Spectroscopy of the Schwarzschild Black Hole at Arbitrary Frequencies

Marc Casals$^{1,2,3}$ and Adrian Ottewill$^3$

$^1$Perimeter Institute for Theoretical Physics, Waterloo, Ontario, Canada N2L 2Y5
$^2$Department of Physics, University of Guelph, Guelph, Ontario, Canada N1G 2W1
$^3$School of Mathematical Sciences and Complex & Adaptive Systems Laboratory, University College Dublin, Belfield, Dublin 4, Ireland

Linear field perturbations of a black hole are described by the Green function of the wave equation that they obey. After Fourier decomposing the Green function, its two natural contributions are given by poles (quasinormal modes) and a largely unexplored branch cut in the complex-frequency plane. We present new analytic methods for calculating the branch cut on a Schwarzschild black hole for arbitrary values of the frequency. The branch cut yields a power-law tail decay for late times in the response of a black hole to an initial perturbation. We determine explicitly the first three orders in the power-law and show that the branch cut also yields a new logarithmic behaviour $T^{-2\ell-5}\ln T$ for late times. Before the tail sets in, the quasinormal modes dominate the black hole response. For electromagnetic perturbations, the quasinormal mode frequencies approach the branch cut at large overtone index $n$. We determine these frequencies up to $n^{-5/2}$ and, formally, to arbitrary order. Highly-damped quasinormal modes are of particular interest in that they have been linked to quantum properties of black holes.

The retarded Green function for linear field perturbations in black hole spacetimes is of central physical importance in classical and quantum gravity. An understanding of the make-up of the Green function is obtained by performing a Fourier transform, thus yielding an integration just above the real-frequency ($\omega$) axis. In his seminal paper, Leaver [1] deformed this real-$\omega$ integration in the case of Schwarzschild spacetime into a contour on the complex-$\omega$ plane. He thus unraveled three contributions making up the Green function: (1) a high-frequency arc, (2) a series over poles of the Green function (quasinormal modes QNMs), and (3) an integral of modes around a branch cut originating at $\omega = 0$ and extending down the negative imaginary axis (NIA), which we refer to as branch cut modes (BCMs). The three contributions dominate the black hole response to an initial perturbation at different time regimes. The high-frequency arc yields a ‘direct’ contribution which is expected to vanish after a certain finite time [2, 3].

The QNM contribution to the Green function dominates the black hole response during ‘intermediate’ times and it has been extensively investigated (e.g., [4] for a review). At ‘late times’ the QNM contribution decays exponentially, with a decay rate given by the overtone number $n \in \mathbb{Z}^+$. QNMs have also triggered numerous interpretations in different contexts in classical and quantum physics, ranging from astrophysical ‘ringdown’ [5] to Hawking radiation [6, 7], the ‘gauge-gravity duality’ ([8] for Schwarzschild black holes which are asymptotically anti-de Sitter and [9] for asymptotically flat ones), black hole area quantization [10–14] and structure of space-time at the shortest length scales [15]. The quantum interpretations are given in the highly-damped limit, i.e., for large $n$. The highly-damped QNM frequencies in Schwarzschild have been calculated up to next-to-leading order in [16–20]. Despite all the efforts, the leading order of the real part of the frequencies for electromagnetic perturbations has remained elusive (only in [21] they find numerical indications that it goes like $n^{-3/2}$).

The contribution from the BCMs, on the other hand, remains largely unexplored. The technical difficulties of its analysis mean that most of the studies have been constrained to large radial coordinate as well as small $\nu = \omega > 0$ along the NIA. An exception is a large-$\nu$ asymptotic analysis of the BCMs in [15] (and near the algebraically-special frequency in [22]) solely for gravitational perturbations. The small-$\nu$ BCMs are known to give rise to a power-law tail decay at ‘late’ times of an initial perturbation [11, 12, 23, 24]. In general, however, there is an appreciable time interval between when the QNM contribution becomes negligible and when the power-law tail starts [25]. The calculation of the BCMs for general values of the frequency (i.e., not in the asymptotically small nor large regimes), to the best of our knowledge has only been attempted in [26, 27] where the radial functions were calculated off the NIA via a numerical integration of the radial ODE [1, 2] followed by extrapolation to the NIA, and only for the gravitational case.

In this Letter we present the following new results:

1. A new analytic method for the calculation of the BCMs directly on the NIA and valid for any value of $\nu$. In particular, this method provides analytic access for the first time to the ‘mid’-$\nu$ regime.

2. A consistent expansion up to 4th order for small-$\nu$ of the BCMs for arbitrary value of the radial coordinate. We explicitly derive a new logarithmic behaviour $T^{-2\ell-5}\ln T$ at late times.

3. A large-$\nu$ asymptotic analysis of the BCMs. It shows a formal divergence, which is expected to be cancelled out by the other contributions to the Green function.

4. A new asymptotic analysis for large-$n$ of the electromagnetic QNMs. The analysis is formally valid up...
to arbitrary order in \( n \); we explicitly calculate the corresponding frequencies up to \( n^{-5/2} \).

Methods in (1)–(3) provide the first full analytic account of the BCMs and they are valid for any spin \( s = 0 \) (scalar), 1 (electromagnetic) and 2 (gravitational) of the field perturbation. For the QNM calculation we focus on spin-1 as this is the least well understood case. We note that spin-1 perturbations are acquiring increasing importance as this is the least well understood case. We note that spin-1 perturbations are acquiring increasing importance as this is the least well understood case.

We present here methods for the analytic calculation across the branch cut.

Methods in (1)–(3) provide the first full analytic account of the BCMs and they are valid for any spin \( s \).

The BCMs are solutions of the radial ODE

\[
\left\{ \frac{d^2}{dr^2} + \omega^2 - \left( 1 - \frac{1}{r} \right) \left[ \frac{\lambda}{r^2} + \frac{(1 - s^2)}{r^3} \right] \right\} \psi_r = 0 \tag{1.2}
\]

where \( r_* \equiv r + \ln(r-1) \) and \( \lambda \equiv \ell(\ell + 1) \). The solutions are uniquely determined when \( \text{Im}(\omega) \geq 0 \) by the boundary conditions: \( f_\ell \sim e^{-i\omega r_*} \) as \( r_* \rightarrow -\infty \) and \( g_\ell \sim e^{+i\omega r_*} \) as \( r_* \rightarrow \infty \). The behaviour of the radial potential at infinity leads to a branch cut in the radial solution \( g_\ell \). The contour of integration in Eq.(1.4) can be deformed in the complex-\( \omega \) plane yielding a contribution from a high-frequency arc, a series over the residues (the QNMs) and a contribution from the branch cut along the NIA:

\[
G^\text{BC \_ret}_\ell (r, r'; t) \equiv \frac{1}{2\pi i} \int_0^\infty \! dv \Delta G_\ell (r, r'; v) e^{-ivt}, \tag{1.3}
\]

where the BCMs are

\[
\Delta G_\ell (r, r'; v) \equiv -\frac{2ivq(v)}{|W(-iv)|^2} f_\ell (r, -iv) f_\ell (r', iv), \tag{1.4}
\]

with \( q(v) \equiv -i\Delta g_\ell (r, v)/g_\ell (r, iv) \) where \( \Delta g_\ell (r, v) \equiv \lim_{\epsilon \rightarrow 0^+} \{ g_\ell (r, \epsilon - iv) - g_\ell (r, \epsilon + iv) \} \) is the discontinuity of \( g_\ell \) across the branch cut.

We present here methods for the analytic calculation of the BCMs. We calculate \( f_\ell \) using the Jaffe series, Eq.39 [32]. The coefficients of this series, which we denote by \( a_k \), satisfy a 3-term recurrence relation. We calculate \( g_\ell \) using the series in Eq.73 [32], which is in terms of the confluent hypergeometric \( U \)-function and the coefficients \( a_k \). This series has seldom been used and one must be aware that, in order for \( g_\ell \) to satisfy the correct boundary condition, we must set \( a_{k=0} = (-2i\omega)^{s+1} \) which itself has a branch cut. To find an expression for \( \Delta G_\ell \) on the NIA we exploit this series by combining it with the known behavior of the \( U \)-function across its branch cut:

\[
\Delta g_\ell (r, v) = \frac{r^{1+s} e^{-\nu r} 2\pi i e^{\pi i(\nu + 1 - 2\nu)}}{(r - 1)^{\nu} \Gamma(1 - 2\nu)} \times \sum_{k=0}^{\infty} \frac{(-1)^k (1 - k + 2\nu) U(s - k + 2\nu, 2s + 1, 2\nu)}{\Gamma(1 + s + k - 2\nu) \Gamma(1 - s + k - 2\nu)} \]  

where we are taking the principal branch both for \( a_{k=0} \) and for the \( U \)-function. In order to check the convergence of this series, we require the behaviour for large-\( k \) of the coefficients \( a_k \). Using the Birkhoff series as in App.B [34], we find the leading order \( a_k \sim k^{-3/4} e^{\nu/2} \) (we have calculated up to four orders higher in [30]) as \( k \rightarrow \infty \). We note that this behaviour corrects Leaver’s Eq.46 [32] in the power ‘-\( \nu \)’ instead of ‘-2\( \nu \)’. The integral test then shows that the series converges for any \( \nu \geq 0 \). Although convergent, the usefulness of [1.5] at small-\( \nu \) is limited since convergence becomes slower as \( \nu \) approaches 0 while, for large-\( \nu \), \( \Delta G_\ell \) grows and oscillates for fixed \( r \) and \( r' \). Therefore we complement our analytic method with asymptotic results for small and large \( \nu \).

The small-\( \omega \) asymptotics are based on an extension of the MST formalism [33, 35]. We start with the ansatz

\[
f_\ell = e^{-i\omega r} \left( 1 - i\omega \right)^{-\ell - 1} \sum_{j=\infty}^\infty a_j^\ell \times \sum_{k=-\infty}^\infty \frac{a_k^\ell \Gamma(k + \mu + s + 1 - i\omega) \Gamma(-k - \mu + s - i\omega)}{\Gamma(1 - 2i\omega)} \times \Gamma(1 + \mu + s - i\omega) \times 2F_1 \left( k + \mu + s + 1 - i\omega, -k - \mu + s - i\omega; 1 - 2i\omega; 1 - r \right), \tag{1.6}
\]

Imposing Eq.(1.2) yields a 3-term recurrence relation for \( a_0^\ell \) and requiring convergence as \( k \rightarrow \pm \infty \) yields an equation for \( \mu \), that may readily be solved perturbatively in \( \omega \) from starting values \( \mu_{\omega=0} = \ell \) and \( \mu_{\omega=0} = -\ell - 1 \). Likewise for the coefficients \( a_0^\mu \), taking \( a_0^\mu = 1 \) we obtain

\[
a_0^\mu = \frac{(\ell + 1 - s)^2}{2(\ell + 1)(2\ell + 1)} \left[ \frac{\omega^2}{\ell + 1} + O(\omega^3) \right]
\]

while \( a_0^\mu_1 \) and \( a_0^\mu_2 \) are given by the corresponding terms with \( \ell \rightarrow -\ell \). (Apparent possible singularities in these coefficients are removable.)

The \( k = 0 \) term in Eq.(1.6) corresponds to Page’s Eq.A.9 [37]. To obtain higher order asymptotics we employ the Barnes integral representation of the hypergeometric functions [38] which involves a contour in the complex \( z \)-plane from \( -i\infty \) to \( i\infty \) threading between the
poles of \(\Gamma(k+\mu+s+1-i\omega+z), \Gamma(-k-\mu+s-i\omega+z)\) and \(\Gamma(-z)\). As \(\omega \to 0\) double poles arise at the non-negative integers from 0 to \(\max(k+\ell-s,-k-\ell-1-s)\), however we may move the contour to the right of all these ambient double poles picking up polynomials in \(r\) with coefficients readily expanded in powers of \(\omega\), leaving a regular contour which admits immediate expansion in powers of \(\nu\).

By the method of MST we can also construct \(q(\nu)\) and hence determine \(q(\nu)\) and \(W\). For compactness, we only give the following small-\(\nu\) expressions for the case \(s = 0\) (cases \(s = 1\) and 2 are presented in [39]),

\[
q(\nu) = \frac{(-1)^\ell \pi}{2^{2\ell-3}} \left( \frac{(2\ell+1)!}{(2(\ell+1))!} \right)^2 \nu^{2\ell+1} - \nu^{2\ell+2} \left( \frac{-32\ell^3 - 63\ell^2 - 7\ell + 23}{2(2\ell + 3)(2\ell + 1)(2\ell - 1)} + 4H_\ell \right)
- \frac{(-1)^\ell \pi}{2^{2\ell-3}} \left( \frac{(2\ell+1)!}{(2(\ell+1))!} \right)^2 \nu^{2\ell+3} \left[ \frac{4(15\ell^2 + 15\ell - 11)}{(2\ell - 1)(2\ell + 1)(2\ell + 3)} (2\ell + 1)(2\ell + 3) \right. \\
\left. \ln(2\nu) + H_\ell - 4H_{2\ell} + \gamma_E \right]
- 4 \left( -8H_\ell^2 + 8H_\ell + 3H_\ell^2 + 2H_\ell^3 \right) + \frac{512\ell^6 + 2016\ell^5 + 1616\ell^4 - 1472\ell^3 - 1128\ell^2 + 722\ell - 59}{(2\ell - 1)^2(2\ell + 1)^2(2\ell + 3)^2} + o(\nu^{2\ell+3})
\]

where \(H_\ell^{(r)}\) is the \(\ell\)-th harmonic number of order \(r\). We note that the \(\ln \nu\) term at second-to-leading order originates both in \(q(\nu)\) and in \(W\). In fact, both functions possess a \(\ln \nu\) already at next-to-leading order for small-\(\nu\), but they cancel each other out in \(q/|W|^2\). Similarly, the coefficient of a potential term in \(q/|W|^2\) of order \(\nu^{2\ell+3} (\ln \nu)^2\) is actually zero.

Let us now investigate the branch cut contribution to the black hole response to an initial perturbation given by the field \(u^{ic}_\ell\) and its time derivative \(\dot{u}^{ic}_\ell\) at \(t = 0\):

\[
u^{2\ell+3} \ln \nu \]  
\[
W^{BC}(r,s,t) = \int_{-\infty}^{+\infty} dr' \left[ G^{BC}_\ell(r,r';t) u^{ic}_\ell(r') \right. + \left. u^{ic}_\ell(r') \partial_z G^{BC}_\ell(r,r';t) \right]
\]

We obtain the asymptotics of the response for late times \(T \equiv t - r - r'\), using Eqs. [1.4] and [1.10] and 1.8]. We note the following features. The orders \(\nu^{2\ell-2}\) and \(\nu^{2\ell+3}\) in the BCMs \(\Delta G^{BC}\) yield tail terms behaving like \(T^{-2\ell-3}\) and \(T^{-2\ell-4}\) respectively. We have thus generalized Leaver's Eq.56 [1] to finite values of \(r\). Furthermore, Eq.56 [1] is an expression containing the leading orders from \(u^{ic}_\ell\) and from \(\dot{u}^{ic}_\ell\). However, the next-to-leading order from \(\dot{u}^{ic}_\ell\) will be of the same order as the leading-order from \(u^{ic}_\ell\). In our approach above we consistently give a series in small-\(\nu\), thus obtaining the correct next-to-leading order term for large-\(T\) in the power-law tail. Importantly, we also obtain the following two orders in the perturbation response: \(T^{-2\ell-5} \ln T\) and \(T^{-2\ell-5}\). We note the interesting \(T^{-2\ell-5} \ln T\) behaviour, which is due to the \(\nu^{2\ell+3} \ln \nu\) term in Eq. [1.8]. To the best of our knowledge, this is the first time in the literature that any of the above features has been obtained. The logarithmic behaviour is not completely surprising given the calculations in [39]. However, one may be led to a wrong logarithmic behaviour [39] if the calculations are not performed in detail. In order to exemplify our results, we give the explicit asymptotic behaviour in the case \(s = 0\) and \(\ell = 1\) and

![Fig. 1. (Color online). Full perturbation response \(u_t\) to the Gaussian described above Eq. (1.10) compared to the late-time asymptotics. Solid-red: numerical solution; dashed-black: Eq. (1.10); lower curves: numerical solution minus the first (green), first 2 (blue) and first 4 (cyan) terms in Eq. (1.10).](image-url)
These asymptotics show a divergence in $G^B_{\ell C}$ when $t < |r_s| + |r'_s|$. They also lead to a divergence in the perturbation response at fixed $t$ and $r$ for a non-compact Gaussian as initial data. Both types of divergences are expected to cancel out with the other contributions to the Green function. We have thus provided a complete account of the BCMs for all frequencies along the NIA; the behaviour is illustrated in Fig.3.

II. SPIN-1 QUASINORMAL MODES

We present here an analysis for large-$n$ of the electromagnetic QNMs. We may find solutions of Eq. (1.2) valid for fixed $v = r^2/\nu/2$ as expansion in powers of $v^{-1/2}$ as $\psi_i = \sum_{k=0}^{\infty} \psi^{(k)}_i$, $i = 1, 2$, starting with the two independent solutions: $\psi^{(0)}_1(v) = (2/v) \sin v$ and $\psi^{(0)}_2(v) = \cosh v$. We may express any higher order solution in terms of the 0th-order Green function as

$$\psi^{(k)}_i(v) = \int_0^v \frac{\sinh(v - u)}{(2u)^{1/2}} \left\{ 8D^2 - 6D - \lambda \right\} \frac{\psi^{(k-1)}(u)}{\nu} - (2u)^{1/2} \left[ 4D^2 - 2D - \lambda \right] \frac{\psi^{(k-2)}(u)}{\nu} \right\} \right) \right) (2.1)$$

where $D = uv \frac{d}{dv}$. From this expression, it follows that

$$\psi^{(k)}_1(e^{\pi v}) = -e^{\pi k/2} \psi^{(k)}_1(v)$$
$$\psi^{(k)}_2(e^{\pi v}) = e^{\pi k/2} \left[ \psi^{(k)}_2(v) - i\alpha^2 \psi^{(k-1)}_1(v) \right]$$

where $\alpha_1 = -\lambda \sqrt{\pi}/2$. In addition, for $\arg(r) = \pi/4$, $e^{-\pi(k-2)/4} \psi^{(k)}_1(v)$ and $e^{-\pi k/4} \left[ \psi^{(k)}_2(v) + \frac{1}{2} i\alpha^2 \psi^{(k-2)}_1(v) \right]$ are both real. It follows that along $\arg(r) = \pi/4$, up to power law corrections,

$$\psi_i \sim A_i e^v + B_i e^{-v} \right) (2.3)$$

$$A_1 = \frac{1}{2} \sum_{k} \alpha_k, \quad A_2 = \frac{1}{2} \sum_{k} \beta_k - i\alpha^2 \alpha_{k-2}$$
$$B_1 = -\frac{1}{2} \sum_{k} \beta_k \alpha_k, \quad B_2 = \frac{1}{2} \sum_{k} \beta^2_k - i\alpha^2 \alpha_{k-2}$$

Equating asymptotic expansions at $\arg(r) = 3\pi/4$ yields $\alpha_k \in \mathbb{R}$, $\beta_k = \alpha^2 \alpha_{k-2}$ and also serves to determine $re^k$ (except when $k = 4p - 2$ for $p \in \mathbb{N}$, which do not contribute to the QNM condition). Along $\arg(v) = \pi/2$ the Green function in Eq. (2.1) is rapidly oscillating and we can obtain the values of $\alpha_k$ directly from a stationary phase analysis [30]:

$$\alpha_0 = 1, \quad \alpha_1 = -\frac{\lambda \sqrt{\pi}}{2}, \quad \alpha_2 = \frac{\lambda^2}{2} - \frac{\lambda}{12}$$
$$\alpha_3 = \frac{\lambda^3}{8} \sqrt{\pi} (4 \ln 2 - \pi) - \frac{11\sqrt{\pi} \lambda^2}{48} + \frac{41 \sqrt{\pi} \lambda}{192} + \frac{\sqrt{\pi}}{16}$$

By matching the $\psi_i$ to WKB solutions along $\arg(r) = \pi/4$ and $3\pi/4$ we are able to find large-$\nu$ asymptotics for $g_t$. Also, we may use the exact monodromy condition, $f_t (r - r_k) e^{2\pi \nu i}, \omega = e^{2\pi} f_t (r - r_k, \omega)$, to obtain large-$\nu$ asymptotics for $f_t$. The asymptotic QNM condition ($W = 0$) in the 4th quadrant then becomes

$$e^{-4\pi \nu i} - 1 = \frac{2 (B_2 B_1^* - B_1 B_2^*)}{(A_1 B_2 - B_1 A_2) B_1} \sum_{k \text{ odd}} \frac{\alpha_k}{\nu^{k/2}} \right) (2.5)$$

It is straightforward to find the QNM frequencies to arbitrary order in $n$ in terms of the $\alpha_k$ by systematically solving Eq. (2.5). Explicitly, using the values in Eq. (2.4), we have

$$\omega_n = -\frac{i n \lambda^2}{2n} + \frac{e^{-\pi\nu/4} \pi^2/\lambda^3}{2n^3/2} + \frac{3\pi \lambda^4}{4n^2} + \frac{e^{\pi\nu/4} \sqrt{\pi} \lambda^2}{2} + O(n^{-3})$$

It is remarkable that the terms in the expansion show the behaviour $e^{i\nu\pi/4}(R)/\nu^{k/2}$ to all orders. In Fig.3 we compare these asymptotics with the numerical data in [31]. In [31] we apply the method used to obtain Eq. (2.6) to the cases $s = 0$ and 2 and we obtain the corresponding QNM frequencies up to order $n^{-1/2}$ and have agreement with [19] [19].

FIG. 2. (Color online). $\Delta G_t$ as a function of $\nu$ for $r_s = 0.1$ and $r'_s = 0.2$. (a) Using Eq. (1.5); dashed-green: $s = \ell = 2$, continuous-blue: $s = 0$, $\ell = 1$, dot-dashed-orange: $s = \ell = 1$. Note the interesting behaviour near the algebraically-special frequency [22] at $\nu = 4$ for $s = 2$. (b) $s = 0$, $\ell = 1$ for small $\nu$; continuous-blue using Eq. (1.5); dashed-red using Eq. (1.8) to $O(\nu^{15})$—see [30]. (c) $s = 0$, $\ell = 1$ for large $\nu$; continuous-blue using Eq. (1.5); dashed-red using the asymptotics of Eq. (1.11).
FIG. 3. (Color online). Log-log plot of QNM frequencies for \( s = \ell = 1 \) from the asymptotics Eq. (2.6) in dashed-green and the numerical data in [41] in dotted-red. The two upper curves correspond to \( 400 |\text{Im}(\omega_{\ell n}) + \frac{n}{2}| \) and the two lower curves to \( \text{Re}(\omega_{\ell n}) \).

ACKNOWLEDGMENTS

We are thankful to Sam Dolan and, particularly, to Barry Wardell for helpful discussions. A.O. thanks Luis Lehner and the Perimeter Institute for Theoretical Physics for hospitality and financial support. M. C. is supported by a IRCSET-Marie Curie International Mobility Fellowship in Science, Engineering and Technology. A.O. acknowledges support from Science Foundation Ireland under grant no 10/RFP/PHY2847.

[1] E. W. Leaver, Phys. Rev. D 34, 384 (1986).
[2] E. S. C. Ching, P. T. Leung, W. M. Suen, and K. Young, Phys. Rev. Lett. 74, 2414 (1995) arXiv:gr-qc/9410044.
[3] E. S. C. Ching, P. T. Leung, W. M. Suen, and K. Young, Phys. Rev. Lett. 74, 4588 (1995).
[4] E. Berti, V. Cardoso, and A. O. Starinets, Class. Quant. Grav. 26, 163001 (2009) arXiv:0905.2975 [gr-qc].
[5] C. V. Vishveshwara, Nature 227, 936 (1970).
[6] J. York Jr., Phys. Rev. D28, 2929 (1983).
[7] U. Keshet and A. Neitzke, Phys. Rev. D78, 044006 (2008) arXiv:0709.1532 [hep-th].
[8] G. T. Horowitz and V. E. Hubeny, Phys. Rev. D62, 24027 (2000) arXiv:hep-th/9909056.
[9] S. Bertini, S. L. Cacciatore, and D. Klemm, Phys. Rev. D85, 064018 (2012) arXiv:1106.0999 [hep-th].
[10] J. D. Bekenstein, Lett. Nuovo Cim. 11, 467 (1974).
[11] J. D. Bekenstein and V. F. Mukhanov, Phys. Lett. B360, 7 (1995) arXiv:gr-qc/9505012.
[12] S. Hod, Phys. Rev. Lett. 81, 4293 (1998) arXiv:gr-qc/9812002.
[13] O. Dreyer, Physical Review Letters 90, 081301 (2003) arXiv:gr-qc/0211076.
[14] M. Maggiore, Phys. Rev. Lett. 100, 141301 (2008) arXiv:0711.3445 [gr-qc].
[15] J. Babb, R. Daghigh, and G. Kunstatter, Phys. Rev. D84, 084031 (2011) arXiv:1106.4357 [gr-qc].
[16] L. Motl and A. Neitzke, Ad. Theor. Math. Phys. 7, 307 (2003).
[17] A. Neitzke, (2003) arXiv:hep-th/0304080.
[18] A. Maassen van den Brink, J. Math. Phys. 45, 327 (2004) arXiv:gr-qc/0303095.
[19] S. Musiri and G. Siopsis, Class. Quant. Grav. 20, L285 (2003) arXiv:hep-th/0308168.
[20] S. Musiri and G. Siopsis, Phys. Lett. B650, 279 (2007).
[21] V. Cardoso, J. P. S. Lemos, and S. Yoshida, Phys. Rev. D69, 044004 (2004) arXiv:gr-qc/0309112.
[22] A. Maassen van den Brink, Phys. Rev. D62, 064009 (2000) arXiv:gr-qc/0001032.
[23] R. H. Price, Phys. Rev. D5, 2419 (1972).
[24] R. H. Price, Phys. Rev. D5, 2439 (1972).
[25] N. Andersson, Phys. Rev. D55, 468 (1997).
[26] P. T. Leung, A. Maassen van den Brink, K. W. Mak, and K. Young, (2003) arXiv:gr-qc/0307024.
[27] P. T. Leung, A. Maassen van den Brink, K. W. Mak, and K. Young, Class. Quant. Grav. 20, L217 (2003) arXiv:gr-qc/0301018.
[28] J. D. Schnittman, Class. Quant. Grav. 28, 094021 (2011) arXiv:1010.3250 [astro-ph.HE].
[29] F. Tamburini, B. Thide, G. Molina-Terriza, and G. Anzolin, Nature Phys. 7, 195 (2011) arXiv:1104.3099 [gr-qc].
[30] M. Casals and A. C. Ottewill, In preparation.
[31] M. Casals and A. C. Ottewill, Phys. Rev. D86, 024021 (2012), arXiv:1112.2695 [gr-qc].
[32] E. W. Leaver, J. Math. Phys. 27, 1238 (1986).
[33] E. S. C. Ching, P. T. Leung, W. M. Suen, and K. Young, Phys. Rev. D52, 2118 (1995) arXiv:gr-qc/9507035.
[34] J. Wimp, Computation with Recurrence Relations (Pitman Advanced Publishing Program, 1984).
[35] S. Mano, H. Suzuki, and E. Takasugi, Prog. Theor. Phys. 95, 1079 (1996).
[36] M. Sasaki and H. Tagoshi, Living Rev. Rel. 6, 6 (2003), arXiv:gr-qc/0306120.
[37] D.N. Page, Phys. Rev. D 13, 198 (1976).
[38] A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, Higher Transcendental Functions Vol I (McGraw-Hill, New York, 1953).
[39] S. Hod, Class. Quant. Grav. 20, L285 (2003) arXiv:hep-th/0308168.
[40] C. M. Bender and S. A. Orszag, Advanced Mathematical Methods for Scientists and Engineers (Springer, 1999).
[41] http://www.phy.olemiss.edu/~berti/qnms.html, http://gamow.ist.utl.pt/~vitor/ringdown.html.