The Impact of Equilibria on the Shape of Hysteresis Loops

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Abstract: Hysteresis is a common phenomenon found in many dynamical systems. It is typically described as a looping behaviour in the systems input-output graph. For a dynamical system to exhibit hysteresis, it must have multiple stable equilibria. This work examines the impact that different types of equilibria can have on the shape of hysteresis loops exhibited in input-output graphs of ordinary differential equations.

Keywords: Hysteresis loops, Dynamic systems, Differential equations, Equilibrium, Stable.

1. INTRODUCTION

The hysteresis phenomena is found in various fields, such as biology (Noori (2014); Mayergoyz and Korman (2019)), magnetism (Bernstein and Oh (2005); Chow and Morris (2014); Bei and Chen (2019)), circuits (Sharifi and Bahre-pour (2015)) and economics (Cross et al. (2009)).

The study of hysteresis is important for understanding the behaviour of a system given changes in the input. The appearance of hysteresis in a dynamical system can make modelling the system more difficult. Consequently, the more hysteresis is understood, the easier it can be to control the hysteresis arising in a given dynamical system. While there is much literature that mentions hysteresis to describe a particular system, the following discuss hysteresis in greater depth by examining its appearance in certain fields and attempts to provide a rigorous definition for it: Bernstein and Oh (2005); Brokate and Sprekels (1996); Ikhouane (2013); Ikhouane and Rodellar (2007); Morris (2011); Noori (2014); Visintin (2005).

Operators are commonly used to describe hysteresis, as well as using notions such as rate-independence, memory and path-independence (Brokate and Sprekels (1996); Cross et al. (2009); Visintin (2005)). However, these definitions are not always tractable, and may exclude systems that exhibit hysteresis. Rather consider the following definitions, which will be used in this paper.

Definition 1. (Morris, 2011, Definition 3)
A hysteretic system is one which has
(i) multiple stable equilibrium points and
(ii) dynamics that are considerably faster than the time scale at which inputs are varied.

Definition 2. Bernstein and Oh (2005)
Hysteresis is a nontrivial closed curve with periodic input in the input-output graph of a system that persists as the frequency of the input approaches 0.

This nontrivial closed curve is called a hysteresis loop.

Nickmand et al. (2013) examine how different porous materials affect the shape of hysteresis loops. Wang and Hui (2017) discusses hysteresis loops that are not simple; that is, loop crosses itself. But in general, there are not many studies in regards to what affects the shape of hysteresis loops. Since multiple stable equilibria are essential to a hysteretic system as noted in definition 1, there is likely a correlation between the equilibria of a system and the shape of the hysteresis loop. This relationship is explored in greater depth in this paper. Additional references that consider the relationship between equilibria and hysteresis are Afshar et al. (2016); Angeli et al. (2004); Chow and Morris (2014); Ikhouane (2013); Morris (2011).

In this paper, hysteresis in dynamical systems of ordinary differential equations (ODEs) are considered. These ODEs have been chosen because they exhibit different types of equilibria including: a continuum of stable equilibria, a finite number of discrete stable equilibria, an infinite number of discrete stable equilibria and the presence or absence of unstable equilibria. These variations affect the shape, such as symmetry, of the corresponding hysteresis loop.

To establish hysteresis in the examples presented in this paper, the following procedure is applied. The equilibria of each system is first determined and then their stability is established. This is a necessary condition of hysteresis as indicated in definition 1. Constructing the input-output graph of each system is also necessary in order to show definition 2 is satisfied. The output is the solution to the ODE. Let u(t) be the input, which is introduced into the system to construct the input-output graphs. These are generated using MATLAB. Figure 1 demonstrates what is meant by the presence versus the absence of looping in the input-output graph of dynamical systems. In this paper, the periodic input u(t) = sin(ωt) is used so that the frequency of the input is ω. It is important to note the presence of looping is not sufficient to conclude hysteresis
but rather, as definition 2 indicates, the loops must persist as \( \omega \) goes to zero.

![Graph](image1.png)

Fig. 1. Two input-output graphs with one displaying looping behaviour and the other not.

When the shape of the hysteresis loop is unchanged as the frequency of the periodic input changes, the arising hysteresis is said to be rate-independent; otherwise, the hysteresis is said to be rate-dependent (Bernstein and Oh (2005)). Rate independent versus rate dependent hysteresis loops are also discussed in this paper. Furthermore, some papers conclude hysteresis can only appear in nonlinear dynamical systems (Iklouane (2013); Bernstein and Oh (2005)). Recent literature indicates linear dynamical systems also exhibit hysteresis, and hence additional linear examples similar to those in Chow and Morris (2014) are presented here.

The shape of the hysteresis loops arising in both linear and nonlinear examples are discussed in the succeeding sections. Observations of these examples and summary discussions are reviewed in the last section.

2. LINEAR EXAMPLES

Consider the linear example \( \dot{x}(t) = u(t) \). The dot notation represents the derivative with respect to time \( t \). When there is no input; that is, \( u(t) = 0 \), and the time derivative is zero, \( x(t) \) is any constant for all \( t \). Trivially this indicates a continuum of stable equilibria. To test for hysteresis as per definition 2, construct the input-output of \( \dot{x}(t) = u(t) \) by letting \( u(t) = \sin(\omega t) \) for various values of \( \omega \). The results are depicted in Figure 2 and show persistent looping in the input-output graph. That means definition 2 is satisfied and suggests \( \dot{x}(t) = u(t) \) exhibits hysteresis.

Consider another linear example

\[
\begin{align*}
\dot{y}(t) + 5\dot{y}(t) &= u(t), \quad (1a) \\
y(0) &= 0, \quad \dot{y}(0) = 0. \quad (1b)
\end{align*}
\]

This example can be found in Chow and Morris (2014) except the first derivative term has been rescaled by a factor of \( \frac{1}{5} \).

In order to verify that (1) has multiple stable equilibria, let \( x_1(t) = \dot{y}(t) \) and \( x_2(t) = y(t) \), so that equation (1) can be rewritten equivalently as a system of coupled linear ODEs:

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= -5x_2(t) + u(t), \\
x_1(0) &= 0, \quad \dot{x}_2(0) = 0.
\end{align*}
\]

If there is no input; that is, \( u(t) = 0 \), and setting the time-derivative to be zero, leads to a continuum of equilibria of the form \((x_1, x_2) = (a, 0)\) where \( a \in \mathbb{R} \) is a constant. The stability of these equilibria are determined by their corresponding eigenvalues, denoted \( \lambda \). In this case, they are \( \lambda_1 = 0 \) and \( \lambda_2 = -5 \), which do not depend on \( a \). If the real part of all the eigenvalues of an associated equilibrium is less than or equal to zero, the equilibrium is said to be stable. Otherwise, it is unstable. This is well known and can be referenced from most undergraduate introductory level textbooks about ODEs. See for instance (Khalil, 2002, Chapter 2) or (Perko, 2002, Chapter 1). Therefore, the continuum of equilibria of (1) are all stable.

![Graph](image2.png)

Fig. 2. The input-output graph of \( \dot{x}(t) = u(t) \) with \( u(t) = \sin(\omega t) \) for various values of \( \omega \), which has continuum of stable equilibria.

![Graph](image3.png)

Fig. 3. The input-output graph of \( u(t) = \sin(\omega t) \) for various values of \( \omega \). Equation (1) has a continuum of stable equilibria.
To test whether (1) exhibits hysteresis, recall definition 2. Construct the input-output of (1) by letting \( u(t) = \sin(\omega t) \). The results are depicted in Figure 3 for various values of \( \omega \) and shows persistent looping. This indicates the system in (1) exhibits hysteresis.

Both linear examples have a continuum of stable equilibria and the shape of their hysteresis loops are similar.

3. NONLINEAR EXAMPLES

Consider the following first order nonlinear ODE
\[
\dot{x}(t) = x(t)(1 - (x(t))^2) + u(t),
\]
(3a)
\[
x(0) = -1.
\]
(3b)

This example can be found in (Perko, 2002, Page 336). For \( u(t) = 0 \), the time derivative is zero when \( x(t) \) equals 1, 0 and \(-1\). Therefore, these are the equilibria of (3). The phase portrait of (3) depicted in Figure 4 indicates 1 and \(-1\) are stable, while 0 is unstable. Notice the symmetry of the equilibria; that is, the two stable equilibria are equidistant from the unstable equilibrium point.

Consider another first order nonlinear ODE
\[
\dot{x}(t) = r(t)x(t)\left(1 - \frac{x(t)}{q}\right) - \frac{(x(t))^2}{1 + (x(t))^2},
\]
(4a)
\[
x(0) = 10.
\]
(4b)

This example can be found in (Murray, 2002, Page 7-8), Ludwig et al. (1978) and represents the classic Spruce Budworm Model. In this model, \( r(t) \) depends on the birth rate of the population and the constant \( q \) depends on the carrying capacity. When \( r(t) \) is set to be the constant input 0.52 and \( q = 25 \), equation (4) exhibits multiple stable equilibria (see Figure 6). In particular, 22.90502477 and 0.9715485792 are stable equilibria while 0 and 1.123426650 are unstable equilibria of (4).

Consider the second order nonlinear ODE
\[
y'(t) + (1 + (y(t))^2)y(t) - y(t) + (y(t))^3 = u(t),
\]
(5a)
\[
y(0) = 1, \quad \dot{y}(0) = 0.
\]
(5b)

This example can be found in (Khalil, 2002, Page 68). As in (1), by rewriting (5) into a system of first order ODEs, the equilibria can be determined. The equilibria of this system are \((1,0), (0,0)\) and \((-1,0)\) with eigenvalues, \(\lambda_{(1,0)} = -1 \pm i\), \(\lambda_{(0,0)} = -1 + \sqrt{7}i\) and \(\lambda_{(-1,0)} = -1 \pm i\), respectively. Therefore, \((1,0)\) and \((-1,0)\) are stable equilibria while...
Consider
\[ \ddot{y}(t) + 5\dot{y}(t) + (y(t))^2(y(t) + 1) = u(t), \] (6a)
\[ y(0) = -1, \dot{y}(0) = 0. \] (6b)
and
\[ \ddot{y}(t) + \dot{y}(t) + ((y(t))^2 - 3)(y(t) - 1) \ln(y(t)) = u(t), \] (7a)
\[ y(0) = 1, \dot{y}(0) = 1. \] (7b)

By re-writing equation (6) into a system of first order ODEs, the equilibria are (-1,0) and (0,0), and the eigenvalues are \( \lambda_{(-1,0)} = -\frac{5+\sqrt{21}}{2} \) and \( \lambda_{(0,0)} = 0, -5 \), respectively. Similarly, the equilibria of (7) are (1,0) and \((\sqrt{3},0)\) with eigenvalues \( \lambda_{(1,0)} = 0, -1 \) and \( \lambda_{(\sqrt{3},0)} = -0.5 \pm \sqrt{4(6-2\sqrt{3})\ln(\sqrt{3})-1} \), respectively. Therefore, these are stable equilibria and furthermore, both equations do not have unstable equilibria. The input-output graphs of (6) and (7) with \( u(t) = \sin(\omega t) \) are shown in Figures 9 and 10, respectively. In comparison to the previous examples, these figures show the size of the hysteresis loops are diminished as \( \omega \) goes to zero.

(0,0) is an unstable equilibria. Notice the symmetry of the equilibria; that is, the two stable equilibria are equidistant from the unstable equilibria.

Figure 8 shows the input-output graph of (5) for different frequencies of the input \( u(t) = \sin(\omega t) \). Notice the symmetry about the origin of the hysteresis loops for smaller values of \( \omega \). This correlates to the symmetry of the equilibria.

The following ODE
\[ \ddot{y}(t) + \dot{y}(t) + \frac{1}{2}\sin(2y(t)) = u(t), \] (8a)
\[ y(0) = 1, \dot{y}(0) = 0. \] (8b)
has infinitely many discrete equilibria of the form \( \frac{n\pi}{2} \) which are stable when \( n \) is even and unstable when \( n \) is odd. This was determined by rewriting (8) into a system of first order ODEs as shown in detail for equation (1). For the stable equilibria, the eigenvalues are \( \frac{-1 \pm i\sqrt{3}}{2} \) and for the unstable equilibria, the eigenvalues are \( \frac{-1 \pm i\sqrt{3}}{2} \). When \( u(t) = \sin(\omega t) \), the hysteresis loops of (8) are illustrated in Figure 11 and are similar in shape and behaviour to the examples with a continuum of stable equilibria (see Figures 2 and 3).
Fig. 10. The input-output graph of (7) with input \( u(t) = \sin(\omega t) \) for various values of \( \omega \). It has two stable equilibria and no unstable equilibria.

Fig. 11. The input-output graph of (8) with input \( u(t) = \sin(\omega t) \) for various values of \( \omega \). Equation (8) has infinitely many stable and unstable discrete equilibria.

The following examples demonstrate rate-independent hysteresis loops and are already well known rate-independent examples. They are included here for a more comprehensive discussion of loop shapes.

From Bernstein and Oh (2005),

\[
\dot{B}(t) = a[H(t)][B(t) - B(t)] + c\dot{H}(t) \quad (9a) \\
B(0) = 0. \quad (9b)
\]

where \( B(t) \) represents magnetic flux, the input is the applied magnetic field, denoted \( H(t) \), and \( a \), \( b \) and \( c \) are parameters of this magnetic model. This means the input is \( u(t) = H(t) \), where \( H(t) = \sin(\omega t) \). When there is no input or a constant input, \( \dot{B}(t) = 0 \), which implies \( B(t) \) is a constant for all \( t \), and hence (9) has a continuum of stable equilibria. Figure 12 depicts the input-output graph of (9) with \( a = 0.02125 \), \( b = 0.1 \) and \( c = 0.04361 \).

Fig. 12. Input-output graphs of (9) with input \( H(t) = \sin(\omega t) \) with \( a = 0.02125 \); \( b = 0.1 \) and \( c = 0.04361 \) for various values of \( \omega \), which demonstrate rate-independent hysteresis loops. Equation (9) has a continuum of stable equilibria.

The Duhem model

\[
\dot{x}(t) = \alpha (u(t))[f(u(t)) - x(t)] + g(u(t))\dot{u}(t) \quad (10a) \\
f(u(t)) = \tanh(u(t)) + 0.1u(t) \quad (10b) \\
g(u(t)) = f'(u(t))(1 - 0.58e^{-|u(t)|}) \quad (10c) \\
x(0) = 0. \quad (10d)
\]

is well known to exhibit hysteresis (Feng et al. (2009); Bernstein and Oh (2005)). Similar to (9), equation (10) has a continuum of stable equilibria. Figure 13 depicts the input-output graphs of (10) with \( u(t) = \sin(\omega t) \).

In both Figures 12 and 13, the shape of the hysteresis loop is unchanged regardless of the value of \( \omega \). This supports the fact that (9) and (10) exhibit rate-independent hysteresis as noted in Bernstein and Oh (2005); Feng et al. (2009).

4. OBSERVATIONS

In this final section, observations and comparisons of the hysteresis loops arising in the preceding linear and nonlinear examples are discussed.

(1) Aside from Figures 12 and 13, all other figures depicted rate-dependent hysteresis loops; that is, the shape of the loops changed as the frequency of the input varied. “Fully” rate-dependent systems are shown in Figures 2, 3 and 11. By fully rate-dependent, this means the shape of the loop changes every time the frequency of the input changes. These fully rate-dependent systems correspond to dynamical systems with either infinitely many discrete stable equilibria or a continuum of stable equilibria. For Figures 5, 7, 8, 9 and 10, rate-independence of the hysteresis loops emerged for small values of \( \omega \).

(2) Figures 5, 7, 8, 9 and 10 correspond to systems with two stable equilibria (with different types of unstable
Their hysteretic loops are distinguished by two “horizontal portions” of stable areas and the vertical “jumps” between them as ω approaches zero. That is, the overall shape of the hysteretic loops are the same as the frequency of the input goes to zero. This may be due to the dynamical systems having exactly two stable equilibria.

(3) The linear ODEs exhibit a continuum of stable equilibria, and the corresponding hysteretic loops shown in Figures 2 and 3 are smooth in that there are no vertical “jumps”. This may be because the system is always transitioning between stable equilibria.

(4) The hysteretic loops in Figures 5 and 8 are symmetric about the origin as ω goes to zero. These correspond to dynamical systems with two stable equilibria that are equidistant from the one unstable equilibrium point at the origin. On the other hand, Figures 9 and 10 do not have symmetry about the origin and the associated dynamical systems do not have equidistant stable equilibria.

These observations suggest shapes of hysteretic loops can be predicted if properties of the dynamical system are known; however, there is still much that needs to be established. For example, perhaps the symmetry of loops can be determined solely by measuring the magnitude of equilibria from one another as well as using the magnitude of the corresponding eigenvalues. The observations made about hysteretic loops here and the definitions available in the reference literature may in the future provide a way to predict or even standardize hysteretic loop shapes. This can also lead to future work on controller design to modify the shape of hysteretic loops by adjusting the types of equilibria that arise in dynamical systems.

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