Asymmetry of Outer Space

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Abstract

We study the asymmetry of the Lipschitz metric $d$ on Outer space. We introduce an (asymmetric) Finsler norm $\| \cdot \|_L$ that induces $d$. There is an $Out(F_n)$-invariant “potential” $\Psi$ defined on Outer space such that when $\| \cdot \|_L$ is corrected by $d\Psi$, the resulting norm is quasi-symmetric. As an application, we give new proofs of two theorems of Handel-Mosher, that $d$ is quasi-symmetric when restricted to a thick part of Outer space, and that there is a uniform bound, depending only on the rank, on the ratio of logs of growth rates of any irreducible $f \in Out(F_n)$ and its inverse.

1 Introduction

Teichmüller space can be equipped with three natural metrics: the Teichmüller metric, the Weil-Petersson metric and Thurston’s Lipschitz metric [Thu]. It is only the latter one that has an analog in Outer space. The first systematic study of the Lipschitz metric in Outer space was conducted by Francaviglia-Martino [FM]. Just like Thurston’s metric, this metric is not symmetric, but it does have many useful properties. For example, it is a geodesic metric, $Out(F_n)$ acts on Outer space by isometries, and if $\Phi \in Out(F_n)$ is a fully irreducible automorphism then the translation distance of $\Phi$ equals $\log \lambda$, the growth rate of $\Phi$. This last property was exploited to give a new proof in [Bes] of the train track theorem [BH92]. Moreover, $\Phi$ acts as a translation by $\log \lambda$ on certain biinfinite geodesics, called axes. In [AK] it is shown that axes are strongly contracting, pointing to negative curvature properties of the Lipschitz metric in these directions.

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In this paper we introduce an asymmetric Finsler norm on the tangent vectors of Outer space that induces the Lipschitz metric. We also show how to correct this norm to make it quasi-symmetric. Our main result explains the lack of quasi-symmetry in terms of a certain potential function.

Main Theorem. There is an $\text{Out}(F_n)$-invariant continuous, piecewise analytic function $\Psi : \mathcal{X}_n \to \mathbb{R}$ and constants $A, B > 0$ (depending only on $n$) such that for every $x, y \in \mathcal{X}_n$ we have

$$d(y, x) \leq A \ d(x, y) + B [\Psi(y) - \Psi(x)]$$

As an application we get a new proof (Theorem 23) of Handel and Mosher’s result [HM07] that the expansion factor of an irreducible automorphism is bounded by a power of the expansion factor of the inverse automorphism (it is well known that in general they need not be equal). We also get an easy proof that in the subspace of Outer Space of the points whose underlying graph has injectivity radius bounded from below, the Lipschitz metric is symmetric up to a multiplicative error (Theorem 24).

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1.1 Outer space and tangent spaces

A graph will always be a finite cell complex of dimension 1 with all vertices of valence $> 2$. A metric on a graph $\Gamma$ is a function $\ell : E(\Gamma) \to [0, 1]$ defined on the set of edges of $\Gamma$ such that

- $\sum_{e \in E(\Gamma)} \ell(e) = 1$, and
- $\cup_{\ell(e)=0} e$ is a forest, i.e. it contains no circles.

The space $\Sigma_\Gamma$ of all metrics $\ell$ on $\Gamma$ is a “simplex with missing faces”; the missing faces correspond to degenerate metrics that vanish on a subgraph which is not a forest.

When $\ell \in \Sigma_\Gamma$, we have the tangent space

$$T_\ell(\Sigma_\Gamma) = \{ \tau : E(\Gamma) \to \mathbb{R} \mid \sum_{e \in E(\Gamma)} \tau(e) = 0 \}$$

If $\ell, \ell'$ are two points in $\Sigma_\Gamma$ the natural identification between $T_\ell(\Sigma_\Gamma)$ and $T_{\ell'}(\Sigma_\Gamma)$ leads to a product decomposition

$$T(\Sigma_\Gamma) \cong \Sigma_\Gamma \times \mathbb{R}^{N-1}$$
of the total tangent space, where \( N \) is the number of edges of \( \Gamma \).

A tangent vector \( \tau \in T_\ell(\Sigma \Gamma) \) is integrable if \( \tau(e) < 0 \) implies \( \ell(e) > 0 \) for all \( e \in E(\Gamma) \). In that case we have the path \( \ell + t\tau \in \Sigma \Gamma \) for small \( t \geq 0 \).

If \( \Gamma' \) is obtained from \( \Gamma \) by collapsing a forest, then we have natural inclusions \( \Sigma \Gamma' \subset \Sigma \Gamma \) and \( T(\Sigma \Gamma') \subset T(\Sigma \Gamma) \) given by considering metrics on \( \Gamma \) that vanish on the forest.

Let \( F_n \) denote the free group of rank \( n \). The rose \( R_n \) is the wedge of \( n \) circles. A marking is a homotopy equivalence \( f : R_n \to \Gamma \) from the rose to a graph. Two marked graphs \( (\Gamma, f) \) and \( (\Gamma', f') \) are equivalent if there is a homeomorphism \( \phi : \Gamma \to \Gamma' \) so that \( \phi f \simeq f' \) (homotopic).

Recall [CV86] that Culler-Vogtmann’s Outer space \( \mathcal{X}_n \) is obtained from the disjoint union

\[ \Pi(\Gamma, f) \Sigma \Gamma \]

by identifying the faces of the simplices along the above inclusions, where the union is taken over the representatives of equivalence classes of marked graphs \( (\Gamma, f) \). Thus a point of \( \mathcal{X}_n \) is represented by a triple \( (\Gamma, f, \ell) \), and we will usually blur the distinction between such a triple and the equivalence class it represents. If \( \alpha \) is an immersed curve in \( \Gamma \) we define \( \ell(\alpha) \) as the sum of the lengths of edges \( \alpha \) crosses, with multiplicity. If \( \alpha \) is not immersed we first tighten to an immersed loop and then compute the length. Similarly, if \( \tau \in \Sigma \Gamma \) and \( \alpha \) a loop in \( \Gamma \) then \( \tau(\alpha) \) is the sum of weights on the edges which are crossed by the immersed loop that is freely homotopic to \( \alpha \).

The outer automorphism group \( \text{Out}(F_n) \) acts on \( \mathcal{X}_n \) on the right by precomposing:

\[ (\Gamma, f, \ell) \cdot \Phi = (\Gamma, f\Phi, \ell) \]

where the group of homotopy equivalences (up to homotopy) of \( R_n \) is identified with \( \text{Out}(F_n) \).

### 1.2 Lipschitz metric on \( \mathcal{X}_n \)

Let \( (\Gamma, f, \ell), (\Gamma', f', \ell') \) represent two points \( x, y \) in \( \mathcal{X}_n \). A difference of markings is a map \( \phi : \Gamma \to \Gamma' \) with \( \phi f \simeq f' \). We will always assume that \( \phi \) is linear on edges. By \( \sigma(\phi) \) denote the largest slope of \( \phi \) over all edges of \( \Gamma \). Define the distance

\[ d(x, y) = \min_{\phi} \log \sigma(\phi) \]

where \( \min \) is taken over all differences of markings (it is attained by Arzela-Ascoli). The following are the basic properties of \( d \) (see e.g. [FM]).
Proposition 1.  (i) \(d(x, y) \geq 0\) with equality only if \(x = y\).

(ii) \(d(x, z) \leq d(x, y) + d(y, z)\) for all \(x, y, z \in X_n\).

(iii) \(d\) is a geodesic metric; for any \(x, y\) there is a path from \(x\) to \(y\) whose length is \(d(x, y)\). Moreover, the path can be taken to be piecewise linear, and in fact linear in each simplex.

(iv) \(\text{Out}(F_n)\) acts on \(X_n\) by isometries.

1.3 Asymmetry and the Main Theorem

However, in general \(d(x, y) \neq d(y, x)\). The following three examples have motivated our Main Theorem.

Example 2. Let \(x_k, k \geq 2\), denote \((R_2, \text{id}, \ell_k)\) where \(\ell_k\) assigns lengths \(\frac{1}{k}\) and \(1 - \frac{1}{k}\) to the two edges of the rose \(R_2\). Then \(d(x_2, x_k) < \log 2\) for all \(k\) while \(d(x_k, x_2) = \log \frac{k}{2} \to \infty\) as \(k \to \infty\). Note that in this case the asymmetry can be explained by the fact that the injectivity radius \(\text{injrad}(x_k)\) of \(x_k\) goes to 0, and in fact \(d(x_k, x_2) \sim -\log \text{injrad}(x_k)\).

Example 3. Let \(\Gamma_{\epsilon, t}\) be the graph consisting of two circles of lengths \(\epsilon\) and 

\(1 - \epsilon - t\) connected by an arc of length \(t\), where \(0 < \epsilon < 0.1\) and \(0 \leq t < 1 - \epsilon\). Then \(d(\Gamma_{\epsilon, 0}, \Gamma_{\epsilon, 1 - 2\epsilon}) = \log(1 + t) < \log 2\) while \(d(\Gamma_{\epsilon, 1 - 2\epsilon}, \Gamma_{\epsilon, 0}) = \log \frac{1 - \epsilon}{\epsilon} \to \infty\) as \(\epsilon \to 0\). In this example both graphs have the same injectivity radius, but one graph has two small loops and the other only one.

Example 4. A more subtle example is illustrated in Figure 1.

![Figure 1](image.png)

Figure 1: The asymmetry of Lipschitz metric is not due to the change in injectivity radius.
Fix $0 < \epsilon < 0.1$ and let $x_t$ denote the graph consisting of two loops of size $\epsilon$ and two arcs connecting them with lengths $t$ and $1 - t - 2\epsilon$, with $0 \leq t < \frac{1}{3}$. Then $d(x_0, x_t) = \log(1 + \frac{t}{\epsilon})$ while $d(x_t, x_0) = \log \frac{1 - \epsilon}{t - \epsilon}$ (see the next section for a calculation of the distances). Thus $d(x_t, x_0)$ is uniformly bounded, but $d(x_0, x_t)$ can be made arbitrarily large by choosing suitable $\epsilon$ and $t$. Here both graphs have the same injectivity radius and both have two small embedded circles, but one graph has a third short (non-embedded) loop.

**Main Theorem.** There is an $\text{Out}(F_n)$-invariant continuous function $\Psi : X_n \to \mathbb{R}$ and constants $A, B > 0$ (depending only on $n$) such that for every $x, y \in X_n$ we have

$$d(y, x) \leq A d(x, y) + B[\Psi(y) - \Psi(x)]$$

### 1.4 Candidates and computing distances

We say that a loop $\alpha$ in $\Gamma$ is a candidate if either

- it is embedded, or
- (figure eight) there are two embedded circles $u, v$ in $\Gamma$ that intersect in one point and $\alpha$ crosses $u, v$ once and does not cross any edges outside of $u$ and $v$, or
- (barbell) there are two disjoint embedded circles $u, v$ in $\Gamma$ connected by an arc $w$ whose interior is disjoint from $u$ and $v$, such that $\alpha$ crosses $u, v$ once, $w$ twice, and no edges outside $u \cup v \cup w$.

The following basic fact which can be found in [FM] allows us to effectively compute the distance between points.

**Proposition 5.** Let $x, y \in X_n, x = (\Gamma, f, \ell), y = (\Gamma', f', \ell')$ and let $\phi : \Gamma \to \Gamma'$ be a difference of markings. Then there is a candidate loop $\alpha$ in $\Gamma$ such that

$$d(x, y) = \log \frac{\ell'(\phi(\alpha))}{\ell(\alpha)}$$

Note that $d(x, y) \geq \log \frac{\ell'(\phi(\alpha))}{\ell(\alpha)}$ for any loop $\alpha$. The right hand side does not depend on a particular choice of $\phi$, so one can effectively compute the distance by maximizing the ratio over the finitely many candidate curves.
For $x, y \in \Sigma \Gamma$ one may ask if $d(x, y)$ may be computed using the same candidate of $x$ when varying $y$ slightly and keeping $x$ fixed. Proposition 6 gives a positive answer to this question under some conditions.

Recall that a closed convex cone in a finite dimensional real vector space $V$ is a closed subset $C \subseteq V$ such that $v, w \in C$ implies $tv + sw \in C$ for all $t, s \in [0, \infty)$. For example, the set of integrable vectors in $T_{\ell} \Sigma \Gamma$ is a closed convex cone.

Notational convention: When we restrict our attention to a specific simplex $\Gamma(\Sigma, f)$ in Outer Space we may identify the point $(\Sigma, f, \ell)$ by only specifying $\ell$.

Proposition 6. (i) Let $\tau \in T_{\ell}(\Sigma \Gamma)$ be an integrable vector. Then there is a candidate loop $\alpha$ in $\Gamma$ such that

$$d(\ell, \ell + t\tau) = \log \left( \frac{(\ell + t\tau)(\alpha)}{\ell(\alpha)} \right)$$

for all sufficiently small $t \geq 0$, i.e. the same $\alpha$ realizes the distance $d(\ell, \ell + t\tau)$ for small $t$. Moreover, $\alpha$ has the property that for any other loop $\beta$, $\frac{\tau(\beta)}{\ell(\beta)} \leq \frac{\tau(\alpha)}{\ell(\alpha)}$ when $t > 0$.

(ii) $\lim_{t \to 0^+} \frac{d(\ell, \ell + t\tau)}{t} = \frac{\tau(\alpha)}{\ell(\alpha)}$ for the loop $\alpha$ in item (i).

(iii) The set of integrable vectors in $T_{\ell}(\Sigma \Gamma)$ is a finite union of closed convex cones $B_1, B_2, \ldots, B_N$ such that for any $B_i$ there is a candidate loop $\alpha_i$ that realizes the distance $d(\ell, \ell + t\tau)$ for any $\tau \in B_i$ and small $t \geq 0$.

Proof. A candidate $\alpha$ realizes $d(\ell, \ell + t\tau)$ if and only if

$$\frac{(\ell + t\tau)(\alpha)}{\ell(\alpha)} \geq \frac{(\ell + t\tau)(\beta)}{\ell(\beta)}$$

for all other candidates $\beta$ in $\Gamma$, which simplifies to $\frac{\tau(\alpha)}{\ell(\alpha)} \geq \frac{\tau(\beta)}{\ell(\beta)}$ when $t > 0$. This is a finite system of linear inequalities which determines a closed convex cone associated to $\alpha$ as in (iii). The inequalities do not depend on $t$, proving (i). Finally (ii) follows from (i) by dividing by $t$ and taking the limit. \qed

2 Finsler metric

Definition 7. Let $\tau \in T_{\ell}(\Sigma \Gamma)$. Define

$$\| (\ell, \tau) \|^L = \sup \left\{ \frac{\tau(\alpha)}{\ell(\alpha)} \mid \alpha \text{ is a loop in } \Gamma \right\}$$

6
Proposition 8. (1) If $\tau$ is integrable, then $\|(\ell, \tau)\|_L^L = \lim_{t \to 0^+} \frac{d(\ell, \ell + t\tau)}{t}$.

(2) The supremum in the definition is achieved on a candidate loop of $\Gamma$.

(3) $\|(\ell, \tau)\|_L^L$ is continuous on $T(\Sigma)$.

(4) $\|(\ell, \tau)\|_L^L \geq 0$ with equality only if $\tau = 0$.

(5) $\|(\ell, \tau_1 + \tau_2)\|_L^L \leq \|(\ell, \tau_1)\|_L^L + \|(\ell, \tau_2)\|_L^L$.

(6) If $c > 0$ then $\|(\ell, c\tau)\|_L^L = c\|(\ell, \tau)\|_L^L$.

Proof. (1) If $\tau$ is integrable, it belongs to the convex cone associated with some candidate $\alpha$. Proposition 6(ii) establishes that $\frac{\tau(\alpha)}{\ell(\alpha)} = \lim_{t \to 0^+} \frac{d(\ell, \ell + t\tau)}{t}$, and by Proposition 6(i) $\frac{\tau(\beta)}{\ell(\beta)} \leq \frac{\tau(\alpha)}{\ell(\alpha)}$ for any other loop $\beta$.

(2) When $\tau$ is integrable this follows from (1) and Proposition 6. Now let $\tau$ be nonintegrable and suppose there is a loop $\alpha$ with $\frac{\tau(\alpha)}{\ell(\alpha)} > \frac{\tau(\beta)}{\ell(\beta)}$ for every candidate $\beta$. Then the same is true after a small perturbation of $\ell$ where $\tau$ becomes integrable, contradiction.

(3) This follows from (2), since in the definition we can replace sup by a maximum over a finite set (of candidates for graphs in $\Sigma$).

(4) If $\tau \neq 0$ we need to produce some loop $\alpha$ so that $\tau(\alpha) > 0$. This statement does not depend on $\ell$ so we may assume that $\tau$ is integrable, and then the statement follows from Proposition 6(i): for small enough $t$: $0 < d(\ell, \ell + t\tau) = \log(1 + t\|(\ell, \tau)\|_L^L)$.

(5),(6) This is evident. 

Thus we have an (asymmetric) norm for the Lipschitz metric (homogeneity (6) only holds for positive scalars). Note first that the norm is not quasi-symmetric by the example in Figure 2.
Figure 2: The labels on the edges of $x$ represent the vector $\tau$ and $-\tau$. $\| (\ell, \tau) \|^L \sim 1$ and $\| (\ell, -\tau) \|^L \sim \frac{1}{\text{length of short loop}} \gg 1$.

Next we analyze the relationship between $\| (\ell, \tau) \|^L$ and $\| (\ell, -\tau) \|^L$. The reader may check that when $\tau$ is integrable $\| (\ell, -\tau) \|^L = \lim_{t \to 0^+} \frac{d(\ell + t\tau, \ell)}{t}$.

3 A corrected Finsler metric

We aim to define a new norm on $T\ell\Sigma\Gamma$ which is quasi-symmetric. The idea is to correct $\| \cdot \|^L$ by adding the directional derivative of a function that, roughly speaking, is the sum of $-\log$’s of the lengths of candidates. Since candidates change from simplex to simplex, we observe that each candidate lifts to an embedded loop in a suitable double cover and the curves we sum over are shortest loops in $\mathbb{Z}_2$-homology classes of all double covers.

First consider a nontrivial homology class $a \in H_1(\Gamma; \mathbb{Z}_2)$. By $\ell(a)$ denote the minimal $\ell(\alpha)$ where $\alpha$ ranges over loops in the class of $a$. Since there are only finitely many loops of $\ell$-length bounded above, this minimum exists, but it might be realized on more than one loop, say $\alpha_1, \ldots, \alpha_k$.

**Proposition 9.** For each $a \in H_1(\Gamma; \mathbb{Z}_2)$ there are finitely many loops $\alpha_1, \ldots, \alpha_k$ so that $\ell(a)$ is realized by some $\alpha_i$ for all $\ell \in \Sigma\Gamma$. Moreover, if $\alpha$ is an embedded loop then for all $\ell \in \Sigma\Gamma$, $\alpha$ is the shortest loop representing $[a]$.

**Proof.** We claim that if $a \in H_1(\Gamma; \mathbb{Z}_2)$ is represented by $\alpha$ which realizes $\ell(a)$ and $\alpha$ crosses the edge $e$ more than once then $\alpha$ crosses $e$ exactly twice in opposite directions and $e$ separates the image of $\alpha$.

To see this we consider two cases. In the first case, suppose $\alpha$ crosses $e$ twice in the same direction. Then up to free homotopy $\alpha = e\beta_1 e\beta_2$. Construct $\alpha' = e\beta_1\beta_2\bar{e}$ which is homotopic to $\beta_1\beta_2$. $\alpha', \alpha$ are $\mathbb{Z}_2$ homologous.
but \( \alpha' \) is strictly shorter than \( \alpha \). In the second case, suppose \( \alpha \) crosses \( e \) twice in opposite directions and \( e \) doesn’t separate the image of \( \alpha \). Then \( \alpha = e\beta_1 \bar{e}\beta_2 \) where \( \text{Im} \beta_1 \cap \text{Im} \beta_2 \neq \emptyset \). Let \( p \in \text{Im} \beta_1 \cap \text{Im} \beta_2 \). Then we can write \( \alpha = \gamma_1 e \gamma_2 \bar{e} \gamma_3 \bar{e} \gamma_4 \) where \( p = i(\gamma_1) = t(\gamma_2) = i(\gamma_3) = t(\gamma_4) \) (here we use \( i(\cdot) \) for the initial point and \( t(\cdot) \) for the terminal point). We also have \( t(\gamma_1) = i(\gamma_4) \) and \( i(\gamma_2) = t(\gamma_3) \). Construct \( \alpha' = \gamma_1 e \bar{e} \gamma_4 \gamma_3 \gamma_2 \sim \gamma_1 \gamma_4 \gamma_3 \gamma_2 \). \( \alpha' \) also represents \( a \) but it has strictly shorter length.

We conclude that if \( \alpha \) is the shortest loop representing \( a \) then for each edge \( e \) in its image, \( \alpha \) either crosses \( e \) once or it crosses \( e \) twice in opposite directions and \( e \) separates the image of \( \alpha \). For each \( a \) there are only finitely many such loops \( \alpha \) (and they don’t depend on \( \ell \)).

For the second part, it is elementary to see that if \( \alpha \) is embedded and \( \beta \) is another loop with \( \beta \) homologous mod \( \mathbb{Z}_2 \) to \( \alpha \) then \( \beta \) crosses all the edges of \( \alpha \). Thus \( \alpha \) is a shortest loop representing its homology class, and any other loop representing the same homology class with the same length must be a reparametrization of \( \alpha \).

The set of linear inequalities \( \ell(\alpha_i) \leq \ell(\alpha_j) \) for the set of \( \alpha_i \)'s in Proposition 9 divides the simplex \( \Sigma_\Gamma \) into closed convex subsets \( C_1, \ldots, C_k \) such that for each \( C_i \) there is an \( \alpha_j \) so that \( \ell(\alpha_i) \leq \ell(\alpha_j) \) for all \( \ell \in C_j \).

**Corollary 10.** A simplex \( \Sigma_\Gamma \) is covered by closed convex subsets \( C_1, \ldots, C_k \) so that for each \( a \in H_1(\Gamma, \mathbb{Z}_2) \) there is a loop \( \alpha_j \) such that \( \ell(a) = \ell(\alpha_j) \) for all \( \ell \in C_j \).

**Corollary 11.** When \( \tau \in T_\ell \Sigma_\Gamma \) is integrable there is a \( j \) such that \( \ell, \ell + \tau \in C_j \) (for all small \( t > 0 \)) and the derivative from the right at 0 of \( t \mapsto (\ell + t\tau)(\alpha) \) is \( \tau(\alpha_j) \). In other words, it equals

\[
\max \{ \tau(\alpha) \mid \alpha \text{ realizes } \ell(a) \}
\]

Let \( \Gamma_i \to \Gamma, i = 1, 2, \ldots, 2^n - 1 \) be the collection of all nontrivial double covers of \( \Gamma \). Any \( \ell \in \Sigma_\Gamma \) induces a metric \( \ell_i \) on each \( \Gamma_i \) by pulling back, and likewise any tangent vector \( \tau \in T_\ell \Sigma_\Gamma \) lifts to a tangent vector in \( T_{\ell_i} \Sigma_{\Gamma_i} \). If \( a \in H_1(\Gamma_i; \mathbb{Z}_2) \) is a given homology class, denote by \( \ell_i(a) \) the length of a shortest loop in \( \Gamma_i \) equipped with \( \ell_i \) that represents \( a \).

**Lemma 12.** If \( \alpha \) is a candidate in \( \Gamma \) then there exists a double cover \( \Gamma_i \to \Gamma \), and a lift \( \bar{\alpha} \) of \( \alpha \) so that \( \bar{\alpha} \) is the unique shortest loop in its (nontrivial) homology class.
Proof. We will show that we can arrange that \( \tilde{\alpha} \) is embedded, and this will guarantee that \( \tilde{\alpha} \) is shortest in its homology class. If \( \alpha \) is embedded, then any double cover to which \( \alpha \) lifts works. If \( \alpha \) is a figure eight or a barbell, take the double cover by cutting and regluing along two points, one in each embedded loop of \( \alpha \) (so \( \alpha \) lifts but the embedded loops don’t).

![Figure 3: The homology class of \( acb\bar{c} \) and \( adb\bar{d} \) are equal. If \( c \) is shorter than \( d \) then \( adb\bar{d} \) will not be a homology representative. However it is the image of a homology representative in some double cover of this graph.](image)

Now we may define the new norm,

**Definition 13.** Let

\[
N(\ell, \tau) = -\sum_{\Gamma_i} \sum_{a \in H_1(\Gamma_i; \mathbb{Z}_2) \setminus \{0\}} \frac{\max \tau(\alpha)}{\ell(a)}
\]  

(1)

where maximum is taken over all loops \( \alpha \) in \( \Gamma_i \) that realize \( \ell(a) \). Note that some of the terms in the sum may be negative (e.g. generically, there is only one \( \alpha \) realizing \( \ell(a) \)).

Define the new norm by

\[
\|(\ell, \tau)\|^N = \|(\ell, \tau)\|^L + \frac{1}{K + 1} N(\ell, \tau)
\]  

(2)

where \( K = (2^n - 1)(2^{2n-1} - 1) \) is the number of summands in (1).

When \( \ell \) can be understood from the context, we will sometimes write \( \|\tau\| \) instead of \( \|(\ell, \tau)\| \) for simplicity.

**Lemma 14.**

\[
\frac{1}{K + 1} \max\{\|\tau\|^L, \|-\tau\|^L\} \leq \|\tau\|^N \leq 2\|\tau\|^L + \|\tau\|^L
\]
Proof. Let $\gamma$ be a loop realizing $\|\tau\|_L$, and $\alpha$ a loop realizing $\|-\tau\|_L$. Recall that for all loops $\beta$ in $\Gamma_i$, $\tau(\beta) \leq \frac{\tau(\alpha)}{\ell(\beta)} \leq \|\tau\|_L$ and $-\tau(\beta) \leq -\tau(\alpha) = \|\tau\|_L$. The max in (1) goes over loops $\beta$ such that $\ell(\beta) = \ell(\alpha)$ thus for each $\beta$ in the sum $\tau(\beta) \leq \|\tau\|_L$ and $-\tau(\beta) \leq \|\tau\|_L$. Hence the right inequality in the statement follows.

Inequality $\frac{1}{K+1}\|\tau\|_L \leq \|\tau\|_N$ is equivalent to $-N(\ell, \tau) \leq K\|\tau\|_L$ which is again evident, since the positive summands on the left hand side are dominated by $\|\tau\|_L$.

Finally, inequality $\frac{1}{K+1}\|\tau\|_L \leq \|\tau\|_N$ can be rewritten as $\|\tau\|_N \leq K+1\|\tau\|_L$

All positive terms in $-N(\ell, \tau)$ are dominated by $\|\tau\|_L$ as before. If $\alpha$ is a candidate that realizes $\|\tau\|_L$ then there is a term in $-N(\ell, \tau)$ of the form $\frac{\tau(\alpha)}{\ell(\alpha)}$ that cancels $\|\tau\|_L = \frac{-\tau(\alpha)}{\ell(\alpha)}$. □

Thus $\|\cdot\|_N$ is a (non-symmetric) norm, just like $\|\cdot\|_L$ (positivity follows from Lemma 11 and subadditivity is evident from the definition). The next corollary states that the new norm, unlike $\|\cdot\|_L$, is quasi-symmetric.

**Corollary 15.** There is a constant $A = 3(K+1)$ so that

$$\|\tau\|_N \leq A \|\tau\|_N$$

Define the map $\Psi : \Sigma_\Gamma \to \mathbb{R}$ by

$$\Psi(\ell) = -\frac{1}{K+1} \sum_{\Gamma_i} \sum_{a \in H_1(\Gamma_i, \mathbb{Z}) \setminus \{0\}} \log \ell_i(a) \tag{3}$$

where $\ell_i$ is the lift of $\ell$ to $\Gamma_i$. Note that $\Psi$ is smooth (even analytic) on each convex set $C_j$ of Corollary 10.

**Proposition 16.** If $\ell \in \Sigma_\Gamma$ and $\tau \in T_\ell \Sigma_\Gamma$ is integrable then

$$\|\tau\|_N = \|\tau\|_L + d_\tau \Psi$$

where the third term is the derivative of $\Psi$ in the direction of $\tau$, i.e. the derivative from the right at $0$ of $t \mapsto \Psi(\ell + t\tau)$.

**Proof.** Applying Corollary 11 to $\Gamma_i, \ell_i$ and the lift $\tau_i$ of $\tau$ to $\Gamma_i$, we obtain that

$$d_\tau \ell_i(a) = \tau_i(\alpha_i)$$
where $\alpha_i$ is a curve that realizes $\ell_i(a)$ on which $\tau_i$ is maximal. Thus

$$d_{\tau_i} \log \ell_i(a) = \frac{\tau(\alpha_i)}{\ell_i(\alpha_i)}$$

and adding gives $d_{\tau} \Psi = \frac{1}{K+1} N(\ell, \tau)$, and the claim follows.

We can easily extend this discussion to the whole Outer space $X_n$. It is easy to see that $\| \cdot \|_L, \| \cdot \|_N$ and $\Psi$ commute with inclusions of simplices corresponding to collapsing forests. If $f : R_n \to \Gamma$ is a marking, $f_* : H_1(R_n; \mathbb{Z}_2) \to H_1(\Gamma; \mathbb{Z}_2)$ is an isomorphism and we identify homology classes in $H_1(\Gamma; \mathbb{Z}_2)$ with homology classes in $H_1(R_n; \mathbb{Z}_2)$. Similarly, cohomology can be identified, i.e. the double covers of $\Gamma$ with double covers of $R_n$, and $f$ lifts to markings of double covers of $\Gamma$ by double covers of $R_n$. This means that $\Psi : X_n \to \mathbb{R}$ can be defined globally. Moreover, changing the marking only permutes the summands in the definition of $\Psi$, so $\Psi$ is $\text{Out}(F_n)$-invariant.

4 Lengths of paths

Let $p : [0, 1] \to X_n$ be a piecewise linear path. In particular, $p$ can be subdivided into finitely many subpaths so that each is contained in one of the convex sets of Corollary 10 on which $\Psi$ is smooth. Then the Lipschitz length of $p$ is

$$\text{len}_L p = \sup \left\{ \sum_{i=1}^{p} d(p(t_{i-1}), p(t_i)) \mid 0 = t_0 < t_1 < \cdots < t_p = 1 \right\}$$

Suppose $\Delta t_i = t_i - t_{i-1}$ is small. Then

$$d(p(t_{i-1}), p(t_i)) = \frac{d(p(t_{i-1}), p(t_{i-1} + \Delta t_i))}{\Delta t_i} \cdot \Delta t_i \sim \| (p(t_{i-1}), \dot{p}(t_{i-1})) \|_L \Delta t_i$$

Thus

$$\text{len}_L p = \int_0^1 \| (p(t), \dot{p}(t)) \|_L dt$$

Define the new length of the path $p$

$$\text{len}_N p = \int_0^1 \| (p(t), \dot{p}(t)) \|_N dt$$
Proposition 17. Let $p : [0, 1] \rightarrow X_n$ be a path from $x$ to $y$ in $X_n$. Then

$$len_N(p) = len_L(p) + \Psi(y) - \Psi(x)$$

Proof. Since $\Psi$ is piecewise differentiable and $p$ is piecewise linear we may apply the Fundamental Theorem of Calculus to $\Psi \circ p$. Thus, by Proposition 16

$$len_N(p) = \int_0^1 \| \dot{p}(t) \|^N dt$$

$$= \int_0^1 \| \dot{p}(t) \|^L + d_{\dot{p}(t)} \Psi dt$$

$$= len_L(p) + \Psi(y) - \Psi(x)$$

Proposition 18. Let $p : [0, 1] \rightarrow X_n$ be a path from $x$ to $y$. Let $-p : [0, 1] \rightarrow X_n$ be the reverse path $-p(t) = p(1 - t)$. Then

$$len_N(-p) \leq A len_N(p)$$

where $A$ is the constant from Corollary 15.

Proof. Since $p$ is piecewise $C^1$, for all but finitely many points $[-p](s) = -\dot{p}(1 - s)$. Thus

$$len_N(-p) = \int_0^1 [\dot{p}(t)]^N ds$$

$$= \int_0^1 \| -\dot{p}(t) \|^N dt$$

$$= \int_0^1 \| -\dot{p}(t) \|^N dt \leq \int_0^1 A \| \dot{p}(t) \|^N dt$$

$$= A \ len_N(p)$$

5 Applications

We now elevate the local results obtained in section 3 to global results on the lengths of paths and distances. Let $A$ be the constant from Corollary 15.

Corollary 19. For any $\phi \in \text{Out}(F_n)$ and any piecewise linear path $p$ from $x$ to $x \cdot \phi$,

$$len_L(p) = len_N(p)$$

Therefore

$$len_L(p) \leq A \ len_L(-p)$$
Thus for every $j$

Claim 22. This follows from

Proof. By Proposition \[17\] $len_N(p) = len_L(p) + \Psi(x \cdot \phi) - \Psi(x)$. But since $\Psi(x) = \Psi(x \cdot \phi)$ we get $len_N(p) = len_L(p)$. \qed

Theorem 20. For any piecewise linear path $p$ from $x$ to $y$

$$len_L(p) \leq A len_L(-p) + (A + 1)[\Psi(x) - \Psi(y)]$$

Proof. By Propositions \[17\] and \[18\] $len_L(p) + \Psi(y) - \Psi(x) = len_N(p) \leq A len_N(-p) + A[\Psi(x) - \Psi(y)]$. \qed

Main Theorem. For any $x, y \in X_n$

$$d(x, y) \leq A \cdot d(y, x) + (A + 1)[\Psi(x) - \Psi(y)]$$

Proof. Apply Theorem 20 to $p$, where $-p$ is a geodesic from $y$ to $x$. \qed

In particular, since $\Psi$ is $Out(F_n)$ invariant, if $x, y \in X_n$ are in the same orbit then $d(x, y) \leq Ad(y, x)$.

Remark 21. The theorem above is equivalent to

$$\max\{d(x, y), d(y, x)\} \asymp \max\{\min\{d(x, y), d(y, x)\}, |\Psi(y) - \Psi(x)|\}$$

This follows from

Claim 22. If $d(x, y) \geq 2A d(y, x)$ then

$$\frac{d(x, y)}{2(A + 1)} \leq \Psi(x) - \Psi(y) \leq d(x, y)$$

Proof. From $0 \leq d(y, x) \leq Ad(x, y) + (A + 1)[\Psi(y) - \Psi(x)]$ we get that $\Psi(x) - \Psi(y) \leq d(x, y)$. From $d(x, y) \leq Ad(y, x) + (A + 1)[\Psi(x) - \Psi(y)]$ we get that $\Psi(x) - \Psi(y) \geq \frac{d(x, y)}{A + 1} (d(x, y) - Ad(y, x)) \geq \frac{d(x, y)}{2(A + 1)}$. \qed

The following theorem is due to Handel-Mosher [HM07].

Theorem 23. For any irreducible automorphism $\Phi \in Out(F_n)$, let $\lambda$ be the expansion factor of $\Phi$ and $\mu$ the expansion factor of $\Phi^{-1}$. Then $\mu \leq \lambda^A$.

Proof. Let $f : \Gamma \to \Gamma$ be a train track representative of $\Phi$ and $g : \Gamma' \to \Gamma'$ a train track representative for $\Phi^{-1}$. Let $D \geq d(\Gamma, \Gamma'), d(\Gamma', \Gamma)$. Then $d(\Gamma \cdot \Phi^j, \Gamma) \leq A \cdot d(\Gamma, \Phi \cdot \Gamma) = A \cdot j \cdot \log \lambda$. On the other hand, $d(\Gamma' \cdot \Phi^j, \Gamma') \geq d(\Gamma' \cdot \Phi^j, \Gamma') - d(\Gamma' \cdot \Phi^j, \Gamma') - d(\Gamma, \Gamma') \geq j \log \mu - 2D$. Therefore

$$Aj \log \lambda \geq j \log \mu - 2D$$

Thus for every $j$, $\log \mu \leq A \log \lambda + \frac{2D}{j}$ which implies $\frac{\log \mu}{\log \lambda} \leq A$. \qed

14
Let \( X_{\geq \varepsilon} \) be the set of all marked metric graphs in \( X_n \) which don’t contain loops shorter than \( \varepsilon \).

**Theorem 24.** For every \( \varepsilon > 0 \) there is a constant \( B \) so that for any \( x, y \in X_{\geq \varepsilon} \) and any piecewise linear path \( p \) from \( x \) to \( y \):

\[
\frac{1}{A} \text{len}(p) - B \leq \text{len}(-p) \leq A \text{len}(p) + B
\]

Moreover, there is a constant \( D \) such that for all \( x, y \in X_{\geq \varepsilon} \)

\[
d(y, x) \leq D d(x, y)
\]

**Proof.** Since \( \Psi \) is continuous and \( X_{\geq \varepsilon}/\text{Out}(F_n) \) is compact, there is a \( C = K \log \frac{1}{\varepsilon} \) so that for every \( x \in X_{\geq \varepsilon} \): \( |\Psi(x)| \leq C \). Then by Propositions 17 and 18

\[
\text{len}_L(p) - 2C \leq \text{len}_L(p) + \Psi(y) - \Psi(x) = \text{len}_N(p) \leq A \text{len}_N(-p)
\]

\[
= A(\text{len}_L(-p) + \Psi(x) - \Psi(y)) \leq A \text{len}_L(-p) + 2AC
\]

From the Main Theorem we see that \( d(y, x) \leq Ad(x, y) + B \) for any \( x, y \in X_{\geq \varepsilon} \). We now need to remove the additive constant. If \( d(x, y) \geq \log 2 \) the additive constant can be absorbed in the multiplicative constant: \( d(y, x) \leq Ad(x, y) + B \leq (A + B/\log 2)d(x, y) \). So suppose \( d(x, y) \leq \log 2 \). Let \( p \) be a geodesic path from \( x \) to \( y \). Then \( p \) must stay inside \( X_{\geq \varepsilon/2} \). By cocompactness, there is some \( M \) so that \( \| - \tau \|^L \leq M \| \tau \|^L \) for all tangent vectors \( \tau \) based at a point in \( X_{\geq \varepsilon/2} \). Thus \( d(y, x) \leq \text{len}_L(-p) \leq M \text{len}_L(p) = Md(x, y) \).

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