Chaos edges of $z$-logistic maps: Connection between the relaxation and sensitivity entropic indices

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Chaos thresholds of the $z$-logistic maps $x_{t+1} = 1 - a|x_t|^z$ ($z > 1; t = 0, 1, 2, ...$) are numerically analysed at accumulation points of cycles 2, 3 and 5. We verify that the nonextensive $q$-generalization of a Pesin-like identity is preserved through averaging over the entire phase space. More precisely, we computationally verify

$$\lim_{t \to \infty} \frac{S_{\text{av} \text{sen}}(t)}{t} = \lim_{t \to \infty} \frac{\ln q_{\text{rel}}(t)}{t} \equiv \lambda_{\text{rel}}^{\text{sen}},$$

where the entropy $S_q \equiv (1 - \sum_i p_i^q) / (q - 1)$ ($S_1 = - \sum_i p_i \ln p_i$), the sensitivity to the initial conditions $\xi \equiv \lim_{\Delta x(0) \to 0} \Delta x(t) / \Delta x(0)$, and $\ln q_x \equiv (x^{1-q} - 1) / (1 - q)$ ($\ln_1 x = \ln x$). The entropic index $q_{\text{rel}}^{\text{sen}} < 1$, and the coefficient $\lambda_{\text{rel}}^{\text{sen}} > 0$ depend on both $z$ and the cycle. We also study the relaxation that occurs if we start with an ensemble of initial conditions homogeneously occupying the entire phase space. The associated Lebesgue measure asymptotically decreases as $1/t^{(5/3q-1)}$ ($q_{\text{rel}} > 1$). These results led to (i) the first illustration of the connection (conjectured by one of us) between sensitivity and relaxation entropic indices, namely $q_{\text{rel}} - 1 \approx A(1 - q_{\text{rel}}^{\text{sen}})^{\alpha}$, where the positive numbers $(A, \alpha)$ depend on the cycle; (ii) an unexpected and new scaling, namely $q_{\text{rel}}^{\text{sen}}(\text{cycle } n) = 2.5 q_{\text{rel}}^{\text{sen}}(\text{cycle } 2) + \epsilon$ ($\epsilon = 0.03$ for $n = 3$, and $\epsilon = 0.03$ for $n = 5$).

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Boltzmann-Gibbs (BG) entropy and corresponding statistical mechanics generically require strong chaos for their applicability and (notorious) usefulness. This type of requirement was first used by Boltzmann himself [1]. Indeed, his “molecular chaos hypothesis” allowed him to arrive to the celebrated distribution of energies at thermal equilibrium, now known as Boltzmann weight. Today, we know that this requirement essentially satisfied when the interactions are short ranged. Such systems typically exhibit three basic exponential functions [2], namely (i) the sensitivity to the initial conditions diverges exponentially with time, (ii) physical quantities exponentially relax with time to their value at the stationary state (thermal equilibrium), and (iii) at thermal equilibrium, the probability of a given microstate exponentially decays with the energy of the microstate. These three exponentials of different, though related, nature can be summarized in the following differential equation:

$$\frac{dy}{dx} = a_1 y \quad (y(0) = 1),$$

whose solution is $y = e^{a_1 x}$ (the subindex 1 will become transparent soon). Let us make explicit the point. The first physical interpretation concerns the sensitivity to the initial conditions of say a one-dimensional case and is defined as

$$\xi(t) \equiv \lim_{\Delta x(0) \to 0} \frac{\Delta x(t)}{\Delta x(0)}.$$
for sensitivity). In other words, we have \((x, y, q, a_q) \equiv (t, \xi, q_{\text{sen}}, \lambda_{q_{\text{sen}}})\). The relaxation is typically expected [6] to be characterized by \(\Omega = -e_{q_{\text{rel}}}^{-t/\tau_{q_{\text{rel}}}}\) (rel stands for relaxation). In other words, in this case we have \((x, y, q, a_q) \equiv (t, \Omega, q_{\text{rel}}, -1/\tau_{q_{\text{rel}}})\). For the long-standing metastable states [7] that precede thermal equilibrium for long-range interacting Hamiltonians, it is expected [8] \(p_i = e_{q_{\text{stat}}}^{-\beta_{q_{\text{stat}}} E_i}/Z_{q_{\text{stat}}} = \sum_{i=1}^{W} e_{q_{\text{stat}}}^{-\beta_{q_{\text{stat}}} E_i}\) (stat stands for stationary). In other words, in this case we have \((x, y, q, a_q) \equiv (E_i, Z_{q_{\text{stat}}}, p_i, q_{\text{stat}}, -\beta_{q_{\text{stat}}})\).

For systems that are at, or close to, the edge of chaos we typically have \(q_{\text{sen}} \leq 1, q_{\text{rel}} \geq 1,\) and \(q_{\text{stat}} \geq 1\). For the BG case, where there is one or more positive Lyapunov exponents, we recover the confluence \(q_{\text{sen}} = q_{\text{rel}} = q_{\text{stat}} = 1\).

One expects the entire q-triplet to be either measurable or calculable for Hamiltonian systems. And indeed it has recently been measured in the solar wind [10]. However, the generic relation among these three q indices is still unknown. For dissipative systems such as say the z-logistic map, no \(q_{\text{stat}}\) exists. Therefore, the problem reduces to only two q indices, namely \(q_{\text{sen}}\) and \(q_{\text{rel}}\). Their generic relation also is unknown. In the present paper, we provide the first (numerical) evidence of such a connection.

Before entering into the details of the present calculation, let us briefly review the connection with the entropy \(S_q\), basis of a current generalization of BG statistical mechanics referred to as nonextensive statistical mechanics [11]. This entropy is defined as follows:

\[
S_q = \frac{1}{q-1} \sum_{i=1}^{W} p_i \ln_q(1/p_i)
\]

where the \(q\)-logarithm function, inverse of the \(q\)-exponential, is defined as \(\ln_q x = \frac{x^{1-1/q} - 1}{1-q}\) (\(\ln_1 x = \ln x\) and \(S_1 = S_{BG} = -\sum_{i=1}^{W} p_i \ln p_i\)).

If we partition the phase space of a one-dimensional map (at its edge of chaos) into \(W\) small intervals, randomly place \(N\) initial conditions into one of those windows, and then run the dynamics for each of those \(N\) points, we get, as time \(t\) evolves, an occupancy characterized by \(\{N_i(t)\} = \{\sum_{i=1}^{W} N_i(t) = N\}\). With \(p_i(t) = N_i(t)/N\) we can calculate \(S_q(t)\) for any value of \(q\). From this, we can calculate the entropy production per unit time [12], defined as follows:

\[
K_q = \lim_{t \to \infty} \lim_{W \to \infty} \lim_{N \to \infty} \frac{S_q(t)}{t}
\]

It has been proved [13] that only \(K_{q_{\text{sen}}} = \text{finite} \) (\(K_q = 0\) for \(q > q_{\text{sen}}\) and \(K_q\) diverges for \(q < q_{\text{sen}}\)). Furthermore, if we consider the upper bound of \(K_{q_{\text{sen}}}\) with regard to the choice of the little window within which we put the \(N\) initial conditions, we obtain the Pesin-like identity \(K_{q_{\text{sen}}} = \lambda_{q_{\text{sen}}}.\) Several aspects of this problem have already been verified for various one-dimensional unimodal maps [14–17]. It was recently studied the influence of averaging [18]. It was verified that, while the \(q\)-generalized Pesin-like identity is preserved, the value of \(q_{\text{sen}}\) is changed into \(q_{\text{sen}}^v\) (av stands for average). The main goal of the present paper is to exhibit that a simple relation exists between \(q_{\text{sen}}^v\) and \(q_{\text{rel}}\) by making use of the z-logistic map family defined as

\[
x_{t+1} = 1 - a|x_t|^z
\]

where \((z > 1; 0 < a < 2; |x_t| \leq 1; t = 0, 1, 2, ...)\).

**TABLE I.** z-logistic map family for cycles 2, 3 and 5.

| \(z\) | cycle | \(a_c\) | \(q_{\text{sen}}^v\) | \(q_{\text{rel}}\) | \(\lambda_{q_{\text{sen}}^v}\) | \(K_{q_{\text{sen}}^v}\) |
|---|---|---|---|---|---|---|
| 1.75 | 2 | 1.355060... | 0.37 ± 0.01 | 2.25 ± 0.02 | 0.26 ± 0.01 | 0.26 ± 0.02 |
| 1.75 | 3 | 1.747303... | 0.92 ± 0.01 | 2.25 ± 0.02 | 0.48 ± 0.01 | 0.47 ± 0.02 |
| 1.75 | 5 | 1.607497... | 0.96 ± 0.01 | 2.25 ± 0.02 | 0.42 ± 0.01 | 0.40 ± 0.02 |
| 2 | 2 | 1.401155... | 0.36 ± 0.01 | 2.41 ± 0.02 | 0.27 ± 0.01 | 0.27 ± 0.02 |
| 2 | 3 | 1.779818... | 0.88 ± 0.01 | 2.41 ± 0.02 | 0.49 ± 0.01 | 0.48 ± 0.02 |
| 2 | 5 | 1.631019... | 0.93 ± 0.01 | 2.41 ± 0.02 | 0.42 ± 0.01 | 0.40 ± 0.02 |
| 2.5 | 2 | 1.470550... | 0.34 ± 0.01 | 2.70 ± 0.02 | 0.28 ± 0.01 | 0.28 ± 0.02 |
| 2.5 | 3 | 1.828863... | 0.82 ± 0.01 | 2.70 ± 0.02 | 0.48 ± 0.01 | 0.47 ± 0.01 |
| 2.5 | 5 | 1.669543... | 0.88 ± 0.01 | 2.70 ± 0.02 | 0.38 ± 0.01 | 0.37 ± 0.01 |
| 3 | 2 | 1.521878... | 0.32 ± 0.01 | 2.94 ± 0.02 | 0.29 ± 0.02 | 0.29 ± 0.03 |
| 3 | 3 | 1.862996... | 0.78 ± 0.01 | 2.94 ± 0.02 | 0.44 ± 0.01 | 0.44 ± 0.01 |
| 3 | 5 | 1.699440... | 0.84 ± 0.01 | 2.94 ± 0.02 | 0.34 ± 0.01 | 0.35 ± 0.01 |
| 5 | 2 | 1.645533... | 0.28 ± 0.01 | 3.53 ± 0.03 | 0.30 ± 0.02 | 0.30 ± 0.03 |
| 5 | 3 | 1.931072... | 0.68 ± 0.01 | 3.53 ± 0.03 | 0.36 ± 0.01 | 0.37 ± 0.01 |
| 5 | 5 | 1.773088... | 0.73 ± 0.01 | 3.53 ± 0.03 | 0.27 ± 0.01 | 0.25 ± 0.02 |
the sensitivity function. In addition to this, we obtained
\( K_{AV}^{\text{ave}} = \lambda_{AV}^{\text{ave}}, \)
which clearly broadens the validity region of the standard Pesin theorem [19].

![Graph showing the volume occupied by the ensemble as a function of discrete time.](image1)

\[ z=2 \; ; \; \text{cycle-3} \]
\[ a_c=1.7798180758 \]

**FIG. 1.** The volume occupied by the ensemble as a function of discrete time. After a transient period, which is the same for all \( N_{\text{box}} \) values, the power-law behavior is evident. For each case, the evolution of a set of 10\( N_{\text{box}} \) identical copies of the system is followed.

In our simulations, we firstly check whether \( q_{rel} \) values of cycles 3 and 5 are different from those of cycle 2 obtained in [6]. To accomplish this task, we analyse the rate of convergence to the critical attractor when an ensemble of initial conditions is uniformly distributed over the entire phase space (the phase space is partitioned \( N_{\text{box}} \) cells of equal size) and we found that, for all cycles that we studied, the volume \( W(t) \) occupied by the ensemble exhibits a power-law decay with the same exponent value for fixed \( z \). As an example, the case of cycle 3 for \( z=2 \) is given in Fig. 1. The same kind of behavior is obtained also for other \( z \) values and cycles, which yields us to conclude that \( q_{rel} \) does not depend on the cycle (see also the Table). Then, we concentrate on the ensemble averages of the sensitivity function \( \xi(t) \) by considering two very close points (throughout this work we take \( \Delta z(0) = 10^{-12} \) and calculating its value from Eq. (2)). This procedure has to be repeated many times with different \( x \) values randomly chosen in the available phase space and finally an average is taken over all \( \ln_q \xi(t) \) values. For cycle 3 and cycle 5, we obtain the behavior of \( \langle \ln_q \xi \rangle(t) \) as a function of \( t \), for various \( z \) values, from where one can deduce \( q_{\text{ave}}^{\text{av}} \) by identifying the linear time dependence as it is seen in Fig. 2. We verify that \( q_{\text{ave}}^{\text{av}} \) and \( q_{\text{ave}}^{\text{ave}} \) values do depend on both \( z \) and the cycle, whereas \( q_{\text{rel}} \) is independent of the cycles. Finally, to investigate the entropy production for cycles 3 and 5, we employ the procedure used so far in [16] for cycle 2 of the \( z \)-logistic map family. It is numerically verified that, as seen for a representative case in Fig. 3, for each \( z \) value and for each cycle, the linear entropy production occurs for a special \( q \) value which coincides with the one obtained from

\[ q_{rel}(\text{cycle } n) - 1 \approx A [1 - q_{\text{ave}}^{\text{ave}}(\text{cycle } n)]^\alpha \]  

(7)

where \( n = 2, 3, 5 \) and the values of \( A \) and \( \alpha \) are given in the caption of Fig. 4; both numbers depend on the cycle. For example, \( \alpha = 5.1 \) for cycle 2, and quickly approaches zero when the cycle increases; \( A \) also decreases when the

![Graph showing the behavior of \( \langle \ln_q \xi \rangle \) as a function of time.](image2)

**FIG. 2.** The behavior of \( \langle \ln_q \xi \rangle \) as a function of time.

![Graph showing the behavior of \( \langle S_q \rangle \) as a function of time.](image3)

**FIG. 3.** The behavior of \( \langle S_q \rangle \) as a function of time.
This kind of relation between these two classes of $q$ index is seen for the first time in a model system. It is clearly consistent with the confluence occurring for BG systems. This is to say, when there is at least one positive Lyapunov exponent, we obtain $q_{rel} = q_{sen}^{av} = 1$.

We also notice (see Fig. 5) a new and unexpected scaling behavior, namely

$$q_{sen}^{av}(cycle \ n) = 2.5 \ q_{sen}^{av}(cycle \ 2) + \epsilon , \quad (8)$$

with $\epsilon = -0.03$ for $n = 3$, and $\epsilon = 0.03$ for $n = 5$.

Summarizing, we have discussed a paradigmatic family of one-dimensional dissipative maps, and have shown that its (averaged) sensitivity to the initial conditions and its relaxation in phase space follow a simple path, which is consistent with current nonextensive statistical mechanical concepts, and which considerably extends the validity of Pesin-like identities. The sensitivity to the initial conditions is characterized by $q_{sen}^{av} < 1$, which monotonically approaches unity with increasing cycle size (at least for the specific cycles that we have studied here), and decreases with $z$. It is further characterized by $q_{rel}^{av}$, which exhibits a maximum both as a function of the cycle size and of $z$. The relaxation is characterized by $q_{rel} > 1$, which monotonically increases with $z$ and does not depend on the cycle. This study has enabled to exhibit two interesting relations, namely Eqs. (7) and (8). This path is expected to appreciably enlighten, among others, the case of long-range-interacting Hamiltonian systems, where the situation is even more complex since a third entropic index, $q_{stat}$, is expected, which would characterize the energy distribution at metastable states. Analytic analysis of the scalings presented here are certainly most welcome.

![Graph showing the relation between cycle size and sensitivity](image)

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