Short-time behaviour of a modified Laplacian coflow of $G_2$-structures

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Abstract

We modify the Laplacian coflow of co-closed $G_2$-structures - $\frac{d}{dt} \psi = \Delta \psi$ where $\psi$ is the closed dual 4-form of a $G_2$-structure $\phi$. The modified flow is now weakly parabolic in the direction of closed forms up to diffeomorphisms. We then prove short time existence and uniqueness of solutions to the modified flow.

1 Introduction

Ever since the Ricci flow has been introduced by Richard Hamilton [15], geometric flows have played an important role in the study of geometric structures. The general idea is to begin with some general geometric structure on a manifold and then use some flow to obtain a more special structure. In the study of $G_2$-structures on 7-dimensional manifolds, an important question is under what conditions is it possible to obtain a torsion-free $G_2$-structure (which corresponds to manifolds with holonomy contained in $G_2$) from a $G_2$-structure with some other torsion class. To do this, one can either attempt a non-infinitesimal deformation of the $G_2$-structure [11, 12, 19], or one can try to construct a flow which interpolates between different torsion classes. The first such flow has been proposed by Bryant [3] - if we start with a closed $G_2$-structure, that is one for which the defining 3-form $\phi$ satisfies

$$d\phi = 0$$

then the Laplacian $\Delta \phi = dd^* \phi$ is an exact form, and hence Laplacian flow of $\phi$

$$\frac{d\phi}{dt} = \Delta \phi$$

preserves the cohomology class, and we get a flow of closed $G_2$-structures within the same cohomology class. Here $\Delta \phi$ denotes the Hodge Laplacian with respect to the metric $g_\phi$ associated to the $G_2$-structure $\phi$.

Suppose now $M$ is a compact 7-dimensional manifold with a $G_2$-structure $\phi$. It is then possible to interpret the flow (1.1) as a gradient flow of the volume functional $V$ [4]

$$V(\phi) = \frac{1}{7} \int_M \phi \wedge *\phi$$

(1.2)
Then, the functional $V$ increases monotonically along this flow. As shown by Hitchin \cite{16}, for a closed $G_2$-structure $\varphi$, $V$ attains a critical point within the fixed cohomology class of $\varphi$ whenever $d \ast \varphi = 0$, that is, if and only if $\varphi$ defines a torsion-free $G_2$-structure. Therefore, it is to be expected that if a long-time smooth solution to (1.1) exists, then it should converge to a torsion-free $G_2$-structure. In \cite{4, 25} it is shown that after applying a version of the DeTurck Trick (named after DeTurck’s proof of short time existence and uniqueness of Ricci flow solutions \cite{7}), the flow (1.1) can be related to a modified flow that is parabolic along closed forms. This was then used to prove short time existence and uniqueness of solutions. Moreover, in \cite{25}, it was shown that if the initial closed $G_2$-structure $\varphi_0$ is near a torsion-free structure $\varphi_1$ then the flow (1.1) converges to a torsion-free $G_2$-structure $\varphi_\infty$ which is related to $\varphi_1$ via a diffeomorphism.

Other flows of $G_2$-structures have also been proposed. In \cite{20}, Karigiannis studied the properties of general flows of $\varphi$ by an arbitrary 3-form. In \cite{23, 24}, Weiss and Witt made significant progress while studying gradient flows of Dirichlet-type functionals for $G_2$-structures. They have obtained short-time existence and uniqueness results as well as long-time convergence to a torsion-free $G_2$-structure if the initial condition is sufficiently close to a torsion-free $G_2$-structure, i.e. stability of the flow.

Most of the flows studied focused on flowing the 3-form $\varphi$. However, a $G_2$-structure can also be defined by the dual 4-form $\ast \varphi$, which we will denote by $\psi$. It is then natural to consider the analogue of the Laplacian flow (1.1), but for the 4-form. Such a flow,

$$\frac{d\psi}{dt} = \Delta_\psi \psi$$

(named the Laplacian coflow of $G_2$-structures, has originally been proposed by Karigiannis, McKay and Tsui in \cite{21}). Here $\Delta_\psi$ is the Hodge Laplacian defined by the metric $g_\psi$ which is the metric associated to the $G_2$-structure defined by $\psi$. Note that in \cite{21}, there was a minus sign on the right hand side of (1.3). The flow (1.3) shares a number of properties with (1.1). In particular, if we start with $\psi$ closed, that is, a co-closed $G_2$-structure, then (1.3) preserves the cohomology class of $\psi$. Thus we get a flow of co-closed $G_2$-structures. The volume functional can be restated in terms of $\psi$, and then it attains a critical point within the cohomology class of $\psi$ whenever $d \ast \psi = d\varphi = 0$. The flow (1.3) can then also be interpreted as a gradient flow of the volume functional, and it is easy to see that the volume grows monotonically along this flow. While qualitatively some of the properties are similar to the Laplacian flow on 3-forms, the initial conditions are completely different - in (1.1) we start from a closed $G_2$-structure, while in (1.3), we start from a co-closed $G_2$-structure.

In this paper we study the analytical properties of the flow (1.3). It turns out that despite the similarities with the 3-form flow (1.1), the 4-form flow cannot be related to a flow that is strictly parabolic in the direction of closed forms using diffeomorphisms. Therefore, we propose a modified version of (1.3), given by

$$\frac{d\psi}{dt} = \Delta_\psi \psi + 2d((A - \text{Tr} T) \varphi)$$

(1.4)

Here $\text{Tr} T$ is the trace of the full torsion tensor $T$ of the $G_2$-structure defined by $\psi$, and $A$ is a positive constant. This flow is now weakly parabolic in the direction of closed forms and hence it is possible to relate it to a strictly parabolic flow using an application of deTurck’s trick. The flow (1.4) still preserves the cohomology class of $\psi$ and if $\text{Tr} T$ is small enough in some sense, the volume functional grows along the flow. We then use the techniques from (4) to show short-time existence and uniqueness for this flow.
The outline of the paper is as follows. In Section 2 we give a brief overview of the properties of $G_2$-structures and their torsion. In Section 3 we consider the deformations of $G_2$-structures in terms of deformations of the 4-form $\psi$, and in Section 4 we consider the properties of $\Delta \psi \psi$, including its linearization. The modified flow $(1.4)$ is then defined in Section 5, and in Section 6 we prove the short time existence and uniqueness of solutions of $(1.4)$. For that we adapt the techniques used by Bryant and Xu in [4] for the Laplacian flow of $\varphi$.

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2 $G_2$-structures and torsion

The 14-dimensional group $G_2$ is the smallest of the five exceptional Lie groups and is closely related to the octonions. In particular, $G_2$ can be defined as the automorphism group of the octonion algebra. Taking the imaginary part of octonion multiplication of the imaginary octonions defines a vector cross product on $V = \mathbb{R}^7$ and the group that preserves the vector cross product is precisely $G_2$. A more detailed account of the relationship between octonions and $G_2$ can be found in [1, 10]. The structure constants of the vector cross product define a 3-form on $\mathbb{R}^7$, hence $G_2$ can alternatively be defined as the subgroup of $GL(7, \mathbb{R})$ that preserves a particular 3-form \cite{18}. In general, given an $n$-dimensional manifold $M$, a $G$-structure on $M$ for some Lie subgroup $G$ of $GL(n, \mathbb{R})$ is a reduction of the frame bundle $F$ over $M$ to a principal subbundle $P$ with fibre $G$. A $G_2$-structure is then a reduction of the frame bundle on a 7-dimensional manifold $M$ to a $G_2$ principal subbundle. It turns out that there is a 1-1 correspondence between $G_2$-structures on a 7-manifold and smooth 3-forms $\varphi$ for which the 7-form-valued bilinear form $B_\varphi$ as defined by (2.1) is positive definite (for more details, see [2] and the arXiv version of [17]).

$$B_\varphi (u, v) = \frac{1}{6} \left( u \lrcorner \varphi \right) \wedge \left( v \lrcorner \varphi \right) \wedge \varphi$$

Here the symbol $\lrcorner$ denotes contraction of a vector with the differential form:

$$(u \lrcorner \varphi)_{mn} = u^a \varphi_{amn}.$$ Note that we will also use this symbol for contractions of differential forms using the metric.

A smooth 3-form $\varphi$ is said to be positive if $B_\varphi$ is the tensor product of a positive-definite bilinear form and a nowhere-vanishing 7-form. In this case, it defines a unique metric $g_\varphi$ and volume form $\text{vol}$ such that for vectors $u$ and $v$, the following holds

$$g_\varphi (u, v) \text{vol} = \frac{1}{6} \left( u \lrcorner \varphi \right) \wedge \left( v \lrcorner \varphi \right) \wedge \varphi$$

In components we can rewrite this as

$$(g_\varphi)_{ab} = (\det s)^{-\frac{1}{2}} s_{ab} \text{ where } s_{ab} = \frac{1}{144} \varphi_{amn} \varphi_{bqp} \varphi_{rst} \hat{\varepsilon}^{mnpqrst}.$$ Here $\hat{\varepsilon}^{mnpqrst}$ is the alternating symbol with $\hat{\varepsilon}^{12...7} = +1$. Following Joyce (\cite{18}), we will adopt the following definition

**Definition 2.1** Let $M$ be an oriented 7-manifold. The pair $(\varphi, g)$ for a positive 3-form $\varphi$ and corresponding metric $g$ defined by (2.2) will be referred to as a $G_2$-structure.
Since a $G_2$-structure defines a metric and an orientation, it also defines a Hodge star. Thus we can construct another $G_2$-invariant object - the 4-form $\ast \varphi$. Since the Hodge star is defined by the metric, which in turn is defined by $\varphi$, the 4-form $\ast \varphi$ depends non-linearly on $\varphi$. For convenience we will usually denote $\ast \varphi$ by $\psi$. We can also write down various contraction identities for a $G_2$-structure $(\varphi, g)$ and its corresponding 4-form $\psi$ [3, 13, 20].

**Proposition 2.2** The 3-form $\varphi$ and the corresponding 4-form $\psi$ satisfy the following identities:

\[
\begin{align*}
\varphi_{abc} \varphi_{mn}^c &= g_{am} g_{bn} - g_{an} g_{bm} + \psi_{abmn} \quad (2.4a) \\
\varphi_{abc} \psi_{mnp}^c &= 3 \left( g_{a[m} \varphi_{np]} b - g_{b[m} \varphi_{np]} a \right) \quad (2.4b) \\
\psi_{abcd} \psi^{mnpq} &= 24 \delta_{a[m} \delta_{b} \delta_{c} \delta_{d]} + 72 \psi_{[ab} \delta_{c} \delta_{d]} - 16 \varphi_{[abc} \varphi^{[mnp] \delta_{d]} \delta_{a]} \quad (2.4c)
\end{align*}
\]

where $[m n p]$ denotes antisymmetrization of indices and $\delta_{a}^{b}$ is the Kronecker delta, with $\delta_{a}^{a} = 1$ if $a = b$ and 0 otherwise.

The above identities can be of course further contracted - the details can be found in [13, 20]. These identities and their contractions are crucial whenever any tensorial calculations involving $\varphi$ and $\psi$ have to be done.

For a general $G$-structure, the spaces of $p$-forms decompose according to irreducible representations of $G$. Given a $G_2$-structure, 2-forms split as $\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14}$, where

$$\Lambda^2_7 = \{ \alpha, \varphi: \text{for a vector field } \alpha \}$$

and

$$\Lambda^2_{14} = \{ \omega \in \Lambda^2: (\omega_{ab}) \in \mathfrak{g}_2 \} = \{ \omega \in \Lambda^2: \omega, \varphi = 0 \} .$$

The 3-forms split as $\Lambda^3 = \Lambda^3_7 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$, where the one-dimensional component consists of forms proportional to $\varphi$, forms in the 7-dimensional component are defined by a vector field $\Lambda^3_7 = \{ \alpha, \psi: \text{for a vector field } \alpha \}$, and forms in the 27-dimensional component are defined by traceless, symmetric matrices:

$$\Lambda^3_{27} = \left\{ \chi \in \Lambda^3: \chi_{abc} = i_\varphi (h) = h_{[a}^d \varphi_{bc]}^d \text{ for } h_{ab} \text{ traceless, symmetric} \right\} .$$

By Hodge duality, similar decompositions exist for $\Lambda^4$ and $\Lambda^5$. In particular, we can define the $\Lambda^4_{27}$ component as

$$\Lambda^4_{27} = \left\{ \chi \in \Lambda^4: \chi_{abcd} = \ast i_\varphi (h) = -\frac{4}{3} h_{[a}^d \psi_{|e|bc]}^d \text{ for } h_{ab} \text{ traceless, symmetric} \right\} .$$

A detailed description of these representations is given in [2, 3]. Also, formulae for projections of differential forms onto the various components are derived in detail in [11, 13, 20]. Note that it is sometimes convenient to consider the 1 and 27-dimensional components together - then in (2.5) and (2.6) we simply drop the condition for $h$ to be traceless. The only difference is that for an arbitrary symmetric $h$,

\[
(\ast i_\varphi (h))_{abcd} = -\frac{4}{3} h_{[a}^d \psi_{|e|bc]}^d + \frac{1}{3} (\text{Tr } h) \psi_{abcd} \quad (2.7)
\]
Also define the operators \( \pi_1, \pi_7, \pi_{14} \) and \( \pi_{27} \) to be the projections of differential forms onto the corresponding representations. Sometimes we will also use \( \pi_{1\oplus 27} \) to denote the projection of 3-forms or 4-forms into \( \Lambda^1_7 \oplus \Lambda^3_7 \) or \( \Lambda^1_1 \oplus \Lambda^1_27 \) respectively. For convenience, when writing out projections of forms, we will sometimes just give the vector that defines the 7-dimensional component, the function that defines the 1-dimensional component or the symmetric 2-tensor that defines the 1 \( \oplus \) 27 component whenever there is no ambiguity. For instance, \( \pi_1 (f \varphi) = f \)
\( \pi_7 (X \varphi)^a = X^a \)
\( \pi_{1\oplus 27} (1 \varphi (h))_{ab} = h_{ab} \)
\( \pi_1 (f \psi) = f \)
\( \pi_7 (X \psi)^a = X^a \)
\( \pi_{1\oplus 27} (\star_1 \varphi (h))_{ab} = h_{ab} \)
\( \pi_7 (X \land \varphi)^a = X^a \)
\( (2.8) \)

The intrinsic torsion of a \( G_2 \)-structure is defined by \( \nabla \varphi \), where \( \nabla \) is the Levi-Civita connection for the metric \( g \) that is defined by \( \varphi \). Following [20], it is easy to see
\( \nabla \varphi \in \Lambda^1_7 \otimes \Lambda^3_7 \cong W. \)
\( (2.9) \)

Here we define \( W \) as the space \( \Lambda^1_7 \otimes \Lambda^3_7 \). Given (2.9), we can write
\( \nabla_a \varphi_{bcd} = T_a^e \psi_{ebcd} \)
\( (2.10) \)

where \( T_{ab} \) is the full torsion tensor. Similarly, we can also write
\( \nabla_a \psi_{bcde} = -4T_a[b \psi_{cde}] \)
\( (2.11) \)

We can also invert (2.10) to get an explicit expression for \( T \)
\( T^m_a = \frac{1}{24} (\nabla_a \varphi_{bcd}) \psi^{mbcd} \)
\( (2.12) \)

This 2-tensor fully defines \( \nabla \varphi \) since pointwise, it has 49 components and the space \( W \) is also 49-dimensional (pointwise). In general we can split \( T_{ab} \) according to representations of \( G_2 \) into torsion components:
\( T = \tau_1 g + \tau_7 \varphi + \tau_{14} + \tau_{27} \)
\( (2.13) \)

where \( \tau_1 \) is a function, and gives the 1 component of \( T \). We also have \( \tau_7 \), which is a 1-form and hence gives the 7 component, and, \( \tau_{14} \in \Lambda^2_{14} \) gives the 14 component and \( \tau_{27} \) is traceless symmetric, giving the 27 component. Hence we can split \( W \) as
\( W = W_1 \oplus W_7 \oplus W_{14} \oplus W_{27}. \)
\( (2.14) \)

As it was originally shown by Fernández and Gray [8], there are in fact a total of 16 torsion classes of \( G_2 \)-structures that arise as the \( G_2 \)-invariant subspaces of \( W \) to which \( \nabla \varphi \) belongs. Moreover, as shown in [20], the torsion components \( \tau_i \) relate directly to the expression for \( d\varphi \) and \( d\psi \). In fact, in our notation,
\( d\varphi = 4\tau_1 \psi - 3\tau_7 \land \varphi - 3 * i_\varphi (\tau_{27}) \)
\( (2.15a) \)
\( d\psi = -4\tau_7 \land \psi - 2 * \tau_{14}. \)
\( (2.15b) \)

Note that in the literature ([3, 4], for example) a slightly different convention for torsion components is sometimes used. Our \( \tau_1 \) then corresponds to \( \frac{1}{4} \tau_0 \), \( \tau_7 \) corresponds to \( -\tau_1 \) in their
notation, \( i_\varphi (\tau_{27}) \) corresponds to \(-\frac{1}{3} \tau_3\) and \( \tau_{14} \) corresponds to \(\frac{1}{2} \tau_2\). Similarly, our torsion classes \( W_1 \oplus W_7 \oplus W_{14} \oplus W_{27} \) correspond to \( W_0 \oplus W_1 \oplus W_2 \oplus W_3 \).

An important special case is when the \( G_2 \)-structure is said to be torsion-free, that is, \( T = 0 \). This is equivalent to \( \nabla \varphi = 0 \) and also equivalent, by Fernández and Gray, to \( d \varphi = d \psi = 0 \). Moreover, a \( G_2 \)-structure is torsion-free if and only if the holonomy of the corresponding metric is contained in \( G_2 \). The holonomy group is then precisely equal to \( G_2 \) if and only if the fundamental group \( \pi_1 \) is finite.

If \( d \varphi = 0 \), then we say \( \varphi \) defines a closed \( G_2 \)-structure. Each of the torsion components in (2.15a) has to vanish separately, so \( \tau_1, \tau_7 \) and \( \tau_{27} \) are all zero, and the only non-zero torsion component remaining is \( \tau_{14} \). This class of \( G_2 \)-structures has been important in the study of Laplacian flows of \( \varphi \) [3,4,25]. If instead, \( d \psi = 0 \), then we say that we have a co-closed \( G_2 \)-structure. In this case, \( \tau_7 \) and \( \tau_{14} \) vanish in (2.15b) and we are left with \( \tau_1 \) and \( \tau_{27} \) components. In particular, the torsion tensor \( T_{ab} \) is now symmetric. In this paper we will mostly be concerned with co-closed \( G_2 \)-structures.

We will also require a number of differential identities for differential forms on manifolds with a co-closed \( G_2 \)-structure. In many ways this is the explicit version of Bryant’s exterior derivative identities [3], but these identities will be useful for us later on. The projections are defined as in (2.8).

Throughout the paper we will be using the following notation. Given a \( p \)-form \( \omega \), the rough Laplacian is defined by

\[
\nabla^2 \omega = g^{ab} \nabla_a \nabla_b \omega = -\nabla^* \nabla \omega. \tag{2.16}
\]

For a vector field \( X \), define the divergence of \( X \) as

\[
\text{div } X = \nabla_a X^a \tag{2.17}
\]

This operator can be extended to a symmetric 2-tensor \( h \)

\[
(\text{div } h)_b = \nabla^a h_{ab} \tag{2.18}
\]

Also, for a vector \( X \), we can use the \( G_2 \)-structure 3-form \( \varphi \) to define a “curl” operator, similar to the standard one on \( \mathbb{R}^3 \):

\[
(\text{curl } X)^a = (\nabla_b X_c) \varphi^{abc} \tag{2.19}
\]

This curl operator can then also be extended to 2-tensor. Given a 2-tensor \( \beta_{ab} \),

\[
(\text{curl } \beta)_{ab} = (\nabla_m \beta_{an}) \varphi^{mn}_{b} \tag{2.20}
\]

From the context it will be clear whether the curl operator is applied to a vector or a 2-tensor. Note that when \( \beta_{ab} \) is symmetric, curl \( \omega \) is traceless. As in [9], we can also use the \( G_2 \)-structure 3-form to define a product \( \alpha \circ \beta \) of two 2-tensors \( \alpha \) and \( \beta \)

\[
(\alpha \circ \beta)_{ab} = \varphi_{amn} \varphi_{pq} \alpha^{mp} \beta^{nq} \tag{2.21}
\]

While the product in (2.21) can be defined for any 2-tensors on a manifold with a \( G_2 \)-structure, for us it will be most useful when restricted to symmetric tensors. It is then easy to see that (2.21) defines a commutative product on the space of symmetric 2-tensors. It is however non-associative, and so defines a non-trivial non-associative algebra on symmetric 2-tensors. For convenience, we will define a standard inner product on symmetric 2-tensors

\[
\langle \alpha, \beta \rangle = \alpha_{ab} \beta_{mn} g^{am} g^{bm} \tag{2.22}
\]
Proposition 2.3 Suppose we have a co-closed $G_2$-structure on a manifold $M$ with 3-form $\varphi$ and dual 4-form $\psi$. Let $\chi \in \Lambda^3$ be given by

$$\chi = X \lrcorner \psi + 3i\varphi(h) \quad (2.23)$$

then, the co-differential $d^* \chi$ is given by

$$(d^* \chi)_{bc} = -((\text{div } h) \lrcorner \varphi)_{bc} - 2(\text{curl } h)_{[bc]} + \nabla_m X_n \psi^{mn}_{bc}$$

$$- (\text{Tr } T) X^a \varphi_{abc} + T_{mn} X^m \varphi^m_{bc} + 2X_m T_n[\psi^{mn}_{bc}] - T_m h^{np} \psi_{mnb} \quad (2.24a)$$

$$(\pi_7 d^* \chi)_a = \frac{2}{3} \left((\text{curl } X)_a - (\text{div } h)_a - \frac{1}{2} \nabla_a \text{Tr } h\right)$$

$$- X_a \text{Tr } T + T_{ab} X^b - \varphi_{abc} T^b_d h^{dc} \quad (2.24b)$$

The type decomposition of the exterior derivative $d\chi$ is

$$\pi_1 d\chi = \frac{4}{7} \left(\text{div } X + \frac{1}{2} \text{Tr } T \text{Tr } h - \frac{1}{2} T_{ab} s^{ab}\right) \quad (2.25a)$$

$$\pi_7 d\chi = \frac{1}{2} \left(\nabla_a \text{Tr } h - (\text{div } h)_a - (\text{curl } X)_a - 2T_{ab} X^b\right) \quad (2.25b)$$

$$\pi_{1\oplus 27} d\chi = 3 \left(-\nabla_{(a} X_{b)} + (\text{curl } h)_{(ab)} + \frac{1}{3} (\text{div } X) g_{ab}\right)$$

$$+ \frac{1}{2} (T \circ h)_{ab} + T_m(a) h^m - \frac{1}{2} (\text{Tr } h) T_{ab}$$

$$- \frac{1}{2} (\text{Tr } T) h_{ab} - \frac{1}{6} g_{ab} (\text{Tr } T) (\text{Tr } h) + \frac{1}{6} (h, T) g_{ab} \quad (2.25c)$$

Also, up to the lower order terms involving the torsion, the $\Lambda^1_1$ and $\Lambda^3_7$ components of $dd^* \chi$ are given by

$$\pi_1 dd^* \chi = -\frac{2}{7} \left(\text{div } (\text{div } h) + \frac{1}{2} \nabla^2 \text{Tr } h\right) + l.o.t. \quad (2.26a)$$

$$\pi_7 dd^* \chi = \frac{1}{2} \left(\nabla_a (\text{div } X) - \nabla^2 X_a - \text{curl } (\text{div } h)_a\right) + l.o.t. \quad (2.26b)$$

Similarly, up to the lower order terms, the $\Lambda^1_3$ and $\Lambda^3_3$ components of $d^* d\chi$ are given by

$$\pi_1 d^* d\chi = \frac{2}{7} \left(\text{div } h - \nabla^2 \text{Tr } h\right) + l.o.t. \quad (2.27a)$$

$$\pi_7 d^* d\chi = \frac{1}{2} \left(\text{curl } (\text{div } h)_a - \nabla_a (\text{div } X) - \nabla^2 X_a\right) + l.o.t. \quad (2.27b)$$

Proof. These identities are found just by manipulating $G_2$ representation components using contraction identities between $\varphi$ and $\psi$. For the second order identities in order to isolate the highest order terms we note that $\pi_7 (\text{Riem})$ and the Ricci tensor are expressed solely in terms of the full torsion tensor $[20]$. □
3 Deformations of $\psi$

Usually deformations of a $G_2$-structure are done via deformations of the 3-form $\varphi$, and from that deformations of associated quantities - $g$, $\psi$ and the torsion are calculated. In particular, infinitesimal deformations of all these quantities have been written down in [20], while the general non-infinitesimal expression were derived in [11]. Since we will be considering a flow of $\psi$, we need to re-derive all the infinitesimal results from [20] using a deformation of $\psi$ as a starting point. Let $(\varphi, g)$ be a $G_2$-structure. Using the metric, define $\psi = *\varphi$. Suppose $\chi \in \Lambda^3$, then $*\chi \in \Lambda^4$. Consider a deformation of the $G_2$-structure via a deformation of $\psi$:

$$\psi \rightarrow \tilde{\psi} = \psi + *\chi \quad (3.1)$$

Assuming $\tilde{\psi}$ remains a positive 4-form, it defines a new $G_2$-structure $(\tilde{\varphi}, \tilde{g})$. The 3-form $\tilde{\varphi}$ is given by:

$$\tilde{\varphi}_{abc} = \frac{1}{(3!)^2} \frac{1}{\sqrt{\text{det} \tilde{g}}} \tilde{\varphi}_{mnpqrst} (\psi_{qrs} + *\chi_{qrs}) \tilde{g}_{ma} \tilde{g}_{nb} \tilde{g}_{pc}$$

$$= \left( \frac{\text{det} g}{\text{det} \tilde{g}} \right) \left( \varphi_{mnp} + \chi_{mnp} \right) \tilde{g}_{ma} \tilde{g}_{nb} \tilde{g}_{pc} \quad (3.2)$$

In particular,

$$\tilde{\varphi}^{\tilde{a}\tilde{b}\tilde{c}} = \left( \frac{\text{det} g}{\text{det} \tilde{g}} \right) \left( \varphi_{mnp} + \chi_{mnp} \right) \quad (3.3)$$

where the raised indices with tildes are raised with the new inverse metric $\tilde{g}^{-1}$. The new metric can be found via the following identity. For any $G_2$-structure $(\varphi, g)$, from (2.4) we find that

$$\psi_{amnp} \psi_{bqrs} \tilde{\varphi}_{mnq} \tilde{\varphi}_{prs} = 16 \varphi_{ap} \varphi_{bp} = -96 g_{ab}$$

Therefore for $(\tilde{\varphi}, \tilde{g})$, we have

$$\tilde{g}_{ab} = -\frac{1}{96} \psi_{amnp} \psi_{bqrs} \tilde{\varphi}_{mnq} \tilde{\varphi}_{prs}$$

$$= -\frac{1}{96} \left( \frac{\text{det} g}{\text{det} \tilde{g}} \right) \left( \psi_{amnp} + *\chi_{amnp} \right) \left( \psi_{bqrs} + *\chi_{bqrs} \right) \left( \varphi_{mnq} + \chi_{mnq} \right) \left( \varphi_{prs} + \chi_{prs} \right) \quad (3.4)$$

Without the determinant factor, this is a 4th order expression in $\chi$. We can also obtain an expression for the inverse metric. Again, from (2.4), we have

$$\varphi^{amn} \varphi^{bpq} \psi_{mnpq} = 4 \varphi^{amn} \varphi^{bmn} = 24 g^{ab}$$

Therefore for $(\tilde{\varphi}, \tilde{g})$, we have

$$\tilde{g}^{\tilde{a}\tilde{b}} = \frac{1}{24} \tilde{\varphi}^{\tilde{a}\tilde{m}\tilde{n}} \tilde{\varphi}^{\tilde{b}\tilde{p}\tilde{q}} \psi_{mnpq}$$

$$= \frac{1}{24} \left( \frac{\text{det} g}{\text{det} \tilde{g}} \right) \left( \varphi^{amn} + \chi^{amn} \right) \left( \varphi^{bpq} + \chi^{bpq} \right) \left( \psi_{mnpq} + *\chi_{mnpq} \right) \quad (3.5)$$
Suppose now $\chi$ is given by

$$\chi = X \cdot \psi + 3i \varphi (h) \quad (3.6)$$

where $v$ is vector and $h$ is a symmetric 2-tensor. Also suppose that we have a one-parameter family $\psi (t)$ given by

$$\frac{d}{dt} \psi = -X \wedge \varphi + 3 \ast i \varphi (h) \quad (3.7)$$

Then the evolution of 3-form, metric, the inverse metric and the volume form is given in the following Proposition.

**Proposition 3.1** Under the flow (3.7), the evolution of related objects is given by:

$$\frac{d}{dt} \sqrt{\det g} = \frac{3}{4} (\text{Tr} h) \sqrt{\det g} \quad (3.9a)$$

$$\frac{d}{dt} g_{ab} = \frac{1}{2} (\text{Tr} h) g_{ab} - 2 h_{ab} \quad (3.9b)$$

$$\frac{d}{dt} \varphi = \frac{3}{4} \pi_1 \chi + \pi_7 \chi - \pi_27 \chi \quad (3.9d)$$

$$\frac{d}{dt} \varphi = \frac{3}{4} \pi_1 \chi + \pi_7 \chi - \pi_27 \chi$$

$$\frac{d}{dt} \varphi = \frac{3}{4} (\text{Tr} h) \varphi + v \cdot \psi - 3i \varphi (h)$$

**Proof.** From (3.4) we find that under the flow (3.7), $g(t)$ is given by

$$g_{ab} (t) = -\frac{1}{96} \left( \frac{\det g}{\det g(t)} \right) (\psi_{anmp} + t \ast \chi_{anmp}) (\psi_{bqrs} + t \ast \chi_{bqrs}) \quad (3.10)$$

$$\times (\varphi^{mnq} + t \chi^{mnq}) (\varphi^{prs} + t \chi^{prs}) + O (t^2)$$

where $g(0) = g$. Now let us find $\frac{d}{dt} \big|_{t=0} g(t)$:

$$\frac{d}{dt} \bigg|_{t=0} g(t) = \left( \frac{d}{dt} \bigg|_{t=0} \det g(t) \right)^{-1} \left( -\frac{\det g}{96} \right) \psi_{anmp} \psi_{bqrs} \varphi^{mnq} \varphi^{prs}$$

$$- \frac{1}{96} \ast X_{anmp} \psi_{bqrs} \varphi^{mnq} \varphi^{prs} - \frac{1}{96} \ast X_{bqrs} \psi_{anmp} \varphi^{mnq} \varphi^{prs}$$

$$- \frac{1}{96} \psi_{anmp} \psi_{bqrs} \chi^{mnq} \chi^{prs} - \frac{1}{96} \psi_{anmp} \psi_{bqrs} \varphi^{mnq} \chi^{prs}$$

$$= -\frac{1}{24} \chi_{anmp} \varphi_{bq} \varphi_{pq} + \frac{1}{24} X_{bqrs} \varphi_{ap} \varphi^{pq} - \frac{1}{24} \psi_{anmp} \psi_{bqrs} \varphi_{pq}$$

$$= -g_{ab} (\det g)^{-1} \left( \frac{d}{dt} \bigg|_{t=0} \det g(t) \right) - \frac{1}{24} \chi_{anmp} \varphi_{bq} \varphi_{pq}$$

where we have used a contracted version of the identity (2.3b). Now if we substitute (3.6) into (3.11), and simplify further using the identities (2.4), we will find that

$$\frac{d}{dt} \bigg|_{t=0} g(t) = - (\det g)^{-1} \left( \frac{d}{dt} \bigg|_{t=0} \det g(t) \right) g_{ab} + 2 (\text{Tr} h) g_{ab} - 2 h_{ab} \quad (3.12a)$$
and similarly, for the inverse metric,

\[
\left. \frac{d}{dt} \right|_{t=0} g(t)^{ab} = -(\det g)^{-1} \left( \left. \frac{d}{dt} \right|_{t=0} \det g(t) \right) g^{ab} + (\Tr h) g^{ab} + 2h^{ab} \tag{3.12b}
\]

Now however, using the fact that the derivative of a determinant gives the trace, we get that

\[
\left( \left. \frac{d}{dt} \right|_{t=0} \det g(t) \right) = -7 \left( \left. \frac{d}{dt} \right|_{t=0} \det g(t) \right) + 12 \det g \Tr h
\]

Hence,

\[
\left( \left. \frac{d}{dt} \right|_{t=0} \det g(t) \right) = \frac{3}{2} \det g \Tr h \tag{3.13}
\]

Therefore, substituting this into (3.12), we obtain (3.9b) and (3.9c). Also, from (3.13) we immediately obtain the expression for \( \frac{d}{dt} \phi \) (3.9a). Now to compute \( \frac{d}{dt} \phi \), we first find from (3.2) that under the flow (3.7)

\[
\phi(t)^{abc} = \left( \frac{\det g}{\det g(t)} \right)^{\frac{1}{2}} (\phi^{mnp} + t\chi^{mnp}) g(t)^{ma} g(t)^{nb} g(t)^{pc} + O(t^2)
\]

Hence,

\[
\left. \frac{d}{dt} \right|_{t=0} \phi(t)^{abc} = -\frac{1}{2} (\det g)^{-1} \left( \left. \frac{d}{dt} \right|_{t=0} \det g(t) \right) \phi^{abc}
\]

\[
+ \chi^{abc} + 3 \left. \frac{d}{dt} \right|_{t=0} g(t)^{ma} \phi^{mb}_{bc}
\]

\[
= \frac{3}{4} (\Tr h) \phi^{abc} + \chi^{abc} + 3i\phi \left( \frac{1}{2} (\Tr h) g - 2h \right)_{abc} \tag{3.14}
\]

This can be rewritten as

\[
\left. \frac{d}{dt} \right|_{t=0} \phi(t) = \frac{3}{4} (\Tr h) \phi + X_a \psi - 3i\phi \phi(h)
\]

\[
= \frac{3}{4} \pi_1 \chi + \pi_7 \chi - \pi_27 \chi
\]

and thus we get (3.9d). \( \blacksquare \)

Now that we know how \( \phi \) evolves, we can easily work out the evolution of the torsion tensor \( T_{ab} \).

**Proposition 3.2** The evolution of the torsion tensor \( T_{ab} \) under the flow (3.7) is given by

\[
\frac{dT_{ab}}{dt} = \frac{1}{4} (\Tr h) T_{ab} - T_a^c h_{cb} - T_a^c X^d \phi_{dcb} + (\curl h)_{ab} + \nabla_a X_b - \frac{1}{4} (\nabla_c \Tr h) \phi^c_{ab} \tag{3.15}
\]

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Proof. From [20] we infer that under a general evolution of $\varphi$ given by

$$\frac{d}{dt}\varphi = X\varphi + 3i\varphi (s)$$

for some symmetric 2-tensor $s$, the evolution of the torsion tensor $T_{ab}$ is given by

$$\frac{dT_{ab}}{dt} = T_{a}^{\ c}s_{cb} - T_{a}^{\ c}X^{d}\varphi_{dcb} - (\nabla_{c}s_{da})\varphi^{cd} + \nabla_{a}X_{b}$$

(3.16)

Note that compared with [20], some of the signs are different due to a different sign convention for $\psi$, which also leads to a different sign for $T$ and $X$. From (3.9d) we have that in our case,

$$s = \frac{1}{4} (\text{Tr} h) g - h$$

Therefore, substituting this into (3.16) we get (3.15). □

A key motivation for studying flows of closed $G_{2}$-structures within a fixed cohomology class of the 3-form $\varphi$ was that a critical point of the volume functional (1.2) within the cohomology class $[\varphi]$ corresponds a torsion-free $G_{2}$-structure. It is trivial to adapt Hitchin’s proof of this fact [12] to co-closed $G_{2}$-structures.

**Proposition 3.3** Let $M$ be a compact 7-manifold, and suppose the 4-form $\psi$ defines a co-closed $G_{2}$-structure on $M$, so that $d\psi = 0$. Let $\varphi = *\psi$ be the corresponding 3-form. Define the volume functional $V$

$$V(\psi) = \frac{1}{7} \int_{M} \varphi \wedge \psi.$$  

(3.17)

Then $\psi$ defines a torsion-free $G_{2}$-structure if and only if it is a critical point of the functional $V$ restricted to the cohomology class $[\psi] \in H^{4}(M, \mathbb{R})$.

Proof. Consider the variation of $V$:

$$\delta V(\dot{\psi}) = \frac{1}{4} \int_{M} \varphi \wedge \dot{\psi} = \frac{1}{4} \int_{M} \varphi \wedge \dot{\psi}$$

(3.18)

where we have used (3.9d). Now suppose $\dot{\psi} = d\eta$ for some 3-form $\eta$, so that we vary in the same cohomology class. Now

$$\delta V(\dot{\psi}) = \frac{1}{4} \int_{M} \varphi \wedge d\eta = \frac{1}{4} \int d\varphi \wedge \eta$$

(3.19)

Thus $\delta V = 0$ for all $\eta$ if and only if

$$d\varphi = 0.$$
Since we already have $d\psi = 0$, this is satisfied if and only if the $G_2$-structure is torsion-free. 

In the arXiv version of [17], Hitchin has shown that critical points of the functional $V(\varphi)$ on 3-forms are non-degenerate in the directions transverse to the action of the diffeomorphism group $Diff(M)$. Here we adapt the proof from [17] to show an analogous result for the functional $V(\psi)$ on 4-forms.

**Proposition 3.4** Suppose the 4-form $\psi$ defines a torsion-free $G_2$-structure on a compact 7-manifold $M$. Then the Hessian of the functional $V$ (3.17) at $\psi$ is non-degenerate transverse to the action of $Diff(M)$.

**Proof.** Since $\psi$ defines a torsion-free $G_2$-structure, it is a critical point of the functional $V$. Let us consider infinitesimal deformations of $\psi$ by an exact form $d\chi$, for $\chi \in \Lambda^3$. Then, from (3.9d), the deformation of the dual 3-form $\varphi$ is given by

$$D(d\chi) = \frac{3}{4} \star \pi_1 d\chi + \star \pi_7 d\chi - \star \pi_2 d\chi.$$  

(3.20)

Now, the tangent space of the orbit of $Diff(M)$ consists of the forms $L_X \psi$ where $X$ is some vector field. Suppose $d\chi = L_X \psi = d(X \lrcorner \psi)$ for some vector field $X$. Then by diffeomorphism invariance, we get that $D(d\chi) = L_X \varphi = d(X \lrcorner \varphi)$ and hence $\varphi$ remains closed along the orbits of $Diff(M)$. So now suppose that $\psi$ is a critical point but which is degenerate in some direction $d\chi$ which is transverse to orbits of $Diff(M)$, that is, $\psi$ remains torsion-free in the direction of $d\chi$. Hence for proof by contradiction, we have to show now that if $d\chi$ orthogonal to $d(X \lrcorner \varphi)$ for any vector field $X$ and $d(D(d\chi)) = 0$, then $d\chi = 0$.

If $d\chi$ is orthogonal to orbits of $Diff(M)$ then for any vector field $X$,

$$0 = \int_M \langle d\chi, L_X \psi \rangle \text{vol} = \int_M \langle d\chi, d(X \lrcorner \psi) \rangle \text{vol} = \int_M \langle d^* d\chi, X \lrcorner \psi \rangle \text{vol}$$  

(3.21)

where $\langle \cdot, \cdot \rangle$ is the standard inner product with respect to the metric $g$ defined by the $G_2$-structure on $M$. Since (3.21) is true for any $X$, this means that

$$\pi_7 d^* d\chi = 0.$$  

(3.22)

Now from the Hodge Theorem, we can write $\chi$ as

$$\chi = H(\chi) + dGd^* \chi + d^* Gd\chi$$  

(3.23)

where $H(\chi)$ gives the harmonic part and $G$ is the Green’s operator for the Hodge Laplacian. An important property of $G$ is that it commutes with $d, d^*$ and the projections onto representation components. Then we have

$$d\chi = dd^* Gd\chi = dGd^* d\chi$$

However, from (3.22),

$$\pi_7 Gd^* d\chi = 0.$$  

(3.24)

So, we can say that $d\chi = d\eta$ for $\eta = Gd^* d\chi \in \Lambda^3_1 \oplus \Lambda^3_{27}$, and moreover

$$d^* \eta = 0$$  

(3.24)
hence we can assume that $\chi \in \Lambda^3_1 \oplus \Lambda^3_{27}$ and $d^* \chi = 0$. In particular, $\pi_7 d^* \chi = 0$. Then if we suppose $\chi$ is given by $\chi = 3i_\varphi(h)$, from Proposition 2.3 we get that

$$(\text{div } h)_a = -\frac{1}{2} \nabla_a \text{Tr } h$$

and therefore, from (2.25),

$$\pi_7 d\chi = \frac{3}{4} d \text{Tr } (h) \wedge \varphi$$

$$= d \left( \frac{3}{4} (\text{Tr } h) \varphi \right)$$

Moreover, since $\chi \in \Lambda^3_1 \oplus \Lambda^3_{27}$, from (2.25) we have that

$$\pi_1 d\chi = 0.$$ 

Thus from the condition $d (D (d\chi)) = 0$ we have

$$d^* d\chi - 2d^* \pi_7 d\chi = 0$$

(3.26)

Now note that

$$\Delta \pi_7 d\chi = dd^* \pi_7 d\chi + d^* d\pi_7 d\chi$$

$$= dd^* \pi_7 d\chi$$

since $\pi_7 d\chi$ is exact. Hence by applying the exterior derivative to (3.26), we find that

$$\Delta d\chi - 2\Delta \pi_7 d\chi = 0$$

However the Hodge Laplacian of a torsion-free $G_2$-structure commutes with the component projections and we recover

$$\pi_27\Delta d\chi - \pi_7 \Delta d\chi = 0.$$ 

Each of the components must vanish individually, and so

$$\Delta d\chi = 0$$

and thus $d\chi = 0$ as required. 

4 Laplacian of $\psi$

Let us now look at the properties of $\Delta \psi$. For now consider a generic $G_2$-structure, so that

$$\Delta \psi = dd^* \psi + d^* d\psi$$

Note that since $\psi = *\varphi$, we have

$$\Delta \psi = *\Delta \varphi$$

so in particular, for the type decomposition of $\Delta \psi$ it is enough to look at $\Delta \varphi$. 

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Proposition 4.1 Suppose $\varphi$ defines a $G_2$-structure. Then $\Delta \varphi$ has the following type decomposition.

\[
\pi_1 \Delta \varphi = \frac{2}{7} \left( -2 \text{Tr} (\text{curl} T) + 2 \text{Tr} (T^d T) + (\text{Tr} T)^2 - \text{Tr} (T^2) - T_{ab} T_{cd} \psi^{abcd} \right) \varphi \quad (4.1a)
\]

\[
\pi_7 \Delta \varphi = \left( -\text{div} T + T_{ab} T^c e_{bc} + (\text{Tr} T) T_{ab} \psi^{abc} \right) \varphi 
\]

\[
\pi_{27} \Delta \varphi = i \varphi \left( -\frac{3}{7} \text{Tr} (\text{curl} T) g_{de} - 3 (\text{curl} T^d) (de) - 3 \psi^{abc} (d T_{ab} T_{c|e}) 
\right.
\]

\[
+ \frac{3}{14} \left( (\text{Tr} T)^2 - \text{Tr} (T^2) - T_{ab} T_{cd} \psi^{abcd} + 2 \text{Tr} (T^d T) + 7 \psi_{mnpq} T^{mnpq} g_{de} 
\right.
\]

\[
\left. - 3 (T \circ T) (de) - 3 \psi^{abc} (d T_{ab} T_{e|c}) - 3 T_{a} T_{ae} \right) \varphi
\]

Proof. This is a straightforward, but rather long calculation. We first expand $\Delta \varphi$ in terms of the covariant derivative

\[
(\Delta \varphi)_{abc} = (dd^* \varphi + d^* d \varphi)_{abc}
\]

and then apply the formula for the covariant derivative of $\varphi$ in terms of $T_{ab}$ \eqref{4.10}. This gives

\[
(\Delta \varphi)_{abc} = -3 \nabla_{[a} \nabla^d \varphi_{d|bc]} - 4 \nabla^d \nabla_{d \varphi_{abc}}
\]

Now, expanding further, and applying \eqref{2.11}, we get a full expression for $\Delta \varphi$ in terms of $T$ and its derivatives. This can then be projected onto the components of $A^3$ to obtain the decomposition \eqref{4.11}. ■

Definition 4.2 Given a differential operator $P$, denote by $D_\varphi P(\chi)$ its linearization at $\varphi$, evaluated at $\chi$:

\[
D_\varphi P(\chi) = \lim_{t \to 0} \left( \frac{P(\varphi + t \chi) - P(\varphi)}{t} \right)
\]

Proposition 4.3 Consider a non-linear differential operator $P_\varphi$, which is defined by $\varphi$, acting on the $G_2$-structure $\varphi$, given to leading order by

\[
\pi_7 (P_\varphi \varphi)_a = -(\text{div} T)_a + \text{l.o.t} \quad (4.2)
\]

\[
\pi_{1\oplus 27} (P_\varphi \varphi)_{ab} = a_{27} (\text{curl} T^d) (ab) + b_{27} (\text{curl} T) (ab) + a_4 \text{Tr} (\text{curl} T) g_{ab} + \text{l.o.t.} \quad (4.3)
\]

for some constants $a_{27}$, $b_{27}$ and $a_4$. Then the linearization of $P_\varphi$ at $\varphi$ is given by

\[
\pi_7 (D_\varphi P_\varphi) (\chi)_b = \text{curl} (\text{div} h)_b - \nabla^2 X_b + \text{l.o.t.} \quad (4.4)
\]

\[
\pi_{1\oplus 27} (D_\varphi P_\varphi) (\chi)_{de} = a_{27} \left( -\nabla_a \nabla_m h_{nb} \psi^{mn}_{(de)} \varphi^{ab}_{e} \right) 
\]

\[
+ b_{27} \left( \nabla^2 h_{de} - \nabla_{(d} (\text{div} h)_{e)} + \nabla_a \nabla_{(d} (\text{div} h)_{e)} \varphi^{ab}_{e} \right) 
\]

\[
+ a_1 \left( \nabla^2 (\text{Tr} h) - \text{div} (\text{div} h) \right) g_{de} + \text{l.o.t.}
\]
Proof. Suppose
\[ \dot{\varphi} = \chi = X^a \psi_{amnp} + 3h^a_{[m} \varphi_{np]a} \]
then from \([20]\) we know that
\[ \dot{g}_{ab} = 2h_{ab}, \quad \dot{g}^{ab} = -2h^{ab} \]
and similarly as in \((3.16)\),
\[ \dot{T}_{ab} = T_a^c h_{cb} - T_a^c X^d \varphi_{dcb} - (\text{curl } h)_{ab} + \nabla_a X_b. \]
Then consider the linearization of \(P\) at \(\varphi\)
\[ \pi_7 (DP_{\varphi}) (\chi) = \text{div} (\text{curl } h)_b - \nabla^2 X_b + l.o.t. \quad (4.6) \]
Note however, that
\[ \text{div} (\text{curl } h)_b = \nabla^a \left( \nabla_m h_{an} \varphi_{b}^{mn} \right) \]
\[ = (\nabla^a \nabla_m h_{an}) \varphi_{b}^{mn} + l.o.t. \]
\[ = \nabla_m \left( (\text{div } h)_n \right) \varphi_{b}^{mn} - (R^c_{an} h_{cn} + R^c_{nam} h^a_n) \varphi_{b}^{mn} + l.o.t. \]
\[ = \text{curl} (\text{div } h)_b + l.o.t. \]
where we have used the Ricci identity for the Riemann tensor in the second to last line. Hence, \((4.6)\) gives us \((4.4)\). Now let us look at the \(\pi_{1\oplus 27}\) component. Substituting \(\dot{T}\) into \((4.3)\) we get
\[ \pi_{1\oplus 27} (DP_{\varphi}) (\chi) = a_{27} \left( -\nabla_a \nabla_m h_{nb} \varphi_{(d}^{mn} \varphi_{e)^{de}} + \nabla_m \nabla_n X_{(d} \varphi_{e)^{mn}} \right) \]
\[ + a_1 \left( -\nabla_a \nabla_m h_{nb} \varphi_{(d}^{mn} \varphi_{e)^{de}} + \nabla_a \nabla_b X_{(d} \varphi_{e)^{mn}} \right) g_{de} \]
\[ + b_{27} \left( -\nabla_a \nabla_m h_{n(\varphi_{b)^{mn}} |d} \varphi_{e)}^{ab} + \nabla_a \nabla_{(d} X_{b)} \varphi_{e)^{mn}} \right) + l.o.t. \]
\[ = a_{27} \left( -\nabla_a \nabla_m h_{nb} \varphi_{(d}^{mn} \varphi_{e)^{de}} \right) \]
\[ + b_{27} \left( \nabla^2 h_{de} - \nabla_a \nabla_{(d} \varphi_{e)^{ab}} \right) \]
\[ + a_1 \left( \nabla^2 (\text{Tr } h) - \nabla_a \nabla_b h^{ab} \right) g_{de} + l.o.t. \]
Using the Ricci identity again to switch the order of the derivatives in the \(b_{27}\) term, we get \((4.5)\).

Note that the Laplacian of a general \(G_2\)-structure \(\varphi\) is given by the above operator \(P\) with \(a_1 = -\frac{3}{7}, a_{27} = -3\) and \(b_{27} = 0\). As an example, consider the well-studied case of the Laplacian \(\Delta_{\varphi}\) of the closed 3-form \(\varphi\). In this case, since \(\varphi\) is closed, \(T = \tau_{14}\) only has a component in the 14-dimensional representation, and is hence anti-symmetric. Moreover, it is a known fact (see e.g. \([11]\)), that in this case
\[ d^* \tau_{14} = 0 \quad (4.7) \]
and hence the highest order term \(\text{div } T\) in \((4.1b)\) vanishes. Moreover, since \(\tau_{14} \in \Lambda^2_{14}\), we also have \((\tau_{14})_{ab} \varphi^{abc} = 0\) (i.e. the projection to \(\Lambda^2_7\) vanishes). Thus the leading order term \(\text{Tr} (\text{curl } T)\)
in (4.1a) becomes:

\[
\text{Tr} (\text{curl} T) = - (\nabla_a T_{bc}) \varphi^{abc} = - \nabla_a \left((\tau_{14})_{bc} \varphi^{abc}\right) + (\tau_{14})_{bc} \nabla_a \varphi^{abc} = - (\tau_{14})_{bc} (\tau_{14})_{ea} \psi^{eabc} = 2 (\tau_{14})_{bc} (\tau_{14})^{bc}
\]

where we have used the identity
\[
\omega_{ab} \psi^{ab} = - 2 \omega_{cd}
\]
for any \( \omega \in \Lambda^2_{14} \). Overall, after similarly simplifying other terms, we obtain

\[
\begin{align*}
\pi_1 \Delta \varphi &= \frac{2}{7} (\tau_{14})_{ab} (\tau_{14})^{ab} \\
\pi_7 \Delta \varphi &= 0 \\
\pi_{27} \Delta \varphi &= 3 \text{curl} (\tau_{14})_{(ad)} - \frac{9}{7} g_{ad} (\tau_{14})_{bc} (\tau_{14})^{bc} + 3 (\tau_{14})_a (\tau_{14})_{bd}
\end{align*}
\]

Thus comparing the highest order terms we find that \( \Delta \varphi \) in this case corresponds to the operator \( P \) in Proposition 4.3 with \( a_1 = a_{27} = 0 \) and \( b_{27} = 3 \). Hence, from Proposition 4.3 we get the linearization:

\[
\begin{align*}
\pi_7 (D \varphi \Delta \varphi) (\chi) &= 0 \\
\pi_{14} \oplus 27 (D \varphi \Delta \varphi) (\chi) &= 3 \left( \nabla^2 h_{de} - \nabla_{(d} (\text{div} h)_{e)} + \nabla_{(d} (\text{curl} X)_{e)} \right) + \text{l.o.t.}
\end{align*}
\]

Let us now assume that \( \chi \) is closed, and is of the form \( \chi = X \cdot \psi + 3i \varphi \). Then from Proposition 2.3, we know the type decomposition of \( d \chi \) and \( d^* d \chi \) up to torsion terms. Since all of these components have to be zero, we have the following relations

\[
\begin{align*}
0 &= \nabla_a (\text{div} X) + \text{l.o.t.} \\
0 &= \nabla_a \text{Tr} h - (\text{div} h)_a - (\text{curl} X)_a + \text{l.o.t.} \\
0 &= \nabla^2 \text{Tr} h - \text{div} (\text{div} h) + \text{l.o.t.} \\
0 &= \nabla^2 X_a - (\text{curl} (\text{div} h))_a + \text{l.o.t.}
\end{align*}
\]

Also note for a vector field \( v \), we have

\[
d (v \cdot \varphi)_{abc} = 3 \nabla_{[a} (v^d \varphi_{bc]d}) = 3 \left( \nabla_{[a} v^d \right) \varphi_{bc]d} + \text{l.o.t.}
\]

Then, the symmetric part of \( \nabla v \) simply gives the \( \Lambda^3_1 \oplus \Lambda^3_{27} \) component of \( d (v \cdot \varphi) \), and the \( \Lambda^2_7 \) part of \( \nabla v \) gives rise to the \( \Lambda^3_2 \) component of \( d (v \cdot \varphi) \). In fact, using identities in \[20\], this can be re-written as

\[
d (v \cdot \varphi) = \frac{1}{2} (\text{curl} v) \cdot \varphi + 3i \varphi \left( \nabla_{(m} v_{n)} \right) + \text{l.o.t.}
\]

Now let \( Y = \nabla \text{Tr} h \) and \( Z = \text{curl} X \). Then, from (4.13), we get

\[
\begin{align*}
\pi_7 d (Y \cdot \varphi) &= \frac{1}{2} \nabla_m (\nabla_n \text{Tr} h) \varphi^{mn} = \text{l.o.t.} \\
\pi_{14} \oplus 27 d (Y \cdot \varphi) &= 3 (\nabla_{(a} \nabla_b) \text{Tr} h) \\
&= 3 \left( \nabla_{(a} \nabla_n X_b) \varphi^{mn} + \nabla_{(a} \nabla_{[m} h^r_{b)} \right) + \text{l.o.t.}
\end{align*}
\]

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where we have applied (4.12b) for the second relation. Similarly, we find that
\[
\pi_7 d (Z \varphi) = \frac{1}{2} \left( \nabla_d \nabla_a X_b \right) \varphi^{ab} \epsilon^{de} p + l.o.t.
\]
\[= \frac{1}{2} \left( \nabla_d \nabla_a X_b \right) \left( -g^{ad} g^{b} p + g^{ap} g^{bd} - \psi^{abd} p \right) + l.o.t.
\]
\[= -\frac{1}{2} \nabla^2 X_p + \frac{1}{2} \nabla_p \left( \nabla_b X^b \right) + \frac{1}{2} \text{Ric}_{pb} X^b + l.o.t.
\]
\[= -\frac{1}{2} \nabla^2 X_p + l.o.t.
\]
(4.16)
\[
\pi_{1 \oplus 27} d (Z \varphi) = 3i \varphi \left( \left( \nabla \left( \nabla d (Z \varphi) \right) \varphi^{ab} \epsilon_e \right) \right) + l.o.t.
\]
(4.17)

where for the first relation we have used (2.4a) to contract the \( \varphi \) terms, and then used (4.12a) and the fact that the Ricci tensor depends only on the torsion.

Now note that
\[2d (Z \varphi) - d (Y \varphi) = - \left( \nabla^2 X \right) \varphi + 3i \varphi \left( - \nabla (\nabla d (Z \varphi)) \varphi^{ab} \epsilon_e \right) + l.o.t.
\]
(4.18)

So in fact,
\[D_\varphi \Delta_\varphi (\chi) = - \left( \nabla^2 X \right) \varphi + 3i \varphi \left( \nabla^2 h_{de} \right) + 2d (Z \varphi) - d (Y \varphi) + l.o.t.
\]
(4.19)

It is easy to see that for any 3-form \( \chi \),
\[\Delta_\varphi \chi = - \left( \nabla^2 X \right) \varphi - 3i \varphi \left( \nabla^2 h_{de} \right) + l.o.t.
\]
(4.20)

Since \( \chi \) is closed, \( \Delta_\varphi \chi \) is exact, so the lower order terms in (4.19) are exact, so we rewrite them as an exterior derivative of some 2-form valued algebraic function of \( \chi \). Thus from (4.19) we conclude that
\[D_\varphi \Delta_\varphi (\chi) = - \Delta_\varphi \chi + 2d (Z \varphi) - d (Y \varphi) + dF (\chi)
\]
(4.21)

Hence we have derived the same result as in [4, 25]:

**Proposition 4.4 ([4, 25])** If \( \varphi \) is a closed \( G_2 \)-structure, then the linearization of the Laplacian \( \Delta_\varphi \) at \( \varphi \), evaluated at a closed form \( \chi \) is given by
\[D_\varphi \Delta_\varphi (\chi) = - \Delta_\varphi \chi - L V_\chi \varphi + dF (\chi)
\]
(4.22)

where \( V_\varphi = -2 \text{curl} X + \nabla \text{Tr} h \) and \( F (\chi) \) is a 2-form valued algebraic function of \( \chi \).

The representation of \( D_\varphi \Delta_\varphi (\chi) \) as (4.22) shows us that we can apply the DeTurck trick - adding \( L V_\varphi \) to (1.22) gives a flow that is elliptic in the direction of closed forms. In [4, 25] this is then used to show short-time existence and uniqueness for the Laplacian flow for 3-forms (1.1).

Now let us consider the Laplacian of \( \varphi \) for a co-closed \( G_2 \)-structure. In this case, \( T_{ab} \) is now symmetric. We can now write
\[d \varphi = 3 \epsilon \varphi \left( - T + \frac{1}{3} (\text{Tr} T) g \right).
\]
(4.23)

From the condition \( d^2 \varphi = 0 \), or equivalently, \( (d^*)^2 \psi = 0 \), the torsion \( T \) satisfies certain Bianchi-type identities.
Lemma 4.5 Suppose \( \varphi \) is a co-closed \( G_2 \)-structure. Then the torsion tensor \( T \) satisfies the following identities

\[
\text{div } T = \nabla \text{Tr } T \\
(\text{curl } T)_{[ab]} = 0
\]

(4.24)

Proof. We have the condition that \( d^* \left( 3i_\varphi \left( -T + \frac{1}{3} \text{Tr } T \right) g \right) = 0 \). Set \( h = -T + \frac{1}{3} \text{Tr } T \) and \( X = 0 \) in Proposition 2.3 Then from (2.24b), we get (4.24) and (4.25) then follows from (2.24a).

Proposition 4.6 Suppose \( \varphi \) is a co-closed \( G_2 \)-structure, then the type decomposition of \( \Delta \varphi \) is given by

\[
\pi_1 \Delta \varphi = \frac{2}{7} |T|^2 + \frac{2}{7} (\text{Tr } T)^2 \tag{4.26a}
\]

\[
(\pi_7 \Delta \varphi) = -\text{div } T = -\nabla \text{Tr } (T) \tag{4.26b}
\]

\[
(\pi_2 \Delta \varphi)_{ad} = -3 (\text{curl } T)_{ad} - \frac{3}{2} (T \circ T)_{ad} - 3T_{ab}T^b_d \\
+ \frac{3}{14} g_{ad} |T|^2 + \frac{3}{14} g_{ad} (\text{Tr } T)^2 \tag{4.26c}
\]

Proof. We obtain this directly from (4.1) by using the fact that \( T \) is symmetric and also applying identity (4.24). Note that from (4.25) it follows that the symmetric part of \( \text{curl } T \) is actually equal to \( \text{curl } T \).

Now suppose we have the flow

\[
\frac{d\psi}{dt} = \Delta \psi = *_{\varphi} \Delta \varphi. \tag{4.27}
\]

Then from (4.26),

\[
\Delta \psi = d \text{Tr } (T) \wedge \varphi + \left( \frac{2}{7} |T|^2 + \frac{2}{7} (\text{Tr } T)^2 \right) \psi \\
+ 3 * i_{\varphi} \left( - (\text{curl } T)_{ad} - \frac{1}{2} (T \circ T)_{ad} - T_{ab}T^b_d \\
+ \frac{1}{14} g_{ad} |T|^2 + \frac{1}{14} g_{ad} (\text{Tr } T)^2 \right) \\
= d \text{Tr } (T) \wedge \varphi + 3 * i_{\varphi} \left( - (\text{curl } T)_{ad} - \frac{1}{2} (T \circ T)_{ad} - T_{ab}T^b_d \\
+ \frac{1}{6} g_{ad} |T|^2 + \frac{1}{6} g_{ad} (\text{Tr } T)^2 \right) \tag{4.28}
\]

The corresponding evolution of the metric now follows from Proposition 3.1 - in this case,

\[
h_{ad} = -(\text{curl } T)_{ad} - \frac{1}{2} (T \circ T)_{ad} - T_{ab}T^b_d + \frac{1}{6} g_{ad} |T|^2 + \frac{1}{6} g_{ad} (\text{Tr } T)^2 \\
\text{Tr } h = \frac{2}{3} \left( |T|^2 + (\text{Tr } T)^2 \right)
\]
hence the evolution of the metric becomes
\[
\frac{dg_{ab}}{dt} = \frac{1}{2} (\text{Tr} h) g_{ab} - 2h_{ab} = 2 (\text{curl} T)_{ab} + (T \circ T)_{ab} + 2T_{ac}T^c_b \tag{4.29}
\]
Note that for a co-closed $G_2$, the Ricci curvature is given by
\[
R_{ad} = -(\text{curl} T)_{ad} - T_{ab}T^b_d + (\text{Tr} T) T_{ad} \tag{4.30}
\]
The details can be found in [11, 20]. Hence we can rewrite (4.29) as
\[
\frac{dg_{ab}}{dt} = -2R_{ab} + (T \circ T)_{ab} + 2(\text{Tr} T)T_{ad} \tag{4.31}
\]
Therefore, same as for the Laplacian flow of the 3-form $\varphi$ [3], the leading term of the metric flow corresponds to the Ricci flow. Also, the evolution of the volume form under this flow is given by
\[
\frac{d}{dt} \sqrt{\det g} = \frac{1}{2} \left( |T|^2 + (\text{Tr} T)^2 \right) \sqrt{\det g} \tag{4.32}
\]
For non-trivial torsion this is always positive and is zero if and only if $T = 0$. Hence the volume functional $V$ [3.17] grows monotonically along the flow (4.27), with an extremum being reached along a flow line if and only if the torsion vanishes. We have already seen in Proposition 3.3 that it is generally true that torsion-free $G_2$-structures correspond to critical points of $V$ when restricted to a fixed cohomology class of $\varphi$. The Laplacian flow of $\varphi$ (1.1) shares this property. As shown in [3], the volume form grows monotonically along the flow (1.1) and in [4], it was interpreted as the gradient flow of the volume functional with respect to an unusual metric. Similarly, we can do the same for (4.27). As in [4], define the following metric on $dC^\infty(M, \Lambda^3(M))$:
\[
\langle \chi_1, \chi_2 \rangle_\psi = \frac{1}{4} \int_M G_\psi \chi_1 \wedge \ast \chi_2.
\]
for any exact 4-forms $\chi_1$ and $\chi_2$. As before, $G_\psi$ is the Green’s operator for the Hodge Laplacian $\Delta_\psi$. From (3.19), we know that for an exact 4-form $\dot{\psi}$, the deformation of $V$ is given by
\[
\delta V (\dot{\psi}) = \frac{1}{4} \int_M \dot{\psi} \wedge \varphi = \frac{1}{4} \int_M \Delta_\psi G_\psi \dot{\psi} \wedge \varphi = \frac{1}{4} \int_M G_\psi \dot{\psi} \wedge \ast \Delta_\psi \dot{\psi} = \langle \dot{\psi}, \Delta_\psi \dot{\psi} \rangle_\psi.
\]
Hence indeed, the gradient flow of $V$ is given by (4.27).

Now consider the gradient flow of $V$ is given by (4.27).

**Proposition 4.7** The linearization of $\Delta_\psi$ at $\psi$ is given by
\[
\pi_7 (D_\psi \Delta_\psi) (\chi) = d (\text{div} X) \wedge \varphi + \text{l.o.t.} \tag{4.33}
\]
\[
\pi_{10:27} (D_\psi \Delta_\psi) (\chi) = 3 \ast \iota_\varphi \left( \nabla^2 h_{de} - \nabla (d \nabla |a| h^a_e) - \nabla_a \nabla (d X_{|b|}) \varphi^{ab} \right) + \frac{1}{4} \nabla_a \nabla_d \text{Tr} h - \frac{1}{4} (\nabla^2 \text{Tr} h) g_{ad} + \text{l.o.t.} \tag{4.34}
\]

where \( \chi \) is a 4-form given by
\[
\chi = *(X, \psi + 3i\varphi (h)) .
\]
Moreover, if \( \chi \) is a closed form, then the 1 \( \oplus \) 27 component of \( D_\psi \Delta_\psi (\chi) \) can written as
\[
\pi_{1 \oplus 27} (D_\psi \Delta_\psi) (\chi) = \frac{3}{2} \star \varphi \left( \nabla^2 h_{de} - \nabla (d, \nabla [a] h^e) - \nabla_a \nabla (d, X) [b] \varphi^{ab} \right) - \nabla_b \nabla m h_{cn} \varphi_{bc}^{mn} (d, \varphi^{mn}) + l.o.t. \tag{4.35}
\]

**Proof.** To find the linearization, we just take the decomposition of \( \Delta_\psi \psi \) \((4.28)\) and use the formula \((3.15)\) for the variation of \( T_{ab} \). So we have
\[
\dot{T}_{ab} = \frac{1}{4} (\text{Tr} \ h) T_{ab} - T_a c h_{cb} - T_a c X^d \varphi_{dcb} + (\text{curl} \ h)_{ab} + \nabla_a X_b - \frac{1}{4} (\text{Tr} \ h) \varphi^{c}_{ab} \tag{4.36}
\]
Now,
\[
\pi_7 (D_\psi \Delta_\psi) (\chi) = d (\text{div} \ X) \wedge \varphi + l.o.t.
\]
\[
\pi_{1 \oplus 27} (D_\psi \Delta_\psi) (\chi) = 3 \star \varphi \left( -\nabla_a \nabla_m h_n (d, \varphi^{mn} [b] \varphi^{ab}) - \nabla_a \nabla (d, X) [b] \varphi^{ab} \right)
+ \frac{1}{4} \nabla_c \left( \nabla m \text{Tr} \ h \right) \varphi^{m (a |b} \varphi^{c b)} + l.o.t.
= 3 \star \varphi \left( \nabla^2 h_{de} - \nabla (d, \nabla [a] h^e) - \nabla_a \nabla (d, X) [b] \varphi^{ab} \right)
+ \frac{1}{4} \nabla a \nabla d \text{Tr} \ h - \frac{1}{4} (\nabla^2 \text{Tr} \ h) g_{ad} + l.o.t.
\]
where we have twice used the contraction identity \((2.43)\) in the last line.

To get the expression \((1.35)\), note that when \( \chi \) is closed, \( \dot{T}_{ab} \) is symmetric, since the torsion class remains in \( W_1 \oplus W_{27} \). Hence the antisymmetric part of \((3.15)\) vanishes and we can write
\[
\dot{T}_{ab} = \frac{1}{4} (\text{Tr} \ h) T_{ab} - T_a c h_{cb} - T_a c X^d \varphi_{dcb} + (\text{curl} \ h)_{ab} + \nabla_a X_b \tag{4.37}
\]
It can been seen explicitly from Proposition \((2.3)\) that the antisymmetric part of \( \dot{T} \) is equal precisely to \( *d\chi \), which is of course zero. Now plugging \((1.37)\) into \((1.28)\) we find that \( \pi_7 (D_\psi \Delta_\psi) (\chi) \) remains the same, and \( \pi_{1 \oplus 27} (D_\psi \Delta_\psi) (\chi) \) becomes as in \((4.35)\). \( \blacksquare \)

To see that this is not a positive operator, consider the principal symbol \( \sigma_\xi (D_\psi \Delta_\psi) (\chi) \) for some vector \( \xi \) :
\[
\pi_7 \sigma_\xi (D_\psi \Delta_\psi) (\chi) = \langle X, \xi \rangle \xi^b \wedge \varphi
\]
\[
\pi_{1 \oplus 27} \sigma_\xi (D_\psi \Delta_\psi) (\chi) = 3 \star \varphi \left( |\xi|^2 h_{de} - \xi (d, \xi [a] h^e) - \xi_a \xi (d, X) [b] \varphi^{ab} \right)
+ \frac{1}{4} \xi_a \xi_d \text{Tr} \ h + l.o.t.
\]
Then, if \( h = 0 \), we have
\[
\langle \sigma_\xi (D_\psi \Delta_\psi) (\chi), \chi \rangle = -4 \langle X, \xi \rangle^2 \leq 0.
\]
We may attempt to use DeTurck’s trick to modify this operator by adding a certain Lie derivative \( \mathcal{L}_{V(\chi)} \psi \) to the flow (4.27), where \( V(\chi) \) is linear in the first derivatives of \( \chi \), so that the modified flow of \( \psi \) is now given by

\[
\frac{d\psi}{dt} = \Delta \psi + \mathcal{L}_{V(\chi)} \psi
\]  

(4.38)

For convenience, denote

\[
Q \psi \psi = \Delta \psi + \mathcal{L}_{V(\chi)} \psi.
\]  

(4.39)

Since \( \psi \) is closed, we have

\[
\mathcal{L}_{V(\chi)} \psi = d (V(\chi) \wedge \psi).
\]

Consider first the type decomposition of \( d (V(\chi) \wedge \psi) \). From Proposition 2.3, we have

\[
\pi_7 d (V \wedge \psi) = -\frac{1}{2} (\text{curl } V) \wedge \varphi + l.o.t.
\]  

(4.40a)

\[
\pi_{1 \oplus 27} d (V \wedge \psi) = 3 \ast i_{\varphi} \left( -\nabla_m V_n + \frac{1}{3} (\text{div } V) g_{mn} \right) + l.o.t.
\]  

(4.40b)

We are only interested in linearizing (4.39) in the direction of closed 4-forms, so suppose \( \chi = \ast (X \wedge \psi + 3 i_{\varphi} (h)) \) is closed, and hence \( \ast \chi = X \wedge \psi + 3 i_{\varphi} (h) \) is co-closed. This requirement gives a number of conditions on \( X \) and \( h \). From Proposition 2.3 we then get the following

\[
-((\text{div } h) \wedge \varphi)_{bc} - 2 (\text{curl } h)_{[bc]} + \nabla_m X_n \psi^{mn}_{bc} = l.o.t.
\]  

(4.41a)

\[
(\text{curl } X)_a - (\text{div } h)_a - \frac{1}{2} \nabla_a \text{Tr } h = l.o.t.
\]  

(4.41b)

The lower order terms in (4.41) are linear in \( X \) and \( h \) and depend only on the torsion. By differentiating (4.42), we get the following relations:

\[
\text{div} \ (\text{div } h) + \frac{1}{2} \nabla^2 \text{Tr } h = l.o.t.
\]  

(4.42a)

\[
\nabla_a (\text{div } h)_b + \frac{1}{2} \nabla_a \nabla_b \text{Tr } h - \left( \nabla_m \nabla_a X_{mn} \right) \varphi^{mn}_{bc} = l.o.t.
\]  

(4.42b)

\[
\nabla_a (\text{div } X) - \nabla^2 X_a = l.o.t.
\]  

(4.42c)

Here, equation (4.42a) is obtained by taking the divergence of (4.41b). Equation (4.42b) is obtained by taking the curl of (4.41a), and then symmetrizing. Similarly, equation (4.42c) is obtained by taking the curl of (4.41b). In both cases the identities (2.4) are used to simplify contractions of \( \varphi \) and \( \psi \).

The only possible vector fields that are linear in covariant derivatives of \( X \) and \( h \) are

\[
Y = \nabla \text{Tr } h
\]  

(4.43)

\[
W = \text{div } h
\]  

(4.44)

\[
Z = \text{curl } X
\]  

(4.45)

However from (4.41b) we see that these vectors are linearly dependent up to lower order terms, so it is sufficient to take a linear combination of any two of them. For convenience we will choose
Y and Z. First consider \(d(Y \lrcorner \psi)\):

\[
\pi_7 d(Y \lrcorner \psi) = \left( -\frac{1}{2} \nabla_m \nabla_n (\text{Tr } h) \varphi_{mn}^a \right) \land \varphi + \text{l.o.t.} = \text{l.o.t.} \quad (4.46a)
\]

\[
\pi_1 \oplus 27 d(Y \lrcorner \psi) = 3 * i_\varphi \left( -\nabla_m \nabla_n \text{Tr } h + \frac{1}{3} \nabla^2 (\text{Tr } h) g_{mn} \right) + \text{l.o.t.} \quad (4.46b)
\]

Similarly, the decomposition of \(d(Z \lrcorner \psi)\) is

\[
\pi_7 d(Z \lrcorner \psi) = \left( -\frac{1}{2} \nabla_m \nabla_c X_d \varphi_{cd}^n \varphi_{mn}^a \right) \land \varphi + \text{l.o.t.} = \text{l.o.t.} \quad (4.47a)
\]

\[
\pi_1 \oplus 27 d(Z \lrcorner \psi) = 3 * i_\varphi \left( -\nabla_m \nabla_n (a X X_d \varphi_{mn}^a) \right) + \text{l.o.t.} \quad (4.47b)
\]

Again, we have used the contraction identity (2.4a) to get to the second line.

Hence, if we take

\[
V = \frac{3}{4} Y - 2 Z
\]

and apply (4.42b), we find that the linearization of \(Q \psi\) is now given

\[
\pi_7 (D \psi Q \psi) (\chi) = \left( -\nabla^2 X + 2 d(\text{div } X) \right) \land \varphi + \text{l.o.t.} \quad (4.49a)
\]

\[
\pi_1 \oplus 27 (D \psi Q \psi) (\chi) = 3 * i_\varphi (\nabla^2 h_{ad}^n) + \text{l.o.t.} \quad (4.49b)
\]

For some vector \(\xi\), consider the principal symbol \(\sigma_\xi (D \psi Q \psi) \chi\), then

\[
\langle \sigma_\xi (D \psi Q \psi) \chi, \chi \rangle = \left\langle \left( -|\xi|^2 X + 2 \xi (\xi, X) \right) \land \varphi, -X \land \varphi \rightangle
\]

\[
+ 9 \left\langle i_\varphi \left( |\xi|^2 h_{ad}^n \right), i_\varphi (h_{ad}^n) \rightangle
\]

\[
= 4 \left( |\xi|^2 |X|^2 - 2 \langle \xi, X \rangle^2 \right) + 2 |\xi|^2 |h|^2
\]

(4.50)

Hence we see that the principal symbol is still indefinite, and so (4.38) is not parabolic. Note that adding more instances of \(L Z \psi\) would not help. To see this, let \(\sigma_\xi (d(Z \lrcorner \psi))\) be the principal symbol of \(d(Z \lrcorner \psi)\) for some vector \(\xi\). Using (4.42c), we get

\[
\langle \sigma_\xi (d(Z \lrcorner \psi)) \chi, \chi \rangle = -2 \xi_m \xi_n h_{mp}^n \varphi_{mp}^a X^a - 2 \xi_m \xi_a X_n \varphi_{mn}^a h_{ad}^n = 0
\]

(4.51)

Therefore, \(L Z \psi\) would not contribute in any way to (4.50). However, overall, we have shown that we can rewrite the linearized operator \(\Delta_\psi \psi\) in the following way.

**Proposition 4.8** The linearization of the operator \(\Delta_\psi \psi\) evaluated at a closed 4-form \(\chi\) given by

\[
\chi = * (X \lrcorner \psi + 3 i_\varphi (h))
\]

is given by

\[
D \psi \Delta_\psi (\chi) = -\Delta_\psi \chi - L V (\chi) \psi + 2 d(\text{div } \varphi) + d F (\chi)
\]

(4.52)

where

\[
V (\chi) = \frac{3}{4} \nabla \text{Tr } h - 2 \text{curl } X
\]

(4.53)

and \(F (\chi)\) is a 3-form-valued algebraic function of \(\chi\).
Proof. From (4.49), and using (4.20) we have
\[ D_\psi \Delta_\psi (\chi) = -\Delta_\psi \chi - L_{V(\chi)} \psi + 2d(\text{div} \, X) \wedge \varphi + \text{l.o.t.} \]
However we can write
\[ (\nabla^2 X) \wedge \varphi = d((\text{div} \, X) \varphi) + \text{l.o.t.} \]
and whence,
\[ D_\psi \Delta_\psi (\chi) = -\Delta_\psi \chi - L_{V_\chi} \psi + 2d((\text{div} \, X) \varphi) + \text{l.o.t.} \tag{4.54} \]
However, for a closed \( \psi \), \( \Delta_\psi \psi \) is exact, and now all the higher order terms are exact, so the lower order terms must be an exact 4-form, and can thus be represented as \( dF(\chi) \) where \( F \) is a 3-form which depends algebraically on \( \chi \).

In both the closed and co-closed case the vector field \( V_\chi \) has been derived by considering linearizations of \( \Delta_\varphi \varphi \) and \( \Delta_\psi \psi \), respectively. It turns out that these vector fields, as well as the vector field used in DeTurck’s trick for Ricci flow, actually have precisely the same origin, even though this is not clear from the above situation, since the specifics of each case are a little different. In general, following DeTurck’s original idea [7], suppose we fix an arbitrary time-independent background metric \( \bar{g} \). Given a geometric flow, this could for example be taken to be the metric at time \( t = 0 \). With respect to this background metric, we can write down a background Levi-Civita connection \( \bar{\nabla} \), using which we get a background Riemann curvature, and in particular, a background Ricci curvature \( \bar{\text{Ric}} \). Now given an evolving metric \( g \), we can express all the quantities related to \( g \) in terms of the background quantities. Let
\[ \bar{h} = g - \bar{g}. \]
Then, it is easy to show that the difference between the corresponding Christoffel symbols \( \Gamma \) and \( \bar{\Gamma} \) is given by
\[ T_{a b c} := \Gamma_{a b c} - \bar{\Gamma}_{a b c} = \frac{1}{2} g^{bd} (\bar{\nabla}_a \bar{h}_{cd} + \bar{\nabla}_c \bar{h}_{ad} - \bar{\nabla}_d \bar{h}_{ac}) \tag{4.55} \]
From this, it is possible to obtain an expression for \( \text{Ric} \) (the details can be found in [5, 22], for example) in terms of the background quantities and difference \( \bar{h} \)
\[ \text{Ric} = \bar{\text{Ric}} - \frac{1}{2} \nabla^2 \bar{h} + \frac{1}{2} \bar{L}_{\bar{\nabla}} \bar{g} + O \left( \|h\|^2 \right) \tag{4.56} \]
where the vector \( \bar{V} \) is given by
\[ \bar{V}^b = g^{ac} T_{a b c} \tag{4.57} \]
In Ricci flow applications, the expression (4.56) is then used to relate the Ricci flow to the strictly parabolic Ricci-DeTurck flow, and hence show short-time existence.

Going back to our 4-form flow (4.27), suppose the background metric is now \( \bar{g} = g_0 \) - the metric associated to the initial \( G_2 \)-structure \( \psi_0 \). We also have
\[ \psi - \psi_0 = \chi \]
where \( \chi \) is some closed 4-form given by \( \chi = * (X \cdot \psi + 3i\varphi (h)) \), as before. From Proposition 3.1
\[ g = \bar{g} + \left( \frac{1}{2} (\text{Tr} h) \bar{g} - 2h \right) + O \left( \|\chi\|^2 \right) \]
where $\text{Tr} h = \bar{g}^{ab} h_{ab}$. Thus from (4.55),
\[
T^b_a c = \frac{1}{2} \bar{g}^{bd} \nabla_a \left( \frac{1}{2} (\text{Tr} h) \bar{g}_{cd} - 2 h_{cd} \right) + \frac{1}{2} \bar{g}^{bd} \nabla_c \left( \frac{1}{2} (\text{Tr} h) \bar{g}_{ad} - 2 h_{ad} \right) \\
- \frac{1}{2} \bar{g}^{bd} \nabla_d \left( \frac{1}{2} (\text{Tr} h) \bar{g}_{ac} - 2 h_{ac} \right) + O \left( |\chi|^2 \right)
\]
\[
= -\bar{g}^{bd} \left( \nabla_a h_{cd} + \nabla_c h_{ad} - \nabla_d h_{ac} \right) \\
+ \frac{1}{4} \left( (\nabla_a \text{Tr} h) \delta^b_c + (\nabla_c \text{Tr} h) \delta^b_a - (\nabla_d \text{Tr} h) \bar{g}_{ac} \right) + O \left( |\chi|^2 \right)
\]
Therefore, in this case the vector $\bar{V}$ becomes
\[
\bar{V}^b = -2 \nabla_c h^c_d - \frac{1}{4} \nabla^b \text{Tr} h + O \left( |\chi|^2 \right)
\] (4.58)
where indices are raised using the background inverse metric $\bar{g}^{-1}$. Note however that $\chi$ is closed, so it satisfies conditions (4.41). Using (4.41b), we can re-write (4.58) as
\[
\bar{V}^b = \frac{3}{4} \bar{V}^b \text{Tr} h - 2 (\nabla_a X) \bar{\varphi}^{acb} + U^b (h, X) + O \left( |\chi|^2 \right)
\] (4.59)
where $U$ is a vector-valued function linear in $h$ and $X$ and which depends only on the torsion of $\bar{\varphi}$. Thus we see that if we are interested in linearization, and only at leading terms, we find that $\bar{V}$ is precisely equal to our $V (\chi)$. A similar argument gives the same result for closed $G_2$-structures as well [4]. This is not surprising, since to leading order, the evolution of the metric is given by the Ricci flow in both cases.

## 5 Modified flow

In the previous section we have seen that the original Laplacian flow of $\psi$ (1.3) is not parabolic, and moreover, unlike the Laplacian flow of $\varphi$ (1.1), it is not even weakly parabolic - as we have seen, the principal symbol of $\Delta_{\psi} \psi$ is indefinite. Hence, unlike the situation with the Ricci flow or the Laplacian flow of $\varphi$, we cannot modify this flow to be parabolic just by adding a Lie derivative along a vector $V$. In fact, in our case, we have seen that we choose $V$ such that the $\Lambda^1_4 \oplus \Lambda^2_7$ component becomes parabolic. However it turns out that this addition cannot fix the highest order terms of the $\Lambda^4_7$ component of the flow. The main motivation for considering the Laplacian flow of $\psi$ is that it gives a flow of co-closed $G_2$-structures with the volume functional increasing monotonically along the flow, which suggests that if a long-time solution to this flow were to exist, it would converge towards a torsion-free $G_2$-structure. So let us modify the flow (1.1) such that it becomes weakly parabolic (before applying the DeTurck trick). However, it is important that the new flow stays within the class of co-closed $G_2$-structures and that the volume functional increases along the flow.

**Theorem 5.1** For a co-closed $G_2$-structure defined by the closed 4-form $\psi$, consider the flow
\[
\frac{d\psi}{dt} = \Delta_{\psi} \psi + 2d ((A - \text{Tr} T) \varphi).
\] (5.1)
where $A$ is a fixed constant. Then, this is a weakly parabolic flow in the direction of closed forms, and its linearization at a closed form $\chi$ is given by

$$\frac{d\chi}{dt} = -\Delta_\psi \chi - L_{V(\chi)} \psi + d\hat{F} (\chi)$$

where $V(\chi)$ is as in Proposition 4.8 and $\hat{F}(\chi)$ is a 3-form-valued function that is algebraic in $\chi$.

Moreover, the evolution of the volume functional (3.14) is given by

$$\frac{dV}{dt} = \frac{1}{2} \int_M \left( |T|^2 + \text{Tr} T (4A - 3 \text{Tr} T) \right) \text{vol.} \quad (5.2)$$

**Proof.** Let $\hat{Q}_\psi$ denote the operator on the right hand side of (5.1). Expand the extra term in $\hat{Q}_\psi$:

$$2d ((A - \text{Tr} T) \varphi) = -2d (\text{Tr} T) \wedge \varphi + 2 (A - \text{Tr} T) d\varphi$$

$$= -2d (\text{Tr} T) \wedge \varphi + 2 (A - \text{Tr} T) (4\tau_1 \psi - 3 * i_\varphi (\tau_{27}))$$

$$= -2d (\text{Tr} T) \wedge \varphi + \frac{8}{7} \text{Tr} T (A - \text{Tr} T) \psi - 6 (A - \text{Tr} T) * i_\varphi (\tau_{27}) \quad (5.3)$$

Thus the only highest order terms are present in the $\Lambda^4$ component. From (3.15)

$$D_\psi (\text{Tr} T) (\chi) = \text{div} X + l.o.t.$$ So, from (4.39) the linearization of $\hat{Q}_\psi$ is now given by

$$\pi_7 \left( D_\psi \hat{Q}_\psi \right) (\chi) = -d (\text{div} X) \wedge \varphi + l.o.t. \quad (5.4)$$

$$\pi_{1\oplus 27} \left( D_\psi \hat{Q}_\psi \right) (\chi) = 3 * i_\varphi \left( \nabla^2 h_{de} - \nabla_{(d} (\text{div} h)_{e)} - \left( \nabla_a \nabla_{(d} X_{b)} \right)^{ab}_{\quad e} \right)$$

$$+ \frac{1}{4} \nabla_d \nabla_e \text{Tr} h - \frac{1}{4} \left( \nabla^2 \text{Tr} h \right) g_{de} \right) + l.o.t. \quad (5.5)$$

From Proposition 4.8 we get that

$$D_\psi \Delta_\psi (\chi) = -\Delta_\psi \chi - L_{V(\chi)} \psi + 2d (\text{div} X) \varphi + dF (\chi)$$

so,

$$D_\psi \hat{Q}_\psi (\chi) = -\Delta_\psi \chi - L_{V(\chi)} \psi + 2d (\text{div} X) \varphi + dF (\chi)$$

$$+ 2d (A \hat{\varphi} - (\text{div} X) \varphi)$$

$$= -\Delta_\psi \chi - L_{V(\chi)} \psi + d\hat{F} (\chi)$$

where $\hat{F} = F + 2A \hat{\varphi}$, so is also a 3-form-valued function that is algebraic in $\chi$. To see that this $\hat{Q}_\psi$ is weakly parabolic, consider the alternative expression (4.35) for the linearization $\hat{Q}_\psi$ at a closed form $\chi$.

$$\pi_7 \left( D_\psi \hat{Q}_\psi \right) (\chi) = -d (\text{div} X) \wedge \varphi + l.o.t.$$  

$$= -\frac{1}{2} \left( \nabla_a (\text{div} X) + \nabla^2 X_a + \text{curl} (\text{div} h)_a \right) \wedge \varphi + l.o.t. \quad (5.6)$$

$$\pi_{1\oplus 27} \left( D_\psi \hat{Q}_\psi \right) (\chi) = \frac{3}{2} * i_\varphi \left( \nabla^2 h_{de} - \nabla_{(d} (\text{div} h)_{e)} - \nabla_a \nabla_{(d} X_{b)}^{\quad ab} 

- (\nabla_b \nabla_m h_{cn}) \varphi_{bc}^{\quad mn} \right) + l.o.t. \quad (5.7)$$
where we have used the relation (4.42c) to rewrite $\pi_7 \left( D_\psi \hat{Q}_\psi \right)$. From this, we can now write down the principal symbol of $D_\psi \hat{Q}_\psi$ evaluated at some vector $\xi$:

$$
\pi_7 \left( \sigma_\xi \left( D_\psi \hat{Q}_\psi \right) \chi \right) = \frac{1}{2} \left( \xi_a \xi_b X^b + |\xi|^2 X_a + \xi_m \xi_n h_{np} \varphi_{mp}^a \right) \wedge \varphi + \text{l.o.t.}
$$

$$
\pi_{1\oplus 27} \left( \sigma_\xi \left( D_\psi \hat{Q}_\psi \right) \chi \right) = \frac{3}{2} \left( (|\xi|^2 h_{de} - \xi \xi_{(d} \xi_{|a|} h^a_{e)}) - \xi \xi_{(d} \xi_{b)} \psi_{e} \right)
$$

$$
= \left( -\xi_b \xi_m h_{cn} \varphi_{bc} \psi_{(d} \varphi_{mn)} + \text{l.o.t.} \right)
$$

Then,

$$
\left\langle \sigma_\xi \left( D_\psi \hat{Q}_\psi \right) \chi, \chi \right\rangle = 2 |\xi|^2 |X|^2 + 2 (\xi_a X^a)^2 + \xi_m \xi_n h_{np} \varphi_{mp}^a X^a
$$

$$
+ |\xi|^2 |h|^2 - \xi a \xi h_{de} - \xi \xi_{b} \xi \xi_{c} \varphi_{bcd} \psi_{mn}
$$

$$
= \frac{7}{4} \left( (|\xi|^2 |X|^2 + (\xi_a X^a)^2) + 2 \omega_{ab} \omega^{ab} \right) \geq 0
$$

where

$$
\omega_{ab} = \xi_m \varphi_{mn} (a h^m) + \frac{1}{2} \xi (a X_b).
$$

Note that if $\chi \in \Lambda_4^1$, then $\left\langle \sigma_\xi \left( D_\psi \hat{Q}_\psi \right) \chi, \chi \right\rangle = 0$, hence $\hat{Q}_\psi$ is indeed only weakly parabolic in the direction of closed forms.

Now consider the evolution of the volume functional:

$$
\frac{dV}{dt} = \int_M \frac{d}{dt} \sqrt{\det g}.
$$

From (4.32) and applying (3.9a) to (5.3), we have

$$
\frac{d}{dt} \sqrt{\det g} = \left( \frac{1}{2} \left( |T|^2 + (\text{Tr } T)^2 \right) + 2 \text{Tr } T (A - \text{Tr } T) \right) \sqrt{\det g}
$$

$$
= \frac{1}{2} \left( |T|^2 + \text{Tr } T (4A - 3 \text{Tr } T) \right) \sqrt{\det g}.
$$

In particular we see that at every time $t$ along the flow the following holds

$$
A \int_M \text{Tr } T \text{vol} \geq \frac{3}{4} \int_M (\text{Tr } T)^2 \text{vol} \quad (5.6)
$$

then the volume functional grows along the flow (5.1). This is of course true if for positive $A$

$$
0 \leq \text{Tr } T \leq \frac{4}{3} A \quad (5.7)
$$

holds everywhere on $M$ for all $t$. So as long as the condition (5.6) holds, our flow satisfies all the desired properties. We can also get an alternative condition for the positivity of $\frac{dV}{dt}$. From (4.30) we have an expression for the scalar curvature $R$ of a co-closed $G_2$-structure:

$$
R = -|T|^2 + (\text{Tr } T)^2 \quad (5.8)
$$
Hence, we can rewrite the evolution of $\sqrt{\det g}$ as

$$\frac{d}{dt} \sqrt{\det g} = \frac{1}{2} (-R + 2 \text{Tr} T (2A - \text{Tr} T)) \sqrt{\det g}.$$  

Thus, $\frac{dV}{dt} \geq 0$ if and only if

$$\int_M R \text{vol} \leq 2 \int_M \text{Tr} T (2A - \text{Tr} T) \text{vol}$$  \hspace{1cm} (5.9)

In order to understand whether any of the conditions (5.6), (5.7) or (5.9) have any hope of holding along the flow, we first need to understand how the torsion evolves along the flow (5.1).

**Proposition 5.2** The evolution of the torsion tensor $T$ under the flow (5.1) is given by

$$\frac{d}{dt} T_{ab} = \Delta_L T_{ab} + (\text{Tr} T - 2A) (\text{curl} T)_{ab} + 4T^{mp} \left( \nabla_n T^{mn} \right) (\varphi_b)_{pm}$$

$$+ 2 \left( \nabla_\alpha T^{\alpha \mu \nu} \right) (\varphi_{b})_{\mu \nu} T_{ab} + 2 (\nabla^m \text{Tr} T) \varphi_{mn(a} T_{b)}^{\alpha} + ((\text{curl} T) \circ T)_{ab}$$

$$+ \frac{2}{3} \psi_{acde} \psi_{bamp} T^{cm} T^{dn} T^{ep} + 2 (T^3)_{ab} + 2 (A - \text{Tr} T) (T^2)_{ab} + \frac{1}{2} \left( |T|^2 - 3 |T|^2 \right) T_{ab}$$

$$+ g_{ab} \left( \frac{1}{2} (\text{Tr} T) |T|^2 - \frac{1}{3} \text{Tr} (T^3) + \frac{1}{6} \langle T \circ T, T \rangle - \frac{1}{6} (\text{Tr} T)^3 \right)$$

where $\Delta_L$ denotes the Lichnerowicz Laplacian, given by

$$\Delta_L T_{ab} = \nabla^2 T_{ab} - 2 R^e_{(b} T_{a)e} + 2 R_{acbd} T^{cd}$$

Also, the evolution of $\text{Tr} T$ under the same flow is given by

$$\frac{d}{dt} \text{Tr} T = \nabla^2 \text{Tr} T - \langle \text{curl} T, T \rangle - \frac{1}{2} \langle T \circ T, T \rangle - \text{Tr} (T^3) - 2 (A - \text{Tr} T) |T|^2$$  \hspace{1cm} (5.11)

**Proof.** From Proposition 3.2 we know that if

$$\frac{d}{dt} \psi = *(X \cdot \psi) + 3 * i_\varphi (h)$$

then,

$$\frac{dT_{ab}}{dt} = \frac{1}{4} (\text{Tr} h) T_{ab} - T_{a c} h_{cb} - T_{a c} X^d \varphi_{deb} + (\text{curl} h)_{ab} + \nabla_a X_b - \frac{1}{4} (\nabla_c \text{Tr} h) \varphi^c_{ab}.$$

Then using (4.26) and (4.23) we find that for the flow (5.1), we have

$$X = \nabla \text{Tr} T$$  \hspace{1cm} (5.12a)

$$h_{ab} = - \text{curl} (T)_{ab} - \frac{1}{2} (T \circ T)_{ab} - (T^2)_{ab} + \frac{1}{6} g_{ad} |T|^2 - 2 (A - \text{Tr} T) T_{ad}$$  \hspace{1cm} (5.12b)

$$+ \frac{1}{6} \text{Tr} T (4A - 3 \text{Tr} T) g_{ad}$$  \hspace{1cm} (5.12c)

$$\text{Tr} h = \frac{2}{3} \left( |T|^2 + \text{Tr} T (4A - 3 \text{Tr} T) \right)$$

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We then substitute $X, h$ and $\text{Tr } h$ into the expression for $\frac{dT_{ab}}{dt}$, and simplify using the contraction identities (2.4), the Ricci identity and the expression for the Ricci curvature (4.30). After a lengthy calculation, the expression (5.10) follows.

The evolution of $\text{Tr } T$ can be derived from (5.10), but an easier way is to calculate first for a general flow using Proposition (3.1) and Proposition 3.2, and then applying (5.12)

$$\frac{d}{dt} \text{Tr } T = \frac{d}{dt} (g_{ab} T_{ab})$$

$$= \left( \frac{d}{dt} g^{ab} \right) T_{ab} + g^{ab} \frac{d}{dt} (T_{ab})$$

$$= \left( -\frac{1}{2} (\text{Tr } h) g^{ab} + 2h^{ab} \right) T_{ab}$$

$$+ g^{ab} \left( \frac{1}{4} (\text{Tr } h) T_{ab} - T_a^c h_{cb} - T_a^c X^d \varphi_{dcb} + (\text{curl } h)_{ab} + \nabla_a X_b - \frac{1}{4} (\nabla_c \text{Tr } h) \varphi_{ac} \right)$$

$$= -\frac{1}{4} \text{Tr } h \text{Tr } T + \langle T, h \rangle + \text{div } X$$

$$= \nabla^2 \text{Tr } T - \langle \text{curl } (T), T \rangle - \frac{1}{2} \langle T \circ T, T \rangle - \text{Tr } (T^2) + \frac{1}{6} |T|^2 \text{Tr } T - 2 (A - \text{Tr } T) |T|^2$$

$$+ \frac{1}{6} (\text{Tr } T)^2 (4A - 3 \text{Tr } T) - \frac{1}{6} \left( |T|^2 + \text{Tr } T (4A - 3 \text{Tr } T) \right)$$

$$= \nabla^2 \text{Tr } T - \langle \text{curl } T, T \rangle - \frac{1}{2} \langle T \circ T, T \rangle - \text{Tr } (T^2) - 2 (A - \text{Tr } T) |T|^2$$

In (5.11), if $A$ is large enough and if $T_{ab}$ is positive definite, then at least the non-derivative terms will be negative. This is due to the following observation regarding the $G_2$ product $\circ$.

**Lemma 5.3** Suppose $A$ and $B$ are positive-definite symmetric 2-tensors. Then the product $A \circ B$ is also positive definite.

**Proof.** The matrices $A$ and $B$ have unique positive-definite symmetric square root matrices $A^{1/2}$ and $B^{1/2}$. So we can rewrite $A \circ B$ as

$$(A \circ B)_{ab} = \varphi_{amn} \varphi_{bpq} \left( A^{1/2} \right)_m^c \left( A^{1/2} \right)_p^d \left( B^{1/2} \right)_n^q \left( B^{1/2} \right)_d^a$$

$$= U_a U_b$$

where

$$U_a = \left( \varphi_{amn} \left( A^{1/2} \right)_m^c \left( B^{1/2} \right)_n^d \right).$$

Hence $A \circ B$ is indeed positive definite. ■

However in order to be able to use the Maximum Principle to conclude that $\text{Tr } T$ is always bounded from above, if it is initially so, we would need to show that a positive definite $T_{ab}$ remains positive definite along the flow and we also need to be able to control the term $\langle \text{curl } T, T \rangle$ in (5.11). Equivalently we would need to be able to control $\langle \text{Ric}, T \rangle$, since $\text{curl } T$ enters the expression for the Ricci curvature (4.30). This is certainly not obvious from (5.10), however given the similarities with Ricci flow, and corresponding results about the evolution of the Ricci and scalar curvatures along the Ricci flow, it seems to be a reasonable conjecture that $T_{ab}$ is positive and $\text{Tr } T$ does in fact satisfy (5.7) along the flow. The properties of the torsion and the curvature along the flow (5.1) will be subject to further research.

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6 Short-time existence and uniqueness

Let us now formulate the initial value problem for which we want to prove short-time existence and uniqueness:

\[
\begin{align*}
\frac{d}{dt} \psi &= \Delta \psi + 2d \left( (A - \text{Tr} T_\psi)^\ast \psi \right) \\
\frac{d\psi}{dt} &= 0 \\
\psi|_{t=0} &= \psi_0
\end{align*}
\]

Here we will adapt the method that was used by Bryant and Xu \cite{4} to show short-time existence and uniqueness for the Laplacian flow of \( \phi \) \cite{11}. Since for closed \( \psi(t) \), the right hand side of the flow is exact, for any \( t \), \( \psi(t) \) stays within the same cohomology class \([\psi_0]\) and hence we can write

\[
\psi(t) = \psi_0 + \chi(t)
\]

where \( \chi(t) \) is an exact form. We can thus rewrite (6.1) in an equivalent way:

\[
\begin{align*}
\frac{d}{dt} \chi &= \Delta \psi + 2d \left( (A - \text{Tr} T_\psi)^\ast \psi \right) \\
\chi \text{ is exact} \\
\chi|_{t=0} &= 0
\end{align*}
\]

As before, suppose \( \chi \) is given by

\[
\chi = \ast(X_\psi + 3i_\psi (h)).
\]

Then, as before, define the vector

\[
V(\chi) = \frac{3}{4} \nabla \text{Tr} h - 2 \text{curl} X
\]

From Theorem 5.1, we can say that the following initial value problem is parabolic in the direction of closed forms

\[
\begin{align*}
\frac{d}{dt} \chi &= \Delta \psi + 2d \left( (A - \text{Tr} T_\psi)^\ast \psi \right) + \mathcal{L}_V(\chi) \psi \\
\chi \text{ is exact} \\
\chi|_{t=0} &= 0
\end{align*}
\]

The idea is to first show short-time existence and uniqueness of solutions for the flow (6.5), and then from these, obtain solutions of (6.3) via diffeomorphisms. Since (6.5) is parabolic only in certain directions, standard theory of parabolic PDEs does not apply, and we thus have to use the Nash-Moser inverse function theorem for tame Fréchet spaces. This technique was first used by Hamilton for the Ricci flow \cite{15, 14} and Bryant and Xu for the Laplacian flow of \( \phi \) \cite{4}. We first review basic definitions as introduced by Hamilton \cite{14}.

**Definition 6.1**

1. A graded Fréchet space \( \mathcal{F} \) is a complete Hausdorff topological vector space with the topology defined by a collection of increasing semi-norms \( \{\|\cdot\|_n\}_{n=1}^\infty \), so that

\[
\|x\|_0 \leq \|x\|_1 \leq \|x\|_2 \leq ...
\]

for every \( x \in \mathcal{F} \), so that a sequence converges if and only if it converges with respect to each semi-norm and a subset \( \mathcal{U} \subset \mathcal{F} \) is open if and only if around every point \( x \in \mathcal{U} \) there exists an open ball contained in \( \mathcal{U} \) with respect to the semi-norm \( \|\cdot\|_n \) for some \( n \). Such a collection of norms is called a grading.
2. Two gradings \( \{ \| \cdot \|_n \}_{n=1}^\infty \) and \( \{ \| \cdot \|'_n \}_{n=1}^\infty \) are said to be tamely equivalent of degree \( r \) and base \( b \), if for all \( x \in F \)

\[
\| x \|_n \leq C(n) \| x \|'_{n+r} \quad \text{and} \quad \| x \|'_n \leq C(n) \| x \|_{n+r}
\]

(6.6)

for all \( n \geq b \).

3. Let \( F \) and \( G \) be graded Fréchet spaces. Then a linear map \( L : F \to G \) is tame if it satisfies a tame estimate of degree \( r \) and base \( b \), that is, for each \( n \geq b \) the following holds:

\[
\| Lx \|_n \leq C(n) \| x \|_{n+r}
\]

(6.7)

for some constant \( C(n) \) that may depend on \( n \). A tame linear map is continuous in the topology of \( F \).

4. Suppose \( P : F \to G \) is a continuous map. Then \( P \) is tame if it satisfies the following tame estimate of degree \( r \) and base \( b \):

\[
\| Px \|_n \leq C(n) \left( 1 + \| x \|_{n+r} \right)
\]

(6.8)

A tame map is said to be smooth tame if all its derivatives are tame.

5. Let \( (B, \| \cdot \|_B) \) be a Banach space, then \( \Sigma(B) \) denotes the graded Fréchet space of all sequences \( \{x_k\}_{k \in \mathbb{N}} \) in \( B \) such that for all \( n \geq 0 \)

\[
\left\| \{x_k\}_{k \in \mathbb{N}} \right\|_n := \sum_{k=0}^\infty e^{nk} \| x_k \|_B < \infty
\]

(6.9)

6. A graded Fréchet space \( F \) is tame if there exists a Banach space \( B \) and two tame linear maps \( L : F \to \Sigma(B) \) and \( M : \Sigma(B) \to F \) such that \( M \circ L \) is the identity on \( F \).

The main reason for introducing the Fréchet space formalism is the Nash-Moser inverse function theorem.

**Theorem 6.2 (Nash-Moser Inverse Function Theorem [14])** Let \( F \) and \( G \) be tame Fréchet spaces, and \( f : U \subset F \to G \) a smooth tame map. Suppose that

1. The derivative \( Df(x) : F \to G \) is a linear isomorphism for all \( x \in U \)
2. The map \( U \times G \to F \) given by

\[
(x, v) \mapsto (Df(x))^{-1} v
\]

is a smooth tame map. Then, \( f \) is locally invertible and each local inverse \( f^{-1} \) is a smooth tame map.

As shown in [14], if \( M \) is a compact manifold, and \( V \) is a vector bundle over \( M \), then the space \( C^\infty(M, V) \) of smooth sections of \( V \) is in fact a tame space. Different tamely equivalent gradings can be chosen, but usually either Sobolev or supremum norms are used. Moreover, suppose \( P \) is a non-linear (smooth) vector bundle differential operator, then it is also shown in
that it is in fact a smooth tame map. Hence Theorem 6.2 is very useful for the study of non-linear PDEs since it provides sufficient conditions for local existence of solutions.

Consider now tame spaces of time-dependent sections of vector bundles, as introduced in [15]. Let $u \in C^\infty([0,T] \times M, V)$ be a time dependent section of the vector bundle $V$ over $M$. Define

$$ |u|^2_n = \int_0^T |u(t)|^2_{H^n} \, dt $$

(6.10)

where $|\cdot|_{H^n}$ is the Sobolev $L^2$ norm of $u$ and its covariant derivatives up to degree $n$. Hence $|u|^n$ only takes into account space derivatives. Now, define

$$ \|u\|_n = \sum_{2j \leq n} \left| \left( \frac{\partial}{\partial t} \right)^j u \right|_{n-2j} $$

(6.11)

This is a weighted norm counting one time derivative equal to two space derivatives. In particular, the grading (6.11) makes $C^\infty([0,T] \times M, V)$ a tame space. We can also define a tamely equivalent grading $|[\cdot]|_n$ given by

$$ |[u]|_n = \sum_{2j \leq n} \left( \left( \frac{\partial}{\partial t} \right)^j u \right)_{n-2j} $$

(6.12)

where $[u]_n$ is the supremum norm of $u$ and its space derivatives up to degree $n$.

Since we are interested in the flow of exact forms (6.5), we introduce the set

$$ \mathcal{U} = \{ \chi \in dC^\infty([0,T] \times M, \Lambda^3(M)) : \psi_0 + \chi \text{ is a definite 4-form} \}. $$

(6.13)

This is an open set of the space $\mathcal{F}$ of time-dependent exact 4-forms on $M$:

$$ \mathcal{F} = dC^\infty([0,T] \times M, \Lambda^3(M)) $$

(6.14)

Also define the space $\mathcal{G}$ of time-independent exact 4-forms:

$$ \mathcal{G} = dC^\infty(M, \Lambda^3(M)) $$

(6.15)

Now let

$$ \mathcal{H} = \mathcal{F} \times \mathcal{G} $$

and consider the map

$$ F : \mathcal{U} \to \mathcal{H} $$

given by

$$ \chi \mapsto \left( \frac{d}{dt} \chi - \Delta_{\psi} \psi - 2d((A - \text{Tr} T_{\psi}) \ast_{\psi} \psi) - \mathcal{L}_{V(\chi)} \psi, \chi|_{t=0} \right) $$

(6.16)

Here $\chi = \psi - \psi_0$. Note that in [4] exactly the same spaces were considered for exact 3-forms rather than 4-forms, and the map involved the Laplacian flow operator for the 3-form $\varphi$. In [4, Proposition 4.2] it was shown that the spaces $dC^\infty([0,T] \times M, \Lambda^2(M))$, $dC^\infty(M, \Lambda^2(M))$ and hence their product are tame. However exactly the same proof applies to exact 3-forms. Thus we have
Proposition 6.3 ([4]) The space $\mathcal{F}$ is a tame Fréchet space with the grading $\|\cdot\|_n$ restricted from $C^\infty([0,T] \times M, \Lambda^3(M))$. The space $\mathcal{G}$ is a tame Fréchet space with the grading $|\cdot|_n$ and the product $\mathcal{H} = \mathcal{F} \times \mathcal{G}$ is a tame Fréchet space with the grading $\|\cdot\|_n + |\cdot|_n$.

The map $F$ is a smooth differential operator, so it is a tame map. Now to apply the Nash-Moser Theorem to the map $F (6.16)$ we thus need to show that the derivative $DF(\chi) : \mathcal{F} \rightarrow \mathcal{H}$ is an isomorphism for all $\chi \in \mathcal{U}$, and moreover that its inverse is smooth tame. Again, the proofs of these facts are almost exactly the same as in [4, Proposition 4.2].

Lemma 6.4 ([4, Lemmas 4.3 and 4.4]) Given the map $F (6.16)$, its derivative $DF(\chi) : \mathcal{F} \rightarrow \mathcal{H}$ is an isomorphism for all $\chi \in \mathcal{U}$.

Proof. From Theorem 5.1, we get that for closed $\theta$, the derivative of the map $F$ is given by

$$DF(\chi) \theta = \left( \frac{d}{dt} \theta + \Delta_\psi \theta + d\hat{F}(\theta), \theta|_{t=0} \right)$$ (6.17)

where $\psi = \psi_0 + \chi$. To show injectivity we need to show that the initial value problem for a closed 4-form $\theta$

$$\begin{cases} \frac{d}{dt} \theta + \Delta_\psi \theta + d\hat{F}(\theta) = 0 \\ \theta|_{t=0} = 0 \end{cases}$$ (6.18)

has a unique solution $\theta = 0$. However this is a linear parabolic PDE, and thus $\theta = 0$ is the unique solution.

To show surjectivity, we have to show that for any time-dependent exact 4-form $\eta$ and any exact 4-form $\theta_0$ there is a time-dependent 3-form $\theta$ which satisfies

$$\begin{cases} \frac{d}{dt} \theta + \Delta_\psi \theta + d\hat{F}(\theta) = \eta \\ \theta|_{t=0} = \theta_0 \end{cases}$$ (6.19)

Since $\theta$ and $\eta$ are all exact, we can write

$$\begin{align*}
\theta &= d\alpha \\
\eta &= d\beta
\end{align*}$$

for time-dependent 3-forms $\alpha$ and $\beta$. Correspondingly, $\theta_0 = d\beta_0$. Now consider the following initial value problem:

$$\begin{cases} \frac{d}{dt} \alpha + \Delta_\psi \alpha + \hat{F}(d\alpha) = \beta \\ \beta|_{t=0} = \beta_0 \end{cases}$$ (6.20)

From standard parabolic theory, there exists a unique solution $\beta(t)$. Hence $\theta = d\beta$ solves (6.19).

Now that we have that $DF(\chi)$ is an isomorphism for each $\chi \in \mathcal{U}$, we can define the family of inverse maps

$$(DF)^{-1} : \mathcal{U} \times \mathcal{H} \rightarrow \mathcal{F}$$

$$(\chi, \theta, \theta_0) \rightarrow (DF(\chi))^{-1}(\theta, \theta_0)$$
Lemma 6.5 The map \((DF)^{-1} : U \times H \to F\) is smooth tame.

Proof. From (6.4) we know that \(\theta = (DF)^{-1}(\chi, \eta, \theta_0)\) is the unique solution to the linear parabolic equation
\[
\begin{align*}
 \frac{d}{dt} \theta + \Delta_{\psi} \theta + d\hat{F}(\theta) &= \eta, \\
 \theta|_{t=0} &= \theta_0
\end{align*}
\] (6.21)
with \(\psi = \psi_0 + \chi\). It then follows from standard linear parabolic theory that \((DF)^{-1}\) is smooth tame. For details we refer the reader to [4, Lemma 4.7], where exactly the same result is proven for 3-forms. It applies in the same way to 4-forms.

Now we can apply the Nash Moser theorem to the map \(F\).

Lemma 6.6 ([4, Lemma 0.2]) Suppose \(\chi \in U\) is a solution to
\[
\begin{align*}
 \frac{d}{dt} \chi - \Delta_{\psi} \psi - 2d((A - \text{Tr} T_{\psi}) *_{\psi} \psi) - L_{\mathcal{V}(\chi)} \psi &= \eta, \\
 \chi|_{t=0} &= \chi_0
\end{align*}
\] (6.22)
Then for \((\bar{\eta}, \bar{\chi}_0) \in H\) sufficiently close to \((\eta, \chi_0)\), there is a unique solution \(\bar{\chi}(t)\) to the system
\[
\begin{align*}
 \frac{d}{dt} \bar{\chi} &- \Delta_{\bar{\psi}} \bar{\psi} - 2d((A - \text{Tr} T_{\bar{\psi}}) *_{\bar{\psi}} \bar{\psi}) - L_{\mathcal{V}(\bar{\chi})} \bar{\psi} = \bar{\eta}, \\
 \bar{\chi}|_{t=0} &= \bar{\chi}_0
\end{align*}
\] (6.23)
where \(\bar{\psi} = \psi_0 + \bar{\chi}\).

Proof. The derivative \(DF\) of the map \(F\) (6.16) satisfies the conditions of Theorem 6.2 and hence \(F\) itself is locally invertible. Therefore, given \((\bar{\eta}, \bar{\chi}_0) \in H\) sufficiently close to \((\eta, \chi_0)\), there exists a solution \(\bar{\chi}(t)\) to the gauge-fixed flow (6.5).

Corollary 6.7 The initial value problem (6.5) has a unique solution for the time period \([0, \varepsilon]\) for some \(\varepsilon > 0\).

Proof. Here we apply the same method as used in [4], and originally by Hamilton in [15]. Let \(\chi(t)\) be a family of 4-forms such that its formal Taylor series at \(t = 0\) is what it must be to solve (6.5) with \(\chi(0) = \chi_0\). This can be done by differentiating through the flow equation (6.5) and solving for \(\frac{d}{dt} \chi|_{t=0}\). Then let
\[
\eta(t) = \frac{d}{dt} \chi - \Delta_{\psi} \psi - 2d((A - \text{Tr} T_{\psi}) *_{\psi} \psi) - L_{\mathcal{V}(\chi)} \psi
\] (6.24)
where \(\psi = \psi_0 + \chi\). It follows that the formal Taylor series of \(\eta\) at \(t = 0\) is identically zero. We then extend \(\eta(t)\) such that \(\eta(t) = 0\) for \(t < 0\). Now define \(\bar{\eta}\) to be the translation of \(\eta\) by some \(\varepsilon > 0\)
\[
\bar{\eta}(t) = \eta(t - \varepsilon).
\]
Thus, we get a 4-form \(\bar{\eta}(t)\) which vanishes for \(t \in [0, \varepsilon]\) for some \(\varepsilon > 0\). We can then apply Lemma 6.6 to the two pairs \((\bar{\eta}, \chi_0)\) and \((\eta, \chi_0)\), possibly for a shorter time period \([0, \varepsilon']\). This
then gives existence of a unique solution $\bar{\chi}(t)$ to the system (6.23). However, since $\bar{\eta}$ vanishes up to time $\varepsilon'$, we get a unique solution to (6.5) for $t \in [0, \varepsilon']$.

Following [4], to prove short-time existence and uniqueness for the initial value problem (6.3), we need to relate (6.5) and (6.3) via diffeomorphisms. Let $\psi(t)$ be a family of $G_2$-structure 4-forms, and $\phi(t)$ a family of diffeomorphisms defined by the evolution equation

$$
\frac{d}{dt}\phi(t) = -V(\bar{\chi}_\phi(t))
$$

where $\bar{\chi}_\phi(t) = (\phi^{-1})^* \psi - \psi_0$ and $V$ is given by (6.4).

**Lemma 6.8** The flow (6.25) is strictly parabolic.

**Proof.** We have to linearize (6.25). Suppose

$$
\frac{\partial \phi}{\partial s} \bigg|_{s=0} = U
$$

for some vector field $U$. Then,

$$
\frac{\partial (\phi^{-1})^* \psi}{\partial s} \bigg|_{s=0} = - (\phi^{-1})_* U
$$

and hence,

$$
\frac{\partial (\phi^{-1})^* \psi}{\partial s} \bigg|_{s=0} = (\phi^{-1})_* \mathcal{L} (\phi^{-1})_* U \psi
$$

$$
= -\mathcal{L}_U (\phi^{-1})^* \psi
$$

(6.28)

Now, define $\bar{\psi} = (\phi^{-1})^* \psi$. This defines now a new $G_2$-structure. So we have

$$
\frac{\partial \bar{\psi}}{\partial s} \bigg|_{s=0} = -\mathcal{L}_U \bar{\psi}
$$

(6.29)

So on the right hand side of (6.25), $\bar{\chi} = \bar{\psi} - \psi_0$ and whence,

$$
- V(\bar{\chi}) = - \left( \frac{3}{4} \bar{\nabla}^a \text{Tr} h + (\overline{\text{curl} X})^a \right)
$$

(6.30)

where everything is with respect to the $G_2$-structure $\bar{\psi}$, and $\bar{\chi}$ is given by

$$
\bar{\chi} = * \left( X \nabla \bar{\psi} + 3i_{\nabla} (h) \right).
$$

Hence

$$
- \frac{\partial V(\bar{\chi})}{\partial s} \bigg|_{s=0} = - \frac{\partial}{\partial s} \left( \frac{3}{4} \bar{\nabla}^a \text{Tr} h + (\overline{\text{curl} X})^a \right) \bigg|_{s=0}
$$

Now,

$$
-\mathcal{L}_U \bar{\psi} = -d \left( U \nabla \bar{\psi} \right) - U \nabla \bar{\psi}
$$

$$
= -d \left( U \nabla \bar{\psi} \right)
$$

[34]
since $\bar{\psi}$ is closed, so from Proposition 2.3 we get the type decomposition of $-\mathcal{L}_U\bar{\psi}$

$$\pi_7 (-\mathcal{L}_U\bar{\psi}) = \frac{1}{2} (\text{curl} U) \wedge \bar{\varphi} + l.o.t.$$ (6.31)

$$\pi_{1\oplus 27} (-\mathcal{L}_U\bar{\psi}) = 3 * i_{\varphi} \left( \nabla_{(m} U_{n)} - \frac{1}{3} (\text{div} U) \bar{g}_{mn} \right) + l.o.t.$$ (6.32)

Since

$$\left. \frac{\partial \bar{X}}{\partial s} \right|_{s=0} = \left. \frac{\partial \bar{\psi}}{\partial s} \right|_{s=0} = -\mathcal{L}_U\bar{\psi}$$

we find that

$$\left. \frac{\partial X_a}{\partial s} \right|_{s=0} = \frac{1}{2} (\text{curl} U)_a + l.o.t.$$ $$\left. \frac{\partial h_{mn}}{\partial s} \right|_{s=0} = \nabla_{(m} U_{n)} - \frac{1}{3} (\text{div} U) \bar{g}_{mn} + l.o.t.$$ and therefore,

$$\left. \frac{\partial \left( \bar{\nabla}^a \text{Tr} h \right) }{\partial s} \right|_{s=0} = -\frac{4}{3} \bar{\nabla}^a \text{div} U + l.o.t.$$ $$\left. \frac{\partial \left( \bar{\nabla}_m X_n \bar{\varphi}^{mna} \right) }{\partial s} \right|_{s=0} = \frac{1}{2} \left( \nabla_{m} \nabla_{p} U_{q} \right) \bar{\varphi}^{pq} \bar{\varphi}^{mna} + l.o.t. \quad = -\frac{1}{2} \nabla^2 U^a + \frac{1}{2} \bar{\nabla}^a \text{div} U + l.o.t.$$ where in the last line we have used once again the identity (2.4a). Whence, the linearization $D_\phi V$ of $V$ at $\phi$ is given by

$$\left( D_\phi V \right) (U) = \frac{1}{2} \nabla^2 U^a + \frac{1}{2} \bar{\nabla}^a \text{div} U + l.o.t.$$ (6.33)

Given some non-zero vector field $\xi$, the principal symbol is now

$$\sigma_\xi \left( D_\phi V \right) (U) = \frac{1}{2} |\xi|^2 U^a + \frac{1}{2} \xi^a \langle \xi, U \rangle$$

and hence

$$\langle \sigma_\xi \left( D_\phi V \right) (U), U \rangle = \frac{1}{2} |\xi|^2 |U|^2 + \frac{1}{2} \langle \xi, U \rangle^2 \geq 0$$

with equality if and only if $U$ is identically zero. So indeed, the flow (6.25) is indeed strictly parabolic.

Now we are finally ready to prove the short-time existence and uniqueness for the initial value problem (6.1). The proof is again modelled on [4].

**Theorem 6.9** The initial value problem (6.1) has a unique solution for $t \in [0, \varepsilon]$. 

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Proof. From Corollary 6.7 we have a short-time solution $\bar{\chi} = \bar{\psi} - \psi_0$ for (6.5). In (6.25) let $\bar{\chi}_\phi(t) = \bar{\chi}(t)$, so that it now becomes an ODE, since $\bar{\chi}(t)$ is now a fixed family of 4-forms, independent of the diffeomorphisms $\phi_t$.

$$\left\{ \begin{array}{l} \frac{d}{dt} \psi(t) = -V(\bar{\chi}(t)) \\ \phi|_{t=0} = Id \end{array} \right. \quad (6.34)$$

The ODE (6.34) has a unique solution $\phi_t$. Now let

$$\psi(t) = (\phi_t)^* \bar{\psi}(t). \quad (6.35)$$

Since $\phi_0 = Id$,

$$\psi(0) = \psi_0.$$

Also,

$$d\psi(t) = d((\phi_t)^* \bar{\psi}(t)) = (\phi_t)^* d\bar{\psi}(t) = 0$$

since diffeomorphisms commute with flow equation (6.1).

Now let us prove uniqueness. Suppose $\psi_1$ and $\psi_2$ are two solutions of (6.25). For $i = 1, 2$ define

$$\bar{\psi}_i = (\phi_i^{-1})^* \psi_i. \quad (6.37)$$

Both $\bar{\psi}_i$ are clearly closed, and satisfy

$$\frac{d}{dt} \bar{\psi}_i(t) = (\phi_i^{-1})^* \left( \mathcal{L}_{-V(\bar{\chi})} \psi_i(t) \right) + (\phi_i^{-1})^* \left( \frac{d}{dt} \psi_i(t) \right) \quad (6.38)$$

$$= \mathcal{L}_{V(\bar{\chi})} \left( (\phi_i^{-1})^* \psi_i \right) + (\phi_i^{-1})^* \left( \Delta_{\bar{\psi}_i} \psi_i + 2d \left( (A - Tr T_{\psi_i}) \star \psi_i \psi_i \right) \right)$$

$$= \mathcal{L}_{V(\bar{\chi})} \bar{\psi}_i + \Delta_{\bar{\psi}_i} \bar{\psi}_i + 2d \left( (A - Tr T_{\bar{\psi}_i}) \star \bar{\psi}_i \bar{\psi}_i \right)$$

Thus indeed, $\bar{\psi}(t)$ solves the system (6.11) for a short time $0 \leq t \leq \varepsilon$. Now let us prove uniqueness. Suppose $\psi_1$ and $\psi_2$ are two solutions of (6.1), and let $\phi_1$ and $\phi_2$ be corresponding solutions of (6.25). For $i = 1, 2$ define

$$\bar{\psi}_i = (\phi_i^{-1})^* \psi_i. \quad (6.37)$$

Both $\bar{\psi}_i$ are clearly closed, and satisfy

$$\frac{d}{dt} \bar{\psi}_i(t) = (\phi_i^{-1})^* \left( \mathcal{L}_{-V(\bar{\chi})} \psi_i(t) \right) + (\phi_i^{-1})^* \left( \frac{d}{dt} \psi_i(t) \right) \quad (6.38)$$

$$= \mathcal{L}_{V(\bar{\chi})} \left( (\phi_i^{-1})^* \psi_i \right) + (\phi_i^{-1})^* \left( \Delta_{\bar{\psi}_i} \psi_i + 2d \left( (A - Tr T_{\psi_i}) \star \psi_i \psi_i \right) \right)$$

$$= \mathcal{L}_{V(\bar{\chi})} \bar{\psi}_i + \Delta_{\bar{\psi}_i} \bar{\psi}_i + 2d \left( (A - Tr T_{\bar{\psi}_i}) \star \bar{\psi}_i \bar{\psi}_i \right)$$
This is precisely the flow equation (6.5). Now, \( \phi_i(0) = \text{Id} \), and \( \psi_i(0) = \psi_0 \), so \( \bar{\psi}_i \) have same initial conditions. However, from Corollary 6.7, we know that the system (6.5) has a unique solution, whence \( \bar{\psi}_1 = \bar{\psi}_2 \). Then, \( \phi_1 \) and \( \phi_2 \) satisfy the same ODE (6.31) with the same initial conditions, so are also equal. Therefore, \( \psi_1 = \psi_2 \). ■

7 Concluding remarks

We have thus found a modified Laplacian coflow of co-closed \( G_2 \)-structures given by

\[
\frac{d}{dt} \psi(t) = \Delta_\psi \psi + 2d((A - \text{Tr} \, T_\psi) \ast_\psi \psi)
\]

for a family of closed 4-forms \( \psi(t) \) and a constant \( A \). Given an initial condition \( \psi(0) = \psi_0 \), this flow was found to have a unique short-time existence. Moreover, if along the flow \( \text{Tr} \, T_\psi \) remains non-negative and less than or equal to \( \frac{4}{3} A \), the volume functional \( V \) (3.17) increases monotonically. An alternative necessary and sufficient condition for this is a bound for the total scalar curvature, in terms of \( \text{Tr} \, T_\psi \):

\[
\int_M R \text{vol} \leq 2 \int_M \text{Tr} \, T (2A - \text{Tr} \, T) \text{vol}
\]

Further questions can be asked about this flow, in particular, the long-term existence of solutions. In general, one would expect singularities to develop in finite time, but perhaps there are some initial conditions which lead to smooth long-term solutions. A more likely property is the stability of solutions - if the initial condition is sufficiently close to a torsion-free \( G_2 \)-structure, whether that would lead to a long-term solution. It is possible that the method used by Xu and Ye in [25] for the Laplacian flow of \( \varphi \) could be adapted in this scenario. This will be the subject of further study. Another interesting question is whether given solutions of the modified flow (6.1) one can find solutions of the original Laplacian coflow of \( \psi \) (1.3). The answers to these questions should lead to a better understanding of the relationships between different torsion classes of \( G_2 \)-structures, and torsion-free \( G_2 \)-structures, in particular.

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