Nonvanishing magnetic field in pure electrostatic systems at rest in the curved space-time of Earth

N.N. Nikolaev and S.N. Vergeles

Landau Institute for Theoretical Physics, Russian Academy of Sciences, Chernogolovka, Moscow region, 142432 Russia

and

Moscow Institute of Physics and Technology, Department of Theoretical Physics, Dolgoprudnyj, Moskow region, 141707 Russia

Solutions of the Maxwell equations for electrostatic systems at rest on the rotating Earths surface are shown to exhibit a nonvanishing magnetic field despite zero electric currents in the system. Such a field is of pure geometric origin, and in contrast to the conventional magnetic field of the Earth it can not be screened away my magnetic shielding. A practical significance of this magnetic field is that its EDM-like background signal would persist in the much discussed searches for the EDM of charged particles of magic energy trapped in all electric storage rings, as well as in the neutron EDM experiments.

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I. INTRODUCTION

In this work we raise a generic issue of Maxwell equations in curved space-time. Evidently, the purely electrostatic system which is at rest on rotating Earth generates electric currents and therefore the magnetic field in the system of fixed stars. Could there be traces of this magnetic field, visible to the experimenter in the laboratory at rest on the rotating Earth? If such a field were possible, then it would radically differ from external fields like the magnetic field of the Earth, as it can not be screened by a magnetic shielding. This property could be of practical importance for super-precision experiments like the search for EDM.

Let’s consider a space-time with stationary metric in the frame of Einstein’s general theory of relativity. The local coordinates are denoted as $x^\mu$, $\mu = 0, 1, 2, 3 = (0, i)$, $i = 1, 2, 3$. The stationarity means that all components of metric tensor $g_{\mu\nu}(x)$ do not depend on the time coordinate, i.e. $(\partial/\partial x^0)g_{\mu\nu} = 0$. It is suggested that

\[ g_{0i} \neq 0. \]  

The outlined coordinate system is denoted as $K$. In the more interesting Example II (see Section IV) the reference frame $K$ is precisely the laboratory frame used in description of the terrestrial experiments. We use also the reference frame of distant stars $K'$. The Earth (and so the system $K$) rotates relative to the system $K'$ with the daily angular velocity $\omega$.

We consider the electrostatic system at rest on the rotating Earth in the system $K$. It is supposed that the electric current is stationary also:

\[ (\partial/\partial x^0)J^\mu(x) = 0, \quad J^0 \neq 0, \quad J^i = 0. \]

We formulate the main result of the paper: For the stationary metric (1.1) in the case of stationary electric current (1.2) with $J^0 \neq 0, J^i = 0$ the magnetic field can not be equal to zero. The reason is that the homogeneous Maxwell equations can not be satisfied for zeroth magnetic field in the case aforesaid.

The article is organized as follows. In Section II the necessary formulae concerning curved geometry and the dynamics in the curved space-time of a charged particle, including the dynamics of its polarisation, are given. In particular, since the space-time is curved, some comments are given regarding what is meant by a magnetic field: the magnetic field is that component of electromagnetic field which induces the magnetic moment precession, even if the particle with magnetic moment is at rest, i.e. if $d x^i / ds = 0$ for the particle world line.

* e-mail:vergeles@itp.ac.ru
In section III we show that in the case of stationary metric with \( g_{0i} \neq 0 \) and for stationary current distribution the magnetic field cannot be equal to zero. This fact follows from Maxwell equations. The effective equation for the magnetic field is obtained.

In section IV three examples are considered. In the first example exact solution for non-zero magnetic field is given. The second example is essentially more complicated and physically interesting because of the EDM registration experiments on purely electric proton accelerator are prepared.

In section V some estimations and conclusion are made.

II. GEOMETRY, MAXWELL AND DYNAMIC EQUATIONS

A. Geometry

We consider stationary metric. Following the textbook let’s diagonalize this quadratic form:

\[
d s^2 = g_{00} \left( d x^0 + \frac{g_{0i} d x^i}{g_{00}} \right)^2 - \left( -g_{ij} + \frac{g_{0i} g_{0j}}{g_{00}} \right) d x^i d x^j = \eta_{ab} \left( e^a_\mu d x^\mu \right) \left( e^b_\nu d x^\nu \right),
\]

where the field \( e^a_\mu(x) \) is called as tetrad, \( a, b, \ldots = 0, 1, 2, 3, a = (0, \alpha), \alpha = 1, 2, 3, \eta_{ab} = \text{diag}(1, -1, -1, -1) \). Two infinitesimally close events are simultaneous if 1-form

\[
e^{\alpha}_0 \ d x^\mu = \sqrt{g_{00}} \left( d x^0 + \frac{g_{0i} d x^i}{g_{00}} \right) = 0.
\]

According to (2.1) and (2.2) the squared interval between simultaneous events is

\[
d s^2 = \left( -g_{ij} + \frac{g_{0i} g_{0j}}{g_{00}} \right) d x^i d x^j = \sum_{\alpha=1}^3 \left( e^\alpha_i \ d x^0 \right) \left( e^\alpha_j \ d x^0 \right).
\]

The local orthonormal basis (ONB) \( \tilde{e}^a_\mu(x) \) is defined by the equations

\[
e^a_\mu(x) \tilde{e}^\mu_\alpha(x) = \delta^a_\alpha, \quad g_{\mu\nu} \tilde{e}^a_\mu \tilde{e}^b_\nu = \eta_{ab}.
\]

Since according to (2.3)

\[
e^0_\alpha = 0,
\]

one readily finds:

\[
\tilde{e}^i_0 = 0, \quad \tilde{e}^0_0 = \left( e^0_0 \right)^{-1}, \quad \tilde{e}^i_\alpha = \delta^i_\alpha, \quad \tilde{e}^\alpha_0 = - \left( e^0_0 \right)^{-1} e^\alpha_i e^i_\alpha.
\]

The rules of the tensor component transitions from coordinate basis to ONB and vice versa are standard. For example

\[
X^a = e^a_\mu X^\mu, \quad X^\mu = e^\mu_a X^a, \quad \xi_a = e^\mu_a \xi_\mu.
\]

In ONB the tensor indices are lowered and raised with the help of metric tensors \( \eta_{ab} \) and \( \eta^{ab} \). The accepted tetrad gives the equalities \( J^0 = 0 \) and \( J^\alpha = 0 \) equivalent:

\[
J^a = e^a_\alpha J^\alpha = \left( e^0_0 J^0, e^\alpha_i J^i \right) = \left( e^0_0 J^0, 0, 0, 0 \right).
\]

The covariant derivatives \( \nabla_\mu \) in the coordinate basis and in ONB are related as

\[
\nabla_\mu X^a = e^a_\nu \nabla_\mu X^\nu = \partial_\mu X^a + \gamma^a_{\beta\mu} X^\beta,
\]

\[
\nabla_c X^a = e^a_\nu \nabla_c X^\nu = e^\mu_c \partial_\mu X^a + \gamma^a_{\beta\mu} X^\beta,
\]

where \( \gamma^a_{\beta\mu} \equiv e^a_\nu \gamma^\nu_{\beta\mu} \) is the totality of the connection coefficients, and

\[
\gamma_{abc} = \eta_{ad} \gamma^d_{bc} = -\gamma_{bac}.
\]
The fact that the connection is free of torsion is fixed by the equations

\[
\partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \gamma^a_{b\mu} e^b_\nu - \gamma^a_{b\nu} e^b_\mu = 0. \tag{2.10}
\]

The connection coefficients are determined identically by Eqs. (2.9) and (2.10):

\[
\gamma_{abc} = \frac{1}{2} (C_{abc} - C_{bac} - C_{cab}),
\]

\[
C_{abc} = \eta_{ad} C^d_{bc}, \quad C^a_{bc} = e^a_{\omega, i} \left( (\varepsilon^i_{b}^e c^e_b - \varepsilon^i_{e}^b e^e_b) \right). \tag{2.11}
\]

To go further, we make the following assumption regarding the metric. Let the coordinate system \( K \) be the system which is rigidly bounded system with rotating Earth or globe. The Earth rotates relative to reference frame of distant stars \( K' \) with constant angular velocity \( \omega \). The local coordinates, vectors etc. in \( K' \) are denoted as \( x^\mu, R' \) and so on. In the system \( K' \) we have the Kerr metric of rotating Earth. For the purposes of our analysis it is sufficient yo use a limit of weak gravity and nonrelativistic rotation velocity. The corresponding expansion of the Kerr metric to linear terms in the Earth gravitational radius \( r_g \) and bilinear in \( r_g \) and \( \omega \) yields:

\[
g'_{00}(R') = 1 - \frac{r_g}{R'}, \quad g'_{0i}(R') = \frac{2kC}{c^2 R^3} [\omega, R']^i = I \frac{r_g R^4_{\oplus}}{R^3} \frac{[\omega, R']^i}{c}, \quad g'_{ij}(R') = - \left( 1 + \frac{r_g}{R'} \right) \delta_{ij}, \tag{2.12}
\]

where \( R' = |R'|, \ k = 6.674 \cdot 10^{-8} \text{cm}^3 \cdot \text{g}^{-1} \cdot \text{sec}^{-2} \) is the gravitation constant, \( M_{\oplus} \) and \( C = IM_{\oplus} R^3_{\oplus} \) are the Earth mass and moment of inertia relative to polar axis, \( I = 0.3307, M_{\oplus} = 5.972 \cdot 10^{27} \text{g}, R_{\oplus} = 6.378 \cdot 10^8 \text{cm}, \ r_g = 2kM_{\oplus}/c^2 = 0.887 \text{cm}. \) The same metric can be obtained with the help of Einstein's linearized equations. Next we transform the metric (2.12) into the metric in the system \( K \) rotating with angular velocity \( \omega \) relative to the system \( K' \). We take the coordinates in \( K \) and \( K' \) having the same origin at the centre of Earth. The local coordinates in the system \( K \) are denoted as \( x^\mu \) and they are connected with coordinates \( x'^\mu \) by definition as follows:

\[
dx^0 = dx^0, \quad dR' = dR + [\omega, R] \frac{dx^0}{c}, \quad |R'| = |R|. \tag{2.13}
\]

With the help of relations (2.13) the metric (2.12) is transformed into the metric \( g_{\mu\nu} \) in the system \( K \):

\[
g_{00} = 1 - \frac{r_g}{R} - \frac{[\omega, R]^2}{c^2}, \quad g_{0i} = \left\{ 1 + \frac{r_g}{R} \left( 1 - I \frac{R^2_{\oplus}}{R^2} \right) \right\} \frac{[\omega, R]^i}{c}, \quad g_{ij} = - \left( 1 + \frac{r_g}{R} \right) \delta_{ij}. \tag{2.14}
\]

Here we have retained the terms linear in \( \omega \) and \( r_g \), bilinear in \( r_g \) and \( \omega \) and quadratic in \( \omega \).

Eqs. (2.11) and (2.14) show that

\[
e^0_0 = \sqrt{g_{00}} = 1 - \frac{r_g}{2R} - \frac{[\omega, R]^2}{2c^2}, \quad e^0_0 = 0,
\]

\[
e_0^0 = \frac{g_{00}}{\sqrt{g_{00}}} = \left\{ 1 + \frac{r_g}{R} \left( \frac{3}{2} - I \frac{R^2_{\oplus}}{R^2} \right) \right\} \frac{[\omega, R]^i}{c},
\]

\[
e_0^i = \left( 1 + \frac{r_g}{2R} \right) \delta^i_0 + \frac{[\omega, R]^i}{2c^2}, \tag{2.15}
\]

in the same approximation. The ONB vector fields are

\[
e_0^0 = 1 + \frac{r_g}{2R} + \frac{[\omega, R]^2}{2c^2}, \quad e_0^0 = 0,
\]

\[
e_0^0 = \left\{ 1 + \frac{r_g}{R} \left( \frac{3}{2} - I \frac{R^2_{\oplus}}{R^2} \right) \right\} \frac{[\omega, R]^i}{c},
\]

\[
e_0^i = \left( 1 - \frac{r_g}{2R} \right) \delta^i_0 + \frac{[\omega, R]^i}{2c^2}, \tag{2.16}
\]

and the inverse metric tensor \( g^{\mu\nu} = \eta^{ab} e^a_\mu e^b_\nu \) is

\[
g^{00} = 1 + \frac{r_g}{R}, \quad g^{0i} = - \left\{ 1 + \frac{r_g}{R} \left( 1 - I \frac{R^2_{\oplus}}{R^2} \right) \right\} \frac{[\omega, R]^i}{c}, \quad g^{ij} = - \left( 1 - \frac{r_g}{R} \right) \delta_{ij} + \frac{[\omega, R]^i [\omega, R]^j}{c^2}. \tag{2.17}
\]
For the correct interpretation of the electromagnetic fields in a curved space-time it is useful to have in mind the form of dynamic equations in the Riemann normal coordinates. The Riemann normal coordinates $y^\mu$ can be introduced in the vicinity of any point $p$, so that $y^\mu(p) = 0$ and

$$e^a_\mu(y) = \delta^a_\mu + \frac{1}{6} \mathcal{R}^a_{\nu\lambda\mu}(p)y^\nu y^\lambda + O(y^3),$$

$$g_{\mu\nu}(y) = \eta_{\mu\nu} + \frac{1}{3} \mathcal{R}_{\mu\lambda\nu\rho}(p)y^\lambda y^\rho + O(y^3), \quad \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1),$$

$$\gamma_{ab\mu}(y) = \frac{1}{2} \mathcal{R}_{ab\nu\mu}(p)y^\nu + O(y^2),$$

(2.18)

where $\mathcal{R}_{ab\nu\mu}$ is the Riemann curvature tensor. Since near the Earth surface $|\mathcal{R}_{ab\nu\mu}| \sim r_g/R^3 \sim 0.5 \cdot 10^{-26}$ cm$^{-2}$, so one can ignore the space-time curvature in the small vicinity of point $p$. This vicinity with Riemann normal coordinates $\{y^\mu\}$ is the mathematical model of a "freely falling lift". Since the scale of the "lift" $a$ can be taken arbitrarily small, then the value of $|a^2 \mathcal{R}_{ab\nu\mu}| \ll 1$ turns out to be arbitrarily small inside the "lift". Therefore one can consider the coordinate system $\{y^\mu\}$ as a Cartesian one inside the "lift". So all dynamic equations are of the same form as in the flat space-time if we use the local coordinates $\{y^\mu\}$ inside the "lift".

B. Maxwell equations

Let the electromagnetic field (2-form) be expressed through 4-potential $A_\mu$ (1-form) in local coordinates as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$  
(2.19)

Then the homogeneous Maxwell equations

$$\varepsilon_{\mu\nu\lambda\rho} \partial_\nu F^{\lambda\rho} = 0$$  
(2.20)

are satisfied automatically. The field (2.19) is called as holonomic field.

We have also inhomogeneous Maxwell equations in local coordinates:

$$\frac{1}{\sqrt{-g}} \partial_\nu \left( \sqrt{-g} F^{\nu\mu} \right) = 4\pi J^\mu, \quad g \equiv \det g_{\mu\nu}, \quad F^{\mu\nu} = g^{\mu\lambda} g^{\nu\rho} F_{\lambda\rho}.$$  
(2.21)

The electric and magnetic fields in ONB are defined by the usual rules:

$$E^\alpha = F^{\alpha0} = e^\alpha_\mu e^0_\nu F^{\mu\nu} = e^\alpha_i e^0_0 F^{ij} + e^\alpha_0 e^0_j F^{ij},$$  
(2.22)

$$\varepsilon_{\alpha\beta\gamma} H^\gamma = -F^{\alpha\beta} = -e^\alpha_\mu e^\beta_\nu F^{\mu\nu} = -e^\alpha_i e^\beta_j F^{ij}, \quad \varepsilon_{123} = 1.$$  
(2.23)

Here we have used the property $e^\alpha_0 = 0$.

C. The dynamic equations

Let $d s = \sqrt{g_{\mu\nu}} d x^\mu d x^\nu$ be infinitely small interval when particle moving along its world line. We use the standard notations $DX^\mu/ds \equiv u^\nu \nabla_\nu X^\mu$, where $u^\nu \equiv d x^\nu/ds$, so $g_{\mu\nu} u^\mu u^\nu = 1$. According to the definition and Eqs. (2.8), (2.9) we have

$$\frac{DX^a}{ds} = \frac{dX^a}{ds} + \left( \gamma^a_{b\rho} \frac{dx^\mu}{ds} \right) X^b = \frac{dX^a}{ds} + \left( \gamma^a_{bc} u^c \right) X^b.$$  
(2.24)

Then equation of motion of a charged particle in local coordinates has the form

$$mc \frac{Du^\mu}{ds} = \frac{q}{c} F^{\mu\nu} u_\nu.$$  
(2.25)
Evidently, inside the "freely falling lift" in the Riemann normal coordinates (2.18) the dynamic equations are of the same form as in a flat space-time with Cartesian coordinates:

\[
\frac{d\gamma}{dt} = \frac{q}{mc} E_\beta, \quad \frac{d(\gamma\beta^\alpha)}{dt} = \frac{q}{mc} (E + [\beta, H])^\alpha, \quad u^a = \gamma(1, \beta), \quad \beta \equiv v/c, \quad ds = \frac{c}{\gamma} dt.
\] (2.26)

Here \(d t\) is the proper time of a laboratory frame of reference relative to which the particle moves with the velocity \(v\).

The spin precession problem is meaningful only for ONB. The complete description of spin precession of relativistic particle in gravity with MDM and EDM is found in works [2]-[10]. If \(S\) denotes spin polarization in the particle rest frame, we have

\[
\frac{dS}{dt} = [\Omega, S], \quad \Omega = \frac{q}{mc} \left\{ - \left( G + \frac{1}{\gamma} \right) H + \gamma G(H\beta) \right\} + \left( G + \frac{1}{\gamma + 1} \right) [\beta, E] + \eta_{\text{EDM}} \left( -E + \frac{\gamma}{\gamma + 1} (E\beta) + [H, \beta] \right) \right\}. \] (2.27)

Here

\[
\mu = (G + 1) \frac{q\hbar}{2mc}, \quad \delta = \eta_{\text{EDM}} \frac{q\hbar}{2mc}
\]

are MDM and EDM, correspondingly.

Thus we see that the fields \(E\) and \(H\) in Eqs. (2.20) and (2.21) have its conventional meaning.

If we come to arbitrary coordinates and ONB, then we must use covariant derivative \(D/ds\) (or \((cD)/(\gamma ds)\)) instead of the derivative \(d/ds\) (or \(d/dt\)) according to Eq. (2.24). Thus the connection appears in equations of motion. It is important that the fields \(E\) and \(H\) differ only by a Lorentz transform in different ONB, so they maintain the status. As a result Eq. (2.26) transforms into the following one:

\[
\frac{d\gamma}{dt} = \frac{q}{mc} E_\beta + c(\gamma_{\alpha0c} u^c) \beta^\alpha, \quad \frac{d(\gamma\beta^\alpha)}{dt} = \frac{q}{mc} (E^\alpha + [\beta, H]^\alpha) + c \left( (\gamma_{\alpha0c} u^c) + (\gamma_{\alpha\beta\gamma} u^\gamma) \beta^\beta \right).
\] (2.28)

Since the polarisation vector \(S\) refer to the particle rest frame, so the connection contribution into precession frequency (2.27) is enough lengthy for relativistic particle. Therefore we give here only the precession frequency for nonrelativistic case:

\[
\Omega^\alpha = -\frac{q}{mc} \left\{ (G + 1)H + \eta_{\text{EDM}} E \right\}^\alpha - \frac{c}{2} \varepsilon_{\alpha\beta\gamma} \gamma_{\beta\delta}.
\] (2.29)

The general conclusion of the subsection is that the fields \(E\) and \(H\) in ONB defined according (2.22) and (2.23) possess all properties of the electric and magnetic fields, correspondingly. The summands in (2.28) and (2.29) proportional to connection are necessary because of the ONB is used in the fairly wide range of space-time where the Riemann normal coordinates become useless. Then the connection coefficients are determined according to (2.11).

### III. MAGNETIC FIELD IN THE SYSTEM \(K\) FOR CURRENT DISTRIBUTION (1.2)

In this Section we show that the Maxwell equations do not have any solution with \(H = 0\) in the case of static metrics with \(g_{0i} \neq 0\) and electrostatic current distribution (1.2). The reason for this is as follows: the homogeneous Maxwell equations (2.20) cannot be satisfied for \(H = 0\) in the interesting case (see Eq. (3.7)).

Equations (2.21) for stationary case with \(\mu = 1, 2, 3\) imply \(\partial_i \left( \sqrt{-g} F^{ij} \right) = 0\). The general solution of the equation is

\[
\sqrt{-g} F^{ij} = \varepsilon_{ijk} \partial_k \psi.
\] (3.1)
According to the rules \( (2.7) \), the field definition \( (2.23) \) and the fact that \( e^0_0 = 0 \), we obtain:

\[
F^{ij} = e^i_0 e^j_b F^{ab} = -\varepsilon_{\alpha\beta\gamma} e^i_\alpha e^j_\beta H^\gamma = -\frac{e^0_0}{\sqrt{-g}} \varepsilon_{ijk} e^0_k H^\alpha. \tag{3.2}
\]

In what follows we use the relations

\[
\varepsilon_{abcd} e^a_i e^b_j = \frac{1}{\sqrt{-g}} \varepsilon_{\mu
u\lambda\rho} e^c_\mu e^d_\nu, \quad \varepsilon_{abcd} \tilde{e}^a_\alpha \tilde{e}^b_\beta \tilde{e}^c_\gamma \tilde{e}^d_\delta = \frac{1}{\sqrt{-g}} \varepsilon_{\mu\nu\lambda\rho} e^0_\mu e^0_\nu e^0_\lambda e^0_\rho,
\]

which imply that

\[
\varepsilon_{\alpha\beta\gamma} e^0_i e^0_j = \frac{e^0_0}{\sqrt{-g}} \varepsilon_{ijk} e^0_k, \quad \varepsilon_{\alpha\beta\gamma} \tilde{e}^0_i \tilde{e}^0_j = \frac{1}{\sqrt{-g}} \varepsilon_{\alpha\beta\gamma} e^0_0 e^0_0 e^0_0 , \tag{3.3}
\]

and so on. Making use of \( (3.1) \) in \( (3.2) \) yields

\[
H^\alpha = - (e^0_0)^{-1} \varepsilon^i_\alpha \partial_i \psi. \tag{3.4}
\]

Now we turn to the homogeneous Maxwell equations \( (2.20) \). We express the holonomic field \( F_{\mu\nu} \) in terms of electric and magnetic fields:

\[
F_{i0} = e^0_i e^0_0 F_{00} = e^0_i e^0_\alpha F_{0\alpha} = -e^0_0 e^0_i E^\alpha = \partial_i A_\alpha, \quad \text{or} \quad E^\alpha = -\frac{\varepsilon}{e^0_0} F_{i0}, \tag{3.5}
\]

\[
F_{ij} = e^i_0 e^j_0 F_{00} = -\varepsilon_{\alpha\beta\gamma} e^0_i e^0_j H^\gamma - (e^0_0)^{-1} e^0_\alpha e^0_\beta \ E^\alpha. \tag{3.6}
\]

Eq. \( (2.20) \) with \( \mu = 0 \) imply the identity \( \varepsilon_{ijk} \partial_k F_{ij} = 0 \). Therefore, applying the operator \( \varepsilon_{ijk} \partial_k \) to the Eq. \( (3.6) \), using \( (3.4) \) and the identity \( \varepsilon_{ijk} \partial_k F_{00} = -\varepsilon_{ijk} \partial_k (e^0_0 e^0_1 E^\alpha) = 0 \) (see \( 3.3 \)), we obtain the equation

\[
\partial_i \left( \sqrt{-g} (e^0_0)^{-2} g^{ij} \partial_j \psi \right) = -\varepsilon_{ijk} e^0_0 E^\alpha \partial_k \left( \frac{e^0_0}{e^0_0} \right). \tag{3.7}
\]

Recall that \( e^0_0 = g_{00} / \sqrt{|g_{00}|} \neq 0 \) according to \( (2.15) \) and \( (1.1) \). Therefore if \( E \neq 0 \), then \( (3.7) \) entails \( \partial_i \psi \neq 0 \), and according to Eq. \( (3.21) \) \( H \neq 0 \) as well. Note that no stationary current distribution can annul the total magnetic field. Indeed, how \( F^{ij} \) (and thus \( H \) according to \( (3.2) \)) can be equal to zero for nonzero right-hand side in Eq. \( (2.21) \) with \( \mu = 1, 2, 3 \)? We could annul the magnetic field only locally by means of a stationary current distribution, but not globally.

Now we formulate the main result of the paper: For the stationary metric \( (1.1) \) in the case of stationary electric current \( (1.2) \) with \( J^0 \neq 0 \), \( J^i = 0 \) the magnetic field can not be equal to zero. The reason is that the homogeneous Maxwell equations can not be satisfied for zeroth magnetic field in the case aforesaid.

Let’s express \( F^{i0} \) in terms of physical fields:

\[
F^{i0} = \varepsilon^i_0 e^0_b F^{0b} = e^0_\alpha e^0_\beta E^\alpha - \varepsilon_{\alpha\beta\gamma} e^0_\alpha e^0_\beta H^\gamma = \frac{1}{e^0_0} \varepsilon^i_\alpha E^\alpha - \frac{1}{\sqrt{-g} e^0_0} \varepsilon_{ijk} e^0_k \partial_k \psi. \tag{3.8}
\]

The substitution of the right-hand side of \( (3.8) \) into Eq. \( (2.21) \) with \( \mu = 0 \) leads to the following one:

\[
\partial_i \left( \sqrt{-g} (e^0_0)^{-2} e^i_\alpha E^\alpha \right) - \varepsilon_{ijk} \partial_i \left( e^0_0 \right) \partial_k \psi = 4\pi \sqrt{-g} J^0. \tag{3.9}
\]

Here we have used representation \( (3.4) \) and one of the relations \( (3.3) \).

The system of equations \( (3.3), (3.7) \) and \( (3.9) \) is complete and exact, i.e. it is true for any metric. The electric field \( E \) is expressed in terms of the potential \( A_\alpha \) with the help of relation \( (3.5) \).

To find electromagnetic fields up to \( \omega, r_g, r_g \omega, \omega \otimes \omega \), let’s rewrite Eqs. \( (3.4), (3.7) \) and \( (3.9) \) with the help of \( (2.14)-(2.17) \). Let’s denote as \( \psi(s) \), \( s = 0, 1, 2 \), the contribution into the field \( \psi \) of the order of zero, one and two
relative to $\omega$, correspondingly. The same notation is used for the rest fields. Since $e^0_j/e^0_i = O(\omega)$, so according to (3.7) $\psi^{(0)} = 0$, $\psi^{(1)} = O(\omega)$. We have also $g = g^{(0)}$ and

$$
\sqrt{-g^{\ddagger}} g^{ij} = \left(1 + \frac{r_g}{R}\right) \delta^i_j + \left(\frac{[\omega, R]}{2c^2} \delta^i_j - \frac{[\omega, R]^{\ddagger}[\omega, R]^{\alpha}}{2c^2}\right).
$$

Equations (3.9) and (3.10) show that $E^{(0)} \neq 0$, $E^{(1)} = 0$, $E^{(2)} \neq 0$, and then Eq. (3.7) guarantees that $\psi^{(2)} = 0$. There is the relation

$$
\frac{\tilde{e}^i}{e_0^i} = \delta^i_j + \left(\frac{[\omega, R]}{2c^2} \delta^i_j - \frac{[\omega, R]^{\ddagger}[\omega, R]^{\alpha}}{2c^2}\right).
$$

Thus according to (3.4)

$$
H_\omega = -\nabla \psi, \quad H_\omega = H^{(1)}_0, \quad \psi \equiv \psi^{(1)},
$$

and $H_\omega^{(0)} = H_\omega^{(2)} = 0$. In what follows the lower index $\omega$ denotes the fact that the rotation of system $K$ relative to the system $K'$ is the source of the field $H_\omega$.

Due to the relations

$$
\sqrt{-g} g^{ij} = -\left(1 + \frac{r_g}{R}\right) \delta^i_j, \quad e^0_0 e^i_j = \delta^i_j + O(\omega \otimes \omega), \quad e^0_j = -\left(1 + \frac{r_g}{R}\right) \left(1 - \frac{IR_\omega}{R^2}\right) \frac{[\omega, R]^{\ddagger}}{c},
$$

we rewrite Eq. (3.7) as follows:

$$
\varepsilon_{ij} \left(1 + \frac{r_g}{R}\right) \partial_i \psi = e_{ij} E^{(0)j\alpha} \delta_{\alpha} \left[1 + \frac{r_g}{R}\left(1 - \frac{IR_\omega}{R^2}\right)\right] \frac{[\omega, R]^{\ddagger}}{c}.
$$

Since $A_0 = A_0^{(0)} + A_0^{(2)}$ and in view of Eq. (3.11) the last equation in (3.5) falls into two equations:

$$
E^{(0)} = -\nabla A_0^{(0)},
$$

$$
E^{(2)} = -\nabla A_0^{(2)} - \left[\frac{[\omega, R]}{2c^2} \nabla A_0^{(0)} \right] + \left(\frac{[\omega, R]^{\ddagger}}{2c^2}\right).
$$

Equation (3.9) also falls into two equations if we take into account Eqs. (3.10), (3.13) and the fact that here $e^0_j/e^0_i = -[\omega, R]/c$ with sufficient accuracy:

$$
-\nabla \left(1 + \frac{r_g}{R}\right) \nabla A_0^{(0)} = 4\pi \left(1 + \frac{r_g}{R}\right) J^0,
$$

$$
-\Delta A_0^{(2)} = \frac{1}{c^2} \nabla \left(\frac{[\omega, R]}{2c^2} \nabla A_0^{(0)} - \left(\frac{[\omega, R]^{\ddagger}}{2c^2}\right) |\omega, R| \right) - \frac{2}{c} (\omega \nabla \psi).
$$

The system of equations (3.12), (3.14)-(3.18) is complete and exact with desired accuracy.

The subsequent calculations are performed under the following assumptions: 1) we neglect the value $r_g/R_\odot \sim 10^{-9}$ (the estimation is true near the Earth surface) and the moment of inertia $I$ (its contribution into metric is commensurable with the contribution of $r_g/R_\odot$ near the Earth surface); 2) all calculations are made up to $O(\omega)$. The last approximation is justified as long as $|\omega, R|/c \ll 1$.

Here $R$ is the radius-vector from the origin of coordinates into observation point.

According to (2.14), (2.17)

$$
g_{00} = g^{00} = 1, \quad g_{0i} = g^{0i} = -\frac{[\omega, R]^{\ddagger}}{c}, \quad g_{ij} = \bar{g}_{ij} = g^{ij} = -\delta^{ij}, \quad g = \bar{g} = -1, \quad e_{00}^{0} = 1, \quad e_{i}^{0} = -\frac{[\omega, R]^{\ddagger}}{c}, \quad e_{i}^{\alpha} = \delta_{i}^{\alpha}, \quad e_{0}^{\alpha} = 0,
$$

$$
e_{0}^{0} = 1, \quad e_{0}^{\alpha} = \frac{[\omega, R]^{\ddagger}}{c}, \quad e_{0}^{\alpha} = \delta_{0}^{\alpha}, \quad e_{0}^{\alpha} = 0.
$$

$$
\frac{c}{2} \varepsilon_{\alpha\beta\rho\gamma} = \omega^{\alpha}.
$$
under outlined assumptions.

Now Eqs. (3.16) and (3.18) becomes excess, Eqs. (3.15) and (3.17) transform into equation

$$\mathbf{E} = -\nabla A_0$$  

(3.20)

and Poisson equation

$$\text{div}\mathbf{E} = -\Delta A_0 = 4\pi J^0,$$  

(3.21)

correspondingly. Equation (3.12) remains invariant, Eq. (3.14) transforms into the following one:

$$\Delta \psi = \frac{2}{c} (\omega \mathbf{E})$$  

(3.22)

In the following section we give two examples.

IV. EXAMPLES

A. Example I. Rotating conducting charged solid sphere

Let’s consider conducting charged solid sphere of radius $a$ which rotates relative to the system $K'$ with angular velocity $\omega$. All calculations are made in the rotating system $K$ in which the sphere is at rest.

For the particle at rest Eq. (2.28) reduces to the following one

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m} \mathbf{E}.$$  

Therefore, as usual in conductors, for the equilibrium of a charged particle it is necessary to have $\mathbf{E} = 0$ inside the sphere. It follows from here according to (3.20) that $A_0 = \text{const}$ inside the sphere. Eqs. (3.20) and (3.21) show that

$$E(r) = \begin{cases} \frac{Qr}{r^3}, & \text{for } r > a, \\ 0, & \text{for } r < a. \end{cases}$$  

(4.1)

Here $Q$ is the charge of the sphere and the radius-vector $r$ connects the centre of the sphere with an observation point. According to (3.22)

$$\psi(r) = -\frac{\omega}{2\pi c} \int \frac{1}{|r - \mathbf{x}|} \mathbf{E}(\mathbf{x}).$$  

(4.2)

The last integral is easily taken:

$$\psi(r) = -\frac{Q(\omega r)}{3c} \begin{cases} \frac{3}{r} - \frac{a^2}{r^3}, & \text{for } r > a, \\ 2/a, & \text{for } r < a. \end{cases}$$  

(4.3)

Now (4.3) and (3.12) lead to

$$H_\omega(r) = \begin{cases} \frac{1}{r^3} \left\{ \frac{3}{c} \{3(\mu n)\mathbf{n} - \mu\} + \frac{1}{c} [\mathbf{E}(r), [\omega, \mathbf{r}]] \right\}, & \mathbf{E}(r) = \frac{Qr}{r^3} \text{ for } r > a, \\ \frac{2Q\omega}{3ca}, & \text{for } r < a, \end{cases}$$  

(4.4)

where

$$\mathbf{n} = \frac{r}{r}, \quad \mu = \frac{Qa^2}{3c} - \omega.$$
Here $\boldsymbol{\mu}$ is the magnetic moment of conducting charged solid sphere of radius $a$ rotating with angular velocity $\omega$ in the system $K'$. The second summand in (4.4) is the familiar motional magnetic field in system $K$ due to existence of electric field in system $K'$.

We see, that for $r > a$ the magnetic field has two components: the first component is the field of magnetic moment $\boldsymbol{\mu}$ in the system $K'$ which does not change as a result of recalculation into system $K$ within the framework of necessary accuracy; the second component is the field arising as a result of transformation of coulomb field $\mathbf{E}$ from system $K'$ onto system $K$. The point $r$ in $K$ has the local velocity relative to the same point in $K'$ equal to $[\omega, \mathbf{r}]$.

It seems that the calculation in the system $K$ straightly with the help of formulaes given in Section III is more reasonable even in the simple considered example than the realization of the following calculational scheme: 1) transition from the system $K$ to the system $K^{(0)}$ (which is identical with the system $K'$ in the case (3.19)) in which $b^{(0)} = 0$; 2) solution of Maxwell equations in the system $K^{(0)}$; 3) transformation of the obtained solution into system $K$.

Indeed, the step 1) leads to a situation when all components of the current $J^{\mu}$ becomes non-zero. Moreover, the current distribution becomes time dependent in a general case. So the step 2) becomes complicated. But there are instances where qualitative discussion of the steps 1) and 2) is useful (see our examples which are given below).

Let’s estimate the significance of $\mathbf{H}_{\omega}$ into spin precession in the case when EDM is zero, electric field is close to the vacuum breakdown and the background magnetic field is absent. Then Eq. (5.29) transforms into

$$\Omega = -(G + 1) \frac{q}{mc} \mathbf{H}_{\omega} - \mathbf{\omega}$$

in the used approximation (3.19). If the electric field is of the order of $10^7 \text{eV/m}$ and $a \sim 10 \text{cm}$, we have the estimations for proton and electron cases as

$$\left(\frac{q|\mathbf{H}_{\omega}|}{m_{p}c}\right)/|\mathbf{\omega}| \sim 10^{-3}, \quad \left(\frac{q|\mathbf{H}_{\omega}|}{m_{e}c}\right)/|\mathbf{\omega}| \sim 1.$$

The given estimations show that the contribution of $\mathbf{H}_{\omega}$ into spin precession of proton is smaller by a factor of $10^{-3}$ than that of the pure rotational contribution, but in the case of electron these two contributions are of the same magnitude.

**B. Example II. Unremovable magnetic field in purely electric proton accelerator**

The second example is essentially more complicated and physically interesting because of the EDM registration experiments on purely electric proton accelerator are prepared. As has been said, the unremovable $\mathbf{H}_{\omega}$ arises due to the Earth daily rotation. In such experiments two infinitesimal parameters compete. One of them is $\eta_{EDM}$ which is thought to be of the order of

$$\eta_{EDM} \sim 10^{-15}.$$  \hspace{1cm} (4.5)

Another of them, designated as $\eta_{\omega}$, is determined below. It is responsible for a smallness of $\mathbf{H}_{\omega}$ in comparison with the electric field $\mathbf{E}$.

The accelerator ring is plano-cylinder condenser having interspace between cylinders $d$ which is much less than the height of cylinders $h$, which in turn is much less than the cylinder radii $r_{1,2} = \rho \pm d/2$:

$$d \ll h \ll \rho.$$

Introduce local Cartesian coordinates $(x, y, z)$ in the system $K$ which is rigidly bounded with the rotating Earth, so that the coordinate centre and the accelerator ring centre are coincident and the vectors $\mathbf{r} = (x, y)$ reside in the central ring plane. Thus the upper and lower edges of cylinders have the coordinates $z = \pm h/2$. The equilibrium particles trajectory passes along the interspace centre between cylinders with coordinates $|\mathbf{r}| = \rho$ and $z = 0$. By virtue of the problem geometry one can neglect the fields dependence on $z$, i.e. one can put $\partial/\partial z = 0$. Thus the problem becomes two-dimensional in the plane $(x, y)$. The electric field is directed toward the center of the ring and it is equal to

$$\mathbf{E}_0 = -\mathbf{e}_0 \frac{\partial \mathbf{r}}{\partial z} = -\nabla A_0(r), \quad A_0(r) = \mathcal{E}_0 \rho \ln \frac{r}{\rho}$$  \hspace{1cm} (4.6)

in the plane $z = 0$ between cylinders.
At first we explain the effect by the example of a ring on North Celestial Pole when $\omega$ is perpendicular to the plane of accelerator ring. Static charges on the cylinders are of the opposite signs. The cylinders rotation with the Earth angular velocity means from the point of view of the observer rigidly bounded with inertial frame of reference $K'$ that there are two opposite currents in the system. These currents generate magnetic fields which are added constructively in the interspace between the cylinders for $r_1 < r < r_2$. In the system $K'$ the linear movement speeds of the charges on cylinders are $v(r_{1,2}) = |\omega, r_{1,2}|$, $|\omega| \sim 7.3 \cdot 10^{-5} \text{sec}^{-1}$. The simple calculation of the fields in the interspace gives an uniform magnetic field which is connected with the electric field by the relation

$$\mathbf{H}_\omega(r) = \frac{1}{c} [\mathbf{v}(r), \mathbf{E}_0(r)]. \quad (4.7)$$

At first sight, this parasitic magnetic field (from a EDM point of view) introduces an asymmetry between the beams which rotate clockwise and anticlockwise in the accelerator ring. But the current picture is more complicated at arbitrary latitude, and the compensation does not work. It is evident that the smallness parameter for the ring with radius $\rho \sim 50 \text{m}$ is

$$\eta_\omega = \frac{|\omega|\rho}{c} \sim 10^{-11}. \quad (4.8)$$

In contrast to the Earth’s magnetic field, the inner parasitic magnetic field $\mathbf{H}_\omega$ cannot be annihilated by a magnetic screening of the proton ring. In spite of infinitesimalty of $\eta_\omega$, an accurate analysis of Maxwell equations including the Earth rotation factor is necessary for interpretation of EDM search experiments with expected sensitivity $\eta_{EDM} \sim 10^{-15}$. Such specific problem about parasitic magnetic fields in the purely electrostatic proton ring has not been discussed previously. Here we perform the analysis of the problem with the help of formulas in Section III which are applied to the system $K'$ rigidly bounded with the rotating ring, so that we have the conditions (1.1), (1.2) and $\frac{\partial}{\partial x^0} = 0$.

In agreement with (4.6) we take the anzats for magnetic potential $\psi$ (see Eq. (3.12)) in the form

$$\psi = f(r) \cdot (\omega_t \mathbf{r}). \quad (4.9)$$

The lower index $t$ means that the vector component in the plane $(x, y)$ is taken. Then we have

$$\mathbf{H}_\omega = -\nabla \psi = -f \cdot \omega_t - f' \cdot (\omega_t \mathbf{r}) \frac{\mathbf{r}}{r}. \quad (4.10)$$

Eq. (3.22) now yields:

$$\Delta \psi = \left(\frac{3f'}{r} + f''\right)(\omega_t \mathbf{r}) = -\frac{2\xi_0\rho}{cr^2}(\omega_t \mathbf{r}). \quad (4.11)$$

The solution of this equation is

$$f(r) = -\frac{\xi_0\rho}{c} \left(\ln \frac{r}{\rho} + \zeta\right) = -\frac{A_\omega(r)}{c} - \frac{\xi_0\rho}{c} \zeta, \quad (4.12)$$

where $\zeta$ is some constant which has to be determined from the boundary conditions. Using the explicit form for $f(r)$ (4.12), let’s rewrite the right-hand side of Eq. (4.10) and decompose it into irreducible tensor structures:

$$\mathbf{H}_\omega^t = \left\{ \frac{A_\omega}{c} \delta_{ij} + \frac{\xi_0\rho}{c} \left[ \frac{1}{2} \delta_{ij} + \frac{1}{2} \left( \delta_{ij} - 2n_i n_j \right) \right] \right\} \omega_t^i. \quad (4.13)$$

where $n = \mathbf{r}/r$.

In order to account for the boundary condition with a view to find out the constant $\eta$ let’s write out the formal solution of Eq. (3.22). In view of (4.6)

$$\psi(r) = \frac{2\omega_t^i}{c} \int \frac{d(3)}{y} \Delta^{-1}(r-y) \mathbf{E}_0^i(y) = -\frac{2\omega_t^i}{c} \int \frac{d(3)}{y} \partial_j \Delta^{-1}(r-y) A_0(y). \quad (4.14)$$

Here $\Delta^{-1}(r) = (1/2\pi) \ln |r|$ is the operator inverse to the Laplace operator. Since the total charge of the condenser system is equal to zero and the potential $A_0(r)$ decreases fairly rapidly far apart the ring (this fact plays the role of
boundary condition), so the surface integral does not arises under transformations of (4.14). According to (4.10) the required magnetic field can be written as

\[ \mathbf{H}_\omega(r) = \frac{2\omega_j^t}{c} A_{ij}(r), \quad A_{ij}(r) = \int d^{(2)}y \, \partial_i \partial_j \Delta^{-1}(r - y) A_0(y). \] (4.15)

Note that the matrix \( A_{ij} \) is symmetrical one and

\[ \text{tr} A(r) = A_0(r). \] (4.16)

We decompose the matrix \( A_{ij} \) into irreducible tensor structures, in addition the Kronecker symbol coefficient is known already from (4.16):

\[ A_{ij}(r) = \frac{1}{2} \delta_{ij} A_0(r) + \sigma(r)(\delta_{ij} - 2n_i n_j). \] (4.17)

The comparison with (4.13) gives immediately

\[ \sigma(r) = \frac{1}{4} \mathcal{E}_0 \rho, \quad \zeta = \frac{1}{2}. \] (4.18)

Hence the problem is solved. In interspace between cylinders we have

\[ \mathbf{H}_\omega = \frac{\mathcal{E}_0 \rho}{c} \left( \ln \left( \frac{r}{\rho} \right) + \delta_{ij} + \left( \frac{1}{2} \delta_{ij} - n_i n_j \right) \right) \omega_j^t, \] (4.19)

C. Example III. Plane condenser

Now we consider the plane condenser with disks having coordinates \( x_1, x_2 = \pm d/2 \). Thus the disks are parallel to the plane \((y, z)\). The considered problem is one-dimensional since only the fields near the condenser centre are interesting here. We have

\[ \mathbf{E} = (\mathcal{E}_0, 0, 0) = \nabla A_0, \quad A_0 = -\mathcal{E}_0 x. \] (4.20)

According to (5.22)

\[ \psi(x) = \frac{2\omega_x}{c} \int dx' \Delta^{-1}(x - x') \mathbf{E}_x(x') = \frac{2\omega_x}{c} \int dx' \frac{d}{dx} \Delta^{-1}(x - x') A_0(x'), \] (4.21)

where \( \Delta^{-1}(x) = |x|/2 \). In agreement with (5.22)

\[ \mathbf{H}_\omega = -\nabla \psi(x) = \left( \frac{2\omega_x}{c} \int dx' \frac{q^2}{d^2 x^2} \Delta^{-1}(x - x') A_0(x'), 0, 0 \right) = \left( \frac{2\omega_x}{c} A_0(x), 0, 0 \right) = \left( -\frac{2\omega_x \mathcal{E}_0}{c} x, 0, 0 \right). \] (4.22)

This example can be interesting in the neutron EDM-registration experiments [11]

V. ESTIMATIONS AND CONCLUSION

Let’s consider a purely electrostatic proton storage ring, so that the current configuration is as in (1.2) and the Earth’s magnetic field is screened completely. But it has been shown that the magnetic field \( \mathbf{H}_\omega \) (4.19) can not be equal to zero. Considering the \( \mathbf{H}_\omega \) contribution into the relativistic spin dynamic equation as a small correction, let’s rewrite equation (2.27) in a convenient form:

\[ \Omega = \frac{q}{mc} \left( G + \frac{1}{\gamma + 1} \right) [\beta, \mathbf{E}_0] + \frac{q}{mc} \left\{ \eta_\text{EDM} \left( -\mathbf{E}_0 + \frac{\gamma}{\gamma + 1} (\mathbf{E}_0 \beta) \right) - \left( G + \frac{1}{\gamma} \right) \mathbf{H}_\omega + \frac{\gamma G(\mathbf{H}_\omega \beta)}{\gamma + 1} \beta \right\} + \ldots \] (5.1)

Here the dots mean the gravitational contribution into angular velocity which is not interesting here. The summand of the order of \( (\eta_\text{EDM} \eta_\omega) \) is also neglected (see estimations (4.5) and (4.8)).
The following fact is important. The electromagnetic contribution into cyclotron frequency $\Omega_c^{(EM)}$ (angular velocity of particle momentum rotation) for relativistic particle in the case $\mathbf{H} = 0$ is [8], [10]

$$\Omega_c^{(EM)} = \frac{q}{mc} \cdot \frac{\gamma}{\gamma^2 - 1} [\beta, \mathbf{E}_0]. \tag{5.2}$$

One can verify directly that for the ”magical” momentum [12]

$$\beta_m^2 = \frac{1}{1 + G} \tag{5.3}$$

the contribution into (5.1)

$$\Delta \Omega \equiv \frac{q}{mc} \left( G + \frac{1}{\gamma + 1} \right) [\beta, \mathbf{E}_0] = \Omega_c^{(EM)}. \tag{5.4}$$

Namely ”magical” momentum is interesting for experimenters. Evidently, the vectors $\Omega_c^{(EM)}$ [5.2] as well as $\Delta \Omega$ [5.4] are directed along $z$-axis. The last remark along with equality [5.4] mean that the angular precession of the polarization component $\mathbf{S}_1$ due to $\Delta \Omega$ coincides with angular precession of particle momentum in the case $\mathbf{H} = 0$. Here $\mathbf{S}_i$ denotes the polarization projection on the plane $(x, y)$. But the $\mathbf{S}_z$-component does not affected by the relatively large $\Delta \Omega$ and therefore $\mathbf{S}_z$-component remains almost constant during cyclotron period $T$. It follows from the above said that besides trivial equalities $(\mathbf{E}_0 | \beta) = \text{const}$, $\oint \mathbf{E}_0 \cdot d\mathbf{t} = 0$ and $\oint \beta \cdot d\mathbf{t} = 0$, we have also $(\beta | \mathbf{S}) \approx \text{const}$, $(\mathbf{E}_0 | \mathbf{S}) \approx \text{const}$ and $\oint \mathbf{S}_i \cdot d\mathbf{t} \approx 0$ along the ring for the ”magical” momentum [5.3]. Everywhere the symbol $\oint ... d\mathbf{t}$ means the integration over cyclotron period $T$. Only the angular difference of vector $\mathbf{S}$ over the cyclotron period is interesting. In summary, since $\mathbf{S}_z \approx \text{const}$ during cyclotron period, the term in (5.1) proportional to $\eta_{EDM}$ effects only on the dynamics of $\mathbf{S}_1$, but not on the dynamics of $\mathbf{S}_z$. On the contrary, the term in (5.1) which is proportional to $\mathbf{H}_\omega$ acts effectively only on $\mathbf{S}'$-component.

Therefore, according to (2.29), (4.19) and (5.1), the second little term in the right-hand side of (5.1) gives the following angular difference order-of-magnitude estimation:

$$\Delta_{EDM} \mathbf{S} \sim -\eta_{EDM} \frac{q}{mc} \oint [\mathbf{E}_0, \mathbf{S}_1] d\mathbf{t} \approx -\eta_{EDM} \frac{qT}{mc} [\mathbf{E}_0, \mathbf{S}_1]. \tag{5.5}$$

$$\Delta_\omega \mathbf{S} \approx \frac{q}{mc} \oint d\mathbf{t} \left\{ - \left( G + \frac{1}{\gamma} \right) \mathbf{H}_\omega + \frac{\gamma G (\mathbf{H}_\omega \beta)}{\gamma + 1} \mathbf{S} \right\} \approx - \frac{q}{mc} \cdot \frac{\mathcal{E}_0}{c} \left( \oint \left\{ \ln \left( \frac{r}{\rho} \right) \omega_t + \left( \frac{1}{2} \omega_t - n(n \omega) \right) \right\} d\mathbf{t}, \mathbf{S} \right]. \tag{5.6}$$

Note that according to (4.19) and the equality $(n | \beta) = 0$ we have

$$(\mathbf{H}_\omega \beta) | \beta = \frac{\mathcal{E}_0}{c} \left[ \frac{1}{2} + \ln \left( \frac{r}{\rho} \right) \right] (\omega_t | \beta) \beta = \frac{\mathcal{E}_0}{2c} (\omega_t | \beta) \beta. \tag{5.7}$$

Here we believed that the proton beam has some width $\rho - \epsilon < r(t) < \rho + \epsilon$ with $\epsilon \sim 1$ cm, so that

$$\left| \ln \left( \frac{r(t)}{\rho} \right) \right| \sim \frac{\epsilon}{\rho} \leq 10^{-3}. \tag{5.8}$$

Taking into account equalities

$$\oint d\mathbf{t} \left( \frac{1}{2} \delta_{ij} - n_i n_j \right) = 0, \quad \oint d\mathbf{t} \beta^j \beta^j = \frac{T}{2} \beta^2 \delta^{ij}; \tag{5.9}$$

and also (4.19) and (5.7), we obtain the following estimation for the integral (5.6):

$$\Delta_\omega \mathbf{S} \approx \frac{q\mathcal{E}_0}{mc^2} \left\{ - \left( G + \frac{1}{\gamma} \right) \ln \left( \frac{r}{\rho} \right) + \frac{\gamma G \beta^2}{4(\gamma + 1)} \right\} [\omega_t, \mathbf{S}] \approx \frac{q\mathcal{E}_0}{mc^2} \cdot \frac{\gamma G \beta^2}{4(\gamma + 1)} [\omega_t, \mathbf{S}]. \tag{5.10}$$

Here we used (5.8) again.

Comparing (5.5) and (5.10) we obtain

$$\frac{|\Delta_\omega \mathbf{S}|}{\Delta_{EDM} \mathbf{S}} \sim \frac{\eta_\omega}{\eta_{EDM}} \gg 1. \tag{5.11}$$

The last estimation is true for the case (4.9) and (4.10).

We see that the magnetic field $\mathbf{H}_\omega$, though it is very small ($\sim 3 \cdot 10^{-13} T$), can create some problems in the process of interpretation of EDM-registration experiments.
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