The low-energy theory of d-wave quasiparticles coupled to fluctuating vortex loops that describes the loss of phase coherence in a two dimensional d-wave superconductor at $T = 0$ is derived from first principles. The theory has the form of 2+1 dimensional quantum electrodynamics ($QED_3$), and is proposed as an effective description of the $T = 0$ superconductor-insulator transition and of the pseudogap phase in underdoped cuprates. The coupling constant ("charge") in this theory is proportional to the dual order parameter of the XY model, which is assumed to be describing fluctuations of the phase of the superconducting order parameter. Finiteness of the charge is then tantamount to the appearance of infinitely large vortex loops, i.e. to the loss of phase coherence in the system. The principal result is that the destruction of superconducting phase coherence in the d-wave superconductors typically, and immediately, leads to the appearance of antiferromagnetism. This transition can be understood in terms of the spontaneous breaking of an approximate "chiral" $SU_c(2)$ symmetry, which may be discerned at low enough energies in the standard d-wave superconductor. The mechanism of this spontaneous symmetry breaking is formally analogous to the dynamical mass generation in the $QED_3$, with the "mass" here being proportional to staggered magnetization. Other phases with broken chiral symmetry include the translationally invariant "d+ip" and "d+is" insulators, and the one-dimensional charge-density and spin-density waves, which are all insulating descendants of the d-wave superconductor.

All the insulating states have the neutral spin-1/2 excitations that one can identify in the superconductor confined by the logarithmic potential. Electron repulsion is in this formalism represented by a particular quartic perturbation to the $QED_3$ action, which breaks the chiral symmetry and selects the antiferromagnet as the preferred broken symmetry state. I formulate the mean-field theory of the antiferromagnetic instability in presence of a short-range repulsive interaction, and find the staggered magnetization to be significantly enhanced deeper inside the insulating state. The theory offers an explanation for the rounded d-wave-like dispersion seen in ARPES experiments on the insulating $Ca_2CuO_2Cl_2$ (F. Ronning et. al., Science 282, 2067 (1998).) Relations to other theoretical approaches to the high-$T_c$ problem are discussed.

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I. INTRODUCTION

Soon after the original discovery, it became well appreciated that the high temperature (high-$T_c$) superconductors are all quasi two-dimensional insulating antiferromagnets that become superconducting with the introduction of holes. The nature of the relationship between antiferromagnetism and high temperature superconductivity has been the central issue in the field. Following the time honored strategy of understanding first the non-superconducting state, most of the approaches to the high-$T_c$ problem focused on finding the mechanism by which doping an antiferromagnet would produce a superconductor [1]. The essential difficulty in pursuing this strategy seems to be that the Mott insulator is itself a non-trivial strongly correlated state, harder to describe in simple terms than the metallic Fermi liquid, which played its role in the BCS theory of the low-temperature superconductivity [2]. The situation becomes only worse away from half-filling, where the ground state of even the simplest models becomes more ambiguous. Experimentally, the cuprates seem to loose their antiferromagnetic ordering with doping before they become superconducting, and many candidates for the intermediate "pseudogap phase" have been discussed in literature. The nature of the non-superconducting state that is supposed to be unstable to superconductivity with doping is at this point, however, far from clear, and may prove to be non-universal. Arguably, the physics of underdoped regime may be the main mystery of high temperature superconductivity.

In a remarkable contrast to the uncertainties inherent to the insulating phase, the superconducting phase of most high-$T_c$ materials is well established to have the d-wave symmetry of the order parameter [3], [4], typically with well-defined, long-lived quasiparticle excitations [5], [6]. This simplicity suggests that an inverted approach to the high-$T_c$ problem may be more natural [7]: if there exists a d-wave state in the phase diagram, which other states can in principle be inferred from it? The purpose of this paper is to establish the theoretical framework for answering this question, answer it, and show how this may help explain some salient features of the cuprates phase diagram and the angle resolved photo emission spectroscopy (ARPES) experiments in the insulating state [8], [9].

Loosely speaking, there are two ways to destroy a superconducting state: 1) by driving the amplitude of the order parameter to zero, which is what is well described by the weak-coupling BCS theory at finite temperature [2], for example. For a d-wave superconductors this process presumably is relevant at large dopings, where weak-coupling treatments of the Hubbard and related models can be trusted, and disorder should eventually force $T_c$ to vanish [10]. 2) Even if the amplitude of the order parameter is large and finite, superconductivity will be lost with the destruction of phase order [11], [12]. There is
evidence that this is what actually occurs in underdoped cuprates, where the superconducting transition temperature ($T_c$) is much lower than the pseudogap temperature $T^*$. Since underdoped cuprates are strongly two-dimensional, at finite temperatures the loss of phase order may be expected to proceed via the Kosterlitz-Thouless transition, and indeed, there are distinct experimental signatures of the fluctuating vortices above $T_c$.

The following question then naturally arises: What is the nature of the $T = 0$ phase that derives from a two-dimensional d-wave superconductor when the phase-coherence is lost, but the order parameter amplitude is still finite? The central thesis of this work is that phase coherence is lost, but the order parameter amplitude is still finite there is a dynamical generation of the mass term $\sim m\bar{\Psi}\Psi$ in (1), which can be identified as the staggered potential felt by the original electrons, i.e. with the SDW order parameter. Quantum fluctuating dSC is thus at $T = 0$ inherently unstable towards SDW ordering once the phase coherence is lost.

The dSC $\rightarrow$ SDW quantum phase transition is an example of spontaneous breaking of a continuous global symmetry in the Eq. (1), which for a lack of better name I will call "chiral" throughout the paper. Chiral symmetry breaking is a well studied field-theoretic phenomenon, believed to be inextricably linked to confinement in the QED$_3$.

Massless QED$_3$ for single species of Dirac fermions has the continuous $U(2) = U(1) \times SU_c(2)$ symmetry, with the generators $I, \gamma_3, \gamma_5,$ and $\gamma_{35} = i\gamma_3\gamma_5$, respectively. In the action in the Eq. (1), the $U(1)$ factor represents the residual spin rotational symmetry left by the choice of representation, as will be explained in detail later. It is the additional $SU_c(2)$ symmetry per Dirac component in the QED$_3$ that will be of central interest here. The fermion mass term $m\bar{\Psi}\Psi$ breaks the $SU_c(2)$ for each Dirac field to $U_c(1)$, and the two broken generators rotate between different insulating states. Chiral $SU_c(2)$ arises as an approximate symmetry of the dSC only at low-energies, and will be manifestly broken, for example, by higher order derivatives omitted in the Eq. (1). It should not be confused with the spin rotational symmetry which is, of course, also, and exactly, present in the dSC. Higher order derivatives and the electron interaction terms reduce the $SU_c(2)$ to its $U_c(1)$ subgroup,
which is related to the spatial translations of the original
electrons. The identification of the approximate chiral
symmetry in the dSC is essential for establishing the con-
nection between the antiferromagnetic and the supercon-
ducting phases advocated in this paper, and represents
one of my central results. The idealized cuprates phase
diagram may be understood in terms of the chiral sym-
metries of different states as depicted in Fig. 1.

Assuming the scale for the SDW transition $T_{SDW}(x)$ in
an anisotropic quasi-two dimensional high-temperature
superconductor to be set by the magnitude of the stag-
gered magnetization at $T = 0$ [2], the present work sug-
gests that near and left of the superconductor-insulator
transition one should expect it to be considerably lower
than the superconducting $T_c(x)$ near and right of the crit-
cal point: $T_{SDW}(x_u - \delta) \ll T_c(x_u + \delta)$, where $x_u$ is the
critical doping for the dSC-SDW transition, and $\delta \ll 1$
(see Fig. 1). This is because the generalized $QED_3$ with
fermion species has a critical point at $N = N_c \approx 3$,
above which there is no dynamical mass generation [20].
The $QED_3$ in (1) has $N = 2$ components, which together
with some numerical factors gives very weak SDW order
near the superconducting phase. The pseudogap phase
in cuprates at $T = 0$ is therefore proposed here to be
actually an extremely fragile SDW, likely to be easily
destroyed by disorder, for example. As half-filling is ap-
proached and the vortex loop condensate $\langle \Phi \rangle$ increases,
the repulsion between electrons also becomes important.
Short-range repulsion is represented in the $QED_3$ by a
particular quartic term, which if weak is irrelevant in the
superconducting state, but which also manifestly breaks
the chiral symmetry of the low-energy theory. I show
that the effect of such a term is first to break the de-
generacy among states with broken chiral symmetry in
favor of the SDW, and then to dramatically increase the
SDW order parameter farther from the dSC. The pic-
ture implied by the $QED_3$ is qualitatively in accord with
the generic phase diagram for the underdoped cuprates,
where the antiferromagnetic transition near half-filling
raises to $\sim 300K$, but is typically unobservably low very
near the superconducting state.

Neutral spinons, which are well defined quasiparticles
in the superconducting state, in the insulator become
broad excitations with the lifetime proportional to the
antiferromagnetic order parameter. At $T = 0$ and at
large distances they become confined by a logarithmic
potential provided by the gauge-field in presence of the
chiral symmetry breaking. Due to the weakness of the
SDW order very near the superconducting transition,
however, spinon confinement is effective only at very
large distances, or equivalently, at very low tempera-
tures. The weak SDW phase therefore appears effectively
deconfined at intermediate length scales. The finite-$T$
pseudogap phase has the gapless spinons strongly scat-
tered by the massless gauge field, in qualitative agree-
ment with the broad spectral features of the electrons
seen in ARPES [13]. Near half-filling the SDW order in-
creases and the bound state of spinons rapidly shrinks,
leaving only magnons in the excitation spectrum.

Confined nature of the standard antiferromagnet close
to half-filling, if postulated, by itself already points to
the $QED_3$ as a viable candidate for the effective the-
ory of underdoped cuprates. If one views the supercon-
ducting state as being spin-charge separated [1], one
needs a mechanism by which spinons would eventually
become confined in the antiferromagnetic phase. The
$QED_3$ provides such a mechanism automatically, since
the massless gauge-field mediates a long-range logarith-
ic interaction between the spinons that binds them at
all energies. Were the gauge-field massive, on the other
hand, the physics would be equivalent to $Z_2$ gauge theory,
and the antiferromagnetic state would be deconfined
and quite different from the usual antiferromagnet [22],
[23]. The very existence of an ordinary antiferromagnet
at, and presumably near, half filling may therefore be
taken as evidence in favor of the type of theory presented
in this paper.

The physical picture of the antiferromagnetic (SDW)
insulator as a phase-disordered d-wave superconductor is
further supported by the ARPES data on the insulating
$Ca_2CuO_2Cl_2$ and $Sr_2CuO_2Cl_2$ [8], [9]. These ex-
periments show two unexpected features of the insulat-
ing state: 1) although the ARPES spectral function is
broad, one can nevertheless identify a remnant of the
"Fermi surface" has a d-wave form, except that it be-
comes rounded and without the characteristic cusp at low
energies. The "relativistic" dispersion for broad quasi-
particle excitations that the $QED_3$ implies in the insu-
lasting state, when measured from the lowest energy given
by the dynamically generated chiral mass, provides a very
good fit to the data (see Fig. 5). The present theory im-
plies that the rounding of the dispersion is controlled by
the size of the sublattice magnetization, and therefore
should decrease with doping, as one approaches the su-
perconducting state. It would be desirable to test this
prediction in future experiments.

In the body of the paper I develop the above picture in
detail. In the next section, I derive the Dirac represen-
tation of the Hamiltonian for low-energy nodal quasipar-
ticles, and discuss the coupling to quantum fluctuating
vortex loops in the section III. A derivation of the dy-
amics of the gauge field starting from the XY model
on a lattice is presented in the sec. IV. This section is
somewhat technical and may be skipped at first read-
ing. Instead, the reader may consult the Appendix B,
where a simpler derivation for finite temperatures is pre-
sented. Dynamical breaking of the chiral symmetry and
the formation of the SDW state is discussed in the sec.
V. More general discussion of the chiral symmetry and
the other ordered states on the chiral manifold is pro-
vided in the sec. VI. The reduction of chiral symmetry
by the irrelevant terms is discussed in sections VII, and
the mean-field theory of the antiferromagnetic instabil-
ity of the $QED_3$ in presence of the electron repulsion is
solved in the sec. VIII. Confinement of spinons in the in-
II. DIRAC THEORY FOR NODAL EXCITATIONS

I begin by assuming that the superconducting state, except from being a d-wave, otherwise exhibits the standard BCS phenomenology. In particular, I take that the quasiparticles are well-defined, long-lived excitations. Generally, the quasiparticle action at $T \neq 0$ may then be taken to be

$$S = T \sum_{k, \sigma, \omega_n} \left[ (i\omega_n - \xi_k) c^\dagger_k(\vec{k}, \omega_n) c_{\sigma}(\vec{k}, \omega_n) - \frac{\sigma}{2} \Delta(\vec{k}) c^\dagger_{\sigma}(\vec{k}, \omega_n) c^\dagger_{\sigma}(\vec{k}, -\omega_n) + h.c. + O(c^4) \right],$$

(2)

where $\Delta(\vec{k})$ has the usual d-wave symmetry, and two spatial dimensions (2D) are assumed. $c$ and $c^\dagger$ are the electron operators, $\sigma = \pm$ labels the $z$-projection of electron spin, and $\omega_n$ are the fermionic Matsubara frequencies. Units are chosen so that $\hbar = c = e = 1$. $O(c^4)$ term stands for all possible short-range interactions between quasiparticles.

We may represent the quasiparticle Hamiltonian in terms of two four-component fields,

$$\Psi^I(q, \omega_n) = (c^\dagger_{\sigma}(\vec{k}, \omega_n), c_{-\sigma}(\vec{k}, -\omega_n)),$$

(3)

where $\vec{Q}_i = 2\vec{K}_i$ is the wave vector that connects the nodes within the diagonal pair $i = 1, 2$, as in Fig. 2. For spinor 1, $\vec{k} = \vec{K}_i + \vec{q}$, with $|\vec{q}| \ll |\vec{K}_i|$, and analogously for the second pair. The construction of the four-component field is not unique. The choice in the Eq. (3) differs from the one made in the ref. [1], for example. I postpone the discussion of the alternative construction used there for the Appendix D. Using the construction in the Eq. (3), and by observing that $\xi_{\vec{k}} = -\xi_{\vec{k} - \vec{Q}_i}$, and $\Delta_{\vec{k}} = \Delta_{\vec{k} - \vec{Q}_i}$, for $\vec{k} \approx \vec{K}_i$, and then by linearizing the spectrum as $\xi_{\vec{k}} = \nu_f q_x + O(q^2)$ and $\Delta_{\vec{k}} = \nu q_y + O(q^2)$, one arrives at the low-energy action

$$S[\Psi^I] = \int d^2q \int_0^{\beta} d\tau \Psi^I_1[\partial_\tau + M_1 v_f \partial_x + M_2 v_x \partial_y] \Psi^I_1 + (1 \to 2, x \leftrightarrow y) + O(\partial \Psi^I \partial \Psi^I, \Psi^4),$$

(4)

with $\beta = 1/T$. The continuous Dirac field $\Psi^I_1(q, \tau)$ is defined as

$$\Psi^I_1(q, \tau) = T \sum_{\omega_n} \int \frac{d^2q}{(2\pi)^2} e^{i\omega_n \tau + i\vec{q} \cdot \vec{r}} \Psi^I(q, \omega_n),$$

(5)

with the integral over momenta performed over $|\vec{q}| < A < T^*$. The $4 \times 4$ matrices in the Eq. (4) are $M_1 = i\sigma_3 \otimes \sigma_3$, and $M_2 = -i\sigma_3 \otimes \sigma_1$. $\sigma$ are the usual Pauli matrices, and the coordinate system has been rotated as in Fig. 2.

To cast the theory in Dirac form we may invoke the matrix $\gamma_0 = \sigma_1 \otimes I$, where $I$ is the 2 $\times$ 2 unit matrix. Then $\gamma_0^2 = I \otimes I$, and $M_1 = \gamma_1 \gamma_0$, with $\gamma_1 = \sigma_2 \otimes \sigma_3$, and $\gamma_2 = -\sigma_2 \otimes \sigma_1$. $\{\gamma_\nu, \gamma_\mu\} = 2\delta_{\nu\mu}, \nu, \mu = 0, 1, 2$, so the $\gamma$-matrices indeed satisfy the Clifford algebra. The quasiparticle action (2) at low energies becomes equivalent to the field theory

$$S[\Psi^I] = \int d^2q \int_0^{\beta} d\tau \Psi^I_1[\gamma_0 \partial_\tau + \gamma_1 v_f \partial_x + \gamma_2 v_x \partial_y] \Psi^I_1 + (1 \to 2, x \leftrightarrow y) + O(\partial \Psi^I \partial \Psi^I, \Psi^4),$$

(6)

where $\Psi^I_1 = \Psi^I_1 \gamma_0$. Weak quartic interactions, as long as they are short-ranged, are irrelevant by simple power counting. This simply reflects the severe phase-space restrictions for scattering of the nodal quasiparticles. I will therefore omit them temporarily, together with the second order derivative terms, to return to their effects in the section VII.

The reader would be correct to note that there is a considerable freedom in selecting the form of the matrix $\gamma_0$. In fact, any $4 \times 4$ matrix that anticommutes with $M_1$ and $M_2$ and squares to unit matrix would yield an equally valid Dirac representation. It is shown later that this freedom will correspond to different "directions" in the space of ordered states with broken chiral symmetry. The specific choice for $\gamma_0$ made here will be analogous to choosing a direction in real space along which to search for a finite magnetization, for example, in the more familiar magnetic phase transitions.

III. COUPLING TO TOPOLOGICAL DEFECTS

The goal in this section will be to find the most economical form of the coupling between nodal excitations

\[ \Psi^I(q, \tau) = T \sum_{\omega_n} \int \frac{d^2q}{(2\pi)^2} e^{i\omega_n \tau + i\vec{q} \cdot \vec{r}} \Psi^I(q, \omega_n) \]
in the dSC and the fluctuations of the phase of the superconducting order parameter. Working assumption is that the amplitude fluctuations are frozen well below the pseudogap temperature $T^*$, so it is only the phase degree of freedom that remains active at low energies. With this in mind I write

$$v_\Delta \to v_\Delta (\vec{r}, \tau) = |v_\Delta| e^{i(\phi_s (\vec{r}, \tau) + \phi_s (\vec{r}, \tau))},$$

(7)

where $\phi_s$ represents the regular (“spin-wave”) part of the order parameter phase, and $\phi_s$ is the singular contribution due to topological defects. At $T = 0$ these would be the vortex loops $\Psi$, or the more familiar vortices and antivortices at $T \neq 0$. At this point it is tempting to transform both spin-up and spin-down fermionic operators by absorbing a half of the total superconducting phase into each. In presence of topological defects, however, this would lead to multivalued fermionic fields and would not be a local change of variables in the partition function. This problem may be circumvented by allowing only vortices of double vorticity $\Phi$, for example, which then leads to the Z$_2$ gauge theory representation of the problem, and a possibility of spin-charge separation in the pseudogap regime $\Phi$. It is the single vortices, however, that first become relevant at the $T \neq 0$ Kosterlitz-Thouless transition $\Phi$, and they should be included in the description of the $T = 0$ transition as well. I will therefore utilize the idea of Franz and Tešanović $\Phi$, who suggested dividing a given vortex configuration into two groups A and B, and transforming the electron operators with spin up and spin down differently. We write

$$\phi_A (\vec{r}, \tau) = \phi_s (\vec{r}, \tau) + \phi_s (\vec{r}, \tau),$$

(8)

and similarly for B. $\phi_A$ is the piece of the singular part of the phase that comes from the defects grouped in A. One may then make a local change of variables by introducing a new Dirac field $\Psi$ as

$$\Psi (\vec{r}, \tau) = U (\vec{r}, \tau) \Psi (\vec{r}, \tau)$$

(9)

where $U = diag (e^{-i\phi_A}, e^{i\phi_B}, e^{-i\phi_A}, e^{i\phi_B})$. Since any given vortex defect is either in group A or B, and therefore associated either with up, or with down spin by the transformation (9), circling around it with the transformed fermion would yield either $2\pi$ or zero of the accumulated phase change. Components of the new field $\Psi$ are therefore single-valued functions.

The gauge-transformed action for the Dirac field $\Psi$ is then

$$S[\Psi] \to S[\Psi, \tilde{a}, \tilde{\nu}] = \int d^2 \vec{r} \int_0^\beta d\tau \tilde{\Psi}_1[\gamma_0 (\partial_x + i a_0) + \gamma_1 v_\nu (\partial_y + i a_y)] + \gamma_2 |v_\Delta| (\partial_y + i a_y) |\Psi_1 + (1 \to 2, x \leftrightarrow y) + i v_\nu J_\nu,$$

with $a_\nu = \partial_\nu (\phi_A - \phi_B)/2$, $v_\nu = \partial_\nu (\phi_A + \phi_B)/2$, and $J_\nu = (\Psi^1 (I \otimes \sigma_3) \Psi_1, v_F \Psi^1 (\sigma_3 \otimes I) \Psi_1, v_F \Psi^1 (\sigma_3 \otimes I) \Psi_2)$.

Since the vector $J_\nu$ is built out only of the products of the creation and the annihilation operators with same spin, it also represents the physical charge current carried by the quasiparticles. On the other hand, since the regular part of the phase $\phi$ was in the Eq. (8) divided equally between spin up and spin down, the Dirac field $\Psi$ is invariant under a regular gauge transformation. Components of $\Psi$ therefore create electrically neutral excitations with spin-1/2 $\Psi$, which may therefore be referred to as spinons.

The action (10) has two rather different gauge symmetries, and it may be worthwhile pausing a little to reflect on them. First, the physical electromagnetic gauge field $A_\mu$ would enter the action (10) by the replacement $v_\mu \to v_\mu + A_\mu$, and couple to the charge current. Under a regular gauge transformation $A_\mu \to A_\mu + \partial_\mu \chi$, the Volovik’s field $\Psi$ remain the same. The action (10) is therefore gauge invariant, in the standard sense. But it also has an additional internal gauge symmetry, under the transformation $a_\mu \to a_\mu + \partial_\mu \chi$, $v_\mu \to v_\mu$, $\Psi \to e^{-i \chi} \Psi$. This reflects the freedom of choice in the Eq. (8); one could have equally well chosen the regular part of $\phi_A$ to be $(\phi_A/2) + \chi$, and of $\phi_B$, $(\phi_B/2) - \chi$. One deals with this gauge freedom, as usual, by eventually introducing the gauge-fixing term for $a_\mu$ that allows one to freely sum over all regular internal gauges $\chi$. Similarly, the division of the singular part of the superconducting phase into that which comes from the defects in the group A, and the defects in the group B, is equally arbitrary. Just like one effectively sums over all regular internal gauges by the introduction of the gauge-fixing term, we will sum over all singular internal gauges by averaging over all possible divisions of defects into two groups. This is explained in the next section, and in the Appendix B. As a byproduct, the averagings over regular and singular internal gauges will insure that up and down spinons are treated equally in the $QED_3$, in respect of the symmetry of the original electronic action (2).

The crucial observation about the action (10) is that the coupling of spinons to phase fluctuations is furnished by two $U(1)$ fields which play quite different roles in the problem. Total superconducting phase determines the Volovik’s field $v_\nu$ and couples to the charge current, in the same way as the true electromagnetic field would. $v_\nu$ will therefore inevitably become massive once the high-energy spinons in the Eq. (10) are begun to be integrated out. Its fluctuations therefore may provide only a short-range interaction between spinons. The gauge field $a_\mu$, on the other hand, enters (10) in a gauge-invariant way, and therefore is protected from acquiring a mass from spinons. Both gauge fields, however, depend on the fluctuating positions of the topological defects, and acquire their dynamics not only from the spinons, but from the defects as well. To determine their dynamics one therefore needs to integrate the defect degrees of freedom out. If $a_\mu$ would stay massless even after this integration is performed, it would mediate a long-range
interaction between the nodal excitations, which, unlike the short-range quartic terms in the Eq. 6, would not be made irrelevant by the phase space restrictions. This, however, depends on the precise way $a_{\mu}$ acquires its dynamics from the fluctuating vortex loops, to which I turn next.

IV. DYNAMICS OF THE GAUGE-FIELDS

The zero-temperature partition function for the coupled system of d-wave quasiparticles and superconducting phase fluctuations is therefore

$$Z = \int D[\Psi, \vec{a}, \vec{v}] e^{-(S[\Psi, \vec{a}, \vec{v}] + S_{U(1)}[\vec{a}, \vec{v}] )},$$

(11)

with $S[\Psi, \vec{a}, \vec{v}]$ defined by the Eq. (10), and with $S_{U(1)}[\vec{a}, \vec{v}]$ to be derived by integrating out the phase fluctuations. For simplicity, I will assume that these may be described the 2 + 1 dimensional XY model. The bare stiffness for the phase fluctuations will be assumed to be provided by the high energy modes that have been integrated out in arriving at the low-energy theory. Our goal will be then to rewrite the partition function for the XY model as the functional integral over the fields $\vec{a}$ and $\vec{v}$. In particular, we want to integrate over the topological defects implicit in the XY model.

I first discretize the space and the imaginary time in writing the partition function of the XY model. This is done to facilitate a more rigorous treatment of the topological defects, and it will prove possible to return to the continuum description we employed until now. On a lattice, in the standard lattice gauge-theory notation [17],

$$Z_{xy} = \int_{0}^{2\pi} \left( \prod_{i} d\phi_{i} \right) \exp \left( \sum_{i,\mu=x,y,z} \cos(\phi_{i+\mu} - \phi_{i}) \right),$$

(12)

where the index $i$ labels the sites of a three (2 + 1) dimensional lattice, and $\vec{x}$ is the lattice unit vector in the $x$ direction. For simplicity, full isotropy in the XY model is assumed. Using the Villain approximation [30] and then integrating over the phases leads to

$$Z = \int_{-\infty}^{\infty} ds \left( \prod_{i} \exp \left( -\frac{1}{2K} \sum_{i} (\nabla \cdot \vec{s}_{i})^{2} + i2\pi \sum_{i} \vec{n}_{i} \cdot \vec{s}_{i} \right) \right),$$

(13)

where $\vec{n}_{i} = (n_{i,x}, n_{i,y}, n_{i,z})$ is an integer vortex-loop vector variable, satisfying the constraint $\nabla \cdot \vec{n}_{i} = 0$ (indicated with the prime on the sum). $\nabla$ and $\nabla \cdot$ should be understood as the lattice gradient and the curl, respectively. Summing over $\vec{n}_{i}$ forces $\vec{s}_{i}$ to take integer values, and the above expression becomes the standard current representation of the XY model [17].

Next, I imagine dividing a given configuration of vortex loops into two arbitrary groups, and write $\vec{n}_{i} = \vec{n}_{A,i} + \vec{n}_{B,i}$, with $\nabla \cdot \vec{n}_{A,i} = \nabla \cdot \vec{n}_{B,i} = 0$. We will want to sum over all integer $\vec{n}_{A,i}$ and $\vec{n}_{B,i}$ in order to average over all possible divisions of vortices into two groups. Introducing the lattice version of the fields $\vec{a}_{i}$ and $\vec{v}_{i}$ as $\vec{B}_{i} + \vec{b}_{i} = 2\pi \vec{n}_{A,i}$, $\vec{B}_{i} - \vec{b}_{i} = 2\pi \vec{n}_{B,i}$, where $\vec{b}_{i} = \nabla \times \vec{a}_{i}$ and $\vec{B}_{i} = \nabla \times \vec{b}_{i}$, I write [31]

$$Z_{xy} = \int_{-\infty}^{\infty} d[\vec{a}, \vec{v}] \sum_{\vec{m}_{A}, \vec{m}_{B}} \exp \left( \sum_{i} \left[ \frac{1}{2K} (\nabla \times \vec{s}_{i})^{2} + i2\pi \vec{n}_{i} \cdot (\vec{n}_{A,i} + \vec{n}_{B,i}) \right] ight)$$

(14)

$$+ \vec{n}_{i} \cdot (\nabla \times (\vec{m}_{A,i} + \vec{m}_{B,i})) + i\vec{a}_{i} \cdot (\nabla \times (\vec{m}_{A,i} - \vec{m}_{B,i})).$$

The summations over $\vec{n}_{A,i}$ and $\vec{n}_{B,i}$ then enforce the constraints $\vec{s}_{i} - \vec{t}_{i} = \vec{m}_{A,i}$, and $\vec{s}_{i} - \vec{r}_{i} = \vec{m}_{B,i}$, where $\vec{m}_{A,i}$ and $\vec{m}_{B,i}$ are new integers. Performing the Gaussian integrals over $\vec{s}_{i}$, yields

$$Z_{xy} = \int_{-\infty}^{\infty} d[\vec{a}, \vec{v}] \sum_{\vec{m}_{A}, \vec{m}_{B}} \exp \left( \sum_{i} \left[ K (\nabla \times \vec{v}_{i})^{2} + i\vec{a}_{i} \cdot (\nabla \times (\vec{m}_{A,i} - \vec{m}_{B,i})) \right] \right)$$

(15)

This can be further simplified by noticing that the action is quadratic in the Volovik’s field $\vec{v}$, which can also be integrated out. In doing so I will neglect the additional coupling of $\vec{v}$ to the charge current $\vec{J}$ in the Eq. (10), which only leads to additional irrelevant interaction between spinons. The integration over $\vec{v}_{i}$ in the last equation then gives

$$Z_{xy} = \int_{-\infty}^{\infty} d\vec{a} \sum_{\vec{m}_{A}, \vec{m}_{B}} \exp \left( \sum_{i} \left[ \frac{1}{8K} (\nabla \times (\vec{m}_{A,i} + \vec{m}_{B,i}))^{2} + i\vec{a}_{i} \cdot (\nabla \times (\vec{m}_{A,i} - \vec{m}_{B,i})) \right] \right)$$

(16)

Integrating over $\vec{a}_{i}$ in (16) would give back the current representation of the XY model, Eq. (13). Alternatively, we can introduce the real variables $\tilde{\Phi}_{+,i}$ and $\tilde{\Phi}_{-,i}$ and write

$$Z_{xy} = \int_{-\infty}^{\infty} d[\vec{a}, \tilde{\Phi}_{+}, \tilde{\Phi}_{-}] \sum_{\vec{t}_{A}, \vec{t}_{B}} \exp \left( \sum_{i} \left[ \frac{1}{8K} (\nabla \times \tilde{\Phi}_{+,i})^{2} + i\vec{a}_{i} \cdot (\nabla \times \tilde{\Phi}_{-,i}) + i2\pi (\tilde{\Phi}_{+,i} \cdot \tilde{\Phi}_{-,i} + \tilde{t}_{A,i} \cdot \tilde{t}_{B,i}) \right] \right)$$

(17)

where $\tilde{\Phi}_{+,i} = \tilde{\Phi}_{A,i} \pm \tilde{\Phi}_{B,i}$. The summations over the auxiliary link variables $\vec{t}_{A,B}$ force $\tilde{\Phi}_{A}$ and $\tilde{\Phi}_{B}$, and therefore $\tilde{\Phi}_{+}$ and $\tilde{\Phi}_{-}$ to be integers. To preserve the gauge invariance ($\tilde{\Phi}_{+,\mu} \rightarrow \tilde{\Phi}_{+,\mu} + \nabla_{\mu} \chi_{+,\mu}$, $\tilde{\Phi}_{-,\mu} \rightarrow \tilde{\Phi}_{-,\mu} + \nabla_{\mu} \chi_{-,\mu}$) of the last expression we must impose $\nabla \cdot \tilde{t}_{A,i} = \nabla \cdot \tilde{t}_{B,i} = 0$ [32] [33]. We may next add a small chemical potential for the link variables $\tilde{t}_{A,B}$ to the action in Eq. (17) as the term $x \sum_{i} (\tilde{t}_{A,i}^{2} + \tilde{t}_{B,i}^{2})$. Up to the
Villain approximation, the last expression is then equal to

\[
Z_{xy} = \lim_{x \to 0} \int_{-\infty}^{\infty} d[\vec{a}, \Phi_A, \Phi_B] \int_{0}^{2\pi} d[\theta_A, \theta_B] \exp \left( -\frac{1}{8K} (\nabla \times \Phi_{+,i})^2 + i\vec{a}_i \cdot (\nabla \times \Phi_{-,i}) \right) \\
- \frac{1}{2\pi} \cos(\theta_{A,i} - \theta_{A,i+\hat{v}} - 2\pi \Phi_{A,i,\nu}) \\
- \frac{1}{2\pi} \cos(\theta_{B,i} - \theta_{B,i+\hat{v}} - 2\pi \Phi_{B,i,\nu})
\]

where I introduced two sets of "dual" angles \(\theta_{A,i}\) and \(\theta_{B,i}\) to insure the gauge invariance, and imposed the "frozen" limit \(x \to 0\). The integration over \(\vec{a}_i\) in the Eq. (18) together with the frozen limit ultimately sets \(\theta_{A,i} \equiv \theta_{B,i}\), so the last equation becomes another representation of the frozen lattice superconductor (FLS), which is well known to be dual to the XY model in three dimensions \([64], [65]\).

In principle, one would like to integrate out all the fields other than \(\vec{a}\) in the Eq. (18), to be left with the effective action \(S_{U(1)}[\vec{a}]\) for \(\vec{a}\) only. The result would be an interacting theory for \(\vec{a}\), which can be expanded in powers of \(\vec{a}\), for example. Instead of doing this, I will approximate the \(S_{U(1)}[\vec{a}]\) with the effective Gaussian action for \(\vec{a}\), that reproduces the gauge-field propagator in the full theory (18). This approximation may be understood as the self-consistent mean field theory for \(\vec{a}\), with the effect of integration over all other fields in (18) lumped into the form of the propagator.

In this approximation the problem of dynamics of the gauge-field \(\vec{a}\) reduces to the computation of the two-point correlation function for \(\vec{a}\) from the representation of the XY model in Eq. (18). I therefore introduce the source term into the last expression by adding \(i \sum_i \vec{j}_i \cdot (\nabla \times \vec{a}_i)\) to the exponent. Then

\[
\langle (\nabla \times \vec{a})_{i,\nu} (\nabla \times \vec{a})_{j,\mu} \rangle = \frac{\partial^2}{\partial j_{i,\nu} \partial j_{j,\mu}} \ln Z_{xy} \big|_{\vec{a} = 0}.
\]

It is convenient then to integrate over \(\vec{a}\) in the \(Z_{xy}\) first. One finds

\[
\langle (\nabla \times \vec{a})_{i,\nu} (\nabla \times \vec{a})_{j,\mu} \rangle = \delta_{i,j} \delta_{\nu,\mu} \lim_{x \to 0} \frac{\pi^2}{x} \langle \cos(\theta_i - \theta_{i+\hat{v}} - 2\pi \Phi_{i,\nu}) \rangle_{FLS},
\]

where the last average is to be taken over the configurations of the FLS

\[
Z_{xy} = \lim_{x \to 0} \int d[\vec{a}, \theta] \exp \left( -\frac{1}{2K} (\nabla \times \Phi)^2 \right) \\
- \frac{1}{x} \langle \cos(\theta_i - \theta_{i+\hat{v}} - 2\pi \Phi_{i,\nu}) \rangle.
\]

It is well established that the lattice superconductor at a small but finite "temperature" \(x\) has a phase transition as \(K\) is varied in the same universality class as in the frozen limit \(x = 0\) \([17], [34], [36]\). We may therefore relax the constraint \(x \to 0\) with impunity and assume \(x\) to be finite. The average that appears on the right hand side of the Eq. (20) can then be computed, for example, by using the mean-field approximation to the FLS action (21) (see Appendix A). This yields

\[
\frac{1}{x} \langle \cos(\theta_i - \theta_{i+\hat{v}} - 2\pi \Phi_{i,\nu}) \rangle_{FLS} \propto |\langle \exp(i\theta_i) \rangle|^2.
\]

This result is quite general, and it simply expresses the fact that in the ordered phase of the theory (21) the dual angles become correlated, while at the same time the gauge-field becomes massive via Meissner effect. The gauge field fluctuations can then be neglected, which makes the requisite average finite when the dual angles \(\theta\), i. e. in the disordered phase of the original XY model.

Returning to the continuum notation, and switching to the Fourier space, the gauge-invariant expression for the correlation function (19) at low momenta is therefore

\[
\langle (\nabla \times \vec{a})_{\nu} (\nabla \times \vec{a})_{\mu} \rangle \propto |\langle \Phi \rangle|^2 + O(q^2)(\delta_{\mu\nu} - q_\mu q_\nu),
\]

where I allowed, in general, for some momentum dependence (the term \(O(q^2)\)). \(O(q^2)\) term should be expected to appear in a more sophisticated approximation for the gauge-field dynamics than provided by the Eq. (20). To the lowest order, the integration over all other fields in (18) effectively yields the Maxwell term for the gauge field \(\vec{a}\), with the stiffness inversely proportional to the expectation value of the dual loop condensate \(\langle \Phi \rangle \sim \langle e^{i\theta} \rangle\) that reflects the phase of the XY model. This is the main result of this section. When the dSC is phase coherent and the vortex loops are finite in size, \(\langle \Phi \rangle = 0\), and \(\vec{a}\) is infinitely stiff, and in first approximation may be considered decoupled from spinons. When vortex loops blow up, \(\langle \Phi \rangle \neq 0\), phase coherence is lost, and the spinons are minimally coupled to a massless gauge field. This is in agreement with the physical arguments advanced in \([15]\).

At high temperature one can neglect the fluctuations in the imaginary time direction and deal with the purely 2D problem of point vortices and antivortices. This simplifies the analysis in that no gauge invariance needs to be insured in the Eq. (17), so no dual angles are required \([15], [17]\). One then ends up with the thermodynamic vortex fugacity playing the role of the dual condensate \([15]\), and with the simpler sine-Gordon theory instead of the FLS. For an alternative derivation of the gauge field dynamics at \(T \neq 0\) and in continuum that is in full accord with the conclusions of this section I direct the reader to the Appendix B.

There is an additional subtlety in going from the lattice to the continuum theory that is worth registering. The partition function in the Villain approximation for the XY model in the Eq. (17) has the symmetry under
\(a_i, \mu \rightarrow a_i, \mu + 2\pi n_i, \mu\), with \(n_i, \mu\) integer, that becomes broken when a small chemical potential \(x \neq 0\) for the link variables \(l_{A, B}\) (in passing to the Eq. (18)) is added. This periodicity would dictate that the summation over the integer vortex variables in Eq. (17) with \(x = 0\) should yield a compact term for \(a\). Absence of the chemical potential \(x\), i.e. of the vortex core energy, in the Eq. (17), on the other hand, must be regarded as an artifact of the Villain representation to the original XY model in which, as well known, vortices do cost finite energy \(^{[37]}\). This is because the Villain approximation reproduces correctly only the long-range part of the vortex interaction, while the short range part needs to be modified "by hand" \(^{[37]}\) in order to obtain the finite core energy. The dynamics of \(a\) should therefore be determined from the theory with \(x \neq 0\), and by the non-compact Maxwell term, as in the Eq. (23). Possible effects of compactness of \(a\) on the picture developed in this paper are discussed in the sec. XII.

V. DYNAMICAL BREAKING OF CHIRAL SYMMETRY

The effective \(T = 0\) low-energy theory for the interacting system of d-wave quasiparticles and fluctuating vortex loops, after the integration over vortex loops is therefore

\[
S[\Psi] = \int d^2 r d\tau \left\{ \bar{\Psi}_I [\gamma_0 (\partial_\tau + ia_\mu) + \gamma_1 v_f (\partial_\tau + ia_\mu) + \gamma_2 \nu_{\Delta}, \nu_y] \Psi_I + (1 \rightarrow 2, x \leftrightarrow y) + \frac{1}{2(|\Psi|^2)} \left( \frac{1}{2} \left( \nabla \times \vec{a} \right)^2 + \left( \nabla \times \vec{a} \right)^2 \right) \right\},
\]

where I omitted the higher derivative terms, and the terms quartic in \(\Psi\). This is the standard three dimensional quantum electrodynamics (\(QED_3\)), with two important caveats: 1) the coordinates \(x\) and \(y\) are exchanged for the second Dirac field, 2) there is an inherent anisotropy in the model, \(v_f \neq v_\Delta \neq c\), where \(c\) is a characteristic velocity for the phase fluctuations \(^{[38]}\). First, let us consider the simpler isotropic limit of the theory, \(v_f = v_\Delta = c\). There are sixteen \(8 \times 8\) matrices then that either commute or anticommute with the three \(8 \times 8\) \(\gamma\)-matrices that appear in the Eq. (24): \(\text{diag}\{\gamma_0, \gamma_0\} \text{diag}\{\gamma_1, \gamma_1\}\), and \(\text{diag}\{\gamma_0, \gamma_0\} \text{diag}\{\gamma_2, \gamma_1\}\) by construction. The unitary transformations in (29) can be shown to form the Lie group \(U(4)\). Following the standard terminology in the field theory literature, I will refer to this symmetry of the \(QED_3\) as "chiral".

As a first step towards understanding of the meaning of the chiral symmetry in the present context, it will prove useful to consider how it may be broken. \(QED_3\) is well known to have the chiral symmetry spontaneously broken \(^{[29]}\), by dynamical generation of the mass term in the action (24):

\[
m \int d^2 r d\tau \sum_{i=1}^2 \bar{\Psi}_i \Psi_i,
\]

with \(m \propto |\langle \Psi \rangle|^2\), i.e. proportional to the effective charge of the \(QED_3\). Containing just a single \(\gamma\)-matrix, the mass term in the Eq. (31) breaks all the anticommuting generators, \(G_i\), with \(i = 5, 6, 7, 8, 13, 14, 15, 16\). The chiral symmetry is reduced from \(U(4)\) to \(U(2) \times U(2)\), with eight generators preserved. The fermion mass is generated dynamically due to the coupling to the gauge field. To see this, neglect the wave-function renormalization and the vertex corrections (which can be rationalized in the limit of a large number of Dirac fields \(N\)), and write the self-energy as

\[
\Sigma(q) = |\langle \Psi \rangle|^2 2 \sqrt{2} \int \frac{d^3 p}{(2\pi)^3} \frac{D_{\nu\mu}(\vec{p} - \vec{q}) \Sigma(p)}{p^2 + \Sigma^2(p)} \gamma_\mu,
\]

where \(\vec{q} = (\omega, \vec{q}_x, q_y)\). The gauge-field propagator in the transverse (Landau) gauge is

\[
D_{\nu\mu}(\vec{p}) = (\delta_{\nu\mu} - \vec{p}_\nu \vec{p}_\mu)/(p^2 + \Pi(p)),
\]
In the large-N approximation \[20\] one finds that as the phase coherence is lost, and the charge of the SDW is set by the vectors \(Q_{\text{ED}}\)

\[
\Pi(p) = \frac{N|\langle \Phi \rangle|^2}{6\pi} \frac{p^2}{m} + O(p^4).
\] (34)

For the polarization at all momenta see the Appendix C. The Eq. (32) was analyzed in \[20\] (see also Appendices C and E), and there is a solution with finite \(m\) for the number of Dirac fields \(N < N_c = 32/\pi^2 = 3.24\). Full numerical solution that includes the wave-function renormalization and vertex corrections confirms that \(N_c \approx 3\) \[39\], almost independently of the choice of vertex. Lattice simulations give \(3 < N_c < 4\) \[41\], or at least that \(N_c > 2\) \[41\]. It therefore seems reasonable to conclude that for \(N = 2\) the chiral symmetry in the isotropic \(Q_{\text{ED}}\) becomes spontaneously broken when the vortex loops unbind and \(\langle \Phi \rangle \neq 0\).

Since the matrix \(\gamma_0\) commutes with the electron-spinon transformation in the Eq. (9), it is easy to rewrite the mass term in the \(Q_{\text{ED}}\) in terms of the original electron operators:

\[
m \sum_{i=1}^{2} \bar{\Psi}_i \Psi_i \rightarrow mT \sum_{\vec{k} \in R_1, \omega_n} \left\{ [c_+^\dagger(\vec{k}, \omega_n)c_+^\dagger(-\vec{k}, \omega_n)] + [c_+^\dagger(\vec{k}, \omega_n)c_-^\dagger(-\vec{k}, \omega_n)] \right\} + (1 \rightarrow 2).
\] (35)

The reader will recognize this as the low-energy part of the staggered potential along the spin z-axis

\[
m \int d^2 \vec{r} d\tau \sum_{\sigma = \pm, i=1,2} \cos(\vec{Q}_i \cdot \vec{r}) \sigma \epsilon_i^\dagger(f, \tau) \sigma \epsilon_i(f, \tau),
\] (36)

so the mass in the \(Q_{\text{ED}}\) is nothing but the spontaneously generated SDW order parameter. The periodicity of the SDW is set by the vectors \(\vec{Q}_i\), and thus tied to the Fermi surface. The SDW order is established as soon as the phase coherence is lost, and the charge \(\langle \Phi \rangle \neq 0\).

In the large-N approximation \[21\] one finds that

\[
m \approx 16|\langle \Phi \rangle|^2 e^{-2\pi/\sqrt{(\sum_{\vec{k}} - 1)}},
\] (37)

Since \(N_c \approx 3\), for \(N = 2\) one finds that \(m \approx 10^{-2}|\langle \Phi \rangle|^2\). This extreme “lightness” of fermions in the \(Q_{\text{ED}}\) derives from the fact that the mass comes solely from the interaction with the soft gauge-field.

Breaking of chiral symmetry in the \(Q_{\text{ED}}\) also implies that the energies of spinons have become complex and finite in the phase incoherent state with \(\langle \Phi \rangle \neq 0\). In the simplest approximation the electron propagator may be computed as a product of the spinon and the gauge-field propagators, so a spinon “gap” should imply a charge gap as well, i.e. the system becomes an insulator \[12\]. In section IX I discuss how spinons should actually be confined in the insulating state. Staggered magnetization, charge gap, and the spinon confinement when taken together imply that the state with broken chiral symmetry is nothing but the standard, albeit a weak, SDW. It seems natural to assume then that this state is continuously connected to the antiferromagnet near half-filling in cuprates. This expectation is further corroborated by considering the effect of Coulomb interactions, which is done in section VIII.

VI. MORE ON CHIRAL SYMMETRY: THE SPACE OF INSULATORS

In discussing the pattern of chiral symmetry breaking in the \(Q_{\text{ED}}\) one needs to distinguish at least two different cases. The isotropic theory \(U_{\Delta} = v_f\) has the full \(U(4)\) symmetry in its massless phase, so the mass term breaks eight of its sixteen generators. In cuprates \[33\], however, \(v_f/v_{\Delta} \sim 10\), and even with \(m = 0\) the symmetry is only \((U(2) \times U(2))\), generated by the block-diagonal \(G_i, i = 1, ... 8\). How such a large anisotropy affects the value of \(N_c\) is a non-trivial problem, and is addressed in a separate publication \[43\]. Here I will consider only the effect of anisotropy on the chiral symmetry, and assume it is reduced to \((U(2) \times U(2))\). It suffices then to look at each Dirac component in the \(Q_{\text{ED}}\) separately, i.e. consider just the \(4 \times 4\) representation of the \(\gamma\)-matrices, as defined right below the Eq. (5).

It can be easily shown that any matrix that anticomutes with both \(M_1\) and \(M_2\) and squares to unit matrix may be chosen as \(\gamma_0\), and will lead to a representation of the \(\gamma\)-matrices like in the Eq. (6). The mass term \(\sim m\langle \Phi \rangle\gamma_0\Psi\) in the action would then gap the quasiparticles, in analogy to the standard relativistic Dirac equation. The problem of different chiral orders is therefore nothing else but finding all the ways in which d-wave quasiparticles can spontaneously acquire such a “relativistic mass”. It will be useful to introduce “directions” in the space of broken symmetry states, as a set of linearly independent matrices that anticommutate with \(M_1\) and \(M_2\), and square to one. It is easy to show that in the \(4 \times 4\) representation there are only four such matrices

\[
\gamma_{0}, \gamma_{3}, \gamma_{5}, i\gamma_{1}\gamma_{2},
\] (38)

with \(\gamma_{0} = \gamma_{0}\), and where \(\gamma_{1} = -iM_{1}\), \(\gamma_{2} = iM_{2}\), \(\gamma_{3} = \sigma_{3} \otimes \sigma_{2}\), and \(\gamma_{5} = \sigma_{3} \otimes I\). In principle, any of these four if used instead of our \(\gamma_{0}\) in the construction of the Dirac theory in the Eq. (6) and in the mass term would give a relativistic gap to Dirac fermions. The last matrix,
$$\dot{\gamma}_1 \gamma_2 = I \otimes \sigma_2,$$  \hfill (39)

however, being a product of two \(\tilde{\gamma}\)-matrices does not break the chiral symmetry, and is believed not to be spontaneously generated in the \(QED_3\) \cite{[40]}. I therefore focus on the remaining three. Choosing one among \(\{\gamma_0, \gamma_3, \gamma_5\}\) as the \(\gamma_0\)-matrix in the mass term reduces the \(SU_c(2)\) subgroup of \(U(2) (= U(1) \times SU_c(2))\), generated by \(\{\gamma_3, \gamma_5, \gamma_{35}\}\), to \(U_c(1)\). The two anticommuting generators of the \(SU_c(2)\) that are broken then rotate the chosen order parameter towards the two remaining "directions" in the chiral space. For example: for our choice of \(\gamma_0 = \gamma_0\), it is \(\gamma_{35}\) that remains unbroken in the cos-SDW phase, whereas the broken generators rotate the cos-SDW order parameter as

$$e^{i\theta \gamma_0} e^{-i\theta \gamma_i} = \cos(2\theta) \gamma_0 - \sin(2\theta) \gamma_i; i = 3, 5.$$  \hfill (40)

Choosing \(i = 3\), for example, for both Dirac fields rotates the cos-SDW in the Eq. (35) into

$$m \int d^2 \vec{r} d\tau \sum_{\sigma = \pm, i = 1, 2} \cos(\vec{Q}_1 \cdot \vec{r} + 2\theta) \sigma c^{\dagger}_\sigma (\vec{r}, \tau)c_\sigma (\vec{r}, \tau).$$  \hfill (41)

Chiral rotations generated by \(\gamma_5\) thus correspond to sliding modes of the SDW. \(\gamma_3\), on the other hand, rotates \(\gamma_0\) towards the direction of \(\gamma_3\), which describes an additional particle-particle pairing between the neutral spinons, with the opposite sign for the diagonally opposed nodes. This may be understood as an additional p-wave pairing between the spinons, so the state described by \(\gamma_0\) order parameter may be called the "\(d+ip\)" state \cite{[40]}. This state preserves the superconducting \(U(1)\) symmetry and the translational invariance, but breaks the spin-rotational invariance and is odd under parity. Since \(\gamma_3\) does not commute with the electron-spinon transformation (9), however, "\(d+ip\)" state can not be that simply expressed in terms of electronic operators, as it proved possible for the SDW states. The relationship between the directions in the order parameter space \(\{\gamma_0, \gamma_3, \gamma_5\}\), and the chiral generators may be summarized pictorially as on Fig. 3.

It is instructive to look more closely at the origin of the \(U(2)\) symmetry (per Dirac component) that appears in the low energy theory of the dSC. First, the transformations in the \(U(1)\) subgroup of \(U(2) = U(1) \times SU_c(2)\) are analogous to the spin rotations around the z-axis. To see this, consider the conserved current that corresponds to the \(U(1)\) subgroup: \(J_{\nu, \mu} = \Psi^\dagger \gamma_{\mu} \gamma_5 \Psi\), so that the conserved charge is simply the z-component of the total spin of the low-energy quasiparticles, one charge per each pair of nodes. Of course, the quasiparticle action (2) has the full \(SO(3)\) spin symmetry, and this is not to imply that a part of it is broken in the dSC. It only means that in writing the full action (2) in terms of the Dirac fields (3) only the subgroup of spin rotations around z-axis is represented by a simple \(4 \times 4\) matrices that act on \(\Psi\). The rest is still present, but not that obvious in our choice of the Dirac fields, which was made to make the chiral symmetry manifest. (For a complementary representation that is fully rotationally symmetric at the expense of chiral symmetry see the Appendix D.) The \(U(1)\) subsymmetry is therefore always present, both in the superconducting and the insulating states. The \(SU_c(2)\) factor is more interesting. The conserved currents (per pair of nodes) in the dSC that correspond to this symmetry are \(J^c_{\nu, \mu} = \Psi^\dagger \gamma_5 \gamma_5 \gamma_3 \Psi\), \(\Gamma = \gamma_3, \gamma_5, \gamma_{35}\). As we have seen already, the \(\gamma_5\) generator simply translates in the diagonal direction. The corresponding conserved charge may be written as

$$Q^c_{\nu} = \int d^2 \vec{r} d\tau J^c_{\nu, 0} = T \sum_{\sigma, \omega, \vec{k} \approx \pm \vec{K}} \pm c^{\dagger}_\sigma (\vec{k}, \omega) c_\sigma (\vec{k}, \omega),$$  \hfill (42)

and may be identified with the quasiparticle momentum along \(\vec{K}_i\). More precisely, under the translation of the original electron operators \(c_\sigma (\vec{k}, \omega) \rightarrow e^{i\vec{k} \cdot \vec{R}} c_\sigma (\vec{k}, \omega)\), the spinon field transforms as

$$\Psi_i (\vec{r}, \tau) \rightarrow e^{i(\vec{K}_i \cdot \vec{R})} \gamma_5 \Psi_i (\vec{r} + \vec{R}, \tau),$$  \hfill (43)

where \(\vec{k} = \vec{K}_i + \vec{q}\). The low-energy theory therefore has more symmetry than the original action (2), as the chiral rotation by \(\gamma_5\) and the translations of \(\Psi\) separately are still the symmetries of the theory (24), while only when combined as above do they represent an ordinary translation. Nevertheless, breaking of chiral generator \(\gamma_5\) implies breaking of the translational symmetry in the theory. The remaining two generators of the chiral \(SU_c(2)\), \(\gamma_3\) and \(\gamma_{35}\), on the other hand, are not related to any spatial symmetry. They should be understood as "internal", and approximate, symmetries of the dSC that emerge at low energies. They rotate the translationally invariant "\(d+ip\)" state into a SDW, and therefore connect the two fundamentally different types of insulators.

![FIG. 3. The corners of the triangle represent the three chiral directions in the space of insulating states that descend from the d-wave superconductor. At the side opposite to a particular direction lies the corresponding unbroken chiral generator, while the remaining two broken generators rotate the chosen insulator towards two other directions.](image-url)
The reader should also note that in the action (24) the order parameter can be rotated independently for the first and the second Dirac field. Any linear combination of $\gamma_0$, $\gamma_3$, and $\gamma_5$ is a regular order parameter too. Since $\gamma_5$ is just a sin-SDW, the fundamentally different states are just the SDW are $d+ip$ state. This, however, already leads to a variety of insulating phases. For example, one can choose the cos-SDW for the first Dirac field $(\bar{Q}^1)$, while being in the $d+ip$ state for the second. This would correspond to a one-dimensional SDW along one of the diagonals.

With the velocity anisotropy neglected the QED$_3$ has a larger $U(4)$ symmetry, with sixteen generators $G_i$. The mass term now breaks all eight anticommuting generators, and the chiral manifold of insulating states becomes larger. For instance, rotating the cos-SDW with $\theta = \pi/4$ and the generator $G = G^1_{13} - G^1_{15}$ leads to a uniform state with an additional "s" component of pairing between spinons, "d+is"$^5$. Interestingly, rotating the cos-SDW by block-off-diagonal generators may also lead to charge stripes. For example, taking $\theta = \pi/4$ and $G^1_{15}$, rotates the $8 \times 8$ cos-SDW order parameter $\bar{I} \otimes \bar{\gamma}_0$ into $(1/2)\sigma_1 \otimes (\gamma_1 + \gamma_2)$. When written in terms of the electronic operators, this order parameter corresponds to the one-dimensional charge density-wave with the periodicity $P_k = K_2 + K_1$, and with residual pairing correlations in the orthogonal a-axis direction. It has been known that stripes indeed occur in some high-$T_c$ materials$^3$. Here they emerge as insulating cousins of the $d$-wave state in the isotropic limit of the theory. It is also interesting that stripes seem always to be accompanied by the residual pairing correlations, so one can think of them as weakly coupled one-dimensional systems on the verge of becoming phase coherent.

VII. REDUCTION OF CHIRAL SYMMETRY BY THE IRRELEVANT TERMS

We saw that the low-energy theory of dSC has the chiral $U(2)$ symmetry per Dirac component, which when spontaneously broken leads to emergence of the SDW or the $d+ip$ insulators. This enlarged symmetry arises only at low-energies, and the irrelevant terms omitted in the Eq. 6 reduce the $U(2)$ to $U(1) \times U_r(1)$. In this section I show the higher order derivatives and the Hubbard repulsion reduce the chiral $SU_c(2)$ symmetry to just translations, generated by $\gamma_5$. However, we will also find that if both perturbations are weak it will actually be the SDW solution that is energetically preferred.

Let us first consider the higher derivative terms in the Eq. (6). Since $\xi(\vec{k} - \vec{Q}_1) = \xi(\vec{q} - \vec{K}_1)$, and $\xi(\vec{q} - \vec{K}_1) = \xi(\vec{K}_1 - \vec{q})$, and analogously for $\Delta(\vec{k})$, one can write the second-order derivatives in the Eq. (6) as

$$S_1 = -i \int d^2\vec{r} dr \bar{\Psi'}_{\gamma_5} \xi''(\gamma_1 \xi''(\vec{q}^2) - \gamma_2 \Delta''(\vec{q}^2)) \Psi'_{\gamma_5}$$

with the average taken over the massive QED$_3$ with $\gamma_0 = \sigma_1 \otimes I$ (cos-SDW). The result, of course, is the same for the sin-SDW, or for any linear combination of the cos-SDW and the sin-SDW. Alternatively, if one assumes the $d+ip$ ordering, one finds that $S_U$ gives then a positive contribution to its energy, to the first order in $U$.

Although $S_U$ is only translationally symmetric, it actually inhibits the formation of the translationally invariant state, and prefers the ordering to be in the "orthogonal" direction, i. e. the SDW.

If both the interaction and the gradient terms are weak, it will therefore always be the SDW solution that is energetically preferred. This is because both the repulsive and the higher derivative terms are equally irrelevant by power counting (and have the engineering dimension $-1$), the gain in energy due to SDW is of first order only in $U$. The gradient terms affect the energy of the SDW.
only to the next order. So at long enough length scales one can alway neglect higher-derivative terms as compared to the repulsive interaction, which then serves to select the SDW over the d+ip insulator.

VIII. MEAN-FIELD THEORY WITH REPULSIVE INTERACTION

The message from the previous section is that the quartic term that represents a short-range repulsion, although irrelevant, at low but finite energies is still finite, and it breaks the chiral symmetry in favor of the SDW state. This is its first important role. The second is that once the chiral symmetry is dynamically broken by unbinding of vortex loops, the quartic term affects the size of the order parameter, and therefore sets the scale for the value of the SDW transition temperature. In this section I formulate the simplest mean-field theory of the chiral symmetry breaking in presence of the repulsion term, and demonstrate that it drastically increases the value of the SDW order parameter at $T = 0$.

We have seen that unbinding of vortex loops leads to weak SDW order, but with the order parameter orders of magnitude smaller than the coupling constant $|\langle \Phi \rangle|^2$. Assuming that the dual condensate as a function of doping $x$ should be of the same order of magnitude as the superfluid density on the other side of the transition (at $x = x_u$), $|\langle \Phi(x_u - \delta) \rangle|^2 \sim \rho_s(x_u + \delta)$, and that Uemura scaling $T_c(x) \propto \rho_s(x)$ is obeyed, the identification of the size of the SDW order parameter with the transition temperature $T_{sdw}(x)$ suggests that $T_{sdw}(x_u - \delta) \ll T_c(x_u + \delta)$. The difference in the relevant scales for the superconducting and the SDW orderings is in accord with the known phase diagram in the underdoped regime. Starting from half-filling, with increased doping the antiferromagnetic order is quickly lost, and only at a larger doping the dSC appears. I at-

FIG. 4. The SDW order parameter $m$ in units of $|\langle \Phi \rangle|^2$ as a function of dimensionless short-range repulsion $g = U |\langle \Phi \rangle|^2/(2\pi)^2$.

with the average to be calculated self-consistently within the theory quadratic in fermionic fields. The above term corresponds to the decoupling in the exchange (Fock) channel, since the direct (Hartree) term vanishes. Therefore in the Hartree-Fock approximation, after the Franz-Tešanović transformation

$S_U \rightarrow U \langle \Psi_i \Psi_i \rangle_0 \int d^2 \vec{r} d\tau \bar{\Psi}_i \Psi_i$, \hspace{1cm} (49)

Assuming a uniform $\chi = -U \langle \Psi_i \Psi_i \rangle_0$, and treating the gauge-field fluctuations in the large-N approximation leads to two coupled equations for $\chi$ and for the momentum dependent fermion self-energy

$\chi = U \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\Sigma(k)}{k^2 + \Sigma^2(k)}$, \hspace{1cm} (50)

with $\vec{p} = \vec{k} - \vec{q}$. When $U = 0$ these reduce to the Eq. (32), which leads to $N_c = 32/\pi^2$. When loops are bound and $\langle \Phi \rangle = 0$, on the other hand, $\Sigma(q) = \chi$, and the Eq. (50) allows a nontrivial solution only when the dimensionless coupling $g = U \Lambda/(2\pi)^2 > 1$, where $\Lambda$ is the UV cutoff, $\Lambda < T^*$. Assuming that long-range SDW order and dSC do not coexist, I take that $g < 1$ in the superconducting phase, so that the quartic coupling is there irrelevant. With $\langle \Phi \rangle \neq 0$, however, small $g$ ceases to be irrelevant, since there is now a small mass scale to effectively cut off its flow. Since $\Sigma(q)$ is quickly damped for $q >> |\langle \Phi \rangle|^2$, one can take the UV cutoff in the above equations to be $\Lambda \sim |\langle \Phi \rangle|^2$. The above equations were solved before [50, 51] in the context of gauged Nambu-Jona Lasinio model of chiral symmetry breaking in particle physics. Here I solve the equations numerically for $N = 2$, as discussed in the Appendix E. The result is presented in Fig. 4. The main point is that as the superconducting phase is more and more disordered and the dual condensate grows, the presence of a moderate repulsion between electrons increases the SDW order parameter at $T = 0$ by one to
two orders of magnitude. Recalling the above argument that compares $T_{SDW}$ to the superconducting $T_c$ on the other side of the superconductor-insulator transition, this appears to be in qualitative agreement with the generic behavior observed in underdoped cuprates.

IX. CONFINEMENT OF SPINONS

In the superconducting state, the electrically neutral low-energy spinons represented by the fermionic field $\Psi$ in the $QED_3$ are well-defined excitations. This effective spin-charge separation implicit in the superconducting state was emphasized in [22], and more recently in [1]. One may therefore naturally wonder if this form of spin-charge separation will survive once the superconductivity is lost via unbinding of vortex loops. The answer seems to be no. It is believed that the chiral symmetry breaking and confinement go together in the $QED_3$ [28]. Qualitative argument why it should be so is provided by the low momentum form of the polarization tensor in the Eq. (33) [53, 54]: $\Pi(q) \sim q^2/m$ for $q \ll m$, so in two dimensions spinons are at large distances bound by a logarithmic potential. One may independently arrive at the same conclusion by analytically continuing the fermion propagator in the broken symmetry phase to real frequencies [53], to find that its poles lie at complex energies with both real and imaginary parts proportional to the chiral mass. The chiral symmetry breaking and confinement of spinons seem therefore to go hand in hand in the $QED_3$, so the states with broken chiral symmetry, including most importantly the SDW, should not have well defined fermionic excitations even above the mass “gap”.

Dissapearance of spinons from the spectrum in the insulating phase, if required, imposes a rather non-trivial constraint on a candidate theory for underdoped cuprates. For example, one could imagine a completely different mechanism of chiral symmetry breaking in dSC: even without the gauge field, simply increasing the quartic coupling $U$ above a certain value ($U, \Lambda/(2\pi)^2 = 1$ in the Hartree-Fock approximation) would open the gap for spinons and lead to SDW order. This would be analogous to the chiral symmetry breaking in the Nambu-Jona Lasinio and related models [60, 7, 58, 59]. The crucial difference, however, is that such a mechanism would yield well defined spinon excitations at energies above the gap, in the insulating state. The integrity of the gaped spinons is assured essentially by the Landau phase space arguments. Such a “deconfined” antiferromagnet was dubbed AF* and studied in [24], for example. From this point of view it becomes a non-trivial problem to understand how spinons could be removed from the spectrum. In the $QED_3$ this is accomplished via the same non-perturbative mechanism that yields chiral symmetry breaking, described by the Eq. (32), for example.

Having said all this, it needs to be realized that in a weak SDW confinement of spinons is effective only over very large distances, $L \gg 1/m$. At intermediate scales, the polarization $\Pi(q) \sim q$, so the potential between spinons is $\sim 1/r$, and at intermediate distances $1/m \gg L \gg 1/|\Phi|^2$ spinons will appear effectively deconfined. In this sense it is still meaningful to think about underdoped cuprates as exhibiting an effective spin-charge separation. Computing the electron spectral function by taking the gauge-field fluctuations into account in large-N approximation [33], which suppresses the dynamical symmetry breaking, for example, gives results in qualitative agreement with the experiment [50]. As one continues to underdope, however, SDW order parameter grows, and spinons become more strongly confined. In the strong antiferromagnet at half-filling therefore, on may expect spinons to be confined already at atomic distances.

X. EXPERIMENT

The principal consequence of the $QED_3$ theory of underdoped cuprates is, of course, the antiferromagnetism itself. All the materials that become d-wave superconductors with doping are insulating antiferromagnets in its parent state. Furthermore, the sharp spectral features in the dSC should become very broad in the insulator, since there is a soft (propagator $\sim 1/q^2$) gauge field in the problem. Nevertheless, an insulator that derives from a dSC should partially inherit the d-wave form for its ”gap”, except for its finite value in the nodal directions. This is in very good agreement with the ARPES measurement on the insulating $\text{Ca}_2\text{CuO}_2\text{Cl}_2$, and $\text{Sr}_2\text{CuO}_2\text{Cl}_2$, in its parent state [8, 9]. In Fig. 5 I compare the ARPES data for the ”gap” measured from the top of the lower Hubbard band in the insulating state with the simplest functional form consistent with the chiral mass: at the remnant Fermi surface $\omega = ((E_{\text{max}}(\cos(k_x) - \cos(k_y))/2 + E_{\text{min}}^2)^{1/2}$, where the chiral mass $m = E_{\text{min}} = 75\text{meV}$ is chosen to be the $T = 0$ sublattice magnetization for $J = 125\text{meV}$. Best fit is obtained then for $E_{\text{max}} = 420\text{meV}$. The quality of the fit is actually not very sensitive to some variations in $E_{\text{min}}$ and the corresponding $E_{\text{max}}$.

The key prediction of this work is that the above ”gaped d-wave” form of the insulating ”gap” is a generic feature of the insulating state. Upon underdoping the ARPES should show the standard d-wave gap for sharp quasiparticles in the superconducting phase, which should evolve into a gaped d-wave form for broad ARPES shape in the insulating state, with the ”gap” increasing as one approaches half filling. The rounding of the data at low energies should therefore be intrinsic to the insulating state, and should weaken with doping. Although the initial experiment on $\text{Ca}_2\text{CuO}_2\text{Cl}_2$ [35] only indicated such rounding, later measurements on $\text{Sr}_2\text{CuO}_2\text{Cl}_2$ with higher resolution [6] clearly showed the deviation from
the simple d-wave cusp at lowest energy. More recent measurements [3] indicate that the rounding of the data at low energies is a robust feature. It would clearly be desirable to perform a systematic study of this effect at variable doping.

It may also be worth mentioning that some signs of the gap rounding in the insulator may be observable already in the superconducting state. In Bi2212 [12], for example, as one underdopes, the d-wave gap continues to show the cusp at zero energy, but with the slope (velocity $v_\Delta$) decreasing, in spite of the increase in the overall gap magnitude in $(\pi, \pi)$ direction. It is tempting to interpret this effect as a precursor of the dynamical mass generation. Detailed study of this effect and of the spectral features in the insulator is deferred to a future work.

**XI. CONCLUSION AND DISCUSSION**

In summary, I have shown that the minimal theory that describes unbinding of vortex defects in the d-wave superconductor at $T = 0$ is the two-component, 2+1 dimensional QED, with the vortex condensate playing the role of “charge”. With the loss of phase coherence, the d-wave superconductor suffers the spontaneous breaking of the low-energy “chiral” symmetry, which results in a weak SDW order. It was argued that with underdoping this SDW smoothly evolves into the strong antiferromagnet near half-filling, with the selection and the increase of the SDW order parameter being provided by the repulsion between electrons. I argued that spinons are marginally confined in a weak SDW, and may appear effectively deconfined over intermediate length scales in the pseudogap regime. Finally, it was proposed that the rounded d-wave form of the “gap” in the insulating $Ca_2CuO_2Cl_2$ observed by Ronning et al. may be a consequence of the chiral mass for the approximate spinon excitations, as implied by the $QED_3$.

The present theory is similar in spirit to the approaches of refs. [5] and [24], in that it attempts to understand the phase diagram of underdoped high temperature superconductors beginning from the superconducting phase. It differs, however, in its conclusions to what the ground state that results from unbinding of topological defects in the d-wave state is. Whereas it was argued in [5] and [24] that the relevant description of this process is provided by the Ising ($Z_2$) gauge theory, and that the resulting state may show spin-charge separation, I argued that unbinding of defects of unit vorticity leads to the dynamical symmetry breaking in the low-energy theory, and the accompanying confinement of spinons in the insulating state. In fact, if one demands that the insulating state near half-filling is the standard antiferromagnet with spin-1 excitations and confined spinons, the form of a single theory that would be able to describe both the dSC and the insulator becomes severely restricted. The $QED_3$ in this paper is one such theory.

A variation of the $QED_3$ as an effective theory for underdoped cuprates has also been considered before [3], [13], [14], [15] as the theory of low-energy fluctuations around the $\pi$-flux phase in the large-N version of the Heisenberg model. In that approach the gauge invariance reflects the local particle number conservation at half-filling, and the gauge-field has no dynamics on its own. As a result, the gauge field is necessarily compact, and the theory is infinitely strongly coupled. Not much is definitely known about such a lattice gauge theory, which greatly diminishes its utility. Nevertheless, it was argued that neglecting the instanton configurations would restore the antiferromagnetic order at half filling, via spontaneous breaking of a different ”chiral” symmetry, which in this case is actually an enlarged spin rotational symmetry [14]. While this logic may at first appear close to the one in the present work, there are crucial differences. First, I begin from the superconducting state, away from half-filling, with the gauge field describing vortex fluctuations. As a result, the gauge field is weakly coupled and non-compact near the dSC-SDW transition. Also, the SDW phase that obtains from chiral symmetry breaking may be incommensurate, and the approximate chiral symmetry of the low-energy theory is unrelated to spin rotations.

Nevertheless, it may be possible to understand the $QED_3$ as a low-energy description of the microscopic t-J model of cuprates. Starting from the mean-field slave-boson theory of the t-J model and integrating the constraints of no double occupancy, for example, leads to an effective theory of the form quite similar to the $QED_3$ [17], but with the Volovik’s field $\vec{v}$ only. Including vortices would then be expected to introduce the gauge field $\vec{a}$, as shown in this paper. The point is that irrespectively from the underlying microscopic model the theory of the fluctuating dSC should assume the $QED_3$ form. Values of the parameters, however, may strongly depend on the microscopic physics: the bare stiffness $K$ in the XY model for the phase fluctuations (Eq. (12)), for example, should be proportional to doping $x$ in the doped Mott insulator [17]. Also, the charge of quasiparticles (the coefficient in the last term in the Eq. (10)) would

![FIG. 5. ARPES results for $Ca_2CuO_2Cl_2$ (bars) and $Sr_2CuO_2Cl_2$ (dots) with $E = E(k) = E(\pi/2, \pi/2)$ in meV. The line is the function described in the text.](image-url)
be expected to change from unity to $\sim x$, at small dopings.

There exist further parallels between the $QED_3$ and the gauge theory of the t-J model. One may formulate a representation of the t-J model with a U(1) gauge field that minimally couples to spinons and holons. It was argued [22] that the effect of holons would be to screen the temporal component of the gauge field, which then may be shown to halve the critical number of spinon species for the chiral instability, $N_c \to N_c/2$. That way one could avoid the chiral transition at $N = 2$ (assuming that $N_c \approx 3$), and have a spin-liquid as the ground state in the underdoped regime instead. The tacit assumption, however, is that uncondensed bosons (holons) at $T = 0$ may exist in a compressible state. If the system becomes insulating with the loss of phase coherence, however, bosons would become incompressible and the above argument breaks down. This is indeed the case in the $QED_3$: with the proliferation of vortices the system becomes insulating, and all the components of the gauge field become massless. The same conclusion would be reached within the gauge theory of the t-J model if one would consider the proliferation of vortices the system becomes insulating, however, bosons would become incompressible and the above argument breaks down. This is indeed the case in the $QED_3$: with the proliferation of vortices the system becomes insulating, and all the components of the gauge field become massless. The same conclusion would be reached within the gauge theory of the t-J model if one would consider the incompressible state of slave bosons [18].

The present work shares the same philosophy with the recent one [15, 16], where the massless U(1) gauge field as an effective description of unbound vortex loops was also considered. While the authors [15] considered the large-N limit of the $QED_3$, and thus precluded the chiral symmetry breaking, my main point is that at $T = 0$ the spontaneous formation of the chiral condensate is nothing else but the SDW instability of the d-wave superconductor. The results of the ref. [15] may therefore be understood as applying to the finite-T phase much below the pseudogap scale $T^*$ in Fig. 1.

The problem of phase disordering of dSC has also been recently studied by Ye [20]. Working in the Anderson gauge [23] in which $\phi_{A} = \phi_s$, $\phi_{B} = 0$ in the Eq. (8), the author concluded that the gauge-field $\vec{a}$ is always massive when charge fluctuations are included. It is easy to see that this is a direct consequence of the gauge choice: in the Anderson gauge $\vec{a} = \vec{v}$, and not only $\vec{v}$, but $\vec{a}$ too is ultimately coupled to the charge current. In my gauge invariant approach, on the other hand, $\vec{a}$ is completely decoupled from charge, and couples only to spin. Inclusion of charge fluctuations therefore does not make $\vec{a}$ massive, but simply adds an irrelevant quartic coupling to the $QED_3$ Lagrangian.

The intimate relationship between the d-wave superconductivity and antiferromagnetism is also the main theme of the $SO(5)$ theory of Zhang [71]. The present work echoes some of that general idea, but is based on entirely different physical principles. In particular, although there should be a direct dSC-SDW transition in the phase diagram, this appears unrelated to the $SO(5)$ symmetry, but comes as a consequence of the chiral symmetry that emerges at low energies in the d-wave superconducting state. It is the spontaneous breaking of this hidden approximate symmetry that implies then the breaking of the spin rotational symmetry in the SDW phase.

Marginal confinement of spinons we found in the weak SDW phase is very much in line with the speculations of Laughlin [71, 72] on parallels between the antiferromagnetism and the confinement in strong interactions. In fact, the $QED_3$ shows precisely how chiral symmetry breaking, i. e. SDW ordering, binds spinons into spin-1 objects. Deconfinement in this theory seems indeed tantamount to the absence of chiral symmetry breaking. In this context, it may be interesting to note that the $d + id$ state, that would correspond to the $i\gamma_1\gamma_2$ matrix in (38), could lead to deconfined spinons. This state is outside of the chiral manifold, and it is believed that it is not spontaneously induced in the $QED_3$ [24] because of the Chern-Simons term that becomes generated for the gauge-field. With the Chern-Simons term, on the other hand, the gauge-field propagator behaves like $\sim q$ at low momenta, and thus spinons may become deconfined [23]. The chiral symmetry breaking in the $QED_3$ is therefore nothing by the effective description of the spinon confinement.

It is also nothing to note that were the critical number of fermions $N_c < 2$, the result of phase disordering of dSC would be quite different. Instead of symmetry breaking and confinement one would find a gapless, chirally symmetric state, in which spinons would be deconfined. This is again because the polarization tensor would then be $\sim q$ at low momenta, i. e. the interaction between spinons would be $\sim 1/r$ at large distances. This state would be similar in spirit to the ”nodal liquid” [7], or analogous to the ”algebraic Fermi liquids” [18, 19], [24] proposed in literature as candidates for the pseudogap phase. It has been proposed recently that $N_c = 3/2$ exactly [7], although all the actual calculations based on Schwinger-Dyson formalism lead to $N_c \geq 3$. If $N_c$ is indeed that low, phase disordering of the dSC would first lead to the deconfined pseudogap phase, which only later would turn into the confined SDW phase, presumably due to the repulsive quartic term which is know to increase $N_c$ [7, 6]. At this time it is hard to say which one of these two scenarios is realized in cuprates.

The main point made in this paper is that unbinding of vortex loops in a d-wave superconductor at $T = 0$ results in SDW order. It then appears natural to assume that the cores of fluctuating vortices are already in the insulating state. This speculation is in accord with the recent STM, neutron scattering, and NMR experiments [74, 75, 76, 77], the SO(5) proposal [78, 80], and the mean-field [81] and the finite size $QED_3$ calculations [82]. The superconductor-insulator transition would then be the result of the decrease of the bare stiffness $K$ in the XY model with underdoping, since $K \sim x$ in the doped Mott insulator [77].
XII. FURTHER PROBLEMS

I finish with a tentative list of problems opened by this work.

1) The role of strong anisotropy $v_f/v_\Delta \gg 1$ that exists in cuprates is unclear. In particular, since anisotropy on the bare level is marginal, it may affect the value of $N_c$. The preliminary results, indicate, however, that this weak anisotropy is irrelevant, so that one would expect $N_c$ to be unaffected by it [14].

2) Nature of the various phase transitions in the theory is also of interest. Whereas one expects that gapless quasiparticles do not change the Kosterlitz-Thouless universality class of the finite temperature superconducting transition, the nature of chiral symmetry breaking at finite temperature and its possible interplay with Neel transition is far less clear [33]. In particular, in relation to the Uemura scaling [19], one would like to understand quantum superconductor-insulator criticality and how it may be affected by gapless spinons.

3) Can long-range SDW and SC order coexist? In the approximation employed in the present work, the gauge-field $\vec{a}$ is considered decoupled from spinons in the dSC phase. This is likely to underestimate the effect of $\vec{a}$, and a better approximation for the gauge-field propagator is needed to study its effect inside the dSC. This could be important in light of the recent experimental data [78], [64] that may be interpreted as indicating the coexistence of the SDW and the SC orders in some compounds [35].

4) The present work also points to a new route towards a deconfined phase in two dimensions: lowering $N_c$ below two would allow for an insulating phase with deconfined spinons. At present, however, it is not clear how to achieve this within the $QED_3$, unless the Schwinger-Dyson equations systematically overestimate $N_c$ [42].

5) The computation of the electron propagator within the $QED_3$ is an important problem [42]. This would be necessary for a detailed comparison of the theory with the ARPES measurements.

6) As mentioned at the end of sec. IV, in the Villain approximation to the XY model, the gauge-field $\vec{a}$ appears to be compact, in contrast to the Volovik’s field $\vec{v}$. Although this should be an artifact of the Villain approximation, it would still be interesting to understand the effect of compactness of $\vec{a}$ on the chiral symmetry breaking in the $QED_3$. It has been argued that the coupling to gapless spinons makes the single instanton anti-instanton pair that derives from compactness of $\vec{a}$ bound above the certain number of spinon components $N_{inst}$ [42], [55]. $N_{inst}$ may be made smaller than $N_c$ for chiral symmetry breaking by a large anisotropy [42], for example. It is unclear, however, whether this conclusion survives the effects of screening by other pairs [57]. Also, even if the instantons can be made irrelevant above $N_c$, below $N_c$ one would expect them to become relevant again with the opening of the spinon "gap". This in turn could have profound consequences for the spinon confinement.

XIII. ACKNOWLEDGEMENT

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XIV. APPENDIX A

I present the self-consistent mean-field theory of the lattice superconductor (20) [44], and use it to approximately compute the correlator appearing in the Eq. (20). By the Bogoliubov inequality:

$$Z_{xy} \geq Z_0 e^{-(H_0 + H_0)\alpha},$$

where $Z_{xy}$ is the partition function in the dual form (21) with a finite "inverse temperature" $x$, and the average in the exponent is performed over a local mean-field Hamiltonian

$$H_0 = -\hbar \sum \cos \theta_i + \frac{1}{8K\pi^2} \sum (\nabla \times \vec{\Phi})^2 + \frac{m^2}{4K\pi^2} \sum \vec{\Phi}^2.$$  \hfill (53)

The optimal values of the parameters $h$ and $m$ that maximize the right hand side in the Bogoliubov inequality are then determined by the equations:

$$h = 6A I_1(h) \frac{x}{I_0(h)},$$

$$m^2 = K\pi^2 \frac{I_1(h)}{3I_0(h)} h,$$

$$A = \exp[-\frac{2K\pi^2}{3} \int \frac{d^3k}{(2\pi)^3} \frac{1}{F(k) + m^2}],$$

where $F(k) = \sum_\nu (e^{ikx} - 1)^2$, and the integral over $\vec{k}$ is taken over $(-\pi, \pi)$. $I_0$ and $I_1$ are the Bessel functions. These equations can be solved graphically, and describe a discontinuous transition from the phase with $h = m = 0$ (bound vortex loops), to the condensed phase $h \neq 0, m \neq 0$ (infinitely large vortex loops) [88].

The requisite average in the Eq. (20) is easy to compute in the mean-field theory that has different sites decoupled:

$$\langle \cos(\theta_i - \theta_{i+\nu} + \Phi_{i,\nu}) \rangle_0 = |\langle \epsilon^{i\theta_i} \rangle_0|^2 \langle e^{-i\Phi_{i,\nu}} \rangle_0.$$  \hfill (57)

Since, $\langle e^{-i\Phi_{i,\nu}} \rangle_0 = A$ and finite, we conclude that

$$\langle \cos(\theta_i - \theta_{i+\nu} - 2\pi \Phi_{i,\nu}) \rangle_0 \propto h^2,$$  \hfill (58)

i.e. finite only in the ordered phase of the dual theory (20), i.e. in the disordered phase of the original XY model.
XV. APPENDIX B

Here I provide a different derivation of the dynamics of the gauge-field $\vec{a}$ at $T \neq 0$ starting from the Hamiltonian for the Coulomb plasma. Assume a collection of $N_+ (N_-)$ vortices (antivortices) at the positions $\{\vec{r}_i\}$. The Hamiltonian of the vortex system is

$$H_v = \frac{1}{2} \sum_{i=1}^{N_+} q_i q_j \delta (\vec{r}_i - \vec{r}_j),$$

where $v(\vec{r}) \approx -\ln |\vec{r}|$, at large distances, and $N = N_+ + N_-$. The partition function of the vortex system $Z_v$ can then be written as

$$Z_v = \sum_{N_{A,B} = 0}^{\infty} \frac{N^+! N^-!}{N_A! N_B!} \left[ \frac{y}{2} \int \prod_{i=1}^{N} d\vec{r}_i e^{-y D_{\vec{r}_i}}, \right]$$

where $N^{+(-)} = N_A^{+(-)} + N_B^{+(-)}$, and $y$ is the bare vortex fugacity. The combinatorial factors serve to ensure that in $Z_v$ one sums over all possible divisions of vortices and antivortices into groups A and B, and divides by the number of combinations. With this symmetrization the symmetry between up and down spin in the original Hamiltonian (2) will be preserved in the Dirac theory for neutral spinons. This also guarantees that on average there is an equal number of vortices (and antivortices) in both groups.

Next, introduce the vorticity densities in $Z_v$ by inserting the unity

$$1 = \int D[\rho_A] \delta (\rho_A(\vec{r}) - \sum_{i=1}^{N_A} q_i A \delta (\vec{r} - \vec{r}_{iA})), \tag{61}$$

and similarly for B. The gauge field then becomes

$$(\nabla \times \vec{a}(\vec{r}))_\tau = \pi (\rho_A(\vec{r}) - \rho_B(\vec{r})), \tag{62}$$

in the transverse gauge $\nabla \cdot \vec{a} = 0$, and the index denotes the $\tau$ component. $\vec{v}$ is defined the same way except with the plus sign between $\rho_A$ and $\rho_B$.

By introducing two auxiliary fields $\Phi_A$ and $\Phi_B$ to enforce the constraints, after the integration over the densities the partition function may be written as

$$Z_v = \sum_{N_{A,B} = 0}^{\infty} \frac{y}{2} \int D[\vec{a}, \vec{v}, \Phi_+, \Phi_-] \tag{63}$$

$$\exp \left[ -\frac{1}{2\pi^2 T} \int d\vec{r} d\vec{r}' B(\vec{r}) v(\vec{r} - \vec{r}') B(\vec{r}') \right]$$

where $\Phi_{+,-} = \Phi_A \pm \Phi_B$, $B(\vec{r}) = (\nabla \times \vec{a})_\tau$, and $b(\vec{r}) = (\nabla \times \vec{a})_\tau$. Performing the summations yields

$$Z_v = \int D[\vec{a}, \vec{v}, \Phi_+, \Phi_-] \tag{64}$$

$$\exp \left[ -\frac{1}{2\pi^2 T} \int d\vec{r} d\vec{r}' B(\vec{r}) v(\vec{r} - \vec{r}') B(\vec{r}') \right]$$

Finally, neglecting the coupling to the charge current, the Gaussian integration over $\vec{v}$ (i.e., $B$) gives

$$Z_v = \int D[\vec{a}, \Phi_+, \Phi_-] \exp \left[ -\frac{1}{2\pi^2 T} \int d\vec{r} \frac{T}{2} (\nabla \Phi_+)^2 \right.$$}

where I also have rescaled the $\Phi$ fields by a factor of two. The last expression is then analogous to the $T = 0$ expression in the Eq. (18) with $x$ finite and without the dual angles $q_{A,B}$. By introducing a source term in the action, $\sim i \int j(r) \Phi(r) / \pi$, and integrating over $b$, one readily finds

$$((\nabla \times \vec{a}(\vec{r}))_\tau (\nabla \times \vec{a}(\vec{r}'))_\tau = \langle y \rangle \delta (\vec{r} - \vec{r}'), \tag{66}$$

where $\langle y \rangle = y \pi^2 (\exp(i \Phi_+))$, with the average to be taken at $\Phi_+ = \vec{a} = 0$. One recognizes $\langle y \rangle$ as the renormalized, or running, fugacity in the Kosterlitz-Thouless scaling, which signals the appearance of free vortices. $\langle y \rangle$ plays the role analogous to the vortex loop condensate in 2+1 dimensions, in providing a mass for the field $\Phi_+$ in the Eq. (65). This implies the Maxwell term at $T \neq 0$ for the $\tau$ component of $\nabla \times \vec{a}$ once fluctuating vortices are integrated out.

XVI. APPENDIX C

For completeness, here I outline the derivation of the result that chiral symmetry in isotropic massless QED3 is spontaneously broken for $N < N_c$, with $N_c$ finite, at any value of the coupling constant.

Rescaling the momenta $p/m \to p$ and self-energies $\Sigma(p)/m \to m$ and $\Pi(p)/m^2 \to \Pi(p)$, after taking the limit $q \to 0$ in the Eq. (32) we find

$$1 = \frac{|\langle \Phi \rangle|^2}{\pi^2 m} \int_0^{\Lambda/m} dp \frac{m^2 + \Sigma(p)}{(p^2 + \Sigma^2(p))^2} \tag{67}$$

where the polarization is now

$$\Pi(p) = \frac{N|\langle \Phi \rangle|^2}{4\pi m} f(p), \tag{68}$$

with
to the leading order in \( N \). We see that the right-hand side of Eq. (67) is a decreasing function of \( m \), so for \( m \neq 0 \) solution to exist we just need \( \text{RHS}(m = 0) > 1 \). This is satisfied for \( N < N_c \), where

\[
N_c = 4 \int_0^\infty \frac{dp}{(p^2 + \Sigma^2(p)) f(p)}.
\]

As defined, \( \Sigma(0) = 1 \), and one expects \( \Sigma(p) \) to vanish at large momenta. Also, \( f(p) \approx \pi p/2 \) for \( p \gg 1 \), so the integrand at large argument behaves like \( \sim \Sigma(p)/p \). \( N_c \) is therefore finite, and independent of the coupling constant \( \Phi \). Its precise value in the large-\( N \)-approximation will depend only on the function \( \Sigma(p) \) at \( N = N_c \), and can be obtained by solving the differential equation equivalent to the integral equation (67) [20] (see Appendix E). This yields \( N_c = 32/\pi^2 \), not far from the results of other more elaborate computations that go beyond the leading order in \( N \) [20].

**XVII. APPENDIX D**

Here I discuss a different representation of the quasiparticle action, more in line with the previous work [2]. This should serve to underline the difference between the approximate chiral \( SU_c(2) \) symmetry, and the exact spin rotational \( SO(3) \), also present in dSC. It is only the latter that will appear in the different version of the theory considered here and in [2], while the chiral symmetry will remain completely obscured.

I start again from the same quasiparticle action in the Eq. (2), but now introduce the four-component field as

\[
\Psi_{1(2)}^\dagger(\vec{q}, \omega_n) = (c_{\sigma}^\dagger(\vec{k}, \omega_n), c_{-}(\vec{k}, -\omega_n)) \tag{71}
\]

By linearizing the spectrum and by retaining only the modes near the four nodes, the continuum theory may again be written as

\[
S[\Psi] = \int d\vec{r} \int_0^\beta d\tau \Psi_{1(2)}^\dagger [\partial_\tau + M_1 v_f \partial_x + M_2 v_\Delta \partial_y] \Psi_{1(2)} + (1 \rightarrow 2, x \leftrightarrow y), \tag{72}
\]

but this time with different form of the matrices \( M_1 \) and \( M_2 \): \( M_1 = -i I \otimes \sigma_3 \), and \( M_2 = i I \otimes \sigma_1 \). Considering \( \gamma_0 = \sigma_3 \otimes \sigma_2 \), for example, the theory becomes

\[
S[\Psi] = \int d\vec{r} \int_0^\beta d\tau \Psi_{1(2)}^\dagger [\gamma_0 \partial_\tau + \gamma_1 v_f \partial_x + \gamma_2 v_\Delta \partial_y] \Psi_{1(2)} + (1 \rightarrow 2, x \leftrightarrow y), \tag{73}
\]

with \( \gamma_1 = \sigma_3 \otimes \sigma_1 \) and \( \gamma_2 = \sigma_3 \otimes \sigma_3 \). It is interesting to consider the generators of the global \( U(2) = U(1) \times SU(2) \) symmetry per Dirac component present in this representation of the theory. They are \( I_1 = I \otimes I \), \( \gamma_3 = \sigma_1 \otimes I \), \( \gamma_5 = -\sigma_2 \otimes I \), and \( \gamma_{35} = \sigma_3 \otimes I \), respectively. One may recognize the \( U(1) \) factor as representing now the continuous translations, since under a translation \( c_\sigma(\vec{k}, \omega) \rightarrow e^{i \vec{r} \cdot \vec{k}} c_\sigma(\vec{k}, \omega) \), the Dirac field now transforms as

\[
\Psi'_{(\vec{r}, \tau)} \rightarrow e^{i \vec{r} \cdot \vec{k}} \Psi_{(\vec{r} + \vec{k}, \tau)} \tag{74}
\]

The \( SU(2) \) operators, on the other hand, are nothing but the spin rotations. In fact, the above \( U(2) \) is an exact symmetry of the Hamiltonian (2), and is present even if all higher order derivatives are retained.

Including the coupling to vortex loops via massless gauge field in the above representation of the problem may spontaneously induce only the d+ip insulator. This breaks two of the above generators, which then simply rotate the spin-axis. Translational symmetry is, on the other hand, always preserved in this formulation, and the SDW remains invisible.

**XVIII. APPENDIX E**

Here I provide the details behind the numerical solution of the Eqs. 50-51. Since we are interested only in the qualitative effect of the U-term, it will suffice to assume that the fermion mass is small, \( m \ll |\langle \Phi \rangle|^2 \), so that one can neglect the \( p^2 \) term compared to \( \Pi(p) \) in the Eq. (51), and take

\[
\Pi(p) = \frac{N|\langle \Phi \rangle|^2}{8} p, \tag{75}
\]

appropriate for \( p \gg m \). This approximation is known to lead to even quantitatively good result for the mass for \( N \) as low as unity, when \( U = 0 \) [20]. Performing the angular integrals then gives

\[
\Sigma(q) = \chi + \frac{8}{N \pi^2 q} \int_0^\Lambda dk \frac{k \Sigma(k)(k - k q) \theta(k - k q)}{k^2 + \Sigma^2(k)}. \tag{76}
\]

Differentiating twice one finds that this integral equation is equivalent to the differential equation [21]:

\[
\frac{d^2}{dq^2}(q^2 \frac{d}{dq}(\Sigma(q))) = -\frac{8}{N \pi^2 q^2 + \Sigma^2(q)}, \tag{77}
\]

with the boundary condition

\[
\Lambda \Sigma'(\Lambda) + \Sigma(\Lambda) = \chi, \tag{78}
\]

and with

\[
\chi = \frac{U}{(2\pi)^2} \int_0^\Lambda dq \frac{q^2 \Sigma(q)}{q^2 + \Sigma^2(q)}. \tag{79}
\]

Here I take \( \Lambda = |\langle \Phi \rangle|^2 \).
The above equations may now be studied by assuming $q \gg \Sigma(q)$, which leads to linear equation that can be exactly solved. This yields, for example, the well known transition line in the $g - N$ plane: $g_c(N) = (1/4)(1 + \sqrt{1 - (N_c/N)^2})$, for $N > N_c$, $g_c \leq 1/4$ for $N = N_c$, with $N_c = 32/\pi^2$. To determine the size of $\Sigma(0)$, however, one needs to solve the full non-linear equation. This may be accomplished, for example, by choosing a value for $\chi$, assuming $\Sigma(\Lambda)$ next, and then iterating back to find $\Sigma(q)$ for $0 < q < \Lambda$. The solution is found by tuning $\Sigma(\Lambda)$ to achieve $\Sigma(0)$ finite. One then computes the value of $g = U\Lambda/(2\pi)^2$ from the assumed $\chi$ and the found $\Sigma(q)$. This procedure leads to the Fig. 4.

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