Exponential ergodicity of branching processes with immigration and competition

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Abstract. We study the ergodic property of a continuous-state branching process with immigration and competition. The exponential ergodicity in a weighted total variation distance is proved under natural assumptions. The main theorem applies to subcritical, critical and supercritical branching mechanisms, including all those of stable types. The proof is based on the construction of a Markov coupling process and the choice of a nonsymmetric control function for the distance. Those are designed to identify and to take the advantage of the dominating factor from the branching, immigration and competition mechanisms in different parts of the state space. The approach provides a way of finding a lower bound of the ergodicity rate.

Résumé. Nous étudions la propriété ergodique d’un processus de branchement en temps et espace continu avec l’immigration et la compétition. L’ergodicité exponentielle dans une distance de variation totale pondérée est prouvée sous des hypothèses naturelles. Le théorème principal s’applique aux mécanismes de branchement sous-critiques, critiques et sur-critiques, y compris tous les types stables. La démonstration est basée sur la construction d’un processus Markovien de couplage et le choix d’une fonction de contrôle non symétrique pour la distance. Ceux-ci sont conçus pour identifier et profiter du facteur dominant des mécanismes de branchement, d’immigration et de compétition dans les parties différentes de l’espace d’états. Cette approche permet de trouver une borne inférieure de la vitesse d’ergodité.

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1. Introduction

Classical Galton–Watson branching processes are Markov processes taking values of nonnegative integers. They are models for the evolution of populations where the progenies of individuals are described by i.i.d. random variables. Standard references on those processes are Athreya and Ney [3] and Harris [31]. The study of continuous-state branching processes (CB-processes) was initiated by Feller [22], who noticed that a one-dimensional diffusion process may arise as the limit of a sequence of rescaled Galton–Watson branching processes. The result was extended by Lamperti [35] to the situation where the limiting process may have discontinuous sample paths; see also Aliev and Shchurenkov [2] and Grimvall [29]. Let $\Psi$ be a function on $[0, \infty)$ with the Lévy–Khintchine representation:

\begin{equation}
\Psi(\lambda) = b\lambda + c\lambda^2 + \int_0^{\infty} \left( e^{-\lambda z} - 1 + \lambda z 1_{\{z \leq 1\}} \right) \mu(dz), \quad \lambda \geq 0,
\end{equation}

where $b \in \mathbb{R}$ and $c \geq 0$ are constants and $(1 \wedge z^2) \mu(dz)$ is a finite measure on $(0, \infty)$. The transition semigroup $(Q_t)_{t \geq 0}$ of a CB-process with branching mechanism $\Psi$ is defined by

\begin{equation}
\int_{\mathbb{R}_+} e^{-\lambda y} Q_t(x, dy) = e^{-x\Psi(\lambda)}, \quad x \geq 0, \lambda > 0,
\end{equation}
where \( t \mapsto v_t(\lambda) \) is the unique strictly positive solution to the differential equation

\[
\frac{\partial}{\partial t} v_t(\lambda) = -\Psi(v_t(\lambda)), \quad v_0(\lambda) = \lambda.
\]

The CB-process is eventually degenerate in the sense that it tends to either zero or infinity as \( t \to \infty \). In fact, the process may explode at a finite time with strictly positive probability. It almost surely has an infinite lifetime if and only if its branching mechanism satisfies the following \textit{conservativeness condition}:

\[
\int_{0+} d\lambda \frac{1}{0+ \cap [-\Psi(\lambda)]} = \infty;
\]

see, e.g., Grey [28, p.670]. In order that the integral on the left-hand side of (1.4) is convergent, we have necessarily that \( \Psi'(0) = -\infty \). Let \( v_t(0) = \lim_{\lambda \to 0+} v_t(\lambda) \) for \( t \geq 0 \). Under condition (1.4), we have \( v_t(0) = 0 \) for every \( t \geq 0 \), which is also the unique solution to (1.3) with \( \lambda = 0 \). The first moments of the transition probabilities of the CB-process are given by

\[
\int_{\mathbb{R}_+} yQ_t(x, dy) = x \exp \left\{ -\Psi'(0+)t \right\}, \quad t, x > 0,
\]

where

\[
\Psi'(0+) = b - \int_{1}^{\infty} z \mu(dz).
\]

The branching mechanism \( \Psi \) is said to be \textit{subcritical}, \textit{critical} or \textit{supercritical} according as \( \Psi'(0+) > 0 \), \( \Psi'(0+) = 0 \) or \( \Psi'(0+) < 0 \), respectively. Let \( \sigma \geq 0 \), \( a \in \mathbb{R} \) and \( 0 < \alpha < 2 \) be constants. The \textit{stable branching mechanism} is defined by

\[
\Psi(\lambda) = \begin{cases} 
  a\lambda + c\lambda^2 + \sigma\lambda^\alpha, & 1 < \alpha < 2, \\
  a\lambda + c\lambda^2 + \sigma\lambda \log \lambda, & \alpha = 1, \\
  a\lambda + c\lambda^2 - \sigma\lambda^\alpha, & 0 < \alpha < 1.
\end{cases}
\]

Clearly, the conservativeness condition (1.4) fails for the stable branching mechanism if \( \sigma > 0 \) and \( 0 < \alpha < 1 \). If \( \sigma < 0 \) and \( 0 < \alpha \leq 1 \), then \( \Psi'(0+) = -\infty \) and the CB-process has infinite first moments by (1.5).

**Example 1.1.** Suppose that \( \Psi(\lambda) = a\lambda + \sigma\lambda^{1+\alpha} \), where \( \sigma \geq 0 \), \( a \in \mathbb{R} \) and \( 1 < \alpha \leq 2 \). Then \( \Psi'(0) = a > -\infty \) and condition (1.4) is satisfied. By solving (1.3) in this case, we see that

\[
v_t(\lambda) = \frac{e^{-at}}{[1 + \sigma q_0(a, t)\lambda^{-1}]^{1/(\alpha-1)}}, \quad t \geq 0, \lambda \geq 0,
\]

where \( q_0(a, t) = a^{-1}(1 - e^{-(\alpha-1)at}) \) for \( a \neq 0 \) and \( q_0(0, t) = (\alpha - 1)t \).

**Example 1.2.** Suppose that \( \Psi(\lambda) = a\lambda + \sigma\lambda \log \lambda \), where \( \sigma \geq 0 \) and \( a \in \mathbb{R} \). Then condition (1.4) is satisfied. In this case, we have

\[
v_t(\lambda) = \exp \left\{ e^{-\sigma t} \log \lambda - a\rho(\sigma, t) \right\}, \quad t \geq 0, \lambda \geq 0,
\]

where \( \rho(\sigma, t) = \sigma^{-1}(1 - e^{-\sigma t}) \) for \( \sigma > 0 \) and \( \rho(0, t) = t \). In particular, if \( \sigma > 0 \), then

\[
\lim_{t \to \infty} v_t(\lambda) = v_\infty := e^{-a/\sigma}, \quad \lambda > 0,
\]

which implies that, by weak convergence on \([0, \infty] \),

\[
\lim_{t \to \infty} Q_t(x, \cdot) = e^{-xv_\infty} \delta_0 + (1 - e^{-xv_\infty})\delta_{\infty}, \quad x \geq 0.
\]

**Example 1.3.** Suppose that \( \Psi(\lambda) = a\lambda + \sigma\lambda^\alpha \), where \( \sigma \geq 0 \), \( a \in \mathbb{R} \) and \( 0 < \alpha < 1 \). Then condition (1.4) is not satisfied. By solving (1.3) in this case, we have

\[
v_t(\lambda) = \left[ \sigma p_0(a, t) + \lambda^{1-\alpha} e^{-(1-\alpha)at} \right]^{1/(1-\alpha)}, \quad t \geq 0, \lambda > 0,
\]
where \( p_\alpha(a, t) = a^{-1}(1 - e^{-(1-\alpha)a}t) \) for \( a \neq 0 \) and \( p_\alpha(0, t) = (1 - \alpha)t \). In particular, if \( a > 0 \), then
\[
v_t(0) := \lim_{\lambda \to 0} v_t(\lambda) = [\sigma p_\alpha(a, t)]^{1/(1-\alpha)}, \quad t \geq 0
\]
and
\[
\lim_{t \to \infty} v_t(\lambda) = v_\infty := \left( \frac{\sigma \phi}{a} \right)^{1/(1-\alpha)}, \quad \lambda \geq 0.
\]
The last limit implies that, by weak convergence on \([0, \infty]\),
\[
\lim_{t \to \infty} Q_t(x, \cdot) = e^{-xv_\infty} \delta_0 + (1 - e^{-xv_\infty})\delta_\infty.
\]

The CB-processes defined by (1.2) and (1.3) involve rich mathematical structures and have been applied to the research in several important areas. In particular, Bertoin and Le Gall [6] gave a representation of the genealogical structure of the general CB-processes by Bochner’s subordination and used the representation to give a precise description of the connection between the so-called Neveu’s CB-process with branching mechanism \( \lambda \mapsto \lambda \log \lambda \) and the coalescent processes introduced by Bolthausen and Sznitman [10] in the study of spin glasses. Berestycki et al. [4] proved Neveu’s CB-process may arise in a limit theorem of some rescaled particle systems. General stochastic flows associated with coalescent processes and CB-processes were studied by Bertoin and Le Gall [7–9]. Constructions of the flows by strong solutions of stochastic equations were given by Dawson and Li [17].

Continuous-state branching processes with immigration (CBI-processes) were introduced by Kawazu and Watanabe [33] in their study of scaling limits of discrete branching processes with immigration; see also Aliev [1]. Let \( \Phi \) be an immigration mechanism, which is a function on \([0, \infty]\) with the Lévy–Khintchine representation:
\[
\Phi(\lambda) = \beta \lambda + \int_0^\infty (1 - e^{-z\lambda})\nu(dz), \quad \lambda \geq 0,
\]
where \( \beta \geq 0 \) and \((1 \wedge z)\nu(dz)\) is a finite measure on \((0, \infty)\). Then a CBI-process with parameters \((\Psi, \Phi)\) has transition semigroup \((Q^\Psi_t)_{t \geq 0}\) characterized by
\[
\int_{\mathbb{R}_+} e^{-\lambda y}Q^\Psi_t(x, dy) = \exp \left\{ -xv_t(\lambda) - \int_0^t \Phi(v_s(\lambda))ds \right\}, \quad \lambda > 0,
\]
where \( t \mapsto v_t(\lambda) \) is defined by (1.3). The CBI-process is a natural model for the study of ergodicity, where the immigration ensures that the stationary distribution is not the degenerate distribution at zero. A sufficient and necessary integrability condition for the ergodicity in weak convergence of the process was announced in Pinsky [52]; see Li [39] for a proof of the result. The exponential ergodicity of the CBI-process in the total variation distance was proved in Li and Ma [42] and the corresponding results for measure-valued processes were established in Friesen et al. [24] and Li [41], which improve the earlier results of Stannat [55, 56] for processes with special branching mechanisms. The exponential ergodicity in a suitably chosen Wasserstein distance was established by Friesen et al. [26] for affine Markov processes, which include finite-dimensional CBI-processes as a special case. The key tools of those explorations are the characterizations of the transition probabilities by Laplace transforms given as in (1.3) and (1.8). Since typical critical or supercritical CBI-processes eventually go to infinity, the studies of their ergodicities have mainly been focused on subcritical mechanisms.

To get the ergodicity in the total variation distance, Li and Ma [42] also assumed the following Grey’s condition: \( \Psi(\lambda) > 0 \) for sufficiently large \( \lambda \) and
\[
\int_0^\infty \frac{d\lambda}{\Psi(\lambda)} < \infty.
\]

Those restrictions unfortunately exclude some interesting branching mechanisms as those in Examples 1.2 and 1.3.

In this work we provide a general framework for exponential ergodicity applicable to branching mechanisms in the full range of criticality. To do so, a natural way is to incorporate a competition mechanism into the model. This consideration has been inspired by the work of Lambert [34], who defined a logistic growth branching process to model the pairwise competition between individuals in the population. The process was constructed in [34] by a random time change from a spectrally positive Ornstein–Uhlenbeck type process. More general branching population systems with competition were studied in depth by Berestycki et al. [5] and Pardoux [51]; see also Foucart [23] and Friesen et al. [25]. Our aim here is to understand whether and how a strong competition could balance the branching and the immigration to guarantee the
exponential ergodicity. These are not clear as the process may have infinite first moments, which means the population could be very large. In fact, to study the ergodic behavior we should first understand whether the competition could prevent the population from exploding at a finite time when (1.4) is not satisfied. The emphasis here is the interplay among the branching, immigration and competition mechanisms.

1.1. Main results

Suppose that \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})\) is a filtered probability space satisfying the usual hypotheses. Let \(\{L(ds, du)\}\) be a spectrally one-sided time-space \((\mathcal{F}_t)\)-Lévy white noise with the Lévy–Itô decomposition:

\[
L(ds, du) = W(ds, du) - bdsdu + \int_0^1 z \dot{M}(ds, dz, du) + \int_1^\infty z M(ds, dz, du),
\]

where \(W(ds, du)\) is a Gaussian white noise on \((0, \infty)^2\) based on \(2cdsdu\) and \(M(ds, dz, du)\) is a Poisson random measure on \((0, \infty)^3\) with intensity \(ds \mu(dz)du\) and compensated measure \(\dot{M}(ds, dz, du) := M(ds, dz, du) - ds \mu(dz)du\). Here and in the sequel, we make the convention that

\[
\int_0^x y = - \int_y^x = \int_{[y,x]} \quad \text{and} \quad \int_x^\infty = \int_{(x, \infty)} , \quad x \geq y \in \mathbb{R}.
\]

Let \(\{\eta(t) : t \geq 0\}\) be an \((\mathcal{F}_t)\)-subordinator defined by

\[
\eta(t) = \beta t + \int_0^t \int_0^\infty z N(ds, dz),
\]

where \(N(ds, dz)\) is a Poisson random measure on \((0, \infty)^2\) with intensity \(ds \nu(dz)\). Suppose that \(\{L(ds, du)\}\) and \(\{\eta(t)\}\) are independent of each other. Let \(g\) be a competition mechanism, which by definition is a nondecreasing and continuous function on \([0, \infty)\) satisfying \(g(0) = 0\). For any \(\mathcal{F}_0\)-measurable random variable \(x(0) \geq 0\), we consider the stochastic equation:

\[
x(t) = x(0) + \int_0^t \int_0^{x(s-)} L(ds, du) - \int_0^t g(x(s)) ds + \eta(t).
\]

We say an \((\mathcal{F}_t)\)-adapted càdlàg process \(\{x(t) : t \geq 0\}\) taking values in \(\mathbb{R}_+ := [0, \infty]\) is a solution to (1.11) if the equation holds almost surely when \(t\) is replaced by \(t \wedge \zeta_n\) for each \(t \geq 0\) and \(n \geq 1\), where \(\zeta_n = \inf\{t \geq 0 : x(t) \geq n\}\) with \(\inf\emptyset = \infty\). We call \(\zeta := \lim_{n \to \infty} \zeta_n\) the lifetime of \(\{x(t)\}\) and make the convention that \(x(t) = \infty\) for \(t \geq \zeta\). If \(\zeta = \infty\) almost surely, we say the process \(\{x(t)\}\) is conservative. We shall prove that there is a pathwise unique solution to (1.11); see Theorem 2.1. Then the solution \(\{x(t)\}\) is a strong Markov process in \(\mathbb{R}_+\). Let \((P_t)_{t \geq 0}\) be the transition semigroup of \(\{x(t)\}\). We shall call any Markov process with transition semigroup \((P_t)_{t \geq 0}\) a continuous-state branching process with immigration and competition (CBIC-process). Constructions of CB- and CBI-processes in terms of similar stochastic equations were suggested by Bertoin and Le Gall [9] and Dawson and Li [16, 17]. The CBIC-process extends the population models of Berestycki et al. [5] and Pardoux [51] by the additional immigration structure. A more general continuous-state population model in random environments was introduced by Palau and Pardo [50] by solving a stochastic equation driven by Brownian motions and Poisson random measures.

We need to introduce some concepts in order to present our main results. Let \(C^2(\mathbb{R}_+)\) be the linear space of twice continuously differentiable functions on \(\mathbb{R}_+\). For \(x \geq 0\) and \(f \in C^2(\mathbb{R}_+)\) write

\[
Lf(x) = cx f''(x) + x \int_0^\infty \left[ \Delta_z f(x) - zf'(x) 1_{\{z \leq 1\}} \right] \mu(dz)
\]

\[
+ [\beta - bx - g(x)] f'(x) + \int_0^\infty \Delta_z f(x) \nu(dz),
\]

where

\[
\Delta_z f(x) = f(x + z) - f(x).
\]

Let \(D(L)\) denote the linear space consisting of functions \(f \in C^2(\mathbb{R}_+)\) such that the two integrals on the right-hand side of (1.12) are convergent and define continuous functions on \(\mathbb{R}_+\). Let \(C^2_b(\mathbb{R}_+)\) be the space of bounded and continuous
functions on $\mathbb{R}_+$ with bounded and continuous derivatives up to the second order. Then $C^2_b(\mathbb{R}_+) \subset \mathcal{D}(L)$. In general, we allow $f$ and $Lf$ to be unbounded functions. We shall see that $(L, \mathcal{D}(L))$ is a restriction of the generator of the CBIC-process.

Given a nonnegative Borel function $V$ on $\mathbb{R}_+$, we denote by $\mathcal{P}_V(\mathbb{R}_+)$ the set of Borel probability measures $\gamma$ on $\mathbb{R}_+$ such that
\[
\int_{\mathbb{R}_+} V(x) \gamma(dx) < \infty.
\]
Let $W_V$ be the $V$-weighted total variation distance on $\mathcal{P}_V(\mathbb{R}_+)$ defined by
\[
W_V(\gamma, \eta) = \int_{\mathbb{R}_+} [1 + V(x)]|\gamma - \eta|(dx), \quad \gamma, \eta \in \mathcal{P}_V(\mathbb{R}_+),
\]
where $|\cdot|$ denotes the total variation measure. We shall see that $W_V$ is actually the Wasserstein distance determined by the metric
\[
d_V(x, y) = [2 + V(x) + V(y)]1_{\{x \neq y\}}, \quad x, y \in \mathbb{R}_+.
\]
More precisely, we have
\[
W_V(\gamma, \eta) = \inf_{\pi \in \mathcal{C}(\gamma, \eta)} \int_{\mathbb{R}_+^2} d_V(x, y) \pi(dx, dy),
\]
where $\mathcal{C}(\gamma, \eta)$ is the collection of all probability measures on $\mathbb{R}_+^2$ with marginals $\gamma$ and $\eta$; see Lemma 5.3. The consideration of this distance was inspired by Hairer and Mattingly [30]; see also [21, 43]. In particular, if $V \equiv 0$, then $W_V$ reduces to the total variation distance. The other two frequently used weight functions are given by
\[
V_1(x) = x \quad \text{and} \quad V_{\log}(x) = \log(1 + x), \quad x \geq 0.
\]
We say a conservative CBIC-process $\{x(t) : t \geq 0\}$ or its transition semigroup $(P_t)_{t \geq 0}$ is exponentially ergodic in the distance $W_V$ with rate $\lambda_*>0$ if it possesses a unique stationary probability distribution $\gamma$ and there is a nonnegative function $\eta \mapsto C(\eta)$ on $\mathcal{P}_V(\mathbb{R}_+)$ such that
\[
W_V(\gamma, \eta P_t) \leq C(\eta)e^{-\lambda_* t}, \quad t \geq 0, \eta \in \mathcal{P}_V(\mathbb{R}_+).
\]
The exponential ergodicity (1.17) follows by standard arguments if there is a constant $K \geq 0$ such that
\[
W_V(P_t(x, \cdot), P_t(y, \cdot)) \leq K e^{-\lambda_* t} d_V(x, y), \quad t \geq 0.
\]
Given $\sigma$-finite measures $\mu$ and $\nu$ on $\mathbb{R}$, we write $\mu \wedge \nu$ for $\mu - (\mu - \nu)_+ = \nu - (\nu - \mu)_+$, where the subscript “$+$” stands for the upper variation of the signed measure in its Jordan decomposition. Let $\mu * \nu$ denote the convolution of $\mu$ and $\nu$ defined by, for all positive Borel functions $f$ on $\mathbb{R}$,
\[
\int f(z)(\mu * \nu)(dz) = \int \mu(dx) \int f(x+y) \nu(dx).
\]
For a nonnegative function $V \in \mathcal{D}(L)$ and constants $C_0, C_1 > 0$ let us consider the inequality
\[
LV(x) \leq C_0 - C_1 V(x), \quad x \geq 0.
\]
Condition 1.1. There exists a constant $\lambda_0 > 0$ such that $\Phi(\lambda_0) > 0$ and $\Phi(\lambda_0) > 0$.
Condition 1.2. One of the following conditions is satisfied: (i) Grey’s condition (1.9); (ii) for some constants $c_0 > 0$ and $\kappa_0 > 0$,
\[
\kappa(x) := [\mu \wedge (\delta_x * \mu)](0, \infty) + [\nu \wedge (\delta_x * \nu)](0, \infty) \geq \kappa_0, \quad |x| \leq c_0.
\]
Condition 1.3. There is a nonnegative function $V \in \mathcal{D}(L)$ satisfying (1.19) and $V(x) \to \infty$ as $x \to \infty$.

We now present the main result of this paper.
Theorem 1.1. Suppose that Conditions 1.1, 1.2 and 1.3 are satisfied. Then the CBIC-process is conservative and there are constants $K > 0$ and $\lambda_s > 0$ such that (1.18) holds. Consequently, the CBIC-process is exponentially ergodic in the $V$-weighted total variation distance with rate $\lambda_s > 0$.

The advantage of Theorem 1.1 is that it works for general branching mechanisms without criticality restriction. In particular, it applies to all the stable branching mechanisms given by (1.6). Furthermore, the proof of the theorem actually provides a way of finding the exponential ergodicity rate $\lambda_s > 0$, which is important in applications; see Remark 5.4. Here, Condition 1.1 is introduced to avoid some extreme cases of (1.11). If the condition fails, then either subordinator $\eta(t)$ vanishes or the Lévy field $L(ds, du)$ only has nonnegative increments. In any of those cases, the immigration essentially plays no role in the ergodic behavior. Conditions like (1.20) have been considered in the study of ergodicities of Ornstein-Uhlenbeck type processes with nonlinear drift; see, e.g., [37, 43, 45, 46, 53]. Our condition (1.20) is actually weaker than the corresponding assumptions for exponential ergodicities in those previous papers, where one usually required $\kappa(x) \to \infty$ as $x \to 0$. In particular, the condition is satisfied if $\mu(dz)$ or $\nu(dz)$ is bounded below by the measure $r 1_{(u,v)}(z)dz$ for some $r > 0$ and $v > u \geq 0$. The existence of a function $V \in D(L)$ with the property like (1.19) has become a standard assumption in the study of uniqueness and ergodicity problems of Markov processes following Chen [11–14]; see also Down et al. [18] and Meyn and Tweedie [47–49]. For the CBIC-process, the condition means roughly that the competition mechanism wins its confrontation against the branching and immigration when the population is large. Typically, this is equivalent to a simple growth condition for the competition mechanism.

Proposition 1.2. Suppose that the measures $\mu$ and $\nu$ satisfy the integrability condition

\begin{equation}
\int_1^\infty z\mu(dz) + \int_1^\infty z\nu(dz) < \infty.
\end{equation}

Then $V_1 \in D(L)$ satisfies (1.19) if and only if

\begin{equation}
\liminf_{x \to \infty} \frac{g(x)}{x} + b - \int_1^\infty z\mu(dz) > 0.
\end{equation}

Proposition 1.3. Suppose that the measures $\mu$ and $\nu$ satisfy the integrability condition

\begin{equation}
\int_1^\infty \log(1+z)\mu(dz) + \int_1^\infty \log(1+z)\nu(dz) < \infty.
\end{equation}

Then $V_{\log} \in D(L)$ satisfies (1.19) if and only if

\begin{equation}
\liminf_{x \to \infty} \left\{ \frac{g(x)}{x \log x} - \frac{x}{\log x} \int_1^\infty \log \left(1 + \frac{z}{1+x}\right)\mu(dz) \right\} > 0.
\end{equation}

Under the integrability condition (1.21), the CBIC-process has finite first moments. In this case, condition (1.22) means the competition mechanism should grow at least linearly with a sufficiently large rate. In particular, if $g(x) = ax$ for $x \geq 0$, the CBIC-process reduces to a CBI-process with branching mechanism $\lambda \to a\lambda + \Psi(\lambda)$ and (1.22) simply means the branching mechanism should be subcritical. This is consistent with the subcriticality condition in [42, Theorem 2.5], but here we do not need to assume Grey’s condition. The conditions of Proposition 1.3 apply to supercritical branching mechanisms including those with infinite first moments but finite logarithmic moments.

1.2. Stable branching CBIC-processes

These processes are solutions of stochastic differential equations driven by spectrally one-sided Lévy processes. Let $m_\alpha$ be the $\sigma$-finite measure on $(0, \infty)$ defined by

\begin{equation}
m_\alpha(dz) = \begin{cases} (\alpha - 1)\Gamma(2 - \alpha)^{-1}z^{1-\alpha}dz, & 1 < \alpha < 2, \\
z^{-2}dz, & \alpha = 1, \\
\Gamma(1 - \alpha)^{-1}z^{1-\alpha}dz, & 0 < \alpha < 1,
\end{cases}
\end{equation}

where $\Gamma$ is the Gamma function. Let $\{M_\alpha(ds, dz)\}$ be a time-space $(\mathcal{F}_t)$-Poisson random measure on $(0, \infty)^2$ with intensity $dsm_\alpha(dz)$ and compensated measure $\bar{M}_\alpha(ds, dz)$. Let $\{z_\alpha(t)\}$ be the spectrally positive $\alpha$-stable Lévy process
and Propositions 1.28 to the stable branching mechanisms, we obtain

\[
(1.27) \quad dz(t) = \sqrt{2cx(t)}dB(t) + \sigma x(t)dz_1(t) + \sigma x(t)\log(\sigma x(t))dt - ax(t)dt - g(x(t))dt + d\eta(t)
\]

and, for \(\alpha \neq 1\),

\[
(1.28) \quad dz(t) = \sqrt{2cx(t)}dB(t) + \sqrt{\alpha\sigma x(t)}dz_\alpha(t) - ax(t)dt - g(x(t))dt + d\eta(t);
\]

see Theorem 2.4. By applying Theorem 1.1 and Propositions 1.2 and 1.3 to the stable branching mechanisms, we obtain the following:

**Corollary 1.4.** Suppose that the branching mechanism is given by (1.6) with \(1 < \alpha < 2\). In addition, assume that

\[
\int_1^\infty z\nu(dz) < \infty \quad \text{and} \quad \liminf_{x \to \infty} \frac{g(x)}{x} > -\alpha.
\]

Then the CBIC-process is exponentially ergodic in the \(V_1\)-weighted total variation distance.

**Corollary 1.5.** Suppose that the branching mechanism is given by (1.6) with \(0 < \alpha \leq 1\) and that \(a > 0\) or \(c > 0\) for \(0 < \alpha < 1\). In addition, assume that

\[
\int_1^\infty \log(1 + z)\nu(dz) < \infty
\]

and

\[
\begin{aligned}
\liminf_{x \to \infty} \frac{g(x)}{x^{2-\alpha}} &> \frac{\sigma\pi}{\Gamma(1-\alpha)\sin(\alpha\pi)}, & 0 < \alpha < 1, \\
\liminf_{x \to \infty} \frac{g(x)}{x\log x} &> \sigma, & \alpha = 1.
\end{aligned}
\]

Then the CBIC-process is exponentially ergodic in the \(V_{\log}\)-weighted total variation distance.

In the situation of Corollary 1.5, the CBIC-process has infinite first moments, so the result has no counterpart in the setting of CBI-processes, where exponential ergodicities can only be considered for subcritical branching mechanisms. The assumption that \(a > 0\) or \(c > 0\) for \(0 < \alpha < 1\) in the corollary comes from Condition 1.1. If this were not satisfied, there would not be sufficient fluctuations in the CBIC-process as the two remaining noise terms in (1.28) are both nondecreasing. As we explained before, if \(\sigma > 0\) and \(0 < \alpha < 1\), then condition (1.4) fails and the corresponding CBI-process without competition explodes at a finite time.

### 1.3. More examples

The following examples concern further applications of the results presented above and show the conditions we introduce are sharp in typical situations.

**Example 1.4.** Consider the CBI-process with branching and immigration mechanisms given respectively by (1.1) and (1.7) with \(b > 0\), \(\beta > 0\) and \(c = \nu(0, \infty) = 0\). We assume that \(\mu(dz)\) is given by

\[
\mu(dz) = \alpha \sum_{k=1}^\infty m_\alpha(k, k+1)\delta_{k+1}(dz) + \alpha \sum_{k=1}^\infty m_\alpha[1/(k+1), 1/k]\delta_{1/k}(dz),
\]
where $1 < \alpha < 2$ and $m_\alpha(dz)$ is the $\sigma$-finite measure on $(0, \infty)$ defined by (1.25). Then the branching mechanism satisfies Grey’s condition (1.9) since

$$
\Psi(\lambda) > b\lambda + \lambda^\alpha = b\lambda + \alpha \int_0^\infty (e^{-\lambda z} - 1 + \lambda z) m_\alpha(dz).
$$

By Proposition 1.2, the CBI-process is exponentially ergodic relative to the $V_1$-weighted total variation distance. The exponential ergodicity in the total variation distance of the process can also be derived from Li and Ma [42, Theorem 2.5]. In this case, Condition 1.2(ii) is not satisfied because $[\mu \wedge (\delta_x + \mu)](0, \infty) = 0$ when $x$ is any irrational number. \hfill \Box

**Example 1.5.** Consider the CBI-process with branching and immigration mechanisms given respectively by (1.1) and (1.7) with $b > 0$, $\beta > 0$ and $c = \nu(0, \infty) = 0$. In addition, assume that $\mu(dz) = r \mathbf{1}_{(u,v)}(z)dz$ for some $r > 0$ and $0 \leq u < v \leq 1$. In this case, the branching mechanism satisfies Condition 1.2(ii). By Proposition 1.2, the CBI-process is exponentially ergodic relative to the $V_1$-weighted total variation distance. However, the branching mechanism does not satisfy Grey’s condition (1.9) as

$$
\lim_{\lambda \to \infty} \lambda^{-1} \Psi(\lambda) = b + \int_0^1 z\mu(dz) = b + \frac{r}{2}(u^2 - u^2).
$$

Therefore the exponentially ergodicity in the total variation distance of the process does not follow from Li and Ma [42, Theorem 2.5]. One can also see that the conditions of Li and Wang [37, Theorem 1.1] are not satisfied. \hfill \Box

**Example 1.6.** Consider the CBI-process with branching and immigration mechanisms given by (1.1) and (1.7), respectively. Suppose that (1.21) holds and the branching mechanism is critical, that is,

$$
b_0 := b - \int_1^\infty z\mu(dz) = 0.
$$

It is known that

$$
\int_{\mathbb{R}_+} yP_t(0, dy) = t\beta + t \int_0^\infty z\nu(dz);
$$

see, e.g., Li [40, p.33]. The right-hand side tends to infinity as $t \to \infty$, so the CBI-process is not exponentially ergodic in the $V_1$-weighted total variation distance. This shows the conditions of Corollary 1.4 are sharp. \hfill \Box

**Example 1.7.** Let $\Psi$ be the stable branching mechanisms given by (1.6). By the proof of Theorem 2.4, the CBIC-process has generator $L$ defined by (1.12) with $b = a + \sigma h_\alpha$ and $\mu(dz) = \alpha \sigma m_\alpha(dz)$, where $m_\alpha$ and $h_\alpha$ are defined by (1.25) and (2.6), respectively. Let us consider the case where $\alpha = 1$ and $b = c = \nu(0, \infty) = 0$. It is easy to show that

$$
LV_{\log}(x) = \frac{\sigma x}{1 + x} [1 + \log(1 + x)] - \frac{g(x)}{1 + x} + \frac{\beta}{1 + x}.
$$

If $\sigma \beta > 0$ and $g(x) = \sigma x \log(1 + x)$ for $x \geq 0$, there is a constant $\lambda > 0$ such that $LV_{\log}(x) \geq \lambda$, and so

$$
\int_{\mathbb{R}_+} V_{\log}(y) P_t(x, dy) = V_{\log}(x) + \int_0^t P_s LV_{\log}(x) ds \geq V_{\log}(x) + \lambda t.
$$

Consequently, the CBIC-process is not exponentially ergodic in the $V_{\log}$-weighted total variation distance. When $0 < \alpha < 1$, a similar example can be given by considering the function $g(x) = [F(1 - \alpha) \sin(\alpha \pi)]^{-1} \sigma \pi x^{2-\alpha}$. Then the conditions of Corollary 1.5 are sharp. \hfill \Box

1.4. The approach of coupling and distance

The characterizations like (1.3) and (1.8) are not available for the CBIC-process. Our proof of Theorem 1.1 is based on the approach of coupling and distance. Those techniques have played important roles in the ergodic theory of Markov processes. We refer to Chen [14, 15] for systematical treatments of the techniques. In particular, a number of results on the exponential ergodicity have been obtained by this method for Ornstein-Uhlenbeck type processes with nonlinear drifts defined by stochastic differential equations of the form:

$$
dx(t) = dL(t) - g(x(t))dt, \quad t \geq 0.
$$
where \( \{ L(t) : t \geq 0 \} \) is a Lévy process and the drift coefficient \( g \) is sufficiently regular so that there is a unique solution to the equation. When the driving noise is a Brownian motion, the exponential ergodicity and related problems were studied by Eberle and his coauthors [19–21] and Luo and Wang [44]. For discontinuous Lévy noises, the ergodicity problem was investigated by Liang et al. [43], Luo and Wang [45], Schilling and Wang [53] and Majka [46]. The feature of those processes is that the random noise in (1.29) is both temporally and spatially homogeneous.

The main difficulty of the proof of Theorem 1.1 is that the branching fluctuations become small and rare when the process is close to zero, which is clear from (1.11) or (1.12). A similar phenomenon has been noticed in nonlinear branching processes by Li and Wang [37]; see also [36, 38]. The difficulty was treated in [37] by introducing conditions on the asymptotics at zero of the branching coefficients. We cannot do that in our setting here since the branching coefficients here have to be linear. The key of our proof is the design of a Markov coupling process and a nonsymmetric control function. More precisely, we construct a two-dimensional Markov coupling process \( \{ (X_t, Y_t) : t \geq 0 \} \) on \( \mathbb{R}_+^2 \), where both \( \{ X_t : t \geq 0 \} \) and \( \{ Y_t : t \geq 0 \} \) are Markov processes with transition semigroup \( \{ P_t \}_{t \geq 0} \). The coupling process satisfies \( X_{T+t} = Y_{T+t} \) for every \( t \geq 0 \), where \( T = \inf \{ t \geq 0 : X_t = Y_t \} \) with \( \inf 0 = \infty \) by convention is the succeeding or coupling time. We need to make the coupling succeed as early as possible, which is realized by reflecting the noises in suitable ways; see Remark 3.6. Then we define a non-symmetric function \( G_0 \) with the exponential contraction property: for \( t \geq 0 \) and \( (x, y) \in \mathbb{R}_+^2 \),

\[
P_t G_0(x, y) \leq e^{-\lambda \cdot t} G_0(x, y),
\]

where \( \{ P_t \}_{t \geq 0} \) is the transition semigroup of the coupling process. The function \( G_0 \) should control the distance \( d_V \) in the sense that there are constants \( c_2 \geq c_1 > 0 \) such that, for \( (x, y) \in \mathbb{R}_+^2 \),

\[
c_1 G_0(x, y) \leq d_V(x, y) \leq c_2 G_0(x, y).
\]

Then we deduce the estimate (1.18) from (1.30) and (1.31). The non-symmetric control function makes it possible to identify the dominating factor from the branching, immigration and competition mechanisms in different parts of the space, which is important in establishing the estimate (1.30). By those devices we not only overcome the difficulty caused by the degeneracy of the branching coefficients but also weaken the fluctuation conditions introduced for the exponential ergodicity of the Ornstein-Uhlenbeck type processes; see, e.g., [37, 43, 45, 46, 53]. In fact, the assumptions on the drift coefficient in Corollaries 1.4 and 1.5 are also weaker than those in the previous papers, where lower bounds of the increment \( g(x) - g(y) \) for \( x \geq y \geq 0 \) were required. We believe that the method used here could be adapted to more general classes of processes with Lévy driving noises.

The rest of the paper is organized as follows. In Section 2, the existence and uniqueness of solutions to the stochastic equations (1.11), (1.27) and (1.28) are proved. In Section 3, we give the construction of the Markov coupling process. In Section 4, some necessary estimates for the coupling generator are established. The proofs of the ergodicity results are given in Section 5.

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2. Stochastic equations of CBIC-processes

In this section, we prove the existence and pathwise uniqueness of solutions to the stochastic equations (1.11), (1.27) and (1.28), which give constructions of the CBIC-processes.

Theorem 2.1. For any \( \mathcal{F}_0 \)-measurable random variable \( x(0) \geq 0 \), there is a pathwise unique solution \( \{ x(t) : t \geq 0 \} \) to (1.11).

Proof. By Theorem 2.5 of Dawson and Li [17], for each \( n \geq 1 \) there is a pathwise unique solution \( \{ x_n(t) : t \geq 0 \} \) to

\[
x(t) = x(0) + \int_0^t \int_0^{x(s-)} W(ds, du) + \int_0^t [\beta - bx(s) - g(x(s))] ds
\]

\[
+ \int_0^t \int_0^1 \int_0^{x(s-)} z \tilde{M}(ds, dz, du) + \int_0^t \int_1^\infty \int_0^{x(s-)} (z \wedge n) M(ds, dz, du)
\]

\[
+ \int_0^t \int_0^\infty (z \wedge n) N(ds, dz).
\]
Let \( \zeta_n = \inf \{ t \geq 0 : x_n(t) \geq n \} \). From the pathwise uniqueness for (2.1) it follows that \( \zeta_n \) is nondecreasing in \( n \geq 1 \) and \( x_{n+1}(t) = x_n(t) \) for \( 0 \leq t < \zeta_n \). Clearly, the pathwise unique solution \( \{ x(t) : t \geq 0 \} \) to (1.11) is given by \( x(t) = x_n(t) \) for \( 0 \leq t < \zeta_n \) and \( x(t) = \infty \) for \( t \geq \zeta := \lim_{n \to \infty} \zeta_n \).

By the above theorem we have given a construction of the CBIC-process. The next result justifies the fact that the operator \((L, D(L))\) defined by (1.12) is a restriction of the generator of the process.

**Theorem 2.2.** Let \( \{ x(t) : t \geq 0 \} \) be the pathwise unique solution to (1.11) and let \( \zeta_n = \inf \{ t \geq 0 : x(t) \geq n \} \). Then for any \( n \geq 1 \) and \( f \in D(L) \) we have

\[
(2.2) \quad f(x(t \wedge \zeta_n)) = f(x(0)) + \int_0^{t \wedge \zeta_n} Lf(x(s))ds + M_n(t),
\]

where \( \{ M_n(t) : t \geq 0 \} \) is a martingale defined by

\[
(2.3) \quad M_n(t) = \int_0^{t \wedge \zeta_n} \int_0^x f'(x(\tau))W(\tau, d\tau) + \int_0^{t \wedge \zeta_n} \int_0^\infty \Delta_x f(x(\tau))\tilde{N}(d\tau, dz) + \int_0^{t \wedge \zeta_n} \int_0^\infty \int_0^x \Delta_x f(x(\tau))\tilde{M}(d\tau, dz, du).
\]

**Proof.** By (1.11) the process \( \{ x(t) : t \geq 0 \} \) is a semimartingale. For any \( f \in D(L) \), we can use Itô’s formula to see that

\[
f(x(t \wedge \zeta_n)) = f(x(0)) - \int_0^{t \wedge \zeta_n} f'(x(s))g(x(s))ds + \int_0^{t \wedge \zeta_n} f'(x(s))L(ds, du)
+ \int_0^{t \wedge \zeta_n} [f''(x(s)) - zf'(x(s)) - f'(x(s))] M(ds, dz, du)
+ \int_0^{t \wedge \zeta_n} [f''(x(s)) - zf'(x(s))] N(ds, dz).
\]

Then (2.2) holds with \( M_n(t) \) defined by (2.3). Since \( x(s-) \leq n \) for \( 0 < s \leq \zeta_n \), it is easy to show that \( \{ M_n(t) : t \geq 0 \} \) is a martingale.

**Proposition 2.3.** Suppose that Condition 1.3 is satisfied. Then the solution \( \{ x(t) : t \geq 0 \} \) to (1.11) is conservative and

\[
(2.4) \quad E[V(x(t))] \leq E[V(x(0))]e^{-C_1 t} + C_0 C_1^{-1} (1 - e^{-C_1 t}), \quad t \geq 0.
\]

**Proof.** There is no loss of generality to assume \( E[V(x(0))] < \infty \). Recall that \( \zeta_n = \inf \{ t \geq 0 : x(t) \geq n \} \) and \( \zeta = \lim_{n \to \infty} \zeta_n \). By applying (2.2) to the function \( V \) and using integration by parts we have

\[
(2.5) \quad e^{C_1 t(t \wedge \zeta_n)} V(x(t \wedge \zeta_n)) = V(x(0)) + \int_0^{t \wedge \zeta_n} e^{C_1 s} [C_1 V(x(s)) + LV(x(s))] ds + M(t),
\]
where
\[
M(t) = \int_0^t \int_0^{x(s)} e^{C_1 s} f'(x(s)) W(ds, du) + \int_0^t \int_0^{x(s)} e^{C_1 s} \Delta z f(x(s)) \tilde{N}(ds, dz) \\
+ \int_0^t \int_0^{x(s)} e^{C_1 s} \Delta z f(x(s)) M(ds, dz, du).
\]

In view of (1.19), we can take the expectations in both sides of (2.5) to obtain
\[
E[e^{C_1 (t \wedge \zeta_n)} V(x(t \wedge \zeta_n))] \leq E[V(x(0))] + C_0 C_1^{-1} e^{C_1 t - 1}.
\]

By Fatou’s lemma it follows that
\[
E[e^{C_1 (t \wedge \zeta)} V(x(t \wedge \zeta))] \leq E[V(x(0))] + C_0 C_1^{-1} e^{C_1 t - 1}
\]
with the convention $V(\infty) = \infty$. Then $P(\zeta \leq t) = 0$ and (2.4) follows. Since $t \geq 0$ was arbitrary, we have $\zeta = \infty$ almost surely. \qed

**Theorem 2.4.** For any $\mathcal{F}_0$-measurable random variable $x(0) \geq 0$, there are pathwise unique solutions to (1.27) and (1.28), and the solutions are CBIC-processes with competition mechanism $g$, stable branching mechanisms $\Psi$ given by (1.6) and immigration mechanism $\Phi$ given by (1.7).

**Proof.** Let \( \{x(t)\} \) be the solution to (1.11) with \( \{M(ds, dz, du)\} \) being a Poisson random measure on \((0, \infty)^3\) with intensity $\alpha \sigma ds m_\alpha(dz)du$, where $m_\alpha(dz)$ is given by (1.25). Moreover, we take $b = a + \sigma h_\alpha$, where

\[ h_\alpha = \begin{cases} 
-\alpha \Gamma(2 - \alpha)^{-1}, & 1 < \alpha < 2, \\
0, & \alpha = 1, \\
\alpha(1 - \alpha)^{-1} \Gamma(1 - \alpha)^{-1}, & 0 < \alpha < 1.
\end{cases} \]

If $c > 0$, one can see that
\[
B(t) = \int_0^t \int_0^{x(s)} \frac{1}{\sqrt{2c x(s)}} 1\{x(s) > 0\} W(ds, du) + \frac{1}{\sqrt{2c}} \int_0^t \int_0^{1} 1\{x(s) = 0\} W(ds, du)
\]
defines a standard Brownian motion and
\[
\int_0^t \sqrt{2c x(s)} dB(s) = \int_0^t \int_0^{x(s)} W(ds, du).
\]

In the sequel, we assume $\sigma > 0$, for otherwise the proof is simpler. Let \( \{M_\alpha(ds, dz)\} \) be the random measure on \((0, \infty)^2\) defined by, for $B \in \mathcal{B}(0, \infty)$,
\[
M_\alpha((0, t] \times B) = \int_0^t \int_0^{x(s)} 1\{x(s) > 0\} 1_B \left( \frac{z}{\sqrt{\alpha \sigma x(s)}} \right) M(ds, dz, du) \\
+ \int_0^t \int_0^{1} 1\{x(s) = 0\} 1_B(z) M(ds, dz, du).
\]

One can show as in Li [39, pp. 287–288] that \( \{M_\alpha(ds, dz)\} \) is a Poisson random measure with intensity $\sigma ds m_\alpha(dz)$. Let \( \{z_\alpha(t)\} \) be the $\alpha$-stable Lévy processes defined by (1.26). It is easy to see that
\[
\int_0^t \sqrt{\alpha \sigma x(s)} dz_\alpha(s) = \int_0^t \int_0^{x(s)} z \tilde{M}(ds, dz, du) + \sigma h_\alpha \int_0^t x(s) ds \\
+ \int_0^t \int_1^{x(s)} z \tilde{M}(ds, dz, du).
\]

Then (1.27) and (1.28) hold. By Fu and Li [27, Theorem 5.1], for any $n \geq 1$ there is a pathwise unique solution to the stochastic equation:
\[
x(t) = x(0) + \int_0^t \sqrt{2c x(s)} dB(s) + \int_0^t \int_0^{x(s)} \sigma x(s) z \tilde{M}_1(ds, dz)
\]
Since $n \geq 1$ is arbitrary, the pathwise uniqueness also holds for (1.27). A similar argument gives the pathwise uniqueness of (1.28). These give the results of the theorem; see, e.g., Situ [54, p.76 and p.104].

3. Construction of the coupling process

Throughout this section, we assume the CBIC-process with an arbitrary initial value is conservative. By Proposition 2.3, this is true if Condition 1.3 is satisfied. We shall give the construction of a Markov coupling of the CBIC-process. Given $x \in \mathbb{R}$ and a general $\sigma$-finite measure $m$ on $(0, \infty)$, we define another $\sigma$-finite measure on $(0, \infty)$ by

\[ m_x(dz) = \frac{1}{2} \left[ m \wedge (\delta_x * m) \right](dz). \]  

(3.1)

It is easy to see that

\[ m_x = \delta_x * m_x - x \] and \[ m_x(0, 0 \lor x) = 0. \] Moreover, we have

\[ m_x(0, \infty) = m_{x-}(0, \infty) \leq \frac{1}{2} m(|x|, \infty). \]  

(3.3)

For any $x, y \geq 0$ write $m_{x,y} = m_{y-x} + m_{x-y}$. One can show that both $x \mapsto m_x(dz)$ and $(x, y) \mapsto m_{x,y}(dz)$ are kernels. For notational convenience, we also write those kernels as $m(x, dz)$ and $m(x, y, dz)$, respectively. We shall use those notations and facts to the Lévy measures $\mu$ and $\nu$ of the CBIC-process. For $x \in \mathbb{R}$ and $z > 0$ we set

\[ \rho_1(x, z) = \frac{\mu(x, dz)}{\mu(dz)}, \quad r_1(x, z) = \frac{\nu(x, dz)}{\nu(dz)}. \]  

(3.4)

By (3.1), the above Radon-Nikodym derivatives exist and satisfy

\[ \sup_{x \in \mathbb{R}} \sup_{z > 0} \rho_1(x, z) \leq \frac{1}{2}, \quad \sup_{x \in \mathbb{R}} \sup_{z > 0} r_1(x, z) \leq \frac{1}{2}. \]

In view of (3.2), we have

\[ \rho_1(x, z) = r_1(x, z) = 0, \quad 0 < z \leq 0 \lor x. \]  

(3.5)

For $x, y \geq 0$ and $z > 0$ let

\[ \rho_2(x, y, z) = \rho_1(x - y, z) + \rho_1(y - x, z) \] and

\[ r_2(x, y, z) = r_1(x - y, z) + r_1(y - x, z). \]

Suppose that $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is a filtered probability space satisfying the usual hypotheses. Let $\{W_0(ds, du)\}$ be a time-space $(\mathcal{F}_t)$-Gaussian white noise on $(0, \infty)^2$ based on $2dudsdu$. Let $\{M_0(ds, dz, du, dv)\}$ be a time-space $(\mathcal{F}_t)$-Poisson random measure on $(0, \infty)^3 \times (0, 1]$ with intensity $d sandbox(dz) dudv$ and compensated measure $\{\tilde{M}_0(ds, dz, du, dv)\}$. Let $\{N_0(ds, dz, dv)\}$ be a time-space $(\mathcal{F}_t)$-Poisson random measure on $(0, \infty)^2 \times (0, 1]$ with intensity $dsdudv$. We assume those random noises are independent of each other. Here the extra component $(0, 1]$ of the Poisson random measures is introduced for the convenience of disassembling the jumps in the construction of the coupling process. Let $\{\eta_t\}$ be the $(\mathcal{F}_t)$-subordinator defined by

\[ \eta_t = \beta t + \int^t_0 \int^\infty_0 \int^1_0 z N_0(ds, dz, dv). \]
Let \( \{L_0(ds, du)\} \) and \( \{L_0^*(ds, du)\} \) be two accompanying spectrally one-sided time-space \((\mathcal{F}_t)\)-Lévy white noises defined by

\[
L_0(ds, du) = W_0(ds, du) - b ds du + \int_{\{0 < z \leq 1\}} \int_{\{0 < v \leq 1\}} z \tilde{M}_0(ds, dz, du, dv)
+ \int_{\{1 < z \leq \infty\}} \int_{\{0 < v \leq 1\}} z \tilde{M}_0(ds, dz, du, dv)
\]

and

\[
L_0^*(ds, du) = -W_0(ds, du) - b ds du + \int_{\{0 < z \leq 1\}} \int_{\{0 < v \leq 1\}} z \tilde{M}_0(ds, dz, du, dv)
+ \int_{\{1 < z \leq \infty\}} \int_{\{0 < v \leq 1\}} z \tilde{M}_0(ds, dz, du, dv).
\]

By Theorem 2.1, for any \( \mathcal{F}_0 \)-measurable random variable \( X_0 \geq 0 \) we can construct a CBIC-process \( \{X_t : t \geq 0\} \) by the pathwise unique solution to the stochastic integral equation:

\[
X_t = X_0 + \int_0^t \int_0^{X_{t-}} L_0(ds, du) - \int_0^t g(X_s) ds + \eta_t.
\]

**Theorem 3.1.** Let \( Y_0 \) be an \( \mathcal{F}_0 \)-measurable random variable satisfying \( X_0 \geq Y_0 \geq 0 \). Then there is a pathwise unique solution \( \{\{Y_t, \xi_t\} : t \geq 0\} \) to the system of stochastic equations:

\[
Y_t = Y_0 + \int_{t \wedge T} \int_0^{Y_{t-}} L_0(ds, du) - \int_0^t g(Y_s) ds + \eta_t
+ \int_0^{t \wedge T} \int_0^{Y_{t-}} L_0^*(ds, du) + \xi_t
\]

and

\[
\xi_t = \int_0^{t \wedge T} \int_0^{Y_{t-}} \int_0^{Y_{t-}} (X_{s-} - Y_{s-}) M_0(ds, dz, du, dv)
+ \int_0^{t \wedge T} \int_0^{Y_{t-}} \int_0^{r_1(Y_{s-} - X_{s-}, z)} (X_{s-} - Y_{s-}) N_0(ds, dz, dv)
+ \int_0^{t \wedge T} \int_0^{Y_{t-}} \int_0^{Y_{t-}} \int_0^{r_2(Y_{s-} - X_{s-}, z)} (Y_{s-} - X_{s-}) M_0(ds, dz, du, dv)
+ \int_0^{t \wedge T} \int_0^{Y_{t-}} \int_0^{r_1(Y_{s-} - X_{s-}, z)} (Y_{s-} - X_{s-}) N_0(ds, dz, dv),
\]

where

\[
T = \inf\{t \geq 0 : X_t \leq Y_t\} = \inf\{t \geq 0 : X_t = Y_t \text{ or } X_{t-} = Y_{t-}\}.
\]

Moreover, we have \( X_{T+t} = Y_{T+t} \) for every \( t \geq 0 \) if \( T < \infty \).

In the proof of the above theorem, we shall see that the pure jump process \( \{\xi_t : t \geq 0\} \) has at most a finite number of jumps. In order to give the proof, we make some preparations. By Theorem 2.1, there is a CBIC-process \( \{Z_0(t) : t \geq 0\} \) defined by the pathwise unique solution to

\[
Z_0(t) = Y_0 + \int_0^t \int_0^{Z_0(s-)} L_0^*(ds, du) - \int_0^t g(Z_0(s)) ds + \eta_t.
\]

**Lemma 3.2.** Let \( U_1 = \inf\{t \geq 0 : X_t \leq Z_0(t)\} \). Under the assumptions of Theorem 3.1, we have

\[
U_1 = \inf\{t \geq 0 : X_t = Z_0(t)\} = \inf\{t \geq 0 : X_{t-} = Z_0(t-)\}.
\]
Proof. From (3.8) and (3.12) it is easy to see that \( X_{t-} = Z_0(t-) \) implies \( X_t = Z_0(t) \). It follows that

\[
U_1 \leq \inf \{ t \geq 0 : X_t = Z_0(t) \} \leq \inf \{ t \geq 0 : X_{t-} = Z_0(t-) \}.
\]

Then (3.13) holds if \( U_1 = \infty \). In the case of \( U_1 < \infty \), we clearly have \( X_{U_1-} \geq Z_0(U_1-) \) and \( X_t > Z_0(t) \) for \( 0 \leq t < U_1 \). By (3.8) and (3.12),

\[
\Delta X_{U_1} = \int_{\{U_1\}} \int_0^\infty \int_0^{X_{U_1-}} \int_0^1 zM_0(ds, dz, du, dv) + \int_{\{U_1\}} \int_0^\infty \int_0^{Z_0(U_1-)} \int_0^1 zN_0(ds, dz, du, dv) \geq \int_{\{U_1\}} \int_0^\infty \int_0^{Z_0(U_1-)} \int_0^1 zM_0(ds, dz, du, dv) + \int_{\{U_1\}} \int_0^\infty \int_0^{Z_0(U_1-)} \int_0^1 zN_0(ds, dz, du, dv) = \Delta Z_0(U_1).
\]

On the other hand, by the right continuity of the processes,

\[
X_{U_1-} + \Delta X_{U_1} = X_{U_1} \leq Z_0(U_1) = Z_0(U_1-) + \Delta Z_0(U_1).
\]

Then we must have \( X_{U_1-} = Z_0(U_1-) \) and \( \Delta X_{U_1} = \Delta Z_0(U_1) \), implying \( X_{U_1} = Z_0(U_1) \). Those yield (3.13). \( \square \)

By (3.13) and the definition of \( U_1 \) we have \( X_{s-} > Z_0(s-) \geq 0 \) for \( 0 < s < U_1 \). Let us consider the pure jump process \( \{\xi_0(t) : t \geq 0\} \) given by

\[
\xi_0(t) = \int_0^{t \land U_1} \int_0^\infty \int_0^{Z_0(s-)} \int_0^{\rho_1(Z_0(s-)-X_{s-},z)} [X_{s-} - Z_0(s-)]M_0(ds, dz, du, dv) + \int_0^{t \land U_1} \int_0^\infty \int_0^{Z_0(s-)} \int_0^{\rho_2(Z_0(s-),z)} [Z_0(s-) - X_{s-}]N_0(ds, dz, du, dv) + \int_0^{t \land U_1} \int_0^\infty \int_0^{Z_0(s-)} \int_0^{\rho_3(Z_0(s-)-X_{s-},z)} [Z_0(s-)-X_{s-}]N_0(ds, dz, du, dv).
\]

The process makes its first jump at time \( \sigma_1 \land \tau_1 \), where

\[
\sigma_1 = \inf \{ t \geq 0 : \Delta \xi_0(t) = X_{t-} - Z_0(t-) \}
\]

and

\[
\tau_1 = \inf \{ t \geq 0 : \Delta \xi_0(t) = Z_0(t-) - X_{t-} \}.
\]

In view of (3.3) and (3.4), for any \( x > y \geq 0 \),

\[
\int_0^\infty \rho_1(x-y,z)\mu(dz) = \int_0^\infty \rho_1(y-x,z)\mu(dz) \leq \frac{1}{2} \mu(x-y, \infty) < \infty
\]

and

\[
\int_0^\infty r_1(x-y,z)\nu(dz) = \int_0^\infty r_1(y-x,z)\nu(dz) \leq \frac{1}{2} \nu(x-y, \infty) < \infty.
\]

Then the stopping times \( \sigma_1 \) and \( \tau_1 \) occur at the same rate

\[
\left[ Z_0(s-) \int_0^\infty \rho_1(X_{s-} - Z_0(s-),z)\mu(dz) + \int_0^\infty r_1(X_{s-} - Z_0(s-),z)\nu(dz) \right] ds.
\]
It follows that
\[
P(\sigma_1 < \tau_1 | \sigma_1 \land \tau_1 < \infty) = P(\tau_1 < \sigma_1 | \sigma_1 \land \tau_1 < \infty) = \frac{1}{2}.
\]

For \( t \geq 0 \) let
\[
Y_0(t) = Z_0(t \land U_1 \land \sigma_1 \land \tau_1) + \xi_0(t \land U_1 \land \sigma_1 \land \tau_1).
\]

**Lemma 3.3.** Under the assumptions of Theorem 3.1, we have \( X_{\sigma_1} = Y_0(\sigma_1) > 0 \) on the event \( \{ \sigma_1 < \tau_1 \leq \infty \} \) and \( X_{\tau_1} > Y_0(\tau_1) > 0 \) on the event \( \{ \tau_1 < \sigma_1 \leq \infty \} \).

**Proof.** (1) On the event \( \{ \sigma_1 < \tau_1 \leq \infty \} \) we have \( \sigma_1 < U_1 \) and so \( \sigma_1 < U_1 \land \tau_1 \). By (3.14) and the definition of \( \sigma_1 \) we have
\[
\Delta \xi_0(\sigma_1) = X_{\sigma_1} - Z_0(\sigma_1) > 0
\]
and
\[
\int_{\{\sigma_1\}}^\infty \int_0^1 M_0(ds, dz, du, dv) + \int_{\{\sigma_1\}}^\infty \int_0^1 N_0(ds, dz, du) = 1.
\]

By the temporarily homogeneous nature of the Poisson random measures, we can enlarge the integration intervals of \( u \) and \( v \) without changing the equality above. In particular, we have
\[
\int_{\{\sigma_1\}}^\infty \int_0^1 M_0(ds, dz, du, dv) + \int_{\{\sigma_1\}}^\infty \int_0^1 N_0(ds, dz, du) = 1.
\]

It follows that
\[
z(\sigma_1) := \int_{\{\sigma_1\}}^\infty \int_0^1 zM_0(ds, dz, du, dv) + \int_{\{\sigma_1\}}^\infty \int_0^1 zN_0(ds, dz, du)
\]
and
\[
z(\sigma_1) = \Delta Z_0(\sigma_1) = \Delta X_{\sigma_1}, \text{ by (3.8) and (3.12). This together with (3.17) yields}
\]
\[
Z_0(\sigma_1) + \xi_0(\sigma_1) = Z_0(\sigma_1) + \Delta Z_0(\sigma_1) + \Delta \xi_0(\sigma_1) = X_{\sigma_1} - X_{\tau_1} > 0.
\]

By (3.16) we have \( Y_0(\sigma_1) = X_{\sigma_1} > 0 \).

(2) On the event \( \{ \tau_1 < \sigma_1 \leq \infty \} \) we have \( \tau_1 < U_1 \) and so \( \tau_1 < U_1 \land \sigma_1 \). By (3.14) and the definition of \( \tau_1 \) we have
\[
\Delta \xi_0(\tau_1) = Z_0(\tau_1) - X_{\tau_1} < 0
\]
and
\[
\int_{\{\tau_1\}}^\infty \int_0^1 M_0(ds, dz, du, dv) + \int_{\{\tau_1\}}^\infty \int_0^1 N_0(ds, dz, du) = 1.
\]

In view of (3.5) and (3.18), for \( 0 < z \leq -\Delta \xi_0(\tau_1) \) we have
\[
\rho_2(X_{\tau_1}, Z_0(\tau_1), z) - \rho_1(Z_0(\tau_1) - X_{\tau_1}, z) = \rho_1(-\Delta \xi_0(\tau_1), z) = 0
\]
and
\[ r_2(X_{\tau_1 -}, Z_0(\tau_1 -), z) - r_1(Z_0(\tau_1) - X_{\tau_1 -}, z) = r_1(- \Delta \xi_0(\tau_1), z) = 0. \]

It follows that
\begin{equation}
(3.19)
\begin{aligned}
z(\tau_1) := & \int_{\{\tau_1\}} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} z_0(ds, dz, du, dv) \\
& + \int_{\{\tau_1\}} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} z_0(ds, dz, du) > - \Delta \xi_0(\tau_1) > 0.
\end{aligned}
\end{equation}

As in Part (1) of this theorem, one sees \( z(\tau_1) = \Delta Z_0(\tau_1) = \Delta X_{\tau_1} \), and so
\[ 0 \leq Z_0(\tau_1) < Z_0(\tau_1) + \Delta Z_0(\tau_1) + \Delta \xi_0(\tau_1) < X_{\tau_1} + \Delta X_{\tau_1} = X_{\tau_1}. \]

Then we have \( 0 < Y_0(\tau_1) < X_{\tau_1} \) by (3.16). \( \square \)

**Proof of Theorem 3.1.** We give an inductive construction of the process \( \{(Y_t, \xi_t) : t \geq 0\} \) in a sequence of steps. The construction also yields the pathwise uniqueness of the solution of the equation system (3.9)-(3.10).

**Step 1.** We start with the process \( \{(X_t, Z_0(t)) : t \geq 0\} \) defined by (3.8) and (3.12). Consider separately the cases \( \sigma_1 = \tau_1 = \infty, \sigma_1 < \tau_1 \) and \( \tau_1 < \sigma_1 \). In the case of \( \sigma_1 = \tau_1 = \infty \), we have \( \xi_0(t) = 0 \) for all \( t \geq 0 \). Then the process \( \{(Y_t, \xi_t) : t \geq 0\} \) is defined by \( T = U_1 \) and
\[ \xi_t = 0, Y_t = Z_0(t \wedge T) + X_t - X_{t\wedge T}. \]

In the case of \( \sigma_1 < \tau_1 \leq \infty \), we have \( X_{\sigma_1} = Y_0(\sigma_1) > 0 \) by Lemma 3.3. Then the process \( \{(Y_t, \xi_t) : t \geq 0\} \) is given by \( T = \sigma_1 \) and
\[ \xi_t = \xi_0(t \wedge T), Y_t = Y_0(t \wedge T) + X_t - X_{t\wedge T}. \]

In case of \( \tau_1 < \sigma_1 \leq \infty \), we clearly have \( Y_t = Z_0(t) \) and \( \xi_t = 0 \) for \( 0 \leq t < \tau_1 \). The continuing construction of \( \{(Y_{\tau_1+t}, \xi_{\tau_1+t}) : t \geq 0\} \) is given in the next step.

**Step 2.** Suppose that \( \tau_1 < \sigma_1 \leq \infty \). In this case, we have \( X_{\tau_1} > Y_0(\tau_1) > 0 \) by Lemma 3.3. Let \( X_1(t) = X_{\tau_1+t} \) for \( t \geq 0 \). Then \( \{X_1(t) : t \geq 0\} \) is also a CBIC-process. From (3.8) it follows that
\begin{equation}
(3.20)
X_1(t) = X_{\tau_1} + \int_{0}^{t} \int_{0}^{\infty} L_1(ds, du) - \int_{0}^{t} g(X_1(s))ds + \eta_{\tau_1+t},
\end{equation}

where \( L_1(ds, du) = L_0(\tau_1 + ds, du) \). By Theorem 2.1 we can construct another CBIC-process \( \{Z_1(t) : t \geq 0\} \) by the pathwise unique solution to
\begin{equation}
(3.21)
Z_1(t) = Y_0(\tau_1) + \int_{0}^{t} \int_{0}^{\infty} L_1^*(ds, du) - \int_{0}^{t} g(Z_1(s))ds + \eta_{\tau_1+t},
\end{equation}

where \( L_1^*(ds, du) = L_0^*(\tau_1 + ds, du) \). Then we repeat the procedure of in the first step for the process \( \{(X_1(t), Z_1(t)) : t \geq 0\} \).

In view of (3.15), we only need a finite number of sequential steps to complete the construction of \( \{(Y_t, \xi_t) : t \geq 0\} \), which means that the process \( \{\xi_t : t \geq 0\} \) has at most a finite number of jumps. By the construction, we have (3.11) and \( X_{T+} = Y_{T+} \) for every \( t \geq 0 \). \( \square \)

**Remark 3.4.** By Theorem 3.1, the process \( \{(X_t, Y_t) : t \geq 0\} \) defined by (3.8) and (3.9) with \( X_0 \geq Y_0 \geq 0 \) actually lives in the space \( D := \{ (x, y) : x \geq y \geq 0 \} \subset \mathbb{R}_+^2 \). Here we think of the coupling time \( T \) as a part of the solution. We may also say that \( \{(X_t, Y_t, \xi_t) : t \geq 0\} \) is a pathwise unique solution to the system of equations (3.8), (3.9) and (3.10).

**Theorem 3.5.** The solution \( \{Y_t : t \geq 0\} \) to (3.9) is a CBIC-process.
Proof. Let \( N(ds, dz) \) be the optional random measure defined by, for \( t \geq 0 \) and \( A \in B(0, \infty) \),

\[
N((0, t] \times A) = \int_0^t \int_0^\infty \int_0 \int_0^\infty \int_0^1 r_1(Y_s - X_s, z) 1_A(z + X_s - Y_s) N_0(ds, dz, dv)
\]

\[
+ \int_0^t \int_0^\infty \int_0^\infty \int_0^1 r_2(X_s - Y_s, z) 1_A(z + Y_s - X_s) N_0(ds, dz, dv)
\]

\[
+ \int_0^t \int_0^\infty \int_0^1 1_A(z) N_0(ds, dz, dv).
\]

Then \( N(ds, dz) \) has predictable compensator \( \hat{N}(ds, dz) \) determined by

\[
\hat{N}((0, t] \times A) = \int_0^t ds \int_0^\infty 1_A(z + X_s - Y_s) r_1(Y_s - X_s, z) \nu(dz)
\]

\[
+ \int_0^t ds \int_0^\infty 1_A(z + Y_s - X_s) r_1(X_s - Y_s, z) \nu(dz)
\]

\[
+ \int_0^t ds \int_0^\infty 1_A(z) [1 - r_2(X_s, Y_s, z)] \nu(dz)
\]

\[
= \int_0^t ds \int_0^\infty 1_A(z + X_s - Y_s) \nu(Y_s - X_s, dz)
\]

\[
+ \int_0^t ds \int_0^\infty 1_A(z + Y_s - X_s) \nu(X_s - Y_s, dz)
\]

\[
+ \int_0^t ds \int_0^\infty 1_A(z) [\nu(dz) - \nu(X_s, Y_s, dz)].
\]

In view of (3.2), for any \( x \geq y \geq 0 \) we have

\[
\int_0^\infty 1_A(z + x - y) \nu(y - x, dz) + \int_0^\infty 1_A(z + y - x) \nu(x - y, dz)
\]

\[
= \int_0^\infty 1_A(z) \nu(y - x, dz) + \int_0^\infty 1_A(z) \nu(y - x, dz)
\]

\[
= \int_0^\infty 1_A(z) \nu(x, y, dz).
\]

Then \( \hat{N}((0, t] \times A) = tv(A) \), and so \( N(ds, dz) \) is a Poisson random measure on \((0, \infty)^2\) with intensity \( ds \nu(dz) \); see, e.g., Theorem III.6.2 in Ikeda and Watanabe [32, p.75]. Let \( M(ds, dz, du) \) be the optional random measure defined by, for \( t \geq 0 \) and \( A, B \in B(0, \infty) \),

\[
M((0, t] \times A \times B) = \int_0^t \int_0^\infty \int_B \int_0^\infty \int_0^1 r_1(Y_s - X_s, z) 1_A(z + X_s - Y_s) M_0(ds, dz, du, dv)
\]

\[
+ \int_0^t \int_0^\infty \int_B \int_0^\infty \int_0^1 r_2(X_s - Y_s, z) 1_A(z + Y_s - X_s) M_0(ds, dz, du, dv)
\]

\[
+ \int_0^t \int_0^\infty \int_B \int_0^1 1_A(z) M_0(ds, dz, du, dv).
\]

By similar calculations as above one can see \( M(ds, dz, du) \) is a Poisson random measure on \((0, \infty)^3\) with intensity \( ds \mu(dz)dz \). Let \( \{\eta(t)\} \) be the \((\mathcal{F}_t)\)-subordinator defined by (1.10). Let \( \{L(ds, du)\} \) be the time-space \((\mathcal{F}_t)\)-Lévy noise defined by

\[
L(ds, du) = -1_{\{s \leq T\}} W_0(ds, du) - b ds du + \int_{\{0 < z \leq 1\}} z M(ds, dz, du)
\]

\[
+ 1_{\{s > T\}} W_0(ds, du) + \int_{\{1 < z < \infty\}} z M(ds, dz, du)
\]

From (3.9) and (3.10) we obtain

\[
Y_t = Y_0 + \int_0^t \int_0^1 Y_s - L(ds, du) - \int_0^t g(Y_s) ds + \eta(t).
\]
Then \( \{ Y_t : t \geq 0 \} \) is a CBIC-process by the uniqueness of the solution to the equation.

The pathwise uniqueness of the solutions to (3.8) and (3.9) implies that \( \{ (X_t, Y_t) : t \geq 0 \} \) is a Markov process. Then it is a Markov coupling of the CBIC-process. Since zero is a trap for \( \{ X_t - Y_t : t \geq 0 \} \), we have

\[
\begin{align*}
X_t - Y_t &= X_0 - Y_0 + \int_0^t \int_{Y_s}^{X_s} W_0(ds, du) + 2 \int_0^t \int_{Y_s}^{X_s-} W_0(ds, du) \\
&\quad - \int_0^t [g(X_s) - g(Y_s)] ds + \int_0^t \int_{Y_s}^{X_s-} \int_0^1 z \tilde{M}_0(ds, dz, du, dv) \\
&\quad + \int_0^t \int_{Y_s}^{X_s-} \int_0^1 z \tilde{M}_0(ds, dz, du, dv) - \xi_t.
\end{align*}
\]

**Remark 3.6.** Concerning the coupling process, what is important to us is its movement before the succeeding time. To establish (1.18) and (1.30), we need to make the coupling succeed as early as possible. The Gaussian and Poissonian integrals on the right-hand side of (3.22) are important in bringing the process to the success. Clearly, the second Gaussian integral comes from the reflection of the Gaussian component in (3.7). The Poisson noises cannot be reflected directly as the CBIC-processes have no negative jumps. The pure jump process \( \{ \xi_t : t \geq 0 \} \) defined by (3.10) gives a proper formulation of the reflections of the discontinuous noises. In view of (3.15), a possible jump of this process at time \( s > 0 \) would take the values \( X_{s-} - Y_{s-} > 0 \) and \( Y_{s-} - X_{s-} < 0 \) with equal probability 1/2. The introduction of the process \( \{ \xi_t : t \geq 0 \} \) was inspired by the construction of a coupling process of Luo and Wang [45]; see also [37, 43]. In fact, if Grey’s condition (1.9) is not satisfied, the success of the coupling process can only come with the first positive jump of \( \{ \xi_t : t \geq 0 \} \).

### 4. The coupling generator

In this section, we assume the CBIC-process with an arbitrary initial value is conservative. We shall give a characterization for the generator of the Markov coupling process \( \{ (X_t, Y_t) : t \geq 0 \} \) defined by (3.8) and (3.9) in terms of a martingale problem. We also establish some estimates for the generator, which provide the basis for the proof of the exponential ergodicity. Recall that \( \{ (X_t, Y_t) : t \geq 0 \} \) has state space \( D := \{(x, y) : x \geq y \geq 0 \} \). Let \( \Delta = \{ (z, z) : z \geq 0 \} \subset D \) and \( \Delta^c = D \setminus \Delta \). Given a function \( F \) on \( D \) twice continuously differentiable on \( \Delta^c \), we write

\[
\begin{align*}
\bar{L}F(x, y) &= \bar{L}_0 F(x, y) + \bar{L}_1 F(x, y), \quad (x, y) \in \Delta^c,
\end{align*}
\]

where

\[
\begin{align*}
\bar{L}_0 F(x, y) &= \beta \left[ F_x(x, y) + F_y(x, y) + cx F'''_{xx}(x, y) + cy F'''_{yy}(x, y) \right] \\
&\quad + 2cy F_{xy}(x, y) - 2cy F_{xy}(x, y) + \int_0^\infty \left[ F(x + z, y + z) - F(x, y) \right] \nu(dz) \\
&\quad + (x - y) \int_0^\infty \left[ F(x + z, y + z) - F(x, y) - F'_x(x, y) z 1_{\{z \leq 1\}} \right] \mu(dz) \\
&\quad + y \int_0^\infty \left[ F(x + z, y + z) - F(x, y) - (F'_x + F'_y)(x, y) z 1_{\{z \leq 1\}} \right] \mu(dz)
\end{align*}
\]

and

\[
\begin{align*}
\bar{L}_1 F(x, y) &= y \int_0^\infty \left[ F(x + z, 2y + z - x) - F(x + z, y + z) \right] \mu_{x-y}(dz) \\
&\quad + y \int_0^\infty \left[ F(x + z, x + z) - F(x + z, y + z) \right] \mu_{y-x}(dz) \\
&\quad + \int_0^\infty \left[ F(x + z, 2y + z - x) - F(x + z, y + z) \right] \nu_{x-y}(dz) \\
&\quad + \int_0^\infty \left[ F(x + z, x + z) - F(x + z, y + z) \right] \nu_{y-x}(dz).
\end{align*}
\]
Let $\mathcal{D}(\hat{L})$ denote the linear space consisting of the functions $F$ such that the integrals in (4.2) and (4.3) are convergent and define functions locally bounded on compact subsets of $\Delta^c$. The operator $\hat{L}$ determines the movement of the coupling process before its succeeding time. In particular, the component $\hat{L}_1$ is induced by the process $\{\xi_t: t \geq 0\}$ defined by (3.10). We call $(\hat{L}, \mathcal{D}(\hat{L}))$ the coupling generator of the CBIC-process. The precise meaning of this terminology is made clear by the martingale problem given in the following:

**Theorem 4.1.** Let $\{(X_t, Y_t): t \geq 0\}$ be the Markov coupling process defined by (3.8) and (3.9) with $X_0 > Y_0 \geq 0$ and let $\zeta^*_n = \inf\{t \geq 0: X_t \geq n \text{ or } X_t - Y_t \leq 1/n\}$. Then for any $n \geq 1$ and $F \in \mathcal{D}(\hat{L})$ we have

\[
F(X_{t \wedge \zeta^*_n}, Y_{t \wedge \zeta^*_n}) = F(X_0, Y_0) + \int_0^{t \wedge \zeta^*_n} \hat{L}F(X_s, Y_s)ds + M_n(t),
\]

where $\{M_n(t): t \geq 0\}$ is a martingale.

**Proof.** By (3.8), (3.9) and (3.10), the process $\{(X_{t \wedge \zeta^*_n}, Y_{t \wedge \zeta^*_n}): t \geq 0\}$ is a semimartingale taking values in $\Delta \cup D_n$, where $D_n = \{(x, y) \in D: x - y \geq 1/n\}$. In view of the three equations, a possible jump $(\Delta X_s, \Delta Y_s)$ of the process at time $s > 0$ is brought about by a point $(s, z, u, v) \in \text{supp}(M_n)$ with $0 < u \leq X_s$ or by a point $(s, z, v) \in \text{supp}(N_0)$. The details of the jump in the two cases are given, respectively, by

\[
(\Delta X_s, \Delta Y_s) = \begin{cases} (z + X_s - Y_s, u \in (0, Y_s), v \in (0, \hat{\rho}_1(s, z)], \\ (z + Y_s - X_s, u \in (0, Y_s), v \in (\hat{\rho}_1(s, z), \hat{\rho}_1(s, z)), \\ (z, z), & u \in (0, Y_s), v \in (\hat{\rho}_1(s, z), 1], \\ (z, 0), & u \in (Y_s, X_s], v \in (0, 1], 
\end{cases}
\]

where $\hat{\rho}_1(s, z) = \rho_1(X_s - Y_s, z)$ and $\hat{\rho}_2(s, z) = \rho_2(X_s - Y_s, z)$, and

\[
(\Delta X_s, \Delta Y_s) = \begin{cases} (z + X_s - Y_s, v \in (0, \hat{r}_1(s, z)], \\ (z + Y_s - X_s, v \in (\hat{r}_1(s, z), 1], \\ (z, z), & v \in (\hat{r}_2(s, z), 1], 
\end{cases}
\]

where $\hat{r}_1(s, z) = r_1(X_s - Y_s, z)$ and $\hat{r}_2(s, z) = r_2(X_s - Y_s, z)$. Then the coupling process may have totally seven different types of jumps. Observe also that $(X_s, Y_s) \in D_n \subset \Delta^c$ when $0 < s \leq \zeta^*_n$. For any $(x, y) \in \Delta^c$ write

\[
\hat{L}_2F(x, y) = cxF''_{xx}(x, y) + cyF''_{yy}(x, y) - 2cyF''_{xy}(x, y) - g(x)F''_x(x, y) - g(y)F''_y(x, y).
\]

We can use Itô’s formula to see

\[
F(X_{t \wedge \zeta^*_n}, Y_{t \wedge \zeta^*_n}) = F(X_0, Y_0) + \int_0^{t \wedge \zeta^*_n} \hat{L}_2F(x, y)ds + \int_0^{t \wedge \zeta^*_n} F'_x(X_s, Y_s)L_0(ds, du) + \int_0^{t \wedge \zeta^*_n} F'_y(X_s, Y_s)L_0(ds, du) + \int_0^{t \wedge \zeta^*_n} (F'_x + F'_y)(X_s, Y_s)M_0(ds, dz, du)
\]

\[
+ \int_0^{t \wedge \zeta^*_n} F'_y(X_s, Y_s)(X_s - Y_s)N_0(ds, dz, du) + \int_0^{t \wedge \zeta^*_n} F'_y(X_s, Y_s)(X_s - Y_s)M_0(ds, dz, du) + \int_0^{t \wedge \zeta^*_n} F'_y(X_s, Y_s)(Y_s - X_s)M_0(ds, dz, du)
\]

\[
+ \int_0^{t \wedge \zeta^*_n} F'_y(X_s, Y_s)(Y_s - X_s)N_0(ds, dz, du) + \int_0^{t \wedge \zeta^*_n} F'_y(X_s, Y_s)(Y_s - X_s)M_0(ds, dz, du) + \int_0^{t \wedge \zeta^*_n} F'_y(X_s, Y_s)(Y_s - X_s)N_0(ds, dz, du)
\]

\[
+ \int_0^{t \wedge \zeta^*_n} \int_0^{Y_s} \int_0^{Y_s} [F(X_s + z, X_s + z) - F(X_s, Y_s)] - zF'_y(X_s, Y_s) - (z + X_s - Y_s)F'_y(X_s, Y_s)M_0(ds, dz, du)
\]

\[
+ \int_0^{t \wedge \zeta^*_n} \int_0^{Y_s} \int_0^{Y_s} [F(X_s + z, 2Y_s + z - X_s) - F(X_s, Y_s)].
\]
\[-zF'_x(X_s, Y_s) - (z + Y_s - X_s)F'_y(X_s, Y_s)] M_0(ds, dz, du, dv) \\
+ \int_0^{t \wedge \zeta^*_n} \int_0^\infty \int_0^{Y_s} \int_0^1 [F(X_s + z, Y_s + z) - F(X_s, Y_s)] \\
- (zF'_x + F'_y)(X_s, Y_s)] M_0(ds, dz, du, dv) \\
+ \int_0^{t \wedge \zeta^*_n} \int_0^\infty \int_0^{X_s} \int_0^1 [F(X_s + z, Y_s) - F(X_s, Y_s)] \\
- zF'_x(X_s, Y_s)] M_0(ds, dz, du, dv) \\
+ \int_0^{t \wedge \zeta^*_n} \int_0^\infty \int_0^{F'_x(s, z)} \int_0^{F'_2(s, z)} [F(X_s + z, 2Y_s + z - X_s) - F(X_s, Y_s)] \\
- zF'_x(X_s, Y_s) - (z + Y_s - X_s)F'_y(X_s, Y_s)] N_0(ds, dz, du, dv) \\
+ \int_0^{t \wedge \zeta^*_n} \int_0^\infty \int_0^{F'_2(s, z)} \int_0^{F'_2(s, z)} [F(X_s + z, 2Y_s + z - X_s) - F(X_s, Y_s)] \\
- zF'_x(X_s, Y_s) - (z + Y_s - X_s)F'_y(X_s, Y_s)] N_0(ds, dz, du, dv) \\
= F(X_0, Y_0) + \int_0^{t \wedge \zeta^*_n} \int_0^\infty \int_0^{X_s} F'_x(X_s, Y_s) ds + \int_0^{t \wedge \zeta^*_n} \int_0^\infty \int_0^{X_s} F'_x(X_s, Y_s) L_0(ds, du, dv) \\
+ \int_0^{t \wedge \zeta^*_n} \int_0^\infty \int_0^{X_s} \int_0^{X_s} \int_0^{X_s} F'_x(X_s, Y_s) L_0(ds, du, dv) + \int_0^{t \wedge \zeta^*_n} \int_0^{t \wedge \zeta^*_n} \int_0^{t \wedge \zeta^*_n} (F'_x + F'_y)(X_s, Y_s) ds, du, dv \\
+ \int_0^{t \wedge \zeta^*_n} \int_0^{t \wedge \zeta^*_n} \int_0^{t \wedge \zeta^*_n} \int_0^{t \wedge \zeta^*_n} [F(X_s + z, X_s + z) - F(X_s, Y_s)] \\
- z(F'_x + F'_y)(X_s, Y_s)] M_0(ds, dz, du, dv) \\
+ \int_0^{t \wedge \zeta^*_n} \int_0^{t \wedge \zeta^*_n} \int_0^{t \wedge \zeta^*_n} \int_0^{t \wedge \zeta^*_n} [F(X_s + z, Y_s + z) - F(X_s, Y_s)] \\
- z(F'_x + F'_y)(X_s, Y_s)] M_0(ds, dz, du, dv) \\
+ \int_0^{t \wedge \zeta^*_n} \int_0^{t \wedge \zeta^*_n} \int_0^{t \wedge \zeta^*_n} \int_0^{t \wedge \zeta^*_n} [F(X_s + z, Y_s + z) - F(X_s, Y_s)] \\
- z(F'_x + F'_y)(X_s, Y_s)] M_0(ds, dz, du, dv) \\
+ \int_0^{t \wedge \zeta^*_n} \int_0^{t \wedge \zeta^*_n} \int_0^{t \wedge \zeta^*_n} \int_0^{t \wedge \zeta^*_n} [F(X_s + z, Y_s + z) - F(X_s, Y_s)] \\
- z(F'_x + F'_y)(X_s, Y_s)] M_0(ds, dz, du, dv) \\
+ \int_0^{t \wedge \zeta^*_n} \int_0^{t \wedge \zeta^*_n} \int_0^{t \wedge \zeta^*_n} \int_0^{t \wedge \zeta^*_n} [F(X_s + z, Y_s + z) - F(X_s, Y_s)] \\
- z(F'_x + F'_y)(X_s, Y_s)] N_0(ds, dz, du, dv) \\
+ \int_0^{t \wedge \zeta^*_n} \int_0^{t \wedge \zeta^*_n} \int_0^{t \wedge \zeta^*_n} \int_0^{t \wedge \zeta^*_n} [F(X_s + z, Y_s + z) - F(X_s, Y_s)] \\
- z(F'_x + F'_y)(X_s, Y_s)] N_0(ds, dz, du, dv)
By some further cancellations of the terms,

\[
F(X_{t\wedge \zeta_n}^*,Y_{t\wedge \zeta_n}) = F(X_0,Y_0) + \int_0^{t\wedge \zeta_n^*} \int_0^1 L_2 F(X_s,Y_s) ds + \int_0^{t\wedge \zeta_n^*} (\beta - bX_s)F'_x(X_s,Y_s) ds \]

\[+ \int_0^{t\wedge \zeta_n^*} \int_0^1 F'_x(X_s,Y_s) W_0(ds,du)\]

\[+ \int_0^{t\wedge \zeta_n^*} \int_0^1 \int_0^1 \int_0^1 F'_x(X_s,Y_s) z \tilde{M}_0(ds,dz,du, dv)\]

\[+ \int_0^{t\wedge \zeta_n^*} (\beta - bY_s) F'_y(X_s,Y_s) ds - \int_0^{t\wedge \zeta_n^*} \int_0^1 Y_s F'_y(X_s,Y_s) W_0(ds,du)\]

\[+ \int_0^{t\wedge \zeta_n^*} \int_0^1 \int_0^1 F'_y(X_s,Y_s) z \tilde{M}_0(ds,dz,du, dv)\]

\[+ \int_0^{t\wedge \zeta_n^*} \int_0^\infty \int_0^1 \int_0^1 [F(X_s + z, X_s + z) - F(X_s,Y_s) - 1_{\{z \leq 1\}} z (F'_x + F'_y)(X_s,Y_s)] \tilde{M}_0(ds,dz,du, dv)\]

\[+ \int_0^{t\wedge \zeta_n^*} \int_0^\infty \int_0^1 \int_0^1 \tilde{\bar{\rho}}_2(s,z) [F(X_s + z, 2Y_s + z - X_s) - F(X_s,Y_s) - 1_{\{z \leq 1\}} z (F'_x + F'_y)(X_s,Y_s)] \tilde{M}_0(ds,dz,du, dv)\]

\[+ \int_0^{t\wedge \zeta_n^*} \int_0^\infty \int_0^1 \int_0^1 [F(X_s + z, Y_s + z) - F(X_s,Y_s) - 1_{\{z \leq 1\}} z F'_x(X_s,Y_s)] \tilde{M}_0(ds,dz,du, dv)\]

\[+ \int_0^{t\wedge \zeta_n^*} \int_0^\infty \int_0^1 \int_0^1 \tilde{\bar{\rho}}_1(s,z) [F(X_s + z, X_s + z) - F(X_s,Y_s)] \tilde{N}_0(ds,dz, dv)\]

\[+ \int_0^{t\wedge \zeta_n^*} \int_0^\infty \int_0^1 \int_0^1 \tilde{\bar{\rho}}_1(s,z) [F(X_s + z, 2Y_s + z - X_s) - F(X_s,Y_s)] \tilde{N}_0(ds,dz, dv)\]

\[+ \int_0^{t\wedge \zeta_n^*} \int_0^\infty \int_0^1 \int_0^1 [F(X_s + z, Y_s + z) - F(X_s,Y_s)] \tilde{N}_0(ds,dz, dv)\]

\[+ \int_0^{t\wedge \zeta_n} \tilde{L}_2 F(X_s,Y_s) ds,\]
Proof. Lemma Then \( R \)

These together with \((4.6)\) define a martingale \( \{M_n(t) : t \geq 0\} \). Then the desired result follows. \( \square \)

We next consider two special forms of the function \( F \in D(\tilde{L}) \). Recall that \( C_b^2(\mathbb{R}_+) \) denotes the space of bounded and continuous functions on \( \mathbb{R}_+ \) with bounded and continuous derivatives up to the second order.

Lemma 4.2. Suppose that \( f \in C_b^2(\mathbb{R}_+) \) is a nonnegative, nondecreasing and concave function. Let

\[
(4.6) 
F(x, y) = f(x - y)1_{\{x \neq y\}}, \quad (x, y) \in D.
\]

Then \( F \in D(\tilde{L}) \) and, for \( (x, y) \in \Delta^c \),

\[
(4.7) 
\tilde{L}F(x, y) \leq (x - y) \left[ cf''(x - y) - bf'(x - y) + \int_0^\infty \left[ f(x - y + z) - f(x - y) \right. \right. \\
\left. \left. - f'(x - y)z1_{\{z \leq 1\}} \right] \mu(dz) \right]
\]

Proof. By \((4.2)\) and \((4.3)\) it is clear that \( F \in D(\tilde{L}) \). Since \( g \) is nondecreasing and \( f \) is nonnegative, nondecreasing and concave, by \((4.2)\) we have, for \( (x, y) \in \Delta^c \),

\[
\tilde{L}_0F(x, y) = -[bx - by + g(x) - g(y)]f'(x - y) + c(x + 3y)f''(x - y) \\
+ (x - y) \int_0^\infty \left[ f(x - y + z) - f(x - y) - f'(x - y)z1_{\{z \leq 1\}} \right] \mu(dz) \\
\leq (x - y) \left[ cf''(x - y) - bf'(x - y) + \int_0^\infty \left[ f(x - y + z) - f(x - y) \right. \right. \\
\left. \left. - f'(x - y)z1_{\{z \leq 1\}} \right] \mu(dz) \right]
\]

On the other hand, since \( f(2z) - 2f(z) \leq 0 \) for any \( z \geq 0 \), by \((4.3)\) it is easy to see that

\[
\tilde{L}_1F(x, y) = y[f(2(x - y)) - 2f(x - y)]\mu_x(0, \infty) + \left. [f(2(x - y)) - 2f(x - y)]\nu_{x-y}(0, \infty) \leq 0 \right).
\]

These together with \((4.1)\) yield \((4.7)\). \( \square \)
Lemma 4.3. Suppose that $f \in \mathcal{C}_b^2(\mathbb{R}_+)$ is a nonnegative, nondecreasing and concave function and $\phi \in \mathcal{C}_b^2(\mathbb{R}_+)$ is a nonnegative nonincreasing function. Let

\[
\begin{align*}
F(x, y) = \phi(x)f(x-y)1_{\{x \neq y\}}, & \quad (x, y) \in D.
\end{align*}
\]

Then $F \in \mathcal{D}(\tilde{L})$ and, for $(x, y) \in \Delta_c$,

\[
\begin{align*}
\tilde{L}F(x, y) & \leq cy\phi(x)f''(x-y) + [\beta - bx - g(x) + y]\phi'(x)f(x-y) \\
& \quad + y\phi(x)[f(2(x-y)) - 2f(x-y)]\mu_{x-y}(0, \infty) \\
& \quad + \phi(x)[f(2(x-y)) - 2f(x-y)]\nu_{x-y}(0, \infty) \\
& \quad + \tilde{A}_1F(x, y) + \tilde{A}_2F(x, y),
\end{align*}
\]

where

\[
\begin{align*}
\tilde{A}_1F(x, y) & = f(x-y)\left[cx\phi''(x) - y\int_0^\infty [\phi(x+z) - \phi(x)]\mu_{y-x}(dz) \\
& \quad + x\int_0^\infty [\phi(x+z) - \phi(x) - \phi'(x)z1_{\{z \leq 1\}}]\mu(dz) \\
& \quad + \int_0^\infty [\phi(x+z) - \phi(x)](\nu - \nu_{y-x})(dz)\right] \\
\end{align*}
\]

and

\[
\begin{align*}
\tilde{A}_2F(x, y) & = (x-y)\phi(x)\left[cf''(x-y) - bf'(x-y) + y\int_0^\infty [f(x-y+z) \\
& \quad - f(x-y) - f'(x-y)z1_{\{z \leq 1\}}]\mu(dz)\right].
\end{align*}
\]

**Proof.** According to the definition of $F$, we have $F(z, z) = 0$ for $z \geq 0$. By (4.2) and (4.3) one can see that $F \in \mathcal{D}(\tilde{L})$ and, for $(x, y) \in \Delta_c$,

\[
\begin{align*}
\tilde{L}_0F(x, y) & = [\beta - bx - g(x)]\phi'(x)f(x-y) + [bx - by + g(x) - g(y)]\phi(x)f'(x-y) \\
& \quad + c(x + 3y)\phi(x)f''(x-y) + 2c(x+y)\phi'(x)f'(x-y) + cx\phi''(x)f(x-y) \\
& \quad + (x-y)\int_0^\infty [\phi(x+z)f(x-y+z) - \phi(x)f(x-y) \\
& \quad - \phi(x)f'(x-y)z1_{\{z \leq 1\}} - \phi'(x)f(x-y)z1_{\{z \leq 1\}}]\mu(dz) \\
& \quad + yf(x-y)\int_0^\infty [\phi(x+z) - \phi(x) - \phi'(x)z1_{\{z \leq 1\}}]\mu(dz) \\
& \quad + f(x-y)\int_0^\infty [\phi(x+z) - \phi(x)]\nu(dz)
\end{align*}
\]

and

\[
\begin{align*}
\tilde{L}_1F(x, y) & = y[f(2(x-y)) - f(x-y)]\int_0^\infty \phi(x+z)\mu_{x-y}(dz) \\
& \quad - yf(x-y)\int_0^\infty \phi(x+z)\mu_{y-x}(dz) - f(x-y)\int_0^\infty \phi(x+z)\nu_{y-x}(dz) \\
& \quad + [f(2(x-y)) - f(x-y)]\int_0^\infty \phi(x+z)\nu_{x-y}(dz)
\end{align*}
\]

Recall that $f$ and $g$ are nondecreasing and $\phi$ is nonincreasing. Then we have

\[
\begin{align*}
\tilde{L}_0F(x, y) & \leq [\beta - bx - g(x)]\phi'(x)f(x-y) - b(x-y)\phi(x)f'(x-y)
\end{align*}
\]
\[ + cx\phi(x)f''(x - y) + cx\phi''(x)f(x - y) \]
\[ + (x - y) \int_0^\infty [\phi(x + z)f(x - y + z) - \phi(x)f(x - y + z) - \phi'(x)f(x - y)z1_{\{z \leq 1\}}] \mu(dz) \]
\[ + (x - y)\phi(x) \int_0^\infty [f(x - y + z) - f(x - y) - f'(x - y)z1_{\{z \leq 1\}}] \mu(dz) \]
\[ + yf(x - y) \int_0^\infty [\phi(x + z) - \phi(x) - \phi'(x)z1_{\{z \leq 1\}}] \mu(dz) \]
\[ + f(x - y) \int_0^\infty [\phi(x + z) - \phi(x)] \nu(dz) \]
\[ \leq [\beta - bx - g(x)]\phi'(x)f(x - y) - b(x - y)\phi(x)f'(x - y) \]
\[ + cx\phi(x)f''(x - y) + cx\phi''(x)f(x - y) \]
\[ + xf(x - y) \int_0^\infty [\phi(x + z) - \phi(x) - \phi'(x)z1_{\{z \leq 1\}}] \mu(dz) \]
\[ + (x - y)\phi(x) \int_0^\infty [f(x - y + z) - f(x - y) - f'(x - y)z1_{\{z \leq 1\}}] \mu(dz) \]
\[ + f(x - y) \int_0^\infty [\phi(x + z) - \phi(x)] \nu(dz) \]

and
\[ \bar{L}1F(x, y) = y\phi(x)\left[f(2(x - y)) - 2f(x - y)\right] \mu_{x-y}(0, \infty) \]
\[ + y\left[f(2(x - y)) - f(x - y)\right] \int_0^\infty [\phi(x + z) - \phi(x)] \mu_{x-y}(dz) \]
\[ - yf(x - y) \int_0^\infty [\phi(x + z) - \phi(x)] \mu_{y-x}(dz) \]
\[ + \phi(x)\left[f(2(x - y)) - 2f(x - y)\right] \nu_{y-x}(0, \infty) \]
\[ + \left[f(2(x - y)) - f(x - y)\right] \int_0^\infty [\phi(x + z) - \phi(x)] \nu_{x-y}(dz) \]
\[ - f(x - y) \int_0^\infty [\phi(x + z) - \phi(x)] \nu_{y-x}(dz) \]
\[ \leq y\phi(x)\left[f(2(x - y)) - 2f(x - y)\right] \mu_{x-y}(0, \infty) \]
\[ - yf(x - y) \int_0^\infty [\phi(x + z) - \phi(x)] \mu_{y-x}(dz) \]
\[ + \phi(x)\left[f(2(x - y)) - 2f(x - y)\right] \nu_{y-x}(0, \infty) \]
\[ - f(x - y) \int_0^\infty [\phi(x + z) - \phi(x)] \nu_{y-x}(dz) \]

where we have used the relation \( \mu_{x-y}(0, \infty) = \mu_{y-x}(0, \infty) \). Returning to (4.1) and reorganizing the terms we get (4.9).

In the sequel, we assume there is some \( \lambda_0 > 0 \) such that \( \Psi(\lambda_0) > 0 \). Under Grey’s condition (1.9), we can define some \( F_0 \in \mathcal{D}(\bar{L}) \) by choosing an explicit form of the function \( f \in C^2_b(\mathbb{R}_+) \) in (4.6). To do so, take any \( \lambda_0 > 0 \) such that \( \Psi(\lambda) > 0 \) for \( \lambda \geq \lambda_0 \) and
\[ \int_{\lambda_0}^{\infty} \frac{1}{\Psi(\lambda)} d\lambda \leq 1. \]

Fix a constant \( \theta \geq 4 \) and define

\[ F_0(x, y) = [\theta + h(x - y)]1_{\{x \neq y\}}, \quad (x, y) \in D, \]

(4.12)
where

\[ h(z) = \int_{\lambda_0}^{\infty} \frac{1 - e^{-\lambda z}}{\psi(\lambda)} \, d\lambda. \]

By Lemma 4.2 we have \( F_0 \in \mathcal{D}(\tilde{L}) \). It is easy to see that

(4.13) \[ \theta \leq F_0(x, y) \leq 1 + \theta, \quad (x, y) \in \Delta^c. \]

**Proposition 4.4.** Suppose that Condition 1.2-(i) is satisfied. Let \( F_0 \in \mathcal{D}(\tilde{L}) \) be defined by (4.12). Then for any \( l \geq 1 \) there is a constant \( \lambda_2 > 0 \) such that

(4.14) \[ \tilde{L}F_0(x, y) \leq -\lambda_2 F_0(x, y)1_{\{x-y \leq l\}}, \quad (x, y) \in \Delta^c. \]

**Proof.** It is clear that \( h \in C^2_b(\mathbb{R}_+) \) and the function is a nonnegative, nondecreasing and concave. By (4.7) one can see that, for \( (x, y) \in \Delta^c \),

\[ \tilde{L}F_0(x, y) \leq - \int_{\lambda_0}^{\infty} (x-y) e^{-\lambda(x-y)} d\lambda = -e^{-\lambda_0(x-y)}. \]

For \( l \geq 1 \) let \( \lambda_2 = (1 + \theta)^{-1} e^{-\theta l} \). Then \( \tilde{L}F_0(x, y) \leq -e^{-\theta l} \leq -\lambda_2 F_0(x, y) \) when \( 0 < x - y \leq l \). \( \square \)

We may also define a function \( F_0 \in \mathcal{D}(\tilde{L}) \) by special choices of the functions \( f \in C^2_b(\mathbb{R}_+) \) and \( \phi \in C^2_b(\mathbb{R}_+) \) in (4.8). To do so, fix \( \lambda_0 > 0 \) such that \( \psi(\lambda_0) > 0 \) and let

(4.15) \[ \psi(x) = 1 - e^{-\lambda_0 x}, \quad x \geq 0. \]

For constants \( 0 < x_0 < 1 \) and \( \theta \geq 4 \) to be specified later, we define

(4.16) \[ \phi(x) = \begin{cases} \theta + (1 - x/x_0)^3, & 0 \leq x < x_0, \\ \theta, & x \geq x_0. \end{cases} \]

Then define the function

(4.17) \[ F_0(x, y) = \phi(x) \left[ 1 + \psi(x - y) \right] 1_{\{x \neq y\}}, \quad (x, y) \in \mathcal{D}. \]

By Lemma 4.3 we have \( F_0 \in \mathcal{D}(\tilde{L}) \). It is easy to see that

(4.18) \[ \theta \leq F_0(x, y) \leq 2(1 + \theta), \quad (x, y) \in \Delta^c. \]

**Lemma 4.5.** Let \( F_0 \in \mathcal{D}(\tilde{L}) \) be a function of the form (4.17). Then for any \( l \geq 1 \) there is a constant \( \lambda_1 > 0 \) such that, for \( (x, y) \in \Delta^c \),

(4.19) \[ \tilde{L}F_0(x, y) \leq -e^{\lambda_1 \theta} y e^{-\lambda_0 (x-y)} - \theta y \mu_{x-y}(0, \infty) - \lambda_1 \theta \psi(x-y) 1_{\{x-y \leq l\}} \]

\[ - \theta \nu_{x-y}(0, \infty) + \left[ y \mu_{x-y}(0, \infty) + I(x) + J(x) \right] \left[ 1 + \psi(x-y) \right] 1_{\{x \leq x_0\}} \]

\[ + \left[ 1 + \psi(x-y) \right] \int_0^\infty \left[ \phi(x+z) - \phi(x) \right] (\nu - \nu_{y-x})(dz) 1_{\{x \leq x_0\}}, \]

where

(4.20) \[ I(x) = [\beta - bx - g(x)] \phi'(x) \]

and

(4.21) \[ J(x) = \frac{3x}{x_0} \left( 2c \int_0^1 z^2 \mu(dz) + \int_0^1 dz \right). \]
Proof. Since \( \psi(0) = 0 \) and \( \psi \) is concave, we have \( \psi(2z) \leq 2\psi(z) \) for \( z \geq 0 \). Then from (4.9) it follows that, for \( (x, y) \in \mathcal{D}^c \),

\[
\tilde{L}F_0(x, y) \leq cy\phi(x)\psi''(x - y) + [\beta - bx - g(x)]\phi'(x)[1 + \psi(x - y)] + y\phi(x)[\psi(2(x - y)) - 2\psi(x - y) - 1]\mu_{x-y}(0, \infty) + \phi(x)[\psi(2(x - y)) - 2\psi(x - y) - 1]\nu_{x-y}(0, \infty) + \tilde{A}_1F_0(x, y) + \tilde{A}_2F_0(x, y)
\]

\[
\leq -c\lambda_0^2\theta y e^{-\lambda_0(x-y)} + I(x)[1 + \psi(x - y)] - \nu_{x-y}(0, \infty) - \theta\nu_{x-y}(0, \infty) + \tilde{A}_1F_0(x, y) + \tilde{A}_2F_0(x, y),
\]

where we have also used the fact that \( \phi(x) \geq \theta \). By the definition of \( \phi \), we know that for all \( x, z \geq 0 \),

\[
0 \leq \phi(x) - \phi(x + z) \leq 1_{\{x \leq x_0\}}, \quad 0 \leq -\phi'(z)z \leq \frac{3z}{x_0}1_{\{x \leq x_0\}},
\]

\[
\phi''(x) \leq \frac{6}{x_0^2}1_{\{x \leq x_0\}}, \quad \phi(x + z) - \phi(x) - \phi'(z)z \leq \frac{3z^2}{x_0^2}1_{\{x \leq x_0\}}.
\]

From (4.10) it follows that

\[
\tilde{A}_1F_0(x, y) = \left[1 + \psi(x - y)\right]\left[\frac{c}{x_0^2}x + y\mu_{x-y}(0, \infty) + x\int_0^1 \frac{3z^2}{x_0^2} \mu(\nu - \nu_{x-y})(d\nu)\right]1_{\{x \leq x_0\}} + \left[1 + \psi(x - y)\right][y\mu_{x-y}(0, \infty) + J(x)]1_{\{x \leq x_0\}} + \left[1 + \psi(x - y)\right]\left[\psi(x + z) - \phi(x)\right](\nu - \nu_{y-x})(d\nu)1_{\{x \leq x_0\}}.
\]

In view of (4.11), we have

\[
\tilde{A}_2F_0(x, y) = (x - y)\phi(x)\left[c\psi''(x - y) - b\psi'(x - y) + \int_0^{\infty} \left[\psi(x + z - y) - \psi(x - y)\right]x_0(z)\phi(x)\right.
\]

\[
\left. - \psi(x - y) - \psi'(x - y)z\right]1_{\{z \leq 1\}}\mu(d\nu)
\]

\[
= - (x - y)\phi(x)e^{-\lambda_0(x-y)}\psi(\lambda_0) - \lambda_0^{-1}(1 - e^{-\lambda_0(x-y)})\phi(x)e^{-\lambda_0x}(\lambda_0)1_{\{x \leq y \leq t\}}.
\]

By putting together the above estimates we get (4.19) with \( \lambda_1 = \lambda_0^{-1}e^{-\lambda_0x}(\lambda_0) > 0 \). \(\square\)

**Proposition 4.6.** Suppose that Condition 1.2-(ii) is satisfied. Let \( F_0 \in \mathcal{D}(\tilde{L}) \) be a function defined by (4.17). Then for any \( l \geq 1 \) there is a constant \( \lambda_2 > 0 \) such that

\[
(4.22) \quad \tilde{L}F_0(x, y) \leq -\lambda_2F_0(x, y)1_{\{x \leq y \leq l\}}, \quad (x, y) \in \mathcal{D}^c.
\]

**Proof.** The idea of the proof is to identify and take the advantage of the dominating factor among the branching, immigration and competition mechanisms in different parts of the space \( \mathcal{D}^c \). Under Condition 1.2-(ii), we can choose constants \( \kappa > 0 \) and \( x_0 \in (0, c_0) \) such that

\[
(4.23) \quad c\lambda_0^2e^{-\lambda_0x} + \mu_2(0, \infty) + \nu_2(0, \infty) \geq 2\kappa, \quad 0 \leq x \leq x_0.
\]
Note that $I(x) = 0$ for all $x > x_0$. By (4.19) we have, for $x > x_0$,

$$
\dot{L}F_0(x, y) \leq -\theta y [c \lambda y e^{-\lambda_0(x-y)} + \mu_{x-y}(0, \infty)] - \theta \nu_{x-y}(0, \infty) \\
- \lambda_1 \theta \psi(x-y) 1_{\{x-y \leq l\}}.
$$

Since $x_0 \leq 1 \leq l$, using (4.19) again we see, for $0 \leq x \leq x_0$,

$$
\dot{L}F_0(x, y) \leq -c \lambda_0^2 \theta y e^{-\lambda_0(x-y)} - y \mu_{x-y}(0, \infty) [\theta - 1 - \psi(x-y)] \\
- \theta \nu_{x-y}(0, \infty) - \lambda_1 \theta \psi(x-y) + [I(x) + J(x)] [1 + \psi(x-y)] \\
+ [1 + \psi(x-y)] \int_0^\infty [\phi(x+z) - \phi(x)] (\nu - \nu_{x-y})(dz).
$$

With these estimates at hand, we prove the desired assertion by considering the following three cases.

(i) We first consider the case of $x > x_0$. We will apply (4.24) and consider separately three subcases. From (4.24) it is easy to see $\dot{L}F_0(x, y) \leq 0$ when $x - y > l$. When $x_0/2 < x - y \leq l$, noticing that $\psi$ is nondecreasing and using the facts $\phi \leq 1 + \theta$ and $\psi \leq 1$, we have

$$
\dot{L}F_0(x, y) \leq -\lambda_1 \theta \psi(x-y) \leq -\frac{\lambda_1 \theta}{2} \psi(x-y) \leq -\frac{\lambda_1 \theta}{2} \psi(x_0/2) [1 + \psi(x-y)] \leq -\frac{\lambda_1 \theta \psi(x_0/2)}{2(1+\theta)} F_0(x, y).
$$

When $x - y \leq x_0/2$, we have $y \geq x_0/2$ and so, by (4.23) and (4.24),

$$
\dot{L}F_0(x, y) \leq -\frac{\theta x_0}{2} [c \lambda_0^2 e^{-\lambda_0(x-y)} + \mu_{x-y}(0, \infty) + \nu_{x-y}(0, \infty)] - \lambda_1 \theta \psi(x-y) \\
\leq -\theta x_0 \kappa - \lambda_1 \theta \psi(x-y) \leq -(x_0 \kappa \wedge \lambda_1) \theta [1 + \psi(x-y)] \\
\leq -(x_0 \kappa \wedge \lambda_1) \frac{\theta}{1+\theta} F_0(x, y).
$$

(ii) Let us consider the case of $x \in (0, r x_0)$, where $r \in (0, 1/2]$ will be specified later. Recall that $\theta \geq 4$ and $\psi \leq 1$. Then, according to (4.25), for $x \in [0, r x_0]$,

$$
\dot{L}F_0(x, y) \leq [I(x) + J(x)] [1 + \psi(x-y)] - \theta \nu_{x-y}(0, \infty) \\
+ [1 + \psi(x-y)] \int_0^\infty [\phi(x+z) - \phi(x)] (\nu - \nu_{x-y})(dz) 1_{\{x \leq x_0\}} \\
\leq [1 + \psi(x-y)] [I(x) + J(x) - \nu_{x-y}(0, \infty)] \\
+ \int_0^\infty [\phi(x+z) - \phi(x)] (\nu - \nu_{x-y})(dz).
$$

Since $0 < x_0 < 1$, we can choose $r_* \in (0, 1/2]$ and $q > 0$ such that, for $0 \leq x \leq r_* x_0$,

$$
I(x) - \nu_{x-y}(0, \infty) + \int_0^\infty [\phi(x+z) - \phi(x)] (\nu - \nu_{x-y})(dz) \\
\leq I(x) - \nu_{x-y}(0, \infty) - \int_0^{x_0-x} \left(1 - \frac{x}{x_0}\right)^3 (\nu - \nu_{x-y})(dz) \\
- \int_0^{x_0-x} \left(1 - \frac{x}{x_0}\right)^3 \left(1 - \frac{x+z}{x_0}\right)^3 (\nu - \nu_{x-y})(dz) \\
\leq \frac{3}{x_0} \left[|b|x + g(x) - \beta\right] \left(1 - \frac{x}{x_0}\right)^2 - \nu_{x-y}(0, \infty) - \frac{1}{8} \int_{x_0-x}^\infty (\nu - \nu_{x-y})(dz) \\
- \frac{1}{x_0^3} \int_0^{x_0-x} z^3 (\nu - \nu_{x-y})(dz)
$$
\[
\frac{3}{x_0} \left[ b|x + g(x) \right] - \frac{3\beta}{4x_0} - \frac{1}{8} \int_0^\infty (1 \wedge z^3) \nu(dz) \leq -q,
\]
where for the last inequality we have used the Condition 1.1 for \( \Phi \) and the fact that \( g \) is a continuous function with \( g(0) = 0 \). Now we take

\[ r = r_s \wedge \left[ \frac{x_0 q}{6} \left( 2c + \int_0^1 z^2 \mu(dz) \right)^{-1} \right]. \tag{4.27} \]

From (4.21) it follows that, for \( x \in [0, rx_0] \),

\[ J(x) \leq \frac{3r}{x_0} \left( 2c + \int_0^1 z^2 \mu(dz) \right) \leq \frac{q}{2}. \]

Recall that \( \theta \geq 4 \) and \( \psi \leq 1 \). Then, for \( x \in [0, rx_0] \),

\[ \tilde{L}F_0(x, y) \leq \left[ g - J(x) \right] \left[ 1 + \psi(x - y) \right] \leq -\frac{q}{2} \left[ 1 + \psi(x - y) \right] \leq -\frac{q}{2(1 + \theta)} F_0(x, y). \]

(iii) We finally consider the case of \( x \in (rx_0, x_0] \). In this case, we have

\[ I(x) \leq \frac{3}{x_0} \left[ b|x_0 + g(x_0) \right], \quad J(x) \leq \frac{3}{x_0} \left( 2c + \int_0^1 z^2 \mu(dz) \right). \]

Now we take

\[ H = \frac{3}{x_0} \left[ 2c + b|x_0 + g(x_0) + \int_0^1 z^2 \mu(dz) \right] \tag{4.28} \]

and

\[ \theta = \max \left\{ 4, \frac{2H}{\lambda_1}, \frac{4H}{r_\kappa x_0}, \frac{8H}{\lambda_1 \psi(rx_0/2)} \right\}. \tag{4.29} \]

From (4.25) it follows that

\[ \tilde{L}F_0(x, y) \leq -c\lambda_0^2 \theta y e^{-\lambda_0(x-y)} - y \mu_{x-y}(0, \infty) \left[ \theta - 1 - \psi(x - y) \right] - \theta \nu_{x-y}(0, \infty) \]
\[ + \psi(x - y) \left[ I(x) + J(x) - \lambda_1 \theta \right] + I(x) + J(x) \]
\[ \leq -c\lambda_0^2 \theta y e^{-\lambda_0(x-y)} - y \mu_{x-y}(0, \infty) \left[ \theta - 1 - \psi(x_0) \right] - \theta \nu_{x-y}(0, \infty) \]
\[ + \psi(x - y) (H - \lambda_1 \theta) + H \]
\[ \leq -c\lambda_0^2 \theta y e^{-\lambda_0(x-y)} - \frac{\theta}{2} y \mu_{x-y}(0, \infty) - \theta \nu_{x-y}(0, \infty) \]
\[ - \frac{\lambda_1 \theta}{2} \psi(x - y) + H. \]

When \( y \geq rx_0/2 \), we have \( x - y \leq rx_0/2 \) and, by (4.23),

\[ -c\lambda_0^2 \theta y e^{-\lambda_0(x-y)} - \frac{\theta}{2} y \mu_{x-y}(0, \infty) - \theta \nu_{x-y}(0, \infty) \leq -\frac{\theta r_\kappa x_0}{2}. \]

Returning to (4.30), we obtain

\[ \tilde{L}F_0(x, y) \leq -\frac{\theta r_\kappa x_0}{2} - \frac{\lambda_1 \theta}{2} \psi(x - y) + H \]
\[ \leq -\frac{\theta r_\kappa x_0}{4} - \frac{\lambda_1 \theta}{2} \psi(x - y) \]
\[ \leq -\left( \frac{r_\kappa x_0}{2} \wedge \lambda_1 \right) \frac{\theta}{2} [1 + \psi(x - y)] \]
\[ \leq -\left( \frac{r_k x_0}{2} \wedge \lambda_1 \right) \frac{\theta}{2(1+\theta)} F_0(x, y). \]

When \( y \in [0, rx_0/2) \), we have \( x - y \geq rx_0/2 \) and hence

\[
\tilde{L}F_0(x, y) \leq -\frac{\lambda_1 \theta}{2} \psi(x - y) + H
\leq -\frac{\lambda_1 \theta}{4} \psi(x - y) - \frac{\lambda_1 \theta}{8} \psi(rx_0/2) + H
\leq -\frac{\lambda_1 \theta \psi(rx_0/2)}{8} [1 + \psi(x - y)]
\leq -\frac{\lambda_1 \theta \psi(rx_0/2)}{8(1+\theta)} F_0(x, y).
\]

Combining all the estimates in the three cases, we obtain (4.22). \( \square \)

5. The exponential ergodicity

In this section, we prove the exponential contraction property (1.30) for a suitable control function. From this property we derive the exponential ergodicity of the CBIC-process. Throughout the section, we assume that Conditions 1.1, 1.2 and 1.3 are satisfied. By Proposition 2.3, the CBIC-process with an arbitrary initial value is conservative. Let

\[ (5.1) \quad V_2(x, y) = V(x) + V(y), \quad V_0(x, y) = V_2(x, y) 1_{\{x \neq y\}}, \quad (x, y) \in D. \]

Let \( F_0 \in \mathcal{D}(\tilde{L}) \) be given as in Propositions 4.4 and 4.6 under Condition 1.2-(i) and Condition 1.2-(ii), respectively. For a constant \( \varepsilon > 0 \) to be specified later, define

\[ (5.2) \quad G_0(x, y) = \varepsilon F_0(x, y) + V_0(x, y), \quad (x, y) \in D. \]

By (4.13) and (4.18) it is easy to show that (1.31) holds.

**Proposition 5.1.** We can define a function \( G_0 \in \mathcal{D}(\tilde{L}) \) by (5.2) such that, for some constant \( \lambda_* > 0 \),

\[ (5.3) \quad \tilde{L}G_0(x, y) \leq -\lambda_* G_0(x, y), \quad (x, y) \in \Delta^c. \]

**Proof.** Clearly, for any \((x, y) \in \Delta^c\) the expression (4.2) of \( \tilde{L}_0 F(x, y) \) only depends on the restriction of \( F \) to \( \Delta^c \). Then \( V_0 \in \mathcal{D}(\tilde{L}) \) and

\[ (5.4) \quad \tilde{L}_0 V_0(x, y) = \tilde{L}_0 V_2(x, y) = L V(x) + L V(y). \]

Since \( 0 = V_0(z, z) \leq V_2(z, z) \) for \( z \geq 0 \), by (4.3) we have \( \tilde{L}_1 V_0(x, y) \leq \tilde{L}_1 V_2(x, y) \). But, by (4.3), (5.1) and the first equality in (3.2),

\[
\tilde{L}_1 V_2(x, y) = y \int_0^\infty \left[ V(2y + z - x) - V(y + z) \right] \mu_{x-y}(dz)
+ y \int_0^\infty \left[ V(x + z) - V(y + z) \right] \mu_{y-x}(dz)
+ \int_0^\infty \left[ V(2y + z - x) - V(y + z) \right] \nu_{x-y}(dz)
+ \int_0^\infty \left[ V(x + z) - V(y + z) \right] \nu_{y-x}(dz) = 0.
\]

It follows that

\[ \tilde{L}G_0(x, y) \leq \varepsilon \tilde{L}F_0(x, y) + L V(x) + L V(y). \]
Let $l \geq 1$ be sufficiently large such that $V(z) > 12C_0/C_I$ for $z > l$ and let $\lambda_2 > 0$ be the corresponding constant given by Propositions 4.4 and 4.6 under Condition 1.2-(i) and Condition 1.2-(ii), respectively. Now let $\varepsilon = 4C_0/(\lambda_2 \theta)$. By (1.19) we have

$$\hat{L}G_0(x, y) \leq -\varepsilon \lambda_2 F_0(x, y)1_{\{x - y \leq l\}} + 2C_0 - C_I [V(x) + V(y)].$$

When $x \geq l$, since $\theta \geq 4$, we can use (4.13) or (4.18) to see

$$\hat{L}G_0(x, y) \leq 2C_0 - C_I [V(x) + V(y)]$$

$$\leq -4C_0 - \frac{C_I}{2} [V(x) + V(y)]$$

$$\leq -\frac{\lambda_2 \theta}{2(1 + \theta)} \varepsilon F_0(x, y) - \frac{C_I}{2} [V(x) + V(y)]$$

$$\leq -\frac{1}{2} \left( \frac{4\lambda_2}{5} \wedge C_I \right) G_0(x, y).$$

When $x \leq l$, using (4.13) or (4.18) again we have

$$\hat{L}G_0(x, y) \leq -\varepsilon \lambda_2 F_0(x, y) + 2C_0 - C_I [V(x) + V(y)]$$

$$\leq -\frac{\varepsilon \lambda_2}{2} F_0(x, y) - \frac{\varepsilon \lambda_2 \theta}{2} + 2C_0 - C_I [V(x) + V(y)]$$

$$\leq -\frac{\varepsilon \lambda_2}{2} F_0(x, y) - C_I [V(x) + V(y)]$$

$$\leq -\left( \frac{\lambda_2}{2} \wedge C_I \right) G_0(x, y).$$

Then (5.3) holds with $\lambda_s = 2(C_I \wedge \lambda_2)/5 > 0$. \hfill $\Box$

**Theorem 5.2.** Let $G_0 \in \mathcal{D}(\hat{L})$ and $\lambda_s > 0$ be given as in Proposition 5.1. Then we have (1.30) for $t \geq 0$ and $(x, y) \in D$.

**Proof.** It suffices to consider $(x, y) \in \Delta'$. Let $\{(X_t, Y_t) : t \geq 0\}$ be the Markov coupling defined by (3.8) and (3.9) with $(X_0, Y_0) = (x, y)$. Recall that $\zeta_n^* = \inf\{t \geq 0 : X_t \geq n \text{ or } X_t - Y_t \leq 1/n\}$. By Proposition 2.3 and Theorem 3.1, the process $\{(X_t, Y_t) : t \geq 0\}$ is conservative. Then (3.11) implies $\lim_{n \to \infty} \zeta_n^* = \lim_{n \to \infty} T_n = T$, where $T_n = \inf\{t \geq 0 : X_t - Y_t \leq 1/n\}$. By Theorem 4.1 and integration by parts, for any $n \geq 1$ we have

$$(5.5) \quad e^{\lambda_s (t \wedge \zeta_n^*)} G_0(X_{t \wedge \zeta_n^*}, Y_{t \wedge \zeta_n^*}) = G_0(x, y) + \int_0^{t \wedge \zeta_n^*} e^{\lambda_s (s \wedge \zeta_n^*)} (\hat{L} + \lambda) G_0(X_s, Y_s) \, ds + M_n(t),$$

where $\{M_n(t) : t \geq 0\}$ is a martingale. From (5.3) and (5.5) it follows that

$$\mathbb{E}[e^{\lambda_s (t \wedge \zeta_n^*)} G_0(X_{t \wedge \zeta_n^*}, Y_{t \wedge \zeta_n^*})] \leq G_0(x, y),$$

and so

$$\mathbb{E}[e^{\lambda_s (t \wedge \zeta_n^*)} G_0(X_{t \wedge \zeta_n^*}, Y_{t \wedge \zeta_n^*}) 1_{\{t < T\}}] \leq G_0(x, y).$$

Since $G_0(x, x) = 0$ for $x \geq 0$ and $X_{T+t} = Y_{T+t}$ for $t \geq 0$, we can let $n \to \infty$ and use Fatou’s lemma to get

$$\mathbb{E}[e^{\lambda_s t} G_0(X_t, Y_t)] = \mathbb{E}[e^{\lambda_s t} G_0(X_t, Y_t) 1_{\{t < T\}}] \leq G_0(x, y),$$

which clearly implies (1.30). \hfill $\Box$

The result of the next lemma should be already known, but we could not find a reference. For the convenience of the reader, we give a simple proof of the result here.

**Lemma 5.3.** The expressions (1.13) and (1.15) for the $V$-weighted total variation distance $W_V$ are equivalent.
Proof. For \( \gamma, \eta \in \mathcal{P}_V(\mathbb{R}_+) \) let \( U = \text{supp}((\gamma - \eta)_-) \) and \( U^c = \mathbb{R}_+ \setminus U \), where \((\gamma - \eta)_- \) denotes the lower variation of the signed measure \( \gamma - \eta \) in its Jordan decomposition. If \( \pi \in \mathcal{C}(\gamma, \eta) \), then

\[
\int_{\mathbb{R}_+^2} dV(x, y)\pi(dx, dy) = \int_{\mathbb{R}_+^2} [2 + V(x) + V(y)]1_{\{x \neq y\}}\pi(dx, dy)
= \int_{\mathbb{R}_+^2} [2 + V(x) + V(y)]\pi(dx, dy) - 2\int_{\mathbb{R}_+^2} [1 + V(x)]1_{\{x = y\}}\pi(dx, dy)
= \int_{\mathbb{R}_+} [1 + V(x)]\gamma(dx) + \int_{\mathbb{R}_+} [1 + V(y)]\eta(dy) - 2\int_{U \times U^c} [1 + V(x)]\pi(dx, dy)
\geq \int_{\mathbb{R}_+} [1 + V(x)]\gamma(dx) + \int_{\mathbb{R}_+} [1 + V(y)]\eta(dy) - 2\int_{U^c} [1 + V(x)]\gamma(dx)
\geq \frac{\int_{\mathbb{R}_+} [1 + V(x)]\gamma(dx) + \int_{\mathbb{R}_+} [1 + V(y)]\eta(dy)}{(\gamma - \eta)_+ + (\gamma - \eta)_-}.
\]

One the other hand, let \( \pi_* \in \mathcal{C}(\gamma, \eta) \) be defined by

\[
\pi_*(dx, dy) = (\gamma \wedge \eta)_*(dx, dy) + \frac{\gamma - \eta)_+ (\gamma - \eta)_- (dy)}{(\gamma - \eta)_+ + (\gamma - \eta)_-}.
\]

where \((\gamma \wedge \eta)_* \) is the image of \( \gamma \wedge \eta \) under the mapping \( x \mapsto (x, x) \) from \( \mathbb{R}_+ \) to \( \mathbb{R}_+^2 \). It is easy to see that

\[
\int_{\mathbb{R}_+^2} dV(x, y)\pi_*(dx, dy) = \frac{1}{(\gamma - \eta)_+ + (\gamma - \eta)_-} \int_{U \times U^c} [1 + V(x)](\gamma - \eta)_+ (dx)(\gamma - \eta)_- (dy)
+ \frac{1}{(\gamma - \eta)_+ + (\gamma - \eta)_-} \int_{U \times U^c} [1 + V(y)](\gamma - \eta)_+ (dx)(\gamma - \eta)_- (dy)
= \int_{U^c} [1 + V(x)](\gamma - \eta)_+ (dx) + \int_{U^c} [1 + V(y)](\gamma - \eta)_- (dy)
= \int_{\mathbb{R}_+} [1 + V(x)]\gamma(dx).
\]

Those clearly imply the desired result. \( \square \)

Proof of Theorem 1.1. By Theorem 5.2, we have (1.30) for \( (x, y) \in D \). It is easy to see that \( \tilde{P}_t((x, y), \cdot) \) is a coupling of \( P_t(x, \cdot) \) and \( P_t(y, \cdot) \). Then (1.18) follows from (1.30) and (1.31) with \( K = c_2/c_1 \). By the convexity of the Wasserstein distance we have, for any \( \gamma, \eta \in \mathcal{P}_V(\mathbb{R}_+) \) and \( \pi \in \mathcal{C}(\gamma, \eta) \),

\[
W_V(\gamma, \eta) \leq \int_{\mathbb{R}_+^2} dV(x, y)\pi(dx, dy) \leq Ke^{-\lambda t} \int_{\mathbb{R}_+^2} dV(x, y)\pi(dx, dy).
\]

see, e.g., Villani [57, Theorem 4.8]. It follows that

\[
W_V(\gamma P_t, \eta) \leq Ke^{-\lambda t} \int_{\mathbb{R}_+^2} dV(x, y)\pi(dx, dy) = Ke^{-\lambda t}W_V(\gamma, \eta).
\]

Then for sufficiently large \( r > 0 \), the operator \( P_r \) on \( \mathcal{P}_V(\mathbb{R}_+) \) is contractive. By the Banach fixed point theorem and the completeness of \( \mathcal{P}_V(\mathbb{R}_+) \), there is a unique \( \gamma_r \in \mathcal{P}_V(\mathbb{R}_+) \) such that \( \gamma_r P_r = \gamma_r \). Now we fix such an \( r > 0 \) and define
γ = r^{-1} \int_{0}^{r} γ_s P_s ds. By the Chapman-Kolmogorov equation, for 0 ≤ t < r we have

\[ γP_t = \frac{1}{r} \int_{t}^{r+t} γ_r P_r ds = \frac{1}{r} \int_{t}^{r} γ_r P_r ds + \frac{1}{r} \int_{0}^{t} γ_r P_{r+s} ds = \frac{1}{r} \int_{t}^{r} γ_r P_s ds + \frac{1}{r} \int_{0}^{t} γ_r P_{s+t} ds = \frac{1}{r} \int_{0}^{r} γ_r P_r ds = γ. \]

More generally, for any t ≥ 0 there is a unique integer k ≥ 0 such that 0 ≤ t − kr < r. Then γP_t = γP_{kr}P_{t−kr} = γP_{t−kr} = γ. By applying (5.6) again we obtain (1.17) with C(η) = K\sqrt{γ, η}). □

**Proofs of Propositions 1.2 and 1.3.** Under the integrability condition (1.21), we have V_1 ∈ D(L). By (1.12) it is easy to see that

\[ LV_1(x) = [β − bx − g(x)] + x \int_{0}^{∞} zμ(dz) + \int_{0}^{∞} zν(dz). \]

Then (1.19) is equivalent to (1.22). The integrability condition (1.23) implies V_0, log ∈ D(L). By (1.12) we have

\[ LV_{log}(x) = -\frac{c}{(1+x)^2} + x \int_{0}^{∞} \left[ \log \left( 1 + \frac{z}{1+x} \right) - \frac{z}{1+x} 1_{\{z<1\}} \right] μ(dz) + \frac{β − bx − g(x)}{1+x} + \int_{0}^{∞} \log \left( 1 + \frac{z}{1+x} \right) ν(dz). \]

By Taylor’s expansion, we have

\[ \frac{z}{1+x} − \log \left( 1 + \frac{z}{1+x} \right) = z^2 \int_{0}^{1} \frac{(1-u)du}{(1+x+uz)^2}. \]

By the dominated convergence theorem,

\[ \lim_{x \to ∞} x^2 \int_{0}^{1} \left[ \frac{z}{1+x} − \log \left( 1 + \frac{z}{1+x} \right) \right] μ(dz) = \frac{1}{2} \int_{0}^{1} z^2 μ(dz). \]

It follows that

\[ \lim_{x \to ∞} \frac{x}{\log x} \int_{0}^{1} \left[ \log \left( 1 + \frac{z}{1+x} \right) − \frac{z}{1+x} \right] μ(dz) = 0. \]

Then (1.19) is equivalent to (1.24). □

**Proofs of Corollaries 1.4 and 1.5.** Corollary 1.4 follows easily by Proposition 1.2. Then it remains to prove Corollary 1.5. By the proof of Theorem 2.4, the CBIC-process with stable branching mechanism has generator L defined by (1.12) with b = a + σh_α and μ(dz) = ασm_α(dz), where m_α and h_α are defined by (1.25) and (2.6), respectively. For α = 1, by elementary calculus we have

\[ \int_{1}^{∞} \log \left( 1 + \frac{z}{1+x} \right) \frac{1}{z} dz = \frac{1}{2} \log(2 + x) + \log \left( \frac{2 + x}{1 + x} \right). \]

For 0 < α < 1, by Zwillinger [58, 3.194.4, p.318] we have

\[ \int_{0}^{∞} \log \left( 1 + \frac{z}{1+x} \right) \frac{1}{z^{1+α}} dz = \frac{π}{α \sin(απ)(1+x)^α}. \]

It follows that

\[ \int_{1}^{∞} \log \left( 1 + \frac{z}{1+x} \right) \frac{1}{z^{1+α}} dz = \frac{π}{α \sin(απ)(1+x)^α} - \int_{0}^{1} \log \left( 1 + \frac{z}{1+x} \right) \frac{1}{z^{1+α}} dz \]

\[ = \int_{0}^{1} \left[ \frac{z}{1+x} − \log \left( 1 + \frac{z}{1+x} \right) \right] \frac{1}{z^{1+α}} dz. \]
Then Corollary 1.5 follows by Proposition 1.3 and (5.7). \hfill \Box

Remark 5.4. Our approach provides a way of finding the exponential ergodicity rate $\lambda_* > 0$. Let $C_0 > 0$ and $C_1 > 0$ be as in (1.19). Then $\lambda_* = 2(C_1 \land \lambda_2)/5$ by the proof of Proposition 5.1. To determine $\lambda_2 > 0$, under Condition 1.2-(i) we follow the proof of Proposition 4.4, while under Condition 1.2-(ii) we follow the steps given below:

1. determine $\kappa = \kappa(\lambda_0, c_0)$ and $x_0 = x_0(\lambda_0, c_0)$ by (4.23);
2. choose $l = l(C_0, C_1)$ as in the proof of Proposition 5.1;
3. let $\lambda_1 = \lambda_0^{-1} e^{-l_{\lambda_0}} \Psi(\lambda_0)$ as in the proof of Lemma 4.5;
4. determine $q = q(x_0)$, $r_* = r_*(x_0)$ and $r = r(x_0)$ by (4.26) and (4.27);
5. define $H = H(x_0)$ and $\theta = \theta(\lambda_1, x_0, \kappa, r)$ by (4.28) and (4.29);
6. choose $\lambda_2 = \lambda_2(\lambda_1, x_0, \kappa, r, \theta, q)$ as indicated in the proof of Proposition 4.6.

Example 5.1. Let $\Psi(\lambda) = \lambda^\alpha$, $\Phi(\lambda) = \lambda$ and $g(x) = x^2$, where $1 < \alpha \leq 2$. In this case, it is easy to see that Conditions 1.1 and 1.2-(i) are satisfied and

$$LV_1(x) = 1 - x^2 \leq 2 - V_1(x), \quad x \geq 0.$$ 

Then (1.19) holds for $V_1$ with $C_0 = 2$ and $C_1 = 1$. Take $\theta = 4$ in (4.12). According the proof of Proposition 5.1, we can choose $l = 12C_0/C_1 = 24$. Next take $\lambda_2 = (1 + \theta)^{-1} e^{-\theta l} = e^{-\theta l} / 5$ as in the proof of Proposition 4.4. Finally, we have $\lambda_* = 2(C_1 \land \lambda_2)/5 = 2e^{-\theta l} / 5$.

The estimates in the procedure of determining the constant $\lambda_* > 0$ are certainly not optimal. It remains an interesting problem to improve the arguments to get the optimal exponential ergodicity rate.

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