Neutrino emission from compact stars and inhomogeneous color superconductivity

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(Dated: March 26, 2022)

We discuss specific heat and neutrino emissivity due to direct Urca processes for quark matter in the color superconductive Larkin-Ovchinnikov-Fulde-Ferrell (LOFF) phase of Quantum-Chromodynamics. We assume that the three light quarks $u, d, s$ are in a color and electrically neutral state and interact by a four fermion Nambu-Jona Lasinio coupling. We study a LOFF state characterized by a single plane wave for each pairing. From the evaluation of neutrino emissivity and fermionic specific heat, the cooling rate of simplified models of compact stars with a quark core in the LOFF state is estimated.

PACS numbers: 12.38.-t, 26.60.+c, 97.60.Jd

I. INTRODUCTION

Neutrino emission due to direct Urca processes, when kinematically allowed, is the most efficient cooling mechanism for a neutron star in the early stage of its lifetime [1]. After a very short epoch, when the temperature of the compact star is of the order of $\sim 10^{11}$K and neutrinos are trapped in the stellar core [2], the temperature drops and neutrinos are able to escape. However, even for smaller temperatures, e.g. below $10^{9}$K, the direct nuclear Urca processes $n \rightarrow p + e^- + \nu_e$ and $e^- + p \rightarrow n + \nu_e$, which would produce rapid cooling, are not kinematically allowed, because energy and momentum cannot be simultaneously conserved. Therefore only modified Urca processes can take place, where a bystander particle, present in the reaction, allows energy-momentum conservation. The resulting cooling is less rapid because neutrino emission rates turn out to be $\varepsilon_\nu \sim T^6$, much smaller than the emission rate $\varepsilon_\nu \sim T^8$ due to direct Urca processes.

These considerations apply to stars containing only nuclear matter. If hadronic densities in the core of neutron stars are sufficiently large, the central region of the star should consist of deconfined quark matter [3] (we do not consider the case of pure quark stars, see e.g. [4]). Therefore direct Urca processes involving quarks, i.e. the processes $d \rightarrow u + e^- + \bar{\nu}_e$ and $u + e^- \rightarrow d + \nu_e$, may take place and largely contribute to the cooling of the star. It has been shown by Iwamoto [5] that quark direct Urca processes are kinematically allowed and the corresponding emission rate for massless quarks is of the order $\alpha_s T^6$, where $\alpha_s$ is the strong coupling constant. This result assumes that quark matter is a normal Fermi liquid. However, since the temperature of aged compact stars is sufficiently low, deconfined quarks in the stellar core are likely to form Cooper pairs and quark matter could be in one of the possible Color Superconductive (CS) phases (see [6], [7] and for reviews [8]). This is due to the fact that the critical temperature of CS matter is of the order of dozens MeV, well above the estimated temperature of the stellar core $\lesssim 100$ KeV.

At asymptotically high densities the energetically favored CS phase is the color-flavor locked (CFL) phase, in which light quarks of any color form Cooper pairs with zero total momentum [9], and all fermionic excitations are gapped. The corresponding neutrino emissivity and specific heat $C$ are suppressed by a factor $e^{-\Delta/T}$, where $\Delta$ is the quasiparticle gap in the CFL phase. Therefore the cooling of quark matter in the CFL phase is distinctly less rapid than in the normal phase. In the CFL phase quark masses can be neglected and color and electric neutrality conditions are automatically implemented. However at densities relevant for compact stars the quark number chemical potential $\mu$ cannot be much larger than 500 MeV and effects due to the strange quark mass $m_s$ must be included. Requiring that bulk quark matter is in weak equilibrium and electrically and color neutral [3, 10, 11, 12, 13], together with $m_s \neq 0$, determines a mismatch $\delta\mu$ between the Fermi momenta of different quarks, with $\delta\mu$ depending on the

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in-medium value of \( m_s \). For values of \( m_s \) less than a critical value, CFL is the energetically favored phase. For larger values the CFL phase cannot be realized and quark matter should pair with a less symmetric pattern.

The ground state of quark matter in these conditions is still a matter of debate and several possible superconductive phases have been suggested (see \[14\] for a review). Recently two superconductive phases characterized by gapless fermionic excitations, i.e. the gapless-2SC (g2SC) phase \[17\] and the gapless-CFL (gCFL) phase \[16, 17, 18\] have been largely discussed. However, it has been shown \[11, 20, 21, 22\] that both are “chromo-magnetically unstable” because the Meissner masses of some of the gluons associated with broken gauge symmetries are imaginary.

Another possibility that has attracted theoretical attention is the Larkin-Ovchinnikov-Fulde-Ferrell (LOFF) state \[23, 24, 25, 26\], where the total momentum of the pair does not vanish and counter propagating color currents are spontaneously generated. A simplified ansatz “crystal” structure with

\[
\langle \psi_{\alpha i}(x) C \gamma_5 \psi_{\beta j}(x) \rangle \propto \sum_{I=1}^{3} \Delta_I e^{2i q_I x} \epsilon_{\alpha \beta I} \epsilon_{ijI},
\]

\((i, j = 1, 2, 3)\) flavor indices, \(\alpha, \beta = 1, 2, 3\) color indices) has been studied in Refs. \[21, 25\] and found energetically favored with respect to the gCFL and the unpaired phases in a certain range of values of \( \delta \mu \). In Eq. (1) \( q_I \) represents the momentum of the Cooper pair and the gap parameters \( \Delta_1, \Delta_2, \Delta_3 \) describe respectively \( d-s, u-s \) and \( u-d \) pairing. For sufficiently large \( \mu \) the energetically favored phase is characterized by \( \Delta_1 = 0, \Delta_2 = \Delta_3 \) and \( q_2 = q_3 \). Gluon Meissner masses corresponding to broken color generators have been evaluated in \[26\] and this phase results to be chromo-magnetically stable. More complex crystal structures have been recently proposed in \[30\]. From the evaluation of the corresponding free-energy in the Ginzburg-Landau approximation one finds that crystal structures with more plane waves are energetically favored with respect to the normal phase and the gCFL phase in a wider range of densities. We note that by (1) we assume attractiveness in the color antisymmetric channel. This follows from the one gluon exchange diagram of QCD. It dominates the asymptotic regime and we assume that it favors attraction also at moderate densities (\( \mu \sim 500 \text{ MeV} \)). In other words, though at moderate density nonperturbative effects can play a role we are assuming that the superconductive ground state is qualitatively similar to that of very high density.

The aim of the present paper is to evaluate the neutrino emission rate and the specific heat of quark matter in the LOFF superconductive phase. We show that, due to the existence of gapless modes in the LOFF phase, a neutron star with a quark LOFF core cools faster than a star made by nuclear matter only. This follows from the fact that in the LOFF phase neutrino emissivity and quark specific heat are parametrically similar to the case of unpaired quark matter (\( \epsilon_\nu \sim T^6 \) and \( C \sim T^3 \) respectively). Therefore the cooling is similar to that of a star comprising unpaired quark matter. Incidentally we note that the dominance of quasiparticle gapless modes has been demonstrated also for the g2SC phase \[31\] and for the gCFL phase \[32\], though in the latter phase the behavior of neutrino emissivity and quark specific heat is different (\( \epsilon_\nu \sim T^{5.5} \) and \( C \sim T^{3.5} \)). For calculation of neutrino emissivity in other models see \[33\].

Our results should be considered as preliminary, since, as we have already pointed out, the simple ansatz \[11\] should be substituted by a more complex behavior as in \[34\]. If the condensate is the sum of more plane waves, the calculation of \( \epsilon_\nu \) and \( C \) is more involved, basically because one should obtain information on the quasiparticle dispersion law in the framework of the Ginzburg Landau expansion. For complex condensate patterns resulting from several plane waves this is a complicated task (a calculation for two flavors is in \[21\]) and we leave it as a future work. Given these limitations it would be fruitless to consider sophisticated star models. Therefore we evaluate the cooling rate by employing toy models of stellar objects, i.e. stars made of nuclear matter with a core in the color superconductive LOFF phase. Also, we will not consider the contribution to the cooling of the compact star due to other processes such as the neutrino pair bremsstrahlung from nuclei in the crust and pionic reactions \[35\] that should also be included in a realistic description of the compact star cooling \[1\].

Our paper is organized as follows. In Section \[11\] we briefly discuss the LOFF phase characterized by the condensate \[11\]. In Section \[11\] we evaluate the neutrino emissivity and in Section \[11\] the specific heat in the LOFF phase. In Section \[11\] we evaluate the effect of \( 1/\mu \) corrections, taking into account the results of \[36\]. In Section \[11\] we estimate the cooling rate of toy models of neutron stars. Given our approximations any comparison with experimental observation, similar e.g. to those already appeared in the literature \[37, 38, 39, 40, 41, 42, 43\], is premature and we limit our analysis to a qualitative comparison among various simple models. In Section \[11\] we draw our conclusions and in the Appendix we report the dispersion laws that are used to compute the quark mixing coefficients and gapless points in the LOFF phase.
II. NEUTRAL LOFF QUARK MATTER

Non interacting quark matter consisting of massless $u$ and $d$ quarks and $s$ quarks with an in-medium mass $m_s$ can be described by the Lagrangian density

$$\mathcal{L}_0 = \bar{\psi}_i \left( i \partial^n_{\alpha} - M^\alpha_{i j} + \mu^\alpha_{i j} \gamma_0 \right) \psi_j ,$$

where $\alpha, \beta = 1, 2, 3$ are flavor indices and $\alpha, \beta = 1, 2, 3$ are color indices; Dirac indices have been suppressed; the mass matrix is given by $M^\alpha_{i j} = \delta^\alpha_{i j} \text{diag}(0, m_s, m_s)$ and $\partial^n_{\alpha} = \partial \delta^\alpha_{i j} \delta_{i j}$. The quark chemical potential matrix is as follows:

$$\mu^\alpha_{i j} = (\mu \delta_{i j} - \mu_c Q_{i j}) \delta^\alpha_{i j} + \delta_{i j} \left( \mu^3 T^\alpha_3 + \frac{2}{\sqrt{3}} \mu_s T^\alpha_8 \right),$$

with $Q_{i j} = \text{diag}(2/3, -1/3, -1/3)_{i j}$ the quark electric-charge matrix and $T_3$ and $T_8$ the Gell-Mann matrices in color space. We are interested in the region close to the second order phase transition point between the normal phase and the Color Superconductive phase, where, to the leading order approximation in $\delta \mu/\mu$, $\mu_3 = \mu_8 = 0$ and $\mu_c = m_s^2/4\mu$ as in the unpaired phase [36]. In all the numerical estimates we use the value $\mu = 500$ MeV.

In order to describe quark interaction we employ a Nambu-Jona Lasinio (NJL) model in a mean field approximation. The use of the NJL model can be motivated by renormalization group analyses [44] showing the dominance of the four-fermion interactions and in particular the numerical importance of the coupling mimicking one gluon exchange. We assume here that these results hold both at high and moderate hadronic densities. The resulting quark condensate is antisymmetric in color and flavor indices and we assume the behavior (1) for it. This phase has been studied assuming that these results hold both at high and moderate hadronic densities. The GL approximation is reliable in a region close to the second order phase transition point [28] where the favored phase and to the normal phase in the range of the chemical potential mismatch $y \in (130, 150)$ MeV, with

$$y = \frac{m_s^2}{\mu} ,$$

Moreover the magnitude of $q$ is given by

$$q = y \frac{y}{8 z_q} , \quad z_q = \frac{1}{1.1997} ,$$

and $\Delta = \Delta_2 = \Delta_3 \leq 0.3 \Delta_0$ where $\Delta_0$ is the value of the gap in the two-flavor homogeneous case [30]. We use the value $\Delta_0 = 25$ MeV. Recent analysis [30] have shown that the range of values of $y$ where the LOFF phase state prevails can be enlarged assuming a more complex pattern of space dependence for $\Delta(y)$. For the purposes of this paper it is however sufficient to consider the simple ansatz (1).

In the basis $A = (1, ..., 9) = (u_r, d_g, s_b, d_r, u_g, s_r, u_b, s_q, d_b)$ the quark chemical potentials can be expressed by

$$\mu_A = \mu + \bar{\mu}_A$$

with

$$\bar{\mu}_1 = -\frac{y}{6} - 2 q z , \quad \bar{\mu}_2 = \frac{y}{12} , \quad \bar{\mu}_3 = -\frac{5 y}{12} , \quad \bar{\mu}_4 = + \frac{y}{12} - q z , \quad \bar{\mu}_5 = - \frac{y}{6} + q z ,$$

$$\bar{\mu}_6 = -\frac{5 y}{12} - q z , \quad \bar{\mu}_7 = -\frac{y}{6} + q z , \quad \bar{\mu}_8 = -\frac{5 y}{12} , \quad \bar{\mu}_9 = \frac{y}{12} .$$

Here

$$z = \cos \theta ,$$

with

$$0 < z < 1$$

and

$$\cos \theta < 0$$

is required.
with θ the angle between the quark momentum and the pair momentum. For later convenience we also define

\[ \mu_A(0) = \mu_A(q = 0) \, . \]  

(10)

We note explicitly that the previous chemical potentials can be obtained by a redefinition of the quark fields in the following way:

\[ \tilde{\psi}_{\alpha i}(x) = e^{i\mathbf{q}_{\alpha i} \cdot \mathbf{x}} \psi_{\alpha i}(x) , \]

(11)

where \( \mathbf{q}_{\alpha i} = \mathbf{q} \) for \( (\alpha, i) = [(r, d), (g, u), (r, s), (b, u)] \), \( \mathbf{q}_{\alpha i} = 2\mathbf{q} \) for \( (\alpha, i) = (r, u) \) and \( \mathbf{q}_{\alpha i} = 0 \) in the other cases. Such a redefinition of the quark fields allows to eliminate the space dependence of the condensate. However quark momenta are shifted as follows:

\[
\begin{align*}
\mathbf{p}_r \to \mathbf{p}_r - 2\mathbf{q} , & \quad \mathbf{p}_d \to \mathbf{p}_d - \mathbf{q} , \quad \mathbf{p}_s \to \mathbf{p}_s , \\
\mathbf{p}_r \to \mathbf{p}_r + \mathbf{q} , & \quad \mathbf{p}_u \to \mathbf{p}_u + \mathbf{q} , \quad \mathbf{p}_s \to \mathbf{p}_s - \mathbf{q} , \quad \mathbf{p}_b \to \mathbf{p}_b - \mathbf{q} \\
\mathbf{p}_s \to \mathbf{p}_s , & \quad \mathbf{p}_d \to \mathbf{p}_d .
\end{align*}
\]

(12)

### III. NEUTRINO EMISSIVITY

The transition rate for the β decay of a down quark \( d_\alpha \), of color \( \alpha = r, g, b \), into an up quark \( u_\alpha \)

\[ d_\alpha(p_1) \to \bar{\nu}_e(p_2) + u_\alpha(p_3) + e^-(p_4) \]  

(13)

is

\[ W_{fi} = V(2\pi)^4 \delta^4(p_1 - p_2 + p_3 - p_4)|\mathcal{M}|^2 \prod_{i=1}^4 \frac{1}{2E_i V} , \]  

(14)

where \( V \) is the available volume and \( \mathcal{M} \) is the invariant amplitude. Neglecting quark masses the squared invariant amplitude averaged over the initial spins and summed over spins in the final state is

\[ |\mathcal{M}|^2 = 64G_F^2 \cos^2 \theta_c (p_1 \cdot p_2)(p_3 \cdot p_4) , \]  

(15)

where \( G_F \) is the Fermi constant and \( \theta_c \) the Cabibbo angle; we will neglect the strange-quark β decay whose contribution is smaller by a factor of \( \tan^2 \theta_c \) in comparison with \([15]\). For relatively aging stars, there is no neutrino trapping \([2]\) which means that neutrino momentum and energy are small. The magnitude of the other momenta is of the order of the corresponding Fermi momenta \( p_F^1, p_F^2 \) and \( p_F^3 \). As discussed in Section II, the electron momentum is of the order \( \mu_e \), which is smaller, but, due to the assumptions discussed in the previous section, still sizeable. On the other hand the neutrino momentum \( p_2 \) is of the order \( k_B T \). It follows that the momentum conservation can be implemented neglecting \( \mathbf{p}_2 \) and one can depict the 3-momentum conservation for the decay \([13]\) as a triangle \([3]\) having for sides \( \mathbf{p}_1, \mathbf{p}_3 \) and \( \mathbf{p}_4 \). It follows that we can approximate

\[ (p_1 \cdot p_2)(p_3 \cdot p_4) \simeq E_1E_2E_3E_4(1 - \cos \theta_{12})(1 - \cos \theta_{34}) \]  

(16)

where \( E_j \) are the energies and \( \theta_{12} \) (resp. \( \theta_{34} \)) is the angle between momenta of the down quark and the neutrino (resp. between the up quark and the electron). Neutrino emissivity \( \varepsilon_\nu \) is defined as the energy loss by neutrino emission per second per unit volume. To compute it, we have to multiply \([15]\) by the neutrino energy and thermal factors, integrate over the available phase space and sum over the three colors \([3]\). Moreover, since the states having definite energy are the quasiquarks involved in the Cooper pairs, one has to multiply the result by two mixing (Bogoliubov) coefficients depending on the quasiparticle dispersion laws \([32]\). Therefore we get

\[ \varepsilon_\nu = \sum_{\alpha=r,g,b} \varepsilon_\nu^\alpha = \sum_{\alpha=r,g,b} \frac{2}{V} \left[ \prod_{i=1}^4 \int \frac{d^3p_i}{(2\pi)^3} \right] E_2W_{fi} \cdot n(p_1) \cdot [1 - n(p_3)] \cdot [1 - n(p_4)] \cdot B_{u_\alpha}^2(p_1)B_{u_\alpha}^2(p_3) \]  

(17)

where \( n(p_j) \) are thermal equilibrium Fermi distributions:

\[ n(p_j) = \frac{1}{1 + \exp \frac{x_j}{T}} , \]  

(18)
They are appropriate here because strong and electromagnetic processes establishing thermal equilibrium are much faster than weak interactions. The overall factor of 2 keeps into account the electron capture process.

Thus far the results are similar to those of [32]. However, in the present case, the phase space integrations and the momentum conservation law must be treated with great care. As a matter of fact, in the LOFF color superconductive state $d_\alpha$ and $u_\alpha$ are in general paired with quarks of another color, and there is breaking of translational invariance because the total momentum of the Cooper pair is $2\mathbf{q}$. Let us choose the $z$–axis along the direction of $\mathbf{q}$ and let $\vartheta_j$ and $\phi_j$ be the polar angles of $\mathbf{p}_j$. Since the temperature is much smaller than the gap parameter, the dominant modes in the $d$– and $u$–momentum integrations are the gapless ones, that we denote as $P_j$ (with $j = 1$ for down quark and $j = 3$ for up quark). They are defined as follows: $P_1$ (resp. $P_3$) is the quark down (resp. quark up) momentum where the corresponding quasi-particle energy vanishes (for more details see the Appendix). On the other hand the relevant momentum for the electron is its Fermi momentum. Therefore we have

$$
\int d^3p_1 \int d^3p_3 \int d^3p_4 \approx \int \mu_\alpha^2 dp_3 d\Omega_4 P_1^2 dp_1 d\Omega_1 P_3^2 dp_3 d\Omega_3 ,
$$

with $d\Omega_j = \sin \vartheta_j d\vartheta_j d\phi_j$. While these approximations are similar to the assumptions used in [32] in dealing with the homogeneous gCFL phase, in the LOFF phase the gapless momenta $P_1$ and $P_3$ depend on the angle $\vartheta_j$ that quark momenta form with the parent momentum $2\mathbf{q}$. In order to simplify the expression of the integral in Eq. (17) the variables $p_j$ can be traded for $x_j$, as in Eq. (19). We put $x_2 = E_2/T$ for the neutrino, $x_4 = (p - \mu_e)/T$ for the electron and $E_j = \mu_j + \epsilon_j(p)$ for the quarks. Expanding around the gapless modes one has $E_j(p) \approx \mu_j + v_j(p - P_j)$ (for $j = 1$ and 3), with the quasiparticle velocity given by

$$
v_j = \frac{\partial E_j}{\partial p} \bigg|_{p=P_j} .
$$

As discussed in detail in the Appendix the dispersion law of each quasiparticle has from one to three gapless modes. Therefore one has to expand the corresponding dispersion laws around each gapless momentum. Another point to be stressed concerns the conservation of three-momentum. As we have noted, to get rid of the space dependence in the condensate, one redefines the quark fields, see Eq. (11), which amounts to a redefinition of momenta as in Eq. (12). Using the new quark fields in the weak decay matrix element [15] adds a momentum $-\mathbf{q}$ to the momentum conservation law that now reads: $p_1 - p_3 - p_4 - \mathbf{q} = 0$ (where the neutrino momentum has been neglected).

Employing the above approximations the neutrino emissivity for each pair of gapless momenta $P_1, P_3$, can be written as

$$
\varepsilon^\alpha_\nu \approx \frac{G_F^2 \cos^2 \theta_c \mu_e^2 T^6}{32 \pi^8} I \int \left( \frac{dr}{2 \pi r^3} \right)^4 \prod_{j=1}^4 \int d\Omega_j \left\{ \frac{P_1^2 P_3^2 B_{d_\alpha}^2 (P_1) B_{u_\alpha}^2 (P_3)}{|v_1| |v_3|} (1 - \cos \theta_{12})(1 - \cos \theta_{13}) e^{-i r (p_1 - p_3 - p_4 - \mathbf{q})} \right. \\
- \left. \times \left( 1 - \frac{P_1^2 + q^2 - 2 q P_1 \cos \vartheta_1 - P_3^2 - \mu_e^2}{2 \mu_e P_3} \right) e^{-i r (p_1 - p_3 - p_4 - \mathbf{q})} \right\} ,
$$

where $P_1$ and $P_3$ depend, respectively, on the angles $\vartheta_1$ and $\vartheta_3$ and $B_{i_\alpha}$ is a mixing coefficient arising from the Bogoliubov-Valatin transformation representing the probability amplitude that the gapless quasiparticle has flavor $i$ and color $\alpha$. Moreover [3, 45]

$$
I = \int_{-\infty}^{+\infty} dx_1 \int_{0}^{+\infty} dx_2 x_2^3 \int_{-\infty}^{+\infty} dx_3 \int_{-\infty}^{+\infty} dx_4 n(x_1)n(-x_3)n(-x_4)\delta(x_1 - x_2 - x_3 - x_4) = \frac{457 \pi^6}{5040} .
$$

Some of the angular integrations appearing in (22) can be performed analytically, with the result

$$
\varepsilon^\alpha_\nu \approx \frac{G_F^2 \cos^2 \theta_c \mu_e^2 T^6}{4 \pi^6} I \sum_{k=1}^{5} \int_{r_k}^{+\infty} dr r^2 \int_{-1}^{1} d(\cos \vartheta) e^{i q r \cos \vartheta} \int_{-\infty_\vartheta}^{+\infty_\vartheta} d(\cos \vartheta_1) e^{-i P_1 r \cos \vartheta \sin \vartheta_1} J_0(P_1 r \sin \vartheta \sin \vartheta_1) f_k(P_1) \frac{P_1^2 B_{d_\alpha}^2 (P_1)}{|v_1|} ,
$$
\[
\times \int_{-1}^{1} d(\cos \varphi) e^{ip_1 r \cos \varphi \cos \vartheta} J_0(p_1 r \sin \vartheta \sin \varphi_3) \frac{P_3^2 B_2^2(P_3)}{|v_3|} \\
\times \int_{-1}^{1} d(\cos \varphi) e^{ip_2 r \cos \varphi \cos \vartheta} J_0(p_2 r \sin \vartheta \sin \varphi) J_0(p_4 r \sin \vartheta \sin \varphi_4) .
\] (24)

Here for convenience we have decomposed the integral as a sum over various terms, with \(f_1 = f_3 = f_4 = 1\), \(f_2 = P_1^2 + q^2\), \(f_5 = -2P_1 q \cos \vartheta_1\), and \(g_1 = 1\), \(g_2 = g_5 = -\frac{1}{2\mu_e P_3}\), \(g_3 = \frac{P_3}{2\mu_e}\), \(g_4 = \frac{\mu_e}{2P_3}\).

In Eq. (24) \(J_0\) is the Bessel function of zeroth order; \(\omega_0\), \(\omega_1\), \(\omega_2\), \(\omega_3\) are appropriate limits taking into account kinematic constraints; \(\vartheta\) is the angle between the directions of \(\mathbf{r}\) and \(\mathbf{q}\). Even if each quasiparticle dispersion law is characterized by various gapless momenta, the number of the gapless momenta relevant for the evaluation of the neutrino emissivity can be reduced observing that the momenta \(\mathbf{p}_1\), \(\mathbf{p}_3\) and \(\mathbf{p}_4\), with \(|\mathbf{p}_4| \approx \mu_e\), must satisfy the condition \(\mathbf{p}_1 - \mathbf{p}_3 - \mathbf{p}_4 - \mathbf{q} = 0\).

We now consider the contributions of the three colors separately. We discuss in detail only the decay of the blue quark that also gives the largest contribution. The decay of the other colors is treated analogously. The blue down quark is unpaired because in the Ginzburg-Landau approximation the gap parameter \(\Delta_1\) vanishes [27]. Therefore, as discussed in the Appendix, the dispersion law is \(E_1(\mu) = p\), with gapless momentum \(P_1 = \mu_0\), mixing coefficient \(B_{d_0}(P_1) = 1\) and \(v_1 = 1\).

Let us now consider the up-blue quark. It is paired with the strange-red quark with gap parameter \(\Delta_2 = \Delta\). In the Appendix (subsection A3) we give the dispersion law and the gapless momentum:

\[
P_3 = \frac{\mu_6 + \mu_7}{2} + \sqrt{\frac{y^2}{64} \left(1 + \frac{\cos \vartheta_3}{z_q}\right)^2 - \Delta^2} .
\] (25)

We do not consider the momentum corresponding to the other gapless mode of Eq. (A21), as it does not satisfy the condition of momentum conservation. The emissivity for the decay of the blue quark is therefore:

\[
\varepsilon_\nu^\text{blue} \simeq \frac{G_F^2}{4\pi^6} \cos^2 \theta \frac{\mu_3^2 \mu_5^2}{T^6} \int_{0}^{\infty} \frac{dr}{r^2} \int_{-1}^{1} d(\cos \varphi) e^{ip_1 r \cos \varphi \cos \vartheta} J_0(p_1 r \sin \vartheta \sin \varphi_3) \\
\times \int_{-1}^{1} d(\cos \varphi) e^{-ip_1 r \cos \varphi \cos \vartheta} J_0(p_1 r \sin \vartheta \sin \varphi_3) \\
\times \int_{-1}^{1} d(\cos \varphi) e^{ip_2 r \cos \varphi \cos \vartheta} J_0(p_2 r \sin \vartheta \sin \varphi_3) g(\cos \vartheta_3, \cos \vartheta_1) \\
\times \int_{-1}^{1} d(\cos \varphi) e^{ip_4 r \cos \varphi \cos \vartheta} J_0(p_4 r \sin \vartheta \sin \varphi_4) ,
\] (26)

where

\[
\omega_2 = \text{Min} \left\{ z_q \left[ \frac{8\Delta}{y} - 1 \right], +1 \right\} ,
\] (27)

\[
g(\cos \vartheta_3, \cos \vartheta_1) = \left(1 - \frac{\mu_3^2 + q^2 - 2\mu_q \cos \vartheta_3 - P_3^2 - \mu_5^2}{2P_3 \mu_e}\right) \left|P_3 B_{d_0}(P_3)\right|^2 \Theta \left(1 - \left|\frac{\mu_3^2 + q^2 - 2\mu_q \cos \vartheta_3 - P_3^2 - \mu_5^2}{2P_3 \mu_e}\right|\right) \] (28)

and

\[
|v_3| = \sqrt{\frac{P_3 - \frac{\mu_6 + \mu_7}{2}}{\left(P_3 - \frac{\mu_6 + \mu_7}{2}\right)^2 + \Delta^2}} .
\] (29)

We note that the presence of \(\omega_2\) implements the existence of the gapless momentum \(P_3\) while the Heaviside function implements the condition \(|\cos \vartheta_3| \leq 1\).

The case of quarks with colors red and green can be treated in a similar way, though the numerical computation is more involved because for these colors neither the down nor the up quarks are unpaired. Because of this one expects that the emissivity of these quarks is smaller than for blue quarks, an expectation confirmed by the numerical analysis. We will report our numerical results in Section [3].
IV. SPECIFIC HEATS

At low temperatures the largest contribution to specific heat \( C \) is determined by the sum of the specific heats of the fermionic quasi-particles. In the three flavor LOFF phase the quasi-particle specific heats are given by

\[
e_j = \frac{\int d\theta}{(2\pi)^3} \frac{\partial m(\epsilon_j)}{\partial T},
\]

where \( j = u_r, d_g, s_b, d_r, u_g, s_r, u_b, s_g, d_b \) and \( \epsilon_j = |E_j(p)| - \mu_s \) are the quasi-particle dispersion laws. Since we work in the regime \( T \ll \Delta \ll \mu \), the contributions of gapped modes are exponentially suppressed and each gapless mode contributes by a factor \( \propto T \). This results from the evaluation of the integral in Eq. (30) employing the saddle point method and assuming that the quasi-particle dispersion laws are linear in the gapless momenta. For the present choice of parameters, the quasi-particle dispersion law is quadratic in the gapless momenta in a negligible range of values of the angle between the direction of quasiparticle momentum and the direction of \( \mathbf{q} \). Such a situation is quite different from the homogeneous gCFL case where a quasiparticle dispersion law is quadratic \([32]\) and gives the choice of parameters, the quasiparticle dispersion law is quadratic in the gapless momenta in a negligible range of values of the angle between the direction of quasiparticle momentum and the direction of \( \mathbf{q} \). Such a situation is quite different from the homogeneous gCFL case where a quasiparticle dispersion law is quadratic \([32]\) and gives the dominant contribution to the specific heat.

Within the above-mentioned approximations the fermionic specific heat is given by

\[
C \simeq \frac{T}{3} \int d\cos\theta \left[ \frac{P^2_{II}}{|v_{II}|} + \frac{P^2_{III}}{|v_{III}|} + \frac{(P^2_{g} - 2\mu_s)}{|v_{g}^2|} + \frac{(P^2_{g} - 2\mu_s)}{|v_{g}^2|} \right]
\]

where the various gapless momenta and Fermi velocities on the r.h.s have been defined in the Appendix and in Eq. (21). In particular \( P_I, P_{II} \) and \( P_{III} \) are the gapless momenta of the sector \( (u_r, d_g, s_b) \) and \( v_I, v_{II} \) and \( v_{III} \) the corresponding velocities. We have also added the electron contribution, though \( \mu_e \) is numerically much smaller than all the quark momenta \( P_g \) (generally of the order of \( \mu \)). We also note that the integration range is in general smaller than \( (-1, +1) \) because the quasi-particle dispersion laws are gapless only in a restricted range of values of the variable \( \cos\theta \).

V. ROLE OF 1/\( \mu \) CORRECTIONS

In previous sections we have evaluated emissivity and specific heat in the three flavor LOFF phase of QCD. All the calculations are based on the High Density Effective Theory, see \([16, 17, 18]\), where one takes the limit \( \mu \to \infty \). In such approximation one neglects the contribution of the antiparticles; moreover the \( m_s \neq 0 \) effects are treated at the leading order by a shift in the strange Fermi momentum \( p_s^F = \mu_s - m_s^2/2\mu \). One can show \([30]\) that, as already stressed in Section II, in this approximation the electron chemical potential \( \mu_e \) is given by \( \mu_e = m_e^2/4\mu \), which implies a symmetric splitting of the \( s \) and \( d \) Fermi surfaces around the \( u \) Fermi surface. Therefore \( \Delta_1 = 0 \) and \( \Delta_2 = \Delta_3 \). If \( \mu \) is not large enough this approximation is not justified and higher order corrections must be included.

The next-to-leading corrections were computed in \([30]\). One expands the strange quark momentum up to the next-leading order:

\[
p_s^F \approx \mu_s - \frac{m_s^2}{2\mu_s} - \frac{1}{2\mu} \left( \frac{m_s^2}{2\mu} \right)^2.
\]

Then one substitutes this expression in the Lagrangian in Eq. (24), getting a correction. It can be proven \([30]\) that other corrections, e.g. the antiparticle contribution, is next-to-next-to-leading in the weak coupling approximation and therefore can be safely neglected.

Keeping only terms of \( \mathcal{O}(\Delta/q)^2 \) in the free energy, the 1/\( \mu \) shift in Eq. (32) results in the chemical potentials

\[
\mu_c = \frac{m_s^2}{4\mu} - \frac{m_s^4}{48\mu^3}, \quad \mu_3, \mu_8 = 0.
\]

The introduction of the 1/\( \mu \) terms in the free energy results in an asymmetric splitting of the Fermi surfaces of the \( u, d \) and \( s \) quarks. As a matter of fact one gets in the normal phase

\[
\frac{\mu_u - p_s^F}{2} = \frac{m_u^2}{8\mu} + \frac{5}{96} \mu^3, \quad \frac{\mu_d - \mu_u}{2} = \frac{m_d^2}{8\mu} - \frac{1}{96} \mu^3.
\]

As a consequence one expects in the LOFF phase still \( \Delta_1 = 0 \), but \( \Delta_2 < \Delta_3 \). For example for \( \mu = 500 \) MeV one finds at \( m_s^2/\mu = 140 \) MeV that \( \Delta_2 = 0 \) and \( \Delta_3 \approx 0.35\Delta_0 \). Thus for this value of \( m_s \) the vacuum consists of LOFF pairs.
of \( u_t - d_g \) and \( u_g - d_r \) quarks and Fermi seas of unpaired \( s \), \( u_b \) and \( d_b \) quarks. This new phase was called LOFF\(^2\)s in Ref. [3] since it is a two flavor LOFF phase whose excitation spectrum consists of quasi-particles with dispersion laws

\[
\epsilon = \frac{\mu_4 - \mu_5}{2} \pm \sqrt{\left( \frac{\epsilon}{\mu_4 + \mu_5} \right)^2 + \Delta_3^2}
\]

(35)

and a sea of unpaired blue and strange quarks.

Motivated by these results we study neutrino emissivity and specific heat of the LOFF\(^2\)s phase. Since blue up and down quarks are unpaired, their contribution to the neutrino emissivity is 1/3 of the neutral unpaired quark matter. Using for it the estimate of Ref. [5] one obtains

\[
\varepsilon^{\text{blue}}_\nu \simeq \frac{1}{2} \times \frac{457}{630} \frac{G_F^2 \cos^2 \theta_c}{\hbar^0 c^6} \alpha_s \mu_a \mu_d \mu_e (k_B T)^6 ,
\]

(36)

where we have restored the correct factor of \( c \), \( \hbar \) and \( k_B \). We note that in this equation \( \alpha_s = g^2 / 4\pi = 4\alpha_c \) where \( g \) gives the coupling among quarks and gluons and \( \alpha_c \) is the coupling constant appearing in Ref. [5]. Notice that the use of this estimate by Iwamoto [5] is only indicative, because at the densities we are considering, perturbative QCD is unreliable. Therefore (36) can provide at best an order of magnitude estimate. Neglecting the contribution of strange quarks, the other contribution to the emissivity is determined by red and green light quarks and can be computed following the same lines of Section III. Since the LOFF pairing results in a restriction of the available phase space, the red and green channels contribute to the total decay rate less than the blue one, similarly to the discussion in Section III.

Next we turn to the specific heat, which is given by

\[
C \simeq \frac{T}{3} \int \frac{d \cos \theta}{2} \left[ 2 \times \left( \frac{\mu_s + \mu_t}{v_+} + \left( \frac{\mu_s - \mu_t}{v_-} \right) \right) + \left( \mu_s \right)^2 + \left( \mu_r \right)^2 + \left( \mu_u \right)^2 + \left( \mu_d \right)^2 + \left( \mu_s \right)^2 + \mu_r^2 \right]
\]

(37)

where the gapless momenta \( P_\pm \) are defined as

\[
P_\pm = \frac{\mu_4 + \mu_5}{2} \pm \sqrt{\left( \frac{\mu_4 - \mu_5}{2} \right)^2 - \Delta_3^2}
\]

(38)

and the Fermi velocities are given by

\[
v_+ = v_- = \left( \frac{\mu_4 - \mu_5}{2} \right) - \frac{\Delta_3^2}{\mu_4 - \mu_5}.
\]

(39)

The factor 2 in the first two addenda of the r.h.s. of Eq. (37) takes into account the fact that the dispersion laws for the \( u_r - d_g \) quasiparticles are the same of the \( u_g - d_r \) ones.

### VI. COOLING BY NEUTRINO EMISSION

Let us assume the presence of quark matter in the LOFF state in the core of a compact star. This will affect the cooling process and we now discuss these effects by comparing various models of stars along lines similar to Ref. [32]. Given the approximations used in the study of the LOFF phase, in particular the use of the simple ansatz II instead of more complex space behavior of the condensate, it would be fruitless to employ sophisticated models. Therefore we will use a simplified approach based on the study of four different star toy models. The first model (denoted as I) is a star consisting of noninteracting "nuclear" matter (neutrons, protons and electrons) with mass \( M = 1.4 M_\odot \), radius \( R = 12 \) km and uniform density \( n = 1.5 n_0 \), where \( n_0 = 0.16 \) fm\(^{-3} \) is the nuclear equilibrium density. The nuclear matter is assumed to be electrically neutral and in beta equilibrium. The second model (II) is a star containing a core of radius \( R_1 = 5 \) km of neutral unpaired quark matter in weak equilibrium at \( \mu = 500 \) MeV, with a mantle of noninteracting nuclear matter with uniform density \( n \). Assuming a star mass \( M = 1.4 M_\odot \) from the solution of the Tolman-Oppenheimer-Volkov equations one gets a star radius \( R_2 = 10 \) km. Finally we discuss two simplified models of compact stars containing a core of electric and color neutral three flavors quark matter in the LOFF phase, with \( \mu = 500 \) MeV and \( m_3^2 / \mu = 140 \) MeV and \( \Delta_0 = 25 \) MeV. Both models have a mantle of noninteracting nuclear
matter and differ by the values of the LOFF gaps. In model III we neglect $O(1/\mu)$ corrections and we take $\Delta_1 = 0$ and $\Delta_2 = \Delta_3 = 0.25 \Delta_0$, the value of the gap parameters in the LOFF phase at $y = 140$ MeV \cite{27}. This is the approximation discussed in Section III. In model IV we include the $1/\mu$ correction discussed in Section V and we take the values of the gaps as in the LOFF2s model discussed in Ref. \cite{36}, i.e. $\Delta_1 = \Delta_2 = 0$, $\Delta_3 = 0.28 \Delta_0$ for $m_\gamma^2/\mu = 140$ MeV. Since the values of the gaps in both LOFF models are small the radii of the star and of the quark core do not differ appreciably from those of a star with a core of unpaired quark matter, i.e. $R_1 = 5$ and $R_2 = 10$ km (also in these cases $M = 1.4M_\odot$).

The main mechanisms of cooling are by neutrino emission and by photon emission, the latter dominating at later ages. Therefore the star cooling is governed by the following differential equation:

$$\frac{dT}{dt} = - \frac{L_\nu + L_\gamma}{V nm c_{V}^{nm} + V qm c_{V}^{qm}} = - \frac{V nm \varepsilon_{\nu}^{nm} + V qm \varepsilon_{\nu}^{qm} + L_\gamma}{V nm c_{V}^{nm} + V qm c_{V}^{qm}}.$$  \hspace{1cm} (40)

Here $T$ is the inner temperature at time $t$, while $L_\nu$ and $L_\gamma$ are neutrino and photon luminosities, i.e. heat losses per unit time. Neutrino luminosity is obtained multiplying the emissivity by the corresponding volume. Therefore one must distinguish between the emissivity $\varepsilon_{\nu}^{nm}$ of nuclear matter, which is present in the volume $V_{nm}$, from the emissivity due to quarks, if they are present in some volume $V_{qm}$; $c_{V}^{nm}$ and $c_{V}^{qm}$ denote specific heats of the two forms of hadronic matter.

For the emissivity of nuclear matter $\varepsilon_{\nu}^{nm}$ we use the standard value \cite{37}:

$$\varepsilon_{\nu}^{nm} = (1.2 \times 10^4 \text{erg cm}^{-3} \text{s}^{-1}) \left( \frac{n}{n_0} \right)^{2/3} \left( \frac{T}{10^7 \text{K}} \right)^8$$ \hspace{1cm} (41)

arising from the analysis of the modified Urca processes $n + X \rightarrow p + X + e + \bar{\nu}$, with $X = p$ or $n$. As for the quark contribution $\varepsilon_{\nu}^{qm}$, we consider both unpaired quarks and quarks in the LOFF state, depending on the model. The former contribution is denoted $\varepsilon_{\nu}^{qm, \text{unpaired}}$ and is given by Eq. (35) multiplied by a factor of 3. The latter have been estimated in Sections III and IV for the two models of LOFF quark matter. For all these stars we assume a common interior temperature $T$, since the matter comprising compact stars is made of good conductors ($p, n$ or quarks).

Let us now turn to cooling by photon emission, the dominant process for sufficiently old stars ($t > 10^6$ years). In this case the luminosity can be estimated in the black-body approximation as follows:

$$L_\gamma = 4\pi R^2 \sigma T_s^4,$$ \hspace{1cm} (42)

where $R$ is the radius of the star, $\sigma$ is the Stefan-Boltzmann constant and

$$T_s = \left( \frac{g_s}{10^{14} \text{cm/s}^2} \right)^{1/4} \left( \frac{T}{10^8 \text{K}} \right)^{0.55},$$ \hspace{1cm} (43)

is the surface temperature \cite{39, 50}, with $g_s = G_N M/R^2$ the surface gravity.

Let us now discuss specific heats. They are given for the LOFF model III and for the LOFF2s model IV by Eqs. (31) and (37) respectively. For nuclear and unpaired quark matter the total specific heat is given by the sum of the specific heats of the different fermionic species. They can be approximated by the fermionic ideal gas result

$$c_V = \frac{k_B^2 T}{3h^2 c} p_F \sqrt{m^2 c^2 + (p_F)^2},$$ \hspace{1cm} (44)

where $m$ is the fermionic mass and $p_F$ the Fermi momentum. For non-interacting nuclear matter, the three species are neutrons, protons and electrons with Fermi momenta evaluated as in neutral matter in weak equilibrium \cite{37}:

$$p_F^n = (340 \text{ MeV}) \left( \frac{n}{n_0} \right)^{1/3},$$

$$p_F^p = p_F^n = (60 \text{ MeV}) \left( \frac{n}{n_0} \right)^{2/3}.$$ \hspace{1cm} (45)

For neutral unpaired quark matter in weak equilibrium, there are nine quark species, with Fermi momenta independent of color and given by $p_F^q = \mu + \frac{m_i^2}{12\mu}$, $p_F^n = \mu - \frac{m_i^2}{6\mu}$, $p_F^n = \mu - \frac{5m_i^2}{12\mu}$.

Eq. (40) is solved imposing a given temperature $T_0$ at a fixed early time $t_0$ (we use $T_0 \rightarrow \infty$ for $t_0 \rightarrow 0$). In Figs. II, II and III the cooling curves for various models of stars are shown. In Fig. II the inner temperature, in Kelvin, as
FIG. 1: (Color online) Inner temperature, in Kelvin, as a function of time, in years, for three toy models of pulsars. Solid black curve refers to model I; dashed line (red online) refers to model II; dotted curve (blue online) refers to model III. Model I is a neutron star formed by nuclear matter with uniform density $n = 0.24$ fm$^{-3}$ and radius $R = 12$ Km; model II corresponds to a star with $R_2 = 10$ km, having a mantle of nuclear matter and a core of radius $R_1 = 5$ Km of unpaired quark matter, interacting via gluon exchange; model III is like model II, but in the core there is quark matter in the LOFF state, see text for more details. All stars have $M = 1.4 M_\odot$. Parameters for the core are $\mu = 500$ MeV and $m_s^2/\mu = 140$ MeV.

A function of time, in years, for three models is reported. Solid line (black online) is for model I (electrically neutral nuclear matter made of non interacting neutrons, protons and electrons in beta equilibrium); dashed curve (red online) refers to model II (nuclear matter mantle and a core of unpaired quark matter, interacting via gluon exchange); the dotted line (blue online) is for model III (nuclear matter mantle and a core of quark matter in the LOFF state in the leading $1/\mu$ approximation, as discussed in Section II). The curve reported in Figs. 1 and 2 for model III corresponds to $m_s = \sqrt{140/\mu}$ MeV, however with increasing values of $m_s$ the neutrino emissivity decreases. This is due to the fact that $\Delta$ decreases as one approaches the second order phase transition to the normal state. On the other hand in this case the quark matter in the star tends to become a normal Fermi liquid, for which the description of Iwamoto which includes Fermi liquid effects, must be adopted.

Fig. 2 gives the star surface temperature as a function of time, as obtained by use of Eq. (43). The three curves refer to the same toy stars as in Fig. 1. Both diagrams are obtained for the following values of the parameters: $\mu = 500$ MeV, $m_s^2/\mu = 140$ MeV, $\Delta_1 = 0$, $\Delta_2 = \Delta_3 \simeq 6$ MeV. For unpaired quark matter we use $\alpha_s \simeq 1$, the value corresponding to $\mu = 500$ MeV and $\Lambda_{QCD} = 250$ MeV. The use of perturbative QCD at such small momentum scales is however questionable. Therefore the results for model II should be considered with some caution and the curve is plotted only to allow a comparison with the other models. In particular the similarity between the LOFF curve and the unpaired quark curve follows from the fact that the LOFF phases are gapless, so that the scaling laws $c_V \sim T$ and $\varepsilon_\nu \sim T^6$ are analogous to those of the unpaired quark matter. However the cooling curve for unpaired quark matter depends on the value we assumed for the strong coupling constant, $\alpha_s \simeq 1$, which is an extrapolation to a regime where perturbative QCD is less reliable. Therefore the similarity between the curves of models II and III might be accidental.

In Fig. 3 we compare two models of a star with a nuclear mantle and a quark core in the LOFF state. The approximations used for these LOFF states were discussed in Sections II and IV. The continuous curve (red online) is for model III and the dashed black curve is for model IV, i.e. with quarks in the LOFF2s state. One can note that both curves for the LOFF models are similar and show a rapid cooling, much faster than for ordinary stars comprising only nuclear matter. This implies that the results for models with a LOFF core are rather robust and, at least for relatively young stars, the presence of quark matter in the LOFF state should offer a signature clearly distinct from that of an ordinary neutron star.
FIG. 2: (Color online) Surface temperature \( T_s \), in Kelvin, as a function of time, in years, for the three toy models of pulsars described in Fig. 1.

FIG. 3: (Color online) Comparison of the cooling curves for the two toy models of stars with a core of quarks in a LOFF state. The curves express the inner temperature, in Kelvin, as a function of time, in years, for model III (solid, red online, curve) and model IV (dashed black curve). Model III is a star with a radius \( R_2 = 10 \) km, having a mantle of nuclear matter and a core of radius \( R_1 = 5 \) Km of quark matter in the LOFF state in the leading approximation \( (\Delta_1 = 0, \Delta_2 = \Delta_3 \simeq 6 \text{ MeV}) \); model IV is analogous to model III, but the core is in the LOFF2s phase \( (\Delta_1 = \Delta_2 = 0, \Delta_3 \simeq 7 \text{ MeV}) \). All stars have \( M = 1.4 M_\odot \). Parameters for the core are \( \mu = 500 \text{ MeV} \) and \( m_s^2/\mu = 140 \text{ MeV} \).

VII. CONCLUSIONS

Our study can be summarized as follows. We have computed neutrino emissivity and specific heat for quark matter in the LOFF state of QCD in presence of three light quarks. Quark matter has been assumed to be in weak equilibrium and in a color and electrically neutral state. We have considered the simplest ansatz for the condensates (single plane wave for all LOFF pairings). We have studied two models of the LOFF state, differing in the approximations used for the Ginzburg Landau evaluation of the free energy and the gap equation. Our analysis shows a similarity between the two approximations, which points to a robustness of our results. We have used this study for an estimate of the cooling of compact stars with a nuclear mantle and a quark core.

Which conclusions can we draw from the present study? From Figs. 1 and 2, we see that stars with a LOFF core cool down faster than ordinary neutron stars. This might have interesting phenomenological consequences. At the present time observational results on the cooling of pulsars are being accumulated at an increasing rate (for compilation of data and comparison between theoretical models and data see, e.g. [37, 38, 39, 40, 41, 42, 43]). Some data indicate that stars with an age in the range \( 10^3 - 10^4 \) years have a temperature significantly smaller than what is expected on the basis of the modified Urca processes. It is difficult however to infer, from these data, predictions on the star.
composition, as the stars may have different masses. As for the impact on our study, it is useful to repeat here that our analysis should be considered as preliminary because the identification of the quasiparticle dispersion laws for the favored crystalline LOFF structures, formed by more plane waves, is still lacking. This is the reason why we have not tried a comparison with observations in this paper. Nevertheless some qualitative assessments can be made. Quantum Chromodynamics predicts that at the densities that can be reached in the core of compact stars deconfined quark matter should be present, and, if so, it should be in a color superconductive state, since Cooper condensation of colored diquarks is energetically favored. Slow cooling is typical of stars containing only nuclear matter or of stars with a color superconductive quark core in a gapped phase (e.g. CFL). If a careful comparison with the data could allow to rule out slow cooling for star masses in the range we have considered, this would favor either the presence of condensed mesons or quark matter in a gapless state in the core, since gapped quarks emit neutrinos very slowly. Meson condensation also might allow rapid cooling. However reliable calculations based on the chiral effective field theory \cite{footnote2} can only be made for \( m_2^2 \simeq \mu \Delta \), i.e. far away from the region where the LOFF state is stable; therefore no direct comparison between the two phases is available. For intermediate densities the quark normal state is less favored than color superconductive states for a wide range of values of the baryonic chemical potential \( \mu \) and the strange quark mass \( m_s \). This leaves us with gapless quark phases and, among them the LOFF state is favored since we know that the gapless phases with homogeneous gap parameters such as, e.g. the gCFL or the g2SC phase are instable, while the LOFF phase does not suffer of a similar instability. Let us finally observe that our conclusions should remain valid, at least qualitatively, also for more complex crystalline patterns of the LOFF condensate. Apparently the fast cooling of relatively young stars with a LOFF quark core is a consequence of the scaling laws for neutrino emissivity and specific heat. They depend on the existence of gapless points and follow from the existence of blocking regions in momentum space. Since this property is typical of the LOFF state, independently of detailed form of the condensate, a rapid cooling should be appropriate not only for the simple ansatz assumed in Eq. \ref{eq:1}, but, more generally, for any LOFF condensate.

**Acknowledgments**

We would like to thank M. Alford, M. Ciminale, R. Gatto, C. Kouvaris, A. Mirizzi, A. Schmitt, I. Shovkovy and Q. Wang for discussions and comments. One of us (MM) would like to thank the Barcelona IEEC for the kind hospitality during the completion of the present work. The work of MM has been supported by the Bruno Rossi fellowship program and by the U.S. Department of Energy (D.O.E.) under cooperative research agreement \#DE-FC02-94ER40818.

**APPENDIX A: QUASIPARTICLE DISPERSION LAWS AND MIXING COEFFICIENTS**

In this appendix we compute the quasiparticle dispersion laws, the gapless points and the Bogoliubov coefficients for the electrically and color neutral LOFF state of QCD with three flavors in beta equilibrium. In the basis \( A = (1, ..., 9) = (u_r, d_g, s_b, d_r, u_g, s_r, u_b, s_g, d_b) \) the gap matrix is as follows:

\[
\Delta_{AB} = \begin{pmatrix}
0 & \Delta_3 & \Delta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Delta_3 & 0 & \Delta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Delta_2 & \Delta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\Delta_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\Delta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\Delta_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\Delta_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\Delta_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\Delta_1 & 0 \\
\end{pmatrix}.
\]

We consider in detail only the solution with \( \Delta_1 = 0 \), \( \Delta_2 = \Delta_3 = \Delta \) (notice that a factor \( e^{2i q_2 \cdot r} = e^{2i q_3 \cdot r} \) has been omitted in the entries \( \Delta_2, \Delta_3 \)). The case of the LOFF2s model \( (\Delta_1 = \Delta_2 = 0, \Delta_3 \neq 0) \), can be treated in a similar way. Since the matrix \[A\] is block-diagonal, we consider its various sectors separately.

1. **Sector** \( A = (u_r, d_g, s_b) \)

To get the dispersion laws of the quasiparticles one has to find the poles of the propagator. This corresponds to solve the equation \( \det S^{-1} = 0 \). The procedure is simplified because one can write the determinant as a product of
various minor determinants. In particular for the quasiparticles given by linear combinations of $u_r, d_g$ and $s_b$ quarks one has to evaluate the determinant of the matrix

$$
\begin{pmatrix}
\omega_1 & \Delta & \Delta \\
\Delta & \bar{\omega}_2 & 0 \\
\Delta & 0 & \bar{\omega}_3
\end{pmatrix},
$$

(A2)

where

$$
\omega_1 = \epsilon - p + \mu_1, \quad \bar{\omega}_2 = \epsilon + p - \mu_2, \quad \bar{\omega}_3 = \epsilon + p - \mu_3.
$$

(A3)

The dispersion laws are obtained by solving the cubic equation in the variable $\epsilon$

$$(\epsilon - p + \mu_1)(\epsilon + p - \mu_2)(\epsilon + p - \mu_3) - \Delta^2[2(\epsilon + p) - \mu_2 - \mu_3] = 0.
$$

(A4)

If we define $y_j = p - \mu_j$ ($j = 1, 2, 3$) and

$$
\begin{align*}
b &= y_1 - y_2 - y_3, \\
c &= -y_1 y_2 - y_1 y_3 - y_2 y_3 - 2\Delta^2, \\
p &= c - \frac{b^2}{3}, \\
q &= d - \frac{bc}{3} + \frac{2b^3}{27}, \\
\zeta &= \sqrt{-\frac{27q}{2} + \frac{3}{2}\sqrt{12p^3 + 81q^2}},
\end{align*}
$$

(A5)

then the roots of Eq. (A3) are as follows:

$$
\epsilon_0 = \frac{\zeta - pw}{3w} + \frac{b}{3}, \quad \epsilon_a = \frac{\zeta w}{3} - \frac{p}{\zeta w} + \frac{b}{3}, \quad \epsilon_b = -\epsilon_0 - \epsilon_a + b,
$$

(A6)

with $w = \frac{-1 + i\sqrt{3}}{2}$. One can show that only the quasiparticle of energy $\epsilon_0(p)$ is gapless, while $\epsilon_{a,b}$ refer to gapped quasiparticles (this is true for the reference value we use, $y = 140$ MeV, and for any $y < 148$ MeV; for $y > 148$ we find that $\epsilon_{a,b}$ are still gapped except for a tiny region around the North and the South poles). There can be one gapless mode ($P_1$) or three ($P_1, P_{11}, P_{111}$) depending on the angle that $2\mathbf{q}$ forms with difference of the pairing quark momenta. The gapless modes are solutions of the equation

$$
(P - \mu_1)(P - \mu_2)(P - \mu_3) + 2\Delta^2 \left( P - \frac{\mu_2 + \mu_3}{2} \right) = 0.
$$

(A7)

The Bogoliubov coefficients given by

$$
\begin{align*}
B_{u_r}^2(\xi) &= \frac{(\bar{\omega}_2 \bar{\omega}_3)^2}{f(\bar{\omega}_2, \bar{\omega}_3, \Delta)}, \\
B_{d_g}^2(\xi) &= \frac{(\Delta \bar{\omega}_3)^2}{f(\bar{\omega}_2, \bar{\omega}_3, \Delta)}, \\
B_{s_b}^2(\xi) &= \frac{(\Delta \bar{\omega}_2)^2}{f(\bar{\omega}_2, \bar{\omega}_3, \Delta)},
\end{align*}
$$

(A8)-(A10)

with $f(\alpha_1, \alpha_2, \alpha_3) = \alpha_1^2\alpha_2^2 + \alpha_1^2\alpha_3^2 + \alpha_2^2\alpha_3^2$, represent the probability that the gapless quasiparticle is respectively $u_r$, $d_g$ or $s_b$. We note that the dispersion laws are anisotropic as they depend on the angle $\theta$ that the quark momentum form with the pair momentum. In Fig. we plot the dispersion law for the gapless quasiparticle for $\cos \theta = 0$. The quark velocities $v_1, v_{11}$ and $v_{111}$ are given by the slopes of the curve near the gapless points.

2. **Sector** $A = (d_r, u_g)$

In this sector quasiparticles have dispersion laws as follows:

$$
|\epsilon^{(d_r, u_g)}_\pm| = |\pm \delta \mu + \sqrt{\left( p - \frac{\mu_2 + \mu_3}{2} \right)^2 + \Delta^2}|.
$$

(A11)
FIG. 4: (Color online) Dispersion laws for the quasiparticles given by linear combinations of $u_r$, $d_g$ and $s_b$ quarks. One of the three laws is gapless (red online) with three gapless momenta. The graph is obtained at $\mu = 500$ MeV, $m^2_s/\mu = 140$ MeV and for $\cos \vartheta = 0$, where $\cos \vartheta$ is the angle between the quark momentum and the pair momentum $2q$.

FIG. 5: (Color online) Solid and double dotted-dashed lines denote gapless dispersion laws (red and black online, respectively for $u_g - d_r$ and $u_b - s_r$), dashed and dot-dashed lines refer to gapped dispersion laws (green and blue online, respectively for $u_g - d_r$ and $u_b - s_r$). Values of the parameters as in Fig. 4.

where $z = \cos \vartheta$ and $\vartheta$ is the angle that the quark momentum form with the pair momentum. Since

$$\delta \mu = \frac{\mu_4 - \mu_5}{2} = \frac{y}{8} - qz = \frac{y}{8} \left( 1 - \frac{z}{z_q} \right),$$  \hspace{1cm} (A12)

with $y$ given in Eq. 4. Gapless modes are present for

$$\delta \mu^2 \equiv \left( \frac{y}{8} - qz \right)^2 > \Delta^2,$$  \hspace{1cm} (A13)

therefore the existence of gapless momenta depends on the value of $z = \cos \vartheta$. The dispersion law $\epsilon_+$ is gapless for $\delta \mu < 0$ whereas $\epsilon_-$ is gapless in the other case. The corresponding gapless momenta are

$$p^u_{\pm d_r} = \frac{\mu_4 + \mu_5}{2} \pm \sqrt{\frac{y^2}{64} \left( 1 - \frac{z}{z_q} \right)^2 - \Delta^2}.$$  \hspace{1cm} (A14)
Denoting the gapless dispersion laws with $\epsilon_g$, the mixing coefficients are given by

\[
B_{d_r} = \frac{\epsilon_g - p + \mu_5}{\sqrt{\Delta^2 + (\epsilon_g - p + \mu_5)^2}}, \quad B_{u_g} = \frac{\Delta}{\sqrt{\Delta^2 + (\epsilon_g - p + \mu_5)^2}}. \tag{A15}
\]

Their values at gapless momenta are

\[
B_{2d_r}(P_{u_g d_r}) = B_{2u_g}(P_{u_g d_r}) = \frac{1}{2} \left( 1 - \sqrt{\frac{\delta \mu^2 - \Delta^2}{\delta \mu}} \right), \tag{A16}
\]

\[
B_{2d_r}(P_{u_g d_r}) = B_{2u_g}(P_{u_g d_r}) = \frac{1}{2} \left( 1 + \sqrt{\frac{\delta \mu^2 - \Delta^2}{\delta \mu}} \right). \tag{A17}
\]

We notice that $\delta \mu > 0$ corresponds to $z < -z_q$ a condition that is true almost everywhere (except in a tiny region around the North pole).

3. Sector $A = (s_r, u_b)$

In this sector the quasiparticles have dispersion laws

\[
|\epsilon_{s_r, u_b}^\pm| = |\delta \mu \pm \sqrt{(p - \frac{\mu_6 + \mu_7}{2})^2 + \Delta^2}|. \tag{A18}
\]

In this case

\[
\delta \mu = \frac{\bar{\mu}_6 - \bar{\mu}_7}{2} = -\frac{y}{8} - q z = -\frac{y}{8} \left( 1 + \frac{z}{z_q} \right). \tag{A19}
\]

Gapless modes are present for

\[
\delta \mu^2 = \left( \frac{y}{8} + q z \right)^2 > \Delta^2. \tag{A20}
\]

The gapless momenta are

\[
P_{s_r u_b}^{\pm} = \frac{\mu_6 + \mu_7}{2} \pm \sqrt{\frac{y^2}{64} \left( 1 + \frac{z}{z_q} \right)^2 - \Delta^2}. \tag{A21}
\]

The values of the Bogoliubov coefficients at the gapless points are

\[
B_{s_r u_b}^2(P_{s_r u_b}^-) = B_{u_b}^2(P_{s_r u_b}^+) = \frac{1}{2} \left( 1 - \sqrt{\frac{\delta \mu^2 - \Delta^2}{\delta \mu}} \right), \tag{A22}
\]

\[
B_{s_r u_b}^2(P_{s_r u_b}^+) = B_{u_b}^2(P_{s_r u_b}^-) = \frac{1}{2} \left( 1 + \sqrt{\frac{\delta \mu^2 - \Delta^2}{\delta \mu}} \right). \tag{A23}
\]

Now $\delta \mu > 0$ corresponds to $z < -z_q$.

4. Sector $A = (s_g, d_b)$

There is no mixing in this sector because $\Delta_1 = 0$, therefore quasiparticles have dispersion laws

\[
|\epsilon_{s_g, d_b}^\pm| = |p - \mu_{8,9}|. \tag{A24}
\]

Gapless momenta are

\[
P_{s_g} = \mu_8 = \mu - \frac{5y}{12}, \tag{A25}
\]

\[
P_{d_b} = \mu_9 = \mu + \frac{y}{12}. \tag{A26}
\]
The mixing coefficients are $B_{ss}(P^u) = B_{dd}(P^d) = 1$ and the corresponding velocities are equal to unity.
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