Collapsibility of simplicial complexes of hypergraphs

Alan Lew

Abstract

Let $H$ be a hypergraph of rank $r$. We show that the simplicial complex whose simplices are the hypergraphs $F \subset H$ with covering number at most $p$ is $((r+p)-1)$-collapsible, and the simplicial complex whose simplices are the pairwise intersecting hypergraphs $F \subset H$ is $\frac{1}{2}(2^r)$-collapsible.

1 Introduction

Let $X$ be a finite simplicial complex. Let $\sigma \in X$ such that $|\sigma| \leq d$ and $\sigma$ is contained in a unique maximal face $\tau \in X$. We say that the complex $X' = X \setminus \{\eta \in X : \sigma \subset \eta \subset \tau\}$ is obtained from $X$ by an elementary $d$-collapse, and we write $X \xrightarrow{\sigma} X'$.

The complex $X$ is $d$-collapsible if there is a sequence of elementary $d$-collapses from $X$ to the void complex $\emptyset$. The sequence $X = X_1 \xrightarrow{\sigma_1} X_2 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_k} X_k = \emptyset$ is called a $d$-collapsing sequence for $X$. The collapsibility of $X$ is the minimal $d$ such that $X$ is $d$-collapsible.

A simple consequence of $d$-collapsibility is the following (see [9]).

Claim 1.1. If $X$ is $d$-collapsible then $X$ collapses to a complex of dimension smaller than $d$. In particular, the homology groups $\tilde{H}_k(X)$ are trivial for $k \geq d$.

Let $\mathcal{H}$ be a finite hypergraph. We identify $\mathcal{H}$ with its edge set. The rank of $\mathcal{H}$ is the maximal size of an edge of $\mathcal{H}$.

A set $C$ is a cover of $\mathcal{H}$ if $A \cap C \neq \emptyset$ for all $A \in \mathcal{H}$. The covering number of $\mathcal{H}$, denoted by $\tau(\mathcal{H})$, is the minimal size of a cover of $\mathcal{H}$.

*Department of Mathematics, Technion, Haifa 32000, Israel. e-mail: alan@campus.technion.ac.il. Supported by ISF grant no. 326/16.
For \( p \in \mathbb{N} \), let
\[
\text{Cov}_{\mathcal{H},p} = \{ \mathcal{F} \subset \mathcal{H} : \tau(\mathcal{F}) \leq p \}.
\]
So \( \text{Cov}_{\mathcal{H},p} \) is a simplicial complex whose vertices are the edges of \( \mathcal{H} \) and whose simplices are the hypergraphs \( \mathcal{F} \subset \mathcal{H} \) that can be covered by a set of size at most \( p \). Some topological properties of the complex \( \text{Cov}_{\binom{r+p}{r},p} \) were studied by Jonsson in [5].

The hypergraph \( \mathcal{H} \) is called pairwise intersecting if \( A \cap B \neq \emptyset \) for all \( A, B \in \mathcal{H} \). Let
\[
\text{Int}_\mathcal{H} = \{ \mathcal{F} \subset \mathcal{H} : A \cap B \neq \emptyset \text{ for all } A, B \in \mathcal{F} \}.
\]
So \( \text{Int}_\mathcal{H} \) is a simplicial complex whose vertices are the edges of \( \mathcal{H} \) and whose simplices are the hypergraphs \( \mathcal{F} \subset \mathcal{H} \) that are pairwise intersecting.

Our main results are the following:

\textbf{Theorem 1.2.} Let \( \mathcal{H} \) be a hypergraph of rank \( r \). Then \( \text{Cov}_{\mathcal{H},p} \) is \( \left( \binom{r+p}{r} - 1 \right) \)-collapsible.

\textbf{Theorem 1.3.} Let \( \mathcal{H} \) be a hypergraph of rank \( r \). Then \( \text{Int}_\mathcal{H} \) is \( \frac{1}{2} \binom{2r}{r} \)-collapsible.

The following examples show that these bounds are sharp:

- Let \( \mathcal{H} = \binom{[r+p]}{r} \) be the complete \( r \)-uniform hypergraph on \( r+p \) vertices. The covering number of \( \mathcal{H} \) is \( p+1 \), but for any \( A \in \mathcal{H} \) the hypergraph \( \mathcal{H} \setminus \{ A \} \) can be covered by a set of size \( p \), namely by \( [r+p] \setminus A \). Therefore the complex \( \text{Cov}_{\binom{r+p}{r},p} \) is the boundary of the \( \left( \binom{r+p}{r} - 1 \right) \)-dimensional simplex, so it is homeomorphic to a \( \left( \binom{r+p}{r} - 2 \right) \)-dimensional sphere. Hence, by Claim 1.1, \( \text{Cov}_{\binom{r+p}{r},p} \) is not \( \left( \binom{r+p}{r} - 2 \right) \)-collapsible.

- Let \( \mathcal{H} = \binom{[2r]}{r} \) be the complete \( r \)-uniform hypergraph on \( 2r \) vertices. Any \( A \in \mathcal{H} \) intersects all the edges of \( \mathcal{H} \) except the edge \( [2r] \setminus A \). Therefore the complex \( \text{Int}_{\binom{[2r]}{r}} \) is the boundary of the \( \frac{1}{2} \binom{2r}{r} \)-dimensional cross-polytope, so it is homeomorphic to a \( \left( \frac{1}{2} \binom{2r}{r} - 1 \right) \)-dimensional sphere. Hence, by Claim 1.1, \( \text{Int}_{\binom{[2r]}{r}} \) is not \( \left( \frac{1}{2} \binom{2r}{r} - 1 \right) \)-collapsible.

The proofs rely on two main ingredients. The first one is a general construction of a \( d \)-collapsing sequence for a simplicial complex (with \( d \) depending on the complex), due essentially to Matoušek and Tancer (who stated it in the special case where the complex is the nerve of a family of finite sets, and used it to prove the case \( p = 1 \) of Theorem 1.2).

The second ingredient is the following combinatorial lemma, proved independently by Frankl and Kalai.
Lemma 1.4 (Frankl \cite{Frankl}, Kalai \cite{Kalai}). Let \( \{A_1, \ldots, A_k\} \) and \( \{B_1, \ldots, B_k\} \) be families of sets such that:

- \( |A_i| \leq r, |B_i| \leq p \) for all \( i \in [k] \),
- \( A_i \cap B_i = \emptyset \) for all \( i \in [k] \),
- \( A_i \cap B_j \neq \emptyset \) for all \( 1 \leq i < j \leq k \).

Then

\[
  k \leq \left( \frac{r + p}{r} \right).
\]

The paper is organized as follows. In Section 2 we present the \( d \)-collapsing sequence of Matoušek and Tancer. In Section 3 we present some results on the collapsibility of independence complexes of graphs. In Section 4 we prove our main results on the collapsibility of complexes of hypergraphs. In Section 5 we present some generalizations of Theorems 1.2 and 1.3, that are obtained by applying different known variants of Lemma 1.4.

2 A \( d \)-collapsing sequence for a simplicial complex

Let \( X \) be a simplicial complex on vertex set \( V \). Fix an arbitrary linear order \( < \) on the vertices \( V \). Let \( \sigma_1, \ldots, \sigma_m \) be the maximal faces of \( X \). For a simplex \( \sigma \in X \) let \( m(\sigma) = \min \{ i \in [m] : \sigma \subset \sigma_i \} \).

Let \( i \in [m] \) and \( \sigma \in X \) such that \( m(\sigma) = i \). We define the minimal exclusion sequence \( \text{mes}(\sigma) = (v_1, \ldots, v_{i-1}) \) as follows: If \( i = 1 \) then \( \text{mes}(\sigma) \) is the empty sequence. If \( i > 1 \) we define the sequence recursively as follows:

Since \( i > 1 \), we must have \( \sigma \not\subseteq \sigma_1 \), hence there is some \( v \in \sigma \) such that \( v \notin \sigma_1 \). Let \( v_1 \) be the minimal such vertex (with respect to the order \( < \)).

Let \( j < i \) and assume that we already defined \( v_1, \ldots, v_{j-1} \). Since \( i > j \), we must have \( \sigma \not\subseteq \sigma_j \), hence there is some \( v \in \sigma \) such that \( v \notin \sigma_j \).

- If there is such a vertex \( v_k \in \{v_1, \ldots, v_{j-1}\} \), let \( v_j \) be such a vertex with minimal \( k \). In this case we call \( v_j \) old at \( j \).
- If \( v_k \in \sigma_j \) for all \( k < j \), then let \( v_j \) be the minimal vertex \( v \in \sigma \) (with respect to the order \( < \)) such that \( v \notin \sigma_j \). In this case we call \( v_j \) new at \( j \).

Let \( M(\sigma) \) be the simplex consisting of all the vertices appearing in the sequence \( \text{mes}(\sigma) \). Note that \( \text{mes}(M(\sigma)) = \text{mes}(\sigma) \). Let

\[
  M_i = \{ M(\sigma) : \sigma \in X, m(\sigma) = i \}
\]

and \( M = \bigcup_{i=1}^m M_i \). Let

\[
  d(X) = \max \{|\eta| : \eta \in M\}.
\]
The following result was stated and proved in [8, Prop. 1.3] in the special case where $X$ is the nerve of a finite family of sets (in our notation, $X = \text{Cov}_{\mathcal{H}, 1}$ for some hypergraph $\mathcal{H}$), but the proof given there can be easily modified to hold in a more general setting.

**Proposition 2.1.** The simplicial complex $X$ is $d(X)$-collapsible.

For completeness we include here the proof.

**Lemma 2.2.** Let $\sigma, \sigma' \in X$. Then $\text{mes}(\sigma) = \text{mes}(\sigma')$ if and only if $M(\sigma) = M(\sigma')$.

**Proof.** If $\text{mes}(\sigma) = \text{mes}(\sigma')$ then clearly $M(\sigma) = M(\sigma')$. Assume $M(\sigma) = M(\sigma')$. Then we have

$$\text{mes}(\sigma) = \text{mes}(M(\sigma)) = \text{mes}(M(\sigma')) = \text{mes}(\sigma').$$

$\square$

We define a linear order $\prec$ on $M$ as follows: First, we order the families $M_i$ by decreasing $i$: The simplices in $M_m$ come first, then the ones in $M_{m-1}$ and so on. Within each $M_i$ we order the simplices lexicographically by their minimal exclusion sequence.

**Lemma 2.3.** Let $\sigma \subset \sigma' \in X$. Then $M(\sigma) \succeq M(\sigma')$.

**Proof.** First we note that since $\sigma \subset \sigma'$, we must have $\sigma \subset \sigma_i$ whenever $\sigma' \subset \sigma_i$. Therefore $m(\sigma) \leq m(\sigma')$. If $m(\sigma) < m(\sigma')$ then $M(\sigma) \in M_i$ and $M(\sigma') \in M_j$ for some $i < j$, hence $M(\sigma) \succ M(\sigma')$.

If $m(\sigma) = m(\sigma')$ then both simplices have a minimal exclusion sequence of the same length. Suppose $\text{mes}(\sigma) \neq \text{mes}(\sigma')$. Let $j$ be the first index where the sequences differ, and let $v_j$ be the vertex at index $j$ in $\text{mes}(\sigma)$ and $v'_j$ be the vertex at index $j$ in $\text{mes}(\sigma')$. Since $v_j \neq v'_j$, then $v_j$ and $v'_j$ must be both new at $j$. So $v_j$ is the minimal vertex in $\sigma$ such that $v_j \notin \sigma_j$ and $v'_j$ is the minimal vertex in $\sigma'$ such that $v'_j \notin \sigma_j$. But since $\sigma \subset \sigma'$, we must have $v'_j < v_j$. So $\text{mes}(\sigma')$ is lexicographically smaller than $\text{mes}(\sigma)$, hence $M(\sigma') \prec M(\sigma)$. $\square$

For each $\eta \in M$ define

$$T(\eta) = \{ v \in V : \text{mes}(\eta \cup \{v\}) = \text{mes}(\eta) \}.$$

Note that $\eta \subset T(\eta)$.

**Lemma 2.4.** Let $\sigma \in X$ and $\eta \in M$. Then $\eta \subset \sigma \subset T(\eta)$ if and only if $M(\sigma) = \eta$. 

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Proof. Assume that $M(\sigma) = \eta$. Then $\eta = M(\sigma) \subset \sigma$. We have to show that $\sigma \subset T(\eta)$. Let $v \in \sigma$. By Lemma 2.3 we have

$$\eta = M(\eta) \geq M(\eta \cup \{v\}) \geq M(\sigma) = \eta.$$ 

So $M(\eta \cup \{v\}) = \eta$, and by Lemma 2.2 $\text{mes}(\eta \cup \{v\}) = \text{mes}(\eta)$. Therefore $v \in T(\eta)$, so $\sigma \subset T(\eta)$.

Assume that $\eta \not\subset T(\eta)$ (and in particular $\eta \not\subset \sigma$). Then $\text{mes}(\eta)$ would be longer than $\text{mes}(\eta \cup \{v\})$, so $\sigma \subset T(\eta)$. Therefore $\text{mes}(\eta \cup \{v\}) = \text{mes}(\eta \cup \{v\})$ and $\text{mes}(T(\eta)) = \text{mes}(\eta)$.

Assume $\eta \in M_i$ for some $i \in [m]$. Then for all $v \in T(\eta)$ we have $v \in \sigma_i$, otherwise $\text{mes}(\eta \cup \{v\})$ would be longer than $\text{mes}(\eta)$. Therefore $T(\eta) \subset \sigma_i$ (and in particular $T(\eta) \in X$). On the other hand, since $\eta \subset T(\eta)$ and $\eta \not\subset \sigma_j$ for $j < i$, then also $T(\eta) \not\subset \sigma_j$. Therefore $m(T(\eta)) = i$, so $\text{mes}(\eta)$ and $\text{mes}(T(\eta))$ have both the same length.

Let $\text{mes}(\eta) = (v_1, \ldots, v_{i-1})$. Assume that $\text{mes}(\eta) \neq \text{mes}(T(\eta))$ and let $j < i$ be the minimal index where the two sequences differ. Then there is some $v \in T(\eta) \setminus \eta$ such that $v \notin \sigma_j$ and $v < v_j$. But then the $j$-th element of $\text{mes}(\eta \cup \{v\})$ is also $v$, a contradiction to $\text{mes}(\eta \cup \{v\}) = \text{mes}(\eta)$.

Therefore $\text{mes}(T(\eta)) = \text{mes}(\eta)$ and, by Lemma 2.2, $M(T(\eta)) = \eta$. □

Lemma 2.5. Let $\eta \in M$ and let $Y = \{\sigma \in X : M(\sigma) \geq \eta\}$. Then

1. The unique maximal face of $Y$ containing $\eta$ is $T(\eta)$.

2. Let $Y' = Y \setminus \{\sigma \in Y : \eta \subset \sigma \subset T(\eta)\}$. If $\eta$ is the maximal element of $M$ with respect to $\prec$, then $Y' = \emptyset$. Otherwise,

$$Y' = \{\sigma \in X : M(\sigma) \geq \eta'\},$$

where $\eta'$ is the element of $M$ succeeding $\eta$ in the order $\prec$.

Proof. 1. Let $\eta \subset \sigma \in Y$. By Lemma 2.3 we have $M(\sigma) \leq \eta$. But since $\sigma \in Y$, $M(\sigma) = \eta$. Hence, by Lemma 2.4 $\sigma \subset T(\eta)$.

2. By Lemma 2.4 we have

$$\{\sigma \in Y : \eta \subset \sigma \subset T(\eta)\} = \{\sigma \in Y : M(\sigma) = \eta\} = \{\sigma \in X : M(\sigma) = \eta\}.$$

Therefore, if $\eta$ is maximal in $M$ then $Y' = \emptyset$, otherwise

$$Y' = \{\sigma \in X : M(\sigma) \geq \eta'\},$$

where $\eta'$ is the element of $M$ succeeding $\eta$ in the order $\prec$. □
Proof of Proposition 2.1. Let \( \eta_1, \ldots, \eta_s \) be the elements of \( M \), written in increasing order. For \( i \in [s] \) let

\[
Y_i = \{ \sigma \in X : M(\sigma) \succeq \eta_i \}.
\]

Note that \( Y_1 = X \). Recall that \( d(X) = \max_{i \in [s]} |\eta_i| \). By Lemma 2.5 we have the following \( d(X) \)-collapsing sequence:

\[
X = Y_1 \xrightarrow{\eta_1} Y_2 \xrightarrow{\eta_2} \cdots \xrightarrow{\eta_{s-1}} Y_s \xrightarrow{\eta_s} \emptyset.
\]

Thus \( X \) is \( d(X) \)-collapsible.

\[\square\]

3 Collapsibility of independence complexes

Let \( G = (V, E) \) be a graph. The independence complex \( I(G) \) is the simplicial complex on vertex set \( V \) whose simplices are the independent sets in \( G \).

Definition 3.1. Let \( k(G) \) be the maximal size of a set \( \{v_1, \ldots, v_k\} \subset V \) that satisfies:

- \( \{v_i, v_j\} \notin E \) for all \( i \neq j \in [k] \),
- There exist \( u_1, \ldots, u_k \in V \) such that
  - \( \{v_i, u_i\} \in E \) for all \( i \in [k] \),
  - \( \{v_i, u_j\} \notin E \) for all \( 1 \leq i < j \leq k \).

Proposition 3.2. The complex \( I(G) \) is \( k(G) \)-collapsible.

Proof. Let \( X = I(G) \), and let \( \sigma_1, \ldots, \sigma_m \) be the maximal faces of \( X \) (i.e. the maximal independent sets of \( G \)). Let \( i \in [m] \) and \( \sigma \in X \) with \( m(\sigma) = i \), that is \( \sigma \subseteq \sigma_i \) and \( \sigma \not\subseteq \sigma_j \) for \( j < i \).

Let \( \text{mes}(\sigma) = (v_1, \ldots, v_{i-1}) \). Then \( M(\sigma) = \{v_{i_1}, \ldots, v_{i_k}\} \) for some \( i_1, \ldots, i_k \in [i-1] \) (these are exactly the indices \( i_j \) such that \( v_{i_j} \) is new at \( i_j \)). For each \( j \in [k] \) we have \( v_{i_j} \notin \sigma_{i_j} \), therefore there is some \( u_{i_j} \in \sigma_{i_j} \) such that \( \{v_{i_j}, u_{i_j}\} \in E \). In addition, since \( v_{i_j} \) is new at \( i_j \), we have \( v_{i_\ell} \in \sigma_{i_\ell} \) for all \( \ell < j \). In particular, \( \{v_{i_\ell}, u_{i_\ell}\} \notin E \) for \( \ell < j \). Also, since \( \sigma \) is an independent set in \( G \), we have \( \{v_{i_\ell}, v_{i_j}\} \notin E \) for all \( j \neq \ell \in [k] \).

Hence \( |M(\sigma)| \leq k(G) \), so \( d(X) \leq k(G) \). Therefore, by Proposition 2.1 \( X \) is \( k(G) \)-collapsible.

As a simple corollary we obtain

Corollary 3.3. The independence complex of a graph \( G = (V, E) \) on \( n \) vertices is \( \left\lfloor \frac{n}{2} \right\rfloor \)-collapsible.

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Proof. Let \( k = k(G) \) and let \( v_1, \ldots, v_k, u_1, \ldots, u_k \in V \) satisfying the conditions in Definition 3.1. Then the vertices \( v_1, \ldots, v_k, u_1, \ldots, u_k \) must be all distinct, therefore \( 2k \leq n \). Thus

\[ k(G) \leq \frac{n}{2}, \]

so the claim follows from Proposition 3.2.

4 Complexes of hypergraphs

Next we prove our main results, Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Let \( H \) be a hypergraph of rank \( r \) on vertex set \([n]\) and let \( X = \text{Cov}_{\mathcal{H}, p} \).

Let \( F_1, \ldots, F_m \subset H \) be the maximal faces of \( X \). For any \( i \in [m] \) there is some \( C_i \subset [n] \) of size at most \( p \) such that for any hypergraph \( F \subset H \), \( F \subset F_i \) if and only if \( C_i \) is a cover of \( F \).

Let \( i \in [m] \) and \( F \in X \) with \( m(F) = i \) and let \( \text{mes}(F) = (A_1, \ldots, A_{i-1}) \) be its minimal exclusion sequence. Then \( M(F) = \{ A_1, \ldots, A_k \} \) for some \( i, \ldots, i_k \in [i-1] \).

By the definition of the minimal exclusion sequence we have \( A_i \cap C_{i_j} = \emptyset \) for all \( j \in [k] \) and \( A_i \cap C_{i_j} \neq \emptyset \) for all \( 1 \leq \ell < j \leq k \).

Also, for all \( j \in [k] \) we have \( A_i \cap C_i \neq \emptyset \), since \( F \in X_i \). Therefore the pair of families

\[ \{A_1, \ldots, A_k, \emptyset\} \]

and

\[ \{C_i, \ldots, C_k, C_i\} \]

satisfies the conditions of Lemma 1.4 hence \( k + 1 \leq \binom{r+p}{r} \). Thus,

\[ |M(F)| = k \leq \binom{r+p}{r} - 1. \]

So \( d(X) \leq \binom{r+p}{r} - 1 \), and by Proposition 2.1 \( X \) is \((\binom{r+p}{r} - 1)\)-collapsible.

Proof of Theorem 1.3. Let \( \mathcal{H} \) be a hypergraph of rank \( r \) and let \( G \) be the graph on vertex set \( \mathcal{H} \) whose edges are the pairs \( \{A, B\} \subset \mathcal{H} \) such that \( A \cap B = \emptyset \). Then \( \text{Int}_\mathcal{H} = I(G) \).

Let \( k = k(G) \) and let \( \{A_1, \ldots, A_k\} \subset \mathcal{H} \) that satisfies the conditions of Definition 3.1. That is,

- \( A_i \cap A_j \neq \emptyset \) for all \( i \neq j \in [k] \),
- There exist \( B_1, \ldots, B_k \in \mathcal{H} \) such that
Then the pair of families
\[
\{A_1, \ldots, A_k, B_k, \ldots, B_1\}
\]
and
\[
\{B_1, \ldots, B_k, A_k, \ldots, A_1\}
\]
satisfies the conditions of Lemma 1.4, therefore \(2k \leq \binom{2r}{r}\). Thus, by Proposition 3.2, \(\text{Int}_H = I(G)\) is \(\frac{1}{2}(2r)\)-collapsible.

5 More complexes of hypergraphs

Let \(H\) be a hypergraph. A set \(C\) is a \(t\)-transversal of \(H\) if \(|A \cap C| \geq t\) for all \(A \in H\). Let \(\tau_t(H)\) be the minimal size of a \(t\)-transversal of \(H\). The hypergraph \(H\) is pairwise \(t\)-intersecting if \(|A \cap B| \geq t\) for all \(A, B \in H\). Let

\[
\text{Cov}_{H,p}^t = \{F \subset H : \tau_t(F) \leq p\}
\]

and

\[
\text{Int}_H^t = \{F \subset H : F \text{ is pairwise } t\text{-intersecting}\}.
\]

The following generalization of Lemma 1.4 was proved by Füredi in [4].

**Lemma 5.1** (Füredi [4]). Let \(\{A_1, \ldots, A_k\}\) and \(\{B_1, \ldots, B_k\}\) be families of sets such that:

- \(|A_i| \leq r, |B_i| \leq p\) for all \(i \in [k]\),
- \(|A_i \cap B_i| \leq t\) for all \(i \in [k]\),
- \(|A_i \cap B_j| > t\) for all \(1 \leq i < j \leq k\).

Then

\[
k \leq \binom{r+p-2t}{r-t}.
\]

We obtain the following:

**Theorem 5.2.** Let \(H\) be a hypergraph of rank \(r\). Then \(\text{Cov}_{H,p}^{t+1}\) is \(\binom{r+p-2t}{r-t} - 1\)-collapsible.

**Theorem 5.3.** Let \(H\) be a hypergraph of rank \(r\). Then \(\text{Int}_H^{t+1}\) is \(\frac{1}{2}(2(r-t))\)-collapsible.
Note that by setting $t = 0$ we recover Theorems 1.2 and 1.3. The proofs are essentially the same as the proofs of Theorems 1.2 and 1.3, only we use Lemma 5.1 instead of Lemma 1.4. The extremal examples are also similar: Let
\[ H_1 = \{ A \cup [t] : A \in \binom{[r + p - t] \setminus [t]}{r - t} \} \]
and
\[ H_2 = \{ A \cup [t] : A \in \binom{[2r - t] \setminus [t]}{r - t} \} \].

The complex $\text{Cov}^{t+1}_{H_1}$ is the boundary of the $\binom{(r+p-2t)}{r-t} - 1$-dimensional simplex, hence it is not $\binom{(r+p-2t)}{r-t} - 2$-collapsible, and the complex $\text{Int}^{t+1}_{H_2}$ is the boundary of the $\frac{1}{2} \binom{2(r-t)}{r-t} - 1$-dimensional cross-polytope, hence it is not $\frac{1}{2} \binom{2(r-t)}{r-t} - 1$-collapsible.

Restricting ourselves to special classes of hypergraphs we may obtain better bounds on the collapsibility of their associated complexes. For example, we may look at $r$-partite $r$-uniform hypergraphs (that is, hypergraphs $\mathcal{H}$ on vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_r$ such that $|A \cap V_i| = 1$ for all $A \in \mathcal{H}$ and $i \in [r]$). In this case we have the following result:

**Theorem 5.4.** Let $\mathcal{H}$ be an $r$-partite $r$-uniform hypergraph. Then $\text{Int}_\mathcal{H}$ is $2^{r-1}$-collapsible.

The next example shows that the bound on the collapsibility of $\text{Int}_\mathcal{H}$ in Theorem 5.4 is tight: Let $\mathcal{H}$ be the complete $r$-partite $r$-uniform hypergraph with all sides of size 2. It has $2^r$ edges, and any edge $A \in \mathcal{H}$ intersects all the edges of $\mathcal{H}$ except its complement. Therefore the complex $\text{Int}_\mathcal{H}$ is the boundary of the $2^{r-1}$-dimensional cross-polytope, so it is homeomorphic to a $(2^{r-1} - 1)$-dimensional sphere. Hence, by Claim 1.1, $\text{Int}_\mathcal{H}$ is not $(2^{r-1} - 1)$-collapsible.

For the proof we need the following Lemma, due to Lovász, Nešetřil and Pultr.

**Lemma 5.5** (Lovász, Nešetřil, Pultr [7, Prop. 5.3]). Let $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_k\}$ be families of subsets of $V = V_1 \cup V_2 \cup \cdots \cup V_r$ such that:
- $|A_i \cap V_j| = 1$, $|B_i \cap V_j| = 1$ for all $i \in [k]$ and $j \in [r]$,
- $A_i \cap B_i = \emptyset$ for all $i \in [k]$,
- $A_i \cap B_j \neq \emptyset$ for all $1 \leq i < j \leq k$.

Then
\[ k \leq 2^r. \]
A common generalization of Lemma 1.4 and Lemma 5.5 was proved by Alon in [2].

The proof of Theorem 5.4 is the same as the proof of Theorem 1.3, except that we replace Lemma 1.4 by Lemma 5.5. A similar argument was also used by Aharoni and Berger ([1, Theorem 5.1]) in order to prove a related result about rainbow matchings in \( r \)-partite \( r \)-uniform hypergraphs.

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