Hidden Killing Fields, Geometric Symmetries and Black Hole Mergers

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Abstract

In the present work, using the recently introduced framework of local geometric deformations, special types of vector fields - so-called hidden Killing vector fields - are constructed, which solve the Killing equation not globally, but only locally, i.e. in local subregions of spacetime. Taking advantage of the fact that the vector fields coincide locally with Killing fields and therefore allow the consideration of integral laws that convert into exact physical conservation laws on local scales, balance laws in dynamical systems without global Killing symmetries are derived that mimic as closely as possible the conservation laws for energy and angular momentum of highly symmetric models. The utility of said balance laws is demonstrated by a concrete geometric example, namely a toy model for the binary merger of two extremal Reissner-Nordström black holes.

Key words: hidden Killing vectors, phantom symmetries, conservation laws

Introduction

Due to a lack of continuous symmetry, the Killing equation cannot be solved in generic geometric settings. Consequently, there are no Killing vector fields in generic spacetimes.

As a result, it turns out that said spacetimes are not endowed with the geometric properties required for the formulation of physical conservation laws for quantities such as energy, momentum and angular momentum.

However, this does not mean that without Killing vectors (KV) it would be impossible to define globally conserved currents in generic spacetimes. On the contrary, even in geometric models without Killing symmetries, there is always an infinite number of such currents known as Komar currents [25], which can be constructed from any given vector field regardless of the concrete geometric structure of spacetime. Furthermore, in special classes of spacetimes that lack not all, but only specific Killing symmetries, there are special types of vector...
fields, such as e.g. Kodama vector fields \[27\] in the case of spherically symmetric spacetimes, which lead to exact conserved quantities and associated exact integral conservation laws.

In the literature, vector fields with the property that they permit the definition of well-defined conservation laws in generic spacetimes (same as the Kodama vector fields mentioned above) are often assigned to the so-called class of generalized Killing vector fields (GKV). The main property of the representatives of this class - of which the most well-known are probably semi-Killing and almost Killing vector fields as well as affine and curvature collineations \[3, 5, 6, 9, 11, 12, 13, 26, 28, 29, 33, 35, 38, 39\] - is that the vector fields in question are, in contrast to traditional KV, not solutions of the Killing equation. Instead, said vector fields represent solutions of less restrictive types of equations, which can be solved under more general geometric circumstances, but which still allow the construction of quantities with similar geometric properties as those available in spacetime with isometries.

Common to all these types of GKV (including Kodama vectors) and their associated conserved currents is that they are defined globally, i.e. with respect to the global symmetry properties of spacetime. Therefore, they prove to be parallel to - or exactly identical with - KV only if the geometry of spacetime has global Killing symmetries.

As a consequence, however, the question arises whether there are generalized KV in generic spacetimes that do not coincide globally but only locally (i.e. in local subregions of spacetime) with solutions of the Killing equation. This question is related to a concrete physical idea, namely to the idea that there should exist classes of spacetimes whose geometries change with time in such a way that their symmetry properties change as well. To understand this, one may recall that it is eventually to be expected that stationary physical systems (which are subject to external disturbances) can become unstable over time or that certain physical processes, such as gravitational collapse, may cause an initially spherically symmetric geometry to lose some of its symmetry properties and change e.g. to an axially symmetric geometry for a certain time. In such a case, the local geometric structure of spacetime would allow for the existence of corresponding KV, whilst its global structure would not allow the same.

To account for this particular aspect of the theory and to show that there are indeed classes of spacetimes that allow for the existence of special types of GKV that coincide with exact KV on local scales, to be referred to as hidden Killing vectors (HKV) from now on, the recently introduced framework of local geometric deformations \[19\] is used in the present work.

This geometric framework is well suited to prove the existence of HKV for several reasons, the most important of which is certainly that it allows the introduction of the concept of a so-called local spacetime (i.e. a spacetime whose local geometric structure differs from that of an associated ambient spacetime). By introducing this particular type of spacetime, the geometric framework mentioned provides a basis for the analysis of the local geometric structure of a given spacetime with respect to its global structure, making it possible (at least in principle) to model complicated geometric transitions of local spacetimes whose
symmetry properties change with time.

The main idea to model such geometric transitions, as explained in the first section of this work, is to deform background spacetimes with certain Killing symmetries in such a way that they lose their symmetry properties, yet using deformation fields with compact supports that vanish in local subregions of spacetime. In this manner, it is ensured by construction that spacetime exhibits HKV. But since spacetime still does not allow for actual Killing symmetries, but only geometric symmetries that correspond to Killing symmetries on local scales, the term **phantom symmetries** is used to indicate that one is not dealing with actual symmetries, but rather with geometric (apparent) symmetries that are confined to a Lorentzian submanifold of spacetime.

As shown in the rear part of the first section of this work, the existence of spacetimes that allow HKV and associated phantom symmetries has an interesting consequence, namely that certain integral laws of the theory reduce to exact conservation laws, resulting in exact expressions for conserved charges for energy and angular momentum on local scales. By means of this insight, physical balance laws for energy and angular momentum are derived in this work, which describe how exactly these locally conserved quantities change over time in the course of the development of a superordinate generic geometric model. The derived balance laws have the interesting property that they are very closely related to the conservation laws of known symmetric models and coincide with these laws except for correction terms resulting from the geometric deformation of the symmetric background geometry, thereby allowing the generalization of the mentioned laws to more general geometric settings.

To demonstrate the utility of said laws, a special geometric model is treated in section two of this work, namely a toy model for a black hole merger of two extremal Reissner-Nordström black holes. To provide an exact characterization of the geometric structure of a merger of this type, a non-stationary axisymmetric geometric model is considered which coincides by construction locally with the (two-body) Majumdar-Papapetrou solution at early times and the Reissner-Nordström solution at late times. As is demonstrated, the geometric structure of the merger field is consistent with Hawking’s area theorem and, moreover, allows the definition of Komar integrals at infinity, which may be used as a starting point for the formulation of the laws of black hole mechanics [2]. A key aspect, in this respect, turns out to be that the (timelike) HKV of the geometry coincides with the (timelike) KV of Majumdar-Papapetrou spacetime at early times and that of Reissner-Nordström spacetime at late times. After demonstrating that the merger geometry admits a HKV with these properties, a local Killing charge is defined and an associated energy balance law is derived, which extends the 'standard' energy conservation laws of known symmetric black hole models. A discussion of all the results obtained forms the conclusion of this treatise.
1 Local Spacetimes, Hidden Killing Fields and Phantom Symmetries

The main objective of this section is to introduce the concept of a HKV and the closely related concept of a phantom symmetry and to show that these concepts can be used to derive balance laws for energy and/or angular momentum in generic non-stationary spacetimes. For the purpose of introducing these concepts, a pair of spacetimes \((M, g)\) and \((\tilde{M}, \tilde{g})\) with Lorentzian manifolds \(M \subseteq \tilde{M}\) shall be considered, whose metrics are subject to the conditions

\[
\tilde{g}_{ab}|_M \equiv g_{ab}, \quad \tilde{g}^{ab}|_M \equiv \tilde{g}^{ab}.
\]

A specific way to meet these conditions, as recently pointed out in [19], is the use of the geometric framework of local metric deformations. This framework deals with metric deformation relations of the form

\[
\tilde{g}_{ab} = g_{ab} + e_{ab},
\]

and

\[
\tilde{g}^{ab} = g^{ab} + f^{ab},
\]

where the deformation tensor fields \(e_{ab}\) and \(f^{ab}\) are assumed to be smooth tensor fields of compact support. In particular, \(\text{supp} e_{ab}, \text{supp} f^{ab} \equiv \tilde{M}\setminus M\) shall apply in this context, where a consistent choice is e.g. \(e_{ab} = \chi e_{ab}\) with \(\chi(x)\) being either an indicator function such as e.g. the Heaviside step function or a smooth non-analytic transition function of the form

\[
\chi(x, x_0) = \frac{\psi \left( \frac{x}{x_0} \right)}{\psi \left( \frac{x}{x_0} \right) - \psi \left( 1 - \frac{x}{x_0} \right)},
\]

which is constructed from a cut-off function

\[
\psi(x) := \begin{cases} 
-\frac{1}{x} & x > 0 \\
0 & x \leq 0
\end{cases}
\]

meeting the conditions \(0 \leq \psi \leq 1\) and \(\psi(x) > 0\) if and only if \(x > 0\). Note that the point \(x_0\) is introduced merely to ensure that the exponent in (5) is dimensionless. Of course, a splitting like this can neither be unique nor coordinate-independent. In fact, there are many different ways to realize splittings of the form (2) and (3). However, assuming for simplicity that \(g_{ab}\) and \(\tilde{g}_{ab}\) are given in the same coordinates, the splitting can be unambiguously performed.

For consistency reasons, the metric \(\tilde{g}_{ab}\) and and its inverse \(\tilde{g}^{ab}\) must fulfill the relations

\[
\tilde{g}_{ab} \tilde{g}^{bc} = \delta^c_a,
\]

which can be re-written in the form

\[
e_a^b + f_a^b + e_a^c f_c^b = 0,
\]
where \( e_a^b = g^{bc}e_{ac} \) and \( f_a^b = g_{ac}f^{bc} \).

Due to the fact that the deformation fields are chosen to be of compact support, it turns out - in the event that consistency conditions (7) are met - that the local expressions

\[
g_{ab} = \tilde{g}_{ab} - e_{ab}, \quad g^{ab} = \tilde{g}^{ab} - f^{ab}
\]

are well-defined tensor fields on \( \tilde{M} \), which coincide locally (i.e. in \( M \)) with the metric \( \tilde{g}_{ab} \) and the inverse metric \( \tilde{g}^{ab} \) of spacetime. For that reason mainly, the pair \( (M, g) \) should not be viewed as a 'proper' spacetime, but rather as a local spacetime, or, more precisely, as a spacetime whose associated Lorentzian manifold \( M \) is contained (and therefore localized) in another Lorentzian manifold \( \tilde{M} \) of the ambient spacetime \( (\tilde{M}, \tilde{g}) \). Ultimately, the use of the term local can be justified in this context by the fact that \( (M, g) \) and \( (\tilde{M}, \tilde{g}) \) are spacetimes, whereas \( (\tilde{M}, g) \), in contrast, is not.

Using decomposition relations (2) and (3), one can calculate the difference tensor

\[
C_a^{\, bc} = \frac{1}{2} \tilde{g}^{ad}(\nabla_b \tilde{g}_{dc} + \nabla_c \tilde{g}_{bd} - \nabla_d \tilde{g}_{bc}) =
\]

\[
= \frac{1}{2}(g^{ad} + f^{ad})(\nabla_b \epsilon_{dc} + \nabla_c \epsilon_{bd} - \nabla_d \epsilon_{bc})
\]

which can be obtained by taking the difference of the Levi-Civita connections of \( (M, g) \) and \( (\tilde{M}, \tilde{g}) \), meaning that \( C_a^{\, bc} = \tilde{\Gamma}_a^{\, bc} - \Gamma_a^{\, bc} \). This difference tensor can be used to decompose the Riemann tensor of \( (\tilde{M}, \tilde{g}) \) in the following way

\[
\tilde{R}^a_{\, bcd} = R^a_{\, bcd} + E^a_{\, bcd},
\]

where \( E^a_{\, bcd} = 2\nabla_c C^a_{\, d|b} + 2C^a_{\, c|d} C^c_{\, d|b} \) holds by definition. By contracting indices, the decomposition

\[
\tilde{R}_{bd} = R_{bd} + E_{bd}
\]

Ricci tensor is obtained, where, of course, \( E_{bd} = \delta^a_c E^a_{\, bcd} = 2\nabla_b C^a_{\, d|c} + 2C^a_{\, c|d} C^c_{\, d|b} \) must apply. By repeating this procedure, the corresponding relation

\[
\tilde{R} = R + g^{bd} E_{bd} + f^{bd} R_{bd} + f^{bd} E_{bd}
\]

for the Ricci scalar is obtained, which, however, allows one to decompose Einstein’s field equations

\[
\tilde{G}_{ab} = 8\pi \tilde{T}_{ab}
\]

in the form

\[
G_{ab} + \rho_{ab} = 8\pi \tilde{T}_{ab}.
\]

Using now the fact that the local field equations

\[
G_{ab} = 8\pi T_{ab}
\]
are met in \((M, g)\), the \textit{deformed Einstein equations}
\[
\rho_{ab} = 8\pi \tau_{ab} 
\]
(16)
can be set up, provided that \(\tau_{ab} = \tilde{T}_{ab} - T_{ab}\).

Given this geometric setting, a HKV is given when a continuous symmetry of \(g_{ab}\) is generated by a Killing vector \(\zeta^a\) (throughout \(M\)), which is not a (global) continuous symmetry to \(\tilde{g}_{ab}\), so that
\[
L_{\zeta} g_{ab} = 2\nabla_{(a} \zeta_{b)} = 0, \quad L_{\zeta} \tilde{g}_{ab} = 2\tilde{\nabla}_{(a} \zeta_{b)} \neq 0
\]
(17) applies due to the fact that \(L_{\zeta} e_{ab} \neq 0\), where \(L_{\zeta}\) is the Lie derivative along \(\zeta^a\). More specifically, a spacetime \((\tilde{M}, \tilde{g})\) has a HKV with respect to \((M, g)\) if there exist local tensor fields \(e_{ab}\) and \(f^{ab}\) such that conditions (1) and (17) are met and therefore a vector field \(\zeta^a\) which has the Killing property only with respect to the \((M, g)\), but not with respect to \((\tilde{M}, \tilde{g})\).

The present definition of a HKV is closely related to the definition of a GKV\(^2\) given in \([14, 15]\). Given in relation to a timelike worldline \(Z\) with unit tangent \(w^a\), a GKV in the mentioned works is defined by the relations
\[
L_{\zeta} g_{ab}|_Z = \nabla_{c} L_{\zeta} g_{ab}|_Z = 0
\]
(18) and proven to be constructible by using Jacobi vector fields, i.e. solutions of the geodesic deviation equation
\[
\ddot{\zeta}^a = \tilde{R}^a_{\ bcd} w^b \zeta^c \zeta^d
\]
(19) with the property that
\[
L_w \zeta^a = 0,
\]
(20) where the overdot is short hand notation for the total derivative \((w \tilde{\nabla}) = w^a \tilde{\nabla}_a\).

An important feature of these GKV is the following: If \((\tilde{M}, \tilde{g})\) has Killing symmetries, it turns out that every GKV of the above form is always an ordinary KV (but, of course, not vice versa).

Although not really obvious at first glance, the definition of a HKV given above is quite similar to that of a GKV, as the geometric setting discussed requires that
\[
L_{\zeta} \tilde{g}_{ab}|_M = \tilde{\nabla}_{c} L_{\zeta} \tilde{g}_{ab}|_M = 0
\]
(21) applies. The main difference between the two definitions is therefore that both the Lie derivative of the metric and its covariant derivative must be zero throughout \(M\) and not exclusively along a single timelike worldline \(Z\). However, since

\(^1\)Regarding this definition, it is important to keep in mind that a HKV and an associated hidden symmetry for \(g_{ab}\) is always given relative to a specific choice for the ambient metric \(\tilde{g}_{ab}\). However, this choice of the ambient metric \(\tilde{g}_{ab}\) is generally not unique, since a multitude of possible ambient spacetimes \((\tilde{M}, \tilde{g})\) exist for a local spacetime \((M, g)\).

\(^2\)Note that various types of definitions of GKV have been given in the literature over the years and that a specific one is considered at this stage.
\( \zeta^a \) must be a KV by construction along any curve (or curve segment) in \( M \subseteq \tilde{M} \), it is, of course, also a GKV.

Anyway, due to the imposed conditions on the geometric structure of \( e_{ab} \) and \( f^{ab} \), it becomes clear that there is a locally conserved current \( j^a = -G^a_{b\zeta^b} \) being subject to the conservation law

\[
\nabla_a j^a = 0, \tag{22}
\]

which, however, holds in this form only in \( M \), i.e. the submanifold of \( \tilde{M} \) in which spacetime geometry becomes symmetric.

As a direct result, considering a submanifold \( D \subset M \) with boundary \( \partial D \), which is contained in a larger submanifold \( \tilde{D} \subset \tilde{M} \) with boundary \( \partial \tilde{D} \) such that \( D \subseteq \tilde{D} \), the Gaussian theorem can be used when integrating relation (22) to obtain

\[
\int_D \nabla_a j^a \, dv = \int_{\partial D} j^a \, d\Sigma_a, \tag{23}
\]

where \( dv \equiv \sqrt{-g} d^4x \) and \( d\Sigma_a = n_a d\sigma \) with \( d\sigma \equiv \sqrt{h} d^3x \) are the corresponding four-volume and induced hypersurface elements, respectively.

On the other hand, by defining the further current \( \tilde{j}^a = -\tilde{G}^a_{b\zeta^b} \) and integrating

\[
\nabla_a \tilde{j}^a = \tilde{G}^a_{b\zeta^b}, \tag{24}
\]

one obtains the result

\[
\int_{\tilde{D}} \nabla_a \tilde{j}^a \, d\tilde{v} = \int_{\tilde{D}} \tilde{G}^a_{b\zeta^b} \, d\tilde{v}, \tag{25}
\]

which holds throughout \( \tilde{D} \) and reduces to the form (23) in \( D \). Consequently, however, the LHS of this relation can be re-written in the form

\[
\int_{\tilde{D}} \nabla_a \tilde{j}^a \, d\tilde{v} = \int_D j^a \, dv + \int_{\tilde{D}\setminus D} \tilde{j}^a \, d\tilde{v}, \tag{26}
\]

and be converted to

\[
\int \partial D j^a d\Sigma_a = \int \partial D j^a d\Sigma_a + \int \partial (\tilde{D}\setminus D) \tilde{j}^a d\tilde{\Sigma}_a, \tag{27}
\]

after repeated application of the Gaussian theorem.

These results, although they appear not to be particularly special or interesting at first sight, have nice applications under special geometric circumstances, i.e. when the ambient spacetime \((\tilde{M}, \tilde{g})\) is a sandwich spacetime with stationary and axisymmetric initial and final geometries \((M, g)\) and \((M', g')\) and a dynamical transition spacetime \((O, \dot{g})\) connecting both of these local spacetimes such
that the manifold structure is \( \tilde{M} = M \cup O \cup M' \): They allow, as shall be demonstrated in the following, the derivation of balance relations that generalize the conservation laws for energy and angular momentum holding with respect to the symmetric parts of spacetime, that is, with respect to the initial and final geometric configurations \((M, g)\) and \((M', g')\) of the generic spacetime \((\tilde{M}, \tilde{g})\).

To prove this assertion, it shall therefore now be assumed that the Lorentzian manifold of \((\tilde{M}, \tilde{g})\) is of the form \(\tilde{M} = M \cup O \cup M'\) and that the metric of spacetime \(\tilde{g}_{ab}\) and its inverse \(\tilde{g}^{ab}\) locally meet the conditions

\[
\tilde{g}_{ab}|_M \equiv g_{ab}, \quad \tilde{g}^{ab}|_M \equiv g^{ab}
\]

and

\[
\tilde{g}_{ab}|_{M'} \equiv g'_{ab}, \quad \tilde{g}^{ab}|_{M'} \equiv g'^{ab},
\]

where \(g_{ab}\) and \(g'_{ab}\) are the metrics of two stationary and axisymmetric local spacetimes \((M, g)\) and \((M', g')\). The listed conditions can be met by using the same methods as above, namely by considering the metric deformation relations

\[
\tilde{g}_{ab} = g_{ab} + e_{ab} = g'_a + e'_a
\]

and

\[
\tilde{g}^{ab} = g^{ab} + f^{ab} = g'^{ab} + f'^{ab},
\]

and by requiring that \(\text{supp } e_{ab}, \text{supp } f^{ab} \equiv \tilde{M} \setminus M\) and \(\text{supp } e'_a, \text{supp } f'^{ab} \equiv \tilde{M} \setminus M'\); geometric conditions that can be met by making the choices \(e_{ab} = \chi e_{ab}\) and \(e'_a = \chi e'_a\) for the corresponding deformation fields in (30), where \(\chi = \chi(x)\) is a transition function of the form (4).

Based on the fact that these specific geometric conditions are met, it is guaranteed that the generic spacetime \((\tilde{M}, \tilde{g})\) allows for the existence of two HKV \(\zeta^a\) and \(\zeta'^a\) such that

\[
L_{\zeta^a} g_{ab} = 2\nabla_{(a} \zeta_{b)} = 0 = 2\nabla'_{(a} \zeta'_{b)} = L_{\zeta^a} g'_a, \quad L_{\zeta^a} \tilde{g}_{ab} \neq 0.
\]

Here, as shall be demonstrated in the next section by a concrete geometric example, it may very well occur that the different HKV \(\zeta^a\) and \(\zeta'^a\) coincide (not in all, but in certain cases of interest), so that \(\zeta^a\) turns out to be the only HKV of the geometry. But this is a special and not the general case, which is why, to continue the main line of argument, two different currents \(j^a = -\tilde{G}^a_b \zeta^b\) and \(j'^a = -\tilde{G}'^a_b \zeta'^b\) shall be defined, which are conserved currents in \(M\) and in \(M'\), so that

\[
\nabla_a j^a = 0
\]

applies in \(M\) and

\[
\nabla'_a j'^a = 0
\]

in \(M'\). Considering then - similar to the above - two submanifolds \(D \subset M\) and \(D' \subset M'\) with boundaries \(\partial D\) and \(\partial D'\), which are contained in an associated
Figure 1: In graphic a) to the left the geometric structure of the ambient spacetime \((\tilde{M}, \tilde{g})\) is depicted. To obtain a sandwich spacetime \((\tilde{M}, \tilde{g})\) of this kind, it is assumed that the 'four-geometric' initial data \((M, g)\) and \((M', g')\) are stationary and axisymmetric local spacetimes in the sense of [19], i.e. spacetimes with independent geometric structures and individual symmetry properties, which do not necessarily have to coincide with those of the ambient spacetime \((\tilde{M}, \tilde{g})\).

The graphic b) to the right shows how a submanifold \(\tilde{D}\) decomposes into three parts based on the decomposition \(\tilde{M} = M \cup \mathcal{O} \cup \tilde{M}\), where \(\mathcal{D}\) lies in \(M\) and \(\mathcal{D}'\) lies in \(M'\).

ambient submanifold \(\tilde{D} \subset \tilde{M}\) with boundary \(\partial\tilde{D}\) such that \(\mathcal{D}, \mathcal{D}' \subseteq \tilde{D}\), one finds by repeating the formal steps described above

\[
\int_{\partial\tilde{D}} \tilde{j}^a d\tilde{\Sigma}_a = \int_{\partial\mathcal{D}} j^a d\Sigma_a + \int_{\partial(\tilde{D} \setminus \mathcal{D})} \tilde{j}^a d\tilde{\Sigma}_a = \int_{\partial\mathcal{D}'} j'^a d\Sigma'_a + \int_{\partial(\tilde{D} \setminus \mathcal{D}') \mathcal{D}'} \tilde{j}^a d\tilde{\Sigma}_a. \tag{35}
\]

This relation may be rewritten more compactly in the form

\[
\tilde{Q} = Q + Q' = Q' + Q', \tag{36}
\]

where \(\tilde{Q} = \int_{\partial\tilde{D}} \tilde{j}^a d\tilde{\Sigma}_a\), \(Q = \int_{\partial\mathcal{D}} j^a d\Sigma_a\), \(Q' = \int_{\partial\mathcal{D}'} j'^a d\Sigma'_a\) and \(Q = \int_{\partial(\tilde{D} \setminus \mathcal{D})} \tilde{j}^a d\tilde{\Sigma}_a\), \(Q' = \int_{\partial(\tilde{D} \setminus \mathcal{D}') \mathcal{D}'} \tilde{j}^a d\tilde{\Sigma}_a\). Thus, provided that the definition \(\mathcal{G} := Q - Q'\) is now used in the present context, one obtains the balance law.
\[ Q' = Q + \mathcal{G} \]  \hfill (37)

from these two decompositions of one and the same quantity \( \tilde{Q} \).

Consequently, considering now the case where spacetime admits only one HKV for both local spacetimes \((M, g)\) and \((M', g')\), there is an interesting conclusion that can be drawn from this derivation of balance law (37): Since \( \zeta^a \equiv \zeta'^a \) may be chosen both in \( M \) and in \( M' \) to be a linear combination of a timelike and an axial Killing vector field such that \( Q \) and \( Q' \) are actual Killing charges associated with the stationary, axisymmetric local spacetimes \((M, g)\) and \((M', g')\) that are constant in time, it can be concluded that also the derived quantity \( \mathcal{G} \), which measures how strongly both types of charges differ from each other, must be constant in time. Formally expressed this means that - on account of the fact that \( L_\zeta Q = L_\zeta Q' = 0 \) applies in the given case - the conclusion can be drawn that \( L_\zeta \mathcal{G} = 0 \) must apply as well as a direct consequence.

As a result, however, despite being formulated in a generic geometric setting, it is found that the balance law (37) closely resembles an actual integral conservation law for energy and/or angular momentum charges of highly symmetric geometric models. This is not least because \( Q \) and \( Q' \) are genuine conserved quantities in \( M \) and \( M' \), respectively, which exist due to the fact that the local spacetimes \((M, g)\) and \((M', g')\) have Killing symmetries, in stark contrast to the ambient spacetime \((\tilde{M}, \tilde{g})\), which has no such symmetries at all. However, since the ambient spacetime \((\tilde{M}, \tilde{g})\) has no exact symmetries whatsoever, relation (37) cannot be an exact physical conservation law, but rather has to be considered a physical balance law, which relates conserved quantities existing before and after the transition from one geometric configuration of spacetime to another takes place.

In view of these results, it becomes clear that any spacetime \((\tilde{M}, \tilde{g})\) with suitable four-geometric initial data, i.e. with stationary and axisymmetric initial and final geometric configurations \((M, g)\) and \((M', g')\), which exhibits one and the same HKV for both of these local geometric configurations, allows for the definition of an apparent conservation law of the form (37). Since the occurrence of an exact conservation law is always accompanied by the occurrence of a corresponding exact symmetry, in the following, whenever a sandwich spacetime with the above properties is given, it shall be said to have a *phantom symmetry* to indicate that spacetime in such a case has no actual, but only apparent geometric symmetries, which arise as a result of imposing special conditions on the geometry of spacetime at early and late times.

Having said that, the next point to be stressed is that there are different types of conserved currents \( \tilde{j}^a \) that can be considered in \((\tilde{M}, \tilde{g})\). Some of these currents are not only locally but globally conserved (in contrast to the types of currents considered above). For the purposes of the present work, however, only one type of conserved current will play a role, namely the Komar current

\[ \tilde{j}^a = \tilde{\nabla}_b \tilde{\nabla}^{[b} \zeta^{a]}, \]  \hfill (38)

which can be constructed on any given spacetime \((\tilde{M}, \tilde{g})\) for any given choice
for the vector field $\zeta^a$. Given a generic spacetime $(\bar{M}, \bar{g})$ with a geometry that allows for the existence of a phantom symmetry, that is, with stationary and axisymmetric initial and final geometric configurations $(M, g)$ and $(M', \bar{g}')$ and a single HKV for both of these local configurations, one may simply select $\zeta^a$ to construct such a Komar current of the form (38). Considering then a submanifold $\bar{C} \subset \bar{M}$ with boundary $\partial \bar{C}$, which in turn is bounded (typically at infinity) by a two-surface $\partial \bar{C}$, the fact that $\bar{\nabla}^{[b} \zeta^{a]}$ is antisymmetric can be used to deduce from (38) the conservation law

$$\int_{\bar{C}} \bar{j}^{a} d\bar{\Sigma}_a = 2 \int_{\partial \bar{C}} \bar{\nabla}^{[a} \zeta^{b]} d\bar{\Sigma}_{ab},$$

where $d\bar{\Sigma}_{ab}$ and $d\bar{\Sigma}_a$ are the surface and hypersurface elements of $\partial \bar{C}$ and $\bar{C}$, respectively. Considering two further submanifolds $C \subset M$ and $C' \subset M'$ with boundaries $\partial C$ and $\partial C'$, which are contained in $\bar{C}$ such that $C, C' \subseteq \bar{C}$, the RHS of (39) reads $\int_{\bar{C}} \bar{j}^{a} d\bar{\Sigma}_a = -\int_{\bar{C}} R^{ab}_{\ z} \zeta^b d\bar{\Sigma}_a$ in $M$ and $\int_{\bar{C}'} \bar{j}^{a} d\bar{\Sigma}_a' = -\int_{\bar{C}'} R^{ab}_{\ z} \zeta^b d\bar{\Sigma}_a'$. Based on this input, a balance law of the form (37) can also be derived in the given case, whereas, of course, the resulting charges are Komar charges in this particular case. In particular, this allows the definition of the Komar expressions for energy and angular momentum and the conclusion that those are conserved in the local regions where spacetime becomes symmetric. Consequently, however, the considered spacetime model is such that locally conservation of energy and angular momentum holds, that is, specifically at early and late times, if it has a hidden (phantom) symmetry.

Now that this all is clarified, the natural question arises what new information can be obtained about the local metrics $g_{ab}$ and $\bar{g}_{ab}$ and their associated ambient metric $\bar{g}_{ab}$ now that it is known that spacetime has a HKV and an associated hidden symmetry?

To answer this question, a brief detour shall be made and the theory of extended irreversible thermodynamics [22, 23, 24, 30] shall be taken into account. In the mentioned theory, which typically deals with the physical behavior of fluid mixtures near thermal equilibrium, three different quantities play a central role, namely the so-called thermal energy-momentum tensor $T^a_b = \epsilon w^a w_b + \omega^a w_b + w^a \omega_b + (p + \Pi) h^a_b + \Pi^a_b$, the so-called particle number current $N^a_A = n_A w^a + \nu^a_A$, defined with respect to a number $A$ of species of particles, and the so-called covariant entropy current $S^a = sw^a + \eta^a$. In this context, the scalar fields $\epsilon, p$ and $s$ represent the energy density, pressure, and total entropy density of the fluid mixture, $n_A$ is the particle density of a number of particle species $A$ and $\pi$ represents the viscous bulk pressure. The vector fields $w^a, \omega^a, \nu^a_A$ and $\eta^a$, on the other hand, represent the four-velocity vector of the system, an energy flux current and so-called particle diffusion and heat conduction fluxes, while the tensor field $\pi^a_A$ represents the trace-free anisotropic viscous stress-tensor. As may be noted, there are different ways to make an ansatz for the vector fields $\omega^a, \nu^a_A$ and $\eta^a$, depending on whether one wants to specify them in the so-called energy or particle reference frames. With re-
spect to one of these choices, the physical behavior of the fluid mixture can be specified the following differential laws

\[ \nabla_a T^a_b = 0, \quad \nabla_a N_A^a = 0, \quad \nabla_a S^a \geq 0. \]  

(40)

The respective laws are the laws of local conservation of energy and particle number and the so-called Clausius-Duhem inequality, which is the differential form of the entropy law of thermodynamics.

The extent to which all this matters for the question raised above of what can be learned about the local and global metrics \( g_{ab}, g'_{ab} \) and \( \tilde{g}_{ab} \) by means of hidden symmetries can now be inferred from the following observation: To be in thermal equilibrium, the matter distribution under consideration must satisfy special equilibrium conditions, where the main equilibrium condition for relativistic viscous fluids in extended irreversible thermodynamics (as well as in classical theory) to be in thermal equilibrium is \( \nabla_a S^a = 0 \). For this condition to be fulfilled and for the corresponding system to actually reach a local thermostatic equilibrium state, the heat exchange between the fluids must cease and the sum of all thermal potentials and chemical reaction rates must be zero. Furthermore, the thermal energy-momentum tensor as well as the particle number and entropy fluxes of the fluid mixture must match those of an ordinary ideal fluid, which means that \( T^a_b = (\epsilon + p)w^aw^b + p\delta^a_b, \quad N_A^a = n_A \cdot w^a \) and \( S^a = s \cdot w^a \). For this to hold, an equilibrium equation of state of the form \( \epsilon + p = T(s + \sum_A \Theta_A n_A) \) must be fulfilled, where each \( \Theta_A \) represents a thermal potential associated with a species \( A \) of particles (characterized in terms of the chemical potential \( \mu_A = \frac{\partial \epsilon}{\partial n_A} \)). Ultimately, however, and this is the crucial point, the motion of the fluids should be rigid in Born’s sense, which means means that the shear tensor constructed from the four-velocity \( w^a \) must vanish, i.e. \( \sigma_{ab} = h^c_a h^d_b \nabla_c (w_d a) = 0 \), which in turn means that \( \nabla_a w_b = 0 \) must be satisfied. However, from this it follows that the matter field under consideration must, at perfect thermal equilibrium, curve spacetime in such a way that a stationary gravitational field is generated with a timelike Killing vector field \( \xi^a \) that is directly proportional to the four-velocity \( w^a \), i.e. \( \xi^a = \beta w^a \) with \( \beta = c \cdot N = c \cdot \sqrt{-\xi_a \xi^a} \) with \( c = \text{const.} \)

Conditions similar to the above must also be satisfied also in more extended theories of irreversible thermodynamics [17, 18, 25, 31, 32], which avoid causality violations by requiring causal propagation of dissipation. Eventually, even when considering types of matter fields other than viscous fluids, it should be reasonable to expect that the realization of thermodynamic equilibria (or rather quasi-equilibria) requires the satisfaction of very similar, or at least comparable, equilibrium conditions and thus the validity of the same type of stationarity of the metric.

Therefore, even though the above results of the theory of extended irreversible thermodynamics may not be applicable to all geometric models of general relativity, it can nevertheless be plausibly concluded from them that in at least some cases of interest a given matter distribution, in order to reach a thermal equilibrium state, must curve spacetime in such a way that its geometry is
(effectively) stationary and thus exhibits time translation symmetry. However, this is only possible if the field of said matter distribution hardly fluctuates, i.e., only in situations where the matter source is not subject to irreversible dissipative processes, such as occur for example in the collision of celestial bodies (or collections of them) or in the continuous accretion of matter by a celestial body over long periods of time.

Conversely, since it is an observational fact that compact massive objects in $N$-body systems undergo irreversible dissipative processes like those mentioned above, just like less dense matter accumulations, it is clear that geometric models in general relativity should be able to account for these types of dynamical behavior of matter sources. As a first step, it might therefore prove to be a useful idealization for the treatment of the dissipative phenomena mentioned above that the associated matter source of the gravitational field does not permanently remain in a thermal equilibrium state with respect to its environment, but only at early and late times, so that the existence of HKV, phantom symmetries and associated metric triples $g_{ab}$, $g'_{ab}$ and $\tilde{g}_{ab}$ as well as changing degrees of symmetry of spacetime occur as natural consequences of this particular aspect of Einstein-Hilbert gravity. After all, given that in everyday life situations it can be observed that bodies tend to reach a thermal equilibrium state before and after undergoing dissipative processes, such an idealization makes far more sense than expecting matter fields to change their thermal properties throughout the entire temporal evolution of spacetime.

Yet, as must be clearly emphasized at this point, a complete theoretical description of the geometrical aspects of dissipative phenomena such as those mentioned above is extraordinarily difficult, even on a purely numerical level, and has thereby proved infeasible to date. Consequently, however, it should be worthwhile to focus instead on a manageable, yet concrete example of a geometric model with hidden symmetries. To this effect, one may now proceed by considering the simple case of Vaidya spacetime, whose metric, in Eddington-Finkelstein coordinates, can be read off from the line element

$$ds^2 = -(1 - \frac{2m}{r})dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$  \hspace{1cm} (41)$$

In contrast to the Schwarzschild spacetime, the mass function $m(v)$ is time-dependent in the given case, so that Vaidya spacetime is generally not a vacuum geometry. Rather, it turns out that the energy-moment tensor of the mentioned geometry characterizes the gravitational field of an uncharged null-fluid source, which is given by the expression

$$\tilde{G}_{ab} = \frac{2\dot{m}}{r^2}dv_adv_b.$$  \hspace{1cm} (42)$$

Due to the fact that $l_a = -dv_a$ is an affine geodesic null vector field, the metric of this spacetime can be cast in Kerr-Schild form, i.e.

$$\tilde{g}_{ab} = \eta_{ab} + \frac{2m}{r}l_al_b.$$  \hspace{1cm} (43)$$
where \( \eta_{ab} \) is the flat Minkowskian metric. By making the specific choice \( m(v) := \chi(v, v_0)m_0 \), where \( \chi(v, v_0) \) is a transition function of the form (4) and \( m_0 = \text{const.} \), one then finds that \( m = 0 \) when \( v \leq 0 \) and \( m = m_0 \) when \( v > 0 \), so that it becomes clear that line element (41) coincides with that of Minkowski spacetime for \( v \leq 0 \) and with that of Schwarzschild spacetime for \( v > v_0 \). The given spacetime therefore represents a toy model which describes how radiation collapses to form a black hole.

By adding and subtracting \( 2m_0/r l_a l_b \) in (43), one therefore obtains two decompositions of \( \tilde{g}_{ab} \) of the form

\[
\tilde{g}_{ab} = \eta_{ab} + \frac{2m}{r} l_a l_b = g_{ab} + \frac{2m'}{r} l_a l_b, \tag{44}
\]

where \( g_{ab} \) is the Schwarzschild metric for mass \( m_0 \) and \( m' = m - m_0 \leq 0 \) applies by definition. Thus, in the given case, one actually obtains a splitting of the form (30) with two spherically symmetric and static local background metrics \( \eta_{ab} \) and \( g_{ab} \) that are given with respect to one and the same HKV \( \zeta^a = \partial_v^a \), which is parallel to the Kodama vector field of the geometry. In fact, due to the fact that the Kodama vector field coincides with \( \zeta^a \) in the static case (and has all the properties described below), the Kodama vector field itself can also be identified as HKV.

Anyhow, due to the fact that \( L_{\zeta} \tilde{g}_{ab} = 2\tilde{\nabla}_a \zeta_b = \frac{2m}{r} l_a l_b \) applies in the given context, one finds that \( \tilde{j}^a = -\tilde{G}^a_{\zeta^b} \zeta^b \) is a globally conserved current in the sense that it gives rise to the conservation law

\[
\tilde{\nabla}_a \tilde{j}^a = 0. \tag{45}
\]

By integrating this relation, one obtains a conserved quasilocal charge that is zero in \( M \) and time independent in \( M' \). Using the fact that \( \sqrt{-\tilde{g}} = \sqrt{-\eta} = r^2 \sin \theta \) applies for the considered (generalized) Kerr-Schild metrics, from which it follows that \( \sqrt{h} = \frac{r^2 \sin \theta}{N} = \frac{r^2 \sin \theta}{\sqrt{1 - 2m/r}} \) and \( d\tilde{\Sigma}^a = \frac{r^2 \sin \theta}{1 - 2m/r} \partial_v^a d^3x \) applies in the given context, one finds the result

\[
\tilde{Q} = 8\pi \dot{m} \int \frac{r \, dr}{r - 2m} = 8\pi \dot{m} \left[ r + 2m \ln |r - 2m| \right]. \tag{46}
\]

In the transition region \( O \), this charge coincides with the Misner-Sharp quasilocal mass and reduces to the ADM mass at spacelike infinity and the Bondi mass at past and future null infinity (but at past null infinity it is zero anyway due to the fact that it is zero in \( M \)). The fact that the Bondi mass increases with time can be interpreted as an indication that the gravitating physical system under consideration absorbs gravitational radiation. Of course, by choosing instead of \( \tilde{j}^a = -\tilde{G}^a_{\zeta^b} \zeta^b \) a current of the form (38), one obtains via (39) also an expression for the Komar mass, which - due to the validity of conditions (28) and (29) - becomes zero in \( M \) and coincides with the Komar mass of Schwarzschild spacetime in \( M' \). The fact that in the present case a geometric transition between static local spacetimes is considered has thus, as was to be expected, an influence on the values and forms of the quasilocal quantities mentioned.
It has to be emphasized that the given example of a geometric transition between Minkowski space and Schwarzschild spacetime, which describes the creation of a black hole due to infalling null radiation, is highly idealized and therefore not of greatest interest from a physical point of view. However, as will be shown in the following, the developed methods can also be used to build physically more interesting models of geometric transitions, such as, for example, one described another toy model (but a more realistic one) for the geometric transition between a (two-body) Majumdar-Papapetrou and a Reissner-Nordström spacetime, which is to be discussed in the next section. This model aims to describe the binary merger of two charged black holes, which remain in a static geometric configuration (described by the two-body Majumdar-Pappapetrou geometry) for a certain period of time, but then change their physical properties with time (one of the black holes accretes null radiation), which forces the entire system to collapse and thus gives rise to a black hole binary merger. At the end of this merger process, both black hole singularities coalesce to a single Reissner-Nordström black hole singularity, thereby leading to the formation of a new black hole geometry with associated mass and charge parameters that are larger than the sum of the masses of the individual black holes before the beginning of the merger and the associated geometric transition process.

What turns out to be special about this particular toy model for a binary black hole merger is that it shows that, despite the fact that there are no exact integral laws for the conservation of energy and angular momentum in black hole mergers (simply because the merger spacetime has no Killing symmetries at all), a balance equation for locally conserved quantities can be formulated - in full agreement with the results presented in this section - which are pretty similar to the standard conservation laws for the total energy existing in spacetime with Killing symmetries. As is argued, the balance laws mentioned therefore represent a reasonable substitute for standard conservation laws in dynamical spacetimes and, in particular, in black hole merger geometries.

2 Black Hole Mergers and Hidden Killing Fields

After having shown in the previous section that there are spacetimes in general relativity whose global geometry does not allow for KV, but does allow for HKV, which have the Killing property only with respect to the locally symmetric geometry of spacetime, this section will now discuss a concrete physical example, namely the geometric model of the binary merger of two extremal Reissner-Nordström black holes.

To describe a binary merger of this kind, a spacetime \((\tilde{M}, \tilde{g})\) with manifold structure \(\tilde{M} = M \cup \tilde{O} \cup M'\) shall be considered, which gives rise to a pair of static spherically local spacetimes \((M, g)\) and \((M', g')\). The first of these local spacetimes shall be assumed to describe the static initial configuration of the geometric field being given by a two-body Majumdar-Papapetrou spacetime with with two black hole singularities. The components of the corresponding 'initial' metric \(g_{ab}\) can be read off the line element.
\[
ds^2 = -\frac{dt^2}{(1 + \frac{m_1}{r_1} + \frac{m_2}{r_2})^2} + (1 + \frac{m_1}{r_1} + \frac{m_2}{r_2})^2(dx^2 + dy^2 + dz^2),
\]

where \( m_i = e_i = \text{const.} \) and \( r_i := |\vec{r} - \vec{r}_i| = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2} \) with \( i = 1, 2 \). By performing a coordinate shift of the type \( x \to x + x_1, \ y \to y + y_1, \ z \to z + z_1, \) it can be achieved that the given line element can be re-written in the form

\[
ds^2 = -\frac{dt^2}{\tilde{U}^2} + \tilde{U}^2(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)),
\]

using the scalar function \( \tilde{U} = 1 + \frac{m_1}{r} + \frac{m_2}{\sqrt{r^2 - 2d_0 \cos \theta r + d_0^2}} \) which is defined with respect to the relative distance \( d_0 = \sqrt{\tilde{d}\tilde{d}} = \text{const.} \) calculated from the time dependent three-vector \( \vec{d} := \vec{r}_2 - \vec{r}_1 \). As may be checked, this is the same form of the line element as presented in [8].

As may be realized, this initial geometry arises as a specific local geometric configuration of an associated ambient geometry, namely that of an ambient spacetime \( (\tilde{M}, \tilde{g}) \) whose metric components can be read off the line element

\[
d\tilde{s}^2 = -\frac{dt^2}{\tilde{U}^2} + \tilde{U}^2(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)),
\]

where \( \tilde{U}(t, r, \theta) = 1 + \frac{m_1}{r} + \frac{m_2}{\sqrt{r^2 - 2d_0 \cos \theta r + d_0^2}} \) applies by definition. While the mass scalars \( m_1 \) and \( m_2 \) occurring in \( \tilde{U}(t, r, \theta) \) are the same as in (48), the remaining scalar functions occurring in the metric components of the ambient spacetime \( (\tilde{M}, \tilde{g}) \) have the form \( m_0(t, t_1) = \chi(t, t_1)m_0 \) with \( m_0 = \text{const.} \) and \( d(t, t_2) = (1 - \chi(t, t_2))d_0 \).

After having now obtained this form of the line element of the ambient spacetime \( (\tilde{M}, \tilde{g}) \), it is necessary to show that the considered toy model is consistent with the geometric deformation approach treated in the previous section. This can easily be done. All one has to do is to add and subtract (in a fixed coordinate chart) the components of the ‘initial’ Majumdar-Papapetrou metric and the ‘final’ extremal Reissner-Nordström metric to/from that of the corresponding ambient metric. By taking these steps, it then becomes clear that the corresponding metric can be written in the form

\[
\tilde{g}_{ab} = g_{ab} + e_{ab} = \tilde{g}'_{ab} + e'_{ab},
\]

where \( g_{ab} \) is the Majumdar-Papapetrou metric and \( \tilde{g}'_{ab} \) is the extremal Reissner-Nordström metric of the black hole resulting from the merger process. Since for the given choice for \( e_{ab} \) and \( e'_{ab} \) it becomes clear that \( e_{ab}|_{\tilde{M}} = 0 \) and \( e'_{ab}|_{\tilde{M}'} = 0 \) such that \( g_{ab} \equiv \tilde{g}_{ab}|_{\tilde{M}} \) and \( \tilde{g}'_{ab} \equiv \tilde{g}_{ab}|_{\tilde{M}'} \), all circumstances are the same as required in the first chapter.

The idea behind considering this type of ambient spacetime is the following: Due to the properties of \( \chi(t, t_i) \) with \( i = 1, 2 \), it becomes clear that the expression \( m_0(t, t_1) \) is zero for \( t \leq 0 \). After this period, said expression starts to grow until it reaches a maximum value of \( m_0 \) in \( t > 0 \). Consequently, however, it becomes
clear that the mass of the first black hole increases in said time period from a value of \( m_1 = \text{const.} \) to the larger value \( m_1 + m_0 = \text{const.} \), so that it can be concluded that one of the two black holes continuously accretes matter and thereby becomes more massive. The corresponding accretion process, which (by assumption) takes place in the time interval \([0, t_1]\), perturbs and disrupts the stable static equilibrium of the two charged black holes. As a result of this disruption, it can be expected that the black hole binary system slowly becomes unstable from a fixed point of time onwards. In the given toy model, which unfortunately is not able to take into account the inspiral which binary black hole systems are expected to undergo under realistic circumstances, this has the sole consequence that the distance between the black hole singularities becomes smaller and smaller due geometric fluctuations that are caused by an increase of the mass of one of the black holes. As may be expected, this process continues until both black hole singularities 'collide' (in the sense that they reach the same position) and 'coalesce' to a single black hole (in the sense that the binary black hole geometry becomes that of a single Reissner-Nordström black hole). Consequently, however, the binary merger process must lead to a static geometric configuration, that is, a local spacetime \((M', g')\), whose metric components can be read off the Reissner-Nordström line element

\[
ds^2 = -(1 - \frac{2m'}{r} + \frac{e'^2}{r'}) dt^2 + \frac{dr^2}{1 - \frac{2m'}{r} + \frac{e'^2}{r'}} + r'^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

where \( m' = m_1 + m_2 + m_0 \) applies in the present context. As can readily be checked, line element (48) reduces to the form

\[
ds^2 = -\frac{dt^2}{U'^2} + U'^2 (dr^2 + r'^2 (d\theta^2 + \sin^2 \theta d\phi^2))
\]

in \( M' \), where \( U' = 1 + \frac{m'}{r} \) applies by definition. Thus, by using now the fact that the black hole geometric configuration resulting from the merger is still an extremal Reissner-Nordström black hole for which \( m' = e' \) applies, it is not difficult to see that the introduction of a new radial coordinate \( r' = r + m' \) allows one to re-write the corresponding line element in the form

\[
ds^2 = -(1 - \frac{m'}{r'})^2 dt^2 + \frac{dr^2}{(1 - \frac{m'}{r'})^2} + r'^2 (d\theta^2 + \sin^2 \theta d\phi^2)),
\]

which, however, is obviously identical to line element (51).

With that settled, it may be noted that the Einstein tensor of this ambient spacetime decomposes according to the rule

\[
\tilde{G}_{ab} = G_{ab} + \rho_{ab} = G'_{ab} + \rho'_{ab},
\]

where \( G_{ab} \) is the Majumdar-Papapetrou Einstein tensor, \( G'_{ab} \) is the Reissner-Nordström Einstein tensor and \( \rho_{ab} \) and \( \rho'_{ab} \) are the dynamical parts of Einstein tensor \( \tilde{G}_{ab} \) of the merger geometry. The first term \( G_{ab} + \rho_{ab} \) has the
simple property that it must necessarily agree with the Einstein-Tensor of the Majumdar-Papapetrou geometry in \( M \), i.e.

\[
\tilde{G}_{ab}\big|_M = G_{ab} = 8\pi T_{ab},
\]

where the corresponding energy-momentum tensor consists of a dust part and a part characterizing the electromagnetic field, i.e.

\[
T_{ab} = \varepsilon u_a u_b + F_{ac} F^c_b - \frac{1}{4} g_{ab} F_{cd} F^{cd}.
\]

Here, \( \varepsilon \) represents the energy density and \( u^a \) the four velocity of the two-body system and \( F_{ab} = 2\nabla_{[a} A_{b]} \) is the Maxwell tensor being defined with respect to the vector potential \( A_b = \frac{1}{2} \partial^t \). To reobtain the associated Einstein tensor of the corresponding local Majumdar-Papapetrou spacetime \((M, g)\) from that of the ambient spacetime \((\tilde{M}, \tilde{g})\) given by expression (54), it becomes clear that the dynamical part of the total Einstein tensor \( \tilde{G}_{ab} \) of the merger geometry must be zero in \( M \), i.e. \( \rho_{ab}\big|_M = 0 \). Of course, the same situation occurs also late times with respect to the local Reissner-Nordström spacetime \((M', g')\), that is in \( M' \), where there must hold

\[
\tilde{G}_{ab}\big|_{M'} = G'_{ab} = 8\pi T'_{ab}
\]

and thus \( \rho_{ab}\big|_{M'} = 0 \) as a direct consequence. The corresponding stress-energy tensor of the merger spacetime thus reads

\[
T'_{ab} = F'_{ac} F'^c_b - \frac{1}{4} g'_{ab} F'_{cd} F'^{cd}.
\]

at late times, from which it can be concluded that both the initial and final geometric configurations prove to be static, spherical spacetimes, whereas the complete merger geometry of spacetime has none of these geometric properties; rather, it proves to be obviously non-stationary and axisymmetric. Here, it may be noted that much about the geometrical structure of axisymmetric solutions of Einstein’s field equations is known in the literature. Accordingly, based on the results of [8] (see pages 142/143), one finds after appropriate identification
of the occurring terms that the only non-vanishing components of \( \rho_{ab} \) are

\[
\rho_{tt} = 3(\partial_t \tilde{U})^2 - (\partial_t U)^2 + \frac{\tilde{U}}{} \left( \frac{2r^2 U d^2 U - 3r^2 (\partial_t U)^2 + U^2}{r^2 U^4} \right) - \frac{U^4}{r^2 U^4} \left( \frac{2U \partial_t U}{r^2 U^4} - (\partial_t U)^2 + 2U \partial_t U \cot \theta \right)
\]

(59)

\[
\rho_{rr} = 2 \partial_r^2 UU + 3(\partial_r U)^2 - 2 \partial_r^2 \tilde{U} \tilde{U} - 3(\partial_r \tilde{U})^2 - \frac{U^4}{r^2 U^4} \left( r^2 (\partial_r U)^2 + U^4 \left( \frac{r^2 (\partial_r U)^2 + (\partial_t U)^2}{r^2 U^4} \right) \right)
\]

(60)

\[
\rho_{\theta\theta} = \rho_{\phi\phi} = 2 \partial_\theta^2 \tilde{U} \tilde{U} + (\partial_\theta U)^2 - 2 \partial_\theta^2 UU - (\partial_\theta U)^2 + \frac{U^4}{r^2 U^4} \left( r^2 (\partial_\theta U)^2 - (\partial_\theta U)^2 \right)
\]

(61)

where the results

\[
\partial_t \tilde{U} = \frac{\tilde{m}_0}{r} - \frac{m_2 \tilde{d}(d - r \cos \theta)}{(r^2 - 2rd \cos \theta + d^2)^2},
\]

(62)

\[
\partial_t^2 \tilde{U} = \frac{\tilde{m}_0}{r} + \frac{3m_2 \tilde{d}^2(d - r \cos \theta)^2}{(r^2 - 2rd \cos \theta + d^2)^2} - \frac{3m_2 \tilde{d}(d - r \cos \theta) + d^2}{(r^2 - 2rd \cos \theta + d^2)^2},
\]

(63)

\[
\partial_r \tilde{U} = - \frac{m_0 + m_1}{r^2} - \frac{m_2 (r - d \cos \theta)}{(r^2 - 2rd \cos \theta + d^2)^2},
\]

(64)

\[
\partial_r^2 \tilde{U} = \frac{2(m_0 + m_1)}{r^3} + \frac{3m_2 (r - d \cos \theta)^2}{(r^2 - 2rd \cos \theta + d^2)^2} - \frac{m_2}{(r^2 - 2rd \cos \theta + d^2)^2},
\]

(65)

\[
\partial_\theta \tilde{U} = - \frac{m_2 dr \sin \theta}{(r^2 - 2rd \cos \theta + d^2)^2},
\]

(66)

\[
\partial_\theta^2 \tilde{U} = \frac{3m_2 d^2 \sin^2 \theta}{(r^2 - 2rd \cos \theta + d^2)^2},
\]

(67)

as well as the relations \( \partial_t U = \partial_t \tilde{U} = 0 \) have been used. Since one also has \( \partial_t U' = \partial_t \tilde{U}' = 0 \), the non-vanishing components of \( \rho'_{ab} \) can simply be obtained by replacing \( U \) by \( U' \) in (59–67). Using then the fact that \( \tilde{\chi} = \chi \) and \( \tilde{\chi} = \chi \), one finds that \( \partial_t U|_M = \partial_t U, \partial_t^2 U|_M = \partial_t^2 U, \partial_t U|_M' = \partial_t U', \partial_t^2 U|_M' = \partial_t^2 U', \partial_r U|_M = \partial_r U, \partial_r^2 U|_M = \partial_r^2 U, \partial_r U|_M' = \partial_r U', \partial_r^2 U|_M' = \partial_r^2 U', \partial_\theta U|_M = \partial_\theta U, \partial_\theta^2 U|_M = \partial_\theta^2 U, \partial_\theta U|_M' = \partial_\theta U', \partial_\theta^2 U|_M' = \partial_\theta^2 U' \), and \( \partial_\theta U|_M = 0, \partial_\theta^2 U|_M = \partial_\theta^2 U = \partial_\theta U|_M' = 0, \partial_\theta^2 U|_M' = 0, \partial_\theta^2 U' = 0 \), which implies that \( \rho_{ab}|_M = 0 \) and \( \rho_{ab}|_M' = \rho_{ab} \) applies as desired.

Having clarified that, it may next be noted that the Majumdar-Papapetrou metric \( g_{ab} \) given by line element (48) coincides locally with that of an one-body extremal Reissner-Nordström black hole system in the limit \( d_0 \to \infty \). A
multipole expansion can therefore be used in $M$ to describe perturbations of this metric due to the second black hole for large enough $d_0$, i.e. in the case of large spatial separation. Since one of the black holes starts to constantly accrete matter in $\tilde{M}\setminus M$, these perturbations grow larger and larger over time, so that the system slowly becomes unstable in the sense that the distance between the singularities of the black holes becomes smaller and smaller as time goes by. This destabilization process continues until both black holes collide and merge with each other. Consequently, however, in order to indicate that a two-body system evolves into a one-body system in the process, it makes sense to re-write metric decomposition relation (50) given above in the form

$$\tilde{g}_{ab} = g^{(j)}_{ab} + e^{(j)}_{ab} = g^{(j-1)}_{ab} + e^{(j-1)}_{ab},$$

(68)

where the index $j$ counts the number of bodies in the system, which, in the given case, is just two. Moreover, in this very relation, which is completely identical to decomposition relation (50), the tensor $g^{(j)}_{ab}$ denotes the local Majumdar-Papapetrou metric, which reduces in the limit of infinite spatial separations, i.e. in the limit $d_0 \to \infty$ to the local metric of an asymptotic extremal Reissner-Nordström black hole either with mass and charge $m_1 = e_1$ or with mass and charge $m_2 = e_2$. On the other hand, $g^{(j-1)}_{ab}$ is the metric of the extremal Reissner-Nordström black hole mass and charge $m' = e'$, which occurs only after the black hole merger has taken place. This local form of the ambient metric $\tilde{g}_{ab}$ is relevant not least because the deformation field $e^{(j-1)}_{ab} = g_{ab} - g^{(j-1)}_{ab}$ is zero in the local subregion $M' \subset \tilde{M}$. In turn, the deformation tensor fields $e^{(j)}_{ab} = g_{ab} - g^{(j)}_{ab}$ become zero in the local subregion $M \subset \tilde{M}$, where the ambient spacetime $(\tilde{M}, \tilde{g})$ coincides with the local Majumdar-Papapetrou configuration $(M, g)$. In the complement of this region, i.e. in $\tilde{M}\setminus M$, the merger geometry $(\tilde{M}, \tilde{g})$ describes how perturbations caused by material accreted by one of the black holes lead to a binary merger of the black holes, whereas, at the end of this merging process, a single Reissner-Nordström black hole is formed whose mass is larger than the sum of the masses of the individual black holes before the beginning of the geometric transition process. That is to say, the merger spacetime $(\tilde{M}, \tilde{g})$ reaches again a static spherically symmetric geometric configuration at late times, i.e. in $M' \subset M$, where the geometry of the ambient spacetime $(\tilde{M}, \tilde{g})$ coincides with that of $(M', g')$.

Consequently, however, although the geometric model considered thus far is only a Mickey Mouse toy model, it turns out to be fully consistent with Hawking’s area theorem, which states that when two black holes merge, the area of the final event horizon is greater than or equal to the sum of the areas of the initial horizons. Moreover, since the two-body system settles down to a stationary spherically symmetric state after the black hole merger, the considered toy model is also consistent with the black hole uniqueness theorems and the no hair theorems [7, 16, 20, 21, 34, 36, 37].

To see that not only the latter, but also the first part of this statement is true, it may be checked that the energy-momentum tensor of the merger geometry
meets the weak null energy condition
\[ \tilde{T}_{ab} K^a K^b \geq 0 \] (69)
for every null vector \( K^a \). In addition, one may consider line element (47), introduce a radial coordinate \( \tilde{r} = r + m_1 \) and calculate the expression
\[ A_1 = \lim_{\tilde{r} \to m_1} \int_0^{2\pi} \int_0^\pi \sqrt{g_{\theta\theta} g_{\phi\phi}} d\theta d\phi = 4\pi m_1^2. \] (70)

Then, by taking into account that all foregoing remarks made with respect to \( m_1 \) can also be made with respect to \( m_2 \) (which is not least because each point \( r_i \) with \( i = 1, 2 \) in line element (47) represents a null hypersurface with an area \( 4\pi m_i \) \( \tilde{M} \)), one may transform line element (48) in suitable coordinates and calculate the further expression
\[ A_2 = \lim_{\tilde{r} \to m_2} \int_0^{2\pi} \int_0^\pi \sqrt{g_{\theta\theta} g_{\phi\phi}} d\theta d\phi = 4\pi m_2^2. \] (71)

Sure enough, the expressions obtained can be combined, i.e.
\[ A_1 + A_2 = 4\pi (m_1^2 + m_2^2). \] (72)

Using next line element (53), which coincides with line element (51), one obtains the result
\[ A' = \lim_{r' \to m'} \int_0^{2\pi} \int_0^\pi \sqrt{\tilde{g}_{\theta\theta} \tilde{g}_{\phi\phi}} d\theta d\phi = 4\pi m'^2. \]

However, since it is known that \( m' > m_1 + m_2 \), this allows one to conclude that Hawking’s area theorem
\[ A_1 + A_2 < A' \] (73)
holds true in the given case even when \( m_0 \ll m_1, m_2 \) and therefore \( m'^2 \simeq (m_1 + m_2)^2 \) applies, which happens to be a case of greatest interest from a physical point of view. It may be noted that exactly the same result was obtained previously in [1].

To proceed, one may take the fact into account that the vector field \( \zeta^a = \partial_i \) living on the ambient spacetime \((\tilde{M}, \tilde{g})\) coincides with the KV of the local spacetimes \((M, g)\) and \((M', g')\). Clearly, this vector field also defines a Papapetrou field in the sense of [4, 10]. Consequently, however, it can be concluded that \( \zeta^a \) is also a HKV in the sense of the previous section satisfying condition (32) and that the ambient spacetime \((\tilde{M}, \tilde{g})\) has phantom symmetry. One is therefore in the favorable situation that the corresponding current \( \tilde{j}^a = \tilde{G}_{ab}^a \zeta^b \) can be used to formulate an integral law, which takes the form (33) in \( M \) and (34) in \( M' \). To see this, one may consider in the same way as in the first section two submanifolds \( D \subset M \) and \( D' \subset M' \) with boundaries \( \partial D \) and \( \partial D' \), which
are contained in an associated ambient submanifold $\tilde{D} \subset \tilde{M}$ with boundary $\partial \tilde{D}$ such that $D, D' \subseteq \tilde{D}$. This geometric setting can be used to deduce the relation

$$\int_{\tilde{D}} \tilde{\nabla}_a j^a d\tilde{v} = 2 \int_{\tilde{D}} \tilde{G}_{ab} \tilde{\nabla}^{(a} \zeta^{b)} d\tilde{v}. \quad (74)$$

Using Gauss' theorem and decomposition relation (35) to convert the LHS of (73), one thus obtains a balance law of the form (37), in relation to which $\mathcal{L}_\zeta Q = \mathcal{L}_\zeta Q' = 0$ applies. However, this implies that also $\mathcal{L}_\zeta g = 0$ applies, from which it can be concluded that a local observer witnessing a geometric transition from $(M, g)$ to $(M', g')$ with respect to the ambient spacetime $(\tilde{M}, \tilde{g})$ could come to the conclusion that the total energy of the system is conserved, although the ambient spacetime lacks the corresponding Killing symmetry! But this seems to suggest that the derived balance law could retain its validity not only in the considered, extremely simplified Mickey Mouse model of a binary black hole merger, but also in more realistic modelings of merger geometries. For this reason, HKV appear to constitute good and reasonable substitutes for KV dynamical spacetimes and, in particular, in black hole merger geometries.

**Conclusion**

In the present work, the geometrical properties of special classes of spacetimes have been studied, whose geometry is not globally, but only locally symmetric, and which therefore admit Killing symmetries only in a local subregion of spacetime, not in the entire Lorentzian manifold. Such spacetimes, as it has been argued, have the advantageous property that they allow one to specify a preferred class of vector fields, the class of so-called HKV, which are geometrically distinguished by coinciding exactly with associated Killing vector fields at local scales, i.e., in the local region where spacetime becomes symmetric. This particular property has been exploited in the present work to show that certain integral laws of the theory reduce to exact conservation laws on local scales, thereby leading to exact expressions for energy and angular momentum in the symmetric parts of spacetime. Using this insight, physical balance laws for energy and angular momentum were derived, which describe how exactly any of these locally conserved quantities change in time during the evolution of a generic ambient geometry that completely lacks any sort of Killing symmetry. The derived balance laws have the interesting property that they are very closely related to the integral conservation laws of known symmetric models and, except for the correction terms resulting from the geometric deformation of the symmetric local background geometry, coincide with these conservation laws even in the face of a generic geometric setting. In order to demonstrate the utility of the laws mentioned, a special geometric model was treated in the second part of this work, namely a toy model for a merger of two extremal Reissner-Nordström black holes. For the exact characterization of the geometric structure of a merger geometry of this type, a non-stationary axially symmetrical...
geometric model was considered, which, by construction, coincides locally with the (two-body) Majumdar-Papapetrou solution at early times and the Reissner-Nordström solution at later times. For this model, the existence of HKV has been demonstrated, and it has been proved that the geometric structure of the merger spacetime proves to be consistent with Hawking’s area theorem and also allows the definition of locally conserved integral expressions, which could prove to be useful for the generalization of the laws of black hole mechanics for merger geometries in the future.

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