DENSE DARK MATTER HAIRS SPREADING OUT FROM EARTH, JUPITER, AND OTHER COMPACT BODIES

G. PRÉZÉAU
Jet Propulsion Laboratory, California Institute of Technology, 4800 Oak Grove Dr., Pasadena, CA 91109, USA

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ABSTRACT

It is shown that compact bodies project out strands of concentrated dark matter filaments, henceforth simply called hairs. These hairs are a consequence of the fine-grained stream structure of dark matter halos, and as such constitute a new physical prediction of ΛCDM. Using both an analytical model of planetary density and numerical simulations utilizing the Fast Accurate Integrand Renormalization algorithm (a fast geodesics calculator described below) with realistic planetary density inputs, dark matter streams moving through a compact body are shown to produce hugely magnified dark matter densities along the stream velocity axis passing through the center of the body. Typical hair density enhancements are 10^7 for Earth and 10^6 for Jupiter. The largest enhancements occur for particles streaming through the core of the body that are mostly focused at a single point called the root of the hair. For the Earth, the root is located at about 10^6 km from the planetary center with a density enhancement of around 10^9 while for a gas giant like Jupiter, the root is located at around 10^3 km with an enhancement of around 10^11. Beyond the root, the hair density precisely reflects the density layers of the body, providing a direct probe of planetary interiors.

Key words: dark matter

1. INTRODUCTION

As the evidence for the existence of dark matter has become overwhelming, understanding its nature and interactions through both direct (from Earth-bound experiments) and indirect (excess γ-ray and positron measurements) detection of dark matter has become an increasingly vigorous field of inquiry. In addition, upper limits on dark matter interactions are now extracted from the analysis of large samples of colliding clusters (Harvey et al. 2015), and constraints on both dark matter stream velocity dispersions and halo formation can be deduced from large structures like the “Field of Streams” (Belokurov et al. 2006). In spite of all these extraordinary advances, little is known about the dark matter particle itself (whether it is a boson, or Dirac or Majorana fermion), its local density and velocity distribution in the solar system, or its interactions. This missing information may remain hidden until a direct detection is made and the interactions of such particles can be studied consistently over time; a longitudinal study could be performed thanks to dark matter production at an accelerator or because a naturally occurring concentrated dark matter source is discovered locally that can be reliably accessed with instruments. Highly concentrated dark matter hairs would be such a source and are in fact a prediction of ΛCDM, a model that fits very well with the most recent Planck cosmic microwave background (CMB) analysis and many other astrophysical data.

For thermally produced cold dark matter (CDM) of the type allowed by ΛCDM, the CDM primordial thermal velocity dispersion is expected to be greatly suppressed as the universe expands and the CDM collisionless gas cools. In particular, for a weakly interacting massive particle (WIMP) with mass 100 GeV that decoupled at 10 MeV, the velocity dispersion is about 3 × 10^{-4} m s^{-1}, while the velocity dispersion for a 10 μeV axion is 10^{-5} m s^{-1} (Sikivie 1999). As the nonlinear effects of gravity become more prominent and the halos grow, a coarse-grained velocity dispersion of the CDM will appear as it orbits the Galaxy (Sikivie et al. 1995); the coarseness of this dispersion will gradually smooth out as the number of orbits increases. In the “Field of Streams” (Belokurov et al. 2006), the effective velocity dispersion is 10 km s^{-1}, providing an experimental upper limit on the velocity dispersion of the fine-grained dark matter streams, each of which should have a tiny primordial velocity dispersion. The “Field of Streams” is an example of a tidal dark matter stream built up from a huge number of fine-grained dark matter streams and distinct from them. A phase-space perspective sheds additional light on the processes affecting the CDM under the influence of gravity.

At the last scattering surface, the CDM occupies a three-dimensional sheet in the six-dimensional phase space since it has a tiny velocity dispersion. The process of galactic halo formation cannot tear this hypersurface filled with the collisionless gas of CDM because of the generalized Liouville theorem that accounts for the time evolution of the metric. Under the influence of gravity, a particular phase-space volume of the hypersurface is stretched and folded with each orbit of the CDM, creating layers of fine-grained dark matter streams, each with a vanishingly small velocity dispersion. These stretches and folds also produce caustics (Tremaine 1999), regions with very high CDM densities that are inversely proportional to the square root of the velocity dispersion, potentially providing large boosts to CDM annihilation into photons. Identifying such boosts is important because it would help to explain the positron excess seen by PAMELA for energies above 10 GeV (Adriani et al. 2013), if such excess stems from WIMP annihilation. Indeed, supersymmetric radiative corrections to the neutralino annihilation positron...
branching ratio can explain the rise in positron excess for rising energies provided that features (like caustics) in the halo structure boost the dark matter density (Bergstrom et al. 2008).

In order to understand how caustics could provide such a boost, Vogelsberger & White (2011, henceforth V&W) integrated the geodesic deviation equation (Vogelsberger et al. 2008) in tandem with the equations of motion of $N$ simulation particles to analyze the fine-grained structure of the halos starting from general $\Lambda$CDM initial conditions. They were able to count all the caustics encountered by a simulation particle along its geodesic and calculate the CDM annihilation rate enhancements; they found that caustics provide at best a boost of 0.1%. The V&W simulations also produced a probability distribution for a fine-grained stream containing a particular fraction of the average local density to pass through a detector on Earth. For example, they found that there was a 20% chance of a local CDM detection where a single stream accounted for 1% of the total signal. These results led V&W to conclude that the CDM velocity distribution was locally smooth and that the fine-grained halo structure would be too difficult to detect on Earth unless the CDM consists of axions.

On the other hand, space-based detectors could discern the local fine-grained structure because the Earth acts as a gravitational lens that separates dark matter streams$^3$ with different, sharpened velocities. In this paper, a Schwarzschild metric is used to show that weakly interacting particles streaming at $220 \text{ km s}^{-1}$ (the approximate orbital velocity of the solar system around the Galactic center (Karachentsev & Makarov 1996)) through a compact body with a small impact parameter (i.e., near the core) will experience huge CDM density enhancements at the root (the nearest focal point composed of particles streaming through the core of the body) of the order of $10^8$ for the Earth and $10^{11}$ for Jupiter. For the Sun, the entire halo from root to tip (the focal point of particles streaming at the surface of a body)$^4$ is inside the solar radius for median CDM velocities of $220 \text{ km s}^{-1}$ so that any direct detection near the Sun would be suppressed by the low probability of finding a high-velocity dark matter stream, not to mention the challenges of close proximity to the Sun. Dense dark matter halos provide an entirely unique opportunity to study dark matter interactions and local stream properties. These CDM density enhancements will boost their direct detection rate for a space-bound experiment without invoking any physics beyond the standard model of particle physics, $\Lambda$CDM, and general relativity, and may be the only way to obtain empirical data on local dark matter streams which would further our understanding of halo structure and formation.

This paper is organized as follows: in Section 2, known results for exterior geodesic solutions near Earth of both massless and massive particles are briefly reviewed for completeness, and a description of the flux magnification for exterior solutions is given. Section 3 derives the analytic, constant-density interior solutions of the geodesic equation (GE), and the density enhancement for massive particles is shown to be huge; the massive particle solution is expanded in powers of the impact parameter $b$, and the focal point is found to be independent of $b$ to leading order with corrections of order $(b/R_E)^2$ where $R_E$ is the radius of the Earth (but could be the radius of any compact body). It is also shown that the non-zero but tiny velocity dispersion of the CDM has a negligible impact on the density enhancements of the dark matter hairs for standard mass assignments of the CDM. In Section 4, the Fast Accurate Integrand Renormalization (FAIR) algorithm for solving the GE for realistic radial densities is described and compared with the analytic constant-density result. Because FAIR is an algorithm that modifies the short-scale radial dependence of the density profile, it can be used to demonstrate the robustness of the density enhancement results. In Section 5, the numerical solutions of the GE for a model Earth with a radial density given by the Preliminary Reference Earth Model (PREM) (Dziewonski & Anderson 1981) are calculated using FAIR, and the hair focal points and density enhancements are plotted as a function of the impact parameter; the same plots are generated for Jupiter with a radial density given by the model J11-4a from Nettelmann et al. (2012). In the last section, the implications of our results are discussed, including the benefits of exploring a hair (which include a powerful new tool to explore the interiors of compact bodies) and the conditions constraining a space mission dedicated to finding such a hair; an equation relating the different parameters of such a mission is also derived.

2. EXTERIOR SOLUTIONS AND DENSITY ENHANCEMENT FOR EARTH

As a weakly interacting particle streams above the surface of the Earth, its trajectory curves toward a point on the axis that passes through the center of mass of the Earth and parallel to the incident velocity at $t \rightarrow -\infty$. The location of this point and the corresponding magnification of flux intensity can be extracted from the known geodesic solutions for massless and massive particles in a Schwarzschild metric. The GE for a massive particle is given by

$$c^2 = \left(1 - \frac{\kappa}{r}\right)c^2\left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{\kappa}{r}\right)^{-1}\left(\frac{d\rho}{d\tau}\right)^2 - r^2 \sin^2\theta \left(\frac{d\phi}{d\tau}\right)^2 - r^2 \left(\frac{d\theta}{d\tau}\right)^2,$$

(1)

while the equation for massless particles is

$$\left(1 - \frac{\kappa}{r}\right)c^2\left(\frac{dt}{d\tau}\right)^2 = \left(1 - \frac{\kappa}{r}\right)^{-1}\left(\frac{d\rho}{d\tau}\right)^2 + r^2 \sin^2\theta \left(\frac{d\phi}{d\tau}\right)^2 + r^2 \left(\frac{d\theta}{d\tau}\right)^2,$$

(2)

where $\kappa = 2GM_E/c^2$ is the Schwarzschild radius, $M_E$ is the Earth’s mass, and $G$ is the gravitational constant. The $r$ derivative of $\theta$ is zero thanks to angular momentum conservation with $\sin \theta = 1$.

The exterior solution for a massless particle with impact parameter $b > R_E$ can be obtained by integrating

$$\frac{d\phi}{du} = -b(1 - b^2u^2 + r_\kappa b^2u^3)^{-\frac{1}{2}},$$

(3)
yields (\frac{R}{r} = \frac{r}{b}) density ratio, density.

All dark matter streams in this paper are de

To calculate the density enhancements caused by a planetary

\begin{equation}
\phi = \phi' + \arccos(bu) + \frac{r_s}{b},
\end{equation}

where \( u = 1/r; \) this corresponds to a deflection angle

\begin{equation}
\phi' = \frac{\pi}{2} - \frac{r_s}{b}.
\end{equation}

Since \( r_s = 9 \times 10^{-3} \text{ m} \) for the Earth and \( b > R_E = 6.371 \times 10^6 \)

for geodesics external to the Earth, the deflection of massless

\begin{equation}
F(b) = \frac{b^2}{2r_s} > 2 \times 10^{15} \text{ m}
\end{equation}

which are completely unreachable. The exterior solutions for

massless particles with small impact parameters \( (b \leq R_E) \)

e xterior to the Earth will be needed below for the initial angle

value of the interior solutions of massless particles. They can

can be obtained by expanding the denominator in Equation (3) and

integrating, yielding

\begin{equation}
\phi = \pi - \frac{r_s}{b} \left( 1 - \sqrt{1 - b^2u^2} \right) - \arcsin(bu).
\end{equation}

For massive particles neglecting the insignificant terms

proportional to \( r_s/r \) the exterior solution is the usual orbit equation

\begin{equation}
\frac{1}{r} = \frac{1}{\xi b^2} \left[ 1 + \sqrt{1 + \xi b^2 \cos(\phi - \phi') \right]}
\end{equation}

\begin{equation}
\phi'_E = \arccos \left( 1 + \xi b^2 \right)^{-\frac{1}{2}},
\end{equation}

\begin{equation}
\xi \equiv \frac{v^2}{GM_E}.
\end{equation}

for particle velocity \( v. \) If \( v = 220 \text{ km h}^{-1},

\begin{equation}
\xi b^2 > \xi b^2 = 6 \times 10^5,
\end{equation}

\begin{equation}
\phi'_E \equiv \arccos \left( \frac{GM_E}{bv^2} \right) \equiv \frac{\pi}{2} - \frac{1}{b \xi}.
\end{equation}

To calculate the density enhancements caused by a planetary

body of weakly interacting particles belonging to a dark matter stream, a circular-disk detector with radius \( R_D \) is taken to be

located on the axis that passes through the center of the planet

and parallel to \( v \) (see Figures 1 and 2). In the Schwarzschild

metric, every particle belonging to that dark matter stream will eventually cross that axis. The density enhancement \( M \) at a detector situated at point \( F(\tilde{b}) \) along the axis is given by the ratio of the surface of the annulus to the surface of the detector,

\begin{equation}
M[F(b)] = \frac{b^2_{\text{max}} - b^2_{\text{min}}}{R_D^2} = \frac{\delta b^2}{R_D^2}.
\end{equation}

Integrating over all possible contributions to a detector located at a point \( \tilde{b} \) yields

\begin{equation}
M(\tilde{b}) = \int_0^{\tilde{b}} \frac{db^2}{R_D^2} \left[ F(b) + \frac{R_D}{\delta \phi(b)} \right]
\end{equation}

\begin{equation}
\int_{\tilde{b}}^{b_{\text{max}}} \frac{db^2}{R_D^2} \left[ F(\tilde{b}) + \frac{R_D}{\delta \phi(\tilde{b})} \right]
\end{equation}

\begin{equation}
\int_0^{\tilde{b}} \frac{db^2}{R_D^2} \left[ F(b) - F(\tilde{b}) \right] \frac{db}{dF} \frac{dF}{b} + 1
\end{equation}

\begin{equation}
\int_{\tilde{b}}^{b_{\text{max}}} \frac{db^2}{R_D^2} \left[ F(\tilde{b}) - F(b) \right] \frac{db}{dF} \frac{dF}{b} + 1
\end{equation}

where the step function \( \theta(x) = 0, 1 \) for \( x < 0 \) and \( x \geq 0 \)

respectively. Particles moving with speed \( v \) with \( \tilde{b} > R_E \)

e xperience a flux enhancement:

\begin{equation}
\delta \phi = \frac{2}{b \xi},
\end{equation}

\begin{equation}
F(\tilde{b}) = \frac{\tilde{b}^2}{2} > 2.5 \times 10^9 \text{ m}.
\end{equation}

\begin{equation}
M(\tilde{b}) = \frac{(\tilde{b} + \delta b)^2 - (\tilde{b} - \delta b)^2}{R_D^2} = 2 \frac{\tilde{b}}{R_D}.
\end{equation}
Although this is a large enhancement for $b > R_E$, the focal point distance increases quadratically with the impact parameter, making it very difficult to benefit from the linearly increasingly magnified flux of streaming massive particles. In addition, the interior solutions of the GE of massive particles streaming through the core will be found to be orders of magnitude larger, with focal points located within a million kilometers of the Earth and less for Jupiter.

3. INTERIOR SOLUTIONS: CONSTANT-DENSITY EARTH

The interior geodesics of massless particles in a general Schwarzschild metric are solutions of

$$e^{-2\phi/c^2} - \frac{b^2}{r^2} = \left[ 1 - \frac{r_s M(r)}{r M_E} \right]^{-1} \left( \frac{dr}{d\phi} \right)^2 \frac{b^2}{c^2 r^4} \quad (19)$$

while the massive particles are solutions of

$$\left( 1 + \frac{v^2}{c^2} \right) e^{-2\phi/c^2} - 1 - \frac{v^2 b^2}{c^2 r^2} = \left[ 1 - \frac{r_s M(r)}{r M_E} \right]^{-1} \left( \frac{dr}{d\phi} \right)^2 \frac{v^2 b^2}{c^2 r^4}, \quad (20)$$

where $\Phi$ is the interior potential of a body. For compact bodies with constant density, $e^{-\Phi/c^2}$ has an analytical form (Wald 1984)

$$e^{\Phi/c^2} = \frac{3}{2} \sqrt{1 - \frac{r_s}{R_E}} - \frac{1}{2} \sqrt{1 - \frac{r_s r^2}{R_E^2}}, \quad (21)$$

$$\Phi \approx -2\pi G \bar{\rho} \left( R_E^2 - \frac{r^2}{3} \right), \quad (22)$$

$$M(r) = \frac{4\pi}{3} \bar{\rho} r^3 \quad (23)$$

with $\bar{\rho}$ the average density of the body.

3.1. Massless Particles

In this analytical case, the differential equation for massless particles is

$$\frac{du}{d\phi} = -\frac{1}{u} \left[ -u^4 + su^2 - \frac{3}{2} \frac{r_s}{b^2 R_E^3} \right]^2, \quad (24)$$

$$s = \left( \frac{1}{b^2} + \frac{3}{2} \frac{r_s}{b^2 R_E} + \frac{r_s}{R_E} \right) \quad (25)$$

with a solution that can be readily written down as

$$\phi = \phi_{1}' + \frac{1}{2} \arccos \left[ \frac{u^2 - \frac{1}{2} \frac{s}{u}}{\sqrt{1 - \frac{3}{2} \frac{r_s}{b^2 R_E^3}}} \right], \quad (26)$$

$$\phi_{1}' = \pi - \frac{r_s}{b} \left( 1 - \sqrt{1 - \frac{b^2}{R_E^2}} \right)$$

$$- \arcsin \left[ \frac{b}{R_E} \left( 1 - \frac{r_s}{2R_E} \right) \right], \quad (27)$$

$$- \frac{1}{2} \arccos \left[ \frac{2/(R_E^2 s) - 1}{\sqrt{1 - 6r_s b^2 / R_E^2}} \right], \quad (28)$$

Note that as $b \to R_E, \phi_{1}' \to \phi'$, and that as $b \to 0, \phi_{1}' \to \pi/2$, as should be the case. Although the flux of massless particles streaming through a constant-density Earth is greatly magnified as seen in Figure 3, their focal points are located far outside the solar system (Figure 4), making explicit the fact that planetary lensing and density enhancement of massless weakly interacting particles cannot be exploited locally for
detection purposes. In Figure 5, the log–log plot of the enhancement as a function of the focal point distance is also given, where the enhancement peak for massless particles streaming near the core is now clearly visible (it is pressed against the ordinate in Figure 3). In addition, the approximate straight line in the log–log plot between the enhancement and the focal distance shows that any exponential density enhancement is accompanied by exponentially increasing distances.

3.2. Massive Particles

Using Equations (20), (22), and (23), the radial equation of motion for a massive particle is given by

\[
\frac{du}{d\phi} = -\frac{1}{u} \left( -u^4 + \alpha u^2 - \beta \right),
\]

\[
\alpha = \frac{1}{b^2} + \frac{3}{2} \frac{r_e}{R_E} \left( 1 + \frac{c^2}{v^2} \right) + \frac{r_e}{R_E},
\]

\[
\beta = \frac{r_e}{2 R_E^2 b^2} \left( 3 + \frac{c^2}{v^2} \right).
\]

This is almost identical to Equation (24) except for the \(c^2/v^2\) that now dominates and enhances the \(r_e\) terms. The massive constant-density interior solution is therefore

\[
\phi = \phi_M' + \frac{1}{2} \text{arccos} \left[ \frac{u^2 - \frac{1}{2} \alpha}{\sqrt{1 - \frac{4}{\alpha^2} \beta}} \right], \quad u^2 \geq 1/R_E^2
\]

The corresponding density enhancement and focal point are plotted against impact parameter in Figures 6 and 7 respectively. The massive particles streaming near the center of the planet clearly experience the largest magnification. To understand why, consider the expansion of \(\phi_M'\) in powers of

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\(^6\) For massless or high-energy neutrinos streaming through the Sun, the focal points are near the orbit of Uranus (Gerver 1988; Escribano et al. 2001).
Density enhancements of massive particles streaming through a constant-density Earth as a function of focal point distance. The enhancement curve is effectively vertical at the root of the hair, suggesting a dense, point-like structure trailed by a sparser tail. Note that this "sparse tail" is still very dense as compared to the ambient density of CDM.

$$b/R_E:$$

$$\delta \phi = \frac{3}{\xi R_E} \left( 1 - \frac{b^2}{4 R_E^2} \right).$$  \hfil (35)

The critical thing to note is that the deflection angle $\delta \phi$ is linear in $b$ with corrections suppressed by factors of $(b/R_E)^2$. As such, near the Earth’s center,

$$F(b) = \frac{\xi R_E^2}{3} \left( 1 + \frac{b^2}{4 R_E^2} \right) \approx 1.6 \times 10^9 \text{ m},$$  \hfil (36)

which is consistent with the exact solution in Figure 7. In other words, the focal point is effectively independent of the impact parameter $b$ for particles streaming through the core within a radius of dozens of kilometers, resulting in a huge density enhancement. This fact is visually clear in Figure 8 where the enhancement is plotted as a function of the focal point where the root of the hair would appear point-like to a space probe. To calculate the density enhancement near the core, it is only necessary to solve for $b$:

$$\delta \phi = F(b) - F(\tilde{b}) = R_D.$$  \hfil (37)

For example, taking $\tilde{b} = 0$ yields

$$M(0) = \left[ \frac{(4R_D R_E^2)^3}{R_D} \right]^2 \approx 3 \times 10^9 \text{ (} R_D = 1 \text{ m).}$$  \hfil (38)

which is consistent with Figure 6 and independent of the dark matter velocity. Velocity independence is important because in a search for hairs the density enhancement will depend only on the density ratio of the dark matter stream, $\rho_f/\langle \rho \rangle$. In addition, the location of the peak density enhancement (which is about twice the value in Equation (38)) is simply

$$b_{\text{peak}} = \left( 4R_D R_E^2 \right)^3 \approx 55 \text{ km for } R_D = 1 \text{ m.}$$  \hfil (39)

Note also that the singularity at $R_D = 0$ does not exist in reality because the non-zero velocity dispersion of the dark matter stream is a hard limit on the focus intensity. For a WIMP dispersion of $3 \times 10^{-4} \text{ m s}^{-1}$ assumed to be perpendicular to the direction of the WIMP, this gives a total perpendicular distance travelled of

$$\delta R_D \sim d_p = \frac{\delta v}{v} F(b) \approx 6 \text{ m at } b \sim 10 \text{ km}$$  \hfil (40)

for a constant-density Earth in the worst-case scenario of an entirely perpendicular dispersion; for a $10 \mu eV$ axion, $d_p$ is a fraction of a millimeter. In other words, reducing the detector radius beyond $\delta R_D$ cannot result in a larger density enhancement.

$$M(0) = \left[ \frac{(4R_D R_E^2)^3}{R_D + \delta R_D} \right]^2.$$  \hfil (41)

The velocity dispersion at present age for WIMPs and axions is of order (Sikivie 1999)

$$\delta v_a \sim 10^{-8} \left( \frac{10^{-5} \text{ eV}}{m_a} \right) \text{ m s}^{-1}$$  \hfil (42)

$$\delta v_W \sim 3 \times 10^{-3} \left( \frac{\text{ GeV}}{m_w} \right) \text{ m s}^{-1}.$$  \hfil (43)

It is therefore seen that for $m_a > 10^{-9}$ eV and $m_w > 10^2$ GeV, the density enhancement observed by a detector with a radius of a few meters will be little affected by the velocity dispersion of the CDM.

The fact that the deflection angle is linear in $b/R_E$ with corrections of order $(b/R_E)^3$ is not unique to constant-density profiles, as will be verified below for the Earth and Jupiter using realistic density profiles from the literature. We now turn to an algorithm that will permit us to calculate density enhancements for arbitrary density profiles.

4. FAIR ALGORITHM

In order to calculate the geodesics for realistic planetary radial density models, it is necessary to quickly and accurately perform integrals of the form

$$\int_{\phi_i}^{\phi_f} d\phi = \int_{R_{\text{ready}}}^{R_{\text{end}}} \frac{\sqrt{b^2/c^2}}{\left[ 1 - \frac{r}{r_e} M(r) \right]^{1/2} \left( 1 + \frac{v^2}{c^2} (e^{-2\phi/c^2} - 1 - \frac{v^2 b^2}{c^2 r^2}) \right)} dr.$$  \hfil (44)
singular at \( r = r_\text{f} \), \( \phi_i \) is difficult to evaluate numerically and the deflection angle may be hard to pin down to sufficient precision.

Early in this work, Runge–Kutta methods were used to solve the GEIs, but they were found to converge too slowly. Other numerical methods found in the literature (Press et al. 2007) were tried with unsatisfactory results because \( \phi_i \) could not be calculated accurately enough near the caustic of the focal point. Indeed, the density enhancement depends on the derivative of the impact parameter with respect to the focal point (see Equation (15)), which is huge for small impact parameters. To get around this problem, a method was developed to perform the integral in a quasi-analytic fashion that relied on the physical intuition that the precise details of the density should have little effect on the overall shape of the geodesics. In other words, the precise form of the density on a scale of a few meters should not significantly impact the interior solution of a large body with a radius of thousands of kilometers. Additionally, none of the planetary models claims to be accurate at small scales, which leaves us with the flexibility to choose a convenient density function that can be solved analytically on the small radial scale.7 This algorithm also allows a test of the robustness of the qualitative features of the density enhancement, like the sharp peak at low impact parameters.

Assuming the prior existence of a radial density model for a particular body, the basic FAIR algorithm works as follows:

1. Split the integrand into \( N \) segments of length much smaller than the radius of the body.
2. Create a density profile within each segment that is analytically solvable. Equate the average density of the density profile in that segment to the known, average, model value within that segment.
3. Starting with the known \( \phi_{i+1} = \phi_i \) from the exterior geodesic equation, use the analytic solution of the integral inside that segment to calculate the next value of the angle \( \phi_i \).
4. Do this until you reach the segment that contains the value of \( r = r_i \) at some layer \( j \). At that point, calculate \( \phi_i \) with

\[
\phi_i = \phi_{i+1} - \left( \text{analytic solution evaluated at } r = r_{j+1} \right).
\] (45)

The FAIR algorithm was verified numerically by comparing the analytic solution of the constant-density Earth with a highly unphysical case where each layer had a short-scale radial dependence of \( 1/r^2 \):

\[
\rho = \sum_{i=0}^{N_{\text{layers}}-1} \frac{\rho_i}{r^3}
\] (46)
subject to the average-density-equality condition

\[
\bar{\rho}_i = \frac{\hat{\rho}_i}{3} \left( \frac{r_{i+1}^3 - r_i^3}{r_i - 1/r_{i+1}} \right),
\] (47)
where \( \hat{\rho}_i \) is the density value extracted from a standard model at the \( i \)th layer. In the constant-density case, \( \bar{\rho}_i = \bar{\rho} \) for all \( i \)’s.

After a bit of algebra, the solution for each layer with the density profile of Equation (46) is found to be

\[
\phi_i = \phi_{i+1} - \arccos \left[ \frac{2\gamma_i^2 u_{i+1} - 1}{\beta_i} \right] - \arcsin \left[ \frac{4\gamma_i^2 \alpha_i}{\beta_i^2} \right] + 1.
\] (48)

\[
\alpha_i = \alpha_{i+1} - \frac{4\pi G}{b^2 v^2} \left( \rho_i - \rho_{i+1} \right),
\] (49)

\[
\beta_i = \frac{8\pi G}{4\pi} \left( \frac{M_{\text{body}}}{b^2 v^2} + \rho_i u_{i+1} \right),
\] (50)

\[
\gamma_i^2 = \frac{4\pi G}{b^2 v^2} \rho_i + 1,
\] (51)

where \( u_i \equiv 1/r_i \) and \( \phi_{i+1} \) was calculated in the previous step. Comparing the results for the density enhancement of the analytic solution to the unphysical \( r^{-4} \) in Figure 9, the agreement improved with the number of layers and was quite good for \( 10^7 \) layers, demonstrating a resolving robustness of the qualitative rapid magnification rise for decreasing impact parameter. The residuals for both the density enhancement and focal point locations are provided in Figure 10, further showing how the FAIR algorithm tends to the right answer with increasing number of layers even for a highly unphysical case. The \( 10^7 \) layer case took about 10 minutes to run and demonstrates that even if the initial guess for the intra-layer density radial dependence is very inaccurate, it is still possible to obtain a very good result compared to the true solution rather quickly.
The simulated results appearing below for PREM and J11-4a were obtained with the geodesolver code based on the above algorithm but with constant intra-density layers more in line with physical expectations. The iteration equations of geodesolver are

\[
\left(\frac{du}{d\phi}\right) = \frac{1}{u^3} \left[ a_0 + a_1 u^2 + a_2 u^3 - u^4 \right]
\]

(52)

\[
a_0 = \frac{4\pi G}{3v^2 b^2} \tilde{\rho}_i
\]

(53)

\[
a_1 \equiv \frac{1}{b^2} + \frac{1}{v^2 b^2} \left( |V_{i+1}| + 4\pi G \tilde{\rho}_i r_{i+1}^2 \right)
\]

(54)

\[
a_2 \equiv \frac{2G}{v^2 b^2} \left( \frac{M_{i-1}}{r_i} - \frac{4\pi}{3} \tilde{\rho}_i r_i^2 \right)
\]

(55)

\[
M_{i-1} \equiv M_i - \frac{4\pi}{3} \tilde{\rho}_i \left( r_i^3 - r_{i-1}^3 \right)
\]

(56)

\[
V_{i+1} \equiv V_{i+2} + 4\pi G \tilde{\rho}_{i+1} \left( r_{i+2}^3 - r_{i+1}^3 \right)
\]

(57)

where the starting values of \( r, M, \) and \( V \) are the radius of the planet, the mass of the planet, and zero, respectively; the \( \tilde{\rho} \)'s are given by the physical planetary model. To solve Equation (52), it is best to remove the \( r^3 \) term with the following trick:

\[
u^3 \approx u_0^3 + 6u^3
\]

(58)

\[
u^3 \approx u_0^3 + 3u^2 (u - u_0)
\]

(59)

and solve for \( u^3 \). Equation (52) now has the same form as Equation (29) and can be solved in the same manner.

The agreement between geodesolver and the analytic solution for the constant-density Earth is excellent: using only \( 3 \times 10^5 \) layers, the density enhancements agree with the constant-density analytical case at the 0.1% level (Figure 11) while the focal points match to six significant digits (Figure 12), including near the critical sharp density enhancement peak at small impact parameter. At \( 10^8 \) layers, running geodesolver takes only a few seconds to calculate each point.

\[
\text{Figure 10. Density enhancement (solid) and focal point (dashed) residuals between the constant-density analytic and interior solutions solved using a } r^{-4}\text{ density renormalization of a constant-density Earth for increasing number of layers. Using } 10^7 \text{ steps, the } r^{-4}\text{ solution is seen to be accurate to about } b = 500 \text{ km, which is orders of magnitude better than the Runge–Kutta solution (not shown) for the same number of steps.}
\]

\[
\text{Figure 11. Density enhancement residual between the analytic and geodesolver interior solutions of a constant-density Earth for } 3 \times 10^5 \text{ layers.}
\]

\[
\text{Figure 12. Focal point residual between the analytic and geodesolver interior solutions of a constant-density Earth.}
\]

5. INTERIOR SOLUTIONS: PRELIMINARY REFERENCE EARTH MODEL AND J11-4A JUPITER

geodesolver was used on realistic planetary models for Earth (PREM; Dziewonski & Anderson 1981) and Jupiter (J11-4a; Nettelmann et al. 2012); these density profiles are plotted in Figures 13 and 18 respectively. The density enhancements and focal points for PREM are plotted in Figures 14–16 while Figures 19–21 are the corresponding plots for J11-4a. The density enhancement plots are particularly striking as they clearly resolve the interior structure of the planets. Additionally, they also exhibit huge density enhancements for small \( b \)'s with point-like root structures as in the constant-density Earth model. Other characteristics of the plots are tabulated in Table 1. The second column of Table 1 refers to the cutoff velocity of the hair, namely the velocity at which the root of the hair is below the surface of the planet. This velocity constrains searches for hairs surrounding massive bodies because entire hairs from root to tip may find themselves inside a body for likely values of the velocity (e.g., the Sun). That cutoff velocity can be calculated numerically using geodesolver but it is also easily obtained from the roots already calculated at
\[ v_c = \sqrt{\frac{R_{\text{body}}}{F_r}}. \]  

The fact that relations derived for a constant-density Earth are applicable to realistic planetary models is explained below. To better understand the behavior of the focal points as \( b \to 0 \), let us enlarge the segment for small impact parameter of the focal point plot in Figure 15 and plot the derivative \( F(b_{i+1}) - F(b_i))/(b_{i+1} - b_i) \) as a function of \( b \) in Figure 17. Thus, even for a realistic model such as the PREM, the derivative of the focal point varies linearly with the impact parameter as \( b \to 0 \). This implies that the focal points are constants save for tiny corrections of the order of \( b^2/R_{\text{body}}^2 \). The fact that this result generally holds for arbitrary radial density.
profiles can be understood by rewriting Equation (20) as

\[ d\phi = \frac{v b d r}{c r^{2} \left( 1 - \frac{r_{s} M(r)}{r M_{E}} \right)^{0.5} \left[ 2 \left( \frac{\Phi}{c^{2}} \right) + \frac{v^{2}}{c^{2}} \left( 1 - \frac{b^{2}}{r^{2}} \right) \right]^{0.5}} \]

(61)

and expanding in powers of \( b \). Note that the singularity of the integrand is not within the integration interval, but at the lower limit where the integrated result is zero since this is the turning point where the radial distance is minimized, and about which the particle trajectory is symmetric. Thus, similar huge density enhancements of coherent particles exiting the core can be expected for any compact body in the Schwarzschild metric, as was verified analytically for the constant-density Earth and numerically for the PREM and J11-4a. Quite generally, the focal points corresponding to impact parameters near the core of a Schwarzschild compact body can be parameterized by

\[ F(b) = F_{r} + \frac{F'_{r}}{2} b^{2}. \]

(62)

In Table 1, these parameters are given in columns 3 and 4 for PREM and J11-4a.

6. DISCUSSION

Key aspects of direct dark matter detection experiments that are not empirically constrained are the incident dark matter flux on Earth and the resulting lack of knowledge of the velocity distribution of the CDM. Traditionally, the gravitationally bound CDM has been modeled as an ideal gas with a
Maxwellian velocity distribution

\[
f_{\text{CDM}}(v) = \frac{4N}{\sqrt{\pi} v_{\text{esc}}^2} \left( \frac{v}{v_{\text{esc}}} \right)^2 \exp \left( - \frac{v^2}{v_{\text{esc}}^2} \right), \quad v < v_{\text{esc}},
\]

where \( N \) is a normalization constant to ensure that the integral over the distribution results in the local CDM density, \( v_{\text{esc}} \) is the Sun’s velocity with respect to a CDM halo taken to be stationary, and \( v_{\text{esc}} \) is the escape velocity of the CDM from the Galaxy. This isotropic, analytically smooth velocity distribution is unrealistic since the CDM had tiny primordial velocity dispersion at the last scattering surface. It only acquired angular momentum and velocity dispersion through their interactions with gravitational wells and tidal forces. The fraction of CDM that would have been thermalized and acquired an approximate Maxwellian velocity distribution is those particles that would have orbited the Milky Way many times. There will also be a fraction of CDM particles that have undergone fewer orbits such that the velocity distribution has peaks (Sikivie et al. 1995).

A number of realistic theoretical and numerical calculations have been performed to try to understand the velocity distribution in the halo (Springel et al. 2008; Hansen 2009; Vogelsberger et al. 2009). In particular, comprehensive and realistic fine-grained simulations (Springel et al. 2008; Vogelsberger et al. 2009; Vogelsberger & White 2011) of the geodesics followed by the CDM produce a distribution of finely parsed streams with discrete velocities that can be treated effectively as smooth on length scales similar to the solar system. Indeed, the original, unclustered, dark matter flows that stretched and folded into fine-grained dark matter streams would have been much larger than a solar system (Natarajan & Sikivie 2008). The original dark matter stream would have had a CDM density near the cosmological average, but the subsequent stretching and folding explains why the particle density in a fine-grained stream is typically small: the total number of collisionless dark matter particles remained constant in an expanding volume. The densest dark matter streams should therefore have the same velocity and density within the solar system, resulting in hairs with nearly identical orientations at different planets.\(^8\)

Locating these hairs is likely to be challenging but may be facilitated by the fact that the density enhancement is independent of the stream velocity, with minimum density enhancements of order \( 10^7 \) for Earth and \( 10^8 \) for Jupiter. In addition, the detailed spherically symmetric density structure of the planetary layers does not impact the qualitative features of a hair such as a sharp rise at low impact parameters.

Although planets are not precisely described by a Schwarzschild metric, slight deformations should not negate the basic arguments presented in this paper. The reason is that planetary deformations due to rotational motion do not break the mirror symmetry of the body across its center of gravity. This mirror symmetry constrains the metric to depend on the square of the angular momentum, since a particle with angular momentum \( L \) and another with angular momentum \(-L\) will meet at the same focal point. Hence, although planets are not precisely spherically symmetric, the hairs will reflect those deformations in a proportional fashion such that a hair’s cross-section will simply represent a two-dimensional image of the planet from that vantage point. The issue of planetary deformation will be explored numerically in a future paper with a generalization of the FAIR algorithm.

Turning now to the parameters necessary for an estimate of the time and resources required for finding a hair, we have:

1. \( p \) represents the percentage of the sphere required for the search area. This number needs to be calculated since the orientation of the hairs is biased by the Galactic orbital motion of the solar system. In addition, hairs with relatively large densities (say \( \rho_{\text{h}} / \rho > 10^{-4} \)) have likely gone through fewer orbits about the Galaxy, and may have a different distribution of orientations from the low-density streams.
2. \( n_{\text{probes}} \) is the number of probes simultaneously conducting a search. In estimating the time required to find a hair, \( n_{\text{probes}} \) will be solved for as an indication of the feasibility of the search.
3. \( T \) is the average transit time of a probe through a hair. This parameter is critical for the strength of the signal in

\[8\] Any observed differences in the hairs between different planets would stem from local corrections of the Schwarzschild metric due to nearby bodies.
the event that the probe does fly through a hair. If the transit time is too short for the sensitivity of the probe, the hair will not be detected.

4. \( h_w \), the average hair width crossed by a transiting probe.

5. \( d_{\text{probe}} \), the distance from the probe where the body is concentrating its search. In practice, one would expect \( d_{\text{probe}} = F \) to afford the best odds.

6. \( N_s \) the number of streams for a particular density ratio \( \rho_s/\langle \rho \rangle \).

The parameters above can be used to estimate the search time. To solve for \( n_{\text{probe}} \), that search time must be limited in some fashion. If one requires the search time to be less than a few years, then

\[
\text{search time} = \frac{p T_i}{n_{\text{probe}} N_s} \left( \frac{4 \pi d_{\text{probe}}^2}{h_w^2} \right) < 10^8 \text{ s} \quad (64)
\]

\[
\zeta \equiv \frac{p T_i}{n_{\text{probe}} N_s h_w^2} < 10^{-11} \text{ m}^2 \text{ s}^{-1} \text{ for Earth or} \quad (65)
\]

\[
< 10^{-9} \text{ m}^2 \text{ s}^{-1} \text{ for Jupiter.} \quad (66)
\]

The parameter \( \zeta \) depends on the details of the experiment, like the choice of target, the target’s size, and the expected event rate. In addition, although the transit time is constrained by the detector’s sensitivity, in practice it will also depend on the details of the probe’s orbit around the planetary body and how much fuel would be available to maneuver it around a desirable area of the focal sphere. These experimental parameters will in turn determine the density enhancement required to detect a hair during an average transit time, \( T_i \). Taking as a requirement that a probe crossing a hair of width \( h_w \) should see the same number of detection events as an Earth-bound mission would see in a year for an equivalent experimental setup, a constraint on \( T_i \) can be imposed:

\[
\frac{\rho_s}{\langle \rho \rangle} \bar{M} \bar{T}_i \sim 10^7 \text{ s with } \frac{\rho_s}{\langle \rho \rangle} \bar{M} > o \quad (67)
\]

where \( \bar{M} \) is the average enhancement in dark matter flux density experienced by the probe of radius \( R_0 \) as it transits across the width \( h_w \) of a hair and \( o \) parameterizes the transit time and is therefore related to the probe’s perpendicular velocity through the hair’s cross-section. Noting from Figure 10 of Vogelsberger & White (2011) that \( \langle \rho_s/\langle \rho \rangle \rangle N_s \sim F(>\rho_s) \), \( \zeta \) can be rewritten as

\[
\zeta \sim \frac{\rho_s}{n_{\text{probe}} \bar{h}_w^2 \bar{M} F(>\rho_s)} \sim 10^7 \text{ s} \quad (68)
\]

where \( F(>\rho_s) \) is the probability that a dark matter particle belongs to a stream with a density greater than \( \rho_s \). As described above, solving this equation involves a number of unknowns that are beyond the scope of the current paper, which is dedicated to describing the new ΛCDM prediction of concentrated dark matter hairs projecting from compact bodies. A comprehensive solution for \( n_{\text{probe}} \) would look at a realistic experimental setup and provide likelihood parameter figures drawn from a sample of simulated planetary hair realizations as well as incorporate orbital/fuel constraints. However, one can get a rough idea of the difficulty of finding a hair by approximating \( \bar{M} \sim M/\bar{h}_w \), and taking \( M \sim 10^8 \) for Jupiter

and \( M \sim 10^7 \) for Earth to obtain

\[
n_{\text{probe}} > \frac{\langle \rho \rangle}{\rho_s} \frac{op}{F(>\rho_s)} \quad \text{for Jupiter} \quad (69)
\]

\[
n_{\text{probe}} > 10^4 \frac{\langle \rho \rangle}{\rho_s} \frac{op}{F(>\rho_s)} \quad \text{for Earth.} \quad (70)
\]

The reason \( \rho_s \) appears in Equations (69) and (70) is because the number of streams with the smallest densities dwarfs the number of streams with the highest densities. For example, the simulations of Vogelsberger & White (2011) see \( 10^{14} \) streams near the Sun but only about \( 10^4 \) are massive. The way to interpret the above equations is as a lower limit on the number of probes required to see hairs generated from streams with density \( \rho_s \). With this understanding, it is seen from Figure 10 of Vogelsberger & White (2011) that the ratio \( \langle \rho \rangle/[\rho_s F(>\rho_s)] \simeq 10^5 \) for the densest streams \( \rho_s/\langle \rho \rangle \gtrsim 10^{-4} \) with the ratio increasing very fast with decreasing \( \rho_s \). In particular, the least dense streams are effectively invisible because their hair width \( h_w \) is tiny as seen from Equation (67). For the dense streams with finite \( h_w \), \( n_{\text{probe}} \) seems prohibitively high, especially since the parameter \( o \) can be large.

This rough estimate based on the simulations of Vogelsberger & White (2011) and assuming Earth-based detector sensitivity therefore appears to put hair detection out of reach, unless (1) \( p \) can be constrained to be quite small, (2) the uncertainty on the radial distance to the hair root can be narrowed enough that density enhancements of \( 10^5–10^9 \) for Earth and \( 10^6–10^{11} \) for Jupiter can be used above, suppressing \( n_{\text{probe}} \) by a factor of \( 10^{-4}$, (3) future theoretical/numerical developments allow the high-\( \rho_s \) slope of Figure 10 from Vogelsberger & White (2011) to be steeper,

9 and (4) a way is found to exploit the characteristics of the space environment (such as microgravity) to design novel direct dark matter detectors, for both WIMPs and axions, with better sensitivity.

One highly developed technology for space missions is refrigeration for experiments requiring very low temperatures typical of axion searches (Asztalos et al. 2010). The CMB Planck satellite mission relied on active cooling to lower the temperature of the High Frequency Instrument bolometer plate to 93 mK (Ade et al. 2011), which is similar to the temperature goal for the ADMX apparatus upgrade. Generally speaking, axion searches and CMB space telescopes broadly share a reliance on microwave science and engineering, which would make it more likely that a space-based axion detection mission could recycle well-tested CMB technology. Low-temperature detectors could also be efficaciously used for a space mission to detect WIMPs thanks to their excellent signal-to-noise ratio and their event-by-event background rejection capabilities that may reduce the shielding requirements for cosmic rays (Guo & McKinsey 2013; Mirabolfathi 2013). Since discovering a hair is a separate goal with different benefits from direct dark matter detection (which may happen on Earth first), it may be possible to transfer improvements in detector sensitivity to a space mission that would greatly increase the odds of finding a hair.

Turning next to whether the compact body hairs could contribute a measurable signal to indirect dark matter searches...
from dark matter annihilation, consider the usual ambient contribution estimated from the equation

$$\Phi(E_\gamma, \psi) = \frac{\langle \sigma v \rangle}{8 \pi m_\chi^2} \delta(E - E_\gamma) \int_{\text{LOS}} \rho^2(l, \psi) dl,$$  

(71)

where a monochromatic $\gamma$-ray decay was assumed, $m_\chi$ is the CDM mass and the astrophysical content of the flux $\Phi$ is contained in the line of sight integral on the end. Including the planetary focusing, the astrophysical integral becomes

$$\int_{\text{LOS}} \rho^2(l, \psi) dl \rightarrow \int_{\text{LOS}} \rho^2(l, \psi) dl + G_L(\phi, \psi)$$  

(72)

$$G_L(\phi, \psi) = \int_{\text{LOS}} d\Omega_{\text{CO}}(l, \phi, \psi) \times \sum_{\text{hair}} P_{\text{hair}}(l, \phi, \psi) \int_{\text{hair}} dt dA(t) \rho_{\text{hair}}^2(l, \psi),$$  

(73)

where $\Omega_{\text{CO}}(l, \phi, \psi)$ is the density of compact objects in the direction $(\phi, \psi)$ that focus the CDM beyond their surface, $\rho_{\text{hair}}(l, \psi)$ is the CDM density at point $t$ along the hair of the gas giant, a distance $l$ away from us with an angle $\psi$ from the Galactic center, and $P_{\text{hair}}$ is the probability of seeing a hair with density $\rho_{\text{hair}}$; the last integral is performed about the volume of the hair with $dA$ the area of the cross-section at $t$. It is immediately clear why that second contribution is negligible: $\Omega_{\text{CO}}$ has units of kpc$^{-2}$ and can be completely ignored in CDM annihilation searches. This conclusion also holds locally because the number of annihilation products produced by each unit volume within the solar system far exceeds the number produced along the hairs of planetary bodies even after accounting for the density enhancement. This result also negates the possibility of detecting dark matter annihilation in hairs surrounding a particular compact body (such as a white dwarf or a neutron star) with a telescope, as it would be swamped by the ambient dark matter annihilation, not to mention the directional suppression stemming from the tiny opening angle of a created photon propagating from the compact body to an observing satellite orbiting the Earth. This is also true for solar system bodies like Jupiter and Earth, which cannot produce detectable amounts of annihilation products.

Lastly consider the possibility of relic neutrino hairs. The relic neutrino energies follow a Fermi–Dirac distribution and cannot form the caustics necessary to produce hairs, since their velocity distribution is completely smooth with a large dispersion.

7. SUMMARY

A new $\Lambda$CDM prediction of concentrated dark matter hairs extending from compact bodies was described. The hairs are caustics of dark matter streams with primordial velocity dispersion passing through compact bodies. These caustics appear for both massless and massive particles but only the massive particles have caustics located near the lensing planet. It was shown that the existence of hairs is robust for realistic radial density profiles. It was also shown that the density enhancements are affected by neither the detailed structure of the body’s radial density profile nor the dark matter stream velocity. Hair density enhancements were calculated for three cases: the analytic constant-density Earth, the PREM, and the J11-4a density model of Jupiter. In all cases, large density enhancements were shown to exist up to $10^5$ for Earth and $10^{11}$ for Jupiter. Furthermore, the density enhancements of the hairs clearly reflected the boundary layers of the density profiles of the planets. The discovery of a hair would therefore be a huge boon for both planetary science and cosmology:

1. A hair would provide a uniquely powerful laboratory to study dark matter interactions.
2. Hairs are unique windows into the fine structure of the local dark matter streams.
3. A hair discovered near Earth (or Jupiter) would find counterparts in all solar system bodies, providing a powerful universal tool to probe the interior of almost any planet or moon.

A set of parameters determining the feasibility of finding a hair was listed and an approximate equation for the likelihood of discovering a hair was written down, given certain constraints and assumptions. Implications for indirect dark matter searches and relic neutrinos were shown to be nonexistent.

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