DEFORMATIONS AND HOMOTOPY OF ROTA-BAXTER OPERATORS AND O-OPERATORS ON LIE ALGEBRAS

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ABSTRACT. This article gives a brief introduction to some recent work on deformation and homotopy theories of Rota-Baxter operators and more generally O-operators on Lie algebras, by means of the differential graded Lie algebra approach. It is further shown that these theories lift the existing connection between O-operators and pre-Lie algebras to the levels of deformations and homotopy.

1. INTRODUCTION

This a abridged survey of some of the recent developments on deformations and homotopy of O-operators and Rota-Baxter operators in relation with pre-Lie algebras. Our focus will be on [31, 32] to which we refer the reader for details.

1.1. Classical theories of deformations and homotopy. Deformation is an omnipresent notion in mathematics and physics. Roughly speaking, a deformation of an object equipped with a certain structure is a perturbation of the object (by a parameter for instance) which has the same kind of structure. In physics, the idea of deformation is behind the perturbative quantum field theory. A regularization procedure in the renormalization of getting rid of a divergency is often a deformation in practice. Deformation quantization has been studied under many contexts in mathematical physics [15, 16, 26, 29].

The foundational work of Kodaira and Spencer [14] for deformations of complex analytic structures led to its generalization in algebraic geometry and number theory. In algebra, deformation theory began with the seminal work of Gerstenhaber [10, 11] for associative algebras, followed by its extension to Lie algebras by Nijenhuis and Richardson [23, 24].

Homotopy is a notion closely related to deformation. Starting with the fundamental procedure in topology describing continuously deforming one function to another, the homotopy of an algebraic structure is obtained when the defining relations of the algebraic structure is relaxed to hold up to a weaker homotopy condition. Thus homotopy is often regarded as deformation from another point of view. This notion abstracted to category theory gives a mathematical context to the fundamental gauge principle in physics that it is more useful to relate two objects by a gauge transformation rather than a strict equality [25]. The processes can be iterated and give rise to higher homotopies and higher gauge transformations.

The first homotopy construction in pure algebra is the $A_\infty$-algebra of Stasheff, arising from his work on homotopy characterization of connected based loop spaces [27]. This was followed by $L_\infty$-algebras, both having many applications in physics, especially in supergravity and string theory [28]. See also [13] for application of homotopy in statistical physics. After deformations and homotopy of other algebraic structures such as pre-Lie algebras were developed (see for example [5, 6]), their general theories for algebraic structures in the context of operads were established [18, 21, 21].

However, these general theories mostly apply only to connected operads, namely the operads whose space of unary operations is no bigger than the linear span of the identity automorphism.
1.2. Rota-Baxter operators and $O$-operators. It is against this background that the characters of our studies in \cite{31,32} took the stage, motivating us to expand the existing literature of algebraic deformations and homotopy. The characters are the $O$-operators and Rota-Baxter operators, with the latter being a special case of the former.

Since this paper is intended to be a brief survey, we will restrict the weight of the $O$-operators and Rota-Baxter operators to be zero, and limit the discussion on $r$-matrices to general remarks.

**Definition 1.1.** Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie algebra.

(i) A linear operator $P : \mathfrak{g} \rightarrow \mathfrak{g}$ is called a **Rota-Baxter operator** (of weight 0) if

\begin{equation}
[P(x), P(y)] = P([P(x), y] + [x, P(y)]), \quad \forall x, y \in \mathfrak{g}.
\end{equation}

(ii) Let $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ on a vector space $V$. An **$O$-operator** on $\mathfrak{g}$ with respect to the representation $(V; \rho)$ is a linear map $T : V \rightarrow \mathfrak{g}$ such that

\begin{equation}
[T u, T v] = T(\rho(T u)(v) − \rho(T v)(u)), \quad \forall u, v \in V.
\end{equation}

Note that when $\rho$ is the adjoint representation of $\mathfrak{g}$, Eq. (2) reduces to Eq. (1), which means that a Rota-Baxter operator is an $O$-operator on $\mathfrak{g}$ with respect to the adjoint representation. Furthermore, a skew-symmetric $r$-matrix corresponds to an $O$-operator on $\mathfrak{g}$ with respect to the coadjoint representation \cite{17}.

The concept of Rota-Baxter operators on associative algebras was introduced in 1960 by G. Baxter \cite{3} in his study of fluctuation theory in probability. Recently it has found many applications, including in Connes-Kreimer’s algebraic approach to the renormalization of perturbative quantum field theory \cite{7,12}. In the algebraic context, a Rota-Baxter operator (of weight 0) was introduced independently in the 1980s as the operator form of the classical Yang-Baxter equation, named after the well-known physicists C. N. Yang and R. J. Baxter \cite{30}.

To better understand the classical Yang-Baxter equation and the related integrable systems, the more general notion of an $O$-operator (later also called a relative Rota-Baxter operator or a generalized Rota-Baxter operator) on a Lie algebra was introduced \cite{4,17}.

1.3. Our approach. Given the importance of Rota-Baxter operators and the more general $O$-operators in mathematics and mathematical physics, it is meaningful to study their deformations and homotopies. However, as noted above, the existing general framework of deformation and homotopy for operads does not apply to Rota-Baxter operators and hence $O$-operators: the operad of Rota-Baxter (associative or Lie) algebras has the Rota-Baxter operator as the non-identity unary operation and hence is not connected and thus is not covered by the general theories of deformations and homotopy.

Instead, we take our approach by general principles of deformation and homotopy theories. As stated in \cite{28}, the following “metatheorem” provides a philosophy (also see the letter of Deligne \cite{8}) for a deformation (or a homotopy) theory:

The deformation theory of any mathematical object, e.g., an associative algebra, a complex manifold, etc., can be described starting from a certain differential graded Lie algebra (dgLa) associated to the mathematical object in question.

Further, the deformations are given as solutions of a “Master Equation”, now known as the Maurer-Cartan equation on the dgLa.

In our constructions of deformation and homotopy theories of Rota-Baxter operators and $O$-operators, we also ensure that the theories are compatible with the recently established relationship of the operators with pre-Lie algebras. To be specific, it has been proved that an $O$-operator on a Lie algebra naturally gives rise to a pre-Lie algebra \cite{3}, whose deformation and homotopy theories have been established \cite{5,6}. Thus a test for our deformation and homotopy theories of $O$-operators
is whether they extend the above connection of \( O \)-operators with pre-Lie algebras to the contexts of deformations and homotopy. We show that this is indeed the case.

The outline of the paper is as follows. In Section \[\text{[1]}\], we first recall background on differential graded Lie algebras (dgLas) and then apply it to define the deformations of \( O \)-operators. We then establish a natural homomorphism from the dgLa defining the deformations of \( O \)-operators to the one defining the deformation of pre-Lie algebras, yielding the desired relationship between the deformation theories. In Section \[\text{[2]}\], we take a similar approach, but work in the framework of symmetric dgLas, for the homotopy of \( O \)-operators including its connection with the homotopy of pre-Lie algebras.

2. DEFORMATIONS OF \( O \)-OPERATORS

We first recall some general notions. Let \( V = \oplus_{k \in \mathbb{Z}} V^k \) be a \( \mathbb{Z} \)-graded vector space. We denote by \( S(V) \) the symmetric algebra of \( V \), i.e. \( S(V) := \oplus_{m \geq 0} S^m(V) \). Denote the product of elements \( v_1, \cdots, v_n \in V \) in \( S^n(V) \) by \( v_1 \cdots v_n \). The degree of \( v_1 \cdots v_n \) is by definition the sum of the degree of \( v_i \). Denote by \( \text{Hom}^n(S(V), V) \) the space of degree \( n \) linear maps from the graded vector space \( S(V) \) to the graded vector space \( V \). Obviously, an element \( f \in \text{Hom}^n(S(V), V) \) is the sum of \( f_i : S^i(V) \rightarrow V \). We will write \( f = \sum_{i=0}^{\infty} f_i \). Set \( C^n(V, V) := \text{Hom}^n(S(V), V) \) and \( C^\cdot(V, V) := \oplus_{n \in \mathbb{Z}} C^n(V, V) \).

A permutation \( \sigma \in \mathbb{S}_n \) is called an \((i, n-i)\)-unshuffle if \( \sigma(1) < \cdots < \sigma(i) \) and \( \sigma(i+1) < \cdots < \sigma(n) \). If \( i = 0 \) or \( n \), we assume \( \sigma = 1d \). The set of all \((i, n-i)\)-unshuffles will be denoted by \( \mathbb{S}_{(i, \cdots, i)} \). The notion of an \((i_1, \cdots, i_k)\)-unshuffle and the set \( \mathbb{S}_{(i_1, \cdots, i_k)} \) are defined analogously.

For a permutation \( \sigma \in \mathbb{S}_n \) and \( v_1, \cdots, v_n \in V \), the Koszul sign \( \varepsilon(\sigma; v_1, \cdots, v_n) \in \{-1, 1\} \) is defined by

\[
\varepsilon(\sigma; v_1, \cdots, v_n) = \varepsilon(\sigma; v_1, \cdots, v_n) v_{\sigma(1)} \cdots v_{\sigma(n)}.
\]

We will abbreviate \( \varepsilon(\sigma; v_1, \cdots, v_n) \) to \( \varepsilon(\sigma) \).

**Definition 2.1.** ([\[\text{[1]}\]]) Let \( g = \oplus_{k \in \mathbb{Z}} g^k \) be a dgLa. A degree 1 element \( \theta \in g^1 \) is called a **Maurer-Cartan element** of \( g \) if it satisfies the following **Maurer-Cartan equation**:

\[
d\theta + \frac{1}{2} [\theta, \theta] = 0. \tag{3}
\]

A gLa is a dgLa with \( d = 0 \). Then we have

**Proposition 2.2.** ([\[\text{[1]}\]]) Let \( g = \oplus_{k \in \mathbb{Z}} g^k \) be a gLa and let \( \mu \in g^1 \) be a Maurer-Cartan element. Then the map

\[
d\mu : g \rightarrow g, \quad d\mu(u) := [\mu, u], \quad \forall u \in g,
\]

is a differential on \( g \). For any \( v \in g_1 \), the sum \( \mu + v \) is a Maurer-Cartan element of the gLa \( g \), if and only if \( v \) is a Maurer-Cartan element of the dgLa \( g \).

Let \( (V; \rho) \) be a representation of a Lie algebra \( g \). Consider the graded vector space \( C^\cdot(V; g) := \oplus_{k \geq 0} \text{Hom}(\wedge^k V, g) \). Define a skew-symmetric bracket operation

\[
[\cdot, \cdot] : \text{Hom}(\wedge^n V, g) \times \text{Hom}(\wedge^m V, g) \rightarrow \text{Hom}(\wedge^{n+m} V, g)
\]

by taking \( f \in \text{Hom}(\wedge^n V, g), \ g \in \text{Hom}(\wedge^m V, g) \) and define

\[
[\cdot, \cdot] := - \sum_{\sigma \in \mathbb{S}_{(m,n,m-1)}} (-1)^{\rho} f(\rho(u_{\sigma(1)}, \cdots, u_{\sigma(m)})) u_{\sigma(m+1)}, u_{\sigma(m+2)}, \cdots, u_{\sigma(m+n)}
\]

\[
+(-1)^{\mu n} \sum_{\sigma \in \mathbb{S}_{(n, m, m)}} (-1)^{\rho} g(\rho(u_{\sigma(1)}, \cdots, u_{\sigma(n)})) u_{\sigma(n+1)}, u_{\sigma(n+2)}, \cdots, u_{\sigma(m+n)}
\]
Proposition 2.3. ([3]) With the above notations, \((C^*(V, g), \llbracket \cdot , \cdot \rrbracket)\) is a gLa. Moreover, its Maurer-Cartan elements are precisely the \(O\)-operators on \(g\) with respect to the representation \((V, \rho)\).

Let \(T : V \to g\) be an \(O\)-operator. Since \(T\) is a Maurer-Cartan element of the gLa \((C^*(V, g), \llbracket \cdot , \cdot \rrbracket)\) by Proposition 2.3, it follows from Proposition 2.2 that \(d_T := [T, \cdot]\) is a graded derivation on the gLa \((C^*(V, g), \llbracket \cdot , \cdot \rrbracket)\) satisfying \(d_T^2 = 0\). Therefore \((C^*(V, g), \llbracket \cdot , \cdot \rrbracket, d_T)\) is a dgLa. This dgLa controls the deformations of \(O\)-operators.

Theorem 2.4. ([3]) Let \(T : V \to g\) be an \(O\)-operator on a Lie algebra \(g\) with respect to a representation \((V, \rho)\). Then for a linear map \(T' : V \to g\), \(T + T'\) is still an \(O\)-operator on the Lie algebra \(g\) with respect to the representation \((V, \rho)\) if and only if \(T'\) is a Maurer-Cartan element of the dgLa \((C^*(V, g), \llbracket \cdot , \cdot \rrbracket, d_T)\).

We next recall the notion of a pre-Lie algebra and the gLa whose Maurer-Cartan elements characterize pre-Lie algebra structures. We show that there is a close relationship between these two graded Lie algebras.

Definition 2.5. ([3]) A pre-Lie algebra is a pair \((V, \cdot_V)\), where \(V\) is a vector space and \(\cdot_V : V \otimes V \to V\) is a bilinear multiplication satisfying that for all \(x, y, z \in V\),

\[
(x \cdot_V y) \cdot_V z = x \cdot_V (y \cdot_V z) - y \cdot_V (x \cdot_V z).
\]

Relating an \(O\)-operator to a pre-Lie algebra, we have

Theorem 2.6. ([3]) Let \(T : V \to g\) be an \(O\)-operator on a Lie algebra \(g\) with respect to a representation \((V, \rho)\). Define a multiplication \(\cdot_T\) on \(V\) by

\[
(\alpha \circ \beta)(u_1, \cdots, u_{m+n+1}) := \sum_{\sigma \in S_{m+1, 1, n-1}} (-1)^{\sigma} \alpha(u_{\sigma(1)}, \cdots, u_{\sigma(n+1)}, u_{\sigma(n+2)}, \cdots, u_{\sigma(m+n)}, u_{m+n+1})
\]

\[
+ (-1)^{mn} \sum_{\sigma \in S_{m, n, 0}} (-1)^{\sigma} \alpha(u_{\sigma(1)}, \cdots, u_{\sigma(n)}, \beta(u_{\sigma(n+1)}, \cdots, u_{\sigma(m+n)}, u_{m+n+1})).
\]

Then the graded vector space \(C^*(V, V) := \bigoplus_{k \geq 0} \text{Hom}(\wedge^k V \otimes V, V)\) is a gLa when it is equipped with the Matsushima-Nijenhuis bracket \([\cdot, \cdot]^C\). ([3, 22, 33]):

\[
[\alpha, \beta]^C := \alpha \circ \beta - (-1)^{mn} \beta \circ \alpha, \quad \forall \alpha \in \text{Hom}(\wedge^n V \otimes V, V), \beta \in \text{Hom}(\wedge^m V \otimes V, V).
\]

Remark 2.7. In fact, \(\alpha \in \text{Hom}(V \otimes V, V)\) defines a pre-Lie algebra structure on \(V\) if and only if \([\alpha, \alpha]^C = 0\), that is, \(\alpha\) is a Maurer-Cartan element of the gLa \((C^*(V, V), [\cdot, \cdot]^C)\).

Define a linear map \(\Phi : \text{Hom}(\wedge^k V, g) \to \text{Hom}(\wedge^k V \otimes V, V), k \geq 0\) by

\[
\Phi(f)(u_1, \cdots, u_k, u_{k+1}) = \rho(f(u_1, \cdots, u_k))(u_{k+1}), \quad \forall f \in \text{Hom}(\wedge^k V, g), u_1, \cdots, u_{k+1} \in V.
\]

Theorem 2.8. ([3]) Let \((V, \rho)\) be a representation of a Lie algebra \(g\). Then \(\Phi\) is a homomorphism of gLas from \((C^*(V, g), \llbracket \cdot, \cdot \rrbracket)\) to \((C^*(V, V), [\cdot, \cdot]^C)\).
Remark 2.9. As a direct consequence of the above proposition, the Maurer-Cartan elements in the first graded Lie algebra are sent to those in the second graded Lie algebra. Thus by Proposition 2.3 and Remark 2.7, the \(O\)-operators are sent to pre-Lie algebra structures on \(V\). Furthermore, two \(O\)-operators are sent to the same pre-Lie algebra if and only if they are in the same fiber of \(\Phi\). This lifts the connection from \(O\)-operators to pre-Lie algebras in Theorem 2.6 to the level of deformations.

3. Homotopy \(O\)-operators on symmetric graded Lie algebras

We first recall the symmetric generalizations the notions of gLas and dgLas \([1]\).

**Definition 3.1.** A symmetric graded Lie algebra (sgLa) is a \(\mathbb{Z}\)-graded vector space \(\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k\) equipped with a bilinear bracket \([\cdot, \cdot]_\mathfrak{g}\) of degree 1 such that

(i) (graded symmetry) \([x, y]_\mathfrak{g} = (-1)^{|x||y|} [y, x]_\mathfrak{g}\),

(ii) (graded Leibniz rule) \([x, [y, z]]_\mathfrak{g} = (-1)^{|x||y|+1} [[x, y], z]_\mathfrak{g} + (-1)^{|x||y|+1} [x, [y, z]]_\mathfrak{g}\).

Here \(x, y, z\) are homogeneous elements in \(\mathfrak{g}\), which also denote their degrees when in exponent.

**Definition 3.2.** A symmetric differential graded Lie algebra (sdgLa) is a symmetric graded Lie algebra \((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})\) equipped with a linear map \(d : \mathfrak{g} \rightarrow \mathfrak{g}\) of degree 1 such that

\[
d[x, y]_\mathfrak{g} = -[dx, y]_\mathfrak{g} - (-1)^{|x|}|x, dy|_\mathfrak{g}.
\]

We also recall the notion of the suspension and desuspension operators. Let \(V = \bigoplus_{i \in \mathbb{Z}} V^i\) be a graded vector space, we define the suspension operator \(s : V \mapsto sV\) by assigning \(V\) to the graded vector space \(sV = \bigoplus_{i \in \mathbb{Z}} (sV)^i\) with \((sV)^i := V^{-i}\). There is a natural degree 1 map \(s : V \mapsto sV\) that is the identity map of the underlying vector space, sending \(v \in V\) to its suspended copy \(sv \in sV\). Likewise, the desuspension operator \(s^{-1}\) changes the grading of \(V\) according to the rule \((s^{-1}V)^i := V^{i+1}\). The degree \(-1\) map \(s^{-1} : V \mapsto s^{-1}V\) is defined in the obvious way.

Let \((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})\) and \((\mathfrak{g}', [\cdot, \cdot]_{\mathfrak{g}'})\) be sgLa’s. A homomorphism from \(g\) to \(g'\) is a linear map \(\phi : \mathfrak{g} \rightarrow \mathfrak{g}'\) of degree 0 such that

\[
\phi([x_1, x_2]_\mathfrak{g}) = [[\phi(x_1), \phi(x_2)]]_{\mathfrak{g}'} , \quad \forall x_1, x_2 \in \mathfrak{g}.
\]

**Definition 3.3.** A representation of an sgLa \((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})\) on a graded vector space \(V\) is a homomorphism of graded vector spaces \(\rho : \mathfrak{g} \mapsto \mathfrak{gl}(V)\) of degree 1 such that \(s^{-1} \circ \rho : \mathfrak{g} \rightarrow s^{-1} \mathfrak{gl}(V)\) is an sgLa homomorphism.

Now we are ready to give the main notion of this section.

**Definition 3.4.** ([32]) Let \(\rho\) be a representation of an sgLa \((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})\) on a graded vector space \(V\). A degree 0 element \(T = \sum_{l=0}^{+\infty} T_l \in \text{Hom}(S(b), \mathfrak{g})\) with \(T_l \in \text{Hom}(S^l(b), \mathfrak{g})\) is called a homotopy \(O\)-operator on an sgLa \((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})\) with respect to the representation \(\rho\) if the following generalized Rota-Baxter identities hold for all \(p \geq 0\) and all homogeneous elements \(v_1, \ldots, v_p \in V\),

\[
\sum_{k+l=p+1} \sum_{\sigma \in S_{l-1,p-k-l+1}} \varepsilon(\sigma) T_{k-1}(\rho(T_l(v_{\sigma(1)}, \cdots, v_{\sigma(l)}))v_{\sigma(l+1)}, v_{\sigma(l+2)}, \cdots, v_{\sigma(p)})
\]

\[
= \frac{1}{2} \sum_{k+l=p+1} \sum_{\sigma \in S_{k-1,l-k}} \varepsilon(\sigma) [T_{k-1}(v_{\sigma(1)}, \cdots, v_{\sigma(k-1)}), T_l(v_{\sigma(k)}, \cdots, v_{\sigma(p)})]_\mathfrak{g}.
\]

**Remark 3.5.** The linear map \(T_0\) is just an element \(\Omega \in \mathfrak{g}^0\). Below are the generalized Rota-Baxter identities for \(p = 0, 1, 2\):

\[
[\Omega, \Omega]_\mathfrak{g} = 0,
\]

\[
T_1(\rho(\Omega)v_1) = [\Omega, T_1(v_1)]_\mathfrak{g},
\]
\[ [T_1(v_1), T_1(v_2)]_3 = T_1(\rho(T_1(v_1))v_2 + (-1)^{v_1v_2}\rho(T_1(v_2))v_1) \\
+ T_2(\rho(\Omega)v_1, v_2) + (-1)^{v_1v_2}T_3(\rho(\Omega)v_2, v_1) - [\Omega, T_2(v_1, v_2)]_3. \]

**Remark 3.6.** If the sgLa reduces to a Lie algebra and the action reduces to a representation of a Lie algebra on a vector space, the above definition reduces to the definition of an \( O \)-operator on a Lie algebra.

**Remark 3.7.** A homotopy Rota-Baxter operator \( R = \sum_{i=0}^{\infty} R_i \in \text{Hom}(S(\mathfrak{g}), \mathfrak{g}) \) on an sgLa \((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})\) is a homotopy \( O \)-operator with respect to the adjoint representation \( \text{ad} \). If moreover the sgLa reduces to a Lie algebra, then the resulting linear operator \( R : \mathfrak{g} \rightarrow \mathfrak{g} \) is a Rota-Baxter operator.

In the sequel, we construct a gLa and show that homotopy \( O \)-operators can be characterized as its Maurer-Cartan elements to justify our definition of homotopy \( O \)-operators.

Let \( \rho \) be a representation of an sgLa \((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})\) on a graded vector space \( V \). Consider the graded vector space \( C^\ast(V, \mathfrak{g}) := \oplus_{n \in \mathbb{Z}} \text{Hom}^n(S(V), \mathfrak{g}) \). Define a graded bracket operation

\[ [\cdot, \cdot] : \text{Hom}^m(S(V), \mathfrak{g}) \times \text{Hom}^n(S(V), \mathfrak{g}) \rightarrow \text{Hom}^{m+n+1}(S(V), \mathfrak{g}) \]

by

\[ [f, g]_\rho(v_1, \ldots, v_p) \]

\[ = - \sum_{k+l=p+1} \sum_{\sigma \in S_{(k-1,l-1)}} \epsilon(\sigma) f_{k-1}(\rho(g_i(v_{\sigma(1)}), \ldots, v_{\sigma(l)}))v_{\sigma(l+1)}, v_{\sigma(l+2)}, \ldots, v_{\sigma(p)} \]

\[ + (-1)^{(m+1)(n+1)} \sum_{k+l=p+1} \sum_{\sigma \in S_{(k-1,1,p-1)}} \epsilon(\sigma) f_{k-1}(\rho(v_{\sigma(1)}, \ldots, v_{\sigma(k-1)}))v_{\sigma(k)}, v_{\sigma(k+1)}, \ldots, v_{\sigma(p)} \]

\[ - \sum_{k+l=p+1} \sum_{\sigma \in S_{(k-1,1,p-1)}} (-1)^{(m+1)\sigma} e(\sigma)(f_{k-1}(v_{\sigma(1)}, \ldots, v_{\sigma(k-1)}), v_{\sigma(k)}, v_{\sigma(k+1)}, \ldots, v_{\sigma(p)}) \]

for all \( f = \sum \rho_i f_i \in \text{Hom}^m(S(V), \mathfrak{g}), g = \sum g_i \in \text{Hom}^n(S(V), \mathfrak{g}) \) with \( f_i, g_i \in \text{Hom}(S^i(V), \mathfrak{g}) \) and \( v_1, \ldots, v_p \in V \). Here we write \([f, g] = \sum_i [f, g]_i \in \text{Hom}(S^i(V), \mathfrak{g}), [f, g]_i \in \text{Hom}(S^i(V), \mathfrak{g})\).

**Theorem 3.8.** ([12]) Let \( \rho \) be a representation of an sgLa \((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})\) on a graded vector space \( V \). Then \((sC^\ast(V, \mathfrak{g}), [\cdot, \cdot])\) is a gLa.

Homotopy \( O \)-operators can be characterized as Maurer-Cartan elements of the above gLa. Note that an element \( T = \sum_{i=0}^{\infty} T_i \in \text{Hom}(S(\mathfrak{g}), \mathfrak{g}) \) is of degree 0 if and only if the corresponding element \( T \in s\text{Hom}(S(\mathfrak{g}), \mathfrak{g}) \) is of degree 1.

**Theorem 3.9.** ([12]) Let \( \rho \) be a representation of an sgLa \((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})\) on a graded vector space \( V \). A degree 0 element \( T = \sum_{i=0}^{\infty} T_i \in \text{Hom}(S(\mathfrak{g}), \mathfrak{g}) \) is a homotopy \( O \)-operator on \( \mathfrak{g} \) with respect to the representation \( \rho \) if and only if \( T = \sum_{i=0}^{\infty} T_i \) is a Maurer-Cartan element of the gLa \((sC^\ast(V, \mathfrak{g}), [\cdot, \cdot])\).

The notion of a pre-Lie\(_{\infty}\)-algebra was introduced in [3] as the homotopy of the pre-Lie algebra. See [13] for applications of pre-Lie\(_{\infty}\)-algebras in geometry.

**Definition 3.10.** A pre-Lie\(_{\infty}\)-algebra is a graded vector space \( V \) equipped with a collection of linear maps \( \theta_k : \otimes^k V \rightarrow V, k \geq 1 \), of degree 1 with the property that, for any homogeneous elements \( v_1, \ldots, v_n \in V \), we have

(i) (graded symmetry) for every \( \sigma \in S_{n-1} \),

\[ \theta_n(v_{\sigma(1)}, \ldots, v_{\sigma(n-1)}, v_n) = \epsilon(\sigma)\theta_n(v_1, \ldots, v_{n-1}, v_n), \]

(ii) for all \( n \geq 1 \),

\[ \sum_{i+j=n+1} \sum_{\sigma \in S_{(i-1,j-1)}(2)} \epsilon(\sigma)\theta_j(v_{\sigma(1)}, \ldots, v_{\sigma(j-1)}, v_{\sigma(i)}, v_{\sigma(i+1)}, \ldots, v_{\sigma(n-1)}, v_n) \]
Therefore, we have \( \theta_j(v_{r(1)}, \cdots, v_{r(j)}, \cdots, v_{r(n-1)}, v_n) = 0, \)
where \( \alpha = v_{r(1)} + v_{r(2)} + \cdots + v_{r(j)}. \)

Pre-Lie\(_{\infty}\)-algebras can be characterized as the Maurer-Cartan elements of a dgLa.

**Proposition 3.11.** (\([\ddagger]\)) Let \( V \) be a \( \mathbb{Z} \)-graded vector space. Denote by
\[
\tilde{C}^n(V, V) := \mathfrak{s}\text{Hom}^{n-1}(S(V), \text{s}^{-1}(V)) = \text{Hom}^n(S(V), \mathfrak{gl}(V)), \quad \tilde{C}(V, V) := \oplus_{n \in \mathbb{Z}} \tilde{C}^n(V, V).
\]
Then \( (\tilde{C}^*(V, V), [, , ]^\mathfrak{g}) \) is a gLa equipped with the graded Lie bracket
\[
[ , , ]^\mathfrak{g} : \tilde{C}^n(V, V) \times \tilde{C}^n(V, V) \to \tilde{C}^{n+\mathfrak{g}}(V, V)
\]
whose details is referred to the original literature.

Moreover, \( L = \sum_{i=0}^{\infty} L_i \in \text{Hom}^1(S(V), \mathfrak{gl}(V)) \) defines a pre-Lie\(_{\infty}\)-algebra structure by
\[
\theta_k(v_1, \cdots, v_k) := L_{k-1}(v_1, \cdots, v_{k-1})v_k, \quad \forall v_1, \cdots, v_k \in V,
\]
on the graded vector space \( V \) if and only if \( L = \sum_{i=0}^{\infty} L_i \) is a Maurer-Cartan element of the gLa \( (\tilde{C}^*(V, V), [, , ]^\mathfrak{g}) \).

In the above result, if the graded vector space \( V \) reduces to a usual vector space, we obtain the **Matsushima-Nijenhuis bracket** \([, , ]^\mathfrak{g}\) given by \([\ddagger]\). See \([\ddagger, \ddagger, \ddagger]\) for more details.

Finally, as promised, we lift the connection from \( O \)-operator to pre-Lie algebras (Theorem \([\ddagger]\)) to the level of homotopy. We first establish the relationship between the two gLas that give Maurer-Cartan characterizations of homotopy \( O \)-operators and pre-Lie\(_{\infty}\)-algebras respectively. Define a graded linear map \( \Psi : \mathfrak{s}C^*(V, \mathfrak{g}) \to \tilde{C}^*(V, V) \) of degree 0 by
\[
\Psi(f) = s^{-1} \circ \rho \circ f, \quad \forall f \in \text{Hom}^n(S(V), \mathfrak{g}).
\]
Therefore, we have \( \Psi(f)_k = s^{-1} \circ \rho \circ f_k. \)

**Theorem 3.12.** (\([\ddagger]\)) Let \( \rho \) be a representation of an sgLa \( (\mathfrak{g}, [, , ]^\mathfrak{g}) \) on a graded vector space \( V \). Then \( \Psi \) is a homomorphism of gLas from \( (\mathfrak{s}C^*(V, \mathfrak{g}), [ , , ]^\mathfrak{g}) \) to \( (\tilde{C}^*(V, V), [, , ]^\mathfrak{g}) \).

By this theorem, we can obtain a pre-Lie\(_{\infty}\)-algebra from a homotopy \( O \)-operator.

**Corollary 3.13.** Let \( T = \sum_{i=0}^{\infty} T_i \in \text{Hom}^0(S(V), \mathfrak{g}) \) be a homotopy \( O \)-operator on an sgLa \( (\mathfrak{g}, [, , ]^\mathfrak{g}) \) with respect to a representation \( \rho : \mathfrak{g} \to \mathfrak{gl}(V) \). Then \( (V, \{ \theta_k \}_{k=0}^\infty) \) is a pre-Lie\(_{\infty}\)-algebra, where \( \theta_k : \mathfrak{g}^k V \to V \) \((k \geq 1)\) are linear maps of degree 1 defined by
\[
\theta_k(v_1, \cdots, v_k) := \rho(T_{k-1}(v_1, \cdots, v_{k-1}))v_k, \quad \forall v_1, \cdots, v_k \in V.
\]

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