Mathematical modeling on minimizing the waiting time in (i,j) preemptive priority queueing system

S Geetha and B Ramesh Kumar
1 Department of Mathematics, National Engineering College, Kovilpatti, Tamil Nadu, India
2 Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology, Avadi, Tamil Nadu, India
E-mail: mathsgeetha@gmail.com, brameshkumar@veltech.edu.in

Abstract

In this paper, we analyze the waiting time in two stage preemptive priority queueing system. The service time considered here is k-phase Erlang distribution. We have assign the priority server to the emergency customer without affecting the regular priority unit. It is possible only if the servers are independent. In this case, we derive the steady state probability of the waiting time customers in ordinary and priority unit. A numerical example is included.

1. Introduction

Due to some special situations the queueing model having some different type of customers. In general, priority system having two types as follows: (i) PP- Pre-emptive Priority (ii) NPP- Non – Pre-emptive Priority. The type (i) is a higher priority customer not interrupted to the entire service time of lower priority customer and otherwise it is called type (ii). Due to the situation the discipline of the priority model is not proper. However, in a situation we separate the service requirements are parallel for each class to improve the system performance level and minimize the waiting time. Kleinrock was first proposed the algorithm and analyzed the discipline of priority customer. Latter many researchers[1-16] have, followed by Kleinrock and proposed different algorithms for priority queueing system and also obtained the system performance. Here, we derive the waiting time probability of priority unit customer and derive the queue length of ordinary unit. In this connection, we consider the system in a general queueing system with k-phase, multi server; arriving customer considered in Poisson model and service time follows two units: (i) OU- Ordinary Unit (postponed) (ii) Priority unit (misplaced).

2. Model description on two server priority Queue

Consider the queueing system in which the service time is not available for immediate because the number of servers is busy and new arrivals are not allowed to begin the service even some channels are idle. Suppose the arrival expects to get an immediate service to a unit, it may either wait for service or else leave the system. Once a unit enters the service, it is not pre-empted. If \( N_i \neq N_j \) the units of the classes ‘i & j’ are served according to the discipline, ‘i’ having non-pre-emptive priority over ‘j’ if \( i < j \). Finally, a class of units queues according to the FIFO discipline. Thus, for a two – class priority process, the discipline of the break-off priority channels is specified as follows:
If less than $N_1$ channels are occupied then the services are taken whether priority unit or ordinary unit. Otherwise, numbers of channels are occupied in greater than $N_1$ the services given only be the priority. But it is not possible to pre-empt the service of low priority units in order to a higher priority unit.

2.1 Probability of waiting time for the Priority Unit

Here, we assumed the priority unit is lost when all the channels are busy. On the other hand the ordinary units wait in the queue when they arrive and find more than $N_1$ channels are busy. To identify the state of the system will require the number of busy channels and the number of ordinary units waiting in the queue. Let $q(m_2,n)$ denotes the steady state probability that there are $m_2$ ordinary units present in the queue and ‘n’ channels are busy.

let $q(^{*},n) = \sum_{m_2=0}^{\infty} q(m_2,n) \quad (1)$

$q(m_2,*) = \sum_{n=N_1}^{m_2} q(m_2,n) \quad (2) \quad m_2 \geq N_1$

The representativeness of the transition rates into out of the states of the system. The difference equation connecting the probabilities defined in (1) and (2) it can be written as from which we obtain the diagram.

$q(0,n) = \frac{\rho^N}{n!} e^{-2} \quad 0 \leq n \leq N_1 \quad (3)$ and $q(^{*},n) = \frac{N! q(^{*},N)}{n! \rho_1^{N-n}} N_1 \leq n \leq N \quad (4)$

Where $e^{-2} = q(0,0)$ is the probability of all the channels being idle.

But, $\lambda_2 q(m_2,*) = N_1 \mu q(m_2 + 1, N_1) \quad (5) \quad m_2 \geq 0$

Take summation of all values of $m_2$, we get $\lambda_2 q(m_2,*) = N_1 \mu q(m_2 + 1, N_1)$

$\lambda_2 \frac{N! q(^{*},N)}{\rho_1^N} \sum_{n=N_1}^{\infty} \frac{\rho^N}{n!} = N_1 \mu \left[ \frac{N! q(^{*},N)}{N_1! \rho_1^{N-N_1}} - \frac{\rho^N e^{-2}}{N_1!} \right] \quad (6)$

And hence

$q(^{*},N) = \frac{\rho^N}{\rho_1(N_1-1)!} \left[ \frac{(\rho_2 - N_1)}{\rho_2} \frac{N_1!}{\rho_1^{N-N_1}} + \sum_{n=0}^{N_1} \frac{\rho^N}{n!} \sum_{n=N_1}^{\infty} \frac{\rho^N}{n!} \right] e^{-2} \quad (7) \quad$ Where $\rho_2 = \frac{\lambda_2}{\mu}$
So that
\[ e^{-12} = \sum_{n=0}^{N} \rho^{n} - \rho^{N} \left\{ (\rho_{z} - N_{i}) \frac{\rho^{N}_{i}}{N_{i}!} + \sum_{n=N_{i}+1}^{N} \frac{\rho^{n}_{i}}{n!} \right\} \]
Thus, \( q^{(*)} \) lies between \( 0 \leq n \leq N_{i} \) and \( q^{(*)} \) lies between \( N_{i} \leq n \leq N \) are completely determined.

The equation (6) represents the loss of priority units and steady state probability that the ordinary unit waits is given by
\[ p(w_{2} > 0) = \frac{N!}{\rho^{N}_{i} q(N)} \sum_{n=N_{i}}^{N} \frac{\rho^{n}_{i}}{n!} \]

3. Derivation of Queue Length for the Ordinary Unit

If the number of \( N_{i} = N \) the priority unit is lost by the proportion of time \( N \) busy channel \( q(N) \).

Consider the arrival unit in both \( \lambda_{i} + \lambda_{z} = \lambda \) and the customer begin the service more than one (\( n > 1 \)). The steady state probability that the \( n \) servers are busy, the probability of all the servers idle denoted by \( e^{-2} \). From the diagram the approximate equation of steady state is as follows:
\[ \frac{\lambda e^{-2}}{\mu} = \mu q(1) \]

\[ -(\lambda + n \mu) q(n) + \lambda q(n-1) + (n+1) \mu q(n+1) = 0 \quad n < N_{i} \quad \text{---- (11)} \]
\[ -(\lambda + N_{i} \mu) q(N_{i}) + \lambda q(N_{i}-1) + (N_{i}+1) \mu q(N_{i}+1) = 0 \quad \text{---- (12)} \]
\[ -(\lambda + n \mu) q(n) + \lambda q(n-1) + (n+1) \mu q(n+1) = 0 \quad n > N_{i} \quad \text{---- (13)} \]
\[ N_{i} \mu q(N) = \lambda q(N-1) \quad \text{---- (14)} \]

From (10), \( q(1) = \frac{\lambda}{\mu} e^{-2} \) where represent the idle servers, it’s denoted as \( q(0) \), we get \( q(1) = \frac{\rho^{1}_{i}}{1!} q(0) \)

and in general \( q(n) = \frac{(\lambda / \mu)^{n}}{n!} q(0) \quad n \leq N_{i} \quad \text{---- (15)} \)

And \( q(n) = \frac{N!}{n! \rho^{N-n}_{i}} q(0) \quad n \leq N \quad \text{---- (16)} \)

Equating (15& 16), we get \( q(N) = \frac{\rho^{N}_{i} \rho^{N-N_{i}}_{i}}{N!} q(0) \quad \text{---- (17)} \)

Hence, \( N_{i} \leq n \leq N \) we get \( q(n) = \frac{\rho^{N}_{i} \rho^{N-N_{i}}_{i}}{n!} q(0) \quad \text{---- (18)} \)

and \( q(0) = \left[ \sum_{n=0}^{N} \frac{\rho^{n}_{i}}{n!} + \frac{\rho^{N}_{i}}{n!} \sum_{n=N_{i}+1}^{N} \frac{\rho^{n}_{i}}{n!} \right] \)

The steady state probability \( l_{1} \) that a probability unit is lost is given by the proportion of time \( N \) channels are busy, and is therefore given by \( q(N) \) for an ordinary unit becomes
\[ l_{2} = \sum_{n=N_{i}}^{N} q(N) = \frac{\rho^{N}_{i}}{\rho^{N}_{i}} \sum_{n=N_{i}}^{N} \frac{\rho^{n}_{i}}{n!} \quad \text{---- (20)} \]

4. Numerical Example

Consider a Breast Cancer center with a branch in a famous city. There is only one Chief Doctor who treats the patients after the consultation with junior doctors in preliminary 3-phases. The arrival pattern of the patients is assumed to have an Erlang 3-distribution with arrival rate \( \lambda \). The service time of the patients is assumed to have an exponential distribution with service rate \( \mu \).

Consider the range of the frequencies of transition from state \( i \) to state \( j \)
(i) \((\rho, \rho_1, \rho_2) = (0.6, 0.2, 0.3)\) (ii) \((\rho, \rho_1, \rho_2) = (0.7, 0.4, 0.5)\) (iii) \((\rho, \rho_1, \rho_2) = (0.8, 0.6, 0.7)\).

Find the probability of ordinary unit waiting time and also find the queue length of priority unit. Using (9) and (20), we obtained \(P(w_2 > 0)\) and \(l_2\). The table value as follows in various range

Table 1 compute the value of \(l_2\) in various ranges.

| \((\rho, \rho_1)\) | \(n!\) | \((N1,N)\) | \(q(0,2)\) | \(q(N)\) | \(L2\) |
|-----------------|-------|-----------|-----------|--------|-------|
| \((0.6,0.2)\)   |       |           |           |        |       |
| 1 (1,2)         | 1     | 0.60241   | 0.03614460| 0.66000|
| 2 (2,3)         | 2     | 0.55804   | 0.00669648| 0.19200|
| 3 (3,4)         | 3     | 0.55012   | 0.00099021| 0.03780|
| 4 (4,5)         | 4     | 0.54896   | 0.00011858| 0.00562|
| 5 (5,6)         | 5     | 0.54883   | 0.00001185| 0.33815|
| \((0.7,0.4)\)   |       |           |           |        |       |
| 1 (1,2)         | 1     | 0.54348   | 0.07608720| 0.84000|
| 2 (2,3)         | 2     | 0.50565   | 0.01651790| 0.27767|
| 3 (3,4)         | 3     | 0.498036  | 0.00284711| 0.0628 |
| 4 (4,5)         | 4     | 0.49678   | 0.00039759| 0.01080|
| 5 (5,6)         | 5     | 0.49661   | 0.00004637| 0.02420|
| \((0.8,0.6)\)   |       |           |           |        |       |
| 1 (1,2)         | 1     | 0.490196  | 0.11764704| 1.04000|
| 2 (2,3)         | 2     | 0.457875  | 0.02930400| 0.38400|
| 3 (3,4)         | 3     | 0.45083   | 0.00577062| 0.09813|
| 4 (4,5)         | 4     | 0.44955   | 0.0009206 | 0.01911|
| 5 (5,6)         | 5     | 0.449357  | 0.00012270| 0.00858|
Table 2 Compute the value of $p(w_2)$ in various ranges

| $(\rho_1, \rho_2, \rho_3)$ | n! | (N1,N) | e-2 | q(*,N) | q(0,N) | p(wq>0) |
|---------------------------|----|--------|-----|--------|--------|---------|
| (0.6, 0.2,0.3) | 1 | (1,2) | 1.98507 | 0.452294 | 0.35731260 | 0.975234 |
|                      | 2 | (2,3) | 1.8286 | 0.074806 | 0.06582960 | 0.196896 |
|                      | 3 | (3,4) | 1.7838 | 0.010527 | 0.00963252 | 0.221067 |
|                      | 4 | (4,5) | 1.82209 | 0.0012641 | 0.00118071 | 0.0328666 |
|                      | 5 | (5,6) | 1.82211 | 0.00011855 | 0.00011807 | 0.00367505 |
| (0.7,0.4,0.5) | 1 | (1,2) | 3.1 | 1.125185 | 0.75950000 | 0.75111 |
|                      | 2 | (2,3) | 2.08746 | 0.147629 | 0.11933313 | 0.2548465 |
|                      | 3 | (3,4) | 2.021999 | 0.023453 | 0.02022841 | 0.257983 |
|                      | 4 | (4,5) | 2.014657 | 0.0031615 | 0.00282170 | 0.04268025 |
|                      | 5 | (5,6) | 2.01384 | 0.000361 | 0.00032906 | 0.0005776 |
| (0.8,0.6,0.7) | 1 | (1,2) | 2.55558 | 6.927235 | 4.01778560 | 0.01801833 |
|                      | 2 | (2,3) | 2.462069 | 0.282641 | 0.21009655 | 0.695846 |
|                      | 3 | (3,4) | 2.254123 | 0.0473 | 0.03847037 | 0.36263333 |
|                      | 4 | (4,5) | 2.22911 | 0.007136 | 0.00608696 | 0.066602667 |
|                      | 5 | (5,6) | 2.223101 | 0.000921187 | 0.00080941 | 0.010133057 |

5. Conclusion

We consider the two stage k-phase priority queueing system for the purpose of minimizing the queue length. In this regard, derived the Waiting time probability of priority unit and Queue Length of Ordinary Unit. Table 1 and 2 has represented to minimize waiting time of length and the probability of waiting time in ordinary unit are very close to zero. So, our proposed method is easy to reduce the queue waiting time and also it’s helpful to the decision making problem.

References

[1] AlHanbali A, Alvarez. E M and Van der Heijden M C 2015, Approximations for the waiting-time distribution in an M/PH/c priority queue, OR Spectrum 37(2), 529-552.
[2] Afeche P and Mendelson H 2014, Pricing and priority auctions in queuing systems with a generalized delay cost structure, Management Sci. 50(7), 869–882.
[3] Ayyappan,G and Thamizhselvi P 2016, Priority Queueing System with a Single Server Serving Two Queues M[X1], M[X2]/G1, G2/1 with Balking and Optional Server Vacation, Applications and Applied Mathematics. 11(1), 61 – 82
[4] Bondi A B and Buzen J P 1981, The Response Times of Priority Classes under Preemptive Resume in M/G/ m Queues, Department of Computer Science Technical Reports. 331.
[5] Cobham A 1954, Priority assignment in waiting line problems, Operations Research. 2, 70-76.
[6] Davis R H 1966, Waiting-time distribution of a multi-server priority Queueing System, Operation Research. 14(1),133-136.
[7] Gross D and Harris C M 1985, Fundamentals of Queueing Theory, 2nd Edition. Wiley,New York.
[8] Haghighi A M and Mishev D 2006, A parallel priority queueing system with finite buffers, Journal of Parallel and Distributed Computing. 66, 379-392
[9] Hawkes A G 1956, Time dependent solution of a priority queue with bulk arrivals, 
*Operations Research*. **34**, pp. 191-198.

[10] Jinting Wang, Shiliang Cui and Zhongbin Wang 2019, Equilibrium Strategies in M/M/1 
Priority Queues with Balking, *Production and Operations Management*. **28(1)**, 43–62

[11] Jiang T and Liu L 2017, The GI/M/1 queue in a multi-phase service environment with disasters 
and working breakdowns. *Int. J. Comput. Math*. **94**, 707–726

[12] Kalidass K and Ramanath K 2012, A queue with working breakdowns. *Comput. Ind. Eng*. **63**, 
779–783.

[13] Thangaraj V and Vanitha S 2010, An M/G/1 queue with two-stage heterogeneous service 
compulsory server vacation and random breakdowns, *Int. J. Contemp. Math. Sciences*. 
**5(7)**, 307 – 322.

[14] Takagi H 1990. Time-dependent analysis of an M/G/1 vacation models with exhaustive 
service, *Queueing Systems*. **6(1)**, 369 – 390.