Darboux transformation for the modified Veselov-Novikov equation

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Abstract

A Darboux transformation is constructed for the modified Veselov-Novikov equation.
I Introduction

Solitons and geometry are closely connected. Many soliton equations or integrable systems have their origins in classical differential geometry. The best-known example, probably the first one, is the celebrated sine-Gordon equation, which was used to describe surfaces with constant negative Gaussian curvature. Another example is the binormal flow of a curve in \( \mathbb{R}^3 \). It essentially appeared in the study of vortex filaments in the paper of da Rios \[1\]. Much later, Hasimoto \[2\] showed the equivalence of this system with the non-linear Schrödinger equation. For more references on the interrelation between geometry and integrable systems, we refer to the recent books \[3\] \[4\] and the references there.

Recently, a class of integrable deformations of surfaces immersed in \( \mathbb{R}^3 \) is defined by using of generalized Weierstrass representation in \[5\]. The mean observation of that paper is that the operator from the linear problem of generalized Weierstrass representation coincides with the operator \( L^{mNV} \) to which the modified Veselov-Novikov (mVN) hierarchy is attached. Thus, the geometrical significance of the mVN equation is established and it is important to construct the explicit solutions for this equation. It is well known that one of the powerful techniques leading to explicit solutions for an integrable equation is Darboux transformation (DT) \[8\]. In this paper we construct a binary DT for the mVN equation.

This paper is organized as follow: In section II we will give a brief review of the mVN equation. In section III, the derivation of the Darboux transformation for the mVN equation will be given and as an example, we generate the solutions by the above Darboux transformation for the simplest case: \( U(x, y, t) = 0 \). We present our conclusions and discussions in section IV.

II mVN equation and its Lax representation

The mVN equation is a natural 2-dimensional generalization of MKdV equation. The MKdV equation reads as

\[
U_t = \frac{1}{4}U_{xxx} + 6U^2U_x,
\]

while the mVN equation is \[3\]

\[
U_t = (U_{xxx} + 3U_x V + \frac{3}{2} UV_z) + (U_{zzz} + 3U_z \bar{V} + \frac{3}{2} U \bar{V}_z), \quad V_{\bar{z}} = (U^2)_{\bar{z}}.
\]

\[2\]
It is known that the mVN equation is represented by a Manakov’s triad, namely it has the following operator formalism

\[ L_t + [L, A] - BL = 0, \]  

(3)

where

\[
L = \begin{pmatrix} \partial - U \\ U \end{pmatrix},
\]

\[
A = \partial^3 + \bar{\partial}^3 + 3 \begin{pmatrix} 0 & -U_z \\ 0 & V \end{pmatrix} \partial + 3 \begin{pmatrix} \bar{V}_z & 0 \\ U_{\bar{z}} & 0 \end{pmatrix} \bar{\partial} + \frac{3}{2} \begin{pmatrix} \bar{V}_z & 2UV \\ -2UV & V_z \end{pmatrix},
\]

\[
B = 3 \begin{pmatrix} 0 & U_z \\ -U_z & 0 \end{pmatrix} \partial + 3 \begin{pmatrix} 0 & U_z \\ -U_{\bar{z}} & 0 \end{pmatrix} \bar{\partial} + 3 \begin{pmatrix} 0 & U_{\bar{z}} + U(\bar{V} - V) \\ -U_{zz} + UV + \bar{U}\bar{V} & 0 \end{pmatrix},
\]

\[\partial = \frac{1}{2}(\partial_x - i\partial_y), \quad \bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y).\]

In [7], it is shown that the system (2) is related to the Veselov-Novikov equation in the similar manner as MKdV related to KdV system.

**Remarks:** 1) if the function \( U \) depends only on one space variable \( x \), the mVN equation (3) reduces to the MKdV equation (1). 2) the field variable \( U \) in the equation (2) is assumed as a real valued function.

Since the mVN equation possesses the operator representation (3), it deforms the kernel of the operator \( L \) via the equation

\[
\begin{aligned}
L\Psi &= 0 \\
\Psi_t &= A\Psi
\end{aligned}
\]

(4)

where

\[
\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}
\]

(5)

is known as a wave function.

### III Darboux transformation of mVN equation

To construct a DT for the mVN equation, we find that it is convenient to transform the Lax pair (3) into the following form

\[
\begin{aligned}
\Psi_x &= J\Psi_y + P\Psi \\
\Psi_t &= -J\Psi_{yyy} - P\Psi_{yy} + Q\Psi_y + S\Psi
\end{aligned}
\]

(6)
where
\[ J = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv i\sigma_3, \quad P = \begin{pmatrix} 0 & 2U \\ -2U & 0 \end{pmatrix} \equiv 2iU\sigma_2, \]
\[ Q = \begin{pmatrix} -iU^2 + 3i\bar{V} & iU_x - 2U_y \\ iU_x + 2U_y & iU^2 - 3i\bar{V} \end{pmatrix}, \]
\[ S = \begin{pmatrix} (-\frac{5}{2}iU_y - \frac{3}{2}U_x)U + \frac{3}{2}\bar{V}_z & -2U^3 - 2U_{yy} + \frac{1}{2}U_{xx} + \frac{i}{2}U_{xy} + 3U(\bar{V} + V) \\ 2U^3 + 2U_{yy} - \frac{1}{2}U_{xx} + \frac{i}{2}U_{xy} - 3U(\bar{V} + V) & \frac{5}{2}iU_y - \frac{3}{2}U_x)U + \frac{3}{2}\bar{V}_z \end{pmatrix}, \]
and
\[ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
are Pauli matrices.

We notice that the matrices \( J, P, Q \) and \( S \) have the following involution property
\[ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} X \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \bar{X}, \tag{7} \]
where \( X \) is one of \( J, P, Q, S \). It is clear that if
\[ \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{8} \]
is a vector solution of (3) then
\[ \Psi^* = \begin{pmatrix} -\bar{\psi}_2 \\ \bar{\psi}_1 \end{pmatrix} \tag{9} \]
also satisfies (3). Hence from a vector solution we obtain a matrix solution of (3)
\[ \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix}, \tag{10} \]
which we also denote by \( \Psi \) for short.

We now consider the construction of a DT for the mVN equation.

To this end, we introduce the linear system formally conjugate to (3)
\[ \begin{cases} \Phi_x = \Phi_y^T J^T + \Phi P^T \\ \Phi_t = -\Phi_{yyy} J^T + \Phi_y P^T + \Phi_y Q^T + \Phi S^T \end{cases} \tag{11} \]
It is easy to see that if \( \Psi \) is a matrix solution of (3) then \( \Phi = \Psi^T \) is a matrix solution of (11).
Now with a solution Ψ of the linear system (6) and a solution Φ of the linear system (11) we introduce a 1-form

\[ \omega(\Phi, \Psi) = \Phi \Psi dy + i \Phi \sigma_3 \Psi dx + \left[ -i(\Phi_{yy}\sigma_3 \Psi + \Phi\sigma_3 \Psi_{yy} - \Phi_y \sigma_3 \Psi_y) \right] dt, \]

where Ψ, Φ are matrix solutions of (6), (11) respectively. It is tedious but otherwise straightforward to show that the 1-form defined above is closed, that is

**Lemma 1.** \( d\omega(\Phi, \Psi) = 0. \)

**Proof:** By straightforward computation.

Thus, the following matrices

\[ \hat{\Omega}(\Phi, \Psi) = \int_{M_0}^M \omega = \int_{(x_0, y_0, z_0)}^{(x, y, z)} \omega, \]

and

\[ \Omega(\Phi, \Psi) = \hat{\Omega}(\Phi, \Psi) + \begin{pmatrix} a + bi & ci \\ ci & a - bi \end{pmatrix} \]

are well defined, where \( a, b \) and \( c \) are real constants. In the sequel, we will take \( \Psi \) a matrix solution of (6) of the form of (10).

Let \( \Psi_0 \) be any matrix solution of (6) of the form of (10), and introduce the matrices \( K \) and \( \sigma \) by

\[ K \equiv \Psi_0 \Omega^{-1}(\Psi_0^T, \Psi_0) = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}, \]

\[ \sigma = \Psi_0 y \Psi_0^{-1}. \]

**Lemma 2.** \( K \) and \( \sigma \) defined above have to be in the following forms:

\[ K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}, \quad \sigma = \begin{pmatrix} \sigma_{11} & -\sigma_{21} \\ \sigma_{21} & \bar{\sigma}_{11} \end{pmatrix} \]

where \( k_{12} = i\lambda \) and \( \lambda \) is a real constant.

**Proof:** From the involution property of \( J, P, Q, S \) and \( \Psi_0 \), we find

\[ K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} K \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]

that is, \( K \) also possesses the involution property. Meanwhile, \( K \) is symmetric:

\[ K = K^T \]
So we have
\[ \bar{k}_{11} = k_{22}, \quad \bar{k}_{21} = -k_{12}, \quad k_{12} = k_{21} = i\lambda. \]

This proves the result for $K$. Similarly, the result for $\sigma$ can be proved.

Our DT is now conveniently formulated as

**Theorem 1.** Let $\Psi$ and $\Psi_0$ be the solutions of the linear system (3). Let $\Omega(\Psi_0^T, \Psi_0)$ and $\Omega(\Psi_0^T, \Psi)$ be given by (15) with $\Phi = \Psi_0^T$, $\Psi = \Psi_0$ and $\Phi = \Psi_0^T$ and $\Psi = \Psi$, respectively. Then if $\Omega^{-1}(\Psi_0^T, \Psi_0)$ is invertible, the new matrix of wave functions defined by

\[ \tilde{\Psi} = \Psi - \Psi_0 \Omega^{-1}(\Psi_0^T, \Psi_0) \Omega(\Psi_0^T, \Psi) \] (17)

satisfies (3) with the potential $U, V$ replaced by

\[ \tilde{U} = U - \lambda = U + ik_{12}, \] (18)
\[ \tilde{V} = V + 2iUk_{12} + \bar{k}_{11}^2 - 2(\sigma_{21}k_{21} + \bar{\sigma}_{11}\bar{k}_{11}). \] (19)

**Proof:** It is quite easy to verify that the transformed quantities do fulfill the first equation of (3). However, the verification of the second equation of (3) is too complex to do by hand. We did check the validity by means of MAPLE.

Thus, we establish a DT for the mVN equation. It is easily seen that this DT is a binary Darboux transformation.

As an example, we generate the solutions by the above Darboux transformation for the simplest case: $U(x, y, t) = 0$. Then $V(x, y, t) = 0$. The linear system (3) in this case is:

\[ \Psi_x = J\Psi_y, \quad \Psi_t = -J\Psi_{yyy}. \] (20)

We take

\[ \Psi(x, y, t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \] (21)
\[ \Psi_0(x, y, t) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} e^{\alpha x - i\alpha y + \alpha^3 t} \\ e^{\beta x + i\beta y + \beta^3 t} \end{pmatrix}, \] (22)

where $\alpha, \beta$ are real constants. The matrix solution is found to be:

\[ \Psi_0(x, y, t) = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix} = \begin{pmatrix} e^{\alpha x - i\alpha y + \alpha^3 t} & -e^{\beta x - i\beta y + \beta^3 t} \\ e^{\beta x + i\beta y + \beta^3 t} & e^{\alpha x + i\alpha y + \alpha^3 t} \end{pmatrix}, \] (23)

we obtain
\[ \tilde{U} = \frac{1}{\det \Omega} \left[ (e^{2(\alpha x^2 + \alpha^3 t)} - e^{2(\beta x^2 + \beta^3 t)}) \left( \frac{2}{\alpha + \beta} - \frac{2}{\alpha + \beta} \cos(\alpha + \beta)ye^{(\alpha + \beta)x + (\alpha^3 + \beta^3)t} + c \right) \\
- \left( \frac{1}{\alpha} - \frac{1}{\beta} - \frac{1}{\alpha} e^{2(\alpha x^2 + \alpha^3 t)} \cos 2\alpha y + \frac{1}{\beta} e^{2(\beta x^2 + \beta^3 t)} \cos 2\beta y - 2b \right) \cos(\beta - \alpha) y \\
- \left( \frac{1}{\alpha} e^{2(\alpha x^2 + \alpha^3 t)} \sin 2\alpha y + \frac{1}{\beta} e^{2(\beta x^2 + \beta^3 t)} \sin 2\beta y + 2a \right) \sin(\beta - \alpha) y \right] \]

with

\[ \det \Omega = \det \Omega(\Psi_0^T, \Psi_0) = \left[ \frac{2}{\alpha + \beta} - \frac{2}{\alpha + \beta} \cos(\alpha + \beta)ye^{(\alpha + \beta)x + (\alpha^3 + \beta^3)t} + c \right]^2 \\
+ \left[ - \frac{1}{2\alpha} + \frac{1}{2\beta} + \frac{1}{2\alpha} e^{2(\alpha x^2 + \alpha^3 t)} \cos 2\alpha y - \frac{1}{2\beta} e^{2(\beta x^2 + \beta^3 t)} \cos 2\beta y + b \right]^2 \\
+ \left[ \frac{1}{2\alpha} e^{2(\alpha x^2 + \alpha^3 t)} \sin 2\alpha y + \frac{1}{2\beta} e^{2(\beta x^2 + \beta^3 t)} \sin 2\beta y + a \right]^2 \]

is a family solution of mVN equation involving three parameters \( a, b, c \).

**IV Conclusions**

In this paper we present a binary Darboux transformation for the mVN equation. We also calculate the solutions of the mVN equation using our DT by dressing the zero background.

Keeping in mind the geometrical background of the mVN equation, it will be interesting to construct solutions based on more sophisticated seeds and study their geometrical implications. This may be considered in a separate work.

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