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A curious formula related to the Euler Gamma function

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Abstract

In this note, we prove that for all $x \in (0, 1)$, we have:

$$\log \Gamma(x) = \frac{1}{2} \log \pi + \pi \eta \left(\frac{1}{2} - x\right) - \frac{1}{2} \log \sin(\pi x) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\log n}{n} \sin(2\pi nx),$$

where $\Gamma$ denotes the Euler Gamma function and

$$\eta := \int_0^1 \log \Gamma(x) \cdot \sin(2\pi x) \, dx = 0.7687478924\ldots$$

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1 Introduction

Throughout this note, we let $\eta := 2 \int_0^1 \log \Gamma(x) \cdot \sin(2\pi x) \, dx$ and we let $\langle x \rangle$ denote the fractional part of a given real number $x$. We let also $B_k$ ($k \in \mathbb{N}$) denote the Bernoulli polynomials which are usually defined by means of the generating function:

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}.$$ 

We just note that each polynomial $B_k$ is monic and has degree $k$ and that the two first Bernoulli polynomials are $B_0(x) = 1$ and $B_1(x) = x - \frac{1}{2}$.

The Euler Gamma function is one of the most important special functions in complex analysis. For complex numbers $z$ with positive real part, it can be defined by the convergent improper integral:

$$\Gamma(z) := \int_0^{+\infty} t^{z-1} e^{-t} \, dt.$$ 

From this, we easily deduce the important functional equation $\Gamma(z + 1) = z \Gamma(z)$ which we use to extend $\Gamma$ by analytic continuation to all complex numbers except the non-positive
integers. The obtained function is meromorphic with simple poles at the non-positive
integers; that is what we call “the Gamma function” in its generality. We also precise
that the Gamma function is an interpolation of the sequence \((n - 1)!\) for all positive integer \(n\). Actually, the interpolation of the factorial sequence is the motivation of Euler that lead him to discover the Gamma function. More
interestingly, the Bohr-Mollerup theorem states that the Gamma function is the unique
function \(f\) satisfying \(f(1) = 1\) and \(f(x + 1) = xf(x)\), which is log-convex on the positive
real axis.

Several important formulas and properties are known for the Gamma function. We
recommend the reader to consult the books [1, 2, 3]. In this note, we just cite the formulas
we use for proving our main result. The first one is the so-called “Euler’s reflection
formula”, which is given (for all \(z \in \mathbb{C} \setminus \mathbb{Z}\)) by:

\[
\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)} \tag{1}
\]

and the second one is “the Gauss multiplication formula” which is given by:

\[
\Gamma(z)\Gamma\left(z + \frac{1}{k}\right) \cdots \Gamma\left(z + \frac{k - 1}{k}\right) = (2\pi)^{\frac{1}{2k}} k^{-z + \frac{1}{2}} \Gamma(kz) \tag{2}
\]

and holds for any positive integer \(k\) and any complex number \(z\) for which the both sides of
(2) are well-defined. The particular case \(k = 2\) of (2) is the so-called “Legendre’s formula
of duplication”.

The aim of this note is to obtain the Fourier expansion of the 1-periodic function
which coincides with the function \(\log \Gamma\) on the interval \((0, 1)\).

Some useful Fourier expansions

The proof of our main result needs the Fourier expansion of the two 1-periodic functions
\(x \mapsto \log |\sin(\pi x)|\) and \(x \mapsto B_1(\langle x \rangle)\). These Fourier expansions are both known.

The Fourier expansions of the functions \(x \mapsto B_k(\langle x \rangle)\) \((k \geq 1)\) was discovered by Hurwitz
in 1890 and are given by the identity:

\[
B_k(\langle x \rangle) = -\frac{2 \cdot k!}{(2\pi)^k} \sum_{n=1}^{\infty} \frac{\cos \left(2\pi nx - \frac{k\pi}{2}\right)}{n^k},
\]

which holds for all \(x \in \mathbb{R} \setminus \mathbb{Z}\) if \(k = 1\) and for all \(x \in \mathbb{R}\) if \(k \geq 2\).

Taking \(k = 1\) in the last identity, we obtain in particular for all \(x \in \mathbb{R} \setminus \mathbb{Z}\):

\[
B_1(\langle x \rangle) = \langle x \rangle - \frac{1}{2} = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n} \tag{3}
\]

On the other hand, the Fourier expansion of the 1-periodic function \(x \mapsto \log |\sin(\pi x)|\) is
well-known and given (for all \(x \in \mathbb{R} \setminus \mathbb{Z}\)) by:

\[
\log |\sin(\pi x)| = -\log 2 - \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{n} \tag{4}
\]

For all these Fourier expansions and trigonometric series in general, we recommend the
reader to consult the book of A. Zygmund [4].
2 The results

Our main result is the following:

**Theorem 1.** For all $x \in (0, 1)$, we have:

$$\log \Gamma(x) = \frac{1}{2} \log \pi + \pi \eta \left(\frac{1}{2} - x\right) - \frac{1}{2} \log \sin(\pi x) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\log n}{n} \sin(2\pi nx),$$

where $\eta := 2 \int_{0}^{1} \log \Gamma(x) \cdot \sin(2\pi x) \, dx = 0.7687478924 \ldots$.

The proof of Theorem 1 requires the following lemma:

**Lemma 2.** For all positive integers $n$ and $k$ and all real number $x$, we have:

$$\sum_{\ell=0}^{k-1} \cos \left\{ 2\pi n \left( x + \frac{\ell}{k} \right) \right\} = \begin{cases} 0 & \text{if } n \not\equiv 0 \pmod{k} \\ k \cos(2\pi nx) & \text{if } n \equiv 0 \pmod{k} \end{cases}$$

and

$$\sum_{\ell=0}^{k-1} \sin \left\{ 2\pi n \left( x + \frac{\ell}{k} \right) \right\} = \begin{cases} 0 & \text{if } n \not\equiv 0 \pmod{k} \\ k \sin(2\pi nx) & \text{if } n \equiv 0 \pmod{k} \end{cases}.$$

**Proof.** Let $n, k$ be positive integers and $x$ be a real number. Define

$$A := \sum_{\ell=0}^{k-1} \cos \left\{ 2\pi n \left( x + \frac{\ell}{k} \right) \right\} \quad \text{and} \quad B := \sum_{\ell=0}^{k-1} \sin \left\{ 2\pi n \left( x + \frac{\ell}{k} \right) \right\}.$$

Then, we have:

$$A + iB = \sum_{\ell=0}^{k-1} e^{2\pi n (x + \frac{\ell}{k}) i} = e^{2\pi n ix} \sum_{\ell=0}^{k-1} \left( e^{2\pi \frac{\ell}{k} i} \right)^{\ell}$$

$$= e^{2\pi nix} \times \begin{cases} 1 - \left( e^{2\pi \frac{\ell}{k} i} \right)^k & \text{if } n \not\equiv 0 \pmod{k} \\ k & \text{if } n \equiv 0 \pmod{k} \end{cases}.$$

Thus:

$$A + iB = \begin{cases} 0 & \text{if } n \not\equiv 0 \pmod{k} \\ ke^{2\pi nix} & \text{if } n \equiv 0 \pmod{k} \end{cases}.$$

The identities of the lemma follow by identifying the real and the imaginary parts of the two sides of the last identity. The lemma is proved. 

**Proof of Theorem 1:**

For the following, we let $\varphi$ denote the 1-periodic function which coincides with the function $\log \Gamma$ on the interval $(0, 1)$. We let also

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi nx) + \sum_{n=1}^{\infty} b_n \sin(2\pi nx)$$
denote the Fourier series associated to \( \varphi \) (where \( a_n, b_n \in \mathbb{R} \)). So, we have
\[
a_0 = \int_0^1 \log \Gamma(x) \, dx \quad \text{and} \quad b_n = 2 \int_0^1 \log \Gamma(x) \cdot \sin(2\pi nx) \, dx.
\]
Remark that all these improper integrals converges because the function \( \log \Gamma \) is continuous on \((0, 1]\) and, at the neighborhood of 0, we have that \( \log \Gamma(x) = \log \Gamma(x + 1) - \log x \sim_0 - \log x \) and that \( \int_0^1 \log x \, dx \) converges. In addition, because \( \varphi \) is of class \( C^1 \) on all the intervals constituting \( \mathbb{R} \setminus \mathbb{Z} \), we have according to the classical Dirichlet theorem:
\[
\varphi(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi nx) + \sum_{n=1}^{\infty} b_n \sin(2\pi nx) \quad (5)
\]
(for all \( x \in \mathbb{R} \setminus \mathbb{Z} \)). Now, we are going to calculate the \( a_n \)'s and the \( b_n \)'s.

**Calculation of the \( a_n \)'s:**
To do this, we lean on the Euler formula \((1)\) and on the Fourier expansion of the function \( x \mapsto \log |\sin(\pi x)| \) given by \((4)\). Using \((5)\), we have for all \( x \in \mathbb{R} \setminus \mathbb{Z} \):
\[
\varphi(x) + \varphi(1-x) = 2a_0 + \sum_{n=1}^{\infty} 2a_n \cos(2\pi nx) \quad (6)
\]
But, on the other hand, according to \((1)\) and \((4)\), we have for all \( x \in (0, 1) \):
\[
\varphi(x) + \varphi(1-x) = \log \Gamma(x) + \log \Gamma(1-x) = \log \pi - \log \sin(\pi x) = \log \pi - \log |\sin(\pi x)|.
\]
But since the two functions \( x \mapsto \varphi(x) + \varphi(1-x) \) and \( x \mapsto \log \pi - \log |\sin(\pi x)| \), which are defined on \( \mathbb{R} \setminus \mathbb{Z} \), are both 1-periodic, we generally have for all \( x \in \mathbb{R} \setminus \mathbb{Z} \):
\[
\varphi(x) + \varphi(1-x) = \log \pi - \log |\sin(\pi x)|.
\]
It follows, according to \((4)\), that for all \( x \in \mathbb{R} \setminus \mathbb{Z} \), we have:
\[
\varphi(x) + \varphi(1-x) = \log(2\pi) + \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{n} \quad (7)
\]
The comparison between \((6)\) and \((7)\) gives (according to the uniqueness of the Fourier expansion):
\[
a_0 = \frac{1}{2} \log(2\pi) \quad \text{and} \quad a_n = \frac{1}{2n} \quad (\forall n \geq 1) \quad (8)
\]
**Calculation of the \( b_n \)'s:**
To do this, we lean on the one hand on Lemma 2 and on the other hand on the Gauss formula \((2)\) and on the Fourier expansion of the function \( x \mapsto B_1((x)) \).
Let \( k \) be a positive integer and let \( \frac{1}{k} \mathbb{Z} \) denote the set \( \{ \frac{n}{k} : n \in \mathbb{Z} \} \). From \((5)\), we have for all \( x \in \mathbb{R} \setminus \frac{1}{k} \mathbb{Z} \):
\[
\varphi(x) + \varphi\left(x + \frac{1}{k}\right) + \cdots + \varphi\left(x + \frac{k-1}{k}\right)
= ka_0 + \sum_{n=1}^{\infty} a_n \left( \sum_{\ell=0}^{k-1} \cos \left\{ 2\pi n \left( x + \frac{\ell}{k} \right) \right \} \right) + \sum_{n=1}^{\infty} b_n \left( \sum_{\ell=0}^{k-1} \sin \left\{ 2\pi n \left( x + \frac{\ell}{k} \right) \right \} \right).
\]
It follows from Lemma 2 that for all \( x \in \mathbb{R} \setminus \frac{1}{k} \mathbb{Z} \):

\[
\varphi(x) + \varphi\left( x + \frac{1}{k} \right) + \cdots + \varphi\left( x + \frac{k-1}{k} \right) = k a_0 + \sum_{n \geq 1}^{\infty} k a_n \cos(2\pi n x) + \sum_{n \equiv 0 (\text{mod } k)}^{\infty} k b_n \sin(2\pi n x),
\]

which amounts to:

\[
\varphi(x) + \varphi\left( x + \frac{1}{k} \right) + \cdots + \varphi\left( x + \frac{k-1}{k} \right) = k a_0 + \sum_{n=1}^{\infty} k a_n \cos(2\pi k n x) + \sum_{n=1}^{\infty} k b_n \sin(2\pi k n x) \quad \text{(9)}
\]

\((\forall x \in \mathbb{R} \setminus \frac{1}{k} \mathbb{Z})\).

But on the other hand, according to (2), we have for all \( x \in (0, \frac{1}{k}) \):

\[
\varphi(x) + \varphi\left( x + \frac{1}{k} \right) + \cdots + \varphi\left( x + \frac{k-1}{k} \right) = \log \Gamma(x) + \log \Gamma\left( x + \frac{1}{k} \right) + \cdots + \log \Gamma\left( x + \frac{k-1}{k} \right) = \frac{k-1}{2} \log(2\pi) + \left( \frac{1}{2} - kx \right) \log k + \log \Gamma(kx) = \frac{k-1}{2} \log(2\pi) - (\log k) B_1((kx)) + \varphi(kx).
\]

But since the two functions \( x \mapsto \varphi(x) + \varphi(x + \frac{1}{k}) + \cdots + \varphi(x + \frac{k-1}{k}) \) and \( x \mapsto \frac{k-1}{2} \log(2\pi) - (\log k) B_1((kx)) + \varphi(kx) \) (which are defined on \( \mathbb{R} \setminus \frac{1}{k} \mathbb{Z} \)) are both \( \frac{1}{k} \)-periodic, we have more generally for all \( x \in \mathbb{R} \setminus \frac{1}{k} \mathbb{Z} \):

\[
\varphi(x) + \varphi\left( x + \frac{1}{k} \right) + \cdots + \varphi\left( x + \frac{k-1}{k} \right) = \frac{k-1}{2} \log(2\pi) - (\log k) B_1 ((kx)) + \varphi(kx). \quad \text{(10)}
\]

Using the Fourier expansions (3) and (5), it follows that for all \( x \in \mathbb{R} \setminus \frac{1}{k} \mathbb{Z} \):

\[
\varphi(x) + \varphi\left( x + \frac{1}{k} \right) + \cdots + \varphi\left( x + \frac{k-1}{k} \right) = \frac{k-1}{2} \log(2\pi) + a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi k n x) + \sum_{n=1}^{\infty} \left( b_n + \frac{\log k}{\pi n} \right) \sin(2\pi k n x) \quad \text{(10)}
\]

Note that the right hand sides of (9) and (10) both represent the Fourier expansion of the \( \frac{1}{k} \)-periodic function \( x \mapsto \varphi(x) + \varphi(x + \frac{1}{k}) + \cdots + \varphi(x + \frac{k-1}{k}) \). So, according to the uniqueness of the Fourier expansion, we have:

\[
ka_0 = \frac{k-1}{2} \log(2\pi) + a_0, \quad ka_k = a_n \quad (\forall n \geq 1) \quad \text{(11)}
\]

and

\[
kb_k = b_n + \frac{\log k}{\pi n} \quad (\forall n \geq 1) \quad \text{(12)}
\]
The relations (11) immediately follow from (8), so they don’t give us any new information. Actually, the information about the \( b_n \)’s follows from (12). Indeed, by taking \( n = 1 \) in (12), we get:

\[
b_k = \frac{\log k}{\pi k} + \frac{b_1}{k} = \frac{\log k}{\pi k} + \frac{\eta}{k}.
\]

Since this holds for any positive integer \( k \), then we have:

\[
b_n = \frac{\log n}{\pi n} + \frac{\eta}{n} \quad (\forall n \geq 1) \tag{13}
\]

**Conclusion:**

By replacing in (5) the values of the \( a_n \)’s and \( b_n \)’s, which are previously obtained in (8) and (13), we get for all \( x \in \mathbb{R} \setminus \mathbb{Z} \):

\[
\varphi(x) = \frac{1}{2} \log(2\pi) + \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{2n} + \sum_{n=1}^{\infty} \left( \frac{\log n}{\pi} + \eta \right) \frac{\sin(2\pi nx)}{n} \tag{14}
\]

Finally, taking \( x \in (0, 1) \), we have \( \varphi(x) = \log \Gamma(x) \), \( \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{n} = -\log 2 - \log \sin(\pi x) \) (according to (4)) and \( \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n} = \pi \left( \frac{1}{2} - x \right) \) (according to (3)); so it follows from (14) that for all \( x \in (0, 1) \), we have:

\[
\log \Gamma(x) = \frac{1}{2} \log \pi + \pi \eta \left( \frac{1}{2} - x \right) - \frac{1}{2} \log \sin(\pi x) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\log n}{n} \sin(2\pi nx),
\]

as required. The theorem is proved.

As an immediate consequence of Theorem 1, we have the following curious corollary which gives the sum of the convergent alternating series \( \sum_{n=0}^{\infty} (-1)^n \frac{\log(2n+1)}{2n+1} \).

**Corollary 3.** We have:

\[
\frac{\log 1}{1} - \frac{\log 3}{3} + \frac{\log 5}{5} - \ldots = \pi \log \Gamma(1/4) - \frac{\pi^2}{4} \eta - \frac{\pi}{2} \log \pi - \frac{\pi}{4} \log 2.
\]

**Proof.** It suffices to take \( x = \frac{1}{4} \) in the formula of Theorem 1.

We finish this note with the following open question:

**Open question:** Is it possible to express the constant \( \eta \) in terms of the known mathematical constants as \( \pi, \log \pi, \log 2, \gamma, \Gamma(1/4), e, \ldots \)?

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