From tunnels to towers: quantum scars from Lie algebras and $q$-deformed Lie algebras

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We present a general symmetry-based framework for obtaining many-body Hamiltonians with scarred eigenstates that do not obey the eigenstate thermalization hypothesis. Our models are derived from parent Hamiltonians with a non-Abelian (or $q$-deformed) symmetry, whose eigenspectra are organized as degenerate multiplets that transform as irreducible representations of the symmetry (‘tunnels’). We show that large classes of perturbations break the symmetry, but in a manner that preserves a particular low-entanglement multiplet of states – thereby giving generic, thermal spectra with a shadow of the broken symmetry in the form of scars. The generators of the Lie algebra furnish operators with ‘spectrum generating algebras’ that can be used to lift the degeneracy of the scar states and promote them to equally spaced ‘towers’. Our framework applies to several known models with scars, but we also introduce new models with scars that transform as irreducible representations of symmetries such as SU(3) and $q$-deformed SU(2), significantly generalizing the types of systems known to harbor this phenomenon. Additionally, we present new examples of generalized AKLT models with scar states that do not transform in an irreducible representation of the relevant symmetry. These are derived from parent Hamiltonians with enhanced symmetries, and bring AKLT-like models into our framework.

I. INTRODUCTION AND GENERAL FRAMEWORK

A central question in non-equilibrium quantum dynamics is whether reversible unitary dynamics in a closed quantum system can establish local thermal equilibrium. Much insight into quantum thermalization follows from the eigenstate thermalization hypothesis (ETH) [1–5], a strong version of which posits that the eigenstate thermalization hypothesis (ETH) [1–5], a strong version of which posits that every finite temperature eigenstate of a thermalizing system reproduces thermal expectation values locally [6]. In contrast, there are classes of interacting, typically disordered, “many-body localized” (MBL) systems that violate the ETH and never thermalize [7–9].

More recently, attention has focused on weak ETH violating systems with so-called ‘many-body quantum scars’ [10–13]. Scars are non-thermal eigenstates embedded within an otherwise thermal eigenspectrum. These typically have sub-thermal entanglement entropy ($\sim O(\log(|A|))$) or $O(|A|)$ for a subsystem $A$ and co-exist at the same energy density as thermal volume-law entangled eigenstates. Scars constitute a vanishing fraction of the eigenspectrum – and hence these systems still obey a weak version of the ETH [14]; nonetheless, their presence can lead to measurable non-thermal dynamical signatures in quenches from atypical but experimentally amenable initial states [12, 15]. Indeed, the recent literature on scars followed from an interesting experimental observation of non-thermal (oscillatory) quench dynamics in a Rydberg atom chain that realizes a constrained ‘PXP’ spin Hamiltonian.

Our understanding of scars in the PXP model is still largely open question, and most of the scarred eigenstates relevant to the quench dynamics are only approximately known [13, 16–26]. In contrast, by now, there are many lattice models with exactly known scar states, ranging from the celebrated AKLT model to deformed topological models [10, 11, 20, 21, 27–37]. Many such examples with exact scar states can be understood via one (or both) of two complementary approaches: the first due to Shiraishi and Mori (SM) [10] relies on local projectors, and the second due to Mark, Lin and Motrunich (MLM) [32] relies on the existence of a spectrum generating algebra (SGA) on the scarred subspace (see also Moudgalya et al. in Ref. [34] for related constructions).

The SM prescription [10] relies on a set of local projectors, $\{ P_i \}$ centered around site $i$, that generically do not commute with one other, in addition to one or more states $|\psi_s\rangle$ that are simultaneously annihilated by all the $P_i$ and span a subspace $S$. Then, the $|\psi_s\rangle$ are scarred eigenstates annihilated by Hamiltonians of the form

$$H_{SM}^i = \sum_i P_i h_i P_i,$$

where the $h_i$ are generic local operators of finite range. The $h_i$ operators ensure that the rest of the spectrum is thermalizing and non-integrable. If, additionally, there exist special Hamiltonians $H'$ that commute with all the $\{ P_i \}$, then these can be added to $H_{SM}^i$ to impart different energies to the states in $S$. Note that $H = H_{SM}^i + H'$ does not have explicit symmetries, but the Hilbert space nevertheless dynamically splits into disconnected ‘Krylov sectors’: the subspaces $S$ and its complement do not mix because $S$ is annihilated by $H_{SM}^i$.

Separately, MLM in Ref. [32] furnished a complimentary framework that unified the existence of ‘towers’ of equally spaced in energy scar states in three different models [11, 21, 28]. In these models, scars $\{ |\psi_n\rangle \}$ were generated by repeatedly acting with a particular opera-
tor $Q^+$ on a particular low entanglement eigenstate of $H$, $|\psi_0\rangle$ so that $|\psi_n\rangle = (Q^+)^n|\psi_0\rangle$. MLM showed that, in all these cases, $Q^+$ acts as a spectrum generating ‘ladder’ operator when restricted to the scarred subspace:

$$\left[(H, Q^+) - \omega Q^+\right]|\psi_n\rangle = 0,$$

which implies that the $|\psi_n\rangle$ are equally spaced energy eigenstates of $H$ with $E_n = \omega n + E_0$. Furthermore, the particular form of the $Q^+$ operator is such that the states $|\psi_n\rangle$ have low entanglement. MLM discussed various example Hamiltonians obeying Eq. (2) which had the form $H = H_{SG} + H_A$, such that $H_{SG}$ has a ‘spectrum generating’ algebra (SGA):

$$[H_{SG}, Q^+] = \omega Q^+,$$

and $H_A$ annihilates the scars, $H_A|\psi_n\rangle = 0$. Similar to SM, these contain a piece that annihilates the scars and one that gives them energy.

While such constructions have been very useful for explicitly deriving and unifying the presence of scars in specific ‘one-off’ models, qualitative ‘pictures’ for when and how scars may arise more generally are still largely missing. For example, it is still largely unclear where and how scars may arise more generally are still largely specific ‘one-off’ models, implicitly deriving and unifying the presence of scars in one that gives them energy.

Our framework makes extensive use of the generators of the Lie algebra of the symmetry group $G$, which furnish a natural set of spectrum generating operators (SGOs) with ‘raising/lowering action’ (for example, operators $\{Q^+, Q^-, Q^z\}$ associated with the SU(2) algebra. (a) $H_{sym}$ has ‘tunnels’ of degenerate eigenstates with the same eigenvalue for the Casimir $Q^2$ but different eigenvalues for $Q^z$. Each tunnel is denoted by a different color. One can move between states in a tunnel using $Q^\pm$. (b) Adding $H_{SG} \propto Q^z$ preserves the eigenstates, but breaks the degeneracy of the tunnels. Instead, states in each tunnel get promoted to ‘towers’ and acquire an evenly spaced harmonic spectrum because of the SGA $[Q^z, Q^\pm] = \pm Q^\pm$. (c) An $H_A$ can be chosen to annihilate a specific tower of states (highlighted) but generically break all symmetries and mix between the other states so as to make the rest of the spectrum thermal. The chosen tower of states are scars in $H = H_{sym} + H_{SG} + H_A$.

Our framework makes extensive use of the generators of the Lie algebra of the symmetry group $G$, which furnish a natural set of spectrum generating operators (SGOs) with ‘raising/lowering action’ (for example, operators $\{Q^+, Q^-, Q^z\}$ associated with an SU(2) symmetry have the SGA: $[Q^z, Q^\pm] = \pm Q^\pm$). We obtain scarred models via a three step process:

- First, $H_{sym}$ contains multiplets of degenerate eigenstates — tunnels — that transform as irreducible representations (irreps) of the symmetry $G$. Each multiplet is labeled by its eigenvalues under the Casimir operators of $G$, and states within the multiplet are distinguished by their eigenvalues under the generators of $G$ in the Cartan subalgebra. Raising operators connect between the states in a multiplet.

As an example, an eigenstate $|\psi_0\rangle$ of an SU(2) symmetric $H_{sym}$ is labeled also by its eigenvalues under the Casimir $Q^2$, and the Cartan generator $Q^z$. Then, $|\psi_n\rangle = (Q^+)^n|\psi_0\rangle$ will be a degenerate eigenstate with the same $Q^2$ but different $Q^z$ eigenvalue since $[H_{sym}, Q^z] = [Q^2, Q^z] = 0$, and $[Q^2, Q^+] = Q^z$.

\[\textbf{FIG. 1. Schematic sketch of the tunnels-to-towers framework for obtaining scars.}\]

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1 We note that $H_{sym}$ by itself is not considered to be scarred because features such as the presence of low entanglement eigenstates result from symmetries of $H_{sym}$; indeed thermalization (and ETH) is always discussed with reference to symmetry appropriate equilibrium ensembles.
• Next, tunnels in \( H_{\text{sym}} \) can be promoted to equally spaced ‘towers’ of non-degenerate eigenstates in the Hamiltonian \( H = H_{\text{sym}} + H_{SG} \). Here \( H_{SG} \) is typically chosen to be a linear combination of the generators in the Cartan subalgebra, which commute with and share the eigenstates of \( H_{\text{sym}} \), but have a SGA with the raising operators (for example, \( H_{SG} = \omega Q^z \) gives the states \( |\psi_n\rangle \) energy \( E_n = E_0 + n\omega \) because \( [H_{SG}, Q^z] = \omega Q^z \)).

In other words, even though the addition of \( H_{SG} \) breaks the symmetry \( G \), the eigenstates of \( H \) and \( H_{sym} \) are still the same and only their energy eigenvalues are different: in particular, degenerate tunnels of states become non-degenerate towers\(^2\).

We will also discuss models where the scar tower does not transform as an irrep of the symmetry and/or where \( H_{SG} \) is not a generator of the symmetry but still has a SGA with the raising operators. This is possible when \( H_{sym} \) has an expanded symmetry, which allows \( H_{sym} \) to be simultaneously diagonalized with \( H_{SG} \) and have tunnels of degenerate eigenstates that do not transform as an irrep.

• Finally, to make \( H \) a scarred model, we introduce symmetry breaking perturbations \( H_A \) that annihilate a particular low entanglement tunnel of states \( \{ |\psi_n\rangle \} \) built upon a particular low entanglement ‘base state’ \( |\psi_0\rangle \). \( H_A \) can typically be chosen to be generic enough to mix states across the various symmetry sectors of \( H_{sym} \) so as to make the rest of the spectrum generic and thermal. In all,

\[
H = H_{\text{sym}} + H_{SG} + H_A
\]  

obeys the MLM condition (2), and has towers (or pyramids) of scar states generated by raising operators of the non-Abelian symmetry \( G \) acting on a low entanglement base state \( |\psi_0\rangle \) which is an eigenstate of each of the three terms in \( H \).

This three-step process is schematically illustrated in Fig. 1. Our picture applies to several exactly solvable scarred models in the literature, and also furnishes a natural way to get many new scarred Hamiltonians derived from various non-Abelian and \( q \)-deformed non-Abelian symmetries. In what follows, we flesh out the ingredients for our framework in more detail in Section II. We then discuss two qualitatively distinct families of scars. In the first, discussed in Section III, the scarred eigenstates inherit the parent symmetry and transform as a single irreducible representation of \( G \) (or \( G_q \)). These represent generalizations of perturbed \( \eta \)-pairing models that have been discussed in the literature [34, 35]. The second, discussed in Section IV, is a generalization of various AKLT like models where the scars do not inherit the symmetry \( G \). However, as we discuss, these can be viewed as arising from parent Hamiltonians with an enhanced symmetry group larger than \( G \). We conclude in Section V, and present various technical details in a series of appendices.

II. INGREDIENTS OF THE FRAMEWORK

We now discuss in more detail our framework for constructing families of Hamiltonians with towers of scarred states. For specificity, we will always consider a one-dimensional chain with \( L \) sites, with a spin -\( S \) degree of freedom (i.e. a \((2S + 1)\)-state Hilbert space) on each site\(^3\). We denote the physical spin operators on site \( j \) as \( S_j^x, S_j^z \), with

\[
S^\pm = \sum_j S_j^\pm, \quad S^z = \sum_j S_j^z, \quad (5)
\]

where \( S_j^\pm \) are the usual spin raising and lowering operators on site \( j \) and \( S_j^z \) measures the \( z \) polarization of the spin. We refer to the resulting SU(2) algebra as the spin-SU(2) algebra, and to any associated symmetry in our model as a spin-SU(2) symmetry.

We will consider Hamiltonians of the form in Eq. 4, where the different pieces needing elaboration are \( H_{sym}, H_{SG} \), and \( H_A \). The tower of scars is built upon some low-entanglement ‘base state’ \( |\psi_0\rangle \) by acting with raising/lowering ‘ladder’ operators \( Q^\pm \) associated with a non-Abelian symmetry \( G \) of \( H_{sym} \). Note that \( H_{sym} \) will not generically have spin-SU(2) symmetry and the \( Q \) operators will generally be distinct from the physical spin operators in Eq. (5).

A. Lie Algebras, Raising Operators and \( H_{SG} \)

Our first task is to characterize a suitable set of ladder operators, \( Q^\pm \), associated with the Lie algebra of a non-Abelian symmetry \( G \) in \( H_{sym} \).

We begin with symmetries \( G \) that act as a product of onsite symmetries, and raising operators that can be expressed as linear combinations of the form,

\[
Q^\pm = \sum_i e^{ik r_i} (Q_i^\pm), \quad Q^z = \sum_i Q_i^z. \quad (6)
\]

where \( k \) is a momentum index, and \( Q_i^\pm \) are raising/lowering operators associated with site \( i \). The operators \( \{Q_i^\pm, Q_j^\pm\} \) are derived from the local generators of the symmetry acting on site \( i \). To ensure that the scar

\(^2\) More generally, we will also consider larger non-Abelian symmetries (such as SU(3)) where the eigenspectra of the multiplets may have more complex ‘pyramidal’ relations.

\(^3\) It is easy to see that the general philosophy of our constructions apply \textit{mutatis mutandis} to systems in higher dimensions.
Notice that symmetry that need not have spin rotation invariance. This choice of operators is invariant under \(G\) and also with operators analogous to the \(G\llcorner\) AKLT model \([11, 27, 32, 33]\) and the spin-1 XY model \([28, 29, 32, 33]\). In a chain with \(L\) sites, these \(\{Q^\pm, Q^z\}\) operators effectively isolate a reducible SU(2) sub-algebra. The remaining generators of the spin-SU(2) algebra can always be combined to furnish one or more \(\{Q^\pm, Q^z\}\) states, and annihilates all other states. This is an example of an embedded SU(2)-sub-algebra of SU(3), and the Gell-Mann matrices provide a natural basis for these embedded sub-algebras of which the choice described in Eq. 8 represents one.

It is natural to also consider other onsite raising operators of the form \(Q^i_+ = (S^i_+)^n\) for \(1 \leq n \leq 2S\). However, for \(n < 2S - 1\) and \(S > 3/2\), these operators do not describe an SU(2) algebra but rather form larger Lie group symmetries. These will be discussed in detail in Sec. III B, but here we review some general features about Lie groups to discuss how these naturally furnish various raising operators.

For any continuous non-Abelian symmetry group \(G\), we can find operators \(Q^\pm\) and \(H_{SG}\) chosen from the associated Lie algebra that satisfy commutation relations corresponding to an SGA. The parent Hamiltonian \(H_{sym}\) is invariant under \(G\), meaning that it commutes with all the generators of \(G\) and also with operators analogous to the total spin, known as the Casimir operators \(C\) for \(G\). Eigenstates of \(H_{sym}\) necessarily come in degenerate multiplets ("tunnels") with the same eigenvalues of \(C\). In this case we find a family of raising operators \(\{Q_\alpha\}\) that connect between the degenerate states, and there are several physically distinct choices for \(H_{SG}\) leading to distinct 'pyramid' structures in the eigenspectrum.

In more detail, consider an \(N\) dimensional semi-simple Lie algebra, with generators \(X_\mu\) where \(\mu = 1, \cdots N\) \([N = m^2 - 1\) for SU(\(m\)]). We denote by \(Q^\mu_\nu\) with \(\mu = 1, \cdots R\) a maximal linearly independent set of commuting Hermitian generators that can be diagonalized simultaneously, called the Cartan subalgebra (CSA). \(R\) is known as the rank of the algebra. The \(N - R\) generators that are not in the CSA can be rearranged into pairs of raising and lowering operators of different SU(2) subalgebras. We denote these collectively as \(\{Q_\alpha\}\), where \(Q_\alpha^\mu = Q_{-\alpha}\). They satisfy the commutation relations

\[
[Q^\mu_\nu, Q_\alpha] = \alpha_\mu Q_\alpha
\]

Here \(\alpha\) is an \(R\)-component vector, known as a root. The set of raising operators can be described by choosing \(\alpha\) to be a positive root, meaning that we include \(\alpha\) but not \(-\alpha\), and that if \(\alpha + \beta\) is a root, with \(\alpha\) and \(\beta\) both positive roots, then \(\alpha + \beta\) is also a positive root.

In a chain with \(L\) sites, we may therefore choose

\[
H_{SG} = \sum_{\mu=1}^{R} h_\mu \sum_j Q^\mu_{\alpha,j}
\]

(10)

\(\text{to be a linear combination of the generators in the CSA. The raising operators}

\[
Q_\alpha = \sum_j Q_{\alpha,j}
\]

(11)

\((\text{with } \alpha \text{ a positive root})\) then satisfy the desired SGA

\[
[H_{SG}, Q_\alpha] = \omega_\alpha Q_\alpha
\]

(12)

where \(\omega_\alpha = \sum_\mu h_\mu \alpha_\mu\). We note that for \(R > 1\) there are multiple linearly independent choices of the coefficients \(h_\mu\), which in general exhibit different spectra for the scar states.

It is convenient to note that the raising and lowering operators obey the commutation relations

\[
[Q_\alpha, Q_\beta] \propto \begin{cases} 
  h_\alpha & \text{if } \alpha = -\beta \\
  Q_{\alpha+\beta} & \text{if } \alpha + \beta \text{ is a root, and } \alpha \neq -\beta \\
  0 & \text{otherwise}
\end{cases}
\]

(13)

where we have defined

\[
h_\alpha = \sum_{\mu} \frac{\alpha_{\mu}}{\alpha \cdot \alpha} Q^\mu_\alpha.
\]

(14)

To summarize: a subset of the generators of a Lie algebra can always be combined to furnish one or more pairs of raising and lowering ladder operators, \(Q_\alpha\), associated with embedded SU(2) subalgebras. The remaining generators \(Q^\mu_\nu\) form the Cartan subalgebra and have spectrum-generating commutation relations with \(Q_\alpha\) (cf. Eq. 9). When \(H_{SG}\) is chosen to be as a linear combination of \(Q^\mu_\nu\), as in Eq. (10), then \(H_{SG}\) can be simultaneously diagonalized with \(H_{sym}\) and the Casimir \(C\), and has
spectral generating commutation relations with \( Q_\alpha \) (cf. Eq. (12)). This immediately implies that specific multiplets of eigenstates of \( H_{\text{sym}} \) with the same eigenvalue of \( C \) but different eigenvalues of \( H_{SG} \) are degenerate. Each of these multiplets forms a “tunnel” in the spectrum of \( H_{\text{sym}} \) that transforms as a single irreducible representation of \( G \), and acting with \( Q_\alpha \) moves between different states in the tunnels (Fig. 1a). When \( H_{SG} \) is added to the Hamiltonian, the degeneracies are broken and the eigenstates in the tunnels acquire energy spacings that are integer superpositions of \( \omega_\alpha \) (cf. Eq. (12)), thereby getting promoted to “towers” (or pyramids) of states (Fig. 1b)). The final step, discussed in the next two subsections, is to add a term \( H_A \) to the Hamiltonian that annihilates a particular “base state” \( |\psi_0\rangle \); \( H_A \) generically breaks all symmetries and mixes between all other states so as to give a thermal spectrum with the chosen states embedded as low entanglement scars.

Before leaving this section, we note two further points elaborated on later. First, it is necessary in some cases, especially in our discussion of generalized AKLT models in Sec. IV, to pick \( H_{SG} \) operators that cannot be expressed in terms of the generators of the CSA, but nevertheless have the desired “raising action” in their commutation relations with \( Q_\alpha \). For example, the total spin-\( z \) operator \( S^z \) obeys the commutation \( [S^z, Q^+] = 2SQ^+ \) for the “raise by \( 2S \)” \( Q^+ \) operators in Eq. (8), but \( S^z \) is linearly independent from \( Q^z \) and does not commute with \( Q^z \) for \( S > 1 \). In these cases, the parent Hamiltonian \( H_{\text{sym}} \) generally has a larger symmetry, so that its eigenstates can still be simultaneously diagonalized with \( H_{SG} \) and the picture of tunnels to towers still applies — however, the states in the tower of scars need not have a definite eigenvalue under \( C \) and are not contained within a single irreducible representation of \( G \). Second, a different context in which operators with suitable commutation relations emerge naturally is in the context of \( q \)-deformed Lie algebras. We discuss the example of \( q \)-deformed SU(2) in detail in Sec. III.C. Importantly, the \( q \)-deformation leaves the commutator between the raising operators \( S^+ \) and \( S^z \) unchanged. However, there is a key difference relative to the Lie algebra case: the commutation relation between the SU(2) raising and lowering operators is altered, such that \( Q_i \) is not a single-site operator but has ‘tails’ to the right and left of \( i \).

B. Base state \( |\psi_0\rangle \)

In order to construct our candidate scar tower, the next ingredient we need is to select a specific multiplet of degenerate tunnel states in \( H_{\text{sym}} \) that will get promoted to form a scar tower. In order for the scars to have low entanglement, the tower should be built by acting with the raising operators \( Q_\alpha^n \) on a particular low entanglement “base” state \( |\psi_0\rangle \). As discussed above, we will require \( |\psi_0\rangle \) to be an eigenstate of \( H_{\text{sym}} \) and \( H_{SG} \). In general, the scar space consists of a discrete set of states of the form:

\[
|\psi_{n_1,n_2,...,n_L}\rangle = Q_{\alpha_1}^{n_1}...Q_{\alpha_2}^{n_2}Q_{\alpha_1}^{n_1}|\psi_0\rangle
\]

where \( \alpha_i \) are positive roots, \( 0 \leq n_i \in \mathbb{Z} \) and \( k = \frac{1}{2}(N - R) \). It is important to note that, as for SU(2), in general on each site \( Q_{\alpha}^{n} = 0 \) as an operator: any state can be raised by at most a fixed amount in any direction. More generally, Eq. (13) implies that any product of positive roots must be 0 as an operator when raised to a sufficiently large power. Thus the number of states in our scar tower grows at most polynomially with the chain length \( L \).

The base states that we consider come in two types. First, \( |\psi_0\rangle \) can be a low-entanglement eigenstate of \( H_{\text{sym}}, H_{SG} \) and the relevant Casimir operators. For \( G=\text{SU}(2) \), one simple choice is the maximally spin-polarized state which is the only state in the symmetry sector labeled by \( (Q = Q_{\text{max}}, s = -Q_{\text{max}}) \). Acting with the raising operator on this state \( n \) times generates the state in the tower that lives in the unique symmetry sector labeled by \( (Q = Q_{\text{max}}, s = -Q_{\text{max}} + n) \). For general \( G \), the analog of the polarized state is obtained as follows. We will work in a basis \( \{|w\rangle\} \) of simultaneous eigenstates of all \( Q_\mu^\pm \). (This is analogous to working in the basis of \( \sigma^z \) eigenstates in the SU(2) case.) Here \( w \), known as the weight vector, describes the eigenvalues of the \( Q_\mu \), via

\[
Q_\mu^w = w_\mu |w\rangle \, .
\]

The commutation relations (9) imply that

\[
Q_\alpha |w\rangle \propto |w + \alpha\rangle
\]

i.e. acting with \( Q_\alpha \) on a state \( |w\rangle \) “raises” the eigenvalue of \( Q_\alpha^w \) by an amount \( \alpha_\mu \) (which can be 0 for some choices of \( \alpha_\mu \)), while preserving the value of the Casimirs. Note that the coefficient of proportionality can be 0, in which case \( |w + \alpha\rangle \) is not a state in our Hilbert space. There is always a unique “lowest weight” state \( |w_{\text{min}}\rangle \) such that \( Q_\alpha |w_{\text{min}}\rangle = 0 \) for any negative root \( \alpha \). (The negative roots are those root vectors that are not positive roots, and are the analog of the lowering operators \( Q^- \) for the SU(2) case.) Thus more generally, we may take \( |\psi_0\rangle = \prod \langle w_{\text{min}},i \rangle \) to be a product of lowest-weight states on each site in our system. By definition, this is an eigenstate of the many-body Casimirs and all \( Q_\mu \). In this case, the scar space contains all states in some irreducible representation (irrep) of the Lie algebra \( G \). The full Hamiltonian can be viewed as a perturbation of the \( G \)-invariant Hamiltonian \( H_{\text{sym}} \) in which the symmetry is generically broken, but preserved in a non-generic way in exactly one of the irreps. We note that for a system with \( L \)
sites, the maximum number of states in any such representation grows only polynomially with $L$, guaranteeing that our scar subspace is sub-extensive. Base states of this form (with $G =$SU(2)) are relevant to the spin-1 $XY$-model [28, 32, 34, 35], as well as the $\eta$-pairing states of the Hubbard model and other electronic models after appropriate mappings from spin lattices to electronic models [35].

Second, we may choose $|\psi_0\rangle$ to be an eigenstate of $H_{SG}$, but not of the relevant Casimirs. In this case the scar pyramid is not contained within a single irreducible representation of $G$, and the associated parent Hamiltonian $H_{sym}$ must have additional degeneracies not explained by the symmetry $G$. This scenario arises in various AKLT-like model Hamiltonians exhibiting exact quantum scars.

We now argue that the states $|\psi_{n_1,n_2,...,n_k}\rangle$ are low entanglement eigenstates of $(H_{sym} + H_{SG})$ and hence good candidate scar states once $H_A$ is added. First, they are eigenstates since

\[
(H_{sym} + H_{SG})|\psi_{n_1,...,n_k}\rangle = (H_{sym} + H_{SG})Q^{n_k}_{\alpha_k}...Q^{n_1}_{\alpha_1}|\psi_0\rangle = Q^{n_k}_{\alpha_k}...Q^{n_1}_{\alpha_1}(H_{sym} + \sum_{i=1}^{k} n_i \omega_{\alpha_i} + H_{SG})|\psi_0\rangle = (E_0 + \sum_{i=1}^{k} n_i \omega_{\alpha_i})Q^{n_k}_{\alpha_k}...Q^{n_1}_{\alpha_1}|\psi_0\rangle, \tag{18}
\]

where the third line follows from Eq. 12 and the fact that $[H_{sym}, Q_{\alpha_k}] = 0$, and the last line follows from the fact that we require $|\psi_0\rangle$ to be an eigenstate of $H_{SG}$ and $H_{sym}$ with eigenvalue $E_0$.

Second, the states $|\psi_{n_1,...,n_k}\rangle$ all have entanglement that grows at most logarithmically in the subsystem size, provided that $|\psi_0\rangle$ has low entanglement. To see this, observe that if $|\psi_0\rangle$ has finite (or $\log(L)$) entanglement, it can be approximated (up to exponentially small corrections) by a matrix product state with bond dimension $d$ for some $d$ that is finite (or $\log(L)$). In fact, the choices of $|\psi_0\rangle$ that we use here will all be exact matrix product states. Further, for all choices of $Q^\dagger$ operators considered in this work - for Lie algebras and $q$-deformed Lie algebras - the operator $(Q^\dagger)^n$ can be expressed as a matrix-product operator (MPO) of bond dimension $n+1$. We show this in Appendix A, by generalizing an argument due to Mougalitya et al in Ref. [27]. Thus the state $Q^{n_k}_{\alpha_k}...Q^{n_1}_{\alpha_1}|\psi_0\rangle$ has entanglement entropy of at most $S \sim \log(d) + \sum_{k} \log(n_k + 1)$. Since the maximum possible value of $n_k$ grows polynomially with $L$, we see that states in our scar tower have entanglement entropy that scales at most logarithmically, rather than linearly, with $L$. This is a defining characteristic of a quantum scar eigenstate.

\section{Annihilation operators $H_A$}

Finally, our construction requires an operator $H_A$ that behaves like a generic, thermal Hamiltonian on the non-scarred eigenstates, but with the special property that

\[
H_A|\psi_{n_1,...,n_k}\rangle = 0 \tag{19}
\]

for any $\{n_\mu\}$ - i.e. it annihilates all states in the scar tower/pyramid.

In general, we will consider two types of $H_A$ operators. The first is of the Shiraishi and Mori form in Eq. (1) which requires a set of local projectors $\{P_i\}$ which annihilate all scar states such that for all $i$ and any set of powers $n_\mu$,

\[
P_i(Q^{n_k}_{\alpha_k}...Q^{n_1}_{\alpha_1})|\psi_0\rangle = 0 \tag{20}
\]

In general, we will restrict ourselves to translation-invariant $h_i$ in Eq. (1), to ensure that eigenstates of $H_{SM}$ are not many-body localized. By choosing these $h_i$ operators sufficiently generically and with sufficiently large (but finite) range $r$, quite generally we expect that $H_A^{SM}$ can be chosen to be ergodic on those states that it does not annihilate.

In many cases, appropriate projectors $P_i$ can be deduced from the properties of the group, say if the scar states are chosen to have the maximum possible eigenvalue under the Casimir. For example, for scarred eigenstates, but with the special property that

\[
P_i(Q^{n_k}_{\alpha_k}...Q^{n_1}_{\alpha_1})|\psi_0\rangle = 0 \tag{20}
\]

At this point, it is also worth commenting on the role of the momentum $k$ in defining SU(2) generators as in Eq. (6). A priori, none of the properties discussed here depend on the choice of $k$. First, the commutation relations are invariant under locally re-defining:

\[
Q_i^+ \rightarrow e^{i\alpha_i} Q_i^+, \quad Q_i^- \rightarrow e^{-i\alpha_i} Q_i^-, \tag{21}
\]

and thus under changes in $k$. It follows that $k$ also does not affect the eigenvalue of $Q^2$. Nevertheless, the scar models that we discuss have particular values of $k$, for example $k = \pi$ for the spin-1 $XY$ and AKLT models. This is because the states in the scar tower, and hence
the choices of annihilating projectors $P_{i,i+1}$, will be $k$-dependent. In general, for certain choices of the momentum $k$, the $P_{i,i+1}$ may not have a simple, physical form in terms of the underlying spin operators.

The second type of term that we include in $H_A$ are “as a sum” annihilators. These are operators of the form

$$H^\Sigma_A = \sum_i \beta_i O_i \tag{22}$$

where $O_i$ is a local operator centered at site $i$ which does not on its own annihilate the scar tower. In this case there is no freedom to adjust the relative coefficients $\beta_i$ at different sites, since only specific superpositions annihilate all scar states. Including such operators is sometimes necessary for understanding the structure of scars in a given model; for example, Ref. [32] worked out a particular $H^\Sigma_A$ for the AKLT model. In other cases, including such terms can lead to physical and potentially experimentally realizable examples of Hamiltonian with scars, such as the one in Eq. (25) presented in Ref. [35]. Additionally, in some cases Hamiltonians of the SM form, Eq. (1), annihilate not only the desired scar tower, but also some of the states outside of the scar tower. Thus in order to ensure that the only non-ergodic states in our spectrum are the scar states, it is also useful to include “as a sum” annihilators in $H_A$.

In order to identify the scarred models described here, we have carried out an exhaustive search for the possible contributions to $H_A$. Specifically, we present a general algorithm which, given a particular set of ‘target’ states, constructs Hamiltonians for which the target states are eigenstates. This is a generalized version of the covariance-matrix algorithm presented in Ref. [40], and we recapitulate some of the main points of the algorithm for completeness. (This method can also be useful for identifying $H_{SG}$ and $H_{sym}$.)

Consider any $m$-dimensional linear space of Hermitian operators of interest $\mathcal{H}$ and construct a Hermitian basis $\{|h_\alpha\}$ for this space. Then, given a target state $|\psi\rangle$, the null space of the $m$ by $m$ matrix

$$C_{\alpha\beta}^{\psi} = \frac{1}{2} \langle \psi | h_\alpha h_\beta + h_\beta h_\alpha | \psi \rangle - \langle \psi | h_\alpha | \psi \rangle \langle \psi | h_\beta | \psi \rangle \tag{23}$$

corresponds to the space of Hermitian operators in $\mathcal{H}$ for which the state $|\psi\rangle$ is an eigenstate. That is, from any vector $\tilde{\psi}$ in the null space, we can construct a Hermitian operator $\sum_\alpha c_\alpha h_\alpha$ with $|\tilde{\psi}\rangle$ as an eigenstate. Because the covariance matrix has non-negative eigenvalues, the null space of a sum of covariance matrices $C_{\alpha\beta}^{(\psi_n)}$ for multiple states $|\psi_n\rangle$ corresponds to the space of Hermitian operators in $\mathcal{H}$ that have all the $|\psi_n\rangle$ as eigenstates. Finally, if one desires Hamiltonians that annihilate the target states, such as $H_A$, then dropping the $-\langle \psi | h_\alpha \psi \rangle \langle \psi | h_\beta | \psi \rangle$ piece of the covariance matrix suffices.

The dimension $m$ of the covariance matrix depends on the size of the space of interest $\mathcal{H}$, but is often quite small. For example, the space of translationally invariant sums of at-most-range-2 operators is just $(2S+1)^2((2S+1)^2-1)$ dimensional, which is independent of $L$. Thus, the null space of the covariance matrix can be computed very quickly. More computational effort is required to calculate the elements of the covariance matrix, and this calculation scales with the size of the eigenstates $|\psi\rangle$. However, in the case of translationally invariant models and states, calculations on a small size chain can capture the null space of the infinite $L$ covariance matrix. Further, when $|\psi_n\rangle$ has a matrix product state (MPS) representation (as is the case for all of the scar states we discuss), MPS techniques are useful to calculate the elements of the covariance matrix.

A complementary algorithm for obtaining as-a-sum annihilators was discussed in reference [35], which specialized to scar towers and translationally invariant operators and relied on matrix product methods. We emphasize that the covariance-matrix algorithm above does not need such specializations, and hence can be used for a wider class of target states and Hamiltonians where matrix product methods may not be readily amenable. Relaxing the restriction on translation invariance allowed us to discover a wider class of nearest-neighbor models with the spin-1 AKLT scar states as eigenstates than had been reported previously in the literature; we discuss this example and its generalizations in Appendix B. We note that this method can also be used to directly search within different classes of operators that may be of interest to different experimental setups. For example, $h_\alpha$ could be chosen to be a staggered field, $h_\alpha = \sum_i S_i^z (-1)^i$ or a particular kind of two or three body interaction. This method is also general enough to find examples of Hamiltonians that embed any specific set of target states of interest – that may or not be derived from symmetries – and hence can be used to construct special ‘one-off’ models with scars.

At this point, we have discussed all the ingredients that enter our framework for constructing scars from symmetries, specifically: (i) $H_{sym}$ with a non-Abelian symmetry $G$, (ii) the ladder operators $Q^\pm$ derived from embedded SU(2) sub-algebras of the Lie algebra, (iii) choices for $H_{SG}$ that may or may not be built from the CSA of the Lie algebra, (iv) choices for the base-state $|\psi_0\rangle$ that may or may not be an eigenstate of the Casimirs of $G$ and (v) choices for $H_A$ that annihilate the tower of scar states. We note that once a particular tower of states has been identified by the action of $Q^\pm$ on $|\psi_0\rangle$, then $H_A$ is the most important piece since it ensures that the Hamiltonian acts non-generically only on the scarred manifold but is well thermalizing on the rest of the spectrum. Indeed, in many cases, the simplest choice of $H_{sym} = 0$ works. Likewise, while $H_{SG}$ is used to give different energies to the scar states, this is not ‘required’ per se and models with degenerate low entanglement are still scarred. In the next two sections, we present several examples of existing and new scarred Hamiltonians that lie within our framework.
III. SYMMETRIC SCARS

In this section, we discuss several examples of models in which the scarred subspace transforms as an irreducible representation of $G$ (or a $q$-deformed version thereof), even though the Hamiltonian as a whole is not invariant under the symmetry. In all such models, the scar tower is obtained by acting with raising operators derived from the generators of $G$ on a ‘base’ low entanglement state that has a definite eigenvalue under all the Casimir operators of $G$ as well as the generators of the Cartan subalgebra of $G$. Section III A presents various examples, several of which have been presented in the literature previously, where the scars are derived from an SU(2) or “η-pairing SU(2)” symmetry. Section III B generalizes to higher Lie groups, while Section III C considers models with $q$-deformed non-Abelian symmetries. For most of our examples, the base state will be a state with maximum eigenvalue under the Casimir, such as a spin-polarized state for SU(2) or its analog for general $G$, but we also present new examples of scar towers built on non-polarized states in Sec. III A 3.

A. Symmetric scars from SU(2) symmetry

1. Spin-SU(2) symmetry

We start with a particularly simple example where the symmetry group $G$ is the spin-SU(2) symmetry represented by spin-1/2 operators on each site. Here $Q^x = S^x$, the only generator in the Cartan subalgebra is $Q^z = S^z$, and the Casimir is $Q^2 = S^2 = 1/2(S^+S^- + S^-S^+) + (S^z)^2$. Any Hamiltonian $H_{\text{sym}}$ with spin-SU(2) symmetry has a tunnel of $(L + 1)$ degenerate states built upon the polarized base state $|\psi_0\rangle = |↓↓↓\cdots↓\rangle$ by acting with $S^+$. Each of these has maximal $S^2$ eigenvalue but different $S^z$ eigenvalues, and take the form

$$|\psi_n\rangle = (Q^+)^n|\psi_0\rangle \propto Q^2 = \frac{L}{2} + n, \quad Q^z = -\frac{L}{2} + n.$$ 

Each $|\psi_n\rangle$ is the unique eigenstate in a particular symmetry sector characterized by $(Q^2 = Q_{\text{max}}(Q_{\text{max}} + 1), Q^z)$ eigenvalues. As discussed above, the form of $Q^+$ ensures that these states have at most logarithmic entanglement.

The degeneracy of these states can be lifted by adding a term $H_{\text{SG}} = \Omega S^2$ to $H_{\text{sym}}$, which promotes the tunnels to towers. Finally, we can consider a Shiraishi-Mori type $H_A$, as in Eq. 1, with projectors onto two-site singlets on neighboring sites $P_{i,i+1} = (1/4 - \tilde{S}_i \cdot \tilde{S}_{i+1})$. Because the $|\psi_n\rangle$ have maximal total spin, they are annihilated by each of these singlet projectors. Indeed, Ref. [18] constructed a model of ‘perfect scars’ of exactly this form:

$$H = \Omega \sum_i S^z_i + \sum_i V_{i,i+1} P_{i,i+1}$$

where $V_{i,j} = \sum_{n,m} J^{n,m}_{i,j} S^m_i S^n_j$ is an arbitrary operator that is used to break the spin-SU(2) symmetry. In this “perfect scar” model, $Q^+ = S^z$, $H_{\text{sym}} = 0$, $H_{\text{SG}} = \Omega S^2$, and $H_A = \sum_i V_{i-1,i+2} P_{i,i+1}$. Note that even though $H_{\text{sym}} = 0$, the action of $H_A$ makes the model well thermalizing outside the scarred subspace, and the scarred states still inherit the SU(2) algebra.

A different example with the same maximal spin scar states is given in reference [35]:

$$H = \sum_i J_1 \tilde{S}_i \cdot \tilde{S}_{i+1} + J_2 \tilde{S}_i \cdot \tilde{S}_{i+2} + D \tilde{z} \cdot (\tilde{S}_i \times \tilde{S}_{i+1})$$

(25)

Unlike the previous model, Eq. 24, this model has a non-trivial $H_{\text{sym}} = J_1 \tilde{S}_i \cdot \tilde{S}_{i+1} + J_2 \tilde{S}_i \cdot \tilde{S}_{i+2}$ with spin-SU(2) symmetry, but has $H_{\text{SG}} = 0$ so that all the scars are degenerate (one could, of course, equally well add a term of the form $H_{\text{SG}} = S^z$). The final term, $H_A = D \sum_i \tilde{z} \cdot (\tilde{S}_i \times \tilde{S}_{i+1})$ breaks the SU(2) symmetry and annihilates the scars, but it is not of the SM form since it only annihilates the scar states as a complete sum, whereas previously each local projector individually annihilated the scar states.

2. Q-SU(2) symmetry

Next, we consider a model where the operators $\{Q^x, Q^y\}$ satisfy SU(2) commutation relations, but are distinct from the spin-SU(2) operators. In particular, we can choose $Q^x, Q^y$ according to Eq. 8 with $k = \pi$ and spin $S = 1$. As before, we use the operators $Q^+$ to construct scar states built upon a base state that is an eigenstate of both $Q^2$ and $Q^z$ so that all of the scars share the same eigenvalue of $Q^2$, but are distinguished by their eigenvalues under $Q^z$.

A particular example of this kind in furnished in the spin-1 XY model:

$$H = \sum_i J(S^z_i S^z_{i+1} + S^y_i S^y_{i+1}) + J_3(S^z_i S^z_{i+3} + S^y_i S^y_{i+3}) + h S^z_i + D(S^z_i)^2.$$ 

(26)

The scars are built by the action of $Q^+$ on the fully polarized down state $|\psi_0\rangle = |↓↓↓\cdots↓\rangle$. Note that the first term $\propto J$ breaks Q-SU(2) symmetry and annihilates the scars, the term $h S^z$ acts as $H_{\text{SG}}$ and gives energy to the scars, while the term $\propto D$ commutes with $Q^2$ and $Q^z$. The third neighbor term is added to further break a non-local SU(2) symmetry that is present in 1D - this nonlocal symmetry has a ladder operator that is the same as $Q^+$ except that it replaces $(-1)^i \rightarrow e^{i\pi S^z_i}$. This ladder operator also generates the same scar tower starting from the same base state, so the scar states would be alone in their symmetry sectors unless this non-local SU(2) and its Casimir are broken.

Similar physics is also at play in the η-pairing states of the Hubbard model on bipartite lattices [41].
Hubbard model has both a spin SU(2) symmetry and an independent \( \eta \)-pairing SU(2) symmetry (which plays the role of the \( Q \)-SU(2) symmetry). The \( \eta \)-pairing states have low-entanglement \cite{42}, and are the unique states in the symmetry sector of maximal \( \eta \)-pairing total spin (i.e., states with maximal eigenvalues under \( Q^2 \)). Analogous to the examples above, the Hubbard model can be perturbed by a suitable \( H_A \) to break the \( \eta \)-pairing SU(2) symmetry while preserving the \( \eta \)-pairing states as scarred eigenstates in the perturbed model \cite{34,35}; the Hirsch model furnishes a notable example \cite{35}. Strikingly, there exists a simple mapping from spin-1 models above to electronic models that allows for translation between the scar states of the spin-1 XY model and the eta-pairing scars of the Hirsch model and some related electronic models \cite{35}.

3. Scar towers from base states of non-maximal spin

The above examples, drawn from previous literature, contain scar towers generated from a fully polarized state for the base state. In each case, this meant that the scar tower transformed in an irreducible representation of \( Q \)-SU(2) with maximal spin. We emphasize, however, that maximal spin (or, more generally, extremal Casimir eigenvalues) are not necessary for scar states, though they are useful for enumerating the bond-wise annihilators.

To demonstrate this, we offer a simple example. Consider a spin-1 chain described by the Hamiltonian \( H = H_{\text{sym}} + H_{SG} + H_A \) with:

\[
H_{\text{sym}} = D(S_i^z)^2 \\
H_{SG} = \hbar S_i^z \\
H_A = \sum_i J_1(\vec{S}_i \cdot \vec{S}_{i+1})^2 + J_2((\vec{S}_i \cdot \vec{S}_{i+2})^2 + \vec{S}_i \cdot \vec{S}_{i+2}) \\
+ B_1((S_i^x)(S_{i+1}^x + S_{i+1}^y) - (S_{i+1}^x + S_{i+1}^y)(S_i^x)^2) \\
+ J_3 S_i^z S_{i+1}^z + B_2(S_i^y (S_{i+1}^y)^2 - (S_{i+1}^z)^2 S_i^z) \tag{27}
\]

This model has a scar tower, generated by acting with \( Q^+ = \sum_i \frac{1}{2}(S_i^z)^2 \), which is of the form in Eq. 8 with \( k = 0 \) and \( S = 1 \) on the base state \( |\psi_0\rangle = \frac{1}{\sqrt{2}}|0 0 \ldots 0\rangle + \frac{1}{\sqrt{2}}|0 - 0 \ldots - 0\rangle \). As promised, \( |\psi_0\rangle \) is an eigenstate of the Casimir \( Q^2 \) with eigenvalue \( L(2L+1) \) which is less than the maximal Casimir \( L/2(L/2 + 1) \).

To see that the terms act as labeled, observe that the terms with proportionality constants \( B_1, B_2, \) and \( J_2 \) all annihilate all states in this scar tower bond-wise, because every state in the scar tower has \( |0\rangle \) on every other site. Similarly, \( J_1 \) is equal to the identity plus three times the projector onto the singlet state and is hence also a bond-wise annihilator on subtracting out the identity. \( J_2 \) is another bond-wise annihilator up to a factor of the identity, as it is equal to a linear combination of the identity and a projector onto the antisymmetric spin-1 states.

We emphasize that the \( J_2 \) term is sensitive to the momentum of \( Q^+ \); further, without this term, any states of \( S^z \) being broken, so those states are also scars. We also note that Eq. (27) contains only a subset of the operators that could be added to \( H_A \) to make the model thermal. Collectively, the terms in \( H_A \) are sufficient to render all but a few of the states outside the scar space thermal, as seen in Figure 2; the fully polarized up and down states remain as eigenstates despite \( S^z \) being broken, so those states are also scars.

More generally, base states with other eigenvalues under \( Q^2 \) and \( Q^z \), such as those with eigenvalue \( (Q_{\text{max}} - p)(Q_{\text{max}} - p + 1) \) under \( Q^2 \), and \( (-Q_{\text{max}} + p) \) under \( Q^z \) for some finite \( p \), will also have low entanglement and can be used to build scar tunnels.

B. Higher Lie group symmetric scars

Another new class of examples that our symmetry-based perspective on scars makes natural is scar states associated with continuous symmetry groups \( G \) other than SU(2). As discussed in Section II, these differ from the SU(2) case in a few important ways. First, in general there are multiple choices of raising operators. Second, there are multiple choices of \( H_{SG} \), which in general satisfy commutation relations of the form (12). Depending on the choice of \( H_{SG} \), we can therefore engineer scar states.
with multiple distinct frequencies, or with exact degeneracies in their spectra that reflect the more complex Lie group symmetry.

The general idea of the construction closely parallels the SU(2) case. Choosing \(|\psi_0\rangle = \prod_i |w_{\text{min}, i}\rangle\) to be a product of the lowest weight state at each site, we have \(Q_{\alpha, i}|w_{\text{min}, i}\rangle = 0\) for any negative root \(\alpha\). This is the analog of the polarized state, and has maximal eigenvalue of \((S^z_i)^2\) by 1. Correspondingly, the eigenvalue of \(S^z\) on the state in Eq. (31) is \(-L + m + n\), while the eigenvalue of \((S^z_i)^2\) is \(L - n + m\).

We now turn to \(H_A\). At the two-site level, the states in the tower will only contain the six symmetric states \(|-\rangle\rangle, |0\rangle\rangle, |00\rangle, |+-\rangle\rangle, |+0\rangle\rangle, |+\rangle\rangle\), and \(|1,1\rangle = |\sqrt{2}((0 + 0) - 0^+)\rangle\) do not appear, and so we can use projectors onto these states as Shiraishi-Mori-type projectors. Correspondingly, there are 9 bond-wise annihilators on a given pair of sites \((i, i + 1)\) that we can use in \(H_A\):

\[
P_{1,i} = |1, -1\rangle\langle 1, -1|, \quad P_{2,i} = |1, 0\rangle\langle 1, 0|, \quad P_{3,i} = |1, 1\rangle\langle 1, 1|,
\]

\[
P_{4,i} = Q_{\alpha, i} = \prod_i Q_{\alpha, i} = |Q_{\alpha, i}\rangle_{\alpha, i}, \quad P_{5,i} = i|1, -1\rangle\langle 1, 0| + h.c., \quad P_{6,i} = |1, -1\rangle\langle 1, 1| + h.c., \quad P_{7,i} = i|1, -1\rangle\langle 1, 1| + h.c., \quad P_{8,i} = |1, 0\rangle\langle 1, 1| + h.c., \quad P_{9,i} = i|1, 0\rangle\langle 1, 1| + h.c.
\]

(34)

We also found eight “as-a-sum” annihilators through the covariance-matrix algorithm discussed above, but we will not discuss these annihilators further here.

FIG. 3. Entanglement entropy in the momentum \(k = 0\) and spatial inversion-symmetric sector of the SU(3)-scared Hamiltonian in Eq. (35) for a system of size \(L = 11\). The scar states are colored according to their values of \(m\) in Eq. (31).

The terms in Eq. (34) break SU(3) symmetry and keep the Hamiltonian from commuting with the two SU(3) Casimirs; hence taking \(H_A\) to be a linear combination of these bond-wise annihilators at each site is sufficient to
We’ve colored the scar states according to their values of \( m \) from equation 31; we have an evenly spaced tower of states for each value of \( m \), with \( n \) ranging from \( m \) to \( L \). As we increase \( m \) by one, the resulting tower has one fewer state than the previous. For the parameters chosen here, increasing \( m \) corresponds to increasing the energy by 2.3, while increasing \( n \) corresponds to decreasing the energy by .3.

1. Higher Lie symmetries from spin operators

A priori, it is not obvious under what conditions higher Lie group symmetries would arise in real solid-state systems, such as spin chains. In fact, however, we are naturally led to these if we consider raising operators of the form

\[
Q_i^+ = \frac{1}{\mathcal{N}} (S_i^z)^n \ , \ Q_i^- = (Q_i^+)^\dagger \ , \ Q_i^z = [Q_i^+, Q_i^-] \ ,
\]

for \( 1 \leq n \leq 2S \), where \( \mathcal{N} \) is a normalization constant. As noted above, for \( n = 1 \) or \( n = 2S \), with a suitable choice of \( \mathcal{N} \) the operators \( Q_i^+, Q_i^- \), and \( Q_i^z \) form an SU(2) algebra. For \( n < 2S - 1 \) and \( S > 3/2 \), however, in general Eq. (36) does not describe an SU(2) algebra. Indeed, the set \( Q_i^+, Q_i^-, Q_i^z \) is not closed under commutation. Closing these operators under commutation leads to a set of raising operators associated with a larger Lie group symmetry.

To illustrate this, we begin with \( S = 5/2 \), and \( n = 2 \) in Eq. (36). We define

\[
P^+ = [Q^z, Q^+], \quad P^- = -[Q^z, Q^-], \quad P^z = [P^+, P^-] \]
\[
R^+ = [P^+, Q^+], \quad R^- = [Q^-, P^-]
\]

Here \( P^+ (P^-) \) is a second, linearly independent operator that raises (lowers) \( S^z \) by 2, and \( R^+ (R^-) \) is a third raising operator, which raises (lowers) \( S^z \) by 4. It is convenient to change basis, defining the raising operators:

\[
Q_{\alpha_1} = \frac{13}{12\sqrt{5}} P^+, \quad Q_{\alpha_1}^z = \frac{1}{1512} (-Q^z + P^z)
\]
\[
Q_{\alpha_2} = \frac{1}{36} (Q^+ + P^+), \quad Q_{\alpha_2}^z = \frac{3037}{7560\sqrt{3}} Q^z - \frac{13}{7560\sqrt{3}} P^z
\]
\[
Q_{\alpha_3} = \frac{1}{36\sqrt{5}} R^+
\]

where we have taken \( \mathcal{N} = \frac{1}{2\sqrt{2}} \), and as usual, the lowering operators are given by \( Q_{-\alpha_j} = Q_{\alpha_j}^\dagger \). As above, the roots \( \alpha_j \) are given by Eq. (30). The 3 raising operators \( Q_{\alpha_j} \) act on our states according to:

\[
Q_{\alpha_1}[5/2,-5/2] = [5/2,-1/2] , \quad Q_{\alpha_1}[5/2,1/2] = [5/2,5/2]
\]
\[
Q_{\alpha_2}[5/2,-1/2] = [5/2,3/2] , \quad Q_{\alpha_2}[5/2,-3/2] = [5/2,1/2]
\]
\[
Q_{\alpha_3}[5/2,-5/2] = [5/2,3/2] , \quad Q_{\alpha_3}[5/2,-3/2] = [5/2,5/2]
\]

where states not shown are annihilated by the raising operator in question.

It is straightforward to check that the raising operators \( \{Q_{\alpha_j}\} \), together with the corresponding lowering operators \( \{Q_{-\alpha_j}\} \), and two diagonal generators \( Q_i^z \), obey the commutation relations of the 8 generators of the Lie group SU(3). Under the action of these 8 matrices, the 6 states in \( s = 5/2 \) split into two sets of three, which are not connected by any raising operator. Thus the representation on each site consists of one copy of the fundamental (triplet) representation of SU(3), containing the states \([5/2,-5/2],[5/2,-1/2] \), and \([5/2,3/2],[5/2,5/2] \), and a copy of the conjugate (anti-fundamental) representation, containing the states \([5/2,-3/2],[5/2,1/2] \), \([5/2,5/2] \).

For general spin \( S \) and \( n = 2 \), one can show the following. For half-integer \( S \), the relevant Lie algebra is SU\((S + 1/2)\), with the 2\(S + 1\) states on each site dividing into a copy of the \((S + 1/2)\)-dimensional fundamental representation, and a copy of its conjugate. For integer \( S \), the algebra can be divided into two sets of operators, which act only on even and odd integer spins, respectively. This leads to a Lie algebra structure SO\((S + 1)\) \(\times\) Sp\((S)\) for even \( S \), and SO\((S)\) \(\times\) Sp\((S + 1)\) for odd \( S \). For even \( S \), the Hilbert space at each site corresponds to a copy of the \((S + 1)\)-dimensional vector representation of SO\((S + 1)\), containing the even-integers, and a copy of the \( S \) dimensional fundamental representation of Sp\((S)\), containing the odd-integers. For odd \( S \) the \( S + 1 \) odd-integer spins transform in the \((S + 1)\) dimensional fundamental representation of Sp\((S + 1)\), while the \( S \) even-integer spins transform in the \( S \)-dimensional vector representation of SO\((S)\).

For these examples, though the Cartan generators \( Q_{\mu,i}^z \) all commute with \( S_i^z \), it is not in general the case that \( S_i^z \) can be expressed as a linear combination of the \( Q_{\mu,i}^z \), since it is not necessarily traceless when acting on each irreducible representation of the relevant Lie group in the Hilbert space. Thus a natural alternative to an \( H_{SG} \) of the form (10) is to take \( H_{SG} = S^2 = h \sum S_i^2 \), which satisfies an SGA commutation relation of the form (12) for all raising operators \( Q_{\alpha_i} \). In this case the value of \( \omega \) is fixed by how much \( Q_{\alpha_i} \) raises \( S^2 \). Thus all frequencies are integer multiple of the elementary frequency \( 2nh \), and in general multiple \( Q_i^+ \) operators will be associated with the same frequency.

With this choice, we find degeneracies in the scar tower characteristic of the underlying larger Lie group symmetry. For example, consider the spin-5/2 system described above, with an SU(3) symmetric scar
tower. The two operators \(Q_{\alpha_1}, Q_{\alpha_2}\) both raise \(S^z\) by 2, while \(Q_{\alpha_1} = Q_{\alpha_2}Q_{\alpha_1}\) raises \(S^z\) by 4. Taking \(|\psi_0\rangle = | -5/2, -5/2, -5/2, \ldots \rangle\) to have energy 0, we see that \((Q_{\alpha_1})^2|\psi_0\rangle\) and \(Q_{\alpha_2}Q_{\alpha_1}|\psi_0\rangle\) are linearly independent states with the same energy of 4\(h\). In contrast, in the SU(2) case, all states in the scar tower have distinct energies, since each power of \(Q^+\) applied to \(|\psi_0\rangle\) necessarily raised the eigenvalue of \(S^z\) by the same amount.

C. \(q\)-deformed towers

In the preceding sections, we considered scar states that transformed in a single irreducible representation of some group. However, we can also consider scar states transforming in representations of “\(q\)-deformed groups”. \(q\)-deformed groups have found many applications, including solving the quantum Yang-Baxter equation \([43]\), describing anyons \([44]\), and phenomenologically describing perturbations to otherwise symmetric models \([45]\). For the purposes of this work, we restrict our attention to SU\(_q\)(2), though we expect that our key results generalize to other \(q\)-deformed groups.

The characteristic feature of \(q\)-deformed groups is a parameter \(q\) that modifies the generator algebra. For example, SU\(_2\)(2) has the following algebra:

\[
[S^z, \tilde{S}^\pm] = \pm \tilde{S}^\pm \quad \text{and} \quad [\tilde{S}^+, \tilde{S}^-] = [2\tilde{S}^z]_q
\]

where

\[
[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}
\]

The deformation is such that \(q \to 1\) returns the algebra to the usual SU(2) algebra.

For real, positive \(q\), the representations of \(q\)-deformed SU(2) that satisfy the algebra share many similarities with the usual representations. The irreducible representations are \(2S + 1\)-dimensional with \(S^z\) independent of \(q\)

\[
\langle \tilde{S}^z \rangle = S^z
\]

and with

\[
(\tilde{S}^\pm)_{m'm} = \sqrt{(S \pm m|q(S \pm m + 1)|q} \delta_{m',m \pm 1}.
\]

The \(\tilde{S}^\pm\) operators are the same as \(S^\pm\) for spin \(S < 1\).

The Casimir operator that commutes with the generators and labels the multiplets is \(\tilde{S}^2 = \tilde{S}^+ \tilde{S}^- + [S^z]_q[S^2 + 1]_q\) with eigenvalues \([S^z]_q[S + 1]_q\); that such an operator commutes with the generators can be checked by explicit computation. We will also define

\[
\tilde{S}^x = \frac{\tilde{S}^+ + \tilde{S}^-}{2}, \quad \tilde{S}^y = \frac{\tilde{S}^+ - \tilde{S}^-}{2i}
\]

for use below.

However, because of the deformation, some of the usual properties of representations of Lie algebras no longer hold. In the regular SU(2) algebra, if we had a representation \(\{S^+, S^-, S^z\}\) we could form a direct product representation,

\[
\{S^+ \otimes I + I \otimes S^+, S^- \otimes I + I \otimes S^-, S^z \otimes I + I \otimes S^z\},
\]

which would also satisfy the algebra. This is how we would describe the action of SU(2) on, say, two spin-\(S\) particles. Such a set of would-be generators generally fail to satisfy the \(q\)-deformed algebra - instead, for SU\(_q\)(2), we have that the operators

\[
\{\tilde{S}^+ \otimes q^{S^z} + q^{-S^z} \otimes \tilde{S}^+, \tilde{S}^- \otimes q^{S^z} + q^{-S^z} \otimes \tilde{S}^-, S^z \otimes I + I \otimes S^z\}
\]

satisfy the deformed algebra if \(\{\tilde{S}^+, \tilde{S}^-, \tilde{S}^z\}\) do. Similarly, \(\tilde{S}^z\) acting on a chain of length \(L\) picks up ‘tails’ of diagonal operators to the left and right for each site:

\[
\tilde{S}^z = \sum_{i=1}^{L} (\tilde{S}^z \otimes I) \otimes \tilde{S}^z \otimes (\otimes_{j=i+1}^{L} q^{S^z})
\]

while \(\tilde{S}^z\) is the same as \(S^z\).

For generating scar towers, we’ll consider a single-site representation of SU\(_q\)(2):

\[
[Q_i^+, \tilde{Q}_i^+] = \pm \tilde{Q}_i^+ \quad \text{and} \quad [\tilde{Q}_i^+, \tilde{Q}_i^-] = [2Q_i^+]_q
\]

From the single-site representation, we can construct a chain-wide representation through

\[
\tilde{Q}_i^+ = \sum_{i=1}^{L} e^{\pm i \phi_i (\otimes_{j=i}^{L} q^{S^z})} \otimes \tilde{Q}_i^+ \otimes (\otimes_{j=i+1}^{L} q^{S^z})
\]

for arbitrary phase-factor \(\phi_i\). The freedom to choose an arbitrary phase factor while maintaining the commutation relations may seem surprising, as \(\tilde{Q}_i^+\) is a sum of tailed operators. Nevertheless, \((\otimes_{j=i}^{L} q^{S^z})\otimes \tilde{Q}_i^+ \otimes (\otimes_{j=i+1}^{L} q^{S^z})\) and \((\otimes_{j=i}^{L} q^{S^z})\otimes \tilde{Q}_m^+ \otimes (\otimes_{j=m+1}^{L} q^{S^z})\) commute for \(i \neq m\), and the phase factors cancel for \(i = m\), so the phases don’t affect the commutation relations.

For \(\phi_i = kr_i\), powers of these operators, \((\tilde{Q}_i^+)^n\), have a translationally-invariant MPO representation with bond-dimension \(n + 1\); see Appendix A. It is striking that the MPO representation is linear in \(n\) rather than exponential in \(n\). This means that \((\tilde{Q}_i^+)^n\) will only increase entanglement entropy by a factor of at most \(O(\log(n))\), rather than by \(n\). Thus, \(\tilde{Q}_i^+\) with an associated \(q\)-deformed symmetry and symmetric base states are good candidates for scar states with additional \(q\)-deformed symmetry relative to the Hamiltonian.

To illustrate some of these ideas, consider the following \(q\)-deformations of previously discovered models. The simplest is a \(q\)-deformation of the model with SU(2) scars in Eq. (24):

\[
H = \Omega \sum_i S_i^z + \sum_{i=1}^{L-1} V_{i-1,i+2} \hat{P}_{i,i+1}
\]
where $\tilde{P}_{i,j+1}$ is a projector onto the two-site $q$-deformed spin-singlet. Explicitly,

$$\tilde{P}_{i,j+1} = \frac{1}{|q|} (|\uparrow\downarrow\rangle - \frac{1}{|q|} |\downarrow\uparrow\rangle)(|\uparrow\downarrow\rangle - \frac{1}{|q|} |\downarrow\uparrow\rangle)$$

(50)

The new scar states are those simultaneous eigenstates of $S^z$ and $\tilde{S}^z$ generated by $S^z$ on the $S^z = -L/2$ state.

We can also modify more complicated models and extract $q$-deformed scar states. We introduce here a deformed version of the spin-1 XY model [28, 32, 34, 35] in Eq. 26,

$$H = \sum_{i=1}^{L-1} J_{i,i+1}(q) + \sum_{i=1}^{L-3} J_{3i,3i+3}(q) + \sum_{i=1}^{L} hS_i^z + D(S_i^z)^2$$

(51)

where

$$\tilde{h}(q)_{i,i+1} = (S^x_i q S^x_{i+1})(q^{-S^z_i S^z_{i+1}}) + (S^y_i q S^y_{i+1})(q^{-S^z_i S^y_{i+1}})$$

(52)

and

$$\tilde{h}(q)_{i,i+3} = (\tilde{S}^x_i q \tilde{S}^x_{i+3})(q^{-3S^z_i \tilde{S}^z_{i+3}}) + (\tilde{S}^y_i q \tilde{S}^y_{i+3})(q^{-3S^z_i \tilde{S}^y_{i+3}})$$

(53)

Here, $\tilde{S}^x$ and $\tilde{S}^y$ are deformed using the value of the deformation parameter given in the superscript of $\tilde{h}$. This model has scar states generated by a deformed version of the raising operator in Eq. (8) with $k = \pi$, $S = 1$ and with a deformation parameter of $\frac{1}{q}$:

$$\tilde{Q}^\pm = \sum_{i=1}^{L} (-1)^i (\otimes_{j=1}^{i-1} q^{2Q^j_i}) \otimes \tilde{Q}^\pm_i \otimes (\otimes_{j=i+1}^{L} q^{-2Q^j_i})$$

(54)

with $\tilde{Q}^\pm_i = \frac{1}{2}(S^\pm_i)^2$. Notice that $\tilde{Q}^\pm_i$ is not $\frac{1}{2}(S^\pm_i)^2$ because $Q_{ij}^\pm$ acts as a reducible ‘doublet’ representation of $SU(2)$ on the $|\pm\rangle$ states and a singlet on the $|0\rangle$ state, and hence its single-site $q$-deformed version is unchanged (cf. Eq. (42)). One can check directly that this $\tilde{Q}^\pm$ is a ladder operator for $SU_1(q^2)$. There’s a separate ladder operator for which $(-1)^i \rightarrow (-1)^{i+1} S^z_i$ that also generates these same scar states. This second operator is a ladder operator for a separate $SU_1(q^2(2))$ symmetry, so we must be careful to break the two different $SU_1(q^2(2)$ Casimirs associated with the different ladder operators. The tower of scar states is annihilated by the $J$ and $J_3$ terms in Eq. (51), and these terms keep the Casimirs from commuting with the Hamiltonian: the nearest neighbor term $J$ is sufficient to violate conservation of the Casimir of the first ladder operator, while the $J_3$ term violates conservation of the Casimir corresponding to the first and second ladder operators. The similarities to the discussion of the original spin-1 XY model in Eqn. 26 should be clear.

In figure 4, we plot the entanglement entropy in the $S^z = -2$ sector of the $q$-deformed XY Hamiltonian in equation 51, for $q = 1.2$ and $J = J_3 = h = D = 1$. This symmetry sector contains only a single scar state circled in orange. The scar state is $(\tilde{Q}^+)^3$ acting on the fully polarized $|---\ldots\rangle$ state with energy $E/L = 8$ on their respective ground states. While $Q_{\text{AKLT}}^+$ is the same as the raising operator associated with $Q$-$SU(2)$ discussed earlier, Eq. (8) with $k = \pi$, the projector onto this asymmetric scarred manifold does not commute with the $Q$-$SU(2)$ symmetry. Previous work showed that the spin-$S$ AKLT model also has an asymmetric tower of scarred eigenstates generated by the same $Q_{\text{AKLT}}^+$ [11, 32].

To understand how this fits with our broader symmetry based picture, we note that Refs. [32, 33] showed that the AKLT model can be deformed to a $H_{\text{sym}}$ with $Q$-$SU(2)$ symmetry while preserving the scars. Now, we generally expect $H_{\text{sym}}$ to have degenerate eigenstates labeled by the same value of the Casimir $Q^2$, but multiplets in different irreps with different values of $C$ will generically not be degenerate. However, the fact that the AKLT ground state is an eigenstate of $H_{\text{sym}}$—despite not being an eigenstate of $Q^2$—implies that tunnels of states

![FIG. 4. Entanglement entropy in the $S^z = -2$ sector of the $q$-deformed spin-1 XY Hamiltonian for $q = 1.2$ and $J = J_3 = h = D = 1$. The scar state within this sector is circled.](image)
with different values of $C$ are degenerate in $H_{\text{sym}}$ so that they can be superposed to give an eigenstate with indefinite $S^2$. This points to an expanded symmetry in $H_{\text{sym}}$ leading to a much larger set of degeneracies. By taking advantage of these, one can prepare base states that are eigenstates of $H_{\text{sym}}$ and $H_{\text{SG}}$, even if they are not eigenstates of $Q^2$ and $Q^2$. Because of such considerations, it is important to systematically study perturbations of the scarred models to connect them to high-symmetry points, and Section IV B considers families of such perturbations with continuously varying scarred states.

The scars in the AKLT model and its generalizations are a consequence of the following general structure, first discussed by Refs. [32, 33]. Consider a Hamiltonian $H = \sum_j h_{j,j+1}$, a base state $|\psi_0\rangle$ with zero energy, and a ladder operator $Q^+ = \sum_j e^{ikj}O_j$. We assume periodic boundary conditions with $kL$ being a multiple of $2\pi$; we discuss the generalization to open boundary conditions briefly in the next subsection and in more detail in Appendix F. We also assume that we can group the two-site Hilbert space into the three disjoint subspaces, $\mathcal{G}$, $\mathcal{R}$, and $M$. The subspace $\mathcal{G}$ (not to be confused with the non-Abelian symmetry group $G$ of $H_{\text{sym}}$) contains all 2-site configurations (which we will refer to as bonds) that are present in the base state. The subspace $\mathcal{R}$ contains the image of all bonds in $\mathcal{G}$ under the action of $q^+_{j,j+1} = O_j + e^{ik}O_{j+1}$, while the subspace $M$ is the complement of $\mathcal{G} \cup \mathcal{R}$. If $h_{j,j+1}$ and $q^+_{j,j+1}$ have the following general forms,

$$h_{j,j+1} = \begin{bmatrix} h_{MM} & 0 & 0 \\ 0 & \omega I & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(56)

$$q^+_{j,j+1} = \begin{bmatrix} h_{RM} & 0 & 0 \\ 0 & h_{RR} & 0 \\ q_{GM} & 0 & 0 \end{bmatrix}$$

(57)

then the model has a scar tower generated by $Q^+$ with energy spacing $2\omega$. The above result follows from explicitly calculating the commutator of $H$ and $Q^+$, as shown in Eq. (61).

That is, if $\mathcal{G}$ and $\mathcal{R}$ are disjoint, and $(q^+_{j,j+1})^2 = 0$ when acting on $\mathcal{G}$, then the Hamiltonian takes the form above, and the model is scarred. We show that the generalized AKLT models in Sec. IV A satisfy these conditions (with $\omega = 1$), and their ground states are not eigenstates of $Q^2$. Hence, they each have asymmetric scar towers.

Remarkably, we can also construct a large, continuously connected class of matrix product states (MPS) such that $\mathcal{G}$ and $\mathcal{R}$ are disjoint for $Q^+ = Q^+_\text{AKLT}$ (see Sec. IV B). Each such state can function as a base state, and we can enumerate the states in $\mathcal{G}$, $\mathcal{R}$, $M$ to construct new scarred Hamiltonians with these base states as ground states. Because the MPS are continuously connected, we can give continuous deformations between scarred Hamiltonians along which the asymmetric scarred states persist and are continuously deformed.

To demonstrate the power of this large class of states, we revisit Ref. [33]’s deformation of the spin-1 AKLT model to an integrable point along a path with (continuously varying) scarred eigenstates. In particular, we show that there are many such deformations between the spin-$S$ AKLT model and corresponding high-symmetry integrable points.

A. The generalized AKLT models

Affleck, Kennedy, Lieb and Tasaki (AKLT) introduced the spin-1 AKLT model to analytically describe the Haldane gap in integer spin chains [46]. Subsequent work discovered that the AKLT chain has fractionalized edge spins in open chains, and is a symmetry-protected topological phase with non-local string order [47, 48]. The spin-1 AKLT chain’s interesting properties prompted many generalizations, including generalizations to spin-$S$, $q$-deformed spin-$S$ [49–53], and other symmetry groups like SO($2S+1$) [54]. These generalizations are all examples of the Haldane phase with exactly known ground states. We demonstrate that these models have a second curious property in common, not directly related to the Haldane phase: they all have scar towers generated by $Q^+_{\text{AKLT}}$ on appropriate ground states.

Each term $h^S_{j,j+1}$ in the Hamiltonian $H^S = \sum_j h^S_{j,j+1}$ of a generalized AKLT model of type $\alpha$ can be expressed as a sum of projectors. For the spin-$S$, $q$-deformed spin-$S$, and the SO($2S+1$) AKLT models, we have:

$$h^S_{j,j+1} = \sum_{t=S+1}^{2S} P_{t,j,j+1}$$

(58)

$$h^S_{j,j+1} = \sum_{t=S+1}^{2S} \tilde{P}_{t,j,j+1}$$

(59)

$$h^{SO(2S+1)}_{j,j+1} = \sum_{k=1}^{S} P^{(2k)}_{j,j+1}$$

(60)

Here, the two-site operators $P^{(t)}_{j,j+1}$ and $\tilde{P}^{(t)}_{j,j+1}$ project onto total spin $t$ and $q$-deformed total spin $t$ respectively. We give their explicit forms in terms of spin operators and $q$-deformed spin operators in Appendix C. The projectors $P^{(t)}_{j,j+1}$ are SU(2) invariant, and thus $H^S$ and $H^{SO(2S+1)}$ are SU(2) invariant$^5$. In comparison, $\tilde{P}^{(t)}_{j,j+1}$ is SU$_q$(2)-symmetric for all $j$ except for $j = L$, implying that $H^S$ is SU$_q$(2)-symmetric with open boundary conditions but not periodic boundary conditions. Notice that $H^S$ and $H^{SO(2S+1)}$ agree for $S = 1$. We also emphasize here

$^5$ The larger SO($2S+1$) symmetry of $H^{SO(2S+1)}$ comes from identifying $\sum_{k=1}^{S} P^{(2k)}_{j,j+1}$ as a projector onto the $2S^2+3S$-dimensional irreducible representation of SO($2S+1$) within the direct product of two fundamental representations.
that although $H^{S_q}$ has $q$-deformed SU(2) symmetry, its scar tower is generated by the “usual” raising operator $Q^+_A$ defined in Eq. 55, and not by $q$-deformed raising operators of the form Eq. (48).

The three models have exactly known matrix product ground states. With periodic boundary conditions, the ground states are frustration-free and unique. With open boundary conditions, the regular and $q$-deformed spin-$S$ AKLT models have $(S + 1)^2$ frustration-free ground states, and the SO$(2S + 1)$ AKLT Hamiltonian has $4^S$ frustration-free ground states.

We briefly comment on the difference between the scar towers in periodic and open chains (see Appendix F for more details). Assume that we have disjoint $G$ and $R$ with $h$ and $q$ as given in Eq. (56) and (57). In periodic chains, we have

$$[H_{pbc}, Q^+] = 2\omega Q^+ + \sum_{j=1}^{L} e^{ikj} A_{j,j+1}$$

(61)

where $A_{j,j+1} = [h_{j,j+1}, q^+_{j,j+1}] - \omega q^+_{j,j+1} + \sum_{j=1}^{L} h_{j,j+1}$. Computing the operator $A_{j,j+1}$ using Eqs. (56) and (57), we see that $A_{j,j+1}$ annihilates all the 2-site configurations on sites $(j, j + 1)$ that appear in the states of the scar tower. The form in Eq. (61) matches the MLM condition in Eq. (2). In open chains, the commutator is modified to be:

$$[H_{obc}, Q^+] = 2\omega Q^+ + \sum_{j=1}^{L-1} e^{ikj} A_{j,j+1} - e^{ik\omega O_1} - e^{ikL\omega O_L}$$

(62)

The critical change is the presence of $O_1$ and $O_L$ acting on the physical edge spins, which restricts which of the ground states in open boundary conditions will be a good base state for the tower of states. We argue in Appendix F that $O_1$ and $O_L$ must individually annihilate the physical edge spins in these models, which we show occurs for $S^2$ out of $(S + 1)^2$ ground states of the regular and $q$-deformed spin-$S$ AKLT models and $4^{S-1}$ out of $4^S$ ground states in the SO$(2S + 1)$ AKLT model. This discussion of open boundary conditions is especially important for the $q$-deformed AKLT models, as the models lose their interesting $SU_q(2)$ symmetry with periodic boundary conditions.

In Appendices D and E, we prove that the $G$ and $R$ subspaces are disjoint, and that Eqs. (56) and (57) hold with $\omega = 1$ and $h_{MM} = I$ for all three models. This furnishes the proof that the spin-$S$, $q$-deformed spin-$S$, and the SO$(2S + 1)$ AKLT models have asymmetric scar towers generated by $Q_A^{\pm}$ on their respective ground states with periodic boundary conditions, while the discussion of open boundary conditions follows additionally from Appendix F.

Fig. 5 shows the eigenstate entanglement entropy of the open $q$-deformed spin-$1$ AKLT model in a fixed $S^2$ sector vs the energy density. In every $S^2 = 1 + 2m$ sector with integer $m \geq 0$, we expect a unique scar state at energy density $2m/L$ generated by the action of $(Q_A^{\pm})^m$ on the $S^2 = 1$ ground state.$^6$ The circled state at $E/L = 0.5$ is thus the predicted scar state in the $S^2 = 7$ sector.

![Fig. 5. Entanglement entropy in the $S^2 = 7$ and $\hat{S}^2 = \{7\}_q$ sector of the $q$-deformed spin-1 AKLT Hamiltonian for $q = 1.2$, $L = 12$, and open boundary conditions. The circled scar state at energy $E/L = 0.5$ is $(Q^+)^5$ on the $S^2 = 1$ ground state.](image)

### B. A large class of Matrix Product States with disjoint $G$ and $R$ subspaces

We now consider consider a broader class of MPSs that can serve as base states for towers generated by $Q_A^{\pm}$. These will also allow us to study deformations of the generalized AKLT models to $Q$-SU$(2)$ symmetric $H_{sym}$ with enhanced symmetries. In Appendix D, we show that any spin-$S$, bond-dimension $S + 1$, translationally invariant Matrix Product State (MPS) of the form:

$$|\psi_0\rangle = \sum_{m} \text{Tr}[A[m_1]A[m_2]...A[m_L]]|m_1...m_L\rangle$$

(63)

with $m_1 = 1, 2, \cdots 2S + 1$ and $A[m]$ being nonzero only on the $m$th diagonal has disjoint $G$ and $R$ under $Q_A^{\pm}$. As $Q_A^{\pm}$ raises the $z$-magnetization $S^z$ by $2S$, all but one of the bonds in $G$ are mapped out of $G$ under the action of $Q_A^{\pm}$. The $\pi$ momentum carried by the $Q_A^{\pm}$ operator furthermore ensures that the remaining bond is also mapped out of $G$. This result was useful in proving that the $q$-deformed AKLT models have scars.

More broadly, we can use the above result to construct a large family of scarred Hamiltonians with an MPS of the form given by Eq. (63) functioning as the base state

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$^6$ As follows from Appendix F, for $S = 1$ in open boundary conditions, the $S^2 = 1$ ground state is the only one hosting a tower of states generated by the action of $Q^+$. 
and $Q^+_{\text{AKLT}}$ functioning as the ladder operator. The Hamiltonians have the general form given by Eq. (56). Different choices of $\omega$ and $h_{MM}$ provide different Hamiltonians with the same scarred manifold. The choice of $\omega$ tunes the energy spacing of the scar tower, while changing $h_{MM}$ corresponds to adding or subtracting bond-wise annihilators of the scar manifold. One simple choice of Hamiltonian is $h_{MM}$ the identity with $\omega = 1$; this choice leads to a frustration-free Hamiltonian with the chosen base MPS as the zero energy ground state.

We can also use the result in Appendix D to construct paths in the parameter space of the Hamiltonian along which the scar manifold varies continuously. For a given spin-$S$, we may obtain different paths in parameter space by varying the coefficients on the $m$th diagonal of $A^{[m]}$ for each $m$. There are thus $(S+1)^2$ complex numbers we can vary continuously for spin $S$, although the number of free parameters will be smaller on taking into account redundancies in MPS descriptions.

As the MPS along the path functions as a base state for the scar tower generated by $Q^+_{\text{AKLT}}$, we can generalize Ref. [33]’s deformation of the spin-1 AKLT model to an integrable point. Ref. [33] described a deformation of the matrices in the spin-1 AKLT ground state to that of an eigenstate of the integrable pure-biquadratic model. That is, they considered

$$A^{[+]} = c_+ \sigma^+, A^{[0]} = c_0 \sigma^z, A^{[-]} = c_- \sigma^-$$

for varying $c_{\pm,0}$. The spin-1 AKLT ground state has coefficients $c_0 = -1$, $c_- = -\sqrt{2}$, $c_+ = \sqrt{2}$, while $c_0 = -1$, $c_- = -i$, $c_+ = i$ corresponds to an eigenstate of the integrable pure-biquadratic model. The authors used “numerical brute force” to verify that $\mathcal{G}$ and $R$ are disjoint, and that Eq. 57 holds for every choice of the $c_{\pm,0}$ coefficients. They thus constructed a family of Hamiltonians with the form in Eq. 56 that connected the spin-1 AKLT model to the pure-biquadratic model. However, we see that numerical brute force is not needed; the conditions on $\mathcal{G}$, $R$ and $q^+$ follow as an immediate corollary of our results on MPS for which $A^{[m]}$ is nonzero only on the $m$th diagonal, as the matrices in Eq. 64 are only nonzero on the correct diagonals.

We can generalize Ref. [33]’s deformation to spin-$S$. We note that the Hamiltonian of the spin-1 integrable pure biquadratic point is equivalent to a sum of projectors onto two-site spin-singlets. A spin-$S$ chain with Hamiltonian given by a sum of projectors onto spin-singlets (the singlet-projector model)

$$H^{\text{SP}} = \sum_j P_{j,j+1}^{(S)}$$

is similarly integrable\(^7\)\([55, 56]\). Furthermore, we note that there is a simple matrix product eigenstate of $H^{\text{SP}}$,

\[^{7}\text{It is Temperley-Lieb equivalent to the Bethe-ansatz solvable XXZ model [55]}\]

which can be written in terms of the matrices $A^{[m]}_{\text{AKLT}}$ that define the ground state of the spin-$S$ AKLT model:

$$A^{[m]}_{\text{SP}} = \sqrt{(-1)^m \left( \frac{2S}{S+m} \right)} A^{[m]}_{\text{AKLT}}$$

It can be shown that this resulting state is annihilated by every two-site spin-singlet projector. Just like $A^{[m]}_{\text{AKLT}}$, $A^{[m]}_{\text{SP}}$ is only nonzero on the $m$th diagonal, and hence serves as a nice endpoint for a deformation between the spin-$S$ AKLT model and the integrable spin-$S$ singlet-projector model. This construction reduces to the form of the MPS in Ref. [33] for $S = 1$.

We note that the spin-$S$ singlet-projector model is spin-SU(2) and $Q$-SU(2) symmetric\(^8\). The $Q$-SU(2) invariance arises because the spin singlet is annihilated by $Q^+$ and $Q^-$, as mentioned in Appendix E. Because the model is $Q$-SU(2) invariant, the scar states all have the same energy. The model in fact corresponds to $\omega = 0$, and $h_{MM}$ zero except for the projector onto the spin-singlet. However, at the cost of breaking the spin-SU(2) and $Q$-SU(2) invariance, we can assign energies to the scar states by setting $\omega > 0$. We can furthermore make the model thermalizing outside the scar manifold by introducing generic $h_{MM}$, e.g. a sum of projectors with unit coefficients.

Between the endpoints of the spin-$S$ AKLT model and singlet-projector model are many different paths along which the scar states deform continuously. For example, one could take the path

$$A^{[m]} = e_m A^{[m]}_{\text{AKLT}}$$

and interpolate $e_m$ between 1 and $\sqrt{(-1)^m \left( \frac{2S}{S+m} \right)}$ as some function of some parameter $\lambda$, where $h_{MM}$ and $\omega$ are functions of $\lambda$.

V. CONCLUSIONS AND OUTLOOK

In this work, we have presented a general framework for understanding how quantum scars emerge from parent Hamiltonians with non-Abelian (and possibly $q$-deformed) symmetries. Generators of the symmetry furnish a natural set of operators with spectrum generating commutation relations, and the parent Hamiltonians have rich structure in their eigenspectrum as a consequence of the symmetry. In particular, the spectrum of $H_{\text{sym}}$ is organized as degenerate multiplets (‘tunnels’\(^9\)) that transform as irreps of the symmetry. Scars emerge when perturbations generically destroy the symmetry and give a thermal spectrum, but do so in a manner

\[^8\text{It is in fact SU(2S+1) symmetric if one uses different representations of SU(2S+1) on alternating sites [56]}\]

\[^9\text{See Eq. (48)}\]
that preserves a shadow of the symmetry so that a particular multiplet of low entanglement states fails to mix with the rest of the Hilbert space. This furnishes one qualitative ‘picture’ for how and when one might expect scars to arise generally, something that has thus far been largely missing in the literature.

Our framework applies to several known models with scars in the literature, but it has allowed us to also introduce several new models with exact quantum scars, significantly generalizing the types of systems known to harbor this phenomenon. These models fall into two broad classes. In the first class, the scar states transform in a single irreducible representation of the symmetry group. Our examples in this class include models where the symmetry is SU(3) rather than SU(2). In the second class, the scar states do not belong to a single representation of the relevant symmetry group, which requires the parent Hamiltonian to have an enhanced symmetry. We have presented examples of this type not previously known in the literature, including generalizations of the AKLT model and families of scarred Hamiltonians that can be smoothly deformed into each other. It is interesting to note that prior studies have tried to explain scars in the PXP model via deformations towards integrable models [17] and those with approximate SU(2) symmetry [18, 26]. Hence this symmetry based framework may prove key to eventually fully understanding scars in the PXP model, too.

Our framework leaves open many important questions about the qualitative features that distinguish Hamiltonians with quantum scars from their ETH-satisfying peers. For example, what distinguishes models in which the symmetry is broken in a generic way from those in which a scarred subspace persists? A key general question concerns the stability of scars to perturbations, and whether scars can survive these perturbations either in an asymptotic or ‘prethermal’ sense [17, 57]. Indeed, to classify scars as a new kind of dynamical phase of matter with ‘intermediate’ thermalization properties - neither fully ergodic nor fully localized - requires scarred models to display some degree of stability in phase space. One important consequence of our picture is that it furnishes a family of scarred models emanating from symmetric parent Hamiltonians, and thus shows that scarred models can have at least some degree of stability to certain classes of perturbations.

Acknowledgments — The authors are grateful to Daniel Renard, Cheng-Ju Lin and Lesik Motrunich, for helpful discussions. A.C. is supported by the Sloan Foundation through a Sloan Research Fellowship. This work was supported by the National Science Foundation through the awards NSF DMR-1752759 (A.C.) and DMR-1928166 (F.J.B.)

Note Added — While we were preparing our manuscript, we became aware of two related works, Refs. [58] and [59], which also apply group theoretic considerations to scarred Hamiltonians. Our results agree where they overlap, although the scope of our work is broader than Ref. [58] which only considers Casimir singlets, and our results on q-deformed and asymmetric scar towers also lie outside the constructions of Ref. [59].

Appendix A: MPO representations for $(Q^+)^n$

In the text, we discuss creating scars by repeated action of some operators on a base state. We show in this appendix that for certain operators $Q^+$, $(Q^+)^n$ has a simple MPO representation with a bond dimension of only $n + 1$. This implies that acting $(Q^+)^{Poly(L)}$ on some base state can only increase the entanglement entropy of the base state by at most $O(\log(L))$. Our approach is inspired by the examples given in [27]. We extend the results there and take a different style of proof.

We will consider $Q^+$ as some momentum $k$ sum of tailed-operators:

$$Q^+(L) = \sum_{m=1}^{L} e^{ikm} (\otimes_{j=1}^{m-1} L_j) \otimes O_m \otimes (\otimes_{j=m+1}^{L} R_j) \quad (A1)$$

We will additionally require that at the single-site level,

$$[L_i, R_i] = 0, \quad L_i O_i^+ = \frac{1}{L} O_i^+ L_i, \quad \text{and} \quad R_i O_i^+ = r O_i^+ R_i \quad (A2)$$

for some arbitrary numbers $r$ and $l$.

As an example, we could have $O^+ = S^+$, $L = R = I$, and $k = 0$; the commutation relations are satisfied with $r, l = 0$. This choice corresponds to $Q^+ = S^+$. As another example, consider $O^+ = S^+, \quad L = q^{-S^2}, \quad R = q^{S^2}, \quad k = 0$; this has $r = q$ and $l = \frac{1}{q}$. This choice corresponds to the ladder operator for SU$_q(2)$.

We’ll prove the case of $k = 0$ first. Define the $n + 1$ by $n + 1$ matrix

$$M_{\alpha,\beta}(X, Y, Z) = \frac{1}{[[\beta - \alpha]]!}(X^{n+1-\beta} Y^{\beta - \alpha} Z^{\alpha - 1}) \quad (A3)$$

for $\beta \geq \alpha$ and 0 otherwise. Here, we’re using the notation, for $m$ and $n$ positive integers,

$$[[n]]_q = \frac{1 - q^n}{1 - q}, \quad [n]_q! = \prod_{m=1}^{n} [[m]]_q. \quad (A4)$$

Additionally define the vectors $(v_r)_i = \delta_{i,1}$ and $(v_l)_i = \delta_{i,n+1}$, and the matrix

$$M_j = M(L_j, O_j, R_j). \quad (A5)$$

Then our claim is

$$v^T (\prod_{i=1}^{L} M_i) v_r = \frac{1}{[[n]]!} (Q^+(L))^n. \quad (A6)$$

Note that we’ve made the length-of-the-chain $L$-dependence of $Q^+$ explicit.
We’ll prove A6 by way of a stronger result:

$$\prod_{i=1}^{L} M_i = M_{\alpha, \beta}(\otimes_{i=1}^{L} L_i, Q^+(L), \otimes_{i=1}^{L} R_i)$$ \hspace{1cm} (A7)$$

This stronger result implies equation A6 by the definition

$$\left( \prod_{i=1}^{L} M_i \right)_{\alpha \beta} = \sum_{\gamma} \left( \prod_{i=1}^{L} M_i \right)_{\alpha \gamma} M_{L, \beta}$$

$$= \sum_{\gamma} \frac{1}{[\gamma - \alpha]_{\gamma}!} \left( \otimes_{i=1}^{L-1} L_i \right)^{n+1-\gamma} (Q^+(L-1))^{\gamma-\alpha} \left( \otimes_{i=1}^{L-1} R_i \right)^{\alpha-1} \frac{1}{[\beta - \gamma]_{\beta}!} L_{L_{L}}^{n+1-\beta} O_{L}^{\beta-\gamma} R_{L}^{\gamma-1}$$

We can simplify the sum on $\gamma$ by noting that for two matrices $A$ and $B$ such that $BA = xAB$,

$$(A + B)^{n} = \sum_{p=0}^{n} \frac{[n]_{x}!}{[n - p]_{x}! [p]_{x}!} A^{p} B^{n-p}. \hspace{1cm} (A9)$$

$$= \frac{1}{[\beta - \alpha]_{\beta}!} \left( \otimes_{i=1}^{L-1} L_i \right)^{n+1-\beta} (Q^+(L-1))^{\beta-\alpha} \left( \otimes_{i=1}^{L-1} R_i \right)^{\alpha-1}$$

This concludes the proof.

For $k \neq 0$, note that $L_{j'} = e^{ik L_{j}}, O_{j'} = e^{ik O_{j}}, R_{j'} = R_{j}$ also satisfy the commutation relations in equation A2. Thus, the proof above holds for these primed single-site operators. This implies that for $(M_{i}')_{\alpha \beta} = M(L_i', O_i', R_i')$,

$$v^{T} \left( \prod_{i=1}^{L} M_{i}' \right) v_{p} = \left( \sum_{j} \left( \otimes_{j'=1}^{j} L_{j'} \right) \otimes O_{j}^{+} \otimes \left( \otimes_{j=1}^{L} R_{j} \right) \right)_{n}$$

$$= \left( \sum_{j} e^{ikj} \left( \otimes_{i=1}^{j} L_{i} \right) \otimes O_{j}^{+} \otimes \left( \otimes_{i=1}^{L} R_{i} \right) \right)$$

i.e. the correct form for a momentum-$k$ sum.

**Appendix B: A new class of as-a-sum annihilators**

We note in this appendix that our covariance-matrix algorithm for finding Hamiltonians with sets of states as eigenstates has found a previously unknown and rather physical as-a-sum annihilator of the tower of states in the spin-1 AKLT model.

This as-a-sum annihilator is the alternating spin-1 AKLT model: $H_{A} = \sum_{j} (-1)^{j} F_{j-1, j}^{(2)}$. This means that the alternating spin-1 AKLT model has the scar states all at zero-energy, and further that the staggered AKLT model $H_{A} = \sum_{j} c_{j} F_{j, j+1}^{(2)}$ with $c_{j} = c_{j+2}$ has scar states separated by energy spacing $c_1 + c_2$. These results are novel and add to the classes of Hamiltonians with the spin-1 AKLT tower of states found in previous papers [33, 35].

We can greatly generalize this as-a-sum annihilator to a large class of models with a quick proof. We will assume that the models have the form discussed in Section IV: namely, they satisfy the “disjoint $G$ and $R$” condition and equations 56 and 57.

As a reminder of the notation, $H = \sum_{j} h_{j, j+1}, Q^{+} = \sqrt{\sum_{j} e^{ikj} O_{j}}, q_{j, j+1}^{+} = O_{j} + e^{ikj} O_{j+1}$, and the base state has zero energy. The form discussed in section IV ensures that $A_{j, j+1} = [h_{j, j+1}, q_{j, j+1}^{+}] - \omega q_{j, j+1}^{+}$ annihilates every
state in the scar tower. Then our claim is that
\[ H_A = \sum_j (-1)^j h_{j,j+1} \] (B1)
annihilates the scar states for all these models. As noted in Section IV, this large class of models includes the spin-\(S\) AKLT chain, the \(q\)-deformed spin-\(S\) AKLT chain, and the SO(2\(S+1\)) AKLT chain.

In periodic boundary conditions, with chain length even and \(kL\) an integer multiple of \(2\pi\), the above assumptions imply that:
\[
[H_A, Q^+]_{pbc} = \sum_{j=1}^{L} \sum_{l=1}^{L} (-1)^j [h_{j,j+1}, e^{ikl}O_l]
= \sum_{j=1}^{L} (-1)^j e^{ikj}[h_{j,j+1}, O_j + e^{ik}O_{j+1}]
= \sum_{j=1}^{L} (-1)^j e^{ikj}(\omega(O_j + e^{ik}O_{j+1}) + A_{j,j+1})
= \omega(\sum_{j=1}^{L} (-1)^j e^{ikj}O_j + \sum_{j=1}^{L} (-1)^j e^{ik(j+1)}O_{j+1})
+ \sum_{j=1}^{L} (-1)^j e^{ikj}A_{j,j+1}
= 0 + \sum_{j=1}^{L} (-1)^j e^{ikj}A_{j,j+1}
\] (B2)

Thus, \([H_A, Q^+]\) is of the form of equation 2 with \(\omega = 0\), while \(H_A|v_0\rangle = 0\). This completes the proof that \(H_A\) is an annihilator in periodic boundary conditions with \(L\) even and \(kL\) a multiple of \(2\pi\). It is certainly an as-a-sum annihilator, as without the alternating sign it would give energy to the scar states.

In open boundary conditions, one will get edge terms \(O_1\) and \(O_L\),
\[
[H_A, Q^+]_{obc} = -\omega(e^{ik}O_1 + (-1)^L e^{ikL}O_L)
+ \sum_{j=1}^{L-1} (-1)^j e^{ikj}A_{j,j+1}.
\] (B3)

These edge terms could potentially spoil this as-a-sum annihilator in open boundary conditions. However, see F for a discussion of choosing the right ground states in the AKLT models to use as the base for a tower of exact eigenstates; these states are exactly the ones for which the edge terms will annihilate the tower of states. Thus \(H_A\) in open boundary conditions satisfies the right commutation relation and has the scar states all at the same energy of 0 for arbitrary \(L\).

Appendix C: Explicit forms for AKLT Projectors

In discussing the AKLT Hamiltonians, we noted that all of them could be written as sums of two-site projectors onto manifolds with various total spin or total \(q\)-deformed spin eigenvalues. We did not need the explicit forms of the projectors for our discussion. In this appendix, we give forms for the AKLT projectors in terms of spin operators, which are useful for generating the AKLT Hamiltonians for exact diagonalization.

For total spin-\(S\), we can project into the total-spin \(t\) manifold by projecting out everything else:
\[
P_{12}^{(t)} = \prod_{s \neq t} (S_1 + S_2)_{s,t} - s(s+1) \frac{t(t+1)-s(s+1)}{t(t+1)-s(s+1)}
\] (C1)
will vanish when acting on a two-site state of total spin \(s\neq t\) and will reduce to 1 acting on a state of total spin \(t\).

Similarly, for the \(q\)-deformed models, we can write
\[
P_{12}^{(t)} = \tilde{P}_{12}^{(t)} = \prod_{s \neq t} \tilde{S}_{12}^{\pm \mp} + S_1^{z}S_2^{z} + [S_1^{z}]_q[S_2^{z} + 1]_q - [s]_q[s+1]_q \frac{t(t+1)-s(s+1)}{t(t+1)-s(s+1)}
\] (C2)
where we have defined
\[
\tilde{S}_{12}^{-} = \tilde{S}_1^{-} \otimes q^{-S_2^{-}} + q^{-S_1^{-}} \otimes \tilde{S}_2^{-}
\] (C3)
and
\[
\tilde{S}_{12}^{+} = \tilde{S}_1^{+} \otimes q^{S_2^{+}} + q^{S_1^{+}} \otimes \tilde{S}_2^{+}
\] (C4)
Here we have used the fact that the Casimir for SU\(_q\)(2) is \(\tilde{S}_1^{\pm} \tilde{S}_2^{\mp} + [S_1^{z}]_q[S_2^{z} + 1]_q\) with eigenvalue \([s]_q[s+1]_q\).

For the explicit form of the single-site \(\tilde{S}_1^{\pm}\), see equations 41 and 42.

Appendix D: Verifying the form of \(q^+\) for the \(q\)-deformed AKLT model

In this appendix, we verify that \(G\) and \(R\) are disjoint under the action of \(Q^+\) via \(\frac{1}{(2S)!} \sum_i (-1)^i(S_i^+)^2\) in the \(q\)-deformed AKLT model. We also show that the form for \(q^+\) matches that of equation 57. As noted in the text, these two facts will complete our proof of scars in the \(q\)-deformed AKLT model in periodic boundary conditions.

We’ll prove these facts for a much wider set of base states than just the \(q\)-deformed spin-\(S\) ground states. Namely, we will show that for towers built with \(Q^+\) on top of a translationally-invariant bond-dimension \(\chi = S+1\) matrix product state
\[
|A\rangle = \sum_m Tr[A^{[m_1]}A^{[m_2]}...A^{[m_L]}]|m_1..m_L\rangle
\] (D1)
for which \(A^{[m]}\) has all its non-zero entries on the \(m\)-th diagonal alone that \(G\) and \(R\) are disjoint and \(q^+\) has the
The space of two-site bonds of some translationally-invariant matrix product state will be contained within the span of the bond-dimension-squared number of states $\langle AA \rangle_{ij} = \sum_{m_1,m_2,k} A^1_{ik} m_1 A^2_{kj} m_2 |m_1 m_2\rangle$. Now, note that the product of a matrix with non-zero elements only on the $k$th diagonal and a matrix with non-zero elements only on the $k$'th diagonal will be a matrix with non-zero elements only on the $k+k'$th diagonal. Thus $\langle AA \rangle_{ij}$ has contributions only from kets with $m_1 + m_2 = j-i$. Then we see we have $-S \leq m_1 + m_2 \leq S$ for all the states in $G$, and there are $S+1 - |m_1 + m_2|$ states with magnetization $m_1 + m_2$.

To see that $G$ and $R$ are disjoint, note that the action of $q^+$ increases the z-magnetization of $m_1 + m_2$ by $2S$. Thus, for the all-but-one states in $G$ with $m_1 + m_2 > -S$, $q^+$ takes the states to those with total z-magnetization $> S$, which is outside of $G$. There’s only one state within $G$ with magnetization $m_1 + m_2 = -S$, $\langle AA \rangle_{S+1,1}$, and there’s only one state in $G$ with magnetization $S$, $\langle AA \rangle_{1,S+1}$. In order to complete the proof of disjoint $G$ and $R$, we have to verify that despite having the same z-magnetization as $\langle AA \rangle_{1,S+1}$, the state $q^+ |AA\rangle_{S+1,1}$ is orthogonal to $\langle AA \rangle_{1,S+1}$.

Within $\langle AA \rangle_{S+1,1}$, all the $|m_1 m_2\rangle$ states are annihilated under $q^+$ except for $| -S0\rangle$ and $|0 -S\rangle$, which are mapped to $|S0\rangle$ and $|0S\rangle$ respectively. Thus, looking at the explicit form of $| AA \rangle_{ij}$, we see $q^+ |AA\rangle_{S+1,1} \propto A^1_{S+1,0} A^2_{11} |S0\rangle - A^1_{S+1,1} A^2_{11} |S0\rangle \propto A^1_{11} |S0\rangle - A^2_{11} |S0\rangle$. Similarly, $|AA\rangle_{1,S+1} \propto A^1_{00} |S0\rangle + A^2_{00} |S0\rangle$, with different $m_1, m_2$, so $q^+ |AA\rangle_{S+1,1}$ and $|AA\rangle_{1,S+1}$ are orthogonal. That is, $|AA\rangle_{S+1}$ is mapped to a state outside of $G$, and hence we’ve verified that all states in $G$ are mapped to states outside of $G$ under the action of $q^+$. We’ve thus shown that $G$ and $R$ are disjoint for towers built with $Q^+$ on top of the matrix product states described above.

Further, noting that $(q^+)^2$ raises the total z-magnetization by $4S$ and hence annihilates all the states in $G$, we see that $q^+$ annihilates the states in $R$. Putting all the information together, we see that the form for $q^+$ (namely, the blocks of zeros) is indeed the one given above.

Thus, the lemma about the forms for $q^+$ and $h$ is proven for the spin-$S$ $q$-deformed AKLT models, completing the proof that the tower of states are indeed eigenstates for these models. We emphasize that disjoint $G$ and $R$, and the form for $q^+$, were all satisfied for the large class of matrix product states in equation D1.
of $A^S$ ground states forming the base of linearly independent scar towers. In past work on the subject, reference [1] described one out of the $S^2$ towers in the regular spin-S AKLT model in OBC.

In order to count the ground states, note that the ground states of the PBC models were frustration free, unique, and could be represented as matrix product states

$$\sum_m T_m[A^{[m_1]}A^{[m_2]}...A^{[m_L]}]|m_1...m_L\rangle$$

(F1)

for some model-dependent $A$ with some model-dependent bond dimension $\chi$. In open boundary conditions, the models enjoy $\chi^2$ frustration-free ground states; i.e.,

$$\sum_m(A^{[m_1]}A^{[m_2]}...A^{[m_L]})_{ij}|m_1...m_L\rangle.$$  

(F2)

Such states are ground states in OBC because they are in the kernel of the projectors in the Hamiltonian: They are composed of the same bonds as in the PBC ground state save for the bond between the edge spins at 1 and $L$. Since the bond dimension of $A$ is $S+1$ for the $q$-deformed and regular AKLT models, while the bond-dimension of $A$ is $2^S$ for the SO$(2S+1)$ models, there are $(S+1)^2$ ground states of the q-deformed and regular AKLT models, while there are $4^S$ ground states of the SO$(2S+1)$ models. A small subtlety is that the SO$(2S+1)$ models need to have long enough chain lengths $L$ for all the ground states found this way to be linearly independent; we will assume that is the case.

The main change for the proof of scars in the above models in open boundary conditions is that the missing $h_{L,1}$ in $H_{obc}$ changes the commutator of $[H, Q^+]$. We had before in equations 61 and 62 that

$$[H_{pbc}, Q^+] = 2\omega Q^+ + \sum_{j=1}^{L} e^{ikj}A_{j,j+1}$$

where $A_{j,j+1}$ annihilated the $j, j+1$ bond in all the states of the tower. Now we’ll have

$$[H_{obc}, Q^+] = 2\omega Q^+ + \sum_{j=1}^{L-1} e^{ikj}A_{j,j+1} - \omega e^{ik}O_1 - \omega e^{ikL}O_L$$

(F4)

where, in the cases discussed here, $k = \pi, \omega = 1$ and $O$ is proportional to $(S^+)^{2^S}$.

The ground states of the OBC Hamiltonian, given in equation F2, and towers of states built on top of them by $Q^+$ will be annihilated by each $A_{j,j+1}$ in $\sum_{j}^{L-1} e^{ikj}A_{j,j+1}$. This follows because the set of bonds in the states and the towers built on top of the states are the same as in PBC; i.e., $G$, $R$, and $M$ are independent of boundary conditions, the towers contain only bonds in $G$ and $R$, and $A_{j,j+1}$ annihilates $G$ and $R$. (Of course, these ground states generically have bonds between $L$ and 1 that are not in $G$ or $R$, but $A_{L,1}$ is not in the sum $\sum_{j}^{L-1} A_{j,j+1}$, so we need not worry about that bond.) However, $\sum_m(A^{[m_1]}A^{[m_2]}...A^{[m_L]})_{ij}|m_1...m_L\rangle$ will not generically be annihilated by $O_1$ and $O_L$.

For the OBC ground states, there’s a simple sufficient condition for the whole tower to be annihilated by $O_1$ and $O_L$: each individual spin-$z$-basis product state within our base state must have edge spins that are annihilated by $O_1$ and $O_L$. It’s clear that this condition ensures the base state is annihilated by $O_1$ and $O_L$. Furthermore, the condition guarantees that the action of $Q^+$ on a given product state within the base state won’t be able to change a given product-state’s edge spins, so each state in the tower generated by $Q^+$ will be composed of product states whose edge spins are still annihilated by $O_1$ and $O_L$. Thus the whole tower of states will be annihilated by $O_1$ and $O_L$ if we satisfy the sufficient condition that the edge spins of all the product states within the base state are annihilated by $O_1$ and $O_L$.

In our cases, $O_1$ and $O_L$ are proportional to $(S^+)^{2^S}$, so satisfying the above condition simply means that the edge spins in the product states comprising the base state can’t be $| - S \rangle$.

For the $q$-deformed AKLT models, $S^2$ out of the $(S + 1)^2$ OBC ground states satisfy this condition and hence host towers of eigenstates. This follows from explicit form of the ground states in equation F2 for the $q$-deformed and regular spin-$S$ AKLT models: $A^{[m]}$ is an $S + 1$ by $S + 1$ dimensional matrix that lives only on the $m$th diagonal, which means that $(A^{[m]}A^{[m]}...)_{ij}$ has left spin between $1 - i$ and $S + 1 - i$ and the right spin between $-S + 1 + j$ and $j - 1$. Thus for the $S^2$ states with both $i < S + 1$ and $j > 1$, the edge spins are not $| - S \rangle$ and hence are annihilated by $O_1$ and $O_L$. That is, for $i < S + 1$ and $j > 1$, the towers built on top of $\sum_m(A^{[m]}A^{[m]}...)_{ij}|m_1...m_L\rangle$ are all eigenstates.

The $q$-deformed AKLT PBC ground state includes contributions from $\sum_m(A^{[m]}A^{[m]}...)_{11}|m_1...m_L\rangle$ and $\sum_m(A^{[m]}A^{[m]}...)_{(S+1)(S+1)}|m_1...m_L\rangle$, which contain some product states with $| - S \rangle$ at the right and left edges respectively. This means that the PBC ground state, despite also being one of the ground states of the OBC Hamiltonian, does not satisfy the above sufficient condition of $O_1$ and $O_L$ annihilating all the edge states.

While we’ve been careful to identify the condition as sufficient but not necessarily necessary, it is indeed necessary for these models. In principle, $O_1$ and $O_L$ need not annihilate each product state separately; instead there could be cancellations where $O_1$ on one product state cancels $O_L$ on some other product state. However, such cancellations are impossible if there’s a product state with an image under $O_{1/L}$ which has an empty preimage under $O_{L/1}$; states failing to obey $O_1$ and $O_L$ annihilating all the ground states are rife with product states suffering such a preimage property. For example, the $L = 3, S = 1, i = 1, j = 1$ OBC $q$-deformed AKLT ground state includes $|0 - +\rangle$, whose image under $O_L$ ($|0 + +\rangle$) has an empty preimage under $O_1$. More generally, product states of the form not $S$ at the left (right) edge and $-S$
at the right (left) edge are also poorly behaved in that the image under $O_L$ ($O_L$) has an empty preimage under $O_L$ ($O_L$). Since for the $g$-deformed and regular spin-$S$ AKLT models, $(A^{[m_1]}A^{[m_2]}...A^{[m_L]})_{ij}$ can have left spin between $1 - i$ and $S + 1 - i$ and the right spin between $-S - 1 + j$ and $j - 1$, the states that don’t satisfy the sufficient condition, i.e. do NOT satisfy $i < S + 1$ or $j > 1$, will generically contain for $L > 2$ some product states of the form with 0 as the left spin and $-S$ as the right spin, or vice versa.

Moreover, we cannot eliminate these poorly behaved product states by taking superpositions of different OBC ground states failing to satisfy $i < S + 1$ or $j > 1$. That’s because we could only bring about such a cancelation between ground states with the same eigenvalue under $S_{\text{stat}}$, and, by inspection, there are at most two such ground states with the same eigenvalue under $S_{\text{tot}}$. When there are two such ground states, one will generically have some product states of the form with 0 as the left spin and $-S$ as the right spin and will necessarily not have any product states of the form with $-S$ as the left spin and 0 as the right spin, while the other ground state with the same eigenvalue under $S_{\text{tot}}$ satisfies the opposite. Thus the above sufficient condition is necessary for these models.

We can make a similar set of arguments for the spin-$S$ SO(2$S$+1) AKLT model to show that $4S_{\text{tot}} - 1$ of the $4S$ OBC ground states host towers of exact eigenstates generated by $Q^+$. For these models, $A^{[0]} = -\otimes_{i=1}^{S} \sigma_i^j$ and for $m > 0$, $A^{[m]} = (\pm)^{m} \sqrt{2} (\otimes_{i=1}^{S} \sigma_i^{j}) \otimes_{j=1-m}^{S} (\otimes_{j=2-m}^{S} \sigma_j^{0})$. This form of the MPS, though not quite explicitly given in Ref. [54] for general $S$, follows from that reference up to similarity transformations of the $A$. This means $A^{[-S]} = (-1)^{S} 2^{S} \otimes_{j=0}^{S} (\otimes_{j=0}^{S} \sigma_j^{0})$, so the only non-zero values in $A^{[-S]}$ fall on the $-2^{S-1}$ diagonal. Then $\sum_{m}(A^{[m]}A^{[m]}...A^{[m]})_{ij}(m_1...m_L)$ doesn’t have $|-S|$ edge spins when $i < 2^{S-1}$ and $j > 2^{S-1}$; all these $4S_{\text{tot}} - 1$ states will thus host towers of exact eigenstates generated by $Q^+$ as they satisfy the sufficient condition.

Similarly to the discussion for the $g$-deformed models of whether $O_L$ and $O_L$ annihilating all the edge spins is necessary and not simply sufficient, this sufficient condition again appears to be necessary for this model. We will omit the proof of necessity -it follows similarly that each of the ground states in equation F2 that fall to satisfy $i \leq 2^{S-1}$ or $j > 2^{S-1}$ will contain poorly behaved product states for large enough $L$, but it is more challenging to prove that it is impossible that a superposition of these ground states could cancel out the poorly behaved product states.

[1] R. V. Jensen and R. Shankar, “Statistical behavior in deterministic quantum systems with few degrees of freedom,” Phys. Rev. Lett. 54, 1879–1882 (1985).
[2] J. M. Deutsch, “Quantum statistical mechanics in a closed system,” Phys. Rev. A 43, 2046–2049 (1991).
[3] Mark Srednicki, “Chaos and quantum thermalization,” Phys. Rev. E 50, 888–901 (1994).
[4] Marcos Rigol, Vanja Dunjko, and Maxim Olshanii, “Thermalization and its mechanism for generic isolated quantum systems,” Nature 452, 854–858 (2008).
[5] Luca D’Alessio, Yariv Kafri, Anatoli Polkovnikov, and Marcos Rigol, “From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics,” Advances in Physics 65, 239–362 (2016).
[6] Hyungwon Kim, Tatsuhiro N. Ikeda, and David A. Huse, “Testing whether all eigenstates obey the eigenstate thermalization hypothesis,” Phys. Rev. E 90, 052105 (2014).
[7] D. M. Basko, I. L. Aleiner, and B. L. Altshuler, “Metal-insulator transition in a weakly interacting many-electron system with localized single-particle states,” Ann. Phys. (Amsterdam) 321, 1126–1205 (2006).
[8] R. Nandkishore and D. A. Huse, “Many-body localization and thermalization in quantum statistical mechanics,” Annu. Rev. Condens. Matter Phys. 6, 15–38 (2015).
[9] D. A. Abanin, E. Altman, I. Bloch, and M. Serbyn, “Colloquium: Many-body localization, thermalization, and entanglement,” Rev. Mod. Phys. 91, 021001 (2019).
[10] Naoto Shiraishi and Takashi Mori, “Systematic construction of counterexamples to the eigenstate thermalization hypothesis,” Phys. Rev. Lett. 119, 030601 (2017).
[11] Sanjay Moudgalya, Stephan Rachel, B. Andrei Bernevig, and Nicolas Regnault, “Exact excited states of noninteracting models,” Phys. Rev. B 98, 235155 (2018).
[12] Hannes Bernien, Sylvain Schwartz, Alexander Keesling, Harry Levine, Ahmed Omran, Hannes Pichler, Soonwon Choi, Alexander S. Zibrov, Manuel Endres, Markus Greiner, Vladan Vuletić, and Mikhail D. Lukin, “Probing many-body dynamics on a 51-atom quantum simulator,” Nature 551, 579–584 (2017).
[13] C. J. Turner, A. A. Michailidis, D. A. Abanin, M. Serbyn, and Z. Papić, “Weak ergodicity breaking from quantum many-body scars,” Nature Physics 14, 745–749 (2018).
[14] Giulio Biroli, Corinna Kollath, and Andreas M. Läuchli, “Effect of rare fluctuations on the thermalization of isolated quantum systems,” Phys. Rev. Lett. 105, 250401 (2010).
[15] Wil Kao, Kuan-Yu Li, Kuan-Yu Lin, Sarang Gopalakrishnan, and Benjamin L. Lev, “Creating quantum many-body scars through topological pumping of a 1D dipolar gas,” arXiv e-prints , arXiv:2002.10475 (2020).
[16] C. J. Turner, A. A. Michailidis, D. A. Abanin, M. Serbyn, and Z. Papić, “Quantum scarred eigenstates in a rydberg atom chain: Entanglement, breakdown of thermalization, and stability to perturbations,” Phys. Rev. B 98, 155134 (2018).
[17] Vedika Khemani, Chris R. Laumann, and Amusha Chandran, “Signatures of integrability in the dynamics of rydberg-blockaded chains,” Phys. Rev. B 99, 161101 (2019).
[18] Soonwon Choi, Christopher J. Turner, Hannes Pichler, Wen Wei Ho, Alexios A. Michailidis, Zlatko Papić, Markos Serbyn, Mikhail D. Lukin, and Dmitry A. Abanin, “Emergent su(2) dynamics and perfect quantum many-body scars,” Phys. Rev. Lett. 122, 220603 (2019).
[19] Wen Wei Ho, Soonwon Choi, Hannes Pichler, and Mikhail D. Lukin, “Periodic orbits, entanglement, and quantum many-body scars in constrained models: Matrix product state approach,” Phys. Rev. Lett. 122, 040603 (2019).

[20] Cheng-Ju Lin and Olexei I. Motrunich, “Exact quantum many-body scar states in the rydberg-blockaded atom chain,” Phys. Rev. Lett. 122, 173401 (2019).

[21] Thomas Iadecola and Michael Schecter, “Quantum many-body scar states with emergent kinetic constraints and finite-entanglement revivals,” Phys. Rev. B 101, 024306 (2020).

[22] Thomas Iadecola, Michael Schecter, and Shenglong Xu, “Quantum many-body scars from magnon condensation,” Phys. Rev. B 100, 184312 (2019).

[23] Andrew J. A. James, Robert M. Konik, and Neil J. Robinson, “Nonthermal states arising from confinement in one and two dimensions,” Phys. Rev. Lett. 122, 130603 (2019).

[24] A. A. Michailidis, C. J. Turner, Z. Papic, D. A. Abanin, and M. Serbyn, “Slow quantum thermalization and many-body revivals from mixed phase space,” Phys. Rev. X 10, 011055 (2020).

[25] Federica M. Surace, Paolo P. Mazza, Giuliano Giudici, Alessio Lerose, Andrea Gambassi, and Marcello Dalmon, “Lattice gauge theories and string dynamics in rydberg atom quantum simulators,” Phys. Rev. X 10, 021041 (2020).

[26] Kieran Bull, Jean-Yves Desaules, and Zlatko Papic, “Quantum scars as embeddings of weakly broken lie algebra representations,” Phys. Rev. B 101, 165139 (2020).

[27] Sanjay Moudgalya, Nicolas Regnault, and B. Andrei Bernevig, “Entanglement of exact excited eigenstates of the Hubbard Model in Arbitrary Dimension,” SciPost Phys. 3, 043 (2017).

[28] Shahn Majid, “Quasitriangular Hopf Algebras and Yang-Baxter Equations,” International Journal of Modern Physics A 5, 1–91 (1990).

[29] Albertio Lerda and Stefano Sciuto, “Anyons and quantum groups,” Nuclear Physics B 401, 613–643 (1993).

[30] Dennis Bonatsos and C Daskaloyannis, “Quantum groups and their applications in nuclear physics,” Progress in particle and nuclear physics 43, 537–618 (1999).

[31] Ian Affleck, Tom Kennedy, Elliott H. Lieb, and Hal Tasaki, “Rigorous results on valence-bond ground states in antiferromagnets,” Phys. Rev. Lett. 59, 799–802 (1987).

[32] Tom Kennedy and Hal Tasaki, “Hidden symmetry breaking and the haldane phase ins=1 quantum spin chains,” Communications in Mathematical Physics 147, 431–484 (1992).

[33] Frank Pollmann, Erez Berg, Ari M. Turner, and Masaki Oshikawa, “Symmetry protection of topological phases in one-dimensional quantum spin systems,” Phys. Rev. B 85, 075125 (2012).

[34] M T Batchelor, L Dezzascale, R I Nepomechie, and V Rittenberg, “q-deformations of the o(3) symmetric spin-1 heisenberg chain,” Journal of Physics A: Mathematical and General 23, L141–L144 (1990).

[35] A. Klümper, A. Schadschneider, and J. Zittartz, “Groundstate properties of a generalized vbs-model,” Zeitschrift für Physik B Condensed Matter 87, 281–287 (1992).

[36] K Totsuka and M Suzuki, “Hidden symmetry breaking in a generalized valence-bond solid model,” Journal of Physics A: Mathematical and General 27, 6443–6456 (1994).

[37] Thomas Quella, “Symmetry protected topological phases beyond groups: The q-deformed AKLT model,” arXiv e-prints, arXiv:2005.09072 (2020).

[38] Raul A Santos, Francis C Paraan, Vladimir E Korepin, and Andreas Klümper, “Entanglement spectra of deformed higher spin VBS states,” Journal of Physics A:
Mathematical and Theoretical 45, 175303 (2012).
[54] Hong-Hao Tu, Guang-Ming Zhang, and Tao Xiang, “Class of exactly solvable $so(n)$ symmetric spin chains with matrix product ground states,” Phys. Rev. B 78, 094404 (2008).
[55] M. T. Batchelor and M. N. Barber, “Spin-s quantum chains and temperley-lieb algebras,” Journal of Physics A: Mathematical and General 23, L15–L21 (1990).
[56] A. Klumper, “The spectra of q-state vertex models and related antiferromagnetic quantum spin chains,” Journal of Physics A: Mathematical and General 23, 809–823 (1990).
[57] Cheng-Ju Lin, Anushya Chandran, and Oleksii I. Motrunich, “Slow thermalization of exact quantum many-body scar states under perturbations,” Phys. Rev. Research 2, 033044 (2020).
[58] Kiryl Pakrouski, Preethi N. Pallegar, Fedor K. Popov, and Igor R. Klebanov, “Many Body Scars as a Group Invariant Sector of Hilbert Space,” arXiv e-prints, arXiv:2007.00845 (2020).
[59] Jie Ren, Chenguang Liang, and Chen Fang, “Quasi-symmetry groups and many-body scar dynamics,” arXiv e-prints, arXiv:2007.10380 (2020).
[60] Anatoli Klimyk and Konrad Schmüdgen, Quantum groups and their representations (Springer Science & Business Media, 2012).