On rational homotopy and minimal models

CHRISTOPH BOCK

MSC 2010: Primary: 55P62; Secondary: 16E45.

Abstract

We prove a result that enables us to calculate the rational homotopy of a wide class of spaces by the theory of minimal models.

1 Introduction

Let \( K \) be a field of characteristic zero. A differential graded algebra (DGA) is a graded \( K \)-algebra \( A = \bigoplus_{i \in \mathbb{N}} A^i \) together with a \( K \)-linear map \( d: A \to A \) such that \( d(A^i) \subset A^{i+1} \) and the following conditions are satisfied:

(i) The \( K \)-algebra structure of \( A \) is given by an inclusion \( K \hookrightarrow A^0 \).

(ii) The multiplication is graded commutative, i.e. for \( a \in A^i \) and \( b \in A^j \) one has \( a \cdot b = (-1)^{ij} b \cdot a \in A^{i+j} \).

(iii) The Leibniz rule holds: \( \forall a \in A^i \forall b \in A \ d(a \cdot b) = d(a) \cdot b + (-1)^i a \cdot d(b) \)

(iv) The map \( d \) is a differential, i.e. \( d^2 = 0 \).

Further, we define \( |a| := i \) for \( a \in A^i \).

Example. Given a manifold \( M \), one can consider the complex of its differential forms \( (\Omega(M), d) \), which has the structure of a differential graded algebra over the field \( \mathbb{R} \).

The \( i \)-th cohomology of a DGA \( (A, d) \) is the algebra

\[
H^i(A, d) := \frac{\ker(d: A^i \to A^{i+1})}{\text{im}(d: A^{i-1} \to A^i)}.
\]

If \( (B, d_B) \) is another DGA, then a \( K \)-linear map \( f: A \to B \) is called morphism if \( f(A^i) \subset B^i \), \( f \) is multiplicative, and \( d_B \circ f = f \circ d_A \). Obviously, any such \( f \) induces a homomorphism \( f^*: H^*(A, d_A) \to H^*(B, d_B) \). A morphism of differential graded algebras inducing an isomorphism on cohomology is called quasi-isomorphism.

Definition 1.1. A DGA \((M, d)\) is said to be minimal if

(i) there is a graded vector space \( V = \bigoplus_{i \in \mathbb{N}_+} V^i \) = Span \( \{a_k \mid k \in I\} \) with homogeneous elements \( a_k \), which we call the generators,
(ii) \( \mathcal{M} = \bigwedge V \),

(iii) the index set \( I \) is well ordered, such that \( k < l \Rightarrow |a_k| \leq |a_l| \) and the expression for \( d(a_k) \) contains only generators \( a_l \) with \( l < k \).

We shall say that \( (\mathcal{M}, d) \) is a minimal model for a differential graded algebra \( (A, d_A) \) if \( (\mathcal{M}, d) \) is minimal and there is a quasi-isomorphism of differential graded algebras \( \rho: (\mathcal{M}, d) \to (A, d_A) \), i.e. \( \rho \) induces an isomorphism \( \rho^*: H^*(\mathcal{M}, d) \to H^*(A, d_A) \) on cohomology.

The importance of minimal models is reflected by the following theorem, which is taken from Sullivan’s work [11, Section 5].

**Theorem 1.2.** A differential graded algebra \( (A, d_A) \) with \( H^0(A, d_A) = \mathbb{K} \) possesses a minimal model. It is unique up to isomorphism of differential graded algebras.

We quote the existence-part of Sullivan’s proof, which gives an explicit construction of the minimal model. Whenever we are going to construct such a model for a given algebra in this note, we will do it as we do it in this proof.

**Proof of the existence.** We need the following algebraic operations to “add” resp. “kill” cohomology.

Let \( (\mathcal{M}, d) \) be a DGA. We “add” cohomology by choosing a new generator \( x \) and setting

\[
\tilde{\mathcal{M}} := \mathcal{M} \otimes \bigwedge \langle x \rangle, \quad \tilde{d}(\mathcal{M}) = d, \quad \tilde{d}(x) = 0,
\]

and “kill” a cohomology class \( [z] \in H^k(\mathcal{M}, d) \) by choosing a new generator \( y \) of degree \( k - 1 \) and setting

\[
\tilde{\mathcal{M}} := \mathcal{M} \otimes \bigwedge \langle y \rangle, \quad \tilde{d}(\mathcal{M}) = d, \quad \tilde{d}(y) = z.
\]

Note that \( z \) is a polynomial in the generators of \( \mathcal{M} \).

Now, let \( (A, d_A) \) a DGA with \( H^0(A, d_A) = \mathbb{K} \). We set \( \mathcal{M}_0 := \mathbb{K}, d_0 := 0 \) and \( \rho_0(x) = x \).

Suppose now \( \rho_k: (\mathcal{M}_k, d_k) \to (A, d_A) \) has been constructed so that \( \rho_k \) induces isomorphisms on cohomology in degrees \( \leq k \) and a monomorphism in degree \( (k + 1) \).

“Add” cohomology in degree \( (k + 1) \) to get a morphism of differential graded algebras \( \rho_{(k+1), 0}: (\mathcal{M}_{(k+1), 0}, d_{(k+1), 0}) \to (A, d_A) \) which induces an isomorphism \( \rho_{(k+1), 0}^* \) on cohomology in degrees \( \leq (k + 1) \). Now, we want to make the induced map \( \rho_{(k+1), 0}^* \) injective on cohomology in degree \( (k + 2) \).

We “kill” the kernel on cohomology in degree \( (k + 2) \) (by non-closed generators of degree \( (k+1) \)) and define \( \rho_{(k+1), 1}: (\mathcal{M}_{(k+1), 1}, d_{(k+1), 1}) \to (A, d_A) \) accordingly. If there are generators of degree one in \( (\mathcal{M}_{(k+1), 0}, d_{(k+1), 0}) \) it is possible that this killing process generates new kernel on cohomology in degree \( (k + 2) \). Therefore, we may have to “kill” the kernel in degree \( (k + 2) \) repeatedly.

We end up with a morphism \( \rho_{(k+1), \infty}: (\mathcal{M}_{(k+1), \infty}, d_{(k+1), \infty}) \to (A, d_A) \) which induces isomorphisms on cohomology in degrees \( \leq (k + 1) \) and a monomorphism in degree \( (k + 2) \). Now, we are going to set \( \rho_{k+1} := \rho_{(k+1), \infty} \) and \( \mathcal{M}_{k+1, d_{k+1}} := (\mathcal{M}_{(k+1), \infty}, d_{(k+1), \infty}) \).

Inductively we get the minimal model \( \rho: (\mathcal{M}, d) \to (A, d_A) \).\(\Box\)
A minimal model \((M_M, d)\) of a connected smooth manifold \(M\) is a minimal model for the de Rahm complex \((\Omega(M), d)\) of differential forms on \(M\). Note that this implies that \((M, d)\) is an algebra over \(\mathbb{R}\). The last theorem implies that every connected smooth manifold possesses a minimal model which is unique up to isomorphism of differential graded algebras.

For a certain class of spaces that includes all nilpotent (and hence all simply-connected) spaces, we can read off the non-torsion part of the homotopy from the generators of the minimal model. The definition of a nilpotent space will be given in the next section.

In general, it is very difficult to calculate the homotopy groups \(\pi_k(X)\) of a given topological space \(X\). However, if one is willing to forget the torsion, with certain assumptions on \(X\), the rational homotopy groups \(\pi_k(X) \otimes \mathbb{Q}\) can be determined by the theory of minimal models.

In order to relate minimal models to rational homotopy theory, we need a differential graded algebra over \(\mathbb{Q}\) to replace the de Rahm algebra.

Let \(\Delta^n\) be a standard simplex in \(\mathbb{R}^{n+1}\) and \((\Omega_{PL}(\Delta^n), d)\) the restriction to \(\Delta^n\) of all differential forms in \(\mathbb{R}^{n+1}\) that can be written as \(\sum P_{i_1 \ldots i_k} dx_{i_1} \ldots dx_{i_k}\), where \(P_{i_1 \ldots i_k} \in \mathbb{Q}[x_1, \ldots, x_{n+1}]\), together with multiplication and differential induced by \(\mathbb{R}^{n+1}\).

Let \(X = \{ (\sigma_i)_{i \in I} \}\) be a path-connected simplicial complex. Set for \(k \in \mathbb{Z}\)
\[
\Omega^k_{PL}(X) := \{ (\alpha_i)_{i \in I} \mid \alpha_i \in \Omega^k_{PL}(\sigma_i) \wedge (\sigma_i \subset \partial \sigma_j \Rightarrow \alpha_j|_{\sigma_i} = \alpha_i) \},
\]
and \(\Omega_{PL}(X) := \bigoplus_{k \in \mathbb{Z}} \Omega^k_{PL}(X)\). It can be verified that the set \(\Omega_{PL}(X)\) of so-called PL forms is a differential graded algebra over \(\mathbb{Q}\) if we use the multiplication and differential on forms componentwise.

**Remark (9, Remark 1.1.6)**. In fact, PL forms can be defined, along with the minimal model, for any CW-complex, say. The process consists of taking the singular complex of the space and treating it as a simplicial set amenable to the PL form construction.

For a CW-complex \(X\), we define the \((\mathbb{Q}-)\)minimal model \(M_{X, \mathbb{Q}}\) of \(X\) to be the minimal model of \((\Omega_{PL}(X), d)\).

## 2 Nilpotent spaces

Already in his paper [11], Sullivan shows that for nilpotent spaces, there is a correspondence between the minimal model and the rational homotopy. To state this result, we need the notion of a nilpotent space resp. nilpotent module.

Let \(G\) be a group, \(H\) be a \(G\)-module, \(\Gamma^0_G H := H\) and
\[
\Gamma^{i+1}_G H := \langle g.h - h \mid g \in G \wedge h \in \Gamma^i_G H \rangle \subset \Gamma^i_G H
\]
for \(i \in \mathbb{N}\).

Then, \(H\) is called a nilpotent module if there is \(n_0 \in \mathbb{N}\) such that \(\Gamma^{n_0}_G H = \{1\}\).

We recall the natural \(\pi_1\)-module structure of the higher homotopy groups \(\pi_n\) of a topological space. For instance, let \((X, x_0)\) be a pointed space with
universal cover \((\tilde{X}, \tilde{x}_0)\). It is well known that \(\pi_1(X, x_0) \cong D(\tilde{X})\), the group of deck transformations of the universal covering. Now, because \(\tilde{X}\) is simply-connected, every free homotopy class of self-maps of \(\tilde{X}\) determines uniquely a class of basepoint preserving self-maps of \(\tilde{X}\) (see e.g. [5, Proposition 4.1.2]). This means that to every homotopy class of deck transformations corresponds a homotopy class of basepoint preserving self-maps (which are, in fact, homotopy equivalences) \((\tilde{X}, \tilde{x}_0) \to (\tilde{X}, \tilde{x}_0)\). These maps provide induced automorphisms of homotopy groups \(\pi_n(\tilde{X}, \tilde{x}_0) \cong \pi_n(X, x_0)\) \((n > 1)\) and this whole process then provides an action of \(\pi_1(X, x_0)\) on \(\pi_n(X, x_0)\).

**Definition 2.1.** A path-connected topological space \(X\) whose universal covering exists is called **nilpotent** if for \(x_0 \in X\) the fundamental group \(\pi_1(X, x_0)\) is a nilpotent group and the higher homotopy groups \(\pi_n(X, x_0)\) are nilpotent \(\pi_1(X, x_0)\)-modules for all \(n \in \mathbb{N}, n \geq 2\). Note, the definition is independent of the choice of the base point.

**Example.**

(i) Simply-connected spaces are nilpotent.

(ii) \(S^1\) is nilpotent.

(iii) The cartesian product of two nilpotent spaces is nilpotent. Therefore, all tori are nilpotent.

(iv) The Klein bottle is not nilpotent.

(v) \(P^n(\mathbb{R})\) is nilpotent if and only if \(n \equiv 1(2)\).

**Proof.** (i) - (iv) are obvious and (v) can be found in Hilton’s book [7] on page 165. \(\square\)

The main theorem on the rational homotopy of nilpotent spaces is the following.

**Theorem 2.2.** Let \(X\) be a path-connected nilpotent CW-complex with finitely generated homotopy groups. If \(\mathcal{M}_{X,\mathbb{Q}} = \bigwedge V\) denotes the \(\mathbb{Q}\)-minimal model, then for all \(k \in \mathbb{N}\) with \(k \geq 2\) holds:

\[
\text{Hom}_\mathbb{Z}(\pi_k(X), \mathbb{Q}) \cong V^k
\]

Using another approach to minimal models (via localisation of spaces and Postnikow towers), this theorem is proved for example in [8]. The proof that we shall give here is new to the author’s knowledge. We will show the following more general result mentioned (but not proved) by Halperin in [4].

**Theorem 2.3.** Let \(X\) be a path-connected triangulable topological space whose universal covering exists. Denote by \(\mathcal{M}_{X,\mathbb{Q}} = \bigwedge V\) the \(\mathbb{Q}\)-minimal model and assume that

(i) each \(\pi_k(X)\) is a finitely generated nilpotent \(\pi_1(X)\)-module for \(k \geq 2\) and
(ii) the minimal model for $K(\pi_1(X),1)$ has no generators in degrees greater than one.

Then for each $k \geq 2$ there is an isomorphism $\text{Hom}_\mathbb{Z}(\pi_k(X), \mathbb{Q}) \cong V^k$.

**Remark.** The homotopy groups of a compact nilpotent smooth manifold are finitely generated:

By [7, Satz 7.22], a nilpotent space has finitely generated homotopy if and only if it has finitely generated homology with $\mathbb{Z}$-coefficients. The latter is satisfied for compact spaces. □

The main tool for the proof of the above theorems is a consequence of the fundamental theorem of Halperin [4]. In the next section, we quote it and use it to prove Theorems 2.2 and 2.3.

### 3 The Halperin-Grivel-Thomas theorem

To state the theorem, let us recall a basic construction for fibrations.

Let $\pi: E \to B$ be a fibration with path-connected basis $B$. Therefore, all fibers $F_b = \pi^{-1}({\{b}\})$ are homotopy equivalent to a fixed fiber $F$ since each path $\gamma$ in $B$ lifts to a homotopy equivalence $L_\gamma: F_{\gamma(0)} \to F_{\gamma(1)}$ between the fibers over the endpoints of $\gamma$. In particular, restricting the paths to loops at a basepoint of $B$ we obtain homotopy equivalences $L_\gamma: F \to F$ for $F$ the fiber over the basepoint $b_0$. One can show that this induces a natural $\pi_1(B,b_0)$-module structure on $H^*(F, \mathbb{Q})$.

**Theorem 3.1** ([9, Theorem 1.4.4]). Let $F, E, B$ be path-connected triangulable topological spaces and $F \to E \to B$ a fibration such that $H^n(F, \mathbb{Q})$ is a nilpotent $\pi_1(B,b_0)$-module for $n \in \mathbb{N}_+$. The fibration induces a sequence

$$
(\Omega_{PL}(B), d_B) \to (\Omega_{PL}(E), d_E) \to (\Omega_{PL}(F), d_F)
$$

of differential graded algebras. Suppose that $H^*(F, \mathbb{Q})$ or $H^*(B, \mathbb{Q})$ is of finite type.

Then there is a quasi-isomorphism $\Psi: (\mathcal{M}_{B,\mathbb{Q}} \otimes \mathcal{M}_{F,\mathbb{Q}}, D) \to (\Omega_{PL}(E), d_E)$ making the following diagram commutative:

\[
\begin{array}{ccc}
\Omega_{PL}(B), d_B & \xrightarrow{\rho_B} & \Omega_{PL}(E), d_E \\
\downarrow \Psi & & \downarrow \rho_F \\
\mathcal{M}_{B,\mathbb{Q}}, D_B & \xrightarrow{\rho} & \mathcal{M}_{B,\mathbb{Q}} \otimes \mathcal{M}_{F,\mathbb{Q}}, D
\end{array}
\]

Furthermore, the left and the right vertical arrows are the minimal models. Moreover, if $\mathcal{M}_F = \bigwedge V_F$, there is an ordered basis $\{v^F_i | i \in I\}$ of $V_F$ such that for all $i, j \in I$ holds $D(v^F_i) \in \mathcal{M}_B \otimes (\mathcal{M}_F)_{<v^F_j}$ and $(v^F_i < v^F_j \Rightarrow |v^F_i| \leq |v^F_j|)$. □

**Remark.** In general, $(\mathcal{M}_{B,\mathbb{Q}} \otimes \mathcal{M}_{F,\mathbb{Q}}, D)$ is not a minimal differential graded algebra and $D|_{\mathcal{M}_{F,\mathbb{Q}} \neq D_F}$ is possible.
We need some further preparations for the proofs of the above theorems. The first is a reformulation of the results 3.8 - 3.10 in [7]. It justifies the statement of the next theorem.

**Proposition 3.2.** Let $G$ be a finitely generated nilpotent group. Then the set $T(G)$ of torsion elements of $G$ is a finite normal subgroup of $G$ and $G/T(G)$ is finitely generated.

**Theorem 3.3.** Let $G$ be a finite generated nilpotent group and denote by $T(G)$ its finite normal torsion group.

Then $K(G, 1)$ and $K(G/T(G), 1)$ share their minimal model.

**Proof.** Since $T(G)$ is finite and $Q$ is a field, we get from [2, Section 4.2] $H^n(K(T(G), 1), Q) = \{0\}$ for $n \in \mathbb{N}_+$. The construction of the minimal model in the proof of Theorem 3.2 implies that $\mathcal{M}_{K(T(G), 1), Q}$ has no generators of degree greater than zero. Now, the theorem follows from the preceding one, applied to the fibration $K(T(G), 1) \to K(G, 1) \to K(G/T(G), 1)$.

**Lemma 3.4.** Let $X$ be topological space with universal covering $p: \tilde{X} \to X$.

Then, up to weak homotopy equivalence of the total space, there is a fibration $\tilde{X} \to X \to K := K(\pi_1(X), 1)$. Moreover, for a class $[\gamma] \in \pi_1(K) \cong \pi_1(X)$ the homotopy equivalences $L_{[\gamma]}: \tilde{X} \to \tilde{X}$ described at the beginning of this section are given by the corresponding deck transformations of $p$.

**Proof.** Denote by $\pi: E \to K(\pi_1(X), 1)$ the universal principal $\pi_1(X)$-bundle. Regard on $E \times \tilde{X}$ the diagonal $\pi_1(X)$-action. Then, the fibre bundle

$$\tilde{X} \to ((E \times \tilde{X})/\pi_1(X)) \to K$$

has the desired properties. □

### 3.1 Proof of Theorem 2.3

Let $X$ be as in the statement of the theorem. For simply-connected spaces, the theorem was proven in [3, Theorem 15.11]. Now, the idea is to use this result and to consider the universal cover $p: \tilde{X} \to X$. Denote by $\mathcal{M}_{\tilde{X}, Q} = \bigwedge \tilde{V}$ and $\mathcal{M}_{X, Q} = \bigwedge V$ the $Q$-minimal models. We shall show

$$\forall k \geq 2 \quad V^k \cong \tilde{V}^k. \quad (1)$$

This and the truth of the theorem for simply-connected spaces implies then the general case

$$\forall k \geq 2 \quad V^k \cong \tilde{V}^k \cong \text{Hom}_Z(\pi_k(\tilde{X}), Q) = \text{Hom}_Z(\pi_k(X), Q).$$

It remains to show (1): Since $X$ is triangulable, $X$ and $\tilde{X}$ can be seen as CW-complexes. Therefore, up to weak homotopy, there is the following fibration of CW-complexes

$$\tilde{X} \to X \xrightarrow{\pi} K(\pi_1(X), 1) =: K.$$
We prove below:

\[ H^*(\tilde{X}, \mathbb{Q}) \text{ is of finite type.} \quad (2) \]

\[ H^*(\tilde{X}, \mathbb{Q}) \text{ is a nilpotent } \pi_1(X)\text{-module.} \quad (3) \]

Then Theorem 3.1 implies the existence of a quasi-isomorphism \( \rho \) such that the following diagram commutes:

\[
\begin{array}{ccc}
(\Omega_{PL}(K), d_K) & \xrightarrow{\rho_K} & (\Omega_{PL}(X), d_X) \\
\downarrow \rho \quad & & \downarrow \rho \\
(\mathcal{M}_{K,Q}, D_K) & \xleftarrow{\rho} & (\mathcal{M}_{K,Q} \otimes \mathcal{M}_{\tilde{X},Q}, D)
\end{array}
\]

Finally, we shall see

\[ (\mathcal{M}_{K,Q} \otimes \mathcal{M}_{\tilde{X},Q}, D) \text{ is a minimal differential graded algebra} \quad (4) \]

and this implies (1) since \( \mathcal{M}_K \) has no generators of degree greater than one by assumption (ii).

We still have to prove (2) - (4):

By assumption (i), \( \pi_k(X) = \pi_k(\tilde{X}) \) is finitely generated for \( k \geq 2 \). Since simply-connected spaces are nilpotent, [7, Satz 7.22] implies the finite generation of \( H_*(\tilde{X}, \mathbb{Z}) \) and (2) follows.

(3) is the statement of Theorem 2.1 \((i) \Rightarrow (ii)\) in [6] - applied to the action of \( \pi_1(X) \) on \( \pi_i(\tilde{X}) \).

ad (4): By assumption (ii), \( \mathcal{M}_K \) has no generators in degrees greater than one, i.e. \( \mathcal{M}_{K,Q} = \bigwedge \{v_i \mid i \in I\} \) with \( |v_i| = 1 \). The construction of the minimal model in the proof of Theorem 3.3 implies that the minimal model of a simply-connected space has no generators in degree one, i.e. \( \mathcal{M}_{\tilde{X},Q} = \bigwedge \{w_j \mid j \in J\} \) with \( |w_j| > 1 \). We expand the well orderings of \( I \) and \( J \) to a well ordering of their union by \( v_{i \in I} \forall j \in J \quad i < j \). Theorem 3.1 implies that \( D(w_j) \) contains only generators which are ordered before \( w_j \). Trivially, \( D(v_i) \) also has this property, so we have shown (4) and the theorem is proved.

\[ \square \]

3.2 Proof of Theorem 2.2:

Let \( X \) be a path-connected nilpotent CW-complex with finitely generated fundamental group and finitely generated homotopy. By Theorem 2.3 we have to show that the minimal model of \( K(\pi_1(X), 1) \) has no generators in degrees greater than one.

Theorem 3.3 implies that it suffices to show that \( K(\pi_1(X)/T, 1) \) has this property, where \( T \) denotes the torsion group of \( \pi_1(X) \). \( \Gamma := \pi_1(X)/T \) is a finitely generated nilpotent group without torsion. By [10, Theorem 2.18], \( \Gamma \) can be embedded as a lattice in a connected and simply-connected nilpotent Lie group \( G \). Therefore, the nilmanifold \( G/\Gamma \) is a \( K(\Gamma, 1) \) and from [11, Theorem 3.11] follows that its minimal model has no generators in degrees greater than one.

\[ \square \]
Acknowledgement. The results presented in this paper are parts of my dissertation that I wrote under the supervision of H. Geiges. I wish to express my sincerest gratitude for his support. Moreover, I wish to thank St. Halperin. I have profited from his suggestions.

Remark. St. Halperin told me that smooth triangulations are unnecessary to do rational homotopy for smooth differential forms, as is already presented in his works [4] and [3].

References

[1] Ch. Bock: On Low-Dimensional Solvmanifolds, arXiv:0903.2926
[2] L. Evens: The Cohomology of Groups, Clarendon Press (1991).
[3] Y. Félix, St. Halperin, J.-C. Thomas: Rational Homotopy Theory, Springer (2001).
[4] St. Halperin: Lectures on Minimal Models, Mém. Soc. Math. France (N.S.), no. 9-10 (1983).
[5] A. Hatcher: Algebraic Topology, Cambridge University Press (2002).
[6] P. J. Hilton: On G-spaces, Bol. Soc. Brasil. Mat. 7 (1976), no. 1, 65–73.
[7] P. J. Hilton: Nilpotente Gruppen und nilpotente Räume, Lecture Notes in Math. 1053, Springer (1984).
[8] D. Lehmann: Théorie homotopique des formes différentielles (d’après D. Sullivan), Astérisque 45 (1977).
[9] J. Oprea, A. Tralle: Symplectic Manifolds with no Kähler Structure, Lecture Notes in Math. 1661, Springer (1997).
[10] M. S. Raghunathan: Discrete Subgroups of Lie Groups, Springer (1972).
[11] D. Sullivan: Infinitesimal computations in topology, Inst. Hautes Études Sci. Publ. Math. 47 (1977), 269–331.

Christoph Bock
Department Mathematik
Universität Erlangen-Nürnberg
Cauerstraße 11
91058 Erlangen
Germany
e-mail: bock@mi.uni-erlangen.de