Criterion for the coincidence of strong and weak Orlicz spaces

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Abstract

We provide necessary and sufficient conditions for the coincidence, up to equivalence of the norms, between strong and weak Orlicz spaces. Roughly speaking, this coincidence holds true only for the so-called exponential spaces.

We find also the exact value of the embedding constant which appears in the corresponding norm inequality.

Key words and phrases:
Measure, Orlicz space, Young-Orlicz function, norm equivalence, tail function and tail norm, expectation, Lorentz spaces, Orlicz-Luxemburg strong and weak norms, embedding constant, Markov-Tchebychev’s inequality.

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1 Notations. Definitions. Statement of the problem.

Let \((X = \{x\}, \mathcal{F}, \mu)\) be a measurable space with atomless sigma-finite non-zero measure \(\mu\). Let \(N = N(u), u \in \mathbb{R}\), be a non-negative numerical-valued Young-Orlicz function. This means that \(N(u)\) is even, continuous, convex, strictly increasing to infinity as \(u \geq 0, u \to \infty\) and such that

\[
\lim_{u \to 0} \frac{N(u)}{u} = 0, \quad \lim_{u \to \infty} \frac{N(u)}{u} = +\infty.
\]

In particular,

\[N(u) = 0 \iff u = 0.\]

Denote by \(M_0 = M_0(X, \mu)\) the set of all numerical-valued measurable functions \(f : X \to \mathbb{R}\), finite almost everywhere. The Orlicz space \(L(N) = L(N; X, \mu)\) consists of all functions \(f : X \to \mathbb{R}\) from
the set $M_0(X, \mu)$ for which the classical Luxemburg norm $\|f\|_{L(N)}$ (equivalent to the Orlicz norm) or, in more detail, the strong Luxemburg norm $\|f\|_{sL(N)}$ defined by
\[
\|f\|_{L(N)} = \|f\|_{sL(N)} := \inf \left\{ k > 0 : \int_X N(|f(x)|/k) \, d\mu(x) \leq 1 \right\}
\]
is finite. Furthermore, if $0 < \|f\|_{L(N)} < \infty$, then
\[
\int_X N\left( \frac{|f(x)|}{\|f\|_{L(N)}} \right) \, d\mu(x) \leq 1.
\] (1.2)

Note that the equality sign occurs in (1.2) if in addition the Young-Orlicz function $N(\cdot)$ satisfies the well known $\Delta_2$-condition. Moreover, if there exists $k_0 > 0$ such that $\int_X N\left( \frac{|f(x)|}{k_0} \right) \, d\mu(x) = 1$, then $f \in L(N)$ and $k_0 = \|f\|_{L(N)}$ (see [15, Chapter 2, Section 9]).

The Orlicz spaces have been extensively investigated by M. M. Rao and Z. D. Ren in [26, 27]; see also [2, 15, 21, 22, 24], etc. Recently in [9] (see also [10]) the authors studied the Gagliardo-Nirenberg inequality in rearrangement invariant Banach function spaces, in particular in Orlicz spaces.

Note that the so-called exponential Orlicz spaces are isomorphic to suitable Grand Lebesgue Spaces, see [3, 13, 14, 22]. For some properties, variants and applications of the classical Grand Lebesgue Spaces see for example [4, 8, 11].

Recall that the Orlicz space $L(N)$ is said to be exponential if there exists $\delta > 0$ such that the generating Young-Orlicz function $N = N(u)$ verifies
\[
\lim_{u \to \infty} \frac{\ln N(u)}{\ln(2 + u)^{1+\delta}} = \infty.
\]

For instance, this condition is satisfied when
\[
N(u) = N^{(m)}(u) = \exp \left( |u|^m / m \right) - 1, \quad m = \text{const} > 0,
\]
as well as for an arbitrary Young-Orlicz function which is equivalent to $N^{(m)}(u)$ or when
\[
N(u) = N_{(\Delta)}(u) \overset{def}{=} \exp \left( \ln(1 + |u|)^\Delta \right) - 1, \quad \Delta = \text{const} > 1.
\]

Denote, as usually, for an arbitrary measurable function $f : X \to \mathbb{R}$ its Lebesgue-Riesz norm
\[
\|f\|_p := \left[ \int_X |f(x)|^p \, d\mu(x) \right]^{1/p}, \quad p \in [1, \infty).
\]

Suppose that the measure $\mu$ is probabilistic (or, more generally, bounded): $\mu(X) = 1$. It is known, see e.g. [23], that the measurable function $f$ (random variable, r.v.) belongs to the space $L(N^m)$, $m = \text{const} > 0$ iff
\[
\sup_{p \geq 1} \left[ \|f\|_p \, p^{-1/m} \right] < \infty.
\]

 Further, the non-zero function $f : X \to \mathbb{R}$ belongs to the Orlicz space $L(N_{(\Delta)})$ iff, for some non-trivial constant $C \in (0, \infty)$,
\[
\sup_{p \geq 1} \left[ \|f\|_p \, \exp \left( -C \, p^{\Delta} \right) \right] < \infty.
\]

Define, as usually, for a function $f : X \to \mathbb{R}$ from the set $M_0(X, \mu)$ its tail function
\[
T[f](t) \overset{def}{=} \mu \{ x : |f(x)| \geq t \}, \quad t \geq 0.
\] (1.3)
We denote this question, i.e., under appropriate simple conditions, may be found in [25]. See also [22, chapter set W are (complete) Banach functional rearrangement invariant spaces? A particular “Are the generalized Lorentz spaces really spaces?”

The following question is formulated in [6] by M. Cwikel, A. Kaminska, L. Maligranda and L. Pick: (triplet value/divides.alt0/divides.alt0

\[ \text{The function defined in (1.3) is also known as “distribution function”, but we prefer the first name since the notion “distribution function” is very used in other sense in the probability theory. An arbitrary tail function is left continuous, monotonically non-increasing, takes values in the interval } [0, \mu(X)] \text{ if } 0 < \mu(X) < \infty \text{ and in the semi-open interval } [0, \mu(X)) \text{ if } \mu(X) = \infty. \text{ Besides, } \\
\lim_{t \to \infty} T[f](t) = 0. \]

The inverse conclusion is also true: such an arbitrary function is the tail function for a suitable measurable finite a.e. map \( f : X \to \mathbb{R} \), defined on a sufficiently rich measurable space.

The set of all tail functions will be denoted by \( W : \)

\[ W = \{ T[f](\cdot), \ f \in M_0(X, \mu) \}. \] (1.4)

There are many rearrangement invariant function spaces in which the norm (or quasi-norm) of the function \( f(\cdot) \) may be expressed by means of its tail function \( T[f](\cdot) \), for example, the well-known Lorentz spaces. For the detailed investigation of the Lorentz spaces we refer the reader, e.g., to [2, 19, 20, 28, 29].

We introduce here a modification of these spaces. Let \( \theta = \theta(t), \ t \geq 0 \), be an arbitrary tail function: \( \theta \in W \). The so-called tail quasi-norm (or for brevity tail norm) \( \|f\|_{\text{Tail}[\theta]} \) of a function \( f \in M_0(X, \mu) \), with respect to the corresponding tail function \( \theta(\cdot) \), is defined by

\[ \|f\|_{\text{Tail}[\theta]} \overset{\text{def}}{=} \inf \{ K > 0 : \forall t > 0 \Rightarrow T[f](t) \leq \theta(t/K) \}. \] (1.5)

It is easily seen that this functional satisfies the following properties:

\[ \|f\|_{\text{Tail}[\theta]} \geq 0; \quad \|f\|_{\text{Tail}[\theta]} = 0 \iff f = 0; \]

\[ \|c \cdot f\|_{\text{Tail}[\theta]} = |c| \cdot \|f\|_{\text{Tail}[\theta]}, \ c = \text{const } \in \mathbb{R}. \]

Correspondingly, the set of all the functions \( f \) belonging to the set \( M_0(X, \mu) \) and having finite value \( \|f\|_{\text{Tail}[\theta]} \) is said to be the tail space Tail[\theta].

The following question is formulated in [6] by M. Cwikel, A. Kaminska, L. Maligranda and L. Pick: “Are the generalized Lorentz spaces really spaces?”, i.e., can these spaces be normed such that they are (complete) Banach functional rearrangement invariant spaces? A particular positive answer on this question, i.e., under appropriate simple conditions, may be found in [25]. See also [22, chapter 1, sections 1,2].

We denote

\[ I(f) = \int f(x) \ d\mu(x) = \int_X f(x) \ d\mu(x); \]

if \( \mu \) is a probability measure, we have \( \mu(X) = 1 \) and we replace \( (X = \{x\}, \mathcal{F}, \mu) \) with the standard triplet \( (\Omega = (\omega), \mathcal{F}, \mathbb{P}) \) and, for any numerical-valued measurable function, i.e., in other words, random variable \( \xi = \xi(\omega) \), we have

\[ \mathbb{E}\xi := I(\xi) = \int_{\Omega} \xi(\omega) \ P(d\omega); \quad T[\xi](t) = \mathbb{P}(|\xi| \geq t), \ t \geq 0. \]

Define now, for an arbitrary Young-Orlicz function \( N = N(u) \), the following tail function from the set \( W \)

\[ V[N](t) = V_N(t) \overset{\text{def}}{=} \min \left( \mu(X), \frac{1}{N(t)} \right). \] (1.6)

Of course, \( \min(c, \infty) = c, \ c \in (0, \infty) \).

Suppose \( 0 \neq f \in L(N) \); then there exists a finite positive constant \( C \) such that \( I(N(|f(\cdot)|/C)) \leq 1 \); one can take for instance \( C = \|f\|_{L(N)} \).
It follows from the classical Markov-Tchebychev’s inequality

\[ T[f](t) \leq V[N](t/C), \ t \geq 0. \quad (1.7) \]

In particular,

\[ T[f](t) \leq V[N]\left(\frac{t}{\|f\|_{sL(N)}}\right), \ t \geq 0. \quad (1.8) \]

In other words, if \( 0 \neq f \in L(N) < \infty \), then the function \( f(\cdot) \), as well as its normed version \( f = f/\|f\|_{L(N)} \), belongs to the suitable tail space:

\[ \|f\|_{\text{Tail}[V[N]]} \leq \|f\|_{L(N)} = \|f\|_{sL(N)}. \quad (1.9) \]

**Definition 1.1.** Let \( N \) be a Young-Orlicz function and \( f \in M_0(X, \mu) \). We say that \( f \) belongs to the weak Orlicz space \( wL(N) \) and we write \( f(\cdot) \in wL(N) \) iff the following condition is satisfied

\[ \|f\|_{\text{Tail}[V[N]]} < \infty \iff f \in \text{Tail}[V[N]]. \quad (1.10) \]

We will write for brevity also

\[ \|f\|_{wL(N)} \overset{\text{def}}{=} \|f\|_{\text{Tail}[V[N]]}. \]

Obviously

\[ \|f\|_{wL(N)} \leq \|f\|_{sL(N)} \]

and

\[ sL(N) \subset wL(N). \]

**Remark 1.1.** Let us emphasize the difference between the general tail space \( \text{Tail}[\theta] \) and the concrete weak Orlicz space \( wL(N) \). In the first case the “parameter” \( \theta \) is an arbitrary element of the tail set \( W \), while for the description of the weak Orlicz space in the definition 1.1 the function \( N(\cdot) \) belongs to the narrow class of Young-Orlicz functions.

The complete review of the theory of these spaces is contained in [18]; see also [16, 17] and the recent paper [12]. It is proved therein, in particular, that these spaces are \( F \)-spaces and may be normed under appropriate conditions, wherein the norm in the corresponding \( F \)-space or Banach space is linear and equivalent to the weak Orlicz norm.

There a natural question appears: under what conditions imposed on the function \( N = N(u) \) can the inequality \( (1.1) \) be reversed, of course, up to a multiplicative constant?

In detail, our aim is to find necessary and sufficient conditions, imposed on the Young-Orlicz function \( N(\cdot) \), under which

\[ Y(N) \overset{\text{def}}{=} \sup_{0 \neq f \in wL(N)} \left\{ \frac{\|f\|_{sL(N)}}{\|f\|_{wL(N)}} \right\} < \infty. \quad (1.12) \]

It is also interesting, by our opinion, to calculate the exact value of the parameter \( Y(N) \) in the case of its finiteness; we will make this computation in Section 3.

**Remark 1.2.** The lower bound in the last relation, namely,

\[ \underline{Y}(N) \overset{\text{def}}{=} \inf_{0 \neq f \in wL(N)} \left\{ \frac{\|f\|_{sL(N)}}{\|f\|_{wL(N)}} \right\}, \]

is known and \( \underline{Y}(N) = 1 \). In detail, it follows from (1.11) that \( Y(N) \leq 1 \); on the other hand, both the norms coincide for the arbitrary indicator function of a measurable set \( A \) having a non-trivial measure: \( 0 < \mu(A) < \infty \) (see [18]).
The comparison theorems between weak as well as between ordinary (strong) Orlicz spaces and other spaces are obtained, in particular, in [2, 3, 11, 14, 21, 28, 29, etc.

In both the next examples the space \((X = \{x\}, \mathcal{F}, P)\) is probabilistic; one can still assume that \(X = [0,1]\), equipped with the ordinary Lebesgue measure \(d\mu(x) = dx\).

**Example 1.1. A negative case.**

Let \(N(u) = N_p(u) = |u|^p, \ p = \text{const} > 1; \) in other words, the Orlicz space \(L(N_p)\) coincides with the classical Lebesgue-Riesz space \(L^p\):

\[ |\xi|_p = [\mathbb{E}|\xi|^p]^{1/p}. \]

The corresponding tail function has the form

\[ V[N_p](t) = \min(1, t^p), \ t > 0. \]

On the other hand, let us introduce the r.v. \(\eta\) such that

\[ T[\eta](t) = V[N_p](t), \ t > 0; \]

then, the r.v. \(\eta\) has unit norm in the corresponding weak Orlicz space \(wL(N_p)\) but

\[ \|\eta\|_p = \infty. \]

In other words \(Y(N_p) = \infty\).

As usual, the classical Lebesgue-Riesz norm \(||\eta||_p, p \geq 1,\) of the random variable \(\eta\) is defined by

\[ ||\eta||_p \overset{\text{def}}{=} [\mathbb{E}|\eta|^p]^{1/p}. \]

**Example 1.2. A positive case.**

Let now

\[ N(u) = N^{(2)}(u) = \exp\left(u^2/2\right) - 1, \ u \in \mathbb{R}, \]

the so-called subgaussian case. It is well-known that the non-zero r.v. \(\zeta\) belongs to the Orlicz space \(L(N^{(2)})\) if and only if there exists \(C = \text{const} > 0\) such that

\[ T[\zeta](t) \leq \exp(-C \ t^2), \ t \geq 0, \]

or equally

\[ \sup_{p \geq 1} [||\zeta||_p/\sqrt{p}] < \infty. \]

Thus, in this case, \(Y(N^{(2)}) < \infty\).

The same conclusion holds true also for the more general so-called exponential Orlicz spaces, which are in turn equivalent to the Grand Lebesgue Spaces, see [13, 14, 24, 22 Chapter 1, Section 1.2].

For instance, this condition is satisfied when

\[ N(u) = N^{(m)}(u) = \exp\left(|u|^m\right) - 1, \ m = \text{const}, > 0 \]

as well as for an arbitrary Young-Orlicz function which is equivalent to \(N^{(m)}(u)\); or when

\[ N(u) = N_{(\Delta)}(u) \overset{\text{def}}{=} \exp\left(\ln(1 + |u|)^\Delta\right) - 1, \ \Delta = \text{const} > 1. \]
2 Main result.

Let \((X = \{x\}, \mathcal{F}, \mu)\) be a measurable space with atomless sigma-finite non-zero measure \(\mu\) and let \(N\) be a Young-Orlicz function. Define an unique value \(t_0 = t_0(\mu(X)) \in [0, \infty)\) by

\[
N(t_0) = \frac{1}{\mu(X)};
\]

in particular, when \(\mu(X) = \infty\), then \(t_0 = 0\).

Denote also

\[
J(N) \stackrel{\text{def}}{=} \inf_{C > 0} \int_0^\infty N(C \cdot t) \mid dV[N](t) \mid = -\sup_{C > 0} \int_0^\infty N(C \cdot t) dV[N](t) = -\sup_{C > 0} \int_0^\infty N(C \cdot t) V'[N](t) dt.
\]

(2.1)

Note that the function \(t \to V[N](t)\) is monotonically non-increasing, therefore \(dV[N](t) = -dV[N](t)\).

Evidently, when \(t_0 > 0\) we have

\[
\int_0^\infty N(C \cdot t) \mid dV[N](t) \mid = -\int_{t_0}^\infty N(C \cdot t) d \left[ \frac{1}{N(t)} \right].
\]

Theorem 2.1. Let \(Y(N)\) and \(J(N)\) be defined respectively by (1.12) and (2.1). The necessary and sufficient condition for the equivalence of the strong and weak Luxemburg-Orlicz’s norms, i.e. \(Y(N) < \infty\), is the following:

\[
J(N) < \infty,
\]

(2.2)

or equivalently

\[
\exists C = C[N] \in (0, \infty) : \int_0^\infty N(C \cdot t) \mid dV[N](t) \mid < \infty.
\]

(2.3)

Remark 2.1. Evidently, if \(C[N] \in (0, \infty)\), then

\[
\forall C_1 \in (0, C[N]) \Rightarrow \int_0^\infty N(C_1 \cdot t) \mid dV[N](t) \mid < \infty.
\]

Proof.

A. First of all, note that

\[
\int_X N(f(x)) \, d\mu(x) = -\int_0^\infty N(t) \, dT[f](t) = \int_0^\infty N(t) \, dT[f](t).
\]

(2.4)

B. An auxiliary tool.

Lemma 2.1. Let \(\xi, \eta\) be non-negative numerical-valued r.v. such that \(T[\xi](t) \leq T[\eta](t), \ t \geq 0\). Let also \(N(u)\) be a non-negative increasing function, \(u \geq 0\). Then

\[
EN(\xi) \leq EN(\eta).
\]

(2.5)
Proof of Lemma 2.1
We can assume as before, without loss of generality, \( X = [0, 1] \) with Lebesgue measure. One can assume also that
\[
\xi(x) = [1 - T[\xi]]^{-1}(x), \quad \eta(x) = [1 - T[\eta]]^{-1}(x),
\]
where \( G^{-1} \) denotes a left-inversion for the function \( G(\cdot) \). Then \( \xi(x) \leq \eta(x) \) and hence \( N(\xi) \leq N(\eta) \), and a fortiori \( EN(\xi) \leq EN(\eta) \).

**Remark 2.2.** Of course, Lemma 2.1 remains true also for non-finite measure \( \mu \), as long as it is sigma-finite.

C. Necessity.

Let us introduce the following non-negative numerical-valued measurable function \( g = g(x), \ x \in X \), for which
\[
T[g](t) = V[N](t), \ t > 0;
\]
then \( g(\cdot) \in wL(N) \) with unit norm in this space. By the condition \( Y(N) < \infty \), the function also \( g \) belongs to the space \( sL(N) \), therefore
\[
\exists C_0 \in (0, \infty) : \gamma(N) = \gamma_{C_0}(N) \stackrel{def}{=} \int_X N(C_0 |g(x)|) \ d\mu(x) < \infty.
\]
We deduce, by virtue of (2.4),
\[
\int_0^\infty N(C_0 \ t) \ |dV[N](t)| = \gamma(N) < \infty,
\]
\[
J(N) = \inf_{C > 0} \int_0^\infty N(C \ t) \ |dV[N](t)| \leq \int_0^\infty N(C_0 \ t) \ |dV[N](t)| = \gamma(N) < \infty.
\]

D. Sufficiency.

Assume that the condition \( J(N) < \infty \) is satisfied. Suppose that the measurable function \( f : X \to \mathbb{R} \) belongs to the weak Orlicz space \( wL(N) \):
\[
T[f](t) \leq V[N](t/C_2), \ t \geq 0,
\]
for some finite positive value \( C_2 \). Let \( C_3 = \text{const} \in (0, \infty) \), its exact value will be clarified below. By using Lemma 2.1 we get
\[
\int_X N(C_3 f(x)) \ d\mu(x) = \int_0^\infty N(C_3 t) dT[f](t)
\leq \int_0^\infty N(C_3 t) \ |dV[N](t/C_2)| = \int_0^\infty N(C_2 C_3 t) \ |dV[N](t)|
= \int_0^\infty N(C_4 t) \ |dV[N](t)| < \infty,
\]
if the (positive) value \( C_4 := C_2 C_3 \) is sufficiently small, for instance \( C_4 \leq C[N] \). Thus, the function \( f(\cdot) \) belongs to the strong Orlicz space \( sL(N) \). □

**Remark 2.3.** The condition of Theorem 2.1 is satisfied for the exponential Orlicz space of the form \( L(N^{(m)}) \), \( m > 0 \), and is not satisfied for the Orlicz space \( L(N(\Delta)) \), \( \Delta > 1 \), also exponential space.
3 Quantitative estimates.

It is interest, by our opinion, to obtain the quantitative estimation of the constant which appears in the norm inequality for the embedding \( wL(N) \subset sL(N) \); namely, our aim is to compute the exact value for \( Y(N) \), defined in (1.12).

In detail, let \( f : X \to \mathbb{R} \) be some function from the space \( wL(N) \); one can suppose, without loss of generality,

\[
T[f](t) \leq V[N](t), \quad t \geq 0 \iff \|f\|_{wL(N)} \leq 1. \tag{3.1}
\]

Assume also that the condition (2.2) is satisfied, namely \( J(N) < \infty \); we want to find the upper estimate for the value \( \|f\|_{sL(N)} \).

Let us introduce the variable

\[
y_0 = y_0(N, \mu(X)) := N^{-1}(1/\mu(X)), \tag{3.2}
\]

so that \( y_0(N, \infty) = 0, \ y_0(N, 1) = N^{-1}(1) \) and define the function

\[
Q(k) = Q[N](k) := \int_{y_0}^{\infty} N(y/k) \left| \frac{1}{N(y)} \right|, \quad k \in (1, \infty]. \tag{3.3}
\]

or equally

\[
Q(k) = \int_{1}^{\infty} N \left( \frac{N^{-1}(w)}{k} \right) \frac{dw}{w^2}.
\]

Of course \( Q(0+) = \infty, \ Q(\infty) = 0 \).

Denote also

\[
k_0[N] := Q^{-1}(1) \in [1, \infty). \tag{3.4}
\]

Notice that the finiteness of the value \( k_0[N] \) is quite equivalent to the condition \( J(N) < \infty \) of Theorem 2.1.

**Theorem 3.1.** Assume that the condition \( J(N) < \infty \) is satisfied. Let \( k_0[N] \) be defined by (3.4).

Then

\[
\|f\|_{sL(N)} \leq k_0[N] \|f\|_{wL(N)}, \tag{3.5}
\]

and the coefficient \( k_0[N] \) is here the best possible. Namely,

\[
Y(N) = \sup_{0 \neq f \in wL(N)} \left\{ \frac{\|f\|_{sL(N)}}{\|f\|_{wL(N)}} \right\} = k_0[N]. \tag{3.6}
\]

In other words, \( k_0[N] \) is the exact value (attainable) of the embedding constant in the inclusion \( wL(N) \subset sL(N) \).

Moreover, there exists a measurable function \( f_0 : X \to \mathbb{R} \), with \( \|f_0\|_{wL(N)} = 1 \), for which the equality in (3.5) holds true:

\[
\|f_0\|_{sL(N)} = k_0[N] \|f_0\|_{wL(N)}. \tag{3.7}
\]

Obviously \( k_0[N] = +\infty \) when \( J(N) = \infty \).

**Proof.**

First of all, note that the function \( k \to Q(k), \ k \in (1, \infty) \) is continuous, strictly monotonically decreasing and herewith

\[
Q(\infty) = \lim_{k \to \infty} Q(k) = 0,
\]

by virtue of dominated convergence theorem; as well as
Let us introduce the following modification of the incomplete beta-function

\[ Q(1+) \overset{\text{def}}{=} \lim_{k \to 1^+} Q(k) = \int_{y_0}^{\infty} N(y) \left| d \frac{1}{N(y)} \right| = \int_{1/\mu(X)}^{\infty} z \left| d \frac{1}{z} \right| = \infty, \]

and the case when \( \mu(X) = \infty \) is not excluded.

Thus, the value \( k_0[N] \) there exists, is unique, positive, and finite: \( k_0[N] \in (1, \infty) \).

Further, assume that the non-zero measurable function \( f : X \to \mathbb{R} \) belongs to the weak Orlicz space \( wL(N) \); one can suppose, without loss of generality, \( \|f\|_{wL(N)} = 1 \):

\[ T[f](t) \leq \min \left( \mu(X), \frac{1}{N(t)} \right) =: T[g](t), \tag{3.8} \]

where \( T[g](t) = V[N](t) \).

We deduce, from the definition of the value \( k_0[N] \) and using once again Lemma 2.1,

\[ \int_X N \left( \frac{f(x)}{k_0[N]} \right) \, d\mu(x) \leq \int_X N \left( \frac{g(x)}{k_0[N]} \right) \, d\mu(x) = \int_{y_0}^{\infty} N \left( \frac{y}{k_0[N]} \right) \left| d \frac{1}{N(y)} \right| = Q(k_0[N]) = 1, \tag{3.9} \]

therefore

\[ \|f\|_{sL(N)} \leq k_0[N] = k_0[N] \|f\|_{wL(N)}. \tag{3.10} \]

So we proved the upper estimate; the unimprovable of ones follows immediately from the relation

\[ \|g\|_{sL(N)} = k_0[N] = k_0[N] \|g\|_{wL(N)}. \tag{3.11} \]

In detail:

\[ \int_X N \left( \frac{g(x)}{k_0[N]} \right) \, d\mu(x) = \int_X N \left( \frac{y}{k_0[N]} \right) \, dV[N](y) = 1 \tag{3.12} \]

in accordance with the choice of the magnitude \( k_0[N] \). Therefore

\[ \|g\|_{sL[N]} = k_0[N] \]

and simultaneously \( \|g\|_{wL[N]} = 1 \). So, in \( \|3.7\) one can choose \( f_0(x) := g(x) \) (attainability).

**Example 3.1.** Let \( (X = \{x\}, \mathcal{F}, \mu) \) be a probability space with atomless sigma-finite measure \( \mu(X) = 1 \). We define the following Young-Orlicz function, more precisely, the following family of Young-Orlicz functions

\[ N(u) = N^{(m)}(u) \overset{\text{def}}{=} \exp \left( |u|^m / m \right) - 1, \quad m = \text{const} > 1. \]

The case \( m = 2 \) is known as subgaussian case. The corresponding tail behavior for non-zero r.v. \( \xi \), having finite weak Orlicz norm in the space \( (X, L(N^{(m)})) \), has the form

\[ T[\xi](t) \leq \exp(-C(m) t^{m'/m'}), \quad t \geq 0, \quad m' \overset{\text{def}}{=} m/(m - 1). \]

Let us introduce the following modification of the incomplete beta-function

\[ B_\gamma(a, b) \overset{\text{def}}{=} \int_0^1 t^{a-1} (1-t)^{b-1} \, dt, \quad \gamma \in (0, 1), \quad a, b = \text{const} \in \mathbb{R}, \quad b > 0, \]

\[ C(m) = \frac{m}{m - 1} \int_0^1 t^{(m - 1)/2} \, dt = \frac{\Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{m}{2} + 1\right)}{\Gamma(m)}, \quad m \geq 1, \quad \Gamma(x) = \int_0^\infty e^{-y} y^{x-1} \, dy, \quad x > 0. \]
and define the variables \( \theta = \theta(k, m) := k^{-m}, \) \( k > 1, \) and the function
\[
G(\alpha) \overset{\text{def}}{=} B_{1/2}(-1, 1 - \alpha) = \int_{1/2}^{1} t^{-2} (1 - t)^{-\alpha} dt \\
= \int_{0}^{1/2} (1 - z)^{-2} z^{-\alpha} dz, \quad \alpha < 1.
\]

With the change of variable \( t = 1 - z \) we have
\[
G(\alpha) = \int_{0}^{1/2} (1 - z)^{-2} z^{-\alpha} dz.
\]
Using the Taylor series expansion
\[
(1 - z)^{-2} = \sum_{n=0}^{\infty} (n + 1) z^n, \quad z \in (-1, 1)
\]
which converges uniformly at least in the closed interval \([0, 1/2] , \) we get
\[
G(\alpha) = \sum_{n=0}^{\infty} (n + 1) \int_{0}^{1/2} z^{n-\alpha} dz,
\]
which gives
\[
G(\alpha) = \sum_{n=0}^{\infty} (n + 1) \frac{2^{-n-1+\alpha}}{n + 1 - \alpha}.
\]

By (3.2) we obtain
\[
y_0 = y_0^{(m)} = N^{-1}(1) = (m \ln 2)^{1/m}
\]
and by (3.3)
\[
Q_m(k) = Q[N^{(m)}](k) = \int_{y_0^{(m)}}^{\infty} \left( e^{y m - k^{-m}/m} - 1 \right) \left| dy \frac{1}{e^{y m - k^{-m}/m} - 1} \right|
\]
\[
= \int_{(m \ln 2)^{1/m}}^{\infty} \left( e^{y m - k^{-m}/m} - 1 \right) \frac{e^{y m - k^{-m}/m} y^{-1}}{(e^{y m - k^{-m}/m} - 1)^2} dy.
\]
Now we put \( x = e^{y m - k^{-m}/m}, \) so \( dx = e^{y m - k^{-m}/m} y^{-1} dy, \ x \in (2, \infty); \) then
\[
Q_m(k) = \int_{2}^{\infty} (x^{k^{-m} - 1})(x - 1)^{-2} dx.
\]
We make another change of variable \( t = 1 - 1/x \quad \Rightarrow \quad x = 1/(1 - t) \quad \Rightarrow \quad dx = \frac{dt}{(1 - t)^2}, \) which yields
\[
Q_m(k) = \int_{1/2}^{1} \frac{1}{(1 - t)^{k^{-m}}} \left( \frac{t}{1 - t} \right)^{-2} \frac{dt}{(1 - t)^2}
\]
\[
= \int_{1/2}^{1} t^{-2} (1 - t)^{-k^{-m}} dt - \int_{1/2}^{1} t^{-2} dt = \int_{0}^{1/2} (1 - z)^{-2} z^{k^{-m}} dz - 1
\]
\[
= G(k^{-m}) - 1 = G(\theta(k, m)) - 1
\]
Therefore, the value \( k_0 = k_0 \left[ N^{(m)} \right] = Q^{-1}(1) \) defined in (3.4) may be found as follows. Define an absolute constant \( \beta_0 \) by means of the relation
\[
\int_{0}^{1/2} (1 - z)^{-2} z^{-\beta_0} dz = G(\beta_0) = 2;
\]
(3.14)
then  \( \beta_0 \approx 0.431870 \)

and

\[ k_0 = k_0 \left( N^{(m)} \right) = [\beta_0]^{-1/m} , \quad (3.15) \]

or equally

\[ G(k_0^m) = 2. \quad (3.16) \]

Note in addition that \( G(0) = 1, G(1^-) = \infty \) and \( G \) is strictly increasing in \((0, 1)\), therefore the value \( \beta_0 \) there exists and it is unique.

Note that

\[ G(\alpha) > \int_{1/2}^1 (1-t)^{-\alpha} \, dt = \frac{2^{\alpha-1}}{1-\alpha}, \quad \alpha \in (0,1), \]

and, when \( \alpha \to 1^- \),

\[ G(\alpha) \sim \frac{1}{1-\alpha}. \]

If \( \alpha \to 0^+ \), by Taylor series expansion we have

\[ G(\alpha) \sim 1 + C_5 \alpha, \]

where

\[ C_5 \overset{\text{def}}{=} \int_0^{1/2} \frac{|\ln z|}{(1-z)^2} \, dz = 2 \ln 2 \approx 1.38629. \]

Indeed, we put

\[ C_6(\varepsilon) := \int_{\varepsilon}^{1/2} \frac{\ln z}{(1-z)^2} \, dz, \]

so that

\[ C_5 = - \lim_{\varepsilon \to 0^+} C_6(\varepsilon), \quad \varepsilon \in (0, 1/2). \]

By means of integration by parts we get

\[ C_6(\varepsilon) = \int_{\varepsilon}^{1/2} \frac{\ln z}{(1-z)^2} \, dz = \int_{\varepsilon}^{1/2} \ln z \, d\left( \frac{1}{1-z} \right) = \]

\[ \frac{\ln(1/2)}{1/2} - \ln \frac{\varepsilon}{(1-\varepsilon)} - C_7, \]

where

\[ C_7 = \int_{\varepsilon}^{1/2} \frac{1}{z(1-z)} \, dz = \int_{\varepsilon}^{1/2} \frac{dz}{z} + \int_{\varepsilon}^{1/2} \frac{dz}{(1-z)} = \]

\[ \ln(1/2) - \ln \varepsilon - \ln(1/2) + \ln(1-\varepsilon) = \ln(1-\varepsilon) - \ln \varepsilon. \]

Therefore

\[ C_6 = -2\ln 2 - \frac{\ln \varepsilon}{1-\varepsilon} - \ln(1-\varepsilon) + \ln \varepsilon \to -2\ln 2, \]

as long as \( \varepsilon \to 0^+ \). Thus, \( C_5 = 2\ln 2 \).

Note that \( \lim_{m \to \infty} k_0 \left[ N^{(m)} \right] = 1. \)

To summarize: denote

\[ k_0 := \inf_N k_0[N], \quad (3.17) \]

where \( \text{" inf" in (3.17) \)} is calculated over all the Young-Orlicz functions \( N(\cdot) \). We actually proved that

\[ k_0 = 1. \quad (3.18) \]
In detail, it follows from (1.11) that
\[ \tau_0 \geq 1. \] (3.19)

On the other hand,
\[ \tau_0 \leq \lim_{m \to \infty} \tau_0 \left[ N^{(m)} \right] = 1. \]

Evidently,
\[ \kappa_0 := \sup_{N} k_0[N] = \infty. \]

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