Schwinger equations for hadronic matrix elements. A lattice study of the quark propagator may enable us to check the validity of the models used in these calculations.

2. $O(a)$-improved quark propagator

All $O(a)$ errors in the fermion action can be removed by adding terms to the Lagrangian

$$
\mathcal{L}(x) = \mathcal{L}^W - \frac{i}{4} \bar{c}_w a \bar{\psi} \sigma_{\mu\nu} F_{\mu\nu} \psi + \frac{b_q a m}{2g_0} \text{Tr}(F_{\mu\nu} F_{\mu\nu}) - b_m a m^2 \bar{\psi} \psi + c_1 a \bar{\psi} \not{D} \psi + c_2 a m \bar{\psi} \not{D} \psi \tag{1}
$$

$m$ should here be taken to be the subtracted bare mass $m \equiv m_0 - m_c$. The $b_q$ and $b_m$ terms correspond to a rescaling of the coupling constant and the mass respectively. The two last terms can be eliminated by a field transformation

$$
\psi \rightarrow \psi' = (1 + b_q a m)(1 - c_q a \not{D}) \psi \equiv L \psi
$$

$$
\bar{\psi} \rightarrow \bar{\psi}' = (1 + b_q a m)\bar{\psi}(1 + c_q a \not{D}) \equiv \bar{\psi} R \tag{2}
$$

The tree level improved action after the transformation (3) has $c_{sw} = 1, b_q = 1, c_q = \frac{1}{2}$. The improved propagator is given by

$$
S(x, y) \equiv \langle \psi'(x) \bar{\psi}'(y) \rangle = L(x) S_0(x, y) R(y) \tag{3}
$$

where $S_0(x, y) \equiv \langle \psi(x) \bar{\psi}(y) \rangle$. Since the propagator $S_0$ is defined as the inverse of the fermion matrix $M(x) \equiv \not{D}(x) + m_0 + O(a)$, we can use this to obtain another, simpler expression for the improved propagator:

$$
S(x, y) = (1 + 2(b_q + c_q) a m) S_0(x, y) - 2a c_q \delta(x - y) + O(a^2) \tag{4}
$$

With this in mind, we define the tree-level improved propagator $S_I(x, y)$ as

$$
S_I(x - y) \equiv (1 + ma) S_0(x - y) - \frac{a}{2} \delta(x - y) \tag{5}
$$

We will also introduce the tree-level ‘rotated’ propagator $S_R(x, y)$ as the special case of the improved propagator in (3),

$$
S_R(x, y) \equiv (1 + \frac{a m}{2}) L' S_0(x, y) R'(y) \tag{6}
$$

where $L' \equiv 1 - a \not{D} / 4$ and $R' \equiv 1 + a \not{D} / 4$.

3. Analysis

The momentum space quark propagator in a particular gauge is given by $S(p) = \sum_x e^{-ipx} S(x, 0)$. We introduce the following ‘lattice momenta’ $k_{\mu} = \frac{a}{2} \sin(p_{\mu} a)$ and $k_{\mu} = \frac{a}{2} \sin(p_{\mu} a / 2)$ which differ by

$$
\frac{a^2}{4} \Delta k^2 \equiv \bar{k}^2 - k^2 = \frac{a^2}{4} \sum_{\mu} p_{\mu}^4 + O(a^4) \tag{7}
$$
In the continuum, the quark propagator has the following general form,

\[ S(p) = \frac{1}{i \not{p} A(p) + B(p)} \equiv \frac{Z^c(p)}{i \not{p} + M^c(p)} \quad (8) \]

We expect the lattice quark propagator to have a similar form, but with \( \not{k} \) replacing \( \not{p} \):

\[ S(p) = \frac{Z(p)}{i \not{k} + M(p)} \quad (9) \]

The dimensionless Wilson fermion propagator at tree level is

\[ S_0^{(0)}(p) = \frac{-i \not{k} a + m_0 a + \frac{1}{2} \tilde{k}^2 a^2}{k^2 a^2 + (m_0 a + \frac{1}{2} \tilde{k}^2 a^2)^2} \quad (10) \]

The tree level ‘improved’ propagator is given by

\[ S_I^{(0)}(p) = (1 + m_0 a) S_0^{(0)}(p) - \frac{1}{2} \quad (11) \]

If we write

\[ \left( S_I^{(0)}(p) \right)^{-1} = \frac{i \not{k} a + m_0 a + a \Delta M^{(0)}(p)}{Z^{(0)}(p)} \quad (12) \]

we find

\[ Z^{(0)}(p) = \frac{k^2 a^2 (1 + m_0 a)^2 + B_I^2}{(1 + m_0 a) D} \quad (13) \]

\[ a \Delta M^{(0)}(p) = \frac{m_0^2 a^2 - a^4 \Delta k^2 + a^4 \hat{k}^4 / 4}{1 + m_0 a} \quad (14) \]

where

\[ D = k^2 a^2 + \left( \frac{m_0 a + \frac{1}{2} \tilde{k}^2 a^2}{2} \right)^2 \quad (15) \]

\[ B_{II} = m_0 a + \frac{m_0^2 a^2}{2} + \frac{a^4 \Delta k^2}{2} - \frac{a^4 \hat{k}^4}{8} \quad (16) \]

The corresponding tree level expressions for \( S_R \) are considerably more complicated than those for \( S_I \), and we will not reproduce them here.

We are here primarily interested in studying the deviation of the quark propagator from its tree level value. Since QCD is asymptotically free, we should at sufficiently high momentum values get \( S(p) \to S^{(0)}(p) \) up to logarithms. We will attempt to separate out the tree level behaviour by writing

\[ S^{-1}(p a) = \frac{i a \not{k} + a M_r(p a) + a \Delta M^{(0)}(p a)}{Z_r(p a) Z^{(0)}(p a)} \quad (17) \]

where \( m_0 \) is replaced by \( m \) in the expressions for \( Z^{(0)} \) and \( \Delta M^{(0)} \). Asymptotically, we expect that \( Z_r(p a) \to 1 \) and \( M_r(p a) \to m \) up to logarithms.

4. Results

The quark propagator is calculated at \( \beta = 6.0 \) on a \( 16^3 \times 48 \) lattice, using the mean-field improved value \( c_{sw} = 1.479 \). The configurations were fixed to Landau gauge with an accuracy of \( \theta_{max} = 10^{-12} \). At \( \kappa = 0.137 \), corresponding to \( ma = 0.0603 \), we have generated both \( S_0 \) and \( S_R \). At \( \kappa = 0.1381 \), corresponding to \( ma = 0.031 \), only \( S_0 \) was generated. \( S_I \) is easily constructed from \( S_0 \).

When we calculate \( Z \) and \( M \) without factoring out the tree-level behaviour, it becomes clear that both improved propagators \( (S_I \) and \( S_R \)) are completely dominated by the unphysical tree level behaviour at high momenta. In particular, \( B \equiv Z M \) computed from \( S_I \) becomes large and negative, approaching a value of \( a B = -3 \). Only in the infrared, below \( pa \lesssim 0.8 \), might we be able to extract physically significant information.

Fig. 1 shows \( Z \) and \( M \) as functions of \( pa \) for both \( S_0 \), \( S_I \) and \( S_R \). In order to ease the comparison and remove residual anisotropy, we have...
selected momenta lying within one unit of spatial momentum of the 4-dimensional diagonal. While all three show a dramatic improvement on the unsubtracted data, the large momentum behaviour of the mass function from $S_I$ is poor as a consequence of the pathological behaviour of $S_I^{(0)}(p)$ at high momenta. This is due to the cancellation of large terms in the subtraction for the mass. It is therefore desirable to use the definition $S_R$ for the improved propagator.

Below $p a \sim 0.9$, the values for $M_r$ agree within errors for the two versions of the improved propagator. In particular, the value for the infrared mass $M_r(0)$ comes out the same. In contrast, $S_0$ yields a mass which is 3-4σ higher.

Fig. 2 shows $M_r(p)$ calculated from $S_I$ for both quark masses. We see that the infrared mass changes only slightly as the bare quark mass is halved, pointing to a dynamically generated quark mass of $(300 \pm 30)$ MeV in the chiral limit. $Z_r$ turns out not to depend on the quark mass.

5. Conclusion and further work

We have used two different definitions of the $\mathcal{O}(a)$ improved quark propagator. We make use of asymptotic freedom to factor out the tree level behaviour, replacing it with the ‘continuum’ tree level behaviour $Z(p) = 1, M(p) = m$. This tree level subtraction dramatically improves the data. The relatively poor behaviour of the tree level subtracted $S_I$ can be put down to the large tree level finite-$a$ effects which require fine tuning to subtract off correctly.

For $p a \lesssim 1$, we see that $M_r(p)$ falls off with $p$ as expected. The values obtained from $S_R$ and from $S_I$ are consistent, while those for the unimproved propagator $S_0$ differ significantly. We find that $M_r(0)$ approaches a value of $300 \pm 30$ MeV in the chiral limit.

We also find a significant dip in the value for $Z_r(p)$ at low momenta. This is more pronounced for the improved propagators than for the unimproved one. The next step will be a quantitative study of the functional form for $Z_r$ and $M_r$.

Finite volume effects have not been studied, but there is no sign of any anisotropy at low $p$, indicating that finite volume effects are small. Repeating these calculations at a different lattice spacing is also essential especially to get reliable results for the quark mass.

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