Counterexamples to strong cosmic censorship in asymptotically flat black hole spacetimes

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The extendibility of spacetime and the existence of weak solutions to the Einstein field equations beyond Cauchy horizons, is a crucial ingredient to examine the limits of General Relativity. The strong cosmic censorship conjecture serves as a firewall for gravitation by demanding inextendibility of spacetime beyond the Cauchy horizon. For asymptotically flat spacetimes, the predominance of the blueshift instability and the subsequent formation of a mass-inflation singularity at the Cauchy horizon have, so far, substantiated the conjecture. Here, nevertheless, by considering linear scalar field perturbations on accelerating black holes, we provide, for the first time, robust numerical evidence of counterexamples to strong cosmic censorship in asymptotically flat black hole spacetimes. In particular, we show that the stability of Cauchy horizons in accelerating charged black holes is connected to quasinormal modes, and we discuss the regularity requirement for which weak solutions to the field equations can exist at the Cauchy horizon and show that the conjecture may be violated near extremality.

I. INTRODUCTION

One of the many fundamental questions concerning black holes (BHs) is their internal anatomy [1]. The inner structure of BHs, together with the nature of singularities [2] lying deep inside, is of paramount importance to understand the global uniqueness of solutions to the Einstein field equations given suitable initial data, as well as the fate of infalling observers.

While, the issue of plunging observers into static and neutral BHs is quite clear [3], their journey takes an unpredictable turn if the BH is charged and/or rotating. In those cases, the observer’s journey seems to continue unaffected through the interior of the BH, eventually emerging into a region where neither the geometry nor the fate of the observer can be determined uniquely by initial data. The boundary of deterministic evolutions is called Cauchy horizon (CH) and marks the division between the region where General Relativity (GR) is able to forecast spacetime developments and the region where predictability of the field equations is lost.

Although Kerr and Reissner-Nordström (RN) BHs are known to have CHs [1], those are highly symmetric solutions which, in more realistic physical grounds, will eventually be perturbed by small time-dependent fluctuations. The relaxation of perturbed BHs leads to the emission of gravitational radiation in the form of damped sinusoid oscillations, described by quasinormal modes (QNMs) [35, 37].

The CH of asymptotically flat BHs is expected to be unstable under such perturbations, yielding a spacetime singularity which effectively seals off the tunnel to regions where the field equations cease to make sense [8]. Any observer approaching the CH would measure a divergent energy flux [9] which leads to the formation of a “mass-inflation” singularity [10, 11], due to the uneven competition between the power-law cutoff of perturbations in the exterior [12, 13] and the exponential blueshift effect triggered at the CH. That this occurs generically is the essence of the Strong Cosmic Censorship (SCC) conjecture [19, 21], which states that weak solutions of the field equations that arise from proper initial data are future inextendible beyond CHs.

If a positive cosmological constant is included in those settings, then the exponential decay rate of perturbations in the exterior [22, 24], which in turn is controlled by the dominant QNMs [25, 28], may possibly counterbalance the blueshift amplification at the CH [29, 32]. This leads to a weaker singularity, where the tidal deformations imposed on the observer there are bounded [33] and weak solutions to the field equations may exist. Then, to test SCC, it all comes down to the regularity of scalar fields at the CH [23, 32, 34] and the calculation of $\beta > -\inf\{\text{Im}(\omega)\}/\kappa_-$, where the numerator captures the decay rate of the dominant QNM $\omega$ and $\kappa_-$ is the surface gravity of the CH, which governs the exponential perturbation growth there. For SCC to hold in these asymptotically de Sitter settings, $\beta < 1/2$, which guarantees the breakdown of field equations at the CH [37]. If on the other hand $\beta > 1/2$, then SCC may be violated.

The conclusions of very recent studies are the following: near-extremally-charged Reissner-Nordström-de Sitter (RNds) BHs violate SCC with scalar [35, 38, 40], Dirac [31, 42] and gravitational perturbations [43], while Kerr-de Sitter BHs do not [44]. For a list of contemporary studies see [45, 65].

Most studies, so far, have been performed under the assumption that BHs are static (or stationary) objects which do not “move” in space. To the contrary, many
BHs are found in binary systems. The gravitational wave emission from these binaries leads to the increment of the BH velocities. Thus, these BHs seem to move and accelerate with respect to our own reference frame.

A natural choice to describe accelerating BHs is the $C$–metric [63] which is an axisymmetric exact solution of the field equations with a boost symmetry [67]. Although its geometrical properties are well known [67], the physics of accelerating BHs has not been so well understood and, in part, this is because an appropriate framework to study their thermodynamics was lacking until recently [68, 69]. Within GR, the $C$–metric has also been used to investigate radiation at infinity [70, 73]. However, it has been beyond classical GR that applications of the $C$–metric had most impact, e.g. in studies about the creation of BH pairs [27], the splitting of cosmic strings [70] and, most notably, in the construction of the black ring solution in five dimensions [72].

The charged version of the $C$–metric possesses a CH, similar to that of RN [73]. The additional feature of the $C$–metric is the existence of an acceleration horizon; a hypersurface beyond which any event is unobservable to the accelerating BH’s light cone. Although this metric has no cosmological constant, its causal structure shares some similarities with RNds spacetimes [78]. However, and most interestingly, the $C$–metric is asymptotically flat [70].

The investigation of SCC in accelerated BHs has been hampered by the fact that, until very recently, there were no studies about the decay of scalar perturbations on those backgrounds. This was partly due to the non-trivial geometry of accelerating BHs and their horizons which are not spherically symmetric. Note though the preliminary study [79], where a simplified analysis was performed using null radiation.

However, a new numerical approach has revealed that scalar perturbations decay exponentially on the charged $C$–metric and, most importantly, their decay is controlled by the dominant QNMs [80]. So, as in RNds, the exponential decay of perturbations in the exterior of the accelerating BH may possibly counterbalance the exponential blueshift at the CH. This is in contrast with the behavior of perturbations on non-accelerating asymptotically flat BHs, which exhibit a power-law cutoff at late times.

Unlike the case of Kerr-dS [27], there is no rigorous mathematical proof yet about the spectral gap and QNM dominance close to the horizons of the charged $C$–metric. However, the methods of [27] for Kerr-dS are expected to apply in our case, as the wave operator involved is considerably simpler in the absence of rotation.

Another ingredient that was missing in order to study SCC in the charged $C$–metric was the calculation of an eventual $\beta$ threshold beyond which the conjecture could be violated. We argue that a threshold indeed exists and is still given by $\beta > 1/2$. Then, by computing $\beta$ we provide clear numerical evidence which indicate that SCC may be violated in near-extremally-charged accelerating BHs. To our knowledge, this is the first counterexample of SCC in asymptotically flat BH spacetimes.

II. THE CHARGED $C$–METRIC IN BRIEF

Spacetimes based on the charged $C$–metric can be interpreted as representing axisymmetric electrically-charged BHs, accelerating along the axis of symmetry due to the presence of a cosmic string [67, 78, 81]. Such BHs are described by the line element

$$ds^2 = \frac{1}{\Omega^2} \left( -f(r)dt^2 + \frac{dr^2}{f(r)} + \frac{r^2d\Omega^2}{P(\theta)} + P(\theta)r^2 \sin^2 \theta d\varphi^2 \right),$$

with $\Omega = 1 - \alpha r \cos \theta$ and

$$f(r) = \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) \left( 1 - \alpha^2 r^2 \right),$$

$$P(\theta) = 1 - 2\alpha M \cos \theta + \alpha^2 Q^2 \cos^2 \theta,$$

where $M$, $Q$ and $\alpha$ are related to the BH mass, charge and acceleration, respectively. The metric (1) asymptotes to the RN solution as $\alpha \to 0$ and to the $C$–metric as $Q \to 0$. There is a curvature singularity at $r = 0$, while the roots of $f(r)$ determine the causal structure of spacetime. There exist three null hypersurfaces at

$$r = r_\alpha := \alpha^{-1}, \quad r = r_\pm := M \pm \sqrt{M^2 - Q^2},$$

called the acceleration horizon $r_\alpha$, event horizon $r_+$ and Cauchy horizon $r_-$, which must satisfy $r_- \leq r_+ \leq r_\alpha$, where $\alpha \leq 1/r_+$. If $M = Q$, then the event horizon coincides with the CH and the BH is extremal. Each horizon has a surface gravity given by [78, 82];

$$\kappa_i = \left| \left. \frac{f'(r)}{2} \right|_{r=r_i} \right., \quad i \in \{-, +, \alpha\}. \quad (5)$$

Conical singularities occur on the axis at $\theta = 0$ and $\theta = \pi$ designating the existence of deficit or excess angles. If we assume that $\varphi \in [0, 2\pi)$, where $C = 1/P(\varphi)$, then we can remove the conical singularity at $\theta = \pi$ to end up with a deficit angle at $\theta = 0$ (see [78, 81] for a detailed analysis). The metric (1) can, therefore, be understood as an RN-like BH accelerating along the axis $\theta = 0$ [78, 83].

We can conformally rescale (1) as $d\tilde{s}^2 = \Omega^2 ds^2$. The conformal structure of the charged $C$–metric was analyzed in [73, 83]. Conformal infinity $\mathcal{I}$ is given by $\Omega = 0$, with $\nabla^\nu \Omega_{I} \neq 0$, and its location depends on the value of $\theta$. It was shown [70, 73] that the topology of $\mathcal{I}$ is $S^2 \times \mathbb{R}$. Thus the charged $C$–metric admits a global $\mathcal{I}$ and is asymptotically flat [70], even though the generators of $\mathcal{I}$ are not complete.

III. QUASINORMAL MODES OF SCALAR FIELDS IN THE CHARGED $C$–METRIC

The charged $C$–metric (1) is a solution to the vacuum Einstein-Maxwell field equations. Therefore, the Ricci
curvature vanishes and the wave equation for a minimally coupled massless scalar field $\phi$ is equivalent to the conformally-invariant wave equation \[ \Box \tilde{g} \phi - \frac{1}{6} \tilde{R} \phi = 0, \] where $\tilde{R}$ is the conformally rescaled Ricci curvature and $\tilde{g}_{\mu\nu} := \tilde{g}_{\mu\nu} \nabla^\mu \nabla^\nu$. We can separate (6) by choosing an ansatz for the scalar field

$$\tilde{\phi} = e^{-i\omega t} e^{im \varphi} \tilde{\Phi}(r) \chi(\theta),$$

where $\omega$ is the QNM and $m = m_0 P(\pi)$ the azimuthal quantum number that guarantees the periodicity of $\varphi$, with $m_0 > 0$ integer. Then, (6) becomes

$$\frac{d^2 \tilde{\Phi}(r)}{dr^2} + \left( \omega^2 - V_r \right) \Phi(r) = 0, \quad \chi(\theta)$$

and $dr_* = dr/f(r)$ and $dz = d\theta/(P(\theta) \sin \theta)$ and

$$V_r = f(r) \left( \frac{\lambda}{r^2} - \frac{f(\theta)}{3r^2} + \frac{f'(\theta)}{3} - \frac{f''(r)}{6} \right),$$

$$V_\theta = P(\theta) \left( \lambda \sin^2 \theta - \frac{P(\theta) \sin^2 \theta}{3} + \frac{\sin \theta \cos \theta P'(\theta)}{2} \right. + \left. \frac{\sin^2 \theta P''(\theta)}{6} \right),$$

with $\lambda$ the separation constant. For QNMs, the boundary conditions are divided in two categories:

$$\Phi(r \to r_+) \sim e^{-i\omega r_*}, \quad \Phi(r \to r_-) \sim e^{i\omega r_*},$$

$$\chi(\theta \to 0) \sim e^{imz}, \quad \chi(\theta \to \pi) \sim e^{-mz}.$$  

Conditions (12) impose purely ingoing (outgoing) waves at the event (acceleration) horizon, while conditions (13) are taken so that the scalar field is bounded at the interval boundaries of $\theta$ \[ [75]. By solving (6), subject to the boundary conditions (13), one obtains the eigenvalues $\lambda$, for a given $m_0$. Then, those eigenvalues can be substituted in (6), subject to the boundary conditions (12), to obtain a discrete set of QNMs $\omega$.

In \[ [27] it was shown that asymptotic solutions to the wave equation in the Kerr-dS metric can be expressed as a sum of QNMs plus other sub-dominant terms. This result provides a mathematical proof for the physical interpretation of QNMs as complex frequencies of (linear scalar) gravitational waves in Kerr-dS. However, there is no such result for the C–metric yet. Instead, in \[ [80], we provided strong numerical evidence that this is the case, i.e. that QNMs indeed dominate the asymptotic dynamics of solutions to the wave equation.

Specifically, the results in \[ [80] indicate that the late-time decay of scalar field perturbations in the charged C–metric follows the numerically extracted exponential law \[ [80] \]

$$\phi \sim e^{-\gamma t}, \quad \text{for} \quad t \to \infty,$$

where $\gamma := -\inf_{\omega} \{ \text{Im}(\omega) \}$ is the smallest (in absolute value) imaginary part of all families of QNMs. This justifies the use of QNMs to study the decay of scalar perturbations in the C–metric and, ultimately, the regularity of the spacetime extensions using, as a proxy, the regularity requirements for the solutions $\phi$ to the wave equation \[ [22].

Although $\phi$ and $g_{\mu\nu}$ may not necessarily be continuously differentiable, heuristically one can still make sense of the Einstein-Maxwell field equations by multiplying with a smooth, compactly supported, test function $\psi$ and integrating in a neighborhood $\mathcal{V}$ around the CH. If the result is finite, we may consider weak solutions to the field equations. The energy momentum tensor of the conformally-coupled scalar field leads, after integration, to the regularity requirement that $\phi$ should belong to the (Sobolev) function space $H^1_{\text{loc}}$ for weak solutions to exist at the CH.

By considering \[ [8] as $r \to r_-$, we find two independent mode solutions

$$\phi_1 \sim e^{-i\omega u}, \quad \phi_2 \sim e^{-i\omega u}\left| r - r_- \right|^{i\omega/\kappa_-},$$

where we have dropped the angular dependence and used outgoing null coordinates $u = t - r_+$, which are regular at the CH. There, $\phi_1$ is smooth, while $\phi_2$ is not necessarily so. Therefore, considering $\phi \in H^1_{\text{loc}}$, we require finiteness of $\int_{\mathcal{V}} (\partial_r \phi_2)^2 dr$ and arrive at the condition

$$\beta := \gamma/\kappa_- > 1/2,$$

which is identical to the one derived rigorously in \[ [22] for Kerr-dS. Since the metric should share similar regularity requirements as $\phi$ \[ [23, 32, 34], then, for $\beta > 1/2$, the corresponding BH spacetime should extend beyond the CH when $\phi \in H^1_{\text{loc}}$, and thus violate the SCC conjecture.

To obtain the separation constant $\lambda$ from (6), for given $m_0$, we use the Mathematica package QNMSpectral developed in \[ [87] (based on pseudospectral collocation methods \[ [88]) and confirm the validity of our results with the Frobenious method \[ [80]. To calculate $\omega$, we use QNMSpectral, and confirm our results with the numerical scheme developed in \[ [17], based on the time-domain integration of (4) and the application of the Prony method \[ [89] to extract the QNMs. For an extensive QNM analysis of the charged C–metric, see \[ [80].

IV. QUASINORMAL MODES AND STRONG COSMIC CENSORSHIP

Our numerics indicate the existence of three families of QNMs which antagonize each other in different regions of the parameter space, as show in Fig. 1 and 2.
The first family of QNMs consists of the usual oscillatory modes obtained with standard WKB tools. We refer to them as “photon surface” (PS) modes (in blue in Fig. 1 and 2). These modes asymptote the oscillatory QNMs of non-accelerating BHs when \( \alpha \to 0 \). The dominant mode of the PS family is obtained at the large \( m \) limit. Our numerics indicate that \( m_0 = 10 \) gives a very good approximation of the actual dominant mode [80].

The second family of modes, the acceleration QNMs (in red in Fig. 1 and 2), is a novel family first found in [80], which depends linearly on the acceleration parameter and vanishes when \( \alpha \to 0 \). The acceleration modes are purely imaginary and arise due to the presence of the acceleration horizon. They are analogous to the de Sitter QNMs found in [35, 37, 39]. Our numerics suggest that the dominant mode of this family is obtained when \( m_0 = 0 \) [80].

The final family of modes is the near extremal (NE) family (in green in Fig. 1 and 2), which consists of purely imaginary modes and dominates the ringdown when the event and CH approach each other. This family asymptotes the NE modes of RN [91] as \( \alpha \to 0 \) and vanishes at extremality. Similar modes have been found in RNdS for scalar [35, 38, 40] and fermionic perturbations [41]. In our case, the dominant mode is obtained when \( m_0 = 0 \) [80].

For completeness, in the right panel of Fig. 1 we provide the temporal response of scalar perturbations for three distinct cases, chosen so that a different family dominates in each case. We observe that the late-time behavior of scalar fields on the charged \( C^- \) metric is, indeed, exponential, while the extracted QNMs at late times match accurately the dominant QNMs derived from the spectral analysis. The parameters for these examples are only indicative, as we find that \([14]\) holds for various cases throughout the sub-extremal parameter space [80].

In Fig. 2 we present the ratio \(-\text{Im}(\omega)/\kappa_\alpha\) determined by the dominant QNMs of each family, in near-extremally-charged accelerating BHs, for various acceleration parameters \( \alpha M \). Then, \( \beta \) is obtained from the smallest contribution of all families of modes (cf. eq. [16]). For slowly accelerating charged BHs, the \( m_0 = 0 \) acceleration modes dominate throughout most of the sub-extremal parameter space. On the other hand, for sufficiently large acceleration the high-frequency PS modes dominate. Independently of the acceleration parameter though, there is always a small, but finite, region in the parameter space, when \( Q \to M \), where the \( m_0 = 0 \) NE family of modes dominates.

For all cases shown, \( \beta \) exceeds 1/2. In fact, \( \beta \) would diverge at extremality if only the PS and acceleration modes were present. However, the NE modes take over to keep \( \beta \) from exceeding unity. Nevertheless, the existence of regions in the parameter space of near-extremally-charged accelerating BHs where \( \beta > 1/2 \), indicates a potential violation of SCC.

V. CONCLUSIONS

Until now, asymptotically flat BHs are known to satisfy SCC. In those cases, linear perturbations decay slowly enough [12] to guarantee that the exponential blueshift effect, triggered at the CH, will turn this region into a mass-inflation singularity, where the field equations cease to make sense [10, 11].

However, a recent study of linear scalar perturbations...
in the charged $C-$metric [80] revealed that the existence of acceleration horizons leads to the exponential decay of perturbations which, in turn, are controlled by the dominant QNM at late times, even though the spacetime remains asymptotically flat [70].

In this article, we exploit this late-time behavior of perturbations to test the modern formulation of SCC [92], using QNMs [35], in accelerating, charged BHs.

We have provided robust numerical evidence supporting the fact that near-extremally-charged accelerating BHs with linear scalar field perturbations violate SCC. To our knowledge, this is the first ever counterexample found in asymptotically flat BH spacetimes. Considering that SCC has been an important mathematical conjecture and a way to test classical GR, then its violation in asymptotically flat spacetimes seems rather problematic and deserves further investigation. Our results appear to open a new road to mathematical and numerical studies of this conjecture in “moving” BH backgrounds, particularly if one takes into account the physical importance of the geometric analogy between highly charged and rapidly rotating accelerating BH in binary mergers [93].

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[1] M. Dafermos, Commun. Pure Appl. Math. 58, 0445 (2005), arXiv:gr-qc/0307013 [gr-qc].
[2] A. Ori, Phys. Rev. D61, 064016 (2000).
[3] J. Shiers. J. Diff. Geom. 108, 319 (2018), arXiv:1507.00601 [gr-qc].
