Synchronization in a class of chaotic systems

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Abstract
In this work, the synchronization problem of a master-slave system of autonomous ordinary differential equations (ODEs) is considered. Here, the systems are, chaotic with a nonlinearity represented by a piecewise linear function, non-identical and linearly coupled.

The idea behind our methodology is quite simple: we couple the systems with a linear function of the difference between the states of the systems and we propose a formal solution for the ODE that governs the evolution of that difference and then we determine what the parameters should be of the coupling function, so that the solution of that ODE is a fixed point close to zero.

As the main result, we obtain conditions for the coupling function that guarantize the synchronization, based on a suitable descomposition of the system joined to a fixed point theorem.

The scheme seems to be valid for a wide class of chaotic systems of great practical utility.

1 INTRODUCTION
Synchronization in dynamical systems describes the correlated evolution of two or more systems under specific conditions for a time interval [18]. We adopt this, as the definition of synchronization although, as suggested by [5], it is very hard to establish a single definition of synchronization that encompasses each and every known example of synchronization. This phenomenon has been studied since the century 17th, when Christiaan Huygens gave a detailed description about the synchronization of dynamical systems formed by two pendulum-clocks hanging from simple support, see for example [29].

Over time, the interest in the study of synchronization has increased. This is due to its relevant manifestation in fields such as Communications [16], Nanotechnology [28] or Biology [24, 25].

The synchronization, in general, can occur spontaneously between interconnected parts of a particular system as in [29], or it can be induced, establishing connections in the systems that promote it, both on localized [23, 1] or spatially extended systems [21, 12, 8]. In [24] and [25], there are two very interesting examples from human physiology.

When considering chaotic dynamical systems that are uncoupled, sensitivity to initial conditions prevents spontaneous synchronization. Thus, a problem of synchronization of chaotic systems can be
posed as the choice of coupling for the systems and the subsequent conditions to obtain the desired correlations as time evolves.

In this direction, the first work that made a very important impact corresponds to Fujisaka and Yamada [11], although the best known reference is by Pecora and Carrol [22]. Since then, the number of works describing fundamental aspects of the phenomenon or its applications has been increasing significantly, see for example the reviews [17, 26] and the references that are found there that add up to several hundred.

In the particular case of synchronization of piece-wise linear chaotic systems, there are many synchronization strategies with a significant number of them based on control strategies or using the Lyapunov approach, see for example [3, 20] and the references found there.

Here, we deal with the problem of establishing conditions that guarantee synchronization between two different chaotic systems. Our scenario, consider a master-slave nonidentical chaotic systems where the nonlinearities are represented by piecewise linear functions and we use as an example the Chua’s equations [7]. These systems are simple electronic circuit that exhibits classic chaotic behavior and we will use them as representatives of a class of chaotic systems with nonlinearity given by piecewise linear functions [27, 4, 19].

The solution to the synchronization problem mentioned above, in this case, begins with the selection of a linear coupling, although we believe that non-linear couplings [10] can be included in this same scheme, function. For this kind of coupling, several strategies have been proposed to achieve synchronization, see for instance [21]

Once the form of the coupling function has been chosen, there are sufficient conditions that the parameters of that function must satisfy, so that synchronization is achieved.

In order to expose our synchronization problem and its solution, we have organized the work as follows:

In section 2, we present the problem to be considered. Specifically the type of dynamical systems and the type of coupling function between them. Section 3 is devoted to develop the synchronization strategy. In section 4, we present the application of the results in Chua’s equations as a representative of a class of piece-wise linear nonlinearity. In section 4, a numerical implementation of the before mentioned results is shown. In section 5, we present some final remarks. There we highlight the characteristics that the systems considered in our work have, and also mention considerations that may lead us to carry out further research. Finally, with the intention of making the exposition more continuous, in the appendix, we offer the proofs for all the propositions, lemmas and the theorem associated with our main result, in cases where it is necessary.

2 The synchronization problem

Let us consider as synchronization problem, the selection of the coupling parameter $\nu$ in the following master-slave system:

$$\begin{align*}
\dot{x} &= f(x, \tilde{\mu}), \\
\dot{y} &= f(y, \mu) + \nu(y - x),
\end{align*}$$

where $\nu$ is a real constant, $\tilde{\mu}, \mu$ are vector parameters in $\mathbb{R}^m$ and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function.

To pose the problem from which we will be able to guarantee conditions that imply the synchronization of the system (1), we will begin by considering a bounded solution $x(t, x_0, \tilde{\mu})$ of (1).
Here \( x(t, x_0, \mu) \) stands by a solution such that at \( t = 0 \) gives \( x(0, x_0, \mu) = x_0 \). Now, consider the following transformation

\[
z = y - x(t, x_0, \mu).
\]  

(3)

If we consider \( y \) as a slave solution, i.e., solution of (2) with input \( x(t, x_0, \mu) \), then the previous transformation yields the non-autonomous equation

\[
\dot{z} = \nu z + f(z + x(t, x_0, \mu), \mu) - f(x(t, x_0, \mu), \mu)
\]  

:= \( F(t, z, \nu, \mu, \bar{\mu}) \).

(4)

We will now focus on the equation

\[
\dot{z} = F(t, z, \nu, \mu, \bar{\mu}),
\]  

(5)

where \( F : [0, \infty) \times \mathbb{R}_n \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n \) is a continuous function. It is assumed that \( F \) can be decomposed as the sum of three functions

\[
F(t, z, \nu, \mu, \bar{\mu}) = Bz + G(t, \mu, \bar{\mu}) + H(t, z, \mu).
\]  

(6)

In what follows, we will choose the conditions that \( F \) must meet and that limits our result to that class of functions and couplings. For the previous decomposition, we establish the following hypotheses \((H_i)\):

\( H_1. \) \( B \) is a constant real \( n \times n \) matrix for which all the eigenvalues have negative real part.

\( H_2. \) \( G : [0, \infty) \times \mathbb{R}^m \to \mathbb{R}^n \) is continuous and satisfies: if given \( \varepsilon > 0 \), then \( \delta > 0 \) exists such that

\[
\|G(t, \mu, \bar{\mu})\| < \varepsilon,
\]  

for any \( t \geq 0 \) and \( \mu, \bar{\mu} \) with \( \|\mu - \bar{\mu}\| < \delta \).

\( H_3. \) \( H : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n \) is a continuous function such that \( H(t, 0, \mu) = 0 \) for any \( \mu \in \mathbb{R}^m \) and \( t \geq 0 \). Also it satisfies the following type of Lipschitz condition: for any \( \mu \in \mathbb{R}^m \) there is a positive constant \( L = L(\mu) \) such that

\[
\|H(t, z_1, \mu) - H(t, z_2, \mu)\| \leq L\|z_1 - z_2\| , \quad t \geq 0.
\]  

These assumptions establish the characteristics of \( F \) and condition our results. Regarding \( H_1 \), the first proposition in this work shows an interesting result that will be useful later.

**Proposition 1.** If \( B \) is a constant real \( n \times n \) matrix for which all the eigenvalues have negative real part, then there are positive constants \( K, \gamma \) such that

\[
\left\| e^{Bt}z \right\| \leq Ke^{-\gamma t}\|z\|, \quad t \geq 0, \quad z \in \mathbb{R}^n.
\]  

(7)

Hypothesis \( H_2 \) tells us that the norm of \( G \), depending on how close the parameters \( \mu \) and \( \bar{\mu} \) are, can be made sufficiently small. \( H_3 \) is a type of condition that is often considered when looking for existence and uniqueness of solutions for differential equations.

Now, in our problem we study the system (5) with the decomposition given in (6) and under the hypotheses \( H_1, H_2 \) and \( H_3 \). We pursue to find solutions of (5) that, when associated with the transformation (3), guarantee synchronization of the master-slave system (1)–(2).
3 The synchronization strategy

The idea behind our approach begin by proposing a formal solution to system (5), in the form of our first lemma here, to later prove that under certain conditions, this solution converges to a fixed point such that the orbit of the slave system is close to the orbit of master system.

Lemma 1. The initial value problem

\[
\begin{aligned}
\dot{z} &= F(t, z, \mu, \bar{\mu}) \\
z(0) &= z_0,
\end{aligned}
\]  

where \(F\) is decomposed as in (6), is equivalent to

\[
z(t) = e^{Bt}z_0 + \int_0^t e^{B(t-s)}N(s, z(s), \mu, \bar{\mu})ds.
\]

Suppose \(K, \gamma\) are the constants appearing in (7). Let \(\rho > 0\) and \(\mu, \bar{\mu} \in \mathbb{R}^m\) such that

\[
\|G(t, \mu, \bar{\mu})\| < \frac{\rho\gamma}{4K}, \quad t \geq 0.
\]

With this choice of \(\rho, \mu\) and \(\bar{\mu}\), and for \(z_0 \in \mathbb{R}^n\) satisfying \(\|z_0\| < \frac{\rho}{2K}\) we define

\[
\mathcal{G}(z_0, \rho, \mu, \bar{\mu}) := \left\{ z \in C_b([0, \infty), \mathbb{R}^n) : \|z\|_\infty := \sup_{t \geq 0} \|z(t)\| \leq \rho \right\}
\]

with \(z(0) = z_0\).

Proposition 2. \(\mathcal{G}(z_0, \rho, \mu, \bar{\mu})\) is a closed subset of the Banach space \(X = C_b([0, \infty), \mathbb{R}^n)\), bounded continuous functions from \([0, \infty)\) to \(\mathbb{R}^n\), with the supremum norm.

Inspired in Lemma 1 for any \(z \in \mathcal{G}(z_0, \rho, \mu, \bar{\mu})\) we define an operator \(Tz\) by

\[
(Tz)(t) = e^{Bt}z_0 + \int_0^t e^{B(t-s)}N(s, z(s), \mu, \bar{\mu})ds,
\]

with \(t \geq 0\).

We have, due to the fact that \(e^{B(\cdot)}z_0\) is continuous and the hypothesis \(H_2, H_3\), that \(Tz\) is a continuous function for \(t \geq 0\).

Now, we state our main theoretical result. It establishes a relationship, although not explicitly given, between the parameters of the coupling function and the quality of synchronization, defined in this case as the mean square error, between the states of the master and slave systems.

Theorem 1. If

\[
\frac{KL}{\gamma} \leq \frac{1}{4},
\]

where \(L = L(\bar{\mu})\) is the constant given in \(H_3\), then \(T\) acts from \(\mathcal{G}(z_0, \rho, \mu, \bar{\mu})\) into itself and also has a unique fixed point in \(\mathcal{G}(z_0, \rho, \mu, \bar{\mu})\).
Lemma 2. Let $M$ be a non-negative constant and let $f$ and $g$ be continuous non-negative functions, for $a \leq t \leq b$, satisfying
\[ f(t) \leq M + \int_a^t f(s)g(s)ds, \quad a \leq t \leq b, \]
then
\[ f(t) \leq Me^{\int_a^t g(s)ds} \quad a \leq t \leq b. \]

If the operator $T$ is considered as a function of $z_0$; that is $T = T_{z_0}$, and $z^*(\cdot, z_0)$ is the unique fixed point, then we have that $z^*(t, z_0)$ is continuous in $z_0$ uniformly with respect to $t$. In fact: Consider the sets $G(z_0, \rho, \mu, \bar{\mu})$, $G^\ast (z_0, \rho, \mu, \bar{\mu})$ and we denote by $z^* := z^*(\cdot, z_0)$ and $\bar{z}^* := \bar{z}^*(\cdot, \bar{z}_0)$ the fixed points of the operator $T$ on $G(z_0, \rho, \mu, \bar{\mu})$ and $G(\bar{z}_0, \rho, \mu, \bar{\mu})$, respectively. Defining $\Delta z = z^*(t, z_0) - \bar{z}^*(t, \bar{z}_0)$ we have
\[ \Delta z = e^{Bt}(z_0 - \bar{z}_0) + \int_0^t e^{B(t-s)} \Delta H^* \, ds. \]  
with, $\Delta H^* = H(s, z^*(s, z_0), \mu) - H(s, \bar{z}^*(s, \bar{z}_0), \mu)$.

Now,
\[ \|\Delta z\| \leq Ke^{-\gamma t}\|z_0 - \bar{z}_0\| + \int_0^t KL e^{-\gamma(t-s)}\|z^*(s, z_0) - \bar{z}^*(s, \bar{z}_0)\| ds. \]  
(14)

By multiplying both members of the previous inequality by $e^{\gamma t}$, we obtain
\[ e^{\gamma t}\|\Delta z\| \leq K\|z_0 - \bar{z}_0\| + \int_0^t KL e^{\gamma s}\|z^*(s, z_0) - \bar{z}^*(s, \bar{z}_0)\| ds. \]  
(15)

For this inequality we have, with $M = K\|z_0 - \bar{z}_0\|$, $f(t) = e^{\gamma t}\|z^*(t, z_0) - \bar{z}^*(t, \bar{z}_0)\|$ and $g(t) = KL$, the hypotheses of Gronwall’s lemma. Therefore, we can conclude that
\[ e^{\gamma t}\|\Delta z\| \leq K\|z_0 - \bar{z}_0\|e^{\int_0^t KLds}, \quad t \geq 0. \]

Thus,
\[ \|\Delta z\| \leq Ke^{(KL-\gamma)\|z_0 - \bar{z}_0\|}, \quad t \geq 0, \]  
(16)

because in the Theorem 1, $\frac{KL}{\gamma} \leq \frac{1}{4}$. Relation (16) implies the result of uniform continuity with respect to $t$. Moreover, $\|z^*(t, z_0) - \bar{z}^*(t, \bar{z}_0)\|$ approaches zero exponentially as $t \to \infty$ and $\|z^*(\cdot, z_0) - \bar{z}^*(\cdot, \bar{z}_0)\|_{\infty} \leq K\|z_0 - \bar{z}_0\|$. 

Next, in order to broaden the perspective of our results and highlight more about the scope of those, we apply the Gronwall’s lemma. Let us recall Gronwall’s lemma:
4 Numerical results

We have selected as an example, an archetype of a piece-wise linear chaotic systems. This class of systems, in addition to simplifying the analytical treatment of problems, has a large number of practical applications. Among them, those associated with cryptography based on synchronization piecewise-linear chaotic systems such as [13] stand out, where an attempt is made to improve the security of digital communications, so necessary in these times.

Chua’s circuit is a bridge to connect the characteristic associated to dynamics of nonlinear phenomena as: stable orbits, bifurcations or attractors, with the study of experimental chaos. This circuit, in its classic configuration, is one of the simplest chaotic systems containing an inductor, two capacitors which are the linear energy-storage elements, a linear resistor and one 2-terminal nonlinear resistor characterized by a current-voltage characteristic which has a negative slope [15]. All circuit elements are passive except for the nonlinear resistor; this element must be active in order for the circuit to become chaotic [7].

In this work, we will use the adimensional form of equation system which is a set of interdependent equations in the form of a 3-dimensional autonomous piece-wise linear ordinary differential equation (flow) described by

\[
\begin{align*}
\dot{x} &= \alpha(y - x - f(x)), \\
\dot{y} &= x - y + z, \\
\dot{z} &= -\beta y,
\end{align*}
\]

where

\[
f(x) = bx + \frac{1}{2} (b - a) \left( |x + 1| - |x - 1| \right),
\]

and \(a, b, \alpha\) and \(\beta\) are real parameter, \(a < b < 0\), \(\alpha > 0\) and \(\beta > 0\). The system (17) is known as Chua’s equation.

This piecewise function can be written as follows

\[
f(x) = \begin{cases} 
bx - (a-b), & \text{if } x \leq -1, \\
a x, & \text{if } -1 \leq x \leq 1, \\
b x + (a-b), & \text{if } x \geq 1.
\end{cases}
\]

This system is the honorary member of a class of chaotic systems to which also belong: Sprott systems [6], Murali-Lakshmanan-Chua circuits [19], memristor based circuits as in [9] or 3-dimensional piecewise-linear system as shown in [27, 30]. That systems can be relatively easily designed through electronic circuits and the corresponding dynamical equations can be numerically analysed and theoretically investigated.

In order to apply the theoretical results in the case of Chua’s systems [17], we consider

\[
z = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}, \quad x(t, x_0, \bar{\mu}) = \begin{pmatrix} x \\ y \\ z \end{pmatrix},
\]

and the parameters,

\[
\mu = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \text{and} \quad \bar{\mu} = \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix}.
\]
We have to find the representation of (4) for the case of (17). For that, first let us compute 
\[ f(z + x(t, x_0, \bar{\mu}), \mu) - f(x(t, x_0, \bar{\mu}), \bar{\mu}) \]. Then,
\[
f(z + x(t, x_0, \bar{\mu}), \mu) = \begin{pmatrix}
\alpha((\bar{y} + y) - (\bar{x} + x) - f(\bar{x} + x)) \\
(\bar{x} + x) - (\bar{y} + y) + (\bar{z} + z) \\
-\beta(\bar{y} + y)
\end{pmatrix},
\]
so that,
\[
f(z + x(t, x_0, \bar{\mu}), \mu) - f(x(t, x_0, \bar{\mu}), \bar{\mu}) \]
results in
\[
\begin{pmatrix}
\alpha((\bar{y} - \bar{x}) - (\bar{y} + z) - f(\bar{x} + x)) \\
(\bar{x} - \bar{y} + \bar{z}) \\
-\beta\bar{y}
\end{pmatrix}
+ \begin{pmatrix}
(\alpha - \bar{\alpha})(y - x - f(x)) \\
0 \\
(\beta - \beta)y
\end{pmatrix}
+ \begin{pmatrix}
\alpha(f(x) - f(\bar{x} + x)) \\
0 \\
0
\end{pmatrix}.
\]

Rewriting the last expression we have
\[
\begin{pmatrix}
-\alpha & \alpha & 0 \\
1 & -1 & 1 \\
0 & -\beta & 0
\end{pmatrix}
\begin{pmatrix}
x \\
\bar{y} \\
\bar{z}
\end{pmatrix}
+ \begin{pmatrix}
(\alpha - \bar{\alpha})(y - x - f(x)) \\
0 \\
(\beta - \beta)y
\end{pmatrix}
+ \begin{pmatrix}
\alpha(f(x) - f(\bar{x} + x)) \\
0 \\
0
\end{pmatrix}.
\]

Now, setting
\[
A = A(\mu) := \begin{pmatrix}
-\alpha & \alpha & 0 \\
1 & -1 & 1 \\
0 & -\beta & 0
\end{pmatrix},
\]
\[
G(t, \mu, \bar{\mu}) := \begin{pmatrix}
(\alpha - \bar{\alpha})(y - x - f(x)) \\
0 \\
(\beta - \beta)y
\end{pmatrix},
\]
and
\[
H(t, \mu, z) = \begin{pmatrix}
\alpha(f(x) - f(\bar{x} + x)) \\
0 \\
0
\end{pmatrix},
\]
the system (1)-(2), that through the transformation (3) lead us to equation (5), in the context of Chua’s equations (17), becomes
\[
\dot{z} = \nu z + Az + G(t, \mu, \bar{\mu}) + H(t, \mu, z) = (A + \nu I)z + G(t, \mu, \bar{\mu}) + H(t, \mu, z).
\]

Notice that we have obtained an expression like the one given in (6). Our next goal is to establish that for the expression given in (22) the hypothesis \(H_1, H_2\) and \(H_3\) are satisfied. To prove \(H_1\), we
consider $B = A + \nu I$ where $B \in \mathcal{M}_{3 \times 3}$ so that, it is important to know some facts about the matrix $A$. For any $\alpha, \beta \in \mathbb{R}$, the eigenvalues of $A$ are given by

$$\lambda_{1A} = \left( - \frac{1}{2} - \frac{\sqrt{3}i}{2} \right) h_2(\alpha, \beta)$$

$$- \left( \frac{\sqrt{3}i}{2} - \frac{1}{2} \right) \left( \frac{\beta}{3} - \frac{(\alpha+1)^2}{9} \right) + \frac{(1) (\alpha + 1)}{3},$$

$$\lambda_{2A} = \left( - \frac{1}{2} + \frac{\sqrt{3}i}{2} \right) h_2(\alpha, \beta)$$

$$- \left( \frac{1}{2} - \frac{\sqrt{3}i}{2} \right) \left( \frac{\beta}{3} + \frac{(-1)(\alpha+1)^2}{9} \right) - \frac{(\alpha + 1)}{3},$$

$$\lambda_{3A} = h_2(\alpha, \beta) - \frac{\beta}{3} + \frac{(-1)(\alpha+1)^2}{9} - \frac{(\alpha + 1)}{3},$$

where

$$h_2(\alpha, \beta) = \left( h_1(\alpha, \beta) + \frac{(\alpha + 1) \beta - 3\alpha \beta}{6} - \frac{(\alpha + 1)\beta}{27} \right)^{\frac{1}{3}},$$

and

$$h_1(\alpha, \beta) = \frac{\sqrt{\beta \left( 4\beta^2 + (8\alpha^2 - 20\alpha - 1) \beta + 4\alpha^4 + 12\alpha^3 + 12\alpha^2 + 4\alpha \right)}}{23^\frac{3}{2}}.$$

**Proposition 3.** $\lambda$ is an eigenvalue of $A$ if and only if $\lambda + \nu$ is an eigenvalue of $B$.

From here, we consider $\alpha = 9$ and $\beta = \frac{100}{7}$ so, the eigenvalues of $A$ are

$$\lambda_{1A} = -0.0639 - 3.608i,$$

$$\lambda_{2A} = -0.0639 + 3.608i,$$

$$\lambda_{3A} = -9.8721.$$

Now, the proposition 3 implies that if $\nu < 0.064$, then, for $\alpha = 9$ and $\beta = \frac{100}{7}$, all the eigenvalues of the matrix $B$ have negative real part. Thus, $H_1$ is satisfied.

Next, we consider $G(t, \mu, \bar{\mu})$ given by (20). Recall that

$$x(t, x_0, \bar{\mu}) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is bounded. This implies that $x, y, f(x)$ are bounded and there exists $R > 0$ such that $|y - x + f(x)| \leq R$ and $|y| \leq R$ for all $t \geq 0$. The next proposition tell us that $H_2$ is satisfied.

**Proposition 4.** Given $\varepsilon > 0$, then

$$\|G(t, \mu, \bar{\mu})\| < \varepsilon$$

for $t \geq 0$ if $\|\mu - \bar{\mu}\| < \frac{\varepsilon}{R}$ holds.
Now, hypothesis $H_3$ is satisfied due to the fact that

**Proposition 5.** The function $H(t, \mu, z)$ given by (21) is globally Lipschitz in $z$. Moreover, the Lipschitz constant $L$ can be chosen as $L = \alpha |a|$, being $a$ as in (18).

We say that the Chua’s System fulfills the conditions $H_1$, $H_2$ and $H_3$. Summarizing for $\alpha = 9, \beta = \frac{100}{7}, a = -\frac{8}{7}$ and $b = -\frac{2}{7}$, we found constants $K, \gamma$, and $L$ such that

$$K = 5.8472, \quad \gamma = 9.936 - 2\nu \quad \text{and} \quad L = 9 | a | = \frac{72}{7}.$$

Thus, for values of $\nu_c < -1.79$, the condition $\frac{KL}{\gamma} < \frac{1}{4}$ is satisfied. Therefore, we can apply the Theorem 1 so that

$$\|y - x(t, x_0, \bar{\mu})\| \leq \rho.$$

The technique exposed in the system (1) – (2) considers two similar copies of the system to be synchronized with different initial conditions on them. Let $x = (x, y, z) \in \mathbb{R}^3$ be the generalized coordinates corresponding to the master system, and $y = (x_s, y_s, z_s) \in \mathbb{R}^3$ those of the slave system. Thus, the whole system is

\[
\begin{align*}
\dot{x} &= \bar{\alpha}(y - x - f(x)), \\
\dot{y} &= x - y + z, \\
\dot{z} &= -\bar{\beta}y, \\
\dot{x}_s &= \alpha(y_s - x_s - f(x_s)) + \nu(x_s - x), \\
\dot{y}_s &= x_s - y_s + z_s + \nu(y_s - y), \\
\dot{z}_s &= -\beta y_s + \nu(z_s - z).
\end{align*}
\]

As we mentioned before, we concentrate our attention on the master–slave system with the usual parameters $\bar{\mu} = (\bar{\alpha}, \bar{\beta}) = (9, \frac{100}{7})$, and $(a, b) = (-\frac{8}{7}, -\frac{2}{7})$. Let us choose $\nu = -2$ since it corresponds to the minor number respect to the negative real part of the eigenvalues mentioned in Proposition 7. Also, let us consider a vector $\mu = (\alpha, \beta) = (8.9, \frac{99}{7})$, closer to $\bar{\mu}$.

To show the performance of the synchronization technique, we starting from two slightly initial conditions $(0.2, 0.4, 1.0)$ and $(0.2, 0.5, 0.9)$ and show in Figure 1 the evolution of the systems.

A quality criteria for synchronization can be settled down by defining the synchronization error $(E)$ in the interval $(t_0, T)$, as

$$E(\nu) = \int_{t_0}^{T} \| (x(t) - y(t)) \|^2 \, dt \quad (23)$$

The following figure shows the dependence of the error of synchronization as a function of the coupling parameter $\nu$.

### 5 Concluding remarks

In this work, a synchronization strategy has been proposed and developed for a class of piecewise-linear chaotic systems.

Our findings can be summarized in the following list:
Figure 1: Evolution of the master and slave systems from different initial conditions. The gray points represent the orbit of the master and the black ones that of the slave system.

Figure 2: Error of synchronization as a function of the coupling parameter, in the case of several sets of values of the slave system \( \mu_1 = (9, \frac{95}{7}), \mu_2 = (8.5, \frac{100}{7}), \mu_3 = (8, \frac{95}{7}) \).

- Our approach appears to be useful in inducing synchronization in the case of a class of systems with many practical applications, among which cryptography or the training of reaction-diffusion neural networks stand out.

- The scheme allows an analytical development of the strategy, which in turn gives us ideas for solving the problem in other contexts.

- The main result is to calibrate the intensity of the coupling, given the difference between the parameters.

The presentation exposes explicitly and in considerable detail not only the results, but the proofs in a way that we hope will also be pedagogical.

Finally, although the study of the problem is outside the scope of the work, we believe that it is possible to extend this result to the case of continuous chaotic systems (ODEs) whose non-linearity is approximated by piecewise linear functions.
Appendix

In the following, the proofs of the propositions, lemmas and theorems previously presented are shown, in the cases in which it is necessary.

A Proof of Proposition 1

Proof. The proof of this result is strongly based on the Jordan canonical form of the matrix \( B \) and can be seen in [14] (Theorem 4.2., (ii)).

B Proof of Lemma 1

Proof. It is a direct consequence of the main Theorem of Calculus.

C Proof of Proposition 2

Proof. We are going to show that \( X - \mathcal{G}(z_0, \rho, \mu, \bar{\mu}) \) is open i.e. given \( \bar{z} \in X - \mathcal{G}(z_0, \rho, \mu, \bar{\mu}) \) there exists an \( \varepsilon \)–neighborhood of \( \bar{z} \) which is contained in \( X - \mathcal{G}(z_0, \rho, \mu, \bar{\mu}) \). If \( \bar{z} \in X - \mathcal{G}(z_0, \rho, \mu, \bar{\mu}) \) then \( \bar{z}(0) \neq z_0 \) or for some \( \bar{t} \geq 0 \) we have that \( \| \bar{z}(\bar{t}) \| > \rho \).

If \( \bar{z}(0) \neq z_0 \), then \( \| \bar{z}(0) - z_0 \| := \bar{r} > 0 \). Choose \( \varepsilon = \frac{\bar{r}}{2} \) and consider the set

\[
\left\{ z \in C_b([0, \infty), \mathbb{R}^n) : \| z - \bar{z} \|_\infty < \varepsilon \right\}.
\]

We have that this \( \varepsilon \)–neighborhood is contained in \( X - \mathcal{G}(z_0, \rho, \mu, \bar{\mu}) \). In fact: If \( z \in \left\{ z \in C_b([0, \infty), \mathbb{R}^n) : \| z - \bar{z} \|_\infty < \varepsilon \right\} \) then

\[
\| z(0) - z_0 \| = \| z(0) - \bar{z}(0) + \bar{z}(0) - z_0 \| \geq \| z(0) - z_0 \| - \| z(0) - \bar{z}(0) \| \Rightarrow \bar{r} - \bar{r} \geq 0.
\]

Thus, \( z(0) \neq z_0 \).

If \( \| \bar{z}(\bar{t}) \| > \rho \) for some \( \bar{t} \geq 0 \), then the \( \varepsilon \)–neighborhood of \( \bar{z} \), with \( \varepsilon = \| \bar{z}(\bar{t}) \| - \rho \), is contained in \( X - \mathcal{G}(z_0, \rho, \mu, \bar{\mu}) \). In fact: If \( z \in \left\{ z \in C_b([0, \infty), \mathbb{R}^n) : \| z - \bar{z} \|_\infty < \varepsilon \right\} \), then

\[
\| z(\bar{t}) \| = \| z(\bar{t}) - \bar{z}(\bar{t}) + \bar{z}(\bar{t}) \| \geq \| \bar{z}(\bar{t}) \| - \| z(\bar{t}) - \bar{z}(\bar{t}) \| = \varepsilon + \rho - \| z(\bar{t}) - \bar{z}(\bar{t}) \| > \rho.
\]

Thus, \( \| z(\bar{t}) \| > \rho \).
D Proof of Lemma 2

Proof. Define \( h(t) = M + \int_a^t f(s)g(s)ds \). We have that \( h(a) = M \) and \( H(t) = f(t)g(t) \). Now, since \( f(t) \leq h(t) \), \( f(t) \geq 0 \) and \( g(t) \geq 0 \) for \( a \leq t \leq b \) it is obtained

\[
\dot{h}(t) = f(t)g(t) \leq h(t)g(t) .
\]

By multiplying both members of this inequality by \( e^{-\int_a^t g(s)ds} \), we obtain

\[
e^{-\int_a^t g(s)ds} \dot{h}(t) \leq e^{-\int_a^t g(s)ds} h(t)g(t) .
\]

Thus,

\[
e^{-\int_a^t g(s)ds} \left( \dot{h}(t) - h(t)g(t) \right) \leq 0 \quad \text{and} \quad \frac{d}{dt} \left( e^{-\int_a^t g(s)ds} h(t) \right) \leq 0 .
\]

Now, integrating from \( a \) to \( t \) we get

\[
e^{-\int_a^t g(s)ds} h(t) - e^{-\int_a^t g(s)ds} h(a) \leq 0 ,
\]

which implies \( e^{-\int_a^t g(s)ds} h(t) \leq h(a) = M \) and \( h(t) \leq Me^\int_a^t g(s)ds \). Finally, the fact that \( f(t) \leq h(t) \) produces the result

\[
f(t) \leq Me^\int_a^t g(s)ds .
\]

\[\square\]

E Proof of Theorem 1

Proof. Let \( z \in G(\mathbf{z}_0, \rho, \mu, \bar{\mu}) \). From \( H_1, [10], [12] \), we obtain

\[
\| (Tz)(t) \| \leq \| e^{B(t-a)} \mathbf{z}_0 \| + \int_0^t \| e^{B(t-s)} N(s, \mathbf{z}(s), \mu, \bar{\mu}) \| ds \\
\leq Ke^{-\gamma t} \| \mathbf{z}_0 \| + K \int_0^t e^{-\gamma(t-s)} \| N(s, \mathbf{z}(s), \mu, \bar{\mu}) \| ds \\
\leq Ke^{-\gamma t} \| \mathbf{z}_0 \| + K \int_0^t e^{-\gamma(t-s)} \| (G(s, \mu, \bar{\mu})) \| ds \\
\quad + K \int_0^t e^{-\gamma(t-s)} \| (H(s, \mathbf{z}(s), \mu)) \| ds \\
\leq \frac{\rho}{2} + \frac{\rho \gamma}{4} \int_0^t e^{-\gamma(t-s)} ds + KL \int_0^t e^{-\gamma(t-s)} \| \mathbf{z}(s) \| ds \\
\leq \frac{\rho}{2} + \frac{\rho \gamma}{4} \int_0^t e^{-\gamma(t-s)} ds + KL \| \mathbf{z} \|_{\infty} \int_0^t e^{-\gamma(t-s)} ds \\
= \frac{\rho}{2} + \left( \frac{\rho \gamma}{4} + KL \| \mathbf{z} \|_{\infty} \right) \frac{1}{\gamma} (1 - e^{-\gamma t}) \\
\leq \frac{\rho}{2} + \left( \frac{\rho \gamma}{4} + KL \| \mathbf{z} \|_{\infty} \right) \frac{1}{\gamma} \\
= \frac{\rho}{2} + \frac{\rho \gamma}{4} + \frac{KL}{\gamma} \| \mathbf{z} \|_{\infty} \\
\leq \rho .
\]
Thus, \( \|Tz\| \leq \rho \) and \( Tz \in G(z, \rho, \mu) \).

Now, let us take \( z_1, z_2 \in G(z, \rho, \mu) \), the same type of estimates yields

\[
\| (Tz_2)(t) - (Tz_1)(t) \| \leq \int_0^t \left\| e^{B(t-s)} \Delta H \right\| ds \\
\leq \int_0^t Ke^{-\gamma(t-s)} \| \Delta H \| ds \\
\leq \int_0^t KLe^{-\gamma(t-s)} \| z_2(s) - z_1(s) \| ds \\
\leq \left( \int_0^t KLe^{-\gamma(t-s)} ds \right) \| z_2 - z_1 \|_{\infty} \\
= \frac{KL}{\gamma} (1 - e^{-\gamma t}) \| z_2 - z_1 \|_{\infty} \\
\leq \frac{KL}{\gamma} \| z_2 - z_1 \|_{\infty} \\
\leq \frac{1}{4} \| z_2 - z_1 \|_{\infty}.
\]

with \( \delta H = H(s, z_2(s), \mu) - H(s, z_1(s), \mu) \)

Thus, \( T \) is a contraction on \( G(z, \rho, \mu) \) and there is a unique fixed in \( G(z, \rho, \mu) \).

\[ \square \]

**F Proof of Proposition 3**

*Proof.* Let \( B \) be a \( 3 \times 3 \) real matrix given by

\[
B = A + \nu I = \begin{pmatrix}
-\alpha + \nu & \alpha & 0 \\
1 & -1 + \nu & 1 \\
0 & -\beta & \nu
\end{pmatrix}.
\]  

(24)

By *Proposition 1*, for any value of \( \alpha, \beta \), the eigenvalues of \( B \) are given by

\[
\lambda_{1B} = \lambda_{1A} + \nu, \quad i = 1, 2, 3,
\]

then for values of \( \alpha = 9 \) and \( \beta = \frac{100}{7} \), the eigenvalues become

\[
\lambda_{1B} = -0.0639 + \nu - 3.6082i, \\
\lambda_{2B} = -0.0639 + \nu + 3.6082i, \\
\lambda_{3B} = -9.8721 + \nu.
\]

By choosen \( \nu < 0.064 \), we have established that all the eigenvalues have negative real part. By *Proposition 3*, \( B \) has the decomposition

\[
B = PJP^{-1},
\]

where \( J \) is given by

\[
J = \begin{pmatrix}
-0.0639 + \nu & -3.6082 & 0 \\
3.6082 & -0.0639 + \nu & 0 \\
0 & 0 & -9.8721 + \nu
\end{pmatrix}.
\]

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By Proposition 2, the eigenvalues of $A$ are eigenvalues for $B$ so $P$ becomes

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0.9929 & -0.4009 & -0.0969 \\ -1.5172 & -3.9579 & -0.1402 \end{pmatrix},$$

the inverse corresponding to this matrix is

$$P^{-1} = \begin{pmatrix} 0.06728 & 0.8135 & -0.0824 \\ -0.0588 & -0.2830 & -0.2240 \\ 0.9327 & -0.8135 & 0.0824 \end{pmatrix}.$$  

We know that

$$e^{Bt} = Pe^{Jt}P^{-1}. \quad (25)$$

By taking the Euclidean norm in both sides, we get

$$\|e^{Bt}\| = \|Pe^{Jt}P^{-1}\| \leq \|P\|\|P^{-1}\|\|e^{Jt}\|$$

$$= \|P\|\|P^{-1}\|\left(\begin{array}{ccc}
(-0.0639+\nu)t & \cos (3.6082t) & 0 \\
-0.0639t & -\sin (3.6082t) & 0 \\
0 & e^{(-0.0639+\nu)t} & 0
\end{array}\right)$$

$$\leq 5.8470 \left(e^{(-9.936+2\nu)t}\right)$$

$$= Ke^{-\gamma t},$$

where $K = 5.8470$ and $\gamma = 9.936 - 2\nu$. 

\[\square\]

**G Proof of Proposition 4**

**Proof.** Let $\varepsilon > 0$ be arbitrary. We have

$$\|G(t, \mu, \bar{\mu})\|^2 = |\alpha - \bar{\alpha}|^2|y - x - f(x)|^2$$

$$+ |\beta - \bar{\beta}|^2|y|^2$$

$$\leq R^2\|\mu - \bar{\mu}\|^2.$$ 

Therefore $\|G(t, \mu, \bar{\mu})\| \leq R\|\mu - \bar{\mu}\|$ and if $\|\mu - \bar{\mu}\| < \frac{\varepsilon}{R}$ then the desired result is obtained. 

\[\square\]

**H Proof of Proposition 5**

**Proof.** Consider

$$z_1 = \left(\begin{array}{c}
\bar{x}_1 \\
\bar{y}_1 \\
\bar{z}_1
\end{array}\right), \quad z_2 = \left(\begin{array}{c}
\bar{x}_2 \\
\bar{y}_2 \\
\bar{z}_2
\end{array}\right)$$

be in $\mathbb{R}^3$. 

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We have
\[ H(t, \mu, z_2) - H(t, \mu, z_1) = \begin{pmatrix} \alpha(f(\bar{x}_1 + x) - f(\bar{x}_2 + x)) \\ 0 \\ 0 \end{pmatrix} \]

From the definition of \( f \), given in (18), we pay attention, according to the values of \( \bar{x}_1 + x \) and \( \bar{x}_2 + x \), to nine cases:

i) \( \bar{x}_1 + x \leq -1, \ \bar{x}_2 + x \leq -1 \)

ii) \( \bar{x}_1 + x \leq -1, \ |\bar{x}_2 + x| \leq 1 \)

iii) \( \bar{x}_1 + x \leq -1, \ 1 \leq \bar{x}_2 + x \)

iv) \( |\bar{x}_1 + x| \leq 1, \ \bar{x}_2 + x \leq -1 \)

v) \( |\bar{x}_1 + x| \leq 1, \ |\bar{x}_2 + x| \leq 1 \)

vi) \( |\bar{x}_1 + x| \leq 1, \ 1 \leq \bar{x}_2 + x \)

vii) \( 1 \leq \bar{x}_1 + x, \ \bar{x}_2 + x \leq -1 \)

viii) \( 1 \leq \bar{x}_1 + x, \ |\bar{x}_2 + x| \leq 1 \)

ix) \( 1 \leq \bar{x}_1 + x, \ 1 \leq \bar{x}_2 + x \).

Cases i) and ix) produce
\[
\alpha(f(\bar{x}_1 + x) - f(\bar{x}_2 + x)) = \alpha b(\bar{x}_1 - \bar{x}_2).
\]

For case v) it is obtained
\[
\alpha(f(\bar{x}_1 + x) - f(\bar{x}_2 + x)) = \alpha a(\bar{x}_1 - \bar{x}_2).
\]

For case ii) we have that \( \bar{x}_2 - \bar{x}_1 \geq 0 \) and
\[
f(\bar{x}_1 + x) - f(\bar{x}_2 + x) = (b - a)(x + 1) + b\bar{x}_1 - a\bar{x}_2.
\]

Now, conditions \( a < b < 0 \) and \( x + 1 \leq -\bar{x}_1 \) imply that
\[
f(\bar{x}_1 + x) - f(\bar{x}_2 + x) \leq a(\bar{x}_1 - \bar{x}_2). \tag{26}
\]

Similarly, from conditions \( a < b < 0 \) and \( x + 1 \geq -\bar{x}_2 \) it is obtained
\[
f(\bar{x}_1 + x) - f(\bar{x}_2 + x) \geq b(\bar{x}_1 - \bar{x}_2). \tag{27}
\]

From (26) and (27), we may conclude that
\[
|\alpha(f(\bar{x}_1 + x) - f(\bar{x}_2 + x))| \leq \alpha|a||\bar{x}_2 - \bar{x}_1|.
\]

A similar treatment, to that given in case ii), is given in cases iv), vi) and viii); and in all of them it can be obtained that
\[
|\alpha(f(\bar{x}_1 + x) - f(\bar{x}_2 + x))| \leq \alpha|a||\bar{x}_2 - \bar{x}_1|.
\]
For case iii) \( f(\bar{x}_1 + x) - f(\bar{x}_2 + x) = b(\bar{x}_1 - \bar{x}_2) - 2(a - b) \). Now, \( a < b < 0 \) implies that
\[
\begin{align*}
b(\bar{x}_1 - \bar{x}_2) - 2(a - b) & \geq b(\bar{x}_1 - \bar{x}_2).
\end{align*}
\] (28)

In this case \( \bar{x}_1 - \bar{x}_2 \leq -2 \). Then
\[
\begin{align*}
b(\bar{x}_1 - \bar{x}_2) - 2(a - b) & \leq b(\bar{x}_1 - \bar{x}_2) + (a - b)(\bar{x}_1 - \bar{x}_2) \\
& = a(\bar{x}_1 - \bar{x}_2).
\end{align*}
\] (29)

From (28) and (29), we may conclude that
\[
|\alpha(f(\bar{x}_1 + x) - f(\bar{x}_2 + x))| \leq \alpha|a||\bar{x}_2 - \bar{x}_1|.
\]

In a similar way the latter estimate is also obtained for the case vii).

Finally, taking into consideration the estimates obtained in each case, we can conclude that \( H \) satisfies
\[
\|H(t, \mu, z_2) - H(t, \mu, z_1)\| \leq L\|z_2 - z_1\|,
\]
with \( L = \alpha \max(|a|, |b|) = \alpha|a| \).

References

[1] A. Acosta and P. García. Synchronization of non-identical chaotic systems: an exponential dichotomies approach. *J. Phys. A: Math. Gen.*, 34(1):9143–9151, 2001.

[2] A. Acosta, P. García, and H. Leiva. Synchronization of non-identical extended chaotic systems. *Applicable Analysis*, 92(4):740–751, 2013.

[3] David I. Rosas Almeida, Joaquín Alvarez, and Juan Gonzalo Barajas. Robust synchronization of sprott circuits using sliding mode control. *Chaos, Solitons and Fractals*, 30(1):11–18, 2006.

[4] R. Brown. Generalizations of the chua equations. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 40(11):878–884, 1993.

[5] Reggie Brown and Ljupco Kocarev. A unifying definition of synchronization for dynamical systems. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 10(2):344–349, 2000.

[6] J. C. Sprott. A new class of chaotic circuit. *Physics Letters A*, 266(1):19–23, 2000.

[7] Leon Chua. The genesis of chua’s circuit. *Archiv für Electronik und übertragungstechnik*, 46:250–257, 1992.

[8] J. De Abreu, P. García, and J. García. A deterministic approach to the synchronization of nonlinear cellular automata. *Advances in Complex Systems*, 20(0):1750006–1 – 1750006–11, 2017.

[9] Yue Deng and Yuxia Li. A memristive conservative chaotic circuit consisting of a memristor and a capacitor. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 30(1):013120, 2020.

[10] P. Feketa, A. Schaum, T. Meurer, D. Michaelis, and K. Ochs. Synchronization of nonlinearly coupled networks of chua oscillators. *IFAC-PapersOnLine*, 52(16):628–633, 2019. 11th IFAC Symposium on Nonlinear Control Systems NOLCOS 2019.
[11] Hirokazu Fujisaka and Tomoji Yamada. Stability Theory of Synchronized Motion in Coupled-Oscillator Systems: Progress of Theoretical Physics, 69(1):32–47, 01 1983.

[12] P. García, A. Acosta, and H. Leiva. Synchronization conditions for master-slave reaction diffusion systems. EPL (Europhysics Letters), 88(6):60006, dec 2009.

[13] Omar Guillén-Fernández, Ashley Meléndez-Cano, Esteban Tlelo-Cuautle, Jose Cruz Núñez-Pérez, and Jose de Jesus Rangel-Magdaleno. On the synchronization techniques of chaotic oscillators and their fpga-based implementation for secure image transmission. PLOS ONE, 14(2):1–34, 02 2019.

[14] Jack K. Hale. Ordinary Differential Equations. Robert E. Krieger, Malabar, Florida 32950, 2nd edition, 1980.

[15] Stefan Irimiciuc, Ovidiu Vasilovici, and Dan-Gheorghe Dimitriu. Chua’s circuit: Control and synchronization. International Journal of Bifurcation and Chaos, 25:1550050, 04 2015.

[16] Lars Keuninckx, Miguel C. Soriano, Ingo Fischer, Claudio R. Mirasso, Romain M. Nguimdo, and Guy Van der Sande. Encryption key distribution via chaos synchronization. Scientific Reports, 7(1):43428, 2017.

[17] Pecora Louis M. and Carroll Thomas L. Synchronization of chaotic systems. Chaos: An Interdisciplinary Journal of Nonlinear Science, 25(9), 2019.

[18] Albert C.J. Luo. A theory for synchronization of dynamical system. Communications in Nonlinear Science and Numerical Simulation, 14(5):1901 – 1951, May 2009.

[19] Lakshmanan M. and Murali K. Nonlinear dynamics of a class of piecewise linear systems. In Adamatzky Andrew and Chen Guanrong, editors, Chaos, CNN, Memristors and Beyond: A Festschrift for Leon Chua, chapter 23, pages 285–306. Worl Scientific, Singapore, 2013.

[20] Hanène Mkaouar and Olfa Boubaker. Chaos synchronization for master slave piecewise linear systems: Application to chua’s circuit. Communications in Nonlinear Science and Numerical Simulation, 17(3):1292–1302, 2012.

[21] M.J. Ogorzalek. Taming chaos. i. synchronization. IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications, 40(10):693–699, 1993.

[22] Louis Pecora and T. Carroll. Synchronization in chaotic system. Physical Review Letters, 64:821, 03 1990.

[23] Louis M. Pecora, Thomas L. Carroll, Gregg A. Johnson, Douglas J. Mar, and James F. Heagy. Fundamentals of synchronization in chaotic systems, concepts, and applications. Chaos: An Interdisciplinary Journal of Nonlinear Science, 7(4):520–543, 1997.

[24] Alejandro Pérez, Manuel Carreiras, and Jon Andoni Duñabeitia. Brain-to-brain entrainment: Eeg interbrain synchronization while speaking and listening. Scientific Reports, 7(1), 2018.

[25] Pauline Pérez, Jens Madsen, Leah Banellis, Basak Türker, Frederico Raimondo, Vincent Perlberg, Melanie Valente, Marie-Cécile Niérat, Louis Puybasset, Lionel Naccache, Thomas Similowski, Damian Cruse, Lucas C. Parra, and Jacobo D. Sitt. Conscious processing of narrative stimuli synchronizes heart rate between individuals. Cell Reports, 36(11):109692, 2021.
[26] Boccaletti S., Kurths J., Osipov G., Valladares D.L., and Zhou C.S. The synchronization of chaotic systems. *Physics Reports*, 366(1):1–101, 2002.

[27] Yang Tao and Chua Leon O. Piecewise-linear chaotic systems with a single equilibrium point. *International Journal of Bifurcation and Chaos*, 10(9):2015–2060, 2000.

[28] Hussein Waried. Chaos synchronization of coupled nano-quantum cascade lasers with negative optoelectronic feedback. *The European Physical Journal D*, 73(39), 2019.

[29] R. Willms Allan, M. Kitanov Petko, and F Langford William. Huygens’ clocks revisited. *R. Soc. open sci.*, 4(170777), 2017.

[30] Elhadj Zeraoulia. A new 3-d piecewise-linear system for chaos generation. *Radioengineering*, 16(2):40–43, 2007.