Abstract

This is the second of a series of papers in which we introduce and study a rigorous "simplicial" realization of the non-Abelian Chern-Simons path integral for manifolds $M$ of the form $M = \Sigma \times S^1$ and arbitrary simply-connected compact structure groups $G$. More precisely, we introduce, for general links $L$ in $M$, a rigorous simplicial version $WLO_{\text{rig}}(L)$ of the corresponding Wilson loop observable $WLO(L)$ in the so-called "torus gauge" by Blau and Thompson (Nucl. Phys. B408(2):345–390, 1993). For a simple class of links $L$ we then evaluate $WLO_{\text{rig}}(L)$ explicitly in a non-perturbative way, finding agreement with Turaev's shadow invariant $|L|$.

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1 Introduction

Recall from [24] that our goal is to find, for manifolds $M$ of the form $M = \Sigma \times S^1$, a rigorous realization of the non-Abelian Chern-Simons path integral in the torus gauge. We want to achieve this with the help of a suitable "simplicial" approach. In [24] we introduced such an approach and stated without proof our main result, i.e. Theorem 6.4 in [24] (= Theorem 3.5 in Sec. 3.10 below, to be proven in Sec. 5 below).

In the present paper we will briefly recall the simplicial approach in [24]. In order to understand the motivation for some of the definitions and constructions (e.g. in Sec. 3.2 and Sec. 3.4 below) the reader will probably find it helpful to have a look at Sec. 5 in [24] (and at Sec. 1 and Sec. 3 of [24]).

The paper is organized as follows:

In Sec. 2 we will recall some of the notation from [24] and we will recall the basic heuristic formula from [24], cf. Eq. (2.7) below. In Sec. 3 we recall the discretization approach of [24] and restate the main result, i.e. Theorem 3.5. In Sec. 4 we study "oscillatory Gauss-type measures" on Euclidean spaces and discuss some of their properties. In Sec. 5 we prove Theorem 3.5. In Sec. 6 we comment briefly on the case of general (simplicial ribbon) links. In Sec. 7 we then give an outlook on some promising further directions within the framework of $BF_3$-theory before we conclude the main part of this paper with a short discussion of our results in Sec. 8.

The present paper has an appendix consisting of four parts: part A contains a list of the Lie theoretic notation which will be relevant in the present paper. (This list is a continuation of the list in Appendix A in [24]). In part B we recall the definition of Turaev's shadow invariant
where \(|L|\) for links \(L\) in 3-manifolds \(M\) of the type \(M = \Sigma \times S^1\). In part [C] we recall the definition of \(BF\)-theory in 3 dimensions and we briefly comment on the relationship between \(BF_3\)-theory and CS theory. In part [D] we sketch possible reformulations/modifications of the discretization approach in Sec. 2.4.

## 2 The basic heuristic formula in [24]

### 2.1 Basic spaces

As in [24] we fix a simply-connected compact Lie group \(G\) and a maximal torus \(T\) of \(G\). By \(g\) and \(t\) we will denote the Lie algebras of \(G\) and \(T\) and by \(\langle \cdot , \cdot \rangle_g\) or simply by \(\langle \cdot , \cdot \rangle\) the unique Ad-invariant scalar product on \(g\) satisfying the normalization condition \(\langle \alpha , \alpha \rangle = 2\) for every short coroot \(\alpha\) w.r.t. \((g, t)\), cf. part [A] of the Appendix. For later use let us also fix a Weyl chamber \(C \subset t\).

Moreover, we will fix a compact oriented 3-manifold \(M\) of the form \(M = \Sigma \times S^1\) where \(\Sigma\) is a (compact oriented) surface, and an ordered oriented link \(L = (l_1, \ldots , l_m)\), \(m \in \mathbb{N}\), in \(M = \Sigma \times S^1\). Each \(l_i\) is “colored” with an irreducible, finite-dimensional, complex representation \(\rho_i\) of \(G\).

As in [24] we will use the following notation:

\[
\begin{align*}
B &= C^\infty(\Sigma, t) \cong \Omega^0(\Sigma, t) \\
A &= \Omega^1(M, g) \\
A_\Sigma &= \Omega^1(\Sigma, g) \\
A_{\Sigma,t} &= \Omega^1(\Sigma, t) \\
A^\perp &= \{ A \in A \mid A(\partial/\partial t) = 0 \} \\
A^\perp_\perp &= \{ A \in A^\perp \mid A^\perp \text{ is constant and } A_{\Sigma,t}\text{-valued} \}
\end{align*}
\]

Here \(\perp\) is the orthogonal complement of \(t\) in \(g\) w.r.t. \(\langle , \rangle\). \(dt\) is the normalized (translation-invariant) volume form on \(S^1\), \(\partial/\partial t\) is the vector field on \(M = \Sigma \times S^1\) obtained by “lifting” the standard vector field \(\partial/\partial t\) on \(S^1\) and in Eqs. (2.1f) and (2.1g) we used the “obvious” identification (cf. Sec. 2.3.1 in [24])

\[
A^\perp \cong C^\infty(S^1, A_{\Sigma})
\]

where \(C^\infty(S^1, A_{\Sigma})\) is the space of maps \(f : S^1 \to A_{\Sigma}\) which are “smooth” in the sense that \(\Sigma \times S^1 \ni (\sigma, t) \mapsto (f(t))(X_{\sigma}) \in g\) is smooth for every smooth vector field \(X\) on \(\Sigma\). It follows from the definitions above that

\[
A^\perp = A^\perp_\perp \oplus A^\perp_\perp
\]

### 2.2 The heuristic Wilson loop observables

Recall that in the special case when \(G\) is simple\(^2\) the Chern-Simons action function \(S_{CS} : A \to \mathbb{R}\) associated to \(M\), \(G\), and the “level” \(k \in \mathbb{Z}\setminus\{0\}\) is given by

\[
S_{CS}(A) = -k \pi \int_M \langle A \wedge dA \rangle + \frac{1}{4} \langle A \wedge [A \wedge A] \rangle, \quad A \in A
\]

where \([\cdot \wedge \cdot]\) denotes the wedge product associated to the Lie bracket \([\cdot , \cdot] : g \times g \to g\) and where \(\langle \cdot \wedge \cdot \rangle\) denotes the wedge product associated to the scalar product \(\langle \cdot , \cdot \rangle : g \times g \to \mathbb{R}\).

\(^1\)recall that \(\Omega^p(N, V)\) denotes the space of \(V\)-valued \(p\)-forms on a smooth manifold \(N\)

\(^2\)see Remark [D] below for the case when \(G\) is not simple
Recall also that the heuristic Wilson loop observable \( WLO(L) \) of a link \( L = (l_1, l_2, \ldots, l_m) \) in \( M \) with “colors” \( (\rho_1, \rho_2, \ldots, \rho_m) \) is given by the informal expression

\[
WLO(L) := \int_A \prod_{i=1} \text{Tr}_{\rho_i}(\text{Hol}_l(A)) \exp(iS_{CS}(A)) dA
\]

(2.5)

where \( \text{Hol}_l(A) \) is the holonomy of \( A \in A \) around the loop \( l \in \{l_1, \ldots, l_m\} \). The following explicit formula for \( \text{Hol}_l(A) \) proved to be useful in [24],

\[
\text{Hol}_l(A) = \lim_{n \to \infty} \prod_{k=1}^n \exp\left(\frac{1}{n} A(l'(t))\right)_{t=k/n}
\]

(2.6)

where \( \exp : g \to G \) is the exponential map of \( G \).

**Remark 2.1** One can assume without loss of generality that \( G \) is a closed subgroup of \( U(N) \) for some \( N \in \mathbb{N} \). In the special case where \( G \) is simple we can then rewrite Eq. (2.4) as

\[
S_{CS}(A) := k\pi \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)
\]

with \( \text{Tr} := c \cdot \text{Tr}_{\text{Mat}(N, \mathbb{C})} \) where \( c \in \mathbb{R} \) is chosen such that \( \langle A, B \rangle = -\text{Tr}(A \cdot B) \) for all \( A, B \in g \subset u(N) \subset \text{Mat}(N, \mathbb{C}) \). Clearly, making this assumption is a bit inelegant but it has some practical advantages, which is why we made use of it in [24]. In the present paper we will use this assumption only at a later stage, namely in part C of the Appendix below (with \( G \) replaced by \( \check{G} \)).

**Remark 2.2** Recall from Remark 2.2 in [24] that if \( G \) is a general simply-connected compact Lie group then \( G \) will be of the form \( G = \prod_{r=1}^r G_i, \ r \in \mathbb{N} \), where each \( G_i \) is a simple simply-connected compact Lie group. We can generalize the definition of \( S_{CS} \) to this general situation by setting – for any fixed sequence \((k_i)_{i \leq r}\) of non-zero integers –

\[
S_{CS}(A) := \sum_{i=1}^r S_{CS,i}(A_i) \quad \forall A \in A
\]

where \( S_{CS,i} \) is the Chern-Simons action function associated to \( M, G_i, \) and \( k_i \) and where \((A_i)_i\) are the components of \( A \) w.r.t. to the decomposition \( g = \bigoplus_{i=1}^r g_i \) (\( g_i \) being the Lie algebra of \( G_i \)).

In the present paper only two special cases will play a role, namely the case \( r = 1 \) (i.e. \( G \) simple) and the case \( r = 2, G_2 = G_1 \) and \( k_2 = -k_1 \), cf. Sec. 7 below.

### 2.3 The basic heuristic formula

The starting point for the main part of [24] was a second heuristic formula for \( WLO(L) \) which one obtains from Eq. (2.5) above by applying “torus gauge fixing”, cf. Sec. 2.2.4 in [24].

Let \( \sigma_0 \in \Sigma \setminus \left( \bigcup_{i=1}^m \text{arc}(l_i^\Sigma) \right) \) be fixed. Here we have set \( l_i^\Sigma := \pi_\Sigma \circ l_i, \ i \leq m \), where \( \pi_\Sigma : \Sigma \times S^1 \to \Sigma \) is the canonical projection. Then we have (cf. Eq. (2.53) in [24])

\[
WLO(L) \sim \sum_{y \in Y} \int_{A^+} \left\{ 1_{C(\Sigma \cup \{y\})}(B) \text{Det}_{FP}(B) \right. \\
\left. \times \left[ \int_{\hat{A}^+} \left( \prod_{i=1}^m \text{Tr}_{\rho_i}(\text{Hol}_l(A_i^+ + A_i^+, B)) \right) \exp(iS_{CS}(A^+, B)) dA^+ \right] \\
\times \exp\left(-2\pi ik\langle y, B(\sigma_0) \rangle \right) \left\} \exp(iS_{CS}(A_i^+, B))(DA^+_i \otimes DB \right) \]

(2.7)
where \( I := \ker(\exp|_t) \subset t \) and \( t_{\text{reg}} := \exp^{-1}(T_{\text{reg}}) \) (with \( T_{\text{reg}} \) denoting the set of regular elements of \( T \)) and where for each \( B \in \mathcal{B}, A^{\perp} \in A^{\perp} \) we have

\[
S_{\text{CS}}(A^{\perp}, B) := S_{\text{CS}}(A^{\perp} + B dt) \tag{2.8}
\]

\[
\text{Hol}_{t}(A^{\perp}, B) := \text{Hol}_{t}(A^{\perp} + B dt) \tag{2.9}
\]

Here \( dt \) is the real-valued 1-form on \( M = \Sigma \times S^1 \) obtained by pulling back the 1-form \( dt \) on \( S^1 \) in the obvious way. Finally, \( \text{Det}_{FP}(B) \) is the informal expression given by

\[
\text{Det}_{FP}(B) := \det(1_t - \exp(\text{ad}(B))|_t) \tag{2.10}
\]

For the rest of this paper we will now fix an auxiliary Riemannian metric \( g \) on \( \Sigma \). After doing so we obtain a scalar product \( \ll \cdot, \cdot \gg_{A^{\perp}} \) on \( A^{\perp} \cong C^\infty(S^1, A_{\Sigma}) \) in a natural way. Moreover, we then have a well-defined Hodge star operator \( \ast : A_{\Sigma} \to A_{\Sigma} \) which induces an operator \( \ast : C^\infty(S^1, A_{\Sigma}) \to C^\infty(S^1, A_{\Sigma}) \) in the obvious way. According to Eq. (2.48) in [24] we then have the following explicit formula

\[
S_{\text{CS}}(A^{\perp}, B) = \pi k \ll A^{\perp}, \ast(\frac{\partial}{\partial t} + \text{ad}(B))A^{\perp} \gg_{A^{\perp}} + 2\pi k \ll \ast A^{\perp}, dB \gg_{A^{\perp}} \tag{2.11}
\]

for all \( B \in \mathcal{B} \) and \( A^{\perp} \in A^{\perp} \), which implies

\[
S_{\text{CS}}(A^{\perp}, B) = \pi k \ll A^{\perp}, \ast(\frac{\partial}{\partial t} + \text{ad}(B))A^{\perp} \gg_{A^{\perp}} \tag{2.12}
\]

\[
S_{\text{CS}}(A^{\perp}, B) = 2\pi k \ll \ast A^{\perp}, dB \gg_{A^{\perp}} \tag{2.13}
\]

for \( B \in \mathcal{B}, A^{\perp} \in A^{\perp} \), and \( A^{\perp} \in A^{\perp} \).

**Remark 2.3** In view of Sec. 3.9 below we recall that – according to Remark 2.7 in [24] – we can replace space \( \mathcal{B} \) appearing in the outer integral \( \int_{A^{\perp} \times \mathcal{B}} \cdots (DA^{\perp}_{\mathcal{B}} \otimes DB) \) in Eq. (2.7) above by the space

\[
\mathcal{B}^{\text{loc}} := \{ B \in \mathcal{B} | B \text{ is locally constant around } \sigma_0 \}
\]

### 3 Simplicial realization of WLO\((L)\)

In the present section we briefly recall the definition of the rigorous simplicial analogue \( \text{WLO}_{\text{rig}}(L) \) for the RHS of Eq. (2.7) above which we gave in Sec. 5 in [24] and we recall the main result of [24], namely Theorem 6.4 (= Theorem 3.5 below).

Anyway, the reader will probably find it useful to have a look at Sec. 4 and Sec. 5 in [24] where we explain in much more detail the motivation of our constructions.

#### 3.0 Review of the simplicial setup in Sec. 4 in [24]

Recall from Sec. 4.1 in [24] that for a finite oriented polyhedral cell complex \( \mathcal{P} \) we denote by \( \mathfrak{F}_p(\mathcal{P}), p \in \mathbb{N}_0 \), the set of \( p \)-faces of \( \mathcal{P} \), and – for every fixed real vector space \( V \) – we denoted by \( C^p(\mathcal{P}, V) \) the space of maps \( \mathfrak{F}_p(\mathcal{P}) \to V \) (“\( V \)-valued \( p \)-cochains of \( \mathcal{P} \)”). The elements of \( \mathfrak{F}_0(\mathcal{P}) \) (resp. \( \mathfrak{F}_1(\mathcal{P}) \)) will be called the “vertices” (resp. “edges”) of \( \mathcal{P} \). Instead of \( C^p(\mathcal{P}, \mathbb{R}) \) we will often write \( C^p(\mathcal{P}) \). By \( d_{\mathcal{P}} \) we will denote the usual coboundary operator \( C^p(\mathcal{P}, V) \to C^{p+1}(\mathcal{P}, V) \).

In [24] we actually only considered the special situation \( \mathcal{P} \in \{ \mathbb{Z}_N, K, K', qK, K \times \mathbb{Z}_N, K' \times \mathbb{Z}_N, qK \times \mathbb{Z}_N \} \) where \( \mathbb{Z}_N, K, K' \), and \( qK \) are given as follows:

Recall that in Sec. 4.4 in [24] we fixed \( N \in \mathbb{N} \) and used the finite cyclic group \( \mathbb{Z}_N \) with the “obvious” \(^5\) (oriented) graph structure as a discrete analogue of the Lie group \( S^1 \).

\(^5\)i.e. the set of edges is given by \( \{(t, t+1) \mid t \in \mathbb{Z}_N \} \)
Moreover, we fixed a finite oriented smooth polyhedral cell decomposition \( C \) of \( \Sigma \). By \( C' \) we denoted the dual polyhedral cell decomposition, equipped with an orientation. By \( K \) and \( K' \) we denoted the corresponding (oriented) polyhedral cell complexes, i.e. \( K := (\Sigma, C) \) and \( K' := (\Sigma, C') \).

Instead of \( K \) (resp. \( K' \)) we usually wrote \( K_1 \) (resp. \( K_2 \)) and we set \( K := (K_1, K_2) \).

We then introduced a joint sub division \( qK := (\Sigma, qC) \) of \( K = K_1 \) and \( K' = K_2 \) which can be characterized by the conditions

\[
\mathfrak{F}_0(qK) = \mathfrak{F}_0(bK), \quad \mathfrak{F}_1(qK) = \mathfrak{F}_1(bK) \setminus \{e \in \mathfrak{F}_1(bK) \mid \text{both endpoints of } e \text{ lie in } \mathfrak{F}_0(K_1) \cup \mathfrak{F}_0(K_2)\}
\]

where \( bK \) is the barycentric sub division of \( K \). The set \( \mathfrak{F}_2(qK) \) is uniquely determined by \( \mathfrak{F}_0(qK) \) and \( \mathfrak{F}_1(qK) \). Observe that each \( F \in \mathfrak{F}_2(qK) \) is a tetragon. We introduced the notation

\[
\mathfrak{F}_0(K_1|K_2) := \mathfrak{F}_0(qK) \setminus (\mathfrak{F}_0(K_1) \cup \mathfrak{F}_0(K_2))
\]

(3.1)

Recall that also the faces of \( qK \) were equipped with an orientation. For convenience we chose the orientation on the edges of \( qK \) to be “compatible”\(^4\) with the orientation on the edges of \( K_1 \) and \( K_2 \).

Recall from Sec. 4.2 in [24] that a “simplicial curve” in \( P \) is a finite sequence \( x = (x^{(k)})_{k \leq n}, n \in \mathbb{N}, \) of vertices in \( P \) such that for every \( k \leq n \) the two vertices \( x^{(k)} \) and \( x^{(k+1)} \) either coincide or are the two endpoints of an edge \( e \in \mathfrak{F}_1(P) \). If \( x^{(n)} = x^{(1)} \) we will call \( x = (x^{(k)})_{k \leq n} \) a “simplicial loop” in \( P \).

Recall also that every simplicial curve \( x = (x^{(k)})_{k \leq n} \) with \( n > 1 \) induces a sequence \( (e^{(k)})_{k \leq n-1} \) of “generalized edges”, i.e. elements of \( \mathfrak{F}_1(P) \cup \{0\} \cup (-\mathfrak{F}_1(P)) \subset C_1(P) \) in a canonical way. Unless \( x = (x^{(k)})_{k \leq n} \) is constant, we can reconstruct \( x = (x^{(k)})_{k \leq n} \) from \( (e^{(k)})_{k \leq n-1} \).

We write \( \bullet e^{(k)} \) instead of \( x^{(k)} \) (for \( k \leq n - 1 \)).

Recall from Sec. 4.3 in [24] that a “(closed)\(^5\) simplicial ribbon” in \( P \) is a finite sequence \( R = (F_i)_{i \leq n} \) of 2-faces of \( P \) such that every \( F_i \) is a tetragon and such that \( F_i \cap F_j = \emptyset \) unless \( i = j \) or \( j = i \pm 1 \) (mod \( n \)). In the latter case \( F_i \) and \( F_j \) intersect in a (full) edge.

**Convention 1** Let \( V \) be a fixed finite-dimensional real vector space.

i) We set \( C_1(K) := C_1(K_1) \oplus C_1(K_2) \) and \( C^1(K, V) := C^1(K_1, V) \oplus C^1(K_2, V). \)

ii) Let \( \psi : C_1(K) \to C_1(qK) \) be the (injective) linear map given by

\[
\psi(e) = e_1 + e_2 \quad \text{for all } e \in \mathfrak{F}_1(K_1) \cup \mathfrak{F}_1(K_2)
\]

where \( e_1 = e_1(e), e_2 = e_2(e) \in \mathfrak{F}_1(qK) \) are the two edges of \( qK \) “contained” in \( e \). In the following we will identify \( C_1(K) \) with the subspace \( \psi(C_1(K)) \) of \( C_1(qK) \). Moreover, using the identifications \( C^1(qK, V) \cong C^1(qK) \otimes_{\mathbb{R}} V \) and \( C^1(K, V) \cong C^1(K) \otimes_{\mathbb{R}} V \) we naturally obtain the linear maps

\[
\psi^V := \psi \otimes \text{id}_V : C^1(K, V) \to C^1(qK, V)
\]

We will identify \( C^1(K, V) \) with the subspace \( \psi^V(C^1(K, V)) \) of \( C^1(qK, V) \).

\(^4\)More precisely, for each \( e \in \mathfrak{F}_1(qK) \) we choose the orientation which is induced by orientation of the unique edge \( e' \in \mathfrak{F}_1(K_1) \cup \mathfrak{F}_1(K_2) \) which contains \( e \)

\(^5\)We will often omit the word “closed”
3.1 The basic spaces

As in Sec. 5.1 in [24] we introduce the following discrete analogues of the spaces $\mathcal{B}$, $\mathcal{A}_\Sigma$ and $\mathcal{A}^\perp$ in Sec. 2.1 above:

\[
\mathcal{B}(qK) := C^0(qK, t) \quad \text{(3.2a)}
\]
\[
\mathcal{A}_\Sigma(qK) := C^1(qK, g) \quad \text{(3.2b)}
\]
\[
\mathcal{A}^\perp(qK) := \text{Map}(\mathbb{Z}_N, \mathcal{A}_\Sigma(qK)) \quad \text{(3.2c)}
\]

Clearly, the scalar product $\langle \cdot, \cdot \rangle_\mathcal{B}$ on $\mathcal{B}(qK)$ induces scalar products $\ll \cdot, \cdot \gg_{\mathcal{B}(qK)}$ and $\ll \cdot, \cdot \gg_{\mathcal{A}_\Sigma(qK)}$ on $\mathcal{B}(qK)$ and $\mathcal{A}_\Sigma(qK)$ in the standard way. We introduce a scalar product $\ll \cdot, \cdot \gg_{\mathcal{A}^\perp(qK)}$ on $\mathcal{A}^\perp(qK) = \text{Map}(\mathbb{Z}_N, \mathcal{A}_\Sigma(qK))$ by

\[
\ll A^\perp_1, A^\perp_2 \gg_{\mathcal{A}^\perp(qK)} = \frac{1}{N} \sum_{t \in \mathbb{Z}_N} \ll A^\perp_1(t), A^\perp_2(t) \gg_{\mathcal{A}_\Sigma(qK)} \quad \text{(3.3)}
\]

for all $A^\perp_1, A^\perp_2 \in \mathcal{A}^\perp(qK)$.

**Convention 2** We identify $\mathcal{A}_\Sigma(qK)$ with the subspace $\{ A^\perp \in \text{Map}(\mathbb{Z}_N, \mathcal{A}_\Sigma(qK)) \mid A^\perp \text{ is constant} \}$ of $\mathcal{A}^\perp(qK)$ in the obvious way.

For technical reasons\(^6\) we will not only work with the full spaces $\mathcal{A}_\Sigma(qK)$ and $\mathcal{A}^\perp(qK)$ but also their subspaces (cf. Convention 1 above) $\mathcal{A}_\Sigma(K)$ and $\mathcal{A}^\perp(K)$ given by

\[
\mathcal{A}_\Sigma(K) := C^1(K_1, g) \oplus C^1(K_2, g) \subset \mathcal{A}_\Sigma(qK) \quad \text{(3.4)}
\]
\[
\mathcal{A}^\perp(K) := \text{Map}(\mathbb{Z}_N, \mathcal{A}_\Sigma(K)) \subset \mathcal{A}^\perp(qK) \quad \text{(3.5)}
\]

**The decomposition** $\mathcal{A}^\perp(K) = \mathcal{A}^\perp(K) \oplus \mathcal{A}^\perp_c(K)$

In order to obtain a discrete analogue of the decomposition $\mathcal{A}^\perp = \mathcal{A}^\perp \oplus \mathcal{A}^\perp_c$ in Eq. (2.3) above let us introduce the following spaces:

\[
\mathcal{A}_{\Sigma,t}(K) := C^1(K_1, t) \oplus C^1(K_2, t) \quad \text{(3.6a)}
\]
\[
\mathcal{A}_\Sigma(K) := C^1(K_1, t) \oplus C^1(K_2, t) \quad \text{(3.6b)}
\]
\[
\mathcal{A}^\perp(K) := \{ A^\perp \in \mathcal{A}^\perp(K) \mid \sum_{t \in \mathbb{Z}_N} A^\perp(t) \in \mathcal{A}_{\Sigma,t}(K) \} \quad \text{(3.6c)}
\]
\[
\mathcal{A}^\perp_c(K) := \{ A^\perp \in \mathcal{A}^\perp(K) \mid A^\perp(\cdot) \text{ is constant and } \mathcal{A}_{\Sigma,t}(K)\text{-valued} \} \cong \mathcal{A}_{\Sigma,t}(K) \quad \text{(3.6d)}
\]

Observe that we have

\[
\mathcal{A}^\perp(K) = \mathcal{A}^\perp(K) \oplus \mathcal{A}^\perp_c(K) \quad \text{(3.7)}
\]

which is indeed a discrete analogue of the decomposition $\mathcal{A}^\perp = \mathcal{A}^\perp \oplus \mathcal{A}^\perp_c$ in Eq. (2.3) above.

### 3.2 Discrete analogue of the operator $\frac{\partial}{\partial t} + \text{ad}(B) : \mathcal{A}^\perp \rightarrow \mathcal{A}^\perp$

Let us recall the definition of the operator $L^{(N)}(B) : \mathcal{A}^\perp(K) \rightarrow \mathcal{A}^\perp(K)$ which we introduced in [24] as the discrete analogue of the continuum operator $\frac{\partial}{\partial t} + \text{ad}(B) : \mathcal{A}^\perp \rightarrow \mathcal{A}^\perp$ in Eq. (2.3) above.

Let $\tau_x$, for $x \in \mathbb{Z}_N$, denote the translation operator $\text{Map}(\mathbb{Z}_N, g) \rightarrow \text{Map}(\mathbb{Z}_N, g)$ given by $(\tau_x f)(t) = f(t + x)$ for all $t \in \mathbb{Z}_N$ and $f \in \text{Map}(\mathbb{Z}_N, g)$. Instead of $\tau_0$ we will simply write 1 in the following.

\(^6\)namely, in order to obtain a nice simplicial analogue of the Hodge star operator, cf. Sec. 3.3 below
In [24] we introduced, for fixed $b \in \mathfrak{t}$, the following natural discrete analogues $L^N(b) : \text{Map}(Z_N, g) \to \text{Map}(Z_N, g)$ of the continuum operator $L(b) := \frac{\partial}{\partial t} + \text{ad}(b) : C^\infty(S^1, g) \to C^\infty(S^1, g)$ (cf. Sec. 5.2 in [24] where we also explain why these operators really are natural):

\[
\hat{L}^N(b) := N(\tau_1 e^{\text{ad}(b)}/N - 1) \quad (3.8a)
\]

\[
\bar{L}^N(b) := N(1 - \tau_1 e^{-\text{ad}(b)/N}) \quad (3.8b)
\]

\[
\tilde{L}^N(b) := \frac{N}{2}(\tau_1 e^{\text{ad}(b)/N} - \tau_1 e^{-\text{ad}(b)/N}) \quad \text{if } N \text{ is even} \quad (3.8c)
\]

Let $B \in B(qK)$. The operator $L^N(B) : \mathcal{A}^\perp(K) \to \mathcal{A}^\perp(K)$ mentioned above is the linear operator which, under the identification

\[
\mathcal{A}^\perp(K) \cong \text{Map}(Z_N, C^1(K_1, g)) \oplus \text{Map}(Z_N, C^1(K_2, g)),
\]

is given by

\[
L^N(B) = \begin{pmatrix}
\hat{L}^N(B) & 0 \\
0 & \bar{L}^N(B)
\end{pmatrix} \quad (3.9)
\]

Here the linear operators $\hat{L}^N(B) : \text{Map}(Z_N, C^1(K_1, g)) \to \text{Map}(Z_N, C^1(K_1, g))$ and $\bar{L}^N(B) : \text{Map}(Z_N, C^1(K_2, g)) \to \text{Map}(Z_N, C^1(K_2, g))$ are given by

\[
\hat{L}^N(B) \cong \oplus_{\bar{e} \in \bar{\mathfrak{g}}_0(K_1|K_2)} \bar{L}^N(B(\bar{e})) \quad (3.10a)
\]

\[
\bar{L}^N(B) \cong \oplus_{\bar{e} \in \bar{\mathfrak{g}}_0(K_1|K_2)} \bar{L}^N(B(\bar{e})) \quad (3.10b)
\]

where $\bar{\mathfrak{g}}_0(K_1|K_2)$ is as in Eq. (3.11) above. In Eqs. (3.10a) and (3.10b) we used the obvious identification

\[
\text{Map}(Z_N, C^1(K_1, g)) \cong \oplus_{\bar{e} \in \bar{\mathfrak{g}}_0(K_1|K_2)} \text{Map}(Z_N, g) \cong \oplus_{\bar{e} \in \bar{\mathfrak{g}}_0(K_1|K_2)} \text{Map}(Z_N, g).
\]

We remark that $L^N(B)$ leaves the subspace $\mathcal{A}^\perp(K)$ of $\mathcal{A}^\perp(K)$ invariant. The restriction of $L^N(B)$ to $\mathcal{A}^\perp(K)$ will also be denoted by $L^N(B)$ in the following.

### 3.3 Definition of $S^\text{disc}_{CS}(A^\perp, B)$

Recall that in [24] we introduced discrete Hodge operators $*_{K_1} : C^1(K_1, g) \to C^1(K_2, g)$ and $*_{K_2} : C^1(K_2, g) \to C^1(K_1, g)$, cf. Sec. 4.5 in [24]. Moreover, we introduced two different operators denoted by $*_{K}$ (cf. Sec. 4.5 and Sec. 5.3 in [24]). Firstly, the operator $*_{K} : \mathcal{A}_\Sigma(K) \to \mathcal{A}_\Sigma(K) = C^1(K_1, g) \oplus C^1(K_2, g)$ given by

\[
*_{K} := \begin{pmatrix}
0 & *_{K_2} \\
*_{K_1} & 0
\end{pmatrix} \quad (3.11)
\]

and, secondly, the operator $*_{K} : \mathcal{A}^\perp(K) \to \mathcal{A}^\perp(K)$ given by

\[
(*_{K}A^\perp)(t) = *_{K}(A^\perp(t)) \quad \forall A^\perp \in \mathcal{A}^\perp(K), t \in Z_N \quad (3.12)
\]

As the discrete analogues of the continuum expression $S_{CS}(A^\perp, B)$ in Eq. (2.11) above we use the expression

\[
S^\text{disc}_{CS}(A^\perp, B) := \pi k \left[ \ll A^\perp, *_{K} L^N(B)A^\perp \gg_{\mathcal{A}^\perp(qK)} + 2 \ll *_{K}A^\perp, d_{qK}B \gg_{\mathcal{A}^\perp(qK)} \right] \quad (3.13a)
\]

for $B \in B(qK), A^\perp \in \mathcal{A}^\perp(K) \subset \mathcal{A}^\perp(qK)$. Observe that this implies

\[
S^\text{disc}_{CS}(\hat{A}_c, B) = 2\pi k \ll *_{K}A^\perp, d_{qK}B \gg_{\mathcal{A}^\perp(qK)} \quad (3.13c)
\]

for $B \in B(qK), \hat{A}_c \in \mathcal{A}^\perp(K), A^\perp \in \mathcal{A}^\perp(K)$.

Recall that we have (cf. Proposition 5.3 in [24]):

**Proposition 3.1** The operator $*_{K} L^N(B) : \mathcal{A}^\perp(K) \to \mathcal{A}^\perp(K)$ is symmetric w.r.t to the scalar product $\ll \cdot, \cdot \gg \mathcal{A}^\perp(qK)$. 
3.4 Definition of $\text{Hol}^\text{disc}_R(A^\perp, B)$

Let $A^\perp \in A^\perp(K) \subset A^\perp(qK)$ and $B \in \mathcal{B}(qK)$. Moreover, let $R = (F_k)_{k \leq n}$, $n \in \mathbb{N}$, be a closed simplicial ribbon in $qK \times \mathbb{Z}_N$. According to Remark 3.6 in [24] $R$ induces a pair $(l, l')$ of simplicial loops $l = (l^{(k)})_{k \leq n}$ and $l' = (l'^{(k)})_{k \leq n}$ in $qK \times \mathbb{Z}_N$ in the obvious way ($l$ and $l'$ are simply the two loops “on the boundary” of $R$). Let $l_S', l_{S1}', l_{S1}$ denote the corresponding “projected” simplicial loops in $qK$ and $\mathbb{Z}_N$, cf. Sec. 4.4.4 in [24].

The simplicial analogue of the continuum expression $\text{Hol}(A^\perp, B)$ we used in [24] (cf. Sec. 5.4 in [24]; here it is very helpful to recall the discussion in Sec. 5.4 in [24] since otherwise the motivation for the RHS of Eq. (3.14) may not become clear) was

$$\text{Hol}^\text{disc}_R(A^\perp, B) := \prod_{k=1}^{n} \exp \left( \frac{1}{2} \left( A^\perp(l^{(k)}_S) (l^{(k)}_{S1}) + \frac{1}{2} (A^\perp(l'^{(k)}_S)) (l'^{(k)}_{S1}) \right) + \frac{1}{2} B(l^{(k)}_S) \cdot dt^N(l^{(k)}_{S1}) + \frac{1}{2} B(l'^{(k)}_S) \cdot dt^N(l'^{(k)}_{S1}) \right)$$

(3.14)

where $dt^N(e) = \frac{1}{N} \forall e \in \mathfrak{H}_1(\mathbb{Z}_N)$

and where we have made the identification $A_S(qK) = C^1(qK, \mathfrak{g}) \cong \text{Hom}(C_1(qK), \mathfrak{g})$.

**Remark 3.2** In view of Remark 3.6 below let us point out that instead of working with simplicial ribbons in $qK \times \mathbb{Z}_N$ (“half ribbons”) one could also work with simplicial ribbons in $K \times \mathbb{Z}_N$ (“full ribbons”). Observe that every closed simplicial ribbon $R$ in $K \times \mathbb{Z}_N$ induces three loops $l^+ = (l^{+(k)})_{k \leq n}$, $l^- = (l^{-(k)})_{k \leq n}$, and $l = (l^{(k)})_{k \leq n}$, $n \in \mathbb{N}$, in $qK \times \mathbb{Z}_N$ in a natural way, $l^+$ and $l^-$ being the two boundary loops and $l$ being the loop “inside” $R$. We now define $\text{Hol}^\text{disc}_R(A^\perp, B)$ by

$$\text{Hol}^\text{disc}_R(A^\perp, B) := \prod_{k=1}^{n} \exp \left( \sum_{\pm} \frac{1}{2} \left( A^\perp(l^{(k)}_S) (l^{(k)}_{S1}) \right) + \frac{1}{2} (A^\perp(l'^{(k)}_S)) (l'^{(k)}_{S1}) \right) + \frac{1}{2} B(l^{(k)}_S) \cdot dt^N(l^{(k)}_{S1}) + \frac{1}{2} B(l'^{(k)}_S) \cdot dt^N(l'^{(k)}_{S1}) \right)$$

(3.16)

where we use the notation $\sum_{\pm} \cdots$ in the obvious way.

3.5 Definition of $\text{Det}^\text{disc}_{FP}(B)$

In [24] our original ansatz for the discrete analogue $\text{Det}^\text{disc}_{FP}(B)$ of the heuristic expression $\text{Det}_{FP}(B) = \det(1_t - \exp(\text{ad}(B)))_{|t}$ given by Eq. (2.10) was

$$\text{Det}^\text{disc}_{FP}(B) := \prod_{x \in \mathfrak{H}_0(qK)} \det(1_t - \exp(\text{ad}(B(x)))_{|t})$$

(3.17)

for every $B \in \mathcal{B}(qK)$.

---

7 observe that every simplicial ribbon $R$ in $K \times \mathbb{Z}_N$ can be considered as the union of two simplicial ribbons $R_+$ and $R_-$ in $qK \times \mathbb{Z}_N$ in a natural way.

8 More precisely, the loop $l$ is the loop “lying on the intersection” of the two associated half ribbons $R_+$ and $R_-$.

9 e.g. $\sum_{\pm} \frac{1}{2} A^\perp(l^{\pm(k)}_S)(l^{\pm(k)}_{S1})$ is a short form of $\frac{1}{2} A^\perp(l^{\pm(k)}_S)(l^{\pm(k)}_{S1}) + \frac{1}{2} A^\perp(l'^{(k)}_S) (l'^{(k)}_{S1})$.

10 Recall that in [24] we later modified this definition; we will do this also in the present paper, cf. Sec. 4.4.4 below.
3.6 Discrete version of $1_{C^\infty(\Sigma, t_{reg})}(B)$

Let us fix a family $(1^{(s)}_{t_{reg}})_{s>0}$ of elements of $C^\infty_{\mathbb{R}}(t)$ with the following properties:

- Image$(1^{(s)}_{t_{reg}}) \subset [0, 1]$, and $\text{supp}(1^{(s)}_{t_{reg}}) \subset t_{reg}$ for each $s > 0$,
- $1^{(s)}_{t_{reg}} \to 1_{t_{reg}}$ pointwise as $s \to 0$,
- Each $1^{(s)}_{t_{reg}}, s > 0$, is invariant under the operation of the affine Weyl group $W_{aff}$ on $t$.

For fixed $s > 0$ and $B \in \mathcal{B}(qK)$ we will now take the expression

$$
\prod_x 1^{(s)}_{t_{reg}}(B(x)) := \prod_{x \in \mathfrak{g}_0(qK)} 1^{(s)}_{t_{reg}}(B(x))
$$

(3.18)

as the discrete analogue of $1_{C^\infty(\Sigma, t_{reg})}(B)$. Later we will let $s \to 0$.

3.7 Discrete versions of the two Gauss-type measures in Eq. (2.7)

Convention 3 In the following we will always consider $\mathcal{B}(qK)$, $\mathcal{A}^\perp(K)$ and their subspaces as Euclidean spaces in the “obvious” way.

i) Let $D\tilde{\mathcal{A}}^\perp$ denote the (normalized) Lebesgue measure on $\tilde{\mathcal{A}}^\perp(K)$. According to Eq. (3.13) the complex measure

$$
\exp(iS_{\mathcal{C}S}^{\mathcal{W}L_{\mathcal{O}}}(\mathcal{A}^\perp, B))D\tilde{\mathcal{A}}^\perp
$$

(3.19)

is a centered oscillatory Gauss type measure on $\tilde{\mathcal{A}}^\perp(K)$ in the sense of Definition 4.1 in Sec. 4 below.

ii) Let $DA_c^\perp$ denote the (normalized) Lebesgue measure on $\mathcal{A}_c^\perp(K)$ and $DB$ the (normalized) Lebesgue measure on $\mathcal{B}(qK)$.

According to Eq. (3.13) above, the complex measure

$$
\exp(iS_{\mathcal{C}S}^{\mathcal{W}L_{\mathcal{O}}}(\mathcal{A}^\perp, B))(DA_c^\perp \otimes DB)
$$

(3.20)

is a centered oscillatory Gauss type measure on $\mathcal{A}_c^\perp(K) \oplus \mathcal{B}(qK)$ in the sense of Definition 4.1 in Sec. 4 below.

3.8 Definition of $WLO_{rig}^{\mathcal{W}L_{\mathcal{O}}}(L)$ and $WLO_{rig}(L)$

For the rest of this paper we will fix a simplicial ribbon link $L = (R_1, R_2, \ldots, R_m)$ in $qK \times \mathbb{Z}_N$ with “colors” $(\rho_1, \rho_2, \ldots, \rho_m)$, $m \in \mathbb{N}$.

Using the definitions of the previous subsections we then arrive at the following simplicial analogue $WLO_{rig}^{\mathcal{W}L_{\mathcal{O}}}(L)$ of the heuristic expression $WLO(L)$ in Eq. (2.7)

$$
WLO^{\mathcal{W}L_{\mathcal{O}}}_{\text{rig}}(L) := \lim_{s \to 0} \sum_{y \in I} \int_{\sim} \left\{ \left( \prod_x 1^{(s)}_{t_{reg}}(B(x)) \right) \text{Det}_{FF}^{\mathcal{W}L_{\mathcal{O}}}(B) \right. \\
\times \left. \left[ \int_{\sim} \left( \prod_{i=1}^m \text{Tr}_{\rho_i} \left( \text{Hol}_{R_i}^{\mathcal{W}L_{\mathcal{O}}}(\tilde{A}^\perp + A_c^\perp, B) \right) \right) \exp(\text{i}S_{\mathcal{C}S}^{\mathcal{W}L_{\mathcal{O}}}(\tilde{A}^\perp, B)) \right. \\
\times \left. \exp(-2\pi i k \langle y, B(\sigma_0) \rangle) \right\} \exp(\text{i}S_{\mathcal{C}S}^{\mathcal{W}L_{\mathcal{O}}}(A_c^\perp, B))(DA_c^\perp \otimes DB) \\
(3.21)
$$

\[\text{More precisely, we will assume that the space } \mathcal{B}(qK) \text{ (or any subspace of } \mathcal{B}(qK)) \text{ is equipped with the (restriction of the) scalar product } \langle \cdot, \cdot \rangle_{\mathcal{B}(qK)} \text{ on } \mathcal{B}(qK), \text{ and the space } \mathcal{A}_c^\perp(K) \text{ (or any subspace of } \mathcal{A}_c^\perp(K)) \text{ is equipped with the restriction of the scalar product } \langle \cdot, \cdot \rangle_{\mathcal{A}_c^\perp(qK)}, \text{ introduced in Sec. 3.1 above.}\]
where we use the notation \( \int_{\cdot} \cdots \) as defined in Definition 4.2 in Sec. 4 below and where \( \sigma_0 \) is an arbitrary fixed point of \( \mathcal{S}_0(qK) \) which does not lie in \( \bigcup_{i \leq m} \text{Image}(R_i) \). Here \( R_i^c \) is the (“reduced”) \( \Sigma \)-projection of the closed simplicial ribbon \( R_i \) (cf. Sec. 4.4.4 in [24]) and we consider each \( R_i^c \) as a map \([0,1] \times S^1 \to \Sigma\), cf. Remark 4.3 in [24].

Finally, we set
\[
\text{WLO}_{\text{rig}}(L) := \frac{\text{WLO}_{\text{disc}}^\text{rig}(L)}{\text{WLO}_{\text{disc}}^\text{rig}(\emptyset)} \tag{3.22}
\]
where \( \emptyset \) is the “empty” link.

3.9 Two modifications

As we discovered in [24] we have to modify our original approach if we want to obtain the correct values for \( \text{WLO}_{\text{rig}}(L) \). In order to do so we will now make two modifications (Mod1) and (Mod2). More precisely, we will redefine \( \text{WLO}_{\text{disc}}^\text{rig}(L) \) according to the modifications (Mod1) and (Mod2). \( \text{WLO}_{\text{rig}}(L) \) will again be given by Eq. (3.22) (with the redefined version of \( \text{WLO}_{\text{disc}}^\text{rig}(L) \) appearing on the RHS).

Modification (Mod1)

Let us now reconsider the question of what a suitable discrete analogue \( \text{Det}^\text{disc}_{FP}(B) \) of the continuum expression \( \text{Det}_{FP}(B) = \text{det}(1_t - \exp(\text{ad}(B)))_{|\emptyset} \) should be. Above we made the ansatz
\[
\text{Det}^\text{disc}_{FP}(B) = \prod_{x \in \mathcal{S}_0(qK)} \text{det}(1_t - \exp(\text{ad}(B(x))))_{|\emptyset} \tag{3.23}
\]
We will now modify Eq. (3.23) and make instead the ansatz
\[
\text{Det}^\text{disc}_{FP}(B) := \prod_{x \in \mathcal{S}_0(qK)} \text{det}^{1/2}(1_t - \exp(\text{ad}(B(x))))_{|\emptyset} \tag{3.24}
\]
where \( \text{det}^{1/2}(1_t - \exp(\text{ad}(\cdot)))_{|\emptyset} : t \to \mathbb{R} \) is any of the two smooth functions \( f : t \to \mathbb{R} \) fulfilling \( \forall b \in t : f(b)^2 = \text{det}(1_t - \exp(\text{ad}(b)))_{|\emptyset} \), and which is given explicitly by
\[
\forall b \in t : \text{det}^{1/2}(1_t - \exp(\text{ad}(b)))_{|\emptyset} = \pm \prod_{\alpha \in \mathcal{R}_+} (2 \sin(\pi(\alpha, b))) \tag{3.25}
\]
where \( \mathcal{R}_+ \) on the RHS is the set of positive real roots associated to \((\mathfrak{g}, t)\) and the Weyl chamber \( C \) fixed above. Without loss of generality we will assume that the + sign on the RHS of Eq. (3.25) holds.

Remark 3.3 The inclusion of the exponent 1/2 on the RHS of Eq. (3.24) is necessary if we want to obtain the correct values for the WLOs. We will see later (cf. Appendix D.2 below) that – after making the transition to the \( BF_3 \)-theoretic setting as explained in Sec. 7 – it may be possible to obtain a better understanding of the origin of this exponent 1/2.

Alternatively, one can simply bypass this point by rewriting the heuristic equation (2.7) in a suitable way before discretizing it, cf. Sec. 2.5 and Sec. 3.6 in [26].
Modification (Mod2)

In the following we will replace\footnote{so, in particular, $DB$ will denote the normalized Lebesgue measure on $B_0(qK)$ where we have equipped $B_0(qK)$ with the scalar product induced by the one on $B(qK)$} the space $B(qK) = C^0(qK, t)$ appearing on the RHS of Eq. (3.21) by a certain subspace $B_0(qK)$ (chosen as naturally as possible) with the property that

$$\ker(\pi \circ (d_{qK})_{|B_0(qK)}) = B_c(qK)$$

holds where $\pi : C^1(qK, t) \to C^1(K, t)$ is the orthogonal projection w.r.t. $\llp \cdot, \cdot \rrp_{A_{2\Sigma}(qK)}$ and where we have set

$$B_c(qK) := \{B \in C^0(qK, t) \mid B \text{ constant}\}$$

(3.27)

We remark that Eq. (3.26) will play a crucial role in the proof of Theorem 3.5. (We also remark that we cannot choose simply $B_0(qK) = B(qK)$ because $\ker(\pi \circ d_{qK}) \neq B_c(qK)$).

The following choice is probably the best when working with simplicial ribbons in $K \times \mathbb{Z}_N$ instead of simplicial ribbons in $qK \times \mathbb{Z}_N$ (cf. Remark 3.2 above, and Remark 3.6 and Sec. 7 below).

**Choice 1** $B_0(qK) := \psi(B(K))$ where $B(K) := C^0(K, t)$ and where $\psi : B(K) \to B(qK)$ is the linear injection which associates to each $B \in B(K)$ the extension $\tilde{B} \in B(qK)$ given by

$$\tilde{B}(x) = \text{mean}_{y \in C(x)} B(y) \quad \text{for all } x \in \mathcal{F}_0(qK)$$

with “mean” referring to the arithmetic mean. Above $C(x)$ denotes the set of all $y \in \mathcal{F}_0(K)$ which lie in the closure of the unique open cell of $K$ containing $x$.

In the main part of the present paper we will work with simplicial ribbons in $qK \times \mathbb{Z}_N$. In this case Choice 1 will not work and we will make the following choice (motivated by Remark 2.3 above):

**Choice 2** $B_0(qK) := B_{\sigma_0}^{loc}(qK) \cap B_{\text{aff}}(qK)$ with

$$B_{\sigma_0}^{loc}(qK) := \{B \in B(qK) \mid B \text{ is constant on } U(\sigma_0) \cap \mathcal{F}_0(qK)\},$$

$$B_{\text{aff}}(qK) := \{B \in B(qK) \mid B \text{ is affine on each } F \in \mathcal{F}_2(qK)\}$$

Here $U(\sigma_0) \subset \Sigma$ is the union of those few $F \in \mathcal{F}_2(qK)$ which contain the point $\sigma_0$ and by “$B$ is affine on $F$” we mean that

$$B(p_1) + B(p_4) = B(p_2) + B(p_3)$$

holds where $p_1, p_2, p_3, p_4$ are the four vertices of $F$ and numbered in such a way that $p_1$ is diagonal to $p_4$ and therefore $p_2$ is diagonal to $p_3$.

**Remark 3.4**

i) It is not difficult to verify that $B_0(qK) = B_{\sigma_0}^{loc}(qK) \cap B_{\text{aff}}(qK)$ indeed satisfies Eq. (3.26) above.

ii) Above we said that we replace the space $B(qK)$ in Eq. (3.21) above by the subspace $B_0(qK)$. In fact, in Sec. 7 below many statements will hold (and many definitions make sense) for all $B \in B(qK)$ and not only for $B \in B_0(qK)$. Only in Sec. 5.2 the use of the space $B_0(qK)$ will be essential. In the other parts of Sec. 5 we can and will work with the full space $B(qK)$.

For later use we will also define the space

$$C_{\text{aff}}^0(qK, \mathbb{R}) := \{f \in C^0(qK, \mathbb{R}) \mid f \text{ is affine on each } F \in \mathcal{F}_2(qK)\}$$

where “$f$ is affine on $F$” we mean that

$$f(p_1) + f(p_4) = f(p_2) + f(p_3)$$

(3.29)

where $p_1, p_2, p_3, p_4$ are the four vertices of $F$ numbered again as above.
3.10 The main result

Recall that in Sec. 3.8 above we fixed a simplicial ribbon link \( L = (R_1, R_2, \ldots, R_m) \) in \( qK \times \mathbb{Z}_N \) with colors \( (\rho_1, \rho_2, \ldots, \rho_m) \). Let \( \Lambda_+ \) denote the set of dominant real weights associated to the pair \((\mathfrak{g}, t)\) and the Weyl chamber \( \mathcal{C} \). For each \( i \leq m \) we denote by \( \lambda_i \in \Lambda_+ \) the highest weight of \( \rho_i \).

We will now restrict ourselves to the special situation where \( L \) fulfills the following two conditions:

\((\text{NCP})'\) The maps \( R^i_{\Sigma} \) neither intersect each other nor themselves.\(^{16}\)

\((\text{NH})'\) Each of the maps \( R^i_{\Sigma}, i \leq m \) is null-homotopic.

Here \( R^i_{\Sigma} \) is as in Sec. 3.8 above and where we consider each \( R^i_{\Sigma} \) as a map \([0,1] \times S^1 \rightarrow \Sigma\), cf. Remark 4.3 in [24].

Recall that in [24] we stated the following theorem which will be proven in Sec. 5 below (and recall also the comments we made in Remark 6.5, Remark 6.6, and Remark 6.7 in [24])

**Theorem 3.5** Assume that the (colored) simplicial ribbon link \( L = (R_1, R_2, \ldots, R_m) \) in \( qK \times \mathbb{Z}_N \) fixed in Sec. 3.8 above fulfills conditions (\(\text{NCP})'\) and (\(\text{NH})'\) above. Assume also that \( k \geq c_{\mathfrak{g}} \) where \( c_{\mathfrak{g}} \) is the dual Coxeter number of \( \mathfrak{g} \) and that \( \lambda_i \in \Lambda^k_+ \) for \( i \leq m \), where \( \Lambda^k_+ \) is as in part A of the Appendix below. Then \( \text{WLO}_{\text{rig}}(L) \) is well-defined and we have

\[
\text{WLO}_{\text{rig}}(L) = \frac{|L|}{|\emptyset|} \tag{3.30}
\]

where \( \emptyset \) is the “empty link” and where \(|·|\) is the shadow invariant associated to \( \mathfrak{g} \) and \( k \), cf. part B of the Appendix.

**Remark 3.6** As mentioned in Remark 3.2 above, instead of working with simplicial ribbons in \( qK \times \mathbb{Z}_N \) (“half ribbons”) one could try to work with simplicial ribbons in \( K \times \mathbb{Z}_N \) (“full ribbons”). This would have several important advantages, cf. Choice 1 above, Remark 5.4 in Sec. 5.5 below, and Remark 6.1 in Sec. 6 below. On the other hand the use of “full ribbons” instead of “half ribbons” would also have an important disadvantage, cf. again Remark 5.4. This is why in Theorem 3.5 we only consider the case of half ribbons. We will come back to the case of full ribbons in Sec. 7 below and in [26].

4 Oscillatory Gauss-type measures on Euclidean spaces

In the present section we will recall two definitions introduced in Sec. 5 in [24] and then derive several elementary results which will play an important role in the proof of Theorem 3.5.

4.1 Basic Definitions

Let us fix a Euclidean vector space \((V, \langle·,·\rangle)\) and set \(d := \dim(V)\).

**Definition 4.1** An “oscillatory Gauss-type measure” on \((V, \langle·,·\rangle)\) is a complex Borel measure \(d\mu\) on \(V\) of the form

\[
d\mu(x) = \frac{1}{2\pi} e^{-\frac{1}{2}(x-m,S(x-m))} dx \tag{4.1}
\]

\(^{16}\) i.e. these maps have pairwise disjoint images and each \( R^i_{\Sigma} \), considered as a continuous map \([0,1] \times S^1 \rightarrow \Sigma\), is an embedding

\(^{17}\) the situation \(0 < k < c_{\mathfrak{g}}\) is not interesting since in this case the set \( \Lambda^k_+ \) is empty, cf. Remark 5.3 below. Accordingly, \(|L| = |\emptyset| = 0\). It turns out that we then also have \( \text{WLO}_{\text{rig}}^{\text{disc}}(L) = \text{WLO}_{\text{rig}}^{\text{disc}}(\emptyset) = 0\)
with \( Z \in \mathbb{C}\setminus\{0\} \), \( m \in V \), and where \( S \) is a symmetric endomorphism of \( V \) and \( dx \) the normalized Lebesgue measure on \( V \). Note that \( Z \), \( m \) and \( S \) are uniquely determined by \( d\mu \) so we can use the notation \( Z_{\mu} \), \( m_{\mu} \) and \( S_{\mu} \) in order to denote these objects.

i) We call \( d\mu \) “centered” iff \( m = 0 \).

ii) We call \( d\mu \) “degenerate” iff \( S \) is not invertible.

iii) We call \( d\mu \) “normalized” iff \( Z = \frac{(2\pi)^{d/2}}{\det^2(iS')} \) where \( S' := S_{\ker(S)^\perp} \). (See Example 4.4 below for the definition of \( \det^2(iS) \) and a motivation for the term “normalized”).

**Definition 4.2** Let \( d\mu \) be an oscillatory Gauss-type measure on \((V, \langle \cdot, \cdot \rangle)\). A (Borel) measurable function \( f : V \to \mathbb{C} \) will be called improperly integrable w.r.t. \( d\mu \) if \(^{19}\)

\[
\int_{V} f d\mu := \int_{V} f(x) d\mu(x) := \lim_{c \to 0} (\frac{2\pi}{\pi})^{n/2} \int f(x) e^{-\langle x \rangle^2} d\mu(x) \tag{4.2}
\]

exists. Here we have set \( n := \dim(\ker(S_{\mu})) \). Note that if \( d\mu \) is non-degenerate we have \( n = 0 \) so the factor \((\frac{2\pi}{\pi})^{n/2}\) is then trivial.

Most of the time we will consider non-degenerate oscillatory Gauss-type measures, the exception being Proposition 4.12 below.

Using a simple analytic continuation argument and the corresponding explicit formulas for Gaussian probability measures we can easily prove the existence of \( \int_{V} f d\mu \) and compute the corresponding value explicitly for a large class of functions \( f \). Let us illustrate this by looking at some simple examples:

**Example 4.3** Consider the special case where \( V = \mathbb{R} \), where \( \langle \cdot, \cdot \rangle \) is the scalar product given by \( \langle x, y \rangle = xy \), and where \( d\mu(x) = \exp(i(x,x)) dx = \exp(ix^2) dx \). Then the improper integrals

\[
\int_{-\infty}^{\infty} 1 \, d\mu(x), \quad \int_{-\infty}^{\infty} x \, d\mu(x), \quad \int_{-\infty}^{\infty} x^2 \, d\mu(x), \quad \int_{-\infty}^{\infty} e^{cx} \, d\mu(x), \quad c \in \mathbb{C}
\]

exist and are given explicitly by

- \( \int_{-\infty}^{\infty} 1 \, d\mu(x) = \sqrt{\pi} = \sqrt{\pi} e^{\frac{n}{2}} i \)
- \( \int_{-\infty}^{\infty} x \, d\mu(x) = 0 \)
- \( \int_{-\infty}^{\infty} x^2 \, d\mu(x) = \frac{i}{2} \sqrt{\pi} \)
- \( \int_{-\infty}^{\infty} e^{cx} \, d\mu(x) = e^{\frac{c^2}{4}} \sqrt{\pi} \)

where \( \sqrt{\cdot} : \mathbb{C}\setminus(-\infty,0) \to \mathbb{C} \) denotes the standard square root.

In order to show the existence (and to compute the explicit value) of \( \int_{-\infty}^{\infty} 1 \, d\mu(x) \) we consider the analytic function \( F : \{ z \mid Re(z) > 0 \} \to \mathbb{C} \) given by \( F(z) := \int \exp(-zx^2) dx \). According to a well-known formula we have \( F(a) = \sqrt{\pi/a} \) for all \( a \in (0,\infty) \). The obvious uniqueness argument for analytic functions now implies that \( F(z) = \sqrt{\pi/z} \) for all \( z \in \mathbb{C} \) with \( Re(z) > 0 \). Thus \( \int_{-\infty}^{\infty} 1 \, d\mu = \lim_{c \to \infty} F(e - i) = \lim_{c \to \infty} \sqrt{\pi/(c - e)} = \sqrt{\pi} \).

The other three integrals can be dealt with in a similar way.

\(^{18}\) i.e. unit hyper-cubes have volume 1 w.r.t. \( dx \)

\(^{19}\) Observe that \( \int_{\ker(S_{\mu})} e^{-\frac{1}{2} \|x\|^2} dx = \left( \frac{2\pi}{\pi} \right)^{-n/2} \). In particular, the factor \((\frac{2\pi}{\pi})^{n/2}\) in Eq. (4.2) above ensures that also for degenerate oscillatory Gauss-type measure the improper integrals \( \int_{-\infty}^{\infty} 1 \, d\mu \) exists, cf. Example 4.3 below.
In the next example \((V, \langle \cdot, \cdot \rangle)\) is again an arbitrary Euclidean space.

**Example 4.4** Let \(d\mu\) be a non-degenerate oscillatory Gauss-type measure on \((V, \langle \cdot, \cdot \rangle)\) with \(S\), \(m\), and \(Z\) given as in Eq. (4.1)

i) We have  
\[
\int_\sim 1 \, d\mu = \frac{1}{Z} \frac{(2\pi)^{d/2}}{\det \frac{1}{2} (iS)}
\]  
where we have set \(\det \frac{1}{2} (iS) := \prod_k \sqrt{i\lambda_k} = e^{\frac{\pi}{4} \sum_k \text{sgn}(\lambda_k) \left( \prod_k |\lambda_k|^{1/2} \right)} \) where \((\lambda_k)_k\) are the (real) eigenvalues of the symmetric matrix \(S\). In particular, \(d\mu\) is normalized in the sense of Definition 4.1 above iff \(\int_\sim 1 \, d\mu = 1\).

ii) In the special case when \(d\mu\) is normalized we have for all \(v, w \in V\)  
\[
\int_\sim \langle v, x \rangle \, d\mu(x) = \langle v, m \rangle, \quad \int_\sim \langle v, x \rangle \langle w, x \rangle \, d\mu(x) = \frac{1}{4} \langle v, S^{-1} w \rangle + \langle v, m \rangle \langle w, m \rangle
\]  

We will not try to identify the largest possible class of functions \(f\) for which \(\int_\sim f \, d\mu\) exists. For our purposes the function algebra \(\mathcal{P}_{\text{exp}}(V)\) defined in the next definition will be sufficient.

**Definition 4.5**

i) Let \(W\) be a finite-dimensional associative \(\mathbb{R}\)-algebra (with the standard topology). By \(\mathcal{P}_{\text{exp}}(V, W)\) we will denote the subalgebra of \(\text{Map}(V, W)\) which is generated by the affine maps \(\varphi : V \to W\) and their “exponentials” \(\exp_W \circ \varphi\). Here \(\exp_W : W \to W\) denotes the exponential map of \(W\).

ii) By \(\mathcal{P}_{\text{exp}}(V)\) we denote the subalgebra of \(\text{Map}(V, \mathbb{C})\) which is generated by the functions of the form \(\theta \circ f\) with \(f \in \mathcal{P}_{\text{exp}}(V, W)\) and \(\theta \in \text{Hom}_{\mathbb{R}}(W, \mathbb{C})\) where \(W\) is any finite-dimensional associative \(\mathbb{R}\)-algebra.

Using analytic continuation arguments, some explicit formulas for Gaussian probability measures, and suitable growth estimates one can prove the following result (which is easy to believe):

**Proposition 4.6** Let \(d\mu\) be a non-degenerate oscillatory Gauss-type measure on \((V, \langle \cdot, \cdot \rangle)\). Then for every \(f \in \mathcal{P}_{\text{exp}}(V)\) the improper integral \(\int_\sim f \, d\mu \in \mathbb{C}\) exists.

4.2 Three propositions

Let us fix for a while a normalized non-degenerate oscillatory Gauss-type measure \(d\mu\) on \((V, \langle \cdot, \cdot \rangle)\) and introduce the notation  
\[
\mathbb{E}_\sim[X] := \int_\sim X \, d\mu \in \mathbb{C} \tag{4.5a}
\]
\[
\text{cov}_\sim(X, X') := \mathbb{E}_\sim[XX'] - \mathbb{E}_\sim[X] \mathbb{E}_\sim[X'] \in \mathbb{C} \tag{4.5b}
\]
for maps \(X, X' \in \mathcal{P}_{\text{exp}}(V)\) (in analogy to the case of (Gaussian or non-Gaussian) probability measures on \(V\)).

**Observation 4.7** Let \(d\mu\) be as above and let \(X_1, X_2, \ldots, X_n\) be a sequence of affine maps \(V \to \mathbb{R}\).

---

20 we remark that if \(d\mu\) is degenerate then an analogous statement will hold with \(S\) replaced by \(S' := S_{|\ker(S)}\)

21 note that we do not keep \(W\) fixed here
i) If $\mathbb{E}_{\sim}[X_i] = 0$ for every $i \leq n$ then
\[
\mathbb{E}_{\sim}\left[\prod_j X_j\right] = \begin{cases} 
\frac{1}{(n/2)!2^{n/2}} \sum_{\sigma \in S_n} \prod_{i=1}^{n/2} \text{cov}_{\sim}(X_{\sigma(2i-1)}, X_{\sigma(2i)}) & \text{if } n \text{ is even} \\
0 & \text{if } n \text{ is odd}
\end{cases} (4.6)
\]

ii) If $\text{cov}_{\sim}(X_i, X_j) = 0$ for all $i, j \leq n$ with $i \neq j$ we have
\[
\mathbb{E}_{\sim}\left[\prod_j X_j\right] = \prod_j \mathbb{E}_{\sim}[X_j] (4.7)
\]

Here Eq. (4.6) follows from the analogous formula for the moments of a Gaussian probability measure and a suitable analytic continuation argument. Clearly, in the special case where $\text{cov}_{\sim}(X_i, X_j) = 0$ for $i, j \leq n$ with $i \neq j$ Eq. (4.6) reduces to $\mathbb{E}_{\sim}\left[\prod_j X_j\right] = 0$. By applying the latter equation to the subsequences of the sequence $X'_1, X'_2, \ldots, X'_n$ given by $X'_i := X_j - \mathbb{E}[X_j]$ we arrive at Eq. (4.7).

Let us now consider the case where $Y : V \to \mathbb{R}$ is an affine map with $\text{cov}_{\sim}(Y, Y) = 0$ and let us consider the trivial sequence $(X_i)_{i=1}^n$ where $X_i = Y$ for each $i \leq n$. Since $\text{cov}(Y, Y)_{\sim} = 0$ we trivially have $\text{cov}_{\sim}(X_i, X_j) = 0$ for $i \neq j$ (and even for $i = j$) so according to Eq. (4.7) we have $\mathbb{E}_{\sim}[Y^n] = \mathbb{E}_{\sim}[Y]^n$, from which we conclude, for example, that

\[
\mathbb{E}_{\sim}[\exp(Y)] = \mathbb{E}_{\sim}\left[\sum_n \frac{Y^n}{n!}\right] = \sum_n \mathbb{E}_{\sim}[\frac{Y^n}{n!}] = \sum_n \frac{\mathbb{E}_{\sim}[Y]^n}{n!} = \exp(\mathbb{E}_{\sim}[Y]) (4.8)
\]

In order to see that step (*) above holds observe that for every fixed $\epsilon > 0$ we have
\[
\int \sum_n \left| \frac{Y(x)^n}{n!} e^{-\epsilon \|x\|^2} \frac{1}{z} e^{-\frac{i}{2} (x-m, S(x-m))} \right| dx \leq \int \sum_n \left| \frac{Y(x)^n}{n!} \frac{1}{2} e^{-\epsilon \|x\|^2} \right| dx < \infty
\]
and we can therefore conclude from the dominated convergence theorem\(^{23}\) that
\[
\int \sum_n \frac{Y(x)^n}{n!} e^{-\epsilon \|x\|^2} d\mu(x) = \sum_n \int \frac{Y(x)^n}{n!} e^{-\epsilon \|x\|^2} d\mu(x)
\]
for each $\epsilon > 0$. Accordingly, in order to prove step (*) it is enough to prove that $\lim_{\epsilon \to 0} \sum_n \frac{I(n, \epsilon)}{n!} = \sum_n \lim_{\epsilon \to 0} \frac{I(n, \epsilon)}{n!}$ where we have set $I(n, \epsilon) := \int Y(x)^n e^{-\epsilon \|x\|^2} d\mu(x)$. The latter claim can easily be proven by computing the integrals $I(n, \epsilon)$ explicitly.

More generally, we obtain\(^{24}\) for every $\Phi \in \mathcal{P}_{\text{exp}}(\mathbb{R})$
\[
\mathbb{E}_{\sim}[\Phi(Y)] = \Phi(\mathbb{E}_{\sim}[Y]) (4.9)
\]

since every such $\Phi$ is necessarily entire analytic and the coefficients $(c_n)_{n}$, given by $\Phi(x) = \sum_{n=0}^{\infty} c_n x^n$ for all $x \in \mathbb{R}$, have the property that $c_n \overset{n \to \infty}{\to} 0$ rapidly enough so that

1. we can again apply the dominated convergence theorem in a similar way as above and prove that the two limit procedures $\int \cdots dx$ and $\sum_n$ can be interchanged

2. we can prove again that the two limit procedures $\lim_{\epsilon \to 0}$ and $\sum_n$ can be interchanged

---

\(^{22}\)Note that the condition $\text{cov}_{\sim}(Y, Y) = 0$ does not imply that $Y$ is a constant map on $V$. This is in sharp contrast to the situation for (Gaussian or non-Gaussian) probability measures where the relation $\text{cov}(Y, Y) = 0$ always implies $Y = \mathbb{E}[Y] \ d\mu$-a.s.

\(^{23}\)applied to the positive measure $dx$ “appearing” in $d\mu$

\(^{24}\)Observe that $\Phi(Y) \in \mathcal{P}_{\text{exp}}(V)$ so the existence of the LHS of Eq. (4.9) is guaranteed by Proposition 4.6
Finally, we can generalize Eq. (4.9) to the case where we have a \( \Phi \in \mathcal{P}_\text{exp}(\mathbb{R}^n) \), \( n \in \mathbb{N} \), and where \( (Y_k)_{k \leq n} \) is a sequence of affine maps \( V \to \mathbb{R} \) such that \( \text{cov}_\sim(Y_i, Y_j) = 0 \) holds for all \( i, j \leq n \). We then arrive at the following result, which will be the key argument in Sec. 5.1 below. In order to make the application of Proposition 4.8 in Sec. 5.1 more transparent we avoid the use of the notation \( \mathbb{E}_\sim[\cdot] \) and \( \text{cov}_\sim(\cdot, \cdot) \) from now on.

**Proposition 4.8** Let \( d\mu \) be a normalized non-degenerate oscillatory Gauss-type measure on \((V, \langle \cdot, \cdot \rangle)\) and let \((Y_k)_{k \leq n}, n \in \mathbb{N}\), be a sequence of affine maps \( V \to \mathbb{R} \) such that

\[
\int \sim Y_i Y_j d\mu = (\int \sim Y_i d\mu)(\int \sim Y_j d\mu)
\]

holds for all \( i, j \leq n \). Then for every \( \Phi \in \mathcal{P}_\text{exp}(\mathbb{R}^n) \) we have

\[
\int \sim \Phi((Y_k) d\mu) = \Phi((\int \sim Y_k d\mu)_k)
\]

**Remark 4.9** In Sec. 5.1 below we will actually apply a reformulation of Proposition 4.8 where the sequence \((Y_k)_{k \leq n}, n \in \mathbb{N}\), is replaced by a family \((Y^{i,a}_k)_{k \leq n, i \leq m, a \leq D}\) of affine maps fulfilling the obvious analogue of Eq. (4.10) above and the function \( \Phi \in \mathcal{P}_\text{exp}(\mathbb{R}^n) \) is replaced by a function \( \Phi \in \mathcal{P}_\text{exp}(\mathbb{R}^{m \times n \times D}) \).

**Example 4.10** Consider the (non-degenerate normalized centered) oscillatory Gauss-type measure \( d\mu(x) := \frac{1}{2\pi} \exp(i\langle x_1, x_2 \rangle) dx_1 dx_2 \) on \( V = \mathbb{R}^2 \). For every \( f \in \mathcal{P}_\text{exp}(\mathbb{R}) \) we have

\[
\int \sim f(x) d\mu(x) = f(0)
\]

This follows by applying Proposition 4.8 with \( \Phi = f \) and \( Y_1(x) := x_1 \). Observe that Eq. (4.10) is indeed fulfilled since according to Example 4.4 we have \( \int \sim Y_1 d\mu = \int \sim \langle x, e_1 \rangle d\mu(x) = 0 \) and \( \int \sim Y_1 Y_1 d\mu = \frac{1}{2}(e_1, S_{\mu}^{-1} e_1) = 0 \) where we have set \( e_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and used that \( S_{\mu} = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

Using a different argument one can easily prove that Eq. (4.12) holds for arbitrary continuous bounded functions \( f \). Moreover, we can include an additional “exponential factor”, and we can in fact consider more general oscillatory Gauss-type measures \( d\mu \), cf. Proposition 4.11 below, which will play a key role in Sec. 5.3 below.

As a preparation for Proposition 4.12 let us first consider the special case where the oscillatory Gauss-type measure \( d\mu \) is non-degenerate:

**Proposition 4.11** Assume that \( V = V_1 \oplus V_2 \) where \( V_1 \) and \( V_2 \) are two isomorphic subspaces of \( V \) which are orthogonal to each other. For each \( j = 1, 2 \) we denote the \( V_j \)-component of \( x \in V \) by \( x_j \). Moreover, let \( d\mu \) be a (non-degenerate centered) normalized oscillatory Gauss-type measure on \((V, \langle \cdot, \cdot \rangle)\) of the form \( d\mu(x) = \frac{1}{2\pi} \exp(i\langle x_2, M x_1 \rangle) dx \) for some linear isomorphism \( M : V_1 \to V_2 \). Then for every bounded continuous function \( f : V_1 \to \mathbb{C} \) and every fixed \( v \in V_2 \) we have

\[
\int \sim f(x_1) \exp(i\langle x_2, v \rangle) d\mu(x) = f(-M^{-1}v)
\]

(In particular, the LHS of Eq. (4.13) exists; note that the present situation is not covered by Proposition 4.6 above).
Proof. Let \( dx_1 \) resp. \( dx_2 \) be the normalized Lebesgue measure on \( V_1 \) resp. \( V_2 \) (equipped with the scalar product induced by the one on \( V \)). We have

\[
\int_{x_1} f(x_1) \exp(i\langle x_2, v \rangle) d\mu(x) = \frac{1}{Z} \lim_{\epsilon \to 0} \int_{V_1} \int_{V_2} e^{-\epsilon(|x_1|^2 + |x_2|^2)} f(x_1) \exp(i\langle x_2, v + Mx_1 \rangle) dx_2 dx_1 = \frac{(2\pi)^{d_2}}{Z} \lim_{\epsilon \to 0} \int_{V_1} e^{-\epsilon|x_1|^2} f(x_1) \delta_\epsilon(v + Mx_1) dx_1 \tag{4.14}
\]

where \( d_2 := \dim(V_2) = d/2 \) and where \( \delta_\epsilon : V_2 \to \mathbb{R} \) is given by

\[
\delta_\epsilon(w) := \frac{1}{(2\pi)^{d_2}} \int_{V_2} e^{-\epsilon|x_2|^2} \exp(i\langle x_2, w \rangle) dx_2 = \frac{1}{(2\pi)^{d_2}} (\pi \epsilon)^{d_2/2} e^{-\frac{1}{2} \frac{|w|^2}{\epsilon}} = \frac{1}{(4\pi)^{d_2/2}} e^{-\frac{|w|^2}{4\epsilon}} \tag{4.15}
\]

for all \( w \in V_2 \). Let us now fix an (arbitrary) isometry \( \psi : V_2 \to V_1 \). Clearly, the pushforward \( (\psi)_* dx_2 \) of \( dx_2 \) coincides with \( dx_1 \) so we have

\[
\int_{V_1} e^{-\epsilon|x_1|^2} f(x_1) \delta_\epsilon(v + Mx_1) dx_1 = \int_{V_2} e^{-\epsilon|x_2|^2} f(\psi x_2) \delta_\epsilon(v + M\psi x_2) dx_2 \tag{4.16}
\]

Making the change of variable \( M\psi x_2 + v \to y_2 \) on the RHS of Eq. (4.16) we obtain

\[
\int_{V_1} e^{-\epsilon|x_1|^2} f(x_1) \delta_\epsilon(v + Mx_1) dx_1 = |\det(M\psi)|^{-1} \int_{V_2} e^{-\epsilon|M^{-1}(y_2 - v)|^2} f(M^{-1}(y_2 - v)) \delta_\epsilon(y_2) dy_2 \tag{4.17}
\]

where \( dy_2 \) is the normalized Lebesgue measure on \( V_2 \). Since \( (\delta_\epsilon)_\epsilon \to \delta_0 \) is an “approximation to the identity” (i.e. converges weakly to the Dirac distribution \( \delta_0 \)) it is therefore clear26 that

\[
\lim_{\epsilon \to 0} \int_{V_1} e^{-\epsilon|x_1|^2} f(x_1) \delta_\epsilon(v + Mx_1) dx_1 = f(-M^{-1}v) \cdot |\det(M\psi)|^{-1} \tag{4.18}
\]

The assertion of the proposition now follows from Eq. (4.12), Eq. (4.18) and the following equation:

\[
\frac{(2\pi)^{d_2}}{Z} = \frac{(2\pi)^{d_2/2}}{Z} (\ast) \det \left( (iS_\mu)^t \right) = \det((M\psi)^t M\psi)^{1/2} = |\det(M\psi)| \tag{4.19}
\]

Here step \((\ast)\) follows from Example 4.4 and the assumption that \( d\mu \) is normalized, step \((\ast\ast)\) follows because26 for each eigenvalue \( \lambda \) of \( S_\mu \) also \(-\lambda\) is an eigenvalue of \( S_\mu \) and has the same multiplicity as \( \lambda \), and step \((\ast\ast\ast)\) follows because after making the identification \( V_2 \cong \psi V_1 \) we have \( S_\mu = -\left( \begin{array}{cc} 0 & M\psi \\ (M\psi)^t & 0 \end{array} \right) \).

\[\square\]

Convention 4 For a continuous function \( f : V_0 \to \mathbb{C} \) on a \( d_0 \)-dimensional Euclidean space \( V_0 \) we set

\[
\int_{V_0} f(x_0) dx_0 := \frac{1}{\pi^{d_0/2}} \lim_{\epsilon \to 0} \epsilon^{d_0/2} \int_{V_0} e^{-\epsilon|x_0|^2} f(x_0) dx_0 \tag{4.20}
\]

provided that the expression on the RHS of the previous equation is well-defined. Here \( dx_0 \) is the normalized Lebesgue measure on \( V_0 \).

---

26 A formal proof of Eq. (4.18) can be obtained after a suitable change of variable and the application of the dominated convergence theorem, cf. the proof of Eq. (4.23) below which generalizes Eq. (4.18).

26 This can be seen, e.g., from the equation \( J^{-1} S_\mu J = -S_\mu \) where \( J := \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right) \), \( 1 \) denoting both the identity in \( \text{End}(V_1) \) and \( \text{End}(V_2) \).
Proposition 4.12 Assume that $V = V_0 \oplus V_1 \oplus V_2$ where $V_0$, $V_1$, $V_2$ are pairwise orthogonal subspaces of $V$. For each $j = 0, 1, 2$ we denote the $V_j$-component of $x \in V$ by $x_j$. Moreover, let $d\mu$ be a (centered) normalized oscillatory Gauss-type measure on $(V, \langle \cdot, \cdot \rangle)$ of the form $d\mu(x) = \frac{1}{z} \exp(i \langle x, Mx \rangle) dx$ for some linear isomorphism $M : V_1 \rightarrow V_2$. Then for every fixed $v \in V_2$ and every bounded uniformly continuous function $F : V_0 \oplus V_1 \rightarrow \mathbb{C}$ the LHS of the following equation exists iff the RHS exists and in this case we have

$$\int_{V_0} F(x_0 + x_1) \exp(i \langle x_2, v \rangle) d\mu(x) = \int_{V_0} F(x_0 - M^{-1}v) dx_0$$

(4.21)

where $dx_0$ is the normalized Lebesgue measure on $V_0$.

**Proof.** We set $d_j := \dim(V_j)$ for $j = 0, 1, 2$. Similarly as in Eq. (4.14) and with $\delta_\epsilon : V_2 \rightarrow \mathbb{R}$ as above we obtain

$$\int_{V_0} F(x_0 + x_1) \exp(i \langle x_2, v \rangle) d\mu(x)$$

$$= \lim_{\epsilon \to 0} \epsilon^{d_0/2} \int_{V_0} \left[ \int_{V_1} \left[ \int_{V_2} e^{-\epsilon (|x_0|^2 + |x_1|^2 + |x_2|^2)} F(x_0 + x_1) \exp(i \langle x_2, v \rangle) \times \right. \right. \right.$$

$$\times \frac{1}{2} \exp(i \langle x_2, Mx_1 \rangle) dx_1 \left. \right] dx_0$$

$$= \frac{1}{\pi^{d_0/2}} \lim_{\epsilon \to 0} \epsilon^{d_0/2} \int_{V_0} \int_{V_1} dx_1 e^{-\epsilon |x_1|^2} F(x_0 + x_1) \delta_\epsilon(v + Mx_1).$$

(4.22)

Let $\psi : V_2 \rightarrow V_1$ be a fixed isometry. From the assumption that $d\mu$ was normalized it follows – using the same argument as in Eq. (4.19) above – that $\langle (2\pi)^d \rangle = \det \frac{1}{z} (i(S_\mu)|V_1 \oplus V_2) = |\det(M\psi)|$. Eq. (4.22) above and the equality just mentioned will therefore imply the assertion of the proposition provided that we can show

$$\lim_{\epsilon \to 0} T(\epsilon) = 0$$

(4.23)

where

$$T(\epsilon) := \epsilon^{d_0/2} \int_{V_0} \int_{V_1} dx_1 e^{-\epsilon |x_1|^2} \left[ F(x_0 - M^{-1}v) - |\det(M\psi)| \right] \int_{V_1} dx_1 e^{-\epsilon |x_1|^2} F(x_0 + x_1) \delta_\epsilon(v + Mx_1)$$

(4.24)

In order to prove (4.23) recall that $\psi_\epsilon(dx_2) = dx_1$ and that $1 = \frac{1}{\pi^{d_2/2}} \int_{V_2} e^{-|y_2|^2} dy_2$ so we obtain

$$T(\epsilon) = \epsilon^{d_0/2} \int_{V_0} \int_{V_2} dy_2 e^{-\epsilon |y_2|^2} F(x_0 - M^{-1}v)$$

$$- |\det(M\psi)| \int_{V_2} dy_2 e^{-\epsilon |\psi y_2|^2} F(x_0 + \psi y_2) \frac{1}{(4\epsilon)^{d_2/2}} e^{-\frac{|v + M\psi y_2|^2}{4\epsilon}}$$

$$\overset{(*)}{=} \frac{1}{\pi^{d_2/2}} \int_{V_0 \oplus V_2} dy_0 dy_2 e^{-\epsilon |y_0|^2 - |y_2|^2} \left[ F(\frac{y_0}{\sqrt{\epsilon}} - M^{-1}v) - \gamma_\epsilon(y_2) F(\frac{y_0}{\sqrt{\epsilon}} - M^{-1}v + \sqrt{\epsilon}2M^{-1}y_2) \right]$$

(4.25)

In Step $(*)$ we have made the changes of variable $\sqrt{\epsilon}y_0 \rightarrow y_0$ and $\frac{1}{\sqrt{\epsilon}^2} (v + M\psi y_2) \rightarrow y_2$ and we have set $\gamma_\epsilon(y_2) := e^{-\epsilon |M^{-1}(2\sqrt{\epsilon}y_2 - v)|^2}$. Relation (4.23) now follows by applying the dominated convergence theorem to the last expression in Eq. (4.25) and taking into account that for all fixed $y_0 \in V_0$ and $y_2 \in V_2$ we have

$$\lim_{\epsilon \to 0} \left[ F\left(\frac{y_0}{\sqrt{\epsilon}} - M^{-1}v\right) - \gamma_\epsilon(y_2) F\left(\frac{y_0}{\sqrt{\epsilon}} - M^{-1}v + \sqrt{\epsilon}2M^{-1}y_2\right) \right]$$

$$= \lim_{\epsilon \to 0} \left[ F\left(\frac{y_0}{\sqrt{\epsilon}} - M^{-1}v\right)(1 - \gamma_\epsilon(y_2)) \right]$$

$$+ \lim_{\epsilon \to 0} \gamma_\epsilon(y_2) \left[ F\left(\frac{y_0}{\sqrt{\epsilon}} - M^{-1}v\right) - F\left(\frac{y_0}{\sqrt{\epsilon}} - M^{-1}v + \sqrt{\epsilon}2M^{-1}y_2\right) \right] = 0 + 0 = 0$$
since \( \lim_{\epsilon \to 0} \gamma_\epsilon(y_2) = 1 \) and, by assumption, \( F \) is bounded and uniformly continuous.

The following remark will be useful in Sec. 5.2 and Sec. 5.4 below.

**Remark 4.13** If \( \Gamma \) is a lattice in \( V \) and \( f : V \to \mathbb{C} \) a \( \Gamma \)-periodic continuous function then \( \int_V \sim f(x)dx \) exists and we have

\[
\int_V \sim f(x)dx = \frac{1}{\text{vol}(Q)} \int_Q f(x)dx
\]

with \( Q := \{ \sum_i x_i e_i \mid 0 \leq x_i \leq 1 \forall i \leq d \} \) where \( (e_i)_{i \leq d} \) is any fixed basis of the lattice \( \Gamma \) and where \( \text{vol}(Q) \) denotes the volume of \( Q \). Observe that Eq. (4.26) implies

\[
\forall y \in V : \int_V \sim f(x)dx = \int_V \sim f(x + y)dx \quad (4.27)
\]

### 5 Proof of Theorem 3.5

Recall that Theorem 3.5 states that in the special situation described above \( \text{WLO}^{\text{rig}}(L) \) is well-defined and has the value \( |L|/|\emptyset| \). In the following we will concentrate on the “computational half” of this statement. That \( \text{WLO}^{\text{rig}}(L) \) is well-defined in the first place will also become clear during the computations, even though we will rarely make explicit statements in this direction.

**Convention 5** In the following \( \sim \) will denote equality up to a multiplicative non-zero constant. This “constant” may depend on \( G, N, K \), and \( k \) but it will never depend on the (simplicial ribbon) link \( L \).

#### 5.0 Some preparations

**a) Computation of \( \det(L(N)(B)) \)**

Recall that in Sec. 3 above we used the notation \( L(N)(B) \) both for the linear operator \( \mathcal{A}^\perp(K) \to \mathcal{A}^\perp(K) \) and the restriction of this operator to the invariant subspace \( \mathcal{A}^\perp(K) \). From now on the notation \( L(N)(B) \) will always refer to the restricted operator.

**Proposition 5.1** For \( B \in B(qK) \) we have

\[
\det(L(N)(B)) = N^d \prod_{\bar{e} \in \mathfrak{g}_0(K_1|K_2)} \det(1_t \! - \! \exp(B(\bar{e})))|_t|^2 \quad (5.1)
\]

where \( d := \dim(A^\perp(K)) \). In particular, if

\[
B \in B_{\text{reg}}(qK) := \{ B \in B(qK) \mid B(x) \in t_{\text{reg}} \text{ for all } x \in \mathfrak{g}_0(qK) \} \quad (5.2)
\]

then we have \( \det(L(N)(B)) \neq 0 \).

According to Eqs. (3.9), (3.10a), (3.10b) in order to prove Proposition 5.1 it will be enough to prove Lemma 11 below.

Recall the definition of the two linear operators \( \hat{L}(N)(b) \) and \( \hat{L}(N)(b) \) on \( \text{Map}(\mathbb{Z}_N, g) \) for fixed \( b \in t \), cf. Eqs. (3.8) above.

\[27\] in order to check well-definedness we should, of course, reverse the order of our considerations/computations: we first check that the expressions appearing in Step 6 are well-defined. Based on this we can verify that also the expressions in Step 5 must be well-defined and so on until we arrive at the expressions in Step 1.
In the following we will consider the restriction of each of these two operators to the orthogonal complement of its kernel. The restrictions will again be denoted by $\hat{L}^{(N)}(b)$ and $\hat{L}^{(N)}(b)$. 

**Lemma 1** We have

$$\det(\hat{L}^{(N)}(b)) = \pm \det(1_t - \exp(\text{ad}(b)))_{|t} \cdot N^d,$$  
(5.3)

$$\det(\hat{L}^{(N)}(b)) = \pm \det(1_t - \exp(\text{ad}(b)))_{|t} \cdot N^d,$$  
(5.4)

where $d := \dim(\text{Map}(Z_N, g)) = N \dim(g)$, $r := \dim(t)$.

**Proof.** Let us prove Eq. (5.3). The proof of Eq. (5.4) is similar. First observe that

$$\det(\hat{L}^{(N)}(b)) = \det(N(\tau_1 e^{\text{ad}(b)/N} - 1)) = N^d \det(\tau_1 - e^{-\text{ad}(b)/N})$$  
(5.5)

where $d' := \dim(\text{Map}'(Z_N, g)) = d - \dim(t)$ and where we have used that $e^{\text{ad}(b)/N}$ is orthogonal. The complexified operator $(\tau_1 - e^{-\text{ad}(b)/N}) \otimes \text{id}_C$ is diagonalizable with eigenvalues

$$\lambda_{k, \alpha} := e^{\frac{2\pi i k}{N} - \frac{-\alpha(b)}{N}}, \quad \text{for each } k \in \{1, 2, \ldots, N\}, \alpha \in \mathcal{R}_C$$

$$\mu_{k, a} := e^{\frac{2\pi i k}{N} - 1}, \quad \text{for each } k \in \{1, 2, \ldots, N - 1\}, a \in \{1, 2, \ldots, r\}$$

where $\mathcal{R}_C$ denotes the set of complex roots of $g$ w.r.t. $t$ (cf. part \(A\) of the Appendix). Using the two polynomial equations $x^N - 1 = \prod_{k=0}^{N-1} (x - e^{\frac{2\pi i k}{N}}) = (-1)^N \prod_{k=0}^{N-1} (e^{\frac{2\pi i k}{N}} - x)$ and $x^N - 2 + \ldots + 1 = \prod_{k=1}^{N-1} (x - e^{\frac{2\pi i k}{N}}) = (-1)^{N-1} \prod_{k=0}^{N-1} (e^{\frac{2\pi i k}{N}} - x)$ we therefore obtain

$$\det(\tau_1 - e^{-\text{ad}(b)/N}) = \det_{\mathcal{C}}((\tau_1 - e^{-\text{ad}(b)/N}) \otimes \text{id}_C)$$

$$= \left(\prod_{\alpha \in \mathcal{R}_C} \prod_k \left(e^{\frac{2\pi i k}{N} - \frac{-\alpha(b)}{N}}\right)\right) \left(\prod_{\alpha \neq 0} (\prod_{k \neq 0} (e^{\frac{2\pi i k}{N}} - 1)\right)$$

$$= \left(\prod_{\alpha \in \mathcal{R}_C} (-1)^N (e^{-\alpha(b)} - 1)\right) \left(\prod_{\alpha = 1} (-1)^{N-1} \{N\}\right) = (-1)^{r(N-1)} N^r \prod_{\alpha \in \mathcal{R}_C} \{e^{-\alpha(b)} - 1\}$$

The assertion now follows by combining the last equation with Eq. (5.5) above and by taking into account the relations $d = d' + r$ and $\prod_{\alpha \in \mathcal{R}_C} (e^{-\alpha(b)} - 1) = \prod_{\alpha \in \mathcal{R}_C} (1 - e^{\alpha(b)}) = \det(1_t - \exp(\text{ad}(b)))_{|t}$.

\[\square\]

**b) Some consequences of conditions (NCP)' and (NH)'**

Let $L = (R_1, R_2, \ldots, R_m)$ be the simplicial ribbon link in $qK \times Z_N$ fixed in Sec. \(3.8\) above. Recall that each $R_i = (F_k)_{k \leq a_i}, a_i \in \mathbb{N}$, “induces”\(28\) two simplicial loops $l_i$ and $l'_i$ in $qK \times Z_N$. In the following we use the short notation $L_{\Sigma}^2 := (l_i)_{\Sigma}, L_{\Sigma}^2 := (l'_i)_{\Sigma}, L_{S^1} := (l_i)_{S^1}$, and $L_{S^1} := (l'_i)_{S^1}$ for the $\Sigma$- or $S^1$-projections of these loops and we will often consider $L_{\Sigma}^2$ and $L_{S^1}^2$ as (piecewise smooth) maps $S^1 \to \Sigma$. Clearly, arc($L_{\Sigma}^2$) and arc($L_{S^1}^2$) can then be considered as subsets of $\Sigma$.

It is not difficult to see that the two conditions (NCP)' and (NH)' on our simplicial ribbon link $L$ imply the following conditions:

---

\[\text{w.r.t. the obvious scalar product on Map}(Z_N, g)\]

\[\text{for } b \in \text{reg}, \text{which is the case relevant for us, we have ker}(\hat{L}^{(N)}(b)) = ker(\hat{L}^{(N)}(b)) = \text{Map}_c(Z_N, t)\]

\[\text{with Map}_c(Z_N, t) := \{ f \in \text{Map}(Z_N, g) \mid f \text{ is a constant function taking values in } t \} \cong t. \text{ The orthogonal complement Map}'(Z_N, g) \text{ of Map}_c(Z_N, t) \text{ is given by Map}'(Z_N, g) = \{ f \in \text{Map}(Z_N, g) \mid \sum f(t) = 0 \}\]

\[\text{the - sign in Eqs. (5.3) and (5.4) holds iff both } r \text{ and } N - 1 \text{ are odd}\]

\[l_i \text{ and } l'_i \text{ are just the two loops on the boundary of } R_i.\]
(FC1) For all $i,j \leq m$ we have $\text{arc}(l^i_\Sigma) \cap \text{arc}(l^j_\Sigma) = \emptyset$ and we also have $\text{arc}(l^i_\Sigma) \cap \text{arc}(l^i_\Sigma) = \emptyset$ if $i \neq j$.

(FC2) For each $i \leq m$ the open region $O_i \subset \Sigma$ “between” $\text{arc}(l^i_\Sigma)$ and $\text{arc}(l^i_\Sigma)$ does not contain an element of $\mathcal{F}_0(qK)$.

(FC3) For each $i \leq m$ and every $F \in \mathcal{F}_2(qK)$ with $F \subset \text{Image}(R^i_\Sigma)$ exactly one of the four sides of (the tetragon) $F$ will lie on $\text{arc}(l^i_\Sigma)$ and exactly one side will lie on $\text{arc}(l^i_\Sigma)$.

(FC4) $l^i_{S_1} = l^i_{S_1}$ is fulfilled for each $i \leq m$.

In order to simplify the notation in Secs 5.1, 5.2 below we will set

$$n := \max_{i \leq m} n_i$$

and we will extend each simplicial loop $l^i_\Sigma$, $l'^i_\Sigma$, $l^i_{S_1}$, $l'^i_{S_1}$ to a simplicial loop with “length” $n$ in a trivial way, i.e. by “adding” $n - n_i$ empty edges. For the extended simplicial loops we will use the same notation.

Finally we set, for each $i \leq m$ and $k \leq n$,

$$\tilde{I}^{(k)}_\Sigma := \pi(I^{(k)}_\Sigma), \quad \text{and} \quad \tilde{I}^{(k)}_\Sigma := \pi(I^{(k)}_\Sigma)$$

where $\pi : C_1(qK) \to C_1(K)(\subset C_1(qK))$ is the orthogonal projection. Observe that for each $i \leq m$ we have

$$\forall k \leq n : \tilde{I}^{(k)}_\Sigma \in C_1(K_1) \quad \text{and} \quad \forall k \leq n : \tilde{I}^{(k)}_\Sigma \in C_1(K_2)$$

(5.7a)

or

$$\forall k \leq n : \tilde{I}^{(k)}_\Sigma \in C_1(K_2) \quad \text{and} \quad \forall k \leq n : \tilde{I}^{(k)}_\Sigma \in C_1(K_1)$$

(5.7b)

From (FC1) and Eqs 5.7 it easily follows that for all $i_1, i_2 \leq m$, and $k_1, k_2 \leq n$ we have

$$\star_K \tilde{I}^{(k_1)}_\Sigma \neq \pm \tilde{I}^{(k_2)}_\Sigma$$

(5.8)

provided that $\tilde{I}^{(k_1)}_\Sigma \neq 0$ and $\tilde{I}^{(k_2)}_\Sigma \neq 0$. Here $\star_K$ is the linear isomorphism on $C_1(K) = C^1(K, \mathbb{R})$ which is defined exactly in the same way as the operator $\star_K$ on $A_K(K) = C^0(K, \mathbb{R})$ which we introduced in Sec. 5.3 above.

5.1 Step 1: Performing the $\int_{\sim} \cdots \exp(iS^{\text{disc}}_{CS}(\bar{A}^1, B))D\bar{A}^1$ integration in Eq. 3.21

Lemma 2 Under the assumptions on the simplicial ribbon link $L = (R_1, \ldots, R_m)$ made above we have for every fixed $A^1_c \in A^1_+(K)$ and $B \in \mathcal{B}_{reg}(qK)$

$$\int_{\sim} \prod_i \text{Tr}_{\rho_i}(\text{Hol}^{\text{disc}}_{R_i}(\bar{A}^1 + A^1_c, B)) \exp(iS^{\text{disc}}_{CS}(\bar{A}^1, B))D\bar{A}^1 = Z^{\text{disc}}_B \prod_i \text{Tr}_{\rho_i}(\text{Hol}^{\text{disc}}_{R_i}(A^1_c, B))$$

(5.9)

where $Z^{\text{disc}}_B := \int_{\sim} \exp(iS^{\text{disc}}_{CS}(\bar{A}^1, B))D\bar{A}^1$

Proof. Let $A^1_c \in A^1_+(K)$ and $B \in \mathcal{B}_{reg}(qK)$ be as in the assertion of the lemma. In order to prove the lemma we will apply Proposition 4.8 (and Remark 4.9 above) to the special situation where (cf. Convention 3 in Sec. 3.7)

---

32 more precisely, $O_i$ is the interior of the subset Image($R^i$) of $\Sigma$

33 or, more precisely, both $\star_K R^i_{\Sigma} \neq \tilde{I}^{(k_2)}_\Sigma$ and $\star_K \tilde{I}^{(k_1)}_\Sigma \neq \pm \tilde{I}^{(k_2)}_\Sigma$ etc.
• $V = \mathcal{A}^\perp(K)$,

• $d\mu = d\nu^\text{disc}_B$ with $d\nu^\text{disc}_B = \frac{1}{Z^\text{disc}_B} \exp(iS^\text{disc}_{CS}(\mathcal{A}^\perp, B))D\mathcal{A}^\perp$,

• $(Y^{i,a}_k)_{i \leq m, k \leq n, a \leq \dim(\mathfrak{g})}$ is the family of maps $Y^{i,a}_k : \mathcal{A}^\perp(K) \to \mathbb{R}$ given by

$$Y^{i,a}_k(\mathcal{A}^\perp) := \left< T_a, \left( \mathcal{A}^\perp(\bullet i_{S^1}^{i(k)}) \right)(\frac{1}{2} i_{\Sigma}^{i(k)} + \frac{1}{2} i'_\Sigma^{i(k)}) + A^\perp(\frac{1}{2} i_{\Sigma}^{i(k)} + \frac{1}{2} i'_\Sigma^{i(k)}) + (\frac{1}{2} B(\bullet i_{\Sigma}^{i(k)}) + \frac{1}{2} B(\bullet i'_\Sigma^{i(k)})) \cdot dt^{(N)}(i_{S^1}^{i(k)}) \right> \quad (5.10)$$

where $(T_a)_{a \leq \dim(\mathfrak{g})}$ is an arbitrary $(\cdot, \cdot)$-ONB of $\mathfrak{g}$ (which will be kept fixed in the following), and

• $\Phi : \mathbb{R}^{m \times n \times \dim(\mathfrak{g})} \to \mathbb{C}$ is given by

$$\Phi((x^i_k)_{i,k,a}) = \prod_{i=1}^m \text{Tr}_{\rho_i}(\prod_{k=1}^n \exp(\sum_{a=1}^{\dim(\mathfrak{g})} T_a x^i_k)) \quad \text{for all } (x^i_k)_{i,k,a} \in \mathbb{R}^{m \times n \times \dim(\mathfrak{g})} \quad (5.11)$$

Observe that

i) $d\nu^\text{disc}_B$ is a well-defined normalized non-degenerate centered oscillatory Gauss-type measure. By assumption $B \in \mathcal{B}_\text{reg}(q\mathcal{K})$ this follows from Eq. (3.13b) and Proposition 3.1 in Sec. 3.3 above and from Proposition 5.1 in Sec. 5.0.

For later use let us mention that according to Example 4.4 above and Eq. (5.1) in Proposition 5.1 above we have

$$Z^\text{disc}_B = \int_{\sim} \exp(iS^\text{disc}_{CS}(\mathcal{A}^\perp, B))D\mathcal{A}^\perp \sim \det(L^{(N)}(B))^{-1/2} \sim \prod_{L \in \mathfrak{g}_0(\mathfrak{K}_1|\mathfrak{K}_2)} \det(1 - \text{ad}(B(\tilde{c})))^{-1} \quad (5.12)$$

ii) $\Phi \in \mathcal{P}_\exp(\mathbb{R}^{m \times n \times \dim(\mathfrak{g})})$ since

$$\Phi((x^i_k)_{i,k,a}) = \prod_i \text{Tr}_{\rho_i}(\prod_k \exp(\sum_a T_a x^i_k)) = \prod_i \text{Tr}_{\text{End}(V_i)}(\prod_k \rho_i(\exp(\sum_a T_a x^i_k)))$$

$$= \prod_i \text{Tr}_{\text{End}(V_i)}(\prod_k \exp_{\text{End}(V_i)}(\sum_a ((\rho_i)_* T_a) x^i_k)) \quad (5.13)$$

where $\exp_{\text{End}(V_i)}$ is exponential map of the associative algebra $\text{End}(V_i)$ and $(\rho_i)_* : \mathfrak{g} \to \text{gl}(V_i)$, for $i \leq m$, is the Lie algebra representation induced by $\rho_i$.

iii) For all $i \leq m$, $k \leq n$, $a \leq \dim(\mathfrak{g})$ we have

$$\int_{\sim} Y^{i,a}_k d\nu^\text{disc}_B = Y^{i,a}_k(0) \quad (5.14)$$

In order to see this let us introduce $j : \mathbb{Z}_N \to C_1(q\mathcal{K})$ by

$$j(t) := \begin{cases} \frac{1}{2} i_{\Sigma}^{i(k)} + \frac{1}{2} i'_\Sigma^{i(k)} & \text{if } t = \bullet i_{S^1}^{i(k)} \\ 0 & \text{if } t \neq \bullet i_{S^1}^{i(k)} \end{cases} \quad (5.15)$$

\footnote{observe that in view of condition (FC4) above the RHS of Eq. (5.10) is very closely related to the RHS of Eq. (3.11) in Sec. 3.4}

\footnote{observe that the vector spaces underlying $\text{End}(V_i)$ and $\text{gl}(V_i)$ coincide}
and set \( \tilde{j}_a := p(T_a j) \) where \( p : \mathcal{A}^\perp(q\mathcal{K}) \to \bar{A}^\perp(K) \) is the \( \ll \cdot, \cdot \gg_{\mathcal{A}^\perp(q\mathcal{K})} \)-orthogonal projection onto the subspace \( \bar{A}^\perp(K) \) of \( \mathcal{A}^\perp(q\mathcal{K}) \) and where \( T_a j := j \otimes T_a \in \mathcal{A}^\perp(q\mathcal{K}) \cong \operatorname{Map}(\mathbb{Z}_N, C_1(q\mathcal{K})) \otimes \mathfrak{g} \). Then

\[
Y^i_{k,a}(\bar{A}^\perp) - Y^i_{k,a}(0) = (T_a, ((\bar{A}^\perp)(\mathfrak{l}^{i(k)}))((\frac{1}{2} l^{i(k)} + \frac{1}{2} l^{i(k)}) \ll \bar{A}^\perp, T_a \gg_{\mathcal{A}^\perp(q\mathcal{K})} = \ll \bar{A}^\perp, \tilde{j}_a \gg_{\mathcal{A}^\perp(q\mathcal{K})}
\]

On the other hand since \( d\nu^\text{disc}_B \) is centered Example 4.4 above implies that \( \int_{\sim} \ll \cdot, \tilde{j}_a \gg_{\mathcal{A}^\perp(q\mathcal{K})} d\nu^\text{disc}_B = 0 \). Since \( d\nu^\text{disc}_B \) is also normalized we obtain Eq. (5.14).

iv) For all \( i, i' \leq m, \ k, k' \leq n, \ a, a' \leq \dim(\mathfrak{g}) \) we have

\[
\int_{\sim} Y^i_{k,a} Y^{i'}_{k',a'} d\nu^\text{disc}_B = \int_{\sim} Y^i_{k,a} d\nu^\text{disc}_B \int_{\sim} Y^{i'}_{k',a'} d\nu^\text{disc}_B
\]

(5.15)

This follows from Eq. (5.14) above and

\[
\int_{\sim} (Y^i_{k,a} - Y^i_{k,a}(0))(Y^{i'}_{k',a'} - Y^{i'}_{k',a'}(0)) d\nu^\text{disc}_B = \int_{\sim} \ll \tilde{j}_a, \cdot \gg_{\mathcal{A}^\perp(q\mathcal{K})} \ll \tilde{j}'_{a'}, \cdot \gg_{\mathcal{A}^\perp(q\mathcal{K})} d\nu^\text{disc}_B
\]

(5.16)

where \( \tilde{j}_a \) is as in point iii) above and where \( \tilde{j}'_{a'} \) is defined in a completely analogous way with \( i, \ k, \) and \( a \) replaced by \( i', \ k', \) and \( a' \). Here step (*) follows from Example 4.4 above and step (***) follows from the inequalities (5.8) appearing at the end of Sec. 5.0 above.

Thus the assumptions of Proposition 4.8 above are fulfilled and we obtain

\[
\frac{1}{Z^\text{disc}_B} \int_{\sim} \prod_i \operatorname{Tr}_{\rho_i} \left( \operatorname{Hol}^\text{disc}_{R_i} \left( \bar{A}^\perp + A^\perp_c, B \right) \right) \exp(iS^\text{disc}_{CS}(\bar{A}^\perp, B)) D\bar{A}^\perp
\]

\[
= (\int_{\sim} \prod_i \operatorname{Tr}_{\rho_i} \left( \prod_k \exp \left( \sum_a T_a Y^i_{k,a} \right) \right) d\nu^\text{disc}_B = \int_{\sim} \Phi((Y^i_{k,a}), i, k, a) d\nu^\text{disc}_B = \Phi((Y^i_{k,a}), i, k, a)\]

\[
= \Phi((Y^i_{k,a}(0)), i, k, a) = \prod_i \operatorname{Tr}_{\rho_i} \left( \prod_k \exp \left( \sum_a T_a Y^i_{k,a}(0) \right) \right) = \prod_i \operatorname{Tr}_{\rho_i} \left( \operatorname{Hol}^\text{disc}_{R_i} (A^\perp_c, B) \right)
\]

(5.17)

Here step (+) follows from the definitions and condition (FC4) in Sec. 5.0 above, step (*) follows from Proposition 4.8 and Remark 4.9 above, and step (***) follows from Eq. (5.14) above.

\[\square\]

Using Lemma 2 and taking into account the implication

\[
\prod_{x} 1^{(s)}_{\mathcal{B}_{\text{reg}}}(B(x)) \neq 0 \implies B \in \mathcal{B}_{\text{reg}}(q\mathcal{K})
\]

for all \( s > 0 \) we now obtain from Eq. (3.21)

\[
\operatorname{WLO}^\text{disc}_{\text{rig}}(L) = \lim_{s \to 0} \sum_{y \in I} \int_{\sim} \left\{ \prod_{x} 1^{(s)}_{\mathcal{B}_{\text{reg}}}(B(x)) \prod_i \operatorname{Tr}_{\rho_i} \left( \operatorname{Hol}^\text{disc}_{R_i} (A^\perp_c, B) \right) \operatorname{Det}^\text{disc}(B) \right\} \times \exp(-2\pi ik\langle y, B(s_0) \rangle) \operatorname{exp}(iS^\text{disc}_{CS}(A^\perp_c, B))(D A^\perp_c \otimes DB)
\]

(5.18)

where we have set

\[
\operatorname{Det}^\text{disc}(B) := \operatorname{Det}^\text{disc}_{FP}(B) Z^\text{disc}_B
\]

(5.19)
5.2 Step 2: Performing the $\int_\sim \cdots \exp(iS_{CS}^{\text{disc}}(A_c^\perp, B))(DA_c^\perp \otimes DB)$-integration in (5.18)

With the help of Proposition 4.12 above let us now evaluate the $\int_\sim \cdots \exp(iS_{CS}^{\text{disc}}(A_c^\perp, B))(DA_c^\perp \otimes DB)$-integral appearing in Eq. (5.18). In order to do so we first rewrite the integrand in Eq. (5.18) in such a way that it assumes the form of the integrand on the LHS of the formula appearing in Proposition 4.12 above. In order to achieve this we will now exploit the fact that all of the remaining fields $A_c^\perp$ and $B$ in Eq. (5.18) take values in the Abelian Lie algebra $\mathfrak{t}$. For fixed $A_c^\perp$ and $B$ we can therefore rewrite $\text{Hol}^{\text{disc}}(A_c^\perp, B)$ as an exponential of a sum, namely as

$$\text{Hol}^{\text{disc}}(A_c^\perp, B) = \exp(\Phi_i(B) + \sum_k A_c^\perp \left( \frac{1}{2} \ell_{\Sigma}^{(k)} + \frac{1}{2} \ell_{\Sigma}^{(i(k))} \right))$$

(cf. Eq. (5.14) above) where we have set for each $i \leq m$

$$\Phi_i(B) := \sum_k \left( \frac{1}{2} B(\bullet \ell_{\Sigma}^{(k)}) + \frac{1}{2} B(\bullet \ell_{\Sigma}^{(i(k))}) \right) \cdot dt^{(N)}(l_{\Sigma}^{(i(k))})$$

Moreover, since $\text{Hol}^{\text{disc}}(A_c^\perp, B) \in T$ we can replace in Eq. (5.18) the characters $\chi_i := \text{Tr}_{\rho_i}$, $i \leq m$, by their restrictions $\chi_i|_T$. But $\chi_i|_T$ is just a linear combination of global weights, more precisely, for every $b \in \mathfrak{t}$ we have

$$\text{Tr}_{\rho_i}(\exp(b)) = \chi_i|_T(\exp(b)) = \sum_{\alpha \in \Lambda} m_{\chi_i}(\alpha) e^{2\pi i(\alpha, b)}$$

where $m_{\chi_i}(\alpha)$ the multiplicity of $\alpha \in \Lambda$ as a weight in $\chi_i$ (here $\Lambda \subset \mathfrak{t}^* \cong \mathfrak{t}$ denotes the lattice of the real weights associated to the pair $(\mathfrak{g}, \mathfrak{t})$, cf. part $\text{A}$ of the Appendix below). Combining Eqs. (5.20) – (5.22) we obtain \(^{36}\)

$$\prod_i \text{Tr}_{\rho_i}(\text{Hol}^{\text{disc}}(A_c^\perp, B))$$

$$= \prod_i \left( \sum_{\alpha_i \in \Lambda} m_{\chi_i}(\alpha_i) \cdot \exp(2\pi i(\alpha_i, \Phi_i(B))) \cdot \exp(2\pi i \sum_k \langle \alpha_i, A_c^\perp \left( \frac{1}{2} \ell_{\Sigma}^{(k)} + \frac{1}{2} \ell_{\Sigma}^{(i(k))} \right) \rangle) \right)$$

$$= \sum_{\alpha_1, \alpha_2, \ldots, \alpha_m \in \Lambda} \left( \prod_i m_{\chi_i}(\alpha_i) \prod_i \exp(2\pi i(\alpha_i, \Phi_i(B))) \exp(2\pi i \ll A_c^\perp, \sum_i \alpha_i \cdot l_{\Sigma}^{(k)} \gg A_c^\perp(y_{gK}) \right)$$

where we have set

$$l_{\Sigma}^{(k)} := \sum_k \frac{1}{2} (\ell_{\Sigma}^{(k)} + \ell_{\Sigma}^{(i(k))}) \in C_1(gK)$$

Let us now set for each $s > 0$, $y \in I$, $(\alpha_i)_i := (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \Lambda^m$, and $B \in B(gK)$:

$$F^{(s)}(\alpha_i)_i, y(B) := \left( \prod_x 1^{(s)} \chi(y, B(x)) \right) \left( \prod_i \exp(2\pi i(\alpha_i, \Phi_i(B))) \right) \text{Det}^{\text{disc}}(B)$$

$$\times \exp(-2\pi ik(y, B(\sigma_0)))$$

Then, according to Eq. (5.23), we can rewrite Eq. (5.18) as

$$\text{WLO}^{\text{disc}}_{\text{rig}}(L) = \lim_{s \to 0} \sum_{(\alpha_i)_i \in \Lambda^m} \left( \prod_i m_{\chi_i}(\alpha_i) \right) \sum_{y \in I} \int_\sim F^{(s)}(\alpha_i)_i, y(B) \exp(2\pi i \ll A_c^\perp, \sum_i \alpha_i \cdot l_{\Sigma}^{(k)} \gg A_c^\perp(y_{gK}) \exp(iS_{CS}^{\text{disc}}(A_c^\perp, B))(DA_c^\perp \otimes DB)$$

(Observe that there are only finitely many $(\alpha_i)_i \in \Lambda^m$ for which the product $\prod_i m_{\chi_i}(\alpha_i)$ does not vanish. Accordingly, the summation $\sum_{(\alpha_i)_i \in \Lambda^m} \left( \prod_i m_{\chi_i}(\alpha_i) \right) \cdots$ above is a finite and we can interchange it with $\sum_{y \in I}$).

Let us now fix for a while $s > 0$, $y \in I$, and $(\alpha_i)_i \in \Lambda^m$ and evaluate the corresponding $\int_\sim \cdots$-integral in Eq. (5.26). In order to do so we will apply Proposition 4.12 above to the special situation where

\(^{36}\text{here we are a bit sloppy and use the letter } i \text{ both for the multiplication index and the imaginary unit} \)
• \( V := A_c^\perp(K) \oplus B_0(qK) \) (cf. Convention 3).

• \( d\mu := d\nu^{\text{disc}} \) where

\[
d\nu^{\text{disc}} := \frac{1}{Z^{\text{disc}}} \exp(i S^{\text{disc}}_C(A_c^\perp, B))(DA_c^\perp \otimes DB)
\]

\[
= \frac{1}{Z^{\text{disc}}} \exp(i \ll A_c^\perp, \lesssim 2\pi k(*_K \circ \pi \circ d_{qK})B \gg A_c^\perp) (DA_c^\perp \otimes DB)
\]

where we have set \( Z^{\text{disc}} := \int \exp(i S^{\text{disc}}_C(A_c^\perp, B))(DA_c^\perp \otimes DB) \) and where \( \pi : A_{\Sigma, i}(qK) \to A_{\Sigma, i}(qK) \cong A_c^\perp(K) \) is the orthogonal projection (cf. also Convention 2 above),

• \( V_1 := \ker(*_K \circ \pi \circ d_{qK})^\perp = \ker(\pi \circ d_{qK}) \perp \) \( \subset B_0(qK) \) where (\( \ast \)) we used Eq. (3.20) in Sec. 3.9 above. (Here and in the next paragraph \( d_{qK} \) is a short notation for \( (d_{qK})|_{B_0(qK)} \)).

• \( V_2 := \text{Image}(*_K \circ \pi \circ d_{qK}) \subset A_c^\perp(K) \)

• \( V_0 := B_c(qK) \oplus (V_2)^\perp \)

where \( (V_2)^\perp \) is the orthogonal complement of \( V_2 \) in \( A_c^\perp(K) \) (cf. Convention 3).

• \( F := F^{(s)}_{(\alpha_i), \gamma} \circ p \) where \( p : V_0 \oplus V_1 = B_0(qK) \oplus (V_2)^\perp \to B_0(qK) \) is the obvious projection.

• \( v := 2\pi \sum_i \alpha_i \cdot 1_{\Sigma} \)

The following remarks show that the assumptions of Proposition 4.12 above are indeed fulfilled:

i) \( d\nu^{\text{disc}} \) is a normalized centered oscillatory Gauss type measure on \( A_c^\perp(K) \oplus B_0(qK) \) which has the form as in Proposition 4.12 with \( V_0, V_1, \) and \( V_2 \) as given above and where \( M : V_1 \to V_2 \) is the well-defined linear isomorphism given by

\[
M = -2\pi k(*_K \circ \pi \circ d_{qK})|_{V_1}
\]

(5.27)

ii) The function \( F \) is bounded and uniformly continuous

iii) In order to see that \( v \) is an element of \( V_2 \) we consider the linear map \( m_{\mathbb{R}} : C^0(qK, \mathbb{R}) \to C^1(qK, \mathbb{R}) \) which is given by

\[
m_{\mathbb{R}} := *_K \circ \pi \circ d_{qK}
\]

where \( d_{qK} : C^0(qK, \mathbb{R}) \to C^1(qK, \mathbb{R}), \pi : C^1(qK, \mathbb{R}) \to C^1(K, \mathbb{R}), \) and \( *_K : C^1(K, \mathbb{R}) \to C^1(K, \mathbb{R}) \) are the “real analogues” of the three maps appearing on the RHS of Eq. (5.27) above. From Lemma 3 in Sec. 5.3 below it follows that for each \( \ell_2^i \in C_1(qK) \) \( = C^1(qK, \mathbb{R}) \) there is a \( f_i \in C^0_{\text{eff}}(qK, \mathbb{R}) \) such that

\[
\ell_2^i = m_{\mathbb{R}} \cdot f_i
\]

(5.28)

holds. Moreover, \( f_i \) can be chosen to be constant on \( U(\sigma_0) \cap \mathcal{Z}_0(qK) \). In the following we will assume that \( f_i \) is chosen in this way. (According to Lemma 3 below this determines \( f_i \) uniquely up to an additive constant which will be fixed later using a suitable normalization condition, see Eq. (5.31) and Eq. (5.34) below). Clearly, this implies that

\[
v = 2\pi \sum \alpha_i \cdot 1_{\Sigma} = 2\pi \sum \alpha_i \cdot m_{\mathbb{R}} \cdot f_i = (*_K \circ \pi \circ d_{qK}) \cdot (2\pi \sum \alpha_i \cdot f_i)
\]

(5.29)

is indeed an element of \( V_2 \).

\[\text{[37 recall Eq. (3.13c) and observe that } \ll *_K A_c^\perp, d_{qK}B \gg A_c^\perp = \ll *_K A_c^\perp, \pi(d_{qK}B) \gg A_c^\perp = - \ll A_c^\perp, *_K (\pi(d_{qK}B)) \gg A_c^\perp]\]
Applying Proposition 4.12 above to the present situation we therefore obtain
\[ \frac{1}{Z_{\text{disc}}} \int_{x_0} F_{(\alpha_i),g}^{(s)} (B) \exp(2\pi i \ll A_\perp, \sum_i \alpha_i \cdot K_\perp \gg A_\perp (q\mathcal{K})) \exp(iS_{CS}^{\text{disc}} (A_\perp, B))(DA_\perp \otimes DB) \]
\[ = \int F(B) \exp(i \ll A_\perp, v \gg A_\perp (q\mathcal{K})) dB^{\text{disc}}(A_\perp, B) \]
\[ \overset{(\dagger)}{\int_{V_0}^{\sim} F(x_0 - M^{-1}v)dx_0} \overset{(s)}{\int_{V_0}^{\sim} F_{(\alpha_i),g}^{(s)} (b - M^{-1}v)db} \]  
\[ (5.30) \]
where the two improper integrals \( \int^\sim \cdots dx_0 \) and \( \int^\sim \cdots db \) are defined according to Convention 4 in Sec. 4 above. In Step (+) we have applied Proposition 4.12. Step (*) above follows because \( F(x_0 - M^{-1}v) \) depends only on the \( \mathcal{B}_c(q\mathcal{K}) \)-component of \( x_0 \in V_0 = \mathcal{B}_c(q\mathcal{K}) \oplus (V_2)^\perp \) and because \( \mathcal{B}_c(q\mathcal{K}) = \{ B \in \mathcal{B}(q\mathcal{K}) \mid B \text{ is constant} \} \cong t. \)

Recall that \( f_i \) as introduced above is uniquely determined up to an additive constant. We can fix this constant by demanding that the normalization condition
\[ \sum_{x \in \delta_0 (q\mathcal{K})} f_i (x) = 0 \]
\[ (5.31) \]
is fulfilled. With this normalization condition we obtain \( \sum_i \alpha_i \cdot f_i \in V_1 = (\mathcal{B}_c(q\mathcal{K}))^\perp. \) From this and Eq. (5.29) above we see that
\[ M^{-1}v = -\frac{1}{t} \sum_i \alpha_i \cdot f_i \]
\[ (5.32) \]
Combining Eqs. (5.25), (5.26), (5.30), and (5.32) we obtain
\[ \text{WLO}_{\text{rig}}^{\text{disc}} (L) \sim \lim_{s \to 0} \sum_{(\alpha_i) \in \Lambda^n} \left( \prod_i m_{X_i}(\alpha_i) \right) \sum_{y \in I} \left[ \exp(\frac{1}{2} i k \langle y, B(\sigma_0) \rangle) \prod_x \exp \left( \frac{1}{2} i k \langle \alpha_i, \Phi_i (B) \rangle \right) \right] \]
\[ \times \left[ \prod_x \exp \left( 2\pi i \langle \alpha_i, \Phi_i (B) \rangle \right) \right] \text{Det}^{\text{disc}} (B) \mid_{B = b + \frac{1}{t} \sum_i \alpha_i f_i} \] 
\[ (5.33) \]
Apart from the remaining \( \int_{t}^\sim \cdots db \)-integration (which will be taken care of in Step 4 below) we have now completed the evaluation of the \( \int \cdots \exp(iS_{CS}^{\text{disc}} (A_\perp, B))(DA_\perp \otimes DB) \)-integral in Eq. (5.18).

**Remark 5.2** It is not difficult to see that Eq. (5.33) also holds if we (re)define \( f_i \) using the following normalization condition (instead of the normalization condition (5.31) above):
\[ f_i (\sigma_0) = 0 \]
\[ (5.34) \]
Since condition (5.34) is technically more convenient than (5.31) we will use the latter normalization condition in the following, i.e. we assume that \( f_i \) is defined as in (5.28) above in combination with (5.34).

---

38 that the last (and therefore also the first) of these two improper integrals is in fact well-defined follows from Remark 4.13 above and a “periodicity argument”, which will be given in Step 4 below.

39 this follows from Eq. (1.27) in Remark 4.13 above and the periodicity properties of the integrand in \( \int_{t}^\sim \cdots db \) (for fixed \( y \) and \( \alpha_1, \ldots, \alpha_m \)), cf. Step 4 below.
5.3 Step 3: Some simplifications

The next lemma will prove a claim made in Sec. 5.2 above and it will also allow us to simplify the RHS of Eq. (5.33) above.

Lemma 3 Assume that \( l_{\Sigma}^0 \in C_1(qK) \equiv C^1(qK, \mathbb{R}) \) is as in Eq. (5.24) above. Then we have:

i) There is a \( f \in C^0_{\text{aff}}(qK, \mathbb{R}) \) such that \( l_{\Sigma}^0 = m_{\mathbb{R}} \cdot f \) and such that \( f \) is constant on \( U(\sigma_0) \cap \mathfrak{K}_0(qK) \) (cf. the end of Sec. 5.2 above). Moreover, these properties determine \( f \) uniquely up to an additive constant.

ii) If \( f \) is as in part i) of the present lemma then the map \( f : \mathfrak{K}_0(qK) \ni \sigma \mapsto f(\sigma) \in \mathbb{R} \) is constant on \( \text{arc}(l_{\Sigma}^0) \cap \mathfrak{K}_0(qK) \) and on \( \text{arc}(l_{\Sigma}^0) \cap \mathfrak{K}_0(qK) \) for all \( j \leq m \).

Proof. From conditions (NCP)' and (NH)' on the simplicial ribbon link \( L \) it follows that there are three connected components \( C_0, C_1 \) and \( C_2 \) of \( \Sigma \setminus (\text{arc}(l_{\Sigma}^0) \cup \text{arc}(l_{\Sigma}^0)) \). In the following we assume that these three connected components are given as in Fig. 1 below.

It follows from Condition (FC2) in Sec. 5.0 that \( \mathfrak{K}_0(qK) \cap C_0 = \emptyset \), or, in other words, that \( \mathfrak{K}_0(qK) \subset \overline{C_1} \cup \overline{C_2} \). Accordingly, the map \( f : \mathfrak{K}_0(qK) \rightarrow \mathbb{R} \) given by

\[
f(p) := \begin{cases} 
  c & \text{if } p \in \overline{C_1} \\
  c \pm 1 & \text{if } p \in \overline{C_2}
\end{cases}
\]  

(5.35)

for all \( p \in \mathfrak{K}_0(qK) \) is well-defined. Here \( c \in \mathbb{R} \) is an arbitrary constant which will be kept fixed in the following and the sign \( \pm \) is \( + \) if for any \( t \in [0, 1] \) in which \( l_{\Sigma}^0 \) is differentiable the normal vector \( n = \star (\frac{d}{dt} l_{\Sigma}^0(t)) \) “points into” \( C_0 \) and “−” otherwise.

Let us first verify that \( f \in C^0_{\text{aff}}(qK, \mathbb{R}) \). Let \( F \in \mathfrak{K}_2(qK) \) and let \( p_1, p_2, p_3, p_4 \) be the four vertices of \( F \) enumerated such that \( p_1 \) is diagonal to \( p_4 \) (and therefore \( p_2 \) is diagonal to \( p_3 \)). We have to show that \( f(p_1) + f(p_4) = f(p_2) + f(p_3) \) (cf. Eq. (3.29)). If \( F \subset \overline{C_1} \) or \( F \subset \overline{C_2} \) this is obvious. If \( F \subset \overline{C_0} \) (for example \( F = F_0 \) where \( F_0 \) is as in Fig. 1 above) then Condition (FC3) implies that \( f(p_1) + f(p_4) = c + (c \pm 1) = f(p_2) + f(p_3) \).

Recall from Sec. 3.8 that \( \sigma_0 \in \mathfrak{K}_0(qK) \) was chosen such that \( \sigma_0 \notin \text{Image}(P_{\Sigma}^i) \). This implies that \( \sigma_0 \in C_1 \) or \( \sigma_0 \in C_2 \) and from the definition of \( U(\sigma_0) \) we obtain \( U(\sigma_0) \subset \overline{C_1} \) or \( U(\sigma_0) \subset \overline{C_2} \) so Eq. (5.35) implies that \( f \) is constant on \( U(\sigma_0) \cap \mathfrak{K}_0(qK) \).
The uniqueness part of the assertion follows by combining the definition of $m_\Sigma$ with the real analogue of Eq. (3.20) in Sec. 3.9 above and the fact that $\star_K : C^1(K, \mathbb{R}) \to C^1(K, \mathbb{R})$ is a bijection.

In order to conclude the proof of part i) of Lemma 3 we have to show that

$$\int_\Sigma = m_\Sigma \cdot f = \star_K (\pi(d_\Sigma f)) \quad (5.36)$$

Observe first that $(d_\Sigma f)(e) = 0$ unless the interior of $e \in \mathcal{F}(qK)$ is contained in $C_0$. In the latter case we have $(d_\Sigma f)(e) = \pm \text{sgn}(e)$ where the sign $\pm$ is the same as in Eq. (5.25) above and where $\text{sgn}(e) = 1$ if the (oriented) edge $e$ “points from” the region $C_1$ to the region $C_2$ and $\text{sgn}(e) = -1$ otherwise.

Next observe that for every $e \in \mathcal{F}(qK)$ whose interior is contained in $C_0$ there exists an index $k \leq n$ such that

$$\star_K (\pi(\pm \text{sgn}(e) \cdot e)) = \begin{cases} \bar{\pi}_\Sigma^{(k)} & \text{if } e \text{ has an endpoint in } \mathcal{F}_0(K_2) \\ \bar{i}_\Sigma^{(k)} & \text{if } e \text{ has an endpoint in } \mathcal{F}_0(K_1) \end{cases} \quad (5.37)$$

and that the map

$$\psi : \{ e \in \mathcal{F}(qK) \mid \text{the interior of } e \text{ is contained in } C_0 \} \to (\{ \bar{\pi}_\Sigma^{(k)} \mid k \leq n \} \cup \{ \bar{i}_\Sigma^{(k)} \mid k \leq n \}) \setminus \{ 0 \}$$

given by $\psi(e) = \star_K (\pi(\pm \text{sgn}(e) \cdot e))$ for all $e \in \text{dom}(\psi)$ is a bijection.

Finally, observe that the sum of the elements of the finite subset $\{ \bar{\pi}_\Sigma^{(k)} \mid k \leq n \} \cup \{ \bar{i}_\Sigma^{(k)} \mid k \leq n \}$ of $C_1(K) \subset C_1(qK)$ equals $l_\Sigma$, cf. Eq. (5.24) above. From this Eq. (5.36) follows.

It remains to show part ii) of Lemma 3. Recall that according to Eq. (5.35), $f$ is constant on $\mathcal{F}_0(qK) \cap C_1$ and also on $\mathcal{F}_0(qK) \cap C_2$. From condition (FC1) and condition (FC2) it follows that for each $j \neq i$ the set $\text{arc}(l_\Sigma^{(1)})$ (considered as a subset of $\Sigma$) either lies entirely in $C_1$ or entirely in $C_2$. Similarly, it follows that for each $j \neq i$ the set $\text{arc}(l_\Sigma^{(j)})$ either lies entirely in $C_1$ or in $C_2$. Since we also have $\text{arc}(l_\Sigma^{(i)}) = \partial C_2 \subset C_2$ and $\text{arc}(l_\Sigma^{(j)}) = \partial C_1 \subset C_1$ part ii) of Lemma 3 now follows.

\[\square\]

**Corollary 5.3** Let $B \in \mathcal{B}(qK)$ be of the form

$$B = b + \frac{1}{k} \sum_{i=1}^{m} \alpha_i f_i \quad (5.38)$$

with $b \in \mathfrak{t}$, $\alpha_i \in \Lambda$ and where $f_i$ is given by Eq. (5.28) above in combination with (5.34). Then the map $\mathcal{F}_0(qK) \ni \sigma \mapsto B(\sigma) \in \mathfrak{t}$ is constant on $\text{arc}(l_\Sigma^{(1)}) \cap \mathcal{F}_0(qK)$ and on $\text{arc}(l_\Sigma^{(1)}) \cap \mathcal{F}_0(qK)$ for all $j \leq m$.

Let us set

$$\sigma_i := \bullet l_\Sigma^{(i)} \in \mathcal{F}_0(qK), \quad \sigma'_i := \bullet l_\Sigma^{(i)} \in \mathcal{F}_0(qK) \quad (5.39)$$

\[\text{40}\]From the definition of $qK$ it follows that for every edge $e$ in $qK$ exactly one endpoint is in $\mathcal{F}_0(K_1|K_2)$ and the other endpoint is either in $\mathcal{F}_0(K_1)$ or in $\mathcal{F}_0(K_2)$

\[\text{41}\]here 0 is the zero element of $C_1(K)$

\[\text{42}\]in order to avoid confusion recall that according to the standard convention for sets we have $\{a, a\} = \{a\}$, $\{a, a, b, b\} = \{a, b\}$ etc. Since $\bar{\pi}_\Sigma^{(2k-1)} = \bar{\pi}_\Sigma^{(2k)}$ for $k \leq n/2$ this means that $\{ \bar{\pi}_\Sigma^{(k)} \mid k \leq n \} = \{ \bar{\pi}_\Sigma^{(k)} \mid k \leq n, k \text{ odd} \}$, which implies that the sum of the elements of $\{ \bar{\pi}_\Sigma^{(k)} \mid k \leq n \}$ equals $\sum_{k=1}^{n} \bar{\pi}_\Sigma^{(k)}$. A similar remark applies to the sum of the set of the elements of $\{ \bar{i}_\Sigma^{(k)} \mid k \leq n \}$
According to Corollary 5.3 we have
\[
B(\sigma_i) = B(\bullet l_i^{(k)}) \quad \forall k \leq n
\]  
(5.40a)
\[
B(\sigma'_i) = B(\bullet l_i^{(k)}) \quad \forall k \leq n
\]  
(5.40b)
for every \(B\) of the form in Eq. (5.38). Next observe that
\[
\epsilon_i := \text{wind}(l_{S_i}) = \sum_k d \lambda^{(N)}(l_{S_i}^{(k)})
\]  
(5.41)
where \(\text{wind}(l_{S_i})\) is the winding number of \(l_{S_i}\).

Combining Eq. (5.33) and Eq. (5.21) with Eqs. (5.40a) – (5.41) and taking into account the normalization condition (5.34) appearing at the end of Sec. 5.2 we obtain
\[
\text{WLO}_{\text{rig}}^{\text{disc}}(L) \sim \lim_{s \to 0} \sum_{(\alpha_i) \in \Lambda^m} \left( \prod_i m_{\chi_i}(\alpha_i) \right) \sum_{y \in I} \int_t^\infty \left[ \exp(-2\pi i k \langle y, b \rangle) \left( \prod_x \frac{1^{(s)}}{v_{\text{reg}}}(B(x)) \right) \right. \\
\left. \times \left( \prod_j \exp(\pi i \epsilon_j \langle \alpha_j, B(\sigma_j) + B(\sigma'_j) \rangle) \right) \right] \Big|_{B=b+\frac{1}{k} \sum_i \alpha_i f_i} \]  
(5.42)

Setting
\[
F_{(\alpha_i)}^{(s)}(b) := \left[ \left( \prod_x \frac{1^{(s)}}{v_{\text{reg}}}(B(x)) \right) \right. \\
\left. \times \left( \prod_j \exp(\pi i \epsilon_j \langle \alpha_j, B(\sigma_j) + B(\sigma'_j) \rangle) \right) \right] \Big|_{B=b+\frac{1}{k} \sum_i \alpha_i f_i}
\]  
(5.43)
we can rewrite Eq. (5.42) as
\[
\text{WLO}_{\text{rig}}^{\text{disc}}(L) \sim \lim_{s \to 0} \sum_{(\alpha_i) \in \Lambda^m} \left( \prod_i m_{\chi_i}(\alpha_i) \right) \sum_{y \in I} \int_t^\infty \int_t^\infty \left[ e^{-2\pi i k \langle y, b \rangle} F_{(\alpha_i)}^{(s)}(b) \right] \]  
(5.44)

### 5.4 Step 4: Performing the remaining limit procedures \(\int_t^\infty \cdots \int_t^\infty db\) in Eq. (5.44) with \(s \to 0\) in Eq. (5.44)

i) Let us first rewrite the \(\int_t^\infty \cdots \int_t^\infty db\) integral. The crucial observation is that for fixed \(y \in I\), \(s > 0\), and \((\alpha_i)_t \in \Lambda^m\) the function \(t \mapsto e^{-2\pi i k \langle y, b \rangle} F_{(\alpha_i)}^{(s)}(b) \in \mathbb{C}\) is invariant under all translations of the form \(b \mapsto b + x\) where \(x \in I = \ker(\exp|_{t}) \cong \mathbb{Z}^{\dim(t)}\).

Indeed, for all \(b \in t\) and \(x \in I\) we have
\[
1^{(s)}_{v_{\text{reg}}}(b+x) = 1^{(s)}_{v_{\text{reg}}}(b) \quad (5.45a)
\]
\[
e^{2\pi i \epsilon(x,a+b)} = e^{2\pi i \epsilon(x,a)} \quad \text{for all } \alpha \in \Lambda, \epsilon \in \mathbb{Z} \quad (5.45b)
\]
\[
\det^{1/2}(1_t - \exp(\text{ad}(b+x))_{|t}) = \det^{1/2}(1_t - \exp(\text{ad}(b))_{|t}) \quad (5.45c)
\]
\[
e^{-2\pi i k(y,b+x)} = e^{-2\pi i k(y,b)} \quad \text{for all } y \in I \quad (5.45d)
\]

The second equation follows because the assumption that \(G\) is simply-connected implies that
\[
I = \Gamma 
\]  
(5.46)

where \(\Gamma \subset t\) is the lattice generated by the real coroots and, by definition, \(\Lambda\) is the lattice dual to \(\Gamma\). The first of these four equations follows from the assumption in Sec. 5.6 that \(1^{(s)}_{v_{\text{reg}}}\) is
According to the general theory of semi-simple Lie algebras we have 2
invariant under $\mathcal{W}_\text{aff}$ (and using again Eq. \ref{5.40} above).
The third equation follows because (cf. Eq. \ref{5.25} above)
\[
\det^{1/2}(1_t - \exp(\text{ad}(b + x))|_t) = \prod_{\alpha \in \mathcal{R}_+} (2 \sin(\pi \langle \alpha, b + x \rangle))
\]
\[
= (-1)\sum_{\alpha \in \mathcal{R}_+} \langle \alpha, x \rangle \prod_{\alpha \in \mathcal{R}_+} (2 \sin(\pi \langle \alpha, b \rangle))
\]
\[
= \prod_{\alpha \in \mathcal{R}_+} (2 \sin(\pi \langle \alpha, b \rangle)) = \det^{1/2}(1_t - \exp(\text{ad}(b))|_t)
\]
where in step (*) we used $\sum_{\alpha \in \mathcal{R}_+} \langle \alpha, x \rangle = 2\langle \rho, x \rangle \in 2\mathbb{Z}$ since $\rho \in \Lambda$. Finally, in order to see that
the fourth equation holds, observe that because of (5.46) it is enough to show that
\[
\langle \check{\alpha}, \check{\beta} \rangle = 2 \quad \text{for all coroots } \check{\alpha}, \check{\beta}
\]
\[
\text{According to Eq. (5.48) we can now rewrite Eq. (5.44) as}
\]
\[
\frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} \langle \alpha, x \rangle = 2 \langle \rho, x \rangle \in 2\mathbb{Z}
\]
\[
\text{Moreover, there are at most two different (co)roots lengths and the quotient between the square lengths of the}
\]
\[
\text{long and short coroots is either 1, 2, or 3. Since the normalization of}\}
\[
\text{of} \langle \cdot, \cdot \rangle \text{was chosen such that}
\]
\[
\langle \check{\alpha}, \check{\alpha} \rangle = 2 \text{ holds if } \check{\alpha} \text{ is a short coroot we therefore have } \langle \check{\alpha}, \check{\alpha} \rangle/2 \in \{1, 2, 3\} \quad \text{and (5.47) follows.}
\]
\]
\[
\text{From Eqs. (5.43), (5.45), and Eq. (5.48) below we conclude that } t \ni b \mapsto e^{-2\pi ik(y,b)} F^{(s)}_{(\alpha_i)}(b) \in \mathbb{C} \text{ is indeed } I\text{-periodic and we can therefore apply Eq. (4.13) in Remark 4.13 above and obtain}
\]
\[
\int \sim db \quad e^{-2\pi ik(y,b)} F^{(s)}_{(\alpha_i)}(b) \sim \int_Q db \quad e^{-2\pi ik(y,b)} F^{(s)}_{(\alpha_i)}(b)
\]
\[
\text{where on the RHS } \int_Q \cdots \text{db is now an ordinary integral and where we have set}
\]
\[
Q := \{ \sum_{i} \lambda_i \epsilon_i | \lambda_i \in (0,1) \text{ for all } i \leq m \} \subset t,
\]
\[
\text{Here } (\epsilon_i)_{i \leq m} \text{ is an (arbitrary) fixed basis of } I.
\]
\[
\text{According to Eq. (5.48) we can now rewrite Eq. (5.44) as}
\]
\[
\text{WLO}^\text{disc}_{\text{rig}}(L) \sim \lim_{s \to 0} \sum_{(\alpha_i)_{i \in \Lambda^m}} \left( \prod_i m_{\chi_i}(\alpha_i) \right) \sum_{y \in I} \int_Q db \quad e^{-2\pi ik(y,b)} F^{(s)}_{(\alpha_i)}(b)
\]
\[
\text{ii) We can now perform the infinite sum } \sum_y \text{ and the } \int \cdots \text{db-integral in Eq. (5.50):}
\]
\[
\text{First recall that, due to Eq. (5.46) above and the definition of } \Lambda, \Lambda \text{ is dual to } I. \quad \text{According to the (rigorous) Poisson summation formula for distributions we therefore have}
\]
\[
\sum_{y \in I} e^{-2\pi ik(y,b)} = c_\Lambda \sum_{x \in \mathbb{R} \Lambda} \delta_x(b)
\]
\[
\text{where } \delta_x \text{ is the delta distribution in } x \in t \text{ and } c_\Lambda \text{ a constant depending on the lattice } \Lambda. \text{ Let us}
\]
\[
\text{now apply Eq. (5.51) to the RHS of Eq. (5.50) above. In order to see that this is possible note}
\]
\[
\text{first that not only } F^{(s)}_{(\alpha_i)} \text{ is smooth but also the product } 1_Q F^{(s)}_{(\alpha_i)} \text{ because } \partial Q \subset t \setminus t_{\text{reg}} \text{ and because}
\]
\[
F^{(s)}_{(\alpha_i)} \text{ vanishes on an open neighborhood of the set } t \setminus t_{\text{reg}} \text{ (cf. the condition supp(1^{(s)}_{t_{\text{reg}}}) \subset t_{\text{reg}}}
\]
\[
\text{in Sec. 3.6 so, according to the definition of } F^{(s)}_{(\alpha_i)}, \text{ and Eq. (5.41), there is a factor } 1^{(s)}_{t_{\text{reg}}}(b)
\]
\[
\text{appearing in } F^{(s)}_{(\alpha_i)}(b). \text{ Moreover, since } Q \text{ is bounded } 1_Q F^{(s)}_{(\alpha_i)} \text{ has compact support. Thus we}
\]
\[
\text{can indeed apply Eq. (5.51) to the RHS of Eq. (5.50) above and we then obtain}
\]
\[
\text{WLO}^\text{disc}_{\text{rig}}(L) \sim \lim_{s \to 0} \sum_{(\alpha_i)_{i \in \Lambda^m}} \left( \prod_i m_{\chi_i}(\alpha_i) \right) \sum_{b \in \mathbb{R} \Lambda} 1_Q(b) F^{(s)}_{(\alpha_i)}(b)
\]
iii) Finally, let us also perform the $s \to 0$ limit. Taking into account that $1_{t_{reg}}^{(s)} \to 1_{t_{reg}}$ pointwise we obtain from Eq. (5.52) and Eq. (5.43) after the change of variable $b \to kb =: \alpha_0$.

\[
\text{WLO}_{\text{rig}}(L) \sim \sum_{\alpha_0, \alpha_1, \ldots, \alpha_m \in \Lambda} 1_{kQ}(\alpha_0) \left( \prod_{i=1}^{m} m_{\chi_i}(\alpha_i) \right) \\
\times \left[ \left( \prod_s 1_{t_{reg}}(B(x)) \right) \prod_{j=1}^{m} \exp(\pi i \epsilon_j(\alpha_j, B(\sigma_j) + B(\sigma_j')) \right] \det_{\text{disc}}(B) \bigg|_{B = \frac{1}{k}(\alpha_0 + \sum_i \alpha_i f_i)}
\]  

(5.53)

5.5 Step 5: Rewriting $\det_{\text{disc}}(B)$ in Eq. (5.53)

In Steps 1–4 we have reduced the original “path integral” expression for $\text{WLO}_{\text{rig}}(L)$ to a “combinatorial expression”, i.e. an expression which does not involve any limit procedure. Let us now have a closer look at $\det_{\text{disc}}(B)$, cf. Eq. (5.19) in Step 1 above.

From assumptions (NCP)' and (NH)' above it follows that the set of connected components of $\Sigma \setminus (\bigcup_j \text{arc}(l^0_j))$ has exactly $m + 1$ elements, which we will denote by $Y_0, Y_1, \ldots, Y_m$ in the following. Moreover, it follows that the also the set of connected components of $\Sigma \setminus \bigcup_j \text{Image}(R^1_j) = \Sigma \setminus \bigcup_j (O_j \cup \text{arc}(l^0_j) \cup \text{arc}(l^1_j))$ has exactly $m + 1$ elements, which we will denote by $Z_0, Z_1, \ldots, Z_m$.

(Here $O_j \subset \Sigma, j \leq m$, denotes the open “region between” $\text{arc}(l^0_j)$ and $\text{arc}(l^1_j)$, cf. condition (FC2) in Sec. 5.0 above.)

In the following we assume without loss of generality that the numeration of the $Z_i, i \leq m$, was chosen such that

\[Z_i \subset Y_i \quad \forall i \in \{0, 1, \ldots, m\}\]

holds. For later use we remark that $\{Z_i \mid 0 \leq i \leq m\} \cup \{O_i \mid i \leq m\}$ is a partition of $\Sigma$. Thus condition (FC2) implies that

\[\mathcal{J}_0(qK) = \bigcup_{i=0}^{m} (\mathcal{J}_0(qK) \cap Z_i) \]

(5.54)

Observe also that

\[\mathcal{J}_0(qK) = \mathcal{J}_0(K_1) \cup \mathcal{J}_0(K_1 \cup K_2) \cup \mathcal{J}_0(K_2) \]

(5.55)

(Here and in the following we use the notation $S \sqcup T$ for the disjoint union of two sets $S$ and $T$).

In the following we assume that $B \in \mathcal{B}(qK)$ is of the form

\[B = \frac{1}{k}(\alpha_0 + \sum_i \alpha_i f_i), \quad \text{with } \alpha_0, \ldots, \alpha_m \in \Lambda\]

(5.56)

and with $f_i$ as in Eq. (5.28) above in combination with (5.34).

**Lemma 4** If $B \in \mathcal{B}(qK)$ is of the form (5.56) then the restriction of $B : \mathcal{J}_0(qK) \to t$ to $\mathcal{J}_0(qK) \cap Z_i$ is constant for each $i$.

**Proof.** This lemma is a generalization of Corollary 5.3 in Sec. 5.3 above and follows easily from (the proof of) Lemma 5 above.

□

In the following we set $B(Y_i) := B(Z_i) := B(x)$ for any $x \in Z_i \cap \mathcal{J}_0(qK)$ (according to Lemma 4 the value of $B(Y_i) = B(Z_i)$ does not depend on the choice of $x \in Z_i \subset Y_i$).

**Lemma 5** For every $B \in \mathcal{B}(qK)$ of the form (5.56) and fulfilling $\prod_s 1_{t_{reg}}(B(x)) \neq 0$ we have

\[\det_{\text{disc}}(B) \sim \prod_{i=0}^{m} \det^{1/2}(1_t - \exp(\text{ad}(B(Y_i))))|_t \chi(Y_i) \]

(5.57)

where $\chi(Y_i)$ is the Euler characteristic of $Y_i$.  

\[^{43}\text{here } Z_i \text{ denotes the closure of } Y_i\]
Proof. From the definition of $\text{Det}^{\text{disc}}(B)$ in Eq. (5.19), from Eq. (5.12) and Eq. (5.24) in Sec. 3 above it follows that

$$
\text{Det}^{\text{disc}}(B) \sim \frac{\prod_{x \in \mathfrak{s}_0(K_1)} \det^{1/2}(1_t - \exp(\text{ad}(B(x)))_{|t}) \prod_{x \in \mathfrak{s}_0(K_2)} \det^{1/2}(1_t - \exp(\text{ad}(B(x)))_{|t})}{\prod_{x \in \mathfrak{s}_0(K_1\cap K_2)} 1_t - \exp(\text{ad}(B(x)))_{|t}}
$$

(5.58)

(observe that the expression on the RHS is well-defined since by assumption $\prod_{x} 1_{t_{\text{reg}}}(B(x)) \neq 0$, which implies that the denominator is non-zero).

According to Lemma 4 and Eqs. (5.54) and (5.55) it is enough to prove that for each $i \in \{0,1,\ldots,m\}$ we have

$$
\chi(Y_i) = \#(\mathfrak{s}_0(K_1) \cap \bar{Z}_i) - \#(\mathfrak{s}_0(K_1|K_2) \cap \bar{Z}_i) + \#(\mathfrak{s}_0(K_2) \cap \bar{Z}_i)
$$

(5.59)

Clearly, $\chi(Y_i) = \chi(\bar{Y}_i)$ where $\bar{Y}_i$ is the closures of $Y_i$. Moreover, $\bar{Y}_i$ is a subcomplex of the CW complex $K_1 = K = (\Sigma,C)$ so setting $\text{Cell}_p(\bar{Y}_i) := \{ \sigma \in \text{Cell}_p(K_1) \mid \sigma \subset \bar{Y}_i \}$ where $\text{Cell}_p(K_1)$ is the set of (open) $p$-cells of $K_1$ we obtain

$$
\chi(\bar{Y}_i) = \sum_{p=0}^{2} (-1)^p \# \text{Cell}_p(\bar{Y}_i) = \#(\mathfrak{s}_0(K_1) \cap \bar{Z}_i) - \#(\mathfrak{s}_0(K_1|K_2) \cap \bar{Z}_i) + \#(\mathfrak{s}_0(K_2) \cap \bar{Z}_i)
$$

(5.60)

(step (*) follows by taking into account the natural 1-1-correspondences $\text{Cell}_0(K_1) \leftrightarrow \mathfrak{s}_0(K_1)$, $\text{Cell}_1(K_1) \leftrightarrow \mathfrak{s}_0(K_1|K_2)$, and $\text{Cell}_2(K_1) \leftrightarrow \mathfrak{s}_0(K_2)$).

In order to complete the proof of Lemma 5 it is therefore enough to show that the RHS of Eq. (5.59) and the RHS of Eq. (5.60) coincide.

In order to see this observe that for each $0 \leq i \leq m$ there is $J \subset \{0,1,\ldots,m\}$ such that

$$
\bar{Y}_i = \bar{Z}_i \cup \bigcup_{j \in J} (O_j \cup \text{arc}(l_{ij}^{'j}))
$$

so our claim follows from

$$
\#(\mathfrak{s}_0(K_1) \cap \text{arc}(l_{ij}^{'j})) = \#(\mathfrak{s}_0(K_1|K_2) \cap \text{arc}(l_{ij}^{'j})) + \#(\mathfrak{s}_0(K_2) \cap \text{arc}(l_{ij}^{'j})) = 0
$$

(5.61a)

and from

$$
\#(\mathfrak{s}_0(K_1) \cap O_j) - \#(\mathfrak{s}_0(K_1|K_2) \cap O_j) + \#(\mathfrak{s}_0(K_2) \cap O_j) = \#\emptyset - \#\emptyset + \#\emptyset = 0
$$

(5.61b)

(cf. condition (FC2) and Eq. (5.54)).

□

Lemma 6 For every $B \in \mathcal{B}(qK)$ of the form (5.56) we have

$$
\prod_{x \in \mathfrak{s}_0(qK)} 1_{t_{\text{reg}}}(B(x)) = \prod_{i=0}^{m} 1_{t_{\text{reg}}}(B(Y_i))
$$

(5.62)

Proof. The assertion follows from Lemma 4 and Eq. (5.54) above.

□

Combining Eq. (5.54) with Lemma 4 and Lemma 6 we arrive at

$$
\text{WLO}_{\text{rig}}^{\text{disc}}(L) \sim \sum_{\alpha_0,\alpha_1,\ldots,\alpha_m \in \Lambda} k_{q}(\alpha_0) (\prod_{i=1}^{m} m_{\chi_i}(\alpha_i)) \times \left[ \prod_{i=0}^{m} 1_{t_{\text{reg}}}(B(Y_i)) \det^{1/2}(1_t - \exp(\text{ad}(B(Y_i)))_{|t})^{\chi(Y_i)} \times \prod_{j=1}^{m} \exp(\pi i e_j(\alpha_j, B(\sigma_j) + B(\sigma_j'))) \right]_{B=\frac{1}{t} (\alpha_0 + \sum_{i} \alpha_i f_i)}
$$

(5.63)
Remark 5.4  What would happen if we had worked with “full ribbons” $\bar{R}_j$ instead of “half ribbons” $R_j$ (cf. Remark 3.2 and Remark 3.6 above)? In this case the RHS of Eq. (7.24) above and therefore also the elements $f_i$ of $C^0(gK, \mathbb{R})$ given by Eq. (5.28) would have different values, which would make it necessary to redefine the sets $Z_i$ appearing above in a suitable way\textsuperscript{44}.

i) Lemma 5 above would then still be true. In fact, the proof would be simpler. Even more importantly, the structure of the proof of the new version of Lemma 5 (or rather, its BF-theoretic analogue) is exactly what is needed when trying to obtain a result like Eq. (7.20) below\textsuperscript{15} for general ribbon links.

ii) On the other hand, for the “new” definition of the sets $Z_i$ mentioned above, Eq. (5.54) would no longer be true and Lemma 6 would no longer hold unless we insert additional indicator functions on the RHS of Eq. (5.63). It is still possible that – by exploiting suitable algebraic identities – one can recover Eq. (5.63) after all (in spite of the additional indicator functions appearing in Eq. (5.62)). If Eq. (5.63) cannot be recovered, which is likely, then one can bypass this complication by using an additional regularization procedure. Since at the moment it is not clear whether such an additional regularization procedure is indeed necessary or not we decided to work only with half ribbons until now. We will begin to work with full ribbons in Sec. 7 below.

5.6  Step 6: Comparison of $\text{WLO}_\text{rig}(L)$ with the shadow invariant $|L|$  

From the computations in Sec. 5 in [14] it follows that the RHS of Eq. (5.63) above coincides with the shadow invariant $|L|$ (associated to $g$ and $k$) up to a multiplicative constant. For the convenience of the reader we will briefly sketch this derivation. In the following we will use the notation of part A and B of the Appendix.

For $\alpha_1, \ldots, \alpha_m \in \Lambda$ and $\alpha_0 \in \Lambda \cap kQ$ set

$$B := \frac{1}{k}(\alpha_0 + \sum_i \alpha_i f_i)$$

and introduce the function $\varphi : \{Y_0, Y_1, \ldots, Y_m\} \to \Lambda$ by

$$\varphi(Y) := kB(Y) - \rho \quad \forall Y \in \{Y_0, Y_1, \ldots, Y_m\}$$

One can show that then (cf. Sec. 5 in [14])

$$\det^{1/2}(1_t - \exp(\text{ad}(B(Y)))) \sim \dim(\varphi(Y))$$

$$\prod_j \exp(\pi i \epsilon_j \langle \alpha_j, B(\sigma_j) + B(\sigma'_j) \rangle) = \prod_Y \exp(\frac{\pi i}{k} \langle \varphi(Y), \varphi(Y) + 2\rho \rangle)$$

Let $P$ be the unique Weyl alcove which is contained in the Weyl chamber $C$ fixed in Sec. 2.1 and which has $0 \in t$ on its boundary. Moreover, let $\Lambda^k_+ \cap \{PK-\rho\}$, the bijectivity of the map $\theta : P \times W_{aff} \ni (b, \sigma) \mapsto \sigma(b) \in t_{reg}$ and the fact that for a suitable finite subset $W_{aff}(\cong W_k)$ we have $\theta(P \times W) = Q \cap t_{reg}$ it follows that there is a natural 1-1-correspondence between the set $\text{col}(L) \times W_{aff} \{Y_0, Y_1, \ldots, Y_m\}$ and the set of those $B$ which are of the form in Eq. (5.64) above (with $\alpha_0 \in \Lambda \cap kQ$ and $\alpha_1, \ldots, \alpha_m \in \Lambda$) and which have the extra property that $\prod_Y 1_{\text{t_{reg}}}(B(Y)) = 1$.

\textsuperscript{44} recall that the “old” sets $Z_i$ are the connected components of $\Sigma \cup \text{Image}(R^c_i)$ where each $R^c_i$ is the (reduced) projection of the “half ribbon” $R_j$ in $gK \times Z_0$. The “new” sets $\bar{Z}_i$ will be the connected components of $\Sigma \cup \text{Image}(\bar{R}^c_i)$ where each $\bar{R}^c_i$ is the (reduced) projection of the “full ribbon” $\bar{R}_j$ in $K \times Z_0$.

\textsuperscript{15} recall that we do not expect that Theorem 5.3 can be generalized successfully to the case of general ribbon links unless we make a transition to the $BF_3$-theoretic point of view, cf. the beginning of Sec. 7 below.
Using this and Eq. (B.7) below plus a suitable symmetry argument based on the group \( W_k \cong W_{\text{aff}} \) (cf. the proof of Theorem 5.1 in \[14\]) one then arrives at

\[
WLO_{\text{rig}}^\text{disc}(L) \sim \sum_{\varphi \in \text{col}(L)} \left( \prod_i N^\varphi(Y_i)^+ \right) \times \left( \prod_Y \text{dim}(\varphi(Y)) \chi(Y) \exp\left( \frac{2i\pi}{k} \langle \varphi(Y), \varphi(Y) + 2\rho \rangle \right) \right) = |L| \quad (5.67)
\]

If we apply Eq. (5.67) to the empty link \( \emptyset \) instead of \( L \) and take into account that \( \sim \) denotes equality up to a multiplicative non-zero constant independent of \( L \) we see that also \( WLO_{\text{rig}}^\text{disc}(\emptyset) \neq 0 \) for \( k \geq c_0 \). Accordingly, \( WLO_{\text{rig}}(L) \) is then well-defined and Eq. (5.67) implies that indeed

\[
WLO_{\text{rig}}(L) = \frac{WLO_{\text{rig}}^\text{disc}(L)}{WLO_{\text{rig}}^\text{disc}(\emptyset)} = \frac{|L|}{|\emptyset|} \quad (6.1)
\]

6 A brief comment regarding general simplicial ribbon links

Let us make some comments regarding the question if it is possible to generalize the computations above to general simplicial ribbon links. The crucial step will be the evaluation of the integral

\[
\int \sum_{i=1}^m \text{Tr}_{\rho_i}(\text{Hol}^{\text{disc}}_{R_i}(A^\perp + A^\perp_i, B)) \exp(iS_{CS}^{\text{disc}}(A^\perp_i, B)) D A^\perp \quad (6.1)
\]

for given \( B \) and \( A^\perp_i \), cf. Eq. (5.9) above. For general simplicial ribbon links this is considerably more difficult than in Sec. 5.1 above. The good news is that the evaluation of the integral (6.1) can be reduced to the computation of the “2-clusters”

\[
\int \{ \rho_i(\exp(\sum_a T_a Y^{i,a}_k)) \otimes \rho_{i'}(\exp(\sum_{a'} T_{a'} Y^{i',a'}_{k'})) \} \exp(iS_{CS}^{\text{disc}}(A^\perp_i, B)) D A^\perp_i \in \text{End}(V_i) \otimes \text{End}(V_{i'}) \quad (6.2)
\]

for the few \( i, i' \leq m \) and \( k, k' \leq n \) for which \( *_k \rho(k) = \pm \overline{\rho(k')} \). Here \( Y^{i,a}_k \) and \( Y^{i',a'}_{k'} \) are as in Eq. (5.10) above and \( V_i, V_{i'} \) are the representation spaces of \( \rho_i, \rho_{i'} \), cf. Sec. 5.1. The integral in (6.1) above can be expressed by these “2-clusters” by a similar formula as Eq. (6.4) in \[19\] (cf. also Sec. 3.3 in \[22\] and \[30\]).

The explicit formula for \( WLO_{\text{rig}}(L) \) for general \( L \) which one obtains in this way should again be a sum over the set of “area colorings” \( \varphi \), but this time every summand will contain an extra factor involving a product \( \prod_{x \in V(L)} \cdots \). One could hope that this factor coincides with the factor \( |L|^\varphi \) (cf. part B of the Appendix for the notation used here).

In order to evaluate the chances for this being the case we can\(^{46}\) consider the case of Abelian structure group \( G = U(1) \). The computations are then analogous to those appearing in the continuum setting in Secs 5.1 and 6.1 in \[22\] (which led to the correct result). However, in these computations there is one crucial difference in comparison to the computations in the continuum computations: there are several factors of \( 1/2 \), coming from the RHS of Eq. (5.14) above, which “spoil” the final result. So ultimately we do not recover the (correct) expressions which appeared

\(^{46}\)the multiplicities \( m_\varphi(\alpha_i), i \leq m \), appearing in Eq. (5.63) lead to the fusion coefficients \( N^\varphi_{\mu \nu} \) appearing in Eq. (5.67) below, cf. the RHS of Eq. (5.7).

\(^{47}\)this follows easily from Eq. (5.9) below after taking into account that the set \( \Lambda^k_\perp \) is not empty if \( k \geq c_0 \), cf. Remark 5.1.

\(^{48}\)in fact, the heuristic equation Eq. (2.4) was only derived for simply-connected compact groups \( G \) and therefore does not include the case \( G = U(1) \) or \( G = U(1) \times U(1) \). However, it is not difficult to see that for \( G = U(1) \times U(1) \) and \( (k_1, k_2) = (k, -k) \) an analogue of Eq. (7.3) below can be derived, cf. Remark 7.1 below.
in the continuum setting. This complication can be resolved\(^{49}\) by making the transition to the “\(BF_3\)-theory point of view”.

**Remark 6.1** As mentioned in Sec. 5.5 above (cf. also Sec. 8 below) there is something else we have to do in order to have a reasonable chance of finding a generalization of Theorem 3.5 to the case of general ribbon links. Instead of working with “half ribbons” we should rather work with “full ribbons”. In Remark 5.4 above we gave already one argument in favor of this claim. Here is another argument:

Let \(L = (R_1, R_2, \ldots, R_m)\) be a simplicial ribbon link in \(qK \times \mathbb{Z}_N\) which does not fulfill condition (NCP)\(^{49}\) above. More precisely, we assume that for some \(i \leq m\) and \(i' \leq m\) the \(\Sigma\)-projections of \(R_i\) and \(R_{i'}\) intersect each other in a 2-face \(F \in \mathfrak{F}_2(qK)\). Let us study the corresponding “2-cluster” given as in Eq. 6.2 above. Observe that the \(\Sigma\)-projections of the boundary loops of \(R_i\) and \(R_{i'}\) intersect in four points\(^{50}\) which contribute to the corresponding 2-cluster. However, these four intersection points are not treated equally. Two of these intersection points do not “detect” an interaction\(^{51}\) while the other two intersection points do. This asymmetry is already a strong indication that things will go wrong and we cannot expect to obtain the correct result when evaluating the 2-cluster explicitly.

Now, if we work with full ribbons then there will be nine intersection points contributing to each 2-cluster because to each of the two intersecting full ribbons we have three associated loops (cf. Remark 3.2 in Sec. 3.4 above). These nine intersection points are again not treated equally. Five intersection points do not detect an interaction (one intersection point inside the intersection 2-face \(F \in \mathfrak{F}_2(K)\) and four on its four vertices) but four intersection points do detect an interaction (namely, the four points in the middle of each edge of \(F\)). This time we have a symmetric situation and can be more optimistic that we obtain the correct result when evaluating the 2-cluster explicitly in the \(BF_3\)-setting.

7 Transition to the “\(BF_3\)-theory point of view”

7.1 Motivation

The simplicial program for Abelian CS theory (cf. Sec. 3 in [21]) was completed successfully by D.H. Adams, see [1, 2]. A crucial step in [1, 2] was the transition to the “BF-theory point of view”, which can be divided into two steps, namely “field doubling”\(^{52}\) followed by a suitable linear change of variables, cf. part C of the Appendix below.

Adams’ results seem to suggest that – if one wants to have a chance of carrying out the simplicial program successfully also for Non-Abelian CS theory – then a similar strategy will have to be used.

So far we have worked with the original CS point of view because this helped us to reduce the lengths of many formulas and because for Theorem 3.5 (which deals only with a special class of simplicial ribbon links \(L\)) the original CS point of view is sufficient. On the other hand, the Abelian “test situation” which we considered at the end of Sec. 6 showed us that we cannot expect to obtain correct results within the original CS point of view when dealing with general

\(^{49}\)observe that in contrast to the RHS of Eq. 3.14 above not all of the summands in the exponential on the RHS of Eq. 7.17 below are multiplied with a weight factor 1/2

\(^{50}\)recall from Sec. 5.4 that each of the two half ribbons \(R_i\) and \(R_{i'}\) induces a pair of loops which will contribute to \(\text{Hol}^{\text{AB}}(A^+, B)\) (with \(j = i, i'\)) and therefore to the 2-cluster; the \(\Sigma\)-projections of these two pairs of loops intersect in four points, which are simply the four vertices of the intersection 2-face \(F\) mentioned above

\(^{51}\)more precisely, the corresponding “covariance expression”, ie the term analogous to the expression in Eq. 5.16 in Sec. 5.1 above will vanish

\(^{52}\)which can come in the form of “group doubling” (see Step 1 below) or “base manifold doubling”; observe that the word “doubling” is slightly misleading because it ignores a sign change: we have \(k_2 = -k_1\) where \(k_1, k_2\) are as in the first paragraph of “Step 1” below
simplicial ribbon links. This is why from now on we will work with the “BF$_3$-theory point of view”.

7.2 The “BF$_3$-theory point of view”

In the following we will make the transition from non-Abelian CS theory in the torus gauge to the corresponding “BF$_3$-theory point of view” at a heuristic level.

Step 1: “Group doubling”

Let us now consider the version of Eq. (2.7) in the special case where $G = \hat{G} \times \tilde{G}$ where $\hat{G}$ is a simple, simply-connected compact Lie group and where $(k_1, k_2)$ fulfills $k_1 = -k_2$, cf. Remark 2.2 above. We set $k := k_1 = -k_2$.

For simplicity, let us consider the special case where each of the representations $\rho_i$ appearing in Eq. (2.7) is of the form $\rho_i(\tilde{g}_1, \tilde{g}_2) = \tilde{\rho}_i(\tilde{g}_1), \tilde{g}_1, \tilde{g}_2 \in \tilde{G}$, for some $\tilde{G}$-representation $\tilde{\rho}_i$. In this situation we should have

$$WLO(L) \sim WLO_{\tilde{G}}(\tilde{L})WLO_{\tilde{G}}(\tilde{0}) \sim |L| \cdot |\tilde{0}| \stackrel{(\ast)}{=} |L| \cdot |\tilde{0}| \tag{7.1}$$

where $WLO_{\tilde{G}}(L)$ on the RHS is defined as $WLO(L)$ in Sec. 2.2 for the group $\tilde{G}$ instead of $G$ and where $| \cdot |$ is now the shadow invariant for $\tilde{g}$ and $k$. In step (\ast) we used the fact that $|\tilde{0}|$ is a real number.

Let us fix a maximal torus $\tilde{T}$ of $\tilde{G}$ and set $T = \tilde{T} \times \tilde{T}$. Let $B, A^\perp, \tilde{A}^\perp, A^\perp_c$, and $\ll \cdot, \gg A^\perp$ be defined as in Sec. 2.1 and Sec. 2.3 for the group $G = \tilde{G} \times \hat{G}$.

In the following $B_1, B_2$ (resp. $A^\perp_1$ and $A^\perp_2$) will denote the two components of $B \in C^\infty(\Sigma, \tilde{t}) = C^\infty(\Sigma, \tilde{t}) \oplus C^\infty(\tilde{\Sigma}, \tilde{t})$ (resp. $A^\perp \in C^\infty(S^1, \tilde{A}_\Sigma) = C^\infty(S^1, \tilde{A}_\Sigma) \oplus C^\infty(S^1, \tilde{A}_\Sigma)$). Moreover, we denote by $\tilde{I}$ the kernel of $\exp_{\tilde{t}}: \tilde{t} \rightarrow \tilde{T}$.

Step 2: Linear change of variable

As we explain in part C of the Appendix, CS theory with group $G = \hat{G} \times \tilde{G}$ and $(k_1, k_2) = (k, -k)$ is equivalent to BF$_3$-theory with group $\tilde{G}$ and “cosmological constant” $\Lambda$ given by $\Lambda = \frac{1}{k^2}$. More precisely, at the heuristic level, these two theories are related by a simple linear change of variables, cf. Eqs. (C.5) or Eqs. (C.10) in part C of the Appendix depending on whether we are dealing with the non-gauge fixed path integral or the path integral in the torus gauge.

In order to simplify the notation a bit (and to avoid the appearance of multiple $k$-factors) we will work with the following simplified change of variable $A^\perp \rightarrow \tilde{A}^\perp, B \rightarrow \tilde{B}$ instead of the one in Eq. (C.10):

$$\tilde{A}^\perp := \left(\frac{A^\perp + A^\perp_c}{2}, \frac{A^\perp - A^\perp_c}{2}\right), \tag{7.2a}$$

$$\tilde{B} := \left(\frac{B_1 + B_2}{2}, \frac{B_1 - B_2}{2}\right) \tag{7.2b}$$

By applying this linear change of variable to the RHS of Eq. (2.7) (in the special case

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$^{53}$the first “$\sim$” follows from a short heuristic computation

$^{54}$observe that $\kappa = \frac{1}{k}$ in Eqs. (C.5) and (C.10)
\[ G = \tilde{G} \times \hat{G}, \quad (k_1, k_2) = (k, -k) \) we arrive at

\[
\text{WLO}(L) \sim \sum_{(y_1, y_2) \in \mathcal{I} \times \mathcal{I}} \int_{\tilde{A}^c_x \times \mathcal{B}^c_y} \prod_{(\tilde{A}^c_x, \mathcal{B}^c_y)} 1_{C^\infty(\Sigma \cup \mathcal{U}_{reg} \times \mathcal{U}_{reg})} ((\tilde{B}_1 + \tilde{B}_2, \mathcal{B}_1 - \mathcal{B}_2)) \det_{FP}((\tilde{B}_1 + \tilde{B}_2, \mathcal{B}_1 - \mathcal{B}_2)) \\
\times \left[ \int_{\tilde{A}^c_x} \prod_{i} \text{Tr}_{\tilde{\rho}_i}(\text{Hol}_{\tilde{\rho}_i}((\tilde{A}_i^c + \tilde{A}_c^c)_{0} + (\tilde{A}^c + \tilde{A}_c^c)_{2}, \tilde{B}_1 + \mathcal{B}_2)) \exp(i\mathcal{S}(\tilde{A}^c, \tilde{B})) D\tilde{A}^c \\
\times \exp(-2\pi i k((y_1, y_2), ((\tilde{B}_1 + \tilde{B}_2)(\sigma_0), (\tilde{B}_1 - \mathcal{B}_2)(\sigma_0)))) \right] \exp(i\mathcal{S}(\tilde{A}_c, \tilde{B}))(D\tilde{A}_c^c \otimes D\tilde{B})
\]

(7.3)

where for reasons of notational consistency, we have written \( \tilde{A}^c \) instead of \( \tilde{A}_c^c, \tilde{A}_c^c \) instead of \( A_c^c, B \) and we have set

\[
\mathcal{S}(\tilde{A}^c, \tilde{B}) := S_{CS}(\tilde{A}^c, B) \\
\mathcal{S}(\tilde{A}_c^c, \tilde{B}) := S_{CS}(A_c^c, B)
\]

More explicitly, we have\(^{56}\)

\[
\mathcal{S}(\tilde{A}^c, \tilde{B}) = S_{CS}((\tilde{A}^c_1 + \tilde{A}^c_2, \tilde{A}_c^c - \tilde{A}_c^c), (\tilde{B}_1 + \tilde{B}_2, \mathcal{B}_1 - \mathcal{B}_2)) \\
= S_{CS}(\tilde{A}^c_1 + \tilde{A}^c_2, \mathcal{B}_1 + \mathcal{B}_2) - S_{CS}(\tilde{A}_c^c + \tilde{A}_c^c, \mathcal{B}_1 - \mathcal{B}_2) \\
= S_{CS}(\tilde{A}^c_1 + \tilde{A}^c_2 + (\tilde{B}_1 + \tilde{B}_2)dt) - S_{CS}(\tilde{A}_c^c + \tilde{A}_c^c + (\mathcal{B}_1 - \mathcal{B}_2)dt) \\
= \pi k \ll (\tilde{A}^c_1, \tilde{A}^c_2), \left(\begin{array}{c}
\begin{array}{*{20}{c}}
\ast & \ast \mathcal{M} + \mathcal{B}_1) \\
\ast \mathcal{M} + \mathcal{B}_2)
\end{array}
\end{array}\right) \cdot (\tilde{A}^c_1, \tilde{A}^c_2) \gg \tilde{A}_c^c
\]

(7.4)

and

\[
\mathcal{S}(\tilde{A}_c^c, \tilde{B}) = S_{CS}((\tilde{A}_c^c)_{1}, (\tilde{A}_c^c)_{2}, (\tilde{A}_c^c)_{1} - (\tilde{A}_c^c)_{2}, (\tilde{B}_1 + \tilde{B}_2, \mathcal{B}_1 - \mathcal{B}_2)) \\
= \ldots \\
= 4\pi k \ll \ast \cdot ((\tilde{A}_c^c)_{2}, (\tilde{A}_c^c)_{1}), (d\tilde{B}_1, d\mathcal{B}_2) \gg \mathcal{A}_{\mathcal{C}_i \mathcal{B}_{\mathcal{E}}}
\]

(7.5)

where \( \tilde{A}_c^c = (\tilde{A}_c^c, \tilde{A}_c^c) \), \( \tilde{A}_c^c = (\tilde{A}_c^c, (\tilde{A}_c^c)_{2}) \), and \( \tilde{B} = (\mathcal{B}_1, \mathcal{B}_2) \).

**Remark 7.1** Recall that above we have assumed that \( \tilde{G} \) is a simple, simply-connected compact (and therefore non-Abelian) Lie group. For sake of completeness (and in view of the discussion at the end of Sec. 6 above and Remark 7.2 and Remark 7.1 below) let us mention that, in fact, one can derive (an analogue of) Eq. (7.3) also if \( \tilde{G} \) is an Abelian compact Lie group. Of course, in this case the RHS of Eq. (7.3) simplifies drastically\(^{57}\).

**Remark 7.2** It might seem surprising that the second of the aforementioned two steps, i.e. the linear change of variable, really makes an essential difference. Clearly, the original heuristic path integral and the heuristic path integral after the application of the change of variable are equivalent. However, once the problem of discretizing the corresponding path integral is considered the difference really matters. A detailed look at \([12] \) will convince the reader that this is indeed the case at least in the Abelian situation.

---

\(^{55}\)here we have used that – according to the assumption made at the beginning of “Step 1” – we have \( \rho_{\tilde{\rho}_j}((\tilde{g}_1 \tilde{g}_2) = \hat{\rho}_j(g_i) \), for all \( \tilde{g}_1, \tilde{g}_2 \in \tilde{G} \), which implies \( \text{Tr}_{\tilde{\rho}_j}(\text{Hol}_{\tilde{\rho}_j}(\tilde{A}^c, \tilde{B})) = \text{Tr}_{\hat{\rho}_j}(\text{Hol}_{\hat{\rho}_j}(A^c, B_1), \text{Hol}_{\hat{\rho}_j}(A^c, B_2)) = \text{Tr}_{\hat{\rho}_j}(\text{Hol}_{\hat{\rho}_j}(A^c, B_1)) \)

\(^{56}\)the first appearance of \( S_{CS} \) on the RHS of the following equation is a shorthand for \( S_{CS}(M, \tilde{G} \times \hat{G}, (k_1, k_2)) \) while the other appearances are a shorthand for \( S_{CS}(M, \tilde{G}), k, \tilde{G} \times \hat{G}, (k_1, k_2)) \)

\(^{57}\)we then have \( \tilde{T} = \tilde{G}, \mathcal{U}_{reg} = \hat{t}, \text{ad}(\mathcal{B}_j) = 0, \text{Det}_{FP}(\tilde{B}_1 + \tilde{B}_2, \mathcal{B}_1 - \mathcal{B}_2) \) is a constant function, and the sum \( \sum_{y_1, y_2} \) is trivial. So we obtain: \( \text{WLO}(L) \sim \int [\int \prod_{i} \text{Tr}_{\tilde{\rho}_i}(\text{Hol}_{\tilde{\rho}_i}(\tilde{A}^c, \tilde{B}, (\tilde{A}^c + \tilde{A}_c^c)_{1} + (\tilde{A}^c + \tilde{A}_c^c)_{2}, \tilde{B}_1 + \mathcal{B}_2)) \exp(i\mathcal{S}(\tilde{A}^c, \tilde{B})) D\tilde{A}^c \}] \exp(i\mathcal{S}(\tilde{A}_c, \tilde{B}))(D\tilde{A}_c^c \otimes D\tilde{B}) \)
That a linear change of variables is useful also for the discretization of non-Abelian CS (with doubled group) is less obvious. Observe, for example, that there is a ⋆-operator on the main diagonal of the $2 \times 2$-matrix appearing in Eq. (7.24) above. Because of this we cannot hope to be able to find a discretized version of the path integral on the RHS of Eq. (7.3) where each of the two components $\tilde{A}_1^±$ and $\tilde{A}_2^±$ “lives” either on $K_1 \times \mathbb{Z}_N$ or on $K_2 \times \mathbb{Z}_N$. Instead, each component $\tilde{A}_1^±$ and $\tilde{A}_2^±$ must be implemented in a “mixed” fashion (which is what we did in the discretization approach of the present paper, cf. Sec. 3). This is a crucial difference compared to the Abelian situation where it was indeed possible to find a non-mixed discretization for the relevant simplicial fields. This difference is one of the reasons why we decided to postpone the transition to the $BF_3$-theory point of view until now.

### 7.3 Simplification of some of the notation in Sec. 7.2

Before we discretize the expression on the RHS of Eq. (7.3) let us first simplify the notation somewhat:

Firstly, we will drop the ⋆-signs appearing in the previous subsection, for example will write $G$ instead of $\tilde{G}$, $B$ instead of $\tilde{B}$, $I$ instead of $\tilde{I}$ and so on. Clearly, we then have

\[
B = C^\infty(\Sigma, tt) \quad (7.6a)
\]
\[
A^± = C^\infty(S^\infty, A_{\Sigma, g=0}) \quad (7.6b)
\]
\[
\tilde{A}^± = \{A^± \in A^± | \int A^±(t)dt \in A_{\Sigma, t=\Sigma}\} \quad (7.6c)
\]
\[
A_c^± = \{A^± \in A^± | A^± \text{ is constant and } A_{\Sigma, t=\Sigma}-\text{valued}\} \quad (7.6d)
\]

Moreover, we will set

\[
B_± := B_1 \pm B_2 \quad \text{for } B = (B_1, B_2) \in B
\]
\[
A_± := A_1^± \pm A_2^± \quad \text{for } A^± = (A_1^±, A_2^±) \in A^±
\]

and use the notation $y_+$ instead of $y_1$ and $y_-$ instead of $y_2$. Then we can rewrite Eq. (7.3) in the following way

\[
\text{WLO}(L) \sim \sum_{y_+, y_- \in I} \int_{A^±} \prod_\pm \left[ 1_{C^\infty(\Sigma, t=\Sigma)}(B_±) \det_{FP}(B_±) \right] \times \left[ \int \prod_\pm \text{Tr}_{\rho_i}(\text{Hol}_{\Sigma, i}((\tilde{A}^± + A_c^±)_+, B_±)) \exp(iS(\tilde{A}^±, B))DA^± \right] \\
\times \exp(-2\pi ik \sum_{y_\pm}(y_\pm, B_±(\sigma_0))) \right] \exp(iS(A_c^±, B))(DA_c^± \otimes DB) \quad (7.7)
\]

where $\prod_\pm \cdots$ (resp. $\sum_\pm \cdots$) is the obvious two term product (resp. sum). Above we have set (cf. Eq. (7.24) and Eq. (7.25) above)

\[
S(\tilde{A}^±, B) := \pi k \ll (\tilde{A}_1^±, \tilde{A}_2^±), \star \left( \frac{\partial}{\partial t} + \text{ad}(B_1) \right) \right) \cdot (\tilde{A}_1^±, \tilde{A}_2^±) \gg A^± \quad (7.8)
\]
\[
S(A_c^±, B) := 4\pi k \ll \star \cdot ((A_c^±)_2, (A_c^±)_1), (dB_1, dB_2) \gg A^± \quad (7.9)
\]

where $\star : A^± \to A^±$ is the Hodge star operator induced by the auxiliary Riemannian metric $g$.

**Remark 7.3** Observe that for each $l \in \{l_1, l_2, \ldots, l_m\}$ we have (cf. Eqs. (7.6) and (7.9) in Sec.
above and Eq. (5.15) in \[24\])

\[
\text{Hol}(A^\perp_+, B_+) = \lim_{n \to \infty} \prod_{k=1}^n \exp \left( \frac{1}{n} (A^\perp_+ + B_+ dt)(l'(t)) \right)_{t=k/n} = \lim_{n \to \infty} \prod_{k=1}^n \exp \left( (A^\perp_+(l_{S^1}(t)) \left( \frac{1}{n} l'_{S^1}(t) \right) + A^\perp_2(l_{S^1}(t))(\frac{1}{n} l'_{S^1}(t)) \right) + B_1(l_{S^1}(t))dt \left( \frac{1}{n} l'_{S^1}(t) \right) + B_2(l_{S^1}(t))dt \left( \frac{1}{n} l'_{S^1}(t) \right) \right)_{t=k/n} (7.10)
\]

for \(A^\perp \in \mathcal{A}^\perp\) and \(B \in \mathcal{B}\).

### 7.4 Discretization of Eq. (7.7)

We will now sketch how – using a suitable discretization of the expression on the RHS of Eq. (7.7) – a rigorous definition of WLO(L) appearing in Eq. (7.7) can be obtained. (In part D of the Appendix we will sketch an alternative way of discretizing the RHS of Eq. (7.7)).

In view of what we have learned in Sec. 5.5 (cf. Remark 5.4) we will now work with simplicial ribbons in \(\mathcal{K} \times \mathcal{Z}_N\) (= “full ribbons”) instead of simplicial ribbons in \(q\mathcal{K} \times \mathcal{Z}_N\) (= “half ribbons”).

We set

\[
\begin{align*}
\mathcal{B}(q\mathcal{K}) &:= C^0(q\mathcal{K}, t \oplus t) \quad (7.11a) \\
\mathcal{A}_\Sigma(q\mathcal{K}) &:= C^1(q\mathcal{K}, \mathfrak{g} \oplus \mathfrak{g}) \quad (7.11b) \\
\mathcal{A}^\perp(q\mathcal{K}) &:= \text{Map}(\mathcal{Z}_N, \mathcal{A}_\Sigma(q\mathcal{K})) \quad (7.11c)
\end{align*}
\]

and introduce the scalar product \(\ll \cdot, \cdot \gg_{\mathcal{A}^\perp(q\mathcal{K})}\) on \(\mathcal{A}^\perp(q\mathcal{K})\) in an analogous way as in Sec. 3.1 above.

For technical reasons we will again introduce the subspaces \(\mathcal{A}_\Sigma(K)\) and \(\mathcal{A}^\perp(K)\) of \(\mathcal{A}_\Sigma(q\mathcal{K})\) and \(\mathcal{A}^\perp(q\mathcal{K})\) given by\(^{58}\)

\[
\begin{align*}
\mathcal{A}_\Sigma(K) &:= \mathcal{A}_{\Sigma, \mathfrak{g} \oplus \mathfrak{g}}(K) \subset \mathcal{A}_\Sigma(q\mathcal{K}) \quad (7.12a) \\
\mathcal{A}^\perp(K) &:= \text{Map}(\mathcal{Z}_N, \mathcal{A}_\Sigma(K)) \subset \mathcal{A}^\perp(q\mathcal{K}) \quad (7.12b)
\end{align*}
\]

Moreover, we will introduce the subspace \(\mathcal{B}_0(q\mathcal{K}) := \psi(\mathcal{B}(\mathcal{K}))\) of \(\mathcal{B}(q\mathcal{K})\) where \(\mathcal{B}(\mathcal{K}) := C^0(K, t \oplus t)\) and where \(\psi : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(q\mathcal{K})\) is given exactly like in Choice \(\mathfrak{I}\) in Sec. 3.9 above with \(t\) replaced by \(t \oplus t\).

As in Sec. 3.3 we have a well-defined operator \(\ast_K : \mathcal{A}^\perp(K) \to \mathcal{A}^\perp(K)\) and as in Sec. 3.1 we have a decomposition

\[
\mathcal{A}^\perp(K) = \mathcal{A}^\perp(K) \oplus \mathcal{A}^\perp_\Xi(K)
\]

where

\[
\begin{align*}
\mathcal{A}^\perp(K) &:= \{ A^\perp \in \mathcal{A}^\perp(K) \mid \sum_{t \in \mathcal{Z}_N} A^\perp(t) \in \mathcal{A}_{\Sigma, t \oplus t}(K) \} \quad (7.13a) \\
\mathcal{A}^\perp_\Xi(K) &:= \{ A^\perp \in \mathcal{A}^\perp(K) \mid A^\perp(\cdot) \text{ is constant and } \mathcal{A}_{\Sigma, t \oplus t}(K)\text{-valued} \} \equiv \mathcal{A}_{\Sigma, t \oplus t}(K) \quad (7.13b)
\end{align*}
\]

Moreover, we set

\[
\begin{align*}
B_\pm &:= B_1 \pm B_2 \quad \text{for } B = (B_1, B_2) \in \mathcal{B}(q\mathcal{K}) \\
A^\perp_\pm &:= A^\perp_1 \pm A^\perp_2 \quad \text{for } A^\perp = (A^\perp_1, A^\perp_2) \in \mathcal{A}^\perp(K)
\end{align*}
\]

\(^{58}\)Here and in the following we use again the notation \(\mathcal{A}_{\Sigma, V}(K) := C^1(K_1, V) \oplus C^1(K_2, V)\) for a finite-dimensional real vector space \(V\).
As the discrete analogues of Eq. (7.8) and Eq. (7.9) above we now take

\[ \mathcal{S}^{\text{disc}}(\tilde{A}^+, B) := \pi k \ll (\tilde{A}_1^+, \tilde{A}_2^+) \cdot \star R(N)(B) \cdot (\tilde{A}_1^+, \tilde{A}_2^+) \gg A^+(qK) \] (7.14)

\[ \mathcal{S}^{\text{disc}}(A_c^+, B) := 4\pi k \ll \star R \cdot ((A_{c1}^+)^2, (A_{c2}^+)^1), (d_{qK}B_1, d_{qK}B_2) \gg A^+(qK) \] (7.15)

where

\[ R(N)(B) := \left( \frac{L(N)(B_+) - L(N)(B_-)}{2} \frac{L(N)(B_+) + L(N)(B_-)}{2} \right) \] (7.16)

As mentioned above we will now work with “full ribbons”, i.e. closed simplicial ribbons \( R \) in \( K \times \mathbb{Z}_N \). Recall from Remark 3.2 that each such \( R \) induces three simplicial loops \( l^+ = (l^+(k))_{k \leq n} \), \( l^- = (l^-(k))_{k \leq n} \), and \( l = (l^k)_{k \leq n}, n \in \mathbb{N} \), in \( qK \times \mathbb{Z}_N \). As the discrete analogue \( \mathcal{H}_{\text{disc}}(A^+_c, B_+) \) of the continuum expression \( \mathcal{H}_R(A^+_c, B_+) \) in Eq. (7.10) above we now take

\[ \mathcal{H}_{\text{disc}}(A^+_c, B_+) := \prod_{k=1}^n \exp \left( \sum_{\pm} \frac{1}{2} A_{c1}^+ (\bullet l_{\Sigma}^{(k)})(l_{\Sigma}^{(k)}) + A_{c2}^+ (\bullet l_{\Sigma}^{(k)})(l_{\Sigma}^{(k)}) \right) \] (7.17)

In view of Eq. (3.16) above and the list of replacements in Sec. 5.4.1 in [24] the ansatz in Eq. (7.17) is quite natural. The only point that requires an explanation is why the field component \( A^+_c \) “interacts” only with the loop \( l \) while \( A^+_1 \) “interacts” with the two loops \( l^+ \) and \( l^- \). We will give this explanation in “Observation 1” in part D.1 of the Appendix.

Remark 7.4 In the special case where \( G \) is Abelian (cf. Remark 7.1 above) there are several simplifications. For example, in the special case where the (complex) representation \( \rho \) of \( G \) is irreducible (and therefore 1-dimensional) we have from Eq. (7.17)

\[ \text{Tr}_\rho(\mathcal{H}_{\text{disc}}(A^+_c, B_+)) = \rho \left( \prod_{k=1}^n \exp \left( \sum_{\pm} \frac{1}{2} A_{c1}^+ (\bullet l_{\Sigma}^{(k)})(l_{\Sigma}^{(k)}) + B_1 (\bullet l_{\Sigma}^{(k)}) dt(N)(l_{\Sigma}^{(k)}) \right) \right) \times \rho \left( \prod_{k=1}^n \exp \left( A_{c2}^+ (\bullet l_{\Sigma}^{(k)})(l_{\Sigma}^{(k)}) + B_2 (\bullet l_{\Sigma}^{(k)}) dt(N)(l_{\Sigma}^{(k)}) \right) \right) \]

We can work with a generalization of the last expression where the second “\( \rho \)” appearing above is replaced by another finite-dimensional representation \( \rho' \) of \( G \). By doing so we obtain a kind of torus gauge “analogue” of the approach in [1, 2].

As the discrete version for the two expressions \( \text{Det}_{FP}(B_+) \) appearing in Eq. (7.14) we choose again (as in Eq. (3.24) in Sec. 3.9 above)

\[ \text{Det}_{\text{disc}}(B_+) := \prod_{x \in \mathbb{B}_0(qK)} \det^{1/2}(1 - \exp(\text{ad}(B_+)(x))) |t| \] (7.18)

---

59 Recall that \( L(N)(B_0) \) is a discrete “approximation” of \( \partial_t + \text{ad}(B_0) \) so \( \frac{1}{2}(L(N)(B_+) + L(N)(B_-)) \) is a discrete analogue of \( \frac{1}{2}(\partial_t + \text{ad}(B_+)) + \frac{1}{2}(\partial_t + \text{ad}(B_-)) = \partial_t + \text{ad}(B_1 + B_2) \); similarly \( \frac{1}{2}(L(N)(B_+) - L(N)(B_-)) \) is a discrete approximation of \( \frac{1}{2}(\partial_t + \text{ad}(B_1 + B_2) - (\partial_t + \text{ad}(B_1 - B_2))) = \text{ad}(B_2) \).

60 One could, of course, ask the analogous question with respect to the field components \( B_1 \) and \( B_2 \). However, the latter question can easily be avoided since it turns out that the value of \( WLO_{\text{rig}}(L) \) as defined in Eq. (7.19) below does not change if in Eq. (7.14) we replace the expression \( \sum_{\pm} \frac{1}{2} B_1 (\bullet l_{\Sigma}^{(k)}) dt(N)(l_{\Sigma}^{(k)}) + B_2 (\bullet l_{\Sigma}^{(k)}) dt(N)(l_{\Sigma}^{(k)}) \) by the symmetric expression \( \sum_{\pm} \left| \frac{1}{2} B_1 (\bullet l_{\Sigma}^{(k)}) dt(N)(l_{\Sigma}^{(k)}) + B_2 (\bullet l_{\Sigma}^{(k)}) dt(N)(l_{\Sigma}^{(k)}) \right| \).

61 Let us emphasize that we do not get a strict analogue: in the case of Abelian \( G \) our approach is less general than [1, 2]. We remark that for an Abelian \( G \) one actually can construct a strict torus gauge analogue of the approach in [1, 2] by modifying our approach in a suitably way. However, this modified approach will not be useful for dealing with the case of non-Abelian \( G \).
The remaining steps for discretizing the RHS of Eq. (7.7) can be carried out easily (as in Secs 3.6–3.8 above). There is only one exception: since we are now working with “full ribbons” it will be necessary to use an additional regularization procedure, cf. Remark 5.4 in Sec. 5.5 above.

We then arrive at a rigorous version \( WLO^{\text{disc}}_{\text{rig}}(L) \) of \( WLO(L) \) and its normalization

\[
WLO_{\text{rig}}(L) := \frac{WLO^{\text{disc}}_{\text{rig}}(L)}{WLO^{\text{disc}}_{\text{rig}}(\emptyset)} \tag{7.19}
\]

In view of the heuristic formula Eq. (7.1) we expect that for \( k \geq c_g \) we have

\[
WLO_{\text{rig}}(L) = \frac{|L|\emptyset}{|\emptyset|\emptyset} = \frac{|L|}{|\emptyset|} \tag{7.20}
\]

and in contrast to the situation in Theorem 3.5 there is now a reasonable chance that Eq. (7.20) even holds for general simplicial ribbon links \( L \) in \( \mathcal{K} \times \mathbb{Z}_N \).

8 Discussion & Outlook

In [24] we proposed a rigorous “simplicial” realization of the torus-gauge-fixed non-Abelian CS-path integral for manifolds \( M \) of the form \( M = \Sigma \times S^1 \). In the present paper we proved the main result of [24], Theorem 3.5 above, which deals with a special class of simplicial ribbon links. During the proof of Theorem 3.5 and in Sec 6 it became clear that in order to have a reasonable chance of generalizing our computations successfully to the case of general simplicial ribbon links it seems to be necessary to make (at least) the following two modifications of our approach:

i) we should make the “transition to the BF-theory point of view” (cf. “Step 1” and “Step 2” in Sec. 7.2 above),

ii) we should work with simplicial ribbons in \( \mathcal{K} \times \mathbb{Z}_N \) (“full ribbons”) rather than with simplicial ribbons in \( q\mathcal{K} \times \mathbb{Z}_N \) (“half ribbons”).

We sketched such a modification of our approach in Sec. 7.4 (and a suitable reformulation of it in part D of the Appendix). In [25] we will study this new approach in more detail. It remains to be seen whether we really obtain the correct values for \( WLO_{\text{rig}}(L) \) in the case of general \( L \). Here are some points which suggest that the chances for this being the case are quite good.

- We mentioned in Sec. 6.2 in [24] that also for simplicial ribbon links \( L \) fulfilling only a weaker version of condition (NCP)’ we can evaluate \( WLO_{\text{rig}}(L) \) explicitly using a suitable modification of the approach used in Sec. 5 above for proving Theorem 3.5 cf. [26].

- The heuristic argument in Appendix B.2 in [24] shows that also for general links we can expect the value of \( WLO(L) \) to be a sum over step functions \( B \). Using a rigorous version of this heuristic argument this should not be difficult to prove that also \( WLO_{\text{rig}}(L) \) defined as in Sec. 3 above will be a sum over step functions \( B \). At the moment it is not yet clear whether these step functions have the correct values (or, rather, “step sizes”) and wheather the symmetry argument based on the affine Weyl group \( \mathcal{W}_\text{aff} \) we referred to in Sec. 5.6 above will lead to a sum over area colorings (cf. the sum \( \sum_{\phi \in \text{col}(L)} \cdots \) in Eq. (B.4) below) also in the case of general \( L \).

---62 Here \(| \cdot |\) is the shadow invariant for \( g \) and \( k \); recall that we write the Lie algebra \( \tilde{g} \) now simply as \( g \), i.e. without the \( \sim \)

---63 In fact, since the computations for general links are quite complicated we will restrict our attention to the special group \( G = SU(2) \) and we will first compute only the low order terms in the expansion of \( WLO_{\text{rig}}(L) \) as an asymptotic series of powers of \( 1/k \) (for \( k \to \infty \))

---64 In this case some of the simplicial ribbons \( R_i, i \leq m \), contained in \( L \) are allowed to be ribbon analogues of non-trivial framed torus knots
• If the sum $\sum_{\varphi \in \text{col}(L)} \cdots$ indeed arises also in the general situation then it is clear\textsuperscript{65} from Sec. 5.5 that also for general $L$ we will again obtain the factor $|L|^2$ appearing in Eq. (3.4) below. Moreover, also the factor $|L|^3$ should appear again. Things are a bit less clear regarding the factor $|L|^2$. What remains to be clarified is whether we obtain the correct gleam expressions gleam$(Y)$ also for general $L$.

• The crucial open question is related to the factor $|L|^4$ in Eq. (3.4), which is definitely the most complicated and interesting factor. At the moment it is completely open whether this factor can be obtained using our approach\textsuperscript{66} by evaluating the appropriate “2-cluster” expressions (which will be very similar to the ones we studied in Sec. 6).

We remark that in the case of Chern-Simons theory on $\mathbb{R}^3$ in axial gauge we already performed similar calculations in \cite{19} (cf. also \cite{13, 30}) and found that the analogous expressions for the “2-clusters” were problematic. One serious problem in \cite{19} was that the values of the 2-clusters turned out to depend on the implementation of a regularization procedure called “loop smearing” in \cite{19}. It is possible that in the continuum approach in \cite{22, 23} to CS theory on $\Sigma \times S^1$ in the torus gauge, which also makes use of “loop smearing”, there will be a similar “loop smearing” dependence. Fortunately, within the simplicial approach developed and studied in \cite{24}, the present paper, and \cite{25} this complication is essentially\textsuperscript{67} absent.

Acknowledgements: I want to thank the anonymous referee of my paper \cite{22} whose comments motivated me to look for an alternative approach for making sense of the RHS of (the original version of) Eq. (2.7), which is less technical than the continuum approach in \cite{20, 22, 23}. This eventually led to \cite{24} and the present paper.

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A Appendix: Lie theoretic notation II

The following two lists extend the two lists in Appendix A in \cite{24}.

A.1 List of notation in the general case

Recall that in Sec. 2.1 we fixed a simply-connected compact Lie group $G$ (with Lie algebra $\mathfrak{g}$), a maximal $T$ of $G$ (with Lie algebra $\mathfrak{t}$), and a Weyl chamber $C \subset \mathfrak{t}$.

Apart from the notation given in Appendix A of \cite{24} we also use the following Lie theoretic notation in the present paper:

• $\langle \cdot, \cdot \rangle$: the unique $\text{Ad}$-invariant scalar product on $\mathfrak{g}$ such that\textsuperscript{68} $\langle \alpha, \alpha \rangle = 2$ holds for every short real coroot $\check{\alpha}$ associated to $(\mathfrak{g}, \mathfrak{t})$. Using $\langle \cdot, \cdot \rangle$ we now make the identification $\mathfrak{t} \cong \mathfrak{t}^\ast$.

• $\mathfrak{t}^\perp$: the $\langle \cdot, \cdot \rangle$-orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$.

\textsuperscript{65}cf. Remark 5.4 above and recall that in \cite{25} we will be working with “full ribbons”

\textsuperscript{66}or a suitable modification of our approach

\textsuperscript{67}in fact, the “half ribbons vs full ribbons” issue mentioned above could be seen as a (fortunately very harmless) simplicial analogue of the issue of “loop smearing dependence” in the continuum setting

\textsuperscript{68}which is equivalent to the condition $\langle \alpha, \alpha \rangle^\ast = 2$ for every long real root $\alpha \in \mathfrak{t}^\ast$ where $\langle \cdot, \cdot \rangle^\ast$ is the scalar product on $\mathfrak{t}^\ast$ induced by $\langle \cdot, \cdot \rangle$
\begin{itemize}
  \item $\mathcal{R}_C$: the set of complex roots $t \rightarrow \mathbb{C}$ associated to $(g, t)$
  \item $\mathcal{R} \subset t^*$: the set $\{ \frac{1}{2 \pi} \alpha_c \mid \alpha_c \in \mathcal{R}_C \}$ of real roots associated to $(g, t)$
  \item $\mathcal{R}_+ \subset \mathcal{R}$: the set of positive (real) roots corresponding to $C$
  \item $\Gamma \subset t$: the lattice generated by the set of real coroots associated to $(g, t)$, i.e. by the set $\{ \tilde{\alpha} \mid \alpha \in \mathcal{R} \}$ where $\tilde{\alpha} = \frac{2 \alpha}{(\alpha, \alpha)} \in t^* \cong t$ is the coroot associated to the root $\alpha \in \mathcal{R}$.
  \item $I \subset t$: the kernel of $\exp : t \rightarrow T$. From the assumption that $G$ is simply-connected it follows that $I = \Gamma$.
  \item $\Lambda \subset t^*(\cong t)$: the real weight lattice associated to $(g, t)$, i.e. $\Lambda$ is the lattice which is dual to $\Gamma$.
  \item $\Lambda_+ \subset \Lambda$: the set of dominant weights corresponding to the Weyl chamber $C$, i.e. $\Lambda_+ := \overline{C} \cap \Lambda$
  \item $\rho$: half sum of positive roots (“Weyl vector”)
  \item $\theta$: unique long root in $\mathcal{C}$.
  \item $c_g = 1 + \langle \theta, \rho \rangle$: the dual Coxeter number of $g$.
  \item $P \subset t$: a fixed Weyl alcove
  \item $Q \subset t$: a subset of $t$ of the form $Q = \{ \sum_i \lambda_i e_i \mid 0 < \lambda_i < 1 \ \forall i \leq \dim(t) \}$ where $(e_i)_{i \leq \dim(t)}$ is a fixed basis of $\Gamma = I$.
  \item $W \subset \text{GL}(t)$: the Weyl group of the pair $(g, t)$
  \item $W_{\text{aff}} \subset \text{Aff}(t)$: the “affine Weyl group of $(g, t)$”, i.e. the subgroup of $\text{Aff}(t)$ generated by $W$ and the set of translations $\{ \tau_x \mid x \in \Gamma \}$ where $\tau_x : t \ni b \mapsto b + x$.
  \item $W_k \subset \text{Aff}(t)$, $k \in \mathbb{N}$: the subgroup of $\text{Aff}(t)$ given by $\{ \psi_k \circ \sigma \circ \psi_k^{-1} \mid \sigma \in W_{\text{aff}} \}$ where $\psi_k : t \ni b \mapsto b \cdot k - \rho \in t$ (the “quantum Weyl group corresponding to the level $l := k - c_g$”)
  \item $\Lambda_+^k \subset \Lambda$, $k \in \mathbb{N}$: the subset of $\Lambda_+$ given by $\Lambda_+^k := \{ \lambda \in \Lambda_+ \mid \langle \lambda, \theta \rangle \leq k - c_g \}$ (the “set of dominant weights which are integrable at level $l = k - c_g$”).
\end{itemize}

In the main text, the number $k \in \mathbb{N}$ appearing above will be the integer $k$ fixed in Sec. 2.1 (which later is assumed to fulfill $k \geq c_g$).

### A.2 List of notation in the special case $G = SU(2)$

Let us now consider the special group $G = SU(2)$ with the standard maximal torus $T = \{ \exp(\theta \tau) \mid \theta \in \mathbb{R} \}$ where

$$
\tau := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}
$$

Then $g = su(2)$ and $t = \mathbb{R} \cdot \tau$. There are two Weyl chambers, namely $C_+$ and $C_-$ where $C_\pm := \pm [0, \infty) \tau$. Let us fix $C := C_+$ in the following.

- $\langle \cdot, \cdot \rangle$ is the scalar product on $g$ given by\(^{69}\)

\[ \langle A, B \rangle = -\frac{1}{4\pi} \text{Tr}_{\text{Mat}(2, \mathbb{C})}(AB) \quad \text{for all } A, B \in g \subset \text{Mat}(2, \mathbb{C}) \]

\(^{69}\) in view of the formula $\tilde{\alpha} = \alpha = 2\pi \tau$ below we see that $\langle \cdot, \cdot \rangle$ indeed fulfills the normalization condition $\langle \tilde{\alpha}, \tilde{\alpha} \rangle = 2$.
\[ \mathcal{R}_c = \{ \alpha_c, -\alpha_c \} \quad \text{where} \quad \alpha_c : t \to \mathbb{C} \\text{is given by} \quad \alpha_c(\tau) = 2i \]

\[ \mathcal{R} = \{ \alpha, -\alpha \} \quad \text{where} \quad \alpha := \frac{1}{2\pi i} \alpha_c. \quad \text{A short computation shows that} \quad \alpha = 2\pi \tau \in t \quad \text{(recall that we made the identification} \quad t \cong t^*) \]

\[ \mathcal{R}_+ = \{ \alpha \} \]

\[ I = \Gamma = \mathbb{Z} \cdot \hat{\alpha} \quad \text{where} \quad \hat{\alpha} = \frac{2\alpha}{(\alpha, \alpha)} = \alpha \]

\[ \Lambda = \mathbb{Z} \cdot \frac{\alpha}{2} \]

\[ \Lambda_+ = \mathbb{N}_0 \cdot \frac{\alpha}{2} \]

\[ \rho = \frac{\alpha}{2} \]

\[ \theta = \alpha \]

\[ c_g = 2 \]

\[ \text{possible choices for} \quad P \quad \text{and} \quad Q \quad \text{are} \quad P = (0, \frac{1}{2}) \alpha \quad \text{and} \quad Q = (0, 1) \alpha \]

\[ \mathcal{W} = \{ 1, \sigma \} \quad \text{where} \quad 1 = \text{id} \quad \text{and} \quad \sigma(b) = -b \quad \text{for} \quad b \in t; \quad \text{using this and the explicit description} \quad \text{of} \quad I = \Gamma \quad \text{above one easily obtains an explicit description of} \quad \mathcal{W}_{\text{aff}} \quad \text{and} \quad \mathcal{W}_k \]

\[ \Lambda_k^+ = \{ 0, \frac{k}{2}, \ldots, \frac{k-2}{2} \alpha \} \quad \text{for} \quad k \in \mathbb{N} \]

**B Appendix: Turaev’s shadow invariant**

Let us briefly recall the definition of Turaev’s shadow invariant in the situation relevant for us, i.e. for manifolds \( M \) of the form \( M = \Sigma \times S^1 \) where \( \Sigma \) is an oriented surface.

Let \( L = (l_1, l_2, \ldots, l_m) \), \( m \in \mathbb{N} \), be a framed piecewise smooth link \( \text{in} \quad M = \Sigma \times S^1 \). For simplicity we will assume that each \( l_i, \quad i \leq m \) is equipped with a “horizontal” framing, cf. Remark 4.5 in Sec. 4.3 in [24]. Let \( V(L) \) denote the set of points \( p \in \Sigma \) where the loops \( l_i \), \( i \leq m \), cross themselves or each other (the “crossing points”) and \( E(L) \) the set of curves in \( \Sigma \) into which the loops \( l_1, l_2, \ldots, l_m \) are decomposed when being “cut” in the points of \( V(L) \). We assume that there are only finitely many connected components \( Y_0, Y_1, Y_2, \ldots, Y_m', m' \in \mathbb{N} \) (“faces”) of \( \Sigma \setminus (\bigcup_i \text{arc}(l_i^)) \) and set

\[ F(L) := \{ Y_0, Y_1, Y_2, \ldots, Y_m' \}. \]

As explained in [38] one can associate in a natural way a half integer gleam \( \text{of} \quad Y \) to each face \( Y \in F(L) \). In the special case where the two conditions (NCP) and (NH) appearing in Sec. 6.1 in [24] are fulfilled \( \text{we have the explicit formula} \]

\[ \text{gleam}(Y) = \sum_{i \text{ with arc}(l_i^) \subset \partial Y} \text{wind}(l_i^) \cdot \text{sgn}(Y; l_i^) \in \mathbb{Z}, \quad \text{(B.1)} \]

where \( \text{wind}(l_i^) \) is the winding number of the loop \( l_i^ \) and where \( \text{sgn}(Y; l_i^) \) is given by

\[ \text{sgn}(Y; l_i^) := \begin{cases} 
1 & \text{if} \quad Y \subset R_i^+ \\
-1 & \text{if} \quad Y \subset R_i^- 
\end{cases} \quad \text{(B.2)} \]

Here \( R_i^+ \) (resp. \( R_i^- \)) is the unique connected component \( R \) of \( \Sigma \setminus \text{arc}(l_i^) \) such that \( l_i^ \) runs around \( R \) in the “positive” (resp. “negative”) direction.

\text{this includes the case of simplicial ribbon links in} \ qK \times \mathbb{Z}_N \text{as a special case, cf. Remark B.2 below} \]

\text{cf. Remark B.2 below for the relevance of these two conditions for the present paper}
Let $G$ be a simply-connected and simple compact Lie group with maximal torus $T$. In the following we will use the notation from part A of the Appendix. In particular, we have

$$\Lambda^k = \{ \lambda \in \Lambda_+ \mid \langle \lambda, \theta \rangle \leq k - c_0 \} \quad (B.3)$$

where $k \in \mathbb{N}$ is as in Sec. 2 above and where $c_0 = 1 + \langle \theta, \rho \rangle$ is the dual Coxeter number of $g$.

**Remark B.1** Observe that for $k < c_0$ the set $\Lambda^k$ is empty so $|L|$ as defined in Eq. (B.4) below will then vanish. If $k = c_0$ then $|L|$ will in general not vanish but will still be rather trivial. Not surprisingly, the definition of $|L|$ in the literature often excludes the situation $k \leq c_0$ so our definition of $|L|$ in Eq. (B.4) below is more general than usually.

Assume that each loop $l_i$ in the link $L$ is equipped with a “color” $\rho_i$, i.e. a finite-dimensional complex representation of $G$. By $\gamma_i \in \Lambda_+$ we denote the highest weight of $\rho_i$ and set $\gamma(e) := \gamma_i$ for each $e \in E(L)$ where $i \leq m$ denotes the unique index such that $\text{arc}(e) \subset \text{arc}(l_i)$. Finally, let $\text{col}(L)$ be the set of all mappings $\varphi : \{Y_0, Y_1, Y_2, \ldots, Y_m\} \rightarrow \Lambda^k_+$ (“area colorings”).

We can now define the “shadow invariant” $|L|$ of the (colored and “horizontally framed”) link $L$ associated to the pair $(g, k)$ by

$$|L| := \sum_{\varphi \in \text{col}(L)} |L|_{\varphi}^0 |L|_{\varphi}^1 |L|_{\varphi}^2 |L|_{\varphi}^3 \quad (B.4)$$

with\(^{72}\)

$$|L|_{\varphi}^0 = \prod_{Y \in F(L)} \text{dim}(\varphi(Y))^{\chi(Y)} \quad (B.5a)$$

$$|L|_{\varphi}^1 = \prod_{Y \in F(L)} \exp\left(\frac{\pi}{k} \langle \varphi(Y), \varphi(Y) + 2\rho \rangle\right)^{\text{gleam}(Y)} \quad (B.5b)$$

$$|L|_{\varphi}^2 = \prod_{e \in E_+(L)} N^e(Y^e) \quad (B.5c)$$

$$|L|_{\varphi}^3 = \left(\prod_{e \in E(L)} S(e, \varphi)\right) \times \left(\prod_{x \in V(L)} T(x, \varphi)\right) \quad (B.5d)$$

Here $Y^e_+$ (resp. $Y^e_-$) denotes the unique face $Y$ such that $\text{arc}(e) \subset \partial Y$ and, additionally, the orientation on $\text{arc}(e)$ described above coincides with (resp. is opposite to) the orientation which is obtained by restricting the orientation on $\partial Y$ to $e$. Moreover, we have set (for $\lambda, \mu, \nu \in \Lambda^k_+$)

$$\text{dim}(\lambda) := \prod_{\alpha \in \mathcal{R}_+} \frac{\sin \frac{\pi(\lambda + \rho, \alpha)}{k}}{\sin \frac{\pi(\rho, \alpha)}{k}} \quad (B.6)$$

$$N_{\mu\nu}^\lambda := \sum_{\tau \in \mathcal{W}_k} \text{sgn}(\tau) m_\mu(\nu - \tau(\lambda)) \quad (B.7)$$

where $m_\mu(\beta)$ is the multiplicity of the weight $\beta$ in the unique (up to equivalence) irreducible representation $\rho_\mu$ with highest weight $\mu$ and $\mathcal{W}_k$ is as in part A of the Appendix. $E_+(L)$ is a suitable subset of $E(L)$ (cf. the notion of “circle-1-strata” in Chap. X, Sec. 1.2 in [37]).

The explicit expression for the factors $T(x, \varphi)$ appearing in $|L|_{\varphi}^3$ above involves the so-called “quantum 6j-symbols” (cf. Chap. X, Sec. 1.2 in [37]) associated to $U_q(\mathfrak{g}_\mathbb{C})$ where $q$ is the root of unity\(^{73}\)

$$q := \exp\left(\frac{2\pi i}{k}\right) \quad (B.8)$$

\(^{72}\)see footnote 93 in [24] for a brief comment on the exact relationship between our formula for $|L|$ and the corresponding formula for $|L|$ in [37].

\(^{73}\)We remark that there are different conventions for the definition of $U_q(\mathfrak{g}_\mathbb{C})$. Accordingly, one finds different formulas for $q$ in the literature. For example, using the convention in [30] one would be led to the formula $q := e^{\frac{i\pi}{D}}$ where $D$ is the quotient of the square lengths of the long and the short roots of $\mathfrak{g}$.
We omit the explicit formulae for $T(x, \varphi)$ and $S(e, \varphi)$ since they irrelevant for the present paper. Indeed, for links $L$ fulfilling the aforementioned conditions (NCP) and (NH) of Sec. 6.1 in \cite{24} the set $V(L)$ is empty and the set $E_+(L)$ coincides with $E(L)$, so Eq. (B.4) (combined with Eqs (B.5)) then reduces to

$$|L| = \sum_{\varphi \in \text{col}(L)} \left( \prod_{i=1}^{m} N^{\varphi(Y_i^-)} \gamma_i(\varphi(Y_i^-)) \right) \left( \prod_{Y \in F(L)} \dim(\varphi(Y)) \chi(Y) \exp(\frac{\chi_1}{\kappa}(\varphi(Y), \varphi(Y)+2\rho)) \langle \text{clean}(Y) \rangle \right)$$

(B.9)

where we have set $Y_i^\pm := Y_i^{\pm_\mathbb{C}}$.

**Remark B.2** The shadow invariant can be defined in a straightforward for every (colored) simplicial ribbon link $L$ in $qK \times \mathbb{Z}_N$ (or in $K \times \mathbb{Z}_N$) in by setting

$$|L| := |L_f|$$

where $L_f$ is the (colored and horizontally framed) piecewise smooth link associated to $L$. Observe that if $L$ fulfills the conditions (NCP) and (NH) of Sec. 6.1 in \cite{24} above then $L_f$ will fulfill the conditions (NCP) and (NH) mentioned above, so in this case Eq. (B.9) above will again hold.

**C Appendix: BF$_3$-theory in the torus gauge**

**C.1 BF$_3$-theory**

Let $M$ be a closed oriented 3-manifold, let $\mathcal{G}$ be a simple simply-connected compact Lie group with Lie algebra $\mathfrak{g}$, and let $\mathcal{G} := C^\infty(M, \mathcal{G})$.

For $\Lambda \in \mathcal{A} := \Omega^1(M, \mathcal{G})$ and $\bar{\Lambda} \in \bar{\mathcal{C}} := \Omega^1(M, \mathfrak{g})$ and $\Lambda \in \mathbb{R}$ (the “cosmological constant”) we define\footnote{Here we assume for simplicity (cf. Remark 2.1 above) that $\mathcal{G}$ is Lie subgroup of $U(\bar{N})$ for some $\bar{N} \in \mathbb{N}$ and we set $\text{Tr} := \tilde{c} \text{Tr}_{\text{Mat}(N, \mathbb{C})}$ where $\tilde{c} \in \mathbb{R}$ is chosen suitably.}

$$S_{BF}(\Lambda, \bar{\Lambda}) := \frac{1}{\pi} \int_M \text{Tr}(F^{\Lambda} \wedge \bar{\Lambda} + \Lambda \bar{\Lambda} + \bar{\Lambda} \wedge \bar{\Lambda})$$

(C.1)

where $F^{\Lambda} := d\Lambda + \Lambda \wedge \Lambda$. Let us assume in the following that $\Lambda \in \mathbb{R}_+$ and set

$$\kappa := \sqrt{\Lambda}$$

(C.2)

Note that $S_{BF} : \mathcal{A} \times \bar{\mathcal{C}} \to \mathbb{C}$ is $\mathcal{G}$-invariant under the $\mathcal{G}$-operation on $\mathcal{A} \times \bar{\mathcal{C}}$ given by $(A, C) \cdot \bar{\Lambda} = (\bar{\Lambda}^{-1}A\bar{\Lambda} + \bar{\Lambda}^{-1}d\bar{\Lambda} + \bar{\Lambda}^{-1}C\bar{\Lambda})$.

It is well-known that in the situation $\kappa \neq 0$ the relation

$$S_{BF}(\Lambda, \bar{\Lambda}) = S_{CS}(\Lambda + \kappa \bar{\Lambda}) - S_{CS}(\Lambda - \kappa \bar{\Lambda})$$

(C.3)

holds with $S_{CS} = S_{CS}(M, G, k)$ where $G := \mathcal{G}$ and $k := \frac{1}{\kappa}$. Using the change of variable $(A, \bar{\Lambda}) \to (A_1, A_2)$ given by

$$A_1 := A + \kappa \bar{\Lambda}, \quad A_2 := A - \kappa \bar{\Lambda}$$

(C.4)

or, equivalently,

$$\Lambda := \frac{1}{\kappa}(A_1 + A_2), \quad \bar{\Lambda} := \frac{1}{\kappa}(A_1 - A_2)$$

(C.5)

we therefore obtain, informally, for every $\chi : \mathcal{A} \times \bar{\mathcal{C}} \to \mathbb{C}$

$$\int \int \chi(\Lambda, \bar{\Lambda}) \exp(iS_{BF}(\Lambda, \bar{\Lambda})) D\Lambda D\bar{\Lambda}$$

$$\sim \int \int \chi((A_1, A_2)) \exp(IS_{CS}(A_1)) \exp(-iS_{CS}(A_2)) DA_1 DA_2$$

(C.6)
where $\chi: A_1 \times A_2 \to \mathbb{C}$ with $A_j := \Omega^1(M, g)$, $j = 1, 2$, is the function given by $\chi((A_1, A_2)) = \tilde{\chi}(\tilde{A}, \tilde{C})$.

If – instead of setting $S_{CS} := S_{CS}(M, G, k)$ with $G := \tilde{G}$ and $k := \frac{1}{\kappa}$ – we use $S_{CS} := S_{CS}(M, G, (k_1, k_2))$ with $G = \tilde{G} \times \tilde{G}$ and $(k_1, k_2) = (1/\kappa, -1/\kappa)$ (cf. Remark 2.2 in [24]) then we can rewrite Eq. (C.6) as

$$\int \chi(\tilde{A}, \tilde{C}) \exp(iS_{BF}(\tilde{A}, \tilde{C})) D\tilde{A}D\tilde{C} \sim \int \chi(A) \exp(iS_{CS}(A)) DA$$

(C.7)

Thus we see that $BF_3$-theory on $M$ with group $\tilde{G}$ and $\kappa \neq 0$ is essentially equivalent to $CS$-theory on $M$ with group $G = \tilde{G} \times \tilde{G}$ and $(k_1, k_2) = (1/\kappa, -1/\kappa)$.

C.2 $BF_3$-theory on $M = \Sigma \times S^1$ “in the torus gauge”

Let us now consider the special case where $M = \Sigma \times S^1$, $\kappa \neq 0$ and $1/\kappa \in \mathbb{N}$, and where $\tilde{\chi}: \tilde{A} \times \tilde{C} \to \mathbb{C}$ is of the form

$$\tilde{\chi}(\tilde{A}, \tilde{C}) = \prod_{i=1}^{m} \operatorname{Tr}_{\rho_i}(\operatorname{Hol}_{\overline{i}}(\tilde{A} + \kappa \tilde{C}, \tilde{A} - \kappa \tilde{C}))$$

(C.8)

(with $(l_1, l_2, \ldots, l_m)$ and $(\rho_1, \rho_2, \ldots, \rho_m)$ as in Sec. 2.1).

Let us now apply “torus gauge fixing” to the expression

$$\int \tilde{\chi}(\tilde{A}, \tilde{C}) \exp(iS_{BF}(\tilde{A}, \tilde{C})) D\tilde{A}D\tilde{C}$$

(C.9)

More precisely, we will perform the following three steps:

- we make a change of variable from “BF-variables” to “CS-variables"\(^\text{70}\) (Step 1)
- we apply torus gauge fixing (Step 2)
- we change back to “BF-variables” (Step 3)

Concretely, these three steps are given as follows:

**Step 1:** We replace the expression (C.9) by the RHS of Eq. (C.7)

**Step 2:** We perform torus gauge fixing on the RHS of Eq. (C.7), i.e. we replace the RHS of Eq. (C.7) by the RHS of Eq. (2.7) in Sec. 2.3 above (in the situation $G = \tilde{G} \times \tilde{G}$, $T := \tilde{T} \times \tilde{T}$, and $(k_1, k_2) = (1/\kappa, -1/\kappa)$ where $\tilde{T}$ is a fixed maximal torus of $\tilde{G}$)

**Step 3:** We apply the change of variable $(A^\perp, B) \to (\tilde{A}^\perp, \tilde{B})$ given by

$$\tilde{A}^\perp := \left( \frac{A^1 + A^\perp}{2}, \frac{A^1 - A^\perp}{2\kappa} \right),$$

(C.10a)

$$\tilde{B} := \left( \frac{B^1 + B^\perp}{2}, \frac{B^1 - B^\perp}{2\kappa} \right)$$

(C.10b)

to the RHS of Eq. (2.7) (in the situation $G = \tilde{G} \times \tilde{G}$, $T := \tilde{T} \times \tilde{T}$, and $(k_1, k_2) = (1/\kappa, -1/\kappa)$)

The expression which we obtain after performing the three steps above is the analogue of the RHS of Eq. (7.3) above where instead of the change of variable (7.2) the change of variable (C.10) is used.

\(^\text{70}\)using $A = A_1 \oplus A_2$ and $DA = DA_1 DA_2$

\(^\text{71}\)i.e. from $(\tilde{A}, \tilde{C})$ to $(A_1, A_2)$

\(^\text{72}\)cf. Remark 2.8 in [24]
Appendix: Some alternatives to the discretization approach in Sec. [7]

D.1 A reformulation of Sec. 7.4 using the spaces $A_{2N}^\perp(K)$, $A_{\text{altern},1}^\perp(K)$, and $A_{\text{altern},2}^\perp(K)$

We will now reformulate/modify the discretization approach in Sec. 7.4 in a suitable way. This will not only lead to certain stylistic improvements but also to several insights which should be useful in [25]. Let us consider the space

$$A_{2N}^\perp(K) := \text{Map}(Z_{2N}, A_{\Sigma,g}(K)) = \text{Map}(Z_{2N}, C^1(K_1, g) \oplus C^2(K_1, g))$$

Clearly, we have

$$A_{2N}^\perp(K) \cong A_{\text{altern},1}^\perp(K) \oplus A_{\text{altern},2}^\perp(K)$$

where

$$A_{\text{altern},1}^\perp(K) := \text{Map}(Z_{2N}^{\text{even}}, C^1(K_1, g)) \oplus \text{Map}(Z_{2N}^{\text{odd}}, C^1(K_2, g))$$

$$A_{\text{altern},2}^\perp(K) := \text{Map}(Z_{2N}^{\text{even}}, C^1(K_2, g)) \oplus \text{Map}(Z_{2N}^{\text{odd}}, C^1(K_1, g))$$

with

$$Z_{2N}^{\text{even}} := \{ \pi(t) \mid t \in \{2, 4, 6, \ldots, 2N\} \} \subset Z_{2N}$$

$$Z_{2N}^{\text{odd}} := \{ \pi(t) \mid t \in \{1, 3, 5, \ldots, 2N - 1\} \} \subset Z_{2N}$$

$\pi : Z \rightarrow Z_{2N}$ being the canonical projection.

Let us now make the connection with the constructions of Sec. 7.4. It is convenient to use the notation $A_{\text{double}}^\perp(K)$ for what was denoted by $A^\perp(K)$ in Sec. 7.4. Observe that

$$A_{\text{double}}^\perp(K) = \text{Map}(Z_N, A_{\Sigma,g,g}(K)) \cong A_1^\perp(K) \oplus A_2^\perp(K)$$

where we have set for $j = 1, 2$

$$A_j^\perp(K) := \text{Map}(Z_N, A_{\Sigma,g}(K)) \cong \text{Map}(Z_N, C^1(K_1, g)) \oplus \text{Map}(Z_N, C^1(K_2, g))$$

We now make the identifications

$$Z_{2N}^{\text{even}} \cong Z_N \cong Z_{2N}^{\text{odd}}$$

which are induced by the bijections $j_{\text{even}} : Z_N \rightarrow Z_{2N}^{\text{even}}$ and $j_{\text{odd}} : Z_N \rightarrow Z_{2N}^{\text{odd}}$ which are given by

$$j_{\text{even}}(\pi(t)) = \pi(2t), \quad j_{\text{odd}}(\pi(t)) = \pi(2t - 1) \quad \forall t \in \{1, 2, \ldots, N\}$$

Clearly, the identifications [D.7] give rise to identifications

$$A_{\text{altern},j}^\perp(K) \cong A_j^\perp(K), \quad j = 1, 2, \quad \text{and} \quad A_{\text{double}}^\perp(K) \cong A_{2N}^\perp(K)$$

Recall that in Sec. 7.4 we worked with “full ribbons” $R$, i.e. closed simplicial ribbons in $K \times Z = K_1 \times Z$ and recall also that such a $R$ induces three loops $l^+$, $l^-$, and $l$ in a natural way, $l^\pm$ being simplicial loops in $K_1 \times Z$ and $l$ being a simplicial loop in $K_2 \times Z$. In view of the identification $Z_N \cong Z_{2N}^{\text{even}}$, we will now consider $R$ as a closed simplicial ribbon in $K_1 \times Z_{2N}^{\text{even}}$ and the loops $l^\pm$ as simplicial loops in $K_1 \times Z_{2N}^{\text{even}}$ and $l$ as a simplicial loop in $K_2 \times Z_{2N}^{\text{even}}$ (here we have equipped $Z_{2N}^{\text{even}}$ with the polyhedral cell complex structure inherited from $Z_N$).

Using the identifications above we can now rewrite the discretization approach in Sec. 7.4. Doing so will not only lead to several stylistic improvements and “insights” (cf. Observation 1
and, possibly, Observation 3) but it also suggests certain modifications which we will incorporate in [25] (cf. Observation 2):

**Observation 1:** Recall that on the RHS of Eq. (7.17) the field component \( A_1^\perp \) “interacts” only with the loops \( l^+ \) and \( l^- \) while \( A_2^\perp \) “interacts” with the loop \( l \). In view of the reformulation we just made this ansatz is very natural: If \( A_1^\perp \) and \( A_2^\perp \) are given as the components w.r.t. the decomposition (D.2) then \( A_1^\perp \) and \( A_2^\perp \) will “live” on different edges. More precisely, for \( t \in \mathbb{Z}_{2N}^{\text{even}} \) we have \( A_1^\perp(t) \in C^1(K_1, g) \) and \( A_2^\perp(t) \in C^1(K_2, g) \). Since the \( l^+ \) and \( l^- \) are simplicial loops in \( K_1 \times \mathbb{Z}_{2N}^{\text{even}} \) and \( l \) is a simplicial loop in \( K_2 \times \mathbb{Z}_{2N}^{\text{even}} \) \( A_1^\perp \) will indeed only interact with \( l^+ \) and \( l^- \) and \( A_2^\perp \) will interact only with \( l \).

**Observation 2:** Recall that the operators \( L^{(N)}(B_{\pm}) \) appearing on the RHS of Eq. (7.16) are given by

\[
L^{(N)}(B_{\pm}) := \begin{pmatrix}
\bar{L}^{(N)}(B_{\pm}) & 0 \\
0 & \bar{L}^{(N)}(B_{\pm})
\end{pmatrix}
\]  

with \( \bar{L}^{(N)}(B_{\pm}) \) and \( \bar{L}^{(N)}(B_{\pm}) \) as in Eq. (3.10a) and Eq. (3.10b) in Sec. 4.2 above (with \( B \) replaced by \( B_{\pm} \) and where the matrix notation refers to the decomposition appearing in Eq. (D.6) above.

It is natural to ask whether it is possible to rewrite or redefine (if not \( L^{(N)}(B_{\pm}) \) itself then at least) the product \( \star_K L^{(N)}(B_{\pm}) \) by a formula which involves the (anti-symmetrized) operators \( \bar{L}^{(2N)}(b) \); \( b \in t \), as in Eq. (3.8a) instead of the operators \( \bar{L}^{(N)}(b) \) and \( \bar{L}^{(N)}(b) \) appearing in Eq. (3.8a) and Eq. (3.8b). And indeed, using the aforementioned reformulation of the discretization approach of Sec. 7.4 this is possible, as we will now explain.

Let \( \star_K^{(2N)} : A_{2N}^\perp(K) \to A_{2N}^\perp(K) \) be the operator defined totally analogously as the operator \( \star_K : A^\perp(K) \to A^\perp(K) \) in Sec. 4.3 above but with \( 2N \) playing the role of \( N \). Moreover, let \( L^{(2N)}(B_{\pm}) \) be the operator on

\[
A_{2N}^\perp(K) \cong (\oplus_{\ell \in \mathfrak{g}_{(K_1,K_2)}} \text{Map}(S_{2N}^\perp,g)) \oplus (\oplus_{\ell \in \mathfrak{g}_{(K_1,K_2)}} \text{Map}(S_{2N}^\perp,g))
\]  

which is given by

\[
L^{(2N)}(B_{\pm}) := (\oplus_{\ell \in \mathfrak{g}_{(K_1,K_2)}} \bar{L}^{(2N)}(B_{\pm}(\ell))) \oplus (\oplus_{\ell \in \mathfrak{g}_{(K_1,K_2)}} \bar{L}^{(2N)}(B_{\pm}(\ell)))
\]  

where each \( \bar{L}^{(2N)}(b) \) with \( b = B_{\pm}(\ell) \) is defined totally analogously as in Eq. (3.8a) above (with \( N \) replaced by \( 2N \)). Observe that, even though neither of the two operators \( \star_K^{(2N)} \) nor \( L^{(2N)}(B_{\pm}) \) leaves the two subspaces \( A_{\text{altern}}^\perp(K) \); \( j = 1, 2 \), of \( A_{2N}^\perp(K) \) invariant the composition \( \star_K^{(2N)} L^{(2N)}(B_{\pm}) : A_{2N}^\perp(K) \to A_{2N}^\perp(K) \) does. It turns out that under the identifications (D.8) the operator \( \star_K^{(2N)} L^{(2N)}(B_{\pm}) \) is similar but does not quite coincide with the operator \( \star_K L^{(N)}(B_{\pm}) \) where \( L^{(N)}(B_{\pm}) \) is as in Eq. (D.9) above. In [25] we will work with the “new” operators \( \star_K^{(2N)} L^{(2N)}(B_{\pm}) \).

**Observation 3:** Using the reformulation of the discretization approach in Sec. 7.4 sketched above it might be possible to obtain a better understanding of the origin or the “meaning” of the 1/2-exponent appearing in Eq. (7.18) in Sec. 7.4 above.

We begin by having a closer look at the spaces which are relevant in continuum BF3-theory [79] (cf. part C of the Appendix) and some candidates for a simplicial realization:

- **Continuum spaces:** The full space is \( A_{\text{double}} = \Omega^1(\Sigma \times S^1, g \oplus g) \). We have \( A_{\text{double}} = A_{\text{double}}^\perp \oplus A_{\text{double}}^{\perp} \) where \( A_{\text{double}}^\perp = \{ A \in A_{\text{double}} \mid A(\partial/\partial t) = 0 \} \cong C^\infty(S^1, \Omega^1(\Sigma, g \oplus g)) \) and \( A_{\text{double}}^{\perp} = \{ A_0 dt \mid A_0 \in C^\infty(\Sigma \times S^1, g \oplus g) \} \cong C^\infty(\Sigma, \Omega^1(\Sigma, g \oplus g)) \). By applying torus gauge fixing we can reduce \( A_{\text{double}}^{\perp} \) essentially [80] to \( B_{\text{double}} = C^\infty(\Sigma, t \oplus t) \) (here we

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78 or rather, the equation which is analogous to Eq. (3.8a) but with \( N \) replaced by \( 2N \)

79 in contrast to part C of the Appendix we write \( G \) and \( T \) etc instead of \( \tilde{G}, \tilde{T} \); moreover, we often use the subscript “double”

80 here we ignore the issue of the well-known topological obstructions, cf. Sec. 2.2.4 in [21]
have identified $B_{\text{double}}$ with the subspace \{BD|BD \in B_{\text{double}}\} of $A_{\text{double}}^\|$. The application of torus gauge fixing “produces” the heuristic determinant $\prod_{\pm} [\text{det}_{FP}(B_{\pm})] = \prod_{\pm} [\text{det}(1_t - \exp(ad(B_{\pm})))_{\tau}] = \prod_{\pm} [\text{det}(1_t - \text{Ad}(\exp(B_{\pm})))_{\tau}]$ where $1_t - \text{Ad}(\exp(B_{\pm}))_{\tau} : C^\infty(\Sigma, t) \rightarrow C^\infty(\Sigma, t)$ (cf. Sec. 2.2.3 in [21]).

- **Simplicial spaces, 1. choice:** A natural choice for the simplicial analogues of the continuum spaces mentioned above are the spaces $A_{\text{double}(qK)} := C^1(qK \times \mathbb{Z}_N, g \oplus g)$, $A_{\text{double}}^\| (qK) := \text{Map}(\mathbb{Z}_N, C^1(qK, g \oplus g))$, and $B_{\text{double}}(qK) := C^0(qK, C^1(\mathbb{Z}_N, g \oplus g))$, and $B_{\text{double}}(qK) := C^0(qK, t \oplus t)$. And in fact, in Sec. 7.3 above we used this definition of $B_{\text{double}}(qK)$ and of $A_{\text{double}}^\| (qK)$ (and for technical reasons we also introduced the subspace $A_{\text{double}}^\| (K):= \text{Map}(\mathbb{Z}_N, C^1(K, g \oplus g))$ of $A_{\text{double}}^\| (qK)$).

One can hope[82] that by applying “discrete torus gauge fixing” the space $A_{\text{double}}^\| (qK)$ can be reduced to the space $B_{\text{double}}(qK)$ (considered as a subspace of $A_{\text{double}}^\| (qK)$ in the obvious way) and that this produces the determinant $\prod_{\pm} [\prod_{x \in \mathbb{Z}_N} \text{det}(1_t - \exp(ad(B_{\pm}(x)))_{\tau})]$. However, even if this argument could be made rigorous it would be totally unclear where the 1/2-exponent appearing in Eq. (7.18) in Sec. 7.4 above should come from. Let us therefore consider an alternative choice for the simplicial spaces.

- **Simplicial spaces, 2. choice:** Above (cf. Observation 1 and Observation 2) we observed[82] that it has several advantages to use the space $A_{\text{double}}^\| (qK) := \text{Map}(\mathbb{Z}_N, C^1(qK, g \oplus g))$ instead of the space $A_{\text{double}}^\| (qK) = \text{Map}(\mathbb{Z}_N, C^1(qK, g \oplus g))$ as the simplicial analogue of the continuum space $A_{\text{double}}^\|$ and to work with a suitable “intertwined/interlinked” direct sum decomposition of $A_{\text{double}}^\| (qK)$. Similarly, we can introduce $A_{\text{double}}^\| (qK) := C^1(qK \times \mathbb{Z}_N, g)$ and $A_{\text{double}}^\| (qK) := C^0(qK, C^1(\mathbb{Z}_N, g))$ as simplicial analogues of the continuum spaces $A_{\text{double}}^\|$ and $A_{\text{double}}^\|$. It is now natural to ask

- if one can find a “good” “intertwined/interlinked” direct sum decomposition of $A_{\text{double}}^\| (qK)$ (possibly, in the spirit of Appendix D.2 below, i.e. involving the root space decomposition (D.12) in some way),
- if one can embed $B_{\text{double}}(qK)$ as a suitable subspace $B_{\text{double}}^{\text{embedded}}(qK)$ of $A_{\text{double}}^\| (qK)$ which respects the “intertwined/interlinked” direct sum decomposition of $A_{\text{double}}^\| (qK)$ and, finally,
- if one can use a version of “discrete torus gauge fixing”[83] in order to reduce the space $A_{\text{double}}^\| (qK)$ to $B_{\text{double}}^{\text{embedded}}(qK)$.

If all this is possible then there might indeed be a chance of obtaining a satisfactory justification for the 1/2-exponent appearing in Eq. (7.18) in Sec. 7.4 above.

### D.2 Another version of Eq. (7.3) in Sec. 7.2

We observed in Remark 7.2 in Sec. 7 above that because of the $*$-operators on the main diagonal of the $2 \times 2$-matrix appearing in Eq. (7.3) above we cannot hope to be able to find a discretized version of the path integral on the RHS of Eq. (7.3) where each of the two components $\hat{A}_1$ and $\hat{A}_2$ “lives” either on $K_1 \times \mathbb{Z}_N$ or on $K_2 \times \mathbb{Z}_N$.

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[81] as mentioned in Appendix D in [21] discrete torus gauge fixing can easily be implemented rigorously when working in a setting where spaces of Lie group valued maps are used; it is not yet clear if/how this is also possible in the present setting which works with spaces consisting of Lie algebra valued maps

[82] in fact, for technical reasons we concentrated on the subspace $A_{\text{double}}^\| (K) := \text{Map}(\mathbb{Z}_N, C^1(K, g))$ of $A_{\text{double}}^\| (qK)$ but this is not essential for the present discussion

[83] cf. footnote 81 above
We will now show that by using a more sophisticated change of variable $A^\perp \to \tilde{A}^\perp$ instead of \eqref{7.2a} the $\star$-operator on the main diagonal can be eliminated after all, cf. Eq. \eqref{D.16} below. This new change of variable, which we will introduce below, is based on the “root space decomposition”

$$\tilde{g} = \mathfrak{i} \oplus \left( \bigoplus_{\alpha \in \tilde{R}_+} \tilde{g}_\alpha \right)$$  \hfill (D.12)

of $\tilde{g}$ where $\tilde{R}_+$ is the set of positive real roots of $\tilde{g}$ associated to $\mathfrak{i}$ (and to a fixed Weyl chamber of $\mathfrak{i}$) and $\tilde{g}_\alpha \cong \mathbb{R}^2$ is the “root space” corresponding to $\alpha \in \tilde{R}_+$. For simplicity let us consider only the special case $\tilde{G} = SU(2)$ and $\tilde{T} = \{ \exp(\theta \tau_0) \mid \theta \in \mathbb{R} \}$ with $\tau_0$ given below. In this case we can rewrite Eq. \eqref{D.12} as

$$\tilde{g} = su(2) = \mathbb{R} \cdot \tau_0 \oplus (\mathbb{R} \cdot \tau_1 \oplus \mathbb{R} \cdot \tau_2)$$  \hfill (D.13)

where

$$\tau_0 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Using the concrete basis $(\tau_0, \tau_1, \tau_2)$ we can identify $\tilde{g} = su(2)$ with $\mathbb{R}^3$ in the obvious way, which in turn leads to the identification

$$C^\infty(S^1, A_{\Sigma, \tilde{g}}) \cong C^\infty(S^1, A_{\Sigma, \mathbb{R}})^3$$  \hfill (D.14)

The new change of variable $A^\perp \to \tilde{A}^\perp$ mentioned above is defined by

$$\tilde{A}^\perp := (\frac{Q}{2} (A_1^\perp + A_2^\perp), \frac{Q}{2}^{-1} (A_1^\perp - A_2^\perp)),$$  \hfill (D.15)

where $Q$ is the operator on $C^\infty(S^1, A_{\Sigma, \tilde{g}}) \cong C^\infty(S^1, A_{\Sigma, \mathbb{R}})^3$ given by

$$Q := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \star \\ 0 & 1 & -\star \end{pmatrix}$$

Here $1$ denotes the identity operator on $C^\infty(S^1, A_{\Sigma, \mathbb{R}})$ and $\star$ the Hodge star operator on $C^\infty(S^1, A_{\Sigma, \mathbb{R}})$ which is induced by the auxiliary Riemannian metric $g$ fixed in Sec. 2.3 above.

Using the changes of variable \eqref{D.15} and \eqref{7.2b} we arrive at the following modification of Eq. \eqref{7.3} above

$$S(\tilde{A}^\perp, \tilde{B}) = \pi k \ll (\tilde{A}^\perp_1, \tilde{A}^\perp_2), \left( \begin{pmatrix} J \text{ad}(\tilde{B}_2) \\ \frac{\partial}{\partial r} + J \text{ad}(\tilde{B}_1) \end{pmatrix} \star \frac{\partial}{\partial r}, J \text{ad}(\tilde{B}_2) \right) \cdot (\tilde{A}^\perp_1, \tilde{A}^\perp_2) \gg \tilde{A}^\perp$$  \hfill (D.16)

where $J$ is the linear operator on $C^\infty(S^1, A_{\Sigma, \tilde{g}})$ which is induced in the obvious way by the linear operator $J_0$ on $\tilde{g} = su(2)$ given by $J_0 \cdot \tau_0 = 0$, $J_0 \cdot \tau_1 = \tau_2$, and $J_0 \cdot \tau_2 = \tau_1$.

Observe that in contrast to Eq. \eqref{7.3} above, there is no $\star$-operator appearing on the main diagonal of the matrix operator in Eq. \eqref{D.16}. It turns out, however, that when trying to discretize the analogue of Eq. \eqref{7.3} which is obtained after applying the change of variable \eqref{D.15} instead of \eqref{7.2a} it is still not possible to to find an implementation where each of the two components $\tilde{A}_1^\perp$ and $\tilde{A}_2^\perp$ “lives” either on $K_1 \times \mathbb{Z}_N$ or on $K_2 \times \mathbb{Z}_N$. In other words, it is still necessary to use a “mixed” implementation (cf. Remark \ref{7.2} above). This time we have more freedom in choosing the “mixed” implementation. It remains to be seen whether this has any advantages compared to the original approach in Sec. \ref{7.4}.
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