Three plots about the Cremona groups

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Abstract. The first group of results of the paper concerns the compressibility of finite subgroups of the Cremona groups. The second concerns the embeddability of other groups in the Cremona groups and, conversely, of the Cremona groups in other groups. The third concerns the connectedness of the Cremona groups.

Keywords: Cremona group, compressibility, Jordan property, connectedness.

To the memory of V. A. Iskovskikh

§1. Introduction

1.1. The Cremona group $\text{Cr}_n(k)$ of rank $n$ over a field $k$ is the group of $k$-automorphisms of the field $k(x_1, \ldots, x_n)$ of rational functions over $k$ in variables $x_1, \ldots, x_n$. It has a geometric interpretation: if we identify the field $k(x_1, \ldots, x_n)$ using a $k$-isomorphism with the field $k(X)$ of an irreducible algebraic variety $X$ defined and rational over $k$, then every element $\sigma$ of the group $\text{Bir}_k(X)$ of all $k$-birational self-maps $X \rightarrow X$ defines an element $\sigma^* \in \text{Cr}_n(k)$,

$$\sigma^*(f) := f \circ \sigma, \quad f \in k(X),$$

(1.1)

and the map $\text{Bir}_k(X) \rightarrow \text{Cr}_n(k), \sigma \mapsto (\sigma^{-1})^*$, is a group isomorphism. For this reason, the group $\text{Bir}_k(X)$ is also called the Cremona group and is denoted by $\text{Cr}_n(k)$. It is usually clear from the context which interpretation of $\text{Cr}_n(k)$ is meant, algebraic or geometric. The notion of morphism of an algebraic variety in the Cremona group (or the notion of algebraic family of elements of the Cremona group), which can be defined in a natural way, enables us to equip the Cremona group with the Zariski topology [1], §1.6 (see also [2]). Along with this property, there are several others that enable us to speak of far-reaching analogies between the Cremona groups and affine algebraic groups; see [3]–[6].

The Cremona groups are classical objects of research, whose intensity has increased significantly in recent years and led to substantial advancements in the understanding of the structure of these groups. Among the most impressive advances, we note the tour de force [7] by I. V. Dolgachev and V. A. Iskovskikh concerning the classification of finite subgroups of $\text{Cr}_2(\mathbb{C})$.

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1.2. In this paper three aspects of the structure of the Cremona groups are explored.

The topic of §2 (the longest section) is a comparison of various finite subgroups of the Cremona group $\text{Cr}_n(k)$, where $k$ stands for an algebraically closed field of characteristic zero. In the investigations of these subgroups, including those in [7], all the subgroups have so far been considered on an equal footing. However, some subgroups should be treated as “non-basic ones”, since they can be obtained from other subgroups by the standard construction of “base change” [8], §3.4. This leads to the problem formulated in [8], §3.4 and [9], Question 1 of finding the subgroups in the classification lists that are obtained by such a non-trivial change or, in other terminology, are non-trivially “compressible” (for the definitions, see §2.1 below).

Developing this topic in §2, we prove a series of statements concerning these subgroups. Some of them are of a general nature, and some concern the cases $n = 1$ and $2$. For example, we obtain the following result (Theorem 2.1), which immediately implies the non-trivial self-compressibility of any finite subgroup $G$ of $\text{Cr}_1$: for the corresponding binary group $\tilde{G}$ of linear transformations of the affine plane we find an infinite increasing sequence of integers $d > 0$ such that $\tilde{G}$ admits a homogeneous polynomial self-compression of degree $d$ which descends to a non-trivial self-compression of $G$. In principle, the proof enables us to define these self-compressions by explicit formulae. For $n = 2$ we prove, for example, that if $G$ is a non-Abelian finite subgroup of $\text{GL}_2(k) \subset \text{Cr}_2(k)$ not isomorphic to a dihedral group, then every finite subgroup of $\text{Cr}_2(k)$ isomorphic to $G$ as an abstract group can be obtained from $G$ by a non-trivial base-change (Theorem 2.19). For other assertions concerning this topic and proved in §2, see Theorems 2.8–2.19 and their corollaries below.

1.3. The topics in §3 are the embeddability of other groups in the Cremona groups and, conversely, the embeddability of the Cremona group in other groups. This topic originates from a question of J.-P. Serre [10], §6, 6.0 on the existence of finite groups not embeddable in $\text{Cr}_3(\mathbb{C})$. Currently (June 2019), significant information has accumulated for this question (including an affirmative answer). The most substantial contribution related to obtaining this information is connected with the Jordan property (see Definition 3.1 below) of the Cremona groups $\text{Cr}_n(k)$, whose complete proof for every $n$ has recently been completed. Although the assertions on embeddings of groups proved in §3 are also related to the Jordan property, which has long been a focus of attention, the author has not met these assertions in the published literature.

The fact that every finite $p$-subgroup of $\text{Cr}_n(k)$ is Abelian for sufficiently large $p$ and $\text{char } k = 0$ follows immediately from the Jordan property of the Cremona groups (this was noted in [10], §6, 6.1). Therefore, every non-Abelian finite $p$-group (such groups exist for every $p$) is not embeddable in $\text{Cr}_n(k)$ for sufficiently large $p$. We prove (Corollary 3.7) that for every Cremona group $\text{Cr}_n(k)$ with $\text{char } k = 0$ there is an integer $b_{n,k} > 0$ such that every product of groups $G_1 \times \cdots \times G_s$, each containing a non-Abelian finite subgroup, is not embeddable in the group $\text{Cr}_n(k)$.

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1In [11], Theorem 1.8, a conditional proof (modulo the so-called BAB conjecture) of the Jordan property of the group $\text{Bir}_k(X)$ was given for every rationally connected algebraic $k$-variety $X$ in the case when $\text{char } k = 0$ (and therefore also a conditional proof of the Jordan property of every Cremona group $\text{Cr}_n(k)$). The BAB conjecture was later proved in [12], Theorem 3.7. This completed the proof of the Jordan property of the groups $\text{Bir}_k(X)$. 

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when \( s > b_{n,k} \). In particular, for every prime \( p \) (rather than only for sufficiently large \( p \)) there is a non-Abelian finite \( p \)-group not embeddable in \( \text{Cr}_n(k) \).

A consideration of \( p \)-subgroups gives invariants which allow us to prove in some cases that some groups cannot be embedded in others. In this way, some applications are obtained.

For example (a special case of Corollary 3.14), we prove that, if \( k \) is an algebraically closed field of characteristic zero and we assign to every \( d > 0 \) an abstract group \( H_d \) in the list

\[
\begin{align*}
(a) & \quad \text{Cr}_d(k); \\
(b) & \quad \text{Aut}(\mathbb{A}^d_k); \\
(c) & \quad \text{a connected affine algebraic group over } k \text{ with maximal tori of dimension } d; \\
(d) & \quad \text{a connected real Lie group with maximal tori of dimension } d,
\end{align*}
\]

then the group \( H_n \) cannot be embedded in \( H_m \) when \( n > m \). In particular, the groups \( H_n \) and \( H_m \) are not isomorphic when \( n \neq m \). For example: \( \text{Cr}_n(k) \) can be embedded in \( \text{Cr}_m(k) \) if and only if \( n \leq m \); in particular, \( \text{Cr}_n \) and \( \text{Cr}_m \) are isomorphic if and only if \( n = m \) (this was proved in [13], Theorem B, [11], Remark 1.11).

Another example (Theorem 3.20): we prove that, if \( M \) is a compact connected \( n \)-dimensional topological manifold and \( B_M \) is the sum of the Betti numbers of \( M \) with respect to the homology with coefficients in \( \mathbb{Z} \), then the Cremona group \( \text{Cr}_d(k) \) cannot be embedded in the group of homeomorphisms of \( M \) when

\[
d > \sqrt[2]{n^2 + 4n(n + 1)B_M + n} + \log_2 B_M.
\]

Concerning other assertions about non-embeddable groups proved in §3, see below Lemma 3.2, Theorems 3.12, 3.13 and 3.21 and their corollaries.

1.4. In §4, we return to Serre’s question concerning the connectedness of the Cremona groups \( \text{Cr}_n(k) \) in the Zariski topology [1], §1.6. An affirmative answer to the question was obtained in [14], where the linear connectedness (and therefore also the connectedness) of the group \( \text{Cr}_n(k) \) is proved in the case of an infinite field \( k \) (for any algebraically closed field \( k \) this was proved earlier in [15]). We give a short new proof in the case of any infinite field \( k \). It is based on an argument ideologically close to one of Alexander, which he applied in [16] to the proof of the connectedness of the group of homeomorphisms of the ball, and was adapted in [17], Lemma 4, [5], Theorem 6, and [6] to proofs of the connectedness of the groups \( \text{Aut}(\mathbb{A}^n) \) and their affine-triangular subgroups, respectively.

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1.5. Notation and conventions. \( \overline{k} \) is a fixed algebraically closed field containing \( k \).

\[
\begin{align*}
\text{Cr}_n := \text{Cr}_n(\overline{k}), \quad \text{Bir}(X) := \text{Bir}_k(X), \quad \text{Aut}(X) := \text{Aut}_k(X). \\
o = (0, \ldots, 0) \in \mathbb{A}^n. \\
\langle S \rangle \text{ is the linear span of a subset } S \text{ of a linear space over } k. \\
\text{Grass}(n, V) \text{ is the Grassmannian of all } n \text{-dimensional linear subspaces of a finite-dimensional linear space } V \text{ over } k. \\
\mathbb{P}(V) := \text{Grass}(1, V). \text{ We set } \mathbb{P}(\{0\}) = \emptyset \text{ and } \dim(\emptyset) = -1.
\end{align*}
\]
$L^\oplus m$ is the direct sum of $m$ copies of a linear space $V$ over $k$ (when $m = 0$ it is assumed to be zero).

$G^s$ is the direct product of $s$ copies of a group $G$.

The term “variety” means “algebraic variety over $k$”. By the irreducibility of a variety we mean its geometric irreducibility, and by points we mean $\overline{k}$-points. The set of $k$-points of a variety $X$ is denoted by $X(k)$.

$\text{Dom}(\varphi)$ is the domain of a rational map $\varphi$.

$T_{a,X}$ is the tangent space of a variety $X$ at a point $a$.

$d_a \varphi : T_{a,X} \to T_{\varphi(a),X}$ is the differential of a rational map $\varphi : X \to X$ at a point $a \in \text{Dom}(\varphi)$.

$k[x_1, \ldots, x_n]_d$ is the space of all forms of degree $d$ in the variables $x_1, \ldots, x_n$ with coefficients in $k$.

$L_d := L \cap k[x_1, \ldots, x_n]_d$ for every $k$-linear subspace $L$ of the algebra $k[x_1, \ldots, x_n]$.

$FH = \{ fh \mid f \in F, h \in H \}$ for every non-empty sets $F, H \subseteq k[x_1, \ldots, x_n]$.

The variables $x_1, \ldots, x_n$ in the definition of the Cremona group are assumed to be the standard coordinate functions on $\mathbb{A}^n$:

$$x_i(a) := a_i, \quad a := (a_1, \ldots, a_n) \in \mathbb{A}^n.$$  

For a rational map $\sigma : \mathbb{A}^n \to \mathbb{A}^n$ we use the notation

$$\sigma = (\sigma_1, \ldots, \sigma_n) : \mathbb{A}^n \to \mathbb{A}^n, \quad \text{where } \sigma_i := \sigma^*(x_i). \quad (1.2)$$

We call $\sigma$ a polynomial homogeneous map of degree $d$ if $\sigma_1, \ldots, \sigma_n \in k[x_1, \ldots, x_n]_d$.

In this notation, if for a rational map $\tau : \mathbb{A}^n \to \mathbb{A}^n$ the composition $\nu := \sigma \circ \tau$ is defined, then it is described by the formula

$$\nu_i = \tau^*(\sigma_i) \quad \text{for all } i, \quad (1.3)$$

that is, $\nu_i$ is obtained from the rational function $\sigma_i$ in $x_1, \ldots, x_n$ by replacing $x_j$ by $\tau_j$ for all $j$.

A map (1.2) is said to be affine (linear, respectively) if all the non-zero functions $\sigma_i$ are polynomials in $x_1, \ldots, x_n$ of degree $\leq 1$ (forms of degree 1 in $x_1, \ldots, x_n$, respectively). The set of all invertible affine (linear, respectively) maps $\mathbb{A}^n \to \mathbb{A}^n$ forms a subgroup $\text{Aff}_n$ ($\text{GL}_n$, respectively) of $\text{Cr}_n$.

§ 2. Compression of finite subgroups of the Cremona groups

In this section, $k = \overline{k}$ and char $k = 0$.

2.1. Terminology. First, we fix the terminology. Unless a special reservation is made, a rational action of a finite group $G$ on an irreducible variety $X$ means a faithful action (that is, having trivial kernel) by birational self-maps of this variety. Specifying such an action is equivalent to specifying a group embedding $\varrho : G \hookrightarrow \text{Bir}(X)$; therefore, below the homomorphism $\varrho$ itself is called a rational action. The number $\dim(X)$ is called the dimension of the action $\varrho$. We call the image $\varrho(G)$ the subgroup of $\text{Bir}(X)$ defined by the action $\varrho$. If $\varrho(G) \subseteq \text{Aut}(X)$, then the action $\varrho$ is said to be regular.

A regularization of an action $\varrho$ is an arbitrary regular action $\rho$ of the group $G$ on an irreducible smooth complete variety $Y$ such that there is a $G$-equivariant
birational isomorphism $X \dasharrow Y$; the combination of results in [18], Theorem 1 and [19] shows that a regularization always exists. When there is a regularization $\rho$ for which $Y^G \neq \emptyset$, we say that $\rho$ has a fixed point.

Consider two rational actions $\varrho_i : G \hookrightarrow \text{Bir}(X_i)$, $i = 1, 2$. Let $\pi_i : X_i \dasharrow X_i/G$, $i = 1, 2$, be the corresponding rational quotients; see [20], §2.4. Suppose that there is a $G$-equivariant dominant rational map $\varphi : X_1 \dasharrow X_2$. Let $\varphi_G : X_1/G \dasharrow X_2/G$ be the dominant rational map induced by $\varphi$. Then the following properties hold (see, for example, [21], §2.6).

First, the commutative diagram

$$
\begin{array}{ccc}
X_1 & \dasharrow & X_2 \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
X_1/G & \dasharrow & X_2/G
\end{array}
$$

(2.1)

is Cartesian, that is, $\pi_1$ is obtained from $\pi_2$ by the base change $\varphi_G$. In particular, $X_1$ is birationally $G$-equivariantly isomorphic to the variety

$$
X_2 \times_{X_2/G} (X_1/G) := \{(x, y) \in \text{Dom}(\pi_2) \times \text{Dom}(\varphi_G) \mid \pi_2(x) = \varphi_G(y)\} \tag{2.2}
$$

(the bar in (2.2) stands for the closure in $X_2 \times X_1/G$ on which $G$ acts via $X_2$).

Second, for any irreducible variety $Z$ and any dominant rational map $\beta : Z \dasharrow X_2/G$ such that the variety $X_2 \times_{X_2/G} Z$ is irreducible, the latter inherits via $X_2$ a rational action of the group $G$ for which the commutative diagram (2.1) holds with $X_1 = X_2 \times_{X_2/G} Z$, $\varphi_G = \beta$ and $\varphi = \text{pr}_1$.

We say [22] that $\varphi$ is a compression of the action $\varrho_1$ into the action $\varrho_2$ (or that $\varrho_2$ is obtained by the compression $\varphi$ from $\varrho_1$) and also [8], §3.4 that $\varrho_1$ is obtained by the base change $\varphi_G$ from $\varrho_2$. A compression that is not (is, respectively) a birational isomorphism is said to be non-trivial (trivial, respectively). In this case, we say that $\varrho_1$ is obtained by a non-trivial (trivial, respectively) base change from $\varrho_2$. When for $\varrho_1$ there is a $\varrho_2$ which is obtained from $\varrho_1$ by a non-trivial compression, we say that $\varrho_1$ is non-trivially compressible, and otherwise that it is incompressible. Similar terminology is applied to groups: if $G_i \subseteq \text{Bir}(X_i)$, $i = 1, 2$, are finite subgroups isomorphic to $G$, then we say that $G_1$ is compressible into $G_2$ if there are rational actions $\varrho_i : G \hookrightarrow \text{Bir}(X_i)$, $i = 1, 2$, such that $\varrho_i(G) = G_i$, $i = 1, 2$, and $\varrho_2$ is obtained by a compression $\varphi$ from $\varrho_1$. If $\varrho_1, \varrho_2$ and $\varphi$ can be chosen in such a way that $\varphi$ is non-trivial, then $G_1$ is said to be non-trivially compressible into $G_2$. If $G_1$ admits no non-trivial compressions into any subgroups of $\text{Bir}(X_2)$, then $G_1$ is said to be incompressible.

When $X_1 = X_2$ and $\varrho_1 = \varrho_2$, we speak of the self-compressions of an action and a group. In particular, when in this case there is a non-trivial compression, we say that $\varrho_1(G)$ is a non-trivially self-compressible subgroup of $\text{Bir}(X)$.

2.2. Self-compressibility of finite subgroups of $\text{Cr}_1$: a reformulation. We first consider the problem of non-trivial self-compressibility of finite subgroups in the Cremona group $\text{Cr}_1$ of rank 1. It can be reformulated as follows.
We assume that \( \mathbb{A}^1 = \{(a_0 : a_1) \in \mathbb{P}^1 \mid a_0 \neq 0\} \) and denote the standard coordinate function \( x_1 \in k[\mathbb{A}^1] \) by \( z \). The elements of every finite subgroup \( G \) of the Cremona group \( \text{Cr}_1 \) are fractional-linear functions in the field \( k(z) \) (regarded as rational maps \( \mathbb{A}^1 \rightarrow \mathbb{A}^1 \)). The restriction to \( \mathbb{A}^1 \) defines a bijection between the set of self-compressions \( \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) of the group \( G \) and the set of rational functions \( f = f(z) \in k(z) \) that are solutions of the following system of functional equations:

\[
f\left(\frac{az + b}{cz + d}\right) = \frac{af + b}{cf + d} \quad \text{for all} \quad \frac{az + b}{cz + d} \in G.
\]

In this setting, the non-triviality of the self-compression defined by the rational function \( f \) is equivalent to the condition \( \deg(f) > 2 \) and, in (2.3), it suffices to consider only the generators of the group \( G \) instead of all functions in it.

Thus, the problem of the non-trivial self-compressibility of the group \( G \) is equivalent to the existence question for a rational function \( f \) of degree \( > 2 \) among the solutions of the system (2.3).

### 2.3. Self-compressibility of binary polyhedral groups: formulation of the result.

An exhaustive answer to the above problem can be successfully obtained for every finite subgroup of the Cremona group \( \text{Cr}_1 \): all of them are non-trivially self-compressible. This answer is an immediate consequence of a more delicate result which we obtain here. Namely, we prove that there are infinitely many homogeneous polynomial self-compressions \( \mathbb{A}^2 \rightarrow \mathbb{A}^2 \) of every binary polyhedral group that descend to non-trivial self-compressions \( \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) of the corresponding polyhedral group. The proof is effective and gives a way of defining these self-compressions by explicit formulae (see Remark (c) in §2.8).

We now give a precise formulation of this result. Let \( G \) be a non-trivial finite subgroup of \( \text{PSL}_2 = \text{Aut}(\mathbb{P}^1) = \text{Cr}_1 \). Consider the canonical homomorphism

\[
\nu: \text{SL}_2 \rightarrow \text{PSL}_2,
\]

whose kernel is the centre \( Z := \langle -\text{id} \rangle \). The group

\[
\tilde{G} := \nu^{-1}(G) \subset \text{SL}_2
\]

is either a binary group of rotations of one of the polyhedra (dihedron, tetrahedron, octahedron or icosahedron) or a cyclic group of even order \( > 4 \).

The subset \( \mathbb{A}^2_0 := \mathbb{A}^2 \setminus o \), which is open in \( \mathbb{A}^2 \), is invariant under the actions of the groups \( \tilde{G} \) and \( T := \{(tx_1, tx_2) \mid t \in k^\times\} \) on \( \mathbb{A}^2 \). Let

\[
\pi: \mathbb{A}^2_0 := \mathbb{A}^2 \setminus o \rightarrow \mathbb{P}^1
\]

be the natural projection. The pair \( (\pi, \mathbb{P}^1) \) is a geometric quotient for the action of the torus \( T \) on \( \mathbb{A}^2_0 \). The morphism \( \pi \) is \( \tilde{G} \)-equivariant if we assume that the action of \( \tilde{G} \) on \( \mathbb{P}^1 \) is the restriction to \( \tilde{G} \) of the homomorphism \( \nu \) (this action is not faithful; its kernel is \( Z \)).

When a self-compression

\[
\tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2): \mathbb{A}^2 \rightarrow \mathbb{A}^2
\]
of $\tilde{G}$ is polynomial and homogeneous of degree $d$, the morphism $\pi \circ \tilde{\varphi}$ is constant on the $T$-orbits in $A_2^g$ and thus it factors through $\pi$, that is, there is a morphism

$$\varphi : P^1 \rightarrow P^1$$

(2.6)
such that $\varphi \circ \pi = \tilde{\varphi} \circ \pi$. It is dominant (and therefore surjective) since the morphism $\tilde{\varphi}$ is. It follows from the $\tilde{G}$-equivariance of the morphisms $\pi$ and $\tilde{\varphi}$ that the morphism $\varphi$ is $G$-equivariant, and therefore also $G$-equivariant. Hence, $\varphi$ is a self-compression of the natural action of the group $G$ on $P^1$. We say that the self-compression $\varphi$ is a descent of the self-compression $\tilde{\varphi}$.

**Theorem 2.1.** Let $G$ be a non-trivial finite subgroup of the Cremona group $Cr_1 = PSL_2 = \text{Aut}(P^1)$. We assign to it a formal power series

$$S_G(t) = \sum_{n \geq 0} s_n t^n \in \mathbb{Z}[t]$$

(2.7)
of the following form.

(a) If $G$ is the group of rotations of a tetrahedron, octahedron or icosahedron, then

$$S_G(t) = t^{2a-1}(1 + t^{4a-6}) \sum_{n \geq 0} t^{2na} \sum_{n \geq 0} t^{(4a-4)n} + t^{4a-5} \sum_{n \geq 0} t^{(4a-4)n},$$

(2.8)
where $a = 3, 4$ or $6$, respectively.

(b) If $G$ is a dihedral group of order $2\ell \geq 4$ or a cyclic group of order $\ell \geq 2$, then

$$S_G(t) = \sum_{n \geq 0} t^{2\ell(n+1)-1}.$$  

(2.9)

Suppose that the coefficient $s_d$ in the series (2.7) is non-zero. Then there is a polynomial homogeneous self-compression (2.5) of the binary group $\tilde{G}$ (see (2.4)) whose degree is equal to $d$, and the descent (2.6) is a non-trivial self-compression of the group $G$.

The proof of Theorem 2.1 is given in §2.7 after proving several necessary auxiliary assertions in §2.6.

2.4. Application: self-compressibility of finite subgroups of $Cr_1$. Theorem 2.1 immediately implies statement (i) of the following theorem.

**Theorem 2.2.** (i) Every finite subgroup of $Cr_1$ is non-trivially self-compressible.

(ii) Every compression of a finite subgroup of $Cr_1$ is a compression $P^1 \rightarrow P^1$ into a conjugate subgroup.

**Proof.** (ii) Since every variety onto which the variety $P^1$ is mapped dominantly is rational, statement (ii) follows from the definition of compression and the well-known fact that two finite subgroups of $Cr_1$ are isomorphic if and only if they are conjugate. □
Remark 2.3. Another proof of statement (i) of Theorem 2.2 is given in [23], Corollary 1.3. This proof consists of presenting explicit formulae, and the reader is supposed to verify by direct calculation that these formulae define $G$-equivariant maps. In [23], there are no comments concerning the origin of these formulae. For example (see [23], Lemma 9.7), when $\omega_5 \in k$ is a primitive root of unity of degree 5 and $G$ is the group of rotations of the icosahedron that is contained in $\text{Cr}_1$ and generated by the fractional-linear transformations

$$\omega_5 z \quad \text{and} \quad \frac{(\omega_5 + \omega_5^{-1})z + 1}{z - (\omega_5 + \omega_5^{-1})},$$

the corresponding formula has the form

$$P^1 \to P^1, \quad (x : y) \mapsto (x^{11} + 66x^6y^5 - 11xy^{10} : -11x^{10}y - 66x^5y^6 + y^{11}).$$

We explain below (see Remark (c) in §2.8) how, in principle, one can find explicit formulae defining an arbitrary self-compression specified in Theorem 2.1.

2.5. Notation. To prove Theorem 2.1, we need some notation.

We denote by $\tilde{G}^\vee$ the set of characters of all simple $k\tilde{G}$-modules.

The action of the group $G$ on the affine plane $A^2$ defines the structure of a $k\tilde{G}$-module on the algebra

$$A := k[A^2] = k[x_1, x_2].$$

This module is graded: every subspace $A_n$ is a $k\tilde{G}$-submodule of $A$.

We denote by $\chi$ the character of the submodule $A_1$. If the group $\tilde{G}$ is not cyclic, then this submodule is simple.

For every simple $k\tilde{G}$-module $M$ with character $\gamma \in \tilde{G}^\vee$ we denote by $A(\gamma)$ the isotypical component of type $M$ in the $k\tilde{G}$-module $A$; this is a graded submodule of $A$. In particular, $A(1)$ is the subalgebra of $\tilde{G}$-invariants in $A$.

We also need the following set of characters:

$$[\chi] := \{ \gamma \in \tilde{G}^\vee \mid \dim(\gamma) = 1, \gamma \chi = \chi \}. \quad (2.10)$$

This set is not empty, since $1 \in [\chi]$.

For every finite-dimensional $k\tilde{G}$-module $L$ we put

$$\text{mult}_\chi(L) := \max\{ d \mid \text{there is an embedding } A_1^\oplus d \hookrightarrow L \text{ of } k\tilde{G}\text{-modules} \}. \quad (2.11)$$

2.6. Auxiliary assertions. We now prove several auxiliary assertions to be used in the proof of Theorem 2.1.

Lemma 2.4. Let $H$ be a subgroup of $\text{GL}_n$ and $L$ finite-dimensional $k$-linear subspace of $k[x_1, \ldots, x_n]$.

(a) The following conditions are equivalent.

(a1) $L$ is a submodule of the $kH$-module $k[x_1, \ldots, x_n]$, and $L$ is isomorphic to the $kH$-module $k[x_1, \ldots, x_n]_1$.

(a2) There is a basis $\sigma_1, \ldots, \sigma_n$ of the linear space $L$ such that the morphism $\sigma := (\sigma_1, \ldots, \sigma_n) : A^n \to A^n$ is $H$-equivariant.
(b) Let the equivalent conditions (a1) and (a2) hold.
   (b1) The morphism \( \sigma \) in (a2) is dominant if and only if \( \sigma_1, \ldots, \sigma_n \) are algebraically independent over \( k \).
   (b2) If \( n = 2 \) and \( \sigma_1, \sigma_2 \in k[x_1, x_2]_d \) for some \( d \), then \( \sigma_1, \sigma_2 \) are algebraically independent over \( k \).

Proof. (a1) \( \Rightarrow \) (a2). Let \( k[x_1, \ldots, x_n]_1 \rightarrow L \) be an isomorphism of \( kH \)-modules and let \( \sigma_i \) be the image of \( x_i \) under this isomorphism. Then the \( H \)-equivariance of \( \sigma \) follows immediately from the definitions and the formulae (1.1) and (1.2).

(a2) \( \Rightarrow \) (a1). It follows from (1.1) and (1.2) that the restriction of \( \sigma^* \) to \( k[x_1, \ldots, x_n]_1 \) is a linear space isomorphism \( k[x_1, \ldots, x_n]_1 \rightarrow L \). This, together with the \( H \)-invariance of \( k[x_1, \ldots, x_n]_1 \), implies the \( H \)-invariance of \( L \), and thus the above restriction is an isomorphism of \( kH \)-modules.

(b1). The dominance of \( \sigma \) is equivalent to the triviality of the kernel of the homomorphism \( \sigma^* \) of the algebra \( k[x_1, \ldots, x_n] \), which, by (1.2), is equivalent to the algebraic independence of \( \sigma_1, \ldots, \sigma_n \) over \( k \).

(b2). Suppose that \( \sigma_1 \) and \( \sigma_2 \) are algebraically dependent over \( k \), that is, there is a non-zero polynomial \( F = F(t_1, t_2) \in k[t_1, t_2] \), where \( t_1 \) and \( t_2 \) are variables, such that

\[
F(\sigma_1, \sigma_2) = 0. \tag{2.12}
\]

Since \( \sigma_1 \) and \( \sigma_2 \) are forms in \( x_1, x_2 \) of equal degrees, we may assume that \( F \) is a form in \( t_1 \) and \( t_2 \), say of degree \( s \):

\[
F(t_1, t_2) = \sum_{i=0}^{s} \alpha_i t_1^{s-i} t_2^i, \quad \alpha_0, \ldots, \alpha_s \in k. \tag{2.13}
\]

By (a2), the polynomial \( \sigma_2 \) is non-zero, and thus we can consider the rational function \( \sigma_1/\sigma_2 \in k(x_1, x_2) \). By (2.12) and (2.13), we obtain for it the relation

\[
0 = \sum_{i=0}^{s} \alpha_i \left( \frac{\sigma_1}{\sigma_2} \right)^{s-i}. \tag{2.14}
\]

It follows from the linear independence of the polynomials \( \sigma_1 \) and \( \sigma_2 \) over \( k \) that the rational function \( \sigma_1/\sigma_2 \) is not a constant in \( k \), and therefore takes infinitely many distinct values on \( \mathbb{A}^2 \). By (2.14), each of these values is the root of the non-zero polynomial \( \sum_{i=0}^{s} \alpha_i t^{s-i} \in k[t] \), where \( t = t_1/t_2 \). The contradiction thus obtained proves the algebraic independence of \( \sigma_1 \) and \( \sigma_2 \) over \( k \). \( \square \)

**Lemma 2.5.** Let (2.5) be a polynomial homogeneous self-compression of degree \( d \) of the group \( \tilde{G} \). Let a form \( a \in A \) be the greatest common divisor of the forms \( \tilde{\varphi}_1 \) and \( \tilde{\varphi}_2 \) defining (2.5). The following properties are equivalent:

(a) the descent (2.6) of the self-compression (2.5) is trivial;
(b) \( \deg(a) = d - 1 \);
(c) there is a character \( \gamma \in [\chi] \) and an element \( s \in A(\gamma)_{d-1} \) for which

\[
\tilde{\varphi}^*(A_1) = sA_1. \tag{2.15}
\]
Proof. (a) ⇔ (b). If $x_1$ and $x_2$ are viewed as homogeneous coordinates on $\mathbb{P}^1$, then the self-compression (2.6) is defined by the formula

$$\varphi = \left( \frac{\bar{\varphi}_1}{a} ; \frac{\bar{\varphi}_2}{a} \right)$$

(2.16)

(see [24], Chap. III, §1, 4). Since every $k$-automorphism of the field of rational functions in one variable over $k$ is a fractional-linear transformation [25], §73, it follows from (2.16) and the inclusion $\bar{\varphi}_1, \bar{\varphi}_2 \in A_d$ that the self-compression $\varphi$ is trivial if and only if the forms $\bar{\varphi}_1/a$ and $\bar{\varphi}_2/a$ are linear, that is, (b) holds.

(b) ⇔ (c). Let (b) hold. Then it follows from the equality

$$\bar{\varphi}^*(A_1) = \langle \bar{\varphi}_1, \bar{\varphi}_2 \rangle$$

(2.17)

and the definition of the form $a$ that the equality (2.15) holds with $s = a$. By the $\tilde{G}$-invariance of the subspaces $\bar{\varphi}^*(A_1)$ and $A_1$, for every $g \in \tilde{G}$ we obtain the following equalities from (2.15):

$$aA_1 = g \cdot (aA_1) = (g \cdot a)(g \cdot A_1) = (g \cdot a)A_1.$$  

(2.18)

Take a linear form $l \in A_1$ whose zero in $\mathbb{P}^1$ differs from any zero of the form $a$. Since, by (2.18), the form $(g \cdot a)l$ is divisible by $a$ and $\deg(g \cdot a) = \deg(a)$, this condition means that the divisors of the forms $a$ and $g \cdot a$ on $\mathbb{P}^1$ coincide, and therefore $\langle a \rangle = \langle g \cdot a \rangle$. Hence, $a$ is a semi-invariant of $\tilde{G}$. Let $\gamma \in \tilde{G}^\vee$ be the character of the one-dimensional $k\tilde{G}$-module $\langle a \rangle$. Then $a \in A(\gamma)_{d-1}$, and $\gamma \chi$ is the character of the $k\tilde{G}$-module $aA_1$. However, since the $k\tilde{G}$-modules $A_1$ and $\bar{\varphi}^*(A_1)$ are isomorphic, it follows from (2.15) that the character of the $k\tilde{G}$-module $aA_1$ is $\chi$. Hence, $\gamma \in [\chi]$. This proves the implication (b) ⇒ (c).

Conversely, if (c) holds, then it follows from (2.15) and (2.17) that $s$ is the greatest common divisor of the forms $\bar{\varphi}_1$ and $\bar{\varphi}_2$, and therefore $\langle s \rangle = \langle a \rangle$, and thus (b) holds. This proves that (c) ⇒ (b). □

Lemma 2.6. Let $H$ be a group, let $L$ be a non-zero $kH$-module of dimension $s < \infty$, and let $m$ be a positive integer. The Grassmannian $\text{Grass}(s, L^\oplus m)$ contains a closed irreducible $(m - 1)$-dimensional subset such that all the $s$-dimensional linear subspaces of the $kH$-module $L^\oplus m$ corresponding to the points of this subset are submodules isomorphic to $L$.

Proof. Let us assign to every non-zero vector $(\lambda_1, \ldots, \lambda_m)$ in $k^m$ the embedding

$$\iota_{(\lambda_1, \ldots, \lambda_m)} : L \hookrightarrow L^\oplus m, \quad v \mapsto (\lambda_1 v, \ldots, \lambda_m v)$$

of $kH$-modules. The images of the embeddings $\iota_{(\lambda_1, \ldots, \lambda_m)}$ and $\iota_{(\mu_1, \ldots, \mu_m)}$ coincide if and only if the vectors $(\lambda_1, \ldots, \lambda_m)$ and $(\mu_1, \ldots, \mu_m)$ are proportional, that is, the corresponding points $(\lambda_1 : \cdots : \lambda_m)$ and $(\mu_1 : \cdots : \mu_m)$ of the projective space $\mathbb{P}(k^m)$ coincide. Hence, the map $\mathbb{P}(k^m) \to \text{Gr}(s, L^\oplus m)$ assigning to every point $(\lambda_1 : \cdots : \lambda_m) \in \mathbb{P}(k^m)$ the image of the embedding $\iota_{(\lambda_1, \ldots, \lambda_m)}$ is injective. It can readily be seen that this map is a morphism. Therefore, image is an irreducible closed subset of dimension $\dim(\mathbb{P}(k^m)) = m - 1$. It is this image that should be taken for the subset indicated in the statement of Lemma 2.6. □
Lemma 2.7. If the following inequality holds for a positive integer \( d \):
\[
\text{mult}_\chi(A_d) > \max\{\dim_k(A(\gamma)_{d-1}) \mid \gamma \in [\chi]\}, \tag{2.19}
\]
then the group \( \tilde{G} \) admits a polynomial homogeneous self-compression (2.5) of degree \( d \) whose descent (2.6) is non-trivial.

Proof. Let the inequality (2.19) hold. For brevity, we put
\[
m := \text{mult}_\chi(A_d). \tag{2.20}
\]
It follows from (2.19) that \( m > 0 \).

By (2.11), there is a submodule \( M \) of the \( k\tilde{G} \)-module \( A_d \) isomorphic to \( A_1^{\oplus m} \). Because of \( \dim(A_1) = 2 \) and Lemma 2.6, there is an irreducible closed subset \( X \) of \( \text{Grass}(2, M) \) such that
\[
\dim(X) = m - 1, \tag{2.21}
\]
and the 2-dimensional linear subspaces of \( M \) corresponding to the points of \( X \) are submodules isomorphic to \( A_1 \). Since \( M \) is a linear subspace of \( A_d \), it follows that the variety \( \text{Grass}(2, M) \), and therefore also \( X \), are closed subsets of \( \text{Grass}(2, A_d) \).

It follows from the definition of the set \( [\chi] \) (see (2.10)) that for every character \( \gamma \in [\chi] \) and every non-zero element \( s \in A(\gamma)_{d-1} \) the linear subspace \( sA_1 \) is a submodule of the \( k\tilde{G} \)-module \( A_d \) isomorphic to \( A_1 \). This submodule is preserved when \( s \) is multiplied by non-zero elements of \( k \), and therefore the assignment of the submodule \( sA_1 \) to an element \( s \) defines a map \( P(A(\gamma)_{d-1}) \to \text{Grass}(2, A_d) \). It can readily be seen that this map is a morphism. Therefore, its image \( Y(\gamma) \) is an irreducible closed subset of \( \text{Grass}(2, A_d) \) and
\[
\dim(Y(\gamma)) \leq \dim(P(A(\gamma)_{d-1})) = \dim_k(A(\gamma)_{d-1}) - 1. \tag{2.22}
\]
Since the set \( [\chi] \) is finite, it follows from (2.19)–(2.22) that
\[
X \setminus \bigcup_{\gamma \in [\chi]} Y(\gamma) \tag{2.23}
\]
is a non-empty subset of the Grassmannian \( \text{Grass}(2, A_d) \).

Consider a point of the set (2.23) and the corresponding two-dimensional linear subspace \( L \) of \( A_d \). Then it follows from the above properties of the sets \( X \) and \( Y(\gamma) \) that

(i) \( L \) is a submodule of the \( k\tilde{G} \)-module \( A_d \) isomorphic to \( A_1 \);
(ii) there are no \( \gamma \in [\chi] \) and \( s \in A(\gamma)_{d-1} \) such that \( L = sA_1 \).

By (i) and Lemma 2.4, there is a basis \( \tilde{\varphi}_1, \tilde{\varphi}_2 \) of \( L \) such that (2.5) is a polynomial homogeneous self-compression of the group \( G \) of degree \( d \) for which \( \tilde{\varphi}^*(A_1) = L \). It follows from (ii) and Lemma 2.5 that the descent (2.6) of this self-compression is non-trivial. \( \square \)
2.7. Proof of Theorem 2.1. The plan of the proof of Theorem 2.1 is as follows. For every non-cyclic finite subgroup \( G \) of \( \text{PSL}_2 = \text{Aut}(P^1) = \text{Cr}_1 \) we will explicitly describe, for \( \widetilde{G} \), the set \([\chi]\) and the Poincaré series
\[
P(\chi, t) := \sum_{n \geq 1} (\text{mult}_\chi (A_n)) t^n, \quad P(\gamma, t) := \sum_{n \geq 0} (\dim_k (A(\gamma)_n)) t^n, \quad \text{where} \ \gamma \in [\chi].
\]
(2.24)

By comparing the coefficients of these series, we will show that, if a coefficient \( s_d \) of the series (2.7) is non-zero, then the inequality (2.19) holds, which implies, by Lemma 2.7, the assertion of Theorem 2.1 for \( G \). The case of a cyclic finite subgroup \( G \) is reduced to the corresponding case of a dihedral group.

Proof of Theorem 2.1. We consider three possible types of the group \( \widetilde{G} \) separately.

(a) \( \widetilde{G} \) is a primitive subgroup of the group \( \text{SL}_2 \), that is, a binary tetrahedral, octahedral or icosahedral group.

By [26], §3.2(a) and the definition of the set \([\chi]\) (see (2.10)), in the case under consideration we have
\[
[\chi] = \{1\}.
\]
(2.25)

It follows from [26], §4.2 that
\[
P(\chi, t) = \frac{t + t^{2a-1} + t^{4a-5} + t^{6a-7}}{(1 - t^{2a})(1 - t^{4a-4})},
\]
\[
P(1, t) = \frac{1 + t^{6a-6}}{(1 - t^{2a})(1 - t^{4a-4})},
\]
(2.26)

where \( a = 3, 4 \) and 6, for the binary tetrahedral, octahedral, and icosahedral group respectively. We then derive from (2.24) and (2.26) the following:
\[
P(\chi, t) - tP(1, t) = \sum_{n \geq 1} (\text{mult}_\chi (A_n) - \dim_k (A(1)_{n-1})) t^n
\]
\[
= t^{2a-1} \left( 1 + t^{2a-4} + t^{4a-6} - t^{4a-4} \right)
\]
\[
= t^{2a-1} \frac{1 + t^{4a-6}}{(1 - t^{2a})(1 - t^{4a-4})} + t^{4a-5} \frac{1}{1 - t^{4a-4}}
\]
\[
= t^{2a-1} (1 + t^{4a-6}) \sum_{n \geq 0} t^{2na} \sum_{n \geq 0} t^{(4a-4)n} + t^{4a-5} \sum_{n \geq 0} t^{(4a-4)n}
\]
\[
= S_G(t).
\]
(2.27)

It follows from (2.27) and (2.7) that \( s_d = \text{mult}_\chi (A_d) - \dim_k (A(1)_{d-1}) \) for every \( d > 0 \). By (2.25), this gives
\[
s_d = \text{mult}_\chi (A_d) - \max \{ \dim_k (A(\gamma)_{d-1}) \mid \gamma \in [\chi] \} \quad \text{for every} \ d.
\]
(2.28)

As (2.8) shows, if \( s_d \neq 0 \), then \( s_d > 0 \). By (2.28) and by Lemma 2.7, this implies the assertion of Theorem 2.1 in the case (a).

(b) \( \widetilde{G} \) is an irreducible imprimitive subgroup of \( \text{SL}_2 \), that is, a binary dihedral group of order \( 4\ell \geq 8 \).
In this case, the McKay correspondence [26], §2 assigns to the group \( \tilde{G} \) the extended Dynkin diagram of type \( D^{(1)}_{\ell+2} \) with \( \ell + 3 \) vertices. By [26], §2.3(a), the vertex assigned to the character \( \chi \) is a branch point of this diagram. There are precisely four one-dimensional characters \( 1, \theta, \theta' \) and \( \theta'' \) in \( G^\vee \) (see [26], §4.3); the vertices of the diagram corresponding to them form the set of all end vertices. By [26], §2.2 and definition (2.10), an end vertex corresponds to a character in \([\chi]\) if and only if this vertex is joined by an edge to the vertex corresponding to the character \( \chi \). Therefore, apart from \( 1 \), \([\chi]\) also contains at least one other character (which we denote by \( \theta \)), and two possibilities occur.

If \( \ell \geq 3 \), then the vertex assigned to the character \( \chi \) is joined by edges to only two end vertices of the diagram, corresponding to which are the one-dimensional characters \( 1 \) and \( \theta \):

\[
\begin{array}{c}
1 \\
\theta
\end{array} \xrightarrow{\chi} \cdots \xrightarrow{\alpha} \begin{array}{c}
\theta' \\
\theta''
\end{array} , \quad \chi \neq \alpha. \tag{2.29}
\]

Thus, we obtain

\[ [\chi] = \{1, \theta\} \quad \text{when } \ell \geq 3. \tag{2.30} \]

If \( \ell = 2 \), then the vertex assigned to \( \chi \) is joined by edges to all four end vertices that are assigned to the one-dimensional characters \( 1, \theta, \theta' \) and \( \theta'' \):

\[
\begin{array}{c}
1 \\
\theta
\end{array} \xrightarrow{\chi} \begin{array}{c}
\theta' \\
\theta''
\end{array} .
\]

Thus, we obtain

\[ [\chi] = \{1, \theta, \theta', \theta''\} \quad \text{when } \ell = 2. \tag{2.31} \]

It follows from [26], (9) that \( \theta \) for every \( \ell \geq 2 \) is the character denoted in [26], p.103 by \( c_1 \). From this and [26], §4.4 we obtain

\[
\begin{align*}
P(\chi, t) & := \frac{t + t^3 + t^{2\ell-1} + t^{2\ell+1}}{(1-t^4)(1-t^{2\ell})}, \\
P(1, t) & := \frac{1 + t^{2\ell+2}}{(1-t^4)(1-t^{2\ell})}, \\
P(\theta, t) & := \frac{t^2 + t^{2\ell}}{(1-t^4)(1-t^{2\ell})}
\end{align*}
\]

for every \( \ell \geq 2 \). \tag{2.32}

Moreover, by [26], §4.4,

\[ P(\theta, t) = P(\theta', t) = P(\theta'', t) \quad \text{for } \ell = 2. \tag{2.33} \]
From (2.24) and (2.32) we obtain
\[ P(\chi, t) - tP(1, t) - tP(\theta, t) = \sum_{n \geq 1} (\text{mult}_\chi (A_n) - \dim_k (A(1)_{n-1}) - \dim_k (A(\theta)_{n-1})) t^n \]
\[ = t^{2\ell-1} \frac{1 - t^4}{(1 - t^4)(1 - t^{2\ell})} = \sum_{n \geq 0} t^{2\ell(n+1)-1} \]
\[ = S_G(t). \tag{2.34} \]

It follows from (2.34) and (2.7) that
\[ s_d = \text{mult}_\chi (A_d) - \dim_k (A(1)_{d-1}) - \dim_k (A(\theta)_{d-1}) \quad \text{for every } d > 0. \]

By (2.30), (2.31), and (2.33), this gives
\[ s_d \leq \text{mult}_\chi (A_d) - \max \{ \dim_k (A(\gamma)_{d-1}) | \gamma \in [\chi] \} \quad \text{for every } d. \tag{2.35} \]

As is shown by (2.9), if \( s_d \neq 0 \), then \( s_d > 0 \). By (2.35) and Lemma 2.7, this implies the assertion of Theorem 2.1 in the case (b).

(c) \( \tilde{G} \) is a cyclic subgroup of the group \( SL_2 \) of order \( 2\ell \geq 4 \).

Since \( \tilde{G} \) is a subgroup of a binary dihedral group of order \( 4\ell \), it follows that every polynomial homogeneous self-compression of the latter group is a self-compression of the group \( G \). Therefore, the assertion of Theorem 2.1 for \( \tilde{G} \) follows from the part of this theorem which has already been proved for binary dihedral groups. Theorem 2.1 is proved. \( \square \)

2.8. Remarks concerning Theorems 2.1 and 2.2. To conclude the discussion of the self-compressibility of finite subgroups of the Cremona group of rank 1, we make several remarks concerning Theorems 2.1 and 2.2.

(a) For every non-trivial finite subgroup \( G \) of the Cremona group \( \text{Cr}_1 \), Theorem 2.1 yields an infinite family of natural integers \( d \) for which there is a polynomial homogeneous self-compression (2.5) of the corresponding binary group \( \tilde{G} \) whose degree is equal to \( d \), and the descent (2.6) is a non-trivial self-compression of \( G \). We obtain the minimal value of these \( d \) from the formulae (2.8) and (2.9): it is equal to 5, 7, 11, respectively, for the rotation groups of the tetrahedron, octahedron, icosahedron, and is equal to \( 2\ell - 1 \) for the dihedral group of order \( 2\ell \geq 4 \) and for the cyclic group of order \( \ell \geq 2 \).

(b) Let \( K \) be a field of algebraic functions in one variable over \( k \). By Theorem 2.2, (i), if the genus of \( K \) is equal to 0, then every finite subgroup of \( \text{Aut}_k (K) \) is non-trivially self-compressible. This is not the case in general for fields \( K \) of genus \( \geq 2 \); see [22], Example 6 and [23].

(c) We briefly explain how we can explicitly find the forms \( \tilde{\varphi}_1 \) and \( \tilde{\varphi}_2 \) defining the homogeneous polynomial self-compressions (2.5) of the group \( \tilde{G} \); their degrees are mentioned in Theorem 2.1.

Let \( d \) be a degree of this kind. It follows from what was said in part (c) of the proof of Theorem 2.1 that we can (and shall) assume that the group \( \tilde{G} \) is not cyclic,
and thus the character $\chi$ is irreducible. Consider the linear space $\mathcal{L}(A_1, A_d)$ of all linear maps $A_1 \rightarrow A_d$. The group $\tilde{G}$ acts linearly on this space by the rule

$$(g\ell)(a) := g(\ell(g^{-1}(a))), \quad g \in \tilde{G}, \quad \ell \in \mathcal{L}(A_1, A_d), \quad a \in A_1,$$  

(2.36)

and the $\tilde{G}$-equivariant maps are precisely the fixed points of this action. They form a linear subspace $\mathcal{L}(A_1, A_d)^G$ of $\mathcal{L}(A_1, A_d)$; see [20], §3.12. The former is the image of the Reynolds operator $|\tilde{G}|^{-1} \sum_{g \in \tilde{G}} g$ for the action (2.36); see [20], §3.4, and therefore can be found effectively. If $\ell_1, \ldots, \ell_m$ is a basis in $\mathcal{L}(A_1, A_d)^G$, then $\langle \bigcup_{i=1}^m \ell_i(A_1) \rangle = A(\chi)_d$. Similarly, the isotypic components $A(\gamma)_{d-1}$ can be found effectively for all $\gamma \in [\chi]$.

By the proof of Theorem 2.1, the set of elements $(\alpha_1, \ldots, \alpha_m) \in k^m$ such that $L := (\sum_{i=1}^m \alpha_i \ell_i)(A_1)$ does not lie in $\langle A(\gamma)_{d-1} A_1 \rangle$ for all $\gamma \in [\chi]$ is non-empty. An effective determination of an $(\alpha_1, \ldots, \alpha_m)$ of this kind reduces to finding, for a non-zero polynomial in $m$ variables with coefficients in $k$, which is described explicitly, some values of these variables for which the polynomial does not vanish.

The linear subspace $L$ is a $k\tilde{G}$-submodule of $A_d$ isomorphic to $A_1$. For a pair of forms $\tilde{\varphi}_1, \tilde{\varphi}_2$ we can now take a basis of this subspace such that the matrices of all elements of $\tilde{G}$ in this basis are the same as those in the basis $x_1, x_2$ of the space $A_1$ (it suffices to satisfy this condition only for a system of generators of the group $\tilde{G}$ which contains two elements for the dihedral group and three for the others; see [26]). An effective determination of this basis reduces to finding a solution of a system of linear equations for the coefficients of the transition matrix which satisfies an inequality equivalent to the non-degeneracy of this matrix.

(d) Theorem 2.2 leads naturally to the question of whether or not its assertions (i) and (ii) remain valid if we replace $\text{Cr}_1$ by $\text{Cr}_2$. As is shown below (see Theorem 2.14), the answer is negative for assertion (ii). The author does not know the answer for assertion (i) at the time of writing (June 2019), and the following question seems to him to be of principal importance.

**Question** (see [9], Question 1). Is there an incompressible rational action of a finite group on $A^n$?

In the case of a positive answer to this question, the problem naturally arises of finding all the incompressible actions in the list in [7].

### 2.9. Self-compressions of linear actions.

The remaining results in §2 are divided into two groups: some deal with the general case and the others with the case of $\text{Cr}_2$. The following theorem is related to the general case.

**Theorem 2.8.** Let $G$ be a finite subgroup of $\text{GL}_n$, $n \geq 1$.

(a) If $k[x_1, \ldots, x_n]^G_d \neq 0$, then $G$ admits a polynomial homogeneous self-compression $A^n \rightarrow A^n$ of degree $d + 1$. It is non-trivial when $d \neq 0$.

(b) If $d$ is divisible by $|G|$, then $k[x_1, \ldots, x_n]^G_d \neq 0$.

**Proof.** (a) Take a non-zero polynomial $f \in k[x_1, \ldots, x_n]^G_d$ and consider the morphism

$$\varphi: A^n \rightarrow A^n, \quad a \mapsto f(a)a.$$  

(2.37)
Since \( f \) is \( G \)-invariant and the action of \( G \) on \( \mathbb{A}^n \) is linear, for every \( g \in G \) and \( a \in \mathbb{A}^n \) we have
\[
\varphi(g \cdot a) \overset{(2.37)}{=} f(g \cdot a)(g \cdot a) = f(a)(g \cdot a) = g \cdot (f(a)a) \overset{(2.37)}{=} g \cdot (\varphi(a)) ,
\]
that is, \( \varphi \) is a \( G \)-equivariant morphism. From (2.37) and \( f \in k[x_1, \ldots, x_n]_d \) we obtain
\[
\varphi(ta) = t^{d+1} f(a) a = t^{d+1} \varphi(a) \text{ for every } a \in \mathbb{A}^n , \; t \in k , \tag{2.38}
\]
and thus \( \varphi \) is a polynomial homogeneous map of degree \( d + 1 \).

It follows from (2.38) that if a line \( L \) in \( \mathbb{A}^n \) contains 0 and a point \( a \in U := \{ c \in \mathbb{A}^n \mid f(c) \neq 0 \} \) distinct from 0, then
(i) \( \varphi(L) = L ; \)
(ii) the degree of the morphism \( \varphi|_L : L \to L \) is equal to \( d + 1 \).

By (i), the image of \( \varphi \) contains a set \( U \) open in \( \mathbb{A}^n \), and therefore \( \varphi \) is dominant. Thus, \( \varphi \) is a self-compression of \( G \).

Suppose that \( \varphi \) is a birational isomorphism. Then the restriction of \( \varphi \) to some non-empty open subset \( U' \) in \( \mathbb{A}^n \) is injective. Since \( \mathbb{A}^n \) is irreducible, we have \( U \cap U' \neq \emptyset \). Let \( a \in U \cap U' \). Then, in the above notation, the degree of the morphism \( \varphi|_L \) is equal to 1 since \( \varphi|_L \) is injective on an open subset \( L \cap U' \) of \( L \).

We then obtain \( d = 0 \) from (ii), which completes the proof of (a).

(b) The kernel of the natural action of \( G \) on \( k[x_1, \ldots, x_n]_1 \) is trivial. Therefore, there is a non-zero linear form \( \ell \in k[x_1, \ldots, x_n]_1 \) whose \( G \)-orbit contains precisely \( |G| \) elements. Hence, \( (\prod_{g \in G} g \cdot \ell)^s \) is a non-zero \( G \)-invariant form of degree \( s|G| \) for every integer \( s \geq 0 \), which proves (b). \( \square \)

**Corollary 2.9.** Every finite subgroup of \( \text{Cr}_n \) conjugate to a subgroup of the group \( \text{GL}_n \) is non-trivially self-compressible.

### 2.10. Compression of actions with a fixed point

The following result is an application of Theorem 2.8.

**Theorem 2.10.** Every (faithful) rational action \( \varrho \) of a finite group \( G \) on an \( n \)-dimensional irreducible variety that has a fixed point, can be obtained by a non-trivial base change from a (faithful) linear action of the group \( G \) on an \( n \)-dimensional linear space.

**Proof.** Let \( Y \) be an irreducible smooth complete variety and let \( G \hookrightarrow \text{Aut}(Y) \) be a regularization of \( \varrho \) such that \( Y^G \neq \emptyset \). Let \( y \in Y^G \) and consider a non-empty open affine subset \( U \) of \( Y \) containing \( y \). Since \( \bigcap_{g \in G} g \cdot U = U \) is a \( G \)-invariant open affine subset, it follows that replacing \( U \) by this subset we may (and shall) assume that \( U \) is \( G \)-invariant. Since \( U \) is dense in \( Y \), it follows that the action of \( G \) on \( U \) is faithful.

Let \( T_{y,U} \) be the tangent space to \( U \) at the point \( y \). The tangent action
\[
\tau : G \to \text{GL}(T_{y,U}) \subset \text{Bir}(T_{y,U})
\]
of the group \( G \) on the space \( T_{y,U} \) is faithful [8], Lemma 4. By [27], Lemma 10.3, there is a \( G \)-equivariant dominant morphism \( \alpha : U \to T_{y,U} \). By Theorem 2.8, it follows from the linearity of the action \( \tau \) that there is a non-trivial self-compression \( \beta : T_{y,U} \rightarrow T_{y,U} \) of the group \( G \) (thus, \( \deg(\beta) > 1 \)). Then \( \beta \circ \alpha : Y \to T_{y,U} \) is a non-trivial compression of the action \( \varrho \) since \( \deg(\beta \circ \alpha) = \deg(\beta) \deg(\alpha) > 1 \). \( \square \)
Corollary 2.11. An incompressible rational action of a finite group on an irreducible variety has no fixed points.

Remark 2.12. All rational actions of finite groups on $\mathbb{A}^2$ having a fixed point were found in [28]. There are rather many of them. By Theorem 2.10, all of them are obtained by non-trivial base changes from linear actions on $\mathbb{A}^2$ (whose classification has long been known; see, for example, [29]).

We recall that every finite Abelian group $G$ decomposes into a direct sum of cyclic subgroups of orders $m_1, \ldots, m_r$, where $m_i$ divides $m_{i+1}$ for $i = 1, \ldots, r-1$, and $m_1 > 1$ when $|G| > 1$. The sequence $m_1, \ldots, m_r$ is uniquely determined by $G$ and is called the sequence of invariant factors of the group $G$. The number $r$ is called its rank and is equal to the minimal number of generators of the group $G$.

For every integer $n \geq r$ we single out the following subgroup of $\text{GL}_n \subset \text{Cr}_n$ isomorphic to $G$:

$$T_n(m_1, \ldots, m_r) := \{(t_1x_1, \ldots, t_r x_r, x_{r+1}, \ldots, x_n) \mid t_i \in k, t_i^{m_i} = 1, 1 \leq i \leq r\}. \tag{2.39}$$

Theorem 2.13. Let $G$ be a finite Abelian group with sequence of invariant factors $m_1, \ldots, m_r$. If a (faithful) rational action $\varrho$ of the group $G$ on an $n$-dimensional irreducible variety has a fixed point, then $n \geq r$ and $\varrho$ is obtained by a non-trivial base change from a linear action $\lambda: G \hookrightarrow \text{GL}_n(k)$ on $\mathbb{A}^n$ such that $\lambda(G) = T_n(m_1, \ldots, m_r)$.

Proof. We use the notation in the proof of Theorem 2.10. Fixing some isomorphism of the space $T_{y,U}$ to $\mathbb{A}^n$, we identify the group $\text{Bir}(T_{y,U})$ with $\text{Cr}_n$. Since the groups $G$ and $\tau(G)$ are isomorphic, they have the same invariant factors. By [4], Theorem 1, every finite Abelian subgroup of $\text{Aff}_n$ with invariant factors $m_1, \ldots, m_r$ can be transformed into $T_n(m_1, \ldots, m_r)$ using conjugation in $\text{Cr}_n$. Thus, the same holds for the subgroup $\tau(G)$. Whence, arguing as in the proof of Theorem 2.10, we obtain the desired assertion. $\square$

2.11. Compression of actions of cyclic groups. By [30], Theorem A, the sets of conjugacy classes of cyclic subgroups of $\text{Cr}_2$ of some fixed orders $d < \infty$ are infinite, and there are even families of these classes depending on parameters (this is the case if $d$ is even and $d/2$ is odd). The following theorem (a special case of which was proved in [22], Example 5) implies that all these subgroups are obtained by base changes from a single subgroup of this kind.

Theorem 2.14. Let $n$, $m$ and $d$ be any positive integers with $n \geq m$. Every faithful rational action $\varrho$ of a finite cyclic group $G$ of order $d$ on an $n$-dimensional irreducible variety $X$ is obtained by a non-trivial base change from a linear action $\lambda: G \hookrightarrow \text{GL}_n(k)$ on $\mathbb{A}^m$ such that $\lambda(G) = T_n(d)$.

Proof. Let $G^\vee$ be the group of all homomorphisms $G \to k^\times$ and let $K := k(X)$. If $\chi \in G^\vee$, then

$$K_\chi := \{f \in K \mid g \cdot f = \chi(g)f \text{ for all } g \in G\} \tag{2.40}$$

is a linear subspace of the linear space $K$ over $k$. Since $\dim((G \cdot f)) < \infty$ and $G$ is Abelian, it follows that $K = \bigoplus_{\chi \in G^\vee} K_\chi$, and (2.40) implies that $\{\chi \in G^\vee \mid K_\chi \neq 0\}$
is a subgroup of $G^\vee$. Since $G^\vee$ is a cyclic group of order $d$ and the action of $G$ on $K$ is faithful, it follows that this subgroup coincides with $G^\vee$.

Let $\chi_0$ be some generator of $G^\vee$ and let $f \in K_{\chi_0}$, $f \neq 0$. We may (and shall) assume that $f \notin k$ (and therefore $f$ is transcendental over $k$): when $d > 1$ this holds automatically and when $d = 1$ it follows from the fact that $K \neq k$.

Since $K/K^G$ is a finite field extension, we have $\text{tr} \deg_k K^G = \text{tr} \deg_k K = n$. Let $h_1, \ldots, h_n \in K^G$ be a transcendence basis of the field $K^G$ over $k$. Then $\text{tr} \deg_k f(h_1, \ldots, h_n) = n$. Since $f$ is transcendental over $k$, this, together with Theorem 1 in Chap. X, §1 of [31], implies that, possibly after a renumbering of the elements $h_1, \ldots, h_n$, the subfield $L := k(f, h_1, \ldots, h_{n-1})$ of $K$ is a purely transcendental extension of the field $k$ of degree $n$. It follows from the construction that $L$ is $G$-invariant and the action of $G$ on $L$ is faithful. It follows from the definition of $\chi_0$ that the linear action $\mu: G \mapsto \text{GL}_n(k)$ on $A^n$ given by the formula $g \cdot (a_1, \ldots, a_n) := (\chi_0(g^{-1})a_1, a_2, \ldots, a_n)$ is faithful, $\mu(G) = T_n(d)$, and the map $\varphi: X \rightarrow A^n$ is a compression of the action $\varphi$ into the action $\mu$. Since $\mu$ is in turn compressing into a linear action of the group $G$ on $A^m$ using the projection $(a_1, \ldots, a_m) \mapsto (a_1, \ldots, a_m)$, it follows that the desired assertion follows now from Corollary 2.9. □

2.12. Auxiliary assertion: embeddings of $G$-modules in coordinate algebras. We shall need the following general assertion.

Lemma 2.15. If a finite group $G$ acts regularly (and faithfully) on an irreducible affine variety $X$, then every finite-dimensional $kG$-module $M$ is isomorphic to a submodule of the $kG$-module $k[X]$.

Proof. We may (and shall) assume that $\dim(X) > 0$. Since

$$\text{tr} \ deg_k (k(X)^G) = \dim(X) - \dim(G) = \dim(X)$$

(see [20], §2.3, Corollary) and $k(X)^G$ is the field of fractions of the algebra $k[X]^G$ (see [20], Lemma 3.2), it follows that $k[X]^G$ is an infinite-dimensional linear space over $k$:

$$\dim_k (k[X]^G) = \infty. \quad (2.41)$$

Since $\text{char} \ k = 0$ and the group $G$ is finite, it follows that the $kG$-modules $M$ and $k[X]$ are completely reducible. Therefore to prove the lemma, it suffices to show that for every non-zero simple $kG$-module $S$ the multiplicity of the occurrence of $S$ in the $S$-isotypic component of the $kG$-module $k[X]$ is infinite, which is equivalent to the fact that this isotypic component is infinite-dimensional as a linear space over $k$. In turn, to this end it suffices to show that this $S$-isotypic component is non-zero. Indeed, the multiplication of functions defines the structure of a $k[X]^G$-module on this component. Therefore, if this $S$-isotypic component contains a non-zero function, then it is infinite-dimensional by (2.41) and the absence of the zero divisors in $k[X]$. Keeping this reduction in mind, we now shall prove that the $S$-isotypic component of the $kG$-module $k[X]$ is really non-zero.

The set of fixed points of every element of $G$ is closed in $X$. Since $G$ is finite, $X$ is irreducible and the action of $G$ on $X$ is faithful, it follows that there is a point $x$
of $X$ with trivial $G$-stabilizer $G_x$. The $G$-orbit $G \cdot x$ is a $G$-invariant and (by the finiteness) a closed subset of $X$. By its closedness, the $kG$-module homomorphism

$$k[X] \to k[G \cdot x], \quad f \mapsto f|_{G \cdot x},$$

is surjective. Therefore, to show that the $S$-isotypic component of the $kG$-module $k[X]$ is non-trivial, it suffices to prove that the $S$-isotypic component of the $kG$-module $k[G \cdot x]$ is non-trivial. However, it follows from Frobenius reciprocity that the multiplicity of the occurrence of the $kG$-module $S$ in the $kG$-module $k[G \cdot x]$ is equal to the dimension of the space of $G_x$-fixed points in the dual $kG$-module $S^*$ (see [20], Theorem 3.12). Since the group $G_x$ is trivial, this shows that the multiplicity in question is equal to $\dim(S) > 0$. Hence, the $S$-isotypic component of the $kG$-module $k[G \cdot x]$ really is non-zero. This completes the proof of Lemma 2.15.

2.13. $\text{rdim}_k(G)$ and the existence of compressions. For every finite group $G$ and every field $\ell$ we write

$$\text{rdim}_\ell(G) := \min\{m \in \mathbb{Z}, m > 0 \mid \text{there is a group embedding } G \hookrightarrow \text{GL}_m(\ell)\}. \quad (2.42)$$

In other words, $\text{rdim}_\ell(G)$ is the minimal dimension of the faithful finite-dimensional linear representations of $G$ over the field $\ell$. Thus, $G$ admits a faithful $n$-dimensional linear representation over $\ell$ if and only if $n \geq \text{rdim}_\ell(G)$. We note that, if the group $G$ is Abelian, then $\text{rdim}_k(G)$ is equal to its rank.

**Theorem 2.16.** Let $\varrho$ be a (faithful) rational action of a finite group $G$ on an $n$-dimensional irreducible variety.

(i) If $\varrho$ has a fixed point, then $n \geq \text{rdim}_k(G)$.

(ii) If $n > \text{rdim}_k(G)$, then $\varrho$ is compressible into a (faithful) rational action of smaller dimension.

(iii) If there is a faithful $n$-dimensional linear representation over $k$,

$$\lambda: G \to \text{GL}_n \subset \text{Aut}(\mathbb{A}^n), \quad (2.43)$$

then $\varrho$ either is compressible into a (faithful) rational action of smaller dimension or is obtained by a non-trivial base change from a linear action $\lambda$ of the group $G$ on $\mathbb{A}^n$.

**Proof.** Let us consider a regular (faithful) action of the group $G$ on an $n$-dimensional irreducible smooth variety $X$ which is a regularization of the action $\varrho$.

(i) If $\varrho$ has a fixed point, then we choose $X$ in such a way that $X^G \neq \emptyset$. Let $x \in X^G$. Since the tangent action

$$G \to \text{GL}(T_{x,X}) \quad (2.44)$$

of the group $G$ on $T_{x,X}$ is faithful [8], Lemma 4, we see that the homomorphism (2.44) is injective. This and (2.42) imply that $n = \dim(X) = \dim(T_{x,X}) \geq \text{rdim}_k(G)$.

(ii) By the inequality $\text{ed}_k(G) \leq \text{rdim}_k(G)$, which follows from (2.42) and the definition of $\text{ed}_k(G)$ (see [32], Theorem 3.1(b)), statement (ii) follows from the inequality $\text{ed}_k(X) \leq \text{ed}_k(G)$ proved in [32], Theorem 3.1(c).
Another proof of statement (ii), which does not use [32], Theorem 3.1(b), (c), is obtained in the course of the proof of statement (iii) below; see Remark 2.17.

(iii) As in the proof of Theorem 2.10, by replacing $X$ by an appropriate invariant open subset, we may (and shall) assume below that $X$ is affine.

Since the representation $\lambda$ is faithful, it follows that the dual representation $\lambda^* : G \to \text{GL}_r(k)$ is also faithful. It follows from (2.42) and Lemma 2.15 that there is a linear subspace $L$ of $k[X]$ having the following properties:

(a) $L$ is $G$-invariant;
(b) $\text{dim}(L) = n$;
(c) the action of $G$ on $L$ is the representation $\lambda^*$.

Consider the $k$-subalgebra $A$ of $k[X]$ generated by the subspace $L$. Since $\text{dim}(L) < \infty$, $A$ is finitely generated, and therefore it is isomorphic to the algebra of regular functions of an affine variety $Y$. It follows from (b) that

$$\text{dim}(Y) \leq n. \quad (2.45)$$

The identity embedding of algebras $A \hookrightarrow k[X]$ defines a dominant morphism $\varphi : X \to Y$. It follows from (a) that $A$ is $G$-invariant. The action of $G$ on $A$ defines a regular action $\vartheta$ of the group $G$ on the variety $Y$. The morphism $\varphi$ is $G$-equivariant with respect to $\vartheta$. By (c), since the representation $\varrho^*$ is faithful, it follows that the action $\vartheta$ is faithful. Therefore, $\varphi$ is a compression of $\varrho$ into $\vartheta$.

Suppose that this compression does not reduce the dimension of the action $\varrho$, that is, $\text{dim}(X) = \text{dim}(Y)$. Then

$$\text{dim}(Y) = n. \quad (2.46)$$

Thus, in this case, every basis of the linear space $L$ over $k$ consists of elements of the algebra $A$ that are algebraically independent over $k$ since, by construction, $A$ is generated by these elements; the transcendental degree of $A$ over $k$ is then equal to the number of these elements:

$$\text{tr deg}_k(A) = \text{dim}(Y) \overset{(2.46)}{=} n \overset{(b)}{=} \text{dim}(L).$$

This proves that there is a $G$-equivariant isomorphism $\alpha : Y \to L^*$, where $L^*$ stands for the $kG$-module dual to $L$. It follows from (c) that the action of $G$ on $L^*$ is the representation $(\lambda^*)^* = \lambda$. By Theorem 2.8, there is a $G$-equivariant dominant morphism $\varepsilon : L^* \to L^*$ which is not a birational isomorphism. Hence, the composition $\varepsilon \circ \alpha \circ \varphi : X \to L^*$ is a non-trivial compression of $\varrho$ into $\lambda$. \(\square\)

Remark 2.17. By (2.42), as in the proof of statement (iii) we can establish (assuming that $X$ is affine) the existence of the following objects:

- an $\text{rdim}_k(G)$-dimensional $kG$-submodule $M$ in $k[X]$ on which the action of $G$ is faithful;
- an affine $G$-variety $Z$ and a dominant $G$-equivariant morphism $\psi : X \to Z$ such that $\psi^*(k[Z])$ is the $k$-subalgebra of $k[X]$ generated by the subspace $M$.

Since the action of $G$ on $Z$ is faithful, it follows that $\psi$ is a compression of the action $\varrho$. If $n > \text{rdim}_k(G)$, then $\psi$ lowers the dimension of $\varrho$, since $\text{dim}(Z) \leq \text{dim}_k(M) = \text{rdim}_k(G)$.

This gives another proof of statement (ii) of Theorem 2.16.
2.14. Compression of Abelian subgroups of rank 2 of the group $\text{Cr}_2$. Theorem 2.14 answers the question of how to construct finite Abelian subgroups of rank 1 of the Cremona group $\text{Cr}_n$ using base changes. When $n = 2$ the following theorem answers the analogous question for subgroups of rank 2, that is, for non-cyclic subgroups isomorphic to $\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}, \ a \geq 2, b \geq 2$.

**Theorem 2.18.** Let $\varrho$ be a (faithful) rational action of a finite Abelian group $G$ of rank 2 on $\mathbb{A}^2$ and let $m_1, m_2$ be the sequence of invariant factors of the group $G$.

(i) In each of the cases

(a) $|G| \neq 4$,

(b) $|G| = 4$ and $\varrho$ has a fixed point,

the rational action $\varrho$ is obtained by a non-trivial base change from a linear action $\lambda: G \hookrightarrow \text{GL}_2(k)$ on $\mathbb{A}^2$ such that $\lambda(G) = T_2(m_1, m_2)$ (see (2.39)). In these cases, the rational action $\varrho$ cannot be compressed into a rational action of smaller dimension.

(ii) If $|G| = 4$ and $\varrho$ has no fixed point, then $G$ is a dihedral group (that is, the Klein four-group), and $\varrho$ is obtained by a base change from an action $\gamma: G \to \text{Cr}_1$ on $\mathbb{P}^1$ for which $\gamma(G)$ is generated by the elements $\sigma, \tau \in \text{Aut}(\mathbb{P}^1)$ given by the formulae

$$
\sigma \cdot (a_0 : a_1) = (a_0 : -a_1), \quad \tau \cdot (a_0 : a_1) = (a_1 : a_0) \quad \text{for all } (a_0 : a_1) \in \mathbb{P}^1. \quad (2.47)
$$

**Proof.** Since $G$ is an Abelian group of rank 2, there is a faithful linear representation (2.43) with $n = 2$ and $\lambda(G) = T_2(m_1, m_2)$.

If $\varrho$ is compressible into a rational action $\vartheta$ of smaller dimension, then $\vartheta$ is a faithful rational action of the group $G$ on an irreducible algebraic curve $C$. Since there is a dominant rational map $\mathbb{A}^2 \dasharrow C$ (a compression of $\varrho$ into $\vartheta$), the curve $C$ is rational. Therefore, we may (and shall) assume that $C = \mathbb{P}^1$, and thus $G$ is isomorphic to a subgroup of $\text{Cr}_1 = \text{Bir}(\mathbb{P}^1) = \text{Aut}(\mathbb{P}^1) = \text{PSL}_2$. It follows from the well-known description of finite subgroups in $\text{PSL}_2$ that among them, the only subgroups that are non-cyclic and Abelian are conjugate to the dihedral subgroup of order 4 generated by the elements $\sigma$ and $\tau$ given by the formulae (2.47). These subgroups have no fixed points on $\mathbb{P}^1$. Thus, $G = D_2$ in this case. By the “going down” property for fixed points (see [33], Proposition A.2), it follows from $(\mathbb{P}^1)^G = \emptyset$ that $\varrho$ has no fixed point.

If $\varrho$ is not compressible into a rational action of smaller dimension, then by Theorem 2.16, (iii), $\varrho$ is obtained by a non-trivial base change from a linear action $\lambda$ of the group $G$ on $\mathbb{A}^2$. This, together with the “going up” property for fixed points (see [33], Proposition A.4), implies that, if in the case in question both the invariant factors $m_1$ and $m_2$ are equal to the same prime integer, then $\varrho$ has a fixed point. In particular, this is the case when $|G| = 4$. □

2.15. Compression of other subgroups. A classification of the finite Abelian subgroups of $\text{Cr}_2$ up to conjugation is given in [30]. By Theorems 2.14 and 2.18, this classification implies that, among these subgroups, only the subgroups isomorphic to

$$
\mathbb{Z}/2d\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2, \quad d \geq 1, \quad (\mathbb{Z}/4\mathbb{Z})^2 \oplus \mathbb{Z}/2\mathbb{Z}, \quad (\mathbb{Z}/3\mathbb{Z})^3, \quad (\mathbb{Z}/2\mathbb{Z})^4
$$
Three plots about the Cremona groups remain to be studied from the point of view of non-trivial compressibility. The ranks of these subgroups are equal to 3, 3, 3, 4, respectively. By Theorem 2.16, (i), none of these subgroups has fixed points.

**Theorem 2.19.** Let $G$ be a non-Abelian finite group that differs from the dihedral group and admits a faithful linear representation $\lambda: G \hookrightarrow \text{GL}_2(k)$. Then every (faithful) rational action of $G$ on $\mathbb{A}^2$ is obtained by a non-trivial base change from the linear action $\lambda$ of $G$ on $\mathbb{A}^2$.

**Proof.** The assertion will follow from Theorem 2.16, (iii) if we can prove that the action $\varrho$ in this theorem cannot be compressed into a faithful rational action of smaller dimension.

Arguing by contradiction, we assume that such a compression exists. Then, as in the proof of Theorem 2.18, we see that $G$ is isomorphic to a subgroup of the group $\text{Aut}(\mathbb{P}^1) = \text{PSL}_2$. Since the only non-cyclic and non-dihedral finite subgroups of $\text{PSL}_2$ are the groups of rotations of the regular tetrahedron, cube and icosahedron, respectively, it follows that $G$ is isomorphic to one of these groups. However, this is impossible since the group of rotations of the icosahedron has no non-trivial two-dimensional representations, and although the groups of rotations of the regular tetrahedron and cube have such representations, the kernels of these representations are non-trivial (their orders are equal to 4); see, for example, [34]. A contradiction. □

§ 3. The Cremona groups and group embeddings

In this section, the characteristic of the field $k$ is equal to zero.

**3.1. Properties of the abstract Jordan groups.** We recall the notions introduced in [35], Definition 2.1 and [8], Definition 1.

For every finite group $H$ put

$$m_H := \min_S [H : S],$$

where $S$ ranges over all normal Abelian subgroups of $H$.

**Definition 3.1.** Let $G$ be a group and let

$$J_G := \sup_F m_F,$$

where $F$ ranges over all finite subgroups of $G$. If $J_G < \infty$, then $G$ is called a Jordan group (in other words, $G$ has the Jordan property) and $J_G$ is called the Jordan constant of $G$.

**Lemma 3.2.** The following inequality holds for all groups $G_1, \ldots, G_s$:

$$J_{G_1 \times \cdots \times G_s} \geq J_{G_1} \cdots J_{G_s}$$

(if $J_{G_i} = \infty$, then, by definition, (3.3) means that $J_{G_1 \times \cdots \times G_s} = \infty$).
Proof. Let $F_i$ be a finite subgroup of $G_i$ and let $N$ be a normal Abelian subgroup of the finite subgroup $F_1 \times \cdots \times F_s$ of the group $G_1 \times \cdots \times G_s$. Let $\pi_i : F_1 \times \cdots \times F_s \to F_i$ be the projection to the $i$th factor. Then $\pi_i(N)$ is a normal Abelian subgroup of the group $F_i$, and therefore the definition (3.1) implies the inequality

$$|\pi_i(N)| \leq \frac{|F_i|}{m_{F_i}}. \quad (3.4)$$

It follows from the inclusion $N \subseteq \pi_1(N) \times \cdots \times \pi_s(N)$ and the inequality (3.4) that

$$|N| \leq |\pi_1(N) \times \cdots \times \pi_s(N)| = \prod_{i=1}^s |\pi_i(N)| \leq \prod_{i=1}^s \frac{|F_i|}{m_{F_i}} = \frac{|F_1 \times \cdots \times F_s|}{m_{F_1} \cdots m_{F_s}}. \quad (3.5)$$

It follows from (3.5) that $|(F_1 \times \cdots \times F_s) : N| \geq m_{F_1} \cdots m_{F_s}$, and hence, by (3.1), we obtain the inequality

$$m_{F_1} \cdots m_{F_s} \geq m_{F_1} \cdots m_{F_s}. \quad (3.6)$$

The inequality (3.3) now follows from (3.6) and (3.2). □

Remark 3.3. We put $j_G := \sup_F \min_A [F : A]$, where $F$ varies over all finite subgroups of $G$, and $A$ over all Abelian (not necessarily normal) subgroups of $F$. It is clear that $j_G \leq J_G$. The conditions $J_G < \infty$ and $j_G < \infty$ are equivalent; see [35], Remark 2.2. Omitting the assumption that the subgroup $N$ is normal from the proof of Lemma 3.2, we obtain for all groups $G_1, \ldots, G_s$ a proof of the inequality

$$j_{G_1 \times \cdots \times G_s} \geq j_{G_1} \cdots j_{G_s}.$$

Theorem 3.4. Let $P$ be a Jordan group and let $Q_1, \ldots, Q_s$ be groups each containing a non-Abelian finite subgroup. Then the group $Q_1 \times \cdots \times Q_s$ cannot be embedded in $P$ if $s > \log_2(J_P)$.

Proof. It follows from Definition 3.1 that $J_P < \infty$ and, if $Q_1 \times \cdots \times Q_s$ can be embedded in $P$, then $J_{Q_1 \times \cdots \times Q_s} \leq J_P$. This, together with Lemma 3.2, implies the inequality

$$J_{Q_1} \cdots J_{Q_s} \leq J_P.$$

However, it follows from (3.1), (3.2) and the condition on $Q_i$ that $J_{Q_i} \geq 2$ for every $i$. Thus, $2^s \leq J_P$, and therefore $s \leq \log_2(J_P)$. □

Remark 3.5. The assertion and proof of Theorem 3.4 remain valid if we replace there $J_P$ by $j_P$ and $J_{Q_i}$ by $j_{Q_i}$.

3.2. Subgroups of the form $G_1 \times \cdots \times G_s$ and $p$-subgroups of the Cremona groups. We now apply the results of §3.1 to the Cremona groups.

Theorem 3.6. Let $X$ be a rationally connected variety defined over $k$. Then there is an integer $b_X$ depending on $X$ such that every product of groups $G_1 \times \cdots \times G_s$ each containing a finite non-Abelian subgroup cannot be embedded in the group $\text{Bir}_k(X)$ when $s > b_X$.

Proof. The assertion follows from Theorem 3.4 and the Jordan property of the group $\text{Bir}_k(X)$ (see the footnote in the introduction). □
Corollary 3.7. Let \( n \) be a positive integer. Then there is an integer \( b_{n,k} \) depending on \( n \) and the field \( k \) such that every product of groups \( G_1 \times \cdots \times G_s \) each containing a finite non-Abelian subgroup cannot be embedded in the Cremona group \( \text{Cr}_n(k) \) when \( s > b_{n,k} \).

Proof. The assertion follows from Theorem 3.6 by the rational connectedness of rational varieties. \( \square \)

Remark 3.8. By Theorem 3.4 and Remark 3.5, we can take \( b_X = \log_2(j_{\text{Bir}_k(X)}) \) in Theorem 3.6. Explicit upper bounds for \( j_{\text{Cr}_2(k)} \) and \( j_{\text{Bir}_k(X)} \) for rationally connected threefolds \( X \), and also the precise values of \( j_{\text{Cr}_2(k)} \) and \( j_{\text{Bir}_k(X)} \) under certain conditions, were found in [10] and [36]. For example, if \( k = \bar{k} \), then \( j_{\text{Cr}_n} = 288 \) and \( 10368 \) when \( n = 2 \) and \( 3 \), respectively.

Corollary 3.9. For every prime integer \( p \) and every rationally connected variety \( X \) defined over \( k \) there is a non-Abelian finite \( p \)-group non-embeddable in \( \text{Bir}_k(X) \). In particular, for every integer \( n > 0 \) there is a non-Abelian finite \( p \)-group non-embeddable in the Cremona group \( \text{Cr}_n(k) \).

Proof. The assertion follows from Theorem 3.6, Corollary 3.7 to this theorem, and the existence of finite non-Abelian \( p \)-groups. \( \square \)

3.3. Applications: \( p \)-rank and embeddings of groups. A consideration of \( p \)-subgroups raises obstacles to the existence of group embeddings. This has some applications.

Namely, let \( p \) be a prime integer. We recall that a finite \( p \)-group is said to be elementary if it is Abelian and all its invariant factors (see §2.10 above) are equal to \( p \).

Definition 3.10. For every group \( G \) and every prime integer \( p \) we call the least upper bound of the ranks of all elementary \( p \)-subgroups of \( G \) by the \( p \)-rank of \( G \) and denote it by \( \text{rk}_p(G) \).

Example 3.11. Let a group \( T \) be an \( n \)-dimensional torus in the category of either affine algebraic groups over \( \bar{k} \) or real Lie groups (that is, \( T \) is isomorphic to the direct product of \( n \) copies of the group \( \mathbb{C}^\times \) in the first case and of the group \( \{ z \in \mathbb{C}^\times \mid |z| = 1 \} \) in the second). It can readily be seen that then \( \text{rk}_p(T) = n \) for every prime integer \( p \).

It is clear that if \( G_1 \) and \( G_2 \) are two groups and \( \text{rk}_p(G_1) > \text{rk}_p(G_2) \) for some \( p \), then \( G_1 \) cannot be embedded in \( G_2 \). Applications of this remark use the fact that in some cases, we can explicitly compute or estimate \( \text{rk}_p(G) \). In particular, this is the case for some subgroups of the Cremona groups.

Theorem 3.12. For every integer \( n > 0 \) there is a constant \( R_n \) depending on \( n \) such that \( \text{rk}_p(H) = n \) for every (not necessarily closed) subgroup \( H \) of \( \text{Cr}_n \) containing an \( n \)-dimensional algebraic torus and for every \( p > R_n \). In particular,

\[
\text{rk}_p(\text{Cr}_n) = \text{rk}_p(\text{Aut}(\mathbb{A}^n)) = n \quad \text{for every} \quad p > R_n.
\]

Proof. Let \( p \) be a prime. It follows from Example 3.11 and the condition on \( H \) that \( \text{rk}_p(H) \geq n \). On the other hand, by combining [11], Theorem 1.10 with [12],
Corollary 1.3, we conclude that if \( p \) exceeds some constant \( R_n \) depending on \( n \), then \( \text{rk}_p(\text{Cr}_n) \leq n \), and therefore \( \text{rk}_p(H) \leq n \). This completes the proof. □

Other examples of groups \( G \) whose \( p \)-rank can successfully be computed are connected affine algebraic groups over \( \overline{K} \) and connected real Lie groups. All maximal tori are conjugate in a group \( G \) of this kind; let

\[
\text{r}(G)
\]
denote the dimension of the maximal tori of \( G \).

We recall that a prime \( p \) is called a torsion prime of the group \( G \) if \( G \) has a finite Abelian \( p \)-subgroup not contained in any maximal torus. The torsion primes of a group \( G \) divide the order of its Weyl group, and therefore the set of all such primes is finite.

**Theorem 3.13.** Let \( G \) be either a connected affine algebraic group over \( \overline{K} \) or a connected real Lie group. Let \( p \) be a prime that is not a torsion prime of \( G \). Then \( \text{rk}_p(H) = \text{r}(G) \) for every (not necessarily closed) subgroup \( H \) of \( G \) containing a maximal torus of the group \( G \).

**Proof.** It follows from Example 3.11 and the condition on \( H \) that the inequality \( \text{rk}_p(H) \geq \text{r}(G) \) holds. On the other hand, if \( F \) is a finite elementary \( p \)-subgroup of \( H \), then, since \( p \) is not a torsion prime of \( G \), it follows that \( F \) is contained in a maximal torus of the group \( G \). By Example 3.11, this implies that the rank of \( F \) does not exceed \( \text{r}(G) \). Therefore, \( \text{rk}_p(H) \leq \text{r}(G) \). This completes the proof. □

Theorems 3.12 and 3.13 imply the following assertion.

**Corollary 3.14.** Let \( k \) be an algebraically closed field of characteristic zero. To every positive integer \( d \) we assign a (not necessarily closed) subgroup \( H_d \) from the following list:

1. a subgroup of the group \( \text{Cr}_d(k) \) containing a \( d \)-dimensional algebraic torus;
2. a subgroup of a connected affine algebraic group \( G \) over \( k \) with \( \text{r}(G) = d \) containing a maximal torus of the group \( G \);
3. a subgroup of a connected real Lie group \( G \) with \( \text{r}(G) = d \) containing a maximal torus of the group \( G \).

Then the group \( H_n \) cannot be embedded in the group \( H_m \) for \( n > m \). In particular, the following properties are equivalent:

(a) the groups \( H_n \) and \( H_m \) are isomorphic;
(b) \( n = m \).

Let us single out three special cases as Corollaries 3.15, 3.16 and 3.17.

**Corollary 3.15** (see [13], Theorem B, [11], Remark 1.11). The group \( \text{Cr}_n \) can be embedded in the group \( \text{Cr}_m \) if and only if \( n \leq m \). In particular, the groups \( \text{Cr}_n \) and \( \text{Cr}_m \) are isomorphic if and only if \( n = m \).

**Corollary 3.16.** The group \( \text{Aut}(\mathbb{A}^n_k) \) can be embedded in the group \( \text{Aut}(\mathbb{A}^m_k) \) if and only if \( n \leq m \). In particular, the groups \( \text{Aut}(\mathbb{A}^n_k) \) and \( \text{Aut}(\mathbb{A}^m_k) \) are isomorphic if and only if \( n = m \).

Let \( K_0 := k \) and \( K_i := k(x_1, \ldots, x_i) \) for \( i = 1, \ldots, n \). For every \( a_i, b_i \in K_{i-1}, a_i \neq 0 \), the map (1.2), where \( \sigma_i = a_i x_i + b_i \) for every \( i \), is an element of \( \text{Cr}_n(k) \), and
the set $B_n(k)$ of all elements of this kind is a subgroup of $Cr_n(k)$. By [6], the group $B_n := B_n(k)$ is a Borel subgroup of $Cr_n$: it contains the $n$-dimensional diagonal torus of the group $GL_n$.

**Corollary 3.17.** The group $B_n$ can be embedded in the group $B_m$ if and only if $n \leq m$. In particular, the groups $B_n$ and $B_m$ are isomorphic if and only if $n = m$.

We obtain the following assertion as another application.

**Corollary 3.18.** If $\varphi : Cr_n \to Cr_m$ is a continuous epimorphism of groups equipped with the Zariski topology, then $n = m$ and $\varphi$ is an automorphism.

**Proof.** By the topological simplicity of the group $Cr_n$ (see [14], Theorem 1), the kernel of the epimorphism $\varphi$ is trivial, and therefore $\varphi$ is an isomorphism of abstract groups. The assertion follows now from Corollary 3.14. □

**Remark 3.19.** By [37], it follows from [38] that there is an epimorphism of abstract groups $Cr_3 \to Cr_2$. Therefore, the continuity assumption in Corollary 3.18 is substantial. On the other hand, $Cr_2$ is an abstract Hopfian group, that is, every (not necessarily continuous) surjective endomorphism of $Cr_2$ is an automorphism [39].

In the following theorem we use an upper bound of the $p$-rank of the group rather than its precise value.

**Theorem 3.20.** Let $M$ be a connected compact $n$-dimensional topological manifold and let $B_M$ be the sum of its Betti numbers with respect to homology with coefficients in $\mathbb{Z}$. If

$$d > \frac{\sqrt{n^2 + 4n(n + 1)B_M} + n}{2} + \log_2 B_M, \tag{3.7}$$

then the Cremona group $Cr_d$ cannot be embedded in the group $\mathcal{H}(M)$ of homeomorphisms of $M$.

**Proof.** Suppose that the inequality (3.7) holds.

Let $p > 2$ be a prime satisfying the following conditions:

(i) $p > R_n$ (see Theorem 3.12);

(ii) $p$ does not divide the order of the finite Abelian group $\bigoplus_{i=0}^n \text{Tors}(H_i(M, \mathbb{Z}))$.

It follows from [40], Theorem 2.5(3) that the rank of every finite elementary $p$-subgroup of the group $\mathcal{H}(M)$ does not exceed

$$\frac{\sqrt{n^2 + 4n(n + 1)B_{M,p}} + n}{2} + \log_2 B_{M,p},$$

where $B_{M,p}$ stands for the sum of the Betti numbers of $M$ with respect to homology with coefficients in $\mathbb{F}_p$. It follows from (ii) and the universal coefficient theorem that $B_{M,p} = B_M$, and hence, by (3.7), we obtain the inequality $d > \text{rk}_p(\mathcal{H}(M))$. This inequality, the condition (i) and Theorem 3.12 imply that $\text{rk}_p(Cr_d) > \text{rk}_p(\mathcal{H}(M))$. This completes the proof of the theorem. □

By [11], Theorem 1.10, [12], Corollary 1.3, and Definition 3.10, the constant $R_n$ in Theorem 3.12 can be chosen in such a way that for every rationally connected $n$-dimensional variety $X$ defined over $k$ and every prime $p > R_n$ the inequality $\text{rk}_p(\text{Bir}_k(X)) \leq n$ is satisfied. This implies another assertion concerning non-embeddable groups.
Theorem 3.21. Let $X$ be a rationally connected $n$-dimensional variety $X$ defined over $k$ and $p$ a prime exceeding the constant $R_n$ of Theorem 3.12. Then every product of groups $G_1 \times \cdots \times G_s$ each containing an element of order $p$ cannot be embedded in the group $\text{Bir}_k(X)$ for $s > n$.

§ 4. Connectedness of the Cremona groups

4.1. A new proof of the connectedness theorem. Two elements $\sigma, \tau \in \text{Cr}_n(k)$ are said to be linearly connected if there are a $k$-defined open subset $U$ of the affine line $\mathbb{A}^1$ and a $k$-morphism $\varphi : U \to \text{Cr}_n$ such that $\sigma, \tau \in \varphi(U(k))$. It can readily be seen that the relation of linear connectedness is an equivalence relation on $\text{Cr}_n(k)$ (see [15], p.363). By definition, the linear connectedness of the group $\text{Cr}_n(k)$ means that this equivalence relation admits only one equivalence class. The linear connectedness of $\text{Cr}_n(k)$ implies its connectedness.

Theorem 4.1 (see [14]). The Cremona group $\text{Cr}_n(k)$ is linearly connected when the field $k$ is infinite.

Proof (which differs from that in [14]).

(a) First, id and every element $\sigma \in \text{Aff}_n(k)$ are linearly connected since $\text{Aff}_n$ is an open subset of the $(n^2 + n)$-dimensional affine space $\mathcal{A}_n$ of all affine maps $\mathbb{A}^n \to \mathbb{A}^n$, and therefore one can take $\varphi$ to be the identity map of the set $U := \ell \cap \text{Aff}_n$, where $\ell$ stands for a line in $\mathcal{A}_n$ containing $\sigma$ and id.

(b) Second, every element $\sigma \in \text{Bir}_k(\mathbb{A}^n) = \text{Cr}_n(k)$ has the form $\sigma = \alpha \circ \theta \circ \tau$, where $\alpha, \tau \in \text{Aff}_n(k)$ and $\theta = (\theta_1, \ldots, \theta_n) \in \text{Cr}_n(k)$ has the following properties:

(i) $\theta$ is defined at $\alpha$;
(ii) $\theta(\alpha) = \alpha$;
(iii) $\theta$ is étale at $\alpha$, and the map $d_\theta : T_{\alpha, \mathbb{A}^n} \to T_{\alpha, \mathbb{A}^n}$ is the identity.

Indeed, since the map $\sigma : \mathbb{A}^n \to \mathbb{A}^n$ is $k$-birational and the field $k$ is infinite, it follows that there is a point $s \in \mathbb{A}^n(k)$ at which $\sigma$ is defined and étale (the existence of $s$ is equivalent to the existence of a point in $\mathbb{A}^n(k)$ which is not a zero of some non-zero polynomial in $k[x_1, \ldots, x_n]$). We can now take $\alpha$ and $\tau$ to be arbitrary elements of $\text{Aff}_n(k)$ for which $\tau^{-1}(s) = s$ and $\alpha^{-1}(\sigma(s)) = o$, and the composition of the maps

$$T_{\alpha, \mathbb{A}^n} \xrightarrow{d_\sigma^{-1}} T_{s, \mathbb{A}^n} \xrightarrow{d_\sigma} T_{\sigma(s), \mathbb{A}^n} \xrightarrow{d_{\sigma(s)}^{-1}} T_{\alpha, \mathbb{A}^n}$$

is the identity map; such elements obviously exist.

(c) We now show that id and the element $\theta \in \text{Cr}_n(k)$ indicated in (b) are linearly connected. By (a) and (b), it is clear that this will complete the proof of Theorem 4.1.

Let $\mathcal{O}$ and $\widehat{\mathcal{O}}$ be the local ring of the variety $\mathbb{A}^n$ at the point $o$ and its completion with respect to the maximal ideal, respectively. The family of functions $x_1, \ldots, x_n$ is a system of local parameters of the variety $\mathbb{A}^n$ at $o$. Therefore, we may (and shall) assume that $\widehat{\mathcal{O}} = k[[x_1, \ldots, x_n]]$, and $\mathcal{O}$ is the subring of $\widehat{\mathcal{O}}$ formed by the Taylor series at the point $o$ of all functions in $\mathcal{O}$ with respect to this system of local parameters. We have $\mathcal{O}_k := \mathcal{O} \cap k(\mathbb{A}^n) \subset k[[x_1, \ldots, x_n]]$.

It follows from (i) that $\theta_i \in \mathcal{O}_k$ for every $i = 1, \ldots, n$, and thus

$$\theta_i = F_i(x_1, \ldots, x_n) \in k[[x_1, \ldots, x_n]]. \quad (4.1)$$
By (ii) and (iii), the series \( F_i(x_1, \ldots, x_n) \) has the form
\[
F_i(x_1, \ldots, x_n) = x_i + \sum_{d \geq 2} F_{i,d}(x_1, \ldots, x_n),
\]
(4.2)
where \( F_{i,d}(x_1, \ldots, x_n) \) is a form of degree \( d \) in \( x_1, \ldots, x_n \) with coefficients in \( k \), and thus
\[
F_{i,d}(tx_1, \ldots, tx_n) = t^d F_{i,d}(x_1, \ldots, x_n) \quad \text{for every } t \in \overline{k}.
\]
(4.3)
It follows from (4.1), (4.2), and (4.3) that for every \( t \in \overline{k} \) the series
\[
 tx_i + \sum_{d \geq 2} t^d F_{i,d}(x_1, \ldots, x_n) \in \mathcal{O}
\]
belongs to \( \mathcal{O} \), and to \( \mathcal{O}_k \) when \( t \in k \). This implies that the series
\[
 x_i + \sum_{d \geq 2} t^{d-1} F_{i,d}(x_1, \ldots, x_n)
\]
also has these properties. Therefore, for every \( t \in \overline{k} \) we obtain a rational map
\[
\rho(t): A^n \to A^n, \quad \rho(t)_i = x_i + \sum_{d \geq 2} t^{d-1} F_{i,d}(x_1, \ldots, x_n), \quad i = 1, \ldots, n. \quad (4.4)
\]
In fact, \( \rho(t) \in \text{Cr}_n \) for every \( t \). Indeed, (4.4) gives
\[
\rho(0) = (x_1, \ldots, x_n) \overset{(1.2)}{=} \text{id} \in \text{Cr}_n.
\]
(4.5)
When \( t \neq 0 \) and \( \vartheta(t) := (tx_1, \ldots, tx_n) \in \text{GL}_n \), it follows from (1.3), (4.1), (4.2) and (4.4) that
\[
\vartheta(t^{-1}) \circ \theta \circ \vartheta(t) = \rho(t).
\]
(4.6)
Since the left-hand side of the equation (4.6) belongs to \( \text{Cr}_n \), it follows that the same holds for the right-hand side.

Thus, a map \( \varphi: A^1 \to \text{Cr}_n, t \mapsto \rho(t) \), arises. By (4.4), this map is a \( k \)-morphism. It follows now from (4.5) and the equality \( \rho(1) = \theta \) (which results from (4.4), (4.2) and (4.1)) that \( \theta \) and \( \text{id} \) are linearly connected. □

4.2. The case of a finite field \( k \). The following examples, which are due to A. Borisov [41], show that the infiniteness condition for \( k \) cannot be omitted in the above proof.

Examples. Let \( k = F_q \) and \( n = 2 \). Then the birational self-map
\[
\tau := \left( x_1, x_2 - \frac{1}{x_1^q - x_1} \right) \in \text{Cr}_2(F_q)
\]
is not defined at all points of the set \( A^2(F_q) \), and the birational self-map
\[
\tau := ((x_1^q - x_1)x_1x_2, (x_1^q - x_1)x_2) \in \text{Cr}_2(F_q)
\]
is not étale at all these points.
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