Jaeger’s Higman-Sims state model and the $B_2$ spider

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Jaeger [6] discovered a remarkable checkerboard state model based on the Higman-Sims graph that yields a value of the Kauffman polynomial, which is a quantum invariant of links. We present a simple argument that the state model has the desired properties using the combinatorial $B_2$ spider [10].

1. INTRODUCTION

Two related approaches to defining quantum topological invariants are skein relations and state models. One important example of the former is the Kauffman polynomial, while an important class of the latter is the class of checkerboard state models.

Given two numbers or indeterminates $Q$ and $d$, the Kauffman polynomial is a function on link projections on the 2-sphere defined axiomatically by the rules:

\[
\begin{align*}
\bigcirc & = (Q - Q^{-1}) \left( \left( - \right) - \left( + \right) \right) \\
\bigcirc & = Q^{d-1} + Q - Q^{-1} - Q^{-d} \\
\bigcirc & = Q^{d-1}
\end{align*}
\]

by the rule that it is invariant under the second and third Reidemeister moves:

\[
\begin{align*}
\bigcirc & = \\
\bigcirc & = \\
\bigcirc & = \bigcirc
\end{align*}
\]

and by the rule that its value at the empty link is 1. These rules are an example of a skein theory, a concept which can be understood with some elementary background.

A (tame) knot or link is represented by a knot projection (a tetravalent graph embedded in the 2-sphere with vertices decorated to distinguish over-crossings from under-crossings); it is known that a function on links which is invariant under the three Reidemeister moves (indicated above) is a function on links. It is further understood that invariance under the second and third Reidemeister moves (regular isotopy invariance) is almost as strong as invariance under all three, in a manner analogous to the difference between linear and projective representations of a group. In this paper, we will loosely call a regular isotopy invariant a link invariant. A skein theory describes a function on knot projections by axioms relating projections that differ only in a small region. Typically one implicitly considers an invariant $I$ and one writes graphical equations where a knot projection $P$ denotes $I(P)$. Furthermore, if an equation involves link fragments (tangles), the fragments should all have the same boundary, and the equation is read as relating link projections that differ only in the indicated fragments. For example, equation (1) is really infinitely many equations relating any quadruple of link projections that differ in only one crossing.

Kauffman [8] proved that the Kauffman polynomial exists uniquely; i.e., that the skein relations are consistent and complete. It is therefore a Laurent polynomial in $Q$ and $Q^d$. (Our parameterization of the Kauffman polynomial is slightly different from Kauffman’s.) The specialization $d = -2$ is called the Kauffman bracket, another polynomial which is up to normalization the same as the Jones polynomial. (Both the Kauffman polynomial and the Kauffman bracket are clearly not invariant under the first Reidemeister move, but a simple extra normalization factor achieves this invariance as well and produces a function fully invariant under link isotopy. This is the main difference between the Kauffman bracket and the Jones polynomial.)

An invariant with a skein theory often has an alternative definition using a state model; we will consider a particular type of state model for link projections called a checkerboard model [7]. A checkerboard model is given by a state set $S$, a number $x$, and two symmetric functions $W_+$ and $W_-$ from $S \times S$ to a commutative ring $R$. Given a link projection on the 2-sphere, a checkerboard coloring is one of the two alternating black-white colorings of the complementary regions:

Given a checkerboard coloring, the crossings can be labelled as positive or negative, as indicated, according to their sense relative to the neighboring black regions. (The labelling depends on an orientation of the 2-sphere.) The black regions are called the atoms of the state model. A state is a function from the atoms to the state set. Given a state, the weight of
a positive (respectively negative) crossing, which takes values in \( \mathbb{C} \) or some other field, is given by some function \( W_+(a,b) \) (resp. \( W_-(a,b) \)) if the two incident atoms are assigned states \( a \) and \( b \); these functions are called interactions. The weight of a state is then the product of the weights of the atoms, and the state sum \( Z \) is the total weight of all states. If \( \chi \) is the total Euler characteristic of all black regions, it may happen that the normalized state sum \( Z' = x^{-2}Z \) is a regular isotopy invariant and in particular one that satisfies skein relations.

For example, the Potts model is a checkerboard model whose normalized state sum is a value of the Kauffman bracket, and in particular is a regular isotopy invariant. Choose a real or complex \( q \) such that \( n = q + 2 + q^{-1} \) is a positive integer and choose a root \( q^{1/4} \). Then the Potts model of order \( n \) has a state set with \( n \) elements and the following weights:

\[
\begin{align*}
W_+(a,a) &= q^{3/4} \\
W_-(a,a) &= q^{-3/4} \\
W_+(a,b) &= -q^{-1/4} \quad \text{when } a \neq b \\
W_-(a,b) &= -q^{1/4} \quad \text{when } a \neq b
\end{align*}
\]

The Euler normalization \( x = -(q^{1/2} + q^{-1/2}) = \pm \sqrt{n} \). One can check that the normalized state sum is then the Kauffman bracket with \( Q = q^{1/4} \).

There are only a few known non-trivial checkerboard models that produce link invariants \([4]\). By far the most interesting of these is the Higman-Sims state model discovered by Jaeger \([3]\). This model produces the value of the Kauffman polynomial at \( Q = \tau \), the golden ratio, and \( d = -4 \). In quantum groups terms, this corresponds to the \( q = \tau^2 \) point of the quantum group \( U_q(\text{sp}(4)) \).

In this paper, we present an alternative argument that the Higman-Sims state model produces a topological invariant and a value of the Kauffman polynomial. The idea is to use the combinatorial \( B_2 \) spider \([10]\), a skein theory of graphs that produces the Kauffman polynomial for arbitrary \( Q \) with \( d = -4 \). The author hopes that this argument will help draw attention to Jaeger’s remarkable state model and help elucidate a mysterious relationship between the Higman-Sims sporadic simple group and the quantum group \( U_q(\text{sp}(4)) \). We also refer the reader to another review of Jaeger’s model by de la Harpe \([3]\).

## 2. SPIDERS, SKEIN MODULES, AND INVARIANTS

The combinatorial \( B_2 \) spider is essentially a skein theory of trivalent, planar graphs (called webs) with unoriented edges of two types, which are called type 1 and type 2 strands and are denoted by single and double edges. Such a skein theory uses the same concepts and allows the same notation as a skein theory for link projections, except that it describes a function on a class of planar graphs and we require no particular topological invariance a priori (other than invariance under isotopy in the 2-sphere). The only allowed vertices in \( B_2 \) webs are those with two single edges and one double edge:

\[
\begin{align*}
\text{The skein relations are:} \\
\tau = -(q^2 + q + q^{-1} + q^{-2}) \\
\lambda = q^3 + q + 1 + q^{-1} + q^{-3} \\
\lambda = 0 \\
\lambda = -(q + 2 + q^{-1})
\end{align*}
\]

Here \( q \) is a complex number or an indeterminate with a preferred square root \( q^{1/2} \). Together with the stipulation that the empty web has value 1, these skein relations again define a unique function on \( B_2 \) webs \([10]\).

To understand something of the relation between the \( B_2 \) spider and the Lie algebra \( B_2 \), we can consider skein modules, which are another general concept in skein theory. Instead of considering functions on link projections or \( B_2 \) webs, we consider formal linear combinations of such objects over a field such as \( \mathbb{C}(q^{1/2}) \), or sometimes over a ring. We quotient the space of all formal linear combinations by the skein relations, more properly interpreted as relations. Discarding stipulations about empty diagrams, we can say that the skein module for the Kauffman polynomial and the skein module for the \( B_2 \) spider are both 1-dimensional. More generally, if we fix a boundary of a tangle or a \( B_2 \) web (meaning that the tangle or web is embedded in a disk and has prespecified univalent endpoints on the boundary of the disk), we can consider the skein module of tangles or webs with this boundary.

In the \( B_2 \) spider, the skein module is called a web space and all of its elements are called webs. If a web space has \( n \) endpoints of type 1 and \( k \) of type 2, then it is isomorphic to the invariant space \( \text{Inv}(V(\lambda_1)^{\otimes n} \otimes V(\lambda_2)^{\otimes m}) \), where \( V(\lambda_1) \) and \( V(\lambda_2) \) are, respectively, the 5-dimensional and 4-dimensional irreducible representations of the quantum group \( U_q(\text{sp}(4)) \) \([10]\). (Such a quantum group is an algebra which specializes to the usual universal enveloping algebra when \( q = 1 \).) Using \( U_q(\text{sp}(4)) \), one can define an algebraic \( B_2 \) spider, and one can say that the algebraic and combinatorial \( B_2 \) spiders are isomorphic when \( q \) is a transcendental element in a field or is not a root of unity.
Working in the $B_2$ spider, we may define particular webs called crossings by the equations

\[
\begin{align*}
\begin{array}{c}
\includegraphics{crossing1} \\
\includegraphics{crossing2}
\end{array}
\end{align*}
\]

These webs then satisfy the regular isotopy equations and yield invariants of framed graphs in $S^3$ or in a ball which take values in the appropriate web spaces. Moreover, a crossing of two type 1 strands satisfies Kauffman’s relations with $Q = q^{1/2}$ and $d = -4$, while a crossing of two type 2 strands satisfies Kauffman’s relation with $Q = q$ and $d = 5$. So one can say that there is a homomorphism (in fact an epimorphism which is far from injective) from a one-variable slice of the Kauffman spider to the $B_2$ spider.

3. THE MODEL

Given a $B_2$ web on the sphere or in a disk, a checkerboard coloring is a coloring of its faces such that two regions that meet at a type 1 strand have opposite colors, while two regions that meet at a type 2 strand have the same color. For technical reasons we only consider webs with no type 2 strands at the boundary and with no closed loops of type 2. We can then write the same $B_2$ skein relations for this class of colored webs, for example:

\[
\begin{align*}
\begin{array}{c}
\includegraphics{checkerboard1} \\
\includegraphics{checkerboard2}
\end{array}
\end{align*}
\]

These relations constitute a perfectly valid skein theory even though the colorings of the faces are somewhat redundant.

We consider state models on checkerboard colorings of $B_2$ webs. Let $S$ be a state set; as before, a state is a function from the black regions (atoms) to the state set. Assuming that there are no crossings, we consider two interactions (functions) $W_H$ and $W_I$ from $S \times S$ to $\mathbb{C}$. The weight of a state has a factor of $W_I(a, b)$ for every type 2 edge that bridges an atom in state $a$ with an atom in state $b$ and a factor of $W_H(a, b)$ for every type 2 edge that lies on the border between an atom in state $a$ from an atom in state $b$:

\[
\begin{align*}
W_I(a, b) & \quad W_H(a, b)
\end{align*}
\]

As before, the state sum $Z$ is the total weight of all states and $Z' = x^{-2}Z$ is the normalized state sum for some constant $x$. Note that for a web in a disk, for every fixed state of all atoms at the boundary, there is a state sum over all states in the interior; we do not sum over the colorings of the boundary regions. If we establish that the state sum $Z'$ satisfies the above skein relations (equations (2) to (3)), then we can say that it is a linear functional on web spaces. We can further define interactions $W_-$ and $W_+$ for crossings by using equation (4) (see also the relation between $W_H$ and $W_I$ below):

\[
W_\pm = -q^{\pm 1/2}xI - \frac{q^{\mp 1}}{q^{1/2} + q^{-1/2}}N - \frac{W_I}{q^{1/2} + q^{-1/2}}
\]

Here and below the interactions are interpreted as $n \times n$ matrices, $I$ is the identity matrix, and $N$ is the matrix of all 1’s. The interactions $W_\pm$ then constitute a checkerboard model for link projections, one whose normalized state sum is automatically a value of the Kauffman polynomial at $d = -4$.

Consider the restrictions on $W_H$, $W_I$, and $x$ given by the skein relations. Assume that the state set $S$ has $n$ elements. Let

\[
\begin{align*}
&h = -(q + 2 + q^{-1}) \\
&\ell = -(q^2 + q + q^{-1} + q^{-2})
\end{align*}
\]

The relations

\[
\begin{align*}
\includegraphics{relation1} & \quad \includegraphics{relation2}
\end{align*}
\]

say that $\ell = \frac{4}{x}$ and $\ell = x$, which implies that $n = \ell^2$. The relation

\[
\includegraphics{relation3}
\]

says that $W_H(a, a) = 0$. The relation

\[
\includegraphics{relation4}
\]

says that $W_H(a, b)$ is either 0 or $h$ for all $a$ and $b$. Since $W_H(a, b)$ is symmetric, it is proportional to the adjacency matrix $A$ of some graph $J$ with vertex set $S$. (The graph $J$ need
A quadratic minimal polynomial is equivalent to the property that a certain graph

\[ H \]

is strongly regular \[6\], or 2-point regular as defined below: If the graph \( J \) of a checkerboard state model is 3-point regular, this sum only depends on which of \( a, b \) and \( c \) are equal and which are connected by edges of \( J \).

**Lemma 3.1.** If the graph \( J \) of a checkerboard state model is 3-point regular, equation (5) is a corollary of the other checkerboard skein relations.

(Conversely, Jaeger \[3\] proved that \( J \) must be 3-point regular if all of the checkerboard skein relations hold.)

**Proof.** For convenience, we define a new type of strand, denoted by dashes, as a linear combination of other webs:

\[ \begin{array}{c}
= - \frac{1}{x}
\end{array} \]

This strand has its own weight matrix \( W_D \) which is a linear combination of \( W_H, N, \) and \( I \); the weights are chosen so that \( W_D(a,b) = 1 \) if \( a \) and \( b \) are distinct but not connected by an edge of \( J \) and \( W_D(a,b) = 0 \) if they are connected by an edge. Then each case of equation (\ref{eq:4}) can be converted to a statement about a state sum of a web on the sphere. For example, if \( a, b, \) and \( c \) form an anti-triangle and \( J \) has \( t \) anti-triangles, then the
left side of equation (7) differs by a factor of $t$ from the state sum of the graph

\[ = 0 \]

Let $w$ be either this web or its counterpart from one of the other cases of equation (7). Given that the $B_2$ skein relations are consistent, and given that $w$ certainly does vanish modulo the skein relations together, it suffices to show that $w$ is a multiple of the empty web modulo all skein relations other than equation (3); the coefficient is then necessarily zero. Finally, since $w$ has at most four black regions, it satisfies this condition by Lemma 3.2.

**Lemma 3.2.** Any colored $B_2$ web $w$ on the sphere with at most seven black regions is proportional to the empty web using only skein relations other than equation (3).

**Proof.** The proof is by induction on the number of vertices. Following the usual description of the $B_2$ spider $\spider$, we assign formal angles of 45, 135 degrees, and 135 degrees to each vertex:

\[
\begin{array}{ccc}
135^\circ & 135^\circ & \\
\end{array}
\]

We use these formal angles to define a non-standard Euler characteristic of each face of $w$ with the property that the total Euler characteristic is 2. (It should not be confused with the ordinary Euler characteristic used above for normalization.) It may be defined for a face as $1 - \frac{\text{exterior angle}}{360^\circ}$, where $A$ is the sum of all exterior angles, and an exterior angle at a given vertex is defined as $180^\circ$ minus the usual interior angle. Alternatively, we may recall that the Euler characteristic of a face or a vertex is 1 and that of an edge is $-1$, and then modify this definition by dividing the Euler characteristic of an edge equally between the two faces that contain it and, at each vertex, giving the right-angled faces $1/4$ of the characteristic of the vertex and the other faces $3/8$ each. This second definition makes clear that the total is still 2.

Note also that the Euler characteristic of any face is a multiple of $1/4$. Therefore $w$ has either a white face with positive Euler characteristic or a black face whose Euler characteristic is at least $1/2$. Modulo webs with fewer vertices, we can apply the exchange equation (3) to all sides of such a face which are type 2 strands, for example:

\[
\begin{array}{c}
\text{This operation does not change the Euler characteristic of the face. The face can then be simplified by one of the other skein relations, for it has at most three sides if it is white and at most two sides if it is black.}
\end{array}
\]

There are only two known graphs $J$ that satisfy all of the above conditions, namely the pentagon and the Higman-Sims graph. The Higman-Sims graph is a graph on $n = 100$ vertices whose symmetry group contains the Higman-Sims sporadic simple group as a subgroup of index two. The symmetry group acts 3-point transitively, so that the graph is 3-point regular. We claim that there is a corresponding state model that satisfies the $B_2$ skein relations with $q^{1/2} = \tau$, which implies that $\ell = -10$ and $h = -5$. Clearly $n = \ell^2$. The graph has no triangles and each vertex has valence 22. Therefore the eigenvalues of its adjacency matrix $A$ are $22, 2, 2, -8$; the image of $N$ is the unique line with eigenvalue 22. Therefore the eigenvalues of $W^2 = hhW_f$ are $-110 + 100 + 10 = 0, -10 + 10 = 0$, and $40 + 10 = 50$, which implies $W_f^2 = xyW_f$, as desired.

### 4. DISCUSSION

The properties of the Higman-Sims state model imply a number of mysterious numerical connections between the Higman-Sims group ($HS$) and the (quantum) representation theory of $sp(4)$. In this discussion, “representation” will in general mean a finite-dimensional linear representation.

The smallest representations of $sp(4)$ are $V(\lambda_1)$, the defining 4-dimensional representation, $V(\lambda_2)$, the 5-dimensional representation that identifies $sp(4)$ with $so(5)$, and $V(2\lambda_1)$, the 10-dimensional representation which is the symmetric tensor square of $V(\lambda_1)$. In quantum representation theory, representations have a quantum dimension, which is a natural generalization of the non-quantum or honest dimension. (It also coincides with the character of a certain circle in the non-quantum representation theory and appears in some proofs of the Weyl dimension formula.) The quantum dimensions of these representations are

- $\dim_q V(\lambda_1) = q^2 + q + q^{-1} + q^{-2}$
- $\dim_q V(\lambda_2) = q^3 + q + 1 + q^{-1} + q^{-3}$
- $\dim_q V(2\lambda_1) = q^4 + q^3 + q^2 + q + 2 + q^{-1} + q^{-2} + q^{-3} + q^{-4}$

At the same time, the quantum dimension is the value of a closed loop in the $B_2$ spider; in the first two cases the loop is a type 1 or 2 strand, and in the third case it is a dashed loop. By the existence of the Higman-Sims state model, these
three numbers must also be 10, the square root of the number of vertices of the Higman-Sims graph; 22, the degree of a vertex; and 77, the anti-degree of a vertex. Now the Higman-Sims graph has a special duality realized by switching colors in the state model, and this duality tells us that 22 and 77 must also be dimensions of representations of HS; as it happens, the two smallest irreducible representations. Thus we learn that quantum dimensions of representations of \( U_{q=\tau^2}(\mathfrak{sp}(4)) \) coincide with honest dimensions of representations of HS. This pattern extends to all representations of \( U_{q=\tau^2}(\mathfrak{sp}(4)) \), except that the corresponding representations of HS are eventually not irreducible.

In fact, the \( B_2 \) spider can be understood as (the Hom spaces of) the representation category of \( U_q(\mathfrak{sp}(4)) \), and the Higman-Sims state model establishes a functor from (the even half of) the representation category of \( U_{q=\tau^2}(\mathfrak{sp}(4)) \) to the representation category of HS. Such a functor would exist if there were an algebra homomorphism from the group algebra \( \mathbb{C}[HS] \) to the quantum group \( U_{q=\tau^2}(\mathfrak{sp}(4)) \), but this possibility is nonsense (as the referee mentioned), because it would relate honest dimensions to honest dimensions and not quantum dimensions to honest dimensions. But perhaps one can construct HS from \( U_{q=\tau^2}(\mathfrak{sp}(4)) \) using this functor.

As a warm-up to this problem, one can try to construct a relationship between the quantum group \( U_q(\mathfrak{sl}(2)) \) and the symmetric group \( S_n \) on \( n = q + 2 + q^{-1} \) letters, which is the symmetry group of the Potts model. The Potts model relates these objects in the same way that the Higman-Sims model relates \( U_{q=\tau^2}(\mathfrak{sp}(4)) \) and HS.

It would be especially interesting if one could construct not only the Higman-Sims group but also other sporadic simple groups using quantum groups at special values of \( q \).

Besides its numerology, the Higman-Sims state model also has the following interesting aspect. The most common axiomatic description of a spider, which is a collection of web spaces for disks with different boundaries but with the same skein relations, is that it is a kind of monoidal category, or a braided category if crossings exist. However, another interesting point of view is that a spider is a certain kind of 2-category with only one 0-morphism (or “object”), with a 1-morphism for every choice of boundary, and such that each web is a 2-morphism. In this setting, a collection of checkerboard skein modules has a suggestive definition also, namely as a 2-category with two 0-morphisms.

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