Finitely generated submonoids

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June 29, 2021

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Abstract
We prove several results concerning finitely generated submonoids of the free monoid. These results generalize those known for free submonoids. We prove in particular that if $X = Y \circ Z$ is a composition of finite sets of words with $Y$ complete, then $d(X) = d(Y)d(Z)$.

1 Introduction

We develop the theory of finitely generated submonoids of a free monoid. This parallels the presentation of [1] made for free submonoids. The main feature is that, at the loss of some technical complications, many notions can be generalized to arbitrary finitely submonoids of a free monoid, be they free or not.

The first step is to set a correspondence with an adequate category of automata. For free submonoids, there is a natural correspondence with unambiguous automata, which is a cornerstone of the theory. The adequate generalization of that of automata recognizing with multiplicities, in the sense that they count the number of factorizations (thus they are unambiguous when the submonoid is free).
We introduce the notion of composition for arbitrary finite sets of words. This is certainly natural as it corresponds to the composition of morphisms, the composition of codes corresponding to the composition of injective morphisms.

The technical difficulties begin with the replacement of unambiguous monoids of relations by arbitrary monoids of relations. Indeed, the multiplication of matrices is then different of their result with integer 0, 1 coefficients. In particular, the representation of maximal subgroups by permutations is still possible but more complicated.

We generalize the notion of the group $G(X)$ and the degree $d(X)$ of a finite set $X$ of words. In this way, as for codes, a set is synchronized if and only if it is of degree 1. We prove that if $X = Y \circ Z$ with $Y$ complete, then $d(X) = d(Y)d(Z)$ (Proposition 8.1).

A first version of this paper was presented at the conference FSTTCS 2020.

## 2 Automata

We denote by $A^*$ the free monoid on a finite alphabet $A$ and by $1$ the empty word.

Let $A = (Q, i, t)$ be an automaton on the alphabet $A$ with $Q$ as set of states, $i$ as initial state and $t$ as terminal state (we will not need here to have several initial or terminal states). We do not specify in the notation the set of edges which are triples $(p, a, q)$ with two states $p, q \in Q$ and a label $a \in A$ denoted $p \xrightarrow{a} q$. We form paths as usual by concatenating consecutive edges.

The language recognized by $A$, denoted $L(A)$, is the set of words in $A^*$ which are labels of paths from $i$ to $t$. There can be several paths from $i$ to $t$ for a given label and this motivates the introduction of multiplicities.

For a semiring $K$, a $K$-subset of $A^*$ is a map from $A^*$ into $K$. The value at $w$ is called its multiplicity. We denote by $K\langle A \rangle$ the semiring of $K$-subsets of finite number of words with nonzero multiplicity (on these notions, see [2]).

If $X, Y$ are $K$-subsets, then $X + Y$ and $XY$ are the $K$-subsets defined by

$$(X + Y, w) = (X, w) + (Y, w), \quad (XY, w) = \sum_{u=v} (X, u)(Y, v).$$

Moreover, if $X$ has no constant term, that is if $(X, 1) = 0$, then $X^*$ is the $K$-subset

$$X^* = 1 + X + X^2 + \ldots$$

Since $X$ has no constant term, for every word $w$, the number of nonzero terms $(X^n, w)$ in the sum above is finite and thus $X^*$ is well defined.

For a set $X \subset A^*$, we denote $X$ the characteristic series of $X$, considered as an $\mathbb{N}$-subset. It is easy to verify that for $X \subset A^*$, the multiplicity of $w \in A^*$ in $X^*$ is the number of factorizations of $w$ in words of $X$.

For an automaton $A = (Q, i, t)$ on the alphabet $A$, we denote by $|A|$ its behaviour, which is an element of $\mathbb{N}\langle A \rangle$. It is the $\mathbb{N}$-subset of $A^*$ such that the multiplicity of $w \in A^*$ in $|A|$ is the number of paths from $i$ to $t$ labeled $w$.  

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We denote by $\mu_A$ the morphism from $A^*$ into the monoid of $Q \times Q$-matrices with integer coefficients defined for $w \in A^*$ by

$$\mu_A(w)_{p,q} = \text{Card}\{a \in A \mid p \xrightarrow{a} q\}.$$  

Thus, the multiplicity of $w$ in $|A|$ is

$$(|A|, w) = \mu_A(w)_{i,t}$$

Given a set $X \subset A^+$ which is the minimal generating set of the submonoid $X^*$, we say that the automaton $A$ recognizes $X^*$ with multiplicities if the behaviour of $A$ is the multiset assigning to $x$ its number of distinct factorizations in $X$. Formally, $A$ recognizes $X$ with multiplicities if

$$|A| = X^*$$

**Example 2.1** Let $X = \{a, a^2\}$. The number of factorizations of $a^n$ in words of $X$ is the Fibonacci number $F_{n+1}$ defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. The automaton $A$ represented in Figure 1 recognizes $X^*$ with multiplicities, that is $|A| = (a + a^2)^*$.  

![Figure 1: An automaton recognizing $X^*$ with multiplicities.](image)

We have indeed for every $n \geq 1$,

$$\mu_A(a^n) = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

For an automaton $A = (Q, i, t)$ on the alphabet $A$, we denote by $\varphi_A$ the morphism from $A^*$ onto the monoid of transitions of $A$. Thus, for $w \in A^*$, $\varphi_A(w)$ is the Boolean $Q \times Q$-matrix defined by

$$\varphi_A(w)_{p,q} = \begin{cases} 1 & \text{if } p \xrightarrow{w} q, \\ 0 & \text{otherwise} \end{cases}$$

Let $X \subset A^+$ be a finite set of words on the alphabet $A$. The flower automaton of $X$ is the following automaton. Its set of states is the subset $Q$ of $A^* \times A^*$ defined as

$$Q = \{(u, v) \in A^+ \times A^+ \mid uv \in X\} \cup (1, 1).$$

We often denote $\omega = (1, 1)$. There are four type of edges labeled by $a \in A$

- $(u, av) \xrightarrow{a} (ua, v)$ for $uv \in X$, $u, v \neq 1$
- $\omega \xrightarrow{a} (a, v)$ for $av \in X$, $v \neq 1$
- $(u, a) \xrightarrow{a} \omega$ for $ua \in X$, $u \neq 1$
- $\omega \xrightarrow{a} \omega$ for $a \in X$.  

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The state $\omega$ is both initial and terminal.

The proof of the following result is straightforward. It generalizes the fact that the flower automaton of a code recognizes $X^*$ and is unambiguous (see [1, Theorem 4.2.2]).

**Proposition 2.2** For any finite set $X \subset A^+$, the flower of automaton of $X$ recognizes $X^*$ with multiplicities.

**Example 2.3** Let $X = \{a, ab, ba\}$. The flower automaton of $X^*$ is represented in Figure 2. As an example, there are 2 paths labeled $aba$ from $\omega$ to $\omega$, corresponding to the 2 factorizations $(a)(ba) = (ab)(a)$.

A more compact version of the flower automaton is the prefix automaton $A = (P, i, t)$ of a finite set $X \subset A^+$. Its set of states is the set $P$ of proper prefixes of $X$ and its edges are the $p \xrightarrow{a} pa$ for every $p \in P$ and $a \in A$ such that $pa \in P$ and the $p \xrightarrow{1} 1$ such that $pa \in X$. It also recognizes $X^*$ with multiplicities.

**Example 2.4** Let $X = \{a^2, a^3\}$. The flower automaton of $X$ is shown in Figure 3 on the left and its prefix automaton on the right.

A reduction from an automaton $A = (P, i, t)$ onto an automaton $B = (Q, j, u)$ is a surjective map $\rho : P \to Q$ such that $\rho(i) = j$, $\rho(t) = u$ and such that for every $q, q' \in Q$ and $w \in A^*$, there is a path $q \xrightarrow{w} q'$ in $B$ if and only if there is a path $p \xrightarrow{w} p'$ in $A$ for some $p, p' \in P$ with $\rho(p) = q$ and $\rho(p') = q'$. (2.1)

The reduction is sharp if $\rho^{-1}(j) = \{i\}$ and $\rho^{-1}(u) = \{t\}$.

**Proposition 2.5** Let $\rho$ be a reduction from $A = (P, i, t)$ onto $B = (Q, j, u)$. Then $L(A) \subset L(B)$ with equality if $\rho$ is sharp.
The term reduction is the one used in [1] and it is not standard but captures the general idea of a covering. The term conformal morphism is the one used in [7]. The following statement replaces [1, Proposition 4.2.5].

**Proposition 2.6** Let $X \subset A^+$ be a finite set which is the minimal generating set of $X^*$. For each trim automaton $B = (Q, i, i)$ recognizing $X^*$ with multiplicities, there is a sharp reduction from the flower automaton of $X$ onto $B$.

**Proof.** Let $A = (P, \omega, \omega)$ be the flower automaton of $X$. We define a map $\rho : P \to Q$ as follows. We set first $\rho(\omega) = i$. Next, if $(u, v) \in P$ with $(u, v) \neq \omega$, then $uv \in X$. Since $X$ is the minimal generating set of $X^*$, there is only one factorization of $uv$ into elements of $X$. Since $B$ recognizes $X$ with multiplicities, there is only one path $i \xrightarrow{u} q \xrightarrow{v} i$ in $B$. We define $\rho(u, v) = q$.

It is straightforward to verify that $\rho$ is a reduction. Assume first that $q \xrightarrow{u} q'$ in $B$. Let $i \xrightarrow{u}$ and $q' \xrightarrow{u'} i$ be simple paths, that is not passing by $i$ except at the origin or the end. Then $i \xrightarrow{uwv'} i$ and thus $uwv' = x_1x_2 \cdots x_n$ with $x_i \in X$, $u$ a proper prefix of $x_1 = uv$ and $v'$ a proper suffix of $x_n = u'v'$. Thus $\rho(u, v) = p$ and $\rho(u', v') = q$. Since $w = vx_2 \cdots x_{n-1}u'$, we have in $A$ a path $(u, v) \xrightarrow{x} (u', v')$.

Conversely, consider a path $(u, v) \xrightarrow{w} (u', v')$ in $A$. If the path does not pass by $\omega$, then $u' = uw$, $v = wv'$ and we have a path $q \xrightarrow{w} q'$ in $B$ with $\rho(u, v) = q$ and $\rho(u', v') = q'$. Otherwise, the path decomposes in $(u, v) \xrightarrow{w} \omega \xrightarrow{x} \omega \xrightarrow{w'} (u', v')$ with $x \in X^*$. Since $B$ recognizes $X^*$, we have a path $i \xrightarrow{x} i$ in $B$ and thus also a path $q \xrightarrow{w} q'$ with $q = \rho(u, v)$ and $q' = \rho(u', v')$.

The statement above is false if $X$ is not the minimal generating set of $X^*$, as shown by the following example.

**Example 2.7** Let $X = \{a, a^2\}$. There is no sharp reduction from the automaton of Figure 1 to the one-state automaton recognizing $X^* = a^*$. The statement is also false if the automaton $B$ does not recognize $X^*$ with multiplicities, as shown by the following example.

**Example 2.8** Let $X = \{a^2\}$. The flower automaton of $X$ is represented in Figure 4 on the left. There is no reduction to the automaton represented on the right which recognizes also $X^*$ (but not with multiplicities).

![Figure 4: Two automata recognizing $X^*$](image-url)
3 Transducers

A literal transducer $T = (Q, i, t)$ on a set of states $Q$ with an input alphabet $A$ and an output alphabet $B$ is defined by a set of edges $E$ which are of the form $p \xrightarrow{(a,v)} q$ with $p, q \in Q$, $a \in A$ and $v \in B \cup \{1\}$. The input automaton associated with a transducer is the automaton with same set of states and edges but with the output labels removed.

The relation realized by the transducer $T$ is the of pairs $(u, v) \in A^* \times B^*$ such that there is a path from $i$ to $t$ labeled $(u, v)$.

We denote by $\varphi_T$ the morphism from $A^*$ to the monoid of $Q \times Q$-matrices with elements in $\mathbb{N}^\langle A \rangle$ defined for $u \in A^*$ and $p, q \in Q$ by

$$\varphi_T(u)_{p,q} = \sum_{p \xrightarrow{a|v} q} v$$

Let $X \subset A^+$ be a finite set. Let $\beta : B^* \to A^*$ be a coding morphism for $X$, that is, a morphism whose restriction to $B$ is a bijection onto $X$.

The decoding relation for $X$ is the relation $\gamma = \{(u, v) \in A^* \times B^* \mid u = \beta(v)\}$. A decoder for $X$ is a literal transducer which realizes the decoding relation.

Let $X \subset A^+$ be a finite set with coding morphism $\beta : B \to A^+$. The flower transducer associated to $\beta$ is the literal transducer build on the flower automaton of $X$ by adding an output label 1 to each edge $\omega \xrightarrow{a} (a, v)$ or $(u, av) \xrightarrow{a} (ua, v)$ and a label $b$ to each edge $\omega \xrightarrow{a} \omega$ such that $a \in X$ with $\beta(b) = a$ or $(u, a) \xrightarrow{a} \omega$ such that $ua = x \in X$ with $\beta(b) = x$.

**Proposition 3.1** For every finite set $X$ with coding morphism $\beta$, the flower transducer associated to $\beta$ is a decoder for $X$.

**Example 3.2** Let $X = \{a, ab, ba\}$ and let $\beta : u \to a, v \to ab, w \to ba$. The flower transducer associated to $\beta$ is represented in Figure 5. One has

$$\varphi_T(a) = \begin{bmatrix} u & 1 & 0 \\ 0 & 0 & 1 \\ w & 0 & 0 \end{bmatrix} \text{ and } \varphi_T(b) = \begin{bmatrix} 0 & 0 & 1 \\ v & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

![Figure 5: The flower transducer associated to $\beta$.](image-url)
The prefix transducer $T = (P, 1, 1)$ is the same modification of the prefix automaton. Its states are the proper prefixes of the elements of $X$. There is an edge $p \xrightarrow{a} pa$ for every prefix $p$ and every letter $a$ and an edge $p \xrightarrow{a b}$ for every prefix $p$ and letter $a$ such that $pa = \beta(b) \in X$. Thus the input automaton of the prefix transducer of $X$ is the prefix automaton of $X$.

Let $B = (Q, j, j)$ be an automaton and let $T = (P, i, i)$ be a literal transducer. We build an automaton $A = B \circ T$ as follows. Its set of states is $Q \times P$ and for every $a \in A$, the matrix $\varphi_A(a)$ is obtained by replacing in $\varphi_T(a)$ the word $w = \varphi_T(a)_P$ by the matrix $\varphi_B(w)$. The initial and terminal state is $(j, i)$.

The automaton $A$ is also called the *wreath product* of $B$ and $T$ (see [3]).

## 4 Composition

Let $Y \subset B^*$ and $Z \subset A^*$ be finite sets of words such that there exists a bijection $\beta : B \to Z$ and that every letter of $B$ appears in a word of $Y$. Two such sets are called *composable*. Then $X = \beta(Y)$ is called the composition of $Y$ and $Z$ through $\beta$. We denote $X = Y \circ \beta Z$. We also denote $X = Y \circ Z$ when $\beta$ is understood. We also say that $X = Y \circ Z$ is a *decomposition* of $X$.

**Example 4.1** Let $Y = \{u, uv, vu\}$ and $Z = \{a, ab, ba\}$ with $\beta : u \to a, v \to ab, w \to ba$. Then $X = \{a, aba\}$.

A decomposition $X = Y \circ \beta Z$ of a finite set $X$ is *trim* if every $x \in X$ has a unique factorization in words of $Z$ or, equivalently, if the restriction of $\beta$ to $Y$ is injective. For any decomposition $X = Y \circ Z$, there is $Y' \subset Y$ such that $X = Y' \circ Z$ is trim. Indeed, if $x \in X$ has two decompositions in words of $Z$ as $x = z_1z_2\cdots z_n = z'_1z'_2\cdots z'_m$, we may remove $\beta^{-1}(z'_1z'_2\cdots z'_m)$ from $Y$ without changing $X$. A finite number of these removals gives a trim decomposition.

A set $X \subset A^*$ is *complete* if any word in $A^*$ is a factor of a word in $X^*$.

**Proposition 4.2** Let $Y \subset B^*$ and $Z \subset A^*$ be two composable finite sets and let $X = Y \circ \beta Z$ be a trim decomposition. Let $B = (Q, 1, 1)$ be the prefix automaton of $Y$ and let $T = (P, 1, 1)$ be the prefix transducer of $Z$. The automaton $A = B \circ T$ recognizes $X^*$ with multiplicities.

If $Y$ is complete, there is a reduction $\rho$ from $A$ onto the prefix automaton of $Z$. Moreover, the automaton $B$ can be identified through $\beta$ with the restriction of $A$ to $\rho^{-1}(1)$.

**Proof.** The simple paths in $A$ have the form

$$(1, 1) \xrightarrow{z_1} (b_1, 1) \xrightarrow{z_2} (b_1b_2, 1) \cdots \xrightarrow{z_n} (1, 1)$$

for $x = z_1 \cdots z_n = \beta(b_1 \cdots b_n)$ in $X$ and $z_i \in Z$. Since the decomposition is trim, there is exactly one such path for every $x \in X$ and thus $A$ recognizes $X^*$ with multiplicities.
Let us show that, if $Y$ is complete, the map $\rho : (q, p) \rightarrow p$ in the prefix automaton $C$ of $Z$ if and only if there exist $q, q' \in Q$ such that $(q, p) \xrightarrow{w} (q', p')$. Assume that $p \xrightarrow{w} p'$ in $C$. Then we have $p \xrightarrow{w} p'$ in the prefix transducer $T$ for some $u \in B^*$. Since $Y$ is complete, there are some $q, q' \in Q$ such that $q \xrightarrow{b} q'$ in $B$. Then $(q, p) \xrightarrow{w} (q', p')$ in $A$. The converse is obvious.

Finally, the edges of the restriction of $A$ to $\rho^{-1}(1)$ are the simple paths $(q, 1) \xrightarrow{z} (q', 1)$ for $z = \beta(b) \in Z$ and $q \xrightarrow{b} q'$ an edge of $B$. This proves the last statement.

Example 4.3 Let $Y = \{u, uv\}$ and $Z = \{a, ab\}$ with $\beta : u \rightarrow a, v \rightarrow ab$. We have, in view of Figure 6,

$$\varphi_A(a) = \begin{bmatrix} \varphi_B(u) & I \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \varphi_A(b) = \begin{bmatrix} 0 & \varphi_B(v) \\ 0 & 0 \end{bmatrix}.$$  

The flower automaton of $Y^*$ is represented in Figure 6. The trim part of the

![Figure 6: The flower automaton of $Y$, the prefix transducer $T$ of $Z$ and the trim part of the automaton $A$.](image)

automaton $A$ is sown in Figure 6 on the right.

5 Monoids of relations

We use monoids of binary relations and prove some technical results on idempotents and groups in such monoids. Few authors have considered the monoids of binary relations. In [9], the Green’s relations in the monoid $B_Q$ of all binary relations on a set $Q$ are considered. It is shown in [5] that any finite group appears as a maximal subgroup of the monoid $B_Q$ (in contrast with the monoid of all partial maps in which all maximal subgroups are symmetric groups).

We write indifferently relations on a set $Q$ as subsets of $Q \times Q$, as boolean $Q \times Q$-matrices or as directed graphs on a set $Q$ of vertices.

The rank of a relation $m$ on $Q$ is the minimal cardinality of a set $R$ such that $m = uv$ with $u$ a $Q \times R$ relation and $v$ an $R \times Q$ relation. Equivalently, the
rank of $m$ is the minimal number of row (resp. column) vectors (perhaps not rows or columns of $m$) which generate the set of rows (resp. columns) of $m$.

For example, the full relation $m = Q \times Q$ has rank 1. In terms of matrices

$$m = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

More generally, the rank of an equivalence relation is equal to the number of its classes.

A fixed point of a relation $m$ on $Q$ is an element $q \in Q$ such that $q \xrightarrow{m} q$.

The following result appears in [8] (see also [4]).

**Proposition 5.1** Let $e$ be an idempotent relation on a finite set $Q$, let $S$ be the set of fixed points of $e$ and let $\Gamma$ be the set of strongly connected components of the restriction of $e$ to $S$.

1. For all $p, q \in Q$ we have $p \xrightarrow{e} q$ if and only if there exists an $s \in S$ such that $p \xrightarrow{e} s$ and $s \xrightarrow{e} q$.

2. We have

$$e = \ell r$$

where $\ell = \{(p, \sigma) \in Q \times \Gamma \mid p \xrightarrow{e} s$ for some $s \in \sigma\}$ and $r = \{(\sigma, q) \in \Gamma \times Q \mid s \xrightarrow{e} q$ for some $s \in \sigma\}$.

**Proof.**

1. Choose $n > \text{Card}(Q)$. Since $p \xrightarrow{e^n} q$ there is some $s \in Q$ such that $p \xrightarrow{e} s$ and

$$s \xrightarrow{e} s \xrightarrow{e} s \xrightarrow{e} \cdots$$

with $i + j + k = n$. Then $p \xrightarrow{e} s \xrightarrow{e} s \xrightarrow{e} q$ and the statement is proved.

2. If $p \xrightarrow{e} q$, let $s \in S$ be such that $p \xrightarrow{e} s$ and let $\sigma$ be the strongly connected component of $s$. Then $p \xrightarrow{\ell} \sigma \xrightarrow{r} q$. Thus $e \leq \ell r$. Conversely, if $p \xrightarrow{\ell} \sigma \xrightarrow{r} q$ there are $s, s' \in \sigma$ such that $p \xrightarrow{e} s$ and $s' \xrightarrow{e} q$. Since $s, s'$ are in the same strongly connected component, we have $s \xrightarrow{e} s'$ and we obtain $p \xrightarrow{\ell} s \xrightarrow{e} s' \xrightarrow{e} q$ whence $p \xrightarrow{e} q$.

The decomposition of $e = \ell r$ given by Equation (5.1) is called the column-row decomposition of $e$. Note that Proposition 5.1 is false without the finiteness hypothesis on $Q$. Indeed, the relation $e = \{(x, y) \in \mathbb{R}^2 \mid x < y\}$ is idempotent but has no fixed points.

**Example 5.2** The matrix

$$m = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

is an idempotent of rank 1.
For an element \( m \) of a monoid \( M \), we denote by \( H(m) \) the \( H \)-class of \( m \), where \( H \) is the Green relation \( H = R \cap L \). It is a group if and only if it contains an idempotent \( e \) (see [1]). In this case, every \( m \in H(e) \) has a unique inverse \( m^{-1} \) in the group \( H(e) \).

The following result is the transposition to monoids of relations of [1, Proposition 9.1.7]. However, the result is restricted to a statement on the group \( H(e) \) instead of the monoid \( eMe \).

**Proposition 5.3** Let \( M \) be a monoid of relations on a finite set \( Q \), let \( e \in M \) be idempotent and let \( \Gamma \) be the set of strongly connected components of the fixed points of \( e \). For \( m \in H(e) \), let \( \gamma_e(m) \) be the relation on \( \Gamma \) defined by \( \gamma_e(m) = \{(\rho, \sigma) \in \Gamma \times \Gamma \mid r \xrightarrow{m} s \xrightarrow{m^{-1}} r \text{ for some } r \in \rho \text{ and } s \in \sigma\} \).

Then \( m \mapsto \gamma_e(m) \) is an isomorphism from \( H(e) \) onto a group of permutations on \( \Gamma \).

**Proof.** First \( \gamma_e(m) \) is a map. Indeed, let \( s \xrightarrow{m} t \xrightarrow{m^{-1}} s \) and \( s' \xrightarrow{m} t' \xrightarrow{m^{-1}} s' \). If \( s \xrightarrow{e} s' \), we have \( t \xrightarrow{m^{-1}} s \xrightarrow{e} s' \xrightarrow{m} t' \) and thus \( t \xrightarrow{e} t' \). By a symmetrical proof, we obtain that \( \gamma_e(m) \) is a permutation.

Next, it is easy to verify that \( \gamma_e \) is a morphism.

Finally, \( \gamma_e \) is injective. Indeed, assume that for \( m, m' \in H(e) \) we have \( \gamma_e(m) = \gamma_e(m') \). Assume first that \( p \) is a fixed point of \( e \), let \( r, r' \) be such that \( p \xrightarrow{m} r \xrightarrow{m^{-1}} p \) and \( p \xrightarrow{m'} r' \xrightarrow{m'^{-1}} p \). Since \( \gamma_e(m) = \gamma_e(m') \), we obtain that \( r, r' \) are in the same element of \( \Gamma \). We conclude that \( p \xrightarrow{m} r' \xrightarrow{e} r \xrightarrow{m^{-1}} p \xrightarrow{m} q \) which implies that \( p \xrightarrow{m} q \).

Now if \( p \) is not a fixed point of \( e \), there is since \( em = m \), an \( r \) such that \( p \xrightarrow{e} r \xrightarrow{m} q \). By Proposition 5.1 there is a fixed point \( r' \) of \( e \) such that \( p \xrightarrow{e} r' \xrightarrow{m} r \xrightarrow{m} q \). Then \( r' \xrightarrow{m} q \) implies \( r' \xrightarrow{m'} q \) by the preceding argument and finally \( p \xrightarrow{m} q \).

We denote \( G_e = \gamma_e(H(e)) \).

**Example 5.4** Let \( M \) be the monoid of all binary relations on \( Q = \{1, 2\} \). For \( e = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \), the group \( H(e) \) is trivial. On the contrary, if

\[
e = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

the group \( H(e) \) has order 2.
The definition of $\gamma_e$ can be formulated differently.

**Proposition 5.5** Let $M$ be a monoid of relations on a finite set $Q$ and let $e \in M$ be an idempotent. Let $\sigma, \tau$ be two distinct connected components of fixed points of $e$ and let $s \in \sigma, t \in \tau$. If $e_{s,t} = 1$, then $m_{t,s} = 0$ for every $m \in H(e)$ and thus $(\sigma, \tau) \notin \gamma_e(m)$. If $e_{s,t} = e_{t,s} = 0$ then $s \rightarrow t$ implies $(\sigma, \tau) \in \gamma_e(m)$.

**Proof.** Assume first that $e_{s,t} = 1$ so that the restriction of $e$ to $\{s,t\}$ is the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. If $m_{t,s} = 1$, then the restriction of $m$ to $\{s,t\}$ is the matrix with all ones, which is impossible since no power of $m$ can be equal to $e$.

If the restriction of $e$ to $\{s,t\}$ is the identity, then the restriction of $m \in H(e)$ is a permutation. Thus $(\sigma, \tau) \in \gamma_e(m)$ if and only if $s \rightarrow t$. $\blacksquare$

The following replaces [1, Proposition 9.1.9]. It uses the Green relation $D = LR = RL$.

**Proposition 5.6** Let $M$ be a monoid of relations on a finite set $Q$ and let $e, e' \in M$ be $D$-equivalent idempotents. Then the groups $G_e$ and $G_{e'}$ are equivalent permutation groups.

**Proof.** Let $(a, a', b, b')$ be a passing system from $e$ to $e'$, that is such that

$$\begin{align*}
eaa' &= e, & \bba'e' &= e', & \eva &= \vb'e'.
\end{align*}$$

We will verify that there is a commutative diagram of isomorphisms

$$\begin{array}{ccc}
H(e) & \xrightarrow{\tau} & H(e') \\
\downarrow{\gamma_e} & & \downarrow{\gamma_{e'}} \\
G_e & \xrightarrow{\tau'} & G_{e'}
\end{array}$$

We define the map $\tau$ by $\tau(m) = bma$. Then it is easy to verify that $\tau$ is a morphism and that $m' \mapsto b'm'a'$ is its inverse. Thus $\tau$ is an isomorphism.

We define $\tau'$ as follows. Let $\Gamma_e, \Gamma_{e'}$ be the sets of strongly connected components of fixed points of $e$ and $e'$ respectively. Let $\theta$ be the relation between $\Gamma_e$ and $\Gamma_{e'}$ defined by $(\sigma, \sigma') \in \theta$ if for some $s \in \sigma$ and $s' \in \sigma'$, we have $s \eas s'$. One may verify that $\theta$ is a bijection between $\Gamma_e$ and $\Gamma_{e'}$. Its inverse is the map on classes induced by $e'be = e'a'e$. Then $\tau'(n) = \theta n \theta$.

We verify that the diagram is commutative. Suppose that for some $m \in H(e)$ $(\sigma_1', \sigma_1) \in \tau'(\gamma_e(m))$. By definition of $\tau'$ there exist $\sigma_1, \sigma_2 \in \Gamma_e$ such that

$$(\sigma_1', \sigma_1) \in \theta', \quad (\sigma_1, \sigma_2) \in \gamma_e(m) \quad \text{and} \quad (\sigma_2, \sigma_2') \in \theta.$$

Then for $s_1 \in \sigma_1, s'_1 \in \sigma_1', s_2' \in \sigma_2'$ and $s_2 \in \sigma_2$, we have

$$s'_1 \xrightarrow{e'be} s_1, \quad s_1 \xrightarrow{m} s_2 \xrightarrow{m^{-1}} s_1, \quad s_2 \xrightarrow{eas} s_2'.$$
Then $s_1' \xrightarrow{bm} s_2' \xrightarrow{bm^{-1}a} s_1'$ showing that $(\sigma_1', \sigma_1') \in \gamma_{e'}(\tau(m))$.

Note that, contrary to the case of a monoid of unambiguous relations, two $D$-equivalent idempotents need not have the same number of fixed points. This is shown in the following example.

**Example 5.7** Let $M$ be the monoid of all relations on $Q = \{1, 2\}$. The two idempotents

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad e' = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

are $D$-equivalent although the first has one fixed point and the second has two.

Let $M$ be a monoid of relations on a finite set $Q$. The **minimal rank** of $M$, denoted $r(M)$ is the minimum of the ranks of the elements of $M$ other than 0. The following statement is [1, Theorem 9.3.10] proved for unambiguous monoids of relations. A $D$-class is **regular** if it contains an idempotent. A monoid of relations on $Q$ is **transitive** if for every $p, q \in Q$, there is an $m \in M$ such that $p \xrightarrow{m} q$.

**Theorem 5.8** Let $M$ be a transitive monoid of relations on a finite set $Q$. The set $K$ of elements of rank $r(M)$ is a regular $D$-class . The groups $G_e$ for $e$ idempotent in $K$ are equivalent transitive permutation groups. Moreover, for $i$ fixed point of $e$, the minimal rank $r(M)$ is the index in $H(e)$ of the subgroup $\{m \in H(e) \mid i \xrightarrow{m} i\}$.

The proof is the same as for the case of an unambiguous monoid of relations except for the last statement. Let $\sigma, \tau$ be two distinct strongly connected components of fixed points of $e$ and let $s \in \sigma, t \in \tau$. Since $M$ is transitive there is an $m \in M$ such that $s \xrightarrow{m} t$. Then $eme$ is not 0 and thus $eme \in H(e)$. Similarly if $n \in M$ is such that $t \xrightarrow{n} s$, then $ene \in H(e)$. This implies by Proposition 5.5 that the restriction of $e$ to $\{s, t\}$ is the identity and that $(\sigma, \tau) \in \gamma_e(eme)$. Thus $G_e$ is transitive. The last statement follows from the fact that for any transitive permutation group on a set $S$, the number of elements of $S$ is equal to the index of the subgroup fixing one of the points of $S$.

The **Sushkevitch group** of $M$ is one of the equivalent groups $G_e$ for $e$ of rank $r(M)$.

### 6 Group and degree of a set

Let $A = (P, i, i)$ and $B = (Q, j, j)$ be automata and let $\rho : P \to Q$ be a reduction. For $m = \varphi_A(w)$, the relation $n = \varphi_B(w)$ is well defined. We denote $n = \hat{\rho}(m)$. Then $\hat{\rho}$ is a morphism from $\varphi_A(A^*)$ onto $\varphi_B(A^*)$ called the **morphism associated** with $\rho$. The following result extends [1] Proposition 9.5.1 to arbitrary finite sets of words.
Proposition 6.1 Let \( X \subset A^+ \) be finite. Let \( A = (P, i, i) \) and \( B = (Q, j, j) \) be trim automata recognizing \( X^* \). Let \( M = \varphi_A(A^*) \) and \( N = \varphi_B(A^*) \). Let \( E \) be the set of idempotents in \( M \) and \( F \) the set of idempotents in \( N \).

Let \( \rho \) be a sharp reduction of \( A \) onto \( B \) and let \( \hat{\rho} : M \to N \) be the morphism associated with \( \rho \). Then

1. \( \hat{\rho}(E) = F \).

2. Let \( e \in E \) and \( f = \hat{\rho}(e) \). The restriction of \( \rho \) to the set \( S \) of fixed points of \( e \) is a bijection on the set of fixed points of \( f \) and the groups \( H_e \) and \( H_f \) are equivalent.

Proof. 1. Let \( e \in E \). Then \( \hat{\rho}(e) \) is idempotent since \( \hat{\rho} \) is a morphism. Thus \( \hat{\rho}(E) \subset F \). Conversely, if \( f \in F \), let \( w \in A^* \) be such that \( \varphi_B(w) = f \). Let \( n \geq 1 \) be such that \( e = \varphi_A(w)^n \) is idempotent. Then \( \hat{\rho}(e) = f \) since \( \hat{\rho} \circ \varphi_A = \varphi_B \).

2. Let \( S \) be the set of fixed points of \( e \) and \( T \) the set of fixed points of \( f \). Consider \( s \in S \) and let \( t = \rho(s) \). From \( s \xrightarrow{\varphi_A} s \), we obtain \( t \xrightarrow{\varphi_B} t \) and thus \( \rho(S) \subset T \). Conversely, let \( t \in T \). The restriction of \( e \) to the set \( R = \rho^{-1}(t) \) is a non zero idempotent. Thus there is some \( s \in R \) which is a fixed point of this idempotent, and thus of \( e \). Thus \( t \in \rho(S) \).

Since \( \hat{\rho} \) is a morphism from \( M \) onto \( N \), we have \( \hat{\rho}(H(e)) = H(f) \). It is clear that \( \rho \) maps a strongly connected component of \( e \) onto a strongly connected component of \( f \). To show that this map is a bijection, consider \( s, s' \in S \) such that \( \rho(s), \rho(s') \) belong to the same connected component. We may assume that \( e \) is not the equality relation. Let \( w \in A^* \) be such that \( \varphi_A(w) = e \). Since \( X \) is finite, there are factorizations \( w = uv = u'v' \) such that \( s \xrightarrow{u} i \xrightarrow{v} s \) and \( s' \xrightarrow{u'} i \xrightarrow{v'} s' \). Then we have \( j \xrightarrow{w} \rho(s) \xrightarrow{w} \rho(s') \xrightarrow{w} j \). Since \( \rho \) is sharp, this implies \( i \xrightarrow{wv^*} i \) and finally \( s \xrightarrow{wvw^*v'} s' \). This shows that \( s \xrightarrow{w} s' \). A similar proof shows that \( s' \xrightarrow{w} s \). Thus, \( s, s' \) belong to the same connected component of \( e \).

Moreover, for every \( m \in H(e) \), one has \( s \xrightarrow{\rho(m)} t \xrightarrow{\rho(m)^{-1}} s \) if and only if \( \rho(s) \xrightarrow{\hat{\rho}(m)} \rho(t) \xrightarrow{\hat{\rho}(m)^{-1}} \rho(s) \). Thus \( H_e \) and \( H_f \) are equivalent permutation groups.

Let \( X \subset A^+ \) be a finite set and let \( A \) be the flower automaton of \( X \). The degree of \( X \), denoted \( d(X) \), is the minimal rank of the monoid \( M = \varphi_A(A^*) \). The group of \( X \) is the Suschkevitch group of \( M \). Proposition 6.1 shows that the definition of the group and of the degree do not depend on the automaton chosen to recognize \( X^* \), provided one takes a trim automaton recognizing \( X^* \) with multiplicities.

7 Synchronization

Let \( X \subset A^+ \) be a finite set of words. A word \( x \in A^* \) is synchronizing for \( X \) if for every \( u, v \in A^* \),

\[ uXv \in X^* \Rightarrow uXv \in X^* \]

The set \( X \) is synchronized if there is a synchronizing word \( x \in X^* \).
Proposition 7.1  A finite set $X$ of words is synchronized if and only if its degree $d(X)$ is 1.

Proof. Let $A = (Q, i, i)$ be a trim automaton recognizing $X^*$.

Assume first that $d(X) = 1$. Let $x \in X^*$ be such that $\varphi_A(x)$ has rank 1. If $uxv \in X^*$, we have $i \xrightarrow{u} p \xrightarrow{x} q \xrightarrow{v} i$ for some $p, q \in Q$. Since $\varphi_A(x)$ has rank 1, we deduce from $i \xrightarrow{x} i$ and $p \xrightarrow{x} q$ that we have also $i \xrightarrow{x} q$ and $p \xrightarrow{x} i$. Thus $ux, xv \in X^*$, showing that $x$ is synchronizing.

Assume conversely that $X$ is synchronized. Let $x \in X^*$ be a synchronizing word. Replacing $x$ by some power, we may assume that $\varphi_A(x)$ is an idempotent $e$. Let $m \in H(e)$ and let $w \in \varphi_A^{-1}(m)$. Since $H(e)$ is finite, there is some $n \geq 1$ such that $m^n = e$. Then $(me)^n = e$ implies that $(wx)^n \in X^*$. Since $x$ is synchronizing, we obtain $wx \in X^*$ and since $\varphi_A(wx) = me = m$, this implies $w \in X^*$. This shows that $H(e)$ is contained in $\varphi_A(X^*)$ and thus that $d(X) = 1$ by Theorem 5.8.

Example 7.2  Consider again $X = \{a, ab, ba\}$ (Example 2.3). The flower automaton of $X$ is represented again for convenience in Figure 7.

![Figure 7: The flower automaton of $X$.](image)

The minimal rank of the elements of $\varphi_A(A^*)$ is 1. Indeed, we have

$$\varphi_A(a^2) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

Accordingly, $aa$ is a synchronizing word.

![Figure 8: The set $K$ of elements of rank 1.](image)
Similarly, we indicate above it the set of states \( q \) such that the column of index \( q \) is nonzero. A \( \ast \) indicates an \( \mathcal{H} \)-class which is a group. Note that

\[
\varphi_A(a^2b) = \begin{bmatrix}
1 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 1
\end{bmatrix}
\]

has two fixed points but only one strongly connected class, in agreement with fact that if is of rank 1.

## 8 Groups and composition

Given a transitive permutation group on a set \( Q \), an \emph{imprimitivity relation} of \( G \) is an equivalence on \( Q \) compatible with the group action. If \( \theta \) is such an equivalence relation, we denote by \( G_\theta \) the permutation group induced by the action of \( G \) on the classes of \( \theta \). The groups induced by the action on the class of an element \( i \in Q \) by the action of the elements of \( G \) stabilizing the class of \( i \) are all equivalent. We denote by \( G^\theta \) one of them.

**Proposition 8.1** Let \( X \subset A^+ \) be a finite set which decomposes into \( X = Y \circ Z \) with \( Y \) complete. There exists an imprimitivity equivalence \( \theta \) of \( G = G(X) \) such that

\[
G^\theta = G(Y), \quad G_\theta = G(Z).
\]

In particular, \( d(X) = d(Y)d(Z) \).

**Proof.** Let \( B = (Q, i, i) \) be the flower automaton of \( Y \) and let \( T \) be the prefix transducer of \( Z \). Let \( A = B \circ T \). By Proposition 4.2 there is a reduction \( \rho \) from \( A = (Q \times P; (i,1), (i,1)) \) onto the prefix automaton \( \mathcal{C} \) of \( Z \).

Let \( e \) be an idempotent of minimal rank in \( \varphi_A(X^+) \). Let \( S \) be the set of fixed points of \( e \) and let \( \Gamma \) be the set of connected components of the elements of \( S \). Let \( \hat{S} \) be the set of fixed points of \( \hat{e} \equiv \hat{\rho}(e) \) and let \( \hat{\Gamma} \) be the set of correpsonding strongly connected components. If \( s, s' \in S \) are in the same strongly connected component, then \( \rho(s), \rho(s') \) are in the same strongly connected component of \( \hat{S} \). Thus, we have a well defined map \( \hat{\rho} : \Gamma \to \hat{\Gamma} \) such that \( s \in \Gamma \) if and only if \( \rho(s) \in \hat{\rho}(\Gamma) \).

We define an equivalence \( \theta \) on \( \Gamma \) by \( \sigma \equiv \sigma' \) if \( \hat{\rho}(\sigma) = \hat{\rho}(\sigma') \). Let \( m \in H(e) \) and suppose that \((\sigma, \tau), (\sigma', \tau') \in \gamma_e(m) \). If \( \sigma \equiv \sigma' \mod \theta \), then \( \tau \equiv \tau' \mod \theta \). Let indeed \( s \in \sigma, s' \in \sigma' \) and \( t \in \tau, t' \in \tau' \). We have by definition of \( \gamma_e \)

\[
s \xrightarrow{m} t \xrightarrow{m^{-1}} s \quad \text{and} \quad s' \xrightarrow{m} t' \xrightarrow{m^{-1}} t'
\]

and thus

\[
\rho(s) \xrightarrow{\hat{\rho}(m)} \rho(t) \xrightarrow{\hat{\rho}(m)^{-1}} \rho(s) \quad \text{and} \quad \rho(s') \xrightarrow{\hat{\rho}(m)} \rho(t') \xrightarrow{\hat{\rho}(m)^{-1}} \rho(t')
\]

This implies that \( \rho(t) \xrightarrow{\hat{\rho}} \rho(t') \) and \( \rho(t') \xrightarrow{\hat{\rho}} \rho(t) \). But since \( \gamma_e(\hat{m}) \) is a permutation, this forces \( \hat{\rho}(\tau) = \hat{\rho}(\tau') \) and finally \( \tau \equiv \tau' \mod \theta \).
Since the action of \( H(e) \) on the classes of \( \theta \) is the same as the action of \( H(\hat{e}) \), we have \( G(Z) = G_a \).

Finally, let \( \sigma \in \Gamma \) be the class of the initial state \((i, 1)\) and let \( I \) be its class \( \text{mod } \theta \). Let \( x \in X^* \) be such that \( \varphi_A(x) = e \) and let \( y = \beta^{-1}(x) \). Then \( f = \varphi_B(y) \) is an idempotent of \( \varphi_B(B^*) \) of rank \( d(Y) \). Moreover the strongly connected components of fixed points of \( f \) can be identified with the elements of \( I \). Thus \( G^\theta = G(Y) \) and \( \text{Card}(I) = d(Y) \), showing finally that \( d(X) = d(Y)d(Z) \).

Example 8.2 Let \( X = \{aa, aab, aba, abab, abba, baa, baab, baba\} \). We have \( X = Z^2 \) where \( Z = \{a, ab, ba\} \) is the set of example 7.2. An automaton recognizing \( X^* \) is represented in Figure 9. There is a reduction \( \rho \) from \( A \) to the automaton recognizing \( Z^* \) of Example 7.2 given by

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\rho & 1 & 2 & 2 & 3 & 2 & 1 & 3 \\
\end{array}
\]

We have \( d(X) = 2 \), in agreement with Proposition 5.1 since \( X = Y \circ Z \) with \( d(Y) = 2 \) and \( d(Z) = 1 \).

\[
\begin{array}{ccc}
1, 5/2 & 3 & 1, 4/6 & 4/7 \\
1, 4/2, 6, 7 & a^2 & a^2b & a^2b^2 \\
1, 5/3, 6 & ba^2 & ba^2b & ba^2b^2 \\
3/5 & b^2a^2 & b^2a^2b & b^2a^2b^2 \\
\end{array}
\]

Figure 10: The set \( K \) of elements of rank 1.

The set \( K \) of elements of \( \varphi_A(A^*) \) of rank 2 is represented in Figure 10. We have indicated only one element of each \( H \)-class (which contain two elements each). It is the direct product of \( Z/2Z \) with the corresponding set \( K \) of Example 7.2.
We finally prove the following result which relates the decompositions of codes and those of arbitrary sets of words.

**Proposition 8.3** Let $X = Y \circ Z$ be a trim decomposition of a finite set $X$. If $X$ is a code and if $Y$ is complete, then $Z$ is a code.

**Proof.** Since $\beta$ is trim, $Y$ is a code. Let $\beta : B \rightarrow Z$ be the coding morphism for $Z$ such that $X = Y \circ_\beta Z$. Assume that $z \in Z^*$ is a word with more than one factorization into words of $Z$. Let $u, v \in B^*$ two distinct elements in $\beta^{-1}(z)$. Let $\mathcal{A}$ be the flower automaton of $Y$. Let $y \in Y^*$ be such that $\varphi_\mathcal{A}(y)$ has minimal rank. Then $yuy, yvy$ are not zero since $Y$ is complete. Thus $\varphi_\mathcal{A}(yuy), \varphi_\mathcal{A}(yvy)$ belong to the $\mathcal{H}$-class of $\varphi_\mathcal{A}(y)$ which is a finite group. Let $e$ be its idempotent. There are integers $n, m, p$ such that $\varphi_\mathcal{A}(y)^n = \varphi_\mathcal{A}(yuy)^m = \varphi_\mathcal{A}(yvy)^p = e$. Since $y \in Y^*$, this implies that $e \in \varphi_\mathcal{A}(Y^*)$ and thus that $(yuy)^m, (yvy)^p$ are in $Y^*$. We conclude that $Y$ is not a code, a contradiction.  

**Acknowledgements** We thank Jean-Eric Pin and Jacques Sakarovitch for references concerning the composition of automata and transducers.

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