LIE-HOPF ALGEBRAS AND LOOP HOMOLOGY OF SUSPENSION SPACES

VICTOR BUCHSTABER AND JELENA GRBIĆ

ABSTRACT. For an arbitrary topological space $X$, the loop space homology $H_*(\Omega \Sigma X; \mathbb{Z})$ is a Hopf algebra. We introduce a new homotopy invariant of a topological space $X$ taking for its value the isomorphism class (over the integers) of the Hopf algebra $H_*(\Omega \Sigma X; \mathbb{Z})$. This invariant is trivial if and only if the Hopf algebra $H_*(\Omega \Sigma X; \mathbb{Z})$ is isomorphic to a Lie-Hopf algebra, that is, to a primitively generated Hopf algebra. We show that for a given $X$ these invariants are obstructions to the existence of a homotopy equivalence $\Sigma X \simeq \Sigma^2 Y$ for some space $Y$. We further investigate relations between this new invariant and well known classical invariants and constructions in homotopy theory.

1. INTRODUCTION

One of the classical problems in homotopy theory is the Milnor double suspension problem, posed in 1961, in which he asked whether the double suspension of a homology sphere is homeomorphic to the standard sphere. This problem can be generalised in homotopy theory to the problem of describing all spaces which after being suspended twice become a triple suspension space.

In this paper we want to study those topological spaces $X$ which after being suspended once become double suspension spaces, that is, $\Sigma X \simeq \Sigma^2 Y$ for some $Y$. Our approach is similar in nature to the one used in the study of $H$-spaces in the sense that we explore certain properties of naturally arising Hopf algebras.

A pointed topological space $X$ is an $H$-space if there is a continuous map $\mu: X \times X \longrightarrow X$ called multiplication such that the base point acts as a left and right unit. $H$-spaces appear in mathematics all the time. The most classical examples are topological groups: spaces $X$ with a group structure such that both the multiplication map $\mu: X \times X \longrightarrow X$ and the inversion map $X \longrightarrow X, x \mapsto x^{-1}$, are continuous. Other examples are based loop spaces, Eilenberg-MacLane spaces, finite $H$-spaces such as Lie groups, and the 7 sphere, the homotopy fibre of an $H$-map and so on.

A significant breakthrough in the study of $H$-spaces was achieved by Hopf and Borel (see [H, B]) who classified graded Hopf algebras which can be realised as the cohomology rings of topological spaces, and consequently gave a necessary condition on a Hopf algebra to be the cohomology algebra of an $H$-space. They showed that the existence of the comultiplication in a Hopf algebra turns out to restrict the multiplicative structure considerably. Using this Hopf algebra classification, it is easy to see that, for example, $\mathbb{C}P^n$ is not an $H$-space. The Hopf and Borel theorems give necessary but not sufficient conditions for a space to be an $H$-space; for example, although the cohomology of $BU(2)$ satisfies the conditions of the Hopf and Borel theorems, $BU(2)$ is not an $H$-space. It is worth noting that Hopf and Borel studied possible multiplications of appropriate Hopf algebras and their results are based on algebra isomorphisms of Hopf algebras while ignoring their coalgebra structures.

There has been an extensive development of the theory of graded connected Hopf algebras (see [MM]) using algebraic and topological tools. Let $A$ be a Hopf algebra. Denote by $P(A)$ the primitives in $A$, by $I(A)$ the augmentation ideal of $A$, and by $Q(A) = I(A)/I(A)^2$ the indecomposables. Then $A$ is primitively generated if the natural map $P(A) \longrightarrow I(A) \longrightarrow Q(A)$ is onto. Such a primitively generated Hopf algebra $A$ is called Lie-Hopf algebra.

Let $R$ be a principal ideal domain and assume we work with a topological space $X$ such that $H^*(X \times X; R) \cong H^*(X; R) \otimes H^*(X; R)$. The diagonal map on $X$ induces the cup product in $H^*(X; R)$ and dually it is closely related to the comultiplication in the Hopf algebra $H_*(\Omega \Sigma X; R)$, presenting in certain ways obstructions to $H_*(\Omega \Sigma X; R)$ be a Lie-Hopf algebra.

Utilising the theory of graded connected Hopf algebras, in particular by studying the properties of comultiplications, we look for a necessary condition for a topological space $X$ to have the property that $\Sigma X \simeq \Sigma^2 Y$ for some $Y$. Our main result is the following theorem.

2000 Mathematics Subject Classification. Primary 55T25, 16T05; Secondary 55P35, 55P40, 57T05.
Key words and phrases. Hopf algebra, Lie-Hopf algebra, loop suspension spaces, double suspension spaces, quasisymmetric functions.
Theorem 1.1. Let $X$ be a space such that $\Sigma X \simeq \Sigma^2 Y$ for some $Y$ and assume that $H_*(X; \mathbb{Z})$ is torsion free. Then the Hopf algebra $H_*(\Omega \Sigma X; \mathbb{Z})$ is isomorphic to a Lie-Hopf algebra.

As will be discussed in Sections 4 and 5, spaces with the property that $\Sigma X \simeq \Sigma^2 Y$ are not rare; they appear, for example, in the context of polyhedral product functors - which are functorial generalisations of moment-angle complexes, complements of hyperplane arrangements, some simply connected 4-manifolds and so on.

Corollary 1.2. Let $Z_K$ be a moment-angle complex. Then the comultiplication in homology, induced by the ring structure of $H^*(Z_K; \mathbb{Z})$, defines a Hopf algebra structure on the tensor algebra $T(H_*(Z_K; \mathbb{Z}))$ which over $\mathbb{Z}$ is isomorphic to a Lie-Hopf algebra.

The strength and beauty of this approach lies in the strong connection between algebra and topology. In Section 6 we detect some properties of the algebra of quasi-symmetric functions by relating it to $H_*(\Omega \Sigma CP^\infty; R)$ and by applying homotopy theoretic decomposition methods.

Let us assume that $\Sigma X \simeq \Sigma Y$ is not imbedded as the equator in $S^2$. Then the quotient space $\Sigma X/\Sigma Y$ for some topological space $Y$.

Milnor considered a homology 3-sphere $M^3$ with $\pi_1(M) \neq 0$ and asked whether the double suspension of $M^3$ is homeomorphic to $S^0$. This was partially proved in 1975 by Edwards [E], and in a sharper form in 1979 by Cannon [C]. It provided the first example of a triangulated manifold which is not locally PL-homeomorphic to Euclidean space. The solution of the Milnor double suspension problem provides us with a non-trivial example of an $S^n$-space.

It is seen readily from the definition that if $X$ is an $S^n$-space, then it is also an $S^{n+1}$-space. Thus a trivial example of $S^n$-spaces is given by a suspension space $X$, that is, $X \simeq \Sigma Y$ for some topological space $Y$.

Returning to topology, our invariant allows us to observe some interesting properties of the cup product in the cohomology of spaces $X$ such that $\Sigma X \simeq \Sigma^2 Y$ for some $Y$. Namely, knowing that $H_*(\Omega \Sigma X; R)$ is isomorphic to a Lie-Hopf algebra we are saying that there is a change of basis in $H_*(X; R)$ such that the reduced diagonal $\Delta : H_*(X; R) \to H_*(X; R) \otimes H_*(X; R)$ becomes trivial, or dually that the cup product in the new basis is trivial.

2. $S^n$-spaces.

We start by introducing the notion of $S^n$-spaces.

**Definition 2.1.** For a given $n \in \mathbb{N} \cup \{0\}$, a CW complex $X$ is said to be an $S^n$-space if there is a CW complex $Y$ such that $\Sigma^n X \simeq \Sigma^{n+1} Y$.

It is seen readily from the definition that if $X$ is an $S^n$-space, then it is also an $S^{n+1}$-space. Thus a trivial example of $S^n$-spaces is given by a suspension space $X$, that is, $X \simeq \Sigma Y$ for some topological space $Y$.

Milnor considered a homology 3-sphere $M^3$ with $\pi_1(M) \neq 0$ and asked whether the double suspension of $M^3$ is homeomorphic to $S^0$. This was partially proved in 1975 by Edwards [E], and in a sharper form in 1979 by Cannon [C]. It provided the first example of a triangulated manifold which is not locally PL-homeomorphic to Euclidean space. The solution of the Milnor double suspension problem provides us with a non-trivial example of an $S^2$-space. In general, Cannon [C] proved that the double suspension $\Sigma^2 S^n_H$ of any homology $n$-sphere $S^n_H$ is homeomorphic to the topological sphere $S^{n+2}$, therefore showing that a homology sphere is an $S^2$-space.

Our study of $S^n$-spaces begins by detecting operations under which the family of $S^n$-spaces is closed.

If we start with a topological pair $(X_1, X_2)$ of $S^n$-spaces, a natural question to ask is whether the quotient space $X_1/X_2$ is an $S^n$-space. The pair $(S^k, S^{k-1})$ where $S^{k-1}$ is imbedded as the equator in $S^k$ gives one of the simplest examples where the quotient $S^k/S^{k-1} \simeq S^k \vee S^k$ is an $S^n$-space for $k \geq 1$.

**Lemma 2.2.** Let $(X_1, X_2)$ be a topological pair of $S^n$-spaces. Then the quotient space $X_1/X_2$ is not in general an $S^n$-space.

**Proof.** We prove the lemma by constructing an example. Let $E \to S^2$ be a disc $D^2$ bundle associated to the Hopf bundle over $S^2$. Then $\partial E = S^3$ and $E/\partial E = \mathbb{C}P^2$. In this way we have constructed a topological pair $(E, \partial E)$ of $S^1$-spaces such that $E/\partial E$ is not an $S^1$-space.

There are several operations under which the property of being an $S^n$-space is preserved.

**Lemma 2.3.** Let $X_1, \ldots, X_k$ be $S^n$-spaces. Then the following spaces are $S^n$-spaces:

1. $X_1 \vee \cdots \vee X_k$,
2. $X_1 \wedge \cdots \wedge X_k$,
3. $X_1 \times \cdots \times X_k$.

**Proof.** Let us assume that $\Sigma^n X_i \simeq \Sigma^{n+1} Y_i$ for some $Y_i$ and $i = 1, \ldots, k$.

1. (1) The following homotopy equivalences

   $$\Sigma^n (X_1 \vee \cdots \vee X_k) \simeq \Sigma^n X_1 \vee \cdots \vee \Sigma^n X_k \simeq \Sigma^{n+1} (Y_1 \vee \cdots \vee Y_k)$$

   give statement (1).
(2) Statement (2) is proved by the following homotopy equivalences

\[ \Sigma^n(X_1 \wedge \ldots \wedge X_k) \simeq \Sigma^{n+1}Y_1 \wedge X_2 \wedge \ldots \wedge X_k \simeq \Sigma Y_1 \wedge \Sigma^{n+1}Y_2 \wedge \ldots \wedge X_k \simeq \Sigma^{n+k}Y_1 \wedge \ldots \wedge Y_k. \]

(3) Notice that \( \Sigma(X_1 \times X_2) \simeq \Sigma X_1 \vee \Sigma X_2 \vee \Sigma(X_1 \wedge X_2). \) Now by statements (1) and (2), \( \Sigma^n(X_1 \times X_2) \simeq \Sigma^nX_1 \vee \Sigma^nX_2 \vee \Sigma^n(X_1 \wedge X_2) \simeq \Sigma^{n+1}Y_1 \vee \Sigma^{n+1}Y_2 \vee \Sigma^{n+2}(Y_1 \wedge Y_2). \) The proof of statement (3) now follows by induction on \( k. \)

A generalisation of Lemma 2.3 can be given in terms of the Whitehead filtration of \( X_1 \times \ldots \times X_k. \) For \( k \) pointed topological spaces \( (X_i, *) \), the Whitehead filtration \( T_k \) is defined in the following way

\[ T_k = \{(x_1, \ldots, x_k) \mid \text{where } x_i = * \text{ for at least } l \text{ coordinates}\}. \]

Notice that \( T_k \simeq X_1 \times \ldots \times X_k, T_{k-1} \) is known as the fat wedge, \( T_k \simeq X_1 \times \ldots \times X_k, \) and \( T_0 = \ast. \)

Lemma 2.4. Let \( X_1, \ldots, X_k \) be \( S^n \)-spaces. Then \( T_k \) is an \( S^n \)-space for \( 0 \leq l \leq k. \)

Proof. We prove the lemma by induction on \( l. \) The statement for \( l = 1 \) is true by Lemma 2.3(1). Let us assume that \( T_k \) is an \( S^n \)-space. Notice that \( T_k \) includes in \( T_{k-1} \) for \( 0 \leq l \leq k-1 \) and that there is a cofibration sequence

\[ T_k \rightarrow T_{k-1} \rightarrow \bigvee_{1 \leq i_1 < \ldots < i_{k-1} \leq k} X_{i_1} \wedge X_{i_2} \wedge \ldots \wedge X_{i_{k-1}} \]

which after being suspended becomes trivial, that is,

\[ \Sigma T_k \simeq \Sigma T_{k-1} \bigvee \left( \bigvee_{1 \leq i_1 < \ldots < i_{k-1} \leq k} \Sigma X_{i_1} \wedge X_{i_2} \wedge \ldots \wedge X_{i_{k-1}} \right). \]

Now by the induction hypothesis and Lemma 2.3(2), we get that \( T_{k-1} \) is an \( S^n \)-space, which proves the lemma. \( \Box \)

Lemma 2.5. Let \( X \) be an \( S^n \)-space. Then

(1) the space \( \Omega \Sigma^nX \) is an \( S^1 \)-space,

(2) the space \( \Omega \Sigma X \) is an \( S^n \)-space.

Proof. Both statements are direct corollaries of the James splitting \( \Pi. \) \( \Omega \Sigma \Omega Z \simeq \bigvee_{k=1}^{\infty} \Sigma Z^{(k)}, \) where \( Z^{(k)} \) denotes the \( k \)th fold smash product. \( \Box \)

The above elementary operations over \( S^n \)-spaces already indicate that the class of \( S^n \)-spaces is wide.

In the remainder of the paper, of special interest to us will be the case \( n = 1. \) We show that the class of \( S^1 \)-spaces is unexpectedly large by describing non-trivial homotopy theoretic constructions which produce \( S^1 \)-spaces.

3. APPLICATIONS OF THE THEORY OF HOPF ALGEBRAS TO \( S^1 \)-SPACES

We start by recalling some fundamental definition of the theory of Hopf algebras. Let \( R \) be a commutative ring.

Definition 3.1. An \( R \)-algebra \((A, m_A, \eta_A)\) is a Hopf algebra if it has an additional structure given by \( R \)-algebra homomorphisms: \( \Delta_A: A \rightarrow A \otimes_R A \) called comultiplication; \( \epsilon_A: A \rightarrow R \) called counit, and an \( R \)-module homomorphism \( S_A: A \rightarrow A \) called antipode that satisfy the following properties

(1) coassociativity:

\[ (\text{Id}_A \otimes \Delta_A) \Delta_A = (\Delta_A \otimes \text{Id}_A) \Delta: A \rightarrow A \otimes A \otimes A; \]

(2) counitarity:

\[ m_A(\text{Id}_A \otimes \epsilon_A) \Delta_A = \text{Id}_A = m_A(\epsilon_A \otimes \text{Id}_A) \Delta_A; \]

(3) antipode property:

\[ m_A(\text{Id}_A \otimes S_A) \Delta_A = \eta_A \epsilon_A = m_A(S_A \otimes \text{Id}_A) \Delta \]

where \( m_A: A \otimes A \rightarrow A \) is the multiplication in \( A \) and \( \eta_A \) is the unit map.

Definition 3.2. Two Hopf algebras \((A, m_A, \Delta_A, \epsilon_A, \eta_A, S_A)\) and \((B, m_B, \Delta_B, \epsilon_B, \eta_B, S_B)\) over \( R \) are isomorphic if there is an algebra homomorphism \( f: (A, m_A, \eta_A) \rightarrow (B, m_B, \eta_B) \) satisfying

i) \( (f \otimes f) \circ \Delta_A = \Delta_B \circ f. \)
ii) $f \circ S_A = S_B \circ f$.

iii) $\epsilon_A = \epsilon_B \circ f$.

Note that $f$ uniquely determines the Hopf algebra structure of $B$ if the Hopf algebra structure of $A$ is given.

**Definition 3.3.** A Hopf algebra $A$ is called a Lie-Hopf algebra if the set of multiplicative generators $\{a_i\}_{i \in \mathbb{N}}$ comprises primitives, that is,

$$\Delta a_i = a_i \otimes 1 + 1 \otimes a_i \text{ for every } i.$$

Let $R$ be a PID and let $X$ be a connected space such that $H_\ast(X; R)$ is torsion free. Then the Bott-Samelson theorem [BS] asserts that the homology $H_\ast(\Omega \Sigma X; R)$ is isomorphic as an algebra to the tensor algebra $T(\tilde{H}_\ast(X; R))$ and the adjoint of the identity on $\Sigma X$, $E: X \to \Omega \Sigma X$ induces the canonical inclusion of $H_\ast(X; R)$ into $T(\tilde{H}_\ast(X; R))$.

This tensor algebra can be given a structure of a Hopf algebra if we take the comultiplication on $\tilde{H}_\ast(X; R)$ induced by the diagonal $\Delta_X: X \to X \times X$ and extend it multiplicatively.

**Proposition 3.4.** The above Hopf algebra structure on $T(\tilde{H}_\ast(X; R))$ coincides with the Hopf algebra structure of $H_\ast(\Omega \Sigma X; R)$ where the comultiplication is induced by the diagonal map $\Delta_{\Omega \Sigma X}: \Omega \Sigma X \to \Omega \Sigma X \otimes \Omega \Sigma X$.

**Proof.** For any $H$-space $H$ with multiplication $\mu_H$, we have the following commutative diagram

$$
\begin{array}{c}
H \times H \xrightarrow{\mu_H} H \\
\downarrow \Delta_H \times Id \downarrow \\
H \times H \times H \xrightarrow{\text{Id} \times T \times \text{Id}} H \times H \times H \xrightarrow{\mu_H \times \mu_H} H \times H
\end{array}
$$

where $T$ is the twist map. Since the composite $(\mu_H \times \mu_H) \circ (\text{Id} \times T \times \text{Id})$ is the natural multiplication on $H \times H$ induced by $\mu_H$, we have that the diagonal $\Delta_H$ is a multiplicative map.

To prove the proposition notice further that $\Omega \Sigma X$ is a universal space in the category of homotopy associative $H$-spaces. The universal property states that any $H$-map from $\Omega \Sigma X$ to a homotopy associative space is determined by its restriction to $X$. Now since the diagonal $\Delta_{\Omega \Sigma X}: \Omega \Sigma X \to \Omega \Sigma X \otimes \Omega \Sigma X$ is an $H$-map, it is determined by its restriction to $X$ which is the composite $X \xrightarrow{\Delta_X} X \times X \xrightarrow{E \times E} \Omega \Sigma X \otimes \Omega \Sigma X$. This proves the proposition.

**Corollary 3.5.** Let $C$ be a co-$H$-space. Then the Hopf algebra $H_\ast(\Omega \Sigma C; R)$ is a Lie-Hopf algebra, that is, it is primitively generated.

**Proof.** By Proposition 3.4, the comultiplication in $H_\ast(\Omega \Sigma C; R)$ is generated by the comultiplication on $\tilde{H}_\ast(C; R)$.

Now directly from the definition of a co-$H$-space, we have that the reduced diagonal $\tilde{\Delta}: C \to C \wedge C$ is trivial, showing that $\tilde{H}_\ast(C; R)$ consists only of primitive elements.

**Theorem 3.6.** Let $X$ be a topological space such that $\Sigma X \simeq \Sigma C$ where $C$ is a co-$H$-space and assume that $H_\ast(X; \mathbb{Z})$ is torsion free. Then over $\mathbb{Z}$ the Hopf algebra $H_\ast(\Omega \Sigma X; \mathbb{Z})$ is isomorphic to a Lie-Hopf algebra, that is, to a primitively generated Hopf algebra.

**Proof.** Let $\varphi: \Sigma X \to \Sigma C$ be a homotopy equivalence. Since $\Omega \varphi$ is an $H$-map it induces a Hopf algebra morphism $\Omega \varphi_*: H_\ast(\Omega \Sigma X) \to H_\ast(\Omega \Sigma C)$ which is an isomorphism of Hopf algebras as $\Omega \varphi$ is a homotopy equivalence. Now the statement of the theorem follows from Corollary 3.5 since $H_\ast(\Omega \Sigma C)$ is a Lie-Hopf algebra.

Throughout the rest of this paper we assume the coefficient ring $R = \mathbb{Z}$ and thus, for the sake of convenience, in integral homology and cohomology we suppress the coefficients from the notation.

Let $\varphi: \Sigma X \to \Sigma^2 Y$ be a homotopy equivalence. Then by the Bott-Samelson theorem, we have

$$H_\ast(\Omega \Sigma X) \simeq T(\tilde{H}_\ast(X)), \quad H_\ast(\Omega \Sigma^2 Y) \simeq T(\tilde{H}_\ast(\Sigma Y)).$$

Let $\{a_i\}$ be an additive basis for $\tilde{H}_\ast(X)$ and let $\{b_i\}$ be an additive basis for $\tilde{H}_\ast(\Sigma Y)$. Then the elements $b_i$ are primitive, that is, $\Delta b_i = 1 \otimes b_i + b_i \otimes 1$. If we know the cohomology ring $H^\ast(X)$, we can calculate the comultiplication $\Delta_X: H_\ast(X) \to H_\ast(X) \otimes H_\ast(X)$.

**Lemma 3.7.** The comultiplication $\Delta_X: H_\ast(X) \to H_\ast(X) \otimes H_\ast(X)$ is determined by the Hopf algebra homomorphism

$$\varphi_*: H_\ast(\Omega \Sigma X) \to H_\ast(\Omega \Sigma^2 Y).$$
Proof. From the defination of a Hopf algebra isomorphism, the comultiplication \( \Delta \) is determined by the formula 
\[
\Delta_{\Omega\Sigma Y}\varphi_s(a) = \varphi_s \otimes \varphi_s(\Delta_{\Omega\Sigma Y} a) \quad \text{where} \quad a \in H_s(X).
\]
\[\square\]

**Proposition 3.8.** The Hopf algebra \( H_s(\Omega\Sigma X) \) for \( X = \Omega\Sigma Z \) with \( Z \) a co-H-space is isomorphic to a Lie-Hopf algebra. The change of homology generators is given by Hopf invariants.

Proof. The proof follows from the James splitting of \( X \), that is, \( \Sigma X \simeq \Sigma \Omega\Sigma Z \simeq \Sigma(\bigvee_{k=1}^{\infty} Z^k) \). Since \( Z \) is a co-H-space, \( H_s(\Omega\Sigma(\bigvee_{k=1}^{\infty} Z^k)) \) is primitively generated. Since the James splitting is given by Hopf invariants, the statement of the proposition follows.

Hopf and Borel used the theory of Hopf algebras to find a necessary condition on homology of \( X \) so that \( X \) is an \( H \)-space. Using the same theory, we give a necessary condition on the homology of \( X \) to be an \( S^1S \)-space.

**Corollary 3.9.** Let \( X \) be a CW-complex such that \( H_s(X) \) is torsion free and let the diagonal \( \Delta : H_s(X) \to H_s(X) \otimes H_s(X) \) define a Hopf algebra structure on \( H_s(\Omega\Sigma X) \cong T(H_s(X)) \). If the Hopf algebra \( H_s(\Omega\Sigma X) \) is not isomorphic to a Lie-Hopf algebra, then \( X \) is not an \( S^1S \)-space.

\[\square\]

4. Examples of \( S^1S \)-spaces

4.1. “Small” CW complexes. The “smallest” topological spaces amongst which we can find non-trivial \( S^1S \)-spaces are two cell complexes. Let \( \alpha : S^k \to S^l \) where \( k > l \) such that \( \Sigma \alpha \simeq * \). Taking its homotopy cofibre, we obtain a space \( X = S^l \cup_{\alpha} e^{k+1} \) which is an \( S^1S \)-space. Slightly more generally start with a wedge of spheres and attach to them a single cell by an attaching map \( \alpha \) such that \( \Sigma \alpha \simeq * \). In this case we also obtain an \( S^1S \)-space. As an explicit example, take the Whitehead product \( S^{m+n-1} \to S^m \vee S^n \) which is the attaching map for \( S^m \times S^n \). As \( \Sigma \omega \simeq * \), there is a stable splitting \( \Sigma(S^m \times S^n) \simeq S^{m+1} \vee S^{n+1} \vee S^{m+n+1} \), proving that \( S^m \times S^n \) is an \( S^1S \)-space.

4.2. Simply-connected 4-dimensional manifolds. We start by showing that \( CP^2 \) is not an \( S^1S \)-space. To prove that, according to Corollary 3.9, we need to show that the Hopf algebra \( H_s(\Omega\Sigma CP^2) \) is not isomorphic to a Lie-Hopf algebra. We first calculate the coproduct structure of \( H_4(\Omega\Sigma CP^2) \). Let \( u_1 \in H_4(\Omega\Sigma CP^2) \) and \( u_2 \in H_4(\Omega\Sigma CP^2) \) be generators. Then the coproduct in \( H_4(\Omega\Sigma CP^2) \) is determined by \( \Delta u_1 = 1 \otimes u_1 + u_1 \otimes 1 \) and \( \Delta u_2 = 1 \otimes u_2 + u_2 \otimes 1 \).

By Proposition 3.8, the coproduct in the Hopf algebra \( H_4(\Omega\Sigma CP^2) \cong T(u_1, u_2) \) is determined by the coproduct in \( H_4(\Omega\Sigma CP^2) \). Thus we have

\[
\Delta u_1 = 1 \otimes u_1 + u_1 \otimes 1, \quad \Delta u_2 = 1 \otimes u_2 + u_2 \otimes 1.
\]

We use the bar notation to denote the tensor product in \( T(u_1, u_2) \). The group \( H_4(\Omega\Sigma CP^2) = \mathbb{Z} \oplus \mathbb{Z} \) is generated by \( u_2 \) and \( u_1 | u_1 \). The only possible change of basis is given by \( u_1 = u_1 \), \( u_2 = u_2 + \lambda u_1 | u_1 \) for some integer \( \lambda \). From the fact that \( w_2 \) is a primitive element, we have the following relation

\[
\Delta w_2 = 1 \otimes w_2 + w_2 \otimes 1 = 1 \otimes u_2 + u_1 \otimes u_1 + u_2 \otimes 1 + \lambda(1 \otimes u_1 + u_1 \otimes 1)|1 \otimes u_1 + u_1 \otimes 1).
\]

Thus we obtain the following condition on \( \lambda \)

\[
1 + 2\lambda = 0.
\]

This equation has no solution over the integers, proving that \( CP^2 \) is not an \( S^1S \)-space as there is no Hopf algebra isomorphism between \( H_4(\Omega\Sigma CP^2) \) and a Lie-Hopf algebra. In this way we have also reproved the classical well known result that the suspension of the Hopf map \( S^1 \to S^2 \) is not null homotopic.

**Remark 4.1.** In an analogous way we can prove that no Hopf invariant one complexes are \( S^1S \)-spaces and as a consequence we see that the suspension of the Hopf map is not null homotopic.

If instead of \( CP^2 \) we take \( S^2 \times S^2 \), then the analogous equation to (1) takes the form \( 1 + \lambda = 0 \) which is solvable over \( \mathbb{Z} \) by taking \( \lambda = -1 \). Thus there is a change of basis in \( H_4(\Omega\Sigma(S^2 \times S^2)) \) which induces a Hopf algebra isomorphism with a Lie-Hopf algebra. Topologically, following the arguments of Subsection 4.1 we also know that \( S^2 \times S^2 \) is an \( S^1S \)-space, as \( \Sigma(S^2 \times S^2) \simeq \mathbb{Z}^2(S^1 \vee S^1 \vee S^3) \).

To approach simply connected 4-dimensional manifolds \( M^4 \) more generally, we use their classfication in terms of the intersection form on \( H^2(M^4) \). Let us denote the intersection form matrix by \( A \). We ask for a condition on \( A \) so that \( M^4 \) is an \( S^1S \)-space. Following our programme we want to construct a looped homotopy equivalence \( \Omega\Sigma M^4 \to \Omega\Sigma^2 Y \) for some CW-complex \( Y \). Let us assume that \( H_2(M^4) \cong \mathbb{Z}^k \) is generated by \( u_i \) for \( 1 \leq i \leq k \).
For dimensional reasons all \( u_i \) are primitive. If the matrix \( A \) is non-trivial, then a generator \( v \in H_4(M^4) \) is not primitive and its coproduct is given by \( \Delta v = v \otimes 1 + 1 \otimes v + u^\top Au \). Thus we are looking for a change of basis

\[
u_i \xrightarrow{f} \lambda_{i1}w_1 + \lambda_{i2}w_2 + \ldots + \lambda_{ik}w_k\]

where all \( w_i \) for \( 1 \leq i \leq k \) are primitive. In matrix form \( f(u) = \Lambda w \) where \( \Lambda \) is the \( k \)-matrix \((\lambda_{ij})\) such that \( \det \Lambda = \pm 1 \). In degree 4, we look to find a primitive generator \( \tilde{w} \in H_4(M^4) \). For dimensional reasons, \( f(v) = \tilde{w} + \sum \gamma_{kl}w_k|w_i \) for some \( \gamma_{kl} \in \mathbb{Z} \); or equivalently, in matrix form \( f(v) = \tilde{w} + w^\top \Gamma w \) for the \( k \)-matrix \( \Gamma = (\gamma_{ij}) \).

For \( M^4 \) to be an \( S^1 \) \( S \)-space, the following diagram needs to commute

\[
\begin{array}{ccc}
v & \xrightarrow{f} & \tilde{w} + w^\top \Gamma w \\
\Delta & & \Delta \\
v \otimes 1 + 1 \otimes v + u^\top Au & \xrightarrow{f} & \tilde{w} \otimes 1 + 1 \otimes \tilde{w} + (w \otimes 1 + 1 \otimes w)^\top \Gamma (w \otimes 1 + 1 \otimes w).
\end{array}
\]

From here we deduce the necessary condition

\[
\Lambda^\top A \Lambda = -\Gamma - \Gamma^\top
\]

or equivalently, if we denote \( \Lambda^{-1} \) by \( L \), then

\[
A = L^\top (-\Gamma - \Gamma^\top)L.
\]

**Remark 4.2.** Since all \( u_i \) are primitive, as a particular case we can choose \( \Lambda \) to be the identity matrix.

For an easy illustrative example, consider \( S^2 \times S^2 \) and its associated quadratic matrix

\[
A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Taking \( \Lambda \) to be \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), then \( \Gamma = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \) satisfies the condition \( A = L^\top (-\Gamma - \Gamma^\top)L \).

### 4.3. The complement of a hyperplane arrangement

Let \( A \) be a complex hyperplane arrangement in \( \mathbb{C}^l \), that is, a finite set of hyperplanes in \( \mathbb{C}^l \). Denote by \( M(A) \) its complement, that is, \( M(A) = \mathbb{C}^l \setminus \text{supp } A \). The cohomology of \( M(A) \) is given by the Orlik-Solomon algebra \( A(A) \) (see for example [OT]). As there are many non-trivial products in this algebra, we can see the Hopf algebra \( H_*(\Omega \Sigma M(A)) \) is not primitively generated. On the other hand, it is well known that \( \Sigma M(A) \) breaks into a wedge of spheres (see for example [S]) and therefore it is an \( S^1 \) \( S \)-space. Thus we can conclude that the Hopf algebra \( H_*(\Omega \Sigma M(A)) \) is isomorphic to the Lie-Hopf algebra \( H_s(\Omega \setminus \bigvee S^{n_\alpha}) \). In a subsequent paper we will study properties of the cup product in \( H^*(M(A)) \), that is, of the Orlik-Solomon algebra \( A(A) \).

### 5. Polyhedral products as \( S^1 \) \( S \)-spaces

A new large family of \( S^1 \) \( S \)-spaces appeared as a result of recent work of Bahri, Bendersky, Cohen, and Gitler [BBCG]. We start by recalling the definition of a polyhedral product functor. Let \( K \) be an abstract simplicial complex on \( m \) vertices, that is, a finite set of subsets of \( [m] = \{1, \ldots, m\} \) which is closed under formation of subsets and includes the empty set. Let \( (X, A) = \{(X_i, A_i, x_i)\}_{i=1}^m \) denote \( m \) choices of connected, pointed pairs of \( CW \)-complexes. Define the functor \( D: K \rightarrow CW_s \) by

\[
D(\sigma) = \prod_{i=1}^m B_i, \text{ where } B_i = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \in [m] \setminus \sigma \end{cases}
\]

with \( D(\emptyset) = A_1 \times \ldots \times A_m \). Then the polyhedral product \( (X, A)^K \) is given by

\[
(X, A)^K = \colim_{\sigma \in \mathcal{K}} D(\sigma).
\]

When \( (X, A) = (D^2, S^1) \), we recover the definition of the moment-angle complex \( Z_K \) introduced by Buchstaber and Panov [BP].
In [BBCG] it was proved that
\[ \Sigma(D^{n+1}, S^n)^K \cong \bigvee_{I \notin K} \Sigma^{2+n|I|} |K_I| \]
which shows that \((D^{n+1}, S^n)^K\), and in particular the moment-angle complexes \(Z_K\), are \(S^1 S\)-spaces. This shows that the collection of \(S^1 S\)-spaces is large and consists of many important spaces which are studied in various mathematical disciplines such as toric topology, complex, symplectic and algebraic geometry, combinatoric and so on.

A natural question that arises is to determine which polyhedral products are \(S^1 S\)-spaces. Recall that for two topological spaces \(X\) and \(Y\), the join \(X \ast Y\) is homotopy equivalent to \(\Sigma X \wedge Y\). We now collect three statements on the stable homotopy type of certain polyhedral product functors proven in [BBCG].

**Theorem 5.1 ([BBCG]).** Let \(K\) be an abstract simplicial complex with \(m\) vertices, and let
\[ (\Sigma, A)^m = \{(X_i, A_i, x_i)\}_{i=1}^m \]
denote \(m\) choices of connected, pointed pairs of CW-complexes with the inclusion \(A_i \subset X_i\) null-homotopic for all \(i\). Then there is a homotopy equivalence
\[ \Sigma(X, A)^K \longrightarrow \Sigma \left( \bigvee_I \left( \bigvee_{\sigma \in K_I} |\Delta(K_I)\ast \hat{D}(\sigma)| \right) \right). \]

**Theorem 5.2 ([BBCG]).** If all of the \(A_i\) are contractible with \(X_i\) and \(A_i\) closed CW-complexes for all \(i\), then there is a homotopy equivalence
\[ \Sigma(X, A)^K \longrightarrow \Sigma \left( \bigvee_I \hat{X}^I \right). \]

**Theorem 5.3 ([BBCG]).** If all of the \(X_i\) in \((\Sigma, A)\) are contractible with \(X_i\) and \(A_i\) closed CW-complexes for all \(i\), then there is a homotopy equivalence
\[ \Sigma(X, A)^K \longrightarrow \Sigma \left( \bigvee_{I \notin K} |K_I| \ast \hat{A}^I \right). \]

**Corollary 5.4.** The polyhedral products considered in Theorems 5.1 and 5.2 are \(S^1 S\)-spaces. For the polyhedral products in Theorem 5.3 to be \(S^1 S\)-spaces we additionally need each \(X_i\) to be a suspension space.

Particularly interesting is the case of moment-angle manifolds \(Z_K\) where \(K\) is the dual \((\partial P^n)^*\) of the boundary of a polytope \(P^n\). The cohomology of the moment-angle manifold is known in terms of the Stanley-Reisner algebra \(\mathbb{Z}[K]\) (see for example [BP, F]). Our approach to the theory of Hopf algebras allows us to give a condition on the algebra necessary for it to be realised as the cohomology of a moment-angle complex. A detailed account of this problem will appear in a subsequent paper.

**Example 5.5.** Let us consider the 6-dimensional moment-angle manifold \(M = (D^2, S^1)^K\) where \(K\) is a square, that is,
\[ K = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\}. \]
From the work of Buchstaber and Panov [BP], we know
\[ H^*(D^2, S^1)^K; \mathbb{Z} = H^*[A[u_1, u_2, u_3, u_4] \otimes \mathbb{Z}[v_1, v_2, v_3, v_4]/(v_1 v_3, v_2 v_4); d] \]
where \(|u_1| = 1, |v_i| = 2\) and the differential \(d\) is given by \(d(u_i) = v_i\) and \(d(v_i) = 0\) for \(1 \leq i \leq 4\). By a straightforward calculation, we find that the 3-dimensional cycles are \(a_1 = u_1 v_3\), \(a_2 = u_2 v_4\), the 6-dimensional cycle is \(b = u_1 u_2 v_3 v_4\), and the intersection form on \(H^2(M^6)\) is given by the following matrix
\[ A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]
Thus in integral homology we have \(H_*(\Omega \Sigma M; \mathbb{Z}) \cong T(a_1, a_2, b)\), where \(|a_1| = |a_2| = 3\) and \(|b| = 6\), and the counit is given by \(\Delta(a_1) = 1 \otimes a_1 + a_1 \otimes 1\) for \(i = 1, 2\) and \(\Delta(b) = 1 \otimes b + a_1 \otimes a_2 - a_2 \otimes a_1 + b \otimes 1\). Therefore
This multiplication rule of compositions is called the five 3-dimensional cycles $u^3$ where all non-trivial cup products are given by $\Delta(u^3) = 1 \otimes u^3 + u^3 \otimes 1 = \Delta(b) + \lambda_1 \Delta(a_1) \Delta(a_2) + \lambda_2 \Delta(a_2) \Delta(a_1) + \lambda_3 \Delta(a_1) \Delta(a_1) + \lambda_4 \Delta(a_2) \Delta(a_2)$.

By a direct calculation, we conclude that for $w_3$ one can take, for example, $w_3 = b + a_2 a_1$. Thus we have confirmed that $H_*(\text{OM}; \mathbb{Z})$ is isomorphic to a Lie-Hopf algebra.

**Example 5.6.** Now let us look at the 7-dimensional moment-angle manifold $M = (D^2, S^1)^K$ where $K$ is a pentagon, that is,

$$K = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{1, 5\}\}.$$

Following [BP], the cohomology algebra is given by

$$H^*((D^2, S^1)^K; \mathbb{Z}) \cong H^*[\Lambda[u_1, u_2, u_3, u_4, u_5] \otimes \mathbb{Z}[v_1, v_2, v_3, v_4, v_5]/(v_1 v_3, v_1 v_4, v_2 v_4, v_2 v_5, v_3 v_5): \delta]$$

where $|u_1| = 1, |v_i| = 2$ and the differential $\delta$ is given by $d(u_i) = v_i$, and $d(v_i) = 0$ for $1 \leq i \leq 5$. There are five 3-dimensional cycles $a_1 = u_1 v_3, a_2 = u_4 v_1, a_3 = u_2 v_4, a_4 = u_5 v_2, a_5 = u_3 v_5$; five 4-dimensional cycles $b_1 = u_4 u_5 v_2, b_2 = u_2 u_3 v_5, b_3 = u_3 u_1 v_3, b_4 = u_3 u_4 v_1, b_5 = u_1 u_2 v_4$ and one 7-dimensional cycle $c = u_1 u_2 u_3 v_1 v_5$.

All non-trivial cup products are given by $a_1 a_i = b_i a_i = c$ for $1 \leq i \leq 5$.

Thus in integral homology we have $H_*(\text{OM}; \mathbb{Z}) \cong T(\{a_i\}_{i=1}^5, c)$, where $|a_i| = 3$, $|b_i| = 4$ and $|c| = 7$, and the comultiplication is given by $\Delta(a_i) = 1 \otimes a_i + a_i \otimes 1, \Delta(b_i) = 1 \otimes b_i + b_i \otimes 1$ for $1 \leq i \leq 5$ and $\Delta(c) = 1 \otimes c + \sum_{i=1}^5 (a_i \otimes b_i + b_i \otimes a_i) + c \otimes 1$.

This Hopf algebra is not primitively generated. We want to show that this Hopf algebra is however isomorphic to a Lie-Hopf algebra by describing the change of this basis to a primitive one. As $a_i$ and $b_i$ are primitive elements, we take for $w_{2i-1} = a_i$ and $w_{2i} = b_i$ for $1 \leq i \leq 5$. For dimensional reasons, $w_{11} = c = \sum_{i=1}^5 (\lambda_i a_i | b_i + \lambda_{i+5} b_i | a_i)$.

As we want $w_{11}$ to be primitive, we need the following relation to hold

$$\Delta(w_{11}) = 1 \otimes w_{11} + w_{11} \otimes 1 = \Delta(c) + \sum_{i=1}^5 (\lambda_i \Delta(a_i) \Delta(b_i) + \lambda_{i+5} \Delta(b_i) \Delta(a_i)).$$

We conclude that for $w_{11}$ one can take, for example, $w_{11} = c - \sum_{i=1}^5 a_i | b_i$. Thus we have confirmed that $H_*(\text{OM}; \mathbb{Z})$ is isomorphic to a Lie-Hopf algebra.

### 6. Applications to Algebra

#### 6.1. Quasi-symmetric polynomials

In this section we use our new homotopy invariant to study the ring of quasi-symmetric functions. We start by recalling the main definitions, following mainly the notation of Hazewinkel [Ha]. For more details on applications of topological methods to the study of quasi-symmetric functions see Buchstaber and Erokhovets [BE].

**Definition 6.1.** A composition $\omega$ of a number $n$ is an ordered set $\omega = (j_1, \ldots, j_k)$, for $j_k > 1$, such that $n = j_1 + \ldots + j_k$. Let us denote $|\omega| = n, l(\omega) = k$. The empty composition of $0$ we denote by $(\cdot)$. Then $|()| = 0, l(()| = 0$.

**Definition 6.2.** Let $t_1, t_2, \ldots$ be a finite or an infinite set of variables of degree 2. For a composition $\omega = (j_1, \ldots, j_k)$, consider a quasi-symmetric monomial

$$M_\omega = \sum_{t_1^{j_1} \ldots t_k^{j_k}} M_{()} = 1.$$

whose degree is equal to $2|\omega| = 2(j_1 + \ldots + j_k)$.

For any two monomials $M_\omega$ and $M_{\omega'}$, their product in the ring of polynomials $\mathbb{Z}[t_1, t_2, \ldots]$ is equal to

$$M_\omega M_{\omega'} = \sum_{\omega} \left( \sum_{\Omega + \Omega' = \omega} 1 \right) M_\omega$$

where for the compositions $\omega = (j_1, \ldots, j_k)$, $\omega' = (j'_1, \ldots, j'_l)$, $\omega'' = (j''_1, \ldots, j''_m)$, $\Omega$ and $\Omega'$ are all the $k$-tuples such that

$$\Omega' = (0, \ldots, j'_1, 0, \ldots, 0), \Omega'' = (0, \ldots, j''_1, 0, \ldots, 0).$$

This multiplication rule of compositions is called the overlapping shuffle multiplication.
Thus finite integer combinations of quasi-symmetric monomials form a ring. This ring is called the ring of quasi-symmetric functions and is denoted by \( Q\text{Symm}[t_1, \ldots, t_n] \), where \( n \) is the number of variables. In the case of an infinite number of variables it is denoted by \( Q\text{Symm}[t_1, t_2, \ldots] \) or \( \text{QSymm} \).

The diagonal map \( \Delta: Q\text{Symm} \to Q\text{Symm} \otimes Q\text{Symm} \) given by

\[
\Delta(M_{a_1, \ldots, a_k}) = \sum_{i=0}^{k} M_{(a_1, \ldots, a_i)} \otimes M_{(a_{i+1}, \ldots, a_k)}
\]
defines on \( Q\text{Symm} \) the structure of a graded Hopf algebra.

In Hazewinkel proved the Ditters conjecture that \( Q\text{Symm}[t_1, t_2, \ldots] \) is a free commutative algebra of polynomials over the integers.

Let \( R \) be a commutative associative ring with unit.

**Definition 6.3.** A Leibnitz-Hopf algebra over the ring \( R \) is an associative Hopf algebra \( H \) over the ring \( R \) with a fixed sequence of a finite or countable number of multiplicative generators \( H_i, i = 1, 2, \ldots \) satisfying the comultiplication formula

\[
\Delta(H_i) = \sum_{i+j=n} H_i \otimes H_j, \quad H_0 = 1.
\]

A universal Leibnitz-Hopf algebra \( A \) over the ring \( R \) is a Leibnitz-Hopf algebra with the universal property: for any Leibnitz-Hopf algebra \( \mathcal{H} \) over the ring \( R \) the correspondence \( A_i \to H_i \) defines a Hopf algebra homomorphism.

Of special interest to us will be the free associative Leibnitz-Hopf algebra over the integers \( \mathbb{Z} = \mathbb{Z}[Z_1, Z_2, \ldots] \) in countably many generators \( Z_i \).

**Definition 6.4.** A universal commutative Leibnitz-Hopf algebra \( C = \mathbb{Z}[C_1, C_2, \ldots] \) is a free commutative polynomial Leibnitz-Hopf algebra in generators \( C_i \) of degree 2\( i \). We have \( C = \mathbb{Z}/J_C \), where the ideal \( J_C \) is generated by the relations \( Z_i Z_j - Z_j Z_i \).

The Leibnitz-Hopf algebra \( C \) is a self-dual Hopf algebra and the graded dual Hopf algebra is naturally isomorphic to the algebra of symmetric functions \( \mathbb{Z}[\sigma_1, \sigma_2, \ldots] = \text{Symm}[t_1, t_2, \ldots] \subset Q\text{Symm}[t_1, t_2, \ldots] \) generated by the symmetric monomials

\[
\sigma_i = M_{\omega_i} = \sum_{l_1 < \ldots < l_i} t_{l_1} \ldots t_{l_i}
\]

where \( \sigma_i = (1, \ldots, 1) \).

The isomorphism \( C = C^* \) is given by the correspondence \( C_i \to \sigma_i \).

The Hopf algebra of symmetric functions \( \text{Symm} \) has a non-commutative analogue \( \text{NSymm} \) obtained by replacing the polynomial algebra in the definition by a free associative algebra \( \text{NSymm} = \mathbb{Z}[\sigma_1, \ldots, \sigma_m, \ldots] \). The diagonal of \( \text{NSymm} \) is defined by the same formula as in \( \text{Symm} \) and is still cocommutative. The dual of \( \text{NSymm} \) is the commutative algebra of quasi-symmetric functions \( \text{QSymm} \).

Looking at the homology of \( \Omega \Sigma \mathbb{C} P^\infty \) with integral coefficients we get a Hopf algebra which is isomorphic to \( \text{NSymm} \) (see [BR]). This was the starting point for Baker and Richter to positively solve Ditters conjecture using topological methods. By calculating the cohomology algebra \( H^*(\Omega \Sigma \mathbb{C} P^\infty) \) they showed that \( Q\text{Symm} \) is a polynomial algebra.

For the sake of completeness and as an illustration of the close relation between topology and algebra, we explicitly calculate Hopf algebras related to the quasi-symmetric algebra \( Q\text{Symm} \) by identifying them with the (co)homology of certain topological spaces.

We have the following Hopf algebras:

I. \( H_i(\mathbb{C} P^\infty) \) is a divided power algebra \( \mathbb{Z}[u_1, u_2, \ldots]/I \), where the ideal \( I \) is generated by the relations \( u_i u_j = \binom{i+j}{i} u_{i+j} \), with the comultiplication

\[
\Delta u_n = \sum_{k=0}^{n} u_k \otimes u_{n-k};
\]

II. \( H_i(\mathbb{C} P^\infty) \) is a divided power algebra \( \mathbb{Z}[u_1, u_2, \ldots]/I \), where the ideal \( I \) is generated by the relations \( u_i u_j = \binom{i+j}{i} u_{i+j} \), with the comultiplication

\[
\Delta u_n = \sum_{k=0}^{n} u_k \otimes u_{n-k};
\]
(2) \(H^*(\mathbb{C}P^\infty) = \mathbb{Z}[u]\) is a polynomial ring with the comultiplication

\[ \Delta u = 1 \otimes u + u \otimes 1. \]

II.

(1) \(H_*(\Omega \Sigma \mathbb{C}P^\infty) \simeq T(\hat{H}_*(\mathbb{C}P^\infty)) = \mathbb{Z}\langle u_1, u_2, \ldots \rangle\) with \(u_i\) being non-commuting variables of degree \(2i\). Thus there is an isomorphism of rings

\[ H_*(\Omega \Sigma \mathbb{C}P^\infty) \simeq \mathbb{Z} \]

under which \(u_n\) corresponds to \(\mathbb{Z}_n\).

The coproduct \(\Delta\) on \(H_*(\Omega \Sigma \mathbb{C}P^\infty)\) induced by the diagonal in \(\Omega \Sigma \mathbb{C}P^\infty\) is compatible with the one in \(\mathbb{Z}\):

\[ \Delta u_n = \sum_{i+j=n} u_i \otimes u_j. \]

Thus there is an isomorphism of graded Hopf algebras.

(2) \(H^*(\Omega \Sigma \mathbb{C}P^\infty)\) is the graded dual Hopf algebra to \(H_*(\Omega \Sigma \mathbb{C}P^\infty)\).

III.

\(H_*(BU) \simeq H^*(BU) \simeq \mathbb{Z}[\sigma_1, \sigma_2, \ldots] \simeq \mathcal{C}\). It is a self-dual Hopf algebra of symmetric functions. In cohomology \(\sigma_i\) are represented by Chern classes.

IV.

(1) \(H_*(\Omega \Sigma S^2) = H_*(\Sigma^3) = \mathbb{Z}[w]\) is a polynomial ring with \(\deg w = 2\) and the comultiplication

\[ \Delta w = 1 \otimes w + w \otimes 1 \]

(2) \(H^*(\Omega \Sigma S^2) = \mathbb{Z}[u_n]/I\) is a divided power algebra. Thus \(H^*(\Omega \Sigma S^2) \simeq H_*(\mathbb{C}P^\infty)\).

V.

\(H_*(\Omega \Sigma (\Omega \Sigma S^2)) \simeq \mathbb{Z}\langle w_1, w_2, \ldots \rangle\). It is a free associative Hopf algebra with the comultiplication

\[ \Delta w_n = \sum_{k=0}^{n} \binom{n}{k} w_k \otimes w_{n-k}. \]

VI.

\(H_*(\Omega (\bigvee_{i=1}^\infty S^{2i})) \simeq \mathbb{Z}\langle \xi_1, \xi_2, \ldots \rangle\). It is a free associative algebra and has the structure of a graded Hopf algebra with the comultiplication

\[ \Delta \xi_n = 1 \otimes \xi_n + \xi_n \otimes 1. \]

Over the rationals NSymm, that is, \(H_*(\Omega \Sigma \mathbb{C}P^\infty; \mathbb{Q})\), is isomorphic to a Lie-Hopf algebra. Following the first example in Section 3.1, we conclude that \(H_*(\Omega \Sigma \mathbb{C}P^\infty)\) is not isomorphic to a Lie-Hopf algebra. We will use topological methods to find a maximal subalgebra of the algebra \(H_*(\Omega \Sigma \mathbb{C}P^\infty)\) which over \(\mathbb{Q}\) is isomorphic to \(H_*(\Omega \Sigma \mathbb{C}P^\infty; \mathbb{Q})\) but over \(\mathbb{Z}\) is a Lie-Hopf algebra.

**Theorem 6.5.** The Hopf algebra \(H_*(\Omega \Sigma \Omega \Sigma S^2)\) is a maximal subHopf algebra of \(H_*(\Omega \Sigma \mathbb{C}P^\infty)\) which is isomorphic to a Lie-Hopf algebra.

**Proof.** Let us start with the map \(f: \Omega \Sigma S^2 \longrightarrow \mathbb{C}P^\infty\) which realises a generator of \(H^2(\Omega \Sigma S^2) \cong \mathbb{Z}\) and which induces an isomorphism in rational homology but which obviously does not induce an isomorphism over the integers (see examples I (1) and IV (1)). By taking the loop suspension of \(f\), we get a loop map \(\Omega f: \Omega \Sigma \Omega \Sigma S^2 \longrightarrow \Omega \Sigma \mathbb{C}P^\infty\) which in rational homology induces a Hopf algebra isomorphism. Using the James-Hopf invariants, we produce a homotopy equivalence between \(\Sigma \Omega \Sigma S^2\) and \(\bigvee_{i=1}^\infty S^{2i}\), showing that \(\Omega \Sigma S^2\) is an \(S^1\)-s-space. Now by Theorem 3.6 we have that \(H_*(\Omega \Sigma \Omega \Sigma S^2)\) is isomorphic as a Hopf algebra to a Lie-Hopf algebra, which finishes the proof. \(\square\)

6.2. **Obstructions to desuspending a map.** In this subsection we explicitly write obstructions to desuspending the homotopy equivalence

\[ \Sigma (\Omega \Sigma S^2) \longrightarrow \Sigma \left( \bigvee_{i=1}^\infty S^{2i} \right). \]
The homotopy equivalence
\[ a: \Omega \Sigma \left( \bigvee_{i=1}^{\infty} S^{2n} \right) \to \Omega \Sigma (\Omega \Sigma S^2) \]
induces an isomorphism of graded Hopf algebras
\[ a_*: \mathbb{Z} \langle \xi_1, \xi_2, \ldots \rangle \to \mathbb{Z} \langle w_1, w_2, \ldots \rangle \]
and its algebraic form is determined by the conditions
\[ \Delta a_* \xi_n = (a_* \otimes a_*)(\Delta \xi_n). \]
For example, \( a_* \xi_1 = w_1, \) \( a_* \xi_2 = w_2 - w_1 | w_1, \) \( a_* \xi_3 = w_3 - 3w_2 | w_1 + 2w_1 | w_1. \)
Thus using topological results we have obtained that two Hopf algebra structures on the free associative algebra with comultiplications \( \xi \) and \( \xi \) are isomorphic over \( \mathbb{Z}. \)
This result is interesting from the topological point of view, since the elements \( (w_n - a_* \xi_n) \) for \( n \geq 2 \) are obstructions to the desuspension of homotopy equivalence \( \xi. \)

References

[BBCG] A. Bahri, M. Bendersky, F. R. Cohen, and S. Gitler, The polyhedral product functor: a method of computation for moment-angle complexes, arrangements and related spaces, [arXiv:0711.4689]
[B] A. Borel, Sur la cohomologie des espaces fibrés principaux, Ann. of Math. 57 (1953), 115-207
[BP] V. M. Buchstaber, and T. E. Panov, Torus actions, combinatorial topology, and homological algebra, Russian Mathematical Surveys 55(5) (2000), 825–921
[BR] R. Bott, and H. Samelson, On the Pontryagin product in spaces of paths, Comment. Math. Helv. 27 (1953), 320–337
[BE] V. Buchstaber, and N. Erokhovets, Ring of Polytopes, Quasi-symmetric functions and Fibonacci numbers, [arXiv:1002.0810]
[C] J. W. Cannon, Shrinking cell-like decompositions of manifolds. Codimension three, Ann. of Math. 110 (1979), 83–112
[E] R. D. Edwards, The double suspension of a certain homology 3-sphere is \( S^5 \), Notices AMS 22 A-334 (1975)
[F] M. Franz, The integral cohomology of toric manifolds, Tr. Mat. Inst. Steklova 252 (2006), 61–70
[Ha] M. Hazewinkel, The algebra of quasi-symmetric functions is free over the integers, Adv. Math. 164 (2001), 283–300
[H] H. Hopf, Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen, Ann. of Math. 42 (1941), 22–52
[J] J. M. James, Reduced product spaces, Ann. of Math. 62 (1955), 170–197
[MM] J. Minkor, and J. Moore, On the structure of Hopf algebras, Ann. of Math. 81 (1965), 211–264
[OT] P. Orlik, and H. Terao, Arrangements of hyperplanes, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] 300, Springer-Verlag, (1992)
[S] Ch. Schaper, Suspensions of affine arrangements, Math. Ann. 309 (1997), 463–473

School of Mathematics, University of Manchester, Manchester M13 9PL, United Kingdom
E-mail address: Jelena.Grbic@manchester.ac.uk
E-mail address: Victor.Buchstaber@manchester.ac.uk