Minimal charts of type $(3, 3)$
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Abstract

Let $\Gamma$ be a chart. For each label $m$, we denote by $\Gamma_m$ the “sub-graph” of $\Gamma$ consisting of all the edges of label $m$ and their vertices. Let $\Gamma$ be a minimal chart of type $(m; 3, 3)$. That is, a minimal chart $\Gamma$ has six white vertices, and both of $\Gamma_m \cap \Gamma_{m+1}$ and $\Gamma_{m+1} \cap \Gamma_{m+2}$ consist of three white vertices. Then $\Gamma$ is C-move equivalent to a minimal chart containing a “subchart” representing a 2-twist spun trefoil or its “reflection”.

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1 Introduction

Charts are oriented labeled graphs in a disk with three kinds of vertices called black vertices, crossings, and white vertices (see Section 2 for the precise definition of charts). Charts correspond to surface braids (see [4, Chapter 14] for the definition of surface braids). The closures of surface braids are embedded oriented closed surfaces in 4-space $\mathbb{R}^4$ (see [4, Chapter 23]). A C-move is a local modification between two charts in a disk. A C-move induces an ambient isotopy between the closures of the corresponding two surface braids. Two charts are said to be C-move equivalent if there exists a finite sequence of C-moves which modifies one of the two charts to the other. We will work in the PL category or smooth category. All submanifolds are assumed to be locally flat.

The 4-chart as shown in Fig. 1(a) represents a 2-twist spun trefoil. It is well known that the 2-knot is not a ribbon 2-knot. On the other hand, Hasegawa showed that if a non-ribbon chart representing a 2-knot is minimal, then the chart must possess at least six white vertices [2] where a minimal chart $\Gamma$ means its complexity $(w(\Gamma), -f(\Gamma))$ is minimal among the charts C-move equivalent to the chart $\Gamma$ with respect to the lexicographic order of pairs of integers, $w(\Gamma)$ is the number of white vertices in $\Gamma$, and $f(\Gamma)$ is the number of free edges in $\Gamma$. Nagase, Ochiai, and Shima showed that there does not exist a minimal chart with five white vertices [9]. Nagase and Shima show that there does not exist a minimal chart with seven white vertices [5], [6], [7], [8]. Ishida, Nagase, and Shima showed that any minimal chart with exactly four white vertices is C-move equivalent to a chart in two kinds of classes [3].

Two charts are said to be lor-equivalent (Label-Orientation-Reflection equivalent) provided that one of the charts is obtained from the other by a finite sequence of the following five modifications:
Figure 1: Charts contained in the lor-family of the 4-chart describing the 2-twist spun trefoil. Here \( k \) is a positive integer.

(i) consider an \( n \)-chart as an \((n+1)\)-chart,

(ii) for an \( n \)-chart, change all the edges of label \( k \) to ones of label \( n - k \) for each \( k = 1, 2, \ldots, n - 1 \), simultaneously (see Fig. 1(b)),

(iii) add a positive constant integer \( k \) to all the labels simultaneously (so that the \( n \)-chart changes to an \((n + k)\)-chart) (see Fig. 1(c)),

(iv) reverse the orientation of all the edges (see Fig. 1(d)),

(v) change a chart by the reflection in the sphere (see Fig. 1(e)).

The set of all the charts lor-equivalent to a chart \( \Gamma \) is called the lor-family of \( \Gamma \). For example, the lor-family of the 4-chart describing the 2-twist spun trefoil contains the charts as shown in Fig. 1.

Let \( \Gamma \) be a chart. For each label \( m \), we denote by \( \Gamma_m \) the “subgraph” of \( \Gamma \) consisting of all the edges of label \( m \) and their vertices.

A chart \( \Gamma \) is of type \((m; n_1, n_2, \ldots, n_k)\) or of type \((n_1, n_2, \ldots, n_k)\) briefly if it satisfies the following three conditions:

(i) For each \( i = 1, 2, \ldots, k \), the chart \( \Gamma \) contains exactly \( n_i \) white vertices in \( \Gamma_{m+i-1} \cap \Gamma_{m+i} \).

(ii) If \( i < 0 \) or \( i > k \), then \( \Gamma_{m+i} \) does not contain any white vertices.

(iii) Both of the two subgraphs \( \Gamma_m \) and \( \Gamma_{m+k} \) contain at least one white vertex.

Note that \( n_1 \geq 1 \) and \( n_k \geq 1 \) by the condition (iii). In this paper we shall show the following theorem:
Theorem 1.1 Let $\Gamma$ be a minimal chart of type $(3, 3)$. Then $\Gamma$ is C-move equivalent to a minimal chart containing a subchart in the lor-family of the 2-twist spun trefoil.

The paper is organized as follows. In Section 2 we define charts. In Section 3 we review a useful lemma for a lens, a disk whose boundary consists of an edge of label $m$ and an edge of label $m + 1$ satisfying some condition. And we review a useful lemma for a loop. In Section 4 we review a $k$-angled disk, a disk whose boundary consists of edges of label $m$ and contains exactly $k$ white vertices. In particular, we prove a lemma for a 2-angled disk. In Section 5 we review three lemmata for a 3-angled disk. In Section 6 we introduce some property of charts. In Section 7 we prove a useful lemma called New Disk Lemma. And we give generalizations of a C-I-M3 move and a C-I-M4 move. In Section 8, we give four examples of charts of type $(3, 3)$. Three of them are non-minimal charts, the last one is a minimal chart C-move equivalent to a chart containing a subchart in the lor-family of the 2-twist spun trefoil. In Section 9 we prove Main Theorem (Theorem 1.1).

2 Preliminaries

Let $n$ be a positive integer. An $n$-chart is an oriented labeled graph in a disk, which may be empty or have closed edges without vertices, called hoops, satisfying the following four conditions:

(i) Every vertex has degree 1, 4, or 6.

(ii) The labels of edges are in \{1, 2, \ldots, n - 1\}.

(iii) In a small neighborhood of each vertex of degree 6, there are six short arcs, three consecutive arcs are oriented inward and the other three are outward, and these six are labeled $i$ and $i + 1$ alternately for some $i$, where the orientation and the label of each arc are inherited from the edge containing the arc.

(iv) For each vertex of degree 4, diagonal edges have the same label and are oriented coherently, and the labels $i$ and $j$ of the diagonals satisfy $|i - j| > 1$.

We call a vertex of degree 1 a black vertex, a vertex of degree 4 a crossing, and a vertex of degree 6 a white vertex respectively (see Fig. 2). Among six short arcs in a small neighborhood of a white vertex, a central arc of each three consecutive arcs oriented inward or outward is called a middle arc at the white vertex (see Fig. 2(c)). There are two middle arcs in a small neighborhood of each white vertex.

Now C-moves are local modifications of charts in a disk as shown in Fig. 3 (see [1], [4], [10] for the precise definition). These C-moves as shown in Fig. 3 are examples of C-moves.
We showed the difference of a chart in a disk and in a 2-sphere (see [5, Lemma 2.1]). This lemma follows from that there exists a natural one-to-one correspondence between \{charts in \(S^2\)/C-moves and \{charts in \(D^2\)/C-moves, conjugations ([4, Chapter 23 and Chapter 25]). To make the argument simple, we assume that the charts lie on the 2-sphere instead of the disk.

**Assumption 1** *In this paper, all charts are contained in the 2-sphere \(S^2\).*

We have the special point in the 2-sphere \(S^2\), called the point at infinity, denoted by \(\infty\). In this paper, all charts are contained in a disk such that the disk does not contain the point at infinity \(\infty\).

Let \(\Gamma\) be a chart. An edge of \(\Gamma\) is the closure of a connected component of the set obtained by taking out all of white vertices and crossings from \(\Gamma\). For each label \(m\), we denote by \(\Gamma_m\) the “subgraph” of \(\Gamma\) consisting of all the edges of label \(m\) and their vertices. We assume
an edge of $\Gamma_m$ is the closure of a connected component of the set obtained by taking out all the white vertices from $\Gamma_m$.

Thus

any vertex of $\Gamma_m$ is a black vertex or a white vertex. Hence any crossing of $\Gamma$ is not considered as a vertex of $\Gamma_m$. In this paper, an edge of label $m$ means an edge of $\Gamma$.

Let $\Gamma$ be a chart, and $m$ a label of $\Gamma$. A hoop is a closed edge of $\Gamma$ without vertices (hence without crossings, neither). A ring is a closed edge of $\Gamma_m$ containing a crossing but not containing any white vertices. Let $e$ be an edge of $\Gamma$ or $\Gamma_m$. The edge $e$ is called a free edge if it has two black vertices. The edge $e$ is called a terminal edge if it has a white vertex and a black vertex. The edge $e$ is a loop if it is a closed edge with only one white vertex. Note that free edges, terminal edges, and loops may contain crossings of $\Gamma$.

A hoop or a ring is said to be simple if one of the two complementary domains of the curve does not contain any white vertices.

We can assume that all minimal charts $\Gamma$ satisfy the following five conditions (see [5],[6],[7]):

**Assumption 2** No terminal edge of $\Gamma_m$ contains a crossing. Hence any terminal edge of $\Gamma_m$ contains a middle arc.

**Assumption 3** No free edge of $\Gamma_m$ contains a crossing. Hence any free edge of $\Gamma_m$ is a free edge of $\Gamma$.

**Assumption 4** All free edges and simple hoops in $\Gamma$ are moved into a small neighborhood $U_\infty$ of the point at infinity $\infty$. Hence we assume that $\Gamma$ does not contain free edges nor simple hoops, otherwise mentioned.

**Assumption 5** Each complementary domain of any ring and hoop must contain at least one white vertex.

**Assumption 6** The point at infinity $\infty$ is moved in any complementary domain of $\Gamma$.

In this paper for a set $X$ in a space we denote the interior of $X$, the boundary of $X$ and the closure of $X$ by $\text{Int}X$, $\partial X$ and $\text{Cl}(X)$ respectively.

### 3 Lenses and loops

Let $\Gamma$ be a chart. Let $D$ be a disk such that

1. $\partial D$ consists of an edge $e_1$ of $\Gamma_m$ and an edge $e_2$ of $\Gamma_{m+1}$, and
2. any edge containing a white vertex in $e_1$ does not intersect the open disk $\text{Int}D$.  

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Note that $\partial D$ may contain crossings. Let $w_1$ and $w_2$ be the white vertices in $e_1$. If the disk $D$ satisfies one of the following conditions, then $D$ is called a lens of type $(m, m + 1)$ (see Fig. 4):

(i) Neither $e_1$ nor $e_2$ contains a middle arc.

(ii) One of the two edges $e_1$ and $e_2$ contains middle arcs at both white vertices $w_1$ and $w_2$ simultaneously.

![Figure 4:](image)

**Lemma 3.1** ([5, Theorem 1.1] and [6, Corollary 1.3]) Let $\Gamma$ be a minimal chart. Then we have the following:

(a) There exist at least three white vertices in the interior of any lens.

(b) If $\Gamma$ contains at most seven white vertices, then there is no lens of $\Gamma$.

Let $X$ be a set in a chart $\Gamma$. Let $w(X) = \text{the number of white vertices in } X,$

$c(X) = \text{the number of crossings in } X.$

Let $\ell$ be a loop of label $m$ in a chart $\Gamma$. Let $e$ be the edge of $\Gamma_m$ containing the white vertex in $\ell$ with $e \neq \ell$. Then the loop $\ell$ bounds two disks on the 2-sphere. One of the two disks does not contain the edge $e$. The disk is called the associated disk of the loop $\ell$ (see Fig. 5).

![Figure 5: The gray region is the associated disk of a loop $\ell$.](image)

**Lemma 3.2** ([6, Lemma 4.2]) Let $\Gamma$ be a minimal chart with a loop $\ell$ of label $m$ with the associated disk $D$. Let $\varepsilon$ be the integer in $\{+1, -1\}$ such that the white vertex in $\ell$ is in $\Gamma_{m+\varepsilon}$. Then $w(\Gamma \cap \text{Int} D) \geq 2$ and $w(\Gamma \cap (S^2 - D)) \geq 2.$
Lemma 3.3 (c.f. [6, Theorem 1.4]) Let $\Gamma$ be a minimal chart with $w(\Gamma) = 6$. If $\Gamma$ contains a loop, then $\Gamma$ is of type $(2,4)$ or $(4,2)$.

In our argument we often construct a chart $\Gamma$. On the construction of a chart $\Gamma$, for a white vertex $w$, among the three edges of $\Gamma_m$ containing $w$, if we have specified two edges and if the last edge of $\Gamma_m$ containing $w$ contains a black vertex (see Fig. 6(a) and (b)), then we remove the edge containing the black vertex and put a black dot at the center of the white vertex as shown in Fig. 6(c).

![Figure 6](image)

Lemma 3.4 (c.f. [7, Lemma 6.1 and Lemma 6.2]) Let $\Gamma$ be a minimal chart. Let $G$ be a connected component of $\Gamma_m$.

(a) If $w(G) \geq 1$, then $w(G) \geq 2$.

(b) If $w(G) = 3$ and $G$ does not contain any loop, then $G$ is the subgraph as shown in Fig. 7.

![Figure 7](image)

Notation. We use the following notation:

In our argument, we often need a name for an unnamed edge by using a given edge and a given white vertex. For the convenience, we use the following naming: Let $e', e_i, e''$ be three consecutive edges containing a white vertex $w_j$. Here, the two edges $e'$ and $e''$ are unnamed edges. There are six arcs in a neighborhood $U$ of the white vertex $w_j$. If the three arcs $e' \cap U$, $e_i \cap U$, $e'' \cap U$ lie anticlockwise around the white vertex $w_j$ in this order, then $e'$ and $e''$ are denoted by $a_{ij}$ and $b_{ij}$ respectively (see Fig. 8). There is a possibility $a_{ij} = b_{ij}$ if they are contained in a loop.
Let $\Gamma$ be a chart, $m$ a label of $\Gamma$, $D$ a disk, and $k$ a positive integer. If $\partial D$ consists of $k$ edges of $\Gamma_m$, then $D$ is called a $k$-angled disk of $\Gamma_m$. Note that the boundary $\partial D$ may contain crossings, and each of two disks bounded by a loop of label $m$ is a 1-angled disk of $\Gamma_m$.

Let $\Gamma$ be a chart, and $m$ a label of $\Gamma$. An edge of $\Gamma_m$ is called a feeler of a $k$-angled disk $D$ of $\Gamma_m$ if the edge intersects $N - \partial D$ where $N$ is a regular neighborhood of $\partial D$ in $D$.

Let $\Gamma$ be a chart. If an object consists of some edges of $\Gamma$, arcs in edges of $\Gamma$ and arcs around white vertices, then the object is called a pseudo chart.

**Lemma 4.1 (\cite{6, Lemma 5.2} and \cite{7, Theorem 1.1})** Let $\Gamma$ be a minimal chart, and $m$ a label of $\Gamma$. Let $D$ be a 2-angled disk of $\Gamma_m$ without feelers. If $w(\Gamma \cap \text{Int} D) = 0$, then a regular neighborhood of $D$ contains the pseudo chart as shown in Fig. 9(a).

Let $\Gamma$ be a chart, and $m$ a label of $\Gamma$. Let $D$ be a $k$-angled disk of $\Gamma_m$, and $G$ a pseudo chart in $D$ with $\partial D \subset G$. Let $r : D \to D$ be a reflection of $D$, and $G^*$ the pseudo chart obtained from $G$ by changing the orientations of all of the edges. Then the set $\{G, G^*, r(G), r(G^*)\}$ is called the RO-family of the pseudo chart $G$. 

![Figure 8:](image)

![Figure 9: The thick lines are edges of label $m$, and $\varepsilon \in \{+1, -1\}$.

![Diagram](image)
Warning. To draw a pseudo chart in a RO-family, we draw a part of \( \Gamma \cap N \) here \( N \) is a regular neighborhood of a \( k \)-angled disk \( D \).

Lemma 4.2 (c.f. [6, Lemma 5.3, Lemma 5.4 and Lemma 5.5]) Let \( \Gamma \) be a minimal chart, and \( m \) a label of \( \Gamma \). Let \( D \) be a 2-angled disk of \( \Gamma_m \) without feelers. Let \( w_1 \) be a white vertex in \( \partial D \), and \( \varepsilon \) the integer in \( \{+1, -1\} \) with \( w_1 \in \Gamma_{m+\varepsilon} \). Suppose that \( \partial D \) is oriented clockwise or anticlockwise. Then we have the following:

(a) \( w(\Gamma \cap \text{Int} D) \geq 1 \).

(b) If \( w(\Gamma \cap \text{Int} D) = 1 \), then \( D \) contains an element of the RO-family of the pseudo chart as shown in Fig. 9(b).

(c) If \( w(\Gamma \cap \text{Int} D) = 2 \) and \( w(\Gamma_{m+\varepsilon} \cap \text{Int} D) = 2 \), then \( D \) contains an element of RO-families of the two pseudo charts as shown in Fig. 9(c) and (d).

Proof. Let \( w_2 \) be the white vertex in \( \partial D \) different from \( w_1 \), and \( e_1, e_2 \) the edges of \( \Gamma_{m+\varepsilon} \) in \( D \) containing \( w_1, w_2 \) respectively.

Suppose \( w(\Gamma \cap \text{Int} D) = 0 \). Since \( \partial D \) is oriented clockwise or anticlockwise, the edge \( e_1 \) is not middle at \( w_1 \). Thus the edge \( e_1 \) is not a terminal edge by Assumption 2. Since \( w(\Gamma \cap \text{Int} D) = 0 \), we have \( e_1 \ni w_2 \), i.e. \( e_1 = e_2 \). Hence the edge \( e_1 \) separates the disk \( D \) into two lenses each of which does not contain any white vertices in its interior. This contradicts Lemma 3.1(a).

Hence \( w(\Gamma \cap \text{Int} D) \geq 1 \). Thus Statement (a) holds.

Suppose \( w(\Gamma \cap \text{Int} D) = 1 \). Similarly we can show that the edge \( e_1 \) is not a terminal edge and \( e_1 \ni w_2 \). Hence \( e_1 \) contains a white vertex in \( \text{Int} D \), say \( w_3 \). Similarly we can show that the edge \( e_2 \) is not a terminal edge and \( e_2 \in w_1 \). Thus \( e_2 \) contains the white vertex \( w_3 \) in \( \text{Int} D \). Hence there exists a terminal edge of label \( m+\varepsilon \) containing \( w_3 \). Thus the disk \( D \) contains the pseudo chart as shown in Fig. 9(b). Hence Statement (b) holds.

Suppose \( w(\Gamma \cap \text{Int} D) = 2 \) and \( w(\Gamma_{m+\varepsilon} \cap \text{Int} D) = 2 \). Similarly we can show that neither \( e_1 \) nor \( e_2 \) is a terminal edge, and \( e_1 \ni w_2, e_2 \ni w_1 \). Let \( w_3 \) be the white vertex in \( \text{Int} D \) with \( e_1 \ni w_3 \). Let \( w_4 \) be the white vertex in \( \text{Int} D \) different from \( w_3 \).

If \( e_2 \ni w_3 \), then there exists a loop of label \( m+\varepsilon \) containing \( w_4 \) bounding a disk \( E \) and \( \text{Int} E \) does not contains any white vertices. This contradicts Lemma 3.2. Thus \( e_2 \ni w_3 \). Since \( e_2 \) is not a terminal edge and since \( e_2 \ni w_1 \), we have \( e_2 \ni w_4 \).

There are three cases:

(1) both of \( w_3, w_4 \) are contained in loops of label \( m+\varepsilon \),

(2) there exists an edge \( e_3 \) of \( \Gamma_{m+\varepsilon} \) containing \( w_3 \) and \( w_4 \), and both of \( w_3, w_4 \) are contained in terminal edges of label \( m+\varepsilon \),

(3) there exists two edges \( e_3, e_4 \) of \( \Gamma_{m+\varepsilon} \) containing \( w_3 \) and \( w_4 \).
Without loss of generality we can assume that $e_1$ is oriented from $w_1$ to $w_3$.

For Case (1), the associated disk of each loops does not contain any white vertices in its interior. This contradicts Lemma 3.2. Hence Case (1) does not occur.

For Case (2), the edge $e_3$ is oriented from $w_4$ to $w_3$ and the edge $e_2$ is oriented from $w_4$ to $w_2$. Hence $D$ contains the pseudo chart as shown in Fig. 9(c).

For Case (3), there exists a 2-angled disk $E$ of $\Gamma_m + \varepsilon$ in $D$ with $\partial E = e_3 \cup e_4$. Since $E$ has no feelers, by Lemma 4.1 we have that either the two edges $e_3, e_4$ are oriented from $w_4$ to $w_3$, or the two edges $e_3, e_4$ are oriented from $w_3$ to $w_4$. Since $e_1$ is oriented from $w_1$ to $w_3$, the two edges $e_3, e_4$ are oriented from $w_3$ to $w_4$. Hence $D$ contains the pseudo chart as shown in Fig. 9(d). Thus Statement (c) holds.

\[\square\]

5 3-angled disks

Lemma 5.1 ([7, Lemma 4.3 and Lemma 4.5]) Let $\Gamma$ be a minimal chart. Let $D$ be a 3-angled disk of $\Gamma_m$ without feelers. If $w(\Gamma \cap \text{Int} D) = 0$, then $D$ contains an element in the RO-families of the two pseudo charts as shown in Fig. 10(a) and (b).

Let $\Gamma$ and $\Gamma'$ be C-move equivalent charts. Suppose that a pseudo chart $X$ of $\Gamma$ is also a pseudo chart of $\Gamma'$. Then we say that $\Gamma$ is modified to $\Gamma'$ by C-moves keeping $X$ fixed. In Fig. 11 we give examples of C-moves keeping pseudo charts fixed.

Let $\Gamma$ be a minimal chart, and $m$ a label of $\Gamma$. Let $D$ be a $k$-angled disk of $\Gamma_m$. The pair of integers $(w(\Gamma \cap \text{Int} D), c(\Gamma \cap \partial D))$ is called the local complexity with respect to $D$. Let $S$ be the set of all minimal charts obtained from $\Gamma$ by C-moves in a regular neighborhood of $D$ keeping $\partial D$ fixed. The chart $\Gamma$ is said to be locally minimal with respect to $D$ if its local complexity with respect to $D$ is minimal among the charts in $S$ with respect to the lexicographic order of pairs of integers.

Now for a chart $\Gamma$, a $k$-angled disk $D$ of $\Gamma_m$ is special provided that any feeler is a terminal edge where a feeler is an edge of $\Gamma_m$ intersecting $\partial D$ and $\text{Int} D$.

Lemma 5.2 ([7, Theorem 1.2]) Let $\Gamma$ be a minimal chart. Let $D$ be a special 3-angled disk of $\Gamma_m$ such that $\Gamma$ is locally minimal with respect to $D$. If $w(\Gamma \cap \text{Int} D) \leq 1$, then $D$ contains an element in the RO-families of the eight pseudo charts as shown in Fig. 10.

Lemma 5.3 (Triangle Lemma) ([8, Lemma 8.3]) For a minimal chart $\Gamma$, if there exists a 3-angled disk $D_1$ of $\Gamma_m$ without feelers in a disk $D$ as shown in Fig. 12, then $w(\Gamma \cap \text{Int} D_1) \geq 1$. 

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Figure 10: The 3-angled disks (g) and (h) have one feeler, the others do not have any feelers.

6 IO-Calculation

Let $\Gamma$ be a chart, and $v$ a vertex. Let $\alpha$ be a short arc of $\Gamma$ in a small neighborhood of $v$ with $v \in \partial \alpha$. If the arc $\alpha$ is oriented to $v$, then $\alpha$ is called an inward arc, and otherwise $\alpha$ is called an outward arc.

Let $\Gamma$ be an $n$-chart. Let $F$ be a closed domain with $\partial F \subset \Gamma_{k-1} \cup \Gamma_k \cup \Gamma_{k+1}$ for some label $k$ of $\Gamma$, where $\Gamma_0 = \emptyset$ and $\Gamma_n = \emptyset$. By Condition (iii) for charts, in a small neighborhood of each white vertex, there are three inward arcs and three outward arcs. Also in a small neighborhood of each black vertex, there exists only one inward arc or one outward arc. We often use the following fact, when we fix (inward or outward) arcs near white vertices and black vertices:

*The number of inward arcs contained in $F \cap \Gamma_k$ is equal to the number of outward arcs in $F \cap \Gamma_k$.*

When we use this fact, we say that we use IO-Calculation with respect to $\Gamma_k$ in $F$. For example, in a minimal chart $\Gamma$ of type $(m; 3, 3)$, consider the pseudo chart as shown in Fig. 13(a) and

(1) $D_1$ is the 2-angled disk of $\Gamma_m$ with $w(\Gamma \cap \text{Int}D_1) = 1$, 

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Figure 11: C-moves keeping thickened figures fixed.

Figure 12: The gray regions are 3-angled disks $D$. The thick lines are edges of label $m$, and $\varepsilon \in \{+1, -1\}$.

(2) $D_3$ is the 3-angled disks of $\Gamma_m$ with $w(\Gamma \cap \text{Int}D_3) = 0$.

Let $e$ be the edge of $\Gamma_m$ oriented from $w_3$ to $w_2$. Let $F$ be the 3-angled disk of $\Gamma_{m+1}$ containing $e$ with $\partial F \ni w_2, w_3, w_4$, and $e_4$ the terminal edge of label $m + 1$ containing $w_4$. Then we can show that $e_4 \notin F$. For if not, then $e_4 \subset F$ (see Fig. 13(b)). Hence in $F$ there are two edges of $\Gamma_{m+2}$ containing $w_4$ but not middle at $w_4$. By Assumption 2, neither two edges are terminal edges. Since $w(\Gamma \cap \text{Int}F) = 0$, the number of inward arcs in $F \cap \Gamma_{m+2}$ is two, but the number of outward arcs in $F \cap \Gamma_{m+2}$ is zero. This is a contradiction. Instead of the above argument, we just say that

(3) $e_4 \notin F$ by IO-Calculation with respect to $\Gamma_{m+2}$ in $F$.

7 C-I-M3 moves and C-I-M4 moves

Let $e$ be an edge or a simple arc. Suppose that $e$ is not a loop, a hoop nor a ring. Let $v, v'$ be the two end points of $e$. Then we denote

$$\partial e = \{v, v'\}, \quad \text{Int}e = e - \partial e.$$ 

Let $\Gamma$ be a chart, and $D$ a disk. Let $\alpha$ be a simple arc in $\partial D$. We call a simple arc $\gamma$ in an edge of $\Gamma_k$ a $(D, \alpha)$-arc of label $k$ provided that $\partial \gamma \subset \text{Int}\alpha$ and $\text{Int}\gamma \subset \text{Int}D$. If there is no $(D, \alpha)$-arc in $\Gamma$, then the chart $\Gamma$ is said to be $(D, \alpha)$-arc free.

**Lemma 7.1** (c.f. [5, Lemma 3.2]) (New Disk Lemma) Let $\Gamma$ be a chart and $D$ a disk whose interior does not contain a white vertex nor a black vertex.
Let $\alpha$ be a simple arc in $\partial D$ such that $\text{Int}\alpha$ does not contain a white vertex nor a black vertex of $\Gamma$. Let $V$ be a regular neighborhood of $\alpha$. Suppose that the arc $\alpha$ satisfies one of the following two conditions:

(a) The arc $\alpha$ is contained in an edge of $\Gamma_k$ for some label $k$ of $\Gamma$.

(b) $\Gamma \cap \partial \alpha = \emptyset$ and if an edge of $\Gamma$ intersects the arc $\alpha$, then the edge transversely intersects the arc $\alpha$.

Then by applying C-I-M2 moves, C-I-R2 moves, and C-I-R3 moves in $V$, there exists a $(D, \alpha)$-arc free chart $\Gamma'$ obtained from the chart $\Gamma$ keeping $\alpha$ fixed (see Fig. 14).

Proof. **Case (a).** We prove by induction on the number of $(D, \alpha)$-arcs. Let $n$ be the number of $(D, \alpha)$-arcs. If $n = 0$, then $\Gamma$ is $(D, \alpha)$-arc free. Suppose $n > 0$. Then there exists a $(D, \alpha)$-arc $L$ such that the disk $D_L$ bounded by $L$ and an arc $L_\alpha$ in $\alpha$ contains no other $(D, \alpha)$-arc. Now for a proper arc of $D_L$ if the arc is in an edge and transversely intersects $\text{Int}L_\alpha$, then it must intersect the arc $L$. Let $s$ be the number of such arcs transversely intersecting $\text{Int}L_\alpha$.

Let $h$ be the label of $L$, and $\tilde{L}$ the connected component of $\Gamma_h \cap (D \cup V)$ containing the arc $L$. If $s > 0$, then by deforming $\tilde{L}$ in $V$ by C-I-R2 moves and C-I-R3 moves, we can push an end point of $L$ near the other end point of $L$ along $L_\alpha$ (see Fig. 14) so that we can assume that $s = 0$.

By applying a C-I-M2 move and a C-I-R2 move, we can split the arc $\tilde{L}$ to a ring (or a hoop) $R$ and an arc $L'$ to get a new chart $\Gamma'$ with $(R \cup L') \cap \alpha = \emptyset$. Hence by induction we can assume that the chart is $(D, \alpha)$-arc free.

Similarly we can show for **Case (b).** □
Figure 14:

**Lemma 7.2** ([1, Lemma 8.2]) Let $\Gamma$ be a chart, $e$ an edge of $\Gamma_m$, and $w_1, w_2$ the white vertices in $e$. Suppose $w_1 \in \Gamma_{m-1}$ and $w_2 \in \Gamma_{m+1}$. Then for any neighborhood $V$ of the edge $e$, there exists a chart $\Gamma'$ obtained from the chart $\Gamma$ by C-I-R2 moves, C-I-R3 moves and C-I-R4 moves in $V$ keeping $\Gamma_{m-1} \cup \Gamma_m \cup \Gamma_{m+1}$ fixed such that the edge $e$ does not contain any crossings.

Let $\Gamma$ be a chart and $k$ a label of $\Gamma$. If a disk $D$ satisfies the following two conditions, then $D$ is called an $M_4$-disk of label $k$ (see Fig. 15(a)).

(i) $\partial D$ consists of four edges $e_1, e_2, e_3, e_4$ of $\Gamma_k$ situated on $\partial D$ in this order.

(ii) Set $w_1 = e_1 \cap e_4, w_2 = e_1 \cap e_2, w_3 = e_2 \cap e_3, w_4 = e_3 \cap e_4$. Then

(a) $D \cap \Gamma_{k-1}$ consists of an edge $e_5$ connecting $w_1$ and $w_3$, and

(b) $D \cap \Gamma_{k+1}$ consists of an edge $e_6$ connecting $w_2$ and $w_4$.

(iii) Int$D$ does not contain any white vertex.

We call the union $X = \cup_{i=1}^6 e_i$ the $M_4$-pseudo chart for the disk $D$, and $(w_1, w_2, w_3, w_4; e_1, e_2, e_3, e_4; e_5, e_6)$ the fundamental information for the $M_4$-disk $D$.

**Lemma 7.3** Let $\Gamma$ be a chart, and $k$ a label of $\Gamma$. Suppose that $D$ is an $M_4$-disk of label $k$ with an $M_4$-pseudo chart $X$. Then by deforming $\Gamma$ in a regular neighbourhood of $D$ without increasing the complexity of $\Gamma$, the chart $\Gamma$ is C-move equivalent to a chart $\Gamma'$ with $D \cap (\cup_{i=k-2}^{k+2} \Gamma'_i) = X$.

**Proof.** Let $(w_1, w_2, w_3, w_4; e_1, e_2, e_3, e_4, e_5, e_6)$ be the fundamental information for the $M_4$-disk $D$. Then
Figure 15: (a) The gray region is the M4-disk. (b) The gray region is the M3-disk.

(1) Int\(D\) does not contain any white vertex.

First of all, applying Lemma 7.2 for \(e_1, e_3\) respectively keeping \(\Gamma_{k-1} \cup \Gamma_k \cup \Gamma_{k+1}\) fixed, we can assume that

(2) neither \(e_1\) nor \(e_3\) contains any crossing.

Secondly, applying New Disk Lemma (Lemma 7.1) for \(e_2, e_4 \subset \partial D\) respectively, the chart \(\Gamma\) is C-move equivalent to a chart \(\Gamma'\) deformed in regular neighbourhoods of \(e_2\) and \(e_4\) so that

(3) the chart \(\Gamma'\) is \((D, e_2)\)-arc free and \((D, e_4)\)-arc free.

By Statement (2) and Statement (3), for each label \(i \neq k\), if an edge of \(\Gamma'_i\) intersects the edge \(e_2\) (resp. \(e_4\)), then it must intersect the edge \(e_4\) (resp. \(e_2\)). Since \(e_5\) is of label \(k-1\) and \(e_6\) of label \(k+1\), and since each of \(e_5\) and \(e_6\) separates the two edges \(e_2\) and \(e_4\) in the disk \(D\), any edge of \(\Gamma'_{k-2}\) or \(\Gamma'_{k+2}\) does not intersect \(e_2 \cup e_4\). Hence Statement (1) implies that each connected component of \(D \cap (\bigcup_{i=k-2}^{k} \Gamma'_i)\) does not contain a hoop nor a ring. Hence we have \(D \cap (\bigcup_{i=k-2}^{k} \Gamma'_i) = X\). Thus we have the result.

Let \(\Gamma\) be a chart, \(k\) a label of \(\Gamma\), and \(\epsilon \in \{+1,-1\}\). A disk \(D\) is called an \(M3\)-disk of type \((k,k+\epsilon)\) provided that (see Fig. 15(b))

(i) \(\partial D\) consists of two edges \(e_1, e_3\) of \(\Gamma_k\),

(ii) \(D \cap \Gamma_k = \partial D\),

(iii) \(D \cap \Gamma_{k+\epsilon}\) consists of an edge \(e_2\) of \(\Gamma_{k+\epsilon}\) with \(\partial e_2 = \partial e_1\),

(iv) Int\(D\) does not contain a white vertex nor a black vertex.

We call the union \(X = e_1 \cup e_2 \cup e_3\) the \(M3\)-pseudo chart for the disk \(D\).

Similarly we can show the following lemma.
Lemma 7.4 Let $\Gamma$ be a chart, and $k$ a label of $\Gamma$. Suppose that $D$ is an $M3$-disk of type $(k, k + \varepsilon)$ with an $M3$-pseudo chart $X$. Then by deforming $\Gamma$ in a regular neighbourhood of $D$ without increasing the complexity of $\Gamma$, the chart $\Gamma$ is C-move equivalent to a chart $\Gamma'$ with $D \cap (\Gamma'_{k-\varepsilon} \cup \Gamma'_{k} \cup \Gamma'_{k+\varepsilon} \cup \Gamma'_{k+2\varepsilon}) = X$.

8 Non-minimal charts with six white vertices

Lemma 8.1 Let $\Gamma$ be a chart with $w(\Gamma) = 6$. If $\Gamma$ contains one of the three subcharts as shown in Fig. 16(a), (b) and (c), then $\Gamma$ is not minimal.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure16.png}
\caption{The thick lines are edges of label $m+1$. (a) The gray region is the 4-angled disk $E$. (b) The gray region is the 6-angled disk $E$. (c) The gray region is the 3-angled disk $E$. (d) The gray region is the 6-angled disk $E$.}
\end{figure}

Proof. Suppose that $\Gamma$ contains the subchart as shown in Fig. 16(a). We use notations in Fig. 16(a). Let $E$ be the 4-angled disk containing the edge $e$ with $\partial E \ni w_2, w_3, w_4, w_6$. Then the disk is an $M4$-disk of label $m+1$. Then by Lemma 7.3 we can apply a C-I-M4 move in a neighborhood of $E$, we obtain a chart containing a subchart as shown in Fig. 17(a). The terminal edge
containing \(w_2\) is not middle at \(w_2\). Applying a C-III move, we can eliminate
the white vertex \(w_2\). Hence \(\Gamma\) is not minimal.

Suppose that \(\Gamma\) contains the subchart as shown in Fig. 16(b). We use
notations in Fig. 16(b), here \(e'^*, e''*\) are the terminal edges of label \(m+1\)
containing \(w_3, w_6\) respectively, \(e_1\) the terminal edge of label \(m\) containing \(w_1\).
Let \(E\) be the 6-angled disk of \(\Gamma_{m+1}\) with \(E \supset e'^* \cup e''*\) but \(E \not\supset e_1\), shown
in Fig. 16(b). By Assumption 4 and Assumption 5, we can assume that \(E\)
does not contain any free edges, hoops nor rings of label \(m, m+1, m+2\).
Let \(\gamma\) be a simple arc connecting two black vertices in \(e'^*\) and \(e''*\) with \((\Gamma_m \cup \Gamma_{m+1} \cup \Gamma_{m+2}) \cap \text{Int}_\gamma = \emptyset\). Applying C-II moves and a C-I-M2 move along \(\gamma\),
we obtain a new free edge of label \(m+1\). Hence \(\Gamma\) is not minimal.

Suppose that \(\Gamma\) contains the subchart as shown in Fig. 16(c). We use
notations in Fig. 16(d), here \(e'_4\) is the terminal edge of label \(m+2\) containing
\(w_4\), and the edges \(e_2, e_3, e', e'', e'''\) are the edges of \(\Gamma_{m+1}\) with \(\partial e_2 = \{w_2, w_5\}\),
\(\partial e_3 = \{w_3, w_6\}\), \(\partial e' = \{w_2, w_4\}\), \(\partial e'' = \{w_3, w_1\}\) and \(\partial e''' = \{w_2, w_3\}\). Further
\(e_1, e_2, e_3\) are the edges of \(\Gamma_{m+2}\) such that \(\partial e_1 = \{w_4, w_3\}\), \(\partial e_2 = \{w_4, w_6\}\),
and \(e_3\) is oriented from \(w_6\) to \(w_5\).

Let \(E\) be the 3-angled disk of \(\Gamma_{m+2}\) bounded by \(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\) with \(E \ni w_2, w_3\).
Set \(X = e' \cup e'' \cup e''' \cup e_2 \cup e_3 \cup e'_4 \cup \partial E\).

Now \((\Gamma_{m+1} \cup \Gamma_{m+2} \cup \Gamma_{m+3}) \cap E = X\). For, by applying Lemma 7.2 to
\(e_2, e_3\) respectively, we can assume that

(1) neither \(e_2\) nor \(e_3\) contains a crossing.

Now \(\partial E \subset \Gamma_{m+2}\) implies that the closure of \((\Gamma_{m+1} \cup \Gamma_{m+2} \cup \Gamma_{m+3}) \cap E - X\)
consists of hoops, and rings. In the disk \(E\) each of the hoops and the rings
is simple by Statement (1). Hence by Assumption 4 and Assumption 5,
we have that \((\Gamma_{m+1} \cup \Gamma_{m+2} \cup \Gamma_{m+3}) \cap E = X\).

Applying C-II moves, we can shift the black vertex of the terminal edge
\(e'_4\) near the edge \(e'''\). Applying a C-III move, we obtain the chart as shown
in Fig. 17(b).

Let \(F\) be the 4-angled disk of \(\Gamma_{m+1}\) as shown in Fig. 17(b). Then \(F\)
is an M4-disk of label \(m+1\). By Lemma 7.3, we can apply a C-I-M4 move for
\(F\) so that we obtain the chart as shown in Fig. 17(c). The terminal edge of
label \(m+1\) containing \(w_4\) is not middle at \(w_4\). Thus by a C-III move, we
obtain a minimal chart with six white vertices and a lens. This contradicts
Lemma 8.1(b). Therefore \(\Gamma\) is not minimal. \(\Box\)

**Lemma 8.2** Let \(\Gamma\) be a minimal chart with \(w(\Gamma) = 6\). If \(\Gamma\) contains
the subchart as shown in Fig. 16(d), then \(\Gamma\) is C-move equivalent to a minimal
chart containing a subchart in the lor-family of the 2-twist spun trefoil.

**Proof.** Let \(\Gamma\) be a minimal chart with \(w(\Gamma) = 6\) containing the subchart
as shown in Fig. 16(d). We use the notations in Fig. 16(d), here \(e'^*, e_4\) are the
terminal edges of label \(m+1\) containing \(w_3, w_4\) respectively, \(e_1\) is a terminal
edge of label \(m\) containing \(w_1\), the edges \(e'_4, e'_4\) are the edges of \(\Gamma_{m+2}\) with
Figure 17: The thick lines are edges of label $m+1$. (b) The gray region is the 4-angled disk $F$. (c) The gray region is a lens of type $(m+1, m+2)$.

$\partial e'_4 = \{w_4, w_5\}$ and $\partial e''_4 = \{w_4, w_6\}$, the edge $e$ is the edge of $\Gamma_m$ oriented from $w_3$ to $w_2$, and $e'_3$ is the edge of $\Gamma_m$ with $\partial e'_3 = \{w_1, w_3\}$. Let $E$ be the 6-angled disk of $\Gamma_{m+1}$ with $w(\Gamma \cap \text{Int}E) = 0$ and $E \supset e^* \cup e_4$ but $E \not\supset e_1$.

Set $X = (e \cup e'_3 \cup e^* \cup e_4 \cup e'_4 \cup e''_4) \cup \partial E$. Since $\partial E \supset \Gamma_{m+1}$, and since $\text{Int}E$ contains no white vertex, each connected component of $\text{Cl}((\Gamma_m \cup \Gamma_{m+1} \cup \Gamma_{m+2}) \cap E - X)$ is a simple hoop or a simple ring. By Assumption 1 and Assumption 5, we have that $(\Gamma_m \cup \Gamma_{m+1} \cup \Gamma_{m+2}) \cap E = X$.

By C-II moves, we move the two black vertices in $e^*, e_4$ near the crossing $e \cap e'_4$. By a C-I-M1 move, we create a hoop of label $m+1$ near the crossing $e \cap e'_4$ (see Fig. 18(a)). Then we apply two C-III moves for the two terminal edges $e^*, e_4$, and then we obtain the chart as shown in Fig. 18(b). By Lemma 7.3, we can apply a C-I-M4 move for the 4-angled disk of $\Gamma_{m+1}$ so that we obtain the chart as shown in Fig. 18(c).

Let $E'$ be the 2-angled disk of $\Gamma_m$ with $w(\Gamma \cap \text{Int}E') = 0$ and $\partial E' \ni w_2, w_4$. Then $E'$ is an M3-disk of type $(m, m+1)$. Thus by Lemma 7.3, we can apply a C-I-M3 move for $E'$ and then we obtain the chart with six white vertices as shown in Fig. 18(d). This chart contains a subchart in the lor-family of the 2-twist spun trefoil. □
9 Proof of Main Theorem

In this section, we assume that

(1) $\Gamma$ is a minimal chart of type $(m; 3, 3)$.

By Lemma 3.3, we can assume that

(2) $\Gamma$ does not contain any loop.

Claim 1. The subgraph $\Gamma_m$ contains a graph $G$ as shown in Fig. 7.

Proof of Claim 1. First of all, we shall show $w(\Gamma_m) = 3$. Since $\Gamma$ is of type $(m; 3, 3)$, we have that $w(\Gamma_m \cap \Gamma_{m+1}) = 3$ and $w(\Gamma_{m-1}) = 0$ by the conditions (i) and (ii) of type of a chart. Since any white vertex in $\Gamma_m$ is contained in $\Gamma_{m-1}$ or $\Gamma_{m+1}$, we have $w(\Gamma_m) = 3$.

Let $G$ be a connected component of $\Gamma_m$ with $w(G) \geq 1$. We shall show $w(G) = 3$. By Lemma 3.4(a), we have $w(G) \geq 2$. If $w(G) = 2$, then there exists a white vertex $w$ in $\Gamma_m$ with $w \notin G$. Since $w(G) = 2$ and $w(\Gamma_m) = 3$, the connected component of $\Gamma_m$ containing $w$ contains exactly one white vertex. This contradicts Lemma 3.4(a). Hence $w(G) = 3$. By Lemma 3.4(b),
the graph $G$ is the graph as shown in Fig. 7. Thus $\Gamma_m$ contains the graph $G$ as shown in Fig. 7.

The closures of connected components of $S^2 - G$ consists of three disks. Let $e_1$ be the terminal edge of label $m$. Let $D_1$ be one of the three disks with $D_1 \cap e_1 = \emptyset$, $D_2$ one of the three disks with $D_2 \supset e_1$, and $D_3$ the last one of the three disks (see Fig. 19(a)). Then $D_1$ is a 2-angled disk without feelers, $D_2$ is a 3-angled disk with one feeler, and $D_3$ is a 3-angled disk without feelers. Let $w_1, w_2, w_3$ be white vertices in $\Gamma_m$ with $w_1 \in e_1$, and $e$ the edge of $\Gamma_m$ with $e = D_1 \cap D_3$. In this section we assume that

(3) $\Gamma$ is locally minimal with respect to $D_2$,

(4) the terminal edge $e_1$ of label $m$ is oriented from $w_1$,

(5) the edge $e$ is oriented from $w_3$ to $w_2$,

(6) the point at infinity $\infty$ is contained in $\text{Int} D_2$.

We use the notations shown in Fig. 19(a).

**Claim 2.** $w(\Gamma \cap \text{Int} D_1) \geq 1$ and $w(\Gamma \cap \text{Int} D_2) \geq 1$.

**Proof of Claim 2.** Since $\partial D_1$ is oriented anticlockwise, by Lemma 4.2(a) we have $w(\Gamma \cap \text{Int} D_1) \geq 1$.

Since $D_2$ is a special 3-angled disk with one feeler, by Lemma 5.2 we have $w(\Gamma \cap \text{Int} D_2) \geq 1$. \qed

Let $w_4, w_5, w_6$ be white vertices in $\Gamma_{m+2}$. By Claim 2, without loss of generality we can assume that $w_4 \in \text{Int} D_1$ and $w_5 \in \text{Int} D_2$.

Figure 19: The thick lines are edges of label $m+1$. (a) The light gray region is the disk $D_1$, and the dark gray region is the disk $D_3$. (b) The gray region is the disk $F$. 

**Claim 2.** $w(\Gamma \cap \text{Int} D_1) \geq 1$ and $w(\Gamma \cap \text{Int} D_2) \geq 1$.

**Proof of Claim 2.** Since $\partial D_1$ is oriented anticlockwise, by Lemma 4.2(a) we have $w(\Gamma \cap \text{Int} D_1) \geq 1$.

Since $D_2$ is a special 3-angled disk with one feeler, by Lemma 5.2 we have $w(\Gamma \cap \text{Int} D_2) \geq 1$. \qed

Let $w_4, w_5, w_6$ be white vertices in $\Gamma_{m+2}$. By Claim 2, without loss of generality we can assume that $w_4 \in \text{Int} D_1$ and $w_5 \in \text{Int} D_2$. 

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Claim 3. \( w_6 \not\in \operatorname{Int}D_1 \).

Proof of Claim 3. Let \( e', e'' \) be the edges of \( \Gamma_{m+1} \) in \( D_1 \) containing \( w_2, w_3 \) respectively. Then both of \( e', e'' \) are oriented inward at white vertices on the boundary. Suppose \( w_6 \in \operatorname{Int}D_1 \). Then \( w(\Gamma \cap \operatorname{Int}D_1) = 2 \) and \( w(\Gamma_{m+1} \cap \operatorname{Int}D_1) = 2 \). Thus by Lemma 4.2(c), the disk \( D_1 \) contains an element of RO-families of the two pseudo charts as shown in Fig. 9(b) and (d). By Lemma 5.1, there exists an edge of \( \Gamma_{m+1} \) oriented outward at the white vertex on the boundary. Hence \( e' \) or \( e'' \) is oriented outward at the white vertex on the boundary \( \partial D_1 \). This is a contradiction. Hence \( w_6 \not\in \operatorname{Int}D_1 \). \( \square \)

Claim 4. \( w_6 \not\in \operatorname{Int}D_2 \).

Proof of Claim 4. Suppose \( w_6 \in \operatorname{Int}D_2 \). Then \( w(\Gamma \cap \operatorname{Int}D_2) = 1 \), \( w(\Gamma \cap \operatorname{Int}D_3) = 2 \) and \( w(\Gamma \cap \operatorname{Int}D_3) = 0 \). By Lemma 4.2(b), the disk \( D_1 \) contains a pseudo chart as shown in Fig. 9(b). By Lemma 5.1, the disk \( D_3 \) contains a pseudo chart as shown in Fig. 10(b). Thus we have the pseudo chart as shown in Fig. 13(a).

Let \( e_3 \) be the terminal edge of label \( m+1 \) containing \( w_4 \). Let \( F \) be the 3-angled disk of \( \Gamma_{m+1} \) containing \( e \) with \( \partial F \ni w_2, w_3, w_4 \). Then \( w(\Gamma \cap \operatorname{Int}F) = 0 \). Now we have the same situation in the example of IO-Calculation in Section 6. Thus we have \( e_4 \not\in F \) by IO-Calculation with respect to \( \Gamma_{m+2} \) in \( F \) (see Statement (3) in Section 6). Hence we have the pseudo chart in Fig. 10(b). We use the notations shown in Fig. 10(b).

Now \( e_2 \) is the edge of \( \Gamma_{m+1} \) in \( D_2 \) containing \( w_2 \). Then \( e_2 \) is not a terminal edge. For, if \( e_2 \) is a terminal edge, then the 3-angled disk \( F \) satisfies the condition in Lemma 5.3. Thus \( w(\Gamma \cap \operatorname{Int}F) \geq 1 \). This contradicts the fact that \( w(\Gamma \cap \operatorname{Int}F) = 0 \). Hence \( e_2 \) is not a terminal edge.

Let \( F' = \operatorname{Cl}(S^2 - F) \). Then \( F' \) is a 3-angled disk of \( \Gamma_{m+1} \). We shall show \( e_2 \not\ni w_3 \). For, if \( e_2 \ni w_3 \), then the edge \( e_2 \) separates \( F' \) into two disks. One of the two disks contains the terminal edge \( e_4 \), say \( D \). Let \( D' \) be the other disk. By IO-Calculation with respect to \( \Gamma_{m+2} \) in \( D \), we have that \( D \) contains \( w_5 \) or \( w_6 \). Further, by IO-Calculation with respect to \( \Gamma_{m+1} \) in \( D' \), we have that \( D' \) contains \( w_5 \) or \( w_6 \). Without loss of generality we can assume that \( w_5 \in D \) and \( w_6 \in D' \). Thus \( w(\Gamma_{m+2} \cap D') = 1 \) and there exists a connected component of \( \Gamma_{m+2} \) containing exactly one white vertex \( w_6 \). This contradicts Lemma 5.4(a). Hence \( e_2 \not\ni w_3 \). Since \( e_2 \neq a_{11} \) and \( e_2 \neq b_{11} \) and since \( e_2 \) is not a terminal edge, we have \( e_2 \ni w_5 \) or \( e_2 \ni w_6 \). Without loss of generality we can assume \( e_2 \ni w_5 \).

Now \( e_3 \) is the edge of \( \Gamma_{m+1} \) in \( D_2 \) containing \( w_3 \). Since the edge \( e_3 \) is not middle at \( w_3 \), by Assumption 2 the edge \( e_3 \) is not a terminal edge. Hence there are four cases: (1) \( e_3 = a_{11} \), (2) \( e_3 = b_{11} \), (3) \( e_3 \ni w_5 \), (4) \( e_3 \ni w_6 \).

For Case (1), there exists a lens of type \((m, m + 1)\) whose boundary contains \( e_3 \). This contradicts Lemma 3.1(b). Hence Case (1) does not occur.

For Case (2), we have \( a_{11} \ni w_6 \) because \( a_{11} \) is not middle at \( w_1 \). Hence there exists a loop of label \( m + 1 \) containing \( w_6 \). This contradicts the fact that \( \Gamma \) does not contain any loop (see Statement (2) in this section). Hence Case (2) does not occur.
For **Case (3)**, let $e_5$ be the edge of $\Gamma_{m+1}$ containing $w_5$ different from $e_2$ and $e_3$. Since $e_2$ is oriented from $w_2$ to $w_5$ and since $e_3$ is oriented from $w_3$ to $w_5$, the edge $e_5$ is not middle at $w_5$. By Assumption 2 the edge $e_5$ is not a terminal edge.

Let $E$ be the 4-angled disk of $\Gamma_{m+1}$ with $\partial E = e' \cup e'' \cup e_2 \cup e_3$ and $E \ni w_1$. If $e_5 \not\in E$, then $w_6 \in e_5$, because $e_5$ is not a terminal edge. Thus there exists a loop of label $m + 1$ containing $w_6$. This contradicts the fact that $\Gamma$ does not contain any loop. Hence $e_5 \subset E$ (see Fig. 20(a)).

Let $e'''$ be the edge of $\Gamma_{m+1}$ with $\partial e''' = \{w_2, w_3\}$. Let $E'$ be the 3-angled disk of $\Gamma_{m+1}$ in $E$ with $\partial E' = e_2 \cup e_3 \cup e'''$. By IO-Calculation with respect to $\Gamma_{m+1}$ in $E'$, we have $w_6 \in E' \subset E$. Thus $w(\Gamma \cap (S^2 - E)) = 0$. However, by IO-Calculation with respect to $\Gamma_{m+2}$ in $Cl(S^2 - E)$, there exists at least one white vertex of $\Gamma_{m+2}$ in $S^2 - E$. This is a contradiction. Hence Case (3) does not occur.

For **Case (4)**, let $e'_2$, $e''_2$ be edges of $\Gamma_{m+1}$ containing $w_5$ different from $e_2$ such that $e_2$, $e'_2$, $e''_2$ lie clockwise around $w_5$ in this order. Let $e'_3$, $e''_3$ be edges of $\Gamma_{m+1}$ containing $w_6$ different from $e_3$ such that $e_3$, $e'_3$, $e''_3$ lie clockwise around $w_6$ in this order (see Fig. 20(b)).

We can show that $a_{11} = e'_3$. If not, then there are three cases: (4-1) $a_{11} = e'_2$, (4-2) $a_{11} = e''_2$, (4-3) $a_{11} = e''_3$.

For **Case (4-1)**, there exists a loop of label $m + 1$ containing $w_6$. This contradicts the fact that $\Gamma$ does not contain any loop.

For **Case (4-2)**, the edge $b_{11}$ is a terminal edge not middle at $w_1$. This contradicts Assumption 2.

For **Case (4-3)**, the edge $e'_3$ is a terminal edge not middle at $w_6$. This contradicts Assumption 2.

Hence we have $a_{11} = e'_3$. Similarly we can show that $b_{11} = e''_2$. Since the edge $e''_2$ is not middle at $w_6$, we have $e''_2 = e'_2$. Hence by the help of New Disk Lemma (Lemma 7.1), the chart $\Gamma$ contains the subchart as shown in Fig. 21(c). By Lemma 8.1 the chart $\Gamma$ is minimal. This is a contradiction. Hence Case (4) does not occur. Therefore we complete the proof of Claim 4.

**Proof of Main Theorem.** Now we start from the pseudo chart as shown in Fig. 16(a). By Claim 2, Claim 3 and Claim 4, we have $w(\Gamma \cap IntD_1) = 1$, $w(\Gamma \cap IntD_2) = 1$ and $w(\Gamma \cap IntD_3) = 1$. Without loss of generality, we can assume that $IntD_1$ contains a white vertex $w_4$, $IntD_2$ contains a white vertex $w_5$, $IntD_3$ contains a white vertex $w_6$.

By Lemma 4.2(b), the disk $D_1$ contains the pseudo chart as shown in Fig. 9(b). Since $D_2$ is a special 3-angled disk with one feeler, by Lemma 5.2 the disk $D_2$ contains an element of RO-families of the two pseudo charts as shown in Fig. 10(g) and (h) (see Fig. 21(a) and (b)).

Suppose that $\Gamma$ contains the pseudo chart as shown in Fig. 21(b). We use the notations shown in Fig. 21(b), here $e$ is the edge of $\Gamma_m$ oriented from $w_3$ to $w_2$, and $e'$, $e''$, $e'''$ are edges of $\Gamma_{m+1}$ with $\partial e' = \{w_2, w_4\}$, $\partial e'' = \{w_3, w_4\}$ and $\partial e''' = \{w_2, w_3\}$. Let $E_1$, $E_2$ be the 3-angled disks of $\Gamma_{m+1}$ with $E_1 \cap E_2 =$
Figure 20: The thick lines are edges of label $m + 1$. (a) The gray region is the disk $E$.

$$\partial E_1 = \partial E_2 = e' \cup e'' \cup e'''$$ and $E_1 \supset e$. Let $e_4$ be the terminal edge of label $m + 1$ containing $w_4$. By IO-Calculation with respect to $\Gamma_{m+2}$ in $E_2$, we have $e_4 \not\subset E_2$. Thus $e_4 \subset E_1$ (see Fig. 21(c)).

Let $D$ be the 2-angled disk of $\Gamma_{m+1}$ in $D_2$ with $\partial D \ni w_1, w_5$ (see Fig. 21(c)). Let $n_I$ be the number of inward arcs of label $m + 2$ in $Cl(E_1 - D)$ and $n_O$ the number of outward arcs of label $m + 2$ in $Cl(E_1 - D)$. We shall count the numbers $n_I$ and $n_O$ as follows.

**Assertion 1.** There are two outward arcs of label $m + 2$ containing $w_5$ in $Cl(E_1 - D)$.

For, the two edges $e'_5, e''_5$ are edges of $\Gamma_{m+2}$ containing $w_5$ in $Cl(E_1 - D)$ (see Fig. 21(c)). Since the two edges are oriented outward at $w_5$, there are two outward arcs of label $m + 2$ containing $w_5$ in $Cl(E_1 - D)$.

**Assertion 2.** There are two inward arcs of label $m + 2$ containing $w_4$ in $Cl(E_1 - D)$.

For, $e_4 \subset E_1$ implies that there are two edges $e'_4, e''_4$ of $\Gamma_{m+2}$ in $Cl(E_1 - D)$ containing $w_4$ (see Fig. 21(c)). Since the terminal edge $e_4$ is oriented inward at $w_4$, the two edges $e'_4, e''_4$ are oriented inward at $w_4$. Thus there are two inward arcs of label $m + 2$ containing $w_4$ in $Cl(E_1 - D)$.

Now for the white vertex $w_6 \in D_3 \subset Cl(E_1 - D)$, there are two cases:

- **Case (b-1)** there is one inward arc of label $m + 2$ containing $w_6$ and there are two outward arcs of label $m + 2$ containing $w_6$,

- **Case (b-2)** there is one outward arc of label $m + 2$ containing $w_6$ and there are two inward arcs of label $m + 2$ containing $w_6$.

For **Case (b-1)**, by Assumption 2 the white vertex $w_6$ is contained in at most one terminal edge of label $m + 2$. Let $e_6$ be the edge of $\Gamma_{m+2}$ middle at $w_6$.

Suppose that $e_6$ is a terminal edge. Then by Condition (b-1) the terminal edge $e_6$ is oriented inward at $w_6$. Thus
Figure 21: The thick lines are edges of label $m + 1$. (b), (c) The dark gray region is the 3-angled disk $E_2$, and the light gray region is the 2-angled disk $D$.

(*) there exists an outward arc of label $m + 2$ containing the black vertex in the terminal edge $e_6$.

Since none of $e'_4, e''_4, e'_5, e''_5$ contain middle arcs at $w_4$ or $w_5$, none of $e'_4, e''_4, e'_5, e''_5$ are terminal edges by Assumption 2. Thus the two edges $e'_4, e''_4$ contain $w_5$ or $w_6$, and the two edges $e'_5, e''_5$ contain $w_4$ or $w_6$. Hence by (*) and Assertion 1 and Assertion 2 we have $n_I = 3$ and $n_O = 5$. This is a contradiction by IO-Calculation with respect to $\Gamma_{m+2}$ in $\text{Cl}(E_1 - D)$.

Similarly for the case that $e_6$ is not a terminal edge, we have $n_I = 3$ and $n_O = 4$. This is a contradiction. Hence Case (b-1) does not occur.

For Case (b-2), in a similar way to Case (b-1), we have $n_I = 4$ or 5 and $n_O = 3$. This is a contradiction by IO-Calculation with respect to $\Gamma_{m+2}$ in $\text{Cl}(E_1 - D)$. Hence Case (b-2) does not occur. Thus $\Gamma$ does not contain the pseudo chart as shown in Fig. 21(b).

Suppose that $\Gamma$ contains the pseudo chart as shown in Fig. 21(a). Since the disk $D_3$ is a 3-angled disk without feelers with $w(\Gamma \cap \text{Int}D_3) = 1$, by Lemma 5.2 there are three cases:
(a-1) $D_3$ contains an element of the RO-family of the pseudo chart as shown in Fig. 10(c) (see Fig. 22(a)).

(a-2) $D_3$ contains an element of RO-family of the pseudo chart as shown in Fig. 10(d) (see Fig. 22(b)).

(a-3) $D_3$ contains an element of RO-family of the pseudo chart as shown in Fig. 10(e) (see Fig. 22(c)).

Let $e_4$ be the terminal edge of label $m+1$ containing $w_4$.

For Case (a-1), let $E$ be the 4-angled disk of $\Gamma_{m+1}$ with $\partial E \ni w_2, w_3, w_4, w_6$ and $E \supset e$. By IO-Calculation with respect to $\Gamma_{m+2}$ in $E$, we have $e_4 \not\subset E$. Thus by the help of New Disk Lemma (Lemma 7.1), the chart $\Gamma$ contains the subchart as shown in Fig. 10(a). By Lemma 8.1 the chart $\Gamma$ is not minimal. This is a contradiction. Hence Case (a-1) does not occur.

For Case (a-2), let $E$ be the 6-angled disk with $E \ni e_1$ and $E \supset e$. By IO-Calculation with respect to $\Gamma_{m+2}$ in $E$, we have $e_4 \subset E$. Thus by the help of New Disk Lemma (Lemma 7.1), the chart $\Gamma$ contains the subchart as shown in Fig. 10(d). By Lemma 8.2, the chart $\Gamma$ is C-move equivalent to a minimal chart containing a subchart in the lor-family of the 2-twist spun trefoil as shown in Fig. 11.

For Case (a-3), let $E$ be the 6-angled disk with $E \ni e_1$ and $E \supset e$. By IO-Calculation with respect to $\Gamma_{m+2}$ in $E$, we have $e_4 \not\subset E$. Thus by the help of New Disk Lemma (Lemma 7.1), the chart $\Gamma$ contains the subchart as shown in Fig. 10(b). By Lemma 8.1 the chart $\Gamma$ is not minimal. This is a contradiction. Hence Case (a-3) does not occur.

Therefore we complete the proof of Main Theorem. □

References

[1] J. S. Carter and M. Saito, ”Knotted surfaces and their diagrams”, Mathematical Surveys and Monographs, 55, American Mathematical Society, Providence, RI, (1998). MR1487374 (98m:57027)

[2] I. Hasegawa, The lower bound of the w-indices of non-ribbon surface-links, Osaka J. Math. 41 (2004), 891–909. MR2116344 (2005k:57045)

[3] S. Ishida, T. Nagase and A. Shima, Minimal n-charts with four white vertices, J. Knot Theory Ramifications 20, 689–711 (2011). MR2806339 (2012e:57044)

[4] S. Kamada, ”Braid and Knot Theory in Dimension Four”, Mathematical Surveys and Monographs, Vol. 95, American Mathematical Society, (2002). MR1900979 (2003d:57050)

[5] T. Nagase and A. Shima, Properties of minimal charts and their applications I, J. Math. Sci. Univ. Tokyo 14 (2007), 69–97. MR2320385 (2008c:57040)
Figure 22: The thick lines are edges of label $m+1$ and the gray region is the disk $E$.

[6] T. Nagase and A. Shima, *Properties of minimal charts and their applications II*, Hiroshima Math. J. **39** (2009), 1–35. MR2499196 (2009k:57040)

[7] T. Nagase and A. Shima, *Properties of minimal charts and their applications III*, Tokyo J. Math. **33** (2010), 373–392. MR2779264 (2012a:57033)

[8] T. Nagase and A. Shima, *Properties of minimal charts and their applications IV: Loops*, arXiv:1603.04639

[9] M. Ochiai, T. Nagase and A. Shima, *There exists no minimal $n$-chart with five white vertices*, Proc. Sch. Sci. TOKAI UNIV., **40** (2005), 1–18. MR2138333 (2006b:57035)

[10] K. Tanaka, *A Note on CI-moves*, Intelligence of Low Dimensional Topology 2006 Eds. J. Scott Carter *et al.* (2006), 307–314. MR2371740 (2009a:57017)
