A Note on the Representation Theory of Fell Bundles

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ABSTRACT. We show that every Fell bundle $B$ over a locally compact group $G$ is proper in a sense recently introduced by Ng. Combining our results with those of Ng we show that if $B$ satisfies the approximation property then it is amenable in the sense that the full and reduced cross-sectional $C^*$-algebras coincide.

1. Introduction.

Let $B = \{B_t\}_{t \in G}$ be a Fell bundle over a locally compact group $G$ (see [FD] for a comprehensive treatment of the theory of Fell bundles, also referred to as $C^*$-algebraic bundles). We denote by $L^2(B)$ the right Hilbert $B_e$-module obtained by completing the space $C_c(B)$ of all continuous compactly supported sections of $B$, under the $B_e$-valued inner product (cf. [E: Section 2], [N: 2.2]) given by

$$
\langle \xi, \eta \rangle_{B_e} = \int \xi(t)^* \eta(t) \, dt, \quad \xi, \eta \in C_c(B).
$$

We should warn the reader that our notation for $L^2(B)$ differs from that used in [N].

The left-regular representation of $B$ is the map (cf. [E: 2.2])

$$
\Lambda : B \to \mathcal{L}(L^2(B)),
$$

where $\mathcal{L}(L^2(B))$ indicates the $C^*$-algebra of all adjointable operators [JT: 1.1.7] on $L^2(B)$, given for any $t$ in $G$ and any $b_t$ in $B_t$, by

$$
\Lambda(b_t)\xi \big|_s = b_t\xi(t^{-1}s), \quad \xi \in L^2(B), \quad s \in G.
$$

Let $\mathcal{C}^*(B)$ be the cross-sectional $C^*$-algebra of $B$ (cf. [FD: VIII.17.2]) defined to be the enveloping $C^*$-algebra of the Banach *-algebra $L_1(B)$ formed by the integrable sections [FD: VIII.5.2]. The integrated form of $\Lambda$, which we also denote by $\Lambda$, is the *-homomorphism

$$
\Lambda : \mathcal{C}^*(B) \to \mathcal{L}(L^2(B))
$$

specified by setting $\Lambda(f)\xi = f \ast \xi$ (cf. [N: 2.10]) for all $f$ in the dense subalgebra $C_c(B) \subseteq \mathcal{C}^*(B)$, and all $\xi \in C_c(B) \subseteq L^2(B)$.

Suppose that we are given a *-representation (cf. [FD: VIII.8.2 and 9.1]) $\pi$ of $B$ on a Hilbert space $H$, i.e, a map $\pi : B \to \mathcal{B}(H)$ that is linear on each fiber, that satisfies

(i) $\pi(b)\pi(c) = \pi(bc),$

(ii) $\pi(b)^* = \pi(b^*),$

for each $b, c \in B$, and that is continuous in the sense that for each $u \in H$, the map

$$
b \in B \mapsto \pi(b)u \in H
$$

is continuous in the norm of $H$.

* Partially supported by CNPq.
We may then form the representation $\pi_\lambda$ of $B$ on $L_2(G) \otimes H = L_2(G, H)$ by setting

$$\pi_\lambda(b_t) = \lambda_t \otimes \pi(b_t), \quad t \in G, \ b_t \in B_t,$$

where $\lambda_t$ refers to the left-regular representation of $G$ on $L_2(G)$. We will also denote by

$$\pi_\lambda : C^*(B) \to B(L_2(G, H))$$

its integrated form [FD: VIII.11.2, 11.4, 17.2].

Generalizing [E: 2.3], Ng defines in [N: 2.11] the reduced cross-sectional $C^*$-algebra of $B$, denoted $C^*_R(B)$, to be $\Lambda(C^*(B))$. Ng also proposes an alternative notion of reduced algebra, namely

$$C^*_R(B) := \pi_\lambda(C^*(B)),$$

where $\pi$ is any faithful $^*$-representation of $B$ on a Hilbert space $H$.

There exists (see below) a unique surjective $^*$-homomorphism $\Psi : C^*_R(B) \to C^*_e(B)$ such that the diagram

$$\begin{array}{ccc}
C^*(B) & \xrightarrow{\pi_\lambda} & C^*_R(B) \\
\downarrow & & \downarrow \Psi \\
C^*_e(B) & \xrightarrow{\Lambda} & C^*_e(B)
\end{array}$$

commutes. Ng thus introduced the notion of *proper* Fell bundles (cf. [N: 2.15]) to single out those for which $\Psi$ is injective. It is noticed in [N] that Theorem 3.3 in [E] implies that Fell bundles over discrete groups are automatically proper. It is also shown that saturated Fell bundles are always proper [N: 2.17], as well as those whose underlying group is compact [N: A.3].

It is the purpose of this note to show that all Fell bundles are proper and hence that the alternative reduced algebra $C^*_R(B)$ proposed by Ng always coincides with $C^*_e(B)$.

One of the main consequences is that the properness hypothesis required in the main result of [N] (Proposition 3.9) becomes superfluous and hence we conclude that all Fell bundles satisfying the approximation property (Definition 3.6 in [N]; see also [E: 4.4]) are amenable in the sense that $\Lambda$ is an isomorphism from $C^*(B)$ to $C^*_e(B)$.

2. Preliminaries.

Let us fix, throughout, a $^*$-representation $\pi : B \to B(H)$. Restricting $\pi$ to $B_e$ we may view $H$ as a left $B_e$-module and hence we may form the tensor product $L_2(B) \otimes_{B_e} H$ (cf. [R: 5.1], [JT: 1.2.3]), which is a Hilbert space under the inner product defined by

$$\langle \xi \otimes u, \eta \otimes v \rangle = \langle u, \pi \left( (\xi, \eta)_{B_e} \right) v \rangle, \quad \xi, \eta \in L_2(B), \ u, v \in H.$$

2.1. Proposition. (Lemma 2.4 in [N]). There exists an isometry

$$V : L_2(B) \otimes_{B_e} H \to L_2(G, H),$$

such that for all $\xi \in L_2(B), u \in H$, and $t \in G$ one has

$$V(\xi \otimes u) |_{t} = \pi(\xi(t))u.$$

Proof. It is obvious that $V$ is balanced with respect to the corresponding actions of $B_e$ and hence it is well defined on the algebraic tensor product $L_2(B) \otimes_{B_e} H$. Now let $\xi, \eta \in L_2(B)$, and $u, v \in H$. We have

$$\langle V(\xi \otimes u), V(\eta \otimes v) \rangle = \int \langle \pi(\xi(t))u, \pi(\eta(t))v \rangle \ dt = \left\langle u, \pi \left( \int \xi(t)^* \eta(t) \ dt \right) v \right\rangle =$$

$$= \langle u, \pi((\xi, \eta)_{B_e})v \rangle = \langle \xi \otimes u, \eta \otimes v \rangle,$$

from which all of the remaining details follow.
It should be observed that $V$ is not necessarily surjective. In fact, note that the vector $V(\xi \otimes u)|_s$, mentioned above, lies in $\pi(B_t)\mathcal{H}$ which is often a proper subset of $\mathcal{H}$. This is related to the notion of saturated representations [N: Definition 2.5] and is one of the main stumbling blocks we must overcome in order to achieve our goals.

2.2. Proposition. (Lemma 1.3 in [N]). If $\pi|_{B_e}$ is injective then so is the *-homomorphism

$$T \in \mathcal{L}(L_2(B)) \mapsto T \otimes 1 \in \mathcal{B}(L_2(B) \otimes_{B_e} \mathcal{H}).$$

Proof. Suppose that $T \otimes 1 = 0$. Then, for all $\xi, \eta \in L_2(B)$, and $u, v \in \mathcal{H}$ we have

$$0 = \langle (T \otimes 1)(\xi \otimes u), \eta \otimes v \rangle = \langle u, \pi(T(\xi), \eta)_{B_e} v \rangle.$$  

Since $u$ and $v$ are arbitrary, and $\pi$ is supposed injective on $B_e$, this implies that $\langle T(\xi), \eta \rangle_{B_e} = 0$ for all $\xi$ and $\eta$, which in turn gives $T = 0$. \qed

In particular, when $\pi|_{B_e}$ is injective, we have by 2.2 that $C^*_r(B)$ is isomorphic to the algebra $\Lambda(C^*(B)) \otimes 1$ of operators on the Hilbert space $L_2(B) \otimes_{B_e} \mathcal{H}$.

2.3. Proposition. For any $b \in B$ the diagram

$$
\begin{array}{ccc}
L_2(B) \otimes_{B_e} \mathcal{H} & \xrightarrow{\Lambda(b) \otimes 1} & L_2(B) \otimes_{B_e} \mathcal{H} \\
V & & \downarrow V \\
L_2(G, \mathcal{H}) & \xrightarrow{\pi_\lambda(b)} & L_2(G, \mathcal{H})
\end{array}
$$

commutes.

Proof. Let $t \in G$ be such that $b \in B_t$. We then have for all $\xi \in L_2(B)$, $u \in \mathcal{H}$, and $s \in G$ that

$$V(\Lambda(b) \otimes 1)(\xi \otimes u)|_s = V(\Lambda(b)\xi \otimes u)|_s = \pi\left(\Lambda(b)\xi\right)_s u = \pi(b\xi(t^{-1}s))u.$$

On the other hand

$$\pi_\lambda(b)V(\xi \otimes u)|_s = \pi(b)\left(V(\xi \otimes u)|_{t^{-1}s}\right) = \pi(b)\pi(\xi(t^{-1}s))u.$$

It follows that the same holds if, in place of the "$b$" in the statement above, we substitute any $a \in C^*(B)$, since the corresponding representations at the level of $C^*(B)$ are integrated from those of $B$. \qed

2.4. Definition. (cf. [N]). Given a *-representation $\pi : B \to \mathcal{H}$ as above we shall denote by $C^*_{R,\pi}(B)$ the algebra $\pi_\lambda(C^*(B))$ of operators on $L_2(G, \mathcal{H})$. The first relationship between $C^*_{R,\pi}(B)$ and $C^*_r(B)$ is given by:

When $\pi|_{B_e}$ is faithful, $C^*_{R,\pi}(B)$ was proposed by Ng [N] as an alternative reduced cross-sectional C*-algebra for $B$. The first relationship between $C^*_{R,\pi}(B)$ and $C^*_r(B)$ is given by:

2.5. Proposition. Suppose that $\pi|_{B_e}$ is injective. Then for any $a \in C^*(B)$ one has that $\|\Lambda(a)\| \leq \|\pi_\lambda(a)\|$. Therefore there exists a unique *-homomorphism $\Psi : C^*_{R,\pi}(B) \to C^*_r(B)$ such that the diagram

$$
\begin{array}{ccc}
C^*_r(B) & \xrightarrow{\Psi} & C^*_{R,\pi}(B) \\
\pi_\lambda \downarrow & & \downarrow \Lambda \\
C^*_r(B) & & C^*_r(B)
\end{array}
$$

commutes.

Proof. By 2.3 we have that $\Lambda \otimes 1$ is equivalent to a subrepresentation of $\pi_\lambda$. Therefore

$$\|\Lambda(a) \otimes 1\| \leq \|\pi_\lambda(a)\|.$$  

Now, by 2.2, we have that $\|\Lambda(a) \otimes 1\| = \|\Lambda(a)\|$. The existence of $\Psi$ now follows by routine arguments. \qed
3. The main result.

As already indicated, we plan to prove that $\Psi$ is an isomorphism under the hypothesis that $\pi|_{B_\ast}$ is injective. This is clearly equivalent to proving that for any $a \in C^*(B)$ one has $\|\Lambda(a)\| = \|\pi_\lambda(a)\|$. The starting point is that, although $\Lambda \otimes 1$ is but a subrepresentation of $\pi_\lambda$, we may “move it around” filling out the whole of the representation space for $\pi_\lambda$. What will do the “moving around” will be the right-regular representation of $G$, namely the unitary representation $\rho$ of $G$ on $L_2(G)$ given by

$$\rho_r(\xi)|_s = \Delta(r)^{1/2}\xi(sr),$$

for $\xi \in L_2(G)$, and $r, s \in G$, where $\Delta$ is, as usual, the modular function for $G$.

3.1. Proposition. For each $r \in G$,

(i) The unitary operator $\rho_r \otimes 1$ on $L_2(G) \otimes \mathcal{H} = L_2(G, \mathcal{H})$ lies in the commutant of $\pi_\lambda(C^*(B))$.

(ii) Consider the isometry

$$V_r : L_2(B) \otimes B_\ast \mathcal{H} \to L_2(G, \mathcal{H}),$$

given by $V_r = (\rho_r \otimes 1)V$. Then for all $a \in C^*(B)$ one has $V_r(\Lambda(a) \otimes 1) = \pi_\lambda(a)V_r$.

(iii) Let $K_r$ be the range of $V_r$. Then $K_r$ is invariant under $\pi_\lambda$ and the restriction of $\pi_\lambda$ to $K_r$ is equivalent to $\Lambda \otimes 1$.

Proof. It is clear that $\rho_r \otimes 1$ commutes with $\pi_\lambda(b_t) = \lambda_t \otimes \pi(b_t)$ for any $b_t \in B_t$. It then follows that $\rho_r \otimes 1$ also commutes with the range of the integrated form of $\pi_\lambda$, whence (i). The second point follows immediately from (i) and 2.3. Finally, (iii) follows from (ii). \square

Our next result is intended to show that the $K_r$’s do indeed fill out the whole of $L_2(G, \mathcal{H})$.

3.2. Proposition. Suppose that $\pi|_{B_\ast}$ is non-degenerate. Then the linear span of $\bigcup_{r \in G} K_r$ is dense in $L_2(G, \mathcal{H})$.

Proof. Let

$$\Gamma = \text{span}\{V_r(\xi \otimes u) : r \in G, \xi \in C_\ast(B), u \in \mathcal{H} \}.$$

Since

$$V_r(\xi \otimes u)|_t = (\rho_r \otimes 1)V(\xi \otimes u)|_t = \Delta(r)^{1/2}V(\xi \otimes u)|_{tr} = \Delta(r)^{1/2}\pi(\xi(tr))u, \quad t \in G,$$

and since we are taking $\xi$ in $C_\ast(B)$ above, it is easy to see that $\Gamma$ is a subset of $C_\ast(G, \mathcal{H})$. Our strategy will be to use [FD: II.15.10] for which we must prove that:

(I) If $f$ is a continuous complex function on $G$ and $\eta \in \Gamma$, then the pointwise product $f\eta$ is in $\Gamma$;

(II) For each $t \in G$ the set $\{\eta(t) : \eta \in \Gamma \}$ is dense in $\mathcal{H}$.

The proof of (I) is elementary in view of the fact that $C_\ast(B)$ is closed under pointwise multiplication by continuous scalar-valued functions [FD: II.13.14]. In order to prove (II) let $v \in \mathcal{H}$ have the form $v = \pi(b)u$, where $b \in B_\ast$ and $u \in \mathcal{H}$. By [FD: II.13.19] let $\xi \in C_\ast(B)$ be such that $\xi(e) = b$. It follows that $\eta_r := V_r(\xi \otimes u)$ is in $\Gamma$ for all $r$. Also note that, setting $r = t^{-1}$, we have

$$\eta_r^{-1}(t) = \Delta(t)^{-1/2}\pi(\xi(e))u = \Delta(t)^{-1/2}\pi(b)u = \Delta(t)^{-1/2}v.$$

This shows that $v \in \{\eta(t) : \eta \in \Gamma \}$. Since the set of such $v$’s is dense in $\mathcal{H}$, because $\pi|_{B_\ast}$ is non-degenerate, we have that (II) is proven.

As already indicated, it now follows from [FD: II.15.10] that $\Gamma$ is dense in $L_2(G, \mathcal{H})$. Since $\Gamma$ is contained in the linear span of $\bigcup_{r \in G} K_r$, the conclusion follows. \square

The following is our main technical result:

3.3. Lemma. For all $a \in C^*(B)$ one has that $\|\pi_\lambda(a)\| \leq \|\Lambda(a)\|$.
Proof. We may clearly suppose, without loss of generality, that \( \pi \) is non-degenerate. By [FD: VIII.9.4] it follows that \( \pi|_{B_e} \) is non-degenerate as well. Under this assumption we claim that for all \( a \in C^*(B) \) one has \[
\Lambda(a) = 0 \implies \pi_\lambda(a) = 0.
\]
In order to see this suppose that \( \Lambda(a) = 0 \). Then for each \( r \in G \) we have by 3.1.(ii) that \( \pi_\lambda(a)V_r = V_r(\Lambda(a) \otimes 1) = 0 \). Therefore \( \pi_\lambda(a) = 0 \) in the range \( K_r \) of \( V_r \). By 3.2 it follows that \( \pi_\lambda(a) = 0 \), thus proving our claim.

Define a map \[
\varphi : C^*_r(B) \longrightarrow B(L_2(G, H))
\]
by \( \varphi(\Lambda(a)) := \pi_\lambda(a) \), for all \( a \) in \( C^*(B) \). By the claim above we have that \( \varphi \) is well defined. Also, it is easy to see that \( \varphi \) is a \(*\)-homomorphism. It follows that \( \varphi \) is contractive and hence that for all \( a \) in \( C^*(B) \)
\[
\|\pi_\lambda(a)\| = \|\varphi(\Lambda(a))\| \leq \|\Lambda(a)\|.
\]
\( \square \)

Our main results follow more or less immediately from 3.3:

3.4. Corollary. Let \( B \) be any Fell bundle over a locally compact group \( G \) and let \( \pi \) be a \(*\)-representation of \( B \) on the Hilbert space \( H \) such that \( \pi|_{B_e} \) is injective. Then the map \( \Psi : C^*_{R,\pi}(B) \rightarrow C^*_r(B) \) defined above is an isomorphism. Therefore \( B \) is always proper in the sense of Ng [N].

3.5. Corollary. Suppose that the Fell bundle \( B \) satisfies the approximation property (Definition 3.6 in [N]; see also [E: 4.4]), then \( B \) is amenable in the sense that \( \Lambda \) is an isomorphism from \( C^*(B) \) to \( C^*_r(B) \).

Proof. Combine Proposition 3.9 in [N] with 3.4. \( \square \)

The following generalizes [P: 7.7.5] to the context of Fell bundles:

3.6. Corollary. Let \( \pi : B \rightarrow B(H) \) be a representation of the Fell bundle \( B \) and let \( \pi_\lambda \) be the representation of \( B \) on \( L_2(G, H) \) given by \( \pi_\lambda(b_t) = \lambda_t \otimes \pi(b_t) \), for \( t \in G \), and \( b_t \in B_t \). Denote also by \( \pi_\lambda \) the representation of \( C^*(B) \) obtained by integrating \( \pi_\lambda \). Then \( \pi_\lambda \) factors through \( C^*_r(B) \). Moreover, in case \( \pi|_{B_e} \) is faithful, the representation of \( C^*_r(B) \) arising from this factorization is also faithful.

Proof. Follows immediately from 2.5 and 3.3. \( \square \)

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