How to produce quantum entanglement for ascertaining incompatible properties in double-slit experiments

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Double-slit experiment very well lends itself in describing the problem of measuring simultaneously incompatible properties. In such a context, we theoretically design an ideal experiment for spin-7/2 particles, able to produce the entanglement which makes possible the detection.

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I. INTRODUCTION

In Quantum Mechanics the algebraic structure of the set of observables is mirrored in the algebraic structure of the self-adjoint operators, which is not commutative; then, whereas in Classical Mechanics all the observables can be measured together on a physical system, this is not the case in Quantum Mechanics [1]. Here such a possibility is restricted to those observables corresponding to commutative self-adjoint operators.

The double-slit experiment is a very effective example in describing such a phenomenon: the incompatibility between the position observables at different times makes impossible to measure simultaneously which slit each particle passes through (WS property) and the localization on the final screen (L property). This notwithstanding, over the years a lot of devices have been conceived in order to reach “indirect” knowledge about WS property by measuring a different property, T, correlated with it [2, 3]. More recently, experiments have been proposed which can give indirect knowledge about either WS property or an incompatible one, according to their set-up [4-6]. The theoretical investigation performed in [6] in the context of double-slit experiment, establishes that, under suitable hypothesis and exploiting entanglement, the question whether experimental situations may be conceived, which make possible to obtain “indirect” knowledge about two incompatible properties (WS and an incompatible one, G), is positively answered, in spite of the predicted impossibility of measuring them simultaneously. Following such a result, in [7] an ideal double-slit experiment for atoms is designed such that the values of two non-commuting quantum properties are inferred simultaneously by revealing photons emitted by single atoms within micro-maser cavities.

Other approaches in literature face the problem of ascertaining simultaneously incompatible properties. The claimed impossibility of producing inferences for more than three observables urges to consider the situation of detecting two incompatible properties, besides WS, together with the measurement of the final impact point. The theoretical investigation in [8] shows that also this question has affirmative answer within this approach; an ideal experiment realizing such a detection is presented in [7].

In the present work, after introducing the problem the simultaneous detection of three incompatible properties, WS, G and L, together with the final impact point (section 2), an ideal double slit experiment is designed which realizes such a detection (section 3). We notice that this ideal experiment correspond to a particular solution of the theoretical investigation in [9] and with the assumptions therein, properties L and G turn out always correlated. Moreover, it must be stressed that such a detection requires an entanglement between the particle and the detector. Section 3 is devoted to show how such an entanglement can be ideally realized.

II. DETECTING INCOMPATIBLE PROPERTIES SIMULTANEOUSLY

We consider a system (e.g. the nucleus of an atom) which can travel towards the two slits, but not elsewhere, whose position is represented, in Heisenberg picture, by an operator in the Hilbert space $\mathcal{H}_I$. It possesses further degrees of freedom (e.g. spin) described in the Hilbert-space $\mathcal{H}_U$; hence, $\mathcal{H} = \mathcal{H}_I \otimes \mathcal{H}_U$ is the Hilbert space describing the entire system. In general, we denote by $A_I$ ($A_U$) an operator of $\mathcal{H}_I$ ($\mathcal{H}_U$). The projection operator representing WS property “the particle passes through slit 1 (2)” has the form $E = E_I \otimes 1_U$ ($E' = (1_I - E_I) \otimes 1_U$). Given any interval $\Delta$ on the final screen, we denote the projection operator representing the property “the particle hits the final screen in a point within $\Delta$” by $F(\Delta)$. Though $E$ and $F(\Delta)$ are both localization projections, they refers to different times – the time $t_1$ of the slits’ crossing for $E$ and the time $t_2 > t_1$ of the final impact for $F(\Delta)$. We suppose that the Hamiltonian operator in independent of the degrees of freedom described by $\mathcal{H}_U$, so that it has the form $H = H_I \otimes 1_U$; hence, it can be shown that $F(\Delta) = F_I(\Delta) \otimes 1_U$ follows, but $[E, F(\Delta)] \neq 0$ must hold too [9]. Thus, it is not generally possible...
to ascertain WS property and the final impact point, by direct localization measurements. Rather than measuring $E$, a property $T = 1_f \otimes T_B$ acting on $\mathcal{H}_B$ can be measured together with $F(\Delta)$, whose outcomes are correlated with the outcome of $E$, as expressed by the following general definition of detector $[10]$

**Definition 1.** A projection operator $S$ of $\mathcal{H}$ is called a detector of a property $R$ with respect to a state $\psi$ if

(i) $[S, F(\Delta)] = 0$,

(ii) $[S, R] = 0$ and $S\psi = R\psi$.

Condition (i) ensures that $S$ can be measured together with $F(\Delta)$; condition (ii) allows us to infer the outcome of $R$ (albeit not measured) from the outcome of $S$ $[11]$. According to Def. 1, if for a given state $\Psi$, a projection operator $T = 1_f \otimes T_B$ exists such that $T\Psi = E\Psi$, then it is possible to detect which slit each particle passes through by means of a measurement of $T$; indeed, since $E = E_f \otimes 1_B$ and $F(\Delta) = F_f \otimes 1_B$, conditions $[T, F(\Delta)] = 0$ and $[T, E] = 0$ are automatically satisfied. Hence, in other words, outcome 1 (0) for $T$ reveals the passage of the particle through slit 1 (2).

We seek for the possibility of detecting two further properties, $G$ and $L$: let $G = G_f \otimes 1_B$ and $L = L_f \otimes 1_B$ be other properties, incompatible with each other and with WS property $E$; i.e. $[L, G] \neq 0$, $[L, E] \neq 0$ and $[E, G] \neq 0$; if for a given state $\Psi$, two commuting projection operators $Y = 1_f \otimes Y_B$ and $W = 1_f \otimes W_B$ exist such that $Y$ is a detector for $G$ and $W$ is a detector for $L$, then the outcomes of $Y$ and $W$ reveal the occurrence of the properties $G$ and $L$ respectively; therefore, the problem is:

**Problem ($P'$).** Given WS property $E = E_f \otimes 1_B$ we have to find

- two projection operators $G_f$ and $L_f$ of $\mathcal{H}_f$,

- three projection operators $T_B$, $Y_B$ and $W_B$ of $\mathcal{H}_B$,

- a state vector $\Psi \in \mathcal{H}_f \otimes \mathcal{H}_B$,

such that the following conditions are satisfied:

(C.1) $[E, G] \neq 0$ i.e. $[E_f, G_f] \neq 0$

(C.2) $[E, L] \neq 0$ i.e. $[E_f, L_f] \neq 0$,

(C.3) $[L, G] \neq 0$ i.e. $[L_f, G_f] \neq 0$,

(C.4) $[T, E] = 0$ and $T\Psi = E\Psi$

(C.5) $[Y, G] = 0$ and $Y\Psi = G\Psi$

(C.6) $[W, L] = 0$ and $W\Psi = L\Psi$

(C.7) $[T, Y] = 0$

(C.8) $[T, W] = 0$

(C.9) $Y W = 0$

(C.10) $\Psi \neq E\Psi \neq 0$, $\Psi \neq G\Psi \neq 0$ and $\Psi \neq L\Psi \neq 0$.

The ideal experiment described in the next section represents a solution of problem ($P'$).

**III. AN IDEAL EXPERIMENT FOR SIMULTANEOUS DETECTION ON INCOMPATIBLE PROPERTIES**

The system of our ideal experiment is a spin-7/2 particle whose position observable is described in a Hilbert space $\mathcal{H}_I$, while the spin observables are described in $\mathcal{H}_B \equiv \mathbb{C}^8$. Let $\psi_i$, (resp., $\psi_{i+5}$), $i = 1, \ldots, 5$ be 5 mutually orthonormal vectors of $\mathcal{H}_I$ localized in slit 1 (resp., slit 2) when the particle crosses the slits’ support, i.e. such that $E_f \psi_i = \psi_i$ (resp., $E_f \psi_{i+5} = 0$). No further condition is required to these vectors. These ten vectors form an orthonormal set. Then we take the Hilbert space $\mathcal{H}_I$ as the space generated by them. This implies that

\[ E_f \varphi = \langle \psi_1 | \varphi \rangle \psi_1 + \langle \psi_2 | \varphi \rangle \psi_2 + \langle \psi_3 | \varphi \rangle \psi_3 + \langle \psi_4 | \varphi \rangle \psi_4 + \langle \psi_5 | \varphi \rangle \psi_5 \quad \text{for every} \quad \varphi \in \mathcal{H}_I. \]  

(1)

There are 8 eigenvectors $\alpha_1 = |7/2\rangle$, $\alpha_2 = |5/2\rangle$, $\alpha_3 = |3/2\rangle$, $\alpha_4 = |1/2\rangle$, $\alpha_5 = | -1/2\rangle$, $\alpha_6 = | -3/2\rangle$, $\alpha_7 = | -5/2\rangle$, $\alpha_8 = | -7/2\rangle \in \mathcal{H}_B$ corresponding to the 8 possible values (in $\hbar$ units) of the spin along direction $z$, represented by the hermitian operator $S_z$ of $\mathbb{C}^8$. 
A straightforward calculation based on (1) and (3) shows that the WS detector.

Let the particle be prepared in the entangled state represented by

$$\Psi = \frac{1}{32} (-\psi_1 - 2\psi_2 + \psi_3 + \psi_4 + \psi_5) |7/2\rangle + \frac{1}{8} \sqrt{\frac{7}{10}} (-\psi_1 + \psi_2 - 2\psi_3 + 3\psi_5) |3/2\rangle + \frac{\sqrt{35}}{32} (-\psi_6 - 2\psi_7 + \psi_8 + \psi_9 + \psi_{10}) |1/2\rangle + \frac{1}{16} \left[ \sqrt{\frac{35}{11}} (\psi_1 + \psi_2 + 3\psi_4) + (4\psi_1 + \psi_2 + 3\psi_3 + 3\psi_5) \right] | -1/2 \rangle + \frac{1}{8} \sqrt{\frac{7}{30}} (-\psi_6 + \psi_7 - 2\psi_8 + 3\psi_9) | -5/2 \rangle + \frac{1}{16} \left[ \frac{1}{\sqrt{11}} (\psi_6 + \psi_7 + 3\psi_9) + \frac{1}{\sqrt{35}} (4\psi_6 + \psi_7 + 3\psi_8 + 3\psi_{10}) \right] | -7/2 \rangle. \tag{2}$$

The 8 projection operators $A^i_{\psi} = |j_i\rangle \langle j_i|$, $i = 1, ..., 8$ represent spin observables pertaining $\mathcal{H}_H$ (if $A = A^i_{\psi}$ has outcome 1, then the particle has spin $j_i = \frac{5}{2} - (i-1)^n$ along $z$ and so on). They trivially commute with both $F(\Delta)$ and $E$. Then also the projection operator $T = A^1 + A^2 + A^3 + A^5 = 1_I \otimes |7/2\rangle \langle 7/2| + |5/2\rangle \langle 5/2| + |3/2\rangle \langle 3/2| + |1/2\rangle \langle -1/2| (-1/2)$ commute with $F(\Delta)$ and $E$. Now, straightforward calculations show that $E \Psi = T \Psi$. Therefore, $T$ turns out to be a WS detector.

Now we introduce a property $G = G_I \otimes 1_H$ incompatible with $E$, which can be detected by means of a suitable detector $Y$ without renouncing to the WS knowledge provided by $T$. Given any $\varphi \in \mathcal{H}_I$, we define

$$G_I \varphi = \langle \psi^{(1)} | \varphi_I \rangle \psi^{(1)} + \langle \psi^{(2)} | \varphi_I \rangle \psi^{(2)} + \langle \psi^{(3)} | \varphi_I \rangle \psi^{(3)} \tag{3}$$

where

$$\psi^{(1)} = \frac{1}{6} \psi_1 - \frac{1}{6} \psi_2 - \frac{1}{6} \psi_1 - \frac{\sqrt{3}}{2} \psi_7 + \frac{1}{2\sqrt{3}} \psi_9 + \frac{1}{2\sqrt{3}} \psi_{10}$$

$$\psi^{(2)} = \frac{\sqrt{3}}{2} \psi_1 + \frac{1}{2\sqrt{3}} \psi_2 + \frac{1}{2\sqrt{3}} \psi_3 - \frac{\sqrt{6}}{4} \psi_6 + \frac{\sqrt{6}}{4} \psi_8 + \frac{1}{2\sqrt{6}} \psi_9 + \frac{1}{2\sqrt{6}} \psi_{10}$$

$$\psi^{(3)} = \frac{\sqrt{2}}{2} \psi_1 - \frac{\sqrt{2}}{2} \psi_2 + \frac{\sqrt{2}}{2} \psi_3 + \frac{\sqrt{2}}{2} \psi_4 + \frac{\sqrt{2}}{2} \psi_5.$$ 

A straightforward calculation based on (1) and (3) shows that $[G_I, E_I] \varphi \neq 0$ so that $G$ and $E$ are incompatible with each other. However, the projection operator $Y = A^1 + A^2 + A^4 + A^5 = 1_I \otimes |7/2\rangle \langle 7/2| + |5/2\rangle \langle 5/2| + |3/2\rangle \langle 3/2| + |1/2\rangle \langle -1/2| \langle -3/2| - 3/2\rangle \langle -3/2|)$ satisfies condition $Y \Psi = G \Psi$ and it trivially commutes with $F(\Delta) = F_I(\Delta) \otimes 1_H$; therefore $Y$ is a detector of $G$.

Now we introduce a further property $L = L_I \otimes 1_H$ incompatible with $E$ and $G$, which can be detected by means of a suitable detector $W$ without renouncing to the WS knowledge provided by $T$ and to the knowledge of $G$ provided by $L$. Given any $\varphi \in \mathcal{H}_I$, we define

$$L_I \varphi = \langle \psi^{[1]} | \varphi_I \rangle \psi^{[1]} + \langle \psi^{[2]} | \varphi_I \rangle \psi^{[2]} + \langle \psi^{[3]} | \varphi_I \rangle \psi^{[3]} + \langle \psi^{[4]} | \varphi_I \rangle \psi^{[4]} + \langle \psi^{[5]} | \varphi_I \rangle \psi^{[5]} \tag{4}$$

where

$$\psi^{[1]} = \frac{1}{5} \sqrt{\frac{11}{15}} \left( -2\psi_1 + 2\psi_2 + 2\psi_3 - \frac{9}{\sqrt{11}} \psi_6 + \frac{9}{\sqrt{11}} \psi_7 + \frac{9}{\sqrt{11}} \psi_{10} \right)$$

$$\psi^{[2]} = \frac{1}{10} \sqrt{\frac{11}{65}} \left( 3\psi_1 - 3\psi_2 - 3\psi_3 - \frac{24}{\sqrt{11}} \psi_6 - \frac{51}{\sqrt{11}} \psi_7 + \frac{25}{\sqrt{11}} \psi_9 + \frac{49}{\sqrt{11}} \psi_{10} \right)$$

$$\psi^{[3]} = \frac{1}{5} \sqrt{\frac{33}{26}} \left( -\psi_1 + \psi_2 + \psi_3 - \frac{17}{3\sqrt{11}} \psi_6 - \frac{14}{3\sqrt{11}} \psi_7 + \frac{65}{6\sqrt{11}} \psi_8 + \frac{5}{2\sqrt{11}} \psi_9 - \frac{11}{2\sqrt{11}} \psi_{10} \right)$$

$$\psi^{[4]} = \frac{1}{\sqrt{15}} (-\psi_1 + \psi_2 - 2\psi_3 + 3\psi_5)$$

$$\psi^{[5]} = \frac{1}{2\sqrt{2}} (-\psi_1 - 2\psi_2 + \psi_3 + \psi_4 + \psi_5).$$
A straightforward calculation shows that \( [G_I, L_I] \varphi \neq 0 \) so that \( G \) and \( L \) are incompatible with each other; furthermore, \( [E_I, L_I] \varphi \neq 0 \) so that \( E \) and \( L \) are incompatible with each other, too. However, the projection operator \( W = A^1 + A^3 + A^4 + A^7 = 1_I \otimes (7/2)(7/2) + 3(3/2)(3/2) + 1(1/2) + 5(5/2) - 5(5/2) \) satisfies condition \( W \Psi = L \Psi \) and it trivially commutes with \( F(\Delta) = F_I(\Delta) \otimes 1_I; \) therefore \( W \) is a detector of \( L \).

Nevertheless, we have that \( W \), \( Y \), and \( T \) pairwise commute with each other. Then all, \( W, Y \), and \( T \) can be simultaneously measured together with the position of the final impact; in other words, properties \( L, G, \) and \( E \), incompatible with each other, can be detected together on each particle localized on the final screen. In the present work, we are not concerned with the question of the physical meaning of \( G \) and \( L \), which evidently depends upon the choice of vectors \( \psi_i, \psi_{i+5}, i = 1, \ldots, 5 \). Thus, we have a solution of problem \( (P') \).

### IV. EXPERIMENTAL ISSUES FOR ENTANGLEMENT

In the perspective of a realization of the detection of \( E, G \), and \( L \), a crucial experimental task is to create the entanglement, encoded in the state vector \( \Psi \) in (2), between the particle and the detectors, before the time \( t_1 \) when the particle reaches the screen supporting the slits. This can be (ideally) realized in three steps. In the first step only particles with the \( x \) component of the spin equal to \( 7/2 \) are selected, for instance by means of a suitable Stern-Gerlach apparatus. Hence, in Schrödinger picture, at this stage - time \( t_0 < t_1 \) - the state vector is of the kind \( \psi|s \rangle \), with \( \psi \in H_I, \| \psi \| = 1 \), and \( S_x|s \rangle = 7/2|s \rangle \), so that we can take \( |s \rangle = \frac{1}{8\sqrt{2}} \{ (7/2) + \sqrt{7}(3/2) + \sqrt{21}(3/2) + \sqrt{35}(1/2) + \sqrt{35} - 1/2 + \sqrt{21} - 3/2 + \sqrt{7} - 5/2 + 1 - 7/2 \}. \)

In the second step, during their flight between times \( t_0 \) and \( t_1 \), the particles undergo the action of another Stern-Gerlach magnet, able to spatially separate the particles with respect to \( S_z \): the particles with \( S_z = 7/2, 5/2, 3/2 \) or \(-1/2, -5/2, -3/2 \) or \( 1/2 \) are forced to travel towards slit 1 (resp. 2), but through two alternative spatial channels according to the value of \( S_z \), so that the dynamical evolution between times \( t_0 \) and a time \( t_{1/2} < t_1 \) is represented by a unitary operator \( U \) such that

\[
\begin{align*}
U(|7/2\rangle) &= \psi_1^{2/7/2}, \\
U(|5/2\rangle) &= \psi_1^{2/5/2}, \\
U(|3/2\rangle) &= \psi_1^{2/3/2}, \\
U(|-1/2\rangle) &= \psi_1^{2/-1/2}, \\
U(|1/2\rangle) &= \psi_2^{2/1/2}.
\end{align*}
\]

where \( \psi_k^{2j/7/2} \) are vectors of \( \mathcal{H}_I \) representing the alternative spatial channels taken by the particles to reach slit \( k \); hence \( \langle \psi_k^{2j_1/7/2} | \psi_k^{2j_2/7/2} \rangle = \delta_{j_1,j_2} \cdot \delta_{j_1,j_2} \) and \( E_I \psi_1^{2j/7/2} = \psi_1^{2j/7/2}, E_I \psi_2^{2j/7/2} = 0 \). Then the outcoming state must be \( \tilde{\Psi} = U(|s \rangle) = \sum_k \psi_k^{2j/7/2} |j \rangle \).

Before reaching the slits, between the times \( t_1/2, t_1 \), the particles undergo the action of a filter blocking the beams of particles corresponding to \( S_z = 5/2 \) and \(-3/2 \): the state \( \tilde{\Psi} \) is of the kind \( \Psi_0 + \Psi_1 \), with \( \Psi_0 = \psi_1^{2j/5/2} + \psi_1^{2j/-3/2} \) and \( \Psi_1 = \sum_k |k=5/2,-3/2,k \rangle \psi_k^{2j/7/2} |j \rangle \); the effect of the filter is represented by the projection operator \( P = |\Psi_1 \rangle \langle \Psi_1 | \) so that the final state vector outcoming the whole preparing procedure is

\[
\Psi = P \tilde{\Psi} = \frac{1}{8\sqrt{2}} \{ (\psi_1^{2j/7/2} |7/2\rangle + \sqrt{21}\psi_1^{2j/3/2} |3/2\rangle + \sqrt{35}\psi_1^{2j/1/2} |1/2\rangle + \sqrt{35}\psi_1^{2j/-1/2} |1/2\rangle + \sqrt{7}\psi_1^{2j/-3/2} |7/2\rangle + \psi_2^{2j/-5/2} |5/2\rangle + \psi_2^{2j/-1/2} |3/2\rangle + \psi_2^{2j/-3/2} |1/2\rangle + \psi_2^{2j/-5/2} |1/2\rangle \}.
\]

Now, if \( \dim(E_I \mathcal{H}_I), \dim((1_I - E_I) \mathcal{H}_I) \geq 5 \), then five mutually orthonormal vectors \( \psi_i^4 \in E_I \mathcal{H}_I, \psi_i^{i+5} \in (1_I - E_I) \mathcal{H}_I \), with \( i = 1, \ldots, 5 \), exist such that

\[
\begin{align*}
\psi_1^{2j/7/2} &= \frac{1}{2\sqrt{2}} (-\psi_1 - 2\psi_2 + \psi_3 + \psi_4 + \psi_5), \\
\psi_1^{2j/5/2} &= \frac{1}{\sqrt{15}} (-\psi_1 + \psi_2 - 2\psi_3 + 3\psi_5), \\
\psi_1^{2j/3/2} &= \frac{1}{2\sqrt{2}} (-\psi_6 - 2\psi_7 + \psi_8 + \psi_9 + \psi_{10}),
\end{align*}
\]
\[ \psi_1 \left[ -\frac{1}{2} \right] = \frac{1}{\sqrt{22}} (\psi_1 + \psi_2 + 3\psi_4) + \frac{1}{\sqrt{70}} (4\psi_1 + \psi_2 + 3\psi_3 + 3\psi_5), \]

\[ \psi_2 \left[ -\frac{1}{2} \right] = \frac{1}{\sqrt{15}} (-\psi_6 + \psi_7 - 2\psi_8 + 3\psi_9), \]

\[ \psi_2 \left[ -\frac{3}{2} \right] = \frac{1}{\sqrt{22}} (\psi_6 + \psi_7 + 3\psi_9) + \frac{1}{\sqrt{70}} (4\psi_6 + \psi_7 + 3\psi_8 + 3\psi_{10}), \]

then the state \( \Psi \) outcoming from this dynamical preparing process must be

\[
\Psi = \frac{1}{32} (-\psi_1 - 2\psi_2 + \psi_3 + \psi_4 + \psi_5) |7/2\rangle + \]

\[
+ \frac{1}{8} \sqrt{\frac{7}{10}} (\psi_1 + \psi_2 - 2\psi_3 + 3\psi_5) |3/2\rangle + \frac{\sqrt{35}}{32} (-\psi_6 - 2\psi_7 + \psi_8 + \psi_9 + \psi_{10}) |1/2\rangle + 
\]

\[
+ \frac{1}{16} \left[ \sqrt{\frac{35}{11}} (\psi_1 + \psi_2 + 3\psi_4) + (4\psi_1 + \psi_2 + 3\psi_3 + 3\psi_5) \right] | -1/2 \rangle + 
\]

\[
+ \frac{1}{8} \sqrt{\frac{7}{30}} (-\psi_6 + \psi_7 - 2\psi_8 + 3\psi_9) | -5/2 \rangle + 
\]

\[
+ \frac{1}{16} \left[ \frac{1}{\sqrt{11}} (\psi_6 + \psi_7 + 3\psi_9) + \frac{1}{\sqrt{35}} (4\psi_6 + \psi_7 + 3\psi_8 + 3\psi_{10}) \right] | -7/2 \rangle,
\]

which is just the state vector (2) which carries the right entanglement allowing for a simultaneous detection of the mutually incompatible properties \( L, G, E \), together with the measurement of the final impact point.

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