Quantum Otto Engine based on Multiple-State Single Fermion in 1D Box System

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Abstract. In this work, a quantum Otto engine (QOE) based on a multiple-state single fermion particle trapped in an one-dimensional potential well has been studied. In this study, we used an analogical model completed with the first law of thermodynamics for the quantum system. The system satisfied the relativistic Hamiltonian, especially Dirac Hamiltonian. The engine consisted of the quantum isochoric and isentropic processes to form an Otto Cycle. The heat input and output were transferred under isochoric processes. As a result, the higher the number of energies was arranged, the lower the efficiency the system has. However, we could increase the efficiency of QOE by controlling the compression ratio. In addition, we also obtained the correspondence of the efficiency ultra-relativistic limits QOE and non-relativistic limits one.

Keywords: Quantum Otto engine, Dirac Hamiltonian, relativistic limits.

1. Introduction
A heat engine is a device for converting the heat energy to work, such as mechanical work, electrical work, chemical work, etc. The heat energy is absorbed from the heat reservoir and converted into work and partly transferred into a cold reservoir. Classically, the heat engine consists of an ideal gas as a working substance that expands and pushes a piston in a cylinder. The main difference between classical and quantum systems is the working substance and it is concerned with the discrete energy level [1–5]. It causes the quantum heat engine to have unique properties.

In recent years, the study of a quantum heat engine increases significantly. Some studies have been focusing on a working substance [4–10], and others on the dynamic of the source of a reservoir energy [11,12] with some models such as a potential well [4–8,13–15]], oscillator harmonic [11,12,16] etc. It has been also inspecting on the kind of heat engines like Carnot Engine [4,8,17,18], Diesel Engine [19–21], Otto engine [14,19,22]. The study has been done by [11] that used a harmonic oscillator model for irreversible cyclic. They have obtained that the efficiency can reach maximum power at a high temperature and quasi-static operating condition. Another study for a reversible cycle, the efficiency of quantum Otto engine has been similar with the classical one. Moreover, the efficiency has relied on the compression adiabatic ratio. Another study investigated the use of a single Dirac particle system for quantum Carnot engine and they obtained the efficiency of quantum Carnot engine which has the similarly with the classical cycle [7]. The optimization of relativistic Carnot Engine [15] and the internal effect process [8] have also been investigated.
For the generalization of Otto engine, however, it is necessary to study about a relativistic Otto Engine. The relativistic Otto engine which is explored uses a single fermion that fulfills the Dirac Hamiltonian. The study focused on evaluating the efficiency of quantum Otto engine (QOE) using a single fermion system as a working substance, trapped in a potential well. The number of energy arranged will also be investigated in order to find out the effect of level state on the performance of QOE in relativistic approach. The performance of relativistic QOE must be valid in a non-relativistic limit. Thus, we can clarify this result with the previous study.

2. Experimental Methods
In this work, we used an analogical model first law of thermodynamics implementation. The physical system that we chose was a single fermion which was trapped in 1 Dimensional system. The implementation of the first law of thermodynamics was used to redefine the external force act to the 1D box system.

Since we used the fermion system, the motion equation is represented by Dirac equation that satisfied relativistic Hamiltonian. After we obtained the solution of Dirac equation, explicitly expectation value of energy would be found. The expectation value of energy was used to describe the quantum thermodynamic processes.

2.1. A Fermion Trapped in the 1D Box System
The fermion which is trapped in the 1D box system can be described by Dirac Hamiltonian operator

\[ \hat{H} = -i\hbar c \hat{\alpha} \cdot \nabla + m_c c^2 \hat{\beta} + V(x) \] (1)

Here \( \hat{\alpha} \) and \( \hat{\beta} \) are Dirac matrices which are expressed by

\[ \hat{\alpha} = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad \hat{\beta} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \]

For the case, 1D box system the potential is bound and given by

\[ V(x) = \begin{cases} 0, & 0 \leq x \leq L \\ -\infty, & x < L \text{ and } x > L \end{cases} \] (2)

From that equation, in region \( 0 \leq x \leq L \) particle is under the free-states because the particle motion is not influenced by the potential. Thus Dirac Hamiltonian operator is expressed by

\[ \hat{H}_0 = -i\hbar c \hat{\alpha} \cdot \nabla + m_c c^2 \hat{\beta} \] (3)

In 1D box system, there is no particle can go through and out. It is guaranteed by the current density in boundary (\( x = 0 \) and \( x = L \)) is zero. Summarizing the Eigen equation of Dirac Hamiltonian for particle trapped in 1D box system is given by [7]

\[ \hat{H} \psi = E \psi, \] (4)

with boundary condition \( j(x = 0 \text{ and } x = L) = 0. \)

According to equation (3), equation (4) will be able to be expressed by

\[ \left( -i\hbar c (\hat{\alpha} \cdot \nabla + \hat{\beta} m_c c^2) \right) \psi(x) = E \psi(x) \] (5)

Since \( \hat{\alpha} \) and \( \hat{\beta} \) are matrices, so the solution of Dirac equation has the column vector is consist of 4-component spinors [9]. Therefore, the solution is expressed by \( \psi(x) = (\varphi, \varphi_2, \chi, \chi_2) \) [4]. In this notation, \( \varphi(x) \) is denoted as the “large” component and \( \chi(x) \) is denoted as the “small” component.
The solution of equation (5) is

\[ \psi = A \begin{pmatrix} 
\sin \left( kx - \frac{kL}{2} \right) \\
0 \\
0 \\
- \frac{i\hbar ck}{E + m_n c^2} \cos \left( kx - \frac{kL}{2} \right) 
\end{pmatrix}, \]  

with \( A \) is \( 2iA_e^{\mu_i} \). That equation associated with discrete energy eigenvalue

\[ E_n = \pm m_n c^2 \left[ \sqrt{1 + \left( \frac{n\lambda}{2L} \right)^2} - 1 \right], \]  

with \( \lambda = 2\pi \hbar / (m_n c) \) is wavelength. The positive and negative sign corresponds with particle and antiparticle solution respectively, however, in this paper, we used a positive solution because by assuming that the particle is trapped in the potential well and hence we would keep the positive eigenvalue [7].

2.2. Quantum Thermodynamic Processes

In the previous study, the definition of force is not well defined to construct thermodynamic processes for quantum system especially for isochoric and isobar process [1,4]. We redefined the force of the system based on the phenomenological representation of infinitesimally changing of the expectation value of Hamiltonian [5]. The infinitesimally changing of the expectation value of Hamiltonian is

\[ dE = \sum_n E_n dp_n + \sum_n p_n dE_n \]  

In the right side of equation (8), the first term indicates the heat absorbed by the system or released to the surroundings and the second term indicates work produced by the system. The second term of equation (8) is the work of the quantum system, \( \sum_n p_n dE_n = -dW = -FdL \). Thus, we have the ensemble average force as,

\[ F = -\sum_n p_n \frac{dE_n}{dL} = -\sum_n p_n F_n \]  

The ensemble average of force \( F \) is the average of \( F_n \) over all the states represented in the ensemble. According to equation (9) the force act on the wall is

\[ F(L) = \left( \frac{m_0 c^2 \lambda^2}{4E_L} \right) \sum_n p_n \frac{n^2}{\sqrt{1 + \left( n\lambda/2L \right)^2}} \]  

2.2.1 Adiabatic Process.

Classically, there is no exchanged heat in the system for the adiabatic process. Therefore, the changing of the internal energy produces work to push or pull the system

\[ dU = -PdV \]  

In quantum adiabatic, since there is no exchanged of heat, the occupation probability remained constant. Thus, according to the first law of thermodynamics, the internal energy changed is
\[ dE = \sum_{n} p_n dE_n \]  

(12)

2.2.2 Isochoric Process.

Classically, there is no changing of the volume of the piston under isochoric process. Consequently, there is no work produced during this process. According to the first law of thermodynamics, all of the heat transferred to the system is used to change the internal energy of the system. In a quantum system, all of the heat transferred to the system is used to change occupation probability. Thus, the particle is exited to the higher level state \( n \). The heat that transferred to the system is

\[ \delta Q = E(L_f) - E(L_i) \]  

(13)

3. Results and Discussion

Otto engine consists of adiabatic compression, isochoric, adiabatic expansion, and back to the initial condition by isochoric. Figure 1 describes the quantum Otto cycle for a single fermion system. The process is started by compressing adiabatic. The occupation probability remains constant under this process. After that, the system is thermally contacted with the hot reservoir by an isochoric process. During this process, the particle is excited to the other \( n \) state. Next, the system is on expansion adiabatic process and the eigenvalue of the system energy decrease along of increasing the width of well. Finally, the cycle of QOE is back to the initial condition by an isochoric process by transferring the residual heat energy to the cold reservoir.

![Figure 1. Diagram F-L for quantum otto cycle with single fermion system](image)

The heat is transferred from hot reservoir \( Q_H \) into the system by an isochoric process. According to the first law of thermodynamics, the heat input to a system is

\[ Q_H = \frac{mc^2}{2L_A} \left[ \sqrt{4L_n^2 + \lambda^2} - \sum_{n} \sqrt{4L_n^2 + n^2 \lambda^2} + 2L_n \left( \sum_{n} p_n - 1 \right) \right] \]  

(14)

Meanwhile, the heat output to the cold reservoir by an isochoric process is

\[ Q_L = \frac{mc^2}{2L_A} \left[ \sqrt{4L_n^2 + \lambda^2} - \sum_{n} \sqrt{4L_n^2 + n^2 \lambda^2} + 2L_n \left( \sum_{n} p_n - 1 \right) \right] \]  

(15)

According to the conservative law of energy, absorbed heat is converted to work and residual heat is transferred to the cold reservoir. The total work for this cycle is expressed by

\[ W = Q_H - Q_L \]  

(16)
The efficiency of quantum Otto engine (QOE) is the ratio between work that is produced by system and heat absorbed. According to equation (14) (15) and (16), we obtained the efficiency of QOE with single fermion system as follow

\[ \eta_{QOE}^{\text{QR}} = 1 - \left( \frac{L_B}{L_A} \right) \left( \frac{\sum n p_n \sqrt{4L_A^2 + n^2 \lambda^2} + 2L_A \left( \sum p_n - 1 \right)}{\sum n p_n \sqrt{4L_B^2 + n^2 \lambda^2} + 2L_B \left( \sum p_n - 1 \right)} \right), \] (17)

with \( L_A / L_B \) is ratio of compression and \( n \) which is integer number is the number of energy arranged.

From equation (17), the efficiency of QOE based on single fermion system depends on compression ratio and the number of energy arranged. Considering the number of energy arranged are \( n = 2, 3, 4 \) (by assuming Fermion excited to the maximum state), we have obtained the efficiency of QOE based on single fermion system which has less number of energy arranged that is the highest efficiency for a certain condition as a Figure 2. However, we can increase the efficiency by controlling the compression ratio. If the system has less number of energy arranged, the compression ratio should be bigger. This result has corresponded with a non-relativistic case [2,4].

\[ \eta = 1 - \left( \frac{L_B}{L_A} \right) \left( \frac{\sum n p_n \sqrt{4L_A^2 + n^2 \lambda^2} + 2L_A \left( \sum p_n - 1 \right)}{\sum n p_n \sqrt{4L_B^2 + n^2 \lambda^2} + 2L_B \left( \sum p_n - 1 \right)} \right), \] (17)

Consider for the ultra-relativistic limit (\( \lambda \to \infty \)), the efficiency of QOE is

\[ \eta_{QOE}^{\text{UR}} = 1 - \frac{L_B}{L_A}, \] (18)

That efficiency does not depend on the number of the level state, but it only depends on ratio compression. If we take for a non-relativistic limit (\( \lambda \to 0 \)) the efficiency of QOE is

\[ \eta_{QOE}^{\text{NR}} = 1 - \left( \frac{L_B}{L_A} \right)^2 \] (19)

Equation (19) has also confirmed the previous study [5,20,23]. Therefore, by exploring a relativistic Otto Engine we can reformulate the efficiency of non-relativistic Otto engine.

If we compare this result with the classical one,

\[ \eta = 1 - \left( \frac{V_B}{V_A} \right)^{\gamma - 1}, \] (20)

with \( \gamma \) is the heat capacity ratio then we also obtained the heat capacity ratio for a fermion system that was trapped in a 1D box. The value of the heat capacity ratio for ultra-relativistic is 2. These results also correlated with Diesel fermion engine [21]. Meanwhile, for a non-relativistic limit, the ratio of the heat capacity is 3 [5,7,22,23]. This difference of the heat capacity ratio for classical, non-relativistic and ultra-relativistic is corresponding with the properties of the system.
4. Conclusion
In this paper, we have explored quantum Otto engine (QOE) based on a single fermion in a 1D box system. The fermion satisfies Dirac Hamiltonian which is a relativistic equation and we have obtained the eigenvalue of relativistic energy for this system. The system occurred by some processes such as isochoric and adiabatic processes. By using the first law of thermodynamics, we have obtained the efficiency of QOE for single fermion system. For certain condition, the higher the level state of the system, the lower the efficiency of QOE is. However, by controlling of ratio compression we can keep the performance of QOE high. By setting the limit $\lambda \to 0$ and $\lambda \to \infty$, we also obtained the efficiency of QOE in non-relativistic limit and ultra-relativistic limit.

References
[1] Bender C M, Brody D C and Meister B K 2000 Quantum Mechanical Carnot Engine J. Phys. A 33 4427–36
[2] Quan H T, Zhang P and Sun C P 2005 Quantum heat engine with multilevel quantum systems Phys Rev E 72 56110
[3] Quan H T, Liu Y, Sun C P and Nori F 2007 Quantum thermodynamic cycles and quantum heat engines Phys Rev E 76 31105
[4] Latifah E and Purwanto A 2011 Multiple-State Quantum Carnot Engine J. Mod. Phys. 2 1366–72
[5] Latifah E and Purwanto A 2013 Quantum Heat Engines; Multiple-State 1D Box System J. Mod. Phys. 04 1108–15
[6] Bender C M, Brody D C and Meister B K 2000 Quantum mechanical Carnot engine J. Phys. Math. Gen. 33 4427–36
[7] Muñoz E and Peña F J 2012 Quantum heat engine in the relativistic limit: The case of a Dirac particle Phys. Rev. E - Stat. Nonlinear Soft Matter Phys. 86 1–11
[8] Purwanto A, Sukamto H, Subagyo B A and Taufiqi M 2016 Two Scenarios on the Relativistic Quantum Heat Engine J. Appl. Math. Phys. 4 1344–53
[9] Ivanchenko E A and Science N 2015 Quantum Otto cycle efficiency on coupled qudits 1–19
[10] Akbar M S, Latifah E, Wisodo H, Hidayat A, Prasetyo D and Qomariyah S N 2016 Proses Adiabatis dan Isovolume Sistem Dua Partikel Simetri JPSE J. Phys. Sci. Eng. 2 55–65
[11] Rezek Y and Kosloff R 2006 Irreversible performance of a quantum harmonic heat engine New J. Phys. 8
[12] Insinga A, Andresen B and Salamon P 2016 Thermodynamical analysis of a quantum heat engine based on harmonic oscillators Phys Rev E 94 012119
[13] Quan H T, Zhang P and Sun C P 2005 Quantum Heat Engine with Multilevel Quatum Systems Phys Rev E 72 56110
[14] Wu F, Yang Z, Yang L, Liu X and Wu S 2010 Work Output and Efficiency of a Reversible Quantum Otto Cycle Therm. Sci. 14 879–86
[15] Pena F J, Ferre M, Orellana P A, Rojas R G and Vargas P 2016 Optimization of a relativistic quantum mechanical engine Phys. Rev. E - Stat. Nonlinear Soft Matter Phys. 94 1–12
[16] Quan H T 2009 Quantum thermodynamic cycles and quantum heat engines. II. 1–10
[17] Quan H T, Liu Y, Sun C P and Nori F 2007 Quantum Thermodynamic Cycles and Quantum Heat Engines Phys. Rev. E 76 031105
[18] Munoz E and Pena F 2012 A single Dirac-fermion Quantum Heat Engine ArXiv Prepr. ArXiv12076149 2 1–8
[19] Latifah E and Purwanto A 2013 Quantum Heat Engines; Multiple-State 1D Box System 2013 1091–8
[20] Wang J, Ma Y and He J 2015 Quantum-mechanical engines working with an ideal gas with a finite number of particles confined in a power-law trap EPL Europhys. Lett. 111 20006
[21] Setyo D P, Latifah E, Wisodo H and Hidayat A 2018 Quantum Relativistic Diesel Engine with
Single Massless Fermion in 1 Dimensional Box System *J. Penelit. Fis. Dan Apl. JPFA* 8 25–32

[22] Wang J, Ma Y and He J 2015 Quantum-Mechanical Engines Working with an Ideal Gas with a Finite Number of Particle Confined in a Power-Law Trap *EPL* 111 2

[23] Wu F, Yang Z, Chen L, Liu X and Wu S 2010 Work output and efficiency of a reversible quantum otto cycle *Therm. Sci.* 14 879–86