TOWARDS A BROADER VIEW OF THEORY OF COMPUTING -
PART 1

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ABSTRACT. Beginning with the projectively invariant method for linear pro-
gramming, interior point methods have led to powerful algorithms for many
difficult computing problems, in combinatorial optimization, logic, number
theory and non-convex optimization. Algorithms for convex optimization ben-
efit from many pre-established ideas from classical mathematics, but non-
convex problems require new concepts.

Lecture series I am presenting at the conference on Foundations of Compu-
tational Mathematics, 2014, outlines some of these concepts– computational
models based on the concept of the continuum, algorithms invariant w.r.t.
projective , bi-rational , and bi-holomorphic transformations on co-ordinate
representation, extended proof systems for more efficient certificates of opti-
mality, extensions of Grassmanns extension theory, efficient evaluation meth-
ods for the effect of exponential number of constraints , theory of connected
sets based on graded connectivity, theory of curved spaces adapted to the
problem data, and concept of relatively algebraic sets in curved space.

Since this conference does not have a proceedings, the purpose of this arti-
cle is to provide the material being presented at the conference in more widely
accessible form.

1. INTRODUCTION

In Part 1 of this lecture series we discuss two topics :

- Computational models based on the concept of the continuum
- Extended proof systems for more efficient certificates of optimality

In the part 2 of this lecture series, we describe efficient evaluation methods for
the effect of exponential number of constraints in the context of continuum based
algorithms. We illustrate these ideas by considering concrete examples of finding
maximum independent set in a graph and the satisfiability problem. Objective
function for these problems is treated by non-convex optimization methods covered
in part 3.

2. Models of Computation

2.1. Introduction. A model of computation provides mathematical abstractions
of basic data objects and operations on those objects, available as building blocks.
This effectively decouples design of algorithms for complex tasks from lower level

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details of how the underlying hardware implements the basic primitives. The Turing machine model uses strings of 0s and 1s and finite state machines. Careful study of the work of early pioneers Turing, Von Neumann and Godel shows that they were acutely aware of the limitations of this model for comprehensive understanding of the fundamental aspects of computation. BSS model (Blum, Shub, Smale) uses real or complex numbers as data objects and algebraic operations (including comparisons in real case). This is more natural for many algorithms in numerical analysis, whereas Turing Machine model might seem more appropriate for discrete applications.

2.2. Applications of computing. Various applications of computing come in both flavours- discrete and continuous. E.g. Many business and information technology applications of computing clearly use discrete models. At the same time, there are many areas of scientific and engineering applications based on continuous models. A few such examples are:

- Numerical simulation of natural phenomena or engineered systems based on differential equations
- Estimation of probabilities, assessing strength of association between various entities in social networks
- Interior point algorithms in optimization which are based on embedding discrete configurations in a multidimensional continuous space and constructing trajectories converging to the solution

Ideas presented in this paper are primarily motivated by such applications. A model of computation is used to map applications of interest on physical machines.

2.3. Examples of physical computers.

- Standard digital computers
- Analog computers explored in the past e.g. Shannon’s differential analyzer
- Quantum computers
- Biological systems that appear to do information processing / computing
- Computers inspired by biological systems and partially mimicking some aspects of those systems
e.g. neural networks, neuromorphic computing, etc.

2.4. The need for a broader approach – Early Pioneers’ views.

- Alan Turing was interested in understanding the origin of intelligence in biological systems. He believed it is due to some form of computing happening in these systems. However, there doesn’t seem to be anything similar to Turing machines in these systems. Many years after developing the discrete model of computation, Turing started exploration of partial differential equations modelling biological functions. These are clearly continuum based models of biological systems.

- Von Neumann defined architecture of a digital computer using many ideas from Turing Machine model. However, he was particularly critical of the limitations imposed on the theory of automata by its foundations in formal logic, combinatorics. He articulated the need for a detailed, highly mathematical, and more specifically, analytical theory of computation based on mathematical analysis.
- **Shannon** worked on analog computer called differential analyzer. However, direct implementation of differentiation in analog system is highly error prone as it involves subtraction of two nearly equal continuous quantities.
2.5. **Physical devices that process information in continuum form.**

- **Role of Integration and Integral Transform**
  - It’s well known that for certain non-linear differential equations which are difficult to solve numerically, it helps to convert them into equivalent integral equations.
  - Biological systems seem to use integral transform. e.g. Human ear essentially computes the magnitude of Fourier Transform of speech signal. The transform of the derivative is then obtained by scaling, a simpler algebraic operation than differentiation in analog setting.
  - Some optical computers are implementing Fourier Transform directly for computing differential operators required in computational fluid dynamic
  - Form of integration or convolution restricted to non-negative real functions is easier to provide in a physical system.

- **Robustness**
  - In digital systems, the mapping from 
    
    **Physical states / quantities** → **information states/quantities**
    
    is many to one, In fact, each information state has infinite pre images.
  - This is to gain robustness in the face of small variations in physical states, due to noise, thermal effects, variations in manufacturing etc.
  - However, this does not necessarily require information states to be discrete. e.g. An ideal low pass filter also provides infinity to one mapping but the output signal still belongs to the continuum.

It is important to include these two mechanisms when constructing machines that support continuum computing

2.6. **Intertwining of discrete and continuous models.** Logical and physical processes underlying computation involves several levels of abstractions, both continuous and discrete. Let us examine a specific application – Solution of a non-linear differential equation.

- This application and algorithm used for solution are both based on continuous model.
- Real numbers in the continuous model are approximated by floating point arithmetic for mapping onto the discrete Turing machine model.
- The computer implementing discrete Turing Machine model is actually a network of transistors. As a circuit, it is modeled by a system of non-linear differential equations. Restricted interpretation is applied to continuous quantities. e.g. Continuous time is divided into clock cycles bigger than settling time of the transient solution to the differential equations, to create discrete time steps. Voltage greater than certain threshold voltage represents logical one. In this way, the discrete logical model is supported by the underlying circuit-level continuous model.
- This continuous circuit model is itself approximating underlying discrete phenomenon:
Towards a broader view of theory of computing

Continuous, fluid-like model of current flow, for approximating the statistical mechanics of large number of discrete particles (electrons and holes in a semiconductor)

- Continuous energy bands (valence and conduction bands) representing large number of closely spaced quantized energy levels

The underlying quantum mechanical phenomenon is modelled mathematically by the concept of Hilbert space which is based on the continuum concept. Similarly, more recent and on-going model of physics is based on string theory, which again requires the concept of the continuum.

2.7. Towards a broader view of theory of computing.

Given this intertwining of discrete and continuous models from top to bottom, it would be more illuminating to take a broader view of theory of computing. In addition to the discrete Turing machine model, one also needs to explore continuous models for a comprehensive understanding of the fundamental aspects of computation. BSS model uses real or complex numbers as data objects and algebraic operations (including comparisons in real case). This is more natural for many algorithms in numerical analysis.

Various computing models can be organized in a similar way as Cantor had organized infinite sets – by cardinal number of the set of all possible machines and data objects in the model. Staying within the same cardinal number, a more powerful approach is to use further extension, e.g. real analytic functions or algebraic closure of meromorphic functions over suitable domains. Operations include algebraic as well as analytic operations. i.e. integration and differentiation regarded as binary operations. (specification of the contour of integration is one of the input operands).

All such models that use both algebraic and analytical operations on continuous data objects are collectively referred to as continuum computing. Our research program is aimed at exploration of what a continuum view of computing might suggest for science of algorithms, relative difficulty of various computing tasks etc.

2.8. Abstract Continuum Computing Model AC^2M or CM for short. The basic data objects and operations in this model are as follows:

- **Basic data objects**
  algebraic closure of meromorphic functions (over suitable domain)

- **Basic unit operations**
  - field operations: $+, -, \times, ÷$
    and additionally for real quantities comparison ($<, =$)
  - analytic operations
    * integration: $\int_C f(z)dz$ this is a binary operation, function $f$ and specification contour $C$ are the two operands
    * differentiation: $\frac{\partial f}{\partial z_i}$ is also a binary operation with inputs $f$ and $z_i$.

(a word of caution when comparing with TM - there is no such thing as “conservation of difficulty” across the models.)
2.9. Extension of $P \neq NP$ conjecture from TM theory.

- Computing models can be organized in the same way as Cantor organized infinite sets i.e. according to the cardinal number of set of all possible data objects and machines in the model.
- At each cardinal number in the sequence $\aleph_0 < \aleph_1 < \aleph_2 < \ldots$, you have models of computation, and corresponding $P \neq NP$ question
  \[
  \begin{align*}
  TM & \rightarrow \aleph_0, \quad \text{cardinalnumber}(\mathbb{Q}) \\
  CM & \rightarrow \aleph_1, \quad \text{cardinalnumber}(\mathbb{R})
  \end{align*}
  \]
- It appears that $P \neq NP$ problem for TM is just the first member in a sequence of strict inclusions
  \[
  P(TM) \subsetneq NP(TM) \subsetneq P(CM) \subsetneq NP(CM) \subsetneq P(\aleph_2) \ldots
  \]
- An interesting question across adjacent level:
  \[
  NP(TM) \subsetneq P(CM)
  \]
  i.e., is non-deterministic computing at any one level no more powerful than deterministic computing at the next level?

2.10. Our Research Program. Is aimed at understanding the following:

- how to construct physical machines supporting Continuum Computing?
- Can non-deterministic computing at the TM level be simulated by deterministic computing at next level i.e. by Continuum Computing?

  and a harder question:

- to what extent can one approximate deterministic Continuum Computing by deterministic computing in TM?
- initially, approximate cross-simulation was meant to be "stop-gap" measure, but now it appears that building continuum machine may take many years, hence simulation becomes more important.
- Floating Point numbers allows approximation of reals by rationals
- Similarly, one can approximate functions by other simpler functions
- In this investigation, length of "binary encoding" of data object does not have the same fundamental significance as in TM theory. Instead, other properties of the problem space seem more important
3. Extensions of proof systems for more efficient proofs of optimality and non-satisfiability

3.1. Introduction. We are interested in converting results of continuum based algorithms so that proofs of optimality or non-existence of solutions can be verified on the standard model.

Both involve proofs of non-negativity of functions.

• Optimality:
  Proving that $x_{\text{min}}$ is a global minimum of $f(x_1, x_2, \ldots, x_n)$ is equivalent to showing that for $f_{\text{min}} = f(x_{\text{min}})$, we have
  \[ f(x) - f_{\text{min}} \geq 0 \quad \forall x \in \mathbb{R}^n \]

• Non-satisfiability: Each variable $x_i = \pm 1$
  let $V$ denote the variety defined by $x_i^2 = 1 \quad \forall i$
  For each clause associate a polynomial, e.g.
  $C = x_i \lor \bar{x}_j \lor x_k$, associated $p_c(x) = [1 - x_i] \cdot [1 + x_j] \cdot [1 - x_k]$

  For a $\pm 1$ vector $x$, $p_c(x) = \begin{cases} 0 & \text{if } x \text{ is a satisfying assignment} \\ 8 & \text{otherwise} \end{cases}$

 Define $f = \sum_{C \in \text{Clauses}} p_c(x)$.
  Then $f(x) > 0 \quad \forall x \in V$ gives proof of non-satisfiability.

3.2. Relation of Positivity Proofs to Hilbert’s 17th problem.

3.2.1. Introduction.

• One approach to proving positivity of a polynomial is to express it as sums of squares of other functions

• Using polynomials:
  Hilbert (1888) realized that it is not always possible, but gave a non-constructive proof of the existence of a counter example.

• Using rational functions:
  Artin’s (1926) solution of Hilbert’s 17th problem.
  But exponentially many terms of high degree required (Pfister).

• Concrete examples of Hilbert’s result took long time to construct.
  First counter example - Motzkin
  \[ f(x, y, z) = z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2 \]

  Further examples - Lam, Robinson and others.

• Computational results:
  For these “counterexamples”, we have computed expressions as sums of squares of polynomials, with a modified interpretation.
  Algorithm underlying the solver was described in previous lecture [Karmarkar, MIT91, IPCO92]
3.2.2. **Computational approach to positivity proofs.**

- Consider a homogeneous polynomial \( f(x_1, x_2, \ldots, x_n) \) of degree \( d \) in \( n \) real variables \( x_1, x_2, \ldots, x_n \), which is non-negative everywhere.

- To show that a function is non-negative on a compact set, it is enough to show that the function is **non-negative** at all its **critical points**.
  (note we have homogeneous polynomials over projective space). Therefore non negativity at set of infinite number of points in \( \mathbb{R}^n \) is implied by non-negativity at a finite subset of points viz the critical points.

- This can be achieved if we
  
  (1) construct a **variety containing all critical points** of the function and  
  (2) construct an expression of the function as sums of squares of polynomials in the **coordinate ring of that variety**, instead of the polynomial ring \( \mathbb{R}(x_1, \ldots, x_n) \).

- Additionally, without loss of generality, we impose a spherical constraint

\[
(3.2.1) \quad x_1^2 + x_2^2 + \ldots + x_n^2 = 1
\]

- Each point in the real projective space is covered twice (by a pair of antipodal points) in this representation.

- A critical point of the function satisfies

\[
(3.2.2) \quad \frac{\partial f}{\partial x_i} = \lambda x_i, \quad i = 1, \ldots, n
\]

where \( \lambda \) is the Lagrange multiplier.

- We now work in \( \mathbb{R}^{n+1} \), and points in this expanded space will be denoted by \((x_1, x_2, \ldots, x_n, \lambda)\)

- The equations (1) and (2) define an algebraic variety \( U \) (i.e an algebraic set – in our terminology a variety does not have to be irreducible).

- Our approach is to construct a variety \( V \) such that \( U \subseteq V \subseteq \mathbb{R}^{n+1} \) and in the co-ordinate ring of \( V \), construct an expression for \( f \) as sums of squares

\[
\sum_i S_i^2(x_1, x_2, \ldots, x_n, \lambda)
\]

- We also produce an explicit expression for

\[
f - \sum_i S_i^2(x_1, x_2, \ldots, x_n, \lambda)
\]

where \( g_j(x_1, x_2, \ldots, x_n, \lambda) = 0 \) are the defining equations for \( V \).
3.2.3. **Samples of computer generated proofs.** For small examples, the proofs are short enough to be displayed on viewgraphs:

Motzkin’s first counter example

\[
\begin{align*}
  f(x, y, z) &= z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2 \\
  f &\equiv \left( \frac{1}{4} S_1^2 + S_2^2 + S_3^2 + S_4^2 + \frac{3}{4} S_5^2 \right) \mod V \\
  S_1 &= xy(x^2 - y^2) \\
  S_2 &= x^4 + y^4 - 2x^2 - 2y^2 + x^2y^2 + 1 \\
  S_3 &= xz(x^2 + 2y^2 - 1) \\
  S_4 &= yz(2x^2 + y^2 - 1) \\
  S_5 &= xy(3x^2 + 3y^2 - 2)
\end{align*}
\]

Robinson’s counter example

\[
\begin{align*}
  f &= x^6 + y^6 + z^6 - (x^4y^2 + x^2y^4 + y^2z^4 + z^4x^2 + z^2x^4) + 3x^2y^2z^2 \\
  f &\equiv \left( S_1^2 + \frac{3}{4} S_2^2 + \frac{1}{4} S_3^2 + S_4^2 + S_5^2 \right) \mod V \\
  S_1 &= -x^3y + xy^3 \\
  S_2 &= -1 + 3x^2 - 2x^4 - 4x^2y^2 + 2y^2 \\
  S_3 &= 1 - x^2 - 2x^4 + 4x^2y^2 - 4y^2 + 4y^4 \\
  S_4 &= -2x^3z - xy^2z + xz \\
  S_5 &= -x^2yz - 2y^2z + yz
\end{align*}
\]

3.3. **Rules for positivity proofs.**

3.3.1. **Basic Rules.**

1. Starting primitives:
   a. **Constant functions**
      
      Let \( \alpha \in \mathbb{R}_+ \) be a positive scalar.
      
      If \( f(x) = \alpha \), then \( f > 0 \).
   
   b. **Square functions**
      
      If \( f(x) = g^2(x) \), then \( f \geq 0 \).

2. **Algebraic operations preserving positivity:**
   
   If \( f \geq 0 \) and \( g \geq 0 \), then \( f + g \geq 0 \)
   
   If \( f \geq 0 \) and \( g \geq 0 \), then \( f \cdot g \geq 0 \)

3. **Positivity restricted to variety defining constraints:**
   
   Suppose there are integrality constraints, \( x_i^2 = 1 \), or other constraints like \( Ax = b \) etc. They define an algebraic set or subvariety \( V \) of \( \mathbb{R}^n \). It is understood that algebraic operations are to be interpreted upto equivalence classes of functions on \( V \).

   There has been extensive exploration of application of rules of this type along with rules for cuts based on integrality constraints. see e.g. Lovász, Schrijver, Grigoriev,Chlamatac,Schoenebeck, Tulsiani,Worah, and references therein.
3.3.2. **New rules.** We strengthen the previous proof systems by adding the following new rules.

(1) **Substitution or Composition rule:**

Let \( f(x_1, \ldots, x_n) \) and \( g_i(y_1, \ldots, y_m) \) for \( 1 \leq i \leq n \) be real polynomials.

Let \( h(y_1, \ldots, y_m) = f(g_1(y_1, \ldots, y_m), \ldots, g_n(y_1, \ldots, y_m)) \).

(a) If \( f(x_1, \ldots, x_n) \geq 0 \) for all \( x \in \mathbb{R}^n \), then \( h(y_1, \ldots, y_m) \geq 0 \).

(b) If \( f(x_1, \ldots, x_n) \geq 0 \) for all \( x \in \mathbb{R}^n_+ \) and \( g_i(y_1, \ldots, y_m) \geq 0 \) for all \( y \in \mathbb{R}^m_+ \), then \( h(y_1, \ldots, y_m) \geq 0 \).

(2) **Division rule:** In some sense, this rule is not new, since Artin’s approach uses rational functions. However, number of rational terms in that approach can be exponential, but having a separate rule enables selective application for more efficient proof.

Suppose \( g(x) > 0, f(x) \geq 0 \) and \( g(x) | f(x) \).

If \( f(x) = g(x) \cdot h(x) \), then \( h(x) \geq 0 \).

This rule reduces the degree.

(3) **Odd Radical rule:**

Suppose \( f(x) \) is a perfect odd \((2k + 1)^{th}\) power. Let \( g(x) \) be the \((2k + 1)^{th}\) root of \( f(x) \), i.e. \( f(x) = g^{2k+1}(x) \).

If \( f(x) \geq 0 \), then \( g(x) \geq 0 \).

This is one of the rare rules that reduces the degree.

**Notes:**

(a) Certain inequalities (e.g. Cauchy-Schwartz) are proved symbolically, and have proof-length \( O(1) \), independent of \( n \). When invoking such inequalities during application of the composition rule, only length involved in specifying the substitution should be counted as addition to the length of the proof.

(b) Lower bounds based on proof system without these rules don’t show intrinsic difficulty of any problem, but only show intrinsic limitations of the proof system used.

4. **Maximum Independent Set**

4.1. **Introduction.**

- Let \( G = (V, E) \) be a graph with vertex set \( V \) and edge set \( E \).
- A subset \( U \subseteq V \) of nodes is called **independent** if for every edge \((i, j) \in E\), either \( i \notin U \) or \( j \notin U \).
- **Problem:** Find the largest independent set.
- \( \pm 1 \) integer programming formulation

  **Constraints:**
  
  \[
  w_i = \pm 1; \quad i = 1, \ldots, n
  \]
  
  \[
  w_i + w_j \leq 0; \quad \forall (i, j) \in E
  \]

  **Objective:** Maximize \( \sum w_i \)

- Relaxation of the feasible set allows \(-1 \leq w_i \leq 1\)
- Each extreme point of an LP has co-ordinates that are \( \pm 1 \) or \( 0 \).
- For a \( \pm 1 \) solution, \( \sum w_i^2 = n \).
4.2. Defining a sharper polytope. There are additional inequalities

- that hold for the solution points, i.e. feasible points with ±1 solution
- but cannot be expressed as non-negative linear combinations of basic inequalities
- number of such inequalities grows exponentially with \( n \)
- significant effort in polyhedral combinatorics is aimed at identifying such constraints (e.g. see Schrijver\(^3\)).

4.3. Example: odd cycle inequalities.

- Consider an odd cycle in the graph, having \( 2k + 1 \) vertices. At most \( k \) of the \( 2k + 1 \) vertices can be in an independent set.
- Let \( i_1, i_2, \ldots, i_{2k+1} \) be the vertices in the cycle; hence \( w_{i_1} + w_{i_2} + \ldots + w_{i_{2k+1}} \leq -1 \).
- This inequality is sharper, since the previous inequalities only imply:
  \[ w_{i_1} + w_{i_2} + \ldots + w_{i_{2k+1}} \leq 0. \]
- Number of odd cycles, and hence the corresponding number of inequalities, can grow exponentially with \( n \).

4.4. Inequalities for other subgraphs.

- Similar to odd cycles, there are constraints based on other types of subgraphs.
- Let
  \[ \sum_{i \in G_0} a_i \cdot w_i \leq b \]
  be a constraint based on a fixed graph \( G_0 \).
- Let \( G \) be a graph obtained by an odd sub-division of the edges of \( G_0 \), i.e. replace an edge by an odd path.
- Then sum of newly added vertices is always non-positive, i.e.
  \[ \sum_{i \in \text{New vertex}} w_i \leq 0. \]
- Furthermore, to achieve the equality, \( \sum_{i \in \text{New vertex}} w_i = 0 \), there are only two ways to assign values to new vertices and each corresponds to a particular feasible assignment to the end points of the original edge.
Using this observation, it is easy to show that the inequality for \( G_0 \) implies a similar inequality for \( \tilde{G} \), i.e.

\[
\sum_{i \in G_0} a_i \cdot w_i \leq b \Rightarrow \sum_{i \in \tilde{G}} \tilde{a}_i \cdot w_i \leq b
\]

• In this context, we are using a more restricted notion of homeomorphism.
• Specifically, in order for two paths to be homeomorphic both must have odd length or both must have even length.
• We call this “parity-respecting” homeomorphisms, and denote it by \( \tilde{G} \cong G_0 (\text{mod}2) \)

(different authors use different terminology/notation, e.g. see [ ])

• Given a fixed graph \( G_0 \) and an input graphs \( G \),
  an inequality for \( G_0 \) ⇒ an inequality for each subgraph \( \tilde{G} \) of \( G \) that is homeomorphic to \( G_0 \).
• Number of such subgraphs can grow exponentially with \( n \).
• As an example, let \( G_0 \) be a triangle and \( G \) be a given graph.
  Then the subgraphs of \( G \) that are homeomorphic to \( G_0 \) respecting parity,
  are exactly the odd cycles of \( G \).
  All odd cycle inequalities for \( G \) are obtained from inequalities for \( G_0 \).
• We will use this example to show how the combined effect of all odd cycle inequalities can be computed in polynomial time in the continuum model.
• Observe that all the inequalities in this example are linear, which simplifies our exposition.
• However, these techniques are not limited to the linear case. Later, we will show an example of non-linear, non-convex inequalities as well.
5. Combining Effect of Exponential Number of Inequalities

5.1. Projectively Invariant Metric.

- Consider the projectively invariant metric we use in linear programming algorithm. Let
  \[ \triangle = \{ x_i \mid x_i \geq 0, \sum x_i = 1 \} \]
  be the simplex used in the algorithm.
- Let \( x, y \epsilon \text{int}(\triangle) \) be two interior points in the simplex.
  Projectively invariant distance between \( x \) and \( y \), based on \( p \)-norms:
  \[ d(x, y) = \frac{1}{2} \left( \sum_i \left( \frac{x_i}{y_i} - \frac{y_i}{x_i} \right)^p \right)^{\frac{1}{p}} \]
  - The infinitesimal version of \( d \) gives the Riemannian metric \( g(x) \) for \( p = 2 \) and Riemann-Finsler metric for \( p > 2 \).

\[ g_{ij}^p(x)dx_i dx_j = \sum_i \left( \frac{dx_i}{x_i} \right)^p \]

The performance of the algorithm depends on curvature in this metric. Karmarkar [4]

- For inequalities in the form \( Ax \leq b \), let \( x, y \) be two interior points, and let \( s, t \) be the corresponding slack variables i.e.
  \[ Ax + s = b, \ Ay + t = b \]
  Then projectively invariant distance between \( x \) and \( y \) is given by
  \[ d^p(x, y) = \sum_i \left[ \frac{1}{2} \left( \frac{s_i}{t_i} - \frac{t_i}{s_i} \right) \right]^p \]
  - Let \( s_0 = \text{slack variable for the current interior point } x_0 \) (constant) and \( s = \text{slack variable for the next (unknown) interior point } x \).
  - Observe the particular “distributive” form of the function \( d^p(x, x_0) \).

  If \( S \) is the set of all slack variables, then \( d^p \) distributes linearly over \( S \).

  \[ d^p(x, x_0) = \sum_{s \epsilon S} \psi(s) \]

  where the \( \psi(s) \) for the individual slack variable is

  \[ \psi(s) = \left( \frac{1}{2} \left( \frac{s}{s_0} - \frac{s_0}{s} \right) \right)^p \]

5.2. Computing exponential sums efficiently.

- Note that the potential function \( \psi \) used in the interior point methods also has the same distributive form,
  where \( \psi \) could be a non-linear rational or transcendental function of individual slack variable.
- We are interested in the case when \( S \) is exponentially large.
- Such sums can be evaluated efficiently for
Large class of problems involving exponential number of inequalities,

- when $\psi(s)$ belongs to certain special parametric family of functions such as exponential function, e.g. $\psi(s) = e^{-zs}$.

- While the actual function $\psi(s)$ of interest is not of this form, it can be expressed as a linear superposition of functions of this form.

- E.g. techniques such as Laplace transform or Fourier transform and their inverse transforms enable expressing $\psi$ as (infinite)superposition of (real or complex) exponentials.

- For approximate cross-simulation on the standard model, we use a suitable finite superposition.

6. Combined Effect of Inequalities for all Odd Cycles in a Graph

6.1. Introduction. Contribution of an individual slack variable $s$: $\psi(s) = e^{-zs}$ where $z \in \mathbb{C}$.

**Goal:** We want to find closed form meromorphic function for total contribution of all slack variables.

**Edge Matrix**

- For a single edge, we have $\frac{w_i + w_j}{2} \leq 0$

  Note: since each node in a cycle has degree 2, we are dividing by 2

  For more general graph minors, the weighting factors are different.

- Introducing slack variable $s_{ij}$, we get $\frac{w_i + w_j}{2} + s_{ij} = 0$. Then,

  $$s_{ij} = \frac{w_i + w_j}{2}, \quad \psi = e^{-z s_{ij}} = e^z \left\{ \frac{w_i + w_j}{2} \right\}$$

- Define edge matrix $A(z, w)$ over the field of meromorphic functions as

  $$A_{ij}(z, w) = \begin{cases} e^z \{w_i + w_j\} & \text{if } (i, j) \text{ is an edge} \\ 0 & \text{otherwise.} \end{cases}$$

- For a single slack variable $s$, there is a relation between the derivative operators $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial z}$

  $$s \frac{\partial \psi}{\partial s} = z \frac{\partial \psi}{\partial z}$$

- For the edge matrix $A(z, w)$, derivative with respect to $w_i$, the co-ordinates of the interior point, are expressed in terms of the "J-products".

  $$\frac{\partial A}{\partial w_p} = \frac{z}{2} \left\{ I \overset{\text{p}}{\circ} A + A \overset{\text{p}}{\circ} I \right\}$$

- This gives an autonomous differential equation for $A$, which allows us to create recurrence relation connecting

  - higher powers of $A$ to lower powers and
  - higher derivatives of $A$ to lower order derivatives.
• These recurrence relations lead to closed form expressions for higher derivatives and higher powers.

6.2. Permitting additional inequalities. We are interested in all odd cycles in the graph.

A odd cycle is a special case of a closed odd walk.

Since a closed odd walk contains an odd cycle as a subgraph, a sharper version of inequality is also valid for closed odd walk.

However this inequality is implied by the odd cycle inequalities.

Therefore, from the point of view of formulation, these additional inequalities are superfluous:

useless → since they are implied by other inequalities but harmless → they are still valid.

However from the point of view of obtaining a closed form expression that can be evaluated in polynomial time, they are essential.

7. Getting the closed form expression for the combined effect of inequalities

7.1. Introduction. Steps:

(1) Get expressions for the effect of the implied inequalities for the following:

(a) For a given walk $W$ of length $l : \psi_W$

(b) For a given pair of nodes $i$ and $j$, all walks of length $l : \psi_{ij}$

(c) In the entire graph, all closed walks of length $l : \psi_l$

(d) In the entire graph, all closed odd walks of length at most $n : \psi$

(2) Transitioning from expressions for implied inequalities to expressions for sharper inequalities for the following:

(a) All closed odd walks of length at most $n$ in the entire graph : $\tilde{\psi}$

7.2. Closed form expression: Step (1a)

. Single walk $W$ of length $l$:

Consider a walk $W$ with $l$ edges $i_1, i_2, \ldots, i_{l+1}$. Summing the edge inequalities, we have

$$\frac{1}{2}w_1 + w_2 + \ldots + w_l + \frac{1}{2}w_{l+1} \leq 0$$

Let $s_w$ be a slack variable for the “implied” inequality for walk $W$.

$$s_w = \sum_{e \in W} s_e$$

$$\psi_W(s_w) = e^{-z s_w} = e^{-z \sum_{e \in W} s_e} = \prod_{e \in W} e^{-z s_e}$$

$$\psi_W = A_{i_1 i_2} A_{i_2 i_3} \cdots A_{i_l i_{l+1}}$$
7.3. Closed form expression: Steps (1b) & (1c)

Between given pair of nodes $i$ and $j$ effect of all walks of length $l$:

Let $i_1 = i$ and $i_{l+1} = j$.

\[ \psi_{ij} = \sum_{i_2, i_3, \ldots, i_l} A_{i_1 i_2} A_{i_2 i_3} \cdots A_{i_l j} \]

\[ \psi_{ij} = A_{ij}^l \]

All closed walks of length $l$ in the entire graph:

\[ \psi_l = \sum_i A_{ii}^l \]

However, a closed walk of length $l$ is counted $2^l$ times in the expression above,

- Due to $l$ vertices of the walk and
- The two senses of traversal along the walk.

Compensating for this repetition, we have

\[ \psi_l = \frac{\text{tr}\{A^l(z, w)\}}{2^l} \]

7.4. Closed form expression: Step (1d)

All closed odd walks of length at most $n$ in the entire graph:

Let $k_{max} = \lfloor \frac{n-1}{2} \rfloor$.

\[
\psi = \sum_{k=1}^{k_{max}} \psi_{2k+1} = \sum_{k=1}^{k_{max}} 2^{k} \cdot (2k+1) \text{tr}\{A^{2k+1}(z, w)\} = \text{tr}\{B(z, w)\}
\]

where

\[ B(z, w) = \sum_{k=1}^{k_{max}} A^{2k+1} \]

7.5. Transitioning from effect of implied inequalities to sharper ones

Let $i_1, i_2, \ldots, i_{l+1}$ be a walk of length $l$; its implied inequality is given by

\[ \frac{w_{i_2} + w_{i_3} + \ldots + w_{i_l} + w_{i_{l+1}}}{2} \leq 0 \]

If $i_{l+1} = i_1$, we have a closed walk with the implied inequality given by

\[ w_{i_1} + w_{i_2} + \ldots + w_{i_l} \leq 0 \]

For closed odd walks ($l = 2k + 1$), we get the following **sharper** inequality.

\[ w_{i_1} + w_{i_2} + \ldots + w_{i_l} \leq -1 \]

since at most $k$ nodes can be in an independent set.
If $s$ is the slack variable for the implied inequality, let $\tilde{s}$ denote the slack variable for the corresponding sharper inequality.

$$\tilde{s} = s - 1$$

$$e^{-z\tilde{s}} = e^z \cdot e^{-s}$$

$$\tilde{\psi}_l = e^z \psi_l$$

7.6. **Closed form expression: Step (2a)**

Transitioning from the expression for implied inequality for a closed odd walk to the sharper inequality involves multiplication by $e^z$, which is independent of the length of the walk.

Therefore, for $l$ odd, $\tilde{\psi}_l$ is given by

$$\tilde{\psi}_l = e^z \cdot \frac{tr\{A^l(z,w)\}}{2l}$$

And $\tilde{\psi}$ is given by

$$\tilde{\psi} = e^z \cdot B(z,w)$$

It is trivial to see that this can be evaluated in polynomial time in the continuum model.

- Straightforward evaluation of $\tilde{\psi}$ as written, will take $O(n^4)$ operations. But with some rearrangement, it is possible to evaluate it with $O(n^3)$ operations.
8. Join Product

8.1. Introduction. We split the tensor product involving contraction into two steps. The first step is similar to join operation on relations in a relational data base. Simple examples of J-products –

1. **Matrix Multiplication**:
   
   \[ C = A \cdot B, \quad C_{pr} = \sum_q A_{pq} B_{qr} \]
   
   Corresponding "join" product (definition):
   
   \[ C_{pqr} = A_{pq} B_{qr} \]
   
   Without the summation (contraction) over q, corresponding "join" product (notation):
   
   \[ C = A \bigotimes_q B \]

2. **Dot Product of vectors**:
   
   \[ \mathbf{c} = \mathbf{a} \cdot \mathbf{b} = \sum_i a_i b^i \]
   
   Corresponding join product (definition):
   
   \[ c_i = a_i b^i \]
   
   Corresponding join product (notation):
   
   \[ \mathbf{c} = \mathbf{a} \bigotimes_i \mathbf{b} \]

3. **Hadamard Product of matrices**:
   
   is an example of J-product with two repeated indices
   
   \[ A_{pq} = B_{pq} C_{pq} \]
   
   \[ A = B \bigotimes_{p, q} C \]

8.2. Tensor contraction

- **Input Tensors**: A, B, with repeated covariant and contravariant indices, and
- **Output Tensor**: C with summation over repeated indices \( r_1, r_2, \ldots, r_m \).

\[ C_{p_1 p_2 \ldots p_k q_1 q_2 \ldots q_l t_1 t_2 \ldots t_u} = \sum_{r_1 r_2 \ldots r_m} A_{p_1 p_2 \ldots p_k q_1 q_2 \ldots q_l r_1 r_2 \ldots r_m} B_{r_1 r_2 \ldots r_m s_1 s_2 \ldots s_n} \]

If there are m repeated indices, rank(C) = rank(A) + rank(B) - 2m

Corresponding **join operator**:
There is no summation over repeated indices

One copy of such indices is present in the output, enclosed in round ()

They don’t have significance as tensor indices but simply denote an indexed family of tensors of the same rank as above

\[
[C_{p_1 p_2 ... p_k s_1 s_2 ... s_n}^{q_1 q_2 ... q_l t_1 t_2 ... t_u}]_{r_1 r_2 ... r_m} = A_{p_1 p_2 ... p_k}^{q_1 q_2 ... q_l} B_{t_1 t_2 ... t_u}^{r_1 r_2 ... r_m s_1 s_2 ... s_n}
\]

Notation :

\[C = A \circ B\]

Sometimes, the repeated indices are noted below \(\circ\) as shown :

\[C = A \circ \circ_{r_1 r_2 ... r_m} B\]

Tensor Product as composition of join and summation operation :

\[A.B = \sum_{r_1 r_2 ... r_m} A \circ \circ_{r_1 r_2 ... r_m} B\]

8.3. Properties of join product.

- Linear in both arguments :
  
  If \(\alpha\) and \(\beta\) are scalars
  
  \[\{\alpha A + \beta B\} \circ C = \alpha(A \circ C) + \beta(B \circ C)\]
  
  If \(A, B, X\) are matrices of compatible dimensions,
  
  \[X(A \circ B) = \{XA\} \circ B\]
  
  \[(A \circ B)X = A \circ \{BX\}\]
  
  Similar rule applies for general compatible tensor multiplication.

- Associative :

  \[A \circ \circ \circ_{\circ \circ} B \circ \circ \circ_{\circ \circ} C = A \circ \circ \circ_{\circ \circ} \circ \circ \circ_{\circ \circ} B \circ \circ \circ_{\circ \circ} \circ \circ \circ_{\circ \circ} C\]

- Derivative rule :

  \[
  \frac{\partial}{\partial x_i} \{A(x_1 x_2...x_n) \circ B(x_1 x_2...x_n)\} = \frac{\partial A}{\partial x_i} \circ B + A \circ \frac{\partial B}{\partial x_i}
  \]

- Transpose :

  \[C_{ij}^T = C_{kji}\]
\[ [A \odot B]^T = [B^T \odot A^T] \]

For symmetric A, B

\[ [A \odot B] = [B \odot A] \]

- Transpose of triple product:

\[ [A \odot B \odot C]^T = C^T \odot B^T \odot A^T \]

Note the reversal of p, q

8.4. Application of J-product in the present context. Derivative of Edge Matrix w.r.t. co-ordinates of interior point can be expressed in terms of J-product, which reduces derivative to an algebraic operation:

\[ \frac{\partial A}{\partial w_p} = \frac{z}{2} \left\{ I \odot_{q} A + A \odot_{q} I \right\} \]

Recurrence relation for derivative of kth power of the Edge Matrix in terms of J-products of lower powers:

\[ \frac{\partial A^k}{\partial w_p} = \frac{z}{2} \text{sym} \left\{ \sum_{i=1}^{k} A^i \odot_{q} A^{k-i} \right\} \]

Recurrence relation for kth derivative of the Edge Matrix in terms of derivatives of lower order e.g. second derivative in terms of first derivative:

\[ \frac{\partial^2 A}{\partial w_p \partial w_q} = \frac{z}{2} \left\{ \frac{\partial A}{\partial w_p} \odot_{q} I + I \odot_{q} \frac{\partial A}{\partial w_p} \right\} \]

8.5. Solution of recurrence relations for higher derivatives.

\[ \frac{\partial A^k}{\partial w_p} = \frac{z}{2} \left\{ W_{\text{rank}(\alpha)}^{k,1} A_{\alpha_1} \odot_{p} A_{\alpha_2} \right\} \]

\[ \frac{\partial^2 A^k}{\partial w_p \partial w_q} = \frac{z^2}{4} \cdot \frac{1}{2!} \left\{ W_{\text{rank}(\alpha)}^{k,2} \left[ A_{\alpha_1} \odot_{p} A_{\alpha_2} \odot_{q} A_{\alpha_3} \right] + A_{\alpha_1} \odot_{q} A_{\alpha_2} \odot_{p} A_{\alpha_3} \right\} \]

where

\[ \alpha = (\alpha_1, \alpha_2) \quad \alpha_1 + \alpha_2 = k, \alpha_i \geq 0 \text{ for (1)} \]

\[ \alpha = (\alpha_1, \alpha_2, \alpha_3) \quad \alpha_1 + \alpha_2 + \alpha_3 = k, \alpha_i \geq 0 \text{ for (2)} \]

\[ \text{rank}(\alpha) = \text{number of non-zero } \alpha'_i \text{s} \]

\[ W_{\text{rank}(\alpha)}^{k,m} = \text{fixed weights depending on rank} \]

Similar formulas hold for higher derivatives.
9. **Approximate Evaluation on the Standard Model**

- We want to illustrate how a function of distributive type over the set of all slack variables can be approximated by superposition of exponentials, for various \( \psi(s) \).
- For simplicity of exposition consider

\[
\psi(s) = \frac{1}{s}, \ s > 0
\]

\[
\frac{1}{s} = \int_0^\infty e^{-sx} \, dx
\]

- In the problem of interest, all slack variable lie in a bounded interval of real axis.
- Upper bound: trivial since \(-1 \leq w_i \leq 1\). Hence \( |a^T w| < |a|_1 \leq n \) for closed walks of length at most \( n \).
- Lower bound: at each iteration we round the interior point to nearest valid \( \pm 1 \) solution by a simple method, until optimal solution is identified.
- Hence the minimum value of the slack variable remains bounded away from zero.

Consider the following nested regions

\[
R_i = \{ x \in \mathbb{R} \mid e^{-i} \leq x \leq e^{j_{\text{max}}} \} \text{ for } i = 1, \ldots, L
\]

so that

\[
R_1 \subset R_2 \subset R_3 \cdots \subset R_L
\]

Note: \( L \) is not known in advance; we increment it dynamically based on the number of iterations so far.

Projective invariance of the algorithm implies invariance w.r.t. simple uniform scaling transformations \( T_k : s \rightarrow e^k s, \ k \in \mathbb{Z} \).

To get an efficient approximation exploiting the scale invariance, we make further substitution

\[
x = e^{-at}
\]

in the integral for \( \frac{1}{s} \),

\[
\frac{1}{s} = \int_{-\infty}^{\infty} a \cdot e^{-at} \cdot e^{-[e^{-at}, s]} \, dt
\]

Approximate the integral by sum, with integer nodes in a bounded range \([-m, M]\), \( m, M \in \mathbb{N} \).

\[
\frac{1}{s} \sim \sum_{i=-m}^{i=M} a \cdot e^{-ia} \cdot e^{-[e^{-ia}, s]}
\]

Let \( \lambda_i = e^{-ia} \)

\[
\frac{1}{s} \sim \sum_{i=-m}^{i=M} a \cdot \lambda_i \cdot e^{-\lambda_i s} \ (*)
\]

As an example, suppose the region of interest is

\[
e^{-30} < s < e^1
\]
Taking \( a = \frac{1}{3}, \quad m = \frac{1}{6} = 2, \quad M = \frac{30}{a} = 60 \), we have 63 terms and this gives the l.h.s. and r.h.s. of \((*)\) which are close when evaluated in standard double precision (64-bit) arithmetic.

10. Superposition of Exponentials

- By similar techniques approximation to the function \( \psi(s) \) of interest is expressed as sum of exponential
  \[
  \psi(s) = \sum_i c_i e^{-\lambda_i s}
  \]
- The main function \( \phi \) (either metric or potential) is expressed in terms of \( \psi(s) \) in distributive form
  \[
  \phi = \sum_{s \in S} \psi(s),
  \]
  \[
  = \sum_{s \in S} \sum_i c_i e^{-\lambda_i s}
  \]
  \[
  = \sum_i c_i \left\{ \sum_{s \in S} e^{-\lambda_i s} \right\}
  \]
  \[
  = \sum_i c_i \cdot e^z \cdot \text{tr} \{ B(z) \} \big|_{z = \lambda_i}
  \]
- Note that the number of terms is \( O(L) \), and evaluation of each term is polynomial in \( n \).

11. An example of application to non-convex, non-linear problem

- Consider set of inequalities of the form
  \[
  (a_i^T x)^2 - (b_i^T x)^2 \leq c, \quad c > 0
  \]
  The set defined by these inequalities is \((2, 2)\)-connected. Karmarkar [2] slack variable \( s_i = c - (a_i^T x)^2 + (b_i^T x)^2 \)
- With the function \( \psi(s) : \mathbb{R} \to \mathbb{R} \) of interest, we associate another function \( \tilde{\psi} : \mathbb{R}^2 \to \mathbb{R} \) as follows
  \[
  \tilde{\psi}(u, v) = \psi(c - u^2 + v^2)
  \]
  Let \( R \) be the region of interest in the \((u, v)\) plane.
  Note: due to the symmetries based on flipping signs of \( u \) and \( v \), we can consider only one quarter of the plane.
- Let \( \chi_R : \text{Characteristic function of } R \). By considering 2-D inverse Laplace transform of \( \tilde{\chi}_R \tilde{\psi}(u, v) \), (or by other means) we can obtain finite approximation based on exponentials functions in the plane
  \[
  \chi_R \tilde{\psi} \sim \sum_{(\mu_k, \lambda_k)} c_k e^{-[\mu_k u + \lambda_k v]}
  \]
- Since \( u = a_i^T x \) and \( v = b_i^T x \) are linear in \( x \), the above expression is of the same type as considered before.
12. Extension to the satisfiability problem

12.1. Introduction to SAT.
- We have boolean variables $x_1, \ldots, x_n$.
- A literal $d$ is a variable or its complement: $d = x$ or $d = \bar{x}$.
- A clause is an OR of literals. Eg. $C = x \lor y \lor \bar{z}$. In 3-SAT, each clause has 3 literals.
- A satisfiability formula $f = C_1 \land C_2 \land \ldots \land C_m$.
- Problem (SAT): Find truth assignments to the variables $x_1, \ldots, x_n$ such that $f$ is true, or prove that no such assignment exists.

12.2. Generation of implied clauses.
- Consider the following pair of clauses
  $C_1 = a \lor b \lor \bar{c}$
  $C_2 = c \lor d \lor e$
- Together they imply the clause $C_3 = a \lor b \lor d \lor e$.
- We refer to this operation as the "join" operation on clauses.
- This operation is fundamental in resolution or elimination methods for solving the SAT problem.
- It is known that the number of such new clauses generated during the resolution process grows exponentially with $n$ for almost all instances of the problem with parameters in a certain range Chvátal and Szemerédi [1].

12.3. Continuum based approach.
- Associate real variables taking ±1 values with the boolean variables.
- Embedding the problem in $\mathbb{R}^n$, allow the variables to take on values in the interval $[-1, 1]$, i.e. $-1 \leq x_i \leq 1$.
- For a literal $l$,
  \[ v(l) = \begin{cases} 
  x & \text{if } l = x \\
  -x & \text{if } l = \bar{x} 
  \end{cases} \]
- Each clause corresponds to an inequality
  \[ C = l_1 \lor l_2 \lor l_3 \rightarrow v(l_1) + v(l_2) + v(l_3) \geq -1 \]
  Explanation: Sum of three ±1 variables can be -3, -1, 1 or 3 of which the value -3 is forbidden since it corresponds to all literals being false.

12.4. Interpreting the join operation in the continuum model

| Clauses | Inequalities |
|---------|-------------|
| $a \lor b \lor \bar{c}$ (C1) | $a + b - c \geq -1$ (1) |
| $c \lor d \lor e$ (C2) | $c + d + e \geq -1$ (2) |
| Join: $a \lor b \lor d \lor e$ (C3) | $a + b + d + e \geq -2$ (3) |

- Observe that the inequality (3) corresponding to the newly generated clause is just the sum of (1) and (2).
- Imposition of constraint (3) does not change the feasible set, hence it is superfluous.
- This is the first reason why the continuum approach is more economical than resolution.
12.5. **A thought experiment.**

- Run the resolution algorithm.
- For each new clause generated, ask if the corresponding inequality is superfluous or essential (i.e. not derivable as a non-negative combination of the previous inequalities).
- Unless the previous inequalities have only vectors with all ±1 co-ordinates as extreme points, an essential constraint must get generated during the course of the algorithm since resolution is a complete method.
- This leads us to the following question: Which sequence of clauses when joined, yields an essential constraint, and is such that no subsequence has the same property?
- To understand this, we introduce the concepts of paths, walks, cycles, etc. formed by subformulas in a SAT problem.

13. **Paths, Cycles and Mobius cycles**

13.1. **(Open) Path.**

- Consider a sequence of clauses of the following type:
  \[ l_1 \lor l_2 \lor \bar{l}_3, \ l_3 \lor l_4 \lor \bar{l}_5, \ l_5 \lor l_6 \lor \bar{l}_7, \ldots, l_{2k-1} \lor l_{2k} \lor \bar{l}_{2k+1} \]
  where each \( l_i \) is a literal based on a distinct variable.
- For any two consecutive clauses in the sequence, the last literal of the earlier clause and the first literal of the latter clause are complementary.
- The above sequence yields the following joined clause:
  \[ l_1 \lor l_2 \lor l_4 \lor l_6 \lor \ldots \lor l_{2k} \lor \bar{l}_{2k+1} \]
  and the corresponding inequality is superfluous.

13.2. **(Ordinary) Cycle.**

- If \( l_{2k+1} = l_1 \) above, then we can join the two ends by normal rules of joining, but the joined clause of the sequence is a tautology (always true) due to the presence of \( l_1 \) and \( \bar{l}_1 \).

13.3. **Mobius cycles.**

- Consider the above sequence of clauses again:
  \[ l_1 \lor l_2 \lor \bar{l}_3, \ l_3 \lor l_4 \lor \bar{l}_5, \ l_5 \lor l_6 \lor \bar{l}_7, \ldots, l_{2k-1} \lor l_{2k} \lor \bar{l}_{2k+1} \]
- If \( l_{2k+1} = \bar{l}_1 \), then the joined clause of the sequence contains two copies of \( l_1 \), but we need only one. Hence, the joined clause is equivalent to \( l_1 \lor l_2 \lor l_4 \lor l_6 \lor \ldots \lor l_{2k} \).
- Corresponding inequality
  \[ v(l_1) + v(l_2) + v(l_4) + \ldots + v(l_{2k}) \geq -k + 1 \]
  where
  \[ v(l) = \begin{cases} 
  x & \text{if } l = x \\
  -x & \text{if } l = \bar{x}
  \end{cases} \]
- This is sharper than the inequality implied by the sum of the constituent inequalities, namely
  \[ 2 \cdot v(l_1) + v(l_2) + v(l_4) + \ldots + v(l_{2k}) \geq -k \]
• We call the cycles of the kind referred to above as “mobius cycles”.
• The mobius cycles are the only mechanism giving rise to new sharper inequalities.
• If there were no mobius cycles, then solving the L.P. corresponding to the SAT problem is sufficient to solve the latter.
• In other words, the special classes of SAT formulas in which mobius cycles are forbidden as subformulas, can be solved in polynomial time.

14. Computing the effect of mobius cycles in the continuum model
• The total number of mobius cycles in the original formula can be exponentially large, but we can compute their net effect in polynomial time.
• The method is similar to the one used for odd cycles in the maximum independent set problem, except that we construct a directed graph as explained below.

2n nodes corresponding to the literals \( x_1, \overline{x_1}, x_2, \overline{x_2}, \ldots, x_n, \overline{x_n} \).
• Let \( C = l_1 \lor l_2 \lor l_3 \) be a clause.
• Inequality corresponding to this clause
  \[ v(l_1) + v(l_2) + v(l_3) \geq -1 \]
• Slack variable \( s = 1 + v(l_1) + v(l_2) + v(l_3) \)
  \[ \psi_C(s) = e^{-z \cdot s} = e^{-z \cdot (v(l_1) + v(l_2) + v(l_3))} \]
• Corresponding to \( C \), put the following 6 edges in the graph.
  \((l_1, \overline{l_2}), (l_1, \overline{l_3}), (l_2, \overline{l_3})
  (\overline{l_1}, l_2), (\overline{l_1}, l_3), (\overline{l_2}, l_3)\)
• Each of the 6 edges corresponding to clause \( C \) is labelled with \( \psi_C(s) \).
• Construct a “clause matrix” \( A(z, x) \)

\[
A_{ij} = \begin{cases} 
\text{sum of all } \psi_C(s) \text{ of all parallel edges} & \text{if any} \\
0 & \text{otherwise}
\end{cases}
\]

• Note that the clause matrix is not symmetric
  \( A(i, j) \neq A(j, i) \)
but it has another kind of symmetry
  \( A(i, j) = A(\overline{j}, \overline{i}) \)

15. Closed form expression for combined effect of all mobius cycles
• As before, we permit self-intersecting, directed, closed walks and thereby include the effect of “useless but harmless” inequalities. This leads to closed form expression for the combined effect as before.
• Consider a mobius cycle \( l_1 \lor l_2 \lor \overline{l_3}, l_3 \lor l_4 \lor \overline{l_5}, l_5 \lor l_6 \lor \overline{l_7}, \ldots, l_{2k-1} \lor l_{2k} \lor l_1 \)
Let $s$ and $\tilde{s}$ be the slack variables for the implied and sharper inequalities respectively.

\[
s = k + 2 \cdot v(l_1) + v(l_2) + v(l_4) + \ldots + v(l_{2k})
\]

\[
\tilde{s} = k - 1 + v(l_1) + v(l_2) + v(l_4) + \ldots + v(l_{2k})
\]

\[
s - \tilde{s} = 1 + v(l_1)
\]

\[
e^z(s - \tilde{s}) = e^z[1 + v(l_1)]
\]

\[
\psi(\tilde{s}) = e^{-z} \cdot e^z[1 + v(l_1)] = e^z[1 + v(l_1)] \cdot \psi(s)
\]

The factor $e^z[1 + v(l_1)]$ is associated with a special edge from $l_1$ to $l_1$. Traversing this edge differentiates mobius cycles from ordinary cycles.

Construct “mobius completion matrix” $M_c$ from special edges as follows:

\[
M_C(z, x)_{i, \bar{i}} = e^{z[1 - x_i]}
\]

\[
M_C(z, x)_{i, i} = e^{z[1 + x_i]}
\]

\[
M_C(z, x)_{i, j} = 0 \text{ if } j \neq \bar{i} \{\text{Note:} \bar{i} = i\}
\]

Define walk matrix $B(z, x)$ as before, except that we include all walks (odd and even) up to length $k_{\max}$, where $k_{\max} =$ number of clauses.

\[
B(z, x) = \sum_{k=2}^{k_{\max}} A^k(z, x) \frac{2k}{k_{\max}}
\]

For each walk, there is a “mirror-image” walk obtained by complementing all the nodes in the walk, hence factor of 2 in the denominator.

Multiplication by $M_c$ also achieves the transition from implied to sharper inequalities. The combined effect is given by

\[
\phi(z, x) = tr\{M_c(z, x) \cdot B(z, x)\}
\]

Lengths of mobius cycles have exponential effect on resolution. In contrast, we are able to include the effect of mobius cycles of all lengths in polynomial number of operations.

The remaining part of computation is similar to the case of maximum independent set problem.

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