The Poisson enveloping algebra and the algebra of Poisson differential operators of a generalized Weyl Poisson algebra

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Abstract

For a generalized Weyl Poisson algebra $A$, explicit sets of generators and defining relations are presented for its Poisson enveloping algebra $U(A)$. Simplicity criteria are given for the algebra $U(A)$ and algebra of Poisson differential operators $PD(A)$ on $A$. The Gelfand-Kirillov dimensions of the algebras $U(A)$ and $PD(A)$ are calculated. It is proven that the algebra $U(A)$ is a domain provided that the coefficient ring $D$ of the generalized Weyl Poisson algebra $A$ is a domain of essentially finite type over a perfect field.

For the algebra $A$, the set of its minimal primes and the prime radical are described and an equidimensionality criterion is given. For the equidimensional algebra $A$ of essentially finite type, two regularity criteria are presented.

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1 Introduction

In this paper, $K$ is a field of arbitrary characteristic, $P_n = K[x_1, \ldots, x_n]$ is a polynomial algebra in $n$ variables over the field $K$, $D$ is a $K$-algebra and

$$A = D[X,Y; a] = D[X_1, \ldots, X_N, Y_1, \ldots, Y_N]/(X_1Y_1 - a_1, \ldots, X_NY_N - a_N)$$
is a **commutative generalized Weyl algebra** (GWA) of rank $N$ where $X = (X_1, \ldots, X_N)$, $Y = (Y_1, \ldots, Y_N)$ and $a = (a_1, \ldots, a_N) \in D^N$.

**The set of minimal primes and an equidimensionality criterion for a GWA $A$.** For an algebra $R$, we denote by $\text{min}(R)$ the set of its minimal primes. Proposition 2.1 gives an explicit description of the set of minimal primes of a commutative generalized Weyl algebra $A$ via the set of minimal primes of the Noetherian algebra $D$, it also describes the prime radical of the algebra $A$ via the prime radical of the algebra $D$. Proposition 2.1 also gives an equidimensionality criterion for the GWA $A$.

**Regularity criteria for the GWA $A$.** In the case when the algebra $D$ is of essentially finite type of pure dimension $d < \infty$ over a perfect field $K$, two regularity criteria are given for the GWA $A = D[X, Y; a]$ of rank $N$, Theorem 2.2 and Theorem 2.3. The first criterion, Theorem 2.2, is given via an explicit ideal of the algebra $D$. The second criterion, Theorem 2.3, is given in terms of a new concept introduced in the paper - a **homological sequence** of algebra (every regular sequence is a homological sequence but not the other way round, in general). Proposition 2.5 collects some of the properties of homological sequences.

**The generalized Weyl Poisson algebras and their Poisson simplicity criterion.** In [6], a new large class of Poisson algebras - the class of generalized Weyl Poisson algebras - is introduced.

An associative commutative algebra $A$ is called a **Poisson algebra** if it is a Lie algebra $(A, \{\cdot, \cdot\})$ that satisfies the **Leibniz’s rule**: For all elements $a, x, y \in D$,

$$\{a, xy\} = \{a, x\}y + x\{a, y\}.$$

The set $\text{PZ}(A) := \{a \in A \mid \{a, x\} = 0 \text{ for all } x \in A\}$ is called the **Poisson centre** of $A$, it is a subalgebra of $A$. Let $\text{Der}_K(A)$ be the set of $K$-derivations of the associative algebra $A$. Then

$$\text{PDer}_K(D) := \{\delta \in \text{Der}_K(D) \mid \delta(\{a, b\}) = \{\delta(a), b\} + \{a, \delta(b)\} \text{ for all } a, b \in D\}$$

is the **set of derivations** of the Poisson algebra $D$. Elements of $\text{PDer}(A)$ are called **Poisson derivations** of $A$.

**Definition, [6].** Let $D$ be a Poisson algebra, $\partial = (\partial_1, \ldots, \partial_n) \in \text{PDer}_K(D)^n$ be an $n$-tuple of commuting derivations of the Poisson algebra $D$, $a = (a_1, \ldots, a_n) \in \text{PZ}(D)^n$ be such that $\partial_i(a_j) = 0$ for all $i \neq j$. The commutative generalized Weyl algebra

$$A = D[X, Y; a] = D[X_1, \ldots, X_n, Y_1, \ldots, Y_n]/(X_1Y_1 - a_1, \ldots, X_nY_n - a_n)$$

admits a Poisson structure which is an extension of the Poisson structure on $D$ and is given by the rule: For all $i, j = 1, \ldots, n$ and $d \in D$,

$$\{Y_i, d\} = \partial_i(d)Y_i, \quad \{X_i, d\} = -\partial_i(d)X_i \quad \text{and} \quad \{Y_i, X_i\} = \partial_i(a_i),$$

(1)
\{X_i, X_j\} = \{X_i, Y_j\} = \{Y_i, Y_j\} = 0 \text{ for all } i \neq j. \quad (2)

The Poisson algebra is denoted by $A = D[X, Y; a, \partial]$ and is called the \textbf{generalized Weyl Poisson algebra} of degree/rank $n$ (or GWPA, for short) where $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$.

Examples of GWPAs are given in Section 3. A Poisson simplicity criterion is given in [6, Theorem 1.1], see Theorem 4.2. This result is used in several theorems of this paper.

\textbf{Generators and defining relations of the Poisson enveloping algebra $U(A)$ of a GWPA $A$.} For an arbitrary Lie algebra $G$, every $G$-module is a module over its \textbf{universal enveloping algebra} $U(G)$, which is an associative algebra, and vice versa. Furthermore, the defining relations of the algebra $U(G)$ are precisely the defining relations of the Lie algebra $G$ where the Lie bracket $[,]_G$ of $G$ is ‘replaced’ by the algebra commutator: If $\{X_i\}_{i \in I}$ is a $K$-basis of the Lie algebra $G$ then the algebra $U(G)$ is generated by the elements $\{X_i\}_{i \in I}$ that subject to the defining relations:

$$X_iX_j - X_jX_i = [X_i, X_j]_G \text{ for all } i, j \in I.$$  

Similarly, for each Poisson algebra $P$ there is a concept of Poisson module over $P$, and every Poisson module over $P$ is a module over, the so-called, Poisson enveloping algebra $U(P)$, which is an associative algebra, and vice versa. In [17], for each Poisson algebra $P$ that is defined by generators abd defining relations, explicit sets of generators and defining relations are given for its Poisson enveloping algebra $U(P)$, [17, Theorem 2.2], see Theorem 3.1. Using this result, for each GWPA $A$, Theorem 3.2 gives explicit sets of generators and defining relations for the algebra $U(A)$. Examples are considered in Proposition 3.3, Corollary 3.4 and Corollary 3.5.

\textbf{The Poincaré-Birkhof-Witt Theorem for Poisson algebras.} The (classical) Poincaré-Birkhof-Witt Theorem states that for each Lie algebra $G$ there is a natural isomorphism of graded algebras,

$$\text{gr } U(G) \simeq \text{Sym}(G),$$

where $\text{gr } U(G)$ is the associated graded algebra of the universal enveloping algebra $U(G)$ of the Lie algebra $G$ and $\text{Sym}(G)$ is the symmetric algebra of $G$. For a smooth Poisson algebra $P$, a similar result holds [18, Theorem 3.1]:

$$\text{gr } U(P) \simeq \text{Sym}_P(\Omega_P),$$

where $\text{gr } U(P)$ is the associated graded algebra of the Poisson enveloping algebra $U(P)$ of the Poisson algebra $P$ and $\text{Sym}_P(\Omega_P)$ is the symmetric algebra of the $P$-module $\Omega_P$ of Kähler differentials of the associative algebra $P$. In fact, [18, Theorem 3.1] holds in sightly more general situation, namely, for the universal enveloping algebra of the Lie-Reinhart algebra. The pair $(P, \Omega_P)$ is an example of a Lie-Reinhart algebra and its universal enveloping algebra $V(P, \Omega_P)$ is isomorphic to the Poisson enveloping algebra (PEA) $U(P)$,
Recently, it was proven that the PBW Theorem holds for certain singular Poisson hypersurfaces. Theorem 3.7. states that the Poincaré-Birkhof-Witt Theorem holds for all Poisson algebras (over an arbitrary field).

In the Poisson enveloping algebra of a Poisson algebra was introduced as a universal object in a certain category and an alternative to Reinhart’s proof of its existence was given. For certain classes of Poisson algebras explicit descriptions of their Poisson enveloping algebras were presented in 17, 19, 20, 13, 14, 12.

Simplicity criterion for the algebra $PD(A)$ of Poisson differential operators on the generalized Weyl Poisson algebra $A$. Corollary 4.3 is a simplicity criterion for the algebra of Poisson differential operators $PD(A)$ on a generalized Weyl Poisson algebra $A$.

Simplicity criterion for the Poisson enveloping algebra $\mathcal{U}(A)$ of the generalized Weyl Poisson algebra $A$. Corollary 4.5 is a simplicity criterion for the algebra $\mathcal{U}(A)$ where $A$ is a generalized Weyl Poisson algebra. If in addition, the coefficient ring $D$ of the generalized Weyl Poisson algebra $A = D[X, Y; a, \partial]$ is a regular domain of essentially finite type over a field of characteristic zero, Theorem 4.9 is an explicit simplicity criterion for the algebra $\mathcal{U}(A)$. Theorem 4.9 (4) is a very efficient tool in proving/disproving simplicity of the algebra $\mathcal{U}(A)$.

The Gelfand-Kirillov dimension of the algebras $\mathcal{U}(A)$, $PD(A)$, $gr\mathcal{U}(A)$ and $Sym_{A}(\Omega_{A})$. Let $A = D[X, Y; a, \partial]$ be a generalized Weyl Poisson algebra where $D$ is a domain of essentially finite type over a perfect field. Let $\Omega_{A}$ be the $A$-module of Kähler differential of the algebra $A$ and $Sym_{A}(\Omega_{A})$ be its symmetric algebra. Corollary 4.11 (resp., Corollary 4.13) gives an exact figure for the Gelfand-Kirillov dimension of the algebras $\mathcal{U}(A)$, $gr\mathcal{U}(A)$ and $Sym_{A}(\Omega_{A})$ (resp., $PD(A)$, char($K$) = 0).

The algebra $\mathcal{U}(A)$ is a domain when $A$ is a regular domain of essentially finite type. Let $A = D[X, Y; a, \partial]$ be a generalized Weyl Poisson algebra where $D$ is a domain of essentially finite type over a perfect field. Corollary 4.15 states that the algebra $\mathcal{U}(A)$ is a domain provided the algebra $A$ is a regular domain.

## 2 Regularity and equidimensionality criteria for commutative generalized Weyl algebras

The aim of this section is to give regularity criteria for a commutative generalized Weyl algebra $A = D[X, Y; a]$ of rank $N$ where $D$ is a commutative algebra of essentially finite type of pure dimension $d < \infty$ over a perfect field $K$. The first criterion, Theorem 2.2 is given in terms of an explicit ideal of the ring $D$. The second criterion, Theorem 2.3
is more conceptual and is given via a new concept - a homological sequence of algebra.
One of the key results in proving Theorem 2.3 is Proposition 2.1 which describes the set of
minimal primes of the algebra \( A \) via the set of minimal primes of the algebra \( D \) and gives
an equidimensionality criterion for the algebra \( A \).

**Generalized Weyl algebras, [1, 2, 3].** Let \( D \) be a ring, \( \sigma = (\sigma_1, \ldots, \sigma_n) \) be an
\( n \)-tuple of commuting automorphisms of \( D \), \( a = (a_1, \ldots, a_n) \) be an \( n \)-tuple of elements
of the centre \( Z(D) \) of \( D \) such that \( \sigma_i(a_j) = a_j \) for all \( i \neq j \). The **generalized Weyl algebra** \( A = D[X, Y; \sigma, a] \) (briefly GWA) of degree/rank \( n \) is a ring generated by \( D \) and
\( 2n \) indeterminates \( X_1, \ldots, X_n, Y_1, \ldots, Y_n \) subject to the defining relations:

\[
Y_iX_i = a_i, \ X_iY_i = \sigma_i(a_i), \ X_id = \sigma_i(d)X_i, \ Y_id = \sigma_i^{-1}(d)Y_i \quad (d \in D),
\]

\[
[X_i, X_j] = [X_i, Y_j] = [Y_i, Y_j] = 0, \quad \text{for all } i \neq j,
\]

where \([x, y] = xy - yx\). We say that \( a \) and \( \sigma \) are the sets of **defining** elements and
automorphisms of the GWA \( A \), respectively.

The \( n \)-th **Weyl algebra**, \( A_n = A_n(K) \) over a field (a ring) \( K \) is an associative \( K \)-algebra
generated by \( 2n \) elements \( X_1, \ldots, X_n, Y_1, \ldots, Y_n \), subject to the relations:

\[
[Y_i, X_i] = \delta_{ij} \quad \text{and} \quad [X_i, X_j] = [Y_i, Y_j] = 0 \quad \text{for all } i, j,
\]

where \( \delta_{ij} \) is the Kronecker delta function. The Weyl algebra \( A_n \) is a generalized Weyl
algebra \( A = D[X, Y; \sigma, a] \) of rank \( n \) where \( D = K[H_1, \ldots, H_n] \) is a polynomial ring in
\( n \) variables with coefficients in \( K \), \( \sigma = (\sigma_1, \ldots, \sigma_n) \) where \( \sigma_i(H_j) = H_j - \delta_{ij} \) and \( a = (H_1, \ldots, H_n) \). The map

\[
A_n \to A, \quad X_i \mapsto X_i, \quad Y_i \mapsto Y_i, \quad i = 1, \ldots, n,
\]

is an algebra isomorphism (notice that \( Y_iX_i \mapsto H_i \)).

It is an experimental fact that many quantum algebras of small Gelfand-Kirillov dimen-
sion are GWAs (eg, \( U(sl_2) \), \( U_q(sl_2) \), the quantum Weyl algebra, the quantum plane, the
Heisenberg algebra and its quantum analogues, the quantum sphere, and many others).

The GWA-construction turns out to be a useful one. Using it for large classes of al-
gebras (including the mention ones above) all the simple modules were classified, explicit
formulae were found for the global and Krull dimensions, their elements were classified in
the sense of Dixmier, [9], etc.

**A \( \mathbb{Z}^n \)-grading on a GWPA.** The GWPA of rank \( n \),

\[
A := D[X, Y; a, \partial] = \bigoplus_{\alpha \in \mathbb{Z}^n} A_\alpha,
\]

is a \( \mathbb{Z}^n \)-graded Poisson algebra where \( A_\alpha = Dv_\alpha, \ v_\alpha = \prod_{i=1}^n v_{\alpha_i}(i) \) and

\[
v_j(i) = \begin{cases} 
X_i^j & \text{if } j > 0, \\
1 & \text{if } j = 0, \\
Y_i^{-j} & \text{if } j < 0.
\end{cases}
\]
So, \( A_\alpha A_\beta \subseteq A_{\alpha+\beta} \) and \( \{A_\alpha, A_\beta\} \subseteq A_{\alpha+\beta} \) for all elements \( \alpha, \beta \in \mathbb{Z}^n \).

The set of minimal primes of a generalized Weyl algebra. For an algebra \( R \), \( \min(R) \) is the set of its minimal primes and \( n_R \) is its prime radical. We say that a commutative algebra \( R \) has pure dimension \( d \) if the Krull dimension of all factor algebras \( R/p \), where \( p \in \min(R) \), is \( d \). We also say the algebra \( R \) is equidimensional. Let \( C_R \) be the set of all regular elements of the algebra \( R \), (i.e. the set of non-zero-divisors).

Proposition 2.1 gives an explicit description of the set of minimal primes of a commutative generalized Weyl algebra \( A \) via the set of minimal primes of the Noetherian algebra \( D \), it also describes the prime radical of the algebra \( A \) via the prime radical of the algebra \( D \).

**Proposition 2.1** Let \( D \) be a commutative Noetherian ring, \( A = D[X,Y,a] \) be a commutative GWA of rank \( N \), for each \( p \in \min(D) \), let

\[
\text{in}(p) = \{ j \mid a_j \in p, 1 \leq j \leq N \} \quad \text{and} \quad \text{out}(p) = \{ j \mid a_j \not\in p, 1 \leq j \leq N \}.
\]

Then

1. \( \min(A) = \coprod_{p \in \min(D)} \min(A, p) \) where

\[
\min(A, p) := \{ P \in \min(A) \mid p \subseteq P \} = \{ P_{p,\varepsilon} \mid \varepsilon = (\varepsilon_1, \ldots, \varepsilon_{\text{in}(p)}) \in \{ \pm \}_{\text{in}(p)} \},
\]

\[
P_{p,\varepsilon} := \begin{cases} (p, v_{\varepsilon_1}(i_1), \ldots, v_{\varepsilon_{\text{in}(p)}}(i_{\text{in}(p)})) & \text{if } \text{in}(p) \neq \emptyset, \\ (p) & \text{if } \text{in}(p) = \emptyset, \end{cases}
\]

\( v_+(j) = X_j \) and \( v_-(j) = Y_j \),

(a) for all \( P \in \min(A, p) \), \( P \cap D = p \),

(b) \( |\min(A, p)| = 2^{\text{in}(p)} \) and \( \min(A) \geq |\min(D)| \),

(c) \( \min(A) = |\min(D)| \) iff \( \text{in}(p) = \emptyset \) for all \( p \in \min(D) \) iff \( \min(A) = \{ Ap \mid p \in \min(D) \} \).

2. \( n_A = An_D \) where \( n_A \) and \( n_D \) are the prime radicals of the rings \( A \) and \( D \), respectively. In particular, the ring \( A \) is reduced iff the ring \( D \) is so.

3. For each minimal prime \( p \in \min(D) \), the set \( \min(A, p) \) is the set of all the minimal primes over the ideal \( Ap \) of \( A \), and the ideal \( Ap = \bigcap_{P \in \min(A, p)} P \) is a semiprime ideal of \( A \).

4. \( A/P_{p,\varepsilon} = D/p[v_{-\varepsilon_1}(i_1), \ldots, v_{-\varepsilon_{\text{in}(p)}}(i_{\text{in}(p)})][(X_{j_1}, \ldots, X_{j_{\text{out}(p)}}), (Y_{j_1}, \ldots, Y_{j_{\text{out}(p)}}); a = (a_{j_1}, \ldots, a_{j_{\text{out}(p)}})] \) is a commutative GWA of rank \( |\text{out}(p)| \) with coefficients in the polynomial algebra \( D/p[v_{-\varepsilon_1}(i_1), \ldots, v_{-\varepsilon_{\text{in}(p)}}(i_{\text{in}(p)})] \)
in \(|\text{in}(p)|\) variables with coefficients in \(D/p\). In particular, the field of fractions of the algebra \(A/P_{p,\varepsilon}\) is

\[ Q(A/P_{p,\varepsilon}) = Q(D/p)(X_1,\ldots,X_N) \]

which is the field of rational functions in the variables \(X_1,\ldots,X_N\) over the field of fractions \(Q(D/p)\) of the algebra \(D/p\), and \(\text{tr.deg}_K Q(A/P_{p,\varepsilon}) = \text{tr.deg}_K Q(D/p) + N\).

5. If, in addition, the algebra \(D\) has pure dimension \(d\) then the algebra \(A\) has pure dimension \(d + N\), and vice versa.

**Proof.**

4. Statement 4 is obvious (use the \(\mathbb{Z}^N\)-grading of the GWA \(A\)).

1. (i) All ideals \(P_{p,\varepsilon}\) of \(A\) are distinct prime ideals: This follows from statement 4.

   (ii) \(P_{p,\varepsilon}\cap D = p\): This follows from the fact that the ideal \(P_{p,\varepsilon}\) is a homogeneous ideal of the GWA \(A\) (with respect of the \(\mathbb{Z}^N\)-grading) and definitions of the sets \(\text{in}(p)\) and \(\text{out}(p)\).

   (iii) The set \(\text{min}(A)\) contains precisely all the ideals \(P_{p,\varepsilon}\): Let \(P\) be a prime ideal of \(A\). Then the intersection \(D\cap P\) is a prime ideal of the algebra \(D\). Hence, \(p \subseteq D\cap P\) for some \(p \in \text{min}(D)\). Let \(\text{in}(p) = \{i_1,\ldots,i_l\}\) and \(\text{out}(p) = \{j_1,\ldots,j_m\}\). Then the factor algebra

\[ A/Ap = D/p[X_{\text{out}(p)}, Y_{\text{out}(p)}; a_{\text{out}}(p)]/[X_{\text{in}(p)}, Y_{\text{in}(p)}; 0] \]

where \(X_{\text{out}}(p) = (X_{j_1},\ldots,X_{j_m}), Y_{\text{out}}(p) = (Y_{j_1},\ldots,Y_{j_m}), X_{\text{in}}(p) = (X_{i_1},\ldots,X_{i_l}), Y_{\text{in}}(p) = (Y_{i_1},\ldots,Y_{i_l})\) and \(a_{\text{out}}(p) = (a_{j_1},\ldots,a_{j_m})\) where \(a_{i_l} = a_{i_l} + p\). Since

\[ X_{i_l}Y_{i_l} = 0,\ldots,X_{i_l}Y_{i_l} = 0, \]

we must have \(P_{p,\varepsilon} \subseteq P\) for some \(\varepsilon\). So, every prime ideal of \(A\) contains some prime ideal \(P_{p,\varepsilon}\) and all the ideals \(P_{p,\varepsilon}\) are distinct. Now, the statement (iii) follows, and as a result we have statement 1.

3. By statement 1, \(Ap = \bigcap_{P \in \text{min}(A,p)} P\), and statement 3 follows.

2. Statement 2 follows from statements 1 and 3:

\[ A_{\text{in}}D = A(\bigcap_{p \in D/\text{min}(D)} P_{p,\varepsilon}) = \bigcap_{p \in D/\text{min}(D)} Ap = \bigcap_{p \in D/\text{min}(D)} \bigcap_{P \in \text{min}(A,p)} P = n_A. \]

The rest is obvious.

5. Statement 5 follows from statement 4: For each minimal prime \(P_{p,\varepsilon}\) of \(A\),

\[ A/P_{p,\varepsilon} = D/p[X_{\text{out}}(p), Y_{\text{out}}(p); a_{\text{out}}(p)][v_{-\varepsilon_1}(i_1),\ldots,v_{-\varepsilon_l}(i_l)] \]

is a polynomial algebra in the variables \(v_{-\varepsilon_1}(i_1),\ldots,v_{-\varepsilon_l}(i_l)\) with coefficients in the GWA \(B = D/p[X_{\text{out}}(p), Y_{\text{out}}(p); a_{\text{out}}(p)]\) of degree \(m = |\text{out}(p)|\) and all the coordinates of the vector \(a_{\text{out}}(p) = (a_{j_1},\ldots,a_{j_m})\) are nonzero (hence regular) elements of the domain \(D/p\). Hence, \(Q(A/P_{p,\varepsilon}) = Q(D/p)(X_1,\ldots,X_m)\). Now,

\[ \text{Kdim}(A/P_{p,\varepsilon}) = \text{Kdim}(B) + l. \]
CLAIM. \( \text{Kdim}(B) = \text{Kdim}(D/p) + m = d + m. \)

By the Claim,
\[
\text{Kdim}(A/P_p) = d + m + l = d + n, \]
as required (since \( m + l = n \)).

To prove the Claim we use induction on \( m \). The result is obvious if \( m = 0 \). Suppose that \( m > 0 \) and that the Claim is true for all \( m' < m \). Then
\[
B = B_{m-1}[X_m, Y_m; a_m] = B_{m-1}[X_m, Y_m]/(X_mY_m - a_m). \]
The polynomial algebra \( B_{m-1}[X_m, Y_m] \) is a Noetherian domain of Krull dimension \( \text{Kdim}(B_{m-1}) + 2 \). By the Krull’s Principal Ideal Theorem, the algebra \( B \) is a Noetherian algebra of pure dimension
\[
\text{Kdim}(B_{m-1}) + 2 - 1 = \text{Kdim}(B_{m-1}) + 1 = d + m - 1 + 1 = d + m \]
since, by induction, \( \text{Kdim}(B_{m-1}) = d + m - 1 \), and the Claim follows. \( \square \)

**Homological sequences.**

*Definition.* Let \( D \) be a regular algebra of essentially finite type. A sequence of elements \( a_1, \ldots, a_N \) in \( D \) is called a **homological sequence of length** \( N \) if the following two conditions hold:

1. the element \( a_1 \) is a **homological sequence of length** 1, that is (by definition) the element \( a_1 \) is a regular element of \( D \) such that the factor algebra \( D/(a_1) \) is regular provided the element \( a_1 \) is not a unit, and
2. the images \( \overline{a}_2, \ldots, \overline{a}_N \) of the elements \( a_2, \ldots, a_N \) in the ring

\[
D_1 := \begin{cases} 
D & \text{if } a_1 \in D^x, \\
D/(a_1) & \text{if } a_1 \notin D^x \cup \{0\}, 
\end{cases}
\]
is a homological sequence of length \( N - 1 \). The algebra \( D_1 \) is called the **algebra of the element** \( a_1 \) in the homological sequence \( s: a_1, \ldots, a_N \) in \( D \). Similarly, the algebra \( D_2 \) of the element \( \overline{a}_2 \) in the homological sequence \( \overline{a}_2, \ldots, \overline{a}_N \) in \( D_1 \) is called the **algebra of the element** \( a_2 \) in the homological sequence \( s \) in \( D \). Finally, the algebra \( D_N \) of the element \( a_N \) in the homological sequence \( s \) is defined. The sequence of algebras \( D_1, \ldots, D_N \) is called the **sequence of algebras of the homological sequence** \( s \).

Notice that the algebras \( D_1, \ldots, D_N \) are regular algebras of essentially finite type and \( \overline{a}_i \in C_{D_{i-1}} \) for all \( i = 1, \ldots, N \). Let
\[
\mathcal{U}(s) := \{ a_i \mid \overline{a}_i \in D^x_{i-1} \} \quad \text{and} \quad \mathcal{N}(s) := \{ a_i \mid \overline{a}_i \notin D^x_{i-1} \}.
\]
The set \( \mathcal{U}(s) \) (resp., \( \mathcal{N}(s) \)) is called the **set of relative units** (resp., **regular non-units**) of \( s \). If \( \mathcal{N}(s) = \{ a_{i_1}, \ldots, a_{i_\nu} \} \) where \( 1 \leq i_1 < \cdots < i_\nu \leq N \) then
\[
D_{i_\mu} = D/(a_{i_1}, \ldots, a_{i_\mu}) \quad \text{for all} \quad \mu = 1, \ldots, \nu. \quad (4)
\]
Furthermore, there is a chain of equalities and epimorphisms:

$$D_0 = D_1 = \cdots = D_{n-1} \to D_i = \cdots = D_{2} \to \cdots \to D_1 = \cdots = D_0$$

where the arrows are epimorphisms (e.g., $D_{i-1} \to D_i \simeq D_{i-1}/(a_i)$) and $D_0 := D$. The elements in $N(s)$ are a regular sequence of the algebra $D$. The algebra $D_i$ has pure dimension $d - u_i$ where $u_i = \#\{ j \mid a_j \in N(s), j \leq i \}$.

**Regularity criteria for a GWA $A$.** Theorem [2.2] is a first regularity criterion for the a GWA $A$.

**Theorem 2.2** Let $D = S^{-1}(P_n/I)$ be a commutative algebra of essentially finite type of pure dimension $d$ (= $n - r$ where $P_n = K[x_1, \ldots, x_n], I = (f_1, \ldots, f_m)$ and $r = r(\frac{\partial f_i}{\partial x_j})$) over a perfect field $K$ and $A = D[X, Y; a]$ be a GWA of rank $N$. Then the algebra $A$ is regular iff

$$\sum_I b_I = D$$

where the sum is taken over all subsets $I \subseteq \{1, \ldots, N\}$ with $|I| \geq N - d$, $CI = \{1, \ldots, N\} \setminus I$ and $b_I := a_\mathcal{C}I a_I$ is an ideal of $D$ where $a_I := \prod_{i \in I} a_i, a_\emptyset := 1$, and the ideal $a_\mathcal{C}I$ of $D$ is generated by all the minors of size $r + |CI|$ of the $(m + |CI|) \times n$ matrix with coefficients in $D$ where the $m + |CI|$ rows are $\text{grad}(f_1), \ldots, \text{grad}(f_m), \text{grad}(a_1), \ldots, \text{grad}(a_{|CI|})$ where $CI = \{ j_1 < \cdots < j_{|CI|} \}$.

**Proof.** By Theorem [2.1](5), the algebra $A$ is an algebra of essentially finite type of pure dimension $d + N$ over a perfect field $K$. By the Jacobian Criterion of Regularity, the algebra $A$ is regular iff the Jacobian ideal $a = a(A)_{d+N}$ of $A$ is equal to $A$. The strategy of the proof of the theorem is as follows. First, we find explicit set of generators of the ideal $a$. Then we show that $D \cap a = \sum_I b_I$, and the theorem follows. Notice that

$$A = S^{-1}K[x_1, \ldots, x_n, X_1, Y_1, \ldots, X_N, Y_N]/(f_1, \ldots, f_m, a_1 - X_1 Y_1, \ldots, a_N - X_N Y_N).$$

The Jacobian matrix of $A$ is of size $(m + N) \times (n + 2N)$ with $m + N$ rows as follows

$$(\text{grad}(f_1), 0, \ldots, 0), \ldots, (\text{grad}(f_m), 0, \ldots, 0),$$

$$(\text{grad}(a_1), -X_1, -Y_1, 0, \ldots, 0), \ldots, (\text{grad}(a_N), 0, \ldots, 0, -X_N, -Y_N).$$

It follows that

$$a = \sum_{I, \varepsilon} a_\mathcal{C}I v_{I, \varepsilon}$$

where $v_{I, \varepsilon} = \prod_{i \in I} v_\varepsilon(i), \varepsilon = (\varepsilon_i)_{i \in I} \in \{ \pm \}^I$ and $v_+(i) = X_i$ and $v_-(i) = Y_i$ and the set $I$ runs through all the subsets of $\{1, \ldots, N\}$ such that $r + N - |I| \leq n$ ($\iff |I| \geq N - d$). The ideal $a$ is a sum of homogeneous ideals w.r.t. the $\mathbb{Z}^N$-grading of the GWA $A$. So,

$$1 \in a \iff 1 \in a \cap D = \sum_{I, \varepsilon} a_\mathcal{C}I v_{I, \varepsilon} v_{I, -\varepsilon} = \sum_I a_\mathcal{C}I a_I,$$

as required. $\square$

Let $S_N$ be the symmetric group, i.e. the group of all permutations of the set $\{1, \ldots, N\}$. Theorem [2.3] is the second regularity criterion for GWAs.
Theorem 2.3 Let $D$ be a commutative algebra of essentially finite type of pure dimension $d$ over a perfect field $K$ and $A = D[X, Y; a]$ be a GWA of rank $N$. Then the following statements are equivalent:

1. The algebra $A$ is a regular algebra.

2. The algebra $D$ is a regular algebra and the sequence $a_1, \ldots, a_N$ is a homological sequence in $D$ of length $N$.

3. The algebra $D$ is a regular algebra and every permutation of the sequence $a_1, \ldots, a_N$ is a homological sequence in $D$ of length $N$.

If one of the equivalent conditions hold then the algebras $D_1, \ldots, D_N$ of the homological sequence $a_1, \ldots, a_N$ are commutative regular equidimensional algebras of essentially finite type. Furthermore, the pure dimension of the algebra $D_i$ is $d - u_i$ where $u_i = \# \{ j \mid a_j \in N(s), 1 \leq j \leq i \}$.

Proof. (1 $\iff$ 2) To prove that the equivalence holds we use induction on $N$. Suppose that $N = 1$. By Theorem 2.2 the algebra $A$ is regular iff $\sum I$ $b_I = D$ where the sum is taken over all the subsets $I$ of the set $\{1\}$ such that $|I| \geq 1 - d$. There are two cases to consider: $d = 0$ and $d > 0$.

Suppose that $d = 0$. Then $|I| \geq 1$, i.e. $I = \{1\}$. Now, $D = \sum_{I=\{1\}} b_I = a_r a_1$ iff $a_r = D$ and $a_1 \in D^\times$.

Suppose that $d > 0$. Then $|I| \geq 1 - 1 = 0$, and so $I = \{1\}$ or $I = \emptyset$. Now,

$$D = \sum_{I\in\emptyset,\{1\}} b_I = a_0 a_1 + a_{\{1\}} a_0 = a_r a_1 + a_{\{1\}} \subseteq a_r \quad (\text{since } a_0 = 1)$$

iff $a_r = D$ and $D = (a_1, a_{\{1\}})$ iff the algebra $D$ is regular and the algebra $D/(a_1)$ is regular since $(a_1, a_{\{1\}})$ is the Jacobian ideal of the $D/(a_1)$ of essentially finite type of pure dimension $d - 1$ (by Krull’s Principal Ideal Theorem). The proof of the equivalence when $N = 1$ is complete.

Suppose that $N > 1$, and the equivalence holds for all natural numbers $N' < N$. Notice that $A = A_{N-1}[X_1, Y_1; a_1]$ is a GWA of rank 1 where $A_{N-1} = D[X', Y'; a']$ is a GWA of degree $N - 1$ where $X' = (X_2, \ldots, X_N)$, $Y' = (Y_2, \ldots, Y_N)$ and $a' = (a_2, \ldots, a_N)$. Recall that the algebra $D$ is an algebra of essentially finite type of pure dimension $d$. By Proposition 2.1.5, the algebra $A_{N-1}$ is an algebra of essentially finite type of pure dimension $d + N - 1$. By the case $N = 1$, $A$ is regular iff the algebra $A_{N-1}$ is regular, $a_1 \in C_{A_{N-1}}$ ($\iff a_1 \in C_D$) and if $a_1 \not\in A_{N-1}^*$ ($\iff a_1 \not\in D^\times$) then the algebra

$$A_{N-1}/(a_1) \simeq D/(a_1)[X', Y'; a']$$

is regular where $\overline{a'} = (a_2 + (a_1), \ldots, a_N + (a_1))$ iff $a_1$ is a homological sequence in $D$ and $\overline{a_2}, \ldots, \overline{a_N}$ is a homological sequence in $D_1$. This means that the sequence $a_1, \ldots, a_N$ is homological sequence in $D$. 

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For any permutation $\sigma \in S_N$, $A \simeq D[XS, YS; a^s]$ where $X^s = (X_{\sigma(1)}, \ldots, X_{\sigma(N)})$ and $a^s = (a_{\sigma(1)}, \ldots, a_{\sigma(N)})$, and the equivalence follows from the fact that $(1 \Leftrightarrow 2)$. □

We say that ideals $a$ and $b$ of an algebra $R$ are co-prime if $a + b = R$.

**Corollary 2.4** Let $D$ be a ring of essentially finite type of pure dimension $d$ over a perfect field $K$ and $A = D[XS, YS; a]$ be a commutative GWA of degree $N$. The following statements are equivalent.

1. The algebra $A$ is a regular algebra.
2. $A \simeq \prod_{P \in \text{min}(A)} A/P$ and the algebras $A/P$ are regular.
3. The algebra $D$ is regular, elements in $\text{min}(A)$ are pairwise co-prime, and for each $P = P_{p, x}$, the GWA
   
   $$D/p[(X_{j1}, \ldots, X_{j|\text{out}(p)|}), (Y_{j1}, \ldots, Y_{j|\text{out}(p)|}); a = (a_{j1}, \ldots, a_{j|\text{out}(p)|})]$$

   is a regular algebra.

**Proof.** 1. $(1 \Leftrightarrow 2)$ The equivalence is true for all algebras of essentially finite type.

$(2 \Leftrightarrow 3)$ The algebra $D$ is reduced ($n_D = 0$) iff the algebra $A$ is so, by Proposition 2.1(2), i.e. $0 = n_A = \bigcap_{P \in \text{min}(A)} P$. So, $A \simeq \prod_{P \in \text{min}(A)} A/P$ iff the minimal primes of the algebra $A$ are pairwise co-prime. Now, the equivalence $(2 \Leftrightarrow 3)$ follows from Proposition 2.1(4). □

**Proposition 2.5** Let $D$ be a commutative regular algebra of essentially finite type of pure dimension $d$ over a perfect field $K$ and $a_1, \ldots, a_N$ be a homological sequence in $D$ of length $N$. Then

1. Every permutation of the sequence $a_1, \ldots, a_N$ is a homological sequence in $D$ of length $N$.
2. There is a permutation, say $a_{i_1}, \ldots, a_{i_N}$, of the sequence $a_1, \ldots, a_N$ such that
   
   (a) the algebras $D_i := D, D_1, \ldots, D_{\nu}$ are regular where $D_i = D/(a_1, \ldots, a_i)$ for $i = 1, \ldots, \nu$,
   
   (b) $\overline{a}_i \in C_D \setminus D_i^\times$ for $i = 1, \ldots, \nu$, and
   
   (c) $\overline{a}_{\nu+1}, \ldots, \overline{a}_N \in D_\times^\times$ where $\overline{a}_j = a_j + (a_1, \ldots, a_\nu)$.

**Proof.** 1. Statement 1 follows from Theorem 2.3.

2. Let $D_1, \ldots, D_{\nu}$ be as in (4). Then $D_{\nu} = D/(a_1, \ldots, a_\nu)$ where $1 \leq i_1 < \cdots < i_\nu \leq N$ and $a_j + (a_1, \ldots, a_\nu) \in D_{\nu}^\times$ for all $j \in \{j_1, \ldots, j_{N-\nu}\} = \{1, \ldots, N\} \setminus \{i_1, \ldots, i_\nu\}$, by Theorem 2.3. Then the sequence $a_{i_1}, \ldots, a_{i_\nu}, a_{j_1}, \ldots, a_{j_{N-\nu}}$ is required permutation of the original sequence. □
3 Generators and defining relations of the Poisson enveloping algebra of a generalized Weyl Poisson algebra

The aim of this section is for an arbitrary generalized Weyl Poisson algebra to give explicit sets of generators and defining relations for its Poisson enveloping algebra (Theorem 3.2). We also consider some interesting examples (Proposition 3.3 and Corollary 3.4). At the beginning of the section, examples of generalized Weyl Poisson algebras are considered.

Examples of GWPAs. 1. If $D$ is a algebra with trivial Poisson bracket $(\{\cdot, \cdot\} = 0)$ then any choice of elements $a = (a_1, \ldots, a_n)$ and $\partial = (\partial_1, \ldots, \partial_n) \in \text{Der}_K(D)^n$ such that $\partial_i(a_j) = 0$ for all $i \neq j$ determines a GWPA $D[X,Y; a, \partial]$ of rank $n$ where $\text{Der}_K(D)$ is the set of $K$-derivations of the algebra $D$. If, in addition, $n = 1$ then there is no restriction on $a_1$ and $\partial_1$.

2. The classical Poisson polynomial algebra $P_{2n} = K[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ $(\{Y_i, X_j\} = \delta_{ij}$ (the Kronecker delta) and $\{X_i, X_j\} = \{X_i, Y_j\} = \{Y_i, Y_j\} = 0$ for all $i \neq j$) is a GWPA

$$P_{2n} = K[H_1, \ldots, H_n][X, Y; a, \partial] \quad (5)$$

where $K[H_1, \ldots, H_n]$ is a Poisson polynomial algebra with trivial Poisson bracket, $a = (H_1, \ldots, H_n)$, $\partial = (\partial_1, \ldots, \partial_n)$ and $\partial_i = \frac{\partial}{\partial H_i}$ (via the isomorphism of Poisson algebras $P_{2n} \rightarrow K[H_1, \ldots, H_n][X, Y; a, \partial]$, $X_i \mapsto X_i$, $Y_i \mapsto Y_i$).

3. $A = D[X,Y; a, \partial]$ where $D = K[H_1, \ldots, H_n]$ is a Poisson polynomial algebra with trivial Poisson bracket, $a = (a_1, \ldots, a_n) \in K[H_1] \times \cdots \times K[H_n]$, $\partial = (\partial_1, \ldots, \partial_n)$ where $\partial_i = b_i \partial_{H_i}$ (where $\partial_{H_i} = \frac{\partial}{\partial H_i}$) and $b_i \in K[H_i]$. In particular, $D[X,Y; (H_1, \ldots, H_n), (\partial_{H_1}, \ldots, \partial_{H_n})] = P_{2n}$ is the classical Poisson polynomial algebra.

Let $S$ be a multiplicative set of $D$. Then $S^{-1} A \simeq (S^{-1}D)[X,Y; a, \partial]$ is a GWPA. In particular, for $S = \{H^\alpha | \alpha \in \mathbb{Z}^n\}$ we have $K[H_1^{\pm 1}, \ldots, H_n^{\pm 1}][X,Y; a, \partial]$. In the case $n = 1$, the Poisson algebra

$$K[H_1^{\pm 1}][X_1, Y_1; a_1, -H_1 \frac{d}{dH_1}]$$

where $a_1 \in K[H_1^{\pm 1}]$ is, in fact, isomorphic to a Poisson algebra in the paper of Cho and Oh [S] which is obtained as a quantization of a certain GWA with respect to the quantum parameter $q$. In [S] Theorem 3.7, a Poisson simplicity criterion is given for this Poisson algebra.

Poisson $\mathcal{P}$-modules. Let $(\mathcal{P}, \{\cdot, \cdot\})$ be a Poisson algebra. A left $\mathcal{P}$-module $M$ is called a Poisson $\mathcal{P}$-module if there is a bilinear map

$$\mathcal{P} \times M \rightarrow M, \ (a, m) \mapsto \delta_a m$$
which is called a Poisson action of \( P \) on \( M \) such that for all elements \( a, b \in P \) and \( m \in M \),

(PM1) \( \delta_{\{a,b\}} = [\delta_a, \delta_b] \),

(PM2) \( [\delta_a, b] = \{a, b\} \), and

(PM3) \( \delta_{ab} = a\delta_b + b\delta_a \).

**Example.** Every Poisson algebra \( P \) is a Poisson \( P \)-module where \( \delta_a = pad_a \).

**The Poisson enveloping algebra \( U(P) \) of a Poisson algebra \( P \).**

For each Poisson algebra \( P \) there is a unique (up to isomorphism) associative algebra \( U(P) \) such that

- every Poisson \( P \)-module is a \( U(P) \)-module, and vice versa.

The algebra \( U(P) \) is called the Poisson enveloping algebra (PEA) of the Poisson algebra \( P \).

**Generators and defining relations of the Poisson enveloping algebra of a generalized Weyl Poisson algebra.** For a Poisson algebra \( P \) which is defined by generators and defining relations (as an associative algebra), Theorem 3.1 gives explicit sets of generators and defining relations for the Poisson enveloping algebra \( U(P) \).

**Theorem 3.1 ([4, Theorem 2.2.(2)])** Let \( P \) be a Poisson algebra, \( U(P) \) be its enveloping algebra (as a Lie algebra) and \( U(P) \) be its Poisson enveloping algebra. Then

1. \( U(P) \simeq P \times_{pad} U(P)/I(P) \) where \( I(P) = (\delta_{ab} - a\delta_b - b\delta_a)_{a,b \in P} \) is the ideal of the algebra \( P \times_{pad} U(P) \) generated by the set \( \{\delta_{ab} - a\delta_b - b\delta_a | a, b \in P\} \).
2. If \( P = S^{-1}K[x_i]_{i \in \Lambda}/(f_s)_{s \in \Gamma} \) where \( S \) is a multiplicative subset of the polynomial algebra \( K[x_i]_{i \in \Lambda} \) (\( \Lambda \) and \( \Gamma \) are index sets). Then the algebra \( U(P) \) is generated by the algebra \( P \) and the elements \( \{\delta_i := \delta_{x_i} | i \in \Lambda\} \) subject to the defining relations (a)–(c): For all elements \( i, j \in \Lambda \) such that \( i \neq j \) and \( s \in \Gamma \),

   (a) \( [\delta_i, \delta_j] = \sum_{k \in \Lambda} \frac{\partial\{x_i, x_j\}}{\partial x_k} \delta_k \),

   (b) \( [\delta_i, x_j] = \{x_i, x_j\} \), and

   (c) \( \sum_{i \in \Lambda} \frac{\partial f_s}{\partial x_i} \delta_i = 0 \).

So, the algebra \( U(P) \) is generated by the algebra \( P \) and the set \( \delta_p = \{\delta_a | a \in P\} \) subject to the defining relations: For all elements \( a, b \in P \) and \( \lambda, \mu \in K \),

(a) \( [\delta_a, \delta_b] = \delta_{\{a,b\}} \),
(b) $[\delta_a, b] = \{a, b\}$,

(c) $\delta_{ab} = a\delta_b + b\delta_a$,

(d) $\delta_{\lambda a + \mu b} = \lambda \delta_a + \mu \delta_b$ and $\delta_1 = 0$.

3. The map $\pi_P : \mathcal{U}(\mathcal{P}) \rightarrow \mathcal{D}(\mathcal{P})$, $a \mapsto a$, $\delta_b \mapsto \text{pad}_b = \{b, \cdot\}$ is an algebra homomorphism where $a, b \in \mathcal{P}$ and its image is the algebra $PD(\mathcal{P})$ of Poisson differential operators of the Poisson algebra $\mathcal{P}$.

4. The algebra $\mathcal{P}$ is a subalgebra of $\mathcal{U}(\mathcal{P})$. Furthermore, $\mathcal{U}(\mathcal{P}) = \mathcal{P} \oplus \text{ann}_\mathcal{U}(\mathcal{P})(1)$ is a direct sum of left $\mathcal{P}$-modules where $\text{ann}_\mathcal{U}(\mathcal{P})(1) = \sum_{i \in \mathcal{I}} \mathcal{U}(\mathcal{P})\delta_i$ is the annihilator of the identity element of the Poisson $\mathcal{P}$-module $\mathcal{P}$. The Poisson $\mathcal{P}$-module structure on the Poisson algebra $\mathcal{P}$ is obtained from the $\mathcal{D}(\mathcal{P})$-module structure on $\mathcal{P}$ by restriction of scalars via $\pi_P$.

**Theorem 3.2** Let $A = D[X, Y; a, \partial]$ be a GWPA of degree $n$ and $D = S^{-1}K[t_\nu]_{\nu \in I}/(f_s)_{s \in S}$. Then the Poisson enveloping algebra $\mathcal{U}(A)$ of the Poisson algebra $A$ is generated by the algebra $A$ and elements $\delta_{X_i}, \delta_{Y_i}, \delta_{X_{ij}}, \delta_{Y_{ij}}$ (where $\nu \in I$) subject to the defining relations given in statements 1-3: For all elements $i, j = 1, \ldots, n$ and $\nu, \mu \in I$ (where $\delta_{ij}$ is the Kronecker delta):

1. $[\delta_{X_i}, \delta_{X_j}] = 0$, $[\delta_{Y_i}, \delta_{Y_j}] = 0$, $[\delta_{Y_i}, \delta_{X_j}] = \delta_{ij} \sum_{\nu \in I} \frac{\partial}{\partial t_\nu} (\partial_i (a_\nu)) \delta_{\nu}$,

2. $[\delta_{X_i}, t_\nu] = -\partial_i (t_\nu) \delta_{X_i} - X_i \sum_{\mu \in I} \frac{\partial}{\partial t_\nu} (\partial_i (t_\nu)) \delta_{\nu}$,

3. $[\delta_{Y_i}, t_\nu] = \partial_i (t_\nu) \delta_{Y_i} + Y_i \sum_{\mu \in I} \frac{\partial}{\partial t_\nu} (\partial_i (t_\nu)) \delta_{\nu}$,

4. $[\delta_{X_i}, X_j] = 0$, $[\delta_{Y_i}, Y_j] = 0$, $[\delta_{Y_i}, X_j] = \delta_{ij} \partial_i (a_\nu)$,

5. $[\delta_{X_i}, Y_j] = 0$, $[\delta_{Y_i}, Y_j] = 0$, $[\delta_{X_i}, t_\nu] = \{t_\nu, t_\mu\}$,

6. $[\delta_{Y_i}, Y_j] = -\partial_i (t_\nu) \delta_{Y_i}$, and

7. $[\delta_{Y_i}, Y_j] = -\partial_i (t_\nu) \delta_{Y_i}$.

Proof. The theorem follows from Theorem 3.1(2), where the relations 1–3 are the relations (a)–(c) of Theorem 3.1(2), respectively, in the case of the GWPA $A$. □

**Proposition 3.3** Let $A = K[H][X, Y; a, \partial = b\partial_H]$ be a GWPA of degree 1 where $K[H]$ is a polynomial algebra in a single variable $H$ with trivial Poisson bracket, $a, b \in K[H]$ and $\partial_H = \frac{d}{dH} = (\cdot)'$. Then the algebra $\mathcal{U}(A)$ is generated by the algebra $A = K[H, X, Y]/(XY - a)$ and the elements $\Delta_H$, $\delta_X$ and $\delta_Y$ subject to the defining relations given in statements 1–3:

1. $[\delta_X, X] = (ba')' \delta_H$, $[\delta_X, \delta_H] = -b\delta_X - Xb' \delta_H$, $[\delta_Y, \delta_H] = b\delta_Y + Yb' \delta_H$,

2. $[\delta_Y, X] = 0$, $[\delta_Y, Y] = 0$, $[\delta_X, H] = -bX$, $[\delta_Y, H] = bY$, $[\delta_X, Y] = -ba'$, $[\delta_Y, X] = ba'$, $[\delta_H, H] = 0$, $[\delta_H, X] = bX$, $[\delta_H, Y] = bY$, and
3. \( Y \delta_X + X \delta_Y = a' \delta_H \).

**Proof.** The proposition follows from Theorem 3.2. The relations in statements 1–3 of the proposition are precisely the relations 1–3 of Theorem 3.2 for the GWPA \( A \). \( \square \)

**Corollary 3.4** Let the Poisson algebra \( A = \bigotimes_{i=1}^n A_i \) be a tensor product of the GWPAs \( A_i = K[H_i][X_i, Y_i; a_i, \partial_i = b_i \partial_{H_i}] \) from Proposition 3.3. Then \( \mathcal{U}(A) = \bigotimes_{i=1}^n \mathcal{U}(A_i) \) and the defining relations for the algebra \( \mathcal{U}(A) \) are the union of the defining relations of all the algebra \( \mathcal{U}(A_i) \) from Proposition 3.3 and obvious commutation relations that come from the tensor product of algebras (i.e. \( st = ts \)).

**Proof.** [7, Proposition 2.7] states that the Poisson enveloping algebra of the tensor product of finitely many Poisson algebras is isomorphic to the tensor product of the corresponding Poisson enveloping algebras. Now, the corollary follows from Proposition 3.2. \( \square \)

**Corollary 3.5** Let \( A = K[H][X, Y; a, \partial = b \partial H] \) be a GWPA of degree 1 as in Proposition 3.3 such that \( a \notin K \cup \{0\} \), \( A_a' \) be the localization of the algebra \( A \) at the powers of the elements \( a' = \frac{da}{\partial H} \). Then the algebra \( \mathcal{U}(A_a') \) is generated by the algebra \( A_a' \) and the elements \( \delta_X \) and \( \delta_Y \) subject to the defining relations:

1. \( [\delta_Y, \delta_X] = \frac{(ba')'}{a'}(Y \delta_X + X \delta_Y) \),
2. \( [\delta_X, X] = 0, [\delta_Y, Y] = 0, [\delta_X, H] = -bX, [\delta_Y, H] = bY, [\delta_X, Y] = -ba', [\delta_Y, X] = ba' \).

**Proof.** Proposition 3.3 gives generators and defining relations of the algebra \( \mathcal{U}(A) \). By [7, Theorem 2.10], \( \mathcal{U}(A_a') \simeq \mathcal{U}(A)_{a'} \). Since \( \delta_H = \frac{1}{a'}(Y \delta_X + X \delta_Y) \), the algebra \( \mathcal{U}(A_{a'}) \) is generated by the algebra \( A_a' \) and the elements \( \delta_X \) and \( \delta_Y \) subject to the defining relations in statements 1 and 2 of the corollary which are precisely the relations 1 and 2 of Proposition 3.3 respectively (by direct computations, all the relations where \( \delta_H \) is involved in Proposition 3.3 are redundant). \( \square \)

4 A simplicity criterion for the Poisson enveloping algebra of a generalized Weyl Poisson algebra

**Simplicity criterion for the algebra \( PD(\mathcal{P}) \) of Poisson differential operators on \( \mathcal{P} \).** An ideal \( I \) of a Poisson algebra \( \mathcal{P} \) is called a **Poisson ideal** if \( \{\mathcal{P}, I\} \subseteq I \). A Poisson algebra \( \mathcal{P} \) is called **Poisson simple** if the ideals 0 and \( \mathcal{P} \) are the only Poisson ideals of the Poisson algebra \( \mathcal{P} \). Let \( \text{Der}_K(\mathcal{P}) \) be the Lie algebra of \( K \)-derivations of the (associative) algebra \( \mathcal{P} \). For each element \( a \in \mathcal{P} \), the derivation \( \text{pad}_a := \{a, \cdot\} \in \text{Der}_K(\mathcal{P}) \) is called the **Hamiltonian vector field** associated with the element \( a \). Then the set of Hamiltonian vector fields

\[ H_\mathcal{P} := \{\text{pad}_a \mid a \in \mathcal{P}\} \]
is a Lie subalgebra of the Lie algebra $\text{Der}_K(\mathcal{P})$.

**Definition.** Let $\mathcal{P}$ be a Poisson algebra and $\mathcal{D}(\mathcal{A})$ be the algebra of differential operators on $\mathcal{P}$. The subalgebra of $\mathcal{D}(\mathcal{P})$,

$$PD(\mathcal{P}) := \langle \mathcal{P}, \mathcal{H}_\mathcal{P} \rangle,$$

is called the **algebra of Poisson differential operators** of the Poisson algebra $\mathcal{P}$.

In general, $PD(\mathcal{P}) \neq \mathcal{D}(\mathcal{P})$. Theorem 4.1 is a simplicity criterion for the algebra $PD(\mathcal{P})$ of Poisson differential operators on $\mathcal{P}$.

**Theorem 4.1** ([7, Theorem 1.1]) Let $\mathcal{P}$ be a Poisson algebra over an arbitrary field $K$. Then the following statements are equivalent:

1. The algebra $PD(\mathcal{P})$ is a simple algebra.
2. The Poisson algebra $\mathcal{P}$ is a Poisson simple algebra.

**Poisson simplicity criterion for generalized Weyl Poisson algebras.** Let $\mathcal{D}$ be a Poisson algebra and $\partial = (\partial_1, \ldots, \partial_n) \in \text{PDer}_K(\mathcal{D})^n$. An ideal $I$ of $\mathcal{D}$ is called $\partial$-**invariant** if $\partial_i(I) \subseteq I$ for all $i = 1, \ldots, n$. The set

$$D^\partial := \{ d \in \mathcal{D} \mid \partial_1(d) = 0, \ldots, \partial_n(d) = 0 \}$$

is called the **algebra of $\partial$-constants** of $\mathcal{D}$, it is a subalgebra of $\mathcal{D}$. Theorem 4.2 is Poisson simplicity criterion for generalized Weyl Poisson algebras.

**Theorem 4.2** ([6, Theorem 1.1]) Let $A = D[X,Y; a, \partial]$ be a GWPA of rank $n$. Then the Poisson algebra $A$ is a simple Poisson algebra iff

1. the Poisson algebra $D$ has no proper $\partial$-invariant Poisson ideals,
2. for all $i = 1, \ldots, n$, $Da_i + D\partial_i(a_i) = D$, and
3. the algebra $\text{PZ}(A)$ is a field ($\Leftrightarrow \text{char}(K) = 0$, $\text{PZ}(D)^\partial$ is a field and $D_\alpha = 0$ for all $\alpha \in \mathbb{Z}^n \setminus \{0\}$ where $D_\alpha := \{ \lambda \in D^\partial \mid \text{pad}_\lambda = \lambda \sum_{i=1}^n \alpha_i \partial_i, \ \lambda \alpha_i \partial_i(a_i) = 0 \text{ for } i = 1, \ldots, n \}$, [6, Proposition 1.2]).

The next corollary follows from Theorem 4.1 and Theorem 4.2.

**Corollary 4.3** Let $A = D[X,Y; a, \partial]$ be a GWPA of rank $n$. Then the following statements are equivalent:

1. The algebra $PD(A)$ is a simple algebra.
2. (a) the Poisson algebra $D$ has no proper $\partial$-invariant Poisson ideals,
(b) for all \( i = 1, \ldots, n \), \( Da_i + D\partial_i(a_i) = D \), and

(c) the algebra \( PZ(A) \) is a field (\( \iff \) \( \text{char}(K) = 0 \), \( PZ(D)^\partial \) is a field and \( D\alpha = 0 \) for all \( \alpha \in \mathbb{Z}^n \backslash \{0\} \) where \( D\alpha := \{ \lambda \in D^\partial \mid \text{pad}_\lambda = \lambda \sum_{i=1}^n \alpha_i \partial_i, \lambda \alpha_i \partial_i(a_i) = 0 \) for \( i = 1, \ldots, n \} \), [6, Proposition 1.2]).

**Simplicity criteria for the Poisson enveloping algebra** \( U(\mathcal{P}) \). Let \( \mathcal{P} \) be a Poisson algebra. The algebra \( \mathcal{P} \) is a \( D(\mathcal{P}) \)-module and hence \( P\mathcal{D}(\mathcal{P}) \) and \( U(\mathcal{P}) \)-module, and so there is a natural algebra epimorphism (Theorem 3.1.(3)):

\[
\pi_\mathcal{P} : U(\mathcal{P}) \rightarrow P\mathcal{D}(\mathcal{P}), \quad a \mapsto a, \quad \delta_b \mapsto \text{pad}_b = \{b, \cdot\}
\]

where \( a, b \in \mathcal{P} \). Theorem 4.4 is a simplicity criteria for the Poisson enveloping algebra \( U(\mathcal{P}) \).

**Theorem 4.4 ([4, Theorem 1.2])** Let \( \mathcal{P} \) be a Poisson algebra over an arbitrary field \( K \). Then the following statements are equivalent:

1. The algebra \( U(\mathcal{P}) \) is a simple algebra.
2. The algebra \( P\mathcal{D}(\mathcal{P}) \) is a simple algebra and \( U(\mathcal{P}) \simeq P\mathcal{D}(\mathcal{P}) \).
3. The Poisson algebra \( \mathcal{P} \) is a Poisson simple algebra and \( \mathcal{P} \) is a faithful left \( U(\mathcal{P}) \)-module.

If one of the equivalent conditions holds then \( U(\mathcal{P}) \simeq P\mathcal{D}(\mathcal{P}) \).

Corollary 4.5 is a simplicity criteria for the Poisson enveloping algebra \( U(\mathcal{A}) \) where \( \mathcal{A} \) is a generalized Weyl Poisson algebra. It follows from Theorem 4.2 and Theorem 4.4.

**Corollary 4.5** Let \( \mathcal{A} = D[X,Y;\alpha,\partial] \) be a GWPA of rank \( n \). Then the following statements are equivalent:

1. The algebra \( U(\mathcal{A}) \) is a simple algebra.
2. (a) the Poisson algebra \( D \) has no proper \( \partial \)-invariant Poisson ideals,
   (b) for all \( i = 1, \ldots, n \), \( Da_i + D\partial_i(a_i) = D \),
   (c) the algebra \( PZ(\mathcal{A}) \) is a field, and
   (d) the algebra \( \mathcal{A} \) is a faithful left \( U(\mathcal{A}) \)-module.

**Generators and defining relations for the \( \mathcal{A} \)-module** \( \text{Der}_K(\mathcal{A}) \) **where** \( \mathcal{A} \) **is a regular domain of essentially finite type.** A localization of an affine commutative algebra is called an **algebra of essentially finite type.** The following notation is fixed: \( P_n = K[x_1, \ldots, x_n] \) is a polynomial algebra over **perfect** field \( K \), \( I = (f_1, \ldots, f_m) \) is a prime but not a maximal ideal of \( P_n \), \( \mathcal{A} = S^{-1}(P_n/I) \) is a domain of essentially finite type and \( Q = Q(\mathcal{A}) \) is its field of fractions, \( r = r(\partial f_i/\partial x_j) \) is the rank (over \( Q \)) of the **Jacobian**
matrix \((\frac{\partial f_r}{\partial x_j})\) of \(\mathcal{A}\), \(a_r\) is the Jacobian ideal of the algebra \(\mathcal{A}\) which is (by definition) generated by all the \(r \times r\) minors of the Jacobian matrix of \(\mathcal{A}\) (the algebra \(\mathcal{A}\) is regular iff \(a_r = \mathcal{A}\), it is the Jacobian criterion of regularity, [10, Corollary 16.20]), \(\Omega_\mathcal{A}\) is the module of Kähler differentials for the algebra \(\mathcal{A}\).

For \(i = (i_1, \ldots, i_r)\) such that \(1 \leq i_1 < \cdots < i_r \leq m\) and \(j = (j_1, \ldots, j_r)\) such that \(1 \leq j_1 < \cdots < j_r \leq n\), \(\Delta(i,j)\) denotes the corresponding minor of the Jacobian matrix of \(\mathcal{A}\), and the \(i\) (resp., \(j\)) is called non-singular if \(\Delta(i,j') \neq 0\) (resp., \(\Delta(i',j') \neq 0\)) for some \(j'\) (resp., \(i'\)). We denote by \(\mathbb{I}_r\) (resp., \(\mathbb{J}_r\)) the set of all the non-singular \(r\)-tuples \(i\) (resp., \(j\)).

Since \(r\) is the rank of the Jacobian matrix of \(\mathcal{A}\), it is easy to show that \(\Delta(i,j) \neq 0\) iff \(i \in \mathbb{I}_r\) and \(j \in \mathbb{J}_r\) ([4, Lemma 2.1]). We denote by \(\mathbb{J}_{r+1}\) the set of all \((r+1)\)-tuples \(j = (j_1, \ldots, j_{r+1})\) such that \(1 \leq j_1 < \cdots < j_{r+1} \leq n\) and when deleting some element, say \(j_\nu\), we have a non-singular \(r\)-tuple \((j_1, \ldots, j_{\nu}, \ldots, j_{r+1}) \in \mathbb{J}_r\) where the hat over a symbol means that the symbol is omitted from the list. The set \(\mathbb{J}_{r+1}\) is called the critical set and any element of it is called a critical singular \((r+1)\)-tuple. \(\text{Der}_K(\mathcal{A})\) is the \(\mathcal{A}\)-module of \(K\)-derivations of the algebra \(\mathcal{A}\).

The next theorem gives a finite set of generators and a finite set of defining relations for the left \(\mathcal{A}\)-module \(\text{Der}_K(\mathcal{A})\) when \(\mathcal{A}\) is a regular algebra.

**Theorem 4.6** ([4, Theorem 4.2] if \(\text{char}(K) = 0\); [5, Theorem 1.1] if \(\text{char}(K) = p > 0\))

Let the algebra \(\mathcal{A}\) be a regular domain of essentially finite type over the perfect field \(K\). Then the left \(\mathcal{A}\)-module \(\text{Der}_K(\mathcal{A})\) is generated by the derivations \(\partial_{ij}, i \in \mathbb{I}_r, j \in \mathbb{J}_{r+1}\) where

\[
\partial_{ij} = \partial_{i_1, \ldots, i_r; j_1, \ldots, j_{r+1}} := \det \begin{pmatrix} \frac{\partial f_1}{\partial x_{j_1}} & \cdots & \frac{\partial f_1}{\partial x_{j_{r+1}}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial x_{j_1}} & \cdots & \frac{\partial f_r}{\partial x_{j_{r+1}}} \\ \frac{\partial f_r}{\partial j_1} & \cdots & \frac{\partial f_r}{\partial j_{r+1}} \end{pmatrix}
\]

that satisfy the following defining relations (as a left \(\mathcal{A}\)-module):

\[
\Delta(i,j)\partial_{ij} = \sum_{l=1}^{s} (-1)^{r+1+\nu_l} \Delta(i',j'_1,\ldots,j'_{\nu_l},\ldots,j'_{r+1})\partial_{i'j'_{\nu_l}} j_{\nu_l} \tag{7}
\]

for all \(i, i' \in \mathbb{I}_r, j = (j_1, \ldots, j_r) \in \mathbb{J}_r, \text{ and } j' = (j'_1, \ldots, j'_{r+1}) \in \mathbb{J}_{r+1}\) where \(\{j'_{\nu_1}, \ldots, j'_{\nu_2}\} = \{j'_1, \ldots, j'_{r+1}\} \backslash \{j_1, \ldots, j_r\}\).

Suppose that the algebra \(\mathcal{A} = S^{-1}(P_n/I)\) is a Poisson domain of essentially finite type. Let \(d = d_{\mathcal{A}} = r(\mathcal{C}_\mathcal{A})\) be the rank of the \(n \times n\) matrix

\[
\mathcal{C}_\mathcal{A} = (\{x_i, x_j\}) \in M_n(\mathcal{A})
\]

over the field \(Q\). For each \(l = 1, \ldots, n\), let

\[
\text{ind}_n(l) = \{i = (i_1, \ldots, i_l) \mid 1 \leq i_1 < \cdots < i_l \leq n\}.
\]
For elements \( i = (i_1, \ldots, i_l) \) and \( j = (j_1, \ldots, j_l) \) of \( \text{ind}_n(l) \), let \( C_A(i, j) = \{ (x_{i_1}, x_{j_1}) \} \) be the \( l \times l \) submatrix of the matrix \( C_A \). So, the rows (resp., the columns) of the matrix \( C_A(i, j) \) are indexed by \( i_1, \ldots, i_l \) (resp., \( j_1, \ldots, j_l \)). The \( (i_\nu, j_\mu)^{th} \) element of the matrix \( C_A(i, j) \) is \( \{ x_{i_\nu}, x_{j_\mu} \} \). Let \( M_{A,l} = \{ C_A(i, j) \mid i, j \in \text{ind}_n(l) \} \) be the set of all \( l \times l \) submatrices of \( C_A \) and

\[
C_{A,l} = \{ \mu(i, j) := \det(C_A(i, j)) \mid i, j \in \text{ind}_n(l) \}
\]

be the set of all \( l \times l \) minors of \( C_A \). Let

\[
\begin{align*}
\mathbb{I}(l) &= \mathbb{I}_A(l) = \{ i \in \text{ind}_n(l) \mid \mu(i, j) \neq 0 \text{ for some } j \in \text{ind}_n(l) \} \\
\mathbb{J}(l) &= \mathbb{J}_A(l) = \{ j \in \text{ind}_n(l) \mid \mu(i, j) \neq 0 \text{ for some } i \in \text{ind}_n(l) \}
\end{align*}
\]

The ideal \( \kappa_A \) of the algebra \( \mathcal{U}(A) \) and its generators \( \delta_{i, i_\nu, j} \). For each pair of elements \( i = (i_1, \ldots, i_d) \) and \( j = (j_1, \ldots, j_d) \) of \( \mathbb{I}_A(d) = \mathbb{J}_A(d) \) and each element \( i_\nu \in \{ i_{d+1}, \ldots, i_1 \} = \{ 1, \ldots, n \} \setminus \{ i_1, \ldots, i_d \} \), let us consider the following elements of the algebra \( \mathcal{U}(A) \), [7],

\[
\delta_{i, i_\nu, j} := \det\begin{pmatrix} x_{i_1}, x_{j_1} & \cdots & x_{i_1}, x_{j_d} & \delta_{i_1} \\
\vdots & \ddots & \vdots & \vdots \\
x_{i_d}, x_{j_1} & \cdots & x_{i_d}, x_{j_d} & \delta_{i_d} \\
x_{i_\nu}, x_{j_1} & \cdots & x_{i_\nu}, x_{j_d} & \delta_{i_\nu} \end{pmatrix} = \mu(i, j)\delta_{i_\nu} + \sum_{s=1}^{d} (-1)^{s+d+1} \mu(i_1, \ldots, i_s, \ldots, j_d, i_\nu; j)\delta_{i_s} \tag{8}
\]

where \( \delta_{i} := dx_i \in \Omega_A \subseteq \mathcal{U}(A) \).

**Definition.** [7] Let \( \kappa_A \) be an ideal of the algebra \( \mathcal{U}(A) \) which is generated by the finite set of elements \( \delta_{i, i_\nu, j} \in \Omega_A \) where \( i, j \in \mathbb{I}_A(d) \), see (8). Then \( \kappa_A \subseteq \ker(\pi_A) \), [7].

**Criteria for** \( \ker(\pi_A) = 0 \). In the case when the Poisson algebra \( A \) is a *regular* domain of essentially finite type, Theorem [4,7] is an efficient explicit criterion for \( \ker(\pi_A) = 0 \), i.e. for the epimorphism \( \pi_A : \mathcal{U}(A) \to PD(A) \) to be an isomorphism.

Since \( \text{Der}_K(A) \approx \text{Hom}_A(\Omega_A, A) \), there is a *pairing* of left \( A \)-modules (which is an \( A \)-bilinear map):

\[
\text{Der}_K(A) \times \Omega_A \to A, \quad (\partial, \omega) \mapsto (\partial, \omega) := \partial(\omega).
\]

**Theorem 4.7 (7, Theorem 1.9)** Let a Poisson algebra \( A = S^{-1}(P_n/I) \) be a regular domain of essentially finite type over the field \( K \) of characteristic zero, \( d = r(C_A) \) and \( r = r(\partial_{i,j}) \). Then the following statements are equivalent (the derivations \( \partial_{i,j} \) of \( A \) are defined in Theorem [4,7]):

1. \( \ker(\pi_A) = 0 \) \((\Leftrightarrow \pi_A : \mathcal{U}(A) \approx PD(A))\).
2. \( \kappa_A = 0 \).
3. \( d = n - r \) and \( \partial_{i,j} \) for all elements \( i \in \mathbb{I}_r, j \in \mathbb{J}_r, i' \in \mathbb{I}_n, j' \in \mathbb{J}_n, \) where for \( j = (j_1, \ldots, j_r) \) and \( i' = (i'_1, \ldots, i'_d) \), \( \{ j_{r+1}, \ldots, j_n \} = \{ 1, \ldots, n \} \setminus \{ j_1, \ldots, j_r \} \) and \( \{ i'_{d+1}, \ldots, i'_n \} = \{ 1, \ldots, n \} \setminus \{ i'_1, \ldots, i'_d \} \).

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GK stands for the **Gelfand-Kirillov dimension** and $\text{Sym}_A(\Omega_A)$ is the **symmetric algebra** of the $A$-module $\Omega_A$ of Kähler differentials of the algebra $A$.

In the case when the Poisson algebra $P = A$ is an algebra of essentially finite type over a field of characteristic zero, Theorem 4.4 can be strengthened, see Theorem 4.8.

**Theorem 4.8** ([7, Theorem 1.3]) Let a Poisson algebra $A$ be an algebra of essentially finite type over the field $K$ of characteristic zero. Then the following statements are equivalent:

1. The algebra $U(A)$ is a simple algebra.
2. The algebra $PD(A)$ is a simple algebra and one of the equivalent conditions of Theorem 4.7 holds.
3. The algebra $A$ is Poisson simple and one of the equivalent conditions of Theorem 4.7 holds.

If one of the equivalent conditions holds then the algebra $A = S^{-1}(P_n/I)$ is a regular, Poisson simple domain of essentially finite type over the field $K$ of characteristic zero, the algebra epimorphism $\pi_A : U(A) \rightarrow PD(A)$ is an isomorphism, $d = n - r$ where $d = r(C_A)$ and $r = r(\frac{\partial f_i}{\partial x_j})$, and the algebra $U(A)$ is a simple Noetherian domain with

$$\text{GK } U(A) = \text{GK } PD(A) = \text{GK } \text{gr } U(A) = \text{GK } \text{Sym}_A(\Omega_A) = 2\text{GK } (A) = 2(n - r).$$

**A simplicity criterion for the PEA $U(A)$ of a GWPA $A$.** Let $A = D[X, Y; \partial, a]$ be a GWPA of degree $N$; a Poisson algebra $D = \mathcal{A} = S^{-1}(P_n/I)$ be a regular domain of essentially finite type; $I = (f_1, \ldots, f_m)$; $r = r(\frac{\partial f_i}{\partial x_j})$ be the rank of the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})$ of the algebra $D$; $d = r(C_A)$ be the rank of the $(n + 2N) \times (n + 2N)$ matrix $C_A = \{s_i, s_j\}$ where $s_i, s_j \in \{x_1, \ldots, x_n, X_1, \ldots, X_N, Y_1, \ldots, Y_N\}$.

Theorem 4.9 is a simplicity criterion for the Poisson enveloping algebra $U(A)$ of a GWPA $A$.

**Theorem 4.9** Let a Poisson algebra $D = S^{-1}(P_n/I)$ be a regular domain of essentially finite type over the field $K$ of characteristic zero and $r = r(\frac{\partial f_i}{\partial x_j})$ be the rank of the Jacobian matrix of $D$, $A = D[X, Y; \partial, a]$ be a GWPA of degree $N$ and $d = r(C_A)$. Then the following statements are equivalent:

1. The algebra $U(A)$ is a simple algebra.
2. The algebra $PD(A)$ is a simple algebra and one of the equivalent conditions of Theorem 4.7 holds.
3. The Poisson algebra $A$ is a Poisson simple algebra and one of the equivalent conditions of Theorem 4.7 holds.
Let a Poisson algebra $A$ be a domain of essentially finite type over a perfect field $K$, $I = (f_1, \ldots, f_m)$ is a prime ideal of $P_n$, and $r = r(\partial f_\alpha/\partial x_j)$ be the rank of the Jacobian matrix $(\partial f_\alpha/\partial x_j)$. Then the algebra $\mathcal{U}(A)$ is a Noetherian algebra with

$$\text{GK } \mathcal{U}(A) = \text{GK } \mathcal{U}(A) = \text{GK } \text{Sym}_A(\Omega_A) = 2\text{GK } (A) = 2(n - r).$$

By Theorem 3.2, the Poisson enveloping algebra $\mathcal{U}(A)$ of the GWPA $A = D[X,Y; a, \partial]$, where $D = S^{-1}(P_n/I)$, admits a filtration by the total degree of the elements $\delta_{x_1}, \ldots, \delta_{x_n}$, $\delta_{x_1}, \ldots, \delta_{y_n}$, $\delta_{y_1}, \ldots, \delta_{y_n}$ (nonzero elements of the algebra $A$ have degree zero). The associated graded algebra of $\mathcal{U}(A)$ is denoted by $\text{gr } \mathcal{U}(A)$. In this case, $\mathcal{U}(A) = PD(A)$ and char($K$) = 0, the filtration coincides with the order filtration on the algebra of Poisson differential operators $PD(A)$ on $A$.

**Corollary 4.11** Let a Poisson algebra $D = S^{-1}(P_n/I)$ be a domain of essentially finite type over a perfect field $K$, $I = (f_1, \ldots, f_m)$, $r = r(\partial f_\alpha/\partial x_j)$ be the rank of the Jacobian matrix $(\partial f_\alpha/\partial x_j)$ of $D$, $A = D[X,Y; \partial, a]$ be a GWPA of degree $N$ with all $a_i \neq 0$ and $d = r(C_A)$. Then the algebra $\mathcal{U}(A)$ is a Noetherian algebra with

$$\text{GK } \mathcal{U}(A) = \text{GK } \text{gr } \mathcal{U}(A) = \text{GK } \text{Sym}_A(\Omega_A) = 2\text{GK } (A) = 2(n - r + N).$$
Proof. Since \( \text{GK} (A) = \text{GK} (D) + N = n - r + N \), the corollary follows from Theorem 4.10. \( \square \)

The Gelfand-Kirillov dimension of the algebra \( PD(A) \) of Poisson differential operators on \( A \). Proposition 4.12 gives the exact figure for the Gelfand-Kirillov dimension of the algebra \( PD(A) \) where the Poisson algebra \( A \) is a domain of essentially finite type over the field \( K \) of characteristic zero.

**Proposition 4.12** ([7, Theorem 1.5]) Let a Poisson algebra \( A \) be a domain of essentially finite type over the field \( K \) of characteristic zero and \( r \) be the rank of Jacobian matrix of \( A \) and \( d = r(C_A) \). Then

\[
\text{GK} (PD(A)) = \text{GK} (A) + d = n - r + d.
\]

**Corollary 4.13** Let a Poisson algebra \( D = S^{-1}(P_n/I) \) be a domain of essentially finite type over the field \( K \) of characteristic zero, \( I = (f_1, \ldots, f_m) \), \( r = r\left(\frac{\partial f_i}{\partial x_j}\right) \) be the rank of the Jacobian matrix \( \left(\frac{\partial f_i}{\partial x_j}\right) \) of \( D \), \( A = D[X,Y; \partial, a] \) be a GWPA of degree \( N \) with all \( a_i \neq 0 \) and \( d = r(C_A) \). Then

\[
\text{GK} (PD(A)) = \text{GK} (A) + d = n - r + N + d.
\]

Proof. The GWPA \( A \) is a domain since the algebra \( D \) is so and all the elements \( a_i \) are not equal to zero. Since \( \text{GK} (A) = \text{GK} (D) + N = n - r + N \), the corollary follows from Theorem 4.12. \( \square \)

The algebra \( U(A) \) is a domain when \( A \) is a regular domain of essentially finite type. Theorem 4.14 states that the algebra \( U(A) \) is a domain provided the algebra \( A \) is a regular domain of essentially finite type.

**Theorem 4.14** ([7, Theorem 4.4]) Let a Poisson algebra \( A = S^{-1}(P_n/I) \) be a regular domain of essentially finite type over the perfect field \( K \) where \( I = (f_1, \ldots, f_m) \) is a prime but not maximal ideal of \( P_n \) and \( r = r(\frac{\partial f_i}{\partial x_j}) \) is the rank of the Jacobian matrix \( \left(\frac{\partial f_i}{\partial x_j}\right) \) over the field of fractions of the domain \( P_n/I \). Then the algebra \( U(A) \) is a Noetherian domain with \( \text{GK} U(A) = \text{GK} \text{gr} U(A) = \text{GK} \text{Sym}_A(\Omega_A) = 2\text{GK} (A) = 2(n - r) \).

**Corollary 4.15** Let a Poisson algebra \( D = S^{-1}(P_n/I) \) be a domain of essentially finite type over the perfect field \( K \), \( I = (f_1, \ldots, f_m) \), \( r = r(\frac{\partial f_i}{\partial x_j}) \) be the rank of the Jacobian matrix \( \left(\frac{\partial f_i}{\partial x_j}\right) \) of \( D \), \( A = D[X,Y; \partial, a] \) be a regular GWPA of degree \( N \) with all \( a_i \neq 0 \) and \( d = r(C_A) \). Then the algebra \( U(A) \) is a Noetherian domain with \( \text{GK} U(A) = \text{GK} \text{gr} U(A) = \text{GK} \text{Sym}_A(\Omega_A) = 2\text{GK} (A) = 2(n - r + N) \).

Proof. Since \( \text{GK} (A) = \text{GK} (D) + N = n - r + N \) and the algebra \( A \) is a domain (since \( D \) is so and all the elements \( a_i \neq 0 \)), the corollary follows from Theorem 4.14. \( \square \)

**Data Availability.** Data sharing is not applicable to this article as no new data were created or analyzed in this study.
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