FANS AND POLYTOPES IN TILTING THEORY I: FOUNDATIONS

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Abstract. For a finite dimensional algebra $A$ over a field $k$, the 2-term silting complexes of $A$ gives a simplicial complex $\Delta(A)$ called the $g$-simplicial complex. We give tilting theoretic interpretations of the $h$-vectors and Dehn-Sommerville equations of $\Delta(A)$. Using $g$-vectors of 2-term silting complexes, $\Delta(A)$ gives a nonsingular fan $\Sigma(A)$ in the real Grothendieck group $K_0(\text{proj} A)_R$ called the $g$-fan. We give several basic properties of $\Sigma(A)$ including sign-coherence, sign decomposition, idempotent reductions, Jasso reductions, pairwise positivity and a connection with Newton polytopes of $A$-modules. Moreover, $\Sigma(A)$ gives a (possibly infinite and non-convex) polytope $P(A)$ in $K_0(\text{proj} A)_R$ called the $g$-polytope of $A$. We call $A$ $g$-convex if $P(A)$ is convex. In this case, we show that it is a reflexive polytope, and that the dual polytope is given by the 2-term simple minded collections of $A$. There are precisely 7 convex $g$-polygons up to isomorphism. We give a classification of algebras whose $g$-polytopes are smooth Fano.

We study $g$-fans and $g$-polytopes of two important classes of algebras. We show that the $g$-fan of a classical or generalized preprojective algebra is given by the Coxeter fan. It is $g$-convex if and only if it is of type $A$ or $B$, and in this case, its $g$-polytope is the dual polytope of the short root polytope. Moreover we classify Brauer graph algebras which are $g$-convex, and describe their $g$-polytopes as the root polytopes of type $A$ or $C$.

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1. Introduction

The notion of tilting objects is basic to study the structure of a given derived category. The set of partial tilting modules over a finite dimensional algebra has a structure of a simplicial complex, and gives rise to a fan in the Grothendieck group. Their structure has been studied by a number of authors including [RS] [U] [HI]. The class of silting objects gives a completion of the class of tilting objects from a point of view of mutation [KV] [AII]. Silting objects correspond bijectively with other important objects in the derived category, including (co-)t-structures and simple-minded collections [AIR] [IT] [KY]. The subset of 2-term silting complexes enjoys especially nice properties, e.g. [An] [As] [AMV] [BY] [DIRRT]. It plays an important role in categorification of cluster algebras of Fomin and Zelevinsky. The 2-term silting version of the simplicial complex and the fan as well as their applications to cluster algebras have been studied e.g. in [Pl] [DF] [B] [DL] [BST] [As2].
We study the dimensional algebra $A$ and the indecomposable direct summands of $T$ and we obtain a nonsingular fan $\Sigma(A)$. Gluing them together, we obtain the $g$-polytope $P(A)$. We study the $g$-simplicial complex $\Delta(A)$, the $g$-fan $\Sigma(A)$ and the $g$-polytope $P(A)$ of a finite dimensional algebra $A$ mainly in the case where $A$ is $g$-finite. We give some examples of $P(A)$.

We denote by $2\text{-silt}A$ the set of isomorphism classes of basic 2-term silting complexes of $A$, which has a natural partial order such that the Hasse quiver $\text{Hasse}(2\text{-silt}A)$ is $n$-regular (see Section 2). The $f$-vector of $\Delta(A)$ gives the number of isomorphism classes of basic 2-term silting complexes with a fixed number of indecomposable direct summands. Our first main result gives the following representation theoretic interpretation of the $h$-vector of $\Delta(A)$.

**Theorem 1.1** (Theorem 3.4). Let $A$ be a finite dimensional algebra over a field $k$ which is $g$-finite, $n := |A|$ and $(h_0, \ldots, h_n)$ the $h$-vector of $\Delta(A)$. Then, for each $0 \leq j \leq n$, we have

$$h_j = \#2\text{-silt}_j A = \#\text{sbrick}_j A,$$

where $2\text{-silt}_j A$ is the set of isomorphism classes of basic 2-term silting complexes $T$ such that precisely $j$ arrows start at $T$ in $\text{Hasse}(2\text{-silt}A)$, and $\text{sbrick}_j A$ is the set of isomorphism classes of basic semibricks $S$ of $A$ satisfying $|S| = j$.

It was shown in [DIRRT] that there is a canonical bijection $2\text{-silt}_j A \simeq 2\text{-silt}_{n-j} A$ between join-irreducible elements in $2\text{-silt}A$ and meet-irreducible elements in $2\text{-silt}A$. We give the following generalization, which categorifies the famous Dehn-Sommerville equations $h_j = h_{n-j}$ for $h$-vectors.

**Theorem 1.2** (Theorem 3.15). Let $A$ be a finite dimensional algebra over a field $k$ which is $g$-finite, $n := |A|$ and $(h_0, \ldots, h_n)$ the $h$-vector of $\Delta(A)$. For $0 \leq j \leq n$, there are canonical bijections

$$\text{sbrick}_j A \simeq \text{sbrick}_{n-j} A \text{ and } 2\text{-silt}_j A \simeq 2\text{-silt}_{n-j} A.$$

In particular, we have $h_j = h_{n-j}$.

The $h$-vectors of boundary complexes of simplicial polytopes are known to be unimodal (see [Zi, Section 8]). Recently, this was generalized to simplicial spheres [Adi]. As an application, we obtain the unimodality result below. It will be interesting to give a representation theoretic proof.
Any cone in $\text{A}^h$ indecomposable direct summands. Using the Theorem 1.8 (Theorem 5.5) $n := |\text{A}|$ of isomorphism classes of 2-term presilting complexes of $\text{A}$ following problem for finite dimensional algebra.

A possible characterization sign-coherent fans in $\text{A}^g$ induced by a natural isomorphism $K_0$ (Theorem 4.25) as an analog of the sign decomposition $[\text{Ao}]$. That there exist a canonical bijection $\text{K}$ and an isomorphism $\text{B}$ as a subspace of $\text{K}$. On the other hand, there exists a finite dimensional algebra $\text{B}$ of the following sense.

In Section 4 we study $g$-fans of finite dimensional algebras. The following are straightforward consequences of known results in tilting theory.

Proposition 1.4 (Propositions 4.2, 4.14, 4.16, 5.8). Let $\text{A}$ be a finite dimensional algebra over a field $\text{k}$ and $\text{n} := |\text{A}|$.

(a) $\Sigma(\text{A})$ is a nonsingular fan in $K_0(\text{proj} \text{A})_\text{R}$.
(b) Any cone in $\Sigma(\text{A})$ is a face of a cone of dimension $\text{n}$.
(c) Any cone in $\Sigma(\text{A})$ of dimension $\text{n} - 1$ is a face of precisely two cones of dimension $\text{n}$.
(d) $\text{A}$ is $g$-finite (or equivalently, $\Sigma(\text{A})$ is finite) if and only if $\Sigma(\text{A})$ is complete.
(e) $\Sigma(\text{A})$ is sign-coherent (see Definition 4.13), ordered (see Definition 4.15), and pairwise positive (see Definition 5.6).

For each idempotent $e \in \text{A}$ and the corresponding subalgebra $e\text{A}e$ of $\text{A}$, we regard $K_0(\text{proj} e\text{A}e)_\text{R}$ as a subspace of $K_0(\text{proj} \text{A})_\text{R}$. Thanks to sign-coherence, one can restrict $\Sigma(\text{A})$ to $K_0(\text{proj} e\text{A}e)_\text{R}$ to get a subfan. It has the following representation theoretic meaning.

Theorem 1.5 (Theorem 4.18). There exists an isomorphism of fans

$$\Sigma(e\text{A}e) \simeq \{\sigma \in \Sigma(\text{A}) \mid \sigma \subset K_0(\text{proj} e\text{A}e)_\text{R}\}.$$}

We also show that the restriction of $\Sigma(\text{A})$ to each orthant can be described by a simpler algebra (Theorem 4.25) as an analog of the sign decomposition $[\text{Ao}]$.

For each 2-term presilting complex $U$ of $\text{A}$, we obtain a new fan $\Sigma(\text{A})/C(U)$ (Definition 4.10). On the other hand, there exists a finite dimensional algebra $\text{B}$ (called Jasso reduction $[\text{J}]$) such that there exist a canonical bijection

$$\{T \in 2\text{-}\text{silt} \text{A} \mid U \in \text{add} T\} \simeq 2\text{-}\text{silt} \text{B}$$

and an isomorphism $K_0(\text{proj} \text{A})_\text{R}/RC(U) \simeq K_0(\text{proj} \text{B})_\text{R}$. These two constructions are compatible in the following sense.

Theorem 1.6 (Theorem 4.11). There exists an isomorphism of fans

$$\Sigma(\text{A})/C(U) \simeq \Sigma(\text{B})$$

induced by a natural isomorphism $K_0(\text{proj} \text{A})_\text{R}/RC(U) \simeq K_0(\text{proj} \text{B})$.

Even in rank 2, there are infinitely many $g$-fans (Example 4.3). It is interesting to classify all possible $g$-fans in $\mathbb{R}^d$. In a forthcoming paper $[\text{AHKM}]$, we will give a complete answer to the following problem for $d = 2$.

Problem 1.7. Characterize sign-coherent fans in $\mathbb{R}^d$ which can be realized as a $g$-fan of some finite dimensional algebra.

In Section 5 using the $g$-fans, we introduce the $g$-polytopes $\mathcal{P}(\text{A})$ of finite dimensional algebras $\text{A}$. We study the Ehrhart series $\text{Ehr}_A(x)$ of $\mathcal{P}(\text{A})$, which is the generating function of the number of isomorphism classes of 2-term presilting complexes of $\text{A}$ with at most $\ell$ (possibly isomorphic) indecomposable direct summands. Using the $h$-vector, we will give the following formula.

Theorem 1.8 (Theorem 5.5). Let $\text{A}$ be a finite dimensional algebra over a field $\text{k}$ which is $g$-finite, $n := |\text{A}|$ and $(h_0, \ldots, h_n)$ the $h$-vector of $\Delta(\text{A})$. Then the Ehrhart series of $\text{A}$ is given by

$$\text{Ehr}_A(x) = \sum_{i=0}^{n} h_i x^i \frac{(1-x)^{n+1}}{(1-x)^{n+1}}.$$
Using silting theory, we give explicit connections between the $g$-fan $\Sigma(A)$ and the normal fans $\Sigma(N(X))$ of the Newton polytopes $N(X)$ of $A$-modules $X$ (Definitions 5.18, 2.4), and between the Hasse quiver $\text{Hasse}(2\text{-silt}A)$ and the 1-skeleton of $N(X)$. In particular, the following result recovers results in [Fe1] for $g$-finite case (see also [BCDMTY, PPPP]).

**Theorem 1.9** (Theorem 5.23, Corollary 5.25). Let $A$ be a finite dimensional algebra over a field $k$ which is $g$-finite. For each $X \in \text{mod } A$, we have

$$\Sigma(N(X)) = \Sigma(A)/\sim_X$$

and $\widetilde{N}_1(X) \simeq \text{Hasse}(2\text{-silt}A)/\sim_X$ (see Definition 5.22). Moreover, there exists $X \in \text{mod } A$ such that

$$\Sigma(N(X)) = \Sigma(A)/\sim_X$$

and $\widetilde{N}_1(X) \simeq \text{Hasse}(2\text{-silt}A)$.

The rest of this paper is devoted to study finite dimensional algebras whose $g$-polytopes are convex. We call $A$ $g$-convex if $P(A)$ is convex. We give characterizations of $g$-convexity (Theorem 5.10). In particular, $g$-convexity implies $g$-finiteness. We introduce the $c$-polytope $P_c(A)$ by using 2-simple-minded collections (Definition 5.13). Using silting-t-structure correspondence, we prove the following result.

**Theorem 1.10** (Theorem 5.14). Let $A$ be a finite dimensional algebra over a field $k$. Then $A$ is $g$-convex if and only if

$$P(A) = (P_c(A))^*.$$  

In this case, both $P(A)$ and $P_c(A)$ are reflexive polytopes.

For each $n \geq 1$, there exists only finitely many convex $g$-polytopes of dimension $n$ up to isomorphisms of $g$-polytopes (see Definition 6.1, Proposition 6.2). It is interesting to know the maximal number of $\#2\text{-silt}A$ for $g$-convex algebras $A$ with $|A| = n$, see Problem 1.15 below.

In Section 6, we give the following classification of convex $g$-polygons by using the well-known list of 16 reflexive polygons [PR].

**Corollary 1.11** (Theorem 6.3). There are precisely 7 convex $g$-polygons up to isomorphism of $g$-polytopes.

The corresponding $c$-polygons are the following.

It is well-known that there are 4319 reflexive polytopes in dimension 3 [KS]. It is interesting to know which one can be realized as a $g$-polytope. In a forthcoming paper [AHIKM2], we will give a complete answer to the following problem for $d = 3$.

**Problem 1.12.** Classify convex $g$-polytopes in $\mathbb{R}^d$.

We also give a classification of algebras whose $g$-polytopes are smooth Fano (Definition 6.4), a much stronger notion than convexity.

**Theorem 1.13** (Theorem 6.6). Let $A$ be a finite dimensional algebra over a field $k$. Then $P(A)$ is a smooth Fano polytope if and only if $A$ is a product of local algebras, algebras of pentagon type and algebras of hexagon type.

In Sections 7–8, we give explicit descriptions of the $g$-fans and/or $g$-polytopes of certain important classes of algebras. In Section 7, we describe $g$-polytopes for classical and generalized preprojective algebras due to Geiss-Leclerc-Schröer [GLS] by using the root polytope.
Theorem 1.14 (Theorem 7.4). Let $\Pi$ be a classical or generalized preprojective algebra of Dynkin type.

(a) $\Sigma(\Pi)$ is the Coxeter fan.

(b) $\Pi$ is $g$-convex if and only if it is either of type $A_n$ or $B_n$. In this case, $P(\Pi)$ is the dual polytope of the short root polytope of type $A_n$ or $B_n$ respectively.

In particular, $\#2$-silt $\Pi$ is the order of the Weyl group. In type $B_n$, it is $2^n n!$ and hence the volume of $P(\Pi)$ is $2^n$. The following is a list of natural questions to study.

Problem 1.15. (a) Is the volume of convex $g$-polytopes of rank $n$ at most $2^n$?

(b) Classify symmetric convex $g$-polytopes.

Note that the part (a) is true for centrally symmetric convex polytopes by Minkowski’s convex body Theorem [Fu, BR, BP] for these materials.

Preliminaries on fans and polytopes.

2.1. Conventions

All modules are right modules. The composition of morphisms $f : X \to Y$ and $g : Y \to Z$ is denoted by $gf : X \to Z$. The composition of arrows $a : i \to j$ and $b : j \to k$ in a quiver is denoted by $ab : i \to k$. Let $A$ be a finite dimensional algebra over a field $k$. We denote by $\mod A$ the category of finitely generated right $A$-modules and by $\proj A$ the category of finitely generated projective $A$-modules. We denote by $D^b(\mod A)$ the bounded derived category of $\mod A$ and by $K^b(\proj A)$ the bounded homotopy category of $\proj A$. For an object $X$ in a Krull-Schmidt category (e.g. $\mod A$, $K^b(\proj A)$, $D^b(\mod A)$), we denote by $|X|$ the number of non-isomorphic indecomposable direct summands of $X$.

2. Preliminaries

2.1. Preliminaries on fans and polytopes. We recall some fundamental materials on fans and polytopes. We refer the reader to e.g. [Ei], [BR], [BP] for these materials.

A convex polyhedral cone $\sigma$ is a set of the form $\sigma = \{ \sum_{i=1}^s r_i v_i \mid r_i \geq 0 \}$, where $v_1, \ldots, v_s \in \mathbb{R}^d$. We denote it by $\sigma = \text{cone}(v_1, \ldots, v_s)$. Note that $\{0\}$ is regarded as a convex polyhedral cone. We collect some notions concerning convex polyhedral cones. Let $\sigma$ be a convex polyhedral cone.

- The dimension of $\sigma$ is the dimension of the linear space generated by $\sigma$.
- We say that $\sigma$ is strongly convex if $\sigma \cap (-\sigma) = \{0\}$ holds, i.e., $\sigma$ does not contain a linear subspace of positive dimension.
We call \( \sigma \) rational if each \( v_i \) can be taken from \( \mathbb{Q}^d \).

We denote by \( \langle \cdot, \cdot \rangle \) the usual inner product. A **supporting hyperplane** of \( \sigma \) is a hyperplane \( \{ v \in \sigma \mid \langle u, v \rangle = 0 \} \) in \( \mathbb{R}^d \) given by some \( u \in \mathbb{R}^d \) satisfying \( \sigma \subset \{ v \in \mathbb{R}^d \mid \langle u, v \rangle \geq 0 \} \).

A face \( \tau \) of \( \sigma \) is the intersection of \( \sigma \) with a supporting hyperplane of \( \sigma \).

If a face \( \tau \) is maximal with respect to the inclusion, then \( \tau \) is called a **facet** of \( \sigma \).

In what follows, a cone means a strongly convex rational polyhedral cone for short.

**Definition 2.1.** A fan \( \Sigma \in \mathbb{R}^d \) is a collection of cones in \( \mathbb{R}^d \) such that

(a) each face of a cone in \( \Sigma \) is also contained in \( \Sigma \), and

(b) the intersection of two cones in \( \Sigma \) is a face of each of those two cones.

In this case, for each \( 0 \leq i \leq d \), we denote by \( \Sigma_i \) the subset of cones of dimension \( i \). For example, \( \Sigma_0 \) consists of the trivial cone \( \{ 0 \} \). We call each element in \( \Sigma_1 \) a ray of \( \Sigma \). We collect some notions concerning fans used in this paper. Let \( \Sigma \) be a fan in \( \mathbb{R}^d \).

- We call \( \Sigma \) **finite** if it consists of a finite number of cones.
- We call \( \Sigma \) **complete** if \( \bigcup_{\sigma \in \Sigma} \sigma = \mathbb{R}^d \).
- We call \( \Sigma \) **nonsingular** (or **smooth**) if each maximal cone in \( \Sigma \) is generated by a \( \mathbb{Z} \)-basis for \( \mathbb{Z}^d \).

We prepare some notions which will be used in this paper.

**Definition 2.2.** Let \( \Sigma \) and \( \Sigma' \) be fans in \( \mathbb{R}^d \) and \( \mathbb{R}^d' \) respectively. An **isomorphism** \( \Sigma \cong \Sigma' \) of fans is an isomorphism \( \mathbb{Z}^d \cong \mathbb{Z}^d' \) of abelian groups such that the induced linear isomorphism \( \mathbb{R}^d \to \mathbb{R}^{d'} \) gives a bijection \( \Sigma \cong \Sigma' \) between cones.

**Definition 2.3.** Let \( \Sigma \) be a finite fan in \( \mathbb{R}^d \), and let \( \sim \) be an equivalence relation on \( \Sigma_d \). We say that \( \sim \) **coarsens** \( \Sigma \) if, for each \( \sigma \in \Sigma_d \), the set \( [\sigma] := \bigcup_{\tau \sim \sigma} \tau \) is convex. In this case, we define a fan \( \Sigma/\sim \) called the **coarsening** of \( \Sigma \) by

\[
\Sigma/\sim := \{ [\sigma_1] \cap \cdots \cap [\sigma_s] \mid s \geq 1, \sigma_1, \ldots, \sigma_s \in \Sigma_d \}.
\]

A polytope \( P \) is a convex hull of a finite subset \( K \) of \( \mathbb{R}^d \). It is called **lattice polytope** if \( K \) is contained in \( \mathbb{Z}^d \). A **supporting hyperplane** of \( P \) is a hyperplane \( \{ v \in P \mid \langle u, v \rangle = a \} \) in \( \mathbb{R}^d \) given by some \( u \in \mathbb{R}^d \) and \( a \in \mathbb{R} \) satisfying \( P \subset \{ v \in \mathbb{R}^d \mid \langle u, v \rangle \geq a \} \). A face of \( P \) is the intersection of \( P \) with a supporting hyperplane of \( P \). A maximal face is called a **facet** of \( P \).

For lattice polytopes \( P \) and \( P' \) in \( \mathbb{Z}^d \) and \( \mathbb{Z}^{d'} \) respectively, an **isomorphism** \( P \cong P' \) of lattice polytopes is an isomorphism \( \mathbb{Z}^d \cong \mathbb{Z}^{d'} \) of abelian groups such that the induced linear isomorphism \( \mathbb{R}^d \to \mathbb{R}^{d'} \) gives a bijection \( P \cong P' \).

**Definition 2.4.** Let \( P \) be a polytope in \( V = \mathbb{R}^d \), and \( V^* \) the dual space of \( V \). For each face \( F \) of \( P \), let \( F_1, \ldots, F_s \) be all facets of \( P \) containing \( F \). For each \( 1 \leq i \leq s \), let \( v_i \in V^* \) be an outer normal vector of \( F_i \), and let

\[
\sigma_F := \text{cone}(v_1, \ldots, v_s).
\]

The normal fan of \( P \) is

\[
\Sigma(P) := \{ \sigma_F \mid F \text{ is a face of } P \}.
\]

If \( P \) has dimension \( d \), then \( \Sigma(P) \) is a finite complete fan in \( \mathbb{R}^d \). Otherwise, the cones of \( \Sigma(P) \) are not strongly convex.

Each element \( f \in V^* \) gives a face of \( P \):

\[
P_f := \{ v \in P \mid f(v) = \max f(P) \}.
\]

For each face \( F \) of \( P \), the corresponding cone \( \sigma_F \in \Sigma(P) \) can be written as

\[
\sigma_F = \{ f \in V^* \mid P_f \supseteq F \}.
\]

Then each \( f \in \sigma_F^\perp \) satisfies \( P_f = F \).
2.2. Preliminaries on tilting theory. We recall basic results on silting theory from [AIR].

We refer to [AIR] for mutation in more general setting. First we recall the definition of 2-term silting objects/complexes.

Definition 2.5. Let \( \mathcal{F} \) be a Krull-Schmidt triangulated category

(a) An object \( T \in \mathcal{F} \) is called presilting if \( \text{Hom}_\mathcal{F}(T,T[1]) = 0 \) for all positive integers \( \ell \).

(b) An object \( T \in \mathcal{F} \) is called silting if it is presilting and \( \mathcal{F} = \text{thick} \, T \).

(c) We denote by \( \text{silt} \, \mathcal{F} \) (respectively, \( \text{psilt} \, \mathcal{F} \)) the set of isomorphism classes of basic silting (respectively, presilting) objects of \( \mathcal{F} \).

(d) For \( T, U \in \text{silt} \, \mathcal{F} \), we write \( T \geq U \) if \( \text{Hom}_\mathcal{F}(T,U[\ell]) = 0 \) holds for all positive integers \( \ell \). Then \( (\text{silt} \, \mathcal{F}, \geq) \) is a partially ordered set [AIR].

(e) For \( T \in \text{silt} \, \mathcal{F} \), let

\[
\begin{align*}
2_T - \text{silt} \, \mathcal{F} & := \{ U \in \text{silt} \, \mathcal{F} \mid T \geq U \geq T[1] \}, \\
2_T - \text{psilt} \, \mathcal{F} & := \{ V \in \mathcal{F} \mid \exists U \in 2_T - \text{silt} \, \mathcal{F} \text{ such that } V \in \text{add} \, U \}, \\
\text{silt} \, A := \text{silt} \, \mathcal{F}, \text{ psilt} \, A := \text{psilt} \, \mathcal{F}, \text{ 2-silt} \, A := 2_A - \text{silt} \, \mathcal{F} \text{ and } 2 - \text{psilt} \, A := 2_A - \text{psilt} \, \mathcal{F}.
\end{align*}
\]

(f) For a ring \( A \), let \( \mathcal{F} := K^b(\text{proj} \, A) \) and

\[
\text{silt} \, A := \text{silt} \, \mathcal{F}, \text{ psilt} \, A := \text{psilt} \, \mathcal{F}, \text{ 2-silt} \, A := 2_A - \text{silt} \, \mathcal{F} \text{ and } 2 - \text{psilt} \, A := 2_A - \text{psilt} \, \mathcal{F}.
\]

We apply the same definitions for a non-positive dg ring \( A \) and \( \mathcal{F} := \text{per} \, A \).

Note that 2-psilt\( A \) consists of 2-term complexes \( T = (T^i, d^i) \), that is, \( T^i = 0 \) for all \( i \neq 0, -1 \). Moreover, \( T \in 2 \text{-psilt} \, A \) is silting if and only if \( |T| = |A| \) holds.

Later we use the following basic fact.

Proposition 2.6. [BY Theorem A.7] Let \( \mathcal{F} \) be a Krull-Schmidt algebraic triangulated category.

For \( T \in \text{silt} \, \mathcal{F} \), let \( A := \text{End}_\mathcal{F}(T) \). Then there exists a triangle functor \( F : \mathcal{F} \to K^b(\text{proj} \, A) \) which sends \( T \) to \( A \) and gives an isomorphism \( K_0(\mathcal{F}) \simeq K_0(\text{proj} \, A) \) and bijections

\[
2_T - \text{silt} \, \mathcal{F} \cong 2_A - \text{silt} \, \mathcal{F} \text{ and } 2_T - \text{psilt} \, \mathcal{F} \cong 2_A - \text{psilt} \, \mathcal{F}.
\]

Moreover, the bijection \( 2_T - \text{silt} \, \mathcal{F} \cong 2_A - \text{silt} \, \mathcal{F} \) commutes with mutation, and the functor \( F \) sends exchange triangles in \( 2_T - \text{silt} \, \mathcal{F} \) to those in \( 2_A - \text{silt} \, \mathcal{F} \).

Proof. This is [BY Theorem A.7]. The last assertion follows from [BY Proposition A.6] and its proof. \( \square \)

A typical setting of Proposition 2.6 is the following.

Example 2.7. Let \( A \) be a non-positive dg ring \( A \), and \( B := H^0(\mathcal{A}) \). Then there exists a triangle functor \( F : \text{per} \, A \to K^b(\text{proj} \, B) \) which sends \( A \) to \( B \) and gives an isomorphism \( K_0(\text{per} \, A) \cong K_0(\text{proj} \, B) \) and bijections

\[
2 \text{-silt} \, A \cong 2 \text{-silt} \, B \text{ and } 2 \text{-psilt} \, A \cong 2 \text{-psilt} \, B.
\]

In the rest, let \( A \) be a finite dimensional algebra over a field \( k \). The subposet \( (\text{silt} \, A, \geq) \) of \( (\text{silt} \, A, \succeq) \) plays a central role in this paper.

Recall that the Hasse quiver Hasse \( P \) of a poset \( P \) has the set \( P \) of vertices, and an arrow \( x \to y \) if \( x > y \) and there does not exist \( z \in P \) satisfying \( x > z > y \). It is known that Hasse(2-silt\( A \)) is \( n \)-regular for \( n := |A| \). More precisely, let \( T = T_1 \oplus \cdots \oplus T_n \in 2 \text{-silt} \, A \) with indecomposable \( T_i \). For each \( 1 \leq i \leq n \), there exists precisely one \( T' \in 2 \text{-silt} \, A \) such that \( T' = T'_i \oplus (\bigoplus_{j \neq i} T'_j) \) for some \( T'_i \neq T_i \). In this case, we call \( T' \) mutation of \( T \) at \( T_i \) and write

\[
T' = \mu_i(T).
\]

In this case, either \( T > T' \) or \( T' < T \) holds. The following result is fundamental in silting theory.

Proposition 2.8. Let \( A \) be a finite dimensional algebra over a field \( k \), and \( T, T' \in 2 \text{-silt} \, A \). Take a decomposition \( T = T_1 \oplus \cdots \oplus T_n \) with indecomposable \( T_i \). Then the following conditions are equivalent.
There is an arrow $X$.

We denote by $s$.

The triangles in (c) and (d) are isomorphic, and called an exchange triangle.

We often identify 2-term silting complexes with support $\tau$-tilting modules via the following bijection.

**Proposition 2.10.** [AIR Theorem 3.2] Let $A$ be a finite dimensional algebra over a field $k$, $\tau$ the Auslander-Reiten translation of $A$, and $X \in \mod A$.

(a) $X$ is called $\tau$-rigid if $\operatorname{Hom}_{A}(X, \tau X) = 0$.
(b) $X$ is called $\tau$-tilting if it is $\tau$-rigid and satisfies $|X| = |A|$.
(c) $X$ is called support $\tau$-tilting if there exists an idempotent $e \in A$ such that $X$ is a $\tau$-tilting $A/e$-module.
(d) We denote by $s\tau$-tilt $A$ the set of isomorphism classes of basic support $\tau$-tilting $A$-modules.

We often identify 2-term silting complexes with support $\tau$-tilting modules via the following bijection.

There is a strong connection between 2-term silting complexes and some important subcategories defined as follows. See section 3 for more details.

**Definition 2.11.** Let $A$ be a finite dimensional algebra over a field $k$. A full subcategory $\mathcal{C}$ of $\mod A$ is called a torsion class (respectively, torsionfree class) if it is closed under extensions and factor modules (respectively, submodules). It is called functorially finite if there exists $M \in \mathcal{C}$ satisfying $\mathcal{C} = \operatorname{Fac} M$ (respectively, $\mathcal{C} = \operatorname{Sub} M$). We denote by $\operatorname{tors} A$ (respectively, $\operatorname{f-tors} A$, $\operatorname{torf} A$, $\operatorname{f-torf} A$) the set of torsion classes (respectively, functorially finite torsion classes, torsionfree classes, functorially finite torsionfree classes) in $\mod A$.

We have mutually inverse bijections

$$\operatorname{tors} A \simeq \operatorname{torf} A, \quad \mathcal{C} \mapsto \mathcal{C}^{\perp} \quad \text{and} \quad \operatorname{torf} A \simeq \operatorname{tors} A, \quad \mathcal{C} \mapsto \perp \mathcal{C}. \quad (2.1)$$

A pair $(\mathcal{T}, \mathcal{F}) \in \operatorname{tors} A \times \operatorname{torf} A$ is called a torsion pair if $\mathcal{T} = \mathcal{F}^{\perp}$, or equivalently, $\mathcal{F} = \perp \mathcal{F}$. In this case, we have functors

$$t_{\mathcal{T}} : \mod A \to \mathcal{T} \quad \text{and} \quad f_{\mathcal{T}} : \mod A \to \mathcal{F}$$

such that each $X \in \mod A$ admits an exact sequence

$$0 \to t_{\mathcal{T}} X \to X \to f_{\mathcal{T}} X \to 0. \quad (2.2)$$

The bijections (2.1) restrict to bijections

$$\operatorname{f-tors} A \simeq \operatorname{f-torf} A \quad \text{and} \quad \operatorname{f-torf} A \simeq \operatorname{f-tors} A.$$

The following bijection is also important.
Definition-Proposition 2.12. [AIR] Proposition 1.2(b), Lemma 3.4] Let \( A \) be a finite dimensional algebra over a field \( k \).
(a) We have surjections
\[
2\text{-silt}A \to f\text{-tors}A, \quad U \mapsto \mathcal{F}_U := \text{Fac} H^0(U), \\
2\text{-silt}A \to f\text{-tors}A, U \mapsto \mathcal{F}_U := H^{-1}(\nu U), \\
2\text{-silt}A \to f\text{-torf}A, U \mapsto \mathcal{F}_U := \text{Sub} H^{-1}(\nu U), \\
2\text{-silt}A \to f\text{-torf}A, U \mapsto \mathcal{F}_U := H^0(U)^{\perp}
\]
such that \( (\mathcal{F}_U, \mathcal{F}_U) \) and \( (\mathcal{F}_U, \mathcal{F}_U) \) form torsion pairs. Thus each \( X \in \text{mod} A \) admits exact sequences
\[
0 \to t_U X \to X \to t_U X \to 0 \text{ for } t_U X := t_{\mathcal{F}_U} X \text{ and } t_U X := f_{\mathcal{F}_U} X, \\
0 \to f_U X \to X \to f_U X \to 0 \text{ for } f_U X := t_{\mathcal{F}_U} X \text{ and } f_U X := f_{\mathcal{F}_U} X.
\]
(b) We regard \( f\text{-tors}A \) and \( f\text{-torf}A \) as posets with respect to the inclusion relation. Then the first two surjections in (a) restrict to the same isomorphism of posets
\[
2\text{-silt}A \cong f\text{-tors}A, \quad T \mapsto \mathcal{F}_T = \mathcal{F}_T.
\]
The last two surjections in (a) restrict to the same anti-isomorphism of posets
\[
2\text{-silt}A \cong f\text{-torf}A, \quad T \mapsto \mathcal{F}_T = \mathcal{F}_T.
\]
The following finiteness condition plays a central role in this paper.

Definition 2.13. Let \( A \) be a finite dimensional algebra over a field \( k \). We say that \( A \) is \( g \)-finite if \( \#2\text{-silt}A < \infty \). (This is called \( \tau \)-tilting finite in [DLJ].)

We give a characterization of \( g \)-finiteness.

Proposition 2.14. [DLJ] Theorem 1.2] Let \( A \) be a finite dimensional algebra over a field \( k \). Then \( A \) is \( g \)-finite if and only if \( \text{tors}A = f\text{-tors}A \) if and only if \( \text{torf}A = f\text{-torf}A \).

There is a strong connection between 2-term silting complexes and the following class of modules.

Definition 2.15. Let \( A \) be a finite dimensional algebra over a field \( k \).
(a) An object \( X = X_1 \oplus \cdots \oplus X_r \in \text{mod} A \) is called a \emph{semibrick} if
\[
\text{Hom}_A(X_i, X_j) = \begin{cases} 
\text{division ring} & (i = j) \\
0 & (j \neq i).
\end{cases}
\] (2.3)

We denote by \( \text{sbrick}A \) the set of isomorphism classes of semibricks in \( \text{mod} A \).
(b) A full subcategory \( \mathcal{C} \) of \( \text{mod} A \) is called \emph{wide} if it is closed under extensions, kernels and cokernels. We denote by \( \text{wide}A \) the set of wide subcategories of \( \text{mod} A \).

Note that the usual definition of a semibrick is more general: a (possibly infinite) set of modules satisfying \( \text{[2.3]} \) \( \text{[A1]} \). In this paper, we only need to consider semibricks in the sense above. It is basic that there is a bijection
\[
\text{sbrick}A \cong \text{wide}A
\]
sending \( X \) to the smallest extension closed subcategory containing \( \text{add} X \) \( \text{[R]} \).

The following notion is a derived category version of semibricks.

Definition 2.16. Let \( A \) be a finite dimensional algebra over a field \( k \). An object \( X = X_1 \oplus \cdots \oplus X_r \in \text{D}^b(\text{mod} A) \) is called a \emph{simple-minded collection} if the following conditions hold.
- \( \text{Hom}_{\text{D}^b(\text{mod} A)}(X, X[\ell]) = 0 \) for all negative integers \( \ell \).
- For \( 1 \leq i, j \leq r \), \( \text{Hom}_{\text{D}^b(\text{mod} A)}(X_i, X_j) = \begin{cases} 
\text{division ring} & (i = j) \\
0 & (j \neq i).
\end{cases} \)
- \( \text{D}^b(\text{mod} A) = \text{thick} X \).
Note that \( r = |A| \) holds in this case. A simple-minded collection \( X \) is called 2-term if \( H^i(X) = 0 \) holds for all integers \( i \neq -1, 0 \). We denote by \( \smc(A) \) (respectively, \( 2\smc(A) \)) the set of isomorphism classes of simple-minded collections (respectively, 2-term simple-minded collections) on \( \mathcal{D}^b(\text{mod } A) \).

We have the following silting-t-structure correspondence.

**Proposition 2.17.** [KY] Let \( A \) be a finite dimensional algebra over a field \( k \) and \( n = |A| \). Then there exists a bijection between \( \silt A \) and \( \smc(A) \) such that \( T = T_1 \oplus \cdots \oplus T_n \in \silt A \) and the corresponding \( S = S_1 \oplus \cdots \oplus S_n \in \smc A \) satisfy

\[
\text{Hom}_{\mathcal{D}^b(\text{mod } A)}(T_i, S_j[p]) = \begin{cases} 
\text{End}_{\mathcal{D}^b(\text{mod } A)}(S_i) & (i = j \text{ and } p = 0) \\
0 & \text{(otherwise).}
\end{cases}
\]

In particular, we have

\[
([T_i], [S_j]) = \delta_{ij} \cdot \dim_k \text{End}_{\mathcal{D}^b(\text{mod } A)}(S_j).
\]

3. \( g \)-simplicial complexes

In this section, we introduce \( g \)-simplicial complexes and study their basic properties. In particular, we give a representation theoretic interpretations of their \( h \)-vectors. Moreover we give a proof of Dehn-Sommerville equations in terms of the representation theory.

Throughout this section, let \( A \) be a finite dimensional algebra over a field \( k \).

**Definition 3.1.** For \( j \geq 0 \), let \( 2\psilt^j A \) be the set of isomorphism classes of basic 2-term presilting complexes \( T \) such that \( |T| = j \). We define a simplicial complex \( \Delta(A) \) called the \( g \)-simplicial complex of \( A \) as follows: The set of \( j \)-simplices is \( 2\psilt^j A \).

We give the following basic properties.

**Proposition 3.2.** Let \( A \) be a finite dimensional algebra over a field \( k \) and \( n := |A| \).

(a) \( \Delta(A) \) is pure of dimension \( n - 1 \), that is, all facets of \( \Delta(A) \) have dimension \( n - 1 \).

(b) Each face of dimension \( n - 2 \) in \( \Delta(A) \) is contained in precisely two facets.

(c) \( \Delta(A) \) is flag, that is, each minimal subset which is not a face of \( \Delta(A) \) consists of two points.

**Proof.** For (a) and (b), see [DIJ]. (c) For distinct elements \( T_1, \ldots, T_j \in 2\psilt^1 A \), their direct sum belongs to \( 2\psilt^j A \) if and only if \( \text{Hom}_{\mathcal{D}^b(\text{proj } A)}(T_i, T_j[1]) = 0 \) for each \( 1 \leq i \neq j \leq j \) if and only if \( T_i \oplus T_j \in 2\psilt^2 A \) for each \( 1 \leq i \neq j \leq j \).

Now we assume that \( A \) is \( g \)-finite. For \( n := |A| \), we denote by

\[
(f_{-1}, f_0, \ldots, f_{n-1}) \quad \text{and} \quad (h_0, h_1, \ldots, h_n)
\]

the \( f \)-vector and the \( h \)-vector of the \( g \)-simplicial complex \( \Delta(A) \). Thus

\[
f_{-1} := 1 \quad \text{and} \quad f_j := \#2\psilt^{j+1} A
\]

is the number of the \( j \)-simplices in \( \Delta(A) \) for \( j \geq 0 \), and \( h_j \) is defined by

\[
h_j = \sum_{i=0}^j (-1)^{j-i} \binom{n-i}{j-i} f_{i-1} \quad \text{for} \quad 0 \leq j \leq n.
\]

In other words, the \( f \)-polynomial and \( h \)-polynomial

\[
f(x) := \sum_{i=0}^n f_{i-1} x^{n-i} \quad \text{and} \quad h(x) := \sum_{i=0}^n h_i x^{n-i}
\]

are related by \( h(x) = f(x - 1) \). Thus the equations [3.1] are equivalent to

\[
f_{j-1} = \sum_{i=0}^j \binom{n-i}{j-i} h_i \quad \text{for} \quad 0 \leq j \leq n,
\]
which recover the $f$-vector from the $h$-vector. We give a representation theoretic meaning of the $f$- and $h$-vectors.

**Definition 3.3.** Let $A$ be a finite dimensional algebra over a field $k$ and $j \geq 0$.

(a) Let $2$-$\text{silt}_j A$ be the set of all $T \in 2$-$\text{silt} A$ such that there exist precisely $j$ arrows starting at $T$ in Hasse($2$-$\text{silt} A$).

(b) Let $\text{sbrick}_j A$ be the set of all $S \in \text{sbrick} A$ such that $|S| = j$.

The following is the first main result in this section.

**Theorem 3.4.** Let $A$ be a finite dimensional algebra over a field $k$ which is $g$-finite. Then for each $0 \leq j \leq |A|$, we have

$$h_j = \#2$-$\text{silt}_j A = \#\text{sbrick}_j A.$$

To prove Theorem 3.4, we need preparations. In the rest, let $A$ be a finite dimensional algebra over a field $k$ (which is not necessarily $g$-finite).

First, we recall the following well-known fact (a), see e.g. [AS, Corollary 2.4(a)].

**Definition-Proposition 3.5.** Let $A$ be a finite dimensional algebra over a field $k$.

(a) Each $X \in \text{mod} A$ has a direct summand $X_{\text{gen}}$ such that a direct summand $Y$ of $X$ satisfies $\text{Fac} X = \text{Fac} Y$ if and only if $Y$ has a direct summand which is isomorphic to $X_{\text{gen}}$.

(b) For a 2-term complex $T \in K^b(\text{proj} A)$, we denote by $T_{\text{gen}}$ a minimal direct summand of $T$ such that $H^0(T)_{\text{gen}} = H^0(T_{\text{gen}})$.

Clearly $X_{\text{gen}}$ and $T_{\text{gen}}$ are unique up to isomorphism.

**Example 3.6.** Consider a finite dimensional algebra $A$ given by

$$Q = \begin{bmatrix} 1 & a & 2 \\ 0 & 2 & b \\ 3 & d & 3 \end{bmatrix} \quad \text{and} \quad A := kQ/\langle ab, dc, ca - bd \rangle.$$

Then $A = P_1 \oplus P_2 \oplus P_3 = \frac{1}{3} \oplus \frac{2}{2} \oplus \frac{3}{3}$. For example, we can take a 2-term silting complex

$$T := [0 \rightarrow P_1] \oplus [P_2 \rightarrow P_1 \oplus P_3] \oplus [0 \rightarrow P_3].$$

Then $H^0(T) = \frac{1}{1} \oplus \frac{1}{2} \oplus \frac{3}{2} \oplus \frac{3}{3}$ and

$$\left(\frac{1}{1} \oplus \frac{1}{2} \oplus \frac{3}{2} \oplus \frac{3}{3}\right)_{\text{gen}} = \frac{1}{1} \oplus \frac{3}{3}.$$

Thus we have $T_{\text{gen}} = [0 \rightarrow P_1] \oplus [0 \rightarrow P_3]$. For $T \in 2$-$\text{silt} A$, we have the following description of $T_{\text{gen}}$ in terms of Hasse($2$-$\text{silt} A$).

**Lemma 3.7.** (a) For $T = T_1 \oplus \cdots \oplus T_n \in 2$-$\text{silt} A$ with indecomposable $T_i$, we have

$$T_{\text{gen}} = \bigoplus_{T > \mu_i(T)} T_i.$$

(b) $2$-$\text{silt}_j A = \{T \in 2$-$\text{silt} A \mid |T_{\text{gen}}| = j\}$.

**Proof.** (a) It is immediate from definition of $T_{\text{gen}}$ that $T_i$ is a direct summand of $T_{\text{gen}}$ if and only if $H^0(T_i) \not\subseteq \text{Fac} H^0(T/T_i)$. This is clearly equivalent to $T > \mu_i(T)$ (see the proof of [IRRT, Theorem 2.7]). Thus the assertion holds.

(b) Immediate from (a). \qed

Recall from Definition-Proposition 2.12 that each $U \in 2$-$\text{psilt} A$ gives torsion classes

$$\mathcal{T}_U = \text{Fac} H^0(U) \subseteq \mathcal{T}_U = \frac{1}{H^{-1}(\mu U)}$$

such that the equality holds if $U \in 2$-$\text{silt} A$. We need the following notions.

**Definition-Proposition 3.8.** Let $U \in 2$-$\text{psilt} A$. 
(a) We call \( T \in 2\text{-silt} A \) a completion of \( U \) if it satisfies \( U \in \text{add} T \).
(b) A completion \( T \) of \( U \) is called minimal if
\[
\mathcal{R}_T = \mathcal{R}_U.
\]
Then \( U \) has a unique minimal completion up to isomorphism, which we denote by \( U_{\min} \).
(c) A completion \( T \) of \( U \) is called maximal (or Bongartz) if
\[
\mathcal{R}_T = \overline{\mathcal{R}_U}.
\]
Then \( U \) has a unique maximal completion up to isomorphism, which we denote by \( U_{\max} \).
(d) (Jasso reduction) We have
\[
U' := \mathcal{F}_U \cap \mathcal{F}_U \in \text{wide} A.
\]
Define a functor
\[
w_U : \text{mod} A \to U_U \quad \text{by} \quad w_U X := U_U X/\tau_U X = t_{\max} X/t_{\min} X.
\]
Let \([U]\) be an ideal of \( \text{End}_{K^0(\text{proj} A)}(U_{\max}) \) consisting of all morphisms factoring through objects in \( \text{add} U \), and
\[
B := \text{End}_{K^0(\text{proj} A)}(U_{\max})/[U].
\]
Then \( |B| = |A| - |U| \) holds, and there exists an equivalence
\[
\text{mod} B \cong U_U \subseteq \text{mod} A
\]
which induces an injective homomorphism \( K_0(\text{mod} B) \subseteq K_0(\text{mod} A) \).

**Proof.** (b) By Definition-Proposition \((2.12a)\), \( \mathcal{R}_U \in \text{f-tors} A \) holds. By Definition-Proposition \((2.12c)\), there exists unique \( T \in 2\text{-silt} A \) satisfying \( \mathcal{R}_T = \mathcal{R}_U \).
(c) This is shown similarly.
(d) This is \([1]\) Theorem 3.8, see also \([\text{DIRRT}]\) Theorem 4.12.

**Lemma 3.9.** Let \( U \in 2\text{-silt} A \). Then we have
\[
(U_{\min})_{\text{gen}} \simeq U_{\gen} \text{ and } (U_{\gen})_{\min} \simeq U_{\min}.
\]

**Proof.** Since \( U_{\min} \) has \( U \) as a direct summand and \( \mathcal{R}_{U_{\min}} = \mathcal{R}_U \), we have \((U_{\min})_{\text{gen}} \simeq U_{\gen} \). Since \( \mathcal{R}_{U_{gen}} = \mathcal{R}_{gen} = \mathcal{R}_U = \mathcal{R}_{U_{min}} \) holds, we have \((U_{gen})_{\min} \simeq U_{\min} \) by Definition-Proposition \((2.12c)\).

**Definition-Proposition 3.10.** For \( 0 \leq j \leq |A| \), let
\[
2\text{-psilt}_j A := \{ U \in 2\text{-psilt}^1 A \mid U_{\min} \in 2\text{-silt}_j A \}
\]
(3.3)
\[
= \{ U \in 2\text{-psilt}^1 A \mid U_{\gen} = U \}. \quad (3.4)
\]

**Proof.** By Lemma \((3.7)\) \( U_{\min} \in 2\text{-silt} A \) holds if and only if \(|(U_{\min})_{\text{gen}}| = j \) holds. By Lemma \((3.9)\) this is equivalent to \(|U_{\gen}| = j \). Since \(|U| = j \) holds and \( U \) has \( U_{\gen} \) as a direct summand, this is equivalent to \( U_{\gen} \simeq U \).

Now we are ready to prove two key results. The first one is the following.

**Theorem 3.11.** Let \( A \) be a finite dimensional algebra over a field \( k \) and \( 0 \leq j \leq |A| \). Then we have bijections
\[
(-)_{\min} : 2\text{-psilt}_j A \simeq 2\text{-silt}_j A,
\]
whose inverse is given by \((-)_{\gen}\).

**Proof.** For each \( U \in 2\text{-psilt}_j A \), we have \( U_{\min} \in 2\text{-silt}_j A \) by \((3.7)\). Thus the map \((-)_{\min} : 2\text{-psilt}_j A \to 2\text{-silt}_j A \) is well-defined. Moreover, \((U_{\min})_{\text{gen}} \simeq U_{\gen} \simeq U \) holds by Lemma \((3.9)\) and \((3.3)\).

For each \( T \in 2\text{-silt}_j A \), let \( U := T_{\gen} \). Then \( U_{\gen} \simeq U \) holds, and moreover \(|U| = j \) holds by Lemma \((3.7)\). Thus \( U \in 2\text{-psilt}_j A \) holds by \((3.4)\), and the map \((-)_{\gen} : 2\text{-silt}_j A \to 2\text{-psilt}_j A \) is well-defined. By Lemma \((3.9)\) we have \((T_{\gen})_{\min} \simeq T_{\min} = T \). Thus the assertion follows.

The second one is the following.
Theorem 3.12. Let $A$ be a finite dimensional algebra over a field $k$ and $0 \leq j \leq |A|$. Then we have a bijection
\[
2\text{-psilt}^jA \cong \bigoplus_{i=0}^j \{(V, W) \in 2\text{-psilt}_{i}A \times 2\text{-psilt}^{j-i}A \mid W \text{ is a direct summand of } V_{\text{min}}/V\}
\] (3.5)
given by $U \mapsto (U_{\text{gen}}, U/U_{\text{gen}})$, and the converse is given by $(V, W) \mapsto V \oplus W$.

Proof. For $U \in 2\text{-psilt}^jA$, let $(V, W) := (U_{\text{gen}}, U/U_{\text{gen}})$ and $i := |U_{\text{gen}}|$. Then $0 \leq i \leq j$ and $V_{\text{gen}} = V$ hold, and hence $V \in 2\text{-psilt}_{i}A$. Clearly $W \in 2\text{-psilt}^{j-i}A$ holds. Since $V_{\text{min}} = (U_{\text{gen}})_{\text{min}} \cong V_{\text{min}}$ holds by Lemma 3.9, $V_{\text{min}}/V \cong U_{\text{gen}}/U_{\text{gen}}$ has $W = U/U_{\text{gen}}$ as a direct summand. Thus the map $U \mapsto (U_{\text{gen}}, U/U_{\text{gen}})$ is well-defined. It is injective since $U \cong U_{\text{gen}} \oplus (U/U_{\text{gen}})$ holds.

To prove surjectivity, take $(V, W) \in 2\text{-psilt}_{i}A \times 2\text{-psilt}^{j-i}A$ such that $W$ is a direct summand of $V_{\text{min}}/V$, and let $U := V \oplus W \in 2\text{-psilt}^jA$. Since $T_V \subseteq T_U \subseteq T_{V_{\text{min}}} = T_U$ holds, we have $T_V = T_U$. Since $V_{\text{gen}} = V$ holds by (3.4), we have $U_{\text{gen}} \cong V$ and hence $(U_{\text{gen}}, U/U_{\text{gen}}) \cong (V, W)$. \qed

We need the following preparation on semibricks.

Definition 3.13. Let $S$ be a semibrick of $A$.
(a) We call $S$ left-finite if the smallest torsion class containing $S$ is functorially finite. We denote by $f_l\text{-sbrick}A$ the set of isomorphism classes of left-finite semibricks of $A$.
(b) We call $S$ right-finite if the smallest torsionfree class containing $S$ is functorially finite. We denote by $f_r\text{-sbrick}A$ the set of isomorphism classes of right-finite semibricks of $A$.

If $A$ is $g$-finite, then all semibricks of $A$ are left-finite and right-finite by Proposition 2.14.

Proposition 3.14. [As1 Theorem 2.3] Let $A$ be a finite dimensional algebra over a field $k$ with $n := |A|$. Then we have the following bijections, where $\nu = - \otimes_A DA : K^b(\text{proj} A) \cong K^b(\text{inj} A)$.

\[
\begin{align*}
2\text{-silt}_iA & \cong f_l\text{-sbrick}_{n-i}A, & T \mapsto H^0(T)/\text{rad}_{\text{End}_A(H^0(T))}H^0(T). \\
2\text{-silt}_jA & \cong f_r\text{-sbrick}_{n-j}A, & T \mapsto \text{soc}_{\text{End}_A(H^{-1}(\nu T))}H^{-1}(\nu T).
\end{align*}
\]

We are ready to prove Theorem 3.4.

Proof of Theorem 3.4 By Theorem 3.12 $f_{j-1} = #2\text{-psilt}^jA$ is equal to the cardinality of the right-hand side of (3.5). For each $V \in 2\text{-psilt}_iA$, there are $\binom{n-i}{j-i}$ choices of $W$. Thus the equality
\[
f_{j-1} = #2\text{-psilt}^jA = \sum_{i=0}^j \binom{n-i}{j-i} #2\text{-psilt}_iA
\]
holds. Comparing with (3.2), we obtain
\[
h_j = #2\text{-silt}_jA,
\]
which is equal to $#2\text{-silt}_jA$ by Theorem 3.11. Finally, $#2\text{-silt}_jA = f_l\text{-sbrick}_jA = f_r\text{-sbrick}_jA$ holds by Proposition 3.14. \qed

Since the $g$-simplicial complex $\Delta(A)$ is a simplicial sphere [DL], Dehn-Sommerville equations
\[
h_j = h_{n-j}
\]
are satisfied [V Theorem 6.8.8] (see also [Za Theorem 8.21]). Our next result categorifies these equations by giving a symmetry of the set 2-silt$A$.

Theorem 3.15. Let $A$ be a finite dimensional algebra over a field $k$ which is $g$-finite. For $0 \leq j \leq n := |A|$, there is a canonical bijection
\[
\text{sbrick}_jA \simeq \text{sbrick}_{n-j}A \text{ and } 2\text{-silt}_jA \simeq 2\text{-silt}_{n-j}A.
\]
In particular, we have $h_j = h_{n-j}$. 

For the case \( j = 1 \), the bijection \( 2\text{-}\text{silt}_t A \cong 2\text{-}\text{silt}_{n-1} A \) between join-irreducible elements in \( 2\text{-}\text{silt} A \) and meet-irreducible elements in \( 2\text{-}\text{silt} A \) was shown in [DIRRT] (see also [HRRT]). To prove Theorem 3.15, we need the following result.

**Proposition 3.16.** [As1] Let \( A \) be a finite dimensional algebra over a field \( k \) with \( n := |A| \).

(a) There exist bijections

\[
H^0 : 2\text{-}\text{smc} A \cong f^L\text{-}\text{sbrick} A \quad \text{and} \quad H^{-1} : 2\text{-}\text{smc} A \cong f^R\text{-}\text{sbrick} A
\]

such that \( S = H^0(S) \oplus H^{-1}(S)[1] \) holds for each \( S \in 2\text{-}\text{smc} A \).

(b) The following diagram commutes.

\[
\begin{array}{ccc}
\text{Prop 3.14} & \cong & \text{Prop 3.14} \\
H^0 & \downarrow & H^{-1} \\
\text{f-L-sbrick} A & \cong & \text{f-R-sbrick} A.
\end{array}
\]

(c) Assume that \( T = T_1 \oplus \cdots \oplus T_n \in 2\text{-}\text{silt} A \) and \( S = S_1 \oplus \cdots \oplus S_n \in 2\text{-}\text{smc} A \) correspond to each other by the bijection in Proposition 3.14. For each \( 1 \leq i \leq n \), \( \mu_i(T) < T \) if and only if \( S_i \in \text{mod} A \) and \( \mu_i(T) > T \) if and only if \( S_i \in (\text{mod} A)[1] \).

**Proof.** (a) and (b) are [As1] Theorem 3.3. To prove (c), it suffices to show the first equivalence. By the left part of the commutative diagram in (b), we have \( H^0(S) = H^0(T)/\text{rad}_{\text{End}_A(H^0(T))} H^0(T) \) and hence

\[
H^0(S_i) = H^0(T_i)/\sum_{f \in \text{rad}_{\text{End}_A(H^0(T))}} \text{Im} f.
\]

Thus \( S_i \in \text{mod} A \) if and only if \( H^0(S_i) \neq 0 \) if and only if \( H^0(T_i) \notin \mathcal{T}_T/T_i \). By Lemma 3.7(a), this is equivalent to \( \mu_i(T) < T \).

We are ready to prove Theorem 3.15.

**Proof of Theorem 3.15.** It suffices to give a bijection \( \text{sbrick}_j A \cong \text{sbrick}_{n-j} A \). Since \( A \) is \( g \)-finite, Proposition 3.16(a) gives bijections \( H^0 : 2\text{-}\text{smc} A \cong \text{sbrick} A \) and \( H^{-1} : 2\text{-}\text{smc} A \cong \text{sbrick} A \) such that \( S = H^0(S) \oplus H^{-1}(S)[1] \) for each \( S \in 2\text{-}\text{smc} A \). Since \( \|H^0(S)\| + \|H^{-1}(S)\| = |S| = n \) holds, they give a bijection \( \text{sbrick}_j A \cong \text{sbrick}_{n-j} A \) for each \( j \).

Using unimodality results of \( h \)-vectors in combinatorics, we obtain the following result as an application. It will be an interesting question if there is a direct proof using tilting theory.

**Corollary 3.17.** Let \( A \) be a finite dimensional algebra over a field \( k \) which is \( g \)-finite, and \( n := |A| \). Then we have

\[
\#\text{sbrick}_1 A \leq \#\text{sbrick}_2 A \leq \cdots \leq \#\text{sbrick}_{\lfloor \frac{n}{2} \rfloor} A \leq \#\text{sbrick}_{\lceil \frac{n}{2} \rceil} A,
\]

\[
\#\text{sbrick}_{\lfloor \frac{n}{2} \rfloor} A \geq \#\text{sbrick}_{\lfloor \frac{n}{2} \rfloor + 1} A \geq \cdots \geq \#\text{sbrick}_{n-1} A \geq \#\text{sbrick}_n A.
\]

**Proof.** The unimodality of \( h \)-vectors was originally proved for boundary complexes of simplicial polytopes, and this is generalized for simplicial spheres (see [AD1]). Since \( \Delta(A) \) gives a simplicial sphere (see [DJ1] Theorem 5.4), the result follows from Theorem 3.14. □

There are several natural problems [PQ] in view of Corollary 3.17 and the fact that \( \Delta(A) \) is flag (Proposition 3.12(c)). The \( \gamma \)-vector \( (\gamma_0, \ldots, \gamma_{\lfloor \frac{n}{2} \rfloor}) \) is defined by the equality

\[
h(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_i x^i (1 + x)^{n-2i}.
\]

**Problem 3.18.** Let \( A \) be a finite dimensional algebra over a field \( k \) which is \( g \)-finite, and let \( h(x) := h(x) \) be the \( h \)-polynomial of \( \Delta(A) \).

(a) Is \( h(x) \) real-rooted (that is, all roots are real numbers)?

(b) Is \( h(x) \) log-concave (that is, \( h_i^2 \geq h_{i-1} h_{i+1} \) holds for each \( i \))?
(c) Is \( h(x) \) \( \gamma \)-nonnegative (that is, \( \gamma_i \geq 0 \) for each \( i \))? 

Gal's conjecture asks if each flag simplicial complex is \( \gamma \)-nonnegative. The following implications are known.

\[
\begin{align*}
\text{real-rooted} & \quad \longrightarrow \quad \text{log-concave} \\
\downarrow & \quad \quad \quad \quad \downarrow \\
\text{\( \gamma \)-nonnegative} & \quad \longrightarrow \quad \text{unimodal}
\end{align*}
\]

4. \( g \)-FANS

4.1. Definition and basic properties. We introduce the \( g \)-fan of a finite dimensional algebra. Let \( A \) be a finite dimensional algebra over a field \( k \). Let \( K_0(\text{proj} \ A) \) be the Grothendieck group of \( K^b(\text{proj} \ A) \) and

\[
K_0(\text{proj} \ A) \mathbb{R} := K_0(\text{proj} \ A) \otimes \mathbb{R} \cong \mathbb{R}^{[A]}.
\]

The \( g \)-simplicial complex has a canonical geometric realization as a fan in the real Grothendieck group \( K_0(\text{proj} \ A) \mathbb{R} \) [DL]. Now we introduce its fan version.

**Definition 4.1.** For \( T = T_1 \oplus \cdots \oplus T_j \in 2\text{-psilt}A \) with indecomposable \( T_i \), let

\[
C(T) := \{ \sum_{i=1}^j a_i[T_i] \mid a_1, \ldots, a_j \geq 0 \} \subset K_0(\text{proj} \ A) \mathbb{R}.
\]

We call the set

\[
\Sigma(A) := \{ C(T) \mid T \in 2\text{-psilt}A \}
\]

of cones the \( g \)-fan of \( A \).

Notice that \( \Sigma(A) \) can be an infinite set. For each \( 0 \leq i \leq |A| \), the subset \( \Sigma_i(A) \) of cones of dimension \( i \) is given by

\[
\Sigma_i(A) = \{ C(U) \mid U \in 2\text{-psilt}^iA \}.
\]

We give the following basic properties of \( g \)-fans.

**Proposition 4.2.** Let \( A \) be a finite dimensional algebra over a field \( k \) and \( n := |A| \).

(a) \( \Sigma(A) \) is a nonsingular fan in \( K_0(\text{proj} \ A) \mathbb{R} \).
(b) Any cone in \( \Sigma(A) \) is a face of a cone of dimension \( n \).
(c) Any cone in \( \Sigma(A) \) of dimension \( n - 1 \) is a face of precisely two cones of dimension \( n \).
(d) \( A \) is \( g \)-finite (or equivalently, \( \Sigma(A) \) is finite) if and only if \( \Sigma(A) \) is complete.

**Proof.** For (a), (b) and (c), see [DL] and Proposition 3.2 (d) is [AS2] Theorem 4.7. \( \square \)

**Example 4.3.** We give examples of \( g \)-fans of algebras of rank 2.

\[
\Sigma(k \times k) = \begin{array}{c}
\includegraphics{example1.png}
\end{array}, \quad \Sigma(k[1 \rightarrow 2]) = \begin{array}{c}
\includegraphics{example2.png}
\end{array}, \quad \Sigma(k[1 \begin{array}{c}
\frac{a}{b} \\
\cdot \\
\cdot \\
\cdot
\end{array} 2]/{ab, ba}) = \begin{array}{c}
\includegraphics{example3.png}
\end{array}
\]

\[
\Sigma(k[1 \begin{array}{c}
\frac{a}{b} \\
\cdot \\
\cdot \\
\cdot
\end{array} 2 \bigcup_c]/{c^2}) = \begin{array}{c}
\includegraphics{example4.png}
\end{array}, \quad \Sigma(k[1 \begin{array}{c}
\frac{a}{b} \\
\cdot \\
\cdot \\
\cdot
\end{array} 2 \bigcup_c]/{ab, ba, cb, c^2}) = \begin{array}{c}
\includegraphics{example5.png}
\end{array}
\]

\[
\Sigma(k[1 \begin{array}{c}
\frac{a}{b} \\
\cdot \\
\cdot \\
\cdot
\end{array} 2 \bigcup_d]/{ab, ba, ad, cb, c^2, d^2, cd, dc}) = \begin{array}{c}
\includegraphics{example6.png}
\end{array}
\]

\[
\Sigma(k[1 \begin{array}{c}
\frac{a}{b} \\
\cdot \\
\cdot \\
\cdot
\end{array} 2 \bigcup_d]/{ab, ba, da, cb, c^2, d^2}) = \begin{array}{c}
\includegraphics{example7.png}
\end{array}
\]

We give examples of more complicated \( g \)-fans.
**Example 4.4.** [Ka Proposition 6.1] For positive integers \( \ell \geq 1, m \geq 1 \), we define an algebra \( A := kQ/I \) as follows. The quiver \( Q \) is the following:

\[
\begin{array}{c}
\vdots \\
1 \\
\vdots
\end{array}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\vdots
\end{array}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\vdots
\end{array}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\vdots
\end{array}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\vdots
\end{array}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\vdots
\end{array}
\]

The ideal \( I \) of \( kQ \) is generated by the following elements for all possible \( i, j \):

\[ a_i a_j \quad (i - j \neq 1), \quad b_i b_j \quad (i - j \neq 1), \quad a_i b_j \quad \text{and} \quad b_i a_j. \]

Then Hasse(2-silt\( A_{\ell,m} \)) is

\[
A = P^{(0)} \oplus Q^{(0)} \xrightarrow{\cdot 1} P^{(1)} \oplus P^{(2)} \xrightarrow{\cdot 2} \cdots \xrightarrow{\cdot \ell} P^{(\ell-1)} \oplus P^{(\ell)} \xrightarrow{\cdot 1} A[1] = P^{(\ell)} \oplus Q^{(m)}
\]

with \( P^{(0)} = P_1 := e_1A, P^{(\ell)} = P_1[1], Q^{(0)} = P_2 := e_2A \) and \( Q^{(m)} = P_2[1] \). Since

\[ [P^{(i)}] = [P_1] - i[P_2] \quad (0 \leq i \leq \ell - 1) \quad \text{and} \quad [Q^{(j)}] = [P_2] - j[P_1] \quad (0 \leq j \leq m - 1), \]

the \( g \)-fan \( \Sigma(A) \) consists of

\[
\text{cone}([P_1], [P_2]), \quad \text{cone}([-P_1], [-P_2]), \\
\text{cone}([P_1] - (i - 1)[P_2], [P_1] - i[P_2]) \quad (1 \leq i \leq \ell - 1), \quad \text{cone}([P_1] - (\ell - 1)[P_2], [-P_2]), \\
\text{cone}([P_2] - (j - 1)[P_1], [P_2] - j[P_1]) \quad (1 \leq j \leq m - 1), \quad \text{cone}([P_2] - (m - 1)[P_1], [-P_1]).
\]

For example, if \( (\ell, m) = (4, 5) \), then \( \Sigma(A) \) is

---

**Example 4.5.** Consider a finite dimensional algebra \( A \) given by

\[
Q = \begin{bmatrix}
1 & 1 & 1 & a \\
2 & 3 & b & c \\
\end{bmatrix}
\quad \text{and} \quad A := kQ/(c^2 bd, abcd, cdab - dabc)
\]

which is a Brauer graph algebra (see Example 8.13). Then \( A = \frac{1}{3} \oplus \frac{2}{3} \oplus \frac{1}{3} \oplus \frac{1}{3} \oplus \frac{1}{3} \oplus \frac{1}{3} \oplus \frac{1}{3} \), and \( \Sigma(A) \) is given by the following.

---

The following general results give canonical isomorphisms between the \( g \)-fans of two algebras.
Proposition 4.6. Let $A$ be a finite dimensional algebra over a field $k$.

(a) The isomorphism $\text{Hom}_A(-, A) : K_0(\underline{\text{proj}} A) \simeq K_0(\underline{\text{proj}} A^{op})$ gives an isomorphism of $g$-fans:

$$\Sigma(A) \simeq \Sigma(A^{op}).$$

(b) Let $K/k$ be a field extension such that the functor $K \otimes_k - : \underline{\text{proj}} A \to \underline{\text{proj}} K \otimes_k A$ preserves the indecomposability. Then the isomorphism $K_0(\underline{\text{proj}} A) \simeq K_0(\underline{\text{proj}} K \otimes_k A)$ gives an isomorphism of $g$-fans:

$$\Sigma(A) \simeq \Sigma(K \otimes_k A).$$

Proof. (a) We have a duality $\underline{\text{RHom}}_A(-, A)[1] : K^b(\underline{\text{proj}} A) \simeq K^b(\underline{\text{proj}} A^{op})$, which gives the isomorphism $- \otimes_A B : K_0(\underline{\text{proj}} A) \simeq K_0(\underline{\text{proj}} A^{op})$ and bijections $2\text{ - silt} A \simeq 2\text{ - silt} A^{op}$ and $2\text{-psilt} A \simeq 2\text{-psilt} A^{op}$. Thus the assertion follows.

(b) The assertion follows from $[IK]$. \hfill \Box

We also study the $g$-fans of arbitrary factor algebras. Let $A$ be a finite dimensional algebra over a field $k$, and $B = A/I$ a factor algebra of $A$. Using the triangle functor $- \otimes_A B : K^b(\underline{\text{proj}} A) \to K^b(\underline{\text{proj}} B)$, we define an equivalence relation $\sim_B$ on $2\text{-silt} A$: For $T, U \in 2\text{-silt} A$, we write $T \sim_B U$ if $\text{add}(T \otimes_A B) = \text{add}(U \otimes_A B)$. Then we have the following description of $\Sigma(B)$, see Definition 2.3 for $\Sigma(A)/\sim_B$.

Proposition 4.7. Let $A$ be a finite dimensional algebra over a field $k$, $I$ an ideal of $A$ contained in rad $A$, and $B = A/I$ a factor algebra of $A$.

(a) $I$ annihilates all bricks of $A$ if and only if the isomorphism $- \otimes_A B : K_0(\underline{\text{proj}} A) \simeq K_0(\underline{\text{proj}} B)$ gives an isomorphism of $g$-fans:

$$\Sigma(A) \simeq \Sigma(B).$$

For example, if $I$ is generated by a set of central nilpotent elements of $A$, then we have the isomorphism of $g$-fans above.

(b) If $A$ is $g$-finite, then the equivalence relation $\sim_B$ coarsens $\Sigma(A)$, and the isomorphism $- \otimes_A B : K_0(\underline{\text{proj}} A) \simeq K_0(\underline{\text{proj}} B)$ gives an isomorphism of $g$-fans:

$$\Sigma(B) = \Sigma(A)/\sim_B.$$

Proof. Since $I \subset \text{rad} A$, $- \otimes_A B : K_0(\underline{\text{proj}} A) \to K_0(\underline{\text{proj}} B)$ is an isomorphism.

(a) The assertion follows from $[DIRRT]$ Theorem 5.12 and $[EJR]$ Theorem 1.

(b) $- \otimes_A B$ gives an injective map $2\text{-silt} A/\sim_B \to 2\text{-silt} B$. Thus for each $T \in 2\text{-silt} A$, we have $\bigcup_{U \in 2\text{-silt} A, U \sim_B T} C(U) \subset C(T \otimes_A B)$. Since $\Sigma(A)$ is complete, the equality holds, and the map $2\text{-silt} A/\sim_B \to 2\text{-silt} B$ is bijective. Thus $\Sigma(B) = \Sigma(A)/\sim_B$ holds. \hfill \Box

Note that one can not drop the $g$-finiteness in (b) above. For example, let $A$ be an $\ell$-Kronecker algebra with $\ell \geq 2$, and $B$ the factor algebra of $A$ by the ideal generated by $\ell - 1$ arrows. Then $\Sigma(B)$ is complete and hence can not be a coarsening of $\Sigma(A)$ which is not complete.

One of the importance of the $g$-fan is that it gives an explicit description of torsion classes given by stability conditions. For each $\theta \in K_0(\underline{\text{proj}} A)_R$, let

$$\mathcal{T}_\theta := \{ X \in \text{mod} A \mid \theta(X) > 0 \text{ for all factor modules } X \neq 0 \text{ of } X \},$$

$$\overline{\mathcal{T}}_\theta := \{ X \in \text{mod} A \mid \theta(X) \geq 0 \text{ for all factor modules } X \text{ of } X \},$$

$$\mathcal{F}_\theta := \{ X \in \text{mod} A \mid \theta(X) < 0 \text{ for all submodules } X \neq 0 \text{ of } X \},$$

$$\overline{\mathcal{F}}_\theta := \{ X \in \text{mod} A \mid \theta(X) \leq 0 \text{ for all submodules } X \text{ of } X \},$$

$$\mathcal{V}_\theta := \mathcal{T}_\theta \cap \overline{\mathcal{F}}_\theta.$$

The following properties are elementary.

Definition-Proposition 4.8. $[BKT] [Kin]$ Let $\theta \in K_0(\underline{\text{proj}} A)_R$. 

(a) \((\mathcal{T}_0, \mathcal{F}_0)\) and \((\mathcal{T}_\theta, \mathcal{F}_\theta)\) are torsion pairs in \(\text{mod} A\) satisfying \(\mathcal{T}_0 \subseteq \mathcal{T}_\theta\) and \(\mathcal{F}_\theta \subseteq \mathcal{F}_0\). In particular, for \(t_\theta := t_{\mathcal{T}_0}, \mathcal{T}_\theta := \mathcal{T}_{t_\theta}, \mathcal{F}_\theta := \mathcal{F}_{t_\theta}\) in \((2.2)\), each \(X \in \text{mod} A\) admits exact sequences
\[
0 \to t_\theta X \to X \to \mathcal{T}_0 X \to 0 \quad \text{with} \quad t_\theta X \in \mathcal{T}_0 \quad \text{and} \quad \mathcal{T}_0 X \in \mathcal{F}_\theta,
\]
\[
0 \to \mathcal{T}_\theta X \to X \to t_\theta X \to 0 \quad \text{with} \quad \mathcal{T}_\theta X \in \mathcal{F}_\theta \quad \text{and} \quad \mathcal{T}_\theta X \in \mathcal{F}_\theta.
\]

(b) \(\mathcal{W}_\theta\) is a wide subcategory of \(\text{mod} A\). Moreover, each \(X \in \text{mod} A\) admits a filtration
\[
0 \subseteq t_\theta X \subseteq \mathcal{T}_\theta X \subseteq X \quad \text{such that} \quad \mathcal{W}_\theta := \mathcal{T}_0 X / t_\theta X \in \mathcal{W}_\theta.
\]

For \(T = T_1 \oplus \cdots \oplus T_j \in 2\text{-psilt} A\) with indecomposable \(T_i\), let
\[
C^+(T) := \{ \sum_{i=1}^j a_i [T_i] \mid a_1, \ldots, a_j > 0 \} \subset C(T).
\]

We have the following descriptions of torsion pairs given by stability conditions in terms of 2-term presilting complexes (see Definition-Propositions 2.12 and 3.8 for functors \(f_U, w_U\) and so on). We refer to \(\text{As2} \text{[As]}\) for more explicit results.

**Proposition 4.9.** [Y] Proposition 3.3 [BST] Proposition 3.27
Let \(U \in 2\text{-psilt} A\) and \(\theta \in C^+(U)\). Then we have
\[
\mathcal{T}_0 = \mathcal{T}_U, \quad \mathcal{T}_\theta = \mathcal{T}_U, \quad \mathcal{F}_0 = \mathcal{F}_U, \quad \mathcal{F}_\theta = \mathcal{F}_U \quad \text{and} \quad \mathcal{W}_\theta = \mathcal{W}_U.
\]

Therefore we have
\[
t_0 = t_U, \quad t_\theta = t_U, \quad f_0 = f_U, \quad f_\theta = f_U, \quad \text{and} \quad \mathcal{W}_\theta = \mathcal{W}_U.
\]

To explain a remarkable property of \(g\)-fans, we introduce the following notion.

**Definition 4.10.** For a fan \(\Sigma \in \mathbb{R}^d\) and \(\sigma \in \Sigma\), we define the reduction of \(\Sigma\) at \(\sigma\) by
\[
\Sigma / \sigma := \{ \pi(\tau) \mid \tau \in \Sigma, \sigma \subseteq \tau\},
\]
where \(\pi: \mathbb{R}^d \to \mathbb{R}^d / \mathbb{R}\sigma\) is a natural projection. Then \(\Sigma / \sigma\) is a fan in \(\mathbb{R}^d / \mathbb{R}\sigma\).

This reduction process of fans corresponds to Jasso reduction given in Definition \(\text{As3}\)(d), see also \(\text{As2} \text{Theorem 4.5}\).

**Theorem 4.11.** Let \(A\) be a finite dimensional algebra over a field \(k\), and \(U \in 2\text{-psilt} A\). For the Bongartz completion \(U_{\text{max}}\) of \(U\), let \(B := \text{End}_{K^0(\text{proj} A)}(U_{\text{max}}) / [U]\).

(a) There exists a triangle functor \(K^b(\text{proj} A) \to K^b(\text{proj} B)\) which sends \(U_{\text{max}}\) to \(B\) and gives an isomorphism \(K_0(\text{proj} A) / K_0(\text{add} U) \simeq K_0(\text{proj} B)\) and bijections
\[
\{ T \in 2\text{-psilt} A \mid U \in \text{add} T \} \simeq \text{psilt} B \quad \text{and} \quad \{ T \in 2\text{-psilt} A \mid U \in \text{add} T \} \simeq 2\text{-psilt} B.
\]

(b) The isomorphism \(K_0(\text{proj} A) / K_0(\text{add} U) \simeq K_0(\text{proj} B)\) gives an isomorphism of fans
\[
\Sigma(A) / C(U) \simeq \Sigma(B).
\]

**Proof.** (a) Let \(\mathcal{T} := K^b(\text{proj} A) / \text{thick} U\) be the Verdier quotient, \(\pi : K^b(\text{proj} A) \to \mathcal{T}\) the canonical functor, and \(V := \pi(U_{\text{max}}) \in \mathcal{T}\). Then we have an isomorphism \(\pi : K_0(\text{proj} A) / K_0(\text{add} U) \simeq K_0(\mathcal{T})\). By [LY] Theorems 3.1, 3.7, Corollary 3.8, \(\mathcal{T}\) is a \(k\)-linear Hom-finite Krull-Schmidt triangulated category such that \(\pi\) gives an isomorphism
\[
\{ T \in \text{silt} A \mid U \in \text{add} T \} \simeq \text{silt} \mathcal{T}
\]
of posets, a bijection
\[
\{ T \in \text{psilt} A \mid U \in \text{add} T \} \simeq \text{psilt} \mathcal{T}
\]
and an isomorphism
\[
B \simeq \text{End}_\mathcal{T}(V)
\]
of \(k\)-algebras. Since \(\pi(A) \simeq V\) holds by [H] Propositions 4.10, \(\pi\) gives bijections
\[
\{ T \in 2\text{-silt} A \mid U \in \text{add} T \} \simeq 2\text{-silt} \mathcal{T}\text{ and } \{ T \in 2\text{-psilt} A \mid U \in \text{add} T \} \simeq 2\text{-psilt} \mathcal{T}.
\]
On the other hand, \( \mathcal{T} \) is algebraic [Ke2] [D]. Applying Proposition 2.10 to \( \mathcal{T} \) and \( V \), there is a triangle functor \( F : \mathcal{T} \to K^b(\text{proj} B) \) which sends \( V \) to \( B \) and gives an isomorphism \( K_0(\mathcal{T}) \cong K_0(\text{proj} B) \) and bijections

\[
2\nu\text{-silt} \mathcal{T} \cong 2\nu\text{-silt} B \text{ and } 2\nu\text{-psilt} \mathcal{T} \cong 2\nu\text{-psilt} B.
\]

Thus the composition \( F \circ \pi : K^b(\text{proj} A) \to K^b(\text{proj} B) \) gives the desired triangle functor.

(b) For \( \ell := |U| \leq j \leq |A| \), the triangle functors \( \pi \) and \( F \) give bijections

\[
\{T \in 2\nu\text{-psilt}^j A \mid U \in \text{add} T\} \cong 2\nu\text{-psilt}^{j-\ell} \mathcal{T} \cong 2\nu\text{-psilt}^{j'-\ell} B
\]

such that \( F \circ \pi(C(T)) = C(F \circ \pi(T)) \) as cones in \( K_0(\text{proj} B)_{\mathbb{R}} \). Thus the assertion follows. \( \Box \)

Example 4.12. Let \( A \) be the algebra in Example 4.5. For \( \sigma := C(e_1 A) \), the cones \( \tau \in \Sigma(A) \) satisfying \( \sigma \subset \tau \) is shown by the left picture below. Moreover, \( \Sigma(A)/\sigma \) is the fan shown by the right picture below.

4.2. Idempotents and \( g \)-fans. In this subsection, we observe that \( g \)-fans of finite dimensional algebras are rather special.

**Definition 4.13.** A sign-coherent fan is a pair \((\Sigma, \sigma_+)\) satisfying the following conditions.

(a) \( \Sigma \) is a nonsingular fan in \( \mathbb{R}^d \), and \( \sigma_+, -\sigma_+ \in \Sigma_d \).

(b) Take a \( \mathbb{Z} \)-basis \( e_1, \ldots, e_d \in \mathbb{Z}^d \) such that \( \sigma_+ = \text{cone}\{e_i \mid 1 \leq i \leq d\} \). Then for each \( \sigma \in \Sigma \), there exists \( e_1, \ldots, e_d \in \{1, -1\} \) such that \( \sigma \subseteq \text{cone}\{e_1 e_1, \ldots, e_d e_d\} \).

(c) Any cone in \( \Sigma \) is a face of a cone of dimension \( d \).

(d) Any cone in \( \Sigma \) of dimension \( d-1 \) is a face of precisely two cones of dimension \( d \).

The following property of \( g \)-fans is basic.

**Proposition 4.14.** For a finite dimensional algebra \( A \) over a field \( k \), \((\Sigma(A), C(A))\) is a sign-coherent fan. In this case, the isomorphism classes of indecomposable projective \( A \)-modules gives the \( \mathbb{Z} \)-basis which generate \( C(A) \) as a cone.

**Proof.** The conditions (a),(c) and (d) follow from Proposition 4.2. The condition (b) is the sign-coherence of \( g \)-vectors [AIR]. \( \Box \)

Now we show that the \( g \)-fans form a very special class of sign-coherent fans.

**Definition 4.15.** An ordered fan is a sign-coherent fan \((\Sigma, \sigma_+)\) satisfying the following conditions.

(a) For each adjacent cones \( \sigma, \sigma' \in \Sigma_d \), a normal vector of \( \sigma \cap \sigma' \) belongs to the interior of \( \sigma_+ \).

(b) There exists a partial order \( \leq \) on \( \Sigma_d \) such that \((\Sigma_d, \leq)\) has the maximum element \( \sigma_+ \) and the minimum element \( -\sigma_+ \).

(c) For any \( \sigma, \sigma' \in \Sigma_d \), the following conditions are equivalent.

- There is an arrow \( \sigma \rightarrow \sigma' \) in the Hasse quiver.
- \( \sigma \) and \( \sigma' \) are adjacent. Moreover, consider the hyperplane \( H := \mathbb{R}(\sigma \cap \sigma') \subset \mathbb{R}^d \). Then \( \sigma \) and \( \sigma_+ \) belong to the same connected component of \( \mathbb{R}^d \setminus H \).
For $d = 2$, it is easy to check that any sign-coherent fan is ordered. For $d \geq 3$, there are many
sign-coherent fans which are not ordered.

Notice that, if $A$ is $g$-finite, then the partial order $\leq$ is uniquely determined by the condition
(c). Therefore the following property of $\Sigma(A)$ is important since it claims that the partial order
on $2$-silt $A$ can be recovered from $\Sigma(A)$ if $A$ is $g$-finite.

**Proposition 4.16.** For a finite dimensional algebra $A$ over a field $k$, $(\Sigma(A), C(A))$ is an ordered
fan.

**Proof.** This is shown in [DIJ, Theorem 6.11]. □

Now we explain idempotent reductions of $g$-fans. We prepare the following general notion.

**Definition 4.17.** Let $(\Sigma, \sigma^+)$ be a sign-coherent fan, and $e_1, \ldots, e_d$ the basis in Definition 4.13(c).
For each subset $I \subset \{1, \ldots, d\}$, consider the subspace
$$R^I = \bigoplus_{i \in I} R e_i \subset R^d.$$ Then the subset
$$\Sigma^I := \{ \sigma \in \Sigma \mid \sigma \subset R^I \}$$ is a subfan of $\Sigma$ thanks to the condition Definition 4.13(c).

Now let $A$ be a basic finite dimensional algebra over a field $k$, and $1 = e_1 + \cdots + e_n$ the orthogonal
primitive idempotents. As in Definition 4.17, for each subset $I \subset \{1, \ldots, n\}$, we obtain a subspace
$$K_0(\text{proj } A)^I_R := \bigoplus_{i \in I} R[e_i A] \subset K_0(\text{proj } A)_R$$ and a subfan of $\Sigma(A)$ given by
$$\Sigma^I(A) := \{ \sigma \in \Sigma(A) \mid \sigma \subset K_0(\text{proj } A)^I_R \}.$$ On the other hand, we consider the idempotent
$$e = e^I := \sum_{i \in I} e_i \in A$$ and the corresponding subalgebra $eAe$ of $A$. Then we have a fully faithful functor
$$- \otimes_{eAe} eA : \text{proj } eAe \to \text{proj } A$$ which induces an isomorphism
$$- \otimes_{eAe} eA : K_0(\text{proj } eAe)_R \simeq K_0(\text{proj } A)^I_R. \quad (4.1)$$ We are ready to state the following result.

**Theorem 4.18.** Let $A$ be a finite dimensional algebra over a field $k$, and $1 = e_1 + \cdots + e_n$ the orthogonal
primitive idempotents. For each subset $I \subset \{1, \ldots, n\}$ and $e := e^I$, the isomorphism (4.1) gives an isomorphism of fans
$$\Sigma(eAe) \simeq \Sigma^I(A).$$

**Proof.** For the thick subcategory $K^b(\text{add } eA)$ of $K^b(\text{proj } A)$, we have a triangle equivalence
$$- \otimes_{eAe} eA : K^b(\text{proj } eAe) \simeq K^b(\text{add } eA) \subset K^b(\text{proj } A).$$ Therefore we have a bijection
$$- \otimes_{eAe} eA : 2\text{-psilt}(eAe) \simeq \{ T \in 2\text{-psilt } A \mid T^0, T^{-1} \in \text{add } eA \}.$$ Since $K_0(\text{add } eA)_R = K_0(\text{proj } A)^I_R$ holds, the assertion follows immediately. □
Example 4.19. Let $A$ be the algebra in Example 4.5. Then the subfans $\Sigma^{(1,2)}(A)$, $\Sigma^{(1,3)}(A)$, $\Sigma^{(2,3)}(A)$ are the following.

We show that the product of fans corresponds to the product of algebras.

Definition 4.20. Let $\Sigma$ and $\Sigma'$ be fans in $\mathbb{R}^d$ and $\mathbb{R}^{d'}$ respectively. We define a product fan $\Sigma \times \Sigma'$ in $\mathbb{R}^{d + d'}$ by
\[
\Sigma \times \Sigma' := \{ \sigma \times \sigma' | \sigma \in \Sigma, \sigma' \in \Sigma' \}.
\]

We say that $\Sigma$ is indecomposable if it can not be written as a product of two fans.

The following result shows that the decomposition of $g$-fans precisely corresponds to the decomposition of algebras.

Theorem 4.21. Let $A$ be a finite dimensional algebra over a field $k$.

(a) If $A = A_1 \times \cdots \times A_\ell$ for a finite dimensional algebra $A_i$, then we have
\[
\Sigma(A) = \Sigma(A_1) \times \cdots \times \Sigma(A_\ell).
\]

(b) In (a), assume that each $A_i$ is ring-indecomposable. Then, for each decomposition $\Sigma(A) = \Sigma_1 \times \cdots \times \Sigma_m$, there exists a decomposition $\{1, \ldots, \ell\} = \bigsqcup_{j=1}^m I_j$ such that $\Sigma_j = \Sigma(\prod_{i \in I_j} A_i)$ for each $1 \leq j \leq m$.

(c) If $A$ is ring-indecomposable, then the fan $\Sigma(A)$ is indecomposable.

Proof of (a). There is a bijection $2$-psilt $A_1 \times \cdots \times 2$-psilt $A_\ell \cong 2$-psilt $(A_1 \times \cdots \times A_\ell)$ given by $(T_1, \ldots, T_\ell) \mapsto T_1 \oplus \cdots \oplus T_\ell$. Since $C(T_1 \oplus \cdots \oplus T_\ell) = C(T_1) \times \cdots \times C(T_\ell)$, we obtain the desired equation. \qed

To prove Theorem 4.21(b)(c), we need the following observation.

Lemma 4.22. Let $e \in A$ be an idempotent and $f := 1 - e$.

(a) We have an injective map
\[
2$-psilt^1(eAe) \sqcup 2$-psilt^1(fAf) \subset 2$-psilt^1A
\]

sending $P \in 2$-psilt$^1(eAe)$ to $P \otimes_{eAe} eA$ and $Q \in 2$-psilt$^1(fAf)$ to $Q \otimes_{fAf} fA$.

(b) The equality holds in (a) if and only if $e$ is a central idempotent.

Proof. (a) Immediate from Theorem 4.18

(b) It suffices to prove “only if” part. Assume that the equality holds. If $e$ is not central, then at least one of Hom$_A(eA, fA)$ and Hom$_A(fA, eA)$ is non-zero. Without loss of generality, assume Hom$_A(eA, fA) \neq 0$. Take an indecomposable direct summand $e_iA$ of $eA$ such that Hom$_A(e_iA, fA) \neq 0$, and take a minimal left (add $fA$)-approximation of $e_iA$
\[
eq e_iA \rightarrow Q \rightarrow X \rightarrow e_iA[1].
\]

Then $X$ is an indecomposable direct summand of $\mu_{eA}^{-1}(eA)$ in $2$-silt$A$, and thus $X \in 2$-psilt$^1A$. Moreover $Q$ is non-zero by our choice, and hence $X$ is not contained in the image of the map in (a). Thus $e$ has to be central. \qed
We are ready to prove Theorem 4.21(b)(c).

Proof of Theorem 4.21(b)(c). It suffices to prove (b). Without loss of generality, we can assume that $A$ is basic. Let $1 = e_1 + \cdots + e_n$ be a primitive orthogonal idempotents and $P_i := e_iA$. By definition, there exist $\sigma^j \in \Sigma^j$ for each $1 \leq j \leq m$ such that

$$C(A) = \sigma^1 \times \cdots \times \sigma^m.$$

Since $C(A) = \text{cone}([P_1], \ldots, [P_n])$, there exists a decomposition $\{1, \ldots, n\} = \bigsqcup_{j=1}^m J_j$ such that

$$\sigma^j = \text{cone}([P_i] \mid i \in J_j)$$

for each $1 \leq j \leq m$. Let

$$e^j := \sum_{i \in J_j} e_i \in A.$$

Then $\Sigma^j$ is a fan in $K_0(\text{proj} A)^{J_j}_R = K_0(\text{proj} e^j A)e^j$ and hence each ray of $\Sigma(A)$ belongs to $K_0(\text{proj} e^j A)$ for some $1 \leq j \leq m$. Thus the map

$$\bigsqcup_{j=1}^m 2\text{-psilt}^1(e^j A)e^j \to 2\text{-psilt}^1 A$$

sending $P \in 2\text{-psilt}^1(e^j A)e^j$ to $P \otimes_{\text{proj} e^j A} e^j A$ is bijective. By Lemma 4.22, $e^j$ is a central idempotent of $A$. Thus there exists a decomposition $\{1, \ldots, \ell\} = \bigsqcup_{j=1}^m I_j$ such that $Ae^j = \prod_{i \in I_j} A_i$, and the assertion follows. \hfill $\square$

Next we explain the sign decomposition of $g$-fans \([Ao]3\). We prepare the following general notion.

**Definition 4.23.** Let $(\Sigma, \sigma_+)$ be a sign-coherent fan in $\mathbb{R}^d$ and $\epsilon \in \{\pm 1\}^d$. Consider the basis $e_1, \ldots, e_d$ of $\mathbb{R}^d$ and the orthant $\mathbb{R}^d_\epsilon$ given in Definition 4.13(b). Define a subfan of $\Sigma$ by

$$\Sigma_\epsilon := \{\sigma \in \Sigma \mid \sigma \subset \mathbb{R}^d_\epsilon\}.$$ 

Thanks to the condition Definition 4.13(b), we have $\Sigma = \bigcup_{\epsilon \in \{\pm 1\}^d} \Sigma_\epsilon$.

Now let $A$ be a basic finite dimensional algebra over a field $k$ with $|A| = n$, and $1 = e_1 + \cdots + e_n$ the orthogonal primitive idempotents. For $\epsilon \in \{\pm 1\}^n$, as in Definition 4.23, we obtain an orthant

$$K_0(\text{proj} A)_{\epsilon,R} := \text{cone}(e_i[\epsilon_i A] \mid i \in \{1, \ldots, n\})$$

and a subfan of $\Sigma(A)$ given by

$$\Sigma_\epsilon(A) := \{\sigma \in \Sigma(A) \mid \sigma \subset K_0(\text{proj} A)_{\epsilon,R}\}.$$ 

We can describe the fan $\Sigma_\epsilon(A)$ by a simpler algebra defined as follows.

**Definition 4.24.** For $\epsilon \in \{\pm 1\}^n$, let

$$e^+_{\epsilon} := \sum_{\epsilon_i = 1} e_i \quad \text{and} \quad e^-_{\epsilon} := \sum_{\epsilon_i = -1} e_i.$$ 

We denote by $A_{\epsilon}$ the subalgebra of $A$ given by

$$A_{\epsilon} := \begin{bmatrix} e^+_\epsilon A e^+_\epsilon & e^+_\epsilon A e^-_\epsilon \\ e^-_\epsilon A e^+_\epsilon & e^-_\epsilon A e^-_\epsilon \end{bmatrix}.$$ 

The functor $- \otimes_{A_{\epsilon}} A : \text{proj} A_{\epsilon} \to \text{proj} A$ gives an isomorphism

$$- \otimes_{A_{\epsilon}} A : K_0(\text{proj} A_{\epsilon})_R \simeq K_0(\text{proj} A)_R \quad (4.2)$$

and a functor

$$- \otimes_{A_{\epsilon}} A : K'(\text{proj} A_{\epsilon}) \to K'(\text{proj} A). \quad (4.3)$$

We denote by $K'(\text{proj} A)$ the full subcategory of $K^b(\text{proj} A)$ consisting of 2-term complexes of the form $P^{-1} \to P^0$ with $P^{-1} \in \text{add} e^- A$ and $P^0 \in \text{add} e^+_A$. Let

$$2\text{-silt}^1 A := 2\text{-silt} A \cap K'(\text{proj} A) \quad \text{and} \quad 2\text{-silt} A := 2\text{-silt} A \cap K'(\text{proj} A).$$
Thus the cones in \( \Sigma_\epsilon(A) \) are given by the elements in \( 2\text{-psilt}_\epsilon A \). Now we state main properties of sign decomposition, where the part (b) is a generalization of [Ao] Theorem 4.5.

**Theorem 4.25.** Let \( A \) be a basic finite dimensional algebra over a field \( k \), and \( 1 = e_1 + \cdots + e_n \) the orthogonal primitive idempotents. Let \( \epsilon \in \{ \pm 1 \}^n \).

(a) [Ao, Proposition 3.2] The subset \( 2\text{-silt}_\epsilon A \) of the poset \( 2\text{-silt} A \) is an interval with a maximal element \( (e_1 + A)_{\text{min}} \) and a minimal element \( (e_n - A)_{\text{max}} \).

(b) The functor (4.3) gives bijections \( 2\text{-psilt}_\epsilon A \) and \( 2\text{-silt}_\epsilon A \), and the isomorphism (4.2) gives an isomorphism of fans

\[ \Sigma_\epsilon(A) \simeq \Sigma_\epsilon(A). \]

**Proof.** (b) The following properties of the functor \( F : K^e(\text{proj} A) \rightarrow K^e(\text{proj} A) \) in (4.3) can be checked easily.

(i) \( F \) is full and dense.

(ii) If \( P \in K^e(\text{proj} A) \) satisfies \( F(P) \simeq 0 \), then \( P \simeq 0 \).

(iii) \( \text{Hom}_{K^e(\text{proj} A)}(P, Q[1]) \simeq \text{Hom}_{K^e(\text{proj} A)}(F(P), F(Q)[1]) \).

By (i) and (ii), \( F \) gives a bijection between the isomorphism classes of indecomposable objects in \( K^e(\text{proj} A) \) and those in \( K^e(\text{proj} A) \). Therefore by (iii), \( F \) gives a bijection

\[ F : 2\text{-psilt}_\epsilon A \simeq 2\text{-psilt}_\epsilon A. \]

Thus the assertion follows. \( \square \)

**Example 4.26.** Let \( A \) be the algebra in Example 4.5. Then \( \Sigma_\epsilon(A) \) is the following. We give another condition for two algebras \( A \) and \( B \) such that \( \Sigma_\epsilon(A) \) and \( \Sigma_\epsilon(B) \) are isomorphic. For each ideal \( I \) of \( A \) contained in \( \text{rad} A \), let \( B := A/I \). Then the functor \( - \otimes_A B : \text{proj} A \rightarrow \text{proj} B \) induces an isomorphism

\[ - \otimes_A B : K_0(\text{proj} A)_\mathbb{R} \simeq K_0(\text{proj} B)_\mathbb{R} \quad (4.4) \]

and a functor

\[ - \otimes_A B : K^e(\text{proj} A) \rightarrow K^e(\text{proj} B). \quad (4.5) \]

We give a sufficient condition for \( I \) such that (4.3) induces an isomorphism \( \Sigma_\epsilon(A) \simeq \Sigma_\epsilon(B) \).
Proposition 4.27. Let $A$ be a basic finite dimensional algebra algebra over a field $k$, $1 = e_1 + \cdots + e_n$ the orthogonal primitive idempotents, and $e \in \{\pm 1\}^n$. Let $I$ be an ideal of $A$, and $B := A/I$. Assume that $I$ satisfies
\[ I \subseteq \mathrm{rad} A, \quad e^+_i I e^-_i = 0, \quad (e^+_i I e^-_i)(e^+_i A e^-_i) = 0 \quad \text{and} \quad (e^+_i A e^-_i)(e^+_i I e^-_i) = 0. \]
Then the functor \((4.4)\) gives bijections $2\text{-}\mathrm{silt}_A \simeq 2\text{-}\mathrm{silt}_B$ and $2\text{-}\mathrm{silt}_A \simeq 2\text{-}\mathrm{silt}_B$, and the isomorphism \((4.3)\) gives an isomorphism of fans
\[ \Sigma_\varepsilon(A) \simeq \Sigma_\varepsilon(B). \]

Proof. By our assumptions on $I$, for each $P^{-1} \in \mathrm{add} e^-_i A$ and $P^0 \in \mathrm{add} e^+_i A$, the functor $- \otimes_A B$ gives an isomorphism $\mathrm{Hom}_A(P^{-1}, P^0) \simeq \mathrm{Hom}_B(P^{-1} \otimes_A B, P^0 \otimes_A B)$. Thus the functor $F := - \otimes_A B : \mathcal{K}(\mathrm{proj} A) \to \mathcal{K}(\mathrm{proj} B)$ satisfies the following conditions

(i) $F$ is full and dense.

(ii) If $P \in \mathcal{K}(\mathrm{proj} A)$ satisfies $F(P) \simeq 0$, then $P \simeq 0$.

(iii) $\mathrm{Hom}_{\mathcal{K}(\mathrm{proj} A)}(P, Q[1]) \simeq \mathrm{Hom}_{\mathcal{K}(\mathrm{proj} B)}(F(P), F(Q)[1])$.

Thus $F$ gives a bijection $F : 2\text{-}\mathrm{silt}_A \cap \mathcal{K}(\mathrm{proj} A) \simeq 2\text{-}\mathrm{silt}_B \cap \mathcal{K}(\mathrm{proj} B)$, and the assertion follows. \hfill \Box

We record the following useful observation.

Example 4.28. In the setting of Definition 4.24 define an ideal $I_\varepsilon$ of $A$, by
\[ I_\varepsilon := \left[ \begin{array}{c} \mathrm{rad}(e^+_i A e^-_i) \cap \mathrm{Ann}(e^+_i A e^-_i) \\ 0 \end{array} \right]. \]
For each ideal $I$ of $A$, contained in $I_\varepsilon$, we have an isomorphism of fans
\[ \Sigma_\varepsilon(A) \simeq \Sigma_\varepsilon(A/e) \simeq \Sigma_\varepsilon(A/I) \]
by Theorem 4.26 and Proposition 4.27.

Definition 4.29. Let $(\Sigma, \sigma_\pm)$ be a sign-coherent fan in $\mathbb{R}^d$, and $e_1, \ldots, e_d$ the basis of $\mathbb{R}^d$ given in Definition 4.13(c). For $1 \leq i \leq d$ and $\delta \in \{\pm 1\}$, consider a half space
\[ \mathbb{R}^d_{\varepsilon, \delta} := \{ x_1 e_1 + \cdots + x_d e_d \in \mathbb{R}^d \mid \delta x_i \geq 0 \} \]
and define a subfan of $\Sigma$ by
\[ \Sigma_{\varepsilon, \delta} := \{ \sigma \in \Sigma \mid \sigma \subset \mathbb{R}^d_{\varepsilon, \delta} \}. \]
Thanks to the condition Definition 4.13(c), we have $\Sigma = \Sigma_{\varepsilon, \pm} \cup \Sigma_{\varepsilon, -}$.

Let $A$ be a basic finite dimensional algebra over a field $k$ with $|A| = n$, and $1 = e_1 + \cdots + e_n$ the orthogonal primitive idempotents. For $1 \leq i \leq d$ and $\delta \in \{\pm 1\}$, as in Definition 4.29, we obtain a half space
\[ K_0(\mathrm{proj} A), \varepsilon, \delta, \varepsilon := \{ x_1 e_1 A + \cdots + x_d e_d A \in K_0(\mathrm{proj} A) \mid \delta x_i \geq 0 \} \]
and a subfan of $\Sigma(A)$ given by
\[ \Sigma_{\varepsilon, \delta}(A) := \{ \sigma \in \Sigma(A) \mid \sigma \subset K_0(\mathrm{proj} A), \varepsilon, \delta, \varepsilon \}. \]
For elements $T \geq T'$ in $\mathrm{silt}A$, we consider the interval
\[ [T', T] := \{ U \in \mathrm{silt}A \mid T \geq U \geq T' \}. \]
The following result gives information how $g$-fans change under mutation.

Theorem 4.30. Let $A$ be a basic finite dimensional algebra over a field $k$ with $|A| = n$, and $1 = e_1 + \cdots + e_n$ the orthogonal primitive idempotents. For $1 \leq i \leq n$, let $B := \mathrm{End}_A(\mu_i(A))$. Then there exists a triangle functor $\mathcal{K}^B(\mathrm{proj} A) \to \mathcal{K}^B(\mathrm{proj} B)$ which sends $\mu_i(A)$ to $B$ and gives an isomorphism $K_0(\mathrm{proj} A) \simeq K_0(\mathrm{proj} B)$, a bijection
\[ [A[1], \mu_i(A)] \simeq [\mu_i(B[1]), B] \]
and an isomorphism of fans
\[ \Sigma_{\varepsilon, -}(A) \simeq \Sigma_{\varepsilon, +}(B). \]
Proof. Applying Proposition 2.6 to $\mathcal{T} := \mathcal{K}^b(\text{proj } A)$ and $T := \mu_i(A)$, we obtain a triangle functor $F : \mathcal{T} \rightarrow \mathcal{K}^b(\text{proj } B)$ satisfying $F(T) = B$ and giving an isomorphism $K_0(\text{proj } A) \simeq K_0(\text{proj } B)$ and bijections

$$F : 2_T\text{-silt} \mathcal{T} \simeq 2\text{-silt} B$$

and

$$F : 2_T\text{-psilt} \mathcal{T} \simeq 2\text{-psilt} B,$$

see Definition 2.5(e) for $2_T\text{-silt}$ and $2_T\text{-psilt}$. Since $F(A[1])$ and $F(T[1]) = B[1]$ have the same direct summands except the $i$-th one, we have $F(A[1]) = \mu_i(B[1])$. Thus the bijection $F : 2_T\text{-silt} \mathcal{T} \simeq 2\text{-silt} B$ restricts to the desired bijection

$$[A[1], T] \simeq [\mu_i(B[1]), B].$$

Since the subfan $\Sigma_i(−)(A)$ (respectively, $\Sigma_i(+)B))$ consists of the cones corresponding to the interval $[A[1], T]$ (respectively, $[\mu_i(B[1]), B]$), we obtain the desired isomorphism $\Sigma_i(−)(A) \simeq \Sigma_i(+)B$ of fans. □

5. $g$-polytopes, $c$-polytopes and Newton polytopes

In this section, we introduce $g$-polytopes of finite dimensional algebras and characterize when they are convex. We show that convex $g$-polytopes are reflexive polytopes, and describe the dual polytopes as $c$-polytopes associated with the set of the 2-term simple minded collections. We also describe Newton polytopes of $A$-modules by using the $g$-fan of $A$.

5.1. Definition and basic properties. With each nonsingular fan, we associate a (not necessarily convex) polytope as follows.

**Definition 5.1.** Let $\Sigma$ be a nonsingular fan in $\mathbb{R}^d$. For each $\sigma \in \Sigma_d$, take a basis $v_1, \ldots, v_d$ of $\mathbb{Z}^d$ such that $\sigma = \text{cone}\{v_1, \ldots, v_d\}$, and let

$$\sigma_{\leq 1} := \text{conv}\{0, v_1, \ldots, v_d\} \subset \mathbb{R}^d$$

be the convex hull. Define a (not necessarily convex) polytope in $\mathbb{R}^d$ by

$$P(\Sigma) := \bigcup_{\sigma \in \Sigma_d} \sigma_{\leq 1}.$$ 

We say that $\Sigma$ is convex if $P(\Sigma)$ is convex.

Applying this construction to $g$-fans, we obtain the following notion.

**Definition 5.2.** Let $A$ be a finite dimensional algebra over a field $k$.

(a) We call $P(A) := P(\Sigma(A))$ the $g$-polytope of $A$. More precisely, for $T = T_1 \oplus \cdots \oplus T_\ell \in 2\text{-psilt} A$ with indecomposable $T_i$, let

$$C_{\leq 1}(T) := \{ \sum_{i=1}^\ell a_i[T_i] \mid a_i \geq 0, \sum_{i=1}^\ell a_i \leq 1 \} \subset K_0(\text{proj } A)_\mathbb{R}.$$ 

Then $P(A)$ is defined by

$$P(A) := \bigcup_{T \in 2\text{-silt} A} C_{\leq 1}(T).$$

(b) We say that $A$ is $g$-convex if the $g$-polytope $P(A)$ is convex (i.e. $\Sigma(A)$ is convex).

**Example 5.3.** For integers $\ell, m \geq 1$, let $A = A_{\ell, m}$ be the algebra in Example 3. Then $A$ is $g$-convex if and only if $\ell \leq 3$ and $m \leq 3$. For example, if $(\ell, m) = (4, 5)$, then $P(A)$ is

Now we study the Ehrhart series of $A$, which is the generating function of the number of 2-term presilting complexes.
Definition 5.4. For each integer $\ell \geq 0$, we denote by
\[
\text{2-psilt}^{\leq \ell} A
\]
the set of isomorphism classes of (not necessarily basic) 2-term presilting complexes of $A$ which have at most $\ell$ indecomposable direct summands. We define the Ehrhart series of $A$ by
\[
\text{Ehr}_A(x) := 1 + \sum_{\ell \geq 1} \#(\text{2-psilt}^{\leq \ell} A)x^\ell.
\]

Since there is a canonical bijection
\[
\text{2-psilt}^{\leq \ell} A \simeq P(A) \cap \frac{1}{\ell}K_0(\text{proj } A) \text{ given by } U \mapsto \frac{[U]}{\ell},
\]
the Ehrhart series $\text{Ehr}_A(x)$ of $A$ coincides with the Ehrhart series of the $g$-polytope $P(A)$ (see [BR]) though $P(A)$ is not necessarily convex. We give the following explicit description of the Ehrhart series by using the $h$-vector.

Theorem 5.5. Let $A$ be a finite dimensional algebra over a field $k$ which is $g$-finite, $n := |A|$ and $(h_0, \ldots, h_n)$ the $h$-vector of $\Delta(A)$. Then the Ehrhart series of $A$ is given by
\[
\text{Ehr}_A(x) = \sum_{i=0}^n h_ix^{i}.\]

In other words, for each $\ell \geq 0$, we have
\[
\#(\text{2-psilt}^{\leq \ell} A) = \sum_{j=0}^n \binom{n+\ell-j}n h_j.\]

Proof. The $g$-fan $\Sigma(A)$ gives a unimodular triangulation (see [BR page 185]) of $P(A)$. Thus the desired assertion is [BR Theorem 10.3]. (Note that, even though $P(A)$ is not necessarily convex, the same proof works). \hfill \square

To characterize $g$-convexity, we introduce two conditions below. The first one is combinatorial.

Definition 5.6. Let $\Sigma$ be a nonsingular fan in $\mathbb{R}^d$. We call $\Sigma$ pairwise positive if the following condition is satisfied.

- For each two adjacent maximal cones $\sigma, \tau \in \Sigma_d$, take $\mathbb{Z}$-basis $\{v_1, \ldots, v_{d-1}, v_d\}$ and $\{v_1, \ldots, v_{d-1}, v'_d\}$ of $\mathbb{Z}^d$ such that $\sigma = \text{cone}\{v_1, \ldots, v_{d-1}, v_d\}$ and $\tau = \text{cone}\{v_1, \ldots, v_{d-1}, v'_d\}$. Then $v_d + v'_d$ belongs to $\text{cone}\{v_1, \ldots, v_{d-1}\}$.

We call $\Sigma$ pairwise convex if the following condition is satisfied.

- Define $v_d$ and $v'_d$ as above. Then $v_d + v'_d$ is either 0, $v_i$ for some $1 \leq i \leq d-1$ or $v_i + v_j$ for some $1 \leq i, j \leq d-1$.

This notion in fact characterizes convexity of $P(\Sigma)$ in the following sense.

Proposition 5.7. Let $\Sigma$ be a nonsingular fan in $\mathbb{R}^d$. Then the following conditions are equivalent.

(a) $\Sigma$ is pairwise convex.
(b) For each two adjacent maximal cones $\sigma$ and $\tau$, the union $\sigma_{\leq 1} \cup \tau_{\leq 1}$ is convex.

Moreover, if $\Sigma$ is finite and complete, then the following condition is also equivalent.

(c) $\Sigma$ is pairwise positive and $P(\Sigma)$ is convex.

Proof. (a)$\Rightarrow$(b) This is [AMN Lemma 2.17].

(b)$\Rightarrow$(a) Since both $\{v_1, \ldots, v_{d-1}, v_d\}$ and $\{v_1, \ldots, v_{d-1}, v'_d\}$ are $\mathbb{Z}$-basis of $\mathbb{Z}^d$, both $v_d$ and $v'_d$ are $\mathbb{Z}$-basis of $\mathbb{Z}^d / \sum_{i=1}^{d-1} \mathbb{Z}v_i \simeq \mathbb{Z}$, and hence $v_d + v'_d$ belongs to $\sum_{i=1}^{d-1} \mathbb{Z}v_i$. Thus $(v_d + v'_d)/2$ belongs to $\sigma_{\leq 1} \cup \tau_{\leq 1}$ since it is convex. Thus
\[
v_d + v'_d \in (\sigma \cup \tau) \cap \left(\sum_{i=1}^{d-1} \mathbb{Z}v_i\right) = \sum_{i=1}^{d-1} \mathbb{Z}_{\geq 0} v_i\]
The following conditions are equivalent.

• For any $v_i$ such that $f(v_i) = 1$ holds for each $1 \leq i \leq d$. Then the hyperplane $f^{-1}(1)$ contains a facet $\text{conv}\{v_i \mid 1 \leq i \leq d\}$ of $\sigma_{\leq 1}$. Since $\sigma_{\leq 1} \cup \tau_{\leq 1}$ is convex, it is contained in the half space $\{x \in \mathbb{R}^d \mid f(x) \leq 1\}$. In particular, $f(v_d + v' d) \leq 2$ holds. Using (5.1), we obtain the assertion.

(a)$\Rightarrow$(c) Clearly $\Sigma$ is pairwise positive. Moreover $P(\Sigma)$ is convex by the same argument as in [AMN] Proposition 2.23.

(c)$\Rightarrow$(a) Since $\Sigma$ is pairwise positive, we have $v_d + v' d \in \sum_{i=1}^{d-1} \mathbb{Z}_{\geq 0} v_i$. Using the latter half of the proof of (b)$\Rightarrow$(a), we obtain that $\Sigma$ is pairwise convex.

Notice that $g$-fans are always pairwise positive.

**Proposition 5.8.** Let $A$ be a finite dimensional algebra over a field $k$. Then $\Sigma(A)$ is pairwise positive.

**Proof.** Let $A = A_1 \oplus \cdots \oplus A_n \in \text{psilt}^1 A$ and $A_n \rightarrow U_n \rightarrow T_n' \rightarrow T_n[1]$ be an exchange triangle. Then $[T_n] + [T_n'] = [U_n]$ belongs to $\sum_{i=1}^{n} \mathbb{Z}_{\geq 0}[T_i]$. □

The pairwise convexity of $g$-fans can be stated as follows.

**Definition 5.9.** Let $A$ be a finite dimensional algebra over a field $k$. We say that $A$ is pairwise $g$-convex if the following condition is satisfied.

- For any $T \in 2\text{-silt}^1 A$ and an indecomposable direct summand $T_i$ of $T$, let
  \[ T_i \rightarrow U_i \rightarrow T_i' \rightarrow T_i[1]. \]
  be an exchange triangle. Then $U_i$ has at most two indecomposable direct summands.

Inspired by [HI2], we give characterizations of the convexity of $g$-polytope.

**Theorem 5.10.** Let $A$ be a finite dimensional algebra over a field $k$.

(a) $A$ is $g$-finite if and only if $P(A)$ contains the origin in its interior. In this case, the origin is a unique lattice point in the interior of $P(A)$.

(b) The following conditions are equivalent.

   (i) $A$ is $g$-convex.

   (ii) $P(A) = \text{conv}\{[U] \mid U \in 2\text{-psilt}^1 A\}$.

   (iii) $\Sigma(A)$ is finite and pairwise convex.

   (iv) $A$ is $g$-finite and pairwise $g$-convex.

**Proof.** (a) The former assertion is immediate from Proposition 4.2(d). The latter one is clear since $P(A)$ is a union of $C_{\leq 1}(T)$ for each $T \in 2\text{-silt}^1 A$.

(b) (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) are clear.

(i)$\Rightarrow$(iii) Since the convex hull of $C_{\leq 1}(A) \cup C_{\leq 1}(A[1])$ contains the origin in its interior, so does $P(A)$. Thus $\Sigma(A)$ is complete and finite by (a), and pairwise positive by Proposition 5.8. By Proposition 5.7(c)$\Rightarrow$(a), $\Sigma(A)$ is pairwise convex.

(iii)$\Rightarrow$(i) By Proposition 4.2(d), $\Sigma(A)$ is complete. Thus the assertion follows from Proposition 5.7(a)$\Rightarrow$(c). □

The following is a polytope analog of Definition 4.20.

**Definition 5.11.** [E] [VK] Let $P$ and $Q$ be polytopes in $\mathbb{R}^d$ and $\mathbb{R}^{d'}$ containing 0 in its interior respectively. We define a polytope $P \oplus Q$ in $\mathbb{R}^{d+d'}$ by

\[ P \oplus Q := \text{conv}(P \cup Q). \]

We give the following easy observations.

**Proposition 5.12.** Let $A$ be a finite dimensional algebra over a field $k$. 

(a) For each idempotent $e \in A$, we have
\[ P(eAe) = P(A) \cap K_0(\proj eAe). \]
In particular, if $A$ is $g$-convex, then so is $eAe$.

(b) Assume $A = A_1 \times \cdots \times A_\ell$ for a finite dimensional $k$-algebra $A_i$. Then $A$ is $g$-convex if and only if each $A_i$ is $g$-convex. In this case, we have
\[ P(A) = P(A_1) \oplus \cdots \oplus P(A_\ell), \]
where we identify $K_0(\proj A)_{\mathbb{R}}$ and $\bigoplus_{i=1}^\ell K_0(\proj A_i)_{\mathbb{R}}$ by (1.1).

Proof. (a) This is immediate from Theorem 4.18.

(b) We have a bijection
\[ 2\silt A_1 \times \cdots \times 2\silt A_\ell \simeq 2\silt A \quad \text{given by} \quad (T_1, \cdots, T_\ell) \mapsto T_1 \oplus \cdots \oplus T_\ell. \]
Moreover the exchange sequences for $A$ are given by the exchange sequences for some $A_i$. Thus $A$ is pairwise $g$-convex if and only if each $A_i$ is pairwise $g$-convex. By Theorem 5.10(b)(iii)$\Rightarrow$(i), we obtain the first assertion. Since $2\silt 1 A = 2\silt 1 A_1 \sqcup \cdots \sqcup 2\silt 1 A_\ell$ holds, Theorem 5.10(b)(ii) implies
\[ P(A) = \conv\{[U] \mid U \in 2\silt 1 A\} = \conv(\conv\{[U] \mid U_i \in 2\silt 1 A_i\} \mid 1 \leq i \leq \ell) = \bigoplus_{i=1}^\ell P(A_i). \]
Then the second assertion follows. $\square$

5.2. Dual polytopes and $c$-polytopes. Let $P$ be a convex polytope $P$ in $V = \mathbb{R}^n$ containing the origin in its interior. For the dual $\mathbb{R}$-vector space $V^* \simeq \mathbb{R}^n$, we denote by $(\cdot, \cdot) : V^* \times V \to \mathbb{R}$ the natural pairing. Recall that the dual polytope is defined by
\[ P^* := \{ x \in V^* \mid \forall y \in P, \ (x, y) \leq 1 \}. \]
Then $P^*$ is again a convex polytope in $V^*$ containing the origin in its interior, and $P^{**} = P$ holds. We call $P$ reflexive if $P$ and $P^*$ are lattice polytopes.

Throughout this subsection, let $A$ be a finite dimensional algebra over a field $k$. We introduce the notion of $c$-polytope as follows.

**Definition 5.13.** For $X \in \mathbb{D}^b(\mod A)$, let
\[ [X]' := (\dim_k \End_{\mathbb{D}^b(\mod A)}(X))^{-1}[X] \in K_0(\mod A)_{\mathbb{R}}. \]
For the simple $A$-modules $S_1, \ldots, S_n$, we define the $k$-Grothendieck group $K_0(\mod A, k)$ of $\mod A$ as the subgroup of $K_0(\mod A)_{\mathbb{R}}$ generated by $[S_1]', \ldots, [S_n]'$.
For $S = S_1 \oplus \cdots \oplus S_n \in \text{smeA}$, let
\[ v_S := \sum_{i=1}^n [S_i]' \in K_0(\mod A)_{\mathbb{R}}. \]
We define the $c$-polytope $P^c(A)$ of $A$ as the convex hull
\[ P^c(A) := \conv\{v_S \mid S \in 2\text{smeA}\} \subset K_0(\mod A)_{\mathbb{R}}. \]

Using the Euler form
\[ K_0(\proj A)_{\mathbb{R}} \times K_0(\mod A)_{\mathbb{R}} \to \mathbb{R} = K_0(\mod k)_{\mathbb{R}} \quad \text{given by} \quad (X, Y) \mapsto [\text{RHom}_A(X, Y)], \]
we identify $(K_0(\proj A)_{\mathbb{R}})^* \times K_0(\mod A)_{\mathbb{R}}$ with $K_0(\mod A)_{\mathbb{R}}$. Using this identification, we can state the following main result in this subsection.
Theorem 5.14. Let $A$ be a finite dimensional algebra over a field $k$. Then $A$ is $g$-convex if and only if

$$P(A) = (P^c(A))^*.$$

In this case, both $P(A)$ and $P^c(A)$ are reflexive polytopes.

We need the following information on silting-t-structure correspondence (Proposition 2.17).

Proposition 5.15. Let $A$ be a finite dimensional algebra over a field $k$.

(a) The abelian groups $K_0(\text{mod} A, k)$ and $K_0(\text{proj} A)$ are dual to each other with respect to the Euler form.

(b) For any $S = S_1 \oplus \cdots \oplus S_n \in \text{smc} A$, the elements $[S_1]', \ldots, [S_n]'$ are the basis of $K_0(\text{mod} A, k)$.

Proof. (a) This is immediate from Proposition 2.17.

(b) Take $T = T_1 \oplus \cdots \oplus T_n \in \text{silt} A$ corresponding to $S$. Then $[T_1], \ldots, [T_n]$ give a basis of $K_0(\text{proj} A)$ by [Am]. Thus the claim follows from (a).

The following simple observation is crucial.

Proposition 5.16. Keep the setting in Proposition 2.17. Then the element $v_S \in K_0(\text{mod} A)_R$ gives a normal vector of the facet of $P(A)$ corresponding to $T$.

Proof. For each $1 \leq i, j \leq n$, (2.5) implies

$$([T_i] - [T_j], v_S) = ([T_i], v_S) - ([T_j], v_S) = 1 - 1 = 0.$$

This shows the assertion.

Now we are ready to prove Theorem 5.14.

Proof of Theorem 5.14. It suffices to show the “only if” part. Assume that $A$ is $g$-convex. For $x \in K_0(\text{mod} A)_R$, we consider the half space

$$H^1_x := \{ y \in K_0(\text{proj} A)_R \mid (y, x) \leq 1 \} \subset K_0(\text{proj} A)_R.$$

Since $P(A)$ is a convex polytope by assumption, Proposition 5.14 shows

$$P(A) = \bigcap_{S \in \text{2-smc} A} H^1_x = (P^c(A))^*,$$

where the first equality follows from [Zi, Theorem 2.15(7)].

5.3. Newton polytopes. Throughout this subsection, let $A$ be a finite dimensional algebra over a field $k$. We study a connection between $g$-fans $\Sigma(A)$ and Newton polytopes (also known as Harder-Narashimhan polytopes) of $A$-modules. In particular, our results recover some of results in [Fe1] (see also [BCDMTY, PPPP]).

Definition 5.18. [BKT, Fe1] The Newton polytope of $X \in \mathcal{A}$ is the convex hull

$$n(X) := \{ [Y] \in K_0(\text{mod} A) \mid Y \text{ is a submodule of } X \} \subset K_0(\text{mod} A),$$

$$N(X) := \text{conv} n(X) \subset K_0(\text{mod} A)_R.$$
The dimension of $N(X)$ clearly equals the number of isomorphism classes of simple $A$-modules appearing in $X$ as composition factors. Let $0 \neq \theta \in K_0(\text{proj} A)$. Each $r \in \mathbb{R}$ give a hyperplane

$$H^\theta_r := \{ x \in K_0(\text{mod } A)_R \mid \theta(x) = r \} \subset K_0(\text{proj } A)_R$$

and a half-space

$$H^\theta_{\leq} := \{ x \in K_0(\text{mod } A)_R \mid \theta(x) \leq r \} \subset K_0(\text{proj } A)_R.$$ 

Let $X \in \text{mod } A$. Clearly we have

$$N(X) = \bigcap_{0 \neq \theta \in K_0(\text{proj } A)} H^\theta_{\leq} \cap \theta(N(X)).$$

Each $0 \neq \theta \in K_0(\text{proj } A)_R$ gives a face of $N(X)$:

$$N(X)_\theta := N(X) \cap H^\theta_{\leq} \cap \theta(N(X)) .$$

Conversely each face of $N(X)$ has a form $N(X)_\theta$ for some $0 \neq \theta \in K_0(\text{proj } A)$.

The following properties are basic to study $N(X)$, see Definition-Proposition 4.13\textsuperscript{[3]} for $t_\theta$ and $\tilde{t}_\theta$.

**Lemma 5.19.** For each $\theta \in K_0(\text{proj } A)_R$ and $X \in \text{mod } A$, the following assertions hold.

(a) $\theta(X) \leq \theta(t_\theta X) = \theta(\tilde{t}_\theta X) = \max \theta(N(X))$.

(b) For each submodule $Y$ of $X$, we have $\theta(Y) \leq \theta(t_\theta X)$.

(c) Let $Y$ be a submodule of $X$. Then $\theta(Y) = \theta(t_\theta X)$ holds if and only if $t_\theta X \subseteq Y \subseteq \tilde{t}_\theta X$ and $Y/t_\theta X \in \mathcal{H}_\theta$ hold.

(d) If a submodule $Y$ of $X$ satisfies $[Y] = [t_\theta X]$ (respectively, $[Y] = [\tilde{t}_\theta X]$), then $Y = t_\theta X$ (respectively, $Y = \tilde{t}_\theta X$).

**Proof.** (a) The exact sequence $0 \to t_\theta X \to X \to \tilde{t}_\theta X \to 0$ shows $\theta(X) = \theta(t_\theta X) + \theta(\tilde{t}_\theta X)$. Since $\theta(\tilde{t}_\theta X) \leq 0$, we obtain the left inequality. Similarly, the exact sequence $0 \to t_\theta Y \to \tilde{t}_\theta X \to t_\theta Y \to 0$ shows $\theta(t_\theta Y) = \theta(t_\theta X) + \theta(\tilde{t}_\theta Y)$, and hence the equality $\theta(\tilde{t}_\theta Y) = 0$ implies the middle equality. To prove the right equality, it suffices to show (b).

(b) By (a), $\theta(Y) \leq \theta(t_\theta X)$ holds. The exact sequence $0 \to t_\theta Y \to t_\theta X \to t_\theta X/t_\theta Y \to 0$ implies $t_\theta X/t_\theta Y \in \mathcal{H}_\theta$ and hence $\theta(t_\theta X/t_\theta Y) \geq 0$. Then the inequality $\theta(t_\theta X/t_\theta Y) \geq 0$ implies $\theta(t_\theta X) \leq \theta(t_\theta X/t_\theta Y)$. Thus the assertion follows.

(c) The “if” part is clear from the exact sequence $0 \to t_\theta X \to Y \to Y/t_\theta X \to 0$ and $\theta(Y/t_\theta X) = 0$. We prove the “only if” part. Since $\theta(Y/t_\theta X) = \theta(\tilde{t}_\theta Y)$, we have $\theta(Y) \leq \theta(t_\theta X) \leq \theta(t_\theta X/t_\theta Y) = \theta(Y/t_\theta X)$. Thus all the 4 equalities hold, and $\theta(t_\theta X/t_\theta Y) = 0$ implies $t_\theta X/t_\theta Y = 0$ and $t_\theta X = t_\theta Y \subseteq Y$. Similarly, since $\theta(Y/t_\theta Y) = \theta(t_\theta Y) \geq 0$ and $\theta(t_\theta X/t_\theta Y) \geq 0$ hold, we have $\theta(Y) \leq \theta(t_\theta Y) \leq \theta(t_\theta X) = \theta(Y/t_\theta X)$. Thus all the 4 equalities hold, and $\theta(t_\theta Y) = 0$ implies $t_\theta Y = 0$ and $Y = \tilde{t}_\theta X$.

Finally, the claim $Y/t_\theta X \in \mathcal{H}_\theta$ follows from $Y/t_\theta X \subseteq \tilde{t}_\theta X$, $\theta(Y/t_\theta X) = 0$ and a general fact: A submodule $Z'$ of $Z \in \mathcal{H}_\theta$ belongs to $\mathcal{H}_\theta$ if and only if $\theta(Z') = 0$.

(d) Since the assumption implies $\theta(Y) = \theta(t_\theta X)$, we have $t_\theta X \subseteq Y \subseteq \tilde{t}_\theta X$ by (c). Thus the assertion clearly holds.

Let $\theta \in K_0(\text{proj } A)_R$. Then the inclusion functor $\mathcal{H}_\theta \to \text{mod } A$ induces morphisms

$$\iota : K_0(\mathcal{H}_\theta) \to K_0(\text{mod } A) \text{ and } \iota : K_0(\mathcal{H}_\theta)_R \to K_0(\text{mod } A)_R.$$ 

As in Definition 5.13\textsuperscript{[3]}, for each $X \in \mathcal{H}_\theta$, let

$$n_{\mathcal{H}_\theta}(X) := \{ [Y] \in K_0(\mathcal{H}_\theta) \mid Y \text{ is a subobject of } X \} \subset K_0(\mathcal{H}_\theta),$$

$$N_{\mathcal{H}_\theta}(X) := \text{conv}(n_{\mathcal{H}_\theta}(X)) \subset K_0(\mathcal{H}_\theta)_R.$$ 

We have the following descriptions of the faces of $N(X)$.

**Lemma 5.20.** Let $X \in \text{mod } A$ and $0 \neq \theta \in K_0(\text{proj } A)_R$. Then we have

$$N(X)_\theta = [t_\theta X] + \iota(N_{\mathcal{H}_\theta}(\text{mod } A)),$$

$$\{ \text{vertices of } N(X) \} = \{ [t_\theta X] \mid 0 \neq \theta \in K_0(\text{proj } A) \}.$$
Moreover, each edge of $N(X)$ has a form $\text{conv}\{[t_\theta X], [\overline{\tau}_\theta X]\}$ for some $0 \neq \theta \in K_0(\text{proj} A)$.

Proof. By Lemma 5.19(a)(c), we have

$$n(X) \cap H^\text{max}_\theta(\text{N}(X)) = [t_\theta X] + i(n_{\mathcal{W}_\theta}(w_\theta X)).$$

Taking the convex hull, we obtain the first equality. In particular, $[t_\theta X]$ is a vertex of $\text{N}(X)_\theta$ and hence of $\text{N}(X)$. Conversely, for each vertex $v$ of $\text{N}(X)$, there exists $0 \neq \theta \in K_0(\text{proj} A)$ such that $\text{N}(X)_\theta = \{v\}$. Then $v = [t_\theta X]$ holds by the first equality.

Since each face of $\text{N}(X)$ has a form $\text{N}(X)_\theta$ for some $0 \neq \theta \in K_0(\text{proj} A)$ and contains $[t_\theta X]$ and $[\overline{\tau}_\theta X]$, the last assertion follows. \hfill \Box

To give a more explicit description of faces of $\text{N}(X)$, we use 2-term presilting complexes. For $0 \neq U \in 2\text{-psilt}A$, define a face of $\text{N}(X)$ by

$$\text{N}(X)_U := \text{N}(X)[U].$$

We have torsion classes $\mathcal{B}_U$, $\mathcal{F}_U$, a wide subcategory $\mathcal{W}_U$ and functors $t_U$, $\overline{\tau}_U$ and $w_U$ (see Definition-Propositions 2.12 and 3.8). For each $Y \in \mathcal{W}_U$, we denote by $s_{\mathcal{W}_U}(Y)$ the number of isomorphism classes of simple objects in $\mathcal{W}_U$ appearing in $Y$ as a composition factor.

**Lemma 5.21.** Let $X \in \text{mod} A$ and $0 \neq U \in 2\text{-psilt}A$.

(a) We have

$$\dim \text{N}(X)_U = s_{\mathcal{W}_U}(w_U X) \leq |A| - |U|.$$

(b) For each $\theta \in \text{C}^+(U)$, we have

$$\text{N}(X)_\theta = \text{N}(X)_U.$$

Proof. (a) By Definition-Proposition 3.8(d), there exists a finite dimensional algebra $B$ such that $\mathcal{W}_U \simeq \text{mod} B$ and $|B| = |A| - |U|$ and $i : K_0(\mathcal{W}_U) \simeq K_0(\text{mod} B) \rightarrow K_0(\text{mod} A)$ is injective. Thus we have

$$\dim \text{N}(X)_\theta \leq \dim \text{N}_{\mathcal{W}_U_i}(w_U X) = s_{\mathcal{W}_U}(w_U X) \leq |B| = |A| - |U|.$$

(b) By Proposition 4.9 we have $t_\theta = t_U = t_{[U]}$, $\mathcal{W}_\theta = \mathcal{W}_U = \mathcal{W}_{[U]}$ and $w_\theta = w_U = w_{[U]}$. By Lemma 5.20 we obtain

$$\text{N}(X)_\theta = [t_\theta X] + i(N_{\mathcal{W}_\theta}(w_\theta X)) = [t_{[U]} X] + i(N_{\mathcal{W}_{[U]}}(w_{[U]} X)) = \text{N}(X)_{[U]} = \text{N}(X)_U.$$

To state our main result, we introduce some notions. Recall that $2\text{-psilt}A$ is the set of isomorphism classes of basic 2-term presilting complex $U \in K^b(\text{proj} A)$ with $|U| = i$ (Definition 5.1).

**Definition 5.22.** Let $A$ be a finite dimensional algebra over a field $k$ and $X \in \text{mod} A$.

(a) An object $X$ in an abelian length category $\mathcal{W}$ is called sincere if each simple object in $\mathcal{W}$ appears in $X$ as a composition factor.

(b) For $0 \leq i \leq n$, let

$$2\text{-psilt}_X A := \{U \in 2\text{-psilt}A \mid w_U X \text{ is sincere in } \mathcal{W}_U\},$$

$$\Sigma_i(A) := \{C(U) \mid U \in 2\text{-psilt}_X A\} \subseteq \Sigma_i(A).$$

By Lemma 5.21(a), $U \in 2\text{-psilt}A$ belongs to $2\text{-psilt}_X A$ if and only if $\dim \text{N}(X)_U = n - i$.

(c) By Definition-Proposition 2.12(b), we have an order preserving map

$$2\text{-silt}A \rightarrow \{\text{submodules of } X\} \text{ given by } U \mapsto t_U X.$$  \hspace{1cm} (5.2)

We define an equivalence relation $\sim_X$ on $2\text{-silt}A$ by

$$T \sim_X U \iff t_T X = t_U X \quad \text{and} \quad t_T X = [t_U X].$$

This gives an equivalence relation $\sim_X$ on $\Sigma_n(A)$ since we have a bijection $2\text{-silt}A \simeq \Sigma_n(A)$ given by $T \mapsto C(T)$.

(d) We regard the vertices and the edges of $\text{N}(X)$ as a graph. Define a quiver $\overline{\mathcal{N}}_\theta(X)$ by regarding each edge $\text{conv}\{[t_\theta X], [\overline{\tau}_\theta X]\}$ (see Lemma 5.20) as an arrow $[\overline{\tau}_\theta X] \rightarrow [t_\theta X]$. 


(c) Define a contraction $\text{Hasse}(2\text{-silt}A)/\sim_X$ of $\text{Hasse}(2\text{-silt}A)$ by identifying all vertices in each equivalence class of $\sim_X$ and removing all loops.

Now we give explicit descriptions of the normal fan $\Sigma(N(X))$ (see Definition 2.41) as a coarsening fan of the $g$-fan $\Sigma(A)$ (see Definition 2.32), and of the 1-skeleton $\tilde{N}_i(X)$ as a contraction of the Hasse quiver $\text{Hasse}(2\text{-silt}A)$. The following result is an explicit version of [Fe2, Propositions 7.4, 8.5] for $g$-finite case.

\textbf{Theorem 5.23.} Let $A$ be a finite dimensional algebra over a field $k$ which is $g$-finite, and $n := |A|$. 
(a) The equivalence relation $\sim_X$ coarsens $\Sigma(A)$, and we have 
$$\Sigma(N(X)) = \Sigma(A)/\sim_X.$$ 

(b) For each $0 \leq i \leq n$, there is a surjection 
$$2\text{-silt}_X^i A \simeq \Sigma_{i,X}(A) \to \{\text{faces of } N(X) \text{ of dimension } n-i\}$$

given by $U \mapsto C(U) \mapsto N(X)_U$, which induces bijections 
$$2\text{-silt}A/\sim_X \simeq \{\text{vertices of } N(X)\},$$
$$2\text{-silt}_X^1 A \simeq \{\text{facets of } N(X)\}.$$ 

(c) We have an isomorphism of quivers 
$$\tilde{N}_1(X) \simeq \text{Hasse}(2\text{-silt}A)/\sim_X.$$ 

To prove this, we need the following preparation.

\textbf{Lemma 5.24.} Let $n := |A|$ and $X \in \text{mod } A$.

(a) Each cone of $\Sigma(A)$ is contained in a cone of $\Sigma(N(X))$.

(b) Let $\sigma, \tau \in \Sigma_n(A)$. Then $\sigma \sim_X \tau$ if and only if $\sigma$ and $\tau$ are contained in the same cone of $\Sigma_n(N(X))$.

\textbf{Proof.} (a) Immediate from Lemma 5.24(b).

(b) Take $T, U \in 2\text{-silt}A$ such that $\sigma = C(T)$ and $\tau = C(U)$. Then $N(X)_T = \{[t_U X]\}$ and $N(X)_U = \{[t_T X]\}$ hold by Lemmas 5.20 and 5.21(a). Thus $\sigma$ and $\tau$ are contained in the same cone of $\Sigma_n(N(X))$ if and only if $[t_T X] = [t_U X]$, that is, $\sigma \sim_X \tau$. \hfill $\square$

We are ready to prove Theorem 5.23.

\textbf{Proof of Theorem 5.23} (a) The assertion follows from Lemma 5.24(a)(b).

(b) The map (5.3) is well-defined by the definition of $2\text{-silt}_X^i A$. We prove that it is surjective. Let $F$ be a face of $N(X) \text{ of dimension } n-i$. Since $\Sigma(A)$ is complete, Lemma 5.24(a) implies that the cone $\sigma_F \in \Sigma(N(X)) \text{ of dimension } i$ is a union of cones in $\Sigma(A)$. Thus there exists $U \in 2\text{-silt}_X^i A$ such that $C(U) \subseteq \sigma_F$. Since $[U] \in \sigma_F$, we have $N(X)_U = F$. Thus $U \in 2\text{-silt}_X^i A$ holds, and the map (5.3) is surjective.

Since $2\text{-silt}_X^1 A = 2\text{-silt}A$, the map (5.3) gives the first bijection by Lemma 5.24(b). It also gives the second bijection since different elements in $\Sigma_{1,X}(A)$ give different facets of $N(X)$.

(c) By (b), both quivers have the set $2\text{-silt}A/\sim_X$ of vertices. We compare their arrows. By Lemma 5.20 we have a surjection 
$$2\text{-silt}_X^{i-1}A \to \{\text{arrows in } \tilde{N}_1(X)\} \text{ given by } U \mapsto [t_U X \to t_U X].$$

On the other hand, by Proposition 2.32 each arrow in $\text{Hasse}(2\text{-silt}A)$ can be written as $U_{\max} \to U_{\min}$ for some $U \in 2\text{-silt}_X^{i-1} A$. It is still an arrow in $\text{Hasse}(2\text{-silt}A)/\sim_X$ if and only if $U_{\max} \neq U_{\min}$ if and only if $t_U X \neq t_U X$ (since $t_U = t_{U_{\min}}$ and $t_U = t_{U_{\max}}$ hold by Definition-Proposition 5.3) if and only if $U \in 2\text{-silt}_X^{i-1} A$. Thus we have a surjection 
$$2\text{-silt}_X^{i-1}A \to \{\text{arrows in } \text{Hasse}(2\text{-silt}A)/\sim_X\} \text{ given by } U \mapsto [U_{\max} \to U_{\min}].$$
Moreover, \( U, V \in 2\text{-psilt}^{n-1}A \) give the same arrow in \( \overline{N}_1(X) \) if and only if \( \overline{t}_U X = \overline{t}_V X \) and \( t_U X = t_V X \) hold if and only if \( U_{\max} \sim_X V_{\max} \) and \( U_{\min} \sim_X V_{\min} \) hold if and only if \( U \) and \( V \) give the same arrow in \( \text{Hasse}(2\text{-silt}A)/\sim_X \). Thus the assertion follows.

\[ \square \]

As an application of Theorem 5.23, we obtain the following result. The part (b) was obtained in [Fe2, Theorem 4.17, Corollary 6.9].

**Corollary 5.25.** Let \( A \) be a finite dimensional algebra over a field \( k \) which is \( g \)-finite.

(a) For \( X \in \text{mod} A \), the following conditions are equivalent.
- (i) \( \Sigma(N(X)) = \Sigma(A) \) holds.
- (ii) \( \overline{N}_1(X) \simeq \text{Hasse}(2\text{-silt}A) \) holds.
- (iii) For each \( U \in 2\text{-psilt}^{n-1}A \), \( w_U X \neq 0 \) holds.

(b) If one of the following conditions is satisfied, then the conditions in (a) are satisfied.
- (i) Each brick of \( A \) is a direct summand of \( X \).
- (ii) For each \( V \in 2\text{-psilt}^1A \), \( H^0(V) \) is a direct summand of \( X \).

**Proof.** (a) By Theorem 5.23(a)(c), the conditions (i) and (ii) are equivalent to that the map (5.2) is injective. This is equivalent to that \( t_T X \neq t_{T'} X \) holds for each arrow \( T \to T' \) in \( \text{Hasse}(2\text{-silt}A) \).

Recall that there is a bijection between \( 2\text{-psilt}^{n-1}A \) and the set of arrows of \( \text{Hasse}(2\text{-silt}A) \) given by \( U \mapsto [U_{\max} \to U_{\min}] \), and we have

\[ w_U X = t_{U_{\max}} X/t_{U_{\min}} X, \]

see Definition-Proposition 3.8(d). Thus the map (5.2) is injective if and only if (iii) holds.

(b) In each case, it suffices to check the condition (a)(iii).

Consider the case (i). For each \( U \in 2\text{-psilt}^{n-1}A \), the wide subcategory \( \mathcal{W}_U = \mathcal{T}_U \cap \mathcal{F}_U \) has a unique simple object \( S \). Then \( w_U S = \overline{U}_S/t_U S = S \neq 0 \) holds. Since \( S \) is a brick and hence a direct summand of \( X \), we have \( w_U X \neq 0 \).

Consider the case (ii). For each \( U \in 2\text{-psilt}^{n-1}A \), let \( U_{\max} = U \oplus V \) for some \( V \in 2\text{-psilt}^1A \). Then \( H^0(V) \) belongs to \( \mathcal{H}_{U_{\max}} = \mathcal{F}_U \) and does not belong to \( \mathcal{H}_{U_{\min}} = \mathcal{F}_U \). Thus \( t_U H^0(V) = H^0(V) \neq t_U H^0(V) \) and hence \( w_U H^0(V) \neq \). Since \( H^0(V) \) is a direct summand of \( X \), we have \( w_U X \neq 0 \).

\[ \square \]

**Example 5.26.** Consider a finite dimensional algebra \( A \) given by

\[
Q = \begin{bmatrix}
a & 0 & 0 & 0 \\
0 & b & 0 & 0 \\
0 & 0 & c & 0 \\
0 & 0 & 0 & d
\end{bmatrix}
\]

and \( A := kQ/\langle ab, dc, ca - bd \rangle \).
Then $A = \frac{1}{1} \oplus \frac{2}{1} \oplus \frac{3}{2} \oplus \frac{3}{2}$. For $X = P_{1}^{p} \oplus P_{2}^{q} \oplus P_{3}^{r} \oplus S_{2}^{s}$ with $p, q, r, s \geq 0$, the Newton polytope $N(X)$ is given by the following.

In particular, $\overrightarrow{N}(X) \simeq \text{Hasse}(2\text{-silt}A)$ holds if all of $p, q, r, s$ are positive.

It is interesting and difficult to realize a fan as a normal fan e.g. [HPS, PPPP]. From this viewpoint, the following useful result follows immediately from Corollary 5.25(a)(i).

**Corollary 5.27.** Let $\Sigma$ be a complete fan in $V = \mathbb{R}^{d}$. If there exists a finite dimensional algebra $A$ over a field $k$ such that $\Sigma \simeq \Sigma(A)$, then $\Sigma$ is a normal fan of a polytope in $V^{*}$.

6. **Convex $g$-polygons and Smooth Fano $g$-polytopes**

In this section, we give complete classifications of convex $g$-polygons and smooth Fano $g$-polytopes. To state our result explicitly, we need the following general notion.

**Definition 6.1.** (a) Let $(\Sigma, \sigma_{+})$ and $(\Sigma', \sigma_{+}')$ be sign-coherent fans in $\mathbb{R}^{d}$ and $\mathbb{R}^{d'}$ respectively. An isomorphism of sign-coherent fans is an isomorphism $f : \Sigma \simeq \Sigma'$ of fans (Definition 2.2) such that $\{f(\sigma_{+}), f(-\sigma_{+})\} = \{\sigma_{+}', -\sigma_{+}'\}$.

(b) Let $A$ and $A'$ be finite dimensional algebras. An isomorphism of $g$-fans is an isomorphism $\Sigma(A) \simeq \Sigma(A')$ of sign-coherent fans. The induced isomorphism $P(A) \simeq P(A')$ is called an isomorphism of $g$-polytopes.

We have the following finiteness of convex $g$-polytopes/$g$-fans.

**Proposition 6.2.** For each $n \geq 1$, there exist only finitely many convex $g$-polytopes (respectively, convex $g$-fans) of dimension $n$ up to isomorphisms of $g$-polytopes (respectively, $g$-fans).

**Proof.** By Theorem 5.10(a), the origin is a unique lattice point in the interior of $P(A)$. By [LZ], there exists only finitely many convex $g$-polytopes of dimension $n$ up to isomorphisms. Each polytope has only finitely many unimodular triangulations, and each nonsingular fan has only finitely many choices of $\sigma_{+}$. Thus the assertion follows. □
6.1. Convex $g$-polygons. In this subsection, we give a classification of convex $g$-polygons. The following result shows that Example 4.3 gives all convex $g$-polygons up to isomorphism of $g$-polygons.

**Theorem 6.3.** There are precisely 7 convex $g$-polygons up to isomorphism of $g$-polytopes (Definition 6.1).

The corresponding $c$-polygons are the following.

Proof. By Example 4.3, these 7 polygons are in fact $g$-polytopes. Thus it suffices to show that they are all. By Theorem 5.14, convex $g$-polytopes are reflexive. It is well-known that there are precisely 16 reflexive polytopes [PR].

On the other hand, $g$-fans are sign-coherent by Proposition 4.14. One can easily check by case-by-case analysis that, among 16 fans given by these 16 reflexive polygons, there are exactly 7 sign-coherent fans listed above up to isomorphism of sign-coherent fans. □

6.2. Smooth Fano $g$-polytopes. The aim of this subsection is to characterize finite dimensional algebras whose $g$-polytopes satisfy the following property.

**Definition 6.4.** Let $d$ be a positive integer. A lattice polytope $P$ in $\mathbb{R}^d$ containing the origin in its interior is called a smooth Fano $d$-polytope if the vertices of every facet $F$ of $P$ is $\mathbb{Z}^d$-basis of the lattice $\mathbb{Z}^d$.

Among 16 reflexive polygons, there are 5 smooth Fano polygons.

The following special class of algebras plays an important role.

**Definition 6.5.** A finite dimensional algebra $A$ over a field $k$ is pentagon type (respectively, hexagon type) if $P(A)$ is the left (respectively, right) polygon below.

We refer to [AHIKMU] for characterizations of these algebras.

The following is a main result of this section.

**Theorem 6.6.** Let $A$ be a finite dimensional algebra over a field $k$. Then $P(A)$ is a smooth Fano polytope if and only if $A$ is a product of local algebras, algebras of pentagon type and algebras of hexagon type.
The rest of this section is devoted to a proof of Theorem 6.6. We start with the following, where we refer to Definition 5.11 for the definition of sums of polytopes.

**Lemma 6.7.** The following assertions hold.

(a) Let $P_i$ be a lattice polytope in $\mathbb{R}^d_i$ for $1 \leq i \leq \ell$. Then $P_1 \oplus \cdots \oplus P_\ell$ is smooth Fano if and only if $P_i$ is smooth Fano for each $1 \leq i \leq \ell$.

(b) Let $A_i$ be a finite dimensional algebra over a field $k$ for $1 \leq i \leq \ell$ and $A = A_1 \times \cdots \times A_\ell$. Then $P(A)$ is smooth Fano if and only if $P(A_i)$ is smooth Fano for each $1 \leq i \leq \ell$.

**Proof.** (a) This is clear.

(b) This is immediate from (a) and Proposition 5.12(b). \qed

The following technical notion plays a central role.

**Definition 6.8.** Let $k \geq 1$. A del Pezzo $2k$-polytope (respectively, pseudo del Pezzo $2k$-polytope) is

$$V_{2k} := \conv\{ \pm e_1, \ldots, \pm e_{2k}, \pm \sum_{i=1}^{2k} e_i \}$$

(respectively, $\tilde{V}_{2k} := \conv\{ \pm e_1, \ldots, \pm e_{2k}, \sum_{i=1}^{2k} e_i \}$).

The following is clear.

**Lemma 6.9.** Let $A$ be a finite dimensional algebra.

(a) $A$ is local if and only if $P(A)$ is isomorphic to a line segment $[e_1, -e_1]$.

(b) $A$ is hexagon type if and only if $P(A) \simeq V_2$.

(c) $A$ is pentagon type if and only if $P(A) \simeq \tilde{V}_2$.

Recall that a polytope $P$ in $\mathbb{R}^d$ is called a symmetric if $v \in P$ implies $-v \in P$. More generally, a polytope $P$ in $\mathbb{R}^d$ is called a pseudo-symmetric [9] if there exists a facet $F$ of $P$ such that $-F$ is also a facet.

The following result provides a classification of pseudo-symmetric smooth Fano $d$-polytopes.

**Proposition 6.10.** Any pseudo-symmetric smooth Fano $d$-polytope $P$ splits into copies of line segments, del Pezzo polytopes and pseudo del Pezzo polytopes. That is, $P$ is isomorphic to a convex polytope

$$\bigoplus_{i=1}^{k} L \oplus \cdots \oplus L \oplus V_{2a_1} \oplus \cdots \oplus V_{2a_j} \oplus \tilde{V}_{2b_1} \oplus \cdots \oplus \tilde{V}_{2b_k},$$

where each $L$ is a lattice convex polytope isomorphic to the line segment $[e_1, -e_1]$ and $i + 2(a_1 + \ldots + a_j + b_1 + \ldots + b_k) = d$.

We also need the following description of the facets of (pseudo) del Pezzo polytopes.

**Lemma 6.11.** Let $s := \sum_{i=1}^{2k} e_i$.

(a) The facets of a del Pezzo $2k$-polytope $V_{2k}$ consist of

$$\conv\{ e_{i_1}, \ldots, e_{i_k}, -e_{j_1}, \ldots, -e_{j_k} \},$$

where $1 \leq i_1 < \cdots < i_k \leq 2k$, $1 \leq j_1 < \cdots < j_k \leq 2k$ and $\{ i_1, \ldots, i_k \} \cap \{ j_1, \ldots, j_k \} = \emptyset$, and

$$\conv\{ e_{i_1}, \ldots, e_{i_k}, -e_{j_1}, \ldots, -e_{j_{k-1}}, s \} \text{ and}$$

$$\conv\{ -e_{i_1}, \ldots, -e_{i_k}, e_{j_1}, \ldots, e_{j_{k-1}}, -s \},$$

where $1 \leq i_1 < \cdots < i_k \leq 2k$, $1 \leq j_1 < \cdots < j_{k-1} \leq 2k$ and $\{ i_1, \ldots, i_k \} \cap \{ j_1, \ldots, j_{k-1} \} = \emptyset$.

(b) The facets of a pseudo del Pezzo $2k$-polytope $\tilde{V}_{2k}$ consist of

$$\conv\{ e_{i_1}, \ldots, e_{i_\ell}, -e_{j_1}, \ldots, -e_{j_{2k-\ell}} \},$$

where $1 \leq i_1 < \cdots < i_\ell \leq 2k$, $1 \leq j_1 < \cdots < j_{2k-\ell} \leq 2k$, $\{ i_1, \ldots, i_\ell \} \cap \{ j_1, \ldots, j_{2k-\ell} \} = \emptyset$ and $\ell \leq k$.\qed
Proof. Although the proof is routine, we include it for convenience of the reader.

(a) Let $F$ be a facet of $V_{2k}$ and let $H$ be the supporting hyperplane of $F$. Then $\pm e_i$ is not contained in $F$ at the same time, and so is $\pm s$.

- When $F$ does not contain $\pm s$, the vertices of $F$ look like, say, $e_1, \ldots, e_\ell, -e_{\ell+1}, \ldots, -e_{2k}$. Then $H$ is defined by the equality $f(x) = 1$ for
  \[ f(x) := x_1 + \cdots + x_\ell - x_{\ell+1} - \cdots - x_{2k} \]
  and $V_{2k}$ is contained in the half space $\{ x \in \mathbb{R}^d \mid f(x) \leq 1 \}$. If $\ell \neq k$, then one of $f(s)$ and $f(-s)$ is greater than 1, a contradiction. Hence, $\ell = k$, i.e., $F$ is of the form (6.1).

- When $F$ contains $\pm s$, the vertices of $F$ look like $\pm (e_1, \ldots, e_\ell, -e_{\ell+1}, \ldots, -e_{2k-1}, s)$. Then $H$ is defined by the equality $f(x) = 1$ for
  \[ f(x) := x_1 + \cdots + x_\ell - x_{\ell+1} - \cdots - x_{2k-1} + (2k - 2\ell)x_{2k} \]
  and $V_{2k}$ is contained in the half space $\{ x \in \mathbb{R}^d \mid f(x) \leq 1 \}$. If $\ell \neq k$, then one of $f(e_{2k})$ and $f(-e_{2k})$ is greater than 1, a contradiction. Hence, $\ell = k$, i.e., $F$ is of the forms (6.2) and (6.3).

(b) Let $F$ be a facet of $\tilde{V}_{2k}$ and let $H$ be the supporting hyperplane of $F$. When $F$ does not contain $s$, the vertices of $F$ look like $e_1, \ldots, e_\ell, -e_{\ell+1}, \ldots, -e_{2k}$. Then $H$ is defined by the equality $f(x) = 1$ for
  \[ f(x) := x_1 + \cdots + x_\ell - x_{\ell+1} - \cdots - x_{2k}. \]
  and $\tilde{V}_{2k}$ is contained in the half space $\{ x \in \mathbb{R}^d \mid f(x) \leq 1 \}$. If $\ell > k$, then $f(s) > 1$, a contradiction. Hence, $\ell \leq k$, i.e., $F$ is of the form (6.4). When $F$ contains $s$, the same proof as above can be applied, and we conclude that $F$ is of the form (6.2).

Let $P$ be a smooth Fano polytope in $\mathbb{R}^d$. Then there exists a unique fan $\Sigma$ in $\mathbb{R}^d$ such that $P = P(\Sigma)$, see Definition 5.3. We call $P$ ordered if there exists $\sigma_+ \in \Sigma_d$ such that $(\Sigma, \sigma_+)$ is an ordered fan in the sense of Definition 4.1.

Now we prove the following key observation.

**Proposition 6.12.** (a) A del Pezzo $2k$-polytope $V_{2k}$ is ordered if and only if $k = 1$.

(b) A pseudo del Pezzo $2k$-polytope $\tilde{V}_{2k}$ is ordered if and only if $k = 1$.

**Proof.** Recall that an ordered polytope $P$ has a facet $F_+$ such that $(\text{cone} F_+)^\circ \cap \text{span} F' = \emptyset$ for each non-maximal face $F'$ of $P$.

(a) ‘if’ part is clear. To prove ‘only if’ part, we assume that $V_{2k}$ is ordered for $k \geq 2$. By Lemma 6.11 (a), we may assume that
  \[ F_+ = \text{conv}\{e_1, \ldots, e_k, -e_{k+1}, \ldots, -e_{2k}\} \]
  without loss of generality. In the first case, since $k \geq 2$, we have
  \[(\text{cone} F_+)^\circ \ni e_1 + \cdots + e_k - e_{k+1} - \cdots - e_{2k} = s + 2(-e_{k+1} - \cdots - e_{2k}) \in \text{span} F'\]
  for the non-maximal face $F' = \text{conv}\{e_{k+1}, \ldots, e_{2k}, s\}$, a contradiction. In the second case, we have
  \[(\text{cone} F_+)^\circ \ni e_1 + \cdots + e_k - e_{k+1} - \cdots - e_{2k-1} + s = 2(e_1 + \cdots + e_k) + e_{2k} \in \text{span} F'\]
  for the non-maximal face $F' = \text{conv}\{e_1, \ldots, e_k, -e_{2k}\}$, a contradiction.

(b) ‘if’ part is clear. To prove ‘only if’ part, we assume that $\tilde{V}_{2k}$ is ordered for $k \geq 2$. By Lemma 6.11 (b), we may assume that
  \[ F_+ = \text{conv}\{e_1, \ldots, e_\ell, -e_{\ell+1}, \ldots, -e_{2k}\} \]
  where $\ell \leq k$. In the first case, we have
  \[(\text{cone} F_+)^\circ \ni e_1 + \cdots + e_\ell - e_{\ell+1} - \cdots - e_{2k} = 2(e_1 + \cdots + e_\ell) - s \in \text{span} F'\]
  for the non-maximal face $F' = \text{conv}\{e_1, \ldots, e_\ell, s\}$, a contradiction. In the second case, we get a contradiction since $-F_+$ is not a facet of $\tilde{V}_{2k}$.

\[\square\]
Now we are ready to give a proof of Theorem 6.6.

**Proof of Theorem 6.6.** “If” part is clear. In fact, each of local algebras, algebras of pentagon type and algebras of hexagon type is smooth Fano, and therefore their product is also smooth Fano by Lemma 6.7(b).

To prove “only if” part, assume that $P(A)$ is smooth Fano. By Lemma 6.7(b), we only have to consider the case $A$ is ring-indecomposable. Then $\Sigma(A)$ is indecomposable by Theorem 3.21(c).

By Propositions 6.10 and 6.12, $P(A)$ is one of the line segments, del Pezzo 2-polytopes and pseudo del Pezzo 2-polytopes. Hence $A$ is the one of the local algebras, algebras of pentagon type and algebras of hexagon type by Lemma 6.9. \[\square\]

7. Preprojective algebras and Coxeter fans

The aim of this section is to show that the $g$-fans of preprojective algebras $\Pi$ of Dynkin type are all the Coxeter fans, and the $c$-polytopes $P_c(\Pi)$ are the short root polytopes. In particular, $\Pi$ is $g$-convex if and only if it is of type $A_n$ or $B_n$. In this case, $P(\Pi)$ is the dual polytope of the short root polytope.

7.1. Classical preprojective algebras. To recall the definition of preprojective algebras associated to symmetrizable generalized Cartan matrices, we start with the definition of hereditary algebras.

**Definition 7.1.** (a) We call a pair $(D_i,iM_j)_{1\leq i,j\leq n}$ $k$-species if

(i) $D_i$ is a finite dimensional division $k$-algebra.

(ii) $iM_j$ is a finitely generated $D_j \otimes_k D_i^{op}$-module. In other words, $iM_j$ is a $(D_i,D_j)$-bimodule and $k$ acts centrally on $iM_j$.

(b) We call a $k$-species $(D_i,iM_j)$ acyclic if there does not exist a sequence $i_1$, $i_2$, $\ldots$, $i_t$, $i_{t+1} = i_1$ such that $i_iM_{i_{i+1}} \neq 0$ for $1 \leq j \leq n$.

Let $(D_i,iM_j)$ be a $k$-species, $D := \prod_{i=1}^n D_i$ and $M := \bigoplus_{1\leq i,j\leq n} iM_j$. We define the tensor algebra by

$$T_D(M) := \bigoplus_{n=0}^{\infty} T^n(M),$$

where $T^0(M) = D$ and $T^n(M) := M \otimes_D \cdots \otimes_D M$ ($n$ times). This is a $k$-algebra, which is finite dimensional if and only if $(D_i,iM_j)$ is acyclic. Moreover, in this case, $T_D(M)$ is hereditary [11, Theorem 2.35].

**Definition 7.2.** (a) A matrix $C = (c_{ij}) \in M_n(\mathbb{Z})$ is a symmetrizable generalized Cartan matrix if the following conditions hold.

(C1) $c_{ii} = 2$ for all $i$;

(C2) $c_{ij} \leq 0$ for all $i \neq j$;

(C3) $c_{ij} \neq 0$ if and only if $c_{ji} \neq 0$.

(C4) There is a diagonal integer matrix $D = \text{diag}(c_1, \cdots, c_n)$ with $c_i \geq 1$ for all $i$ such that $CD$ is symmetric. It is called a symmetrizer of $C$.

(b) Let $(D_i,iM_j)$ be an acyclic $k$-species. We define the matrix $C = (c_{ij})_{1\leq i,j\leq n}$ associated to it as follows: We define $c_{ii} = 2$ for any $i$. If $i \neq j$, then we define $c_{ij}$ and $c_{ji}$ as follows.

(i) If $iM_j = 0$, then $c_{ij} = 0 = c_{ji}$.

(ii) If $iM_j \neq 0$, then $c_{ij} := -\dim(iM_j)_{D_i}$ and $c_{ji} := -\dim(D_i(iM_j))$.

This is well-defined since the acyclicity implies that at least one of the $iM_j$ or $iM_i$ is zero.

Then the matrix $C$ is a symmetrizable generalized Cartan matrix. Indeed, let $c_i := \dim_k D_i$ and $D := \text{diag}(c_1, \cdots, c_n)$. Then if $iM_j \neq 0$, then we have $c_{ij}c_j = -\dim_k(iM_j) = c_{ji}c_i$ and hence $CD$ is symmetric. The other cases are similar.
(c) Let $C = (c_{ij})$ be the symmetrizable generalized Cartan matrix corresponding an acyclic $k$-species $(D_{i}, I_{M_{j}})$. We call $H := T_D(M)$ is Dynkin type if the matrix $C$ is Dynkin, that is, one of the type $A_n, B_n, \ldots, G_2$.

For example, the Cartan matrix of type $A_n$ and $B_n$ are, respectively,

$$
\begin{pmatrix}
2 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
0 & -1 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & -1 & 2 \\
0 & \cdots & 0 & 0 & -1
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
2 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
0 & -1 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & -1 & 2 \\
0 & \cdots & 0 & 0 & -1
\end{pmatrix}
$$

Definition 7.3. Let $H^\ast := H \otimes H^{op}$ and $E := \text{Ext}^1_H(H, H^\ast) \in \text{mod } H^\ast$. We define the tensor algebra $\Pi := T_H(E)$ and call it the preprojective algebra of $H$. We remark that it is a $\mathbb{Z}$-graded algebra with $\Pi_i = E_{i-1}$, and $(\Pi_1)H \simeq \tau^{-1}H$ in $\text{mod } H$ and $H(\Pi_1) \simeq \tau^{-1}H$ in $\text{mod } H^{op}$.

Let $C$ be a symmetrizable Cartan matrix of Dynkin type of rank $n$, that is, one of the type $A_n, B_n, \ldots, G_2$. Let $\Phi = \Phi(C)$ be the root system of $C$. Let $\{\alpha_1, \ldots, \alpha_n\} \subset \Phi$ be a set of simple roots and $L$ the root lattice. We let $V := L \otimes_\mathbb{Z} R$ and denote by $V^*$ the dual of $V$ with the basis $\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*$. We denote by the natural pairing $(v^*, v)$ for $v \in V$ and $v^* \in V^*$.

Define a reflection $s_i : V \to V$ by

$$s_i(\alpha_j) := \alpha_j - c_{ij} \alpha_i.$$

The Weyl group is defined as a subgroup

$$W = W(C) = \langle s_1, \ldots, s_n \rangle$$

of $\text{GL}(V)$. Then $W$ acts also on $V^*$ by

$$(wf)(v) = (f, w^{-1}v) \quad \text{for } f \in V^*, \ v \in V.$$

Define the dominant chamber as follows:

$$D := \bigcap_{i \in Q_0} \{v^* \in V^* \mid (v^*, \alpha_i) \geq 0\} = \{\sum_{i=1}^n a_i \alpha_i^* \mid a_i \geq 0\}.$$ 

Then the set $\{wD \mid w \in W\}$ of all cones $wD$ and their faces consists of a fan in $V^*$ and we call the Coxeter fan, see e.g. [Kaz2].

Recall from Definition 5.13 that, for $X \in D^b(\text{mod } \Pi)$, let

$$[X]' := (\dim_k \text{End}_{D^b(\text{mod } \Pi)}(X))^{-1}[X] \in K_0(\text{mod } \Pi)_\mathbb{R}.$$

For the simple $\Pi$-modules $D_1, \ldots, D_n$, we define the isomorphism

$$\iota : K_0(\text{mod } \Pi)_\mathbb{R} \simeq V \quad \text{by} \quad \iota([D_i]') = \alpha_i,$$

and we identify $K_0(\text{mod } \Pi)_\mathbb{R}$ with $V$ via $\iota$. Moreover, for $X = X_1 \oplus \cdots \oplus X_n \in \text{smc } \Pi$, let

$$v_X := \iota\left(\sum_{i=1}^n [X_i]'\right) \in V.$$

We have an action of $W$ on $K_0(\text{proj } \Pi)_\mathbb{R}$ via the isomorphism $\iota^* : V^* \simeq K_0(\text{proj } \Pi)_\mathbb{R}$.

The following is the main result of this subsection.

Theorem 7.4. Let $H$ be a finite dimensional hereditary $k$-algebra of Dynkin type and $\Pi$ the preprojective algebra of $H$.

(a) The $g$-fan $\Sigma(\Pi)$ is the Coxeter fan and we have

$$P(\Pi) = \bigcup_{w \in W} wC_{\leq 1}(\Pi).$$
If there exists an arrow 

The split. By Proposition 7.5, if we have

Proposition 7.5 also shows that we have

almost split sequence

Moreover we have

We denote the multiplicity of

Proof. From now on, we will give a proof of Theorem 7.4. For this purpose, following [ARS], we introduce some notations. Let \( X, Y \in \text{mod } H \) be indecomposable modules. In the Auslander-Reiten quiver (AR quiver, for short), we write

\[ X \xrightarrow{(a,b)} Y \]

if there is a minimal right almost split \( X^a \oplus M \to Y \) such that \( M \) contains no summands isomorphic to \( X \), and there is a left right almost split \( X \to Y^b \oplus N \) such that \( N \) contains no summands isomorphic to \( Y \). Let \( D_X := \text{End}_H(X)/\text{rad}_H(X, X) \) and \( D_Y := \text{End}_H(Y)/\text{rad}_H(Y, Y) \). Then we have the following basic result.

**Proposition 7.5.** [ARS] section VII

(a) If there exists \( X \xrightarrow{(a,b)} Y \) in the AR quiver of \( \text{mod } H \), then we have

\[ \dim(\text{rad}_H(X, Y)/\text{rad}_H^2(X, Y)) \leq a \quad \text{and} \quad \dim(\text{rad}_H(X, Y)/\text{rad}_H^2(X, Y)) = b. \]

Moreover we have

\[ a \dim_k(D_X) = b \dim_k(D_Y) = \dim_k(\text{rad}_H(X, Y)/\text{rad}_H^2(X, Y)). \]

(b) If there exists an arrow \( Y \xrightarrow{(b,a)} Z \) in the AR quiver with non-projective \( Z \), then there exists the arrow \( \tau Z \xrightarrow{(a,b)} Y \).

Then we show the following proposition.

**Proposition 7.6.** Let \( H \) be a finite dimensional hereditary \( k \)-algebra and \( e_i \) a primitive idempotent of \( H \) and \( Q_i := e_i H \) the indecomposable projective \( H \)-module. Then there exists the following almost split sequence

\[ 0 \to Q_i \to \bigoplus_{i \neq j, M_i \neq 0} Q_j^{-e_{ij}} \to \bigoplus_{i \neq j, M_i \neq 0} \tau^{-1} Q_k^{-e_{ik}} \to \tau^{-1} Q_i \to 0. \]

**Proof.** We denote the multiplicity of \( X \) in \( M \) by \( m_X(M) \). Let \( Q_i \to E \) be a minimal left almost split. By Proposition 7.5, if we have \( Q_i \xrightarrow{(a,b)} \tau^{-1} Q_k \), then we have \( Q_k \xrightarrow{(a,b)} Q_i \) in the AR quiver. Since \( \text{rad} Q_i \to Q_i \) is a minimal right almost split, we have \( m_{-\text{rad} Q_i}(E) = m_{Q_k}(\text{rad} Q_i) \). Thus we have

\[ m_{Q_k}(\text{rad} Q_i) = \dim((\text{rad} Q_i/\text{rad}^2 Q_i)e_k)_{D_k} = \dim(M_k)_{D_k} = -c_{ik}. \]

On the other hand, Proposition 7.5 implies that, for \( Q_i \xrightarrow{(a,b)} Q_j \), we have \( a = m_{Q_i}(\text{rad} Q_j) \), \( b = m_{Q_j}(E) \) and

\[ \dim_k(\text{rad}_H(Q_i, Q_j)/\text{rad}_H^2(Q_i, Q_j)) = a \dim_k(D_{Q_i}) = b \dim_k(D_{Q_j}). \]

Since \( H \) is acyclic, we have \( \dim_k(D_{Q_i}) = \dim_k(D_i) = c_i \) and \( \dim_k(D_{Q_j}) = \dim_k(D_j) = c_j \). Thus, Proposition 7.5 also shows that we have

\[ m_{Q_j}(E) = m_{Q_j}(\text{rad} Q_j)c_i/c_j = \dim(M_i)_{D_i}c_i/c_j = -c_{ij}c_i/c_j = -c_{ij}. \]
and the conclusion follows. □

Let \( \Pi \) be the preprojective algebra of \( H \) and we denote by \( \{e_1, \ldots, e_n\} \) be a complete set of primitive orthogonal idempotents of \( \Pi \). For \( 1 \leq i \leq n \), we denote by \( i^\pm := \{1 \leq j \leq n \mid \imath_j \neq 0 \text{ or } \imath_j \neq 0\} \).

**Proposition 7.7.** Let \( H \) be a finite dimensional hereditary \( k \)-algebra of non-Dynkin type and \( \Pi \) the preprojective algebra of \( H \). For any \( 1 \leq i \leq n \), we have the following minimal projective resolution

\[
0 \longrightarrow e_i \Pi \longrightarrow \bigoplus_{j \in i^\pm} e_j \Pi^{\ell - c_{ij}} \longrightarrow e_i \Pi \longrightarrow D_i \longrightarrow 0.
\]

Although this is quite standard fact, we give a sketch for the convenience of the reader.

**Proof.** Since the preprojective algebra \( \Pi \) does not depend on the orientation of \((D_i, \imath_j)\), we choose an orientation such that \( \imath_M = 0 \) for any \( k \) and hence \( Q_k \) is simple. Then, a minimal projective resolution of \( Q_k \) is obtained as \( \mathbb{Z} \)-graded modules and its \( n \)-degree is given by applying \( \tau^{-n} \) to the exact sequence of Proposition 7.7, see e.g. [MY, Proposition 3.7], [GI, Theorem 4.12], [SS, Proposition 6.8]. Since \((e_i \Pi)_n \cong \tau^{-n}(Q_k)\), we have an exact sequence

\[
0 \longrightarrow e_i \Pi \longrightarrow \bigoplus_{j \in i^\pm} e_j \Pi^{\ell - c_{ij}} \longrightarrow e_i \Pi(1) \longrightarrow D_i(1) \longrightarrow 0
\]

as \( \mathbb{Z} \)-graded modules and this is a minimal projective resolution of \( D_i \). □

**Lemma 7.8.** Let \( H \) be a finite dimensional hereditary \( k \)-algebra of non-Dynkin type, \( \Pi \) the preprojective algebra of \( H \) and \( I_i := \Pi(1 - e_i)\Pi \).

(a) \( I_i \) is a classical tilting \( \Pi \)-module and we have an equivalence

\[
- \otimes \Pi \mathbb{L} I_i : \mathbb{D}^b(\mod \Pi) \to \mathbb{D}^b(\mod \Pi).
\]

(b) The action \( R_i := - \otimes \Pi I_i : K_0(\proj \Pi) \to K_0(\proj \Pi) \), \( [P] \mapsto [\imath \otimes \Pi I_i] \) satisfies the following

\[
R_i(e_i \Pi) := \begin{cases} [e_i \Pi], & \ell \neq i, \\ [-e_i \Pi] + \sum_{j=1}^n c_{ij} [e_j \Pi], & \ell = i. \end{cases}
\]

**Proof.** By Proposition 6.8, we have an exact sequence

\[
0 \longrightarrow e_i \Pi \longrightarrow \bigoplus_{j \in i^\pm} e_j \Pi^{\ell - c_{ij}} \longrightarrow e_i I_i \longrightarrow 0.
\]

From this sequence, the results follow from by the same argument of [BIRS, section II.1] (some general situation is also discussed in [IR, section 6]). □

From now on, let \( H := T_D(M) \) be a finite dimensional hereditary \( k \)-algebra of Dynkin type and we denote by the corresponding matrix by \( C \). Let \( \Pi \) be the preprojective algebra of \( H \) and \( I_i := \Pi(1 - e_i)\Pi \), where \( e_i \) the primitive idempotent of \( \Pi \). We denote by \( \langle I_1, \ldots, I_n \rangle \) the set of ideals of \( \Pi \) which can be written as

\[
I_{i_1} I_{i_2} \cdots I_{i_\ell}
\]

for some \( \ell \geq 0 \) and \( i_1, \ldots, i_\ell \in Q_0 \). Then we have the following results.

**Theorem 7.9.** There exists a bijection \( W \to \langle I_1, \ldots, I_n \rangle \), which is given by \( w \to I_w = I_{i_1} I_{i_{\ell-1}} \cdots I_{i_1} \), for any reduced expression \( w = s_{i_1} \cdots s_{i_\ell} \). Moreover the map gives an anti-isomorphism of posets

\[
W \longrightarrow \text{sr-tilt} \Pi,
\]

where we regard \( W \) as a poset with weak order, and also \( \text{sr-tilt} \Pi \) as a poset via the bijection \( 2-\text{silt} \Pi \cong \text{sr-tilt} \Pi \).
Proof. In simply-laced case, the result is [M Theorem 2.21]. In non-simply-laced case, results of [IR] implies that the map is bijection. Using Proposition 7.7, it is also easy to check that the same argument of [M] shows that they are support τ-tilting modules. □

Moreover, we prepare the following set up. Let \( \hat{H} \) be a finite dimensional hereditary \( k \)-algebra of affine type \( \hat{C} \) whose restriction to 1 to \( n \) columns and 1 to \( n \) rows is \( C \). Let \( \tilde{\Pi} \) be the preprojective algebra of \( \hat{H} \) such that \( \tilde{\Pi}/(e_{n+1}) \simeq \Pi \) and \( \tilde{W} \) the Coxeter group of \( \hat{C} \). As same as Dynkin type, we define \( \tilde{I}_i := \Pi(1-e_i)\Pi \) and \( \tilde{I}_w \) for \( w \in \tilde{W} \). We naturally regard \( W = \langle s_1, \ldots, s_n \rangle \) as a subgroup of \( \tilde{W} = \langle s_1, \ldots, s_n, s_{n+1} \rangle \). Then we recall the following lemma.

**Proposition 7.10.** (a) We have

\[
2\text{-silt}\Pi = \{ \tilde{I}_w \otimes \frac{L}{\Pi} \Pi \mid w \in W \}.
\]

(b) For any \( w \in W \), we have

\[
[c_i, \tilde{I}_w \otimes \frac{L}{\Pi} \Pi] = w\alpha_i^*.
\]

**Proof.** (a) Proposition 7.7 shows \( \text{sr-}\text{tilt}\Pi = \{ I_w \mid w \in W \} \). Then [M Proposition 5.2] (which works for arbitrary Dynkin type) implies that the correspondence \( I_w \mapsto \tilde{I}_w \otimes \frac{L}{\Pi} \Pi \) gives a bijective map \( \text{sr-}\text{tilt}\Pi \rightarrow 2\text{-silt}\Pi \) of [AIR Theorem 3.2].

(b) Since \( \tilde{I}_w \) is a classical tilting module, we have the following projective resolution

\[
0 \longrightarrow \tilde{P}^1 \longrightarrow \tilde{P}^0 \longrightarrow c_i \tilde{I}_w \longrightarrow 0,
\]
where \( \tilde{P}^0, \tilde{P}^1 \in \text{proj} \tilde{\Pi} \). Therefore, we have \( [c_i, \tilde{I}_w \otimes \frac{L}{\Pi} \Pi] = [\tilde{P}^0 \otimes \frac{L}{\Pi} \Pi] - [\tilde{P}^1 \otimes \frac{L}{\Pi} \Pi] \). Since \( \Pi \simeq \Pi/(e_{n+1}) \), \( [\tilde{P}^0 \otimes \frac{L}{\Pi} \Pi] \) is given by the restriction of \( [\tilde{P}^0] \) to \( K_0(\text{proj} \Pi) \) for \( j \in \{0, 1\} \). Hence, Lemma 7.8 implies that we have \( [c_i, \tilde{I}_w \otimes \frac{L}{\Pi} \Pi] = w\alpha_i^* \in K_0(\text{proj} \Pi) \).

Then we are ready to give a proof Theorem 7.4 (a) and (b).

**Proof of Theorem 7.4 (a) and (b).** (a) Recall that \( C(T) := \{ \sum_{i=1}^n a_i[T_i] \mid a_i \geq 0 \} \) for \( T = T_1 \oplus \cdots \oplus T_n \in 2\text{-silt}\Pi \) with indecomposable \( T_i \). Hence we have \( C(\Pi) = \{ \sum_{i=1}^n a_i[T_i] \mid a_i \geq 0 \} = \{ \sum_{i=1}^n a_i\alpha_i^* \mid a_i \geq 0 \} = D \). Then, by Proposition 7.10 (b), we have

\[
C(\tilde{I}_w \otimes \frac{L}{\Pi} \Pi) = \{ \sum_{i=1}^n a_i[c_i, \tilde{I}_w \otimes \frac{L}{\Pi} \Pi] \mid a_i \geq 0 \} = \{ \sum_{i=1}^n a_iw\alpha_i^* \mid a_i \geq 0 \} = w\{ \sum_{i=1}^n a_i\alpha_i^* \mid a_i \geq 0 \} = wD.
\]

Therefore, Proposition 7.10 (a) implies

\[
\bigcup_{T \in 2\text{-silt}\Pi} C(T) = \bigcup_{w \in W} wD.
\]

As a consequence, the \( g \)-fan \( \Sigma(\Pi) := \{ C(T) \mid T \in 2\text{-silt}\Pi \} \) is the set of all cones of \( wD \) and their faces, and we are done. By the same argument, we have \( C_{\leq 1}(\tilde{I}_w \otimes \frac{L}{\Pi} \Pi) = wC_{\leq 1}(\Pi) \) and hence

\[
P(\Pi) = \bigcup_{T \in 2\text{-silt}\Pi} C_{\leq 1}(T) = \bigcup_{w \in W} wC_{\leq 1}(\Pi).
\]

(b) Since

\[
\pi := - \otimes \frac{L}{\Pi} : K_0(\text{proj} \tilde{\Pi}) \rightarrow K_0(\text{proj} \Pi)
\]
is compatible with the action of \( W \), Theorem 5.10 and Proposition 7.10 implies that \( P(\Pi) \) is convex if and only if \( \sum_{j \in \pm 1} |c_{ij}| \leq 2 \). This is equivalent to saying that the Cartan matrix \( C \) is type \( A_n \) or \( B_n \). □

For a proof of Theorem 7.4 (c), we use the following well-known facts.
Lemma 7.11. (a) All roots of a given length are conjugate under W.
(b) The sum of all simple roots is a short root.

Proof. (a) is [Hu, III.10.4 Lemma C]. (b) is also well-known and it can be checked by case-by-case analysis. □

Then we give a proof of Theorem 7.4 (c).

Proof of Theorem 7.4 (c). Recall that 2-siltII = \{ \tilde{I}_w \otimes \Pi | w \in W \} from Proposition 7.11. For simplicity, we let \( P(w) := \tilde{I}_w \otimes \Pi \) and \( P(w)_i := e_i \tilde{I}_w \otimes \Pi \). By Proposition 2.17 there exists \( S(w) \in 2\text{-smcII} \) and \( 2\text{-smcII} = \{ S(w) \mid w \in W \} \) such that \( S(w) = S(w)_1 \oplus \cdots \oplus S(w)_n \) satisfying
\[
([P(w)_i], [S(w)_j]) = \delta_{ij}.
\]

On the other hand, Proposition 7.11 implies that we have \([P(w)_i] = w\alpha_i^*\). Therefore, we have
\[
(w\alpha_i^*, [S(w)_j]) = \delta_{ij}.
\]

Then, since the bilinear form \((-,-)\) is non-degenerate, we have \([S(w)_j]' = w\alpha_j\) and hence
\[
\{ [S(w)_j]' \mid 1 \leq j \leq n, w \in W \} = \{ w\alpha_j \mid 1 \leq j \leq n, w \in W \}.
\]
Thus (i) follows. Moreover, by this argument, we have
\[
v_{S(w)} = \sum_{i=1}^n [S(w)_i]' = w \sum_{i=1}^n \alpha_i.
\]
Then, Lemma 7.11 shows that \{ \{ \sum_{i=1}^n \alpha_i \} \mid w \in W \} coincide with all short roots of \( \Phi \) and we get (ii). Finally, (iii) immediately follows from (b) and Theorem 5.11. □

Example 7.12. (a) Let \( H = \left( \begin{smallmatrix} 3 & 0 \\ 0 & 2 \end{smallmatrix} \right) \). Then the corresponding Cartan matrix is \( \begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix} \). Let \( \Pi \) be the preprojective algebra of \( H \). We obtain the Hasse quiver of \( \mathfrak{s}_\tau\text{-tilt}\Pi \) as follows.

![Hasse Quiver](image)

On the other hand, the corresponding 2-simple minded collections are shown as follows.

![2-Simple Minded Collections](image)

Then, by taking \([-']\), we have the set of roots as follows
\[
\alpha_1, \alpha_2 \rightarrow -\alpha_1, \alpha_1 + \alpha_2 \rightarrow \alpha_1 + 2\alpha_2, -(\alpha_1 + \alpha_2) \rightarrow -(\alpha_1 + 2\alpha_2), \alpha_2
\]
\[
\alpha_1, \alpha_2 \rightarrow \alpha_1 + 2\alpha_2, -\alpha_2 \rightarrow -\alpha_1, -\alpha_2
\]
and hence we have
\[
\{ v_X \mid X \in 2\text{-smcII} \} = \{ \pm (\alpha_2, \alpha_1 + \alpha_2) \}.
\]
Thus $c$-polytope $P^c(\Pi)$ is illustrated as a dotted line in the left picture below and the $g$-polytope $P(\Pi)$ is illustrated in the right picture below.

\[
\begin{array}{c}
\begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_1 + \alpha_2 \\
\alpha_1 + 2\alpha_2
\end{array}
\end{array}
\quad
\begin{array}{c}
P_1 \\
P_2
\end{array}
\]

(b) Let $H = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)$. Then the corresponding Cartan matrix is $\left(\begin{smallmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{smallmatrix}\right)$. Let $\Pi$ be the preprojective algebra of $H$.

The root system of type $B_3$ is illustrated in the left picture below and the $g$-polytope $P(\Pi(B_3)) = (P^c(\Pi(B_3)))^*$ is illustrated in the right picture below.

\[
\begin{array}{c}
\begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_1 + \alpha_2 \\
\alpha_2 + \alpha_3 \\
\alpha_1 + 2\alpha_2
\end{array}
\end{array}
\quad
\begin{array}{c}
P_1 \\
P_2
\end{array}
\]

7.2. Generalized preprojective algebras. In this subsection, we study $g$-polytopes of generalized preprojective algebras introduced by [GLS]. We show that the $g$-polytope is convex if and only if the Cartan matrices is type $A_n$ or $B_n$.

First we introduce the notion of generalized preprojective algebras associated with symmetrizable generalized Cartan matrices [GLS].

Let $C = (c_{ij}) \in M_n(\mathbb{Z})$ be a symmetrizable generalized Cartan matrix. We denote by $g_{ij} := |\gcd(c_{ij}, c_{ji})|$ and $f_{ij} := |c_{ij}|/g_{ij}$.

An orientation of $C$ is a subset $\Omega$ of $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ such that the followings hold:

(i) $\{i, j\} \cap \Omega \neq \emptyset$ if and only if $c_{ij} < 0$;
(ii) For each sequence $((i_1, i_2), (i_2, i_3), \ldots, (i_t, i_1))$ with $t \geq 1$ and $(i_s, i_{s+1}) \in \Omega$ for all $1 \leq s \leq t$, we have $i_1 \neq i_t$.

The opposite orientation of an orientation $\Omega$ is defined as $\Omega^* := \{(j, i) \mid (i, j) \in \Omega\}$. Let $\overline{\Omega} := \Omega \cup \Omega^*$ and we define

$$\overline{\Omega}(-, i) := \{j \in Q_0 \mid (j, i) \in \overline{\Omega}\}.$$

For an orientation $\Omega$ of $C$, define the quiver $Q := Q(C, \Omega) := (Q_0, Q_1)$ with the set of vertices $Q_0 := \{1, \ldots, n\}$ and with the set of arrows

$$Q_1 := \{a_{ij}^g : j \rightarrow i \mid (i, j) \in \Omega, 1 \leq g \leq g_{ij}\} \cup \{\epsilon_i : i \rightarrow i \mid 1 \leq i \leq n\}.$$

We call $Q$ a quiver of type $C$. Let $Q^* := Q^*(C, \Omega)$ be the quiver obtained from $Q$ by deleting all loops $\epsilon_i$.

Then we define the generalized preprojective algebra associated to $C$ as follows.

Definition 7.13. Let $C$ be a symmetrizable Cartan matrix with a symmetrizer $D$.

For $(i, j) \in \Omega$, define

$$\text{sgn}(i, j) := \begin{cases} 1 & \text{if } (i, j) \in \Omega, \\ -1 & \text{if } (i, j) \in \Omega^*. \end{cases}$$

For $Q = Q(C, \Omega)$ and a symmetrizer $D = \text{diag}(c_1, \ldots, c_n)$ of $C$, we define an algebra

$$\Pi = \Pi(C, D, \Omega) := K\overline{\Omega}/\mathcal{T}$$
as follows. The double quiver $\overline{Q} = \overline{Q}(C)$ is obtained from $Q$ by adding a new arrow $\alpha^{(g)}_{ji} : j \to i$ for each arrow $\alpha^{(g)}_{ij} : i \to j$ of $Q^g$.

The ideal $\mathcal{I}$ of the path algebra $KQ$ is defined by the following relations:

(P1) For each $i \in Q_0$, we have the nilpotency relation
$$e_i^2 = 0.$$  

(P2) For each $(i, j) \in \Omega$ and each $1 \leq g \leq g_{ij}$, we have the commutativity relation
$$\alpha^{(g)}_{ij} f_i^{g_{ij}} = e_i^{f_i^{g_{ij}}} \alpha^{(g)}_{ij}.$$  

(P3) For each $i \in Q_0$, we have the mesh relation
$$\sum_{j \in \Omega(\pm i)} g_{ji} f_i^{g_{ij}} \sum_{g=1}^{f_i^{g_{ij}}} \text{sgn}(i, j) \epsilon_i^{f_i^{g_{ij}}-g} \alpha^{(g)}_{ij} \alpha^{(g)}_{ij} e_i^{f_i^{g_{ij}}} = 0.$$  

Remark 7.14. Our definition of the preprojective algebra is slightly different from the original one given by [GLS] (which we denote by $\Pi^{GLS}$), but two definitions are essentially the same objects. More precisely, we have $\Pi(C, D) = \Pi^{GLS}(tC, D)$, where $tC$ is the transposed Cartan matrix.

We remark that $\Pi$ does not depend on the orientation $\Omega$ of $C$, so that we can write $\Pi = \Pi(C, D)$.

For the one dimensional simple $\Pi$-modules $S_1, \ldots, S_n$, we define the $\iota : K_0(\text{mod } \Pi) \cong V$ by
$$\iota([S_i]) = e_i \text{ and, for } X = \bigoplus_{i=1}^n [S_i] \in \text{smc } \Pi,$$
$$v_X := \iota\left(\sum_{i=1}^n [X_i]\right) \in V.$$  

Then we have the following analogous result of Theorem 7.4, where the only difference is that $[S]$ in Theorem 7.4(c)(i) is replaced by $[S]$.

**Theorem 7.15.** Let $C$ be a symmetrizable Cartan matrix of Dynkin type with a symmetrizer $D$ and $\Pi = \Pi(C, D)$.

(a) $\Sigma(\Pi)$ is the Coxeter fan and we have
$$P(\Pi) = \bigcup_{w \in W} wC_{\leq 1}(\Pi).$$  

(b) $P(\Pi)$ is convex if and only if the Cartan matrix $C$ is type $A_n$ or $B_n$.

(c) (i) We have
$$\{[S] \mid S \in \text{ind-2-smc } \Pi\} = \Phi.$$  

(ii) Let $\Phi_{\text{short}}$ be the set of short roots of $\Phi$. We have
$$\{v_X \mid X \in 2\cdot \text{smc } \Pi\} = \Phi_{\text{short}} \text{ and } P^c(\Pi) = \text{conv}(\Phi_{\text{short}}).$$  

(iii) If the Cartan matrix $C$ is type $A_n$ or $B_n$, then $P(\Pi)$ is the dual polytope of $P^c(\Pi) = \text{conv}(\Phi_{\text{short}})$.

The proof of Theorem 7.15 is completely parallel to that of Theorem 7.4. Below we only highlight the differences. Instead of Proposition 7.7, we use the following result.

**Proposition 7.16.** [FG] Let $C$ be a symmetrizable Cartan matrix of non-Dynkin type with a symmetrizer $D$ and $\Pi = \Pi(C, D)$. For any $1 \leq i \leq n$, we have the following minimal projective resolution
$$0 \longrightarrow e_i \Pi \longrightarrow \bigoplus_{j \in \Omega} e_j \Pi^{-c_{ij}} \longrightarrow e_i \Pi \longrightarrow \hat{S}_i \longrightarrow 0,$$  

Because of our definition, we use the index $c_{ij}$ instead of $c_{ji}$ used in [FG].
where \( \hat{S}_i \) denotes by the generalized simple module. In particular, we have an exact sequence

\[
0 \rightarrow e_i \Pi \rightarrow \bigoplus_{j \in \hat{S}} e_j \Pi^{-e_{ij}} \rightarrow I_i \rightarrow 0.
\]

Then, by combining with results of [M, FG], Theorem 7.15 (a) and (b) follows from the same argument of subsection 7.1. To show Theorem 7.15 (c), it is enough to show the following result.

**Lemma 7.17.** For any \( S \in \text{ind-2-smc} \Pi \), we have

\[
\text{End}_{\text{D}^b(\text{mod} \Pi)}(S) \cong k.
\]

In particular,

\[
[S] = [S']'.
\]

For a proof, we recall results of [KM] and prepare the following setting. Let \( \hat{C} \) be an affine Cartan matrix whose restriction to 1 to \( n \) columns and 1 to \( n \) rows is \( C \). Let \( \hat{\Pi} \) be the complete preprojective algebra of \( \hat{C} \) with a symmetrizer. Then we have the following result.

**Proposition 7.18.** For any \( T \in \text{2-silt} \hat{\Pi} \), we have

\[
\text{End}_{\text{D}^b(\text{mod} \Pi)}(T) \cong \hat{\Pi}.
\]

Moreover, for any brick \( S \), we have \( \text{End}_{\hat{\Pi}}(S) \cong k \).

**Proof.** Recall from [KM] that any two-term silting complex is tilting complex and it belongs to one of the two connected components of mutation (which also works for non-simply laced cases by [FG]). One component contains \( \hat{\Pi} \) and the other contains \( \hat{\Pi}[1] \). Then by [FG, BIRS], the endomorphism algebra of them is isomorphic to \( \hat{\Pi} \). Thus, for \( T_i \in \text{2-psilt} \hat{\Pi} \), we have \( \text{End}_{\text{D}^b(\text{mod} \hat{\Pi})}(T_i) \cong e_i \hat{\Pi} e_i \) for an idempotent \( e_i \) of \( \hat{\Pi} \). Hence \( \text{End}_{\text{D}^b(\text{mod} \hat{\Pi})}(T_i)/\text{rad}(\text{End}_{\text{D}^b(\text{mod} \hat{\Pi})}(T_i)) \cong k \). On the other hand, for any \( S \in \text{ind-2-smc} \hat{\Pi} \), there exists \( T_i \in \text{2-psilt} \hat{\Pi} \) such that

\[
\text{End}_{\text{D}^b(\text{mod} \hat{\Pi})}(T_i)/\text{rad}(\text{End}_{\text{D}^b(\text{mod} \hat{\Pi})}(T_i)) \cong \text{End}_{\text{D}^b(\text{mod} \Pi)}(S).
\]

Therefore, we get \( \text{End}_{\text{D}^b(\text{mod} \Pi)}(S) \cong k \). \( \square \)

We give a proof of Lemma 7.17.

**Proof of Lemma 7.17.** Because \( \Pi = \hat{\Pi}/e_{n+1} \), mod \( \Pi \) is a full subcategory of mod \( \hat{\Pi} \). Therefore, Proposition 7.18 implies that we conclude \( \text{End}_{\hat{\Pi}}(S) \cong k \) for any brick \( S \) of mod \( \Pi \). \( \square \)

Using these preparations, one can prove Theorem 7.15 by an argument completely parallel to the proof of Theorem 7.1.

### 7.3. \( h \)-vectors of the simplicial complexes of preprojective algebras

In this subsection, we discuss \( h \)-vectors of the simplicial complexes of preprojective algebras. Let \( C \in M_n(\mathbb{Z}) \) be a symmetrizable Cartan matrix, and \( W = W(C) \) the corresponding Weyl group.

**Definition 7.19.** For \( w \in W \), we define

\[
\text{Des}(w) = \{ s_i \mid \ell(w) > \ell(s_i w) \} \quad \text{and} \quad \text{des}(w) = |\text{Des}(w)|.
\]

Moreover, we define

\[
E(W, j) := |\{ w \in W \mid \text{des}(w) = j \}|
\]

and it is called \( W \)-Eulerian numbers.

Then we get the following conclusion.

**Theorem 7.20.** Let \( C \in M_n(\mathbb{Z}) \) be a symmetrizable Cartan matrix, and \( \Pi := \Pi(C) \) the (classical or generalized) preprojective algebra. Then the \( h \)-vectors of \( \Delta(\Pi) \) is given by \( W \)-Eulerian numbers, that is, for each \( 0 \leq j \leq n \), we have

\[
h_j = E(W, j).
\]
Proof. By Theorem 3.4, $h_j$ is the cardinality of the set of all $T \in \text{2-silt}\Pi$ such that there exist precisely $j$ arrows starting at $T$ in Hasse(2-silt)$\Pi$. By using $h_j = h_{n-j}$ and the anti-isomorphism of posets in Theorem 7.9 and its variation for generalized preprojective algebras [FG], this equals $|\{w \in W \mid \text{des}(w) = j\}| = E(W, j)$. □

Table 1. The Eulerian numbers of type $A_n$.

| $n \setminus k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-----------------|---|---|---|---|---|---|---|
| 1               | 1 | 1 |   |   |   |   |   |
| 2               | 1 | 4 | 1 |   |   |   |   |
| 3               | 1 | 11| 11| 1 |   |   |   |
| 4               | 1 | 26| 66| 26| 1 |   |   |
| 5               | 1 | 57| 302|302|57|1 |   |
| 6               | 1 | 120|1191|2416|1191|120|1 |

Table 2. The Eulerian numbers of type $D_n$.

| $n \setminus k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|---|---|---|---|---|---|---|---|---|
| 4               | 1 | 44|102|44|1 |   |   |   |   |
| 5               | 1 | 157|802|802|157|1 |   |   |   |
| 6               | 1 | 530|5551|10876|5551|530|1 |   |   |
| 7               | 1 | 1731|35121|124427|124427|35121|1731|1 |   |
| 8               | 1 | 5528|208732|1265704|2201030|1265704|208732|5528|1 |

8. Brauer graph algebras and Root polytopes

In this section, we study $g$-polytopes of Brauer graph algebras. We show that every $g$-finite Brauer graph algebra is $g$-convex and the $g$-polytope is isomorphic to the root polytope of type $A_n$ or $C_n$ (Theorem 8.4), where $n$ is the number of edges of the associated Brauer graph.

8.1. Definition and main result. A Brauer graph algebra is defined from a combinatorial object called Brauer graph. We refer to a survey paper [Sc] for the background and basic properties of Brauer graph algebras.

Definition 8.1. A ribbon graph is a triple $\Gamma = (H, \sigma, (\ ))$, where $H$ is a non-empty finite set, $\sigma$ is a permutation on $H$, and $(\ ) : H \to H$ is a fixed-point free involution.

- Each element of $H$ is called half-edge of $\Gamma$.
- Let $H \to H/(\ )$ be a canonical surjection. Each element of $H/(\ )$ is called edge of $\Gamma$. We denote by $[h]$ the edge containing $h \in H$. Then, we have $[h] = [\overline{h}]$.
- Let $s : H \to H/\langle \sigma \rangle$ be a canonical surjection. Each element of $H/\langle \sigma \rangle$ is called vertex of $\Gamma$.
- For each vertex $v$ of $\Gamma$, the $\sigma$-orbit $(h, \sigma(h), \ldots, \sigma^{\ell-1}(h))$ incident to $v$ is called the cyclic ordering around $v$, where $\ell$ is the cardinality of this orbit.

A Brauer graph is a ribbon graph $\Gamma$ equipped with a multiplicity function, which assigns a positive integer $m(v) > 0$, called multiplicity, for every vertex $v$ of $\Gamma$.

In order to describe a given Brauer/ribbon graph $\Gamma = (H, \sigma, (\ ))$, we usually use its geometric realization, that is, a graph obtained from $\Gamma$ by gluing half-edges $h$ and $\overline{h}$ together to form a line whose endpoints are $s(h)$ and $s(\overline{h})$. When we describe the cyclic ordering $(h, \sigma(h), \ldots, \sigma^{\ell-1}(h))$
of half-edges around a vertex $v$, we draw them in the plane locally so that the half-edges appear around $v$ in this order counterclockwisely. See the following figure.

We also notice that a geometric realization naturally provides an undirected graph, which we call the underlying graph of $\Gamma$. From now on, we assume that every ribbon graph is connected, that is, the underlying graph is connected.

**Definition 8.2.** Let $\Gamma = (H, \sigma, \ell)$ be a Brauer graph with multiplicity function $m$. Let $Q_{\Gamma}$ be a finite quiver given as follows:

- The set of vertices is the set $E$ of edges of $\Gamma$.
- We draw an arrow $a_h: [h] \to [\sigma(h)]$ for every $h \in H$.

We define the algebra $B_{\Gamma} := kQ_{\Gamma}/I_{\Gamma}$, where $I_{\Gamma}$ is a two-sided ideal in the path algebra $kQ_{\Gamma}$ generated by all relations of the following forms:

$$a_{\sigma^{-1}(h)}a_{\bar{h}}: [\sigma^{-1}(h)] \xrightarrow{\sigma^{-1}(h)} [h] \xrightarrow{\sigma(h)} [\sigma(h)] \quad \text{and}$$

$$C_h^{m(s(h))} - C_{\bar{h}}^{m(s(\bar{h}))}$$

for all $h \in H$. Here, $C_h$ denotes a cycle

$$C_h: [h] \xrightarrow{a_h} [\sigma(h)] \xrightarrow{\sigma(h)} [\sigma^2(h)] \xrightarrow{\sigma(\ell)(h)} [\sigma^{\ell}(h)] = [h]$$

in $Q_{\Gamma}$ and $\ell$ denotes the cardinality of the $\sigma$-orbit incident to $s(h)$. We call $B_{\Gamma}$ the **Brauer graph algebra** associated to $\Gamma$.

It is well-known (see [Sc] for example) that Brauer graph algebras are finite dimensional symmetric algebras, which are special biserial. We give an example of some Brauer graphs and their associated Brauer graph algebras in Example 8.14.

The following special classes of Brauer graph algebras play a central role in this section. Now, we say that a **cycle graph** of length $\ell$ is an undirected graph which consists of a single cycle having $\ell$ edges. An **odd cycle** (respectively, **even cycle**) is a cycle graph of odd (respectively, even) length.

**Definition 8.3.** A Brauer graph $\Gamma$ is called **Brauer tree** if its underlying graph is a tree, and called **Brauer odd-cycle** if its underlying graph contains precisely one odd cycle and no even cycles. In this case, $B_{\Gamma}$ is called **Brauer tree algebra** and **Brauer odd-cycle algebra** respectively.

Brauer trees and Brauer odd-cycles have canonical embeddings into the plane so that the cyclic ordering of the half-edges incident to each vertex is described in counterclockwise direction. See Figure 1.
The following is a main result in this section, where the equivalence of (i) and (iii) in (a) below was shown in [AAC] (see also [STV]), and (b) for Brauer tree algebras was shown in [AMN].

**Theorem 8.4.** Let $\Gamma$ be a Brauer graph with $n$ edges and $B_\Gamma$ the Brauer graph algebra of $\Gamma$.

(a) $B_\Gamma$ is pairwise $g$-convex. Moreover, the following conditions are equivalent.

(i) $B_\Gamma$ is $g$-finite,
(ii) $B_\Gamma$ is $g$-convex,
(iii) $\Gamma$ is either a Brauer tree or a Brauer odd-cycle.

(b) Let $\Phi$ be a root system of type $A_n$ or $C_n$ and $L$ the root lattice of $\Phi$. If one of the equivalent conditions of (a) hold, then we have an isomorphism $K_0(\text{proj } B_\Gamma) \simeq L$ which restricts to a bijection $[2\text{-psilt}^1 B_\Gamma] \simeq \Phi$:

$$K_0(\text{proj } B_\Gamma) \longrightarrow L$$

where $\Phi$ is of type $A_n$ (respectively, $C_n$) if $\Gamma$ is a Brauer tree (respectively, Brauer odd-cycle). Therefore, it induces an isomorphism

$$P(B_\Gamma) \cong P_\Phi$$

of lattice polytopes, and the $f$-vector and the $h$-vector are given by

$$f_{A_n}(x) := \sum_{m=0}^{n} \frac{(n+m)!}{m!m!(n-m)!} x^{n-m}, \quad h_{A_n}(x) := \sum_{k=0}^{n} \binom{n}{k} x^k$$

(respectively, $f_{C_n}(x) := \sum_{m=0}^{n} \frac{2m}{n+m} \binom{n+m}{2m} x^{n-m}, \quad h_{C_n}(x) := \sum_{k=0}^{n} \binom{2n}{2k} x^k$).

We note that two $g$-convex Brauer graph algebras may have the same $g$-polytope even though they are not derived equivalent, see Example 8.4. We also note that one can describe the $g$-fan $\Sigma(B_\Gamma)$ in terms of shear coordinate of laminations corresponding to $\Gamma$ [AY] (see also [FT]).

To prove Theorem 8.4 we need some preparations. It is mentioned in [EJR] (see also [AAC]) that the set 2-silt $B_\Gamma$ only depends on the ribbon graph of $\Gamma$ and is independent of the multiplicity function $\mathbf{m}$. For simplicity, we may replace $\mathbf{m}$ to a multiplicity function identically equal to 1, and identify $\Gamma$ with its ribbon graph naturally.

### 8.2. Signed half-walks.

In this subsection, we recall a classification of 2-term silting complexes over Brauer graph algebras due to [AAC]. Let $\Gamma = (H, \sigma, (\cdot))$ be an arbitrary Brauer graph with $n$ edges.

**Definition 8.5.** (1) A half-walk of $\Gamma$ is a sequence of half-edges $w := (h_1, \ldots, h_m)$ such that $s(h_i) = s(h_{i+1})$ and $\overline{h}_i \neq h_{i+1}$ for all $i \in \{1, \ldots, m-1\}$. Defining $\overline{\mathbf{m}} = (\overline{h}_m, \ldots, \overline{h}_1)$ makes ( ) an involution on the set of signed half-walks.

(2) A walk of $\Gamma$ is the unordered pair $W = \{w, \overline{\mathbf{m}}\}$ of half-walks. For a walk $W = \{w = (h_1, \ldots, h_m), \overline{\mathbf{m}}\}$, we define $s(w) := s(h_1)$ and $s(\overline{\mathbf{m}}) = s(\overline{h}_m)$ as the endpoints of $W$.

(3) A signature on a walk $W = \{w := (h_1, \ldots, h_m), \overline{\mathbf{m}}\}$ is an assignment of signs $\epsilon(h) = \epsilon(\overline{h}_i) \in \{\pm 1\}$ for the half-edges appearing in $W$ such that $\epsilon(h_i) \neq \epsilon(h_{i+1})$ for all $i \in \{1, \ldots, m-1\}$.

A half-walk $w$ equipped with a signature $\epsilon$ on a walk $w \in W$ is called signed half-walk and written by $w^\epsilon$ or $(h_1, \ldots, h_m; \epsilon)$. A signed walk is the unordered pair $W^\epsilon = \{w^\epsilon, \overline{\mathbf{m}}\}$ of signed half-walks.

Remark that one can not always find a signature for a given walk $W = \{w := (h_1, \ldots, h_m), \overline{\mathbf{m}}\}$. However, if one finds a signature on $W$, then there are precisely two signatures $\pm \epsilon$, giving signed half-walks $w^\epsilon = (h_1, \ldots, h_m; \epsilon)$ and $w^{-\epsilon} = (h_1, \ldots, h_m; -\epsilon)$. 
Similar to a walk on undirected graphs, we can describe a given (signed) half-walk graphically as follows:

\[ w := (h_1, h_2, \ldots, h_m) \]

\[ s(w) = \cdot h_1 \quad h_2 \quad \ldots \quad h_m \quad \cdot s(w) \]

Here, we need to take in account of loops carefully. For example, \( w_1 \neq w_2 \) as half-walks in the next example:

\[ w_1 := (h_1, h_2, h_3) \quad w_2 := (h_1, h_2, h_3) \]

\[ \bullet h_1 \quad \bullet h_2 \quad \bullet h_3 \]

\[ \bullet h_1 \quad \bullet h_2 \quad \bullet h_3 \]

We need to attach some extra data to each endpoint of a signed walk, which is uniquely determined by its signature.

**Definition 8.6.** A virtual (half-)edge is an element of the set of the symbols \( \{ \operatorname{vr}_+ (h), \operatorname{vr}_- (h) \mid h \in H \} \). We extend the map \( s \) to virtual edges by \( s(\operatorname{vr}_+ (h)) = s(h) \) for all \( h \in H \). For a vertex \( v \), we extend the cyclic ordering \( (h, \sigma(h), \ldots, \sigma^{\ell-1}(h)) \) around \( v \) to the cyclic ordering around \( v \) accounting the virtual edges by

\[ (\operatorname{vr}_-(h), h, \operatorname{vr}_+(h), \operatorname{vr}-(\sigma(h)), \sigma(h), \operatorname{vr}_+(\sigma(h)), \ldots, \operatorname{vr}-(\sigma^{\ell-1}(h)), \sigma^{\ell-1}(h), \operatorname{vr}_+(\sigma^{\ell-1}(h))) \]  

(8.2)

A subsequence of (8.2) is called a cyclic subordering around \( v \) accounting the virtual edges. To each signed walk \( W^\epsilon = \{ w^\epsilon = (h_1, \ldots, h_m; \epsilon), \overline{w^\epsilon} \} \), we attach virtual edges given by

\[ h_0 = h_0 := \operatorname{vr}_-(h_1)(h_1) \quad \text{and} \quad h_{r+1} = h_{r+1} := \operatorname{vr}_-(h_m)(h_m) \]

with signs \( \epsilon(h_0) = \epsilon(h_1) \) and \( \epsilon(h_{m+1}) = \epsilon(h_m) \).

Now we define the admissibility of signed walks by the non-crossing conditions (NC0)-(NC3) below. Fix a pair of (not necessarily distinct) signed walks \( W^\epsilon = \{ w^\epsilon = (h_1, \ldots, h_m; \epsilon), \overline{w^\epsilon} \} \) and \( W^{\epsilon'} = \{ w^{\epsilon'} = (h'_1, \ldots, h'_m; \epsilon'), \overline{w^{\epsilon'}} \} \). Notice that there are virtual edges \( h_0, h_{\ell+1}, h'_0, h'_{m+1} \), etc. with signs attached to their endpoints.

**Definition 8.7.** [AAC] Define 2.7 We say that \( W^\epsilon \) and \( W^{\epsilon'} \) satisfies (NC0) (it was called the sign condition in [AAC]) if the following condition is satisfied:

(NC0) Whenever \( W^\epsilon \) and \( W^{\epsilon'} \) have a common endpoint \( v \), the signatures on the half-edges of \( W^\epsilon \) and of \( W^{\epsilon'} \) incident to \( v \) are the same.

Next, we say that a maximal common subwalk of \( W \) and \( W' \) is a walk \( Z = \{ z, \overline{z} \} \) such that \( z \) is a common continuous subsequence of some \( x \in W \) and \( y \in W' \), and there are no common continuous subsequences of \( x \) and \( y \) properly containing \( z \) as a continuous subsequence. It is said to be proper if no endpoints of \( Z \) are the common endpoint(s) of \( W \) and \( W' \). Notice that there are several maximal common subwalks of \( W \) and \( W' \) in general.

**Definition 8.8.** [AAC] Define 2.8 Let \( Z \) be a maximal common subwalk of \( W \) and \( W' \) given by a half-walk \( z = (t_1, \ldots, t_r) \) so that \( t_k = h_{i+k-1} = h'_{i+k-1} \) for all \( k = 1, \ldots, r \). Let \( u := s(t_1) \) and \( v := s(\overline{t_r}) \) be endpoints of \( Z \). We call the pair of cyclic subordering on \( \{ h_{i-1}, h'_{i-1}, t_1 \} \) around \( u \) and \( \{ h_{i+r}, h'_{i+r}, \overline{t_r} \} \) around \( v \) accounting virtual edges the neighbourhood cyclic ordering of \( Z \). Then, we say that \( W^\epsilon \) and \( W^{\epsilon'} \) satisfies (NC1) and (NC2) at \( Z \) if the following condition is satisfied respectively.

(NC1) \( \epsilon(t_k) = \epsilon'(t_k) \) for all \( k = 1, \ldots, r \).
If $Z$ is proper, then the neighbourhood cyclic ordering of $Z$ is either

\[ \begin{array}{ccc}
 h_{i+1} & h_{i+r} & h_i \\
 t_1 & t_2 & \ldots & t_r \\
 h'_{j-1} & h'_{j+r} & h'_i \\
 u & v & t_1 & t_2 & \ldots & t_r \\
 \end{array} \]

or

\[ \begin{array}{ccc}
 h'_{j-1} & h'_{j+r} & h_i \\
 t_1 & t_2 & \ldots & t_r \\
 h_{i+1} & h_{i+r} & h_i \\
 u & v & t_1 & t_2 & \ldots & t_r \\
 \end{array} \]

Notice that the condition (NC1) is automatically satisfied for any signed walk with itself by the definition of the signature (Definition 8.5(3)).

Lastly, for a vertex $v$ and an integer $i \in \{1, \ldots, r\}$ such that $v = s(h_i)$, we refer to the set $\{h_{i-1}, h_i\}$ as a neighborhood of $v$ in $W$. Notice that half-edges appearing in the neighborhood at a vertex can be virtual.

Definition 8.9. \(\text{[AAC, Definition 2.9]}\) For a vertex $v$, suppose that $\{a, b\}$ and $\{c, d\}$ are neighborhoods of $v$ in $W$ and $W'$ respectively. Then, we say that $v$ is the intersecting vertex of $W$ and $W'$ if $a, b, c, d$ are pairwise distinct. We say that $W$ and $W'$ satisfies (NC3) at the intersecting vertex $v$ if the following condition is satisfied.

(NC3) If $v$ is an intersecting vertex with respect to the neighborhoods $\{a, b\}$ in $W$ and $\{c, d\}$ in $W'$, and at most one of $a, b, c, d$ is virtual, then the cyclic subordering around $v$ accounting virtual edges and signatures are either

\[ \begin{array}{ccc}
 b^- & a^+ & c^- \\
 e^- & a^- & d^- \\
 \end{array} \]

or

\[ \begin{array}{ccc}
 b^- & a^+ & c^- \\
 e^- & a^- & d^- \\
 \end{array} \]

Definition 8.10. We say that two signed walks $W'$ and $W''$ are admissible if they satisfy (NC0), (NC1) and (NC2) at all maximal common subwalks, and (NC3) at all intersecting vertices. An admissible signed walk is a signed walk that is admissible with itself.

Now, we denote by $\text{AW}(\Gamma)$ the set of admissible signed walks of $\Gamma$, by $\text{CW}(\Gamma)$ the set of maximal collections consisting of admissible signed walks of $\Gamma$ which are pairwise admissible.

Theorem 8.11. \(\text{[AAC, Theorem 4.6]}\) Let $\Gamma$ be a Brauer graph and $B\Gamma$ the Brauer graph algebra of $\Gamma$. Then, there are bijections

\[ 2\text{-psilt}^1 B\Gamma \xrightarrow{\sim} \text{AW}(\Gamma) \text{ and } 2\text{-silt}B\Gamma \xrightarrow{\sim} \text{CW}(\Gamma). \]  \hspace{1cm} (8.3)

From the definition of Brauer graph algebras, the Grothendieck group $K_0(\text{proj } B\Gamma)$ is canonically isomorphic to the free $\mathbb{Z}$-module $\mathbb{E}$ over the set $E$ of edges of $\Gamma$. For a signed walk $W^\epsilon = \{w^\epsilon = (h_1, \ldots, h_m; \epsilon), \mathbb{w}^\epsilon\}$, one can define

\[ [W^\epsilon] := \sum_{i=1}^r \epsilon(h_i)[h_i] \in \mathbb{E}^\epsilon. \]  \hspace{1cm} (8.4)

In fact, it is independent of a choice of $w \in W$. In addition, let

\[ [\text{AW}(\Gamma)] := \{[W^\epsilon] \mid W^\epsilon \in \text{AW}(\Gamma)\} \subset \mathbb{E}. \]

Then we have the following.

Proposition 8.12. We have the following commutative diagram

\[ K_0(\text{proj } B\Gamma) \xrightarrow{\sim} \mathbb{E} \]

\[ \cup \]

\[ [2\text{-psilt}^1 B\Gamma] \xrightarrow{\sim} [\text{AW}(\Gamma)]. \]  \hspace{1cm} (8.5)
Proof. Let $T \in 2$-psilt$^1 \mathcal{B}_\Gamma$ and $W^e \in \mathcal{AW}(\Gamma)$ the signed walk corresponding to $T$ under the bijection $2$-psilt$^1 \mathcal{B}_\Gamma \simeq \mathcal{AW}(\Gamma)$ in \cite[8.3]{AAC}. According to \cite[Section 4]{AAC}, one can see that the class $[T]$ is sent to $[W^e]$ under the canonical isomorphism $K_0(\text{proj} \mathcal{B}_\Gamma) \simeq \mathbb{Z}E$. \hfill $\square$

Example 8.13. For an edge $X = \{h, \overline{h}\} \in E$, the element $X \in \mathbb{Z}E$ belongs to $[\mathcal{AW}(\Gamma)]$. In fact, a signed half-walk $W^+_X = \{(h; +), (\overline{h}; +)\}$ is clearly admissible and satisfies $[W^+_X] = X$ in $\mathbb{Z}E$. On the other hand, let $P_X$ be the indecomposable projective $\mathcal{B}_\Gamma$-module corresponding to $X$. Then $P_X \in 2$-psilt$^1 \mathcal{B}_\Gamma$ is sent to $W^+_X$ by the bijection in Theorem \ref{thm:8.11}.

Example 8.14. We depict the $g$-polytope and the corresponding root polytope for a class of Brauer graph algebras with small number of edges.

(a) For $n = 2$, there are only one Brauer tree and one Brauer odd-cycle described as follows.

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {$1$}; \node (2) at (1,0) {$2$};
\draw (1) -- (2);
\end{tikzpicture}
\end{center}

Then the associated Brauer graph algebras are presented by the following quivers with relations:

\[ k\left(\begin{array}{c}
1 & b \\
\frac{1}{a} & 2
\end{array}\right) / \langle aba, bab \rangle \quad \text{and} \quad k\left(\begin{array}{c}
1 \\
\frac{1}{a} & 2 & b
\end{array}\right) / \langle ac, b^2, abca, bca - cab \rangle \]

The $g$-polytopes of their associated Brauer graph algebras are

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {$1$}; \node (2) at (1,0) {$2$};
\draw (1) -- (2);
\end{tikzpicture}
\end{center}

which clearly correspond to root polytopes of type $A_2$ and $C_2$ respectively.

(b) Let $\Gamma_1$ be the following Brauer tree having 3 edges.

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {$1$}; \node (2) at (1,0) {$2$}; \node (3) at (2,0) {$3$};
\draw (1) -- (2) -- (3) -- (1);
\end{tikzpicture}
\end{center}

Then $B_{\Gamma_1}$ is the algebra given in Example \ref{example:5.26}. There are 12 elements in $2$-psilt$^1 B_{\Gamma_1}$. We describe the corresponding $g$-vectors and admissible signed walks in the following table.

| $[W^e]$ | $(1, 0, 0)$ | $(0, 1, 0)$ | $(0, 0, 1)$ | $(1, -1, 0)$ | $(0, 1, -1)$ | $(1, -1, 1)$ |
|---------|-------------|-------------|-------------|-------------|-------------|-------------|
| $W^e$   | $\bullet \ldots \cdot \cdot \cdot$ | $\cdot \ldots \cdot \cdot \cdot$ | $\ldots \cdot \cdot \cdot \cdot \cdot \cdot$ | $\cdot \cdot \cdot \cdot \cdot \cdot$ | $\cdot \cdot \cdot \cdot \cdot \cdot$ | $\cdot \cdot \cdot \cdot \cdot \cdot$ |
| $(-1, 0, 0)$ | $(0, -1, 0)$ | $(0, 0, -1)$ | $(-1, 1, 0)$ | $(0, -1, -1)$ | $(-1, 1, 1)$ | $(-1, -1, -1)$ |
| $\cdot \ldots \cdot \cdot \cdot \cdot \cdot \cdot$ | $\cdot \ldots \cdot \cdot \cdot \cdot \cdot \cdot$ | $\ldots \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$ | $\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$ | $\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$ | $\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$ | $\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$ |

By Proposition \ref{prop:5.10}(b), the $g$-polytope of $B_{\Gamma_1}$ is given by the convex hull in $\mathbb{R}^3$ of all integer vectors appearing in the above table and described as follows.

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {$1$}; \node (2) at (1,0) {$2$}; \node (3) at (2,0) {$3$};
\draw (1) -- (2) -- (3) -- (1);
\end{tikzpicture}
\end{center}

It is isomorphic to the root polytope of type $A_3$ up to the linear isomorphism by \[ \left(\begin{array}{c}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{array}\right) \].
(c) Let $\Gamma_2$ be the following Brauer odd-cycle having 3 edges.

$$
\begin{align*}
&1 \\
\end{align*}
\begin{align*}
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qu
It implies that $B_{r_2}$ and $B_{r_3}$ determine the same g-polytope, while they do not determine the same g-fan, see Figure 2. Notice that they are not derived equivalent.

8.3. Signed half-walks and root lattices. Let $\Gamma = (H, \sigma, (\cdot))$ be a Brauer graph having $n$ edges, which is either a Brauer tree or a Brauer odd-cycle. In this subsection, we associate the root system $\Phi(\Gamma)$ to $\Gamma$ and show that $\Phi(\Gamma)$ is in bijection with the set $\mathcal{AW}(\Gamma)$ of admissible signed walks of $\Gamma$ (Proposition 8.16). Using this result, we prove Theorem 8.4.

Definition 8.15. Let $V$ be the set of vertices of $\Gamma$ and $RV$ the vector space with basis $V$. Then

$$\dim \mathbb{R}V = \begin{cases} 
  n+1 & \text{if } \Gamma \text{ is a Brauer tree,} \\
  n & \text{if } \Gamma \text{ is a Brauer odd-cycle.}
\end{cases}$$

We define the inner product on $\mathbb{R}V$ which makes $V$ an orthonormal basis. We define the root system associated to $\Gamma$ by

$$\Phi(\Gamma) := \begin{cases} 
  \Phi_{An} = \{ u - v \mid (u,v) \in V \times V, u \neq v \} & \text{if } \Gamma \text{ is a Brauer tree,} \\
  \Phi_{Cn} = \{ \pm u \pm v \mid (u,v) \in V \times V, u \neq v \} \cup \{ \pm 2u \mid u \in V \} & \text{if } \Gamma \text{ is a Brauer odd-cycle.}
\end{cases}$$

We denote the root lattice of $\Phi(\Gamma)$ by $L(\Gamma)$. The root polytope $P_{\Phi(\Gamma)}$ is defined as the convex hull of $\Phi(\Gamma)$ in $\mathbb{R} \otimes \mathbb{Z} L(\Gamma)$.

Proposition 8.16. Let $\Gamma$ be a Brauer graph which is either a Brauer tree or a Brauer odd-cycle. Let $E$ be the set of edges of $\Gamma$. Then, we have an isomorphism $\partial: \mathbb{Z}E \simeq L(\Gamma)$ which restricts to a bijection $[\mathcal{AW}(\Gamma)] \simeq \Phi(\Gamma)$.

$$\partial: \mathbb{Z}E \overset{\sim}{\longrightarrow} L(\Gamma)$$  \hfill (8.6)

$$\bigcup \bigcup [\mathcal{AW}(\Gamma)] \overset{\sim}{\longrightarrow} \Phi(\Gamma).$$

From now on, we prove Proposition 8.16. We need to fix a spanning tree $\Gamma_{sp}$ of $\Gamma$, that is, $\Gamma_{sp}$ is one of subtrees which include all vertices of $\Gamma$. Let $H_{sp}$ be the set of half-edges of $\Gamma_{sp}$ and $H_{sp}^c$ the complement of $H_{sp}$. If $\Gamma$ is a Brauer tree, then $H_{sp}^c = \emptyset$; Otherwise, $H_{sp}^c$ consists of two half-edges which form an edge lying in the unique odd cycle of $\Gamma$.

Definition 8.17. An orientation of $\Gamma_{sp}$ is a complete set $\sigma$ of representatives of $H_{sp}/(\cdot)$ in $H_{sp}$. For such $\sigma$, we say that a vertex $v$ is a source (respectively, a sink) if there are no half-edges $h \in \sigma$ such that $v = s(h)$ (resp., $v = s(h)$). A bipartite orientation is an orientation such that every vertex of $\Gamma_{sp}$ is either a source or a sink.
By definition, we have $C$, which is clearly a basis of $L$. Take one of such an orientation $o$ of $\Gamma$. We have the following commutative diagram.

**Proposition 8.18.** We have the following commutative diagram.

![Diagram](image)

Moreover, for any $h \in H$, we have

$$\partial([h]) = \begin{cases} 
\sigma(s(h)))(s(h) - s(h)) & \text{if } h \in H, \\
\sigma(s(h))(s(h) + s(h)) & \text{if } h \in H^c.
\end{cases} \quad (8.7)$$

**Proof.** For the former assertion, it suffices to show $\delta(\ker\pi) = 0$. By definition, the kernel $\ker\pi$ is generated by elements of the forms $h + h$ with $h \in H$ and $h - h$ with $h \in H^c$. Then, we have

$$\delta(h + h) = (s(h) - s(h)) + (s(h) - s(h)) = 0 \quad \text{for all } h \in H,$$

$$\delta(h - h) = (s(h) + s(h)) - (s(h) + s(h)) = 0 \quad \text{for all } h \in H^c.$$

Thus, we get the former assertion. The latter one is clear.

**Proposition 8.19.** The map $\partial$ restricts to an isomorphism $\partial : \mathbb{Z}E \simeq L(\Gamma)$.

**Proof.** As we mentioned in Example 8.13, a basis $E$ of $\mathbb{Z}E$ is included in $[AW(\Gamma)]$. We first assume that $\Gamma$ is a Brauer tree and hence $\Phi(\Gamma) = \Phi_{\text{A}}$. By $[6,7]$, $E$ is bijectively sent to

$$\partial(E) = \{s(h) - s(h) \mid h \in \sigma\} \subseteq \Phi_{\text{A}},$$

which is clearly a basis of $L(\Gamma)$. Next, we assume that $\Gamma$ is a Brauer odd-cycle and hence $\Phi(\Gamma) = \Phi_{\text{C}}$. We denote by $X = \{h_0, h_0\}$ the unique edge which does not lie in $\Gamma$. Then, $E$ is bijectively sent to

$$\partial(E) \supseteq \{s(h) - s(h) \mid h \in \sigma\} \cup \{\sigma(s(h_0))(s(h_0) + s(h_0))\} \subseteq \Phi_{\text{C}},$$

which is easily shown to be a basis of $L(\Gamma)$.

![Figure 3. A bipartite orientation on the spanning tree.](image)
Now, we study signed walks of \( \Gamma \). The following is basic.

**Lemma 8.20.** Let \( \Gamma \) be a Brauer tree or a Brauer odd-cycle.

(a) Let \( w^e = (h_1, \ldots, h_m; \epsilon) \) be a signed half-walk. For any edge \( X \) lying in the odd cycle in \( \Gamma \), there is at most one \( 1 \leq i \leq m \) such that \( h_i \in X \). For a signed half-walk \( w^e \) of \( \Gamma \), every half-edge lying in the odd cycle in \( \Gamma \) appears at most once in \( w^e \).

(b) Every signed walk \( W^e \) of \( \Gamma \) satisfies (NC0) and (NC1).

**Proof.** (a) We only have to consider a Brauer odd-cycle \( \Gamma \). We denote by \( Z \) the unique odd cycle in \( \Gamma \), by \( \ell \) the length of \( Z \). Take an integers \( 1 \leq j < k \leq m \) such that \( h_j, h_k \in X \) and there are no \( j < i < k \) satisfying \( h_i \in X \). Since \( \Gamma \) has no cycle except for \( Z \), we have \( h_j = h_k \) and \( k - j = \ell \). Since \( \ell \) is odd, we have \( \epsilon(h_j) = \epsilon(h_k) = (-1)^{k-j}\epsilon(h_j) = (-1)^{\ell}\epsilon(h_j) = -\epsilon(h_j) \), a contradiction.

(b) The condition (NC1) is automatic for any signed walk.

Let \( W^e = \{w^e = (h_1, \ldots, h_m; \epsilon), \overline{w} \} \) be a signed walk with endpoints \( u := s(h_1) \) and \( v := s(h_m) \). If \( u \neq v \), the condition (NC0) is automatic. Assume that \( u = v \). In this case, \( \Gamma \) is a Brauer odd-cycle. We denote by \( Z \) the unique odd cycle in \( \Gamma \), by \( \ell \) the length of \( Z \). Since \( \Gamma \) has no cycles except for \( Z \), using (a), we can write \( w = p \overline{\pi} \), where \( p = (h_1, \ldots, h_r) \) with \( r \geq 0 \) and \( z = (h_{r+1}, \ldots, h_{r+\ell}) \) forms the cycle \( Z \).

\[
\begin{array}{c}
\cdot \ldots \cdot \\
| | | | \\
h_m \quad \cdots \quad h_{r+1} \quad \cdots \quad h_{r+\ell} \\
\cdot \ldots \cdot \\
u \ 
\end{array}
\]

Since \( \ell \) is odd, so is \( m = 2r + \ell \). Thus, \( \epsilon(h_m) = (-1)^{m-1}\epsilon(h_1) = \epsilon(h_1) \) as desired. \( \square \)

By Lemma 8.20(b), for a signed walk \( W = \{w^e = (h_1, \ldots, h_m; \epsilon), \overline{w} \} \), one can extend the signature \( \epsilon \) to the endpoints \( s(w) = s(h_1) \) and \( s(\overline{w}) = s(h_m) \) of \( W \) by \( \epsilon(s(w)) := \epsilon(h_1) \) and \( \epsilon(s(\overline{w})) := \epsilon(h_m) \). Moreover, one can assign the sign \( \sigma(s(w)) := \sigma(s(w))\epsilon(s(w)) \in \{\pm 1\} \) on \( s(w) \), and similarly \( \sigma(s(\overline{w})) \) on \( s(\overline{w}) \).

The following result means that the element \( \partial([W^e]) \) is completely determined by the endpoints of \( W \) and the signature on them.

**Lemma 8.21.** For a signed walk \( W^e = \{w^e, \overline{w} \} \) of \( \Gamma \), the following holds.

\[
\partial([W^e]) = \begin{cases} 
\sigma(s(w))(s(w) - s(\overline{w})) & \text{if } W \text{ is a walk of } \Gamma_{sp}, \\
\sigma(s(w))(s(w) + s(\overline{w})) & \text{else.}
\end{cases}
\]  

(8.8)

**Proof.** Let \( W^e = \{w^e = (h_1, \ldots, h_m; \epsilon), \overline{w} \} \) be a signed walk. By (8.4) and (8.7), we have

\[
[W^e] = \sum_{i=1}^{m} \epsilon(h_i)[h_i] \in \mathbb{Z}E \quad \text{and}
\]

\[
\partial([h_i]) = \begin{cases} 
\sigma(s(h_i))(s(h_i) - s(h_{i+1})) & \text{if } h_i \in H_{sp}, \\
\sigma(s(h_i))(s(h_i) + s(h_{i+1})) & \text{if } h_i \in H_c
\end{cases}
\]

where \( 1 \leq i \leq m \) and \( s(h_{m+1}) = s(h_m) \). If \( W \) is a walk of \( \Gamma_{sp} \) (that is, \( h_i \in H_{sp} \) for all \( 1 \leq i \leq m \)), we have

\[
\partial([W^e]) = \sum_{i=1}^{m} \epsilon(h_i)\partial([h_i]) = \sigma(s(h_1))\sum_{i=1}^{m} (s(h_i) - s(h_{i+1}))
\]

\[
= \sigma(s(h_1))(s(h_1) - s(h_m)) = \sigma(s(w))(s(w) - s(\overline{w})).
\]
Otherwise, by Lemma 8.20(a), there exists a unique integer $1 \leq j \leq m$ such that $h_j \in H_{\text{sp}}$. Then we have

$$\partial([W_\varepsilon]) = \sum_{i=1}^{m} \varepsilon(h_i) \partial([h_i])$$

$$= \alpha(s(h_1)) \left( \sum_{i=1}^{j-1} (s(h_i) - s(h_{i+1})) + (s(h_j) + s(h_{j+1})) - \sum_{i=j+1}^{m} (s(h_i) - s(h_{i+1})) \right)$$

$$= \alpha(s(h_1)) (s(h_1) + s(h_m)) = \alpha(s(w) (s(w) + s(\overline{w}))) .$$

Thus, we obtain the desired equalities (8.8). □

Lemma 8.22. Let $T$ be a subtree of $\Gamma$. Then, the following hold.

(a) Every signed walk of $T$ is admissible on $\Gamma$.

(b) For two vertices $u, v$ of $T$ with $u \neq v$, there exists a unique walk $W$ having $u, v$ as endpoints and it gives rise to admissible signed walks $W_{\pm \varepsilon}$. Conversely, every admissible signed walks of $T$ can be obtained in this way.

Proof. (a) Let $W_\varepsilon$ be a signed walk of $T$. By Lemma 8.20(b), $W_\varepsilon$ satisfies the conditions (NC0) and (NC1). Since it has no proper maximal common subwalks and no intersecting vertices with itself, (NC2) and (NC3) are automatic. We conclude that $W_\varepsilon$ is admissible on $T$, and also on $\Gamma$.

(b) Let $u, v$ be distinct vertices of $T$. Since $T$ is a tree, there exists a unique half-walk $w = (h_1, \ldots, h_m)$ such that $s(h_1) = u$ and $s(h_m) = v$ and it gives rise to a walk $W = \{w, \overline{w}\}$ having $u, v$ as endpoints. In this case, $h_1, ..., h_m$ are pairwise distinct because $T$ is a tree. Then one can find a signature $\pm \varepsilon$ on $W$, and both of $W_{\pm \varepsilon}$ are admissible by (a). Conversely, it is easy to see that every admissible signed walk of $T$ can be obtained in this way. □

Now, we are ready prove Proposition 8.10.

Proof of Proposition 8.10. In the above notations, we show that the isomorphism

$$\partial: ZE \xrightarrow{\sim} L(\Gamma)$$

in Proposition 8.19 restricts to a bijection

$$\partial|_{[AW(\Gamma)]: [AW(\Gamma)] \xrightarrow{\sim} \Phi(\Gamma)} .$$

(8.9)

We first consider the case when $\Gamma$ is a Brauer tree and hence $\Phi(\Gamma) = \Phi_A$. From 8.8, we have $\partial([AW(\Gamma)]) \subseteq \Phi_A$ since every signed walk of $\Gamma$ is admissible and has distinct endpoints by Lemma 8.22(b). On the other hand, by Lemma 8.22(b), for two distinct vertices $u, v$ of $\Gamma$, we have two admissible signed walks $W_{\pm \varepsilon}$ having $u, v$ as the endpoints and

$$\partial(\{W_\varepsilon, W^-\varepsilon\}) = \{\pm(u - v)\} .$$

Therefore, we get the assertion in this case.

Next, we assume that $\Gamma$ is a Brauer odd-cycle and hence $\Phi(\Gamma) = \Phi_C$. We denote by $Z$ the unique odd cycle in $\Gamma$, by $\ell$ the length of $Z$. For each vertex $v \in V$, we define a subtree $T_v$ of $\Gamma$ as follows: Consider a graph obtained from $\Gamma$ by deleting all (half-)edges lying in $Z$ and then $T_v$ is defined to be its connected component containing the vertex $v$. We write $v$ for the the unique common vertex of $T_v$ and $Z$.

By 8.8, we have $\partial([W_\varepsilon]) \in \Phi_C$ for any signed walk $W_\varepsilon$ of $\Gamma$. In fact, for the endpoints $u, v$ of $W_\varepsilon$, we have $\partial([W_\varepsilon]) = \pm u \pm v$ if $u \neq v$, $\partial([W^+_\varepsilon]) = \pm 2u$ by (NC0) otherwise. Thus, $\partial([AW(\Gamma)]) \subseteq \Phi_C$. In order to show the bijectivity, it is enough to show $\Phi_C \subseteq \partial([AW(\Gamma)])$.

(a) Firstly, we consider the elements of the form $\pm(u - v) \in \Phi_C$ with $u \neq v$. Let $u, v$ be distinct vertices of $\Gamma$. Applying Lemma 8.22(b) to the spanning tree $\Gamma_{\text{sp}}$, there are two admissible signed walks $W_{\pm \varepsilon}$ having $u, v$ as endpoints. By 8.8, we have

$$\partial(\{W_\varepsilon, W^-\varepsilon\}) = \{\pm(u - v)\} .$$
(b) Secondly, we show that \( \pm(u+v) \) belong to \( \partial(\prescript{\epsilon}{w}w(\Gamma)) \) for vertices \( u \neq v \) with \( T_v \neq T_u \). Recall from Lemma 8.22 that we have a unique half-walk \( w \) of \( \Gamma_{\text{up}} \) such that \( u = s(w) \) and \( v = s(w) \). Since \( \Gamma \) contains no cycles except for \( Z \), \( w \) can be written as \( w = pq \), where

- \( p = (p_1, \ldots, p_m) \) \((m \geq 0)\) is a unique half-walk on \( T_u \) such that \( u = s(p_1) \) and \( \bar{u} = s(p_m) \),
- \( z := (z_1, \ldots, z_j) \) \((1 \leq j < \ell)\) is a half-walk consisting of half-edges appearing in \( Z \) such that \( \bar{u} = s(z_1) \) and \( \bar{\bar{v}} = s(z_j) \) with \( z_j \in H_{sp} \) for all \( 1 \leq i \leq j \), and
- \( q = (q_1, \ldots, q_r) \) \((r \geq 0)\) is a unique half-walk on \( T_v \) such that \( \bar{v} = s(q_1) \) and \( v = s(q_r) \).

For a half-walk \( z \), there is a half-walk \( z' = (z_{j+1}, \ldots, z_\ell) \) such that \( z z' = (z_1, \ldots, z_{j}, z_{j+1}, \ldots, z_\ell) \) forms the cycle \( Z \). Since \( \bar{u} = s(z_1) \) and \( s(z_{j+1}) = \bar{v} \), we obtain a half-walk \( w' := p z q \) and a walk \( W' := \{w', \bar{w}'\} \) of \( \Gamma \).

From our construction, \( W' \) is not a walk of \( \Gamma_{\text{up}} \) but a walk of a certain subtree of \( \Gamma \). By Lemma 8.22(b), it gives rise to two admissible signed walks \( W'^{\pm, \epsilon} \). By (8.8), we have

\[ \partial(\{W'^{\epsilon} \}) = \{\pm(u+v)\} \]

(c) Thirdly, we consider vertices \( u \neq v \) with \( T_u = T_v \), i.e., \( \bar{u} = \bar{v} \). We will construct \( X, Y \in \prescript{\epsilon}{w}w(\Gamma) \) such that

\[ \partial(\{X, Y\}) = \{\pm(u+v)\}. \quad (8.10) \]

We may assume \( u \neq v \) by replacing \( u \) and \( v \) if necessary. Then, one can find a half-walk \( w = (h_1, \ldots, h_m) \) of the form \( w := pq \), where

- \( p = (p_1, \ldots, p_m) \) \((m \geq 1)\) is a unique half-walk on \( T_u \) such that \( u = s(p_1) \) and \( \bar{u} = s(p_m) \),
- \( z = (z_1, \ldots, z_j) \) is a half-walk which forms the cycle \( Z \) so that \( \bar{u} = s(z_1) = s(z_j) \), and
- \( q = (q_1, \ldots, q_r) \) \((r \geq 0)\) is a unique half-walk on \( T_v \) such that \( \bar{v} = s(q_1) \) and \( v = s(q_r) \).

On the other hand, let \( w' \) be a half-walk of \( \Gamma \) defined by \( w' := p \overline{z} q \). Now, we fix a signature \( \epsilon \) on a walk \( w \in W' \). Then, it gives a signature on \( w' \in W' \). By Lemma 8.20(b), all \( W^{k, \epsilon}, W'^{k, \epsilon} \) satisfy (NC0) and (NC1). For the admissibility, we need to check the conditions (NC2) and (NC3). Depending on the structure of \( \Gamma \), we have three cases (c-i)-(c-iii) as follows.

(c-i) We first consider the case when \( r \geq 1 \) and \( p_m = q_1 \). In this case, (NC3) is automatic. For a signed walk \( W'^{\epsilon} \), take an integer \( k > 0 \) such that \( t_1 := p_{m-k+1} = t \) for all \( i \in \{1, \ldots, k\} \) but \( p_{m-k} \neq q_k+1 \). Since \( u \neq v \), at most one of \( p_{m-k} \) and \( q_k+1 \) is the virtual edge attached to \( W'^{\epsilon} \). Then, \( (t_1, t_k) \) gives a unique proper maximal common subwalk of \( W \) with itself. By definition, \( W'^{\epsilon} \) satisfies (NC2) if and only if the cyclic subordering around \( \bar{u} \) and around \( s := s(t_1) \) accounting the virtual edges is either

\[
\begin{align*}
\begin{array}{cccccccc}
\bullet & t_1 & t_2 & \cdots & t_k & \circ \quad \text{or} \quad \circ & t_1 & t_2 & \cdots & t_k
\end{array}
\end{align*}
\]

By the cyclic ordering around \( \bar{u} \), precisely one of \( W'^{\epsilon} \) and \( W'^{\epsilon} \) satisfies (NC2) by taking in account of virtual edges attached to them. Similarly, precisely one of \( W'^{-\epsilon} \) and \( W'^{-\epsilon} \) satisfies (NC2). As a consequence, we obtain the desired admissible signed walks \( X \) and \( Y \) satisfying (8.10).

(c-ii) Next, we assume that \( r \geq 1 \) and \( p_m \neq q_1 \). In this case, the condition (NC2) is automatic. For a signed walk \( W'^{\epsilon}, \bar{u} \) is the intersecting vertex of \( W \) with itself, whose neighbourhoods are
From the cyclic ordering around \( \bar{u} \), precisely one of \( W^\varepsilon \) and \( W'^\varepsilon \) satisfies (NC3). Similarly, precisely one of \( W^{-\varepsilon} \) and \( W'^{-\varepsilon} \) satisfies (NC3). Then, we obtain the desired pair \( X \) and \( Y \) satisfying (S.10).

(c-iii) Lastly, we assume that \( r = 0 \), i.e., \( v = \bar{u} \). In this case, by letting \( q_1 := z_{r+1} \) be the virtual edge attached to \( W^\varepsilon \), a similar argument as in (c-ii) gives the desired pair \( X \) and \( Y \) satisfying (S.10).

(d) Finally, we show that \( \pm 2u \in \partial[[\mathbf{AW}(\Gamma)]] \) for any vertex \( u \in V \). By the same way as a proof of Lemma (S.20(b)), one can find a half-walk \( w = (h_1, \ldots, h_m) \) of the form \( w = pz^p \), where

- \( p = (h_1, \ldots, h_r) \) (\( r \geq 0 \)) is a unique half-walk on \( T_u \) such that \( u = s(h_1) \) and \( \bar{u} = s(h_r) \) and
- \( z = (h_{r+1}, \ldots, h_{r+\ell}) \) is a half-walk which forms the cycle \( Z \) so that \( \bar{u} = s(h_{r+1}) = s(h_{r+\ell}) \).

Then, it gives rise to a walk \( W \) equipped with signatures \( \pm \varepsilon \). By Lemma (S.20(b)), it satisfies the conditions (NC0) and (NC1). Since the endpoints of \( W \) are the same, the condition (NC2) is automatic. Now, we consider (NC3). This is automatic if \( u \neq \bar{u} \). Assume that \( u = \bar{u} \) and let \( \{a, b\} = \{h_r, h_{r+1}\} \) and \( \{c, d\} = \{h_{r+\ell}, h_{r+\ell+1}\} \) be neighbourhoods of \( u \) in \( \Gamma \). Since two of \( a, b, c, d \) are virtual, the condition (NC3) is satisfied. Therefore, both of \( W^\pm \varepsilon \) are admissible. By (S.8), we have

\[
\partial([W^\varepsilon, W'^{-\varepsilon}]) = \{\pm 2u\}.
\]

By (a)-(d), we conclude that the map (S.9) is bijective. \( \Box \)

We end this subsection with a proof of Theorem 8.4.

**Proof of Theorem 8.4** Let \( \Gamma \) be a Brauer graph having \( n \) edges and \( B := B_T \) the Brauer graph algebra of \( \Gamma \).

(a) Thanks to Proposition 4.6(b), we can assume that a base field \( k \) is algebraically closed. In order to show that the algebra \( B \) is pairwise \( g \)-convex, we first consider a left mutation of \( B \) with respect to the indecomposable projective \( B \)-module \( P \) so that the exchange triangle is

\[
P \to U \to P' \to P[1]. \tag{8.11}
\]

The triangle (8.11) is explicitly described in [Ai] Section 6], in particular, the number of indecomposable direct summands of \( U \) is at most two.

Next, let \( T \in 2\text{-}\text{silt}B \). Then \( T \) is a tilting complex since \( B \) is a symmetric algebra [Ai] Example 2.8]. We have a triangle equivalence

\[
F : \text{D}^b(\text{mod } B) \to \text{D}^b(\text{mod } \text{End}_{\text{D}_B}(\text{mod } B)(T))
\]

mapping \( T \) to \( \text{End}_{\text{D}_B}(\text{mod } B)(T) \). By [AZ] Corollary 1.3, we have \( \text{End}_{\text{D}_B}(\text{mod } B)(T) \cong B_\Gamma \) for some Brauer graph \( \Gamma \) having \( n \) edges. Since \( F \) sends an exchange triangle \( X \to U' \to Y \to X[1] \) of \( T \) to an exchange triangle of \( B_\Gamma \), by applying the argument above to \( B_\Gamma \), the number of indecomposable direct summands of \( U' \) is at most two.

Consequently, \( B \) is pairwise \( g \)-convex.

We show the latter assertion. (i)\(\Leftrightarrow\)(iii) is [AAC] Theorem 6.7. (i)\(\Leftrightarrow\)(ii) follows from Theorem 5.10(b).
(b) Assume that $\Gamma$ is a Brauer tree or a Brauer odd-cycle. Combining (8.5) and (8.6), we obtain the following commutative diagram.

$$
\begin{array}{c}
K_0(\text{proj } B_{\Gamma}) \cong \bigcup Z E \cong \bigcup L(\Gamma) \\
\cong \bigcup [2-\text{psilt}^1 B_{\Gamma}] \cong \bigcup [\text{AW}(\Gamma)] \cong \Phi(\Gamma).
\end{array}
$$

Thus, we obtain (8.1), which clearly gives rise to the desired isomorphism $P(B_{\Gamma}) \cong P(\Phi(\Gamma))$ of lattice polytopes. The last assertion follows from [ABHPS, Theorem 1].

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