Non-Quadratic Pseudo Dual Potentials for Plastic Flow Modeling

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Abstract. The existence of dual flow potentials is well established in mathematical theory of plasticity since the seminal work by Hill in 1987. For a metal undergoing plastic flow, a flow stress potential is used to compute its plastic strain increments when the applied yield stress is known. On the other hand, the corresponding dual flow strain-rate potential is used to compute the stress on the flow surface when the plastic strain increments are given. This work examines some issues associated with plasticity modeling using non-quadratic dual flow potentials. Unlike the quadratic case where flow stress and strain-rate potentials are the exact dual to each other, it is often difficult if not impossible to obtain analytically the dual of a non-quadratic flow stress or strain-rate potential. The study instead focuses on formulating and assessing various non-quadratic pseudo dual flow potentials that approximate the actual flow surfaces in either stress or strain-rate space. The difference and connection between the yield surface and flow surface in non-associated plasticity are also investigated. Although only one of the dual flow potentials is actually needed for their applications in associated and non-associated plasticity modeling, the unique advantage of having both dual flow potentials on hand even in their pseudo forms is pointed out for new computational analyses.

1. Introduction
In classical mathematical theory of plasticity \cite{1, 2, 3}, two positive scalars called equivalent flow stress \(\bar{\sigma}\) and plastic strain-rate \(\bar{\varepsilon}\) are introduced to define the plastic work-rate \(\dot{w}^p\) of a plastically deforming material element with a plastic strain-rate \(\dot{\varepsilon}^p\) under the applied Cauchy stress \(\sigma\)

\[
\dot{w}^p(\sigma; \dot{\varepsilon}^p) = \bar{\sigma} \varepsilon = f(\sigma) \dot{\varepsilon}^p = \sigma f(\dot{\varepsilon}^p) = \sigma : \dot{\varepsilon}^p > 0, \quad \sigma \neq 0, \quad \dot{\varepsilon}^p \neq 0,
\]

where \(\bar{\sigma} = f(\sigma)\) and \(\bar{\varepsilon} = q(\dot{\varepsilon}^p)\) are the flow stress function and plastic strain-rate function respectively\(^1\). They are positive convex homogeneous functions of degree one with respect to positive multipliers and are what Hill calls dual plastic potentials or polars to each other\(^4\).

Consider a flat thin sheet modeled by Hill’s 1948 associated quadratic anisotropic plasticity \cite{5} with either its yield/flow stress function (YLD) given in plane stress \(\sigma = (\sigma_x, \sigma_y, \tau_{xy})\)

\[
\Phi_h(\sigma_x, \sigma_y, \tau_{xy}) = \bar{\sigma}^2 = f^2(\sigma) = A_1 \sigma_x^2 + A_2 \sigma_x \sigma_y + A_3 \sigma_y^2 + A_4 \tau_{xy}^2,
\]

\(^1\) It is understood here that the use of a set of dual variables \((\sigma; \dot{\varepsilon})\) for a function \(\dot{w}^p(\sigma; \dot{\varepsilon})\) emphasizes only either \(\sigma\) or \(\dot{\varepsilon}\) but not both may be independent variables. In the following, either \(\dot{w}_p(\sigma)\) or \(\dot{w}_p(\dot{\varepsilon})\) will be used as two specific examples of \(\dot{w}^p(\sigma; \dot{\varepsilon})\).
or its (plastic work rate conjugate) plastic strain-rate function (SRP) with the imposed planar plastic strain increments $\dot{e}^p = (\dot{e}_{x}^p, \dot{e}_{y}^p, \dot{e}_{z}^p)$

$$\Psi_h(\dot{e}_{x}^p, \dot{e}_{y}^p, \dot{e}_{z}^p) = q^p(\dot{e}^p) = B_1(\dot{e}_{x}^p)^2 + B_2\dot{e}_{x}^p\dot{e}_{y}^p + B_3(\dot{e}_{y}^p)^2 + B_4(\dot{e}_{z}^p)^2.$$  (3)

A flow strain-rate or stress rule is then used to compute the plastic strain increments $\dot{e}^p$ or Cauchy stress $\sigma$ respectively as $(\dot{e} = q(\dot{e}^p))$

$$\dot{e}^p = \dot{e}^p \frac{\partial f(\sigma)}{\partial \sigma} = \dot{e}^p \frac{\partial \Psi_h(\sigma)}{\partial \sigma}, \quad \sigma = \sigma \frac{\partial q(\dot{e}^p)}{\partial \dot{e}^p} = \sigma \frac{\partial \Psi_h(\dot{e}^p)}{\partial \dot{e}^p}. \quad (4)$$

By using the above associated flow rules and plastic work rate equivalency, one can show that the polynomial coefficients $(A_1, A_2, A_3, A_4)$ of the quadratic yield function and $(B_1, B_2, B_3, B_4)$ of the quadratic strain-rate function are directly related algebraically via [4, 6, 7]

$$A_1 = \frac{4B_3}{4B_1B_3 - B_2^2}, \quad A_2 = -\frac{4B_2}{4B_1B_3 - B_2^2}, \quad A_3 = \frac{4B_1}{4B_1B_3 - B_2^2}, \quad A_4 = \frac{1}{B_4}, \quad (5)$$

$$B_1 = \frac{4A_3}{4A_1A_3 - A_2^2}, \quad B_2 = -\frac{4A_2}{4A_1A_3 - A_2^2}, \quad B_3 = \frac{4A_1}{4A_1A_3 - A_2^2}, \quad B_4 = \frac{1}{A_4}. \quad (6)$$

It is noted that both $4A_1A_3 - A_2^2 > 0$ and $4B_1B_3 - B_2^2 > 0$ per strict convexity requirements [8, 9, 10].

As pointed out by Hill [4], when the yield stress function $f(\sigma)$ and the plastic strain-rate function $q(\dot{e}^p)$ are entirely smooth and strictly convex (obviously they should be positive too as $\sigma > 0$ and $\dot{e}^p > 0$), there is a one-to-one correspondence between the Cauchy stress $\sigma$ and plastic strain-rate $\dot{e}^p$. Consequently, only the yield stress (or plastic strain-rate) function is needed to be specified in an associated plasticity theory as the corresponding plastic strain-rate (or yield stress) function can be derived in principle from the flow or normality rules Eq.(4).

2. Non-Quadratic Isotropic Flow Potentials

Unlike the quadratic case above where flow stress and strain-rate potentials are the exact dual to each other, it is often difficult if not impossible to obtain analytically the dual of a non-quadratic flow stress or strain-rate potential even for isotropic materials. We will examine several commonly used ones to identify their pseudo dual potentials instead.

2.1. Hershey-Hosford Flow Potentials

The non-quadratic Hershey-Hosford yield function generalizes the quadratic von Mises yield function [11, 12]. In terms of three principal Cauchy stresses $(\sigma_1, \sigma_2, \sigma_3)$, it has the following form with the exponent $a > 1$

$$f^H_H(\sigma_d) = \frac{|\sigma_1 - \sigma_2|^a + |\sigma_2 - \sigma_3|^a + |\sigma_3 - \sigma_1|^a}{2}. \quad (7)$$

Flow stress surfaces $f^H_H(\sigma_d) = \sigma_f(\dot{e}^p)$ with $a = 6, 8,$ and $20$ are shown in Figure 1(a) in terms of biaxial deviatoric stresses $\sigma_d = (\sigma_1 - \sigma_3, \sigma_2 - \sigma_3)$.

The plastic principal strain increments are readily obtained per the associated flow rule of Eq.(4). Numerically generated flow surfaces with $a = 6, 8,$ and $20$ are shown in terms of biaxial plastic strain increments $\dot{e}^p_d = (\dot{e}_{1}^p, \dot{e}_{2}^p)$ in Figure 1(b). It is noted that the flow surface is given in the parametric form in Cauchy stress on the yield surface per the flow rule $\dot{e}^p_d = \dot{e}^p_d(\sigma)$. Except for $a = 2$ or $a = 4$, an analytical strain-rate function $q^H_H(\dot{e}^p_d)$ as the exact dual plastic potential of Hershey-Hosford yield stress function cannot be obtained in general. Instead, the
Figure 1. (a) Flow surfaces in stress space given by Hershey-Hosford yield function with $a = 6, 8, \text{ and } 20$; (b) the corresponding flow surfaces in strain-rate space given numerically by the associated flow rule.

Following strain-rate function of the Hershey-Hosford form is used as an approximate or pseudo dual potential (noting $\dot{\varepsilon}_3 = -\dot{\varepsilon}_1 - \dot{\varepsilon}_2$ due to plastic incompressibility)

$$\tilde{q}_H(\dot{\varepsilon}^p_d) = \frac{|\dot{\varepsilon}_1^p|^b + |\dot{\varepsilon}_2^p|^b + |\dot{\varepsilon}_3^p|^b}{1 + 2^{1-b}} = \frac{|\dot{\varepsilon}_1^p|^b + |\dot{\varepsilon}_2^p|^b + |\dot{\varepsilon}_1^p + \dot{\varepsilon}_2^p|^b}{1 + 2^{1-b}}. \quad (8)$$

where the exponent $b = a$ is assumed first for simplicity. As shown in Figs. 2(a)-2(c), the approximate flow surface $\tilde{q}_H(\dot{\varepsilon}^p_d) = \dot{\varepsilon}^p$ with $b = a$ gives quite close matching of actual flow surfaces.

For a better approximation, one may find the least-square best-fit $b$ numerically for a given value of stress exponent $a$ by minimizing the sum-squared errors

$$SSD_H = \frac{1}{K} \sum_{k=1}^{K} (\tilde{q}_H(\dot{\varepsilon}^p_1(k)/\dot{\varepsilon}^p, \dot{\varepsilon}^p_2(k)/\dot{\varepsilon}^p) - 1)^2, \quad (9)$$

where $(\dot{\varepsilon}^p_1(k)/\dot{\varepsilon}^p, \dot{\varepsilon}^p_2(k)/\dot{\varepsilon}^p)$ are the numerical values of plastic strain increments computed from the flow rule of Hershey-Hosford yield function given in the parametric form in Cauchy stress on the yield surface with a total of $K = 1440$ discrete points. As an example (for the case of $a = 6$), the best-fit $b$ is found to be 1.52234 with the resulting deviation $\delta_H(k)$ being shown in Figure 3 (where the largest absolute deviation is about 0.00218319). More detailed best-fit results are summarized in Table 1 for $a = 6, 8, \ldots, 100$. Besides the corresponding best-fit $b$ values and resulting sum-squared errors $SSD_H$, the cases of $b = a$ and $b = a/(a-2)$ and their sum-squared errors $SSD_H$ are also listed in the table. All three choices of the strain-rate exponent $b$ of the pseudo dual potential Eq.(8) give rather small sum-of-square errors.

2.2. Drucker’s 1949 Sixth-Order Yield Function

We consider next the flow stress surface given by Drucker’s 1949 sixth-order yield function for isotropic materials [13]

$$f_D^6(\sigma_d) = 27\alpha [J_2^3(\sigma_1, \sigma_2, \sigma_3) - cJ_3^2(\sigma_1, \sigma_2, \sigma_3)] = \sigma_f^6, \quad (10)$$
Figure 2. Comparison of the approximate flow surface based on the pseudo dual strain-rate function \( \tilde{q}_H(\dot{\varepsilon}) = \dot{\varepsilon}^p \) in solid lines with the actual flow surface per flow rule in dashed lines: (a) \( b = 6 \); (b) \( b = 8 \); (c) \( b = 20 \).

Figure 3. Deviations of the approximate from actual flow surfaces \( \delta_H = \tilde{q}_H(\dot{\varepsilon})/\dot{\varepsilon}^p - 1 \) along the stress states of the yield surface: (a) \( b = 6 \); (b) \( b = 8 \).

Table 1. A summary of exponents \( a \) and \( b \) values plus deviations in flow surfaces.

| \( a \) | 6     | 8     | 12    | 20    | 50    | 100   |
|--------|-------|-------|-------|-------|-------|-------|
| \( b = \text{best-fit} \) | 1.52234 | 1.3467 | 1.2059 | 1.11314 | 1.04196 | 1.02048 |
| \( SSD_H \) | \( 1.1 \times 10^{-6} \) | \( 9.0 \times 10^{-7} \) | \( 2.6 \times 10^{-7} \) | \( 2.9 \times 10^{-8} \) | \( 3.4 \times 10^{-10} \) | \( 1.1 \times 10^{-11} \) |
| \( b = a \) | 6     | 8     | 12    | 20    | 50    | 100   |
| \( SSD_H \) | \( 3.8 \times 10^{-6} \) | \( 2.1 \times 10^{-5} \) | \( 3.7 \times 10^{-5} \) | \( 2.6 \times 10^{-5} \) | \( 5.0 \times 10^{-6} \) | \( 1.0 \times 10^{-6} \) |
| \( b = a/(a-2) \) | 1.5   | 1.3333 | 1.2   | 1.1111 | 1.04167 | 1.02041 |
| \( SSD_H \) | \( 3.8 \times 10^{-6} \) | \( 3.3 \times 10^{-6} \) | \( 1.3 \times 10^{-6} \) | \( 2.5 \times 10^{-7} \) | \( 8.3 \times 10^{-9} \) | \( 5.5 \times 10^{-10} \)

where

\[
6J_2(\sigma_1, \sigma_2, \sigma_3) = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2, \\
27J_3(\sigma_1, \sigma_2, \sigma_3) = (2\sigma_1 - \sigma_2 - \sigma_3)(2\sigma_2 - \sigma_3 - \sigma_1)(2\sigma_3 - \sigma_1 - \sigma_2),
\]

(11)

\( c \) is an adjustable material constant, \( \alpha = 27/(27 - 4c) \) and \( \sigma_f \) is the current yield strength of the material in uniaxial tension. Dodd and Naruse [14] subsequently showed that \( c < 27/4 \) and \(-27/8 \leq c \leq 9/4 \) are the necessary and sufficient conditions that are required to ensure Drucker’s yield function be positive and convex respectively. Flow stress surfaces \( f_D(\sigma_d) = \sigma_f(\dot{\varepsilon}^p) \) with \( c = -27/8 \) and \( 9/4 \) are shown in Figure 4(a) in terms of biaxial deviatoric stresses \( \sigma_d = (\sigma_1 - \sigma_3, \sigma_2 - \sigma_3) \) and the corresponding flow strain-rate surfaces numerically generated per the associated flow rule are shown in Figure 4(b).

Guided by the results obtained for the Hershey-Hosford yield function above, the following
Figure 4. (a) Flow surfaces given by Drucker’s 1949 yield function with $c = -27/8$ (solid lines) and $9/4$ (dashed lines): (a) in stress space; (b) in strain-rate space.

strain-rate function of the Drucker’s 1949 form is proposed as an approximate or pseudo dual potential

$$\tilde{q}_D(\dot{\varepsilon}^p) = 27\beta[I^3_2(\dot{\varepsilon}^p_1, \dot{\varepsilon}^p_2, \dot{\varepsilon}^p_3) - dI^3_3(\dot{\varepsilon}^p_1, \dot{\varepsilon}^p_2, \dot{\varepsilon}^p_3)],$$  

where

$$6I_2(\dot{\varepsilon}^p_1, \dot{\varepsilon}^p_2, \dot{\varepsilon}^p_3) = (\dot{\varepsilon}^p_1 - \dot{\varepsilon}^p_2)^2 + (\dot{\varepsilon}^p_2 - \dot{\varepsilon}^p_3)^2 + (\dot{\varepsilon}^p_3 - \dot{\varepsilon}^p_1)^2,$$

$$27I_3(\dot{\varepsilon}^p_1, \dot{\varepsilon}^p_2, \dot{\varepsilon}^p_3) = (2\dot{\varepsilon}^p_1 - \dot{\varepsilon}^p_2 - \dot{\varepsilon}^p_3)(2\dot{\varepsilon}^p_2 - \dot{\varepsilon}^p_3 - \dot{\varepsilon}^p_1)(2\dot{\varepsilon}^p_3 - \dot{\varepsilon}^p_1 - \dot{\varepsilon}^p_2),$$  

$d$ is an adjustable material constant between $-27/8$ and $9/4$ per the convexity requirement and $\beta = 64/(27(27 - 4d))$.

Clearly, the value of constant $d$ is dependent on the value of constant $c$. One may find the least-square best-fit $d$ numerically for a given value of $c$ by minimizing the sum-squared errors

$$SSD_D = \frac{1}{K} \sum_{k=1}^{K} \delta^p_D(k) = \frac{1}{K} \sum_{k=1}^{K} (\tilde{q}_D(\dot{\varepsilon}^p(k)/\dot{\varepsilon}^p, \varepsilon^p_2(k)/\dot{\varepsilon}^p) - 1)^2, \quad -27/8 \leq d \leq 9/4,$$

where $(\dot{\varepsilon}^p(k)/\dot{\varepsilon}^p, \varepsilon^p_2(k)/\dot{\varepsilon}^p)$ are the numerical values of plastic strain increments computed from the flow rule of Drucker’s 1949 yield function given in the parametric form in Cauchy stress on the yield surface with a total of $K = 1440$ discrete points. Table 2 summarized some typical results for selected $c$ values between $-27/8$ and $9/4$ and the corresponding best-fit $d$ values and resulting sum-squared errors $SSD_D$. As shown in Figs.5(a) and 5(b), the approximate flow strain-rate surfaces $\tilde{q}_D(\dot{\varepsilon}^p) = \dot{\varepsilon}^p$ with $d = 9/4$ and $-27/8$ are compared with the numerically generated ones based on the flow rule applied to $f_D(\sigma_d) = \sigma_f(\dot{\varepsilon}^p)$ with $c = -27/8$ and $9/4$.

Table 2. A summary of constants $c$ and $d$ values plus deviations in flow surfaces.

| $c$ | -27/8 | -2 | -1 | 0 | 1 | 9/4 |
|-----|-------|----|----|---|---|-----|
| best-fit $d$ | 9/4 | 1.65891 | 0.909222 | 0 | -1.11579 | -3.0723 |
| $SSD_D$ | $4.8 \times 10^{-5}$ | $6.7 \times 10^{-6}$ | $5.5 \times 10^{-7}$ | 0 | $8.7 \times 10^{-7}$ | $2.9 \times 10^{-5}$ |
For a calibrated Gotoh’s yield function (i.e., with a set of known coefficients as convex conditions have recently been given in [19] for biaxial loading on their five polynomial shear stress or strain rate components. It is noted that simple algebraic necessary and sufficient the initial effort, we limit our attention to the special case of biaxial loading only without any pseudo dual to Gotoh’s fourth-order yield stress function for some selected sheet metals. As [16, 17, 18].

From texture functions was subsequently applied to effectively model polycrystalline sheet metals order plastic strain-rate potential with orthotropic symmetry and its analytical calculations (yield function), they are exact dual plastic potentials. Nevertheless, a complete 3D fourth-order plastic strain-rate potential with orthotropic symmetry and its analytical calculations from texture functions was subsequently applied to effectively model polycrystalline sheet metals [16, 17, 18].

Here we examine the degree of approximation of a fourth-order strain-rate function as a pseudo dual strain-rate in solid lines with the actual flow surface per flow rule in dashed lines: (a) $c=27/8$, $d=9/4$; (b) $c=9/4$, $d=-27/8$.

### 3. Non-Quadratic Orthotropic Flow Potentials

For modeling plastic flow in sheet metals, orthotropic yield functions and plastic potentials are often used. The earliest and perhaps also simplest non-quadratic orthotropic yield function that has been fully applied to sheet metal plasticity modeling is Gotoh’s 1977 quartic yield function [15] in the following form in plane stress

$$f^1_G(\sigma) = A_1\sigma_x^4 + A_2\sigma_x^3\sigma_y + A_3\sigma_x^2\sigma_y^2 + A_4\sigma_x\sigma_y^3 + A_5\sigma_y^4 + A_6\sigma_x^2\tau_{xy} + A_7\sigma_x\sigma_y\tau_{xy} + A_8\sigma_y^2\tau_{xy} + A_9\tau_{xy}^4,$$

(15)

where $(A_1, A_2, ..., A_9)$ are nine polynomial coefficients (material constants). Gotoh considered also a fourth-order strain-rate potential

$$\tilde{\eta}_G(\dot{\varepsilon}) = B_1\dot{\varepsilon}_x^4 + B_2\dot{\varepsilon}_x^3\dot{\varepsilon}_y + B_3\dot{\varepsilon}_x^2\dot{\varepsilon}_y^2 + B_4\dot{\varepsilon}_x\dot{\varepsilon}_y^3 + B_5\dot{\varepsilon}_y^4 + B_6\dot{\varepsilon}_x^2\dot{\varepsilon}_{xy} + B_7\dot{\varepsilon}_x\dot{\varepsilon}_y\dot{\varepsilon}_{xy} + B_8\dot{\varepsilon}_y^2\dot{\varepsilon}_{xy} + B_9\dot{\varepsilon}_{xy}^4,$$

(16)

where $(B_1, B_2, ..., B_9)^2$ are its nine polynomial coefficients. He showed that the fourth-order stress-based yield function and the fourth-order strain-rate-based plastic potential are not equivalent in general. That is, only under some special occasions (e.g., square of Hill’s 1948 yield function), they are exact dual plastic potentials. Nevertheless, a complete 3D fourth-order plastic strain-rate potential with orthotropic symmetry and its analytical calculations from texture functions was subsequently applied to effectively model polycrystalline sheet metals [16, 17, 18].

Here we examine the degree of approximation of a fourth-order strain-rate function as a pseudo dual to Gotoh’s fourth-order yield stress function for some selected sheet metals. As the initial effort, we limit our attention to the special case of biaxial loading only without any shear stress or strain rate components. It is noted that simple algebraic necessary and sufficient convex conditions have recently been given in [19] for biaxial loading on their five polynomial coefficients as

$$\begin{pmatrix} 12A_1 & 3A_2 & 2A_3 \\ 3A_2 & 2A_3 & 3A_4 \\ 2A_3 & 3A_4 & 12A_5 \end{pmatrix} \geq 0, \quad \begin{pmatrix} 12B_1 & 3B_2 & 2B_3 \\ 3B_2 & 2B_3 & 3B_4 \\ 2B_3 & 3B_4 & 12B_5 \end{pmatrix} \geq 0. \quad (17)$$

For a calibrated Gotoh’s yield function (i.e., with a set of known $A_1, A_2, ..., A_5$), one may find the least-square best-fit set of $(B_1, B_2, ..., B_5)$ by minimizing the sum-squared errors

$$SSD_G = \frac{1}{K} \sum_{k=1}^{K} \tilde{\eta}_G^2(k) = \frac{1}{K} \sum_{k=1}^{K} (\tilde{\eta}_G(\dot{\varepsilon}_1^p(k)/\dot{\varepsilon}_1^p, \dot{\varepsilon}_2^p(k)/\dot{\varepsilon}_2^p) - 1)^2, \quad (18)$$

$^2$ Coefficients $(E_1, ..., E_9)$ were used instead in Eq.(19) of [15].
where \( \dot{\varepsilon}_p^1(k)/\dot{\varepsilon}_P, \dot{\varepsilon}_p^2(k)/\dot{\varepsilon}_P \) are the numerical values of plastic strain increments computed from the flow rule of Gotoh’s 1977 yield function given in the parametric form in Cauchy stress on the yield surface with a total of \( K = 1440 \) discrete points.

Biaxial flow stress surfaces and the corresponding flow strain-rate surfaces are shown in Fig.6(a) and (b) for an aluminum sheet and a steel sheet respectively. Table 3 summarized the best-fit results for the two selected sheet metals and the resulting sum-squared errors \( SSD_G \). The convexity conditions (positive-semidefinite on those two 3-by-3 matrices in Eq.(17)) were verified numerically. As shown in Figs.7(a) and 7(b), the approximate flow strain-rate surfaces \( \tilde{\dot{q}}_G(\dot{\varepsilon}_p^d) = \dot{\varepsilon}_P \) are compared very well with the numerically generated ones based on the flow rule applied to \( f_G(\sigma_d) = \sigma_f(\dot{\varepsilon}_P) \) for the aluminum and steel sheets.

**Figure 6.** (a) Flow stress surfaces (solid lines) and flow strain-rate surfaces (dashed lines) given by Gotoh’s 1977 yield function: (a) an aluminum 60616-T4 sheet [10]; (b) an IF steel sheet [10].

**Figure 7.** Comparison of the approximate flow surface based on the pseudo dual strain-rate function \( \tilde{\dot{q}}_G(\dot{\varepsilon}_p^d) = \dot{\varepsilon}_P \) in solid lines with the actual flow surface per flow rule in dashed lines: (a) an aluminum 60616-T4 sheet [10]; (b) an IF steel sheet [10].

### 4. Discussion and Conclusions

Historically, two forms of the same plastic work rate in plasticity are the dual (the function and its conjugate) as noted much earlier by Hill in 1956 [20]

\[
\frac{1}{2} \dot{\psi}_P(\sigma) + \frac{1}{2} \dot{\psi}_P^*(\dot{\varepsilon}_P) = \frac{1}{2} \dot{\varepsilon}_P f(\sigma) + \frac{1}{2} \sigma_f q(\dot{\varepsilon}_P) = \sigma : \dot{\varepsilon}_P, \tag{19}
\]
Table 3. A summary of polynomial coefficient values plus deviations in flow surfaces.

|          | \( A_1 \) | \( A_2 \) | \( A_3 \) | \( A_4 \) | \( A_5 \) |
|----------|-----------|-----------|-----------|-----------|-----------|
| AA6016-T4 | 1         | -1.7273   | 2.3675    | -1.9619   | 1.2945    |
| IF steel  | 1         | -2.7179   | 4.1770    | -3.0688   | 1.0327    |
| best-fit  | \( B_1 \) | \( B_2 \) | \( B_3 \) | \( B_4 \) | \( B_5 \) | \( SSD_{G} \) |
| AA6016-T4 | 1.78004   | 3.9593    | 6.22603   | 3.4712    | 1.37219   | 2.0\times10^{-6} |
| IF steel  | 2.95682   | 8.11137   | 13.2123   | 11.006    | 4.28155   | 9.4\times10^{-7} |

where the first and second terms on the left-hand side of the above equation have been called by Hill as work and complementary functions respectively. In fact, dual plastic potentials may also be regarded as examples of Legendre dual transformation on convex differentiable functions or their conjugates [21, 22]. One function may be obtained from the other function via the Legendre transform [23]

\[
\frac{1}{2} \sigma_f q(\dot{\epsilon}^p) = \sigma(\dot{\epsilon}^p) - \frac{1}{2} \dot{\epsilon}^p f(\sigma(\dot{\epsilon}^p)), \quad \dot{\epsilon}^p = \dot{\epsilon}^p \frac{\partial f(\sigma)}{\partial \sigma} \Rightarrow \sigma(\dot{\epsilon}^p), \quad (20)
\]

\[
\frac{1}{2} \dot{\epsilon}^p f(\sigma) = \sigma : \dot{\epsilon}^p(\sigma) - \frac{1}{2} \sigma_f q(\dot{\epsilon}^p(\sigma)), \quad \sigma = \sigma_f \frac{\partial q(\dot{\epsilon}^p)}{\partial \dot{\epsilon}^p} \Rightarrow \dot{\epsilon}^p(\sigma). \quad (21)
\]

Although these two Legendre transforms appear at the first glance to be different from the early introduction on the dual plastic potentials given by Eq.(1)-Eq.(6), they are in fact equivalent as \( \sigma_f q(\dot{\epsilon}^p) = \dot{\epsilon}^p f(\sigma) = \sigma : \dot{\epsilon}^p \) due to the two particular positive scalars \( \sigma_f \) and \( \dot{\epsilon}^p \) used here.

As a convex even-order homogeneous polynomial [5, 1, 15, 10] is often used to define the yield stress function, one may obtain directly the strain-rate function \( g(\dot{\epsilon}^p) \) using the normalized Cauchy stress \( \sigma/\sigma_f \) in terms of the normalized plastic strain-rate \( \dot{\epsilon}^p/\dot{\epsilon}^p \) from the flow strain-rate rule

\[
\dot{\epsilon}^p/\dot{\epsilon}^p = \frac{\partial f(\sigma)}{\partial \sigma} = r(\sigma/\sigma_f) \Rightarrow \sigma = r^{-1}(\dot{\epsilon}^p/\dot{\epsilon}^p) \Rightarrow g(\dot{\epsilon}^p) = f(r^{-1}(\dot{\epsilon}^p/\dot{\epsilon}^p)) = 1. \quad (22)
\]

An analytical solution of \( \sigma \) in terms of \( \dot{\epsilon}^p \) from inverting the given flow rule result of Eq.(4)_1 may not always be feasible in general even when the yield stress function \( f(\sigma) \) is both smooth and convex. The resulting plastic strain-rate surface may thus only strictly be in the discrete numerical form for non-quadratic yield stress function for either isotropic or anisotropic materials. Nevertheless, results from our current study as presented here indicate that a plastic strain-rate function formulated simply in the same form as the non-quadratic yields stress function may be used as its pseudo dual with good approximation of the actual flow surface.

So far in this study the dual plastic potentials have been presented and discussed in the framework of associated plasticity where yield and flow surfaces are treated to be identical and exchangeable. In non-associated anisotropic plasticity modeling of sheet metals, the yield surface and flow surface in the stress space are distinct as they are defined by a yield stress function \( f(\sigma) \) and a flow stress function \( g(\sigma) \) respectively where \( g(\sigma) \neq f(\sigma). \) The notion of a dual plastic strain-rate potential is applied to the flow stress function only with the flow strain-rate and stress rules as

\[
\dot{\epsilon}^p = \lambda \frac{\partial g(\sigma)}{\partial \sigma}, \quad \sigma = \tilde{\sigma} \frac{\partial g(\dot{\epsilon}^p)}{\partial \dot{\epsilon}^p}, \quad \lambda(\dot{\epsilon}^p) > 0 \text{ if } \dot{\epsilon}^p \neq 0, \quad \tilde{\sigma}(\sigma) > 0 \text{ if } \sigma \neq 0, \quad (23)
\]

where \( \lambda(\dot{\epsilon}^p) \) is not necessarily the equivalent plastic strain rate \( \dot{\epsilon} \) and \( \tilde{\sigma}(\sigma) \) is not necessarily the equivalent yield/flow stress \( \sigma(\dot{\epsilon}) \) (which has been set to be \( f(\sigma) = \sigma_f \) with \( \dot{\epsilon} = \dot{\epsilon}^p \) for associated
plasticity in this study per Eq.(1)). In fact, one has per the plastic work rate equivalency

\[ \sigma : \dot{\varepsilon}^p = \sigma \dot{\varepsilon} = g(\sigma) \lambda = \tilde{\sigma} q(\dot{\varepsilon}^p) \Rightarrow \dot{\lambda} = \frac{\tilde{\sigma}}{g(\sigma)} \dot{\varepsilon}, \quad \tilde{\sigma} = \frac{\dot{\varepsilon}}{q(\dot{\varepsilon}^p)} \sigma. \quad (24) \]

For simplicity, one may choose \( \tilde{\sigma} = \sigma = g(\sigma) \) so \( \dot{\lambda} = \dot{\varepsilon} = q(\dot{\varepsilon}^p) \) for non-associated plasticity.

It is noted here that the flow surface in the strain-rate space \( q(\dot{\varepsilon}^p) = \dot{\varepsilon} \) is no longer dual to the yield surface \( f(\sigma) = \sigma_f \). As illustrated in Figure 8 in biaxial stress space, the yield surface \( f(\sigma) = \sigma_f \) is defined in terms of the actual Cauchy stress \( \sigma \) while the nominal flow surface \( g(\sigma^*) = \sigma_f \) is defined in terms of the nominal Cauchy stress \( \sigma^* \) instead. It is this nominal flow surface that is associated with the dual plastic strain-rate potential \( q(\dot{\varepsilon}^p) \). This can be seen from the flow stress rule of Eq.(23)

\[ \sigma = g(\sigma) \frac{\partial q(\dot{\varepsilon}^p)}{\partial \dot{\varepsilon}^p} \Rightarrow \sigma^* = \sigma_f \frac{\partial q(\dot{\varepsilon}^p)}{\partial \dot{\varepsilon}^p}, \quad \sigma^* = \eta \sigma \quad \text{and} \quad g(\sigma^*) = \eta g(\sigma) = \sigma_f, \quad (25) \]

where the positive scalar \( \eta \) can be obtained from the flow stress rule and yield condition as

\[ f(\sigma) = f(g(\sigma) \frac{\partial q(\dot{\varepsilon}^p)}{\partial \dot{\varepsilon}^p}) = g(\sigma) f(\frac{\partial q(\dot{\varepsilon}^p)}{\partial \dot{\varepsilon}^p}) = \sigma_f \Rightarrow \eta = f(\frac{\partial q(\dot{\varepsilon}^p)}{\partial \dot{\varepsilon}^p}). \quad (26) \]

Although the development of a particular plasticity model and its implementation in a finite element code need only either flow stress or plastic strain-rate function, there are certain advantages to have both functions on hand. For example, when using a texture-adjusted plastic strain-rate potential in sheet metal forming modeling applications, the principle of maximal work has been used to test whether a given stress state lies within the yield surface instead of invoking directly the yield condition [17, 18]. When one obtains in very good approximation the pseudo dual yield stress function from its plastic strain-rate potential, one may then use the existing finite element code commonly based on the yield stress function where the yield condition can be used directly to check a given stress state in the numerical simulation.

In summary, the existence of a dual plastic strain-rate potential has been well established for any flow stress function in plasticity [4] and its applications appear increasingly in anisotropic plasticity modeling of crystalline metals [16, 17, 18, 24, 25, 26]. For example, a yield polyhedron and its dual were derived for a single crystal [24] and a textured polycrystal [25]. The strain rate potential Srp2004-18p analogous to Yld2004-18p was constructed based on experimental measurements [26]. Rabahallah et al. [27] considered several advanced strain-rate potentials (quartus, Srp93, Srp2004-18p) identified with crystallographic texture data and mechanical testing inputs. Analytical derivation of the exact dual potential may be only feasible in general for quadratic yield functions. Numerical results presented in this study show that one

![Figure 8. An illustration of yield surface \( f(\sigma) = \sigma_f \) (solid line) and the corresponding flow surface \( g(\sigma^*) = \sigma_f \) (dashed line) for non-associated plasticity \( f \neq g \): where the nominal stress is given as \( \sigma^* = \eta \sigma, \eta = \sigma_f / g(\sigma) > 0 \).]
may nevertheless obtain for both isotropic and anisotropic materials a very good pseudo-dual non-quadratic strain-rate potential of the same mathematical form as the given non-quadratic flow stress function. As our work so far has only studied three simple non-quadratic yield stress functions, our future effort will focus on pseudo-dual strain-rate potentials of more advanced yield stress functions such as Yld2000-2d, Yld2004-18p and CB2001 and its variants [28, 29, 30].

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