Harmonic $p$-forms on Finsler manifolds

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Abstract

The adjoint horizontal differential and co-differential operators and a horizontal Laplacian are defined on a Finsler manifold. These operators yield a natural definition for harmonic $p$-forms on a Finsler manifold in the sense that a horizontal $p$-form is harmonic if and only if the horizontal Laplacian vanishes. This approach permits to define a harmonic vector field on a Finsler manifold in a more natural sense and study its fundamental properties. Finally, a classification of harmonic vector fields based on the curvature tensor is obtained.

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1 Introduction

The existence of harmonic vector fields on Riemannian manifolds are directly related to the sign of Ricci tensor. Bochner and Yano have studied nonexistence of harmonic vector fields on compact Riemannian manifolds with positive Ricci curvature based on the Laplace-Beltrami operator [4]. They proved that if the Ricci curvature on $M$ is positive-definite, then a harmonic vector field other than zero does not exist on $M$. Yano proved that a vector field $X$ is harmonic, if and only if the Laplacian of its corresponding 1-form vanishes. Akbar-Zadeh defined the divergence of horizontal and vertical 1-forms on

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SM for a Finsler manifold [1]. Harmonic forms in Finsler geometry studied by several authors [3, 7, 8]. Recently the second author has studied a harmonic vector field on a Finsler manifold, cf. [5, 6], see Remark 5.1.

In the present work, the horizontal differential operator \( d_H \) and the horizontal co-differential operator \( \delta_H \), are defined and it shown they are adjoint operators, see Theorem 1.1. Next the horizontal Laplacian \( \Delta_H \) is defined on \( SM \) and it’s shown that if \( \omega \) is a p-form on \( SM \), then we have the familiar equivalence relation \( \Delta_H \omega = 0 \) if and only if \( d_H \omega = 0 \) and \( \delta_H \omega = 0 \).

Using these operators, the horizontal harmonic p-form on \( SM \) is studied and it is shown that a p-form \( \varphi \) is horizontally harmonic, if and only if its horizontal Laplacian vanishes.

Definition of harmonic p-forms on \( SM \) leads to the natural definition of Finslerian harmonic vector fields in the sense that a vector field is harmonic if and only if the horizontal Laplacian vanishes.

**Theorem 1.1.** Let \((M, F)\) be a closed Finsler manifold. If \( \omega \) is a p-form on \( SM \), then
\[
\Delta_H \omega = 0 \quad \text{if and only if} \quad d_H \omega = 0, \quad \text{and} \quad \delta_H \omega = 0. \tag{1.1}
\]

Finally a classification of harmonic vector fields is obtained in the following sense.

**Theorem 1.2.** Let \((M, F)\) be a closed Finsler manifold and \( X \) a harmonic vector field on \( M \).

1. If \( K = 0 \), then \( X \) is parallel.
2. If \( K > 0 \), then \( X \) vanishes,

where, \( K \) is a scalar function determined by the curvature tensor.

This theorem is an extension of a result obtained by Bochner and Yano.

## 2 Preliminaries on Cartan connection

Let \( M \) be a connected differentiable manifold of dimension \( n \), \( \pi : TM_0 \to M \) the bundle of non-zero tangent vector where \( TM_0 = TM \setminus 0 \) is the entire slit tangent bundle. A point
of $TM$ is denoted by $z = (x, y)$, where $x \in M$ and $y \in T_x M$. Let $(x^i)$ be a local chart with the domain $U \subseteq M$ and $(x^i, y^i)$ the induced local coordinates on $\pi^{-1}(U)$, where $y^i = y^i \frac{\partial}{\partial x^i} \in T_{x^i} M$, and $i$ running over the range $1, 2, ..., n$. A (globally defined) Finsler structure on $M$ is a function $F : TM \rightarrow [0, \infty)$ with the following properties; $F$ is $C^\infty$ on the entire slit tangent bundle $TM\setminus 0$; $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$; the $n \times n$ Hessian matrix $(g_{ij}) = \frac{1}{2}([F^2]_{y^i y^j})$ is positive-definite at every point of $TM_0$. The pair $(M, g)$ is called a Finsler manifold [2]. Denote by $TTM_0$ and $SM$ the tangent bundle of $TM_0$ and the sphere bundle respectively, where $SM := \bigcup_{x \in M} S_x M$ and $S_x M := \{y \in T_x M | F(y) = 1\}$. 

Let us consider the natural projection $p : SM \rightarrow M$ which pulls back the tangent bundle $TM$ to an $n$-dimensional vector bundle $p^*TM$ over the $(2n-1)$-dimensional base $SM$. Given the natural induced coordinates $(x_i, y_i)$ on $TM$, the coefficients of spray vector field are defined by

$$G^i := \frac{1}{4} g^{ih} \left( \frac{\partial^2 F^2}{\partial y^h \partial x^j} y^j - \frac{\partial F^2}{\partial x^h} \right).$$

The pair $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ forms a horizontal and vertical frame for $TTM$, where $\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}$, and $N^j_i := \frac{\partial G^j}{\partial y^i}$ are called the coefficients of nonlinear connection. The tangent bundle $TTM_0$ of $TM_0$ can be split to the direct sum of the horizontal part $HTM$ spanned by $\{\frac{\delta}{\delta x^i}\}$ and the vertical part $VTM$ spanned by $\{\frac{\partial}{\partial y^i}\}$. The dual basis of $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ is $\{dx^i, \delta y^i\}$, where

$$\delta y^i := dy^i + N^j_i dx^j,$$

and we have the following Whitney sum

$$TTM_0 = HTM \oplus VTM = \text{span}\{\frac{\delta}{\delta x^i}\} \oplus \text{span}\{\frac{\partial}{\partial y^i}\},$$

$$T^*TM_0 = H^*TM \oplus V^*TM = \text{span}\{dx^i\} \oplus \text{span}\{\delta y^i\}.$$ 

The Cartan connection is a natural extension of Riemannian connection which is metric compatible and semi-torsion free. For a global approach to the Cartan connection one can refer to [1]. According to the definition, the 1-forms of Cartan connection with respect to the dual basis $\{dx^i, \delta y^i\}$ are given by

$$\omega^i_j := \Gamma^i_{jk} dx^k + C^i_{jk} \delta y^k,$$ 

(2.4)
where, $\Gamma^i_{jk}$ and $C^i_{jk}$ are the horizontal and vertical coefficients of Cartan connection respectively defined by

$$\Gamma^i_{jk} := \frac{1}{2} g^{il} (\delta_j g_{lk} + \delta_k g_{jl} - \delta_l g_{jk}),$$

$$C^i_{jk} := \frac{1}{2} g^{il} \partial_l g_{jk},$$

and $\delta_i := \frac{\delta}{\delta x^i}, \dot{\delta}_i := \frac{\partial}{\partial y^i}$. In local coordinates we have

$$\nabla_k \dot{\delta}_j = \Gamma^i_{jk} \dot{\delta}_j, \quad \nabla_k \dot{\delta}_j = C^i_{jk} \dot{\delta}_j,$$

where in, $\nabla_k := \nabla_{\frac{\partial}{\partial x^k}}, \quad \dot{\nabla}_k := \nabla_{\frac{\partial}{\partial y^k}}$.

Let us consider the components of an arbitrary $(2,2)$-tensor field $T^i_{jk}$ on $TM$. The horizontal and vertical components of the Cartan connection of $T^i_{jk}$ in a local coordinates are given respectively by

$$\nabla^h T^i_{jk} = \delta_i T^i_{jk} - T^i_{jk} \Gamma^p_{ph} + T^i_{jk} \Gamma^p_{ih} + T^i_{jk} \Gamma^p_{sh} + T^i_{jk} \Gamma^p_{lq} C^q_{ph},$$

$$\dot{\nabla}^h T^i_{jk} = \dot{\delta}_i T^i_{jk} - T^i_{jk} \Gamma^p_{ph} + T^i_{jk} \Gamma^p_{ih} + T^i_{jk} \Gamma^p_{sh} + T^i_{jk} \Gamma^p_{lq} C^q_{ph}.$$

The curvature tensor in Cartan connection is given by the $hh$-curvature, $hv$-curvature and $vv$-curvature with the following components, cf. [1];

$$R^h \Gamma_i \Gamma_j = \delta_i \Gamma^h \Gamma_j - \delta_j \Gamma^h \Gamma_i + \Gamma^p \Gamma^h \Gamma_i \Gamma_j + R^i \Gamma^h_{lk},$$

$$P^h \Gamma_i \Gamma_j = \dot{\delta}_i \Gamma^h \Gamma_j + \Gamma^r \Gamma^h \Gamma_i \Gamma_j - \Gamma^h \Gamma^r \Gamma_i \Gamma_j + \dot{\delta}_j \Gamma^r \Gamma_i \Gamma_j,$$

$$Q^h \Gamma_i \Gamma_j = C^p_{rj} C^h_{ki} - C^h_{ri} C^p_{kj},$$

respectively where,

$$R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j} = y^m R^i_{mjk}. \quad (2.5)$$

Trace of the $hh$-curvature of Cartan connection is denoted by $R_{ij} := R_{iilj}$, which is not symmetric in general.

Let $(M,F)$ be a Finsler manifold, $\pi : TM_0 \rightarrow M$ the bundle of non-zero tangent vectors and $\pi^*TM$ the pullback bundle. The tangent space $T_x M, x \in M$ can be considered as a fiber of the pullback bundle $\pi^*TM$. Therefore a section $X$ on $\pi^*TM$ is denoted
by $X = X^i(x, y) \frac{\partial}{\partial x^i}$. The Ricci identity for Cartan connection is given by the following equation, cf. [1].

$$\nabla_k \nabla_h X^i - \nabla_h \nabla_k X^i = X^r R_{rkh}^i - \dot{\nabla}_r X^i R_{rkh}^i. \quad (2.6)$$

Now we are in a position to define some basic notions on harmonic forms on Finsler manifolds.

## 3 The p-forms and horizontal operators

Here and everywhere in this paper we assume the differential manifold $M$ is compact and without boundary or simply closed. Let $(M,F)$ be a closed Finsler manifold, $u : M \to SM$ a unitary vector field and $\omega = u_i dx^i$ the corresponding 1-form on $M$. A volume element on $SM$ is given by $\eta = (-1)^{\frac{n(n-1)}{2}} \omega \wedge (d\omega)^{n-1}$, cf. [1]. We denote the space of all horizontal $p$-forms on $SM$ by $\Lambda^H_p(SM)$ or simply $\Lambda^H_p$,

$$\Lambda^H_p(SM) := \{ \varphi_{i_1 i_2 \ldots i_p}(z) dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p} | \varphi_{i_1 i_2 \ldots i_p} \in C^\infty(SM) \}. \quad (3.1)$$

Let $\pi = a_i(z) dx^i$ be a horizontal 1-form on $SM$. The co-differential or divergence of $\pi$ with respect to the Cartan connection is defined by

$$\delta \pi = -\left( \nabla^j a_j - a_j \nabla_0 T^j \right), \quad (3.2)$$

where, $T_{kij} = C_{kij} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}$, are the components of Cartan tensors and $\nabla_0 = y^i \nabla_i$ cf. [1, p. 223]. Also we have

$$\int_{SM} \delta \pi \eta = - \int_{SM} (\nabla^j a_j - a_j \nabla_0 T^j) \eta = - \int_{SM} (\nabla_j a^j - a^j \nabla_0 T_j) \eta = 0, \quad (3.3)$$

where $a^j = g^{ij} a_j$, cf. [1, p. 67]. Let us denote the horizontal part of the differential $d\pi$ by

$$Hd\pi := \frac{1}{2} (\nabla_i a_j - \nabla_j a_i)(z) \ dx^i \wedge dx^j,$$

cf. [1, p. 224]. According to the above discussion we are in a position to define a horizontal differential operator in the following sense.
Definition 3.1. Let \((M,F)\) be a Finsler manifold and \(\varphi = \frac{1}{p!}\varphi_{i_1...i_p}(z)dx^{i_1}\wedge...\wedge dx^{i_p} \in \Lambda_p^H\) a horizontal p-form on \(SM\). A horizontal differential operator is a differential operator on \(SM\) given by
\[
d_H : \Lambda_p^H \rightarrow \Lambda_{p+1}^H,
\]
\[
\varphi \rightarrow d_H \varphi,
\]
where, for \(1 \leq i, i_k \leq n\) and \(1 \leq k \leq p\) we have
\[
d_H \varphi = \frac{1}{(p+1)!}(\nabla_i \varphi_{i_1...i_p} - \nabla_{i_1} \varphi_{i_2...i_p} - ... - \nabla_{i_p} \varphi_{i_1...i_{p-1}})dx^i \wedge dx^{i_1} \wedge ... \wedge dx^{i_p}.
\]

Let \(\varphi\) and \(\pi\) be the two arbitraries horizontal p-forms on \(SM\) with the components \(\varphi_{i_1...i_p}\) and \(\pi_{i_1...i_p}\), respectively. We consider an inner product \((.,.)\) on \(\Lambda_p^H\) as follows
\[
(\varphi, \pi) := \int_{SM} \frac{1}{p!} \varphi_{i_1...i_p} \pi_{i_1...i_p} \eta,
\]
where, \(\varphi_{i_1...i_p} = g^{i_1j_1}...g^{i_pj_p} \varphi_{j_1...j_p}\).

4 The horizontal Laplacian and harmonic p-forms

Using the above notions we give a definition for horizontal Laplacian. This definition of Laplacian is different by those given in [1, 9]. Let \((M,F)\) be a Finsler manifold and \(\psi\) a horizontal \(p+1\)-form on \(SM\), given by
\[
\psi = \frac{1}{(p+1)!} \psi_{i_1...i_p}dx^i \wedge dx^{i_1} \wedge ... \wedge dx^{i_p}.
\]

We define the horizontal divergence (co-differential) of \(\psi\) by
\[
(\delta_H \psi)_{j_1...j_p} := -g^{ij}(\nabla_i \psi_{j_{j_1}...j_p} - \psi_{jj_1...j_p} \nabla_0 T_i).
\]

Remark 4.1. If \(\varphi\) is a horizontal 1-form on \(SM\), then \(\delta_H\) reduces to \(\delta\) and we have
\[
\delta_H \varphi = \delta \varphi = -(\nabla^j \varphi_j - \varphi_j \nabla_0 T^j).
\]

Definition 4.2. Let \((M,F)\) be a Finsler manifold. A horizontal Laplacian on \((M,F)\) is defined by
\[
\Delta_H := d_H \delta_H + \delta_H d_H.
\]
where \( d_H \) and \( \delta_H \) are horizontal differential and horizontal co-differential operators on \( SM \), respectively.

Now, we are in a position to show the basic equivalence relation

\[
\Delta_H \omega = 0 \quad \text{if and only if} \quad d_H \omega = 0, \quad \text{and} \quad \delta_H \omega = 0,
\]

in the following theorem.

**Proof of Theorem 1.1.** It is clear that if \( \delta_H = 0 \) and \( d_H = 0 \), then we have \( \Delta_H \omega = 0 \).

Conversely, let \( \varphi = \frac{1}{p!} \varphi_{i_1 \ldots i_p}(z) dx^{i_1} \wedge \ldots \wedge dx^{i_p} \in \Lambda^H_p \) be a horizontal p-form on \( SM \) and \( \psi \) a horizontal \((p+1)\)-form on \( SM \), given by

\[
\psi = \frac{1}{(p+1)!} \psi_{i_1 \ldots i_p} dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}.
\]

Antisymmetric property of p-forms yield

\[
\nabla_{i_k} \varphi_{i_1 \ldots i_{k-1} i_{k+1} \ldots i_p} \psi^{i_1 \ldots i_p} = \nabla_i \varphi_{i_1 \ldots i_{k-1} i_{k+1} \ldots i_p} \psi^{i_1 \ldots i_i \ldots i_{k-1} i_{k+1} \ldots i_p} = (-1)^{k(k-1)} \nabla_i \varphi_{i_1 \ldots i_{k-1} i_{k+1} \ldots i_p} \psi^{i_1 \ldots i_i \ldots i_{k-1} i_{k+1} \ldots i_p} = -\nabla_i \varphi_{i_1 \ldots i_{k-1} i_{k+1} \ldots i_p} \psi^{i_1 \ldots i_i \ldots i_{k-1} i_{k+1} \ldots i_p}.
\]

Using the last equation we have

\[
(d_H \varphi, \psi) = \int_{SM} \frac{1}{(p+1)!} \left( \nabla_i \varphi_{i_1 \ldots i_p} - \ldots - \nabla_{i_p} \varphi_{i_1 \ldots i_{p-1} i} \right) \psi^{i_1 \ldots i_p} \eta.
\]

Letting \( a^i = \varphi_{i_1 \ldots i_p} \psi^{i_1 \ldots i_p} \), equation (3.3) yields

\[
\int_{SM} \nabla_i (\varphi_{i_1 \ldots i_p} \psi^{i_1 \ldots i_p}) \eta = \int_{SM} \varphi_{i_1 \ldots i_p} \psi^{i_1 \ldots i_p} \nabla_0 T_i \eta.
\]
Replacing (4.6) in (4.5) and using metric compatibility of Cartan connection yields

$$p!(d_H \phi, \psi) = \int_{SM} \nabla_i (\varphi_{i_1...i_p} \psi^{i_1...i_p}) \eta - \int_{SM} \varphi_{i_1...i_p} \nabla_i \psi^{i_1...i_p} \eta$$

$$= \int_{SM} \varphi_{i_1...i_p} \psi^{i_1...i_p} \nabla_0 T_i \eta - \int_{SM} \varphi_{i_1...i_p} \nabla_i \psi^{i_1...i_p} \eta$$

$$= -\int_{SM} (\nabla_i \psi^{i_1...i_p} - \psi^{i_1...i_p} \nabla_0 T_i) \varphi_{i_1...i_p} \eta$$

$$= \int_{SM} g^{ij} g^{i_1j_1}...g^{i_pj_p} (\nabla_i \psi^{j_1...j_p} - \psi^{j_1...j_p} \nabla_0 T_i) \varphi_{i_1...i_p} \eta. \quad (4.7)$$

Therefore (4.5) becomes

$$p!(d_H \phi, \psi) = \int_{SM} g^{i_1j_1}...g^{i_pj_p} (\delta_H \psi)_{j_1...j_p} \ varphi_{i_1...i_p} \ eta$$

$$= p!(\delta_H \psi, \phi),$$

then it yields

$$(d_H \phi, \psi) = (\phi, \delta_H \psi). \quad (4.8)$$

If $\phi = \omega$ is a p-form and $\psi = d_H \omega$, the equation (4.8) yields

$$(d_H \omega, d_H \omega) = (\omega, \delta_H d_H \omega). \quad (4.9)$$

If $\phi = \delta_H \omega$ and $\psi = \omega$, using (4.8) we have

$$(d_H \delta_H \omega, \omega) = (\delta_H \omega, \delta_H \omega). \quad (4.10)$$

By (4.9) and (4.10) and using (4.3) we have

$$(\Delta_H \omega, \omega) = (d_H \delta_H \omega, \omega) + (\delta_H d_H \omega, \omega)$$

$$= (\delta_H \omega, \delta_H \omega) + (d_H \omega, d_H \omega) \geq 0.$$
4.1 Horizontal Laplacian of p-forms

Let \( \varphi \) be a horizontal p-form on \( SM \), by definitions of horizontal differential and codifferential we can easily see that

\[
\delta_{H} d_{H} \varphi = -\frac{1}{p!}[(g^{rs}(\nabla_{r} \nabla_{s} \varphi_{i_{1} \ldots i_{p}} - \nabla_{s} \varphi_{i_{1} \ldots i_{p}} \nabla_{0} T_{r})
- g^{rs}(\nabla_{r} \varphi_{i_{1} \ldots i_{p}} - \nabla_{i_{1}} \varphi_{i_{2} \ldots i_{p}} \nabla_{0} T_{r})
- g^{rs}(\nabla_{r} \varphi_{i_{1} \ldots i_{p}} - \nabla_{i_{1}} \varphi_{i_{2} \ldots i_{p}} \nabla_{0} T_{r}) - \ldots
- g^{rs}(\nabla_{r} \varphi_{i_{1} \ldots i_{p}} - \nabla_{i_{1}} \varphi_{i_{2} \ldots i_{p}} \nabla_{0} T_{r})] dx^{i_{1}} \wedge \ldots \wedge dx^{i_{p}},
\]

and

\[
\delta_{H} \varphi = -\frac{1}{(p-1)!}g^{rs}(\nabla_{r} \varphi_{i_{1} \ldots i_{p}} - \varphi_{i_{1} \ldots i_{p}} \nabla_{0} T_{r}) dx^{i_{2}} \wedge \ldots \wedge dx^{i_{p}}.
\]

On the other hand by definition we have

\[
d_{H} \delta_{H} \varphi = -\frac{1}{p!}[(g^{rs}(\nabla_{r} \varphi_{i_{1} \ldots i_{p}} - \nabla_{i_{1}} \varphi_{i_{2} \ldots i_{p}} \nabla_{0} T_{r})
- g^{rs}(\nabla_{r} \varphi_{i_{1} \ldots i_{p}} - \nabla_{i_{1}} \varphi_{i_{2} \ldots i_{p}} \nabla_{0} T_{r})) - \ldots
- g^{rs}(\nabla_{r} \varphi_{i_{1} \ldots i_{p}} - \nabla_{i_{1}} \varphi_{i_{2} \ldots i_{p}} \nabla_{0} T_{r})] dx^{i_{1}} \wedge \ldots \wedge dx^{i_{p}}.
\]

The equations (4.11) and (4.12) yield

\[
(\delta_{H} d_{H} + d_{H} \delta_{H}) \varphi = -\frac{1}{p!}[(g^{rs}(\nabla_{r} \nabla_{s} \varphi_{i_{1} \ldots i_{p}} - \nabla_{s} \varphi_{i_{1} \ldots i_{p}} \nabla_{0} T_{r})
- g^{rs}(\nabla_{r} \varphi_{i_{1} \ldots i_{p}} - \nabla_{i_{1}} \varphi_{i_{2} \ldots i_{p}} \nabla_{0} T_{r})
- g^{rs}(\nabla_{r} \varphi_{i_{1} \ldots i_{p}} - \nabla_{i_{1}} \varphi_{i_{2} \ldots i_{p}} \nabla_{0} T_{r}) - \ldots
- g^{rs}(\nabla_{r} \varphi_{i_{1} \ldots i_{p}} - \nabla_{i_{1}} \varphi_{i_{2} \ldots i_{p}} \nabla_{0} T_{r})] dx^{i_{1}} \wedge \ldots \wedge dx^{i_{p}}.
\]

In the special case of an arbitrary horizontal 1-form \( \varphi = \varphi_{i}(z) dx^{i} \) on \( SM \), the above equation reduces to

\[
(\delta_{H} d_{H} + d_{H} \delta_{H}) \varphi = -[g^{rs}(\nabla_{r} \varphi_{i} - \nabla_{s} \varphi_{i} \nabla_{0} T_{r})
- g^{rs}(\nabla_{r} \varphi_{s} - \nabla_{s} \varphi_{r})
- g^{rs}(\varphi_{s} \nabla_{0} T_{r})] dx^{i}.
\]
This fact gives rise to a new definition of horizontal harmonic vector fields on Finsler manifolds.

**Definition 4.3.** A horizontal p-form $\varphi$ on $SM$ is called *horizontally harmonic* if we have

$$\Delta_h \varphi = 0.$$ 

Horizontally harmonic p-forms are denoted in the sequel by $h$-harmonic.

## 5 The harmonic vector fields on Finsler manifolds

Recently, one of the present authors has introduced in a joint work a definition for harmonic vector fields on Finsler manifolds using the Cartan and Berwald connections in the following sense.

**Remark 5.1.** Let $(M, F)$ be a closed Finsler manifold. A vector field $X = X^i \frac{\partial}{\partial x^i}$ on $M$ is called harmonic if its corresponding horizontal 1-form $X = X_i(z)dx^i$ on $SM$ satisfies $\Delta X = 0$ or $dX = 0$ and $\delta X = 0$, where

$$dX = \frac{1}{2}(D_iX_j - D_jX_i)dx^i \wedge dx^j - \frac{\partial X_i}{\partial y^j}dx^i \wedge dy^j,$$

$$\delta X = -(\nabla^jX_j - X_j\nabla^0T^j) = -g^{ij}D_iX_j,$$

and $\nabla$ and $D$ are the covariant derivatives of Cartan and Berwald connections, respectively, cf. [5, 6].

The above definition of harmonic vector fields and the corresponding harmonic 1-forms, has some inconveniences. First of all it could not be easily extend for definition of harmonic p-forms on Finsler manifolds. In fact, appearance of the mixed terms of differential and co-differential could not be easily determined. Moreover, one has to consider both the Berwald and the Cartan covariant derivatives on his computations. Finally, contrary to the definition of harmonic vector fields on Riemannian manifolds, we do not have the following useful bilateral relation in general;

$$\Delta \varphi = d\delta \varphi + \delta d\varphi = 0 \iff d\varphi = 0 \text{ and } \delta \varphi = 0.$$ (5.2)
The remedy lies in a slight modification of definition in the following sense. Let \( X = X^i(x) \frac{\partial}{\partial x^i} \) be a vector field on \( M \). One can associate to \( X \) a 1-form \( \tilde{X} \) on \( SM \) defined by
\[
\tilde{X} = X_i(z)dx^i + \frac{\dot{X}_i}{F},
\]
where \( \dot{X}_i = \frac{1}{F} (\nabla_0 X_i - y_i \nabla_0 (y^j X_j) F^{-2}) \), and \( z \in SM \) [1]. The horizontal part of associated 1-form on \( SM \) is denoted by \( X = X_i(z)dx^i \).

**Definition 5.2.** Let \((M, F)\) be a Finsler manifold. A vector field \( X = X^i(x) \frac{\partial}{\partial x^i} \) on \( M \) is said to be harmonic if its corresponding horizontal 1-form is h-harmonic on \( SM \).

**Theorem 5.3.** Let \((M, F)\) be a closed Finsler manifold. A vector field \( \varphi = \varphi^i \frac{\partial}{\partial x^i} \) on \( M \) is harmonic if and only if
\[
g^{rs}(\nabla_r \nabla_s \varphi_i - \nabla_s \varphi_i \nabla_0 T_r) = \varphi^t R_{ti} - \hat{\nabla}_t \varphi^r R^t_{ri} + \varphi^i \nabla_i \nabla_0 T_r. \tag{5.3}
\]

**Proof.** The Ricci identity (2.6) yields
\[
g^{rs}(\nabla_r \nabla_s \varphi_i - \nabla_s \varphi_i \nabla_0 T_r) = \nabla_r \nabla_i \varphi^r - \nabla_i \nabla_r \varphi^r
\]
\[
= \varphi^t R^r_{tri} - \hat{\nabla}_t \varphi^r R^t_{ri} \tag{5.4}
\]
\[
= \varphi^t R_{ti} - \hat{\nabla}_t \varphi^r R^t_{ri}.
\]
Substituting the last equation in (4.14) we get the result. \( \square \)

**Corollary 5.4.** Let \((M, F)\) be a closed Landsberg manifold. A vector field \( \varphi = \varphi^i \frac{\partial}{\partial x^i} \) on \( M \) is harmonic if and only if
\[
g^{rs} \nabla_r \nabla_s \varphi_i = \varphi^t R_{ti} - \hat{\nabla}_t \varphi^r R^t_{ri}.
\]

If \((M, F)\) is Riemannian, then the above equation reduces to the following well known form.
\[
g^{rs} \nabla_r \nabla_s \varphi_i = \varphi^t R_{ti}.
\]

Inspiring [5, 6] we define the scalar function \( K \) as
\[
K := X^k X^t R_{tk} - X^k \hat{\nabla}_r X^j R^r_{jk} - X^k \nabla_k X^j \nabla_0 T_j, \tag{5.5}
\]
and obtain a classification result given in Theorem 1.2.

**Proof of Theorem 1.2.** Let $X = X^i(x)\frac{\partial}{\partial x^i}$ be a vector field on $(M,F)$ and $Y$ and $Z$ two 1-forms on $SM$ defined at $z \in SM$ by $Y = (X^k\nabla_k X_i)(z)dx^i$ and $Z = (X_i\nabla_j X^j)(z)dx^i$, respectively. By means of (3.2) we have

$$\delta Y = -\nabla_j (X^k\nabla_k X^j) + X^k\nabla_k X^j \nabla_0 T_j$$

and similarly

$$\delta Z = -\nabla_k X^k \nabla_j X^j - X^k\nabla_k \nabla_j X^j + X^k\nabla_k X^j \nabla_0 T_k$$

$$= \nabla_k X^k \delta X - X^k \nabla_k \nabla_j X^j.$$  

The difference of $\delta Z$ and $\delta Y$ yields

$$\delta Z - \delta Y = \nabla_k X^k \delta X + X^k (\nabla_j \nabla_k X^j - \nabla_k \nabla_j X^j)$$

$$+ \nabla_j X^k \nabla_k X^j - X^k \nabla_k X^j \nabla_0 T_j.$$  

On the other hand we have

$$d_H X = \frac{1}{2}(\nabla_i X_j - \nabla_j X_i)dx^i \wedge dx^j,$$

from which

$$||d_H X||^2 = \frac{1}{4}(\nabla_i X_j - \nabla_j X_i)(\nabla^i X^j - \nabla^j X^i)$$

$$= \frac{1}{4}[(\nabla_i X_j)(\nabla^i X^j) - (\nabla_i X_j)(\nabla^j X^i) - (\nabla_j X_i)(\nabla^i X^j) + (\nabla_j X_i)(\nabla^j X^i)]$$

$$= \frac{1}{2}[||\nabla X||^2 - (\nabla_i X_j)(\nabla^i X^j)].$$

Therefore

$$\nabla_j X^k \nabla_k X^j = ||\nabla X||^2 - 2 ||d_H X||^2.$$  

Substituting (5.9) and (2.6) in (5.8) we obtain

$$\delta Z - \delta Y = \nabla_k X^k \delta X + X^k X^i R_{tk} - X^k \nabla_r X^j R_{jk}$$

$$+ ||\nabla X||^2 - 2 ||d_H X||^2 - X^k \nabla_k X^j \nabla_0 T_j.$$  

(5.10)
If $X$ is a harmonic vector field, then by definition of $K$ given by (5.5) the last equation becomes

$$\delta Z - \delta Y = \|\nabla X\|^2 + K.$$ 

By integration over $SM$ and using (3.3), we obtain

$$\int_{SM} (K + \|\nabla X\|^2)\eta = 0.$$ \hfill (5.11)

If $K = 0$, or

$$X^kX^tR_{tk} = X^k\hat{\nabla}_rX^jR^r_{jk} + X^k\nabla_kX^j\nabla_0T_j,$$

then (5.11) yields the first assertion. If $K > 0$, that is, if we have

$$X^kX^tR_{tk} > X^k\hat{\nabla}_rX^jR^r_{jk} + X^k\nabla_kX^j\nabla_0T_j,$$

then using the equation (5.11) we get the second assertion. \hfill $\square$

**Corollary 5.5.** Let $(M, F)$ be a closed Landsberg manifold and $X$ a harmonic vector field on $M$.

1. If $X^kX^tR_{tk} = X^k\hat{\nabla}_rX^jR^r_{jk}$, then $X$ is parallel.

2. If $X^kX^tR_{tk} > X^k\hat{\nabla}_rX^jR^r_{jk}$, then $X$ vanishes.

Using Theorem 1.2 we obtain the following theorem which is an extension of a well known Riemannian theorem.

**Corollary 5.6.** Let $(M, F)$ be a closed Riemannian manifold and $X$ a harmonic vector field on $M$.

1. If $X^kX^tR_{tk} = 0$, then $X$ is parallel.

2. If $X^kX^tR_{tk} > 0$, then $X$ vanishes.
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