Analysis and applicability of a new quartic polynomial one-step method for solving COVID-19 model

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Abstract. This paper presents the Quartic Polynomial One-Step Method (QPSOM) based on quartic polynomial interpolating function for solving first order Ordinary Differential Equations (ODEs). The validity of the paper is analysed through consistency, order of convergence and stability. Also, the stability polynomial of this method is derived and the corresponding stability region is obtained. The applicability of the method has been demonstrated by COVID-19 model.

Keywords: COVID-19 model, one-step method, ordinary differential equations, quartic polynomial, stability polynomial, stability region

1. Introduction

There are several numerical methods developed for the solution of the physical and biological models in Engineering and Sciences that are emanated or arising from differential equations, using prescribed conditions. These differential equations are difficult to solve or cannot be solved conventionally. In the framework of initial value problems (IVPs), Fadugba [1] has developed an improved numerical integration approach via the transcendental function of the exponential form:

\[ y' = f(x,y), y(0) = y_0, a \leq x \leq b, -\infty < y < \infty \] (1)

in ODEs. Fadugba et al. [2] examined the second order numerical method for solving IVPs. Niekerk [3] proposed explicit non-linear methods of first, second and third order for singular and stiff IVPs. The algorithms depend on the finite continued fraction representation of the solution. In Abolarin and Akingbade [4], the authors derived the fourth stage inverse polynomial scheme for solving initial value problems. Vinci and Emimal [5] developed a new one-step method for solving stiff and non-stiff delay differential equations using Lagrange interpolation. Analysis of the composite Runge-Kutta solution methods and the relatively new one-step technique for stiff delay differential equations was considered by [6]. Islam [7] compared the numerical solutions of IVPs for ODEs with Euler's and Runge-Kutta methods. Vinci and Emimal [8] derived generalized rational multi-step method for solving delay differential equations. Other authors that have studied the numerical solutions of ordinary differential equations are [9-13], just to mention a few.
Coronavirus disease (COVID-19) is one of the viral infectious diseases caused by a newly discovered virus namely coronavirus. Most people are affected by COVID-19 in different ways. Without special care, most affected individuals may show mild to moderate symptoms and fully recover. Fever, dry cough, and tiredness are the most frequent indications, and aches and pains, sore throat, diarrhoea, conjunctivitis, headache, lack of taste or odor are lesser symptoms. Whenever an infected person sneezes, coughs or exhales, this virus is mainly spread by droplets produced. These droplets are very large to stay in the air and crash onto surfaces immediately[14]. The disease COVID-19 was first detected in Wuhan, the capital of the province of Hubei, China, in late December 2019, sparking the first pandemic of this century[15]. Since January 2020, the infectious disease COVID-19 has started to spread globally and resulting in almost 14,334,754 confirmed cases, 601,990 deaths and 8,553,614 recovered, till 18th July, 2020. Also there are 9,155,604 closed cases with 8,553,614 (93%) recovered/ discharged and 601,990 (7%) deaths. Hence, there are 5,179,150 currently infected patients with 5,119,282 (99%) in mild condition and 59,868 (1%) serious or critical condition [16]. According to [17], the summary of COVID-19 cases India as at July, 18th 2020 is analysed in the Figures 1-6:
Figure 3: Total Coronavirus deaths in India

Figure 4: Newly infected cases against newly recovered cases in India

Figure 5: Outcome of total closed cases (recovery rate against death rate) in India
In this paper, a new quartic polynomial one-step method is developed for the solution of IVPs in ODEs. The organization of the rest of the paper is presented as follows: Section 2 presents the derivation of the new method, "Quartic polynomial one-step method", via interpolating function of polynomial form. In Section 3, the properties of the method were analysed. Section 4 presents the applicability of a new quartic polynomial one-step method for solving COVID-19 model. Section 5 concludes the paper with the discussion of results.

2. Derivation of the Proposed New Method
Consider the quartic interpolating function as follows
\[ F(x_n) = q_4 x^4 - 3q_3 x^3 - 12q_2 x^2 + 54q_1 x - 40q_0 \]  
Expanding (2) at the points \( x = x_n \) and \( x = x_{n+1} \) yields, respectively;
\[ F(x_n) = q_4 x_n^4 - 3q_3 x_n^3 - 12q_2 x_n^2 + 54q_1 x_n - 40q_0 \]  
and
\[ F(x_{n+1}) = q_4 x_{n+1}^4 - 3q_3 x_{n+1}^3 - 12q_2 x_{n+1}^2 + 54q_1 x_{n+1} - 40q_0 \]  
Then, subtracting (3) from (4), one gets
\[ F(x_{n+1}) - F(x_n) = q_4 (x_{n+1}^4 - x_n^4) - 3q_3 (x_{n+1}^3 - x_n^3) - 12q_2 (x_{n+1}^2 - x_n^2) + 54q_1 (x_{n+1} - x_n) \]  
From the definition of mesh point \( x_n \), we have that
\[ x_n = a + nh \]  
or
\[ x_{n+1} = a + (n + 1)h \]  
Suppose that \( a = 0 \), and using (6) and (7), we have the following;
\[ x_{n+1}^4 - x_n^4 = (4n^3 + 6n^2 + 4n + 1)h^4 \]  
\[ x_{n+1}^3 - x_n^3 = (3n^2 + 3n + 1)h^3 \]  
\[ x_{n+1}^2 - x_n^2 = (2n + 1)h^2 \]  
\[ x_{n+1} - x_n = h \]  
Substituting (8), (9), (10) and (11) into (5), one gets
\[ F(x_{n+1}) - F(x_n) = q_4 (4n^3 + 6n^2 + 4n + 1)h^4 - 3q_3 (3n^2 + 3n + 1)h^3 - 12q_2 (2n + 1)h^2 + 54q_1 h \]  
Setting \( F(x_{n+1}) - F(x_n) = y_{n+1} - y_n \)
Therefore, (12) becomes
\[ y_{n+1} - y_n = q_4 (4n^3 + 6n^2 + 4n + 1)h^4 - 3q_3 (3n^2 + 3n + 1)h^3 - 12q_2 (2n + 1)h^2 + 54q_1 h \]  
To obtain the values of \( q_1, q_2, q_3 \) and \( q_4 \), differentiating (3) four times and setting
We obtain the following
\begin{align}
F'(x_n) &= f_n \\
F''(x_n) &= f_n^{(1)} \\
F'''(x_n) &= f_n^{(2)} \\
F''''(x_n) &= f_n^{(3)}
\end{align}
(14)

We obtain the following
\begin{align}
f_n &= 4q_4 x_n^3 - 9q_3 x_n^2 - 24q_2 x_n + 54q_1 \\
f_n^{(1)} &= 12q_4 x_n^2 - 18q_3 x_n - 24q_2 \\
f_n^{(2)} &= 24q_4 x_n - 18q_3 \\
f_n^{(3)} &= 24q_4 \\
\end{align}
(15)-(18)

Solving (15), (16), (17) and (18), yields
\begin{align}
q_1 &= \frac{1}{324} (nh)^3 f_n^{(3)} + \frac{1}{108} (nh)^2 f_n^{(2)} - \frac{1}{54} (nh) f_n^{(1)} + \frac{1}{54} f_n \\
q_2 &= -\frac{1}{48} (nh)^2 f_n^{(3)} + \frac{1}{24} (nh) f_n^{(2)} - \frac{1}{24} f_n^{(1)} \\
q_3 &= \frac{1}{18} (nh) f_n^{(3)} - \frac{1}{18} f_n^{(2)} \\
q_4 &= \frac{1}{24} f_n^{(3)}
\end{align}
(19)-(22)

Substituting (19), (20), (21) and (22) into (13), one obtains
\begin{align}
y_{n+1} - y_n &= \frac{1}{24} f_n^{(3)} (4n^3 + 6n^2 + 4n + 1)h^4 - \frac{3}{10} (nh) f_n^{(3)} - f_n^{(2)} (3n^2 + 3n + 1)h^3 \\
&\quad - \frac{12}{18} (-(nh)^2 f_n^{(3)} + 2(nh) f_n^{(2)} - 2f_n^{(1)}) 12q_2 (2n + 1)h^2 \\
&\quad + \frac{54}{324} (-(nh)^3 f_n^{(3)} - 3(nh)^2 f_n^{(2)} - 6(nh) f_n^{(1)} + 6f_n) h
\end{align}
(23)

Solving (23) further yields
\begin{align}
y_{n+1} - y_n = hf_n + \left[ \frac{h^2}{2} (2n + 1) - nh^2 \right] f_n^{(1)} \\
&\quad + \left[ \frac{3}{18} (3n^2 + 3n + 1)h^3 - \frac{1}{2} (2n + 1)nh^3 - \frac{1}{2} n^2 h^3 \right] f_n^{(2)} \\
&\quad + \left[ \frac{1}{24} (4n^3 + 6n^2 + 4n + 1)h^4 - \frac{3}{8} (3n^2 + 3n + 1) h^4 \right] f_n^{(3)} \\
&\quad + \frac{12}{48} (2n + 1)h^4 - \frac{54}{324} n^3 h^4
\end{align}
(24)

Hence,
\begin{align}
y_{n+1} - y_n = hf_n + \left[ \frac{h^2}{2} f_n^{(1)} + \frac{h^3}{6} f_n^{(2)} + \frac{h^4}{24} f_n^{(3)} \right]
\end{align}
(25)

Equation (25) is called the quartic polynomial one-step method.

3. Analysis of the Properties of the Quartic Polynomial One-Step Method

The properties of the method such as order of accuracy, consistency, stability and convergence were discussed in this section.

3.1. Order of Accuracy of the method

To check for the order of accuracy of the scheme, consider the Taylor’s series expansion of \(y(x_n + h)\) of the form
\begin{align}
y(x_n + h) = y(x_n) + hf(x_n, y(x_n)) + \frac{h^2}{2!} f^{(1)}(x_n, y(x_n)) + \frac{h^3}{3!} f^{(2)}(x_n, y(x_n)) \\
+ \frac{h^4}{4!} f^{(3)}(x_n, y(x_n)) + O(h^5)
\end{align}
(26)

From the new scheme (25), we have that
\begin{align}
y_{n+1} = y_n + hf_n + \left[ \frac{h^2}{2} f_n^{(1)} + \frac{h^3}{6} f_n^{(2)} + \frac{h^4}{24} f_n^{(3)} \right]
\end{align}

From the definition of a local truncation error, we write that
\begin{align}
T_n(h) = y(x_n + h) - y_{n+1}
\end{align}
(27)

Substituting (25) and (26) into (27) and by means of the localizing assumption (see [18]), we obtain
\begin{align}
T_n(h) = O(h^5)
\end{align}
(28)
It is clearly from the local truncation error given by (28) that the quartic polynomial one-step method is of order 4. Hence, the new method has an accuracy of order four.

3.2. Consistency property of the Method

According to [18], a one-step method is said to be consistency if

(i) it has at least order \( p = 1 \),
(ii) the increment function \( \phi(x_n, y_n; h) = f_n = f(x_n, y_n) \)

We conclude that the new method is consistency since it is of order four and that
\[
\phi(x_n, y_n; h) = f_n \quad \text{as} \quad h \to 0
\]

That is
\[
\phi(x_n, y_n; h) = f_n + \left( \frac{h}{2} \right) f_n^{(1)} + \left( \frac{h^2}{6} \right) f_n^{(2)} + \left( \frac{h^3}{24} \right) f_n^{(3)}
\]

As \( h \to 0 \)
\[
\phi(x_n, y_n; 0) = f_n = f(x_n, y_n)
\]

Alternatively, the consistency of a one-step method can be checked as follows:
\[
\lim_{h \to 0} \left( \frac{T_n(h)}{h} \right) = 0
\]

From (28), we obtain
\[
\lim_{h \to 0} \left( \frac{\theta(h^5)}{h} \right) = 0
\]

3.3. Stability of the Method

According to [18], a one-step method is said to be numerically stable if it is capable of damping out small fluctuations carried out in input data. To discuss the stability of the quartic polynomial method, consider the Dahlquist test problem of the form
\[
y' = \lambda y, \quad y(0) = 1, \quad \lambda < 0
\]
whose exact solution is obtained as
\[
y(x) = e^{\lambda x}
\]

Evaluating (32) at the points \( x = x_{n+1} \) and using the fact that \( h = x_{n+1} - x_n \), one gets
\[
y(x_{n+1}) = y(x_n) e^{\lambda h}
\]

Using (25), the numerical approximation of (32) is obtained as
\[
y_{n+1} = y_n \left( 1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{6} + \frac{(\lambda h)^4}{24} \right)
\]

Setting
\[
\theta = 1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{6} + \frac{(\lambda h)^4}{24}
\]

Thus
\[
y_{n+1} = \theta y_n
\]

It is clearly seen that (35) is the fifth term of the series expansion of (32). Hence, the stability of the scheme requires that
\[
\|\theta\| < 1
\]

Also, setting \( z = \lambda h \) in (35), the region of stability of the scheme satisfies
\[
\theta = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}
\]

The stability region of the quartic polynomial one-step method is plotted in the Figure 7 below.
4. Numerical Simulation

The numerical simulation of COVID-19 model is analysed via the quartic polynomial one-step method as follows:

Consider the system of first order differential equations for the COVID-19 model [15]

\[
\begin{align*}
\frac{dS}{dt} &= A - \beta(1 - \rho_1)(1 - \rho_2)SE + b_1Q - dS - pSM \\
\frac{dE}{dt} &= \beta(1 - \rho_1)(1 - \rho_2)SE + b_2E - \alphaE - \sigmaE - dE \\
\frac{dQ}{dt} &= b_2E - b_1Q - cQ - dQ \\
\frac{dI}{dt} &= \alphaE + cQ - (\eta + d + \delta)I \\
\frac{dR}{dt} &= \etaI + \sigmaE - dR + pSM
\end{align*}
\] (38)

where \(S=S(t)\), \(E=E(t)\), \(Q=Q(t)\), \(I=I(t)\), \(R=R(t)\) denote susceptible cases, exposed cases, hospitalized infected cases, quarantine cases and recovered cases at time \(t\) respectively. The definitions and the values of the parameters which are used in this model are summarized in the Table 1.

**Table 1: The variables and parameters [15]**

| Parameters | Explanation | Value |
|------------|-------------|-------|
| \(A\)      | Total population | 50    |
| \(\beta\)  | Rate Transmission | [0.5,2.3] |
| \(\rho_1\) | Amount of S associates with E | (0,1) |
| \(\rho_2\) | Amount of E associates with S | (0,1) |
| \(d\)      | Death rate    | 0.2   |
| \(b_1\)    | The rate at which Q becomes S | 0.25  |
| \(b_2\)    | The rate at which E becomes Q | 0.8   |
| \(\alpha\) | The rate at which E becomes I | 0.3   |
| \(\eta\)   | The rate at which I becomes R naturally | 0.25  |
| \(\sigma\) | The rate at which E becomes R naturally | 0.2   |
| \(c\)      | The rate at which Q becomes I | 0.12  |
with the basic reproduction number $R_0$ given by, (see [15])

$$R_0 = \frac{\alpha \beta (1 - \rho_1)(1 - \rho_2)}{(d + \nu M)(\beta_2 + \sigma + \delta)}$$  \hspace{1cm} (39)

The comparative results analysis of the quartic polynomial one-step method (QPOSM) and [15] are presented in the Figures 8-12 below.
5. Discussion of Results and Concluding Remarks

In this paper, we have presented the summary of the COVID-19 cases in India as at July, 18th 2020 as shown in Figures 1-6. We have derived a new quartic polynomial one-step method for solving the COVID-19 model. Also, the order of accuracy, consistency and stability of the method were derived. Moreover, errors as well as the stability analysis are presented. Also, the corresponding stability
region is plotted in the Figure 7. Finally, the applicability of the new quartic polynomial one-step method for solving the COVID-19 model is analysed. The three most COVID-19 affected states in India namely Delhi, Maharashtra and Tamilnadu were considered and also the dynamical behaviour of Disease Free Equilibrium (DFE) when the basic reproduction number $R_0<1$ is presented. The numerical simulations of COVID 19 are well comparable with [15] as seen in Figures 8-12. Therefore, it is evident that the method is suitable for solving realistic problems existing in various fields of science and technology. Hence, it is concluded that the quartic polynomial one-step method is consistent, fourth order convergent, stable and easy to implement.

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