Symmetries of the Bell correlation inequalities

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Abstract

Bell correlation inequalities for two sites and $2 + n$ or $3 + 3$ two-way measurements ("dichotomic observables") are considered. In the $2 + n$ case, any facet of the classical experience polytope is defined by a CHSH inequality involving only two pairs of the observables. In the $3 + 3$ case, contrary to earlier results, the action of the symmetry group reduces the set of all Bell inequalities to just 3 orbits, only one of them being "new" (not known from the $2 + 2$ case). A detailed calculation for the singlet state of two qubits reveals the configurations of a maximal violation for this class of inequalities.

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Although the notion of a general Bell inequality seems to be well established by now, examples that can be found in the present literature rarely go beyond the well-known Clauser-Horne-Shimony-Holt (CHSH) inequality,

\[ \frac{1}{2} [E(A_1B_1) + E(A_1B_2) + E(A_2B_1) - E(A_2B_2)] \leq 1. \]  

(1)

It is so because, despite the great simplicity of the basic idea, any attempt to find effectively a complete set of Bell inequalities for a given set of observables encounters computational difficulties that can be overcome only for a few simple cases. Two such cases has been considered in [2], namely the case of two observables at each of three sites and three observables at each of two sites, with observables having two possible values. On the other hand, depending on the configuration of the observables, the problem admits a number of symmetries, like permuting observables, sites with identical configuration of observables or values of a given observable. It is not clear whether this symmetries can be employed to reduce the complexity of the convex-hull problem. However, what one wants to find, are the classes of equivalent facets of the correlation polytope, rather than all the facets themselves. Therefore, following a convex hull computation, one should split the set of all Bell inequalities into orbits of the symmetry group, and take just one representative from each orbit.

We can assume the two values of the observables to be \{+1, −1\}. Under this assumption the notation \((2, 2)\) and \((2, 2, 2)\) or \((3, 3)\) unambiguously describes the case of and the two cases considered in [2]. Let us label the sites \(X\) with letters \(A, B, C\), and denote the observables respectively by \(A_i, B_j, C_k\). With this notation, the symmetries are: \(X_i \rightarrow −X_i\), \(X_i \rightarrow X_i\), \(X_i \rightarrow X_i\), where the \(σ\) denotes a permutation. To illustrate a method of systematic treatment of the no-signaling constraints, instead of the unknown probability distribution on the space of all possible classical configurations \(Λ\) we consider as coordinates all the correlations of the form

\[ E_{i_1i_2\ldots} = E(A_{i_1}^{-1}B_{i_2}^{-1}\ldots), \quad n_X = 1, 2, \ldots, |\text{Sp}(X_{i_X})|. \]  

(2)

For the simplest cases considered in [2], the coordinates are the expected values \(E(X_i), E(X_iY_j), E(X_iY_jZ_k)\), where \(X_iY_j\) denotes the product of the two (fictitious) classical random variables that we associate with the outcome of (independent) measurements at the sites \(X\) and \(Y\). Now, Bell inequalities correspond to the facets of the (purely classical) correlation
polytope" that is obtained by projecting the simplex of classical probability distributions on $\Lambda$ onto the coordinates introduced by [2]. This procedure respects the no-signaling constraints and the uncertainty principle that let us measure simultaneously only one observable at each site, and preserves all the information available from a measurement (further projection, e.g. taking into account only full correlations [1, section 3.2], leads to other types of Bell inequalities). In short, the vertices of the correlation polytope are obtained by enumerating the elements $\lambda \in \Lambda$, and calculating for each $\lambda$ all the products that appear in (2).

The case $(2, 2)$ has been discussed already by [3]. Any Bell inequality is of the form (1). The extremality of this inequality manifests through a feature that is usually silently ignored: the expression

$$f_2(a_1, a_2; b_1, b_2) = \frac{1}{2} (a_1 b_1 + a_1 b_2 + a_2 b_1 - a_2 b_2)$$

(3)

takes only the extremal values $\{+1, -1\}$ if the arguments do so. Hence, it may be considered a boolean function of the four arguments. In this approach, instead of checking that the values are in the required range, one checks that the expected value of $f_2$ is well-defined in quantum mechanics, i.e. that $f_2$ is a sum of terms none of which depends on both $a_1$ and $a_2$, or both $b_1$ and $b_2$. Moreover, up to the symmetries $a_1 \leftrightarrow a_2$, $b_1 \leftrightarrow b_2$, $a_i \rightarrow -a_i$, $b_i \rightarrow -b_i$, $(a_1, a_2) \leftrightarrow (b_1, b_2)$, it is the only boolean function with these properties that non-trivially depends on all the four arguments. Other such functions can be obtained by substitutions, like $a_1 \rightarrow +1, b_2 \rightarrow b_1$, etc. Let us mention that this includes any boolean function of two arguments, like $f(a_1; b_1)$, in the form $f(a; b) = \pm f_2(a, \pm 1; b, \pm 1)$, $f(a; b) = \pm f_2(a, +1; b, b)$.

The results obtained for the $(3, 3)$ case are definitely interesting. Among all the 684 inequalities, there are 576 of the form

$$0 \leq 4 + E(A_1) + E(A_2) + E(B_1) + E(B_2)$$

(4)

$$+ E(A_1 B_1) + E(A_1 B_2) + E(A_2 B_1) + E(A_2 B_2)$$

$$+ E(A_3 B_1) - E(A_3 B_2) + E(A_1 B_3) - E(A_2 B_3),$$

72 are the variations of the CHSH ineq. [1] corresponding to choosing two observables at each site, and the remaining 36 guarantee that the probability distribution for the result of any joint measurement is positive. One can easily check the counts to convince oneself that, as expected, the orbits of the symmetry group action are full. A convenient way to handle the symmetries is to organize the coordinates into a rank-2 tensor $[E(\tilde{A}_i, \tilde{B}_j)]_{ij}$, with
\[(X_i) = (1, X_1, X_2, X_3)\]. Then, the 3 + 3 Bell inequality takes the form

\[0 \leq \alpha_{ij} E(\hat{A}_i, \hat{B}_j), \quad (\alpha_{ij}) = \begin{pmatrix} 4 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}. \tag{5}\]

Certainly, it is the possibility of a quantum violation that makes the ineq. (4) a true Bell inequality. It is quite easy to demonstrate this possibility for a singlet state of two qubits, \(|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)\). The well-known relations for Pauli matrices let us write any observable \(\hat{A} \in B(H_2)\), \(\hat{A}^2 = 1\) in the form \(\hat{A} = a \cdot \hat{\sigma} = a_x \hat{\sigma}_x + a_y \hat{\sigma}_y + a_z \hat{\sigma}_z\), where \(a\) is a unit vector, \(\|a\|^2 = a_x^2 + a_y^2 + a_z^2 = 1\). The R.H.S of (4) is equal to the expected value of \(\hat{F}(a_i, b_j) = 4 + (\hat{A}_1 + \hat{A}_2) + (\hat{B}_1 + \hat{B}_2) + (\hat{A}_1 + \hat{A}_2)(\hat{B}_1 + \hat{B}_2) + \hat{A}_3(\hat{B}_1 - \hat{B}_2) + (\hat{A}_1 - \hat{A}_2)\hat{B}_3\), where the unit vectors \(a_i, b_j\) have been introduced to describe what observables \(\hat{A}_i, \hat{B}_j\) are to be measured at each site. In terms of \(a_i\) and \(b_j\), the measured correlations are \(\langle \psi | \hat{A}_i | \psi \rangle = 0\), \(\langle \psi | \hat{B}_j | \psi \rangle = 0\), and \(\langle \psi | \hat{A}_i \hat{B}_j | \psi \rangle = -a_i \cdot b_j\), thus

\[\langle \psi | \hat{F}(a_i, b_j) | \psi \rangle = 4 - a_3 \cdot (b_1 - b_2) - a_1 \cdot (b_1 + b_2 + b_3) - a_2 \cdot (b_1 + b_2 - b_3). \tag{6}\]

For given \(b_j\)'s, the extremal value,

\[4 - \|b_1 - b_2\| - \|b_1 + b_2 + b_3\| - \|b_1 + b_2 - b_3\|, \tag{7}\]

is attained when \(a_1 \propto (b_1 + b_2 + b_3)\), etc. Fortunately, \((b_1 + b_2)\) and \((b_1 - b_2)\) are orthogonal. Let \(\beta\) be the angle between \((b_1 + b_2)\) and \(b_3\). Hence, \(\|b_1 - b_2\| = 2 \cos \alpha, \|b_1 + b_2\| = 2 \sin \alpha, \|b_1 + b_2 \pm b_3\| = \sqrt{(2 \sin \alpha \cos \beta)^2 + \sin^2 \beta}\). We obtain the extremal violation (by -1) for \(\alpha = \pi/3, \beta = \pi/2\), but there is still one degree of freedom in choosing the extremal configuration: the angle between \((b_1 - b_2)\) and \(b_3\). In the spherical coordinates, \(a = (a_x, a_y, a_z) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)\), the extremal configurations are (up to a rotation/unitary transform in \(H_2\)): \(\theta_{a1} = \theta_{a2} = \theta_{b1} = \theta_{b2} = \pi/6, \theta_{a3} = \theta_{b3} = \pi/2, \phi_{a1} = \phi_{a2} + \pi = \phi_{b3}, \phi_{b1} = \phi_{b2} + \pi = \phi_{a3}\).

Now that we have a Bell inequality for 3 + 3 observables, one asks the question what we gain using it instead of (4). Comparing the raw numbers of maximum violation is certainly meaningless, unless some normalization is defined. One can normalize, e.g., the free numeric coefficient or the classical range of the inequality’s R.H.S., \(E = \sum_{ij} a_{ij} E_{ij}\). The natural
choice is, however, to normalize the standard deviation

$$\Delta E = \sqrt{\sum_{ij} \alpha_{ij} (\Delta E_{ij})^2}, \quad (8)$$

where $\Delta E_{ij}$ is the standard deviation of the measurement outcome for $E(\hat{A}_i, \hat{B}_j)$. In the simplest case, each $E_{ij}$ is determined by a separate sequence of measurements, with an equal number of individual measurements for each configuration $(i, j)$. In this case,

$$ (\Delta E_{ij})^2 \propto \left[ \langle \psi | (\hat{A}_i\hat{B}_j)^2 | \psi \rangle - \left( \langle \psi | \hat{A}_i\hat{B}_j | \psi \rangle \right)^2 \right] \quad (9) $$

(the possible optimization is to derive $E_{1j}$ and $E_{i1}$ from the remaining $E_{ij}$’s, but this would result in correlations between the components, invalidating the above formula). The maximal ratio $|E/\Delta E|$ is (proportional to) 0.585786 for (11), and 0.342997 for (14). Therefore, for the singlet state, the well-known CHSH-inequality is not only significantly simpler, but also stronger than (14). Nevertheless, since a complete set of Bell inequalities for 3+3 observables includes (14), there may exist states that satisfy (11), even for any $\hat{A}_i, \hat{B}_j$, but violate (14).

Let us now consider the $(2, n)$ case. This corresponds to a $d$-dimensional convex-hull problem, with $d = 3n + 2$. Any facet of the correlation polytope coincides with $d$ vertices in general position, i.e. the vectors from a chosen reference vertex to the remaining vertices must span a subspace of dimension $d - 1$. Let $\alpha_{ij}$ be the coefficients of the corresponding inequality. Alternatively, one can classify the vertices by the values $(a_1, a_2)$ of $(A_1, A_2)$, and choose one reference vertex in each of the classes. Then, if there are two vertices in the class $(a_1, a_2)$ that differ in $b_j$, $\alpha_{0j} + a_1\alpha_{1j} + a_2\alpha_{2j} = 0$. The number of such $j$’s in a class is the dimension of the subspace spanned by the corresponding vectors. The sum of this numbers must be at least $d - 4$, as there is one reference vertex in each class and there can be at most 4 classes. Hence, since $3n - 2 > 2n$ for $n > 2$, for some $j$ there are 3 such equations. Consequently, $\alpha_{0j} = \alpha_{1j} = \alpha_{2j} = 0$, and $B_j$ does not appear in the corresponding inequality. By induction, any Bell inequality for $2 + n$ variables $(A_1, A_2, B_1, B_2, \ldots, B_n)$ involves only one pair of $B_j$’s, and therefore is of the CHSH-form.

A slightly more general argument can be applied to the case $(k, n)$, with $k > 2$ observables at one site to show that there are no Bell inequalities beyond the limit $n \leq 2^k - 2$. Since this upper bound depends exponentially on $k$, its practical significance may probably be limited.

Similar results have been obtained for other cases. In particular, for the case $(2, 2, 2)$ of three pairs of observables, there are 46 classes of inequalities. They are listed in Table 11.
Although it is possible to reconstruct a number of them by chaining the boolean function $f_2$, the interpretation and structure of those inequalities remain mostly unknown.

In summary, a method of systematic treatment of the Bell correlation inequalities and their symmetries has been proposed. Detailed results for several special cases have been presented, including some combinatorial properties of the CHSH-inequality.

[1] R. F. Werner and M. M. Wolf, *Bell inequalities and Entanglement*, Quant. Inf. Comp. 1 (3), 1 (2002)
[2] I. Pitowsky and K. Svozil, *Optimal tests of quantum nonlocality*, Phys. Rev. A 64, 014102 (2001).
[3] A. Fine, *Hidden Variables, Joint Probability, and the Bell Inequalities*, Phys. Rev. Letters 48, 291 (1982).
| Class | Representative |
|-------|----------------|
| 1     | $0 \leq 1 - E(A_1) - E(B_1) + E(A_1B_1) - E(C_1) + E(A_1C_1) + E(B_1C_1) - E(A_1B_1C_1)$ |
| 2     | $0 \leq 2 - E(A_1B_1C_1) - E(A_2B_2C_1) - E(A_2B_1C_2) + E(A_1B_2C_2)$ |
| 3     | $0 \leq 2 - E(A_1B_1C_1) - E(A_2B_1C_1) - E(A_1B_2C_2) + E(A_2B_2C_2)$ |
| 4     | $0 \leq 2 - 2E(A_1) - E(B_1C_1) + E(A_1B_1C_1) - E(B_2C_1) + E(A_1B_2C_1) - E(B_1C_2) + E(A_1B_1C_2) + E(B_2C_2) - E(A_1B_2C_2)$ |
| 5     | $0 \leq 3 - E(A_1) - E(B_1) - E(A_2B_1) - E(A_1B_2) + E(A_2B_2) - E(C_1) - E(A_2B_2) + E(A_1B_1C_1) + E(A_1B_2C_1) - E(B_2C_1) + E(A_1B_2C_1) - E(B_1C_2) + E(A_1B_1C_2) + E(B_2C_2) - E(A_1B_2C_2)$ |
| 6     | $0 \leq 3 - E(A_1) - E(B_1) - E(A_1B_1) - E(C_1) - E(A_2B_2) - E(A_1B_1C_1) + E(A_2B_1C_1) - E(B_2C_1) + E(A_1B_2C_1) - E(A_1C_2) + E(A_2C_2) + B_1(B_2C_2) + E(A_1B_1C_2) - E(B_2C_2) + E(A_1B_2C_2)$ |
| 7     | $0 \leq 4 - 3E(A_1B_1C_1) - E(A_2B_1C_1) - E(A_1B_2C_1) + E(A_2B_2C_1) - E(A_1B_1C_2) + E(A_2B_1C_2) + E(A_1B_2C_2) - E(A_2B_2C_2)$ |
| 8     | $0 \leq 4 - E(A_1B_1) - E(A_2B_1) - E(A_1B_2) - E(A_2B_2) - 2E(A_1B_1C_1) + 2E(A_2B_2C_1) - E(A_1B_1C_2) + E(A_2B_1C_2) + E(A_1B_2C_2) - E(A_2B_2C_2)$ |
| 9     | $0 \leq 4 - E(A_1B_1) - E(A_2B_1) - E(A_1B_2) - E(A_2B_2) - 2E(A_1B_1C_1) + 2E(A_1B_2C_1) - E(A_1B_1C_2) + E(A_2B_1C_2) - E(A_1B_2C_2) + E(A_2B_2C_2)$ |
| 10    | $0 \leq 4 - E(A_1B_1) - E(A_2B_1) - E(A_1B_2) - E(A_2B_2) - E(A_1B_1C_1) + E(A_2B_1C_1) - E(B_1C_1) - E(A_1B_1C_2) + E(B_2C_1) + E(A_2B_2C_1) - E(A_1B_2C_2) - E(B_2C_2) + E(A_1B_2C_2)$ |
| 11    | $0 \leq 4 - 2E(A_1B_1) - 2E(A_2B_2) - E(A_1B_1C_1) - E(A_2B_1C_1) + E(A_1B_2C_1) + E(A_2B_2C_1) - E(A_1B_1C_2) + E(A_2B_1C_2) - E(A_1B_2C_2) + E(A_2B_2C_2)$ |
| 12    | $0 \leq 4 - 2E(A_1B_1) - 2E(A_2B_2) - E(A_1B_1C_1) - E(A_2B_1C_1) + E(A_1B_2C_1) + E(B_1C_1) - E(A_2B_1C_1) + E(B_2C_1) + E(A_1B_2C_1) - E(A_1C_2) - E(A_2C_2) + E(B_1C_2) + E(A_2B_1C_2) + E(B_2C_2) - E(A_1B_2C_2)$ |
| Class | Representative |
|-------|----------------|
| 13    | \( 0 \leq 4 - 2E(A_1B_1) - 2E(A_2B_1) - E(A_1B_1C_1) + E(A_2B_1C_1) - E(A_1B_2C_1) + E(A_2B_2C_1) - E(A_1B_1C_2) + E(A_2B_1C_2) + E(A_1B_2C_2) - E(A_2B_2C_2) \) |
| 14    | \( 0 \leq 4 - 2E(A_1B_1) - 2E(A_2B_1) - E(A_1C_1) + E(A_2C_1) - E(A_1B_2C_1) + E(A_2B_2C_1) - E(A_1C_2) + E(A_2C_2) + E(A_1B_2C_2) - E(A_2B_2C_2) \) |
| 15    | \( 0 \leq 4 - 2E(A_1B_1) - 2E(A_2B_1) - E(A_1B_1) - E(A_1B_1C_1) + E(A_2B_2C_1) - E(A_1B_2C_1) + E(A_2B_2C_1) - E(A_1B_2C_2) + E(A_2B_2C_2) \) |
| 16    | \( 0 \leq 4 - E(A_1) - E(A_2) - E(A_1B_1) - E(A_2B_1) - E(A_1C_1) + E(A_2B_1C_1) + E(A_1B_1C_2) - E(A_2B_1C_2) - E(A_1B_2C_2) + E(A_1B_2C_2) + E(A_2B_2C_2) \) |
| 17    | \( 0 \leq 4 - E(A_1) - E(A_2) - E(A_1B_1) - E(A_2B_1) - E(A_1C_1) - E(A_2C_1) + E(A_1B_1C_1) + E(A_2B_1C_1) - 2E(A_1B_2C_2) + 2E(A_2B_2C_2) \) |
| 18    | \( 0 \leq 4 - E(A_1) - E(A_2) - E(A_1B_1) - E(A_2B_1) - E(A_1C_1) - E(A_2C_1) + 2E(B_1C_1) - E(A_1B_2C_1) + E(A_2B_2C_1) - E(A_1B_1C_2) + E(A_2B_1C_2) - 2E(B_2C_2) + E(A_1B_2C_2) + E(A_2B_2C_2) \) |
| 19    | \( 0 \leq 4 - E(A_1) - E(A_2) - E(A_1B_1) - E(A_2B_1) - E(A_1C_1) - E(A_2C_1) + 2E(B_1C_1) - 2E(B_2C_1) + E(A_1B_2C_1) + E(A_2B_2C_1) - E(A_1B_1C_2) + E(A_2B_1C_2) + E(A_2B_2C_2) \) |
| 20    | \( 0 \leq 4 - E(A_1) - E(A_2) - E(A_1B_1) + E(A_2B_1) - E(A_1B_2) + E(A_2B_2) - E(A_1C_1) + E(A_2C_1) + E(B_1C_1) - E(A_1B_1C_1) + E(A_2B_1C_1) + E(B_2C_1) - E(A_1B_2C_1) - E(A_2B_2C_1) - E(B_1C_2) + E(A_1B_1C_2) + E(A_2B_1C_2) + E(B_2C_2) - E(A_1B_2C_2) - E(A_2B_2C_2) \) |
| 21    | \( 0 \leq 4 - E(A_1) - E(A_2) - E(B_1) - E(A_1B_1) - E(B_2) + E(A_2B_2) - E(A_1C_1) - E(A_2C_1) - E(B_1C_1) + 2E(A_1B_1C_1) + E(A_2B_1C_1) - E(B_2C_1) + E(A_1B_2C_1) - E(A_1B_1C_2) + E(A_2B_1C_2) + E(A_1B_2C_2) - E(A_2B_2C_2) \) |
| Class | Representative |
|-------|----------------|
| 22    | \( 0 \leq 4 - E(A_1) - E(A_2) - E(B_1) - E(A_1B_1) - E(B_2) + E(A_2B_2) - E(C_1) \) - \( E(A_1C_1) - E(B_1C_1) + 2E(A_1B_1C_1) + E(A_1B_1C_1) + E(A_1B_2C_1) - 
\( E(A_2B_2C_1) - E(C_2) + E(A_2C_2) + E(A_1B_1C_2) - E(A_1B_1C_2) + E(B_2C_2) - 
E(A_1B_2C_2) \) |
| 23    | \( 0 \leq 4 - E(A_1) - E(A_2) - E(B_1) + E(A_1B_1) + E(A_2B_1) - E(B_2) + 
E(A_1B_2) + E(A_2B_2) - E(A_1C_1) + E(A_2C_1) + E(A_1B_1C_1) - E(A_2B_1C_1) + 
E(A_1B_2C_1) - E(A_2B_2C_1) - E(B_1C_2) + E(A_1B_1C_2) + E(A_2B_1C_2) + 
E(B_2C_2) - E(A_1B_2C_2) - E(A_2B_2C_2) \) |
| 24    | \( 0 \leq 5 - E(A_1) - E(B_1) - E(A_2B_1) - E(B_1C_1) - E(A_1B_2) - E(A_2B_2) - E(C_1) - 
E(A_2B_1) + E(B_1C_1) - 2E(A_1B_1C_1) + E(A_2B_1C_1) + 2E(A_2B_2C_1) - 
E(A_1C_1) - E(A_2C_2) + 2E(A_2B_1C_2) + E(A_1B_2C_2) - E(A_2B_2C_2) \) |
| 25    | \( 0 \leq 5 - E(A_1) - E(B_1) - E(A_2B_1) - E(A_1B_1) - 2E(A_2B_2) - E(C_1) - E(A_1C_1) - 
E(B_1C_1) + E(A_1B_1C_1) + 2E(A_2B_2C_1) - 2E(A_2B_1C_2) + 2E(A_2B_1C_2) + 
2E(B_2C_2) - 2E(A_1B_2C_2) \) |
| 26    | \( 0 \leq 5 - 2E(A_1) - E(A_2) - E(B_1) + E(A_1B_1) - E(A_1B_2) - E(A_2B_2) - E(C_1) + 
E(A_1C_1) - 2E(A_1B_1C_1) + 2E(A_2B_1C_1) - E(B_2C_1) + E(A_1B_2C_1) - E(A_2C_2) - 
E(B_1C_2) + E(A_1B_1C_2) - E(B_2C_2) + 2E(A_1B_2C_2) + E(A_2B_2C_2) \) |
| 27    | \( 0 \leq 6 - E(A_1) - E(A_2) - E(A_1B_1) + E(A_2B_1) - E(A_1C_1) + E(A_2C_1) + E(B_1C_1) - 
2E(A_1B_1C_1) - E(A_2B_1C_1) - E(B_2C_1) + E(A_1B_2C_1) + 2E(A_2B_2C_1) - 
E(B_1C_2) + E(A_1B_1C_2) + 2E(A_2B_1C_2) - E(B_2C_2) + 3E(A_1B_2C_2) \) |
| 28    | \( 0 \leq 6 - E(A_1) - E(A_2) - E(A_1B_1) + E(A_2B_1) - E(A_1C_1) + E(A_2C_1) + E(B_1C_1) - 
2E(A_1B_1C_1) - E(A_2B_1C_1) - E(B_2C_1) + E(A_1B_2C_1) + 2E(A_2B_2C_1) - 
E(B_1C_2) + 3E(A_1B_1C_2) - E(B_2C_2) + E(A_1B_2C_2) + 2E(A_2B_2C_2) \) |
| Class | Representative |
|-------|----------------|
| 30    | \(0 \leq 6 - E(A_1) - E(A_2) - 2E(A_1B_1) + 2E(A_2B_1) - E(A_1B_2) + E(A_2B_2) - \) \(E(A_1C_1) + E(A_2C_1) + E(B_1C_1) - 2E(A_1B_1C_1) - E(A_2B_1C_1) + \) \(E(B_2C_1) - E(A_1B_2C_1) - 2E(A_2B_2C_1) - E(B_1C_2) + 2E(A_1B_1C_2) + \) \(E(A_2B_1C_2) + E(B_2C_2) - 2E(A_1B_2C_2) - E(A_2B_2C_2)\) |
| 31    | \(0 \leq 6 - E(A_1) - E(A_2) - E(B_1) + E(A_2B_1) - E(B_2) + E(A_1B_2) - E(A_1C_1) + \) \(E(A_2C_1) - 2E(A_2B_1C_1) + E(A_1B_2C_1) - 3E(A_2B_2C_1) - E(B_1C_2) + \) \(2E(A_1B_1C_2) + E(A_2B_1C_2) + E(B_2C_2) - 2E(A_1B_2C_2) - E(A_2B_2C_2)\) |
| 32    | \(0 \leq 6 - E(A_1) - E(A_2) - E(B_1) + E(A_2B_1) - E(B_2) + E(A_1B_2) - E(C_1) + \) \(E(A_2C_1) - 2E(A_2B_1C_1) + E(B_2C_1) - 2E(A_1B_2C_1) - E(A_2B_2C_1) - \) \(E(C_2) + E(A_1C_2) + E(B_1C_2) - 2E(A_1B_1C_2) - E(A_2B_1C_2) - \) \(E(A_1B_2C_2) + 3E(A_2B_2C_2)\) |
| 33    | \(0 \leq 6 - E(A_1) - E(A_2) - E(B_1) + E(A_2B_1) - E(B_2) + E(A_1B_2) - E(C_1) + \) \(E(A_2C_1) + E(B_1C_1) + 2E(A_1B_1C_1) - E(A_2B_1C_1) + 2E(B_2C_1) - \) \(2E(A_1B_2C_1) - 2E(A_2B_2C_1) - E(C_2) + E(A_1C_2) + 2E(B_1C_2) + E(B_2C_2) - \) \(E(A_1B_2C_2) + 2E(A_2B_2C_2)\) |
| 34    | \(0 \leq 6 - E(A_1) - E(A_2) - E(B_1) + E(A_2B_1) - E(B_2) + E(A_1B_2) - E(C_1) + \) \(E(A_2C_1) + E(B_1C_1) + 2E(A_1B_1C_1) - E(A_2B_1C_1) + 2E(B_2C_1) - \) \(2E(A_1B_2C_1) - 2E(A_2B_2C_1) - E(C_2) + E(A_1C_2) + 2E(B_1C_2) + 2E(B_2C_2) - \) \(E(A_1B_2C_2) + 2E(A_2B_2C_2)\) |
| 35    | \(0 \leq 6 - E(A_1) - E(A_2) - E(B_1) + E(A_2B_1) + 2E(A_2B_1) - E(B_2) + \) \(2E(A_1B_2) + E(A_2B_2) - E(A_1C_1) + E(A_2C_1) + E(A_1B_1C_1) - \) \(E(A_2B_1C_1) + 2E(A_1B_2C_1) - 2E(A_2B_2C_1) - E(B_1C_2) + 2E(A_1B_1C_2) + \) \(E(A_2B_1C_2) + E(B_2C_2) - 2E(A_1B_2C_2) - E(A_2B_2C_2)\) |
| 36    | \(0 \leq 6 - 2E(A_1) - E(A_1B_1) - E(A_2B_1) - E(A_2B_2) - E(A_1C_1) - \) \(E(A_2C_1) - E(B_1C_1) + 2E(A_1B_1C_1) - E(A_2B_1C_1) + E(B_2C_1) - \) \(E(A_1B_2C_1) + 2E(A_2B_2C_1) - E(A_1C_2) - E(A_2C_2) + E(B_1C_2) - \) \(E(A_1B_1C_2) + 2E(A_2B_1C_2) + E(B_2C_2) - 2E(A_1B_2C_2) + E(A_2B_2C_2)\) |
| Class | Representative |
|-------|----------------|
| 37    | $0 \leq 6 - 2E(A_1) - E(A_1B_1) - E(A_2B_1) - E(A_1B_2) - E(A_2B_2) - E(A_2C_1) - E(A_1B_1C_1) + 3E(A_1B_1C_1) + E(B_2C_1) - 2E(A_1B_2C_1) + E(A_2B_2C_1) - E(A_1C_2) - E(A_2C_2) + E(B_1C_2) - 2E(A_1B_1C_2) + E(A_2B_1C_2) + E(B_2C_2) - E(A_1B_2C_2) + 2E(A_2B_2C_2)$ |
| 38    | $0 \leq 6 - 2E(A_1) - 2E(A_1B_1) - 2E(A_2B_1) - E(A_1B_1) + E(A_1B_1C_1) + E(A_1B_1C_1) + 2E(A_2B_1C_1) - E(B_2C_1) + E(A_1B_1C_1) + 2E(A_2B_1C_1) - 2E(A_1B_1C_1) + E(A_1B_1C_1) + E(A_2B_2C_1) - E(A_2B_2C_1) + 2E(A_2B_2C_1) - E(A_1C_2) - E(A_2C_2) + E(B_1C_2) - E(A_1B_1C_2) + 2E(A_1B_2C_2) - E(B_2C_2) - E(A_1B_2C_2) + E(A_2B_2C_2)$ |
| 39    | $0 \leq 6 - 2E(A_1) - 2E(A_2) - 2E(B_1) + E(A_1B_1) + E(A_2B_1) - E(A_1B_2) - E(A_2B_2) - E(A_1C_2) - E(A_2B_1) - E(A_2B_1C_1) + 2E(A_2B_1C_1) - 2E(A_1B_1C_1) + E(A_1B_1C_1) + E(A_1B_1C_1) + E(A_2B_2C_1) - E(A_1B_1C_1) + 2E(B_1C_2) - E(B_2C_1) + 2E(A_1B_1C_2) - E(A_1B_1C_2) + 2E(A_1B_2C_2) - E(A_1C_2) + E(A_1B_2C_2) - 2E(B_2C_2) - E(A_1B_2C_2) + E(A_2B_2C_2)$ |
| 40    | $0 \leq 7 - E(A_1) - E(B_1) - E(A_1B_1) - E(A_2B_1) - E(C_1) - E(A_2C_1) + 3E(A_1B_1C_1) + E(A_2B_1C_1) - E(B_2C_1) + E(A_1B_2C_1) + 2E(A_2B_2C_1) - E(A_1C_2) + E(A_2B_1C_2) - E(A_2B_1C_2) + 4E(A_1B_1C_2) - E(A_2B_1C_2) + E(B_2C_2) - E(A_1B_2C_2) - 2E(A_2B_2C_2)$ |
| 41    | $0 \leq 8 - E(A_1) - E(A_2) - E(B_1) - E(A_1B_1) - E(A_1B_2) - E(A_2B_2) + E(A_2B_2) - E(A_1C_1) + E(A_2B_1) - E(B_1C_1) + 2E(A_1B_1C_1) + E(A_2B_1C_1) + E(B_2C_1) + E(A_1B_2C_1) - 4E(A_2B_2C_1) - 2E(A_2C_2) + E(A_1B_1C_2) + 3E(A_2B_1C_2) - 2E(B_2C_2) + 3E(A_1B_2C_2) + E(A_2B_2C_2)$ |
| 42    | $0 \leq 8 - 2E(A_1) - 2E(B_1) + E(A_1B_1) - E(A_2B_1) - E(A_1B_2) - E(A_2B_2) - E(A_1C_1) - E(A_2B_1) - E(B_1C_1) + 2E(A_1B_1C_1) + 3E(A_2B_1C_1) + E(B_2C_1) - E(A_1B_2C_1) - 2E(A_2B_2C_1) - E(A_1C_2) + E(A_2C_2) - E(B_1C_2) + 3E(A_1B_1C_2) - E(B_2C_2) + 4E(A_1B_2C_2) - E(A_2B_2C_2)$ |
| Class | Representative |
|-------|----------------|
| 44    | $0 \leq 8 - 2E(A_1) - 2E(A_2) - 2E(A_1B_1) + 2E(A_2B_1) - E(A_1C_1) +$  
|       | $E(A_2C_1) + 2E(B_1C_1) - 2E(A_1B_1C_1) - 2E(A_2B_1C_1) - 2E(B_2C_1) +$  
|       | $E(A_1B_2C_1) + 3E(A_2B_2C_1) - E(A_1C_2) + E(A_2C_2) + 2E(B_1C_2) -$  
|       | $2E(A_1B_1C_2) - 2E(A_2B_1C_2) + 2E(B_2C_2) - 3E(A_1B_2C_2) - E(A_2B_2C_2)$ |
| 45    | $0 \leq 8 - 3E(A_1) - E(A_2) - 2E(A_1B_1) + 2E(A_2B_1) - E(A_1B_2) + E(A_2B_2) -$  
|       | $2E(A_1C_1) + 2E(A_2C_1) + 2E(B_1C_1) - 2E(A_1B_1C_1) - 2E(A_2B_1C_1) +$  
|       | $2E(B_2C_1) - 2E(A_1B_2C_1) - 2E(A_2B_2C_1) - E(A_1C_2) + E(A_2C_2) + 2E(B_1C_2) -$  
|       | $2E(A_1B_1C_2) - 2E(A_2B_1C_2) - 2E(B_2C_2) + 3E(A_1B_2C_2) + E(A_2B_2C_2)$ |
| 46    | $0 \leq 10 - 3E(A_1) - E(A_2) - 3E(B_1) + 2E(A_1B_1) + E(A_2B_1) - E(B_2) +$  
|       | $E(A_1B_2) + 2E(A_2B_2) - 2E(A_1C_1) + 2E(A_2C_1) - E(B_1C_1) +$  
|       | $3E(A_1B_1C_1) - 4E(A_2B_1C_1) - E(B_2C_1) + E(A_1B_2C_1) - 2E(A_2B_2C_1) -$  
|       | $E(A_1C_2) - E(A_2C_2) - 2E(B_1C_2) + 3E(A_1B_1C_2) + E(A_2B_1C_2) +$  
|       | $2E(B_2C_2) - 4E(A_1B_2C_2) - 2E(A_2B_2C_2)$ |