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Dynamical behavior of a higher order stochastically perturbed SIRI epidemic model with relapse and media coverage

Qun Liu\textsuperscript{a}, Daqing Jiang\textsuperscript{b,c,d,e}, Tasawar Hayat\textsuperscript{d,c}, Ahmed Alsaedi\textsuperscript{d}, Bashir Ahmad\textsuperscript{d}

\textsuperscript{a} School of Mathematics and Statistics, Key Laboratory of Applied Statistics of MOE Northeast Normal University Changchun Jilin Province 130024 PR China
\textsuperscript{b} Key Laboratory of Unconventional Oil and Gas Development China University of Petroleum (East China) Ministry of Education Qingdao 266580 PR China
\textsuperscript{c} College of Science China University of Petroleum Qingdao Shandong Province 266580 PR China
\textsuperscript{d} Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group King Abdulaziz University Jeddah Saudi Arabia
\textsuperscript{e} Department of Mathematics Quaid-i-Azam University 45220 Islamabad 44000 Pakistan

\begin{abstract}
This paper is intended to explore a higher order stochastically perturbed SIRI epidemic model with relapse and media coverage. Firstly, we derive sufficient criteria for the existence and uniqueness of an ergodic stationary distribution of positive solutions to the system by establishing a suitable stochastic Lyapunov function. Then we obtain adequate conditions for complete eradication and wiping out of the infectious disease. In a biological interpretation, the existence of a stationary distribution implies that the disease will prevail and persist in the long term. Finally, the theoretical results are illustrated by computer simulations, including two examples based on real-life disease.
\end{abstract}

1. Introduction

In modern society, there is no doubt that public health issue has become a great threat to the global personal and property security, we must take some self-protection measures during the epidemic period. The probe of mathematical modeling is a good way to describe the transmission dynamics of infectious disease and provide effective control strategies \[1\]. Recently, many scholars have constructed an idea of mathematical models to investigate the dynamics of infectious disease which is named as compartmental model. In the epidemic models the total population is classified as the susceptible compartment \(S\), the infected compartment \(I\) and the recovered compartment \(R\). There exist few occurrences where susceptibles become infectious, then are recovered with temporary immunity and then become infectious again which is called as SIRI model. This recurrence of disease is an important characteristic of some animal and human diseases, for example, tuberculosis including human and bovine and herpes \[2,3\].

In case of the breakout of infectious disease in a particular region, the immediate endeavour of disease control authorities is to make a ultimate venture to contain the spread of disease. Educating people about the disease via numerous agencies such as mass media, etc, is one of the significant preventive measures. Mass media plays an important role in propagating the nature and the cause of assorted deadly diseases such as respiratory disease, hepatitis diseases, human immunodeficiency syndrome (HIV), Avian influenza A (H7N9), human tuberculosis (TB), acquired immunodeficiency syndrome (AIDS), severe acute respiratory syndrome (SARS), Ebola virus disease (EVD), etc. Recently, numerous mathematical models have been developed to study the influence of media coverage on the dynamics of infection disease \[4–10\]. Caraballo et al. \[10\], Sun et al. \[11\] and Li and Cui \[12\] employed deterministic models to study the impacts of media coverage on the transmission dynamics, in view of the following incidence

\[
g(S, I) = \left( \beta_1 - \frac{\beta_2 I}{m + I} \right) SI,
\]

where \(\beta_1\) indicates the contact rate after and before media alert respectively; the term \(\frac{\beta_2 I}{m + I}\) represents the diminished value of the transmission rate when infectious individuals are taken into account. If the infectives are adequately large then the diminished value of the transmission rate tends to its maximum \(\beta_2\), and infectives reaches \(m\), the diminished value of the transmission rate equals half of the maximum \(\beta_2\). Due to the inability of the coverage report to prevent disease, spreading rampantly, we have
\[ \beta_1 \geq \beta_2 > 0, \ m \text{ denotes reactive velocity of the people and media coverage to the disease. The term } \frac{1}{m+1} \text{ is a continuous bounded function that describes psychological influences or disease saturation. Then the deterministic SIR epidemic model with relapse and media coverage can be expressed as follows:} \]

\[ \begin{align*}
\frac{dS}{dt} &= \Lambda - \left( \beta_1 - \frac{\beta_2 I}{m+1} \right) SI - \mu_1 S, \\
\frac{dI}{dt} &= \left( \beta_1 - \frac{\beta_2 I}{m+1} \right) SI - (\mu_2 + \gamma) I + \delta R, \\
\frac{dR}{dt} &= \gamma I - (\mu_3 + \delta) R. 
\end{align*} \tag{1.1} \]

\[ \text{where } \Lambda, \mu_1, \mu_2, \beta_1, \beta_2, m, \gamma, \delta \text{ are all positive constants, } S \text{ denotes the numbers of the individuals susceptible to the disease, } I \text{ denotes the infected members and } R \text{ is the members who have recovered from the infection. The parameters have the following biological meanings: } \Lambda \text{ represents the recruitment rate, } \mu_1, \mu_2, \mu_3 \text{ are the natural death rate of susceptible, infected and recovered compartments respectively. The parameter } \gamma \text{ reflects recover rate with temporary immunity and } \delta \text{ represents the relapse rate. The basic reproduction number } R_0 = \frac{\beta_1 \Lambda}{\mu_1 (\mu_2 + \gamma - \frac{\delta \gamma}{\mu_1 \mu_3})} \]

\[ \text{has a great importance in epidemiology since it is a threshold quantity which determines whether an epidemic occurs or the disease dies out. The following behaviors of solutions according to the value of the threshold } R_0 \text{ can be given:} \]

- If } R_0 \leq 1, \text{ then system (1.1) has only the disease-free equilibrium } \]

\[ E_0 = \left( \frac{\Lambda}{\mu_1}, 0, 0 \right), \]

\text{and it is globally asymptotically stable in the invariant set } \Theta.

- If } R_0 > 1, \text{ then } E_0 \text{ is unstable, and there is a unique endemic equilibrium } E^* = (S^*, I^*, R^*) \text{ with } S^* > 0, I^* > 0 \text{ and } R^* > 0 \text{ which is globally asymptotically stable in the interior of } \Theta, \text{ where} \]

\[ \Theta := \left\{ (S, I, R) : S > 0, I \geq 0, R \geq 0, S + I + R \leq \frac{\Lambda}{\min\{\mu_1, \mu_2, \mu_3\}} \right\}. \]

\[ \text{However, it is well established that epidemic systems are always subjected to environmental white noise and the influence of fluctuating environmental white noise is not inclusive in the deterministic models [15]. Thus, the deterministic epidemic model has some restrictions to predict the future dynamics accurately [16]. Recently, stochastic differential equation models [17–22] have played an important role in various branches of applied sciences including infection dynamics and population dynamics, since they can provide some additional degree of realism compared to their deterministic counterparts [23]. And it is hence interesting to introduce the stochastic perturbation into the deterministic system to reveal the effect of environmental white noise on human disease [18,21,24–27]. For example, Gray et al. [18] investigated a stochastic SIS epidemic model with constant population size and degenerate diffusion. They established conditions for extinction and persistence of the disease. Moreover, in the case of persistence, they showed the existence of a stationary distribution and obtained expressions for its mean and variance. In [21], Liu et al. studied the dynamical behavior of a stochastic SIRI epidemic model with relapse and constant population size. The authors used the Markov semigroup theory to obtain the existence of a stable stationary distribution. Caraballo [27] analyzed a stochastic epidemic model with isolation and nonlinear incidence. They established sufficient condition for extinction and obtained necessary and sufficient conditions for persistence in mean of the disease. In particular, they derived the stochastic threshold for the model. However, in the literatures [18,21] and [27], the authors didn't consider the effect of the media coverage. In addition, in the literatures [18] and [21], the diffusion matrix is degenerate, the authors cannot use the Has’minskii’s theory to obtain the ergodicity of a stationary distribution, while in the literature [27], although the diffusion matrix is nondegenerate, the authors didn't study the existence and uniqueness of an ergodic stationary distribution which implies stochastic weak stability. Accordingly, it is necessary for us to consider both of them together.} \]

\[ \text{To incorporate the effect of environmental white noise, in this paper, motivated by the approach proposed by Liu and Jiang [28], we construct a stochastic differential equation model by introducing the nonlinear perturbation into each equation of system (1.1) as the stochastic perturbation may be dependent on square of the variables } S, I \text{ and } R, \text{ respectively. Then to make model (1.1) more realistic and reasonable, we introduce a corresponding stochastic model as follows:} \]

\[ \begin{align*}
\frac{dS}{dt} &= \left[ \Lambda - \left( \beta_1 - \frac{\beta_2 I}{m+1} \right) SI - \mu_1 S \right] \ dt + (\sigma_1 S + \sigma_12) dB_1(t), \\
\frac{dI}{dt} &= \left[ \left( \beta_1 - \frac{\beta_2 I}{m+1} \right) SI - (\mu_2 + \gamma) I + \delta R \right] \ dt + (\sigma_21 + \sigma_22) dB_2(t), \\
\frac{dR}{dt} &= \gamma I - (\mu_3 + \delta) R \ dt + (\sigma_31 + \sigma_32 ) dB_3(t). 
\end{align*} \tag{1.2} \]

\[ \text{subject to the initial conditions} \]

\[ S(0) = S_0 > 0, \ I(0) = I_0 \geq 0, \ R(0) = R_0 \geq 0. \]

\[ \text{Here } B_i(t) \text{ denote mutually independent standard Brownian motions defined on a complete probability space } (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \text{ with a filtration } \{\mathcal{F}_t\}_{t \geq 0} \text{ satisfying the usual conditions [29], } i = 1, 2, 3, \sigma_1 > 0, \sigma_2 > 0, \sigma_3 > 0 \text{ denote the intensities of the environmental random disturbance. Note that this model appears when we assume that the parameters } \mu_1, \mu_2 \text{ and } \mu_3 \text{ are disturbed by some stochastic perturbation in each equation, that is to say, we replace } \mu_1, \mu_2 \text{ and } \mu_3 \text{ by} \]

\[ \mu_1 \to \mu_1 - (\sigma_11 S + \sigma_12), \mu_2 \to \mu_2 - (\sigma_21 + \sigma_22) I, \text{ and } \mu_3 \to \mu_3 - (\sigma_31 + \sigma_32) R, \]

\[ \text{in each equation, respectively. A large number of scholars concentrate only on media coverage and they have not considered media coverage with relapse and higher order perturbation [4–9,30]. Yet another group of scholars focus only on higher order perturbation [19,28,31–33]. Therefore, it is necessitated that we focus on them together. Investigating such problem is important and meaningful.} \]

\[ \text{The structure of this paper is as follows. In Section 2, we establish sufficient criteria for the existence and uniqueness of an ergodic stationary distribution of positive solutions to the stochastic system (1.2). In Section 3, we obtain sufficient criteria for extinction of the disease. In Section 4, the presented results are demonstrated and confirmed by numerical simulations. Finally, conclusion is provided to end this paper.} \]

\[ \text{Throughout this paper, if } G \text{ is a matrix, its transpose is denoted by } G^T. \text{ Moreover, we introduce the following notations:} \]

\[ \mathbb{R}_+^d = \{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_i > 0, 1 \leq i \leq d \} \text{ and } \mathbb{R}_i^d = \{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_i \geq 0, 1 \leq i \leq d \}. \]

\[ \text{To proceed, we should first give some condition under which system (1.2) has a unique global positive solution. Since the proof is similar to the statement of Lemma 1.1 in Liu and Jiang [28], we only state the result without proof.} \]

\[ \text{Lemma 1.1. For any initial value } S_0 > 0, \ I_0 \geq 0, \ R_0 \geq 0, \text{ there is a unique solution } (S(t), I(t), R(t)) \text{ to system (1.2) on } t \geq 0 \text{ and the solution will remain in } \mathbb{R}_+^d \text{ with probability one, namely,} \]

\[ (S(t), I(t), R(t)) \in \mathbb{R}_+^d \text{ for all } t \geq 0 \text{ almost surely (a.s.).} \]
2. Existence of ergodic stationary distribution

In this section, we will establish sufficient criteria for the existence and uniqueness of an ergodic stationary distribution of positive solutions to the stochastic system (1.2). In a biological viewpoint, the existence of a stationary distribution implies that the disease will be prevalent and persistent when the intensities of stochastic perturbations are adequately small.

Let $X(t)$ be a regular time-homogeneous Markov process in $\mathbb{R}^d$ described by the stochastic differential equation

$$
\text{d}X(t) = f(X(t))\text{d}t + \sum_{r=1}^{k}g_r(X(t))\text{d}B_r(t).
$$

The diffusion matrix of the process $X(t)$ is defined as follows

$$
A(x) = (a_{ij}(x)), \quad a_{ij}(x) = \sum_{r=1}^{k}g_r(x)g_r^j(x).
$$

**Lemma 2.1.** [34]. The Markov process $X(t)$ has a unique ergodic stationary distribution $\pi(\cdot)$ if there exists a bounded open domain $U \subset \mathbb{R}^d$ with regular boundary $\Gamma$, having the following properties:

$A_1$: the diffusion matrix $A(x)$ is strictly positive definite for all $x \in U$.

$A_2$: there exists a nonnegative $C^2$-function $V$ such that $LV$ is negative for any $\mathbb{R}^d \setminus U$.

**Lemma 2.2.** For any $x \geq 0$, the following two inequalities hold

(a). $x^3 \geq \left(x - \frac{1}{2}\right)(x^2 + 1)$; (b). $x^4 \geq \left(\frac{3}{4}x^2 - \frac{1}{4}\right)(x^2 + 1)$.

The proofs are based on the following facts

(i) $2x^3 - (2x - 1)(x^2 + 1) = 2x^3 - 2x^3 - 2x + x^2 + 1 = (x - 1)^2 \geq 0,$

(ii) $4x^4 - (3x^2 - 1)(x^2 + 1) = 4x^4 - 3x^4 - 3x^2 + x^2 + 1 = (x^2 - 1)^2 \geq 0.$

**Theorem 2.1.** Assume that $\mathcal{R}_0^5 := \frac{\beta_1A}{(\mu_1 + \frac{\sigma_1^2}{2} + 2\sqrt{\frac{\Lambda\sigma_1}{1-p}})(\mu_2 + \frac{\sigma_2^2}{2} + 2\sqrt{\frac{\Lambda\sigma_2}{1-p}})} > 1$, then system (1.2) admits a unique stationary distribution $\pi(\cdot)$ and it has the ergodic property.

**Proof.** To prove Theorem 2.1, we only need to verify conditions $A_1$ and $A_2$ in Lemma 2.1. We first verify the condition $A_1$. The diffusion matrix of system (1.2) is given by

$$
A = \begin{pmatrix}
(\sigma_1S + \sigma_{12})^2S^2 & 0 & 0 \\
0 & (\sigma_{21} + \sigma_{22}I)^2 & 0 \\
0 & 0 & (\sigma_{31} + \sigma_{32}R)^2R^2
\end{pmatrix}.
$$

Obviously, the matrix $A$ is positive definite for any compact subset of $\mathbb{R}^3_+$, so the condition $A_1$ in Lemma 2.1 holds.

Now we verify the condition $A_2$. For any adequately small constant $p \in (0, 1)$, define

$$
\mathcal{R}_0^5(p) = \frac{\beta_1A}{(\mu_1 + \frac{\sigma_1^2}{2} + 2\sqrt{\frac{\Lambda\sigma_1}{1-p}})(\mu_2 + \frac{\sigma_2^2}{2} + 2\sqrt{\frac{\Lambda\sigma_2}{1-p}})}.
$$

Evidently, $\lim_{p \to 0^+} \mathcal{R}_0^5(p) = \mathcal{R}_0^5$. Due to the continuity of the function $\mathcal{R}_0^5(p)$ and $\mathcal{R}_0^5 > 1$, we can pick $p$ adequately small such that $\mathcal{R}_0^5(p) > 1$. In view of system (1.2), we obtain

$$
L(-\ln S) = -\frac{\Lambda}{S} + \left(\beta_1 - \frac{\beta_2I}{m+1}\right)I + \mu_1 + \frac{\sigma_1^2}{2} + \sigma_{12}S + \frac{\sigma_{11}^2S^2}{2}
\leq -\frac{\Lambda}{S} + \beta_1I + \mu_1 + \frac{\sigma_{12}^2}{2} + \sigma_{12}S + \frac{\sigma_{11}^2S^2}{2},
\tag{2.1}
$$

$$
L(-\ln I) = -\beta_1S + \frac{\beta_2SI}{m+1} - \frac{\delta R}{T} + \mu_2 + \frac{\sigma_2^2}{2} + \sigma_{21}I + \frac{\sigma_{22}^2I^2}{2}
\leq -\beta_1S + \frac{\beta_2S}{m+1} - \frac{\delta R}{T} + \mu_2 + \frac{\sigma_{21}^2}{2} + \sigma_{21}I + \frac{\sigma_{22}^2I^2}{2},
\tag{2.2}
$$

and

$$
L(-\ln R) = \frac{\gamma I}{R} + \mu_3 + \delta + \frac{\sigma_3^2}{2} + \sigma_{31}SR + \frac{\sigma_{32}^2R^2}{2}.
\tag{2.3}
$$
Define
\[ V_1(S) = \sum_{i=1}^{2} \frac{a_i(S + b_i)^p}{p}, \quad V_2(S, I) = c_1S + \frac{c_2(I + c_3)^p}{p}, \quad V_3(S) = -\ln S + V_1(S). \]

\[ V_2(S, I, R) = -\ln I + V_2(S, I) + \frac{c_2e^{\varepsilon I} \delta}{\mu_3 + \delta} R, \quad V_3(R) = -\ln R + \frac{d_1(\sigma_{31} + \sigma_{32} R)^p}{p} + \frac{\sigma_{31} \sigma_{32} R}{\mu_3 + \delta} R, \]

\[ V_4(S, I, R) = \left(\frac{\sigma_{11} S + \sigma_{12}}{p}\right)^\delta + \left(\frac{\sigma_{21} + \sigma_{22}}{p}\right)^\delta + \left(\frac{\sigma_{31} + \sigma_{32} R}{p}\right)^\delta, \quad V_5(S, I, R) = V_2(S, I, R) + d_2 V_1(S) + d_3 V_3(R). \]

where \(a_1, a_2, b_1, b_2, c_1, c_2, c_3, d_1, d_2\) and \(d_3\) are positive constants which will be determined later. Applying Itô’s formula to \(V_1\), we get

\[ IV_1 = \sum_{i=1}^{2} \frac{a_i(S + b_i)^{p-1}}{p} \left( A - \left(\beta_1 - \frac{\beta_3 I}{m + 1}\right) SI - \mu_1 S \right) \leq \frac{a_1(1-p)}{2(S + b_1)^{2-p}} \left(\sigma_{11} S + \sigma_{12}\right)^2 S^2, \]

\[ \leq \frac{a_1 A}{b_1^{p-1}} - \frac{a_1(1-p)b_1^{2-2p} \sigma_{11}^2 S^4}{2(\frac{S}{b_1} + 1)^{2-p}}, \quad \leq \frac{a_2 A}{b_2^{p-1}} - \frac{a_2(1-p)b_2^{2-2p} \sigma_{11}^2 \sigma_{12}^3}{2(\frac{S}{b_2} + 1)^{2-p}} \]

where in the third inequality, we have used Lemma 2.2. Choose

\[ a_1 = \frac{8}{3(1-p)\beta_1^{2p}}, \quad a_2 = \frac{2}{(1-p)b_2^{2p}}, \quad b_1 = 2\sqrt{\frac{A}{(1-p)^2}}, \quad b_2 = 2\sqrt{\frac{A}{(1-p)^2}}, \]

then we have

\[ IV_1 \leq 2\sqrt{\frac{\Lambda \sigma_{11} \sigma_{12}}{1-p} + 2\sqrt{\frac{\Lambda^2 \sigma_{11}^2}{(1-p)^2} - \sigma_{11}^2 S^2}}. \]

Thus, from (2.1) and (2.4) it follows that

\[ I V_2 = c_1 \left( A - \left(\beta_1 - \frac{\beta_3 I}{m + 1}\right) SI - \mu_1 S \right) + c_2 (I + c_3)^{p-1} \left[ \left(\beta_1 - \frac{\beta_3 I}{m + 1}\right) SI - (\mu_2 + \gamma) I + \delta R \right] \]

\[ \leq c_1 A + (c_2 c_3^{p-1} - c_1) \left(\beta_1 - \frac{\beta_3 I}{m + 1}\right) SI + c_2 (I + c_3)^{p-1} \delta R \leq \frac{c_2(1-p)c_3^{p-2} \sigma_{11}^2 S^2}{2(\frac{S}{b_1} + 1)^{2-p}}, \]

\[ \leq c_1 A + (c_2 c_3^{p-1} - c_1) \left(\beta_1 - \frac{\beta_3 I}{m + 1}\right) SI + c_2 (I + c_3)^{p-1} \delta R \leq \frac{c_2(1-p)c_3^{p-2} \sigma_{11}^2 \sigma_{12}^2 (\frac{S}{b_2})^4}{2(\frac{S}{b_2} + 1)^{2-p}}, \]

\[ \leq c_1 A + (c_2 c_3^{p-1} - c_1) \left(\beta_1 - \frac{\beta_3 I}{m + 1}\right) SI + c_2 (I + c_3)^{p-1} \delta R \leq \frac{c_2(1-p)c_3^{p-2} \sigma_{11}^2 \sigma_{12}^2}{4(\frac{S}{b_2} + 1)^{2-p}}, \]

\[ = c_1 A + (c_2 c_3^{p-1} - c_1) \left(\beta_1 - \frac{\beta_3 I}{m + 1}\right) SI + c_2 (I + c_3)^{p-1} \delta R \leq \frac{c_2(1-p)c_3^{p-2} \sigma_{11}^2 \sigma_{12}^2}{16} + \frac{c_2(1-p)c_3^{p-2} \sigma_{11}^2 \sigma_{12}^2}{16}. \]
where in the third inequality, we have used Lemma 2.2. Choose
\[ c_1 = c_2c_3^{-1}, \quad c_2 = \frac{8}{3(1-p)c_3^2}, \quad c_3 = 2\sqrt[3]{\frac{\Lambda}{(1-p)\sigma_{22}^2}}, \]
then we obtain
\[ L_{V_2} \leq 2\sqrt[3]{\frac{\Lambda^2\sigma_{22}^2}{(1-p)^2}} + c_2c_3^{-1}\delta R - \frac{\sigma_1^2}{2}p^2. \]
(2.6)

Therefore, by (2.2) and (2.6), we have
\[ L_{V_2} \leq -\beta_1 SL + \frac{\beta_2}{m}SI - \frac{\delta R}{T} + \mu_2 + \gamma + \sigma_1^2 + \sigma_{21}\sigma_{22}l + \frac{\sigma_1^2}{2}l + 2\sqrt[3]{\frac{\Lambda^2\sigma_{22}^2}{(1-p)^2}} + c_2c_3^{-1}\delta R - \frac{\sigma_1^2}{2}p^2 \\
+ \frac{c_2c_3^{-1}\delta}{\mu_3 + \delta}\gamma I - (\mu_3 + \delta)R \]
\[ = -\beta_1 S - \frac{\delta R}{T} + \mu_2 + \gamma + \sigma_1^2 + \frac{\sigma_{21}}{2} + \frac{\sigma_{22}\gamma}{\mu_3 + \delta} - 2\sqrt[3]{\frac{\Lambda^2\sigma_{22}^2}{(1-p)^2}} + \frac{\beta_2}{m}SI + \left(\sigma_{21}\sigma_{22} + \frac{c_2c_3^{-1}\delta}{\mu_3 + \delta}\gamma I\right). \]
(2.7)

Moreover, according to (2.3), we derive
\[ L_{V_3} = \frac{-\gamma I}{R} + \mu_3 + \delta + \frac{\sigma_1^2}{2} + \frac{\sigma_{21}}{2} + \frac{\sigma_{31}\sigma_{32}\gamma}{\mu_3 + \delta} - 2\sqrt[3]{\frac{\Lambda^2\sigma_{22}^2}{(1-p)^2}} + \frac{\sigma_{22}\gamma}{\mu_3 + \delta} - \frac{d_1\sigma_{31}^2 (1-p)\sigma_{31}}{2}R. \]
Choose
\[ d_1 = \frac{1}{(1-p)\sigma_{31}^2}, \]
then
\[ L_{V_3} \leq -\gamma I + \mu_3 + \delta + \frac{\sigma_1^2}{2} + \frac{\sigma_{31}\sigma_{32}\gamma}{\mu_3 + \delta} + d_1\sigma_{32}\sigma_{31}^{-1}\gamma I. \]
(2.8)

Analogously, we have
\[ L_{V_4} = \sigma_{11}(\sigma_{11}S + \sigma_{12})^{-1}\left(\Lambda - \left(\beta_1 - \frac{\beta_2}{m}I\right)SI - \mu_1 S\right) - \frac{\sigma_1^2}{2}(1-p)(\sigma_{11}S + \sigma_{12})R^{-1}
\times \left(\beta_1 - \frac{\beta_2}{m}I\right)SI - (\mu_2 + \gamma) + \delta R - \frac{\sigma_1^2}{2}l + \frac{\sigma_{22}\gamma}{\mu_3 + \delta} - \frac{d_1\sigma_{31}^2 (1-p)\sigma_{31}}{2}R. \]
(2.9)

Consequently, in view of (2.5), (2.7) and (2.8), we obtain
\[ L_{V_2} \leq -\beta_1 S - \frac{d_2\Lambda}{S} - \frac{\delta R}{T} - \frac{d_3\gamma I}{R} + \mu_2 + \gamma + \frac{\sigma_1^2}{2} + 2\sqrt[3]{\frac{\Lambda^2\sigma_{22}^2}{(1-p)^2}} + d_2\left(\mu_1 + \frac{\sigma_1^2}{2} + 2\sqrt[3]{\frac{\Lambda\sigma_{11}\sigma_{12}}{1-p}} + 2\sqrt[3]{\frac{\Lambda^2\sigma_{11}^2}{(1-p)^2}}\right) \\
+ d_3\left(\mu_3 + \delta + \frac{\sigma_1^2}{2}\right) + \frac{\beta_2}{m}SI + \left(\sigma_{21}\sigma_{22} + \frac{c_2c_3^{-1}\delta}{\mu_3 + \delta}\gamma I\right) + d_2\beta_1 I + d_3\left(\sigma_{31}\sigma_{32}\gamma + d_1\sigma_{32}\sigma_{31}^{-1}\gamma I\right) \\
\leq -2\sqrt[3]{\frac{\beta_1\Lambda}{d_2}} - 2\sqrt[3]{\delta\gamma d_3} + d_2\left(\mu_1 + \frac{\sigma_1^2}{2} + 2\sqrt[3]{\frac{\Lambda\sigma_{11}\sigma_{12}}{1-p}} + 2\sqrt[3]{\frac{\Lambda^2\sigma_{11}^2}{(1-p)^2}}\right) + d_3\left(\mu_3 + \delta + \frac{\sigma_1^2}{2}\right) + \mu_2 + \gamma \\
+ \frac{\sigma_1^2}{2} + 2\sqrt[3]{\frac{\Lambda^2\sigma_{22}^2}{(1-p)^2}} + \frac{\beta_2}{m}SI + \left(\sigma_{21}\sigma_{22} + \frac{c_2c_3^{-1}\delta}{\mu_3 + \delta}\gamma I\right) + d_2\beta_1 I + d_3\sigma_{31}\sigma_{32}\gamma + d_1d_3\sigma_{32}\sigma_{31}^{-1}\gamma I + \sigma_{21}\sigma_{22}. \]
Choose
\[ d_2 = \frac{\mu_1 + \frac{\sigma_1^2}{2} + 2\sqrt[3]{\frac{\Lambda\sigma_{11}\sigma_{12}}{1-p}} + 2\sqrt[3]{\frac{\Lambda^2\sigma_{11}^2}{(1-p)^2}}}{\mu_3 + \delta + \frac{\sigma_1^2}{2}}. \]
then we obtain
\[
LV_3 \leq -\frac{\beta_1 \Lambda}{\mu_1 + \frac{\sigma_1^2}{2} + 2 \sqrt{\frac{\Lambda \sigma_1 \sigma_2}{1 - p}} + 2 \sqrt{\frac{\Lambda \sigma_2^2}{1 - p}} p} + \mu_2 + \gamma + \frac{\sigma_2^2}{2} + 2 \sqrt{\frac{\Lambda \sigma_2^2}{1 - p}} \frac{\delta y}{\mu_3 + \delta + \frac{\sigma_1^2}{2}} + \beta_2 SL
\]
\[
+ \left( \frac{c_2 c_3^{-1} y}{\mu_3 + \delta} + d_2 \beta_1 + d_3 \sigma_2 \sigma_2 \gamma \right) + d_3 \sigma_3 \sigma_3 \sigma_3 \sigma_3 \gamma + \sigma_2 \sigma_2 \gamma \right) I
\]
\[
= -\left( \mu_2 + \gamma + \frac{\sigma_2^2}{2} + 2 \sqrt{\frac{\Lambda \sigma_2^2}{1 - p}} \frac{\delta y}{\mu_3 + \delta + \frac{\sigma_1^2}{2}} (R^2_0(p) - 1) + \beta_2 SL + \left( \frac{c_2 c_3^{-1} y}{\mu_3 + \delta} + d_2 \beta_1 + d_3 \sigma_3 \sigma_3 \sigma_3 \gamma \right) \right)
\]
\[
+ d_3 \sigma_3 \sigma_3 \sigma_3 \sigma_3 \gamma + \sigma_2 \sigma_2 \gamma \right) I.
\]
(2.10)
where
\[
R^2_0(p) := \frac{\beta_1 \Lambda}{\mu_1 + \frac{\sigma_1^2}{2} + 2 \sqrt{\frac{\Lambda \sigma_1 \sigma_2}{1 - p}} + 2 \sqrt{\frac{\Lambda \sigma_2^2}{1 - p}}} (\mu_2 + \gamma + \frac{\sigma_2^2}{2} + 2 \sqrt{\frac{\Lambda \sigma_2^2}{1 - p}} \frac{\delta y}{\mu_3 + \delta + \frac{\sigma_1^2}{2}}).
\]
Define a C^2-function \( \bar{V} : \mathbb{R}^3_+ \rightarrow \mathbb{R} \) in the following form
\[
\bar{V}(S, I, R) = M V_3(S, I, R) - \ln S - \ln R + V_4(S, I, R),
\]
where \( M > 0 \) is a sufficiently large number satisfying the following condition
\[
-M \left( \mu_2 + \gamma + \frac{\sigma_2^2}{2} + 2 \sqrt{\frac{\Lambda \sigma_2^2}{1 - p}} \frac{\delta y}{\mu_3 + \delta + \frac{\sigma_1^2}{2}} (R^2_0(p) - 1) + J_1 \right) \leq -3
\]
(2.11)
and
\[
J_1 = \sup_{(S, I, R) \in \mathbb{R}^3_+} \left\{ -\frac{1}{4} p \sigma_1^2 + \frac{1}{4} p \sigma_2^2 + \frac{1}{4} p \sigma_3^2 \right\} (\mu_2 + \gamma + \frac{\sigma_2^2}{2} + 2 \sqrt{\frac{\Lambda \sigma_2^2}{1 - p}} \frac{\delta y}{\mu_3 + \delta + \frac{\sigma_1^2}{2}}) R + \mu_1 + \mu_3 + \delta + \sigma_2 \sigma_2 \sigma_2 \sigma_2 \sigma_2 \Lambda + \frac{\sigma_2^2}{2} + \frac{\sigma_1^2}{2}
\]
< \infty.
In addition, notice that \( \bar{V}(S, I, R) \) is not only continuous, but also tends to \( \infty \) as \( (S, I, R) \) approaches the boundary of \( \mathbb{R}^3_+ \). So it should be lower bounded and achieve this lower bound at a point \((S_0, I_0, R_0)\) in the interior of \( \mathbb{R}^3_+ \). Then we define a C^2-function \( V : \mathbb{R}^3_+ \rightarrow \mathbb{R}_+ \) as follows
\[
V(S, I, R) = M V_3(S, I, R) - \ln S - \ln R + V_4(S, I, R) - \bar{V}(S_0, I_0, R_0).
\]
From (2.1), (2.3), (2.9) and (2.10) it follows that
\[
IV \leq -M \left( \mu_2 + \gamma + \frac{\sigma_2^2}{2} + 2 \sqrt{\frac{\Lambda \sigma_2^2}{1 - p}} \frac{\delta y}{\mu_3 + \delta + \frac{\sigma_1^2}{2}} (R^2_0(p) - 1) + M \beta_2 SL + M \left( \frac{c_2 c_3^{-1} y}{\mu_3 + \delta} + d_2 \beta_1 \right)
\]
\[
+ d_3 \sigma_3 \sigma_3 \sigma_3 \sigma_3 \gamma + d_3 \sigma_3 \sigma_3 \sigma_3 \sigma_3 \gamma + \sigma_2 \sigma_2 \gamma \right) I - \frac{\Lambda}{S} - \frac{\gamma I}{R} - \frac{1}{2} p \sigma_1^2 p \sigma_2^2 + \frac{1}{2} p \sigma_2^2 p \sigma_2^2 + \frac{1}{2} p \sigma_2^2 p \sigma_2^2
\]
\[
+ \frac{\sigma_1^2}{2} R^2 + \frac{\sigma_2^2}{2} \right) R + \sigma_2 \sigma_2 \sigma_2 \sigma_2 \sigma_3 \sigma_3 \sigma_3 \sigma_3 \sigma_3 \gamma + \sigma_2 \sigma_2 \gamma \right) R + \mu_1 + \mu_3 + \delta + \sigma_2 \sigma_2 \sigma_2 \sigma_2 \sigma_2 \Lambda + \frac{\sigma_2^2}{2} + \frac{\sigma_1^2}{2}
\]
\[
+ \delta + \sigma_2 \sigma_2 \sigma_2 \sigma_2 \sigma_2 \Lambda + \frac{\sigma_2^2}{2} + \frac{\sigma_1^2}{2}.
\]
(2.12)
Now we are in the position to construct a bounded open domain \( U_\epsilon \) such that the condition \( A_2 \) in Lemma 2.1 holds. Define a bounded open set as follows
\[
U_\epsilon = \left\{ (S, I, R) \in \mathbb{R}^3_+ : \epsilon < S < \frac{1}{\epsilon^2}, \epsilon^2 < I < \frac{1}{\epsilon^2}, \epsilon^3 < R < \frac{1}{\epsilon^2} \right\}
\]
where \( 0 < \epsilon < 1 \) is a small enough number. In the set \( \mathbb{R}^3_+ \setminus U_\epsilon \), we can select \( \epsilon \) small enough such that the following conditions hold
\[
\frac{\Lambda}{\epsilon} + J_2 \leq -1,
\]
(2.13)
\[
\frac{\sigma_1^2 (1 - p)}{4 \epsilon p} + J_2 \leq -1.
\]
(2.14)
\[
\epsilon \leq \frac{m}{\epsilon B_2}.
\]
(2.15)
where $J_2$ is a positive constant which will be given explicitly in expression (2.21). For the sake of convenience, we can divide $\mathbb{R}^3_+ \setminus U_\epsilon$ into six domains,

\[
U_1 = \{(S, I, R) \in \mathbb{R}^3_+ : S \leq \epsilon\}, \quad U_2 = \{(S, I, R) \in \mathbb{R}^3_+ : S \geq \frac{1}{\epsilon}\}, \quad U_3 = \{(S, I, R) \in \mathbb{R}^3_+ : S < \frac{1}{\epsilon}, \ I \leq \epsilon^2\}.
\]

\[
U_4 = \{(S, I, R) \in \mathbb{R}^3_+ : I > \epsilon^2, R \leq \epsilon^3\}, \quad U_5 = \{(S, I, R) \in \mathbb{R}^3_+ : I \geq \frac{1}{\epsilon^2}\}, \quad U_6 = \{(S, I, R) \in \mathbb{R}^3_+ : R \geq \frac{1}{\epsilon^3}\}.
\]

Evidently, $U_\epsilon = \mathbb{R}^3 \setminus U_\epsilon = U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5 \cup U_6$. Next, we will show that $LV = -1$ for any $(S, I, R) \in U_\epsilon$, which is equivalent to verifying it on the above six domains, respectively.

Case 1. For any $(S, I, R) \in U_1$, in view of (2.12), we obtain

\[
LV \leq -\frac{\Lambda}{S} + \frac{\beta_2 S}{m} SI + M\left(\frac{c_2 c_3^{-1} \delta \gamma}{\mu_3 + \delta} + d_2 \beta_1 + \frac{d_1 d_3 c_2 \sigma_{31}^{p-1} Y + \sigma_{21} \sigma_{32}}{\mu_3 + \delta} + d_1 d_3 c_2 \sigma_{31}^{p-1} Y + \sigma_{21} \sigma_{32}\right)I - \frac{1 - p}{4} \sigma_{31}^{p-1} S^{p+2} - \frac{1 - p}{4 \sigma_{32}^{p-1} S^{p+2}} - \frac{1 - p}{4 \sigma_{32}^{p-1} S^{p+2}} - \frac{\sigma_{31}^2 S^2}{2} + \frac{\sigma_{32}^2 S^2}{2} + \sigma_{21}^2 S + (\beta_1 + \sigma_{32} \sigma_{31}^{p-1} Y) I + (\sigma_{31} \sigma_{32} + \sigma_{21} \sigma_{31}^{p-1} S) R + \mu_1 + \mu_3 + \delta + \sigma_{11} \sigma_{12}^{-1} \Lambda + \frac{\sigma_{32}^2}{2} + \frac{\sigma_{31}^2}{2}
\]

\[
\leq -\frac{\Lambda}{S} + J_2
\]

\[
\leq -\frac{\Lambda}{\epsilon} + J_2
\]

\[
\leq -1.
\]

which follows from (2.13) and

\[
J_2 := \sup_{(S, I, R) \in U_1} \left\{\frac{\beta_2 S}{m} SI + M\left(\frac{c_2 c_3^{-1} \delta \gamma}{\mu_3 + \delta} + d_2 \beta_1 + \frac{d_1 d_3 c_2 \sigma_{31}^{p-1} Y + \sigma_{21} \sigma_{32}}{\mu_3 + \delta} + d_1 d_3 c_2 \sigma_{31}^{p-1} Y + \sigma_{21} \sigma_{32}\right)I - \frac{1 - p}{4} \sigma_{31}^{p-1} S^{p+2} - \frac{1 - p}{4 \sigma_{32}^{p-1} S^{p+2}} - \frac{1 - p}{4 \sigma_{32}^{p-1} S^{p+2}} - \frac{\sigma_{31}^2 S^2}{2} + \frac{\sigma_{32}^2 S^2}{2} + \sigma_{21}^2 S + (\beta_1 + \sigma_{32} \sigma_{31}^{p-1} Y) I + (\sigma_{31} \sigma_{32} + \sigma_{21} \sigma_{31}^{p-1} S) R + \mu_1 + \mu_3 + \delta + \sigma_{11} \sigma_{12}^{-1} \Lambda + \frac{\sigma_{32}^2}{2} + \frac{\sigma_{31}^2}{2}\right\}
\]

\[
< \infty.
\]

Case 2. For any $(S, I, R) \in U_2$, from (2.12) it follows that

\[
LV \leq -\frac{1 - p}{4} \sigma_{31}^{p-1} S^{p+2} + \frac{\beta_2 S}{m} SI + M\left(\frac{c_2 c_3^{-1} \delta \gamma}{\mu_3 + \delta} + d_2 \beta_1 + \frac{d_1 d_3 c_2 \sigma_{31}^{p-1} Y + \sigma_{21} \sigma_{32}}{\mu_3 + \delta} + d_1 d_3 c_2 \sigma_{31}^{p-1} Y + \sigma_{21} \sigma_{32}\right)I - \frac{1 - p}{4} \sigma_{31}^{p-1} S^{p+2} - \frac{1 - p}{4 \sigma_{32}^{p-1} S^{p+2}} - \frac{1 - p}{4 \sigma_{32}^{p-1} S^{p+2}} - \frac{\sigma_{31}^2 S^2}{2} + \frac{\sigma_{32}^2 S^2}{2} + \sigma_{21}^2 S + (\beta_1 + \sigma_{32} \sigma_{31}^{p-1} Y) I + (\sigma_{31} \sigma_{32} + \sigma_{21} \sigma_{31}^{p-1} S) R + \mu_1 + \mu_3 + \delta + \sigma_{11} \sigma_{12}^{-1} \Lambda + \frac{\sigma_{32}^2}{2} + \frac{\sigma_{31}^2}{2}
\]

\[
\leq -\frac{1 - p}{4} \sigma_{31}^{p-1} S^{p+2} + J_2
\]

\[
\leq -\frac{\sigma_{31}^2}{4} \left(1 - p\right) + J_2
\]

\[
\leq -1.
\]

which follows from (2.14).
Case 3. For any \((S, I, R) \in U_3\), by (2.12), we have
\[
IV \leq -M\left(\mu_2 + \gamma + \frac{\sigma_2^3}{2} + 2\sqrt{\frac{\Lambda^2 \sigma_2^2}{(1-p)^2} - \frac{\delta \gamma}{\mu_3 + \delta + \sigma_2^3}}\right)(R_5^2(p) - 1) + \frac{M\beta_2}{m}\sigma_1 \delta + M\left(\frac{c_2}{m} \frac{1}{\delta \gamma} \sigma_2^2 \right)
+ d_1 \sigma_1 \sigma_2 \gamma + d_3 \sigma_3 \sigma_2 \gamma + (\gamma + \sigma_2 \gamma + \sigma_2 \gamma) I - \frac{1-p}{4p} \sigma_{p+2}^2 \sigma_{p+2}^2 - \frac{1-p}{4p} \sigma_{p+2}^2 \sigma_{p+2}^2 - \frac{1-p}{4p} \sigma_{p+2}^2 \sigma_{p+2}^2
+ \frac{\sigma_1^2}{2} \sigma_2^2 R + \sigma_3 \sigma_2 \gamma + (\beta_1 + \sigma_2 \sigma_2 \gamma) I + (\sigma_3 \sigma_2 + \sigma_3 \sigma_2 \gamma) R + \mu_1 + \mu_3
+ \delta + \sigma_1 \sigma_2 \gamma \Lambda + \sigma_1^2 + \frac{\sigma_3^2}{2}
\leq -M\left(\mu_2 + \gamma + \frac{\sigma_2^3}{2} + 2\sqrt{\frac{\Lambda^2 \sigma_2^2}{(1-p)^2} - \frac{\delta \gamma}{\mu_3 + \delta + \sigma_2^3}}\right)(R_5^2(p) - 1) + \frac{M\beta_2}{m}\sigma_1 \delta + M\left(\frac{c_2}{m} \frac{1}{\delta \gamma} \sigma_2^2 \right)
+ d_1 \sigma_1 \sigma_2 \gamma + d_3 \sigma_3 \sigma_2 \gamma + (\gamma + \sigma_2 \gamma + \sigma_2 \gamma) I - \frac{1-p}{4p} \sigma_{p+2}^2 \sigma_{p+2}^2 - \frac{1-p}{4p} \sigma_{p+2}^2 \sigma_{p+2}^2 - \frac{1-p}{4p} \sigma_{p+2}^2 \sigma_{p+2}^2
+ \frac{\sigma_1^2}{2} \sigma_2^2 R + \sigma_3 \sigma_2 \gamma + (\beta_1 + \sigma_2 \sigma_2 \gamma) I + (\sigma_3 \sigma_2 + \sigma_3 \sigma_2 \gamma) R + \mu_1 + \mu_3
+ \delta + \sigma_1 \sigma_2 \gamma \Lambda + \sigma_1^2 + \frac{\sigma_3^2}{2}
\leq -3 + 1 + 1
= -1.\] (2.23)

which follows from (2.11), (2.15) and (2.16) and

\[
J_1 = \sup_{(S, I, R) \in U_1} \left\{ \frac{1 - p}{4} \sigma_1^2 \sigma_{p+2}^2 - \frac{1 - p}{4} \sigma_2^2 \sigma_{p+2}^2 \right\}
+ (\sigma_3 \sigma_2 + \sigma_3 \sigma_2 \gamma) R + \mu_1 + \mu_3 + \delta + \sigma_1 \sigma_2 \gamma \Lambda + \sigma_1^2 + \frac{\sigma_3^2}{2}
< \infty.
\]

Case 4. For any \((S, I, R) \in U_4\), according to (2.12), we get
\[
IV \leq -M \frac{1}{R} \frac{\beta_2}{m} \sigma_1 \delta + M\left(\frac{c_2}{m} \frac{1}{\delta \gamma} \sigma_2^2 \right)
+ \frac{d_1}{\mu_3 + \delta} \sigma_1 \sigma_2 \gamma + \frac{d_3}{\mu_3 + \delta} \sigma_3 \sigma_2 \gamma + (\gamma + \sigma_2 \gamma + \sigma_2 \gamma) I - \frac{1-p}{4p} \sigma_{p+2}^2 \sigma_{p+2}^2
+ \frac{\sigma_1^2}{2} \sigma_2^2 R + \sigma_3 \sigma_2 \gamma + (\beta_1 + \sigma_2 \sigma_2 \gamma) I + (\sigma_3 \sigma_2 + \sigma_3 \sigma_2 \gamma) R + \mu_1 + \mu_3 + \delta + \sigma_1 \sigma_2 \gamma \Lambda + \sigma_1^2 + \frac{\sigma_3^2}{2}
\leq -M \frac{1}{R} \frac{\beta_2}{m} \sigma_1 \delta + M\left(\frac{c_2}{m} \frac{1}{\delta \gamma} \sigma_2^2 \right)
+ \frac{d_1}{\mu_3 + \delta} \sigma_1 \sigma_2 \gamma + \frac{d_3}{\mu_3 + \delta} \sigma_3 \sigma_2 \gamma + (\gamma + \sigma_2 \gamma + \sigma_2 \gamma) I - \frac{1-p}{4p} \sigma_{p+2}^2 \sigma_{p+2}^2
+ \frac{\sigma_1^2}{2} \sigma_2^2 R + \sigma_3 \sigma_2 \gamma + (\beta_1 + \sigma_2 \sigma_2 \gamma) I + (\sigma_3 \sigma_2 + \sigma_3 \sigma_2 \gamma) R + \mu_1 + \mu_3 + \delta + \sigma_1 \sigma_2 \gamma \Lambda + \sigma_1^2 + \frac{\sigma_3^2}{2}
\leq -M \frac{1}{R} \frac{\beta_2}{m} \sigma_1 \delta + M\left(\frac{c_2}{m} \frac{1}{\delta \gamma} \sigma_2^2 \right)
+ \frac{d_1}{\mu_3 + \delta} \sigma_1 \sigma_2 \gamma + \frac{d_3}{\mu_3 + \delta} \sigma_3 \sigma_2 \gamma + (\gamma + \sigma_2 \gamma + \sigma_2 \gamma) I - \frac{1-p}{4p} \sigma_{p+2}^2 \sigma_{p+2}^2
+ \frac{\sigma_1^2}{2} \sigma_2^2 R + \sigma_3 \sigma_2 \gamma + (\beta_1 + \sigma_2 \sigma_2 \gamma) I + (\sigma_3 \sigma_2 + \sigma_3 \sigma_2 \gamma) R + \mu_1 + \mu_3 + \delta + \sigma_1 \sigma_2 \gamma \Lambda + \sigma_1^2 + \frac{\sigma_3^2}{2}
\leq -1.\] (2.24)

which follows from (2.17).

Case 5. For any \((S, I, R) \in U_5\), by (2.12), we derive
\[
IV \leq -\frac{1-p}{4} \sigma_{p+2}^2 \sigma_{p+2}^2 + \frac{M\beta_2}{m} \sigma_1 \delta + M\left(\frac{c_2}{m} \frac{1}{\delta \gamma} \sigma_2^2 \right)
+ \frac{d_1}{\mu_3 + \delta} \sigma_1 \sigma_2 \gamma + \frac{d_3}{\mu_3 + \delta} \sigma_3 \sigma_2 \gamma + (\gamma + \sigma_2 \gamma + \sigma_2 \gamma) I - \frac{1-p}{4p} \sigma_{p+2}^2 \sigma_{p+2}^2
+ \frac{\sigma_1^2}{2} \sigma_2^2 R + \sigma_3 \sigma_2 \gamma + (\beta_1 + \sigma_2 \sigma_2 \gamma) I + (\sigma_3 \sigma_2 + \sigma_3 \sigma_2 \gamma) R + \mu_1 + \mu_3 + \delta + \sigma_1 \sigma_2 \gamma \Lambda + \sigma_1^2 + \frac{\sigma_3^2}{2}
\leq -\frac{1-p}{4} \sigma_{p+2}^2 \sigma_{p+2}^2 + \frac{M\beta_2}{m} \sigma_1 \delta + M\left(\frac{c_2}{m} \frac{1}{\delta \gamma} \sigma_2^2 \right)
+ \frac{d_1}{\mu_3 + \delta} \sigma_1 \sigma_2 \gamma + \frac{d_3}{\mu_3 + \delta} \sigma_3 \sigma_2 \gamma + (\gamma + \sigma_2 \gamma + \sigma_2 \gamma) I - \frac{1-p}{4p} \sigma_{p+2}^2 \sigma_{p+2}^2
+ \frac{\sigma_1^2}{2} \sigma_2^2 R + \sigma_3 \sigma_2 \gamma + (\beta_1 + \sigma_2 \sigma_2 \gamma) I + (\sigma_3 \sigma_2 + \sigma_3 \sigma_2 \gamma) R + \mu_1 + \mu_3 + \delta + \sigma_1 \sigma_2 \gamma \Lambda + \sigma_1^2 + \frac{\sigma_3^2}{2}
\leq -1.\] (2.25)

which follows from (2.18).
Case 6. For any \((S, I, R) \in U_6\), from (2.12) it follows that

\[
LV \leq - \frac{1}{4} p \sigma_{11}^{p-2} S R^{p+2} + \frac{M_1}{
}\]

\[
\left(2 \gamma \frac{\sigma_{21}}{\mu_3 + \delta} \right) + d_2 \beta_1 + \frac{d_3 \gamma \sigma_{12} \gamma}{\mu_3 + \delta} + d_4 \sigma_{12} \sigma_{21} \gamma + \sigma_{21} \sigma_{22} \right)
\]

\[
- \frac{1}{4} p \sigma_{11}^{p-2} S R^{p+2} - \frac{1}{4} p \sigma_{22}^{p-2} R^{p+2} - \frac{1}{4} p \sigma_{22}^{p-2} R^{p+2} + \frac{\sigma_{21}^2 \gamma^2}{2} + \frac{\sigma_{21}^2 \gamma^2}{2} R^2 + \sigma_{22} \sigma_{21} \mu_1 S + \sigma_{11} \sigma_{12} \gamma
\]

\[
+ \left( \beta_1 + \sigma_{31} \sigma_{21} \right) R + \left( \sigma_{31} \sigma_{32} + \sigma_{22} \sigma_{21} \mu_1 \right) + \sigma_{11} \sigma_{12} \gamma + \frac{\sigma_{21}^2 \gamma^2}{2} + \frac{\sigma_{21}^2 \gamma^2}{2}
\]

\[
\leq \frac{1}{4} p \sigma_{32}^{p-2} R^{p+2} + f_2
\]

\[
\leq \frac{\sigma_{32} \gamma^2 \left( 1 - p \right)}{4 \epsilon \left( p + 2 \right)} + f_2
\]

\[
\leq -1.
\]

(2.26)

which follows from (2.19).

Accordingly, from (2.20), (2.22), (2.23), (2.24), (2.25) and (2.26) it follows that for a small enough \(\epsilon\),

\[
LV \leq -1 \quad \text{for all } (S, I, R) \in \mathbb{R}_+^3 \setminus U_6
\]

Hence the condition \(A_2\) of Lemma 2.1 also holds. By Lemma 2.1, we derive that system (1.2) admits a unique stationary distribution \(\pi(\cdot)\) and it has the ergodic property. This completes the proof. \(\square\)

Remark 2.1. From the expression of \(R_0^-\), we can obtain that if there is no stochastic perturbation in system (1.2), then \(R_0^- = R_0\), so \(R_0^- > 1\) is a generalized result determining the persistence of the disease. In addition, if we only consider the linear perturbation, i.e., \(\sigma_{11} = \sigma_{22} = \sigma_2 = 0\), then

\[
R_0^- := \frac{\beta_1 \Lambda}{\left( \mu_1 + \sigma_{21}^3 \right) \left( \mu_2 + \gamma \right) + \sigma_{21}^3 \left( \frac{\delta}{\mu_1 + \delta} \right) - \frac{\sigma_{21}^3 \gamma^3}{\mu_1 + \delta} + \frac{\sigma_{21}^3 \gamma^3}{\mu_1 + \delta}}.
\]

Therefore, \(R_0^- > 1\) can be regarded as a generalized result of Caraballo et al. [10].

3. Extinction

In this section, we will obtain sufficient criteria for extinction of the disease. To this end, we establish the following theorem.

Theorem 3.1. Let \((S(t), I(t), R(t))\) be the solution to system (1.2) with any initial value \(S_0 > 0\), \(I_0 \geq 0\), \(R_0 \geq 0\), then for almost \(\omega \in \Omega\), the solution has the following property:

\[
\limsup_{t \to \infty} \mathbb{E} \left[ \int_0^t \left( \frac{\omega_1}{\mu_2 + \gamma} - \frac{\omega_2}{\mu_3 + \delta} \right) R(t) \right] \leq v \quad \text{a.s.,}
\]

where

\[
R_0 = \frac{\beta_1 \Lambda}{\mu_1 (\mu_2 + \gamma - \frac{\delta \gamma}{\mu_1 + \delta})}, \quad \omega_1 = \frac{\gamma}{\mu_3 + \delta}, \quad \omega_2 = \frac{\delta \gamma}{(\mu_3 + \delta) (\mu_2 + \gamma - \frac{\delta \Lambda}{\mu_1})},
\]

\[
v := \beta_1 \int_0^\infty \left( \mu_2 + \gamma - \frac{\delta \Lambda}{\mu_1}, \mu_3 + \delta \right) \left( \frac{\delta \gamma}{(\mu_3 + \delta) (\mu_2 + \gamma - \frac{\delta \Lambda}{\mu_1})} - 1 \right) 1_{\{\sigma_2 < 1\}}
\]

In particular, if \(v < 0\), then the disease \(I\) will die out exponentially with probability one, i.e.,

\[
\lim_{t \to \infty} I(t) = 0 \quad \text{a.s.}
\]

In addition, the distribution of \(S(t)\) converges weakly to the measure which has the density

\[
\pi(x) = \frac{x^{2 - \frac{2 \sigma_{11} \sigma_{12} \gamma (2 + \sigma_{11} \sigma_{12} \gamma)}{\mu_1 + \sigma_{11} \sigma_{12} \gamma}} e^{-\frac{2 \sigma_{11} \sigma_{12} \gamma (2 + \sigma_{11} \sigma_{12} \gamma)}{\mu_1 + \sigma_{11} \sigma_{12} \gamma}}}{\mathbb{Q}} \quad x \in (0, \infty),
\]

where \(\mathbb{Q}\) is a constant such that \(\int_0^\infty \pi(x) dx = 1\).

Proof. Consider the following auxiliary logistic equation with stochastic perturbation

\[
\frac{dX}{dt} = \left[ \Lambda - \mu_1 X \right] dt + \left( \sigma_{11} X + \sigma_{12} \right) X dB_1(t)
\]

with initial value \(X_0 = S_0 > 0\).

From Theorem 3.1 of Liu and Jiang [28], we can obtain that system (3.2) has the ergodic property and the invariant density is given by

\[
\pi(x) = \frac{x^{- \frac{2 \sigma_{11} \sigma_{12} \gamma (2 + \sigma_{11} \sigma_{12} \gamma)}{\mu_1 + \sigma_{11} \sigma_{12} \gamma}} e^{-\frac{2 \sigma_{11} \sigma_{12} \gamma (2 + \sigma_{11} \sigma_{12} \gamma)}{\mu_1 + \sigma_{11} \sigma_{12} \gamma}}}{\mathbb{Q}} \quad x \in (0, \infty),
\]
where Q is a constant such that
\[ \int_0^\infty \pi(x)dx = 1. \]
Let \( X(t) \) be the solution to (3.2) with initial value \( X_0 = S_0 > 0 \), then employing the comparison theorem of 1-dimensional stochastic differential equation [35], we obtain \( S(t) \leq X(t) \) for any \( t \geq 0 \) a.s.

On the other hand, by Theorem 1.4 of [36], p.27, we can derive that there exists a left eigenvector of
\[ M_0 = \begin{pmatrix} 0 & \delta \\ \gamma & \mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1} \\ \frac{1}{\mu_3 + \delta} & 0 \end{pmatrix} \]
corresponding to \( \rho(M_0) = \sqrt{\frac{\delta \gamma}{(\mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1})}} \) (the spectral radius of \( M_0 \)), which is denoted as \((\omega_1, \omega_2) = \left( \frac{\gamma - \delta}{\mu_1}, \sqrt{\frac{\delta \gamma}{(\mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1})}} \right)\).

Define a \( C^2 \)-function \( \bar{V} : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+ \) by
\[ \bar{V}(l, R) = \alpha_1 l + \alpha_2 R, \]
where \( \alpha_1 = \frac{\omega_1}{\mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1}} \) and \( \alpha_2 = \frac{\omega_2}{\mu_1} \). Applying Itô's formula [29] to \( \ln \bar{V} \), we obtain
\[
\begin{align*}
\frac{d(\ln \bar{V})}{\bar{V}} &= \frac{\alpha_1 (\sigma_{21} + \sigma_{22} l)I}{\bar{V}} dB_2(t) + \frac{\alpha_2 (\sigma_{31} + \sigma_{32} R)}{\bar{V}} dB_3(t),
\end{align*}
\]
where
\[
L(\ln \bar{V}) = \frac{\alpha_1}{\bar{V}} \left[ \left( \beta_1 I - \beta_2 l I \right) I - (\mu_2 + \gamma) I + \delta R \right] + \frac{\alpha_2}{\bar{V}} \left[ 2 \gamma I - (\mu_3 + \delta) R \right] - \frac{\alpha_1^2 (\sigma_{21} + \sigma_{22} l)^2 I^2}{2 \bar{V}^2} - \frac{\alpha_2^2 (\sigma_{31} + \sigma_{32} R)^2 R^2}{2 \bar{V}^2}.
\]
In addition, by Cauchy inequality, we have
\[
\bar{V}^2 = \left( \alpha_1 \sigma_{21} l \frac{1}{\sigma_{21}} + \alpha_2 \sigma_{31} R \frac{1}{\sigma_{31}} \right)^2 \leq \left( \alpha_1^2 \sigma_{21}^2 l^2 + \alpha_2^2 \sigma_{31}^2 R^2 \right) \left( \frac{1}{\sigma_{21}^2} + \frac{1}{\sigma_{31}^2} \right).
\]
(3.4)

Therefore
\[
L(\ln \bar{V}) \leq \beta_1 \left| X - \frac{\Delta}{\mu_1} \right| + \min \left\{ \mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1}, \mu_3 + \delta \right\} \left( \sqrt{\frac{\delta \gamma}{(\mu_3 + \delta)(\mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1})}} - 1 \right) 1_{\{\tau_0 \leq 1\}}
\]
\[+ \max \left\{ \mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1}, \mu_3 + \delta \right\} \left( \sqrt{\frac{\delta \gamma}{(\mu_3 + \delta)(\mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1})}} - 1 \right) 1_{\{\tau_0 > 1\}} - \frac{1}{2(\sigma_{21}^2 + \sigma_{31}^2)} \frac{\alpha_1^2 \sigma_{21}^2 l^4}{2V^2} \frac{\alpha_2^2 \sigma_{31}^2 R^4}{2V^2}.
\]
(3.5)

In view of (3.3), (3.4) and (3.5), we obtain
\[
ds (\ln \bar{V}(t)) \leq \left[ \beta_1 \left| X - \frac{\Delta}{\mu_1} \right| + \min \left\{ \mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1}, \mu_3 + \delta \right\} \left( \sqrt{\frac{\delta \gamma}{(\mu_3 + \delta)(\mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1})}} - 1 \right) 1_{\{\tau_0 \leq 1\}}
\]
\[+ \max \left\{ \mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1}, \mu_3 + \delta \right\} \left( \sqrt{\frac{\delta \gamma}{(\mu_3 + \delta)(\mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1})}} - 1 \right) 1_{\{\tau_0 > 1\}} - \frac{1}{2(\sigma_{21}^2 + \sigma_{31}^2)} \frac{\alpha_1^2 \sigma_{21}^2 l^4}{2V^2} \]  
\[- \frac{\alpha_2^2 \sigma_{31}^2 R^4}{2V^2} \right\} dt + \frac{\alpha_1(\sigma_{21} + \sigma_{22})l}{V} d\mathcal{B}_2(t) + \frac{\alpha_2(\sigma_{31} + \sigma_{32})R}{V} d\mathcal{B}_3(t).
\]
(3.6)

Integrating (3.6) from 0 to t and then dividing by t on both sides, we get
\[
\frac{\ln \bar{V}(t)}{t} \leq \frac{\ln \bar{V}(0)}{t} + \min \left\{ \mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1}, \mu_3 + \delta \right\} \left( \sqrt{\frac{\delta \gamma}{(\mu_3 + \delta)(\mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1})}} - 1 \right) 1_{\{\tau_0 \leq 1\}}
\]
\[+ \max \left\{ \mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1}, \mu_3 + \delta \right\} \left( \sqrt{\frac{\delta \gamma}{(\mu_3 + \delta)(\mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1})}} - 1 \right) 1_{\{\tau_0 > 1\}} - \frac{1}{2(\sigma_{21}^2 + \sigma_{31}^2)} \frac{\alpha_1^2 \sigma_{21}^2 l^4}{2V^2} \]  
\[- \frac{\alpha_2^2 \sigma_{31}^2 R^4}{2V^2} \right\} dt + \frac{\alpha_1(\sigma_{21} + \sigma_{22})l}{V} d\mathcal{B}_2(t) + \frac{\alpha_2(\sigma_{31} + \sigma_{32})R}{V} d\mathcal{B}_3(t).
\]
(3.7)

where

\[M_1(t) := \int_0^t \frac{\alpha_1 \sigma_{21} l(s)}{V(s)} d\mathcal{B}_2(s), \quad M_2(t) := \int_0^t \frac{\alpha_2 \sigma_{31} R(s)}{V(s)} d\mathcal{B}_3(s).\]

\[M_3(t) := \int_0^t \frac{\alpha_1 \sigma_{22} l^2(s)}{V(s)} d\mathcal{B}_2(s), \quad M_4(t) := \int_0^t \frac{\alpha_2 \sigma_{32} R^2(s)}{V(s)} d\mathcal{B}_3(s)\]

are local martingales whose quadratic variations are respectively

\[\langle M_1, M_1 \rangle(t) = \int_0^t \left( \frac{\alpha_1 \sigma_{21} l(s)}{V(s)} \right)^2 ds \leq \sigma_{21}^2 l^2 t, \quad \langle M_2, M_2 \rangle(t) = \int_0^t \left( \frac{\alpha_2 \sigma_{31} R(s)}{V(s)} \right)^2 ds \leq \sigma_{31}^2 R^2 t.\]

\[\langle M_3, M_3 \rangle(t) = \int_0^t \left( \frac{\alpha_1 \sigma_{22} l^2(s)}{V(s)} \right)^2 ds \quad \text{and} \quad \langle M_4, M_4 \rangle(t) = \int_0^t \left( \frac{\alpha_2 \sigma_{32} R^2(s)}{V(s)} \right)^2 ds.\]

From the strong large numbers theorem it follows that

\[\lim_{t \to \infty} \frac{M_i(t)}{t} = 0 \quad \text{a.s.,} \quad i = 1, 2.\]
(3.8)
Furthermore, in view of the exponential martingales inequality [29], for any positive constants $T, \alpha$ and $\beta$, we have
\[
P(\sup_{0 \leq t \leq T} [M_j(t) - \frac{\alpha}{2} \langle M_j, M_j \rangle(t)] > \beta) \leq e^{-\alpha \beta}, \quad j = 3, 4.
\]
Choose $T = k$, $\alpha = 1$, $\beta = 2 \ln k$, we obtain
\[
P(\sup_{0 \leq t \leq k} [M_j(t) - \frac{1}{2} \langle M_j, M_j \rangle(t)] > 2 \ln k) \leq \frac{1}{k^2}, \quad j = 3, 4.
\]
By the Borel-Cantelli Lemma [29], we obtain that for almost all $\omega \in \Omega$, there is a random integer $k_0 = k_0(\omega)$ such that for $k \geq k_0$, we derive
\[
\sup_{0 \leq t \leq k} [M_j(t) - \frac{1}{2} \langle M_j, M_j \rangle(t)] \leq 2 \ln k, \quad j = 3, 4.
\]
That is
\[
M_3(t) \leq 2 \ln k + \frac{1}{2} \langle M_3, M_3 \rangle(t) = 2 \ln k + \frac{1}{2} \int_0^t \left( \frac{\alpha_1 \sigma_2 \tilde{S}(s)}{\tilde{V}(s)} \right)^2 ds
\]
and
\[
M_4(t) \leq 2 \ln k + \frac{1}{2} \langle M_4, M_4 \rangle(t) = 2 \ln k + \frac{1}{2} \int_0^t \left( \frac{\alpha_2 \sigma_2 \tilde{S}(s)}{\tilde{V}(s)} \right)^2 ds
\]
for all $0 \leq t \leq k$, $k \geq k_0 \text{ a.s.}$ Substituting (3.9) and (3.10) into (3.7) leads to
\[
\frac{\ln \tilde{V}(t)}{t} \leq \frac{\ln \tilde{V}(0)}{t} + \min \left\{ \mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1}, \mu_3 + \delta \right\} \left( \sqrt{\frac{\delta \gamma}{(\mu_3 + \delta)(\mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1})}} - 1 \right) 1_{\{|\tilde{R}_3| > 1\}} - \frac{1}{2(\sigma_2^2 + \sigma_3^2)}
\]
\[
+ \max \left\{ \mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1}, \mu_3 + \delta \right\} \left( \sqrt{\frac{\delta \gamma}{(\mu_3 + \delta)(\mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1})}} - 1 \right) 1_{\{|\tilde{R}_3| > 1\}} + \frac{\beta_1}{\mu_1} \int_0^t |X(s) - \Lambda| ds
\]
\[
+ \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{4 \ln k}{t} + \frac{4 \ln k}{k - 1}
\]
for all $0 \leq t \leq k$, $k \geq k_0 \text{ a.s.}$ In other words, we have shown that for $0 \leq k - 1 \leq t \leq k$,
\[
\frac{\ln \tilde{V}(t)}{t} \leq \frac{\ln \tilde{V}(0)}{t} + \min \left\{ \mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1}, \mu_3 + \delta \right\} \left( \sqrt{\frac{\delta \gamma}{(\mu_3 + \delta)(\mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1})}} - 1 \right) 1_{\{|\tilde{R}_3| > 1\}} - \frac{1}{2(\sigma_2^2 + \sigma_3^2)}
\]
\[
+ \max \left\{ \mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1}, \mu_3 + \delta \right\} \left( \sqrt{\frac{\delta \gamma}{(\mu_3 + \delta)(\mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1})}} - 1 \right) 1_{\{|\tilde{R}_3| > 1\}} + \frac{\beta_1}{\mu_1} \int_0^t |X(s) - \Lambda| ds
\]
\[
+ \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{4 \ln k}{t} + \frac{4 \ln k}{k - 1}
\]
(3.11)
Since $X(t)$ is ergodic and $\int_0^\infty \pi(x) dx < \infty$ a.s., we have
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t |X(s) - \frac{\Lambda}{\mu_1}| ds = \int_0^\infty |x - \frac{\Lambda}{\mu_1}| \pi(x) dx.
\]
Taking the superior limit on both sides of (3.11) and combining with (3.8) and (3.12), we obtain
\[
\limsup_{t \to \infty} \frac{\ln \tilde{V}(t)}{t} \leq \beta_1 \int_0^\infty |x - \frac{\Lambda}{\mu_1}| \pi(x) dx + \min \left\{ \mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1}, \mu_3 + \delta \right\} \left( \sqrt{\frac{\delta \gamma}{(\mu_3 + \delta)(\mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1})}} - 1 \right) 1_{\{|\tilde{R}_3| > 1\}}
\]
\[
+ \max \left\{ \mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1}, \mu_3 + \delta \right\} \left( \sqrt{\frac{\delta \gamma}{(\mu_3 + \delta)(\mu_2 + \gamma - \frac{\beta_1 \Lambda}{\mu_1})}} - 1 \right) 1_{\{|\tilde{R}_3| > 1\}} - \frac{1}{2(\sigma_2^2 + \sigma_3^2)}
\]
\[
:= v \text{ a.s.},
\]
which is the required assertion (3.1). In addition, if $v < 0$, we can easily obtain that
\[
\limsup_{t \to \infty} \frac{\ln \tilde{V}(t)}{t} < 0 \text{ and } \limsup_{t \to \infty} \frac{\ln R(t)}{t} < 0 \text{ a.s.},
\]
which implies that $\lim_{t \to \infty} I(t) = 0$ and $\lim_{t \to \infty} R(t) = 0$ a.s. In other words, the disease $I$ will die out exponentially with probability one.

Therefore, for any small $\epsilon > 0$ there are $t_0$ and a set $\Omega_1 \subset \Omega$ such that $\mathbb{P}(\Omega_1) > 1 - \epsilon$ and $-(\beta_1 - \frac{\beta_1}{\mu_1^2})S \geq \beta_1 S > -\beta_1 \epsilon S$ for $t \geq t_0$ and $\omega \in \Omega_1$. Now from
\[
[\Lambda - \mu S - \beta_1 eS]dt + (\sigma_1 S + \sigma_1 S)SdB(t) \leq \Lambda dt + (\sigma_1 S + \sigma_1 S)SdB(t),
\]
it follows that the distribution of the process $S(t)$ converges weakly to the measure with the density $\pi$. This completes the proof. \(\square\)

**Remark 3.1.** From Theorem 3.1, we can derive that if $R_0 > 1$, $\sigma_2^2$ and $\sigma_3^2$ are sufficiently large such that $v < 0$, then the disease will die out exponentially a.s. However, as far as we know, in the deterministic system (1.1), if $R_0 > 1$, then the endemic equilibrium $E^\ast$ is globally asymptotically stable. This shows that the disease will prevail and persist in the long term. Hence our result is very different from the one of the deterministic system (1.1). This shows that large white noise will lead to the eradication of the infectious disease.
Thus,\\[
\nu = \sigma R_0 \left( \frac{4R_0}{\mu + \delta} \right) = 0.1000 < 1
\]

and hence the system is locally asymptotically stable. As there is no unstable manifold, there is no other possible outcome. The disease dies out and the population returns to the disease-free equilibrium.\\

\[v := \int_0^\infty \left| x - \frac{\Lambda}{\mu_1} \right| \pi(x) dx = \frac{\beta_1 \mu_1}{\mu_2 + \gamma} \left( 1 - \frac{\delta \gamma}{(\mu_2 + \gamma) (\mu_2 + \gamma - \beta_1 \mu_1)} \right) \frac{1}{2(\sigma_{21}^2 + \sigma_{31}^2)} < 0\]

Thus, the condition of Theorem 3.1 is satisfied. That is to say, the disease will be extinct a.s. Fig. 2 illustrates this.
Fig. 1. The left column shows the paths of $S(t)$, $I(t)$ and $R(t)$ of system (1.2) with initial values $S_0 = 0.2; I_0 = 0.3; R_0 = 0.1$ under the noise intensities $\sigma_{11} = 0.01, \sigma_{12} = 0.004, \sigma_{21} = 0.09672245, \sigma_{22} = 0.01, \sigma_{23} = 0.1286$ and $\sigma_{32} = 0.08$. Other parameter values are given in Table 1. The blue lines represent the solution to system (1.2) and the red lines represent the solution to the corresponding undisturbed system (1.1). The right column displays the histogram of the probability density functions of $S$, $I$, $R$ populations. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
5. Conclusion

In this paper, we have studied the salient features of a higher order stochastically perturbed SIRI epidemic model with relapse and media coverage. We first establish sufficient criteria for the existence and uniqueness of an ergodic stationary distribution of positive solutions to the stochastic system (1.2) by constructing a suitable stochastic Lyapunov function. Then we make up adequate conditions for complete eradication and wiping out of the infectious disease. In a biological viewpoint, the existence of a stationary distribution implies that the disease will be prevalent and persistent in the long term. Moreover, it is a meaningful topic to study whether or not the method used in this paper can be also applicable to other stochastic multi-dimensional epidemic models, such as rabies transmission model, malaria transmission model and syphilis transmission model. These works could be hopping taken up by the future studies and aptly solved.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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