On the duality gap and Gale’s example in conic linear programming

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Abstract

The aim of this paper is to revisit some duality results in conic linear programming and to answer an open problem related to the duality gap function for Gale’s example.

1 Introduction

In this paper we are mainly concerned by duality results for the following conic linear programming problem

\[(P_{c,b}) \text{ minimize } c(x) \text{ s.t. } x \in P, Ax - b \in Q,\]

and its dual

\[(P^*_{c,b}) \text{ maximize } y^*(b) \text{ s.t. } y^* \in Q^+, A^*y^* - c \in -P^+,\]

where \(X, Y\) are Hausdorff locally convex spaces, \(X^*\) and \(Y^*\) are their topological dual spaces, \(A : X \to Y\) is a continuous linear operator, \(A^* : Y^* \to X^*\) is the adjoint of \(A\), \(P \subset X\) and \(Q \subset Y\) are convex cones, \(P^+ \subset X^*\) and \(Q^+ \subset Y^*\) are the positive dual cones of \(P\) and \(Q\), \(b \in Y\) and \(c \in X^*\).

In dealing with duality results for the problem \((P_{c,b})\), one uses several approaches in the literature. For example, in the paper [9], the book [11], as well as in the recent paper [8], one associates certain convex sets to \((P_{c,b})\) and \((P^*_{c,b})\) and studies their relationships; in [12] one uses the Lagrangian function associated to problem \((P_{c,b})\) and several results from convex analysis.

In this paper we derive the main duality results for problems \((P_{c,b})\) and \((P^*_{c,b})\) using Rockafellar’s perturbation method (see [11], [16]). Then, we give an answer to the open problem concerning the perturbed Gale’s example considered in [14, p. 12], followed by a discussion of what seems to be the first version of Gale’s example.

Below, we introduce some notions, notations and preliminary results.

Having \(X\) a Hausdorff locally convex space (H.L.C.S. for short), \(X^*\) is its topological dual endowed with its weakly-star topology \(w^* := \sigma(X^*, X)\). The value \(x^*(x)\) of \(x^* \in X^*\) at \(x \in X\) is denoted by \(\langle x, x^* \rangle\). It is well known that \((X^*, w^*)^*\) can be identified with \(X\), what we do in the sequel. For \(E \subset X\), by span \(E\), aff \(E\), int \(E\) and cl \(E\) one denotes the linear hull, the

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affine hull, the interior and the closure of $E$, respectively; moreover, the intrinsic core (or relative algebraic interior) of $\emptyset \neq E \subset X$ is the set

$$
icr E := \{x_0 \in X \mid \forall x \in \text{aff } E, \exists \delta \in \mathbb{P}, \forall \lambda \in \mathbb{R} : |\lambda| \leq \delta \Rightarrow (1 - \lambda)x_0 + \lambda x \in E\},$$

where $\mathbb{P} := ]0, \infty[; the core (or algebraic interior) of $E$, denoted $\text{cor } E$, is $\nicr E$ if $\text{aff } E = X$ and the empty set otherwise. Having $(\emptyset \neq) K \subset X$ a convex cone (that is, $x + x' \in K$ and $tx \in K$ for all $x, x' \in K$ and $t \in \mathbb{R}_+ := [0, \infty[$), we set $x \leq_K x'$ (equivalently $x' \geq_K x$) for $x, x' \in X$ with $x' - x \in K$; clearly, $\leq_K$ is a preorder on $X$, that is, $\leq_K$ is reflexive and transitive. For $\emptyset \neq A \subset X$ (and similarly for $\emptyset \neq B \subset X^*$) we set $A^+ := \{x^* \in X^* \mid \forall x \in A : \langle x, x^* \rangle \geq 0\}$ for the positive dual cone of $A$; it is well known that $A^+ (\subset X^*)$ is a $w^*$-closed convex cone and $(K^+)^+ = \text{cl } K$ if $K \subset X$ is a convex cone.}

Having a function $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, its domain is the set $\text{dom } f := \{x \in X \mid f(x) < \infty\}$; $f$ is proper if $\text{dom } f \neq \emptyset$ and $f(x) \neq -\infty$ for all $x \in X$; $f$ is convex if its epigraph $\text{epi } f := \{(x, t) \in X \times \mathbb{R} \mid f(x) \leq t\}$ is convex; $f$ is positively homogeneous if $f(tx) = tf(x)$ for all $t \in \mathbb{P}$ and $x \in X$; $f$ is subadditive if $f(x + x') \leq f(x) + f(x')$ for all $x, x' \in \text{dom } f$; $f$ is sublinear if $f$ is positively homogeneous, subadditive and $f(0) = 0$; $f$ is lower semicontinuous (l.s.c. for short) at $x \in X$ if $\lim_{x' \to x} f(x') \geq f(x)$, where $\overline{\mathbb{R}}$ is endowed with its usual topology and partial order; $f$ is l.s.c. if $f$ is l.s.c. at any $x \in X$; the l.s.c. envelope of $f$ is the function $\overline{f} : X \to \overline{\mathbb{R}}$ such that $\text{epi } \overline{f} = \text{cl } (\text{epi } f)$, and so $\overline{f}$ is convex if $f$ is so; the subdifferential of $f$ at $x \in X$ with $f(x) \in \overline{\mathbb{R}}$ is the set

$$
\partial f(x) := \{x^* \in X^* \mid \forall x' \in X : \langle x' - x, x^* \rangle \leq f(x') - f(x)\}
$$

and $\partial f(x) := \emptyset$ if $f(x) \notin \mathbb{R}$; $f$ is subdifferentiable at $x \in X$ if $\partial f(x) \neq \emptyset$. The conjugate of $f$ is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined by

$$
f^*(x^*) := \sup \{\langle x, x^* \rangle - f(x) \mid x \in X\} = \sup \{\langle x, x^* \rangle - f(x) \mid x \in \text{dom } f\} \quad (x^* \in X^*),
$$

where $\sup \emptyset := -\infty$; clearly, $f^*$ is a $w^*$-l.s.c. convex function. Having $g : X^* \to \overline{\mathbb{R}}$, its conjugate $g^* : X \to \overline{\mathbb{R}}$ is defined similarly. Notice that $f^* = (\overline{f})^*$; moreover, for $x \in X$ and $x^* \in X^*$ one has

$$
x^* \in \partial f(x) \iff [f(x) \in \mathbb{R} \land f(x) + f^*(x^*) = \langle x, x^* \rangle] \Rightarrow \overline{f}(x) = f(x) \in \mathbb{R} \Rightarrow \partial f(x) = \overline{\partial f(x)}.
$$

As in [16], the class of proper convex functions defined on $X$ is denoted by $\Lambda(X)$. It is worth observing that, for $f : X \to \overline{\mathbb{R}}$ a convex function, one has

$$
[f(x_0) = -\infty \quad \text{and} \quad x \in \text{icr } (\text{dom } f)] \Rightarrow f(x) = -\infty \quad (1)
$$

for $x_0 \in X$ (by [16, Prop. 2.1.4]) and

$$
\overline{f} \in \Lambda(X) \iff \exists x \in X : \overline{f}(x) \in \mathbb{R} \iff \exists x^* \in X^* : f^*(x^*) \in \mathbb{R} \Rightarrow f^{**} = \overline{f},
$$

$$
[\exists x \in X : \overline{f}(x) = -\infty] \iff [\forall x \in \text{cl } (\text{dom } f) : \overline{f}(x) = -\infty] \iff f^* = \infty \iff f^{**} = -\infty,
$$

by [16 Ths. 2.2.6, 2.3.4]. The directional derivative of $f \in \Lambda(X)$ at $x \in \text{dom } f$ is

$$
\begin{align*}
f^+_+(x, \cdot) : X & \to \overline{\mathbb{R}}, \\
f^+_+(x, u) & := \lim_{t \to 0^+} \frac{f(x + tu) - f(x)}{t} = \inf_{t > 0} \frac{f(x + tu) - f(x)}{t}.
\end{align*}
$$

One has \( f'_+(x, 0) = 0 \), \( f'_+(x, s u) = s f'_+(x, u) \) and \( f'_+(x, u + u') \leq f'_+(x, u) + f'_+(x, u') \) for \( s \in \mathbb{P} \) and \( u, u' \in X \); hence \( f'_+(x, \cdot) \) is sublinear. It follows easily that
\[
f'_+(x, u) \leq f(x + u) - f(x), \quad \partial f(x) = \partial f'_+(x, \cdot)(0) \quad \forall f \in \Lambda(X), \ x \in \text{dom} \ f, \ u \in X.
\]

The \textit{indicator function} of \( E \subset X \) is \( \iota_E : X \to \overline{\mathbb{R}} \) defined by \( \iota_E(x) := 0 \) for \( x \in E \) and \( \iota_E(x) := \infty \) for \( x \in X \setminus E \); notice that \( \iota_E \) is l.s.c. iff \( E \) is closed, and \( \iota_E \) is convex iff \( E \) is convex.

\section{Alternative proofs for some duality results in conic linear programming}

In the sequel, we consider the H.I.c.s. \( X \) and \( Y \), the continuous linear operator \( A : X \to Y \), its adjoint \( A^* : Y^* \to X^* \) defined by \( A^* y^* := y^* \circ A \) (and so \( \langle Ax, y^* \rangle = \langle x, A^* y^* \rangle \) for \( x \in X \), \( y^* \in Y^* \)), the convex cones \( P \subset X, Q \subset Y \), as well as their positive dual cones \( P^+ \subset X^* \) and \( Q^+ \subset Y^* \). The preorders defined by \( P, Q, P^+ \) and \( Q^+ \) are simply denoted by \( \leq \).

For \( c \in X^* \) we associate the mapping \( \Phi_c : X \times Y \to \overline{\mathbb{R}} \), defined by
\[
\Phi_c(x, y) := \langle x, c \rangle + \iota_P(x) + \iota_Q(Ax - y) > -\infty \quad ((x, y) \in X \times Y);
\]
\( \Phi_c \) is a proper sublinear function because \( \iota_P \) and \( \iota_Q \) are so.

For \( (x^*, y^*) \in X^* \times Y^* \) one has
\[
\Phi^*_c(x^*, y^*) = \sup \{ \langle x, x^* \rangle + \langle y, y^* \rangle - \langle x, c^* \rangle - \iota_P(x) - \iota_Q(Ax - y) \mid x \in X, \ y \in Y \}
\[
= \sup \{ \langle x, x^* - c^* \rangle + \langle Ax - q, y^* \rangle \mid x \in P, \ q \in Q \}
\]
\[
= \sup \{ \langle x, x^* - c^* + A^* y^* \rangle - \langle q, y^* \rangle \mid x \in P, \ q \in Q \}
\]
\[
= \begin{cases} 
0 & \text{if } y^* \in Q^+ \text{ and } c^* - A^* y^* - x^* \in P^+, \\
\infty & \text{otherwise,} \\
\iota_{P^+}(c^* - A^* y^* - x^*) + \iota_{Q^+}(y^*). 
\end{cases}
\]

The \textit{marginal (value) function} associated to \( \Phi_c \) is
\[
h_c : Y \to \overline{\mathbb{R}}, \quad h_c(y) := \inf \{ \Phi_c(x, y) \mid x \in X \} = \inf \{ \langle x, c \rangle \mid x \geq 0, \ Ax \geq y \},
\]
where \( \inf \emptyset := \infty \); hence \( \text{dom} \ h_c = A(P) - Q \).

Clearly, \( h_c(0) \leq \Phi_c(0, 0) = 0 \) and \( h_c \) is increasing, that is, \( h_c(y_1) \leq h_c(y_2) \) for \( y_1 \leq y_2 \). Because \( \Phi_c \) is convex, so is \( h_c \). Moreover, for \( \alpha > 0 \) and \( y \in Y \) one has
\[
h_c(\alpha y) = \inf_{x \in X} \Phi_c(x, \alpha y) = \inf_{x \in X} \alpha \Phi_c(\alpha^{-1} x, y) = \alpha \inf_{x' \in X} \Phi_c(x', y) = \alpha h_c(y),
\]
and so \( h_c \) is positively homogeneous; it follows that \( h_c(0) \in \{ 0, -\infty \} \).

If \( h_c(0) = -\infty \), then \( h_c(y) = h_c(y + 0) \leq h_c(y) + h_c(0) = -\infty \) for every \( y \in \text{dom} \ h_c \); moreover, \( \partial h_c(0) = \emptyset \) (by definition), \( h_c^* = \infty \) and \( h_c^{**} = -\infty \).

Assume that \( h_c(0) = 0 \); then
\[
y^* \in \partial h_c(0) \iff [\forall y \in \text{dom} \ h_c : \langle y, y^* \rangle \leq h_c(y)] \iff [\forall x \in P, \ \forall z \in Q : \langle Ax - z, y^* \rangle \leq \langle x, c \rangle] \\
\iff [y^* \in Q^+ \land [\forall x \in P : 0 \leq \langle x, c - A^* y^* \rangle]] \iff y^* \in Q^+ \cap (A^*)^{-1}(c - P^+),
\]
and so $\partial h_c(0) = Q^+ \cap (A^*)^{-1}(c - P^+)$.

It follows easily from the definition of $h_c$ (or from [16 Th. 2.6.1(i)]) that

$$h_c^*(y^*) = \Phi^*_c(0, y^*) = \begin{cases} 0 & \text{if } y^* \in Q^+ \text{ and } A^*y^* - c \in (-P^+), \\ \infty & \text{otherwise,} \end{cases}$$

and so $\dom h_c^* = Q^+ \cap (A^*)^{-1}(c - P^+)$. Consequently, if $h_c^*$ is proper (hence $h_c(0) = 0$), then

$$\partial h_c(0) = \dom h_c^* = Q^+ \cap (A^*)^{-1}(c - P^+) \neq \emptyset, \quad h_c^* = \imath_{\partial h_c(0)} \quad \text{and} \quad \nabla h_c = h_{c,b}^*.$$

On the other hand,

$$h_c(0) = 0 \iff [x \in P \cap A^{-1}(Q) \Rightarrow (x, c) \geq 0] \iff c \in [P \cap A^{-1}(Q)]^+.$$ 

If $P$ and $Q$ are closed, then $(A^{-1}(Q))^+ = \cl_{w^*} A^*(Q^+)$ by [9 Lem. 4] (see also [15 Lem. 1]), and so

$$[P \cap A^{-1}(Q)]^+ = \cl_{w^*} [P^+ + (A^{-1}(Q))^+] = \cl_{w^*} [P^+ + A^*(Q^+)].$$

For $c \in X^*$ and $b \in Y$, consider

$$\Phi_{c,b} : X \times Y \to \R, \quad \Phi_{c,b}(x, y) := \Phi_c(x, y + b);$$

hence

$$\forall y \in Y : h_{c,b}(y) = h_c(y + b) \land \nabla h_{c,b}(y) = \nabla h_c(y + b) \land \partial h_{c,b}(y) = \partial h_c(y + b),$$

where $h_{c,b}$ is the value function associated to $\Phi_{c,b}$. It follows that $\Phi_{c,b}^*(x^*, y^*) = \Phi_c^*(x^*, y^*) - \langle b, y^* \rangle$ for $(x^*, y^*) \in X^* \times Y^*$, whence

$$\forall y^* \in Y^* : h_{c,b}^*(y^*) = \Phi_{c,b}^*(0, y^*) = \Phi_c^*(0, y^*) - \langle b, y^* \rangle = h_c^*(y^*) - \langle b, y^* \rangle.$$

From (2) and (5), one has $\Phi_{c,b}(x, 0) = \Phi_c(x, b) = \langle x, c \rangle$ if $Ax \geq b$, $x \geq 0$, and $\Phi_{c,b}(x, 0) = \infty$ otherwise, and so the problem $(P_{c,b})$ becomes

$$(P_{c,b}) \quad \text{minimize } \Phi_{c,b}(x, 0) \quad \text{s.t. } x \in X.$$ 

On the other hand, from (3) and (7) one has $\Phi_{c,b}^*(0, y^*) = -\langle b, y^* \rangle$ if $A^*y^* \leq c$, $y^* \geq 0$, and $\Phi_{c,b}^*(0, y^*) = \infty$ otherwise, and so the problem $(P_{c,b}^*)$ becomes

$$(P_{c,b}^*) \quad \text{maximize } -\Phi_{c,b}^*(0, y^*) \quad \text{s.t. } y^* \in Y^*.$$ 

For further use, similarly to the notations from [14], we set:

- $F(b) := \{x \in X \mid Ax \geq b, \ x \geq 0\} - \text{the feasible set of } (P_{c,b});$
- $\varphi(c, b) := \inf \{\langle x, c \rangle \mid x \in F(b)\} = h_c(b) \in \R - \text{the value of } (P_{c,b});$
- $F^*(c) := \{y^* \in Y^* \mid A^*y^* \leq c, \ y^* \geq 0\} - \text{the feasible set of } (P_{c,b}^*);$
- $\psi(c, b) := \sup \{\langle b, y^* \rangle \mid y^* \in F^*(c)\} = h_{c,b}^*(b) \in \R - \text{the value of } (P_{c,b}^*);$
- $\Lambda := \{(c, b) \in X^* \times Y \mid F^*(c) \neq \emptyset, \ F(b) \neq \emptyset\} = (P^+ + A^*(Q^+)) \times (A(P) - Q);$
- $g : \Lambda \to \R, \ g(c, b) := \varphi(c, b) - \psi(c, b) - \text{the duality gap function of } (P_{c,b}).$

Applying [16 Th. 2.6.1] for $\Phi_{c,b}$ one gets the following result.
Proposition 1 The following assertions hold:

(i) \( h_c(b) = \text{val}(P_{c,b}) \); \( h_c^*(b) = \text{val}(P_{c,b}^*) \); \( \text{val}(P_{c,b}) \geq \text{val}(P_{c,b}^*) \) for all \( c \in X^* \) and \( b \in Y \); \( \text{val}(P_{c,b}) < \infty \) \( \iff \) \( b \in A(P) - Q \); \( \text{val}(P_{c,b}^*) > - \infty \) \( \iff \) \( c \in P^* + A^*(Q^+) \); consequently, \( g(c,b) \in \mathbb{R}_+ \) for all \( (c,b) \in \Lambda \).

(ii) \( h_c(b) \in \mathbb{R} \) and \( h_c \) is l.s.c. at \( b \) \( \iff \) \( \text{val}(P_{c,b}) = \text{val}(P_{c,b}^*) \in \mathbb{R} \).

(iii) \( \partial h_c(b) \neq \emptyset \) \( \iff \) \( \{ \text{val}(P_{c,b}) = \text{val}(P_{c,b}^*) \} \in \mathbb{R} \) and \( (P_{c,b}^*) \) has optimal solutions; moreover, \( \text{Sol}(P_{c,b}^*) = \partial h_c(b) \) if \( \partial h_c(b) \neq \emptyset \), where \( \text{Sol}(P_{c,b}^*) \) is the set of optimal solutions of problem \( (P_{c,b}^*) \).

(iv) \( \overline{h}_c \) is proper \( \iff \) \( h_c^* \) is proper \( \iff \) \( \text{dom} h_c^* \neq \emptyset \) \( \iff \) \( c \in P^* + A^*(Q^+) \).

Proof. We apply [16, Th. 2.6.1] for \( \Phi_{c,b} \) and its associated value function \( h_{c,b} \). Having in view the equality \( h_{c,b}(\cdot) = h_c(\cdot + b) \) from [16], one has \( h_{c,b}^*(\cdot) = h_c^*(\cdot + b) \), too. Hence

\[ h_{c,b}(0) = h_c(b), \quad h_{c,b}^*(0) = h_c^*(b), \quad \partial h_{c,b}(0) = \partial h_c(b), \quad h_{c,b}^* = h_c^* - \langle b, \cdot \rangle. \tag{8} \]

(i) Using [3] and [16, Th. 2.6.1(iii)] one gets

\[ \text{val}(P_{c,b}) = h_{c,b}(0) = h_c(b) \geq h_{c,b}^*(b) = h_{c,b}^*(0) = \text{val}(P_{c,b}). \]

The conclusion follows observing that \( [\text{val}(P_{c,b}) < \infty \iff \text{val}(P_{c,b}) \geq \infty \iff F^*(c) \neq \emptyset] \).

(ii) and (iii) Having in view [3], these assertions are nothing else than assertions (v) and (vi) from [16, Th. 2.6.1] in the present case, respectively.

(iv) Having in view [3], the first equivalence is provided by [16, Th. 2.6.1(vii)], while the other two are immediate from [4]. \( \square \)

Notice that the equality \( h_{c,b}^*(b) = \text{val}(P_{c,b}^*) \) from Proposition [1](i) is established in [12, Prop. 2.2], the other assertions from (i) being well known; moreover, the assertion (iii) is established in [12, Prop. 2.5], while the equivalence in (iii) is also established in [5, Th 1].

The next result provides sufficient interiority conditions for the subdifferentiability of \( h_c \) at \( b \).

Proposition 2 Let \( Y_0 := \text{span}(A(P) - Q) \) be endowed with the induced topology. Assume that one of the following conditions is verified:

(i) there exists \( \lambda_0 \in \mathbb{R} \) such that \( N_0 := \{ y \in Y \mid \exists x \in P : Ax - y \in Q \text{ and } \langle x, c \rangle \leq \lambda_0 \} \) is a neighborhood of \( b \) in \( Y_0 \);

(ii) there exists \( x_0 \in P \) such that \( Ax_0 - b \in \text{int} Y_0 Q \);

(iii) \( X \) and \( Y \) are Fréchet spaces, \( P, Q \) and \( Y_0 \) are closed, and \( b \in \text{icr}(A(P) - Q) \);

(iv) \( \text{dim} Y_0 < \infty \) and \( b \in \text{icr}(A(P) - Q) \).

Then either (a) \( h_c(y) = - \infty \) for every \( y \in \text{icr}(A(P) - Q) \), and so \( \text{val}(P_{c,b}) = \text{val}(P_{c,b}^*) = - \infty \), or (b) \( c \in P^* + A^*(Q^+) \) and \( h_c|_{Y_0} \) is continuous at \( b \). Consequently, in the case (b), \( \partial h_c(b) \neq \emptyset \), whence \( \text{val}(P_{c,b}) = \text{val}(P_{c,b}^*) \in \mathbb{R} \) and \( (P_{c,b}^*) \) has optimal solutions; moreover, if \( Y_0 = Y \), then \( \text{Sol}(P_{c,b}^*) \) is \( w^* \)-compact.

Proof. We apply [16, Th. 2.7.1] for \( \Phi := \Phi_{c,b} \) and its associated value function \( h_{c,b} \).

Clearly, \( \text{dom} \Phi_{c,b} = \bigcup_{x \in P} \{ x \} \times (Ax - Q - b) \), and so \( \text{dom} h_{c,b} = \text{Pr}_{Y} \text{dom} \Phi_{c,b} = A(P) - Q - b = \text{dom} h_c - b \). Observe that each of the conditions (i)–(iv) implies that \( b \in \text{icr}(A(P) - Q) = \text{dom} h_c - b \), and so \( h_c(b) < \infty \).
Because \( b \in A(P) - Q \) and \( P, Q \) are convex cones, it follows that
\[
\text{span } (\text{Pr}_{Y}(\text{dom } \Phi_{c,b})) = \text{span } (A(P) - Q) = Y_0 = A(P - P) + Q - Q \supseteq Q \cup A(P) \cup \{b\}.
\]

Observe that, if (i), (ii), (iii) or (iv) is verified, then condition (i), (iii), (vii) or (viii) from [16 Th. 2.7.1] is verified, respectively; using this theorem, one obtains that either \( h_{c,b}(0) = -\infty \), or \( h_{c,b}(0) \in \mathbb{R} \) and \( h_{c,b}|_{Y_0} \) is continuous at 0, or, equivalently, either \( h_c(b) = -\infty \), or \( h_c(b) \in \mathbb{R} \) and \( h_c|_{Y_0} \) is continuous at \( b \).

If \( h_c(b) = -\infty \), then \( h_c(y) = -\infty \) for every \( y \in \text{icr}(\text{dom } h_c) \) \( [= \text{icr}(A(P) - Q)] \) and \( \text{val}(P_{c,b}) = \text{val}(P_{c,b}^*) = -\infty \) by Proposition [1(i)]; hence (a) holds.

If \( h_c(b) \in \mathbb{R} \), then \( h_c|_{Y_0} \) is proper, whence \( \partial h_c(b) \neq \emptyset \) by [16 Th. 2.4.12] (that is (b) holds), and so \( \text{val}(P_{c,b}) = \text{val}(P_{c,b}^*) \in \mathbb{R} \) and \( (P_{c,b}^*) \) has optimal solutions by Proposition [1(iii)]. Moreover, if \( Y_0 = Y \) then \( h_c \) is continuous at \( b \), and so \( \partial h_c(b) \) is \( w^* \)-compact by [16 Th. 2.4.9]. \( \Box \)

Observe that Kretschmer obtained, in [9 Th. 3], that \( g(c,b) = 0 \) and that \( (P_{c,b}^*) \) has optimal solutions when \( Y_0 = Y, P, Q \) are closed, \( \text{val}(P_{c,b}) \in \mathbb{R} \), and condition (iii) above holds; the version of [9 Th. 3] for the dual problem is stated in [9 Cor. 3.1]. Notice also that the equality \( g(c,b) = 0 \) is established for \( Y_0 = Y \) and \( \text{val}(P_{c,b}) \in \mathbb{R} \) in Theorems 3.11–3.13 from [1]: more precisely, one asks: (iii) above in [1 Th. 3.13]; \( Q = \{0\} \) and (i) holds in [1 Th. 3.11]; \( Q = \{0\} \), \( X, Y \) are Banach spaces and \( \text{int } P \neq \emptyset \) in [1 Th. 3.12]. In fact, in the latter case it is shown that (i) holds. Moreover, the version of [1 Th. 3.13] for the dual problem, when \( P \) and \( Q \) are closed, is stated in [1 Cor. 3.14]. In [1 Sect. 3.8] one finds more references concerning the absence of duality gap in linear programming. Furthermore, [12 Prop. 2.9] states that \( \text{val}(P_{c,b}) = \text{val}(P_{c,b}^*) \) and \( \text{Sol}(P_{c,b}^*) \) is nonempty and bounded provided that \( X, Y \) are Banach spaces and \( b \in \text{int } (A(P) - Q) \); of course, \( Y_0 = Y \) in this case.

In the last 20 years, the interest for studying optimization problems in the algebraic framework increased. In fact, several results established in this framework can be deduced from the corresponding topological ones; the next corollary is a sample. Actually, having a real linear space \( E \), the strongest locally convex topology on \( E \), denoted \( \tau_c \) and called the convex core topology, is generated by the family of all the semi-norms defined on \( E \) (see, e.g., [6 Exer. 2.10]); notice that the family \( \{N \subset E \mid N \text{ is convex and } \text{cor } N \neq \emptyset \} \) is a neighborhood base of 0 \( \in E \) for \( \tau_c \). If \( \dim E < \infty \), \( \tau_c \) coincides with any Hausdorff linear topology on \( E \). Moreover, \( (E, \tau_c)^* = E' \), where \( E' \) is the algebraic dual of \( E \), and so \( (E', \sigma(E', E))^* = E; \) hence \( \tau_c \) coincides with the Mackey topology of \( E \) for the dual pair \( (E, E') \). Because \( (E', \tau_c)^* = (E')' \), one has \( w^* \neq \tau_c \) if \( \dim E = \infty \). Having another real linear space \( F \), any linear operator \( B : E \to F \) is continuous when \( E \) and \( F \) are endowed with their convex core topologies. Proposition 6.3.1 from [7] collects several results concerning \( \tau_c \).

**Corollary 3** Let the real linear spaces \( X \) and \( Y \) be endowed with their convex core topologies. Assume that \( b \in \text{icr}(A(P) - Q) \). Then the conclusions of Proposition [2] hold.

Proof. In the proof of Proposition [2] we observed that \( \text{dom } h_c = A(P) - Q \), \( h_c(y) = -\infty \) for every \( y \in \text{icr}(\text{dom } h_c) \) if \( h_c(b) = -\infty \), and that \( h_c|_{Y_0} \) is proper if \( h_c(b) \in \mathbb{R} \), whence \( h_c \) is subdifferentiable on \( \text{icr}(\text{dom } h_c) \) by [7 Prop. 6.3.1(v)]. \( \Box \)

In the case \( Q := \{0\} \), Corollary 3 is comparable with [10 Th. 4]. More precisely, the main part of [10 Th. 4], stated in the proof of Point 1, establishes (in our terms) that \( \text{val}(P_{c,b}) = \text{val}(P_{c,b}^*) \) whenever \( \text{cor } P \neq \emptyset \) (with \( P := C \)), \( Q := \{0\} \) and \( \text{val}(P_{c,b}) \in \mathbb{R} \). Observe
that Corollary 3 may not be used for $b \in A(P) \setminus \text{icr } A(P)$ even if $\text{cor } P \neq \emptyset$. Based on the following example, we assert that the conclusion of [10 Th. 4] might be false for $b \in A(P) \setminus \text{icr } A(P)$. The next example can be found in [17].

Example 4 (See [17 Examp. 4].) Consider $X_0 := \mathbb{R}^2$, $Y := \mathbb{R}^3$, $A_0 : X_0 \to Y$ with $A_0(x_1, x_2) := (x_1, x_2, 0)$,

$$P_0 := \mathbb{R} \times \mathbb{R}^+,$$ 

$$Q_0 := \{(y_1, y_2, y_3) \in Y \mid y_1, y_3 \in \mathbb{R}^+, (y_2)^2 \leq 2y_1y_3\},$$

the conic linear programming problem

$$(P_y)_y \text{ minimize } x_2 \text{ s.t. } x := (x_1, x_2) \in P_0, A_0x - y \in Q_0,$$

and $h_0 : Y \to \mathbb{R}$ defined by $h_0(y) := \text{val}(P_y)$. Then $h_0(y) = y_2$ if $y := (y_1, y_2, y_3) \in (\mathbb{R} \times \mathbb{R}^+ \times \{0\})$, $h_0(y) = 0$ for $y \in \mathbb{R} \times \mathbb{R} \times (-\mathbb{P})$, and $h_0(y) = \infty$ elsewhere.

Observe that the problem $(P_y)$ is equivalent with the following one:

$$(P'_y) \text{ minimize } x_2 \text{ s.t. } (x, v) \in P := P_0 \times Q_0, A(x, v) = y,$$

where $A : X := X_0 \times Y \to Y$ is defined by $A(x, v) := A_0x - v$. Clearly, the value function associated to problem $(P'_y)$ is $h_0$. By Proposition 1(i) one has that $\text{val}(P_{c,b}) = h_0(b) = b_2 > 0 = h_0^*(b) = \text{val}(P_{c,b}^*)$ for $b \in \mathbb{R} \times \mathbb{R} \times (-\mathbb{P}) = A(P) \setminus \text{icr } A(P)$.

Corollary 5 Let $(c, b) \in \Lambda$ and set $Y_0 := \text{span}(A(P) - Q)$. Assume that one of the conditions (i)–(iv) of Proposition 2 holds. Then $\text{int}_{Y_0}(\text{dom } h_c) = \text{int}_{Y_0}(A(P) - Q) \neq \emptyset$ and $h_c|_{Y_0}$ is finite and continuous at every $y \in \text{int}_{Y_0}(\text{dom } h_c)$; consequently, all the other conclusions of Proposition 3 hold at $y$. Moreover, assume that $y \in \text{dom } h_c \setminus \text{int}_{Y_0}(\text{dom } h_c)$ is such that $(h_c)'_+(y, \cdot)$ is proper; then $\partial h_c(y) \neq \emptyset$, and so $\text{val}(P_{c,y}) = \text{val}(P_{c,y}^*)$ and $\text{Sol}(P_{c,y}) = \partial h_c(y)$.

Proof. Recall that having a proper convex function $f : Y \to \mathbb{R}$ which is continuous at some $y_0 \in \text{dom } f$ (or, equivalently, $f$ is bounded above on some nonempty open subset of $\text{dom } f$), then $\text{int}(\text{dom } f) \neq \emptyset$ and $f$ is continuous at any $y \in \text{int}(\text{dom } f)$ (see, e.g., [16 Th. 2.2.9]).

Take $b' \in \text{int}_{Y_0}(\text{dom } h_c)$; because, in the present situation, $h_c|_{Y_0}$ is continuous at $b$, it follows that $h_c|_{Y_0}$ is continuous at $b'$, and so the other conclusions of Proposition 2 hold at $b'$.

Assume now that $b \in \text{dom } h_c \setminus \text{int}_{Y_0}(\text{dom } h_c)$ is such that $(h_c)'_+(b, \cdot)$ is proper. Consider $f := h_c|_{Y_0}$; of course, $f$ is finite and continuous at $b$ and $f'_+(\vec{b}, \cdot) := (h_c)'_+(\vec{b}, \cdot)|_{Y_0}$ is proper. Applying [2 Prop. 2.134(ii)] we get $(\partial h_c|_{Y_0}(\vec{b}) = ) \partial f(\vec{b}) \neq \emptyset$, whence $\partial h_c(\vec{b}) \neq \emptyset$.

Remark 6 Notice that, besides the fact that $g(c, b) = 0$ for $(c, b) \in (P^+ + A^*(Q^+)) \times \text{icr } (A(P) - Q)$, Corollary 5 shows that $\text{Sol}(P_{c,b}^*)$ is nonempty for such $(c, b)$.

Observe that Shapiro [12 Prop. 2.8], when $h_c(b) \in \mathbb{R}$ and either dim $Y < \infty$ or $h_c$ is bounded above on a nonempty open set, established that $|\text{val}(P_{c,b}) = \text{val}(P_{c,b}^*)$ and $\text{Sol}(P_{c,b}^*) \neq \emptyset|$ iff $(h_c)'_+(b, \cdot)$ is proper (that is, $\partial h_c(b) \neq \emptyset \iff (h_c)'_+(b, \cdot)$ is proper).

Notice also that one uses the same hypotheses on $A$ and $b$ as in Theorems 3.11 and 3.12 from [1] for getting $g(c, b) = 0$ for $(c, b) \in (P^+ + A^*(Q^+)) \times \text{cor } (A(P) - Q)$ in Theorems 3.1 and 3.2 from [14], respectively; having in view the discussion after the proof of Proposition 2 the conclusions of [14 Ths. 3.1, 3.2] follow by using Corollary 5.
3 On Gale’s example in conic linear programming

We have seen (see Example 4) that, even in finite-dimensional conic linear programming, it is possible to have problems with positive duality gap; of course such examples exist also in the infinite-dimensional case. The main aim of this section is to answer the open problem from [14, p. 12] concerning the perturbed Gale’s example.

Gale’s example is presented in Section 3.4.1 of the book [11] as being the problem:

\[(PG) \text{ minimize } x_0 \text{ s.t. } x_0 + \sum_{k=1}^{\infty} kx_k = 1, \sum_{k=1}^{\infty} x_k = 0, x_k \geq 0, \ k = 0, 1, 2, \ldots \]

In this example, the primal variable space is

\[X := \mathbb{R}^{(n)} := \{(x_k)_{k \in \mathbb{N}} \subset \mathbb{R} \mid \exists k_0 \in \mathbb{N}, \forall k \geq k_0 : x_k = 0\},\]

called the generalized finite sequence space; in the sequel \(\mathbb{N} := \{0, 1, 2, \ldots\}\). The dual of \((PG)\) is

\[(PG^*) \text{ maximize } y_1 \text{ s.t. } y_1 \leq 1, ky_1 + y_2 \leq 0, k = 1, 2, \ldots,\]

in which the dual variable space is \(\mathbb{R}^2\).

As mentioned in [1], the only feasible (and so, optimal) solution of \((PG)\) is \(x = (1, 0, 0, \ldots)\); hence \(\text{val}(PG) = 1\). Moreover, \((y_1, y_2) \in \mathbb{R}^2\) is feasible for \((PG^*)\) iff \(y_1 \leq 0, y_1 + y_2 \leq 0\), and so \((0, 0)\) is an optimal solution of \((PG^*)\). Hence \(\text{val}(PG^*) = 0\), and so there is a positive duality gap.

Observing that \((e_n)_{n \in \mathbb{N}}\) is an algebraic (Hamel) basis of \(X\), where \(e_n := (\delta_{nk})_{k \in \mathbb{N}}\) (\(\delta_{nk}\) being the Kronecker symbol), the algebraic dual \(X'\) of \(X\) is \(\mathbb{R}^n\), the space of all real sequences \(x' := (x_k)_{k \geq 0}\), with

\[x'(x) := \langle x, x' \rangle := \sum_{k=0}^{\infty} x_k x'_k \quad (x := (x_k)_{k \in \mathbb{N}} \in X).\]

Take \(c := e_0 \in X', b := (1, 0) \in \mathbb{R}^2\),

\[P := \mathbb{R}^{(n)}_+ := \{x \in \mathbb{R}^{(n)} \mid \forall k \in \mathbb{N} : x_k \geq 0\}, \quad Q := \{0\} \subset \mathbb{R}^2, \quad A : \mathbb{R}^{(n)} \to \mathbb{R}^2, \quad A(x) := \left( x_0 + \sum_{k=1}^{\infty} kx_k, \sum_{k=1}^{\infty} x_k \right); \quad (9)\]

then \(P\) and \(Q\) are convex cones and \(A\) is a linear operator.

We endow \(Y := \mathbb{R}^2\) with its usual inner product, and so \(Y'\) is identified with \(Y\); moreover, we endow \(X := \mathbb{R}^{(n)}\) with the convex core topology \(\tau_c\), and so \(X' = \mathbb{R}^n\) is the topological dual of \(X\). So, the operator \(A\) from \((PG)\) is (now) a continuous linear operator whose adjoint is

\[A^* : Y = \mathbb{R}^2 \to X' = \mathbb{R}^n, \quad A^*(u, v) = (u, u + v, \ldots, ku + v, \ldots),\]

and so \((P^*_{c,b})\) becomes problem \((PG^*)\).

At the end of the previous section, we mentioned that Theorems 3.1 and 3.2 from [14] provide conditions which ensure that \(g(c, b) = 0\) when \((c, b) \in (P^+ + A^*(Q^+)) \times \text{cor}(A(P) - Q)\) for the problem \((P_{c,b})\) with \(Q = \{0\}\). Besides such results, in [14], one considers “the example of Gale with both \(b\) and \(c\) being perturbed” to illustrate [14, Th. 3.1]; the problems \((P_{c,b})\) and \((P^*_{c,b})\) with \(X := \mathbb{R}^{(n)}, Y := \mathbb{R}^2, P\) and \(Q\) as in (9), \(A\) as in (10) and arbitrary \((c, b) \in X' \times Y\), are denoted by \((PG_{c,b})\) and \((PG^*_{c,b})\), respectively.
On page 12 of [14], one mentions: “In connection with Claim 2, we observe that the question whether the result is valid without the extra assumption (6) remains open”. Our aim in this section is to answer this open problem.

In the rest of this section, $F(b), \varphi(c,b), F^*(c), \psi(c,b), \Lambda$ and $g$ refer to the problems $(PG_{c,b})$ and $(PG_{c,b}^*)$. It follows that $A(P) = \{(b_1, b_2) \in \mathbb{R}^2 \mid b_1 \geq b_2 \geq 0\}$ and $c \in \mathbb{P}^+ + \text{Im} A^*$ if and only if there exist $(u,v) \in \mathbb{R}^2$ and $(\beta_k)_{k \in \mathbb{N}} \in \mathbb{R}_+^N$ such that

$$c_0 = u + \beta_0, \quad c_1 = u + v + \beta_1, \quad \ldots, \quad c_k = ku + v + \beta_k, \quad \ldots$$

(11)

**Lemma 7** Let $c := (c_k)_{k \in \mathbb{N}} \in \mathbb{R}^\mathbb{N}$, $(u,v) \in \mathbb{R}^2$ and $(\beta_k)_{k \in \mathbb{N}} \in \mathbb{R}_+^N$ verify (11), $\beta' \in [0,\beta_0]$ and $v' \in \mathbb{R}$. Then $(u,v) \in F^*(c)$; moreover

$$(u + \beta', v') \in F^*(c) \iff v' \leq v + \inf \{\beta_k - \beta'k \mid k \geq 1\} \Rightarrow \beta' \leq \overline{\beta},$$

(12)

where

$$\overline{\beta} := \liminf_{k \to \infty} \beta_k/k \in [0,\infty].$$

(13)

Furthermore, if $\beta' < \overline{\beta}$, then there exists $v'' \in \mathbb{R}$ such that $(u + \beta', v'') \in F^*(c)$.

Proof. Because (11) is verified, $A^*(u,v) - c \leq 0$, and so $(u,v) \in F^*(c)$. Set $v'' := v' - v$; because $u + \beta' \leq c_0$, one has that

$$(u + \beta', v') \in F^*(c) \iff \forall k \geq 1 : k(u + \beta') + v' \leq c_k \iff \forall k \geq 1 : \beta_k \geq \beta'k + v'$$

$$(u + \beta', v') \in F^*(c) \iff v' \leq v + \inf \{\beta_k - \beta'k \mid k \geq 1\},$$

whence the equivalence from (12) follows. Assume now that $\beta_k \geq \beta'k + v''$ for $k \geq 1$; then $\beta_k/k \geq \beta' + v''/k$ for $k \geq 1$, whence

$$\overline{\beta} = \liminf_{k \to \infty} \beta_k/k \geq \liminf_{k \to \infty} (\beta'k + v''/k) = \beta',$$

and so the (last) implication from (12) holds, too.

Assume that $\beta' < \overline{\beta}$; then there exists $k_0 \geq 1$ such that $\beta' \leq \beta_k/k$ for $k \geq k_0$, whence $k(u + \beta') + v = ku + v + k\beta' \leq k\overline{\beta} + v + \beta_k = c_k$ for $k \geq k_0$. Setting

$$v_0 := \min \{c_k - k(u + \beta') \mid k \in \overline{1,k_0}\}, \quad v'' := \min \{v,v_0\},$$

one has $k(u + \beta') + v'' \leq c_k$ for $k \geq 1$; because $\beta' \leq \beta_0$ and $v'' \leq v$ we get $k(u + \beta') + v'' \leq c_k$ for $k \geq 0$, whence $(u + \beta', v'') \in F^*(c)$.

Claim 2 from [14] asserts, equivalently, the following:

*Suppose that $b = (b_1,0)$ with $b_1 > 0$ and $c \in \mathbb{P}^+ + \text{Im} A^*$ is of the form (11) such that $\overline{\beta} = 0$, where $\overline{\beta}$ is defined in (13). Then, $g(c,b) > 0$ if $\beta_0 > 0$ and $g(c,b) = 0$ if $\beta_0 = 0$.***

As recalled above, on page 12 of [14] it is said: “In connection with Claim 2, we observe that the question whether the result is valid without the extra assumption (6) remains open”.

Related to [14] Claim 2 we have the following result:

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1Assumption (6) from [14] is equivalent to $\overline{\beta} = 0$, where $\overline{\beta}$ is defined in (13).
Proposition 8 Consider $b := (b_1, 0)$ with $b_1 > 0$ and
\[ c = (u + \beta_0, u + v + \beta_1, \ldots, ku + v + \beta_k, \ldots) \in \mathbb{R}^N \]
with $(u, v) \in \mathbb{R}^2$ and $(\beta_k)_{k \in \mathbb{N}} \in \mathbb{R}_+^N$; hence $(c, b) \in \Lambda$. Then $F(b) = \{b_1 e_0\}$, and
\[ \varphi(c, b) = b_1 c_0, \quad \psi(c, b) = b_1 \overline{u}, \quad g(c, b) = b_1 \max\{0, \beta_0 - \overline{\beta}\}, \quad \tag{14} \]
where $\overline{\beta}$ is defined in (13) and $\overline{u} := u + \min\{\beta_0, \beta\}$; moreover, $(u, v)$ is an optimal solution of $(PG_{c,b}^e)$ if $\min\{\beta_0, \beta\} = 0$, and $(PG_{c,b}^e)$ has optimal solutions if $0 < \beta_0 < \overline{\beta}$. Furthermore, if $\beta_0 \geq \overline{\beta} > 0$, $(PG_{c,b}^e)$ has optimal solutions if and only if $\inf\{\beta_k - \overline{\beta} k \mid k \geq 1\} \in \mathbb{R}$.

Proof. Observe first that $F(b) = \{b_1 e_0\}$, whence $\varphi(c, b) = b_1 c_0$, and $(u, v) \in F^*(c)$.

Consider $(u', v') \in F^*(c)$; then $u' \leq c_0 = u + \beta_0$ and $ku' + v' \leq c_k = ku + v + \beta_k$ for $k \geq 1$.

It follows that $u' + v'/k \leq u + v/k + \beta/k$ for $k \geq 1$, whence $u' \leq u + \overline{\beta}$, where $\overline{\beta}$ is defined in (13). It follows that $u' \leq \overline{u} \leq c_0$ for all $(u', v') \in F^*(c)$.

Because $\psi(c, b) = \sup\{b_1 u' \mid (u', v') \in F^*(c)\}$ and $(u, v) \in F^*(c)$, we get
\[ b_1 u \leq \psi(c, b) \leq b_1 \overline{u} \leq b_1 c_0 = \varphi(c, b). \quad \tag{15} \]

Assume first that $\min\{\beta_0, \overline{\beta}\} = 0$; then $u = \overline{u}$, whence $\psi(c, b) = b_1 u = b_1 \overline{u}$ by (15), and so $(u, v)$ is an optimal solution of $(PG_{c,b}^e)$.

Assume now that $\min\{\beta_0, \overline{\beta}\} > 0$; take $0 < \beta' < \min\{\beta_0, \overline{\beta}\}$. Using the last part of Lemma 7 it follows that there exists $v'' \in \mathbb{R}$ such that $(u + \beta', v'') \in F^*(c)$, and so $\psi(c, b) \geq b_1 (u + \beta')$. Because $\beta'$ is arbitrary in $[0, \min\{\beta_0, \overline{\beta}\}]$, one obtains that $\psi(c, b) \geq b_1 \overline{u}$. Having in view (15), one gets $\psi(c, b) = b_1 \overline{u}$. Hence $\varphi(c, b) = b_1 c_0$ and $\psi(c, b) = b_1 \overline{u}$, and so
\[ g(c, b) := \varphi(c, b) - \psi(c, b) = b_1 (c_0 - \overline{u}) = b_1 (\beta_0 - \min\{\beta_0, \overline{\beta}\}) = b_1 \max\{0, \beta_0 - \overline{\beta}\}; \]
therefore, (14) holds.

As seen above, $(u, v)$ is an optimal solution of $(PG_{c,b}^e)$ when $\min\{\beta_0, \overline{\beta}\} = 0$. Assume that $0 < \beta_0 < \overline{\beta}$. Using again the last part of Lemma 7 with $\beta_0$ instead of $\beta'$, one gets $v'' \in \mathbb{R}$ such that $F^*(c) \ni (u + \beta_0, v'') = (\overline{u}, v'')$, and so $(\overline{u}, v'')$ is an optimal solution of $(PG_{c,b}^e)$.

Assume that $\beta_0 \geq \overline{\beta} > 0$. Because $\psi(c, b) = b_1 \overline{u} = b_1 (u + \beta)$, $(PG_{c,b}^e)$ has optimal solutions if and only if there exists $v' \in \mathbb{R}$ such that $(\overline{u} + \overline{\beta}, v') \in F^*(c)$, and this is equivalent with $\inf\{\beta_k - \overline{\beta} k \mid k \geq 1\} \in \mathbb{R}$ by the equivalence in (12). \hfill \Box

Claim 4 from (14) asserts, equivalently, the following:
For every $c \in P^+ + \text{Im } A^*$ and $b = (b_1, b_2)$ with $b_1 > b_2 > 0$, one has $g(c, b) = 0$.

The proof of Claim 4 is quite involved in (14); for its proof one uses (14), Th. 3.1. In fact the conclusion of this assertion follows immediately using Corollary 5. Indeed, because $A(P) = A(P) - Q = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq y_2 \geq 0\}$, cor $A(P) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 > y_2 > 0\}$, and so condition (iv) of Proposition 2 is verified; as observed in Remark 6, $g(c, b) = 0$ and sol$(PG_{c,b}^e) \neq \emptyset$ for all $(c, b) \in (P^+ + A^*(Q^+)) \times \text{icr } A(P)$.

In (8), only $b$ is perturbed in Gale’s example; hence $c := e_0$. So, one may take $u := v := 0$, $\beta_0 := 1$ and $\beta_k := 0$ for $k \geq 1$ in (11), and so $\overline{\beta} = 0$ in (13). The results on problem $(PG_{e_0,b})$ and $(PG_{e_0}^*)$ from (8, Claims 2, 3) confirm those obtained in (14) for this case; even more, it is obtained that both problems have optimal solutions for $b \in A(P)$. 

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4 On Gale’s example from Van Slyke and Wets paper [13]

At the end of Section 3.8 of Anderson & Nash book [1] it is mentioned that “The example of Section 3.5.1 is due to Gale and appears in a paper by Van Slyke and Wets (1968).”

Indeed, in Van Slyke and Wets paper [13], one finds the following text containing also the problem to which Gale’s example refers:

“Our purpose is to obtain characterizations of the optimal solutions to

\[ \text{Find inf } z = c(x) \text{ subject to } \langle A, x \rangle = b, \ x \in X \subseteq \mathcal{X} \]  

(3.1)

where \( c(x) \) is a convex functional defined everywhere but possibly with values \(+\infty\) or \(-\infty\) for some \( x \) in \( \mathcal{X} \), \( A \) is a continuous linear operator from \( \mathcal{X} \) into \( \mathcal{Y} \), both locally convex linear Hausdorff topological spaces and \( X \) is a convex subset of \( \mathcal{X} \). By \( \langle \cdot, \cdot \rangle \) we denote linear composition. It is easy to see that (3.1) is equivalent to

\[ \text{Find inf } \eta \text{ such that } (\eta, y) \in C \cap L \subset \mathbb{R} \times \mathcal{Y}, \]  

(3.2)

where \( \mathbb{R} \) denotes the real line (with its natural topology),

\( C = \{(\eta, y) \mid \eta \geq c(x), \ y = b - \langle A, x \rangle \text{ for some } x \in X\} \) and

\( L = \{(\eta, 0) \mid \eta \in \mathbb{R}, \ 0 \in \mathcal{Y}\} \).

... the following simple example, due to David Gale (which is a variant of the one given in [9]).

EXAMPLE (3.5). Consider the semi-infinite program

\[ \text{Find inf } x_0 \text{ subject to } x_0 + \sum_{n=1}^{\infty} nx_n = 1, \ \sum_{n=1}^{\infty} x_n = 0, \ x_n \geq 0, \ n = 0, 1, ..., \]  

and \( \mathcal{X} = \ell_p, \ 1 \leq p < \infty \).

Now

\[ C = \{(\eta, y_1, y_2) \mid \eta \geq x_0, \ y_1 = 1 - x_0 - \sum_{n=1}^{\infty} nx_n, \ y_2 = 0 - \sum_{n=1}^{\infty} x_n, \ x_n \geq 0, \ n = 0, 1, ..., \}

We first observe that

\[ \overline{C} \cap L = C \cap L = \{(\eta, 0, 0) \mid \eta \geq 0\}. \]

But since the closure of \( C \) is

\[ \overline{C} = \{(\eta, y_1, y_2) \mid \eta \geq x_0, \ y_1 = 1 - x_0 - \sum_{n=1}^{\infty} nx_n - z, \ y_2 = 0 - \sum_{n=1}^{\infty} x_n, \ x_n \geq 0, \ n = 0, 1, ..., \}

we have that \( L \cap \overline{C} = \{(\eta, 0, 0) \mid \eta \geq 0\}. \) Note that in this example \( L \cap C \) is closed and thus the inf is attained.”

From the above text, we understand that \( \mathcal{X}, \mathcal{Y} \) are real H.l.c.s., \( A : \mathcal{X} \to \mathcal{Y} \) is a continuous linear operator, \( c : \mathcal{X} \to \mathbb{R} \) is convex and \( X \subseteq \mathcal{X} \) is a convex set; we set \( Ax \) instead of \( \langle A, x \rangle \). Moreover,

\[ C = \{(\eta, b - Ax) \mid x \in X, \ \eta \in \mathbb{R} \cap [c(x), \infty)\} \subset \mathbb{R} \times \mathcal{Y}, \quad L = \mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathcal{Y}. \]

Below, we refer to Example (3.5) from [13] without mentioning it explicitly. Having in view the formulation of this example, in the sequel, all the sequences are indexed by \( n \in \mathbb{N} \); for example, by \( x := (x_n) \) we mean \( x := (x_n)_{n \in \mathbb{N}} \in \mathbb{R}^\mathbb{N} \).

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1 In fact, it is Section 3.4.1, pages 42, 43.
2 This is our reference [4].
So, $X := \ell_p$ with $1 \leq p < \infty$ and $Y := \mathbb{R}^2$; consequently, $X^*$ is identified with $\ell_q$, where $q := p/(p-1) \in ]1,\infty[ \cap ]1,\infty[ \cap ]1,\infty[$, $q := \infty$ for $p = 1$, and $(x,y) = \sum_{n \geq 0} x_n y_n$ for $x := (x_n) \in X$ and $y := (y_n) \in X^*$. Moreover, $Y^*$ is identified with $\mathbb{R}^2 (= Y)$.

For $z := (z_n) \in \mathbb{R}^N$, by $\sum_{n=0}^{\infty} z_n \in \mathbb{R}$ we mean that the series $\sum_{n=0}^{\infty} z_n$ is convergent in $\mathbb{R}$ (and so its sum belongs to $\mathbb{R}$ and $z \in c_0 := \{ x \in \mathbb{R}^N \mid x_n \to 0 \}$).

**Remark 9** Consider the sets

$$
D_0^a := \{ z \in \mathbb{R}^N \mid \sum_{n=0}^{\infty} n |z_n| \in \mathbb{R} \}, \quad D_0 := \{ z \in \mathbb{R}^N \mid \sum_{n=0}^{\infty} n z_n \in \mathbb{R} \},
$$

$$
D_1 := \{ z \in \mathbb{R}^N \mid \sum_{n=0}^{\infty} n^1 z_n \in \mathbb{R} \}.
$$

Then $D_0^a$, $D_0$ and $D_1$ are linear spaces such that $\mathbb{R}(\mathbb{N}) \subset D_0^a \subset D_0 \cap \ell_1 \subset c_0$ and $D_0 \subset D_1 \cap \ell_r$ for every $r \in ]1,\infty[ \setminus \mathbb{N}$, each inclusion being strict; moreover, $D_0 \not\subset \ell_1$.

Proof. To check that $D_0^a$, $D_0$ and $D_1$ are linear spaces is routine.

The inclusions $\mathbb{R}(\mathbb{N}) \subset D_0^a \subset D_0 \cap \ell_1 \subset c_0$ are obvious; moreover, $(2^{-n}) \in D_0^a \setminus \mathbb{R}(\mathbb{N})$, 

$$
\left( \frac{(-1)^n}{n+1} \right) \in (D_0 \cap \ell_1) \setminus D_0^a \quad \text{and} \quad \left( \frac{1}{n+1} \right) \in c_0 \setminus \ell_1.
$$

The inclusion $D_0 \subset D_1$ follows by Abel’s test. Take $z := (z_n) \in D_0$ and $r \in ]1,\infty[$. Then $(n z_n)$ is bounded, whence there exists $\mu > 0$ such that $|n z_n| \leq \mu$ for $n \in \mathbb{N}$, and so $|z_n|^r \leq \mu^r/n^r$ for $n \geq 1$; hence $\sum_{n=0}^{\infty} |z_n|^r$ is convergent. Moreover, 

$$
\left( \frac{(-1)^n}{n+1} \right) \in (D_1 \cap \ell_r) \setminus D_0
$$

for $r \in ]1,\infty[$. Furthermore, take $z := (z_n)$ defined by $z_0 := z_1 := 0$, $z_n := \frac{(-1)^n}{n \ln n}$ for $n \geq 2$; then 

$$
\sum_{n=0}^{\infty} n z_n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} \in \mathbb{R}
$$

by Leibniz’s test, but $\sum_{n=0}^{\infty} |z_n| = \sum_{n=2}^{\infty} \frac{1}{n \ln n} = \infty$, and so $z \in D_0 \setminus \ell_1$. 

Having in view the description of the data considered in (3.1), in Example (3.5), for $x := (x_n)$, one must have that $c(x) := x_0$, 

$$
A x := \left( x_0 + \sum_{n=1}^{\infty} n x_n, \sum_{n=1}^{\infty} x_n \right) \in \mathbb{R}^2, \quad (16)
$$

$b := (1,0) \in \mathbb{R}^2$ and $X := P := (\ell_p)_+ := \ell_p \cap \mathbb{R}^N_+$. It follows that $c(x) = \langle x, e_0 \rangle$ for $x \in X$, and $P^+ = (\ell_q)_+$.

**Remark 10** Let $D_A := \{ x \in X \mid Ax \in Y \}$ be the (maximal) domain of $A$ as an operator from $X$ to $Y$. Then $D_A = X \cap D_0$, $D_A$ is a dense linear subspace of $X$, and $A : D_A \to Y$ is a linear operator; moreover, $\text{gph} A$ is not closed. The same conclusions hold for $D_0^a := X \cap D_0^a$ and $A|_{D_0^a}$.

Proof. From Remark 9 and (16) we obtain that $D_A = X \cap D_0 \cap D_1 = X \cap D_0 \supset \mathbb{R}(\mathbb{N})$, and so $D_A$ is a dense linear subspace of $X$. From the expression of $Ax$, it follows immediately that $A$ is linear.

Assume that $G := \text{gph} A (= \{ (x,Ax) \mid x \in D_A \})$ is a closed subset of $X \times Y$, and so $G$ is a closed linear subspace of the normed vector space $X \times Y$. Because $D_A \neq X$, there exists $x \in X \setminus D_A$, and so $x \neq 0$, $E := \{ x \} \times Y$ is a closed convex subset of $X \times Y$ and $E \cap G = \emptyset$; moreover, $E$ is (clearly) locally compact because $\text{dim} (\text{span} E) = 3$. Furthermore, the asymptotic cones of $E$ and $G$ are $C_E := \{ 0 \} \times Y$ and $C_G := G$, and so $C_E \cap C_G = \{ (0,0) \}$. 

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Applying [3, Prop. 1], the set $E - G (\subset X \times Y)$ is a closed convex set with $(0, 0) \not\in E - G$. Therefore, there exist $(u, v) \in X^* \times Y^*$ and $\gamma \in \mathbb{R}$ such that

$$0 > \gamma \geq \langle (\overline{x} - x, u) + (y - Ax, v) = \langle \overline{x}, u \rangle + \langle y, v \rangle - \langle x, u \rangle - \langle Ax, v \rangle \rangle \forall x \in D_A, \ y \in Y.$$  

Because $D_A$ and $Y$ are linear spaces, one obtains that $\langle y, v \rangle = 0$ for $y \in Y$ and $\langle x, u \rangle + \langle Ax, v \rangle = 0$ for $x \in D_A$, and so $\langle \overline{x}, u \rangle < 0$, $v = 0$ and $\langle x, u \rangle = 0$ for every $x \in D_A$. Since $D_A$ is dense in $X$ and $u \in X^*$, we get $u = 0$, contradicting the fact that $\langle \overline{x}, u \rangle < 0$. Hence $\text{gph} A$ is not closed. 

Remark [10] shows that Example (3.5) is not adequate for illustrating the problem (3.1).

Having in view Remark [10] the definition of $C$ in Example (3.5) is not clear (because $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} nx_n$ do not make sense for every $x \in X$). Let us consider that $C$ is the set

$$C' = \{(\eta, y) \in \mathbb{R} \times Y : \exists x \in \mathbb{R}^{n} : \eta \in [x_0, \infty[, y_1 = 1 - x_0 - \sum_{n=1}^{\infty} nx_n, y_2 = -\sum_{n=1}^{\infty} x_n \in \mathbb{R}\},$$

that is, $C'$ is the biggest set for which all the elements from the description of $C$ are well defined [10], hence $x$ from the definition of $C'$ belongs to $\mathbb{R}^{n} \cap D_A$.

Let $(\eta, y) \in C'$, and take $x \in \mathbb{R}^{n}$ such that $\eta \in [x_0, \infty[, y_1 = 1 - x_0 - \sum_{n=1}^{\infty} nx_n \in \mathbb{R}$ and $y_2 = -\sum_{n=1}^{\infty} x_n \in \mathbb{R}$. Then $x \in \ell_1$ because $x_n \geq 0$ for $n \geq 0$ and $\sum_{n=1}^{\infty} x_n \in \mathbb{R}$, and so $x \in \ell_r$ for $r \in [1, \infty[$; in particular, $x \in X$. It follows that $y_2 \leq 0$ and

$$y_1 = 1 - x_0 + y_2 - \sum_{n=2}^{\infty} (n - 1)x_n \leq 1 - x_0 + y_2 \leq 1 + y_2,$$

whence $y_1 \leq 1 + y_2 \leq 1$ and $x_0 \leq 1 + y_2 - y_1$; moreover, if $y_2 = 0$, then $x_n = 0$ for $n \geq 1$ and so $y_1 = 1 - x_0$, whence $\eta \geq x_0 = 1 - y_1$. Therefore, $C' \subset C_0 \cup C_1$, where

$$C_0 := \{(\eta, y_1, 0) \mid y_1 \leq 1, \ \eta \geq 1 - y_1\}, \ \ C_1 := \{(\eta, y_1, y_2) \mid y_1 - 1 \leq y_2 < 0, \ \eta \geq 0\}. \quad (17)$$

Conversely, take first $(\eta, y_1, y_2) \in C_0$. Then $y_2 = 0$, $y_1 \leq 1$ and $\eta \geq 1 - y_1 (\geq 0)$, and so $x := (1 - y_1, 0, 0, 0, 0, \ldots)$ verifies the conditions from the definition of $C'$; hence $(\eta, y_1, y_2) \in C'$. Take now $(\eta, y_1, y_2) \in C_1$, then $\eta \geq \beta > 0$, where $\alpha := 1 - y_1$ and $\beta := -y_2 > 0$. Take $n \in \mathbb{N}$ with

$$\overline{\eta} \geq \max\{2, \alpha/\beta\}, \ x_1 := (\overline{\eta} \beta - \alpha) / (\overline{\eta} - 1), \ x_\alpha := (\alpha - \beta) / (\overline{\eta} - 1) \text{ and } x_n := 0 \text{ for } n \in \mathbb{N} \setminus \{1, \overline{\eta}\};$$

then this $x$ verifies the conditions from the definition of $C'$, and so $(\eta, y_1, y_2) \in C'$. Therefore, $C' = C_0 \cup C_1$, where $C_0$ and $C_1$ are defined in (17).

Having in view the description of $C$ and $\overline{C}$ in [13] Example (3.5), in the case in which these sets were correctly defined, one would have $\overline{C} = C - (\{0\} \times \mathbb{R}_+ \times \{0\})$. However, we have that

$$\text{cl} C' = \{(\eta, y_1, y_2) \mid y_1 - 1 \leq y_2 \leq 0, \ \eta \geq 0\}.$$  

Consequently, $\overline{C' = \text{cl} C'} = \{(\eta, 0, 0) \mid \eta \geq 1\}$ and $\text{cl} C' = \{(\eta, 0, 0) \mid \eta \geq 0\}$.

In conclusion, Example (3.5) is not adequate for illustrating the problem (3.1) from [13] because the operator $A$ is not a continuous linear operator from $X$ into $Y$; moreover, because the graph of $A$ is not closed, one can not speak of its adjoint, which is necessary when defining (using) the dual problem. However, the sets $\overline{C'}$ and $\text{cl} C'$ are those indicated in [13] Example (3.5).\footnote{This agrees with the following text from [4]: “Definition 2. The infinite program $P$ is said to be consistent if there exists a sequence $(x_1, x_2, \ldots)$ for which all of the series in question converge and which satisfies all of the constraints. Such a sequence is termed feasible.”.}
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