GEOMETRIZING RATES OF CONVERGENCE UNDER DIFFERENTIAL PRIVACY CONSTRAINTS∗

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We study estimation of a functional \( \theta(\mathcal{P}) \) of an unknown probability distribution \( \mathcal{P} \in \mathcal{P} \) in which the original iid sample \( X_1, \ldots, X_n \) is kept private even from the statistician via an \( \alpha \)-local differential privacy constraint. Let \( \omega_1 \) denote the modulus of continuity of the functional \( \theta \) over \( \mathcal{P} \), with respect to total variation distance. For a large class of loss functions \( l \), we prove that the privatized minimax risk is equivalent to \( l(\omega_1(n^{-1/2})) \) to within constants, under regularity conditions that are satisfied, in particular, if \( \theta \) is linear and \( \mathcal{P} \) is convex. Our results complement the theory developed by Donoho and Liu (1991) with the nowadays highly relevant case of privatized data. Somewhat surprisingly, the difficulty of the estimation problem in the private case is characterized by \( \omega_1 \), whereas, it is characterized by the Hellinger modulus of continuity if the original data \( X_1, \ldots, X_n \) are available. We also provide a general recipe for constructing rate optimal privatization mechanisms and illustrate the general theory in numerous examples. Our theory allows to quantify the price to be paid for local differential privacy in a large class of estimation problems.

1. Introduction. One of the many new challenges for statistical inference in the information age is the increasing concern of data privacy protection. Nowadays, massive amounts of data, such as medical records, smart phone user behavior or social media activity, are routinely being collected and stored. On the other side of this trend is an increasing reluctance and discomfort of individuals to share this sometimes sensitive information with companies or state officials. Over the last few decades, the problem of constructing privacy preserving data release mechanisms has produced a vast literature, predominantly in computer science. One particularly fruitful approach to data protection that is insusceptible to privacy breaches is the concept of differential privacy (see Dinur and Nissim, 2003; Dwork, 2008; Dwork and Nissim, 2004; Evfimievski et al., 2003). In a nutshell, differential privacy is a form of randomization, where, instead of the original data, a perturbed version of the data is released, offering plausible deniability to the data providers, who can always argue that their true answer was different from the one that was actually provided. Aside from the academic discussion, (local) differential privacy has also found its way into real world applications. For instance, the Apple Inc. privacy statement explains the notion quite succinctly as follows.

“It is a technique that enables Apple to learn about the user community without learning about individuals in the community. Differential privacy transforms the information shared with Apple before it ever leaves the user’s device such that Apple can

∗Supported by the DFG Research Grant RO 3766/4-1.
Here, the qualification of ‘local’ differential privacy refers to a procedure which randomizes the original data already on the user’s ‘local’ machine and the original data is never released, whereas (non-local) differential privacy may also be employed to privatize and release an entire database that was previously compiled by a trusted curator. Here, we focus only on the local version of differential privacy.

More recently, differential privacy has also received some attention from a statistical inference perspective (see, e.g., Duchi et al., 2013,a,b; Dwork and Smith, 2010; Smith, 2008, 2011; Wasserman and Zhou, 2010; Ye and Barg, 2017). In this line of research, the focus is more on the inherent trade-off between privacy protection and efficient statistical inference and the question what optimal privacy preservation mechanisms may look like. Duchi et al. (2013,a,b, 2017) introduced new variants of the LeCam, Fano and Assouad techniques to derive lower bounds on the privatized minimax risk. On this way, they were the first to provide minimax rates of convergence for specific estimation problems under privacy constraints in a very insightful case by case study. Here, we develop a general theory, in the spirit of Donoho and Liu (1991), to characterize the differentially private minimax rate of convergence. Characterizing the minimax rate of convergence under differential privacy, and comparing it to the minimax risk in the non-private case, is one way to quantify the price, in terms of statistical accuracy, that has to be paid for privacy protection. It also allows us to develop (asymptotically) minimax optimal privatization schemes for a large class of estimation problems.

To be more precise, consider $n$ individuals who possess data $X_1, \ldots, X_n$, assumed to be iid from some probability distribution $P \in \mathcal{P}$. However, the statistician does not get to see the original data $X_1, \ldots, X_n$, but only a privatized version of observations $Z$. The conditional distribution of $Z$ given $X = (X_1, \ldots, X_n)'$ is denoted by $Q$ and referred to as a channel distribution or a privatization scheme, i.e. $\Pr(Z \in A|X = x) = Q(A|x)$. For $\alpha \in (0, \infty)$, the channel $Q$ is said to provide $\alpha$-differential privacy if

\begin{equation}
\sup_A \sup_{x, x': \|x - x'\|_0 = 1} \frac{\Pr(Z \in A|X = x)}{\Pr(Z \in A|X = x')} \leq e^\alpha,
\end{equation}

where the first supremum runs over all measurable sets and $\|\cdot\|_0$ denotes the number of non-zero entries. This definition captures the idea that the distribution of the observation $Z$ does not change too much if the data of any single individual in the database is changed, thereby protecting its privacy. The smaller $\alpha \in (0, \infty)$, the stronger is the privacy constraint (1.1). More formally, (if we consider the original data $X$ as fixed) Wasserman and Zhou (2010, Theorem 2.4) show that under $\alpha$-differential privacy, any level-$\gamma$ test using $Z$ to test $H_0: X = x$ versus $H_1: X = x'$ has power bounded by $\gamma e^\alpha$. In this paper we focus on a special case of differential privacy, namely, local differential privacy. A channel satisfying (1.1) is said to provide

\footnote{https://images.apple.com/privacy/docs/Differential_Privacy_Overview.pdf}
α-local differential privacy, if $Z = (Z_1, \ldots, Z_n)'$ is a random $n$-vector, and for all $i$, $Z_i$ is conditionally independent of $(X_j)_{j \neq i}$, given $Z_1, \ldots, Z_{i-1}, X_i, Z_{i+1}, \ldots, Z_n$, that is, if the $i$-th individual can generate its privatized data $Z_i$ using only its original data $X_i$ and possibly the privatized information of other individuals.

Suppose now that we want to estimate a real parameter $\theta(\mathbb{P})$ based on the privatized observation vector $Z$, whose unconditional distribution is equal to $Q^{\otimes n}(dz) := \int Q(dz|x) \mathbb{P}^{\otimes n}(dx)$, where $\mathbb{P}^{\otimes n}$ is the $n$-fold product measure of $\mathbb{P}$. The $Q$-privatized minimax risk of estimation under a loss function $l : \mathbb{R} \to \mathbb{R}$ is therefore given by

\[
M_n(Q, \mathcal{P}) := \inf_{\hat{\theta}_n} \sup_{P \in \mathcal{P}} \mathbb{E}_Q[ l(\hat{\theta}_n - \theta(P)) ],
\]

where the infimum runs over all estimators $\hat{\theta}_n$ taking $Z$ as input data. Note that if the channel $Q$ is given by $Q(Z \in S|X = x) = 1_S(x)$, then there is no privatization at all and the privatized minimax risk reduces to the conventional minimax risk of estimating $\theta(\mathbb{P})$. If we want to guarantee $\alpha$-differential privacy, then we may choose any channel $Q$ that satisfies (1.1) and we will try to make (1.2) as small as possible. This leads us to the $\alpha$-private minimax risk

\[
M_n(\alpha, \mathcal{P}) := \inf_{\mathcal{Q} \in \mathcal{Q}_\alpha} M_n(Q, \mathcal{P}),
\]

where the infimum runs over (some set of) $\alpha$-differentially private channels. In this paper we focus on the collection $\mathcal{Q}_\alpha$ that consists of all $\alpha$-locally differentially private channels. A channel $Q' \in \mathcal{Q}_\alpha$, for which $M_n(Q', \mathcal{P})$ is of the order of $M_n(\alpha, \mathcal{P})$, is referred to as a minimax rate optimal channel and may depend on the specific estimation problem under consideration, i.e., on $\theta$ and $\mathcal{P}$. Therefore, this situation is different from the statistical inverse problem setting. We write $M_n(\infty, \mathcal{P})$ for the classical (non-private) minimax risk.

The new contribution of this article is to characterize the rate at which $M_n(\alpha, \mathcal{P})$ converges to zero as $n \to \infty$, in high generality, and to provide concrete minimax rate optimal $\alpha$-locally differentially private channel distributions. To this end, we utilize the modulus of continuity of the functional $\theta : \mathcal{P} \to \mathbb{R}$ with respect to the total variation distance $d_{TV}(\mathbb{P}_0, \mathbb{P}_1)$, that is,

\[
\omega_1(\varepsilon) := \sup \{ |\theta(\mathbb{P}_0) - \theta(\mathbb{P}_1)| : d_{TV}(\mathbb{P}_0, \mathbb{P}_1) \leq \varepsilon, \mathbb{P}_0, \mathbb{P}_1 \in \mathcal{P} \},
\]

and we show that for any fixed $\alpha \in (0, \infty)$,

\[
M_n(\alpha, \mathcal{P}) \asymp l \left( \omega_1 \left( n^{-1/2} \right) \right).
\]

Here, $a_n \asymp b_n$ means that there exist constants $0 < c_0 < c_1 < \infty$ and $n_0 \in \mathbb{N}$, not depending on $n$, so that $c_0 b_n \leq a_n \leq c_1 b_n$, for all $n \geq n_0$. The lower bound on $M_n(\alpha, \mathcal{P})$ holds in full generality, whereas, in order to obtain a matching upper bound, it is necessary to impose some regularity conditions on $\mathcal{P}$ and $\theta$. These will be satisfied, in particular, if $\mathcal{P}$ is convex and dominated, $\theta$ is linear and bounded and $\omega_1(\varepsilon) \asymp \varepsilon^\gamma$, as $\varepsilon \to 0$, is of Hölderian form, but also hold in some important
cases of non-convex and non-dominated $\mathcal{P}$. It is important to compare (1.3) to the analogous result for the non-private minimax risk. This was established in the seminal paper by Donoho and Liu (1991), who, under regularity conditions similar to those imposed here, showed that

\begin{equation}
(1.4) \quad \mathcal{M}_n(\infty, \mathcal{P}) \approx l \left( \omega_H \left( n^{-1/2} \right) \right),
\end{equation}

where $\omega_H(\varepsilon) = \sup\{ |\theta(P_0) - \theta(P_1)| : H(P_0, P_1) \leq \varepsilon, P_0, P_1 \in \mathcal{P} \}$ and $H$ is the Hellinger distance. Comparing (1.4) to (1.3), we notice that the Hellinger modulus $\omega_H$ of $\theta$ is replaced by the total variation modulus $\omega_1$. This may, and typically will, lead to different rates of convergence in private and non-private problems. Note that in (1.3) we have suppressed constants that depend on $\alpha$. Our results will even reveal that if $\alpha$ is small, the effective sample size reduces from $n$ to $n(e^\alpha - 1)^2$ when $\alpha$-differential privacy is required. That differential privacy leads to slower minimax rates of convergence was already observed by Duchi et al. (2013,a, 2017), for specific estimation problems. Here, we develop a unifying general theory to quantify the privatized minimax rates of convergence in a large class of different estimation problems, including (even irregular) parametric and non-parametric cases. This is also the first step towards a fundamental theory of adaptive estimation under differential privacy that will be pursued elsewhere.

We also exhibit a general construction scheme for minimax rate optimal $\alpha$-locally differentially private channels that applies in many classical estimation problems. Suppose that for some $s \geq 0, t > 0$, there is an estimator of the form $\frac{1}{n} \sum_{i=1}^{n} \ell_h(X_i)$ in the direct (non-private) estimation problem, that has a bias which decays at least as fast as $h^t$, as the ‘bandwidth parameter’ $h \to 0$, and such that $\|\ell_h\|_\infty \lesssim h^{-s}$. If $\omega_1(\varepsilon) \approx \varepsilon^{\frac{s}{s+t}}$ as $\varepsilon \to 0$, and the regularity conditions on $\theta$ and $\mathcal{P}$ are satisfied, then generating $Z_i$ independently and binary distributed on $\{-z_0, z_0\}$, with

\[ Pr(Z_i = z_0|X_i = x_i) = \frac{1}{2} \left( 1 + \frac{\ell_h(x_i)}{z_0} \right), \quad h_n = \left( \frac{e^\alpha + 1}{\sqrt{n}(e^\alpha - 1)} \right)^{\frac{s}{s+t}} \]

and $z_0 = \|\ell_h\|_\infty \frac{e^\alpha + 1}{\sqrt{n}(e^\alpha - 1)}$, yields an $\alpha$-locally differentially private channel that attains the minimax rate in (1.3). We also treat the unisotropic multivariate case, where $h_n$ may be a vector of bandwidth parameters. The conditions on $\ell_h$ are satisfied in many classical moment or density estimation problems (cf. Section 5).

The paper is organized as follows. In the next section (Section 2), we formally introduce the private estimation problem, several classes of channel distributions and a few tools required for the analysis of the $\alpha$-private minimax risk $\mathcal{M}_n(\alpha, \mathcal{P})$. Section 3 contains the derivation of a general lower bound on $\mathcal{M}_n(\alpha, \mathcal{P})$. That this lower bound is attained in many interesting cases is then established in Section 4. This section is divided into two subsections. The first one establishes attainability per se, in a non-constructive way. In the second subsection, under somewhat different assumptions, we then exhibit feasible, minimax rate optimal channel distributions and estimators. We also illustrate the general theory by a number of concrete examples in Section 5.
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2. Preliminaries and notation. Let \( \mathcal{X} \subseteq \mathbb{R}^d \) be equipped with the Borel sigma field \( \mathcal{B}(\mathcal{X}) \) with respect to the usual topology and \( \mathcal{P} \) be a set of probability measures on the measurable space \( (\mathcal{X}, \mathcal{B}(\mathcal{X})) \). Let \( \theta : \mathcal{P} \rightarrow \mathbb{R} \) be a functional of interest. We are given the privatized data \( Z_1, \ldots, Z_n \) on the measurable space \( (\mathcal{Z}^n, \mathcal{B}(\mathcal{Z}^n)) \), \( \mathcal{Z} = \mathbb{R}^q \). The conditional distribution of the observations \( Z = (Z_1, \ldots, Z_n)' \) given the original sample \( X = (X_1, \ldots, X_n)' \) is described by the channel distribution \( Q \). That is, \( Q \) is a probability kernel from \( (\mathcal{X}^n, \mathcal{B}(\mathcal{X}^n)) \) to \( (\mathcal{Z}^n, \mathcal{B}(\mathcal{Z}^n)) \). For ease of notation we suppress its dependence on \( n \). Hence, the joint distribution of the observation vector \( Z = (Z_1, \ldots, Z_n)' \) on \( \mathcal{Z}^n \) is given by \( Q \mathcal{P}^\otimes n \), i.e., the measure \( A \mapsto \int_{\mathcal{X}^n} Q(A|x)d\mathcal{P}^\otimes n(x) \).

2.1. Locally differentially private minimax risk. Recall that for \( \alpha \in (0, \infty) \), a channel distribution \( Q \) is called \( \alpha \)-differentially private, if

\[
\sup_{A \in \mathcal{B}({\mathcal{Z}^n})} \sup_{x, x' \in \mathcal{X}^n} \frac{Q(A|x)}{Q(A|x')} \leq e^\alpha.
\]

(2.1)

Note that for this definition to make sense, the probability measures \( Q(\cdot|x) \), for different \( x \in \mathcal{R}^n \), have to be equivalent and we interpret \( \frac{0}{0} \) as equal to 1.

Next, we introduce two types of local differential privacy. A channel distribution \( Q : \mathcal{B}(\mathcal{Z}^n) \times \mathcal{X}^n \rightarrow [0, 1] \) is said to be \( \alpha \)-sequentially interactive (or provides \( \alpha \)-sequentially interactive differential privacy) if the following two conditions are satisfied. First, we have for all \( A \in \mathcal{B}(\mathcal{Z}^n) \) and \( x_i \in \mathcal{X} \),

\[
Q \left( A \mid x_1, \ldots, x_n \right)
\]

(2.2)

\[
= \int_\mathcal{Z} \cdots \int_\mathcal{Z} Q_n(A_{z_1:n-1}|x_n, z_{1:n-1})Q_{n-1}(dz_{n-1}|x_{n-1}, z_{1:n-2}) \cdots Q_1(dz_1|x_1),
\]

where, for each \( i = 1, \ldots, n \), \( Q_i \) is a channel from \( \mathcal{X} \times \mathcal{Z}^{i-1} \) to \( \mathcal{Z} \). Here, \( z_{1:n} = (z_1, \ldots, z_n)' \) and \( A_{z_1:n-1} = \{ z \in \mathcal{Z} : (z_1, \ldots, z_{n-1}, z)' \in A \} \) is the \( z_{1:n-1} \)-section of \( A \). Second, we require that the conditional distributions \( Q_i \) satisfy

\[
\sup_{A \in \mathcal{B}(\mathcal{Z})} \sup_{x_i, x_i', z_{1:i-1}} \frac{Q_i(A|x_i, z_1, \ldots, z_{i-1})}{Q_i(A|x_i', z_1, \ldots, z_{i-1})} \leq e^\alpha \quad \forall i = 1, \ldots, n.
\]

(2.3)

By the usual approximation of integrands by simple functions, it is easy to see that (2.2) and (2.3) imply (2.1), so that \( \alpha \)-sequentially interactivity is a special case of \( \alpha \)-differential privacy. This notion coincides with the definition of sequentially interactive channels introduced in Duchi et al. (2013b, page 2). We note that (2.3) only makes sense if for any \( x_i^* \in \mathcal{X} \) and for all \( x_i, z_1, \ldots, z_{i-1} \), the probability measure \( Q_i(\cdot|x_i^*, z_{1:i-1}) \) is absolutely continuous with respect to \( Q_i(\cdot|x_i^*, z_{1:i-1}) \). Sequentially interactive differential privacy is a special case of local differential privacy as defined in the introduction. Here, the idea is that individuals \( i \) can
only use previous $Z_j$, $j < i$, in their local privacy mechanism, thus leading to the sequential structure in the above definition. In the rest of the paper we only consider $\alpha$-sequentially interactive channels, to which we also refer simply as $\alpha$-private channels.

An important special case of $\alpha$-private channels are the so-called non-interactive channels $Q$ that are of product form

\begin{equation}
Q \left( A_1 \times \cdots \times A_n \middle\vert x_1, \ldots, x_n \right) = \prod_{i=1}^{n} Q_i(A_i|x_i), \quad \forall A_i \in \mathcal{B}(Z), x_i \in \mathcal{X}.
\end{equation}

Clearly, a non-interactive channel $Q$ satisfies (2.1) if, and only if,

\[ \sup_{A \in \mathcal{B}(Z)} \sup_{x, x' \in \mathcal{X}} Q_i(A \mid x) \leq e^\alpha \quad \forall i = 1, \ldots, n. \]

If we measure the error of estimation by the measurable loss function $l : \mathbb{R}_+ \to \mathbb{R}_+$, the minimax risk of this estimation problem is given by

\begin{equation}
M_n(Q, \mathcal{P}) = \inf_{\theta_n \in \mathcal{P}} \mathbb{E}_{Q \otimes \cdots \otimes Q} \left[ l \left( \hat{\theta}_n - \theta(P) \right) \right],
\end{equation}

where the infimum runs over all estimators $\hat{\theta}_n : Z^n \to \mathbb{R}$. Finally, define the set of $\alpha$-private channels

\begin{equation}
\mathcal{Q}_\alpha := \bigcup_{q \in \mathbb{N}} \{ Q : Q \text{ is } \alpha\text{-sequentially interactive from } \mathcal{X}^n \text{ to } Z^n = \mathbb{R}^{n \times q} \}.
\end{equation}

Therefore, the $\alpha$-private minimax risk is given by

\begin{equation}
M_n(\alpha, \mathcal{P}) = \inf_{Q \in \mathcal{Q}_\alpha} M_n(Q, \mathcal{P}).
\end{equation}

Note that the above infimum runs also over all possible dimensions $q$ in $Z = \mathbb{R}^q$.

2.2. **Testing affinities and minimax identities.** Let $\mathcal{P}$, $\mathcal{P}_0$ and $\mathcal{P}_1$ be sets of probability measures on a measurable space $(\Omega, \mathcal{F})$ and for $P_0 \in \mathcal{P}_0$, $P_1 \in \mathcal{P}_1$, define the testing affinity

\begin{equation}
\pi(P_0, P_1) = \inf_{\phi, \psi} \mathbb{E}_{P_0}[\phi] + \mathbb{E}_{P_1}[1 - \phi],
\end{equation}

where the infimum runs over all (randomized) tests $\phi : \Omega \to [0, 1]$. Moreover, we write

\begin{equation}
\pi(P_0, P_1) = \sup_{P_j \in \mathcal{P}_j, j=0,1} \pi(P_0, P_1).
\end{equation}

Throughout, we follow the usual conventions that $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$. If $\theta : \mathcal{P} \to \mathbb{R}$ is a functional of interest, then for $t \in \mathbb{R}$ and $\Delta > 0$, denote
\( \mathcal{P}_{\leq t} := \{ \mathbb{P} \in \mathcal{P} : \theta(\mathbb{P}) \leq t \} \) and \( \mathcal{P}_{\geq t+\Delta} := \{ \mathbb{P} \in \mathcal{P} : \theta(\mathbb{P}) \geq t + \Delta \} \) and let \( \mathcal{P}_{\leq t}^{(n)} \) and \( \mathcal{P}_{\geq t+\Delta}^{(n)} \) the sets of \( n \)-fold product measures with identical marginals from \( \mathcal{P}_{\leq t} \) and \( \mathcal{P}_{\geq t+\Delta} \), respectively. If \( Q \) is a channel distribution, then we write \( Q\mathcal{P}^{(n)} \) for the set of all probability measures of the form \( Q\mathcal{P}^{\otimes n} \), where \( \mathbb{P} \in \mathcal{P} \). Note that if \( Q \) is \( \alpha \)-differentially private, then \( Q\mathcal{P}^{(n)} \) is necessarily dominated even if \( \mathcal{P}^{(n)} \) is not. Recall that a family of measures on a common probability space is dominated if there exists a \( \sigma \)-finite measure \( \mu \) such that every element of that family is absolutely continuous with respect to \( \mu \). We define the convex hull \( \text{conv}(\mathcal{P}) \) in the usual way to be the set of all finite convex combinations \( \sum_{i=1}^{m} \lambda_{i} \mathbb{P}_{i} \), for \( \lambda_{i} \geq 0 \), \( \sum_{i=1}^{m} \lambda_{i} = 1 \) and \( \mathbb{P}_{i} \in \mathcal{P} \). For \( \mathbb{P}_{0}, \mathbb{P}_{1} \in \mathcal{P} \), we consider the Hellinger distance

\[
H(\mathbb{P}_{0}, \mathbb{P}_{1}) := \sqrt{\int_{\Omega} \left( \sqrt{p_{0}(x)} - \sqrt{p_{1}(x)} \right)^{2} \, d\mu(x)},
\]

where \( p_{0} \) and \( p_{1} \) are densities of \( \mathbb{P}_{0} \) and \( \mathbb{P}_{1} \) with respect to some dominating measure \( \mu \) (e.g., \( \mu = \mathbb{P}_{0} + \mathbb{P}_{1} \)), and the total variation distance is defined as \( d_{TV}(\mathbb{P}_{0}, \mathbb{P}_{1}) := \sup_{A \in \mathcal{F}} |\mathbb{P}_{0}(A) - \mathbb{P}_{1}(A)| \). Furthermore, for a monotone function \( g : \mathbb{R} \to \mathbb{R} \), we write \( g(x^{-}) = \lim_{x \downarrow} g(y) \) and \( g(x^{+}) = \lim_{x \uparrow} g(y) \), for the left and right limits of \( g \) at \( x \in \mathbb{R} \), respectively, and we write \( g(\infty^{-}) = \lim_{x \to -\infty} g(x) \) and \( g(\infty^{+}) = \lim_{x \to -\infty} g(x) \). We also make use of the abbreviations \( a \vee b = \max(a, b) \) and \( a \wedge b = \min(a, b) \).

Next, we define the upper affinity

\[
\eta_{A}^{(n)}(Q, \Delta) = \sup_{t \in \mathbb{R}} \pi \left( \text{conv} \left( Q\mathcal{P}_{\leq t}^{(n)} \right), \text{conv} \left( Q\mathcal{P}_{\geq t+\Delta}^{(n)} \right) \right)
\]

and its generalized inverse for \( \eta \in [0, 1] \),

\[
\Delta_{A}^{(n)}(Q, \eta) = \sup \{ \Delta \geq 0 : \eta_{A}^{(n)}(Q, \Delta) > \eta \}.
\]

Note that for \( \eta < 1 \) the set in the previous display is never empty, since \( \eta_{A}^{(n)}(Q, 0) = 1 \), and thus \( \Delta_{A}^{(n)}(Q, \eta) \geq 0 \). Also note that \( \Delta \mapsto \eta_{A}^{(n)}(Q, \Delta) \) is non-increasing.

In order to show that our subsequent lower bounds on \( \mathcal{M}_{\alpha}(\alpha, \mathcal{P}) \) are attained, we will need the following consequence of a fundamental minimax theorem of Sion (1958, Corollary 3.3). See Appendix B.1 for the proof.

**Proposition 2.1.** Fix constants \( -\infty < a \leq b < \infty \). Let \( \mathbb{S} \) be a convex set of finite signed measures on a measurable space \( (\Omega, \mathcal{F}) \), so that \( \mathbb{S} \) is dominated by a \( \sigma \)-finite measure \( \mu \). Furthermore, let \( T = \{ \phi \in L_{\infty}(\Omega, \mathcal{F}, \mu) : a \leq \int_{\Omega} \phi \, d\mu \leq b, \forall f \in L_{1}(\Omega, \mathcal{F}, \mu) : ||f||_{L_{1}} \leq 1 \} \). Then

\[
\sup_{\phi \in T} \inf_{\sigma \in \mathbb{S}} \int_{\Omega} \phi \, d\sigma = \inf_{\sigma \in \mathbb{S}} \sup_{\phi \in T} \int_{\Omega} \phi \, d\sigma.
\]

Proposition 2.1 implies that for arbitrary subsets \( \mathcal{P}_{0} \) and \( \mathcal{P}_{1} \) of \( \mathcal{P} \), and if the class \( Q\mathcal{P}^{(n)} \) is dominated by some \( \sigma \)-finite measure (note that this is always the
case if $Q$ is $\alpha$-private), we have the identity

\begin{equation}
\inf_{\text{tests } \phi} \sup_{P_0 \in QP_0^{(n)}} \mathbb{E}_{P_0}[\phi] + \mathbb{E}_{P_1}[1 - \phi] = \sup_{P_0 \in \text{conv}(QP_0^{(n)})} \inf_{\text{tests } \phi} \mathbb{E}_{P_0}[\phi] + \mathbb{E}_{P_1}[1 - \phi]
\end{equation}

\begin{equation}
= \pi \left( \text{conv} \left( QP_0^{(n)} \right), \text{conv} \left( QP_1^{(n)} \right) \right).
\end{equation}

To see this, note that the left-hand side of (2.12) does not change if we replace $QP_i^{(n)}$ by its convex hull, for $r = 0, 1$, because for $P_{r,i} \in QP_i^{(n)}$,

\begin{equation}
\mathbb{E}_{\sum_{i=1}^{n} \alpha_i P_{0,i}}[\phi] + \mathbb{E}_{\sum_{j=1}^{n} \beta_j P_{1,j}}[1 - \phi] = \sum_{i,j} \alpha_i \beta_j \left( \mathbb{E}_{P_{0,i}}[\phi] + \mathbb{E}_{P_{1,j}}[1 - \phi] \right)
\end{equation}

\begin{equation}
\leq \sup_{P_0 \in \text{conv}(QP_0^{(n)})} \mathbb{E}_{P_0}[\phi] + \mathbb{E}_{P_1}[1 - \phi].
\end{equation}

Now apply Proposition 2.1 with $S = \{P_0 - P_1 : P_r \in \text{conv}(QP_r^{(n)}), r = 0, 1\}$ and $a = 0, b = 1$.

The identity (2.12) was prominently used by Donoho and Liu (1991) in the case where $Q(A(x) = 1_A(x)$ (non-private case), in order to derive their lower bounds on the minimax risk. It is due to C. Kraft and L. Le Cam (Theorem 5 of Kraft (1955), see also page 40 of LeCam (1973)). We will also make use of (2.12) to derive lower bounds (see the proof of Theorem 3.1). However, in order to show that there exist channel distributions $Q'$ so that $M_n(Q', P)$ attains the rate of $M_n(\alpha, P)$, we need the generality of Proposition 2.1 (see Section 4.1).

3. A general lower bound on the $\alpha$-private minimax risk. In this section we establish a lower bound on $M_n(\alpha, P) = \inf_{Q \in Q_n} \mathcal{M}_n(Q, P)$, $\alpha \in (0, \infty)$, in terms of the total variation and Hellinger moduli of continuity $\omega_1$ and $\omega_H$ of the functional $\theta : P \to \mathbb{R}$. We also bridge the gap to the non-private case $\alpha = \infty$ in which the rate is characterized by $\omega_H$ only, and therefore, we substantially extend results of Donoho and Liu (1991) to the case of privatized data. Under suitable regularity conditions, these lower bounds are shown to be rate optimal in the next section.

From now on, we let $P$ denote our model of data generating distributions on $X \subseteq \mathbb{R}^d$, so that $P^{(n)}$ is the corresponding model of $n$-fold product measures on the sample space $X^n$, and $\theta : P \to \mathbb{R}$ is the functional of interest. As a first step to lower bound $M_n(\alpha, P)$, we extend the result of Donoho and Liu (1991, Theorem 2.1) (see also Tsybakov, 2009, Theorem 2.14) to the case of privatized observations. In our case, where $Q \in Q_\alpha$, $\alpha \in (0, \infty)$, it holds without any assumptions on $\theta$ and $P$. Recall that the dominatedness condition on $QP^{(n)}$ is always satisfied if $Q$ is an $\alpha$-private channel. See Appendix A.1 for the proof.
\textbf{Theorem 3.1.} Let $\eta \in (0, 1)$ be fixed and let $l: \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing loss function. If $Q^{P^{(n)}}$ is dominated, then

$$M_n(Q, P) := \inf_{\theta_n} \sup_{P \in \mathcal{P}} \mathbb{E}_{Q^{P^{(n)}}} \left[ l \left( \hat{\theta}_n(Z) - \theta(P) \right) \right] \geq l \left( \frac{1}{2} \Delta_A^{(n)}(Q, \eta)^- \right) \frac{\eta}{2}.$$ 

As pointed out by Donoho and Liu (1991) in the non-private case, the quantity $\Delta_A^{(n)}(Q, \eta)$ is not easy to calculate in general. However, in this case, where there is no privatization and $Q(x)$ is the Dirac measure at $x$, these authors derive a general lower bound on $\Delta_A^{(n)}(Q, \eta)$ in terms of the Hellinger modulus of continuity of $\theta$, i.e.,

$$\omega_H(\varepsilon) := \sup \{ |\theta(P_0) - \theta(P_1)| : H(P_0, P_1) \leq \varepsilon, P_j \in \mathcal{P}, j = 0, 1 \},$$

where $H(P_0, P_1)$ is the Hellinger distance. The Hellinger distance turns out to be exactly the right metric to characterize the minimax rate in the non-private case, because of its relation to the testing affinity (2.8) and its convenient behavior under product measures. In particular, we have the well known identities

\begin{equation}
H^2 = 2(1 - \rho) \quad \text{and} \quad \rho(P_0^{\otimes n}, P_1^{\otimes n}) = \rho(P_0, P_1)^n,
\end{equation}

where $\rho(P_0, P_1) = \int \sqrt{p_0 p_1}$ is the Hellinger affinity, and

\begin{equation}
\pi \leq \rho \leq \sqrt{\pi(2 - \pi)} = \sqrt{(1 - d_{TV})(1 + d_{TV})} = \sqrt{1 - d_{TV}^2},
\end{equation}

where $\pi$, $\rho$ and $d_{TV}$ are abbreviations for $\pi(P_0, P_1)$, $\rho(P_0, P_1)$ and $d_{TV}(P_0, P_1)$, respectively; cf. Equation (3.7) of Donoho and Liu (1991). In that reference, the authors show that

$$\Delta_A^{(n)}(Q, \eta) \geq \omega_H \left( c \sqrt{\frac{|\log \eta|}{n}} \right),$$

for all small $\eta > 0$, all large $n \in \mathbb{N}$, and in the special case where $Q(A|x) = \mathbb{1}_A(x)$ is the channel that returns the original observations without privatization. In the privatized case, and if $Q(A_1 \times \cdots \times A_n|x_1, \ldots, x_n) = \prod_{i=1}^n Q_1(A_i|x_i)$ is a non-interactive channel with identical marginals $Q_1$, one can follow the same strategy to obtain a bound of the form

\begin{equation}
\Delta_A^{(n)}(Q, \eta) \geq \omega_H^{(Q_1)} \left( c \sqrt{\frac{|\log \eta|}{n}} \right),
\end{equation}

where $\omega_H^{(Q_1)}(\varepsilon) := \sup \{ |\theta(P_0) - \theta(P_1)| : H(Q_1 P_0, Q_1 P_1) \leq \varepsilon, P_j \in \mathcal{P}, j = 0, 1 \}$. Moreover, if $Q_1$ is $\alpha$-private, one can use Theorem 1 of Duchi et al. (2013b) (see Remark 4.2 below for details) to show that

\begin{equation}
\omega_H^{(Q_1)}(n^{-1/2}) \geq \omega_1 \left( \frac{1}{2 \sqrt{n}(e^\alpha - 1)} \right),
\end{equation}
where \( \omega(\varepsilon) = \sup \{ \theta(P_0) - \theta(P_1) : d_{TV}(P_0, P_1) \leq \varepsilon, P_j \in \mathcal{P}, j = 0, 1 \} \) is the total variation (or \( L_1 \)) modulus of continuity of \( \theta \). This strategy, however, applies only to non-interactive channels with identical marginals. A more general approach can be based on the remarkable inequality

\[
d_{TV}(Q_{0}^{n}, Q_{1}^{n}) \leq \sqrt{2n(e^n - 1)}d_{TV}(P_0, P_1),
\]

which was first observed in Duchi et al. (2013b, their Corollary 1 combined with Pinsker’s inequality), and which holds for all \( \alpha \)-sequentially interactive channels \( Q \). Of course, for interactive channels, an inequality as in (3.3) cannot be obtained.

The next step is, still for fixed \( Q \), to pass over from \( \Delta_{2}^{(n)}(Q, \eta) \) to the moduli of continuity \( \omega_{1} \) and \( \omega_{H} \) of \( \theta \).

**Lemma 3.2.** Fix \( \eta \in (0, 1) \) and a channel distribution \( Q \). Then

\[
\omega_{1} (g_{1}(Q, \eta)^{-}) \leq \Delta_{2}^{(n)}(Q, \eta), \quad \text{and}
\]

\[
\omega_{H} (g_{H}(Q, \eta)^{-}) \leq \Delta_{2}^{(n)}(Q, \eta),
\]

where

\[
g_{1}(Q, \eta) := \inf\{ d_{TV}(P_1, P_0) : \pi(Q_{0}^{n}, Q_{1}^{n}) \leq \eta, P_j \in \mathcal{P}, j = 0, 1 \}, \quad \text{and}
\]

\[
g_{H}(Q, \eta) := \inf\{ H(P_1, P_0) : \pi(Q_{0}^{n}, Q_{1}^{n}) \leq \eta, P_j \in \mathcal{P}, j = 0, 1 \}.
\]

**Proof.** For \( \delta > 0 \), set \( C := C(Q, \eta, \delta) := \{ \theta(P_1) - \theta(P_0) : d_{TV}(P_1, P_0) \leq g_{1}(Q, \eta) - \delta, P_j \in \mathcal{P} \} \), so that \( \sup C = \omega_{1}(g_{1}(Q, \eta) - \delta) \). If \( C = \emptyset \), then the desired inequality is trivial. So let \( \Delta \in C \). Then there exist \( P_0, P_1 \in \mathcal{P} \), such that \( \Delta = |\theta(P_1) - \theta(P_0)| \) and \( d_{TV}(P_1, P_0) \leq g_{1}(Q, \eta) - \delta \). But this entails that

\[
\pi(Q_{0}^{n}, Q_{1}^{n}) > \eta, \quad \text{or otherwise} \quad g_{1} \text{ could not be the infimum. Now, without}\n\]

loss of generality, let \( t_{0} := \theta(P_0) \leq \theta(P_1) \), so that \( P_0 \in \mathcal{P}_{\leq t_{0}} \) and \( P_1 \in \mathcal{P}_{\geq t_{0} + \Delta} \). Thus,

\[
\eta_{2}^{(n)}(Q, \Delta) = \sup_{t \in \mathbb{R}} \pi \left( Q_{0}^{n}, Q_{1}^{n} \right) > \eta. \quad \text{So we have established that}
\]

\[
C = C(Q, \eta, \delta) \subseteq \{ \Delta \geq 0 : \eta_{2}^{(n)}(Q, \Delta) > \eta \} \quad \text{and therefore,} \quad \sup_{C} \leq \Delta_{2}^{(n)}(Q, \eta).
\]

Now let \( \delta \to 0 \). The result for \( \omega_{H} \) is established in an analogous way. \( \square \)
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It remains to derive lower bounds on $g_1$ and $g_H$. See Appendix A.2 for the proof of the following result.

**Theorem 3.3.** Fix $n \in \mathbb{N}$, $\eta \in (0, 1)$, $\alpha \in (0, \infty)$ and let $Q$ be an $\alpha$-sequentially interactive channel as in (2.2) and (2.3). Then

$$g_1(Q, \eta) \geq \sqrt{\frac{(1 - \eta)}{2n(e^\alpha - 1)^2}},$$

where $g_1$ is as in Lemma 3.2. Consequently, we have

$$\Delta^{(n)}_\alpha(Q, \eta) \geq \omega_1 \left( \sqrt{\frac{1 - \eta}{2n(e^\alpha - 1)^2}} \right)^{\frac{1}{2}} \vee \omega_H \left( c \sqrt{\frac{|\log \eta|}{n}} \right)^{\frac{1}{2}}.$$

Moreover, for all $\eta_0 \in (0, 1)$ and every $\varepsilon_0 > 0$, there exists a finite positive constant $c > 0$, so that for all $\eta \in (0, \eta_0)$, for all $n > |\log \eta|/\varepsilon_0$ and for all channels $R$,

$$g_H(R, \eta) \geq c \sqrt{\frac{|\log \eta|}{n}},$$

where $g_H$ is as in Lemma 3.2. Consequently, for such $\eta$, $n$ and $\alpha$-private channel $Q$, we have

$$\Delta^{(n)}_\alpha(Q, \eta) \geq \omega_1 \left( \sqrt{\frac{1 - \eta}{2n(e^\alpha - 1)^2}} \right)^{\frac{1}{2}} \vee \omega_H \left( c \sqrt{\frac{|\log \eta|}{n}} \right)^{\frac{1}{2}} \eta^{\frac{1}{2}}.$$

We thus conclude with the following corollary.

**Corollary 3.4.** Fix $\eta_0, \varepsilon_0 \in (0, 1)$, $\alpha \in (0, \infty)$ and let $l : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing loss function. Then there exists a positive finite constant $c = c(\eta_0, \varepsilon_0)$, so that for all $\eta \in (0, \eta_0)$ and for all $n > |\log \eta|/\varepsilon_0$,

$$M_n(\alpha, \mathcal{P}) := \inf_{Q \in \mathcal{Q}_\alpha} M_n(Q, \mathcal{P})$$

$$\geq l \left( \frac{1}{2} \omega_1 \left( \sqrt{\frac{1 - \eta}{2n(e^\alpha - 1)^2}} \right)^{\frac{1}{2}} \vee \omega_H \left( c \sqrt{\frac{|\log \eta|}{n}} \right)^{\frac{1}{2}} \eta^{\frac{1}{2}},

$$

where $\mathcal{Q}_\alpha$ is the set of $\alpha$-sequentially interactive channels $Q$ as in (2.6).

Corollary 3.4 extends the lower bound of Donoho and Liu (1991) to privatized data. In general, we have $\omega_H(\varepsilon) \leq \omega_1(\varepsilon)$, because $d_{TV}(\mathbb{P}_0, \mathbb{P}_1) \leq H(\mathbb{P}_0, \mathbb{P}_1)$. Therefore, privatization typically leads to a larger lower bound compared to the direct case. However, if $\alpha$ is sufficiently large, i.e., the privatization constraint is weak,
then the lower bound of Corollary 3.4 reduces to the classical lower bound in the case of direct estimation derived by Donoho and Liu (1991).

In our theory we consider the class $Q_\alpha$ of $\alpha$-sequentially interactive channels, because those admit a reasonably simple and attainable lower bound (cf. Duchi et al., 2013b) and they comprise a relevant class of local differential privatization mechanisms. In the next section, we show that for estimation of linear functionals $\theta$ over convex parameter spaces $P$ (and also for more general, but sufficiently regular $\theta$ and $P$), the rate of our lower bound is attained even within the much smaller class of non-interactive channels. So within the class of sequentially interactive channels, the non-interactive channels already lead to rate optimal private estimation of linear functionals over convex parameter spaces.

Remark 3.5. Corollary 3.4 does not restrict the values of $\alpha \in (0, \infty)$ and is formulated for any sample size $n$. In particular, it continues to hold if $\alpha$ is replaced by an arbitrary sequence $\alpha_n \in (0, \infty)$. The choice of this sequence has a fundamental impact on the private minimax rate of convergence. For example, if we consider the highly privatized case where $\alpha_n \approx n^{-1/2}$, then $n(e^{\alpha_n} - 1)^2$ is bounded and the $\alpha_n$- privatized minimax risk no longer converges to zero as $n \to \infty$.

4. Attainability of lower bounds for regular functionals and parameter spaces. To establish tight upper bounds on the minimax risk, some regularity conditions are needed. In the case where the channel $Q$ is fixed, the main ingredients for a characterization of $M_n(Q, P)$ are a certain (near) minimax identity (see Theorem 4.1 below) and a Hölderian behavior of the privatized Hellinger modulus

$$\omega^{(Q)}_{H}(\epsilon) = \sup \{ |\theta(P_0) - \theta(P_1)| : H(QP_0, QP_1) \leq \epsilon, P_0, P_1 \in P \},$$

where here $Q : \mathcal{B}(\mathcal{Z}) \times \mathcal{X} \to [0, 1]$. If $Q$ is a non-interactive channel with identical marginals, so that $Q(A_1 \times \cdots \times A_n|x_1, \ldots, x_n) = \prod_{i=1}^n Q_1(A_i|x_i)$, then we also write $\omega^{(Q)}_{H} = \omega^{(Q_1)}_{H}$. Throughout this section, we make the following assumptions.

A) The functional $\theta : P \to \mathbb{R}$ of interest is bounded, i.e., $M := \sup_{P \in P} |\theta(P)| < \infty$.

B) Let the non-decreasing loss function $l : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $l(0) = 0$ and $l(\frac{1}{2}t) \leq a l(t)$, for some $a \in (1, \infty)$ and for every $t \in \mathbb{R}_+$.

The boundedness assumption A is also maintained in Donoho and Liu (1991). However, in their context, it is actually not necessary in some special cases such as the location model. On the other hand, the boundedness of $\theta$ appears to be much more fundamental in the case of private estimation. See, for example, Section G in Duchi et al. (2013b), who show that in the privatized location model under squared error loss, Assumption A is necessary in order to obtain finiteness of the $\alpha$-private minimax risk $M_n(\alpha, P)$. Assumption B is also taken from Donoho and Liu (1991). It is satisfied for many common loss functions, such as $l_\gamma(t) = t^\gamma$, with $\gamma > 0$, or
the Huber loss $l_\gamma(t) = I_{[0,\gamma]}(t)^2/2 + I_{[\gamma,\infty]}(t)(t-\gamma/2)$, which satisfies B with $a = 9/2$.

The following theorem, whose proof is deferred to Appendix A.3, provides general assumptions on the sequence of channels $Q^n$, the model $P$ and the functional $\theta$, so that the privatized minimax risk $\mathcal{M}_n(Q^n,P)$ is upper bounded by a constant multiple of $l \cdot \omega_H^{(Q^n)}(n^{-1/2})$. This is a strict generalization of results of Donoho and Liu (1991) to cover also the case where $Q^n$ is an arbitrary non-interactive channel with identical marginals and not necessarily equal to $I_A(x)$. Concerning its proof, we introduce a binary search estimator different to the one used by Donoho and Liu (1991), which, in particular, takes the privatized data as input data. The binary search estimator is based on minimax tests whose existence is verified in Donoho and Liu (1991), which, in particular, takes the privatized data as input data. The binary search estimator is based on minimax tests whose existence is verified in a non-constructive way. Therefore, it is not available for practical purposes. Subsequently, we will exhibit sequences of non-interactive $\alpha$-private channels $Q^n$ that satisfy the imposed assumptions and are such that $\omega_H^{(Q^n)}(n^{-1/2})$ is of the same order as $\omega_1(n^{-1/2})$. This shows that the lower bound of the previous section can be attained.

**Theorem 4.1.** Suppose that A and B hold and that there exist positive finite constants $r, \varepsilon_0 > 0$, $0 < A_0 \leq A_1 < \infty$ and a collection of non-interactive channels $\{Q^\varepsilon : \varepsilon > 0\}$ with identical marginals $Q^\varepsilon_1$, such that for every $\varepsilon \in (0, \varepsilon_0 \land 1]$, $Q^\varepsilon_1 P$ is dominated (by a $\sigma$-finite measure), and

$$\omega_H^{(Q^n)}(\varepsilon) = \sup_{\theta_0, \theta_1} \mathbb{E}_{Q^n} \left[ l\left(\hat{\theta}_n - \theta(P)\right)\right] \leq C_0 \cdot l \left(\omega_H^{(Q^n)} \left(n^{-1/2}\right)\right),$$

for all $n \geq n_0$, where $C_0 = [1 + 16a]a^{\log(C)/\log(3/2)}$ and $a > 1$ is the constant from Condition B. Here, the constants $C = C(r, \varepsilon_0, A_0, A_1, a, M)$ and $n_0 = n_0(\varepsilon_0)$ can be chosen as

$$C = \max \left\{ \frac{2A_1}{A_0}, \left(\frac{r \log 2a}{A_0(\varepsilon_0 \land 1)^r}\right)^{\frac{1}{r}}, \frac{2M}{A_0(\varepsilon_0 \land 1)^r} \right\} \text{ and } n_0 = (\varepsilon_0 \land 1)^{-2}.$$
Remark 4.2. Condition (4.1) is the privatized version of the analogous condition maintained in Donoho and Liu (1991). It is instrumental in the current strategy of proof because it simplifies. The lower bound in Condition (4.1) of Theorem 4.1 is satisfied for \( A_0 = \tilde{A}_0/2(\varepsilon^a - 1)\) and \( \varepsilon_0 = (\varepsilon_0 \wedge 1)2(\varepsilon^a - 1)\), provided that \( Q^r \) is \( \alpha \)-sequentially interactive, and the \( L_1 \)-modulus \( \omega_1 \) admits a Hölderian lower bound \( \omega_1(\varepsilon) \geq \tilde{A}_0\varepsilon^r \), for all \( \varepsilon \in (0, \varepsilon_0 \wedge 1] \). To see this, note that the Hellinger distance is bounded by the square root of the Kullback-Leibler divergence and use Theorem 1 of Duchi et al. (2013b) to show that \( H(Q^r P_0, Q^r P_1) \leq 2(\varepsilon^a - 1)d_{TV}(P_0, P_1) \). Thus, we obtain the bounds

\[
\frac{A_0}{2^r(\varepsilon^a - 1)^r} \varepsilon^r \leq \omega_1 \left( \frac{\varepsilon}{2(\varepsilon^a - 1)} \right) \leq \omega_1^{(Q^r)}(\varepsilon) \quad \forall \varepsilon \in (0, \varepsilon_0 \wedge 1].
\]

The upper bound in (4.1) will be verified, if \( \omega_1(\varepsilon) \) also admits a Hölderian upper bound with the same exponent \( r \) and if we can exhibit an \( \alpha \)-sequentially interactive channel \( Q^r \) so that \( \omega_1^{(Q^r)}(\varepsilon) \) is of the same order as \( \omega_1(\varepsilon) \) as \( \varepsilon \to 0 \) (cf. the discussion leading up to Theorem 4.1). This is our main objective for the remainder of this section.

Remark 4.3. Condition (4.3) is the privatized version of Condition (4.2) in Donoho and Liu (1991). It is instructive to study Section 4 of that reference to gain some intuition on the mechanism that leads to the attainability result. This discussion, extended to the privatized case, applies also in the present paper, but we do not repeat it here. We only point out, that under (4.3), the quantities \( \Delta_{\alpha}^{(n)}(Q^r, \eta) \) and \( \Delta_{\alpha}^{(n)}(Q^r, \eta) \) defined in (2.11) and (3.6) coincide, so that the bound \( \Delta_{\alpha}^{(n)}(Q^r, \eta) \leq \Delta_{\alpha}^{(n)}(Q^r, \eta) \), which was instrumental in the previous section, is actually tight.

Remark 4.4. Condition (4.2) is an alternative to Condition (4.3) and a version of it in the non-private case also appears in Donoho and Liu (1991, Lemma 4.3). Note that, unlike (4.3), the suprema in (4.2) are only over sets of one-dimensional marginal distributions, which may be convex even if the sets of corresponding product distributions are not. Moreover, it is important to note that Condition (4.2) is satisfied, in particular, if \( Q_1^r P_{\leq s} \) and \( Q_1^r P_{\geq t} \) are convex. This follows, for instance, if \( \theta : P \to \mathbb{R} \) is linear and \( P \) is convex, because then \( P_{\leq s} \) and \( P_{\geq t} \) are both convex, and so are \( Q_1^r P_{\leq s} \) and \( Q_1^r P_{\geq t} \). However, because of the privatization effect, \( Q_1^r P_{\leq s} \) and \( Q_1^r P_{\geq t} \) may be convex even though \( P_{\leq s} \) and \( P_{\geq t} \) are not. In Section 5.4 we present an example where \( P_{\leq s} \) and \( P_{\geq t} \) are non-convex, but \( Q_1^r P_{\leq s} \) and \( Q_1^r P_{\geq t} \) are convex, so that (4.2) is still satisfied.

Remark 4.5. In Theorem 4.1, the requirement that \( r \leq 2 \) is no real restriction, because even in the non-private case (where \( Q^r(\Delta|x) = 1_{A}(x) \)), if \( \omega_1^{(Q_1^r)}(\varepsilon) = \omega_1(\varepsilon) \) converges to zero faster than \( \varepsilon^2 \), as \( \varepsilon \to 0 \), and if \( P \) is convex, then \( \theta \) must be constant and the result is trivial (cf. Lemma B.2 in Appendix B). Subsequently, we will exhibit \( \alpha \)-private channels \( Q^r \), so that \( \omega_1^{(Q_1^r)}(\varepsilon) \) is of the same order as \( \omega_1(\varepsilon) \), for \( \varepsilon \to 0 \). Therefore, Condition (4.1) reduces to the requirement of a Hölderian behavior of \( \omega_1 \). We point out that in this case, and if \( P \) is convex, then Theorem 4.1
is non-trivial only for \( r \leq 1 \), because, if \( P \) is convex, then \( \omega_1(\varepsilon)/\varepsilon \to 0 \) as \( \varepsilon \to 0 \), is equivalent to \( \theta \) being constant on \( P \) (cf. Lemma B.2).

The challenge in deriving rate optimal upper bounds on the \( \alpha \)-private minimax risk \( \mathcal{M}_n(\alpha, P) \) is now to find \( \alpha \)-sequentially interactive channel distributions \( Q \), such that the upper bound of the form \( l \circ \omega^{(Q)}_H(n^{-1/2}) \) on \( \mathcal{M}_n(Q, P) \), obtained in Theorem 4.1, matches the rate of the lower bound

\[
l \left( \frac{1}{2} \omega_1 \left( \sqrt{\frac{(1 - \eta)}{2n(e^\alpha - 1)^2}} \right) \right)
\]

of Corollary 3.4. It turns out that non-interactive channels with identical binary marginals lead to rate optimal procedures for \( \alpha \)-private estimation of a large class of functionals (bounded ones with a H"olderian \( L_1 \)-modulus \( \omega_1 \) satisfying either (4.3) or (4.2)). More precisely, we suggest to use a channel with binary marginals

\[
Q^{(\alpha, \ell)}_1(\{\pm z_0\} | x) = \frac{1}{2} \left( 1 \pm \frac{\ell(x)}{z_0} \right),
\]

(4.5)

where \( z_0 := \|\ell\|_{\infty} e^{\alpha + 1} \) and where \( \ell : X \to \mathbb{R} \) is an appropriate measurable and bounded function. Note that

\[
\sup_{S \in B(\mathbb{R})} \frac{Q^{(\alpha, \ell)}_1(S | x_1)}{Q^{(\alpha, \ell)}_1(S | x_2)} = \max \left( \frac{1 + \frac{\ell(x_1)}{\|\ell\|_{\infty} e^{\alpha + 1}}}{1 + \frac{\ell(x_2)}{\|\ell\|_{\infty} e^{\alpha + 1}}}, \frac{1 - \frac{\ell(x_1)}{\|\ell\|_{\infty} e^{\alpha + 1}}}{1 - \frac{\ell(x_2)}{\|\ell\|_{\infty} e^{\alpha + 1}}} \right) \leq 1 + \frac{e^{\alpha - 1}}{e^{\alpha + 1}} = e^\alpha,
\]

so that a non-interactive channel distribution with identical marginals (4.5) is \( \alpha \)-private. Actually, the support \( \{-z_0, z_0\} \) of \( Q^{(\alpha, \ell)}_1 \) has no effect on its privacy provisions. However, with this specific choice of its support, the channel

\[
Q^{(\alpha, \ell)}(S_1 \times \cdots \times S_n | x_1, \ldots, x_n) = \prod_{i=1}^n Q^{(\alpha, \ell)}_1(S_i | x_i),
\]

has the property that the conditional expectation of \( Z_i \) given \( X_i \) under \( Q^{(\alpha, \ell)} \) is equal to \( \ell(X_i) \).

We have thus reduced the problem to finding a sequence \( (\ell_n) \) in \( L_\infty \) for which

\[
\omega^{(Q^{(\alpha, \ell_n)})}_H(n^{-1/2}) \lesssim \omega_1 \left( \sqrt{\frac{1}{n(e^\alpha - 1)^2}} \right).
\]

(4.6)

If such a bound holds, then, together with Remark 4.2, the condition (4.1) can be replaced by the assumption that \( \omega_1(\varepsilon) \asymp \varepsilon^r \), as \( \varepsilon \to 0 \), for some \( r > 0 \). This H"olderian behavior of the \( L_1 \)-modulus of continuity will be used throughout.
Subsequently, in Section 4.1, we establish a general result, stating that under some regularity conditions on $P$ and $\theta$ (which are particularly satisfied if $P$ is dominated and convex, and $\theta$ is linear), we have

\begin{equation}
\inf_{\ell \in L_\infty} \omega_H(Q^{(\alpha,\ell)}) (n^{-1/2}) \lesssim \omega_1 \left( \frac{1}{\sqrt{n(e^\alpha - 1)^2}} \right),
\end{equation}

thereby showing that the privatized lower bound of Corollary 3.4 can always be attained for linear $\theta$, convex and dominated $P$ and for a Hölderian $L_1$-modulus $\omega_1$.

This result shows existence of a sequence of functions $\ell_n$ so that (4.6) holds, but does not provide an explicit construction of such an $\ell_n$. Therefore, in Section 4.2, we formulate sufficient conditions under which we provide such an explicit construction.

The starting point for both, the results of Section 4.1 and Section 4.2, is the following observation. The channel (4.5) has the nice feature that for $P_0, P_1 \in P$ with densities $p_0$ and $p_1$ with respect to $\mu = P_0 + P_1$,

\begin{equation}
d_{\text{TV}}(Q^{(\alpha,\ell)}_{1, P_0}, Q^{(\alpha,\ell)}_{1, P_1}) = \sup_{A \in B(\mathbb{R})} \left| \int_{X} Q^{(\alpha,\ell)}_{1}(A|x)p_0(x) \, d\mu(x) - \int_{X} Q^{(\alpha,\ell)}_{1}(A|x)p_1(x) \, d\mu(x) \right|
\end{equation}

\begin{align*}
&= \max \left\{ \left| \int_{X} \frac{1}{2} \left( 1 + \frac{\ell(x)}{z_0} \right) [p_0(x) - p_1(x)] \, d\mu(x) \right|, \\
&\quad \left| \int_{X} \frac{1}{2} \left( 1 - \frac{\ell(x)}{z_0} \right) [p_0(x) - p_1(x)] \, d\mu(x) \right| \right\} \\
&= \left| \int_{X} \frac{\ell(x)}{2z_0} [p_0(x) - p_1(x)] \, d\mu(x) \right| = \frac{1}{2z_0} |E_{\mathcal{F}_0}[\ell] - E_{\mathcal{F}_1}[\ell]|.
\end{align*}

4.1. An attainability result for convex $P$ and linear $\theta$. In this subsection we show that if $P$ is convex and dominated and if $\theta$ is linear, then the previously derived lower bound on $\mathcal{M}_n(\alpha, P)$ of Section 3 can be attained. However, these results are non-constructive in the sense that they do not exhibit a sequence of $\alpha$-private channels $Q^{(n)}$ so that

$$\mathcal{M}_n(Q^{(n)}, P) \lesssim \mathcal{M}_n(\alpha, P).$$

This problem is then solved subsequently in Subsection 4.2 where the conditions on $P$ and $\theta$ are relaxed, but it is assumed that there exists an appropriate estimator in the direct (non-private) estimation problem. The following result realizes the claim of (4.7). At this point we make use of the minimax identity in Proposition 2.1.

**Theorem 4.6.** Suppose that $P$ is convex and dominated by a $\sigma$-finite measure $\mu$ and that $\theta : P \to \mathbb{R}$ is linear. Fix $\alpha \in (0, \infty)$ and $\ell \in L_\infty(\mu)$, and let $Q^{(\alpha,\ell)}_1$ be the non-interactive $\alpha$-private channel with identical marginals $Q^{(\alpha,\ell)}_1$ as in (4.5). Then,
for $\mathbb{T} = \{\ell \in L_\infty(\mu) : \|\ell\|_\infty \leq 1\}$ and for every $\varepsilon \in [0, \infty)$, we have
\[
\inf_{\ell \in \mathbb{T}} \omega_H^{Q(\alpha, \ell)}(\varepsilon) \leq \omega_1 \left( \frac{2\varepsilon^{\alpha} + 1}{\varepsilon^{\alpha} - 1} \right).
\]

**Proof.** For $\ell \in \mathbb{T}$ and $\eta \geq 0$, define
\[
\Phi_\ell(\eta) := \sup_{\mathcal{P}} \{ \theta(\mathcal{P}_0) - \theta(\mathcal{P}_1) : \mathcal{P}_0, \mathcal{P}_1 \in \mathcal{P}, \|E_{\mathcal{P}_0}[\ell] - E_{\mathcal{P}_1}[\ell]\| \leq \eta \},
\]
and note that since $d_{TV} \leq H$ and by (4.8), we have
\[
\omega_H^{Q(\alpha, \ell)}(\varepsilon) \leq \Phi(2\varepsilon^{\alpha} + 1).
\]

Clearly, the function $\Phi_{\ell}$ is non-decreasing. In order to minimize the upper bound in $\ell$ we consider the dual problem. For $\delta \geq 0$, define $\Psi_{\ell}(\delta) := \inf\{ \eta \geq 0 : \Phi_{\ell}(\eta) \geq \delta \}$. The functions $\Phi_{\ell}$ and $\Psi_{\ell}$ have the following properties.

\[\Psi_{\ell}(\delta) > \eta \Rightarrow \Phi_{\ell}(\eta) \leq \delta, \quad \sup_{\ell \in \mathbb{T}} \Psi_{\ell}(\delta) > \eta \Rightarrow \inf_{\ell \in \mathbb{T}} \Phi_{\ell}(\eta) \leq \delta,\]

(4.9) $\Psi_{\ell}(\delta) > \eta \Rightarrow \Phi_{\ell}(\eta) \leq \delta$, 

(4.10) $\Psi_{\ell}(\delta) \geq \inf\{ \{E_{\mathcal{P}_0}[\ell] - E_{\mathcal{P}_1}[\ell] : \theta(\mathcal{P}_0) - \theta(\mathcal{P}_1) \geq \delta \}\}$

The first two are obvious. To establish (4.10), set $A_{\ell}(\delta) := \{ \eta \geq 0 : \Phi_{\ell}(\eta) \geq \delta \}$ and $B_{\ell}(\delta) := \{ \|E_{\mathcal{P}_0}[\ell] - E_{\mathcal{P}_1}[\ell]\| : \theta(\mathcal{P}_0) - \theta(\mathcal{P}_1) \geq \delta \}$ and note that for $A_{\ell}(\delta) = \emptyset$ the claim is trivial. So take $\eta \in A_{\ell}(\delta)$. Then $\Phi_{\ell}(\eta) > \delta$, which implies that there are $\mathcal{P}_0, \mathcal{P}_1 \in \mathcal{P}$ with $\|E_{\mathcal{P}_0}[\ell] - E_{\mathcal{P}_1}[\ell]\| \leq \eta$ and $\theta(\mathcal{P}_0) - \theta(\mathcal{P}_1) \geq \delta$. Thus, $\nu := \|E_{\mathcal{P}_0}[\ell] - E_{\mathcal{P}_1}[\ell]\| \leq \eta$ and $\nu \in B_{\ell}(\delta)$. We have just shown that for every $\eta \in A_{\ell}(\delta)$ there exists a $\nu \in B_{\ell}(\delta)$ with $\nu \leq \eta$. But this clearly means that $\Psi_{\ell}(\delta) = \inf A_{\ell}(\delta) \geq \inf B_{\ell}(\delta)$, as required. Next, we can extend the linear functional $\theta : \mathcal{P} \to \mathbb{R}$ to signed measures of the form $\mathcal{P}_0 - \mathcal{P}_1, \mathcal{P}_0, \mathcal{P}_1 \in \mathcal{P}$ and it is still linear on this set. Thus, using convexity and dominatedness of $\mathcal{P}$, we see that $S_{\mathcal{P}} := \{ \theta(\mathcal{P}_0) - \theta(\mathcal{P}_1) : \mathcal{P}_0, \mathcal{P}_1 \in \mathcal{P} \}$ defines a dominated convex set of finite signed measures. Also note that $\{ \phi : \mathcal{P} \to \mathbb{R} : \|\phi\|_1 \leq 1 \} = \{ \theta \in L_\infty(\Omega, \mathcal{F}, \mu) : \|\phi\|_1 \leq 1 \}$. Hence, for all $\xi_1, \xi_2 > 0$, (4.10) together with Proposition 2.1 with $a = 1$, $b = 1$, and $\delta := \omega_1(\eta + \xi_1) + \xi_2 = \sup\{ \theta(\mathcal{P}_0 - \mathcal{P}_1) : d_{TV}(\mathcal{P}_0, \mathcal{P}_1) \leq \eta + \xi_1 + \xi_2 \}$, yields

\[
\sup_{\ell \in \mathbb{T}} \Psi_{\ell}(\delta) \geq \sup_{\ell \in \mathbb{T}} \inf_{\sigma \in \mathbb{S}_\mathcal{P}} \left\| \int_X \ell d\sigma \right\| \geq \sup_{\ell \in \mathbb{T}} \inf_{\sigma \in \mathbb{S}_\mathcal{P}} \int_X \ell d\sigma = \inf_{\ell \in \mathbb{T}} \inf_{\sigma \in \mathbb{S}_\mathcal{P}} \int_X \ell d\sigma \geq \inf_{\ell \in \mathcal{P}} \|\sigma\|_{TV} = \inf\{ d_{TV}(\mathcal{P}_0, \mathcal{P}_1) : \theta(\mathcal{P}_0 - \mathcal{P}_1) \geq \delta \}
\]

\[
\geq \inf\{ d_{TV}(\mathcal{P}_0, \mathcal{P}_1) : \theta(\mathcal{P}_0 - \mathcal{P}_1) \geq \omega_1(\eta + \xi_1) \} \geq \eta + \xi_1 + \xi_2.
\]

Now (4.9) implies $\inf_{\ell \in \mathbb{T}} \Phi_{\ell}(\eta) \leq \delta = \omega_1(\eta + \xi_1) + \xi_2$, for all $\xi_1, \xi_2 > 0$. Therefore, the proof is finished upon setting $\eta = 2\varepsilon^{\alpha} + 1$ and taking the limits $\xi_1, \xi_2 \to 0$. \qed

**Remark 4.7.** From the proof of Theorem 4.6, it is easy to see that the assumptions of convexity of $\mathcal{P}$ together with the linearity of $\theta$, can be replaced by the
following, more general, condition (see Proposition 2.1). For all $\delta > 0$, we have

$$\sup_{\ell \in T} \inf_{\sigma \in S_\delta} \int_X \ell \, d\sigma = \inf_{\sigma \in S_\delta} \sup_{\ell \in T} \int_X \ell \, d\sigma,$$

where $S_\delta = \{ P_0 - P_1 : \theta(P_0) - \theta(P_1) \geq \delta, P_0, P_1 \in \mathcal{P} \}$ and $T$ is as in the theorem.

The proof of the next corollary is deferred to Appendix A.4.

**Corollary 4.8.** Suppose that Assumptions A and B hold, that $\mathcal{P}$ is convex and dominated by a $\sigma$-finite measure $\mu$, that $\theta : \mathcal{P} \to \mathbb{R}$ is linear, and that there exist positive finite constants $\bar{\epsilon}_0 > 0, r \in (0, 2]$ and $\bar{A}_0 \leq \bar{A}_1$, such that the $L_1$-modulus $\omega_1$ satisfies

$$\bar{A}_0 \epsilon^r \leq \omega_1(\epsilon) \leq \bar{A}_1 \epsilon^r \quad \forall \epsilon \in (0, \bar{\epsilon}_0 \wedge 1].$$

Then, for every $\alpha > 0$, there exist constants $n_0 \in \mathbb{N}$ and $C_0 \in (0, \infty)$, so that for every $n \geq n_0$,

$$M_n(\alpha, \mathcal{P}) := \inf_{Q \in Q_\alpha} M_n(Q, \mathcal{P}) \leq C_0 \cdot l \circ \omega_1 \left( \sqrt{\frac{1}{n(e^\alpha - 1)^2}} \right),$$

where $Q_\alpha$ is the collection of $\alpha$-sequentially interactive channels as in (2.6). The constants $C_0$ and $n_0$ can be chosen so that $n_0 = (\bar{\epsilon}_0 \wedge 1) - 16 \log a \log(3/2)$, and

$$C_0 = \bar{C}(e^\alpha + 1)^{2r} \frac{\log a}{\log(3/2)},$$

for a constant $\bar{C} = \bar{C}(\bar{A}_0, \bar{A}_1, a, r, M, \bar{\epsilon}_0)$ that does not depend on $\alpha$.

Summarizing, under the conditions of Corollary 4.8 and invoking Corollary 3.4, we obtain the characterization (1.3) announced in the introduction, i.e., for any fixed $\alpha \in (0, \infty)$,

$$M_n(\alpha, \mathcal{P}) \asymp l \circ \omega_1 \left( n^{-1/2} \right).$$

More precisely, we even find that

$$\frac{1}{4} l \left( \frac{\bar{A}_0}{2^{r+1}} \omega_1 \left( \sqrt{\frac{1}{n(e^\alpha - 1)^2}} \right) \right) \leq M_n(\alpha, \mathcal{P}) \leq C_0 \cdot l \left( \omega_1 \left( \sqrt{\frac{1}{n(e^\alpha - 1)^2}} \right) \right),$$

for all $n \geq n_0 = n_0(\alpha)$. This also shows that in the private case the effective sample size reduces from $n$ to $n\alpha^2$, for $\alpha$ small.
4.2. Constructing a rate optimal channel. In this subsection, we complement the general attainability result of Subsection 4.1 by constructing minimax rate optimal channel distributions that satisfy (4.6). We now replace the assumptions that \( P \) is convex and dominated and that \( \theta \) is linear by an alternative high level condition (Condition C below), which, however, turns out to be easily verifiable in many practical examples. Our starting point for the construction of an optimal \( \alpha \)-private estimation procedure is a measurable function \( \ell_h : \mathcal{X} \to \mathbb{R} \), that depends on a possibly vector valued tuning parameter \( h \in \mathbb{R}^k \). The idea is now to use the channel \( Q^{(\alpha,\ell_h)} \) with identical marginals as in (4.5) to generate privatized observations \( Z_i \) from the non-private data \( \ell_h(X_i) \), so that \( E[|Z_i|] = \ell_h(X_i) \). We then simply take the sample mean \( \bar{Z}_n(h) = \frac{1}{n} \sum_{i=1}^n Z_i \) as our private estimator and determine the optimal value of \( h = h_n \) from a bias-variance trade-off.

It is remarkable, and maybe somewhat surprising, that a private estimation procedure as simple as the one described above, can be rate optimal in such a broad class of different estimation problems. In particular, the sample mean \( \bar{Z}_n = \frac{1}{n} \sum_{i=1}^n Z_i \) can never achieve a faster rate than \( n^{-1/2} \), which is not optimal in some direct (non-private) estimation problems such as the estimation of the endpoint of a uniform distribution (see Section 5.4). However, Lemma B.2 in Appendix B, in conjunction with the lower bound of Corollary 3.4, suggests that \( n^{-1/2} \) is the best possible rate of convergence in locally differentially private estimation problems.

In the case \( k = 1, h \in \mathbb{R} \), the mentioned regularity condition C below states that the collection of measurable functions \( \ell_h : \mathcal{X} \to \mathbb{R} \), \( h > 0 \), satisfies \( \| \ell_h \|_{\infty} \leq h^{-s} \), for some \( s \geq 0 \), and is such that the worst case absolute bias

\[
B_{P,\theta}(\ell_h) := \sup_{P \in \mathcal{P}} |E_P[\ell_h] - \theta(P)|
\]

of the estimator \( \bar{Z}_n(h) \) in the private problem (which coincides with the worst case absolute bias of the estimator \( \frac{1}{n} \sum_{i=1}^n \ell_h(X_i) \) in the non-private problem) is bounded by an expression of the order \( h^t \), as \( h \to 0 \), for some \( t > 0 \). We show that for the choice of tuning parameter

\[
h = h_n = \left( \frac{1}{n} \frac{e^{\alpha} + 1}{e^{\alpha} - 1} \right)^{\frac{1}{s+t}},
\]

the above privatization and estimation protocol is \( \alpha \)-private minimax rate optimal if \( e^r \gtrsim \omega_1(e) \) for \( r = t/(s + t) \). This consideration misleadingly suggests that the estimator \( \frac{1}{n} \sum_{i=1}^n \ell_h(X_i) \) is minimax optimal in the non-private case for a specific choice of \( h = h_n \). Although this appears to be correct in Examples 5.1, 5.2 and 5.3, it is not true in general (see Example 5.4 where the minimax rate optimal estimator in the direct problem is not even of linear form).

For some estimation problems, such as estimating a multivariate anisotropic density at a point (cf. Section 5.3), the case \( k = 1 \) is not sufficient and we need the full flexibility of Condition C.
C) Suppose that $\mathcal{P}$ and $\theta$ are such that there exists $k \in \mathbb{N}$, $t \in (0, \infty)^k$, $s \in [0, \infty)^k$, $C_0 \in (0, \infty)$ and $h_0 \in (0, 1]$ and a class of measurable functions $\ell_h : \mathcal{X} \to \mathbb{R}$ indexed by $h \in \mathbb{R}^k$, such that for all $h \in (0, h_0]^k$,

$$
(4.12) \quad \|\ell_h\|_\infty \leq C_0 \prod_{j=1}^k h_j^{s_j}, \quad \text{and} \quad B_{\mathcal{P}, \theta}(\ell_h) \leq C_0 \frac{1}{k} \sum_{j=1}^k h_j^{t_j}.
$$

**Remark 4.9.** Note that Condition C implies Condition A, because without A $B_{\mathcal{P}, \theta}(\ell)$ is infinite whenever $\ell$ is bounded.

The proof of the following theorem is deferred to Section A.5.

**Theorem 4.10.** Suppose that Conditions B and C hold and set $\bar{r} = \sum_{j=1}^k t_j$. For $\alpha \in (0, \infty)$, let $Q^{(\alpha, \ell)}$ be the $\alpha$-private channel with identical marginals (4.5) and set $h_n = (h_{n,1}, \ldots, h_{n,k})$ and

$$
h_{n,j} = \left( \frac{1}{\sqrt{n}} e^{\alpha} - 1 \right)^{\frac{1}{1+\bar{r}}}.
$$

Then the sample mean $Z_n = \frac{1}{n} \sum_{i=1}^n Z_{i,n}$ based on the privatized observations $Z = (Z_{1,n}, \ldots, Z_{n,n})'$ generated from $Q^{(\alpha, h_n)}$ satisfies

$$
(4.13) \quad \sup_{\mathcal{P} \in \mathcal{P}} \mathbb{E}_{Q^{(\alpha, h_n)}} [l(\|\bar{Z}_n - \theta(\mathcal{P})\|)] \leq C_0 \cdot l \left( \left( \frac{e^{\alpha} + 1}{\sqrt{n}(e^{\alpha} - 1)} \right)^{\frac{1}{\bar{r} + 1}} \right),
$$

for all $n \in \mathbb{N}$ and a positive finite constant $C_0$ that depends only on $a$ and $C_0$.

The private estimator $\bar{Z}_n$ of Theorem 4.10 is $\alpha$-private minimax rate optimal if the derived upper bound (4.13) on the worst case risk is of the same order as the lower bound of Corollary 3.4. The latter is true if $A_0 \varepsilon^{\frac{1}{1+\bar{r}}} \leq \omega_1(\varepsilon)$, for all small $\varepsilon > 0$.

4.2.1. **Further comments.** In the rest of this section, we provide some intuition for the role of Condition C in Theorem 4.10 and explore some of this conditions further consequences. In particular, we show that a collection $\ell_{h}$ as in Condition C satisfies (4.6) with optimal choice of tuning parameter $h = h_n$ as in Theorem 4.10 and $\ell_n = \ell_{h_n}$, provided that $A_0 \varepsilon^{\frac{1}{1+\bar{r}}} \leq \omega_1(\varepsilon)$.

In order to upper bound the modulus

$$
\omega^Q_{H_{1\to 1}}(\varepsilon) = \sup \left\{ \theta(\mathcal{P}_0) - \theta(\mathcal{P}_1) : H \left( Q_0^{(\alpha, \ell)}, Q_1^{(\alpha, \ell)} \right) \leq \varepsilon, \mathcal{P}_0, \mathcal{P}_1 \in \mathcal{P} \right\},
$$

we simply observe that $|\mathbb{E}_{\mathcal{P}_0}[\ell] - \mathbb{E}_{\mathcal{P}_1}[\ell]| \geq |\theta(\mathcal{P}_0) - \theta(\mathcal{P}_1)| - 2B_{\mathcal{P}, \theta}(\ell)$. If we now use (4.8) and $d_{TV} \leq H$ to arrive at $H \left( Q_0^{(\alpha, \ell)}, Q_1^{(\alpha, \ell)} \right) 2\varepsilon_0 \geq |\theta(\mathcal{P}_0) - \theta(\mathcal{P}_1)|$.
2B_{P, \theta}(\ell), then, from the definition of \( z_0 \) right after (4.5), we can conclude that

\[
\omega_H^{(Q, \alpha, h^*)}(\varepsilon) \leq 2\varepsilon\|\ell_h\|_\infty \frac{e^\alpha + 1}{e^\alpha - 1} + 2B_{P, \theta}(\ell_h).
\]

Setting \( \varepsilon = n^{-1/2} \), we see that minimizing the upper bound in (4.14) constitutes a bias-variance trade-off of the private estimator \( \tilde{Z}_n(h) \), because

\[
\text{Var}_{Q^{(\alpha, h)}}[\tilde{Z}_n(h)] = \frac{1}{n} \text{Var}_{Q^{(\alpha, h)}}[Z_1] \leq \frac{1}{n} \mathbb{E}_{Q^{(\alpha, h)}}[Z_1^2] = \frac{\omega^2}{n}
\]

\[
= n^{-1}\|\ell_h\|_\infty^2 \frac{(e^\alpha + 1)^2}{(e^\alpha - 1)^2}.
\]

Now, under Condition C, the bias-variance trade-off on the right-hand-side of (4.14) can be optimized. The following theorem summarizes these considerations and also shows that Condition C implies the upper bound \( \omega_1(\varepsilon) \leq \hat{A}_1 \varepsilon^{1+r} \) corresponding to the hypothesized lower bound \( A_0 \varepsilon^{1+r} \leq \omega_1(\varepsilon) \).

**Proposition 4.11.** Fix \( \alpha \in (0, \infty) \) and suppose that Condition C is satisfied. Then, for \( h^* = h^*(\varepsilon) = (h_1^*, \ldots, h_k^*)^\top \), \( h_j^* := \left(\frac{e^\alpha + 1}{e^\alpha - 1}\varepsilon\right)^{\frac{1}{1+r(j)}} \) and \( \bar{r} = \sum_{j=1}^k \frac{z_j}{n} \), we have

\[
\omega_H^{(Q, \alpha, h^*)}(\varepsilon) \leq 4\hat{C}_0 \left(\frac{e^\alpha + 1}{e^\alpha - 1}\varepsilon\right)^{\frac{1}{1+r}},
\]

provided that \( 0 < \varepsilon \leq \frac{e^\alpha - 1}{e^\alpha + 1} \bar{r}^{-1}[1+\bar{r}(\max_j t_j)] \), and

\[
\omega_1(\varepsilon) \leq 4\hat{C}_0 \varepsilon^{\frac{1}{1+r}} \quad \forall \varepsilon \in \left(0, \frac{1+\bar{r}(\max_j t_j)}{\bar{r}}\right].
\]

If, in addition to Assumption C, \( B_{P, \theta}(\ell_h) = 0 \) for all \( h \in (0, \bar{r})^k \), then, for all \( \varepsilon > 0 \) and for \( h_0 = (\bar{r}_0, \ldots, \bar{r}_0)^\top \in \mathbb{R}^k \), then

\[
\omega_H^{(Q, \alpha, h_0)}(\varepsilon) \leq 2\varepsilon \frac{e^\alpha + 1}{e^\alpha - 1} \hat{C}_0 \bar{r}^{-\sum_{j=1}^k s_j} \varepsilon, \quad \text{and} \quad \omega_1(\varepsilon) \leq 2\hat{C}_0 \bar{r}^{-\sum_{j=1}^k s_j} \varepsilon.
\]

**Proof.** The bounds on the privatized Hellinger modulus follow immediately from (4.14). For the \( L_1 \)-modulus, note that for \( \varepsilon \) as in the proposition and for \( d_{TV}(P_0, P_1) \leq \varepsilon \),

\[
|\theta(P_0) - \theta(P_1)| \leq |\mathbb{E}_{P_0}[\ell_{h^*}] - \mathbb{E}_{P_1}[\ell_{h^*}]| + |\theta(P_0) - \mathbb{E}_{P_0}[\ell_{h^*}]| + |\theta(P_1) - \mathbb{E}_{P_1}[\ell_{h^*}]| \leq \int_X |\ell_{h^*} - d(P_0 - P_1)| + 2B(\ell_{h^*}) \leq 2d_{TV}(P_0, P_1)\hat{C}_0 \prod_{j=1}^k (h_j^*)^{-s_j} + 2\hat{C}_0 \frac{1}{k} \sum_{j=1}^k (h_j^*)^{t_j} \leq 4\hat{C}_0 \varepsilon^{\frac{1}{1+r}}.
\]
5. Examples. In this section, we discuss several concrete estimation problems for which we derive lower bounds on the total variation modulus

\[ \omega_1(\varepsilon) = \sup \{|\theta(P_1) - \theta(P_0)| : d_{TV}(P_1, P_0) \leq \varepsilon, P_j \in \mathcal{P}\} \]

and exhibit families of functions \( \ell_h \) which, in conjunction with the binary construction in (4.5) and an appropriate choice of tuning parameters, lead to minimax rate optimal channel distributions. The previously derived private minimax rates of convergence

\[ l \circ \omega_1 \left( n^{-1/2} \right) \]

are then compared to their non-private counterparts

\[ l \circ \omega_H \left( n^{-1/2} \right) \]

to evaluate the cost of privatization in each example (see Table 1 below). Even in cases where the moduli of continuity are hard to evaluate explicitly, the following relationship is always true,

\[ \omega_H(\varepsilon) \leq \omega_1(\varepsilon) \leq \omega_H(\sqrt{2\varepsilon}) \quad \forall \varepsilon > 0, \]

because \( d_{TV} \leq H \leq \sqrt{2d_{TV}} \) (cf. for instance Tsybakov, 2009, Lemma 2.3). This shows that in the worst case, the private minimax rate of estimation is the square root of the non-private minimax rate, whereas the private rate can never be better than the non-private one. Both extremal cases can occur, see examples below.

To exclude trivialities, throughout this section we assume that \( \theta \) is not constant on \( \mathcal{P} \). The proofs of all claims made in this section are deferred to Appendix C. Our list of examples is far from being exhaustive, but due to space constraints we present only a few cases for which the non-private rate is well known.

5.1. Estimating moment functionals. Let \( \mathcal{X} \subseteq \mathbb{R} \) and consider estimation of an integral functional \( \theta(P) = \mathbb{E}_P[f] \) for some measurable \( f : \mathcal{X} \to \mathbb{R} \), such that either (a) \( \text{Im}(|f|) \supseteq (0, \infty) \), or (b) \( \|f\|_\infty < \infty \). For instance, \( f(x) = x^m \), for moment estimation, or \( f(x) = e^{sx} \), for estimation of the moment generating function at the point \( s \in \mathbb{R} \). For \( \kappa \in (1, \infty) \) and \( C > 0 \), consider the class \( \mathcal{P} = \mathcal{P}_\kappa(C) \) of all probability measures \( \mathbb{P} \) on \( \mathcal{B}(\mathcal{X}) \) such that \( \mathbb{E}_\mathbb{P}[|f|^\kappa] \leq C \). Clearly, the parameter space \( \mathcal{P}_\kappa(C) \) is convex and \( \theta \) is bounded on \( \mathcal{P}_\kappa(C) \), because \( \sup_{\mathbb{P} \in \mathcal{P}_\kappa(C)} |\theta(\mathbb{P})| \leq \sup_{\mathbb{P} \in \mathcal{P}_\kappa(C)} \mathbb{E}_\mathbb{P}[|f|^\kappa]^\frac{1}{\kappa} \leq C\frac{1}{\kappa} < \infty \).

In case (a), the total variation modulus \( \omega_1 \) of \( \theta \) over \( \mathcal{P}_\kappa(C) \) satisfies, for all \( \varepsilon \in (0, 1) \),

\[ \omega_1(\varepsilon) \geq (C/2)^{\frac{1}{\kappa}} \varepsilon^{-\frac{1}{\kappa-1}}. \]
Furthermore, $\ell_h(x) := f(x)\mathbb{1}_{|f(x)| \leq h}$ satisfies Condition C with $k = 1$, $s_1 = 1$, $t_1 = \kappa - 1 > 0$, $C_0 = C \lor 1$ and $\bar{h}_0 = 1$. Thus, for sufficiently regular loss functions $l$ as in Assumption B and for $h_n = \left(\frac{e^\alpha + 1}{\sqrt{n}(e^\alpha - 1)}\right)^{\frac{\alpha}{2}}$, combining Theorem 3.1, Theorem 3.3 (with $\eta = 1/2$) and Theorem 4.10, we get

$$\frac{1}{4} \cdot l\left(\frac{(C/2)^{\frac{\alpha}{2}}}{4} \left[\frac{1}{\sqrt{n}(e^\alpha - 1)}\right]^{\frac{\alpha - 1}{2}}\right) \leq \inf_{Q \in \mathcal{Q}_\alpha} \mathcal{M}_n(Q, \mathcal{P}_\kappa(C))$$

$$\leq \mathcal{M}_n\left(Q^{(\alpha, h_0)}, \mathcal{P}_\kappa(C)\right) \leq C_0 \cdot l\left(\left[\frac{e^\alpha + 1}{\sqrt{n}(e^\alpha - 1)}\right]^{\frac{\alpha - 1}{2}}\right),$$

for all $n \in \mathbb{N}$ with $4n(e^\alpha - 1)^2 > 1$, $C_0 = C_0(a, C)$ as in Theorem 4.10, and where $Q^{(\alpha, l)}$ is the channel with identical marginals (4.5) and $\mathcal{Q}_\alpha$ is the set of all sequentially interactive $\alpha$-private channels. In particular, the sample mean $\bar{Z}_n = \frac{1}{n} \sum_{i=1}^{n} Z_i$, based on the privatized observations $Z_i$ from $Q^{(\alpha, h_0)} \otimes^n$, is rate optimal. Note that this private minimax rate of convergence was already discovered by Duchi et al. (2013b) in the case $f$ equals identity but with a rate optimal channel sequence different to ours.

In case (b), there exist positive finite constants $\bar{A}_0$ and $\bar{e}_0$, so that the total variation modulus $\omega_1$ of $\theta$ over $\mathcal{P}_\kappa(C)$ satisfies

$$\omega_1(\varepsilon) \geq \bar{A}_0 \varepsilon, \quad \forall \varepsilon \in [0, \bar{e}_0].$$

Furthermore, with the definition of $\ell_h$ as in case (a), Condition C holds with $k = 1$, $s_1 = 0$, $t_1 = \kappa - 1$, $C_0 = C \lor \|f\|_\infty$ and $\bar{h}_0 = 1$. Thus, for sufficiently regular loss functions $l$ as in Assumption B and for

$$h_n = \left(\frac{e^\alpha + 1}{\sqrt{n}(e^\alpha - 1)}\right)^{\frac{1}{\alpha}},$$

combining Theorem 3.1, Theorem 3.3 and Theorem 4.10, we get a positive finite constant $C_1 = C_1(a, C, \|f\|_\infty)$, such that for every $n \in \mathbb{N}$ with $4n(e^\alpha - 1)^2 \geq \bar{e}_0^{-2}$,

$$\frac{1}{4} \cdot l\left(\frac{A_0}{4} \frac{1}{\sqrt{n}(e^\alpha - 1)}\right) \leq \inf_{Q \in \mathcal{Q}_\alpha} \mathcal{M}_n(Q, \mathcal{P}_\kappa(C))$$

$$\leq \mathcal{M}_n\left(Q^{(\alpha, h_0)}, \mathcal{P}_\kappa(C)\right) \leq C_1 \cdot l\left(\frac{e^\alpha + 1}{\sqrt{n}(e^\alpha - 1)}\right).$$

Again, the sample mean $\bar{Z}_n = \frac{1}{n} \sum_{i=1}^{n} Z_i$, based on the privatized observations $Z_i$ from $Q^{(\alpha, h_0)} \otimes^n$, is rate optimal.

5.2. **Estimating the derivative of the density at a point.** Let $X = \mathbb{R}$ and, for $\beta, C > 0$, consider the Hölder class $\mathcal{H}_{\beta, C}^\infty = \mathcal{H}_{\beta, C}^\infty(\mathbb{R})$ of all Lebesgue densities $p$.
on $\mathbb{R}$ that are $b := |\beta|$ times differentiable and whose $b$-th derivative $p^{(b)}$ satisfies

$$|p^{(b)}(x) - p^{(b)}(y)| \leq C|x - y|^{|\beta - b|}, \quad \forall x, y \in \mathbb{R}.$$ 

We consider estimation of the $m$-th derivative of the density at a point $x_0 \in \mathbb{R}$, i.e., for $p \in \mathcal{H}^{<\lambda}_{\beta,C}$, we consider the linear functional $\theta(p) = p^{(m)}(x_0)$, where $0 \leq m < \beta$. This functional is uniformly bounded on $\mathcal{H}^{<\lambda}_{\beta,C}$ (see, e.g., Tsybakov, 2009, Equation (1.9)). Clearly, $\mathcal{H}^{<\lambda}_{\beta,C}$ is convex.

There exist positive finite constants $\tilde{A}_0$ and $\tilde{\varepsilon}_0$, depending only on $C$, $\beta$ and $m$, so that the total variation modulus $\omega_1$ of $\theta$ over $\mathcal{H}^{<\lambda}_{\beta,C}$ satisfies

$$\omega_1(\varepsilon) \geq \tilde{A}_0 \varepsilon^\frac{\beta - m}{\beta + 1}, \quad \forall \varepsilon \in [0, \tilde{\varepsilon}_0].$$

Let $K : [-1, 1] \to \mathbb{R}$ be a kernel of order $b - m$ that is $m$-times continuously differentiable and satisfies $K^{(j)}(1) = K^{(j)}(-1) = 0$, for $j = 0, 1, \ldots, m - 1$, $C_1 := \int_{-1}^{1} |u|^{\beta - m} K(u) \, du < \infty$ and $\int_{-1}^{1} K(x) \, dx = (-1)^m$ (for a construction see Hansen, 2005; Müller, 1984). Then $\ell_h = \kappa_h^{(m)} = \mathbf{1}_{[x_0 - h, x_0 + h]}$, where

$$\kappa_h(x) = \frac{1}{h} K \left( \frac{x - x_0}{h} \right),$$

satisfies Condition C with $k = 1$, $s_1 = m + 1$, $t_1 = \beta - m$, $\bar{C}_0 = \|K^{(m)}\|_\infty \vee \frac{C_1}{\beta - m}^{1/n}$ and $\bar{h}_0 = 1$. Thus, for sufficiently regular loss functions $l$ as in Assumption B and for $h_n = \left( \frac{e^n + 1}{\sqrt{n} (e^n - 1)} \right)^{\frac{1}{\beta + 1}}$, combining Theorem 3.1, Theorem 3.3 and Theorem 4.10, we get

$$\frac{1}{4} \cdot l \left( \frac{\tilde{A}_0}{4} \left[ \frac{1}{\sqrt{n} (e^n - 1)} \right]^\frac{\beta - m}{\beta + 1} \right) \leq \inf_{Q \in \mathcal{Q}_n} \mathcal{M}_n(Q, \mathcal{H}^{<\lambda}_{\beta,C}(\mathbb{R})) \leq \mathcal{M}_n \left( Q^{(\alpha,\ell_n)}; \mathcal{H}^{<\lambda}_{\beta,C}(\mathbb{R}) \right) \leq C_0 \cdot l \left( \left[ \frac{e^n + 1}{\sqrt{n} (e^n - 1)} \right]^\frac{\beta - m}{\beta + 1} \right),$$

for all $n \in \mathbb{N}$ with $4n(e^n - 1)^2 \geq \varepsilon_0^{-2}$, and where $C_0$ is as in Theorem 4.10 and $Q^{(\alpha,\ell)}$ is the channel with identical marginals (4.5). The sample mean $\bar{Z}_n = \frac{1}{n} \sum_{i=1}^{n} Z_i$, based on the privatized observations $Z_i$ from $Q^{(\alpha,\ell_n)} \mathbb{P}^\otimes n$, is rate optimal.

5.3. Multivariate density estimation at a point. In this example, for $\beta \in (0, 1]^d$ and $C \in (0, \infty]^d$, we consider the anisotropic Hölder-class $\mathcal{H}_{\beta,C}(\mathbb{R}^d)$ of Lebesgue densities $p$ on $\mathbb{R}^d$, such that for every $j \in \{1, \ldots, d\}$ and every $x, \bar{x} \in \mathbb{R}^d$,

$$|p(x_1, \ldots, x_{j-1}, \bar{x}_j, x_{j+1}, \ldots, x_d) - p(x)| \leq C_j |\bar{x}_j - x_j|^{\beta_j}.$$
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For $x^{(0)} \in \mathbb{R}^d$, the functional of interest is $\theta(p) = p(x^{(0)})$, which is linear. Clearly, the anisotropic Hölder-class is convex.

There exist positive finite constants $\tilde{A}_0$ and $\tilde{\varepsilon}_0$, so that for $\bar{r} = \sum_{j=1}^d \frac{1}{\tilde{h}_j}$,

$$\omega_1(\varepsilon) \geq \tilde{A}_0 \varepsilon^{\bar{r} - \frac{1}{\varepsilon}} \quad \forall \varepsilon \in (0, \tilde{\varepsilon}_0].$$

Let $K : \mathbb{R} \to \mathbb{R}$ be a bounded kernel that satisfies

$$\int_{\mathbb{R}} K(u) \, du = 1, \quad \tilde{c}_j := \int_{\mathbb{R}} |K(u)||u|^{\tilde{\beta}_j} \, du < \infty, \quad \forall j = 1, \ldots, d.$$

Then for $h \in (0, \infty)^d$ and $x \in \mathbb{R}^d$, the function

$$l_h(x) = \prod_{j=1}^d \frac{1}{\bar{h}_j} K \left( \frac{x_j - x^{(0)}_j}{\bar{h}_j} \right)$$

satisfies Condition C with $k = d$, $s_j = 1$, $\bar{t}_j = \tilde{\beta}_j$, $\tilde{C}_0 = \|K\|_{\infty} \vee \max_j (dC_j \tilde{c}_j)$ and $\bar{h}_0 = 1$. Thus, for sufficiently regular loss functions $l$ as in Assumption B and for

$$h_{n,j} = \left( \frac{e^{\alpha} + 1}{\sqrt{n}(e^{\alpha} - 1)} \right)^{\tilde{s}_j(1+\tilde{\beta}_j)},$$

combining Theorem 3.1, Theorem 3.3 and Theorem 4.10, we get

$$\frac{1}{4} \cdot l \left( \frac{\tilde{A}_0}{4} \left[ \frac{1}{\sqrt{n}(e^{\alpha} - 1)} \right]^{\bar{r} - \frac{1}{\varepsilon}} \right) \leq \inf_{Q \in \mathcal{Q}_n} \mathcal{M}_n(Q, \mathcal{H}^{\leq \lambda}_{\tilde{\beta},C}(\mathbb{R}^d)) \leq \mathcal{M}_n \left( Q^{(\alpha,\tilde{t}_n)}, \mathcal{H}^{\leq \lambda}_{\tilde{\beta},C}(\mathbb{R}^d) \right) \leq C_0 \cdot l \left( \left[ \frac{e^{\alpha} + 1}{\sqrt{n}(e^{\alpha} - 1)} \right]^{\bar{r} - \frac{1}{\varepsilon}} \right),$$

for all $n \in \mathbb{N}$ with $4n(e^{\alpha} - 1)^2 \geq \tilde{\varepsilon}_0^{-2}$, and where $C_0$ is as in Theorem 4.10 and $Q^{(\alpha,\tilde{t})}$ is the channel with identical marginals (4.5). The sample mean $\bar{Z}_n = \frac{1}{n} \sum_{i=1}^n Z_i$, based on the privatized observations $Z_i$ from $Q^{(\alpha,\tilde{t}_n)} \otimes_{\delta_n}$, is rate optimal.

5.4. Estimating the endpoint of a uniform distribution. Fix $M \geq 1$ and consider the class of distributions $\mathcal{P} = \mathcal{P}_M = \{P_\theta : \theta \in [0,M]\}$, where $P_\theta$ denotes the uniform distribution on $[0, \theta]$. We may take $X = [0,M]$. Clearly, the functional $\theta(P_\theta) = 2 \int_{[0,\theta]} x \, dP_\theta(x) = \theta$ is linear, but $\mathcal{P}$ is not convex. Nevertheless, if we take $\ell_h(x) = 2x$, then $Q_1^{(\alpha,\tilde{t}_n)} \otimes_{\delta}$ has support $\{ -z_0, z_0 \}$, with $z_0 = 2M \frac{e^{\alpha} + 1}{e^{\alpha} - 1}$, and

$$p_\theta := Q_1^{(\alpha,\tilde{t}_n)} \otimes_{\delta} (\{ z_0 \}) = \frac{1}{2} \left( 1 + \int_{[0,M]} x \frac{e^{\alpha} + 1}{M e^{\alpha} - 1} \, dP_\theta(x) \right) = \frac{1}{2} \left( 1 + \frac{1}{2} \frac{\theta}{M e^{\alpha} - 1} \right).$$
In other words, the set $Q_1^{(\alpha,\ell_0)} P_{\le t_1} \cap \mathbb{P}_{\le t}$ is the collection of all binary distributions supported on $\{-z_0, z_0\}$ under which $\{z_0\}$ has probability $p_0$, for some $\vartheta \in [0, t]$, and is thus convex. By the same argument, also $Q_1^{(\alpha,\ell_0)} P_{\ge s_1}$ is convex, and therefore (4.2) holds true.

It is also clear that Condition C holds with $k = 1$, $s_1 = 0$, $\bar{C}_0 = 2M$, and any $t_1 > 0$. Moreover, since for $\varepsilon \in (0, 1)$, $d_{TV}(P_{\omega_0}, P_{\omega_1}) = \varepsilon$, we have $\omega_1(\varepsilon) \ge \varepsilon/2$. Thus, for sufficiently regular loss functions $l$ as in Assumption B and for $h_n = \left( e^{\alpha + 1} \sqrt{n(e^\alpha - 1)} \right)$, combining Theorem 3.1, Theorem 3.3 and Theorem 4.10, we get

$$\frac{1}{4} \cdot l \left( \frac{1}{8} \sqrt{n(e^\alpha - 1)} \right) \le \inf_{Q \in \mathcal{Q}_n} M_n(Q, \mathcal{P}_M) \le M_n \left( Q^{(\alpha,\ell_0)}, \mathcal{P}_M \right) \le C_0 \cdot l \left( \frac{e^{\alpha + 1}}{\sqrt{n(e^\alpha - 1)}} \right),$$

for all $n \in \mathbb{N}$ with $4n(e^\alpha - 1)^2 > 1$, and where $C_0$ is as in Theorem 4.10. This rate should be compared with the well known rate of $n^{-1}$ from the case of direct observations. Even though $\mathcal{P}_M$ is not convex, the rate of direct estimation, in this case, is characterized by the Hellinger modulus, as one easily shows that $\varepsilon^2 (1 - \varepsilon^2/4) \le \omega_H(\varepsilon) \le M \varepsilon^2$, using that $H^2(P_{\omega_0}, P_{\omega_1}) = 2 \left( 1 - \frac{\vartheta_0 \wedge \vartheta_1}{\sqrt{n(\vartheta_0 \vartheta_1)}} \right)$.

| $\mathcal{P}$ | $\theta : \mathcal{P} \to \mathbb{R}$ | $\omega_H(\varepsilon)$ | $\omega_1(\varepsilon)$ |
|----------------|-------------------|-----------------|-----------------|
| $\mathbb{P} : \mathbb{E}_{|f|^n} \le C$ | $\mathbb{P} \mapsto \mathbb{E}_f[f]$ | $\text{Im}(|f|) \ge (0, \infty)$ | $\varepsilon(2 \vartheta - 1)^\lambda \wedge 1$ | $\varepsilon^{\frac{\vartheta - 1}{\vartheta}}$ |
| $C > 0, \kappa > 1$ | $||f||_\infty < \infty$ | $\varepsilon$ | $\varepsilon$ |
| $\mathcal{H}_{\beta,\mathcal{C}}(\mathbb{R})$ | $\mathbb{P} \mapsto p^{(m)}(x_0)$ | $\vartheta \in (0, \mathcal{C})$ | $\varepsilon^{\frac{\vartheta - m}{\vartheta}}$ | $\varepsilon^{\frac{\vartheta - m}{\vartheta}}$ |
| $C > 0, \beta > 0$ | $\bar{r} = \sum_{j=1}^d \frac{1}{\beta_j}$ | $\varepsilon^{\frac{1}{1+\beta}}$ | $\varepsilon^{\frac{1}{1+\beta}}$ |
| $\mathcal{H}_{\beta,\mathcal{C}}(\mathbb{R}^d)$ | $\mathbb{P} \mapsto p(x_0)$ | $\bar{r} = \sum_{j=1}^d \frac{1}{\beta_j}$ | $\varepsilon^{\frac{1}{1+\beta}}$ | $\varepsilon^{\frac{1}{1+\beta}}$ |
| $C \in \mathbb{R}_+$, $\beta \in [0, 1]^d$ | $\bar{r} = \sum_{j=1}^d \frac{1}{\beta_j}$ | $\varepsilon^{\frac{1}{1+\beta}}$ | $\varepsilon^{\frac{1}{1+\beta}}$ |

Unif$(0, \theta) : \theta \in [0, M]$ | $\mathbb{P} \mapsto \theta$ | $\varepsilon^2$ | $\varepsilon$

Table 1

Comparison of Hellinger (non-private) and total variation (private) moduli of continuity for several estimation problems. The minimax rate of convergence (for fixed $\alpha$) in each problem is given by $l \circ \omega(n^{-1/2})$, where $l$ is the loss function.
A.1. Proof of Theorem 3.1. If $\Delta_A^{(n)}(Q, \eta) = 0$, then the bound holds because $l \left( |\hat{\theta}_n(Z) - \theta(P)| \right) \geq l(0) \frac{\eta}{2} = l \left( \frac{1}{2} \Delta_A^{(n)}(Q, \eta) \right) \frac{\eta}{2}$, in view of the monotonicity of $l$. Now, for arbitrary $\Delta \in [0, \infty)$, define the sets $S := \{ z \in \mathbb{Z}^n : |\hat{\theta}_n(z) - \theta(P)| \geq \Delta \}$, $S_1 := \{ z \in \mathbb{Z}^n : \hat{\theta}_n(z) \geq t + \Delta, \theta(P) \leq t \}$ and $S_2 := \{ z \in \mathbb{Z}^n : \hat{\theta}_n(z) < t + \Delta, \theta(P) \geq t + 2\Delta \}$, which obey the inclusions $S_j \subseteq S$, for $j = 1, 2$. Therefore, we obtain the lower bound

$$\sup_{P \in \mathcal{P}} \mathbb{P}^{\otimes n}(S) \geq \max \left\{ \sup_{P \in \mathcal{P}_{\leq t}} \mathbb{P}^{\otimes n} \left( \hat{\theta}_n \geq t + \Delta \right), \sup_{P \in \mathcal{P}_{t+2\Delta}} \mathbb{P}^{\otimes n} \left( \hat{\theta}_n < t + \Delta \right) \right\}$$

$$\geq \frac{1}{2} \sup_{P_0 \in \mathcal{P}_{\leq t}, P_1 \in \mathcal{P}_{t+2\Delta}} \mathbb{P}^{\otimes n} \left( \hat{\theta}_n \geq t + \Delta \right) \mathbb{P}^{\otimes n} \left( \hat{\theta}_n < t + \Delta \right)$$

$$\geq \frac{1}{2} \inf_{\phi} \sup_{P_0 \in \mathcal{P}_{\leq t}, P_1 \in \mathcal{P}_{t+2\Delta}} \mathbb{E}_{\mathbb{P}_0^{\otimes n}}[\phi] + \mathbb{E}_{\mathbb{P}_1^{\otimes n}}[1 - \phi]$$

$$= \frac{1}{2} \inf_{\phi} \sup_{P_0 \in \mathcal{P}_{\leq t}, P_1 \in \mathcal{P}_{t+2\Delta}} \mathbb{E}_{\mathbb{P}_0}[\phi] + \mathbb{E}_{\mathbb{P}_1}[1 - \phi],$$

which holds for any $t \in \mathbb{R}$. Since $\mathbb{P}^{(n)}$ is dominated by assumption, we can use (2.12) to obtain

(A.1) \[ \sup_{P \in \mathcal{P}} \mathbb{P}^{\otimes n} \left( |\hat{\theta}_n - \theta(P)| \geq \Delta \right) \geq \frac{1}{2} \eta_A^{(n)}(Q, 2\Delta). \]

If $\Delta_A^{(n)}(Q, \eta) \in (0, \infty)$, then take $\varepsilon > 0$ such that $\Delta_0 := \frac{1}{2} |\Delta_A^{(n)}(Q, \eta) - \varepsilon| > 0$. Since $\Delta \mapsto \eta_A^{(n)}(Q, \Delta)$ is non-increasing, the set $D := \{ \Delta \geq 0 : \eta_A^{(n)}(Q, \Delta) > \eta \}$ is of interval form $D = [0, \Delta_A^{(n)}(Q, \eta)]$ and thus $2\Delta_0 \in D$, so that $\eta_A^{(n)}(Q, 2\Delta_0) > \eta$. Thus, from (A.1), we obtain

$$\sup_{P \in \mathcal{P}} \mathbb{P}^{\otimes n} \left( |\hat{\theta}_n - \theta(P)| \geq \frac{1}{2} |\Delta_A^{(n)}(Q, \eta) - \varepsilon| \right) \geq \frac{\eta}{2}.$$

If $l \left( \frac{1}{2} \Delta_A^{(n)}(Q, \eta) \right) = 0$, then the claimed lower bound is trivial. Otherwise, the result follows from Markov’s inequality, since $\varepsilon > 0$ was arbitrarily small.

If $\Delta_A^{(n)}(Q, \eta) = \infty$, then $\eta_A^{(n)}(Q, \Delta) > \eta > 0$ for all $\Delta \geq 0$, and the inequality (A.1) together with Markov’s inequality, yields

$$\mathcal{M}_n(Q, \mathcal{P}) \geq \inf_{\hat{\theta}_n} \sup_{P \in \mathcal{P}} \mathbb{P}^{\otimes n} \left( |\hat{\theta}_n - \theta(P)| \geq \Delta \right) l(\Delta) \geq \frac{\eta}{2} l(\Delta).$$

Since $\Delta \geq 0$ was arbitrary, the claim follows. \qed
A.2. Proof of Theorem 3.3. Using (3.5), we obtain
\[ d_{TV}(Q^{P_{1}^{\otimes n}}, Q^{P_{0}^{\otimes n}}) \leq \sqrt{2n(e^{n} - 1)d_{TV}(P_{1}, P_{0})}, \]
for all \( \alpha \)-sequentially interactive channels \( Q \) and every \( P_{1}, P_{0} \in \mathcal{P} \). On the other hand, we have the well known identity
\[ \pi(Q^{P_{1}^{\otimes n}}, Q^{P_{0}^{\otimes n}}) = 1 - \sup_{\text{tests } \phi} \left( \mathbb{E}_{Q^{P_{0}^{\otimes n}}} [\phi] - \mathbb{E}_{Q^{P_{1}^{\otimes n}}} [\phi] \right) \]
so that
\[ d_{TV}(P_{1}, P_{0}) \geq \frac{1 - \pi(Q^{P_{1}^{\otimes n}}, Q^{P_{0}^{\otimes n}})}{\sqrt{2n(e^{n} - 1)}}. \]
The first result now follows from the definition of \( g_{1} \). For the lower bound on \( g_{H} \), we use the Hellinger identities in (3.1) and (3.2), as well as Proposition B.1, to obtain
\[ H^{2}(P_{1}, P_{0}) = 2 \left( 1 - \rho(P_{1}^{\otimes n}, P_{0}^{\otimes n})^{1/n} \right) \geq 2 \left( 1 - \rho(R_{P_{1}^{\otimes n}}, R_{P_{0}^{\otimes n}})^{1/n} \right) \]
\[ \geq 2 \left( 1 - \pi(R_{P_{1}^{\otimes n}}, R_{P_{0}^{\otimes n}}) [2 - \pi(R_{P_{1}^{\otimes n}}, R_{P_{0}^{\otimes n}})]^{1/2} \right). \]
Thus, by definition of \( g_{H} \), we arrive at the lower bound
\[ g_{H}(R, \eta) \geq \sqrt{2 \left( 1 - (\eta [2 - \eta])^{1/2} \right)}. \]
The result now follows from Lemma 3.3 of Donoho and Liu (1991). \( \square \)

A.3. Proof of Theorem 4.1. We follow essentially the same arguments as in Lemma 2.2, Theorem 2.3 and Theorem 2.4 of Donoho and Liu (1991), but we directly focus on the modulus \( \omega_{(Q)}^{(n)} \) rather than on \( \Delta_{A}^{(n)}(Q, \eta) \). First, we propose an alternative version of the binary search estimator of Donoho and Liu (1991), which is particularly designed for the privatized setting (cf. Lemma 2.2 of that reference).

**Lemma A.1.** Fix a finite constant \( \Delta > 0 \) and suppose that \( M := \sup_{P \in \mathcal{P}} |\theta(P)| < \infty \). Let \( Q : \mathcal{B}(Z^{n}) \times X^{n} \to [0, 1] \) be a non-interactive channel distribution with identical marginals \( Q_{1} \). Moreover, let \( N = N(M, \Delta) \) be the smallest integer such that \( N\Delta > 2M \). For \( l \in N_{0}, \) set \( \eta = (l + 1)\Delta. \) If \( Q_{1}P \) is dominated (by a \( \sigma \)-finite measure), then there exists an estimator \( \hat{\theta}_{\Delta}^{(n)} : Z^{n} \to \mathbb{R} \) with tuning parameter \( \Delta \) \( (\hat{\theta}_{\Delta}^{(n)} \text{ taking values in the set } \{j\Delta - M : j \in \{1, \ldots, N - 1\}\} \), such that for every \( l \in N_{0}, \)
\[ \sup_{P \in \mathcal{P}} Q^{P^{\otimes n}} \left( z \in Z^{n} : \left| \hat{\theta}_{\Delta}^{(n)}(z) - \theta(P) \right| > \eta \right) \leq 4 \sum_{k=l+1}^{N-2} \eta_{A}^{(n)}(Q, k\Delta), \]
and an empty sum is interpreted as equal to zero.
Proof. Without loss of generality, we may assume that $M > 0$, since otherwise the result is trivial for $\hat{\theta}_n^\Delta \equiv 0$. Furthermore, we may assume that $0 \leq \theta(\mathbb{P}) \leq 2M$, for all $\mathbb{P} \in \mathcal{P}$, by estimating $\theta(\mathbb{P}) + M$ instead of $\theta(\mathbb{P})$.

To rigorously introduce the binary search estimator, consider first the case where $\Delta > 0$ is such that $N = N(\Delta, M) \leq 2$. In that case, we set $\hat{\theta}_n^\Delta \equiv M$, which satisfies the desired inequality trivially, because in this case $\Delta > M$ which implies $|\hat{\theta}_n^\Delta(z) - \theta(\mathbb{P})| = |M - \theta(\mathbb{P})| \leq M < \Delta = \eta_0 \leq \eta_l$. If $N \geq 3$, the estimator $\hat{\theta}_n^\Delta$ takes values in the set $\{j\Delta : j = 1, \ldots, N - 1\}$.

![Fig 1. An example of the interval construction for the binary search estimator.](image-url)
length $\Delta$, i.e., $[0, \Delta)$ or $[(N - 1)\Delta, N\Delta)$, to produce two new intervals $[l_{2,1}, h_{2,1}) = [0, (N - 1)\Delta)$ and $[l_{2,2}, h_{2,2}) = [\Delta, N\Delta)$, each of length $(N - 1)\Delta$. Then proceed in the same way again to produce three (note that removing the left-most subinterval in the first step and then removing the right-most in the second step results in the same interval as if we had removed them in the opposite order) new intervals $[l_{3,1}, h_{3,1})$, $[l_{3,2}, h_{3,2})$, $[l_{3,3}, h_{3,3})$, each of length $(N - 2)\Delta$. Continue this process for $N - 2$ steps to arrive at the intervals $[l_{N-1,j}, h_{N-1,j})$, $j = 1, \ldots, N - 1$, of length $2\Delta$ whose midpoints are exactly the values $j\Delta$.

Formally, for $k \in \{1, \ldots, N - 1\}$ and $j \in \{1, \ldots, k\}$, we set $l_{k,j} = (j - 1)\Delta$, $h_{k,j} = l_{k,j} + (N - k + 1)\Delta$, and we also define $a_{k,j} = l_{k,j} + \Delta$ and $b_{k,j} = h_{k,j} - \Delta$, so that $b_{k,j} - a_{k,j} = (N - k - 1)\Delta =: d_k$. With each pair $(k,j)$ as before, we associate a (randomized) minimax test $\xi_{k,j} : \mathcal{Z}^n \to [0,1]$ for $H_0 : Q_1 P_{\leq a_{k,j}}$ against $H_1 : Q_1 P_{\geq b_{k,j}}$. Recall that such a minimax test has the property that

$$\sup_{P_0 \in [Q_1 P_{\leq a_{k,j}}]} \mathbb{E}_{P_0}(\xi_{k,j}) + \mathbb{E}_{P_1}(1 - \xi_{k,j}) = \inf_{\phi \in \{Q_1 P_{\geq b_{k,j}}\}} \sup_{P_0 \in [Q_1 P_{\leq a_{k,j}}]} \mathbb{E}_{P_0}(\phi) + \mathbb{E}_{P_1}(1 - \phi).$$

Existence is well known (see Lemma B.4 in Appendix B, which is a minor modification of a result by Krafft and Witting (1967), see also Lehmann and Romano (2005, Problem 8.1 and Theorem A.5.1)). To obtain a non-randomized test from $\xi_{k,j}$, we set $\xi_{k,j}^* = \mathbb{1}_{(1/2,1]}(\xi_{k,j})$. Since $\mathbb{E}_P[\xi_{k,j}^*] = P(\xi_{k,j} > 1/2) \leq 2\mathbb{E}_P[\xi_{k,j}]$ and $\mathbb{E}_P[1 - \xi_{k,j}^*] = P(\xi_{k,j} \leq 1/2) = P(1 - \xi_{k,j} \geq 1/2) \leq 2\mathbb{E}_P[1 - \xi_{k,j}]$, we see that
the worst case risk of $\xi_{k,j}^*$ is not larger than twice the minimax risk of testing $H_0 : \mathcal{P}_{\leq \alpha k,j}$ against $H_1 : \mathcal{P}_{\geq \beta k,j}$. Thus, in view of (2.12), we get

$$
\sup_{P_0 \in [\mathcal{P}_{\leq \alpha k,j}], P_1 \in [\mathcal{P}_{\geq \beta k,j}]} E_{P_0}(\xi_{k,j}^*) + E_{P_1}(1 - \xi_{k,j}^*) \leq 2 \pi \left( Q \mathcal{P}_{\leq \alpha k,j}^{(n)}, Q \mathcal{P}_{\geq \beta k,j}^{(n)} \right)
$$

(A.2) 

$$
\leq 2 \eta_A^{(n)}(Q, \alpha k,j - \beta k,j) = 2 \eta_A^{(n)}(Q, d_k).
$$

If one of $\mathcal{P}_{\leq \alpha k,j}$ or $\mathcal{P}_{\geq \beta k,j}$ is empty, then any test is trivially minimax for $H_0 : \mathcal{P}_{\leq \alpha k,j}$ against $H_1 : \mathcal{P}_{\geq \beta k,j}$, because we have defined the supremum of the empty set to be $-\infty$. If exactly one of the two hypotheses is empty, we take as $\hat{\xi}_{k,j}$ the test that always decides for the non-empty hypothesis. If both hypotheses to be tested are empty, then we may take any test, e.g., we may always decide for $\xi_{k,j}^* = 0$.

To determine the value of the binary search estimator $\hat{\theta}^A_n(z)$ for a given observation $z \in \mathbb{Z}^n$, we perform a stepwise testing procedure (cf. Figure 2). Starting at the full interval $[0, N\Delta)$, we always remove the outer-most subinterval of length $\Delta$ that was rejected by the test $\hat{\xi}_{k,j}^*$. Formally, set $j_1(z) = 1$ and for $k \in \{2, \ldots, N - 1\}$, set $j_k(z) = j_{k-1}(z) + \xi_{k-1,j_{k-1}(z)}^*$, i.e., $j_k(z)$ is the index of the test to be performed on level $k$. Then $\hat{\theta}^A_n(z) = (h_{N-1,j_{N-1}(z)} + l_{N-1,j_{N-1}(z)})/2 = j_{N-1}(z)\Delta$.

We now analyze the estimation error of $\hat{\theta}^A_n$. Fix $P \in \mathcal{P}$ and $z \in \mathbb{Z}^n$. We say that the test $\hat{\xi}_{k,j_{k}(z)}^*(z)$ decided incorrectly, if its decision lead to the removal of a length-$\Delta$ subinterval that actually contained $\theta(P)$. Formally, $\xi_{k,j_{k}(z)}^*(z)$ decided incorrectly if $\xi_{k,j_{k}(z)}^*(z) = 0$ and $\theta(P) \in [h_{k,j_{k}(z)}, \alpha_{k,j_{k}(z)}]$, or $\xi_{k,j_{k}(z)}^*(z) = 1$ and $\theta(P) \in [l_{k,j_{k}(z)}, \beta_{k,j_{k}(z)}]$. Note that the test $\xi_{k,j_{k}(z)}^*(z)$ cannot decide incorrectly if $\theta(P) \notin [l_{k,j_{k}(z)}, \beta_{k,j_{k}(z)}]$. If, for some $l \in \{0, \ldots, N - 3\}$, all the tests $\xi_{k,j_{k}(z)}^*(z), k = 1, \ldots, N - 2 - l$, decide correctly, then $\theta(P) \in [l_{N-1-l,j_{N-1-l}(z)}, h_{N-1-l,j_{N-1-l}(z)}]$.

Since, by construction, we have $\hat{\theta}_n(z) \in [a_{N-1-l,j_{N-1-l}(z)}, b_{N-1-l,j_{N-1-l}(z)}]$, and the latter interval has length $d_{N-1-l} = l\Delta$, this means that $|\hat{\theta}^A_n(z) - \theta(P)| \leq (l+1)\Delta = \eta_l$. Therefore, if $|\hat{\theta}^A_n(z) - \theta(P)| > \eta_l$, then there exists $k \in \{1, \ldots, N - 2 - l\}$, so that $\xi_{k,j_{k}(z)}^*(z)$ decided incorrectly. If $\xi_{k,j_{k}(z)}^*(z)$ incorrectly decided for $H_0$, then $\theta(P) \in [h_{k,j_{k}(z)}, \alpha_{k,j_{k}(z)}]$. But by disjointness there is at most one index $j^*_1 = j^*_1(P) \in \{1, \ldots, k\}$, so that $\theta(P) \in [h_{k,j^*_1}, \alpha_{k,j^*_1}]$. Thus, $j_k(z) = j^*_1$, $\xi_{k,j_1}^*(z) = 0$ and $\theta(P) \in [h_{k,j^*_1}, \alpha_{k,j^*_1}]$. If, on the other hand, $\xi_{k,j_{k}(z)}^*(z)$ incorrectly decided for $H_1$, then $\theta(P) \in [h_{k,j_{k}(z)}, \beta_{k,j_{k}(z)}]$. But again, by disjointness there is at most one index $j^*_2 = j^*_2(P) \in \{1, \ldots, k\}$, so that $\theta(P) \in [l_{k,j^*_2}, \alpha_{k,j^*_2}]$. Thus, $\xi_{k,j^*_2}^*(z) = 1$ and $\theta(P) \in [l_{k,j^*_2}, \alpha_{k,j^*_2}]$. This fact, that at any level $k$ there are at most two tests that can decide incorrectly, is the crucial point of our construction. Consequently, for
simply write and But both

$$\sum_{k=1}^{N-2-l} Q_{P_{k}}^n \left( \xi_{k,j}^* (z) \text{ decides correctly} \right)$$

$$\leq \sum_{k=1}^{N-2-l} \left[ Q_{P_{k}}^n \left( \xi_{k,j}^* (z) = 0, \theta(P) \in [b_{k,j}, h_{k,j}] \right) + Q_{P_{k}}^n \left( \xi_{k,j}^* (z) = 1, \theta(P) \in [l_{k,j}, a_{k,j}] \right) \right].$$

But both

$$Q_{P_{k}}^n \left( \xi_{k,j}^* (z) = 0, \theta(P) \in [b_{k,j}, h_{k,j}] \right)$$

and

$$Q_{P_{k}}^n \left( \xi_{k,j}^* (z) = 1, \theta(P) \in [l_{k,j}, a_{k,j}] \right)$$

is bounded by the worst case risk of the respective test and thus, in view of (A.2), they are both bounded by $2 \eta_A^{(n)}(Q, d_k)$. We conclude that

$$Q_{P_{k}}^n \left( |\hat{\theta}_n^A - \theta(P)| > \eta \right) \leq 4 \sum_{k=1}^{N-2-l} \eta_A^{(n)}(Q, d_k) = 4 \sum_{k=l+1}^{N-2} \eta_A^{(n)}(Q, k\Delta).$$

We apply Lemma A.1 with $\Delta := C^{2} \omega_{d}^{(Q_{r})}(\epsilon)$ and $\epsilon = \epsilon_n = n^{-1/2}$, and we simply write $Q = Q_{r}$ and $Q_{1} = Q_{r}^{1}$. As in Donoho and Liu (1991) we establish bounds on $\sum_{k=l+1}^{N-2} \eta_A^{(n)}(Q, k\Delta)$. Without loss of generality, we assume $\epsilon_0 \leq 1$. We fix $k \in \{1, \ldots, N-2\}$ and $n \geq n_0$, and we first show that

(A.3) $\eta_A^{(n)}(Q, k\Delta) \leq \sup_{t \in P_{l+\Delta}} \left( 1 - \frac{1}{2} H^2(Q_1 P_1, Q_1 P_0) \right)^n.$

If (4.3) holds, then, using (3.2) and (3.1), we obtain

$$\eta_A^{(n)}(Q, k\Delta) = \sup_{t \in P_{l+\Delta}} \sup_{P_1 \in \text{conv}(Q_{P_{l+\Delta}}^n)} \pi(P_1, P_0) = \sup_{t \in \text{conv}(Q_{P_{l+\Delta}}^n)} \sup_{P_1 \in Q_{P_{l+\Delta}}^n} \pi(P_1, P_0)$$

$$\leq \sup_{t \in \text{conv}(Q_{P_{l+\Delta}}^n)} \sup_{P_1 \in Q_{P_{l+\Delta}}^n} \rho(P_1, P_0)$$

$$= \sup_{t \in \text{conv}(Q_{P_{l+\Delta}}^n)} \sup_{P_1 \in Q_{P_{l+\Delta}}^n} \rho(P_1, P_0)^n.$$
If, on the other hand, (4.2) holds, then, using (3.2) and Lemma 2 of LeCam (1986, page 477), we have

\[ \eta_A^{(n)}(Q, k\Delta) = \sup_t \sup_{\mathbb{P}_1 \in \text{conv}(Q \mathbb{P}_1^{(n)}_{\leq t})} \pi(\mathbb{P}_1, \mathbb{P}_0) \]

\[ \leq \sup_t \sup_{\mathbb{P}_1 \in \text{conv}(Q \mathbb{P}_1^{(n)}_{\leq t})} \rho(\mathbb{P}_1, \mathbb{P}_0) \]

\[ \leq \sup_t \sup_{\mathbb{P}_1 \in \text{conv}(Q \mathbb{P}_1^{(n)}_{\leq t})} \rho(\mathbb{P}_1, \mathbb{P}_0)^n \]

\[ = \sup_t \sup_{\mathbb{P}_1 \in \text{conv}(Q \mathbb{P}_1^{(n)}_{\leq t})} \rho(\mathbb{P}_1, \mathbb{P}_0)^n. \]

In view of (3.1), the expression on the last line of the two previous displays is equal to

\[ \sup_t \sup_{\mathbb{P}_1 \in \mathcal{P}_{\leq t}} \sup_{\mathbb{P}_0 \in \mathcal{P}_{\geq t + k\Delta}} \left( 1 - \frac{1}{2} H^2(Q_1 \mathbb{P}_1, Q_1 \mathbb{P}_0) \right)^n. \]

This establishes (A.3).

Next, define \( \xi_k := k^{1/r} C^{1/r} n^{-1/2} \) and note that from (4.1), by our choice of \( C \) and \( n_0 \), and for \( n \geq n_0 \), we obtain the lower bound

\[ k\Delta = kC^2 \omega_H(Q_1) (n^{-1/2}) \geq CA_0 kC n^{-r/2} = CA_0 \xi_k = \frac{CA_0}{A_1} A_1 \xi_k > A_1 \xi_k. \]

Furthermore, by definition of \( N = N(\Delta, M) \) in Lemma A.1, we have \( (N - 1)\Delta \leq 2M \). Hence, using (4.1) and our choice of \( C \), we obtain for \( k \leq N - 2 \) and \( n \geq n_0 \),

\[ \xi_k = \left( kC n^{-r/2} \right)^{1/r} \leq \left( \frac{C}{A_0} (N - 1) A_0 \left( \frac{1}{\sqrt{n}} \right) \right)^{1/r} \]

\[ \leq \left( \frac{N - 1}{CA_0} C^2 \omega_H(Q) (n^{-1/2}) \right)^{1/r} \]

\[ = \left( \frac{(N - 1)\Delta}{CA_0} \right)^{1/r} \]

\[ \leq \left( \frac{2M}{CA_0 \varepsilon_0} \right)^{1/r} \varepsilon_0 \]

\[ \leq \varepsilon_0. \]

But if \( \xi_k \leq \varepsilon_0 \), then (4.1) implies \( k\Delta > A_1 \xi_k \geq \omega_H(Q)(\xi_k) \), which, together with our previous bound (A.3), implies that \( \eta_A^{(n)}(Q, k\Delta) \leq \left( 1 - \frac{1}{2} \xi_k^2 \right)^n \). But since \( \log(1 + t) \leq \sqrt{2t} \) for all \( t \geq 0 \), we have

\[ \eta_A^{(n)}(Q, k\Delta) \leq \left( 1 - \frac{1}{2} \xi_k^2 \right)^n \leq \left( 1 - \frac{1}{2} \varepsilon_0^2 \right)^n. \]

Notice the typo in the formulation of that Lemma.
for $t > -1$, this can further be upper bounded by $e^{-n\xi_t^2}$, because $\varepsilon_0 \leq 1$. Finally, $n\xi_t^2/2 = 1/2(kC)^2/r \geq k\frac{1}{2}C^2/r$, because $r \leq 2$, and
\[
\eta_A^{(n)}(Q, k\Delta) \leq \zeta_0^k,
\]
for $\zeta := \exp(-\frac{1}{2}C^2/r) < 1$.

Using these considerations, we can now derive an upper bound on the sum of $\eta_A^{(n)}(Q, k\Delta)$ in Lemma A.1, namely,
\[
\sum_{k=j}^{N-2} \eta_A^{(n)}(Q, k\Delta) \leq \sum_{k=j}^{N-2} \zeta^k \leq \sum_{k=j}^{\infty} \zeta^k = \frac{\zeta^j}{1-\zeta}.
\]
Also note that $C \geq (r \log 2a)^{r/2}$ and $a > 1$ imply $a\zeta = a \exp(-C^2/r) \leq 1/2$ and $\zeta \leq 1/2$, so that we have $\frac{1}{1-a\zeta} \leq 2$ and $1 - \zeta \geq 1/2$. Consequently, from Lemma A.1, using Condition B together with Lemma B.3 and setting $\eta_{-1} = 0$, we get
\[
\sup_{\mathcal{P} \in \mathcal{P}} \mathbb{E}_{Q \otimes \mathbb{P}^{\otimes n}} [ l \left( \left| \hat{\theta}^\Delta_n - \theta(\mathbb{P}) \right| \right)] 
\leq \sup_{\mathcal{P} \in \mathcal{P}} \mathbb{E}_{Q \otimes \mathbb{P}^{\otimes n}} \left[ \sum_{j=0}^{\infty} l(\eta_j) \mathbb{1} \{ \eta_{j-1} < |\hat{\theta}^\Delta_n - \theta(\mathbb{P})| \leq \eta_j \} \right]
\leq \sum_{j=0}^{\infty} l(\eta_j) \sup_{\mathcal{P} \in \mathcal{P}} \mathbb{E}_{Q \otimes \mathbb{P}^{\otimes n}} \left( |\hat{\theta}^\Delta_n(Z) - \theta(\mathbb{P})| > \eta_{j-1} \right)
\leq l(\Delta) + 4 \sum_{j=1}^{\infty} l((j+1)\Delta) \sum_{k=j}^{N-2} \eta_A^{(n)}(Q, k\Delta)
\leq l(\Delta) + 4 \sum_{j=1}^{\infty} l((3/2)^{j+1} \Delta) \frac{\zeta^j}{1-\zeta}
\leq l(\Delta) + 4 \sum_{j=1}^{\infty} (a^{j+1}l(\Delta)) \frac{\zeta^j}{1-\zeta}
\leq l(\Delta) \left[ 1 + \frac{4a}{1-\zeta} \sum_{j=0}^{\infty} (a\zeta)^j \right]
\leq l(\Delta) \left[ 1 + \frac{4a}{1-\zeta} \frac{1}{1-a\zeta} \right]
\leq l(\Delta) \left[ 1 + 16a \right].
\]

Now
\[
\sup_{\mathcal{P} \in \mathcal{P}} \mathbb{E}_{Q \otimes \mathbb{P}^{\otimes n}} [ l \left( \left| \hat{\theta}^\Delta_n(Z) - \theta(\mathbb{P}) \right| \right)] \leq l(\Delta) \left[ 1 + 16a \right] = l(C^2 \omega_H(Q)(n^{-1/2})) \left[ 1 + 16a \right]
\leq [1 + 16a] a^{\log(C)/\log(3/2)}(\omega_H(Q)(n^{-1/2}))
\]
where we have used the regularity condition B of the loss function $l$ again. □
A.4. Proof of Corollary 4.8. Fix $\alpha \in (0, \infty)$ and $\varepsilon \in (0, \varepsilon_0 \land 1]$, where $\varepsilon_0 := (\bar{\varepsilon}_0 \land 1)^{\frac{e^\alpha - 1}{4(e^\alpha + 1)}}$. By Theorem 4.6 and Assumption (4.11), there exists a function $\ell(\varepsilon) \in L_\infty(\mu)$, with $\|\ell(\varepsilon)\|_\infty \leq 1$, so that the $\alpha$-private channel $Q^\varepsilon := Q^{(\alpha, \ell(\varepsilon))}$ with identical marginals as in (4.5) satisfies

$$\omega_H^{(Q^\varepsilon)}(\varepsilon) \leq 2\omega_1 \left( \frac{2\varepsilon e^\alpha + 1}{e^\alpha - 1} \right)^{\frac{r}{2}} \leq 2\bar{A}_12^r\frac{(e^\alpha + 1)^r}{(e^\alpha - 1)^r}\varepsilon^r.$$

Thus, by Remark 4.2, Assumption (4.1) of Theorem 4.1 is satisfied with $A_0 = \bar{A}_0/[2^r(e^\alpha - 1)^r]$, $A_1 = \bar{A}_12^{r+1}(e^\alpha + 1)^r/(e^\alpha - 1)^r$ and $\varepsilon_0$ as defined above, because $\varepsilon_0 < (\bar{\varepsilon}_0 \land 1)(e^\alpha - 1)$. Clearly, $Q_\varepsilon^P$ is dominated by a two point measure. Since $P$ is convex and $\theta$ is linear, Condition (4.2) of Theorem 4.1 is satisfied (cf. Remark 4.4). Thus, if we take

$$C' := \max \left\{ \frac{2\bar{A}_1}{A_0}, (r \log 2a)^{r/2}, \frac{2M}{\bar{A}_0(\varepsilon_0 \land 1)^r} \right\},$$

$$n_0 := (\varepsilon_0 \land 1)^{-2} = (\bar{\varepsilon}_0 \land 1)^{-2}16(e^\alpha + 1)^2/(e^\alpha - 1)^2$$

and $\varepsilon_n = n^{-1/2}$, then Theorem 4.1 yields

$$\inf_{Q \in \mathcal{Q}_n} M_n(Q, P) \leq C'_0 \cdot l \circ \omega_H^{(Q^\varepsilon)}(\varepsilon_n)$$

$$\leq C'_0 \cdot l \left( \frac{\bar{A}_1}{A_0}2^{r+1}(e^\alpha + 1)^r \bar{A}_0 \left[ \frac{1}{\sqrt{n}(e^\alpha - 1)} \right] \right)^r,$$

for all $n \geq n_0$, where $C'_0 = [1 + 16a]a^{[\log(C')/\log(3/2)]}$. Since, for

$$k := [\log((\bar{A}_1/A_0)2^{r+1}(e^\alpha + 1)^r)/\log(3/2)],$$

we have $(\bar{A}_1/A_0)2^{r+1}(e^\alpha + 1)^r \leq (3/2)^k$ and since $n \geq n_0 \geq (\bar{\varepsilon}_0 \land 1)^{-2}(e^\alpha - 1)^{-2}$, using (4.11) and Condition B, the upper bound of the previous display can be further bounded by

$$C'_0 \cdot l \left( \left( \frac{3}{2} \right)^k \bar{A}_0 \left[ \frac{1}{\sqrt{n}(e^\alpha - 1)} \right] \right)^r \leq C'_0a^k \cdot l \circ \omega_1 \left( \frac{1}{\sqrt{n}(e^\alpha - 1)} \right),$$

for $n \geq n_0$. Finally, to simplify the constant, note that

$$C'_0a^k \leq a^2(1 + 16a)(C')^{\frac{\log a}{\log a + \log 3/2}} \left[ (\bar{A}_1/A_0)2^{r+1}(e^\alpha + 1)^r \right]^{\frac{\log a}{\log a + \log 3/2}} = \bar{C}(e^\alpha + 1)^{2r}{\frac{\log a}{\log a + \log 3/2}},$$

where $\bar{C} = \bar{C}(\bar{A}_0, \bar{A}_1, a, r, M, \bar{\varepsilon}_0)$ is a constant that does not depend on $\alpha$. \hfill \Box

A.5. Proof of Theorem 4.10. Throughout this proof, we abbreviate $P_n = Q^{(\alpha, \ell_{n})}_{\mathbb{P}^\otimes n}$, $\mathbb{E}_n = \mathbb{E}_{P_n}$, $Z_i = Z_{i,n}$ and

$$\Delta_n = \left( \frac{e^\alpha + 1}{\sqrt{n}(e^\alpha - 1)} \right)^{\frac{1}{e^\alpha}}.$$
As a preliminary consideration, note that by definition of \( Z_i \), \( \|Z_i\| = z_0 = \|\ell_{h_n}\| \leq \frac{\alpha + 1}{e^\alpha - 1} \), and for \( p \geq 1 \), \( V_i := Z_i - E[Z_i] \) satisfies

\[
|V_i|^p \leq 2^{p-1}(|Z_i|^p + E[Z_i]^p) \leq \left(2\|\ell_{h_n}\| \frac{e^\alpha + 1}{e^\alpha - 1}\right)^p \leq n^{p/2} \left(2\bar{C}_0 \sqrt[n]{e^\alpha - 1} \left(\frac{e^\alpha + 1}{\sqrt[n]{e^\alpha - 1}}\right)^{-\frac{1}{1+p}}\right)^p \leq B_p n^{-p/2} \left(\frac{e^\alpha + 1}{\sqrt{n(e^\alpha - 1)}}\right)^p, 
\]

in view of Condition C. Therefore, if \( p \geq 2 \), by the Marcinkiewicz-Zygmund inequality (cf. Theorem 2 in Section 10.3 of Chow and Teicher, 1997) and using Jensen’s inequality (for the sample mean), we have

\[
E_n[|\bar{Z}_n - E[Z_1]|^p] = n^{-p}E_n\left[\sum_{i=1}^n V_i^p\right] \leq B_p n^{-p}E_n\left[\sum_{i=1}^n V_i^{2p/2}\right] \leq B_p n^{-p/2}E_n\left[\left(\frac{1}{n} \sum_{i=1}^n V_i^2\right)^{p/2}\right] \leq B_p n^{-p/2}E_n[|V_i|^p] \leq B_p (2\bar{C}_0)^p \left(\frac{e^\alpha + 1}{\sqrt{n(e^\alpha - 1)}}\right)^{p/2} \leq B_p 2^{p} \bar{C}_0^p \Delta_n^p, 
\]

where \( B_p \) is a constant that depends only on \( p \). For \( a > 1 \) from Condition B, let \( q = q(a) \geq 2 \) be so that \( \left(\frac{a}{2}\right)^a a < 1 \). For \( \mathbb{P} \in \mathcal{P} \) and \( n \in \mathbb{N} \), write \( \nabla_n = |\bar{Z}_n - \theta(\mathbb{P})| \) and set \( \eta_0 = 0 \) and \( \eta_k = \left(\frac{a}{2}\right)^{k-1} \Delta_n \), for \( k \in \mathbb{N} \). Recall that \( E_n[Z_1] = E[\ell_{h_n}] \) and

\[
B_{\mathbb{P},\theta}(\ell_{h_n}) = \sup_{\mathbb{P} \in \mathcal{P}}|E[\ell_{h_n}] - \theta(\mathbb{P})| \leq \bar{C}_0 \left(\frac{e^\alpha + 1}{\sqrt{n(e^\alpha - 1)}}\right)^{\frac{1}{1+p}} \leq \bar{C}_0 \Delta_n, 
\]
by C. Then, by the monotone convergence theorem,

\[ E_n[l(\nabla_n)] \leq E_n\left[\sum_{k=0}^{\infty} l(\eta_{k+1}) \mathbb{I}_{[\eta_k, \eta_{k+1})}(\nabla_n)\right] \leq \sum_{k=0}^{\infty} l(\eta_{k+1}) P_n(\nabla_n \geq \eta_k) \]

\[ \leq l(\eta_1) + \sum_{k=1}^{\infty} l(\eta_{k+1}) \frac{E_n[\nabla_n]}{\eta_k^q} \]

\[ \leq l(\Delta_n) + l(\Delta_n) \frac{E_n[\nabla_n]}{\Delta_n^q} \sum_{k=0}^{\infty} a^k \left(\frac{2}{3}\right)^q \]

\[ \leq l(\Delta_n) \left(1 + \frac{2^{q-1}}{1 - a(2/3)^q} \frac{E_n[|\tilde{Z}_n - E_n[Z_1]|^q] + |E_n[Z_1] - \theta(|\mathbb{F}|^q)}{\Delta_n^q}\right) \]

\[ \leq l(\Delta_n) \left(1 + \frac{2^{q-1}}{1 - a(2/3)^q} (B_q 2^q \tilde{C}_0^q + C_0^q)\right). \]

\[ \square \]

APPENDIX B: AUXILIARY RESULTS AND PROOFS

PROPOSITION B.1. Consider two measurable spaces \((X, \mathcal{X})\) and \((Z, \mathcal{Z})\), a Markov kernel \(Q : Z \times X \to [0, 1]\) and two finite measures \(P_0, P_1\) on \((X, \mathcal{X})\). Then \(QP_0\) and \(QP_1\) are finite measures and

\[ \rho(P_0, P_1) \leq \rho(QP_0, QP_1) \quad \text{and} \quad H(QP_0, QP_1) \leq H(P_0, P_1). \]

PROOF. The result is an immediate consequence of Proposition 1.1 in Del Moral et al. (2003), because \(\Phi(x, y) := (\sqrt{x} - \sqrt{y})^2\) is convex on \(\mathbb{R}^2_+\). For the convenience of the reader, we include a direct proof below.

Finiteness of \(QP_0\) and \(QP_1\) is obvious. Clearly, the two remaining conclusions are equivalent, because \(H^2 = 2(1 - \rho)\). Set \(Q_0 = QP_0\), \(Q_1 = QP_1\), \(\mu = P_0 + P_1\) and \(\nu = Q_0 + Q_1\), and let \(p_0\), \(p_1\) and \(q_0\), \(q_1\) denote the corresponding densities. Consider the Lebesgue decomposition (cf. Klenke, 2008, Theorem 7.33) of \(P_0\) with respect to \(P_1\), i.e.,

\[ P_0 = P_0^A + P_0^\perp, \]

where \(P_0^A \ll P_1\) and \(P_0^\perp \perp P_1\). Clearly, \(P_0^A\) and \(P_0^\perp\) are absolutely continuous with respect to \(\mu\) and we write \(p_0^A\) and \(p_0^\perp\) for corresponding \(\mu\)-densities, which satisfy \(p_0 = p_0^A + p_0^\perp\). For \(D \in \mathcal{Z}\), define the (finite) measures \(Q_0^A(D) := \int_X Q(D|x) d\mu_0^A\) and \(Q_0^\perp(D) := \int_X Q(D|x) d\mu_0^\perp\) and note that \(Q_0 = Q_0^A + Q_0^\perp\), so that \(Q_0^A\) and \(Q_0^\perp\) are absolutely continuous with respect to \(\nu\) and we write \(q_0^A\) and \(q_0^\perp\) for corresponding \(\nu\)-densities, which satisfy \(q_0 = q_0^A + q_0^\perp\). Now, by singularity of \(P_1^\perp\) and \(P_1\), there exists a set \(S \in \mathcal{X}\), such that \(p_0^\perp\) is \(\mu\)-a.e. equal to zero on \(S\) and \(p_1\) is \(\mu\)-a.e. equal
Next, define the measures to zero on \( S \). Therefore,

\[
\rho(P_0, P_1) = \int_X \sqrt{p_0 p_1} \, d\mu = \int_S \sqrt{p_0^A p_1 + p_0^\perp p_1} \, d\mu + \int_S \sqrt{p_0^A p_1} \, d\mu \\
= \int_S \sqrt{p_0^A p_1} \, d\mu \leq \int_X \sqrt{p_0^A p_1} \, d\mu = \rho(P_0^A, P_1).
\]

On the other hand, we have

\[
\rho(Q_0, Q_1) = \int_Z \sqrt{q_0 q_1} \, d\nu \geq \int_Z \sqrt{q_0^A q_1} \, d\nu = \rho(Q_0^A, Q_1).
\]

Thus, it remains to show that \( \rho(P_0^A, P_1) \leq \rho(Q_0^A, Q_1) \). To this end, consider a \( P_1 \)-density \( \tilde{p}_0 \) of \( P_0^A \). Clearly, the function \( \tilde{p}_1 \equiv 1 \) is a \( P_1 \)-density of \( P_1 \). Thus, we have

\[
\rho(P_0^A, P_1) = \int_X \sqrt{p_0 dP} \quad \text{and} \quad \rho(Q_0^A, Q_1) = \int_X Q(D|x) p_0(x) \, dP_1(x), \quad Q_1(D) = \int_X Q(D|x) dP_1(x), \quad \text{so that} \quad Q_0^A \ll Q_1, \quad \text{and we let} \quad \tilde{q}_0 \text{ denote a corresponding} \quad Q_1 \text{ density}. \]

Therefore, it remains to show that

\[
\int_X \sqrt{p_0} \, dP_1 \leq \int_Z \sqrt{\tilde{q}_0} \, dQ_1.
\]

In fact, we will show slightly more than that. For a Markov kernel \( Q : Z \times X \to [0, 1] \), a finite measure \( P \) on \((X, \mathcal{X})\) and a non-negative function \( p \in \mathcal{L}_1(X, \mathcal{X}, P) \), we show that

\[
(B.1) \quad \int_X \sqrt{p} \, dP \leq \int_Z \sqrt{q} \, dQ,
\]

where \( Q := Q^P \) dominates the finite measure \( Q^A(dz) := \int_X Q(dz|x) p(x) \, dP(x) \) on \((Z, Z)\) and \( q : Z \to [0, \infty) \) is a corresponding \( Q \)-density. We establish this fact first for simple functions \( p = \sum_{i=1}^n \alpha_i 1_{A_i} \), where \( \alpha_i \in (0, \infty) \) and the \( A_1, \ldots, A_n \in \mathcal{X} \) are pairwise disjoint. By disjointness, we easily see that

\[
(B.2) \quad \int_X \sqrt{p} \, dP = \int_X \sqrt{\sum_{i=1}^n \alpha_i 1_{A_i}} \, dP = \sum_{i=1}^n \sqrt{\alpha_i} \, \mathbb{P}(A_i).
\]

Next, define the measures \( Q_i^A(dz) := \int_{A_i} Q(dz|x) \, dP(x) \) and note that for any \( D \in Z \),

\[
\int_D q \, dQ = Q^A(D) = \int_Z Q(D|x) \, p(x) \, dP(x) = \sum_{i=1}^n \alpha_i q^A_i(D).
\]

Since \( \alpha_i > 0 \), we have \( Q_i^A \ll Q^A \ll Q \), and we write \( q_i \) for corresponding, finite \( Q \)-densities, so that \( \int_D q \, dQ = \int_D \sum_{i=1}^n \alpha_i q_i \, dQ \), for every \( D \in Z \), which implies that \( q = \sum_{i=1}^n \alpha_i q_i \), \( Q \)-almost everywhere. Now, set \( r(z) := \sum_{i=1}^n q_i(z) \) and \( R := \{ z \in Z : r(z) \in (0, \infty) \} \), and note that for every \( D \in Z \),

\[
\int_D 1 \, dQ = Q(D) = \int_X Q(D|x) \, dP \geq \sum_{i=1}^n \int_{A_i} Q(D|x) \, dP = \sum_{i=1}^n \int_D q_i \, dQ = \int_D r \, dQ,
\]
by disjointness of the $A_i$, so that $r \leq 1$, $Q$-almost everywhere. Thus, using Jensen’s inequality, we obtain the lower bound
\[
\int_Z \sqrt{q} \, dQ \geq \int_R \sqrt{\sum_{i=1}^n \alpha_i q_i} \, dQ = \int_R \sqrt{\sum_{i=1}^n \alpha_i} \frac{q_i}{r} \, dQ 
\]
\[
\geq \int_R \sqrt{\sum_{i=1}^n \alpha_i} \frac{q_i}{r} \, dQ \geq \int_R \sum_{i=1}^n \sqrt{\alpha_i} q_i \, dQ = \sum_{i=1}^n \sqrt{\alpha_i} Q_i^4(R).
\]
But clearly, $Q_i^4(R) = Q_i^4(Z) - Q_i^4(R^c) = Q_i^4(Z) = P(A_i)$. So, in view of (B.2), we have established (B.1) for simple $p$. Now, for general $p$, let $(p^{(n)})_{n\in\mathbb{N}}$ be a sequence of simple functions such that $p^{(n)}(z) \uparrow p(z)$. If $q^{(n)}$ denotes a $Q$-density of the finite measure $\int_X Q(dz|x)p^{(n)}(x) \, dP(x)$, then it is easy to see that $q^{(n)} \leq q$, $Q$-almost everywhere. Thus, from (B.1) for simple functions, we conclude that
\[
\int_X \sqrt{p^{(n)}} \, dP \leq \int_Z \sqrt{q^{(n)}} \, dQ \leq \int_Z \sqrt{q} \, dQ.
\]
Relation (B.1) now follows from the monotone convergence theorem. \hfill \square

**Lemma B.2.** Let $\mathcal{P}$ be a non-empty set of probability measures on some measurable space $(\Omega, \mathcal{F})$ and let $\theta : \mathcal{P} \to \mathbb{R}$ be a functional. If $\mathcal{P}$ is convex, then the following statements are equivalent.

i) $\theta : \mathcal{P} \to \mathbb{R}$ is constant.

ii) $\frac{\omega_1(\varepsilon)}{\varepsilon} \to 0$, as $\varepsilon \to 0$.

iii) $\frac{\omega_H(\varepsilon)}{\varepsilon^2} \to 0$, as $\varepsilon \to 0$.

**Proof.** (i) $\Rightarrow$ (ii) and (iii): This is immediate, because for constant $\theta$, we have $\omega_1(\varepsilon) = \sup \{\theta(P_0) - \theta(P_1) : d_{TV}(P_0, P_1) \leq \varepsilon, P_j \in \mathcal{P}\} = 0$ and $\omega_H(\varepsilon) = 0$.

(ii) or (iii) $\Rightarrow$ (i): Note that if $\mathcal{P}$ contains only one element, then $\theta$ is necessarily constant. So it is no loss of generality to assume that $\mathcal{P}$ contains at least two distinct elements. Take such $P_0, P_1 \in \mathcal{P}$, $P_0 \neq P_1$, fix $\lambda \in [0, 1]$ and define $P_\lambda := \lambda P_1 + (1 - \lambda)P_0$. Now, for $\lambda_0 \in [0, 1]$, note that
\[
\begin{align*}
d_{TV}(P_{\lambda_0}, P_\lambda) &= \sup_{A \in \mathcal{F}} \{[\lambda_0 P_1 + (1 - \lambda_0)P_0](A) - [\lambda P_1 + (1 - \lambda)P_0](A)\} \\
&= \sup_{A \in \mathcal{F}} \{[\lambda_0 - \lambda]P_1(A) - (\lambda_0 - \lambda)P_0(A)\} \\
&= |\lambda_0 - \lambda| d_{TV}(P_1, P_0) \xrightarrow{\lambda \to \lambda_0} 0.
\end{align*}
\]
Now, by convexity of $\mathcal{P}$, $P_\lambda \in \mathcal{P}$ and $f(\lambda) := \theta(P_\lambda) \in \mathbb{R}$ is well defined, for every $\lambda \in [0, 1]$. Thus, since $\frac{1}{2} H^2(P_{\lambda_0}, P_\lambda) \leq d_{TV}(P_{\lambda_0}, P_\lambda)$, we have
\[
\left| \frac{f(\lambda_0) - f(\lambda)}{\lambda_0 - \lambda} \right| = d_{TV}(P_0, P_1) \frac{|\theta(P_{\lambda_0}) - \theta(P_\lambda)|}{d_{TV}(P_{\lambda_0}, P_\lambda)} \leq 2 d_{TV}(P_0, P_1) \frac{\omega_H(H(P_{\lambda_0}, P_\lambda))}{H^2(P_{\lambda_0}, P_\lambda)}.
\]

and
\[
\frac{|f(\lambda_0) - f(\lambda)|}{\lambda_0 - \lambda} = d_{TV}(P_0, P_1) \frac{\theta'(P_{\lambda_0}) - \theta'(P_{\lambda})}{d_{TV}(P_{\lambda_0}, P_{\lambda})} \leq d_{TV}(P_0, P_1) \frac{\omega_1 d_{TV}(P_{\lambda_0}, P_{\lambda})}{d_{TV}(P_{\lambda_0}, P_{\lambda})},
\]

But at least one of these upper bounds converges to zero if \(\lambda \to \lambda_0\), by assumption. Thus, \(f : [0, 1] \to \mathbb{R}\) has derivative constant equal to zero and is thus constant on its domain, implying \(\theta(P_0) = f(0) = f(1) = \theta(P_1)\). Since \(P_0\) and \(P_1\) were arbitrary, \(\theta\) is constant on \(\mathcal{P}\).

\textbf{Lemma B.3.}
\[\forall x \in \mathbb{R} : \left(\frac{3}{2}\right)^x \geq x.\]

\textbf{Proof.} For \(x \in \mathbb{R}\), we set \(f(x) := (3/2)^x - x\) and show that \(f(x) \geq 0\). Since \(f(x) = \exp(x \log(3/2) - x\), we have \(f'(x) = (3/2)^x \log(3/2) - 1\) and \(f''(x) = (3/2)^x (\log(3/2))^2 > 0\). Thus, \(f\) is strictly convex and has its unique minimum at \(x_0 = -\frac{\log \log(3/2)}{\log(3/2)}\), because \(f'(x_0) = \exp(x_0 \log(3/2)) \log(3/2) - 1 = 0\). But \(1 < 3/2 \leq e\), so that \(0 < \log(3/2) \leq 1\) and \(\log \log(3/2) > -1\). Therefore,
\[f(x_0) = \frac{1}{\log(3/2)} + \frac{\log \log(3/2)}{\log(3/2)} > 0.\]

\textbf{Lemma B.4 (Krafft and Witting (1967)).} Let \(\mathcal{P}_0\) and \(\mathcal{P}_1\) be two sets of probability measures on a measurable space \((\Omega, \mathcal{F})\) and \(\mu\) a \(\sigma\)-finite measure on \((\Omega, \mathcal{F})\) that dominates \(\mathcal{P}_0 \cup \mathcal{P}_1\). Let \(\mathcal{T}\) be the collection of all randomized test functions, i.e., all measurable functions \(\phi : \Omega \to [0, 1]\). Then there exists a minimax test, i.e., an element \(\phi^* \in \mathcal{T}\), so that
\[
\sup_{P_0 \in \mathcal{P}_0} \mathbb{E}_{P_0}[\phi^*] + \mathbb{E}_{P_1}[1 - \phi^*] = \inf_{\phi^* \in \mathcal{T}} \sup_{P_0 \in \mathcal{P}_0} \mathbb{E}_{P_0}[\phi] + \mathbb{E}_{P_1}[1 - \phi].
\]

\textbf{Proof.} Without loss of generality, we may assume that \(\mathcal{P}_0\) and \(\mathcal{P}_1\) are non-empty, because otherwise any test \(\phi \in \mathcal{T}\) is minimax. For \(\phi \in \mathcal{T}\), \(P_0 \in \mathcal{P}_0\) and \(P_1 \in \mathcal{P}_1\), set
\[
\pi(P_0, P_1, \phi) := \mathbb{E}_{P_0}[\phi] + \mathbb{E}_{P_1}[1 - \phi],
\]
and
\[
R(\phi) := \sup_{P_0 \in \mathcal{P}_0} \pi(P_0, P_1, \phi),
\]

\[\begin{align*}
\text{ROHDE, A. AND STEINBERGER, L.}
\end{align*}\]
and let \( \phi_n \in \mathcal{T} \) be a sequence of tests, such that \( R(\phi_n) \rightarrow \inf_{\phi \in \mathcal{T}} R(\phi) =: \rho \), as \( n \rightarrow \infty \). From the weak sequential compactness of \( \mathcal{T} \) (cf. Nöle and Plachky, 1967), it follows that there exists a subsequence \((\phi_{n_m})_{m \in \mathbb{N}}\) of \((\phi_n)_{n \in \mathbb{N}}\) and a test \( \phi^* \in \mathcal{T} \), so that

\[
\int_{\Omega} \phi_{n_m} f \, d\mu \xrightarrow{m \rightarrow \infty} \int_{\Omega} \phi^* f \, d\mu,
\]

for every \( f \in L_1(\Omega, \mathcal{F}, \mu) \). This entails, in particular, that \( E_P[\phi_{n_m}] \rightarrow E_P[\phi^*] \), for every \( P \in \mathcal{P}_0 \cup \mathcal{P}_1 \). Consequently, for every \( P_0 \in \mathcal{P}_0 \) and \( P_1 \in \mathcal{P}_1 \),

\[
\pi(P_0, P_1, \phi^*) = \lim_{m \rightarrow \infty} \pi(P_0, P_1, \phi_{n_m}) \leq \lim_{m \rightarrow \infty} R(\phi_{n_m}) = \lim_{n \rightarrow \infty} R(\phi_n) = \rho.
\]

But this entails that \( R(\phi^*) \leq \rho \), whereas \( R(\phi^*) \geq \rho \) holds trivially. \( \square \)

**B.1. Proof of Proposition 2.1.** Set \( M = \max(|a|, |b|) \). We check the conditions of Corollary 3.3 in Sion (1958). Clearly, \( S \) and \( \mathcal{T} \) are convex sets and the function \( F(\phi, \sigma) := \int_{\Omega} f \, d\sigma \) on \( \mathcal{T} \times \mathcal{S} \) is quasi-concave-convex, because it is linear in both arguments. We equip \( \mathcal{S} \) with the topology induced by \( \| \cdot \|_{TV} \) and note that this makes \( \sigma \mapsto F(\phi, \sigma) \) continuous, for every \( \phi \in \mathcal{T} \), because

\[
|F(\phi, \sigma_1) - F(\phi, \sigma_2)| \leq \sup_{\| \phi \|_{TV} \leq M} \left| \int_{\Omega} \phi d(\sigma_1 - \sigma_2) \right| \leq 2M \| \sigma_1 - \sigma_2 \|_{TV}.
\]

The proof is finished if we can find a topology \( \tau \) for \( L_\infty = L_\infty(\Omega, \mathcal{F}, \mu) \) in which \( \mathcal{T} \) is compact and \( \phi \mapsto F(\phi, \sigma) \) is continuous, for every \( \sigma \in \mathcal{S} \). Abbreviate the space of equivalence classes of \( \mu \)-integrable functions by \( L_1 = L_1(\Omega, \mathcal{F}, \mu) \) and its topological dual by \( L_1^* = L_1(\Omega, \mathcal{F}, \mu) \). For \( f \in L_1 \), let \( E_f : L_1^* \rightarrow \mathbb{R} \) denote the evaluation functional on the dual space \( L_1^* \), i.e., \( E_f(\psi) = \psi(f) \). Set \( V_M = \{ f \in L_1 : \| f \|_{L_1} \leq 1/M \} \) and \( K = \{ \psi \in L_1^* : |\psi(f)| \leq M, \forall f \in V_1 \} = \{ \psi \in L_1^* : |\psi(f)| \leq 1, \forall f \in V_M \} \). By the Banach-Alaoglu theorem (Rudin, 1973, Section 3.15), the set \( K \) is compact in the weak*-topology on \( L_1^* \), i.e., the weakest topology \( \tau^* \) on \( L_1^* \) for which all the evaluation functionals \( E_f, f \in L_1 \), are continuous. Next, we use the fact that \( L_\infty \) is the dual of \( L_1 \) (Dunford and Schwartz, 1957, Theorem IV.8.5). Let \( \Psi : (L_\infty, \| \cdot \|_\infty) \rightarrow (L_1^*, \| \cdot \|_{L_1}^*) \) denote the isometric isomorphism that associates each \( \phi \in L_\infty \) with the linear functional \( f \mapsto \int_{\Omega} \phi f \, d\mu \). Therefore, we can map the weak*-topology \( \tau^* \) to a topology \( \tau = \{ \Psi^{-1}(O) : O \in \tau^* \} \) on \( L_\infty \), so that all the functions \( E_f \circ \Psi : (L_\infty, \tau) \rightarrow \mathbb{R}, f \in L_1 \), are continuous and

\[
\Psi^{-1}(K) = \left\{ \phi \in L_\infty : \left| \int_{\Omega} \phi f \, d\mu \right| \leq M, \forall f \in V_1 \right\} = \left\{ \phi \in L_\infty : \| \phi \|_{TV} \leq M \right\}
\]

is \( \tau \)-compact, because \( \Psi \) is an isomorphism and therefore \( \Psi^{-1} \) is continuous. If \( f \) is a \( \mu \)-density of the (finite) measure \( \sigma \in \mathcal{S} \), then \( f \in L_1 \) and \( E_f \circ \Psi(\phi) = \int_{\Omega} \phi f \, d\mu = F(\phi, \sigma) \). Thus, we see that \( \phi \mapsto F(\phi, \sigma) \) is \( \tau \)-continuous for every \( \sigma \in \mathcal{S} \). It remains
to show that $T \subseteq \Psi^{-1}(K)$ is $\tau$-closed. But clearly

$$T^c = \left\{ \phi \in L_\infty : a \leq \int_\Omega \phi \, d\mu \leq b, \forall f \in V_1 \right\}^c$$

$$= \bigcup_{f \in V_1} \left( \left[ E_f \circ \Psi \right]^{-1}((\infty, a)) \cup \left[ E_f \circ \Psi \right]^{-1}((b, \infty)) \right) \in \tau.$$

\[ \square \]

\section*{APPENDIX C: PROOFS OF EXAMPLE SECTION}

\subsection*{C.1. Proofs of Section 5.1.}

We begin with the case $(a)$. For the lower bound on $\omega_1$, fix $\varepsilon \in (0, 1)$ and $\delta \in (0, (C/2)^{1/\alpha})$. By our assumption on $|f|$, there exist $x_\varepsilon, x_\delta \in X$, such that $|f(x_\varepsilon)| = \delta$ and $|f(x_\delta)| = (C/(2\varepsilon))^{1/\alpha}$. Now let $P_0$ be a dirac measure at the point $\{x_\delta\}$ and let $P_1(\{x_\delta\}) = 1 - \varepsilon$ and $P_1(\{x_\varepsilon\}) = \varepsilon$. Then $E_{P_0}[|f|^\alpha] = \delta^\alpha \leq C/2 \leq C$ and $E_{P_1}[|f|^\alpha] = \delta^\alpha (1 - \varepsilon) + (C/(2\varepsilon)) \varepsilon \leq C/2 + C/2 = C$. So both $P_0$ and $P_1$ belong to $P_\alpha(C)$. Furthermore, $d_{TV}(P_0, P_1) = \varepsilon$ and

$$|E_{P_0}[f] - E_{P_1}[f]| = |f(x_\varepsilon)(1 - \varepsilon) + f(x_\delta)\varepsilon - f(x_\delta)| \xrightarrow{\delta \to 0} |f(x_\varepsilon)|\varepsilon = (C/2)^{1/\alpha} \varepsilon^{1/\alpha}.$$

Thus, we have exhibited a sequence in the set $\{\theta(P_0) - \theta(P_1) : d_{TV}(P_0, P_1) \leq \varepsilon, P_i \in P_\alpha(C)\}$ that converges to $(C/2)^{1/\alpha} \varepsilon^{1/\alpha}$, and the supremum can not be less than that quantity.

To verify (4.12), note that

$$B(\ell_h) = \sup_{P \in P_\alpha(C)} |\theta(P) - E_P[\ell_h]| = \sup_{P \in P_\alpha(C)} \left| E_P[f] - E_P\left[f I_{|f| \leq \frac{1}{h}} \right] \right|$$

$$\leq \sup_{P \in P_\alpha(C)} E_P\left[|f| I_{|f| > \frac{1}{h}}\right] \leq \sup_{P \in P_\alpha(C)} (E_P[|f|^\alpha])^{1/\alpha} P(|f| > 1/h)^{1 - \frac{1}{\alpha}}$$

$$\leq C^{1/\alpha} \sup_{P \in P_\alpha(C)} (h^{\alpha} E_P[|f|^\alpha])^{\frac{1}{\alpha - 1}} \leq Ch^{\alpha - 1}.$$

Finally, we compute the squared bias and variance of $\hat{\theta}_n$ under $Q^{*P^{\otimes n}}$, where $Q^* = Q^{(\alpha, \ell_h, \varepsilon)}$. For the bias, as above, we obtain

$$\left| E_{Q^{*P^{\otimes n}}}[\hat{\theta}_n] - \theta(P) \right|^2 \leq B(\ell_{h_n})^2 \leq C^2 \left( \frac{(e^\alpha + 1)^2}{n(e^\alpha - 1)^2} \right)^{\frac{1}{\alpha - 1}},$$

and for the variance, we see that

$$\text{Var}_{Q^{*P^{\otimes n}}}[\hat{\theta}_n] = \frac{1}{n} \text{Var}_{Q^{*P^{\otimes n}}}[Z_1] \leq \frac{1}{n} E_{Q^{*P^{\otimes n}}}[Z_1^2] = \frac{1}{n} \|\ell_{h_n}\|_\infty^2 \left( \frac{(e^\alpha + 1)^2}{n(e^\alpha - 1)^2} \right)^{\frac{1}{\alpha - 1}}$$

$$= \left( \frac{(e^\alpha + 1)^2}{h_n^2 n(e^\alpha - 1)^2} \right)^{\frac{1}{\alpha - 1}} = \left( \frac{(e^\alpha + 1)^2}{n(e^\alpha - 1)^2} \right)^{\frac{1}{\alpha - 1}}.$$
To establish the lower bound on \( \omega_1 \) in case (b), first note that if \(|f(x)|^\kappa > C\), for all \( x \in \mathcal{X} \), then \( \mathcal{P}_\kappa(C) = \emptyset \). Suppose now that \(|f(x_0)|^\kappa = C\) for some \( x_0 \in \mathcal{X} \) and that \(|f(x)|^\kappa > C\) for all \( x \neq x_0 \). Then the only probability distribution \( \mathbb{P} \) on \((\mathcal{X}, \mathcal{B}(\mathcal{X}))\) for which \( \mathbb{E}_\mathbb{P}[|f|^\kappa] \leq C \) holds, is the Dirac point mass at \( x_0 \). But this contradicts that \( \theta \) is not constant on \( \mathcal{P}_\kappa(C) \). Thus, either (i) \(|f(x_0)|^\kappa < C\), or (ii) there exists \( x_1 \neq x_0 \), such that \(|f(x_j)|^\kappa = C\), for \( j = 0, 1 \). In case (ii), suppose that \(|f(x)|^\kappa \geq C\), for all \( x \in \mathcal{X} \). Otherwise, we are in case (i). Let \( A = \{ x \in \mathcal{X} : |f(x)|^\kappa = C \} \) and note that all elements of \( \mathcal{P}_\kappa(C) \) must be supported on \( A \). If \( f \) is constant on \( A \), then \( \mathbb{P} \mapsto \theta(\mathbb{P}) = \mathbb{E}_\mathbb{P}[f] \) is constant on \( \mathcal{P}_\kappa(C) \). A contradiction.

Thus, in case (ii), there exist \( x_0, x_1 \in A \), such that \( f(x_0) \neq f(x_1) \).

Now, in case (i), let \( x_0 \) be such that \(|f(x_0)|^\kappa < C\) and take \( x_1 \in \mathcal{X} \), so that \( f(x_1) \neq f(x_0) \). In case (ii), let \( x_0, x_1 \in A \), such that \( f(x_0) \neq f(x_1) \). In either case, for \( \varepsilon \in (0, 1) \), set \( \mathbb{P}_0 = \delta_{x_0} \), the Dirac point mass at \( x_0 \) and set \( \mathbb{P}_1 = (1-\varepsilon)\delta_{x_0} + \varepsilon \delta_{x_1} \). Then, in case (ii), we have \( \mathbb{E}_{\mathbb{P}_0}[f]^\kappa = |f(x_0)|^\kappa \leq C \) and \( \mathbb{E}_{\mathbb{P}_1}[f]^\kappa = (1-\varepsilon)|f(x_0)|^\kappa + \varepsilon |f(x_1)|^\kappa \leq C \). In case (i), we have \( \mathbb{E}_{\mathbb{P}_0}[f]^\kappa < C \), and there exists \( \varepsilon_0 \in (0, 1) \), so that for all \( \varepsilon \in [0, \varepsilon_0] \), we have \( \mathbb{E}_{\mathbb{P}_\varepsilon}[f]^\kappa = (1-\varepsilon)|f(x_0)|^\kappa + \varepsilon |f(x_1)|^\kappa < C \). Hence, in both cases we have \( \mathbb{P}_j \in \mathcal{P}_\kappa(C) \), for \( j = 0, 1 \), at least for \( \varepsilon \in [0, \varepsilon_0] \). Now, it is easy to compute \( |\theta(\mathbb{P}_0) - \theta(\mathbb{P}_1)| = \varepsilon |f(x_0) - f(x_1)| \) which is non-zero in both cases.

Since \( d_{\text{TV}}(\mathbb{P}_0, \mathbb{P}_1) = \varepsilon \), we arrive at the lower bound \( \omega_1(\varepsilon) \geq |f(x_0) - f(x_1)|\varepsilon \), for all \( \varepsilon \in [0, \varepsilon_0] \).

The bias condition (4.12) has already been verified above. Squared bias and variance of \( \hat{\theta}_n \) can be computed as in case (a), except that we now use \( \|\ell_n\|_\infty \leq \|f\|_\infty < \infty \). \( \square \)

**C.2. Proofs of Section 5.2.** We begin with the lower bound on \( \omega_1 \). Let

\[
\kappa_0(u) = \exp\left(-\frac{1}{1-4u^2}\right)1_{[-1/2, 1/2]}(u)
\]

and let \( \kappa = a_0\kappa_0 \), for an \( a_0 = a_0(\beta) > 0 \), so that the \( b \)-th derivative of \( \kappa \) is Hölder continuous with exponent \( \beta - b \) and constant 1/2. Similarly, by appropriate rescaling and shifting of \( \kappa_0 \), we obtain a density \( p_0 \in \mathcal{H}^{\leq \beta/2}(\mathbb{R}) \), such that for constants \( \delta_0, \delta_1 > 0 \), depending only on \( \beta \) and \( C \), we have \( p_0(x) \geq \delta_0 \) for all \( x \in (x_0 - \delta_1, x_0 + \delta_1) \). Now, for \( x, y \in \mathbb{R} \) and \( h > 0 \), set \( g(y) = \kappa(y+1) - \kappa(y) \) and

\[
p_1(x) = p_0(x) + \frac{C}{2} h^{\beta} g\left(\frac{x-x_0}{h}\right).
\]

It follows that

\[
|p_1^{(b)}(x) - p_1^{(b)}(y)| \leq |p_0^{(b)}(x) - p_0^{(b)}(y)| + \frac{C}{2} h^{\beta-b} \left| g^{(b)}\left(\frac{x-x_0}{h}\right) - g^{(b)}\left(\frac{y-x_0}{h}\right) \right|
\]

\[
\leq \frac{C}{2} |x-y|^{\beta-b} + \frac{C}{2} h^{\beta-b} \left| \frac{x-x_0}{h} - \frac{y-x_0}{h} \right|^{\beta-b} = C|x-y|^{\beta-b}.
\]
Furthermore, since $g((x - x_0)/h) < 0$ if, and only if, $x \in (x_0 - h/2, x_0 + h/2)$, we see that $p_1(x) \geq 0$, for all $x \in \mathbb{R}$, if $h < 2\delta_1$ and $h^\beta < \delta_0/(C\|\kappa\|_\infty)$. Now

\begin{equation}
\l|\theta(p_0) - \theta(p_1)\r| = \frac{C}{2} h^{\beta-m} |g^{(m)}(0)|,
\end{equation}

and

\[ d_{TV}(P_0, P_1) = \frac{1}{2} \| p_0 - p_1 \|_{L_1} = \frac{C}{4} h^\beta \int g \left( \frac{x - x_0}{h} \right) \, dx = \frac{C}{2} h^{\beta+1} \|\kappa\|_{L_1}. \]

Thus, if $d_{TV}(P_0, P_1) \leq \varepsilon$, the maximum value of $h$ we can choose is $h = (2\varepsilon/(C\|\kappa\|_{L_1}))^{1/\beta+1}$, which obeys our restrictions on $h$, if $\varepsilon \leq \varepsilon_0$, for an appropriate choice of $\varepsilon_0$. Plugging this back into (C.1) yields the claimed lower bound.

Next, we verify (4.12). Since $\kappa_h^{(m)}(x) = h^{-(m+1)} K^{(m)} \left( \frac{x - x_0}{h} \right)$, we have $\|\ell_h\|_\infty \leq \|K^{(m)}\|_\infty h^{-(m+1)}$. Furthermore, $p^{(m)}$ is $(b - m)$-times continuously differentiable, so that

\[ p^{(m)}(x_0 + hu) = p^{(m)}(x_0) + hu \cdot p^{(m+1)}(x_0) + \cdots + \frac{(hu)^{b-m}}{(b-m)!} p^{(b)}(x_0 + \tau hu), \]

for some $\tau \in [0,1]$. Thus, from the properties of the kernel $K$ and integration by parts, we get

\[ |E_p [\ell_h] - \theta(p)| = \left| \int_{x_0-h}^{x_0+h} \ell^{(m)}_h(x) p(x) \, dx - p^{(m)}(x_0) \right| \]

\[ = \left| (-1)^m \int_{x_0-h}^{x_0+h} \ell_h(x) p^{(m)}(x) \, dx - p^{(m)}(x_0) \right| \]

\[ = \left| \int_{-1}^{1} K(u) \left[ p^{(m)}(x_0 + hu) - p^{(m)}(x_0) \right] \, du \right| \]

\[ = \left| \int_{-1}^{1} K(u) \frac{(hu)^{b-m}}{(b-m)!} p^{(b)}(x_0 + \tau hu) \, du \right| \]

\[ = \left| \int_{-1}^{1} K(u) \frac{(hu)^{b-m}}{(b-m)!} \left[ p^{(b)}(x_0 + \tau hu) - p^{(b)}(x_0) \right] \, du \right| \]

\[ \leq \int_{-1}^{1} |K(u)| \frac{|hu|^{b-m}}{(b-m)!} C|\tau hu|^{\beta-b} \, du \leq \frac{C_0 C}{(b-m)!} \frac{h^{\beta-m}}{\tau^{\beta+1}}. \]

Finally, for the mean squared error, we compute squared bias and variance as follows. For the bias, as above, we obtain

\[ |E_{\mathbb{P}^* \times \mathbb{P}^*} [\hat{\theta}_n] - \theta(\mathbb{P})|^2 \leq B(\ell_h)_n^2 \leq \left( \frac{C_0 C}{(b-m)!} \right)^2 \left( \frac{(e^{\alpha} + 1)^2}{n(e^{\alpha} - 1)^2} \right)^{\frac{\beta-m}{\beta+1}}, \]
and for the variance, we see that
\[
\text{Var}_Q P^{\otimes n}[\hat{\theta}_n] = \frac{1}{n} \text{Var}_Q P^{\otimes n}[Z_1] = \frac{1}{n} \|\ell_{h_n}\|_\infty^2 \frac{(e^\alpha + 1)^2}{(e^\alpha - 1)^2} \leq \|K^{(m)}\|_\infty^2 \frac{(e^\alpha + 1)^2}{h_n^{2(m+1)} n (e^\alpha - 1)^2} = \|K^{(m)}\|_\infty^2 \left(\frac{(e^\alpha + 1)^2}{n(e^\alpha - 1)^2}\right)^{\frac{\beta - m}{\beta + 1}}.
\]

\[\square\]

C.3. Proofs of Section 5.3. For \( j = 1, \ldots, d \), we use the same construction as in Section C.2 to obtain kernels \( \kappa_j \in \mathcal{H}_{\beta_j, 1/2}(\mathbb{R}) \) and set \( g_j(y) = \kappa_j(y + 1) - \kappa_j(y) \). Moreover, we take
\[
p_0(x) = \prod_{j=1}^d (2\pi \sigma^2)^{-1/2} \exp\left(-\frac{(x_j - x_j^{(0)})^2}{2\sigma^2}\right),
\]
which satisfies \( p_0 \in \mathcal{H}^{< \lambda}_{\beta, C/2}(\mathbb{R}^d) \) for some sufficiently large \( \sigma > 0 \) depending only on \( C = (C_1, \ldots, C_d)' \). Furthermore, for \( h, h_1, \ldots, h_d > 0 \), define
\[
p_1(x) = p_0(x) + h \prod_{j=1}^d C_j \|g_j\|_\infty \frac{g_j \left(\frac{x_j - x_j^{(0)}}{h_j}\right)}{2}.
\]
Since the mappings \( x_j \mapsto g_j((x_j - x_j^{(0)})/h_j) \) take only negative values if \( x_j^{(0)} - h_j/2 < x < x_j^{(0)} + h_j/2 \), we see that \( p_1(x) \) is non-negative for all \( x \in \mathbb{R}^d \), provided that
\[
h \prod_{j=1}^d C_j \|g_j\|_\infty \leq q_d((h_1, \ldots, h_d)/2),
\]
where \( q_d \) is the density of the \( \mathcal{N}_d(0, \sigma^2 I_d) \) distribution. Thus, \( p_1 \) is a probability density, if \( \max_j h_j \leq 2 \) and
\[
h \leq \frac{q_d(1, \ldots, 1)}{\prod_{j=1}^d C_j \|g_j\|_\infty}.
\]
Next, observe that for $x \in \mathbb{R}^d$ and $\bar{x}_j \in \mathbb{R}$,

$$|p_1(x_1, \ldots, x_{j-1}, \bar{x}_j, x_{j+1}, \ldots, x_d) - p_0(x)|$$

$$\leq |p_0(x_1, \ldots, x_{j-1}, \bar{x}_j, x_{j+1}, \ldots, x_d) - p_1(x)|$$

$$+ h \left| \prod_{k=1 \atop k \neq j}^{d} C_k \right| \frac{g_k \left( \frac{x_k - x_k^{(0)}}{h_k} \right)}{2} \left| \prod_{k=1 \atop k \neq j}^{d} C_k \right| \frac{g_j \left( \frac{\bar{x}_j - x_j^{(0)}}{h_j} \right)}{2} - g_j \left( \frac{x_j - x_j^{(0)}}{h_j} \right)$$

$$\leq C_j \left| \bar{x}_j - x_j \right|^{\beta_j} + h \left| \prod_{k=1 \atop k \neq j}^{d} C_k \right| \|g_k\|_\infty \left| \prod_{k=1 \atop k \neq j}^{d} C_k \right| \frac{\bar{x}_j - x_j^{(0)}}{h_j} - \frac{x_j - x_j^{(0)}}{h_j}$$

$$\leq C_j |\bar{x}_j - x_j|^{\beta_j},$$

provided that

$$h_j^{-\beta_j} h \leq \left| \prod_{k=1 \atop k \neq j}^{d} C_k \right| \|g_k\|_\infty \left| \prod_{k=1 \atop k \neq j}^{d} C_k \right|^{-1} =: \tilde{c}_j,$$

Consequently, we see that $p_1 \in \mathcal{H}_{\beta,C}^{\leq \lambda}(\mathbb{R}^d)$, if $h_j^{-\beta_j} h \leq c_0 := \min_j \tilde{c}_j$, for all $j = 1, \ldots, d$. Now, $\theta(p_0) - \theta(p_1) = h \prod_{j=1}^{d} C_j \|g_j\|_\infty = h c_1$, and $d_{TV}(\mathbb{P}_0,\mathbb{P}_1) = h \prod_{j=1}^{d} \frac{h_j C_j \|g_j\|_\infty}{2} = h \prod_{j=1}^{d} h_j^\beta \tilde{c}_2$. Thus, solving the system

$$h_j^{-\beta_j} h = c_0, \quad c_2 h \prod_{j=1}^{d} h_j = \varepsilon,$$

yields $h_j = (h/c_0)^{1/\beta_j}$ and

$$h = \left( \frac{\varepsilon c_0^{\tilde{r}}}{c_2} \right)^{1/\tilde{r}},$$

where, $\tilde{r} = \sum_{j=1}^{d} \frac{1}{\beta_j}$, which establishes the claimed lower bound on $\omega_1(\varepsilon)$ for all small $\varepsilon > 0$.

The first part of (4.12) is trivial. To verify the bias condition, we note that for $p \in \mathcal{H}_{\beta,C}^{\leq \lambda}(\mathbb{R}^d)$, $H = \text{diag}(h_1, \ldots, h_d)$ and with the substitution $T(x) = H^{-1}(x - x^{(0)})$, $|DT^{-1}(u)| = |H| = \prod_{j=1}^{d} h_j$, we have

$$\left| \int_{\mathbb{R}^d} \ell_k(x)p(x) \, dx - p(x^{(0)}) \right| \leq \int_{\mathbb{R}^d} \prod_{j=1}^{d} |K(u_j)| \left| p(Hu + x^{(0)}) - p(x^{(0)}) \right| \, du$$

$$\leq \sum_{j=1}^{d} \int_{\mathbb{R}^d} |K(u_j)| C_j |h_j u_j|^{\beta_j} \, du = \sum_{j=1}^{d} C_j \tilde{c}_j h_j^{\beta_j}.$$
Thus, (4.12) holds with $k = d$, $s_j = 1$, $t_j = \beta_j$, $\tilde{C}_0 = \|K\|_\infty \lor \max_j (dC_j \bar{c}_j)$ and $\tilde{h}_0 = 1$.

The bias and variance computations for $\hat{\theta}_n$ are analogous to those of Section C.2.

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