Coarsening in the 2D incompressible Toner-Tu equation: Signatures of turbulence

Navdeep Rana and Prasad Perlekar
Tata Institute of Fundamental Research, Centre for Interdisciplinary Sciences, Hyderabad, India

We investigate coarsening dynamics in the two-dimensional, incompressible Toner-Tu equation. We show that coarsening proceeds via a vortex merger events, and the dynamics crucially depend on the Reynolds number (Re). For low Re, the coarsening process has similarities with Ginzburg-Landau dynamics. On the other hand, for high Reynolds number, coarsening shows signatures of turbulence. In particular, we show the presence of an enstrophy cascade from the inter-vortex separation scale to the dissipation scale.

Active matter theories have made remarkable progress in understanding the dynamics of suspension of active polar particles (SPP) such as fish school, locust swarm, and bird flock [1-3]. The particle based Vicsek model [4] and the hydrodynamic Toner-Tu (TT) equation [5] provide the simplest setting to investigate dynamics of SPP. Variants of the TT equation have been used to model bacterial turbulence [6] and pattern formation in active fluids [7-10]. An important prediction of these theories is the presence of an liquid-gas like transition from a disordered gas phase to an orientationally-ordered liquid phase [11, 12]. This picture is dramatically altered if the density fluctuations are suppressed by imposing an order parameter and argued that the accelerated dynamics are because of the advective nonlinearity in the TT equation. However, how nonlinearity alters energy transfer between different scales remains unanswered.

In the following, we quantity how vortex dynamics contribute to the acceleration. We initialize our simulations with a disordered configuration, randomly oriented velocity vectors drawn from a Gaussian distribution with zero mean and standard deviation $\sigma = U/3$, and monitor the coarsening dynamics. Our main findings are summarized below:

(i) Coarsening proceeds via vortex mergers.

(ii) For low-Re, advective non-linearities can be ignored and the dynamics resembles coarsening in the GL equation.

(iii) For high-Re, we find signatures of two-dimensional (2D) turbulence and the coarsening accelerates with increasing Re. We also provide evidence of a forward enstrophy cascade which is hallmark of 2D turbulence.

In the following, we quantity how vortex dynamics control coarsening. The pseudocolor plot of the vorticity

$$\partial_t u + \lambda u \cdot \nabla u = -\nabla P + \nu \nabla^2 u + f,$$

where $u(x, t)$ is the velocity field at position $x$ and time $t$, $\lambda$ is the advection coefficient, $\nu$ is the viscosity, and $f \equiv (\alpha - \beta |u|^2)u$ is the active driving term with coefficients $\alpha, \beta > 0$, and the pressure $P(x, t)$ enforces incompressibility criteria $\nabla \cdot u = 0$. We do not consider random driving term in (1) because we are interested in coarsening under a sudden quench to zero noise. For $\lambda = 0$ and in absence of pressure term, (1) reduces to the Ginzburg-Landau (GL) equation. On the other hand, (1) reduces to the Navier-Stokes (NS) equation on fixing $\alpha = 0, \beta = 0, \lambda = 1$. Since most studies of dry active matter are done on a substrate, we investigate coarsening in two space dimensions (2D).

We use a pseudo-spectral method [27-28] to perform direct numerical simulation (DNS) of (1) in a periodic square box of length $L$. The simulation domain is discretized with $N^2$ collocation points. We use a second order ET2D scheme [29] for time marching. Unless stated otherwise, we set $L = 2\pi$ and $N = 2048$.

By rescaling $x \to x/L$, $t \to \alpha t$, $u \to u/U$ and $P \to P/\alpha UL$, we find that the Reynolds number $Re = \lambda UL/\nu$ and the Cahn number $Cn = l_c/L$ completely characterize the flow. Here $U = \sqrt{\alpha/\beta}$ is the characteristic speed, and $l_c = \sqrt{\nu/\alpha}$ is the length scale above which fluctuations in the disordered state $u = 0$ are linearly unstable.

We initialize our simulations with a disordered configuration, randomly oriented velocity vectors drawn from a Gaussian distribution with zero mean and standard deviation $\sigma = U/3$, and monitor the coarsening dynamics. Our main findings are summarized below:

(i) Coarsening proceeds via vortex mergers.

(ii) For low-Re, advective non-linearities can be ignored and the dynamics resembles coarsening in the GL equation.

(iii) For high-Re, we find signatures of two-dimensional (2D) turbulence and the coarsening accelerates with increasing Re. We also provide evidence of a forward enstrophy cascade which is hallmark of 2D turbulence.

In the following, we quantity how vortex dynamics control coarsening. The pseudocolor plot of the vorticity
FIG. 1. (a,c) Pseudocolor plots of the vorticity field $\omega = \hat{z} \cdot \nabla \times \mathbf{u}$ superimposed with the velocity streamlines at different times for $\text{Re} = 2\pi \times 10^2$ (a) and $\text{Re} = 2\pi \times 10^4$ (c) in coarsening regime. (b,d) Contour plots of the vorticity field $\omega$ showing merger of two isolated counter-rotating vortices (vortex-saddle-vortex configuration) at $\text{Re} = 2\pi \times 10^2$ (b) and $\text{Re} = 2\pi \times 10^4$ (d).

field in Fig. 1(a) and (c) shows different stages of coarsening at low $\text{Re} = 2\pi \times 10^2$ and high $\text{Re} = 2\pi \times 10^4$. During coarsening vortices merge and inter-vortex spacing goes on increasing. For low $\text{Re} = 2\pi \times 10^2$ [see Fig. 1(a)] the dynamics in the coarsening regime resembles defect dynamics in the Ginzburg-Landau equation [18, 21, 30]. On the other hand, for high $\text{Re} = 2\pi \times 10^4$, vorticity snapshots resemble 2D turbulence. In particular, similar to vortex merger events in 2D [31, 32], it is easy to identify a pair of co-rotating vortices undergoing a merger and the surrounding filamentary structure.

To further investigate vortex merger, we perform DNS of isolated vortex-saddle-vortex configuration at various Reynolds number. For these simulations we use $N = 4096$ collocation points. Furthermore, to minimize effect of periodic boundaries, we set $\alpha = -10$ for $r > 0.9L/2$ and keep $\alpha = 1$ otherwise, where $r \equiv \sqrt{(x - L/2)^2 + (y - L/2)^2}$. This ensures that the velocity decays to zero for $r \geq 0.9L/2$. Note that a vortex in the 2D ITT equation is a point defect with unit topological charge and core radius $l_c$ (see Supplemental Material).

We observe that during the evolution of a vortex-saddle-vortex configuration [see Fig. 1(b) and (c)]: (i)
Similar to defect dynamics in the GL equation [30, 33–34], each vortex gets attracted to the saddle due to the opposite topological charge; (ii) The two vortices rotate around each other similar to convective merging in NS [31, 32]; and (iii) The flexure of the vortex trajectory depends on the Re (see Supplemental Material). Thus a vortex merger event in the two-dimensional ITT equation has ingredients both from the NS and GL equations. In the Supplemental Material, we provide a more detailed investigation of vortex merger with varying Re.

To further quantify coarsening dynamics, we conduct a series of high-resolution DNS ($N = 2048$) of the ITT equation by varying Re while keeping $C_n = 1/(100L)$ fixed. For ensemble averaging, we evolve 48 independent realizations at every Re. We monitor the evolution of the energy spectrum $E_k(t) \equiv \frac{1}{2} \sum_{k-1/2 \leq p < k+1/2} |u_p(t)|^2$, and the energy dissipation rate (or equivalently the excess free energy) $\epsilon(t) \equiv \langle 2\nu \sum_k k^2 E_k(t) \rangle$. Here $\dot{u}_k(t) \equiv \sum_x u(x,t) \exp(-ik\cdot x)$, $i = \sqrt{-1}$, and the angular brackets indicate ensemble average [35].

The time evolution of the energy dissipation rate $\epsilon(t)$ is shown in Fig. 2. For the initial disordered configuration, because the statistics of velocity separation is Gaussian, we approximate the fourth-order correlations in terms of product of second-order correlations to get the following equation for the early time evolution of the energy spectrum [36]

$$\partial_t E_k(t) \approx [2\alpha - 8\beta E(t)]E_k(t) - 2\nu k^2 E_k(t), \quad (2)$$

where $E(t) = \sum_k E_k(t)$. In Fig. 2 we show that the early-time evolution of energy dissipation rate $\epsilon(t)$ obtained from (2) is in good agreement with the DNS.

For late times, coarsening proceeds via vortex (defect) mergers. For GL equations in two dimensions, Refs. [33, 37] show that $\epsilon(t) \propto \frac{t}{\log(t)}$ with $\delta = 1$. We show that for low Re, where the effect of the advective nonlinearity can be ignored, GL scaling is recovered. However, on increasing the Re coarsening dynamics is accelerated with $\delta \sim 1 + cRe^{0.6\pm0.1}$ [see inset in Fig. 2(a)].

The plot in Fig. 2(b,c) shows the energy spectrum $E_k$ versus $k$ at different times for low Re $= 2\pi \times 10^2$ and high Re $= 2\pi \times 10^4$. In both cases, energy spectrum in the coarsening regime show a power law scaling $E_k(t) \sim k^{-3}$. We define the coarsening length $L(t) \equiv 2\pi \sum_k E_k(t)/\sum_k kE_k(t)$ and find that, consistent with the dynamic scaling hypothesis [18], the scaled spectrum collapses [see insets in Fig. 2(b,c)]. Physically $L(t)$ is an indicator of the average inter-defect separation.

The observed $k^{-3}$ scaling can appear because of: (a) the modulation of the velocity field around the topological defects (Porod’s tail) [30], and (b) the enstrophy cascade, similar to two-dimensional turbulence, due to the advective nonlinearity in (1).

To investigate the dominate balances between different scales, we use the scale-by-scale enstrophy budget equation

$$\partial_t \Omega_k(t) + T_k(t) = -2\nu k^2 \Omega_k(t) + F_k(t), \quad (3)$$

where $\Omega_k \equiv k^2 E_k$ is the enstrophy, $F_k(t) \equiv k^2(\dot{u}_k \cdot \hat{f}_k + \dot{u}_k \cdot \hat{f}_-k)$ is the net enstrophy injected because of active driving, $T_k \equiv dZ_k(t)/dt$ is the enstrophy transfer func-
Plot of different terms in the enstrophy budget for low $Re = 2\pi \times 10^3$. (Inset) Time evolution of $Z^m(t)$. (b) Enstrophy budget: Plot of the transfer function $F_k(t)$, the enstrophy injection due to the active driving $F_k(t)$ and the enstrophy dissipation $D_k(t)$ for $Re = 2\pi \times 10^3$ and at time $t = 7$ in the coarsening regime. (Inset) Plot of different terms in the enstrophy budget for low $Re = 2\pi \times 10^3$ and at time $t = 25$ in the coarsening regime.

The classical theory of 2D turbulence [38–43] assumes presence of an inertial range with constant enstrophy flux at scales smaller than the forcing scale and larger than the dissipation scale. Indeed for high $Re = 2\pi \times 10^4$, in Fig. 3(a) we confirm the presence of a positive enstrophy flux $Z_k$ between wave-number $k_L \equiv 1/L$ corresponding to the inter-vortex separation and the dissipation wave-number $k_d \equiv (8\nu^2/Z_m)^{-1/6}$ for $2 \leq t < 30$ in the coarsening regime. As the coarsening proceeds, the region of positive flux becomes broader and $k_L$ shifts to smaller wave-numbers but the maximum value of the flux $Z^m(t)$ decreases (Fig. 3a,inset). In Fig. 3(b) we plot different terms in the enstrophy budget equation (3). We find that the active driving primarily injects enstrophy ($F_k > 0$) around wave-number $k_L$ but, unlike classical turbulence, it is not zero in the region of constant enstrophy flux ($k_L < k < k_d$). Viscous dissipation is only active at small-scales $k \geq k_d$. At late times $t > 30$, the enstrophy flux is negligible (Fig. 3a,inset).

For low $Re$, the enstrophy transfer $T_k$ is negligible and the enstrophy dissipation $D_k(t)$ balances the injection because of the active driving $F_k(t)$ (see Fig. 3b,inset). Therefore, the $k^{-3}$ scaling in the energy spectrum (Fig. 2c) is due to the Porod’s tail.

The real-space measures of enstrophy flux in 2D turbulence is the following exact relation for the inertial range scaling of the third-order velocity structure function:

$$S_3(r,t) = \frac{1}{8} Z_{k \sim 1/r} r^3. \quad (4)$$

Here $S_3(r,t) \equiv \langle [\delta_\nu^3] \rangle$, $\delta_\nu \equiv [u(x+r,t) - u(x,t)] \cdot \hat{r}$, and the angular brackets indicate spatial and ensemble averaging [44, 45]. In the statistically steady turbulence, enstrophy flux $Z_k$ is constant in the inertial range and is equal to the enstrophy dissipation rate. During coarsening in ITT, we observe a nearly uniform flux $Z_k$ for $k_L \leq k \leq k_d$, albeit with decreasing magnitude [see Fig. 3(a)]. Therefore, for ITT we choose $Z_{k \sim 1/r} = Z^m(t)$ in (4). In Fig. 4 we show the compensated plot of $S_3(r,t)$ in the coarsening regime and find the inertial range scaling to be consistent with the exact result (4).

In conclusion, we have investigated coarsening dynamics in ITT equations. We find that at low Reynolds number dynamics is similar to coarsening in Ginzburg-Landau equation whereas, for high Reynolds number coarsening shows signatures of 2D turbulence. Specifically, for high Reynolds numbers, we show the presence of an enstrophy cascade and verify exact relation for the structure function.
The authors acknowledge support from intramural funds at TIFR Hyderabad from the Department of Atomic Energy (DAE), India.

**SUPPLEMENTAL MATERIAL FOR COARSENING IN THE 2D INCOMPRESSIBLE TONER-TU EQUATION: SIGNATURES OF TURBULENCE**

**DIMENSIONLESS ITT EQUATION**

Consider the incompressible Toner-Tu (ITT) equation

\[
\partial_t \mathbf{u} + \lambda \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u} + \left( \alpha - \beta |\mathbf{u}|^2 \right) \mathbf{u}.
\]

By rescaling the space \(x' \rightarrow x/L\), the time \( t' \rightarrow \alpha t \), the pressure \( P' \rightarrow P/\alpha L U \), and the velocity field \( \mathbf{u}' \rightarrow \mathbf{u}/U \), the ITT equation becomes

\[
\alpha U \partial_{t'} \mathbf{u}' + \frac{\lambda U^2}{L} \mathbf{u}' \cdot \nabla \mathbf{u}' = -\alpha U \nabla' P' + \frac{\nu U}{L^2} \nabla'^2 \mathbf{u}' + \left( \alpha - \beta U^2 |\mathbf{u}'|^2 \right) \mathbf{u}' U,
\]

where \( U^2 = \alpha/\beta \). Ignoring the primed index for convenience, we arrive at the dimensionless form of the ITT equation

\[
\partial_t \mathbf{u} + \text{Re} Cn \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + Cn \nabla^2 \mathbf{u} + \left(1 - |\mathbf{u}|^2 \right) \mathbf{u}.
\]

Here \( \text{Re} \equiv \lambda U L/\nu \) is the Reynolds number, \( Cn \equiv l_c^2/2L^2 \) is the Cahn number, and \( l_c = \sqrt{\nu/\alpha} \) is the length scale above which fluctuations in the homogeneous disordered state \( \mathbf{u} = 0 \) are linearly unstable.

**VORTEX SOLUTION**

Consider the radially symmetric velocity field of an isolated unbounded vortex \( \mathbf{u}(x, t) \equiv f(r) \hat{\theta} \), where \( \hat{\theta} \) is the unit vector along the angular direction, \( f(0) = 0 \), and \( f'(1) = 0 \). Substituting in the ITT equation, we get the following equations

\[
\left( f'' + \frac{f'}{r} - \frac{f}{r^2} \right) = \frac{1}{\text{Cn}^2} (f'^2 - 1) f, \quad (S5)
\]

\[
P = \text{Re} Cn \int_0^r \frac{f^2(r')}{r'} dr', \quad (S6)
\]

where the superscript \( t \) indicates derivative w.r.t. \( r \). Note that Eq. (S5) does not depend on \( \text{Re} \) and is identical to the equation of a defect in Ginzburg-Landau equation [30]. In [Fig. 5] we plot numerical solution of \( f(r) \) for different values of \( Cn \). For \( Cn < 1 \), a regular perturbation analysis reveals that \( f(r) \rightarrow An (1 - r^2/8Cn^2) \).

**VORTEX MERGER**

To investigate the merger of two co-rotating vortices, we perform DNS of an isolated vortex-saddle-vortex configuration at various Reynolds numbers. We use a square domain of area \( L^2 = 4\pi^2 \) and discretize it with \( N^2 = 4096^2 \) collocation points. Furthermore, to minimize effect of periodic boundaries, we set \( \alpha = -10 \) for \( r > 0.9L/2 \) and keep \( \alpha = 1 \) otherwise, where \( r \equiv \sqrt{(x - L/2)^2 + (y - L/2)^2} \). This ensures that the velocity decays to zero for \( r \geq 0.9L/2 \). The initial condition constitutes a saddle at the center the square domain, and two vortices placed at coordinates \([(L-1)/2, L/2]\) and \([(L+1)/2, L/2]\). As discussed in the main text, it is important to note that:

- Similar to the GL equation [30][33], vortices in ITT have a topological charge,
- Similar to the NS equation [10], the ITT equation has an advective nonlinearity and the presence of pressure leads to non-local interactions.

In [Fig. 6(a-c)], we plot vorticity contours during different stages of the vortex merger for different \( \text{Re} \). Since the saddle is at an equal distance away from the two-vortices, its position does not change during evolution. For low \( \text{Re} = 0 \), the vortex dynamics has similarities with the over-damped motion of defects with opposite topological charge in the Ginzburg-Landau equation. Vortices get attracted towards the saddle and move along a straight-line path. On increasing the \( \text{Re} \geq 2\pi \times 10^2 \), similar to Navier-Stokes, advective nonlinearity in the ITT becomes crucial. Not only do the vortices are attracted to the saddle, but they also go around each other.

In [Fig. 7(a)] we plot the inter-vortex separation for different \( \text{Re} \). For \( \text{Re} = 0 \) the inter-vortex separation \( d(t) \sim 1/\sqrt{t} \) decreases monotonically. On increasing the \( \text{Re} \) number, inertia becomes dominant and \( d(t) \) decreases...
FIG. 6. (a-e) Contour plots of the vorticity field $\omega$ at various times during merger process for different values of Reynolds number $Re = 0, 2\pi \times 10^2, 2\pi \times 10^3, \pi \times 10^4,$ and $2\pi \times 10^4$. 

in an oscillatory manner. The time for the merger decreases with increasing $Re$ (see Fig. 7(b)). 

[1] S. Ramaswamy, Annu. Rev. Condens. Matter Phys. 1, 323 (2010).
[2] M. C. Marchetti, J. F. Joanny, S. Ramaswamy, T. B. Liverpool, J. Prost, M. Rao, and R. A. Simha, Rev. Mod. Phys. 85, 1143 (2013).
[3] S. Ramaswamy, Nat. Rev. Phys. 1, 640 (2019).
[4] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, Phys. Rev. Lett. 75, 1226 (1995).
[5] J. Toner and Y. Tu, Phys. Rev. E 58, 4828 (1998).
[6] H. H. Wensink, J. Dunkel, S. Heidenreich, K. Drescher, R. E. Goldstein, H. Löwen, and J. M. Yeomans, PNAS (2012), 10.1073/pnas.1202032109.
[7] S. Sankararaman, G. I. Menon, and P. B. Sunil Kumar, Phys. Rev. E 70, 031905 (2004).
[8] K. Gowrishankar and M. Rao, Soft Matter 12, 2040 (2016).
[9] T. L. Goff, B. Liebchen, and D. Marenduzzo, Phys. Rev. Lett. 117, 238002 (2016).
[10] K. Husain and M. Rao, Phys. Rev. Lett. 118, 078104 (2017).
[11] M. Cates and J. Tailleur, Annu. Rev. Condens. Matter Phys. 6, 219 (2015).
[12] H. Chaté, Annu. Rev. Condens. Matter Phys. 11, 189 (2020) https://doi.org/10.1146/annurev-conmatphys-031119-050752.
[13] L. Chen, C. F. Lee, and J. Toner, New J. Phys. 17, 042002 (2015).
[14] L. Chen, C. F. Lee, and J. Toner, Nat. Comm. 7, 12215 (2016).
[15] A. Bricard, J.-B. Caussin, N. Desreumaux, O. Dauchot,
FIG. 7. (a) Plot of inter vortex distance $d(t)$ vs. time $t$ at various Reynolds number. Time axis is scaled by merger time $t_0$. (b) Plot of merger time $t_0$ versus Re. As Re increases merger time decreases.

[16] A. Maitra, P. Srivastava, M. C. Marchetti, S. Ramaswamy, and M. Lenz, arXiv:1901.01069 [cond-mat, physics:physics] (2019) arXiv: 1901.01069.

[17] T. Kibble, Physics Reports 67, 183 (1980).

[18] A. J. Bray, Adv. Phys. 43, 357 (1994).

[19] I. Chuang, R. Durrer, N. Turok, and B. Yurke, Science 251, 1336 (1991).

[20] K. Damle, S. Majumdar, and S. Sachdev, Phys. Rev. A 54, 5037 (1996).

[21] S. Puri, in Kinetics of Phase Transitions, Vol. 6, edited by S. Puri and V. Wadhawan (CRC Press, Boca Raton, US, 2009) p. 437.

[22] P. Perlekar, J. Fluid Mech. 873, 459 (2019).

[23] A. Tiribocchi, R. Wittkowski, D. Marenduzzo, and M. E. Cates, Phys. Rev. Lett. 115, 188302 (2015).

[24] S. Mishra, A. Baskaran, and M. C. Marchetti, Phys. Rev. E 81, 061916 (2010).

[25] N. Katyal, S. Dey, D. Das, and S. Puri, Eur. Phys. J. E 43, 1 (2020).

[26] P. Perlekar and R. Pandit, New J. Phys. 11, 073003 (2009).

[27] P. Perlekar, D. Mitra, and R. Pandit, Phys. Rev. E 82, 066313 (2010).

[28] S. M. Cox and P. C. Matthews, Journal of Computational Physics 176, 430 (2002).

[29] A. Onuki, Phase Transition Dynamics (Cambridge University Press, Cambridge, UK, 2002).

[30] T. Leweke, S. Le Dizès, and C. H. Williamson, Annu. Rev. Fluid. Mech. 48, 507 (2016)

[31] R. V. Swaminathan, S. Ravichandran, P. Perlekar, and R. Govindarajan, Phys. Rev. E 94, 013105 (2016).

[32] B. Yurke, A. N. Pargellis, T. Kovacs, and D. A. Huse, Phys. Rev. E 47, 1525 (1993).

[33] P. Chaikin and T. Lubensky, Principles of condensed matter physics (Cambridge, Cambridge University Press, UK, 1998).

[34] The energy spectrum $E_k$ and the structure factor $S_k$ are related to each other as $E_k = k^{d-1} S_k$. [35] V. Bratanov, F. Jenko, and E. Frey, PNAS 112, 15048 (2015).

[36] R. Kraichnan, Physics of Fluids 10, 1417 (1967).

[37] C. Leith, Physics of Fluids 11, 671 (1968).

[38] G. K. Batchelor, Phys. Fluids Suppl. II 12, 233 (1969).

[39] R. Pandit, P. Perlekar, and S. S. Ray, Pramana 73, 179 (2009).

[40] G. Boffetta and R. E. Ecke, J. Fluid Mech. 44, 427 (2012).

[41] R. Pandit, D. Banerjee, A. Bhatnagar, M. Brachet, A. Gupta, D. Mitra, N. Pal, P. Perlekar, S. Ray, V. Shukla, and D. Vincenzi, Phys. Fluids 29, 111112 (2017).

[42] R. T. Cerbus and P. Chakraborty, Phys. Fluids 29, 111110 (2017).

[43] E. Lindborg, J. Fluid Mech. 326, 343 (1996).

[44] C. Doering and J. Gibbon, Applied Analysis of the Navier-Stokes equations (Cambridge University Press, Cambridge, 1995).