Universal Random Access Error Exponent for Codebooks with Different Word-Lengths

Lóránt Farkas  
BME Department of Analysis  
Email: lfarkas@math.bme.hu

Tamás Kói  
BME Department of Stochastics  
Email: koitomi@math.bme.hu

Abstract—Csiszár’s channel coding theorem for multiple codebooks is generalized allowing the codeword lengths differ across codebooks. It is shown that simultaneously for each codebook an error exponent can be achieved that equals the random coding error exponent for this codebook alone, with possible exception of codebooks with small rates.

Index Terms—multiple codebook, error exponent, variable length, joint source-channel coding, random access

I. INTRODUCTION

This paper generalizes the discrete memoryless channel (DMC) coding theorem for multiple codebooks of [4]. Similar models have been analyzed recently in the context of random access communication, see [3], [12], [6] (typically for multiple access scenarios, not entered here), and unequal error protection [2].

It is assumed that the sender has a codebook library consisting of several codebooks. Each codebook consists of codewords of the same length and type. As a new feature compared to [4], here not only the type but also the length of the codewords may vary across codebooks, thus a model in between fixed length and variable length coding is addressed. The sender uses the codebooks alternately, in any order he chooses. The receiver is not aware of the codebook choices of the sender. The different codeword lengths cause a certain asynchronism at the receiver, who should also estimate the boundaries of the codewords and avoid error propagation. A maximal mutual information based universal decoder is proposed to meet these challenges. The main theorem shows that simultaneously for each codebook, the same error exponent can be achieved as the random coding error exponent for this codebook alone provided that it does not exceed the rate of this codebook. The method of types, more exactly the subtypes technique of [7] is used along with second order types (see [2] for a detailed explanation).

In a related work [11] the problem of transmitting a discrete memoryless source over a DMC with variable-length codes is analyzed. Using random coding argument along with maximum likelihood decoder [11] shows that variable-length source-channel codes achieve an error exponent equal to the random coding exponent of the channel evaluated at the source entropy. A channel coding problem as in this work is not explicitly discussed in [11].

In this paper, to keep the discussion simple, all codeword length ratios are assumed to be in $\left(\frac{1}{2}, 2\right)$. This assumption could be relaxed replacing 2 by an arbitrary constant, then our main theorem could be used to give an alternative proof of this result of [1]. The advantage of this approach would be the universality of the decoder. Due to the length-constraint, we omit the elaboration of this idea.

Note that the topic of this paper is also connected to strong asynchronism (11), (10).

II. NOTATIONS

The notation $\text{subexp}(n)$ denotes a quantity growing subexponentially as $n \to \infty$, that could be given explicitly. For some subexponential sequences individual notations are used and the parameters on which these sequences depend will be indicated in parentheses. Denote the set $\{1, 2, \ldots, M\}$ by $[M]$. Notations like $T^n_p$ (type class), $P^n(M)$ (family of $n$-types), $H_V(X, Y), I_V(X \wedge Y)$ (information measures under distribution $V$) follow [3], [9] and [7]. Denote $I(x \wedge y)$ the mutual information $I_V(X \wedge Y)$ with $V$ equal to the joint type of $(x, y)$. If $V = V^{XY}$ is a distribution on $X \times Y$ then $V_X$, $V_Y$ and $V_{Y|X}$ denote the associated marginal resp. conditional distributions. The concatenation of an $n_1$-type $V_1 \in P^{n_1}(X)$ and an $n_2$-type $V_2 \in P^{n_2}(X)$ is the $(n_1 + n_2)$-type $V_1 \oplus V_2 \in P^{n_1+n_2}(X)$ with

$$V_1 \oplus V_2(x) = \frac{n_1}{n_1+n_2}V_1(x) + \frac{n_2}{n_1+n_2}V_2(x).$$

Let $V_1, V_2, V_3$ be $n_1, n_2, n_3$-type respectively and let $n_{12} = n_1 + n_2, n_{123} = n_1 + n_2 + n_3, P_{12} = V_1 \oplus V_2, P_{23} = V_2 \oplus V_3$ then

$$F(V_1, V_2) = H(P_{12}) - \frac{n_1}{n_{12}}H(V_1) - \frac{n_2}{n_{12}}H(V_2).$$

$$F(V_1, V_2, V_3) = H(P_{23}) - \sum_{i=1}^{3} \frac{n_i}{n_{123}}H(V_i).$$

Note that the nonnegative quantities (2) and (3) are frequently called Jensen-Shannon divergence.

For a given $V \in P^n$ and $P \in P^n(M)$, let $\tilde{V} \in P^{n-n_1}(M')$ denote the type for which $V \oplus \tilde{V} = P$. As $P$ is always clear from the context it is not indicated in the notation.

Finally we define the notion of second order type following [5]. For a sequence $x = x_1 \ldots x_n \in X^n$ let $P_x^{2} \in P^{n-1}(M' \times X')$ defined by

$$P_x^{2}(a, b) = \frac{1}{n-1} |\{ i : x_i = a, x_{i+1} = b \}|.$$
Let $T_{V,a}^{n,2}$ denote the second order type class $\{x : x \in X^n, P_{X}^n = V, x_1 = a\}$. We cite from [5] that

$$|T_{V,a}^{n,2}| \leq 2^n H_V(\hat{x}|X).$$

(5)

III. THE MODEL

A DMC $W : X \rightarrow Y$ is given. The transmitter has a codebook library with multiple constant composition codebooks. The codewords length and type are fixed within codebooks, but can vary from codebook to codebook.

**Definition 1.** Let positive integers $n, M, l^1, l^2, \ldots, l^M$ with $\frac{n}{l^j} < l^j \leq n$ for all $j \in [M]$, distributions $\{P^{i} \in \mathcal{P}^{l^j}(X), i \in [M]\}$ and rates $\{R^i, i \in [M]\}$ be given parameters. A codebook library with the above parameters, denoted by $A$, consists of constant composition codebooks $\{A^1, \ldots, A^M\}$ such that $A^i = \{x_1^i, x_2^i, \ldots, x_{l^i}^i\}$ with $x_n^i \in T_{P^i}^{n,l^i}$, $i \in [M]$, $N^i = \frac{2^n}{l^i R^i}$, $a \in [N^i]$. In the sequel, $n$ will be referred to as length-bound.

The transmitter continuously sends messages to the receiver through channel $W$. Before sending a message, the transmitter arbitrarily chooses one codebook of the library. This choice is not known to the receiver. The performance of the following decoder is analyzed.

The output symbols of the channel are denoted by the infinite sequence $y_1, y_2, \ldots$. Assume that decoding related to symbols $y_1, \ldots, y_{l^h-1}$ is already performed and now the position of $y_{l^h}$ is analyzed. In the first stage of decoding the decoder tries to find indices $\hat{h}, \hat{c}$ which uniquely maximize

$$l^h \left( I(x^{\hat{h}}_c \wedge y_{r} \cdot y_{r+1} \cdots y_{r+l^h-1}) - R^{l^h} \right).$$

(6)

If the decoder successfully finds a unique triplet $\hat{h}, \hat{c}, \hat{s}$, the second stage of decoding starts. In this stage, if for all $h, c$ and $d \in \{r - l^h + 1, \ldots, r - 1\} \cup \{r + 1, \ldots, r + l^h - 1\}$ the maximum of (6) is strictly larger than

$$l^h \left( I(x^{\hat{h}}_c \wedge y_d \cdot y_{d+l^h-1}) - R^{l^h} \right),$$

(7)

the decoder decodes $x^{\hat{h}}_c$ as the codeword sent in the positions of $y_{r} \cdot y_{r+1} \cdots y_{r+l^h-1}$ and jumps to the position of $y_{r+l^h}$, there the same but shifted procedure is performed. Otherwise the decoder goes to the position of $y_{r+1}$ without decoding at the position of $y_r$. See Fig. 1.

The output of this decoder can be described by a sequence of triplets $(\hat{h}, \hat{c}, \hat{s})$, where the first coordinate is a codebook index, the second one is a message index related to this codebook, and the third refers to the starting position of the codeword. This sequence is denoted by $\hat{o}$.

As mentioned above, the transmitter arbitrarily chooses codebooks. His choices are described by an infinite codebook index sequence $h = h_1, h_2, h_3, \ldots$ where $h_j \in [M]$. To each fixed $h$, there corresponds a sequence $M_1, M_2, \ldots$ of mutually independent random messages, where $M_j$ is uniformly distributed on $[N^{h_j}]$. Let $s_j = \sum_{k=1}^{j-1} l^{h_k} + 1$ be the starting position of the $j$'th message. The average decoding error probability of the $j$'th message is defined by

$$Err^j_h = Pr \left( (h_j, M_j, s_j) \notin \hat{O} \right),$$

(8)

where the probability is calculated over the random choice of the messages, and the channel transitions. Capital letters are used to indicate randomness.

**Remark 1.** In practice, typically also the sequence $h$ is random: The messages at any time instant $j$ may be one of $M$ different kinds, with $N^i$ equiprobable messages of kind $i$. Of course, this scenario is covered by our main theorem, as our bound does not depend on $h$.

IV. MAIN THEOREM

**Theorem 1.** For each $n$ let codebook library parameters as in Definition 7 be given with length-bound $n$ and with $\frac{1}{n} \log M \to 0$ as $n \to \infty$. Then there exist a sequence $\delta_n(|X||Y|, \{M\}_{n=1}^{\infty})$ with $\frac{1}{n} \log \delta_n \to 0$ and for each $n$ a codebook library $A$ with the given parameters such that for all infinite codebook index sequences $h$ and index $j$

$$Err^j_h \leq \delta_n \cdot 2^{-1/l^h} \min_{P \in \mathcal{P}^{l^h}(X,Y)} D(V_{X,Y} || W|| P) + |I_Y(X \wedge Y) - R^+|,$$

(9)

where $E_r(R,P,W)$ is the random coding error exponent function, i.e., it is equal to

$$\min_{P \in \mathcal{P}^{r}(X,Y)} D(V_{X,Y} || W|| P) + |I_Y(X \wedge Y) - R^+|.$$  (10)

**Remark 2.** To avoid additional technical difficulties it is assumed in the calculation below that all three consecutive messages are different (see the terms below the sum in (12) and (13)). The minimum in (9) is present due to this assumption.

The next packing lemma provides the appropriate codebook library for Theorem 1. Note that the constructed codebook library works simultaneously for all infinite codebook index sequences $h$. The notations are explained on Fig. 2.

![Figure 2: Different types of packings along with the explanation of notations $q, n_1, n_2, n_3, V_1, V_2$ and $V_3$.](image-url)
Lemma 1. Let a sequence of codebook library parameters be given as in Theorem 1. Then there exist a sequence \( \delta_n(|X^i|, |Y|, |M|) \) with \( \frac{1}{n} \log \delta_n \to 0 \) and for each \( n \) a codebook-library \( A \) with the given parameters such that the following bounds hold:

**T1:** for any \( k_1 \in [M], k \in [M] \) with \( i_k \leq k_1 \), \( q \in \{1, 2, \ldots, k_1 - i_k + 1\} \) and for all joint types \( V_1 = V_1^{XX} \in V_T^{k_1,k} \)

\[
K_{T1,q}^{k_1,k}[V_1] = \sum_{a \in [N_{k_1}], \dot{a} \in [N_k]} 1_{T_1,q}(x_{a_1}^1, x_{a_2}^1, x_{\dot{a}_1}^1, x_{\dot{a}_2}^1) 
\leq \delta_n \cdot 2^{-n_1 V_1(X \land X) - t_i k + t_i k + \frac{n}{i} k + i_k R_k + i_k R_k}
\]

Here \( n_1 = \frac{i_k}{k} \), the indicator function \( 1_{T1,q}(x_{a_1}^1, x_{a_2}^1, x_{\dot{a}_1}^1, x_{\dot{a}_2}^1) \) equals 1 if filling the pattern \( T_1 \) of Fig. 2 by \( x_{a_1}^1 \) and \( x_{a_2}^1 \) the joint type of the indicated subsequences equals \( V_1 \). Furthermore, \( V_T^{k_1,k} \)

denotes the set of all joint type pairs that may occur in this way.

**T2:** for any \( k_1 \in [M], k_2 \in [M], k \in [M] \), \( q \in \{l_{k_1} \} \) with \( l_{k_1} < q + 1 \leq l_{k_1} + l_{k_2} \) and for all \( (V_1, V_2) = (V_1^{XX}, V_2^{XX}) \in V_T^{k_1,k_2,k} \)

\[
K_{T2,q}^{k_1,k_2,k}[V_1, V_2] = \sum_{a \in [N_{k_1}], \dot{a} \in [N_{k_2}], \ddot{a} \in [N_{k_2}]} 1_{T_2,q}(x_{a_1}^1, x_{a_2}^2, x_{\dot{a}_1}^2, x_{\dot{a}_2}^2, x_{\ddot{a}_1}^2, x_{\ddot{a}_2}^2) 
\leq \delta_n \cdot 2^{-n_1 V_1(X \land X) - l_{k_1} F(V_1, V_2, V_3) + \sum_{i=1}^{3} i_{k_1} k_{i} R_k + i_{k_1} k_{i} R_k} \]

Here \( n_1 = l_{k_1} - q + 1, n_2 = l_{k_2} - n_1, n_2 = l_{k_2} - n_1 \), the indicator function \( 1_{T_2,q}(x_{a_1}^1, x_{a_2}^2, x_{\dot{a}_1}^2, x_{\dot{a}_2}^2, x_{\ddot{a}_1}^2, x_{\ddot{a}_2}^2) \) equals 1 if filling the pattern \( T_2 \) of Fig. 2 by \( x_{a_1}^1 \), \( x_{a_2}^2 \) and \( x_{\dot{a}_1}^2 \) the joint types of the indicated subsequences equal \( V_1 \) and 1 respectively. Furthermore, \( V_T^{k_1,k_2,k} \)

denotes the set of all joint type triplets that may occur in this way.

**T3:** for any \( k_1 \in [M], k_2 \in [M], k_3 \in [M], \hat{k} \in [M] \), \( q \in \{l_{k_1} \} \) with \( l_{k_1} + l_{k_2} < q + 1 \leq l_{k_1} + l_{k_2} + l_{k_3} \) and for all \( (V_1, V_2, V_3) = (V_1^{XX}, V_2^{XX}, V_3^{XX}) \in V_T^{k_1,k_2,k_3,k} \)

\[
K_{T3,q}^{k_1,k_2,k_3,k}[V_1, V_2, V_3] = \sum_{a \in [N_{k_1}], \dot{a} \in [N_{k_2}], \ddot{a} \in [N_{k_2}], \dddot{a} \in [N_{k_2}]} 1_{T_3,q}(x_{a_1}^1, x_{a_2}^2, x_{\dot{a}_1}^2, x_{\dot{a}_2}^2, x_{\ddot{a}_1}^2, x_{\ddot{a}_2}^2, x_{\dddot{a}_1}^2, x_{\dddot{a}_2}^2) 
\leq \delta_n \cdot 2^{-n_1 V_1(X \land X) - i_{k_1} F(V_1, V_2, V_3) + \sum_{i=1}^{3} i_{k_1} k_{i} R_k + i_{k_1} k_{i} R_k} \]

Here \( n_1 = l_{k_1} - q + 1, n_2 = l_{k_2} - n_1, n_3 = l_{k_3} - n_2, \)

the indicator function \( 1_{T_3,q}(x_{a_1}^1, x_{a_2}^2, x_{\dot{a}_1}^2, x_{\dot{a}_2}^2, x_{\ddot{a}_1}^2, x_{\ddot{a}_2}^2, x_{\dddot{a}_1}^2, x_{\dddot{a}_2}^2) \) equals 1 if filling the pattern \( T_3 \) of Fig. 2 by \( x_{a_1}^1 \), \( x_{a_2}^2 \), \( x_{\dot{a}_1}^2 \) and \( x_{\ddot{a}_1}^2 \) the joint types of the indicated subsequences equal \( V_1, V_2 \) and \( V_3 \) respectively. Furthermore, \( V_T^{k_1,k_2,k_3,k} \)

denotes the set of all joint type triplets that may occur in this way.

Proof: Choose the codebook library \( A \) at random, i.e., for all \( i \in [M] \) the codewords of \( A' \) are chosen independently and uniformly from \( T_i^{k_i} \). Denote the exponents in upper-bounds (11), (12) and (13) by \( E_{T1,q}^{k_1,k}, E_{T2,q}^{k_1,k_2,k}, E_{T3,q}^{k_1,k_2,k_3,k} \) respectively. We first claim that under this random selection the expected value of the expressions

\[
K_{T1,q}^{k_1,k}[V_1] = E_{T1,q}^{k_1,k}[V_1] \leq \text{subexp}(n) 
\]

\[
K_{T2,q}^{k_1,k_2,k}[V_1, V_2] = E_{T2,q}^{k_1,k_2,k}[V_1, V_2] \leq \text{subexp}(n) 
\]

(14)

(15)

They are bounded above by a subexponential function of \( n \) that depends only on the alphabet size \( X \).

We establish this claim for (15), the other cases are similar or easier. Assume that \( k_1 \in [M], k_2 \in [M], k_2 \in [M] \), \( k \in [M] \) and \( q \in \{l_{k_1} \} \) fulfills the requirements of case T2. First let \( a, b \) be indices with \( (k_1, a) \neq (k_2, b) \neq (k, d) \). In this case

\[
E_{T1,q}^{k_1,k_2,k}[V_1, V_2] \leq \text{subexp}(n) 
\]

(16)

Then with some algebraic rearrangement we get:

\[
E_{T2,q}^{k_1,k_2,k}[V_1, V_2] \leq \text{subexp}(n) 
\]

(17)

(18)

(19)

(20)

(21)

(22)

Now let \( a, b, d \) be indices with \( (k_1, a) \neq (k_2, b) \) and \( (k_2, b) \neq (k, d) \) (the case \( (k_1, a) = (k, d) \) can be treated similarly). Then

\[
E_{T2,q}^{k_1,k_2,k_2}[V_1, V_2, V_3] \leq \text{subexp}(n) \cdot 2^{-(n_1 + n_2) H(p_{k_1}) + 2^{-l_{k_1} H(p_{k_1})}} 
\]

(23)

In this case upper bounding \#_{T_{2,q,a,b,d}}^{k_1,k_2,k_3,k}[V_1, V_2, V_3] \) is a bit technical. We have to separately investigate two cases: \( n_1 \geq \log n \), \( n_1 < \log n \). For didactic purposes the second case is divided further into two subcases: \( n_1 = 1, 1 < n_1 < \log n \).

\( (a), (d), (k, d) = (k_2, b), n_1 \geq \log n \): In this case we divide the block corresponding to \( V_2 \) into consecutive subblocks of
length $\log n$ (the length of the last block, denoted by $n_{2s}$, is at most $\log n$). We perform the counting from left to right (see part (a) of Fig. 3):

$$\#_{T2,q,a,b,d}(V_1, V_2) \leq \sum_{V_{21}, \ldots, V_{2s}} 2^{(l^{k_1} - n_1)} H_{V_1}(X)$$

where

$$H_{V_1}(X) = \log(n)$$

Figure 3: Solutions for technical difficulties

In (28) again the sum is over subtype sequences corresponding to the division in part (c) of Fig. 3 whose convex combination equals $V_2$. In (28) again the concavity of the entropy and the fact that the number of subtype sequences in the sum is subexponential in $n$ are used. Substituting (28) into (27) and proceeding similarly as in (20)-(21) we get (22).

As (22) is fulfilled in all cases, taking into account the definition of $N_{k_1}^{N_{k_2}}$, $N_{k_2}^{N_{k_3}}$ and $N^{k_3}$ we proved the claim for (15).

Next, denote by $S$ the sum of form (14) for all possible $k_1, k_2, k_3, k, q$ and joint types $V_1, V_2, V_3$. As $M$ grows at most subexponentially and the number of types is polynomial it follows that $E(S) \leq \delta_n'$ for suitable $\delta_n'(|X|, |Y|, \{M\})$ with $\frac{1}{2} \log \delta_n' \to 0$. Hence, there exists a realization of the codebook library with $S \leq \delta_n'$.

**Remark 3.** In the proof of Lemma 1 the separate investigation of cases $n_1 < \log n$ and $n_2 \geq \log n$ ensures that the number of terms of the sums in both (24) and (27) is subexponential.

**Proof of Theorem 1.** We prove that the codebook library $A$ provided by Lemma 1 with the decoder specified in Section III fulfills (4).

Let an infinite codebook sequence $h$ and an index $j$ be given. Let $Y_{s_1}, Y_{s_1+1}, \ldots, Y_{s_j+1}$ denote the output symbols affected by the $j$-th message. For $k \in [M]$ and $d \in \{s_j - l^k + 1, s_j - l^k + 2, \ldots, s_j + l^k - 1\}$, let $E_j^h(k, d)$ be following event

$$\left\{ \begin{array}{l}
\left( x_{s_1} \wedge Y_{d+1} \ldots Y_{d+l^k-1} - R^k \right) \\
\geq t_j^h \left( x_{s_j} \wedge Y_{s_j+1} \ldots Y_{s_j+l^k-1} - R^h \right)
\end{array} \right\}$$

for some $c \in [N^k]$ ($c \neq M_j$ if $k = h_j$ and $d = s_j$),

and let $\text{Err}_j^h(k, d)$ denote its probability. Then

$$\text{Err}_j^h(k, d) \leq \sum_{(k,d)} \text{Err}_j^h(k, d).$$

For $k = h_j$ and $d = s_j$ by standard argument

$$\text{Err}_j^h(k, d) \leq \text{subexp}(n) \cdot 2^{-t_j^h \cdot \text{min}(E, R^{h_j}, R^h, W)},$$

$$\text{Err}_j^h(k, d) \leq \text{subexp}(n) \cdot 2^{-t_j^h \cdot \text{min}(E, R^{h_j}, R^h, W), R^h)}.$$

To prove the theorem it is enough to show that (32) holds for all $k \in [M]$ and $d \in \{s_j - l^k + 1, s_j - l^k + 2, \ldots, s_j + l^k - 1\}$ (the number of such $(k, d)$ pairs is subexponential). Fix such a $(k, d)$ pair. Without any loss in generality assume that $d \leq s_j$ and that both the $j-2$th and $j-1$th messages affect outputs $Y_{d+1} \ldots Y_{s_j+l^k-1}$ (see Fig. 4). The analyzes of other cases are similar. Assume further that $h_{j-2} \neq h_{j-1} \neq h_j$. Let $N = 2^{l^h_{j-2} l^h_{j-1} + 2 l^{h_j} - 1} 2^{l^h_{j-1} - 1} 2^{l^h_{j-2} - 1}$. Then $\text{Err}_j^h(k, d)$ can be upper-bounded by

$$N^{-1} \sum_{a \in [N^{h_j}]} \Pr\{\text{Err}_j^h(k, d) = a, M_{j-2} = b, M_{j-1} = c \}$$

Here in (27) again (5) is used and the sum is over type sequences corresponding to the division in part (c) of Fig. 3 whose convex combination equals $V_2$. In (28) again the concavity of the entropy and the fact that the number of subtype sequences in the sum is subexponential in $n$ are used. Substituting (28) into (27) and proceeding similarly as in (20)-(21) we get (22).
Let $k_1 = h_{j-2}$, $k_2 = h_{j-1}$, $k_3 = h_j$, $\hat{k} = k$ and $q = |s_j - d|$. (Lemma 1 will be used with these choices) and $V_{MT_{2k}}$ be equal to

$$\begin{align*}
\mathcal{V} = (V_1, V_2, V_3) : V_4 = V_4^{XY} \in \mathcal{P}_{n_4}(X \times Y) \setminus \mathcal{P}_n(X \times X)
\end{align*}$$

then using (13) we get that $V_{MT_{2k}}$ can be bounded from below by

$$\begin{align*}
\sum_{i=1}^{n_i} I_{V_i} \left( X \wedge Y \right) + \sum_{i=1}^{3} n_i F(V_i^X, V_2^X, V_3^X) - l^k R^k
\end{align*}$$

Using convexity, $\mathcal{V}$ can be further upper-bounded by

$$\begin{align*}
\sup_{V \in \mathcal{V}} \left| \sum_{i=1}^{n_i} D(V_i^X || W^X) - l^k R^k \right|
\end{align*}$$

where the indicator function $\mathbb{I}_Y \left\{ x_a, x_b, x_c, x_d, y \right\}$ equals 1 if filling the pattern of Fig. 4 by $x_a$, $x_b$, $x_c$, $x_d$ and $y$ the joint types of the indicated subwindows equal $V_1$, $V_2$, $V_3$ and $V_4$ respectively. We can upper-bound the set size in (35) two different ways. The first bound is $2^{n_4} \sum_{i=1}^{n_i} H_{V_i}(Y|X)$. Let $V_i^X = V_i^{X_k}$, $1 \leq i \leq 3$. The second bound is

$$\begin{align*}
2^{n_4} \sum_{i=1}^{n_i} H_{V_i}(Y|X) + \sum_{i=1}^{n_i} H_{V_i}(Y|X) - \sum_{i=1}^{n_i} D(V_i^X || W^X)
\end{align*}$$

Substituting these bounds into (35) and using (13) we get that $Err_j(k, d)$ can be upper-bounded by

$$\begin{align*}
\sup_{V \in \mathcal{V}} \left| \sum_{i=1}^{n_i} I_{V_i} (X \land Y) + \sum_{i=1}^{n_i} n_i F \left( V_i^X, V_2^X, V_3^X \right) - l^k R^k \right|
\end{align*}$$

Let $l = \sum_{i=1}^{n_i} n_i$ be the length of the analyzed output window.

![Figure 4:](image)

The configuration of messages and outputs in the analyzed case along with explanation of some notations.

(see again Fig. 4). Then (33) can be further upper-bounded by

$$\begin{align*}
\sum_{V \in \mathcal{V}} N^{-1} \sum_{a \in [a]} \prod_{i=1}^{4} \left( 2^{-n_i} D(V_i^X || W^X) \right) \cdot \mathbb{I}_F \{ x_a, x_b, x_c, x_d, y \} = 1 \text{ for some } d \in [N^d],
\end{align*}$$

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