THE MÖBIUS FUNCTION ON IMPLICATION SUBLATTICES OF A
BOOLEAN ALGEBRA

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Abstract. Let $B$ be a finite Boolean algebra. Let $A$ be the partial order of all implication
sublattices of $B$. We will compute the Möbius function on $A$ in two different ways.

1. Introduction

Let $B$ be a finite Boolean algebra.

Definition 1.1. An implication subalgebra of $B$ is a subset closed under $\rightarrow$.
An implication sublattice of $B$ is a subset closed under $\rightarrow$ and $\land$.

Implication algebras are developed in [1]. In this paper we wish to examine the poset
of implication sublattices of a finite Boolean algebra. By the general theory of implication
algebras we know that an implication sublattice is exactly a Boolean subalgebra of $[a, 1]$ for
some $a \in B$, and so we are really considering certain partial partitions of the atoms of
$B$.

Let $\mathcal{A}$ be the partial order of all implication sublattices of $B$ ordered by inclusion. Of
course $\mathcal{A}$ is a finite lattice.

Our interest is in understanding the Möbius function of the poset $\mathcal{A}$. We consider two
methods of finding it. Both methods use a closure operators that provide two ways of
closing an implication sublattice to a Boolean subalgebra.

2. Method One

We wish to compute the Möbius function on $\mathcal{A}$. So let $A_1$ and $A_2$ be two implication
sublattices of $B$. Since $A_2$ is a Boolean algebra and $A_1$ is an implication sublattice of $A_2$
we may assume that $A_2 = B$. Let $A = A_1$ and $a = \min A$.

First we define a closure operator on $\mathcal{A}$.

Definition 2.1. Let $A \in \mathcal{A}$. Let $A^c = \{ \overline{x} \mid x \in A \}$.
Let $\overline{A} = A \cup A^c$.

Lemma 2.2. If $A \in \mathcal{A}$ then $\overline{A}$ is a Boolean subalgebra of $B$.

Proof. Clearly $\overline{A}$ is closed under complements. As $1 \in A$ we have $1 \in \overline{A}$ and $0 \in \overline{A}$. It
suffices to show closure under joins.

If $x, y \in A$ then $x \lor y \in A$ and $x \land y \in A$ as $A$ is an implication lattice. Thus if $x, y \in A^c$
then $x \lor y = \overline{x \land y} \in A^c$.

If $x \in A$ and $y \in A^c$ then $x \lor y = x \lor \overline{y} = \overline{y} \rightarrow x \in A$ as $\overline{y} \in A$ and $A$ is $\rightarrow$-closed. \qed

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Lemma 2.3. Let $A \in \mathcal{A}$. Then $\overline{A} = A$ if $A$ is a Boolean subalgebra of $B$.

Proof. By the last lemma $\overline{A}$ is a Boolean subalgebra.

If $A$ is a Boolean subalgebra then $A^c \subseteq A$ so that $\overline{A} = A$. □

Lemma 2.4. $A \mapsto \overline{A}$ is a closure operator on $\mathcal{A}$.

Proof. Clearly $A \subseteq \overline{A}$. As $A$ is a Boolean subalgebra we have (by the last lemma) $A = \overline{A}$. □

Now we recall the closure theorem for Möbius functions – see [2] Proposition 2.1.19.

Theorem 2.5. Let $X$ be a locally finite partial order and $x \mapsto \overline{x}$ be a closure operator on $X$. Let $\overline{X}$ be the suborder of all closed elements of $X$ and $y, z$ be in $X$. Then

$$\sum_{x \in \overline{X}} \mu(y, x) = \begin{cases} \mu(x, \overline{y}) & \text{if } y \in \overline{X} \\ 0 & \text{otherwise} \end{cases}$$

Proof. See [2]. □

Lemma 2.6. Let $C \in \mathcal{A}$. Then $\overline{C} = B$ if $C = B$ or $C$ is an ultrafilter of $B$.

Proof. Suppose that $C$ is an ultrafilter. Then for any $x \in B$ we have $x \in C$ or $x \in C^c$ so that $x \in \overline{C}$. Thus $\overline{C} = B$.

Suppose that $\overline{C} = B$ and $C \neq B$. We first show that $C$ is upwards-closed. Indeed, if not, then there is some $x \in B$ and $b < x < 1$ with $b \in C$ and $x \notin C$. As $\overline{C} = B$ we have $\overline{x} \in C$ so that $\overline{x} \rightarrow b \in C$. But $\overline{x} \rightarrow b = \overline{x} \lor b = x \lor b = x$ – contradiction.

Thus $C$ is upwards-closed, and meet and join-closed, so $C$ is a filter. As $C \neq B$ we know that $0 \notin C$. Also $C \cup C^c = B$ so that for all $x \in C$ either $x \in C$ or $\overline{x} \in C$. Thus $C$ is an ultrafilter. □

This lemma together with the closure theorem allow us to use an induction argument to compute the Möbius function. The induction comes from the following lemma.

Lemma 2.7. Let $A \in \mathcal{A}$ and $a = \min A$. Let $c_1$ and $c_2$ be any atoms below $a$. Then

$$[A, [c_1, 1]]_{\mathcal{A}} \cong [A, [c_2, 1]]_{\mathcal{A}}.$$

Proof. Let $\tau$ be the permutation of the atoms of $B$ that exchanges $c_1$ and $c_2$. Then $\tau$ induces an automorphism of $B$ and that induces an automorphism of $\mathcal{A}$. It is clear that this induces the desired isomorphism between $[A, [c_1, 1]]_{\mathcal{A}}$ and $[A, [c_2, 1]]_{\mathcal{A}}$. □

Now suppose that $A \in \mathcal{A}$ and $a = \min A > 0$. Then we have

$$\mu(A, B) + \sum_{c \in \mathcal{B} \text{ atom}} \mu(A, [c, 1]) = 0$$

by the closure theorem and lemma 2.6. Thus

$$\mu(A, B) = - \sum_{c \in \mathcal{B} \text{ atom}} \mu(A, [c, 1])$$

$$= -|a| \mu(A, [c, 1])$$
where $c$ is any $B$-atom below $a$. $|a|$ is the rank of $a$ in $B$ and equals the number of atoms below $a$. As we now have a reduction in rank (of $a$ in $[c, 1]$) we see that we can proceed inductively to get

$$= -1^{\text{id}} |a|! \mu(A, [a, 1]).$$

So we are left with the case that $A$ is in fact a Boolean subalgebra of $B$. We note that in this case, if $C \in [A, B]_A$ then $C$ is also a Boolean subalgebra. We also note that any subalgebra is determined by its set of atoms and these form a partition of $n$. So the lattice of subalgebras of the Boolean algebra $2^n$ is isomorphic to the lattice of partitions of $n$ and the Möbius function of this is well known. This gives us the final result that

$$\mu(A, B) = (-1)^{|d + w(A) - w(B)|} |a|! \prod_{c \text{ is an } A\text{-atom}} (|c| - |a| - 1)!$$

where $w(A)$ is the number of atoms of $A$ and $|c|$ is the $B$-rank of $c$.

3. Method Two

Consider any implication sublattice $A$ of $B$. Then $A$ is a Boolean subalgebra of $[a = \min A, 1]$. This means we can take any extension $A \subseteq C \subseteq B$ and factor $C$ into $([a, 1] \cap C, [0, a] \cap C)$ and this pairing completely determines $C$.

It follows that the interval $[A, B]$ is isomorphic to a product of two partial orders:

$$P_1 = [C \mid C \text{ is a Boolean subalgebra of } [a, 1] \text{ containing } [a, 1] \cap A]$$

$$P_2 = [C \mid C \text{ is an implication sublattice of } [0, a]].$$

$P_1$ is known as a partition lattice. $P_2$ is essentially the same as the interval we are considering with the assumption that $A = [1]$.

So we will compute $\mu([1], B)$.

**Definition 3.1.** Let $C$ be any implication sublattice of $B$. Then

$$C \uparrow = \{x \mid \exists c \in C \ x \geq c\}$$

is the upwards-closure of $C$. Note that $C \uparrow = [\min c, 1]$.

**Lemma 3.2.** $C \mapsto C \uparrow$ is a closure operator and $C \uparrow = B$ iff $\min C = 0$.

**Proof.** This is immediate. □

It follows from this lemma that $C \uparrow = B$ iff $C$ is a Boolean subalgebra of $B$.

**Lemma 3.3.** Let $C_1$ and $C_2$ be two Boolean subalgebras of $B$. Then

$$[(1), C_1] \simeq [(1), C_2] \text{ iff } C_1 \simeq C_2.$$

**Proof.** The left-to-right direction is clear.

Conversely, if $1 > s_1 > s_2 > \cdots > s_i = 0$ is a maximal chain in $C_i$ then the set $\{[s_{ij}, 1] \mid 1 \leq j \leq j_i\}$ is a maximal chain in $[(1), C_i]$ -- since $[s_{i(j+1)}, 1]$ has one more atom than $[s_{ij}, 1]$.

Thus $j_1 = j_2$ and so $C_1 \simeq C_2$. □

We recall that there are $S_{n,k}$ Boolean subalgebras of $B$ that have $k$ atoms -- here $n$ is the number of atoms that $B$ has and $S_{n,k}$ is a Stirling number of the second kind, counting the number of partitions of $n$ into $k$ pieces.
Let $C_k$ be any Boolean subalgebra of $B$ with $k$ atoms. We can now apply the lemma and the closure theorem to see that

$$
\mu(\{1\} , B) = - \sum_{C \subseteq B} \mu(\{1\} , C) \\
= - \sum_{n > k \geq 1} S_{n,k} \mu(\{1\} , C_k) \\
= \sum_{\Gamma \text{ a chain in } [1,n]} (-1)^p S_{n_0,n_1} \cdots S_{n_{p-1},n_p} \mu(\{1\} , C_1) \\
= \sum_{\Gamma \text{ a chain in } [1,n]} (-1)^{p+1} S_{n_0,n_1} \cdots S_{n_{p-1},n_p} \\
$$

3.1. An Identity. We can put these two methods together to see that

$$
\mu(\{1\} , B) = (-1)^n n! = \sum_{\Gamma \text{ a chain in } [1,n]} (-1)^{p+1} S_{n_0,n_1} \cdots S_{n_{p-1},n_p}.
$$

3.2. Conclusion & Beyond. We see from the above results that the poset $\mathcal{A}$ is close to the poset of partitions of a set. The analysis we’ve undertaken shows this in two distinct ways – via the closure operators. In future work we plan to apply these results to an analysis of the subalgebras of cubic implication algebras.

References

[1] J. C. Abbott, Sets, Lattices, and Boolean Algebras, Allyn and Bacon, Boston, MA, 1969.
[2] E Spiegel and C. J. O’Donnell, Incidence Algebras, Marcel Dekker Inc., 1997.

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