An asymptotic expansion for the Stieltjes constants

R. B. Paris

Division of Computing and Mathematics,
University of Abertay Dundee, Dundee DD1 1HG, UK

Abstract

The Stieltjes constants $\gamma_n$ appear in the coefficients in the Laurent expansion of the Riemann zeta function $\zeta(s)$ about the simple pole $s = 1$. We present an asymptotic expansion for $\gamma_n$ as $n \to \infty$ based on the approach described by Knessl and Coffey [Math. Comput. 80 (2011) 379–386]. A truncated form of this expansion with explicit coefficients is also given. Numerical results are presented that illustrate the accuracy achievable with our expansion.

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1. Introduction

The Stieltjes constants $\gamma_n$ appear in the coefficients in the Laurent expansion of the Riemann zeta function $\zeta(s)$ about the point $s = 1$ given by

$$\zeta(s) = \frac{1}{s - 1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s - 1)^n,$$

where $\gamma_0 = 0.577216 \ldots$ is the well-known Euler-Mascheroni constant. Some historical notes and numerical values of $\gamma_n$ for $n \leq 20$ are given in [3]. Recent high-precision evaluations of $\gamma_n$ based on numerical integration have been described in [5, 8]. In [5], Keiper lists various $\gamma_n$ up to $n = 150$, whereas in [8], Kreminski has computed values to several thousand digits for $n \leq 10^4$ and for further selected values (accurate to $10^3$ digits) up to $n = 5 \times 10^4$. All values up to $n = 10^5$ have been computed by Johansson in [4] to about $10^4$ digits.

Upper bounds for $|\gamma_n|$ in the form

$$|\gamma_n| \leq \{3 + (-)^n\} \frac{\lambda_n \Gamma(n)}{\pi^n},$$

have been obtained by Berndt [1] with $\lambda_n = 1$, and by Zhang and Williams [13] with $\lambda_n = (2/n)^n \pi^{-\frac{1}{2}} \Gamma(n + \frac{1}{2}) \sim \sqrt{2/(2/e)}^n$ for $n \to \infty$. On the other hand,
Matsuoka [9] has shown that
\[ |\gamma_n| \leq 10^{-4} e^{n \log \log n} \quad (n \geq 10). \]
However, all these bounds grossly overestimate the growth of \(|\gamma_n|\) for large values of \(n\). An asymptotic approximation for \(\gamma_n\) has recently been given by Knessl and Coffey [6] in the form
\[ \gamma_n \sim \frac{Be^{nA}}{\sqrt{n}} \cos (na + b) \quad (n \to +\infty), \tag{1.1} \]
where \(A, B, a\) and \(b\) are functions that depend weakly on \(n\); see Section 2 for the definition of these quantities. Knessl and Coffey have verified numerically that for \(n \leq 3.5 \times 10^4\) the above formula accounts for the asymptotic growth and oscillatory pattern of \(\gamma_n\), with the exception of \(n = 137\) where the cosine factor in (1.1) becomes very small.

The aim in this note is to extend the analysis in [6] to generate an asymptotic expansion for \(\gamma_n\) as \(n \to +\infty\). The coefficients in this expansion are determined numerically by application of Wojdylo’s formulation [14] for the coefficients in the expansion of a Laplace-type integral. An explicit evaluation of the coefficients is obtained in the case of the expansion truncated after three terms. This approximation is extended to the more general Stieltjes constants \(\gamma_n(\alpha)\) appearing in the Laurent expansion of the Hurwitz zeta function \(\zeta(s, \alpha)\). Numerical results are presented in Section 3 to demonstrate the accuracy of our expansion compared to that in (1.1).

2. Asymptotic expansion for \(\gamma_n\) We start with the integral representation for \(n \geq 1\) given in [13]
\[ \gamma_n = \int_1^\infty B_1(x - [x]) \log^{n-1} \frac{x}{x^2} (n - \log x) \, dx, \]
where \(B_1(x - [x]) = -\sum_{j=1}^\infty \frac{\sin 2\pi j x}{j} \) is the first periodic Bernoulli polynomial. With the change of variable \(t = \log x\), we obtain [6, Eq. (2.3)]
\[ \gamma_n = -3 \left\{ \sum_{k=1}^\infty \frac{1}{\pi k} \int_0^\infty \exp [2\pi i k t + n \log t - t] \left( \frac{n}{t} - 1 \right) dt \right\}. \]
Following the approach used in [6], we define
\[ \psi_k(t) \equiv \psi_k(t; n) = -\frac{2\pi ik}{n} e^t - \log t, \quad f(t) \equiv f(t; n) = e^{-t} \left( 1 - \frac{t}{n} \right) \tag{2.1} \]
and write
\[ \gamma_n = -3 \sum_{k=1}^\infty J_k, \quad J_k := \frac{n}{\pi k} \int_0^\infty e^{-n\psi_k(t)} f(t) \, dt. \tag{2.2} \]
We employ the method of steepest descents to estimate the integrals $J_k$ for large values of $n$. Saddle points of the exponential factor occur at the zeros of $\psi'_k(t) = 0$; that is, they satisfy

$$te^t = \frac{ni}{2\pi k}.$$  \hfill (2.3)

There is an infinite string of saddle points, which is approximately parallel to the imaginary $t$-axis, given by \cite{6}

$$t_m = \log \frac{n}{2\pi k} - \log \log n + (2m + \frac{1}{2})\pi i + O\left(\frac{\log \log n}{\log n}\right)$$

for $m = 0, \pm 1, \pm 2, \ldots$ and large $n$. For fixed $k$ and $m$, the value of $\Re \psi_k(t_m)$ is then

$$-\Re \psi_k(t_m) = \log \log n - \frac{1}{\log n}(1 + \log (2\pi k \log n)) + O((\log n)^{-2})$$

as $n \to \infty$, where the dependence on $m$ is contained in the order term. This shows that the heights of the saddles corresponding to $k \geq 2$ are exponentially smaller as $n \to \infty$ than the saddle with $k = 1$, so that to within exponentially small correction terms we may neglect the contribution in (2.2) arising from $k$ values corresponding to $k \geq 2$; but see the discussion in Section 3. From hereon, we shall drop the subscript $k$ and write $\psi_1(t) \equiv \psi(t)$.

Typical paths of steepest descent and ascent through the saddles $t_0$ and $t_1$ are shown in Fig. 1. Steepest descent and ascent paths terminate at infinity in the right-half plane in the directions $\Im(t) = (2j + \frac{1}{2})\pi$ and $\Im(t) = (2j + \frac{3}{2})\pi$ ($j = 0, \pm 1, \pm 2, \ldots$), respectively. The steepest descent paths through $t_0$ and $t_1$ emanate from the origin and pass to infinity in the directions $\Im(t) = \frac{1}{2}\pi$ and $\frac{5}{2}\pi$, respectively. Similarly, the steepest descent path through $t_{-1}$ (not shown) emanates from the origin and passes to infinity in the direction $\Im(t) = -\frac{3}{2}\pi$.

The integration path in (2.2) can then be deformed to pass through the saddle $t_0$ as shown in Fig. 1.

Application of the method of steepest descents (see, for example, \cite[p. 127]{10} and \cite[p. 11]{11}) then yields

$$J_1 \sim \frac{n}{\pi} \sqrt{2\pi e^{-n\psi(t_0)-t_0}} \left(1 - \frac{t_0}{n}\right) \sum_{s=0}^{\infty} \frac{\hat{c}_{2s} \left(\frac{1}{2}\right)_s}{n^{s+1/2}},$$  \hfill (2.4)

where $(a)_s = \Gamma(a + s)/\Gamma(a)$ is the Pochhammer symbol and $\hat{c}_0 = 1$. The normalised coefficients $\hat{c}_{2s}$ can be obtained by an inversion process and are listed for $s \leq 4$ in \cite[p. 119]{2} and for $s \leq 2$ in \cite[p. 13]{11}; see below. Alternatively, they can be obtained by an expansion process to yield Wojdylo’s formula \cite{14} given by

$$\hat{c}_s = \alpha_0^{-s/2} \sum_{k=0}^{s} \frac{\beta_{s-k}}{\beta_0} \sum_{j=0}^{k} \frac{(-)^j (\frac{1}{2}s + \frac{1}{2})_j}{j!} \beta_0^j \beta_0^j \mathcal{B}_{kj};$$  \hfill (2.5)
Figure 1: Paths of steepest descent and ascent through the saddles $t_0$ and $t_1$ when $n = 100$ and $k = 1$. The steepest paths through the saddle $t_{-1}$ (not shown) in the lower half-plane are similar to those through $t_1$. The arrows indicate the direction of integration.

see also [12, p. 25]. Here $B_{kj} \equiv B_{kj}(\alpha_1, \alpha_2, \ldots, \alpha_{k-j+1})$ are the partial ordinary Bell polynomials generated by the recursion

$$B_{kj} = \sum_{r=1}^{k-j+1} \alpha_r B_{k-r,j-1}, \quad B_{k0} = \delta_{k0},$$

where $\delta_{mn}$ is the Kronecker symbol, and the coefficients $\alpha_r$ and $\beta_r$ appear in

$$\psi(t) - \psi(t_0) = \sum_{r=0}^{\infty} \alpha_r (t - t_0)^{r+2}, \quad f(t) = \sum_{r=0}^{\infty} \beta_r (t - t_0)^r$$

valid in a neighbourhood of the saddle $t = t_0$.

Following [6], we put $t_0 = u + iv$, where $u, v$ are real, and write $-\psi(t_0) = A + ia$, where

$$A := \Re(\log t_0 - 1/t_0) = \frac{1}{2}\log(u^2 + v^2) - \frac{u}{u^2 + v^2}, \quad a := \Im(\log t_0 - 1/t_0) = \arctan\left(\frac{v}{u}\right) + \arctan\left(\frac{v}{u^2 + v^2}\right).$$

We have $\psi''(t_0) = (1 + t_0)/t_0^2$ and accordingly define

$$B := 2\sqrt{2\pi} \left| \frac{t_0}{\sqrt{1 + t_0}} \right|, \quad b := \frac{1}{2}\pi - v - \arctan\left(\frac{v}{1 + u}\right).$$

1For example, this generates the values $B_{41} = \alpha_4$, $B_{42} = \alpha_3^2 + 2\alpha_1\alpha_3$, $B_{43} = 3\alpha_1^2\alpha_2$ and $B_{44} = \alpha_3^4$.

2In [6], the quantity $\frac{1}{2}\pi - v$ appearing in the definition of $b$ is written as $\arctan(v/u)$ by virtue of the first relation in (2.9).
A simple calculation using (2.3) with $k = 1$ shows that
\[ \tan v = \frac{u}{v}, \quad e^{-u} = \frac{2\pi|t_0|}{n}. \] (2.9)

Then, from (2.2) with $k = 1$, (2.4) and the second relation in (2.9), we find upon incorporating the factor $1 - t_0/n$ into the asymptotic series that
\[ \gamma_n \sim B e^{nA} \sqrt{n} \Re \left\{ \hat{c}^{(na+b)} \sum_{s=0}^{\infty} \frac{\hat{c}_{2s}(1/2)_s}{n^s} \right\}, \]
where
\[ \hat{c}_{2s} = \hat{c}_{2s} - \frac{2t_0}{2s-1} \hat{c}_{2s-2} \quad (s \geq 1). \] (2.10)

If we now introduce the real and imaginary parts of the coefficients $\hat{c}_{2s}$ by
\[ \hat{c}_{2s} := C_s + iD_s \quad (s \geq 1), \quad C_0 = 1, \quad D_0 = 0, \] (2.11)
where we recall that $C_s$ and $D_s$ contain an $n$-dependence, then we have the expansion of $\gamma_n$ given by the following theorem.

**Theorem 1.** Let the quantities $A$, $B$, $a$ and $b$, and the coefficients $C_s$, $D_s$, be as defined in (2.7), (2.8) and (2.11). Then, neglecting exponentially smaller terms, we have
\[ \gamma_n \sim B e^{nA} \sqrt{n} \Re \left\{ \cos (na + b) \sum_{s=0}^{\infty} \frac{C_s(1/2)_s}{n^s} - \sin (na + b) \sum_{s=1}^{\infty} \frac{D_s(1/2)_s}{n^s} \right\} \] (2.12)
as $n \to \infty$.

We note that to leading order $A \sim \log \log n$ and $B \sim (8\pi \log n)^{1/2}$ for large $n$.

A simpler form of the expansion (2.12) can be given by truncating the above series at $s = 2$ and use of the form of the normalised coefficients $\hat{c}_{2s}$ in (2.4) expressed in the form
\[ \hat{c}_2 = \frac{1}{2\psi''(t_0)} \{ 2F_2 - 2\Psi_3 F_3 + \frac{2}{3} \Psi_3^2 F_1 + \frac{5}{6} \Psi_3^2 - \frac{1}{3} \Psi_4 \}, \]
\[ \hat{c}_4 = \frac{1}{(2\psi''(t_0))^2} \{ \frac{2}{3} F_4 - \frac{20}{9} \Psi_3 F_3 + \frac{5}{3} (\frac{2}{3} \Psi_3^2 - \Psi_4) F_2 - \frac{35}{9} (\Psi_3^2 - \Psi_3 \Psi_4 + \frac{6}{35} \Psi_5) F_1 \]
\[ + \frac{33}{2} (\frac{11}{21} \Psi_3^4 - \frac{2}{5} (\Psi_3^2 - \frac{1}{5} \Psi_4) \Psi_5 + \frac{1}{5} \Psi_3 \Psi_5 - \frac{1}{5} \Psi_6) \}
\]
where, for brevity, we have defined
\[ \Psi_m := \frac{\psi^{(m)}(t_0)}{\psi''(t_0)} \quad (m \geq 3), \quad F_m := \frac{f^{(m)}(t_0)}{f(t_0)} \quad (m \geq 1); \]
see [2, p. 119], [11, pp. 13–14].

From (2.1) and (2.10), use of Mathematica shows that

\[ c'_2 = \frac{\varphi_2(t_0)}{12(1 + t_0)^3} + \frac{(4 + 3t_0)t_0^3}{n(1 + t_0)^2} + O(n^{-2}), \quad c'_4 = \frac{\varphi_4(t_0)}{864(1 + t_0)^6} + O(n^{-1}), \]

where

\[ \varphi_2(t_0) = 2 - 18t_0 - 20t_0^2 - 3t_0^3 + 2t_0^4, \]
\[ \varphi_4(t_0) = 4 - 72t_0 - 332t_0^2 - 8028t_0^3 - 19644t_0^4 - 20280t_0^5 - 9911t_0^6 - 1884t_0^7 + 4t_0^8. \]

Then we obtain the following result.

**Theorem 2.** Let the quantities \( A, B, a \) and \( b \) be as defined in (2.7) and (2.8). Then, with

\[ c_1 + id_1 = \frac{\varphi_2(t_0)}{24(1 + t_0)^3}, \quad c_2 + id_2 = \frac{\varphi_4(t_0)}{1152(1 + t_0)^6} + \frac{(4 + 3t_0)t_0^3}{2(1 + t_0)^2}, \]

where \( c_s, d_s \) (\( s = 1, 2 \)) are real (and independent of \( n \)) and \( t_0 \) is the saddle point given by the principal solution of (2.3) with \( k = 1 \), we have the asymptotic approximation

\[ \gamma_n \sim \frac{Be^{nA}}{\sqrt{n}} \left\{ \cos (na + b) \left( 1 + \frac{c_1}{n} + \frac{c_2}{n^2} \right) - \sin (na + b) \left( \frac{d_1}{n} + \frac{d_2}{n^2} \right) \right\} \quad (2.13) \]

as \( n \to \infty \).

We remark that the expansion of the integrals \( J_k \) for fixed \( k \geq 2 \) follows the same procedure. If we still refer to the real and imaginary parts of the contributory saddle \( t_0 \) (when \( k \geq 2 \)) as \( u \) and \( v \), the second relation in (2.9) is now replaced by \( e^{-u} = 2\pi k|t_0|/n \). It then follows that the form of the expansion for \(-\Im J_k\) is given by (2.12), provided the quantities \( A, B, a \) and \( b \), and the coefficients \( C_s, D_s \), are interpreted as corresponding to the saddle \( t_0 \) with the \( k \)-value under consideration.

3. **Numerical results and concluding remarks** We discuss numerical computations carried out using the expansions given in Theorems 1 and 2. For a given value of \( n \) the saddle \( t_0 \) is computed from (2.3) with \( k = 1 \) to the desired accuracy. Mathematica is used to determine the coefficients \( \alpha_r \) and \( \beta_r \) in (2.6) for \( 0 \leq r \leq 2s_0 \), where in the present computations \( s_0 = 6 \). The coefficients \( C_s \) and \( D_s \) can then be calculated for \( 0 \leq s \leq s_0 \) from (2.5), (2.10) and (2.11).

We display the computed values of \( C_s \) and \( D_s \) for two values of \( n \) in Table 1. We repeat that these coefficients contain an \( n \)-dependence and so have to be computed for each value of \( n \) chosen. In Table 2, the values of the absolute
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Table 1: Values of the coefficients $C_s$ and $D_s$ (to 10 dp) for $1 \leq s \leq 6$ for two values of $n$.

|   | $n = 100$ | $n = 1000$ |
|---|-----------|------------|
|   | $C_s$     | $D_s$      | $C_s$     | $D_s$      |
| 1 | -0.3158578918 | +0.1626819326 | -0.0885061806 | +0.1958085240 |
| 2 | -2.9096870797 | -2.1947177121 | -6.5840165991 | -2.6459812815 |
| 3 | -0.3804847598 | -3.3953890569 | -9.4682639154 | -10.09635962642 |
| 4 | +1.4820479884 | -0.1130053628 | -1.3074432243 | -11.3104092292 |
| 5 | -0.2630549338 | +0.9253656779 | +4.9469591967 | -1.67819725309 |
| 6 | -0.3783700609 | -0.3119889058 | +0.8180579543 | +3.98701271605 |

Table 2: Values of the absolute relative error in the computation of $\gamma_n$ from (2.12) as a function of the truncation index $s$ for different $n$.

|   | $n = 75$       | $n = 100$       | $n = 137$       | $n = 1000$       |
|---|----------------|-----------------|-----------------|------------------|
| 0 | $1.759 \times 10^{-3}$ | $1.412 \times 10^{-3}$ | $-$             | $1.597 \times 10^{-4}$ |
| 1 | $6.503 \times 10^{-4}$ | $3.226 \times 10^{-4}$ | $2.701 \times 10^{-1}$ | $2.649 \times 10^{-6}$ |
| 2 | $1.244 \times 10^{-5}$ | $4.472 \times 10^{-6}$ | $8.775 \times 10^{-2}$ | $4.125 \times 10^{-9}$ |
| 3 | $3.063 \times 10^{-7}$ | $9.370 \times 10^{-8}$ | $3.811 \times 10^{-5}$ | $7.711 \times 10^{-11}$ |
| 4 | $2.535 \times 10^{-9}$ | $7.850 \times 10^{-10}$ | $2.183 \times 10^{-6}$ | $2.026 \times 10^{-13}$ |
| 5 | $5.101 \times 10^{-10}$ | $9.022 \times 10^{-11}$ | $1.248 \times 10^{-8}$ | $6.157 \times 10^{-16}$ |
| 6 | $1.850 \times 10^{-11}$ | $1.982 \times 10^{-12}$ | $9.415 \times 10^{-10}$ | $2.743 \times 10^{-18}$ |

Relative error in the computation of $\gamma_n$ from the expansion (2.12) are presented as a function of the truncation index $s$ for several values of $n$.

The case $n = 137$ has been included in Table 2 since this corresponds to the factor $\cos(na+b)$ possessing the very small value $\simeq 1.69881 \times 10^{-4}$. The leading term approximation in (1.1), and (2.12) (with $s = 0$), yields an incorrect sign, namely $+3.89874 \times 10^{27}$ when $\gamma_{137} = -7.99522199 \ldots \times 10^{27}$. According to [4], this is the only instance for $n \leq 10^5$ when the leading approximation produces the wrong sign. It is seen that inclusion of the higher order correction terms with $s \leq 6$ yields a relative error of order $10^{-10}$ in this case. When $n = 10^5$, [4] gives the value

$$\gamma_{100000} = 1.99192730631254109565822724315 \ldots \times 10^{53432}.$$

The expansion (2.12) for this value of $n$ with truncation index $s = 6$ is found to yield a relative error of order $10^{-30}$; that is, the expansion correctly reproduces
all the digits displayed above.

Table 3: Values of the absolute relative error in the computation of $\gamma_n$ from (2.12) with $k = 1$ and $k \leq 2$ as a function of the truncation index $s$ for $n = 25$.

| $s$ | $k = 1$       | $k \leq 2$         |
|-----|---------------|---------------------|
| 0   | $1.051 \times 10^{-2}$ | $1.052 \times 10^{-2}$ |
| 1   | $2.909 \times 10^{-3}$ | $2.894 \times 10^{-3}$ |
| 2   | $2.608 \times 10^{-4}$ | $2.460 \times 10^{-4}$ |
| 3   | $2.390 \times 10^{-6}$ | $1.723 \times 10^{-5}$ |
| 4   | $1.518 \times 10^{-5}$ | $3.412 \times 10^{-7}$ |
| 5   | $1.495 \times 10^{-5}$ | $1.160 \times 10^{-7}$ |
| 6   | $1.482 \times 10^{-5}$ | $1.189 \times 10^{-8}$ |

For the smallest value $n = 75$ presented in Table 2, it is found numerically that the contribution to (2.2) corresponding to $k = 2$ is about 11 orders of magnitude smaller than the dominant term with $k = 1$. For the larger $n$ values, this contribution is even smaller and the terms with $k \geq 2$ can be safely neglected. However, for smaller $n$ this is no longer the case and a meaningful approximation has to take into account the contribution from other $k \geq 2$ values.

In Table 3, we illustrate this situation by taking $n = 25$. The second column shows the absolute relative error in the computation of $\gamma_n$ with $k = 1$ for different truncation index $s$; that is, with the approximation $\gamma_n \simeq -3J_1$. For $4 \leq s \leq 6$, this error is seen to remain essentially constant at $O(10^{-5})$. The contribution with $k = 2$ is about 5 orders of magnitude smaller than the $k = 1$ contribution, so that this additional contribution needs to be included for larger index $s$. The absolute relative error including the contribution with $k = 2$ is shown in the third column; that is, with the approximation $\gamma_n \simeq -3(J_1 + J_2)$. The expansion with $k = 3$ is about 8 orders of magnitude smaller than the $k = 1$ contribution, so this would only begin to make a significant contribution for $s \geq 6$. This problem becomes even more acute for smaller $n$ values, say $n = 10$, where higher $k$ values need to be retained. However, the chief interest in the asymptotic expansion in (2.12) is for large $n$, where this problem is of no real concern.

In Table 4 we show some examples of the asymptotic approximation given in (2.13). We compare these with the values produced by the leading approximation (1.1) and the exact value of $\gamma_n$ obtained from Mathematica using the command StieltjesGamma[n]. It will be observed that for $n = 500$ the approximation (2.13) yields nine significant figures.
Table 4: Values for $\gamma_n$ obtained from (1.1) and (2.13) compared with the exact value.

| $n$  | Eq. (1.1)            | Eq. (2.13)            | Exact $\gamma_n$ |
|------|----------------------|----------------------|------------------|
| 10   | $+2.105395 \times 10^{-4}$ | $+2.04713213 \times 10^{-4}$ | $+2.05332814 \ldots \times 10^{-4}$ |
| 50   | $+1.275493 \times 10^{2}$  | $+1.26823798 \times 10^{2}$  | $+1.26823602 \ldots \times 10^{2}$  |
| 80   | $+2.514857 \times 10^{10}$ | $+2.51633995 \times 10^{10}$ | $+2.51634410 \ldots \times 10^{10}$ |
| 100  | $-4.259408 \times 10^{17}$ | $-4.25340036 \times 10^{17}$ | $-4.25340157 \ldots \times 10^{17}$ |
| 137  | $+3.898740 \times 10^{27}$ | $-7.99377883 \times 10^{27}$ | $-7.99522199 \ldots \times 10^{27}$ |
| 200  | $-7.060244 \times 10^{55}$ | $-6.97465335 \times 10^{55}$ | $-6.97464971 \ldots \times 10^{55}$ |
| 500  | $-1.165662 \times 10^{204}$ | $-1.16550527 \times 10^{204}$ | $-1.16550527 \ldots \times 10^{204}$ |

Finally, we remark that the analysis in Section 2 is immediately applicable to the more general Stieltjes constants $\gamma_n(\alpha)$ appearing in the Laurent expansion for the Hurwitz zeta function $\zeta(s, \alpha)$ about the point $s = 1$. These constants are defined by

$$
\zeta(s, \alpha) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(\alpha) (s-1)^n,
$$

where $\gamma_0(\alpha) = -\Gamma'(\alpha)/\Gamma(\alpha)$ and $\gamma_n(1) = \gamma_n$. It is shown in [7, Eq. (2.9)] that

$$
C_n(\alpha) := \gamma_n(\alpha) - \frac{1}{\alpha} e^{n \log \log \alpha} = -\Im \sum_{k=1}^{\infty} e^{-2\pi i k \alpha} J_k.
$$

Then it follows that the expansions in Theorems 1 and 2 are modified only in the argument of the trigonometric functions appearing therein, which become $na + b - 2\pi \alpha$. Thus, for example, from (2.13) we have

$$
C_n(\alpha) \sim \frac{B e^{nA}}{\sqrt{n}} \left\{ \cos (na+b-2\pi \alpha) \left(1 + \frac{c_1}{n} + \frac{c_2}{n^2}\right) - \sin (na+b-2\pi \alpha) \left(\frac{d_1}{n} + \frac{d_2}{n^2}\right) \right\}
$$

as $n \to \infty$, where the quantities $A$, $B$, $a$, $b$ and the coefficients $c_s, d_s$ ($s = 1, 2$) are as specified in Theorem 2. The leading approximation agrees with that obtained in [7, Eq. (2.4)].

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