Double field theory inspired cosmology

Houwen Wu and Haitang Yang

Center for theoretical physics, College of Physical Science and Technology, Sichuan University, Chengdu, 610064, China

E-mail: 2013222020003@stu.scu.edu.cn, hyanga@scu.edu.cn

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Abstract. Double field theory proposes a generalized spacetime action possessing manifest T-duality on the level of component fields. We calculate the cosmological solutions of double field theory with vanishing Kalb-Ramond field. It turns out that double field theory provides a more consistent way to construct cosmological solutions than the standard string cosmology. We construct solutions for vanishing and non-vanishing symmetry preserving dilaton potentials. The solutions assemble the pre- and post-big bang evolutions in one single line element. Our results show a smooth evolution from an anisotropic early stage to an isotropic phase without any special initial conditions in contrast to previous models. In addition, we demonstrate that the contraction of the dual space automatically leads to both an inflation phase and a decelerated expansion of the ordinary space during different evolution stages.

Keywords: string theory and cosmology, physics of the early universe

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1 Introduction

Double field theory (DFT) was first proposed to realize T-duality explicitly at the level of component fields of closed string field theory [1–3]. Earlier efforts are traced back to [4, 5]. The equivalence of spacetime momenta and winding numbers in the string spectra leads to an introduction of a set of dual coordinates \( \tilde{x}^i \), conjugated to winding numbers. These dual coordinates are treated on the same footing as the usual coordinates \( x^i \). The dimensionality of spacetime is then doubled from \( D \) to \( D + D \). T-duality is manifested as an \( O(D,D) \) symmetry in the action. The full set of coordinates are denoted as \( X^M = (\tilde{x}^i, x^i) \), where \( M = 1, 2, \ldots, 2D \) is the \( O(D,D) \) index, \( x^i (i = 1, 2, \ldots, D) \) is the usual spacetime coordinate and \( \tilde{x}^i \) represents the dual coordinate. All the spacetime component fields are dependent on both the usual and the dual coordinates: \( \phi_I(\tilde{x}^i, x^i) \).

A full construction of DFT is still an open question. The current research focuses on the massless sector of closed string spectra. The field content includes a \( D \) dimensional metric \( g_{ij} \), a scalar dilaton \( \phi \) and the anti-symmetric Kalb-Ramond field \( b_{ij} \). The manifestly \( O(D,D) \) invariant action built on the generalized \( O(D,D) \) metric tensor \( \mathcal{H}^{MN} \) is

\[
S = \int d^Dx d\tilde{x} e^{-2d} \left( \frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL} - \frac{1}{2} \mathcal{H}^{MN} \partial_N \mathcal{H}^{KL} \partial_L \mathcal{H}_{MK} - \partial_M d \partial_N \mathcal{H}^{MN} + 4 \mathcal{H}^{MN} \partial_M d \partial_N d \right),
\]

where the dilaton \( d = \phi - \frac{1}{4} \ln g \) is an \( O(D,D) \) scalar. The indices \( M, N, \ldots \) are raised and lowered by the off-diagonal \( O(D,D) \) metric and are all contracted. The component fields \( g^{ij} \) and \( b^{ij} \) enter the action via the definition

\[
\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix}.
\]
The level matching condition in closed string theory imposes the weak constraint 
\[ \partial \tilde{\partial} \phi (x, \tilde{x}) = 0 \] 
for an arbitrary field \( \phi (x, \tilde{x}) \). The action is invariant under the gauge transformation 
\[ \delta \tilde{H}^{MN} = \hat{\mathcal{L}}_{\xi} H^{MN} \equiv \xi^P \partial_P H^{MN} + (\partial^M \xi_P - \partial_P \xi^M) H^{PN} + (\partial^N \xi_P - \partial_P \xi^N) H^{MP}, \] 
with gauge parameters \( \xi^M = (\tilde{\xi}^i, \xi^i) \) and “generalized Lie derivatives” \( \hat{\mathcal{L}}_{\xi} \). The gauge transformation \( \xi_i \) is the traditional diffeomorphism and \( \tilde{\xi}^i \) is the dual diffeomorphism. To ensure the action is locally equivalent to the low energy effective string action, a strong constraint is needed: \( \partial \tilde{\partial} = 0 \) as an operator equation, acting on any products of the fields. This strong constraint is also sufficient in the construction of DFT based on closed string field theory beyond cubic order. The low energy effective action of closed string theory is 
\[ S^* = \int d^D x \sqrt{-g} e^{-2\phi} \left[ R + 4 (\partial_\mu \phi)^2 - \frac{1}{12} H_{ijk} H^{ijk} \right], \]
where \( R \) is the Ricci scalar constructed from the string metric \( g_{\mu \nu} \), \( \phi \) is the usual diffeomorphic dilaton and \( H_{ijk} = 3 \partial_i b_{jk} \) is the field strength of the Kalb-Ramond \( b_{ij} \) field. This action, also named as tree-level string action, is the foundation of tree level string cosmology.

Since the pioneer work of Hull and Zwiebach \([2]\), many progresses have been achieved. Good reviews are referred to \([6–8]\) and various developments can be found in \([9–50]\). However, to our knowledge, solutions of the action (1.1) have not been constructed. The main purpose of this paper is to find cosmological solutions of DFT. In the traditional string cosmology, various solutions are constructed by the scale factor duality and time reversal symmetry. However, all the solutions are self-contained. There exists no natural way to combine two solutions together to cover the whole spacetime of the pre- and post-big bangs. It is of interest to note that the scale factor duality is an intrinsic property of DFT when applied to a FRW like metric. This observation makes it possible to include the pre- and post-big bangs in one single line element. We will show that this unification of the pre- and post-big bangs manifests the existence of extra dimensions. Moreover, we demonstrate that the universe starts from an visibly anisotropic phase in the pre-big bang, evolves to an isotropic big bang and continues to an asymptotically isotropic post-big bang. This whole process needs no free parameters.

In order to simplify the story, in this paper, we set the Kalb-Ramond field \( b_{ij} = 0 \).\(^1\) We suppose the line element is FRW like
\[ ds^2 = \tilde{g}^{ij} d\tilde{x}_i d\tilde{x}_j + g_{ij} dx^i dx^j \]
\[ = -dt^2 + \tilde{a} (t, \tilde{t})^2 \delta^{ij} d\tilde{x}_i d\tilde{x}_j \]
\[ -dt^2 + a (t, \tilde{t})^2 \delta_{ij} dx^i dx^j, \]
where we put tildes on \( \tilde{g}^{ij} = \tilde{g}^{ij} \) to remind us that it is related to \( \tilde{x} \) for convenience of our calculation. This notation is introduced in section 3.2. One can see that the non-vanishing

\(^1\)In a follow-up work, we will incorporate non-vanishing \( b \) field. From the off-diagonal non-vanishing components in the generalized metric (1.2), one can expect the appearance of cross terms \( dx^i d\tilde{x}^j \). These cross terms can not be found by the traditional string cosmology and are novel.
components of the generalized metric $H_{MN}$ in (1.2) are only $g_{ij}$ and $g^{ij}$. Therefore, we conclude $\tilde{a}(t, \tilde{t}) = a^{-1}(t, \tilde{t})$.

For closed strings live on a torus, all states ought to respect the level matching condition:

$$L_0 - \bar{L}_0 = N - \bar{N} - p_i w^i = 0,$$

where $p_i$ is the momentum operator and $w^i$ is the winding number. If we restrict ourselves to the massless sector, the level matching condition becomes

$$L_0 - \bar{L}_0 = -p_i w^i = 0,$$

where $N = \bar{N} = 1$. In the language of DFT, the level matching condition is

$$\partial^M \partial_M A = 0,$$

for an arbitrary field $A$,

(1.8)

where $\partial_M = (\tilde{\partial}^i, \partial_i)$ and $M$ runs from 1 to $2D$. This condition is usually named as weak constraint. However, it turns out that the weak constraint is insufficient to make the theory consistent, since the weak constraint can not guarantee the gauge invariance

$$\tilde{\partial}^i \partial_i (\delta \Phi) = \tilde{\partial}^i \partial_i (\xi \cdot \Phi) \neq 0$$

(1.9)

where $\Phi$ is a DFT field and $\xi$ is a gauge parameter. In order to solve this problem, a more strict condition “strong constraint” has to be introduced

$$\partial^M \partial_M (\cdot) = 0,$$

(1.10)

where $\cdot$ stands for any product of fields or gauge parameters. This condition requires all fields and parameters to be dependent on only one set of coordinates, $x$ or $\tilde{x}$ but not both.

In recent works, there are various progresses to relax the strong constraint for DFT living on a torus background, massive type IIA and gauged supergravity [19, 21, 23]. In particular, the works [21, 23, 46] make efforts to relax even the weak constraint. In order to relax the strong constraint, two conditions must be satisfied [8]. The first one is to find the action and the gauge transformations, which only require weak constraint $\partial^M \partial_M A = 0$ to preserve a gauge invariance of the action and a closure of the gauge algebra. The second requirement is that the symmetry variations must be compatible with weak constraint. For example, considering the bosonic sector of Type II double field theory, after rewriting the gauge transformations as $\delta \xi \chi = \xi \delta \chi$, one only needs the weak constraint $\partial^M \partial_M A = 0$, $A = \{ \chi, \lambda, \xi^M \}$, but not strong constraint $\partial^M \partial_M (A \cdot B) = 0$ to close the gauge algebra and gauge invariance of RR action. However, the symmetry variation is incompatible with the weak constraint in this case, since $\tilde{\partial}^i \partial_i (\delta \chi) \neq 0$. Instead adopting the strong constraint, one can use a weakened strong constraint. In this weakened strong constraint, dependence on $\tilde{x}$ only shows up linearly and has no cross with $x$ for RR fields [8]. Therefore, the second requirement is also respected. In [21, 23, 46], the authors discussed the Scherk-Schwarz compactification of double field theory. In this scenario, the corresponding gauged supergravity is gauge invariant and consistent with a relaxation of constraint. The detailed discussions on strong constraint and its relaxation are summarized in [8, 30].

In this paper, since we begin with a non-compact background, the strong constraint can not be relaxed. Therefore, although at the first place, for usefulness during the analysis, we assume the metric components depending on two sets of coordinates $(x, \tilde{x})$, we will only
select solutions satisfying the strong constraint, depending on only half of the coordinates. This choice also avoids the double time trouble.

There are three dilatons in double field theory. The $O(D,D)$ scalar dilaton $d$ is invariant under $O(D,D)$ transformation. The traditional diffeomorphic scalar dilaton $\phi$ is invariant under gauge transformation $\xi$. The dual diffeomorphic scalar dilaton $\tilde{\phi}$ is invariant under dual gauge transformation $\tilde{\xi}$. With the metric (1.5), the relationship between the three dilatons is

\[ d = \phi - \frac{D-1}{2} \ln a, \]

\[ \phi = \tilde{\phi} + (D-1) \ln a. \]  

(1.11)

We will show that the dilaton $d$ is precisely the “shifted dilaton” in string cosmology. The second equation of (1.11) is nothing but the scale factor duality of the dilatons in the standard string cosmology.

Before devoted to calculations, we clarify that the continuous $O(D,D)$ symmetry is a very fundamental symmetry. When we compactify $d$ dimensions of $D = n + d$, this symmetry breaks to $O(n,n) \times O(d,d;Z)$, where $O(n,n)$ is still a continuous group and $O(d,d;Z)$ represents T-duality in the compactified background. Since no compactification presents in string cosmology, the scale factor duality in string cosmology is not T-duality but a realization of the continuous $O(D,D)$ symmetry specifically for the FRW metric. The $O(D,D)$ symmetry enables us to easily find solutions of DFT from these of string cosmology [51].

The main purpose of this paper is to put forward cosmological solutions of DFT and discuss their physical interpretations for two scenarios. We first start from the action (1.1). After substituting the metric ansatz (1.5) into the equations of motion (EOM) derived from the DFT action (1.1), we obtain two distinct metrics, the pre-big bang metric $dS^2_{\text{pre}}$ for $t < 0$ and the post-big bang metric $dS^2_{\text{post}}$ for $t > 0$. Each of the solutions consists of a pair of $O(D,D)$ connected solutions of string cosmology. This is different from the story in string cosmology, where the four line elements are completely disconnected. Both solutions unify contracting and expanding dimensions. To cover the whole spacetime and have a clearer physical picture, we introduce an $O(D,D)$ invariant dilaton potential to smooth out the singularity. We then have a unique line element describing the whole geometry, from the far past pre-big bang to the post-big bang, in contrast to string cosmology where two disjointed metrics exist. Remarkably, in both scenarios, $V(d) = 0$ and $V(d) \neq 0$, the solutions manifest that extra dimensions are naturally hidden in the far past and far future. Moreover, we explicitly show that the solutions reveal an intrinsic evolution of the universe from an isotropic pre-big bang phase to an anisotropic big bang and then to an isotropic post-big bang phase. All these new features originate from the $O(D,D)$ symmetry of the theory. This is consistent with the modern point of view that symmetries dictate physics. We further demonstrate that both the inflation and decelerated expansion of space can be triggered by the contraction of the dual space.

The reminder of this paper is outlined as follows. In section 2, we give a brief review on the relevant results we need in string cosmology. Section 3 refers to the EOM of the generalized double action for generic metrics and the FRW like metric. In section 4, we give the cosmological solutions of double field theory with vanishing and non-vanishing dilaton potential. Section 5 is our conclusion and discussions. We put some details of the calculation in the appendix.

2 A brief review of string cosmology
Since our discussions are closely related to the standard string cosmology, in this section, we briefly review the tree level results in string cosmology. A comprehensive treatment is referred to [52], on which our review is based and references therein.

We start with the tree level string action. For the reason of simplicity, we only consider the gravi-dilaton system without any matter sources. The anti-symmetric Kalb-Ramond field $b_{ij}$ is set to vanish. The action is given by

$$S = \frac{1}{2\kappa^2} \int d^Dx \sqrt{-g} e^{-2\phi} \left[ R + 4 (\partial_\mu \phi)^2 \right],$$  

(2.1)

where $D$ is the spacetime dimension, $\phi$ represents the dilaton which is a function of $t$, and $g_{\mu\nu}$ is the string metric. Note that the string metric is related to Einstein metric by $g^{\text{E}}_{\mu\nu} = \exp \left( -\frac{4}{D-1} \phi \right) g_{\mu\nu}$. The EOM are

$$R_{\mu\nu} + 2 \nabla_\mu \nabla_\nu \phi = 0,$$

$$\nabla^2 \phi - 2 (\partial_\mu \phi)^2 = 0.$$  

(2.2)

An isotropic metric is adopted to study string cosmology

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j, \quad H(t) \equiv \frac{\dot{a}(t)}{a(t)}.$$  

(2.3)

With this metric, from (2.2), the EOM of the graviton and the dilaton take the form

$$\ddot{\phi} - 2 \dot{\phi}^2 + (D - 1) H \dot{\phi} = 0,$$

$$2 \ddot{\phi} - (D - 1) \left( \dot{H} + H^2 \right) = 0,$$

$$\dot{H} + (D - 1) H^2 - 2H \dot{\phi} = 0.$$  

(2.4)

In the convention of string cosmology, it is convenient to introduce the “shifted dilaton”

$$\psi = 2\phi - (D - 1) \ln a.$$  

(2.5)

We will show that this “shifted dilaton” $\psi$ is the $O(D, D)$ scalar dilaton $2\phi$ in double field theory. Therefore, (2.4) becomes

$$2 \ddot{\psi} - \dot{\psi}^2 - (D - 1) H \dot{\psi} = 0,$$

$$-(D - 1) H^2 + \ddot{\psi} = 0,$$

$$\dot{H} - \dot{\psi} H = 0.$$  

(2.6)

These equations are invariant under the transformation of the famous scale factor duality

$$a \to \tilde{a} = a^{-1}, \quad \phi \to \tilde{\phi} = \phi - (D - 1) \ln a,$$  

(2.7)

which leads to

$$\tilde{H} = -H, \quad \tilde{\psi} = 2\tilde{\phi} - (D - 1) \ln a^{-1} = \psi.$$  

(2.8)

There are only two independent equations in (2.6). After applying the scale factor duality on the solution of (2.6), one hence obtains

$$1: \quad a(t) = \left( \frac{t}{t_0} \right)^{1/\sqrt{D-1}}, \quad \psi = -\ln \left( \frac{t}{t_0} \right).$$  

(2.9)
\[ \dot{a}(t) > 0, \text{ expansion} \quad \ddot{a}(t) < 0, \text{ decelerated} \quad \dot{H} < 0, \text{ decreasing curvature} \quad \text{post-big bang} \]

\[ \dot{a}(t) < 0, \text{ contraction} \quad \ddot{a}(t) > 0, \text{ decelerated} \quad \dot{H} > 0, \text{ decreasing curvature} \quad \text{post-big bang} \]

\[ \dot{a}(-t) < 0, \text{ contraction} \quad \ddot{a}(-t) < 0, \text{ accelerated} \quad \dot{H} < 0, \text{ increasing curvature} \quad \text{pre-big bang} \]

\[ \dot{a}(-t) > 0, \text{ expansion} \quad \ddot{a}(-t) > 0, \text{ accelerated} \quad \dot{H} > 0, \text{ increasing curvature} \quad \text{pre-big bang} \]

Table 1. The properties of the solutions in tree level string cosmology.

|   | \( \dot{a}(t) \) | \( \ddot{a}(t) \) | \( \dot{H} \) | Post-Bang \n|---|---|---|---|---|
| 1 | > 0 | < 0 | < 0 | expanding, decreasing curvature |
| 2 | < 0 | > 0 | > 0 | contracting, decreasing curvature |
| 3 | < 0 | < 0 | < 0 | contracting, increasing curvature |
| 4 | > 0 | > 0 | > 0 | expanding, increasing curvature |

Figure 1. Hubble parameters in four solutions.

and its dual solution

\[ 2 : \quad \tilde{a}(t) = \left( \frac{t}{t_0} \right)^{-1/\sqrt{D-1}}, \quad \tilde{\psi} = -\ln \left( \frac{t}{t_0} \right). \quad (2.10) \]

The equations (2.6) also possesses a “time reversal” symmetry \( t \rightarrow -t \). Therefore, there are two more solutions

\[ 3 : \quad \tilde{a}(-t) = \left( \frac{-t}{t_0} \right)^{1/\sqrt{D-1}}, \quad \tilde{\psi} = -\ln \left( \frac{-t}{t_0} \right) \quad (2.11) \]

and

\[ 4 : \quad \tilde{a}(-t) = \left( \frac{-t}{t_0} \right)^{-1/\sqrt{D-1}}, \quad \tilde{\psi} = -\ln \left( \frac{-t}{t_0} \right) \quad (2.12) \]

In summary, four branches are found. The properties of the solutions are listed in table 1.

Note that deceleration occurs when sign \( \dot{a} = -\text{sign} \ddot{a} \), acceleration occurs when sign \( \dot{a} = \text{sign} \ddot{a} \). When \( H^2 \) or \( \dot{H}^2 \) is growing with time, the curvature is increasing, otherwise, the curvature is decreasing. Moreover, when \( H > 0 \), the universe is expanding, otherwise, the universe is contracting. All these solutions share the curvature singularity located at \( |t| \rightarrow 0 \), as illustrated in figure 1.

In order to group the solutions, a “self-dual” \( \tilde{a}(t) = a^{-1}(-t) \) is introduced. This duality pairs solution 1 with 4, an accelerated expansion followed by a decelerated expansion; solution
2 with 3, an accelerated contraction followed by a decelerated contraction. Each pair covers the whole spacetime except the singularity. It is natural to name the $t < 0$ phase as the pre-big bang and the $t > 0$ region as the post-big bang.

It is not surprising that, by including some matter sources or dilaton potentials, one can smooth the singularity to connect the pre- and post-big bangs, as illustrated in figure 2. These models are of great help to understand the physics in the region $t \sim 0$.

However, it should be noted that only one pair of solutions can be chosen to describe the evolution of the universe, between the two choices: $4 \rightarrow 1$ or $3 \rightarrow 2$. This observation makes it difficult for the standard string cosmology to embody curved up extra dimensions and the widely accepted anisotropy in the early stage of the universe. We are going to show that DFT cosmology provides a natural way to solve these two problems in section 4.

3 Equations of motion of double field theory

The $O(D, D)$ invariance of the action enables us to construct infinitely many solutions from a single solution. However, general solutions of DFT, dependent both on $x$ and $\tilde{x}$, are beyond the solution space of the low energy effective action, and can only be calculated from the EOM of DFT. Therefore, it is of importance to present the EOM of DFT for general metrics and the FRW like metric.

3.1 Equations of motion for general metrics

Before we calculate the cosmological solutions, it is of help to review the EOM of action (1.1) for general metrics. The EOM of the dilaton can be obtained directly by variation to the action:

$$\frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL} - \frac{1}{2} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_K \mathcal{H}_{NL}$$

$$- \partial_M \partial_N \mathcal{H}^{MN} - 4 \mathcal{H}^{MN} \partial_M \partial_N d + 4 \partial_M \mathcal{H}^{MN} \partial_N d + 4 \mathcal{H}^{MN} \partial_M \partial_N d = 0. \quad (3.1)$$

Next, we calculate the EOM of the graviton. Varying the action with respect to the generalized metric $\mathcal{H}^{MN}$, we get

$$\delta \mathcal{H} S = \int dx d\tilde{x} e^{-2d} \delta \mathcal{H}^{MN} \mathcal{K}_{MN}, \quad (3.2)$$
where

\[ K_{MN} = \frac{1}{8} \partial_M H^{KL} \partial_N H_{KL} - \frac{1}{4} (\partial_L - 2 \partial_L d) (H^{LK} \partial_K H_{MN}) + 2 \partial_M \partial_N d \\
- \frac{1}{2} \partial_K H^{KL} \partial_L H_{MK} + \frac{1}{2} (\partial_L - 2 \partial_L d) \left( H^{KL} \partial_{(N} H_{M)K} + H^K (M \partial_K H^L N) \right). \]  

(3.3)

However, \( K_{MN} = 0 \) is not the field equation, since one needs to check the \( O(D, D) \) symmetry of this term. Following the definitions in [3], we take the notations \( H \equiv H^{**} \) and \( \eta \equiv \eta^{**} \). \( H \) should satisfy \( H \eta H = \eta^{-1} \) to respect the \( O(D, D) \) symmetry. It is easy to see that the variation of \( H \) satisfies the condition

\[ \delta H \eta H + H \eta \delta H = 0. \]  

(3.4)

It is convenient to define

\[ S_{MN} = H_{MN} = \eta^{MP} H_{PN} = \eta^{MP} H_{PN}, \]  

(3.5)

Using the notation \( S \equiv S^{**} = H \eta \), one finds

\[ \delta H S^t + S \delta H = 0. \]  

(3.6)

Then since \( S^2 = 1 \), one has

\[ \delta H = -S \delta H S^t. \]  

(3.7)

It can be rewritten as

\[ \delta H = \frac{1}{4} (1 + S) \mathcal{M} (1 - S^t) + \frac{1}{4} (1 - S) \mathcal{M} (1 + S^t), \]  

(3.8)

where \( \mathcal{M} \) is an arbitrary symmetric matrix to guarantee the symmetry of \( \delta H \). Substituting it back into the variation of the action (3.2), we find

\[ \frac{1}{4} (1 - S^t) K (1 + S) + \frac{1}{4} (1 + S^t) K (1 - S) = 0. \]  

(3.9)

Inserting \( O(D, D) \) indices \( M \) and \( N \) to rewrite the equation above, the field equation of \( H^{MN} \) is obtained

\[ R_{MN} = \frac{1}{4} (\delta_{MP} - S^P M) K_{PQ} \left( \delta^Q_N + S^Q N \right) + \frac{1}{4} (\delta_{MP} + S^P M) K_{PQ} \left( \delta^Q_N - S^Q N \right) \]  

\[ = \frac{1}{2} K_{MN} - \frac{1}{2} S^P M K_{PQ} S^Q N. \]  

(3.10)

From this equation, one finds the field equations of the metric \( g \) and the \( b \) field. In summary, the EOM of the dilaton and the graviton are

\[ \frac{1}{8} H^{MN} \partial_M H^{KL} \partial_N H_{KL} - \frac{1}{2} H^{MN} \partial_M H^{KL} \partial_K H_{NL} - \partial_M \partial_N H^{MN} \\
- 4 H^{MN} \partial_M d \partial_N d + 4 \partial_M H^{MN} \partial_N d + 4 H^{MN} \partial_M \partial_N d = 0, \]  

(3.11)

\[ K_{MN} - S^P M K_{PQ} S^Q N = 0. \]  

(3.12)
3.2 Some notations and definitions

We want to introduce some notations and calculation rules before performing the calculation. This helps us to track the double coordinates when performing derivatives. For convenience, we use block matrices to rewrite the vectors, the dual vectors and the generalized metric

\[ \partial_M = \left( \frac{\partial}{\partial \tilde{x}_i}, \frac{\partial}{\partial \tilde{x}_j} \right), \quad dX^M = \left( \frac{dX^1}{dX^2} \right), \quad \mathcal{H}_{MN} = \left( \mathcal{H}_{11} \mathcal{H}_{12}, \mathcal{H}_{21} \mathcal{H}_{22} \right), \quad \mathcal{H}^{MN} = \left( \mathcal{H}_{11}^{12}, \mathcal{H}_{12}^{12}, \mathcal{H}_{21}^{12}, \mathcal{H}_{22}^{12} \right). \]  

(3.13)

Here 1 represents the dual coordinate \( \tilde{x}_i \) and 2 corresponds to the usual coordinate \( x^i \). The components of the generalized metric are divided into four parts. Each of them defines the metric of spacetime and its dual as shown in [33]

\[ \mathcal{H}_{11} \left( \frac{\partial}{\partial \tilde{x}_i}, \frac{\partial}{\partial \tilde{x}_j} \right) = g^{ij}, \quad \mathcal{H}_{12} \left( \frac{\partial}{\partial \tilde{x}_i}, \frac{\partial}{\partial \tilde{x}_j} \right) = -g^{ik}b_{kj}, \quad \mathcal{H}_{21} \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = b_{ik}g^{kj}, \quad \mathcal{H}_{22} \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \eta_{ij} - b_{ik}g^{kl}b_{lj}. \]  

(3.14)

The generalized line element is

\[ dS^2 = \mathcal{H}_{11}dX^1dX^1 + \mathcal{H}_{12}dX^1dX^2 + \mathcal{H}_{21}dX^2dX^1 + \mathcal{H}_{22}dX^2dX^2, \]

\[ = g^{ij}\tilde{x}_i\tilde{x}_j - g^{ik}b_{kj}\tilde{x}_i\tilde{x}_j + b_{ik}g^{kj}dx^i\tilde{x}_j + \left( \eta_{ij} - b_{ik}g^{kl}b_{lj} \right) dx^i dx^j. \]  

(3.15)

In this paper, we consider the simplest situation with \( b_{ij} = 0 \). The line element is simplified to

\[ dS^2 = \mathcal{H}_{11}dX^1dX^1 + \mathcal{H}_{22}dX^2dX^2 = \tilde{g}^{ij}\tilde{x}_i\tilde{x}_j + g_{ij}dx^i dx^j. \]  

(3.16)

To exhibit the contraction of the \( O(D, D) \) indices, we introduce extra indices \( \mathcal{H}^{(i)1(j)} \) to denote elements of block matrices. Therefore, the generalized metric is rewritten with the extra indices

\[ \mathcal{H}_{M(i)N(j)} = \left( \begin{array}{cc} \mathcal{H}^{(1)(1)}_{11} & 0 \\ 0 & \mathcal{H}^{(2)(2)}_{22} \end{array} \right) = \left( \begin{array}{cc} \tilde{g}^{ij} & 0 \\ 0 & g_{ij} \end{array} \right), \]  

(3.17)

where \( b_{ij} = 0 \) is assumed. Now, there exist two sets of indices: the block matrix notations \( M, N = 1, 2 \) and the indices \( i, j = 1, 2, \cdots D \) of the components of block matrices. A contraction of \( M, N \) is given by

\[ \mathcal{H}^{MN}_{M(i)N(j)} \partial_M d\partial_N d = \mathcal{H}^{11}_{11}\partial_1 d\partial_1 d + \mathcal{H}^{22}_{22}\partial_2 d\partial_2 d, \]  

(3.18)

and a contraction of \( i, j \) takes the form

\[ \mathcal{H}^{M(i)N(j)}_{M(i)N(j)} \partial_M d\partial_N d = \mathcal{H}^{11}_{11}\partial_1 d\partial_1 d + \mathcal{H}^{22}_{22}\partial_2 d\partial_2 d \]

\[ = \tilde{g}^{ij}\tilde{\partial}^i d\tilde{\partial}^j d + g^{ij}\partial_i d\partial_j d. \]  

(3.19)

On the right hand side of (3.17), we introduce a tilde \( \tilde{g}^{ij} = g^{ij} \). The purpose of this notation is to help us calculate the derivatives of the metric, since there are two operators \( \tilde{\partial}^i \) and \( \partial_i \). When \( \partial_1 \) acts on the metric, we use the tilde \( \tilde{g} \); when \( \partial_2 \) acts on the metric, we use the untide \( g \). We give some illustrations

\[ \partial_{1(k)} \mathcal{H}^{2(i)2(j)} = \tilde{\partial}^k g^{ij} = \tilde{\partial}^k \tilde{g}^{ij}, \]

\[ \partial_{2(k)} \mathcal{H}^{1(i)1(j)} = \partial_k g_{ij} = \partial_k g_{ij}, \]

\[ \partial_{1(q)} \mathcal{H}^{1(i)1(j)} \partial_{2(q)} \mathcal{H}^{1(i)1(j)} = \tilde{\partial}^p \tilde{g}_{ij} \partial_q g^{ij} = \tilde{\partial}^p \tilde{g}_{ij} \partial_q g^{ij}. \]  

(3.20)
and
\[ g_{ij} \dot{g}^j = \partial_j, \quad g^{ij} \partial_k = \partial^j, \quad g_{ij} g^{jk} = g_{ij} \tilde{g}^{jk} = \delta_i^k. \] (3.21)

Two other useful relations are
\[ \partial_2(\dot{\mathcal{H}})^{(i)} = \partial_k \tilde{g}_{ij} = \partial_k g_{ij}, \]
\[ \partial_1(\dot{\mathcal{H}})^{(i)} = \ddot{\delta}^{k}g^{ij} = \ddot{\delta}^{k} \tilde{g}^{ij}. \] (3.22)

### 3.3 Equations of motion for FRW like metric

We first consider the equation of motion of the dilaton (3.11) and expand it in components. For reference, we rewrite it
\[ \frac{1}{8} \dot{\mathcal{H}}^{MN} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL} - \frac{1}{2} \dot{\mathcal{H}}^{MN} \partial_M \mathcal{H}^{KL} \partial_K \mathcal{H}_{NL} - \partial_M \partial_N \dot{\mathcal{H}}^{MN} - 4 \dot{\mathcal{H}}^{MN} \partial_M \partial_N d + 4 \partial_M \mathcal{H}^{MN} \partial_N d + 4 \dot{\mathcal{H}}^{MN} \partial_M \partial_N d = 0. \] (3.23)

Substituting the matrix components defined in (3.17), after some simplifications, we get
\[ \frac{1}{8} \tilde{g}_{ij} \ddot{\tilde{g}}^{ijkl} \ddot{\tilde{g}}^{ij} + \frac{1}{8} \tilde{g}_{ij} \ddot{\delta}^{ijkl} \ddot{g}_{kl} + \frac{1}{8} g^{ij} \partial_k \tilde{g}_{ij} \partial_M \partial_N d + 4 \partial_M \mathcal{H}^{MN} \partial_N d + 4 \dot{\mathcal{H}}^{MN} \partial_M \partial_N d = 0. \] (3.24)

In double field theory, we know that the relationship of the three dilatons is
\[ e^{-2d} = \sqrt{g} e^{-2\tilde{\phi}} = \sqrt{g} e^{-2\phi}. \] (3.25)

When applied to the metric ansatz (1.5), we find
\[ \phi = \tilde{\phi} + (D - 1) \ln a, \quad d = \phi - \frac{D - 1}{2} \ln a = \frac{\tilde{\phi} - (D - 1)}{2} \ln \tilde{a}. \] (3.26)

The first equation is precisely the so-called “scale-factor duality” transformations (2.7) in string cosmology. The second equation proves our claim that the “shifted dilaton” in string cosmology, defined in (2.5), is 2d. Then, with the metric ansatz (1.5), the EOM of the dilaton is
\[ -(D - 1) \frac{\ddot{a}}{a} + \left( (D - 1) \frac{\dot{a}}{a} \right)^2 - 4(D - 1) \frac{a}{\dot{a}} \phi + 4 \dot{\phi}^2 + 2(D - 1) \frac{\dot{a}}{a} - 4 \dot{\phi} = 0, \]
\[ -(D - 1) \frac{\ddot{a}}{a^3} + \left( (D - 1) \frac{\dot{a}}{a} \right)^2 - 4(D - 1) \frac{a}{\dot{a}} \phi + 4 \dot{\phi}^2 + 2(D - 1) \frac{\dot{a}}{a} - 4 \dot{\phi} = 0, \] (3.27)

where \( \dot{a} = \frac{da}{dt} \) and \( \ddot{a} = \frac{d^2a}{dt^2} \). In order to compare the equations from string cosmology, we used \( \phi \) and \( \tilde{\phi} \) but not \( d \). To avoid confusion of \( a \) and \( \tilde{a} \), we replace \( \dot{a} = a^{-1} \) in the EOM and define the Hubble parameters
\[ H = \frac{\dot{a}}{a}, \quad \tilde{H} = \frac{\dot{\tilde{a}}}{a}. \] (3.28)
Then the EOM of the dilaton for our metric ansatz (1.5) is
\[
\left( (D - 1) \ddot{\tilde{H}} \right)^2 + 4 (D - 1) \ddot{\tilde{H}} \ddot{\phi} + 4 \dddot{\phi}^2 - 2 (D - 1) \dot{\tilde{H}} + (D - 1) \tilde{H}^2 - 4 \dddot{\phi} = 0.
\]
(3.29)

The EOM of the graviton is given in (3.12). We rewrite it for reference
\[
\mathcal{R}_{MN} = \mathcal{K}_{MN} - S^P M \mathcal{K}_{PQ} S^Q_N = 0.
\]
(3.30)

Refer to the appendix, we find there exist symmetries between the components of the general-
zied Ricci tensor
\[
\mathcal{R}_{1(\rho)1(\sigma)} \leftrightarrow g^{\rho \sigma}, \quad \tilde{\phi} \leftrightarrow \phi \quad \mathcal{R}_{2(\rho)2(\sigma)},
\]
\[
\mathcal{R}_{1(\rho)2(\sigma)} \leftrightarrow g^{\rho \sigma}, \quad \tilde{\phi} \leftrightarrow \phi \quad \mathcal{R}_{2(\rho)1(\sigma)}. \quad (3.31)
\]

Also in the appendix, we present the lengthy calculation process and find the EOM of the graviton
\[
\mathcal{R}_{2(i)2(i)} = - (D - 1) \frac{\ddot{a}}{a} + 2 \dddot{\phi} + (D - 1) \frac{\ddot{a}}{a} - 2 \dddot{\phi}
\]
\[
= - (D - 1) \left( \dot{\tilde{H}} + \dot{H}^2 \right) + 2 \dddot{\phi} + (D - 1) \left( - \dddot{H} + \dot{H}^2 \right) - 2 \dddot{\phi} = 0,
\]
(3.32)

Including the EOM of the dilaton (3.29), the set of equations we need to solve is
\[
\left( (D - 1) \ddot{\tilde{H}} \right)^2 + 4 (D - 1) \ddot{\tilde{H}} \ddot{\phi} + 4 \dddot{\phi}^2 - 2 (D - 1) \dot{\tilde{H}} + (D - 1) \tilde{H}^2 - 4 \dddot{\phi} = 0,
\]
\[
+ ((D - 1) \tilde{H})^2 - 4 (D - 1) \tilde{H} \ddot{\phi} + 4 \dddot{\phi}^2 + 2 (D - 1) \dot{\tilde{H}} + (D - 1) \tilde{H}^2 - 4 \dddot{\phi} = 0,
\]
\[
- (D - 1) \left( \dot{\tilde{H}} + \dot{H}^2 \right) + 2 \dddot{\phi} + (D - 1) \left( - \dddot{H} + \dot{H}^2 \right) - 2 \dddot{\phi} = 0,
\]
\[
\left( \dot{\tilde{H}} - (D - 1) \tilde{H}^2 - 2 \dddot{\phi} \hat{\tilde{H}} \right) - \left( - \dddot{H} - (D - 1) \tilde{H}^2 + 2 \dddot{\phi} \hat{\tilde{H}} \right) = 0. \quad (3.33)
\]

Now we replace \( \phi \) and \( \ddot{\phi} \) by the \( O(D, D) \) scalar dilaton \( d \)
\[
\phi = d + \frac{1}{2} (D - 1) \ln a, \quad \ddot{\phi} = d - \frac{1}{2} (D - 1) \ln a. \quad (3.34)
\]

Eventually, the EOM are
\[
\left( 4 \partial_i \partial_i d - 4 \left( \partial_i d \right)^2 - (D - 1) \hat{H}^3 \right) + \left( 4 \partial_i \partial_i d - 4 \left( \partial_i d \right)^2 - (D - 1) \hat{H}^2 \right) = 0,
\]
\[
- (D - 1) \hat{H}^2 + 2 \partial_i \partial_i d \right) - \left( - (D - 1) \hat{H}^2 + 2 \partial_i \partial_i d \right) = 0,
\]
\[
\left( \hat{H} - 2 \hat{H} \partial_i d \right) + \left( \hat{H} - 2 \hat{H} \partial_i d \right) = 0. \quad (3.35)
\]
Clearly, the tilde part and untilde parts are identical, being the same as the EOM (2.4) in string cosmology. This indicates that if \( a(t, \tilde{t}) \) is a solution, an \( O(D, D) \) rotation of \( a(t, \tilde{t}) \) is also a solution, as one can expects from the explicit \( O(D, D) \) invariance in the action. One can easily check that

\[
\begin{align*}
    a_{\pm}(\tilde{t}, t) &= |t|^{\pm1/\sqrt{D-1}}, & d(t) &= -\frac{1}{2} \ln |t|, \\
    a_{\pm}(\tilde{t}, t) &= |\tilde{t}|^{\pm1/\sqrt{D-1}}, & d(\tilde{t}) &= -\frac{1}{2} \ln |\tilde{t}|,
\end{align*}
\]

are solutions of the EOM. These solutions are beyond the solution space of string cosmology and only exist in DFT cosmology. However, as we explained in the introduction, they are excluded by the constraints.

4 Cosmological solutions

In this section, we address solutions dependent on only one set of coordinates, one time-like coordinate in particular. Then, the action takes a form

\[
S = \int d^{D}x d^{D-1}\tilde{x}L(x).
\]

4.1 Solutions with \( V(d) = 0 \)

The solutions are easily obtained from those of string cosmology

\[
\begin{align*}
    a(t) &= |t|^{1/\sqrt{D-1}}, & d(t) &= -\frac{1}{2} \ln |t|,
\end{align*}
\]

where we set the initial time to the unity. In this work, we choose \( D = 4 \). The metrics (1.5) become

\[
\begin{align*}
    dS^2_{\text{pre}} &= -dt^2 + a(-t)^{-2}\left(dx_2^2 + dx_3^2 + dx_4^2\right) + a(-t)^2\left(d\tilde{x}_2^2 + d\tilde{x}_3^2 + d\tilde{x}_4^2\right), & t < 0, \\
    dS^2_{\text{post}} &= -dt^2 + a(t)^2\left(dx_2^2 + dx_3^2 + dx_4^2\right) + a(t)^{-2}\left(d\tilde{x}_2^2 + d\tilde{x}_3^2 + d\tilde{x}_4^2\right), & t > 0.
\end{align*}
\]

where, we have selected the “self-dual” evolutions: an accelerated expansion followed by a decelerated expansion of \( x \), an accelerated contraction evolving to a decelerated contraction of \( \tilde{x} \), as illustrated in figure (3) New physics already show up evidently in the solutions. However, we will put off the discussions to the \( V(d) \neq 0 \) scenario. Since the singularity can be smoothed out by a nontrivial dilaton potential, we can see the novel features more clearly. Moreover, the reason to choose the self-dual evolution will be justified.

4.2 Solutions with \( V(d) \neq 0 \)

To remove the singularity, we introduce a dilaton potential

\[
V(d) = V_0 e^{8d}.
\]

where \( V_0 > 0 \). This non-local potential represents the backreactions of higher loop corrections [53]. It is worth to note that \( V_0 \) includes proper volume which makes dilaton potential
be a scalar under generalised diffeomorphisms. This potential certainly respect the $O(D,D)$ symmetry. In physics, we always give symmetries the highest priority. Therefore, its presence in the action is well justified. Anyhow, the primary physics are not affected by the potential. It is easy to figure out that the physically relevant solution is

$$a(t) = a_0 \left[ t + (t_0^2 + t^2)^{1/3} \right] \sqrt{V_0 - t_0^2}, \quad d(t) = -\frac{1}{4} \ln \left[ \frac{\sqrt{V_0}}{t_0} (t_0^2 + t^2) \right].$$

where $a_0$ and $t_0$ are integration constants. Setting $a_0 = 1$, $t_0 \to 0$, $V_0 \to 0$ and keeping $\sqrt{V_0/t_0} = 1$, we recover the solution (4.2) without the potential. We set $a_0 = t_0 = V_0 = 1$ and $D = 4$ for plotting the curves. The unified line element of DFT cosmology is

$$dS^2 = -dt^2 + a_1(t)^2 (dx_1^2 + dx_2^2 + dx_3^2) + a_2(t)^2 (d\tilde{x}_1^2 + d\tilde{x}_2^2 + d\tilde{x}_3^2).$$

where $a_1(t) = a(t)$ and $a_2(t) = a(t)^{-1}$. This solution is unique in the sense that $x$ and $\tilde{x}$ are completely equivalent. The evolutions of the scale factors and Hubble parameters are illustrated in figure (4).

In the pre-big bang region ($t \to -\infty$)

$$a_1(t) \sim (-t)^{-\frac{1}{\sqrt{3}}}, \quad a_2(t) \sim (-t)^{\frac{1}{\sqrt{3}}}.$$  

The ordinary spatial dimensions $x_i$’s are hidden, while the dual spatial dimensions $\tilde{x}_i$’s are visible.

In the post-big bang region ($t \to \infty$)

$$a_1(t) \sim t^{\frac{1}{3\sqrt{3}}}, \quad a_2(t) \sim t^{-\frac{1}{3\sqrt{3}}}.$$  

Figure 3. The evolutions for $V(d) = 0$. The left hand side figure describes the scale factors and the right hand side one represents the Hubble parameters. The solid line stands for the evolutions of the ordinary coordinates $x_i$ and the dashed line is the evolutions of the dual coordinates $\tilde{x}_i$.

Figure 4. The evolutions for $V(d) = V_0 e^{8d}$. The left hand side figure describes the scale factors and the right hand side one represents the Hubble parameters. The solid line stands for the evolutions of the ordinary coordinates $x_i$ and the dashed line is the evolutions of the dual coordinates $\tilde{x}_i$.  

$$dS^2 = -dt^2 + a_1(t)^2 (dx_1^2 + dx_2^2 + dx_3^2) + a_2(t)^2 (d\tilde{x}_1^2 + d\tilde{x}_2^2 + d\tilde{x}_3^2).$$

where $a_1(t) = a(t)$ and $a_2(t) = a(t)^{-1}$. This solution is unique in the sense that $x$ and $\tilde{x}$ are completely equivalent. The evolutions of the scale factors and Hubble parameters are illustrated in figure (4).
It is obvious that \( x_i \)’s expand to be visible and effectively isotropic. Dual coordinates \( \tilde{x}_i \)’s contract to extra dimensions.

**In the big bang region \((t \sim 0)\)**

\[
a_1(t) = \kappa_0 + \kappa_1 \left( t - \frac{1}{\sqrt{2}} \right) - \kappa_3 \left( t - \frac{1}{\sqrt{2}} \right)^3 + \mathcal{O} \left( t - \frac{1}{\sqrt{2}} \right)^4, \\
a_2(t) = \kappa_0 - \kappa_1 \left( t + \frac{1}{\sqrt{2}} \right) + \kappa_3 \left( t + \frac{1}{\sqrt{2}} \right)^3 + \mathcal{O} \left( t + \frac{1}{\sqrt{2}} \right)^4, \tag{4.9}
\]

where \( \kappa_0, \kappa_1 \) and \( \kappa_3 \) are positive numbers. One can easily see that the ordinary spatial dimensions \( x_i \)’s inflate all the way from \( t \to -\infty \) to \( t = 1/\sqrt{2} \). After that, \( x_i \)’s start a decelerated expansion. On the other hand, all \( \tilde{x}_i \)’s first experience an accelerated contraction until \( t = -1/\sqrt{2} \), followed by a decelerated contraction to \( t \to \infty \). This picture confirms our “self-dual” choice in the \( V(d) = 0 \) scenario.

We see that the absence of the singularity makes the physical picture much clearer. Two novel physical features come out immediately

1. Extra dimensions are intrinsically hidden in the asymptotic regions \((|t| \to \infty)\).
2. Without any fine-tunning of parameters, we have a natural evolution from a visibly isotropic pre-big bang to an evidently anisotropic big bang and then again to an isotropic universe at present time.

Bear in mind that in string theory, in order to cancel the anomalies, the dimensionality of spacetime is fixed. However, the traditional string cosmology can have only one of the evolutions, always expanding or contracting. Therefore, one has to set up certain mechanism to hide the extra dimensions. While, in DFT cosmology, thanks to the \( O(D, D) \) symmetry, half of the spatial dimensions are naturally hidden in the far past and far future without any pre-assumptions. This fact again asserts the importance of symmetries in physics.

It is worth noting that the string coupling constant

\[
g_s = e^\phi = \frac{a_0^{3/2} t_0^{1/4} (t + \sqrt{t^2 + t_0^2})^{\sqrt{3}/2}}{V_0^{1/8} (t_0^2 + t^2)^{1/4}} \sim \frac{a_0^{3/2} t_0^{1/4}}{V_0^{1/8}} t^{-\frac{\sqrt{3}}{2}}, \quad \text{for} \quad t \gg t_0. \tag{4.10}
\]

A large value of \( V_0 \) can make the results trustable in sizable region. However, in the strong coupled phase as \( t \to \infty \), the tree level effective theory breaks down and thereby the solutions are not reliable. One has to seek for higher order corrections or even more complete theory to solve the problem.

It is of interest to compare DFT cosmology with compactified Kaluza-Klein gravity in the post-big bang region, since they have similar metrics. The higher dimensional Kaluza-Klein gravity takes a form \([54]\)

\[
ds_{KK}^2 = -dt^2 + t^\alpha (dx_2^2 + dx_3^2 + dx_4^2) + t^{-\alpha} dy^2, \tag{4.11}
\]

where \( \alpha \) is constant and \( y \) represents an extra dimension. The five dimensional Brans–Dicke Theory also has the same line element \([55]\). It is immediately to see that DFT cosmology possesses all the properties of these theory, but without man-made setups.

Furthermore, our solutions show that, when traced back along the evolution, our current universe was totally hidden in the pre-big bang. While the visible dimensions \((d\tilde{x}_2^2, d\tilde{x}_3^2, d\tilde{x}_4^2)\)
in the pre-big bang become extra dimensions at present time. Two groups of spaces have interactions around the big bang region. Therefore, exploring the extra dimensions could reveal the existence and information of the pre-big bang.

In [56], the authors showed that the shear of the contracting dimensions causes a decelerated expansion of other dimensions. In [57], Levin demonstrated that if the dimensionality of the contracting extra dimensions is larger than 1, a kinetic inflation purely driven by the contraction of extra dimensions is also possible. However, the number of expanding/contracting dimensions has to be specified as initial conditions. These models further suffer graceful exit and isotropy problem. Remarkably, as we see in DFT cosmology, both scenarios are automatically achieved without any initial data. In the domain $-\infty < t < 1/\sqrt{2}$, the contraction of $\tilde{x}$ inflates $x$ and in the domain $1/\sqrt{2} < t < \infty$, the contraction of $\tilde{x}$ makes $x$ expanding in an accelerated pace.

5 Conclusion and Discussions

In this paper, we calculated the cosmological solutions of double field theory. We set the anti-symmetrical Kalb-Ramond field vanishing for simplicity. When taking the FRW-like metric ansatz, we demonstrated that, the scale factor dual dilatons, $\phi = \hat{\phi} + (D - 1) \ln a$, in string cosmology are exactly the diffeomorphic and dual diffeomorphic dilatons in double field theory. The “shifted dilaton” in string cosmology is really the $O(D,D)$ scalar dilaton in double field theory with $2d = \psi = \tilde{\psi}$.

We found two cosmological solutions, with and without an $O(D,D)$ invariant dilaton potential. In the $V(d) = 0$ scenario, solutions have two distinct metrics, representing the pre- and post-big bangs, respectively. Each of them unifies contracting and expanding dimensions. As $t \to 0$, all solutions approach the big bang singularity. To understand the physics around the singularity more clearly, we make use of an $O(D,D)$ invariant dilaton potential, $V(d) = V_0 e^{8d}$, $V_0 > 0$, which does not affect the main conclusions. Not only does this potential preserve the symmetry, it has physical origin from the backreactions of higher loop corrections. With this potential, the big bang singularity disappears. We thus have a single line element which unifies originally disconnected pre- and post-big bang metrics. The visible pre-big bang dimensions contract to invisible extra dimensions. While extra dimensions in the pre-big bang expand all the way to the visible dimensions of the present universe. Due to this observation, one can expect that detection of extra dimension will reveal information of the pre-big bang.

In addition, we showed that the contraction of the dual dimensions causes both an inflation and a decelerated expansion of the ordinary dimensions in different time domains. The advantage of DFT cosmology is that no initial conditions are needed to set up such scenarios.

The solutions we have obtained are special ones of EOM (3.35). We also presented some constraint violating solutions. However, the physical implications of these solutions are unclear. Though it looks not easy to figure out other more nontrivial solutions, it would be of interest if one can find some.

Looking for other solutions of the generalized action, say black holes, is of interest. However, the physical interpretations are blurry. One has to be careful to deal with the gauge constraint and identify the parameters in the solutions.

In the formulation of DFT, the weak and strong constraints are sufficient but not necessary for the consistency of the theory. It is possible to relax these constraints in some
scenarios of flux compactification and dimensional reduction [19, 21, 23, 30, 32, 35, 46]. It is of interest to investigate the relevance of our solutions to these compactifications in the future works.

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A Equations of motion of the gravitons

Recall the generalized Ricci tensor (3.10),

\[ R_{MN} = \mathcal{K}_{MN} - S^P_M \mathcal{K}_{PQ} S^Q_N = 0. \]  

(A.1)

To simplify the calculations, we separate \( \mathcal{K}_{MN} \) into two parts

\[ \mathcal{K}_{MN} = * \mathcal{K}_{MN} + * \mathcal{K}_{MN}, \]  

(A.2)

where

\[ * \mathcal{K}_{MN} = \frac{1}{8} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL} - \frac{1}{4} \partial_L \left( \mathcal{H}^{LK} \partial_K \mathcal{H}_{MN} \right) \]

\[- \frac{1}{2} \partial_N \mathcal{H}^{KL} \partial_L \mathcal{H}_{(M)} + \frac{1}{2} \partial_L \left( \mathcal{H}^{KL} \partial_N \mathcal{H}_{M} + \mathcal{H}^K_{(M} \partial_K \mathcal{H}^L_{N)} \right), \]  

(A.3)

and

\[ * \mathcal{K}_{MN} = \frac{1}{2} \partial_L d \left( \mathcal{H}^{LK} \partial_K \mathcal{H}_{MN} \right) + 2 \partial_M \partial_N d - \partial_L d \left( \mathcal{H}^{KL} \partial_N \mathcal{H}_{M} + \mathcal{H}^K_{(M} \partial_K \mathcal{H}^L_{N)} \right). \]  

(A.4)

Therefore, \( R_{MN} = 0 \) can be put into two parts

\[ R_{MN} = * R_{MN} + * R_{MN} = 0, \]  

(A.5)

with

\[ * R_{MN} = * \mathcal{K}_{MN} - S^P_M * \mathcal{K}_{PQ} S^Q_N, \]  

\[ * R_{MN} = * \mathcal{K}_{MN} - S^P_M * \mathcal{K}_{PQ} S^Q_N. \]  

(A.6)

Calculation of \( * R_{MN}. \)

\[ * R_{MN} = * \mathcal{K}_{MN} - S^P_M * \mathcal{K}_{PQ} S^Q_N \]

\[ = \frac{1}{8} \partial_M \tilde{g}_{ij} \partial_N \tilde{g}^{ij} + \frac{1}{8} \partial_M \tilde{g}^{ij} \partial_N \tilde{g}_{ij} \]

\[ - \frac{1}{4} \tilde{\delta}^i \left( \tilde{g}_{ij} \tilde{\delta}^j M_N \right) - \frac{1}{4} \delta_i \left( \tilde{g}^{ij} \partial_j M_N \right) \]
\[-\frac{1}{4} \partial_N \tilde{g}_{ij} \partial^j \mathcal{H}_M 1 \left( i \right) - \frac{1}{4} \partial_M \tilde{g}_{ij} \partial^j \mathcal{H}_N 1 \left( i \right) \]
\[-\frac{1}{4} \partial_N g^{ij} \partial_j \mathcal{H}_M 2 \left( i \right) - \frac{1}{4} \partial_M g^{ij} \partial_j \mathcal{H}_N 2 \left( i \right) \]
\[+ \frac{1}{4} \partial^j \left( \tilde{g}_{ji} \partial_N \mathcal{H}_M 1 \left( j \right) \right) + \frac{1}{4} \partial^j \left( \tilde{g}_{ji} \partial_M \mathcal{H}_N 1 \left( j \right) \right) \]
\[+ \frac{1}{4} \partial_i \left( g^{ij} \partial_N \mathcal{H}_M 2 \left( i \right) \right) + \frac{1}{4} \partial_i \left( g^{ij} \partial_M \mathcal{H}_N 2 \left( i \right) \right) \]
\[+ \frac{1}{4} \partial^i \left( \mathcal{H}^1 \left( j \right) M \left( i \right) \partial^j \mathcal{H}^1 \left( i \right) \right) + \frac{1}{4} \partial^i \left( \mathcal{H}^1 \left( j \right) N \left( i \right) \partial^j \mathcal{H}^1 \left( i \right) \right) \]
\[+ \frac{1}{4} \partial^i \left( H^2 \left( j \right) M \left( i \right) \partial_j \mathcal{H}^2 \left( i \right) \right) + \frac{1}{4} \partial^i \left( H^2 \left( j \right) N \partial_j \mathcal{H}^2 \left( i \right) \right) \]
\[+ \frac{1}{4} \partial_i \left( \mathcal{H}^1 \left( j \right) M \partial_j \mathcal{H}^2 \left( i \right) \right) + \frac{1}{4} \partial_i \left( \mathcal{H}^2 \left( j \right) N \partial_j \mathcal{H}^2 \left( i \right) \right) \]
\[= \frac{1}{8} S_1 \left( i \right) M \partial^i \tilde{g}_{kl} \partial^j \tilde{g}^{jkl} S_1 \left( i \right) N - \frac{1}{8} S_1 \left( i \right) M \partial^i \tilde{g}^{jkl} \partial_j \tilde{g}_{kl} S_1 \left( i \right) N \]
\[= \frac{1}{8} S_1 \left( i \right) M \partial^i \tilde{g}_{kl} \partial^j \tilde{g}^{jkl} S_2 \left( i \right) N - \frac{1}{8} S_1 \left( i \right) M \partial^i \tilde{g}^{jkl} \partial_j \tilde{g}_{kl} S_2 \left( i \right) N \]
\[= \frac{1}{8} S_1 \left( i \right) M \partial^i \tilde{g}_{kl} \partial^j \tilde{g}^{jkl} S_1 \left( i \right) N - \frac{1}{8} S_1 \left( i \right) M \partial^i \tilde{g}^{jkl} \partial_j \tilde{g}_{kl} S_1 \left( i \right) N \]
\[+ \frac{1}{4} S_1 \left( i \right) M \partial^i \left( \tilde{g}_{kl} \partial^j \tilde{g}^{jkl} S_2 \left( i \right) \right) + \frac{1}{4} S_1 \left( i \right) M \partial^i \left( \tilde{g}^{jkl} \partial_j \tilde{g}_{kl} S_2 \left( i \right) \right) \]
\[+ \frac{1}{4} S_2 \left( i \right) M \partial^i \tilde{g}_{kl} \partial^j \tilde{g}^{jkl} S_1 \left( i \right) N + \frac{1}{4} S_2 \left( i \right) M \partial^i \tilde{g}^{jkl} \partial_j \tilde{g}_{kl} S_1 \left( i \right) N \]
\[+ \frac{1}{4} S_2 \left( i \right) M \partial^i \tilde{g}_{kl} \partial^j \tilde{g}^{jkl} S_2 \left( i \right) N + \frac{1}{4} S_2 \left( i \right) M \partial^i \tilde{g}^{jkl} \partial_j \tilde{g}_{kl} S_2 \left( i \right) N \]
\[= \frac{1}{4} S_1 \left( i \right) M \partial^i \left( \tilde{g}_{kl} \partial^j \tilde{g}^{jkl} S_1 \left( i \right) \right) + \frac{1}{4} S_1 \left( i \right) M \partial^i \left( \tilde{g}^{jkl} \partial_j \tilde{g}_{kl} S_1 \left( i \right) \right) \]
\[= \frac{1}{4} S_1 \left( i \right) M \partial^i \left( \tilde{g}_{kl} \partial^j \tilde{g}^{jkl} S_2 \left( i \right) \right) + \frac{1}{4} S_1 \left( i \right) M \partial^i \left( \tilde{g}^{jkl} \partial_j \tilde{g}_{kl} S_2 \left( i \right) \right) \]
\[= \frac{1}{4} S_1 \left( i \right) M \partial^i \left( \tilde{g}_{kl} \partial^j \tilde{g}^{jkl} S_1 \left( i \right) \right) - \frac{1}{4} S_1 \left( i \right) M \partial^i \left( \tilde{g}^{jkl} \partial_j \tilde{g}_{kl} S_1 \left( i \right) \right) \]
\[+ \frac{1}{4} S_1 \left( i \right) M \partial^i \left( \tilde{g}_{kl} \partial^j \tilde{g}^{jkl} S_2 \left( i \right) \right) - \frac{1}{4} S_1 \left( i \right) M \partial^i \left( \tilde{g}^{jkl} \partial_j \tilde{g}_{kl} S_2 \left( i \right) \right) \]
\[= \frac{1}{4} S_1 \left( i \right) M \partial^i \left( \tilde{g}_{kl} \partial^j \tilde{g}^{jkl} S_1 \left( i \right) \right) - \frac{1}{4} S_1 \left( i \right) M \partial^i \left( \tilde{g}^{jkl} \partial_j \tilde{g}_{kl} S_1 \left( i \right) \right) \]
\[+ \frac{1}{4} S_1 \left( i \right) M \partial^i \left( \tilde{g}_{kl} \partial^j \tilde{g}^{jkl} S_2 \left( i \right) \right) - \frac{1}{4} S_1 \left( i \right) M \partial^i \left( \tilde{g}^{jkl} \partial_j \tilde{g}_{kl} S_2 \left( i \right) \right) \]
\[= \frac{1}{4} S_1 \left( i \right) M \partial^i \left( \tilde{g}_{kl} \partial^j \tilde{g}^{jkl} S_1 \left( i \right) \right) - \frac{1}{4} S_1 \left( i \right) M \partial^i \left( \tilde{g}^{jkl} \partial_j \tilde{g}_{kl} S_1 \left( i \right) \right) \]
\[+ \frac{1}{4} S_1 \left( i \right) M \partial^i \left( \tilde{g}_{kl} \partial^j \tilde{g}^{jkl} S_2 \left( i \right) \right) - \frac{1}{4} S_1 \left( i \right) M \partial^i \left( \tilde{g}^{jkl} \partial_j \tilde{g}_{kl} S_2 \left( i \right) \right) \]
\[= \frac{1}{4} S_1 \left( i \right) M \partial^i \left( \tilde{g}_{kl} \partial^j \tilde{g}^{jkl} S_1 \left( i \right) \right) - \frac{1}{4} S_1 \left( i \right) M \partial^i \left( \tilde{g}^{jkl} \partial_j \tilde{g}_{kl} S_1 \left( i \right) \right) \]
\[+ \frac{1}{4} S_1 \left( i \right) M \partial^i \left( \tilde{g}_{kl} \partial^j \tilde{g}^{jkl} S_2 \left( i \right) \right) - \frac{1}{4} S_1 \left( i \right) M \partial^i \left( \tilde{g}^{jkl} \partial_j \tilde{g}_{kl} S_2 \left( i \right) \right) \]
\[-\frac{1}{4} S^{2(i)}_{\text{M}} \partial^i \left( g_{ki} \partial^k g_{ij} \right) S^{2(j)}_{\text{N}} - \frac{1}{4} S^{2(i)}_{\text{M}} \partial^i \left( g_{ki} \partial^k g_{ia} \right) S^{2(j)}_{\text{N}}. \]  

(A.7)

The components are

\[ \star R_{(p)1(q)} = \frac{1}{8} \partial^p g_{ij} \partial^q g_{ij} + \frac{1}{8} \partial^p g_{ij} \partial^q g_{ij} - \frac{1}{4} \partial^i \left( \tilde{g}_{ij} \partial^j g_{pq} \right) - \frac{1}{4} \partial^i \left( g_{ij} \partial^j g_{pq} \right) + \frac{1}{4} \partial_i \left( g^{ij} \partial_j g^{pq} \right) - \frac{1}{8} \partial^{pq} g_{kl} \partial_{jk} g_{ij} - \frac{1}{4} \partial_i \left( g^{ij} \partial_j g_{pq} \right) + \frac{1}{4} \partial_i \left( g^{ij} \partial_j g_{pq} \right) - \frac{1}{8} \partial^{pq} g_{kl} \partial_{jk} g_{ij} \]

\[ \star R_{(p)2(q)} = \frac{1}{8} \partial^p g_{ij} \partial^q g_{ij} + \frac{1}{8} \partial^p g_{ij} \partial^q g_{ij} - \frac{1}{4} \partial^i \left( \tilde{g}_{ij} \partial^j g_{pq} \right) - \frac{1}{4} \partial^i \left( g_{ij} \partial^j g_{pq} \right) + \frac{1}{4} \partial_i \left( g^{ij} \partial_j g^{pq} \right) - \frac{1}{8} \partial^{pq} g_{kl} \partial_{jk} g_{ij} - \frac{1}{4} \partial_i \left( g^{ij} \partial_j g^{pq} \right) + \frac{1}{4} \partial_i \left( g^{ij} \partial_j g^{pq} \right) - \frac{1}{8} \partial^{pq} g_{kl} \partial_{jk} g_{ij} \]

\[ \star R_{(p)1(q)} = \frac{1}{8} \partial^p g_{ij} \partial^q g_{ij} + \frac{1}{8} \partial^p g_{ij} \partial^q g_{ij} - \frac{1}{4} \partial^i \left( \tilde{g}_{ij} \partial^j g_{pq} \right) - \frac{1}{4} \partial^i \left( g_{ij} \partial^j g_{pq} \right) + \frac{1}{4} \partial_i \left( g^{ij} \partial_j g^{pq} \right) - \frac{1}{8} \partial^{pq} g_{kl} \partial_{jk} g_{ij} - \frac{1}{4} \partial_i \left( g^{ij} \partial_j g^{pq} \right) + \frac{1}{4} \partial_i \left( g^{ij} \partial_j g^{pq} \right) - \frac{1}{8} \partial^{pq} g_{kl} \partial_{jk} g_{ij} \]

\[ \star R_{(p)2(q)} = \frac{1}{8} \partial^p g_{ij} \partial^q g_{ij} + \frac{1}{8} \partial^p g_{ij} \partial^q g_{ij} - \frac{1}{4} \partial^i \left( \tilde{g}_{ij} \partial^j g_{pq} \right) - \frac{1}{4} \partial^i \left( g_{ij} \partial^j g_{pq} \right) + \frac{1}{4} \partial_i \left( g^{ij} \partial_j g^{pq} \right) - \frac{1}{8} \partial^{pq} g_{kl} \partial_{jk} g_{ij} - \frac{1}{4} \partial_i \left( g^{ij} \partial_j g^{pq} \right) + \frac{1}{4} \partial_i \left( g^{ij} \partial_j g^{pq} \right) - \frac{1}{8} \partial^{pq} g_{kl} \partial_{jk} g_{ij} \]

(A.8)
It is straightforward to verify the symmetry
\[ *R_{1(p)1(q)} \mathcal{g}^{\bullet \bullet} \leftrightarrow \mathcal{g}_{\bullet \bullet}, \quad \tilde{\partial}^\bullet \leftrightarrow \tilde{\partial}_\bullet, \quad *R_{2(p)2(q)}, \]
\[ *R_{1(p)2(q)} \mathcal{g}^{\bullet \bullet} \leftrightarrow \mathcal{g}_{\bullet \bullet}, \quad \tilde{\partial}^\bullet \leftrightarrow \tilde{\partial}_\bullet, \quad *R_{2(p)1(q)}. \] (A.9)

With our metric ansatz (1.5) and calculation rules in section (4.1), we obtain
\[ *R_{2(t)2(t)} = -(D-1) \frac{\dot{a}^2}{a^2} + (D-1) \frac{\ddot{a}^2}{a^2}, \]
\[ *R_{2(t)2(t)} = \frac{1}{a^4} (\ddot{a}^2 - \dddot{a}^2) - (\ddot{a}^2 - a \dddot{a}), \]
\[ *R_{1(t)2(t)} = 0. \] (A.10)

Calculation of \( *R_{MN} \).
\[ *R_{MN} = *K_{MN} - S^P_M *K_{PQ} S^Q_N \]
\[ = \frac{1}{2} \left( -\frac{1}{4} g_{ab} \tilde{\partial}^i g^{ab} + \tilde{\partial}^i \phi \right) \left( \tilde{\partial}_j \tilde{\partial}^j \mathcal{H}_{MN} \right) \]
\[ + \frac{1}{2} \left( -\frac{1}{4} g^{ab} \partial_i g_{ab} + \partial_i \phi \right) \left( g^{ij} \partial_j \mathcal{H}_{MN} \right) + 2 \partial_M \partial_N d \]
\[ - \left( -\frac{1}{4} g_{ab} \tilde{\partial}^i g^{ab} + \tilde{\partial}^i \phi \right) \left( \mathcal{H}^{1(j)}_{(M} \tilde{\partial}^j \mathcal{H}^{1(i)}_{N)} \right) \]
\[ - \left( -\frac{1}{4} g^{ab} \partial_i g_{ab} + \partial_i \phi \right) \left( \mathcal{H}^{2(j)}_{(M} \partial_j \mathcal{H}^{1(i)}_{N)} \right) \]
\[ - \left( -\frac{1}{4} g_{ab} \tilde{\partial}^i g^{ab} + \tilde{\partial}^i \phi \right) \left( \tilde{\partial}_j \partial(N \mathcal{H})_{M} 1(j) \right) \]
\[ - \left( -\frac{1}{4} g^{ab} \partial_i g_{ab} + \partial_i \phi \right) \left( g^{ij} \partial(N \mathcal{H})_{M} 2(j) \right) \]
\[ - \frac{1}{2} S^{1(j)}_{(M} \left( -\frac{1}{4} g_{ab} \tilde{\partial}^i g^{ab} + \tilde{\partial}^i \phi \right) \left( \tilde{\partial}_k \tilde{\partial}^k g^{1(j)}_{N)} \right) \]
\[ - \frac{1}{2} S^{1(j)}_{(M} \left( -\frac{1}{4} g^{ab} \partial_i g_{ab} + \partial_i \phi \right) \left( g^{lk} \partial_k g^{1(j)} N \right) \]
\[ - \frac{1}{2} S^{2(j)}_{(M} \left( -\frac{1}{4} g_{ab} \tilde{\partial}^i g^{ab} + \tilde{\partial}^i \phi \right) \left( \tilde{\partial}_k \tilde{\partial}^k g^{2(j)} N \right) \]
\[ - \frac{1}{2} S^{2(j)}_{(M} \left( -\frac{1}{4} g^{ab} \partial_i g_{ab} + \partial_i \phi \right) \left( g^{lk} \partial_k g^{2(j)} N \right) \]
\[ - 2 S^{1(j)}_{(M} \partial_1(j) \partial_1(j) d S^{1(j)}_{N)} - 2 S^{1(j)}_{(M} \partial_1(j) \partial_2(j) d S^{2(j)}_{N)} \]
\[ - 2 S^{2(j)}_{(M} \partial_2(j) \partial_1(j) d S^{1(j)}_{N)} - 2 S^{2(j)}_{(M} \partial_2(j) \partial_2(j) d S^{2(j)}_{N)} \]
\[ + \frac{1}{2} S^{1(j)}_{(M} \left( -\frac{1}{4} g_{ab} \tilde{\partial}^i g^{ab} + \tilde{\partial}^i \phi \right) \left( \tilde{\partial}_k \tilde{\partial}^k g^{1(j)} N \right) \]

\[ - 19 - \]
Thus the components are

\[ *R_{1(p)1(q)} = \frac{1}{2} \left( -\frac{1}{4} g_{ab} \partial^i \tilde{g}^{ab} + \partial^i \tilde{\phi} \right) (\tilde{g}_{ij} \tilde{g}^{jq} \partial_i \tilde{g}^{pq}) + \frac{1}{2} \left( -\frac{1}{4} g_{ab} \partial_i g_{ab} + \partial_i \tilde{\phi} \right) (g^{ij} \partial_j \tilde{g}^{pq}) \]

- \frac{1}{2} \partial^p \tilde{g}_{aa} \tilde{g}^{aa} - \frac{1}{2} \tilde{g}_{aa} \tilde{g}^{pp} \tilde{g}^{aa} + \tilde{g}^{pp} \tilde{\phi} \\
- 2g^{ip} \left( -\frac{1}{4} g_{ab} \partial_j g_{aa} - \frac{1}{4} g^{aa} \partial_i \partial_j ga + \partial_i \partial_j \phi \right) g^{jq} \\
- \frac{1}{2} \left( -\frac{1}{4} g_{ab} \partial_i g_{ab} + \partial_i \phi \right) (g^{ip} \partial_j g^{jq}) - \frac{1}{2} \left( -\frac{1}{4} g_{ab} \partial_i g_{ab} + \partial_i \phi \right) (g^{ip} \partial_j g^{jq}) \\
- \frac{1}{2} \left( -\frac{1}{4} g_{ab} \tilde{g}^{ab} + \partial^i \tilde{\phi} \right) (\tilde{g}_{ij} \tilde{g}^{ji} \tilde{g}^{pq}) - \frac{1}{2} \left( -\frac{1}{4} g_{ab} \tilde{g}^{ab} + \partial^i \tilde{\phi} \right) (\tilde{g}_{ij} \tilde{g}^{ji} \tilde{g}^{pq}) \\
- \frac{1}{2} g^{ip} \left( -\frac{1}{4} g_{ab} \tilde{g}^{ab} + \partial^i \tilde{\phi} \right) (\tilde{g}_{ij} \tilde{g}^{ji} \tilde{g}^{pq}) \]

(A.11)
\[
-\frac{1}{2} g^{ip} \left( -\frac{1}{4} g^{ab} \partial_i g_{ab} + \partial_i \phi \right) \left( g^{k\ell} \partial_k g_{ij} \right) g^{jq} \\
+ \frac{1}{2} g^{ip} \left( -\frac{1}{4} g^{ab} \partial_i g_{ab} + \partial_i \phi \right) \left( g^{k\ell} \partial_j g_{ik} \right) g^{jq} \\
+ \frac{1}{2} g^{ip} \left( -\frac{1}{4} g^{ab} \partial_j g_{ab} + \partial_j \phi \right) \left( g^{k\ell} \partial_i g_{jk} \right) g^{jq} \\
+ \frac{1}{2} g^{ip} \left( -\frac{1}{4} \tilde{g}_{ab} \partial^i \tilde{g}^{ab} + \partial^i \tilde{\phi} \right) \left( g_{kj} \partial^k g_{ij} \right) g^{jq} \\
+ \frac{1}{2} g^{ip} \left( -\frac{1}{4} \tilde{g}_{ab} \partial^j \tilde{g}^{ab} + \partial^j \tilde{\phi} \right) \left( g_{kj} \partial^k g_{ij} \right) g^{jq},
\]

\[
* \mathcal{R}_{2(p)2(q)} = \frac{1}{2} \left( -\frac{1}{4} \tilde{g}_{ab} \partial^j \tilde{g}^{ab} + \partial^j \tilde{\phi} \right) \left( \tilde{g}_{ij} \partial^j g_{pq} \right) + \frac{1}{2} \left( -\frac{1}{4} g^{ab} \partial_i g_{ab} + \partial_i \phi \right) \left( g^{ij} \partial_j g_{pq} \right) \\
- \frac{1}{2} \partial_j g^{aa} \partial_i g_{aa} - \frac{1}{2} g^{aa} \partial_p \partial_q g_{aa} + 2 \partial_p \partial_q \phi \\
- 2 g^{iq} \left( -\frac{1}{4} \tilde{g}_{aa} \partial^i \tilde{g}^{aa} - \frac{1}{4} \tilde{g}_{aa} \partial^j \tilde{g}^{aa} + \partial^i \tilde{\phi} \right) g_{jq} \\
- \frac{1}{2} \left( -\frac{1}{4} \tilde{g}_{ab} \partial^i \tilde{g}^{ab} + \partial^i \tilde{\phi} \right) \left( g_{jp} \partial^j g_{iq} \right) - \frac{1}{2} \left( -\frac{1}{4} \tilde{g}_{ab} \partial^j \tilde{g}^{ab} + \partial^j \tilde{\phi} \right) \left( g_{jq} \partial^j g_{ip} \right) \\
- \frac{1}{2} \left( -\frac{1}{4} g^{ab} \partial_i g_{ab} + \partial_i \phi \right) \left( g^{ij} \partial_q g_{pj} \right) - \frac{1}{2} \left( -\frac{1}{4} g^{ab} \partial_j g_{ab} + \partial_j \phi \right) \left( g^{ij} \partial_p g_{qj} \right) \\
- \frac{1}{2} g^{ip} \left( -\frac{1}{4} \tilde{g}_{ab} \partial^i \tilde{g}^{ab} + \partial^i \tilde{\phi} \right) \left( \tilde{g}_{ik} \partial^k g^{ij} \right) g_{jq} \\
- \frac{1}{2} g^{ip} \left( -\frac{1}{4} g^{ab} \partial_i g_{ab} + \partial_i \phi \right) \left( g^{jk} \partial_k g^{ij} \right) g_{jq} \\
+ \frac{1}{2} g^{ip} \left( -\frac{1}{4} \tilde{g}_{ab} \partial^j \tilde{g}^{ab} + \partial^j \tilde{\phi} \right) \left( \tilde{g}_{ik} \partial^i g^{jk} \right) g_{jq} \\
+ \frac{1}{2} g^{ip} \left( -\frac{1}{4} g^{ab} \partial_j g_{ab} + \partial_j \phi \right) \left( g^{ki} \partial_k g^{ij} \right) g_{jq} \\
+ \frac{1}{2} g^{ip} \left( -\frac{1}{4} \tilde{g}_{ab} \partial_i \tilde{g}^{ab} + \partial_i \tilde{\phi} \right) \left( \tilde{g}_{ik} \partial^k g^{ij} \right) g_{jq},
\]

\[
* \mathcal{R}_{1(p)2(q)} = 2 \tilde{\partial}^p \partial_q d - 2 \tilde{g}^{ip} \partial_i \tilde{g}^d g_{jq} \\
- \frac{1}{2} \left( -\frac{1}{4} \tilde{g}_{ab} \partial^i \tilde{g}^{ab} + \partial^i \tilde{\phi} \right) \left( \tilde{g}^{jp} \partial_j g_{iq} \right) - \frac{1}{2} \left( -\frac{1}{4} g^{ab} \partial_i g_{ab} + \partial_i \phi \right) \left( g_{jq} \partial^j g^{ip} \right) \\
- \frac{1}{2} \left( -\frac{1}{4} \tilde{g}_{ab} \partial^j \tilde{g}^{ab} + \partial^j \tilde{\phi} \right) \left( \tilde{g}_{ij} \partial^j g^{pq} \right) - \frac{1}{2} \left( -\frac{1}{4} g^{ab} \partial_j g_{ab} + \partial_j \phi \right) \left( g^{ij} \partial^p g_{qj} \right) \\
+ \frac{1}{2} g^{ip} \left( -\frac{1}{4} \tilde{g}_{ab} \partial^i \tilde{g}^{ab} + \partial^i \tilde{\phi} \right) \left( \tilde{g}_{ik} \partial^k g^{ij} \right) g_{jq} \\
+ \frac{1}{2} g^{ip} \left( -\frac{1}{4} g^{ab} \partial_j g_{ab} + \partial_j \phi \right) \left( g^{ki} \partial_k g^{ij} \right) g_{jq}
\begin{align}
&+ \frac{1}{2} g^{ip} \left( -\frac{1}{4} \tilde{g}_{ab} \tilde{\partial}^i \tilde{g}^{ab} + \tilde{\partial}^i \tilde{\phi} \right) \left( \tilde{g}^{kj} \partial_k g_{li} \right) g_{jq} \\
&+ \frac{1}{2} g^{ip} \left( -\frac{1}{4} g^{ab} \partial_l g_{ab} + \partial_l \phi \right) \left( g_{ki} \tilde{\partial}^k \tilde{g}^{ij} \right) g_{jq},
\end{align}

\[\ast R_{(p)1(q)} = +2 \partial_p \tilde{\partial}^q d - 2 \tilde{g}_{ip} \tilde{\partial}^j \partial_j g^{jq} - \frac{1}{2} \left( -\frac{1}{4} g^{ab} \partial_l g_{ab} + \partial_l \phi \right) \left( \tilde{g}_{ip} \tilde{\partial}^j \tilde{g}^{jq} \right) \]

\[-\frac{1}{2} \left( -\frac{1}{4} \tilde{g}_{ab} \tilde{\partial}^i \tilde{g}^{ab} + \tilde{\partial}^i \tilde{\phi} \right) \left( \tilde{g}^{kj} \partial_k g_{li} \right) g_{jq} - \frac{1}{2} \left( -\frac{1}{4} g^{ab} \partial_l g_{ab} + \partial_l \phi \right) \left( g^{ji} \tilde{\partial}^k \tilde{g}^{ij} \right) g_{jq} + \frac{1}{2} \tilde{g}_{ip} \left( -\frac{1}{4} \tilde{g}_{ab} \tilde{\partial}^i \tilde{g}^{ab} + \tilde{\partial}^i \tilde{\phi} \right) \left( g_{ki} \tilde{\partial}^k \tilde{g}^{ij} \right) g^{jq}. \tag{A.12}\]

There also exist similar symmetries as in \(\ast R\)

\[
\ast R_{(p)1(q)} \leftrightarrow \tilde{g}^{**}, \quad \tilde{\partial}^i \leftrightarrow \partial_i, \quad \tilde{\phi} \leftrightarrow \phi \quad \Rightarrow \ast R_{(p)2(q)} \]

\[
\ast R_{(p)2(q)} \leftrightarrow \tilde{g}^{**}, \quad \tilde{\partial}^i \leftrightarrow \partial_i, \quad \tilde{\phi} \leftrightarrow \phi \quad \Rightarrow \ast R_{(p)1(q)}. \tag{A.13}\]

Similarly, from the metric ansatz \((1.5)\) and calculation rules in section \((4.1)\),

\[
\ast R_{(2)2(1)} = (D - 1) \frac{\ddot{a}^2}{a^2} - (D - 1) \frac{\ddot{a}}{a} + 2 \ddot{\phi} - (D - 1) \frac{\dot{a}^2}{a^2} + (D - 1) \frac{\dot{a}}{a} - 2 \ddot{\phi},
\]

\[
\ast R_{(2)2(1)} = \frac{1}{a^4} \left( - (D - 1) \dot{a}^2 + 2 \ddot{\phi} \dot{a} \dot{a} \right) - \left( - (D - 1) \ddot{a}^2 + 2 \dddot{\phi} \dot{a} \ddot{a} \right),
\]

\[
\ast R_{(1)2(1)} = 0. \tag{A.14}\]

Finally, combining two parts of the generalized Ricci tensor,

\[
\mathcal{R}_{MN} = \ast \mathcal{R}_{MN} + \ast \mathcal{R}_{MN} = 0, \tag{A.15}\]

we get

\[
\mathcal{R}_{(2)2(1)} = - (D - 1) \left( \dot{H} + H^2 \right) + 2 \dot{\phi} + (D - 1) \left( - \ddot{H} + \dddot{H} \right) - 2 \ddot{\phi},
\]

\[
\mathcal{R}_{(2)2(1)} = \frac{1}{a^2} \left( \dot{H} - (D - 1) \dot{H} - 2 \ddot{\phi} \dot{H} \right) - a^2 \left( - \dddot{H} - (D - 1) H^2 + 2 \dddot{\phi} \right),
\]

\[
\mathcal{R}_{(1)2(1)} = 0. \tag{A.16}\]

with symmetry

\[
\mathcal{R}_{(p)1(q)} \leftrightarrow \tilde{g}^{**}, \quad \tilde{\partial}^i \leftrightarrow \partial_i, \quad \tilde{\phi} \leftrightarrow \phi \quad \Rightarrow \mathcal{R}_{(p)2(q)} \]

\[
\mathcal{R}_{(p)2(q)} \leftrightarrow \tilde{g}^{**}, \quad \tilde{\partial}^i \leftrightarrow \partial_i, \quad \tilde{\phi} \leftrightarrow \phi \quad \Rightarrow \mathcal{R}_{(p)1(q)}. \tag{A.17}\]
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