ON TRILINEAR OSCILLATORY INTEGRAL INEQUALITIES AND RELATED TOPICS

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In memoriam Elias M. Stein

Abstract. Inequalities are established for certain trilinear scalar-valued functionals. These functionals act on measurable functions of one real variable, are defined by integration over two- or three-dimensional spaces, and are controlled in terms of Lebesgue space norms of the functions, and in terms of negative powers of large parameters describing a degree of oscillation. Related sublevel set inequalities are a central element of the analysis.

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Date: February 16, 2022.
2010 Mathematics Subject Classification. 42B20, 26D15.
Key words and phrases. Multilinear functionals, oscillatory integrals, sublevel sets.
Research supported in part by NSF grants DMS-13363724 and DMS-1901413.
1. Introduction

Oscillatory integral operators, inequalities governing them, curvature, and the interrelationships between these topics are pervasive themes in the work of E. M. Stein, as for instance in [27, 24, 25]. In the present paper, we investigate scalar-valued multilinear oscillatory integral forms

\[ T_\lambda^\phi(f) = \int_B e^{i\lambda \phi(x)} \prod_{j \in J} f_j(x_j) \, dx, \]

along with related forms and inequalities. Here \( B \) is a ball or product of balls in \( (\mathbb{R}^d)^J \), \( J \) is a finite index set of cardinality \( |J| \geq 2 \), \( f = (f_j : j \in J) \) is a tuple of rather arbitrary functions \( f_j : \mathbb{R}^d \to \mathbb{C}, \ x = (x_j : j \in J) \in (\mathbb{R}^d)^J \), \( \phi : (\mathbb{R}^d)^J \to \mathbb{R} \) is a \( C^\infty \) function, and \( \lambda \in \mathbb{R} \) is a large parameter. More generally, one can form

\[ S_\lambda(f) = \int_{B \subset \mathbb{R}^d} e^{i\lambda \phi(x)} \prod_{j \in J} (f_j \circ \varphi_j)(x) \, dx, \]

with \( B \subset \mathbb{R}^D \) a ball, \( \varphi_j : B \to \mathbb{R}^d \) smooth submersions, and with the cardinality of \( J \) finite but \( |J|d \) possibly large relative to \( D \). We seek upper bounds, for \( |T_\lambda^\phi(f)| \) and for \( |S_\lambda(f)| \), that are small when \( |\lambda| \) is large, require no smoothness hypothesis on \( f \), are uniform over a large class of \( f \), reflect cancellation due to oscillation of \( e^{i\lambda \phi} \) when \( |\lambda| \) is large, and also reflect the influence of geometric and algebraic effects implicit in \( (\phi, (\varphi_j : j \in J)) \). In this paper we establish such bounds, and deduce an application to related multilinear forms in which no oscillatory factors are overtly present.

1.1. Background. Inequalities of the form

\[ |T_\lambda^\phi(f)| \leq C|\lambda|^{-\gamma} \prod_{j \in J} \|f_j\|_{L^{p_j}(\mathbb{R}^d)}, \]

with \( \gamma > 0 \) and \( C < \infty \) dependent on \( \phi \) and on \( \eta \), have been analyzed in various works. Hörmander [16] established the fundamental upper bound \( O(|\lambda|^{-d/2}\|f_1\|\|f_2\|) \) for the bilinear case \( |J| = 2 \), with the mixed Hessian \( \frac{\partial^2 \phi(x,y)}{\partial x \partial y} \) everywhere nonsingular. The bilinear case, with \( x, y \) in spaces of unequal dimensions, has been intensively studied in connection with Fourier restriction inequalities. Likewise, in connection with Fourier restriction, multilinear forms have been investigated, in which each function \( f_j \) is individually acted upon by a linear oscillatory integral operator, and the product of the resulting functions is integrated. Such multilinear forms are not studied here.

Forms \( |T_\lambda^\phi(f)| \) and \( |S_\lambda(f)| \), with \( |J| \geq 3 \), have been studied by Phong-Stein-Sturm [23], Gilula-Gressman-Xiao [13], and others. An introductory treatment can be found in the book [24] of Stein. The works [23] and [13] deal with general phase functions \( \phi \), and seek optimal relationships between \( \phi \), decay exponents \( \gamma \), and Lebesgue exponents \( p_j \). [23] also emphasizes stability — whether the optimal exponent \( \gamma \) is a lower semicontinuous function of \( \phi \).

The regime \( |J| > D/d \) is singular, in the sense that the integral extends only over a positive codimension subvariety of the Cartesian product of the domains of the functions \( f_j \). Variants of the form (1.2), in the singular regime and with all mappings \( \varphi_j \) linear, were investigated by Li, Tao, Thiele and the present author [12]. They established conditions under which there exists an exponent \( \gamma > 0 \) for which a corresponding inequality holds. One of the results of the present paper relaxes the assumption of linearity. Another treats
certain cases with \( \varphi_j \) linear that were not treated in [12], and provides an alternative proof of one of the main results of that work.

Results of this type under lower bounds for certain partial derivatives of the phase function, but with no upper bounds at all, have been investigated by Carbery and Wright [5] for \(|J| \geq 3\), building on earlier work [4] for the bilinear case \(|J| = 2\). This thread is not developed further in the present paper, in which upper bounds are implicit through smoothness hypotheses on phase functions.

For certain ranges of exponent tuples \( p = (p_j : j \in J) \), the works [23] and [13] establish upper bounds for (1.1) with optimal exponents \( \gamma \) as \(|\lambda| \to \infty\), up to powers of \( \log(1 + |\lambda|) \), where “optimal” means largest possible under indicated hypotheses on \( \varphi \) for given \( p \). We will not review the hypotheses of those works precisely, but their general form is significant for our discussion. For each \( \alpha \) (with \( \alpha_j \neq 0 \) for at least two distinct indices \( j \)) there are certain parameters \( p, \gamma \) for which nonvanishing of \( \partial^\alpha \varphi / \partial x^\alpha \) at \( x_0 \) implies validity of (1.3), with \( B = B(x_0, r) \) for sufficiently small \( r > 0 \). This conclusion is independent of other coefficients in the Taylor expansion of \( \varphi \) about \( x_0 \). Thus any other nonvanishing coefficients imply corresponding inequalities, and interpolation of the resulting inequalities yields further inequalities. All bounds obtained in these cited works are consequences of bounds obtained in this way, together with inclusions among \( L^p \) spaces resulting from Hölder’s inequality.

1.2. Four questions.

**Question 1.1.** Let \( \varphi \) be a real analytic, real-valued phase function. Let \( x_0 \in (\mathbb{R}^d)^J \). For which \( \gamma > 0 \) do there exist \( C < \infty \) and a neighborhood \( B \) of \( x_0 \) such that
\[
|T_\lambda^\varphi(f)| \leq C|\lambda|^{-\gamma} \prod_{j \in J} \|f_j\|_{L^\infty} \quad \text{for all functions } f_j \in L^\infty,
\]
for every \( \lambda \in \mathbb{R} \)?

For multilinear forms of the more general type (1.2), a less precise question is at present appropriate.

**Question 1.2.** Let \( \varphi \) be a real analytic, real-valued phase function. Let \( \varphi_j : \mathbb{R}^D \to \mathbb{R}^d \) be real analytic submersions. Let \( x_0 \in \mathbb{R}^D \). Under what conditions on \( \varphi \) and on \( (\varphi_j : j \in J) \) does there exist \( \gamma > 0 \) such that
\[
|S_\lambda(f)| \leq C|\lambda|^{-\gamma} \prod_{j \in J} \|f_j\|_{L^\infty} \quad \text{for all functions } f_j \in L^\infty,
\]
for every \( \lambda \in \mathbb{R} \)?

Roughly speaking, the difficulty in establishing inequalities (1.5) increases as the ratio \( d/D \) increases.

In the formulation (1.4), the main structural hypothesis is that the nonoscillatory part of the integrand is a product of factors \( f_j(x_j) \). No smoothness is required of these factors, and the strongest possible size restriction, \( L^\infty \), is imposed. As a refinement, one could ask to what degree the \( L^\infty \) norms could be replaced by weaker \( L^p \) norms without reducing \( \gamma \). One focus of the present paper is on obtaining a comparatively large exponent \( \gamma \), rather than on weakening the hypotheses under which it is obtained. Although we are not able to determine optimal exponents \( \gamma \) in Question 1.1, we improve on the largest exponent previously known for generic real analytic phases in the trilinear case with \( d = 1 \). We show
that interactions between monomial terms can give rise to upper bounds not obtainable from monomial-based inequalities.

A second focus, for related functionals, is on obtaining some decay inequality of power law type, for situations in which no decay bound was previously known, without attention to the value of the exponent $\gamma$. See for instance Theorem 4.4.

Oscillation can arise implicitly through the presence of high frequency Fourier components in the factors $f_j$, instead of explicitly through the presence of overt oscillatory factors $e^{i\lambda \phi}$. This suggests a third question.

**Question 1.3.** Let $\eta$ be smooth and compactly supported. Let $J$ be a finite index set. Let $\varphi_j : \mathbb{R}^D \to \mathbb{R}^d$ be real analytic submersions. Under what circumstances can the quantity $\int \prod_{j \in J} (f_j \circ \varphi_j)(x) \eta(x) \, dx$ be majorized by a product of strictly negative Sobolev norms of the factors $f_j$?

In analyzing these questions about oscillatory integral forms, we are led to questions about sublevel sets. Let $\varphi_j : [0,1]^2 \to \mathbb{R}^1$ and $a_j : [0,1]^2 \to \mathbb{R}^1$ be real analytic. To an ordered triple $f$ of Lebesgue measurable $f_j : \mathbb{R} \to \mathbb{R}$, and to $\varepsilon > 0$, associate the sublevel set

$$S(f, \varepsilon) = \{ x \in [0,1]^2 : \left| \sum_{j=1}^3 a_j(x)(f_j \circ \varphi_j)(x) \right| < \varepsilon \}.$$  

**Question 1.4.** Under what hypotheses on $(\varphi_j, a_j : j \in \{1,2,3\})$ and what conditions on $f$ do there exist $\gamma > 0$ and $C < \infty$ such that for every small $\varepsilon > 0$,

$$|S(f, \varepsilon)| \leq C\varepsilon^\gamma?$$

Some condition on $f$ is needed to exclude trivial solutions with $f \equiv 0$ or with every $|f_j|$ small pointwise. Another necessary condition for an inequality (1.7) is that any exact smooth solution $f$ of $\sum_j a_j \cdot (f_j \circ \varphi_j) \equiv 0$, in an arbitrary nonempty open set, should vanish identically. Situations in which there is a small family of such exact solutions, e.g. constant $f$ or affine $f$, are also of interest. In such a situation, one asks instead whether the inequality (1.7) can fail to hold only for those $f$ that are closely approximable by elements of the family of exact solutions.

One of the themes of this work is the web of interconnections between these three questions and their variants. For instance, variants of (1.7), in which the coefficients $a_j$ are vector-valued, arise naturally in our investigation of Questions 1.1 and 1.2.

**1.3. Content of paper.** We begin with remarks and examples placing Question 1.1 better in context. We then focus on the trilinear case, with $d = 1$. In all previous results for this case known to this author, the exponent $\gamma$ obtained for (1.4) has been no greater than $\frac{1}{2}$, and we seek to surpass that threshold. We introduce a condition for $\phi$, whose negation we call rank one degeneracy on some hypersurface, or simply rank one degeneracy. We prove that if $\phi \in C^\omega$ is not rank one degenerate and satisfies an auxiliary hypothesis, then the inequality (1.4) holds for some $\gamma$ strictly greater than $\frac{1}{2}$. Conversely, if $\phi \in C^\omega$ is rank one degenerate on some hypersurface, then there exists no $\gamma > \frac{1}{2}$ for which (1.4) holds. In a sense clarified in §16, the nondegeneracy hypothesis is satisfied by generic $C^\omega$ phase functions. We also explore the connection between multilinear oscillatory forms (1.2) and the multilinear oscillatory forms studied in [12].

Two applications to Question 1.3 are derived. The first is to products $\prod_{j=1}^3 (f_j \circ \varphi_j)$, for functions $f_j : \mathbb{R}^1 \to \mathbb{C}$ and for systems of mappings $\varphi_j : \mathbb{R}^2 \to \mathbb{R}^1$ satisfying an appropriate
curvature condition. We show that these are indeed well-defined as distributions when $f_j$ lie in Sobolev spaces of slightly negative orders. The second, a consequence of the first, is an alternative proof of a theorem of Joly, Métivier, and Rauch [17] on weak convergence of products $\prod_{j=1}^{3}(f_j \circ \varphi_j)$ when the functions $f_j$ are weakly convergent and the system of mappings $(\varphi_j : j \in \{1, 2, 3\})$ satisfies a suitable curvature hypothesis. We exploit the improvement of the exponent $\gamma$ beyond the threshold $\frac{1}{2}$ in (1.4) in these applications.

Our main results concerning (1.4) and (1.5), respectively, are Theorem 4.1 and Theorem 4.2. Their proofs are based on decomposition in phase space, a dichotomy between structure and pseudo-randomness, a two scale analysis, and a connection with sublevel sets.

In §17 we use the same method to give an alternative proof of a theorem of Li, Tao, Thiele, and the author [12], and to establish an extension.

The machinery developed here establishes, and utilizes, upper bounds of the form (1.7) for Lebesgue measures of sublevel sets associated to certain vector-valued functions, in situations in which having each $f_j$ vector-valued is an advantage. However, we also study the scalar-valued case. In §18 we consider sublevel sets of the type (1.6), with constant coefficients $a_j$, and establish upper bounds for their Lebesgue measures under natural hypotheses on $(\varphi_j : j \in \{1, 2, 3\})$ and appropriate nonconstancy hypotheses on $f$. In §19 we develop a rather different method to study sublevel set inequalities for nonconstant coefficients $a_j$, in the special case in which the mappings $\varphi_j$ are all linear. In §21 we construct an example demonstrating optimality, for one of the simplest possible vector-valued instances of (1.7), of the apparently crude bound that our method yields. This example is based on multiprogressions of rank greater than 1. Finally, §22 is devoted to further remarks and questions concerning sublevel set inequalities.

The case $|J| = 2$ of (1.1) is already well understood. We focus primarily on the next simplest case, in which $|J| = 3$ and $d = 1$, although these restrictions are relaxed in some of our results. The techniques used here are developed further and applied in two sequels, with Durcik and Roos [10] in work on multilinear singular integral operators, and with Durcik, Kovač, and Roos [11] in work on averages associated to $\mathbb{R}$–actions. The author plans to treat more singular cases, such as $|J| \geq 4$ in Theorem 4.2 in future work via a further extension of the method.

In most of the paper, we assume phase functions and mappings $\varphi_j$ to be real analytic, rather than merely infinitely differentiable. This is done primarily because hypotheses can be formulated more simply in the $C^\infty$ case, with its natural dichotomy between functions that vanish identically, and those that vanish to finite order. Extensions of two of the main theorems to the $C^\infty$ case are formulated in §5. Another extension will be developed in the forthcoming dissertation of Zirui Zhou.

There are connections between the results and methods in this paper, a much earlier work of Bourgain [2], and recent works of Peluse and Prendiville [19], [20], [21] involving quantitative nonlinear analogues of Roth’s theorem, cut norms, and degree reduction. See also [28] for an exposition of some of these ideas.

The author is indebted to Zirui Zhou for corrections and useful comments on the exposition, to Philip Gressman for a useful conversation, and to Terence Tao for pointing out the connection with the works of Bourgain, Peluse, and Prendiville. He thanks Craig Evans for serendipitously acquainting him with the work of Joly, Métivier, and Rauch, and for stimulating discussion.
2. Examples

In all of the examples of this section, and most of the main results of this paper, $d = 1$. $B$ is often replaced by $[0, 1]^d$, so $T^\phi_\lambda(f) = \int_{[0, 1]^d} e^{i\lambda\phi(x)} \prod_{j \in J} f_j(x_j) \, dx$. In these examples, excepting Example 2.3, $T^\phi_\lambda$ is trilinear.

Example 2.1. For $|J| = 3$ and $\phi(x_1, x_2, x_3) = x_2(x_1 + x_3)$, the inequality (1.4) holds with $\gamma = 1$. To justify this, write

$$T^\phi_\lambda(f) = |\lambda|^{-1/2} \int_{[0, 1]^2} f_1(x) F(y) e^{-i\lambda xy} \, dx \, dy$$

where $F(y) = c f_2(y)|\lambda|^{1/2} \hat{f}_3(\lambda y)$ for a certain harmless constant $c \neq 0$. This function satisfies $\|F\|_2 \lesssim \|f_2\|_\infty \|f_3\|_2 \leq \|f_2\|_\infty \|f_3\|_\infty$. One factor of $|\lambda|^{-1/2}$ has already been gained.

The remaining integral is $O(|\lambda|^{-1/2} \|f_1\|_2 \|F\|_2)$ by Plancherel’s theorem and a change of variables.

This gives $|T^\phi_\lambda(f)| \leq C |\lambda|^{-1} \|f_1\|_2 \|f_2\|_\infty \|f_3\|_2$. For $p \in [2, \infty)$, the optimal bound in terms of $\|f_1\|_\infty \|f_2\|_2 \|f_3\|_\infty$ is $O(|\lambda|^{-1} |\lambda|^{1/p})$. This can be seen by considering $f_j(x_j) = e^{-i\lambda x_j} \mathbf{1}_{[0, 1]}(x_j)$ for $j = 1, 3$. Integrating with respect to $x_1, x_3$ then leaves a function of $x_2$ whose real part is bounded below on $[0, \frac{\pi}{4} |\lambda|^{-1}]$ by a positive constant independent of $\lambda$.

Choose $f_2$ to be the indicator function of $[0, \frac{\pi}{4} |\lambda|^{-1}]$.

However, for $\phi = x_1 x_2 + x_2 x_3$, the situation changes if $\|f_2\|_\infty$ is replaced by $\|f_2\|_2$. The inequality $|T^\phi_\lambda(f)| \leq C |\lambda|^{-\gamma} \|f_2\|_2 \|f_1\|_\infty \|f_3\|_\infty$ holds for $\gamma = \frac{1}{2}$, but not for any strictly larger exponent. This is seen by choosing $f_2$ to be the indicator function of $[0, \frac{\pi}{4} |\lambda|^{-1}]$ and $f_1, f_3$ each to be the indicator function of $[0, 1]$.

This example will illustrate subtle points regarding the necessity of hypotheses in some of our main results, below.

Example 2.2. More generally, consider $\int_{[0, 1]^n} e^{i\lambda \phi(x)} \prod_{j=1}^n f_j(x_j) \, dx$. If (1.4) holds then $\gamma \leq \frac{1}{2}(n - 1)$. Indeed, by a change of variables, one can replace $[0, 1]$ by $[-1, 1]$. Define $a_j = \partial \phi / \partial x_j(0)$. For each $j \leq n - 1$ define

$$f_j(x_j) = e^{i\tau x_j^2/2} e^{-i\lambda a_j x_j} \eta(x_j)$$

where $\eta$ is a $C^\infty$ function supported in a small neighborhood of 0, satisfying $\eta(0) \neq 0$. If $\tau$ is a sufficiently large constant, depending on $\phi$ but not on $\lambda$, and if $\eta$ is supported in a sufficiently small neighborhood of 0, then by the method of stationary phase, for a certain constant $c \neq 0$, as $\lambda \to +\infty$,

$$\left| \int_{[0, 1]^{n-1}} e^{i\lambda \phi(x)} \prod_{j=1}^{n-1} f_j(x_j) \, dx_1 \, dx_2 \cdots \, dx_{n-1} \right| = c \lambda^{-(n-1)/2} + O(\lambda^{-(n-3)/2})$$

uniformly for all $x_n$ in some neighborhood $V$ of 0 independent of $\lambda$. Choose $f_n(x_n)$ to vanish outside of $V$ and to be equal to $e^{i h(x_n)}$ with $h$ real-valued so that

$$e^{i h(x_n)} \int_{[0, 1]^{n-1}} e^{i\lambda \phi(x)} \prod_{j=1}^{n-1} f_j(x_j) \, dx_1 \, dx_2 \cdots \, dx_{n-1} \geq 0$$

for each $x_n \in V$. Thus $|T^\phi_\lambda(f)| = c' \lambda^{-(n-1)/2} + O(\lambda^{-(n-3)/2})$ with $c' \neq 0$. 
Example 2.3. For any \( n \geq 3 \), the exponent \( \gamma = (n - 1)/2 \) is realized for
\[
\phi(x_1, \ldots, x_n) = x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n.
\]
This follows from the same reasoning as for \( n = 3 \).

Example 2.4. For \( \phi(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_3x_1 \), the optimal exponent is \( \gamma = \frac{1}{2} \).
Choosing \( f_j(x) = e^{i\lambda x_j^2/2} \), the integrand becomes \( e^{i\lambda \psi} \) with \( \psi(x) = (x_1 + x_2 + x_3)^2 \). This net phase function \( \psi \) factors through a submersion from \( \mathbb{R}^3 \) to \( \mathbb{R}^1 \), and has a critical point.

This example, contrasted with \( \phi = x_1x_2 + x_2x_3 \), for which the optimal exponent is 1, demonstrates that enlarging the set of monomials that occur with nonzero coefficients can cause the optimal exponent \( \gamma \) to decrease, in contrast to the theory for a restricted range of parameters developed in [23] and [13].

Example 2.5. More generally, for any parameter \( r > 0 \), the optimal exponent is \( \frac{r}{2} \) for
\[
\phi(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + rx_3x_1.
\]
Thus the optimal exponent is not lower semicontinuous with respect to \( \phi \). This also suggests that the exponent \( \gamma = (n - 1)/2 \) is rarely attained.

Example 2.6. For \( \phi(x) = x_1x_2 x_3 \), the exponent \( \gamma = \frac{1}{2} \) is again optimal. Choosing \( f_j(x) = e^{-i\lambda \ln(x) \frac{1}{|x|^2}} (x) \), the net oscillatory factor becomes \( e^{i\lambda \psi} \) with \( \psi(x) = x_1x_2x_3 - \ln(x_1x_2x_3). \)
The gradient of \( \psi \) vanishes identically on the hypersurface \( x_1x_2x_3 = 1 \), and the integral is no better than \( O(|\lambda|^{-1/2}) \).

Example 2.7. \( \phi(x) = (x_1 + x_2)x_3 \). This is merely Example 2.4 with the indices 1, 2, 3 permuted. Thus we have already observed that the inequality (1.4) holds with \( \gamma = 1 \). Here we reexamine that example from an alternative perspective. Integrating with respect to \( x_3 \) leads to
\[
\lambda^{-1/2} \int_{[0,1]^2} f_1(x)f_2(y)F_3(x+y) \, dx \, dy
\]
where \( F_3 \) depends on \( \lambda \) and satisfies \( \|F_3\|_{L^2} = O(\|f_3\|_{L^2}) \), but no stronger inequality for any \( L^p \) norm of \( F_3 \) is available. No oscillatory factor remains, yet we have already shown in Example 2.4 that an upper bound with another factor of \( \lambda^{-1/2} \) does hold.

Example 2.8. Consider \( \phi(x) = x_3 \varphi(x_1, x_2) + \psi(x_1, x_2) \) where \( \psi \) is a polynomial and \( (x, y) \mapsto \varphi(x, y) \) is a linear function that is a scalar multiple neither of \( x_1 \) nor of \( x_2 \). The same analysis as above leads to \( \lambda^{-1/2} \) multiplied by
\[
(2.1) \int_{[0,1]^2} f_1(x)f_2(y)F(\varphi(x, y))e^{i\lambda \psi(x,y)} \, dx \, dy
\]
with \( \|F\|_2 = O(\|f_3\|_2) \). Suppose that \( (\varphi, \psi) \) is nondegenerate in the sense that there do not exist polynomials \( h_j \) satisfying
\[
\psi(x, y) = h_1(x) + h_2(y) + h_3(\varphi(x, y)) \quad \forall (x, y).
\]
Then according to an inequality\(^1\) of Li-Tao-Thiele and the present author \(^2\), the integral (2.1) satisfies an upper bound of the form \( O(\lambda^{-\delta}\|f_1\|_2\|f_2\|_2\|F\|_2) \) for some \( \delta(\varphi, \psi) > 0 \). Thus \( |T_{\lambda}^{\psi}(f)| \leq C|\lambda|^{-\delta} \prod_{j=1}^3 \|f_j\|_{\infty} \).

\(^1\) An alternative proof of this inequality is developed in §17.
\(^2\) The imprecise methods of [12] very rarely yield optimal exponents.
Moreover, the analysis of [12] implicitly proves that this bound holds uniformly for all sufficiently small perturbations of \( \varphi, \psi \).

**Example 2.9.** Consider \( \phi(x) = x_1 x_2 + x_2 x_3^k \), with \( N \geq k \geq 2 \). For \( k = 2 \), \((1.4)\) holds for every \( \gamma \) strictly less than 1. For \( k \geq 3 \), it holds for every \( \gamma \leq \frac{1}{2} + \frac{1}{k} \), and this exponent is optimal. This can be shown by substituting \( x_3^k = \tilde{x}_3 \) and using

\[
\left| \int_{[0,1]^3} e^{i\lambda(x_1 x_2 + x_2 x_3^k)} \prod_{j=1}^3 g_j(x_j) \, dx \right| \leq C|\lambda|^{-1/2}\|g_1\|_{\infty}\|g_2\|_{\infty}\|g_3\|_{2}
\]

with \( g_3(y) = f_3(y^{1/k})y^{\frac{1}{k}-1} \). That this exponent cannot be improved when \( k \geq 3 \) can be shown by considering \( f_3 \) equal to the indicator function of \([0, \frac{4}{3}\lambda^{-1/k}]\).

**Example 2.10.** Let \( \phi(x,y) = x^2 y - xy^2 \), or more generally, any homogeneous cubic polynomial that is not a linear combination of \( x^3, y^3, (x + y)^3 \). Then

\[
\left| \int_{[0,1]^2} e^{i\lambda\phi(x,y)} f_1(x)f_2(y)f_3(x + y) \, dx \, dy \right| \leq C|\lambda|^{-\gamma} \prod_j \|f_j\|_{\infty}
\]

holds for \( \gamma = \frac{1}{4} \) \([15]\). Even for this simplest trilinear case of \((1.2)\), the optimal exponent remains unknown.

### 3. Nondegeneracy and Curvature

For convenience we integrate over \([0,1]^3\), rather than over a ball, though this makes no effective difference. The two formulations are equivalent, by simple and well known arguments involving partitions of unity and expansion of cutoff functions in Fourier series, resulting in unimodular factors that can be absorbed into the functions \( f_j \).

Thus we study functionals

\[
T^\phi_\lambda(f_1, f_2, f_3) = \int_{[0,1]^3} e^{i\lambda\phi(x)} \prod_{j=1}^3 f_j(x_j) \, dx
\]

with \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \) and \( f_j : [0,1] \to \mathbb{C} \), and associated inequalities

\[
|T^\phi_\lambda(f)| \leq C|\lambda|^{-\gamma} \prod_{j=1}^3 \|f_j\|_{L^\infty}.
\]

We assume throughout the discussion that \( \lambda \) is positive (as may be achieved by complex conjugation if \( \lambda \) is initially negative) and that \( \lambda \geq 1 \). For this situation, none of the results in \([23]\) and \([13]\) yield any exponent \( \gamma \) strictly greater than \( \frac{1}{2} \), and we focus on exceeding this benchmark exponent \( \frac{1}{2} \).

In results of this type, \( \tilde{\phi} \) should be regarded as an equivalence class of functions. If \( \tilde{\phi} \) takes the form \( \tilde{\phi}(x) = \phi(x) - \sum_{j=1}^3 h_j(x_j) \) with all functions \( h_j \) real-valued and Lebesgue measurable, then

\[
\sup_{\|f_j\|_{\infty} \leq 1} |T^\phi_\lambda(f)| = \sup_{\|f_j\|_{\infty} \leq 1} |T^\tilde{\phi}_\lambda(f)|
\]

since each function \( f_j \) can be replaced by \( f_j e^{-i\lambda h_j} \). Thus \( \tilde{\phi} \) is equivalent to \( \phi \), so far as the inequality \((3.2)\) is concerned. On the other hand, it is natural to require that the functions \( h_j \) possess the same degree of regularity as is required of \( \phi \). The next definition is
formulated in terms of maximally regular $h_j$, but the minimally regular situation inevitably arises in the analysis.

The examples in §2 suggest a notion of degeneracy for phases $\phi$, or equivalently, for such equivalence classes. Write $\pi_j(x_1, x_2, x_3) = x_j$.

**Definition 3.1.** Let $U \subset \mathbb{R}^3$ be open and nonempty. Let $\phi : U \to \mathbb{R}$ be $C^\omega$. Let $H \subset U$ be a $C^\omega$ hypersurface. $\phi$ is rank one degenerate on $H$ if there exist $C^\omega$ functions $h_j$ defined in $\pi_j(U)$ such that the associated net phase function $\tilde{\phi} = \phi - \sum_{j=1}^3 (h_j \circ \pi_j)$ satisfies

$$ (\nabla \tilde{\phi})|_H \equiv 0. $$

In this definition, $H$ may be defined merely in some small subset of $U$. $\phi : U \to \mathbb{R}$ is said to be rank one degenerate on some hypersurface, or simply rank one degenerate, if there exist $H \subset U$ and functions $h_j$ such that (3.3) holds. $\phi : [0,1]^3 \to \mathbb{R}$ is said to be rank one degenerate on some hypersurface if this holds for the restriction of $\phi$ to $(0,1)^3$.

If (3.3) holds, then the Hessian matrix of $\tilde{\phi}$ has rank less than or equal to 1 at each point of $H$, whence the term “rank one”. It is the restriction to $H$ of the full gradient that is assumed to vanish in (3.3), rather than the gradient of the restriction.

**Example 3.2.** $\phi(x) = x_1 x_2 + x_2 x_3 + x_3 x_1$ is rank one degenerate on the hypersurface $H$ defined by $x_1 + x_2 + x_3 = 0$. Choosing $h_j(x_j) = -x_j^2/2$ gives $\phi(x) = (x_1 + x_2 + x_3)^2/2$, whose gradient vanishes on $H$.

More generally, for $r \neq 0$, the rank one degenerate phases $\phi_r(x) = x_1 x_2 + x_2 x_3 + r x_3 x_1$ are equivalent to phases $\tilde{\phi}$ whose gradients vanish along hyperplanes $H_r$ defined by $x_2 = -r(x_1 + x_3)$.

**Example 3.3.** Let $r \in \mathbb{R}$. $\phi(x) = x_3(x_1 + x_2) + r x_1 x_2 x_3$ is not rank one degenerate. $\phi(x) = (x_1 + x_2 + x_3)^2 + r x_1 x_2 x_3$ is rank one degenerate if and only if $r = 0$.

**Proposition 3.1.** If $\phi \in C^\omega$ is rank one degenerate on a hypersurface $H$, then the inequality (1.4) cannot hold for any $\gamma$ strictly greater than $1/2$ on any ball $B$ containing $H$.

*Proof.* If the Hessian of $\phi$ does not vanish identically on $H$ then there exists a relatively open subset $\tilde{H}$ of $H$ on which this Hessian has rank 1. Choose $f_j(x_j) = e^{i\lambda h_j(x_j)}$, multiplied by cutoff functions that localize $\prod_j f_j(x_j)$ to a neighborhood of $\tilde{H}$, and invoke asymptotics provided by the method of stationary phase. The same reasoning applies so long as $\phi$ is not an affine function on $[0, 1]^3$, by fibering a neighborhood of a point of $H$ by line segments transverse to $H$, evaluating the asymptotic contribution of each line segment as $|\lambda| \to \infty$, and integrating with respect to a transverse parameter.

**Example 3.4.** For any multi-index $\alpha \in \mathbb{N}^3$, the phase function $\phi(x) = x^\alpha = \prod_{j=1}^3 x_j^{\alpha_j}$ is rank one degenerate on every open subset of $(\mathbb{R} \setminus \{0\})^3$. Therefore $\phi$ does not satisfy (1.4) with $\gamma > 1/2$ on any domain $B$.

We will also study integrals of the form

$$ (3.4) \quad \int_{\mathbb{R}^2} e^{i\lambda \psi(x)} \prod_{j=1}^3 (f_j \circ \varphi_j)(x) \eta(x) \, dx $$

with $\psi, \varphi_j : \mathbb{R}^2 \to \mathbb{R}$ real analytic, $\lambda \in \mathbb{R}$, and $f_j$ in Lebesgue spaces or Sobolev spaces of negative order. $\eta \in C^\infty(\mathbb{R}^2)$ will be a compactly supported smooth cutoff function. Both the situations in which $\lambda$ is a large parameter, and that in which $\lambda = 0$, are of interest.
The concepts of a 3-web, and its curvature, are relevant here. A 3-web in \( \mathbb{R}^2 \) is by definition a 3-tuple of pairwise transverse smooth foliations of a connected open subset of \( \mathbb{R}^2 \) \([1, 17]\). The leaves of each foliation are one-dimensional. If \( \varphi_j : \mathbb{R}^2 \to \mathbb{R}^1 \) are smooth functions, and if \( \nabla \varphi_j(x) \) and \( \nabla \varphi_k(x) \) are linearly independent at every point \( x \) for each pair of distinct indices \( j, k \), then the datum \( (\varphi_j : j \in \{1, 2, 3\}) \) defines a 3-web, whose leaves are level sets of these functions. Conversely, any 3-web is locally defined by such a tuple of functions. If \( \varphi \) and \( \tilde{\varphi} \) define the same foliation, then any \( f \circ \varphi \) can be written as \( \tilde{f} \circ \tilde{\varphi} \), where \( \tilde{f} \) has \( L^p \) and \( W^{s,p} \) Sobolev norms comparable to those of \( f \). Thus the inequalities that we will study will depend on the underlying web, rather than on the tuple \( (\varphi_j) \) used to describe it.

Associated to a 3-web on an open set \( U \) is its curvature, a real-valued function with domain \( U \) defined by Blaschke, and discussed in \([17]\) and references cited there. This curvature vanishes at a point \( x_0 \) if and only if there exist smooth functions \( f_j : \mathbb{R} \to \mathbb{R} \) satisfying \( f_j'((\varphi_j(x_0))) \neq 0 \) for at least one index \( j \), such that the associated function \( F = \sum_{j=1}^{3} f_j \circ \varphi_j \) satisfies \( F(x) - F(x_0) = O(|x - x_0|^4) \) as \( x \to x_0 \). This condition depends only on the underlying 3-web, not otherwise on associated functions \( \varphi_j \). It is invariant under local diffeomorphism of the ambient space \( \mathbb{R}^2 \). The equivalence of this condition with vanishing curvature can be shown via a short calculation in local coordinates chosen so that \( \varphi_j(x_1, x_2) \equiv x_j \) for \( j = 1, 2 \).

If \( \varphi_j(x) = x_j \) for \( j = 1, 2 \), then a web defined by \( (\varphi_j : j \in \{1, 2, 3\}) \) has curvature identically zero in an open set if and only if the ratio \( \frac{\partial \varphi_1 / \partial x_1}{\partial \varphi_2 / \partial x_2} \) factors locally as the product of a function of \( x_1 \) alone with a function of \( x_2 \) alone \([17]\).

If there exist \( f_j \) such that \( F \) vanishes identically in a neighborhood of \( x_0 \) then necessarily \( f_j'((\varphi_j(x_0))) \neq 0 \) and the change of variables \( x \leftrightarrow (f_1 \circ \varphi_1(x), f_2 \circ \varphi_2(x)) \) and the substitution \( \varphi_3 \mapsto f_3 \circ \varphi_3 \) transform all three functions \( \varphi_i \) into affine functions. If \( \varphi_j(x_j) \equiv x_j \) for \( j = 1, 2 \), and if \( \frac{\partial^2 \varphi_3}{\partial x_1 \partial x_2} \) vanishes identically in a neighborhood of \( x_0 \), then \( \varphi_3 \) is a sum of functions of the individual coordinates. Therefore the curvature vanishes identically in a neighborhood of \( x_0 \).

We say that \( (\varphi_j : j \in \{1, 2, 3\}) \) is equivalent to a linear system if there exist \( C^\infty \) real-valued functions \( H_j \), each with derivatives that do not vanish identically in any neighborhood of \( \varphi_j([0, 1]^2) \), satisfying \( \sum_{j=1}^{3} H_j \circ \varphi_j \equiv 0 \). In this situation, \( (\tilde{\varphi}_j = H_j \circ \varphi_j : j \in \{1, 2, 3\}) \) defines the same 3-web as does \( (\varphi_j) \). Taking \( \tilde{\varphi}_j \) as coordinates for \( j = 1, 2 \), all three functions \( \tilde{\varphi}_j \) become linear.

If \( \nabla \varphi_j, \nabla \varphi_k \) are linearly independent at \( x_0 \) for each pair of distinct indices \( j \neq k \), then the curvature of the 3-web defined by \((\varphi_j)\) vanishes identically in a neighborhood of \( x_0 \) if and only if \( (\varphi_j : j \in \{1, 2, 3\}) \) is equivalent to a linear system in a neighborhood of \( x_0 \).

The following lemma connects two notions of curvature/nondegeneracy, and will be used in the proof of Theorem \ref{thm:main}.

**Lemma 3.2.** Suppose that in some nonempty open subset \( U \subset \mathbb{R}^2 \), \( \varphi \in C^\infty \), \( \partial \varphi / \partial x_i \) vanishes nowhere for \( i = 1, 2 \), and the 3-web associated to \((x_1, x_2, \varphi(x_1, x_2))\) has nowhere vanishing curvature. Then the phase function \( \phi(x_1, x_2, x_3) = x_3 \varphi(x_1, x_2) \) is not rank one degenerate in any open subset of \( U \times (\mathbb{R} \setminus \{0\}) \).

\[3\]Assuming pairwise transversality of the foliations, there always exist \( f_j \) such that \( F(x) - F(x_0) = O(|x - x_0|^3) \).
Proof. Write \( \varphi_i = \partial \varphi / \partial x_i \) for \( i = 1, 2 \). Suppose that \( \hat{\phi} = x_3 \varphi(x_1, x_2) - \sum_{j=1}^{3} h_j(x_j) \) has gradient identically vanishing on a smooth hypersurface \( H \). If \( H \) can be expressed in some nonempty open set in the form \( x_3 = F(x_1, x_2) \), then \( x_3 \varphi_i(x_1, x_2) \equiv h'_i(x_i) \) for \( i = 1, 2 \). It is given that \( \varphi_i \) does not vanish. Therefore we may form the ratio of partial derivatives \( \varphi_1 / \varphi_2 \) and conclude that it can be expressed, in some nonempty open subset of \( U \), as a product of a function of \( x_1 \) with a function of \( x_2 \). This contradicts the hypothesis of nonvanishing curvature, as shown in [17].

If \( \hat{\phi} \) has gradient identically vanishing on some smooth hypersurface \( H \) that cannot be expressed in the above form on any nonempty open set, then \( H \) must take the form \( \Gamma \times I \) for some nonconstant curve \( \Gamma \subset \mathbb{R} \) and some interval \( I \subset \mathbb{R} \) of positive length. The equations \( x_3 \varphi_i(x_1, x_2) \equiv h'_i(x_i) \) force \( \varphi_i(x_1, x_2) \equiv 0 \) on \( \Gamma \), contradicting the assumption that \( \varphi_i = \partial \varphi / \partial x_i \) vanishes nowhere. \( \square \)

4. Formulations of some results

The first main result of this paper is concerned with multilinear expressions

\begin{equation}
(4.1) \quad T_\lambda^\phi(f) = \int_{[0,1]^3} e^{i\lambda \phi(x)} \prod_{j=1}^{3} f_j(x_j) \, dx,
\end{equation}

restricting attention to three functions \( f_j : [0,1] \to \mathbb{C} \), and integrating over \([0,1]^3\) rather than over a ball.

**Theorem 4.1.** Let \( \phi \) be a real analytic, real-valued function in a neighborhood \( U \) of \([0,1]^3\). Suppose that \( \phi \) is not rank one degenerate on any hypersurface in \( U \). Suppose that for each pair of distinct indices \( j \neq k \in \{1,2,3\} \), \( \partial^2 \phi / \partial x_j \partial x_k \) vanishes nowhere on \([0,1]^3\). Then there exist \( \gamma > \frac{1}{2} \) and \( C < \infty \) such that the operators defined in (4.1) satisfy

\begin{equation}
(4.2) \quad |T_\lambda^\phi(f)| \leq C|\lambda|^{-\gamma} \prod_j \|f_j\|_2
\end{equation}

uniformly for all functions \( f_j \in L^2(\mathbb{R}^1) \) and all \( \lambda \in \mathbb{R} \).

The condition that a single partial derivative \( \partial^2 \phi / \partial x_j \partial x_k \) vanishes nowhere suffices, for \( C^\infty \) phases \( \phi \) without other hypotheses, to ensure that

\[
\int_{[0,1]^2} e^{i\lambda \phi(x_1,x_2)} \prod_{j=1}^{2} f_j(x_j) \, dx_1 \, dx_2 = O(|\lambda|^{-1/2} \|f_1\|_2 \|f_2\|_2)
\]

uniformly in \( x_3 \) [16]. Consequently

\[
T_\lambda^\phi(f) = O(|\lambda|^{-1/2} \|f_1\|_2 \|f_2\|_2 \|f_3\|_1).
\]

The content of Theorem 4.1 is the improvement, with appropriate norms on the right-hand side, of the exponent beyond \( \frac{1}{2} \).

The set of all \( \phi \) that satisfy the hypotheses of Theorem 4.1 is nonempty, and is open with respect to the \( C^3 \) topology. The set of all 3–jets for \( \phi \) at \( x_0 \) that guarantee validity of the hypotheses in some small neighborhood of \( x_0 \) is open and dense. Moreover, its complement is contained in a \( C^\infty \) variety of positive codimension in the space of jets. This is shown in [16].

The theorem is not valid for \( C^\infty \) phases \( \phi \) as stated. If \( \phi \) were merely \( C^\infty \), then \( \phi \) could vanish to infinite order at a single point, without any equivalent phase \( \hat{\phi} \) satisfying...
\( \nabla \phi |_{\text{H}} \equiv 0 \) for any hypersurface \( \text{H} \). Infinite order degeneracy at a point implies that (1.2) does not hold for any \( \gamma > 0 \), even with \( L^\infty \) norms on the right-hand side of the inequality. Corresponding remarks apply to other results formulated in this paper.

The norms appearing on the right-hand side of (4.2) are \( L^2 \) norms, rather than \( L^\infty \). Thus phases that satisfy the hypotheses of the theorem enjoy stronger bounds on \( L^2 \times L^2 \times L^2 \) than does the example \( \phi(x) = x_3(x_1 + x_2) \), which attains the largest possible exponent, \( \gamma = 1 \), on \( L^\infty \times L^\infty \times L^\infty \), but only \( \gamma = \frac{3}{2} \) on \( L^2 \times L^2 \times L^2 \). This phase satisfies the main hypothesis of rank one nondegeneracy, but fails to satisfy the auxiliary hypothesis of three nonvanishing mixed second partial derivatives.

We believe that under the rank one nondegeneracy hypothesis, the conclusion holds if one of the three mixed second partial derivatives vanishes nowhere, but the other two are merely assumed not to vanish identically. Theorem 5.1 below, supports this belief.

Functions associated to \( \phi \) by solutions of certain implicit equations arise naturally in our analysis, so it is not natural to restrict attention to polynomial phases in the formulation of the theorems, as is sometimes done in works on this topic. Example 2.6 also demonstrates that for polynomial phases, it is not always natural to restrict to polynomial functions \( h_j \) in formulating the equivalence relation between phases or the notion of rank one degeneracy.

Oscillatory factors do not appear explicitly in the formulation of our second main result, Theorem 4.2, which is concerned with conditions under which the integral of \( \prod_{j \in J} (f_j \circ \varphi_j) \) is well-defined. If \( \eta \in C^0 \) has compact support in \( \mathbb{R}^2 \), and if \( \nabla \varphi_j \) and \( \nabla \varphi_k \) are linearly independent at each point in the support of \( \eta \) for every pair of distinct indices \( j, k \in \{1, 2, 3\} \), and if each \( f_j \in L^{3/2}(\mathbb{R}^1) \), then the product \( \eta(x) \prod_{j=1}^3 f_j \circ \varphi_j \) belongs to \( L^1(\mathbb{R}^2) \). This is a simple consequence of complex interpolation, since the product belongs to \( L^1 \) whenever two of the three functions belong to \( L^1(\mathbb{R}^1) \) and the third belongs to \( L^\infty \). The exponent \( \frac{3}{2} \) is optimal in this respect.

The answer is negative without further hypotheses. In particular, it is negative whenever all \( \varphi_j \) are linear. But inequalities (1.3) do hold under suitable conditions.

**Question 4.1.** Let \( J \) be a finite index set. Let \( U \subset \mathbb{R}^2 \) nonempty and open. For \( j \in J \), let \( \varphi_j : U \to \mathbb{R} \) be \( C^\infty \) with nowhere vanishing gradient. Suppose that for any \( j \neq k \in J \), \( \nabla \varphi_j \) and \( \nabla \varphi_k \) are linearly independent at almost every point in \( U \). Let \( \eta \in C^0_0(U) \). Do there exist \( s < 0 \), \( p < \infty \), and \( C < \infty \) such that

\[
(4.3) \quad \left| \int_{\mathbb{R}^2} \eta \cdot \prod_{j \in J} (f_j \circ \varphi_j) \right| \leq C \prod_{j \in J} \|f_j\|_{W^{s,p}}
\]

for all functions \( f_j \in C^1(\varphi_j(U)) \)?

The answer is negative without further hypotheses. In particular, it is negative whenever all \( \varphi_j \) are linear. But inequalities (1.3) do hold under suitable conditions.

**Theorem 4.2.** Let \( \varphi_j \in C^\infty \) for each \( j \in \{1, 2, 3\} \). Suppose that for every pair of distinct indices \( j \neq k \in \{1, 2, 3\} \), \( \nabla \varphi_j \) and \( \nabla \varphi_k \) are linearly independent at \( x_0 \). Suppose that the curvature of the web defined by \( (\varphi_1, \varphi_2, \varphi_3) \) does not vanish at \( x_0 \). Then there exist \( \eta \in C^0_0 \) satisfying \( \eta(x_0) \neq 0 \) such that for any exponent \( p > \frac{3}{2} \), there exist \( C < \infty \) and \( s < 0 \) such that

\[
(4.4) \quad \left| \int_{\mathbb{R}^2} \prod_{j=1}^3 (f_j \circ \varphi_j) \eta \right| \leq C \prod_{j} \|f_j\|_{W^{s,p}} \quad \text{for all } f \in (L^{3/2}(\mathbb{R}^1))^3.
\]
The assumption that \( f \in L^{3/2} \) guarantees absolute convergence of the integral. The particular instance of Theorem 4.2 with the ordered triple \( (x_1, x_2) \mapsto (x_1, x_1 + x_2, x_1 + x_2^2) \) of mappings was treated by Bourgain [2] in 1988.

The proof will implicitly establish a formally stronger inequality. Let \( \gamma \in (0, 1) \). Let \( \lambda \in (0, \infty) \) be large. Suppose that the Fourier transform of at least one of the functions \( f_j \) is supported in the region in which the Fourier variable satisfies \( |\xi| \geq \lambda \). Partition a sufficiently small neighborhood of the support of \( \eta \) into cubes \( Q_n \), each of sidelength \( \lambda - \gamma \).

Then

\[
\sum_n \left| \int_{Q_n} \prod_{j=1}^3 (f_j \circ \varphi_j) \right| \leq C \lambda^s \prod_j \|f_j\|_{L^p}.
\]

Theorem 4.2 is a simple consequence of Theorem 4.1, with the validity of the inequality (4.2) for some exponent strictly greater than \( \frac{1}{2} \) being crucial in the analysis. It is worth noting that the deduction relies on the appearance of \( L^2 \) norms, rather than merely \( L^\infty \) norms, on the right-hand side of (4.2). The tuple \( (\varphi_1, \varphi_2, \varphi_3) = (x_1, x_2, x_1 + x_2) \) illustrates this relatively delicate distinction. This example does not satisfy the inequality (4.4). When the analysis used below to reduce Theorem 4.2 to (4.2) is applied to it, the phase that arises is \( \Phi(x_1, x_2, x_3) = x_3(x_1 + x_2) \). This is Example 2.1, for which the \( L^\infty \) inequality holds with \( \gamma = 1 \), but the \( L^2 \) inequality (4.2) holds only for \( \gamma = \frac{1}{2} \), not for any larger exponent.

Theorem 4.2 has the following immediate consequence for the weak convergence of products of weakly convergent factors.

**Corollary 4.3.** Let \( \eta, \varphi_j \) satisfy the hypotheses of Theorem 4.2. Let \( p > \frac{3}{2} \). For \( \nu \in \mathbb{N} \) let \( f_\nu^j \in L^p \) have uniformly bounded \( L^p \) norms. If \( f_\nu^j \) converges weakly to \( f_j \) as \( \nu \to \infty \) for \( j = 1, 2, 3 \) then

\[
\prod_{j=1}^3 (f_\nu^j \circ \varphi) \text{ converges weakly to } \prod_{j=1}^3 (f_j \circ \varphi) \text{ as } \nu \to \infty
\]

in a neighborhood of \( x_0 \).

That is,

\[
\int \eta \prod_{j=1}^3 (f_\nu^j \circ \varphi) \to \int \eta \prod_{j=1}^3 (f_j \circ \varphi) \text{ as } \nu \to \infty
\]

for every function \( \eta \in C^\infty \) supported in a sufficiently small neighborhood of \( x_0 \).

Corollary 4.3 is a slight variant of a result established by Joly-Métilvier-Rauch [17] using semiclassical defect measures. In [17], each \( f_j \circ \varphi_j \) is replaced by a function that possesses some quantitative smoothness along the level curves of \( \varphi_j \) but need not be constant. Such an extension is a simple consequence of Theorem 4.2, and is formulated and proved below as Theorem 15.1 and Corollary 15.2.

Consider functionals of the form

\[
S_\lambda(f) = \int_{[0,1]^2} e^{i\lambda \psi(x)} \prod_{j=1}^3 f_j(\varphi_j(x)) \, dx.
\]

In [17] the functions \( \varphi_j \) are \( C^\infty \) rather than \( C^\omega \). In Theorem 2.2.1 of [17] it is assumed that the curvature is nonzero at \( x_0 \), while in Theorem 2.2.3 the curvature is allowed to vanish on any set of Lebesgue measure zero, but a stronger hypothesis is imposed on \( f_j \).
Theorem 4.4. Let \( \varphi_j : [0,1]^2 \to \mathbb{R} \) and \( \psi : [0,1]^2 \to \mathbb{R} \) be real analytic. Suppose that for any two indices \( j, k \in \{1, 2, 3\} \), the Jacobian determinant of the mapping \( [0,1]^2 \ni x \mapsto (\varphi_j(x), \varphi_k(x)) \in \mathbb{R}^2 \) does not vanish identically. Suppose that there exist no nonempty open subset \( U \subset (0,1)^2 \) and \( C^\omega \) functions \( h_j : \varphi_j(U) \to \mathbb{R} \) satisfying

\[
\psi(x) = \sum_{j=1}^{3} h_j(\varphi_j(x)) \quad \text{for all } x \in U.
\]

Then there exist \( \delta > 0 \) and \( C < \infty \) satisfying

\[
|S_\lambda(f)| \leq C|\lambda|^{-\delta} \prod_{j=1}^{3} \|f_j\|_{L^2} \quad \text{for all } f \text{ and all } \lambda \in \mathbb{R}.
\]

For linear mappings \( \varphi_j \), two different generalizations of this inequality were proved in \cite{12}. For this linear case, and for any tuple \( \{\varphi_j : j \in \{1, 2, 3\}\} \) reducible to a linear tuple by a change of variables, Theorem 4.4 is a special case of results obtained in that work. While the method of analysis in \cite{12} exploited linearity of \( \varphi_j \), in \cite{17} we sketch an alternative proof of one of the two main results of \cite{12} by the method developed here that allows an extension to the nonlinear case.

If the Jacobian determinant of \( x \mapsto (\varphi_j(x), \varphi_k(x)) \) vanishes nowhere for each pair of distinct indices \( j, k \), then \( |S_\lambda(f)| \leq C\|f_1\|_1\|f_2\|_1\|f_3\|_\infty \) for any permutation \( (i, j, k) \) of \( (1, 2, 3) \). Thus by interpolation, it suffices to prove \( (4.9) \) with the \( L^2 \) norms replaced by \( L^\infty \) norms on the right-hand side.

Example 4.2. For \( (\varphi_1, \varphi_2, \varphi_3) = (x_1, x_2, x_1 + x_2) \) and \( \psi(x) = x_1^2 x_2 \), and with the \( L^2 \) norms on the right-hand side replaced by \( L^\infty \) norms, the inequality for \( S_\lambda(f) \) holds with \( \delta = \frac{1}{3} \), and fails for \( \delta > \frac{1}{3} \) \cite{15}. The optimal exponent, for \( L^\infty \) norms, is unknown for even this (simplest) example.

Conjecture 4.5. Let \( J \) be a finite set of indices. Let \( D \geq 2 \), and let \( d_j \geq 1 \) for \( j \in J \). Let \( B \subset \mathbb{R}^D \) be a ball of finite radius. For each \( j \in J \), let \( \varphi_j \in C^\omega(B, \mathbb{R}^{d_j}) \) be nonconstant. Likewise, Let \( \psi \in C^\omega(B, \mathbb{R}) \). Suppose that \( \psi \) cannot be expressed as \( \psi = \sum_{j \in J} h_j \circ \varphi_j \) in any open subset of \( B \), with \( h_j \in C^\omega \). Then there exists \( \gamma > 0 \) such that for all \( \lambda \in \mathbb{R} \) and all continuous functions \( f_j \),

\[
|\int_B e^{i\lambda \psi} \prod_{j \in J} (f_j \circ \varphi_j)| \leq C|\lambda|^{-\delta} \prod_{j \in J} \|f_j\|_{L^\infty}.
\]

For the case in which \( D = 2, d_j = 1 \), all \( \varphi_j \) are linear, and \( \psi \) is a polynomial, this is proved in \cite{12}.

This paper is organized so that Theorem 4.2 is proved along with related results, including Theorems 4.1 and 5.1. A more direct and somewhat simpler proof of Theorem 4.2 can be extracted from the discussion.

5. Variants and Extensions

We next formulate a result for the special case in which \( \phi \) is an affine function of \( x_3 \); thus \( \phi(x) = x_3 \varphi_3(x_1, x_2) + \psi(x_1, x_2) \). The proof developed below for this special case is a simplification of the proof of Theorem 4.1 and relies on Theorem 4.4, thus bringing to light connections between these results.
Theorem 5.1. Let $J = \{1, 2, 3\}$ and $d = 1$. Let
\begin{equation}
\phi(x_1, x_2, x_3) = x_3 \varphi(x_1, x_2) + \psi(x_1, x_2)
\end{equation}
where $\varphi, \psi$ are real-valued real analytic functions defined in a neighborhood of $[0, 1]^2$. Suppose that $\partial \varphi / \partial x_1$ and $\partial \psi / \partial x_2$ vanish nowhere on $[0, 1]^2$. Suppose that there exists no open subset of $[0, 1]^2$ in which $\psi$ can be expressed in the form
\begin{equation}
\psi(x_1, x_2) = Q_1(x_1) + Q_2(x_2) + (Q_3 \circ \varphi)(x_1, x_2)
\end{equation}
for $C^\omega$ functions $Q_1, Q_2, Q_3$. Then there exist $\gamma > \frac{1}{2}$ and $C < \infty$ satisfying
\begin{equation}
|T_\chi^\phi(f)| \leq C|\lambda|^{-\gamma} \prod_{j=1}^{3} \|f_j\|_2
\end{equation}
uniformly for all functions $f_j \in L^2(\mathbb{R})$ and all $\lambda \in \mathbb{R}$.

Theorem 5.1 is not quite a special case of Theorem 4.1 because it is not assumed here that $\frac{\partial^2 \psi}{\partial x_1 \partial x_2}$ is nonzero, and therefore $\frac{\partial^2 \psi}{\partial x_1 \partial x_2}(0)$ could vanish. The hypothesis that $\psi$ cannot be expressed in the form (5.2) is not necessary for the conclusion to hold, as shown by the example $\phi(x) = x_3(x_1 + x_2)$, for which $\psi \equiv 0$. In this respect, Theorem 4.1 is more satisfactory. A more typical example excluded by this hypothesis is $(\varphi, \psi) = (x_1 + x_2, x_1 x_2)$, for which the conclusion (5.3) does indeed fail.

Theorem 4.4 directly implies Theorem 5.1. Indeed, set $\varphi_j(x_1, x_2) = x_j$ for $j = 1, 2$, and $\varphi_3 = \varphi$. Then
\begin{equation}
T_\chi^\phi(g) = |\lambda|^{-1/2}S_\lambda(f)
\end{equation}
where $f_j = g_j$ for $j = 1, 2$, and $f_3(t) = |\lambda|^{1/2}g(t)$. Then $f_3$ satisfies $\|f_3\|_2 = O(\|g_3\|_2)$.

The next result combines oscillation with negative order Sobolev norms in the context of Theorem 4.2.

Theorem 5.2. Consider $S_\lambda$ with $J = \{1, 2, 3\}$, $d = 1$, and $D = 2$. Let $\varphi_j \in C^\omega$ for each $j \in \{1, 2, 3\}$. Suppose that for any two indices $j \neq k$, $\nabla \varphi_j$ and $\nabla \varphi_k$ are linearly independent at every point. Suppose that there exist no nonempty open subset $U \subset (0, 1)^2$ and $C^\omega$ functions $h_j : \varphi_j(U) \to \mathbb{R}$ satisfying
\begin{equation}
\psi(x) = \sum_{j=1}^{3} h_j(\varphi_j(x)) \quad \text{for} \ x \in U.
\end{equation}

Suppose also that $(\varphi_1, \varphi_2, \varphi_3)$ is not equivalent to a linear system in any nonempty open set. Then for each $p > \frac{3}{2}$ there exist $C < \infty$, $\delta > 0$, and $s < 0$ such that
\begin{equation}
|S_\lambda(f)| \leq C(1 + |\lambda|)^{-\delta} \prod_{j=1}^{3} \|f_j\|_{W^{s,p}} \quad \text{for all} \ f \in (L^p)^3 \text{ and all } \lambda \in \mathbb{R}.
\end{equation}

Theorem 4.2, Corollary 4.3, and Theorem 4.4 imply straightforward generalizations to integrals over $[0, 1]^d$ with products of $d + 1$ functions for arbitrary $d \geq 2$. Such generalizations are obtained by changing variables and regarding the domain of integration as the union of a $d - 2$-dimensional family of two-dimensional slices, in such a way that $d - 2$ of the factors are constant along slices. Under appropriate hypotheses, the results of this paper can be applied to each slice.\footnote{Theorem 4.1 generalizes in the same way, but the threshold exponent $\gamma = \frac{1}{2}$ is less natural for $d > 3$.}
Here is such an analogue of Theorem 4.2. Let $d \geq 3$, and let $J$ be an index set of cardinality $|J| = d + 1$. Let $\varphi_j \in C^\infty$ for each $j \in J$. Suppose that $(\nabla \varphi_j : j \in J)$ are transverse, in the sense that for any subset $\tilde{J} \subset J$ of cardinality $d$, $\{\nabla \varphi_j : j \in \tilde{J}\}$ are linearly independent at $x_0$. For each subset $J' \subset J$ of cardinality $d - 2 = |J| - 3$ consider the foliation of a neighborhood of $x_0$ in $\mathbb{R}^d$ with 2-dimensional leaves $L_t = \{x : \varphi_j(x) = t_j \forall j \in J'\}$, where $t \in \mathbb{R}^{J'}$. For each such $t$, restriction of the family of three functions $\{\varphi_i : i \in J \setminus J'\}$ to $L_t$ defines a web on $L_t$ for each $t$.

**Theorem 5.3.** Let $d \geq 3$ and $|J| = d + 1$. Let $x_0 \in \mathbb{R}^d$, and let $V$ be a neighborhood of $x_0$. Suppose that at $x_0$, $(\varphi_j : j \in J)$ satisfies the transversality hypothesis introduced above. Suppose that for each $i \in J$ there exists a subset $J_i' \subset J$ satisfying $|J_i'| = d - 2$, with $i \notin J'$, such that for $t \in \mathbb{R}^{J'}$ defined by $\varphi_j(x_0) = t_j$ for each $j \in J'$, the curvature of the web defined above on $L_t$ does not vanish at $x_0$. Then there exist $\eta \in C^\infty_0$ satisfying $\eta(x_0) \neq 0$, $C < \infty$, and $s < 0$ satisfying

$$\left|\int_{\mathbb{R}^d} \prod_{j=1}^3 (f_j \circ \varphi_j) \eta\right| \leq C \prod_{j \in J} \|f_j\|_{W^{s,2}} \text{ for all } f \in (L^2(\mathbb{R}^d))^J.$$ 

This hypothesis, application of Theorem 4.2 to integrals over two-dimensional slices defined by $L_j(x) = t_j$ for $j \in J'$, and integration with respect to $t$ over a bounded subset of $\mathbb{R}^{J'}$ yield an upper bound of the form $C \|f_i\|_{W^{s,2}} \prod_{j \notin J_i} \|f_j\|_{L^2}$ for some $s < 0$, for each $i \in J$. Interpolation of these bounds then produces an upper bound of the desired form $C \prod_{j \in J} \|f_j\|_{W^{s,2}}$, with $s$ replaced by $s/|J|$.

Theorem 5.3 in turn implies a corresponding extension of Corollary 4.3.

All of our results have extensions to the case of $C^\infty$ phase functions, but the hypothesis of rank one nondegeneracy must be reformulated. For Theorem 4.1, such an extension can be phrased as follows.

**Theorem 5.4.** Let $J = \{1, 2, 3\}$, and $d = 1$. Let $\phi \in C^\infty$ be real-valued and defined in a neighborhood $U$ of $[0, 1]^3$. Suppose that there do not exist a point $z \in U$, a germ $M$ of $C^\infty$ manifold $M$ of dimension 2 at $z$, and $C^\infty$ functions $h_j$ such that the restriction to $M$ of the gradient of $\phi(x) = \phi(x) - \sum_{j=1}^3 h_j(x_j)$ vanishes to infinite order at $z$.

Suppose that for each pair of distinct indices $j \neq k \in \{1, 2, 3\}$, $\frac{\partial^2 \phi}{\partial x_j \partial x_k}$ vanishes nowhere on $[0, 1]^3$. Then there exist $\gamma > \frac{1}{2}$ and $C < \infty$ satisfying

$$|T^\phi_\lambda(f)| \leq C|\lambda|^{-\gamma} \prod_j \|f_j\|_2$$

uniformly for all functions $f_j \in L^2(\mathbb{R})$ and all $\lambda \in \mathbb{R}$.

A corresponding modification of the hypotheses of Theorem 4.4 is needed for a $C^\infty$ analogue. Consider functionals of the form $S_{\lambda}(f) = \int_{[0, 1]^2} e^{i\lambda \psi(x)} \prod_{j=1}^3 f_j(\varphi_j(x)) \, dx$ as in Theorem 4.4, where $\varphi_j : U \to \mathbb{R}$ and $\psi : U \to \mathbb{R}$ are $C^\infty$ functions defined in some neighborhood $U$ of $[0, 1]^2$.

**Theorem 5.5.** Suppose that for any two indices $j, k \in \{1, 2, 3\}$, the Jacobian determinant of the mapping $[0, 1]^2 \ni x \mapsto (\varphi_j(x), \varphi_k(x)) \in \mathbb{R}^2$ does not vanish identically. Suppose that there do not exist $C^\infty$ functions $h_j$ defined in neighborhoods of the closure of $\varphi_j(U)$ and a point $z \in U$ such that $\psi(x) = \sum_{j=1}^3 h_j(\varphi_j(x))$ vanishes to infinite order at $z$. Then there
exist $\delta > 0$ and $C < \infty$ satisfying

$$|S_\lambda(f)| \leq C|\lambda|^{-\delta} \prod_{j=1}^3 \|f_j\|_{L^2} \quad \text{for all } f \text{ and all } \lambda \in \mathbb{R}.$$ 

The proofs of Theorems 5.4 and 5.5 are the same as those of the corresponding results for the $C^\infty$ case, with small modifications in the concluding sublevel set analysis. The details of these modifications are omitted.

The next part of the paper is organized as follows. We begin the proofs with Theorem 5.1, reducing it in §6 to a special case of Theorem 4.4, which we then prove in §§7, 8, 9, and 11.

Theorem 4.1 is proved in §12 and §13 by elaborating on that analysis. We establish Theorem 4.4 in its full generality, and derive Theorems 4.2 and 5.2 from these methods and results, in §14. In §15 we enunciate and prove extensions to the Joly-Métivier-Rauch framework, in which the condition that the factors $f_j$ be constant along leaves of foliations is replaced by smoothness along those leaves. §16 contains remarks concerning the hypotheses, demonstrating that these are satisfied generically, in an appropriate sense.

6. Reductions

We begin by showing how Theorem 5.1 follows from a bandlimited case of Theorem 4.4. Let $(\varphi, \psi)$ satisfy its hypotheses. There are two cases, depending on whether or not $\varphi$ can be expressed in the form

$$\varphi(x, y) \equiv H(h_1(x) + h_2(y)) \quad \text{on } [0, 1]^2$$

with $H, h_1, h_2 \in C^\infty$. If $\varphi$ does take the form (6.1) then a $C^\infty$ change of variables with respect to $x$ and to $y$, together with replacement of $\varphi$ by $\tilde{H} \circ \varphi$ for appropriate $\tilde{H}$, reduces matters to the case in which $h_1, h_2$ are linear. In these new coordinates, $\psi$ remains $C^\infty$, and (5.2) continues to hold. This places us in the setting of Example 2.8 which was treated above as a consequence of the results of [12]. We restrict attention henceforth to the second case, in which $\varphi$ cannot be expressed in the form (6.1).

Integrate with respect to $x_3$ to reexpress

$$\int_{[0, 1]^3} e^{i\lambda x_3 \varphi(x_1, x_2)} e^{i\lambda \psi(x_1, x_2)} \prod_{j=1}^3 f_j(x_j) \, dx_1 \, dx_2 \, dx_3$$

$$= |\lambda|^{-1/2} \int_{[0, 1]^2} e^{i\lambda \psi(x_1, x_2)} f_1(x_1) f_2(x_2) F_3(\varphi(x_1, x_2)) \, dx_1 \, dx_2$$

with $F_3(t) = |\lambda|^{1/2} \tilde{f}_3(\lambda t)$ satisfying $\|F_3\|_2 = c\|f_3\|_2$. Thus

$$T_\lambda^\varphi(f) = c|\lambda|^{-1/2} S_\lambda(f_1, f_2, F_3)$$

with $S_\lambda$ defined in terms of the phase function $\psi$, and with the ordered triple of mappings

$$(\varphi_1, \varphi_2, \varphi_3)(x_1, x_2) = (x_1, x_2, \varphi(x_1, x_2)).$$

The hypotheses of Theorem 4.4 are satisfied by $(\psi, \varphi_1, \varphi_2, \varphi_3)$. Therefore the conclusion of Theorem 5.1 for $T_\lambda^\varphi(f)$ is a consequence of the conclusion of Theorem 4.4 which yields a factor of $|\lambda|^{-\delta}$ with $\delta > 0$, supplementing the factor $|\lambda|^{-1/2}$ that is already present.

The function $F_3$ is $|\lambda|$-bandlimited, that is, its Fourier transform was supported in $[-|\lambda|, |\lambda|]$. Thus this proof relies only on this bandlimited case of Theorem 4.4.
In the following sections we will establish the conclusion of Theorem 4.4 in the $O(|\lambda|)$–bandlimited case, thus completing the proof of Theorem 5.1. The general case of Theorem 4.4 will be treated later, in §14. Theorem 5.1 will be used in the proof of the general case. Our treatment of the bandlimited case of Theorem 4.4 will not rely on Theorem 5.1 so the reasoning is not circular.

We begin the proof of Theorem 4.4 for general $(\psi, \varphi_1, \varphi_2, \varphi_3)$ satisfying its hypotheses, without any bandlimitedness hypothesis for the present. Thus it is given that for each $j \neq k \in \{1, 2, 3\}$, $\nabla \varphi_j, \nabla \varphi_k$ are linearly independent on the complement of a analytic variety of positive codimension. If $(\varphi_j : j \in \{1, 2, 3\})$ is equivalent to a linear system, then the conclusion (4.9) holds. Indeed, suppose that $\sum_j H_j \circ \varphi_j \equiv 0$. Supposing initially that the derivatives of $H_j$ vanish nowhere, the change of variables $x \mapsto (H_1 \circ \varphi_1(x), H_2 \circ \varphi_2(x))$ reduces matters to the case in which $\varphi_j(x) \equiv x_j$ for $j = 1, 2$. Replace $\varphi_3$ by $\tilde{\varphi}_3 = -H_3 \circ \varphi_3$. In these new coordinates, $\tilde{\varphi}_3(x) = x_1 + x_2$, and the nondegeneracy hypothesis for $\psi$ continues to hold. For this situation, the conclusion (4.3) was established in [12].

In the more general case in which derivatives $H'_j$ are permitted to vanish at isolated points, and gradients $\nabla \varphi_j$ are permitted to be pairwise linearly dependent on analytic varieties of positive codimensions, the same conclusion is reached by partitioning $[0, 1]^2$ into finitely many good rectangles, on each of which each derivative has absolute value bounded below by $|\lambda|^{-\delta}$, together with a bad set of Lebesgue measure $O(|\lambda|^{-\delta'})$ for small exponents $\delta, \delta' > 0$. The reasoning of the preceding paragraph gives the desired bound for the contribution of each good rectangle, while the contribution of the remaining bad set is majorized by a constant multiple of its Lebesgue measure.

We claim further that in order to prove Theorem 4.4 it suffices to treat the special case in which $\varphi_j(x_1, x_2) \equiv x_j$ for $j = 1, 2$, neither partial derivative $\frac{\partial \varphi_j}{\partial x_j}$ with $j = 1, 2$ vanishes at any point of $[0, 1]^2$, and $(\varphi_j : j \in \{1, 2, 3\})$ is not equivalent to a linear system. To justify this claim, let $\varepsilon > 0$ be a small auxiliary parameter, and partition $[0, 1]^2$ into subcubes of sidelengths comparable to $\lambda^{-\varepsilon}$. Discard every subcube on which any one of the three Jacobian determinants fails to have magnitude greater than $\lambda^{-\varepsilon}$. The sum of the measures of these discarded subcubes is $O(\lambda^{-\delta})$ for some $\delta = \delta(\varepsilon) > 0$. Treat each of the remaining subcubes by reducing it to $[0, 1]^2$ via an affine change of variables. This replaces $\lambda$ by a positive power of $\lambda$, and likewise modifies $\varphi_j, \psi$.

Next, make the change of variables $x = (x_1, x_2) \mapsto \phi(x) = (\varphi_1(x), \varphi_2(x))$, which is a local diffeomorphism because of the nonvanishing Jacobian condition. Replace $\varphi_3$ by $\varphi_3 \circ \phi^{-1}$, replace $\varphi_j(x)$ by $x_j$ for $j = 1, 2$, and replace $\psi$ by $\psi \circ \phi^{-1}$. The hypotheses of Theorem 4.4 continue to hold for this new system of data. $\phi([0, 1]^2)$ is no longer equal to $[0, 1]^2$, but is contained in a finite union of rectangles, in each of which the hypotheses of the theorem hold after affine changes of variables.

This change of variables introduces a Jacobian factor, which is a function of $x$ rather than of individual coordinates. This Jacobian can be expanded into a Fourier series, expressing it as an absolutely convergent linear combination of products of unimodular functions of the individual coordinates. These factors can be absorbed into the functions $f_j$. The case in which $\varphi_j : j \in \{1, 2, 3\}$ is equivalent to a linear system has already been treated.

Write $D = \frac{\partial}{\partial x}$. 

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Definition 6.1. Let $\lambda \in (0, \infty)$ and $N \in \mathbb{N}$. $\| \cdot \|_{N, \lambda}$ is the norm on the Banach space of $N$ times continuously differentiable functions on $[0, 1]$ given by

$$
\| f \|_{N, \lambda} = \sum_{k=0}^{N} \lambda^{-k} \| D^k f \|_{L^\infty([0,1])}.
$$

\[6.2\]

Sections 8, 9, and 11 are devoted to the proof of the following lemma.

Lemma 6.1. Suppose that $\varphi_j(x_j) \equiv x_j$ for $j = 1, 2$, that $\varphi_3$ is not expressible in the form $h_1(x_1) + h_2(x_2)$, and that $\psi$ is not expressible in the form (4.8). Then there exist $N, C, \delta$ such that for all $f$ and every $\lambda \geq 1$,

$$
\| S_\lambda(f) \| \leq C \lambda^{-\delta} \| f_1 \|_\infty \| f_2 \|_\infty \| f_3 \|_{N, \lambda}.
$$

(6.3)

We have observed that

$$
T^\phi_\lambda(g) = \lambda^{-1/2} S_\lambda(f)
$$

with $f_j = g_j$ for $j = 1, 2$, $\| f_3 \|_2 \leq C \| g_3 \|_2$, and $f_3$ is $|\lambda|$-bandlimited. Therefore in order to complete the proof of Theorem 5.1, it suffices to prove that $|S_\lambda(f)| \leq C |\lambda|^{-1/2} \| f_1 \|_\infty \| f_2 \|_\infty \| f_3 \|_2$ under this bandlimitedness assumption on $f_3$.

Theorem 5.1 follows from this lemma. Indeed, we may assume without loss of generality that $\lambda > 0$, by replacing $\psi$ by $-\psi$ if $\lambda$ is initially negative. Since $f_3$ is $\lambda$-bandlimited, we may express $f_3 = P_\lambda f_3$, where $P_\lambda$ are linear smoothing operators that satisfy

$$
\| \nabla^k P_\lambda f \|_q \leq C_{q,k} \lambda^k \| f \|_q
$$

for all $f \in L^q$ uniformly for all $q \in [1, \infty]$ and $\lambda > 0$, for each $k \in \{0, 1, 2, \ldots\}$. Thus

$$
\| P_\lambda f \|_{N, \lambda} \leq C_N \| f \|_\infty
$$

uniformly for all $\lambda > 0$ and $f \in L^\infty$.

The hypothesis that $\partial \varphi / \partial x_1$ does not vanish leads immediately to an upper bound

$$
|S_\lambda(f)| \leq C \| f_1 \|_\infty \| f_2 \|_1 \| f_3 \|_1,
$$

and interchanging the roles of the coordinates gives a bound $C \| f_1 \|_\infty \| f_j \|_1 \| f_k \|_1$ for any permutation $(i, j, k)$ of $(1, 2, 3)$. Therefore by interpolation, since $P_\lambda$ is bounded on $L^q$ for all $q$ uniformly in $\lambda$,

$$
|S_\lambda(f_1, f_2, P_\lambda(f_3))| \leq C \lambda^{-\delta} \| f_1 \|_\infty \| f_2 \|_\infty \| f_3 \|_2.
$$

(6.7)

By (6.4), this completes the proof of Theorem 5.1.



7. Microlocal decomposition

We decompose each $f_j$ in phase space into summands that are essentially supported in rectangles of dimensions $(\lambda^{-1/2}, \lambda^{1/2})$ in $[0, 1]_x \times \mathbb{R}_\xi$. To do this, partition $[0, 1]$ into $\lambda^{1/2}$ intervals $I_m$ of lengths $|I_m| = \lambda^{-1/2}$. Let $\eta_m$ be $C^\infty$ functions with each $\eta_m$ supported on the interval $I_m^*$ of length $2\lambda^{-1/2}$ concentric with $I_m$, with $\sum_m \eta_m^2 \equiv 1$ on $[0, 1]$, and with $d^k \eta_m / dx^k = O(\lambda^{k/2})$ for each $k \geq 0$.

For $\nu = (m_1, m_2)$ let $Q_\nu = I_m \times I_m \subset [0, 1] \times [0, 1]$. Let $z_\nu$ be the center of $Q_\nu$. To each $\nu$ are associated those intervals $I_{m_3}$ for which there exists at least one point $x = (x_1, x_2) \in Q_\nu$ such that $\varphi(x) \in I_{m_3}^*$. Because each partial derivative $\partial \varphi / \partial x_j$ vanishes nowhere, the number of such indices $m_3$ is majorized by a constant independent of $\lambda, \nu$. 

Let $\sigma \in (0, 1]$ be a small quantity to be chosen at the very end of the analysis. For each interval $I_m$, decompose $f_j \eta_m^2$ as

$$f_j \eta_m^2 = g_{j,m} + h_{j,m}$$

with $g_{j,m}, h_{j,m}$ identically zero outside of $I_m$,

$$g_{j,m}(x) = \eta_m(x) \sum_{k=1}^{N} a_{j,m,k} e^{i \xi_{j,m,k} x}$$

$$|a_{j,m,k}| = O(\|f_j\|_\infty),$$

$$\xi_{j,m,k} \in [\pi \lambda^{1/2} \mathbb{Z}],$$

$$N = \lfloor \lambda^{2\sigma} \rfloor.$$

while

$$h_{j,m}(x) = \eta_m(x) \sum_{n \in \mathbb{Z}} b_{j,m,n} e^{i \pi \lambda^{1/2} n x}$$

$$\left( \sum_n |b_{j,m,n}|^2 \right)^{1/2} = O(\|f_j\|_\infty)$$

$$|b_{j,m,n}| = O(\lambda^{-\sigma} \|f_j\|_\infty).$$

Decompositions of this type were used by the author and J. Holmer, in unpublished work circa 2009, to prove upper bounds for certain generalizations of twisted convolution inequalities.

This is achieved by expanding $f_j \eta_m$ into Fourier series

$$f_j(x) \eta_m(x) = 1_{I_m^*}(x) \sum_{n \in \mathbb{Z}} c_n e^{i \pi \lambda^{1/2} n x},$$

with coefficients $c_n$ that depend also on the indices $j, m$. Define $g_{j,m}$ to be the sum of all terms with $|c_n| > \lambda^{-\sigma} \|f_j\|_\infty$, multiplied by $\eta_m$. Define $h_{j,m} = f_j \eta_m - g_{j,m}$. By Parseval’s identity, there are at most $\lfloor \lambda^{2\sigma} \rfloor$ values of $n$ for which $|c_n| > \lambda^{-\sigma}$. Define the frequencies $\xi_{j,m,k}$ and associated coefficients $a_{j,m,k}$ to be those frequencies $\pi \lambda^{1/2} n$ and associated coefficients $c_n$ that satisfy $|c_n| > \lambda^{-\sigma} \|f_j\|_\infty$, with some arbitrary ordering. If there are fewer than $N$ indices $n$ for which $|c_n| > \lambda^{-\sigma} \|f_j\|_\infty$, then augment this list by introducing extra indices $k$ so that there are exactly $N$ terms, and set some $a_{j,m,k} = 0$ for each of these extra indices. This is done purely for convenience of notation.

Define

$$g_j = \sum_m g_{j,m} \quad \text{and} \quad h_j = \sum_m h_{j,m} \quad \text{for} \ j \in \{1, 2\}.$$

For $j = 3$, this construction is modified in order to exploit the bandlimited character of $f_3$. Let $\rho > 0$ be another small parameter. It follows from $N$-fold integration by parts that

$$|\widehat{f_3 \eta_m}(\xi)| \leq C_N \lambda^N \xi^{-N} \|f_3\|_{N, \lambda} \forall \xi.$$

If $N$ is chosen to satisfy $N \geq \rho^{-1}$, it follows that

$$|\widehat{f_3 \eta_m}(\xi)| \leq C_N \lambda^{-1} \quad \text{whenever} \ |\xi| \geq \lambda^{1+\rho}.$$

One may think of $\rho$ as being arbitrarily small, but of $\sigma$ as moderate in size. Thus factors such as $\lambda^{-\sigma + C\rho}$ will be small for large $\lambda$, so long as $C$ remains constant.
Therefore the frequencies $\xi_{3,m,k}$ defined above satisfy
\begin{equation}
|\xi_{3,m,k}| \leq \lambda^{1+\rho}.
\end{equation}

Moreover, if $N$ is chosen sufficiently large as a function of $\rho$, then the contribution made to $h_3$ by all terms $b_{3,m,k}e^{ikx}$ with $|k| \geq \lambda^{1+\rho}$ has $L^2$ norm $O(\lambda^{-1})$. Define $F_3$ to be the sum of all of these terms. Then $f_3$ is decomposed as
\begin{equation}
f_3 = g_3 + h_3 + F_3,
\end{equation}
with
\begin{equation}
\|F_3\|_\infty = O(\lambda^{-1}),
\end{equation}
with $g_3, h_3$ enjoying all of the properties indicated above for $j = 1, 2$, and with the supplementary bandlimitedness property
\[|n| \leq \lambda^{1+\rho}\]
for all frequencies $n$ appearing in terms $\eta_m(x)b_{3,m,n}e^{inx}$, as well as for all frequencies $\xi_{3,m,k}$.

8. Local bound

Recall that
\[S_\lambda(F_1, F_2, F_3) = \int_{\mathbb{R}^2} F_1(x_1)F_2(x_2)F_3(\varphi(x_1, x_2)) e^{i\lambda \psi(x_1, x_2)} \, dx_1 \, dx_2.\]

Let $m = (m_1, m_2, m_3) \in \mathbb{Z}^3$. Write $\|a_r\|_{\ell^p} = (\sum_n |a_{r,n}|^p)^{1/p}$, with the usual limiting interpretation for $p = \infty$.

**Lemma 8.1.** Let $\rho > 0$ be a small auxiliary parameter. Let $f_j$ be functions of the form
\[f_j(x) = \sum_{n \in \mathbb{Z}} a_{j,n} e^{i\pi \lambda^{1/2} nx} \]
with $a_{j,n} \in \mathbb{C}$ and $|a_{3,n}| = 0$ for all $|n| > \lambda^{1+\rho}$. Then for each $m$ and any permutation $(j,k,l)$ of $(1,2,3)$,
\begin{equation}
|S_\lambda(f_1\eta_{m_1}, f_2\eta_{m_2}, f_3\eta_{m_3})| \leq C\lambda^{-1+2\rho}\|a_j\|_{\ell^2}\|a_k\|_{\ell^2}\|a_l\|_{\ell^\infty}
\end{equation}

**Proof.** For $i = 1, 2$ we write $\varphi_i, \psi_i$ as shorthand for $\partial \varphi/\partial x_i, \partial \psi/\partial x_i$, respectively. Write $\nu = (m_1, m_2)$, and recall that $z_\nu$ denotes the center of $Q_\nu = I_{m_1} \times I_{m_2}$. Write $x = (x_1, x_2)$.

Let $\xi = (\xi_j : j \in \{1,2,3\}) \in \mathbb{R}^3$, and suppose that
\begin{equation}
\max_j |\xi_j| \leq \lambda^{1+\rho}.
\end{equation}

Consider
\begin{equation}
I(\xi) = \int_{\mathbb{R}^2} e^{i\xi_1 x_1} e^{i\xi_2 x_2} e^{i\xi_3 \varphi(x)} e^{i\lambda \psi(x)} \eta_{m_1}(x_1)\eta_{m_2}(x_2)\eta_{m_3}(\varphi(x)) \, dx.
\end{equation}
The net phase function in this integral is
\begin{equation}
\Phi(x) = \xi_1 x_1 + \xi_2 x_2 + \xi_3 \varphi(x) + \lambda \psi(x),
\end{equation}
whose gradient is
\[\nabla \Phi(x) = \left(\begin{array}{c}
\xi_1 + \xi_3 \varphi_1(x) + \lambda \psi_1(x) \\
\xi_2 + \xi_3 \varphi_2(x) + \lambda \psi_2(x)
\end{array}\right).\]

If
\begin{equation}
|\nabla \Phi(z_\nu)| \geq \lambda^{\rho} \lambda^{1/2}
\end{equation}
Thus matters are reduced to the analysis of

\[ |I| \leq C_{p,K} \lambda^{-K} \text{ for every } K < \infty. \]

Indeed, if \((8.5)\) holds then \(\frac{\partial \Phi}{\partial x_i}(z_\nu) \geq \frac{1}{2} \lambda^2 \lambda^{1/2} \) for at least one index \(i \in \{1, 2\}\). Suppose without loss of generality that this holds for \(i = 1\). Then

\[ (8.7) \quad \left| \frac{\partial \Phi}{\partial x_1}(u_1, u_2) \right| \geq \lambda^\rho \lambda^{1/2} \text{ for every point } (u_1, u_2) \in Q^*_{\nu} = I^*_{m_1} \times I^*_{m_2}. \]

This holds because the function \(\lambda \psi_1\) varies by at most \(O(\lambda \cdot \lambda^{-1/2})\) over \(Q^*_{\nu}\), while the assumption \((8.2)\) guarantees that \(\xi_3 \varphi_1\) varies by at most \(O(\lambda^{1+\rho} \lambda^{-1/2}) = (\lambda^{2+\rho})\). Integrating by parts \(C K \rho^{-1}\) times with respect to the \(x_1\) coordinate and invoking \((8.6)\) then yields \((8.7)\).

Now writing \(n = (n_1, n_2, n_3) \in \mathbb{Z}^3\),

\[ S_\lambda(f_1 \eta_{m_1}, f_2 \eta_{m_2}, f_3 \eta_{m_3}) = \sum_{n_1, n_2, n_3} a_{n_1} a_{n_2} a_{n_3} I(\pi \lambda^{1/2} n_1, \pi \lambda^{1/2} n_2, \pi \lambda^{1/2} n_3). \]

For any \(n = (n_1, n_2, n_3)\) there is the trivial bound \(|I(n)| = O(|Q^*_{\nu}|) = O(\lambda^{-1})\). On the other hand, by \((8.6)\), the \(n\)-th term is \(O(\lambda^{-K})\) if the associated phase function \(\Phi\) defined by \((8.4)\) with \(\xi = \pi \lambda^{1/2} n\) satisfies \(|\nabla \Phi(z_\nu)| \geq \lambda^{1/2+2\rho}\).

For each \(n_1\), there are \(O(\lambda^{2\rho})\) pairs \((n_2, n_3)\) for which \(\xi = \pi \lambda^{1/2} n\) fails to satisfy \((8.5)\). This follows from the form of \(\nabla \Phi\) and the assumption that both partial derivatives \(\varphi_1, \varphi_2\) are nowhere vanishing. The same holds with the roles of \(n_1, n_2, n_3\) permuted in an arbitrary way. Since the total number of all tuples \((m, n)\) is \(O(\lambda^{3+2\rho})\), the conclusion of the lemma follows directly from these facts by invoking \((8.6)\) with \(K\) sufficiently large. \(\square\)

9. Reduction to sublevel set bound

Let \(f_1, f_2\) be decomposed as \(f_j = g_j + h_j\) as in \((7.1), (7.2)\), and let \(f_3\) have the modified form \(f_3 = g_3 + h_3 + F_3\) of \((7.5)\), with the restriction \((7.4)\). Then \(S_\lambda(f_1, f_2, F_3) = O(\lambda^{-1} \prod_{j=1}^{3} \|f_j\|_{\infty})\), so the contribution of \(F_3\) can be disregarded and \(f_3\) may be replaced by \(\tilde{f}_3 = g_3 + h_3\). By summing over all cubes \(Q_\nu\) we conclude from Lemma \((8.1)\) that

\[ (9.1) \quad |S_\lambda(h_1, f_2, \tilde{f}_3)| \leq C \lambda^{-\sigma} \lambda^{2\rho} \prod_{j=1}^{3} \|f_j\|_{\infty}. \]

In the same way,

\[ (9.2) \quad |S_\lambda(g_1, h_2, \tilde{f}_3)| + |S_\lambda(g_1, g_2, h_3)| \leq C \lambda^{-\sigma} \lambda^{2\rho} \prod_{j=1}^{3} \|f_j\|_{\infty} \]

so that

\[ (9.3) \quad |S_\lambda(f_1, f_2, \tilde{f}_3)| \leq |S_\lambda(g_1, g_2, g_3)| + C \lambda^{-\sigma} \lambda^{2\rho} \prod_{j=1}^{3} \|f_j\|_{\infty}. \]

Thus matters are reduced to the analysis of \(S_\lambda(g_1, g_2, g_3)\).

To complete the proof, we analyze functions of the special form

\[ (9.4) \quad G_j(x) = \sum_m \eta_m(x) a_{j,m} e^{ix \cdot \xi_{j,m}} \]
with each $a_{j,m} \in \mathbb{C}$ satisfying $|a_{j,m}| \leq 1$, and each $\xi_{j,m} \in \mathbb{R}$. For $j = 3$, we also assume

\begin{equation}
|\xi_{3,m}| \leq \lambda^{1+\rho}.
\end{equation}

For each index $j$, $g_j$ is expressed as a sum over $k_j \in \{1, 2, \ldots, N\}$ of functions $G_j$ of the form (9.4), multiplied by $O(\|f_j\|_\infty)$. Moreover, each summand $G_3$ is bandlimited in the sense (9.5).

\begin{equation}
S_\lambda(g_1, g_2, g_3) = O(\prod_{j=1}^{3} \|f_j\|_\infty) \cdot \sum_{(k_1, k_2, k_3) \in \{1, 2, \ldots, N\}^3} |S_\lambda(G_{1,k_1}, G_{2,k_2}, G_{3,k_3})|
\end{equation}

with $N^3$ terms in the sum.

We will prove:

**Lemma 9.1.** There exist $\tau_0 > 0$ and $C < \infty$ such that for all functions of the form (9.4) satisfying also (9.5),

\begin{equation}
|S_\lambda(G_1, G_2, G_3)| \leq C\lambda^{-\tau_0}
\end{equation}

uniformly for all real $\lambda \geq 1$.

Taking Lemma 9.1 for granted for the present, we can now complete the proof of Theorem 4.4 in the $O(|\lambda|)$–bandlimited case, and hence the proof of Theorem 5.1. Applying Lemma 9.1 to each of the $N^3$ summands in (9.6) yields

\begin{equation}
|S_\lambda(g_1, g_2, g_3)| \leq CN^3\lambda^{-\tau_0} \prod_{j=1}^{3} \|f_j\|_\infty.
\end{equation}

In all,

\begin{equation}
|S_\lambda(f_1, f_2, f_3)| \leq (CN^3\lambda^{-\tau_0} + C\lambda^{-\sigma} \lambda^{2\rho}) \prod_{j=1}^{3} \|f_j\|_\infty
\end{equation}

\begin{align*}
&\leq (C\lambda^{6\sigma} \lambda^{-\tau_0} + C\lambda^{-\sigma} \lambda^{2\rho}) \prod_{j=1}^{3} \|f_j\|_\infty,
\end{align*}

where $C < \infty$ depends only on $\varphi, \psi$ and the auxiliary parameters $\sigma, \rho > 0$. The exponent $\sigma$ remains at our disposal, while $\rho$ may be taken to be arbitrarily small. Choosing $\sigma = \tau_0 / 7$ gives

\begin{equation}
|S_\lambda(f_1, f_2, f_3)| \leq C\lambda^{-\tau} \prod_{j=1}^{3} \|f_j\|_\infty
\end{equation}

for every $\tau < \tau_0 / 7$. \hfill \Box

We next reduce Lemma 9.1 to a sublevel set bound. Let $G_j$ have the above form for $j \in \{1, 2, 3\}$. By decomposing $G_3$ as a sum of $O(1)$ subsums, we may assume that for each $\nu = (m_1, m_2)$ there exists at most one index $m_3 = m_3(\nu)$ for which the product $\eta_{m_1}(x_1)\eta_{m_2}(x_2)\eta_{m_3}(\varphi(x_1, x_2))$ does not vanish identically.

For $\nu = (m_1, m_2)$ and for $m = (m_1, m_2, m_3(\nu))$, for each $(x_1, x_2) \in Q_\nu$ define

$$\Phi_\nu(x_1, x_2) = \xi_{1,m_1}x_1 + \xi_{2,m_2}x_2 + \xi_{3,m_3}\varphi(x_1, x_2) + \lambda\psi(x_1, x_2).$$
Decompose $S_{\lambda}(G_1, G_2, G_3)$ as
\[ (9.11) \quad \sum_{\nu} a_{1,m_1} a_{2,m_2} a_{3,m_3} \int e^{i\Phi_\nu(x_1,x_2)} \eta_{m_1}(x_1) \eta_{m_2}(x_2) \eta_{m_3}(\varphi(x_1,x_2)) \, dx \]
with $\nu, m = (m_1, m_2, m_3)$ related as above. This sum is effectively taken over either a single index $m_3 = m_3(\nu)$, or over an empty set of indices $m_3$. Indices $\nu$ of the latter type may be dropped.

For each remaining $\nu$, the integral in (9.11) is $O(\lambda^{-K})$ for every $K$ unless $|\nabla \Phi_\nu(z_\nu)| \leq \lambda^{2\rho} \lambda^{1/2}$.

**Definition 9.1.** The sublevel set $E^\nu(K)$ is the union of all $Q_\nu$ for which $|\nabla \Phi_\nu(z_\nu)| \leq \lambda^{2\rho} \lambda^{1/2}$.

The contribution of each such $Q_\nu$ to $S_{\lambda}(G_1, G_2, G_3)$ is $O(|Q_\nu| \prod_j \|f_j\|_{\infty})$. Therefore
\[ (9.12) \quad |S_{\lambda}(G_1, G_2, G_3)| = O(\lambda^{-K} + |E^\nu|) \prod_j \|f_j\|_{\infty}. \]

To complete the proof of Lemma 9.1 and hence the proofs of Theorems 4.4 and 5.1 it suffices to show that there exists $\tau_0 > 0$ such that
\[ (9.13) \quad |E^\nu| = O(\lambda^{-\tau_0}) \]
uniformly in all possible choices of functions $m_j \mapsto \xi_{j,m_j}$.

10. Interlude

A connection between oscillatory integral bounds of the form
\[ (10.1) \quad | \int_B e^{i\lambda \varphi} \prod_{j \in J} (f_j \circ \varphi_j) | \leq \Theta(\lambda) \prod_{j \in J} \|f_j\|_{\infty}, \]
where $\Theta(\lambda) \to 0$ as $|\lambda| \to \infty$, and bounds for Lebesgue measures of sublevel sets
\[ (10.2) \quad E = \{ x \in B : |\psi(x) - \sum_j (g_j \circ \varphi_j)(x) | < \varepsilon \}, \]
of the form
\[ (10.3) \quad |E| \leq \theta(\varepsilon) \]
where $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ with $\theta(\varepsilon)$ independent of $(g_j)$, is well known. The former implies the latter: Fix an auxiliary compactly supported $C^\infty$ function $\zeta : \mathbb{R} \to [0, \infty)$ satisfying $\zeta(t) = 1$ for $|t| \leq 1$. Then
\[ |E| \leq \int_B \zeta(\varepsilon^{-1}(\psi - \sum_j (g_j \circ \varphi_j))) = \int_{\mathbb{R}} \zeta(t) \left( \int_B e^{2\pi i (t/\varepsilon) \varphi(x)} \prod_{j \in J} (f_{j,t} \circ \varphi_j)(x) \, dx \right) \, dt \]
with $f_{j,t} = e^{-2\pi i (t/\varepsilon) g_j}$. Rewriting this as
\[ \int_{\mathbb{R}} e^{\zeta(\varepsilon \lambda)} \left( \int_B e^{2\pi i \lambda \psi(x)} \prod_{j \in J} (f_{j,\varepsilon \lambda} \circ \varphi_j)(x) \, dx \right) \, d\lambda \]
and invoking (10.1) gives (10.3).

The analysis in this paper proceeds primarily in the opposite sense, using sublevel set bounds to deduce bounds for oscillatory integrals. However, the sublevel sets that arise here are variants of those defined by (10.2), in which $\nabla \varphi$ appears, rather than $\varphi$ itself. The reasoning in the preceding paragraph is elaborated in (18) to establish an inverse theorem,
roughly characterizing tuples \((g_j : \mathbb{R}^2 \to \mathbb{R})\) for which associated sublevel sets \(E = \{x \in B : |\sum_{j=1}^{3}(g_j \circ \varphi_j(x))| < \varepsilon\}\) are relatively large.

Sublevel set bounds of the type (10.3), with \(E\) defined by (10.2) have been established in certain cases [6], with \(\varphi_j : \mathbb{R}^D \to \mathbb{R}\) and \(|J|\) arbitrarily large relative to \(D\), as consequences of an extension of Szemerédi’s theorem due to Furstenberg and Katznelson.

11. Proof of a sublevel set bound

Continue to denote by \(\varphi, \psi\) the partial derivatives of \(\varphi, \psi\) with respect to \(x_j\) for \(j = 1, 2\), respectively. The following lemma is essentially a restatement of the desired bound \(|E| = O(\lambda^{-\gamma_0})\), with the substitutions

\[
h_j = \lambda^{-1} \sum_{m} \xi_{j,m} 1_m
\]

and \(\varepsilon = \lambda^{\delta_0}\).

**Lemma 11.1.** Let \((\varphi, \psi)\) satisfy the hypotheses of Theorem 4.4. Suppose that the ordered triple of mappings \((x_1, x_2) \mapsto (x_1, x_2, \varphi(x_1, x_2))\) is not equivalent to a linear system. Then there exist \(C < \infty\) and \(\rho > 0\) with the following property. Let \(h_j\) be real-valued Lebesgue measurable functions, and let \(\varepsilon \in (0, 1]\). Let \(E\) be the set of all \((x, y) \in [0, 1]^2\) that satisfy

\[
\begin{align*}
|h_1(x) + \varphi_1(x, y)h_3(\varphi(x, y)) + \psi_1(x, y)| &\leq \varepsilon \\
|h_2(y) + \varphi_2(x, y)h_3(\varphi(x, y)) + \psi_2(x, y)| &\leq \varepsilon.
\end{align*}
\]

Then

\[
|E| \leq C \varepsilon^\rho.
\]

The upper bound (9.13) for the measure of the set \(E\) of Definition defn:sublevelset is an immediate consequence of (11.3), so Lemma 11.1 suffices to complete the proof of Theorem 4.4.

The proof of Lemma 11.1 relies on the next lemma, which should be regarded as being well known, though it is more often formulated only for the special case of families of polynomials of bounded degree. We write \(F_\omega(x) = F(x, \omega)\).

**Lemma 11.2.** Let \(\Omega\) be a compact topological space, let \(K \subset \mathbb{R}^D\) be a compact convex set with nonempty interior, and let \(V \subset \mathbb{R}^D\) be an open set containing \(K\). Assume that \(F_\omega \in C^\omega(V)\) for each \(\omega \in \Omega\), and that the mappings \((x, \omega) \mapsto \partial_\omega^\alpha F(x, \omega)\) are continuous for every multi-index \(\alpha\). Suppose further that none of the functions \(F_\omega\) vanish identically on \(K\). Then there exist \(\tau > 0\) and \(C < \infty\) such that for every \(\varepsilon > 0\) and every \(\omega \in \Omega\),

\[
|\{x \in K : |F_\omega(x)| < \varepsilon\}| \leq C \varepsilon^\tau.
\]

**Proof.** A simple compactness and slicing argument reduces matters to the case in which \(D = 1\) and \(K\) has a single element. A proof for that case is implicit in proofs of van der Corput’s lemma concerning one-dimensional oscillatory integrals, for instance in [24] and [30]. For a derivation as a corollary of bounds for oscillatory integrals, see [5], page 14. \(\square\)

The following simple result will be used repeatedly.

**Lemma 11.3.** Let \((X, \mu)\) and \((Y, \nu)\) be probability spaces. Let \(\lambda = \mu \times \nu\). Let \(E \subset X \times Y\) satisfy \(\lambda(E) > 0\). Define

\[
\tilde{E} = \{x \in X : \nu(\{y : (x, y) \in E\}) \geq \frac{1}{2} \lambda(E)\}.
\]
There exists \( y_0 \in Y \) such that
\[
\lambda(\{(x, y) \in E : x \in \tilde{E} \text{ and } (x, y_0) \in E\}) \geq \frac{1}{8} \lambda(\tilde{E})^2.
\]

**Proof.** Since
\[
\lambda(\tilde{E} \setminus (E \cap (\tilde{E} \times Y))) \leq \int_{Y} \frac{1}{2} \lambda(E) \, d\nu = \frac{1}{2} \lambda(\tilde{E}),
\]
one has \( \lambda(\tilde{E} \setminus (E \times Y)) \geq \frac{1}{2} \lambda(E) \) and therefore \( \mu(\tilde{E}) \geq \frac{1}{4} \lambda(E) \).
Consider
\[
E^* = \{(x, y, y') : x \in \tilde{E}, (x, y) \in E, \text{ and } (x, y') \in E,\}
\]
which satisfies \( (\mu \times \nu \times \nu)(E^*) \geq \frac{1}{4} \lambda(E)^2 \). Indeed, by the Cauchy-Schwarz inequality,
\[
\frac{1}{4} \lambda(E)^2 \leq \lambda(E \cap (\tilde{E} \times Y))^2
\]
\[
= \left( \int_{\tilde{E}} \int_{Y} 1_E(x, y) \, d\nu(y) \, d\mu(x) \right)^2
\]
\[
\leq \int_{\tilde{E}} \int_{Y} 1_E(x, y) \, d\nu(y) \int_{Y} 1_E(x, y') \, d\nu(y')
\]
\[
= (\mu \times \nu \times \nu)(E^*).
\]
The stated conclusion now follows from Fubini’s theorem. \(\square\)

**Proof of Lemma [1.1.1]**. There is a \( C^\omega \) function \( \kappa_1(x, t) \) satisfying
\[
\varphi(x, \kappa_1(x, t)) \equiv t.
\]
The hypothesis that \( \partial \varphi / \partial x_2 \) vanishes nowhere implies that uniformly for all Lebesgue measurable sets \( A \), \( |\{(x, t) : (x, \kappa_1(x, t)) \in A\}| \) is comparable to \( |A| \). Likewise, there exists \( \kappa_2 \) satisfying
\[
\varphi(\kappa_2(t, y), y) \equiv t
\]
with \( |\{(y, t) : (\kappa_2(t, y), y) \in A\}| \) comparable to \( |A| \) for all measurable \( A \).

Define
\[
E_0 = \{(x, y) \in E : |h_3(\varphi(x, y))| \leq 1\}.
\]

For \( N \geq k > 0 \) let
\[
E_k = \{(x, y) \in E : 2^{k-1} < |h_3(\varphi(x, y))| \leq 2^k\}.
\]
It follows immediately from [1.1.2] that \( |h_1(x)| \) and \( |h_2(y)| \) are \( O(2^k) \) whenever \( (x, y) \in E_k \).
We will show that \( |E_k| = O(2^{-k\varepsilon \varepsilon}) \). Summation with respect to \( k \) then yields [1.1.3].

Consider first \( E_0 \). Define
\[
E'_0 = \{x \in [0, 1] : |\{(t : (x, \kappa_1(x, t))) \in E_0\}| \geq c_0 |E_0|\}.
\]

By Lemma [1.1.3] there exists \( t_0 \) such that the set
\[
E''_0 = \{(x, t) : x \in E'_0 \text{ and } (x, \kappa_1(x, t)) \in E_0 \text{ and } (x, \kappa_1(x, t_0)) \in E_0\}
\]
satisfies \( |E''_0| \geq c |E_0|^2 \), where \( c > 0 \) is a constant that depends on the function \( \kappa_1 \), but not on \( |E_0| \).

Define \( \alpha = h_3(t_0) \). By definition of \( E_0 \), \( \alpha \in [-1, 1] \). For every \( (x, t) \in E_0 \),
\[
|h_1(x) + \alpha \varphi_1(x, \kappa_1(x, t_0)) + \psi_1(x, \kappa_1(x, t_0))| \leq \varepsilon.
\]
Define
\[
\tilde{h}_1(x) = -\alpha \varphi_1(x, \kappa_1(x, t_0)) - \psi_1(x, \kappa_1(x, t_0)).
\]
For any \((x, t) \in E_0'\),
\[
|h_1(x) + \varphi_1(x, \kappa_1(x, t))h_3(t) + \psi_1(x, \kappa_1(x, t))| \leq 2\varepsilon
\]
by (11.6), the inequality
\[
|h_1(x) + \varphi_1(x, \kappa_1(x, t))h_3(t) + \psi_1(x, \kappa_1(x, t))| \leq \varepsilon \text{ whenever } (x, \kappa_1(x, t)) \in E_0,
\]
and the triangle inequality.

The function \(h_1\) belongs to a compact family of \(C^\omega\) functions of \(x \in [0, 1]\), parametrized by \(\alpha, t_0\). This family is defined solely in terms of \(\varphi, \psi\). Defining
\[
E_0^{(1)} = \{(x, \kappa_1(x, t)) : (x, t) \in E_0'\},
\]
one has \(|E_0^{(1)}| \geq c|E_0|^2\) and
\[
|h_1(x) + \varphi_1(x, y)h_3(\varphi(x, y)) + \psi_1(x, y)| \leq 2\varepsilon \text{ for all } (x, y) \in E_0^{(1)}.
\]

Repeating this reasoning with the roles of the two coordinates \(x, y\) interchanged and with \(E_0\) replaced by \(E_0^{(1)}\), we conclude that there exist a subset \(E_0^{(2)} \subset E_0^{(1)} \subset [0, 1]^2\) satisfying \(|E_0^{(2)}| \geq |E_0|^4\), and a function \(h_2\) belonging to a compact family of \(C^\omega\) functions defined solely in terms of \(\varphi, \psi\), that satisfy
\[
|h_2(y) + \varphi_2(x, y)h_3(\varphi(x, y)) + \psi_2(x, y)| \leq 2\varepsilon \text{ for all } (x, y) \in E_0^{(2)}.
\]
The condition that \((x, y) \in E_0\) directly provides an upper bound \(|h_3(\varphi(x, y))| \leq 1\). It also implies upper bounds for \(|h_j(x, y)| \leq C < \infty\) for \(j = 1, 2\) via the inequalities (11.2) and the assumption that \(\varepsilon \leq 1\).

A third iteration of this reasoning yields a set \(E_0^{(3)} \subset E_0^{(2)}\) and a function \(h_3\) of the special form
\[
\tilde{h}_3(x) = -[\varphi_1(\kappa_2(x, s), s)]^{-1}(\alpha - \psi_1(\kappa_2(x, s), s))
\]
for some parameters \(s \in [0, 1]\) and \(\alpha \in \mathbb{R}\), satisfying
\[
\begin{align*}
|h_1(x) + \varphi_1(x, y)\tilde{h}_3(\varphi(x, y)) + \psi_1(x, y)| & \leq C\varepsilon \\
|h_2(y) + \varphi_2(x, y)\tilde{h}_3(\varphi(x, y)) + \psi_2(x, y)| & \leq C\varepsilon
\end{align*}
\]
for all \((x, y) \in E_0^{(3)}\), with \(|E_0^{(3)}| \geq c|E_0|^8\). Again, \(\tilde{h}_3\) belongs to a compact family of \(C^\omega\) functions that is defined in terms of \(\varphi, \psi\) alone.

Define
\[
\begin{align*}
h_1^{s, \alpha}(x) &= -\alpha\varphi_1(x, \kappa_1(x, s))h_3(s) - \psi_1(x, \kappa_1(x, s)) \\
h_2^{s, \alpha}(y) &= -\alpha\varphi_2(\kappa_2(y, s), y)h_3(s) + \psi_2(\kappa_2(y, s), y) \\
h_3^{s, \alpha}(u) &= -\varphi_1(\kappa_2(u, s), s)]^{-1}(\alpha - \psi_1(\kappa_2(u, s), s)),
\end{align*}
\]
for some appropriate \(C < \infty\), defined by
\[
F_{(s, \alpha)}(x, y) = \left(\begin{array}{c}
h_1^{s, \alpha}(x) + \varphi_1(x, y)h_3^{s, \alpha}(\varphi(x, y)) + \psi_1(x, y) \\
h_2^{s, \alpha}(y) + \varphi_2(x, y)h_3^{s, \alpha}(\varphi(x, y)) + \psi_2(x, y)
\end{array}\right).
\]

There exist no real-valued functions \(h_3\) in \(C^1\) that satisfy
\[
\begin{align*}
h_1^{\beta}(x) + \varphi_1(x, y)h_3^{\beta}(\varphi(x, y)) + \psi_1(x, y) & \equiv 0 \\
h_2^{\beta}(y) + \varphi_2(x, y)h_3^{\beta}(\varphi(x, y)) + \psi_2(x, y) & \equiv 0
\end{align*}
\]
on $[0,1]^2$. For if there were, then defining $H_j$ to be an antiderivative of $h_j^s$, one would have
\[
\nabla_{x,y}(\psi(x,y) - H_1(x) - H_2(y) - H_3(\varphi(x,y))) \equiv 0,
\]
contradicting the nondegeneracy hypothesis on $(\varphi, \psi)$. Therefore for any $(s, \alpha)$, the function $F(s, \alpha)$ does not vanish identically as a function of $(x, y) \in [0,1]^2$. Lemma [11.2] can now be applied to conclude that $|E_0^{(3)}| \leq C\varepsilon^\tau$, with $C < \infty$ and $\tau > 0$ depending only on $\varphi, \psi$. Therefore
\[
|E_0| \leq C\varepsilon^{\tau/8}
\]
for another constant $C' < \infty$. This completes the analysis of $E_0$.

The same analysis yields an upper bound of the form $|E_k| \leq C2^{-k}e^\varepsilon \varepsilon'$, uniformly for all $k > 0$. Indeed, define $\tilde{h}_j = 2^{-k}h_j$ for $j \in \{1, 2, 3\}$, and set $\tilde{\varepsilon} = 2^{-k}e\varepsilon$, to obtain
\[
\begin{align*}
|\tilde{h}_1(x) + \varphi(x,y)\tilde{h}_3(\varphi(x,y)) + 2^{-k}\psi_1(x,y)| & \leq \tilde{\varepsilon} \\
|\tilde{h}_2(y) + \varphi_2(x,y)\tilde{h}_3(\varphi(x,y)) + 2^{-k}\psi_2(x,y)| & \leq \tilde{\varepsilon}
\end{align*}
\]
for all $(x,y) \in E_k$.

Compactify by considering the system of inequalities
\[
\begin{align*}
|h_1(x) + \varphi(x,y)h_3(\varphi(x,y)) + r\psi_1(x,y)| & \leq \varepsilon' \\
h_2(y) + \varphi_2(x,y)h_3(\varphi(x,y)) + r\psi_2(x,y)| & \leq \varepsilon'
\end{align*}
\]
for arbitrary $r \in [0,1]$ and $\varepsilon' \in [0,\varepsilon_0]$. We may assume that $\varepsilon_0$ is as small as desired.

The situation differs from the analysis of $E_0$ in one respect: For $(x,y) \in E_k$,
\[
\frac{1}{2} \leq |h_3(\varphi(x,y))| \leq 1.
\]
The lower bound, of which we had no analogue in the analysis of $E_0$, will be crucial below.

By repeating the above reasoning, we find that if $h_j$ satisfy (11.17) and (11.18) on some set $\mathcal{E}'$ then there exist functions $\tilde{h}_j$ drawn from a compact family of $C^\omega$ functions associated to $\varphi, \psi$, that satisfy
\[
\begin{align*}
|\tilde{h}_1(x) + \varphi(x,y)\tilde{h}_3(\varphi(x,y)) + r\psi_1(x,y)| & \leq C\varepsilon' \\
|\tilde{h}_2(y) + \varphi_2(x,y)\tilde{h}_3(\varphi(x,y)) + r\psi_2(x,y)| & \leq C\varepsilon'
\end{align*}
\]
for all $(x,y) \in \mathcal{E}'$, with $|\mathcal{E}'| \geq c|\mathcal{E}|^{\tau/8}$. Moreover, the lower bound (11.18) implies that $\|h_3\|_{C^0} \geq \frac{1}{2}$, provided that $\varepsilon_0$ is sufficiently small.

There exists no solution $(\tilde{h}_j : j \in \{1, 2, 3\})$ of the system of equations
\[
\begin{align*}
\tilde{h}_1(x) + \varphi(x,y)\tilde{h}_3(\varphi(x,y)) + r\psi_1(x,y) = 0 \\
\tilde{h}_2(y) + \varphi_2(x,y)\tilde{h}_3(\varphi(x,y)) + r\psi_2(x,y) = 0
\end{align*}
\]
on $[0,1]^2$.

For $r \neq 0$, this follows from the same reasoning as given above for $r = 1$ in the analysis of $E_0$. For $r = 0$, the simplified system
\[
\begin{align*}
 h_1(x) + \varphi(x,y)h_3(\varphi(x,y)) & \equiv 0 \\
h_2(y) + \varphi_2(x,y)h_3(\varphi(x,y)) & \equiv 0
\end{align*}
\]

admits no solutions with $h_3$ vanishing nowhere. For if there were such a solution, defining $H_j$ to be an antiderivative of $\tilde{h}_j$ and adjusting $H_1$ by an appropriate additive constant,
\[
H_3(\varphi(x,y)) + H_1(x) + H_2(y) \equiv 0.
\]
If \( H'_3 = h_3 \) vanishes nowhere, this contradicts the hypothesis that \((x_1, x_2, \varphi(x_1, x_2))\) is not equivalent to a linear system. Thus \( h_3 \) must vanish, contradicting the lower bound (11.18).

By the same reasoning as in the case \( k = 0 \), it follows that \(|E'| \leq C(e')^q\) for a certain exponent \( q > 0 \). Applying this with \( E' = E_k \) and \( e' = 2^{-k} \varepsilon \) gives \(|E_k| \leq C2^{-k}q\varepsilon^2\). Summing over all \( k \geq 0 \) completes the proof of the lemma.

12. Proof of Theorem 4.1

In the deduction of Theorem 5.1 from Theorem 4.4 we were able to immediately gain a factor of \( |\lambda|^{-1/2} \) upon integration with respect to \( x_3 \), reducing matters to a self-contained situation in which a supplementary factor of \( |\lambda|^{-\delta} \) was to be gained. In the framework of Theorem 4.1, the analysis does not split cleanly into two separate steps.

Let \( \tilde{\eta} \) be a \( C_0^\infty \) cutoff function supported in a small neighborhood of \([0, 1]^3\) and identically equal to 1 on \([0, 1]^3\), such that \( \phi \) is real analytic and continues to satisfy the linear independence hypotheses of the theorem in a neighborhood of the support of \( \tilde{\eta} \). Modify the definition of \( T^\phi_\lambda \) to

\[
T^\phi_\lambda(f) = \int_{\mathbb{R}^3} e^{i\lambda \phi(x)} \prod_{j=1}^{3} f_j(x_j) \tilde{\eta}(x) \, dx.
\]

We will show that this modified form satisfies the indicated upper bound as \( \lambda \to +\infty \).

It suffices to prove the conclusion (11.2) with \( \prod_j \|f_j\|_2 \) replaced by \( \prod_j \|f_j\|_\infty \) on the right-hand side. Indeed, the assumption that \( \frac{\partial^2 \phi}{\partial x_1 \partial x_2} \) vanishes nowhere implies that

\[
\left| \int_{[0,1]^2} e^{i\lambda \phi(x_1, x_2, x_3)} f_1(x_1) f_2(x_2) \, dx_1 \, dx_2 \right| \leq C |\lambda|^{-1/2} \|f_1\|_2 \|f_2\|_2
\]

uniformly for all \( x_3 \). Therefore

\[
|T^\phi_\lambda(f)| \leq C |\lambda|^{-1/2} \|f_1\|_2 \|f_2\|_2 \|f_3\|_1.
\]

Therefore by interpolation, it suffices to establish the conclusion with \( \|f_1\|_2 \|f_2\|_2 \|f_3\|_\infty \) on the right-hand side and some exponent \( \gamma > \frac{1}{2} \). By repeating this reduction with the roles of \( f_2, f_3 \) interchanged, interpolating between bounds in terms of \( \|f_1\|_2 \|f_2\|_1 \|f_3\|_\infty \) and \( \|f_1\|_2 \|f_2\|_\infty \|f_3\|_\infty \) to conclude a bound in terms of \( \|f_1\|_2 \|f_2\|_2 \|f_3\|_\infty \), we infer that it suffices to establish the conclusion in terms of \( \|f_1\|_2 \|f_2\|_\infty \|f_3\|_\infty \). Repeating this step once more reduces matters to a bound in terms of the product of \( L^\infty \) norms. Note that this reasoning requires nonvanishing of all three mixed second partial derivatives \( \frac{\partial^2 \phi}{\partial x_1 \partial x_k} \), hence does not apply to \( \phi = x_1 x_2 + x_2 x_3 \).

Write \( e_\xi(x) = e^{i\xi x} \). There exists a constant \( A \) depending only on \( \phi \) and on the choice of \( \tilde{\eta} \) such that

\[
|T^\phi_\lambda(e_\xi, f_2, f_3)| \leq C_N |\xi|^{-N} \|f_2\|_1 \|f_3\|_1 \quad \text{for every } |\xi| \geq A \lambda
\]

for every \( N < \infty \) and every \( \lambda \geq 1 \). This is proved by writing

\[
e^{i\xi x_1 + i\lambda \phi(x)} = \left( i\xi + i\lambda \frac{\partial \phi}{\partial x_1}(x) \right)^{-1} \frac{\partial}{\partial x_1} \right)^N e^{i\xi x_1 + i\lambda \phi(x)}
\]

and integrating by parts \( N \) times with respect to \( x_1 \) while holding \( x_2, x_3 \) fixed. The same holds with the role of \( x_1 \) taken by \( x_2 \) or \( x_3 \). As a consequence, it suffices to analyze \( T^\phi_\lambda(f) \) under the bandlimitedness assumption that for each \( j \in \{1, 2, 3\} \), \( \tilde{f}_j(\xi) = 0 \) whenever \( |\xi| \geq A \lambda \). We assume this for the remainder of the proof of Theorem 4.1.
Suppose that each function \( f_j \) satisfies \( \|f_j\|_\infty \leq 1 \). Expand each \( f_j \) in the form
\[
f_j(x) = \sum_m \eta_m(x) \sum_{k \in \mathbb{Z}} a_{j,m,k} e^{i\pi \lambda^{1/2} kx}
\]
with
\[
\sum_k |a_{j,m,k}|^2 \leq C < \infty \quad \text{uniformly in } j, m, \lambda.
\]
Decompose \( f_j = g_j + h_j + F_j \) where \( F_j \) is the sum of those terms with \( |k| > \lambda^{1/2} \lambda^\rho \), \( h_j \) is the sum of those terms with \( |k| \leq \lambda^{1/2} \lambda^\rho \) and \( |a_{j,m,k}| \leq \lambda^{-\sigma} \), and \( g_j \) is the sum of all remaining terms. From the \( O(\lambda) \)–bandlimitedness condition of the preceding paragraph, it follows that
\[
\|F_j\|_2 = O(\lambda^{-N}) \quad \text{for every } N < \infty.
\]
\( T_j^\phi(f) \) equals \( T_j^\phi(g_1 + h_1 + g_2 + h_2 + g_3 + h_3) \) plus terms involving one or more of the functions \( F_j \). Each of the latter terms is \( O(\lambda^{-N}) \) for every \( N < \infty \), and may consequently be disregarded henceforth. Thus henceforth, \( f_j = g_j + h_j \) and \( |k| \leq \lambda^{1/2} \lambda^\rho \) in (12.2).

Expand
\[
T_j^\phi(f) = \sum_m \sum_k \prod_{j=1}^3 a_{j,m,k_j} \int e^{i\phi_k(x)} \eta_m(x) \, dx
\]
with \( \eta_m(x) = \prod_{l=1}^3 \eta_l(x_l) \) and with the net phase function
\[
\Phi_k(x) = \pi \lambda^{1/2} k \cdot x + \lambda \phi(x),
\]
whose partial derivatives satisfy
\[
\lambda^{-1/2} \frac{\partial \Phi_k}{\partial x_j} = \pi k_j + \lambda^{1/2} \frac{\partial \phi}{\partial x_j} \quad \text{for each } j \in \{1, 2, 3\}.
\]
\( \frac{\partial \phi}{\partial x_j} (x) \) depends only on the single component \( k_j \) of \( k = (k_1, k_2, k_3) \); this will be exploited. We will establish an upper bound for the sum of absolute values
\[
\sum_m \sum_k \prod_{j=1}^3 |a_{j,m,k,j}| \cdot \left| \int e^{i\phi_k(x)} \eta_m(x) \, dx \right|.
\]
For any \((m,k)\),
\[
\int e^{i\phi_k(x)} \eta_m(x) \, dx = O(\lambda^{-3/2}).
\]
A tuple of indices \((m,k)\) is said to be nonstationary if
\[
|\nabla \Phi_k(z_m)| \geq \lambda^\rho \lambda^{1/2},
\]
and otherwise is said to be stationary. For any nonstationary \((m,k)\), repeated integration by parts gives
\[
\int e^{i\phi_k(x)} \eta_m(x) \, dx = O(\lambda^{-N}) \quad \text{for every } N < \infty.
\]
The total number of ordered pairs \((m,k)\) is \( O((\lambda^{1/2})^6) = O(\lambda^3) \). Therefore the total contribution made to (12.8) by all nonstationary \((m,k)\) is \( O(\lambda^{-M}) \) for all \( M < \infty \).
For each \((m_1, m_2, k_1)\) there are at most \(O(\lambda^\rho)\) values of \(m_3\) that satisfy
\[
(12.12) \quad \left| \frac{\partial \Phi_k}{\partial x_1}(z_m) \right| \leq \lambda^{1/2} \lambda^\rho,
\]
with the standing notation \(m = (m_1, m_2, m_3)\). The condition (12.12) is independent of \(k_2, k_3\), since \(\frac{\partial \Phi_k}{\partial x_1}(z_m)\) does not depend on these quantities. The derivative \(\frac{\partial}{\partial x_1} \Phi_k\) vanishes nowhere and has absolute value \(\geq \epsilon \lambda\). Therefore for each \((x_1, x_2, k_1)\), \(\left| \frac{\partial \Phi_k}{\partial x_1}(x_1, x_2, x_3) \right| \leq \lambda^{1/2} \lambda^\rho\) only on a single interval whose length is \(O(\lambda^{-1/2} \lambda^\rho)\). Such an interval intersects the support of \(\eta_{m_3}\) for at most \(O(\lambda^\rho)\) values of \(m_3\). Thus for each \((m_1, m_2, k_1)\), for every \(m_3\) with at most \(O(\lambda^\rho)\) exceptions, \((m, k)\) is nonstationary for every choice of \(k_2, k_3\).

Likewise, for any \((m_1, m_2, k_1, m_3)\), there are most \(O(\lambda^\rho)\) values of \(k_2\) for which \(\left| \frac{\partial \Phi_k}{\partial x_2}(z_m) \right| \leq \lambda^{1/2} \lambda^\rho\), and most \(O(\lambda^\rho)\) values of \(k_3\) for which \(\left| \frac{\partial \Phi_k}{\partial x_3}(z_m) \right| \leq \lambda^{1/2} \lambda^\rho\). Thus for each \((m_1, m_2, k_1, m_3)\) there are at most \(O(\lambda^\rho)\) values of \(k_2\) for which there exists \(k_3\) such that \((m, k)\) is stationary; and for any such \(k_2\), there are at most \(O(\lambda^\rho)\) such \(k_3\). Therefore for each \((m_1, m_2, k_1)\), there are at most \(O(\lambda^{\rho^2})\) values of \(k_2\) for which there exists \((m_3, k_3)\) such that \((m, k)\) is stationary; and for any such \(k_2\), there are at most \(O(\lambda^{\rho^2})\) such pairs \((m_3, k_3)\).

Decompose \(T^\phi_\lambda(f) = T^\phi_\lambda(f_1, f_2, g_3) + T^\phi_\lambda(f_1, f_2, h_3)\) and consider the second summand. All coefficients arising in the expansion of \(h_3\) satisfy \(|a_{3, m_3, k_3}| \leq \lambda^{-\rho}\). Therefore
\[
|T^\phi_\lambda(f_1, f_2, h_3)| \leq O(\lambda^{-N}) + C\lambda^{-3/2} \sum_{m_1, m_2} \sum_{k_1} \sum_{m_3, k_2, k_3} |a_{1, m_1, k_1} a_{2, m_2, k_2} a_{3, m_3, k_3}|
\]
\[
\leq O(\lambda^{-N}) + C\lambda^{-3/2} \lambda^{-\rho} \sum_{m_1, m_2} \sum_{k_1} \sum_{m_3, k_2, k_3} |a_{1, m_1, k_1} a_{2, m_2, k_2}|
\]
for every \(N < \infty\), with the inner sums over \(m_3, k_2, k_3\) extending only over those indices such that \((m, k)\) is stationary. Thus
\[
(12.13) \quad |T^\phi_\lambda(f_1, f_2, h_3)| \leq O(\lambda^{-N}) + O(\lambda^{-3/2} \lambda^{\rho^2 - \rho}) \sum_{m_1, m_2} \sum_{k_1, k_2} |a_{1, m_1, k_1} a_{2, m_2, k_2}|,
\]
with the inner sum taken only over those \((k_1, k_2)\) for which there exist \(m_3, k_3\) such that \((m, k)\) is stationary.

For each \((m_1, m_2, k_1)\), at most \(O(\lambda^{2\rho})\) indices \(k_2\) appear in this sum. Likewise, for each \((m_1, m_2, k_2)\), at most \(O(\lambda^{2\rho})\) indices \(k_1\) appear. For each \(j, m_j\), the sequence \(a_{j, m_j, k_j}\) belongs to \(l^2\) with respect to \(k_j\), with norm \(O(1)\). Therefore an application of Cauchy-Schwarz to the inner sum gives an upper bound
\[
O(\lambda^{-3/2} \lambda^{\rho^2 - \rho}) \sum_{m_1, m_2} O(1) + O(\lambda^{-N}),
\]
which is \(O(\lambda^{-1/2} \lambda^{\rho^2 - \rho}) + O(\lambda^{-N})\) since there are \(O(\lambda^{3/2})\) ordered pairs \((m_1, m_2)\). The conclusion is that
\[
(12.14) \quad |T^\phi_\lambda(f_1, f_2, f_3)| \leq |T^\phi_\lambda(f_1, f_2, g_3)| + O(\lambda^{-1/2} \lambda^{\rho^2 - \rho}).
\]

Repeating this reasoning with indices permuted gives
\[
|T^\phi_\lambda(f_1, f_2, g_3)| \leq |T^\phi_\lambda(f_1, g_2, g_3)| + O(\lambda^{-1/2} \lambda^{\rho^2 - \rho}),
\]
and after one more repetition,
\[
(12.15) \quad |T^\phi_\lambda(f)| \leq |T^\phi_\lambda(g)| + O(\lambda^{-1/2} \lambda^{\rho^2 - \rho}),
\]
where each component of $\mathbf{g} = (g_1, g_2, g_3)$ satisfies (12.2) with at most $O(\lambda^{2\rho})$ nonzero coefficients $a_{j,m_j,k_j}$ for each $m_j$.

It remains to treat $T^\phi_\chi(\mathbf{g})$. By the same reasoning as in the proof of Theorem 4.4 in order to complete the proof of Theorem 4.1 it now suffices to prove an appropriate upper bound for measures of associated sublevel sets, formulated below as Lemma 13.1.

13. Sublevel set analysis for Theorem 4.1

Write $\nabla_j = \frac{\partial}{\partial x_j}$ and $\nabla_j^2 = \frac{\partial^2}{\partial x_j \partial x_k}$. 

**Lemma 13.1.** Suppose that for every distinct pair of indices $j \neq k \in \{1, 2, 3\}$, the mixed partial derivative $\frac{\partial^2 \phi}{\partial x_j \partial x_k}$ vanishes nowhere on the support of $\bar{\eta}$. Then there exist $\delta > 0$ and $C < \infty$ such that for any $\epsilon \in (0, 1]$ and any Lebesgue measurable real-valued functions $h_1, h_2, h_3$, the sublevel set 

$$\mathcal{E} = \{ x : |\nabla_j \phi(x) - h_j(x_j)| \leq \epsilon \text{ for each } j \in \{1, 2, 3\} \}$$

satisfies 

$$|\mathcal{E}| \leq C \epsilon^{1+\delta}. \quad (13.1)$$

Here we seek a bound with an exponent strictly greater than 1, whereas in Lemma 11.1 above, we merely sought an exponent greater than 0. Invoking Lemma 13.1 with $\epsilon = \lambda^{-1/2} \lambda^{C \rho}$, for $\rho$ sufficiently small relative to $\delta$, completes the proof of Theorem 4.1.

We may assume that $h_j(x_j)$ belongs to the range of $\nabla_j \phi$ for each index $j$. By the implicit function theorem together with the hypothesis $\frac{\partial^2 \phi}{\partial x_j \partial x_k} \neq 0$, there exists a $C^\omega$ function $\kappa_0$ satisfying 

$$\nabla \phi(x_1, x_2, \kappa_0(x_1, x_2, x_3)) = x_3. \quad (13.3)$$

Differentiating this equation with respect to $x_2$ gives

$$\frac{\partial \kappa_0(x)}{\partial x_2} = \frac{-\nabla^2 \phi_{1,2}}{\nabla^2 \phi_{1,3}}(x_1, x_2, \kappa_0(x)).$$

Therefore since $\nabla^2 \phi_{1,2}$ never vanishes, the mapping $x \mapsto (x_1, \kappa_0(x), x_3)$ is locally invertible.

Define 

$$\kappa(x_1, x_2) = \kappa_0(x_1, x_2, t) \text{ with } t = h_1(x_1). \quad (13.4)$$

Thus for each $x_1, x_2 \mapsto \kappa(x_1, x_2)$ is a $C^\omega$ function that satisfies 

$$\nabla \phi(x_1, x_2, \kappa(x_1, x_2)) = h_1(x_1). \quad (13.5)$$

This function of $x_2$ is drawn from a compact family of $C^\omega$ functions that is specified in terms of $\phi$ and is parametrized by $(x_1, t)$ with $x_1 \in [0, 1]$ and $|t| \leq \|\nabla \phi\|_{C^\omega([0, 1])} + 1$.

Write $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$. By the nonvanishing of $\frac{\partial}{\partial y_3} \nabla \phi = \nabla^2 \phi_{1,3}$, the relation $
abla \phi(y_1, y_2, x_3) = h_1(y_1) + O(\epsilon)$ implies that $|x_3 - \kappa(\mathbf{y})| = O(\epsilon)$. Thus $|\mathcal{E}| = O(\epsilon)$.

Define 

$$\mathcal{E}' = \{ \mathbf{y} \in [0, 1]^2 : |(\nabla_j \phi)(\mathbf{y}, \kappa(\mathbf{y})) - h_j(y_j)| \leq C_0 \epsilon \text{ for each } j \in \{2, 3\} \} \quad (13.6)$$

with the convention $y_3 = \kappa(\mathbf{y})$ and with $C_0$ a sufficiently large constant. Then 

$$\mathcal{E} \subset \{(x_1, x_2, x_3) : (x_1, x_2) \in \mathcal{E}' \text{ and } |x_3 - \kappa(x_1, x_2)| \leq C_0 \epsilon \}.$$
Define $\mathcal{E}_t'$ in the same way that $\mathcal{E}_t'$ was defined, but with the roles of the coordinates $x_1$ and $x_2$ interchanged, relying on the assumption that $\nabla^2_{2,3}\phi$ never vanishes and replacing $\kappa$ in the construction by the corresponding function $\kappa_2$ defined by

$$\kappa_2(x_1, x_2) = \kappa_0(x_1, x_2, t) \text{ with } t = h_2(x_2).$$

Then

$$\mathcal{E} \subset \{(x_1, x_2, x_3) : (x_1, x_2) \in \mathcal{E}_t' \text{ and } |x_3 - \kappa_2(x_1, x_2)| \leq C_0 \varepsilon\}.$$

Define

$$\mathcal{E}' = \mathcal{E}_1' \cap \mathcal{E}_2'.$$

Then

$$\mathcal{E} \subset \{(x_1, x_2, x_3) : (x_1, x_2) \in \mathcal{E}' \text{ and } |x_3 - \kappa_2(x_1, x_2)| + |x_3 - \kappa_2(x_1, x_2)| \leq 2C_0 \varepsilon\},$$

whence

$$|\mathcal{E}| \leq C \varepsilon |\mathcal{E}'|.$$

In order to complete the proof of Lemma 13.1, it remains only to show that $|\mathcal{E}'|$ is suitably small, as asserted in the next lemma.

**Lemma 13.2.** Suppose that $\phi \in C^\omega$ is not rank one degenerate. Suppose that for every pair of distinct indices $j \neq k$, $\nabla^2_{j,k} \phi$ vanishes nowhere in a neighborhood of $[0, 1]^3$. Then there exists $\delta$ such that for any measurable functions $h_j$ and any $\varepsilon > 0$, the set $\mathcal{E}'$ introduced in (13.6) satisfies $|\mathcal{E}'| = O(\varepsilon^\delta)$.

**Proof.** The first step is to replace $h_2$ by a $C^\omega$ function, drawn from a compact family specified in terms of $\phi$ alone. There exists a set $E \subset \mathbb{R}^1$ satisfying $|E| \gtrsim |\mathcal{E}'|$ such that for each $y_2 \in E$,

$$|\{(y_1 : (y_1, y_2) \in \mathcal{E}')| \gtrsim |\mathcal{E}'|.$$

Therefore the subset $\mathcal{E}'' \subset \mathcal{E}'$ defined by $\mathcal{E}'' = \{(y_1, y_2) \in \mathcal{E}' : y_2 \in E\}$ satisfies $|\mathcal{E}''| \gtrsim |\mathcal{E}'|^2$.

By Fubini’s theorem, there exists $\bar{y}_1$ such that

$$|\{y_2 \in E : (\bar{y}_1, y_2) \in \mathcal{E}'\}| \gtrsim |E| \gtrsim |\mathcal{E}'|.$$

Consider the relation $\nabla_2 \phi(\bar{y}_1, y_2, \kappa(\bar{y}_1, y_2)) = h_2(y_2) + O(\varepsilon)$ for those $y_2 \in E$ satisfying $(\bar{y}_1, y_2) \in \mathcal{E}'$. Since $\phi \in C^\omega$ and $\kappa(\bar{y}_1, y_2)$ is a $C^\omega$ function of $y_2$, drawn from a compact family specified in terms of $\phi$ alone, this relation expresses $h_2(y_2)$ as $h_2(y_2) + O(\varepsilon)$ for these values of $y_2$, with $h_2$ drawn from another compact family of $C^\omega$ functions. Therefore $h_2$ can be replaced by $h_2$ in the definition of $\mathcal{E}'$, at the cost of replacing $\mathcal{E}'$ by its subset $\mathcal{E}''$ and modifying the constant $C_0$ in that definition.

In the preceding two paragraphs, the roles of the variables $y_1$ and $y_2$ can be interchanged, since the definition of $\mathcal{E}' = \mathcal{E}_1' \cap \mathcal{E}_2'$ is invariant under this interchange. Therefore by replacing $\mathcal{E}''$ by an appropriate subset $\mathcal{E}'''$, satisfying $|\mathcal{E}'''| \gtrsim |\mathcal{E}''|^2 \gtrsim |\mathcal{E}'|^4$, we can reduce matters to the case in which $h_1$ is also drawn from a compact set of $C^\omega$ functions specified solely in terms of $\phi$.

Return to the equation $\nabla_1 \phi(x_1, x_2, \kappa(x_1, x_2)) = h_1(x_1)$, restricted now to $(x_1, x_2) \in \mathcal{E}'''$. Since the right-hand side differs from a $C^\omega$ function by $O(\varepsilon)$ on $\mathcal{E}'''$, and since $\nabla_1 \phi$ never vanishes, the implicit function theorem can now be applied to conclude that $\kappa$ differs on $\mathcal{E}'''$ by $O(\varepsilon)$ from a $C^\omega$ function, drawn from an appropriate compact family. Therefore by (13.5), $\kappa$ can in turn be replaced by a $C^\omega$ function of $y \in [0, 1]^2$, at the price of replacing $C_0$ by a yet larger constant.
k was defined by the relation \( \nabla_1 \phi(x_1, x_2, \kappa(x_1, x_2)) - h_1(x_1) = 0 \). Differentiating this equation with respect to \( x_2 \) gives

\[
\nabla_{1,2}^2 \phi(x_1, x_2, \kappa(x_1, x_2)) + \nabla_{1,3}^2 \phi(x_1, x_2, \kappa(x_1, x_2)) \frac{\partial \kappa(x_1, x_2)}{\partial x_2} = 0.
\]

Since \( \nabla_{1,2}^2 \phi \) vanishes nowhere by hypothesis, it follows that \( \frac{\partial \kappa(x_1, x_2)}{\partial x_2} \) vanishes nowhere. Therefore the relation

\[
x_3 = \kappa(x_1, x_2) \Leftrightarrow x_2 = \tilde{\kappa}(x_1, x_3)
\]

defines a \( C^\omega \) function \( \tilde{\kappa} \).

The relation

\[
\nabla_3 \phi(x_1, x_2, \kappa(x_1, x_2)) = h_3(\kappa(x_1, x_2)) + O(\varepsilon) \quad \text{for } (x_1, x_2) \in \mathcal{E}^{m}
\]
can be rewritten with the aid of \( \tilde{\kappa} \) as

\[
(13.11) \quad \nabla_3 \phi(x_1, \tilde{\kappa}(x_1, x_3), x_3)) = h_3(x_3) + O(\varepsilon) \quad \text{when } (x_1, \tilde{\kappa}(x_3)) \in \mathcal{E}^{m'}.
\]

Therefore \( h_3 \) can likewise be replaced by a \( C^\omega \) function drawn from an appropriate compact set.

We have thus shown that under the hypotheses of Lemma \( 13.1 \), there exist \( \mathcal{E}^{m'} \subset \mathbb{R}^2 \) satisfying \( |\mathcal{E}| \leq C \varepsilon |\mathcal{E}^{m'}|^{1/4} \) and \( C^\omega \) functions \( h_j, \kappa \) belonging to appropriate compact families such that with \( x_3 = \kappa(x_1, x_2) \), \( |\nabla_j(\phi(x) - \tilde{h}_j(x_j))| = O(\varepsilon) \) for all \( x \in \mathcal{E}^{m'} \) for each \( j \in \{1, 2, 3\} \).

With this analyticity in hand, Lemma \( 11.2 \) gives \( |\mathcal{E}^{m'}| \lesssim \varepsilon^\delta \) unless there exists a choice of \( C^\omega \) functions \( h_j, \kappa \) in the indicated families satisfying the exact equations

\[
(13.12) \quad \nabla_j \phi(x) = h_j(x_j) \quad \text{for } j \in \{1, 2, 3\},
\]

with \( x_3 = \kappa(x_1, x_2) \), identically in \( [0, 1]^2 \). If such \( h_j, \kappa \) do exist, then for each index \( j \), define \( H_j \) to be an antiderivative of \( h_j \). Define \( \tilde{\phi} : [0, 1]^3 \rightarrow \mathbb{R} \) by

\[
\tilde{\phi}(x) = \phi(x) - H_1(x_1) - H_2(x_2) - H_3(x_3).
\]

The equations \( (13.12) \) imply that \( \nabla \tilde{\phi} \equiv 0 \) on the graph \( x_3 = \kappa(x_1, x_2) \). Thus \( \phi \) is rank one degenerate, contradicting a hypothesis of Theorem \( 4.1 \).

Therefore \( |\mathcal{E}^{m'}| \lesssim \varepsilon^\delta \) and consequently \( |\mathcal{E}| \lesssim \varepsilon^{\delta/4} \), completing the proof of Lemma \( 13.2 \).

Therefore Lemma \( 13.1 \) is proved, as well. \( \square \)

14. Completion of proofs of Theorems \(4.4, 5.2 \) and \(4.2\)

**Conclusion of proof of Theorem \( 4.4 \)**: This theorem has been reduced to the special case in which \((\varphi_1, \varphi_2, \varphi_3)(x_1, x_2) = (x_1, x_2, \varphi(x_1, x_2))\) and in which \((\varphi_1, \varphi_2, \varphi_3)\) is not equivalent to a linear system. We write \( S_{\lambda}^{(\varphi, \psi)} \). That subcase has been proved under a supplementary bandlimitness hypothesis on \( f_3 \).

Therefore by choosing \( \tau \) to be a positive integral power of \( 2 \) and summing, it suffices to analyze \( S_{\lambda}^{(\varphi, \psi)}(f_1, f_2, g) \) with \( \tilde{g} \) supported in \([\tau, 2\tau]\) with \( \tau \geq \lambda^{1+\rho/2} \). Assume that \( \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} \) does not vanish identically. We will prove that

\[
(14.1) \quad |S_{\lambda}^{(\varphi, \psi)}(f_1, f_2, g)| \leq C \tau^{-\delta} \|g\|^2 \prod_{j=1}^{2} \|f_j\|^2
\]

under this hypothesis, completing the proof of Theorem \(4.4\).
By Plancherel’s theorem and an affine change of variables, we may express
\[ g(t) = \tau^{1/2} e^{it} \int_{[0,1]} f_3(x_3) e^{itx_3} \, dx_3 \]
with \( \|f_3\|_2 = c\|g\|_2 \). Thus
\[ S_{(\varphi,\psi)}(f_1, f_2, g) = ce^{it} \tau^{1/2} \int_{[0,1]^3} e^{i\lambda\psi(x_1,x_2)} e^{itx_3\varphi(x_1,x_2)} \prod_{j=1}^3 f_j(x_j) \, dx. \]
Thus
\[ |S_{(\varphi,\psi)}(f_1, f_2, g)| = c\tau^{1/2} T^{\Psi,\lambda}_\tau (f) \]
with
\[ \Psi_{\tau,\lambda}(x) = x_3\varphi(x_1,x_2) + \tau^{-1}\lambda\psi(x_1,x_2). \]
The factor \( \tau^{-1}\lambda \) is \( < \lambda^{-\nu/2} \ll 1 \) for large \( \lambda \). Provided that \( \lambda \) is large, \( \Psi_{\tau,\lambda} \) is well approximated by \( x_3\varphi(x_1,x_2) \).

If \( \psi \) is any \( C^\omega \) function and the partial derivatives \( \frac{\partial^\omega \varphi}{\partial x_j} \) for \( j = 1,2 \) and \( \frac{\partial^\omega \varphi}{\partial x_1\partial x_2} \) vanish nowhere on \( [0,1]^2 \), then \( \Psi_{\tau,\lambda} \) satisfies all hypotheses of Theorem 5.1 uniformly for all sufficiently large \( \lambda \) and all \( \tau \geq \lambda^{1+\mu/2} \). The proof of Theorem 5.1 relied only on a special bandlimited case of Theorem 4.4 that has already proved in full, so we may invoke Theorem 5.1 here without circularity in the reasoning. We conclude that for \( |\lambda| \) sufficiently large,
\[ |T^{\Psi,\lambda}_\tau(f)| \leq C\tau^{-\gamma} \prod_j \|f_j\|_2 \]
with \( C < \infty \) and \( \gamma > \frac{1}{2} \) independent of \( \lambda, \tau \). This establishes (14.1) with \( \delta = \gamma - \frac{1}{2} > 0 \), completing the proof of Theorem 4.4 under the supplemental hypothesis that the mixed second derivative \( \frac{\partial^2 \varphi}{\partial x_1\partial x_2} \) vanishes nowhere on \( [0,1]^2 \).

This nonvanishing hypothesis can be weakened; it suffices to assume that the partial derivative does not vanish identically on any open set. Indeed, we have already implicitly proved a more quantitative result, namely an upper bound of the form
\[ C(1 + |\lambda|)^{-\delta} \left( \min_{j \neq k} \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \right)^{-N} \]
for some \( N, C < \infty \) provided that \( \varphi, \psi \) lie in some compact (with respect to the \( C^3 \) norm) family of \( C^\omega \) functions.

Let \( \varepsilon_0 \) be a sufficiently small positive number, depending only on \( \varphi, \psi \). Partition a neighborhood of the support of the cutoff function \( \eta \) into squares of sidelengths \( |\lambda|^{-\varepsilon_0} \). The union of those squares on which some mixed second partial derivative of \( \varphi \) has magnitude \( < |\lambda|^{-\varepsilon_0} \) has Lebesgue measure \( O(|\lambda|^{-\varepsilon}) \) for some \( \varepsilon > 0 \) that depends only on \( \varphi, \psi \) and the choice of \( \varepsilon_0 \). The number of remaining squares is \( O(|\lambda|^{2\varepsilon_0}) \). The contribution of each such square can be analyzed by making an affine change of variables that converts it to \( [0,1]^2 \).

Invoking the more quantitative result produces a bound of the form \( C|\lambda|^\delta - C\varepsilon_0 \) for each. If \( \varepsilon_0 \) is sufficiently small, the result follows. \( \square \)

**Conclusion of proof of Theorem 5.2** The roles of the indices 1,2,3 in Theorem 4.4 can be freely permuted by making changes of coordinates \( (x_1,x_2) \mapsto (x_1,\varphi(x_1,x_2)) \) and \( (x_2,\varphi(x_1,x_2)) \). Therefore the roles of the three functions can be freely interchanged in (14.1). Theorem 5.2 is an immediate consequence for \( p = 2 \). For \( p \in (0,2) \) it is obtained by interpolating between this result for \( p = 2 \) and the elementary result for \( (p,s) = (0,0) \). \( \square \)
**Proof of Theorem 4.2.** It suffices to analyze the case in which two functions are in $L^2$ and one is in a negative order Sobolev space, that is, to prove that

\[
(14.2) \quad \int \eta \cdot \prod_{j=1}^{3} (f_j \circ \varphi_j) = O\left(\|f_1\|_2 \|f_2\|_2 \|f_3\|_{H^s}\right).
\]

A simple interpolation then completes the proof.

By introducing a partition of unity and making local changes of coordinates, we may reduce matters to the case in which $\varphi(x) = x_i$ for $i = 1, 2$, and $\varphi = \varphi_3$ has a mixed second partial derivative $\frac{\partial^2 \varphi}{\partial x_1 \partial x_2}$ that vanishes nowhere on the support of $\eta$.

Express

\[
f_3(\varphi(x_1, x_2)) = c_0 \int_{\mathbb{R}} e^{i\tau \varphi(x_1, x_2)} \mathcal{F}_3(\tau) \, d\tau.
\]

It suffices to show that for large positive $\lambda$, the contribution of the interval $\tau \in [\lambda, 2\lambda]$ is $O(\lambda^{-\delta})$ for some $\delta > 0$.

Substituting $\tau = \lambda x_3$, with $x_3 \in [1, 2]$, expresses this contribution as a constant multiple of

\[
\lambda^{1/2} \int_{\mathbb{R}^2 \times [1, 2]} e^{i\lambda \psi(x)} \prod_{j=1}^{3} g_j(x_j) \eta(x_1, x_2) \, dx
\]

with

\[
\psi(x) = x_3 \varphi(x_1, x_2),
\]

$g_i = f_i$ for $i = 1, 2$, and $g_3(t) = \lambda^{-1/2} \mathcal{F}_3(\lambda^{-1}t)$. The function $g_3$ satisfies

\[
\|g_3\|_2 \leq C \lambda^{-s} \|f_3\|_{H^s}.
\]

According to Lemma 3.2 $\psi$ is not rank one degenerate on the product of the support of $\eta$ with $[1, 2]$. Moreover, for any pair of distinct indices $j \neq k \in \{1, 2, 3\}$, $\frac{\partial^2 \psi}{\partial x_j \partial x_k}$ vanishes nowhere on the domain of integration. For $\frac{\partial^2 \psi}{\partial x_j \partial x_k}$, this is equivalent to nonvanishing of $\frac{\partial \psi}{\partial x_j}$, which is a hypothesis. For $\frac{\partial^2 \psi}{\partial x_j \partial x_3}$, it follows from the nonvanishing of $\frac{\partial^2 \varphi}{\partial x_j \partial x_k}$ and of $x_3$. Thus $\psi$ satisfies all hypotheses of Theorem 4.1. Therefore

\[
\left| \int_{\mathbb{R}^2 \times [1, 2]} e^{i\lambda \psi(x)} \prod_{j=1}^{3} g_j(x_j) \eta(x_1, x_2) \, dx \right| \leq C \lambda^{-\gamma} \prod_{j=1}^{3} \|g_j\|_2
\]

\[
\leq C \lambda^{-\gamma + \frac{1}{2} - s} \|f_1\|_2 \|f_2\|_2 \|f_3\|_{H^s}
\]

for some $\gamma > \frac{1}{2}$. If $s < 0$ is sufficiently close to 0, then $\gamma > \frac{1}{2} - s$, and the proof is complete. \qed

**15. Yet another variant**

Let $U \subset \mathbb{R}^2$ be a nonempty open set. For $j \in \{1, 2, 3\}$, let $X_j$ be a $C^\infty$ nowhere vanishing vector field in $U$. Suppose that for any distinct indices $j \neq k \in \{1, 2, 3\}$, all integral curves of $X_j, X_k$ intersect transversely at every point of $U$.

The weak convergence theorem of [17] is concerned with functions that satisfy $g_j \in L^2(U)$ and $X_j g_j \in L^2(U)$, whereas the results stated above in [44] are concerned with the special case in which $X_j g_j \equiv 0$. Here we extend those results to this more general situation.
Theorem 15.1. Let \((X_j : j \in \{1, 2, 3\})\) be as above. Suppose that the curvature of the 3-web associated to \((X_j : j \in \{1, 2, 3\})\) does not vanish at any point of \(U\). Then for any exponent \(p > \frac{3}{2}\) and any auxiliary function \(\eta \in C_0^\infty(U)\), there exist \(C < \infty\) and \(s < 0\) such that
\[
\left| \int_{\mathbb{R}^2} \eta \prod_{j=1}^{3} g_j \right| \leq C \prod_{j} \left( \|g_j\|_{W^{s,p}} + \|X_j g_j\|_{W^{s,p}} \right) \quad \text{for all } g_j \in C^1(\mathbb{R}^2).
\]

Corollary 15.2. Let \((X_j : j \in \{1, 2, 3\})\) be as above. Let \(p > \frac{3}{2}\). Let \(g_j^{\nu}, X_j g_j^{\nu} \in L^p(\mathbb{R}^2)\) be uniformly bounded, and suppose that \(g_j^{\nu} \rightharpoonup g_j\) weakly as \(\nu \to \infty\) for \(j = 1, 2, 3\). Then
\[
\prod_{j=1}^{3} g_j^{\nu} \rightharpoonup \prod_{j=1}^{3} g_j \quad \text{weakly as } \nu \to \infty
\]
in every relatively compact open subset of \(U\).

To deduce Theorem 15.1 from the results proved above, introduce \(C^\omega\) diffeomorphisms \(\phi_j = (\varphi_j^1, \varphi_j^2)\) from \(U\) to open subsets of \(\mathbb{R}^2\), such that the curves \(\{x : \varphi_j^1(x) = t\}\) are the integral curves of \(X_j\). Write \(g_j = F_j \circ \phi_j\). Then the \(W^{s,p}\) norms of \(g_j\) and of \(X_j g_j\) together control the \(W^{s,p}\) norms of \(F_j\) and of \(\frac{\partial F_j}{\partial y_M}\). By simple decomposition and interpolation, it suffices to bound the integral under the assumption that for each \(j\),
\[
\|F_j\|_{W^{s,p}} + \|\frac{\partial M F_j}{\partial y_M}\|_{W^{s,p}} \leq 1
\]
for some \(M < \infty\); we may choose \(M\) as large as may be desired.

Expand \(F_j\) in Fourier series with respect to the second variable:
\[
F_j(y, t) = \sum_{n \in \mathbb{Z}} f_{j,n}(y) e^{int}.
\]

Then
\[
\|f_{j,n}\|_{W^{s,p}} = O(1 + |n|)^{-N}
\]
with \(N\) as large as may be desired. Thus we are led to
\[
\sum_{n \in \mathbb{Z}^3} \int \eta(x) e_n(x) \prod_{j} (f_{j,n} \circ \varphi_j^1)
\]
with
\[
e_n(x) = \prod_{k=1}^{3} e^{i n_k \varphi_k^2(x)}.
\]
Set \(\eta_n = \eta e_n\). These functions satisfy
\[
\|\eta_n\|_{C^K} = O(1 + |n|)^{K-N}
\]
for any \(K < \infty\).

Thus it suffices to invoke a small improvement on Theorem 4.2, under the hypotheses of that theorem, there exists \(K < \infty\) such that
\[
\left| \int \eta \prod_{j} (f_j \circ \varphi_j) \right| \leq C \|\eta\|_{C^K} \prod_{j} \|f_j\|_{W^{s,p}},
\]
uniformly for all \(C^K\) functions \(\eta\) supported in a fixed compact region in which the hypotheses hold. This can be deduced from the formally more restrictive result already proved, by
introducing a $C^\infty$ partition of unity $\{\zeta^\alpha_3\}$ to reduce to the case in which $\varphi_j(x) \equiv x_j$ for $j = 1, 2$ for each $\alpha$, then expanding $\zeta_\alpha \cdot \eta$ in Fourier series and incorporating factors $e^{in_jx_j}$ into $f_j$. \hfill $\square$

16. Remarks on the nondegeneracy hypotheses

(1) Phases $\phi$ that satisfy the hypotheses of Theorem 11 exist in profusion. Given a point $\bar{x} \in [0, 1]^3$, for generic tuples $(a_{j,k}, b_{j,k})$ of real numbers satisfying the natural symmetry conditions, any phase satisfying

$$
\frac{\partial^2 \phi}{\partial x_j \partial x_k}(\bar{x}) = a_{j,k} \quad \text{and} \quad \frac{\partial^3 \phi}{\partial x_i \partial x_j \partial x_k}(\bar{x}) = b_{i,j,k}
$$

is rank one nondegenerate in some neighborhood of $\bar{x}$. In other words, we claim that if $\phi$ is rank one degenerate in every neighborhood of $\bar{x}$, then its second and third order partial derivatives at $\bar{x}$ must satisfy certain algebraic relations (16.3).

Restrict attention to phases whose mixed second order partial derivatives $\frac{\partial^2 \phi}{\partial x_j \partial x_k}, j \neq k,$ are all nonzero at $x$. Suppose that $H$ is a small $C^\omega$ hypersurface containing $x$, on which $\nabla \phi$ vanishes identically, with $\phi(x) = \phi(x) - \sum_j h_j(x_j)$. Suppose that $H$ is represented by an equation $x_3 = \kappa(x_1, x_2)$ in a neighborhood of $(\bar{x}_1, \bar{x}_2)$, with $\kappa$ smooth. Thus $\kappa(\bar{x}_1, \bar{x}_2) = \bar{x}_3$.

Write $\phi_j$ for $\frac{\partial \phi}{\partial x_j}$, $\phi_{j,k}$ for the corresponding second partial derivatives, and $\phi_{i,j,k}$ for third order derivatives. Denote partial derivatives of $\kappa$ by $\kappa_j$, for $j = 1, 2$.

The vanishing of $\nabla \phi\bar{j}$ at $(x_1, x_2, \kappa(x_1, x_2))$ for $j = 1, 2$ implies that $\phi_1(x_1, x_2, \kappa(x_1, x_2))$ is independent of $x_2$ in a neighborhood of $(\bar{x}_1, \bar{x}_2)$. Therefore

$$\phi_{1,2}(x_1, x_2, \kappa(x_1, x_2)) + \kappa_2(x_1, x_2)\phi_{1,3}(x_1, x_2, \kappa(x_1, x_2)) = 0$$

in a neighborhood of $(\bar{x}_1, \bar{x}_2)$. We write this relation as $\phi_{1,2} + \kappa_2\phi_{1,3} = 0$, leaving it understood that $\phi$ and its partial derivatives are evaluated at $(x_1, x_2, \kappa(x_1, x_2))$ while $\kappa$ is evaluated at $(x_1, x_2)$, and that $(x_1, x_2)$ varies within a small neighborhood of $(\bar{x}_1, \bar{x}_2)$. Likewise, $\phi_{2,1} + \kappa_1\phi_{2,3} = 0$. Thus

$$\kappa_2 = -\phi_{1,2}\phi_{1,3}^{-1} \quad \text{and} \quad \kappa_1 = -\phi_{2,1}\phi_{2,3}^{-1}.$$  

Differentiating the first of these relations with respect to $x_1$ and the second with respect to $x_2$, and invoking the relation $\kappa_{2,1} = \kappa_{1,2}$, we find that

$$\frac{\partial^2 \phi}{\partial x_1 \partial x_2}(\bar{x}) = (\phi_{1,2}^2)(\phi_{1,2,1}\phi_{1,3} - \phi_{1,2}\phi_{1,3,1}) \equiv (\phi_{1,2}^2)\phi_{1,2,2}\phi_{2,3} - \phi_{1,2}\phi_{2,3,2}$$

at $(x_1, x_2, \kappa(x_1, x_2))$, for all $(x_1, x_2)$ in a neighborhood of $(\bar{x}_1, \bar{x}_2)$. In particular, (16.3) holds.

(16.3) was derived under the assumption that the third coordinate vector does not belong to the tangent space to $H$ at $\bar{x}$. Thus without that assumption, we conclude that if $\phi$ is rank one degenerate in every neighborhood of $\bar{x}$, then at least one of three variants of (16.3), obtained from (16.3) by permuting the three coordinate variables, must hold for the partial derivatives of $\phi$ at $\bar{x}$. Rank one nondegeneracy therefore holds in all sufficiently small neighborhoods of $\bar{x}$, for generic values of second and third partial derivatives of $\phi$ at $\bar{x}$.

(2) The hypotheses of Theorem 11, taken as a whole rather than individually, are stable with respect to small perturbations of $\phi$. Indeed, the hypothesis that all three mixed second partial derivatives are nowhere vanishing is manifestly stable. A phase $\phi$ satisfying this auxiliary hypothesis is rank one degenerate if and only if there exist $C^\omega$ functions...
\( h_j(x_j) \) such that \( \nabla_j \phi(x) = h_j(x_j) \) for every \( x \in H \), for some piece of \( C^\omega \) hypersurface \( H \subset (0,1)^3 \).

An exhaustive class of candidate hypersurfaces \( H \) can be constructed, in terms of \( \phi \), as follows. Fix a base point \( \bar{x} \) and consider hypersurfaces \( H \ni \bar{x} \) such that the third coordinate vector does not lie in the tangent space to \( H \) at \( \bar{x} \). Express \( H \) locally as a graph \( x_3 = \kappa(x_1, x_2) \). Determine \( \kappa(x_1, \bar{x}_2) \) by solving the differential equation
\[
\frac{\partial \kappa}{\partial x_1}(x_1, \bar{x}_2) = -\phi_{2,1} \phi_{2,3}^{-1}(x_1, \bar{x}_2)
\]
derived above, with initial condition \( \kappa(\bar{x}_1, \bar{x}_2) = \bar{x}_3 \). Recall that the mixed second partial derivative \( \phi_{2,3} \) vanishes nowhere, by hypothesis.

For each \( x_1 \) in a small neighborhood of \( \bar{x}_1 \), determine \( \kappa(x_1, x_2) \) by solving
\[
\frac{\partial \kappa}{\partial x_2}(x_1, x_2) = -\phi_{1,2} \phi_{1,3}^{-1}(x_1, x_2)
\]
with the initial condition \( \kappa(x_1, \bar{x}_2) \) determined in the preceding step. This defines a \( C^\omega \) hypersurface \( H \) containing \( \bar{x} \), and this is locally the only such hypersurface passing through \( \bar{x} \) whose tangent space does not contain the third coordinate vector and that could potentially satisfy the condition in the definition of rank one degeneracy of \( \phi \). Repeating this construction twice more with suitable permutations of the coordinate indices yields three (or fewer) candidate hypersurfaces for each point \( \bar{x} \). Plainly this construction is continuous with respect to \( \phi, \bar{x} \).

Once a hypersurface \( H \) is specified, the vanishing of the gradient of \( \tilde{\phi}(x) = \phi(x) - \sum_{j=1}^3 h_j(x_j) \) at each point of \( H \) determines the derivative \( h'_j \) at each point of \( \mathbb{R} \) sufficiently close to \( x_j \). Thus the functions \( h_j \) are completely determined in a neighborhood of \( \bar{x} \), up to additive constants. Again, these depend continuously on \( \phi, \bar{x} \).

If \( \phi \) satisfies the hypotheses of Theorem 4.1, if \( \bar{x} \in [0,1]^3 \), and if \( H \) and associated functions \( h_j \) are as above, then \( \nabla \tilde{\phi} \) fails to vanish identically on \( H \), so by real analyticity, some partial derivative along \( H \) of \( \nabla \phi \) fails to vanish at \( \bar{x} \). This nonvanishing is stable under small perturbations of \( \phi, \bar{x} \).

(3) The examples also demonstrate that the optimal exponent \( 1+\delta \) in Lemma 13.1 is not stable with respect to perturbations of \( \phi \), if the auxiliary hypothesis on the nonvanishing of all three mixed partial derivatives is relaxed.

17. More on integrals with oscillatory factors

Li, Tao, Thiele, and the present author investigated multilinear functionals
\[
S_\lambda(f) = \int_{\mathbb{R}^D} e^{i\lambda \psi_j} \prod_{j \in J} (f_j \circ \varphi_j)
\]
with \( \varphi_j : \mathbb{R}^D \to \mathbb{R}^{d_j} \) linear, and established bounds of the type \( O(|\lambda|^{-\gamma} \prod_{j \in J} \|f_j\|_\infty) \), for certain tuples \( (\psi, (\varphi_j : j \in J)) \), under two different sets of hypotheses. Both sets of hypotheses were rather restrictive. In one set, it was required that \( d_j = D - 1 \) for every \( j \in J \). In the other, \( d_j = 1 \) for every \( j \), and \( |J| < 2D \). The latter result was invoked in the discussion above.

The method developed above yields an alternative proof of these results, and thus our discussion can be modified to be self-contained, with no invocation of results from [12]. More significantly, the method makes it possible to remove the assumption that \( d_j = 1 \), as we now show.
Let \( D > d \in \mathbb{N} \). Let \( \{\varphi_j : j \in J\} \) be a family of surjective linear mappings from \( \mathbb{R}^D \) to \( \mathbb{R}^d \). Such a family is said to be in general position if for any subset \( \tilde{J} \subset J \) satisfying \( 0 < |\tilde{J}| \leq D/d \), the linear mapping

\[
(17.1) \quad \mathbb{R}^D \ni x \mapsto (\varphi_j(x) : j \in \tilde{J}) \in (\mathbb{R}^d)^{\tilde{J}}
\]
is injective.

**Theorem 17.1.** Let \( d, D \in \mathbb{N} \) with \( D/d \in \mathbb{N} \). Let \( \eta \in C_0^\infty(\mathbb{R}^D) \). Let \( \psi \) be a real-valued \( C^\omega \) function defined in a neighborhood \( U \) of the support of \( \eta \). Let \( J \) be a finite index set of cardinality \( |J| \) satisfying \( 1 \leq |J| < 2D/d \).

Let \( \{\varphi_j : j \in J\} \) be a family of surjective linear mappings \( \varphi_j : \mathbb{R}^D \to \mathbb{R}^d \) in general position. Suppose that \( \psi \) cannot be expressed in the form \( \psi = \sum_{j \in J} h_j \circ \varphi_j \) in any nonempty open set, with \( h_j \in C^\omega \).

Then there exist \( \delta > 0 \) and \( C < \infty \) such that for all \( \lambda \in \mathbb{R} \) and all functions \( f_j \in L^\infty(\mathbb{R}^d) \), the form

\[
(17.2) \quad S_\lambda(f) = \int_{\mathbb{R}^D} e^{i\lambda \psi} \prod_{j \in J} (f_j \circ \varphi_j) \eta
\]
satisfies

\[
(17.3) \quad |S_\lambda(f)| \leq C|\lambda|^{-\delta} \prod_{j \in J} \|f_j\|_{L^\infty}.
\]

This extends Theorem 2.1 of [12], in which it is assumed that \( d = 1 \), and that \( \psi \) is a polynomial. The polynomial hypothesis is not essential to the proof given in [12], but the restriction \( d = 1 \) is.

The simplest instance of Theorem 17.1 with \( d > 1 \) is as follows. Let \( B \subset \mathbb{R}^d \) be a ball centered at 0. Let \( Q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) be a homogeneous quadratic real-valued polynomial. To \( Q \), associate its antisymmetric part \( Q^*(x,y) = \frac{1}{2}(Q(x,y) - Q(y,x)) \). Denote by \( \| \cdot \| \) any norm on the vector space of all antisymmetric quadratic real-valued polynomials.

**Corollary 17.2.** Let \( d \geq 2 \). There exist \( C < \infty \) and \( \gamma > 0 \) such that for all functions \( f_j \in L^2 \),

\[
(17.4) \quad \left| \int_B \int_B e^{iQ(x,y)} f_1(x)f_2(y)f_3(x+y) \, dx \, dy \right| \leq C\|Q^*\|^{-\gamma} \prod_j \|f_j\|_{L^2}.
\]

**Example 17.1.** Let \( d = 2 \) and \( Q((x_1, x_2), (y_1, y_2)) = x_1y_2 \). Then

\[
(17.5) \quad \left| \int_{[0,1]^2} \int_{[0,1]^2} e^{i\lambda x_1y_2} f_1(x)f_2(y)f_3(x+y) \, dx \, dy \right| \leq C|\lambda|^{-\gamma} \prod_j \|f_j\|_{L^2}.
\]

**Proof of Theorem 17.1** The proof of Theorem 17.1 has the same overarching structure as the analysis developed above for Theorem 4.4. However, one step of the proof of Theorem 4.4 broke down when the mappings \( \varphi_j \) were linear, and Theorem 2.1 of [12] was invoked, in a black box spirit, to treat the linear case. Much of the proof of Theorem 17.1 closely follows arguments above and hence will be merely sketched, but we will show in more detail how the problematic step, which arises near the end of the analysis, can be modified to handle linear mappings.

Let \( \rho > 0 \) be a small exponent, which will ultimately depend on another exponent \( \sigma \) introduced below, which in turn will depend on an exponent \( \tau \) in a sublevel set bound
(17.17). Assume without loss of generality that \( \lambda \geq 1 \) and that \( \|f_j\|_\infty \leq 1 \). Decompose

\[
(17.6) \quad f_j(y) = \sum_m \eta_m(y) \sum_{k \in \mathbb{Z}^d} a_{j,m,k} e^{i \lambda^{1/2} k \cdot y}
\]

with each \( \eta_m \) supported on the double of a cube of sidelength \( \lambda^{-1/2} \) and \( |\eta_m| + |\lambda^{-1/2} \nabla \eta_m| = O(1) \), and

\[
\sum_k |a_{j,m,k}|^2 = O(1)
\]

uniformly in \( j, m, \lambda \). Decompose \( f_j = g_j + h_j + F_j \) where \( \|F_j\|_\infty = O(\lambda^{-N}) \) for all \( N < \infty \), \( h_j \) is the sum over \( m, k \) of those terms satisfying \( |a_{j,m,k}| \leq \lambda^{-\sigma} \), and \( g_j \) has an expansion of the same type with \( a_{j,m,k} = 0 \) for all but at most \( O(\lambda^{2\sigma}) \) indices \( k \) for each \( j, m \). The contributions of all \( F_j \) are negligible, and we may therefore henceforth replace \( f_j \) by \( g_j + h_j \) for each index \( j \).

Write \( k = (k_j : j \in J) \in (\mathbb{Z}^d)^J \). Define the linear mapping \( L : (\mathbb{Z}^d)^J \to \mathbb{R}^D \) to be the transpose of \( x \mapsto (\varphi_j(x) : j \in J) \); thus

\[
(17.7) \quad L(k) = \sum_{j \in J} \varphi_j^*(k_j)
\]

where \( \varphi_j^* \) denotes the transpose of the linear mapping \( \varphi_j \). Writing \( m = (m_j : j \in J) \) and \( x \in \mathbb{R}^D \), our functional can be expanded as

\[
S_\lambda(f) = \sum_m \sum_k \prod_{j \in J} a_{j,m_j,k_j} I(m,k)
\]

with

\[
I(m,k) = \int e^{i \lambda \Phi_k(x)} \zeta_m(x) \, dx
\]

\[
\Phi_k(x) = \psi(x) + \pi \lambda^{-1/2} L(k) \cdot x
\]

\[
\zeta_m(x) = \prod_{j \in J} \eta_{m_j}(\varphi_j(x)).
\]

While the number of indices \( m \) in play is comparable to \( (\lambda^{d/2})^{|J|} \), there are only \( O(\lambda^{D/2}) \) indices \( m \) for which \( \zeta_m \) does not vanish identically. We claim that there exists \( \theta \in (0,1) \), which depends only on the ratio \( D/(d|J|) \), such that for any \( m \) and any sequences of scalars \( b_j(\cdot) \),

\[
(17.8) \quad \sum_k \prod_{j \in J} |b_{j}(k_j)| \cdot |I(m,k)| \leq C \lambda^{C \rho \lambda^{-D/2}} \prod_{j \in J} \|b_j\|_{L^\infty}^{1-\theta} \|b_j\|_{L^\infty}^{\theta}
\]

uniformly in \( m, \lambda \). Indeed,

\[
(17.9) \quad |I(m,k)| = O(\lambda^{-D/2})
\]

uniformly in \( m, \lambda \). For each \( m \) for which \( \zeta_m \) does not vanish identically, choose \( z_m \) in the support of \( \zeta_m \). Integrating by parts sufficiently many times gives

\[
(17.10) \quad |I(m,k)| \leq C_N \lambda^{-N} (1 + |\nabla \psi(z_m) + L(k)|)^{-N} \text{ for every } N < \infty
\]

unless

\[
(17.11) \quad |\nabla \Phi_k(z_m)| \leq \lambda^\rho.
\]
Recalling that $|J| \geq D/d$, consider any subset $S \subset J$ of cardinality equal to $D/d$. If \( \sum_{j \in S} \varphi_j(k_j) = 0 \) then $k_j = 0$ for every $j \in S$, since the mapping $x \mapsto (\varphi_j(x) : j \in S)$ is bijective by the general position hypothesis. Therefore if $N$ is chosen to be sufficiently large then the summation over all vectors $(k_j : j \in S)$ of $\min(\lambda^{-D/2}, \lambda^{-N}(1 + |\nabla \psi(z_m) + L(k)|))^{-N}$ is $O(\lambda^{-D/2} \lambda^{C\rho})$, uniformly for all vectors $(k_j : j \in J \setminus S)$. It follows that

$$
\sum_k \prod_{j \in S} |b_j(k_j)| \cdot |I(m, k)| \leq C \lambda^{C\rho} \lambda^{-D/2} \prod_{j \in J \setminus S} \|b_j\|_1 \prod_{j \in S} \|b_j\|_{l^\infty}
$$

by (17.9), (17.10), and the general position assumption (17.1). Since $|J| < 2d^{-1}D$, it follows by interpolation that

$$
\sum_k \prod_{j \in J} |b_j(k_j)| \cdot |I(m, k)| \leq C \lambda^{C\rho} \lambda^{-D/2} \prod_{j \in J} \|b_j\|_{l^q}
$$

for some exponent $q > 2$. Then $\|b_j\|_{l^q} \leq \|b_j\|_{l^2}^{1-\theta} \|b_j\|_{l^\infty}^\theta$, for some $\theta = \theta(q) > 0$, yielding (17.8).

From (17.9) and (17.8), for $f_j = g_j + h_j$ with the properties indicated above, there follows

$$
|S_\lambda(f)| \leq |S_\lambda(g)| + O(\lambda^{\sigma+C\rho}).
$$

Therefore, choosing $\rho$ to be sufficiently small relative to $\sigma$, it suffices to analyze $S_\lambda(g)$.

The quantity $S_\lambda(g)$ can in turn be expressed as a sum of $O(\lambda^{C\sigma})$ terms, in each of which each function $g_j$ takes the simple form

$$
g_j(x) = \sum_{m_j} a_j, m_j e^{i \pi \lambda^{1/2} k_j, m_j} \zeta_m(x)
$$

with $|a_j, m_j| = O(1)$. We assume this form henceforth, at the expense of a factor $O(\lambda^{C\sigma})$. This factor can be absorbed at the end of the proof, by choosing $\sigma$ sufficiently small relative to the exponent $\tau$ that appears below, just as was done in other proofs earlier in the paper. Thus

$$
|S_\lambda(g)| \leq C \sum_m |I(m, k_m)|
$$

with $k_m = (k_j, m_j : j \in J)$.

Define

$$
\Phi(x) = \psi(x) + \pi \lambda^{-1/2} L(k) \cdot x.
$$

Consider those $m$ that are stationary in the sense that $|\nabla \Phi(z_m)| \geq \lambda^0$. By (17.10), the sum of the contributions of all such $m$ is $O(\lambda^{-N})$ for every $N < \infty$. Therefore in order to complete the analysis, it suffices to show that the number of $m$ for which $\Phi$ is nonstationary, is $O(\lambda^{-\tau} \lambda^{D/2})$ for some exponent $\tau > 0$.

Let $B \subset \mathbb{R}^D$ be any ball of finite radius. Let $h_j : \mathbb{R}^d \to \mathbb{R}^d$ be arbitrary Lebesgue measurable functions. Define

$$
\mathcal{E} = \{x \in B : |\nabla \psi(x) - \sum_{j \in J} (h_j \circ \varphi_j)(x) \cdot D\varphi_j| < \varepsilon\}.
$$

Here, $h_j$ takes values in $\mathbb{R}^d$, and $(h_j \circ \varphi_j) \cdot D\varphi_j$ takes values in $\mathbb{R}^D$. To complete the proof of Theorem 17.1 it now suffices to show that there exist $C < \infty$ and $\tau > 0$ such that

$$
|\mathcal{E}| \leq C \varepsilon^\tau
$$

uniformly for all $\varepsilon \in (0, 1]$ and all functions $h_j$. 
Assume temporarily that $|J| \geq D/d$. Let $\tilde{J} \subset J$ be any subset of cardinality $|\tilde{J}| = D/d$. The general position hypothesis ensures that there exists a linear subspace $V \subset \mathbb{R}^D$ of dimension $d$ such that kernel($\varphi_j$) $\subset V$ for each $j \in J \setminus \tilde{J}$, but for each $j \in \tilde{J}$, $\varphi_j|_V$ is an invertible linear system of $d$ equations with $d$ unknowns $h_j(\varphi_j(x)) \cdot D\varphi_j$, with the index $j$ running over $\tilde{J}$.

If the system of equations $\nabla \psi(x) - \sum_{j \in J}(h_j \circ \varphi_j)(x) \cdot D\varphi_j | < \varepsilon \} = 0$ is restricted to any translate $V + y$ of $V$, those terms $h_j \circ \varphi_j$ with $j \in J \setminus \tilde{J}$ become constant functions of $x \in V$. For any $x \in V$, what results is an invertible linear system of $d$ equations for $d$ unknowns $h_j(\varphi_j(x)) \cdot D\varphi_j$, with the index $j$ running over $\tilde{J}$.

By the same reasoning as developed in the analyses of upper bounds for measures of sublevel sets above, we may conclude that there exist functions of the form $H_j + r_j$, where $H_j : \mathbb{R}^d \to \mathbb{R}^d$ are drawn from a compact family of $C^\omega$ functions specified in terms of $\psi$, $\{\varphi_j : j \in J\}$ alone and $r_j \in \mathbb{R}^d$ are constant vectors, and a set $\tilde{E}$ satisfying $|\tilde{E}| \geq c|E|^C$, such that

$$|\nabla \psi(x) - \sum_{j \in J}((H_j \circ \varphi_j)(x) + r_j) \cdot D\varphi_j| < C\varepsilon \quad \text{for all } x \in \tilde{E}. \tag{17.18}$$

We have reached the point at which the proof of Theorem 4.4 must be augmented in order to treat Theorem 17.1. Let $C_0$ be some finite constant. If

$$|r_j| \leq C_0 \quad \text{for all } j \in J \tag{17.19}$$

then the functions $\tilde{H}_j = H_j + r_j$ are drawn from a compact family of $C^\omega$ functions, and the same reasoning as in the proof of Theorem 4.4 can be applied to conclude that $|\tilde{E}| \leq C\varepsilon^r$, and hence that the same holds for $|E|$ with modified constants $C, \tau$.

However, it is not true that there exists $C_0$ such that (17.19) holds. Indeed, if $G_j : \mathbb{R}^d \to \mathbb{R}$ are linear functions satisfying $\sum_{j \in J} G_j \circ \varphi_j \equiv 0$, then for any $t \in (0, \infty)$, replacement of $h_j$ by $h_j + t\nabla G_j$ does not change the quantity $\nabla \psi - \sum_{j \in J}(h_j \circ \varphi_j) \cdot D\varphi_j$, and consequently does not change $E$. If $|J| > D/d$ then there exists a linear solution of $\sum_{j \in J} G_j \circ \varphi_j \equiv 0$, with at least one $G_j$ not identically zero. By taking $t$ arbitrarily large, one finds that no uniform a priori bound (17.19) is available for the functions $h_j$ in terms of $\psi$, $(\varphi_j : j \in J)$, and $\varepsilon$ alone.

If $\varepsilon \leq 1$, as we may assume, and if $E$ is nonempty, then while the individual quantities $r_j$ may be large,

$$\sum_{j \in J} r_j \cdot D\varphi_j = O(1). \tag{17.20}$$

This follows from the condition

$$|\nabla \psi(x) - \sum_{j}(H_j \circ \varphi_j) \cdot D\varphi_j - (\sum_{j} r_j \cdot D\varphi_j)\bigg| < \varepsilon \leq 1$$

by the triangle inequality, since $\nabla \psi$ and $H_j$ are uniformly bounded. There exist $\tilde{r}_j \in \mathbb{R}^D$ satisfying

$$\sum_{j \in J} \tilde{r}_j \cdot D\varphi_j = \sum_{j \in J} r_j \cdot D\varphi_j$$

and $|\tilde{r}_j| = O(1)$ for every $j \in J$. Define $\tilde{H}_j = H_j + \tilde{r}_j$. These modified functions define the same sublevel set as do the original $h_j$, since

$$\sum_{j}(\tilde{H}_j \circ \varphi_j) \cdot D\varphi_j \equiv \sum_{j}(h_j \circ \varphi_j) \cdot D\varphi_j \quad \text{on } \tilde{E}.$$
Thus we may replace $h_j$ by $\tilde{h}_j$ for all indices $j$. The functions $\tilde{h}_j$ are now drawn from a compact family of $C^\omega$ functions determined by $\psi, \{\varphi_j : j \in J\}$. The same reasoning as in the above analyses of sublevel sets completes the proof of Theorem 17.1. □

The less singular case in which $|J| < D/d$ can be treated by a simplified form of this reasoning. Details are omitted.

The intermediate conclusion that $h_j = H_j + r_j$ on a large set, with $H_j$ uniformly bounded and $r_j$ constant though not necessarily uniformly bounded, breaks down without the restriction $|J| < 2D/d$. For an example, consider $d = 1, D = 2$, and $|J| = 4$ with mappings $\varphi_1(x) = \varphi_1(x_1, x_2) = x_1, \varphi_2(x) = x_2, \varphi_3(x) = x_1 + x_2, \varphi_4(x) = x_1 - x_2$. Set $G_j(x) = 2x^2$ for $j = 1, 2$, and $= -x^2$ for $j = 3, 4$. Then $\sum_{j=1}^4 G_j \circ \varphi_j \equiv 0$. Therefore $g_j = G'_j$ satisfy $\sum_{j=1}^4 (g_j \circ \varphi_j) \cdot D\varphi_j \equiv 0$. Therefore any tuple of functions $h_j$ could be replaced by $h_j + t g_j$ for any parameter $t \in \mathbb{R}$, without changing the associated sublevel set $\mathcal{E}$. Thus no upper bound at all holds for $(h_j : j \in J)$ modulo constants $r_j$, as in the above argument.

Conversely, in the context of the preceding paragraph, if $\sum_j G_j \circ \varphi_j \equiv 0$ then each $G_j$ must be a polynomial of degree at most 2. Thus each $g_j$ is a polynomial of degree at most 1, though not necessarily constant. This suggests that when $|J| \geq 2D/d$, the reasoning should be modified by applying difference operators to $\nabla \psi - \sum_j (h_j \circ \varphi_j) \cdot D\varphi_j$, and that difference operators of higher degrees should be required as $D/d$ increases.

18. A SCALAR-VALUED SUBLEVEL SET INEQUALITY

Let $B \subset \mathbb{R}^2$ be a ball of positive radius, and let $\varphi_j : B \to \mathbb{R}^1$ be real analytic for $j \in \{1, 2, 3\}$. Suppose that $\nabla \varphi_j$ are pairwise linearly independent at each point in $B$. Let $0 \leq \eta \in C^\infty(B)$.

The functional equation $f(x) + g(y) + h(x+y) = 0$, has been widely studied. Its solutions are the ordered triples $(f(x), g(y), h(x+y)) = (ax + c_1, ay + c_2, a(x+y) - c_1 - c_2)$ with $a, c_1, c_2$ all constant, and no others. Approximate solutions, in a certain sense, have been studied in [7]. We consider here the more general functional equation

$$\sum_{j=1}^3 (f_j \circ \varphi_j) = 0 \text{ almost everywhere} \tag{18.1}$$

where the mappings $\varphi_j$ need not be linear, and the functions $f_j$ are real-valued. We discuss related sublevel sets

$$S(f, r) = \{x \in B : |\sum_{j=1}^3 (f_j \circ \varphi)(x)| \leq r\} \tag{18.2}$$

associated to ordered triples $f$ of scalar-valued functions. The inequality (18.2) differs from corresponding inequalities studied and exploited in various proofs above in two ways: it is homogeneous rather than inhomogeneous, and it is a single scalar inequality, rather than a system of two scalar inequalities.

Theorem 18.2 has the following implication concerning the nonexistence of nontrivial solutions of (18.1).

**Corollary 18.1.** Let $B \subset \mathbb{R}^2$ be a closed ball of positive, finite radius. For $j \in \{1, 2, 3\}$ let $\varphi_j \in C^\omega$ map a neighborhood of $B$ to $\mathbb{R}$, and suppose that $\nabla \varphi_j$ are pairwise linearly independent at each point of $B$. Suppose that the curvature of the web defined by $(\varphi_j : j \in$
\{1, 2, 3\}) does not vanish identically on \(B\). Let \(f\) be an ordered triple of Lebesgue measurable real-valued functions. Suppose that for each index \(j\) and each \(t \in \mathbb{R}\),

\[
|\{x : f_j(x) = t\}| = 0.
\]

(18.3)

If \(f\) is a solution of the functional equation (18.1) then each function \(f_j\) is constant.

In particular, all \(C^\omega\) solutions \(f\) of (18.1) are constants. Indeed, one of the three component functions \(f_j\) must fail to satisfy the hypothesis (18.3), and hence must be constant. It follows immediately from the functional equation (18.1) that the other two component functions are also constant. \(\square\)

A more quantitative statement is as follows.

**Corollary 18.1.** Let \(B \subset \mathbb{R}^2\) be a closed ball of positive, finite radius. For \(j \in \{1, 2, 3\}\) let \(\varphi_j \in C^\omega\) map a neighborhood of \(B\) to \(\mathbb{R}\), and suppose that \(\nabla \varphi_j\) are pairwise linearly independent at each point of \(B\). Suppose that the curvature of the web defined by \((\varphi : j \in \{1, 2, 3\})\) does not vanish identically on \(B\). There exist \(\delta > 0\) and \(C < \infty\) such that for any ordered triple \(f\) of Lebesgue measurable real-valued functions and any \(r \in (0, \infty)\), the sublevel set \(S(f, r)\) satisfies

\[
|S(f, r)| \leq C \sup_{t \in \mathbb{R}} |\{x \in \varphi_j(B) : |f_j(x) - t| \leq r\}|^\delta
\]

for each \(j \in \{1, 2, 3\}\).

In §19 we discuss a related inequality for sublevel sets associated to expressions \(\sum_{j=1}^3 a_j(x)(f_j \circ \varphi_j)(x)\) with nonconstant coefficients \(a_j\), in the special case in which the mappings \(\varphi_j\) are all linear.

Returning to the two corollaries formulated above, we will first prove Corollary 18.2, then will indicate how a modification of the proof gives Corollary 18.1. The following lemma will be used.

**Lemma 18.3.** Let \(\sigma < 0\). Let \(I \subset \mathbb{R}\) be a bounded interval. Then there exists \(C < \infty\) such that for any real-valued function \(f \in L^2(\mathbb{R})\) supported in a fixed bounded set, for any \(A \in (0, \infty)\),

\[
\int_{A < \lambda \leq 1} \|1_I e^{i\lambda f}\|^2_{H^\sigma} \, d\lambda \leq C A \sup_{t \in \mathbb{R}} |\{x \in I : |f(x) - t| \leq A^{-1}\}|^{\sigma}.
\]

(18.5)

**Proof.** It suffices to treat the case \(A = 1\), since the substitution \(\lambda = A \tau\) reduces the general case to this one.

Let \(h\) be a nonnegative Schwartz function satisfying \(h(y) \geq 1\) for all \(y \in [-1, 1]\), with \(\widehat{h}\) supported in \([-1, 1]\).

\[
\int_{\lambda \leq 1} \|1_I e^{i\lambda f}\|^2_{H^\sigma} \, d\lambda \leq \int h(\lambda) \|1_I e^{i\lambda f}\|^2_{H^\sigma} \, d\lambda
\]

\[
= \int h(\lambda) \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{i\lambda f(x)} e^{-ix\xi} 1_I(x) \, dx \right|^2 (1 + \xi^2)^\sigma \, d\xi \, d\lambda
\]

\[
= \int h(\lambda) \int_{\mathbb{R}} \int_{I \times I} e^{i\lambda f(x) - f(y)} e^{-i(x-y)\xi} \, dx \, dy (1 + \xi^2)^\sigma \, d\xi \, d\lambda
\]

\[
= \int_{I \times I} \int_{\mathbb{R}} e^{-i(x-y)\xi} (1 + \xi^2)^\sigma \, d\xi \, \widehat{h}(A(f(y) - f(x))) \, dx \, dy
\]

\[
\leq CA \int_{I \times I} |x - y|^{-1 - \sigma} \left| \widehat{h}(A(f(y) - f(x))) \right| \, dx \, dy.
\]
Since $\sigma < 0$, this is majorized by
\[
CA \int_{\mathbb{R}^2} |x - y|^{-1+|\sigma|} 1_{|f(x) - f(y)| \leq A^{-1} (x, y)} \, dx \, dy
\]
\[
\leq CA \sup_{y \in I} \int_{I} |x - y|^{-1+|\sigma|} 1_{|f(x) - f(y)| \leq A^{-1} (x)} \, dx
\]
\[
\leq CA \sup_{t} \{|x \in I : |f(x) - t| \leq A^{-1}|\sigma|\}.
\]

Proof of Corollary 18.2. It suffices to establish the conclusion in the special case in which $r = 1$, since replacing $f_j$ by $r^{-1} f_j$ reduces the general case to this one.

Fix a nonnegative $C_0^\infty$ cutoff function $\zeta$. We aim for an upper bound for $\int_{\mathbb{R}^2} 1_{S(f, 1)} \cdot \zeta \, dx \, dy$. Let $h : \mathbb{R} \to [0, \infty)$ be $C^\infty$ and compactly supported, and be $\equiv 1$ on $[-1, 1]$. Consider instead the majorant
\[
\int \int h((\sum_j (f_j \circ \varphi_j)) \cdot \zeta).
\]
By implementing a partition of unity, we may introduce $C_0^\infty$ cutoff functions satisfying $\prod_{j=1}^{3} \eta_j(\varphi_j(x, y)) \equiv 1$ on the support of $\zeta$, with $\eta_j$ supported on an interval $I_j$. Then (18.6) is equal to
\[
c \int_{\mathbb{R}} \hat{h}(\lambda) \left( \int_{\mathbb{R}^2} \prod_{j} (\eta_j \circ \varphi_j) e^{i\lambda f_j \circ \varphi_j} \zeta(x, y) \, dx \, dy \right) \, d\lambda.
\]
By Theorem 4.2, there exists $\sigma < 0$ for which (18.7) is majorized by
\[
Cr \int_{\mathbb{R}} (1 + \lambda)^{-2} \prod_{j=1}^{3} \|\eta_j e^{i\lambda f_j}\|_{H^s} \, d\lambda \leq C \prod_{j=1}^{3} \left( \int_{\mathbb{R}} (1 + \lambda)^{-2} \|\eta_j e^{i\lambda f_j}\|_{H^s}^3 \, d\lambda \right)^{1/3}
\]
\[
\leq C \prod_{j=1}^{3} \left( \int_{\mathbb{R}} (1 + \lambda)^{-2} \|\eta_j e^{i\lambda f_j}\|_{H^s}^2 \, d\lambda \right)^{1/3}
\]
since $\|\eta_j e^{i\lambda f_j}\|_{H^s} \leq \|\eta_j e^{i\lambda f_j}\|_{L_2} = O(1)$ uniformly in all parameters because each $f_j$ is real-valued and $\eta_j$ has bounded support.

For any index $j \in \{1, 2, 3\}$,
\[
\int_{\mathbb{R}} (1 + \lambda)^{-2} \|\eta_j e^{i\lambda f_j}\|_{H^s}^2 \, d\lambda \leq C \sum_{k=0}^{\infty} 2^{-2k} \int_{|\lambda| \leq 2^k} \|\eta_j e^{i\lambda f_j}\|_{H^s}^2 \, d\lambda.
\]
To each term in this sum, apply Lemma 18.3 with $A = 2^k$ to obtain a majorization by
\[
C \sum_{k=0}^{\infty} 2^{-2k} \cdot 2^k \sup_{t} \{|x \in I_j : |f_j(x) - t| \leq 2^{-k}\} |\sigma| \leq C \sup_{t} \{|x \in I_j : |f_j(x) - t| \leq 1\} |\sigma|.
\]
Inserting this bound into (18.8) gives
\[
|S(f, 1)| \leq C \prod_{j=1}^{3} \sup_{t_j \in \mathbb{R}} \{|x \in \varphi_j(B) : |f_j(x) - t_j| \leq 1\} |\sigma|^{1/3}.
\]

\[\square\]
We have implicitly proved a lemma that may be useful in future work:

**Lemma 18.4.** Let \( \sigma < 0 \). Let \( \eta \in C^\infty(\mathbb{R}) \) be supported in a closed bounded interval \( I \subset \mathbb{R} \). There exists \( C < \infty \), depending on \( \sigma, \eta, |I| \), such that for any measurable function \( f : \mathbb{R} \to \mathbb{R} \),

\[
\int_\mathbb{R} (1 + \lambda^2)^{-1/2} \| \eta e^{i\lambda f} \|_{H^\sigma}^2 \, d\lambda \leq C \sup_t \{ |x - y| : |f(x) - t| \leq 1 \}^{|\sigma|}.
\]

**Proof of Corollary 18.4.** Defining a measure \( \mu \) on \( I^2 \) by \( d\mu(x,y) = |x - y|^{-1+\gamma} \, dx \, dy \), we have shown that

\[
\int_{|r| \leq 2^{k+1}} \| \eta_j e^{i\lambda f_j} \|_{H^\sigma}^2 \, d\lambda \leq C 2^k r^{-1} \mu(\{(x,y) : |f_j(x) - f_j(y)| \leq 2^{-k} r\})
\]

By summing over all nonnegative integers \( k \) we deduce that

\[
\int_{|r| \leq 2} r (1 + r\lambda)^{-2} \| \eta_j e^{i\lambda f_j} \|_{H^\sigma}^2 \, d\lambda \leq C \mu(\{(x,y) \in I^2 : |f_j(x) - f_j(y)| \leq r\})
\]

If \( f_j \) satisfies the hypothesis \( 18.3 \), then \( \mu(\{(x,y) \in I^2 : |f_j(x) - f_j(y)| \leq r\}) \to 0 \) as \( r \to 0^+ \). Therefore \( |S(f,r)| \to 0 \) as \( r \to 0^+ \). Therefore the set of points at which the equation \( 18.1 \) holds is a Lebesgue null set. \( \square \)

**19. A SCALAR SUBLEVEL SET INEQUALITY WITH VARIABLE COEFFICIENTS**

Throughout this section, \( \varphi_j : \mathbb{R}^2 \to \mathbb{R}^1 \) are assumed to be linear and surjective. Let \( \Omega \subset \mathbb{R}^2 \) be a nonempty bounded open ball or parallelepiped. For \( j \in \{1,2,3\} \) let \( a_j : \Omega \to \mathbb{R} \) be \( C^\omega \) functions. By this we mean that \( a_j \) extends to a real analytic function defined in some neighborhood of \( \bar{\Omega} \). To any three-tuple \( f = (f_j : j \in \{1,2,3\}) \) of Lebesgue measurable functions \( f_j : \Omega \to \mathbb{R} \), and to any \( \varepsilon > 0 \), associate the sublevel set

\[
S(f,\varepsilon) = \{ x \in \Omega : \left| \sum_{j=1}^3 a_j(x)(f_j \circ \varphi_j)(x) \right| < \varepsilon \}.
\]

**Theorem 19.1.** Let \( \varphi_j : \mathbb{R}^2 \to \mathbb{R}^1 \) be pairwise linearly independent linear mappings. Let \( \Omega, a_j \) be as above. Suppose that for each \( j \in \{1,2,3\}, a_j(x) \neq 0 \) for every \( x \in \Omega \). Finally, suppose that for any nonempty open set \( U \subset \Omega \) and any \( C^\omega \) functions \( F_j : U \to \mathbb{R} \) satisfying

\[
\sum_{j=1}^3 a_j(x)(F_j \circ \varphi_j)(x) = 0 \quad \text{for every} \ x \in U,
\]

all three functions \( F_j \) vanish identically on \( U \). Then there exist \( \gamma > 0 \) and \( C < \infty \) such that for every \( \varepsilon > 0 \) and every three-tuple \( f \) of Lebesgue measurable functions satisfying

\[
|f_1(y)| \geq 1 \ \forall \ y \in \varphi_1(\Omega),
\]

the sublevel set \( S(f,\varepsilon) \) satisfies

\[
|S(f,\varepsilon)| \leq C \varepsilon^\gamma.
\]

The conclusion seems likely to remain valid if the hypothesis that \( a_j \) vanish nowhere, is relaxed to \( a_j \) not vanishing identically. We emphasize that the mappings \( \varphi_j \) are assumed in Theorem 19.1 to be linear.

Several results related to Theorem 19.1 are known, besides those in 18. If each \( a_j \) is constant and the mappings \( \varphi_j \) are linear, then whenever \( \sum_j f_j \circ \varphi_j \) vanishes Lebesgue almost everywhere, each \( f_j \) must agree almost everywhere with an affine function. If \( | \sum_j f_j \circ \varphi_j(x) | \leq \varepsilon \) for all \( x \in \Omega \setminus E \), and if \( |E| \) is sufficiently small, then there exist affine functions...
For any $x, y, s \in \mathcal{E}_2$, 
\begin{equation}
    f_3(x + y) + a(x - s, y + s) f_1(x - s) = b(x - s, y + s) f_2(y + s) + O(\varepsilon) \quad \forall (x, y, s) \in \mathcal{E}_2.
\end{equation}
For any \((x, y, s) \in \mathcal{E}_2\) we have the two approximate relations (19.5), (19.8). The contributions of \(f_3\) cancel when these two relations are subtracted, leaving

\[
(a(x - s, y + s) - a(x, y)) f_1(x)
= b(x - s, y + s) f_2(y + s) - b(x, y) f_2(y) + O(\varepsilon) \quad \forall (x, y, s) \in \mathcal{E}_2.
\]

The set \(\mathcal{E}_3\) of all \((x', s, y) \in \mathbb{R}^4\) such that both \((x, y, s)\) and \((x', y, s)\) belong to \(\mathcal{E}_2\) satisfies

\[
|\mathcal{E}_3| \gtrsim |\mathcal{E}_2|^2 \gtrsim |\mathcal{E}|^8 \gtrsim \varepsilon^{8\delta_0}.
\]

Consider any such \((x', s, y)\). Consider the conjunction of (19.9) with the corresponding relation with \((x, y, s)\) replaced by \((x', y, s)\). Express this pair of relations as the approximate matrix equation

\[
B(x, x', s, y) \begin{pmatrix} f_2(y) \\ f_2(y + s) \end{pmatrix} = A(x, x', s, y) + O(\varepsilon)
\]

in which the coefficient matrices \(A, B\) are the square matrix

\[
B(x, x', s, y) = \begin{pmatrix} b(x - s, y + s) & -b(x, y) \\ b(x' - s, y + s) & -b(x', y) \end{pmatrix}
\]

and the column matrix

\[
A(x, x', s, y) = \begin{pmatrix} a(x - s, y + s) f_1(x - s) - a(x, y) f_1(x) \\ a(x' - s, y + s) f_1(x' - s) - a(x', y) f_1(x') \end{pmatrix},
\]

respectively.

**Lemma 19.2.** As a function of \((x, x', s, y)\), the determinant \(\det(B)\) does not vanish identically.

**Proof.** Assume to the contrary that \(\det(B) \equiv 0\). Then the ratio \(b(x - s, y + s) / b(x' - s, y + s)\) is independent of \(s\), whence

\[
\frac{\partial^2}{\partial s \partial x} \ln |b(x - s, y + s)| \equiv 0.
\]

Therefore \(b\) takes the form

\[
b(x, y) \equiv h(x + y) \cdot k(y)
\]

for some smooth nowhere vanishing functions \(h, k\).

Choosing \(f_1(x) \equiv 0, f_2(y) = k(y)^{-1}\), and \(f_3(z) = h(z)\), we have

\[
f_3(x + y) + a(x, y) f_1(x) \equiv b(x, y) f_2(y)
\]

on a nonempty open set. This contradicts the hypothesis of Theorem 19.1 that the functional equation has no solution except the trivial solution \(f_1 \equiv f_2 \equiv f_3 \equiv 0\).

For any \((x, x', s, y)\), multiply both sides of the approximate matrix equation (19.11) by the cofactor matrix of \(B(x, x', s, y)\) to conclude that

\[
\det(B)(x, x', s, y) \cdot g(y) = A(x, x', s, y) + O(\varepsilon),
\]

where \(A(x, x', s, y)\) is one of the two components of the product of the cofactor matrix of \(B(x, x', s, y)\) with \(A(x, x', s, y)\). Thus \(A\) is a linear combination of products of the given coefficients \(a, b\), evaluated at points that are functions of \((x, x', s, y)\), with coefficients in \([-2, 2]^4\). Those coefficients are the quantities \(f_1(x - s), f_1(x), f_1(x' - s), f_1(x')\), whose dependence on \((x, x', s)\) is merely Lebesgue measurable and is unknown. However, \(A\) depends linearly, hence real analytically, on those coefficients.
By partitioning $[0,1]^2$ into finitely many smaller cubes, and identifying each subcube again with $[0,1]^2$ via an affine change of variables, we may assume that each coefficient $a_j$ is defined and analytic in a large fixed ball that contains $[0,1]^2$. Define $K \subset \mathbb{R}^7$ to be the set of all tuples $\theta = (x, x', s, r) = (x, x', s, r_1, r_2, r_3, r_4)$ such that $r \in [-2, 2]^4$, $(x, x') \in [0, 1]^2$, and $s \in [-2, 2]$. $K$ is compact and connected. Let $(y, \theta)$ vary over $[0,1] \times K$. \eqref{eq19.15} can thus be written as

\begin{equation}
\det(B)(y, \theta) \cdot f_2(y) = \mathcal{A}^*(y, \theta) + O(\varepsilon) \tag{19.16}
\end{equation}

for all $(y, \theta) \in K$ for which $(x, x', s, y) \in \mathcal{E}_3$, with $\mathcal{A}^*$ a real analytic function of $(y, \theta)$ in a neighborhood of $[0, 1] \times K$.

The set of all $(x, x', s, y) \in \mathcal{E}_3$ has Lebesgue measure $\geq |\mathcal{E}|^{C_0} \geq \varepsilon^{C_0 \delta_0}$. On the other hand, since the $C^\omega$ function $(x, x', s, y) \mapsto \det(B)(x, x', s, y)$ does not vanish identically, there exists $\eta > 0$ such that

\begin{equation}
|\{(x, x', s, y) : |\det(B)(x, x', s, y)| \leq r \}| \lesssim r^\eta \forall r \in (0, 1). \tag{19.17}
\end{equation}

Choose a constant $C_1 \in \mathbb{R}^+$ that satisfies $\eta \cdot C_1 > C_0$. Applying the preceding inequality with $r = \varepsilon^{C_1 \delta_0}$, $r^\eta$ is small relative to $\varepsilon^{C_0 \delta_0}$, and thus we may conclude that there exists $(x, x', s)$ satisfying

\begin{equation}
|\{y \in [0,1] : (x, x', s, y) \in \mathcal{E}_3 \text{ and } |\det(B)(x, x', s, y)| \geq \varepsilon^{C_1 \delta_0}\}| \gtrsim \varepsilon^{\delta_0}. \tag{19.18}
\end{equation}

The conclusion is that there exists $\bar{\theta} = \bar{\theta}(f, \varepsilon) \in K$ satisfying \eqref{eq19.16} for the indicated set of pairs $(y, \theta)$, with

\begin{equation}
|\det(B)(x, x', s, y)| \geq \varepsilon^{C_1 \delta_0}. \tag{19.19}
\end{equation}

For such $\bar{\theta}$,

\begin{equation}
|f_2(y) - \det(B)(y, \bar{\theta})^{-1} \mathcal{A}^*(y, \bar{\theta})| = O(\varepsilon) \tag{19.20}
\end{equation}

for all $y$ in a set of measure $\gtrsim \varepsilon^{\delta_0}$.

Revert to the initial notation, with mappings $\varphi_j$ and coefficients $a_j$. The conclusion proved thus far can be summarized as follows. Let $a_j, \varphi_j$ satisfy the hypotheses of Theorem \ref{thm19.1}. Let $\delta_0, \varepsilon_0 > 0$ be sufficiently small. There exist a compact connected set $K \subset \mathbb{R}^7$, and a function $F_2 : [0,1] \times K \to \mathbb{R}$ that extends meromorphically to a neighborhood of $[0,1] \times K$, with the following property. Let $f$ and $\varepsilon \in (0, \varepsilon_0]$ satisfy the hypotheses of the theorem, as well as the auxiliary condition $|S(f, \varepsilon)| \geq \varepsilon^{\delta_0}$. Then there exist $\mathcal{E}' \subset S(f, \varepsilon)$ satisfying $|\mathcal{E}'| \gtrsim |S(f, \varepsilon)|^C$, and $\bar{\theta} = \bar{\theta}(f, \varepsilon) \in K$, such that the triple $(f_1, \tilde{f}_2, f_3)$ defined by $\tilde{f}_2(y) = F_2(y, \bar{\theta})$ satisfies

\begin{equation}
|a_2(x) \tilde{f}_2(\varphi_2(x)) + \sum_{j \neq 2} a_j(x) f_j(\varphi_j(x))| = O(\varepsilon) \forall x \in \mathcal{E}' \tag{19.21}
\end{equation}

and

\begin{equation}
|f_2(y) - \tilde{f}_2(y)| = O(\varepsilon) \forall y \in \varphi_2(\mathcal{E}'). \tag{19.22}
\end{equation}

Moreover, the function $F_2$ factors as $F_2(y, \theta) = \alpha(y, \theta)/\beta(y, \theta)$ with $\alpha, \beta$ both analytic in a neighborhood of $[0,1] \times K$ and satisfying

\begin{equation}
|\beta(y, \bar{\theta})| \geq \varepsilon^{C_1 \delta_0} \forall y \in \varphi_2(\mathcal{E}'). \tag{19.23}
\end{equation}

This reasoning can be applied twice more in succession, with the roles of the indices $j \in \{1, 2, 3\}$ permuted, to approximate each of $f_1, f_3$ by $C^\omega$ functions in the same way as has been done for $f_2$. With each iteration, $\mathcal{E}$ is replaced by a subset, and one of the
functions $f_k$ is replaced by an approximating meromorphic function $\tilde{f}_k$; these replacements are retained through subsequent iterations. The conclusion may be summarized as follows, incorporating a change in the meaning of the auxiliary space $K$.

Let $a_j, \varphi_j$ be as in the statement of Theorem 19.1. Let $\delta_0, \varepsilon_0 > 0$ be sufficiently small. There exist a compact connected set $K \subset \mathbb{R}^{21} = (\mathbb{R}^7)^3$ and three $C^\omega$ functions $F_j : [0, 1] \times K \to \mathbb{R}$, such that for any $f$ and any $\varepsilon \in (0, \varepsilon_0]$ satisfying the hypotheses of the theorem with associated sublevel set $S(f, \varepsilon)$ satisfying $|S(f, \varepsilon)| \geq \varepsilon^{\delta_0}$, there exist a subset $\mathcal{E}' \subset S(f, \varepsilon) \subset [0, 1]^2$ satisfying $|\mathcal{E}'| \geq |S(f, \varepsilon)|^C$ and an associated parameter $\tilde{\theta} = \tilde{\theta}(f, \varepsilon) \in K$, such that the ordered triple of approximating functions $(\tilde{f}_j : j \in \{1, 2, 3\})$ defined by $\tilde{f}_j(y) = F_j(y, \tilde{\theta})$ satisfies

\begin{equation}
| \sum_{j=1}^{3} a_j(x) \tilde{f}_j(\varphi_j(x)) | = O(\varepsilon) \quad \forall x \in \mathcal{E}'
\end{equation}

and

\begin{equation}
| f_j(y) - \tilde{f}_j(y) | = O(\varepsilon) \quad \forall y \in \varphi_j(\mathcal{E}').
\end{equation}

Moreover, for each $j \in \{1, 2, 3\}$, the function $F_j$ factors almost everywhere in its domain $[0, 1] \times K$ as

\[ F_j(y, \theta) = \alpha_j(y, \theta) / \beta_j(y, \theta) \]

with $\alpha_j, \beta_j$ analytic in a neighborhood of $[0, 1] \times K$. The denominators $\beta_j$ satisfy

\begin{equation}
| \beta_j(y, \tilde{\theta}(f, \varepsilon)) | \geq \varepsilon^{C_1 \delta_0} \quad \forall y \in \varphi_j(\mathcal{E}').
\end{equation}

The exponents $C, C_1$ depend only on the data $a_j, \varphi_j$ and the choice of $\varepsilon_0, \delta_0$.

Consider the function of $(x, \theta) \in [0, 1]^2 \times K$ defined by

\begin{equation}
H(x, \theta) = \sum_{j=1}^{3} a_j(x) \cdot \alpha_j(\varphi_j(x), \theta) \cdot \prod_{i \neq j} \beta_i(\varphi_i(x), \theta)
\end{equation}

along with the partial derivatives $\frac{\partial^\alpha}{\partial x^\alpha} H(x, \theta)$ with respect to $x$ of $H$, indexed by $\alpha \in \{0, 1, 2, \ldots \}^3$. This function $H$ is arrived at by multiplying $\sum_{j=1}^{3} a_j(x) F_j(\varphi_j(x), \theta)$ by $\prod_{i=1}^{3} \beta_i(\varphi_i(x), \theta)$ in order to arrive at a function that is holomorphic, rather than merely meromorphic.

If $x \in \mathcal{E}'$ then

\[ | H(x, \theta) | \lesssim | \sum_j a_j(x) F_j(\varphi_j(x, \theta)) | = O(\varepsilon) \]

since the functions $\beta_i$ are bounded. Thus in order to majorize the Lebesgue measure of the sublevel set $S(f, \varepsilon)$, it will suffice to produce a satisfactory majorization of the measure of a sublevel set of $x \mapsto H(x, \tilde{\theta}(f, \varepsilon))$.

To analyze sublevel sets associated to $H$ requires information concerning $H$, and information concerning $\tilde{\theta}(f, \varepsilon)$. But first, we review a happy general property of real analytic functions that depend real analytically on auxiliary parameters. See Bourgain [3], and Stein and Street [26]. There exist $N, C < \infty$ such that for any multi-index satisfying $|\alpha| = N + 1$, for every $(x, \theta) \in [0, 1]^2 \times K$,

\begin{equation}
| \frac{\partial^\alpha}{\partial x^\alpha} H(x, \theta) | \leq C \sum_{|\beta| \leq N} | \frac{\partial^3}{\partial x^\beta} H(x, \theta) |.
\end{equation}
Introducing the nonnegative $C^\omega$ function
\[(19.29) \quad \tilde{H}(x, \theta) = \sum_{|\beta| \leq N} \left| \frac{\partial^\alpha H(x, \theta)}{\partial x^\alpha} \right|^2,
\]
it follows from the Cauchy-Schwarz inequality that $\tilde{H}$ satisfies the differential inequality
\[(19.30) \quad |\nabla_x \tilde{H}(x, \theta)| \leq C|\tilde{H}(x, \theta)|
\]
uniformly for all $(x, \theta) \in [0, 1]^2 \times K$. This differential inequality allows us to replace $\tilde{H}(x, \theta)$ by a function of $\theta$ alone; it implies that there exists $C \in (0, \infty)$ such that the function $G(\theta) = \tilde{H}((0, 0), \theta)$ satisfies
\[(19.31) \quad C^{-1} G(\theta) \leq \tilde{H}(x, \theta) \leq CG(\theta)
\]
uniformly for all $(x, \theta) \in [0, 1]^2 \times K$.

$G \in C^\omega$ in a neighborhood of $K$, and $H(x, \theta) = 0$ for every $x \in [0, 1]^2$ if and only if $G(\theta) = 0$.

The following result, a variant of a lemma often attributed to van der Corput, is essentially well known.

**Lemma 19.3.** Let $N < \infty$. Let $C_1, C_2 \in (0, \infty)$. There exist $C < \infty$ and $\rho > 0$ with the following property. Let $\psi \in C^{N+1}([0, 1]^2)$ satisfy $\|\psi\|_{C^{N+1}} \leq C_2$ and
\[
\sum_{0 \leq |\alpha| \leq N} |\partial^\alpha \psi(x)| \geq C_1 \forall x \in [0, 1]^2.
\]

Then for any $\epsilon > 0$,
\[(19.32) \quad |\{x \in [0, 1]^2 : |\psi(x)| \leq \epsilon\}| \leq C \epsilon^\rho.
\]

The upper bound on the $C^{N+1}$ norm cannot be dispensed with entirely in this formulation. Consider for instance the example $\psi(x) = \epsilon \sin(\epsilon^{-1} x_1)$, with $N = 2$. A consequence of the lemma is for any $\theta$ for which $G(\theta) \neq 0$, for any $\eta \in (0, \infty)$,
\[(19.33) \quad |\{x \in [0, 1]^2 : |H(x, \theta)| \leq \eta G(\theta)\}| \leq C \eta^\rho.
\]

To complete the proof of the theorem, it would be desirable to know that $G$ does not vanish identically on $K$. We will not actually prove that this is the case. Instead, note that if $|S(f, \epsilon)| \leq \epsilon^{\delta_0}$ for every datum $(f, \epsilon)$ satisfying the hypotheses of the theorem, then the desired conclusion holds with $\gamma = \delta_0$. Thus it suffices to treat the case in which there exists at least one datum $(f, \epsilon)$ that satisfies the reverse inequality $|S(f, \epsilon)| > \epsilon^{\delta_0}$, along with the hypotheses of the theorem. We will prove that $G(\tilde{\theta}(f, \epsilon)) \neq 0$ for any such datum, and hence may assume in the remainder of the proof that $G$ does not vanish identically on $K$.

To prove that $G(\tilde{\theta}) \neq 0$ in this situation, with $\tilde{\theta} = \tilde{\theta}(f, \epsilon)$, observe first that none of the factors $\beta_j(y, \theta)$ vanishes identically as a function of $y$. Indeed, each such factor is $\succeq \epsilon^{C_1 \delta_0}$ on a set whose Lebesgue measure is minorized by a positive quantity. By dividing by $\prod_j \beta_j(\varphi_j(x), \theta)$ in the definition of $H$, we conclude that if $G(\tilde{\theta}) = 0$ then $\sum_{j=1}^3 a_j(x) F_j(\varphi_j(x), \tilde{\theta}) = 0$ almost everywhere as a function of $x \in [0, 1]^2$. By the main hypothesis of Theorem 19.1, $\sum_{j=1}^3 a_j(x) F_j(\varphi_j(x), \tilde{\theta})$ vanishes on an open set of values of $x$ only if each function $x \mapsto F_j(\varphi_j(x), \tilde{\theta})$ vanishes identically. However, the construction has $|f_1(y) - F_1(y, \tilde{\theta})| = O(\epsilon)$ for $y$ in a subset of positive measure, and by hypothesis, $|f_1(y)| \in [1, 2]$ for almost every $y$. Therefore $F_1(y, \tilde{\theta}) \neq 0$.

Define the zero variety
\[(19.34) \quad Z = \{\theta \in K : G(\theta) = 0\}.
\]
G is $C\omega$ and nonnegative in a neighborhood of $K$, G does not vanish identically on $K$, and $K$ is connected. Therefore by a theorem of Lojasiewicz [18], there exist $\epsilon, \tau > 0$ such that
\begin{equation}
G(\theta) \geq c \text{ distance}(\theta, Z)^\tau \forall \theta \in K.
\end{equation}

If $f, \varepsilon, S(f, \varepsilon)$ satisfy the hypotheses, then $\tilde{\theta} = \tilde{\theta}(f, \varepsilon)$ satisfies $\text{distance}(\tilde{\theta}, Z) \geq \varepsilon^{C\delta_0}$. Indeed, consider any $x \in \mathcal{E}'$. Then for $y = \varphi_1(x)$, $|f_1(y) - F_1(y, \tilde{\theta})| = O(\varepsilon)$ and $|f_1(y)| \in [1, 2]$, so $|F_1(y, \tilde{\theta})| \geq 1 - O(\varepsilon) \geq \frac{1}{2}$. Since $F_1 = \alpha_1/\beta_1$, it follows that
\begin{equation}
|\alpha_1(y, \tilde{\theta})| \geq \frac{1}{2}|\beta_1(y, \tilde{\theta})| \geq \varepsilon^{C\delta_0}.
\end{equation}
The function $\alpha_1$ is real analytic with respect to both variables, hence is Lipschitz, and vanishes identically on $Z$. Therefore $\text{distance}(\tilde{\theta}, Z) \geq \varepsilon^{C\delta_0}$, and consequently $G(\theta) \geq \varepsilon^{C\delta_0}$. Applying (19.33) gives
\begin{equation}
|\{x \in [0, 1]^2 : |H(x, \tilde{\theta})| = O(\varepsilon)\}| = O((\varepsilon^{-1}C\delta_0)^\rho).
\end{equation}

If $\delta_0$ is chosen to be sufficiently small then $1 - C\delta_0 > 0$, so this inequality becomes
\begin{equation}
|\{x \in [0, 1]^2 : |H(x, \tilde{\theta})| = O(\varepsilon)\}| = O(\varepsilon^\gamma),
\end{equation}
for a certain exponent $\gamma > 0$ that depends only on the coefficients $a_j$ and the mappings $\varphi_j$. This completes the proof of Theorem [19.1].

20. A REMARK AND A QUESTION

Continuing to assume linearity of the mappings $\varphi_j$, more can be deduced from the analysis in [19]. Drop the assumption that no nontrivial solution exists, and ask whether for any $f = (f_1, f_2, f_3)$ and any $\varepsilon, f$ can be approximated within $O(\varepsilon)$ on some subset $S' \subset S(f, \varepsilon)$ satisfying $|S'| \geq |S(f, \varepsilon)|^C$, by an $\mathbb{R}^3$–valued function $g$ drawn from a finite-dimensional family of $C^\omega$ functions that depends only on the data $(\varphi_i, a_i : i \in \{1, 2, 3\})$.

More generally, in light of that analysis, we allow meromorphic approximants by asking whether there exist $g_j$ and $\beta_j$, drawn from such a family, such that $\beta_j$ does not vanish identically and $\beta_j f_j - g_j = O(\varepsilon^{1-\rho})$ on $\varphi_j(S')$. We refer to this as the approximability property.

It suffices to approximate $f_k$ by a component $g_k$ of such a $g$ for a single index $k$, for then a rather simple analysis can be applied to the relation $\sum_{j \neq k} a_j (f_j \circ \varphi_j) = -a_k (g_k \circ \varphi_k) + O(\varepsilon)$; restrict this equation to level curves of $\varphi_i$ for each of the two indices $i \neq k$ in turn and exploit the transversality hypothesis.

The analysis in [19] shows that $f_2$ can be so approximated, except possibly in the special case in which $a_2(x, y)/a_3(x, y)$ can be factored in the form $h(x + y)/k(y)$, that is, $(h \circ \varphi_3)/(k \circ \varphi_2)$. This reasoning can be repeated for any permutation of the indices 1, 2, 3. The conclusion, in invariant form with the mappings $\varphi_j$ assumed to be linear, is that the approximability property holds, and follows from the analysis sketched, for all but a small family of exceptional cases. Each of those exceptional cases can be transformed, by application of symmetries of the problem, to one of the two examples
\begin{align}
(20.1) & \quad f_1(x) + f_2(y) + f_3(x + y) = 0. \\
(20.2) & \quad f_1(x) + f_2(y) + \varepsilon^\tau f_3(x + y) = 0.
\end{align}

These symmetries are linear changes of variables in $\mathbb{R}^2$ and in the domains $\mathbb{R}^1$ of the three mappings $\varphi_j$, multiplication of the equation by an arbitrary nowhere vanishing $C^\omega$ function $b(x, y)$, and incorporation of coefficients into functions $f_j$ via multiplicative substitutions $f_j(x) = f_j(x)u_j(x)$, with $u_j \in C^\omega$ vanishing nowhere in the relevant domain. The equation
(20.2) has a two-dimensional space of solutions \( f \), with \( f_3(x) = c_1 e^{-x} + c_2 \) for arbitrary coefficients \( c_1, c_2 \in \mathbb{R} \). The approximability property does not hold for either (20.1) or (20.2); counterexamples can be constructed by exploiting multiprogressions of arbitrarily high rank.

**Question 20.1.** Let \( \epsilon > 0 \), and let \( f \) be measurable. Let \( \varphi_j(x, y) = x, = y, \) and \( = x + y \) for \( j = 1, 2, 3 \), respectively. Let \( S(f, \epsilon) \) be the set of all \((x, y) \in B\) satisfying \(|f_1(x) + f_2(y) + \epsilon^2 f_3(x + y)| < \epsilon\).

Do there exist an exact \( C^\omega \) solution \( f^\ast \) of (20.2) and a subset \( S' \subseteq S(f, \epsilon) \) satisfying \(|S'| \geq c|S(f, \epsilon)|^C\) such that \(|f_j \circ \varphi_j(x, y) - f_j^\ast \circ \varphi_j(x, y)| \leq C \epsilon\) for every \((x, y) \in S'\)? The constants \( c, C \) are to be independent of \( f, \epsilon \).

The answer is negative for the equation (20.1).

**Question 20.2.** Does Theorem 19.1 remain valid if the mappings \( \varphi_j \) are assumed to be merely real analytic with pairwise transverse gradients, rather than linear?

A manuscript answering Question 20.2 in the affirmative, under certain auxiliary hypotheses, is in progress [8]. That result is used to establish a quadrilinear analogue of Theorem 4.2 — again, under auxiliary hypotheses — in [9]. It would be desirable to go further, dropping the hypothesis that no exact \( C^\omega \) solutions of the underlying equation exist, and weakening the conclusion to approximability by exact solutions, as in Question 20.1.

## 21. Large sublevel sets: An example

Consider the ordered triple of submersions \([0, 1]^2 \to \mathbb{R}\) defined by \((x, y) \mapsto x, \mapsto y, \) and \( \mapsto x + y \). To any ordered triple \((f, g, h)\) of Lebesgue measurable functions associate the sublevel set

\[
(21.1) \quad \mathcal{E} = \{(x, y) \in [0, 1]^2 : |g(x) - h(x + y)| < \epsilon \quad \text{and} \quad |y - f(x) - h(x + y)| < \epsilon\}
\]

defined by the indicated inhomogeneous system of two inequalities for \((f, g, h)\). The reasoning developed above, for instance in §11, demonstrates that

\[
(21.2) \quad |\mathcal{E}| = O(\epsilon^{1/2}).
\]

That reasoning may appear to have been wasteful, and indeed, \(|\mathcal{E}| = O(\epsilon)\) uniformly for all affine functions \(f, g, h\). Here we show, via a construction based on multiprogressions of rank 2, that the exponent \(1/2\) in (21.2) cannot be improved.

Let \( \epsilon > 0 \) be small, with \( \epsilon^{-1/2} \in \mathbb{N} \). Set \( N = \epsilon^{-1/2} \). For each \( k \in \mathbb{Z} \), define

\[
(21.3) \quad f(x) = k \epsilon^{1/2} - x \quad \text{whenever} \quad |x - k \epsilon^{1/2}| < \frac{1}{2} \epsilon^{1/2}.
\]

Define

\[
(21.4) \quad g(y) = k \epsilon^{1/2} + k \epsilon \quad \text{whenever} \quad |y - k \epsilon^{1/2}| < \frac{1}{2} \epsilon^{1/2}.
\]

For each \( t \in \mathbb{R} \) there exist unique \( k, n \in \mathbb{Z} \) with \( 0 \leq n < N \) such that \(|t - (k \epsilon^{1/2} + n \epsilon)| < \frac{1}{2} \epsilon\). Define

\[
(21.5) \quad h(t) = n \epsilon^{1/2} + n \epsilon \quad \text{whenever} \quad |t - (k \epsilon^{1/2} + n \epsilon)| < \frac{1}{2} \epsilon.
\]
For \( m, n \in \mathbb{Z} \) satisfying \( 0 \leq n < N \), define \( E(m, n) \) to be the set of all \((x, y) \in \mathbb{R}^2 \) that satisfy the three inequalities

\[
\begin{align*}
|y - n \varepsilon^{1/2}| &< \frac{1}{2} \varepsilon^{1/2}, \\
|x - (m - n) \varepsilon^{1/2}| &< \frac{1}{2} \varepsilon^{1/2}, \\
|x + y - (m \varepsilon^{1/2} + n \varepsilon)| &< \frac{1}{2} \varepsilon.
\end{align*}
\]

The sets \( E(m, n) \) are pairwise disjoint and satisfy

\[
|E(m, n)| = \varepsilon^{3/2} + O(\varepsilon^2).
\]

The number of indices \((m, n) \in \mathbb{Z} \times \{0, 1, 2, \ldots, N - 1\}\) for which \( E(m, n) \subset [0, 1]^2 \) is \( \geq c \varepsilon^{-1} \).

If \( E(m, n) \subset [0, 1]^2 \), then \( E(m, n) \subset E \). Indeed, let \((x, y) \in E(m, n)\). Firstly,

\[
g(y) - h(x + y) = 0
\]
since both \( g(y) \) and \( h(x + y) \) are defined to be \( n \varepsilon^{1/2} + n \varepsilon \) in this region. Secondly,

\[
f(x) + h(x + y) - y = ((m - n) \varepsilon^{1/2} - x) + h(x + y) - y = -((x + y - m \varepsilon^{1/2} - n \varepsilon) + (h(x + y) - n \varepsilon^{1/2} - n \varepsilon)).
\]

Since \( x + y \) lies in the strip indicated in the definition of \( E(m, n) \),

\[|x + y - m \varepsilon^{1/2} - n \varepsilon| < \frac{1}{2} \varepsilon \text{ and } h(x + y) = n \varepsilon^{1/2} + n \varepsilon.\]

Consequently

\[
|y - f(x) - h(x + y)| < \frac{1}{2} \varepsilon.
\]

Thus \( E(m, n) \subset E \) whenever \( E(m, n) \subset [0, 1]^2 \). There are \( \geq c \varepsilon^{-1} \) such sets, pairwise disjoint and satisfying \( |E(m, n)| \geq \varepsilon^{3/2} + O(\varepsilon^2) \). Therefore

\[
|E| \geq c' \varepsilon^{1/2}
\]
for a certain constant \( c' > 0 \).

\[\square\]

22. Remarks on sublevel sets

Implicit in the discussion is a variant of the usual notion of a sublevel set bound. Let \( d \geq 1 \) be an arbitrary dimension. Let \( \varepsilon, \delta > 0 \) and \( N \in \mathbb{N} \) be parameters.

Let \( S \) be the collection of all sets \( S \subset \delta \mathbb{Z} = \{\delta n : n \in \mathbb{Z}\} \) of cardinality exactly \(|S| = N\). Let there be given \( d \) functions \( h_j \), each with domain \([0, 1]\) and with range in \( S \). Codomains consisting of sets of cardinality \( N \), rather than of \( N \)-tuples, are natural in the variant that we seek to formulate. Set \( h = (h_j : j \in \{1, 2, \ldots, d\}) \).

Let \( \phi : [0, 1]^d \to \mathbb{R} \) be \( C^1 \). Define \( E_N(\phi, h) \subset [0, 1]^d \) to be the set of all \( x \in [0, 1]^d \) for which there exists \((s_1, \ldots, s_d) \in (\delta \mathbb{Z})^d \), with each \( s_j \in h_j(x_j) \), satisfying

\[
|\nabla_j \phi(x) - s_j| \leq \varepsilon.
\]

Define

\[
\Lambda_N(\phi) = \sup_h |E_N(\phi, h)|.
\]

Question 22.1. For \( \phi \) or for a class of functions \( \phi \), what upper bounds are valid for \( \Lambda_N(\phi) \)?
In the special case $N = 1$, in which $h_j(x_j)$ can be regarded as a scalar rather than a set, we are asking for an upper bound for $\left|\{x : |\nabla(\tilde{\phi}) < \varepsilon\}\right|$, with $\tilde{\phi}(x) = \phi(x) - \sum_j H_j(x_j)$ and $H_j' = h_j$. There is a trivial majorization

\begin{equation}
\Lambda_N(\phi) \leq N^d \Lambda_1(\phi),
\end{equation}

obtained by regarding each $h_j$ as a collection of $N$ real-valued functions $h_{j,i}$, leading to an inclusion

$$E_N(\phi, h) \subset \bigcup_{i_1, \ldots, i_N} E_1\left(\phi, (h_{i_1}, \ldots, h_{i_N})\right).$$

Thus

$$\Lambda_N(\phi) \leq N^d \Lambda_1(\phi).$$

We hope that for large $N$, for natural classes of $\phi$ such as compact families of $C^\omega$ functions, stronger bounds hold for $\Lambda_N(\phi)$.

This is a simplification of the issue that arose, with $N$ comparable to $\lambda'$ for a certain positive exponent $t$, in the proof of Theorem 4.1. Let $L_j : [0, 1]^2 \to \mathbb{R}$ be submersions, for $j \in \{1, 2, 3\}$, with no two of these having linearly dependent differentials at any $x \in [0, 1]^2$.

Let $\varepsilon, \delta, N, S$ be as above. Let $M \in \mathbb{N}$ be another parameter.

Let $\phi : [0, 1]^2 \to \mathbb{R}$ be $C^1$. Let $h$ be as above. Define $M(x)$ to be the number of tuples $(s_1, s_2, s_3)$ with each $s_j \in h_j(L_j(x))$ that satisfy $|\nabla \phi(x) - s_j| < \varepsilon$. Let

$$E(\phi, h) = \{x : M(x) \geq M\}.$$

Let

$$\Lambda(\phi) = \sup_h |E(\phi, h)|.$$

**Question 22.2.** For $\phi$ and $\{L_j\}$ or for a class of such functions, what upper bounds does $\Lambda(\phi)$ satisfy in terms of $\varepsilon, N, M$?

A multitude of variants and generalizations of sublevel set inequalities are likely to be relevant to future investigations of oscillatory integral inequalities. With

$$S(f, \varepsilon) = \left\{x \in S : \left|\sum_{j \in J} a_j(x)(f_j \circ \varphi_j)(x)\right| < \varepsilon\right\},$$

the cardinality $|J|$ of the index set $J$, the dimension $D$ of the ambient set $S$, the dimension $d$ of the codomain of the mappings $\varphi_j$, the nature of the coefficient functions $a_j$ (which may be scalar- or matrix-valued, $C^\omega$ or $C^\infty$, and so on), the dimension of the codomain of vector-valued functions $f_j$ can all be varied. Mappings $\varphi_j$ that are homogeneous of degree one with respect to a subset of the coordinates for $S$ arise naturally, as Cauchy-Schwarz/TT$^*$ reasoning leads naturally to factors $f_k(\varphi_k(x')) J_k(\varphi_k(x))$, and the substitution $x' = x + t$ and Taylor expansion then lead to $F_k(\psi_k(x, t)) = f_k(\varphi_k(x) + tD\varphi_k(x)) J_k(\varphi_k(x))$. Sublevel sets of the more general type

$$S(f, \varepsilon) = \left\{x \in S : \left|\sum_{j \in J} \sum_{\alpha \in A} a_{j, \alpha}(x)(f_{j, \alpha} \circ \varphi_j)(x)\right| < \varepsilon\right\},$$

with $A$ another finite index set and with the mappings $\varphi_j$ independent of the index $\alpha \in A$, arise upon consideration of the formal gradient of $\sum_{j \in J} a_j \cdot (f_j \circ \varphi_j)$. 
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