Symmetries of geometric flows

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January 11, 2010

Abstract

By applying the theory of group-invariant solutions we investigate the symmetries of Ricci flow and hyperbolic geometric flow both on Riemann surfaces. The warped products on $S^{n+1}$ of both flows are also studied.

§1 Introduction

The Ricci flow is the geometric evolution equation in which one starts with a smooth Riemannian manifold $(M^n, g_0)$ and evolves its metric by the equation

$$\frac{\partial}{\partial t} g = -2Rc,$$

where $Rc$ denotes the Ricci tensor of the metric $g$. The Ricci flow has been exhaustively studied and successfully applied to solve the famous Poincaré’s Conjecture [2]. Recently, Kong and Liu [9] introduced the hyperbolic geometric flow which is the hyperbolic version of Ricci flow

$$\frac{\partial^2}{\partial t^2} g = -2Rc,$$

which shows different behavior with the original Ricci flow.

On Riemann surfaces $(M^2, g)$, equations (1.1) and (1.2) can be simplified to scalar equations

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\[ u_t = \Delta \ln u, \quad \text{(1.3)} \]
\[ u_{tt} = \Delta \ln u, \quad \text{(1.4)} \]

where function \( u(x, y, t) \) is the conformal factor of \( g \):

\[ g_{ij} = u(x, y, t) \delta_{ij} \]

Later we will use the theory of group-invariant solutions to investigate (1.3) and (1.4). As we will see, the sets of symmetries of the two equations are quite large and we expect to find large classes of exact solutions to both flows on Riemann surfaces and the symmetries dependent on the solution of two-dimensional Laplace equation.

The technique here we use to investigate the symmetries and exact solutions of the equations is the theory of group-invariant solutions for differential equations which applies Lie group, Lie algebra and adjoint representation to differential equations. For most cases, there is a one-to-one correspondence between different symmetries of an equation and the conjugate classes of subgroups of its one-parameter transformation group. So finally through the classification of subalgebras of the Lie algebra of the transformation group, we are able to classify all the symmetries of the equations. We will introduce this technique briefly later in this paper. For more details, see [11].

We will also study warped products on \( S^{n+1} \) or \( SO(n + 1) \)-invariant metrics on \( S^{n+1} \) of both flows on the set \( (-1, 1) \times S^n \):

\[ g = \varphi^2(x, t)dx^2 + \psi^2(x, t)g_{\text{can}}, \]

where \( g_{\text{can}} \) denotes the canonical metric on \( S^n \). This metric under Ricci flow was studied in [1]. Analyzing its asymptotic behavior leads to significant information about the nechpinch which is important in the surgery of Ricci flow. By a change of coordinate

\[ s(x) = \int_0^x \varphi(x)dx, \]

the evolutions of \( \varphi(s, t) \) and \( \psi(s, t) \) under Ricci flow and hyperbolic geometric flow are the followings respectively:

\[ \begin{cases} 
\varphi_t = n \frac{\psi_{ss}}{\psi} \varphi \\
\psi_t = \psi_{ss} - (n - 1) \frac{1 - \psi^2}{\psi} \end{cases} \quad \text{(1.5)} \]
under Ricci flow, and
\[
\begin{align*}
\varphi_t &= n \frac{\psi_s^s}{\psi} \varphi - \frac{\varphi^2}{\psi} \\
\psi_t &= \psi_{ss} - (n - 1) \frac{1 - \psi^2}{\psi} - \frac{\psi^2}{\psi}
\end{align*}
\tag{1.6}
\]
under hyperbolic geometric flow. In contrast to (1.3) and (1.4), equations (1.5) and (1.6) have few symmetries especially in higher dimensions.

This paper is organized as follows: We would begin with the theory of group-invariant solutions for differential equations in Section 2. In Section 3, we will study the symmetries and exact solutions of Ricci flow on surfaces. In Section 4, we investigate hyperbolic geometric flow on surfaces. In Section 5, the warped product of $S^{n+1}$ on both flows are studied. In Section 6, we give some further discussions. Finally, in section 7, we derive the evolutions of warped products on both flow.

Acknowledgement. The author thanks the Center of Mathematical Sciences at Zhejiang University where he wrote this paper during the summer of 2009.

§2 Theory of group-invariant solutions for differential equations

In this section, we briefly introduce the theory of group-invariant solutions for differential equations. The following main definitions and theorems are cited from [11].

First we introduce the jet space. Given
\[
u_t = \Delta \ln u,
\]
let $w = \ln u$, so
\[
e^w w_t - w_{xx} - w_{yy} = 0. \tag{2.1}
\]
We regard $w$ and its derivatives as variables in (2.1), so (2.1) can be regarded as defined on
\[
X \times U^{(2)} = \{(x, y, t; w; w_x, w_y, w_t; w_{xx}, w_{xy}, w_{xt}, w_{yy}, w_{yt}, w_{tt})\},
\]
where $X = \{(x, y, t)\}$ is the space of independent variables $(x, y, t)$. 

In general, we denote an $n$-th order differential equation of $w$ with independent variables $x = (x^1, ..., x^p)$ by
\[
\Delta(x, w^{(n)}) = 0.
\]
Thus $\Delta$ can be regarded as a smooth map from the jet space $X \times U^{(n)}$ to $\mathbb{R}$
\[
\Delta : X \times U^{(n)} \to \mathbb{R},
\]
and the differential equation tells where the given map $\Delta$ vanishes on $X \times U^{(n)}$, thus determines a subvariety
\[
\mathcal{I}_\Delta = \{(x, w^{(n)}) : \Delta(x, w^{(n)}) = 0\} \subset X \times U^{(n)}
\]
of the total jet space.

**Definition 2.1.** Let $\mathcal{S}$ be a system of differential equations. A symmetry group of the system $\mathcal{S}$ is a local group of transformations $G$ acting on an open subset $M$ of the space of independent and dependent variables for the system with the property that whenever $u = f(x)$ is a solution of $\mathcal{S}$, and whenever $g \cdot f$ is defined for $g \in G$, then $u = g \cdot f(x)$ is also a solution of the system.

**Theorem 2.2.** Let $M$ be an open subset of $X \times U^{(n)}$ and suppose $\Delta(x, w^{(n)}) = 0$ is an $n$-th order equation defined over $M$, with corresponding subvariety $\mathcal{I}_\Delta \subset M$. Suppose $G$ is a local group of transformations acting on $M$ which leaves $\mathcal{I}_\Delta$ invariant, meaning that whenever $(x, w^{(n)}) \in \mathcal{I}_\Delta$, we have $g \cdot (x, w^{(n)}) \in \mathcal{I}_\Delta$ for all $g \in G$ such that this is defined. Then $G$ is a symmetry group of the equation in the sense of Definition 2.1.

Next we introduce the prolongation of vector fields corresponding to one-parameter transformation group acting on $M \subset X \times U = \{(x, w)\}$. We only state its formula here for our use. The interested reader can see [11].

**Theorem 2.3.** Let
\[ \mathbf{v} = \sum_{i=1}^{p} \xi^i(x, w) \frac{\partial}{\partial x^i} + \phi(x, w) \frac{\partial}{\partial w} \]

be a vector field defined on an open subset \( M \subset X \times U \). The \( n \)-th prolongation of \( \mathbf{v} \) is the vector field

\[ \text{pr}^{(n)} \mathbf{v} = \mathbf{v} + \sum_{J} \phi^J(x, w^{(n)}) \frac{\partial}{\partial w^J} \]

defined on the corresponding jet space \( M^{(n)} \subset X \times U^{(n)} \), the summation being over all (unordered) multi-indices \( J = (j_1, \ldots, j_k) \), with \( 1 \leq j_k \leq p, 1 \leq k \leq n \). The coefficient functions \( \phi^J \) of \( \text{pr}^{(n)} \mathbf{v} \) are given by the following formula:

\[ \phi^J(x, w^{(n)}) = D^J(\phi - \sum_{i=1}^{p} \xi^i w_i) + \sum_{i=1}^{p} \xi^i w_{J,i}, \]

where \( D \) is the total derivative operator, and \( w_i = \partial w / \partial x^i \), \( w_{J,i} = \partial w^J / \partial x^i \).

We state two important definitions which play important role in the theory.

**Definition 2.4.** Let

\[ \triangle(x, w^{(n)}) = 0, \]

be a differential equation. The equation is said to be of maximal rank if the Jacobian matrix

\[ \mathcal{J}_{\triangle}(x, w^{(n)}) = \left( \frac{\partial \triangle}{\partial x^i}, \frac{\partial \triangle}{\partial w^J} \right) \]

of \( \triangle \) with respect to all the variables \( (x, w^{(n)}) \) is of rank 1 whenever \( \triangle(x, w^{(n)}) = 0 \).

For example, consider (2.1), the corresponding Jacobian matrix is

\[ \mathcal{J}_{\triangle}(x, y, t; w; w_x, w_y, w_t; w_{xx}, w_{xy}, w_{xt}, w_{yy}, w_{yt}, w_{tt}) = (0, 0, 0; e^w w_t; 0, 0, e^w; -1, 0, 0, -1, 0, 0), \]

which is of rank 1 whenever \( \triangle(x, y, t, w^{(2)}) = 0 \). So (2.1) is of maximal rank.
Definition 2.5. An $n$-th order differential equation $\triangle(x, w^{(n)}) = 0$ is locally solvable at the point $(x_0, w_0^{(n)}) \in \mathcal{S}_\triangle = \{(x, w^{(n)}) : \triangle(x, w^{(n)}) = 0\}$ if there exists a smooth solution $u = f(x)$ of the equation, defined for $x$ in a neighborhood of $x_0$, which has the prescribed initial condition $w_0^{(n)} = \text{pr}^{(n)}f(x_0)$, where $\text{pr}^{(n)}f(x_0)$ means $f$ and all its derivatives up to order $n$ at point $x_0$. The equation is locally solvable if it is locally solvable at every point of $\mathcal{S}_\triangle$. A differential equation is nondegenerate if at every point $(x_0, w_0^{(n)}) \in \mathcal{S}_\triangle$ it is both locally solvable and of maximal rank.

The main theorem we will use is the following:

Theorem 2.6. Let $\triangle(x, w^{(n)}) = 0$ be a nondegenerate differential equation. A connected local group of transformations $G$ acting on an open subset $M \subset X \times U$ is a symmetry group of the equation if and only if

$$\text{pr}^{(n)}v[\triangle(x, w^{(n)})] = 0,$$

whenever $\triangle(x, w^{(n)}) = 0$, for every infinitesimal generator $v$ of $G$.

We calculate some prolongation formulas here that we will use later. On $M \subset X \times U$, given a vector

$$v_1 = (x, y, t, w) \frac{\partial}{\partial x} + \zeta(x, y, t, w) \frac{\partial}{\partial y} + \tau(x, y, t, w) \frac{\partial}{\partial t} + \phi(x, y, t, w) \frac{\partial}{\partial w},$$

its second order prolongation is

$$\text{pr}^{(2)}v_1 = v_1 + \phi^x \frac{\partial}{\partial w_x} + \phi^y \frac{\partial}{\partial w_y} + \phi^t \frac{\partial}{\partial w_t} + \phi^{xx} \frac{\partial}{\partial w_{xx}} + \phi^{xy} \frac{\partial}{\partial w_{xy}} + \phi^{xt} \frac{\partial}{\partial w_{xt}} + \phi^{yy} \frac{\partial}{\partial w_{yy}} + \phi^{yt} \frac{\partial}{\partial w_{yt}} + \phi^{tt} \frac{\partial}{\partial w_{tt}}.$$

We will use the followings:

$$\phi^t = \phi_t - \xi_t w_x - \eta_t w_y + (\phi_w - \tau_t)w_t - \xi_w w_x w_t - \eta_w w_y w_t - \tau_w w_t^2,$$

$$\phi^x = \phi_x + (\phi_w - \xi_x)w_x - \eta_x w_y - \tau_x w_t - \xi_x w_x^2 - \eta_x w_x w_y - \tau_x w_x w_t,$$
\[ \phi^t = \phi_t - \xi_t \psi_s + (\phi_\psi - \tau_\psi) \psi_t - \xi_\psi \psi_s \psi_t - \tau_\psi \psi_t^2, \]

\[ \phi^s = \phi_s + (\phi_\psi - \xi_\psi) \psi_s - \tau_s \psi_t - \xi_\psi \psi_s^2 - \tau_\psi \psi_s \psi_t, \]

Next, given

\[ \mathbf{v}_2 = \xi(s, t, \psi) \frac{\partial}{\partial s} + \tau(s, t, \psi) \frac{\partial}{\partial t} + \phi(s, t, \psi) \frac{\partial}{\partial \psi}, \]

its second order prolongation is

\[ pr^{(2)} \mathbf{v}_2 = \mathbf{v}_2 + \phi^s \frac{\partial}{\partial \psi_s} + \phi^t \frac{\partial}{\partial \psi_t} + \phi^{st} \frac{\partial}{\partial \psi_{st}} + \phi^{ss} \frac{\partial}{\partial \psi_{ss}} + \phi^{tt} \frac{\partial}{\partial \psi_{tt}}. \]
\[
\phi_{tt} = \phi_{tt} + (2\phi_{t\psi} - \tau_{tt})\psi_t - \xi_{tt}\psi_s + (\phi_{\psi\psi} - 2\tau_{t\psi})\psi_t^2 - 2\xi_{t\psi}\psi_s\psi_t \\
-\tau_{\psi\psi}\psi_t^3 - \xi_{\psi\psi}\psi_s^2 + (\phi_{\psi} - 2\tau_t)\psi_{tt} - 2\xi_t\psi_{st} - 3\tau_{\psi}\psi_{tt}
\]
\[
-\xi_{\psi}\psi_s\psi_{tt} - 2\xi_{\psi}\psi_t\psi_{st},
\]

\[
\phi_{ss} = \phi_{ss} + (2\phi_{s\psi} - \xi_{ss})\psi_s - \tau_{ss}\psi_t + (\phi_{\psi\psi} - 2\xi_{s\psi})\psi_s^2 - 2\tau_{s\psi}\psi_s\psi_t \\
-\xi_{\psi\psi}\psi_s^3 - \tau_{\psi\psi}\psi_s^2\psi_t + (\phi_{\psi} - 2\xi_s)\psi_{ss} - 2\tau_s\psi_{st} - 3\xi_{\psi}\psi_s\psi_{ss} \\
-\tau_{\psi}\psi_{t\psi\psi}\psi_{ss} - 2\tau_{\psi}\psi_s\psi_{st}.
\]

When we solve out for example \(\xi(x, y, t, w), \eta(x, y, t, w), \tau(x, y, t, w), \phi(x, y, t, w)\) for \(v_1\), we get some vectors
\[
v_1, \cdots, v_k.
\]

These vectors generate the Lie algebra of the transformation group \(G\). To classify its subalgebras, we need to calculate the structure constants
\[
[v_i, v_j] = C^l_{ij}v_l,
\]
and the adjoint representations
\[
Ad(\exp(\varepsilon v_1))v_j = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(ad v_1)^n(v_j)
\]
\[
= v_j - \varepsilon[v_1, v_j] + \frac{\varepsilon^2}{2}[v_1, [v_1, v_j]] - \cdots.
\]

After the classification of subalgebras and subgroups of \(G\), we get an optimal system for the equation. By constructing invariants from \(v\) in these subalgebras, we can simplify the equation to ODE or lower order PDE, thus we expect to find exact symmetric solutions to the original equation. These will be investigated in detail for our equations in the following sections.
§3 Ricci flow on Riemann surfaces

On a surface, all of the information about curvature is contained in the scalar curvature function $R$. The Ricci curvature is given by

$$ R_{ij} = \frac{1}{2} R g_{ij}, $$

and the Ricci flow equation can be simplified to

$$ \frac{\partial}{\partial t} g_{ij} = -R g_{ij}. $$

The metric for a surface can always be written (at least locally) in the following form

$$ g_{ij} = u(x, y, t) \delta_{ij}, $$

where $u(x, y, t) > 0$. Therefore, we have

$$ R = -\frac{\Delta \ln u}{u}. $$

Thus

$$ \frac{\partial}{\partial t} u = \frac{\Delta \ln u}{u} \cdot u, $$

namely,

$$ u_t - \Delta \ln u = 0. \quad (3.1) $$

Denote $w = \ln u$, thus

$$ \Delta(x, y, t; w^{(2)}) = w_{xx} - w_{yy} - e^w w_t = 0 \quad (3.2) $$

we will use the techniques developed in section 2 to analyze (3.2).

First note that the Jacobian matrix of (3.2) is

$$ \mathcal{J}_{\Delta}(x, y, t; w, w_x, w_y, w_t, w_{xx}, w_{xy}, w_{xt}, w_{yy}, w_{yt}, w_{tt}) = (0, 0, 0; -e^w w_t; 0, 0, -e^w; 1, 0, 0, 1, 0, 0), $$
which is obviously of rank 1 in $\mathcal{S}_{\triangle}$ and (3.2) is obviously locally solvable. So we can apply Theorem 2.6.

Given a vector

$$v = \xi(x, y, t, w) \frac{\partial}{\partial x} + \eta(x, y, t, w) \frac{\partial}{\partial y} + \tau(x, y, t, w) \frac{\partial}{\partial t} + \phi(x, y, t, w) \frac{\partial}{\partial w},$$

we have

$$pr^{(2)}v(\triangle(x, y, t, w^{(2)})) = -\phi e^w w_t - \phi^x e^w + \phi^{yy}.$$

We apply the formulas for $\phi^t$, $\phi^{xx}$, $\phi^{yy}$ derived in the above section, since $pr^{(2)}v(\triangle(x, y, t, w^{(2)})) = 0$ whenever (3.2) holds, we use $w_t = e^{-w}(w_{xx} + w_{yy})$ to cancel $w_t$ and set coefficients of every monomial zero. For example the coefficient of $w_y w_{yt}$ is $-2\tau_w$. So $\tau_w = 0$, i.e. $\tau = \tau(x, y, t)$. See the following table for all the coefficients.

| monomial | coefficient | monomial | coefficient |
|----------|-------------|----------|-------------|
| $e^{-w}w_{xx}$ | $-\tau_{xx} - \tau_{yy} = 0$ | $e^{-w}w_{yy}$ | $-\tau_{xx} - \tau_{yy} = 0$ |
| $e^{-w}w_xw_{xx}$ | $-2\tau_{xx} = 0$ | $e^{-w}w_xw_{yy}$ | $-2\tau_{xx} = 0$ |
| $e^{-w}w_x^2w_{xx}$ | $-\tau_{ww} = 0$ | $e^{-w}w_x^2w_{yy}$ | $-\tau_{ww} = 0$ |
| $e^{-w}w_yw_{xx}$ | $-\tau_{ww} = 0$ | $e^{-w}w_y^2w_{xx}$ | $-\tau_{ww} = 0$ |
| $w_x$ | $-\phi = \tau_t - 2\xi_x = 0$ | $w_y$ | $-\phi + \tau_t - 2\eta_y = 0$ |
| $w_xw_{xx}$ | $-2\xi_w = 0$ | $w_yw_{yy}$ | $-2\eta_w = 0$ |
| $w_x$ | $2\phi_{xx} - \xi_{xx} - \xi_{yy} = 0$ | $w_y$ | $2\phi_{yy} - \eta_{xx} - \eta_{yy} = 0$ |
| $w_x^2$ | $\phi_{ww} - 2\xi_{xx} = 0$ | $w_xw_y$ | $-2\eta_{xx} - 2\xi_{yy} = 0$ |
| $w_x^3$ | $-\xi_{ww} = 0$ | $w_x^2w_y$ | $-\eta_{ww} = 0$ |
| $w_xt$ | $-\tau_x = 0$ | $w_xw_y$ | $-2\eta_x - 2\xi_y = 0$ |
| $w_xw_{xy}$ | $-2\eta_w = 0$ | $w_xw_{xt}$ | $-2\tau_w = 0$ |
| $w_x^2$ | $\phi_{ww} - 2\eta_{yw} = 0$ | $w_x^3$ | $-\eta_{ww} = 0$ |
| $w_yw_{xx}$ | $-\xi_{ww} = 0$ | $w_yt$ | $-2\tau_y = 0$ |
| $w_yw_{xy}$ | $-2\xi_w = 0$ | $w_yw_{yt}$ | $-2\tau_w = 0$ |

Thus we finally get
\[
\begin{align*}
\frac{\xi}{\eta} &= \xi(x, y) \\
\eta &= \eta(x, y) \\
\tau &= c_1 + c_2 t \\
\phi &= c_2 - 2\xi_x \\
\xi_x - \eta_y &= 0 \\
\eta_x + \xi_y &= 0
\end{align*}
\] (3.3)

From the last two relations in (3.3), we have
\[
\begin{align*}
\xi_{xx} + \xi_{yy} &= 0 \\
\eta_{xx} + \eta_{yy} &= 0.
\end{align*}
\]

By solving the two-dimensional Laplace equation, we can get a large number of solutions to the system (3.3). We first look at one simple case
\[
\begin{align*}
\tau &= c_1 + c_2 t \\
\xi &= c_3 + c_4 x + c_5 y \\
\eta &= c_6 - c_5 x + c_4 y \\
\phi &= c_2 - 2c_4
\end{align*}
\] (3.4)

Thus we get
\[
\begin{align*}
\mathbf{v}_1 &= \frac{\partial}{\partial t} \\
\mathbf{v}_2 &= \frac{\partial}{\partial x} \\
\mathbf{v}_3 &= \frac{\partial}{\partial y} \\
\mathbf{v}_4 &= t \frac{\partial}{\partial t} + \frac{\partial}{\partial w} \\
\mathbf{v}_5 &= y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \\
\mathbf{v}_6 &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - 2 \frac{\partial}{\partial w}
\end{align*}
\] (3.5)

The corresponding one-parameter transformation groups are
\[
\begin{align*}
G_1 : (x, y, t + \varepsilon, w) \\
G_2 : (x + \varepsilon, y, t, w) \\
G_3 : (x, y + \varepsilon, t, w) \\
G_4 : (x, y, e^\varepsilon t, w + \varepsilon) \\
G_5 : (x + \varepsilon y, y - \varepsilon x, t, w) \\
G_6 : (e^\varepsilon x, e^\varepsilon y, t, w - 2\varepsilon)
\end{align*}
\]

Equivalently, if \( w = f(x, y, t) \) is a solution to (3.2), then the following are also solutions
to (3.2):

\[
\begin{align*}
  w^{(1)} &= f(x, y, t - \varepsilon) \\
  w^{(2)} &= f(x - \varepsilon, y, t) \\
  w^{(3)} &= f(x, y - \varepsilon, t) \\
  w^{(4)} &= f(x, y, e^{-\varepsilon}t) + \varepsilon \\
  w^{(5)} &= f(x - \varepsilon y, y + \varepsilon x, (1 + \varepsilon^2)t) \\
  w^{(6)} &= f(e^{-\varepsilon}x, e^{-\varepsilon}y, t) - 2\varepsilon
\end{align*}
\]

For example we examine \( w^{(6)} \),

\[
\begin{align*}
  (e^{w^{(6)}})_t - w^{(6)}_{xx} - w^{(6)}_{yy} &= (e^{-2\varepsilon}e^f)_t - e^{-2\varepsilon}f_{xx} - e^{-2\varepsilon}f_{yy} \\
  &= e^{-2\varepsilon}((e^f)_t - f_{xx} - f_{yy}) \\
  &= 0.
\end{align*}
\]

Next we have the following structure constants table such that the entry in \( i \)-row and \( j \)-volume represents \([v_i, v_j]\):

| Lie  | \( v_1 \) | \( v_2 \) | \( v_3 \) | \( v_4 \) | \( v_5 \) | \( v_6 \) |
|------|-----------|-----------|-----------|-----------|-----------|-----------|
| \( v_1 \) | 0 | 0 | 0 | \( v_1 \) | 0 | 0 |
| \( v_2 \) | 0 | 0 | 0 | 0 | \( -v_3 \) | \( v_2 \) |
| \( v_3 \) | 0 | 0 | 0 | 0 | \( v_2 \) | \( v_3 \) |
| \( v_4 \) | \( -v_1 \) | 0 | 0 | 0 | 0 | 0 |
| \( v_5 \) | 0 | \( v_3 \) | \( -v_2 \) | 0 | 0 | 0 |
| \( v_6 \) | 0 | \( -v_2 \) | \( -v_3 \) | 0 | 0 | 0 |

Using the formula

\[
Ad(\exp(\varepsilon v_1))v_j = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(ad v_1)^n(v_j) = v_j - \varepsilon[v_i, v_j] + \frac{\varepsilon^2}{2}[v_i, [v_i, v_j]] - \cdots
\]

We get the adjoint representation table for (3.5):
Now we use the adjoint representation table to give the classification of subalgebras of (3.5). Given a vector

\[ \mathbf{v} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + a_4 \mathbf{v}_4 + a_5 \mathbf{v}_5 + a_6 \mathbf{v}_6, \]

we first assume \( a_6 \neq 0 \), so after scaling, we can make \( a_6 = 1 \):

\[ \mathbf{v} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + a_4 \mathbf{v}_4 + a_5 \mathbf{v}_5 + \mathbf{v}_6. \]

If we act on \( \mathbf{v} \) by \( \text{Ad}(\exp((a_2 - a_5(a_3 + a_2a_5))\mathbf{v}_2)) \) and \( \text{Ad}(\exp((a_3 + a_2a_5)\mathbf{v}_3)) \) respectively, we can make the coefficients of \( \mathbf{v}_2 \) and \( \mathbf{v}_3 \) vanish:

\[ \mathbf{v}^{(1)} = \text{Ad}(\exp((a_3+a_2a_5)\mathbf{v}_3)) \circ \text{Ad}(\exp((a_2-a_5(a_3+a_2a_5))\mathbf{v}_2)) \mathbf{v} = a_1 \mathbf{v}_1 + a_4 \mathbf{v}_4 + a_5 \mathbf{v}_5 + \mathbf{v}_6. \]

Next we act on \( \mathbf{v}^{(1)} \) by \( \text{Ad}(\exp(a_1\mathbf{v}_1)) \) to cancel the the coefficient of \( \mathbf{v}_1 \), so finally \( \mathbf{v} \) is equivalent to \( \mathbf{v}^{(2)} = a_4 \mathbf{v}_4 + a_5 \mathbf{v}_5 + a_6 \) under the adjoint representation. In other words, every one-dimensional subalgebra generated by \( \mathbf{v} \) with \( a_6 \neq 0 \) is equivalent to the subalgebra spanned by \( a_4 \mathbf{v}_4 + a_5 \mathbf{v}_5 + a_6 \).

The remaining one-dimensional subalgebras are spanned by vector with \( a_6 = 0 \). If \( a_5 \neq 0 \), by scaling we make \( a_5 = 1 \), and then act on \( \mathbf{v} \) by \( \text{Ad}(\exp(-a_3\mathbf{v}_2)) \) and \( \text{Ad}(\exp(a_2\mathbf{v}_3)) \) respectively so that \( \mathbf{v} \) is equivalent to \( \mathbf{v}^{(1)} = a_4 \mathbf{v}_4 + \mathbf{v}_5. \)

Next, if \( a_5 = a_6 = 0 \) and \( a_4 \neq 0 \), consider \( \mathbf{v} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \mathbf{v}_4 \). First act on it by \( \text{Ad}(\exp(a_1\mathbf{v}_1)) \),

\[ \mathbf{v}^{(1)} = \text{Ad}(\exp(a_1\mathbf{v}_1)) \mathbf{v} = a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \mathbf{v}_4. \]
If \( a_2 = 0 \), then \( v^{(1)} = a_3 v_3 + v_4 \). Otherwise,

\[
v^{(2)} = \text{Ad}(\exp(\arctan(a_3/a_2)v_5))v^{(1)} = lv_2 + v_4,
\]

where \( l = l(a_2, a_3) \). Further we can use and act on \( v^1 \) and \( v^{(2)} \) respectively to scale \( a_3 \) and \( l \). Thus any one-dimensional subalgebra spanned by \( v \) with \( a_5 = a_6 = 0 \) and \( a_4 \neq 0 \) is equivalent to the subalgebra spanned by either \( v_4, v_4 + v_2, v_4 - v_2, v_4 + v_3 \) or \( v_4 - v_3 \).

If \( a_4 = a_5 = a_6 = 0 \) and \( a_1 \neq 0 \), let \( v = v_1 + a_2 v_2 + a_3 v_3 \). If \( a_2 = 0 \), then \( v = v_1 + a_3 v_3 \). Otherwise

\[
v^{(1)} = \text{Ad}(\exp(\arctan(a_3/a_2)v_5))v = bfv_1 + lv_2,
\]

where \( l = l(a_2, a_3) \). By further scaling \( a_3 \) and \( l \) using the adjoint representation, we finally get the result that when \( a_4 = a_5 = a_6 = 0 \) and \( a_1 \neq 0 \), the subalgebra spanned by \( v \) is equivalent to either \( v_1, v_1 + v_2, v_1 - v_2, v_1 + v_3, v_1 - v_3 \).

If \( a_1 = a_4 = a_5 = a_6 = 0 \), then the subalgebra spanned by \( v \) is equivalent to either \( v_2 + a_3 v_3 \) or \( v_3 \).

If we further allow the discrete symmetry for example \( (\xi, \eta, \tau, \phi) \mapsto (-\xi, \eta, \tau, \phi) \), then \( v_1 - v_2 \) is mapped to \( v_1 + v_2 \). So the following theorem holds:

**Theorem 3.1.** The operators in (3.5) generate an optimal system \( S \)

(a) \( v_6 + a_4 v_4 + a_5 v_5, a_6 \neq 0 \);
(b) \( v_5 + a_4 v_4, a_6 = 0, a_5 \neq 0 \);
(c1) \( v_4, a_5 = a_6 = 0, a_4 \neq 0 \);
(c2) \( v_4 + v_2, a_5 = a_6 = 0, a_4 \neq 0 \);
(c3) \( v_4 + v_3, a_5 = a_6 = 0, a_4 \neq 0 \);
(d1) \( v_1, a_4 = a_5 = a_6 = 0, a_1 \neq 0 \);
(d2) \( v_1 + v_2, a_4 = a_5 = a_6 = 0, a_1 \neq 0 \);
(d3) \( v_1 - v_3, a_4 = a_5 = a_6 = 0, a_1 \neq 0 \);
(e) $v_2 + a_3 v_3$, $a_1 = a_4 = a_5 = a_6 = 0$, $a_2 \neq 0$;

(f) $v_3$, $a_1 = a_2 = a_4 = a_5 = a_6 = 0$.

We calculate two examples to show how to use the subalgebras to find symmetric solutions of (3.2) or (3.1).

From $v = v_4 + v_2 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$, its characteristics are derived from

$$\frac{dt}{t} = \frac{dx}{t} = \frac{dw}{t}.$$ 

So let $z = e^y/t$ and $e^w = t \omega(z)$, then we have

$$z^2 \omega'' - z^2 (\omega')^2 + z \omega^2 \omega' - \omega^3 = 0. \quad (3.8)$$

From $v = 2v_4 + v_5 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$, let $\varepsilon = \frac{x}{\sqrt{t}}$, $\eta = \frac{y}{\sqrt{t}}$, and $e^w = \omega(\varepsilon, \eta)$, then we have

$$\omega_{\varepsilon \varepsilon} + \omega_{\eta \eta} = \frac{\omega_{\varepsilon}^2 + \omega_{\eta}^2}{\omega} - \frac{1}{2} \omega (\varepsilon \omega_{\varepsilon} + \eta \omega_{\eta}). \quad (3.9)$$

Note that we only used a quite simple solution (3.4) of (3.3) to analyze the symmetries of (3.2). If we use other solutions of (3.3), we expect to find many more symmetries of (3.2). The whole question lies in finding solutions for the two-dimensional Laplace equation. In contrast to the linear heat equation, amazingly (3.1) has various symmetries. We believe study of (3.3) will lead to some significant results for Ricci flow on surfaces. For example, we can consider

$$\begin{cases} 
\tau = c_1 + c_2 t \\
\xi = c_3 (x^2 - y^2) + c_4 xy + c_5 \\
\eta = \frac{1}{2} c_4 (y^2 - x^2) + 2 c_3 xy + c_6 \\
\phi = c_2 - 4 c_3 x - 2 c_4 y,
\end{cases} \quad (3.10)$$

or more complicated case
\[
\left\{
\begin{array}{l}
\tau = c_1 + c_2 t \\
\xi = (c_3 \cos x + c_4 \sin x)e^y + (c_4 \cos y + c_6 \sin y)e^x + c_7 \\
\eta = (c_4 \cos x - c_3 \sin x)e^y + (-c_6 \cos y + c_5 \sin y)e^x + c_8 \\
\phi = c_2 + 2c_3 e^y \sin x - 2c_4 e^y \cos x - 2c_5 e^x \cos y - 2c_6 e^x \sin y.
\end{array}
\right.
\] (3.11)

\section{Hyperbolic geometric flow on Riemann surfaces}

In this section, we consider the hyperbolic geometric flow on Riemann surfaces. By the same statements as in the beginning of section 3, the hyperbolic geometric flow

\[\frac{\partial^2}{\partial t^2}g = -2Rc\]

can be simplified to

\[u_{tt} = \Delta \ln u,\] (4.1)

Let \(w = \ln u\), then

\[\Delta(x, y, t, w^{(2)}) = e^w w_{tt} + e^w w_t^2 - w_{xx} - w_{yy} = 0.\] (4.2)

The following initial problem has been studied in \([10]\),

\[
\left\{
\begin{array}{l}
u_{tt} - (\ln u)_{xx} = 0 \\
t = 0 : \quad u = u_0(x), \quad u_t = u_t(x).
\end{array}
\right.
\]

Given any initial metric only depending on one space variable, if the initial velocity is large enough, then the solution of (4.1) exists for all positive time and the scalar curvature is uniformly bounded. Otherwise, the solution exists only finite time and the scalar curvature goes to infinity as \(t\) goes to the maximal time.

The Jacobian matrix of (4.2) is

\[\mathcal{J}_\Delta(x, y, t; w; w_x, w_y, w_t; w_{xx}, w_{xy}, w_{xt}, w_{yy}, w_{yt}, w_{tt}) = (0, 0, 0; e^w (w_{tt} + w_t^2), 0, 0, 2e^w w_t; -1, 0, 0, -1, 0, e^w).\]

Thus the original equation (4.2) is of maximal rank everywhere in \(\mathcal{J}_\Delta\). Obviously (4.2)
is locally solvable. So (4.2) is nondegenerate and we can apply Theorem 2.6.

Given a vector

\[ v = \xi(x, y, t, w) \frac{\partial}{\partial x} + \eta(x, y, t, w) \frac{\partial}{\partial y} + \tau(x, y, t, w) \frac{\partial}{\partial t} + \phi(x, y, t, w) \frac{\partial}{\partial w}, \]

we have

\[ pr(2)\phi = \phi^w (w_{tt} + w_t^2) + 2\phi^t w_t + \phi^w w - \phi^{xx} - \phi^{yy}. \]

We apply the formulas for \( \phi^t, \phi^w, \phi^{xx}, \phi^{yy} \) derived in section 2, since \( pr(2)\phi = 0 \) whenever (4.2) holds, we use \( w_{yy} = \phi^w w_{tt} + e^w w_t^2 - w_{xx} \) to cancel \( w_{yy} \) and set coefficients of every monomial zero. The coefficient table is

| monomial \( e^w \) | coefficient |
|------------------|-------------|
| \( w_{tt} \)    | \( \phi - 2\tau + 2\eta_y = 0 \) |
| \( w_t \)       | \( 2\phi_t + 2\phi_{tw} - \tau_t = 0 \) |
| \( w_t w_y \)   | \( -2\eta_t - 2\eta_{tw} = 0 \) |
| \( w_y w_t \)   | \( \eta_w - \eta_{ww} = 0 \) |
| \( w_x \)       | \( \phi_{tt} = 0 \) |
| \( w_x \)       | \( -\xi_{tt} = 0 \) |
| \( w_y \)       | \( -2\eta_t = 0 \) |
| \( w_{yt} \)    | \( e^w w_{yt} = 0 \) |
| \( w_{yt} \)    | \( 2\tau_{yw} = 0 \) |
| \( w_{x,y} \)   | \( -2\xi = 0 \) |
| \( w_{yt} \)    | \( 1 \) |
| \( w_y \)       | \( -2\phi_{yw} + \eta_{xx} + \eta_{yy} = 0 \) |
| \( w_t \)       | \( \tau_{xx} + \tau_{yy} = 0 \) |
| \( w_{x,y} \)   | \( 2\xi_{yw} + 2\eta_{xx} = 0 \) |
| \( w_{yy} \)    | \( \eta_{ww} = 0 \) |
| \( w_{x} \)     | \( \tau_{ww} = 0 \) |
| \( w_{yt} \)    | \( 2\tau_y = 0 \) |
| \( w_{yt} \)    | \( w_{xy} = 0 \) |
| \( w_{yt} \)    | \( 2\xi_x = 0 \) |
| \( w_{yt} \)    | \( 2\xi_y + 2\eta_x = 0 \) |
| \( w_{yt} \)    | \( 2\tau_y = 0 \) |
| \( w_{yt} \)    | \( 2\tau_y = 0 \) |
| \( w_{yt} \)    | \( 2\tau_y = 0 \) |
| \( w_{yt} \)    | \( 2\tau_y = 0 \) |
| \( w_{yt} \)    | \( 2\tau_y = 0 \) |
| \( w_{yt} \)    | \( 2\tau_y = 0 \) |
Finally, we get

\[
\begin{align*}
\xi &= \xi(x, y) \\
\eta &= \eta(x, y) \\
\tau &= c_1 + c_2 t \\
\phi &= 2c_2 - 2\xi_x \\
\xi_x - \eta_y &= 0 \\
\eta_x + \xi_y &= 0
\end{align*}
\]

This is the same as (3.3) except the coefficient 2 of \(c_2\) in \(\phi\). This is due to the fact that in wave type equation (4.2), we differentiate \(w\) twice along the time variable \(t\).

If we choose

\[
\begin{align*}
\tau &= c_1 + c_2 t \\
\xi &= c_3 + c_4 x + c_5 y \\
\eta &= c_6 - c_5 x + c_4 y \\
\phi &= 2c_2 - 2c_4
\end{align*}
\]

Then we get

\[
\begin{align*}
v_1 &= \frac{\partial}{\partial t} \\
v_2 &= \frac{\partial}{\partial x} \\
v_3 &= \frac{\partial}{\partial y} \\
v_4 &= t\frac{\partial}{\partial t} + 2\frac{\partial}{\partial w} \\
v_5 &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \\
v_6 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2\frac{\partial}{\partial w}
\end{align*}
\]

The corresponding one-parameter transformation groups are

\[
\begin{align*}
G_1 &: (x, y, t + \varepsilon, w) \\
G_2 &: (x + \varepsilon, y, t, w) \\
G_3 &: (x, y + \varepsilon, t, w) \\
G_4 &: (x, y, e^\varepsilon t, w + 2\varepsilon) \\
G_5 &: (x + \varepsilon y, y - \varepsilon x, t, w) \\
G_6 &: (e^\varepsilon x, e^\varepsilon y, t, w - 2\varepsilon)
\end{align*}
\]

Equivalently, if \(w = f(x, y, t)\) is a solution to (4.2), then the following are also solutions to (4.2):
The commutator table and the adjoint representation of (4.5) are the same as those of (3.5), namely (3.6) and (3.7), so we omit them here.

Similarly, we have

**Theorem 4.1.** The operators in (4.5) generate an optimal system $S$

(a) $v_6 + a_4 v_4 + a_5 v_5, a_6 \neq 0$;

(b) $v_5 + a_4 v_4, a_6 = 0, a_5 \neq 0$;

(c) $v_4, a_5 = a_6 = 0, a_4 \neq 0$;

(d) $v_4 + v_2, a_5 = a_6 = 0, a_4 \neq 0$;

(e) $v_4 + v_3, a_5 = a_6 = 0, a_4 \neq 0$;

(f) $v_1, a_4 = a_5 = a_6 = 0, a_1 \neq 0$;

(g) $v_1 + v_2, a_4 = a_5 = a_6 = 0, a_1 \neq 0$;

(h) $v_1 - v_3, a_4 = a_5 = a_6 = 0, a_1 \neq 0$;

(i) $v_2 + a_3 v_3, a_1 = a_4 = a_5 = a_6 = 0, a_2 \neq 0$;

(j) $v_3, a_1 = a_2 = a_4 = a_5 = a_6 = 0$.

From $v = v_4 + v_2 = t \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + 2 \frac{\partial}{\partial w}$; its characteristics are derived from

$$\frac{dt}{t} = \frac{dx}{1} = \frac{dw}{2}.$$ 

So let $z = e^y/t$ and $e^w = t^2 \omega(z)$, then we have

$$z^2(\omega^2 - \omega)'' - z(2\omega^2 + \omega)\omega' + z^2(\omega')^2 + 2\omega^3 = 0. \quad (4.6)$$
From $v = v_4 + v_5 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$, let $\varepsilon = \frac{x}{t}$, $\eta = \frac{y}{t}$, and $e^w = \omega(\varepsilon, \eta)$, then we have

\[
(\varepsilon^2 \omega^2 - \omega) \omega_{\varepsilon \varepsilon} + (\eta^2 \omega^2 - \omega) \omega_{\eta \eta} + 2 \varepsilon \eta \omega^2 \omega_{\varepsilon \eta} + \omega_{\varepsilon}^2 + \omega_{\eta}^2 = 0. \tag{4.7}
\]

§5 Warped products on $S^{n+1}$ of both flows

In contrast to the Ricci flow and hyperbolic geometric flow on surfaces, the warped products on $S^{n+1}$ of both flows admit few symmetries. We omit the somewhat laboring calculations and only state the results here.

Recall that warped product on $S^{n+1}$ is of the form

\[ g = \varphi^2(x, t)dx^2 + \psi^2(x, t)g_{can}, \]

where $g_{can}$ denotes the canonical metric on $S^n$. By a change of coordinate

\[ s(x) = \int_0^x \varphi(x)dx, \tag{5.1} \]

the evolutions of $\varphi(s, t)$ and $\psi(s, t)$ under Ricci flow and hyperbolic geometric flow are the followings respectively:

\[
\begin{cases}
\varphi_t = n \frac{\psi_s}{\psi} \varphi \\
\psi_t = \psi_{ss} - (n - 1) \frac{1 - \psi^2}{\psi}
\end{cases} \tag{5.2}
\]

under Ricci flow, and

\[
\begin{cases}
\varphi_{tt} = n \frac{\psi_{ss}}{\psi} \varphi - \frac{\varphi^2}{\psi} \\
\psi_{tt} = \psi_{ss} - (n - 1) \frac{1 - \psi^2}{\psi} - \frac{\psi^2}{\psi}
\end{cases} \tag{5.3}
\]

under hyperbolic geometric flow. Due to the change of coordinate (5.1), we can only consider the second equations of (5.2) and (5.3), namely

\[ \psi_t = \psi_{ss} - (n - 1) \frac{1 - \psi^2}{\psi}, \tag{5.4} \]
and
\[ \psi_{tt} = \psi_{ss} - (n - 1) \frac{1 - \psi^2_s}{\psi} - \frac{\psi_t^2}{\psi}. \] (5.5)

Given a vector
\[ \mathbf{v} = \xi(s, t, \psi) \frac{\partial}{\partial s} + \tau(s, t, \psi) \frac{\partial}{\partial t} + \phi(s, t, \psi) \frac{\partial}{\partial \psi} \]
We have the following two theorems:

**Theorem 5.1 (symmetries for warped product of Ricci flow).** For equation (5.4), we have

- when \( n = 1 \):
  \[ \begin{align*}
  v_1 &= \frac{\partial}{\partial s} \\
  v_2 &= \frac{\partial}{\partial t} \\
  v_3 &= s \frac{\partial}{\partial s} + t \frac{\partial}{\partial t}
  \end{align*} \]
  These are translations and dilatation.

- when \( n = 2 \):
  \[ \begin{align*}
  v_1 &= \frac{\partial}{\partial s} \\
  v_2 &= \frac{\partial}{\partial t} \\
  v_3 &= t \frac{\partial}{\partial s} + s \frac{\partial}{\partial t}
  \end{align*} \]
  These are translations and hyperbolic rotation.

- when \( n > 2 \):
  \[ \begin{align*}
  v_1 &= \frac{\partial}{\partial s} \\
  v_2 &= \frac{\partial}{\partial t}
  \end{align*} \]
  These are only translations.

**Theorem 5.2 (symmetries for warped product of hyperbolic geometric flow).** For equation (5.5), we have

- when \( n = 1 \): the equation becomes linear heat equation, so
Symmetries of geometric flows

\[
\begin{align*}
v_1 &= \frac{\partial}{\partial s} \\
v_2 &= \frac{\partial}{\partial t} \\
v_3 &= \psi \frac{\partial}{\partial \psi} \\
v_4 &= s \frac{\partial}{\partial s} + 2t \frac{\partial}{\partial t} \\
v_5 &= 2t \frac{\partial}{\partial s} - s \psi \frac{\partial}{\partial \psi} \\
v_6 &= 4ts \frac{\partial}{\partial s} + 4t^2 \frac{\partial}{\partial t} - (s^2 + 2t) \psi \frac{\partial}{\partial \psi},
\end{align*}
\]

and the infinite-dimensional subalgebra

\[
v_\alpha = \alpha(s,t) \frac{\partial}{\partial \psi},
\]

where \(\alpha\) is an arbitrary solution of the heat equation.

when \(n > 1\):

\[
\begin{align*}
v_1 &= \frac{\partial}{\partial s} \\
v_2 &= \frac{\partial}{\partial t}
\end{align*}
\]

§6 Further discussion

Ricci flow is a powerful tool to understand the geometry and topology of Riemann manifolds. Any symmetry and exact solution of its equation will help us understand its behavior for general cases and the singularity formation, further the basic topological and geometrical properties as well as analytic properties of the underlying manifolds. The hyperbolic geometric flow is the hyperbolic version of Ricci flow. It is also closely related to the Einstein equation. Any symmetry and exact solution of it can help us find new solutions of the Einstein Equation which plays significant role in general relativity and modern theoretical physics.

The techniques we use in this paper, namely the theory of group-invariant solutions for differential equations is a powerful tool to analyze various differential equations. In fact, it can also be used in normal functions and systems of differential equations, see [11]. We hope in the future this method can be generalized to tensor equations so that we can use it to analyze complicated systems on manifolds. This theory of group-invariant solutions is an application of Lie groups to differential equations. Lie group plays a fundamental role in modern mathematics since it has significant influence on almost all the branches of mathematics.
§7 Appendix: Derivation of the evolutions of warped products on $S^{n+1}$

In this appendix, we derive evolution equations from the metrics on $S^{n+1}$:

$$g = \varphi(x)^2 dx \otimes dx + \psi(x)^2 ds_n^2$$

where $ds_n^2$ denotes the canonical metric of constant curvature 1 on $S^n$.

By introducing a new coordinate the distance $s$ to the equator given by

$$s(x) = \int_0^x \varphi(x) dx,$$

the metrics can be simplified to

$$g = ds^2 + \psi(s)^2 ds_n^2 = ds^2 + g_s.$$  \hspace{1cm} (7.1)

In the following we will consider (7.1).

Metrics (7.1) are standard warped products which are studied in details in [12]. Our discussion follows closely with Chapter 3 of that book.

Given a Riemannian manifold $(\mathcal{M}, g)$, first we define the Hessian function of $f$ as a symmetric $(0, 2)$-tensor

$$Hess f(X, Y) = \nabla_X \nabla_Y f - \nabla_{\nabla_X Y} f,$$

where $X, Y$ are vector fields on $\mathcal{M}$. Thus

$$Hess f(X, Y) = g(\nabla_X (\nabla f), Y).$$

Second we will say that $s : U \to \mathbb{R}$, where $U \subset (\mathcal{M}, g)$ is open, is a distance function if $|\nabla s| \equiv 1$ on $U$. We shall use the following Gauss equation about distance function

$$g(R(X, Y)Z, W) = g_s(R^s(X, Y)Z, W) - II(Y, Z)II(X, W) + II(X, Z)II(Y, W),$$
here $X, Y, Z, W$ are tangent to the level sets $U_s$ and

$$II(U, V) = Hess s(U, V)$$

is the classical second fundamental form.

Particularly in our case, for the rotationally symmetric metrics (7.1),

$$2Hess s = L_{\partial_s g_s} = L_{\partial_s (\psi^2 ds_n^2)} = \partial_s (\psi^2) ds_n^2 + \psi^2 L_{\partial_s (ds_n^2)} = 2\psi (\partial_s \psi) ds_n^2 = 2\frac{\partial_s \psi}{\psi} g_s.$$

Thus

$$Hess s = \frac{\partial_s \psi}{\psi} g_s.$$

Using that $g_s$ is the metric of curvature $\frac{1}{\psi^2}$ on the sphere, we get

$$g_s(R_s^s(X, Y)V, W) = \frac{1}{\psi^2} g_s(X \wedge Y, W \wedge V).$$

Combining this with $II = Hess s$, from the Guass equation we obtain

$$g(R(X, Y)V, W) = \frac{1 - (\partial_s \psi)^2}{\psi^2} g_s(X \wedge Y, W \wedge V). \quad (7.2)$$

From another important formula

$$(\nabla_{\partial_s} Hess s)(X, Y) + Hess^2 s(X, Y) = -R(X, \partial_s, \partial_s, Y),$$

and in our case
\[ \nabla_{\partial_s} \text{Hess } s = \nabla_{\partial_s} \left( \frac{\partial_s \psi}{\psi} g_s \right) = \partial_s \left( \frac{\partial_s \psi}{\psi} \right) g_s + \frac{\partial_s \psi}{\psi} \nabla_{\partial_s} (g_s) = \frac{1}{\psi^2} \left( \frac{\partial^2 \psi}{\psi} - \left( \frac{\partial_s \psi}{\psi} \right)^2 \right) g_s = \frac{\partial^2 \psi}{\psi} g_s - (\frac{\partial_s \psi}{\psi})^2 g_s = \frac{\partial^2 \psi}{\psi} g_s - \text{Hess}^2 s, \]

we obtain

\[ R(\cdot, \partial_s, \partial_s, \cdot) = -\frac{\partial^2 \psi}{\psi}. \tag{7.3} \]

(7.2) and (7.3) are just

\[ K_0 = -\frac{\psi_{ss}}{\psi}, \quad K_1 = \frac{1 - \psi^2}{\psi^2}, \]

where \( K_0 \) are the sectional curvatures of the 2-planes perpendicular to the spheres \( \{x\} \times S^n \), and \( K_1 \) those of the 2-planes tangential to these spheres.

Hence for tangential vector \( X \),

\[ \text{Ric}(X) = \sum_{i=1}^{n+1} R(X, E_i) E_i = \sum_{i=1}^{n} R(X, E_i) E_i + R(X, \partial_s) \partial_s = ((n - 1)K_1 + K_0)X. \]

And

\[ \text{Ric}(\partial_s) = nK_0. \]

Since the metrics (7.1) are Einstein metrics, we write

\[ \text{Rc} = nK_0 ds^2 + (n - 1)K_1 + K_0)g_s. \]

Finally we obtain

\[ \text{Rc}[g] = -n \frac{\psi_{ss}}{\psi} ds^2 + [-\psi\psi_{ss} - (n - 1)\psi^2 + n - 1]ds_n^2, \tag{7.4} \]
Differentiate $g$ along $t$, we get
\[
\frac{\partial}{\partial t} g = 2\varphi_t \phi ds^2 + 2\psi\psi_t ds_n^2,
\]
and
\[
\frac{\partial^2}{\partial t^2} g = \frac{2(\varphi\varphi_t + \varphi_t^2)}{\varphi^2} ds^2 + 2(\psi\psi_t + \psi_t^2) ds_n^2.
\]
So for Ricci flow $\frac{\partial}{\partial t} g = -2Rc$,
\[
\begin{cases}
\frac{\partial}{\partial t} \phi = n\frac{\psi_{ss}}{\psi}\phi \\
\frac{\partial}{\partial t} \psi = \psi_{ss} - (n - 1)\frac{1}{\psi^2}\phi^2.
\end{cases}
\]
For hyperbolic geometric flow $\frac{\partial^2}{\partial t^2} g = -2Rc$,
\[
\begin{cases}
\frac{\partial^2}{\partial t^2} \phi = n\frac{\psi_{ss}}{\psi}\phi - \frac{\psi_t^2}{\psi} \\
\frac{\partial^2}{\partial t^2} \psi = \psi_{ss} - (n - 1)\frac{1}{\psi^2}\phi^2 - \frac{\psi_t^2}{\psi}.
\end{cases}
\]

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