On boundaries of geodesically complete CAT(0) spaces

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Abstract

We give concrete, “infinitesimal” conditions for a proper geodesically complete CAT(0) space to have semistable fundamental group at infinity.

1 Introduction

For a CAT(0) space $X$ there is a notion of boundary $\partial X$; details are discussed later in this paper, but see also [4]. If a group $G$ acts properly and cocompactly by isometries on $X$ then $G$ is called a CAT(0) group. If one strengthens the CAT(0) assumption to Gromov hyperbolicity it is well-known that the boundaries of any two such spaces $X$ on which $G$ acts must have boundaries that are homeomorphic, so there is a topologically well-defined notion of the boundary of a hyperbolic group. It was shown by Swarup ([17]) in 1996 that connected boundaries of Gromov Hyperbolic groups must be Peano continua (see also [2], [3]). In contrast, Croke-Kleiner ([6]) showed in 2000 that the same group may act properly and cocompactly on homeomorphic CAT(0) spaces with non-homomeomorphic boundaries, and a definitive statement on the topological structure of boundaries of one-ended (i.e. with connected boundary) CAT(0) spaces remains elusive. On the one hand, arbitrary metric compacta can be realized as boundaries of CAT(0) spaces (attributed to Gromov with a proof sketched in [11], Proposition 2). But if $X$ is a cocompact, proper CAT(0) space then there are the following known constraints: According to Swenson ([18]), $\partial X$ must be finite dimensional. According to Geoghagen-Ontandeda ([11]) if the dimension of $\partial X$ is $d$ then the $d$-dimensional Čech cohomology with integer coefficients is non-trivial. In the same paper, the authors show that cocompact proper CAT(0) spaces must be “almost geodesically complete” in a sense attributed to Michael Mihalik that extends the following notion of geodesically complete.
complete (also known as the geodesic extension property): Every geodesic extends to a geodesic defined for all \( \mathbb{R} \).

A natural candidate for a definitive general topological statement about boundaries of proper cocompact CAT(0) spaces is that they are always “pointed 1-moveable”, a concept from classical shape theory. The reason for this is that Geoghegan-Swenson ([9], Theorem 3.1) showed that a one ended proper CAT(0) space has semistable fundamental group at infinity if and only if the boundary is pointed 1-movable, and it is a long-standing open question whether proper, cocompact CAT(0) spaces all have semistable fundamental group at infinity (or simply are “semistable at infinity”). In the compact metric case, pointed 1-movable is equivalent to the notion of “weakly chained” introduced in [16]. The later is very simple to define, but we do not need definitions of any of these concepts here; rather we use Theorem 1 from [16], stated as Theorem 1 below, which only involves the following new definition from [16].

**Theorem 1** Let \( X \) be a proper, geodesically complete CAT(0) space with connected boundary and \( x_0 \in X \). Suppose there exist some \( K > 0 \) and a positive real function \( \iota \), called the refining increment, such that for all sufficiently large \( t \),

1. \( \lim_{s \to t^+} \iota(s) > 0 \) (in particular if \( \iota \) is lower semicontinuous from the right) and

2. if \( d(x, y) < \iota(t) \) and \( (x, y) \) is a sink in \( \Sigma_{x_0}(t) \) then \( x, y \) may be joined by a curve in \( X_t \cap B(x, K) \cap B(y, K) \).

Then \( \partial X \) is weakly chained (hence \( \partial X \) is pointed 1-movable and \( X \) is semistable at infinity).

We need the following notations to state our main application—we give more details later in the paper. The \( \pi \)-truncated metric of any metric space is the minimum of \( \pi \) and the original metric. For any \( x \) in a metric space \( X \), let \( S_x \) denote the space of directions at \( x \) with the angle \( \alpha \) as metric, the completion of which is known to be a CAT(1) space when \( X \) is CAT(\( k \)). A local cone in a CAT(\( k \)) space \( X \) is a closed metric ball \( C = B(o, \rho_o) \) for some \( o \in X \) called the apex, such that there is an isometry from \( C \) into the \( k \)-cone \( C_k(S_o) \) that takes \( o \) to the apex 0 of the \( k \)-cone. The number \( \rho_o > 0 \) is called the cone radius at \( o \). We say that a geodesic space \( X \) is (resp. uniformly) locally conical if every \( x \in X \) is the apex of a local cone \( B(x, \rho_x) \) (resp. and the cover \( C \) by the interiors of the local cones has a Lebesgue number \( \rho > 0 \)). Such a cover \( C \) is called a uniform cone cover of \( X \). It is easy to check that if \( X \) is a locally conical geodesic space then \( X \) is uniformly locally conical if either there is a positive lower bound on the cone radii at all points or \( X \) is cocompact.
Theorem 2  If $X$ is a proper, geodesically complete CAT(0) space with connected boundary and a uniform cone covering $C$ such that for all $o \in A_C$, the complement of every $\frac{\pi}{2}$-ball in $S_o$ is connected, then $\partial X$ is weakly chained. This condition on $S_o$ is in particular true when

1. $S_o$ (with the angle metric) is a geodesic space or
2. $S_o$ has no cut points and is itself locally conical with all cone radii at least $\frac{\pi}{2}$.

With a minor caveat, the geodesic completeness of $X$ is equivalent to $S_o$ being geodesically complete for all $o$ (Lemma 13). In other words, once $X$ is known to be proper, uniformly locally conical and having connected boundary, the hypotheses of Theorem 2 reduce entirely to “infinitesimal” questions about the space of directions at each apex.

If $K$ is an $M_k$-polyhedral complex, we always assume it has a specific geometric type or “shape” assigned to each cell, and the set $\text{Shapes}(K)$ of these shapes is finite. Bridson showed in his thesis (exposition in [4]) that when $\text{Shapes}(K)$ is finite, $K$ has a natural geodesic metric induced by this choice of shapes, and we will always take this metric on $K$. The space of directions at any point is a spherical ($k = 1$) polyhedral complex called the link $Lk(x, K)$. In fact, the angle metric on $Lk(x, K)$ is precisely the $\pi$-truncated metric of the induced geodesic metric when $Lk(x, K)$ is considered as a spherical polyhedral complex. Each vertex $v$ of $K$ is the apex of a local cone $B(v, \rho)$, where $\rho$ is at least the infimum of distances to any face not containing $v$ of a cell that does contain $v$. Every $x \in K$ is contained in a ball $B(x, \varepsilon)$ that is isometric to a ball of the same radius in some $B(v, \rho)$ with $v$ a vertex such that $B(x, \varepsilon)$ is a local cone, and this $\varepsilon$ has a positive uniform lower bound (Theorem I.7.39, Lemma I.7.54, [4]). That is, $K$ is uniformly locally conical. Now if $x \in B(v, \rho)$ where $v \neq x$, then $Lk(x, K)$ is isometric to the spherical suspension of $Lk(u, Lk(v, K))$, where $u$ is the direction of the geodesic from $v$ to $x$ (Lemma 7). Now suppose that $Lk(v, K)$ has at least two points and no free faces. By Lemma 13 $Lk(v, K)$ is geodesically complete, and since it is locally conical, $Lk(u, Lk(v, K))$ is also geodesically complete by Lemma 14. Moreover, by Lemma 13 and Lemma 9 the conditions of Theorem 2 are always satisfied for any non-vertex $x$ if they are satisfied for every vertex. Putting all of this together we obtain:

Theorem 3 Let $K$ be a CAT(0) Euclidean polyhedral complex with $\text{Shapes}(K)$ finite and connected boundary. If for each vertex $v$ in $K$, $Lk(v, K)$ has at least two points and no free faces, and the complements of all $\frac{\pi}{2}$-balls in $Lk(v, K)$ are connected, then $\partial K$ is weakly chained. In particular this is true when $Lk(v, K)$ satisfies condition (1) or (2) in Theorem 2 at each vertex $v$.

Example 4 The well-known torus complexes of Croke-Kleiner ([6]) are locally isometric to two or four Euclidean half-spaces glued along a line, so the link at every vertex consists geometrically of semicircles (length $\pi$) attached “in parallel” to one another at their endpoints. In particular, the links are geodesic.
spaces with the angle metric and have no free faces. Theorem 3 now provides an easy proof that the fundamental groups of torus complexes are semistable at infinity. This is a known result—we do not have a reference but Michael Mihalik explained that this can be read off from a presentation of the group using some of his earlier results on the subject. Indeed many of the results of this sort are based on assumptions about how the groups are presented, for example for Coxeter and Artin groups (15). In contrast, Theorem 3 requires no knowledge of a group acting on $K$ and in fact $X$ is not even required to be cocompact.

We say that a metrized CAT(0) polyhedral complex $K$ with $\text{Shapes}(K)$ finite satisfies Moussong’s condition if for every vertex $v$, every edge in $Lk(v, K)$ has length at least $\frac{\pi}{2}$. Spherical geometry then implies that $Lk(v, K)$ is locally conical at every vertex with cone radius at least $\frac{\pi}{2}$. In this case, Theorem 2 gives a purely combinatorial sufficient condition on the link at each vertex:

**Corollary 5** Let $K$ be a CAT(0) Euclidean polyhedral complex with $\text{Shapes}(K)$ finite and connected boundary that satisfies Moussong’s condition. If the link at each vertex has at least two points and no free faces or cut points, then $\partial K$ is weakly chained.

**Example 6** Moussong showed (13) that the Davis complex (7) of any Coxeter group has a non-positively curved metric satisfying Moussong’s condition. See also 5 for background on Coxeter groups. Mihalik showed in 1996 (13) that all Coxeter groups are semistable at infinity, but Corollary 5 gives an alternative proof when the link has no free faces or cut points. However, since any polyhedron may be the link of the Davis complex of a Coxeter group (see Lemma 7.2.2, 8), Corollary 5 does not apply in this way to all Coxeter groups.

**Example 7 (Geoghagen)** Take a unit square. Attach 16 unit squares around the boundary wrapping around twice (topologically attaching a Moebius band via its median circle to the boundary of the square). Add 32 unit squares wrapping twice around the new boundary. Continue this process to create an infinite square complex. Up to isometry there are only two kinds of vertices: “corner vertices”, which lie in seven squares and “side vertices”, which lie in six squares. Therefore this complex is uniformly locally conical. The link at any side vertex is isometric to three semicircles glued at their endpoints with the induced geodesic metric, which has no free faces and the complements of $\frac{\pi}{2}$-balls are connected. The link at the corner vertices is topologically the same, but geometrically consists of two segments of length $\frac{3\pi}{2}$ and one segment of length $\frac{\pi}{2}$. If $u$ is the point at the center of the latter segment then the complement of $B(u, \frac{\pi}{2})$ has two components. Note also that the induced geodesic metric and the angle metric on the link do not coincide at corner vertices; the angle metric is not geodesic. Moreover, Moussong’s condition is not satisfied, although none of the links has a cut point. The boundary of this space is the 2-adic solenoid, which is known to topologists to not be pointed 1-connected, and was alternatively shown in 16 to not be weakly chained. This shows that the condition about $\frac{\pi}{2}$-balls in Theorem 4 cannot simply be removed. Also note that the failure of Theorem
2 Cones and suspensions

We recall a couple of special cases of Berestovskii’s construction of metric cones and suspensions and establish some basic results for which we have no references. Let $S$ be a metric space with distance $\rho$ and $\pi$-truncated metric denoted by $\alpha$. The Euclidean cone $C_0(S)$ consists of $[0, \infty) \times S$ with all points of the form $(0, v)$ identified to a single point called the apex. We will denote the equivalence class of the point $(t, v)$ by $tv$, and the apex will be denoted by 0 or 0$^e$ depending on the situation. Note that with this notation $S$ is naturally identified with the set of all all $1v$ in $X$, which we will denote by 1$S$. However this identification is not generally an isometry and therefore we will distinguish notationally between $v$ and $v^e$. $S$ showed in 1983 that cones $C_kX$ for other curvatures $k$ are primarily interested in of all all $1$ $Z$, $\forall x, y$ $\in M$, the diameter of $X$ and $v$ in $S$. If $u(s)$ is a curve in $S$ then the curve $tu(s)$ in $X$ will be denoted simply by $tu$. $X$ is metrized analogously to how $\mathbb{R}^2$ is metrized as the cone of the unit circle with angle as metric. That is, for $s, t > 0$ and $v, w \in S$, $d(tv, sw)^2 = s^2 + t^2 - 2st \cos \alpha(v, w)$ (cones $C_kX$ for other curvatures $k$ use the corresponding cosine laws).

Let $M_k^2$ denote the 2-dimensional space form of constant curvature $k$. We are primarily interested in $k = 0, 1$, so $M_k^2$ is the plane and $M_k^2$ is the sphere of curvature $1$. The former has diameter $\infty$ and the latter has (intrinsic!) diameter $\pi$. Recall that a CAT($k$) space $X$ is a metric space such that if $d(x, y)$ is less than the diameter of $M_k^2$ then $x, y$ are joined by a geodesic and if a geodesic triangle has a comparison triangle in $M_k^2$ (no restriction for $k \leq 0$) then Alexandrov’s comparisons for curvature $\leq k$ hold (see [4] for more details). Berestovskii ([1]) showed in 1983 that $S$ is a CAT(1) space if and only if $C_0(S)$ is a CAT(0) space (and more strongly the same is true for $k$-cones for any $k \in \mathbb{R}$). We assume now that $S$ is a CAT(1) space and review a few facts about geodesics in $C_0S$. Suppose that $u : [0, K] \to S$ is an arclength parameterized geodesic from $v$ to $w$ in $S$ of length $K < \pi$. For simplicity we consider the constant map to be a geodesic from $v$ to $v$. By definition of the metric, the function $f_\alpha(s, t) = su(t)$ is an isometry from a Euclidean sector $E(v, w)$ of angle $K$ in the plane parameterized with polar coordinates, to the set $Z(v, w) = \{su(t) : 0 \leq s \leq \infty, 0 \leq t \leq K\}$. Therefore the curves in $Z(v, w)$ corresponding to line segments in the Euclidean sector are geodesics in $Z$.

A geodesic in the case $v = w$ is called a radial geodesic, i.e of the form $\gamma_w(t) = tw$, $0 \leq t < \infty$. Now suppose that $\gamma$ is a geodesic from $tv$ to $sw$ that does not meet the apex. Since geodesics are unique, the concatenation of the radial geodesics between the apex and $tv$ and $sw$ cannot be a geodesic and we conclude that $\alpha(v, w) < \pi$. This means that $\gamma$ corresponds to a line in $E(v, w)$.

When $\alpha(v, w) = \pi$, for any $0 < r_1, r_2$, by definition $d(r_1v, r_2w) = |r_1 - r_2|$ and therefore the concatenation of the radial geodesic from $r_1v$ to 0 with the
radial geodesic from 0 to \( r_2 w \) is a geodesic from \( r_1 v \) to \( r_2 w \). All of these geodesics in \( X \) are unique since \( X \) is a CAT(0) space.

**Example 8** The Euclidean cone of an arbitrary metric space \( S \) is “sink-free” in the sense that it has no sinks (\cite{10}). In fact one can move any pair of points towards the apex, strictly decreasing the distance between them (and any non-apex point may similarly move towards the apex). Such cones need not be locally path connected (e.g. when \( S \) is a Cantor set).

The spherical suspension \( \Sigma S \) of a metric space \( S \) is defined analogously using the \( \pi \)-truncated metric, taking the product of the space with \([0, \pi]\), identifying each \( 0 \times S \) and \( \pi \times S \) with points \( \bar{0} \) and \( \bar{\pi} \), respectively. The space is metrized using the spherical cosine law, i.e. as \( S^2 \) as metrized as the suspension of a circle of length \( 2\pi \). For \( u \in S \) and \( \theta \in (0, \pi) \) we denote the point corresponding to the ordered pair \((u, \theta)\) by \( u_{\theta} \).

**Lemma 9** If \( S \) is a CAT(1) space then the spaces of directions at a point \( t_0 v \in C_0 S \) is isometric to \( S \) when \( t_0 = 0 \). If \( t_0 > 0 \) then \( S_{t_0 v} \) is isometric to the spherical suspension of \( S_v \), and if \( B(v, \rho) \) is a local cone in \( S \) then \( B(t_0 v, \rho) \) is a local cone in \( C_0 S \).

**Sketch of Proof.** The case when \( t_0 = 0 \) simply follows from the definition of the cone metric. Suppose \( t_0 > 0 \) and let \( \Gamma_0 \) denote the segment of the radial geodesic of \( v \) outward from \( t_0 v \). Suppose that \( \gamma \) is a geodesic in \( S \) starting at \( v \). By definition of the cone metric, every geodesic in \( C_0 S \) starting at \( t_0 v \) in the same sector as \( \gamma \) is uniquely determined by the angle \( \theta \) between it and \( \Gamma_0 \); we will denote any such geodesic by \( \gamma_{\theta} \). This identifies \( S_{t_0 v} \) (as a set) with the spherical suspension of \( S_v \), with \( \bar{0} \) corresponding to the direction of \( \Gamma_0 \).

Suppose that \( \gamma^1_{\theta_1} \) and \( \gamma^2_{\theta_2} \) are geodesics starting at \( t_0 v \), with directions \( v_1, v_2 \), respectively at \( t_0 v \). The cases when \( \theta \) equal 0 or \( \pi \) are trivial; suppose \( 0 < \theta_1, \theta_2 < \pi \). Let \( \tilde{w}_i \) be unit vectors in the \((x, y)\) plane with \( \langle \tilde{w}_1, \tilde{w}_2 \rangle = \cos \angle(\tilde{w}_1, \tilde{w}_2) = \cos \angle(\gamma^1, \gamma^2) \). Let \( \overline{\tilde{w}_i} \) be unit vectors whose orthogonal projections onto the \((x, y)\)-plane are parallel to \( \tilde{w}_i \) and \( \alpha(\overline{\tilde{w}_i}, \tilde{w}_i) = \theta_i \). Writing \( \tilde{w}_i = \overline{\tilde{w}_i} + (\overline{\tilde{w}_i} - \tilde{w}_i) \) and cancelling orthogonal terms we have

\[
\langle v_1, v_2 \rangle = \langle \tilde{w}_1, \tilde{w}_2 \rangle + \langle \overline{\tilde{w}_1} - \overline{\tilde{w}_2}, \overline{\tilde{w}_2} - \overline{\tilde{w}_1} \rangle = \langle \tilde{w}_1, \tilde{w}_2 \rangle \pm \| \overline{\tilde{w}_1} - \overline{\tilde{w}_2} \| \| \overline{\tilde{w}_2} - \overline{\tilde{w}_1} \| (1)
\]

with the sign of the last term depending on whether \( \theta \) are on the same side of \( \pi \) (in which case it is +). This shows that \( \cos \angle(\overline{\tilde{w}_1}, \overline{\tilde{w}_2}) \), which is what we need to show is equal to \( \cos \alpha(\gamma_{\theta_1}, \gamma_{\theta_2}) \) may be calculated using only \( \langle \tilde{w}_1, \tilde{w}_2 \rangle \) and lengths of vectors in the \( z \)-direction. Now

\[
\cos \angle(\gamma^1_{\theta_1}, \gamma^2_{\theta_2}) = \lim_{t \rightarrow 0} - \frac{d(\gamma^1_{\theta_1}(t), \gamma^2_{\theta_2}(t))^2}{2t^2}.
\]

By definition of the cone metric, the right term may be computed from \( d(\gamma^1(t), \gamma^2(t)) \) using the Euclidean Formula\( \square \) and

\[
\lim_{t \rightarrow 0} - \frac{d(\gamma^1(t), \gamma^2(t))^2}{2t^2} = \cos \angle(\gamma^1, \gamma^2) = \langle \tilde{w}_1, \tilde{w}_2 \rangle.
\]
This shows the first part of the lemma. For the second part note that if $B(v, \rho)$ is a local cone in $S$ then for all $0 < t \leq \rho$,

$$d(\gamma^1(t), \gamma^2(t))^2 = 2t^2(1 - \cos \angle(\gamma^1, \gamma^2)).$$

Therefore the above limit is constant and we see that

$$d(\gamma^1_0(t), \gamma^2_0(t))^2 = 2t^2(1 - \cos \angle(\gamma^1_0, \gamma^2_0)),$$

i.e. $B(t_0v, \rho)$ is a local cone in $C_0S$.

**Lemma 10 (Radial Geodesics Don’t Bifurcate)** Let $S$ be a CAT(1) space. If $\gamma$ is a geodesic in $C_0S$ that intersects a radial geodesic $\beta$ in more than one point then $\gamma$ is a (possibly infinite) segment of $\beta$.

**Proof.** Let $\beta(t) = tu$ for some $u \in S$. By assumption, $\gamma$ contains two points $t_1u, t_2u$ with $0 \leq t_1 < t_2$. By uniqueness, $\gamma = \beta$ between those two points. Now suppose that another point $sw$ lies on $\gamma$ with $w \neq u$ and $s > t_2$. Then the segment of $\gamma$ from $t_1u$ to $sw$ lies in $Z(u, w)$ and the segment of $\gamma$ from $t_1u$ to $t_2u$ also lies in $Z(u, w)$. But his means that one segment of $\gamma$ in $Z(u, w)$ is radial and another segment is not, which is impossible in Euclidean geometry. The proof for $s < t_1$ is similar, showing that $\gamma(t) = tu$ where ever it is defined.

**Remark 11** One known consequence of the above lemma is that there is the continuous “radial retraction” from any $B(0, \rho) \setminus \{0\}$ onto the sphere $\Sigma_0(\rho)$, which just takes every $tv$ with $0 < t \leq \rho$ to $\rho \cdot v$.

**Definition 12** Let $X$ be a metric space. A curve $c$ in $X$ is called a local geodesic if the restriction of $c$ to any sufficiently small closed interval is a geodesic (so in a Riemannian manifold this would simply be what is normally referred to as a “geodesic”). $X$ is called geodesically complete if for every non-constant geodesic $\gamma : [a, b] \to X$ there is some $\varepsilon > 0$ such that $\gamma$ extends to a curve $\gamma^\varepsilon : [a, b + \varepsilon] \to X$ such that the restriction of $\gamma^\varepsilon$ to $[b - \varepsilon, b + \varepsilon]$ is a geodesic.

Note that if $X$ is a complete metric space then $X$ is geodesically complete if and only if every local geodesic extends to a local geodesic defined on all of $\mathbb{R}$.

**Lemma 13** If $S$ is a complete CAT(1) space with at least two points then the following are equivalent:

1. $X = C_0(S)$ is geodesically complete.
2. $S$ is geodesically complete.
3. Every non-trivial geodesic in $S$ extends to a geodesic of length at least $\pi$.  

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Proof. Since there is a 1-1 correspondence between non-radial geodesics in X and geodesics in S, if X is geodesically complete then so is S. Suppose S is geodesically complete, u ∈ S and γ is a non-trivial geodesic in S starting at u. Then γ extends as a local geodesic of length π to a point v. We claim that d(u, v) = π, which means that this extension is in fact a geodesic, completing the proof of the third part. We assume that γ : [0, π] → X is parameterized by arclength. Consider the following statement S(t) : d(u, γ(t)) = t. Since γ is a local geodesic, S(t) is true for small positive t. By continuity of the distance function, if S(s) is true for all s ≤ t then S(t) is true. Therefore we need only show that if S(t) is true for some t < π then S(t + ε) is true for some ε > 0. Since γ is a local geodesic, there is some ε > 0 such that the restrictions of γ to [t, t + ε] and [t − ε, t + ε] are geodesics. The first of these statements implies that the angle between the reversal of γ starting at γ(t) and the restriction of γ starting at γ(t) is π. Since both segments are bona fide geodesics we may apply the CAT(1) condition to conclude that d(u, γ(t + ε)) = t + ε, completing the proof of 2 → 3.

If the third part is true then by the 1-1 correspondence mentioned above, every non-radial geodesic in X extends to a geodesic defined on ℝ, and all radial geodesics by definition extend outwards from the apex. The only remaining question is whether the reversal any radial geodesic γ₀ extends through the apex. Since S has at least two points there is some v ̸= u in S. If α(u, v) = π then ∠(γ₀, γ₀) = π and γ₀ extends π₀ as a geodesic beyond the apex. If α(u, v) < π then u and v are joined by a geodesic in S, which extends to a geodesic to some v such that α(u, v) = π, completing the proof. ■

Corollary 14 If S is a geodesically complete CAT(1) space then the spherical suspension ΣS of S is geodesically complete and the complement of every π/2-ball in ΣS is (path) connected.

Proof. That ΣS is geodesically complete follows from Lemma 13 and the definition of the spherical suspension metric. Let v₀ ∈ ΣS; without loss of generality suppose θ ≤ π/2. We will show that if w₁ ̸= π with d(w₁, v₀) ≥ π/2 then there is a path from w₁ to π that stays outside B(v₀, π/2). If v = w then by definition of the spherical suspension metric there is a uniquely determined isometric circle determined by v = w and we may move in either direction from w₁ (depending on whether θ > π) to π, staying outside B(v₀, π/2). If w ̸= v then the geodesic from v to w extends to length π. Therefore we may move from w₁ away from v₀ along the corresponding geodesic to the “antipodal point” of v₀ and proceed as in the first step. ■

3 Mₖ-polyhedral complexes

Recall that a free face in an Mₖ-polyhedral complex is a face that lies in exactly one cell of higher dimension. The proof of the next lemma involves Proposition II.5.10, [4] (and uses some similar arguments), which states that an Mₖ-polyhedral complex with curvature bounded above and finite shapes is geodesi-
cally complete if and only if it has no free faces. This statement is not quite correct according to the traditional definitions because discrete complexes are geodesically complete (there are no non-trivial local geodesics), and even if one considers the empty set as a face of dimension $-1$, strictly speaking it is free if and only if the complex consists of exactly one vertex. And for example discrete spherical complexes of curvature $\leq 1$ occur as the space of directions in 1-dimensional Euclidean complexes. The proof of Proposition II.5.10 is by induction on dimension starting with $n = 0$, which is precisely when the statement is not true. But this minor issue is easily solved by starting with $n = 1$ and handling the discrete case as a special case.

**Lemma 15** If $K$ is an $M_k$-polyhedral complex with non-positive curvature and Shapes($K$) finite then $K$ is geodesically complete if and only if the link at each vertex has at least two vertices and is either discrete or has no free faces.

**Proof.** Since $K$ has non-positive curvature, the space of directions at each point, hence the link at each vertex $v$ is a CAT(1) space. Suppose $X$ is not geodesically complete. Since $X$ has non-positive curvature it is connected and hence this is equivalent to having a free face $F$. If $F$ is a vertex then at that vertex the link is a single point. If is $F$ higher dimensional then let $v$ be any vertex of $F$ and $E_1$ be an edge containing $v$, not contained $F$ but contained in the unique higher dimensional cell containing $F$. Let $E_2$ be an edge in $F$ containing $v$. Then the geodesic in $Lk(v, K)$ from the directions $u_1, u_2$ corresponding to $E_1, E_2$, respectively, cannot be extended beyond $u_2$. That is, $Lk(v, K)$ is not geodesically complete, and since it is not discrete it has a free face.

Conversely, if $Lk(v, K)$ is a single vertex for some vertex $v$ in $K$ then there is an edge in that direction. Moreover, there is no edge having angle $\pi$ with that edge, so the edge cannot be extended as a geodesic past $v$. That is, $K$ is not geodesically complete. Finally, suppose there is some vertex $v$ such that $Lk(v, K)$ is not discrete and has a free face. Since $Lk(v, K)$ is not discrete, it is not geodesically complete. But some $B(v, \varepsilon)$ is isometric to $B(0, \varepsilon)$ in $C_0Lk(v, K)$ and the proof is finished by Lemma 13.

**Example 16** Let $S$ be a complete $\pi$-geodesic space of diameter $\pi$ (e.g. a CAT(1) space with $\pi$-truncated metric $\alpha$). Then $S$ is a length space if and only if $S$ is sink-free. Necessity follows from Example 28 in \[12\]. For the converse, we need only consider the case $d(x, y) = \pi$. If $(x, y)$ is not a sink, there exist points $x', y'$ arbitrarily close to $x, y$ such that $d(x', y') < \pi$ and since $\alpha$ is a $\pi$-geodesic metric, $x', y'$ are joined by a geodesic. Then a midpoint between $x', y'$ is an “almost midpoint” for $x, y$, which is classically known to be sufficient to show that $X$ is a length space (cf. \[12\], Proposition 7 for an exposition). In particular, if $S$ is compact then $S$ is geodesic if and only if it is sink-free.

**Proposition 17** Suppose $X$ is a geodesically complete CAT(0) space, $x_0 \in X$, $r > 0$, and $(x, y)$ is a sink in $\Sigma_{x_0}(r)$. In addition, suppose that $x, y$ lie in a local cone with apex $\alpha$ and cone radius $\rho > 0$. Then
1. the geodesics $\gamma_x, \gamma_y$ coincide up to $o$,
2. $d(x, o) = d(y, o) = \frac{d(x, y)}{2}$ and
3. $\rho \geq \frac{d(x, y)}{2}$.

**Proof.** Let $\delta := d(x, y)$, $\beta = \gamma_{xy}$ and $\gamma^x, \gamma^y$ be geodesics starting at $x, y$, respectively, that extend $\gamma_x, \gamma_y$ to geodesic rays. Suppose first that $\angle(\gamma^x, \beta) < \pi$. Since $S_x$ is $\pi$-geodesic, there is a geodesic $\xi$ starting at $x$ such that $\angle(\gamma^x, \xi), \angle(\xi, \beta) < \frac{\pi}{2}$. By the triangle inequality in $S_x$, $\angle(\gamma^x, \xi) > \frac{\pi}{2}$ and so by the CAT(0) condition all points on $\xi$ lie strictly outside $\Sigma_{x_0}(r)$. Since $\angle(\xi, \beta) < \frac{\pi}{2}$, by the “single-sided limit” method to measure angles (c.f. Proposition 3.5 in [4]), any point $x'$ sufficiently close to $x$ on $\xi$ satisfies $d(x', y) < d(x, y)$. In the plane, consider the comparison triangle with corners $X_0, X', Y$ corresponding to the one determined by $x_0, x', y$. Since

$$d(X_0, X') = d(x_0, x') > r = d(x_0, y) = d(X_0, Y),$$

by elementary geometry, if $Z$ is the point on the segment $X_0X'$ with $d(X_0, Z) = r$, $d(Z, Y) < d(X', Y)$. By the CAT(0) condition, if $z$ is the projection of $x'$ onto $\Sigma_{x_0}(r)$ then $d(z, y) \leq d(Z, Y) < d(X', Y) = d(x', y)$. That is, $z$ is arbitrarily close to $x$ but $d(z, y) < d(x, y)$. By definition, $(x, y)$ is not a sink in $\Sigma_{x_0}(r)$, a contradiction.

Therefore we may assume that

$$\angle(\gamma^x, \beta) = \angle(\gamma^y, \beta) = \pi$$

and by Equation 2 $\gamma := \gamma^x * \beta * \gamma^y$ is a geodesic. Assume first that $o$ does not lie on $\gamma$ and consider the Euclidean sector $E$ determined by $\gamma$ in the local cone. At $x$ and $y$ let $\beta_x$ and $\beta_y$ be geodesics corresponding to lines in $E$ that are perpendicular the line corresponding to $\gamma$ with the same orientation. That is, moving an arbitrarily small but equal amount along $\beta_x$ and $\beta_y$ to points $x', y'$ we have $d(x', y') = d(x, y)$. Since $\angle(\gamma^x, \beta_x) = \pi$ and $\angle(\gamma^x, \beta_x) = \frac{\pi}{2}$, by the triangle inequality $\angle(\beta_x, \gamma^x) \geq \frac{\pi}{2}$, and similarly $\angle(\beta_y, \gamma^y) \geq \frac{\pi}{2}$. By the CAT(0) condition, $d(x_0, x'), d(x_0, y') > r$. Projecting $x', y'$ onto $\Sigma_{x_0}(r)$ strictly reduces $d(x', y') = d(x, y)$, showing $(x, y)$ is not a sink in $\Sigma_{x_0}(r)$, a contradiction.

Therefore $o$ must lie on $\gamma$, hence on $\gamma^x, \beta, \text{ or } \gamma^y$. Suppose $o$ lies on $\gamma^x$ and $o \neq x$. Then $\gamma_{ox} * \beta$ is radial and $\gamma_{ox} * \gamma^x$ is radial as long as it lies in the local cone at $o$. Since $y$ is also in the local cone, $\gamma_{ox} * \gamma^x$ lies in the local cone for at least length $d(x, y)$ beyond $x$. By the triangle inequality, the midpoint $m$ of $\beta$ satisfies $d(x, m) \geq r - \frac{d(x, y)}{2}$. Since $\gamma^x$ is a geodesic to $x_0$ and $d(x_0, y) = r$, $\gamma^x$ and $\beta$ cannot coincide for the entire length of $\beta$. By definition this means that the radial geodesic $\gamma_{ox} * \beta$ bifurcates inside the local cone, a contradiction to Lemma 10.

Suppose that $o = x$. Then $\beta * \gamma^y$ is a radial geodesic that coincides with $\gamma_y * \gamma^y$ on $\gamma^y$. But the former does not pass through $x$ and hence the radial geodesic $\gamma_y * \gamma^y$ bifurcates somewhere along $\beta$, which is in the local cone, a contradiction. Similarly $o$ does not lie on $\gamma^y$. 

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Next suppose that \( o \) lies on \( \beta \), so \( \gamma_{ox} \) is radial. By a similar argument to what we have used above, the fact that radial geodesics do not bifurcate implies that \( \gamma_x \) must coincide with \( \gamma_{ox} \) from \( o \) to \( x \). By symmetry, \( \gamma_y \) must coincide with \( \gamma_{oy} \) from \( o \) to \( y \) and \( o = m \). Since \( \beta \) is a geodesic, \( d(x, o) = d(y, o) = \frac{d(x, y)}{2} \).

Since \( x \) and \( y \) lie in the local cone and \( o \) is the apex, \( \rho \geq \frac{d(x, y)}{2} \). ■

The next lemma is probably known but we do not have a reference. It is useful because it allows us to replace a given cover by metric balls with a cover by metric balls of uniformly large size.

**Lemma 18** Let \( \mathcal{C} = \{B(x, r_x)\}_{x \in X} \) be a covering of a geodesic space \( X \) by metric balls. If \( \mathcal{U} \) has a Lebesgue number \( \lambda > 0 \) then there is a subcovering \( \mathcal{C}' = \{B(x, r_x)\}_{x \in X'} \) of \( X \) with the following properties:

1. For all \( \alpha \in \Lambda \), \( r_{\alpha} \geq \frac{1}{4} \) also covers \( X \).
2. \( \mathcal{C}' \) has Lebesgue number \( \frac{1}{2} \).

**Proof.** Without loss of generality, if \( X \) has finite diameter then we can assume that \( \lambda \) is less than the diameter \( D \) of \( X \); if \( X \) is unbounded let \( D > \lambda \) be arbitrary. In either case, since \( X \) is a geodesic space, there exist \( a, b \) such that \( d(a, b) = D \). Then for any \( x \in X \) \( d(x, a) \geq \frac{D}{2} \) or \( d(x, b) \geq \frac{D}{2} \). Moving along the geodesic \( \gamma_xa \) or \( \gammaxb \) there is some point \( y \in X \) such that \( d(x, y) = \frac{\lambda}{2} < \lambda \).

Therefore there is some \( B(x, r_x) \) containing both \( x \) and \( y \). But since \( d(x, y) = \frac{\lambda}{2} \) by the triangle inequality, \( \frac{\lambda}{2} \leq 2r_{\alpha} \), showing that the balls of radius \( \frac{\lambda}{2} \) in \( \mathcal{C} \) cover \( X \). Now suppose that \( d(x, y) < \frac{\lambda}{2} \). If \( z \) is the midpoint of a geodesic from \( x \) to \( y \), by what we have just shown there is some \( B(x, r_x) \) containing \( z \) with \( r_{\alpha} \geq \frac{\lambda}{4} \), and by this ball must contain both \( x \) and \( y \). ■

**Proof of Theorem 2** Let \( \rho \) be a Lebesgue number for a uniform cone cover of \( X \). By Lemma 18 we can assume that the cone radii for the local cones are all at least \( \frac{\lambda}{2} \). For every \( t > 0 \) let

\[
\iota(t) := 2 \left( t - \sqrt{t^2 - \frac{\rho^2}{4}} \right) > 0
\]

and assume that \( t \) is large enough that \( \iota(t) < \rho \). Since \( \iota \) is positive and continuous we need only verify Theorem 2 for any \( t \). Suppose that \( d(x, y) < \iota(t) \) and \( (x, y) \) is a sink in \( \Sigma_{x_0}(t) \). Since \( d(x, y) < \rho \), \( x, y \) lie in a local cone with vertex \( o \).

Since \( (x, y) \) is a sink, by Proposition \( \gamma_x, \gamma_y \) coincide up to \( o \) and \( d(o, x) = d(o, y) = \frac{d(x, y)}{2} < t - \sqrt{t^2 - \frac{\rho^2}{4}} \). We also have

\[
d(x_0, o) = t - \frac{\iota(t)}{2} = \sqrt{t^2 - \frac{\rho^2}{4}}.
\]

Moreover, in \( S_o \), \( \alpha(\gamma_{ox}, \overrightarrow{o}) = \alpha(\gamma_{oy}, \overrightarrow{o}) = \pi \geq \frac{\pi}{2} \). By assumption there is a curve \( c \) in \( S_o \) from \( \gamma_{ox} \) to \( \gamma_{oy} \) such that for all \( q, \alpha(\overrightarrow{q}, c(q)) \geq \frac{\pi}{2} \). Since radial geodesics do not bifurcate, and the cone radius is \( \frac{\rho}{2} \), there is a point \( x' \) on the
unique (inside the cone) extension of $\gamma_{ox}$ of distance $\frac{\rho}{2}$ from $o$, and an analogous point $y'$. Now the curve $\frac{\rho}{2} \cdot c$ is defined from $x'$ to $y'$. By the CAT(0) inequality, for every $s$, by Equation (3)

$$d\left(\frac{\rho}{2} \cdot c(s), x_0\right)^2 \geq d(x_0, o)^2 + \frac{\rho^2}{4} = t^2.$$ 

Therefore the projection $\tilde{c}$ of $\frac{\rho}{2} \cdot c$ onto $\Sigma_{x_0}(t)$ is defined and joins $x$ and $y$. Moreover, since $\tilde{c}$ remains inside $B(o, \frac{\rho}{2})$ by the triangle inequality it remains inside $B(x, \rho) \cap B(y, \rho)$ and we may take $K = \rho$ to finish the proof of the first part of the theorem.

Suppose that $S_o$ with the angle metric is a geodesic space. Suppose that $a, b, c \in S_o$ with $a, b \notin B(c, \frac{\pi}{2})$. Since $X$ is geodesically complete and locally conical, Lemma 13 implies that $S_o$ is also geodesically complete. By the same lemma, the geodesics $\gamma_{ca}$ and $\gamma_{cb}$ extend to geodesics of length $\pi$ to points $a', b'$, respectively. Since $d(a', b') \leq \pi$, any geodesic from $a'$ to $b'$ must remain outside $B(c, \frac{\pi}{2})$ by the triangle inequality. The extensions of $\gamma_{ca}$ and $\gamma_{cb}$ also remain outside $B(c, \frac{\pi}{2})$ and therefore there is a path from $a$ to $a'$ then $a'$ to $b'$ then $b'$ to $b$ that stays outside $B(c, \frac{\pi}{2})$.

Now suppose $S_o$ has no cut points and is locally conical with all cone radii at least $\frac{\pi}{2}$. Suppose $a, b, c$ are as in the previous paragraph. Since $c$ is not a cut point, the complement of $c$ is connected, hence path connected. That is, there is a curve from $a$ to $b$ that misses $c$. Now any segment of $c$ that enters $B(c, \frac{\pi}{2})$ can be homotoped onto $\Sigma_c(\frac{\pi}{2})$ using the radial retraction (Remark 11), resulting in a curve from $a$ to $b$ that stays outside $B(c, \frac{\pi}{2})$. \qed

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References

[1] Berestovskiı, V.; Borsuk’s problem on metrization of a polyhedron, Dokl. Akad. Nauk. SSSR 27 (1983), no. 1, 56-59.

[2] Bestvina, Mladen and Mess, Geoffrey; The boundary of negatively curved groups. J. Amer. Math. Soc. 4 (1991), no. 3, 469–481.

[3] Bowditch, B. H. Connectedness properties of limit sets. Trans. Amer. Math. Soc. 351 (1999), no. 9, 3673–3686.

[4] Bridson, Martin and Haefliger, Andr´e, Metric Spaces of Non-positive curvature, Grundlehren der Mathematischen Wissenschaften, 319, Springer-Verlag, Berlin, 1999.

[5] Conner, G.; Mihalik, M.; Tschantz, S. Homotopy of ends and boundaries of CAT(0) groups. Geom. Dedicata 120 (2006), 1–17.
[6] Croke, C; Kleiner, B. Spaces with nonpositive curvature and their ideal boundaries. Topology 39 (2000), no. 3, 549–556.

[7] Davis, M. Buildings are CAT(0), Geometry and Cohomology in Group Theory, LMS Lecture Note Series 252, Cambridge U. Press, Cambridge 1988.

[8] Davis, M. The Geometry and Topology of Coxeter Groups, Princeton University Press 2007.

[9] Geoghegan, R.; Swenson, E., On semistability of CAT(0) groups. Groups Geom. Dyn. 13 (2019), no. 2, 695–705.

[10] Geoghegan, R., Topological methods in group theory, Graduate Texts in Mathematics, vol. 243, Springer, New York, 2008.

[11] Geoghegan, R.; Ontaneda, P. Boundaries of cocompact proper CAT(0) spaces. Topology 46 (2007), no. 2, 129–137.

[12] Mihalik, M. Semistability at the end of a group extension. Trans. Amer. Math. Soc. 277 (1983), no. 1, 307–321.

[13] Mihalik, M. Semistability of Artin and Coxeter groups. J. Pure Appl. Algebra 111 (1996), no. 1-3, 205–211.

[14] Moussong, G. PhD thesis, Ohio State University 1988.

[15] Plaut, C. Metric spaces of curvature $\geq k$. Handbook of geometric topology, 819–898, North-Holland, Amsterdam, 2002.

[16] Plaut, C. Weakly chained spaces, preprint.

[17] Swarup, G. A. On the cut point conjecture. Electron. Res. Announc. Amer. Math. Soc. 2 (1996), no. 2, 98–100.

[18] Swenson, E. A cut point theorem for CAT(0) groups. J. Differential Geom. 53 (1999), no. 2, 327–358.