Adaptive Methods for PDE’s
Wavelets or Mesh Refinement?

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Abstract

Adaptive mesh refinement techniques are nowadays an established and powerful tool for the numerical discretization of PDE’s. In recent years, wavelet bases have been proposed as an alternative to these techniques. The main motivation for the use of such bases in this context is their good performances in data compression and the approximation theoretic foundations which allow to analyze and optimize these performances. We shall discuss these theoretical foundations, as well as one of the approaches which has been followed in developing efficient adaptive wavelet solvers. We shall also discuss the similarities and differences between wavelet methods and adaptive mesh refinement.

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1. Introduction

Among those relevant phenomenons which are modelled by partial differential or integral equations, countless are the instances where the mathematical solutions exhibit singularities. Perhaps the most classical examples are elliptic equations on domains with re-entrant corners, or nonlinear hyperbolic systems of conservation laws. While such singularities are sources of obvious theoretical difficulties—classical solutions should be abandoned to the profit of weak solutions—they are also an obstacle to the convergence of numerical approximation methods, in the sense that they deteriorate the rate of decay of the error with respect to the size of the discrete problem: achieving a prescribed accuracy will typically require finer resolution and therefore heavier computational cost and memory storage, in comparison to the approximation of smooth solutions. Let us remark that singularities

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often have a physical relevance: they represent the concentration of stress in elasticity, boundary layers in viscous fluid flows, shock waves in gas dynamics... It is therefore a legitimate requirement that they should be accurately resolved by the numerical method.

In this context, the use of adaptive methods, appears as a natural solution to improve the approximation at a reasonable computational cost. Here, the word *adaptivity* has a twofold meaning: (i) the discretization is allowed to be refined only locally, in particular near the singularities of the solution, and (ii) the resolution algorithm uses information gained during a given stage of the computation in order to derive a new refined discretization for the next stage. The most typical example is *adaptive mesh refinement* based on *a-posteriori* error estimates in the finite element context. While these methods have proved to be computationally successful, the theory describing their advantages over their non-adaptive counterpart is far from being complete. In particular, the *rate of convergence* of the adaptive algorithm, which describes the trade-off between the accuracy and complexity of the approximation, is not clearly understood.

In recent years, wavelet bases have been proposed as an alternative to adaptive mesh refinement, motivated by their good performances in data (more specifically image) compression. In wavelet-based adaptive schemes, the set of basis functions which describe the approximate solution is updated at each stage of the computation. Intuitively, the selection of the appropriate basis functions plays a similar role as the selection of the mesh points in adaptive finite element methods, and one could therefore expect similar performances from both approaches. On a more rigorous level, a specific feature of the wavelet approach is the emergence of a sound theoretical setting which allows to tackle foundational questions such as the rate of convergence of the adaptive method.

The goal of this paper is to give some elements of comparison between adaptive wavelet and mesh refinement methods from this perspective. We shall first describe in §2 a general setting which leads us in §3 to a first comparison between wavelets and adaptive finite elements from the point of view of approximation theory. We discuss in §4 the relation between these results and adaptive algorithms for PDE’s. After recalling in §5 the classical approach in the finite element context, we present in §6 an adaptive wavelet strategy which has been applied to various problems, and discuss its foundational specificities. Finally, we shall conclude in §7 by pointing out some intrinsic shortcomings of wavelet-based adaptive methods.

2. A general framework

Approximation theory is the branch of mathematics which studies the process of approximating general functions by simple functions such as polynomials, finite elements or Fourier series. It plays therefore a central role in the accuracy analysis of numerical methods. Numerous problems of approximation theory have in common the following general setting: we are given a family of subspaces \( (S_N)_{N \geq 0} \) of a
normed space $X$, and for $f \in X$, we consider the best approximation error
\[
\sigma_N(f) := \inf_{g \in S_N} \|f - g\|_X. \tag{1}
\]
Typically, $N$ represents the number of parameters which are needed to describe an element in $S_N$, and in most cases of interest, $\sigma_N(f)$ goes to zero as this number tends to infinity. If in addition $\sigma_N(f) \leq CN^{-s}$ for some $s > 0$, we say that $f$ is approximated at rate $s$.

Given such a setting, the central problem of approximation theory is to characterize by some analytic (typically smoothness) condition those functions $f$ which are approximated at some prescribed rate $s > 0$. Another important problem is how to design simple approximation procedures $f \mapsto f_N \in \Sigma_N$ which avoid solving the minimization problem (1), while remaining near optimal in the sense that
\[
\|f - f_N\|_X \leq C\sigma_N(f), \tag{2}
\]
for some constant $C$ independent of $N$ and $f$.

As an example, consider approximation by finite element spaces $V_h$ defined from regular conforming partitions $T_h$ of a domain $\Omega \subset \mathbb{R}^d$ into simplices with uniform mesh size $h$. The approximation theory for such spaces is quite classical, see e.g. [12], and can be summarized in the following way. If $W^{\ell,p}$ denotes the classical Sobolev space, consisting of those functions $f \in L^p$ such that $D^\alpha f \in L^p$ for $|\alpha| \leq \ell$, we typically have
\[
f \in W^{\ell+r,p} \Rightarrow \inf_{g \in V_h} \|f - g\|_{W^{\ell,p}} \leq Ch^r \tag{3}
\]
provided that $V_h$ is contained in $W^{\ell,p}$ and that $V_h$ has approximation order larger than $\ell + r$, i.e. contains all polynomials of degree strictly less than $\ell + r$. Such classical results also hold for fractional smoothness. We can express them in terms of the number of parameters, remarking that $N := \dim(V_h) \sim h^{-d}$, so that if we set $X = W^{\ell,p}$ and $S_N := V_h$ with $h := N^{-1/d}$, we have obtained
\[
f \in W^{\ell+r,p} \Rightarrow \sigma_N(f) \leq CN^{-\ell+r}. \tag{4}
\]
We have thus identified an analytic condition which ensures the rate $s = r/d$. Note that this is not a characterization (we only have an implication), yet a deeper analysis shows that an “if and only if” result holds if we slightly modify the notion of Sobolev smoothness (using Besov classes, see [13]). In summary, the rate of approximation in $W^{s,p}$ is governed by the approximation order of the $V_h$ spaces, the dimension $d$ and the level of smoothness of $f$ measured in $L^p$. Let us finally remark that near-optimal approximation procedures can be obtained if we can find a sequence of finite element projectors $P_N : X \mapsto S_N$ such that $\|P_N\|_{X \rightarrow X} \leq K$ with $K$ independent of $N$. In this case, we simply take $f_N = P_N f$ and remark that $\|f - f_N\|_X \leq (1 + K)\sigma_N(f)$.

In the following we shall address the same questions in the cases of adaptive finite element and wavelet approximation. As we shall see, a specific feature to such cases is that the spaces $\Sigma_N$ are not linear vector spaces.
3. Adaptive finite elements and wavelets

In the adaptive finite element setting, the number of parameters $N$ is proportional to the number of triangles, but for a given budget $N$ the partition $\mathcal{T}$ and the finite element space $V_T$ are allowed to be locally refined in a way which depends on the function $f$ to be approximated. It is therefore natural to define the approximation spaces $S_N$ as

$$S_N := \bigcup \{ V_T \mid \#(\mathcal{T}) \leq N \}. \quad (5)$$

It should be well understood that the $S_N$ are not linear vector spaces (the sum of two elements does not in general fall in $S_N$ when their triangulation do not match) but any $g \in S_N$ is still described by $O(N)$ parameters, which encode both its triangulation $\mathcal{T}$ and its coordinates in $V_T$. The requirement of adaptivity has thus led us to the concept of nonlinear approximation.

Wavelet bases offer another track toward nonlinear adaptive approximation. The simplest prototype of a wavelet basis is the Haar system. Let us describe this system in the case of expanding a function $f$ defined on $[0,1]$: the first component in this expansion is simply the average of $f$, i.e. the orthogonal projection $\langle f, e_0 \rangle e_0$ onto the function $e_0 = \chi_{[0,1]}$. The approximation is then refined into the average of $f$ on the two half intervals of equal size. This refinement amounts in adding the orthogonal projection $\langle f, e_1 \rangle e_1$ onto the function $e_1 = \chi_{[0,1/2]} - \chi_{[1/2,1]}$. Iterating this refinement process, we see that the next components have the same form as $e_1$ up to a change of scale: at refinement level $j$, we are adding the projection onto the functions

$$\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k), \ k = 0, \ldots, 2^j - 1, \quad (6)$$

where $\psi = e_1$. Since all these functions are orthogonal to the previous ones, letting $j$ go to $+\infty$, we obtain the expansion of $f$ into an orthonormal basis of $L^2([0,1])$

$$f = \sum_{\lambda \in \nabla} f_\lambda \psi_\lambda, \quad (7)$$

with $f_\lambda := \langle f, \psi_\lambda \rangle$. In the above notation $\lambda$ concatenates the scale and space parameters $j$ and $k$, and $\nabla$ is the set of all indices (including also the first function $e_0$). In order to keep track of the scale $j$ corresponding to an index $\lambda = (j,k)$ we shall use the notation $|\lambda| = j$. More general wavelet systems in one or several space dimension are built from similar nested approximation processes, involving e.g. spline functions or finite elements in place of piecewise constant functions (see [23] or [13] for a general presentation).

This brief description suggests that a natural construction of adaptive wavelet approximations is obtained by using only a limited set of indices $\lambda$ as the scale $|\lambda|$ grows, which depends on the function to be approximated and typically corresponds to those wavelets whose supports are close to its singularities. It is therefore natural to define the approximation spaces $S_N$ as the set of all $N$ terms combinations

$$S_N := \{ \sum_{\lambda \in \Lambda} d_\lambda \psi_\lambda : \#(\Lambda) \leq N \}. \quad (8)$$
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Again this is obviously not a linear space, since we allow to approximate a function by choosing the best $N$ terms which differ from one function to another. Note that we still do have $S_N + S_N = S_{2N}$.

Both adaptive finite element and wavelet framework have obvious similarities. However, the answer to the two basic questions raised in the previous section—what are the properties of $f$ which govern the decay of $\sigma_N(f)$ and how to compute in a simple way a near optimal approximation of $f$ in $\Sigma_N$—is only fully understood in the wavelet framework. Concerning the first question, a striking result by DeVore and his collaborators [24] is the following: with $X := W_{t,p}^r$, best $N$-term wavelet approximation satisfies

$$f \in W_{t+q,r}^r \Rightarrow \sigma_N(f) \leq C N^{-r/d},$$

with $q$ and $r$ connected by the relation $1/q = 1/p + r/d$, assuming that the multiresolution approximation spaces associated to the wavelet basis are in $W_{t,p}^r$ and contain the polynomials of degree strictly less than $t + r$.

Such an estimate should be compared with the linear estimate (4): the same convergence rate is governed by a much weaker smoothness assumption on $f$ since $q < p$ (as in the linear case, an “iff and only if” result can be obtained up to slight technical modifications in the statement of (8)). This result gives a precise mathematical meaning to the spatial adaptation properties of best $N$-term wavelet approximation: a function $f$ having isolated discontinuity, has usually a smaller amount of smoothness $s + t$ when measured in $L^p$ than when measured in $L^q$ with $1/q = 1/p + t/d$, and therefore $\sigma_N(f)$ might decrease significantly faster than $\epsilon_N(f)$.

The answer to the second question is given by a result due to Temlyakov : if $f = \sum_{\lambda \in \mathcal{V}} d_\lambda \psi_\lambda$, and if we measure the approximation error in $X = W_{t,p}^r$, a near optimal strategy when $1 < p < \infty$ consists in the thresholding procedure which retains the $N$ largest contributions $\|d_\lambda \psi_\lambda\|_X$ : if $A_N$ is the corresponding set of indices, one can prove that there exists $C > 0$ independent of $N$ and $f$ such that

$$\|f - \sum_{\lambda \in A_N} d_\lambda \psi_\lambda\|_X \leq C \sigma_N(f).$$

This fact is obvious when $X = L^2$ using the orthonormal basis property. It is a remarkable property of wavelet bases that it also holds for more general function spaces. In summary, thresholding plays for best $N$-term wavelet approximation an analogous role as projection for linear finite element approximation.

In the adaptive finite element framework, a similar theory is far from being complete. Partial answers to the basic questions are available if one chooses to consider adaptive partitions with shape constraints in terms of a uniform bound on the aspect ratio of the elements

$$\max_{K \in \mathcal{T}} \left( \frac{\text{Diam}(K)}{\text{vol}(K)} \right)^d \leq C.$$
finite elements of degree \( m \): if for any given tolerance \( \varepsilon > 0 \), one is able to build a partition \( T = T(\varepsilon) \) of cardinality \( N = N(\varepsilon) \) such that on each \( K \in T \) the local error of approximation by polynomials satisfies

\[
\frac{\varepsilon}{2} \leq \inf_{p \in \Pi_m} \|f - p\|_{W^{r,p}(K)} \leq \varepsilon,
\]

then we can build global approximants \( f_N \in V_T \subset S_N \) such that

\[
f \in W^{t+r,q} \Rightarrow \|f - f_N\|_X \leq CN^{-r/d},
\]

with \( q \) and \( r \) connected by the relation \( 1/q = 1/p + r/d \) and assuming \( s + t < m \).

The effective construction of \( T(\varepsilon) \) is not always feasible, in particular due to the conformity constraints on the partition which does not allow to connect very coarse and very fine elements without intermediate grading. However, this result shows that from an intuitive point of view, the adaptive finite element counterpart to wavelet thresholding amounts in equilibrating the local error over the partition. One can actually use these ideas in order to obtain the estimate (12) for adaptive finite elements under the more restrictive assumption that \( 1/q < 1/p + r/d \). Let us finally mention that the approximation theory for adaptive finite elements without shape constraints is an open problem.

4. Nonlinear approximation and PDE’s

Nonlinear approximation theory has opened new lines of research on the theory of PDE’s and their numerical discretization. On the one hand, it is worth revisiting the regularity theory of certain PDE’s for which the solutions develop singularities but might possess significantly higher smoothness in the scale of function spaces which govern the rate of nonlinear approximation in a given norm than in the scale which govern the rate of linear approximation in the same norm. Results of this type have been proved in particular for elliptic problems on nonsmooth domains [20] and for scalar 1D conservation laws [25]. These results show that if \( u \) is the solution of such equations, the rate of decay of \( \sigma_N(u) \) is significantly higher for best \( N \)-term approximation than for the projection on uniform finite element spaces, therefore advocating for the use of adaptive discretizations of such PDE’s.

On the other hand these results also provide with an ideal benchmark for adaptive discretizations of the equation, since \( \sigma_N(u) \) represents the best accuracy which can be achieved by \( N \) parameters. In the wavelet case these parameters are typically the \( N \) largest coefficients of the exact solution \( u \). However, in the practice of solving a PDE, these coefficients are not known, and neither is the set \( \Lambda \) corresponding to the indices of the \( N \) largest contributions \( \|d_\lambda \psi_\lambda\| \). It is therefore needed to develop appropriate adaptive resolution strategies as a substitute to the thresholding procedure. Such strategies aim at detecting the indices of the largest coefficients of the solutions and to compute them accurately, in a similar way that adaptive mesh refinement strategies aim at constructing the optimal mesh for finite element approximation. In both contexts, we could hope for an algorithm which builds approximations \( u_N \in \Sigma_N \) such that \( \|u - u_N\|_X \) is bounded up to a fixed
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multiplicative constant by $\sigma_N(u)$ for a given norm of interest, but this requirement is so far out of reach. A more reasonable goal is that the adaptive strategy exhibits the optimal rate of approximation: if $\sigma_N(u) \leq C N^{-s}$ for some $s > 0$, then $\|u - u_N\|_X \leq C N^{-s}$ up to a change in the constant. Another requirement is that the adaptive algorithm should be scalable, i.e. the number of elementary operations in order to compute $u_N$ remains proportional to $N$. Let us finally remark that the norm $\| \cdot \|_X$ for which error estimates can be obtained is often dictated by the nature of the equation (for example $X = H^1$ in the case of a second order elliptic problem) and that additional difficulties can be expected if one searches for estimates in a different norm.

5. The classical approach

The classical approach to numerically solving linear and nonlinear partial differential or integral equations $\mathcal{F}(u) = 0$ by the finite element method is typically concerned with the following issues:

(c1) Well-posedness of the equation, i.e. existence, uniqueness and stability of the solution.
(c2) Discretization into a finite element problem $\mathcal{F}_T(u_T) = 0$ by the Galerkin method with $u_T \in V_T$, analysis of well-posedness and of the approximation error $\|u - u_T\|_X$.
(c3) Numerical resolution of the finite dimensional system.
(c4) Mesh refinement based on a-posteriori error estimators in the case of adaptive finite element methods.

Several difficulties are associated to each of these steps. First of all, note that the well-posedness of the finite element problem is in general not a consequence of the well-posedness of the continuous problem. Typical examples even in the linear case are saddle point problems. For such problems, it is well known that, for Galerkin discretizations to be stable, the finite element spaces for the different solution components have to satisfy certain compatibility conditions (LBB or Inf-Sup condition), which are also crucial in the derivation of optimal error estimates. Thus the discrete problem does not necessarily inherit the “nice properties” of the original infinite dimensional problem. Concerning the numerical resolution of the discrete system, a typical source of trouble is its possible ill-conditioning, which interferes with the typical need to resort on iterative solvers in high dimension. An additional difficulty occurring in the case of integral equations is the manipulation of matrices which are densely populated.

Finally, let us elaborate more on the adaptivity step. Since more than two decades, the understanding and practical realization of adaptive refinement schemes in a finite element context has been documented in numerous publications [1, 2, 3, 27, 33]. Key ingredients in most adaptive algorithms are a-posteriori error estimators which are typically derived from the current residual $\mathcal{F}(u_T)$: in the case where the Fréchet derivative $D\mathcal{F}(u)$ is an isomorphism between Banach function spaces $X$ to $Y$, one can hope to estimate the error $\|u - u_T\|_X$ by the evaluation of
∥F(uT)∥Y. The rule of thumb is then to decompose ∥F(uT)∥Y into computable local error indicators ηK which aim to describe as accurately as possible the local error on each element K ∈ T. In the case of elliptic problems, these indicators typically consist of local residuals and other quantities such as jumps of derivatives across the interface between adjacent elements. A typical refinement algorithm will subdivide those elements K for which the error indicator ηK is larger than a prescribed tolerance ε resulting in a new mesh ˜T. Note that this strategy is theoretically in accordance with our remarks in §3 on adaptive finite element approximation, since it tends to equilibrate the local error. Two other frequently used strategies consist in refining a fixed proportion of the elements corresponding to the largest ηK, or the smallest number of elements K for which the ηK contribute to the global error up to a fixed proportion. It is therefore hoped that the iteration of this process from an initial mesh T_0 will produce optimal meshes (T_n)_{n ≥ 0} in the sense that the associated solutions u_n := u_{T_n} ∈ V_{T_n} converge to u at the optimal rate:

σN(u) ≤ CN^{−r} ⇒ ∥u − u_n∥_X ≤ C[♯(T_n)]^{−r},

(14)

up to a change in the constant C. Unfortunately, severe obstructions appear when trying to prove (14) even in the simplest model situations. One of them is that ηK is in general not an estimate by above of the local error, reducing the chances to derive the optimal rate. For most adaptive refinement algorithms, the theoretical situation is actually even worse in the sense that it cannot even be proved that the refinement step actually results in a reduction of the error by a fixed amount and that u_n converges to u as n grows. Only recently [26, 30] have proof of convergence appeared for certain type of adaptive finite element methods, yet without convergence rate and therefore no guaranteed advantage over their non-adaptive counterparts.

6. A new paradigm

Wavelet methods vary from finite element method in that they can be viewed as solving systems that are finite sections of one fixed infinite dimensional system corresponding to the discretization of the equation in the full basis. This observation has led to a new paradigm which has been explored in [15] for linear variational problems. It aims at closely intertwining the analysis—discretization—solution process. The basic steps there read as follows:

(n1) Well-posedness of the variational problem.
(n2) Discretization into an equivalent infinite dimensional problem which is well posed in ℓ^2.
(n3) Devise an iterative scheme for the ℓ^2-problem that exhibits a fixed error reduction per iteration step.
(n4) Numerical realization of the iterative scheme by means of an adaptive application of the involved infinite dimensional operators within some dynamically updated accuracy tolerances.

Thus the starting point (n1) is the same. The main difference is that one aims at staying as long as possible with the infinite dimensional problem. Only at the very
end, when it comes to applying the operators in the ideal iteration scheme (n4), one enters the finite dimensional realm. However, the finite number of degrees of freedom is determined at each stage by the adaptive application of the operator, so that at no stage any specific trial space is fixed.

The simplest example is provided by the Poisson equation 

$$-\Delta u = f \text{ on a domain } \Omega \text{ with homogeneous boundary conditions},$$

for which the variational formulation in $X = H^1_0$ reads: find $u \in X$ such that

$$a(u, v) = L(v), \text{ for all } v \in X,$$

with $a(u, v) := \int_\Omega \nabla u \nabla v$ and $L(v) := \int_\Omega f v$. The well-posedness for a data $f \in X' = H^{-1}$ is ensured by the Lax-Milgram lemma. In the analysis of the wavelet discretization of this problem, we shall invoke the fact that wavelet bases provide norm equivalence for Sobolev spaces in terms of weighted $\ell^2$ norms of the coefficients: if $u = \sum \lambda u_\lambda \psi_\lambda$, one has

$$\|u\|_{H^s}^2 \sim \sum \lambda \|u_\lambda \psi_\lambda\|_{H^s}^2 \sim \sum \lambda 2^{2s|\lambda|} |u_\lambda|^2.$$

We refer to [13] and [21] for the general mechanism allowing to derive these equivalences, in particular for Sobolev spaces on domains with boundary conditions such as $H^1_0$. Therefore, if we renormalize our system in such a way that $\|\psi_\lambda\|_X = 1$, we obtain the norm equivalence

$$\|u\|_X^2 \sim \|U\|^2,$$

where $U := (u_\lambda)_{\lambda \in \mathfrak{v}}$ and $\| \cdot \|$ denotes the $\ell^2$ norm. By duality, one also easily obtains

$$\|f\|_{X'}^2 \sim \|F\|^2,$$

with $F := ((f, \psi_\lambda))_{\lambda \in \mathfrak{v}}$. The equivalent $\ell^2$ system is thus given by

$$AU = F,$$

where $A(\lambda, \mu) = a(\psi_\lambda, \psi_\mu)$ is a symmetric positive definite matrix which is continuous and coercive in $\ell^2$. In this case, a converging infinite dimensional algorithm can simply be obtained by the Richardson iteration

$$U^n := U^{n-1} + \tau (F - AU^{n-1})$$

with $0 < \tau < 2|\lambda_{\text{max}}(A)|^{-1}$ and $U^0 = 0$, which guarantees the reduction rate $\|U - U^n\| \leq \rho \|U - U^{n-1}\|$ with $\rho = \max\{1 - \tau \lambda_{\text{min}}(A), \tau \lambda_{\text{max}}(A) - 1\}$. Note that renormalizing the wavelet system plays the role of a multiscale preconditioning, similar to multigrid yet operated at the infinite dimensional level.

At this stage, one enters finite dimensional adaptive computation by modifying the Richardson iteration up to a prescribed tolerance according to

$$U^n := U^{n-1} + \tau (\text{COARSE}(F, \varepsilon) - \text{APPROX}(AU^{n-1}, \varepsilon))$$

where $\|F - \text{COARSE}(F, \varepsilon)\| \leq \varepsilon$ and $\|AU - \text{APPROX}(AU^{n-1}, \varepsilon)\| \leq \varepsilon$, and the $U^n$ are now finite dimensional vector supported by adaptive sets of indices $\Lambda_n$. The
procedure **COARSE**, which simply corresponds to thresholding the data vector $F$ at a level corresponding to accuracy $\varepsilon$, can be practically achieved without the full knowledge of $F$ by using some a-priori bounds on the size of the coefficients $\langle f, \psi_\lambda \rangle$, exploiting the local smoothness of $f$ and the oscillation properties of the wavelets. The procedure **APPROX** deserves more attention. In order to limitate the spreading effect of the matrix $A$, one invokes its *compressibility* properties, namely the possibly to truncate it into a matrix $A_N$ with $N$ non-zero entries per rows and columns in such a way that

$$\|A - A_N\|_{\ell^2 \to \ell^2} \leq CN^{-s}. \quad (22)$$

The rate of compressibility $s$ depends on the available a-priori estimates on the off-diagonal entries $A(\lambda, \mu) := \int_{\Omega} \nabla \psi_\lambda \nabla \psi_\mu$ which are consequences of the smoothness and vanishing moment properties of the wavelet system, see [14]. Once these properties are established, a first possibility is thus to choose

$$\text{APPROX}(AU^{n-1}, \varepsilon) = A_N U^{n-1} \quad (23)$$

with $N$ large enough so that accuracy $\varepsilon$ is ensured. Clearly the modified iteration [21] satisfies $\|U - U^n\| \leq \rho \|U - U^{n-1}\| + 2\tau \varepsilon$, and therefore ensures a fixed reduction rate until the error is of the order $\sqrt{\frac{2\tau \|A\|}{1-\rho}} \varepsilon$, or until the residual $F - AU^n$ is of order $\frac{2\tau \|A\|}{1-\rho} \varepsilon$. A natural idea is therefore to update dynamically the tolerance $\varepsilon$, which is first set to 1 and divided by 2 each time the approximate residual **COARSE**($F, \varepsilon$) – **APPROX**($AU^{n-1}, \varepsilon$) is below $[\frac{2\tau \|A\|}{1-\rho} + 3] \varepsilon$ (which is ensured to happen after a fixed number of steps).

We therefore obtain a converging adaptive strategy, so far without information about the convergence rate. It turns out that the optimal convergence rate can also be proved, with a more careful tuning of the adaptive algorithm. Two additional ingredients are involved in this tuning.

Firstly, the adaptive matrix vector multiplication **APPROX** has to be designed in a more elaborate way than (23) which could have the effect of inflating too much the sets $\Lambda_n$. Instead, one defines for a finite length vector $V$

$$\text{APPROX}(AV, \varepsilon) = \sum_{l=0}^j A_{2j-l} [V_{2^l} - V_{2^{l-1}}] \quad (24)$$

where $V_N$ denotes the restriction of $V$ to its $N$ largest components (with the notation $V_{1/2} = 0$), and $j$ is the smallest positive integer such that the residual $\sum_{l=0}^j \|A - A_{2j-l}\| \|V_{2^l} - V_{2^{l-1}}\| + \|A\| \|V - V_{2^0}\|$ is less than $\varepsilon$. In this procedure, the spreading of the operator is more important on the largest coefficients which are less in number, resulting in a significant gain in the complexity of the outcome.

Secondly, additional coarsening steps are needed in order to further limitate the spreading of the sets $\Lambda_n$ and preserve the optimal rate of convergence. More precisely, the procedure **COARSE** is applied to $U^n$ with a tolerance proportional to $\varepsilon$, for those $n$ such that $\varepsilon$ will be updated at the next iteration.
With such additional ingredients, it was proved in [15] that the error has the optimal rate of decay in the sense that

$$\sigma_N(u) \leq C N^{-s} \Rightarrow \| u - u_n \|_X \sim \| U - U^n \| \leq C[\#(\Lambda_n)]^{-s},$$

and that moreover, the computational cost of producing $u_n$ remains proportional to $\#(\Lambda_n)$. It is interesting to note that this strategy extends to non-elliptic problems such as saddle-points problems, without the need for compatibility conditions, since one inherits the well-posedness of the continuous problem which allows to obtain a converging infinite dimensional iteration, such the Uzawa algorithm or a gradient descent applied to the least-square system (see also [15, 19]). The extension to nonlinear variational problems, based on infinite dimensional relaxation or Newton iterations, has also been considered in [14]. It requires a specific procedure for the application of the nonlinear operator in the wavelet coefficients domain which generalizes (23). It should also be mentioned that matrix compressibility also applies in the case of integral operators which have quasi-sparse wavelet discretizations. Therefore several of the obstructions from the classical approach—conditioning, compatibility, dense matrices—have disappeared in the wavelet approach.

Let us finally mention that the coarsening steps are not really needed in the practical implementations of the adaptive wavelet method (for those problems which have been considered so far) which still does exhibit optimal convergence rate. However, we do not know how to prove (25) without these coarsening steps. There seems to be a similar situation in the finite element context : it has recently been proved in [14] that (14) can be achieved by an adaptive mesh refinement algorithm which incorporates coarsening steps, while these steps are not needed in practice.

7. Conclusions and shortcomings

There exist other approaches for the development of efficient wavelet-based adaptive schemes. In particular, an substantial research activity has recently been devoted to multiresolution adaptive processing techniques, following the line of idea introduced in [28, 29]. In this approach, one starts from a classical and reliable scheme on a uniform grid (finite element, finite difference or finite volume) and applies a discrete multiresolution decomposition to the numerical data in order to compress the computational time and memory space while preserving the accuracy of the initial scheme. Here the adaptive sets $\Lambda_n$ are therefore limited within the resolution level of the uniform grid where the classical scheme operates. This approach seems more appropriate for hyperbolic initial value problems [17, 22], in which a straightforward wavelet discretization might fail to converge properly. It should again be compared to its adaptive mesh refinement counterpart such as in [1, 1].

Let us conclude by saying that despite its theoretical success, in the sense of achieving for certain classes of problems the optimal convergence rate with respect to the number of degrees of freedom, the wavelet-based approach to adaptive numerical simulation suffers from three major curses.
The curse of geometry: while the construction of wavelet bases on a rectangular domains is fairly simple—one can use tensor product techniques and inherit the simplicity of the univariate construction—it is by far less trivial for domains with complicated geometries. Several approaches have been proposed to deal with this situation, in particular domain decomposition into rectangular patches or hierarchical finite element spaces, see [13, 21], and concrete implementations are nowadays available, but they result in an unavoidable loss of structural simplicity in comparison to the basic Haar system of §3.

The curse of data structure: encoding and manipulating the adaptive wavelet approximations $U^n$ to the solution means that we both store the coefficients and the indices of the adaptive set $\Lambda_n$ which should be dynamically updated. The same goes for the indices of the matrix $A$ which are used in the matrix-vector algorithm (24) at each step of the algorithm. This dynamical adaptation, which requires appropriate data structure, results in major overheads in the computational cost which are observed in practice: the numerical results in [4] reveal that while the wavelet adaptive algorithm indeed exhibits the optimal rate of convergence and slightly outperforms adaptive finite element algorithms from this perspective, the latter remains significantly more efficient from the point of view of computational time.

The curse of anisotropy: adaptive wavelet approximation has roughly speaking the same properties as isotropic refinement. However, many instances of singularities such as boundary layers and shock waves, have anisotropic features which suggests that the refinement should be more pronounced in one particular direction.

From a theoretical point of view, the following example illustrate the weakness of wavelet bases in this situation: if $f = \chi_\Omega$ with $\Omega \subset \mathbb{R}^d$ a smooth domain, then the rate of best $N$-term approximation in $X = L^2$ is limited to $r = 1/(2d - 2)$ and therefore deteriorates as the dimension grows. Wavelet bases should therefore be reconsidered if one wants to obtain better rates which take some advantage of the geometric smoothness of the curves of discontinuites. On the adaptive finite element side, anisotropic refinement has been considered and practically implemented, yet without a clean theory available for the design of an optimal mesh.

The significance of wavelets in numerical analysis remains therefore tied to these curses and future breakthroughs are to be expected once simple and appropriate solutions are proposed in order to deal with them.

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