Abstract: Reversible dynamics is well-known to obey variational principles based on the action being the time integral of a Lagrangian with time-reversal symmetry. The purpose of the present paper is to find dissipative Lagrangians giving variational principles in dissipative dynamics with broken time-reversal symmetry. Conceptually the present theory insists on new Least Dissipation & Work Principles (LDWP) based on variational integrals weighted in time, in order to unify variational principles of physics including irreversible processes on many-body systems. This is also closely related to the present author’s theory of entropy production in Kubo’s scheme of transport phenomena and all nonlinear responses beyond. Through these investigations, we can understand the meaning (as a variational integral) of the action of reversible dynamics.

Keywords: variational principles, irreversibility, dissipative Lagrangians, entropy production, Kubo scheme, least dissipation, work principles

1. Introduction

Variational principles play an important role to describe physical laws in a conceptually unified way. Namely, natural phenomena are considered to occur according to principles of least action. However, until now there has been found no ‘physical’ principle of least action to describe even simple dissipative dynamics described by the equation

\[ m \ddot{x}(t) + \zeta \dot{x}(t) = F(x(t)); \quad \dot{x}(t) = \frac{dx(t)}{dt}, \]

and

\[ F(x) = -\frac{dV(x)}{dx}, \tag{1} \]

where \( m \) denotes the mass of the relevant particle, \( \zeta \) its friction constant (which will be generalized later to \( \zeta = \zeta(\dot{x}(t), x(t), t) \) and to many-body systems) and \( F(x(t)) \), the external force at the position \( x(t) \). By the word ‘physical’ in the above statement, we mean that the action of the relevant dissipative system should be given by such a ‘dissipative’ Lagrangian as contains explicitly heat generation. As far as the present author knows, no body has found such a dissipative Lagrangian as derives Eq. [1] logically. For example, the Rayleigh dissipation function, \( R = \frac{1}{2} \zeta \dot{x}^2(t) \) may be relevant to the present problem. However, this is an artificial formulation which does not give the variational principle. This is one of our motivations. Other more serious ones are explained below. Although many papers have already been published concerning variational formulations in various kinds of dissipative systems including viscous fluid, there remains a basic problem to unify variational principles of reversible and irreversible dynamics and to clarify explicitly and conceptually the mathematical and physical relationship between these two variational principles. For this purpose, we first consider the simple model [1], and later we extend it to many-body systems. Clearly, it includes two terms of different time-reversal symmetries. Thus, we try to find such a new formulation as yields an explicit variational integral giving Eq. [1] by its minimization. In fact, we present some explicit analytic calculations of variational integrals (for example, in the case of viscous motion of a falling body under the gravitational force).

Reversible dynamics without the friction term, \( \zeta \dot{x}(t) \), in Eq. [1] is governed by the time-reversal symmetric equation,
\[ m \ddot{x}(t) = F(x(t)). \]  

As is well-known, this is derived by the variational principles based on the action
\[ I = \int_0^T \mathcal{L}_{\text{dyn}}(t) dt, \]
with the time-reversal symmetric dynamical Lagrangian
\[ \mathcal{L}_{\text{dyn}}(t) = T(\dot{x}(t)) - V(x(t)); \]
\[ T(\dot{x}(t)) = \frac{m}{2} \dot{x}^2(t), \quad V(x) = -\int_0^x F(x') dx'. \]

The present variational problem on the dissipative dynamics [1] has the following two key features:
(a) It contains two kinds of time symmetries, and
(b) our new dissipative Lagrangian is not an ordinary function of \( x(t) \) and \( \dot{x}(t) \) (as in Eq. [4]), but is expected to be a functional of \( x(t) \), because it contains heat generation, \( Q(t) \), which is expressed by the time integral (or path integral)
\[ Q(t) = \int_0^t \zeta \dot{x}^2(t') dt'. \]

When \( \zeta \) depends on time \( t \), as in Section 4.2, the variable \( \zeta \) in Eq. [5] should read as \( \zeta(t') \).

Concerning the above second feature (b), there arises a mathematical problem, that there exists no general variational theory on functional integrals. However, fortunately, this mathematical problem can be solved by finding some recursive multiple integral identities or by repeating partial integration, as will be shown later.

This kind of difficulty does not appear in the variational theories of stationary irreversible transport phenomena (or entropy production) \( \sigma_{\text{hp}}(\text{or } \sigma_S) = T \sigma_S = d(\text{Tr} \mathcal{H}_0 \rho_{\text{sym}}(t))/dt = \sigma_F F^2 > 0 \), at the temperature \( T \) of the medium. Here, \( \rho_{\text{sym}}(t) \) denotes the symmetric part of the density matrix \( \rho(t) \), and \( \mathcal{H}_0 \) denotes the internal energy of the relevant system. The positivity of the transport coefficient, \( \sigma_F \), for an external force \( F \), assures the positive entropy production (or irreversibility), which holds only in the thermodynamic limit of taking the infinite volume before the limit \( t \to \infty \). In the stationary state, the heat generation, \( \sigma_{\text{hp}} \), is equal to the work, \( W = \langle J \rangle F \), for Kubo’s current operator \( J \) defined by \( J = A \) with the operator \( A \) conjugate to the external force \( F \), and consequently, the Onsager-Prigogine minimum principles of \( \sigma_{\text{hp}} \) or \( \sigma_S \) hold at least, in the stationary linear regime.

In this situation, the main purpose of the present paper is to find a variational theory to describe non-stationary irreversible dynamics as well as the stationary one.

In our strategy to find a dissipative Lagrangian for Eq. [1], we try first to derive it intuitively by considering a ‘time-dependent mass model’ of the dissipative dynamics, and then using a ‘generalized Newton equation’. This will be useful for understanding conceptually our variational formulations. Secondly, we prove mathematically the fact that our dissipative equation [1] is derived rigorously from a new time-integral of our dissipative Lagrangian, as shown in the text and in Appendices A and B. Thirdly we discuss the physical meaning of our new variational theory on dissipative dynamics to arrive at the crucial conclusion that our dissipative dynamics [1] is governed by ‘Least Dissipation & Work Principles’ (LDWP). We then generalize our theory to more general cases of the time-dependent and nonlinear viscosity, \( \zeta(\dot{x}(t), x(t), t) \), and many-body dynamics with interactions. Finally we give in Appendix C, some simple applications of the present variational theory on dissipative dynamics, in order to show how effective it is.

2. Time-dependent mass dynamics and generalized Newton equation to give ad hoc dissipative Lagrangians

In the present paper, we first try to find variational principles deriving the above non-stationary dissipative system [1], using a time-dependent mass model which has been inspired by the Einstein theory of Brownian motion. That is, the noise is taken into account effectively on average by treating continuous collision with small particles in medium. This remark is crucial in discussing irreversibility, which comes from the infinity of the medium which assures the randomness of noise. This part plays the most important intuitive role in deriving the present dissipative Lagrangian weighted in time.

The energy conservation law for Eq. [1] is clearly given by
\[ T(\dot{x}(t)) + V(x(t)) + Q(t) = \text{constant} \]
with the heat generation, \( Q(t) \), defined by [5]. Following the classical method to construct the dynamical Lagrangian [4] from the energy conservation law, \( T(\dot{x}(t)) + V(x(t)) = \text{constant} \), in the case
of $\zeta = 0$, we might guess the dissipative Lagrangian, $L_{\text{diss}}(t)$, for Eq. [1] in the form

$$L_{\text{diss}}(t) = T(\dot{x}(t)) - (V(x(t)) + Q(t)). \quad [7]$$

Unfortunately, this conjecture has been found to be incorrect, because the derived Euler-Lagrange equation contains time-dependent mass. An extra term, $-\zeta(\tau - t)\ddot{x}(t)$, appears. Here we omit detailed arguments, which contain renormalization group (or recursive) procedures. Anyway, they have given the author several useful hints to proceed further correctly, as shown below.

Corresponding to the random noise in Einstein theory, we consider a dissipative model in which the relevant particle repeats to collide completely inelastically with other small particles in the medium and, consequently, the mass of the relevant particle increases like a snowball as $m(t) = me^{\zeta t}$ with the growing rate $\gamma = \zeta/m$. As the total momentum is conserved, even in such an inelastic collision, we may use the extended Newton equation,

$$\frac{d}{dt}(m(t)v(t)) = F(x(t); t) = e^{\zeta t}F(x(t)), \quad [8]$$

which gives immediately the following equation:

$$m(t)\ddot{v}(t) + m(t)\dot{v}(t) = e^{\zeta t}F(x(t)). \quad [9]$$

Here, we have assumed that the external force is proportional to $m(t)$. Equation [9] is found to be completely equivalent to the original dissipative system [1] by noting

$$\dot{m}(t) = \zeta e^{\zeta t}. \quad [10]$$

Multiplying both sides of Eq. [9] by $v(t)$ and integrating in time, we obtain the following generalized energy conservation law:

$$\frac{1}{2}m(t)v^2(t) + \frac{1}{2}\int_0^t e^{\gamma s}\zeta v^2(s)ds$$

$$- \int_0^t F(x(s); s)v(s)ds = \text{constant}, \quad [11]$$

where we have used the relation [10]. One of the remarkable features in the above formulation of dissipative dynamics is that the heat generation and work effects are expressed by ‘weighted time-integrals’ (in my introduced terminology). Furthermore, the heat-generation effect is partitioned into the first and second terms of the left-hand side of Eq. [11]. (This is the reason why the factor $1/2$ appears in the dissipative term of Eq. [11] or Eq. [16] in our formulation.) From this form of generalized energy conservation law, we may construct the dissipative Lagrangian for the time-dependent mass $m(t)$, $L_{\text{diss}}(t; m(t))$, in the form

$$L_{\text{diss}}(t; m(t)) = \frac{1}{2}m(t)v^2(t) - \frac{1}{2}\int_0^t e^{\gamma s}\zeta v^2(s)ds$$

$$+ \int_0^t F(x(s); s)v(s)ds. \quad [12]$$

Our final ‘physical’ dissipative Lagrangian for the original mass $m$ is expected to be given in the form

$$L_{\text{diss}}(t) = \frac{m}{m(t)}L_{\text{diss}}(t; m(t))$$

$$= T(\dot{x}(t)) - \zeta \int_0^t e^{-\gamma(t-s)}\dot{x}^2(s)ds$$

$$+ \int_0^t e^{-\gamma(t-s)}F(x(s))\dot{x}(s)ds, \quad [13]$$

by introducing the normalization factor, $m/m(t)$. The second term in Eq. [13] clearly expresses the heat generation (or entropy production multiplied by the temperature), weighted in time. This is the most important result in our theory.

Here, we define the following variational integral:

$$S_{\gamma, \tau} = \int_0^\tau L_{\text{diss}}(t)\gamma dt. \quad [14]$$

This corresponds essentially to the action in the ordinary reversible dynamics, but is different from it by physical dimensionality. The variational integral $S_{\gamma, \tau}$ has the dimensionality of energy or the Lagrangian, $L_{\text{diss}}(t)$. Clearly, $S_{\gamma, \tau}$ plays a role of action in the variational theory.

The above argument to get the dissipative Lagrangian [13] seems to be very intuitive. However, fortunately, it is rigorously proved in Section 4 and in Appendix A, that the variation of $S_{\gamma, \tau}$ in Eq. [14] with $L_{\text{diss}}(t)$ in Eq. [13] gives our desired dissipative equation [1]. In general, the positivity of $\delta^2 S_{\gamma, \tau}$ holds, if $\delta^2 L_{\text{diss}}(t)$ is positive (at least $d^2V(x) \leq 0$), as can be seen from Integral Identity I, which will be given in the succeeding section.

3. More transparent representations of the dissipative Lagrangian, $L_{\text{diss}}(t)$, and its time-integral, $S_{\gamma, \tau}$

It may be useful to give some alternative formulations of our new dissipative Lagrangian [13].

First we transform the weighted work, namely the third term of the last expression in Eq. [13] into the following form:
\[ W(x(t); \gamma) = \int_0^t e^{-\gamma(t-s)} F(x(s)) \dot{x}(s) ds \]
\[ = - \int_0^t e^{-\gamma(t-s)} \frac{dV(x(s))}{ds} ds \]
\[ = -V(x(t)) + \gamma \int_0^t e^{-\gamma(t-s)} V(x(s)) ds; \]
\[ V(x(0)) = 0. \quad [15] \]

Here we have used the relation [1] between the force, \( F(x) \), and the potential, \( V(x) \). The third equality of Eq. [15] is shown in Appendix B (see Eq. [B-2]).

Thus, our dissipative Lagrangian is also expressed in the form

\[ L_{\text{diss}}(t) = L_{\text{dyn}}(t) - \frac{\zeta}{2} \int_0^t e^{-\gamma(t-s)} \dot{x}^2(s) ds + \gamma \int_0^t e^{-\gamma(t-s)} V(x(s)) ds. \quad [16] \]

We should emphasize again that the second term of Eq. [16] expresses the effect of entropy production weighted in time. The corresponding variational integral, \( S_{\gamma,\tau} \), is expressed by

\[ S_{\gamma,\tau} = \int_0^\tau L_{\text{diss}}(t) \gamma dt = \frac{\zeta}{2} \int_0^\tau e^{\gamma(t-\tau)} \dot{x}^2(t) dt + \int_0^\tau e^{\gamma(t-\tau)} W(x(t)) \gamma dt, \quad [17] \]

where \( W(x(t)) = V(x(0)) - V(x(t)) = -V(x(t)) \) for \( V(x(0)) = 0 \). In deriving the above compact form [17], we have used the following multiple integral identity.

**Integral Identity I:** For any \( \gamma \) and \( \tau \), and for an arbitrary function \( f(t) \), we have

\[ \int_0^\tau \left( f(t) - \gamma \int_0^t e^{\gamma(s-t)} f(s) ds \right) dt = \int_0^\tau e^{\gamma(t-\tau)} f(t) dt. \quad [18] \]

The proof of this integral identity is given in Appendix B, together with some other multiple integral identities. They are expected to be useful in extending the present theory to more complicated dissipative systems, as will be reported in the future.

Physically speaking, the first term in Eq. [17] yields the ‘weighted heat generation’ and the second term in Eq. [17] the ‘weighted work’. The weight factor, \( \exp(\gamma(t-\tau)) = \exp(-\gamma(\tau-t)) \), indicates that physical quantities should be counted more as the time \( t \) approaches the end time \( \tau \) (observation time). This is a big contrast to the relaxation mechanism described by the convolution form with the factor \( \exp(-\gamma t) \) expressed by the opposite sign of time \( t \). Thus, our new variational principles insist that the real dissipative process occurs when minimum is the sum of heat generation and work weighted in time. This situation becomes much clearer in treating stationary states for the limit \( \gamma \tau \to \infty \), in which the weighted factor \( \exp(\gamma(t-\tau)) \) approaches the delta function, \( \delta(\gamma(t-\tau)) \), with respect to the integration \( \int_0^\tau \ldots \gamma dt \):

\[ \int_0^\tau e^{\gamma(t-\tau)} f(t) \gamma dt \to f(\tau) \quad [19] \]

for any function \( f(t) \). Thus, in the limit \( \tau \to \infty \), we obtain

\[ S_{\gamma,\tau} \to \frac{m}{2} \ddot{x}^2(\tau) + W(x(\tau)). \quad [20] \]

Near the real stationary state, the first term in Eq. [20] almost does not change, but the second term, \( W(x(\tau)) \), in Eq. [20] changes into heat. In fact, when the force, \( F(x) \), is a constant \( F \), the work, \( W(x(\tau)) \), becomes \( \int_0^\tau F(x) \dot{x} dt = F \dot{\bar{v}}_s \tau \) and balances the heat generation (due to the friction force, \( \zeta \dot{v}_s \)), \( Q(\tau) = \zeta \dot{v}_s^2 \tau \), for the stationary velocity \( \dot{v}_s \) defined in the balance equation \( \zeta \dot{v}_s = F \). This remark confirms again our ‘Least Dissipation & Work Principles’ in dissipative dynamics.

As is easily seen in exactly soluble linear systems (say, for a constant field, \( F \), such as the gravitational force, \( F = mg \), for a falling body discussed in Appendix C), we find that the ‘dissipative effect’ term of \( S_{\gamma,\tau} \), namely the first term of the last expression in Eq. [17] plays an important role in the variational treatment, as has been discussed already in detail by the present author. Briefly speaking, the mechanism of the above process, the transient kinetic energy changes into heat in some intermediate time regions.\(^2\)

4. Generalization of Euler variational method to functionals and direct proof of Eq. [16]

4.1. Generalized Euler equations in the variational method of functionals. In the present section, we formulate a new variational scheme of the action, \( I \), defined by the integral of a functional of the form

\[ I = \int_0^\tau F[x(t), \dot{t}] dt, \quad [21] \]

where

\[ F[x(t), \dot{t}] = \int_0^t f(x(s), \dot{x}(s), s; x(t), \dot{x}(t), t) ds. \quad [22] \]
The variation, $\delta I$, is given in the form

$$\delta I = \int_0^\tau \delta F[x(t), t]dt; \quad \delta F = (\delta F)_1 + (\delta F)_2, \tag{23}$$

where

$$(\delta F)_1 = \int_0^t (f_{\dot{x}}(s)\delta \dot{x}(s) + f_x(s)\delta x(s))ds \tag{24}$$

and

$$(\delta F)_2 = \left( \int_0^t f_{\dot{x}}(s)ds \right) \delta \dot{x}(t) + \left( \int_0^t f_x(s)ds \right) \delta x(t). \tag{25}$$

The second variational term, $(\delta F)_2$, in Eq. [23] can be reduced to the ordinary Euler variation scheme. Therefore, we have only to study the first variational term, $(\delta F)_1$, which is expressed by

$$(\delta F)_1 = \int_0^\tau g(s, t)\delta \dot{x}(s)ds + \int_0^\tau h(s, t)\delta x(s)ds, \tag{26}$$

where

$$
\begin{align*}
  g(s, t) &= f_{\dot{x}}(x(s), \dot{x}(s), s; x(t), \dot{x}(t), t), \\
  h(s, t) &= f_x(x(s), \dot{x}(s), s; x(t), \dot{x}(t), t).
\end{align*} \tag{27}
$$

Our strategy is to transform the variation, $\delta I$, defined by Eq. [23], into the Euler form,

$$\delta I = \int_0^\tau \tilde{F}[x(t), t]\delta x(t)dt, \tag{28}$$

by repeating partial integration. If it is possible, we obtain the generalized Euler equation,

$$\tilde{F}[x(t), t] \equiv \tilde{F}(x(t), \dot{x}(t), \ddot{x}(t), t; \tau) = 0. \tag{29}$$

There appears, as shown explicitly later, the new feature that the above generalized Euler equation [29] contains the boundary parameter, $\tau$. As physically required, Eq. [29] should hold for any value of $\tau$. This yields rather strong conditions on $\tilde{F}[x(t), t]$, that is, Eq. [29] is decomposed into some simultaneous differential equations on the function $x(t)$.

It should be remarked that the above strategy does not always seem to be possible, because there appears a new type of double integral containing $\delta \dot{x}(s)$ or $\delta x(s)$ by partial integration. For example, we obtain

$$
\begin{align*}
  \int_0^\tau dt \int_0^t h(s, t)\delta x(s)ds &= \int_0^\tau (\tau h(t, \tau) - th(t, t))\delta x(t)dt \\
  &\quad - \int_0^\tau t \left( \int_0^t h_1(s, t)\delta x(s)ds \right)dt.
\end{align*} \tag{30}
$$

by partial integration. However, this difficulty can be resolved by introducing the separated product representation of the form

$$f(x(s), \dot{x}(s), s; x(t), \dot{x}(t), t)$$

$$= \sum_j k_j(x(t), \dot{x}(t), t)\ell_j(x(s), \dot{x}(s), s). \tag{31}$$

This decomposition is always possible by permitting an infinite series of sum.

Thus, we consider here a single separated product form, $f = k(t)\ell(x(s), \dot{x}(s), s)$, in Eq. [31], where $k(t) = k(x(t), \dot{x}(t), t)$. Then, the variational term, $(\delta F)_1$, is given by

$$(\delta I)_1 = \int_0^\tau (\delta F)_1 dt = \int_0^\tau dtk(t) \int_0^t g(s)\delta \dot{x}(s)ds$$

$$+ \int_0^\tau dtk(t) \int_0^t h(s)\delta x(s)ds, \tag{32}$$

where

$$g(s) = \frac{d}{d\dot{x}(s)} \ell(x(s), \dot{x}(s), s), \quad \text{and}$$

$$h(s) = \frac{d}{dx(s)} \ell(x(s), \dot{x}(s), s). \tag{33}$$

By partial integration, $(\delta I)_1$ is easily transformed into the Euler form

$$(\delta I)_1 = \int_0^\tau dt((K(\tau) - K(t))(h(t) - g'(t))$$

$$+ k(t)g(t)\delta \dot{x}(t), \quad \tag{34}$$

where

$$K(t) = \int_0^t k(s)ds. \tag{35}$$

If the action, $I$, contains the ordinary reversible term of $(\delta I)_0 = \int_0^\tau dtf_0(t)\delta x(t)$ (ex. $f_0(t) = -m\ddot{x}(t) + F(x(t))$), then we have the following generalized Euler equation

$$(K(\tau) - K(t))(h(t) - g'(t)) + k(t)g(t) + f_0(t) = 0 \tag{36}$$

from the condition that $(\delta I)_0 + (\delta I)_1 = 0$, by assuming that $k(t)$ does not contain the variables $\dot{x}(t)$ and $x(t)$, that is, by neglecting $(\delta F)_2$.

As mentioned before, Eq. [36] should hold for any value of $\tau$, and, consequently, we obtain the following simultaneous equations

$$h(t) = g'(t) \quad \text{and} \quad k(t)g(t) + f_0(t) = 0. \tag{37}$$

Fortunately only two kinds of separated product are sufficient in our irreversible dynamics, as will be shown in the succeeding subsection.
4.2. Direct proof of Eq. [16] using generalized Euler method and extension to the case of time-dependent $\zeta(t)$. In this subsection, we first prove Eq. [16] using the above generalized variational theory on functionals.

The variation $\delta I$ for Eq. [16] is easily given by

$$\delta I = \int_0^\tau \delta L_{\text{diss}}(t)dt = \int_0^\tau \left( k(t) \int_0^t (g(s)\delta \dot{x}(s)) + h(s)\delta x(s) \right)ds + f_0(t)\delta x(t)dt,$$

where

$$k(t) = e^{-\gamma t}, \quad g(s) = -\zeta e^{\gamma s}\dot{x}(s),$$

$$h(s) = -\gamma e^{\gamma s}F(x(s)),$$

and

$$f_0(t) = -m\dot{x}(t) + F(x(t)).$$

Using the relations [39] and $\zeta = \gamma x$, we easily find that Eq. [1] is the solution of the generalized variational theory on the dissipative system with the nonlinear viscosity $\zeta(t)$.

The above proof is also extended to the case of a time-dependent viscosity, $\zeta(t)$, namely to the following equation:

$$m\ddot{x}(t) + \zeta(t)\dot{x}(t) = F(x(t)).$$

The dissipative Lagrangian to derive Eq. [40] by the above generalized variational method is easily found to be given by

$$L_{\text{diss}}(t) = L_{\text{dyn}}(t) - \frac{\gamma(t)}{2} \int_0^t G(s,t)\dot{x}^2(s)ds$$

$$+ \frac{\zeta(t)}{2} \int_0^t G(s,t)V(x(s))ds,$$

corresponding to Eq. [16], where $\gamma(t) = \zeta(t)/m$ and

$$G(s,t) = \exp\left( \int_s^t \gamma(u)du \right) = \exp\left( -\int_s^t \zeta(u)du \right).$$

By the way, it is instructive to remark that the above integral identity is easily extended as follows:

**Integral Identity I**: For any $\tau$ and for arbitrary functions $\gamma(t)$ and $f(t)$, we have

$$\int_0^\tau \left( f(t) - \gamma(t) \int_0^t G(s,t)f(s)ds \right)dt = \int_0^\tau G(t,\tau)f(t)dt.$$

This identity is useful in an alternative proof of the above variational theory on the dissipative dynamics [40] with a time-dependent viscosity.

5. Extensions to coupled dissipative dynamics on many-body systems

In the present section, we extend our dissipative variational theory to the following dissipative many-body system with interaction:

$$m_j\ddot{x}_j(t) + \zeta_j(t)\dot{x}_j(t) = F_j(\{x_k(t)\});$$

$$F_j(\{x_k(t)\}) = -\frac{\partial V(\{x_k\})}{\partial x_j},$$

for $j = 1, 2, \ldots, n$, under the condition that $\zeta_j(t)/m_j = \gamma_j(t)$ for all $j$.

Using the generalized variational theory on functionals [22], we find that the dissipative Lagrangian of the above many-body system is given by

$$L_{\text{diss}}(t) = L_{\text{dyn}}(t) - \frac{\gamma(t)}{2} \int_0^t G(s,t)L_{\text{diss}}(s)ds,$$

where $G(s,t)$ is defined by Eq. [42], and

$$L_{\text{diss}} = \frac{1}{2} \sum_{j=1}^n m_j\dot{x}_j^2(t) - V(x_1(t), \ldots, x_n(t)).$$

The proof of this proposition is also easily given by the integral identity [43].

6. Extension to coupled systems with nonlinear viscosity $\zeta_j(x_j(t), \dot{x}_j(t), t)$ for $j = 1, 2, \ldots, n$

The coupled system [44] in the preceding section is extended to the following more complicated dissipative system with the nonlinear viscosity coefficient $\zeta_j(x_j(t), \dot{x}_j(t), t)$,

$$m_j\ddot{x}_j(t) + \zeta_j(x_j(t), \dot{x}_j(t), t)\dot{x}_j(t) = F_j(\{x_k\}),$$

for $j = 1, 2, \ldots, n$ with the interaction, $F_j(\{x_k\})$, defined by the potential $V(\{x_k\})$.

Our strategy to construct a variational scheme is to reduce this coupled system to Eq. [44], by replacing $\zeta_j(x_j(t), \dot{x}_j(t), t)$ by

$$\zeta_j(x_j^*(t), \dot{x}_j^*(t), t) \equiv \zeta_j(t)$$

for fixed functions $\{x_j^*(t)\}$. Here, $\{x_j^*(t)\}$ should be required to be equal to the true solution of Eq. [47] after performing the variational procedure.

7. Summary and discussion

In the present paper, we have intuitively derived the variational principles of dissipative dynamics described by Eq. [1], and these principles have beenproved rigorously by transforming the variational double integral, $S_{\gamma,\tau}$, into a single-integral form, in which the ordinary Euler-Lagrange method can be utilized to derive Eq. [1]. A new variational theory of...
functionals has also been proposed to derive dissipative Lagrangians.

Physically, these variational principles express Least Dissipation & Work more weighted in more recent time in dissipative processes. This aspect of variation based on integrals \textit{weighted in time} is quite new and very crucial in transient (or relaxation) processes. It is consistent that the present variational principles are reduced to the Onsager-Prigogine principle of minimum instantaneous entropy production in the case of linear stationary processes.\(^{10-12}\)

The present author insists that dissipative processes are more fundamental than reversible dynamics from the viewpoint of the present variational theory, because even free motion (namely constant-velocity motion or inertial motion) can be interpreted to occur owing to the least dissipation principles in an infinitesimally small limit of the viscosity coefficient, \(\zeta\). (Note that the action of the kinetic energy, \(T(\dot{x}(t))\), is proportional to the heat generation, \(Q(t)\), in Eq. [5].) That is, the physical meaning (as a variational integral) of the action of reversible dynamics is now clearly understood by the present unified variational theory.

In particular, Newton’s first law on inertial motion can be replaced by one of our conclusions based on the variational theory of least dissipation principles on free particles discussed in Appendix C, which yields the equation of motion, \(\dot{x}(t) = 0\), for a free particle by minimizing the infinitesimal heat generation in Eq. [5] under given boundary conditions.

An explicit heat-generation form of the Lagrangian \(^{41}\) is derived in Appendix D.

We also remark that our dissipative Lagrangians will be useful in quantizing the dissipative system described by Eq. [1] using Feynman’s path integral formulation.\(^{13,14}\) In particular, Heisenberg’s uncertainly relation, \(\Delta x \Delta p \geq \frac{1}{2} \hbar\), will be weakened\(^ {13,14}\) by the dissipative effect as \(\Delta x \Delta p \geq \frac{1}{2} \hbar \exp (-\gamma (\tau - t))\). This lower bound goes to zero in the limit \(\gamma \tau \rightarrow \infty\). This is quite reasonable, because this limit corresponds to classical situations.

An extension of the present variational theory to living things will be published elsewhere in the future.

Acknowledgment

The author would like to thank Prof. T. Yamazaki, M.I.A., for useful discussion and encouragement. The author also thanks Dr. Y. Hashizume for discussion and digitization of the present manuscript.

**Appendix A. Mathematical proof of the variational theory based on Eq. [13]**

At present, there exists no general theory on variational functional integrals. Thus, we try to transform such a variational double integral into an ordinary variational single integral, and make use of the Euler-Lagrange method to derive Eq. [1] from such a transformed variational integral.

First note that, in connection with Eq. [13], we have

\[
S^{(T)}_{\gamma, \tau} = \int_0^\tau \left( T(\dot{x}(t)) - \frac{\zeta}{2} \int_0^t e^{-\gamma (\tau - s)} \dot{x}^2(s) \, ds \right) \gamma \, dt
\]

\[
= \int_0^\tau e^{-\gamma (\tau - t)} T(\dot{x}(t)) \gamma \, dt; \quad \zeta = m \gamma, \quad [A-1]
\]

using the double-integral identity \(^{18}\). Similarly, we have

\[
S^{(V)}_{\gamma, \tau} = \int_0^\tau \gamma \, dt \int_0^t e^{-\gamma (\tau - s)} F(x(s)) \dot{x}(s) \, ds
\]

\[
= - \int_0^\tau \gamma \, dt \int_0^t e^{-\gamma (\tau - s)} \frac{dV(x(s))}{ds} \, ds
\]

\[
= - \int_0^\tau e^{\gamma (\tau - t)} V(x(t)) \gamma \, dt, \quad [A-2]
\]

using the double-integral identity \(^{B-3}\). The variational integral, \(S \equiv S^{(T)}_{\gamma, \tau} + S^{(V)}_{\gamma, \tau}\), thus transformed into a simple but mathematical form yields easily Eq. [1] from the Euler-Lagrange equation,

\[
\frac{d}{dt} f_x - f_x = \ddot{x} f_{xx} + \dot{x} f_{x} + f_{xt} - f_x = 0, \quad [A-3]
\]

on the variational integral,

\[
S = \int_{t_0}^{t_1} f(x(t), \dot{x}(t), t) \, dt, \quad [A-4]
\]

with the fixed boundary condition on \(x(t)\) at time \(t = t_0\) and time \(t = t_1\), \(i.e., \), the infinitesimal variation, \(\delta x(t),\) satisfies the conditions that \(\delta x(t_0) = \delta x(t_1) = 0\).

In this context, one might consider that the function \(L^{(\text{math})}(t)\) defined by

\[
L^{(\text{math})}(t) = e^{-\gamma (\tau - t)} L_{\text{dyn}}(t) \quad [A-5]
\]

may play the role of a dissipative Lagrangian to derive Eq. [1]. However, the present author believes that this is physically inappropriate, since it does not explicitly contain the heat generation \(Q(t)\) defined by Eq. [5]. Of course, it is very interesting...
to note that the relation
\[ \int_0^\tau L_{\text{diss}}(t) dt = \int_0^\tau L^{(\text{math})}(t) dt \] [A-6]
holds for any value of \( \tau \), though \( L_{\text{diss}}(t) \) and \( L^{(\text{math})}(t) \) are quite different formally. (It should be noted that the latter does not contain heat generation explicitly. Such variational treatments without explicit heat generation or entropy production in formulations of dissipative Lagrangians have often been reported, particularly in the field of fluid dynamics.)

**Appendix B. Multiple integral identities**

First we prove here **Integral Identity I** (Eq. [18]). We define the difference of both sides of Eq. [18] by \( h(\tau) \). Then, it is easy to confirm

\[ \frac{dh(\tau)}{d\tau} = 0 \quad \text{and} \quad h(0) = 0. \] [B-1]

This yields immediately the result \( h(\tau) \equiv 0 \), and consequently leads to Eq. [18].

Similarly we have the following integral identity.

**Integral Identity II:** For any \( t \) and any function \( \gamma(t) \), and for an arbitrary function \( V(x(t)) \) satisfying \( V(x(0)) = 0 \), we have

\[ \int_0^t G(s, t) \frac{dV(x(s))}{ds} ds = V(x(t)) - \gamma(t) \int_0^t G(s, t)V(x(s)) ds. \] [B-2]

This is easily derived by the partial integration method. This integral identity was used in finding the final expression of Eq. [41].

From Integral Identities I and II, we obtain the following convenient identity:

**Integral Identity III:** For any \( \gamma \) and \( \tau \), and for an arbitrary function \( V(x(t)) \) satisfying \( V(x(0)) = 0 \), we have

\[ \int_0^\tau e^{-\tau t} \left( \int_0^t e^{\gamma s} \frac{dV(x(s))}{ds} ds \right) dt = \int_0^\tau e^{\gamma(t-\tau)} V(x(t)) dt. \] [B-3]

We can also derive the following double-integral identity, using the above identity [B-3].

**Integral Identity IV:** For any \( \gamma \) and \( \tau \), and for an arbitrary function \( f(t) \), we have

\[ \int_0^\tau e^{-\tau t} \left( \int_0^t e^{\gamma s} f(s) ds \right) dt = \int_0^\tau e^{\gamma(t-\tau)} \left( \int_0^t f(s) ds \right) dt. \] [B-4]

A direct proof of this identity is given similarly to that of Integral Identity I. We define the difference between both sides of Eq. [B-4] by \( h(\tau) \). Then, we easily find that

\[ \frac{d}{d\tau} \left( e^{\gamma \tau} \frac{d}{d\tau} h(\tau) \right) \equiv 0; \quad h'(0) = h(0) = 0. \] [B-5]

This immediately yields the results \( h'(\tau) \equiv 0 \), and consequently leads to Eq. [B-4].

By the way, the above result, \( h'(\tau) \equiv 0 \), together with Integral Identity IV yields the following interesting double-integral identity.

**Integral Identity V:** For any \( \gamma \) and \( \tau \), and for an arbitrary function \( f(t) \), we have

\[ \int_0^\tau e^{\gamma \tau} \left( \int_0^t f(s) ds \right) dt = \int_0^\tau \left( e^{\gamma \tau} - e^{\gamma s} / \gamma \right) f(t) dt. \] [B-6]

Once the above identity has been found, it is much easier to prove it directly by the differential method, as usual.

Furthermore, by taking the limit \( \gamma \to 0 \) in Eq. [B-6], we find the following well-known integral identity:

**Integral Identity VI:** For any \( \tau \) and for an arbitrary function \( f(t) \), we have

\[ \int_0^\tau \left( \int_0^t f(s) ds \right) dt = \int_0^\tau (\tau - t) f(t) dt. \] [B-7]

This is the simplest example of the following multiple integral transformation identity:

**Integral Identity VII:** For any \( \tau \) and positive integer \( n \), and for an arbitrary function \( f(t) \), we have

\[ \int_0^\tau dt \int_0^t dt_1 \hdots \int_0^{t_{n-1}} f(t_n) dt_n = \frac{1}{n!} \int_0^\tau (\tau - t)^n f(t) dt. \] [B-8]

This is easily proved by mathematical induction. This integral identity has been used effectively in trying to find the dissipative Lagrangian [16], by using the renormalization procedures with respect to the dissipative parameter \( \gamma \), as was already mentioned briefly in Section 2. It will be instructive to remark here that the weight factor, \( \exp(-\gamma(t-s)) \), in Eq. [15] or Eq. [16] comes mathematically from the summation of an infinite series of our perturbational expansion,

\[ \sum_{n=0}^\infty \frac{(-\gamma)^n}{n!} \int_0^\tau dt \int_0^t dt_1 \hdots \int_0^{t_{n-1}} f(t_n) dt_n, \] [B-9]

for \( f(t) = \frac{d^2}{dt^2} (t) \) or \( f(t) = V(x(t)) \), etc.
By using the integral identity [B-8], we find that each term of the expansion series [B-9] diverges for \( \tau \to \infty \). Consequently, its summation up to infinite order using the integral identity [B-8], as mentioned in the text, is inevitable in order to describe stationary states.

The above integral identities are so powerful that they may yield such a new viewpoint of the dissipative variational theory, as shown in the text.

**Appendix C. Some explicit analytic calculations of \( S_{\gamma,\tau} \) to show how physically the present theory works**

The present explicit calculations of the variational integral, \( S_{\gamma,\tau} \), for some simple soluble examples give us clear physical pictures showing how effectively it works.

**Viscous motion without force.** First we discuss the following simplest case of relaxation only:

\[
m \ddot{x}(t) + \zeta \dot{x}(t) = 0. \tag{C-1}
\]

The physical solution of Eq. [C-1] is given easily by

\[
\dot{x}_p(t) = \dot{x}_p(0)e^{-\gamma t}; \quad \gamma = \frac{\zeta}{m}. \tag{C-2}
\]

Thus, we obtain

\[
\begin{align*}
x_p(t) &= x_p(\infty)(1 - e^{-\gamma t}); \\
x_p(0) &= 0 \text{ and } \dot{x}_p(0) = \gamma x_p(\infty). \tag{C-3}
\end{align*}
\]

Now, we study here explicitly how \( S_{\gamma,\tau} \) changes in a single parameter space characterizing the variational function \( x(t) \). For the purpose of including the true solution [C-3], we assume that

\[
x(t) = b(1 - e^{-at}), \tag{C-4}
\]

which includes the physical solution [C-3], under the boundary conditions that

\[
x(0) = 0 \text{ and } x(\tau) = x_p(\tau) = x_p(\infty)(1 - e^{-\gamma \tau}). \tag{C-5}
\]

Thus, the parameter \( b \) in Eq. [C-4] is given by

\[
b = b(a) = x_p(\infty) \times \frac{1 - e^{-\gamma \tau}}{1 - e^{-at}}. \tag{C-6}
\]

Consequently our desired variational integral, \( S_{\gamma,\tau} \), is expressed by the single variational parameter, \( a \), as

\[
S_{\gamma,\tau}(a) = \int_0^\tau L_{\text{dis}}(t)\gamma dt = \left( \frac{mx^2}{2} \right) \times \left( \frac{a^2\gamma}{2a - \gamma} \right) \times \left( \frac{e^{(2a-\gamma)\tau} - 1}{(e^{at} - 1)^2} \right), \tag{C-7}
\]

using our general formula [17] for the dissipative variational theory. Clearly, the above integral, \( S_{\gamma,\tau}(a) \), becomes minimum at \( a = \gamma \), when the variational function, \( x(t) \), is equal to the real orbit, \( x_p(t) \).

The proof of this statement is rather lengthy, because of the existence of the third exponential factor in Eq. [C-7]. In the limit \( \tau \to \infty \), the third factor becomes independent of the variable \( a \), and consequently \( S_{\gamma,\tau}(a) \) is proportional to the simple function \( a^2\gamma/(2a - \gamma) \), which is clearly minimum at \( a = \gamma \). (Interestingly, the third factor becomes maximum just at \( a = \gamma \), but this does not change the above variational conclusion, because the minimization effect of the second factor is stronger than the maximization effect of the third one. The positivity of \( \delta^2 S_{\gamma,\tau} \) is easily seen for Eq. [A-2], irrespectively of the choice of \( x(t) \) near \( x_p(t) \), since the sufficient condition \( d^2V(x) \leq 0 \) is satisfied for Eq. [A-2].)

Anyway, our variational theory shows clearly, in this simple example, that the real orbit of Eq. [C-1] appears so that the dissipation may become minimum, as it should be.

One might say that the dissipative system [1] obeys the maximum dissipation principles because of the negative sign of the second term in Eq. [16]. However, this is quite wrong. In fact, as is easily seen from the above explicit calculations, we find that the contribution of the second dissipative term in Eq. [16] to \( S_{\gamma,\tau} \), is expressed as

\[
S_{\gamma,\tau}^{(2)} = -S_{\gamma,\tau}^{(1)} + Q_{\gamma,\tau}(t); \quad S_{\gamma,\tau}^{(1)} > Q_{\gamma,\tau}(t) > 0, \tag{C-8}
\]

where

\[
S_{\gamma,\tau}^{(1)} \equiv \int_0^\tau T(\dot{x}(t))\gamma dt. \tag{C-9}
\]

Thus, we obtain

\[
S_{\gamma,\tau} = S_{\gamma,\tau}^{(1)} + S_{\gamma,\tau}^{(2)} = Q_{\gamma,\tau}(t) > 0. \tag{C-10}
\]

This relation is nothing but a physical consequence of Integral Identity I. The dissipative effect is expressed by the positive integral \( S_{\gamma,\tau} \).

One reason why the present author has explained explicitly the above simple dissipative system is that even the ‘inertial motion’ of a free particle described by \( m\ddot{x}(t) = 0 \) is found to be given by the least dissipation principles in the infinitesimally small limit of the friction constant \( \zeta \to +0 \), according to the present unified theory of variational principles of work and entropy production minimum.

Thus, we conclude that ‘Least Dissipation & Work Principles’ are more universal than the least action principles.
Inertial motion and stationary motion in a simple dissipative system with a constant force. We now consider the following dissipative system with a constant force, \( F \) (ex. for a falling body under the gravitational force \( F = mg \)):
\[
m\ddot{x}(t) + \zeta \dot{x}(t) = F. \tag{C-11}
\]
In this system, there exists a stationary solution given by
\[
\dot{x}_{st} = \frac{F}{\zeta} \equiv v_0 \quad \text{or} \quad x_{st} = v_0 t. \tag{C-12}
\]
Any real solution of Eq. \( \text{[C-11]} \) approaches this stationary solution in the limit \( t \to \infty \). Thus, for convenience we restrict our variational function, \( x(t) \), in a parabolic function which includes the true stationary solution \( \text{[C-12]} \):
\[
x(t) = at^2 + bt; \quad v_0 = a r^2 + b \tau \quad \text{or} \quad b = v_0 - a \tau. \tag{C-13}
\]
After lengthy calculations, we find that
\[
S^{(T)}_{\gamma, \tau} = \frac{\zeta}{2} \int_0^\tau e^{-\gamma(t-t')} \dot{x}^2(t') dt'
= ka^2 + qa + \frac{\zeta v_0^2}{2\gamma} (1 - e^{-\gamma}), \tag{C-14}
\]
and
\[
S^{(V)}_{\gamma, \tau} = \frac{\zeta}{2} \int_0^\tau e^{\gamma(t-t')} Fx(t) \gamma dt'
= -qa + Fv_0 \left( \tau - \frac{1}{\gamma} (1 - e^{-\gamma}) \right), \tag{C-15}
\]
where
\[
k = \frac{\zeta}{2\gamma^3} [(8 + (\gamma \tau)^2)(1 - e^{-\gamma}) - 4\gamma \tau (1 + e^{-\gamma})] > 0 \tag{C-16}
\]
and
\[
q = \frac{\zeta v_0}{\gamma^2} (2\gamma \tau - (2 + \gamma \tau)(1 - e^{-\gamma})). \tag{C-17}
\]
Therefore, we obtain the following expression of the variational integral:
\[
S_{\gamma, \tau} = S^{(T)}_{\gamma, \tau} + S^{(V)}_{\gamma, \tau} = ka^2 + \left( \zeta v_0^2 \tau - \frac{mv_0^2}{2} (1 - e^{-\gamma}) \right); \quad k > 0. \tag{C-18}
\]
Thus, the variational integral \( S_{\gamma, \tau} \) becomes minimum at \( a = 0 \), when \( x(t) \) becomes the stationary solution \( x(t) = v_0 t \), as it should be.

Appendix D. Explicit heat generation form of the dissipative Lagrangian \[41\]

It will be instructive to rewrite the Lagrangian \[41\] into the following explicit form of heat generation:
\[
\mathcal{L}_{\text{diss}}(t) = \mathcal{L}_{\text{dynam}}(t) + \mathcal{L}_{\text{heat}}(t) + \mathcal{L}_{\text{work}}(t). \tag{D-1}
\]
Here, \( \mathcal{L}_{\text{dynam}}(t) \) is defined by Eq. \[4\], and \( \mathcal{L}_{\text{work}}(t) \) is defined by the third term of the right-hand side of \[41\]. The second term, \( \mathcal{L}_{\text{heat}}(t) \), is easily shown, by partial integration, to be expressed by an explicit form of heat generation:
\[
\mathcal{L}_{\text{heat}}(t) = -Q_{\text{whg}}(t) - \gamma(t) \int_0^t \frac{\dot{q}(s)}{\gamma^2(s)} Q_{\text{whg}}(s) ds, \tag{D-2}
\]
where \( \gamma(t) = \zeta(t)/m \) and \( Q_{\text{whg}}(t) \) denotes the weighted heat generation defined by
\[
Q_{\text{whg}}(t) = \frac{1}{2} \int_0^t \mathcal{G}(s,t) \mathcal{G}(s) \dot{x}^2(s) ds \tag{D-3}
\]
with \( \mathcal{G}(s,t) \) defined by Eq. \[42\]. Thus, our dissipative Lagrangian \[41\] is also expressed in terms of the weighted heat generation \[D-3\].

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(Received Oct. 19, 2018; accepted May 24, 2019)