I. INTRODUCTION

Description of a detailed structure of the set of states of quantum system is one of the most important problems in quantum theory. From the general point of view, the state of the system can be considered as the probability measure \( m \) on the lattice of projectors \( \mathcal{P}(\mathcal{H}) \) defined on the Hilbert space \( \mathcal{H} \) of the system. By the Gleason theorem [1], such a measure can be characterized by the trace class nonnegative operator \( \rho \) satisfying \( \text{tr} \rho = 1 \), and

\[
m(P) = \text{tr}(\rho P), \quad P \in \mathcal{P}(\mathcal{H})
\]

Thus, the set of states can be identified with convex set of all nonnegative operators with trace equal to one, with extremal points corresponding to pure vector states. The convex structure of the set of states can be exploited to get some knowledge about the quantum states (see e.g. [2]), but the structure of this set, even in the case of \( n \)-level quantum system, is completely known only for \( n = 2 \).

The simplest bipartite quantum system in which the phenomenon of entanglement occurs consists of two two-level systems (qubits). The structure of resulting four-level quantum system was the subject of many studies, specially in the context of characterizing of entanglement and entanglement measures (see e.g. [3, 4, 5]) and the geometry of quantum system was the subject of many studies, specially in the context of characterizing of entanglement and entanglement measures (see e.g. [3, 4, 5]).

The important problem of evolution of entanglement in realistic quantum systems interacting with their environments, is also modeled by the dynamics of two-qubit states. In that case the processes of degradation or creation of entanglement can be studied in details (see e.g. [6, 7, 8]). The important problem of evolution of entanglement given by concurrence [4, 5].

Much more complex and interesting are processes of disentanglement or creation of entanglement involving multilevel atoms. In such cases, quantum interference between different radiative transitions can influence the dynamics of the system. For a pair of three-level atoms the role of interference was studied in Ref. [14]. It is obvious that detailed knowledge of the set of states is crucial for systematic analysis of such phenomena. Even when we focus on two three-level systems (two qutrits), analytic description of entanglement of general states is not possible, so we can try to solve this problem for specific classes of states. In this context, the interesting study of two-qutrit states appeared recently in [15], where the analog of the set of Bell-diagonal states for two qubits - the simplex \( \mathcal{M} \), was investigated. Since \( \mathcal{M} \) lives in nine-dimensional real linear space (instead of eighty-dimensional space for all two-qutrit states), the investigation of its properties is simpler and one can use symmetries and equivalences inside \( \mathcal{M} \) to classify interesting quantum states and discuss the geometry of entanglement.

In the present paper we also study the properties of the simplex \( \mathcal{M} \), but we concentrate on the computation of the measure of entanglement, given by negativity, for some classes of states in \( \mathcal{M} \). Using the symmetry of \( \mathcal{M} \) we can classify mixed states inside \( \mathcal{M} \) with respect to their local equivalence. As follows from [15], all mixtures of two Bell states (for fixed mixing probabilities) are equivalent and there are two classes of mixtures of three Bell states and two of four states. For those states we compute its negativity analytically or numerically. To obtain some information about general mixed states from \( \mathcal{M} \), we consider two parameters: negativity and degree of mixture given by linear entropy. On the entropy - negativity plane, the simplex \( \mathcal{M} \) is represented by the region bounded by two curves. It turns out that on the upper curve lie all Werner states i.e. mixtures of maximally mixed state and any maximally entangled Bell state. It means that within \( \mathcal{M} \), Werner states have maximal allowed negativity for given linear entropy. By numerical generation of states lying above the curve of Werner states we show, that similarly as in the case of two qubits, Werner states do not maximize entanglement for given entropy.

II. A CLASS OF TWO-QUTRIT STATES

We start with the construction of simplex \( \mathcal{M} \). Let us fix the basis \( |0\rangle, |1\rangle, |2\rangle \) for one-qutrit space \( \mathbb{C}^3 \). In the space of two qutrits \( \mathbb{C}^3 \otimes \mathbb{C}^3 \), consider the maximally entangled pure state of the form

\[
\Psi_{0,0} = \frac{1}{\sqrt{3}} \sum_{k=0}^{2} |k\rangle \otimes |k\rangle \quad (\text{II.1})
\]

To construct the basis of \( \mathbb{C}^3 \otimes \mathbb{C}^3 \) consisting of maximally entangled pure “Bell-like” states, we can proceed as follows [15]. Let \( \mathcal{M} \) be the set of pairs of indices \( (m,n) \), where

\[
\begin{align*}
\text{m} + \text{n} &\leq 2, \\
\text{m} &\leq 2, \\
\text{n} &\leq 2
\end{align*}
\]

where\( m, n \in \mathbb{N} \). Within this set we can define new Bell-like states of the form

\[
\Psi_{m,n} = \frac{1}{\sqrt{3}} \sum_{k=0}^{2} |m,k\rangle \otimes |n,k\rangle
\]

where \( m, n \in \mathcal{M} \).
Then to each point pure states paper, contains all Bell-diagonal states i.e. the mixtures of which form a group of affine transformations considered as transformations of the set of pairs of indices, of states. As was shown in Ref. [15], such mappings can be local operations which do not change entanglement properties all "rectangles" \( \ell \) where

\[
\rho_{\ell} = \sum_{\alpha \in \ell} p_{\alpha} P_{\alpha}
\]

for any line \( \ell \) in \( \mathbb{M} \). Finally, there are two classes of states corresponding to four points:

\[
\rho_{Q} = \sum_{\alpha \in Q} p_{\alpha} P_{\alpha}
\]

for any rectangle \( Q \), and

\[
\rho_{\Gamma} = \sum_{\alpha \in \Gamma} p_{\alpha} P_{\alpha}
\]

for any set \( \Gamma = \ell \cup \{ \alpha \} \).

## III. NPPT STATES AND NEGATIVITY

In the case of compound quantum systems, the main problem is how to distinguish between separable and entangled states. Pure states of the system are entangled if the parts of the system do not have pure states of their own, but for mixed states one cannot deduce if they are entangled by considering partial states. For two qubits there is a simple necessary and sufficient criterion for entanglement: states which are non positive after partial transposition (NPPT states) are entangled [3, 16]. In the case of two qutrits, we only know that all NPPT states are entangled, but there are entangled states which are positive after this operation [17]. For NPPT states, the natural measure of entanglement is based on the trace norm of the partial transposition \( \rho^{PT} \) of the state [18]. One defines *negativity* of the state \( \rho \) as

\[
N(\rho) = \frac{||\rho^{PT}||_1 - 1}{2}
\]

\( N(\rho) \) is equal to the absolute value of the sum of negative eigenvalues of \( \rho^{PT} \) and is an entanglement monotone [18]. Notice that (III.1) is normalized such that \( N(\rho) = 1 \) if and only if \( \rho \) is maximally entangled.
A. Negativity for $\rho \in \mathbb{M}$

To get some insight into the properties of this measure of entanglement, we compute it analytically or numerically for some states from the simplex $\mathbb{M}$.

For any state (II.6) corresponding to the pair of points in $\mathbb{M}$, there is a simple formula for negativity

$$ N(\rho_{(\alpha,\beta)}) = \sqrt{1 - 3p_\alpha p_\beta}, \quad p_\alpha + p_\beta = 1 \quad \text{(III.2)} $$

Notice that the states (II.6) are always entangled and minimal value of negativity is attained for symmetric combination of pure states (FIG.1). For three points lying on some line $\ell$, one can prove by direct computation that

$$ N(\rho_\ell) = \frac{1}{2} \sum_{\alpha,\beta,\ell} (p_\alpha - p_\beta)^2 \quad \text{(III.3)} $$

In that case, symmetric combination of pure states i.e. the state (II.8) such that $p_\alpha = p_\beta = p_\gamma = 1/3$ produces separable state. For the remaining states belonging to other class (II.7), we can compute its negativity numerically and we get

$$ N(\rho_{(\alpha,\beta,\gamma)}) \geq N(\rho_\ell) $$

for every state $\rho_{(\alpha,\beta,\gamma)}$ given by convex combination (II.7), where $\alpha, \beta, \gamma$ do not belong to the line $\ell$ but the coefficients $p_\alpha, p_\beta, p_\gamma$ are the same as in the decomposition (II.8) of $\rho_\ell$. Moreover, $N(\rho_{(\alpha,\beta,\gamma)})$ is always greater then zero (FIG.2).

For nonequivalent states corresponding to four points in $\mathbb{M}$, we are only able to compute its negativity numerically. In particular, we study the states

$$ \rho_Q = p_\alpha P_{(00)} + p_\beta P_{(10)} + p_\gamma P_{(11)} + p_\delta P_{(01)} \quad \text{(III.4)} $$

and

$$ \rho_R = p_\alpha P_{(00)} + p_\beta P_{(10)} + p_\gamma P_{(20)} + p_\delta P_{(21)} \quad \text{(III.5)} $$

We check numerically that

$$ N(\rho_Q) \geq N(\rho_R) $$

for all such states. We also plot the surfaces of negativity for fixed $p_\gamma$ (see FIG.3).

![FIG. 1: Plot of $N(\rho_{(\alpha,\beta)})$](image1)

![FIG. 2: Plot of $N(\rho_\ell)$ (light gray surface) and $N(\rho_{(\alpha,\beta,\gamma)})$ (gray surface)](image2)

![FIG. 3: Negativity for $p_\gamma = 0.5$: gray surface - the states (III.3), light gray surface - the states (III.5)](image3)

B. Negativity versus entropy

General mixed states from the simplex $\mathbb{M}$ can be to some extent characterized by two parameters: degree of entanglement and mixture. To quantify the first we take negativity, and the second is given by normalized linear entropy

$$ S_L(\rho) = \frac{9}{8} \text{tr}(\rho - \rho^2) \quad \text{(III.6)} $$

which vanishes for all pure states and equals to 1 for maximally mixed state

$$ \rho_m = \frac{1}{9} \mathbb{I}_9 \quad \text{(III.7)} $$

One can find general bound on (III.6) for states which are positive after partial transposition (PPT states), by applying the result: for $N \times N$ bipartite quantum system if $\rho$ satisfies

$$ \text{tr} \rho^2 \leq \frac{1}{N^2 - 1} $$

then $\rho$ is PPT state [19]. Thus for two-qutrits we obtain the bound: if $S_L(\rho) \geq \frac{9}{8}$ then $\rho$ is PPT state and NPPT states can only have linear entropies less than this number.

It would be interesting to find the physically allowed degree of entanglement and mixture for NPPT states of two qutrits. In two-qubit case, the subset of the entropy - concurrence plane...
corresponding to possible physical states was characterized by Munro et al. \[20\]. In particular, the states lying on the boundary of this set were identified with maximally entangled mixed states. Similar characterization for two qutrits (in terms of entropy and negativity) is not easy and we restrict our investigations to the case of states $\rho \in \mathcal{M}$. Since these states are diagonal in the basis (II.3) \[
abla \left( \rho_{l} \right) = \frac{9}{8} \left( 1 - \sum_{\alpha \in \mathcal{M}} \rho_{l}^{\alpha} \right) \quad \text{(III.9)}
\]

Explicit relation between linear entropy and negativity can be established for the class of states (II.8). Combining (II.3) with (III.9), one finds that all states $\rho_{l}$ lie on the curve on the entropy - negativity plane, given by the equation
\[
n = \sqrt{1 - \frac{4}{3}s}, \quad s \in [0,3/4] \quad \text{(III.10)}
\]

For the Werner states
\[
\rho_{W} = (1-p)\rho_{\infty} + p\rho_{\alpha}, \quad \alpha \in \mathcal{M}, \ p \in [0,1] \quad \text{(III.11)}
\]
direct calculations show that
\[
N(\rho_{W}) = \frac{1}{3}(4p-1), \quad p > \frac{1}{4} \quad \text{(III.12)}
\]
and
\[
S_{L}(\rho_{W}) = 1 - p^{2} \quad \text{(III.13)}
\]
thus all states (III.11) lie on the curve
\[
n = \frac{1}{3} \left( 4\sqrt{1-s} - 1 \right), \quad s \in [0,15/16] \quad \text{(III.14)}
\]
It turns out that the curves (III.10) and (III.14) form the boundary of the region on the entropy - negativity plane that is occupied by the states $\rho \in \mathcal{M}$ (FIG.4). It means that in the class $\mathcal{M}$, Werner states have maximal allowed negativity for a given entropy, whereas the states $\rho_{l}$ have a minimal negativity under the same conditions. It is obvious that the whole region below the curve (III.10) corresponds to physical states of two qutrits. On the other hand, since the maximal allowed linear entropy of Werner states is less than the value which follows from the bound (III.8), there should exist physical states above the curve (III.14). In the other words, the Werner states are not maximally entangled states for given linear entropy. This conjecture was confirmed by numerical generation of two-qutrit states lying above the curve of Werner states (FIG.4). It is difficult to generate such states, and we were able to find them only for limited values of negativity. For this reason, we cannot predict the curve of maximally entangled mixed states.

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