INTEGRATION ON THE SURREALS: A CONJECTURE OF CONWAY, KRUSKAL AND NORTON.

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ABSTRACT. In 1976 Conway introduced the surreal number system \( \mathbb{N}_o \), containing the reals, the ordinals, and numbers such as \(-\omega, 1/\omega, \sqrt{\omega}, \ln \omega\). \( \mathbb{N}_o \) is a real closed ordered field, with much additional structure. Surreal theory is conveniently developed within the class theory NBG, a conservative extension of ZFC.

A longstanding aim has been to develop analysis on \( \mathbb{N}_o \) as a powerful extension of ordinary analysis on \( \mathbb{R} \). This entails finding a natural way of extending important functions \( f : \mathbb{R} \to \mathbb{R} \) to functions \( f^* : \mathbb{N}_o \to \mathbb{N}_o \), and naturally defining integration on the \( f^* \). The usual square root, \( \log : \mathbb{R} \to \mathbb{R} \), and \( \exp : \mathbb{R} \to \mathbb{R} \) were naturally extended to \( \mathbb{N}_o \) by Bach, Conway, Kruskal, and Norton, retaining their usual properties. Later Norton also proposed a treatment of integration, but Kruskal discovered flaws. The search for natural extensions from \( \mathbb{R} \) to \( \mathbb{N}_o \), and natural integration on \( \mathbb{N}_o \) continues. This paper addresses this and related unresolved issues with positive and negative results.

In the positive direction, we show that Écalle-Borel transseriable functions extend naturally to \( \mathbb{N}_o \), and an integral with good properties exists on them. Transseriable functions include semi-algebraic, semi-analytic, analytic, and meromorphic ones as well as solutions of systems of linear and nonlinear systems of ODEs with possible irregular singularities as in [12]. In particular, most classical special functions (such as Airy, Bessel, Ei, erf, Gamma, Painlevé and so on) extend naturally (and are integrable) from finite to infinite values of the variable.

In the negative direction, we show there is a fundamental obstruction to naturally extending many larger families of functions to \( \mathbb{N}_o \) and to defining integration on surreal functions. We show that there are no descriptions of operators which can be proved within NBG to have the basic properties of integration, even on highly restricted families of real-valued entire functions.

Keywords: surreal numbers, surreal integration, divergent asymptotic series, transseries.

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1. Introduction

In his seminal work On Numbers and Games [9], J. H. Conway introduced a real-closed field $\mathbb{No}$ of surreal numbers, containing the reals and the ordinals $0, ..., \omega, ...$, as well as a great many new numbers, including $-\omega, \omega/2, 1/\omega, \sqrt{\omega}, e^\omega, \log \omega$ and $\sin (1/\omega)$ to name only a few. Motivated in part by the hope of providing a new foundation for asymptotic analysis, there has been a longstanding program, initiated by Conway, Kruskal and Norton, to develop analysis on $\mathbb{No}$ as a powerful extension of ordinary analysis on the ordered field $\mathbb{R}$ of reals. This entails finding a reasonable way of extending important functions from $\mathbb{R}$ to $\mathbb{No}$, and to define integration on the extensions.

In classical analysis over $\mathbb{R}$, many transcendental integrals such as $\int_a^x t^{-1}e^t dt$ and special functions that arise as solutions of ODEs have divergent asymptotic series as $x \to \infty$, roughly of the form $\sum c_k x^{-k}$, where the $c_k$ grow factorially. In $\mathbb{No}$, on the other hand, the same sums converge in a natural sense that we call absolute convergence in the sense of Conway (see §3) for all $x \gg 1$. Part of the original motivation for developing surreal integration was the expectation of finding new and more powerful methods for solving such ODEs and summing such divergent series.

There was initial success with the first part of the program when the square root, log, and exp were extended to $\mathbb{No}$ in a property–preserving fashion [45] by...
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Bach, Norton, Conway, Kruskal and Gonshor ([10] pages 22, 38, [29] Ch. 10). Norton’s proposed “genetic” definition of integration [10] page 227 addressed the second part of the program. However, Kruskal subsequently showed the definition is flawed [10] page 228. Despite this disappointment, the search for a theory of surreal integration has continued [26], [36]. Indeed, in his recent survey [42] page 438, Siegel characterizes the question of the existence of a reasonable definition of surreal integration as “perhaps the most important open problem in the theory of surreal numbers”.

Despite being flawed, Norton’s integral is highly suggestive: indeed, for a fairly wide range of functions arising from applications, such as solving ODEs, we show that the use of inequalities coming from transseries and exponential asymptotics leads to a correct genetic integral. To prove the integral is well defined (in contradistinction to Norton’s [10] page 228) we use Écalle analyzability results, (results that are not required, however, to calculate the integral). Our constructions also provide a method for solving ODEs in No. However, for more general functions, a substantial obstruction arises which involves considerations from the foundations of mathematics. In particular, we will show that, in a sense made precise, there is no description which, provably in NBG, defines an integral (of Norton’s type or otherwise) from finite to the infinite domain, even on spaces of entire functions that rapidly decay towards +∞.

1.1. Definitions and notation. We use the space $T[R]$ of all functions $f$ whose domain $\text{dom}(f)$ is an interval in $R \cup \{-\infty, \infty\}$ and whose range $\text{ran}(f)$ is a subset of $R$. Empty and singleton intervals are likewise allowed. We define $\lambda f$ and $f + g$ as usual, where $\text{dom}(f + g) = \text{dom}(f) \cap \text{dom}(g)$. For such intervals $I$ in $R \cup \{-\infty, \infty\}$, $I^*$ is the corresponding interval in No with the same real inf and sup where appropriate and $\pm On$ in place of $\pm \infty$, respectively. $T[No]$ is the analogous space of all functions whose domain is an interval of No whose values lie in No. We say that $f \in T[R]$ is extended by $f^* \in T[No]$ if and only if for all $x \in \text{dom}(f)$, $f(x) = f^*(x)$, and $\text{dom}(f^*) = \text{dom}(f)^*$. If $F \in T[No]$ we denote by $F|_R$ its restriction to $\text{dom}(F) \cap R$.

In the positive results in our paper, the integral will be defined from a linear antiderivative. The integral is in a sense a generalization of the Hadamard partie finie integral from infinity. The negative results only need to hold on restricted spaces, and we choose subspaces of

$$E := \{ f : R \to R : \exists g : C \to C \text{ such that } g \text{ is entire and } g|R = f \}.$$ 

These functions have a common domain, and an integral from zero gives rise to a linear antidifferentiation operator.

Building on results of the first and third authors in [15], the purpose of this paper is to address these two issues:

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1In surreal theory, the field operations on No as well as the square root, log and exp functions are defined by induction on the complexity of the surreals. Conway has dubbed such definitions “genetic definitions” [10] pages 27, 225, 227. In the integration program originally envisioned by Conway, Kruskal and Norton, the functions as well as their integrals were intended to be genetically defined. At present, however, there is no adequate theory of genetic definitions in the literature, though [29] and [27] contain some useful preliminary remarks. Nevertheless, we will freely refer to various definitions that we define in terms of the complexity of the surreals as “genetic” (even when they are not inductively defined) since for the most part they conform to the way this notion is informally understood in the literature.
1. Which natural families of functions in $T[\mathbb{R}]$ can be naturally lifted to families of $f^* \in T[\mathbb{No}]$, where each $f^*$ extends $f$?

2. Is there a natural antidifferentiation operator on the families of $f^* \in T[\mathbb{No}]$, arising from 1?

As we show, the existence of a natural antidifferentiation leads to the existence of a natural integral. This integral follows Norton’s scheme.

Both of these questions are treated here in terms of operators on spaces of real functions.

**Definition 1.** An extension operator on $K \subseteq T[\mathbb{R}]$ is a function $E : K \rightarrow T[\mathbb{No}]$ such that for all $f, g \in K$:

i. $E(f)$ extends $f$;
ii. $E(\lambda f) = \lambda E(f), E(f + g) = E(f) + E(g)$;
iii. If $f$ is the real monomial $x^n$ on $\text{dom}(f)$, then $E(f)$ is the surreal monomial $x^n$ on $\text{dom}(f)^*$, for all $n \in \mathbb{N}$;
iv. If $f \in K$ is the real exponential on $\text{dom}(f)$, then $E(f)$ is the surreal exponential on $\text{dom}(f)^*$.

To formulate the appropriate notion of an antidifferentiation operator, we require a generalization of the idea of a derivative of a function at a point.

**Definition 2.** Mimicking the usual definition, we say that $f$ is differentiable at $a$ in an ordered field $V$ if there is an $f'(a) \in V$ such that $(\forall \varepsilon > 0 \in V)(\exists \delta > 0 \in V)$ such that $(\forall x \in V) (|x - a| < \delta \Rightarrow |(f(x) - f(a))/(x - a) - f'(a)| < \varepsilon)$. As usual, $f'(a)$ is said to be the derivative of $f$ at $a$.

**Definition 3.** An antidifferentiation operator on $K \subseteq T[\mathbb{R}]$ is a function $A : K \rightarrow T[\mathbb{No}]$ such that for all $f, g \in K$:

i. $A(f)'$ extends $f$;
ii. $A(\lambda f) = \lambda A(f), A(f + g) = A(f) + A(g)$;
iii. If $f$ is the real monomial $x^n$, then $A(f)$ is the surreal monomial $x^{n+1}/(n+1)$ on $\text{dom}(f)^*$;
iv. If $f \in K$ is the real exponential on $\text{dom}(f)$, then $A(f)$ is the surreal exponential on $\text{dom}(f)^*$.

For suitable integrals to exist, we need the “second half” of the fundamental theorem of calculus to hold. This is the motivation for the following strengthening of Definition 3.

**Definition 4.** A strong antidifferentiation operator on $K \subseteq T[\mathbb{R}]$ is an antidifferentiation operator $A$ such that if $F \in T[\mathbb{No}]$ and $(F|_{\mathbb{R}})' = f \in K$ exists, then there is a $C \in \mathbb{No}$ such that $A(f) = F + C$.

Our first group of results show that there are (unnatural) extension and antidifferentiation operators on $T[\mathbb{R}, \mathbb{R}] := \{f \in T[\mathbb{R}] : \text{dom}(f) = \mathbb{R}\}$ correctly acting on finitely many, or even all monomials. For finitely many monomials, the proof is constructive. For infinitely many, the proof uses the classical result that every vector space has a basis extending any given linearly independent set. Of course, such a proof does not lead to any reasonably defined examples of extension or antidifferentiation operators. In fact, our negative results, discussed below, establish that this drawback is unavoidable—even for much more restrictive $K$’s.
Our second group of results are negative, and establish, in several related senses, that there are no reasonable extension or antidifferentiation operators on the space of all real-valued entire functions, even if we restrict ourselves to those with slow growth. Specifically, we show that there are no descriptions, which, provably, uniquely define such extensions or antidifferentiations, even in the presence of the axiom of choice. Our negative results show that, in order to naturally extend families of real-valued entire functions past $\infty$ into the infinite surreals, critical information about the behavior of the functions at $\infty$ is required. For families of real-valued entire functions characterized by growth rates only, we show that polynomial growth forms a threshold for extendability (in a sense made precise). As a consequence of Liouville’s theorem, this entails that such classes of functions consist entirely of polynomials. Analogous results are also shown to apply to antidifferentiation.

Our third group of results construct natural extension and strong antidifferentiation operators with very nice features. Proposition 34 below shows that the latter suffices for the existence of an actual integral. In fact, the antiderivative defines an integral with essentially all the properties of the usual integral. Moreover, when restricted to $\mathbb{R}$, the antiderivative is a proper extension of the Hadamard finite part at $\infty$; see [15]. This group of results is applicable to sets of functions that, at $\infty$, are meromorphic, semi-analytic, Borel summable, or lie in a class of Écalle-Borel transseriable functions; see [13],[18]. As such, the sets of functions to which our positive results apply include most sets of standard classical functions. We argue that for all “practical purposes” in applied analysis, integrals and natural extensions from the reals to the surreals with good properties exist. In particular, most classical special functions (such as Airy, Bessel, Ei, erf, Gamma, Painlevé,...) extend naturally (and are integrable) from finite to infinite values of the variable.

In §9, Logical Issues, we discuss appropriate formal systems in which to cast the results. We identify two main approaches to formalization: literal and classes. NBG is the principal system for the classes approach, which is the approach used in the body of the paper. For the literal approach, we use ZCI = ZC + “there exists a strongly inaccessible cardinal”. Both approaches have their advantages and disadvantages. The underlying mathematical developments are sufficiently robust as to be unaffected by these logical issues. As a consequence, the reader unconcerned with logical issues need not read §9.

This paper is the result of an interdisciplinary collaboration making use of the first author’s expertise in standard and non-standard analysis, exponential asymptotics and Borel summability, the first and second authors’ expertise in surreal numbers, and the third author’s expertise in mathematical logic and the foundations of mathematics.

Organization of the paper. The paper relies on general notions, constructions and results from the theory of surreal numbers. An overview of those that are most relevant to this paper are given in [3]. For the most general positive results, the paper also uses transseries and Écalle-Borel summability. These are reviewed in [4].

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2Integration, for functions with convergent expansions, has been studied in the context of the non-Archimedean ordered field of left-finite power series with real coefficients and rational exponents in [39] and [40].
2. Main results

Our first set of results prove the existence of surreal antiderivatives within NBG (which includes the axiom of choice). Another set of results show that NBG− + DC cannot prove their existence, even in sharply weakened forms. Here NBG− is NBG without any form of the axiom of choice, and DC is dependent choice, which is an important weak form of the axiom of choice; see §9. Thus, in fairly general settings, the existence of antidifferentiation operators in the surreals is neither provable nor refutable in NBG− + DC.

We also prove that there are no explicit descriptions that can be proved to uniquely define such operators, even in NBG. On the other hand, we show, in NBG−, that genetically defined integrals (see footnote 1) exist for functions arising in all “practical applications.”

2.1. Unnatural positive results. The results in this section do not lead to natural operators or even natural functionals. Specifically, while these operators or functionals assume, by construction, the expected values on simple functions, for general functions their existence relies on the axiom of choice and as such their values for specific functions are intrinsically ad hoc.

Theorem 5. Let \{1, x, ..., x_n\} be the set of monomials of degree < n + 1, and W be the linear space generated by them (i.e., the polynomials of degree < n + 1). Then there is an antidifferentiation operator \(A\) on \(T[\mathbb{R}, \mathbb{R}]\) such that \(A(x^{j-1}) = x^j/j\), \(j = 1, ..., n + 1\).

Note 6. The proof of Theorem 4 is carried out as an explicit construction in NBG−. Finitely many other natural functions with natural derivatives can be incorporated.

The next result uses the axiom of choice.

Theorem 7. There is an antidifferentiation operator \(A\) on \(T[\mathbb{R}, \mathbb{R}]\) such that \(A(x^{j-1}) = x^j/j\) for each \(j \in \mathbb{Z}^+\). More generally, if \(W_1 \subset T[\mathbb{R}, \mathbb{R}]\) is a space of functions that naturally extend to No in such a manner that \(F^*(x) - F^*(0)\) is a natural linear antiderivative of \(f(x)\), then \(A(f)(x) = F^*(x) - F^*(0)\). Moreover, \(W_1\) would include the real exp, \(1/(x + 1)\) and other elementary functions.

Theorem 8. There exist \(2^c\) extension operators \(E\) on the set of all functions \(f : \mathbb{R} \rightarrow \mathbb{R}\) which preserve polynomials and other known functions.

Proof. This follows from Theorem 7 by letting \(E(f) := (A(f))'\). □

Note 9. The unnatural positive results above which do not involve exponentials extend to general non-Archimedean fields.

2.2. Obstructions.

The next set of results are negative. We present different types of obstructions to integration with the aim of clarifying the nature of these obstructions. The first group of results (Theorem 12 - Theorem 24) are targeted primarily (though not exclusively) to the proposed Conway-Kruskal-Norton program of surreal integration based on “genetic” definitions; see footnote 1. These results suggest that for their integration program on No to succeed, even for rapidly decaying entire functions, stringent conditions governing the behavior of the functions at \(\infty\) are needed.
The remaining negative results hold for any non-Archimedean ordered field $V$ with specified properties.

Throughout this section, the space $E$ is defined as in [I]. $V$ (which may be a set or a proper class) is a non-Archimedean ordered field extending $\mathbb{R}$, and $\omega$ is a positive infinite element of $V$. If $V = \text{No}$, then $\omega$ may be taken to be $\{\mathbb{N}\}$.

We start with a standard definition.

**Definition 10.** A set of reals is Baire measurable if and only if its symmetric difference with some open set is meager. This is often called the “property of Baire.”

Let $E_b = \{ f \in E : \|f\|_{\infty} < \infty \}$, where, as usual, $\|f\|_{\infty} = \sup_{x \in \mathbb{R}} |f|$. For Theorems 12 and 13 below we could further restrict the space to that of entire functions of exponential order at most two; moreover, compatibility with all monomials is not required.

We remind the reader that a Banach limit is a linear functional $\varphi$ on the space $\ell^\infty = \{s : \mathbb{N} \to \mathbb{C} | \sup_n |s_n| < \infty \}$ such that $\varphi(Ts) = \varphi(s)$ where $T(s_0, s_1, ...) = (s_1, s_2, ...)$ and $\varphi(s) \in \inf_{n \in \mathbb{N}} s_n, \sup_{n \in \mathbb{N}} s_n$, if $\text{ran}(s) \subset \mathbb{R}$. We also remind the reader that NBG, which is conservative over ZF, includes no form of choice.

The following definition makes use of the conceptions introduced in Definitions 11 and 23 from [21].

**Definition 11.** $\mathcal{E}_b = \{ f \in \mathcal{E}_\omega : (h \in \mathcal{E}_b, h \geq 0) \Rightarrow f(h) \geq 0 \}$. $\mathfrak{A}_\omega^+$ is similarly defined.

**Theorem 12.** NBG proves that if $\mathcal{E}_b^+ \cup \mathfrak{A}_\omega^+ \neq \emptyset$, then Banach limits exist.

**Proof.** The proof for $\mathfrak{A}_\omega^+$ is a corollary of the following one about $\mathcal{E}_b^+$. It is enough to define a Banach limit $\varphi$ on real sequences. Let $l_n$ be 0 for $x \leq 0$ and the linear interpolation between 0 and $s_0$ on $[0, 1]$ and between $s_n$ and $s_{n+1}$ on $[n+1, n+2]$ \(\forall n \in \mathbb{N}\). Now, $\forall s \in \ell^\infty$ and $\forall \varepsilon > 0$, $\exists f_{s, x} \in \mathcal{E}_b$ (obtained by mollifying $l_n$) such that $\|f_{s, x} - l_n\|_{\infty} < \varepsilon$. Let $F_{s, x}(x) := \int_0^x f_{s, x}$ and $L_n(x) := \int_0^x l_n$. Note that $\forall x \in \mathbb{R}^+$ we have (*): $|F_{s, x}(x) - L_n(x)| \leq \varepsilon x$. Let $E \in \mathcal{E}_b^+$. Define $\varphi$ on $\ell^\infty$ by $\varphi(s) := \lim_{x \to 0^+} \text{re} (\omega^{-1}(E(F_{s, x}))(\omega))$, where $\text{re}(x)$ denotes the real part of a surreal number $x$ written in normal form (see [39]; this exists by (*)). By mimicking Robinson’s non-standard analysis construction (cf. [31] page 54)), it is easy to check that $\varphi$ is a Banach limit. (The calculations are straightforward; see [46], [47] for the details.)

**Theorem 13.**

1. NBG proves that the existence of Banach limits implies the existence of a subset of $\mathbb{R}$ not having the property of Baire.
2. NBG + DC does not prove the existence of a subset of $\mathbb{R}$ not having the property of Baire.
3. $\mathcal{E}_b^+ \cup \mathfrak{A}_\omega^+ \neq \emptyset$ is independent of NBG + DC.

**Proof.**

1. ZF and, thus, NBG proves that the existence of Banach limits implies the existence of a subset of $\mathbb{R}$ not having the property of Baire; see [37] pages 611-612 and [38].
2. ZFDC does not prove the existence of a subset of $\mathbb{R}$ not having the property of Baire; see [43] and [44]. Since NBG + DC is a conservative extension of ZFDC, (2) follows.
3. This follows from (2) and Theorem 12.

\(\square\)
The above results indicate the nature of the obstructions to the existence of a general “genetic” integral (see footnote 1): it is implicit in the (informal) description of such an integral that it should be explicitly constructed in NBG− and, hence, without any form of choice. However, such an integral exists at most on those functions whose behavior at ∞ ensures that an explicitly constructed Banach limit exists, as in Theorem 12. Even this condition is insufficient, as we shall now see. The class of functions for which genetic definitions exist is in fact highly restricted. See [24].

The next sequence of results shows that the obstructions persist even if we gradually impose more constraints on the functions, until the constraints are so restrictive that the remaining functions are simply polynomials.

In the following definition, $\mathcal{E}_1$ is a space of real functions, exponentially decaying on $\mathbb{R}^+$, that extend to entire functions of exponential order one. In classical analysis, growth conditions (certainly exponential decay) at a singular point of an otherwise smooth function would ensure integrability at that point. This however is not the case in the surreal world.

**Definition 14.** Let $\mathcal{E}_1 := \left\{ f \in \mathcal{E} : \| f \| := \sup_{z \in \mathbb{C}} |e^{-2|z|} f(z)| + \sup_{x \in \mathbb{R}^+} |e^{x/2} f(x)| < \infty \right\}$.

∀$n \in \mathbb{Z}^+$, let $f_n(z) = ne^{-z}(1 - e^{-z/n})$ and $\mathcal{E}_e \subset \mathcal{E}$ be the closed subspace generated by $\{f_n\}_{n \in \mathbb{Z}^+}$.

**Lemma 15.** $\mathcal{E}_1$ is a Banach space, and ∀$n \in \mathbb{Z}^+$, $\| f_n \| < 2$. In addition, $\mathcal{E}_e$ is a separable Banach space and thus a Polish space.

**Proof.** For $K > 0$, let $\mathcal{D}_K = \{ z : |z| < K \}$. On any $\mathcal{D}_K$, $\| f \|$ is manifestly equivalent to the sup norm. A uniform limit on $\mathcal{D}_K$ of functions in $\mathcal{E}$ is analytic in $\mathcal{D}_K$ and real valued on $[0, K)$, thus $\mathcal{E}$ is a Banach space. The rest is calculus. □

The functionals in $\mathcal{E}$ ($\mathcal{F}$, respectively) below have properties expected of any extension to $\omega$ (integration on $[0, \omega)$, respectively).

**Definition 16.** $E \in \mathcal{E}$ if $E$ is linear on $\mathcal{E}_e$, and ∀$a \in \mathbb{R}$, $E(e^{ax}) = e^{a \omega}$ when $e^{ax} \in \mathcal{E}_e$, $J \in \mathcal{F}$ if $J$ is linear on $\mathcal{E}_e$, and ∀$a \in \mathbb{R}$, $J(e^{ax}) = a^{-1}(e^{a \omega} - 1)$ when $e^{ax} \in \mathcal{E}_e$.

**Lemma 17.** If $\mathcal{E} \cup \mathcal{F} \neq \emptyset$, then there is a discontinuous homomorphism $\varphi : \mathcal{E} \to \mathbb{R}$.

**Proof.** Assume $E \in \mathcal{E}$. Define $\varphi : \mathcal{E}_e \to \mathbb{R}$ by $\varphi(f) = \text{re}(e^{a \omega} E(f))$, where $\text{re}(e^{a \omega} E(f))$ is the real part of the normal form of the surreal number $e^{a \omega} E(f)$; see 34. Clearly, $\varphi$ is linear. Moreover, since $\lim_{n \to \infty} \| e^{-1} f_n \| = 0$ and $\varphi(e^{-1} f_n) = 1$, $\varphi$ is discontinuous. Similarly, assuming that $J \in \mathcal{F}$, $\varphi(f) = \text{re}(e^{a J(f)})$ is discontinuous. □

In virtue of Pettis’s Theorem 35. Solovay’s results 33, §4, p. 55, Thomas-Zapletal’s Theorem 1.4 44 and the equiconsistency of ZFDC and NBG− + DC, the following holds.

**Theorem 18** (44). The following is consistent with NBG− + DC. For all Polish groups $G, H$, every homomorphism $\varphi : G \to H$ is continuous.

**Corollary 19.** $\mathcal{E} \cup \mathcal{F} = \emptyset$ is consistent with NBG− + DC.

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3Note that $\mathcal{E}_1$ is not separable. Indeed, for any $u_1 \neq u_2 \in S^1$, $\| e^{-u_1 z} - e^{-u_2 z} \| = 1$, where $f_{u_1} = e^{u_1 z}$, the same holds for $f_{u_1} + \frac{1}{e^{u_1 z}} \in \mathcal{E}_1$ if $\arg u_i \in I = (\frac{3\pi}{4}, \frac{7\pi}{4})$, and $I$ is of course uncountable.
Proof. This is an immediate consequence of Lemmas 15 and 17, and Theorem 18.

Theorem 20. $\mathcal{E} \cup \mathcal{F} \neq \emptyset$ is independent of NBG$^{-} + DC$.

Proof. This is obtained by combining Corollary 19 with Theorem 7.

In virtue of the above, it is clear that if there is a genetic definition of integration on No, its validity cannot be established in NBG$^{-} + DC$. This being the case, it is intuitively obvious that by supplementing NBG$^{-}$ with either the axiom of choice, the axiom of global choice or, say, the Continuum Hypothesis cannot help prove that a concrete integration definition has the intended properties.

Note 21. It is clear from the proofs that the results above would be unchanged if $e^{-|z|}$ is replaced by some $e^{-|z/n|}, n \in \mathbb{Z}^{+}$, and in fact similar results carry over to other types of bounds or decay. For functions of half exponential order, the family $\cosh(\sqrt{z}/n)$ can be used instead of $e^{-z/n}$; for arbitrary types of growth, the argument requires several steps, since special function are to be avoided, and as such, different foundational tools need to be used. In any case, any growth bounds, short of polynomial bounds, yield the same result. Polynomial bounds entail that the functions are polynomials, in which case integration exists trivially. Furthermore, these obstructions continue to hold for more general non-Archimedean fields, and we provide the results and detailed proofs for the more general case.

2.2.1. Further obstructions. We now obtain sharper negative results based on descriptive set-theoretic tools adapted to the questions at hand. By sharper negative results, we mean negative results that require fewer properties of the integral, allow for “nicer” functions and which, in some settings, provide both necessary and sufficient conditions for the existence of extension or antiderivative functionals. Theorem 24 is a consequence of the core descriptive set-theoretic result, Theorem 64, that does not rely on any substantial analytic arguments. Theorem 28 is a deeper consequence of Theorem 64, showing that, in the class of entire functions bounded in some weighted $L^\infty$ norm in $\mathbb{C}$, integrals exist if and only if the weight allows only for polynomials.

We begin with the definitions of $\theta$-extensions and $\theta$-antiderivatives in the setting of ordered fields.

Definition 22. A $\theta$-extension from $K \subseteq T[\mathbb{R}, \mathbb{R}]$ into No is a function $e_\theta : K \to No$ such that
i. $e_\theta$ is linear;
ii. If $f \in K$ is the monomial $x^n$, then $e_\theta(f) = \theta^n$.
We denote by $\mathcal{E}_\theta$ the collection of $\theta$-extensions.

Definition 23. A $\theta$-antidifferentiation from $K \subseteq T[\mathbb{R}, \mathbb{R}]$ into No is a function $a_\theta : K \to No$ such that
i. $a_\theta$ is linear;
ii. If $f \in K$ is the monomial $x^{j-1}, j > 0$, then $a_\theta(f) = \theta^j/j$.
We denote by $\mathcal{A}_\theta$ the collection of $\theta$-antiderivatives.

There is no immediate relation between $\theta$-extensions and $\theta$-antiderivatives.
Theorem 24. i. \( \text{NBG}^- + \text{DC} \) proves the following. If there exists an ordered field \( V \) extending \( \mathbb{R} \), a positive infinite \( \theta \in V \), and a \( \theta \)-extension or a \( \theta \)-antiderivative functional from \( \mathcal{E} \) into \( V \), then there is a set of reals that is not Baire measurable.

ii. \( \text{NBG}^- + \text{DC} \) does not prove that there exists an ordered field \( V \) extending \( \mathbb{R} \), a positive infinite \( \theta \in V \), and a \( \theta \)-extension or a \( \theta \)-antiderivative functional from \( \mathcal{E} \) into \( V \).

iii. There are no three set-theoretic descriptions with parameters such that \( \text{NBG} \) proves the following. There is an assignment of real numbers to the parameters used in all three, such that the first description uniquely defines an ordered field extending \( \mathbb{R} \), the second description uniquely defines a positive infinite \( \theta \in V \), and the third description uniquely defines a \( \theta \)-extension functional or a \( \theta \)-antiderivative functional from \( \mathcal{E} \) into \( V \).

Part iii also holds for extensions of \( \text{NBG} \) by standard large cardinal hypotheses.

Note 25. In Theorem 24, ii follows immediately from i, and the well known fact that \( \text{NBG}^- + \text{DC} + \) “every set of reals is has the Baire property” is consistent; see [43] and [41].

2.3. More general negative results. We explore further spaces without regularity, and obtain stronger versions of the results above, in more generality than No.

We view any \( W : [0, \infty) \to \mathbb{R}^+ \) as a weight.

Definition 26 (Growth class \( W \)). Let \( W : [0, \infty) \to \mathbb{R}^+ \). Define
\[
\mathcal{E}_W = \{ f \in \text{Entire}, f(\mathbb{R}) \subseteq \mathbb{R} : \sup_{z \in \mathbb{C}} |f|/W(|z|) \leq C < \infty \}.
\]

We now state an important if and only if result that provides a dichotomy. Our formulation requires the notion of a continuous weight given by an arithmetically presented code.

Definition 27. Let \( W : \mathbb{R} \to \mathbb{R}^+ \) be a continuous weight. We say that \( E \subset \mathbb{Z}^4 \) codes \( W \) if and only if \( \forall (a, b, c, d) \in \mathbb{Z}^4, a/b < W(c/d) \) just in case \( (a, b, c, d) \in E \). An arithmetic presentation of \( E \subset \mathbb{Z}^4 \) takes the form \( \{(a, b, c, d) \in \mathbb{Z}^4 : \varphi\} \), where \( \varphi \) is a formula involving \( \forall, \exists, \neg, \vee, \wedge, +, -, \cdot, <, 0, 1 \), variables ranging over \( \mathbb{Z} \), with at most the free variables \( a, b, c, d \).

Standard elementary and special functions with rational parameters are continuous functions that can be given by arithmetically presented codes. Also, compositions of continuous functions that are given by arithmetically presented codes are continuous functions that are given by arithmetically presented codes.

Theorem 28. Let \( W \) be a continuous weight given by an arithmetically presented code. The following are equivalent.

\[\text{Note 25:}\text{Note that here and in [15], we consider only arithmetically presented continuous functions. There is a broader notion of “arithmetically presented function” that includes many discontinuous Borel measurable functions, which we do not need here or in [15]. However, in [15] page 4738, paragraph 3], the first and third authors erroneously confused the two notions. That entire paragraph should be replaced with the paragraph to which this note is affixed.}\]

\[\text{Note 26:}\text{More precisely, we start with an arithmetically presented } E \text{ which, provably in ZFC (or equivalently in ZF, NBG, NBG^-) codes a continuous weight } W.\]
(a) NBG\(^-\) proves that \(E_W\) consists entirely of polynomials from \(\mathbb{R}\) into \(\mathbb{R}\).

(b) There are three set-theoretic descriptions with parameters such that NBG proves the following. There is an assignment of real numbers to the parameters used in all three, such that the first description uniquely defines an ordered field extending \(\mathbb{R}\), the second description uniquely defines a positive infinite \(\theta \in V\), and the third description uniquely defines a \(\theta\)-extension functional or a \(\theta\)-antiderivative functional from \(E_W\) into \(V\).

The equivalence holds if NBG\(^-\) is replaced by NBG or if both NBG\(^-\) and NBG are replaced by any common extension of NBG with standard large cardinal hypotheses.

The next question is: can we integrate (or extend past \(\infty\)) real-analytic functions for which the real integral \(\int_1^\infty |f| < \infty\), or that decrease much faster, to ordered fields such as \(\mathbb{N}_\infty\)? To help prepare the way for answering this question we require a series of definitions, some of which make use of our generalization of the notion of a derivative of a function at a point; see Definition 2.

**Definition 29.** An exponentially adequate ordered field is an ordered field \(V\) extending \(\mathbb{R}\) together with an order preserving mapping \(\exp = (x \mapsto e^x)\) (from the additive group of \(V\) onto the multiplicative group of positive elements of \(V\)) with the properties \(e^x \gg x^n\) for all positive infinite \(x\) and \((e^x)' = e^x\).

**Definition 30.** For each \(m \in \mathbb{Z}^+\), let \(\exp_m\) be the \(m\)-th compositional iterate of \(\exp, W_m : \mathbb{R} \to \mathbb{R}^+\) be the weight \(1/\exp_m\), and \(A_{W_m} = \{f \in E : \sup_{\mathbb{R}^+} |f|/W_m < \infty\}\).

The functions in the \(A_{W_m}\) “decrease superexponentially”.

**Definition 31.** Exponentially adequate \(\theta\)-extension functionals and exponentially adequate \(\theta\)-antidifferentiation functionals are \(\theta\)-extension functionals and \(\theta\)-antidifferentiation functionals obeying the following additional condition for all \(m, n \in \mathbb{Z}^+\).

If \(f = x^n e^{-W_m W_m'}(1 - n/(x W_m'))\), then \(e_{\theta, V}(f) = f(\theta)\) and \(a_{\theta, V}(f) = -\theta^n e^{-W_m(\theta)}\).

This corresponds to the antiderivative of \(f\) without constant term.

**Theorem 32.**

i. NBG\(^-\) + DC proves the following. If there exists an exponentially adequate ordered field \(V\), a positive infinite \(\theta\) in \(V\), and an exponentially adequate \(\theta\)-extension or \(\theta\)-antiderivative functional from some \(A_{W_m}\) into \(V\), then there is a set of reals that is not Baire measurable.

ii. NBG\(^-\) + DC does not prove that there exists an exponentially adequate ordered field \(V\), a positive infinite \(\theta\) in \(V\), and an exponentially adequate \(\theta\)-extension or \(\theta\)-antiderivative functional from some \(A_{W_m}\) into \(V\).

iii. There are no three set-theoretic descriptions with parameters such that NBG proves the following. There is an assignment of real numbers to the parameters used in all three, such that the first description uniquely defines an exponentially adequate ordered field, the second description uniquely defines a positive infinite \(\theta\) in \(V\), and the third description uniquely defines an exponentially adequate \(\theta\)-extension functional or \(\theta\)-antiderivative function from some \(A_{W_m}\) into \(V\).

Part iii also holds for extensions of NBG by standard large cardinal hypotheses.

2.4. Positive results about extensions and integrals in \(\mathbb{N}_\infty\) for functions with regularity conditions at infinity.
In the previous sections we have obtained ad hoc extension and antidifferentiation operators and showed that there are no natural ones unless there are restrictions on the behavior of the functions at the endpoints.

In this section, we show that when certain natural restrictions are present, genetically defined (see footnote 1) integral operators, and extensions with good properties do exist. Integrals are defined after constructing linear antidifferentiation operators, see Definition 3; also see Proposition 34 and Note 36 below. Moreover, as we will see, for all special functions for which a genetically defined integral exists, its value at $\omega$ determines the transseries of the function, which in turn completely determines the function. The domain of applicability of these genetically defined integral operators, and extensions includes a wide range of functions arising in applications.

Note 33. In the following, without loss of generality, we assume that the functions of interest are defined on $(x_0, \infty)$, where $x_0$, which is assumed to be in $\mathbb{R} \cup \{-\infty\}$, may depend on the function and we seek extensions and integrals thereof to $\{x \in \mathbb{N}_0 : x > x_0\}$. See \S 4.5.1 for details.

Our positive results apply when there is complete information about the behavior of functions at $+\infty$ in the sense described above. We denote the family of such functions $\mathcal{F}$. It is convenient to first treat a proper subclass $\mathcal{F}_a$ of $\mathcal{F}$ that is simpler to analyze than $\mathcal{F}$ itself.

(a) $\mathcal{F}_a$ consists of functions which have convergent Puiseux-Frobenius power series in integer or noninteger powers of $1/x$ at $+\infty$. Namely, for some $k \in \mathbb{Q}$, $r \in \mathbb{R}$, and all $x > r$, we have

$$f(x) = \sum_{j=-M}^{\infty} c_k x^{-j/k}.$$  

Such is the case of functions which at $+\infty$ are semialgebraic, analytic, meromorphic and semi-analytic, to name some of the most familiar ones. By \[10\], semi-analytic functions at $+\infty$ have convergent Puiseux series in powers of $x^{1/k}$ for some $k \in \mathbb{Z}^+$. We could allow (3) to also contain exponentials and logs, if convergence is preserved.

(b) The class $\mathcal{F}$ is much more general. It consists of functions which, after changes of variables, have for $x > x_0$ Borel-Écalle transseries of the form

$$\tilde{T} = \sum_{\mathbf{k}, \mathbf{l} \in \mathbb{N}} c_{\mathbf{k}, \mathbf{l}} e^{-k \beta \lambda x^{-l}} x^{-\beta \lambda x^{-l}},$$

where $\mathbf{k}, \mathbf{l} \in \mathbb{Z}^n, \beta = (\beta_1, \ldots, \beta_n), \lambda = (\lambda_1, \ldots, \lambda_n)$ and $c_{\mathbf{k}, \mathbf{l}}, \beta_j, \lambda_j \in \mathbb{R}$. We assume $k_j \lambda_j$ be present for infinitely many $k_j$ only if $\lambda_j > 0$. For a technical reason—being able to employ Catalan averages—we assume as in [12] that all $\beta_i \leq 1$; it can then be arranged that $\beta_i \in (0, 1]$. We arrange that the critical Écalle time is $x$ ([12] [13] [14] [17] [18]).

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\textsuperscript{6}We could allow the constants $c_{\mathbf{k}, \mathbf{l}}, \beta$ and $\lambda$ to be complex-valued—and then we would end up with a class of sur-complex functions. We would then require $\text{Re}\beta_j < 1$ ([12] [14]), it being understood that $\text{Re}$ denotes the real part in the sense of complex analysis (which should not be confused with “re” which denotes the real part of a surreal number written in normal form; see [9]).
\( \mathcal{F} \) contains the solutions of linear or nonlinear systems of ODEs or of difference equations for which, after possible changes of variables, \(+\infty\) is at worst an irregular singularity of Poincaré rank one. We will impose still further technical restrictions, to keep the analysis simple. The setting is essentially that of [12, 14] for ODEs and [7] for difference equations. More generally, \( \mathcal{F} \) includes elementary functions and all named functions arising in the analysis of ODEs, difference equations and PDEs. The named classical functions in analysis that satisfy some differential or difference equation such as Airy Ai and Bi, Bessel \( H, J, K, Y, \Gamma, {}_2 F_1 \), and so on are real analytic in \( x > 0 \) and belong to \( \mathcal{F} \).

**Proposition 34** *(Existence of an integral operator).* Let \( A \) be a strong antidifferentiation operator on \( K \). Then there exists an integral operator on \( K \), meaning a function of three variables, \( x, y \in \mathbb{N} \) and \( f \in K \), denoted as usual \( \int_x^y f \), with the following properties:

(a) \( \left( \int_a^x f \right)' = f \ \forall a, x \in \text{dom}(f) \);  
(b) \( \int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g \ \forall a, b \in \text{dom}(f) \cap \text{dom}(g), (\alpha, \beta) \in \mathbb{R}^2 \);  
(c) \( \int_a^b f'(x) = f(b) - f(a) \);  
(d) \( \int_a^b f + \int_a^{a_3} f = \int_a^{a_3} f \ \forall a_1, a_2, a_3 \in \text{dom}(f) \);  
(e) \( \int_a^b f'g = fg|_a^b - \int_a^b f'g \) if \( f, g \) are differentiable and \( a, b \in \text{dom}(f) \cap \text{dom}(g) \);  
(f) \( \int_a^b f(g(s))g'(s)ds = \int_{g(a)}^{g(b)} f(s)ds \), whenever \( g \in K \) is differentiable, \( a, x \in \text{dom}(f) \), and \( g([a, x]) \subset \text{dom}(f) \cap \text{dom}(Af) \);  
(g) \( f \geq 0 \) and \( b > a \) are in \( \text{dom}(f) \) imply \( \int_a^b f \geq 0 \).

**Proof.** Define \( \int_a^b f = A(f)(y) - A(f(x)) \). The properties follow straightforwardly. (c) is the definition of a strong antiderivative, and (c), (e) and (f) are equivalent. If \( f \geq 0 \), then \( A(f)|_\mathbb{R} \) is non-decreasing and thus \( \int_a^b f = A(f(b)) - A(f(a)) \geq 0 \). \( \square \)

**Note 35.** The functions in \( \mathcal{F}_a \) are much easier to deal with. They can be extended past \(+\infty\) essentially by interpreting their Puiseux series at \(+\infty\) as a normal form of a surreal variable. The Puiseux series can then be integrated term by term, resulting in a “good” integral. A genetic definition may then be obtained by minor adaptations of the construction in [26].

For the general family \( \mathcal{F} \) we need the machinery of generalized Écalle-Borel summability of transseries [13 [17 [18] and Catalan averages [32] to establish the properties of the extensions and integrals but not necessarily to calculate them. We will separate the technical constructions accordingly, to enhance readability.
Note 36. Since the surreal exponential exp (see §3) is surreal-analytic at any point, it is clear that any of its antiderivatives is of the form exp + C where C is a constant, at least locally. We show that an antidifferentiation operator is defined on a wide class of functions; this antidifferentiation operator gives C = 0 everywhere for exp. Via Proposition 34, C = 0 translates into \( \int_0^\omega e^s \, ds = e^\omega - 1 \) as expected. We note that this stands in contrast to Norton’s aforementioned proposed definition of integration which was shown by Kruskal to integrate \( e^s \) over the range \([0, \omega]\) to the wrong value \( e^\omega \) [10, page 228].

Note 37 (Cautionary Note). A general \( C^\infty \) function \( f \) cannot be (correctly) extended in an infinitesimal neighborhood of a point by its Taylor series. This is the case even if the Taylor series converges–unless, of course, the series converges to \( f \), in which case \( f \) is analytic at that point. An example is \( e^{-1/x^2} \) extended by zero at zero. The Taylor series at 0 is convergent (trivially) since it is the zero series. But \( e^{-1/x^2} \) is not 0 in \( \text{No} \) for infinitesimal arguments. This function has a convergent transseries, also trivially, since \( e^{-1/x^2} \) is its own transseries, and therefore provides a correct extension.

The precise technical setting is as follows.

Theorem 38 (Existence of extensions and strong antiderivatives). On \( \mathcal{F} \) there exist

(a) an extension operator \( E \) from \( \mathcal{F} \) to \( \mathcal{F}^* = \{ f^* : f \in \mathcal{F} \} \) (see §1.1) which is linear, multiplicative and preserves exp, log and \( x^r \) for \( r \in \mathbb{R} \).

(b) a linear operator \( A : E(\mathcal{F}) \to \mathcal{F}^* \) with the properties

1. \([A(f)]' = f\);
2. If \( F \in T[\text{No}] \) and \((F|_R)' = f \in \mathcal{F} \) exists, then there is a \( C \in \text{No} \) such that \( A(f) = F + C \).

The definitions of \( A \) and \( E \) in the just-stated theorem follow Norton’s original integral scheme, using inequalities with respect to earlier defined functions and earlier values of the integral. Unlike Norton’s definition which was found to be intensional [10, page 228], ours are shown to only depend on the values of the functions involved.

For instance the definition of \( E_i \) is

\[
E_i(x) = \left\{ e^x \sum_{k \in \ell(x)} \frac{k!}{x^{k+1}} - C x^{-1/2}, S^L \right\} \left( e^x \sum_{k \in \ell(x)} \frac{k!}{x^{k+1}} + C x^{-1/2}, S^R \right)
\]

Here \( \ell(x) \) is the “least term” set of indices

\[
\ell(x) := \{ k \in \mathbb{N} : k \leq x \}.
\]

If \( x \) is finite, \( \ell(x) \) is a finite set and \( E_i \) equals the least term summation of the series, see Note 39 below. If \( x \) is infinite, then clearly \( \ell(x) = \mathbb{N} \). In this case, the sum is to be interpreted as an absolutely convergent series in the sense of Conway (see [3]). The bounds in the sets \( S^L \) and \( S^R \) are based on local Taylor polynomials, as in [26]; see [48] for details.
Note 39 (Least term summation). More specifically, when a series \( \sum_{n=0}^{\infty} c_n z^n \) has zero radius of convergence, then \( \limsup_n |c_n z^n| = \infty, \forall z \neq 0 \). Say \( c_n = n! \) and \( z = 1/100 \). Then, using Stirling’s formula for the factorial, it is easy to see that \( n! z^n \) decreases in \( n \) until approximately \( n = 100 \), and then increases in \( n \). In this case, the least term set of indices \( \ell(z) \) is \( \{0, 1, \ldots, 100\} \). The least term is reached approximately at \( n = 100 \) and is of the order \( O(\sqrt{\pi e^{-1/|z|}}) \). Summing to the least term means summing the first 100 terms when \( z = 1/100 \), or the first \( N \) terms if \( z = 1/N \). The error obtained in this way is proved in [16] to be exponentially small, rather than polynomially small as would be the case when summing a fixed number of terms of the series.

See §8.5 for further examples.

3. Surreal numbers: overview and preliminaries

There are a variety of recursive [10], [25], [20] also see, [2], [3] and non-recursive [10, page 65; also see, [29]; [22, page 242] constructions of the class \( \text{No} \) of surreal numbers, each with their own virtues. For the sake of brevity, here we adopt Conway’s construction based on sign-sequences [10, page 65], which has been made popular by Gonshor [29].

In accordance with this approach, a surreal number is a function \( a : \lambda \to \{-, +\} \) where \( \lambda \) is an ordinal called the length of \( a \). The class \( \text{No} \) of surreal numbers so defined carries a canonical linear ordering: \( a < b \) if and only if \( a \) is (lexicographically) less than \( b \) with respect to the linear ordering on \( \{-, +\} \), it being understood that \( - < \text{undefined} < + \). As in [23], we define the canonical partial ordering \( <_s \) on \( \text{No} \) by: \( a <_s b \) (“\( a \) is simpler than \( b \)”)

A tree \( \langle A, <_A \rangle \) is a partially ordered class such that for each \( x \in A \), the class \( \{y \in A : y <_A x\} \) of predecessors of \( x \) is a set well ordered by \( <_A \). If each member of \( A \) has two immediate successors and every chain in \( A \) of limit length (including the empty chain) has one immediate successor, the tree is said to be a full binary tree. Since a full binary tree has a level for each ordinal, the universe of a full binary tree is a proper class.

Proposition 40. \( \langle \text{No}, <, <_s \rangle \) is a lexicographically ordered full binary tree.

Central to the algebraico-tree-theoretic development of the theory of surreal numbers is the following consequence of Proposition 40.

Proposition 41. If \( L \) and \( R \) are (possibly empty) subsets of \( \text{No} \) for which every member of \( L \) precedes every member of \( R \) (written \( L < R \)), there is a simplest member of \( \text{No} \) lying between the members of \( L \) and the members of \( R \) [23, pages 1234-1235] and [25, Proposition 2.4].

Co-opting notation introduced by Conway, the simplest member of \( \text{No} \) lying between the members of \( L \) and the members of \( R \) is denoted by the expression \( \{L|R\} \).

Following Conway [10, page 4], if \( x = \{L|R\} \), we write \( x^L \) for the typical member of \( L \) and \( x^R \) for the typical member of \( R \); for \( x \) itself, we write \( \{x^L|x^R\} \); and
$x = \{a, b, c, \ldots | d, e, f, \ldots\}$ means that $x = \{L|\{R\}\}$ where $a, b, c, \ldots$ are the typical members of $L$ and $d, e, f, \ldots$ are the typical members of $R$. In accordance with these conventions, if $L$ or $R$ is empty, reference to the corresponding typical members may be deleted. So, for example, in place of $0 = \{\varnothing|\varnothing\}$, one may write $0 = \{\}$. Each $x \in \mathbf{No}$ has a canonical representation as the simplest member of $\mathbf{No}$ lying between its predecessors on the left and its predecessors on the right, i.e.

$$x = \{L_{s(x)}|R_{s(x)}\},$$

where $L_{s(x)} = \{a \in \mathbf{No} : a <_s x \text{ and } a < x\}$ and $R_{s(x)} = \{a \in \mathbf{No} : a <_s x \text{ and } x < a\}$.

By now letting $x = \{L_{s(x)}|R_{s(x)}\}$ and $y = \{L_{s(y)}|R_{s(y)}\}$, $+, -, \cdot$ are defined by recursion for all $x, y \in \mathbf{No}$ as follows, where the typical members $x^L, x^R,$ $y^L$ and $y^R$ are understood to range over the members of $L_{s(x)}, R_{s(x)}, L_{s(y)}$ and $R_{s(y)}$, respectively.

**Definition of** $x + y$.

$$x + y = \{x^L + y, x + y^L, x^R + y, x + y^R\}.$$  

**Definition of** $-x$.

$$-x = \{-x^R - x^L\}.$$  

**Definition of** $xy$.

$$xy = \{x^L y + xy^L - x^L y^L, x^R y + xy^R - x^R y^R, x^L y + xy^L - x^R y^R\}.$$  

Despite their cryptic appearance, the definitions of sums and products on $\mathbf{No}$ have natural interpretations that essentially assert that the sums and products of elements of $\mathbf{No}$ are the simplest possible elements of $\mathbf{No}$ consistent with $\mathbf{No}$’s structure as an ordered field [22, page 1236], [23, pages 252-253]. The constraint on additive inverses, which is a consequence of the definition of addition [23, page 1237], ensures that the portion of the surreal number tree less than 0 is (in absolute value) a mirror image of the portion of the surreal number tree greater than 0, 0 being the simplest element of the surreal number tree.

A subclass $A$ of $\mathbf{No}$ is said to be initial if $b \in A$ whenever $a \in A$ and $b <_s a$. Although there are many isomorphic copies of the order field of reals in $\mathbf{No}$, only one is initial. This ordered field, which we denote $\mathbb{R}$, plays the role of the reals in $\mathbf{No}$. Similarly, while there are many well-ordered proper subclasses of $\mathbf{No}$ in which $x < y$ if and only if $x <_s y$, only one is initial. The latter, which consists of the outermost right branch of $\langle \mathbf{No}, <, _s\rangle$, is identified as $\mathbf{No}$’s ordered class $\mathit{On}$ of ordinals. See Figure 1.

A subclass $B$ of an ordered class $\langle A, <\rangle$ is said to be convex, if $z \in B$ whenever $x, y \in B$ and $x < z < y$.

The following are consequence $\mathbf{No}$’s structure as a lexicographically ordered binary tree.

**Proposition 42** ([23]: Theorems 1 and 2). (i) Every non-empty convex subclass of $\mathbf{No}$ has a simplest member; (ii) if $x \in \mathbf{No}$ and $L, R$ are a pair of subsets of $\mathbf{No}$ for which $L < R$, then $x = \{L|\{R\}\}$ if and only if $L < \{x\} < R$ and $\{a \in \mathbf{No} : L < \{a\} < R\} \subseteq \{a \in \mathbf{No} : L_{s(x)} < \{a\} < R_{s(x)}\}$.  


The non-zero elements of an ordered group can be partitioned into equivalence classes consisting of all members of the group that mutually satisfy the Archimedean condition. If \( a \) and \( b \) are members of distinct Archimedean classes and \( |a| < |b| \), then we write \( a \ll b \) and \( a \) is said to be infinitesimal (in absolute value) relative to \( b \).

An element of \( \textbf{No} \) is said to be a leader if it is the simplest member of the positive elements of an Archimedean class of \( \textbf{No} \). Since the class of positive elements of an Archimedean class of \( \textbf{No} \) is convex, the concept of a leader is well defined. There is a unique mapping—the \( \omega \)-map—from \( \textbf{No} \) onto the ordered class of leaders that preserves both \(<\) and \(<_s\). The image of \( y \) under the \( \omega \)-map is denoted \( \omega^y \), and in virtue of its order preserving nature, we have: for all \( x, y \in \textbf{No} \),

\[
\omega^x \ll \omega^y \quad \text{if and only if} \quad x < y.
\]

Using the \( \omega \)-map along with other aspects of \( \textbf{No} \)’s \( s \)-hierarchical structure and its structure as a vector space over \( \mathbb{R} \), every surreal number can be assigned a canonical “proper name” or normal form that is a reflection of its characteristic \( s \)-hierarchical properties. These normal forms are expressed as sums of the form

\[
\sum_{\alpha<\beta} \omega^{y_\alpha} r_\alpha
\]

where \( \beta \) is an ordinal, \( (y_\alpha)_{\alpha<\beta} \) is a strictly decreasing sequence of surreals, and \( (r_\alpha)_{\alpha<\beta} \) is a sequence of non-zero real numbers. Every such expression is in fact the normal form of some surreal number, the normal form of an ordinal being just its Cantor normal form \([10\) pages 31-33\], \([23\) §3.1 and §5\], \([24\).

Making use of these normal forms, Figure 1 offers a glimpse of the some of the early stages of the recursive unfolding of \( \textbf{No} \).

Figure 1. Early stages of the recursive unfolding of \( \textbf{No} \)
When surreal numbers are represented by their normal forms, the sums, products and order on \( \text{No} \) assume the following more tractable termwise and lexicographical forms, where “dummy” terms with zeros for coefficients are understood to be inserted and deleted as needed.

**Proposition 43** ([23], Theorem 16; also see, [29], pages 67-70 and [1], pages 235-246).

\[
\sum_{y \in \text{No}} \omega^y \cdot a_y + \sum_{y \in \text{No}} \omega^y \cdot b_y = \sum_{y \in \text{No}} \omega^y \cdot (a_y + b_y),
\]

\[
\sum_{y \in \text{No}} \omega^y \cdot a_y \cdot \sum_{y \in \text{No}} \omega^y \cdot b_y = \sum_{y \in \text{No}} \omega^y \left[ \sum_{(\mu, \nu) \in \text{No} \times \text{No}} a_\mu b_\nu \right],
\]

\[
\sum_{y \in \text{No}} \omega^y \cdot a_y < \sum_{y \in \text{No}} \omega^y \cdot b_y, \text{ if } a_y = b_y \text{ for all } y > \text{ some } x \in \text{No} \text{ and } a_x < b_x.
\]

The following result is an immediate consequence of Conway’s definition of normal forms and ([10, pages 32-33] and [23, page 1247]) their lexicographical ordering.

**Proposition 44.** For all \( x \in \text{No} \) and all positive \( y \in \text{No} \), \( x = \{x - y | x + y\} \), wherever all of the exponents in the normal form of \( x \) are greater than all of the exponents in the normal form of \( y \).

Since every ordered field \( A \) contains a unique isomorphic copy, \( \mathbb{Q}_A \), of the ordered field of rational numbers, \( a \in A \) may be said to be infinite (infinitesimal) if \(|a|\) is greater than (less than) every positive member of \( \mathbb{Q}_A \). Thus, in virtue of the lexicographical ordering on normal forms, a surreal number is infinite (infinitesimal) just in case the greatest (i.e. the zeroth) exponent in its normal is greater than (less than) 0. As such, each surreal number \( x \) has a canonical decomposition into its purely infinite part \( \Pi(x) \), its real part \( \text{re}(x) \), and its infinitesimal part \( \text{II}(x) \), consisting of the portions of its normal form all of whose exponents are \( > 0 \), \( = 0 \), and \( < 0 \), respectively.

There is a notion of convergence in \( \text{No} \) for sequences and series of surreals that can be conveniently expressed using normal forms written as above with dummy terms. Let \( x \in \text{No} \) and for each \( y \in \text{No} \), let \( r_y(x) \) be the coefficient of \( \omega^y \) in the normal form of \( x \), it being understood that \( r_y(x) = 0 \), if \( \omega^y \) does not occur. Also let \( \{x_n : n \in \mathbb{N}\} \) be a sequence of surreals written in normal form. Following Siegel [42, page 432], we write

\[
x = \lim_{n \to \infty} x_n
\]

to mean

\[
r_y(x) = \lim_{n \to \infty} r_y(x_n), \text{ for all } y \in \text{No},
\]

and say that \( x_n \) converges to \( x \). We also write

\[
x = \sum_{n=0}^{\infty} x_n
\]

to mean the partial sums of the series converge to \( x \).
Among the convergent sequences and series of surreals are those whose mode of convergence is quite distinctive. In particular, for each $y \in \text{No}$, there is an $m \in \mathbb{N}$ such that $r_y(x_n) = r_y(x_m)$ for all $n \geq m$. Thus, for each $y \in \text{No}$,

$$r_y(x) = \lim_{n \to \infty} r_y(x_n) = r_y(x_m),$$

where $m$ depends on $y$. Following Conway, we call this mode of convergence absolute convergence. Moreover, we will call the normal form to which an absolutely convergent series $x_n$ of normal forms converges the Limit of the series and write $\text{Lim}_{n \to \infty} x_n$. We use “Limit” as opposed to “limit” to distinguish the surreal notion from its classical counterpart.

Relying on the above and classical combinatorial results of Neumann ([33] pages 206-209), [42] Lemma 3.2, [1] pages 260-266), one may prove [42] pages 432-434) the following theorem of Conway [10] page 40], which is a straightforward application to No of a classical result of Neumann [33] page 210], [1] page 267].

**Proposition 45.** Let $f$ be a formal power series with real coefficients, i.e. let

$$f(x) = \sum_{n=0}^{\infty} x^n r_n.$$

Then $f(\zeta)$ is absolutely convergent for all infinitesimals $\zeta$ in No.

Conway’s theorem also has the following multivariate formulation [42] page 435].

**Proposition 46.** Let $f$ be a formal power series in $k$ variables with real coefficients, i.e. let

$$f(x_1, ..., x_k) \in \mathbb{R}[[x_1, ..., x_k]].$$

Then $f(\zeta_1, ..., \zeta_k)$ is absolutely convergent for every choice of infinitesimals $\zeta_1, ..., \zeta_k$ in No.

This can also be written in the following useful form.

**Proposition 47.** Let $\{c_k : k \in \mathbb{N}^n\}$ be any multisequence of real numbers and $h_1, ..., h_m$ be infinitesimals. Also let $h^k = h_1^{k_1} \cdots h_m^{k_m}$. Then

$$(7) \sum_{|k| \geq 0} c_k h^k$$

is a well-defined element of No.

The following result, in which $x_n$ and $y_n$ are absolutely convergent series of normal forms, collects together some elementary properties of absolute convergence in No. Many are very similar to the properties of the usual limits.
Proposition 48. Let $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$, and further let $h \ll 1$, $\tau > 0$ and $A, B \in \mathbb{N}$. Then

(a) $\lim_{n \to \infty} (Ax_n + By_n) = Ax + By$;
(b) $\lim_{n \to \infty} x_n y_n = xy$;

(8) $x \neq 0 \Rightarrow \lim_{n \to \infty} \frac{1}{x_n} = \frac{1}{x}$;
(d) $(\exists K)(\forall n)(|x_n| < K)$;
(e) $\lim_{n \to \infty} h^n = 0$;
(f) $(\forall n)(|x_n| \leq \tau) \Rightarrow |x| \leq \tau$.

Proof of Proposition 48. (a) and (b) are proved in [1, page 271], (d) is evident since no set is cofinal with $\mathbb{N}$, (e) follows from Proposition 41 and (f) follows from (e).

For (c), since $\lim_{n \to \infty} x_n \neq 0$, there is a greatest $y \in \mathbb{N}$ such that $r_y(x_n)$ is not eventually zero. Thus, for sufficiently large $n$, $x_n = r_y(1 + h_n)$, where $h_n$ is infinitesimal, and, so, it suffices to establish the result for $x_n$ of the form $\frac{1}{1 + h_n}$.

Since $\frac{1}{1 + h_n} - 1 = -h_n(1 + h_n)^{-1}$ and $\lim_{n \to \infty} h^n = 0$ the coefficients of leaders in $h_n$ eventually vanish, and, as such, eventually vanish for $-h_n(1 + h_n)^{-1}$.

No admits an inductively defined exponential function $\exp : \mathbb{N} \to \mathbb{N}$ together with a natural interpretation of real analytic functions restricted to the finite (i.e. non-infinite) surreals that makes it a model of the theory of real numbers endowed with the exponential function $e^x$ and all real analytic functions restricted to a compact box. The exponential function, which was introduced by Kruskal, is developed in detail by Gonshor in [29, Chapter 10]. Norton and Kruskal independently provided inductive definitions of the inverse function $\ln$, but thus far only an inductive definition of $\ln$ for surreals of the form $\omega^y$ has appeared in print; see [29, page 161] and below. Nevertheless, since each positive surreal $x$, written in normal form, has a unique decomposition of the form

$$x = \omega^y r(1 + \varepsilon),$$

where $\omega^y$ is a leader, $r$ is a positive member of $\mathbb{R}$ and $\varepsilon$ is an infinitesimal, $\ln(x)$ may be obtained for an arbitrary positive surreal $x$ from the equation

$$\ln(x) = \ln(\omega^y) + \ln(r) + \ln(1 + \varepsilon),$$

where $\ln(\omega^y)$ is inductively defined by

$$\{ \ln(\omega^k) + n, \ln(\omega^y) - \omega^{y-r y}/n \} \ln(\omega^y) - n, \ln(\omega^y) + \omega^{y-r y}/n \},$$

and

$$\ln(1 + \varepsilon) = \sum_{k=1}^{\infty} \frac{(-1)^k \varepsilon^k}{k}.$$

Moreover, since $\ln$ is analytic, a genetic definition of $\ln(1 + z)$ for infinitesimal values of the variable can be provided in the manner discussed in detail in the work of Fornasiero [28, Pages 72-74], and sketched below in Definition 77.

Readers seeking additional background in the theory of surreal numbers may consult [1], [10], [23], [25], [29], [35] and [42].
4. Transseries. Écalle-Borel (EB) transseriable functions.

This section reviews some classical results in EB summability theory. Except for the conventions and notation it can be skipped by readers familiar with these notions. For more details on transseries see [13, 19, 11, 47, 5].

Borel and EB summability. EB summability applies to series of the form

$$\hat{f} := \sum_{k=-M}^{\infty} c_k x^{-(k+1)\beta}, \text{ Re } \beta > 0.$$  

Since the sum from $-M$ to 0 is finite we can assume without loss of generality that $M = 0$. Typically, the definitions assume that $\beta = 1$, and as we will see there is essentially no loss of generality in so doing.

4.1. Classical Borel summation of series.

Definition 49. The Borel sum of a formal series $\hat{f}$, denoted $f = \mathcal{B} \hat{f}$ (where $\mathcal{L}$ is the Laplace transform and $\mathcal{B}$ the Borel transform), exists when steps (ii) and (iii) in the following process can be carried out.

(i) Take the Borel transform $\hat{F} = \mathcal{B} \hat{f}$ of $\hat{f}$. ($\hat{F}$ is still a formal power series defined as the term-by-term inverse Laplace transform of $\hat{f}$: $\mathcal{B} \sum_{k=0}^{\infty} c_k x^{-(k+1)\beta} = \sum_{k=0}^{\infty} c_k p^{-\beta} - 1 / \Gamma((k + 1)\beta)$.)

(ii) Assuming $\mathcal{B} \hat{F}$ converges to $F$, analytically continue $F$ on $\mathbb{R}^+$ assuming this is possible.

(iii) Take the Laplace transform, $f = \mathcal{L} F$, provided exponential bounds exist for $F$, say, if $\exists \nu > 0$ such that $\sup_{x > \nu} |e^{-\nu x} F(x)| < \infty$.

For example,

$$\mathcal{L} \sum_{k=0}^{\infty} k! (-x)^{-k-1} = \mathcal{L} (1 + p)^{-1} = -e^x \text{Ei}(-x).$$

For further details on Borel sums, see [10].

If $\beta = 1$, then $\mathcal{B} \hat{f}$ is analytic at $p = 0$; otherwise it is ramified-analytic, $\mathcal{B} \hat{f} = 1/p A(p^{\beta})$ where $A$ is analytic. Given this trivial transformation, we will simply take $\beta = 1$.

Analyticity. Standard complex analysis arguments (e.g. combining Morera’s theorem with Fubini) show that $f = \mathcal{L} \mathcal{B} \hat{f}$ is real analytic for large $x \in \mathbb{R} \cap (\nu, \infty)$.

Definition 50. The power series

$$\sum_{k=-M}^{\infty} \frac{c_k}{x^{k+1}}$$

is Gevrey-one if there are $C, \rho > 0$ such that for all $k$, $|c_k| \leq k! C \rho^{-k}$.

Note 51. It is known [13] pages 104–109 that Borel summable formal series form a differential field, isomorphic to the field of Borel summed series. It is also known [16] Theorem 2 that Borel summable functions, as well as EB-summable functions originating in generic linear or nonlinear systems of ODEs or difference equations,

---

*We emphasize that the proofs in [16] do not use the origin of the transseries.*
have the property that the difference between the function \( f \) and its asymptotic series truncated to its least term \( c_k x^{-k} \) (\( k \) is dependent on \( x \), and is roughly \( k = \lfloor \rho x \rfloor \)) satisfies

\[
|f(x) - \sum_{k \in \ell(x)} c_k x^{-k}| \leq Ce^{-|\rho x| x^b}
\]

where \( C, \rho, b \) are constants that can be estimated relatively easily in concrete examples.

Inequality (10), when the constants are sharp, is the summation to the least term estimate; see Note 39. Summation to the least term, introduced by Cauchy and Stokes, was further developed by Berry [6], and later extended by Berry, Delabaere, Howls, Olde Daalhuis and others; for references, see [34]. Here, however, we only need the classical notion.

4.2. Transseries. Somewhat informally, transseries were already used in the late 19th century since they arise naturally in the study of differential equations. Let’s start with the simplest nontrivial differential equations in a neighborhood of \( x = 0 \):

\[
\begin{align*}
  a_1(x) y'' + (x^2 - x) y(x) + y(x) &= 0; \\
  a_1(x) &= 1, \quad a_2(x) = x^2, \quad a_3(x) = x^3.
\end{align*}
\]

To study the properties of (11) it is useful to first divide the equation by \( a_i \). When \( i = 1 \), the resulting equation has analytic coefficients, and by the general theory of ODEs, it has an analytic fundamental system of solutions at zero (which are entire since the equation has no singularities in \( \mathbb{C} \)). In particular, it can be solved by a convergent power series of the form,

\[
y = A \left( 1 - \frac{1}{2} z^2 - \frac{1}{12} z^4 + \cdots \right) + B \left( z - \frac{1}{12} z^4 - \frac{1}{120} z^6 + \cdots \right).
\]

When \( i = 2 \), \( z = 0 \) is a point where the coefficients are meromorphic, but not analytic. The order of the pole is sufficiently low and the singular point is regular in the sense of Frobenius. This being the case, there still exists a fundamental set of solutions as convergent series, though the powers are not necessarily integer anymore and, non-generically, logs may get mixed in. In our example, we obtain

\[
y = A \sqrt{z} \left( 1 - \frac{1}{2} z - \frac{1}{8} z^2 + \cdots \right) + B \left( z - \frac{1}{12} z^4 - \frac{1}{120} z^6 + \cdots \right).
\]

The situation changes abruptly when the order of the poles exceeds the order of the equation, that is, when \( i = 3 \). The singular point becomes irregular. Now the space of formal power series solutions is just one-dimensional, that is

\[
y = A \sum_{k=0}^{\infty} k! z^{k+1}
\]

whereas a second order ODE must have a two-dimensional space of solutions; furthermore, series (14) is divergent. The second family of solutions is not a power series at zero but rather

\[
y = B e^{-1/z}.
\]

The general solution is thus

\[
y = A \sum_{k=0}^{\infty} k! z^{k+1} + Be^{-1/z}.
\]
However, since the power series in \(14\) has radius of convergence zero, \(y\) in \(16\) –perhaps the simplest nontrivial transseries at zero– is now only a formal solution.

Imagine now that we have a singular nonlinear ODE. One of the simplest irregularly singular nonlinear ODE is obtained by the change of dependent variable \(h(z) = 1/y(z)\) in \(11\). Clearly, the general formal solution for the just-said equation is

\[
D \sum_{k=0}^{\infty} k! z^{k+1} + C e^{-1/z} =: D y + C_2 e^{-1/z}.
\]

If \(z > 0\), then \(e^{-1/z} \ll z\), and hence \(e^{-1/z} \ll y_1\). We let \(y_2 = y_1(z)/z\) where now \(y_2 = 1 + O(z)\), and further let \(\tilde{y} = 1/y_2\), which can be re-expanded as a power series at zero; moreover, we can further expand, simply using the geometric series,

\[
\frac{D}{y_1 + C e^{-1/z}} =: \frac{D \tilde{y}}{z(1 + C_2 e^{-1/z} \tilde{y})} = \frac{D}{z} \left( \tilde{y} - C_2 \tilde{y}^2 e^{-1/z} + C_2^2 \tilde{y}^3 e^{-2/z} + \cdots \right) = z^{-1} \sum_{k=0}^{\infty} \tilde{y}_j C^j e^{-j/z}
\]

where \(\tilde{y}_j\) are formal power series. This is a more general level one transseries at \(0^+\).

**Definition 52** (Informal definition). A transseries with generators \(\mu_1, \mu_2, \ldots, \mu_n\) considered as “variables \(\ll 1\)” is a formal sum

\[
\sum_{k_1, k_2, \ldots, k_n > -M} c_{k_1, k_2, \ldots, k_n} \mu_1^{k_1} \mu_2^{k_2} \cdots \mu_n^{k_n} =: \sum_{k > -M} c_k \mu^k.
\]

**Note 53.** It is customary to work with the case when the variable tends to \(\infty\) rather than to 0. This is achieved simply by taking \(x = 1/z\). One reason for adopting this convention is that equations arising in practice most often have their worst singularities at \(\infty\). Other reasons relate to algebraic simplicity. For instance, repeated differentiation of \(e^{-1/z}\) obviously leads to more complicated expressions than the repeated differentiation of \(e^{-x}\).

**Example.** Note that \(18\) is a transseries with generators \(\mu_1 = z\) and \(\mu_2 = e^{-1/z}\). Also note that \(1\) has generators \(\mu_1 = 1/x, \ldots, \mu_j = x^{\beta_j} e^{-\lambda_j x}\).

The survey paper [19] is an excellent self-contained introduction to transseries and their topology, with connections to normal forms of surreal numbers, as well as ample references.

**Definition 54.** The transseries topology (see [13, 19]) is defined by the following convergence notion. Let \(\sum_{k > -M} c_k^{[m]} \mu^k\) be a sequence of transseries, where the superscript \([m]\) designates the \(m\)th element of the sequence and \(c_k^{[m]}\) designates the sequences of coefficients of the \(m\)th element. Then,

\[
\lim_{m \to \infty} \sum_{k > -M} c_k^{[m]} \mu^k = \sum_{k > -M} c_k \mu^k\text{ if and only if }\forall k \exists M \text{ such that } \forall m > M, c_k^{[m]} = c_k,
\]

i.e., if and only if all the coefficients eventually become those of the limit transseries (rather than merely converge to them).
Definition 55 (Borel summable transseries). Rewriting (4) as

\[ \tilde{T} = \sum_{k \geq k_0, l \in \mathbb{N}} c_{k,l} x^{\beta k} e^{-k \lambda x} x^{-l} = \sum_{k \geq k_0} x^{\beta k+1} e^{-k \lambda x} \tilde{y}_k(x) \]

where \( \tilde{y}_k(x) = \sum_{l=0}^{\infty} c_{k,l} x^{-l-1} \) are formal power series, we say that the transseries \( \tilde{T} \) is Borel-summable if there is a \( k \)-independent exponential bound (the same constant \( \nu \) in Definition 49 for all \( k \)) such that all the \( \tilde{y}_k(x) \) satisfy Definition 49 and there are \( A, B, r > 0 \) such that for all \( \Re x > r \) we have \( LB \tilde{y}_k(x) \leq AB^k \). When the conditions above are met, \( \tilde{T} \) is called a Borel summable transseries, and \( LB \tilde{T} \) is its Borel sum.

These estimates imply

Lemma 56.

\[ T := LB \tilde{T} = \sum_{k \geq k_0} x^{\beta k+1} e^{-k \lambda x} LB \tilde{y}_k = \sum_{k \geq k_0} x^{\beta k} e^{-k \lambda x} y_k \]

is a function series converging to an analytic function for \( |x^{\beta k} e^{-k \lambda x}| < 1/B \).

4.3. EB summation of series. Even in simple cases classical Borel summation fails because of singularities. For instance \( LB \sum_{k=0}^{\infty} k! x^{-k-1} = \mathcal{L}(1-p)^{-1} \), which is undefined as presented.

\'{E}calle introduced significant improvements over Borel summation. Among them are the concepts of critical times and acceleration/deceleration to deal with mixed powers of the factorial divergence. Last but not least, and the only additional ingredient we will need, is that of averaging.

In linear problems, to avoid the singularities of the integrand on \( \mathbb{R}^+ \) (when present), one can take the half-half average of the Laplace transforms above and below \( \mathbb{R}^+ \). In nonlinear problems, on the other hand, the average of two solutions is not a solution. Though \'{E}calle found constructive, universal and simple looking averages, notably the Catalan averages, which successfully replace the naive half-half averages mentioned above, it is altogether nontrivial to construct them and show that they work \[32\]. Invoking Borel transforms followed by analytic continuation along paths avoiding singularities, followed by taking the Catalan averages of these continuations and finally applying the Laplace transforms yields a differential field algebra \[32\].

\'{E}calle introduced other important improvements in asymptotics, such as general transseries, extending EB summation to general transseries, and \'{E}calle cohesive continuation, which allows for a good continuation through “natural natural” boundaries.

Proposition 57. Borel summation and more generally EB summation (see Definition 58) is a differential algebra isomorphism between level one transseries arising in nonlinear systems of ODEs under the assumptions of \[12\] and actual functions in \( \mathbb{C} \) or \( \mathbb{R} \).

Proof of Proposition 57. The Catalan average, \[32\] \( \text{§IV 2, page 604} \), an average in the Borel plane (the space obtained after taking the Borel transform) commutes with convolution and preserves lateral growth (the exponential bounds needed to take Laplace transforms), entailing the commutation of EB summation of a formal series with multiplication. The isomorphism between EB summable series and
functions thus follows directly from [32]. For transseries, we note that the Catalan average and the balanced average used in ODEs (cf. [14]) coincide. Indeed, the first pair of Catalan weights is simply $\frac{1}{2}, \frac{1}{2}$, the same as those of balanced averages, and the first pair $\frac{1}{2}, \frac{1}{2}$ determines all the other weights of a balanced average (cf. [14]). The isomorphism between balanced-Borel summable transseries and their sums is proved in [14], under assumptions more general than those in [12].

\[\square\]

**Definition 58.** By abuse of language, a Borel summable function is understood to be a function that is equal to the Borel sum of its own asymptotic series. Following Écalle, functions that are EB sums of their own asymptotic series or transseries are called analyzable. Borel summable functions are in particular analyzable.

### 4.4. General conditions for level-one EB summable transseries in ODEs.

To apply the same procedure for general solutions of nonlinear ODEs at irregular singular points, we rely on the transseries obtained in [12, 14].

As is well known, by relatively simple algebraic transformations, a higher order differential equation can be turned into a first order vectorial equation (differential system) and vice versa [8]. The vectorial form has some technical advantages.

Furthermore, as discussed in [13], the equations of classical functions, Painlevé equations and others are amenable to the form:

\[
y' = f(x, y) \quad y \in \mathbb{C}^n
\]

under the following assumptions:

(a1) The function $f$ is analytic at $(\infty, 0)$;

(a2) A condition of nonresonance holds: the numbers $\arg \lambda_j, j = 1, ..., n$ are distinct, where $\lambda_j$, all of which are nonzero, are the eigenvalues of the linearization

\[
\hat{\Lambda} := -\left( \frac{\partial f_i}{\partial y_j}(\infty, 0) \right)_{i,j=1,2,...,n}.
\]

We impose, in fact, a stronger condition, namely the condition in the paragraph containing (26) below. Writing out explicitly a few terms in the expansion of $f$, relevant to leading order asymptotics, we get

\[
y' = f_0(x) - \hat{\Lambda}y + \frac{1}{x} \hat{\Lambda}y + g(x, y)
\]

where $g$ is analytic at $(\infty, 0)$ and $g(x, y) = O(x^{-2}, |y|^2, x^{-2}y)$.

**Nonresonance.** For any $\theta > 0$, denote by $\mathbb{H}_\theta$ the open half-plane centered on $e^{i\theta}$. Consider the eigenvalues contained in $\mathbb{H}_\theta$, written as a vector $\hat{\lambda} = (\lambda_{i_1}, ..., \lambda_{i_n})$; for these $\lambda$s we have $\arg \lambda_i - \theta \in (-\pi/2, \pi/2)$.

We require that for all $\theta$, the numbers in the finite set

\[
\{N_{j, k} = \lambda_j - k \cdot \hat{\lambda} : N_{j, k} \in \mathbb{H}_\theta, k \in \mathbb{N}^n, j = i_1, ..., i_n\}
\]

have distinct complex arguments.

Let $d_{i, k}$ be the direction of $N_{i, k}$, that is the set $\{z : \arg z = \arg(N_{i, k})\}$. We note that the opposite directions, $-d_{i, k}$ are Stokes rays along which the Borel transforms are singular.

---

They are sometimes called Stokes lines, and often, in older literature, antistokes lines.
Lemma 60. There is a linear antiderivative on restricted-domain functions follow straightforwardly from the 0-extension ones.

Proof. If \( A \) is a linear antiderivative \( (A(f))'(x_R) = f(x_R) \) for all \( x_R \in \mathbb{R} \).

Proof. Let \( I \subset \mathbb{R} \) be a bounded interval and \( I^* \) be the corresponding interval in \( V \). Also let \( x \in I^* \). Then there is a unique \( x_R \in I \) such that \( x - x_R \) is infinitesimal. Moreover, \( x_R = \inf\{y \in I : y > x\} = \sup\{y \in I : y < x\} \). We define \( A_T(f)(x) = f(x_R)(x - x_R) \) for all infinitesimal \( x - x_R \). Then \( A_T \) is linear, and \((A_T(f))' = f^* \). For \(|x| > \infty \) define \( f^*(x) = 0 \) and \( A_T f(x) = 0 \).

Note that the following proofs of Lemma 60 and Theorem 5 do not require the axiom of choice.

Lemma 61. Let \( W \) be a subspace of \( T[\mathbb{R}, \mathbb{R}] \), \( W_1 \) be a subspace on which there is a linear antiderivative \( A_1 \) and suppose \( P \) is a projector on \( W_1 \). Then there exists an antiderivative \( A \) on \( T[\mathbb{R}, \mathbb{R}] \) that extends \( A_1 \).

Proof. We simply write (uniquely) \( f \in T([\mathbb{R}, \mathbb{R}]) \) as \( f = Pf + (1 - P)f = f_1 + f_2 \). We then simply check that \( A(f) := A_1(f) + A_T(f) \) has the desired properties; see Lemma 59.

Note 61. If \( W_1 \) is finite-dimensional, the existence of \( P \) does not depend on the axiom of choice.

Proof of Theorem 3. This follows from Lemma 59 and the concrete choice of \( P \). We take \( P \) to be the Lagrange interpolation polynomial with nodes \( x_i = i \) for \( i = 0, ..., n \):

\[
L(f)(x) = \sum_{j=0}^{n} f(j) \ell_j(x); \quad \ell_j(x) := \prod_{0 \leq m \leq k, m \neq j} \frac{x - m}{j - m}
\]

\( L(P) = P \) for any \( P \) of degree \( < n + 1 \), by the uniqueness of such a polynomial passing through \( n + 1 \) points (cf. [1]).

Proof of Theorem 4. This follows from Lemma 59 with \( W_1 = \{f \in T[\mathbb{R}, \mathbb{R}] : \exists F \in T[\mathbb{N}] \text{ such that } F \text{ is differentiable and } (F|_{\mathbb{R}})' = f \} \) and the linear choice \( A(f) = F(x) - F(0) \) for \( f \in W_1 \). The existence of such a projector uses standard linear algebra and the axiom of choice.

Note 62. One can add a finite set of other “good” functions with known antiderivatives, if they exist. The construction of the Lagrange interpolation polynomial amounts to solving systems of linear equations which, for distinct nodes, have nonzero Vandermonde determinants. For other functions, the nodes have to be chosen carefully. We do not pursue this, as \( e^x \) and \( e^{2x} \), for example, count as distinct functions, and a finite number of these would not be a significant improvement.
4.5.1. Reduction to the case when the domain is $(x_0, \infty)$.

(1) The functions for which we obtain “good” positive results are analytic except at the endpoints (which we understand to include $\pm \infty$).

(2) Assuming $f : (a, b) \to \mathbb{R}$ is analytic on $(a - \varepsilon, b)$, we ask whether it can be extended to $(a, b) \in \mathbb{No}$. If $b < \infty$ is a point of analyticity, then it is known that an extension exists, using the power series centered at $b$ [20]. Otherwise, the change of variables $g(x) = f((b - x)/(x - a + 1))$ reduces the question to extending $g$ to positive infinitesimal $x$, which by the further transformation $g(x) = h(1/x)$ maps the problem to extensions from some $x_0 \in \mathbb{R} \cup \{-\infty\}$ past the gap at $+\infty$. Similar reasoning applies to antiderivatives.

(3) For the just-said reason, without loss of generality, we will assume that the intervals of interest are of the form $(x_0, \infty), x_0 \in \mathbb{R} \cup \{-\infty\}$, that the singularity is at $+\infty$ and we will seek extensions and integrals to positive infinite $x \in \mathbb{No}$.

4.6. Details of the proof of Theorem [12] Let

\[ \nu_\varepsilon := \nu = 2\|s\|_\infty \pi^{-\frac{1}{2}} \varepsilon^{-1} \]

and consider the mollification $f_{s, \varepsilon}(x) := \int_{-\infty}^{\infty} e^{-\nu^2 (x - t)^2} l_s(t) \, dt$. By standard complex analysis, $f_{s, \varepsilon}$ is entire, and straightforward estimates show that $\sup_{x \in \mathbb{C}} |e^{-\nu^2 |z|^2} f_{s, \varepsilon}(z)| < \infty$. Note that, by construction, $\sup_{(t, x) \in \mathbb{R}^2} |l_s(t) - l_s(x)| \leq 2\|s\|_\infty |t - x|$. Thus, (27) implies

\[ |f_{s, \varepsilon}(x) - l_s(x)| = \pi^{-\frac{1}{2}} \nu \int_{-\infty}^{\infty} e^{-\nu^2 (x - t)^2} (l_s(t) - l_s(x)) \, dt \]
\[ \leq 2\|s\|_\infty \pi^{-\frac{1}{2}} \nu \int_{-\infty}^{\infty} e^{-\nu^2 |t|^2} |t| \, dt = 2\|s\|_\infty \pi^{-\frac{1}{2}} \nu^{-1} \leq \varepsilon. \]

Hence $|F_{s, \varepsilon}(x) - L_s(x)| \leq \int_0^\infty |f_{s, \varepsilon} - l_s| \, dv < \varepsilon$. Note that $\inf_j s_j \leq L_s(x) \leq \sup_j s_j$. Since $E$ is nonnegative, $\inf_j s_j - \varepsilon \leq \omega^{-1}(E(F_{s, \varepsilon})(\omega)) \leq \sup_j s_j + \varepsilon$. Note also that $|L_{TS}(x) - L_s(x)| \leq 2\|s\|$. This implies $\pm |F_{s, \varepsilon} - F_{s, \varepsilon} - L_{TS} + L_{TS} - L_s + L_s - F_{s, \varepsilon}| \leq 2\|s\| + 2\varepsilon$. Hence $\Re (\omega^{-1} E(F_{s, \varepsilon} - F_{s, \varepsilon})(\omega)) < 2\varepsilon$. Since $l_s$ is $\varepsilon$-independent, an $\varepsilon/2$ argument and (28) imply $|F_{s, \varepsilon} - F_{s, \varepsilon}| \leq 2|\varepsilon| - \varepsilon|x|$. Then $\varphi(s) := \lim_{\varepsilon \to 0} \Re (\omega^{-1} E(F_{s, \varepsilon})(\omega))$ exists, is linear, shift invariant, and lies between $\inf_j s_j$ and $\sup_j s_j$.

5. Descriptive set-theoretic results

In this section we prove a result of mathematical logic that is instrumental for establishing some of our more general negative results from [22] and [23]. We apply it to obtain negative results on extensions and integrals of familiar analytic functions. The bridge between mathematical foundations, in this higher generality, and analysis is illustrated in the proof of Theorem [21]. As was mentioned in [22], we show that without stringent conditions governing the behavior of functions at $\infty$, extension or integration functionals exist only in “trivial cases”. Indeed, based on growth conditions alone, in the class of analytic functions outside some ball bounded by some $w$ in $\mathbb{C}$ they exist if and only if $W$ is polynomially bounded, in which case the functions are just polynomials. Moreover, there is no analog of $L^1$ in
No in the sense that even rapidly decaying entire functions do not suffice to ensure the existence of a linear antiderivative to some \( \theta > \infty \).

5.1. Positivity sets. The most primitive form of our negative results concern what we call positivity sets in the Cantor space \( \{-2,-1,0,1,2\}^\mathbb{N} \).

**Definition 63.** \( X \) is the Cantor space \( \{-2,-1,0,1,2\}^\mathbb{N} \). A positivity set in \( X \) is an \( S \subseteq X \) such that the following holds for all \( x, y, z \in X \).

i. If \( x + y = z \) and \( x, y \in S \), then \( z \in S \).

ii. If \( x + y = z \) and \( x, y \notin S \), then \( z \notin S \).

iii. Suppose \( x \in X \) is eventually zero. Then \( x \in S \) if and only if \( x \) has a positive last nonzero term.

**Theorem 64.** i. \( \text{NBG}^- + \text{DC} \) proves that there is no Borel measurable positivity set in \( X \).

ii. \( \text{NBG}^- + \text{DC} \) proves that if there exists a positivity set in \( X \), then there is a set of reals that is not Baire measurable.

iii. \( \text{NBG}^- + \text{DC} \) does not prove that there exists a positivity set in \( X \).

iv. There is no set-theoretic description with parameters such that \( \text{NBG} \) proves the following. There is an assignment of real numbers to the parameters such that the description uniquely defines a positivity set in \( X \). This also holds for extensions of \( \text{NBG} \) by standard large cardinal hypotheses.

**Proof.** For i and ii, we argue in \( \text{NBG}^- + \text{DC} \), and fix a positivity set \( S \subseteq X = \{-2,-1,0,1,2\}^\mathbb{N} \). We will show that \( S' = S \cap \{-1,0,1\}^\mathbb{N} \) is not Baire measurable in the Cantor space \( \{-1,0,1\}^\mathbb{N} \). Suppose \( S' \) is Baire measurable in \( \{-1,0,1\}^\mathbb{N} \). Let \( V \subseteq \{-1,0,1\}^\mathbb{N} \) be open, where \( V \Delta S' \) is meager.

**Lemma 65.** \( V \neq \emptyset \).

**Proof.** Suppose \( V = \emptyset \). Then \( S' \) is meager in \( \{-1,0,1\}^\mathbb{N} \). Hence any bicontinuous permutation of \( \{-1,0,1\}^\mathbb{N} \) sends \( S' \) onto a meager set in \( \{-1,0,1\}^\mathbb{N} \). The finite (even countable) intersection of comeager sets in \( \{-1,0,1\}^\mathbb{N} \) is comeager in \( \{-1,0,1\}^\mathbb{N} \), and by the Baire category theorem, comeager sets in \( \{-1,0,1\}^\mathbb{N} \) are nonempty. We use the six bicontinuous permutations of \( \{-1,0,1\}^\mathbb{N} \) that act coordinatewise and are the identity at coordinates after the first coordinate, and the six bicontinuous permutations of \( \{-1,0,1\}^\mathbb{N} \) that act coordinatewise and are the minus function at coordinates after the first coordinate. There must be an element of \( \{-1,0,1\}^\mathbb{N} \setminus S' \) whose image under all twelve of these bicontinuous permutations lies in \( \{-1,0,1\}^\mathbb{N} \setminus S' \). In particular, let \( f_1, f_2, f_3, g_1, g_2, g_3 \in \{-1,0,1\}^\mathbb{N} \setminus S' \), where \( f_1, f_2, f_3 \) agree on \( \mathbb{N} \setminus \{0\} \), \( g_1, g_2, g_3 = -f_1 \) on \( \mathbb{N} \setminus \{0\} \) and \( f_3(0), f_2(0), f_3(0), g_1(0), g_2(0), g_3(0) \) are \(-1,0,1,-1,0,1\), respectively. Then \( f_2 + g_3 = (1,0,0,...) \in \{-1,0,1\}^\mathbb{N} \), and so \( f_2 + g_3 = (1,0,0,...) \in \{-1,0,1\}^\mathbb{N} \setminus S' \), and has a positive last nonzero term, contradicting iii in the definition of positivity set.

**Lemma 66.** \( S' = S \cap \{-1,0,1\}^\mathbb{N} \) is not Baire measurable in the Cantor space \( \{-1,0,1\}^\mathbb{N} \).

**Proof.** Since \( V \neq \emptyset \), let \( \alpha = (\alpha_0,...,\alpha_k) \in \{-1,0,1\}^{k+1} \) be such that every \( f \in \{-1,0,1\}^\mathbb{N} \) extending \( \alpha \) lies in \( V \). We work in the Cantor space \( T(\alpha) = \{ f \in \{-1,0,1\}^\mathbb{N} : f \text{ extends } \alpha \} \). Since \( V \Delta S' \) is meager, \( (V \cap T(\alpha)) \Delta (S \cap T(\alpha)) \) is meager in \( T(\alpha) \). That is, \( T(\alpha) \Delta (S \cap T(\alpha)) \) is meager in \( T(\alpha) \), and so \( S^* = S \cap T(\alpha) \) is
comeager in $T(\alpha)$. Using coordinatewise bicontinuous bijections of $T(\alpha)$ as before, let $f_1, f_2, f_3, g_1, g_2, g_3 \in S^r$, where $f_1, f_2, f_3$ agree on $\{k + 2, k + 3, \ldots\}$, $g_1, g_2, g_3 = -f_1$ on $\{k + 2, k + 3, \ldots\}$, and $f_1(k+1), f_2(k+1), f_3(k+1), g_1(k+1), g_2(k+1), g_3(k+1)$ are $-1, 0, 1, -1, 0, 1$, respectively. Then $f_1 + g_1 = (2\alpha_0, \ldots, 2\alpha_k, -2, 0, 0, \ldots) \in X$, and so $f_1 + g_1 = (2\alpha_0, \ldots, 2\alpha_k, -2, 0, 0, \ldots) \in S$ and has a negative last nonzero term, contradicting iii in the definition of positivity set. \hfill \Box

We have thus shown that $S \cap \{-1, 0, 1\}^N$ is not Baire measurable in $\{-1, 0, 1\}^N$, and hence not Borel measurable. It is immediate that $S \subseteq \{-2, -1, 0, 1, 2\}^N$ is not Borel measurable, establishing i.

Now since $\{-1, 0, 1\}^N$ and $\{0, 1\}^N$ are homeomorphic, there is a non Baire measurable set in $\{0, 1\}^N$. As in [39], let $T \subseteq \{0, 1\}^N$ be the set that results from removing from $\{0, 1\}^N$ the elements that are eventually constant. Then $T$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q} \subseteq \mathbb{R}$. Also since we have removed only countably many points from $\{0, 1\}^N$, there is a subset of $T$ that is not Baire measurable in $T$. Hence there is a subset of $\mathbb{R} \setminus \mathbb{Q}$ that is not Baire measurable in $\mathbb{R} \setminus \mathbb{Q}$. Hence there is a subset of $\mathbb{R}$ that is not Baire measurable in $\mathbb{R}$. This (well-known argument, already given in [39]) establishes ii.

**Lemma 67.** NBG$^- +$ DC + “all sets of reals are Baire measurable” is consistent.

**Proof.** Since NBG$^- +$ DC is conservative over ZFDC, it suffices to prove this with ZFDC. See Solovay [43] and Shelah [41]. Solovay relies on the consistency of ZFC + “there exists a strongly inaccessible cardinal”, whereas Shelah later only relies on the consistency of ZFC. \hfill \Box

We have thus established iii using ii.

**Lemma 68.** NBG $+$ “every set of reals set-theoretically definable with real parameters is Baire measurable” is consistent. Here the statement in quotes is formulated as a scheme. This holds even if NBG is augmented with standard large cardinal hypotheses (assuming that they are consistent).

**Proof.** This is proved in Solovay [43] and Shelah [41] with NBG replaced by ZFC. This is equivalent in light of the conservativity of NBG over ZFC. \hfill \Box

We now establish iv. We use ZFC below in light of the conservativity of NBG over ZFC.

Let $\varphi(x, v_1, \ldots, v_k)$ be a formula of ZFC, and assume that ZFC proves

1) $\exists v_1, \ldots, v_k \in \mathbb{R})((\exists! x)(\varphi(x, v_1, \ldots, v_k)) \land (\exists x)(\varphi(x, v_1, \ldots, v_k) \land x$ is a positivity set in $X))$.

By Lemma 68, let $M$ be a model of $T +$ “every set of reals set theoretically definable with real parameters is Baire measurable”, where $T$ is either ZFC or an extension of ZFC with standard large cardinal hypotheses. Since $M$ satisfies ZFC, 1 holds in $M$. Working in $M$, let $x$ be a positively set $S \subseteq X$ that is set-theoretically definable with real parameters. From the proof of ii, we:

a. Extract a set that is not Baire measurable in $\{-1, 0, 1\}^N$ from $S$.
b. Extract a set that is not Baire measurable in $\{0, 1\}^N$ from a.
c. Extract a set of reals that is not Baire measurable from b.

Thus in $M$, we have a set of reals, definable with real parameters, that is not Baire measurable. This contradicts the choice of $M$. \hfill \Box
5.2. **Proof of Theorem 24**

In this subsection we prove Theorem 24, using the positivity sets of Theorem 64 to conclude that the space of real-valued entire functions cannot be naturally extended to the surreals. This is based on a fixed convenient power series, which we view as a particularly transparent toy model in advance of our main negative results.

In the next subsection we show that, in order to naturally extend families of real-valued entire functions past $\infty$ into the infinite surreals, strong conditions governing the behavior of the functions at $\infty$ is required.

We begin by recalling the definitions of extension functional and antiderivation functional introduced in the Introduction.

Let $T[\mathbb{R}, \mathbb{R}]$ be the real vector space of all $f : \mathbb{R} \to \mathbb{R}$ and $E$ be defined by (1).

We now apply Theorem 64 to establish the following negative results for the space $E$ making use of Definitions 22 and 23 of $\theta$-extension and $\theta$-antiderivation functionals.

(a) We use the following map $\rho : X \to E$, where $X = \{-2, -1, 0, 1, 2\}$ as in §5.1. For $\xi \in X$, $\rho(\xi)$ is the real-valued entire function given by the infinite radius of convergence power series

\[
\sum_{n \geq 0} \frac{\xi^n}{n!}.
\]

The $n!$ above can be replaced by larger convenient expressions, resulting in slower rates of growth. Let $L_f$ be a $\theta$-extension or $\theta$-antiderivation operator on $E$, where $\theta$ is infinite.

(b) Let

\[S = \{\xi \in X : L_f(\rho(\xi)) > 0\} \text{.}\]

If $\xi + \eta = \zeta$ from $X$, then $L_f(\rho(\xi)) + L_f(\rho(\eta)) = L_f(\rho(\zeta))$. Hence if, furthermore, $\xi, \eta \in S$, then $\zeta \in S$.

Also if, furthermore, $\xi, \eta \notin S$, then $\zeta \notin S$. Also, if any $\xi \in X$ has a positive last nonzero term then $\rho(\xi)$ is a polynomial with a positive leading coefficient. Hence $L_f(\rho(\xi))$ is $P(\theta)$ or $A_0P(\theta)$, where $A_0P$ is the antiderivative of $P$ with constant term 0. Hence in either case, $L_f(\rho(\xi)) > 0$, and so $f \in S$. Therefore $S$ is a positivity set.

We now complete the proof of Theorem 24. To obtain i, ii, we simply cite Theorem 64 ii, iii. To obtain iii, suppose we have such a set-theoretic description in iii with provability in NBG. Then we obtain a corresponding set-theoretic description in Theorem 64 iv with provability in NBG, contradicting Theorem 64 iv. As in Theorem 64 iv, we can use NBG augmented by standard large cardinal hypotheses.

6. **Proof of Theorem 28**

The forward direction (a) implies (b) is immediate. We now show that (b) implies (a).

The essence of the proof is to work in NBG, starting with any weight $W$ and three descriptions as in (b), without regard to the NBG provability condition in
We give an explicit construction, from these three givens, of a set $X$ and prove in NBG that if $\mathcal{E}_W$ does not consist entirely of polynomials, then $X$ is a positivity set. This explicit construction is given below.

We now show that (b) implies (a), given this explicit construction. Let $W$ be as given. Assume (b). Since NBG\textsuperscript{−} is a conservative extension of ZF, it suffices to show that ZF proves that $\mathcal{E}_W$ (see Definition 20) consists entirely of polynomials. By set-theoretic absoluteness, it suffices to show that ZFC proves that $\mathcal{E}_W$ consists entirely of polynomials. Suppose this is false, and let $M$ be a countable model of ZFC in which $\mathcal{E}_W$ does not consist entirely of polynomials. Let $M'$ be an extension of $M$ satisfying NBG in which all sets of reals set-theoretically definable with real parameters have the property of Baire. Let $M'$ be an extension of $M'$ satisfying NBG in which all sets of reals set-theoretically definable with real parameters have the property of Baire, and in which $\mathcal{E}_W$ does not consist entirely of polynomials.

We now make the explicit construction given below in $M'$. We obtain a positivity set which, in $M$, is set-theoretically definable with real parameters, and therefore has the property of Baire. This contradicts Theorem 64.

On the analytic side, the first step consists of constructing a rich enough family of functions of a given rate of growth. This is achieved, as illustrated in the previous section, by taking a sample function $f$ with positive Taylor coefficients and the prescribed rate of growth, and generating from it a set of functions, belonging to the same growth rate class, obtained by multiplying, in all possible ways, the coefficients of the sample function with $-2, -1, 0, 1, 2$. This is an analytic analog of the space $f^{\infty}$ of [15]; in the analytic case $f^{\infty}$ was generated via Weierstrass products whereas here the similar space $f^{\infty}$ is generated by manipulating power series as illustrated in the previous section.

**Proposition 69.** For any weight $W$ such that $\mathcal{E}_W$ contains non-polynomial entire functions, there is a $\varphi_W \in C^\omega(\mathbb{C})$ with strictly positive Taylor coefficients at zero (which is thus not a polynomial), explicitly constructed from $W$, such that $\varphi_W \in \mathcal{E}_W$, and a positivity set explicitly constructed out of $W$.

**Proof.** Let $b_k = \inf_{x \geq 1} x^{-k}W(x)$. We first show that $b_k > 0$. To get a contradiction assume that $b_k = 0$ for some $k$. Since $W > 0$ this means that there exists a sequence $(r_j)_{j \in \mathbb{N}}$ such that $\lim_{j \to \infty} r_j = 0$. Let $f \in \mathcal{E}_W$. By Cauchy’s formula applied on circles of radii $r_j$ it follows that $f^{(n)}(0) = 0$ for all $n > k$, i.e., $f$ is a polynomial.

Let $a_k = b_k 2^{-k}$ and $\varphi(z) = \sum_{k=0}^{\infty} a_k z^k$. Since $b_k > 0$, we have $\varphi(z) \leq \varphi(|z|)$. Let $|z| = \rho$. Then, for all $\rho > 0$ we have $\varphi(\rho) = \sum_{k=0}^{\infty} W(\rho) 2^{-k} \rho^k = W(\rho)$. In particular, (32) holds with $C = 1$, and the series of $f$ converges absolutely (in the classical sense) for any $z$.

**Definition 70.** If $c \in \{-2, -1, 0, 1, 2\}^\mathbb{N}$ and $\varphi(z) = \sum_{k=0}^{\infty} c_k z^k$, then we define

$$(c * \varphi)(z) = \sum_{k=0}^{\infty} c_n a_n z^n.$$

We also define

$$(30) \quad \varphi^c = \{c * \varphi : c \in X\}; \quad X := \{-2, -1, 0, 1, 2\}^\mathbb{N}.$$

**Note 71.** For any $c \in \{-2, -1, 0, 1, 2\}^\mathbb{N}$, $\sup_{z \in \mathbb{C}} |c * \varphi|(z) \leq 2W$ and thus for any $c \in \{-2, -1, 0, 1, 2\}^\mathbb{N}$, $c * \varphi \in \mathcal{E}_W$. 
The next step is to create a positivity set in the sense of Definition 63 from $\varphi$. Let $\theta > \infty$. Then,

$$c^+ := \{ c \in X : c \ast \varphi(\theta) > 0 \}$$

is a positivity set. Indeed, a polynomial with a positive real leading coefficient is manifestly positive at $\theta$, and the two linearity conditions are immediate.

The rest of the proof of part (a) of Theorem 28 shadows the one in (b) of §5.2. □

Note 72. The conclusion is, essentially, that for extensions and antiderivatives to exist, $W$ should be such that only polynomials are allowed.

7. Proof of Theorem 32

This is similar to the other negative proofs: Let $\tilde{W}(x) = x^{\ln x}$ (superpolynomial but sub-exponential), let $\varphi_{\tilde{W}}$ be as in Proposition 69 and $\varphi_{\tilde{W}}$ be as in (30). Let $A$ be an antidifferentiation functional on $e^{-W} \varphi_{\tilde{W}}$. Note that $e^{-W} x^n \in e^{-W} \varphi_{\tilde{W}}$. Straightforward verification shows that $\{ \xi \in X : Af(\theta) < 0 \}$ is a positivity set on $X$. The proof continues in the same way a that of Theorem 28.

Note 73. Our negative results rule out natural ways of going beyond infinity that do not require further details about the nature of the singularity at infinity.

For part iii, once more we mimic the proof of Theorems 18 and 19 in [15].

8. Proof of Theorem 38

8.1. The class $F_a$. Existence of extensions.

Definition 74 (Extension by analytic continuation in the finite domain). Let $f$ be real-analytic on $[a, \infty) \subset \mathbb{R}^+$, meaning that for any $x_0 \in (a, \infty)$ the series

$$f(x_0 + h) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} h^k =: \sum_{k=0}^{\infty} c_k h^k$$

has a positive radius of convergence.

8.1.1. Extensions for finite $x \in \mathbb{N}$. Using the fact that we can uniquely write $x = x_0 + s$, where $x_0 \in \mathbb{R}$ and $|s| \ll 1$, we define the extension $E(f)$ on $(a, \infty) \subset \mathbb{N}$ by

$$E(f)(x_0 + s) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} s^k =: \sum_{k=0}^{\infty} c_k s^k.$$ 

8.1.2. Extensions for $x > \infty$.

Note 75. Taking $z = x^{-1/k}$ in (39), the functions in $F_a$ have series of the form

$$P(1/z) + \sum_{k=1}^{\infty} c_j z^j$$

where $\sum_{k=0}^{\infty} c_j z^j$ has a positive radius of convergence and $P$ is a polynomial. Thus the question of extensions on $F_a$ reduces to extending analytic functions. Note that, of course, (34) is also absolutely convergent in the sense of Conway.

Lemma 76. The extension operator $E$ is linear. Moreover, $E$ has a genetic definition.
Proof. The linearity of $E$ is immediate. The existence of an equivalent genetic definition is known and is discussed in detail in the work of Fornasiero [26, Pages 72-74], a sketch of which is contained in the following definition.

**Definition 77.** (i) In [26], the definition of the extension is given by an expression of the form

$$\{S^L|S^R\}$$

where $S^L$, $S^R$ are left (resp. right) options in the list of Taylor polynomials

$$p_n(x, x^o) = \sum_{j=0}^{n} \frac{f^{(j)}(x^o)}{j!}(x - x^o)^j$$

where $x^o$ are left or right options for $x$. We can assume without loss of generality that $f$ is not a polynomial, otherwise the extension is trivial. Let $N = \min\{j > n : f^{(N)}(x^o)\neq 0\}$. Then, $p_n \in S^L$ if $f^{(N)}(x^o)(x - x^o)^N > 0$ and $p_n \in S^R$ otherwise.

(ii) We will use expressions of the form (35) also in the following setting: $x$ is infinite, $(x - x^o)$ is finite and $f^{(j)}(x^o)/f^{(j+1)}(x^o)$ is infinitesimal.

$\square$

8.2. The class $F_a$. Existence of antiderivatives.

8.2.1. Antiderivatives for finite $x \in \text{No}$. Let $f$ be a real analytic function and $F$ be a real antiderivative of $f$. Of course $F$, the antiderivative of $f$, is also real analytic, and the problem of defining the integral of $f$ reduces to that of the extension of the real analytic function $F$, since we define the surreal integral of $f$ between finite $a, x \in \text{No}$ by

$$\int_a^x f(t)dt = E(F)(x) - E(F)(a).$$

The right side of (39) is defined in the previous section, and it is straightforward to check that (39) provides an integral, with finite endpoints, having the properties specified in Proposition 34, where the positivity of the integral holds for finite surreal numbers.

8.2.2. Antiderivatives for $x > \infty$. Here we use series (3) from §3.4.1. Assume (3) converges for $x > r$, $r_1 \in (r, \infty)$ and write

$$\int_a^x f(t)dt = \int_a^{r_1} f(t)dt + \left( c_{-k} \ln t + \sum_{j \neq -k} \frac{c_j}{1 - j/k} x^{-j/k+1} \right) \bigg|_a^{x}$$

$$= \int_a^\infty \left( f(t) - \sum_{j \leq -k} c_j t^{-j/k} \right) dt + c_{-k} \ln x + \sum_{j = -M}^{-k} \frac{c_j}{1 - j/k} x^{-j/k+1}$$

where the infinite sum is a usual convergent series at $r_1$ treated as a normal form for $x > \infty$. For $a \in \mathbb{R}^+$, $\int_a^\infty$ is the usual integral from $a$ to $\infty$ of a real function.

If the series in $1/x$ converges starting at $a$, one may readily check that the above amounts to termwise integration of the series. Using the same transformation, $x^{-1/k} = z$, as in the previous section, we can transform this definition into a genetic one, using the genetic definition of log.
Proposition 78. The integral thus defined has properties (a)-(g) specified in Proposition 34; moreover, item (g) holds for any finite \( a, b \in \mathbb{N}_0 \).

subsection Extensions and antiderivatives of EB transseriable functions

Definition 79. (i) Lemma 56 shows that EB transseriable functions are analytic for large enough \( x \). Then the definition of \( E(f) \) for finite \( x \in \mathbb{N}_0 \) follows the same procedure that we used for \( F_a \) for finite \( x \in \mathbb{N}_0 \).

(ii) For \( x > \infty \) we simply define

\[
E(T)(x) = T^s(x) := \sum_{k \geq k_0, l \in \mathbb{N}} c_{k,l}x^{\beta_k}e^{-k\lambda x}x^{-l}
\]

where \( x \) is an infinite surreal number written in normal form.

In Definition 79 above, the infinite sum is an absolutely convergent series of normal forms, and thus has a Limit; see \( \S 2 \). Accordingly, for infinite surreal \( x \), Definition 79 in effect defines \( E(f)(x) \) to be the Limit of the absolutely convergent series of normal forms that arises on the right side of equation 38 by writing \( x \) in normal form.

Proposition 80. Formal level one log-free transseries, see (4), with real coefficients correspond 1-1 to surreal functions defined for \( x > \infty \), and the structures are isomorphic as differential algebras.

Proof of Proposition 80. An infinite series is a limit of polynomials. Convergence in the sense of transseries implies absolute convergence in the sense of Conway, when the variable in the transseries is replaced by an infinite surreal number written in normal form. Of course, surreal polynomials with real coefficients have the same algebraic properties as their real counterparts.

For differentiation, we take the restriction of the surreal function to the surreals of length \(< \alpha \) (see \( \S 2 \)), where \( \alpha \) is some limit ordinal. We take \( \alpha_1 \) to be the least ordinal greater than \( n\alpha \) for all \( n \) and \( \alpha_2 \) to be the least ordinal greater than \( n\alpha_1 \) for all \( n \). Consider the infinitesimal \( \delta \) whose sign expansion consists of a single plus followed by a string of \( \alpha_2 \) minuses. It is trivial to show that the reexpansion of \( \sum_{k=0}^{\infty} c_k(h+\delta)^k \), where we expand \((h+\delta)^k\) by the binomial formula, is absolutely convergent in the sense of Conway, since the supports of the coefficients of \( \delta^m S_m(h) \) are disjoint for different values of \( m \). By support we understand as usual the set of ordinals for which \( r_{\alpha} \neq 0 \). The rest is a straightforward calculation.

Proposition 81. (i) The extension operator \( E \), is a differential field isomorphism between EB transseriable functions and their respective images.

(ii) The definition can be recast as a genetic one.

Proof. (i) is straightforward from Propositions 57 and 80 and (ii) is proved in the subsequent sections.

8.2.3. The integral. We postpone the definition of the integral of EB transseriable functions to \( \S 8.4.4 \). For the time being, assume \( f \) is a Borel summable function. We first write \( f = c/x + g(x) \), where now \( g = O(1/x^2) \), and then we write \( \int_x^a f = c\ln s^x_a + \int_x^\infty g - \int_x^\infty g \).

Lemma 82. \( \int_x^\infty g \) is also Borel summable.
Proposition 87. This is also easy to check.

Proof. First, let \( a \) be such that there is an element of \( S \) greater than the corresponding sums. Indeed, for finite \( a \), we have

\[
\int_a^x f = c \ln s_a^x + \int_a^\infty g - E(\int_x^\infty g).
\]

Proposition 84. The integral in Definition 83 has properties (a)–(g) in Proposition 49.

Proof. This follows straightforwardly from Lemma 52 and the properties of \( E \).

8.3. The genetic construction. Borel summable series. Clearly, we only need to show \( E \) has a genetic definition. We arrange that the series begin with \( k = 0 \), that \( \beta = 0 \), and we let \( x > \nu_1 \) where \( \nu_1 \) is both greater than \( \nu \) (cf. 44) and large enough so that the results in 10 apply. By Borel summability, there are \( C, \rho \in \mathbb{R}^+ \) such that \( |c_k| \leq B_k := Ck\rho^{-k} \). The genetic definitions of the exponential and log (for leaders) are standard [29, chapter 10] and we will take those for granted.

Definition 85. Write (cf. Note 51)

\[
(40) \quad \hat{E}(f)(x) = \left\{ \sum_{k \in \ell(px)} \frac{c_k}{x^{k+1}} - Ce^{-\rho x} x^b, S^L \right\} + \left\{ \sum_{k \in \ell(px)} \frac{c_k}{x^{k+1}} + Ce^{-\rho x} x^b, S^R \right\},
\]

where \( S^L, S^R \) are as in Definition 77 and \( C \) is greater than the maximum of 1 and

\[
\sup_{x \in \mathbb{R}, x > 1} x^b e^{\rho x} \left| f(x) - \sum_{k \in \ell(px)} \frac{c_k}{x^{k+1}} \right|,
\]

the latter of which is finite by 10.

For general \( \ell \), the sums in the definition of \( \hat{E} \) are normal forms, but they coincide with the usual sums when \( x < \infty \) since then \( \ell(px) \) is a finite set.

Note 86. For \( x < \infty \), the sets \( S^L \) and \( S^R \) in Equation 40 provide tighter bounds than the corresponding sums. Indeed, for finite \( x \), it is easy to show that some elements of \( S^L(S^R) \) are always greater (resp. less) than the left (resp. right) sum. The relation between the sums and \( S^L, S^R \) is the opposite for \( x > \infty \), unless \( \Pi(x) \) is very small. This is also easy to check.

Proposition 87. \( E = \hat{E} \).

Proof. First, let \( x_0 > \infty \) be such that \( \Pi(x_0) > e^{-ax_0} \) for all \( a \in \mathbb{R}^+ \). In this case, there is an \( a \in \mathbb{R}^+ \) such that \( |x_0 - x_0^a| \neq a \) and thus \( \text{dist}(S^L, S^R) > e^{-ax_0} \) for all \( a \in \mathbb{R}^+ \). Therefore,

\[
(41) \quad \left\{ \sum_{\ell(px_0)} \frac{c_k}{x_0^{k+1}} - 2Ce^{-\rho x_0} x_0^b, S^L \right\} + \left\{ \sum_{\ell(px_0)} \frac{c_k}{x_0^{k+1}} + 2Ce^{-\rho x_0} x_0^b, S^R \right\},
\]

which, by Proposition 44, equals

\[
(42) \quad \left\{ \sum_{k=0}^\infty \frac{c_k}{x_0^{k+1}} - 2Ce^{-\rho x_0} x_0^b \right\} + \left\{ \sum_{k=0}^\infty \frac{c_k}{x_0^{k+1}} + 2Ce^{-\rho x_0} x_0^b \right\} = \sum_{k=0}^\infty c_k x_0^{-k},
\]
which in turn implies
\[(43) \hat{E}(f)(x_0) = \sum_{k=0}^{\infty} \frac{c_k}{x_0^{k+1}} = E(f)(x_0).\]

Now, for \(x = x_0 + \varepsilon\), where \(\Pi(x_0) \gg e^{-ax_0}\) for all \(a \in \mathbb{R}^+\) and \(\varepsilon = \Pi(\varepsilon)\), we use \(S_L\) and \(S_R\) as in Definition 77 to obtain
\[(44) \hat{E}(f)(x_0 + \varepsilon) = \hat{E}(f)(x_0) + \sum_{k=0}^{\infty} \hat{E}(f)^{(k)}(x_0) \varepsilon^k,\]

where
\[
\hat{E}(f)^{(k)}(x_0)
\]
is simply the formal \(k\)th derivative of the formal series \(\sum_{k=0}^{\infty} \frac{c_k}{x_0^{k+1}}\) where we take \(x = x_0\) and interpret the sum as a (clearly well-defined) normal form. We note that, exactly as in routine manipulations of formal series, combining (43) with (44) leads to
\[(45) \hat{E}(f)(x) = \sum_{k \in \mathbb{N}} \frac{c_k}{x^{k+1}} = E(f)(x) \text{ for all } x > \infty.\]

It remains to check that (44) coincides with the value that \(S_L, S_R\) give. For this, one can mimic the steps of [26, pages 72–74], since the arguments employed there, while presented for convergent series do not rely on convergence.

8.4. The genetic definition of \(E\) and of integrals of EB transseriable functions.

8.4.1. Extensions of functions that have (EB)-summable transseries at \(\infty\). Here we rely on Écalle analyzability and transseries. See [13] for more details and references.

Note 88. We limit our analysis to Gevrey-one EB summable transseries. In fact, the definition of EB-summable transseries allows for any mixture of Gevrey types and any combination of power series and exponentials. We do not pursue this generalization here. One reason is that it would require to a momentous undertaking due to the many cases that need to be covered and the technical tools involved (Écalle acceleration, Écalle cohesive continuation and so on). Secondly, some functions arising in applications may exhibit oscillatory behavior, and a general \(\mathbb{C}\)-valued theory of transseries is not expected to exist in any generality; for instance oscillatory transseries do not, in general, have reciprocals. But in limited contexts such as those in [12, 14], they do exist and are well behaved under all operations except division.

8.4.2. Extensions of functions that have EB-summable level one transseries at \(\infty\).

Proposition 89. (4) converges in the formal multiseries topology (see Definition 54) and is also well-defined as a normal form (with obvious generalizations for sur-complex \(\lambda_j\)).

Proof. For transseries, this is proved in [14] and also in [19]. For normal forms, in the real-valued case this follows from Proposition 47 by taking \(h_1 = x^{-1}, h_2 = x^{\beta_1} e^{-\lambda_1 x}, \ldots, h_{m+1} = x^{\beta_m} e^{-\lambda_m x}\). The surcomplex extension is straightforward. □
Allowing for complex valued transseries, one can analyze all classical functions in analysis, general solutions of linear or nonlinear ODEs which are regular at $\infty$, or have a regular singular point at $\infty$, or can be normalized in such a way that they have a rank one singularity at $\infty$ [14].

It is shown in [14] that there is a one-to-one correspondence between solutions of (25) that go to zero as $x \to \infty$ and EB summable transseries of the form
\begin{equation}
y = \sum_{k \in \mathbb{N}^n} c_k x^k \beta e^{-k \lambda} y_k(x),
\end{equation}
where the $y_k$ are either zero, or of the form $L Y_k(p)$, where $L$ is the Catalan averaged Laplace transform; see the proof of Proposition 57 as well as §2.2 for the conventions governing $\beta$ and $\lambda$.

Also, in accordance with Definition 55, it is shown in [12, 14] that
\begin{equation}
|c_k| \sup_{x > x_0} |y_k(x)| < \mu |k|
\end{equation}
for some $\mu > 0$.

Also,
\begin{equation}
y_0 = O(x^{-M})
\end{equation}
where $M \in \mathbb{N}$ can be chosen, via elementary transformations, to be as large as needed in the cases of interest, [12, 14].

**Note 90.** For simplicity, we assume that all coefficients and functions $y$ are real-valued. This condition can be easily eliminated considering surcomplex numbers, and writing inequalities separately for the real and imaginary parts of the functions involved.

### 8.4.3. Further assumption.

The assumption adopted here (and in [12], where, perhaps confusingly, however, $\beta$ is denoted $-\beta$) is that $\text{Re} \beta \in [-1, 0)$. This holds for Painlevé equations and can be always arranged for in all linear ODEs, though it cannot be simply made to hold in general nonlinear ODEs. In [14] this assumption was dropped, but to work in that generality and make the link to the theory of Catalan averages, one needs Écalle pseudo-decelerations, which would lead to very cumbersome, albeit straightforward, calculations and we will not treat this case here.

BE transseriable functions are real-analytic by Lemma 56 so the definition of $E$ for finite $x \in \mathbb{N}$ mimics that of $F$.

The extension beyond $\infty$ of (46) is an immediate generalization of (41). Namely, we note that the function series in (46) is classically convergent. To each of the component functions $y_k$ [10] applies—in fact uniformly in $k$. Equation (47) implies that for large $x_0$, all $x > x_0$ and all $N$, there is a $C_N$ such that
\begin{equation}
T - \sum_{|k| \leq N} c_k x^k \beta e^{-k \lambda} y_k(x) \leq \max_{|k| \geq N} C_N x^k \beta e^{-k \lambda} x =: E_N(x).
\end{equation}
Since the $y_k$ are EB-summable series, their genetic definition is provided by Definition 85 and Proposition 87. This being the case, we may suppose the $y_k$ have already been defined, in which case we may write
Definition 91.
(50)
\[ \hat{E}(T) = \left\{ \sum_{|k| \leq N} c_k e^{k \beta} e^{-k \lambda x} y_k - 2|E_N|, S^L \right\} \]
where \( S^L, S^R \) are defined as in 8.3 above.

Proposition 92. \( \hat{E} = \hat{E} \).

Proof. The proof mimics the one in 8.3. Writing \( x_0 = \text{re}(x_0) + (x_0 - \text{re}(x_0)) \) and noting that \( e^{-k \lambda x_0} \) is a real constant, we can absorb \( e^{-k \lambda x_0} \) into the \( c_k \) and assume \( \text{re}(x_0) = 0 \). Now take the first \( x_0 \) such that \( \Pi(x_0) = 0 \). Then, using the fact that \( e^{-x} \ll x \) for all infinite \( x \), we see that all the leaders in the normal form of \( e^{-k \lambda x} x_k^k \beta \delta y_k \) are much smaller than any leader in \( e^{-k' \lambda x} x_k^k \beta \delta y_k' \ll x \) if \( k' \cdot \lambda > k \cdot \lambda \). The rest of the proof, which concerns the case where \( \Pi(x_0) = 0 \), is the same as in 8.3. If \( x = x_0 + \varepsilon \) with \( \Pi(x_0) = 0 \) (and \( \text{re}(x) = \text{re}(x_0) = 0 \) as above), we write \( e^{-k' \lambda x} = e^{-k \lambda x_0} \left[ 1 + k \cdot \lambda \varepsilon + (k \cdot \lambda \varepsilon)^2/2 + \cdots \right] \) and note that the power series can be absorbed in that of \( y_k(x_0 + \varepsilon) \) and once more the proof continues exactly as in 8.3. \( \square \)

8.4.4. Integration of surreal transseriable functions.

Proposition 93. A transseries of the form (40), under the assumptions given there, can be integrated term-by-term, resulting in an \( \mathcal{E} \)-Bolel summable transseries again of form (40).

Note 94. The interpretation of a termwise integral on transseriable functions in the usual domain is a generalized integral from \( \infty \), which is the “common point”. For the surreal, the integral is the extension of this usual integral and of the Hadamard finite part at \( \infty \).

Proof. Integrating termwise a uniformly and absolutely convergent (in the classical sense) function series is justified by the dominated convergence theorem. For a vector \( y \), integration is carried out component-wise so we can assume with no loss of generality that we are dealing with scalars. For \( y_0 \), we simply use definition 8.3.

For higher indices \( k \), we need to solve ODEs of the type
\[ f' = x^k e^{-a x} y(x). \]
After substituting \( f = x^b e^{-a x} g \), we obtain
\[ (a + p)G' = (b - 1)G - Y' \Rightarrow G = -(a + p)^{-b} Y - bY \ast (a + p)^{-b - 1}. \]
By 8.2, the above operations of multiplication by bounded analytic functions and convolution commute with the Catalan averages, and therefore the integral of an EB-transseriable function \( T \) is again an EB-transseriable function \( T_1 \), and we define \( \int T = \mathcal{E}(T_1) \). Using Proposition 8.1 the rest is straightforward. \( \square \)

8.4.5. Classical functions covered. The following class of functions is amenable to extensions to \( x \in \mathbb{N} \) where \( x > \infty \): Solutions of generic linear or nonlinear systems of ODEs (with an irregular singularity of rank 1 at \( \infty \)) which, after normalization, satisfy the conditions in [14]. These solutions may, of course, be complex-valued. As special cases we have the classical special functions in analysis: Airy \( \text{Ai} \) and \( \text{Bi} \), Bessel \( I_\alpha \), \( J_\alpha \), \( Y_\alpha \), and \( K_\alpha \), Erf, Ei, Painlevé and so on, possibly after changes
of variables of the form \( f \mapsto x^a f(x^b) \). Defining an integral amounts to solving a differential equation \( y' = f \). If \( f \) is a simple function, the equation reduces to the equations studied in [14].

8.5. Some examples.

8.5.1. Ei. Let \( C \in \mathbb{R} \) be given by

\[
C := \sup_{x \in \mathbb{R}, x > 1} x^{1/2} \left| Ei(x) - e^x \sum_{k \in \ell(x)} \frac{k!}{x^{k+1}} \right|,
\]

where \( \ell(x) \) is the least term summation of the series defined in Equation (53). By [16], the sup is finite; in fact, \( C = 3.54 \cdots \). Then, for a surreal \( x > 1 \), where \( E(x) = 0 \) our definition of Ei is:

\[
(53) \quad Ei(x) = \left\{ e^x \sum_{k \in \ell(x)} \frac{k!}{x^{k+1}} - e^x \sum_{k \in \ell(x)} \frac{k!}{x^{k+1}} + C x^{-1/2}, S^L \right\}.
\]

Of course, if \( S^L(Ei(x)) = \emptyset \) (resp. \( S^R(Ei(x)) = \emptyset \)), then \( S^L(Ei(x)) \pm \varepsilon = \emptyset \) (resp. \( S^R(Ei(x)) \pm \varepsilon = \emptyset \)). The sums in (53) are interpreted as normal forms, after possible reexpansion. Indeed, they have finitely many terms if \( x \in [1, \infty) \) and otherwise they are the Limits of series of normal forms that are absolutely convergent in the sense of Conway; see Propositions 45-47. Similarly,

\[
\int_1^x t^{-1} e^t dt = Ei(1, x)
\]

is given by

\[
Ei(1, x) = \left\{ e^x \sum_{k \in \ell(x)} \frac{k!}{x^{k+1}} - e^x \sum_{k \in \ell(x)} \frac{k!}{x^{k+1}} - Ei(1) + C x^{-1/2}, S^R \right\},
\]

which leads to

\[
Ei(\omega) = e^\omega \sum_{k=0}^{\infty} \frac{k!}{\omega^{k+1}}; \quad Ei(1, \omega) = e^\omega \sum_{k=0}^{\infty} \frac{k!}{\omega^{k+1}} - Ei(1).
\]

The just-said value of Ei(\( \omega \)) is (after the reexpansion of terms) a purely infinite Conway normal form with no constant added. Since Ei is an antiderivative, it is defined up to a constant to be determined from the initial conditions. Our calculation shows that for Ei this constant is 0. The fact that Ei corresponds to a purely infinite Conway normal form with constant 0 at \( \infty \) was conjectured by Conway and Kruskal.

8.5.2. Erfi. To calculate

\[
(54) \quad \int_0^x e^{s^2} ds = \frac{\sqrt{\pi}}{2} \text{erfi}(x)
\]

we solve \( f' = e^{s^2} \), where \( f(0) = 0 \). With the change of variables \( f(s) = s \exp(s^2)g(s) \) with \( s = \sqrt{t} \) we get

\[
(55) \quad g' + \left( 1 + \frac{1}{2t} \right) g = \frac{1}{2t}, \quad \text{where} \quad g(z) = 1 + o(z) \text{ as } z \to 0,
\]
whose Borel transform is
\[(p - 1)G' + \frac{1}{2}G = 0,\]
where \(G(0) = 1/2\). But this implies that
\[
\frac{1}{4} \left( \int_0^{\infty - \imath 0} + \int_0^{\infty + \imath 0} \right) \frac{e^{-tp}}{\sqrt{1 - p}} dp = \frac{1}{2} \int_0^{1} \frac{e^{-tp}}{\sqrt{1 - p}} dp,
\]
and hence that the value of \(f\) for \(x > \infty\) is
\[
\int_0^{\infty} e^{x^2} ds = \frac{e^{x^2}}{x\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2x^2)^n}.
\]

8.5.3. The Gamma function. Based on the recurrence of the factorial, it is shown in [13, page 99] that \(\ln \Gamma\) has a Borel summed representation given by,
\[
\ln \Gamma(n) = n(n - 1) - \frac{1}{2} \ln n + \frac{1}{2} \ln(2\pi) + \int_0^{\infty} \frac{1 - p}{p^2} \frac{1}{e^{-p} - 1} - \frac{1}{e^{-p} - 1} dp,
\]
where the integrand is analytic at zero, as is seen by power series reexpansion at zero. It follows from our results that
\[
\ln \Gamma(\omega) = \omega \ln(\omega) - \frac{1}{2} \ln \left( \frac{\omega}{2\pi} \right) + \frac{1}{12\omega} - \frac{1}{360\omega^2} + \frac{1}{1260\omega^3} + \cdots
\]
where the coefficient of \(\omega^{-k-1}\) is \(\frac{B_k}{\kappa(k-1)}\) and the \(B_k\) are the Bernoulli numbers. By reexpansion we arrive at
\[
\Gamma(\omega) = \omega^{\omega-\frac{1}{2}e^{-\omega}\sqrt{2\pi}} \left( 1 + \frac{1}{12\omega} + \frac{1}{288\omega^2} - \frac{139}{51840\omega^3} - \frac{571}{2488320\omega^4} + \cdots \right),
\]
where there is a closed form expression for the coefficients that is more intricate than that which can be obtained from (60). This is just Stirling’s formula. Our results show that (after reexpansion of the series as a normal form) this is valid for all \(x > \infty\).

Note 95. Clearly, one can the employ the approach used in this paper to define infinite sums. For example, since in classical mathematics \(\ln \Gamma\) is a sum of logs indexed by ordinary positive integers, what we have calculated above might reasonably be denoted by
\[
\sum_{k=1}^{\omega} \ln k.
\]

9. Logical issues

Most of the paper directly involves \(\textbf{No}\), which is a highly set-theoretic object. Therefore we need to be far more careful about what axiom systems our developments live in than we would for most mathematical developments. The reader who is not interested in logical issues can safely skip this section.

The issue arises immediately as in almost all mathematical papers, the underlying formal system can be taken to be ZFC. In fact, for most papers, it can be very conveniently taken to be Z (Zermelo set theory), which is ZFC without the axiom of choice or replacement. Yet here \(\textbf{No}\) is taken to be a proper class, too big to be a set.
We delineate two different approaches, each with their advantages and disadvantages.

1. Literal. Here we use absolutely no coding devices in order to simulate classes set theoretically, or to simulate classes of classes as classes, and so forth. We take all of the objects “as they come” without reworking them. This issue arises most vividly with partial functions \( f : T[\mathbb{R}] \to T[\mathbb{No}] \). \( T[\mathbb{R}] \) is unproblematic as it is a set. \( T[\mathbb{No}] \) however consists of (sets and) proper classes, and so under the literal approach, we already need classes of proper classes. There is even a problem with individual such \( f \), which from the literal point of view, is perhaps even more of a problem. This is because \( f \) is a set of ordered pairs - typically \((g, h) = \{\{g\}, \{g, h\}\}\), where \( g \) is a set and \( h \) is a proper class. Note that \((g, h)\) is a class of classes of classes. It is clear that NBG is far from sufficient for the Literal approach.

2. Classes. Here we insist that we work only with the usual sets and classes underlying the usual formal system NBG. We take NBG to include the global axiom of choice, as is customary. We use NBG\(^-\) for NBG without any axiom of choice, even for sets. In this approach, again there is no issue with regard to the set \( T[\mathbb{R}] \) and its subsets. Also, there is no problem with the proper class \( \mathbb{No} \). However, the ordered field \((\mathbb{No}, <, +, \cdot)\) has a problem. If taken literally, it is an ordered quadruple of proper classes, and thus at least as complex as a class of classes of classes, depending on how quadruples are handled. However, there is a well known coding device that appropriately simulates any finite sequence of classes as a class. More generally, there is a well known coding device that appropriately simulates any class valued function whose domain is a set, as a class. This takes care of partial functions \( f : T[\mathbb{R}] \to T[\mathbb{No}] \). However, it does not literally handle \( T[\mathbb{No}] \) itself, as it is a class of classes. In class theory, we treat \( T[\mathbb{No}] \) as a virtual class of classes, just as we would treat \( \mathbb{No} \) as a virtual class in the theory of sets. Adhering to set theory creates some awkwardness that we wish to avoid, because of \( T[\mathbb{No}] \) and partial functions \( f : T[\mathbb{R}] \to T[\mathbb{No}] \). For the Literal approach, we use ZCI, which is Zermelo set theory with the axiom of choice together with “there exists a strongly inaccessible cardinal”.

Under the literal approach, for the purpose of this paper, the sets are the elements of the cumulative hierarchy \( V(\theta) \) up to the first strongly inaccessible cardinal \( \theta \). The classes are the elements of \( V(\theta + 1) \), the classes of classes are the elements of \( V(\theta + 2) \), and so forth. Each \( V(\theta + n) \) can be proved to exist in ZCI, but not \( V(\theta + \omega) \).

Under either approach, DC is a weak form of the axiom of choice for sets, called dependent choice. It asserts that for every binary relation \( R \) and every set \( x \), there is a sequence \( x = x_1, x_2, ... \) such that for all \( i \geq 1 \), if there exists \( y \) such that \( R(x_i, y) \), then \( R(x_i, x_{i+1}) \). NBG\(^-\) (or even Z) proves that DC is equivalent to the Baire category theorem.

NBG, NBG\(^-\), NBG\(^+\) + DC are conservative extensions of ZFC, ZF, ZFDC = ZF + DC, respectively. This means that the former systems are extensions of the latter systems, and the former systems prove the same sentences of set theory as the latter systems, respectively.

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