ASYMPTOTIC STABILITY OF $N$-SOLITONS IN THE CUBIC NLS EQUATION

AARON SAALMANN

ABSTRACT. In this article we consider the Cauchy problem for the cubic focusing nonlinear Schrödinger (NLS) equation on the line with initial datum close to a particular $N$-soliton. Using inverse scattering and the $\mathcal{G}$ method we establish the decay of the $L^\infty(\mathbb{R})$ norm of the residual term in time.

1. INTRODUCTION

We study the cubic focusing nonlinear Schrödinger (NLS) equation

$$\tag{1.1} iu_t + u_{xx} + 2|u|^2 u = 0,$$

on $\mathbb{R}$ where $u(x,t): \mathbb{R} \times \mathbb{R} \to \mathbb{C}$. The initial value problem for (1.1) is globally well posed in $L^2(\mathbb{R})$ due to the results of Tsutsumi [Tsu87]. The linear Schrödinger equation $iq_t + q_{xx} = 0$ is dispersive. Here, dispersion means that any solution decomposes into a solitary wave and a dispersive part as $t \to \infty$. Instead of dispersion, in the NLS equation we have that any solution decomposes into a solitary wave and a dispersive part as $t \to \infty$. Additionally we have

$$\tag{1.2} \epsilon := \|u_0(\cdot) - u^{(sol)}(\cdot,t = 0)\|_{L^{2,s}(\mathbb{R})} < \epsilon_0,$$

the solution of the initial value problem $u(\cdot,0) = u_0$ for (1.1) satisfies

$$\tag{1.3} \left\| u(\cdot,t) - u^{(sol)}_\pm(\cdot,t) \right\|_{L^\infty(\mathbb{R})} < C\epsilon|t|^{-\frac{1}{2}}$$

for all $\pm t \geq T$. Additionally we have

$$\tag{1.4} |z_j - z'_j| + |c_j - c'_j| < C\epsilon$$

for all $1 \leq j \leq N$.

In the proof of Theorem 1.1 we will compute the parameters of $u^{(sol)}_\pm$ explicitly and it will turn out that the $z'_j$ are given by the poles of $u_0$. The coupling constants $c'_j$ can also be derived from the scattering data of $u_0$ via $c'_j = c_j^\pm (\Lambda_j^\pm)^2$, where

$$\tag{1.5} \Lambda_j^\pm := \exp \left( \pm \frac{1}{2\pi i} \int_{\pm \infty}^{\text{Re}(z'_j)} \log(1 + |r(\xi)|^2) \frac{d\xi}{\xi - z'_j} \right).$$

In [CP14a] Contreras and Pelinovsky establish the orbital stability of $N$-solitons in the $L^2(\mathbb{R})$ space under the assumption (1.2). As mentioned by these authors they believed that also the (stronger) result (1.3) holds. For the proof we consider the Riemann Hilbert problem associated to the NLS.
equation and its solution $m$. Then motivated by the paper of Cuccagna and Jenkins \cite{CJ14} we define modifications $m \rightarrow m^{(1)} \rightarrow m^{(2)} \rightarrow m^{(3)} \rightarrow m^{(4)} \rightarrow m^{(5)} \rightarrow m^{(6)} \rightarrow m^{(7)}$ such that in the end $m^{(7)}$ is either trivial or corresponds to the 1-soliton or to a breather solution. These modifications contain the Parabolic Cylinder RHP and the $\partial$-method but not the dressing transformation like in \cite{CP14a}.

The paper is organized as follows: Section 2 gives some information about the direct in inverse scattering transform. Sections 3 - 6 are devoted to the chain of manipulations $m \rightarrow \ldots \rightarrow m^{(7)}$. Finally in Section 7 the results will be collected in order to prove Theorem 1.1.

Acknowledgments. I wish to thank Prof. Scipio Cuccagna and Prof. Markus Kunze for useful discussions.

2. The Inverse Scattering Transform and Soliton Solutions

A key ingredient for many results on stability of solitary waves comes from the methods of inverse scattering. The following theorem summarizes the theory:

Theorem 2.1. Let $s \in (1/2, 1]$. There exist open sets $\mathcal{G}_N \subset L^1(\mathbb{R})$ ($N \in \mathbb{N} \cup \{0\}$) and transformations

$$\mathcal{S}_N : \quad L^2^s(\mathbb{R}) \cap \mathcal{G}_N \to H^s(\mathbb{R}) \times \mathbb{C}_+^N \times \mathbb{C}_-^N$$

$$u_0 \mapsto (r(z); z_1, \ldots, z_N; c_1, \ldots, c_N)$$

such that:

(i) $\mathcal{G} := \bigcup_{N \in \mathbb{N} \cup \{0\}} \mathcal{G}_N$ is dense in $L^1$;
(ii) The maps $\mathcal{S}_N$ are locally Lipschitz and one-to-one;
(iii) The solution of \eqref{eq:main} with $u(x, 0) = u_0(x)$ and $u_0 \in H^1(\mathbb{R}) \cap L^2^s(\mathbb{R}) \cap \mathcal{G}_N$ can be obtained by the following three steps:

1. Step: Calculate the scattering data associated with $u_0$, i.e. $(r(z); z_1, \ldots, z_N; c_1, \ldots, c_N) := \mathcal{S}_N(u_0)$.
2. Step: Solve the following Riemann Hilbert problem:
### RHP[NLS]:

Find for each \((x,t) \in \mathbb{R} \times \mathbb{R}\) a \(2 \times 2\)-matrix valued function \(\mathbb{C} \ni z \mapsto m(z;x,t)\) which satisfies

(i) \(m(z;x,t)\) is meromorphic in \(\mathbb{C} \setminus \mathbb{R}\) (with respect to the parameter \(z\)).

(ii) \(m(z;x,t) = 1 + \mathcal{O}\left(\frac{1}{z}\right)\) as \(|z| \to \infty\).

(iii) The non-tangential boundary values \(m_\pm(z;x,t)\) exist for \(z \in \mathbb{R}\) and satisfy the jump relation \(m_+ = m_- V(r)\), where

\[
V(z;x,t) := \begin{pmatrix}
1 + |r(z)|^2 & e^{\phi(z)} \bar{\sigma}(z) \\
e^{\phi(z)} r(z) & 1
\end{pmatrix}
\]

with

\[
\phi(z) := 2iz + 4iz^2 t.
\]

(iv) \(m\) has simple poles at \(z_1, \ldots, z_N, \bar{z}_1, \ldots, \bar{z}_N\) with

\[
\text{Res} m(z;x,t) = \lim_{z \to z_k} m(z;x,t) \begin{pmatrix} 0 & 0 \\ c_k e^{\phi_k} & 0 \end{pmatrix},
\]

\[
\text{Res} m(z;x,t) = \lim_{z \to \bar{z}_k} m(z;x,t) \begin{pmatrix} 0 & -\bar{c}_k e^{\bar{\phi}_k} \\ 0 & 0 \end{pmatrix}.
\]

Here we set

\[
\phi_k := \phi(z_k) \quad (k = 1, \ldots, N).
\]

#### 3. Step:

Calculate the required solution via

\[
u(x,t) := 2i \lim_{z \to \infty} [m(z;x,t)]_{12}.
\]

Here \([\cdot]_{12}\) denotes the 1-2-component of the matrix in the brackets.

The fact that RHP[NLS] is uniquely solvable is pointed out by Deift and Park in [DP11]. For the convenience of the reader we show roughly how the scattering maps \(\mathcal{S}_N\) are defined: Given a function \(u(x)\) we set

\[
P(z;x) := \begin{pmatrix}
-iz & u(x) \\ \bar{\sigma}(x) & iz
\end{pmatrix}
\]

and consider the ODE

\[(2.6) \quad v_x(z;x) = P(z;x) v(z;x).
\]

We define \(\psi_j^{(\pm)} (j = 1, 2)\) to be the unique \(C^2\)-valued solutions of (2.6) with the boundary conditions

\[
\lim_{x \to \pm \infty} \psi_j^{(\pm)}(z;x) e^{\pm i x z} = e_j, \quad j = 1, 2,
\]

where \(e_1 = (1, 0)^T\) and \(e_2 = (0, 1)^T\). In general if \(u(\cdot) \in L^1(\mathbb{R})\), the functions \(\psi_1^{(-)}\) and \(\psi_2^{(+)}\) exist for \(\Im z \geq 0\) whereas \(\psi_1^{(+)}\) and \(\psi_2^{(-)}\) exist for \(\Im z \leq 0\) (see [APT04]). In both cases the dependence on \(z\) is analytic. Due to \(\text{tr} P = 0\), expressions such as \(\det [\psi_1^{(-)} | \psi_2^{(+)})\) or \(\det [\psi_1^{(+)} | \psi_1^{(-)}]\) do not depend on \(x\). We set

\[
a(z) := \det [\psi_1^{(-)}(z;x) | \psi_2^{(+)}, \\
b(z) := \det [\psi_1^{(+)}(z;x) | \psi_1^{(-)}(z;x)].
\]
such that $a$ is defined for $z \in \mathbb{C}_+$ and $b$ is defined for $z \in \mathbb{R}$. Additionally the map $z \mapsto a(z)$ is analytic in the upper plane $\mathbb{C}_+$. The sets $\mathcal{G}_N$ stated in Theorem 2.1 are now defined by the number of zeros of $a$:

$$
\mathcal{G}_N := \{ u \in L^1(\mathbb{R}) | a \text{ admits exactly } N \text{ simple zeros } z_1, \ldots, z_N \in \mathbb{C}_+ \}.
$$

In [BC84] Beals and Coifman show that the $\mathcal{G}_N$ are indeed open. Furthermore they prove statement (i) of Theorem 2.1. Now we amount to the definition of the scattering data $(r(z); z_1, \ldots, z_N; c_1, \ldots, c_N)$:

**Reflection coefficient:** The so-called reflection coefficient $r$ is given by

$$
r(z) := \frac{b(z)}{a(z)} \quad z \in \mathbb{R}.
$$

As it is shown in [CP14b] by Cuccagna, we have $r \in H^s(\mathbb{R})$ in the case of $u \in L^{2,s}(\mathbb{R})$. Note the analogy to the Fourier transform. See also [Zho98] for more general results.

**Poles:** The $z_k$ are defined to be the simple zeros of $a$. Hence, we have $a(z_k) = 0$ but $a'(z_k) \neq 0$ (the $'$ indicates the derivative with respect to the complex parameter $z$). We will refer to them as poles and we will denote the set $\{z_1, \ldots, z_N\}$ by $\mathcal{Z}_+$. Furthermore we set $\mathcal{Z}_- := \{\bar{z}_1, \ldots, \bar{z}_N\}$ and $\mathcal{Z} := \mathcal{Z}_+ \cup \mathcal{Z}_-$.

**Norming constants:** The so-called norming constants $c_1, \ldots, c_N$ are given by $c_k := \gamma_k / a'(z_k)$, where $\gamma_k$ are defined by the equations $
abla_1^-(z_k;x) = \gamma_k \nabla_2^+(z_k;x)$. Due to

$$
\det[\nabla_1^-(z_k;x)|\nabla_2^+(z_k;x)] = a(z_k) = 0
$$

the two vectors $\nabla_1^-(z_k;x)$ and $\nabla_2^+(z_k;x)$ are indeed linearly dependent, which implies that the numbers $\gamma_k$ exist. They do not depend on $x$ which is verified by differentiation.

Now we turn to the explanation of the second step, stated in Theorem 2.1 (iii). For $u \in \mathcal{G}_N$ it is an elementary calculation (see [APT04]) to show that

$$
m(z;x) := \begin{cases} 
\begin{bmatrix} 
\nabla_1^-(z;x)e^{izx} \\
\nabla_2^+(z;x)e^{-izx}
\end{bmatrix} & \frac{1}{a(z)} \begin{bmatrix} 
\nabla_2^+(z;x)e^{-izx} \\
\nabla_2^+(z;x)e^{-izx}
\end{bmatrix}, & \text{if } z \in \mathbb{C}_+,
\end{cases}
$$

solves the following Riemann Hilbert problem:
Thus, we have the miraculous fact is the following: if \( u(x,t) \) is meromorphic in \( x \) on \( \mathbb{C} \setminus \mathbb{R} \),

(i) \( m(z;x) = 1 + \mathcal{O}(\frac{1}{z}) \) as \( |z| \to \infty \).

(ii) The non-tangential boundary values \( m_{\pm}(z;x) \) exist for \( z \in \mathbb{R} \) and satisfy the jump relation

\[
m_+ = m_- V
\]

where

\[
V(z;x) = \begin{pmatrix} 1 + |r(z)|^2 & e^{-2izr(z)} \\ e^{2izr(z)} & 1 \end{pmatrix}.
\]

(iii) The relation \( m_{\pm}(z;x) \) solve the scattering data.

(iv) \( m(z;x) \) is meromorphic in \( z \) with

\[
\text{Res} m(z) = \lim_{z \to \pm a_k} m(z) = \begin{pmatrix} 0 & 0 \\ c_k e^{2iz a_k} & 0 \end{pmatrix},
\]

\[
\text{Res} m(z) = \lim_{z \to \pm a_k} m(z) = \begin{pmatrix} 0 & -e^{-2iz a_k} \\ 0 & 0 \end{pmatrix}.
\]

From the differential equation (2.6) one can obtain the asymptotic behavior of the functions \( \psi_j^\pm(z;x) \) as \( z \to \infty \). For instance we have (see page 25 in [APT04])

\[
\psi_j^\pm(\mp 1) e^{-ix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2iz} \left( \int_{x}^{\pm \infty} \left| u(x) \right|^2 \right) dy + \mathcal{O}\left( \frac{1}{z^2} \right),
\]

which is equivalent to the following important formula:

\[
u(x) = 2i \lim_{z \to \infty} z[m(z;x)]_{12}.
\]

Here \( m(z;x) \) is the matrix defined from the functions \( \psi_j^\pm \) as above. So far we have described the forward scattering and the inverse scattering, since we can reconstruct the function \( u \) from its scattering data.

Now we are going to take into account the time dependence into (2.8) and (2.9) we end up exactly with (2.11) and (2.13). Summarized the

\[
\text{RHP:}
\]

Find a 2 \times 2-matrix valued function \( z \mapsto m(z;x) \) which satisfies

(i) \( m(z;x) \) is meromorphic in \( z \) on \( \mathbb{C} \setminus \mathbb{R} \).

(ii) \( m(z;x) = 1 + \mathcal{O}(\frac{1}{z}) \) as \( |z| \to \infty \).

(iii) The non-tangential boundary values \( m_{\pm}(z;x) \) exist for \( z \in \mathbb{R} \) and satisfy the jump relation

\[
m_+ = m_- V
\]

where

\[
V(z;x) = \begin{pmatrix} 1 + |r(z)|^2 & e^{-2izr(z)} \\ e^{2izr(z)} & 1 \end{pmatrix}.
\]

(iv) \( m(z;x) \) is meromorphic in \( z \) with

\[
\text{Res} m(z) = \lim_{z \to \pm a_k} m(z) = \begin{pmatrix} 0 & 0 \\ c_k e^{2iz a_k} & 0 \end{pmatrix},
\]

\[
\text{Res} m(z) = \lim_{z \to \pm a_k} m(z) = \begin{pmatrix} 0 & -e^{-2iz a_k} \\ 0 & 0 \end{pmatrix}.
\]
method of (inverse) scattering works as follows:

\begin{equation}
(2.11) \quad u_0 \in L^2(\mathbb{R}) \cap \mathcal{F}_N \quad \xrightarrow{\mathcal{F}_N} \quad (r(z); z_1, \ldots, z_N; c_1, \ldots, c_N)
\end{equation}

We now give a definition of N-solitons in terms of the scattering data:

**Definition 2.2.** A solution \( u \) of (1.1) is called an \( N \)-soliton or multi-soliton if the initial datum \( u_0 \) belongs to \( \mathcal{F}_N \) and the corresponding reflection coefficient vanishes \( (r(z) \equiv 0) \).

If \( u_0 \in \mathcal{F}_0 \) and \( r \equiv 0 \) the Riemann Hilbert problem \( \text{RHP[NLS]} \) (see Theorem 2.1) then reduces to: (i) \( m(z; x, t) \) is entire (with respect to \( z \)); (ii) \( m(z; x, t) = 1 + \mathcal{O}(z^{-1}) \) as \( |z| \to \infty \). By Liouville’s Theorem it follows that \( m(z; x, t) \equiv 1 \) and applying (2.5) we obtain \( u(x, t) \equiv 0 \).

In the case of \( N = 1 \) the ansatz

\[ m(z; x, t) = 1 + \frac{A(x, t)}{z - z_1} + \frac{\tilde{A}(x, t)}{z - \overline{z}_1}, \]

reduces \( \text{RHP[NLS]} \) to an algebraic system, which is solved by

\begin{equation}
(2.12) \quad A(x, t) = \begin{pmatrix}
\frac{2i|c_1|^2 e^{2\mathfrak{R}(\phi_1(x,t))} \mathfrak{I}(z_1)}{c_1 e^{2\mathfrak{R}(\phi_1(x,t))} + 4\mathfrak{I}(z_1)} & 0 \\
\mathfrak{I}(z_1) e^{i[\mathfrak{R}(c_1) + 4\mathfrak{I}(z_1)]} \sech \left[ -2\mathfrak{R}(\phi_1(x,t)) - \frac{|c_1|}{2\mathfrak{I}(z_1)} \mathfrak{R}(\phi_1(x,t)) - \ln \left( \frac{|c_1|}{2\mathfrak{I}(z_1)} \right) \right] & 0
\end{pmatrix},
\end{equation}

\begin{equation}
(2.13) \quad \text{sol}_{z_1, c_1}^1(x, t) := -2i \mathfrak{I}(z_1) e^{-i[\mathfrak{R}(c_1) + 4\mathfrak{I}(z_1)] t} \times \sech \left[ 2\mathfrak{I}(z_1)(x + 4\mathfrak{R}(z_1) t) - \ln \left( \frac{|c_1|}{2\mathfrak{I}(z_1)} \right) \right].
\end{equation}

It describes a single wave packet which is centered at

\begin{equation}
(2.14) \quad x_0 = \left( 2\mathfrak{I}(z_1) \right)^{-1} \ln \left( \frac{|c_1|}{2\mathfrak{I}(z_1)} \right) - 4\mathfrak{R}(z_1) t.
\end{equation}

So we see, that the wave is propagating with the velocity \( v = -4\mathfrak{R}(z_1) \). In doing so, its envelope remains undistorted. Thus \( \text{sol}_{z_1, c_1}^1(x, t) \) is indeed a soliton in the sense of the definition of Drazin and Johnson (see Section 1.2 in [DJ89]). Multisolitons are not solitons in the sense of D. and J. but it can be shown that for \( \mathfrak{R}(z_j) \neq \mathfrak{R}(z_k) \) \( (j \neq k) \) a \( N \)-soliton splits into \( N \) individual 1-solitons (see [ZS72]).

### 3. Separating the Poles

The quintessence of Lemmata [3.1] and [3.2] of this section we will be the following observation: the set of those poles who will contribute to the solution \( u(x, t) \) depends on the ratio \( -x/(4t) \).

For the parameter

\begin{equation}
(3.1) \quad \xi := \frac{-x}{4t}
\end{equation}
we find
\[ \Re \phi(z; x, t) = 8 \Im (z) t (\xi - \Re (z)). \]
and we conclude for \( t > 0 \):
\[ \begin{align*}
\Re \phi(z; x, t) > 0, & \quad \text{if } \{ \Im (z) > 0 \text{ and } \Re (z) < \xi, \\
\Re \phi(z; x, t) < 0, & \quad \text{if } \{ \Im (z) < 0 \text{ and } \Re (z) > \xi, \\
\end{align*} \]
For the \( \phi_k \) defined in (2.4) we have
\[ (3.2) \lim_{t \to \infty} |e^{\phi_k}| = \begin{cases} 
0, & \text{if } \Re z_k > \xi, \\
\infty, & \text{if } \Re z_k < \xi,
\end{cases} \]
and
\[ |e^{\phi_k}| = 1 \text{ if } \Re z_k = \xi. \]
Hence for a fixed \( \xi \) the poles \( z_1, \ldots, z_N \) are split in two classes. We set:
\[ (3.3) \begin{align*}
\nabla(\xi) & := \{ k \in \{1, \ldots, N\} \mid \Re z_k < \xi \}, \\
\Delta(\xi) & := \{ k \in \{1, \ldots, N\} \mid \Re z_k \geq \xi \}.
\end{align*} \]
Since we do not exclude the case where two poles have the same real part, we have to label the poles in a new matter. We group the poles with respect to theirs real parts:
\[ (3.4) \begin{align*}
\nabla(\xi) & := \{ z_1, \ldots, z_N \} = \left\{ z_1^{(1)}, \ldots, z_{m_1}^{(1)}, z_1^{(2)}, \ldots, z_{m_2}^{(2)}, \ldots, z_1^{(K)}, \ldots, z_{m_K}^{(K)} \right\}, \\
m_l \geq 1, & \quad \sum_{l=1}^K m_l = N, \\
\Re z_j^{(l)} = \Re z_h^{(p)} & \iff l = p.
\end{align*} \]
For \( t \) sufficiently large the set
\[ (3.5) \Box(\xi) := \{ z \in \mathcal{Z} \mid |\Re (z) - \xi| \leq 1/\sqrt{t} \} \]
depends only on \( \xi \) and is either empty or equals exactly \( \{ z_1^{(l)}, \ldots, z_{m_1}^{(l)}, z_1^{(l)}, \ldots, z_{m_l}^{(l)} \} \) for one certain \( l \). Now we define the contour
\[ (3.6) \Sigma^{(1)}(x, t) := \bigcup_{z \in \mathcal{Z} / \Box(\xi)} \partial B_{1/\sqrt{t}}(z), \]
Next we set
\[ (3.7) T(z; x, t) := \prod_{k \in \nabla(\xi)} \frac{z - z_k}{z - z_k}, \]
and
\[ (3.8) D(z; x, t) := T(z; x, t)^{\sigma_3} := \begin{pmatrix} T(z; x, t) & 0 \\ 0 & T(z; x, t)^{-1} \end{pmatrix}, \]
such that we are now in a position to formulate the first modification of $\text{RHP}[\text{NLS}]$. From now on we will often drop the dependence on $x$ and $t$. For $m : \mathbb{C} \to \mathbb{C}^{2 \times 2}$ we set

$$ m^{(1)}(z) := \begin{cases} 
    m(z) \begin{pmatrix} 1 & -\frac{z-z_k}{cz_k e^{\phi_k}} \\ 0 & 1 \end{pmatrix} D(z), & \text{if } z \in B_{1/\sqrt{\tau}}(z_k), k \in \bigtriangledown (\xi), z_k \not\in \Box (\xi), \\
    m(z) \begin{pmatrix} 1 & -\frac{z-z_k}{cz_k e^{\phi_k}} \\ 0 & 1 \end{pmatrix} D(z), & \text{if } z \in B_{1/\sqrt{\tau}}(z_k), k \in \bigtriangleup (\xi), z_k \not\in \Box (\xi), \\
    m(z) \begin{pmatrix} 1 & 0 \\ \frac{z-z_k}{cz_k e^{\phi_k}} & 1 \end{pmatrix} D(z), & \text{if } z \in B_{1/\sqrt{\tau}}(z_k), k \in \bigtriangledown (\xi), \overline{z}_k \not\in \Box (\xi), \\
    m(z) \begin{pmatrix} 1 & \frac{z-z_k}{cz_k e^{\phi_k}} \\ 0 & 1 \end{pmatrix} D(z), & \text{if } z \in B_{1/\sqrt{\tau}}(z_k), k \in \bigtriangleup (\xi), \overline{z}_k \not\in \Box (\xi), \\
    m(z) D(z), & \text{else.} 
\end{cases} $$

**Lemma 3.1.** If $m(z)$ solves $\text{RHP}[\text{NLS}]$, then $m^{(1)}(z)$ defined in (3.9) is a solution to the following RHP:
\begin{proof}

(i) \( m^{(1)}(z) \) is meromorphic in \( \mathbb{C} \setminus (\Sigma^{(1)} \cup \mathbb{R}) \).

(ii) \( m^{(1)}(z) = 1 + \mathcal{O}\left(\frac{1}{z}\right) \) as \( |z| \to \infty \).

(iii) If \( \Box(\xi) = \emptyset \), \( m^{(1)} \) has no poles (i.e. \( m^{(1)} \) is analytic on \( \mathbb{C} \setminus (\Sigma^{(1)} \cup \mathbb{R}) \)). If \( \Box(\xi) \) consists of certain \( z_k \) and \( \overline{z}_k \) such that \( k \in \bigtriangleup(\xi) \), \( m^{(1)} \) has simple poles at these \( z_k \) and \( \overline{z}_k \) with:

\[
\text{Res}_{z=z_k}^{m^{(1)}(z)} = \lim_{z \to z_k} m^{(1)}(z) \left( \begin{array}{cc}
0 & 1 \\
\frac{1}{c_k e^{\phi_k(T(z_k))^2}} & 0 \\
0 & 0
\end{array} \right),
\]

(3.10)

\[
\text{Res}_{z=\overline{z}_k}^{m^{(1)}(z)} = \lim_{z \to \overline{z}_k} m^{(1)}(z) \left( \begin{array}{cc}
0 & 0 \\
\frac{-1}{\overline{\tau}_k e^{\phi_k(T(z_k))^2}} & 0 \\
0 & 0
\end{array} \right).
\]

If \( \Box(\xi) \) consists of certain \( z_k \) and \( \overline{z}_k \) such that \( k \in \bigtriangleup(\xi) \), \( m^{(1)} \) has simple poles at these \( z_k \) and \( \overline{z}_k \) with:

\[
\text{Res}_{z=z_k}^{m^{(1)}(z)} = \lim_{z \to z_k} m^{(1)}(z) \left( \begin{array}{cc}
0 & 0 \\
c_k e^{\phi_k(T(z_k))^2} & 0 \\
0 & 0
\end{array} \right),
\]

(3.11)

\[
\text{Res}_{z=\overline{z}_k}^{m^{(1)}(z)} = \lim_{z \to \overline{z}_k} m^{(1)}(z) \left( \begin{array}{cc}
0 & 0 \\
0 & -\overline{\tau}_k e^{\phi_k(T(z_k))^2}
\end{array} \right).
\]

(iv) The non-tangential boundary values \( m^{(1)}_{\pm}(z) \) exist for \( z \in \Sigma^{(1)} \cup \mathbb{R} \) and satisfy the jump relation \( m^{(1)}_{\pm} = m^{(1)} V^{(1)} \), where

\[
V^{(1)}(z) = \begin{cases}
\begin{array}{cc}
1 & \frac{z-z_k}{c_k e^{\phi_k(T(z))^2}} \\
0 & 1
\end{array}, & \text{if } z \in \partial B_{1/\sqrt{T}}(z_k), k \in \bigtriangleup(\xi), \overline{z}_k \notin \Box(\xi), \\
\frac{1}{c_k e^{\phi_k(T(z))^2}} & 0 \\
0 & 1
\end{cases}, \\
\begin{array}{cc}
1 & \frac{1}{\tau_k e^{\phi_k(T(z))^2}} \\
0 & 1
\end{array}, & \text{if } z \in \partial B_{1/\sqrt{T}}(\overline{z}_k), k \in \bigtriangleup(\xi), z_k \notin \Box(\xi), \\
\frac{1}{\tau_k e^{\phi_k(T(\overline{z}_k))^2}} & 0 \\
0 & 1
\end{cases}, \\
D^{-1}(z)V(z)D(z), & \text{if } z \in \mathbb{R}.
\]

(3.12)

\end{proof}

\( \text{Proof.} \) (i) is trivial, (ii) is a consequence of

\[
D(z) = 1 + \mathcal{O}\left(\frac{1}{z}\right) \text{ as } |z| \to \infty.
\]

(iv) is also elementary. It remains to show, that (iii) holds. We have therefore to show that (3.10) and (3.11) are correct and moreover we have to show that the poles at \( z_k \) and \( \overline{z}_k \) are indeed removed in the case of \( z_k, \overline{z}_k \notin \Box(\xi) \). Firstly we consider \( m^{(1)} \) close to \( z_k \) in the case where \( z_k \notin \Box(\xi) \) and \( k \in \bigtriangleup(\xi) \): Let \( m \) be a solution of \textbf{RHP[NLS]}. Then we have

\[
m(z) = \frac{A_k}{z-z_k} + B_k + \mathcal{O}(|z-z_k|) \quad \text{as } z \to z_k.
\]
with suitable matrices $A_k = A_k(x,t)$ and $B_k = B_k(x,t)$. The residua conditions in \textbf{RHP[NLS]} then yield the following two relations:

\begin{align}
(3.13) \quad A_k \begin{pmatrix} 0 & 0 \\ c_k e^{\phi_k} & 0 \end{pmatrix} &= 0, \\
(3.14) \quad A_k = B_k \begin{pmatrix} 0 & 0 \\ c_k e^{\phi_k} & 0 \end{pmatrix}.
\end{align}

By definition, (3.13) and (3.14) we get

\[
m^{(1)}(z) = m(z) \begin{pmatrix} 1 & -\frac{z-z_k}{c_k e^{\phi_k}} \\ 0 & 1 \end{pmatrix} D(z) \\
= \left[ \frac{A_k}{z-z_k} + B_k + \mathcal{O}(|z-z_k|) \right] \left[ 1 + \begin{pmatrix} 0 & \frac{-1}{c_k e^{\phi_k}} \\ 0 & 0 \end{pmatrix} (z-z_k) \right] \\
+ \begin{pmatrix} 0 & 0 \\ \frac{T'(z_k)}{z-z_k} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{O}(|z-z_k|)
\]

and it follows that there is no pole at $z_k$. In the case of $k \in \triangle(\xi)$ we find:

\[
m(z) \begin{pmatrix} 1 & 0 \\ -\frac{z-z_k}{c_k e^{\phi_k}} & 1 \end{pmatrix} = \left[ \frac{A_k}{z-z_k} + B_k + \mathcal{O}(|z-z_k|) \right] \left[ 1 + \begin{pmatrix} 0 & \frac{-1}{c_k e^{\phi_k}} \\ 0 & 0 \end{pmatrix} \frac{z-z_k}{z-z_k} \right] \\
= \frac{A_k}{(z-z_k)^2} + \frac{B_k}{z-z_k} + \frac{A_k}{z-z_k} + \mathcal{O}(1)
\]

\[\{3.13\&3.14\} \quad \mathcal{O}(1)\]

Since $D(z)$ has no pole at $z_k$ ($k \in \triangle(\xi)$), it is clear that also $m^{(1)}(z) = \mathcal{O}(1)$ as $z \to z_k$. The calculations for $z_k \not\in \square(\xi)$ ($k \in (\nabla(\xi) \cup \triangle(\xi))$ are similar. Now we turn to establish the first line of (3.10): Let us assume $z_k \in \square(\xi)$ and $k \in \nabla(\xi)$. We use

\[m(z) = \frac{A_k}{z-z_k} + B_k + C_k (z-z_k) + \mathcal{O}(|z-z_k|^2)\]

and

\[D(z) = \begin{pmatrix} 0 & 0 \\ \frac{1}{T'(z_k)} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} T'(z_k) & 0 \\ 0 & 0 \end{pmatrix} (z-z_k) + \mathcal{O}(|z-z_k|^2)\]

to obtain for $z$ close to $z_k$

\[
m^{(1)}(z) \begin{pmatrix} 3.13 \\ B_k \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{T'(z_k)} + A_k \begin{pmatrix} T'(z_k) & 0 \\ 0 & 0 \end{pmatrix} + B_k \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + C_k \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{T'(z_k)} + \mathcal{O}(|z-z_k|)\]
\]
On the one hand, from this expansion we find

\[(3.15) \quad \text{Res} \left( m^{(1)}(z) \right)_{z=k} = B_k \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{T'(z_k)} \end{pmatrix} \]

and on the other hand

\[(3.16) \quad \lim_{z \to z_k} m^{(1)}(z) \begin{pmatrix} 0 & \frac{1}{c_k e^{\phi_k (T'(z_k))^2}} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_k & 0 \\ 0 & B_k \end{pmatrix} \begin{pmatrix} T'(z_k) & 0 \\ 0 & * \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & C_k(D(z_k)) \end{pmatrix} + C_k(D(z_k)) \begin{pmatrix} 0 & \frac{1}{T'(z_k)} \\ 0 & 0 \end{pmatrix} \]

\[(3.15) \quad \text{and} \quad (3.16) \quad \text{prove the first line of (3.10). The second line follows from analog calculations.} \]

Alternatively we can say that the first line of (3.10) implies the second since \(m^{(1)}\) obeys the symmetry

\[(3.17) \quad \overline{m^{(1)}(z)} = \overline{\sigma_2 m^{(1)}(z)} \overline{\sigma_2}, \]

which can be derived from the symmetries \(\overline{m(z)} = \sigma_2(\overline{z}) \sigma_2\) and \(\overline{D(z)} = \sigma_2(\overline{D(z)}) \sigma_2\). Now we prove (3.11). Let \(z_k \in \mathbb{D}(\xi)\) and \(k \in \triangle(\xi)\).

\[
\begin{align*}
\text{Res} \left( m^{(1)}(z) \right)_{z=z_k} &= \left[ \text{Res} \left( m(z) \right)_{z=z_k} \right] D(z_k) \\
&= \lim_{z \to z_k} \begin{pmatrix} m(z) & 0 \\ 0 & c_k e^{\phi_k} \end{pmatrix} D(z_k) \\
&= \lim_{z \to z_k} \begin{pmatrix} m(1)(z)D(z_k) & 0 \\ 0 & c_k e^{\phi_k} \end{pmatrix} D(z_k) \\
&= \lim_{z \to z_k} \begin{pmatrix} m(1)(z) & 0 \\ 0 & c_k e^{\phi_k (T'(z_k))^2} \end{pmatrix} D(z_k) \\
&= \lim_{z \to z_k} \begin{pmatrix} m(1)(z) & 0 \\ 0 & c_k e^{\phi_k (T'(z_k))^2} \end{pmatrix} D(z_k) \\
&= \lim_{z \to z_k} \begin{pmatrix} m(1)(z) & 0 \\ 0 & \overline{c_k e^{\phi_k (T'(z_k))^2}} \end{pmatrix} D(z_k) \\
&= \lim_{z \to z_k} \begin{pmatrix} m(1)(z) & 0 \\ 0 & \overline{c_k e^{\phi_k (T'(z_k))^2}} \end{pmatrix} D(z_k) \\
&= \lim_{z \to z_k} \begin{pmatrix} m(1)(z) & 0 \\ 0 & \overline{c_k e^{\phi_k (T'(z_k))^2}} \end{pmatrix} D(z_k) \\
\end{align*}
\]

The last step is possible, because of the symmetry \(\overline{T(z)} = \frac{1}{T(z)}\). \(\square\)
We have used the function \( T(z,x,t) \) to define the transformation \( m \mapsto m^{(1)} \). As a consequence the poles at \( z_k \) (and \( \overline{z}_k \), respectively) are removed and instead a jump on the correspondent disk boundaries appears. Next we are going to prove rigorously the fact that this jump \( V^{(1)} \) on \( \Sigma^{(1)} \) defined in (3.12) does not meaningfully contribute to the solution of \( \text{RHP}[1] \) as \( t \to \infty \). Therefore we consider again a Riemann Hilbert problem:

\[
\text{RHP}[2]: \quad \text{Find a } 2 \times 2\text{-matrix valued function } \mathbb{C} \ni z \mapsto m^{(2)}(z) \text{ which satisfies}
\]

(i) \( m^{(2)}(z) \) is meromorphic in \( \mathbb{C} \setminus \mathbb{R} \),

(ii) \( m^{(2)}(z) = 1 + \mathcal{O}\left( \frac{1}{z^2} \right) \) as \( |z| \to \infty \),

(iii) If \( \Box(\xi) = \emptyset, m^{(2)} \) has no poles (i.e. \( m^{(2)} \) is analytic in \( \mathbb{C} \setminus \mathbb{R} \)). If \( \Box(\xi) \) consists of certain \( z_k \) and \( \overline{z}_k \) such that \( k \in \triangle(\xi) \), \( m^{(2)} \) has simple poles at these \( z_k \) and \( \overline{z}_k \) with:

\[
\begin{align*}
\text{Res} m^{(2)}(z) &= \lim_{z \to z_k} m^{(2)}(z) \begin{pmatrix} 0 & \frac{1}{c_k e^{\varphi_k(T(z_k))^2}} \\ 0 & 0 \end{pmatrix}, \\
\text{Res} m^{(2)}(z) &= \lim_{z \to \overline{z}_k} m^{(2)}(z) \begin{pmatrix} 0 & -\frac{1}{c_k e^{\overline{\varphi}_k(T(z_k))^2}} \\ 0 & 0 \end{pmatrix}.
\end{align*}
\]

(3.18)

If \( \Box(\xi) \) consists of certain \( z_k \) and \( \overline{z}_k \) such that \( k \in \Delta(\xi) \), \( m^{(2)} \) has simple poles at these \( z_k \) and \( \overline{z}_k \) with:

\[
\begin{align*}
\text{Res} m^{(2)}(z) &= \lim_{z \to z_k} m^{(2)}(z) \begin{pmatrix} 0 & 0 \\ c_k e^{\varphi_k(T(z_k))^2} & 0 \end{pmatrix}, \\
\text{Res} m^{(2)}(z) &= \lim_{z \to \overline{z}_k} m^{(2)}(z) \begin{pmatrix} -\overline{c}_k e^{\overline{\varphi}_k(T(z_k))^2} & 0 \\ 0 & 0 \end{pmatrix}.
\end{align*}
\]

(3.19)

(iv) The non-tangential boundary values \( m^{(2)}_\pm(z) \) exist for \( z \in \mathbb{R} \) and satisfy the jump relation

\[
m^{(2)}_+ = m^{(2)}_- V(z),
\]

(3.20)

\[
V(z) = D^{-1}(z)V(z)D(z)
\]

\( \text{RHP}[2] \) can be viewed as \( \text{RHP}[1] \) with \( V^{(1)}|_{\Sigma^{(1)}} \equiv 1 \). Since \( \lim_{t \to \infty} V^{(1)}(z) = 1 \) for \( z \in \Sigma^{(1)} \), due to (3.22), it is not surprising that somehow the solution of \( \text{RHP}[1] \) is converging to that of \( \text{RHP}[2] \) as \( t \to \infty \). Indeed, we have:

**Lemma 3.2.** There is a matrix \( C_1(x,t) \) for which

\[
\|C_1\| \leq ce^{-8\sqrt{t}} \quad (t > 0)
\]

(with \( c > 0 \) independent of \( x \)) holds and such that

\[
m^{(1)}(z) = \left[ 1 + \frac{C_1}{z} + \mathcal{O}\left( \frac{1}{z^2} \right) \right] m^{(2)}(z)
\]

(3.21)

as \( |z| \to \infty \). As indicated by the notation, here \( m^{(1)} \) solves \( \text{RHP}[1] \) and \( m^{(2)} \) is a solution to \( \text{RHP}[2] \), respectively.

**Proof.** We claim, that in each of the two cases \( \Box(\xi) = \emptyset \) and \( \Box(\xi) \neq \emptyset \) the matrix valued function

\[
C(z) := m^{(1)}(z) \begin{pmatrix} m^{(2)}(z) \end{pmatrix}^{-1}
\]

is a solution to
RHP[C]
(i) $C$ is analytic in $\mathbb{C} \setminus \Sigma^{(1)}$.
(ii) $C(z) = 1 + O\left(\frac{1}{z}\right)$ as $|z| \to \infty$.
(iii) The non-tangential boundary values $C_{\pm}(z)$ exist for $z \in \Sigma^{(1)}$ and satisfy the jump relation $C_{+} = C_{-} V^{(C)}$, where

$$V^{(C)}(z) = m^{(2)}(z) V^{(1)}|_{\Sigma^{(1)}}(z) \left[ m^{(2)}(z) \right]^{-1}.$$ 

In order to prove (i), we have to show that

$$C(z) = O(1) \quad \text{as} \quad z \to z_k, \bar{z}_k$$

(if $z_k, \bar{z}_k \in \triangle(\xi)$). We begin with $k \in \bigtriangledown$ and consider $C(z)$ close to $z_k$. By $\det m^{(1)} = \det m^{(2)} \equiv 1$, (3.10) and (6.3) (see also (3.13) and (8.14)) we have:

$$m^{(1)}(z) = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \frac{1}{z - z_k} + \begin{pmatrix} \alpha/\eta_k & *_{12} \\ \beta/\eta_k & *_{22} \end{pmatrix} + O(|z - z_k|)$$

$$\left[ m^{(2)}(z) \right]^{-1} = \begin{pmatrix} \tilde{\beta} & -\tilde{\alpha} \\ 0 & 0 \end{pmatrix} \frac{1}{z - z_k} + \begin{pmatrix} *_{22} & *_{12} \\ -\tilde{\beta}/\eta_k & \tilde{\alpha}/\eta_k \end{pmatrix} + O(|z - z_k|)$$

with suitable numbers $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}$ and $\eta_k := \frac{1}{c_{\epsilon}e^{\eta_k(T(z_k))^2}}$. After multiplication we arrive at

$$C(z) = \begin{pmatrix} \alpha\tilde{\beta}/\eta_k & -\alpha\tilde{\alpha}/\eta_k \\ \beta\tilde{\beta}/\eta_k & -\tilde{\alpha}\beta/\eta_k \end{pmatrix} + O(1)$$

$$C(z) = O(1), \quad (z \to z_k).$$

The cases $z \to \bar{z}_k$ and $k \in \bigtriangleup$ are similar. (ii) and (iii) of RHP[err] are obvious.

Now we turn to the analysis of RHP[err]. First of all we state two properties of the jump matrix $V^{(C)}$:

$$\|V^{(C)} - 1\|_{L^\infty(\Sigma^{(1)})} \leq ce^{\sqrt{7}e^{-8\sqrt{7}}}$$

(3.22)

$$\|V^{(C)} - 1\|_{L^2(\Sigma^{(1)})} \leq ce^{1/4}e^{-8\sqrt{7}}$$

(3.23)

These two estimates follow directly from the definition of $V^{(1)}|_{\Sigma^{(1)}}$ and $\|m^{(2)}(z)\| \leq G$ for $z \in \Sigma^{(1)}$ with a bound $G$, which does not depend on $x$ and $t$. It is a fact (see Chapter 7 in [AF03]), that the solution of RHP[err] is given by

$$C(z) = 1 + \frac{1}{2\pi i} \int_{\Sigma^{(1)}} \frac{\mu(\xi)(V^{(C)}(\xi) - 1)}{\zeta - \bar{z}} d\xi,$$

where $\mu \in L^2(\Sigma^{(1)})$ is the unique solution of

$$1 - C_{\mu} \mu = 1$$

(3.24)
with $C_V : L^2(\Sigma^{(1)}) \to L^2(\Sigma^{(1)})$ defined by
\[
(C_V f)(x) := \lim_{z \to x \in \Theta} \frac{1}{2\pi i} \int_{\Sigma^{(1)}} \frac{f(\zeta)(V^C(\zeta) - 1)}{\zeta - z} d\zeta.
\]

By $z \in \Theta$ we indicate that the limit is to be taken non-tangentially from the minus (right) side of the (counter-clockwise) orientated contour $\Sigma^{(1)}$. In other words we set:
\[
\Theta := \bigcup_{z \in \mathbb{R} / \mathbb{N}(\xi)} B_{1/\sqrt{t}}(z), \quad \Theta := \mathbb{C} \setminus \Theta.
\]

We can write $C_V$ in terms of the Cauchy projection operator $C_{\Sigma^{(1)}}^- : L^2(\Sigma^{(1)}) \to L^2(\Sigma^{(1)})$ which is defined by
\[
(C_{\Sigma^{(1)}}^- g)(x) := \lim_{z \to x \in \Theta} \frac{1}{2\pi i} \int_{\Sigma^{(1)}} \frac{g(\zeta)}{\zeta - z} d\zeta
\]
and which has finite $L^2 \to L^2$ operator norm. Moreover the operator norm is independent of $x$ and $t$. We have $C_V f = C_{\Sigma^{(1)}}^- (f(V^C - 1))$ and thus for any $f \in L^2(\Sigma^{(1)})$
\[
\|C_V f\|_{L^2(\Sigma^{(1)})} \leq \text{const.} \|f(V^C - 1)\|_{L^2(\Sigma^{(1)})} \leq \text{const.} \|V^C - 1\|_{L^\infty(\Sigma^{(1)})} \|f\|_{L^2(\Sigma^{(1)})}.
\]

From (3.22) it follows that
\[
\|C_V\|_{L^2(\Sigma^{(1)}) \to L^2(\Sigma^{(1)})} \leq c \sqrt{t} e^{-8\sqrt{t}}
\]
with $c > 0$ independent of $x$. We conclude that for $t$ sufficiently large $1 - C_V$ is invertible and
\[
\|(1 - C_V)^{-1}\|_{L^2(\Sigma^{(1)}) \to L^2(\Sigma^{(1)})} \leq \tilde{c}
\]
with $\tilde{c} > 0$ independent of $x$ and $t$. This implies for large $t$ that $\mu$ defined by equation (3.25) exists and satisfies
\[(3.26) \quad \|\mu\|_{L^2(\Sigma^{(1)})} \leq ct^{-1/4},
\]
where we have to take into account $\|1\|_{L^2(\Sigma^{(1)})} = 4\pi wt^{-1/4}$ for some integer $0 \leq w \leq N$. Equation (3.24) yields for large $z \in \mathbb{C}$
\[
C(z) = 1 - \frac{1}{2\pi i} \int_{\Sigma^{(1)}} \frac{\mu(\zeta)(V^C(\zeta) - 1)}{1 - \frac{\zeta}{z}} d\zeta
\]
and thus we know how to choose the desired $C_1$ in (5.3):
\[
C_1 = -\frac{1}{2\pi i} \int_{\Sigma^{(1)}} \mu(\zeta)(V^C(\zeta) - 1) d\zeta.
\]

Making use of (3.23), (3.26) and the Hölder inequality we conclude $\|C_1\| \leq ce^{-8\sqrt{t}}$. \qed
4. The Parabolic Cylinder RHP

The goal of our next modification \( m^{(2)} \mapsto m^{(3)} \) is the removal of the discontinuity on \( \mathbb{R} \). We will use the same technique presented for example in \([CP14b]\), \([DM08]\) and \([CJ14]\). The first step is the decomposition of the jump condition \( V^{(2)} \) (see (3.20)). We write

\[
V^{(2)}(z) := \left( 1 + \frac{|r^{(2)}(z)|^2}{e^{\varphi(z)}r^{(2)}(z)} \right) e^{-\varphi(z)\bar{r}^{(2)}(z)} \frac{1}{1}, \quad \text{with } r^{(2)}(z) := r(z) \prod_{k \in \gamma(\xi)} \left( \frac{z - z_k}{|z - z_k|^2} \right)
\]

and decompose now as follows:

\[
V^{(2)}(z;x,t) = \begin{cases} \tilde{U}_L \tilde{U}_0 \tilde{U}_R, & \text{for } z < \xi \\ \tilde{W}_L \tilde{W}_R, & \text{for } z > \xi \end{cases},
\]

where

\[
\tilde{U}_L := \begin{pmatrix} 1 & 0 \\ e^{\varphi(z;x,t)} \tilde{R}_4(z) & 0 \end{pmatrix}, \quad \tilde{U}_0 := \begin{pmatrix} 1 + |r(z)|^2 \sigma_3 \end{pmatrix}, \quad \tilde{U}_R := \begin{pmatrix} 1 & e^{-\varphi(z;x,t)} \tilde{R}_3(z) \\ 0 & 1 \end{pmatrix},
\]

\[
\tilde{W}_L := \begin{pmatrix} 1 & 0 \\ e^{-\varphi(z;x,t)} \tilde{R}_6(z) & 0 \end{pmatrix}, \quad \tilde{W}_R := \begin{pmatrix} 1 & e^{\varphi(z;x,t)} \tilde{R}_1(z) \\ 0 & 1 \end{pmatrix},
\]

and

\[
\tilde{R}_4(z) := \frac{r^{(2)}(z)}{1 + |r(z)|^2}, \quad \tilde{R}_3(z) := \frac{\overline{r^{(2)}(z)}}{1 + |r(z)|^2}, \quad \tilde{R}_6(z) := \overline{r^{(2)}(z)}, \quad \tilde{R}_1(z) := r^{(2)}(z).
\]

Note that \(|r(z)| = |\overline{r^{(2)}(z)}| \quad (z \in \mathbb{R}) \) and moreover \( c_1 \| r \|_{H^1(\mathbb{R})} \leq \| \overline{r^{(2)}(z)} \|_{H^1(\mathbb{R})} \leq c_2 \| r \|_{H^1(\mathbb{R})} \).

We will extend the functions \( \tilde{R}_j \) to special domains \( \Omega_j \) which we define to be

\[
\Omega_1 := \{ z \in \mathbb{C} \mid \arg(z - \xi) \in \left(0, \frac{\pi}{4}\right) \},
\]

\[
\Omega_2 := \{ z \in \mathbb{C} \mid \arg(z - \xi) \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right) \},
\]

\[
\Omega_3 := \{ z \in \mathbb{C} \mid \arg(z - \xi) \in \left(\frac{3\pi}{4}, \pi\right) \},
\]

\[
\Omega_4 := \{ z \in \mathbb{C} \mid \arg(z - \xi) \in \left(\pi, \frac{5\pi}{4}\right) \},
\]

\[
\Omega_5 := \{ z \in \mathbb{C} \mid \arg(z - \xi) \in \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right) \},
\]

\[
\Omega_6 := \{ z \in \mathbb{C} \mid \arg(z - \xi) \in \left(\frac{7\pi}{4}, 2\pi\right) \}.
\]
Let now $R_j$ be the extensions of $\tilde{R}_j$ into $\Omega_j$ (for $j \in \{1, 3, 4, 6\}$).

\[
R_1(z) := \cos(2\arg(z - \xi))r(z) + [1 - \cos(2\arg(z - \xi))](z - \xi)^{-2i\nu_{0}\tilde{r}_0}\delta^2(z),
\]
\[
R_3(z) := \cos(2\arg(z - \xi))\frac{\phi(z)}{1 + |\phi(z)|^2} + [1 - \cos(2\arg(z - \xi))](z - \xi)^{-2i\nu_{0}\tilde{r}_0}\delta^2(z),
\]
\[
R_4(z) := \cos(2\arg(z - \xi))\frac{r(z)}{1 + |r(z)|^2} + [1 - \cos(2\arg(z - \xi))](z - \xi)^{-2i\nu_{0}\tilde{r}_0}\delta^2(z),
\]
\[
R_6(z) := \cos(2\arg(z - \xi))\phi(z) + [1 - \cos(2\arg(z - \xi))](z - \xi)^{-2i\nu_{0}\tilde{r}_0}\delta^2(z).
\]

Here we set
\[
r(z) := \begin{cases} 
    r(\Re z), & \text{if } \Im z = 0, \\
    \varphi_{\nu_{0}} \ast r(\Re z), & \text{if } \Im z \neq 0,
\end{cases}
\]
where $\varphi \in C_{0}^{\infty}(\mathbb{R}, \mathbb{R})$ is of compact support and satisfies $\int \varphi dx = 1$. We set $\varphi_0(x) := e^{-1}\varphi(e^{-1}x)$. $\varphi_{\nu_{0}} \ast r$ denotes the convolution of $\varphi$ and $r$. Further definitions are
\[
v_0(x, t) := -\frac{1}{2\pi} \log(1 + |\nu_{0}(y)|^2)
\]
\[
\tilde{r}_0(x, t) := r(\xi)e^{-2i\nu_{0}t - 2\beta_0}
\]
\[
\delta(z; x, t) := \exp\left(\frac{1}{2\pi} \int_{-\infty}^{x} \frac{\log(1 + |r(y)|^2)}{y - z} dy\right)
\]
\[
\beta_0(x, t) := \frac{1}{2\pi} \int_{-\infty}^{x} \frac{\log(1 + |r(y)|^2)}{y - \xi} dy
\]
\[
+ \int_{-\infty}^{x} \frac{\log(1 + |r(y)|^2) - \log(1 + |r(\xi)|^2)}{y - \xi} dy - \frac{v_0}{2\pi i}
\]

By replacing $\tilde{R}_j$ with $R_j$ in (4.2) we can obtain matrices $U_L, U_R, W_L$ and $W_R$, which are extensions into the same domains $\Omega_j$. Using these extensions we now define our third modification by:

\[
m^{(3)}(z) := \begin{cases} 
    m^{(2)}(z)W_L(z)^{-1}\delta^{-\sigma_3}(z), & \text{for } z \in \Omega_1, \\
    m^{(2)}(z)\delta^{-\sigma_3}(z), & \text{for } z \in \Omega_2, \\
    m^{(2)}(z)U_R(z)^{-1}\delta^{-\sigma_3}(z), & \text{for } z \in \Omega_3, \\
    m^{(2)}(z)U_L(z)\delta^{-\sigma_3}(z), & \text{for } z \in \Omega_4, \\
    m^{(2)}(z)\delta^{-\sigma_3}(z), & \text{for } z \in \Omega_5, \\
    m^{(2)}(z)W_L(z)\delta^{-\sigma_3}(z), & \text{for } z \in \Omega_6.
\end{cases}
\]

The price of this modification will be the loss of analyticity in $\Omega_1 \cup \Omega_3 \cup \Omega_4 \cup \Omega_6$ and a jump on
\[
\Sigma^{(3)}(x, t) := \bigcup_{n=1}^{4} \Sigma_n^{(3)} \cup \{\xi\}
\]
with
\[
\Sigma_1^{(3)} = e^{i\pi/4}\mathbb{R}_{+} + \xi,
\Sigma_2^{(3)} = e^{-i\pi/4}\mathbb{R}_{-} + \xi,
\Sigma_3^{(3)} = e^{i\pi/4}\mathbb{R}_{-} + \xi,
\Sigma_4^{(3)} = e^{-i\pi/4}\mathbb{R}_{+} + \xi,
\]
heriting the orientation of $\mathbb{R}_{\pm}$. In exchange for that the jump on $\mathbb{R}$ is removed by (4.7). In order to measure the non-analyticity of $m^{(3)}$ we use the operator $\sigma := \frac{1}{2}(\partial_{\Re z} + i\partial_{\Im z})$: 
Lemma 4.1. If $m^{(2)}(z)$ solves RHP[2], then $m^{(3)}(z)$ defined in (4.7) is a solution to the following $\overline{\partial}$-RHP:

$$\overline{\partial} \text{-RHP}[3]:$$
Find for each $(x,t) \in \mathbb{R} \times \mathbb{R}$ a 2 x 2-matrix valued function $\mathbb{C} \ni z \mapsto m^{(3)}(z;x,t)$ which satisfies

(i) $m^{(3)}(z;x,t)$ is meromorphic in $\Omega_2 \cup \Omega_5$ and continuous in $\Omega_1 \cup \Omega_3 \cup \Omega_4 \cup \Omega_6 \cup \mathbb{R}$ (with respect to the parameter $z$) and $\overline{\partial} m^{(3)} = m^{(3)} W^{(3)}$, where

$$W^{(3)}(z) = \begin{cases} 
\begin{pmatrix} 0 & 0 \\
-e^{\phi(z)} \delta^{-2}(z) & \overline{\partial} R_1(z) \\
0 & -e^{\phi(z)} \delta^2(z) \overline{\partial} R_3(z) 
\end{pmatrix}, & \text{for } z \in \Omega_1, \\
\begin{pmatrix} 0 & 0 \\
e^{\phi(z)} \delta^{-2}(z) & \overline{\partial} R_4(z) \\
0 & e^{\phi(z)} \delta^2(z) \overline{\partial} R_6(z) 
\end{pmatrix}, & \text{for } z \in \Omega_4,
\end{cases}$$

(4.9)

(ii) $m^{(3)}(z;x,t) = 1 + \mathcal{O}\left( \frac{1}{z} \right)$ as $|z| \to \infty$.

(iii) If $\Box(\xi) = \emptyset$, $m^{(2)}$ has no poles (i.e. $m^{(2)}$ is analytic in $\Omega_2 \cup \Omega_5$). If $\Box(\xi)$ consists of certain $z_k$ and $\overline{z}_k$ such that $k \in \bigtriangledown(\xi)$, $m^{(2)}$ has simple poles at these $z_k$ and $\overline{z}_k$ with:

$$\text{Res} m^{(2)}(z) = \lim_{z \to z_k} m^{(2)}(z) \begin{pmatrix} 0 & 0 \\
0 & -1 
\end{pmatrix},$$

(4.10)

$$\text{Res} m^{(2)}(z) = \lim_{z \to \overline{z}_k} m^{(2)}(z) \begin{pmatrix} 0 & 0 \\
0 & 1 
\end{pmatrix}. $$

(iv) The non-tangential boundary values $m^{(3)}_{\pm}(z)$ exist for $z \in \Sigma^{(3)}$ and satisfy the jump relation $m^{(3)}_+ = m^{(3)} V^{(3)}$, where

$$V^{(3)}(z) = \begin{cases} 
\delta^{\sigma_3}(z) W_R(z) \delta^{-\sigma_3}(z), & \text{for } z \in \Sigma^{(3)}_1, \\
\delta^{\sigma_3}(z) U_R(z) \delta^{-\sigma_3}(z), & \text{for } z \in \Sigma^{(3)}_2, \\
\delta^{\sigma_3}(z) U_L(z) \delta^{-\sigma_3}(z), & \text{for } z \in \Sigma^{(3)}_3, \\
\delta^{\sigma_3}(z) W_L(z) \delta^{-\sigma_3}(z), & \text{for } z \in \Sigma^{(3)}_4.
\end{cases}$$

(4.12)

Proof. For $z \in \Omega_1$ we have

$$\overline{\partial} m^{(3)} = m^{(2)} \overline{\partial} W_R^{-1} \delta^{-\sigma_3} = m^{(3)} \delta^{\sigma_3} W_R \overline{\partial} W_R^{-1} \delta^{-\sigma_3} = m^{(3)} W^{(3)}.$$

The same calculation verifies (4.9) for $z \in \Omega_3 \cup \Omega_4 \cup \Omega_6$. The analyticity of $\delta(z)$ implies that $m^{(3)}$ is meromorphic on $\Omega_2 \cup \Omega_5$ if $m^{(2)}$ is meromorphic. Hence, in order to prove (i) it remains to show...
that the jump of \( m^{(2)} \) on \( \mathbb{R} \) is indeed removed (i.e. \( m^{(3)} \) is continuous on \( \mathbb{R} \)). For \( z > \xi \) we have

\[
m_+^{(3)}(z) = m_+^{(2)}(z)\tilde{W}_R^{-1}(z)\delta^{-\sigma_3}(z), \quad m_-^{(3)}(z) = m_-^{(2)}(z)\tilde{W}_L(z)\delta^{-\sigma_3}(z).
\]

Taking into account that \( \delta \) is analytic for \( z > \xi \) and \( m_+^{(2)} = m_-^{(2)}\tilde{W}_L\tilde{W}_R \) (see (4.11)) we find \( m_+^{(3)} = m_-^{(3)} \). For \( z < \xi \) the function \( \delta \) has a jump and satisfies \( \delta_+ = \delta_-(1 + |r|^2) \). This is a consequence of the Plemelj formulae (see [AF03]). Thus we have \( \delta_+^{\sigma_3} = \delta_-^{\sigma_3}\tilde{U}_0 \) for \( z < \xi \) and accordingly

\[
m_+^{(3)} = m_+^{(2)}\tilde{U}_R^{-1}\delta_-^{\sigma_3} = m_-^{(2)}[\tilde{U}_L\tilde{U}_0\tilde{U}_R]\tilde{U}_R^{-1}[\delta_-^{\sigma_3}\tilde{U}_0]^{-1} = m_-^{(2)}\tilde{U}_L\delta_-^{\sigma_3} = m_-^{(3)},
\]

which completes the proof of (i). (ii) follows from \( \delta(z) = 1 + \mathcal{O}\left(\frac{1}{z}\right) \) as \( |z| \to \infty \). (iii) follows easily from the definition (4.7) and the last point (vi) is also obvious. \( \square \)

Our next goal is the elimination of the discontinuity of \( m^{(3)} \) on \( \Sigma^{(3)} \). The idea is very simple: We set

\[(4.13) \quad m^{(4)}(z) := m^{(3)}(z)[D(z)]^{-1},\]

where \( D \) is chosen such that it admits the same jump on \( \Sigma^{(3)} \) as \( m^{(3)} \) and leaves other properties of \( m^{(3)} \) untouched. To be precise we take the solution of the following Riemann Hilbert problem:

**RHP[D]:**

Find for each \((x,t) \in \mathbb{R} \times \mathbb{R}\) a 2 \times 2-matrix valued function \( \mathbb{C} \ni z \mapsto D(z;x,t) \) which satisfies

(i) \( D(z;x,t) \) is analytic in \( \mathbb{C} \setminus \Sigma^{(3)} \) (with respect to the parameter \( z \)).

(ii) \( D(z;x,t) = 1 + \mathcal{O}\left(\frac{1}{z}\right) \) as \( |z| \to \infty \).

(iii) The non-tangential boundary values \( D_\pm(z;x,t) \) exist for \( z \in \Sigma^{(3)} \) and satisfy the jump relation \( D_+ = D_-V^{(3)} \).

As a consequence we have

\[
m^{(4)}_+ = m^{(3)}_+[D_+]^{-1} = m^{(3)}_-V^{(3)}[D_-V^{(3)}]^{-1} = m^{(3)}_-[D_-]^{-1} = m^{(4)}_-, \quad z \in \Sigma^{(3)},
\]

thus \( m^{(4)} \) is indeed continuous on \( \Sigma^{(3)} \). Furthermore the following lemma holds:

**Lemma 4.2.** If \( m^{(3)}(z) \) solves RHP[3], then \( m^{(4)}(z) \) defined in (4.13) is a solution to the following \( \overline{\partial} \)-RHP:
Lemma 4.3. \( \overline{\text{RHP}[4]} \):
Find for each \((x, t) \in \mathbb{R} \times \mathbb{R}\) a \(2 \times 2\)-matrix valued function \(\mathbb{C} \ni z \mapsto m^{(4)}(z; x, t)\) which satisfies

(i) \(m^{(4)}(z; x, t)\) is meromorphic in \(\Omega_2 \cup \Omega_5\) and continuous in \(\Omega_1 \cup \Omega_3 \cup \Omega_4 \cup \Omega_6 \cup \mathbb{R} \cup \Sigma^{(3)}\) (with respect to the parameter \(z\)) and \(\overline{\text{m}}^{(4)} = m^{(4)}W^{(4)}\), where

\[
W^{(4)}(z) = D(z)W^{(3)}(z)[D(z)]^{-1}
\]

(ii) \(m^{(4)}(z; x, t) = 1 + O\left(z^{-1}\right)\) as \(|z| \to \infty\).

(iii) If \(\square(\xi) = \emptyset\), \(m^{(4)}\) has no poles (i.e. \(m^{(4)}\) is analytic in \(\Omega_2 \cup \Omega_5\)). If \(\square(\xi)\) consists of certain \(z_k\) and \(\overline{z}_k\) such that \(k \in \nabla(\xi)\), \(m^{(4)}\) has simple poles at these \(z_k\) and \(\overline{z}_k\) with:

\[
\operatorname{Res} m^{(4)}(z) = \lim_{z \to z_k} m^{(4)}(z)D(z_k)\begin{pmatrix} 0 & \frac{1}{c_k e^{\theta_k} T(z_k)^2 \delta(z_k)} \\ 0 & 0 \end{pmatrix} [D(z_k)]^{-1},
\]

\[
\operatorname{Res} m^{(4)}(z) = \lim_{z \to \overline{z}_k} m^{(4)}(z)D(\overline{z}_k)\begin{pmatrix} 0 & 0 \\ -\frac{1}{\tau_k e^{-\phi_k} T(\overline{z}_k)^2 \delta(\overline{z}_k)} & 0 \end{pmatrix} [D(\overline{z}_k)]^{-1}.
\]

If \(\square(\xi)\) consists of certain \(z_k\) and \(\overline{z}_k\) such that \(k \in \Delta(\xi)\), \(m^{(4)}\) has simple poles at these \(z_k\) and \(\overline{z}_k\) with:

\[
\operatorname{Res} m^{(4)}(z) = \lim_{z \to z_k} m^{(4)}(z)D(z_k)\begin{pmatrix} 0 & 0 \\ c_k e^{\theta_k} T(z_k)^2 \delta(z_k) & 0 \end{pmatrix} [D(z_k)]^{-1},
\]

\[
\operatorname{Res} m^{(4)}(z) = \lim_{z \to \overline{z}_k} m^{(4)}(z)D(\overline{z}_k)\begin{pmatrix} 0 & -\tau_k e^{-\phi_k} T(\overline{z}_k)^2 \delta(\overline{z}_k) \\ 0 & 0 \end{pmatrix} [D(\overline{z}_k)]^{-1}.
\]

The proof of this lemma is elementary and we will skip it here. Instead we have to say a word on \(\text{RHP}[D]\). It can be solved explicitly and the solution has been worked out for example in [CP145, DZ94, DM08] or [IMIT].

Lemma 4.3. \( \text{(1) RHP}[D]\) has an unique solution,

(2) \(|D(z; x, t)|_{L^\infty(\mathbb{C})} \leq C\) (\(C\) does not depend on \(x\) and \(t\))

(3) \(D(z; x, t) = 1 + \frac{D_1(xt)}{z} + O(z^{-2})\) as \(|z| \to \infty\) and \(|D_1(x, t)| \leq c\xi^{-1/2}\)

Using the transformation \(\zeta \leftrightarrow \sqrt{\xi}(z - z_0)\) we can transform \(\text{RHP}[D]\) into the Parabolic Cylinder RHP:

\[
\begin{cases}
P_+(\zeta) = P_-(\zeta)V_D(\zeta) & \text{for arg}(\zeta) \notin \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}, \\
P_+(\zeta) = P_-(\zeta)V_D(\zeta) & \text{for arg}(\zeta) \in \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}, \\
P(\zeta) \to 1 & \text{as } \zeta \to \infty,
\end{cases}
\]

where

\[
V_D(\zeta) := \begin{cases}
\begin{pmatrix} 1 & 0 \\ r_0 \zeta^{-2i\nu_0\delta^2/2} & 1 \end{pmatrix}, & \text{for arg}(\zeta) = \pi/4, \\
\begin{pmatrix} 1 & 0 \\ \frac{\tau_0}{1 + |\rho_0|^2} \zeta^{2i\nu_0 e^{-i\delta^2/2}} & 1 \end{pmatrix}, & \text{for arg}(\zeta) = 3\pi/4, \\
\begin{pmatrix} 1 & 0 \\ \frac{\tau_0}{1 + |\rho_0|^2} \zeta^{-2i\nu_0\delta^2/2} & 1 \end{pmatrix}, & \text{for arg}(\zeta) = 5\pi/4, \\
\begin{pmatrix} 1 & 0 \\ \tau_0 e^{2i\nu_0 e^{-i\delta^2/2}} & 1 \end{pmatrix}, & \text{for arg}(\zeta) = 7\pi/4.
\end{cases}
\]
The statements of Lemma 4.3 on $D(z) = P(\sqrt{\text{Re}(z-z_0)})$ are consequences of analogous statements on $P$ which are well known and derived in the references mentioned above.

5. The $\overline{\partial}$-method

In this section we show that for large $t$ we can forget about the $\overline{\partial}$ part. The proof is taken from [DM08]. We consider $\overline{\partial}$-RHP[4] with $W(4) \equiv 0$:

**RHP[5]:**
Find for each $(x, t) \in \mathbb{R} \times \mathbb{R}$ a $2 \times 2$-matrix valued function $\mathbb{C} \ni z \mapsto m^{(4)}(z; x, t)$ which satisfies

(i) $m^{(5)}(z; x, t)$ is meromorphic in $\mathbb{C}$.
(ii) $m^{(5)}(z; x, t) = 1 + O\left(\frac{1}{z^2}\right)$ as $|z| \to \infty$.
(iii) If $\Box(\xi) = \varnothing$, $m^{(5)}$ has no poles (i.e. $m^{(5)}$ is analytic in $\Omega_2 \cup \Omega_5$). If $\Box(\xi)$ consists of certain $z_k$ and $z_k$ such that $k \in \nabla(\xi)$, $m^{(5)}$ has simple poles at these $z_k$ and $z_k$ with:

$$\text{Res}_{z=z_k} m^{(5)}(z) = \lim_{z \to z_k} m^{(5)}(z)D(z_k) \begin{pmatrix} 0 & \frac{1}{c_k e^{\phi_k T(z_k)^2} \delta(z_k)^{-2}} \\ 0 & 0 \end{pmatrix} [D(z_k)]^{-1},$$

$$\text{Res} m^{(5)}(z) = \lim_{z \to z_k} m^{(5)}(z)D(z_k) \begin{pmatrix} 0 & 0 \\ \frac{-1}{\tau e^{\phi_k T(z_k)^2} \delta(z_k)^2} & 0 \end{pmatrix} [D(z_k)]^{-1}.$$

If $\Box(\xi)$ consists of certain $z_k$ and $z_k$ such that $k \in \Delta(\xi)$, $m^{(5)}$ has simple poles at these $z_k$ and $z_k$ with:

$$\text{Res}_{z=z_k} m^{(5)}(z) = \lim_{z \to z_k} m^{(5)}(z)D(z_k) \begin{pmatrix} 0 & \frac{1}{c_k e^{\phi_k T(z_k)^2} \delta(z_k)^{-2}} \\ 0 & 0 \end{pmatrix} [D(z_k)]^{-1},$$

$$\text{Res} m^{(5)}(z) = \lim_{z \to z_k} m^{(5)}(z)D(z_k) \begin{pmatrix} 0 & 0 \\ -\frac{-1}{\tau e^{\phi_k T(z_k)^2} \delta(z_k)^2} & 0 \end{pmatrix} [D(z_k)]^{-1}.$$

**Lemma 5.1.** Let $m^{(4)}$ solve $\overline{\partial}$-RHP[4] and $m^{(5)}$ be a solution to RHP[5]. Then there is a matrix $E_1(x, t)$ for which

$$\|E_1\| \leq ct^{-1/2} \quad (t > 0)$$

(with $c > 0$ independent of $x$) holds and such that

$$m^{(4)}(z) = \left[1 + E_1(z) + O\left(\frac{1}{z^2}\right)\right] m^{(5)}(z)$$

as $|z| \to \infty$.

**Proof.** It can be easily verified that $E(z) := m^{(4)}(z)[m^{(5)}(z)]^{-1}$ solves the following $\overline{\partial}$-problem:

**$\overline{\partial}$-problem for $E$:**
Find for each $(x, t) \in \mathbb{R} \times \mathbb{R}$ a $2 \times 2$-matrix valued function $\mathbb{C} \ni z \mapsto E(z; x, t)$ which satisfies

(i) $E(z)$ is continuous in $\mathbb{C}$,
(ii) $E(z; x, t) = 1 + O\left(\frac{1}{z^2}\right)$ as $|z| \to \infty$, $z \in \Omega_2 \cup \Omega_5$,
(iii) $\overline{\partial}E = EW$ with $W = m^{(5)}W^{(4)}[m^{(5)}]^{-1}$. 


As described in [CP14a, Section 3], the solution \( E \) is obtained by taking the unique solution of 
\[
E = 1 + J(E).
\]
The operator \( J : L^\infty(\mathbb{C}) \to L^\infty(\mathbb{C}) \cap C^0(\mathbb{C}) \) is defined by
\[
JH(z) := \frac{1}{\pi} \int_{\mathbb{C}} \frac{H(\zeta)W(\zeta)}{\zeta - z} dA(\zeta).
\]

Using estimates on \( \partial R_j \) (see Proposition 3.6 in [CP14b]) it can be proved that 
\[
\| J \|_{L^\infty(\mathbb{C}) \to L^\infty(\mathbb{C})} \leq c t^{1 - \frac{2s}{4}} (c \text{ independent of } x).
\]
Hence, 
\[
\| E \|_{L^\infty(\mathbb{C})} = \| (1 - J)^{-1} \|_{L^\infty(\mathbb{C})} \text{ is bounded uniformly in } (x,t) \text{ for sufficiently large } t.
\]
As a consequence we find
\[
| E_1 | = \left| \frac{1}{\pi} \int_{\mathbb{C}} EWdA \right| \leq c \sum_{j \in \{1,3,4,6\}} \int_{\Omega_j} |W|dA \leq c t^{1 - \frac{1+2s}{4}}.
\]

For the latter inequality see the calculations in the proof of Lemma 3.9 in [CP14b].  

\[\Box\]

6. The last step

Note that \( \text{RHP}[5] \) does not describe a soliton or breather because the residuum conditions are 
not those of solitons. However, we have \( P(z_k) \to 1 \) and \( \delta(z_k) \to 1/\Lambda_k^+ \) for \( |\xi - \Re(z_k)| < 1/\sqrt{t} \)
and \( t \to \infty \). For a small \( \rho > 0 \) such that \( \bigcap_{z \in \mathbb{X}} B_\rho(z) = \emptyset \) we set:

\[
m^{(6)}(z) := \begin{cases} 
m^{(5)}(z)D(z_k) \left( \delta(z_k)\Lambda_k^+ \right)^{\sigma_1}, & \text{if } z \in B_\rho(z_k), z \in \Box(\xi), \\
m^{(5)}(z)D(\overline{z}_k) \left( \delta(\overline{z}_k)/\Lambda_k^+ \right)^{\sigma_3}, & \text{if } z \in B_\rho(\overline{z}_k), \overline{z}_k \in \Box(\xi), \\
m^{(5)}(z), & \text{else.} 
\end{cases}
\]

Note that \( m^{(6)} \) differs from \( m^{(5)} \) only if \( \Box(\xi) \neq \emptyset \). For \( \Box(\xi) \neq \emptyset \) a discontinuity appears on
\[
\Sigma^{(6)}(\xi) = \bigcup_{z \in \Box(\xi)} \partial B_\rho(z).
\]

**Lemma 6.1.** If \( m^{(5)}(z) \) solves \( \text{RHP}[5] \), then \( m^{(6)}(z) \) defined in (6.1) is a solution to the following \( \text{RHP} \):
We are now arrived at our last step. Later in Lemma 6.3 we will show that we may replace $m^{(5)}$ by $m^{(4)}$ as $|z| \to \infty$.

(iii) If $\square(\xi) = \emptyset$, $m^{(6)}$ has no poles. If $\square(\xi)$ consists of certain $z_k$ and $\bar{z}_k$ such that $k \in \nabla(\xi)$, $m^{(5)}$ has simple poles at these $z_k$ and $\bar{z}_k$ with:

$$\text{Res}_{z} m^{(6)}(z) = \lim_{z \to z_k} m^{(6)}(z) \begin{pmatrix} 0 & \frac{1}{c_k(\Lambda_k^+)^2\bar{T}(z_k)^2} \\ 0 & 0 \end{pmatrix},$$

(6.2)

$$\text{Res}_{z} m^{(6)}(z) = \lim_{z \to \bar{z}_k} m^{(6)}(z) \begin{pmatrix} 0 & 0 \\ -\frac{1}{\tau(\Lambda_k)^2\bar{T}(z_k)^2} & 0 \end{pmatrix}.$$  

If $\square(\xi)$ consists of certain $z_k$ and $\bar{z}_k$ such that $k \in \Delta(\xi)$, $m^{(6)}$ has simple poles at these $z_k$ and $\bar{z}_k$ with:

$$\text{Res}_{z} m^{(6)}(z) = \lim_{z \to z_k} m^{(6)}(z) \begin{pmatrix} 0 & 0 \\ c_k(\Lambda_k^+)^2\bar{T}(z_k)^2 & 0 \end{pmatrix},$$

(6.3)

$$\text{Res}_{z} m^{(6)}(z) = \lim_{z \to \bar{z}_k} m^{(6)}(z) \begin{pmatrix} 0 & 0 \\ -\tau_k(\Lambda_k)^2\bar{T}(z_k)^2 & 0 \end{pmatrix}.$$  

(iv) The non-tangential boundary values $m^{(6)}(z)$ exist for $z \in \Sigma^{(6)}$ and satisfy the jump relation $m_+^{(6)} = m_-^{(6)}V^{(6)}$, where

$$V^{(6)}(z) = \begin{cases} D(z_k) \left( \delta(z_k)\Lambda_k^+ \right)^{\sigma_1}, & \text{if } z \in \partial B_{\rho}(z_k), z_k \in \square(\xi), \\
D(\bar{z}_k) \left( \delta(\bar{z}_k)\bar{\Lambda}_k^+ \right)^{\sigma_1}, & \text{if } z \in B_{\rho}(\bar{z}_k), \bar{z}_k \in \square(\xi). \end{cases}$$

(6.4)

The proof is elementary. We are now arrived at our last step. Later in Lemma 6.3 we will show that we may replace $V^{(6)}$ in (6.2) by 1 which is a consequence of $P(z_k) \to 1$ and $\delta(z_k) \to 1/\Lambda_k^+$ for $|\xi - \Re(z_k)| < 1/\sqrt{T}$ and $t \to \infty$. Since the condition $|\xi - \Re(z_k)| < 1/\sqrt{T}$ is fulfilled whenever $\square(\xi) \neq \emptyset$ we thus have $V^{(6)} \to 1$.

**Proposition 6.2.** There exist constants $c, T > 0$ such that

$$\|V^{(6)} - 1\|_{L^2(\Sigma^{(6)})} \leq ct^{-1/2}, \quad \|V^{(6)} - 1\|_{L^2(\Sigma^{(6)})} \leq ct^{-1/2},$$

for $t > T$.

**Proof.** Obviously the $L^2$-estimate of (6.5) follows from the $L^\infty$-estimate, due to $\text{meas}(\Sigma^{(6)}) < \infty$. Furthermore the proposition is trivial in the case of $\square(\xi) = \emptyset$, where we have $\Sigma^{(6)} = \emptyset$. Let us now assume that $z_k \in \square(\xi) \cap D_+$ and thus $|\xi - \Re(z_k)| < 1/\sqrt{T}$:

$$|\delta(z_k)\Lambda_k^+ - 1| = \left| \exp\left( \frac{1}{2\pi i} \int_{\Gamma_k(z_k)} \frac{\log(1 + |r(\xi)|^2)}{\xi - z} d\xi \right) - 1 \right|$$

$$\leq c \int_{\Gamma_k(z_k)} \frac{\log(1 + |r(\xi)|^2)}{\xi - z} d\xi$$

$$\leq \frac{|\xi - \Re(z_k)|}{\Im(z_k)} \log(1 + ||r||_{H^\infty}) \leq ct^{-1/2}$$
Analogously we have $|\delta(z_k)/\Lambda_k^+ - 1| \leq ct^{-1/2}$. Additionally we take $|D(z_k) - 1| \leq ct^{-1/2}$ and $|D(z_k) - 1| \leq ct^{-1/2}$ from Lemma 4.3 and thus the proof is completed.

If we omit the jump on $\Sigma^6$ in RHP[6] we get:

**RHP[7]:**

Find for each $(x, t) \in \mathbb{R} \times \mathbb{R}$ a $2 \times 2$-matrix valued function $\mathbb{C} \ni z \mapsto m^{(7)}(z; x, t)$ which satisfies

(i) $m^{(7)}(z; x, t)$ is meromorphic in $\mathbb{C}$.

(ii) $m^{(7)}(z; x, t) = 1 + O\left(\frac{1}{z}\right)$ as $|z| \to \infty$.

(iii) If $\Box(\xi) = \emptyset$, $m^{(7)}$ has no poles (i.e. $m^{(7)}$ is entire). If $\Box(\xi)$ consists of certain $z_k$ and $\overline{z}_k$ such that $k \in \nabla(\xi)$, $m^{(5)}$ has simple poles at these $z_k$ and $\overline{z}_k$ with:

$$\text{Res}_{z=z_k} m^{(7)}(z) = \lim_{z \to z_k} m^{(7)}(z) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{1}{c_k(\Lambda_k^-)^2 e^{\phi_k T(z_k)^2}}.$$

$$\text{Res}_{z=\overline{z}_k} m^{(7)}(z) = \lim_{z \to \overline{z}_k} m^{(7)}(z) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{\overline{c}_k(\overline{\Lambda}_k^-)^2 e^{\overline{\phi}_k T(z_k)^2}}.$$

If $\Box(\xi)$ consists of certain $z_k$ and $\overline{z}_k$ such that $k \in \triangle(\xi)$, $m^{(7)}$ has simple poles at these $z_k$ and $\overline{z}_k$ with:

$$\text{Res}_{z=z_k} m^{(7)}(z) = \lim_{z \to z_k} m^{(7)}(z) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{c_k(\Lambda_k^-)^2 e^{\phi_k T(z_k)^2}},$$

$$\text{Res}_{z=\overline{z}_k} m^{(7)}(z) = \lim_{z \to \overline{z}_k} m^{(7)}(z) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{\overline{c}_k(\overline{\Lambda}_k^-)^2 e^{\overline{\phi}_k T(z_k)^2}}.$$

The following Lemma is comparable to Lemma 3.2

**Lemma 6.3.** Let $m^{(6)}$ solve RHP[6] and $m^{(7)}$ be a solution to RHP[7]. Then there is a matrix $F_1(x, t)$ for which

$$\|F_1\| \leq ct^{-1/2} \quad (t > 0)$$

(with $c > 0$ independent of $x$) holds and such that

$$m^{(6)}(z) = \left[1 + \frac{F_1}{z} + O\left(\frac{1}{z^2}\right)\right] m^{(7)}(z)$$

as $|z| \to \infty$.

**Proof.** We set $F(z) := m^{(6)}(z)[m^{(7)}(z)]^{-1}$ which admits a solution of the following Riemann Hilbert problem:

**RHP[F]**

(i) $F$ is analytic in $\mathbb{C} \setminus \Sigma^{(1)}$.

(ii) $F(z) = 1 + O\left(\frac{1}{z}\right)$ as $|z| \to \infty$.

(iii) The non-tangential boundary values $F_{\pm}(z)$ exist for $z \in \Sigma^{(6)}$ and satisfy the jump relation $F_+ = F_- V^{(F)}$, where

$$V^{(F)}(z) = m^{(7)}(z) V^{(6)}(z) \left[m^{(7)}(z)\right]^{-1}.$$
Now we proceed as in the proof of Lemma 3.2. That is firstly to find \( \eta \in L^2(\Sigma^{(6)}) \) such that

\[
\eta(x) = 1 + \lim_{z \to \Sigma^{(6)}} \frac{1}{2\pi i} \int_{\Sigma^{(6)}} \frac{\eta(\zeta)(V^{(F)}(\zeta) - 1)}{\zeta - z} d\zeta, \quad z \in \Sigma^{(6)}.
\]

The next step is to observe that

\[
F_1 = -\frac{1}{2\pi i} \int_{\Sigma^{(6)}} \eta(\zeta)(V^{(F)}(\zeta) - 1) d\zeta.
\]

Proposition 6.2 ensures the existence of \( \eta \) and the required estimate \( \|F_1\| \leq c_1^{-1/2} \). Note that (6.5) is also true for \( V^{(F)} \) instead of \( V^{(6)} \) since \( \|m^{(7)}(\cdot, x, t)\|_{L^\infty(\Sigma^{(6)})} \leq C \) with \( C \) independent of \( x \) and \( t \).

\[ \square \]

### 7. PROOF OF THEOREM 1.1

In order to prove Theorem 1.1 we firstly assume \( u_0 \in H^1(\mathbb{R}) \cap L^2(\mathbb{R}) \). Furthermore we assume (1.2), which gives us \( u_0 \in \mathcal{G}_N \) (see Theorem 2.1 (i)). Then the scattering data \( (r; \zeta', \zeta'', c'_1, \ldots, c'_N) \) of \( u_0 \) can be calculated and the solution \( u(x, t) \) can be obtained by applying the reconstruction formula (2.5) to the solution \( m \) of RHP[NLS]. Starting from this \( m \) we consider our chain of manipulations \( m \to m^{(1)} \to \ldots \to m^{(7)} \) and in each step we calculate the associated potential

\[
u^{(j)}(x, t) := 2i \lim_{|z| \to \infty} z[m^{(j)}(z; x, t)]_{12} \quad j = 1, \ldots, 7.
\]

Applying successively (3.9), Lemma 3.2, (4.7), (4.13), Lemma 5.1, (6.1) and finally Lemma 6.3 we arrive at

\[
u(x, t) = \nu^{(1)}(x, t)
\]

\[
= \nu^{(2)}(x, t) + 2i[C_{1}(x, t)]_{12}
\]

\[
= \nu^{(3)}(x, t) + 2i[C_{1}(x, t)]_{12}
\]

\[
= \nu^{(4)}(x, t) + 2i[C_{1}(x, t) + D_{1}(x, t)]_{12}
\]

\[
= \nu^{(5)}(x, t) + 2i[C_{1}(x, t) + D_{1}(x, t) + E_{1}(x, t)]_{12}
\]

\[
= \nu^{(6)}(x, t) + 2i[C_{1}(x, t) + D_{1}(x, t) + E_{1}(x, t)]_{12}
\]

\[
= \nu^{(7)}(x, t) + 2i[C_{1}(x, t) + D_{1}(x, t) + E_{1}(x, t) + F_{1}(x, t)]_{12}
\]

The estimates of Lemmata 3.2, 4.3, 5.1 and 6.3 yield

\[
\|\nu(\cdot, t) - \nu^{(7)}(\cdot, t)\|_{L^\infty(\mathbb{R})} < C|t|^{-\frac{1}{2}}.
\]

Now the remaining question is whether \( \nu^{(7)} \) approximates a \( N \)-soliton \( u^{(sol)}_+ \) and we have to specify its parameters. We claim that the poles of the approximating soliton \( u^{(sol)}_+ \) are the same of \( u_0 \) and the coupling constants are given by \( c^{\pm}_j = c_j(A^{\pm}_j)^2 \) (where \( c_j \) are the coupling constants of \( u_0 \)). The proof is easy if we use again the above manipulations. Therefore we consider the solution \( \tilde{m} \) of RHP[NLS] with parameters \( (0; \zeta', \zeta'', c^+_1, \ldots, c^+_N) \) such that \( u^{(sol)}_+ = 2i\lim z[\tilde{m}]_{12} \). Starting from \( \tilde{m} \) our manipulations \( \tilde{m} \to \ldots \to \tilde{m}^{(7)} \) then yield

\[
u^{(sol)}_+ = \tilde{u}^{(7)} + 2i[\tilde{C}_1]_{12}
\]
and moreover $\tilde{u}^{(7)} = u^{(7)}$. Thus (1.3) follows. $|z_j - z'_j| < C\varepsilon$ and $|c_j - c'_j| < C\varepsilon$ are consequences of the Lipschitz continuity of the scattering transformation. $|c_j - c'_j| < C\varepsilon$ follows if we also use

$$1 - \Lambda_j^+ \leq C \int_{-\infty}^{\text{Re}(z_j)} \log \left( 1 + \frac{|r(\zeta)|^2}{\zeta - z'_j} \right) d\zeta \leq C \|r\|_{L^2(\mathbb{R})}^2 \leq C\varepsilon.$$  

Thus the proof of our main result is done for $u_0 \in H^1(\mathbb{R})$ and $t \to +\infty$. Density arguments like those in [CP14b] prove the statement for $u_0 \in H^1(\mathbb{R}) \cap L^{2,x}(\mathbb{R})$ but $u_0 \notin H^1(\mathbb{R})$.

The case $t \to -\infty$ can be handled as follows. If $u(x,t)$ solves (1.1) then $\tilde{u}(x,t) := \mathcal{F}(x,-t)$ is also a solution to the NLS equation with $\tilde{u}(x,0) = \bar{u}_0$. Assuming that $(r(z); z_1^1, \ldots, z_N^1; c_1^+, \ldots, c_N^+)$ are the scattering data of $u_0$, we know due to the symmetry of (2.6) that $\bar{u}_0$ admits scattering data $(\bar{r}; \bar{z}_1, \ldots, \bar{z}_N; \bar{c}_1, \ldots, \bar{c}_N)$ with

$$\bar{r}(z) = r(-z), \quad \bar{z}_j = -z_j, \quad \bar{c}_j = -c'_j.$$  

By the above calculations we know that

$$\|\tilde{u}(\cdot,t) - \tilde{u}_+(\cdot,t)\|_{L^\infty(\mathbb{R})} < C\varepsilon t^{-1/2} \quad \text{as } t \to +\infty,$$

where $\tilde{u}_+(\cdot,t)$ is the soliton associated to the scattering data $(0; z_1, \ldots, z_N; c_1^+, \ldots, c_N^+)$ with

$$(7.2) \quad \bar{c}_j^+ = \bar{c}_j(\bar{\Lambda}_j^+)^2, \quad \bar{\Lambda}_j^+ := \exp \left( -\frac{1}{2\pi i} \int_{-\infty}^{\text{Re}(\bar{z}_j)} \log \left( 1 + \frac{|r(\zeta)|^2}{\zeta - z'_j} \right) d\zeta \right).$$

After inverse transformation we arrive at

$$\|u(\cdot,t) - u_(\cdot,t)\|_{L^\infty(\mathbb{R})} < C\varepsilon t^{-1/2} \quad \text{as } t \to -\infty,$$

where $u(\cdot,t) = u_+(\cdot,t)$. Making again use of the symmetry of (2.6), we know that $u_+(\cdot,t)$ admits scattering data $(0; z_1^-, \ldots, z_N^-, c_1^-, \ldots, c_N^-)$ where

$$c_j^- = -\bar{c}_j^- = c'_j(\bar{\Lambda}_j^-)^2 = c'_j \exp \left( \frac{1}{\pi i} \int_{-\infty}^{\text{Re}(\bar{z}_j)} \frac{\log \left( 1 + |r(\zeta)|^2 \right)}{\zeta - z'_j} d\zeta \right).$$

The latter equality can be obtained easily from (7.2) and shows us that (1.5) is true. Thus the proof of Theorem 1.1 is completed.

**Remark 7.1.** The two ground states $u_{\pm}^{(sol)}$ are in general distinct which follows immediately from the distinct expressions for $\Lambda_j^+$ and $\Lambda_j^-$, respectively (see (1.5)).

**REFERENCES**

[AF03] M.J. Ablowitz and A.S. Fokas. *Complex Variables: Introduction and Applications*. Cambridge Texts in Applied Mathematics. Cambridge University Press, 2003.

[APT04] J. Ablowitz, B. Prinari, and A.D. Trubatch. *Discrete and Continuous Nonlinear Schrödinger Systems*. Cambridge University Press, 2004.

[BC84] R. Beals and R. R. Coifman. Scattering and inverse scattering for first order systems. *Communications on Pure and Applied Mathematics*, 37(1):39–90, 1984.

[BJM16] Micheal Borghese, Robert Jenkins, and K.D.T-R McLaughlin. Long time asymptotic behavior of the focusing nonlinear Schrödinger equation. 2016. arXiv:1604.07436.

[CJ14] Scipio Cuccagna and Robert Jenkins. On asymptotic stability of $N$-solitons of the Gross-Pitaevskii equation. 2014. arXiv:1410.6887v1.

[CP14a] Andres Contreras and Dmitry Pelinovsky. Stability of multi-solitons in the cubic NLS equation. *Journal of Hyperbolic Differential Equations*, 11(02):329–353, 2014.

[CP14b] Scipio Cuccagna and Dmitry E. Pelinovsky. The asymptotic stability of solitons in the cubic NLS equation on the line. *Applicable Analysis*, 93(4):791–822, 2014.
[DJ89] P.G. Drazin and R.S. Johnson. *Solitons: An Introduction*. Cambridge Computer Science Texts. Cambridge University Press, 1989.

[DM08] Momar Dieng and K.D.T-R McLaughlin. Long-time asymptotics for solutions of the NLS equation via $\partial^-$ methods. 2008. arXiv:0805.2807.

[DP11] Percy Deift and Jungwoon Park. Long-time asymptotics for solutions of the NLS equation with a delta potential and even initial data. *International Mathematics Research Notices*, 2011(24):5505–5624, 2011.

[DJ94] Percy Deift and Xin Zhou. Long-time behavior of the non-focusing nonlinear Schrödinger equation, a case study. *New Series: Lectures in Mathematical Sciences*, 5, 1994.

[JM11] Robert Jenkins and K.D.T-R McLaughlin. The semi-classical limit of focusing NLS for a family of non-analytic initial data. 2011. arXiv:1106.1699v1.

[Tsu87] Y. Tsutsumi. $L^2$ solutions for the nonlinear Schrödinger equation and nonlinear groups. *Funkcial. Ekvac.*, 30:115–125, 1987.

[Zho98] Xin Zhou. $L^2$-sobolev space bijectivity of the scattering and inverse scattering transforms. *Communications on Pure and Applied Mathematics*, 51(7):697–731, 1998.

[ZS72] V.E. Zakharov and A.B. Shabat. Exact theory of two-dimensional self-focusig and one-dimensional self-modulation of waves in nonlinear media. *Soviet Physics JETP*, 34:62–69, 1972.

*MATHEMATISCHES INSTITUT, UNIVERSITÄT ZU KÖLN, 50931 KÖLN, GERMANY*

*E-mail address, A. Saalmann: asaalmann@math.uni-koeln.de*