Optical Spin Transport in Ultracold Quantum Gases

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Measurement of frequency-resolved spin transport is a subject of much interest in condensed matter physics. Here we show that the optical spin conductivity being a small AC response of a spin current can be measured with existing methods in ultracold atom experiments. We point out that once interatomic interactions are turned on, the optical spin conductivity becomes nontrivial even in clean ultracold atomic gases and thereby can be a probe of generic quantum states of matter. This is a sharp contrast to the optical mass conductivity which becomes trivial in typical cold-atom systems without disorder and lattice potential. For systems with arbitrary spin degrees of freedom, we construct a formalism of the optical spin conductivity and derive the f-sum rule. To demonstrate the availability of the optical spin conductivity, our formalism is applied to a spin-1/2 Fermi superfluid and a spin-1 Bose-Einstein condensate. It turns out that both superfluids show nontrivial responses that cannot be captured with the Drude conductivity. The application of our proposed method to generic ultracold atomic gases with spin degrees of freedom is feasible.

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Introduction—Transport plays crucial roles in understanding states of matter and paves the way to application such as control of matter and device fabrication. In solid state physics, the main bearer of transport is an electron and the properties of the electric current have conventionally been investigated. Subsequently, the spin current being a flow of electric spin has attracted attention since the discovery of the giant magnetoresistance and the tunneling magnetoresistance. More recently, due to the progress in nanofabrication technology of devices, physics in spin currents has also been widespread over materials with spin-Hall effects, and topological insulators.

One of the hot topics in the rapid growth of the so-called spintronics is to measure AC spin currents in a direct manner. Such AC currents are detected in junction systems, and thus the determination of an AC conductivity of bulk spin transport is not easy in solid state systems. To address this transport property of spin, here we shed light on ultracold atoms being an ideal platform for quantum simulation of condensed matter systems. Recently, the cold-atom analog of electronics referred to as atomtronics has attracted widespread attention, and transport measurements with ultracold atoms have been done with bulk and mesoscopic setups. One of the advantages of ultracold atoms is that spin-selective manipulation and probe are allowed, which opens up the possibility of precise measurements of spin transport.

In this Letter, we propose that ultracold atoms provide us with a simple way to measure an optical spin conductivity, which characterizes an AC response of bulk spin transport. It turns out that in controllable ultracold-atom experiments the optical spin conductivity can be extracted without relying on exchange coupling of spins with a local magnetization, spin-orbit coupling, and spin-rotational coupling. In addition, we construct a general formalism of the optical spin conductivity for systems with arbitrary spin degrees of freedom. To demonstrate the availability of the optical spin conductivity, we apply the formalism to a spin-1/2 superfluid Fermi gas and a spin-1 Bose-Einstein condensate (BEC). We find that reflecting on nontrivial spin-excitation properties the optical spin conductivities in such superfluids cannot be captured with the Drude conductivity. The application of the interaction-sensitive probe to other quantum states of matter realized in ultracold atoms is also promising. In what follows, we set \( h = k_B = 1 \).

Optical spin conductivity—We consider a system with spin \( S = 1/2, 1, 3/2, \cdots \) whose Hamiltonian is given by \( H(t) = H_1 + H_{\text{int}} + \delta H(t) \). In the first quantization...
formalism, the single-particle term has the form of $H_1 = \sum_{\{s_{z,i}\}}[p_{s_{z,i}}^2/(2m) + V_{s_z}(r_{s_{z,i}})]$, where $m$ is a mass and labels of particles take $s_z = -S, -S + 1, \cdots, S$ and $i = 1, 2, \cdots, N_{s_z}$ with $N_{s_z}$ being the particle number in the $s_z$ component. Each particle has the coordinate $r_{s_{z,i}}$ and momentum $p_{s_{z,i}}$. The interaction term $H_{\text{int}}$ is given in terms of spin-dependent potentials $U_{s_{z,s_{z}'}}(r)$. The time-dependent perturbation is generated by

$$\delta H(t) = - \int dr \, f(t) \cdot r S_z(r),$$

where $f(t)$ provides a driving force coupled to the spin density $S_z = \sum_{\{s_{z,i}\}} s_z \delta(r - r_{s_{z,i}})$. This perturbation generates a spin current whose corresponding operator is $J_S = \sum_{\{s_{z,i}\}} s_z p_{s_{z,i}}/m$. By the linear response theory, the optical spin conductivity $\sigma_{\alpha\beta}^{(S)}(\omega)$ is given by

$$\langle \tilde{J}_{S,\alpha}(\omega) \rangle_{\text{neq}} = \sigma_{\alpha\beta}^{(S)}(\omega) \tilde{f}_\beta(\omega),$$

where $\alpha$ and $\beta$ denote Cartesian components, $\tilde{J}_{S,\alpha}(\omega)$ and $\tilde{f}_\alpha(\omega)$ are the Fourier transforms of $J_{S,\alpha}(t)$ and $f_\alpha(t)$, respectively, and $\langle \cdots \rangle_{\text{neq}}$ denotes the expectation value with respect to a given nonequilibrium state. We note that $\sigma_{\alpha\beta}^{(S)}(\omega)$ is the response not of a spin current density but of a total spin current.

For ultracold atomic gases, we have several ways to induce the perturbation in Eq. (14). The most straightforward way is to apply a time-dependent gradient of a magnetic field $B(r,t) \propto f(t) \cdot r$ along the $z$ axis \cite{33}. Such a gradient potential can also be produced by the optical Stern-Gerlach effect \cite{35}. In the presence of a harmonic trapping potential $V_{s_z}(r) = m\omega_z^2 r^2/2$, oscillation of the trap center $V_{s_z}(r) \rightarrow V_{s_z}(r - s_z, g(t))$ leads to a spin bias $f(t) \propto g(t)$, which is a spin-dependent extension of the method to generate a mass current \cite{31,36}. Furthermore, ultracold atoms have an advantage to observe the spin current. In the Heisenberg picture, $J_S$ is rewritten as

$$J_S(t) = \sum_{s_z} s_z N_{s_z} \frac{dR_{s_z}(t)}{dt},$$

where $R_{s_z} = \sum_{i=1}^{N_{s_z}} r_{s_{z,i}}/N_{s_z}$ is the spin-resolved center-of-mass coordinate. Since $R_{s_z}(t)$ is observable in ultracold atom experiments \cite{34,37}, $J_S(t)$ can be measured from the dynamics of $R_{s_z}(t)$. Therefore, ultracold atoms allow us to directly observe the frequency dependence of the spin conductivity in Eq. (15).

We point out a difference from the mass current induced by a spin-independent perturbation. In clean cold atomic gases trapped in a box or harmonic potential, the total center-of-mass motion is independent of quantum states of matter due to Kohn’s theorem \cite{38,40}. Thus, a system that breaks prior conditions of Kohn’s theorem such as optical lattice or disordered systems must be prepared to obtain a nontrivial mass response, which has recently been confirmed \cite{31}. In contrast, the relative motion between spin components relevant to the optical spin conductivity can show a nontrivial response, once interatomic interactions are present.

We now formulate the general framework to compute $\sigma_{\alpha\beta}^{(S)}(\omega)$. The optical spin conductivity can be expressed in terms of a current-current correlation function in a similar way as for electric and mass transport. By the Kubo formula, $\sigma_{\alpha\beta}^{(S)}(\omega)$ in Eq. (13) is given in terms of the Fourier transforms of $\langle J_{S,\alpha}(t), J_{S,\beta}(0) \rangle$, where $\theta(t)$ is the Heaviside step function and $\langle \cdots \rangle$ denotes the thermal average. Performing integrations by parts and using Eq. (17) and the canonical commutation relations between $r_{s_{z,i}}$ and $p_{s_{z,i}}$, we can obtain

$$\sigma_{\alpha\beta}^{(S)}(\omega) = \frac{i}{\omega} \left[ \delta_{\alpha\beta} \sum_{s_z} \frac{s_z^2 N_{s_z}}{m} + \chi_{\alpha\beta}(\omega) \right],$$

where $\chi_{\alpha\beta}(\omega) = -i \int_0^\infty dt e^{i\omega t} \langle [J_{S,\alpha}(t), J_{S,\beta}(0)] \rangle$ is the retarded response function for the spin current.

By definition of $\chi_{\alpha\beta}(\omega)$, the real and imaginary parts of the longitudinal conductivity ($\alpha = \beta$) are even and odd functions of $\omega$, respectively, and they are related to each other through the Kramers-Kronig relations \cite{42}. Furthermore, the integral of the real part over $\omega$ is exactly related to $N_{s_z}$. By using the Lehmann representation of $\sigma_{\alpha\alpha}^{(S)}(\omega)$, the Heisenberg equation of $R_{s_{z},\alpha}(t)$, and the canonical commutation relations between $r_{s_{z,i}}$ and $p_{s_{z,i}}$, the following $f$-sum rule is obtained \cite{41}:

$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi} \text{Re} \sigma_{\alpha\alpha}^{(S)}(\omega) = \sum_{s_z} \frac{s_z^2 N_{s_z}}{m}.$$
scattering length \([ (k_F a)^{-1} = 0] \) is called the unitary limit, where a Fermi gas is strongly correlated without any characteristic scales of the interaction and has attracted much attention \([47]\). Above the superfluid transition temperature, \( \sigma_{\alpha \alpha}^{(S)} (\omega) \) in the unitary limit has been investigated in Ref. \([43]\).

We employ the BCS-Leggett mean-field theory at zero temperature \([1,2]\). For given \( \alpha \) and \( N \), the chemical potential \( \mu \) and the superfluid order parameter \( \Delta > 0 \) are determined by solving the gap equation with the equation to fix \( N \). As the attraction becomes stronger, \( \mu \) monotonically decreases from \( \mu \simeq E_F \) in the BCS limit \([ (k_F a)^{-1} \to -\infty] \) to \( \mu \simeq -E_b/2 \) in the BEC limit \([ (k_F a)^{-1} \to +\infty] \), where \( E_F = k_F^2/(2m) \) is the Fermi energy and \( E_b = 1/(ma^2) \) is the binding energy of a molecule.

We now turn to \( \sigma_{\alpha \alpha}^{(S)} (\omega) \) of the superfluid Fermi gas. Because of the rotational symmetry of the system, the optical spin conductivity can take a nonzero value only for longitudinal components \( \sigma_{\alpha \alpha}^{(S)} (\omega) \), which are independent of \( \alpha = x, y, z \). In addition, \( \text{Im} \sigma_{\alpha \alpha}^{(S)} (\omega) \) can be reconstructed from \( \text{Re} \sigma_{\alpha \alpha}^{(S)} (\omega) \) as explained previously. For these reasons, we below focus on the real part of \( \sigma_{xx}^{(S)} (\omega) \).

Within the mean-field theory, \( \chi_{xx} (\omega) \) can be straightforwardly evaluated as \([41]\):

\[
\chi_{xx} (\omega) = \sum_k \frac{\Delta^2 k_x^2}{4m^2 E_{k,F}} \left( \frac{1}{\omega_+ - 2E_{k,F}} - \frac{1}{\omega_+ + 2E_{k,F}} \right),
\]

where \( E_{k,F} = \sqrt{\varepsilon_k^2 - \mu^2 + \Delta^2} \) with \( \varepsilon_k = k^2/(2m) \) is the energy of a quasiparticle with momentum \( k \) and \( \omega_+ = \omega + i0^+ \). Substituting this into Eq. \((23)\) yields

\[
\text{Re} \sigma_{xx}^{(S)} (\omega) = \sum_k \frac{\pi \Delta^2 k_x^2}{m^2 |\omega|^3} \delta (|\omega| - 2E_{k,F}).
\]

Because of the delta function in Eq. \((42)\), \( \text{Re} \sigma_{xx}^{(S)} (\omega) \) strongly reflects the structure of the quasiparticle spectrum. In particular, \( \text{Re} \sigma_{xx}^{(S)} (\omega) \) vanishes for \( |\omega| < 2E_{\text{gap}} \) with \( E_{\text{gap}} \equiv \min_k (E_{k,F}) \) being the energy gap. We note that the form of the energy gap depends on the sign of \( \mu \), i.e., \( E_{\text{gap}} = \Delta \) for \( \mu > 0 \) and \( E_{\text{gap}} = \sqrt{\Delta^2 + \mu^2} \) for \( \mu < 0 \). Equation \((42)\) reflects the fact that spin excitations are associated with the dissociation of spin-singlet Cooper pairs or molecules and require energy being larger than \( 2E_{\text{gap}} \). Above the threshold \(|\omega| > 2E_{\text{gap}}\), Eq. \((42)\) reads

\[
\text{Re} \sigma_{xx}^{(S)} (\omega) = \frac{\sqrt{m \Delta^4 \Omega}}{12\pi} \left[ \varepsilon_+ (\omega) \right]^2 + \theta (\varepsilon_- (\omega)) \left[ \varepsilon_- (\omega) \right]^2 \]

\[
\omega^2 \sqrt{\omega^2 - 4\Delta^2}
\]

with \( \varepsilon_\pm (\omega) \equiv 2\mu \pm \sqrt{\omega^2 - 4\Delta^2} \). In Eq. \((5)\), \( \varepsilon_- (\omega) \) is relevant for \( 2\Delta < |\omega| < 2\sqrt{\mu^2 + \Delta^2} \) with \( \mu > 0 \).

Figure 2 shows the spectra of the optical spin conductivity for different interaction strengths. The behavior of \( \text{Re} \sigma_{xx}^{(S)} (\omega) \) near the threshold \(|\omega| \to 2E_{\text{gap}} \) depends on the sign of the chemical potential. In the case of \( \mu > 0 \) \([ (k_F a)^{-1} = -1, 0] \) in Fig. 2, the flat band at \(|k| = \sqrt{2m\mu} \) results in the divergent behavior \( \text{Re} \sigma_{xx}^{(S)} (\omega) \sim 1/|\omega| - 2E_{\text{gap}} \), which is the so-called coherence peak \([51]\). On the other hand, \( E_{k,F} \) is a monotonically increasing function of \(|k| \) on the BEC side with \( \mu < 0 \) \([ (k_F a)^{-1} = 1] \) in Fig. 2 and the optical spin conductivity decreases as \( \text{Re} \sigma_{xx}^{(S)} (\omega) \sim (|\omega| - 2E_{\text{gap}})^{3/2} \). In this way, the optical spin conductivity proves the single-particle properties and the aspects of a spin insulator in the superfluid Fermi gas.

The optical spin conductivity is also connected with a central physical quantity of the system called Tan’s contact \( C \), which measures the probability that a pair of particles approach each other \([52]\). In the high-frequency limit \( \omega \to \pm \infty \), \( \text{Re} \sigma_{xx}^{(S)} (\omega) \) is exactly related to \( C \) by the power law \( \text{Re} \sigma_{xx}^{(S)} (\omega) = C \Omega/[12\pi (m|\omega|)^{3/2}] \) \([43,48,52]\). Indeed, our result in Eq. \((8)\) demonstrates this behavior with the mean-field value of the contact \( C = m^2 \Delta^2 \).

In addition, we checked the validity of the obtained result via the exact \( f \)-sum rule. At zero temperature, Eq. \((28)\) combined with Eq. \((19)\) is equivalent to the equation to fix \( N \), which is one of the starting points in our self-consistent approach \([11]\). We note that the BCS-Leggett theory employed in this Letter gives semi-quantitative descriptions of physical quantities throughout the BCS-BEC crossover at \( T = 0 \) regardless of the presence of the strong interaction \([54]\). Indeed, at zero temperature and unitarity, the contact \( C/k_F^2 = 0.118 \) within the BCS-Leggett theory \([55]\) is close to the recent experimental results \( C_{\text{exp}}/k_F^2 \simeq 0.1 \) \([54,56,57]\).
Spin-1 polar condensate—The optical spin conductivity can also be useful for bosonic systems. To see this, we next investigate spin transport for a spin-1 ($S = 1$) BEC within the Bogoliubov theory [4]. The single-particle potential given by $V_{s_0}(r_{s_{0,i}}) = q s_z^2$ reflects the quadratic Zeeman effect [50]. The interaction potentials $U_{s_{0}s'}(r)$ take contact potentials characterized by the spin-independent $c_0 > 0$ and spin-dependent $c_1$ coupling constants [60, 61]. To show a nontrivial optical response, here we focus on the polar phase in the plain of $(q, n_0c_1)$ satisfying $q + n_0c_1 > n_0|c_1|$, where $n_0$ is the condensate fraction. In this phase realized with $^{23}$Na and $^{87}$Rb, bosons condense only in the $s_z = 0$ channel, which is decoupled from the spin channels ($s_z = \pm 1$) [4]. From Eq. (17), only quasiparticles in the spin channels contribute to spin transport.

We now turn to the optical spin conductivity for the spinor BEC. For the same reason as for the superfluid Fermi gas, we below focus on $\Re \sigma_{xx}^{(S)}(\omega)$. Within the Bogoliubov theory, $\chi_{xx}(\omega)$ can be straightforwardly evaluated as [41]

$$
\chi_{xx}(\omega) = \sum_{k} \frac{n_0^2 c_1^2 k_x^2}{m^2 E_{k,s}} \left( \frac{1}{\omega + 2E_{k,s}} - \frac{1}{\omega + 2E_{k,s}} \right),
$$

(9)

where $E_{k,s} = \sqrt{(\epsilon_k + q)(\epsilon_k + q + 2n_0c_1)}$ is the quasiparticle energy in the spin channels. This expression of the response function is similar to Eq. (11) for the superfluid Fermi gas. Substituting Eq. (57) into Eq. (23) yields

$$
\Re \sigma_{xx}^{(S)}(\omega) = \sum_{k} \frac{4\pi n_0^2 c_1^2 k_x^2}{m^2|\omega|^3} \delta(|\omega| - 2E_{k,s}),
$$

(10)

Because of the delta function in Eq. (58), the optical spin conductivity is sensitive to whether spin excitations are gapped or gapless.

Inside the polar phase, the spin excitations are gapped with the spin gap $E_{\text{gap}} = \sqrt{q(q + 2n_0c_1)}$, and thus $\Re \sigma_{xx}^{(S)}(\omega)$ vanishes for $|\omega| < 2E_{\text{gap}}$ (see Fig. 3b). For $|\omega| > 2E_{\text{gap}}$, Eq. (58) reads

$$
\Re \sigma_{xx}^{(S)}(\omega) = \frac{\sqrt{m} n_0^2 c_1^2 \Omega}{3\pi \omega^2 \sqrt{\omega^2 + 4n_0^2 c_1^2}} |\epsilon_s(\omega)|^2,
$$

(11)

with $\epsilon_s(\omega) = \sqrt{\omega^2 + 4n_0^2 c_1^2} - 2(q + n_0c_1)$. Near the threshold ($|\omega| \rightarrow 2E_{\text{gap}} + 0$), the optical spin conductivity decreases in a similar way as for the superfluid Fermi gas on the BEC side because $E_{k,s}$ and $E_{k,F}$ for $\mu < 0$ are both monotonically increasing functions of $|k|$. On the other hand, spin excitations on the phase boundaries ($q + n_0c_1 = n_0|c_1|$) are gapless and thus Eq. (11) holds for any $\omega$. In particular, the collective spin excitations result in the linear behavior $\Re \sigma_{xx}^{(S)}(\omega) = |\omega|/(48\pi v_s)$ in the low-frequency region ($|\omega| \ll n_0|c_1|$) with the spin velocity $v_s = \sqrt{n_0|c_1|/m}$ (see Fig. 3b). In this way, the optical spin conductivity detects the properties of the spin excitations. In the limit of $\omega \rightarrow \pm \infty$, the optical spin conductivity has the power-law tail $\Re \sigma_{xx}^{(S)}(\omega) = \sqrt{m(n_0c_1)^2 \Omega/(3\pi|\omega|^{3/2})}$ in a similar way as for the superfluid Fermi gas.

With the use of the $f$-sum rule in Eq. (25), the optical spin conductivity can also probe the effects of quantum corrections in the spinor BEC. In the polar phase, the condensate appears only in the $s_z = 0$ channel, and thus the integral of $\Re \sigma_{xx}^{(S)}(\omega)$ over $\omega$ directly measures the particle number $N_1 + N_{-1}$ in the spin channels resulting from the quantum depletion. Indeed, substituting Eq. (55) into Eq. (28) yields the Lee-Huang-Yang corrections to $N_1 + N_{-1}$, which is consistent with the previous result [3, 41]. In addition, $N_1 + N_{-1}$ is known as a probe to the beyond-mean-field effect on the phase transition at $q = -2n_0c_1$ [62].

Conclusion—We discussed the optical spin conductivity $\sigma_{xx}^{(S)}(\omega)$, which serves as a valuable probe to examine many-body interacting systems with spin degrees of freedom. We propose that the frequency-resolved spin transport can be measured with existing methods in ultracold atoms. Our derived general formulae of $\sigma_{xx}^{(S)}(\omega)$ in Eqs. (23) and (25) were applied to spin-1/2 and spin-1 superfluids as typical and experimentally relevant examples. For the superfluid Fermi gas, the gapped single-particle excitations result in the gap of the spectrum of $\Re \sigma_{xx}^{(S)}(\omega)$ and the flat band in the case of $\mu > 0$ leads to the coherence peak. For the spinor BEC, $\Re \sigma_{xx}^{(S)}(\omega)$ detects gapped spin excitations in the polar phase as well as gapless spin excitations on the phase boundaries. In addition, we point out that the spectra reflect the Lee-
Huang-Yang correction.

The optical spin conductivity discussed in this Letter has a variety of prospects. For a spin-1/2 Fermi gas, it is interesting to see how the beyond-mean-field effects in the superfluid phase and the formation of a pseudogap in the normal phase affect the frequency dependence of spin transport. For a bosonic system such as a two-component BEC, since the optical spin conductivity captures the effect of the Lee-Huang-Yang correction, it is interesting to examine a response near the droplet transition induced by the quantum correction. In addition, the optical spin conductivity in one-dimensional systems could be interesting in terms of the spin-charge separation phenomena. Finally, we can also extend our formalism to a system with a spin-orbit coupling, which can be realized by using artificial gauge fields. In this case, searches on nontrivial transverse transport as well as a shift current arising from the quadratic response are intriguing future works.

Note added—When this paper was being finalized, there appeared a preprint, where a spin drag effect and a shift current arising from the quadratic response are intriguing future works.

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SUPPLEMENTAL MATERIAL

I. Formalism

We consider spin transport in a system with spin $S = 1/2, 1, 3/2, \cdots$. We hereafter use the unit system of $k_B = \hbar = 1$. In this section, we employ the first quantization formalism to clarify the connection between the spin current and the spin-resolved center-of-mass motion. The Hamiltonian of the system is given by

$$H = \sum_{(s_z,i)} \left( \frac{p_{s_z,i}^2}{2m} + V_s(r_{s_z,i}) \right) + \frac{1}{2} \sum_{(s_z,i)} \sum_{(s'_z,i')} U_{s_z,s'_z}(r_{s_z,i} - r_{s'_z,i'}) \left( 1 - \delta_{s_z,s'_z} \delta_{ii'} \right),$$

where $m$ is a mass and labels of particles take $s_z = -S, -S + 1, \cdots, S$ and $i = 1, 2, \cdots, N_{s_z}$ with $N_{s_z}$ being the particle number in the $s_z$ component. The operators $r_{s_z,i}$ and $p_{s_z,i}$ denote the coordinate and momentum operators of the particle with a label $(s_z, i)$ and they satisfy the following canonical commutation relations:

$$[(r_{s_z,i})_\alpha, (r_{s'_z,i'})_\beta] = 0, \quad \{ (r_{s_z,i})_\alpha, (p_{s'_z,i'})_\beta \} = i\delta_{s_z,s'_z} \delta_{ii'} \delta_{\alpha\beta},$$

where $\alpha$ and $\beta$ denote Cartesian components. The functions $V_s(r)$ and $U_{s_z,s'_z}(r)$ are single-particle and interatomic potentials, respectively. The time-dependent perturbation term is given by

$$\delta H(t) = \int dr \, f(t) \cdot r S_z(r),$$

where $f(t)$ provides a driving force coupled to the spin density $S_z(r) = \sum_{(s_z,i)} s_z \delta(r - r_{s_z,i})$. Performing the integration, this reduces to

$$\delta H(t) = -f(t) \cdot \sum_{s_z} s_z N_{s_z} R_{s_z},$$

where $R_{s_z} = \sum_{i=1}^{N_{s_z}} r_{s_z,i}/N_{s_z}$ is the spin-resolved center-of-mass coordinate. This perturbation generates a spin current defined as

$$J_S = \sum_{(s_z,i)} s_z \frac{p_{s_z,i}}{m}.$$  

In the Heisenberg picture, we have $p_{s_z,i}(t)/m = dr_{s_z,i}(t)/dt$, leading to

$$J_S(t) = \sum_{s_z} s_z N_{s_z} \frac{dR_{s_z}(t)}{dt}.$$  

A. Optical spin conductivity

We now derive the expression of the optical spin conductivity $\sigma_{\alpha\beta}^{(S)}(\omega)$ in terms of a retarded response function for a spin current. The optical spin conductivity $\sigma_{\alpha\beta}^{(S)}(\omega)$ is given as the linear response of the spin current to the driving force:

$$\langle \tilde{J}_{S,\alpha}(\omega) \rangle_{neq} = \sigma_{\alpha\beta}^{(S)}(\omega) \tilde{f}_\beta(\omega),$$

where $\tilde{J}_{S,\alpha}(\omega)$ and $\tilde{f}_\alpha(\omega)$ are the Fourier transforms of $J_{S,\alpha}(t)$ and $f_\alpha(t)$, respectively, and $\langle \cdot \cdot \rangle_{neq}$ denotes the expectation value with respect to a given nonequilibrium state. The Kubo formula provides

$$\sigma_{\alpha\beta}^{(S)}(\omega) = i \int_{-\infty}^{\infty} dt \, e^{i\omega t} \langle [J_{S,\alpha}(t), X_{S,\beta}(0)] \rangle,$$
where \( X_S(t) = \sum_{s_z} s_z N_{s_z} R_{s_z}(t) \), \( \theta(t) \) is the Heaviside step function, and \( \langle \cdots \rangle \) denotes the thermal average. Since Eq. (17) is rewritten as \( J_S(t) = dX_S(t)/dt \) and we have \( \langle [J_{S,\alpha}(t), X_{S,\beta}(0)] \rangle = \langle [J_{S,\alpha}(0), X_{S,\beta}(-t)] \rangle \), performing the integration by parts yields

\[
\sigma_{\alpha\beta}^{(S)}(\omega) = -\frac{1}{\omega} \langle [J_{S,\alpha}(0), X_{S,\beta}(0)] \rangle + \frac{i}{\omega} \chi_{\alpha\beta}(\omega),
\]

where

\[
\chi_{\alpha\beta}(\omega) = -i \int_{-\infty}^{\infty} dt \, e^{i\omega t} \theta(t) \langle [J_{S,\alpha}(t), J_{S,\beta}(0)] \rangle
\]

is the retarded response function for the spin current. Using Eqs. (13) and (16), we obtain

\[
\langle [J_{S,\alpha}(0), X_{S,\beta}(0)] \rangle = -i \delta_{\alpha\beta} \sum_{s_z} \frac{s_z^2 N_{s_z}}{m}.
\]

Substituting this into Eq. (20), we finally find

\[
\sigma_{\alpha\beta}^{(S)}(\omega) = \frac{i}{\omega} \left( \delta_{\alpha\beta} \sum_{s_z} \frac{s_z^2 N_{s_z}}{m} + \chi_{\alpha\beta}(\omega) \right).
\]

**B. f-sum rule**

We next derive the \( f \)-sum rule, which is the exact constraint on the integral of \( \text{Re} \sigma_{\alpha\alpha}^{(S)}(\omega) \) over \( \omega \). For simplicity, we focus on a canonical ensemble at temperature \( T \). (The derivation of the \( f \)-sum rule below can be straightforwardly extended to a grand canonical ensemble.) The thermal average is given by \( \langle \cdots \rangle = \sum_{\mu} \langle \mu | \cdots | \mu \rangle e^{-E_{\mu}/T} / Z \), where \( |\mu\rangle \) denotes the eigenstate of \( H \) with the eigenvalue \( E_{\mu} \) and \( Z = \sum_{\mu} e^{-E_{\mu}/T} \) is the partition function.

With the use of Eq. (23), the Lehmann representation of \( \text{Re} \sigma_{\alpha\alpha}^{(S)}(\omega) \) is

\[
\text{Re} \sigma_{\alpha\alpha}^{(S)}(\omega) = -\pi \sum_{\mu,\nu} \frac{|\langle \mu | J_{S,\alpha} | \nu \rangle|^2}{Z} \frac{e^{-E_{\nu}/T} - e^{-E_{\nu}/T}}{E_{\mu} - E_{\nu}} \delta(\omega + E_{\mu} - E_{\nu}).
\]

Using

\[
|\langle \mu | J_{S,\alpha} | \nu \rangle|^2 = |\langle \mu | J_{S,\alpha} | \nu \rangle| \langle \nu |(-i)[X_{S,\alpha}, H]|\mu \rangle
\]

\[
= -i(E_{\mu} - E_{\nu}) \langle \mu | J_{S,\alpha} | \nu \rangle \langle \nu | X_{S,\alpha} | \mu \rangle,
\]

we obtain

\[
\text{Re} \sigma_{\alpha\alpha}^{(S)}(\omega) = \pi i \sum_{\mu,\nu} \frac{\langle \mu | J_{S,\alpha} | \nu \rangle \langle \nu | X_{S,\alpha} | \mu \rangle}{Z} \left( e^{-E_{\nu}/T} - e^{-E_{\nu}/T} \right) \delta(\omega + E_{\mu} - E_{\nu}).
\]

Integrating both sides over \( \omega \) yields

\[
\int_{-\infty}^{\infty} \frac{d\omega}{\pi} \text{Re} \sigma_{\alpha\alpha}^{(S)}(\omega) = i \langle [J_{S,\alpha}, X_{S,\alpha}] \rangle,
\]

where \( \sum_{\nu} |\nu\rangle \langle \nu| = 1 \) was used. Substituting Eq. (22) into this, we finally obtain the following \( f \)-sum rule:

\[
\int_{-\infty}^{\infty} \frac{d\omega}{\pi} \text{Re} \sigma_{\alpha\alpha}^{(S)}(\omega) = \sum_{s_z} \frac{s_z^2 N_{s_z}}{m}.
\]
II. SPIN-1/2 SUPERFLUID FERMI GAS

Here we study the spin transport for a spin-1/2 Fermi superfluid at zero temperature within the BCS-Leggett mean-field theory \[1,2\]. In what follows, the \( s_z = +1/2 \) (\( s_z = -1/2 \)) component is referred to as \( \uparrow \) (\( \downarrow \)). The ground canonical Hamiltonian of this system in the second quantization formalism is given by

\[
H - \mu N = \sum_{k} \sum_{\sigma=\uparrow,\downarrow} (\varepsilon_k - \mu) c_{k,\sigma}^\dagger c_{k,\sigma} - \frac{g}{\Omega} \sum_{k,p,p'} c_{k+2+p,\uparrow}^\dagger c_{k-2-p,\downarrow}^\dagger c_{k,\uparrow} c_{k+p',\downarrow},
\]

where \( \varepsilon_k = k^2/(2m) \), \( \mu \) is the chemical potential, \( c_{k,\sigma} \) is the annihilation operator of a Fermi atom with spin \( \sigma \), and \( \Omega \) is the volume. The coupling constant \( g > 0 \) is related to the scattering length \( a \) by

\[
\frac{1}{g} = \frac{1}{\Omega} \sum_{|k|<\Lambda} \frac{m}{k^2} - \frac{m}{4\pi a}
\]

with the momentum cutoff \( \Lambda \).

In the mean-field theory, \( H - \mu N \) reduces to

\[
(H - \mu N)_{\text{BGS}} = E_{\text{GS}} + \sum_{k,\sigma} E_{k,F} \gamma_{k,\sigma}^\dagger \gamma_{k,\sigma},
\]

where \( E_{k,F} = \sqrt{(\varepsilon_k - \mu)^2 + \Delta^2} \) is the quasiparticle energy with the superfluid order parameter \( \Delta \). Since the ground state energy \( E_{\text{GS}} \) does not contribute to spin transport, we do not provide its explicit form. The creation and annihilation operators \( \gamma_{k,\sigma}^\dagger, \gamma_{k,\sigma} \) of quasiparticles are given by the Bogoliubov transformation:

\[
\begin{pmatrix}
\gamma_{k,\uparrow}^\dagger \\
\gamma_{k,\downarrow}^\dagger
\end{pmatrix}
= \begin{pmatrix}
u_{k,F} & -v_{k,F} \\
v_{k,F} & u_{k,F}
\end{pmatrix}
\begin{pmatrix}
\gamma_{-k,\uparrow} \\
\gamma_{-k,\downarrow}
\end{pmatrix}
\]

with

\[
u_{k,F} = \sqrt{\frac{1}{2} \left( 1 + \frac{\varepsilon_k - \mu}{E_{k,F}} \right)}, \quad u_{k,F} = \sqrt{\frac{1}{2} \left( 1 - \frac{\varepsilon_k - \mu}{E_{k,F}} \right)}.
\]

The operators \( \gamma_{k,\sigma}^\dagger, \gamma_{k,\sigma} \) satisfy the following anticommutation relations:

\[
\{\gamma_{k,\sigma}, \gamma_{k',\sigma'}\} = \{\gamma_{k,\sigma}^\dagger, \gamma_{k',\sigma'}^\dagger\} = 0, \quad \{\gamma_{k,\sigma}, \gamma_{k',\sigma'}^\dagger\} = \delta_{k,k'} \delta_{\sigma,\sigma'}.
\]

In the mean field approximation, \( \Delta \) and \( \mu \) for given \( a \) and \( N \) are determined by self-consistently solving the following gap and particle number equations:

\[
\frac{\pi}{a} = \int_{0}^{\infty} d\varepsilon \sqrt{2m\varepsilon} \left( \frac{1}{\varepsilon - \frac{1}{\sqrt{(\varepsilon - \mu)^2 + \Delta^2}}} \right),
\]

\[
N = \frac{m\Omega}{2\pi^2} \int_{0}^{\infty} d\varepsilon \sqrt{2m\varepsilon} \left( 1 - \frac{\varepsilon - \mu}{\sqrt{(\varepsilon - \mu)^2 + \Delta^2}} \right).
\]

A. Current correlation function

Here, we calculate the correlation function \( \chi_{\alpha\beta}(\omega) \) in Eq. \[21\] for the superfluid Fermi gas. In the second quantization formalism, \( J_S \) in Eq. \[16\] is rewritten as

\[
J_S = \sum_k \frac{k}{2m} \left( c_{k,\uparrow}^\dagger c_{k,\uparrow} - c_{k,\downarrow}^\dagger c_{k,\downarrow} \right).
\]

Substituting the inverse of the Bogoliubov transformation in Eq. \[32\] into this yields

\[
J_S = \sum_k \frac{k}{2m} \left[ (u_{k,F}^2 - v_{k,F}^2)(\gamma_{k,\uparrow}^\dagger \gamma_{k,\uparrow} + \gamma_{-k,\downarrow}^\dagger \gamma_{-k,\downarrow}) + 2v_{k,F}^2 + 2u_{k,F} v_{k,F}(\gamma_{k,\uparrow}^\dagger \gamma_{-k,\downarrow} + \gamma_{-k,\downarrow} \gamma_{k,\uparrow}) \right].
\]
In the Heisenberg picture, \( J_S(t) = U_{\text{BCS}}^\dagger(t)J_S U_{\text{BCS}}(t) \) with \( U_{\text{BCS}}(t) = \exp[-i(H - \mu N)_{\text{BCS}} t] \) reads

\[
J_S(t) = \sum_k \frac{k}{2m} \left[ (u_{k,F}^2 - v_{k,F}^2)(\gamma_{k,\uparrow} \gamma_{k,\uparrow}^\dagger + \gamma_{k,\downarrow}^\dagger \gamma_{k,\downarrow}) + 2v_{k,F}^2 + 2u_{k,F} v_{k,F} (e^{2iE_{k,F} t} \gamma_{k,\uparrow}^\dagger \gamma_{k,\downarrow} + e^{-2iE_{k,F} t} \gamma_{k,\downarrow}^\dagger \gamma_{k,\uparrow}) \right] ,
\]

(39)

where \( \gamma_{k,\sigma}(t) = U_{\text{BCS}}^\dagger(t) \gamma_{k,\sigma} U_{\text{BCS}}(t) = \gamma_{k,\sigma} e^{-iE_{k,F} t} \) was used.

Let us now evaluate the correlation function in Eq. (21) at zero temperature. Using Eqs. (33), (34), and (39) as well as \( \langle \gamma_{k,\sigma}^\dagger \gamma_{k,\sigma} \rangle = 0 \) at zero temperature, the expectation value in Eq. (21) is

\[
\langle [J_{S,\alpha}(t), J_{S,\beta}(0)] \rangle = \sum_k \frac{\Delta^2 k_{\alpha}}{4m^2 E_{k,F}^2} \left( e^{-2iE_{k,F} t} - e^{2iE_{k,F} t} \right).
\]

(40)

Therefore, the correlation function for the superfluid Fermi gas is found to be

\[
\chi_{\alpha\beta}(\omega) = \delta_{\alpha\beta} \sum_k \frac{\Delta^2 k_{\alpha}^2}{4m^2 E_{k,F}^2} \left( \frac{1}{\omega_+ - 2E_{k,F}} - \frac{1}{\omega_+ + 2E_{k,F}} \right)
\]

(41)

with \( \omega_+ \equiv \omega + i0^+ \).

### B. f-sum rule

To demonstrate the validity of our calculation within the mean-field theory, we now evaluate the left-hand side of the f-sum rule [Eq. (28)] and show that the f-sum rule is consistent with the particle number equation [Eq. (36)]. Here, we evaluate the f-sum in the case of the superfluid Fermi gas. From Eqs. (23) and (41), the real part of \( \sigma_{xx}^{(S)}(\omega) \) is given by

\[
\text{Re} \sigma_{xx}^{(S)}(\omega) = \sum_k \frac{\pi \Delta^2 k_{\alpha}^2}{m^2 |\omega|^3} \delta(|\omega| - 2E_{k,F}).
\]

(42)

By substituting this into the left-hand side of Eq. (28) and replacing \( \sum_k \rightarrow \Omega \int d^3k/(2\pi)^3 \), the f-sum rule provides the total particle number as

\[
N_{f,\text{sum}} \equiv 4m \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \text{Re} \sigma_{xx}^{(S)}(\omega) = \Omega \int d^3k \frac{\Delta^2 k_{\alpha}^2}{(2\pi)^3 m E_{k,F}^3}.
\]

(43)

Changing the integration variable \( |k| \rightarrow \sqrt{2m \varepsilon} \) and performing the integration over angles of \( k \), we obtain

\[
N_{f,\text{sum}} = \frac{\Omega}{6\pi^2} \int_0^{\infty} d\varepsilon \frac{\Delta^2 (2m \varepsilon)^{3/2}}{[(\varepsilon - \mu)^2 + \Delta^2]^{3/2}}.
\]

(44)

Performing the integration by parts yields

\[
N_{f,\text{sum}} = \frac{m \Omega}{2\pi^2} \int_0^\infty d\varepsilon \sqrt{2m \varepsilon} \left( 1 - \frac{\varepsilon - \mu}{\sqrt{(\varepsilon - \mu)^2 + \Delta^2}} \right).
\]

(45)

Comparing this with the particle number equation in Eq. (36), we find that the particle number \( N_{f,\text{sum}} \) provided by the f-sum rule is equivalent to \( N \) in Eq. (36), which demonstrates the validity of our calculation in the mean-field theory.

### III. SPIN-1 POLAR BOSE-EINSTEIN CONDENSATE

Here we study the spin transport for a spin-1 Bose-Einstein condensate (BEC) at zero temperature within the Bogoliubov theory [3]. The grand canonical Hamiltonian of the system in the second quantization formalism is given
by \[3\]

\[
H - \mu N = \sum_k \sum_{s_z = 0, \pm 1}(\varepsilon_k + q s_z^2 - \mu) a_{k,s_z}^\dagger a_{k,s_z} + \frac{c_0}{2\Omega} \sum_{k, p, p'} \sum_{s_z', s_z''} a_{k/2+p,s_z}^\dagger a_{k/2-p',s_z'} a_{k/2-p',s_z''} a_{k/2+p',s_z''},
\]

+ \frac{c_1}{2\Omega} \sum_{k, p, p'} \sum_{s_z', s_z''} S_{s_z', s_z''} \epsilon_{k/2+p,s_z}^\dagger a_{k/2-p',s_z'} a_{k/2-p',s_z''} a_{k/2+p',s_z''},
\]

(46)

where \( q \) characterizes the quadratic Zeeman effect, \( \mu \) is the chemical potential, \( a_{k,s_z} \) is the annihilation operator of a Bose atom with spin \( s_z \), and \( \Omega \) is the volume. In a spin-1 BEC, the interatomic interactions can be characterized by the spin-independent coupling constant \( c_0 > 0 \) and spin-dependent coupling constant \( c_1 \). The spin-1 matrices \( S_{s_z', s_z''} = (S_{x,s_z', s_z''}, S_{y,s_z', s_z''}, S_{z,s_z', s_z''}) \) are given by

\[
S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

(47)

In the polar phase, the condensate is characterized by \( a_{k=0,s_z} = \sqrt{n_0}\delta_{s_z,0} \) with the condensate fraction \( n_0 \) and is stabilized in the plain of \((q, n_0c_1)\) satisfying \( q + n_0c_1 > n_0|c_1| \). By using the Bogoliubov theory, where the effect of \( a_{k\neq 0} \) is incorporated up to quadratic order, \( H - \mu N \) reduces to \[4\]

\[
H_{\text{Bog}} = E_{\text{GS}} + \sum_{k=0} \left[ E_{k,d} \beta_{k,d}^{\dagger} \beta_{k,d} + E_{k,s} \left( \beta_{k,s}^{\dagger} \beta_{k,s} + \beta_{k,s}^{\dagger} \beta_{k,s} \right) \right].
\]

(48)

Since the ground-state energy \( E_{\text{GS}} \) does not contribute to spin transport, we do not show its explicit form. The quasiparticle energies in the density \((d)\) and spin \((s)\) channels are given by \( E_{k,d} = \sqrt{\varepsilon_k(q + 2n_0c_1)} \) and \( E_{k,s} = \sqrt{\varepsilon_k(q + 2n_0c_1)} \), respectively. The operators \( \beta_{k,d} \), \( \beta_{k,s} \), and \( \beta_{k,s}^{\dagger} \) denotes the annihilation operators of quasiparticles, which are related to

\[
b_{k,d} = a_{k,0}, \quad b_{k,s} = \frac{1}{\sqrt{2}}(a_{k,1} + a_{k,-1}), \quad b_{k,s}^{\dagger} = \frac{i}{\sqrt{2}}(a_{k,1} - a_{k,-1})
\]

by the Bogoliubov transformations:

\[
b_{k,d} = u_{k,d} \beta_{k,d} - v_{k,d} \beta_{-k,d}^{\dagger}, \quad b_{k,s} = u_{k,s} \beta_{k,s} - v_{k,s} \beta_{-k,s}^{\dagger}, \quad b_{k,s}^{\dagger} = u_{k,s} \beta_{k,s}^{\dagger} - v_{k,s} \beta_{-k,s}^{\dagger}
\]

with

\[
u_{k,d} = \sqrt{\frac{\varepsilon_k + n_0c_0 - E_{k,d}}{2E_{k,d}}}, \quad v_{k,d} = \sqrt{\frac{\varepsilon_k + n_0c_0 - E_{k,d}}{2E_{k,d}}}, \quad u_{k,s} = \sqrt{\frac{\varepsilon_k + q + n_0c_1 - E_{k,s}}{2E_{k,s}}}, \quad v_{k,s} = \sqrt{\frac{\varepsilon_k + q + n_0c_1 - E_{k,s}}{2E_{k,s}}},
\]

(51a)

(51b)

Since the density channel does not contribute to spin transport, we below consider the spin channels. The annihilation and creation operators of quasiparticles in the spin channels satisfy the following commutation relations:

\[
[\beta_{k,s,j}, \beta_{k,s,j'}^{\dagger}] = [\beta_{k,s,j'}^{\dagger}, \beta_{k,s,j}^{\dagger}] = 0, \quad [\beta_{k,s,j}, \beta_{k,s,j'}^{\dagger}] = \delta_{kk'}\delta_{sj,s'j'},
\]

(52)

with \( j, j' \in \{x, y\} \).

A. Current correlation function

Here, we calculate the correlation function \( \chi_{\alpha\beta}(\omega) \) in Eq. \[21\] for the spinor BEC in the polar phase. In the second quantization formalism, \( J_S \) in Eq. \[10\] is rewritten as

\[
J_S = \sum_k \frac{k}{m} (a_{k,1}^\dagger a_{k,1} - a_{k,-1}^\dagger a_{k,-1})
\]

\[
= -i \sum_k \frac{k}{m} (b_{k,s}^\dagger b_{k,s} - b_{k,s} b_{k,s}^\dagger),
\]

(53)
where Eq. (19) was used at the last line. Substituting Eqs. (50) into this and using Eq. (52), we obtain

$$J_s = -i \sum_k \frac{k}{m} \left[ (u_{k,s}^2 + v_{k,s}^2) (\beta_{k,s}^\dagger \beta_{k,s} - \beta_{k,s}^\dagger \beta_{k,s}) - 2u_{k,s} v_{k,s} (\beta_{k,s}^\dagger \beta_{k,s} - \beta_{k,s}^\dagger \beta_{k,s}) \right].$$  

(54)

In the Heisenberg picture, $J_s(t) = U_{\text{Bog}}(t) J_s U_{\text{Bog}}(t)$ with $U_{\text{Bog}}(t) = \exp[-i(H - \mu N)_{\text{Bog}} t]$ reads

$$J_s(t) = -i \sum_k \frac{k}{m} \left[ (u_{k,s}^2 + v_{k,s}^2) (\beta_{k,s}^\dagger \beta_{k,s} - \beta_{k,s}^\dagger \beta_{k,s}) - 2u_{k,s} v_{k,s} (e^{2iE_{k,s} t} \beta_{k,s}^\dagger \beta_{k,s} - e^{-2iE_{k,s} t} \beta_{k,s}^\dagger \beta_{k,s}) \right].$$  

(55)

where $\beta_{k,s}(t) = U_{\text{Bog}}^\dagger(t) \beta_{k,s} U_{\text{Bog}}(t) = \beta_{k,s} e^{-iE_{k,s} t}$ was used.

Let us now evaluate the correlation function in Eq. (21) at zero temperature. Using Eqs. (51), (52), and (55) as well as $\langle \beta_{k,s}^\dagger \beta_{k,s} \rangle = 0$ at zero temperature, the expectation value in Eq. (21) is

$$\langle [J_s,\alpha](t),J_s,\beta(0) \rangle = \frac{n_0^2 c_k^2 k k}{m^2 E_{k,s}^2} \left( e^{-2iE_{k,s} t} - e^{2iE_{k,s} t} \right).$$  

(56)

Therefore, the correlation function for the polar BEC is found to be

$$\chi_{\alpha\beta}(\omega) = \delta_{\alpha\beta} \sum_k \frac{n_0^2 c_k^2 k k}{m^2 E_{k,s}^2} \left( \frac{1}{\omega - 2E_{k,s}} - \frac{1}{\omega + 2E_{k,s}} \right).$$  

(57)

B. f-sum rule

To demonstrate the validity of our calculation within the Bogoliubov theory, we now evaluate the left-hand side of the f-sum rule [Eq. (28)] and show that the obtained $N_1 + N_{-1}$ is consistent with the previous result [4]. By substituting Eq. (57) into Eq. (23), the real part of $\sigma^{(S)}_{xx}(\omega)$ is found to be

$$\text{Re} \sigma^{(S)}_{xx}(\omega) = \sum_k \frac{4 \pi n_0^2 c_k^2 k k}{m^2 |\omega|^3} \delta(|\omega| - 2E_{k,s}).$$  

(58)

By substituting this into the left-hand side of Eq. (28) and replacing $\sum_k \rightarrow \Omega \int d^3k/(2\pi)^3$, the f-sum rule provides the particle number in the spin channels $s_z = \pm 1$ as

$$(N_1 + N_{-1})_{f\text{-sum}} \equiv m \int_{-\infty}^\infty \frac{d\omega}{\pi} \text{Re} \sigma^{(S)}_{xx}(\omega) = \Omega \int \frac{d^3k}{(2\pi)^3} \frac{n_0^2 c_k^2 k k}{m E_{k,s}^3}.$$  

(59)

Performing the integration over angles of $k$ yields

$$(N_1 + N_{-1})_{f\text{-sum}} = \frac{\Omega}{6\pi^2} \int_0^\infty dk \frac{n_0^2 c_k^2 k^4}{m E_{k,s}^3}$$  

(60)

with $k = |k|$. Changing the integration variable $k \rightarrow \sqrt{2m n_0 |c_1|} x$ and performing the integration by parts, we obtain

$$(N_1 + N_{-1})_{f\text{-sum}} = \frac{\sqrt{2m^3 n_0^3 c_1 |c_1|^3 \Omega}}{\pi^2} \int_0^\infty dx x^2 \left( \frac{x^2 + \bar{q} + \text{sgn}(c_1)}{\sqrt{(x^2 + \bar{q}) (x^2 + \bar{q} + 2 \text{sgn}(c_1))}} - 1 \right)$$  

(61)

with $\bar{q} = q/(n_0 |c_1|)$. This is consistent with the previous result of the Lee-Huang-Yang correction to $N_1 + N_{-1}$ (see the second term in Eq. (55) in Ref. [4]).

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