Necessary condition on weights for maximal and integral operators with rough kernels

by

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Abstract. Let \(0 \leq \alpha < n\), \(m \in \mathbb{N}\) and let \(T_{\alpha,m}\) be an integral operator given by a kernel of the form

\[K(x, y) = k_1(x - A_1 y)k_2(x - A_2 y) \ldots k_m(x - A_m y),\]

where the \(A_i\) are invertible matrices and each \(k_i\) satisfies a fractional size condition and a generalized fractional Hörmander condition. Ibañez-Firnkorn and Riveros (2018) proved that \(T_{\alpha,m}\) is controlled in \(L^p(w)\)-norms, \(w \in A_\infty\), by the sum of maximal operators \(M_{A^{-1},\alpha}\). In this paper we present a class \(A_{A,p,q}\) of weights, where \(A\) is an invertible matrix. These weights are appropriate for weak-type estimates of \(M_{A^{-1},\alpha}\). For certain kernels \(k_i\) we can characterize the weights yielding strong-type estimates of \(T_{\alpha,m}\). Also, we give a strong-type estimate using testing conditions.

1. Introduction. In this paper we will study the weights, \(0 \leq w \in L^1_{\text{loc}}(\mathbb{R}^n)\), for integral operators of the form

\[T_{\alpha,m}f(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy,\]

with

\[K(x, y) = k_1(x - A_1 y)k_2(x - A_2 y) \ldots k_m(x - A_m y), \quad m \in \mathbb{N},\]

where the \(k_i, i = 1, \ldots, m\), satisfy certain Hörmander and size conditions, \(A_i\) are invertible matrices and \(f \in L^\infty_{\text{loc}}(\mathbb{R}^n)\). See definitions in Section 2. In [15] the authors studied the \(L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})\) boundedness for

\[K(x, y) = \frac{1}{|x - y|^{\alpha}} \frac{1}{|x + y|^{1-\alpha}}, \quad 0 < \alpha < 1.\]
In several articles, different authors studied the boundedness in Lebesgue spaces and weighted Lebesgue spaces for several kinds of kernels \( k_i, \ i = 1, \ldots, m \); see for example [5, 7, 17, 18].

Let \( 0 \leq \alpha < n \) and \( 0 < \alpha_i < n, \ 1 \leq i \leq m \), with \( \alpha_1 + \cdots + \alpha_m = n - \alpha \). Also let \( A_i \) be matrices such that

(H) \( A_i \) and \( A_i - A_j \) are invertible for \( 1 \leq i, j \leq m \) and \( i \neq j \).

If

\[
    k_i(x, y) = \frac{1}{|x - A_i y|^{\alpha_i}}
\]

(see [16]) the integral operator \( T_{\alpha,m} \) satisfies the Coifman–Fefferman inequality

\[
    \left( \int_{\mathbb{R}^n} |T_{\alpha,m}(f)(x)|^q w(x)^q \, dx \right)^{1/q} \leq C_{w,q} \sum_{i=1}^m \int_{\mathbb{R}^n} |M_{\alpha,A_i^{-1}} f(x)|^q w(x)^q \, dx,
\]

for all \( 0 < q < \infty \), \( w^q \) a weight in the \( A_\infty \) Muckenhoupt class and where \( M_{\alpha,A^{-1}} \) is the maximal operator defined by

\[
    M_{\alpha,A^{-1}} f(x) = M_\alpha f(A^{-1} x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q f(y) \, dy.
\]

By a change of variable,

\[
    \left( \int_{\mathbb{R}^n} |M_{\alpha,A^{-1}} f(x)|^q w(x)^q \, dx \right)^{1/q} = |\det A| \left( \int_{\mathbb{R}^n} |M_\alpha f(x)|^q w(Ax)^q \, dx \right)^{1/q}.
\]

Now let \( 1 \leq p \leq q < \infty \), and \( w \in A_{p,q} \), i.e.

\[
    [w]_{A_{p,q}} := \sup_Q \left( \frac{1}{|Q|} \int_Q w(x)^q \, dx \right)^{1/q} \left( \frac{1}{|Q|} \int_Q w(x)^{-p'} \, dx \right)^{1/p'} < \infty \quad \text{for} \quad 1 < p,
\]

\[
    [w]_{A_{1,q}} := \sup_Q \left( \frac{1}{|Q|} \int_Q w(x)^q \, dx \right)^{1/q} \|w^{-1}\|_{\infty,Q} < \infty \quad \text{for} \quad 1 = p.
\]

We say that \( w^p \in A_p \) if \( w \in A_{p,p}, \ 1 \leq p < \infty \), and \( A_\infty = \bigcup_{p \geq 1} A_p \).

For \( p > 1 \), if we require that \( w(Ax) \leq Cw(x) \) and \( w \in A_{p,q} \), then from (1.4) we get

\[
    \left( \int_{\mathbb{R}^n} |M_{\alpha,A^{-1}} f(x)|^q w(x)^q \, dx \right)^{1/q} \leq C_{A_{p,q,p},w} \left( \int_{\mathbb{R}^n} |f|^p w(x)^p \, dx \right)^{1/p}.
\]

Hence from (1.2), if \( w \in A_{p,q} \) and \( w_{A_i}(x) := w(A_i x) \leq Cw(x) \) for all \( 1 \leq i \leq m \), we have (see [17])

\[
    \left( \int_{\mathbb{R}^n} |T_{\alpha,m}(f)(x)|^q w(x)^q \, dx \right)^{1/q} \leq C_{A_{p,q,p},w} \left( \int_{\mathbb{R}^n} |f|^p w(x)^p \, dx \right)^{1/p}.
\]

Therefore, it is natural to consider the following questions. Is there a characterization of the weights giving the boundedness of \( M_{\alpha,A^{-1}} \) defined
Weights for maximal and integral operators

in (1.3)? If this characterization is settled, are we able to obtain some weighted bounds as in (1.6)? For inequality (1.6), \( w^q \in A_\infty \) is required. Can such a condition be avoided?

To give answers to some of these questions we will use an important technique introduced in the last years, the sparse domination of an operator, introducing a version of it appropriate for operators we are considering.

The paper continues in the following way. In Section 2 we give preliminaries and definitions and we state the main results. We prove these results in Section 3 and an appropriate sparse domination in Section 4. Finally, in the Appendix we make extra comments on the \( A_{A,p,q} \) classes of weights.

Throughout this paper, \( c \) and \( C \) will denote positive constants, not the same at each occurrence.

2. Preliminaries and main results. We now define the fractional size and Hörmander conditions for kernels, which we will be working with. Let us introduce the following notation. For \( 1 \leq r < \infty \), we set

\[
\| f \|_{r,B} = \left( \frac{1}{|B|} \int_B |f(x)|^r \, dx \right)^{1/r},
\]

where \( B \) is a ball. Observe that in the averages the balls \( B \) can be replaced by a measurable subset \( X \) with \(|X| > 0\). The notation \(|x| \sim t\) means \( t < |x| \leq 2t \) and we write

\[
\| f \|_{r,|\cdot|\sim t} = \| f \chi_{|\cdot|\sim t} \|_{r,B(0,2t)}.
\]

Let \( 0 \leq \alpha < n \) and \( 1 \leq r \leq \infty \). The function \( K_\alpha \) is said to satisfy the fractional size condition \( \( K_\alpha \in S_{\alpha,r} \) \) if there exists a constant \( C > 0 \) such that

\[
\| K_\alpha \|_{r,|\cdot|\sim t} \leq Ct^{\alpha-n}.
\]

For \( r = 1 \) we write \( S_{\alpha,r} = S_{\alpha} \). Observe that if \( K_\alpha \in S_{\alpha} \), then there exists a constant \( c > 0 \) such that

\[
\int_{|x|\sim t} |K_\alpha(x)| \, dx \leq ct^\alpha.
\]

For \( \alpha = 0 \) we write \( S_{0,r} = S_r \).

The function \( K_\alpha \) satisfies the \( L^{\alpha,r}\)-Hörmander condition \( \( K_\alpha \in H_{\alpha,r} \) \) if there exist \( c_r > 1 \) and \( C_r > 0 \) such that for all \( x \) and \( R > c_r|x| \),

\[
\sum_{m=1}^{\infty} (2^m R)^{n-\alpha} \| K_\alpha(\cdot - x) - K_\alpha(\cdot) \|_{r,|\cdot|\sim 2^m R} \leq C_r.
\]

We say that \( K_\alpha \in H_{\alpha,\infty} \) if \( K_\alpha \) satisfies \( H_{\alpha,r} \) with \( \| \cdot \|_{L^{\infty,|\cdot|\sim 2^m R}} \) in place of \( \| \cdot \|_{r,|\cdot|\sim 2^m R} \). For \( \alpha = 0 \) we write \( H_{0,r} = H_r \), the classical \( L^r \)-Hörmander condition.
Remark 2.1. Observe that if $K_\alpha(x) = |x|^{n-\alpha}$ then $T_\alpha = I_\alpha$ is the fractional integral and $K_\alpha \in S_{\alpha,\infty} \cap H_{\alpha,\infty}$.

Remark 2.2. Let $0 \leq \alpha < n$, $m \in \mathbb{N}$ and $1 \leq i \leq m$. Let $1 < r_i \leq \infty$ and $s \geq 1$ be defined by $1/r_1 + \cdots + 1/r_m + 1/s = 1$ and $0 \leq \alpha_i < n$ with $\alpha_1 + \cdots + \alpha_m = n - \alpha$. If $k_i \in S_{n-\alpha_i, r_i}$ for $1 \leq i \leq m$ then $K \in S_\alpha$ for $K(x,y) = k_1(x-A_1y)k_2(x-A_2y)\ldots k_m(x-A_my)$. See details in [7].

Now, we define the following classes of weights. Let $A$ be an invertible matrix and $1 \leq p \leq q < \infty$. A weight $w$ is in the class $A_{A,p,q}$ if

\begin{equation}
[w]_{A_{A,p,q}} := \sup_Q \left( \frac{1}{|Q|} \int_Q w_A(x)^q \, dx \right)^{1/q} \left( \frac{1}{|Q|} \int_Q w(x)^{-p'} \, dx \right)^{1/p'} < \infty \quad \text{for } p > 1, \\
[w]_{A_{A,1,q}} := \sup_Q \left( \frac{1}{|Q|} \int_Q w_A(x)^q \, dx \right)^{1/q} \|w^{-1}\|_{\infty,Q} < \infty \quad \text{for } p = 1.
\end{equation}

2.1. Results for the maximal operator $M_{\alpha,A^{-1}}$. We obtain the following weak type $(p,q)$ characterization:

Theorem 2.3. Let $0 \leq \alpha < n$, $1 \leq p < n/\alpha$ and $1/q = 1/p - \alpha/n$. The maximal operator $M_{\alpha,A^{-1}}$ is bounded from $L^p(w^p)$ into $L^{q,\infty}(w^q)$ if and only if $w \in A_{A,p,q}$. Even more, $\|M_{\alpha,A^{-1}}\|_{L^p(w^p) \to L^{q,\infty}(w^q)} \leq C[w]_{A_{A,p,q}}$.

In [19] E. Sawyer introduced the following definition:

Definition 2.4. Let $0 \leq \alpha < n$, $1 < p \leq q < \infty$, and let $(u,v)$ be a pair of weights. The pair $(u,v)$ is in $M_{\alpha,p,q}$ if it satisfies the testing condition

\[ [u,v]_{M_{\alpha,p,q}} = \sup_Q v(Q)^{-1/p} \left( \int_Q M_\alpha(\chi_Q v)^q u \right)^{1/q} < \infty. \]

For the classical maximal operator $M_\alpha$ the following two-weight inequality was proved by Sawyer [19] and quantitative version by Moen [12].

Theorem 2.5 ([19] [12]). Let $0 \leq \alpha < n$, $1 < p \leq q < \infty$ and let $(u,v)$ be a pair of weights. The following statements are equivalent:

(i) $(u,v) \in M_{\alpha,p,q}$.

(ii) For every $f \in L^p(v)$,

\[ \left( \int_{\mathbb{R}^n} M_\alpha(fv)^q u \right)^{1/q} \leq C_{n,p,\alpha}[u,v]_{M_{\alpha,p,q}} \left( \int_{\mathbb{R}^n} |f|^p v \right)^{1/p}. \]

Definition 2.6. Let $w$ be a weight. We say $w \in M_{\alpha,A,p,q}$ if $(w_A^q, w^{-p'}) \in M_{A,p,q}$ and $w \in M^*_{\alpha,A,q',p'}$ if $(w^{-p'}, w_A^{q'}) \in M_{\alpha,q',p'}$. For $\alpha = 0$ and $p = q$, we put $w \in M_{A,p} := M_{0,A,p,p}$.
Remark 2.7. Also if \( w \in M_{\alpha,A,p,q} \) and \( 1/p - 1/q = \alpha/n \), we have \( w \in A_{A,p,q} \) and \( [w]_{A_{A,p,q}} \leq [w]_{M_{\alpha,A,p,q}} \).

The previous theorem yields

Corollary 2.8. Let \( 0 \leq \alpha < n \), \( 1 < p < n/\alpha \) and \( 1/q = 1/p - \alpha/n \). The weight \( w \) is in \( M_{\alpha,A,p,q} \) if and only if

\[
\left( \int_{\mathbb{R}^n} M_{\alpha,A^{-1}}(g)^q w^q \right)^{1/q} \leq C_{n,p,\alpha} [w]_{M_{\alpha,A,p,q}} \left( \int_{\mathbb{R}^n} |g|^p w^p \right)^{1/p},
\]

for \( g = f v = f w^{-p'} \in L^p(w^p) \), where \([w]_{M_{\alpha,A,p,q}} := [w^q, w^{-p'}]_{M_{\alpha,p,q}}\).

2.2. Results for the integral operator \( T_{\alpha,m} \). In this paper we will prove \( L^p(w^p) \rightarrow L^q(w^q) \) bounds for integral operator \( T_{\alpha,m} \) defined in (1.1), with kernels satisfying a fractional size condition \( S_{\alpha,r} \), and a fractional Hörmander condition \( H_{\alpha,r} \) (for definitions see Section 2.1), without using \( w \in A_{\infty} \). To obtain these results we will use the sparse domination technique. In the last years this technique has been used to obtain sharp weighted norm inequalities for singular or fractional integral operators; for example see [10].

In the case of the integral operator \( T_{\alpha,m} \) with some particular matrices \( A_i \) we obtain a norm estimate relative to the constant of the weight.

Theorem 2.9. Let \( 0 \leq \alpha < n \), \( m \in \mathbb{N} \) and let \( T_{\alpha,m} \) be the integral operator defined by (1.1). For \( 1 \leq i \leq m \), let \( n/\alpha_i < r_i \leq \infty \) and \( 0 \leq \alpha_i < n \) with \( \alpha_1 + \cdots + \alpha_m = n - \alpha \). Let \( k_i \in S_{n-\alpha_i,r_i} \cap H_{n-\alpha_i,r_i} \) and let the matrices \( A_i \) satisfy hypothesis (H). If \( \alpha = 0 \), suppose that \( T_{0,m} \) is of strong type \((p_0,p_0)\) for some \( 1 < p_0 < \infty \).

Suppose that there exists \( 1 \leq s < n/\alpha \) such that \( 1/r_1 + \cdots + 1/r_m + 1/s = 1 \), \( A_j = A_i^{-1} \) for some \( j \neq i \), and \( w^s \in \bigcap_{i=1}^m A_{A_i,p/s,q/s} \), where \( s < p < n/\alpha \), \( 1/q = 1/p - \alpha/n \). Then there exists \( C > 0 \) such that

\[
\|T_{\alpha,m}f\|_{L^q(w^q)} \leq C\|f\|_{L^p(w^p)} \sum_{i=1}^m [w^s]_{A_{A_i,p/s,q/s}}^{\max\{1-\alpha/n, (p/s)\ '(1-\alpha s/n)\}}
\]

for all \( f \in L^p(w^p) \).

Remark 2.10. Observe that if \( A_j = A_i^{-1} \) for some \( j \neq i \) and \( w^s \in A_{A_i,p/s,q/s} \cap A_{A_j,p/s,q/s} \), then \( w^s \in A_{p/s,q/s} \) and \( w_{A_j} \simeq w \). Hence, in this case, \( w^q \) and \( w^{-s(p/s)'} \) belong to \( A_{\infty} \).

For the integral operator

\[
T_{\alpha,2} f(x) = \int \frac{f(y)}{|x-A_1 y|^{\alpha_1} |x-A_2 y|^{\alpha_2}} dy,
\]

we have the following characterization.
**Theorem 2.11.** Let $0 \leq \alpha < n$, $0 \leq \alpha_1, \alpha_2 < n$ with $\alpha_1 + \alpha_2 = n - \alpha$. Let $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Let $A_1, A_2$ be invertible matrices such that $A_1 - A_2$ is invertible. Let $T_{\alpha,2}$ be the integral operator defined by (2.2). Let $w$ be a weight. If $T_{\alpha,2}$ is bounded from $L^p(w^p)$ into $L^q(w^q)$ then $w \in A_{A_1,p,q} \cap A_{A_2,p,q}$.

Furthermore, if $A_2 = A_1^{-1}$ or $A_1 = -I$ and $A_2 = I$, then $T_{\alpha,2}$ is bounded from $L^p(w^p)$ into $L^q(w^q)$ if and only if $w \in A_{A_1,p,q} \cap A_{A_2,p,q}$.

**Remark 2.12.** This result contains [4, Theorem 3.2 and Corollary 3.3], where the authors consider $p = q$, $\alpha = 0$, $w(x)^p = |x|^\beta \in A_p$, $A_1 = -I$ and $A_2 = I$.

**Remark 2.13.** As is well known, the $A_p$ condition, for $1 < p \leq \infty$, is also necessary for Calderón–Zygmund operators, in the following way: if $w^p$ is a weight such that all the Riesz transforms are strong $(p,p)$, then $w^p \in A_p$. In a similar way we see in this paper that the $A_{A_p,q}$ classes of weights are also necessary to obtain strong $(p,q)$ bounds for certain integral operators.

### 2.2.1. Some results with testing conditions.

To obtain a two-weight bound for the integral operator $T_{\alpha,m}$ in (1.1), we define the testing constants for a pair of weights $(u,v)$ following [9]. Let $D$ be a dyadic lattice, $1 \leq r < n/\alpha$, $1 < p < q < n/\alpha$. Set

$$T_{A,r,\text{out},D} := \sup_{R \in D} \int_{Q \subset R_q} |Q|^{\alpha/n-1/r} \left( \int_Q vQR \chi_Q \right)^{1/r} L^q(uA),$$

$$T_{A,r,\text{out},D}^* := \sup_{R \in D} \int_{Q \subset R_q} |Q|^{\alpha/n-1/r} \left( \int_Q uAQR \chi_Q \right)^{1/r} L^q(v),$$

$$T_{A,r,\text{in},D} := \sup_{R \in D} \int_{Q \subset R_q} |Q|^{\alpha/n-1/r} \left( \int_Q vQR \chi_Q \right)^{1/r} L^q(uA),$$

$$T_{A,r,\text{in},D}^* := \sup_{R \in D} \int_{Q \subset R_q} |Q|^{\alpha/n-1/r} \left( \int_Q uAQR \chi_Q \right)^{1/r} L^q(v).$$

We will also use the following notation:

$$T_{r,\text{out},D} := T_{I,r,\text{out},D}, \quad T_{r,\text{out},D}^* := T_{I,r,\text{out},D}^*,$$

$$T_{r,\text{in},D} := T_{I,r,\text{in},D}, \quad T_{r,\text{in},D}^* := T_{I,r,\text{in},D}^*,$$

where $I$ is the identity matrix. Also for $r = 1$,

$$T_{A,\text{out},D} := T_{A,1,\text{out},D}, \quad T_{A,\text{out},D}^* := T_{A,1,\text{out},D}^*,$$

$$T_{A,\text{in},D} := T_{A,1,\text{in},D}, \quad T_{A,\text{in},D}^* := T_{A,1,\text{in},D}^*.$$

For testing conditions we have the following theorem:

**Theorem 2.14.** Let $0 \leq \alpha < n$, $m \in \mathbb{N}$ and let $T_{\alpha,m}$ be the integral operator defined by (1.1). For $1 \leq i \leq m$, let $n/\alpha_i < r_i \leq \infty$ and $0 \leq \alpha_i < n$. 

...
with $\alpha_1 + \cdots + \alpha_m = n - \alpha$. Let $k_i \in S_{n-\alpha_i, r_i} \cap H_{n-\alpha_i, r_i}$ and let the matrices $A_i$ satisfy hypothesis (H). If $\alpha = 0$, suppose that $T_{0,m}$ is of strong type $(p_0, p_0)$ for some $1 < p_0 < \infty$.

Suppose that there exists $1 \leq s < n/\alpha$ such that $1/r_1 + \cdots + 1/r_m + 1/s = 1$ and that $(u, v)$ are weights such that for any dyadic lattice $D$, the testing constants $T_{A_i, s, \text{out}, D}, T_{A_i, s, \text{in}, D}^*$ are finite for each $1 \leq i \leq m$. Then for $1 \leq s < p < n/\alpha$ and $1/q = 1/p - \alpha/n$, there exists a constant $c > 0$ independent of $f \in L^p(\sigma)$ and of $(u, v)$ such that

$$
\|T_{\alpha,m}(f \sigma)\|_{L^q(u)} \leq c\|f\|_{L^p(\sigma)} \sup_D \sum_{i=1}^m (T_{A_i, s, \text{out}, D} + T_{A_i, s, \text{in}, D}^*),
$$

where $\sigma = v^{p'/((p/s)'}, and the supremum is taken over all dyadic lattices of cubes with edges parallel to coordinate axes.

An analogous result with the conditions $T_{A_i, s, \text{in}, D}, T_{A_i, s, \text{out}, D}^*$ in place of $T_{A_i, s, \text{out}, D}, T_{A_i, s, \text{in}, D}^*$ can be proved as well.

In the case $r_i = \infty$ for all $1 \leq i \leq m$, for the integral operator $T_{\alpha,m}$ we obtain a new two-weight result.

**Theorem 2.15.** Let $0 \leq \alpha < n$, $m \in \mathbb{N}$ and let $T_{\alpha,m}$ be the integral operator defined by (1.1). For $1 \leq i \leq m$, let $0 \leq \alpha_i < n$ with $\alpha_1 + \cdots + \alpha_m = n - \alpha$. Let $k_i \in S_{n-\alpha_i, \infty} \cap H_{n-\alpha_i, \infty}$ and let the matrices $A_i$ satisfy hypothesis (H). If $\alpha = 0$, suppose that $T_{0,m}$ is of strong type $(p_0, p_0)$ for some $1 < p_0 < \infty$.

Let $1 < p < q < n/\alpha$. If $(u, v)$ is a pair of weights such that $(u_{A_i}, v) \in \mathcal{M}_{\alpha, p, q}$ and $(v, u_{A_i}) \in \mathcal{M}_{\alpha, q', p'}$ for $i = 1, \ldots, m$, then $T_{\alpha,m}$ is bounded from $L^p(v)$ into $L^q(u)$.

Given a weight $w$ such that $(u, v) = (w_{A_i}^q, w^{-p'})$ we get

**Corollary 2.16.** Let $0 \leq \alpha < n$, $m \in \mathbb{N}$ and let $T_{\alpha,m}$ be the integral operator defined by (1.1). For $1 \leq i \leq m$, let $0 \leq \alpha_i < n$ with $\alpha_1 + \cdots + \alpha_m = n - \alpha$. Let $k_i \in S_{n-\alpha_i, \infty} \cap H_{n-\alpha_i, \infty}$ and let the matrices $A_i$ satisfy hypothesis (H). If $\alpha = 0$, suppose that $T_{0,m}$ is of strong type $(p_0, p_0)$ for some $1 < p_0 < \infty$.

Let $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. If $w \in \mathcal{M}_{\alpha, A_i, p, q} \cap \mathcal{M}_{\alpha, A_i, q', p'}^*$ for $i = 1, \ldots, m$, then $T_{\alpha,m}$ is bounded from $L^p(w^p)$ into $L^q(w^q)$.

**Remark 2.17.** By Remark 2.7 if $w \in \bigcap_{i=1}^m \mathcal{M}_{\alpha, A_i, p, q}$ then we have $w \in \bigcap_{i=1}^m \mathcal{A}_{A_i, p, q}$ and this implies $w(A_i x) \leq cw(x)$ a.e. $x \in \mathbb{R}^n$, for all $1 \leq i \leq m$ (see Proposition 5.1).

In [17] the conclusion of Corollary 2.16 was proved under the hypothesis $w \in \mathcal{A}_{p, q}$ and $w(A_i x) \leq cw(x)$ a.e. $x \in \mathbb{R}^n$ for all $1 \leq i \leq m$. The hypothesis $w \in \mathcal{A}_{p, q}$ and $w(A x) \leq cw(x)$ implies that $\mathcal{M}_{\alpha, A_i, p}$ is bounded from $L^p(w^p)$
into $L^q(w^q)$. As stated in Corollary 2.8, if $w \in M_{\alpha,A,p,q}$, then $w \in A_{A,p,q}$. In other words,

$$A_{p,q} \cap \{ w : w_A \lesssim w \} \subset \{ w : w \text{ satisfies the testing condition for } M_{\alpha,A-1} \} \subset A_{A,p,q}.$$ 

Therefore we obtain a different proof without using $w \in A_\infty$ essentially.

3. Proofs of the main results

3.1. Proof of the results for the maximal operator $M_{\alpha,A-1}$

Proof of Theorem 2.3. This result follows in the same way as the classical one in [14, 13] taking into account that

$$(3.1) \quad \{ x : M_{\alpha,A-1}f(x) > \lambda \} = w^q_{A} \{ x : M_{\alpha}f(x) > \lambda \}.$$ 

Let $w$ be a weight such that $M_{\alpha,A-1}$ is bounded from $L^p(w^p)$ into $L^{q,\infty}(w^q)$. We take $f \geq 0$ and a cube $Q$ such that $f(Q) = \int_Q f > 0$. If $\lambda < f(Q)/|Q|^{1-\alpha/n}$, then

$$AQ \subset \{ x : M_{\alpha,A-1}f(x) > \lambda \},$$ 

so

$$w(AQ)^q \leq \frac{c}{\lambda^q} \left( \int |f(x)|^p w(x)^p \, dx \right)^{q/p},$$

thus

$$(3.2) \quad w(AQ)^q \left( \frac{f(Q)}{|Q|^{1-\alpha/n}} \right)^q \leq c \left( \int |f(x)|^p w(x)^p \, dx \right)^{q/p}.$$ 

If $p > 1$, we choose $f = w^{1-p} \chi_Q$ and using $1/q + 1/p' = 1 - \alpha/n$, we obtain

$$\left( \frac{1}{|Q|} \int_Q w(Ax)^q dx \right)^{1/q} \left( \frac{1}{|Q|} \int_Q w(x)^{-p'} dx \right)^{1/p'} \leq c.$$ 

If $p = 1$ and $q = 1 - \alpha/n$, let $\xi = \text{ess inf} \{ w(x) : x \in Q \}$. Let $S_\epsilon \subset Q$ such that $|S_\epsilon| > 0$ and $w(x) < \xi + \epsilon$ for all $x \in S_\epsilon$. We take $f = \chi_{S_\epsilon}$ in (3.2), so

$$\frac{w(AQ)^q}{|Q|} \leq c \left( \frac{w(S_\epsilon)}{|S_\epsilon|} \right)^q \leq c(\xi + \epsilon)^q, \quad \forall \epsilon > 0,$$

then

$$\left( \frac{w(AQ)^q}{|Q|} \right) \| w^{-1} \|_{\infty,Q} \leq c.$$ 

Now we suppose $w \in A_{A,p,q}$. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ be a positive function. We consider the disjoint dyadic cubes $\{ Q_j \}$ given by Calderón–Zygmund decomposition for $f$ at height $\lambda/2^{2^n-\alpha}$. We have

$$\{ x : M_{\alpha}f(x) > \lambda \} \subset \bigcup 3Q_j$$
and
\[ \lambda < \frac{1}{|Q_j|^{1-\alpha/n}} \int_{Q_j} f < 2^{n-\alpha} \lambda, \]
where $3Q_j$ is the cube with the same center as $Q_j$, with triple side length. If $p > 1$ then, by (3.1),
\[
\lambda^q w^q \{ x : M_{\alpha,A^{-1}} f(x) > \lambda \} \leq \lambda^q \sum_j \int_{3Q_j} w_A(x)^q \, dx
\]
\[
\leq 3^n \sum_j \left( \frac{1}{|Q_j|^{1-\alpha/n}} \int_{Q_j} f w^{-1} \right)^q |Q_j| \frac{1}{|3Q_j|} \int_{3Q_j} w_A(x)^q \, dx
\]
\[
\leq 3^n \sum_j |Q_j|^{q/p'} \left( \int_{Q_j} f^p w^p \right)^{q/p} \left( \int_{3Q_j} w^{-p'} \right)^{q/p'} \frac{1}{|3Q_j|} \int_{3Q_j} w_A(x)^q \, dx
\]
\[
\leq 3^{n+q/p'} [w]_{A_{\alpha,p,q}}^q \| f \|_{L^p(w^p)}^q.
\]
If $p = 1$, in the third inequality we get $(\int_{Q_j} f w)^q \| w^{-1} \|_{\infty,Q_j}^q$ and the result is straightforward.

3.2. Proof of Theorem 2.11
Let $B = B(c_B, R)$ and $B_i = A_i^{-1} B_i$, $i = 1, 2$. Let $f = \chi_{B_1}$ and suppose that $T_{\alpha,2}(\cdot,v)$ is bounded from $L^p(v)$ into $L^q(u)$. Then
\[
v(B_1)^{-1/p} \left( \int_{T_{\alpha,2}(\chi_{B_1} v)} (x)^q u(x) \, dx \right)^{1/q} < \infty.
\]
If $x \in B$ and $y \in B_1$, then
\[
|x - A_1 y| \leq |x - c_B| + |c_B - A_1 y| \leq R + C_A R = (1 + C_A) R.
\]
If $|x - A_2 y| \leq |x - A_1 y|$, then
\[
|x - A_1 y|, |x - A_2 y| \leq R,
\]
and since $\alpha_1 + \alpha_2 = n - \alpha$ we have
\[
\frac{v(y)}{|x - A_1 y|^\alpha_1 |x - A_2 y|^\alpha_2} \geq \frac{v(y)}{|x - A_1 y|^{n-\alpha}} \geq C \frac{v(y)}{R^{n-\alpha}}.
\]
If $|x - A_1 y| \leq |x - A_2 y|$, then
\[
\frac{v(y)}{|x - A_1 y|^\alpha_1 |x - A_2 y|^\alpha_2} \geq \frac{v(y)}{|x - A_2 y|^{n-\alpha}}.
\]
If $2^j |x - A_1 y| \leq |x - A_2 y| \leq 2^{j+1} |x - A_1 y|$, then
\[
\frac{1}{|x - A_2 y|^{n-\alpha}} \geq 2^{(\alpha-n)(j+1)} \frac{1}{|x - A_1 y|^{n-\alpha}} \geq 2^{(\alpha-n)(j+1)} \frac{1}{R^{n-\alpha}}.
\]
In the case that \( y \in B_1 \) with \( |x - A_1 y| \leq |x - A_2 y| \), we have

\[
\frac{v(y) \chi_{B_1 \cap \{ |x - A_1 y| \leq |x - A_2 y| \}}}{|x - A_1 y|^{\alpha_1} |x - A_2 y|^{\alpha_2}} = \sum_{j=1}^{\infty} \frac{v(y) \chi_{\{ |x - A_1 y| \leq 2^{j+1} |x - A_1 y| \}}}{|x - A_1 y|^{\alpha_1} |x - A_2 y|^{\alpha_2}} \\
\geq C \frac{v(y)}{R^{n-\alpha}}.
\]

Hence, if \( x \in B \) and \( y \in B_1 \), then

\[
\frac{v(y)}{|x - A_1 y|^{\alpha_1} |x - A_2 y|^{\alpha_2}} \geq C_{n,\alpha,A} \frac{v(y)}{R^{n-\alpha}}.
\]

We have an analogous result if \( y \in B_2 \).

If \( x \in B \) then

\[
T_{\alpha,2}(\chi_{B_1} v)(x) \geq R^{-n} v(B_1) = |B|^{-n} v(B_1).
\]

Therefore

\[
v(B_1)^{-1/p} \left( \int_B T_{\alpha,2}(\chi_{B_1} v)(x)^q u(x) \, dx \right)^{1/q} \\
\geq v(B_1)^{-1/p} \left( \int_B T_{\alpha,2}(\chi_{B_1} v)(x)^q u(x) \, dx \right)^{1/q} \\
\geq v(B_1)^{-1/p} \left( \int_B |B|^{q(\alpha/n - 1)} v(B_1)^q u(x) \, dx \right)^{1/q} \\
\geq v(B_1)^{-1/p} |B|^{\alpha/n - 1} v(B_1) u(B)^{1/q} \\
= |\det A_1|^{-1/p'} |B|^{1/q} u(B)^{1/q} B_1^{1/p'} v(B_1)^{1/p'}.
\]

If we take \( f = \chi_{B_2} \), in an analogous way we have

\[
|B|^{\alpha/n - 1} u(B)^{1/q} v_{A_2^{-1}}(B)^{1/p'} < \infty.
\]

If \( u = w^q \) and \( v = w^{-p'} \) we conclude that if \( T_{\alpha,2} \) is bounded from \( L^p(w^p) \) into \( L^q(w^q) \) then \( w \in \cap_{i=1}^2 A_{A_i,p,q} \).

Furthermore, consider the case \( A_1 = A \) and \( A_2 = A^{-1} \). If \( w \in \mathcal{A}_{A,p,q} \cap \mathcal{A}_{A^{-1},p,q} \) then \( w_A \sim w \) and \( w \in \mathcal{A}_{p,q} \). Therefore \( T_{\alpha,2} \) is bounded from \( L^p(w^p) \) into \( L^q(w^q) \). Now for \( A_1 = -I \) and \( A_2 = I \), if \( w \in \mathcal{A}_{-I,p,q} \cap \mathcal{A}_{p,q} \) then \( w_{-I} \sim w \) and \( T_{\alpha,2} \) is bounded from \( L^p(w^p) \) into \( L^q(w^q) \) (see [17]).

### 3.3. Proofs of the results for the integral operator \( T_{\alpha,m} \)

To prove the boundedness of \( T_{\alpha,m} \) for general kernels we will use appropriate sparse domination.

Given a cube \( Q \in \mathbb{R}^n \), we denote by \( \mathcal{D}(Q) \) the family of all dyadic cubes with respect to \( Q \), that is, the cubes obtained by subdividing \( Q \) and each of its descendants into \( 2^n \) subcubes of the same side lengths.
Given a dyadic lattice $\mathcal{D}$ we say that a collection of cubes $\mathcal{S} \subset \mathcal{D}$ is an $\eta$-sparse family with $0 < \eta < 1$ if for every $Q \in \mathcal{S}$, there exists a measurable set $E_Q \subset Q$ such that $\eta |Q| \leq |E_Q|$ and the family $\{E_Q\}_{Q \in \mathcal{S}}$ is pairwise disjoint.

The following theorem will be proved in Section 4.

**Theorem 3.1.** Let $0 \leq \alpha < n$, $m \in \mathbb{N}$ and let $T_{\alpha,m}$ be the integral operator defined by (1.1). For $1 \leq i \leq m$, let $n/\alpha_i < r_i \leq \infty$ and $s \geq 1$ with $1/r_1 + \cdots + 1/r_m + 1/s = 1$ and let $0 \leq \alpha_i < n$ with $\alpha_1 + \cdots + \alpha_m = n - \alpha$. Let $k_i \in S_{n-\alpha_i} \cap H_{n-\alpha_i,r_i}$ and let the matrices $A_i$ satisfy hypothesis (H). If $\alpha = 0$, suppose that $T_{0,m}$ is of strong type $(p_0,p_0)$ for some $1 < p_0 < \infty$.

There exist $c > 0$ and $3^n \frac{1}{2^{yn}}$-sparse families, $\{\mathcal{S}_j\}_{j=1}^{3^n}$, such that for $f \in L_c^\infty(\mathbb{R}^n)$ and $a.e. \, x \in \mathbb{R}^n$, we have

$$|T_{\alpha,m}(f)(x)| \leq c \sum_{j=1}^{3^n} \sum_{i=1}^{m} A_{\alpha,s,\mathcal{S}_j} f(A_i^{-1} x),$$

where $A_{\alpha,s,\mathcal{S}_j}$ is the sparse operator defined by

$$A_{\alpha,s,\mathcal{S}_j} f(y) := \sum_{Q \in \mathcal{S}_j} |Q|^{\alpha/n} \|f\|_{s,Q} \chi_Q(y).$$

To prove the boundedness of $T_{\alpha,m}$ it is enough to show the boundedness of $A_{\alpha,s,\mathcal{S}_j}$.

**Theorem 3.2.** Let $0 \leq \alpha < n$ and $1 < r < p < q < \infty$. Let $A$ be an invertible matrix and $\mathcal{S}$ be a sparse family from a dyadic lattice $\mathcal{D}$. Let $(u,v)$ a pair of weights and $\sigma = v^{(p/r)q}$. If $(u,v)$ satisfies the local testing conditions $T_{r,\text{out},D}, T_{r,\text{out},D}^* < \infty$, then

$$\left( \int_{\mathbb{R}^n} A_{\alpha,r,\mathcal{S}} (f \sigma)^qu \right)^{1/q} \leq C_{n,p,\alpha}(T_{r,\text{out},D} + T_{r,\text{out},D}^*) \left( \int_{\mathbb{R}^n} |f|^p \sigma \right)^{1/p}.$$ 

If $T_{r,\text{out},D}^* < \infty$ then $A_{\alpha,r,\mathcal{S}}$ is bounded from $L^p(\sigma)$ into $L^{q,\infty}(u)$. Furthermore, if $u = w^p_A$ and $v = w^{-r(p/r)'}$, then $\sigma = w^{-p'}$ and $A_{\alpha,r,\mathcal{S}}$ is bounded from $L^p(w^p)$ into $L^{q}(w^q_A)$.

There is an analogous result for the global testing conditions, $T_{r,\text{in},D}$ and $T_{r,\text{in},D}^*$.

We consider the following sparse operator defined in [3], for $\mathcal{S}$ a sparse family, $0 < t < \infty$ and $0 < \beta \leq 1$:

$$\tilde{A}^{\beta}_{t,\mathcal{S}} g(x) = \left( \sum_{Q \in \mathcal{S}} |Q|^{-\beta} \left( \int_Q g \right)^t \chi_Q(x) \right)^{1/t}.$$ 

Observe that

$$A_{\alpha,r,\mathcal{S}}(f) = (\tilde{A}^{1-r\alpha/n}_{1/r,\mathcal{S}} (f^r))^{1/r}.$$
To prove Theorem 3.2 we need the following lemmas. In [9], the authors proved the following characterization for positive dyadic operators.

**Lemma 3.3 (9).** Let $1 < p \leq q < \infty$ and $u, v$ be weights. For a collection $\mathcal{D}$ of dyadic cubes and non-negative real $(\tau_Q)_{Q \in \mathcal{D}}$ consider

$$T(f) = \sum_{Q \in \mathcal{D}} \tau_Q \left( \frac{1}{|Q|} \int_{Q} |f| \right) \chi_Q.$$ 

Then

$$\|T(\cdot, v)\|_{L^p(v) \to L^q(u)} \simeq \sup_{R \in \mathcal{D}} u(R)^{-1/q} \left\| \sum_{Q \in \mathcal{D}: R \subset Q} \tau_Q \left( \frac{1}{|Q|} \int_{Q} v \right) \chi_Q \right\|_{L^{p'}(v)}$$

$$+ \sup_{R \in \mathcal{D}} v(R)^{-1/p} \left\| \sum_{Q \in \mathcal{D}: R \subset Q} \tau_Q \left( \frac{1}{|Q|} \int_{Q} u \right) \chi_Q \right\|_{L^{q'}(u)}.$$

**Lemma 3.4.** Let $1 < p \leq q < \infty$, $t \in (0, p)$, $\beta \in (0, 1]$, let $\mathcal{I}$ be a sparse collection of dyadic cubes and let $(u, v)$ be a pair of weights. Define the testing constants

$$\tilde{T}_{t, \text{out}} := \sup_{R \in \mathcal{I}} v(R)^{-\beta t} \left\| \sum_{Q \in \mathcal{I}: R \subset Q} |Q|^{-\beta t} \left( \int_{Q} v \chi_R \right)^t \chi_Q \right\|_{L^{q/t}(u)},$$

$$\tilde{T}^*_{t, \text{out}} := \sup_{R \in \mathcal{I}} u(R)^{-1/(q/t)} \left\| \sum_{Q \in \mathcal{I}: R \subset Q} |Q|^{-\beta t} v(Q)^{t-1} \left( \int_{Q} u \chi_R \right) \chi_Q \right\|_{L^{p/(t')}(v)}.$$ 

Then

$$\|\tilde{A}^\beta_{t, \mathcal{I}}(v\cdot)\|_{L^p(v) \to L^q(u)} \lesssim \tilde{T}_{t, \text{out}} + \tilde{T}^*_{t, \text{out}}.$$ 

The version with $\tilde{T}_{t, \text{in}}$ was proved in [3].

**Proof of Lemma 3.4.** The proof follows the scheme of [3]. Let $t < p \leq q < \infty$. We apply [3] Lemma 3.2 for $c_Q = |Q|^{-\alpha t}$ which reduces the estimate to an estimate for $T(\cdot, v)$ from $L^{p/t}(v)$ into $L^{q/t}(u)$, where

$$T(f) = \sum_{Q \in \mathcal{I}} |Q|^{-\alpha t} \sigma(Q)^{t-1} \left( \frac{1}{|Q|} \int_{Q} |f| \right) \chi_Q.$$ 

Now, with $\tau_Q = |Q|^{-\alpha t} \sigma(Q)^{t-1}$ we are in the setting of Lemma 3.3. Putting everything together, we get

$$\|\tilde{A}^\beta_{t, \mathcal{I}}(v\cdot)\|_{L^p(v) \to L^q(u)} \simeq \|T(v\cdot)\|_{L^{p/t}(v) \to L^{q/t}(u)} \simeq \tilde{T}_{t, \text{out}} + \tilde{T}^*_{t, \text{out}}.$$ 

**Proof of Theorem 3.2.** Using (3.3), we have

$$\left( \int_{\mathbb{R}^n} A_{\alpha, r, \mathcal{I}} (f \sigma)^{q} u \right)^{1/q} = \left( \int_{\mathbb{R}^n} (\tilde{A}_{1/r, \mathcal{I}}^{1-r\alpha/n} (f^{r} \sigma^{r}))^{q/r} u \right)^{1/q}$$

$$= \left( \int_{\mathbb{R}^n} (\tilde{A}_{1/r, \mathcal{I}}^{1-r\alpha/n} (f^{r} \sigma^{r} v^{-1} u))^{q/r} u \right)^{1/q}.$$
If \((u,v)\) satisfies the testing conditions \(\tilde{T}_{1/r,\text{out}}, \tilde{T}_{1/r,\text{out}} < \infty\) with \(p/r, q/r\) and \(\beta = 1 - \alpha/n\), then, by Lemma 3.4 we have
\[
\left( \int_{\mathbb{R}^n} A_{\alpha,r,\mathcal{R}}(f\sigma)^q u \right)^{1/q} = \left( \int_{\mathbb{R}^n} (A_{1/r,\mathcal{R}}^{1-r\alpha/n}(f^{r \sigma^r v^{-1}}) u)^{1/r} \right)^{r/q} \leq C_{n,p,\alpha}(\tilde{T}_{1/r,\text{out}} + \tilde{T}_{1/r,\text{out}}^*)(\int_{\mathbb{R}^n} |f|^p \sigma^p v^{-1}|p/r v\right)^{r/p} \leq C_{n,p,\alpha}(\tilde{T}_{1/r,\text{out}} + \tilde{T}_{1/r,\text{out}}^*)(\int_{\mathbb{R}^n} |f|^p \sigma^p v^{-1}|p/r v\right)^{1/p}.
\]

As \(v = \sigma^{r(p/r')/p'}\), we have
\[
\left( \int_{\mathbb{R}^n} A_{\alpha,r,\mathcal{R}}(f\sigma)^q u \right)^{1/q} = C_{n,p,\alpha}(\tilde{T}_{1/r,\text{out}} + \tilde{T}_{1/r,\text{out}}^*)(\int_{\mathbb{R}^n} |f|^p \sigma^p v^{-1}|p/r v\right)^{1/p}.
\]

Observe that if the pair \((u,v)\) satisfies the testing conditions \(\mathcal{T}_{r,\text{out},D}, \mathcal{T}_{r,\text{out},D}^* < \infty\), with \(\alpha, p, q\) then the pair \((u,v)\) satisfies the testing conditions \(\tilde{T}_{1/r,\text{out}}, \tilde{T}_{1/r,\text{out}}^* < \infty\) with \(\beta = 1 - \alpha/n, p/r\) and \(q/r\). Moreover \(\tilde{T}_{1/r,\text{out}} \leq \mathcal{T}_{r,\text{out},D}\) and \(\tilde{T}_{1/r,\text{out}}^* \leq \mathcal{T}_{r,\text{out},D}^*\).

Therefore, we get
\[
\left( \int_{\mathbb{R}^n} A_{\alpha,r,\mathcal{R}}(f\sigma)^q u \right)^{1/q} \leq C_{n,p,\alpha}(\mathcal{T}_{r,\text{out},D} + \mathcal{T}_{r,\text{out},D}^*)(\int_{\mathbb{R}^n} |f|^p \sigma^p v^{-1}|p/r v\right)^{1/p}.
\]

If we consider the testing constants \(\mathcal{T}_{r,\text{in},D}\) and \(\mathcal{T}_{r,\text{in},D}^*\), the proof is similar, using ideas in [9].

**Proof of Theorem 2.14** By hypothesis and using Theorem 3.1 we have
\[
|T_{\alpha,m}f(x)| \leq c \sum_{j=1}^{3^n} \sum_{i=1}^m A_{\alpha,s,\mathcal{R}_j} f(A_i^{-1} x).
\]

Hence
\[
\|T_{\alpha,m}(f\sigma)\|_{L^q(u)} \leq c \sum_{j=1}^{3^n} \sum_{i=1}^m \|A_{\alpha,s,\mathcal{R}_j}(f\sigma)\|_{L^q(u_{A_i})}.
\]

Since \(\mathcal{T}_{A_i,s,\text{out},D_j}, \mathcal{T}_{A_i,s,\text{out},D_j}^* < \infty\), for \(1 \leq i \leq m\) and \(1 \leq j \leq 3^n\), by Theorem 3.2 we get
\[
\|A_{\alpha,s,\mathcal{R}_j}(f\sigma)\|_{L^q(u_{A_i})} \lesssim (\mathcal{T}_{A_i,s,\text{out},D_j} + \mathcal{T}_{A_i,s,\text{out},D_j}^*)\|f\|_{L^p(\sigma)},
\]

hence
\[
\|T_{\alpha,m}(f\sigma)\|_{L^q(u)} \leq c\|f\|_{L^p(\sigma)} \sum_{j=1}^{3^n} \sum_{i=1}^m (\mathcal{T}_{A_i,s,\text{out},D_j} + \mathcal{T}_{A_i,s,\text{out},D_j}^*) \leq c\|f\|_{L^p(\sigma)} \sup_{D} \sum_{i=1}^m (\mathcal{T}_{A_i,s,\text{out},D} + \mathcal{T}_{A_i,s,\text{out},D}^*). \]

For the proof of Theorem 2.15 we need the following results:

**Lemma 3.5.** If \((u, v)\) is a pair of weights such that \((u_A, v) \in M_{\alpha, p, q}\) and \((v, u_A) \in M_{\alpha, q', p'}\), then \(\mathcal{T}_{\alpha, \text{out}, D}, \mathcal{T}_{\alpha, \text{out}, D}^* < \infty\) for any dyadic lattice \(D\).

**Proof.** We will only see that \(\mathcal{T}_{\alpha, \text{out}, D} < \infty\); the other case is handled in a similar way.

Let \(R\) be a cube and \(x \in R\), and let \(Q_k \in D\) be such that \(R \subset Q_k\). Then

\[
|Q|^{\alpha/n-1} \left( \int_Q u \, dx \right) \chi_Q(x) = \sum_{k=0}^{\infty} |Q_k|^{\alpha/n-1} \left( \int_{Q_k} v \, dx \right) \chi_{Q_k}(x)
\]

\[
= |R|^{\alpha/n-1} \left( \int_R v \, dx \right) \sum_{k=0}^{\infty} 2^{k(\alpha/n-1)}
\]

\[
\leq CM_{\alpha}(v \chi_R)(x) \chi_R(x).
\]

Therefore

\[
\mathcal{T}_{\alpha, \text{out}, D} \leq C \sup_{R \in D} R \in D v(R)^{-1/p} \| M_{\alpha}(v \chi_R)(x) \chi_R(x) \|_{L^q(u_A)}
\]

\[
= C[u_A, v]_{M_{\alpha, p, q}}.
\]

**Proof of Theorem 2.15** The conclusion follows from Theorem 2.14 and the previous lemma.

**Proof of Theorem 2.9** Using sparse domination (Theorem 3.1), we have

\[
\|T_{\alpha, m} f\|_{L^q(w^q)} \leq C \sum_{j=1}^{3^n} \|A_{\alpha, r, \mathcal{S}, f}\|_{L^q(w_{A_j}^q)}.
\]

Since \(A_j = A_i^{-1}\) and \(w^s \in \bigcap_{i=1}^m A_{A_i, p/s, q/s}\), we have \(w \in A_{p/s, q/s}\) and \(w_{A_j} \lesssim w\), \(w_{A_i} \lesssim w\) for \(i \neq j\). So, \(w^q, w^{-s(p/s)'} \in A_{\infty}\). On the other hand, let \(A\) be an invertible matrix; if \(w^s \in A_{A_i, p/s, q/s}\) then the pair \((w_{A_i}^s, w^{-s})\) satisfies the \(A_{p/s, q/s}\) condition.

Since \((w_{A_i}^s, w^{-s})\) satisfies the \(A_{p/s, q/s}\) condition and \(w^q, w^{-s(p/s)'} \in A_{\infty}\), we obtain

\[
\|A_{\alpha, s, \mathcal{S}, f}\|_{L^q(w_{A_i}^q)} \leq c_n \max_{A_{A_i, p/s, q/s}} \|f\|_{L^p(w^p)},
\]

and the exponent is sharp; the proof is analogous to the one in [8].

Using (3.4) and (3.5), we have

\[
\|T_{\alpha, m} f\|_{L^q(w^q)} \leq C \|f\|_{L^p(w^p)} \sum_{i=1}^{m} \max_{A_{A_i, p/s, q/s}} \|f\|_{L^p(w^p)}.
\]
4. Proof of sparse domination. To prove sparse domination we follow the approach in [10]. For this we need to prove some end-point estimates for the grand maximal operator $M_{T_{\alpha},m}$ defined by

$$M_{T_{\alpha},m}f(x) = \sup_{Q_i \supset A_i^{-1}x} \operatorname{ess sup}_{1 \leq i \leq m} \left| T_{\alpha,m}(f \chi_{\mathbb{R}^n \setminus \bigcup_{i=1}^m Q_i})(x) \right|,$$

and the local version

$$M_{T_{\alpha},m}\bigcup_{i=1}^m Q_i^0 f(x) = \sup_{Q^0_0 \supset A_i^{-1}x} \operatorname{ess sup}_{1 \leq i \leq m} \left| T_{\alpha,m}(f \chi_{\bigcup_{i=1}^m Q^0_0 \setminus \bigcup_{i=1}^m Q_i})(x) \right|,$$

where the supremum is taken over all cubes $Q_i$ in $Q^0_i$ for $1 \leq i \leq m$.

Let $k_i \in S_{n-\alpha_i,r_i} \cap H_{n-\alpha_i,r_i}$. We define

$$\tilde{T}_{\alpha,m} f(x) = \int |K(x,y)| f(y) \, dy,$$

where $K(x,y) = k_1(x-A_1y)k_2(x-A_2y)\ldots k_m(x-A_my)$. Observe that if $k_i \in S_{n-\alpha_i,r_i} \cap H_{n-\alpha_i,r_i}$ then $|k_i| \in S_{n-\alpha_i,r_i} \cap H_{n-\alpha_i,r_i}$.

**Lemma 4.1.** Let $0 < \alpha < n$ and $0 < \alpha_i < n$ with $\alpha_1 + \cdots + \alpha_m = n - \alpha$. Let $\frac{n}{\alpha_i} < r_i \leq \infty$ and $s \geq 1$ with $1/r_1 + \cdots + 1/r_m + 1/s = 1$. For each $1 \leq i \leq m$, let $A_i$ be an invertible matrix satisfying hypothesis (H) and $k_i \in S_{n-\alpha_i,r_i} \cap H_{n-\alpha_i,r_i}$. The following estimates hold:

(i) for a.e. $A_i^{-1}x \in Q^0_i$,

$$|T_{\alpha,m}(f \chi_{\bigcup_{i=1}^m Q^0_0})(x)| \leq M_{T_{\alpha},m}\bigcup_{i=1}^m Q^0_0 f(x),$$

(ii) for all $x \in \mathbb{R}^n$,

$$M_{T_{\alpha},m}(f)(x) \lesssim \sum_{i=1}^m \alpha_s(f)(A_i^{-1}x) + \tilde{T}_{\alpha,m}(|f|)(x).$$

Therefore,

$$|\{x \in \mathbb{R}^n : M_{T_{\alpha},m}(f)(x) > \lambda\}|^{\frac{n-\alpha}{n}} \leq c^s \int_{\{x \in Q : f(x) \geq \lambda |Q|^\alpha/n \}} \left( \frac{|f(x)|}{\lambda |Q|^\alpha/n} \right)^s dx.$$

For the case $\alpha = 0$, we have the following lemma:

**Lemma 4.2.** Let $0 < \alpha_i < n$ with $\alpha_1 + \cdots + \alpha_m = n$. Let $\frac{n}{\alpha_i} < r_i \leq \infty$ and $s \geq 1$ with $1/r_1 + \cdots + 1/r_m + 1/s = 1$. For each $1 \leq i \leq m$, let $A_i$ be an invertible matrix satisfying hypothesis (H) and $k_i \in S_{n-\alpha_i,r_i} \cap H_{n-\alpha_i,r_i}$. Let $T = T_{0,m}$ be of strong type $(p_0,p_0)$, $1 < p_0 < \infty$. The following estimates hold:

(i) for a.e. $A_i^{-1}x \in Q^0_0$,

$$|T(f \chi_{\bigcup_{i=1}^m Q^0_0})(x)| \leq \|T\|_{L^1 \rightarrow L^{1,\infty}} \sum_{i=1}^m |f(A_i^{-1}x)| + M_{T,\bigcup_{i=1}^m Q^0_0} f(x),$$

(ii) for all $x \in \mathbb{R}^n$,

$$M_{T,\bigcup_{i=1}^m Q^0_0}(f)(x) \lesssim \sum_{i=1}^m \alpha_s(f)(A_i^{-1}x) + \tilde{T}_{\alpha,m}(|f|)(x).$$
(ii) for all \( x \in \mathbb{R}^n \),

\[
M_T(f)(x) \lesssim \sum_{i=1}^{m} [M_s f(A_i^{-1} x) + \|T\|_{L^1 \to L^{1,\infty}} M f(A_i^{-1} x)] + M_\delta(T f)(x).
\]

Therefore

\[
|\{ x \in \mathbb{R}^n : M_T(f)(x) > \lambda \} | \leq c s \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^s dx.
\]

The following lemma is the so-called \( 3^n \) dyadic lattices trick; it was established in [11, Theorem 3.1].

**Lemma 4.3 ([11]).** Given a dyadic lattice \( \mathcal{D} \) there exist \( 3^n \) dyadic lattices \( \mathcal{D}_j \) such that

\[
\{3Q : Q \in \mathcal{D}\} = \bigcup_{j=1}^{3^n} \mathcal{D}_j,
\]

and for every cube \( Q \in \mathcal{D} \) and \( j = 1, \ldots, 3^n \) we can find a cube \( R_Q \in \mathcal{D}_j \) such that \( Q \subset R_Q \) and \( 3l_Q = l_{R_Q} \).

We are ready to prove sparse domination for the integral operator \( T_{\alpha,m} \).

**Proof of Theorem 3.1.** We follow the ideas of [6, 10].

We claim that for any \( Q_0^1, \ldots, Q_0^m \), there exist \( \frac{1}{2} \)-sparse families \( \mathcal{F}_i \subset \mathcal{D}(Q_0^i), i = 1, \ldots, m \), such that for a.e. \( x \in \bigcup_{i=1}^{m} A_i Q_0^i \),

\[
|T_{\alpha,m}(f\chi_{\bigcup_{i=1}^{m} Q_0^i})(x)| \leq c \sum_{i=1}^{m} \sum_{Q \in \mathcal{F}_i} |3Q|^{\alpha/n} \|f\|_{s,3Q} \chi_Q(A_i^{-1} x).
\]

Suppose that we have already proved (4.1). We take a partition of \( \mathbb{R}^n \) into cubes \( Q_j \) such that supp \( f \subset 3Q_j \) for each \( j \), proceeding as follows. We start with \( Q_0 \) such that supp \( f \subset Q_0 \) and cover \( 3Q_0 \setminus Q_0 \) by \( 3^n - 1 \) congruent cubes \( Q_j \). Each of them satisfies \( Q_0 \subset 3Q_j \). We do the same for \( 9Q_0 \setminus 3Q_0 \) and so on. The union of all of those cubes, including \( Q_0 \), will have the desired properties.

We apply the claim to each cube \( Q_j \), in the following way. Let \( Q_0^1 = \cdots = Q_0^m = Q_j \). Then there exists a \( \frac{1}{2} \)-sparse family \( \mathcal{F}_j \subset \mathcal{D}(Q_j) \) such that for a.e. \( x \in \bigcup_{i=1}^{m} A_i Q_0^i \),

\[
|T_{\alpha,m}(f\chi_{3Q_j})(x)| \chi_{\bigcup_{i=1}^{m} A_i Q_j}(x) \leq c \sum_{i=1}^{m} \sum_{Q \in \mathcal{F}_j} |3Q|^{\alpha/n} \|f\|_{s,3Q} \chi_Q(A_i^{-1} x).
\]

Observe that \( \sum_j \chi_{\bigcup_{i=1}^{m} A_i Q_j}(x) \leq m \) since the \( Q_j \) are disjoint.
The family $F = \bigcup F_j$ is $\frac{1}{2}$-sparse. Thus,
\[
|T_{\alpha,m}(f)(x)| \leq c \sum_{i=1}^{m} \sum_{Q \in F} |3Q|^{\alpha/n} \|f\|_{s,3Q} \chi_Q(A_i^{-1}x).
\]

Using Lemma 4.3, we can take $R_Q \in D_j$ such that $\{3Q : Q \in F\} \subset \bigcup_{j=1}^{3^n} D_j$ and let
\[
S_j = \{R_Q \in D_j : Q \in F\}.
\]
Since $F$ is $\frac{1}{2}$-sparse, each $S_j$ is $\frac{1}{2^{2m}}$-sparse. Hence
\[
|T_{\alpha,m}(f)(x)| \leq c \sum_{j=1}^{3^n} \sum_{i=1}^{m} A_{\alpha,s,j} f(A_i^{-1}x).
\]

Now to prove (4.1) it suffices to show the following recursive estimate: for each $1 \leq i \leq m$ there exists a countable family $\{P_j^i\}_{j}$ of pairwise disjoint cubes in $D(Q_0^i)$ such that $\sum_j |P_j^i| \leq \frac{1}{2}|Q_0^i|$ and
\[
|T_{\alpha,m}(f\chi_{\bigcup_{i=1}^{m} A_i Q_0^i})(x)|\chi_{\bigcup_{i=1}^{m} A_i Q_0^i}(x) \leq c \sum_{i=1}^{m} \sum_{Q \in F_i} |3Q|^{\alpha/n} \|f\|_{s,3Q} \chi_Q(A_i^{-1}x)
\]
\[+ |T_{\alpha,m}(f\chi_{\bigcup_{i=1}^{m} P_j^i})(x)|\chi_{\bigcup_{i=1}^{m} A_i P_j^i}(x),
\]
for a.e. $x \in \bigcup_{i=1}^{m} A_i Q_0^i$. Iterating this estimate we obtain (4.1) with $F_i = \{P_j^{i,k}\}$ where $\{P_j^{i,0}\} = \{Q_0^i\}$, $\{P_j^{i,0}\} = \{P_j^i\}$ and $\{P_j^{i,k}\}$ are the cubes obtained at the $k$th stage of the iterative process. Each family $F_i$ is $\frac{1}{2}$-sparse. Indeed, for each $P_j^{i,k}$ it suffices to choose
\[
E_{P_j^{i,k}} = P_j^{i,k} \setminus \bigcup_j P_j^{i,k+1}.
\]

Observe that for any family $\{P_j^i\}_{j} \subset D(Q_0^i)$ of pairwise disjoint cubes, we have
\[
|T_{\alpha,m}(f\chi_{\bigcup_{i=1}^{m} Q_0^i})(x)|\chi_{\bigcup_{i=1}^{m} A_i Q_0^i}(x)
\]
\[\leq |T_{\alpha,m}(f\chi_{\bigcup_{i=1}^{m} Q_0^i})(x)|\chi_{\bigcup_{i=1}^{m} A_i(Q_0^i \setminus \bigcup_j P_j^i)}(x)
\]
\[+ \sum_j |T_{\alpha,m}(f\chi_{\bigcup_{i=1}^{m} Q_0^i})(x)|\chi_{\bigcup_{i=1}^{m} A_i P_j^i}(x)
\]
\[\leq |T_{\alpha,m}(f\chi_{\bigcup_{i=1}^{m} Q_0^i})(x)|\chi_{\bigcup_{i=1}^{m} A_i(Q_0^i \setminus \bigcup_j P_j^i)}(x)
\]
\[+ \sum_j |T_{\alpha,m}(f\chi_{\bigcup_{i=1}^{m} Q_0^i})(x)|\chi_{\bigcup_{i=1}^{m} A_i P_j^i}(x)
\]
\[+ \sum_j |T_{\alpha,m}(f\chi_{\bigcup_{i=1}^{m} P_j^i})(x)|\chi_{\bigcup_{i=1}^{m} A_i P_j^i}(x).
\]
for almost every $x \in \mathbb{R}^n$. So it suffices to show that we can choose a countable family \(\{P_j^i\}_j\) of pairwise disjoint cubes in \(D(Q_0^i)\) such that \(\sum_j |P_j^i| \leq \frac{1}{2} |Q_0^i|\) and for a.e. $x \in \bigcup_{i=1}^m A_i Q_0^i$,

\[
|T_{\alpha,m}(f \chi_{3 \bigcup_{i=1}^m Q_0^i})(x)| x_{\bigcup_{i=1}^m A_i(Q_0^i \setminus P_j^i)}(x) 
+ \sum_j |T_{\alpha,m}(f \chi_{3(Q_0^i \setminus P_j^i)})(x)| x_{\bigcup_{i=1}^m A_i P_j^i}(x)
\leq c \sum_{i=1}^m |3Q_0^i|^{\alpha/n} \|f\|_{s,3Q_0^i} \chi_{Q_0^i}(A_i^{-1} x).
\]

To prove this we follow the ideas in [1, 6, 10] with $E = \bigcup_{i=1}^m E_\alpha^i$ defined by

\[
E_0^i = \{ x \in Q_0^i : |f| > \gamma_n \|f\|_{s,3Q_0^i} \}
\cup \{ x \in Q_0^i : M_{T_{\alpha,m},\cup_i Q_0^i}(f) > \gamma_n c \sum_{i=1}^m \|f\|_{s,3Q_0^i} \},
\]

if $\alpha = 0$, and

\[
E_\alpha^i = \{ x \in Q_0^i : M_{T_{\alpha,m},\cup_i Q_0^i}(f) > \gamma_n c \sum_{i=1}^m |3Q_0^i|^{\alpha/n} \|f\|_{s,3Q_0^i} \},
\]

if $\alpha > 0$.

Now, we prove that there exists $\gamma_n$ such that

\[
|E_\alpha^i| \leq \frac{1}{2^{n+2}} \sum_{i=1}^m |Q_0^i|.
\]

If $\alpha = 0$, using Lemma 4.2, we have

\[
|E_0^i| \leq \frac{\int_{Q_0^i} |f(x)| \, dx}{\gamma_n \|f\|_{s,3Q_0^i}} + c \int_{\bigcup_{i=1}^m Q_0^i} \left( \frac{|f(x)|}{\gamma_n c \sum_{i=1}^m \|f\|_{s,3Q_0^i}} \right)^s \, dx
\leq |3Q_0^i| \frac{1}{|3Q_0^i|} \int_{3Q_0^i} |f(x)| \, dx + c \sum_{i=1}^m \frac{1}{\gamma_n c^s} \int_{3Q_0^i} \left( \frac{|f(x)|}{\|f\|_{s,3Q_0^i}} \right)^s \, dx
\leq \frac{|3Q_0^i|}{\gamma_n} + c \sum_{i=1}^m \frac{|3Q_0^i|}{\gamma_n c^s} \left( \frac{1}{\|f\|_{s,3Q_0^i}} \right)^s \int_{3Q_0^i} \left( \frac{|f(x)|}{\|f\|_{s,3Q_0^i}} \right)^s \, dx
\leq \frac{|3Q_0^i|}{\gamma_n} + c \sum_{i=1}^m \frac{|3Q_0^i|}{\gamma_n c^s} \leq \left( \frac{1}{\gamma_n} + \frac{c^{1-s}}{\gamma_n^s} \right) \sum_{i=1}^m |3Q_0^i|.
\]

Thus, we can choose $\gamma_n$ such that

\[
m \left( \frac{1}{\gamma_n} + \frac{c^{1-s}}{\gamma_n^s} \right) \leq \frac{1}{2^{n+2}}.
\]
In the case of $\alpha > 0$, by Lemma 4.1 we have
\[
|E^i_\alpha|^{\frac{n-\alpha s}{n}} \leq c_1 \int_{3Q^i_0} \left( \frac{|f(x)|}{\gamma_n c_2 |3Q^i_0|^{|\alpha/n|} \|f\|_{s,3Q^i_0}} \right)^s \, dx
\]
\[
\leq C \sum_{i=1}^m \frac{1}{\gamma_n |3Q^i_0|^{|\alpha/n|}} \int_{3Q^i_0} \left( \frac{|f(x)|}{\|f\|_{s,3Q^i_0}} \right)^s \, dx
\]
\[
\leq C \sum_{i=1}^m \frac{|3Q^i_0|^s}{\gamma_n |3Q^i_0|^{|\alpha/n|}} \frac{1}{|3Q^i_0|^s} \int_{3Q^i_0} \left( \frac{|f(x)|}{\|f\|_{s,3Q^i_0}} \right)^s \, dx
\]
\[
= C \sum_{i=1}^m \frac{|3Q^i_0|^{1-\alpha s/n}}{\gamma_n s}.
\]
Thus,
\[
|E^i_\alpha| \leq \frac{C3^n}{\gamma_n^{n-\alpha s}} \sum_{i=1}^m |Q^i_0|,
\]
so it is enough to take $\gamma_n$ such that
\[
mC3^n \gamma_n^{-\frac{s n}{n-\alpha s}} \leq \frac{1}{2^{n+2}}.
\]
Now we apply Calderón–Zygmund decomposition to the function $\chi_{E^i_\alpha}$ on $Q^i_0$ at height $\lambda = 1/2^{n+1}$. We obtain pairwise disjoint cubes $P^i_j \in \mathcal{D}(Q^i_0)$ such that
\[
\chi_{E^i_\alpha}(x) \leq \frac{1}{2^{n+1}} \quad \text{a.e. } x \notin \bigcup_{j} P^i_j.
\]
Also $|E^i_\alpha \setminus \bigcup_j P^i_j| = 0$ and
\[
\sum_j |P^i_j| = \left| \bigcup_j P^i_j \right| \leq 2^{n+1} |E^i_\alpha| \leq 2^{n+1} |E| \leq \frac{1}{2} \sum_{i=1}^m |Q^i_0|,
\]
and
\[
\frac{1}{2^{n+1}} \leq \frac{1}{|P^i_j|} \int_{P^i_j} \chi_{E^i_\alpha}(x) \, dx = \frac{|P^i_j \cap E^i_\alpha|}{|P^i_j|} \leq \frac{1}{2}.
\]
From the last estimate it follows that $|P^i_j \cap (E^i_\alpha)^c| > 0$. Indeed,
\[
|P^i_j| = |P^i_j \cap E^i_\alpha| + |P^i_j \cap (E^i_\alpha)^c| \leq \frac{1}{2} |P^i_j| + |P^i_j \cap (E^i_\alpha)^c|,
\]
Thus $\frac{1}{2} |P^i_j| < |P^i_j \cap (E^i_\alpha)^c|$. For $i = 1, \ldots, m$, since $P^i_j \cap (E^i_\alpha)^c \neq \emptyset$, we have $M_{T,\cup_i Q^i_0}(f)(x) \leq \gamma_n c \sum_{i=1}^m \|f\|_{s,3Q^i_0}$ for some $x \in A_i P^i_j$ and so
\[
\text{ess sup}_{\xi \in P^i_j} |T_{\alpha,m}(f|_{\cup_i Q^i_0 \setminus P^i_j})(\xi)| \leq \gamma_n c \|f\|_{s,3Q^i_0}.
\]
Thus,
\[
\text{ess sup}_{\xi \in \bigcup_{i=1}^{m} P_j} |T_{\alpha,m}(f\chi_{\bigcup_i 3(Q^i_0 \setminus P^i_j)})(\xi)| \leq \gamma_n c \sum_{i=1}^{m} \|f\|_{s,3Q^i_0},
\]
which allows us to control the summation in (4.2).

On the other hand, if \( \alpha = 0 \), by Lemma 4.2(i) we know that for a.e. \( x \in A_i Q^i_0 \):
\[
|T_{0,m}(f\chi_{\bigcup_i Q^i_0})(x)| \leq \|T_{0,m}\|_{L^1 \rightarrow L^1, \infty} \sum_{i=1}^{m} |f(A_i^{-1}x)| + M_{T_{0,m} \cup_i Q^i_0} f(x).
\]
If \( x \in Q^i_0 \setminus \bigcup_j P^j_i \) then since \( |E^i_0 \setminus \bigcup_j P^j_i| = 0 \) we have, by the definition of \( E^i_0 \),
\[
|f(x)| \leq \gamma_n \|f\|_{s,3Q^i_0} \leq \gamma_n \sum_{i=1}^{m} \|f\|_{s,3Q^i_0}
\]
for a.e. \( x \), and also \( M_{T_{0,m} \cup_i Q^i_0} f(x) \leq \gamma_n \sum_{i=1}^{m} \|f\|_{s,3Q^i_0} \) for a.e. \( x \). Consequently,
\[
|T_{\alpha,m}(f\chi_{\bigcup_i Q^i_0})(x)| \leq \gamma_n c \sum_{i=1}^{m} \|f\|_{s,3Q^i_0}.
\]
Those estimates allow us to control the remaining terms in (4.2) for \( \alpha = 0 \).

If \( \alpha > 0 \), by Lemma 4.1(i) we know that for a.e. \( A_i^{-1}x \in Q^i_0 \),
\[
|T_{\alpha,m}(f\chi_{\bigcup_i Q^i_0})(x)| \leq M_{T_{\alpha,m} \cup_i Q^i_0} f(x);
\]
then proceeding as above, we prove (4.2).

4.1. Proofs of lemmas. In this subsection we prove Lemmas 4.1 and 4.2. In the proofs we will consider the case \( m = 2 \); the proofs in the general case are analogous. For \( \alpha \geq 0 \), we will write \( T_\alpha := T_{\alpha,2} \) and \( T := T_{0,2} \).

Proof of Lemma 4.1. (i) For \( i = 1,2 \), let \( \tilde{Q}^i \) be a cube centered at \( A_i^{-1}x \) with edge length \( t \) such that \( \tilde{Q}^i \subset Q^i_0 \). Then
\[
|T_\alpha(f\chi_{3(Q^i_0 \cup \tilde{Q}^2)})(x)| \leq |T_\alpha(f\chi_{3(\tilde{Q}^i \cup \tilde{Q}^2)})(x)| + |T_\alpha(f\chi_{3(Q^i_0 \cup \tilde{Q}^2 \setminus 3(\tilde{Q}^i \cup \tilde{Q}^2))})(x)|.
\]

Let \( B^i \) be a ball with center at \( A_i^{-1}x \) and radius \( R = \frac{3}{2} \sqrt{n} t \). Then \( 3\tilde{Q}^i \subset B^i \) and
\[
|T_\alpha(f\chi_{3(\tilde{Q}^i \cup \tilde{Q}^2)})(x)| \leq \int_{3(\tilde{Q}^i \cup \tilde{Q}^2)} |K(x,y)||f(y)| \, dy \leq \int_{B^1 \cup B^2} |K(x,y)||f(y)| \, dy \leq \int_{B^1} |K(x,y)||f(y)| \, dy + \int_{B^2} |K(x,y)||f(y)| \, dy.
\]
For $B^1$, we consider the sets

$$X^1 = B^1 \cap \{ z : |x - A_1 z| \leq |x - A_2 z| \}, \quad X^2 = B^1 \setminus X^1.$$  

For $X^1$, we decompose the set in the following way:

$$C_j^1 = \{ z : |x - A_1 z| \sim 2^{-j} R \| A_1 \| \},$$

where $\| A_1 \| = \sup_{x \neq 0} \frac{|A_1 x|}{|x|}$. Observe that $X^1 \subset \bigcup_{j=0}^{\infty} C_j^1$. Let $\tilde{B}_j = A_1^{-1} B(x, 2^{-j} R \| A_1 \|)$. Then

$$\int_{X^1} |k_1(x - A_1 z)| |k_2(x - A_2 z)| |f(z)| \, dz$$

\[ \leq \sum_{j=0}^{\infty} \frac{\tilde{B}_{j-1}}{|\tilde{B}_{j-1}|} \int_{C_j^1} |k_1(x - A_1 z)| |k_2(x - A_2 z)| |f(z)| \, dz \]

\[ \leq \sum_{j=0}^{\infty} |\tilde{B}_{j-1}| \| k_1(x - A_1 \cdot) \chi_{C_j^1} \|_{r_1, \tilde{B}_{j-1}} \| k_2(x - A_2 \cdot) \chi_{C_j^1} \|_{r_2, \tilde{B}_{j-1}} \| f \|_{s, \tilde{B}_{j-1}}. \]

Since $k_1 \in S_{n-\alpha_1, r_1}$ we have

$$\| k_1(x - A_1 \cdot) \chi_{C_j^1} \|_{r_1, \tilde{B}_{j-1}} \lesssim (2^{-j} R \| A_1 \|)^{-\alpha_2}.$$  

Since $z \in C_j^1$ and $2^{-j} R \| A_1 \| \leq |x - A_1 z| \leq |x - A_2 z|$, it follows that

$$C_j^1 \subset \bigcup_{k \geq 0} \{ z \in C_j^1 : |x - A_2 z| \sim 2^{k-j} R \| A_1 \| \} = \bigcup_{k \geq 0} (C_j^1)_k.$$

Let $\tilde{B}_k^2 = A_2^{-1} B(x, 2^k R \| A_1 \|)$. Observe that $(C_j^1)_k \subset \tilde{B}_k^{2-j+1}$. Then, since $k_2 \in S_{n-\alpha_2, r_2}$,

$$\| k_2(x - A_2 \cdot) \chi_{C_j^1} \|_{r_2, \tilde{B}_{j-1}} = \left( \frac{1}{|\tilde{B}_{j-1}|} \int_{C_j^1} |k_2(x - A_2 z)|_2^p \, dz \right)^{1/r_2}$$

\[ \leq \sum_{k \geq 0} \left( \frac{|\tilde{B}_k^{2-j+1}|}{|\tilde{B}_{k-1}|} \frac{1}{|\tilde{B}_k^{2-j+1}|} \int_{(C_j^1)_k} |k_2(x - A_2 z)|_2^p \, dz \right)^{1/r_2} \]

\[ = \sum_{k \geq 0} \frac{\det A_2^{-1}}{|\det A_2^{-1}|}^{1/r_2} 2^{kn/r_2} \left( \frac{1}{|\tilde{B}_k^{2-j+1}|} \int_{(C_j^1)_k} |k_2(x - A_2 z)|_2^p \, dz \right)^{1/r_2} \]

\[ \leq c \sum_{k \geq 0} \left( 2^{kn} \frac{\det A_2^{-1}}{|\det A_2^{-1}|} \right)^{1/r_2} (2^{k-j} R \| A_1 \|)^{-\alpha_2} \]

\[ = c \left( \frac{\det A_2^{-1}}{|\det A_2^{-1}|} \right)^{1/r_2} (2^{-j} R \| A_1 \|)^{-\alpha_2} \sum_{k \geq 0} 2^{k(n/r_2 - \alpha_2)}. \]
Since \( n/r_2 - \alpha_2 < 0 \),
\[
\|k_2(x - A_{2} \cdot )\chi_{C_j^1}\|_{r_2, \hat{B}_{j-1}} \lesssim (2^{-j} R\|A_1\|)^{-\alpha_2}.
\]
(4.3)

If we decompose \( X^2 \subset \bigcup_{j=1}^{\infty} C_j^2 \) where \( C_j^2 = \{ z \in X^2 : |x - A_2 z| \sim 2^{-j} R\|A_2\| \} \) and \( \|A_2\| = \sup_{x \neq 0} \sup_{|A_2 x|} |x| \), we can proceed as above to get
\[
\int_{X^2} |k_1(x - A_1 z)| |k_2(x - A_2 z)| |f(z)| \, dz \leq c M_s(f)(A_2^{-1} x) R^\alpha.
\]

Hence, for all \( R > 0 \),
\[
|T_\alpha(f \chi_{3(\hat{Q}_1 \cup \hat{Q}_2)})(x)|
\leq c(M_s(f)(A_1^{-1} x) + M_s(f)(A_2^{-1} x)) R^\alpha + M_{T_\alpha, Q_0^1 \cup Q_0^2} f(x).
\]

Letting \( t \to 0 \), we see that \( R \to 0 \), and so
\[
|T_\alpha(f \chi_{3(\hat{Q}_1 \cup \hat{Q}_2)})(x)| \leq M_{T_\alpha, Q_0^1 \cup Q_0^2} f(x).
\]

(ii) We are going to follow the ideas in [6, 7, 8]. Let \( x \in \mathbb{R}^n \); for \( i = 1, 2 \) let \( Q_i \) be a cube containing \( A_i^{-1} x \). Let \( B = B(x, 2R) \) be a ball with radius \( R = \max \{\|A_1\| \text{diam}(Q_1), \|A_2\| \text{diam}(Q_2)\} \) and center \( x \). Let \( B_i = A_i^{-1} B \); observe that \( 3Q_i \subset B_i \). For every \( \xi \in Q_1 \cup Q_2 \), we have

\[
|T_\alpha(f \chi_{\mathbb{R}^n \setminus \bigcup_{3Q_i}})(\xi)|
\leq |T_\alpha(f \chi_{\mathbb{R}^n \setminus (B_i \cup B_j)})(\xi) - T_\alpha(f \chi_{\mathbb{R}^n \setminus (B_i \cup B_j)})(x)|
+ |T_\alpha(f \chi_{(B_i \cup B_j) \setminus 3Q_i})(\xi)| + |T_\alpha(f \chi_{\mathbb{R}^n \setminus (B_i \cup B_j)})(x)|
\lesssim |T_\alpha(f \chi_{\mathbb{R}^n \setminus (B_i \cup B_j)})(\xi) - T_\alpha(f \chi_{\mathbb{R}^n \setminus (B_i \cup B_j)})(x)|
+ |T_\alpha(f \chi_{(B_i \cup B_j) \setminus 3Q_i})(\xi)| + \tilde{T}_\alpha(|f|)(x).
\]

Let \( Z^1 = (B^1 \cup B^2)^c \cap \{ z : |x - A_1 z| \leq |x - A_2 z| \} \) and \( Z^2 = (B^1 \cup B^2)^c \cap \{ z : |x - A_2 z| \leq |x - A_1 z| \} \). Then
\[
|T_\alpha(f \chi_{\mathbb{R}^n \setminus (B_i \cup B_j)})(\xi) - T_\alpha(f \chi_{\mathbb{R}^n \setminus (B_i \cup B_j)})(x)|
\leq \int_{(B^1 \cup B^2)^c} |K(\xi, y) - K(x, y)| |f(y)| \, dy
\leq \int_{Z^1} |K(\xi, y) - K(x, y)| |f(y)| \, dy + \int_{Z^2} |K(\xi, y) - K(x, y)| |f(y)| \, dy.
\]

For \( j \in \mathbb{N} \), define
\[
D_j^1 = \{ z \in Z^1 : |x - A_1 z| \sim 2^{j+1} R \}.
\]

Observe that
\[
D_j^1 \subset \{ z : |x - A_1 z| \sim 2^{j+1} R \} \subset A_1^{-1} B(x, 2^{j+2} R) =: \hat{B}_{1,j},
\]
and \( Z^1 \subset \bigcup_{j \in \mathbb{N}} D_j^1 \).
Then

\[
\int_{Z^1} |k_1(\xi - A_1 y) - k_1(x - A_1 y)| |k_2(\xi - A_2 y)| |f(y)|\,dy
\]

\[
\leq \sum_{j=1}^{\infty} |\tilde{B}_{1,j}| \int_{\tilde{B}_{1,j}} \chi_{D^1_j}(y) |k_1(\xi - A_1 y) - k_1(x - A_1 y)| |k_2(\xi - A_2 y)| |f(y)|\,dy
\]

\[
\leq \sum_{j=1}^{\infty} |\tilde{B}_{1,j}| \|(k_1(\xi - A_1 \cdot) - k_1(x - A_1 \cdot))\chi_{D^1_j}\|_{r_1,\tilde{B}_{1,j}}
\]

\[
\times |k_2(\xi - A_2 \cdot)\chi_{D^1_j}\|_{r_2,\tilde{B}_{1,j}} \|f\|_{s,\tilde{B}_{1,j}}.
\]

If \(z \in D^1_j\) then \(|x - A_2 z| \geq |x - A_1 z| \geq 2^{j+1} R\). So we write \(D^1_j = \bigcup_{k \geq 0} (D^1_{j,k})_{k,2}\) where

\[
(D^1_{j,k})_{k,2} = \{z \in D^1_j : |x - A_2 z| \sim 2^{k+j+1} R\}
\]

and set \(\tilde{B}_{2,k} = A_2^{-1} B(x, 2^k R)\). Since \(k_2 \in S_{n-\alpha_2, r_2}\), we have

\[
|k_2(\xi - A_2 \cdot)\chi_{D^1_j}\|_{r_2,\tilde{B}_{1,j}} \lesssim \sum_{k \geq 0} 2^{kn/r_2}|k_2(\xi - A_2 \cdot)\chi_{(D^1_{j,k})_{k,2}}|_{r_2,\tilde{B}_{2,k+j+2}}
\]

\[
\lesssim (2^j R)^{-\alpha_2}.
\]

Since \(k_1 \in H_{n-\alpha_1, r_1}\) and \(\alpha_1 + \alpha_2 = n - \alpha_1\),

\[
\int_{Z^1} |k_1(\xi - A_1 y) - k_1(x - A_1 y)| |k_2(\xi - A_2 y)| |f(y)|\,dy
\]

\[
\lesssim M_{\alpha,s} f(A_1^{-1} x) \sum_{j=1}^{\infty} |\tilde{B}_{1,j}|^{1-\alpha/n-\alpha_2/n} \|(k_1(\xi - A_1 \cdot) - k_1(x - A_1 \cdot))\chi_{D^1_j}\|_{r_1,\tilde{B}_{1,j}}
\]

\[
\lesssim M_{\alpha,s} f(A_1^{-1} x).
\]

We proceed as above, changing the roles of \(k_1\) and \(k_2\), and obtain

\[
\int_{Z^1} |k_1(x - A_1 y)| |k_2(\xi - A_2 y) - k_2(x - A_2 y)| |f(y)|\,dy \lesssim M_{\alpha,s} f(A_1^{-1} x).
\]

Hence,

\[
\int_{Z^1} |K(\xi, y) - K(x, y)| |f(y)|\,dy \leq M_{\alpha,s} f(A_1^{-1} x).
\]
In an analogous way we obtain
\[
\int_{\mathbb{Z}^2} |K(\xi, y) - K(x, y)| |f(y)| \, dy \leq M_{\alpha,s}f(A_2^{-1}x).
\]
Hence
\[
(4.5) \quad |T_{\alpha}(f\chi_{\mathbb{R}^n \setminus (B^1 \cup B^2)})(\xi) - T_{\alpha}(f\chi_{\mathbb{R}^n \setminus (B^1 \cup B^2)})(x)|
\leq M_{\alpha,s}f(A_1^{-1}x) + M_{\alpha,s}f(A_2^{-1}x).
\]
For the second term of (4.4), we have
\[
|T_{\alpha}(f\chi_{(B^1 \cup B^2) \setminus (3Q_1 \cup Q_2)})|(\xi)|
\leq \int_{(B^1 \cup B^2) \setminus (3Q_1 \cup Q_2)} |k_1(\xi - A_1y)||k_2(\xi - A_2y)||f(y)| \, dy
\leq \int_{Y^1} |k_1(\xi - A_1y)||k_2(\xi - A_2y)||f(y)| \, dy
+ \int_{Y^2} |k_1(\xi - A_1y)||k_2(\xi - A_2y)||f(y)| \, dy,
\]
where
\[
Y^1 = \left( (B^1 \cup B^2) \setminus 3(Q_1 \cup Q_2) \right) \cap \{ z : |x - A_2z| \leq |x - A_1z| \},
\]
\[
Y^2 = \left( (B^1 \cup B^2) \setminus 3(Q_1 \cup Q_2) \right) \cap \{ z : |x - A_1z| \leq |x - A_2z| \}.
\]
Let $B_j^i := A_i^{-1}B(x, 2^jR)$. Observe that for $i = 1, 2, Y^i \subset B_1^i \setminus (B_{-1}^1 \cup B_{-1}^2)$ where $l$ is the smallest positive integer such that $\sqrt{n} \max \{ ||A_1||, ||A_2|| \} \leq 2^l$.
Then, by Hölder’s inequality,
\[
\int_{Y^1} |k_1(\xi - A_1y)||k_2(\xi - A_2y)||f(y)| \, dy
\leq \sum_{j=0}^{l} \left| \frac{B_{-j+1}^1}{B_{-j+1}^1 \setminus B_{-j}^1} \right| \int_{B_{-j+1}^1 \setminus B_{-j}^1} \chi_{(B_j^2)^c}(y)|k_1(\xi - A_1y)||k_2(\xi - A_2y)||f(y)| \, dy
\leq \sum_{j=0}^{l} \left| \frac{B_{-j+1}^1}{B_{-j+1}^1 \setminus B_{-j}^1} \right| ||k_1(\xi - A_1 \cdot)\chi_{B_{-j+1}^1 \setminus B_{-j}^1}||_{r_1, B_{-j+1}^1} 
\times ||\chi_{(B_j^2)^c}k_2(\xi - A_2 \cdot)\chi_{B_{-j+1}^1 \setminus B_{-j}^1}||_{r_2, B_{-j+1}^1} ||f||_{s, B_{-j+1}^1}.
\]
Since $y \in (B_j^2)^c$, we have $|x - A_2y| \geq 2^{-l}R$. Using $k_2 \in S_{n-\alpha_2, r_2}$, we get
\[
||\chi_{(B_j^2)^c}k_2(\xi - A_2 \cdot)\chi_{B_{-j+1}^1 \setminus B_{-j}^1}||_{r_2, B_{-j+1}^1} 
\leq \sum_{k=0}^{l} \left| \frac{B_{-k-j+1}^2}{B_{-k-j+1}^2 \setminus B_{-k-j}^1} \right|^{1/r_2} ||k_2(\xi - A_2 \cdot)\chi_{B_{-k-j+1}^2 \setminus B_{-k-j}^2}||_{r_2, B_{-k-j+1}^2}.
\]
\[ \leq \sum_{k=0}^{l} \frac{|\det A_{2}^{-1}|^{1/r_{2}}}{|\det A_{1}^{-1}|^{1/r_{2}}} 2^{-kn/r_{2}} (2^{-j-k}R)^{-\alpha_{2}} \]

\[ = \frac{|\det A_{2}^{-1}|^{1/r_{2}}}{|\det A_{1}^{-1}|^{1/r_{2}}} (2^{-j}R)^{-\alpha_{2}} \sum_{k=0}^{l} 2^{-k(n/r_{2}-\alpha_{2})} \leq c|B_{-j+1}|^{-\alpha_{2}/n}, \]

where the constant \( c \) only depends of \( \alpha_{2}, r_{2}, A_{1}, A_{2} \) and \( n \).

Then, since \( k_{1} \in S_{n-\alpha_{1},r_{1}} \) and \( \alpha_{1} + \alpha_{2} = n - \alpha \), we get

\[ \int_{Y^{1}} |k_{1}(\xi - A_{1}y)| |k_{2}(\xi - A_{2}y)| |f(y)| dy \]

\[ \leq cM_{\alpha,s}f(A_{1}^{-1}x) \sum_{j=0}^{l} |B_{-j+1}^{1}|^{-\alpha/n-\alpha_{2}/n} \|k_{1}(\xi - A_{1}y)\chi_{B_{-j+1}^{1}B_{-j}^{1}}\|_{r_{1},B_{-j}^{1}} \]

\[ \leq c(l + 1)M_{\alpha,s}f(A_{1}^{-1}x). \]

In an analogous way,

\[ \int_{Y^{2}} |k_{1}(\xi - A_{1}y)| |k_{2}(\xi - A_{2}y)| |f(y)| dy \lesssim M_{\alpha,s}f(A_{2}^{-1}x). \]

By \((4.4), (4.5)\) and the last inequalities, we obtain

\[ |T_{\alpha}(f\chi_{R^{n}\setminus(3Q_{1}\cup Q_{2})})(\xi)| \lesssim M_{\alpha,s}f(A_{1}^{-1}x) + M_{\alpha,s}f(A_{2}^{-1}x) + \tilde{T}_{\alpha}(|f|)(x). \]

By the Coifman–Fefferman inequality \([7]\) for \( \tilde{T}_{\alpha}(|f|) \) and the fact that \( M_{\alpha,s} \) is bounded from \( L^{s} \) to \( L^{n-\alpha,s,\infty} \), we obtain the desired inequality. \( \blacksquare \)

**Proof of Lemma 4.2** We follow the ideas in \([10]\). We only indicate the changes in the proof.

(i) For \( i = 1, 2 \), let \( A_{i}^{-1}x \in \text{int } Q_{0}^{i} \) and suppose that \( A_{i}^{-1}x \) is a point of approximate continuity of \( T(f\chi_{3Q_{0}^{i}}) \) (see \([2]\)). Then for every \( \epsilon > 0 \) and \( t > 0 \), let

\[ E_{t}^{i} = \{ y \in B(A_{i}^{-1}x,t) : |T(f\chi_{3Q_{0}^{i}})(y) - T(f\chi_{3Q_{0}^{i}})(x)| < \epsilon/2 \}. \]

Let \( Q(A_{i}^{-1}x,t) \) be the smallest cube centered at \( A_{i}^{-1}x \) containing \( B(A_{i}^{-1}x,t) \). Take \( t \) such that \( Q(A_{i}^{-1}x,t) \subset Q_{0}^{i} \). Then for a.e. \( y \in E_{t}^{1} \cup E_{t}^{2} \) we have

\[ |T(f\chi_{3(Q_{0}^{1}\cup Q_{0}^{2})})(x)| \leq \|T\|_{L^{1}\rightarrow L^{1,\infty}} \sum_{i=1}^{m} |f(A_{i}^{-1}x)| + M_{T,Q_{0}^{1}\cup Q_{0}^{2}}f(x). \]

(ii) Let \( Q_{1} \) and \( Q_{2} \) be cubes such that \( A_{i}^{-1}x \in Q_{i} \) and let \( \xi \in Q_{1} \cup Q_{2} \). Then
\[ |T(f\chi_{\mathbb{R}^n\setminus\{Q_1\cup Q_2\}}(\xi))| \leq |T(f\chi_{\mathbb{R}^n\setminus\{Q_1\cup Q_2\}}(\xi) - T(f\chi_{\mathbb{R}^n\setminus\{Q_1\cup Q_2\}}(x'))| + |T(f(x')) + |T(f\chi_{\mathbb{R}^n\setminus\{Q_1\cup Q_2\}}(x'))| \]
\[ \leq |T(f\chi_{\mathbb{R}^n\setminus\{Q_1\cup Q_2\}}(\xi) - T(f\chi_{\mathbb{R}^n\setminus\{Q_1\cup Q_2\}}(x'))| + |T(f(x')) + |T(f\chi_{\mathbb{R}^n}(x'))| + |T(f\chi_{\mathbb{R}^n}(x')).| \]

As in the fractional case, since \( k_i \in S_{n-\alpha_i,r_i} \cap H_{n-\alpha_i,r_i} \) for \( i = 1, 2 \), we have
\[ |T(f\chi_{\mathbb{R}^n\setminus\{Q_1\cup Q_2\}}(\xi)) - T(f\chi_{\mathbb{R}^n\setminus\{Q_1\cup Q_2\}}(x'))| \leq c(M_s f(A_1^{-1}x) + M_s f(A_2^{-1}x)). \]

Now, let \( Q \) be a cube such that \( Q_1 \cup Q_2 \subset Q \) and \( x \in Q \). Taking average in \( (L^\delta(Q), \frac{dx'}{|Q|}) \) with \( 0 < \delta < 1 \), we have
\[ |T(f\chi_{\mathbb{R}^n\setminus\{Q_1\cup Q_2\}}(\xi))| \leq cM_s f(A_2^{-1}x) + cM_s f(A_1^{-1}x) + M_\delta(Tf)(x) \]
\[ + \left( \frac{1}{|Q|} \int_Q |T(f\chi_{Q_1})(x')|^{\delta} dx' \right)^{1/\delta} + \left( \frac{1}{|Q|} \int_Q |T(f\chi_{Q_2})(x')|^{\delta} dx' \right)^{1/\delta}. \]

For the last term, for \( i = 1, 2 \), by Kolmogorov’s inequality we have
\[ \left( \frac{1}{|Q|} \int_Q |T(f\chi_{Q_i})(x')|^{\delta} dx' \right)^{1/\delta} \leq c\|T\|_{L^1 \rightarrow L^{1,\infty}} \frac{1}{|Q|} \int_Q |f\chi_{Q_i}(x')| dx' \]
\[ \leq c\|T\|_{L^1 \rightarrow L^{1,\infty}} \frac{1}{|Q_i|} \int_{3Q_i} |f(x')| dx' \leq c\|T\|_{L^1 \rightarrow L^{1,\infty}} M f(A_i^{-1}x). \]

Then
\[ M_T(f)(x) \lesssim \|T\|_{L^1 \rightarrow L^{1,\infty}} (M f(A_1^{-1}x) + M f(A_2^{-1}x)) \]
\[ + M_s f(A_1^{-1}x) + M_s f(A_2^{-1}x) + M_\delta(Tf)(x). \]

Now, we prove the endpoint estimate. Observe that for \( p = p_0 \),
\[ \|M_\delta(Tf)\|_{L^p,\infty} = \|M(|Tf|^\delta)^{1/\delta}\|_{L^p/\delta,\infty} \leq c\|Tf\|_{L^p,\infty}^{1/\delta} = c\|Tf\|_{L^p,\infty} \leq c\|T\|_{L^1 \rightarrow L^{1,\infty}} \|f\|_{L^p}. \]

Then, using \([6] \text{ Lemma 4.4} \),
\[ |\{ x \in \mathbb{R}^n : M_\delta(Tf)(x) > \lambda \}| \lesssim \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{s} dx. \]

Since \( M_s \) is bounded from \( L^s \) in \( L^{s,\infty} \), we obtain the desired inequality. \( \blacksquare \)

5. Appendix: Properties of \( A_{A,p,q} \). In this section, we present some properties of and remarks about the new class of weights.

**Proposition 5.1.** Let \( 0 \leq \alpha < n, \; 1 < p < n/\alpha \) and \( 1/q = 1/p - \alpha/n \). If \( w \in A_{A,p,q} \), then \( w(Ax) \leq [w]_{A_{A,p,q}} w(x) \) for a.e. \( x \in \mathbb{R}^n \).
We say that a weight \( w \) is in \( A_{A,p} \) if \( w^p \in A_{A,p,p} \). The class \( A_{A,p} \) has some properties of Muckenhoupt weights.

**Proposition 5.2.** Let \( w \) be a weight.

(i) If \( p < q \) then
\[
w \in A_{A,p} \implies w \in A_{A,q}.
\]

(ii) If \( w_0 \in A_{A,1} \) and \( w_1 \in A_{A^{-1},1} \) then \( w = w_0 w_1^{1-p} \in A_{A,p} \).

**Proposition 5.3.** Let \( A \) be an invertible matrix and \( w \in A_{A,p} \). Then

\[
|\det A|^{-1} \sup_Q \left( \frac{|AQ \cap Q|}{|Q|} \right)^p \leq [w]_{A_{A,p}},
\]

and we have the “A-doubling” property: for all \( \lambda \geq 1 \) and all \( Q \) we have
\[
w_A(\lambda Q) \leq \lambda^{np} [w]_{A_{A,p}} w(Q)
\]
where \( \lambda Q \) is the cube with same center as \( Q \) and side length \( \lambda \) times that of \( Q \).

**Remark 5.4.** Observe that (5.1) does not imply that the constant must be greater than 1.

**Proposition 5.5.** Let \( 1 < p < \infty \), \( w \) be a weight, \( \sigma = w^{-p'/p} \) and \( A, B \) be invertible matrices.

(i) If \( w \in A_{A,p} \cap A_{A^{-1},p} \) then \( w \in A_p \) and \( [w]_{A,p} \leq [w]_{A_{A,p}} [w]_{A_{A^{-1},p}} \).

(ii) If \( w \in A_{A,p} \cap A_{B,p} \) then \( w \in A_{AB,p} \) and \( [w]_{A_{AB,p}} \leq [w]_{A_{A,p}} [w]_{A_{B,p}} \).

(iii) Let \( Q \) be a cube. If \( w \in A_{A,p} \) then \( \frac{1}{|Q|} \left( \frac{1}{|Q|} \right)^p \leq [w]_{A_{A,p}}^p \).

The following results show in some cases a relation of this class to the Muckenhoupt class and the condition \( w_A \lesssim w \), i.e. \( w(Ax) \leq cw(x) \) a.e. \( x \in \mathbb{R}^n \).

**Proposition 5.6.** Let \( A \) be an invertible matrix and \( w \) be a weight. If \( w \in A_p \) and \( w_A \lesssim w \) then \( w \in A_{A,p} \).

**Theorem 5.7.** Let \( w \) be a weight and \( A \) be an invertible matrix. Then
\[
w \in A_{A,p} \cap A_{A^{-1},p} \quad \text{if and only if} \quad w \in A_p \quad \text{and} \quad w_A \sim w.
\]

**Remark 5.8.** Observe that
\[
A_p \cap \{ w : w_A \lesssim w \} \subset \{ w : w \text{ satisfies the testing condition for } M_{A^{-1}} \} \subset A_{A,p}.
\]

Now, we present examples of matrices such that \( A_{A,p} \) is a subclass of the Muckenhoupt class \( A_p \).
Corollary 5.9. Let $1 < p < \infty$, $w$ be a weight and $A$ be an invertible matrix.

(i) If $A^{-1} = A$ and $w \in A_{A,p}$ then $w \in A_p$ and $[w]_{A_p} \lesssim [w]^{2}_{A_{A,p}}$.

(ii) If $A^N = A$ for some $N \in \mathbb{N}$ and $w \in A_{A,p}$ then $w \in A_p$ and $[w]_{A_p} \lesssim [w]^{N}_{A_{A,p}}$.

An open question is if there exists a matrix $A$ such that $A_{A,p}$ is greater than $A_p \cap \{w : w_A \lesssim w\}$.

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