Invariants of Tutte Partitions and a \( q \)-Analogue

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Abstract

We describe a construction of the Tutte polynomial for both matroids and \( q \)-matroids based on an appropriate partition of the underlying support lattice into intervals that correspond to prime-free minors, which we call a Tutte partition. We show that such partitions in the matroid case include the class of partitions arising in Crapo’s definition of the Tutte polynomial, while not representing a direct \( q \)-analogue of such partitions. We propose axioms of a \( q \)-Tutte-Grothendieck invariant and show that this yields a \( q \)-analogue of a Tutte-Grothendieck invariant. We establish the connection between the rank generating polynomial and the Tutte polynomial, showing that one can be obtained from the other by convolution.

Keywords. \( q \)-analogue, \( q \)-matroid, Tutte partition, Tutte polynomial, rank generating function

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1 Introduction

The Tutte polynomial of a matroid is a fundamental invariant from which several other matroid invariants may be obtained. It is very well-studied and there are numerous papers on the topic; see, for example, [2, 4–6, 10]. In particular, those functions of a matroid \( M \) that can be expressed recursively as functions on a pair of disjoint minors of \( M \) found by deletion and contraction of \( M \), respectively, may be expressed in terms of the Tutte polynomial of \( M \). The Tutte polynomial of \( M \) carries information common to all matroids in the same isomorphism class of \( M \). For example, the number of bases, number of independent sets, and number of spanning sets of a matroid can be found by taking different evaluations of its Tutte polynomial. The characteristic polynomial of \( M \) is obtained by an evaluation in one of the variables of its Tutte polynomial, while its rank function is obtained by a linear substitution of variables. The Tutte polynomial is an invariant of matroid duality: that is the Tutte polynomial of the dual matroid \( M^* \) determined by the Tutte polynomial of \( M \). In the case of representable matroids, the celebrated MacWilliams duality theorem in coding theory can be retrieved as a special case of a duality result on the Tutte polynomial.

While the origins of the Tutte polynomial lie in graph theory, Crapo defined this invariant for matroids by extending the notions of internal and external activity of the bases of a graph, defined originally by Tutte [14, 32], to the setting of matroids. One way of looking at Crapo’s Tutte polynomial is to see it as a sum of monomial terms, each associated with a different element of a partition of the underlying Boolean lattice into intervals that contains a unique basis of the matroid. That different partitions of the lattice yield the same polynomial can be seen by the connection of this polynomial to the rank generating polynomial of the matroid via an invertible linear transformation, as shown by Crapo in [14].

In [7], Brylawski defined the Tutte-Grothendieck ring in order to study functions invariant under certain types of decompositions. In the category with objects multisets of isomorphism classes of matroids and morphisms these matroid decompositions, the polynomial defined by a universal property coincides with the Tutte polynomial as defined earlier by Crapo. In this context, the Tutte-Grothendieck invariants satisfy two fundamental recursions with respect to taking matroid minors.

Since the publication of [25] and [19, 28] shortly thereafter, there has been a good deal of interest in studying \( q \)-analogues of matroids and polymatroids [8, 11, 15–17, 19, 22, 28]. A \( q \)-matroid is defined in the same way as a matroid: it comprises a modular lattice endowed with a rank function satisfying certain rank axioms. The concept of such generalisations of matroids were already evident in Crapo’s early work [13]. The main difference between the two objects is that while a finite matroid consists of a rank function defined on the lattice of subsets of a finite set, the underlying lattice in the case of a \( q \)-matroid is the lattice of subspaces of a finite-dimensional vector space. We have chosen to present our results to encompass both the matroid and \( q \)-matroid case. That is, for most of the paper, \( M \) denotes either a matroid or a \( q \)-matroid, with the matroid case often emerging as a result of the specialization \( q \rightarrow 1 \).

A first attempt to define the Tutte polynomial of a \( q \)-matroid has been given in [3]. Building on this preliminary work, in this paper we describe a construction of the Tutte polynomial based on an appropriate partition of the underlying support lattice into intervals that correspond to prime-free minors, which we call a Tutte partition. Somewhat surprisingly, we show that such partitions in the matroid case include the class of partitions arising in [14], while not representing a direct \( q \)-analogue of such partitions the latter forming a strict subclass of the Tutte partitions defined here. Our description yields a Tutte polynomial both for matroids and \( q \)-matroids. This is our principal contribution, which is motivated by the need to address this fundamental gap in \( q \)-matroid theory. Having said that, our proof of the existence of a Tutte partition for a given
A q-matroid depends on the existence of a proper interval decomposition (defined in [21]), which is known to not exist for some classes of subspace lattices. We posit that the notion of a Tutte partition is an important step towards a definition of a Tutte polynomial that encompasses all q-matroids, which is a point we outline at the end of this paper with the help of an example.

We furthermore consider the notion of a Tutte-Grothendieck invariant. We define a new set of axioms and call the functions on the equivalence classes of matroids or q-matroids that satisfy this set of axioms of a q-Tutte-Grothendieck invariant. We show that in the case of a finite, Boolean lattice (the matroid case) these axioms, which are defined in reference to the concept of a matroid being prime-free or not prime-free, are equivalent to the classical axioms of a Tutte-Grothendieck invariant defined by Brylawski.

We consider the relation between the Tutte polynomial we define here and the rank generating polynomial and prove that one can be obtained from the other by convolution. We show that while the Tutte polynomial is a q-Tutte-Grothendieck invariant both in the matroid and q-matroid case, in the case of q-matroid, the rank generating polynomial is not.

This paper is organised as follows. In Section 2 we gather together basic results on lattices, matroidal \{0, 1\}-weightings (bicolourings) and rank functions. In Section 3 we define the notion of a Tutte partition (in fact we give two equivalent definitions) and describe its properties. With this notion in place, we hence define a q-analogue of the Tutte polynomial with respect to a given Tutte partition. In Section 4, we give an inversion formula, which describes a change of basis in the polynomial ring \(\mathbb{Z}[x, y]\). That is, we present a pair of mutually inversive maps \(\alpha, \beta\) on \(\mathbb{Z}[x, y]\). This answers a conjecture given in [3]. In Section 5, we show that the rank generating polynomial and any Tutte polynomial defined as in Section 3 can be retrieved from each other by applying the maps \(\alpha\) or \(\beta\) of Section 4, as appropriate. From this it follows that the Tutte polynomial defined with respect to a given Tutte partition is independent of the choice of Tutte partition and further, that we retrieve classical Tutte polynomial in the matroid case. For this reason, the Tutte polynomial we define may be referred to simply as the Tutte polynomial of the given matroid or q-matroid. In Section 6 we consider the functions on the class of either matroids or q-matroids that satisfy certain recursion formulae with respect to minors. We introduce the notion of a q-Tutte-Grothendieck invariant, and give a set of axioms defining any such invariant. We show that while the Tutte polynomial of a q-matroid is a q-Tutte-Grothendieck invariant, (unlike the matroid case) the rank generating polynomial of a q-matroid is not. In Section 7, we specialize to the matroid case. We first consider the Tutte partition introduced in Section 3 and compare this with the partition of the matroid into minors that arises in Crapo’s description of the Tutte polynomial given in [14], which we call a Crapo-Tutte partition. We show that while every Crapo-Tutte partition is also a Tutte partition, the converse does not hold: for matroids the Tutte partitions form a strictly larger class of partitions than those defined by Crapo, which extend Tutte’s definition of the internal and external activity of the bases of a graph. We also compare the axioms of a q-Tutte-Grothendieck invariant with those of a Tutte-Grothendieck invariant [5] and show that for matroids, both axiom systems are equivalent. Finally, in Section 8, we consider some basic properties of the Tutte and rank generating polynomials and some alternative representations of them as linear maps.

2 Preliminaries

We introduce some notation that will be used throughout this paper.

**Notation 1.** Throughout this paper, we let \(q\) denote a prime power and we let \(\mathbb{F}_q\) denote the finite field of order \(q\). For any positive integer \(n\), we write \([n] := \{1, \ldots, n\}\).

**Definition 2.** Let \(n, k\) be non-negative integers. Let \(q \in \mathbb{Z}\).
1. If \( q \neq 1 \), then \( [n]_q := \frac{q^n - 1}{q - 1} \); if \( q = 1 \), then \( [n]_q := n \).

2. The \( q \)-factorial of \( n \) is defined by \( [n]_q! := \prod_{k=1}^{n} [k]_q \).

3. The \( q \)-binomial coefficient is defined by

\[
\binom{n}{k}_q := \begin{cases} 
\frac{[n]_q!}{[n-k]_q! [k]_q!} & \text{if } k \leq n \\
0 & \text{if } k > n
\end{cases}
\]

The \( q \)-analogue of a binomial coefficient is called the \( q \)-binomial coefficient or Gaussian coefficient; \( \binom{n}{k}_q \) counts the number of \( k \)-dimensional subspaces of the vector space \( \mathbb{F}_q^n \). We recall the Pascal and \( q \)-Pascal identities for binomial and \( q \)-binomial coefficients respectively, and the \( q \)-binomial theorem. These well-known identities, as well as the \( q \)-Binomial Theorem, can be found in texts such as [12, 29].

**Theorem 3.** For \( k, n \in \mathbb{Z} \), if \( 0 \leq k \leq n \), the following identities hold:

\[
\binom{n}{k} = \binom{n}{k-1} + \binom{n-1}{k-1},
\]

\[
\binom{n}{k}_q = q^k \binom{n}{k-1}_q + \binom{n-1}{k-1}_q,
\]

\[
\binom{n}{k}_q = \binom{n}{k-1}_q + q^{n-k} \binom{n-1}{k-1}_q.
\]

**Theorem 4 (The \( q \)-Binomial Theorem).** For \( n, t \in \mathbb{Z} \), the following identity holds:

\[
\prod_{k=0}^{n-1} (1 - q^k t) = \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \binom{n}{k}_q t^k.
\]

The following well-known theorem can be read, for example, in [30, Theorem 7].

**Theorem 5.** Let \( V \) be an \( n \)-dimensional vector space over \( \mathbb{F}_q \) and let \( W \) be an \( m \)-dimensional subspace of \( V \). Then \( |\{U \leq V : U \oplus W = V\}| = q^{m(n-m)} \).

All of the above formulae and identities have classical counterparts, which can be retrieved by setting \( q = 1 \). In addition, we will use the following notation.

**Definition 6.** Let \( n, k \) be nonnegative integers. Then

\[
\binom{-n}{k} := \frac{\prod_{j=0}^{k-1} (-n - j)}{k!} = (-1)^k \binom{n + k - 1}{k}.
\]

We fix notation and recall some basic definitions on lattices and posets. While it is beyond the scope of this paper to give a review of lattices to those who are new to the theory, we present some of the relevant terminology for the convenience of the reader. Throughout this paper, we will assume that all posets and lattices are finite, unless explicitly stated otherwise. The reader is referred to [20, 27] for further reading on lattices.
**Definition 7.** Let $(P, \leq)$ be a poset. The poset $P$ is called a lattice if every finite subset of $P$ has a well-defined meet and join. That is, for any finite subset $S$ of $P$ the following holds. There exists $b \in P$, called the meet or greatest lower bound of $S$ in $P$ with the property that $b \leq a$ for all $a \in S$ and if there exists $c \in P$ such that $c \leq a$ for all $a \in S$ then $c \leq b$. There exists $b \in P$, called the join or least upper bound of $S$ in $P$ with the property that $a \leq b$ for all $a \in S$ and if there exists $c \in P$ such that $a \leq c$ for all $a \in S$ then $b \leq c$.

**Definition 8.** Let $(L_1, \lor_1, \land_1), (L_2, \lor_2, \land_2)$ be a pair of lattices. A lattice isomorphism between $L_1$ and $L_2$ is a bijective function $\varphi : L_1 \to L_2$ that is order-preserving and that preserves the meet and join; that is, for all $x, y \in L_1$ we have that

1. $x \leq_1 y \implies \varphi(x) \leq_2 \varphi(y)$,
2. $\varphi(x \land_1 y) = \varphi(x) \land_2 \varphi(y)$,
3. $\varphi(x \lor_1 y) = \varphi(x) \lor_2 \varphi(y)$.

It is straightforward to check that reversing the ordering of a lattice yields a lattice, with the meet and join interchanged. Combining this operation with a lattice isomorphism gives a lattice anti-isomorphism.

**Definition 9.** Let $(L_1, \lor_1, \land_1), (L_2, \lor_2, \land_2)$ be a pair of lattices. A lattice anti-isomorphism between $L_1$ and $L_2$ is a bijective function $\psi : L_1 \to L_2$ that is order-reversing and that interchanges the meet and join; that is, for all $x, y \in L_1$ we have:

1. $x \leq_1 y \implies \psi(y) \leq_2 \psi(x)$,
2. $\psi(x \land_1 y) = \psi(x) \lor_2 \psi(y)$,
3. $\psi(x \lor_1 y) = \psi(x) \land_2 \psi(y)$.

**Definition 10.** Let $L$ be a lattice with meet $\land$ and join $\lor$. Let $0$ and $1$ denote the minimal and maximal element of $L$, respectively. Let $a, b \in L$ such that $a \leq b$.

1. An interval $[a, b] \subseteq L$ is the set of all $x \in L$ such that $a \leq x \leq b$. It defines the interval sublattice $([a, b], \leq, \lor, \land)$.
2. Let $b \in L$. We say that $b$ is a complement of $a$ if $a \land b = 0$ and $a \lor b = 1$.
3. $L$ is called relatively complemented if every $x \in [a, b]$ has a complement in $([a, b], \leq, \lor, \land)$ and is called complemented if every $x \in L$ has a complement in $L$.
4. If $[a, b] \subseteq L$ such that $x \in [a, b]$ implies that $x = a$ or $x = b$ for any $x \in L$, then $b$ is called a cover of $a$ and we write $a \ll b$. We also say that $b$ covers $a$ or that $a$ is covered by $b$. If $a \ll b$, we also call the interval $[a, b]$ a cover in $L$. We write $C(L) := \{[a, b] \subseteq L : a \ll b\}$ to denote the set of (interval) covers of $L$.
5. An atom or point of $L$ is any element that is a cover of $0$. A coatom or copoint of $L$ is any element that is covered by $1$.
6. A chain from $a$ to $b$ is a sequence of the form $a < x_1 < x_2 < \ldots < x_k = b$ with $x_j \in L$ for $j = 1, \ldots, k$. We say that the chain has length $k$. The chain is called maximal if $a < x_1 < x_2 < \ldots < x_k = x$. We write $\ell([a, b])$ to denote the maximum length of any chain from $a$ to $b$. 

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7. Let \( x \in \mathcal{L} \). The *height* of \( x \), which we denote by \( h(x) \), is the maximum length of any chain from \( 0 \) to \( x \).

8. The interval \([a, b]\) is called a *diamond* if \( \ell([a, b]) = 2 \).

9. \( \mathcal{L} \) is called *modular* if for all \( a, b, c \in \mathcal{L} \), we have that \( a \geq c \implies (a \land b) \lor c = a \land (b \lor c) \).

We remark that a cover is an interval of length one. We summarize some well-known facts on modular lattices; the reader is referred to [27, Chapter 4] for the details.

**Theorem 11.** Let \( \mathcal{L} \) be a (finite) modular lattice and let \( x, y, z \in \mathcal{L} \). The following hold.

1. If \( x \preceq y \), then \( x \lor z \leq y \lor z \) and \( x \land z \leq y \land z \).
2. We have that \( x \land y \preceq x \) if and only if \( y \preceq x \lor y \).
3. All maximal chains from \( x \) to \( y \) have the same length (i.e. \( \mathcal{L} \) satisfies the Jordan-Dedekind property).
4. We have that \( h(a \lor b) + h(a \land b) = h(a) + h(b) \).

Equivalent to Theorem 11 (2), \([x \land y, y]\) is said to project up to \([x, x \lor y]\) and \([x, x \lor y]\) projects down to \([x \land y, y]\). We then write \([x \land y, y]\) \( \preceq \) \([x, x \lor y]\) and \([x, x \lor y]\) \( \succeq \) \([x \land y, y]\). We will use Theorem 11 (2) repeatedly throughout this paper.

Lemma 12 was shown in [3, Proposition 2]. We provide a proof for the convenience of the reader.

**Lemma 12.** Let \( \mathcal{L} \) be a modular lattice and let \( x, y, z \in \mathcal{L} \) such that \([x, y]\) is a cover in \( \mathcal{L} \).

Exactly one of the following holds:

1. \( x \lor z \preceq y \lor z \) and \( y \land z = x \land z \), or
2. \( y \lor z = x \lor z \) and \( x \land z \preceq y \land z \).

**Proof.** By Theorem 11 (4), we have that \( h(y \lor z) - h(x \lor z) = h(y) - h(x) - h(y \land z) + h(x \land z) \).

By (3) of the same theorem, for any \( a, b \in \mathcal{L} \), \( 1 = h(b) - h(a) \) if and only if \( a \preceq b \). Therefore, \( h(y \lor z) - h(x \lor z) = 1 - h(y \land z) + h(x \land z) \). It follows that either \( x \lor z \preceq y \lor z \) and \( x \land z = y \land z \) or \( x \lor z = y \lor z \) and \( x \land z \preceq y \land z \). \(\square\)

**Notation 13.** Throughout this paper, \( \mathcal{L} \) will denote a modular lattice ordered under \( \preceq \) with meet \( \land \) and join \( \lor \). We will write \( \mathcal{C} \) to denote the set of covers in \( \mathcal{L} \). We write \( 0 := 0_{\mathcal{L}} \) and \( 1 := 1_{\mathcal{L}} \). A complement of \( v \in \mathcal{L} \) will be denoted by \( v^c \). We will let \( V \) denote an \( \mathbb{F}_q \)-vector space of dimension \( n \). We will write \( \mathcal{L}(V) \) to denote the lattice of subspaces of \( V \), ordered by inclusion.

For the objects under consideration in this paper, one of the most relevant lattices will be the lattice of subspaces the finite-dimensional vector space \( V \). The lattice \( \mathcal{L}(V) \) has meet operation given by intersection and has join operation given by the vector space sum. The height of an element in this lattice is its dimension as a subspace of \( V \). For \( x \preceq y \), the length of the interval \([x, y]\) is the dimension of the quotient space \( y/x \) and \([x, y]\) is a cover if \( x \) has codimension 1 in \( y \). If \( S \) is a finite set, we write \( \mathcal{L}(S) \) to denote the Boolean lattice with greatest element \( S \). The height of an element of \( \mathcal{L}(S) \) is given by its cardinality and for \( x \preceq y \) the length of the interval \([x, y] \subseteq \mathcal{L}(S) \) is \( |x - y| \).

There are numerous ways to define a \( q \)-matroid, [8, 25]. We will first define the concept of a \( q \)-matroid as a \( \{0, 1\} \)-*weighting* of a lattice. This is essentially the same concept as a *bicolouring*, as described in [3].
Definition 14. A \(\{0,1\}\)-weighting of \(\mathcal{L}\) is a map \(w : \mathcal{C} \rightarrow \{0,1\}\). We say that \(w([a,b])\) is the weight of \(L\). For any maximal chain \(x_0 \lessdot x_1 \lessdot x_2 \lessdot \ldots \lessdot x_k\), we define the weight of the chain to be \(\sum_{j=0}^{k-1} w([x_i, x_{i+1}])\). If \([a,b]\) is a diamond of \(\mathcal{L}\), we say that it has has type full, empty, mixed or prime with respect to \(w\) if it satisfies one of the following descriptions.

**Full:** \(w([u,v]) = 1\) for each \([u,v] \in \mathcal{C}([a,b])\).

**Empty:** \(w([u,v]) = 0\) for each \([u,v] \in \mathcal{C}([a,b])\).

**Mixed:** There exists \(x \in [a,b], x \neq a, b\) such that \(w([x,b]) = 1\) and \(w([a,x]) = 0\), while \(w([y,b]) = 0\), and \(w([a,y]) = 1\) for every \(y \in [a,b], y \neq x, a, b\).

**Prime:** \(w([a,x]) = 1\) and \(w([x,b]) = 0\) for any \(x \in [a,b], x \neq a, b\).

![Figure 1: The four matroidal diamonds.](image)

Definition 15. A \(\{0,1\}\)-weighting \(w\) of \(\mathcal{L}\) is called matroidal if each diamond in \(\mathcal{L}\) is of type full, empty, mixed, or prime.

The following lemma can be shown by applying an inductive argument to Definition 15.

Lemma 16 ([3]). Let \([a,b],[c,d]\) be covers in \(\mathcal{L}\), endowed with a matroidal \(\{0,1\}\)-weighting. The following hold:

1. If \(w([a,b]) = 0\) and \([a,b] \lessdot [c,d]\) then \(w([c,d]) = 0\).
2. If \(w([a,b]) = 1\) and \([a,b] \lessdot [c,d]\) then \(w([c,d]) = 1\).

Corollary 17. Let \(\mathcal{L}\) be a modular lattice and let \(x, y, z \in \mathcal{L}\) such that \([x,y]\) is a cover in \(\mathcal{L}\). Then the following hold.

1. If \(y \land z = x \land z\) and \(w([x,y]) = 0\) then \(w([x \lor z, y \lor z]) = 0\).
2. If \(y \lor z = x \lor z\) and \(w([x,y]) = 1\) then \(w([x \land z, y \land z]) = 1\).

Proof. The result immediately follows from Lemma 12 and Lemma 16. \(\square\)

Definition 18. Let \(r : \mathcal{L} \rightarrow \mathbb{Z}_{\geq 0}\) be a function. We define the following rank axioms.

(R1) \(0 \leq r(x) \leq h(x)\) for all \(x \in \mathcal{L}\).

(R2) For all \(x, y \in \mathcal{L}\), \(r(x) \leq r(y)\) whenever \(x \leq y\).

(R3) For all \(x, y \in \mathcal{L}\), \(r(x \lor y) + r(x \land y) \leq r(x) + r(y)\).

We say that a rank function is matroidal if it satisfies the rank axioms (R1)-(R3).

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Definition 19. Let $\mathcal{L}$ be a lattice and $r : \mathcal{L} \to \mathbb{Z}_{\geq 0}$ be a matroidal rank function. The lattice $\mathcal{L}$ is called the support lattice of $M = (\mathcal{L}, r)$. The elements of $\mathcal{L}$ are called the support elements of $M$.

Note that if $\mathcal{L}$ is the Boolean lattice $\mathcal{L}(S)$ for a finite set $S$ then $(\mathcal{L}(S), r)$ is a matroid, while if $\mathcal{L} = \mathcal{L}(V)$, then $(\mathcal{L}(V), r)$ is a $q$-matroid.

Definition 20. Let $M = (\mathcal{L}, r)$ be a matroid or a $q$-matroid. Let $a, b \in \mathcal{L}$ such that $a \leq b$. The function $r_{[a,b]} : [a,b] \to \mathbb{Z}$ is defined by $r_{[a,b]}(x) = r(x) - r(a)$ for all $x \in [a,b]$. The function $r_{[a,b]}$ is a matroidal rank function on $[a,b] \subseteq \mathcal{L}$ and $([a,b], r_{[a,b]})$ is called a minor of $M$.

Notation 21. Let $M = (\mathcal{L}, r)$ be a matroid or $q$-matroid and let $a, b \in \mathcal{L}$ with $a \leq b$. For $[a,b] \subseteq \mathcal{L}$ we write $M([a,b])$ to denote the minor $([a,b], r_{[a,b]})$. We will also define $M(a) := M([0,a])$, and define $M/a := M([a,1])$. If $M$ is a matroid, we use $M - a$ to denote the restriction of $M$ to $[0,a^c] \subseteq \mathcal{L}$.

Definition 22. Let $r$ be a matroidal rank function on a support lattice $\mathcal{L}$. For any $x \in \mathcal{L}$, the nullity of $x$ is defined to be $\nu(x) = h(x) - r(x)$. For an interval $[a,b] \subseteq \mathcal{L}$ the function $\nu_{[a,b]} : [a,b] \to \mathbb{Z}$ is defined by $\nu_{[a,b]}(x) = \nu(x) - \nu(a)$ for all $x \in [a,b]$.

Definition 23. Let $M = (\mathcal{L}, r)$ be a matroid or a $q$-matroid and let $x \in \mathcal{L}$. If $h(x) = 1$ and $r(x) = 0$, then $x$ is called a loop of $M$. If $x$ is a coatom of $\mathcal{L}$ such that $r(x) = r(M) - 1$, then $x$ is called a coloop of $M$. If there is an atom $e \in \mathcal{L}$ such that $e \leq y$ for any $y \in \mathcal{L}$ such that $r(y) = r(1)$, then $e$ is called an isthmus of $M$.

Remark 24. In the case of a matroid, a coloop is the dual of an isthmus in the dual matroid. However, this connection does not hold for $q$-matroids. Moreover, as shown in [24, Theorem 5.4], no $q$-matroid other than one whose rank equals its height contains an isthmus. Therefore, while the notion of an isthmus is frequently used in the context of the Tutte polynomial in matroid theory, we do not use it in $q$-matroid theory; instead we use the notion of a coloop.

A cryptomorphism between $q$-matroids defined via a rank function or via a $\{0,1\}$-weighting on the subspace lattice was shown in [3, Theorem 2]. It’s easy to see that the same result holds for matroids. We hence have the following statement.

Theorem 25 ([3]).

1. Let $r : \mathcal{L} \to \mathbb{Z}_{\geq 0}$ be a matroidal rank function. Define a map $w_r$ on the set of covers $\mathcal{C}$ of $\mathcal{L}$ by $w_r([a,b]) := r(b) - r(a)$ for all $[a,b] \in \mathcal{C}$. Then $w_r$ is a matroidal $\{0,1\}$-weighting of $\mathcal{L}$.

2. Let $w : \mathcal{C} \to \{0,1\}$ be a matroidal $\{0,1\}$-weighting. Define a map $r_w : \mathcal{L} \to \mathbb{Z}_{\geq 0}$ by $r_w(x) := \sum_{i=1}^{t} w([x_{i-1}, x_i])$, where $0 = x_0 < x_1 < \cdots < x_t = x$ is a maximal chain in $[0,x]$. Then $r_w$ is a matroidal rank function.

3. Furthermore, $w_{r_w} = w$ and $r_{w_r} = r$.

Definition 26. Let $r$ be a matroidal rank function with support lattice $\mathcal{L}$.

1. For any $v \in \mathcal{L}$, we define the closure of $v$ with respect to $r$ to be $\bigvee \{e \in \mathcal{L} : h(e) = 1, r(v \lor e) = r(v)\}$, and we denote it by $\text{cl}(v)$.

2. For any $v \in \mathcal{L}$, we say that $v$ is independent with respect to $r$ if $h(v) = r(v)$. Otherwise we say that $v$ is dependent.

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3. For any \( v \in \mathcal{L} \), we say that \( v \) is a basis with respect to \( r \) if \( v \) is independent and \( r(v) = r(1) \).

Definition 27. Let \( r_1 \) and \( r_2 \) be a pair of matroidal rank functions with support lattices \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \), respectively. We say that \( r_1 \) and \( r_2 \) are lattice-equivalent if there exists a lattice isomorphism \( \varphi : \mathcal{L}_1 \rightarrow \mathcal{L}_2 \) such that \( r_1(x) = r_2(\varphi(x)) \) for all \( x \in \mathcal{L}_1 \). In this case we write \( M_1 \cong M_2 \), where \( M_1 = (\mathcal{L}_1, r_1) \) and \( M_2 = (\mathcal{L}_2, r_2) \).

Definition 28. Let \( M = (\mathcal{L}, r) \) be a matroid or \( q \)-matroid and let \( w \) be the corresponding matroidal \( \{0, 1\} \)-weighting of \( \mathcal{L} \). Let \( \varphi \) be a lattice anti-isomorphism of \( \mathcal{L} \). We write \( r^* \) to denote the matroidal rank function on \( \varphi(\mathcal{L}) \), which is defined by

\[
r^*(\varphi(x)) := h(1) - h(x) - (r(1) - r(x)),
\]

for all \( x \in \mathcal{L} \). We write \( w^* \) to denote the corresponding matroidal \( \{0, 1\} \)-weighting of \( \varphi(\mathcal{L}) \). The dual of \( M \) is the matroid or \( q \)-matroid \( (\varphi(\mathcal{L}), r^*) \) and is denoted by \( M^* \).

It is straightforward to check that if \( w \) is a matroidal \( \{0, 1\} \)-weighting of \( \mathcal{L} \) then its dual satisfies \( w^*((\varphi(y), \varphi(x))) = 1 - w([x, y]) \) for every cover \( [x, y] \in \mathcal{C} \). Observe also that \( r^*(\varphi(x)) = \nu(1) - \nu(x) \) for all \( x \in \mathcal{L} \).

Remark 30. The definition of the dual of a matroidal rank function \( r \) on \( \mathcal{L} \) depends on the choice of anti-isomorphism \( \varphi \) on \( \mathcal{L} \). However, it is clear that all such duals are lattice-equivalent.

Remark 31. If \( S \) is a finite set then there is a unique anti-automorphism induced by complementing, namely, \( \varphi : \mathcal{L}(S) \rightarrow \mathcal{L}(S) : x \mapsto x^c \). For \( \mathcal{L} = \mathcal{L}(V) \), any non-degenerate bilinear form on \( V \) yields an anti-automorphism of \( \mathcal{L}(V) \) by the map that assigns \( x \in \mathcal{L}(V) \) to its orthogonal complement with respect to that bilinear form.

In several instances throughout this paper, it will be convenient to use the notion of duality typically used in matroid theory. We recall the following lemma (c.f. [9], [25], and [26]).

Lemma 31. Let \( \varphi : \mathcal{L} \rightarrow \mathcal{L} \) be a lattice anti-automorphism. For each \( a \in \mathcal{L} \), let \( \phi_a \) be an anti-automorphism of the interval \( [0, \varphi(a)] \). Then \( \phi_a \circ \varphi : [a, 1] \rightarrow [0, \varphi(a)] \) is a lattice isomorphism. If \( r \) is a matroidal rank function with support lattice \( \mathcal{L} \), then

\[
M([a, 1])^* \cong M^*([0, \varphi(a)]) \text{ and } M^*([a, 1]) \cong M([0, \varphi(a)])^*.
\]

For the remainder of this paper, unless explicitly stated otherwise, \( r \) will denote a fixed but arbitrary matroidal rank function on \( \mathcal{L} \). We will fix \( M := (\mathcal{L}, r) \), so \( M \) will be understood to be a matroid if \( \mathcal{L} = \mathcal{L}(S) \) for the set \( S \) of cardinality \( n \) and will be understood to be a \( q \)-matroid in \( \mathcal{L} = \mathcal{L}(V) \) for the \( n \)-dimensional \( \mathbb{F}_q \)-subspace \( V \). In respect of this approach, where formulas involving \( q \)-binomial coefficients and \( q \)-factorial numbers arise, for the matroid case these should be interpreted as binomial coefficients and factorials, respectively, or those found by letting \( q \rightarrow 1 \), as appropriate. It is our intention that this approach will result in a simpler and more unified presentation of the theory.

3 Tutte partitions

The purpose of this section is to provide a definition of the Tutte polynomial that can apply to both matroids and \( q \)-matroids alike. This definition relies on the existence of a type of partition of the support lattice of a matroid or \( q \)-matroid, which we call a Tutte partition. The main result of this section, Theorem 51, gives that a Tutte partition always exists on the support lattice of a matroid or \( q \)-matroid, as long as the support lattice has a proper interval partition [21]. This provides a generalisation of the definition of the Tutte polynomial given in [14], and follows [3] in its use of prime-free minors.
Definition 32. We say that $M$ is prime-free if $M$ has no prime diamonds in $\mathcal{L}$.

Definition 33. Let $\mathcal{P}$ be a partition of the elements of $\mathcal{L}$ such that every element of $\mathcal{P}$ is an interval. We say that $\mathcal{P}$ is a Tutte partition of $\mathcal{L}$ if the following properties hold for every $[a, b] \in \mathcal{P}$:

1. $r(a) = h(a)$ (i.e. $a$ is independent),
2. $r(b) = r(1)$ (i.e. $b$ is spanning),
3. $[a, b]$ is prime-free.

Example 34. In Figure 2 we have an instance of a $q$-matroid that is not prime-free, as the interval $[0, (e_2, e_3)]$ (among others) is a prime diamond. An example of a Tutte partition of this $q$-matroid is

$$
\{[0, (e_1, e_2)], [(e_2 + e_3), E], [(e_1 + e_2 + e_3), (e_1 + e_3, e_2)], [(e_3), (e_1 + e_2, e_3)], [(e_1 + e_3), (e_1, e_3)]\}.
$$

The subspace $(e_2, e_3)$ is the only coloop in this $q$-matroid. There are no loops in this $q$-matroid. A diagram of this partition is given in Figure 3.

![Figure 2](image1.png)

Figure 2: A $q$-matroid on $F_2^3$. The notation $^{100}_{011}$, for instance, denotes the subspace $(e_1, e_2 + e_3)$. The weight-1 covers have been coloured red, and the weight-0 covers have been coloured green.

![Figure 3](image2.png)

Figure 3: A Tutte partition of the $q$-matroid given in Figure 2.

Central to our strategy to define the Tutte polynomial of a $q$-matroid is the notion of a totally clopen element. Such an element only exists if $M$ is prime-free, in which case it determines the rank of every element of $M$.

Definition 35. An element $z \in \mathcal{L}$ is called a clopen element of $M$ if for every $a \in \mathcal{L}$ such that $z$ covers $a$ we have $w([a, z]) = 0$ (i.e. $z$ is open) and for every cover $b$ of $z$ we have $w([z, b]) = 1$ (i.e. $z$ is closed). We say that $z$ is a totally clopen element of $M$ if any cover $[x, y] \subset \mathcal{L}$ satisfies $w([x, y]) = 1$ if $x \geq z$, and $w([x, y]) = 0$ if $y \leq z$. 

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For example, the only clopen elements of the $q$-matroid in Figure 2 are 0 and $\langle e_2, e_3 \rangle$, and there is no totally clopen element in this $q$-matroid.

The following lemma follows immediately from the definition of a totally clopen element.

**Lemma 36.** Let $z \in \mathcal{L}$ be a totally clopen element of $M$. The following hold:

1. $r(z) = 0$,
2. $\nu(z) = h(z) = \nu(1)$,
3. $h(z) = h(1) - r(1)$.

The following is a modification of [3, Proposition 3] that incorporates the notion of a totally clopen element, which we assert is a necessary concept for the statement to hold.

**Lemma 37.** Suppose that $z \in \mathcal{L}$ is a totally clopen element of $M$. Then for any cover $[x, y]$ in $\mathcal{L}$ we have:

$$w([x, y]) = \begin{cases} 0 & \text{if } x \land z \leq y \land z, \\ 1 & \text{if } x \lor z \leq y \lor z. \end{cases}$$

Furthermore, $z$ is the unique totally clopen element of $M$ and the complements of $z$ are the bases of $M$.

**Proof.** Let $[x, y]$ be a cover in $\mathcal{L}$. By Lemma 12, either $[x \land z, y \land z]$ is a cover, or $[x \lor z, y \lor z]$ is a cover. If $[x \lor z, y \lor z]$ is a cover, then it has weight 1 since $x \lor z \geq z$. Moreover, it is clear that $[x \lor z, y \lor z] \sim [x, y]$ in this case, and so $w([x, y]) = 1$. Similarly, if $[x \land z, y \land z]$ is a cover, then it has weight 0 since $y \land z \leq z$, and since $[x \land z, y \land z] \sim [x, y]$, we have $w([x, y]) = 0$.

We now show that $z$ is the unique totally clopen element of $M$. Suppose that $w \in \mathcal{L}$ is another totally clopen element of $M$. Consider that $h(z) = h(w) = \nu(1)$, so if $w \neq z$, then $z \land w \leq z$ and $z \land w \leq w$. Therefore, there exists an atom $a \leq z$ such that $a \not\leq w$. Consider the cover $c = [z \land w, (z \land w) \lor a]$. Since $z$ is totally clopen and $(z \land w) \lor a \leq z$, we have $w(c) = 0$. On the other hand, as $w$ is totally clopen, we have $w([w, a \lor w]) = 1$ and since $[w, a \lor w] \sim [z \land w, (z \land w) \lor a]$, by Lemma 16 it follows that $w(c) = 1$, which yields a contradiction.

Finally, to show that the complements of $z$ are the bases of $M$, first let $z^c$ be a complement of $z$. Then $z \land z^c = 0$ and so for any cover $[x, y] \subseteq [0, z^c]$, we have $x \land z = 0 = y \land z$, which means, by Lemma 12, that $[x \lor z, y \lor z]$ is a cover, and so $[x, y]$ has weight 1. In particular, every cover of a maximal chain in $[0, z^c]$ has weight 1 and so $r(z^c) = h(z^c)$, which means that $z^c$ is a basis of $M$. Conversely, let $b \in \mathcal{L}$ be a basis of $M$. Since $r(z) = 0$, we have $z \land b = 0$. From Lemma 36, we have $h(z) = h(1) - r(1) = h(1) - h(b)$, and thus $h(z \lor b) = h(z) + h(b) - h(z \land b) = h(z) + h(b) = h(1)$, which gives $z \lor b = 1$. It follows that $b$ is a complement of $z$. 

**Remark 38.** Suppose that $z \in \mathcal{L}$ is a totally clopen element of $M$. Then $z = cl(0)$. To see this, note the following. As $r(z) = 0$ we have $z \leq cl(0)$. If $[x, y]$ is a cover in $\mathcal{L}$ such that $x \geq cl(0)$ then as $z$ is totally clopen we have $w([x, y]) = 1$. Clearly every cover in $[0, cl(0)]$ has weight 0 and so it follows that $cl(0)$ is totally clopen in $M$. By the uniqueness of $z$ established in Lemma 37, we deduce that $z = cl(0)$. On the other hand, for arbitrary $M$, it is not the case that $cl(0)$ is necessarily totally clopen. In particular, in a prime diamond there is no totally clopen element (see the Figure 1).

We give a precise characterization of $M$ for which a totally clopen element exists.

**Theorem 39.** There exists a totally clopen element $z \in \mathcal{L}$ in $M$ if and only if $M$ is prime-free.
Proof. Let \( z \in \mathcal{L} \) be totally clopen. Assume that there exists a prime diamond \( D = [d_b, d_t] \) in \( \mathcal{L} \). By Lemma 37 and Lemma 12, for any cover \([d_b, x] \subseteq D\), \( d_b \wedge z = x \wedge z \). Similarly, for any cover \([x, d_t] \subseteq D\), \( x \vee z = d_t \vee z \). By Lemma 16 this means that \([d_b \vee z, x \vee z] = [d_b \vee z, d_t \vee z] \) is a weight-1 cover and \([x \wedge z, d_t \wedge z] = [d_b \wedge z, d_t \wedge z] \) is a weight-0 cover. By the modular law on \( \mathcal{L} \) we have that \( ([d_b \vee z] \wedge d_t, (d_t \vee z) \wedge d_t] = [d_b \vee (z \wedge d_t), d_t] \). If \( d_b \vee (z \wedge d_t) < d_t \), then \( [d_b \vee z, d_t \vee z] \not\supset [d_b \vee (z \wedge d_t), d_t] \), which would make \([d_b \vee (z \wedge d_t), d_t] \) a weight-1 cover in \( D \) containing \( d_t \), contradicting \( D \) being prime. Therefore, \( d_b \vee (z \wedge d_t) = d_t \). Similarly, \([d_b \vee (z \wedge d_b), d_b \vee (z \wedge d_t)] = [d_b, (d_b \vee z) \wedge d_t] \) implies that \( d_b = (d_b \vee z) \wedge d_t \), since otherwise \([d_b, (d_b \vee z) \wedge d_t]) \) would be a weight-0 cover in \( D \) containing \( d_b \), contradicting \( D \) being prime. This gives that \( d_b = (d_b \vee z) \wedge d_t = d_b \vee (z \wedge d_t) = d_t \), which is impossible, and hence we conclude that \( M \) has no prime diamonds.

Conversely, suppose that \( M \) has no prime diamonds. If \( r(M) = h(\mathcal{L}) \), then \( 0 \in \mathcal{L} \) is totally clopen in \( M \), so suppose otherwise. Let \( z = \text{cl}(0) \). We claim that \( z \) is totally clopen. Since \( z \) is closed, we have that \( w([z, x_i]) = 1 \) whenever \( z \prec x_i \in \mathcal{L} \). Fix such an \( x_1 \) that covers \( z \) and let \( x_1 \prec x_2 \in \mathcal{L} \). If \( w([x_1, x_2]) = 0 \), then \( [z, x_2] \) must be a prime diamond, which contradicts our assumption that \( M \) is prime-free. Therefore, \( w([x_1, x_2]) = 1 \). Since our choices of \( x_1 \) and \( x_2 \) were arbitrary, this process can be repeated until we get a maximal chain \( z = x_0 \prec x_1 \prec \cdots \prec x_k = 1 \) such that \( w([x_{i-1}, x_i]) = 1 \) for \( i = 1, \ldots, k \). Therefore, \( r(M) = h(\mathcal{L}) - h(z) \), which means that all covers \([a, b] \subseteq \mathcal{L} \) such that \( z \leq a \) must have weight 1.

In order for a partition \( \mathcal{P} \) of \( \mathcal{L} \) to satisfy the properties of Definition 33, we must have that all loops of \( M \) are contained in an interval of the form \([0, b] \subseteq \mathcal{L} \) and all coloops of \( M \) are contained in an interval of the form \([a, 1] \subseteq \mathcal{L} \). Otherwise, it would be impossible for \( \mathcal{P} \) to satisfy all of the properties of a Tutte partition, as either a loop would be below an interval in \( \mathcal{P} \), or a coloop would be above an interval in \( \mathcal{P} \). We address these requirements with the following results.

Lemma 40.

1. There exists \( y \in \mathcal{L} \) of rank zero such that every loop of \( M \) is contained in \([0, y]\).

2. There exists \( y \in \mathcal{L} \) such that \([y, 1]\) contains every coloop of \( M \) and \( r(1) - r(y) = h(1) - h(y) = \ell([y, 1])\).

Proof. Clearly, there exists \( y \in \mathcal{L} \) such that \( r(y) = 0 \) and every loop of \( M \) is contained in the interval \([0, y]\), i.e. \( y = \text{cl}(0) \). Let \( \varphi \) be an anti-isomorphism of \( \mathcal{L} \) and let \( x \) be the closure of the minimal element of \( \varphi(\mathcal{L}) \) in \( M^* \) and let \( y \in \mathcal{L} \) such that \( x = \varphi(y) \). If \( a \) is loop of \( M^* \) then \( a \leq x \) and so if \( b := \varphi^{-1}(a) \geq y \), then \( r(b) = r(1) - (h(1) - h(b)) = r(1) - 1 \) and hence \( b \) is a coloop of \( M \). That is, there is a one-to-one correspondence between the loops of \( M^* \) contained in \( x \) and the coloops of \( M \) that contain \( y \). Finally, \( r(y) = r^*(x) - h(1) + r(1) + h(y) = r(1) + h(y) - h(1) \) and so the result follows.

Corollary 41. Suppose that \( 0 < r(1) < h(1) \).

1. There exists a coatom \( c \in \mathcal{L} \) such that every loop of \( M \) is contained in \([0, c]\).

2. There exists an atom \( a \in \mathcal{L} \) such that every coloop of \( M \) is contained in \([a, 1]\).

Proof. Since \( r(1) \neq 0 \), we have that \( \text{cl}(0) < 1 \), which means that there exists a coatom \( c \) such that \( \text{cl}(0) \leq c \). By duality, the result follows.
Definition 42. For an interval \([a, b] \subseteq \mathcal{L}\) we say that a chain \(C(a, b) \subseteq [a, b]\) is a clopen chain if there exists \(c \in C(a, b)\) such that for any cover \([x, y] \subseteq [a, b]\), we have:

\[
  w([x, y]) = \begin{cases} 
    1 & \text{if } x \geq c \\
    0 & \text{if } y \leq c 
  \end{cases}
\]

Furthermore, we say that the clopen chain \(C(a, b)\) is extendable over \(\mathcal{L}\) if there exists \(x \in \mathcal{L}\) such that either \([b, x]\) is a weight-1 cover, or \([x, a]\) is a weight-0 cover. Otherwise, we say that \(C(a, b)\) is non-extendable over \(\mathcal{L}\).

Note that if \(M\) is prime-free, then there exists a clopen chain \(C(0, 1) \subseteq \mathcal{L}\) such that the totally clopen element \(c \in \mathcal{L}\) is in \(C(0, 1)\).

Lemma 43. Let \(C(a, b) \subseteq \mathcal{L}\) be a clopen chain. Then \(C(a, b)\) is non-extendable over \(\mathcal{L}\) if and only if for every cover \([c, d]\) in \(\mathcal{L}\), if \(b \leq c\) then \([c, d]\) has weight 0, and if \(d \leq a\) then \([c, d]\) has weight 1.

Proof. If \(C(a, b)\) is non-extendable over \(\mathcal{L}\), then each cover of the form \([b, x]\) has weight 0, and each cover of the form \([x, a]\) has weight 1. Let \(S = \{x \in [b, 1] : h(x) = h(b) + 1\}\). By the submodularity of any matroidal rank function, we have that \(r_{[b, 1]}(1) \leq \sum_{x \in S} r_{[b, 1]}(x) = 0\), which implies that all covers \([c, d]\) with \(b \leq c\) have weight 0. By Lemma 16, since all covers of the form \([x, a]\) have weight 1 by assumption, we have that all covers \([c, d]\) with \(d \leq a\) have weight 1.

Conversely, if all covers \([x, y] \subseteq \mathcal{L}\) with \(y \leq a\) have weight 1 and all covers \([x', y'] \subseteq \mathcal{L}\) with \(b \leq x'\) have weight 0, then clearly all covers \([x, a]\) have weight 1 and all covers \([b, y']\) have weight 0. Therefore, \(C(a, b)\) is non-extendable over \(\mathcal{L}\).

Lemma 44. Let \(M\) contain a prime diamond. There then exists an atom \(a \in \mathcal{L}\) and a minor \([a, 1] \subseteq \mathcal{L}\) such that all coloops of \(M\) are contained in \([a, 1]\) and \(w([0, a]) = 1\).

Proof. This statement clearly holds if \(h(1) = 2\). Now let \(h(1) = n \in \mathbb{N}\) and assume the statement holds for all matroids of length \(\leq n - 1\). By Corollary 41, there exists an atom \(a \in \mathcal{L}\) such that all coloops of \(M\) are contained in \([a, 1]\). If \([a, 1]\) is prime-free, then by Theorem 39 there is a totally clopen element in \([a, 1]\), and thus there is a clopen chain \(C(a, 1)\). Since \(M\) contains a prime diamond, it does not have a totally clopen lattice element, and thus \(C(a, 1)\) is non-extendable over \(\mathcal{L}\). Therefore, by Lemma 43, \(w([0, a]) = 1\). Assume then that \([a, 1]\) contains a prime diamond. The induction hypothesis then gives us a minor \([f, 1] \subseteq [a, 1]\) such that all coloops in \([a, 1]\), and thus in \(\mathcal{L}\), are contained in \([f, 1]\) and furthermore, \(w([a, f]) = 1\). If we assume that \(w([0, a]) = 0\), then \([0, f]\) is a mixed diamond (since it contains a weight-1 cover as well as a loop), so we can choose an \(e' \in [0, f]\) such that \(w([0, e']) = 1\), and clearly \([e', 1]\) contains all the coloops of \(\mathcal{L}\) since \([f, 1] \subseteq [e', 1]\).

Lemma 45. Let \(M\) contain a prime diamond. There then exists an atom \(a \in \mathcal{L}\) and copoint \(c \in \mathcal{L}\) such that every loop of \(M\) is contained in \([0, c]\), every coloop of \(M\) is contained in \([a, 1]\), and \([0, c] \cap [a, 1] = \emptyset\).

Proof. By Corollary 41 and Corollary 41 we know that such an atom \(a\) and copoint \(c\) exist, so all that remains is to show that they can be chosen such that \(a \nleq c\). By Lemma 44 we can choose \(a \in \mathcal{L}\) such that \([0, a]\) has weight 1. As observed in Lemma 40, for \(y = cl(0)\), the minor \([0, y]\) contains all the loops of \(M\) and clearly \(a \nleq y\). If all copoints of \(M\) containing \(y\) also contain \(a\), then their meet also contains \(a\), but their meet is \(y\), a contradiction. Therefore, there exists a copoint \(c \in \mathcal{L}\) that contains \(y\) and does not contain \(a\), and the result follows.
Now we give an alternative definition of a Tutte partition (referred to below as a Tutte-compatible partition) of $M$. The reason we do this is for convenience in proving the main result of this section. As we’ll shortly see, both definitions are in fact equivalent.

**Definition 46.** A Tutte-compatible partition is a partition of $M$ into prime-free minors, such that for any minor $[a, b] \subseteq \mathcal{L}$ in this partition, any clopen chain $C(a, b) \in [a, b]$ is non-extendable over $\mathcal{L}$.

**Example 47.** Consider the $q$-matroid given in Figure 2, which is not prime-free. Recall that a Tutte partition of this $q$-matroid into prime-free minors is shown in Figure 3. The cover $[(e_1, e_2), E]$ has weight zero, so the clopen chain $C(0, \langle e_1, e_2 \rangle)$ is non-extendable over $\mathcal{L}$. By inspection, one can see that the maximal clopen chains of each interval of this partition are non-extendable over $\mathcal{L}$. Therefore, this Tutte partition is also a Tutte-compatible partition.

**Lemma 48.** A partition of $M$ is a Tutte partition if and only if it is a Tutte-compatible partition.

**Proof.** Let $\mathcal{P}$ be a partition of $\mathcal{L}$. If $\mathcal{P}$ is a Tutte partition of $M$, then for any $[a, b] \in \mathcal{P}$, we have that $r(a) = h(a)$, $r(b) = r(1)$, and $[a, b]$ is prime-free. Therefore, $[a, b]$ contains a lattice element $z$ that is totally clopen in $[a, b]$ and any chain of the form $a \lessdot \cdots \lessdot z \lessdot \cdots \lessdot b$ is totally clopen in $[a, b]$. Since $r(a) = h(a)$ we have that $w([c, d]) = 1$ for any cover $[c, d]$ of the lattice such that $d \leq a$. Since $r(b) = r(1)$ we have that $w([c, d]) = 0$ for any cover $[c, d]$ of the lattice such that $c \geq b$. In particular, by Lemma 43, any totally clopen chain in $[a, b]$ is non-extendable over $\mathcal{L}$.

Conversely, suppose that $\mathcal{P}$ is a Tutte-compatible partition. Since for any $[a, b] \in \mathcal{P}$ there is a clopen chain $C(a, b)$, by Theorem 39 we have that $[a, b]$ is prime-free. Since $C(a, b)$ is non-extendable over $\mathcal{L}$, by Lemma 43, the weight of any cover in $[0, a]$ is 1 and the weight of and cover in $[b, 1]$. It follows that $r(a) = h(a)$ and $r(b) = r(1)$.

We now describe a partition of $\mathcal{L}$, called an proper interval decomposition. The definition of a proper interval decomposition comes from the following. The Goldman-Rota recurrence [18]:

$$G_{n+1} = 2G_n + (q^n - 1)G_{n-1},$$

for the Galois numbers $G_n := \sum_{j=0}^{n} \binom{n}{j} \frac{1}{q^j}$, has inspired the question as to whether this recurrence could be regarded as a consequence of the existence of a partition of the lattice of subspaces into two intervals of length $n - 1$ and $q^n - 1$ intervals of length $n - 2$. It has been shown in [21, Theorem 1], that such a partition is equivalent to the existence of particular linear forms. While there are partial results, a full classification of the subspace lattices that admit a proper interval decomposition is open. Nonetheless, our approach relies on the existence of such a lattice partition.

**Definition 49** ([21]). Let $e \in \mathcal{L}$ be an atom and let $e^c \in \mathcal{L}$ be a complement of $e$. Let $\mathcal{P}$ be a partition of $\mathcal{L}$ of the form

$$[e, 1] \cup [0, e^c] \cup [e_1, \bar{e}_1] \cup \cdots \cup [e_{q^n-1}, \bar{e}_{q^n-1}],$$

where for each $k$, $e_k \leq \bar{e}_k$ for an atom $e_k$ and a coatom $\bar{e}_k$. We say that $\mathcal{P}$ is a proper interval decomposition of $\mathcal{L}$ and denote such a partition by $\mathcal{P}(e)$.

Note that $\mathcal{P}(e)$ is not necessarily unique for each atom $e$, but it is sufficient for our purposes to choose one such partition for a given $e$. Note that $\mathcal{P}(e)$ is a non-trivial partition of $\mathcal{L}$ into intervals with a minimal number of elements.
Definition 55. Let $M$ be a polynomial of this section is Corollary 62, which we will use in Section 5 to show that the rank generating function of $M$ can be derived from a Tutte polynomial of $M$ and vice-versa.

We will now use Definition 46 and Lemma 48 to prove the following theorem. More explicitly, we will show that a proper interval decomposition $P(e)$, with $e$ and $e'$ selected as described in the statement of Lemma 45, has a refinement that is a Tutte-compatible partition.

Theorem 51. Let $L$ have a proper interval decomposition. Then $M$ has a Tutte partition.

Proof. If $h(1) = 2$ the statement clearly holds (note that intervals may have a length of zero), so assume that it holds for all $M = (L, r)$ such that $h(1) \leq n - 1$. Let $h(1) = n$. If $M$ is prime-free then there is nothing to show, so assume that $M$ contains a prime diamond. By Lemma 45, we can choose an atom $e \in L$ and a proper interval decomposition $P(e)$ such that all coloops of $M$ are in $[e, 1]$ and all loops of $M$ are in $[0, e']$. Let $\{[e_i, e_j] : 1 \leq i \leq q^{n-1}-1\}$ be the set of remaining intervals of $P(e)$. Since the length of each of these minors is at most $n - 1$, the induction hypothesis states that each of these minors has a Tutte partition, and hence by Lemma 48 each element of each such partition has a non-extendable clopen chain. By construction, the covers $[e', 1], [e_1, 1], \ldots, [e_k, 1]$ all have weight 0 and the covers $[0, e], [0, e_1], \ldots, [0, e_k]$ all have weight 1. Therefore, if $C(x, y) \subset L$ is a clopen chain in one of the intervals $A \in P(e)$ and is non-extendable over $A$, it must be that $C(x, y)$ is also non-extendable over $L$ by Lemma 43 since $r(x) = h(x)$ and $r(y) = r(1)$. The result now follows. \qed

Notation 52. For an interval $A = [a, b] \subseteq L$, we write $1_A := b$ and $0_A := a$.

We are now ready to give a definition of a Tutte polynomial.

Definition 53. Let $L$ have a proper interval decomposition. Let $P$ be a Tutte partition of $M$. We define the Tutte polynomial of $M$ with respect to the partition $P$ to be

$$
\tau_P(M; x, y) = \sum_{A \in P} x^{r_A(1_A)} y^{\nu_A(1_A)}.
$$

While the definition we give here is given with respect to a partition $P$, we will see in Section 5 that the Tutte polynomial is independent of the choice of Tutte partition.

Example 54. Let $M = (L(F_3), r)$ be the $q$-matroid with support lattice shown on the left of Figure 4. On the right of Figure 4, we see a Tutte partition $P$ of $M$ into 5 prime-free minors. The diamond $[e, 1] = [(101), 1]$ is empty and has rank 0 and nullity 2, which contributes the term $y^2$ to $\tau_P(M, x, y)$. The minor with support lattice $[0, e'] = [0, (100, 010)]$ has rank 1 and nullity 1, which corresponds to the term $xy$. The remaining 3 intervals support minors of length 1, rank 0, and nullity 1.

4 An inversion formula

We establish an inversion formula, which may be of independent interest. The main result of this section is Corollary 62, which we will use in Section 5 to show that the rank generating polynomial of $M$ can be derived from a Tutte polynomial of $M$ and vice-versa.

Definition 55. Let $n, m, i, j$ be non-negative integers. We define:

$$
\alpha_q(n, m; i, j) := \binom{n}{i}_q \binom{m}{j}_q q^{(n-i)(m-j)}.
$$
Figure 4: The Tutte Polynomial $\tau_P(M; x, y) = xy + y^2 + 3y$.

**Definition 56.** Let $\gamma(i, j; a, b)$ be non-negative integers for all non-negative integers $i, j, a, b$. We define $\gamma : \mathbb{Z}[x, y] \rightarrow \mathbb{Z}[x, y]$ by:

$$\gamma \ast P := \sum_{i=0}^{u} \sum_{j=0}^{v} \sum_{a=0}^{i} \sum_{b=0}^{j} \gamma(i, j; a, b)x^{a}y^{b},$$

for all $P(x, y) = \sum_{i=0}^{u} \sum_{j=0}^{v} p_{i,j}x^{i}y^{j} \in \mathbb{Z}[x, y]$.

We will now establish the existence of a map $\beta : \mathbb{Z}[x, y] \rightarrow \mathbb{Z}[x, y]$ such that $\alpha \ast \beta = \beta \ast \alpha$ is the identity map on $\mathbb{Z}[x, y]$. A candidate for such a map $\beta$ was suggested in [3, Conjecture 1] on the basis of experimental computations. We provide a proof of that conjecture in Theorem 61.

**Definition 57.** The Kronecker delta is the binary function on $\mathbb{R} \times \mathbb{R}$ defined by:

$$\delta_{a}^{b} = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise}. \end{cases}$$

**Definition 58.** Let $a, b, c, d$ be non-negative integers. Define:

$$\beta_q(a, b; c, d) := (-1)^{(a-c)+(b-d)} \binom{a}{c} \binom{b}{d} q^{((a-c)-(b-d))/2} (1 + q^{(a-c)-(b-d)}) - q^{\max(a-c,b-d)}.$$

We will show that the following relation holds, showing that $\beta$ is the required inverse map of $\alpha$.

$$\sum_{c=0}^{a} \sum_{d=0}^{b} \alpha_q(a, b; c, d) \beta_q(c, d; u, v) = \delta_u^{a} \delta_v^{b}. \quad (1)$$

We will use the following identity, which is an easy consequence of the $q$-binomial theorem. Recall Definition 6 for the binomial coefficient, which appears in the following.

**Lemma 59.** For any non-negative value of $n$, and any integer $s$ with $0 \leq s < n$,

$$\sum_{i=0}^{n} (-1)^{i} q^{\binom{i-s}{2}} \binom{n}{i} q = \delta_{0}^{n}.$$

**Proof.** The $n = 0$ case is trivial. Next observe that

$$\binom{i-s}{2} = \frac{(i-s)(i-s-1)}{2} = \frac{i^2 - is - i - is + s(s + 1)}{2} = \binom{i}{2} - si + \binom{s+1}{2}$$
to get
\[\sum_{i=0}^{n} (-1)^i q^{\binom{i}{2}} \left[ \begin{array}{c} n \\ i \end{array} \right] q^i = q^{\binom{n+1}{2}} \sum_{i=0}^{n} (-1)^i q^{\binom{i}{2}} \left[ \begin{array}{c} n \\ i \end{array} \right] q^{-Si}.\]
The result now follows. \qed

Note that \(\alpha_q(a, b; c, d) = 0\) whenever \(c > a\) or \(d > b\), while \(\alpha_q(a, b; a, d) = 1 = \alpha_q(a, b; c, b)\). Also \(\beta_q(c, d; u, v) = 0\) whenever \(u > c\) or \(v > d\). Therefore, we assume \(c \leq a, u\) and \(d \leq b, v\) and set \(c = u + i\) and \(d = v + j\) for non-negative integers \(i, j\), which yields \(\alpha_q(a, b; c, d)\beta_q(c, d; u, v) = \left[ \begin{array}{c} a \\ u \end{array} \right] \left[ \begin{array}{c} b \\ v \end{array} \right] \left[ \begin{array}{c} a - u \\ i \end{array} \right] \left[ \begin{array}{c} b - v \\ j \end{array} \right] q^{(a-u-i)(b-v-j)} q^{\binom{i-j}{2}} (1 + q^{i-j} - q^{\max(i,j)}).\)

Since the summands in (1) are now enumerated by \(i\) and \(j\), and \(\left[ \begin{array}{c} a \\ u \end{array} \right] \left[ \begin{array}{c} b \\ v \end{array} \right] \left[ \begin{array}{c} a - u \\ i \end{array} \right] \left[ \begin{array}{c} b - v \\ j \end{array} \right] q\) is a nonzero common factor, we can quotient out this common factor without loss of generality.

**Definition 60.** Let \(a, b, c, d, u, v\) be non-negative integers satisfying \(u \leq a\) and \(v \leq b\). We define:

\[N_{i,j}(a - u, b - v) := \alpha_q(a, b; c, d)\beta_q(c, d; u, v) \left( \left[ \begin{array}{c} a \\ u \end{array} \right] \left[ \begin{array}{c} b \\ v \end{array} \right] \right)^{-1}.\]

**Theorem 61.** Let \(a, b\) be non-negative integers. We have:

\[\sum_{i=0}^{a} \sum_{j=0}^{b} N_{i,j}(a, b) = \delta^a_0 \delta^b_0.\]

**Proof.** Let \(b > a\), then consider the \(i\)-th row sum

\[\sum_{j=0}^{b} N_{i,j}(a, b) = (-1)^i \left[ \begin{array}{c} a \\ i \end{array} \right] \sum_{j=0}^{b} (-1)^j \left[ \begin{array}{c} b \\ j \end{array} \right] q^{(a-i)(b-j)} q^{\binom{i-j}{2}} (1 + q^{i-j} - q^{\max(i,j)})\]

\[= (-1)^i \left[ \begin{array}{c} a \\ i \end{array} \right] (S_{1,i} + S_{2,i} - S_{3,i}),\]

where

\[S_{1,i} := \sum_{j=0}^{b} (-1)^j \left[ \begin{array}{c} b \\ j \end{array} \right] q^{(a-i)(b-j)} q^{\binom{i-j}{2}},\]

\[S_{2,i} := \sum_{j=0}^{b} (-1)^j \left[ \begin{array}{c} b \\ j \end{array} \right] q^{(a-i)(b-j)} q^{\binom{i-j}{2}} q^{i-j},\]

\[S_{3,i} := \sum_{j=0}^{b} (-1)^j \left[ \begin{array}{c} b \\ j \end{array} \right] q^{(a-i)(b-j)} q^{\binom{i-j}{2}} q^{\max(i,j)}.\]

We will now express the \(S_{i,j}\) in more convenient forms. First focusing on the exponents of \(q\), we will consider the cases \(i \geq j\) and \(j \leq i\) separately.
Suppose first that \( i \geq j \) in \( S_{1,i} \). We have

\[
(a - i)(b - j) + \binom{i - j}{2} = ab - aj - bi + ij + \frac{1}{2}(i - j)(i - j - 1)
\]

\[
= \frac{1}{2}(2ab - 2aj - 2bi + 2ij + i^2 - 2ij - i + j^2 + j)
\]

\[
= \frac{(j - (a - 1))(j - (a - 1) - 1)}{2} - \frac{a(a - 1)}{2} + \frac{i(i - 1)}{2} + b(a - i)
\]

\[
= \left( \frac{j - (a - 1)}{2} \right) - \left( \frac{a}{2} \right) + \left( \frac{i}{2} \right) + b(a - i).
\]

For the case \( i \leq j \) in \( S_{1,i} \) we have:

\[
(a - i)(b - j) + \binom{j - i}{2} = ab - aj - bi + ij + \frac{1}{2}(j - i)(j - i - 1)
\]

\[
= \frac{1}{2}(2ab - 2aj - 2bi + 2ij + i^2 - 2ij + i + j^2 - j)
\]

\[
= \frac{(j - a)(j - a - 1)}{2} - \frac{a(a + 1)}{2} + \frac{i(i + 1)}{2} + b(a - i)
\]

\[
= \left( \frac{j - a}{2} \right) - \left( \frac{a + 1}{2} \right) + \left( \frac{i + 1}{2} \right) + b(a - i).
\]

For the exponents of \( q \) in \( S_{2,i} \) and \( S_{3,i} \) we can just add the relevant term to the expressions above, the explicit computations of which are omitted, since it is clear at this point (eg. add \( j - i \) to the last expression above to get the exponent part of \( S_{2,i} \) when \( i \leq j \)). Now we see that we have

\[
S_{1,i} = q^{(\frac{i}{2}) - \binom{i}{2} + b(a - i)} \left( \sum_{j=0}^{i-1} (-1)^j \begin{bmatrix} b \\ j \end{bmatrix} q^{\binom{j}{2} - \binom{a-1}{2}} \right) + q^{\binom{i+1}{2} - \binom{a+1}{2} + b(a - i)} \left( \sum_{j=i}^{b} (-1)^j \begin{bmatrix} b \\ j \end{bmatrix} q^{\binom{j}{2} - \binom{a}{2}} \right),
\]

\[
S_{2,i} = q^{\binom{i+1}{2} - \binom{a+1}{2} + b(a - i)} \left( \sum_{j=0}^{i-1} (-1)^j \begin{bmatrix} b \\ j \end{bmatrix} q^{\binom{j}{2} - \binom{a}{2}} \right) + q^{\binom{i}{2} - \binom{a}{2} + b(a - i)} \left( \sum_{j=i}^{b} (-1)^j \begin{bmatrix} b \\ j \end{bmatrix} q^{\binom{j}{2} - \binom{a-1}{2}} \right),
\]

\[
S_{3,i} = q^{\binom{i+1}{2} - \binom{a}{2} + b(a - i)} \left( \sum_{j=0}^{i-1} (-1)^j \begin{bmatrix} b \\ j \end{bmatrix} q^{\binom{j}{2} - \binom{a-1}{2}} \right) = 0.
\]

Now add \( S_{1,i} \) and \( S_{2,i} \) and apply Lemma 59 to obtain:

\[
S_{1,i} + S_{2,i} = q^{\binom{i}{2} - \binom{a}{2} + b(a - i)} \sum_{j=0}^{b} (-1)^j \begin{bmatrix} b \\ j \end{bmatrix} q^{\binom{j}{2} - \binom{a-1}{2}} + q^{\binom{i+1}{2} - \binom{a+1}{2} + b(a - i)} \sum_{j=0}^{b} (-1)^j \begin{bmatrix} b \\ j \end{bmatrix} q^{\binom{j}{2} - \binom{a}{2}}
\]

\[
= 0.
\]

It now follows that:

\[
\sum_{i=0}^{a} \sum_{j=0}^{b} N_{i,j}(a, b) = \sum_{i=0}^{a} (-1)^i \begin{bmatrix} a \\ i \end{bmatrix} (S_{1,i} + S_{2,i} - S_{3,i}) = 0.
\]

Since relabelling makes the above proof work for \( a > b \), all that remains to check is when \( a = b \).
Now set \( a = b \), with \( S_{1,i}, S_{2,i}, S_{3,i} \), defined as before (writing \( a \) instead of \( b \))

\[
S_{1,i} + S_{2,i} = q^{(i+1) - (\frac{i+1}{2}) + a(a-i)} \sum_{j=0}^{a} (-1)^j \left[ \frac{a}{j} \right] q^{(j-a)} \\
= q^{(i+1) - (\frac{i+1}{2}) + a(a-i)} f(a), \\
S_{3,i} = 0.
\]

where \( f(a) = \sum_{j=0}^{a} (-1)^j \left[ \frac{a}{j} \right] q^{(j-a)} \). Putting things back together, using Lemma 59, we have

\[
\sum_{i=0}^{a} \sum_{j=0}^{a} N_{i,j}(a,a) = \sum_{i=0}^{a} (-1)^i \left[ \frac{a}{i} \right] q^{(i+1) - (\frac{i+1}{2}) + a(a-i)} f(a) \\
= q^{a-1} f(a) \sum_{i=0}^{a} (-1)^i \left[ \frac{a}{i} \right] q^{(i-a-1)} \\
= 0,
\]

where the second to last equality comes from similar manipulations of the exponent of \( q \) as was done for the \( a < b \) case, and the last equality coming from an application of Lemma 59. \( \square \)

We now immediately obtain the following corollary.

**Corollary 62.** Let \( a, b, c, d, u, v \) be non-negative integers. Then

\[
\sum_{c=0}^{a} \sum_{d=0}^{b} \alpha_q(a,b;c,d)\beta_q(c,d;u,v) = \delta_u^a \delta_v^b.
\]

In particular, for any \( P \in \mathbb{Z}[x,y] \), we have

\[
\alpha_q \ast (\beta_q \ast P) = \beta_q \ast (\alpha_q \ast P).
\]

**Remark 63.** The mutually inverse transformations \( \alpha \) and \( \beta \) are indeed \( q \)-analogues of the classical transformations associated with the rank generating polynomial and Tutte polynomial of a matroid. Let \( P(x,y) = \sum_{i=0}^{a} \sum_{j=0}^{b} p_{i,j} x^i y^j \in \mathbb{Z}[x,y] \). If we set \( q = 1 \) in \( \alpha_q \) and \( \beta_q \) we obtain:

\[
\alpha_1(i,j;a,b) := \binom{i}{a} \binom{j}{b} \text{ and } \beta_1(i,j;a,b) := (-1)^{i-a}(-1)^{j-b} \binom{i}{a} \binom{j}{b}.
\]

Therefore,

\[
\alpha_1 \ast P := \sum_{i=0}^{a} \sum_{j=0}^{b} \sum_{a=0}^{i} \sum_{b=0}^{j} \binom{i}{a} \binom{j}{b} x^a y^b = \sum_{i=0}^{a} \sum_{j=0}^{b} p_{i,j} (x+1)^a (y+1)^b.
\]

and

\[
\beta_1 \ast P := \sum_{i=0}^{a} \sum_{j=0}^{b} \sum_{a=0}^{i} \sum_{b=0}^{j} (-1)^{i-a}(-1)^{j-b} \binom{i}{a} \binom{j}{b} x^a y^b = \sum_{i=0}^{a} \sum_{j=0}^{b} p_{i,j} (x-1)^a (y-1)^b.
\]

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5 The rank generating polynomial and the Tutte polynomial

One of the fundamental properties of the Tutte polynomial in classical matroid theory is its connection to the rank generating polynomial. In this section we establish the connection between the rank generating polynomial of $M$ and $\tau_P(M; x, y)$ for any Tutte partition $P$. Explicitly, we will show that the rank generating polynomial is given by $\alpha_q \ast \tau_P(M; x, y)$ and conversely, any $\tau_P(M; x, y)$ can be derived by applying the transformation $\beta$ to the rank generating polynomial of $M$. This leads to another important result of this section, Corollary 68, which gives that $\tau_P(M; x, y)$ is uniquely determined and independent of the choice of Tutte partition $P$.

**Definition 64.** The rank generating polynomial of $M$ is defined to be

$$\rho(M; x, y) := \sum_{z \in \mathcal{L}} x^{r(1) - r(z)} y^{\nu(z)} \in \mathbb{Z}[x, y].$$

We remark that a 4-variable definition of the rank generating polynomial was given in [28], where it was used to obtain a generalisation of Greene’s theorem. We will use the following result (see [3, Equation (2)]). We include the proof for completeness. This result will allow us to deduce that the rank generating polynomial of a prime-free matroid or $q$-matroid depends only on its rank and nullity, which is perhaps not surprising in the light of Lemma 37. Note that the rank-lack of $x \in \mathcal{L}$ is defined to be $r(1) - r(x)$. We recall the notion of a relatively complemented lattice, which means that for all intervals $[a, b] \subseteq \mathcal{L}$ and $x \in [a, b]$, there exists $y \in [a, b]$ such that $a = x \land y$ and $b = x \lor y$. It is well known that $\mathcal{L}$ is relatively complemented. This notion is used in the following proof.

**Proposition 65 (3).** Let $M$ be prime-free. Let $h(1) = n$, $r(1) = \rho$, and $\nu(1) = \nu$. The number of elements in $\mathcal{L}$ with rank-lack $i$ and nullity $j$ is $\alpha_q(\rho; \nu; i, j)$.

**Proof.** Let $c$ be the unique totally clopen element in $\mathcal{L}$, which has height $\nu = n - \rho$. Let $[a, b] \subseteq \mathcal{L}$ be such that $c \in [a, b]$ and let $h(a) = j \leq \nu$ and let $h(b) = n - i$ for $i \leq \rho$. Since $\mathcal{L}$ is relatively complemented, there exists $x \in [a, b]$ such that $a = x \land c$ and $b = x \lor c$. Since $\mathcal{L}$ is prime-free, we have that $[a, b]$ is prime-free with $c$ as its unique totally clopen element. By Lemma 37 we have that $x$ is a basis of the minor $M([a, b])$. Since $r(x \land c) = 0$ and $h(x) - h(x \land c) = h(x \lor c) - h(c)$, we have that

$$r(x) = r(x \land c) + r_{[x \land c, x \lor c]}(x) = r_{[x \land c, x \lor c]}(x) = h(x \lor c) - h(c) = n - i - \nu = n - i - (n - \rho) = \rho - i. \quad (2)$$

This means that $x$ has rank-lack $i$ in $M$. Clearly, $\nu(x) = h(x \land c) = j$.

For any $x \in \mathcal{L}$, it is clear that $x$ is a relative complement of $c$ in the interval $[x \land c, x \lor c]$. Therefore, by (2), the rank-lack and nullity of $x$ in $M$ are fully determined by $h(x \land c)$ and $h(x \lor c)$.

To count the number of elements of $\mathcal{L}$ with rank-lack $i$ and nullity $j$, it thus suffices for us to compute

$$|\{(x, a, b) : c \in [a, b], h(a) = j, h(b) = n - i, x \land c = a, x \lor c = b\}|.$$

The number of elements $a \leq c$ with $h(a) = j$ is the number of elements of height $j$ in the interval $[0, c]$, which is $\left[\frac{h(c)}{j}; q\right]$. The number of elements $b \geq c$ with $h(b) = n - i$ is the number of elements of height $n - i - h(c)$ in the interval $[c, 1]$, which is $\left[\frac{n - \nu}{n - i - h(c)}; q\right]$. Therefore, the number of elements of rank $i$ and nullity $j$ is given by

$$\sum_{c \in [a, b]} \left[\frac{h(c)}{j}; q\right] \left[\frac{n - \nu}{n - i - h(c)}; q\right].$$
By Theorem 5, for any such interval \([a, b]\), the number of elements \(x \in [a, b]\) such that \(a = x \land c\) and \(b = x \lor c\) is 
\[q^{(\nu-j)(n-i-j-(\nu-j))} = q^{(\nu-j)(\rho-i)}\] 
Therefore, the number of elements in \(L\) with rank-lack \(i\) and nullity \(j\) is given by 
\[\sum_{i=0}^{\rho} \sum_{j=0}^{\nu} \alpha_q(\rho, \nu; i, j) x^i y^j\]

**Corollary 66.** Let \(M\) be a prime-free and have rank \(\rho\) and nullity \(\nu\). Then the rank generating polynomial of \(M\) is:

\[\rho(M; x, y) = \sum_{i=0}^{\rho} \sum_{j=0}^{\nu} \alpha_q(\rho, \nu; i, j)x^i y^j\]

**Proposition 67.** Let \(P\) be a Tutte partition of \(M\) and let \(\tau_P(M; x, y) = \sum_{i=0}^{\rho} \sum_{j=0}^{\nu} \tau_{i,j} x^i y^j\). Then the rank generating polynomial of \(M\) is:

\[\rho(M; x, y) = \alpha_q * \tau_P(M; x, y)\]

**Proof.** By definition, we have

\[\tau_P(M; x, y) = \sum_{A \in \mathcal{P}} \alpha_q(r(A), \nu(A)) x^{r(A)} y^{\nu(A)} = \sum_{i=0}^{\rho} \sum_{j=0}^{\nu} \tau_{i,j} x^i y^j\]

which means that \(\tau_{i,j}\) denotes the number of elements of \(P\) with rank \(i\) and nullity \(j\). Let \(A \in \mathcal{P}\).

By Corollary 66, we have that \(\rho(A; x, y) = \sum_{a=0}^{r(A)} \sum_{b=0}^{\nu(A)} \alpha_q(r_A(A), \nu_A(A); a, b) x^a y^b\).

Since \(P\) is a Tutte partition of \(M\), we have \(r(0_A) = h(0_A)\) and \(r(1_A) = r(1)\). For any \(z \in A\) we have that \(r_A(1_A) - r_A(z) = r(1) - r(z)\) and \(\nu_A(z) = \nu(z)\). Therefore,

\[\rho(M; x, y) = \sum_{z \in \mathcal{L}} x^{r(1)-r(z)} y^{\nu(z)} = \sum_{A \in \mathcal{P}} \sum_{z \in A} x^{r(1)-r(z)} y^{\nu(z)} = \sum_{A \in \mathcal{P}} \rho(A; x, y) = \sum_{A \in \mathcal{P}} \sum_{z \in A} x^{r_A(1_A)-r_A(z)} y^{\nu_A(A)} = \sum_{A \in \mathcal{P}} \rho(A; x, y) = \sum_{A \in \mathcal{P}} \sum_{a=0}^{r_A(1_A)} \sum_{b=0}^{\nu_A(1_A)} \alpha_q(r_A(1_A), \nu_A(1_A); a, b) x^a y^b = \sum_{i=0}^{\rho} \sum_{j=0}^{\nu} \tau_{i,j} x^i y^j\]

The result now follows.

From Corollary 62, we obtain that for any Tutte partition \(\mathcal{P}\), \(\tau_P(M; x, y)\) is determined by the rank generating polynomial of \(M\).

---

1The statement of this proposition appeared as part of [3, Theorem 3]. However, the statement does not hold for arbitrary prime-free minors. We refer the reader to [3, Section 10] for a discussion on this matter.
Corollary 68. Let $\mathcal{P}$ be a Tutte partition of $M$. Then
\[ \beta_q \ast \rho(M; x, y) = \tau_{\mathcal{P}}(M; x, y). \]
In particular, the Tutte polynomial $\tau_{\mathcal{P}}(M; x, y)$ of $M$ is independent of the choice of Tutte partition $\mathcal{P}$.

Proof. From Proposition 67 and Corollary 62 we have
\[ \rho(M; x, y) = \alpha_q \ast \tau(M; x, y), \]
\[ \Rightarrow \beta_q \ast \rho(M; x, y) = \beta_q \ast \alpha_q \ast \tau(M; x, y) = \tau_{\mathcal{P}}(M; x, y). \]
That $\tau_{\mathcal{P}}(M; x, y)$ is independent of the choice of $\mathcal{P}$ now follows immediately by the uniqueness of the rank generating polynomial $\rho(M; x, y)$. \qed

Therefore, for the remainder of this paper, we dispense with the explicit reference to a Tutte-partition $\mathcal{P}$ and simply write $\tau(M; x, y)$.

If we specialize to the case for which $M$ is a matroid, we see that the Tutte polynomial as defined in this paper coincides with the Tutte polynomial as defined by Crapo in [14], albeit defined in a different way. For this reason we will for the remainder of this paper, call $\tau(M; x, y)$ the Tutte polynomial of $M$, for both the matroid and $q$-matroid case. We will revisit the matroid case in Section 7.

Example 69. In Figure 5 we compute the Tutte polynomial and rank generating polynomial of some of the length-3 $q$-matroids listed in the appendix of [11]. On the left hand side are the support lattices of the $q$-matroids, assigned a bicolouring consistent with its matroidal $\{0,1\}$-weighting: weight-0 covers are coloured green and weight-1 covers are coloured red. On the right-hand-side are corresponding Tutte partitions of each $q$-matroid. Under each support lattice the rank generating polynomial of that $q$-matroid is written, and under each Tutte partition is its Tutte polynomial.

Example 70. Using the same naming convention as in Example 69, we have
\[ \rho(P^2; x, y) = \sum_{a=0}^{\rho} \sum_{b=0}^{\nu} \rho_{a,b} x^a y^b = x + 3xy + 4 + xy^2 + 6y + y^2. \]
By Theorem 61,
\[ \tau(P^2; x, y) = \sum_{a=0}^{\rho} \sum_{b=0}^{\nu} \rho_{a,b} \sum_{i=0}^{a} \sum_{j=0}^{b} \beta_q(a, b; i, j)x^i y^j \]
\[ = (x - 1) + 3(-x - y + xy) + 4 + (1 + 2x - y^2 - 3xy + xy^2) \]
\[ + 6(-1 + y) + (2 - 3y + y^2) \]
\[ = xy^2, \]
which of course agrees with the lattice-theoretic computation of $\tau(P^2; x, y)$.

6 $q$-Tutte-Grothendieck invariants

In the classical theory of matroids, the notion of a Tutte-Grothendieck (T-G) invariant is fundamental to the study of the Tutte polynomial [5-7]. A T-G invariant is based upon a category

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theoretic construction by Brylawski, which is based upon a ring theoretic construction given by Tutte [31]. Such a construction cannot apply to $q$-matroids as it is too strict. The main contribution of this section is a weakening of this construction, which we call a $q$-T-G invariant, in Definition 74 (in fact, we show in Section 7.2 that in the case of matroids both constructions are equivalent). The main result of Section 6, Theorem 76, provides a definition of the Tutte polynomial for both matroids and $q$-matroids alike, analogous to Brylawski’s definition. In Section 6.1, we provide some results describing how the rank generating polynomial decomposes and is not a $q$-T-G invariant, which contrasts to the case for matroids. Explicitly, in [5, Section 6.1] the notion of a T-G invariant is defined as follows. Recall that $M(e)$ is the restriction of $M$ to the interval $[0, e]$ for an atom $e \in \mathcal{L}$.

**Definition 71.** A T-G invariant is a function $f$ defined on the class of matroids that satisfies the following three properties:

(TG1) For any pair of equivalent matroids, we have $f(M) = f(M')$.

(TG2) If $e$ is a loop or an isthmus of $M$ then $f(M) = f(M(e))f(M - e)$.

(TG3) For any atom $e$ that is neither a loop nor an isthmus in $M$, we have

$$f(M) = f(M - e) + f(M/e).$$

In [5, Theorem 6.2.2] the Tutte polynomial is defined in terms of the T-G invariant properties as follows.

**Theorem 72** (Brylawski, 1972). The Tutte polynomial is the unique function $\tau$ from the class of matroids to the ring $\mathbb{Z}[x, y]$ satisfying the following properties:

1. $\tau(M(e)) = x$ if $e$ is independent, and $\tau(M(e)) = y$ if $e$ is a loop,
2. \( \tau \) is a T-G invariant.

We will define a notion of such invariants in the more general setting of this paper and will show later that we recover the original T-G invariants for matroids by letting \( q \to 1 \).

Recall that if \( M \) is not prime-free, by Lemma 45 there then exists an atom \( e \in L \) such that every coloop of \( M \) is contained in \([e, 1]\) and every loop of \( M \) is contained in \([0, e^2]\), which extends to a partition of the form \( P(e) \).

**Notation 73.** We let \( M \) denote either the class of all \( q \)-matroids or the class of all matroids.

**Definition 74.** Let \( f \) be a function defined on \( M \). We say that \( f \) is a \( q \)-Tutte-Grothendieck invariant (q-T-G) if it satisfies the following three properties:

(q-P1) For all \( M_1, M_2 \in M \), if \( M_1 \cong M_2 \) then \( f(M_1) = f(M_2) \),

(q-P2) If \( M \) is prime-free, then \( f(M) = f(M/e)f(M/e) \) for any atom \( e \in L \),

(q-P3) If \( M \) is not prime-free, and \( e \) is an atom of \( M \) that is independent and contained in every coloop of \( M \), then there exists a proper interval decomposition \( P(e) \) such that

\[
f(M) = \sum_{I \in P(e)} f(M(I)) = f(M(e)) + f(M/e) + \sum_{k=1}^{q^{n-1}-1} f(M([e_k, \bar{e}_k])).
\]

**Proposition 75.** The Tutte polynomial \( \tau(M; x, y) \) is a q-T-G invariant.

**Proof.** It is clear that \( \tau(M; x, y) \) satisfies (q-P1). If \( M \) is prime-free, then \( \tau(M; x, y) = x^{r(M)}y^{\nu(M)} \), and since any minor of \( M \) is also prime-free, for any atom \( e \in M \) we have that

\[
\tau(M/e; x, y) = x^{r(M)-r(e)}y^{\nu(M)-\nu(e)} \text{ and } \tau(M(e); x, y) = x^{r(e)}y^{\nu(e)}.
\]

Therefore, \( \tau(M; x, y) = \tau(M(e); x, y)\tau(M/e; x, y) \), which shows that \( \tau(M; x, y) \) satisfies (q-P2). If \( M \) is not prime-free, then as in the proof of Theorem 51, if \( e \in L \) is an independent atom that is contained in all coloops of \( M \), then there a Tutte partition \( P \) that is a refinement of a proper interval decomposition \( P(e) \). That is, \( M \) has a Tutte partition of the form

\[
P = P' \cup P'' \cup P'_1 \cup \cdots \cup P_{q^{n-1}-1}
\]

where \( P' \) is a Tutte partition of the minor \( M(e^2) \), \( P'' \) is a Tutte partition of the minor \( M/e \), and \( P_k \) is a Tutte partition of the minor \( M([e_k, \bar{e}_k]) \) for each \( k \). Therefore,

\[
\tau(M; x, y) = \tau_P(M; x, y)
\]

\[
= \sum_{P \in P} x^{r_P(M)}y^{\nu_P(M)}
\]

\[
= \sum_{P \in P'} x^{r_P(M)}y^{\nu_P(M)} + \sum_{P \in P''} x^{r_P(M)}y^{\nu_P(M)} + \sum_{k=1}^{q^{n-1}-1} \sum_{P \in P_k} x^{r_P(M)}y^{\nu_P(M)}
\]

\[
= \tau(M(e^2); x, y) + \tau(M/e; x, y) + \sum_{k=1}^{q^{n-1}-1} \tau(M([e_k, \bar{e}_k]); x, y),
\]

which shows that \( \tau(M; x, y) \) satisfies (q-P3). \( \square \)

We now arrive at a definition of the Tutte polynomial as a q-T-G invariant.
**Theorem 76.** The Tutte polynomial is the unique function $\tau$ from $M$ to the ring $\mathbb{Z}[x,y]$ satisfying the following properties:

1. $\tau(M(e);x,y) = x$ if $e \in \mathcal{L}$ is independent in $M$, and $\tau(M(e);x,y) = y$ if $e \in \mathcal{L}$ is a loop of $M$.
2. $\tau(M;x,y)$ is a $q$-T-G invariant.

**Proof.** From Definition 53 and Proposition 75, it is clear that $\tau$ satisfies properties (1) and (2), so all that remains to show is uniqueness. For $h(1) \leq 2$ uniqueness clearly holds, so assume that it holds for any member of $\mathcal{M}$ whose support lattice has length at most $n-1$. Now let $h(1) = n$ and suppose that $\gamma : \mathcal{M} \rightarrow \mathbb{Z}[x,y]$ is a function that satisfies properties (1) and (2). Any proper minor of $M$ has support lattice of length at most $n-1$ and so by hypothesis the Tutte polynomial defined on these minors is the unique $q$-T-G invariant satisfying properties (1) and (2) of the theorem statement. It follows that $\gamma(M(e);x,y) = \tau(M(e);x,y)$ and $\gamma(M/e;x,y) = \tau(M/e;x,y)$ for any atom $e \in \mathcal{L}$, so in particular, if $M$ is prime-free we have

$$\gamma(M;x,y) = \gamma(M(e);x,y)\gamma(M/e;x,y) = \tau(M(e);x,y)\tau(M/e;x,y) = \tau(M;x,y),$$

for any atom $e \in M$. If $M$ is not prime-free, then by property (2) (and specifically, (q-P3)) we have

$$\gamma(M;x,y) = \gamma(M(e^c);x,y) + \sum_{k=1}^{q^n-1} \gamma(M([e_k,\bar{e}_k];x,y)$$

$$= \tau(M(e^c);x,y) + \tau(M/e;x,y) + \sum_{k=1}^{q^n-1} \tau(M([e_k,\bar{e}_k]);x,y)$$

$$= \tau(M;x,y). \quad \square$$

**Remark 77.** A recursive definition of the characteristic polynomial of a $q$-matroid is shown in [23, Theorem 5.14] and in [1, Corollary 3.4], wherein the characteristic polynomial is expressed in terms of characteristic polynomials of minors of the $q$-matroid that are defined with respect to a partition of the one-dimensional spaces of ground vector space $V$. While this resembles the definition of a $q$-Tutte-Grothendieck invariant in this paper, the collection of intervals involved is different.

### 6.1 $q$-Tutte-Grothendieck invariants and the rank generating polynomial

One of the fundamental properties of the rank generating polynomial for matroids is its decomposition as is shown in [5] and [14]. In particular, the rank generating polynomial of a matroid is a $T$-$G$ invariant in the sense of Definition 71. We will see in Section 7.2 that for matroids, the $q$-T-G invariants and the $T$-$G$ invariants coincide but for now we focus on $q$-T-G invariants. We will see a divergence between behaviour of the rank function of a matroid and that of a $q$-matroid: more precisely, we will see that in general, the rank function of a $q$-matroid is not a $q$-T-G invariant.

For the remainder of this section, we assume that $\mathcal{L}$ has a proper interval decomposition, that is, for any atom $e \in \mathcal{L}$, we have a partition of $\mathcal{L}$ into disjoint intervals:

$$\mathcal{P}(e) = [e,1] \cup [0,e^c] \cup [e_1,\bar{e}_1] \cup \cdots \cup [e_{q^n-1},\bar{e}_{q^n-1}],$$

such that $h(e_k) = 1$, $h(\bar{e}_k) = n-1$, and $e_k \leq \bar{e}_k$. Note that we make no assumptions about the atom $e$ here.
Proposition 78. Let $e \in \mathcal{L}$ be an atom and let $\mathcal{P}(e)$ be the proper interval decomposition of $\mathcal{L}$, defined as in (3). Then the rank generating polynomial $\rho(M; x, y)$ satisfies the following identity:

$$\rho(M; x, y) = x^{r(1)-r(e)} \rho(M(e^c); x, y) + \sum_{k=1}^{q^{n-1}-1} x^{r(1)-r(\bar{e}_k)} y^{r(\bar{e}_k)} \rho(M([e_k, \bar{e}_k]); x, y).$$

Proof. For any interval $[a, b] \subseteq \mathcal{L}$, we have that

$$\sum_{z \in [a,b]} x^{r(1)-r(z)} y^{\nu(z)} = x^{r(1)-r(b)} y^{\nu(a)} \sum_{z \in [a,b]} x^{r(1)-r(\bar{a}, \bar{b})-r([a,b])(z)} y^{\nu([a,b])(z)} = x^{r(1)-r(b)} y^{\nu(a)} \rho(M([a, b]); x, y).$$

We then have

$$\rho(M; x, y) = \sum_{z \in \mathcal{L}} x^{r(1)-r(z)} y^{\nu(z)}$$

$$= \sum_{z \in [0,e]} x^{r(1)-r(z)} y^{\nu(z)} + \sum_{z \in [e,1]} x^{r(1)-r(z)} y^{\nu(z)} + \sum_{k=1}^{q^{n-1}-1} \sum_{z \in [e_k, \bar{e}_k]} x^{r(1)-r(z)} y^{\nu(z)}$$

$$= y^{\nu(e)} \rho(M/e; x, y) + x^{r(1)-r(e^c)} \rho(M(e^c); x, y)$$

$$+ \sum_{k=1}^{q^{n-1}-1} x^{r(1)-r(\bar{e}_k)} y^{r(\bar{e}_k)} \rho(M([e_k, \bar{e}_k]); x, y).$$

Example 79. Let $M$ be the $q$-matroid of Example 54. Let $\mathcal{P}(e)$ be the proper interval decomposition determined by $e = \langle 101 \rangle$, $e' = \langle 100, 010 \rangle$, $e_1 = \langle 111 \rangle$, $e_2 = \langle 011 \rangle$, $e_3 = \langle 001 \rangle$, $\bar{e}_1 = \langle 100, 011 \rangle$, $\bar{e}_2 = \langle 010, 001 \rangle$, and $\bar{e}_3 = \langle 110, 001 \rangle$, which is shown on the right-hand side of Figure 4. We express $\rho(M; x, y)$ as a sum of rank generating polynomials of the minors supported on the distinct elements of $\mathcal{P}(e)$.

$$\rho(M; x, y) = x + xy + 7y + y^2 + 6$$

$$= (x + xy + y + 2) + (y^2 + 3y + 1) + 3(y + 1)$$

$$= \rho(M(e^c); x, y) + \rho(M/e; x, y) + \sum_{k=1}^{3} \rho(M([e_k, \bar{e}_k]); x, y).$$

Remark 80. Let $\mathcal{P}$ be a collection of intervals that forms a partition of $\mathcal{L}$. Suppose that for each $A \in \mathcal{P}$ it holds that $r(0_A) = h(0_A)$ and $r(1_A) = r(1)$. Note that we are not assuming that the elements of $\mathcal{P}$ are prime-free and hence $\mathcal{P}$ need not be a Tutte partition. Then we have that

$$\rho(M; x, y) = \sum_{A \in \mathcal{P}} \rho(A; x, y).$$

To see this, let $A \subseteq \mathcal{L}$ be any interval. We have $r_A(1_A) - r_A(x) = r(1_A) - r(x)$, and $\nu_A(x) = h(x) - h(0_A) - r_A(x)$ for all $x \in A$. Therefore, if $r(0_A) = h(0_A)$ and $r(1_A) = r(1)$,
then \( r(1) - r(x) = r_A(1_A) - r_A(x) \) and \( \nu(x) = \nu_A(x) \) for all \( x \in A \). This now gives us

\[
\rho(M; x, y) = \sum_{z \in \mathcal{L}} x^{r(1)-r(z)} y^{\nu(z)}
\]

\[
= \sum_{A \in \mathcal{P}} \sum_{z \in A} x^{r(1)-r(z)} y^{\nu(z)}
\]

\[
= \sum_{A \in \mathcal{P}} \sum_{z \in A} x^{r_A(1_A)-r_A(z)} y^{\nu_A(z)}
\]

\[
= \sum_{A \in \mathcal{P}} \rho(A; x, y).
\]

It is clear that the rank generating polynomial of a \( q \)-matroid satisfies \((q-P1)\). By Proposition 78 we see that \( \rho(M; x, y) \) also satisfies \((q-P3)\). However, if \( M \) is a \( q \)-matroid, in general its rank generating polynomial does not satisfy \((q-P2)\).

**Proposition 81.** Let \( \mathcal{M} \) denote the class of \( q \)-matroids. The rank function \( \rho : \mathcal{M} \rightarrow \mathbb{Z}[x, y] \) is not a \( q \)-T-G invariant.

**Proof.** Suppose that \( M \) is prime-free and let \( e \) be an atom of \( \mathcal{L} \). Then \( \rho(M/e; x, y) \) is also prime free and is supported on a lattice of height \( n - 1 \). We have \( \rho(M; 1, 1) = \sum_{k=0}^{n} \binom{n}{k} q^k \), since this counts the number of elements of \( \mathcal{L} \). However,

\[
\rho(M; 1, 1) \rho(M/e; 1, 1) = 2 \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} q^k \neq \sum_{k=0}^{n} \binom{n}{k} q^k = \rho(M; 1, 1).
\]

Therefore, \( \rho(M; x, y) \) does not satisfy \((q-P2)\) and hence the rank generating polynomial of a \( q \)-matroid is not a \( q \)-T-G invariant. \qed

# 7 A return to the matroid case

In this section, we restrict to the matroid case. Firstly, in Section 7.1, we will look at the notion of a Tutte partition as defined in Section 3 and compare this with the partition arising in Crapo’s definition of the Tutte polynomial [14]. The main result of Section 7.1 is Corollary 88, which shows that Crapo’s definition is a special case of Definition 33.

Next, in Section 7.2, we will consider the axioms of a \( q \)-Tutte Grothendieck invariant (Definition 74), and compare these with the usual Tutte-Grothendieck axioms (Definition 71). The main result of Section 7.2 is Theorem 91, which shows that Definition 74 gives a \( q \)-analogue of a T-G invariant.

**Notation 82.** Throughout this section \( M \) denotes an arbitrary matroid with Boolean support lattice \( \mathcal{L} = \mathcal{L}(S) \) for some finite set \( S \) of cardinality \( n \) and \( \mathcal{B} \) denotes the collection of bases of \( M \). \( \mathcal{M} \) now refers to the class of matroids.

## 7.1 Crapo’s definition of the Tutte polynomial

Recall that by Corollary 68, which holds both for matroids and \( q \)-matroids, the Tutte polynomial of Definition 53 in the matroid case is necessarily the Tutte polynomial of [14]. The definition of the Tutte polynomial of a matroid given in [14] uses a partition \( \mathcal{P} \) of \( \mathcal{L} \) into intervals such
that each $A \in \mathcal{P}$ contains exactly one basis of $M$. This leads to the question of the equivalence the partition of $\mathcal{L}$ given in [14] and the Tutte partition of Definition 33.

We now briefly describe the partition of $\mathcal{L}$ into intervals arising in Crapo’s paper. For each $b \in \mathcal{B}$, we define the interval $[b^-, b^+]$ in the following way. Let $\beta$ denote any total ordering of the bases of $M$ derived from an arbitrary total ordering of the atoms as in [14]. For any interval $[x, y] \subseteq \mathcal{L}$, let the $\beta$-max (respectively, $\beta$-min) basis of $[x, y]$ refer to the maximum (respectively, minimum) basis in $[x, y]$ according to this $\beta$. By [14, Lemmas 7 and 8] we have that the basis $b$ is the $\beta$-max basis of $[b^-, 1]$ and $b$ is the $\beta$-min basis of $[0, b^+]$. We summarize these notions in the form of the following definition.

**Definition 83.** Let $\beta$ be an arbitrary total ordering of the bases of $M$. Let $\mathcal{P}$ be a partition of $M$ into a union of disjoint minors. We say that $\mathcal{P}$ is a Crapo-Tutte partition if for every $[x, y] \in \mathcal{P}$ the following hold.

1. There is exactly one basis $b$ of $M$ in $[x, y]$.
2. The basis $b \in [x, y]$ is the $\beta$-max basis of $[x, 1]$.
3. The basis $b \in [x, y]$ is the $\beta$-min basis of $[0, y]$.

We denote the elements of a Crapo-Tutte partition by $[b^-, b^+]$ for each $b \in \mathcal{B}$ and note that each such interval is uniquely determined by the ordering $\beta$. The definition of the Tutte polynomial given in [14] is as follows.

**Definition 84.** Let $M$ be a matroid with set of bases $\mathcal{B}$. The Tutte polynomial of $M$ is

$$\tau(M; x, y) = \sum_{b \in \mathcal{B}} x^{[b^-, b^+]} y^{[b^+, b^-]}.$$

**Remark 85.** This definition does not immediately generalise to $q$-matroids: in the matroid case, by [14, Lemma 8], the collection of intervals $\{[b^-, b^+] : b \in \mathcal{B}\}$ form a partition of $\mathcal{L}$, whereas such a construction does not yield a partition in the event that the support lattice is the subspace lattice.

**Lemma 86.** Let $A \in \mathcal{L}$. If each of the atoms in $A$ is either a loop or isthmus, then $M/A \cong M - A$.

**Proof.** Define the lattice isomorphism $\varphi : M - A \to M/A$ by $\varphi(x) = x \cup A$. Clearly, if $a$ is an isthmus or loop then $r(a \cup B) = r(a) + r(B)$ for any $B \in \mathcal{L}$ such that $a \notin B$. Therefore, for any $x \subseteq A^c$ we have that $r(x \cup A) = r(x) + r(A)$. We thus have that $r_{M/A}(\varphi(x)) = r(\varphi(x)) - r(A) = r(x) = r_{M-A}(x)$ for any $x \subseteq A^c$. \hfill \Box

**Lemma 87.** For any $[x, y] \subseteq \mathcal{L}$, $M([x, y])$ has only one basis if and only if $[x, y]$ is prime-free.

**Proof.** For one of the implications, it suffices to show that any matroid with only one basis is prime-free, so suppose that $M$ has only one basis. Suppose now that $M$ has a prime diamond $D$, say $D = [A, A \cup x \cup y]$ for some $A, x, y \in \mathcal{L}$ with $h(x) = h(y) = 1$ and $x \notin y \cup A$ and $y \notin x \cup A$. Since $M$ has only one basis, each atom in $\mathcal{L}$ is either a loop or isomorus of $M$. Then from Lemma 86, $M/A \cong M - A$. If there is a basis $B$ of $M/A$ such that $x \cup A$ and $y \cup A$ are both contained in $B$, then $A \cup x \cup y \subseteq B$, which contradicts the fact that $D$ is a prime diamond. Therefore, there are at least two bases in $M/A$ and so there are at least two bases in $M - A$. But any basis of $M - A$ corresponds to a unique basis of $M$, which yields a contradiction.

Conversely, suppose that $M$ contains two bases, $b_1, b_2 \in \mathcal{B}$. By the basis-exchange property of matroids, there exist $b_3 \in \mathcal{B}$ and atoms $e_1 \in b_1 - b_2$ and $e_2 \in b_2 - b_1$ such that $b_3 = (b_1 - e_1) \cup e_2$. This clearly implies that $[b_1 - e_1, b_1 \cup e_2]$ is a prime diamond. \hfill \Box

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Recalling Definition 33, we get the following corollary.

**Corollary 88.** Every Crapo-Tutte partition is a Tutte partition.

**Proof.** Let \( \mathcal{P} \) be a Crapo-Tutte partition of \( \mathcal{L} \) with respect to some ordering of \( \mathcal{B} \). Since \( b^- \subseteq b \) and \( b \subseteq b^+ \) for any \( b \in \mathcal{B} \), we have that \( r(b^-) = h(b^-) \) and \( r(b^+) = r(M) \). Furthermore, by Lemma 87 we have that \([b^-, b^+]\) is prime-free. Therefore, \( \mathcal{P} = \{[b^-, b^+] : b \in \mathcal{B}\} \) is a Tutte partition.

An obvious question now arises, namely, whether or not every Tutte partition of a matroid is also a Crapo-Tutte partition. In fact, it is not the case that any Tutte partitions of a matroid can be represented as a disjoint union of intervals of the form \([b^-, b^+]\), \( b \in \mathcal{B} \) for some total ordering of bases. The following example shows this.

**Example 89.** Let \( M = U_{3,6} \) (the uniform matroid of rank three on six elements). For brevity, we write 123 to denote the set \{1, 2, 3\} etc. Consider the following Tutte partition \( \mathcal{P} \) of \( M \), the elements of which are listed below.

\[
\begin{align*}
[456, 1] \\
[124, 12346], [125, 12345], [126, 12356] \\
[134, 1345], [135, 1356], [136, 1346], [234, 2435], [235, 2356], [236, 2346] \\
[24, 245], [25, 256], [26, 246], [34, 346], [35, 345], [36, 356] \\
[4, 145], [5, 156], [6, 146] \\
[0, 123].
\end{align*}
\]

(Note that for each \([a, b] \in \mathcal{P}\), we have that \( a \) is independent, and \( b \) is spanning, and that \( \mathcal{P} \) is indeed a partition, thus satisfying the properties of a Tutte partition).

Suppose that \( \mathcal{P} \) is a Crapo-Tutte partition and let \( \leq_\beta \) denote its total order on the bases of \( M \), as described in Definition 83. Consider the intervals \([4, 145],[5, 156],[6, 146]\). By Property 2 of Definition 83, we have that 145 is the \( \beta \)-max basis of \([4, 1]\), 156 is the \( \beta \)-max basis of \([5, 1]\), and 146 is the \( \beta \)-max basis of \([6, 1]\). Since we also have that 146 \( \in [4, 1]\), 145 \( \in [5, 1]\), and 156 \( \in [5, 1]\), we respectively have that 146 \( \leq_\beta 145 \), 145 \( \leq_\beta 156 \), and 156 \( \leq_\beta 146 \). This clearly yields a contradiction, which means that \( \mathcal{P} \) cannot be a Crapo-Tutte partition.

### 7.2 Tutte-Grothendieck invariants

We now compare the axioms of a \( q \)-Tutte-Grothendieck invariant (Axioms (q-P1)-(q-P3)) and the classical axioms of a Tutte-Grothendieck invariant (Axioms (TG1)-(TG3)). We will show that for matroids, these sets of axioms are equivalent.

**Remark 90.** We point out that for any \( d, e \in \mathcal{L} \), if \( e \leq d \), then \((M/e)/d = M/d\). To see this, first note that both matroids are defined on the interval \([d, 1]\) and then observe that for any \( x \in [d, 1] \), \( r_{(M/e)/d}(x) = r_{M/e}(x) - r_{M/e}(d) = r(x) - r(e) - (r(d) - r(e)) = r(x) - r(d) = r_{M/d}(x) \).

**Theorem 91.** Let \( f : M \rightarrow \mathbb{Z}[x, y] \). Then \( f \) is a \( q \)-T-G invariant if and only if \( f \) is a T-G invariant.

**Proof.** Assume first that \( f \) is a T-G invariant. It is immediate that (q-P1) is satisfied. If \( M \) is prime-free, then by Lemma 87 it contains a unique basis, which means that each atom of \( M \) is either a loop or an isthmus. Therefore, using Lemma 86, since \( f \) satisfies property (TG2), it also satisfies property (q-P2). If \( M \) contains a prime diamond and \( e \) is an independent atom that is contained in all coloops of \( M \), then \( e \) is neither a loop nor an isthmus and so
since \( f \) satisfies property \((\text{TG}3)\), it necessarily satisfies property \((q\text{-P}3)\) (since the summation \(\sum_{k=1}^{q-1} f(M(|e_k, e_k|))\) is empty with \(q = 1\)). It follows that any T-G invariant is also a \(q\text{-T-G}\) invariant.

Conversely, assume \( f \) is a \(q\text{-T-G}\) invariant, so that \( f \) satisfies properties \((q\text{-P}1)\), \((q\text{-P}2)\), and \((q\text{-P}3)\). Clearly, if \( f \) satisfies \((q\text{-P}1)\), then it also satisfies \((\text{TG}1)\). If \( M \) contains an atom that is neither a loop nor an isthmus, then by Lemma \(87\), \( M \) contains a prime diamond, and thus \((\text{TG}3)\) is clearly satisfied. Therefore, we need only address the case for which \( M \) is not prime-free, and \( e \in M \) is a loop or an isthmus. Namely, we must show that in such a case

\[
f(M) = f(M(e)) f(M/e).
\]

If \( h(1) = 2 \), then this holds trivially, so assume that this holds whenever \( h(1) \leq n - 1 \). Now suppose that \( M \) contains a prime diamond and \( h(1) = 1 \). If \( M \) contains a loop (respectively, an isthmus) \( e \), then by \((q\text{-P}2)\) there exists an independent atom \( a \) such that \( e \in M - a \) and

\[
f(M) = f(M - a) + f(M/a).
\]

Clearly \( e \) is a loop (respectively, an isthmus) of \( M - a \), and \( e \cup a \) is a loop (respectively, an isthmus) of \( M/a \). By the induction hypothesis \( f(M - a) = f((M - a)(e)) f((M - a(e) - e) = f((M - a(e)) f(M - (e \cup a))) \) and \( f(M/a) = f((M/a)(e \cup a)) f((M/e) - (e \cup a)) \).

Since \( e \) is a loop (respectively, an isthmus) we have that \( (M/e) = (M/a)(e) \equiv (M/a)(e \cup a) \), which means that \( f(M(e)) = f((M - a)(e)) = f((M/a)(e \cup a)) \) by \((q\text{-P}1)\). It follows that

\[
f(M) = f(M(e))(f((M - a) - e) + f((M/a) - (e \cup a))).
\]

Since \( a \) is not a loop (respectively, not an isthmus) in \( M \) and all (respectively, none) of the bases are in \( M/e \), \( e \cup a \) is not a loop (respectively, an isthmus) of \( M/e \), and \( M/e \) is not prime-free. By \((q\text{-P}3)\) we have that \( f(M/e) = f((M/e) - (e \cup a)) + f((M/e)/(e \cup a)) \). Since \( e \) is dependent (respectively, independent) to any lattice element in \( M - a \), by Lemma \(86\) we have that \( (M/e) - (e \cup a) \cong M - (e \cup a) \) and \( (M/e)/(e \cup a) \cong (M/a)/(e \cup a) \cong (M/a) - (e \cup a) \).

As a result, we have that

\[
f(M) = f(M(e)) (f((M - a) - (e \cup a)) + f((M/a)/(e \cup a))
\]

\[
= f(M(e)) (f((M/e) - (e \cup a)) + f((M/e)/(e \cup a)))
\]

\[
= f(M(e)) f(M/e).
\]

\[
\square
\]

8 Further properties of the Tutte and rank generating polynomial

In this section again \( M \) may be either a matroid or a \( q \)-matroid. We present a number of properties of the Tutte polynomial of a \( q \)-matroid that are the \( q \)-analogues of already established properties of the Tutte polynomial for matroids.

8.1 Duality properties

Let \( \varphi : \mathcal{L} \rightarrow \mathcal{L}^* \) be a lattice anti-isomorphism. Recall that the dual of \( M \) is defined to be \( M^* = (\mathcal{L}^*, r^*) \) where \( r^*(\varphi(x)) := h(1) - h(x) - r(M) + r(x) \). Moreover, the nullity function of \( M^* \) is defined by \( \nu^*(\varphi(x)) = r(1) - r(x) \) for each \( x \in \mathcal{L} \).

Proposition 92.

1. \( \rho(M^*; x, y) = \rho(M; y, x) \).

2. Let \( M \) have a Tutte partition \( \mathcal{P} \). Then \( \tau(M^*; x, y) = \tau(M; y, x) \).
Proof. For the rank generating polynomial of \( M^* \), we have that:

\[
\rho(M^*; x, y) = \sum_{z \in \mathcal{L}} x^{r^\star(\varphi(z))} y^{r_{\mathcal{L}}(\varphi(z))} = \sum_{z \in \mathcal{L}} x^{h(z) - r(z)} y^{r(M) - r(z)} = \rho(M; y, x).
\]

We claim that \( \mathcal{P}^* \), the image of \( \mathcal{P} \) under \( \varphi \), is a Tutte partition of \( M^* \). For any interval \([a, b] \) supporting an element of \( \mathcal{P} \), \( r(a) = h(a) \) and \( r(b) = r(M) \). Therefore,

\[
\begin{align*}
  r^\star(\varphi(a)) &= h(1) - h(a) - r(M) + r(a) = h(1) - r(1) = \nu(1), \\
  r^\star(\varphi(b)) &= h(1) - h(b) - r(M) + r(b) = h(1) - h(b) = h(\varphi(b)).
\end{align*}
\]

It remains to check is that \([\varphi(b), \varphi(a)] \) is prime-free. Since the image of any diamond in \( \mathcal{L} \) is a diamond in \( \mathcal{L}^* \), we need only consider the duals of matroidal diamonds. Clearly, the only diamond whose dual is prime is a prime itself and so \([\varphi(b), \varphi(a)] \) is prime-free. It follows that \( \mathcal{P}^* \) is a Tutte partition. For each member \( P \in \mathcal{P} \), let \( \mathcal{L}^* \) denote its image under \( \varphi \) in \( \mathcal{P}^* \). Then \( \tau^\star_{\mathcal{P}^*}(P^*) = \nu(P) \) and \( \nu^\star_{\mathcal{P}^*}(P^*) = r_P(P) \) and so we have:

\[
\tau^\star_{\mathcal{P}^*}(M^*; x, y) = \sum_{P^* \in \mathcal{P}^*} x^{r^\star_{\mathcal{P}^*}(M^*(P^*))} y^{\nu^\star_{\mathcal{P}^*}(M^*(P^*))} = \sum_{P \in \mathcal{P}} x^{\nu_P(M(P))} y^{r_P(M(P))} = \tau_P(M; y, x).
\]

The result now follows by the invariance of \( \tau_P(M; x, y) \) under the choice of \( \mathcal{P} \).

Remark 93. In fact, Part 2 of Proposition 92 follows easily from Part 1, since \( \beta_\mathcal{P} * \rho(M^*; x, y) = \beta_\mathcal{P} * \rho(M; y, x) \). However, the proof above is independent of the rank generating polynomial.

8.2 Representations and evaluations

We consider now structural properties of \( M \) that can be extracted from its Tutte polynomial.

Lemma 94. Every Tutte partition of \( M \) has the same cardinality.

Proof. Let \( \mathcal{P} \) be a Tutte partition of \( M \). Then

\[
\tau(M; 1, 1) = \tau_P(M; 1, 1) = \sum_{P \in \mathcal{P}} 1^{r_P(M(P))} 1^{\nu_P(M(P))} = |\mathcal{P}|.
\]

An immediate corollary of Lemma 94 and Corollary 88, is that in the case that \( M \) is a matroid, \( \tau(M; 1, 1) \) counts the number of bases of \( M \). However, in the case that \( M \) is a \( q \)-matroid the evaluation \( \tau(M; 1, 1) \) does not return the number of bases. On the other hand, the evaluation of the rank generating function of \( M \) at \((0, 0) \) is given by \( \rho(M; 0, 0) = |\{a \in \mathcal{L} : r(a) = r(1), \nu(a) = 0\}| \), which is precisely the number of bases of \( M \), just as in the case that \( M \) is a matroid [5, Proposition 6.2.1]. Similarly, \( \rho(M; 1, 0) \) gives the number of independent elements of \( M \) in \( \mathcal{L} \).

Another way to represent the Tutte polynomial of \( M \) is as a matrix. If we gather together terms corresponding to prime-free minors of the same rank and nullity, we obtain:

\[
\tau(M; x, y) = \sum_{i=0}^{r(1) \nu(1)} \sum_{j=0}^{i} \tau_{i,j} x^{i} y^{j}.
\]

Clearly, for each \((i, j) \in [r(1) + 1] \times [\nu(1) + 1] \), we have \( \tau_{i,j} = |\{P \in \mathcal{P} : r_P(P) = i, \nu_P(P) = j\}| \) for any Tutte partition \( \mathcal{P} \); that is, the coefficients of \( \tau(M; x, y) \) count the number of prime-free minors of a Tutte partition of a given rank and nullity. Now consider the representation of these Tutte coefficients as an \((r(1) + 1) \times (\nu(1) + 1) \) matrix \( \tau = (\tau_{i,j}) \).

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**Definition 95.** Let $\mathbb{F}$ be a field and let $s,t$ be positive integers. We define a bilinear form $\langle \cdot, \cdot \rangle : \mathbb{F}^{s \times t} \times \mathbb{F}^{s \times t} \to \mathbb{F}$ by $\langle A, B \rangle := \text{Tr}(AB^T)$ for all $A, B \in \mathbb{F}^{s \times t}$.

It is straightforward to check that for any $A, B \in \mathbb{F}^{s \times t}$, we have $\langle A, B \rangle := \sum_{i=1}^{s} \sum_{j=1}^{t} A_{i,j} B_{i,j}$.

We’ll show that we can retrieve the number of bases of $M$ by an evaluation of the linear form $\tau := \langle \tau, \cdot \rangle$.

**Proposition 96.** Let $Q$ be the $(r(1) + 1) \times (\nu(1) + 1)$ matrix $Q = (q^{ij})$. Then the number of bases of $M$ is $\langle \tau, Q \rangle$.

**Proof.** Let $\mathcal{P}$ be a Tutte partition of $M$ and let $P \in \mathcal{P}$. Then $P$ has a unique totally clopen element $c$, and every complement of $c$ in $P$ is a basis in $P$. Moreover, every such basis is also a basis in $M$, since $r(P) = r(1)$. If $P$ has rank $i$ and nullity $j$, then the number of complements of $c$ in $P$ is $q^{ij}$. It follows that the total number of bases of $M$ is

$$
\sum_{i=0}^{r(1)} \sum_{j=0}^{\nu(1)} \tau_{i,j} q^{ij} \ (\text{by definition}).
$$

Note that in the case that $M$ is a matroid, if we let $q \to 1$ we retrieve that the number of bases of $M$ is $\langle \tau, J \rangle = \tau(M; 1, 1)$, where $J$ denotes the all-ones matrix.

Similarly, we have $\rho(M; x, y) = \sum_{a=0}^{r(1)} \sum_{b=0}^{\nu(1)} \rho_{a,b} x^a y^b$, where $\rho_{a,b}$ counts the number of elements of $\mathcal{L}$ of rank-lack $a$ and nullity $b$ in $M$, so we may represent the rank coefficients as the $(r(1) + 1) \times (\nu(1) + 1)$ matrix $\rho = (\rho_{a,b})$. If $E_{i,j}$ is the matrix with $(i, j)$-entry 1 and zeroes elsewhere, then we have that the number of bases of $M$ is:

$$
\langle \tau, Q \rangle = \langle \beta_q * \rho, Q \rangle = \langle \alpha_q * \tau, E_{0,0} \rangle = \langle \rho, E_{0,0} \rangle.
$$

Proposition 92 now says that the Tutte and rank functions of $M^*$ are obtained by matrix transposition $\tau^T$ and $\rho^T$ of the Tutte and rank functions of $M$, respectively.

### 9 Towards a combinatorial characterisation of the Tutte polynomial for all $q$-matroids

Theorem 61 gives a characterisation of a Tutte polynomial as a certain convolution of the rank generating polynomial of any $q$-matroid. However, Theorem 51 yields a combinatorial characterisation of the Tutte polynomial only for $(q)$-matroids whose support lattices admit a proper interval decomposition. In this section, we outline a limitation of Definition 33, but we show a possible generalisation.

It is shown in [21] that there does not exist a proper interval decomposition of $\mathcal{L}(\mathbb{F}_q^n)$ for $n \geq 5$. Since our proof of the existence of a Tutte partition used proper interval decompositions, we cannot thus conclude that a Tutte partition always exists for $q$-matroids on such lattices. In Example 99 we give a $q$-matroid that does not admit a Tutte partition. For this example, we need the following definition of the direct sum of $q$-matroids. Assume that all vector spaces in Definition 97 and Notation 98 are over the same field.

**Definition 97 (11).** Let $M_1 = (\mathcal{L}(V_1), r_1)$ and $M_2 = (\mathcal{L}(V_2), r_2)$ be $q$-matroids such that $V_1$ has dimension $n_1$ and $V_2$ has dimension $n_2$. Let $V$ be an $(n_1 + n_2)$-dimensional vector space
and let $V_1$ and $V_2$ be identified with their embeddings in $V$ such that $V = V_1 \oplus V_2$. The direct sum of $M_1$ and $M_2$, denoted $M_1 \oplus M_2$, is the $q$-matroid $(\mathcal{L}(V), r)$ such that
\[
  r(A) = \min_{X \subseteq A} \{ r_1(\pi_1(X)) + r_2(\pi_2(X)) + \dim(A) - \dim(X) \}
\]
where $\pi_i : V \to V_i$ is the canonical projection for $i = 1, 2$.

We will also use the following convenient notation.

**Notation 98.** Let $V_1, V_2, \pi_1, \pi_2$ and $V$ be as in Definition 97. Let $X_1 \subseteq V_1$ and $X_2 \subseteq V_2$. We write $X_1 \oplus X_2$ to denote the subspace of $V$ such that $\pi_i(X_1 \oplus X_2) = X_i$ for $i = 1, 2$. We use the notation $\mathcal{U}_{k,n}(q)$ to denote the uniform $q$-matroid $(\mathcal{L}(\mathbb{F}_q^n), r)$, which means that $r(A) = \dim(A)$ if $\dim(A) \leq k$ and has value $k$ otherwise. In particular, $r(1) = k$ in $\mathcal{U}_{k,n}(q)$.

**Example 99.** Let $\mathcal{U}_{1,2}(2)$ and $\mathcal{U}_{0,3}(2)$ be defined on the spaces $\langle e_1, e_2 \rangle$ and $\langle e_3, e_4, e_5 \rangle$ respectively. Let $M = (\mathcal{L}, r) := \mathcal{U}_{1,2}(2) \oplus \mathcal{U}_{0,3}(2)$. Suppose that $M$ admits a Tutte partition $\mathcal{P}$. Since all loops must be contained in an interval in a Tutte partition, we must have that $[0, 0 \oplus \langle e_3, e_4, e_5 \rangle] \subseteq I$ for some $I \in \mathcal{P}$ since $r(0 \oplus \langle e_3, e_4, e_5 \rangle) = 0$. Since the maximal element of any interval in a Tutte partition must be spanning and since $r(\langle x \rangle \oplus \langle e_3, e_4, e_5 \rangle) = 1 = r(1)$ for any $x \leq \langle e_1, e_2 \rangle$, we may assume, without loss of generality, that $[0, \langle e_2 \rangle \oplus \langle e_3, e_4, e_5 \rangle] \subseteq I$. Since $M$ is not prime-free and $\langle e_2 \rangle \oplus \langle e_3, e_4, e_5 \rangle$ is a coatom, we deduce that $[0, \langle e_2 \rangle \oplus \langle e_3, e_4, e_5 \rangle] = I$. Since $r(1) = 1$ and all of the loops of $M$ are contained in $I$, we have that the chain $C(a, 1)$ is a clopen chain for any atom $a \in \mathcal{L}$ not in $I$. We can thus say, without loss of generality, that $J := [\langle e_1 \rangle \oplus 0, 1]$ is in $\mathcal{P}$. By the same reasoning, the chain $C(a', c)$ is clopen for any atom $a' \in \mathcal{L}$ and coatom $c \in \mathcal{L}$ not in $I$ or $J$ (with $a' \leq c$). Therefore, all other intervals in $\mathcal{P}$ must have length three and be isomorphic to $\mathcal{U}_{0,3}(2)$. We thus conclude that $\mathcal{P}$ is a proper interval decomposition of the subspace lattice of $\mathbb{F}_2^n$, which contradicts [21, Theorem 3]. Therefore, $M$ does not admit a Tutte partition.

Example 99 shows us that we indeed need a more general definition of the Tutte polynomial in order for it to apply to all $q$-matroids. We conjecture that the condition of the Tutte partition being a *partition* is too strong. We give a proposed remedy for Example 99 in the following example, which may lead to the desired definition of the Tutte polynomial.

**Example 100.** Let $M$ be the $q$-matroid given in Example 99. Computing the rank generating polynomial of $M$ we get $\rho(M; x, y) = 24 + x + 7xy + 7xy^2 + xy^3 + 148y + 154y^2 + 31y^3 + y^4$. After we apply the convolution $\beta$ to $\rho(M; x, y)$ we get $\beta \ast \rho(M; x, y) = xy^3 + y^4 + 15y^3$. It is possible, using Hall’s theorem, to get a matching between the atoms and coatoms of $\mathcal{L}$ that are not contained in the intervals $I$ and $J$, which gives us a set of 15 intervals, all of which are isomorphic as minors of $M$ to $\mathcal{U}_{0,3}(2)$. By [21, Theorem 3], there must be some non-empty intersections between these intervals. Nonetheless, we clearly get a polynomial that coincides with $\beta \ast \rho(M; x, y)$ when we take the sum of the Tutte polynomials of these intervals with those of $I$ and $J$.

We have not in Example 100 proposed a suggestion of a formal definition of the Tutte polynomial. However, this example suggests that it may be possible to successfully generalise the notion of a Tutte partition, which would give a combinatorial definition of the Tutte polynomial for arbitrary $q$-matroids.

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