The Dynamics of the Forward Interest Rate Curve with Stochastic String Shocks*

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First Version: March 1997
This Version: March 24, 2022

Abstract

This paper offers a new class of models of the term structure of interest rates. We allow each instantaneous forward rate to be driven by a different stochastic shock, constrained in such a way as to keep the forward rate curve continuous. We term the process followed by the shocks to the forward curve “stochastic strings”, and construct them as the solution to stochastic partial differential equations, that allow us to offer a variety of interesting parametrizations. The models can produce, with parsimony, any sort of correlation pattern among forward rates of different maturities. This feature makes the models consistent with any panel dataset of bond prices, not requiring the addition of error terms in econometric models. Interest rate options can easily be priced by simulation. However, options can only be perfectly hedged by trading in bonds of all maturities available.

*Thanks to seminar participants at UCLA and Yale, and to Michael Blank, Michael Brennan, Uriel Frisch, Mark Grinblatt, Jon Ingersoll, Andrew Jeffrey, Olivier Ledoit, and Eduardo Schwartz, for helpful conversations.
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1 Introduction

In this paper, we develop a new class of bond pricing models that greatly extends the framework of Heath, Jarrow and Morton (HJM, 1992). Our model is as parsimonious and tractable as the traditional HJM model, but is capable of generating a much richer class of dynamics and shapes of the forward rate curve. Our main innovation consists in having each instantaneous forward rate driven by its own shock, while constraining these shocks in such a way as to keep the curve continuous. This shock to the curve is termed a “stochastic string”, following the physical analogy of a string whose shape changes stochastically through time.

Using stochastic strings as noise source for the dynamics of the forward curve has important economic advantages over previous approaches. Existing term structure models have the same set of shocks affect all forward rates. This feature constrains the correlations between bond prices, and, therefore, the set of admissible shapes and dynamics of the yield curve. In term structure models with state variables (e.g. Vasicek, 1977 and Cox, Ingersoll and Ross, CIR, 1985), having complex shapes of the forward curve and corresponding elaborate dynamics requires the introduction of a large number of state variables. This makes the models very unparsimonious and virtually impossible to estimate. A consequence of the same problem is the difficulty that these models have in fitting the term structure at any given point in time, for given parameters estimated from a previous time series of data.

Similar problems affect the traditional HJM model, where fitting closely the initial term structure imposes strong constraints on the dynamics of the forward rates, that are in general not verified as the model is successively fitted through time.

In contrast to these models, in our approach, any (finite) set of bond prices is imperfectly correlated. The model is thus fully compatible with any given panel dataset of bond prices, or, to put it differently, any (finite) set of bond prices observed at some (finite) sampling frequency is consistent with the model. There is thus no need to add observation noise when estimating the model.

Another interesting characteristic of our approach is that, in general, it is necessary to use a portfolio with an infinite number of bonds to replicate interest rate contingent claims. However pricing remains simple: interest rate options can in general be priced by simulation and, in some cases, in closed form. As with HJM, our approach does not allow the formulation of a partial differential equation to price derivatives.

We provide a detailed treatment of stochastic strings, and their stochastic calculus. The framework we use is that of stochastic partial differential equations (SPDE’s) and our main tool is the calculus of Dirac distributions. Methodologically, we attempt to present the results and their derivations in the simplest and most intuitive way, rather than emphasize mathematical rigor.

Kennedy (1997) and, in recent independent work, Goldstein (1997), propose a similar approach to modeling forward rates. Kennedy (1997) simply models the forward rate curve as a Gaussian random field. Goldstein (1997) uses a model similar to ours, letting the forward rate curve be shocked by two of the strings that we analyse as special examples of our framework.

The paper is organized as follows. In the next section we define the primitives of our approach and solve a simple model where forward rates are driven by Brownian motion, to compare with the
models driven by stochastic strings. Section 3 defines string shocks and their properties and offers the general no-arbitrage condition for forward rate dynamics driven by stochastic strings. Section 4 discusses the construction of stochastic strings as solutions of SPDE’s. In section 5, we present a collection of examples of stochastic strings, that present interesting properties. Section 6 presents a discussion of the empirical implications of the model, in comparison with traditional factor models. In section 7, we show how interest rate derivatives can be priced and hedged in the model. Section 8 concludes with some directions for future work.

2 The Traditional Model: Brownian Motion as Noise Source

We postulate the existence of a stochastic discount factor (SDF) that prices all assets in this economy, and denote it by $M$. This SDF can be thought of as the nominal, intertemporal, marginal rate of substitution for consumption of a representative agent in an exchange economy. It is well known that assuming that no dynamic arbitrage trading strategies can be implemented by trading in the financial securities issued in the economy is roughly equivalent to the existence of a strictly positive SDF. For no arbitrage opportunities to exist, and under an adequate definition of the space of admissible trading strategies, the product $MV$ must be a martingale, where $V$ is the value process of any admissible self-financing trading strategy implemented by trading on financial securities. Then,

$$V(t) = E_t \left[ V(s) \frac{M(s)}{M(t)} \right],$$

where $s$ is a future date and $E_t[x]$ denotes the mathematical expectation of $x$ taken at time $t$. In particular, we require that a bank account and zero-coupon discount bonds of all maturities satisfy this condition.

A security is referred to as a (floating-rate) bank account, if it is “locally riskless”. Thus, the value at time $t$, of an initial investment of $B(0)$ units in the bank account that is continuously reinvested, is given by

$$B(t) = B(0) \exp \left\{ \int_0^t r(s) ds \right\},$$

where $r(t)$ is the instantaneous nominal interest rate.

We further assume that at any time $t$ riskless discount bonds of all maturity dates $s$ trade in this economy and let $P(t, s)$ denote the time $t$ price of the $s$ maturity bond. We require that $P(s, s) = 1$, that $P(t, s) > 0$ and that $\partial P(t, s)/\partial s$ exists.

Instantaneous forward rates at time $t$ for all times-to-maturity $x > 0$, $f(t, x)$, are defined by

$$f(t, x) = -\frac{\partial \log P(t, t + x)}{\partial x},$$

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1 This process is also termed the pricing kernel, the pricing operator, or the state price density. We use these terms interchangeably. See Duffie (1996) for the theory behind SDF’s.

2 A security is “locally riskless” if, over an instantaneous time interval, its value varies deterministically. It may still be random, but there is no Brownian term in its dynamics.

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which is the rate that can be contracted at time $t$ for instantaneous borrowing or lending at time $t + x$. We require that the initial forward curve $f(0, x)$, for all $x$, be continuous.

Equivalently, from the knowledge of the instantaneous forward rates for all times-to-maturity between 0 and time $s - t$, the price at time $t$ of a bond with maturity $s$ can be obtained by

$$P(t, s) = \exp \left\{ - \int_0^{s-t} dx f(t, x) \right\} . \quad (4)$$

Forward rates thus fully represent the information in the prices of all zero-coupon bonds.

The spot interest rate at time $t$, $r(t)$, is the instantaneous forward rate at time $t$ with time-to-maturity 0,

$$r(t) = f(t, 0) . \quad (5)$$

We use forward rates with fixed time-to-maturity rather than fixed maturity date. Modelling forward rates with fixed time-to-maturity is more natural for thinking of the dynamics of the entire forward curve as the shape of a string evolving in time. In contrast, in HJM, forward rate processes disappear as time reaches their maturities. Note, however, that we still impose the martingale condition on bonds with fixed maturity date, since these are the financial instruments that are actually traded.

We now introduce a simple model of forward rate dynamics, which is a variation of HJM introduced by Musiela (1993) and Brace and Musiela (1994), both for completeness and as a benchmark against which to check the models in later sections.

Let us first simply model all forward rates by

$$df(t, x) = \alpha(t, x)dt + \sigma(t, x)dW(t) . \quad (6)$$

or, in integral form,

$$f(t, x) = f(0, x) + \int_0^t dv \alpha(v, x) + \int_0^t dW(v) \sigma(v, x) \quad (7)$$

From the definition of bond prices, we get at fixed $s$ that

$$d\log P(t, s) = f(t, s - t) \, dt - \int_0^{s-t} dy \, df(t, y) . \quad (8)$$

\(^3\)The model of HJM starts from processes for forward rates with a fixed maturity date. This is different from what we do. If we use a “hat” to denote the forward rates modeled by HJM,

$$\hat{f}(t, s) = f(t, s - t)$$

or, equivalently,

$$f(t, x) = \hat{f}(t, t + x)$$

for fixed $s$. Musiela (1993) and Brace and Musiela (1994) define forward rates in the same fashion. Miltersen, Sandmann and Sondermann (1997), and Brace, Gatarek and Musiela (1995) use definitions of forward rates similar to ours, albeit for non-instantaneous forward rates.
With Ito’s Lemma, we can calculate the dynamics of the prices of bonds with fixed maturity date \( s \),

\[
\frac{dP(t,s)}{P(t,s)} = \left[ f(t,s-t) - \int_0^{s-t} dy \alpha(t,y) + \frac{1}{2} \left( \int_0^{s-t} dy \sigma(t,y) \right)^2 \right] dt - \left( \int_0^{s-t} dy \sigma(t,y) \right) dW(t)
\]  

(9)

Finally, we postulate the following dynamics for the SDF\(^4\)

\[
\frac{dM(t)}{M(t)} = -r(t) \ dt + \phi(t) \ dW(t) \ .
\]  

(10)

The drift of \( M \) is justified from the well-known martingale condition on the product of the bank account with the SDF. The process \( \phi \) denotes the market price of risk, as measured by the covariance of asset returns with the SDF.

The no-arbitrage condition for buying and holding bonds implies that \( PM \) be a martingale in time, for any bond price \( P \). Technically, this amounts to imposing that the drift of \( PM \) be zero,

\[
-r(t) + f(t,s-t) - \int_0^{s-t} dy \alpha(t,y) + \frac{1}{2} \left( \int_0^{s-t} dy \sigma(t,y) \right)^2 - \phi(t) \left( \int_0^{s-t} dy \sigma(t,y) \right) = 0 \ .
\]  

(11)

Equivalently, for all \( t \) and \( s \), with \( x \equiv s-t \)

\[
f(t,x) = r(t) + \int_0^x dy \alpha(t,y) - \frac{1}{2} \left( \int_0^x dy \sigma(t,y) \right)^2 + \phi(t) \left( \int_0^x dy \sigma(t,y) \right)
\]  

(12)

We can differentiate this no-arbitrage condition with respect to \( x \), and obtain

\[
\alpha(t,x) = \frac{\partial}{\partial x} \left\{ f(t,x) + \frac{1}{2} \left( \int_0^x dy \sigma(t,y) \right)^2 - \phi(t) \left( \int_0^x dy \sigma(t,y) \right) \right\} ,
\]  

(13)

or,

\[
\alpha(t,x) = \frac{\partial f(t,x)}{\partial x} + \sigma(t,x) \left( \int_0^x dy \sigma(t,y) - \phi(t) \right) .
\]  

(14)

where \( \frac{\partial f(t,x)}{\partial x} \) is the slope of the forward curve, at time \( t \) for time-to-maturity \( x \). This no-arbitrage condition clearly shows that the diffusion function \( \sigma \) must go to zero when the maturity date goes to infinity to ensure the finiteness of \( \alpha \). The term \( \frac{\partial f(t,x)}{\partial x} \) stems from the parametrization in terms of the time-to-maturity. It would be absent in the usual HJM fixed time-of-maturity parametrization. Let us stress that the two formulations are completely equivalent.

We thus get the following arbitrage-free model of forward rate dynamics

\[
dt f(t,x) = \left( \frac{\partial f(t,x)}{\partial x} + a(t,x) \right) dt + \sigma(t,x) \ dW(t) ,
\]  

(15)

\(^4\) \( M \) can have other forms than (10). The extension to several Brownian motion terms is trivial, along the lines of HJM. More importantly, the SDF can have jumps - that is a Poisson process term - due to discrete changes in the arrival of information.

\(^5\) See Dybvig, Ingersoll and Ross (1996) and Jeffrey (1997).
denoting
\[ a(t, x) = \sigma(t, x) \left( \int_0^x dy \, \sigma(t, y) - \phi(t) \right) \]  
(16)

This condition can also be found in Musiela (1993) and Brace and Musiela (1994).

We see that, at some date \( t \), estimating the drift and volatility functions of any forward rate, along with the slope of the forward rate curve at that point, recovers the market price of risk process \( \phi \).

The use of a SDF for pricing is just a useful presentation device. The no-arbitrage condition for a single forward rate allows us to construct the dynamics of all forward rates from the knowledge of their volatilities.

In appendix A, we solve equation (15) by first rewriting it as
\[ \frac{\partial f(t, x)}{\partial t} - \frac{\partial f(t, x)}{\partial x} = a(t, x) + \sigma(t, x) \frac{dW(t)}{dt}, \]  
(17)
where \( \frac{dW(t)}{dt} \) is such that, when integrated over time from 0 to \( t \), we obtain \( \int_0^t dv \frac{dW(v)}{dv} = \int_0^t dW(v) \).

We show that the solution of (17) is
\[ f(t, x) = f(0, x + t - x) + \int_0^t dv a(v, x + t - v) + \int_0^t dW(v) \sigma(v, x + t - v). \]  
(18)

Notice that if \( \sigma(t, x) = 0 \) (no fluctuations), we recover the natural no-arbitrage condition \( f(t, x) = f(0, x + t) \) stating that the instantaneous forward rate is the same at time 0 and at time \( t \) for the same maturity date \( t + x \).

### 3 A Better Model: Stochastic String Shocks

In this section we develop a more general model of the forward rate curve driven by a stochastic string. Instead of the traditional model (5), we now model the dynamics of the forward rates by
\[ df(t, x) = \alpha(t, x)dt + \sigma(t, x)dZ(t, x), \]  
(19)
or, in integral form, by
\[ f(t, x) = f(0, x) + \int_0^t dv \alpha(v, x) + \int_0^t dZ(v, x) \sigma(v, x). \]  
(20)

where the stochastic process \( Z(t, x) \) generalizes to two dimensions the previous one-dimensional Brownian motion \( W(t) \). The important innovation in (19) is that the stochastic process depends not only on time \( t \) but also on time-to-maturity \( x \). The notation \( dZ(t, x) \) denotes a stochastic perturbation to the forward rate curve at time \( t \), with different magnitudes for forward rates with different times-to-maturity. It is straightforward to extend the model to include more than one string shock or to combine string shocks with Brownian motion shocks.

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6This extends naturally to the case where there is a finite number of Brownian increments in the SDF and the forward rate processes. In this case, we would need information about the drifts and volatilities of as many forward rates as Brownian increments in the model.

7In the equation, \( \frac{dW(t)}{dt} \) is an alternative notation for the noise term \( \eta \) defined later in (34) with covariance (35).
We stress that (19) is not the infinite-dimensional generalization of the multi-factor HJM model in which all forward rates are subjected to the same (possibly infinite-dimensional) set of stochastic processes. We have one stochastic process per time-to-maturity. Our process is infinite dimensional in the sense that the set of times-to-maturity in the continuous description of the yield curve has the order of infinity of the continuum.

We impose several requirements on $Z$ to qualify as a string shock to the forward rate curve:

1. $Z(t, x)$ is continuous in $x$ at all times $t$;
2. $Z(t, x)$ is continuous in $t$ for all $x$;
3. The string is a martingale in time $t$, $E[d_tZ(t, x)] = 0$, for all $x$;
4. The variance of the increments, $\text{Var}[d_tZ(t, x)]$, does not depend on $t$ or $x$;
5. The correlation of the increments, $\text{Corr}[d_tZ(t, x), d_tZ(t, x')]$, does not depend on $t$.

We will see that the first two conditions are automatically satisfied by taking $Z$ to be the solution of a SPDE with at least one partial derivative in $x$ and $t$. Condition 3 captures the unforecastable aspect of shocks. Condition 4 corresponds to making all shocks affecting the forward curve have the same intensity. This intensity can then be changed, or “modulated”, by the function $\sigma(t, x)$ taking different values at different times to maturity $x$. The final condition imposes that the correlation between shocks two forward rates of different maturities depend only on these maturities. Conditions 3, 4 and 5 together make the strings Markovian. We will see that all string shocks produced as solutions of SPDE’s have Gaussian distributions, which are completely characterized by the first two moments.

To have each forward rate, for each time-to-maturity, be driven by its own noise process, we could simply use infinite-dimensional Brownian motion, using a simple Brownian motion as a shock to each forward rate, independent from the Brownian motion used to perturb forward rates with different times-to-maturity. However, such a model makes the forward rate curve discontinuous as a function of time-to-maturity, so that, over time, any two forward rates can become very distant from each other. Although a discontinuous forward curve violates no arbitrage conditions, it is intuitively unlikely.

We develop the model, assuming the existence of a string shock that satisfies our requirements. In the next sections, we show how such string shocks can be constructed and examine several parametric examples.

We now take the process for the SDF given by

$$\frac{dM(t)}{M(t)} = -r(t) \, dt + \int_0^\infty dy \, \phi(t, y) \, d_tZ(t, y)$$

(21)

where the pricing kernel is driven by a “weighted sum” of the shocks that affect the forward rate curve, and the market prices of risk can in principle be different for each shock to the curve.
We follow the steps of the derivation of the no-arbitrage condition on the drift of forward rates of section 2. We have
\[ y(t, s) \equiv d_t \log P(t, s) = \left[ f(t, s - t) - \int_0^{s-t} \alpha(t, y) \, dy \right] \, dt - \int_0^{s-t} dy \, \sigma(t, y) \, d_t Z(t, y) . \] (22)

We need the expression of \( dP(t, s) / P(t, s) \) which is obtained from (22) using Ito’s calculus. In order to get Ito’s term in the drift, recall that it results from the fact that, with \( y \) stochastic,
\[ d_t F(y) = \frac{\partial F}{\partial y} d_t y + \frac{1}{2} \frac{\partial^2 F}{\partial y^2} \text{Var} \left[ d_t y \right] \] (23)

so that the differential \( d_t F \) up to order \( dt \) contains contributions from all the covariances of the form \( \text{Cov} \left[ d_t Z(t, x), d_t Z(t, x') \right] \) appearing in \( \text{Var} \left[ d_t y(t, s) \right] \).

Taking the expectation of the square of the stochastic term in the r.h.s. of (22), bond price dynamics can be written as
\[ \frac{dP(t, s)}{P(t, s)} = \left[ f(t, s - t) - \int_0^{s-t} \alpha(t, y) \, dy + \frac{1}{2} \int_0^{s-t} dy \int_0^{s-t} dy' c(t, y, y') \sigma(t, y') \sigma(t, y) \right] \, dt \\
- \int_0^{s-t} dy \, \sigma(t, y) \, d_t Z(t, y) . \] (24)

where
\[ c(t, y, y') \equiv \text{Corr} \left[ d_t Z(t, y), d_t Z(t, y') \right] . \] (25)

and we make use of \( \text{Var} \left[ d_t Z(t, x) \right] = dt \). In the next sections, we offer several examples of stochastic strings and the correlation functions of their time increments.

The no-arbitrage condition becomes
\[ -r(t) + f(t, x) - \int_0^x dy \, \alpha(t, y) + \frac{1}{2} \int_0^x dy \int_0^x dy' c(t, y, y') \sigma(t, y) \sigma(t, y') \\
- \int_0^\infty dy \int_0^x dy' c(t, y, y') \phi(t, y) \sigma(t, y') = 0 \] (26)

that we can differentiate with respect to \( x \) to obtain
\[ \alpha(t, x) = \frac{\partial f(t, x)}{\partial x} + \sigma(t, x) \left( \int_0^x dy \, c(t, x, y) \sigma(t, y) - \int_0^\infty dy \, c(t, x, y) \phi(t, y) \right) \] (27)

If \( c(t, x, y) \) in (27) is one, we recover the traditional model of a single Brownian motion.

We thus get the following arbitrage-free model of forward rate dynamics
\[ \frac{dt}{d_t} f(t, x) = \left[ \frac{\partial f(t, x)}{\partial x} + \sigma(t, x) \left( \int_0^x dy \, c(t, x, y) \sigma(t, y) - \int_0^\infty dy \, c(t, x, y) \phi(t, y) \right) \right] \, dt \]

\(^8\)In principle, differentiating (24) is only warranted if \( f(t, x) \) is differentiable in \( x \). This is the case for some strings constructed below, such as the integrated O-U sheet, but is not true for others, such as the O-U sheet. In these cases, \( \frac{\partial f(t, x)}{\partial x} \) is not a function but is similar to an increment of a Brownian motion. However, the results below still hold, by reading \( \frac{\partial f(t, x)}{\partial x} \) as a convenient notation for the formal integral calculus used.
Expression (28) can be solved in a similar way to section 2. We denote

\[ A(t, x) = \sigma(t, x) \left( \int_0^x dy \ c(t, x, y) \sigma(t, y) - \int_0^\infty dy \ c(t, x, y) \phi(t, y) \right) . \]  

(29)

to finally obtain

\[ f(t, x) = f(0, t + x) + \int_0^t dv \ A(v, t + x - v) + \int_0^t dv Z(v, t + x - v) \sigma(v, t + x - v) . \]  

(30)

We see that the correlation function of the time increments of the strings is sufficient (together with the risk premia) to characterize arbitrage-free dynamics of the forward curve.

We can still invert the forward curve to obtain the market prices of risk needed for pricing derivatives. However, we now need to use the entire forward rate curve rather than a small number of forward rates (equal to the number of Brownian motions driving the curve in the traditional model).

Note that if we assume that \( \phi(t, x) = \phi(t) \), thus constraining the SDF (21), the “inversion” of the forward rate curve to extract the market prices of risk becomes much easier.

4 Construction of Stochastic String Shocks

This section shows how to construct stochastic string shocks that can be used to perturb the forward rate curve.

We start by reviewing the construction of Brownian motion from a probabilistic point of view and then as the solution to a stochastic ordinary differential equation (SODE). This allows us to obtain a natural generalization of Brownian motion to string motion as the solution of stochastic partial differential equations (SPDE’s). We then formulate the solutions to these SPDE’s in terms of the Green function of the equation and find constraints on this function to impose our requirements for string shocks.

4.1 Brownian motion as solution of a SODE

The usual description of stochastic processes in Finance is based on the Brownian motion \( W \) defined by

\[ W(t) = \int_0^t d_u W(u) , \]  

(31)

normally distributed with

\[ E \left[ d_t W(t) \right] = 0 , \]  

(32)

and

\[ \text{Var} \left[ d_t W(t) \right] = dt . \]  

(33)
Alternatively, $W$ can be defined as the solution of the following SODE:

$$\frac{dW(t)}{dt} = \eta(t) \ .$$

(34)

where $\eta$ is white noise, characterized by the covariance

$$\text{Cov} [\eta(t), \eta(t')] = \delta(t - t') \ ,$$

(35)

and $\delta$ designates the Dirac function.

The SODE (34) describes a particle at position $W$ which is incessantly subjected to random forces leading to random variations $\eta$ of its velocity.

The solution of (34) is formally

$$W(t) = \int_0^t dv \ \eta(v) \ .$$

(36)

which shows that $dt \ \eta(t)$ is simply a notation for $d_t W(t)$, and, as usual, has rigorous mathematical meaning only under the integral representation. From (34), we see (by the Central Limit Theorem) that $W$ will be Gaussian, with mean zero, and we can calculate easily the covariance

$$\text{Cov} [W(t), W(t')] = \int_0^t dv \int_0^{t'} dv' \ \text{Cov} [\eta(v), \eta(v')] = \int_0^t dv \int_0^{t'} dv' \ \delta(v - v') = (t \wedge t') \ ,$$

(37)

where $(t \wedge t')$ stand for $\min(t, t')$.

Finally, we define $d_t W(t)$ as the limit of $[W(t + \delta t) - W(t)]$ when the small but finite increment of time $\delta t$, typical of what we have to deal with in real time series, becomes the infinitesimal $dt$. Using (37), we get $\text{Var} [W(t + \delta t) - W(t)] = \delta t$ which recovers (33) in the infinitesimal time increment limit.

The definition of the Brownian motion as the solution of a SODE is very useful to generalize to other processes. Maybe the simplest extension is the well-known Ornstein-Uhlenbeck (O-U) process $U(t)$ which can be defined as the solution of the following SODE:

$$\frac{dU(t)}{dt} = -\kappa U(t) + \eta(t) \ ,$$

(38)

where $\kappa$ is a positive constant. In addition to the random variations $\eta$ of the velocity, the Brownian particle is now subjected to a restoring force tending to bring it back to the origin, i.e. mean reversion. The formal solution reads

$$U(t) = \int_0^t dv \ \eta(v) \ e^{-\kappa(t-v)} \ .$$

(39)

Its covariance is

$$\text{Cov} [U(t), U(t')] = \frac{1}{2\kappa} \left( e^{-\kappa|t-t'|} - e^{-\kappa(t+t')} \right) \ ,$$

(40)

### Notes:

9 The Dirac function is a distribution in the sense of Schwartz’s theory of distributions, or generalized functions. It is such that, for arbitrary $f(x)$, $\int_{x_1}^{x_2} f(x) \delta(x-x_0) = f(x_0)$ if $x_1 < x_0 < x_2$, $\frac{1}{2} f(x_0)$ if $x_1 = x_0$ or $x_0 = x_2$ and zero otherwise. $\delta(x)$ is obtained as the limit of functions when their width goes to zero and their height goes to infinity in such a way that their integral remains one. An example is the Gaussian $\lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{x^2}{2\sigma^2}} = \delta(x)$. 

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which goes to $\frac{1}{2\kappa}e^{-\kappa|t-t'|}$ at large times. By adding more terms in (38), more complex stochastic processes can be easily generated.

### 4.2 Stochastic strings as solutions of SPDE’s

We now generalize the stochastic processes driving the uncertainty in the forward rate curve from solutions of SODE’s to solutions of SPDE’s. In the context of differential calculus, this is the most natural and general extension that can be performed. The introduction of partial differential equations is called for in order to account for the continuity condition and to deal with the two variables, time and time-to-maturity, and their interplay.

Brownian motion depicts, in physical terms, the motion of a particle subjected to random velocity variations. The analogous physical system described by SPDE’s is a stochastic string. In our application of stochastic strings as the noise source in the dynamics of the forward rate curve, the length along the string is the time-to-maturity, and the string configuration gives the amplitude of the shocks at a given time, for each time-to-maturity. The set of admissible dynamics of the configuration of the string as a function of time depends on the specification of the SPDE.

In the present paper, we restrict our attention to linear SPDE’s, in which the highest derivative is, in most cases, second order. This second order derivative has a simple physical interpretation: the string is subjected to a tension, like a piano chord, that tends to bring it back to zero deformation. This tension forces the “coupling” among different times-to-maturity so that the forward rate curve remains continuous. In principle, a more general formulation would consider SPDE’s with terms of arbitrary derivative orders. However, it is easy to show that the tension term is the dominating restoring force, when present, for deformations of the string (forward rate curve) at long “wavelengths” along the time-to-maturity axis. Second order SPDE’s are thus generic in the sense of a systematic expansion.

The general form of second-order linear SPDE’s reads

$$a(t, x)\frac{\partial^2 X(t, x)}{\partial t^2} + 2b(t, x)\frac{\partial^2 X(t, x)}{\partial t \partial x} + c(t, x)\frac{\partial^2 X(t, x)}{\partial x^2} = f \left( t, x, X(t, x), \frac{\partial X(t, x)}{\partial t}, \frac{\partial X(t, x)}{\partial x} \right),$$

where $X$ is the stochastic string shock we want to characterize. In the present paper, we restrict our discussion to linear equations, where $F$ has the form

$$f \left( t, x, X(t, x), \frac{\partial X(t, x)}{\partial t}, \frac{\partial X(t, x)}{\partial x} \right) = d(t, x)\frac{\partial X(t, x)}{\partial t} + e(t, x)\frac{\partial X(t, x)}{\partial x} + g(t, x)X(t, x) + s(t, x).$$

---

10 Using $2(t \wedge t') = t + t' - |t - t'|$.
11 Further extensions will include fractional differential equations and integro-differential equations, including jump processes.
12 Higher order derivatives also have an intuitive physical interpretation. For instance, going up to fourth order derivatives in the SPDE correspond to the dynamics of a beam, which has bending elastic modulus tending to restore the beam back to zero deformation, even in absence of tension.
13 There are situations where the tension can be made to vanish (for instance in the presence of a rotational symmetry) and then the leading term in the SPDE becomes the fourth order “beam” term.
s is the “source” term that will be generally taken to be Gaussian white noise $\eta$ characterized by the covariance

$$\Cov [\eta(t, x), \eta(t', x')] = \delta(t - t') \delta(x - x'),$$

where $\delta$ denotes, as before, the Dirac distribution. Expression (41) together with (42) is the most general linear second-order SPDE in two variables. The solution $X$ exists and its uniqueness is warranted once “boundary” conditions are given, such as, for instance, the initial value of the string $X(0, x)$, for all $x$, as well as any constraints on the particular form of equation (41).

### 4.3 Green function formulation

All solutions of the above linear SPDE can be characterized by

$$X(t, x) = X(0, x) + \int_0^t dv \int_{-\infty}^{\infty} dy \ G(t, x|v, y) \ \eta(v, y),$$

where the Green function $G$ contains all the information in the underlying SPDE and thus of the coupling between the different times-to-maturity $x$.

Consider a single impulsive source term $\eta(v, y) = \delta(v - t_0)\delta(y - x_0)$. Then,

$$X(t, x) = X(0, x) + G(t, t_0, x_0).$$

The Green function $G$ thus describes the deterministic evolution in the future of the process $X$ at all times-to-maturity due to an impulsive perturbation that occurs at time $t_0$ at the time-to-maturity $x_0$. It thus embodies all the coupling between different times-to-maturities.

We have

$$\Var [X(t, x)] = \int_0^t dv \int_{-\infty}^{\infty} dy \ [G(t, x|v, y)]^2,$$

and

$$\Cov [d_tX(t, x), d_tX(t, x')] = \lim_{\delta t \to 0} \int_t^{t+\delta t} dv \int_{-\infty}^{\infty} dy \ G(t + \delta t, x|v, y) \ G(t + \delta t, x'|v, y)$$

$$+ \int_0^t dv \int_{-\infty}^{\infty} dy [G(t + \delta t, x|v, y) - G(t, x|v, y)] [G(t + \delta t, x'|v, y) - G(t, x'|v, y)].$$

However, this general solution does not satisfy all our conditions for a stochastic string shock to the forward curve. We want to impose in addition that $X(t, x)$ is a martingale, and that the variance of its increments, $\Var [d_tX(t, x)]$, does not depend on $x$. In appendix B, we show that strings that satisfy our requirements can be written as

$$Z(t, x) = Z(0, x) + \int_0^t dv \int_{x_0}^{x} dy \ h(v, x, y)\eta(v, y),$$

\footnote{See for instance Morse and Feshbach (1953).}
where $g$, $j$ and $h$ are functions to be constrained in order to have the variance of the increments independent of $x$, and the lower bound can be adjusted in order to get rid of convergence problems at infinity. This leads to

$$d_t Z(t, x) = dt \int_{x_0}^{j(x)} dy \, h(t, x, y) \eta(t, y) ,$$

so that

$$\text{Var} [d_t Z(t, x)] = dt \int_{x_0}^{j(x)} dy \, [h(t, x, y)]^2 .$$

The additional condition that $\text{Var} [d_t Z(t, x)]$ does not depend on $x$ leads to several solutions.

A first class of solutions is obtained if we assume that $h(t, x, y)$ is independent of $y$. The integration gives then simply $j(x)[h(t, x)]^2$. Equating to a constant (equal to 1 without loss of generality), this gives

$$h(x) = \frac{1}{\sqrt{j(x)}} .$$

The corresponding stochastic string is thus defined by

$$d_t Z(t, x) = dt \frac{1}{\sqrt{j(x)}} \int_0^{j(x)} dy \, \eta(t, y) .$$

and the correlation of the increments is

$$c(t, x, x') \equiv \text{Corr} [d_t Z(t, x), d_t Z(t, x')] = \sqrt{\frac{j(x)}{j(x')}} ,$$

if $j(x) < j(x')$. The role of $x$ and $x'$ in (53) are inversed if $j(x) > j(x')$. The corresponding SPDE reads

$$\frac{\partial}{\partial x} \left( \sqrt{j(x)} \frac{\partial Z(t, x)}{\partial t} \right) = \sqrt{\frac{dj(x)}{dx}} \eta(t, x) .$$

An alternative class of solutions can be obtained by noting that

$$[h(x)]^2 = \frac{1}{x} \frac{d}{dy} [j]^{-1}(y) ,$$

where $[j]^{-1}$ is the inverse function of $j$ defined by $j([j]^{-1}(y)) = y$, verifies the constraint that $\text{Var} [d_t Z(t, x)]$ be independent of $x$. This corresponds to

$$d_t Z(t, x) = dt \frac{1}{\sqrt{x}} \int_0^{j(x)} dy \left( \frac{d}{dy} [j]^{-1}(y) \right)^{1/2} \eta(t, y) .$$

A generalization of (55) is

$$[h(x)]^2 = \frac{1}{x^\alpha} \frac{d}{dy} ([j]^{-1}(y))^{\alpha} ,$$

where $\alpha > 0$ is arbitrary. In fact, even more generally, consider an arbitrary monotonous function $l$. Then,

$$[h(x)]^2 = \frac{1}{l(x)} \frac{d}{dy} l([j]^{-1})(y) .$$

12
satisfies the constraint that $\text{Var} \left[ d_t Z(t, x) \right]$ does not depend on $x$. This leads to

$$
d_t Z(t, x) = dt \frac{1}{\sqrt{l(x)}} \int_0^{j(x)} dy \sqrt{d \frac{d}{dy} \left( (j)^{-1} \right)(y)} \eta(t, y) . \tag{59}
$$

The correlation of the increments is

$$
c(t, x, x') \equiv \text{Corr} \left[ d_t Z(t, x), d_t Z(t, x') \right] = \sqrt{\frac{l(x)}{l(x')}} , \tag{60}
$$

if $j(x) < j(x')$. The role of $x$ and $x'$ in (59) are inversed if $j(x) > j(x')$. This provides a slight generalization to (53) since a different function appears in the correlation function and in the inequality condition on $x$ and $x'$.

The corresponding SPDE is

$$
\frac{\partial}{\partial x} \left( \sqrt{l(x)} \frac{\partial Z(t, x)}{\partial t} \right) = \sqrt{\frac{dj(x)}{dx}} \sqrt{\frac{dl(x)}{dx}} \eta(t, x) . \tag{61}
$$

### 4.4 Differentiability in time-to-maturity

All the processes studied above are continuous both in $t$ and $x$ but not differentiable in either $t$ or $x$. Of special interest is the non-differentiability in $x$.\footnote{We show in appendix C how to determine the differentiability of string shocks.}

We would like now to discuss a class of processes which are non-differentiable in time but differentiable in time-to-maturity, while still keeping an infinite set of shock processes, one for each time-to-maturity $x$.\footnote{HJM provides an obvious example of a stochastic process which is non-differentiable in time and differentiable in $x$, but this is a degenerate case as the same stochastic process drives all times-to-maturities.}

Economics has little to say with respect to the differentiability of forward rate curves. Furthermore, this issue cannot be resolved empirically, since forward rates of very close maturities cannot be observed. The use of string shocks that produce differentiable curves is thus fundamentally a matter of taste.

An intuitive strategy to obtain a shock differentiable in time-to-maturity is to integrate any of the previously defined strings in the $x$-variable\footnote{Smother strings can be obtained by higher-order integration.}

$$
Y(t, x) = \int_0^x dy \ Z(t, y) . \tag{62}
$$

By definition, $\frac{\partial Y(t, x)}{\partial x} = Z(t, x)$. Since $Z(t, x)$ is continuous in $x$, $\frac{\partial Y(t, x)}{\partial x}$ is also continuous in $x$ and $Y(t, x)$ is thus differentiable with respect to $x$.

Since $Z(t, y)$ satisfies a SPDE, we see that, by replacing $Z(t, y)$ by $\frac{\partial Y(t, x)}{\partial x}$ in this equation, that $Y(t, x)$ is also the solution of a SPDE.

The correlation of the increments is

$$
c(t, x, x') \equiv \text{Corr} \left[ d_t Y(t, x), d_t Y(t, x') \right] = x \left[ \int_0^x \sqrt{\frac{l(y)}{l(x)}} + \int_x^{x'} \sqrt{\frac{l(x)}{l(y)}} \right] , \tag{63}
$$

\footnote{We show in appendix C how to determine the differentiability of string shocks.}
for \( x < x' \), assuming that \( j(x) \) is monotonous increasing. The corresponding expressions for other cases are straightforwardly derived.

The variance is

\[
\text{Var} [d_t Y(t, x)] = dt \ x \int_0^x dy \ \sqrt{\frac{l(y)}{l(x)}},
\]

which is independent of \( x \) if \( l(x) = c_1 e^{c_2(x-1/x)} \), where \( c_1 \) and \( c_2 \) are two arbitrary constants.

5 Parametric Examples

In this section we examine several examples of stochastic strings that are solutions to SPDE’s, and consider their interest as a noise source in a model of the dynamics of the forward rate curve.

5.1 The Brownian sheet

We start by discussing the simplest example of a SPDE, which however does not satisfy our criteria. The example is however useful as it is the basis for the derivation of other string processes.

The simplest SPDE in the class given by (41) and (42) is

\[
\frac{\partial^2 W(t, x)}{\partial t \partial x} = \eta(t, x),
\]

where \( \eta(t, x) \) is a white noise both in time and \( x \), characterized by the covariance function (43). By inspection of (65), the first order time derivative ensures that \( W \) will be a Brownian motion in time at fixed \( x \), while the introduction of the partial derivative with respect to \( x \) ensures the continuity of \( W \) with respect to \( x \). The solution of (65) reads

\[
W(t, x) = \int_0^x dy \int_0^t dv \ \eta(v, y). 
\]

This process (66) is known as the Brownian sheet\(^{18}\) and has the following covariance, which can be readily obtained using (43) with (66):

\[
\text{Cov} [W(t, x), W(t', x')] = (t \land t') \ (x \land x'), 
\]

where \( (t \land t') \) is the minimum of \( t \) and \( t' \).

For the calculus used above in the implementation of the no-arbitrage condition on forward rate dynamics, we need the correlation of the time increments of the string. The covariance is the limit when \( \delta t \to dt \) of \( \text{Cov} [W(t + \delta t, x) - W(t, x), W(t + \delta t, x') - W(t, x')] \). Its calculation reduces to that of four terms of the form (67) that simplify to

\[
\text{Cov} [d_t W(t, x), d_t W(t, x')] = (x \land x') \ dt. 
\]

\(^{18}\)See Walsh (1986).
or, stated in terms of the correlation,
\[ c(t, x, x') \equiv \text{Corr} \left[ d_t W(t, x), d_t W(t, x') \right] = (x \wedge x') . \] (69)

The property that the variance of \( W(t, x) \) is not homogeneous in \( x \), makes this process unsatisfactory as a stochastic perturbation in (19): the instantaneous spot rate \((x = 0)\) would have no volatility in this model since \( W(t, 0) = 0 \), and the variance of forward rates would increase linearly with \( x \).

A simple modification of the Brownian sheet can be obtained by making \( j(x) = x \) in (52), leading to
\[ d_t \hat{W}(t, x) = dt \frac{1}{\sqrt{x}} \int_0^x dy \eta(t, y) . \] (70)

This process is just like the Brownian sheet, however rescaled by the factor \( \frac{1}{\sqrt{x}} \) to ensure homogeneity in \( x \). This form could have been guessed directly from the expression (67) of the correlation of the Brownian sheet. The corresponding SPDE is
\[ \frac{\partial}{\partial x} \left( \sqrt{j(x)} \frac{\partial \hat{W}(t, x)}{\partial t} \right) = \eta(t, x) , \] (71)

and
\[ \text{Corr} \left[ d_t \hat{W}(t, x), d_t \hat{W}(t, x') \right] = \sqrt{x \over x'} , \] (72)
for \( x < x' \) and \( \sqrt{x \over x'} \) for \( x > x' \).

5.2 The Ornstein-Uhlenbeck sheet

A natural model for \( Z(t, x) \) is one similar to the Brownian sheet but in which the variance is the same for all times-to-maturity. We look for the string with the same variance for all \( x \) and covariance between \( x \) and \( x' \) that depends only on the difference \((x - x')\).

The O-U string satisfies (52) with \( j(x) = e^{2\kappa x} \), which leads to
\[ U(t, x) = e^{-\kappa x} W(t, e^{2\kappa x}) = e^{-\kappa x} \int_0^{e^{2\kappa x}} dy \int_0^t dv \eta(v, y) . \] (73)

The corresponding SPDE is
\[ \frac{\partial}{\partial x} \left( \sqrt{j(x)} \frac{\partial Z(t, x)}{\partial t} \right) = \sqrt{\frac{dj(x)}{dx}} \eta(t, x) . \] (74)

which we can simplify to
\[ \frac{\partial^2 U(t, x)}{\partial x \partial t} + \kappa \frac{\partial U(t, x)}{\partial t} = \sqrt{2\kappa} \eta(t, x) . \] (75)

By inspection of (73), we get the following properties
\[ \text{Cov} \left[ U(t, x), U(t', x') \right] = (t \wedge t') e^{-\kappa |x-x'|} , \] (76)
and
\[ c(t, x, x') \equiv \text{Corr}[d_t U(t, x), d_t U(t, x')] = e^{-\kappa|x-x'|} . \] (77)

Expressions (76) and (77) both show that, for fixed \( x \) (\( x = x' \)), \( U(t, x) \) is a Brownian motion in time with variance independent of \( x \). This is the property that we were looking for, so that all times-to-maturity have a priori the same volatility for the driving stochastic process. We also learn that the time increments \( d_t U(t, x) \) and \( d_t U(t, x') \) are correlated across times-to-maturity: this is necessary to obtain the continuity condition along \( x \). Again, if the variations along \( x \) were delta-correlated, the process would consist of independent one-dimensional Brownian motions, one for each \( x \), thus leading to an almost everywhere discontinuous process. The correlations along \( x \) are the strongest between close times-to-maturity and vanish exponentially between distant times-to-maturity. The parameter \( \kappa \) allows us to span a set of models, from simple Brownian motion to a stochastic string. Indeed, for \( \kappa \to 0 \), all times-to-maturity are so strongly coupled that they all consist of the same single dimensional Brownian motion. This limit \( \kappa \to 0 \) thus recovers the single factor model driving simultaneously all times-to-maturities. Figure 1 shows an example of the increment in the O-U sheet for different values of \( \kappa \). We see that, as \( \kappa \) increases, the shape of the string has less and less humps.

There is another way to construct the O-U sheet. Let us come back to the definition (73) of \( U(t, x) \) and make the change of variable \( y \to e^{2\kappa z} \) in the integral. We get
\[ U(t, x) = 2\kappa e^{-\kappa x} \int_{-\infty}^{x} dz e^{2\kappa z} \int_{0}^{t} dv \eta(v, e^{2\kappa z}) . \] (78)

We can write \( \eta(v, e^{2\kappa z}) = \frac{e^{-\kappa z} \hat{\eta}(t, z)}{\sqrt{2\kappa}} \), where \( \hat{\eta}(t, z) \) is still white noise \( \text{Cov}[\hat{\eta}(t, z), \hat{\eta}(t', z')] = \delta(t - t') \delta(z - z') \). Dropping the hat on the noise, we get
\[ U(t, x) = \sqrt{2\kappa} e^{-\kappa x} \int_{-\infty}^{x} dz e^{\kappa z} \int_{0}^{t} dv \eta(v, z) . \] (79)

We then write \( \int_{-\infty}^{x} = \int_{-\infty}^{0} + \int_{0}^{x} \). The first integral gives a contribution
\[ e^{-\kappa x} \int_{0}^{t} dv \int_{-\infty}^{0} dz e^{\kappa z} \sqrt{2\kappa} \eta(v, z) = e^{-\kappa x} \int_{0}^{t} dv \eta(v) , \] (80)

where \( \eta(v) \equiv \sqrt{2\kappa} \int_{-\infty}^{0} dz e^{\kappa z} \eta(v, z) \) is a simple Brownian process of unit variance. We have kept the same notation \( \eta \), but it is clear that it now depends only on time. This is thus the noise process that is common to all times-to-maturity. If considered alone, this would recover the single factor model driving simultaneously all times-to-maturities. Putting everything together, we obtain the alternative formulation of the O-U sheet
\[ U(t, x) = e^{-\kappa x} \int_{0}^{t} dv \eta(v) + \sqrt{2\kappa} e^{-\kappa x} \int_{0}^{t} dv \int_{0}^{x} dz e^{\kappa z} \eta(v, z) . \] (81)

This form exemplifies the relationship with the standard one-dimensional O-U process, in the sense that it shows the exponential dependence of the process on innovations at different times-to-maturity. This is thus the generalization in the time-to-maturity dimension of the classical O-U

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\(^{19}\)Different volatilities for forward rates with different times-to-maturity can be obtained by choice of the weighting factor \( \sigma(t, x) \) in (13).
process usually defined in the time direction. The analogy with the one-dimensional O-U process is also clear from the SPDE formulation (75) where it is seen that the exponential dependence is reflected in the mean reversion term $\frac{\partial U}{\partial t}$, which plays a role similar to the mean reversion term $-\kappa U$ in (88).

This process, written in the form (81), has been used by Kennedy (1997) and Goldstein (1997) to model forward rates. Our approach clarifies the special role played by this process and its relationship with SPDE’s.

5.3 The integrated Ornstein-Uhlenbeck sheet

As an interesting example, introduce the “integrated O-U sheet”, defined by

$$V_1(t, x) = \int_0^x dy \ U(t, y) ,$$

(82)

where $U(t, y)$ is the O-U sheet defined in (73). By definition, $\frac{\partial V_1(t, x)}{\partial x} = U(t, x)$. Since $U(t, x)$ is continuous in $x$, $\frac{\partial V_1(t, x)}{\partial x}$ is also continuous in $x$ and $V_1(t, x)$ is thus differentiable with respect to $x$.

Since $U(t, y)$ satisfies the SPDE (75), we see, by replacing $U(t, y)$ by $\frac{\partial V_1(t, x)}{\partial x}$ in this equation, that $V_1(t, x)$ is the solution of the following SPDE

$$\frac{\partial^3 V_1(t, x)}{\partial x^2 \partial t} + \kappa \frac{\partial^2 V_1(t, x)}{\partial x \partial t} = \sqrt{2\kappa} \eta(t, x) .$$

(83)

This is a SPDE of order three.

We obtain

$$c(t, x, x') \equiv \text{Corr} [dV_1(t, x), dV_1(t, x')] = \int_0^x dy \int_0^{x'} dy' \text{Corr} [U(t, y), U(t, y')]$$

$$= \frac{1}{\kappa} \left( 2(x \wedge x') - \frac{1}{\kappa} (1 - e^{-\kappa x})(1 - e^{-\kappa x'}) \right) ,$$

(84)

where we have used (77).

Inspired by the alternative formulation (81) of the O-U sheet $U(t, y)$ in terms of the exponential kernel, we define another integrated O-U sheet $V_2(t, x)$ by

$$V_2(t, x) = \kappa \sqrt{2} \int_0^x dy \ e^{\kappa y} U(t, y) ,$$

(85)

Again by construction, $\frac{\partial V_2(t, x)}{\partial x}$ is continuous and thus $V_2(t, x)$ is differentiable in $x$. By differentiation of (83) with respect to $x$, we get

$$\frac{\partial V_2(t, x)}{\partial x} = -\kappa V(t, x) + \kappa \sqrt{\sigma} U(t, x) .$$

(86)

Kennedy (1997) only considers the case of a constant volatility function for the forward rates, whereas Goldstein (1997) allows for a general volatility function. In contrast to our approach, both authors model forward rates of fixed maturity date.
For instance, take $j$.

This process has been independently used by Goldstein (1997) to model forward rates. Our approach shows that it is one particular example among an arbitrarily large class of processes exhibiting similar properties.

5.4 The term structure of correlations

Empirical observation shows that correlation between two forward rates, with maturities separated by a given interval, increases with maturity. What are the string processes that exhibit this property? Let us start from the parametrization obtained by assuming that $h(t, x, y)$ is independent of $y$, which leads to (53). Consider two time-to-maturities $x$ and $x' = x + \delta x$, where $\delta x > 0$ is small compared to $x$. We can then expand $j(x') = j(x) + \delta x \frac{d j(x)}{d x} + \text{higher order terms}$. Expanding the square root in (52), we get

$$c(t, x, x') \equiv \text{Corr} \left[ d_t Z(t, x), d_t Z(t, x') \right] \approx \left( 1 - \frac{\delta x \log j(x)}{2} \right).$$

We see that $\text{Corr} [d_t Z(t, x), d_t Z(t, x')]$ increases with $x$ if $\frac{d \log j(x)}{dx}$ decreases (while remaining positive) as $x$ increases. An interesting situation is when $\frac{d \log j(x)}{dx} \to 0^+$ as $x$ gets large. In this case, the increments become perfectly correlated for large time-to-maturities.

Parametrizing $j(x) = e^{i(x)}$, the above condition implies that $i(x)$ must be a function which increase slower than $x$, in such a way that its derivative decreases with $x$, that is, $i(x)$ must be concave. In order words, $j(x)$ must grow slower than an exponential. This is not the case of the O-U sheet or the integrated O-U sheet. However, the previous example (72), where $j(x) = x$, qualifies. In fact, any expression like $j(x) = e^{x^\alpha}$ with an exponent $0 < \alpha < 1$ qualifies. This corresponds to

$$c(t, x, y) \equiv \text{Corr} \left[ d_t Z(t, x), d_t Z(t, y) \right] = e^{-\kappa|x-y|}. \quad (89)$$

For instance, take $j(x) = e^{2\kappa \sqrt{x}}$. This corresponds to a correlation function

$$c(t, x, y) \equiv \text{Corr} \left[ d_t Z(t, x), d_t Z(t, y) \right] = e^{-\kappa \sqrt{t_2} - \sqrt{t_1}} \quad (90)$$

which again works fine. Figure 3 shows the correlations between forward rates at maturities spaced by different time intervals, as the maturities increase. We see that correlation increases both with
the proximity of the maturities of the forward rates and with the times-to-maturity of the forward
rates.

These models make it very easy to fit any covariance matrix of the increments of instantaneous
forward rates. The function $\sigma$ can be used to fit the variances, and the parameters $\kappa$ in (90) or $\kappa$
and $\alpha$ in (89) can be used to fit the correlations.

6 Empirical Implications

Traditional factor models of the term structure, are only compatible with samples that include
at most as many bonds as there are factors in the model. In order to be able to use data on
more bond prices to estimate these models, it is necessary to add error terms to their econometric
specification. For econometric tractability, these errors must be assumed to be independent from
the factors\textsuperscript{21}. This approach is correct if the incompatibility of the data with the model is due
to observation error, possibly due to bid-ask spreads or nonsynchronous observations. However, if
the incompatibility is due to a misspecified model, the error terms will not be independent of the
factors and the econometric model will be inappropriate.

Models with stochastic string shocks can be estimated without any such error terms. The models
are compatible with any sample of forward rates (or bond prices) of finitely spaced maturities,
taken at some finite sampling interval. This is so, because, for any parametric specification, there
is always a possible path for the string shock over a finite interval that can lead from the forward
curve at the beginning of the interval to the forward curve at the end of the interval. This realization
may be highly unlikely, but it is always possible. The estimation exercise is thus one of finding the
most likely parameters, given a set of movements of the forward curve over time.

7 Option Pricing and Replication

In our model, derivatives can in general be priced by simulation. We just need to simulate the
dynamics of the forward curve under the risk adjusted probability measure. Define a new string
$Z^*$ with dynamics

$$d_t Z^*(t, y) = d_t Z(t, y) - \int_0^\infty dy \ c(t, x, y) \phi(t, y).$$

Under the risk adjusted measure, this string is a martingale. Asset prices discounted at the bank ac-
count are also martingales under this probability measure. Thus, the price at time $t$ of a (European,
path-independent) contingent claim with payoff $\Phi(s)$ at time $s > t$ is

$$\Phi(t) = E_t^*[\Phi(s) e^{-\int_t^s du r(u)}].$$

It is sometimes convenient to use the $s$-maturity bond price as numeraire. In order to change drifts

\textsuperscript{21}See Pearson and Sun (1994) or Chen and Scott (1993).
for the $s$-forward risk neutral probability measure, note that

\[ v(t, s) = \int_0^{s-t} dx \int_0^{s-t} dy \ c(t, x, y) \sigma(t, x) \sigma(t, y) \]  

(93)

is the instantaneous variance of the $s$-maturity bond price. Then, defining a new string $Z^s$ with dynamics

\[ d_t Z^s(t, y) = d_t Z^*(t, y) + v(t, s) \ dt , \]  

(94)

we can price the contingent claim as

\[ \Phi(t) = P(t, s) \ E_t^s [\Phi(s)] . \]  

(95)

Now the expectation is taken with respect to the probability measure under which $Z^s$ is a martingale.

Unfortunately, we cannot use the Feynman-Kac Theorem to obtain a PDE for pricing contingent claims. In our model, there is no finite set of state variables that determine the value of interest rate derivatives. Derivatives will in general depend on the full history of (string) shocks to the forward rate curve. This shows that to hedge them, it is necessary to use the full set of bonds of all maturities.

Closed form solutions for option prices can be obtained for Gaussian cases, where the volatilities of forward rates are deterministic. Kennedy (1994) gives a formula for pricing European call options on bonds when the string used is the O-U sheet and $\sigma(t, x) = \sigma e^{-ax}$, with $\sigma$ and $a$ positive constants.

8 Future Work

In future work, we intend to estimate and test different parametrizations of our model. Another avenue of research that we are currently exploring is the modelization of forward rates directly as a stochastic string, i.e., as the solution of a SPDE, rather than the approach followed in the present paper of using stochastic strings multiplied by volatility functions to shock forward rates. This new approach is particularly promising for a parsimonious description of forward rate curves with non-linear dynamics.
A Solution of Traditional Model

In this appendix, we obtain the arbitrage-free process for forward rates of the traditional model of section 2.

We perform a change of variable \((t, x) \rightarrow (\tau \equiv x + t, \xi \equiv x - t)\) and denote \(\hat{f}(\tau, \xi) \equiv f(t, x)\). We then have

\[
\frac{\partial f(t, x)}{\partial t} = \frac{\partial \tau}{\partial t} \frac{\partial \hat{f}(\tau, \xi)}{\partial \tau} + \frac{\partial \xi}{\partial t} \frac{\partial \hat{f}(\tau, \xi)}{\partial \xi} = \frac{\partial \hat{f}(\tau, \xi)}{\partial \tau} - \frac{\partial \hat{f}(\tau, \xi)}{\partial \xi} .
\]  

(96)

and, similarly,

\[
\frac{\partial f(t, x)}{\partial x} = \frac{\partial \tau}{\partial x} \frac{\partial \hat{f}(\tau, \xi)}{\partial \tau} + \frac{\partial \xi}{\partial x} \frac{\partial \hat{f}(\tau, \xi)}{\partial \xi} = \frac{\partial \hat{f}(\tau, \xi)}{\partial \tau} + \frac{\partial \hat{f}(\tau, \xi)}{\partial \xi} .
\]

(97)

Replacing these in the l.h.s. of (100), we get

\[
\frac{\partial f(t, x)}{\partial t} - \frac{\partial f(t, x)}{\partial x} = -2 \frac{\partial \hat{f}(\tau, \xi)}{\partial \xi} .
\]

(98)

Equation (17) thus becomes

\[
\frac{\partial \hat{f}(\tau, \xi)}{\partial \xi} = -\frac{1}{2} a \left( \frac{\tau - \xi}{2}, \frac{\tau + \xi}{2} \right) - \frac{1}{2} \sigma \left( \frac{\tau - \xi}{2}, \frac{\tau + \xi}{2} \right) \frac{dW(t)}{dt} \bigg|_{t=\tau+\xi} ,
\]

where we have expressed the values of \(t\) and \(x\) in terms of the new variables \(\tau\) and \(\xi\): \(t = \frac{\tau + \xi}{2}\) and \(x = \frac{\tau - \xi}{2}\). This change of variable allows us to get an ODE from the initial PDE and its straightforward integration gives

\[
\hat{f}(\tau, \xi) - \hat{f}(\tau, \xi_0) = -\frac{1}{2} \int_{\xi_0}^{\xi} dw \, a \left( \frac{\tau - w}{2}, \frac{\tau + w}{2} \right) - \frac{1}{2} \int_{\xi_0}^{\xi} dW(w) \, \sigma \left( \frac{\tau - w}{2}, \frac{\tau + w}{2} \right) .
\]

(100)

Now, by definition of the change of variables, \(\hat{f}(\tau, \xi) = f(t, x)\). We have a free choice for \(\xi_0\). Since we want to express \(f(t, x)\) as a function of \(f(0, .)\), \(\xi_0\) is chosen such that it corresponds to \(t_0 = 0\). To see what this leads to, we use the definition \(\hat{f}(\tau, \xi_0) = f(t_0 = \frac{\tau - \xi_0}{2}, \frac{\tau + \xi_0}{2})\). For \(t_0\) to be zero, this implies that \(\xi_0 = \tau\) and thus \(\frac{\tau + \xi_0}{2} = \tau = x + t\). The l.h.s. of (100) thus reads

\[
\hat{f}(\tau, \xi) - \hat{f}(\tau, \xi_0) = f(t, x) - f(0, x + t)
\]

(101)

To tackle the r.h.s. of (100), we perform the change of variable from \(w\) to \(v = \frac{\tau - w}{2}\). Then \(\frac{\tau + w}{2} = \tau - v\) and \(\int_{\xi_0=\tau=x+t}^{\xi} dw = -2 f_0^t dv\). Finally, we get

\[
f(t, x) = f(0, x + t) + \int_0^t dv \, a(v, t + x - v) + \int_0^t dW(v) \, \sigma(v, t + x - v) .
\]

(102)

Let us now comment on the validity of this derivation when \(f(t, x)\) is not differentiable in \(x\) (and in \(t\) as it is usually). As a teaching example, consider the SPDE

\[
\frac{\partial W}{\partial t} - \frac{\partial W}{\partial x} = \eta(t, x) .
\]

(103)
Performing the same change of variable as above, we get

$$\frac{\partial \hat{W}(\tau, \xi)}{\partial \xi} = -\frac{1}{2} \hat{\eta}(\tau, \xi) ,$$  \hspace{1cm} (104)

where

$$\hat{\eta}(\tau, \xi) \equiv \eta((\tau - \xi)/2, (\tau + \xi)/2) .$$  \hspace{1cm} (105)

The solution of (104) reads

$$\hat{W}(\tau, \xi) = -\frac{1}{2} \int_0^\xi dy \hat{\eta}(\tau, y) = -\frac{1}{2} \int_{x-t}^{x} dy \eta((x + t - y)/2, (x + t + y)/2) .$$  \hspace{1cm} (106)

One readily checks that (106) obeys (103). The terms formally involving the derivatives of the noise \(\eta\) would take sense when discretizing the equations.

**B Requirements for String Shocks**

**B.1 Martingale condition for strings**

We want \(E_t[d_tX(t, x)]\) to be equal to zero. From (44), we have

$$d_tX(t, x) = \int_0^t dv \int_{-\infty}^{\infty} \tilde{\eta}(v, y) \frac{\partial G(t, x|v, y)}{\partial t} \eta(v, y) + \int_{-\infty}^{\infty} dy G(t, x|t, y) \eta(t, y) .$$  \hspace{1cm} (107)

The expectation \(E_t[d_tX(t, x)]\) conditional on the realization of \(X(t, x) = g(x)\), where \(g(x)\) is a specified function, can be written as

$$E_t[d_tX(t, x)] = \int \cdots \int \mathcal{D}\{\eta\} \mathcal{P}(\{\eta\}) d_tX(t, x) \delta\left(X(t, x) - g(x)\right) ,$$  \hspace{1cm} (108)

where \(\int \cdots \int \mathcal{D}\{\eta\}\) denotes the functional integral over all possible realizations of the noise \(\eta\), \(\mathcal{P}(\{\eta\})\) is the probability density function weighting each realization of the noise, and the term \(\delta\left(X(t, x) - g(x)\right)\) ensures that the functional integral is carried over all possible \(\eta\)'s restricted to satisfy the specific realization of \(X(t, x)\). With delta correlated shocks \(\eta\), \(\mathcal{P}(\{\eta\})\) is Gaussian, so that

$$\mathcal{P}(\{\eta\}) = P_0 e^{-\int dv \int dy C^{-1}(v, y) [\eta(v, y)]^2} ,$$  \hspace{1cm} (109)

where \(2C^{-1}(v, y)\) is the inverse of the variance of \(\eta(v, y)\) and we allow it to vary with time and time-to-maturity.

Taking the Fourier transform of \(E_t[d_tX(t, x)]\) with respect to \(g(x)\) gives

$$\tilde{E}_t[d_tX(t, x, k)] = \int \cdots \int \mathcal{D}\{\eta\} \mathcal{P}(\{\eta\}) d_tX(t, x) e^{ik \cdot X(t, x)} ,$$  \hspace{1cm} (110)

---

\(^{22}\)To get an intuitive representation, imagine that time is discrete. Then, the integrals are carried out over the variables \(\eta_i\) for time \(i = 1 \text{ to } t\).
where \( k \) is the conjugate of \( g(x) \) in the Fourier transform. We notice that the second term in the r.h.s. of (107) does not contribute to \( \hat{E}_t[d_t X(t,x,k)] \) since it contains the innovations \( \eta(t,y) \) that are posterior to those contributing to \( X(t,x) \). The gaussian integrals over the centered innovations \( \eta(t,y) \) thus vanish.

The integrand in (110) contains the exponential term
\[
\exp\left(-\int dv \int dy \left[ C^{-1}(v,y) \left[ \eta(v,y) \right]^2 - i k \left( G(t,x|v,y) \eta(v,y) \right) \right] \right).
\]
\( C^{-1}(v,y) \left[ \eta(v,y) \right]^2 - i k \left( G(t,x|v,y) \eta(v,y) \right) \) can be factorized as
\[
C^{-1}(v,y) \left( \eta(v,y) - i k \frac{G(t,x|v,y)}{2C^{-1}(v,y)} \right)^2 + \frac{k^2 \left[ G(t,x|v,y) \right]^2}{4C^{-1}(v,y)}.
\]

The Gaussian integrals can be carried over \( \hat{\eta}(v,y) = \eta(v,y) - i k \frac{G(t,x|v,y)}{2C^{-1}(v,y)} \) which leads to
\[
\hat{E}_t[d_t X(t,x,k)] = \int_0^t dv \int_{-\infty}^{+\infty} dy \frac{i k}{C^{-1}(v,y)} \frac{\partial \left[ G(t,x|v,y) \right]^2}{\partial t} e^{-\frac{k^2}{4} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ G(t,x|v,y) \right]^2}.
\]

We look for the condition that the Green function \( G(t,x|v,y) \) must satisfy for this expression to be identically zero for any \( t \) and \( x \).

The condition is simply that
\[
\frac{\partial G(t,x|v,y)}{\partial t} = 0,
\]
for all \( t \) and \( x \), i.e. \( G(t,x|v,y) \) must be independent of \( t \). We note that the dependence in \( t \) can still appear through dependence in time of possible upper and lower bounds in the integral over \( y \). We verify that this condition is observed for the parametric examples of stochastic string shocks given in the text.

The condition (113) implies that the Fourier transform (110) of \( E_t[d_t X(t,x)] \) is zero, and thus \( E_t[d_t X(t,x)] \) is zero itself. This is a Martingale condition.

### B.2 Markov condition for strings

An additional requirement (more stringent than the previous one) is that \( X(t,x) \) be Markovian, i.e. that its increment \( d_t X(t,x) \) be completely uncorrelated from past time innovations. The general form that the stochastic string can take in order to fulfill this condition is
\[
X(t,x) = X(0,x) + \int_0^t dv \ g(v) \int_{-\infty}^{j(x)} dy \ h(v,x,y) \eta(v,y),
\]
\[23\] If the Fourier transform is zero, the function is zero.

24
where \( g(v), j(x) \) and \( h(v, x, y) \) are arbitrary functions. This expression obeys the condition (113) and in addition time does not appear in the bounds of the integral over \( y \). As a consequence, from (114), we get
\[
d_t X(t, x) = dt \int_0^{j(x)} dy \ h(t, x, y) \eta(t, y) .
\] (115)
The fact that \( \eta(t, y) \) is immediately posterior to \( t \) ensures that the expectation of \( d_t X(t, x) \) conditional to the realization of \( X(t, x) \) is identically zero. This recovers a result of Kennedy (1997), obtained in a different setup.

B.3 Constraints on the covariance of the increments

The variance \( \text{Var}[d_t X(t, x)] \) of the increments must not depend on \( x \). This condition requires some additional condition on the functions \( j(x) \) and \( h(v, x, y) \) that we now derive. Notice that \( \text{Cov}[d_t X(t, x)d_t X(t, x')] \) does not depend on \( t \) if \( g(t) \) is constant.

From (115), we get
\[
\text{Var}[d_t X(t, x)] = dt \int_0^{j(x)} dy \ [h(t, x, y)]^2 ,
\] (116)
putting \( g(t) = 1 \). We have modified the lower bound of the integral in \( y \) so as not to have to worry about additional constraints in order to ensure convergence at \(-\infty\). The condition to impose is that (116) be independent of \( x \), which leads to the solutions given in section 4.3.

C Differentiability in Time-to-Maturity

The string processes studied in section 4.3 are continuous both in \( t \) and \( x \) but not differentiable either in \( t \) or \( x \). Of special interest is the non-differentiability in \( x \). Consider first the Brownian sheet process (66) and let us consider \( \frac{\partial W(t, x)}{\partial x} = \int_0^t \eta(v, x) \). This integral is a continuous sum of random variables. The sum of Gaussian random variables is itself Gaussian with variance equal to \( t \). Thus, \( \frac{\partial W(t, x)}{\partial x} \) is not a function but should rather be interpreted as a noise process, discontinuous almost everywhere with respect to \( x \). The derivative with respect to \( x \) does not exist strictly speaking, but only in the sense that its integral is well defined. This non-differentiability holds true for a large class of noise processes, not necessarily Gaussian.

A similar reasoning holds for all the string processes defined in (48).

All the stochastic partial differential equations that we discuss in this paper are linear. For such linear equations, a particularly convenient mathematical tool is the Fourier or Laplace transform.

Let \( Z(t, x) \) be one of the stochastic strings. Its Fourier transform with respect to \( x \) is
\[
\hat{Z}(t, k) \equiv \int_{-\infty}^{+\infty} dx \ e^{ikx} \ Z(t, x) .
\] (117)
The inverse Fourier transform retrieving \( Z(t, x) \) from \( \hat{Z}(t, k) \) is
\[
Z(t, x) = \int_{-\infty}^{+\infty} dk \ e^{-ikx} \hat{Z}(t, k) .
\] (118)
We have the following useful property:
\[
\int_{-\infty}^{+\infty} dx \ e^{ikx} \frac{\partial^p Z(t, x)}{\partial x^p} = (-ik)^p \hat{Z}(t, k) ,
\]
for an arbitrary positive integer \( p \). If \( \eta(t, x) \) is a Gaussian white noise, we have
\[
\text{Cov} [\hat{\eta}(t, k), \hat{\eta}(t', k')] = \delta(t - t') \delta(k - k') .
\]

Consider first the Brownian sheet \((65)\). Taking the Fourier transform of \((65)\) leads to
\[
\frac{\partial \hat{W}(t, k)}{\partial t} = i \frac{k}{k} \hat{\eta}(t, k) ,
\]
whose solution is simply
\[
\hat{W}(t, k) = \hat{W}(0, k) + i k \int_0^t d\tau \ \hat{\eta}(\tau, k) .
\]
We thus get
\[
\text{Cov} [\hat{W}(t, k)\hat{W}(t', k')] = -\frac{\delta(k - k')}{k^2} (t \wedge t') .
\]
This expression allows us to calculate any desired quantity. For instance, we find
\[
\text{Cov} \left[ \frac{\partial W(t, x)}{\partial x}, \frac{\partial W(t', x')}{\partial x'} \right] = (t \wedge t') \delta(x - x') ,
\]
which shows that \( E \left[ \left| \frac{\partial W(t, x)}{\partial x} \right|^2 \right] \) is infinite, the signature that \( W(t, x) \) is not differentiable in \( x \).

Consider now the O-U process \((73)\). Taking the Fourier transform of the SPDE defining the O-U process, leads to
\[
(-ik + \kappa) \frac{\partial \hat{Z}(t, k)}{\partial t} = \sqrt{2\kappa} \hat{\eta}(t, k) .
\]
Its solution is
\[
\hat{Z}(t, k) = \hat{Z}(0, k) + \frac{\sqrt{2\kappa}}{-ik + \kappa} \int_0^t d\tau \ \hat{\eta}(\tau, k) .
\]
Following the same steps as before, we get
\[
\text{Cov} \left[ \frac{\partial Z(t, x)}{\partial x}, \frac{\partial Z(t', x')}{\partial x'} \right] = (t \wedge t') \int_{-\infty}^{+\infty} dk \ e^{ik(x - x')} \frac{2\kappa}{k^2 + \kappa^2} .
\]
The integral still diverges for \(|k| \to \infty\), again expressing the non-differentiability of \( Z(t, x) \).

In contrast, the integrated process \((82)\) which obeys the SPDE \((83)\) has its Fourier transform \( \hat{Y}(t, k) \) satisfies
\[
(k^2 + ik\kappa) \frac{\partial \hat{Y}(t, k)}{\partial t} = -\sqrt{2\kappa} \hat{\eta}(t, k) .
\]
24 Derivatives are taken in the sense of distributions and we use the usual definition of the Fourier transform of distributions.
The solution is
\[
\hat{Y}(t, k) = \hat{Y}(0, k) \cdot \frac{\sqrt{2\kappa}}{k^2 + ik\kappa} \int_0^t d\tau \ \hat{\eta}(\tau, k),
\]
leading to
\[
\text{Cov} \left[ \hat{Y}(t, k), \hat{Y}(t', k') \right] = (t \wedge t') \cdot \frac{2\kappa \delta(k - k')}{k^4 + k^2 \kappa^2}.
\]
We thus see that
\[
\text{Cov} \left[ \frac{\partial Y(t, x)}{\partial x}, \frac{\partial Y(t', x')}{\partial x'} \right] = 2\kappa \cdot (t \wedge t') \int_{-\infty}^{+\infty} dk \ \frac{e^{ik(x-x')}}{k^2 + \kappa^2} = \pi \cdot (t \wedge t') \cdot e^{-\kappa|x-x'|},
\]
which is finite, signaling as expected the differentiability of \(Y(t, x)\) with respect to \(x\).
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FIGURE CAPTIONS

Figure 1: Sample increment to the O-U sheet, with correlation function \( (77) \) for different values of the parameter \( \kappa \). The time increment is taken to be 1, measured in the same units of time-to-maturity.

Figure 2: Sample increments to the integrated O-U sheet, with correlation function \( (87) \) for different values of the parameter \( \kappa \). The time increment is taken to be 1, measured in the same units of time-to-maturity.

Figure 3: Correlations between forward rates with times-to-maturity separated by different intervals, for the string shock with correlation function \( (90) \). The parameter \( \kappa \) is taken to be 1.
Term Structure of Correlations

c(x,x+y)

x

y=1
y=2
y=5
y=10
Integrated O–U Sheet

- Dotted line: $k=0.001$
- Dashed line: $k=0.01$
- Solid line: $k=0.1$

$dV(.,x)$

$x$ range: 0 to 500

$y$ range: $-2.5$ to $2.0$
