ON UNITARITY OF SOME REPRESENTATIONS OF CLASSICAL 
p-ADIC GROUPS II

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Abstract. C. Jantzen has defined in [2] a correspondence which attaches to an irreducible representation of a classical \( p \)-adic group, a finite set of irreducible representations of classical \( p \)-adic groups supported in a single or in two cuspidal lines (the case of the single cuspidal lines is interesting for the unitarizability). It would be important to know if this correspondence preserves the unitarizability (in both directions). The main aim of this paper is to complete the proof started in [15] of the fact that if we have an irreducible unitarizable representation \( \pi \) of a classical \( p \)-adic group whose one attached representation \( X_{\rho}(\pi) \) supported by a cuspidal line, has the same infinitesimal character as the generalized Steinberg representation supported in that line, then \( X_{\rho}(\pi) \) is unitarizable.

1. Introduction

To an irreducible representation \( \pi \) of a classical \( p \)-adic group, C. Jantzen has attached in [2] a finite set of irreducible representations \( X_{\rho}(\pi) \) of classical \( p \)-adic groups supported in two cuspidal lines \( \{ \nu^x \rho; x \in \mathbb{R} \} \cup \{ \nu^x \tilde{\rho}; x \in \mathbb{R} \} \). These attached representations completely determine \( \pi \). It would be very important to know if this correspondence preserves the unitarizability (in both directions). In [12] we have reduced the problem of the unitarizability to the case when only selfcontragredient cuspidal lines show up (i.e. \( \rho \cong \tilde{\rho} \)).

In this paper we complete a proof a very special case related to the question of this preservation of the unitarizability. We complete a proof of the following

Theorem 1.1. Suppose that \( \pi \) is an irreducible unitary representation of a classical \( p \)-adic group, and suppose that the infinitesimal character of some \( X_{\rho}(\pi) \) is the same as the infinitesimal character of a generalized Steinberg representation\footnote{Generalized Steinberg representations are defined in [10].}. Then \( X_{\rho}(\pi) \) is unitarizable\footnote{We prove that \( \pi_L \) is equivalent to the generalized Steinberg representation, or its Aubert dual.}.

We are particularly thankful to C. Jantzen for reading the second section of this paper, where we present a his main results from [2] in a slightly reformulated form and gave his suggestions. We are also thankful to M. Hanzer, E. Lapid and A. Moy for useful discussions during the writing of this paper.

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We shall now briefly review the contents of the paper. We continue with the notation introduced in [15]. In the second section we recall of the Jantzen decomposition of an irreducible representation of a $p$-adic classical group in a slightly modified version, while the third section discusses the decomposition into the cuspidal lines. In the fourth section we give the proof of the main theorem, while in the fifth section we show that the unitarizability is preserved in the case of the irreducible generic representations of classical $p$-adic groups. In a similar way, using [7], one can see also the unitarizability is preserved for the irreducible unramified representations of the classical groups considered in [7] (i.e. for split classical $p$-adic groups). In the last section we formulate a question if the unitarizability for the irreducible representations of classical groups supported by a single cuspidal line depends only on the reducibility point (i.e., not on the particular cuspidal representations which have that reducibility).

2. Jantzen decomposition

Below we shall recall of the basic results of C. Jantzen from [2]. We shall write them in a slightly different way than in [2]. They are written there for the symplectic and the split odd-orthogonal series of groups. Since the Jantzen's paper is based on the formal properties of the representation theory of these groups (contained essentially in the structure of the twisted Hopf module which exists on the representations of these groups - see [9]), the results of [2] apply also whenever this structure is established. Therefore, it also holds for all the classical $p$-adic groups considered in [6].

First we shall recall of some definitions. For of an irreducible representation $\gamma$ of a classical group there exists an irreducible cuspidal representation $\gamma_{cusp}$ of a classical group and an irreducible representation $\pi$ of a general linear group such that

$$\gamma \hookrightarrow \pi \rtimes \gamma_{cusp}.$$ 

Then $\gamma_{cusp}$ is uniquely determined by this requirement (up to an equivalence), and it is called a partial cuspidal support of $\gamma$. Now the sum of all the terms in $\mu^*(\pi)$ whose right hand side tensor factor is precisely $\gamma_{cusp}$, will be denoted by

$$s_{GL}(\gamma).$$

A representation $\rho \in \mathcal{C}$ is called a factor of an irreducible representation $\gamma$ of a classical group, if there exists an irreducible subquotient $\tau \otimes \gamma_{cusp}$ of $s_{GL}(\gamma)$ such that $\rho$ is in the support of $\tau$.

\footnote{In the case of unitary groups one needs to replace usual contragredient by the contragredient twisted by the non-trivial element of the Galois group of the involved quadratic extension (see [6]). The case of disconnected even split orthogonal group is considered in [6].}
We shall fix below an irreducible cuspidal representation \( \sigma \) of a classical group. Above we have already used the well known notion of (cuspidal) support of an irreducible representation of a general linear group introduced by J. Bernstein and A. V. Zelevinsky. Let \( X \subseteq C \) and suppose that \( X \) is self contragredient, i.e. that
\[
\tilde{X} = X,
\]
where \( \tilde{X} = \{ \tilde{\rho} : \rho \in X \} \). Following C. Jantzen, one says that an irreducible representation \( \gamma \) of a classical group is supported by \( X \cup \{ \sigma \} \) if there exist \( \rho_1, \ldots, \rho_k \) from \( X \) such that
\[
\gamma \leq \rho_1 \times \ldots \times \rho_k \rtimes \sigma.
\]
For not-necessarily irreducible representation \( \pi \) of a classical group, one says that it is supported by \( X \cup \{ \sigma \} \) if each irreducible subquotient of it is supported by that set.

**Definition 2.1.** Let
\[
X = X_1 \cup X_2
\]
be a partition of a selfcontragredient \( X \subseteq C \). We shall say that this partition is regular if \( X_1 \) is self contragredient\(^4\) and if among \( X_1 \) and \( X_2 \) there is no reducibility, i.e. if
\[
\rho \in X_1 \implies \nu \rho \notin X_2.
\]
This is equivalent to say that \( \rho_1 \times \rho_2 \) is irreducible for all \( \rho_1 \in X_1 \) and \( \rho_2 \in X_2 \).

For a partition \( X = X_1 \cup \cdots \cup X_k \) we define to be regular in an analogous way.

**Definition 2.2.** Let \( \pi \) be a representation of \( S_n \) supported in \( X \cup \{ \sigma \} \). Suppose that \( X_1 \cup X_2 \) is a regular partition of a selfcontragredient \( X \subseteq C \). Write \( \mu^*(\pi) = \sum_i \beta_i \otimes \gamma_i \), a sum of irreducible representations in \( R \otimes R[S] \). Let \( \mu^*_{X_1}(\pi) \) denote the sum of every \( \beta_i \otimes \gamma_i \) in \( \mu^*(\pi) \) such that the support of \( \beta_i \) is contained in \( X_1 \) and the support of \( \gamma_i \) is contained in \( X_2 \cup \{ \sigma \} \).

Now we recall below the main results of [2]. As we have already mentioned, our presentation is slightly different from the presentation in [2]. In the rest of this section, \( X_1 \cup X_2 \) will be a regular partition of a selfcontragredient \( X \subseteq C \).

**Lemma 2.3.** If \( \pi \) has support contained in \( X \cup \{ \sigma \} \), then \( \mu^*_{X_1}(\pi) \) is nonzero.

**Definition 2.4.** Suppose \( \beta \) is a representation of a general linear group supported in \( X_1 \). Write \( M^*(\beta) = \sum_i \tau_i \otimes \tau'_i \), a sum of irreducible representations in \( R \otimes R \). Let \( M^*_{X_1}(\beta) \) denote the sum of every summand \( \tau_i \otimes \tau'_i \) in \( M^*(\beta) \) such that the support of \( \tau_i \) is contained in \( X_1 \) and the support of \( \tau'_i \) is contained in \( X_2 \).

**Proposition 2.5.** Suppose \( \beta \) is a representation of a general linear group with support contained in \( X \) and \( \gamma \) a representation of \( S_k \) with support contained in \( X \cup \{ \sigma \} \). Then,
\[
\mu^*_{X_1}(\beta \rtimes \gamma) = M^*_{X_1}(\beta) \rtimes \mu^*_{X_1}(\gamma).
\]

\(^4\)Then \( X_2 \) is also self contragredient
Corollary 2.6. Suppose $\beta$ has support contained in $X_1$ and $\gamma$ has support contained in $X_2 \cup \{\sigma\}$. Then

\begin{enumerate}
\item $\mu^*_{X_1}(\beta \bowtie \gamma) = M^*_{GL}(\beta) \otimes \gamma$.
\item Write $s_{GL}(\gamma) = \Xi \otimes \sigma$ in the Grothendieck group. Then $\mu^*_{X_2}(\beta \bowtie \gamma) = \Xi \otimes \beta \bowtie \sigma$.
\end{enumerate}

Definition 2.7. Suppose $\pi$ is an irreducible representation of $S_n$ supported in $X \cup \{\sigma\}$. Fix $i \in \{1, 2\}$. Then there exists an irreducible $\beta_i \bowtie \gamma_i$ with $\beta_i$ supported on $X_{3-i}$ and $\gamma_i$ supported on $X_i \cup \{\sigma\}$ such that $\pi \hookrightarrow \beta_i \bowtie \gamma_i$.

The representation $\gamma_i$ is uniquely determined by the above requirement, and it is denoted by $X_i(\pi)$.

Further,

\begin{equation}
\mu^*_{X_{3-i}}(\pi) \leq \mu^*_{X_{3-i}}(\beta_i \bowtie \gamma_i) = M^*_{GL}(\beta_i) \otimes \gamma_i.
\end{equation}

Now we shall recall of the key theorem from the Jantzen’s paper [2]:

Theorem 2.8. (Jantzen) Suppose that $X_1 \cup X_2$ is a regular partition of a selfcontragredient subset $X$ of $C$, and $\sigma$ an irreducible cuspidal representation of $S_r$. Let $\text{Irr}(X_i; \sigma)$ denote the set of all irreducible representations of all $S_n$, $n \geq 0$, supported on $X_i \cup \{\sigma\}$, and similarly for $\text{Irr}(X; \sigma)$.

Then the map

$$
\text{Irr}(X; \sigma) \longrightarrow \text{Irr}(X_1; \sigma) \times \text{Irr}(X_2; \sigma),
\pi \longmapsto (X_1(\pi), X_2(\pi))
$$

is a bijective correspondence. Denote the inverse mapping by

$$
\Psi_{X_1, X_2}.
$$

For $\gamma_i \in \text{Irr}(X_i; \sigma)$ these bijective correspondence have the following properties:

\begin{enumerate}
\item If $\gamma_i$ is a representation of $S_{n_1+r}$, then $\pi = \Psi_{X_1, X_2}(\gamma_1, \gamma_2)$ is a representation of $S_{n_1+n_2+r}$
\item $\Psi_{X_1, X_2}(\gamma_1, \gamma_2) = \Psi_{X_1, X_2}(\tilde{\gamma_1}, \tilde{\gamma_2})$ and $X_1(\tilde{\pi}) = X_i(\pi)$, where $\tilde{}$ denotes contragredient.
\end{enumerate}

\footnote{Clearly, $\Xi$ does not need to be irreducible.}
(3) $\Psi_{x_1,x_2}(\gamma_1, \gamma_2)^t = \Psi_{x_1,x_2}(\gamma_1^t, \gamma_2^t)$ and $X_i(\pi^t) = X_i(\pi)^t$, where $^t$ denotes the involution of Aubert.

(4) Suppose that
$$s_{GL}(\gamma_i) = \sum_j c_j(X_i)\tau_j(X_i) \otimes \sigma,$$
where $\tau_j(X_i)$ is an irreducible representation and $c_j(X_i)$ its multiplicity. Then
$$\mu^s_{X_i}(\Psi_{x_1,x_2}(\gamma_1, \gamma_2)) = \sum_j c_j(X_i)\tau_j(X_i) \otimes \gamma_{3-i}$$

(5) Let $\beta = \beta(X_1) \times \beta(X_2)$ be an irreducible representation of a general linear group with support of $\beta(X_i)$ contained in $X_i$, $i = 1, 2$, and $\Psi = \Psi_{x_1,x_2}(\gamma_1, \gamma_2)$ an irreducible representation of $S_k$ with support contained in $X \cup \{\sigma\}$. (We allow the possibility that $\beta(X_i) = 1$ or $\gamma_i = \sigma$.) Suppose
$$\beta(X_i) \rtimes \gamma_i = \sum_j m_j(X_i)\gamma_j(X_i; \sigma),$$
with $\gamma_j(X_i; \sigma)$ irreducible and $m_j(X_i)$ its multiplicity. Then,
$$\beta \rtimes \Psi = \sum_{j_1,j_2} (m_{j_1}(X_1)m_{j_2}(X_2))\Psi_{x_1,x_2}(\gamma_{j_1}(X_1; \sigma), \gamma_{j_2}(X_2; \sigma)).$$

(6) $\Psi_{x_1,x_2}(\gamma_1, \gamma_2)$ is tempered (resp. square-integrable) if and only if $\gamma_1, \gamma_2$ are both tempered (resp. square-integrable).

(7) Suppose, in the subrepresentation setting in "tempered" formulation of the Langlands classification,
$$\gamma_i = L(\nu^{\alpha_1}\tau_1(X_i), \ldots, \nu^{\alpha_t}\tau_t(X_i); T(X_i; \sigma))$$
for $i = 1, 2$ (n.b. recall that $\tau_j(X_i)$ may be the trivial representation of $GL(0, F); T(X_i; \sigma)$ may just be $\sigma$). Then,
$$\Psi_{x_1,x_2}(\gamma_1, \gamma_2) = L(\nu^{\alpha_1}\tau_1(X_1) \times \nu^{\alpha_1}\tau_1(X_2), \ldots, \nu^{\alpha_t}\tau_t(X_1) \times \nu^{\alpha_t}\tau_t(X_2); \Psi_{x_1,x_2}(T(X_1; \sigma), T(X_2; \sigma))).$$

In the other direction, if
$$\pi = L(\nu^{\alpha_1}\tau_1(X_1) \times \nu^{\alpha_1}\tau_1(X_2), \ldots, \nu^{\alpha_t}\tau_t(X_1) \times \nu^{\alpha_t}\tau_t(X_2); T(X; \sigma)),$$
then
$$X_i(\pi) = L(\nu^{\alpha_1}\tau_1(X_i), \ldots, \nu^{\alpha_t}\tau_t(X_i); X_i(T(X; \sigma))).$$
(In the quotient setting of the Langlands classification, the same results hold.)

(8) Suppose,
$$\mu^*(\gamma_i) = \sum_j n_j(X_i)\eta_j(X_i) \otimes \theta_j(X_i; \sigma),$$
with $\eta_j(X_i) \otimes \theta_j(X_i; \sigma)$ irreducible and $n_j(X_i)$ its multiplicity. Then,

$$
\mu^*(\Psi_{X_1,X_2}(\gamma_1, \gamma_2)) = \sum_{j_1,j_2} (n_{j_1}(X_1)n_{j_2}(X_2))(\eta_{j_1}(X_1) \times \eta_{j_2}(X_2)) \otimes \Psi_{X_1,X_2}(\theta_{j_1}(X_1; \sigma), \theta_{j_2}(X_2; \sigma)).
$$

(9) Let $X = X_1 \cup X_2 \cup X_3$ be a regular partition and $\pi \in \text{Irr}(X; \sigma)$. Then

$$
X_1((X_1 \cup X_2)(\pi)) = X_1((X_1 \cup X_3)(\pi)).
$$

In the other direction we have

$$
\Psi_{X_1 \cup X_2, X_3}(\Psi_{X_1, X_2}(\pi_1, \pi_2), \pi_3) = \Psi_{X_1, X_2 \cup X_3}(\pi_1, \Psi_{X_2, X_3}(\pi_2, \pi_3))
$$

for $\pi_i \in \text{Irr}(X_i; \sigma)$.

Remark 2.9. (1) Let $\beta_i$ be an irreducible representation of a general linear group supported in $X_i$, $i = 1, 2$, and let $\gamma_i$ be an irreducible representation of a classical $p$-adic group supported in $X_i \cup \{\sigma\}$, $i = 1, 2$. Then (5) of the above theorem implies

$$
(\beta_1 \times \beta_2) \times \Psi_{X_1, X_2}(\gamma_1, \gamma_2) \text{ is irreducible} \iff \text{both } \beta_i \times \gamma_i \text{ are irreducible}.
$$

(2) One can express the above theorem without the last claim, in a natural way for a regular partition in more than two pieces.

3. Cuspidal lines

Let $\rho$ be an irreducible unitarizable cuspidal representation of a general linear group. Denote

$$
X_\rho = \{\nu^x \rho; x \in \mathbb{R}\} \cup \{\nu^x \tilde{\rho}; x \in \mathbb{R}\},
$$

$$
X^c_\rho = C \backslash X_\rho.
$$

For an irreducible representation $\pi$ of a classical $p$-adic group take any finite set of different classes $\rho_1, \ldots, \rho_k \in C_u$ such that $\rho_i \not\sim \rho_j$ for any $i \neq j$, and that $\pi$ is supported in

$$
X_{\rho_1} \cup \cdots \cup X_{\rho_k} \cup \{\sigma\}.
$$

Then $\pi$ is uniquely determined by

$$
(X_{\rho_1}(\pi), \ldots, X_{\rho_k}(\pi)).
$$

Now we have a natural

Preservation question: Let $\pi$ be an irreducible representation of a classical $p$-adic group. Is $\pi$ unitarizable if and only if all $X_{\rho_i}(\pi)$ are unitarizable?
4. Proof of the main result

**Theorem 4.1.** Suppose that \( \theta \) is an irreducible unitarizable representation of a classical group, and suppose that the infinitesimal character of some \( X_\rho(\theta) \) is the same as the infinitesimal character of a generalized Steinberg representation supported in \( X_\rho \cup \{ \sigma \} \). Then \( X_\rho(\theta) \) is unitarizable.

*Proof.* Denote 

\[
\theta_\rho = X_\rho(\theta), \quad \theta_\rho^c = X_\rho^c(\theta).
\]

Then 

\[
\theta = \Psi_{X_\rho,X_\rho^c}(\theta_\rho, \theta_\rho^c).
\]

Suppose that \( \theta_\rho \) is not unitarizable. Then obviously \( \theta_\rho \) is not the generalized Steinberg representation. Further, \([1]\) implies that \( \theta_\rho \) is not its Aubert dual. Now Proposition 3.2 of \([15]\) implies that there exists a selfcontragredient unitarizable representation \( \pi \) of a general linear group supported in \( X_\rho \) such that the length of 

\[
\pi \rtimes \theta_\rho
\]

is at least 5, and that the multiplicity of \( \pi \otimes \theta_\rho \) in the Jacquet module of \( \pi \rtimes \theta_\rho \) is at most 4.

Consider now 

\[
\pi \rtimes \theta = \pi \rtimes \Psi_{X_\rho,X_\rho^c}(\theta_\rho, \theta_\rho^c).
\]

Then this representation is of length \( \geq 5 \) (take in (5) of Jantzen theorem \( \beta(X_\rho) = \pi, \beta(X_\rho^c) = 1 \), and multiply it by the representation \( \Psi_{X_\rho,X_\rho^c}(\theta_\rho, \theta_\rho^c) \)).

We shall now use the assumption that \( \theta = \Psi_{X_\rho,X_\rho^c}(\theta_\rho, \theta_\rho^c) \) is unitarizable. From the fact that the length of \( \pi \otimes \Psi_{X_\rho,X_\rho^c}(\theta_\rho, \theta_\rho^c) \) is at least 5 and the exactness of the Jacquet module functor, it follows that the multiplicity of \( \pi \otimes \Psi_{X_\rho,X_\rho^c}(\theta_\rho, \theta_\rho^c) \) in \( \mu^*(\pi \rtimes \Psi_{X_\rho,X_\rho^c}(\theta_\rho, \theta_\rho^c)) \) is at least five.

By the definition of \( \theta_\rho \), we can choose an irreducible representation \( \phi \) of a general linear group supported in \( X_\rho \) such that 

\[
\Psi_{X_\rho,X_\rho^c}(\theta_\rho, \theta_\rho^c) \hookrightarrow \phi \rtimes \theta_\rho.
\]

By the Frobenius reciprocity, \( \phi \otimes \theta_\rho \) is a sub quotient of the Jacquet module of \( \Psi_{X_\rho,X_\rho^c}(\theta_\rho, \theta_\rho^c) \). Denote its multiplicity by \( k \). This implies that the multiplicity of \( \pi \otimes \phi \otimes \theta_\rho \) in \( \mu^*(\pi \rtimes \Psi_{X_\rho,X_\rho^c}(\theta_\rho, \theta_\rho^c)) \) is at least \( 5k \).

Recall that the support of \( \pi \) is in \( X_\rho \) and the support of \( \phi \) is in \( X_\rho^c \). Let \( \Pi \) be an irreducible representation of a general linear group which has in its Jacquet module \( \pi \otimes \phi \). Then \( \Pi \cong \pi' \rtimes \phi' \), where the support of \( \pi' \) is in \( X_\rho \) and the support of \( \phi' \) is in \( X_\rho^c \). Further, \( \pi \) and

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6We prove below that \( X_\rho(\theta) \) is equivalent to the generalized Steinberg representation, or its Aubert dual.
\( \pi' \) are representations of the same group (as well as \( \phi \) and \( \phi' \)). Frobenius reciprocity implies that \( \pi' \otimes \phi' \) is in the Jacquet module of \( \Pi \). Further, the formula \( m^* (\Pi) = m^*(\pi) \times m^*(\phi) \) implies that if we have in the Jacquet module of \( \Pi \) an irreducible representation of the form \( \pi'' \otimes \phi'' \), where the support of \( \pi'' \) is in \( X_\rho \) and the support of \( \phi'' \) is in \( X_\rho^c \), then \( \pi'' \cong \pi' \), \( \phi'' \cong \phi' \), and the multiplicity of \( \pi' \times \phi' \) in the Jacquet module of \( \Pi \) is one.

This first implies that \( \Pi \cong \pi \times \phi \), then that the only irreducible representation of a general linear group which has in its Jacquet module \( \pi \otimes \phi \) is \( \pi \times \phi \), and further that the multiplicity of \( \phi \otimes \pi \) in the Jacquet module \( \pi \times \phi \) is one.

This and the transitivity of the Jacquet modules imply that the multiplicity of \( \phi \otimes \pi \otimes \theta_\rho \) in the Jacquet module of \( \mu^*(\pi \times \Psi_{X_\rho, X_\rho^c}(\theta_\rho, \theta_\rho^c)) \) is at least \( 5k \).

Now we examine in a different way the multiplicity of \( \phi \otimes \pi \otimes \theta_\rho \) in the Jacquet module of \( \mu^*(\pi \times \Psi_{X_\rho, X_\rho^c}(\theta_\rho, \theta_\rho^c)) \). Observe that \( \phi \otimes \pi \otimes \theta_\rho \) must be a sub quotient of a Jacquet module of the following part

\[
\mu^*_{X_\rho^c}(\pi \times \Psi_{X_\rho, X_\rho^c}(\theta_\rho, \theta_\rho^c)) = (1 \otimes \pi) \times \mu^*_{X_\rho^c}(\Psi_{X_\rho, X_\rho^c}(\theta_\rho, \theta_\rho^c))
\]

of \( \mu^*(\pi \times \Psi_{X_\rho, X_\rho^c}(\theta_\rho, \theta_\rho^c)) \). Recall that by (2.1), \( \mu^*_{X_\rho^c}(\Psi_{X_\rho, X_\rho^c}(\theta_\rho, \theta_\rho^c)) \) is of the form \( \ast \otimes \theta_\rho \). If we want to get \( \phi \otimes \pi \otimes \theta_\rho \) from a term from here, it must be \( \phi \otimes \theta_\rho \). Recall that we have this term with multiplicity \( k \) here. Therefore, we need to see the multiplicity of \( \phi \otimes \pi \otimes \theta_\rho \) in the Jacquet module of \( k \cdot (1 \otimes \pi) \times (\phi \otimes \theta_\rho) = k \cdot (\phi \otimes \pi \times \theta_\rho) \). We know that this multiplicity is at most \( 4k \). Therefore, \( 5k \leq 4k \) (and \( k \geq 1 \)). This is a contradiction. Therefore, \( \theta_\rho \) is unitarizable.

\[\square\]

5. Irreducible generic and irreducible unramified representations

One can find in \[5\] more detailed exposition of the facts about irreducible generic representations and unitarizable subclasses that we shall use here. We shall recall here only very briefly of some of that facts.

Let \( \gamma \) be an irreducible representation of a classical group. Let \( X_1 \cup X_2 \) be a regular partition of \( C \). Now \[8\] directly implies that \( \gamma \) is generic if and only if \( X_1(\gamma) \) and \( X_2(\gamma) \) are generic. Therefore,

\[\gamma \text{ is generic if and only if all } X_\rho(\tau) \text{ are generic, } \rho \in \mathcal{C}_u.\]  

Analogous statement holds for temperness by (6) of Theorem 2.8.

Recall that by (5) of Theorem 2.8 if support of some irreducible representation \( \beta \) of a general linear group is contained in \( X_\rho' \), then holds

\[\beta \cong \gamma \text{ is irreducible } \iff \beta \times X_\rho'(\gamma) \text{ is irreducible.}\]
Denote by $C_u'$ any subset of $C_u$ satisfying:

$$C_u' \cup (C_u')^\sim = C_u \quad \text{and} \quad \rho \in C_u' \cap (C_u')^\sim \implies \rho \cong \tilde{\rho}.$$ 

Let $\pi$ be an irreducible generic representation of a classical group. We can write $\pi$ uniquely as

$$\pi \cong \delta_1 \times \cdots \times \delta_k \rtimes \tau \quad (5.4)$$

where the $\delta_i$'s are irreducible essentially square-integrable representations of general linear groups which satisfy

$$e(\delta_1) \geq \cdots \geq e(\delta_k) > 0, \quad (5.5)$$

and $\tau$ is a generic irreducible tempered representation of a classical group.

For $\rho' \in C_u'$ chose some irreducible representation $\Gamma_{\rho'}^c$ of a general linear group such that

$$\tau \hookrightarrow \Gamma_{\rho'}^c \rtimes X_{\rho'}(\tau),$$

and that $\Gamma_{\rho'}^c$ is supported out of $X_{\rho'}$. Observe that

$$\pi \cong \left( \prod_{\rho \in C_u' \setminus \{\rho'\}} \left( \prod_{\text{supp}(\delta) \subseteq X_{\rho}} \delta_i \right) \right) \rtimes \tau \hookrightarrow \left( \prod_{\rho \in C_u'} \left( \prod_{\text{supp}(\delta) \subseteq X_{\rho}} \delta_i \right) \right) \times \Gamma_{\rho'}^c \rtimes X_{\rho'}(\tau)$$

$$\cong \left( \prod_{\rho \in C_u' \setminus \{\rho'\}} \left( \prod_{\text{supp}(\delta) \subseteq X_{\rho}} \delta_i \right) \right) \times \Gamma_{\rho'}^c \times \left( \prod_{\text{supp}(\delta) \subseteq X_{\rho'}} \delta_i \right) \rtimes X_{\rho'}(\tau).$$

One easily sees that there exists an irreducible subquotient $\Pi_{\rho'}^c$ of

$$\left( \prod_{\rho \in C_u' \setminus \{\rho'\}} \left( \prod_{\text{supp}(\delta) \subseteq X_{\rho}} \delta_i \right) \right) \times \Gamma_{\rho'}^c$$

such that

$$\pi \hookrightarrow \Pi_{\rho'}^c \times \left( \prod_{\text{supp}(\delta) \subseteq X_{\rho'}} \delta_i \right) \rtimes X_{\rho'}(\tau).$$

Since $\Pi_{\rho'}^c$ is supported out of $X_{\rho'}$ and $\left( \prod_{\text{supp}(\delta) \subseteq X_{\rho'}} \delta_i \right) \rtimes X_{\rho'}(\tau)$ is irreducible and supported in $X_{\rho'} \cup \{\sigma\}$, we get that

$$X_{\rho'}(\pi) = \left( \prod_{\text{supp}(\delta) \subseteq X_{\rho'}} \delta_i \right) \rtimes X_{\rho'}(\tau). \quad (5.6)$$

Let $\pi \cong \delta_1 \times \cdots \times \delta_k \rtimes \tau$ be as in (5.4). Then for any square-integrable representation $\delta$ of a general linear group denote by $\mathcal{E}_\pi(\delta)$ the multiset of exponents $e(\delta_i)$ for those $i$ such that $\delta_i^u \cong \delta$. We denote below by $1_G$ the trivial one-dimensional representation of a group $G$. Now we recall of the solution of the unitarizability problem for irreducible generic representations of classical $p$-adic groups obtained in [5].
Theorem 5.1. Let $\pi$ be given as in (5.4). Then $\pi$ is unitarizable if and only if for all irreducible square integrable representations $\delta$ of general linear groups hold

1. $E_\pi(\tilde{\delta}) = E_\pi(\delta)$, i.e. $\pi$ is Hermitian.
2. If either $\delta \not\cong \tilde{\delta}$ or $\nu^2 \delta \times 1_{G_0}$ is reducible then $0 < \alpha < \frac{1}{2}$ for all $\alpha \in E_\pi(\delta)$.
3. If $\tilde{\delta} \cong \delta$ and $\nu^2 \delta \times 1_{G_0}$ is irreducible then $E_\pi(\delta)$ satisfies Barbasch’ conditions, i.e. we have $E_\pi(\delta) = \{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l\}$ with

   $0 < \alpha_1 \leq \cdots \leq \alpha_k < \frac{1}{2} < \beta_1 < \cdots < \beta_l < 1$

   such that
   (a) $\alpha_i + \beta_j \neq 1$ for all $i = 1, \ldots, k$, $j = 1, \ldots, l$; $\alpha_{k-1} \neq \frac{1}{2}$ if $k > 1$.
   (b) $\#\{1 \leq i \leq k : \alpha_i > 1 - \beta_1\}$ is even if $l > 0$.
   (c) $\#\{1 \leq i \leq k : 1 - \beta_j > \alpha_i > 1 - \beta_{j+1}\}$ is odd for $j = 1, \ldots, l - 1$.
   (d) $k + l$ is even if $\delta \times \tau$ is reducible.

Observe that (5.3) implies that if $\text{supp}(\delta_i) \subset X_{\rho'}$, then

(5.7) $\delta_i \times \tau$ is irreducible $\iff$ $\delta_i \times X_{\rho'}(\tau)$ is irreducible.

Let $\pi$ be a generic representation. We can then present it by the formula (5.4)

Suppose that $\pi$ is unitarizable. This implies that $\pi$ satisfies the above theorem. Now from (5.7), the above theorem implies that $X_{\rho'}(\pi) \cong \left(\prod_{\text{supp}(\delta) \subset X_{\rho'}} \delta\right) \times X_{\rho'}(\tau)$ is unitarizable (we need (5.7) only for (d) of (3) in the above theorem).

Suppose now that all $X_{\rho'}(\pi) = \left(\prod_{\text{supp}(\delta) \subset X_{\rho'}} \delta\right) \times X_{\rho'}(\tau)$, $\rho \in C_u$, are unitarizable. Then each of them satisfy the above theorem. Now the above theorem and (5.7) imply that $\pi$ is unitarizable.

Therefore, we have proved the following

Corollary 5.2. For an irreducible generic representation $\pi$ of a classical group holds

$\pi$ is unitarizable $\iff$ all $X_{\rho}(\pi)$, $\rho \in C_u$, are unitarizable. □

In a similar way, using the classification of the irreducible unitarizable unramified representations of classical $p$-adic groups in [7] (or as it is stated in [13]), we get that the above fact holds for irreducible unramified representations of classical $p$-adic groups.
Let \( \rho \) and \( \sigma \) be irreducible unitarizable cuspidal representations of a general linear and a classical group respectively. If there exists a non-negative \( \alpha_{\rho,\sigma} \in \frac{1}{2} \mathbb{Z} \) such that

\[
\nu^{\alpha_{\rho,\sigma}} \rho \cong \sigma
\]

reduces, then this \( \alpha \) will be denoted also by \( \alpha_{\rho,\sigma} \).

By a \( \mathbb{Z} \)-segment in \( \mathbb{R} \) we shall mean a subset of form \( \{x, x + 1, \ldots, x + l\} \) of \( \mathbb{R} \). We shall denote this subset by \( [k, k + l] \). For such a segment \( \Delta \), we denote

\[
\Delta^{(\rho)} = \{\nu^x; x \in \Delta\}.
\]

We shall take two pairs \( \rho_i, \sigma_i \) as above, such that

\[
\alpha_{\rho_1,\sigma_1} = \alpha_{\rho_2,\sigma_2} = \alpha.
\]

We shall construct a natural bijection

\[
E_{1,2} : \text{Irr}(X_{\rho_1}; \sigma_1) \to \text{Irr}(X_{\rho_2}; \sigma_2),
\]

which will be canonical, except in the case when \( \alpha = 0 \). First we shall define \( E_{1,2} \) on the irreducible square integrable representations.

A classification of irreducible square integrable representations of classical \( p \)-adic groups modulo cuspidal data is completed in [6]. We shall freely use notation of that paper, and also of [11]. We shall very briefly recall of parameters of irreducible square integrable representations in \( \text{Irr}(X_\rho; \sigma) \) (one can find more details in [11], sections 16 and 17). Below \( (\rho, \sigma) \) will denote \( (\rho_1, \sigma_1) \) or \( (\rho_2, \sigma_2) \).

An irreducible square integrable representation \( \pi \in \text{Irr}(X_\rho; \sigma) \) is parameterized by Jordan blocks \( \text{Jord}_\rho(\pi) = \{\Delta_1^{(\rho)}, \ldots, \Delta_k^{(\rho)}\} \), where \( \Delta_i \) are \( \mathbb{Z} \)-segments contained in \( \alpha + \mathbb{Z} \), and by a partially defined function \( \epsilon_\rho(\pi) \) (partial cuspidal support is \( \sigma \)). Since \( \{\Delta_1^{(\rho)}, \ldots, \Delta_k^{(\rho)}\} \) and \( \{\Delta_1, \ldots, \Delta_k\} \) are in a natural bijective correspondence, we can view \( \epsilon_\rho(\pi) \) as defined (appropriately) on \( \{\Delta_1, \ldots, \Delta_k\} \) (which means that \( \epsilon_\rho(\pi) \) is independent of particular \( \rho \)).

In sections 16 and 17 of [11], it is explained how \( \pi \) and the triple

\[
(\{\Delta_1^{(\rho)}, \ldots, \Delta_k^{(\rho)}\}, \epsilon_\rho(\pi), \sigma)
\]

are related. In this case we shall write

\[
(6.8) \quad \pi \leftrightarrow (\{\Delta_1^{(\rho)}, \ldots, \Delta_k^{(\rho)}\}, \epsilon_\rho(\pi), \sigma).
\]

Take irreducible square integrable representations \( \pi_i \in \text{Irr}(X_\rho; \sigma) \), \( i = 1, 2 \). Suppose

\[
(6.9) \quad \pi_1 \leftrightarrow (\{\Delta_1^{(\rho_1)}, \ldots, \Delta_k^{(\rho_1)}\}, \epsilon_{\rho_1}(\pi_1), \sigma_1).
\]

Then we define

\[
E_{1,2}(\pi_1) = \pi_2
\]
if

$$\pi_2 \leftrightarrow (\{\Delta_1^{(\rho_2)}, \ldots, \Delta_k^{(\rho_2)}\}, \epsilon_{\rho_1}(\pi_1), \sigma_2).$$

For defining $E_{1,2}$ on the whole $Irr(X_{\rho_1}; \sigma_1)$, the key step is an extension of $E_{1,2}$ from the square integrable classes to the tempered classes. For this, we shall use parameterization of irreducible tempered representations obtained in [13].

Let $\pi \in Irr(X_{\rho_1}; \sigma)$ be square integrable and let $\delta := \delta(\Delta^{(\rho)})$ be an irreducible (unitarizable) square integrable representation of a general linear group, where $\Delta$ is a segment in $\alpha + \mathbb{Z}$ such that $\delta \propto \pi$ reduces (one directly reads from the invariants (6.8) when this happens). Now Theorem 1.2 of [14] defines the irreducible tempered subrepresentation $\pi_\delta$ of $\delta \propto \pi$. The other irreducible summand is denoted by $\pi_{-\delta}$.

Let $\pi \in Irr(X_{\rho_1}; \sigma)$ be square integrable, let $\delta_i := \delta(\Delta_i^{(\rho)})$ be different irreducible (unitarizable) square integrable representations of general linear groups, where $\Delta_i$ are $\mathbb{Z}$-segments contained in $\alpha + \mathbb{Z}$ such that all $\delta_i \propto \pi$ reduce, and let $j_i \in \{\pm 1\}$, $i = 1, \ldots, n$. Then there exists a unique (tempered) irreducible representation $\pi'$ of a classical group such that

$$\pi' \leftrightarrow \delta_1 \times \ldots \times \delta_{i-1} \times \delta_{i+1} \times \ldots \times \delta_n \propto \pi_{j_i \delta_i},$$

for all $i$. Then we denote

$$\pi' = \pi_{j_1 \delta_1}, \ldots, j_n \delta_n.$$

In the situation as above we define

$$E_{1,2}(\pi_{j_1 \delta(\Delta_1^{(\rho_1)}), \ldots, j_n \delta(\Delta_1^{(\rho_1)})}) = E_{1,2}(\pi_{j_1 \delta(\Delta_2^{(\rho_2)}), \ldots, j_n \delta(\Delta_2^{(\rho_2)})}.$$  

Let additionally $\Gamma_{1}^{(\rho)}, \ldots, \Gamma_{m}^{(\rho)}$ be segments of cuspidal representations such that for each $i$, either $\Gamma_i$ is among $\Delta_j$’s, or $\delta(\Gamma_i^{(\rho)}) \propto \pi$ is irreducible, and $-\Gamma_i = \Gamma_i$. Then the tempered representation

$$(6.10) \quad \delta(\Gamma_1^{(\rho)}) \times \ldots \times \delta(\Gamma_m^{(\rho)}) \propto \pi_{j_1 \delta(\Delta_1^{(\rho)}), \ldots, j_n \delta(\Delta_1^{(\rho)})}$$

is irreducible. We define

$$E_{1,2}(\delta(\Gamma_1^{(\rho_1)}) \times \ldots \times \delta(\Gamma_m^{(\rho_1)}) \propto \pi_{j_1 \delta(\Delta_1^{(\rho_1)}), \ldots, j_n \delta(\Delta_1^{(\rho_1)})}) = \delta(\Gamma_1^{(\rho_2)}) \times \ldots \times \delta(\Gamma_m^{(\rho_2)}) \propto E_{1,2}(\pi_{j_1 \delta_1}, \ldots, j_n \delta_n).$$

In this way we have define $E_{1,2}$ on the subset of all the tempered classes in $Irr(X_{\rho_1}; \sigma)$.

Let now $\pi$ be any element of $Irr(X_{\rho_1}; \sigma)$. Write

$$L(\Delta_1^{(\rho_1)}, \ldots, \Delta_k^{(\rho_1)}; \tau)$$

as a Langlands quotient ($\Delta_i$ are $\mathbb{Z}$ segments in $\mathbb{R}$ and $\tau$ is a tempered class in $Irr(X_{\rho_1}; \sigma)$). Then we define

$$E_{1,2}(L(\Delta_1^{(\rho_1)}, \ldots, \Delta_k^{(\rho_1)}; \tau)) = L(\Delta_1^{(\rho_2)}, \ldots, \Delta_k^{(\rho_2)}; E_{1,2}(\tau)).$$

Another possibility would be to use the Jantzen’s parameterization obtained in [4] (we do not know if using [4] would result with the same mapping $E_{1,2}$).
Independence question: Let \( \pi \in \text{Irr}(X_{\rho, \sigma}) \). Is \( \pi \) unitarizable if and only if \( E_{1,2}(\pi) \) is unitarizable?

One can also ask if the other important representation theoretic data are preserved by \( E_{1,2} \) (Jacquet modules, irreducibilities of parabolically induced representations, Kazhdan-Lusztig multiplicities etc.).

Remark 6.1. We continue with the previous notation. Let \( \delta := \delta(\Delta^{(\rho)}) \) be an irreducible (unitarizable) square integrable representation of a general linear group, where \( \Delta \) is a segment in \( \alpha + \mathbb{Z} \).

Then we know that \( \nu^{1/2} \delta(\Delta^{(\rho)}) \rtimes 1_{G_0} \) reduces if and only if

1. \( \text{card}(\Delta) \) is odd if \( \alpha \notin \mathbb{Z} \);
2. \( \text{card}(\Delta) \) is even if \( \alpha \in \mathbb{Z} \).

(above \( G_{0,i} \) denotes the group of split rank 0 from the corresponding series of the groups). Therefore the conditions of reducibility of \( \nu^{1/2} \delta(\Delta^{(\rho)}) \rtimes 1_{G_{0,i}} \) in (2) and irreducibility in (3) of Theorem [5,4] does not depend on \( \rho \), but only on \( \Delta \) and \( \alpha \).

Further, let \( \tau \) be the representation in [6,10]. Now \( \delta(\Delta^{(\rho)}) \rtimes \tau \) is reducible if and only if

(i) \( \alpha \in \Delta \);
(ii) \( \Delta \notin \{\Delta_1, \ldots, \Delta_n\} \) (recall that \( \Delta_1, \ldots, \Delta_n \) form the Jordan block of \( \pi \) along \( \rho \));
(1) \( \Delta \notin \{\Gamma_1, \ldots, \Gamma_m\} \).

Obviously, these conditions again does not depend on \( \rho, \sigma \), but on \( \alpha \) and parameters which are preserved by \( E_{1,2} \). Therefore now Theorem [5,4] implies that the above independence question has positive answer for the irreducible generic representations, i.e. the unitarizability in this case does not depend on particular \( \rho \) and \( \sigma \), but only on \( \alpha = \alpha_{\rho, \sigma} \).

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