Reflection equation and link polynomials
for arbitrary genus solid tori

C. SCHWIEBERT *

Uji Research Center
Yukawa Institute for Theoretical Physics
Kyoto University, Uji 611, Japan

Abstract

The correspondence between the braid group on a solid torus of arbitrary genus and the algebra of Yang-Baxter and reflection equation operators is shown. A representation of this braid group in terms of $R$-matrices is given. The characteristic equation of the reflection equation matrix is considered as an additional skein relation. This could lead to an intrinsic definition of invariant link polynomials on solid tori and, via Heegaard splitting, to invariant link polynomials on arbitrary three-manifolds without boundary.

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1. Introduction

In this article we should like to point out a relation between the reflection equation (RE) and the braid group on a 3-manifold of arbitrary genus with boundary.

We shall begin by introducing the RE as a quantum group (QG) comodule, then derive its properties as a non-commutative associative algebra and mention several applications of the RE. Concerning the braid group we start from the results of A. B. Sossinsky who defined the braid group on a solid handlebody of arbitrary genus. It consists of the usual generators for genus zero 3-manifolds and additional ones implementing windings around handles. We show that the relations they obey are precisely the Yang-Baxter equation (YBE) and the RE. We find an explicit representation of this algebra in terms of $R$-matrices and quantum algebra generators. We derive quadratic characteristic equations for the additional generators and suggest their interpretation as new skein relations. In principle this defines invariant link polynomials for closed braids in arbitrary genus 3-manifolds with boundary. This in turn, via Heegaard splitting, might pave the way for constructing invariant polynomials for links in arbitrary 3-manifolds without boundary.

2. The Reflection Equation

In this section we introduce the reflection equation, discuss its properties and point out some applications. Many of them were considered earlier in [1], but some have been obtained only afterwards. We will not say anything about one-dimensional representations of the RE algebra [1] as for the braid group at least two-dimensional representations are needed. Furthermore, throughout this paper we only discuss the standard example of $sl_q(2)$. We assume familiarity of the reader with this and basic quantum group terminology as introduced in [2] for example.

We start from scratch and ask for the possibility of generalizing statistics of (space-time) coordinates, i.e. introduce 2-`spinors' $x^i = \begin{pmatrix} u \\ v \end{pmatrix}$, $i = 1, 2$ with commutation relations

$$uv = qvu,$$  \hspace{1cm} (1)
where \( q \in \mathbb{C} \) instead of \( q = 1 \) for bosonic or \( q = -1 \) for fermionic coordinates. We set up a covariant notation and rewrite (1) as

\[
x^i x^j = q^{-1} R^{ji}_{\ kl} x^k x^l,
\]

(2)

where we have to introduce the \( R \)-matrix ((\( ji \)) labeling columns and (\( kl \)) rows with natural index order (11, 12, 21, 22); vanishing entries are omitted)

\[
R^{ji}_{\ kl} = \begin{pmatrix} q & 1 & \omega \\ 1 & 1 & q \\ \omega & q & 1 \\ q & \omega & 1 \\ 1 & \omega & q \end{pmatrix}, \quad \omega = q - q^{-1}
\]

(3)

and for consistency of higher order relations it has to satisfy the Yang-Baxter equation

\[
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.
\]

(4)

Here it was possible to hide all indices by introducing the standard matrix notation \( R_{12} = R \otimes I \) etc., with \( I \) the identity matrix, acting in a triple product of vector spaces [2]. Above construction is just the quantum plane as introduced in [3] and generalized in [4].

We define a transformation of \( x^i \) as

\[
x'^i = T^i_{\ j} x^j,
\]

(5)

and ask what restrictions on the \( T^i_{\ j} \) result if we demand invariance of the basic relation (2). The outcome is the celebrated QG equation

\[
RT_1 T_2 = T_2 T_1 R,
\]

(6)

and again it was possible to get rid of indices by using the notation \( T_1 = T \otimes I \), \( T_2 = I \otimes T \). We do not bother to write out (6) for the entries of \( T \) as we will
not need it. The QG $SL_q(2)$ is defined by (6) if we set equal to one the quadratic central element corresponding to the generalized determinant. In fact, this is a non-commutative and non-cocommutative Hopf algebra, not a Lie group but a deformation of $SL(2)$ [2, 5, 6].

We further introduce a second quantum plane $y_i$, however with lower index and transforming by $T^{-1}$ (the antipode of $T$)

$$y'_i = y_j(T^{-1})^j_i.$$  \hfill (7)

The commutation relations of $y_i$ are completely fixed, using an ansatz $y_iy_j = \alpha y_ky_lM^{kl}_{ij}, \alpha = \text{const.}$, we transform according to (7) and use (6) to get

$$y_iy_j = q^{-1}y_ky_lR^{kl}_{ji}. $$  \hfill (8)

We can even calculate the commutation relations between $x^i$ and $y_j$ this way

$$x^iy_j = \alpha y_kx^lR^{lk}_{ij}.$$  \hfill (9)

this time however the constant $\alpha$ is not fixed by covariance, we do not need it and may set $\alpha = 1$. The inverse of (9) is

$$y_jx^i = \alpha^{-1}(((R^{t_2})^{-1})^{t_2})^{il}_{kj}x^ky_l,$$  \hfill (10)

where the superscript $t_2$ means transposition in the second space in which $R$ acts, i.e. interchange of $(i \leftrightarrow l)$ in (3).

Next we consider a product of the two quantum planes and define the matrix $K^{i,j} = x^iy_j$. It transpires that $K$ transforms as $K' = TKT^{-1}$, i.e. multiplying together quantum planes we can construct tensors of arbitrary rank covariant w.r.t. the QG coaction. They can straightforwardly be $q$-(anti)symmetrized in analogy to the non-deformed case. Again, the commutation relations of $K^{i,j}$ are completely
determined by (2), (8), (9) and (10)

\[ K_{ij} K_{mn} = q^{-2}((R t_2 t_2)^{-1})^m_{l} R^{k_{l}l} R^{s'}_{ml} R^{t'r'}_{s'r'} K^{k'}_{r'} K^{s'}_{s}. \]  

(11)

Hearty readers may check that the entries of \( K = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) satisfy the commutation relations

\[
\begin{align*}
ab &= q^{-2}ba, \\
ac &= q^2ca, \\
bd - db &= -q^{-1}\omega ab, \\
b - cb &= q^{-1}\omega(ad - a^2), \\
bd - db &= -q^{-1}\omega ab, \\
bc - cb &= q^{-1}\omega(ad - a^2), \\
bd - db &= -q^{-1}\omega ab, \\
bc - cb &= q^{-1}\omega(ad - a^2).
\end{align*}
\]  

(12)

This algebra has two central elements, the quantum trace and the quantum determinant which we set equal to one

\[
c_1 = a + q^2d, \quad c_2 = ad - q^2cb \equiv 1. \]  

(13)

The algebra depends only on \( q^2 \) and one may rescale \( q^2 \rightarrow \tilde{q} \), then if \( \tilde{q} \) is a phase, \( \tilde{q}^N = 1 \), we find that \( a^N \) is a further central element.

We can define the ‘antipode’ \( S(K) \equiv K^{-1} \) as

\[
\begin{align*}
S(a) &= q^2d - q\omega a, \\
S(b) &= -q^2b, \\
S(c) &= -q^2c, \\
S(d) &= a.
\end{align*}
\]  

(14)

Then we easily establish a relation (characteristic equation) between \( K \) and \( K^{-1} \)

\[
K^{-1} = -q^2K + c_1I,
\]  

(15)

which will play the role of a skein relation later on.

If we impose suitable reality conditions on \( x^i, y_j \) and hence \( K_{ij} \) then a linear combination of the elements of (12) is just the \( q \)-deformed Minkowski space \([7, 8],\).
where $c_1$ is the time coordinate and $c_2$ the invariant length. Various reality conditions are discussed in [9].

Truncation of algebra (12) by $c_1 = 0$ can be shown to lead to the quantum 2-sphere of Podles, a quantum analogue of homogeneous spaces [10].

Notice also the similarity of (11) to the basic relation (2) if one reads the four $R$-matrices on the RHS as a single one with four pairs of indices [7]. It is simple to introduce an index free notation for quantum planes and extend it to the $K$-matrix, but we shall not go into this here [9, 11].

A basic observation is now that relations (12) can be encoded in a matrix equation

$$RK_1\tilde{R}K_2 = K_2RK_1\tilde{R},$$

where $\tilde{R} = PRP$ with $P$ the permutation operator. This is one of the RE discussed in [1]. It is easy to establish invariance of (16) under the QG coaction $K' = TKT^{-1}$. The other RE discussed extensively in [1] is invariant under $K' = TKT^t$, where the superscript $t$ denotes the transpose of $T$. For the special choice of (3) as $R$-matrix its algebra is isomorphic to (12). We do not write it down here as it is irrelevant for our purposes, but one should keep in mind that different types of QG covariant tensors can be constructed.

It is remarkable that one can define a twisted (or braided) ‘coproduct’ of the same form as for the QG, i.e. $\Delta(K) = K \otimes K$, however with non-commutativity between elements of different spaces. We avoid tensor product notation and distinguish elements of different spaces by a prime. So it is easy to prove the following: Given two different solutions $K$ and $K'$ of (16) then

$$(i) \quad \tilde{K} = KK', \quad \text{and} \quad (ii) \quad \tilde{K} = KK'K^{-1}$$

are also solutions of (16) provided $K$ and $K'$ commute as follows

$$RK_1R^{-1}K_2' = K_2'RK_1R^{-1}. \quad (18)$$

This gives 16 commutation relations between the elements of $K$ and $K'$, we do not
bother to show them explicitly here.

This process of building up new solutions can obviously be continued, but some care has to be taken to keep track of the ordering and multiplying from the correct side as (18) is not symmetric under exchange of $K$ and $K'$. Due to (17) we may interprete (12) as an algebra having a braided (adjoint) coaction on QG comodules. Algebra (12) was also constructed in [12] from the point of view of braided tensor categories.

An important point is that the central elements of $K$ and $K'$ are mutually central in both algebras, i.e.

$$[K^i_j, c'_m] = [K'^i_j, c_m] = 0, \quad m = 1, 2. \quad (19)$$

It is obvious that we have central elements for the combined solutions and also characteristic equations, for example

$$(KK')^{-1} = -q^2 KK' + C_1 I, \quad (20)$$

where $C_1 = aa' + bc' + q^2 (cb' + dd')$.

A further property of the RE will be needed later on. The $sl_q(2)$ algebra dual to the QG (6) can be written in matrix form as [2]

$$\tilde{R}L_1^{\varepsilon_1}L_2^{\varepsilon_2} = L_2^{\varepsilon_2}L_1^{\varepsilon_1}\tilde{R}, \quad (\varepsilon_1, \varepsilon_2) = \{(+, +), (+, -), (-, -)\} \quad (21)$$

where

$$L^+ = \begin{pmatrix} q^{H/2} & q^{-1/2}\omega X^- \\ 0 & q^{-H/2} \end{pmatrix}, \quad L^- = \begin{pmatrix} q^{-H/2} & 0 \\ -q^{1/2}\omega X^+ & q^{H/2} \end{pmatrix} \quad (22)$$

and this gives the $sl_q(2)$ algebra

$$[H, X^\pm] = \pm 2X^\pm, \quad [X^+, X^-] = \omega^{-1}(q^H - q^{-H}) \quad (23)$$

with antipode $S(H) = -H$, $S(X^\pm) = -q^\mp 1X^\pm$ and coproduct $\Delta(L^\pm) = L^\pm \otimes L^\pm$. 

- 7 -
It is easy to show using (21) that $K = S(L^-)L^+$ is a solution of the RE [13], explicitly $K$ is given by

$$K = \begin{pmatrix} q^H & q^{-1/2} \omega q^{H/2} X^- \\ q^{-1/2} \omega X^+ q^{H/2} & q^{-H} + q^{-1} \omega^2 X^+ X^- \end{pmatrix}. \tag{24}$$

The RE algebra hence plays different roles, it is a comodule w.r.t. the QG (with a ‘coaction’ of it on QG comodules, see above) and on the other hand it acts via (24) on representations of the quantum algebra dual to the QG. Formally, there is also an action of the quantum algebra generators $L^\pm$ on the RE algebra [9, 12]. However, interpretations of the RE algebra are different in each case. We refer the reader to [1] for further applications of the RE and historical development, we are quite confident that still more applications can be uncovered.

3. **Braid group on solid tori**

The braid group $B^n_g$ in a solid handlebody $H_g$ of genus $g$ was derived in [14]. In addition to the generators $\sigma_i, i = 1, \ldots, n - 1$ of the braid group $B_n$ defined on a 3-dimensional manifold of genus zero there are generators $\tau_\alpha, \alpha = 1, \ldots, g$ implementing windings around the $g$ handles. The algebra is given as

$$\begin{align*}
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, & i &= 1, \ldots, n - 1 \\
\sigma_i \sigma_j &= \sigma_j \sigma_i, & |i - j| &\geq 2 \\
\sigma_i \tau_\alpha &= \tau_\alpha \sigma_i, & i &\geq 2, \alpha = 1, \ldots, g \\
\sigma_1 \tau_\alpha \sigma_1 \tau_\alpha &= \tau_\alpha \sigma_1 \tau_\alpha \sigma_1, & \alpha &= 1, \ldots, g \\
\sigma_1 \tau_\alpha \sigma_1^{-1} \tau_\beta &= \tau_\beta \sigma_1 \tau_\alpha \sigma_1^{-1}, & \alpha < \beta, \alpha, \beta &= 1, \ldots, g
\end{align*} \tag{25}$$

and the first two relations define the well known Artin braid group [15]. This group $B^n_g$ is a subgroup of $B_{g+n}$ as explained in [14]. We refer to this paper for further details and references. Here we only explain conventions* briefly which should make (25) fairly transparent.

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* We found it necessary to have conventions slightly different from [14], especially in the last formula of (25) the condition in [14] is $\beta < \alpha$, and for lines leaving the unit cube $z_1 > z_2$ (see text below).
On the handlebody (Fig.1), without loss of generality, we prescribe a fixed ordering of the points where the strands begin (resp. end) having coordinates $P_i^{(1)} = \left( \frac{i}{n+1}, \frac{1}{2}, 1 \right)$ (resp. $P_j^{(0)} = \left( \frac{j}{n+1}, \frac{1}{2}, 0 \right)$), $i, j = 1, \ldots, n$ in a lefthanded $(x, y, z)$-coordinate system. So the unit cube in the positive octant is contained in $H_g$ and the usual braids are obtained by connecting points $P_i^{(1)}$ and $P_j^{(0)}$ by strands confined to the unit cube. The braid diagram is obtained by projecting on the $x$-$z$-plane. The handles are positioned, say, to the left of the unit cube around coordinates $h_\alpha = \left( \frac{-\alpha}{g+1}, y, 1 - \frac{\alpha}{g+1} \right)$, $\alpha = 1, \ldots, g$. For the braid group on $H_g$ the strands are allowed to leave the unit cube at height $z$ and go around the handle $h_\alpha$ counterclockwise for $\tau_\alpha$ (clockwise for $\tau_\alpha^{-1}$) and then come back to the unit cube at height $z - \delta$, $\delta$ small. The convention is that strands leaving or entering the unit cube at height $z_1$ should be under those doing so at $z_2$ in the projection onto the $x$-$z$-plane if $z_2 > z_1$. Within the unit cube strands can only go downward in the negative $z$-direction. This definition can be further formalized, but all this is rather intuitive.

Fig.1: The 2-braid $\tau_2^{-1}\sigma_1^{-2}\tau_1$ and its closure (dotted lines)
The strands leaving the unit cube always belong to the first space $V_1$ of the tensor product $V(n) = V_1 \otimes \ldots \otimes V_n$ on which the $\sigma_i$ act, and this explains why only $\sigma_1$ is non-commuting with $\tau_\alpha$. For our arguments it is more appropriate to think of piercing long needles through the handles and after that forget about them. Then, if we rotate the needles by $\pi/4$ around the $x$-axis counterclockwise to $h'_\alpha = (\frac{\alpha}{g+1}, \frac{\alpha}{g+1} - 1, z)$ we can depict the braiding in a suggestive way by projecting on the $x$-$z$-plane (Fig.2). All relations of (25) and all those involving $K$ like (16), (17) and (18) can be represented in terms of diagrams as in Fig.2 and they are proven easily this way.

A $g$-link $L_g$ on $H_g$ is obtained by connecting $P_{i}^{(0)}$ with $P_{i}^{(1)}$ outside the unit cube in the $x > 0$ region. Then, citing a theorem [14], every $g$-link can be obtained as the closure of a $g$-braid. However, the Markov theorem (Markov moves for $B_{n}^{g}$ can be defined completely analogous to the usual ones) was only stated as a conjecture in [14].

In [1] it was already pointed out that for genus one the fourth equation of (25) is just RE (16) if we identify $\sigma_i = PR \equiv \tilde{R}$ and $\tau = K_1$. Knowing now that a solution $K$ of (16) can be extended to a set of solutions $K^{(\alpha)}$ having nontrivial commutation relations (18) we see immediately that (25) is equivalent to YBE (4),
RE (16) and commutation relations (18) if we identify
\[ \sigma_i = -q^{i} \otimes \ldots \otimes \hat{R}_{i,i+1} \otimes \ldots \otimes 1, \quad i = 1, \ldots, n-1 \]
\[ \tau_\alpha = K^{(\alpha)} \otimes 1 \otimes 1, \quad \alpha = 1, \ldots, g \]
with two more rather obvious consistency conditions as given in (25).

Thus \( \sigma_i \) has an explicit matrix representation, but what about \( \tau_\alpha \)? Fig.2 suggests to represent the effect of a handle on a strand carrying a QG representation by \( K^{(\alpha)}_{ij} = (\hat{R}^2_{(\alpha)}_{ij})^{m_n} \). Here \((m,n)\) are the indices of the QG representation of the first strand ‘interacting’ with the handle, which is characterized by the special indices \((i,j)\). Therefore we have a two-dimensional representation of (12), but for \[\{a \leftrightarrow d, \ b \leftrightarrow c\}\]. To be consistent we adopt the convention to read crossings involving thick lines corresponding to handles as \( \hat{R} = P \hat{R} P = RP \) instead of \( \hat{R} \). So we put \( K^{(\alpha)}_{ij} = (\hat{R}^2_{(\alpha)}_{ij})^{m_n} \) and from (3) we calculate
\[ a^m_n = \begin{pmatrix} q^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad b^m_n = \begin{pmatrix} 0 & 0 \\ \omega & 0 \end{pmatrix}, \]
\[ c^n_m = \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}, \quad d^m_n = \begin{pmatrix} 1 + \omega^2 & 0 \\ 0 & q^2 \end{pmatrix}, \]
which indeed satisfy (12). It is easy to generalize this to arbitrary representations the strands may be carrying. In the fundamental representation we get from (24) just \( S(L^-)L^+|_{\rho_{fund}} = q^{-1} \hat{R}^2 \). Thus we represent \( \tau_\alpha \) as
\[ \tau_\alpha = q S(L^-)L^+|_{\rho} \otimes 1 \otimes 1 \ldots 1, \]
where \( K = q S(L^-)L^+ \) is in an arbitrary representation \( \rho \) and the index \( \alpha \) only keeps track of handle numbering. This operator appeared also in the context of conformal field theory [13] and was used there, for example, in connection with topology changing amplitudes in Chern-Simons field theory.

4. IN Variant LINK POLYNOMIALS ON SOLID TORI

There are several approaches to the construction of link polynomials, one may roughly distinguish them in the following way (a convenient access to literature is [16]). It is well known that the expression of \( \sigma_i \) in terms of \( \hat{R} \) gives rise to a Hecke algebra representation of the braid group \( B_n \) and the characteristic equation
\( \hat{R}^2 = \omega \hat{R} + 1 \) of the \( R \)-matrix together with the first two equations of (25) comprise just the relations of the Hecke algebra \( H(q^2, n) \) with generators \( \sigma_i \). One defines a linear functional on \( H(q^2, n) \), the Ocneanu trace, which is the main ingredient in the definition of the invariant link polynomial [17]. Alternatively one may use the matrix trace of the braid group generators represented by \( \hat{R} \) and then prove invariance w.r.t Markov moves [18, 19]. Further it is possible to define link polynomials recursively using skein relations [20, 21, 22]. Finally, there is the Chern-Simons field theory approach [23].

In view of this we may expect that the explicit representation (26) and (27) can be used to define an invariant link polynomial on \( H_g \) by means of (quantum) traces of generators \( \sigma_i \) and \( \tau_a \). However we shall outline the simpler approach via skein relations here.

Above characteristic equation of the \( R \)-matrix is equivalent to the skein relation of the Jones polynomial. It means that the invariant polynomial satisfies for each oriented link \( L \)

\[
\alpha P_+(L) + \beta P_-(L) + \gamma P_0(L) = 0, \quad \alpha, \beta, \gamma = \text{const.} \tag{28}
\]

Generally, the invariant linear functional \( P(L) \) may depend on one or more parameters and \( P_{\pm}, P_0 \) are its value for a link \( L \) which is different in each case only at a single crossing as shown in Fig.3:

\[
\begin{align*}
\init &\quad + \:\init &\quad + \:\init \\
\alpha &\quad + \:\beta &\quad + \:\gamma
\end{align*}
\]

Fig.3: Skein relation of type A

One may read these crossings as \( \hat{R}, \hat{R}^{-1} \) and \( I \) for example. We keep \( \alpha, \beta, \gamma \) arbitrary as we do not want to specify a certain polynomial, nor do we fix the type of isotopy (i.e. ambient or regular). We only assume we are given a skein relation and have normalized the polynomial of the unknot. This is sufficient to construct
the polynomial of any link uniquely, starting from the unknot, or vice versa. Our strategy is to use this procedure to unknot any link on $H_g$ completely which is clearly possible. Then we end up with unknots which do

(i) not go around a handle,
(ii) go around a handle once,
(iii) go several times around a handle,
(iv) go around two (or more) handles once (or several times),

we depict them in Fig.4:

(a) ![Diagram](image1.png) (b) ![Diagram](image2.png) (c) ![Diagram](image3.png) (d) ![Diagram](image4.png) (e) ![Diagram](image5.png)

**Fig.4:** Some examples of cases (i) - (iv) for genus two

Our assertion is that all cases can be reduced to (i) by using (15). This relation can be represented as in Fig.5 and we refer to it as type B skein relation henceforth.

\[
-1 = q^2 + c_1 \cdot 0
\]

**Fig.5:** Skein relation of type B

To unknot any link in $H_g$ we fix the procedure as follows:
1. Eliminate $\tau_a^{-1}$ from the link using type B skein relations

2. Unknot the link using type A skein relations

3. Relate cases (iv), (iii), (ii) $\rightarrow$ (i)

After step 2 we arrive at cases (i) - (iv) described above. Step 1 guarantees that we have only links going around handles counterclockwise since step 2 is orientation preserving. The reduction from (ii) $\rightarrow$ (i) is easily performed. We simply close the braids in Fig.5 and use the type I Reidemeister move depicted in Fig.6a, where $\delta = 1$ for ambient isotopy.

![Reidemeister moves of type I](a) \quad \text{(b)}

We get the result shown in Fig.7, where the first link has different orientation compared to the others.

$$\delta^{-1} \begin{array}{c} \bullet \\ \text{q}^2 \end{array} + \begin{array}{c} \bullet \\ - c_1 \bullet \end{array} = 0$$

Fig.7

We might pick up a phase factor when reversing the orientation of a unknot going around a handle. Denoting this as $\varepsilon$, and further taking into account the normalization of the unknot in topologically trivial regions which we denote by $N$ we finally obtain the result given in Fig.8:
\[ \frac{c_1}{q^2 + \varepsilon \delta^{-1}} = N \]

Fig.8

So this gives the desired expression for case (ii) unknots. It is clear that a case (iii) unknot like the one in Fig.4c gives the square of the above result (with \( c_1 \rightarrow c_1' \)). Then, case (iv) unknots like the one in Fig.4d which corresponds to the product \( \tau_1 \tau_2 \) gives the same result as above, however with \( c_1 \rightarrow C_1 \), due to RE multiplication properties (17) and type B skein relation (20). It is then obvious how to treat the unknot in Fig.4e. We leave all constants unspecified since we do not want to embark upon explicit calculations here.

5. **Summary**

We set out to explain in this paper the braid group on a solid handlebody and its equivalence to the YB and RE operator algebra. Further we suggested an explicit representation which could possibly be used to define invariant link polynomials. We found a way to construct the polynomial of any link on the handlebody recursively via new skein relations. These additional skein relations allowed us to relate the unknot going around a handle to the unknot in topologically trivial region which is normalized to a constant.

It was not our intention to give rigorous derivations or proofs of the assertions in section four, rather we would like to draw attention to the subject, especially it is interesting to see whether this can be used to construct invariant link polynomials on arbitrary 3-manifolds as mentioned in the introduction. Then, of course, can this be applied to conformal and Chern-Simons field theory?

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