Comment on “How (not) to renormalize integral equations with singular potentials in effective field theory”

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[I critically discuss two of the potential inconsistencies pointed out in the recent manuscript by Epelbaum, Gasparyan, Gegelia and Meißner, published in Eur. Phys. J. A54, 186 (2018); the conclusion is that these inconsistencies do not happen.]

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In a recent manuscript Epelbaum et al. [1] discuss non-perturbative renormalization within effective field theory (EFT). The term refers mostly to renormalization as applied in a purely non-perturbative context, but tangentially includes the mixture of perturbative and non-perturbative renormalization advocated, for instance, by Nogga et al. [2]. The idea is that for really hard cut-offs this type of renormalization will lead to a series of (apparent) inconsistencies. Two possible inconsistencies discussed in detail are: (i) a possible mismatch between the $h$ expansion of the scattering amplitude and renormalization, (ii) an impossibility of non-perturbative renormalization to deal with repulsive singular interactions. Here I review these inconsistencies.

First, I will discuss the $h$ mismatch. It can be explained as follows: we consider a scattering amplitude $T$ in a theory with a cutoff $\Lambda$. The amplitude $T$ is generated from the iteration of the leading order potential $V = C_0 + C_2(p^2 + p'^2)$. $T$ has a well-defined $\Lambda \to \infty$ limit, from which we are tempted to conclude that it has been renormalized in agreement with EFT principles. But when we consider its $h$ expansion, i.e. $T = T^{[0]} + hT^{[1]} + O(h^2)$, a closer inspection reveals that $T^{[1]}$ contains a linear divergence. That is, according to Ref. [1] the amplitude $T$ has not been renormalized in agreement with the EFT principles (notice that this is controversial because $T^{[1]}$ is not observable, but let us concede that there is an inconsistency in what follows). The diagnosis of Ref. [1] is that the problem lies in a misguided insistence in taking the $\Lambda \to \infty$ limit, which is not necessary within the EFT framework and can potentially lead to more harm than good. Though the inconsistency might indeed be there, this diagnosis is incorrect owing to a subtlety in the renormalization process that is regularly ignored in cutoff regularization (because, except for formal settings as this one, it is generally of no consequence). This subtlety is the existence of two contributions to the EFT counterterms, one that contains physical information and one that doesn’t. I refer to the second type of contribution as redundant counterterms (RC), which are discussed in Ref. [3]. The RCs are there to cancel the residual cutoff dependence, but can be ignored in practice when $\Lambda \to \infty$. RCs are however regularly included in EFTs using power divergence subtraction (PDS) regularization [4], for instance. In cutoff EFT a trivial way to remove the divergences in the $h$ expansion is to explicitly include the RCs, which for the $T$ matrix discussed here take the form $V^R = \sum_{n\geq 2} C_n^R(p^{2n} + p'^{2n})$, with $C_n^R = \sum_k h^k C_{n+k}^R$. The number of RCs is infinite, but this is inconsequential because they are not observable and carry no physical information; instead, they are an analysis tool. Concrete calculations for this redundant potential are easy to perform with PDS, where explicit solutions exist for the $C_n^R$ and where it can be explicitly shown that no positive power of $\Lambda_{\text{PDS}}$ (i.e. the PDS cutoff) appear in the $h$ expansion of $T$. With a sharp cutoff in momentum space, calculations are more involved and cannot be solved in closed form, yet it is trivial to check that $C_4^{[1]}$ removes the $h$ inconsistency.

Second, I will discuss the non-perturbative renormalization of the interesting toy model of Epelbaum et al. [1], which contains a repulsive singular interaction at leading order. The potential in this toy model contains a long- and short-range piece, $V(r) = V_L + V_S$, with $V_L$ singular and repulsive and $V_S$ attractive. Despite the singular nature of $V_L$, the full potential $V$ is regular, where the details of the toy model can be consulted in Ref. [1]. As shown by explicit calculations $V_L$ is not renormalizable if
counting we use for the toy model, the repulsive singular nature of the leading order potential. In the power counting employed: EFT requires to iterate according to power counting, but repulsive singular interaction likely entail a demotion of the contact interactions, not a promotion, see the discussion below (or see Ref. [6] for a different opinion). The authors of Ref. [2] put into question that a mixture of non-perturbative and perturbative methods could reproduce the fundamental theory.

This viewpoint, though sensible, is premature: here I present calculations for this toy model within a mixture of perturbative and non-perturbative renormalization. For that I simply include the long range potential $V_L$ as the leading order of the calculation and include contact interactions according to their power counting, which I determine by adapting the ideas of Ref. [1]. The outcome is that $V_L$ will be counted as $Q^{-1}$ (to justify its iteration), while the contact-range couplings will be demoted by one order with respect to naive dimensional analysis, i.e. $C_2 n (p^{2 n} + p'^{2 n})$ enters at order $Q^{2 n + 1}$. Concrete calculations of the phase shifts are shown in Fig. 1 where we can see that the perturbative expansion indeed converges well. We regularize the long-range potential with a Gaussian regulator in coordinate space

$$V_L(r) \rightarrow V_L(r; R_c) = V_L(r) (1 - e^{-r/R_c})^2,$$

while for the contact-range potential we regularize the Dirac-delta as

$$\delta(r) \rightarrow \delta(r; r_c) = \frac{1}{\pi^{1/2} R_c^3} e^{-(r/R_c)^2},$$

plus similar expressions for its derivatives, where a local contact-range potential is used: $C_2 n q^{2 n}$. The regularization is slightly different than in the original manuscript, but it is certainly simpler and nonetheless equivalent. The details of the calculation are analogous to those of Ref. [7], but extended to higher orders. The $C_2 n$ couplings are determined by fitting to the toy model phase shifts in the 20 – 80 MeV (80 – 200 MeV) range for $\nu = 1, 3$ ($\nu = 5, 7$). Calculations are shown for the cutoffs $R_c = 0.3, 0.6, 1.2$ and 1.8 fm up to order $Q^7$ (N^3LO) in the EFT expansion. The conclusion is that the standard EFT approach of Ref. [2] is perfectly able to describe the physics of the toy model of Epelbaum et al. [1]. In addition it improves over the proposal of Ref. [1] (namely, a purely non-perturbative approach with a judiciously chosen cutoff), in the sense that there are no strong restrictions on the cutoff (besides the numerical ones, $R_c \geq 0.3$ fm in this case), which can be taken harder than the breakdown scale if one wishes to. Notice that even though the existence of the $R_c \rightarrow 0$ limit has not been proven, this is not a necessary condition for the present approach to be useful.

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**Fig. 1.** Phase shifts for the toy model of Ref. [1] computed within a suitable EFT which takes into account the repulsive singular nature of the leading order potential. In the power counting we use for the toy model, $C_0 \delta(r)$ enters at order $Q^3$, $C_2 \nabla^2 \delta(r)$ at order $Q^5$, $C_4 \nabla^4 \delta(r)$ at order $Q^7$ and $C_6 \nabla^6 \delta(r)$ at order $Q^9$. The $Q^7$ calculation usually falls on top of the full one. The contact interactions are iterated according to the counting, e.g. at order $Q^3$, $C_0$ is iterated once and $C_2$ enters at tree level.
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