DEEP RELAXATION OF CONTROLLED STOCHASTIC GRADIENT DESCENT VIA SINGULAR PERTURBATIONS

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Abstract. We consider a singularly perturbed system of stochastic differential equations proposed by Chaudhari et al. (Res. Math. Sci. 2018) to approximate the Entropic Gradient Descent in the optimization of deep neural networks, via homogenisation. We embed it in a much larger class of two-scale stochastic control problems and rely on convergence results for Hamilton-Jacobi-Bellman equations with unbounded data proved recently by ourselves (ESAIM Control Optim. Calc. Var. 2023). We show that the limit of the value functions is itself the value function of an effective control problem with extended controls, and that the trajectories of the perturbed system converge in a suitable sense to the trajectories of the limiting effective control system. These rigorous results improve the understanding of the convergence of the algorithms used by Chaudhari et al., as well as of their possible extensions where some tuning parameters are modeled as dynamic controls.

1. Introduction

This paper develops some stochastic control and Hamilton-Jacobi-Bellman methods related to general unconstrained non-convex optimization problems

\[
\min_{x \in \mathbb{R}^n} \phi(x)
\]

where \( \phi \) is a scalar function, usually highly nonlinear. This very classical problem received recently a considerable amount of new attention in connection with its role in training deep neural networks. We do not enter here in the numerous practical
applications of these problems and in the particular forms that the loss function $\phi$ may have, and refer instead to the survey [25], the papers [2, 14, 15, 26, 31] and the references therein. Let us just recall that in real-world applications the dimension $n$ of the unknown variable is extremely large, so that classical algorithms are often not efficient enough because of the well-known curse of dimensionality, and the function $\phi$ may have poor smoothness properties.

Gradient descent in continuous version, given by

$$\dot{X}_t = -\nabla \phi(X_t),$$

is known to converge to global minima for smooth convex functions $\phi$, otherwise to merely local minima or even saddles. The starting point of most modern algorithms is stochastic gradient descent (SGD)

$$dX_t = -\nabla \phi(X_t)dt + \sigma dW_t$$

where $W$ is a Wiener process and the stochasticity arises from computing $\nabla \phi$ only on a lower dimensional set of randomly sampled directions, called mini-batch.

The papers [2, 14] proposed to modify the loss function $\phi$ by an associated local entropy $\phi_\gamma$, see (1.2) below. This was shown to be effective in training high dimensional deep neural networks, see [14, 15, 30, 31]. The gradient of $\phi_\gamma$ is an average in space with respect to a Gibbs measure, see (1.4), (1.5), and such measure can be approximated by the long time average of an auxiliary stochastic differential equation (1.6). Chaudhari, Oberman, Osher, Soatto, and Carlier [15] show that the convergence of the resulting algorithm can be better understood by means of the homogenisation theory for two-scale stochastic differential equation. A first goal of this paper is to put this singular perturbation result on a more general and solid mathematical ground. We recall that, according to [15], “Deep neural networks achieved remarkable success in a number of domains(...). A rigorous understanding of the roots of this success, however, remains elusive”.

A second goal of the paper is to add a control variable to the entropic-SGD system and study the singular perturbation in the framework of stochastic control. As a first example, we follow Li, Tai and E [26] and consider the so-called learning rate of the algorithm as a control. However, our methods work for much more general control systems. Our main result does not require a gradient structure for the drift and allows a nonlinear dependence on a vectorial control. The rest of this Introduction gives more details on the problems, our results, and some related literature.

1.1. Local entropy and deep relaxation. Following [14] we consider $\phi_\gamma$, a regularization of the loss function $\phi$, such that

$$\phi_\gamma(x) := -\frac{1}{\beta} \log \left( G_{\beta, 1, \gamma} \ast \exp(-\beta \phi(x)) \right)$$

where

$$G_{\beta, 1, \gamma}(x) := (2\pi \gamma)^{-n/2} \exp \left( -\frac{\beta}{2\gamma} |x|^2 \right)$$

is the heat kernel, and $\beta, \gamma > 0$ are fixed parameter. The function $\phi_\gamma$ plays the role of a local entropy and it is a smooth approximation of $\phi$ as $\gamma \to 0$. The parameter
\( \beta \) corresponds in physics to the inverse of the temperature and it will remain fixed in our analysis. Note that, by setting

\[
\Phi(y, x) := \phi(y) + \frac{1}{2\gamma} |x - y|^2,
\]

we can write the local entropy as

\[
\phi_\gamma(x) = -\frac{1}{\beta} \log \left( \frac{1}{(2\pi \gamma)^{-n/2}} \int_{\mathbb{R}^n} \exp(-\beta \Phi(y, x)) \, dy \right).
\]

Then it is well-defined if \( \Phi \) is a confining potential for all \( x \), i.e., \( \exp(-\beta \Phi(\cdot, x)) \in L^1(\mathbb{R}^n) \) and \( \Phi(y, x) \to +\infty \) as \( |y| \to \infty \). And this is true if \( \phi \) has at most quadratic growth and \( \gamma \) is chosen sufficiently small.

An additional feature of this regularization procedure is that it “provides a way of picking large, approximately flat, regions of the landscape over sharp, narrow valleys, in spite of the latter possibly having a lower loss” [14]. This is important because in practice one is mostly interested in robust minima lying in wide valleys. A further analysis and numerical validations of entropic gradient descent can be found in the recent paper [31].

A direct computation shows that the gradient of the local entropy has the following nice structure [15, Lemma 1]

\[
\nabla \phi_\gamma(x) = \int_{\mathbb{R}^n} \frac{x - y}{\gamma} \rho^\infty(dy; x)
\]

where, for a suitable normalizing constant \( Z(x) \),

\[
\rho^\infty(y; x) := \frac{1}{Z(x)} \exp \left( -\beta \left( \phi(y) + \frac{1}{2\gamma} |x - y|^2 \right) \right).
\]

Next we recognize that \( \rho^\infty \) is the density of the Gibbs invariant measure of the process

\[
dY_s = -\nabla_y \Phi(Y_s, X_s) \, ds + \sqrt{\frac{2}{\beta}} \, dW_s,
\]

where \( W_s \) is a \( n \)-dimensional Brownian motion and \( x \) is frozen, provided \( \Phi \) is smooth enough. We will assume that \( \phi \in C^1 \) with a Lipschitz gradient, then this fact can be found in [8], whereas for \( \phi \in C^2 \) one can check directly that \( \mathcal{L}^* \rho^\infty = 0 \), where \( \mathcal{L} \) is the generator of the process (1.6).

Now the fact that \( \nabla \phi_\gamma \) in (1.3) is the average of \( y \mapsto \frac{1}{\gamma}(x - y) \) over a Gibbs measure is reminiscent of what often happens in homogenization and in singular perturbations. Indeed, the authors of [15] introduce the following system of singularly perturbed SDEs

\[
\begin{align*}
\frac{dX_s}{ds} &= -\nabla_x \Phi(Y_s, X_s) \, ds, \quad X_0 = x \in \mathbb{R}^n \\
\frac{dY_s}{ds} &= -\frac{1}{\varepsilon} \nabla_y \Phi(Y_s, X_s) \, ds + \sqrt{\frac{2}{\varepsilon \beta}} \, dW_s, \quad Y_0 = y \in \mathbb{R}^n.
\end{align*}
\]

They claim that such system converges, as \( \varepsilon \to 0 \), to

\[
\frac{d\hat{X}_s}{ds} = \int_{\mathbb{R}^n} -\frac{1}{\gamma}(X_s - y) \rho^\infty(dy; X_s) \, ds, \quad \hat{X}_0 = x \in \mathbb{R}^n
\]
which also writes, by (1.4),
\begin{equation}
\dot{X}_s = -\nabla \phi_\gamma(\hat{X}_s) \, ds, \quad \hat{X}_0 = x \in \mathbb{R}^n,
\end{equation}
that is the gradient descent of the regularized loss function.

The result we get on this simple model, as a consequence of our more general analysis for controlled processes, is the following.

**Corollary 1.1.** Let \( \phi \in C^1(\mathbb{R}^n) \) with \( \nabla \phi \) Lipschitz continuous. Then, for all \( T > 0 \), for \( \gamma \) in (1.3) small enough,

(i) for any \( y \in \mathbb{R}^n \) the \( x \)-component of the trajectory \((X^\varepsilon, Y^\varepsilon)\) of (1.7) converges to the solution of (1.8) in the sense
\[
\lim_{\varepsilon \to 0} \left( \int_0^T \mathbb{E} \left[ |X^\varepsilon_s - \hat{X}_s|^2 \right] \, ds + \mathbb{E} \left[ |X^\varepsilon_T - \hat{X}_T|^2 \right] \right) = 0,
\]
(ii) if for a sequence of processes \((X^{\varepsilon_n}, Y^{\varepsilon_n})\) solving (1.7), with \( \varepsilon_n \to 0 \), there is a deterministic process \( \bar{x} \cdot \) such that
\[
\lim_{\varepsilon_n \to 0} \int_0^T \mathbb{E} \left[ |X^{\varepsilon_n}_s - \bar{x}_s|^p \right] \, ds = 0,
\]
for some \( p \in [1, 2] \), then \( \bar{x} \cdot \) satisfies (1.8).

The proof is postponed to Section 4.2.

Note that the entropic gradient descent (1.8) does not involve the gradient of the loss function \( \phi \), which is usually hard to compute, because \( \nabla \phi_\gamma \) depends on \( \phi \) only via the Gibbs measure \( \rho^\infty \). In practical algorithms such measure is computed via long time averages of the process (1.6), and in the discrete-time scheme \( \nabla \phi \) is approximated by suitably chosen partial gradients, called *mini-batches*.

### 1.2. Deep relaxation with control of the learning rate.

Following the model in [26, §4], we introduce in (1.7) a control parameter \( u_s \) playing the role of a Learning Rate. Choosing this rate optimally allows to control in a dynamic way to what extent the process \( X^\varepsilon \) (and its limit \( \hat{X} \)) should follow the gradient descent, in other words, how trustful the gradient descent direction is. Usually the control \( u \) takes values in \([0, 1]\). In the sequel we consider \( U \subseteq \mathbb{R} \) as a compact set of values that the control \( u \) can take, and we write the system of *singularly perturbed controlled SDEs*
\begin{align}
\text{d}X^\varepsilon_s &= -u_s \nabla_x \Phi(Y^\varepsilon_s, X^\varepsilon_s) \, ds + \sqrt{2\sigma(X^\varepsilon_s, Y^\varepsilon_s, u_s)} \, dW_s, \\
\text{d}Y^\varepsilon_s &= -\frac{1}{\varepsilon} \nabla_y \Phi(Y^\varepsilon_s, X^\varepsilon_s) \, ds + \sqrt{\frac{2}{\varepsilon\beta}} \, dW_s,
\end{align}
where \( \Phi \) is defined in (1.3), and \( \sigma \) is a diffusion term that we add for the sake of generality and is allowed to be zero. The optimal learning rate should provide a balance between *exploitation* (how fast at each step should we follow the drift) and *exploration* (how much at each step should we diffuse and look around). Given
an appropriate payoff function for the problem of tuning the learning rate, we can write an optimal control problem of the form

\[
\min_u E \left[ g(X_T^\varepsilon) e^{\lambda(t-T)} + \int_t^T \ell(s, X_s^\varepsilon, Y_s^\varepsilon, u_s) e^{\lambda(t-s)} \, ds \mid X_t^\varepsilon = x, Y_t^\varepsilon = y \right]
\]

subject to (1.9), where \( \lambda \) is a non-negative constant, and \( g, \ell \) satisfy some growth assumptions that we will later make precise in section 2.2 and can be chosen according to the performance we seek (e.g., minimizing \( E[\phi(X_T^\varepsilon)] \) or the expected distance of \( X^\varepsilon \) from a reference trajectory).

Our first main result, Theorem 3.1, says that the problem (1.9)-(1.10) converges in a variational sense (i.e., value function goes to value function) as \( \varepsilon \to 0 \), to a limit effective control problem with extended controls whose dynamics is

\[
d\hat{X}_s = \mathcal{F}(\hat{X}_s, \nu_s) \, ds + \sqrt{2\sigma}(\hat{X}_s, \nu_s) \, dW_s, \quad \hat{X}_0 = x \in \mathbb{R}^n,
\]

\[
\mathcal{F}(\hat{x}, \nu) := -\int_{\mathbb{R}^n} \frac{\hat{x} - y}{\gamma} \nu(y) \rho^\infty(\,dy; \hat{x}), \quad \nu \in U^{\infty} := L^{\infty}(\mathbb{R}^m, U),
\]

(1.12)

\[
\mathcal{G}(\hat{x}, \nu) := \sqrt{\sum_{s,t} \sigma(\hat{x}, y, \nu(y)) \rho^\infty(\,dy; \hat{x}),
\]

where \( \sqrt{M} \) denotes the matrix square root of a positive semi-definite matrix \( M \), and the payoff functional has the same form as (1.10) with the effective running payoff

\[
\mathcal{G}(s, \hat{x}, \nu) := \int_{\mathbb{R}^n} \ell(s, \hat{x}, y, \nu(y)) \rho^\infty(\,dy; \hat{x}).
\]

Note that if \( \nu_t(y) = u_t \) is constant in \( y \), then \( \mathcal{F}(x_t, \nu_t) = -u_t \nabla \phi_\gamma(x_t) \), where \( \phi_\gamma \) is the local entropy associated to \( \phi \), see (1.2), and its gradient is given by (1.3). Therefore, by taking \( \sigma \equiv 0 \) and \( U = \{1\} \), we recover (1.8) as a particular case.

Let us illustrate such result on a practical example. Let \( \mathcal{U} \) be the set of measurable functions \([0, T] \to U \). Consider the value function of the deterministic control problem consisting of minimizing the loss function via trajectories of the entropic gradient descent controlled by the learning rate, i.e.,

\[
\mathcal{V}(x) := \inf_{u \in \mathcal{U}} \phi(X_T^\varepsilon)
\]

\[
s.t. \quad \dot{X}_t = -u_t \nabla \phi_\gamma(X_t), \quad t \in [0, T], \quad X_0 = x \in \mathbb{R}^n.
\]

Its deep relaxation is the singularly perturbed optimal control problem

\[
\mathcal{V}_\varepsilon(x, y) := \inf_{u \in \mathcal{U}} E[\phi(X_T^\varepsilon)]
\]

\[
s.t. \quad dX_t^\varepsilon = -u_t \frac{X_t^\varepsilon - Y_t^\varepsilon}{\gamma} \, dt,
\]

\[
dY_t^\varepsilon = -\frac{1}{\varepsilon} \left( \nabla \phi(Y_t^\varepsilon) - \frac{X_t^\varepsilon - Y_t^\varepsilon}{\gamma} \right) \, dt + \sqrt{\frac{2}{\varepsilon \beta}} \, dW_t,
\]

\[
X_0^\varepsilon = x \in \mathbb{R}^n, \quad Y_0^\varepsilon = y \in \mathbb{R}^n, \quad t \in [0, T],
\]

where \( y \) is arbitrary.
Theorem 1.1. Let \( \phi \in C^1(\mathbb{R}^n) \) with \( \nabla \phi \) Lipschitz continuous. Then for all \( \gamma \) in (1.3) small enough
\[
\lim_{\varepsilon \to 0} V^\varepsilon(x, y) \leq V(x)
\]
locally uniformly in \( x, y \in \mathbb{R}^n \), i.e., the perturbed dynamics yields a value not larger than the one with a controlled full gradient descent.

The proof is based on the fact that \( \lim_{\varepsilon \to 0} V^\varepsilon \) is the following value function:
\[
V(x) := \inf_{\nu \in U^\varepsilon} \phi(\hat{X}_T) \quad \text{s.t.} \quad \frac{d}{ds} \hat{X}_s = \tilde{f}(\hat{X}_s, \nu_s), \quad t \in [0, T] \quad X_0 = x \in \mathbb{R}^n,
\]
where \( U^\varepsilon \) is the set of measurable functions \( [0, T] \to U^\varepsilon \) and \( \tilde{f} \) is the effective dynamics (1.12). The details are postponed to Section 4.2.

The other main results of the paper, Theorems 4.1 and 4.2, imply that for \( \sigma \equiv 0 \) also the trajectories of (1.9) converge to those of (1.11) in a sense similar to the preceding Corollary 1.1. A consequence of these properties is the possibility of approximating optimal trajectories of the effective problem (1.16) by sub-optimal trajectories for the perturbed problem (1.15). By this we mean pairs \( (X^\varepsilon_n, Y^\varepsilon_n) \) such that
\[
E[\phi(X^\varepsilon_n_T)] \leq V^\varepsilon_n(x, y) + o(1), \quad \text{as } \varepsilon_n \to 0.
\]

Corollary 1.2. Under the assumptions of Theorem 1.1 with \( U \) convex, let \( (X^\varepsilon_n, Y^\varepsilon_n) \) be trajectories of (1.15) satisfying (1.17),
\[
\lim_{\varepsilon_n \to 0} \int_0^T E[|X^\varepsilon_n_s - \bar{x}_s|^p] \, ds = 0,
\]
for a deterministic process \( \bar{x}_s \) and some \( p \in [1, 2] \), and
\[
\limsup_{\varepsilon_n \to 0} E[\phi(X^\varepsilon_n_T)] \geq \phi(\bar{x}_T).
\]
Then \( \bar{x}_s \) is optimal for the problem (1.10), i.e., for some \( \nu \in U^\varepsilon \) it is a trajectory of the effective system in (1.10) and \( \phi(\bar{x}_T) = V(x) \).

The proof can be found in Section 4.2.

1.3. General singular perturbations in stochastic control. We are going to embed the deep relaxation problems described so far in a more general setting of singularly perturbed stochastic control systems, namely,
\[
dX_s = f(X_s, Y_s, u_s) \, ds + \sqrt{2} \sigma(X_s, Y_s, u_s) \, dW_s,
\]
\[
dY_s = \frac{1}{\varepsilon} b(X_s, Y_s) \, ds + \sqrt{\frac{2}{\varepsilon}} \phi(X_s, Y_s) \, dW_s,
\]
where \( X_s \in \mathbb{R}^n \) is the slow dynamics, \( Y_s \in \mathbb{R}^m \) is the fast dynamics, \( u_s \) is the control taking values in a given compact set \( U \), and \( W_s \) is a multidimensional Brownian motion. We will make suitable regularity and growth assumptions on the data of such system, but we will not require the gradient structure of the drifts in (1.9). The diffusion coefficient of the process \( X_s \) can be degenerate (i.e. \( \sigma^\varepsilon = 0 \) is allowed), whereas the one of \( Y_s \) is required to be non-degenerate. The precise assumptions
are given in Section 2. We will consider optimization problems with cost/payoff function of the general form

\begin{equation}
J(t, x, y, u) := E \left[ e^{\lambda(t-T)} g(X_T, Y_T) + \int_t^T \ell(s, X_s, Y_s, u_s) e^{\lambda(t-s)}\, ds \mid X_t = x, Y_t = y \right].
\end{equation}

The value function \( V^\varepsilon(t, x, y) \) of such problem is known to solve in the viscosity sense a fully nonlinear, degenerate parabolic PDE of Hamilton-Jacobi-Bellman type in \((0, T) \times \mathbb{R}^n \times \mathbb{R}^m\). From our results in the companion paper [5] we have the convergence of \( V^\varepsilon \) to the solution \( V(t, x) \) of a suitable HJB PDE in \((0, T) \times \mathbb{R}^n\), as \( \varepsilon \to 0 \). Here we first show that \( V \) is the value function of a limit effective optimal control problem in reduced dimension \( n \), but with a larger set of extended controls \( U^{ex} \). This limit problem is based on averaging with respect to a probability measure that is fully characterized as invariant measure of an ergodic process, although in general it is not as explicit as the Gibbs measure in (1.11), and (1.13). Then we make suitable choices of \( \ell \) and \( g \) in the functional \( J \) (1.20) to deduce two convergence theorems for trajectories of (1.19) to trajectories of the effective system. These results are new also for problems with data bounded in the fast variables \( Y \), for which the convergence in HJB equations was already proved in [3, 4].

The generality of the system (1.19) is motivated by applications different from the Deep Learning problems studied here. The results of the present paper can be used for models of pricing and trading derivative securities in financial markets with stochastic volatility, as in [4] and [21], or for applications in economics and advertising theory as in [3].

There is a wide literature on singular perturbations for control systems that goes back to the late 60’s [22, 24], and also for diffusion processes, with and without control, and many different models with fast variables have been studied since then, both in deterministic and stochastic settings and using methods of probability, analysis, measure theory, or control. We refer the reader to the introductions of [3, 4] and the large but non-exhaustive list of references therein. Let us mention some additional papers on singular perturbation problems for stochastic differential equations: for the case without control the papers [27, 29] by Pardoux and Veretennikov; for problems with control the work of Borkar and Gaitsgory [12, 13] where the control acts on both the slow and the fast variables, and is analyzed by means of Limit Occupational Measures.

The paper is organized as follows. In Section 2 we list the basic assumptions on (1.19) and (1.20), then recall our results from [5] on the HJB equations and the convergence of their solutions that are crucial for our subsequent analysis. In Section 3 we study a new optimal control problem, the effective problem, and show that the limit \( V \) of the value functions \( V^\varepsilon \) is indeed the value function of such problem. In Section 4 we study the convergence of the trajectories, and finally apply all this to the deep relaxation problems described in Sections 1.1 and 1.2.
2. The two scale stochastic control problem

2.1. The system. Let \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) be a complete filtered probability space and let \((W_t)\) be an \(\mathcal{F}_t\)-adapted standard \(r\)-dimensional Brownian motion. We consider the stochastic control system \([1.19]\), where \(u_t\) is in the set of admissible control functions \(U\), i.e., the set of \(\mathcal{F}_t\)-progressively measurable processes taking values in \(U\). Note that it is much more general than the systems \([1.7]\) and \([1.9]\) arising in deep relaxation, in particular the vector fields do not have a gradient structure. However, we will not present the results in the full generality of our paper \([5]\), instead we strengthen a bit some conditions in order to simplify the statements of this section. In Section 3 we will strengthen further the assumptions to get detailed results on the effective control problem. We denote by \(M_{n,m}(\text{resp. } S_n)\) the set of matrices of \(n\) rows and \(m\) columns (resp. the subset of \(n\)-dimensional squared symmetric matrices).

Assumptions (A):

(A1) For a given compact set \(U\), \(f : \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{R}^n\), \(\sigma^\varepsilon : \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{M}^{n,r}\), and \(b : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m\) are continuous functions, Lipschitz continuous in \((x, y)\) uniformly with respect to \(u \in U\) and \(\varepsilon > 0\).

(A2) The diffusion \(\sigma^\varepsilon\) driving the slow variables \(X_t\) satisfies \(\lim_{\varepsilon \to 0} \sigma^\varepsilon(x, y, u) = \sigma(x, y, u)\) locally uniformly, where \(\sigma : \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{M}^{n,r}\) satisfies the same conditions as \(\sigma^\varepsilon\).

(A3) The diffusion \(\bar{\rho}\) is constant such that \(\bar{\rho} \bar{\rho}^\top = \bar{\rho} I_m\) where \(\bar{\rho} > 0\) is a constant and \(I_m\) is the identity matrix.

(A4) The drift \(b\) satisfies the strong monotonicity condition

\[
\exists \kappa > 0 \quad \text{s.t.} \quad (b(x, y_1) - b(x, y_2)) \cdot (y_1 - y_2) \leq -\kappa |y_1 - y_2|^2, \quad \forall x, y_1, y_2.
\]

Note that assumption (A1) implies that \(f, \sigma^\varepsilon, \) and \(b\) have linear growth in both \(x\) and \(y\), that is, for some positive constant \(C\),

\[
|f(x, y, u)|, \|\sigma^\varepsilon(x, y, u)\| \leq C(1 + |x| + |y|), \quad \forall x, y, u, \forall \varepsilon > 0,
\]

\[
|b(x, y)| \leq C(1 + |x| + |y|), \quad \forall x, y.
\]

We remark that we will not make any non-degeneracy assumption on the matrices \(\sigma^\varepsilon, \sigma\), so the cases \(\sigma^\varepsilon, \sigma \equiv 0\) are allowed.

2.2. The optimal control problem and HJB equation. The functional \(J\) with finite horizon on the time interval \([t, T]\), \(t > 0\), and discount factor \(\lambda \geq 0\), is defined by \([1.20]\). In this section, for consistency with our paper \([5]\), we consider \(J\) as a payoff and the optimal control problem consists of maximizing it. The minimization problem of the previous section is easily recovered by changing signs to \(\ell\) and \(g\). Then the value function is

\[
(OCP(\varepsilon)) \quad V^\varepsilon(t, x, y) := \sup_{u \in U(t, x, y, u)} \text{subject to } (1.19).
\]

The following assumption concerns the utility function \(g\) and the running payoff \(\ell\):
Assumption (B): The discount factor is $\lambda \geq 0$, and the utility function $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and running payoff $\ell : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{R}$ are continuous, Lipschitz in $y$ uniformly in their other arguments, and satisfy

$$
(2.4) \quad \exists K > 0 \text{ s.t. } |g(x, y)|, |\ell(s, x, y, u)| \leq K(1 + |x|^2 + |y|), \quad \forall s \in [0, T], x, y, u.
$$

The HJB equation associated via Dynamic Programming to the value function $V^\varepsilon$ is

$$
(2.5) \quad -V^\varepsilon_t + F^\varepsilon \left( t, x, y, V^\varepsilon, D_x V^\varepsilon, \frac{D_y V^\varepsilon}{\varepsilon}, \frac{D^2_{yy} V^\varepsilon}{\varepsilon}, \frac{D^2_{xy} V^\varepsilon}{\varepsilon} \right) = 0,
$$

in $(0, T) \times \mathbb{R}^n \times \mathbb{R}^m$, complemented with the terminal condition

$$
(2.6) \quad V^\varepsilon(T, x, y) = g(x, y).
$$

The Hamiltonian $F^\varepsilon : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times S^n \times S^m \times M^{n,m} \to \mathbb{R}$ is

$$
F^\varepsilon(t, x, y, p, q, M, N, Z) := \min_{u \in U} \left\{ -\text{trace}(\sigma^\varepsilon \sigma^\varepsilon^\top M) - \ell \cdot p - 2\text{trace}(\sigma^\varepsilon \sigma^\varepsilon^\top Z^\top) - \ell \right\}
$$

where $\sigma^\varepsilon, f$ are computed at $(x, y, u)$, $\ell = \ell(t, x, y, u)$, and

$$
\mathcal{L}(x, y, q, N) := b(x, y) \cdot q + \text{trace}(qq^\top N).
$$

This is a fully nonlinear degenerate parabolic equation (strictly parabolic in the $y$ variables by the assumption (A3)). We define also the Hamiltonian $H$ as $H^\varepsilon$ when $\sigma^\varepsilon$ is replaced by $\sigma$:

$$
(2.7) \quad H(t, x, y, p, M, Z) := \min_{u \in U} \left\{ -\text{trace}(\sigma \sigma^\top M) - f \cdot p - 2\text{trace}(\sigma \sigma^\top Z^\top) - \ell \right\}.
$$

The next result is standard, see, e.g., [4, Proposition 3.1] or in [3, Prop. 2.1].

**Proposition 2.1.** Assume (A) and (B). Then for any $\varepsilon > 0$, the function $V^\varepsilon$ in $(OCP(\varepsilon))$ is the unique continuous viscosity solution to the Cauchy problem $(2.5)-(2.6)$ with at most quadratic growth in $x$ and $y$, i.e.,

$$
\exists K > 0 \text{ such that } |V^\varepsilon(t, x, y)| \leq K(1 + |x|^2 + |y|^2), \quad \forall t \in [0, T], x \in \mathbb{R}^n, y \in \mathbb{R}^m.
$$

Moreover the functions $V^\varepsilon$ are locally equibounded.

We remark that $V^\varepsilon$ is not bounded in $y$, contrary to what occurs in [3,4], but it has at most quadratic growth. This comes from the assumptions 2.2 and 2.4.

### 2.3. The limit PDE and convergence of the value functions.

Consider the fast subsystem,

$$
(2.8) \quad dY_t = b(x, Y_t) \, dt + \sqrt{2} \, dW_t, \quad Y_0 = y \in \mathbb{R}^m
$$

obtained by putting $\varepsilon = 1$ in $(1.19)$, freezing $x \in \mathbb{R}^n$, and applying (A3). It is a non-degenerate diffusion process that is known to be *ergodic* by the monotonicity condition (A4), see [5, §3].
Proposition 2.2. Under the assumptions (A1), (A3), (A4), for all \( x \in \mathbb{R}^n \) the process (2.8) has a unique invariant probability measure \( \mu_x \), which has finite moments of any order. Moreover, \( x \mapsto \mu_x \) is Lipschitz with respect to the 2-Wasserstein distance.

Proof. For existence and uniqueness, we refer to [5 §3.1] and the references therein. See also [11 Examples 5.1 & 5.5]. The moments being finite is [5 Lemma 3.1]. For Lipschitz continuity, see (2.11)-(2.12) below or [5 Lemma 4.1 & eq. (4.9)]. \( \square \)

For \( 1 \leq p < +\infty \), let \( m_p(x) \) denote the \( p \)-moment of the invariant probability measure \( \mu_x \)

\[
m_p(x) := \int_{\mathbb{R}^n} |y|^p \, d\mu_x(y).
\]

Proposition 2.3. Assume (A1), (A3) and (A4) hold. Let \( \mu_{x_1}, \mu_{x_2} \) be the unique invariant probability measures corresponding to (2.8) with \( x_1, x_2 \in \mathbb{R}^n \). Then

\[
\left| m^{1/p}_p(x_1) - m^{1/p}_p(x_2) \right| \leq W_p(\mu_{x_1}, \mu_{x_2}).
\]

In particular, there exists a constant \( C > 0 \) such that

\[
|\mu_1(x_1) - \mu_1(x_2)| + \left| \mu^{1/2}_2(x_1) - \mu^{1/2}_2(x_2) \right| \leq C |x_1 - x_2|,
\]

and for all \( x \in \mathbb{R}^n \)

\[
\mu_1(x) + \mu^{1/2}_2(x) \leq C(1 + |x|).
\]

Proof. It is known that any measure \( \pi \in \mathcal{P}(\mathbb{R}^m \times \mathbb{R}^m) \) whose marginals are some measure \( \mu \in \mathcal{P}(\mathbb{R}^m) \) and \( \delta_{a} \) a Dirac measure supported in some \( a \in \mathbb{R}^m \), is such that \( \pi = \mu \otimes \delta_{a} \). That is, such subset of \( \mathcal{P}(\mathbb{R}^m \times \mathbb{R}^m) \) is the singleton \( \{ \mu \otimes \delta_{a} \} \).

Hence

\[
W_p^p(\mu, \delta_{a}) = \int_{\mathbb{R}^m} |y - a|^p \, d\mu(y).
\]

So when \( a = 0 \) we have \( W_p^p(\mu_{x}, \delta_{0}) = m_p(x) \). Using the triangle inequality, one gets

\[
m^{1/p}_p(x_1) \leq W_p(\mu_{x_1}, \mu_{x_2}) + W_p(\mu_{x_2}, \delta_{0}) = W_p(\mu_{x_1}, \mu_{x_2}) + m^{1/p}_p(x_2)
\]

and then \( m^{1/p}_p(x_1) - m^{1/p}_p(x_2) \leq W_p(\mu_{x_1}, \mu_{x_2}) \). Exchanging the roles of \( x_1, x_2 \) yields the desired result.

To prove the second statement, we recall from [11 Corollary 1] the estimate

\[
W_2(\mu_{x_1}, \mu_{x_2}) \leq C \left( \int_{\mathbb{R}^m} \left| b(x_1, y) - b(x_2, y) \right|^2 d\mu_{x_2}(y) \right)^{1/2}.
\]

Then, with the Lipschitz continuity of \( b \), one gets

\[
W_2(\mu_{x_1}, \mu_{x_2}) \leq C|x_1 - x_2|.
\]

Recalling \( W_1(\mu_{x_1}, \mu_{x_2}) \leq W_2(\mu_{x_1}, \mu_{x_2}) \), one has

\[
\text{for } p = 1, 2 : \quad \left| m^{1/p}_p(x_1) - m^{1/p}_p(x_2) \right| \leq C|x_1 - x_2|.
\]
Now let $x_1 = x$, $x_2 = 0$ and $p = 1, 2$. One has

$$m_1(x) \leq C|x| + m_1(0) \quad \text{and} \quad m_2^{1/2}(x) \leq C|x| + m_2^{1/2}(0).$$

Then $m_1(x) + m_2^{1/2}(x) \leq C(1 + |x|)$ for some $C > 0$ constant depending only on $b(0, \cdot), \bar{g}$ in \((2.8)\). \hfill \Box

Now we can define the effective Hamiltonian $\overline{H}$ and effective initial data $\overline{\sigma}$ associated to the singularly perturbed PDE \((2.6)\) as follows

$$\overline{H}(t, x, p, P) := \int_{\mathbb{R}^m} H(t, x, y, p, P, 0) d\mu(y), \quad \overline{\sigma}(x) := \int_{\mathbb{R}^m} g(x, y) d\mu(y).$$

Note that they are finite for all entries by the growth conditions \((2.2)\) and \((2.4)\). We expect that the value function $V(t, x)$ independent of $y$ which is the unique solution of the Cauchy problem

\begin{equation}
\begin{aligned}
-V_t + \overline{H}(t, x, D_x V, D^2_{xx} V) + \lambda V(x) &= 0, & \text{in } (0, T) \times \mathbb{R}^n, \\
V(T, x) &= \overline{\sigma}(x), & \text{in } \mathbb{R}^n.
\end{aligned}
\end{equation}

This is the statement of the next theorem for which we need the following

**Assumption \((C)\):** For $\Sigma := \sigma \sigma^\top(x, y, u)$, one of these two conditions is satisfied

(a) $\Sigma$ is independent of $y$ and $u$, i.e. $\sigma = \sigma(x)$; or

(b) the drift of the fast process is independent of $x$, i.e. $b = b(y)$, and the matrix function $x \mapsto \Sigma(x, y, u)$ has two continuous spatial derivatives satisfying

(i) there exists $\theta(\cdot) : \mathbb{R}^m \to \mathbb{R}$ such that $\forall u \in U, x \in \mathbb{R}^n$, and $y \in \mathbb{R}^m$

$$\max_{1 \leq i \leq n} \left| \frac{\partial}{\partial x_i} \Sigma(x, y, u) \xi \cdot \xi \right| \leq \theta(y) |\xi|^2, \quad \xi \in \mathbb{R}^n, \quad \int_{\mathbb{R}^m} |\theta(y)| d\mu(y) < +\infty,$$

where $\mu$ in the unique invariant probability measure of the fast process \((2.8)\) (that is now independent of $x$),

(ii) $\exists K > 0$ constant such that for all $u \in U$, $x \in \mathbb{R}^n$, and $y \in \mathbb{R}^m$

$$\max_{1 \leq i \leq n} \left| \frac{\partial^2}{\partial x_i^2} \Sigma(x, y, u) \xi \cdot \xi \right| \leq K |\xi|^2, \quad \forall \xi \in \mathbb{R}^n.$$

Note that in the case (b), thanks to Proposition \((2.2)\), it is sufficient to have $\theta$ with at most a polynomial growth to satisfy the integrability assumption.

**Theorem 2.1.** Assume \((A)\), \((B)\), and \((C)\). Then the solution $V^\varepsilon$ to \((2.3)\) converges uniformly on compact subsets of $[0, T) \times \mathbb{R}^n \times \mathbb{R}^m$ to the unique continuous viscosity solution of the limit problem \((2.14)\) satisfying a quadratic growth condition in $x$, i.e.

\begin{equation}
\exists K > 0 \text{ such that } |V(t, x)| \leq K(1 + |x|^2), \quad \forall (t, x) \in [0, T) \times \mathbb{R}^n.
\end{equation}

Moreover, the convergence is uniform up to time $t = T$ if $g$ is independent of $y$, i.e., $\overline{\sigma}(x) = g(x)$.

**Proof.** See [5, Theorem 4.4]. \hfill \Box
In the next sections we provide an optimal control representation of the solution to this limit problem and prove the convergence of the trajectories of (1.19) to a suitable limit system.

3. The effective control problem

The previous result states the convergence of the value function $V^\epsilon$ to the solution $V$ of a Cauchy problem of an effective PDE. In this section we first show that such PDE is the HJB equation for a suitable optimal control problem, that we call effective. Next we show, under slightly stronger assumptions, that $V$ is indeed the value function of such problem. Since the conditions on the payoff/cost functional are very general, this can be interpreted as a weak “variational” convergence result for the trajectories of (1.19) to the trajectories of the effective system. Such convergence will be made more precise in the final Section 4.

3.1. A control interpretation of the limit PDE. The following result is [5, Proposition 3.6], and allows to represent the effective Hamiltonian (2.13) as a Bellman Hamiltonian associated to an effective optimal control problem, where the control set $U$ is replaced by the extended control set $U^\text{ex} := \{\nu : \mathbb{R}^m \to U \text{ measurable}\}$.

Proposition 3.1. Assume (A) and (B). Then the effective Hamiltonian (2.13) satisfies

$$ (3.1) \quad \mathcal{H}(t, x, p, P) = \min_{\nu \in U^\text{ex}} \int_{\mathbb{R}^m} \left[ -\text{trace}(\sigma \sigma^\top P) - f \cdot p - \ell \right] \, d\mu_x(y) $$

where $\sigma$ and $f$ are computed in $(x, y, \nu(y))$, and $\ell$ in $(t, x, y, \nu(y))$.

Remark 3.1. The extended control set $U^\text{ex}$ contains a copy of $U$, the constant functions, and it coincides with $L^p((\mathbb{R}^m, \mu_x), U)$ for all $x$ and all $p \in [1, +\infty]$, since $\mu_x$ is finite and $U$ is compact.

Remark 3.2. When the controls are decoupled from the fast variables, i.e.,

$$ f(x, y, u) = f_1(x, u) + f_2(x, y), \quad \sigma(x, y, u) = \sigma_1(x, u) + \sigma_2(x, y), $$

$$ \ell(x, y, u) = \ell_1(x, u) + \ell_2(x, y), $$

and $\sigma_1(x, u)^\top \sigma_2(x, y) = 0$ for all $x, y, u$, then repeating the same proof of Proposition 3.1 shows that extended controls are not necessary because $\mathcal{H}$ holds with $\nu(y) = y$ and $U^\text{ex}$ replaced by $U$. This was done in [24] and [3], but the decoupling of $f$ does not occur in our motivating model (1.15). Note that the condition $\sigma_1(x, u)^\top \sigma_2(x, y) = 0$ is satisfied if the diffusion term $\sigma dW_s$ is of the form $\sigma_1 dW^1_s + \sigma_2 dW^2_s$ with $W^1$ and $W^2$ independent.

3.2. The limit $V$ is a value function. In view of the last result it is natural to define the effective drift and diffusion as

$$ (3.2) \quad \mathcal{f}(x, \nu) := \int_{\mathbb{R}^m} f(x, y, \nu(y)) \, d\mu_x(y), \quad \mathcal{\sigma}(x, \nu) := \sqrt{\int_{\mathbb{R}^m} \sigma \sigma^\top(x, y, \nu(y)) \, d\mu_x(y)}, $$
the measure $\mu_x$ being defined in Proposition 2.2 and consider as effective control system

$$\begin{aligned}
\begin{cases}
\dot{X}_s = \bar{f}(X_s, \nu_s) ds + \sqrt{2\sigma}(X_s, \nu_s) dW_s, \\
\nu_s \in \mathcal{U}^{ex}, \quad \text{and} \quad \dot{X}_t = x \in \mathbb{R}^n,
\end{cases}
\end{aligned}$$

(3.3)

where $\mathcal{U}^{ex}$ is the set of progressively measurable processes taking values in the extended control set $U^{ex}$. Define also the effective data for the payoff functional as

$$\begin{aligned}
\bar{g}(x) := \int_{\mathbb{R}^n} g(x, y) d\mu_x(y) \quad \text{and} \quad \bar{\ell}(s, x, \nu) := \int_{\mathbb{R}^m} \ell(s, x, y, \nu(y)) d\mu_x(y).
\end{aligned}$$

(3.4)

Note that we can now rewrite the effective Hamiltonian $\mathcal{H}$ as the Bellman Hamiltonian of such effective optimal control problem

$$\mathcal{H}(t, x, p, P) = \min_{\nu \in \mathcal{U}^{ex}} \left[ -\text{trace}(\bar{\sigma}(x, \nu)^T P) - \bar{f}(x, \nu) \cdot p - \bar{\ell}(t, x, \nu) \right].$$

We want to identify the solution of (2.13) with the value function of the effective optimal control problem. In order to fit the classical setting of stochastic optimal control, we need to strengthen our previous assumptions as follows.

**Assumption (A'):** It includes all the conditions in (A), with $U \subseteq \mathbb{R}^k$ for some $k$ and (A1) complemented by the existence of constants $0 < \alpha \leq 1$, $C'_R \geq 0$ such that for all $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $u_i \in U$, $i = 1, 2$, $|x| \leq R$, $\omega$ and $\mu_x$ depending on $R > 0$, $\bar{C} > 0$, and a modulus $\omega$, s.t. for all $t_i \in [0, T]$, $x_i \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $u_i \in U$, $i = 1, 2$, $|x_1| \leq R$,

$$|f(x, y, u_1) - f(x, y, u_2)| \leq C'_R (1 + |y|) |u_1 - u_2|^{\alpha},$$

(3.5)

$$|\sigma(x, y, u_1) - \sigma(x, y, u_2)| \leq C'_R (1 + |y|)|u_1 - u_2|^{\alpha}.$$

(3.6)

This assumption adds to (A) the Hölder continuity in $u$ of the drift $f$ and of the diffusion $\sigma$, locally in $(x, y)$.

**Assumption (B'):** It includes all the conditions in (B), with $U \subseteq \mathbb{R}^k$, and complemented by the existence of constants $\alpha, \beta \in (0, 1]$, $C_R, C'_R$ depending on $R > 0$, $\bar{C} > 0$, and a modulus $\omega$, s.t. for all $t_i \in [0, T]$, $x_i \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $u_i \in U$, $i = 1, 2$, $|x_1| \leq R$,

$$|\ell(t_1, x_1, y, u_1) - \ell(t_2, x_2, y, u_2)| \leq \bar{C}_R (1 + |y|) \omega(|t_1 - t_2| + |x_1 - x_2|)$$

(3.7)

$$|\ell(t_1, x_1, y, u_1) - \ell(t_2, x_2, y, u_2)| \leq C'_R (1 + |y|)|u_1 - u_2|^{\alpha},$$

This assumption adds to (B) a suitable uniform continuity or local Hölder continuity of $\ell$ and $g$.

The next assumption concerns the invariant probability measure $\mu_x$ introduced in Proposition 2.2.

**Assumption (D):** For each fixed $x$, the invariant measure satisfies $\mu_x(dy) = m(x, y)dy$ and its density $m(x, y)$ is such that for all $R > 0$, $\exists C_R > 0$ such that

$$0 \leq m(x, y) \leq C_R m_0(y), \quad \forall |x| \leq R$$

(3.8)

where $m_0$ is the density of a positive finite measure.
The existence of a density \( m(x, y) \) for the invariant measure \( \mu_x(dy) \) is guaranteed with assumption (A); see [9, Corollary 1.4]. Then assumption (D) is automatically satisfied in the case of assumption (C)(b), because \( m \) is constant in \( x \).

**Remark 3.3.** In the motivating model problems of Sections 1.1 and 1.2 all these assumptions are satisfied if \( \nabla \phi \) is Lipschitz. The density in assumption (D) is \( m_\phi(y) = \exp(-\beta \phi(y)) \). A more general case where (3.8) holds is when the drift \( \nabla m \) is a gradient, i.e., \( \alpha \equiv \beta \). Finally, the control set we shall consider is

\[
\begin{align*}
U^{ex} &= L^\infty(\mathbb{R}^m, U) \\
\text{endowed with the norm } \| \cdot \|_{L^2_{m_\phi}},
\end{align*}
\]

where \( m_\phi \) is defined in assumption (D), and \( \alpha \) is the one in (3.3). We recall the weighted Lebesgue space for \( r \in [1, \infty) \)

\[
L^r_{m_\phi} := \left\{ \varphi : \mathbb{R}^m \to \mathbb{R} \ \text{s.t.} \ \| \varphi \|_{L^r_{m_\phi}} := \int_{\mathbb{R}^m} |\varphi(y)|^r m_\phi(y) \, dy < +\infty \right\}.
\]

Note that \( (U^{ex}, \| \cdot \|_{L^2_{m_\phi}}) \) is a separable complete normed space, and recall \( U^{ex} \) is the set of progressively measurable processes taking values in \( U^{ex} \).

**Proposition 3.2.** Assume (A'), (C), and (D). Then there exist constants \( C \) and, for all \( R > 0 \), \( K_R \) such that

\[
\begin{align*}
|\bar{f}(x_1, \nu_1) - \bar{f}(x_2, \nu_2)| &\leq C |x_1 - x_2| + K_R \| \nu_1 - \nu_2 \|_{L^2_{m_\phi}}, \quad \text{for } |x_1|, |x_2| \leq R, \\
|\bar{\sigma}(x_1, \nu_1) - \bar{\sigma}(x_2, \nu_2)| &\leq C |x_1 - x_2| + K_R^{1/2} \| \nu_1 - \nu_2 \|_{L^2_{m_\phi}}^{1/2}, \quad \text{for } |x_1|, |x_2| \leq R, \\
|\bar{f}(x, \nu)| &\leq C(1 + |x|), \quad |\bar{\sigma}(x, \nu)| \leq C(1 + |x|) \quad \forall x, \nu.
\end{align*}
\]

Consequently, for each \( \nu, \in U^{ex} \) the stochastic differential equation (3.3) has a unique strong solution.

**Proof.** Here \( C \) denotes any positive constant which may change from line to line and from an estimate to another, and is only depending on the data (through the constants in the standing assumptions). We begin with the effective drift.

Let \( (x_i, \nu_i) \in \mathbb{R}^n \times L^\infty(\mathbb{R}^m, U), \ i = 1, 2 \). We have

\[
\begin{align*}
\bar{f}(x_1, \nu_1) - \bar{f}(x_2, \nu_2) &= \int_{\mathbb{R}^m} f(x_1, y, \nu_1(y)) \, d\mu_{x_1}(y) - \int_{\mathbb{R}^m} f(x_2, y, \nu_1(y)) \, d\mu_{x_1}(y) \\
&\quad + \int_{\mathbb{R}^m} f(x_2, y, \nu_1(y)) \, d\mu_{x_1}(y) - \int_{\mathbb{R}^m} f(x_2, y, \nu_2(y)) \, d\mu_{x_1}(y) \\
&\quad + \int_{\mathbb{R}^m} f(x_2, y, \nu_2(y)) \, d\mu_{x_1}(y) - \int_{\mathbb{R}^m} f(x_2, y, \nu_2(y)) \, d\mu_{x_2}(y) \\
&=: (I) + (II) + (III).
\end{align*}
\]
We have the following

\[ |I| \leq \int_{\mathbb{R}^m} |f(x_1, y, \nu_1(y)) - f(x_2, y, \nu_1(y))| \, d\mu_x(y) \leq C |x_1 - x_2|. \]

Next, for \( R > 0 \) such that \( |x_1|, |x_2| \leq R \), using assumptions (A') and (D), as well as Cauchy-Schwarz inequality, we get

\[ |II| \leq \int_{\mathbb{R}^m} |f(x_2, y, \nu_1(y)) - f(x_2, y, \nu_2(y))| \, d\mu_x(y) \]
\[ \leq C_R \int_{\mathbb{R}^m} (1 + |y|)|\nu_1(y) - \nu_2(y)| \, d\mu_x(y) \]
\[ \leq C_R C_R \left( \|\nu_1 - \nu_2\|_{L_{\infty}^m} + m_2^{1/2}(x_1) \left( \int_{\mathbb{R}^m} |\nu_1(y) - \nu_2(y)|^{2\alpha} m_\infty(y) \, dy \right)^{\frac{1}{\alpha}} \right) \]
\[ \leq K_R \|\nu_1 - \nu_2\|_{L_{\infty}^m} \]

where \( K_R > 0 \) is a constant depending on \( R \), and where we have used \( (2.10) \) and the fact that \( \|\cdot\|_{L_{\infty}^m} \leq C \|\cdot\|_{L_{\infty}^m} \) with \( C > 0 \) depending only on the total mass of \( m_\infty \) assumed to be finite with assumption (D).

To estimate (III) it is useful to introduce the densities and write \( d\mu_x(y) = m_1(y)dy \), \( d\mu_x(y) = m_2(y)dy \).

\[ |III| \leq \int_{\mathbb{R}^m} |f(x_2, y, \nu_2(y))(m_1 - m_2)(y)| \, dy \]
\[ \leq C \int_{\mathbb{R}^m} (1 + |x_2| + |y|)(m_1 - m_2)^+(y) \, dy \]
\[ + C \int_{\mathbb{R}^m} (1 + |x_2| + |y|)(m_1 - m_2)^-(y) \, dy =: (III.1) + (III.2). \]

Call \( \Omega := \{ y \in \mathbb{R}^m : m_1(y) \geq m_2(y) \} \) and \( \phi(y) := C(1 + |x_2| + |y|) \). Let \( \pi \) be any probability measure on \( \mathbb{R}^m \times \mathbb{R}^m \) with marginals \( \mu_{x_1} \) and \( \mu_{x_2} \). Then

\[ (III.1) = \int_{\Omega} \phi(y) (m_1 - m_2)(y) \, dy = \int_{\Omega \times \Omega} (\phi(y) - \phi(y')) \, d\pi(y, y') \]
\[ \leq C \int_{\mathbb{R}^m \times \mathbb{R}^m} |y - y'| \, d\pi(y, y'). \]

Since \( \pi \) is arbitrary with the given marginals we get

\[ (III.1) \leq CW_1(\mu_{x_1}, \mu_{x_2}) \leq CW_2(\mu_{x_1}, \mu_{x_2}). \]

Using [10, Corollary 1], we have

\[ W_2(\mu_{x_1}, \mu_{x_2}) \leq C \left( \int_{\mathbb{R}^m} |b(x_1, y) - b(x_2, y)|^2 \, d\mu_{x_2}(y) \right)^{\frac{1}{2}} \]

hence, using Lipschitz continuity of the drift \( b \) in \( x \), we get

\[ (III.1) \leq C |x_1 - x_2|. \]

The term (III.2) can be estimated in the same way by replacing \( \Omega \) with \( \{ m_1 \leq m_2 \} \).

Finally, by exchanging the roles of \( (x_1, \nu_1) \) and \( (x_2, \nu_2) \) and putting the estimates together, we obtain \( (3.10) \).
Next we check the Lipschitz property of the effective diffusion. In case of assumption (C)(a) we have \( \bar{\sigma}(x) = \sigma(x) \) which is Lipschitz by Assumption (A). In the case (C)(b), the invariant measure \( \mu \) is independent of \( x \), so we have

\[
\bar{\sigma}\bar{\sigma}^T(x, \nu) := \Sigma(x, \nu) := \int_{\mathbb{R}^m} \sigma\sigma^T(x, y, \nu(y)) \, d\mu(y) = \int_{\mathbb{R}^m} \Sigma(x, y, \nu(y)) \, d\mu(y).
\]

By the condition (C)(b)(i) we can differentiate the integral w.r.t. \( x \) under the integral sign (by means of dominated convergence theorem). Then (C)(b)(ii) ensures that \( \Sigma \) has bounded second derivatives in \( x \), uniformly in \( \nu \). Therefore it has a square root \( \bar{\sigma} \), Lipschitz in \( x \) uniformly in \( \nu \), by [32, Theorem 5.2.3, p. 132]. The linear growth in \( x \) uniform in \( \nu \) follows easily.

Now we turn to the continuity of \( \bar{\sigma} \) with respect to \( \nu \). In the case (C)(a), \( \Sigma \) does not depend on \( \nu \) and so \( \bar{\sigma} \) is constant in \( \nu \). For the case (C)(b), given \( R > 0 \), we fix \( x \in \mathbb{R} \) such that \( |x| \leq R \). Let \( \nu_i \in U^{ce} \), \( i = 1, 2 \). Noting that \( \bar{\Sigma} \) is a symmetric positive semidefinite matrix, the following inequality holds (see, e.g. [33, inequality (3.2)] or [6, inequality (X.2), p. 290]):

\[
|\bar{\sigma}(x, \nu_1) - \bar{\sigma}(x, \nu_2)| = \left| \Sigma^\top(x, \nu_1) - \Sigma^\top(x, \nu_2) \right| \leq C \left| \Sigma(x, \nu_1) - \Sigma(x, \nu_2) \right|^{\frac{1}{2}}
\]

where \( C > 0 \) is a constant depending on the dimension \( n \) only, and the norm is \( |\Sigma|^2 = \text{trace}(\Sigma\Sigma^\top) \). Therefore we have

\[
|\Sigma(x, \nu_1) - \Sigma(x, \nu_2)|
\]

\[
\leq \left| \int_{\mathbb{R}^m} \sigma\sigma^T(x, y, \nu_1(y)) \, d\mu(y) - \int_{\mathbb{R}^m} \sigma(x, y, \nu_1(y))\sigma^T(x, y, \nu_2(y)) \, d\mu(y) \right|
\]

\[
+ \left| \int_{\mathbb{R}^m} \sigma(x, y, \nu_1(y))\sigma^T(x, y, \nu_2(y)) \, d\mu(y) - \int_{\mathbb{R}^m} \sigma\sigma^T(x, y, \nu_2(y)) \, d\mu(y) \right|
\]

\[
\leq \int_{\mathbb{R}^m} C_R (1 + |y|)|\sigma(x, y, \nu_1(y)) - \sigma(x, y, \nu_2(y))| \, d\mu(y)
\]

\[
+ \int_{\mathbb{R}^m} C_R (1 + |y|)|\sigma(x, y, \nu_1(y)) - \sigma(x, y, \nu_2(y))| \, d\mu(y)
\]

\[
\leq \int_{\mathbb{R}^m} 2C_R C_R (1 + |y|)^2 |\nu_1(y) - \nu_2(y)| \, d\mu(y)
\]

where we have used, respectively, (2.2) and (3.6) in the last two inequalities. We conclude using Cauchy-Schwarz inequality together with (3.8) which yield

\[
|\Sigma(x, \nu_1) - \Sigma(x, \nu_2)| \leq K_R \|\nu_1 - \nu_2\|_{L^2_{\mu_0}}^{1/2}
\]

where \( K_R > 0 \) is a constant depending on \( R \) and on the 4th moment of \( \mu \) (the latter being independent of \( x \) thanks to assumption (C)(b)). Back to (3.15), we finally have the desired inequality

\[
|\bar{\sigma}(x, \nu_1) - \bar{\sigma}(x, \nu_2)| \leq K_R^{1/2} \|\nu_1 - \nu_2\|_{L^2_{\mu_0}}^{\alpha/2}.
\]

To prove (3.12) we use (2.2) to get

\[
|f(x, \nu)| \leq |f(0, \nu)| + C|x| \leq \int_{\mathbb{R}^m} C(1 + |y|) d\mu_0(y) + C|x| \leq C + m_1(0) + C|x|,
\]

and \( |\bar{\sigma}\bar{\sigma}^T(x, \nu)| \leq |\bar{\sigma}\bar{\sigma}^T(0, \nu)| + C^2|x|^2 \leq C^2 + m_2(0) + C|x|^2 \).
Finally, the well-posedness of the SDE in (3.3) is a direct consequence of the latter properties; see e.g. [34, Corollary 6.4 in Chap. 1, p. 44]. □

**Proposition 3.3.** Assume (B') and (D). Then the effective data defined by (3.3) satisfy the following inequalities, for $|x_1|, |x_2| \leq R$,
\[(3.16) \quad \bar{\ell}(t_1, x_1, \nu_1) - \bar{\ell}(t_2, x_2, \nu_2) \leq C_R(\nu_1 - \nu_2)\]
\[(3.17) \quad |\bar{\ell}(t, x, \nu)| \leq C(1 + |x|)^2, \quad \bar{\ell}(t, x, \nu) = (I) + (II) + (III),\]
\[(3.18) \quad |\bar{g}(x_1) - \bar{g}(x_2)| \leq C(1 + |x_1| \vee |x_2|)|x_1 - x_2|^\beta, \quad |\bar{g}(x)| \leq C(1 + |x|^2).
\]

**Proof.** Here $C_R$ denotes a constant which may change from line to line, and only depends on the data of the problem and on $R$. For $(t_i, x_i, \nu_i) \in [0, T] \times \mathbb{R}^n \times L^\infty(\mathbb{R}^m, U), i = 1, 2$, $|x_i| \leq R$, we have
\[
\bar{\ell}(t_1, x_1, \nu_1) - \bar{\ell}(t_2, x_2, \nu_2) = \int_{\mathbb{R}^m} \bar{\ell}(t_1, x_1, \nu_1) \, d\mu(x_1) - \int_{\mathbb{R}^m} \bar{\ell}(t_2, x_2, \nu_2) \, d\mu(x_2)
\]
\[
= \int_{\mathbb{R}^m} \bar{\ell}(t_1, x_1, \nu_1) \, d\mu(x_1) - \int_{\mathbb{R}^m} \bar{\ell}(t_2, x_2, \nu_2) \, d\mu(x_2)
\]
\[
= (I) + (II) + (III).
\]

By (3.7) and (3.10)
\[(I) \leq C_R(1 + m_1(x_1)) \omega(|t_1 - t_2| + |x_1 - x_2|) \leq C_R(\omega(|t_1 - t_2| + |x_1 - x_2|).
\]

Next, we use assumption (D) as in the estimate of the term (II) in the proof of Proposition 3.2 to get
\[(II) \leq C_R \int_{\mathbb{R}^m} (1 + |y|)|\nu_1(y) - \nu_2(y)|^\alpha \, d\mu(x_1(y) \leq K_R||\nu_1 - \nu_2||_{L^\infty}^{\alpha}.
\]

For the last term we use (2.4) and proceed as for the term (III) in the proof of Proposition 3.2 by means of the densities $m_1, m_2$:
\[
|\bar{\ell}(t_1, x_1, \nu_1) - \bar{\ell}(t_2, x_2, \nu_2)| \leq K \int_{\mathbb{R}^m} (1 + |x_2|^2 + |y|)(m_1 - m_2)^+ (y) \, dy
\]
\[
+ K \int_{\mathbb{R}^m} (1 + |x_2|^2 + |y|)(m_1 - m_2)^- (y) \, dy =: (III.1) + (III.2).
\]

The integrals in (III.1) and (III.2) are estimated as the corresponding terms in the proof of Proposition 3.2 by $C|x_1 - x_2|$. Summing up the upperbounds of the three terms (I), (II), and (III), and exchanging the roles of $(t_i, x_i, \nu_i), i = 1, 2$, we get
\[
|\bar{\ell}(t_1, x_1, \nu_1) - \bar{\ell}(t_2, x_2, \nu_2)| \leq C_R(\omega(|t_1 - t_2| + |x_1 - x_2|) + |x_1 - x_2| + ||\nu_1 - \nu_2||_{L^\infty}^{\alpha}).
\]
To check the growth (3.17), let \((t, x, \nu) \in [0, T] \times \mathbb{R}^m \times U^{ex}\) and use (2.4) to get
\[
|\bar{\ell}(t, x, \nu)| \leq \int_{\mathbb{R}^m} |\ell(t, x, y, \nu(y))| \, d\mu_x(y) \leq K \left( 1 + |x|^2 + \int_{\mathbb{R}^m} |y| \, d\mu_x(y) \right) = K(1 + |x|^2 + m_1(x)) \leq C(1 + |x|^2)
\]
where in the last inequality we have used (2.10).

The proof of (3.18) is analogous, by using the second inequality in (3.7).

We can now define the effective optimal control problem
\[
\text{(OCP)} \quad V(t, x) := \sup_{\nu \in U^{ex}} J(t, x, \nu), \quad \text{subject to } (3.3)
\]
where \(U^{ex}\) is the set of progressively measurable processes taking values in the extended control set \(U^{ex}\) defined in (3.9), the effective payoff is
\[
(3.19) \quad J(t, x, \nu(\cdot)) := E \left[ e^{\lambda(T-t)} g(\hat{X}_T) + \int_t^T \ell(s, \hat{X}_s, \nu_s)e^{\lambda(s-t)} \, ds \mid \hat{X}_t = x \right],
\]
and the process \(\hat{X}_s\) solves (3.3).

**Theorem 3.1.** Assume \((A'), (B'), (C),\) and \((D)\). Then the value function \(V(t, x)\) of \text{(OCP)} is the unique continuous viscosity solution to the Cauchy problem (2.14) satisfying the growth condition (2.15). In particular, it is the limit of the value functions \(V^\varepsilon\) defined in \text{(OCP}(\varepsilon)\).

We need the following lemma to prove the latter theorem.

**lem 3.1.** Under the assumptions of Theorem 3.1 the value function \(V(t, x)\) satisfies, for some constant \(C\),
\[
(3.20) \quad |V(t, x)| \leq C(1 + |x|^2) \quad \forall \,(t, x) \in [0, T] \times \mathbb{R}^n,
\]
and
\[
(3.21) \quad \lim_{(t, x) \to (T, \tilde{x})} V(t, x) = \bar{g}(\tilde{x}).
\]

**Proof.** The properties of \(\bar{f}\) and \(\bar{\sigma}\) in Proposition 3.2 allow to use standard estimates on the moments of the process defined by (3.3), see e.g. [20, Appendix D], and imply the following inequalities, for suitable constants \(C\) depending only on the data,
\[
(3.22) \quad E \left[ \sup_{s \in [t, T]} |\tilde{X}_s - x|^2 \right] \leq C e^{C(T-t)} \int_t^T (C + |x|^2) \, ds \leq C_T(T - t)(1 + |x|^2),
\]
\[
(3.23) \quad E \left[ \sup_{s \in [t, T]} |\tilde{X}_s|^2 \right] \leq C(1 + |x|^2).
\]

Then (3.17) and the second inequality in (3.18), together with (3.23) give (3.20). To prove (3.21) we assume for simplicity \(\lambda = 0\) and compute
\[
|J(t, x, \nu) - \bar{g}(\tilde{x})| \leq E \left[ |\bar{g}(\tilde{x}) - \bar{g}(x)| + |\bar{g}(\tilde{X}_T) - \bar{g}(x)| + \int_t^T |\ell(s, \tilde{X}_s, \nu_s)| \, ds \right] =: (I) + (II) + (III).
\]
By \((3.18)\), we have \(|\bar{g}(\bar{x}) - \bar{g}(x)| \leq C(1 + |\bar{x}|) |\bar{x} - x|^\beta\), so \((I) \to 0\) as \(x \to \bar{x}\). Moreover, by Hölder inequality with \(p = 2/\beta\),

\[
\begin{align*}
(II) & \leq E \left[ C(1 + |\hat{X}_T|) |\hat{X}_T - x|^\beta \right] \leq CE \left[ (1 + |\hat{X}_T|) p' \right]^{\frac{1}{p'}} E \left[ |\hat{X}_T - x|^2 \right]^{\frac{\beta}{p'}} \\
& \leq C \left( 1 + |x|^{p'} + E \left[ |\hat{X}_T|^2 \right] \right) \leq C(1 + |x|^2) E \left[ |\hat{X}_T - x|^2 \right]^{\frac{\beta}{p'}} ,
\end{align*}
\]

where we used that \(p' \leq 2\) and \((3.23)\). Then \((II) \to 0\) as \(t \to T_-\) by \((3.22)\).

Finally we use \((3.17)\) and \((3.23)\) to estimate

\[
(III) \leq (T - t) C \left( 1 + E \left[ \sup_{s \in [t,T]} |\hat{X}_s|^2 \right] \right) \leq (T - t) C (1 + |x|^2),
\]

and so also \((III) \to 0\) as \(t \to T_-\), which completes the proof of \((3.21)\) by the arbitrariness of \(\nu \in \mathcal{U}^{ex}\).

\[\square\]

**Proof (of Theorem 3.7).** The proof is based on the Dynamic Programming Principle, namely

\[
V(t, x) := \sup_{\nu \in \mathcal{U}^{ex}} E \left[ e^{\lambda(t-s)} V(\theta, \hat{X}_s) + \int_t^T \mathcal{F}(s, \hat{X}_s, \nu_s) e^{\lambda(t-s)} ds \mid \hat{X}_t = x \right],
\]

for all stopping times \(\theta\) valued in \([t, T]\). This is usually proved for the weak formulation of the control problem and then extended to the strong formulation by proving that the value functions coincide. This is well-known in the compact case: for unbounded problems including ours we refer to [19] and to the very general treatment in [18].

Next, one deduces from the two inequalities in \((3.24)\) that the lower semicontinuous envelope \(V_*\) is a supersolution of the HJB equation in \((2.14)\) and the upper semicontinuous envelope \(V^*\) is a subsolution, see, e.g., [17] or [34]. Moreover they satisfy \(V_*(T, x) = V^*(T, x)\) for all \(x\) by \((3.21)\), and have at most quadratic growth by \((3.20)\). Then we can use the comparison Theorem 2.1 in [17] to get \(V^* \leq V_*\), which implies that \(V\) is continuous and the unique solution of the Cauchy problem \((2.14)\) satisfying \((2.15)\). The convergence to \(V\) of the value functions \(V^{\epsilon}\) now follows from Theorem 2.1.

\[\square\]

**Remark 3.4.** The system \((3.3)\) can also be restricted to standard control functions \(u \in \mathcal{U}\), because they are extended controls constant in \(y\). Consider the corresponding value function \(W(t, x) := \sup_{u \in \mathcal{U}} E \mathcal{F}(t, x, u)\). Then

\[
\lim_{\epsilon \to 0} V^{\epsilon}(t, x, y) \geq W(t, x)
\]

because the left hand side is \(V(t, x)\), which is larger than \(W(t, x)\), being a sup over a larger set \(\mathcal{U}^{ex} \supseteq \mathcal{U}\). This means that, when using standard controls, the perturbed system \((1.19)\) can give a better performance than the limit one \((\text{OCP})\). Theorem 1.1 that we prove in Section 4.2 is based on this remark.

In some cases one can have \(V = W\) and the extended controls are not necessary: this occurs when the controls are decoupled from the fast variables \(Y\) in the data \(f, \sigma,\) and \(\ell\), as in Remark 3.2.
4. Convergence of the trajectories

We have shown so far that the value function $V^\epsilon$ in $OCP(\epsilon)$ converges locally uniformly to the value function $V$ in $OCP$ as $\epsilon \to 0$. In this section, we are interested in the link between the singularly perturbed dynamics (1.19) and the corresponding effective one (3.3). Mainly we will show that, under the standing assumptions and if $\sigma = 0$ in (A2), as $\epsilon \to 0$ every solution to (3.3) is approximated by a sequence of processes of the form (1.19), in a sense that we make precise, and, conversely, the limit of any converging sequence of trajectories of (1.19) solves a relaxation of (3.3).

4.1. Convergence of trajectories with vanishing diffusion. In this subsection, we will assume, besides the standing assumptions of §2, that $\lambda = 0$ and the limit in (A2) is null, that is,

$$\lim_{\epsilon \to 0} \sigma^\epsilon(x, y, u) = 0 \text{ locally uniformly}. \quad (4.1)$$

In this case, assumption (C) is satisfied and the effective dynamics (3.3) becomes the deterministic control system

$$\begin{cases}
\frac{d\hat{x}_t}{dt} = \int_{\mathbb{R}^m} f(\hat{x}_t, y, \nu_t(y))d\mu_{\hat{x}_t}(y) \\
\nu \in U_{\text{ex}}, \quad \text{and} \quad \hat{x}_0 = x \in \mathbb{R}^n.
\end{cases} \quad (4.2)$$

Note that the right hand side of the ODE is $f(\hat{x}_t, \nu_t)$, and, since there is no randomness, $U_{\text{ex}}$ is the set of Lebesgue measurable functions $[0, T] \to U_{\text{ex}}$, where $U_{\text{ex}}$ is as defined in (3.9). Now the effective control problem $OCP_{\text{d}}$ is deterministic and its value function simplifies to

$$( OCP_{\text{d}} ) \quad V(t, x) := \sup_{\nu \in U_{\text{ex}}} \left\{ g(\hat{x}_T) + \int_t^T \tilde{\ell}(s, \hat{x}_s, \nu_s) \, ds \right\}, \quad \text{subject to } (4.2)$$

If we define

$$\overline{F}(x) := \overline{f}(x, U_{\text{ex}}) \quad (4.3)$$

the effective dynamics (4.2) can be equivalently expressed by

$$\hat{x}_{t_2} - \hat{x}_{t_1} = \int_{t_1}^{t_2} \overline{F}(\hat{x}_s) \, ds. \quad (4.4)$$

The next two theorems connect the trajectories of this system to the process

$$\begin{cases}
dX_t = f(X_t, Y_t, u_t) \, dt + \sqrt{2} \sigma^\epsilon(X_t, Y_t, u_t) \, dW_t, \quad X_0 = x \in \mathbb{R}^n, \\
dY_t = \frac{1}{\epsilon} b(X_t, Y_t) \, dt + \sqrt{\frac{2}{\epsilon}} \, \tilde{\sigma} \, dW_t, \quad Y_0 = y \in \mathbb{R}^m.
\end{cases} \quad (4.5)$$

**Theorem 4.1.** Assume (A'), (D) and (4.1). Then, for all $y \in \mathbb{R}^m$, any solution $\hat{x}$ to the controlled effective dynamics (4.2) has the following property: for all $\epsilon > 0$ there is a control $u^\epsilon \in U$ such that the $x$-component of the corresponding trajectory $(X^\epsilon, Y^\epsilon)$ of (4.3) converges to $\hat{x}$ in the sense

$$\lim_{\epsilon \to 0} \int_0^T \mathbb{E} \left[ |X_t^\epsilon - \hat{x}_s|^2 \right] \, ds + \mathbb{E} \left[ |X_T^\epsilon - \hat{x}_T|^2 \right] = 0.$$
Theorem 4.2. Assume (A'), (D) and (4.1). If for a sequence of controlled processes \((X_\varepsilon^n, Y_\varepsilon^n)\) of (4.5) with \(\varepsilon_n \to 0\) there is a deterministic process \(\bar{\pi}\), such that

\[
\lim_{\varepsilon_n \to 0} \int_0^T \mathbb{E} \left[ |X_{\varepsilon_n}^n - \pi_s|^p \right] \, ds = 0,
\]

for some \(p \in [1, 2]\), then \(\bar{\pi}\) satisfies

\[
\pi_s \in \overline{\mathcal{C}}(\bar{\pi}_s), \quad \text{a.e. } s \in [0, T],
\]

where \(\overline{\mathcal{C}}\) is defined by (3.2) and (4.3), and \(\overline{\mathcal{C}}\) denotes the closed convex hull.
Proof. Fix a sequence \( X^\varepsilon \) solution to (1.19) converging to \( \overline{x} \) in the sense of (1.7). We are going to show that the limit process \( \overline{x} \) can be approximated by a sequence of trajectories solving (4.4). We consider an optimal control problem of the form \( \text{OCP}(\varepsilon) \) where the running payoff and final utility function are

\[
\ell(s, z, u) = -|z - \overline{u}|^p, \quad \text{and} \quad g \equiv 0.
\]

Since \( X^\varepsilon \) is an admissible solution to \( \text{OCP}(\varepsilon) \), we have

\[
\int_0^T -\mathbb{E} \left[ |X^\varepsilon_s - \overline{u}|^p \right] ds \leq V^\varepsilon(0, x, y) \leq 0.
\]

We deduce from (1.7) that \( V^\varepsilon(0, x, y) \) converges to 0 as \( \varepsilon \to 0 \). This means that the limit value function \( V(0, x) \) of the effective optimal control problem \( \text{OCP}_d(x) \) also equals 0. Hence, one can consider a minimizing sequence \( \{x^k\} \) of the effective problem \( \text{OCP}_d(x) \) with \( g \) and \( \ell \) as above such that

\[
\int_0^T |x^k_s - \overline{u}|^p ds \xrightarrow{k \to +\infty} 0
\]

which yields (up to extracting a subsequence) \( \lim_{k \to +\infty} |x^k_s - \overline{u}|^p = 0 \), a.e. \( s \in [0, T] \). Since \( x^k \) solves (4.4), by a compactness theorem for differential inclusions, e.g., [16] Theorem 4.1.11, p.186, we get a subsequence (again denoted by \( x^k \)) that converges uniformly to \( z \) and whose derivatives converge weakly to \( \dot{z} \), where

\[
\dot{z}_s \in \overline{\mathcal{C}}(z_s), \quad \text{a.e. } s \in [0, T],
\]

which is the convexification of (4.4). The latter theorem holds true because, for every \( x \), \( \overline{\mathcal{C}}(x) \) is a nonempty compact convex set, moreover \( \overline{\mathcal{C}}(x) \) is upper semi-continuous as a direct consequence of [1] Proposition 1.4.14, p.47, hence also \( \overline{\mathcal{C}}(x) \) is u.s.c., and finally every element of \( \overline{\mathcal{C}}(x) \) is upper bounded by an affine function of \( |x| \) by (2.2). Therefore, and when \( p \geq 1 \), one has

\[
|x_s - \overline{u}|^p \leq |x^k - z|^p + |x^k_s - \overline{u}|^p \xrightarrow{k \to +\infty} 0
\]

for almost every \( s \in [0, T] \). Then \( z_s = \overline{u} \) and \( \overline{x} \) satisfies (4.8). \( \square \)

The last result in this subsection concerns the quasi-optimality of the approximating sequence in Theorem 4.4.1 under the additional conditions that \( g \) and \( \ell \) are bounded and depend only on \( x \). By quasi-optimal we mean a sequence of trajectories \( (X^{\varepsilon_n}, Y^{\varepsilon_n}) \) which satisfy

\[
V^{\varepsilon_n}(t, x, y) \leq E \left[ g(X_T^{\varepsilon_n}) + \int_t^T \ell(X^{\varepsilon_n}_s) ds \right] + o(1), \quad \text{as } \varepsilon_n \to 0.
\]

**Theorem 4.3.** Assume (A), (D), (1.1), and that the utility function \( g = g(x) \) and the payoff \( \ell = \ell(x) \) are continuous bounded functions of \( x \) only. Then, for a solution \((\hat{x}, \hat{y})\) of (1.2) that is optimal for the effective problem \( \text{OCP}_d(x) \), the approximating sequence of trajectories \((X^{\varepsilon}, Y^{\varepsilon})\) found in Theorem 4.1 has in addition the quasi-optimality property (4.4).\[1\]

---

\[1\] \( F \) is upper semicontinuous in \( x \) if \( \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } |x - x'| \leq \delta \Rightarrow F(x') \subset F(x) + \varepsilon B \) where \( B \) is the unit ball (see [1] page 39).
Proof: Let \((\hat{x}, v)\) be an optimal solution of \((\text{OCP} d)\). Let us introduce the path space \(C_n = C([t, T], \mathbb{R}^n)\) of continuous functions from \([t, T]\) to \(\mathbb{R}^n\) endowed with the uniform topology where the distance between two functions \(f\) and \(g\) of \(s \in [t, T]\) is

\[
\rho(f, g) = \sup_{s \in [t, T]} \max_{1 \leq i \leq n} |f_i(s) - g_i(s)|.
\]

A stochastic process \(X\) is then a random function, that is a mapping from \((\Omega, \mathcal{F}, \mathbb{P})\) into \(C_n\), and such that for each \(\omega \in \Omega\), \(X(\omega)\) is an element of \(C_n\), i.e., a continuous function on \([t, T]\), whose value at \(s\) is denoted by \(X_s(\omega)\). And for fixed \(s\), let \(X_s\) denote the real function on \(\Omega\) with value \(X_s(\omega)\) at \(\omega\). Then \(\{X_s^\varepsilon\}_{\varepsilon > 0} := \{(X_s^\varepsilon, t \leq s \leq T)\}_{\varepsilon > 0}\) is a sequence of random functions. We also denote by \(\hat{x}\) the path \(\hat{x} := (\hat{x}_s, t \leq s \leq T)\). From Theorem 4.1 (with the initial time 0 replaced by \(t \in [0, T]\)) we have \((X^\varepsilon, Y^\varepsilon)\) solution of \((1.19)\) with \(X^\varepsilon = x, Y^\varepsilon = y\), such that

\[
\lim_{\varepsilon \to 0} \int_t^T \mathbb{E}\left[|X_s^\varepsilon - \hat{x}_s|^2\right] \, ds = 0
\]

which is the \(L^2\) convergence in the path space \(C_n\) for the random function \(X^\varepsilon\) to \(\hat{x}\). In other words, \((1.10)\) can be written as

\[
\lim_{\varepsilon \to 0} \mathbb{E}[|X^\varepsilon - \hat{x}|^2] = 0
\]

where, for a given random function \(Z := (Z_s, t \leq s \leq T)\), we used the notation

\[
\mathbb{E}[Z] := \frac{1}{T - t} \int_t^T \mathbb{E}[Z_s] \, ds = \frac{1}{T - t} \int_t^T \int_\Omega Z_s(\omega) \, d\mathbb{P}(\omega) \, ds = \int_\Omega Z_s(\omega) \, d\mathbb{P} \otimes \lambda(s, \omega)
\]

the last integral being computed over \(\Omega \times [t, T]\) and \(\lambda\) is the uniform probability measure on \([t, T]\). A direct application of Markov’s inequality \([7, (21.12), p.276]\) to \((4.11)\) ensures that \(X^\varepsilon\) converges to \(\hat{x}\) in probability, which in turn implies the convergence in distribution \([7, \text{Thm. 25.2, p. 330}]\), that gives

\[
\lim_{\varepsilon \to 0} \mathbb{E}[h(X^\varepsilon)] \to \mathbb{E}[h(\hat{x})] = \frac{1}{T - t} \int_t^T h(\hat{x})(s) \, ds
\]

for every \(h\) continuous and bounded function of \(C_n\) into \(C_1\). Now define \(h(X)(s) = \ell(X_s)\). Then \((4.12)\) implies

\[
\lim_{\varepsilon \to 0} \int_t^T \mathbb{E}[\ell(X^\varepsilon_s)] \, ds \to \int_t^T \mathbb{E}[\ell(\hat{x}_s)] \, ds = \int_t^T \ell(\hat{x}_s) \, ds.
\]

For the convergence of the final utility function, recall from Theorem 4.1 that \(\lim_{\varepsilon \to 0} \mathbb{E}[|X_T^{\varepsilon} - \hat{x}_T|^2] = 0\). This implies the convergence in probability via Markov’s inequality and hence convergence in distribution. Since \(g\) is bounded and continuous, we get that \(\lim_{\varepsilon \to 0} \mathbb{E}[g(X_T^{\varepsilon})] = g(\hat{x}_T)\). Combining this with \((4.19)\) and the optimality of \(\hat{x}\) we obtain

\[
\lim_{\varepsilon \to 0} \mathbb{E}\left[g(X_T^{\varepsilon}) + \int_t^T \ell(X_s^\varepsilon) \, ds\right] = V(t, x) = g(\hat{x}_T) + \int_t^T \ell(\hat{x}_s) \, ds.
\]

Finally we recall from Theorem 2.1 that \(V^\varepsilon(t, x, y) \to V(t, x)\) locally uniformly in \(x, y\) as \(\varepsilon \to 0\). Then by a triangular inequality we easily get \((4.13)\). \(\square\)

Remark 4.1. The proof of this theorem works as well if \((\hat{x}, \hat{v})\) is merely quasi-optimal, i.e., it misses \(V(t, x)\) by a quantity \(\eta > 0\), and gives an approximating
sequence that is quasi-optimal for the perturbed problem, in the sense that (1.9) holds with the addition of \( \eta \) on the right hand side.

4.2. The model problem: deep relaxation with learning rate. Here we apply the results of Sections 3.2 and 4.1 to the motivating examples of Sections 1.1 and 1.2. Given \( \phi : \mathbb{R}^n \to \mathbb{R} \), consider its relaxed gradient descent (1.15), where the control \( u_t \in U \subseteq \mathbb{R} \) is the learning rate of the SGD algorithm. Then the effective control problem arising in the singular perturbation limit is (1.16), i.e.,

\[
\mathcal{V}(x) := \min_{x \in \mathbb{R}^n} \phi(x) \quad \text{s.t.} \quad \mathbf{d}t = -\int_{\mathbb{R}^n} ^{\mathbb{R}^n} \mathbf{v}_t(y) \mathbf{X}_t - y \rho^\infty(dy, \mathbf{X}_t) \, dt \quad \text{and} \quad \mathbf{X}_0 = x \in \mathbb{R}^n, \quad t \in [0, T].
\]

where \( \rho^\infty \) is the Gibbs measure defined by (1.5). Recall that if \( \mathbf{v}_t(y) = u_t \) is constant in \( y \), then the dynamics of this system is \( \mathbf{j}(x_t, \mathbf{v}_t) = -u_t \nabla \phi_\gamma(x_t) \), where \( \phi_\gamma \) is the local entropy associated to \( \phi \), see (1.2). We will assume that

\[
\phi \in C^1(\mathbb{R}^n), \nabla \phi \text{ is Lipschitz continuous with constant } L, \quad 0 < \gamma < \frac{1}{L}.
\]

**Corollary 4.1.** If (1.15) holds, then the control system in (1.15) satisfies (A') and (C), and \( \rho^\infty(y, x) = \mu_x(y) \) satisfies (D), where \( \mu_x \) is the invariant probability measure of the fast subsystem (2.8) (see Prop. 2.2). Moreover, for any functional \( J \) of the form (1.20) satisfying (B'), the conclusions of Theorem 2.1 and Theorem 3.1 hold true. In particular,

\[
\lim_{\varepsilon \to 0} \mathcal{V}^\varepsilon(x, y) = \mathcal{V}(x), \quad \text{locally uniformly.}
\]

**Proof.** The drift of the control system in (1.15) is given by

\[
f(x, y, u) = -u(x - y)/\gamma, \quad b(x, y) = -\nabla \phi(y) + (x - y)/\gamma.
\]

The conditions (A2), (A3), and (C) are trivial. For (A1), if \( \bar{u} := \max\{|u| : u \in U\} \), \( f \) is Lipschitz in \( x, y \) with constant \( \bar{u}/\gamma \) and satisfies (2.2) with \( C = \bar{u}/\gamma \). Similarly, \( b \) is Lipschitz with constant \( L + 1/\gamma \) and satisfies (2.3) with \( C = \nabla \phi(0) + L + 1/\gamma \).

To check (A4), we set \( \kappa := L - 1/\gamma \), and observe that

\[
(b(x, y) - b(x, z)) \cdot (y - z) = (\nabla \phi(z) - \nabla \phi(y)) \cdot (y - z) - \frac{|y - z|^2}{\gamma} \leq -\kappa |y - z|^2.
\]

Under the current assumptions on \( \phi \) the equality \( \rho^\infty(y, x) = \mu_x(y) \) can be found, e.g., in [8], and assumption (D) is satisfied following Remark 3.3. Then the assumptions of Theorem 2.1 and Theorem 3.1 are verified, because \( g = \phi \) satisfies condition (B').

**Proof.** (of Theorem 4.1) The set of admissible controls \( U \) in the definition of the value function \( \mathcal{V} \) is a subset of the extended control set \( U^\infty \), since the latter contains all controls which are constant with respect to \( y \) and \( \mu_x \) is a probability measure (see Remark 5.1). Hence by Corollary 4.1 we have \( \lim_{\varepsilon \to 0} \mathcal{V}^\varepsilon(x, y) = \mathcal{V}(x) \leq \mathcal{V}(x) \).

**Proof.** (of Corollary 4.1) By Corollary 4.1 the assumptions of Theorem 4.1 and Theorem 4.2 are verified. Since the control set \( U \) is a singleton we immediately deduce from them the conclusions.
The next result says that optimal trajectories in (4.14) can be recovered as \( \varepsilon \to 0 \) by a sequence of controlled trajectories \( X_\varepsilon \) that are quasi-optimal in (1.15).

**Corollary 4.2.** Assume (4.15). Then, for any controlled trajectory \( \hat{x} \) of the system in problem (4.14), there exist a sequence of controlled trajectories \( (X_\varepsilon, Y_\varepsilon)_{\varepsilon > 0} \) of the system in (1.15) such that

\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \left| X_\varepsilon^T - \hat{x}^T \right|^2 \right] + \int_0^T \mathbb{E} \left[ \left| X_\varepsilon^s - \hat{x}^s \right|^2 \right] \, ds = 0.
\]

If, moreover, \( \phi \) is bounded and \((\hat{x}, \hat{\nu})\) is optimal for (4.14), then this sequence is quasi-optimal in the sense that

\[
\mathbb{E}[\phi(X_\varepsilon^T)] \leq \mathcal{V}_\varepsilon(x, y) + o(1), \quad \text{as} \quad \varepsilon \to 0.
\]

**Proof.** By Corollary 4.1 we can invoke Theorem 4.1. The quasi-optimality of the approximating sequence is a consequence of Theorem 4.3. \( \square \)

**Proof.** (of Corollary 1.2) The assumptions of Theorem 1.2 are verified by Corollary 4.1. Then we know that \( \overline{\mathcal{F}}(x, U^{ex}) \) for a.e. \( s \in [t, T] \), where

\[
\overline{\mathcal{F}}(x, \nu) = -\int \nu(y) \frac{x-y}{\gamma} \, d\mu_x(y),
\]

which is linear in \( \nu \in U^{ex} \). Since \( U \) is convex, also \( U^{ex} \) is convex. Then \( \overline{\mathcal{F}}(x, U^{ex}) = \overline{\mathcal{F}}(x, U^{ex}) \) and so \( x_\varepsilon \) is a trajectory of the effective system in (1.16). Therefore \( \phi(\overline{x}_T) \geq \overline{\mathcal{V}}(x) \). Next, recall from the proof of Theorem 1.1 that \( \lim_{\varepsilon \to 0} \mathcal{V}_\varepsilon(x, y) = \overline{\mathcal{V}}(x) \). Then (1.17) and (1.18) imply \( \phi(\overline{x}_T) \leq \overline{\mathcal{V}}(x) \), which gives the conclusion. \( \square \)

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**References**

[1] J.-P. Aubin and H. Frankowska, *Set-valued analysis*, Birkhäuser, Boston, 1990.
[2] C. Baldassi, A. Ingrosso, C. Lucibello, L. Saglietti, and R. Zecchina, *Subdominant dense clusters allow for simple learning and high computational performance in neural networks with discrete synapses*, Physical Review Letters 115 (2015), no. 12, 128101.
[3] M. Bardi and A. Cesaroni, *Optimal control with random parameters: a multiscale approach*, European journal of control 17 (2011), no. 1, 30–45.
[4] M. Bardi, A. Cesaroni, and L. Manca, *Convergence by viscosity methods in multiscale financial models with stochastic volatility*, SIAM Journal on Financial Mathematics 1 (2010), no. 1, 230–265.
[5] M. Bardi and H. Kouhkouh, *Singular perturbations in stochastic optimal control with unbounded data*, ESAIM: Control, Optimisation and Calculus of Variations 29 (2023), 52.
[6] R. Bhatia, *Matrix Analysis*, Springer-Verlag, New York, 1997.
[7] P. Billingsley, *Probability and measure*, John Wiley & Sons, 2008.
[8] V. Bogachev, A. Kirillov, and S. Shaposhnikov, *Invariant measures of diffusions with gradient drifts*, Doklady mathematics, 2010, pp. 790–793.
[9] V. I Bogachev and M. Röckner, *A generalization of Hasminski’s theorem on existence of invariant measures for locally integrable drifts*, Theory Probab Appl 45 (2000), no. 3.
[10] V. I. Bogachev, A. I. Kirillov, and S. V. Shaposhnikov, The Kantorovich and variation distances between invariant measures of diffusions and nonlinear stationary Fokker-Planck-Kolmogorov equations, Mathematical Notes 96 (2014), 855–863.

[11] V. I. Bogachev, M Rockner, and W Stannat, Uniqueness of solutions of elliptic equations and uniqueness of invariant measures of diffusions, Sbornik: Mathematics 193 (2002), no. 7, 945.

[12] V. Borkar and V. Gaitsgory, Averaging of singularly perturbed controlled stochastic differential equations, Applied mathematics and optimization 56 (2007), no. 2, 109–209.

[13] V. S Borkar and V. Gaitsgory, Singular perturbations in ergodic control of diffusions, SIAM journal on control and optimization 46 (2007), no. 5, 1562–1577.

[14] P. Chaudhari, A. Choromanska, S. Soatto, Y. LeCun, C. Baldassi, C. Borgs, J. Chayes, L. Sagun, and R. Zecchina, Entropy-SGD: Biasing gradient descent into wide valleys, Journal of Statistical Mechanics: Theory and Experiment 2019 (2019), no. 12, 124018.

[15] P. Chaudhari, A. Oberman, S. Osher, S. Soatto, and G. Carlier, Deep relaxation: partial differential equations for optimizing deep neural networks, Research in the Mathematical Sciences 5 (2018), no. 3, 1–30.

[16] F. H Clarke, Y. S Ledyaev, R. J Stern, and P. R Wolenski, Nonsmooth analysis and control theory, Springer-Verlag, New York, 1998.

[17] F. Da Lio and O. Ley, Uniqueness results for second-order Bellman–Isaacs equations under quadratic growth assumptions and applications, SIAM journal on control and optimization 45 (2006), no. 1, 74–106.

[18] M. F. Djete, D. Possamai, and X. Tan, McKean–Vlasov optimal control: the dynamic programming principle, The Annals of Probability 50 (2022), no. 2, 791–833.

[19] N. El Karoui and X. Tan, Capacities, measurable selection and dynamic programming. Part II: application in stochastic control problems, arXiv preprint arXiv:1310.3364 (2015).

[20] W. H Fleming and H. M. Soner, Controlled Markov processes and viscosity solutions, 2nd edition. Springer, New York, 2006.

[21] J.-P. Fouque, G. Papanicolaou, R. Sircar, and K. Sølna, Multiscale stochastic volatility for equity, interest rate, and credit derivatives, Cambridge University Press, 2011.

[22] P. Kokotović, H. K Khalil, and J. O’Reilly, Singular perturbation methods in control: analysis and design, Academic Press, London, 1986.

[23] H. Kouhkhouh, Some asymptotic problems for Hamilton-Jacobi-Bellman equations and applications to global optimization (2022). PhD thesis, University of Padova. Available online https://hdl.handle.net/11577/3444759.

[24] H. Kushner, Weak convergence methods and singularly perturbed stochastic control and filtering problems, Birkhäuser, Boston, 1990.

[25] Y. LeCun, Y. Bengio, and G. Hinton, Deep learning, Nature 521 (2015), no. 7553, 436–444.

[26] Q. Li, C. Tai, and W. E, Stochastic modified equations and adaptive stochastic gradient algorithms, International conference on machine learning, 2017, pp. 2101–2110.

[27] E Pardoux and A Y. Veretennikov, On the Poisson equation and diffusion approximation, 1, Annals of probability (2001), 1061–1085.

[28] E Pardoux and A Y. Veretennikov, On Poisson equation and diffusion approximation 2, The Annals of Probability 31 (2003), no. 3, 1166–1192.

[29] E. Pardoux and A Y. Veretennikov, On the Poisson equation and diffusion approximation 3, The Annals of Probability 33 (2005), no. 3, 1111–1133.

[30] M. Pavon, On local entropy, stochastic control, and deep neural networks, IEEE Control Systems Letters 7 (2022), 437–441.

[31] F. Pittorino, C. Lucibello, C. Feinauer, G. Perugini, C. Baldassi, E. Demyanenko, and R. Zecchina, Entropic gradient descent algorithms and wide flat minima, Journal of Statistical Mechanics: Theory and Experiment 2021 (2021), no. 12, 124015.

[32] D. W Stroock and S. S. Varadhan, Multidimensional diffusion processes, Vol. 233, Springer-Verlag, Berlin-New York, 1979.

[33] T. P Wihler, On the Hölder continuity of matrix functions for normal matrices, J. Inequal. Pure Appl. Math. 10 (2009), no. 4, Article 91, 5 pp.

[34] J. Yong and X. Y. Zhou, Stochastic controls: Hamiltonian systems and HJB equations, Springer-Verlag, New York, 1999.
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