Omnibus goodness-of-fit tests for count distributions

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A consistent omnibus goodness-of-fit test for count distributions is proposed. The test is of wide applicability since any count distribution indexed by a $k$-variate parameter with finite moment of order $2k$ can be considered under the null hypothesis. The test statistic is based on the probability generating function and, in addition to have a rather simple form, it is asymptotically normally distributed, allowing a straightforward implementation of the test. The finite-sample properties of the test are investigated by means of an extensive simulation study, where the empirical power is evaluated against some common alternative distributions and against contiguous alternatives. The test shows an empirical significance level always very close to the nominal one already for moderate sample sizes and the empirical power is rather satisfactory, also compared to that of the chi-squared test.

Keywords: consistent test; contiguous alternatives; count distributions; goodness-of-fit test; omnibus test

1. Introduction

Count data naturally arise in many applied disciplines such as actuarial science, medicine, biology and economics, among many others. The Poisson distribution is likely to be the most popular model for such type of data mainly for its simplicity. Nevertheless, observations may exhibit over-dispersion, under-dispersion, zero-inflation or heavy tails, thus precluding the use of the Poisson model as a suitable model. A plethora of count distributions have been introduced that can model these features (e.g., [19]). Classical examples are the Negative Binomial for over-dispersion and the zero-inflated Poisson for excesses of zeroes. The Poisson-Tweedie family of distributions, which has been studied by several authors with different parametrization (e.g., [13], [2], [1] [3]), is able to fit a wide range of mean-variance ratio and tail heaviness. Moreover, in order to model over-dispersed data, Tsylova and Ekgauz [37] and Castellares et al. [9] introduced the one-parameter Bell family of distributions on the basis of the well-known Bell series expansion ([5]). These laws have many appealing properties, since they are members of the one-parameter exponential family and are infinitely divisible. Subsequently, Castellares et al. [10] introduced the two-parameter Bell family — also known as the Bell-Touchard family — which has been adopted for modelling actuarial data ([7]). A further distribution which has many interesting applications in the setting of queueing theory and branching processes ([19]) is the Borel law ([8]).

A challenging aspect of data analysis consists in testing the goodness-of-fit to a parametric family of count distributions. Many testing procedures dealing with count data are based on the properties of the probability generating function (p.g.f.) and on the corresponding empirical p.g.f. Indeed, the p.g.f. fully characterizes the distribution, it is sometimes simpler than the corresponding probability
mass function (p.m.f.) and possesses convenient features, since it is a real-valued continuous function always defined in the range \([0, 1]\). The use of the p.g.f. in testing the fit of discrete distributions has a long-standing tradition (e.g., [22], [31]). In particular, Rueda et al. [31] introduced a test for the Poisson distribution with known parameter, extended by Rueda and O’Reilly [32] to the case of unknown parameter and to the negative Binomial distribution. As to testing Poissonity, Nakamura and Pérez-Abrera [28] proposed a test based on the empirical p.g.f., while Meintanis and Nikitin [26] and Puig and Weiss [29] introduced tests based on different characterizations of the p.g.f. against alternatives belonging to a large family. In a more general framework, Jiménez-Gamero and Batsidis [17] presented a test statistic based on a distance between the empirical p.g.f. and the p.g.f. of the model under the null hypothesis, together with a weighted bootstrap estimator of its distribution. Moreover, Jiménez-Gamero et al. [18] introduced a computationally convenient test for the Poisson–Tweedie distribution, while Batsidis et al. [4] suggested a test for the Bell distribution, which is consistent against fixed alternatives.

In this paper, a novel omnibus test based on the p.g.f. is proposed. The test, which can be applied with all the families of distributions indexed by a \(k\)-variate parameter and having finite moment of order \(2k\), stands in the long tradition of testing procedures based on distances, such as the Pearson chi-squared test. In particular, the proposed test statistic is based on the \(L_1\)-distance of a function of the p.g.f. of the model and the p.g.f. of the random variable actually generating data. This measure depends on the difference of the normalized variations of the two p.g.f.s and has an interesting interpretation, since the normalized variation is usually adopted to characterize the p.g.f. of infinitely divisible families of distributions (e.g., [34]). The test statistic has a manageable expression and depends on the empirical p.g.f. solely through its value in zero, thus avoiding the complexities of handling the whole empirical functional. In addition, we prove that the test statistic has an asymptotic normal distribution, which allows for a straightforward implementation of the test, without demanding intensive resampling methods. The discriminatory capability of the test statistic is also investigated under non-trivial contiguous alternatives, such as suitable mixtures of distributions and distributions obtained by means of the Binomial thinning operator. Finally, we prove the test is consistent.

Section 2 contains some preliminaries about the null hypothesis to be tested and about the considered distance criterion. In Section 3, we propose the new goodness-of-fit test and its asymptotic properties are proven. Section 4 deals with the test statistics for some well-known families of count distributions. In Section 5 the asymptotic behaviour of the proposed test is investigated under contiguous alternatives. A Monte Carlo simulation to assess the finite-sample performance of the test is described in Section 6. Some concluding remarks are given in Section 7.

2. Preliminaries

Let \(\Theta\) be a subset of \(\mathbb{R}^k\) and \(\{\mathcal{M}_\theta\}_{\theta \in \Theta}\) be a family of distributions concentrated into \(\mathbb{N}\) and with \(\sum_n n^{2k} \mathcal{M}_\theta(\{n\}) < \infty\). Without loss of generality, it is not restrictive to assume \(\mathcal{M}_\theta(\{0\}) \neq 0\), i.e. the singleton \(\{0\}\) is not negligible with respect to \(\mathcal{M}_\theta\). Some relevant and obvious - examples of widely applied one-parameter families of type \(\mathcal{M}_\theta\) are the Poisson family \(\{P(\lambda)\}_{\lambda \in \mathbb{R}^+}\), the Geometric family \(\{G(p)\}_{p \in [0, 1]}\), the one-parameter Bell family \(\{B_c(\theta)\}_{\theta \geq 0}\) and the shifted Borel family \(\{B_0(\lambda)\}_{\lambda > 0}\). In addition, examples of two-parameter families are the Bell family \(\{B_c(\alpha, \theta)\}_{\alpha, \theta \geq 0}\) and the shifted Borel-Tanner family \(\{BT(\lambda, r)\}_{\lambda \in [0, 1], r \in \mathbb{N} \setminus \{0\}}\).

In many contexts, interest is in assessing if a non-negative integer-valued random variable (r.v.) \(X\) is distributed according to the model \(\mathcal{M}_\theta\), i.e., formally, in considering the hypothesis system

\[
\begin{align*}
H_0 : & X \sim \mathcal{M}_\theta \quad \exists \theta \in \Theta \\
H_1 : & X \sim \mathcal{M}_\theta \quad \forall \theta \in \Theta
\end{align*}
\]  

(1)
Goodness-of-fit tests

The literature related to the so-called “goodness-of-fit” inferential setting is huge (e.g., [30] or, more recently, [24] and [21]) and suitable test statistics for assessing (1) are often based on the properties of the probability generating function (p.g.f.) (e.g., [31], [32], [25], [17], [4], [29]). Meintainis and Nikitin [26] have proposed families of consistent test statistics for Poissonity based on differential equations concerning the p.g.f. In the present paper, we aim to introduce, in the same vein, a family of test statistics based on the p.g.f. for assessing the general hypothesis system (1). For the sake of simplicity, we focus on constructing tests based on the p.g.f. for one- and two-parameter families of type $M_{\theta}$, that is for $k = 1$ and $k = 2$, respectively, even if the achieved results can be extended to $k > 2$.

Let us denote $p_{\theta}$ the p.m.f. under the model $M_{\theta}$. In addition, let us assume that

$$h : \theta \mapsto \sum_{n=0}^{\infty} np_{\theta}(n)$$

is a strictly monotone and differentiable function for $k = 1$ and

$$h : (\theta_1, \theta_2) \mapsto \left( \sum_{n=0}^{\infty} n p_{\theta}(n), \sum_{n=0}^{\infty} n^2 p_{\theta}(n) \right)$$

is an invertible and differentiable function for $k = 2$. Then, the model can be parametrized by using the first moment $\mu$ or the first two moments $\mu$ and $\mu_2$, i.e., the model can be parametrized as $M_{h^{-1}(\mu)}$ for $k = 1$ and as $M_{h^{-1}(\mu,\mu_2)}$ for $k = 2$. It is worth noting that both the Geometric, the shifted Borel family and the one-parameter Bell family can be parametrized through their mean $\mu$ since $p = (\mu + 1)^{-1}$, $\lambda = \frac{1}{1+\mu}$ and $\theta = h^{-1}(\mu)$, where $h^{-1}$ is the inverse function of $\theta \mapsto \theta e^\theta$. Moreover, the two-parameter Bell family and the shifted Borel-Tanner distributions can be parametrized using the first two moments, as

$$\theta = (\mu_2 - \mu) \mu^{-1} - 1, \quad \alpha = \mu^{-1} e^{-\theta} \quad \text{and} \quad \lambda = 1 - \frac{\sqrt{\mu}}{\sqrt{\mu_2 - \mu^2}}, \quad r = \frac{\sqrt{\mu^3}}{\sqrt{\mu_2 - \mu^2} - \sqrt{\mu}},$$

respectively. Now, let us assume that $f$ represents the p.g.f. of the r.v. $X$. Moreover, let $g_{\mu}$ be the p.g.f. under the model $M_{h^{-1}(\mu)}$ and $g_{\mu,\mu_2}$ be the p.g.f. under the model $M_{h^{-1}(\mu,\mu_2)}$. In order to assess (1), a suitable discrepancy measure between $f$ and the p.g.f. under the model has to be determined. To this aim, there exist various proposals in literature (e.g., [33] and references therein). Focusing on $M_{h^{-1}(\mu)}$, a further sensible measure could be based on the ratio $f/g_{\mu}$ or on the corresponding derivative $(f/g_{\mu})'$. Actually, under $H_0$ it holds $(f/g_{\mu})(s) = 1$ and $(f/g_{\mu})'(s) = 0$ for any $s \in [0, 1]$. However, since

$$\left( \frac{f}{g_{\mu}} \right)' = \frac{f}{g_{\mu}} \left( \frac{f'}{f} - \frac{g_{\mu}'}{g_{\mu}} \right),$$

the derivative could be considered as a “weighted”version of the original ratio, where the weight is given by the difference of the normalized variation of the single p.g.f.s. It should be noted that in literature the normalized variation is used to give an interesting characterization of infinitely divisible p.g.f.s (see Theorem 4.2 in [34]). Moreover, the difference of the normalized variation (with the appropriate sign) could be more effective to detect small discrepancies between $f$ and $g_{\mu}$. Hence, denoting by

$$D_{\mu}(s) = \left( \frac{f}{g_{\mu}} \right)'(s),$$
the $L^1([0,1])$ distance of $D_\mu(s)$ from the null function can be considered, since such a distance is zero under $H_0$ while positive values evidence departures from $H_0$. If $f$ and $g_\mu$ are such that $D_\mu(s)$ is non-negative or non-positive for any $s \in [0,1]$, the $L^1([0,1])$ distance turns out to be

$$\int_0^1 |D_\mu(s)| ds = \int_0^1 D_\mu(s) ds = \left| \frac{f}{g_\mu} (1) - \frac{f}{g_\mu} (0) \right| = \left| g_\mu (0) - P(X = 0) \right|.$$  \hspace{1cm} (2)

It must be pointed out that $D_\mu(s)$ has constant sign for any $s \in [0,1]$ in many contexts where the assessment of the null hypothesis may be difficult. In particular, $H_0$ may be hard to assess when $f$ is equal to $g_\mu w$ and $w$ is in turn a p.g.f., with $\mu_0 \geq 0$. Obviously, this implies that under $H_1$ the reference random variable $X$ is given by $X_0 + Y$, where $Y$ is a random variable with p.g.f. $w$, independent of $X_0$ and $X_0 \sim M_{h^{-1}(\mu_0)}$, $\mu_0 = E[X_0]$. For example, a possible choice for $w$ could be $w : s \mapsto p + (1 - p)s$ with $p \in [0,1]$, i.e. $w$ is the p.g.f. of a Bernoulli r.v. $Y$ of parameter $p$. An other choice could be $w : s \mapsto \exp(-\lambda(1 - s))$ with $\lambda > 0$, i.e. in this case $w$ is the p.g.f. of a Poisson r.v. $Y$ of parameter $\lambda$.

Sometimes instead of specifying $w$ it could be preferable to define the r.v. $Y$. An interesting case is obtained when $Y = I\{Z \geq 1\} \sum_{n=1}^\infty Y_n$, i.e. $X_0 + Y$ is the random sum $\sum_{n=0}^\infty X_n$ where $(X_n)_n$ is a sequence of independent r.v.s with $X_n \sim X_0$ and $Z$ is a non-negative integer-valued r.v. independent of $X_n$ for any natural $n$. Obviously, $f$ is a bona fide p.g.f. and – loosely speaking – $f$ constitutes a “perturbation” of $g_\mu$, i.e. $f$ nearly resembles $g_\mu$ when $w(0)$ is close to one (hence, as $p$ approaches one in the case of the Bernoulli law, as $\lambda$ approaches zero in the case of the Poisson law and as $E[Z(1 + Z)^{-1}]$ approaches 0 for the random sum). As $f = g_\mu w$, $g_\mu \geq f$ and $D_\mu(s)$ is non-negative for any $s \in [0,1]$. Moreover

$$\int_0^1 |D_{\mu_0}(s)| ds = \int_0^1 w'(s) ds = 1 - w(0) = \frac{g_{\mu_0}(0) - f(0)}{g_{\mu_0}(0)}.$$  \hspace{1cm} (3)

The null hypothesis can be difficult to assess also when $f$ is the p.g.f. of the r.v. $\alpha$-fraction of $X_0$ given by $\sum_{n=1}^{X_0} Y_n$, where $(Y_n)_n$ are i.i.d. Bernoulli r.v.s with parameter $\alpha \in [0,1]$, independent of $X_0$. Following [34], the $\alpha$-fraction of $X_0$ is defined by means of the so-called binomial thinning operator. It is worth noting that $g_{\mu_0} \leq f$ and $\sum_{n=1}^{X_0} Y_n$ converges almost surely to $X_0$ for $\alpha$ approaching 1. Furthermore, $D_{\mu_0}(s)$ is non-negative for any $s \in [0,1]$, and then equation (3) holds, for $X_0$ belonging to many families of type $M_{h^{-1}(\mu_0)}$, such as those of the Binomial, the Negative Binomial, the Logarithmic, the Sibuya, the discrete stable, the discrete Linnik.

Hence, owing to the effectiveness of $D_\mu(s)$ in highlighting discrepancies between $f$ and $g_\mu$ and its being non-negative or non-positive for any $s \in [0,1]$ in many circumstances where $f$ nearly resembles $g_\mu$, a sensible test statistic for assessing $H_0$ should be based on (2). Accordingly, it is straightforward to introduce a family of test statistics which depend on suitable estimators of $\mu$ and $P(X = 0)$.

Finally, all the previous considerations obviously hold also for $k = 2$. In this case the $L^1([0,1])$ distance is given by

$$\int_0^1 |D_{\mu,\mu_2}(s)| ds = \left| \frac{g_{\mu,\mu_2}(0) - P(X = 0)}{g_{\mu,\mu_2}(0)} \right|.$$  \hspace{1cm} (4)

and the family of test statistics depend on suitable estimators of $(\mu, \mu_2)$ and $P(X = 0)$.

3. The test statistics

Given a random sample $X_1, \ldots, X_n$ from $X$, denoting by $\hat{\mu}_n$ and $\hat{P}_n(0)$ the sample mean and the sample proportion of observations equal to zero, that is
Theorem to understand some mild conditions on $\psi$. Goodness-of-fit tests

The first part of the proposition is proven. Now, let $X$ be a r.v. such that $r_0 \neq 0$, $|\hat{T}_0|$ converges in probability to $\infty$ because $\tau(X_1) + \ldots + \tau(X_n) + o_P(1)$ is bounded in probability and $\sqrt{n}|r_0|$ converges to $\infty$. The second part of the proposition is so proven.

In order to obtain a test statistic, $\sigma^2$ can be estimated by means of the plug-in estimator.
\[ \hat{\sigma}_n^2 = (\psi_0')^2(\hat{\mu}_n)v(\hat{\mu}_n) + 2\hat{\mu}_n\psi_0(\hat{\mu}_n)\psi_0'(\hat{\mu}_n) + \psi_0(\hat{\mu}_n)(1 - \psi_0(\hat{\mu}_n)) \] (5)

where

\[ v(\hat{\mu}_n) = \sum_{j=0}^{\infty} (j - \hat{\mu}_n)^2 p_{\hat{\mu}_n}^{-1}(j). \]

Since \( \hat{\sigma}_n^2 \) converges a.s. and in quadratic mean to \( \sigma^2 \), the test statistic turns out to be

\[ Z_n = \frac{\hat{T}_0}{\hat{\sigma}_n}. \]

It is at once apparent that an \( \alpha \)-level large-sample test rejects \( H_0 \) for realizations of the test statistic whose absolute values are greater than \( z_{1-\alpha/2, (1-\alpha/2)} \)-quantile of the standard normal distribution.

When dealing with two-parameter families of distributions \( \mathcal{M}_{h^{-1}(\mu, \mu_2)} \), a suitable test statistic can be based on

\[ \hat{T}_0 = \sqrt{n}(g_{\hat{\mu}_n, \hat{\mu}_2,n}(0) - \hat{P}_n(0)) = \sqrt{n}(\psi_0(\hat{\mu}_n, \hat{\mu}_2,n) - \hat{P}_n(0)) \]

where \( \hat{\mu}_2,n = (X_1^2 + \ldots + X_n^2)/n \) and \( \psi_0 : (\mu, \mu_2) \mapsto \mathcal{M}_{h^{-1}(\mu, \mu_2)} \{(0)\} \).

Similarly, if \( \psi_0 \) is a \( C^1 \) function with bounded first-order partial derivatives, it can be straightforwardly proven that, under the null hypothesis, \( \hat{T}_0 \) converges in distribution to \( \mathcal{N}(0, \sigma^2) \) as \( n \to \infty \), where in this case

\[ \sigma^2 = \text{Var} \left[ \frac{\partial \psi_0}{\partial \mu}(\mu, \mu_2)X + \frac{\partial \psi_0}{\partial \mu_2}(\mu, \mu_2)X^2 - I\{X=0\} \right] \]

\[ = v(\mu, \mu_2) + 2\left( \frac{\partial \psi_0}{\partial \mu}(\mu, \mu_2)\mu + \frac{\partial \psi_0}{\partial \mu_2}(\mu, \mu_2)\mu_2 \right)\psi_0(\mu, \mu_2) + \psi_0(\mu, \mu_2)(1 - \psi_0(\mu, \mu_2)), \]

and \( v(\mu, \mu_2) = \text{Var} \left[ \frac{\partial \psi_0}{\partial \mu_2}(\mu, \mu_2)X + \frac{\partial \psi_0}{\partial \mu_2}(\mu, \mu_2)X^2 \right] \). Finally, \( \sigma^2 \) can be estimated by

\[ \hat{\sigma}_n^2 = \hat{s}_n + \hat{s}_n + \psi_0(\hat{\mu}_n, \hat{\mu}_2,n)(1 - \psi_0(\hat{\mu}_n, \hat{\mu}_2,n)) \] (6)

where

\[ \hat{s}_n = 2 \left( \frac{\partial \psi_0}{\partial \mu}(\hat{\mu}_n, \hat{\mu}_2,n)\hat{\mu}_n + \frac{\partial \psi_0}{\partial \mu_2}(\hat{\mu}_n, \hat{\mu}_2,n)\hat{\mu}_2,n \right)\psi_0(\hat{\mu}_n, \hat{\mu}_2,n) \]

and \( \hat{s}_n \) is an estimator of \( v(\mu, \mu_2) \). In general, if the closed form expressions of the third or the fourth moment of \( X \) are not known or are not easy to compute, a consistent and computationally convenient estimator of \( v(\mu, \mu_2) \) is given by

\[ \hat{v}_n = \frac{1}{n-1} \sum_{i=1}^{n} \left( \frac{\partial \psi_0}{\partial \mu}(\hat{\mu}_i, \hat{\mu}_2,n)X_i + \frac{\partial \psi_0}{\partial \mu_2}(\hat{\mu}_i, \hat{\mu}_2,n)X_i^2 - M_n \right)^2, \] (7)

where \( M_n = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial \psi_0}{\partial \mu}(\hat{\mu}_i, \hat{\mu}_2,n)X_i + \frac{\partial \psi_0}{\partial \mu_2}(\hat{\mu}_i, \hat{\mu}_2,n)X_i^2 \).

Obviously, alternative suitable estimators of \( \sigma^2 \) are possible for particular families of distributions, or by using maximum likelihood estimators of \( \mu \) and \( \mu_2 \) in (6). If \( \hat{v}_n \) is a consistent estimator of \( v(\mu, \mu_2) \), \( Z_n \) converges in distribution to \( \mathcal{N}(0, \sigma^2) \) and the test can be implemented as in the one-parameter case.
4. The test statistic for some families of distributions

4.1. Shifted Borel family

The Borel family arises in the context of queueing theory and branching processes. More precisely, the Borel distribution, obtained by Borel in 1942 ([8]), describes the distribution of the total number of customers served before a queue vanishes, given a single queue with Poisson random arrival of customers and a constant time in serving each customer, when there is initially one customer in the queue. The Borel distribution has parameter $\lambda$ if the constant time is 1 and the constant rate of arrivals is $\lambda$. Equivalently, the Borel distribution is the distribution of the total progeny of a Galton–Watson branching process where each individual has $\mathcal{P}(\lambda)$ children (e.g., [8], [16], [19]). In particular, these distributions are concentrated on $N$ when $\lambda \leq 1$ and have finite moment of any order when $\lambda < 1$.

In the following, the family of shifted Borel distributions $\{\text{Bo}(\lambda)\}_{\lambda \in [0,1]}$ with values in $\mathbb{N}$ and p.m.f.

$$P(X = n) = e^{-\lambda(n+1)} \frac{(\lambda(n+1))^n}{(n+1)!} \quad n \in \mathbb{N}$$

is considered. Since $\mu = \frac{\lambda}{1-\lambda} = h(\lambda)$, then $h^{-1}(\mu) = \frac{\mu}{1 + \mu}$. Moreover, $P(X = 0) = \psi_0(\mu) = e^{-\frac{\mu}{1 + \mu}}$ and

$$\hat{T}_0 = \sqrt{n} \left( e^{-\frac{\hat{\mu}_n}{1 + \hat{\mu}_n}} - \hat{P}_n(0) \right).$$

Owing to Proposition 1, $\hat{T}_0$ converges in distribution to $\mathcal{N}(0, \sigma^2)$ as $n \to \infty$, where

$$\sigma^2 = \text{Var} \left[ e^{-\frac{\mu}{1+\mu}} X + I_{\{X=0\}} \right]$$

and, since $\text{Var}[X] = \frac{\lambda}{(1-\lambda)p^2} = \mu(1 + \mu)^2$, from (4)

$$\sigma^2 = e^{-\frac{2\mu}{1+\mu}} \left( e^{\frac{\mu}{1+\mu}} - 1 - \frac{\mu}{(1 + \mu)^2} \right)$$

and from (5)

$$\sigma_n^2 = e^{-\frac{2\hat{\mu}_n}{1 + \hat{\mu}_n}} \left( e^{\frac{\hat{\mu}_n}{1 + \hat{\mu}_n}} - 1 \right).$$

4.2. Geometric family

Let us consider the well-known family of Geometric distributions $\{\mathcal{G}(p)\}_{p \in [0,1]}$. The random variable $X$ has p.m.f. given by

$$P(X = n) = p(1-p)^n \quad n \in \mathbb{N}.$$

Since $\mu = \frac{1-p}{p} = h(p)$, it holds $h^{-1}(\mu) = \frac{1}{\mu + 1}$ and $P(X = 0) = \psi_0(\mu) = \frac{1}{\mu + 1}$. Then

$$\hat{T}_0 = \sqrt{n} \left( \frac{1}{\hat{\mu}_n + 1} - \hat{P}_n(0) \right).$$
and $\hat{T}_0$ converges in distribution to $\mathcal{N}(0, \sigma^2)$ as $n \to \infty$, where
\[
\sigma^2 = \text{Var} \left[ \frac{X}{(\mu + 1)^2} + I_{\{X=0\}} \right].
\]

From Proposition 1, since $\text{Var}[X] = \frac{1-\mu}{\mu^2} = \mu(1 + \mu)$, from (4) and from (5) it follows
\[
\sigma^2 = \frac{\mu^2}{(\mu + 1)^3}
\]
and
\[
\hat{\sigma}_n^2 = \frac{\hat{\mu}_n^2}{(\hat{\mu}_n + 1)^3}.
\]

### 4.3. One-parameter Bell family

The one-parameter Bell family of distributions $\{B_{e}(\theta)\}_{\theta>0}$ has been recently introduced by Tsylova and Ekgauz [37] and Castellares et al. [9]. Bell distributions have many interesting properties. Indeed, the Bell family belongs to the one-parameter exponential family and it is infinitely divisible. Moreover, the Poisson distribution is not nested in the Bell family but it can be approximated for small values of the parameter by the Bell distribution. This family is rather flexible for fitting a wide spectrum of count data which may present over-dispersion and it may be an alternative model to the very familiar Poisson and Negative Binomial models in several areas. For example, owing to its flexibility, the Bell distribution is used to model the number of insurance claims over a fixed period of time. For further applications see [4].

The Bell p.m.f. with parameter $\theta$ has a very simple form and it is given by
\[
P(X = n) = \frac{\theta^n B_n e^{1-e^{\theta}}}{n!} \quad n \in \mathbb{N},
\]
where the Bell number $B_n$ (see [5]) is the number of partitions of a set of size $n$ and is equal to the $n$-th moment of a Poisson r.v. with $\mu = 1$. Since $\mu = \theta e^{\theta} = h(\theta)$, $h^{-1}(\mu)$ does not have closed form. Thus $P(X = 0) = \psi_0(\mu) = e^{1-e^{h^{-1}(\mu)}}$ and
\[
\hat{T}_0 = \sqrt{n}(e^{1-e^{h^{-1}(\hat{\mu}_n)}} - \hat{P}_n(0)).
\]

From Proposition 1, $\hat{T}_0$ converges in distribution to $\mathcal{N}(0, \sigma^2)$ as $n \to \infty$, where
\[
\sigma^2 = \text{Var} \left[ \frac{e^{1-e^{h^{-1}(\mu)}}}{(1 + h^{-1}(\mu))} X + I_{\{X=0\}} \right].
\]

Moreover, since $\text{Var}[X] = \theta e^{\theta}(1 + \theta) = \mu(1 + h^{-1}(\mu))$, from (4) and from (5) it follows
\[
\sigma^2 = e^{1-e^{h^{-1}(\mu)}} \left( 1 - \frac{h^{-1}(\mu)e^{1+h^{-1}(\mu)} - e^{h^{-1}(\mu)}}{1 + h^{-1}(\mu)} \right).
\]
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4.4. Two-parameter Bell family

The two-parameter Bell family of distributions \( \{Be(\alpha, \theta)\}_{\alpha, \theta > 0} \), also called the Bell-Touchard family, has been proposed by Castellares et al. \[10\] using the series expansion presented in \[36\]. This family of infinitely divisible distributions is very simple to deal with, since its p.m.f. does not contain any complicated function. Further, it is very flexible and it can be used quite effectively for modelling count data which present too many zeros and/or over-dispersion. Insurance is an important field of application of both the one- and the two-parameter Bell distributions. The Bell p.m.f. with parameters \( \alpha \) and \( \theta \) is given by

\[
P(X = n) = \frac{\theta^n T_n(\alpha) e^{\alpha(1-e^\theta)}}{n!} \quad n \in \mathbb{N},
\]

where \( T_n(\alpha) \) is the Touchard polynomial defined by

\[
T_n(\alpha) = \frac{e^{-\alpha} \sum_{k=0}^{\infty} k^n \alpha^k}{n!} = \sum_{j=0}^{n} \alpha^j \binom{n}{j}
\]

where \( \left\{ \frac{n}{j} \right\} \) are the Stirling numbers of the second kind, i.e. the number of possible ways to partition a set of \( n \) elements into \( j \) non-empty subsets ([11] and [10]). The first- and second-order moments are \( (\mu, \mu_2) = (\alpha \theta e^\theta, \alpha \theta e^\theta (\alpha \theta e^\theta + 1 + \theta)) = h(\alpha, \theta) \) by Bhati and Calderín-Ojeda [7] and Castellares et al. [10]. Since

\[
h^{-1}(\mu, \mu_2) = \left( -\frac{\mu^2}{\mu^2 - \mu_2 + \mu}, -\frac{\mu^2}{\mu_2 - \mu + \mu} \right)
\]

and \( P(X = 0) = \psi_0(\mu, \mu_2) = \exp \left( \frac{\mu^2}{\mu_2 - \mu^2} \left( e^{-\mu_2} - 1 \right) \right) \), then

\[
\hat{T}_0 = \sqrt{n} \left( \exp \left( \frac{\hat{\mu}_n^2}{\hat{\mu}_n^2 - \mu_2} \left( e^{-\hat{\mu}_n^2} - 1 \right) \right) - \hat{P}_n(0) \right).
\]

Moreover, since

\[
\frac{\partial \psi_0}{\partial \mu}(\mu, \mu_2) = \psi_0(\mu, \mu_2) \left( \frac{\mu (2 \mu_2 - \mu)}{(\mu_2 - \mu^2 - \mu)} \left( e^{-\mu_2} - 1 \right) + \frac{(\mu_2 + \mu) e^{-\mu_2} - \mu_2}{(\mu_2 - \mu^2 - \mu)} \right)
\]

and

\[
\frac{\partial \psi_0}{\partial \mu_2}(\mu, \mu_2) = \psi_0(\mu, \mu_2) \left( \frac{-\mu_2}{(\mu_2 - \mu^2 - \mu)} \left( e^{-\mu_2} - 1 \right) - \frac{\mu e^{-\mu_2} - \mu_2}{(\mu_2 - \mu^2 - \mu)} \right),
\]

from (6)

\[
\hat{\sigma}_n^2 = \hat{\sigma}_n + \hat{s}_n + \psi_0(\hat{\mu}_n, \hat{\mu}_2, n)(1 - \psi_0(\hat{\mu}_n, \hat{\mu}_2, n)),
\]
Proof. Consider the difference between

\[ R_n = \bar{T}_0 - \bar{\tilde{T}}_0 - \lambda\left( P(X_1 = 0) - P(Y_1 = 0) \right) \]

5. Asymptotic behaviour under contiguous alternatives

The asymptotic behaviour of the test statistic \( Z_n \) is investigated under suitable contiguous alternatives, obtained by mixtures of distributions. The concept of contiguity, developed by Le Cam in [23], is frequently applied in many asymptotic settings (e.g., [38], [12], [6], [20], [27] among others). In particular, let \( \{ A_{t,n} \}_{t=1}^{\cdots,n} \) be a triangular array of independent events and \( (Y_n)_{n \geq 1} \) be a sequence of i.i.d. non-negative integer-valued r.v.s with \( E[Y_1^2] < \infty \). Moreover, suppose \( (I_{A_{t,n}})_{t=1}^{\cdots,n} \) and \( Y_1, \ldots, Y_n \) to be mutually independent and also independent of the i.i.d. random variables \( X_1, \ldots, X_n \), where now \( X_1 \sim M_{h^{-1}(\mu)} \) with \( \mu = E[X_1] \). For \( l = 1, \ldots, n \), denote by

\[ X'_{l,n} = I_{A_{l,n}} X_l + I_{A_{l,n}^c} Y_l \] (8)

with \( P(A_{t,n}^c) = \frac{\lambda}{\sqrt{n}} > 0 \). Given the random sample \( X'_{1,n}, \ldots, X'_{n,n} \), let

\[ \bar{\mu} = \frac{X'_{1,n} + \ldots + X'_{n,n}}{n}, \quad \bar{P}_n(0) = \frac{I_{\{X'_{1,n} = 0\}} + \ldots + I_{\{X'_{n,n} = 0\}}}{n} \]

be the corresponding sample mean and sample proportion. It is at once apparent that also \( X'_{l,n} \) is a non-negative integer-valued r.v. which converges to \( X_l \) for \( n \) approaching infinity.

The following result is useful to highlight the discriminatory capability of the test statistic under non-trivial contiguous alternatives.

Proposition 2. Let \( \psi_0 : \mu \mapsto M_{h^{-1}(\mu)}(\{0\}) \) be a \( C^1 \) function with bounded first-order derivative. If \( E[Y_1] = E[X_1] \) then

\[ \bar{T}_0 - \lambda(P(X_1 = 0) - P(Y_1 = 0)) \xrightarrow{d} \mathcal{N}(0, 1) \] (9)

where

\[ \bar{T}_0 = \sqrt{n}(\psi_0(\bar{\mu}_n) - \bar{P}_n(0)) \]

and

\[ \sigma^2_n = (\psi_0' \bar{\mu}_n)^2 + \psi_0(\bar{\mu}_n) - 2 \bar{\mu}_n \psi_0(\bar{\mu}_n) \psi_0(\bar{\nu}_n) + \psi_0(\bar{\nu}_n) (1 - \psi_0(\bar{\mu}_n)) \]

Proof. Consider the difference between \( \bar{T}_0 - \lambda(P(X_1 = 0) - P(Y_1 = 0)) \) and \( \bar{\tilde{T}}_0 \), that is

\[ R_n = \bar{\tilde{T}}_0 - \bar{T}_0 - \lambda(P(X_1 = 0) - P(Y_1 = 0)) \]
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Note that
\[ R_n = \sqrt{n} \left( \psi_0(\hat{\mu}_n) - \hat{P}_n(0) \right) - \sqrt{n} \left( \psi_0(\hat{\mu}_n) - \hat{P}_n(0) \right) - \lambda \left( P(X_1 = 0) - P(Y_1 = 0) \right) \\
= \psi_0'(\hat{\mu}_n)(\hat{\mu}_n - \hat{\mu}_n) - (\hat{P}_n(0) - \hat{P}_n(0)) - \lambda \left( P(X_1 = 0) - P(Y_1 = 0) \right) + o_P(1). \]

Since
\[ \hat{\mu}_n - \hat{\mu}_n = \frac{\sum_{l=1}^n I_{A_{i,n}}(Y_l - X_l)}{n} \]
and
\[ \hat{P}_n(0) - \hat{P}_n(0) = \frac{\sum_{l=1}^n I_{A_{i,n}}(I_{\{Y_l=0\}} - I_{\{X_l=0\}})}{n}, \]
then
\[ R_n = \psi_0'(\hat{\mu}_n) \frac{\sum_{l=1}^n I_{A_{i,n}}(Y_l - X_l)}{\sqrt{n}} - \frac{\sum_{l=1}^n \left( I_{A_{i,n}}(I_{\{Y_l=0\}} - I_{\{X_l=0\}}) - a_n \right)}{\sqrt{n}} + o_P(1), \]
where \( a_n = \lambda \left( P(Y_1 = 0) - P(X_1 = 0) \right) / \sqrt{n} = E[I_{A_{i,n}}(I_{\{Y_l=0\}} - I_{\{X_l=0\}})]. \)

Moreover, as
\[ E \left[ \left( \frac{\sum_{l=1}^n I_{A_{i,n}}(Y_l - X_l)}{\sqrt{n}} \right)^2 \right] = \frac{\lambda E[(Y_1 - X_1)^2]}{\sqrt{n}} \]
and
\[ E \left[ \left( \frac{\sum_{l=1}^n \left( I_{A_{i,n}}(I_{\{Y_l=0\}} - I_{\{X_l=0\}}) - a_n \right) \right)^2}{\sqrt{n}} \right] = \frac{\lambda \text{Var}[I_{A_{i,n}}(I_{\{Y_l=0\}} - I_{\{X_l=0\}})]}{\sqrt{n}} \leq \frac{\lambda}{\sqrt{n}}, \]
\( R_n \) converges in probability to 0. Therefore, also \( R_n/\hat{\sigma}_n \) converges in probability to 0 as \( \hat{\sigma}_n \) converges in probability to \( \text{Var}[\psi_0'(\hat{\mu})X_1 - I_{\{X_1=0\}}] \) and
\[ \frac{\tilde{T}_0 - \lambda(P(X_1 = 0) - P(Y_1 = 0))}{\hat{\sigma}_n} \]
has the same asymptotic distribution of \( \tilde{T}_0/\hat{\sigma}_n \). As \( \hat{\sigma}_n \) and \( \hat{\sigma}_n \) are asymptotically equivalent, since they both converge in probability to \( \text{Var}[\psi_0'(\mu)X_1 - I_{\{X_1=0\}}] \), the convergence of (9) is proven. \( \square \)

Remark 1. Given a family of i.i.d. Bernoulli r.v.s \( (Y_{j,t})_{j,t} \) with parameter \( 1 - \frac{\lambda}{\sqrt{n}} \) independent of \( (X_{j,t})_n \), let \( X'_{l,n} = \sum_{j=1}^{X_l} Y_{j,l} \) be the \( \alpha \)-fraction of \( X_l \), with \( l = 1, \ldots, n \). Since \( E[X'_{l,n}] - \mu \) is equal to \( -\mu \lambda / \sqrt{n} \), following the same reasoning of Proposition 2,
\[ \frac{\tilde{T}_0 + \lambda(P(X_1 = 1) - \hat{\mu}_n)}{\hat{\sigma}_n} \overset{d}{\to} \mathcal{N}(0,1). \]

The previous proposition can be straightforwardly generalized to the case of two-parameter families of distributions. In the following let \( \hat{\mu}_{2,n} = (X_{1,n}^2 + \ldots + X_{n,n}^2) / n. \)
Proposition 3. Let $\psi_0 : (\mu, \mu_2) \rightarrow \mathcal{M}_{h^{-1}(\mu_n, \mu_2,n)}(\{0\})$ be a C$^1$ function with bounded first-order partial derivatives. If $E[Y_1] = E[X_1]$ and $E[Y_1^4] < \infty$, then

$$\frac{\bar{T}_0 - \lambda(P(X_1 = 0) - P(Y_1 = 0) + \frac{\partial \psi_0}{\partial \mu_2}(\bar{\mu}_n, \bar{\mu}_2, n)(E[Y_1^2] - E[X_1^2])}{\sigma_n} \rightarrow \mathcal{N}(0,1)$$

where

$$\bar{T}_0 = \sqrt{n}(\psi_0(\bar{\mu}_n, \bar{\mu}_2, n) - \hat{P}_n(0))$$

and

$$\bar{\sigma}_n^2 = \bar{\sigma}_n^2 + \hat{\sigma}_n + \psi_0(\mu_n, \mu_2, n)(1 - \psi_0(\mu_n, \mu_2, n))$$

with $\hat{\sigma}_n = 2\left(\frac{\partial \psi_0}{\partial \mu}(\mu_n, \mu_2, n)\mu_n + \frac{\partial \psi_0}{\partial \mu_2}(\mu_n, \mu_2, n)\mu_2, n\right)\psi_0(\mu_n, \mu_2, n)$ and $\hat{\sigma}_n$ is a consistent estimator of $\text{Var} \left[ \frac{\partial \psi_0}{\partial \mu}(\mu, \mu_2)X_1 + \frac{\partial \psi_0}{\partial \mu}(\mu, \mu_2)X_1^2 \right]$.

Proof. Following the same reasoning of the proof of Proposition 2, let

$$R_n = \bar{T}_0 - \hat{T}_0 - \lambda(P(X_1 = 0) - P(Y_1 = 0) + \frac{\partial \psi_0}{\partial \mu_2}(\mu, \mu_2)(E[Y_1^2] - E[X_1^2]))$$

$$= \frac{\partial \psi_0}{\partial \mu}(\mu, \mu_2)(\bar{\mu}_n - \hat{\mu}_n) + \frac{\partial \psi_0}{\partial \mu_2}(\mu, \mu_2)(\bar{\mu}_2, n - \hat{\mu}_2, n) - (\hat{P}_n(0) - \hat{P}_n(0)) - \lambda(P(X_1 = 0) - P(Y_1 = 0) + \frac{\partial \psi_0}{\partial \mu_2}(\mu, \mu_2)(E[Y_1^2] - E[X_1^2])) + o_P(1).$$

Since

$$\bar{\mu}_2, n - \hat{\mu}_2, n = \frac{\sum_{l=1}^{n} I_{A_{l,n}^c}(Y_l^2 - X_l^2)}{n},$$

then

$$R_n = \frac{\partial \psi_0}{\partial \mu}(\mu, \mu_2)\frac{\sum_{l=1}^{n} I_{A_{l,n}^c}(Y_l - X_l)}{\sqrt{n}} + \frac{\partial \psi_0}{\partial \mu_2}(\mu, \mu_2)\frac{\sum_{l=1}^{n} I_{A_{l,n}^c}(Y_l^2 - X_l^2) - \frac{\lambda}{\sqrt{n}}(E[Y_1^2] - E[X_1^2])}{\sqrt{n}}$$

$$- \frac{\sum_{l=1}^{n} (I_{A_{l,n}^c}(I_{\{Y_l=0\}} - I_{\{X_l=0\}})(-a_n))}{\sqrt{n}} + o_p(1),$$

where $a_n = E[I_{A_{l,n}^c}(I_{\{Y_l=0\}} - I_{\{X_l=0\}})] = \lambda(P(Y_1 = 0) - P(X_1 = 0))/\sqrt{n}$. From the proof of Proposition 2 and since $E[Y_1^4] < \infty$, $R_n$ converges in probability to 0. Therefore, also $R_n/\bar{\sigma}_n$ converges in probability to 0 as $\bar{\sigma}_n$ converges in probability to $\text{Var} \left[ \frac{\partial \psi_0}{\partial \mu}(\mu, \mu_2)X_1 + \frac{\partial \psi_0}{\partial \mu_2}(\mu, \mu_2)X_1^2 - I_{\{X_1=0\}} \right]$ and

$$\frac{\bar{T}_0 - \lambda(P(X_1 = 0) - P(Y_1 = 0) + \frac{\partial \psi_0}{\partial \mu_2}(\bar{\mu}_n, \bar{\mu}_2, n)(E[Y_1^2] - E[X_1^2]))}{\bar{\sigma}_n}$$

has the same asymptotic distribution of $\bar{T}_0/\bar{\sigma}_n$. As $\bar{\sigma}_n$ and $\bar{\sigma}_n$ are asymptotically equivalent, the thesis is proven. 

\[\square\]
Remark 2. In both propositions the asymptotic behaviour still holds also removing the condition of bounded first-order (partial) derivates even if, in this case, the order of convergence of $\hat{\sigma}_n$ could be considerably reduced.

6. Simulation study

The performance of the proposed test has been assessed and compared to that of the chi-squared goodness-of-fit test by means of an extensive Monte Carlo simulation. Indeed, the chi-squared test is suitable for the very general hypothesis system (1) and, similarly to our proposal, it is based on a test statistic having known asymptotic distribution and not requiring intensive resampling methods. It is worth noting that the comparison is also meaningful since both tests rely on distance-based statistics.

As to the chi-squared test, it is well-known that the asymptotic approximation is usually satisfactory if each expected frequency is large enough, with some authors suggesting the minimum value of 5 and some others of 3. Therefore, there is no way of avoiding the arbitrariness of grouping classes to compute the chi-squared statistic. [14]. Moreover, the standard implementation considers the sample maximum and the sample minimum as extreme values for the test statistic computation and thanks to simulation studies it is well-known that this choice is not enough in order to capture a sufficient probability mass under $H_0$. Hence, to ensure a reliable approximation, an ad-hoc general version of the chi-squared test is implemented. In particular, denoting by \{ $C_1, \ldots, C_k$ \} the set of classes to be considered, $C_1$ is the set of natural numbers smaller than the largest integer not greater than $\mu - 3\sqrt{\mu}$, $C_k$ is the set of natural numbers greater than the smallest integer not less than $\mu + 3\sqrt{\mu}$ and \{ $C_2, \ldots, C_{k-1}$ \} are the singletons not included in $C_1$ and $C_k$. Obviously expected frequencies are obtained by plugging the parameter estimates, also adopted in $Z_n$, in the null distribution. The simulation is implemented by using R [35].

6.1. Empirical significance level

First of all, we focus on empirically evaluating the actual significance level of the test. To this purpose, fixed the nominal level $\alpha = 0.05$, 5000 samples of size $n = 30, 50$ are independently generated from shifted Borel, Geometric, one-parameter and two-parameter Bell distributions and the empirical significance level is computed as the proportion of rejections of the null hypothesis both for $Z_n$ and the chi-squared statistic $Q_n$. In particular, Figure 1 and Figure 2 display the empirical significance level for the shifted Borel and the Geometric distribution, respectively, for values of $\mu$ varying from 0.5 to 15 by 1. Figure 3 and Figure 4 show the empirical significance level for the one-parameter and two-parameter Bell distribution, respectively. For the Bell distributions, $P(X = 0)$ is reported in the abscissa as the two-parameter distribution depends also on $\mu_2$. In particular, for the first distribution the parameter is chosen to have $P(X = 0)$ varying from 0.05 to 0.95 by 0.07, while for the second one both parameters are chosen to vary from 0.3 to 1.3 by 0.2 and all the resulting pairs of values are considered.

In Figure 1 the proposed test shows an empirical significance level almost equal to the nominal one for any $\mu$ already for the smaller sample size. On the other hand, the chi-squared test does not have a satisfactory behaviour even for the larger sample size, especially for large values of $\mu$, probably owing to the slow rate of convergence of $Q_n$. In Figure 2, for $n = 30$, $Z_n$ shows conservativeness for larger $\mu$ values while the empirical level of $Q_n$ is greater than the nominal one and increases as $\mu$ increases. On the other hand, for $n = 50$ the empirical level reached by $Z_n$ is almost indistinguishable from the nominal one and, even if the performance of $Q_n$ greatly improves, it is not completely satisfactory especially for the larger values of $\mu$. It is worth noting that under Geometric distribution $Z_n$ has a very simple expression leading to a straightforward implementation of the test.
Figure 1. Empirical significance level under shifted Borel distribution for $n = 30$ (left panel) and $n = 50$ (right panel). In the abscissa $\mu$ values.

Figure 2. Empirical significance level under Geometric distribution for $n = 30$ (left panel) and $n = 50$ (right panel). In the abscissa $\mu$ values.

From Figure 3, it is at once apparent that the performances of both tests are quite satisfactory when $P(X = 0)$ ranges between 0.1 and 0.8. Moreover, for $P(X = 0)$ near to 0.9, while the empirical level of $Z_n$ remains rather close to the nominal one, especially for $n = 50$, the empirical level of $Q_n$ dramatically increases.

Figure 4 shows how, under the two-parameter Bell distribution, the discriminatory capability of both tests reduces, especially for large values of $P(X = 0)$, but while the performance of $Q_n$ is not
Figure 3. Empirical significance level under one-parameter Bell distribution for $n = 30$ (left panel) and $n = 50$ (right panel). In the abscissa values of $P(X = 0)$.

Figure 4. Empirical significance level under two-parameter Bell distribution for $n = 30$ (left panel) and $n = 50$ (right panel). In the abscissa values of $P(X = 0)$.

satisfactory at all, the empirical level of $Z_n$ is rather close to the nominal one when $n = 50$ and for values of $P(X = 0) < 0.7$. To analyse the behaviour of both tests for larger sample sizes, Figure 5 displays the empirical level for $n = 100$ and $n = 200$, highlighting that the performance of $Q_n$ remains
6.2. Empirical power

The empirical power of the test based on $Z_n$ is investigated and compared to that of the chi-squared test considering under the null hypothesis the same distributions already considered for assessing the empirical significance level. The power behaviour is assessed against some common alternative distributions with various parameter values and against contiguous alternatives, as introduced in Section 5.

Alternative distributions include overdispersed and underdispersed, mixtures and zero-inflated distributions together with distributions having mean close to variance. In particular, as well as in [15] and [26], we consider

- **Poisson** denoted by $\mathcal{P}(\lambda)$
- **Mixture of two Poisson** denoted by $\mathcal{M}\mathcal{P}(\lambda_1, \lambda_2)$ with mixture weight 0.5
- **Binomial** denoted by $\mathcal{B}(k, p)$
- **Negative Binomial** denoted by $\mathcal{N}\mathcal{B}(k, p)$
- **Generalized Hermite** denoted by $\mathcal{G}\mathcal{H}(a, b, k)$,
- **Discrete Uniform in $\{0, 1, \ldots, \nu\}$** denoted by $\mathcal{D}\mathcal{U}(\nu)$
- **Logarithmic Series** denoted by $\mathcal{L}\mathcal{S}(\theta)$
- **Generalized Poisson** denoted by $\mathcal{G}\mathcal{P}(\lambda_1, \lambda_2)$
- **Zero-inflated Binomial** denoted by $\mathcal{Z}\mathcal{B}(k, p_1, p_2)$
- **Zero-inflated Negative Binomial** denoted by $\mathcal{Z}\mathcal{N}\mathcal{B}(k, p_1, p_2)$
- **Zero-inflated Poisson** denoted by $\mathcal{Z}\mathcal{P}(\lambda_1, \lambda_2)$

where various parameters values are considered (see Table 1 and 2). From each distribution, 5000 samples of size $n = 30, 50$ are independently generated and on each sample the tests based on $Z_n$ completely unsatisfactory, while that of $Z_n$ improves since the empirical level are stably close to the nominal one for $P(X = 0)$ up to 0.8.

**Figure 5.** Empirical significance level under two-parameter Bell distribution for $n = 100$ (left panel) and $n = 200$ (right panel). In the abscissa values of $P(X = 0)$. 
Goodness-of-fit tests

and $Q_n$ are performed at the nominal significance level $\alpha = 0.05$. The empirical power of each test is computed as the proportion of rejections of the null hypothesis. Table 1 reports, for each one-parameter model specified under null hypothesis and for each alternative distribution, the empirical powers and achieved by the test based on $Z_n$ and $Q_n$, respectively, for $n = 30$, while Table 2 contains the same information for $n = 50$.

Not surprisingly, none of the two tests shows better performance with all models and all alternative distributions. Indeed, the power crucially depends both on the model specified under the null and alternative hypothesis and on the set of parameter values for alternatives in the same class. In particular, when under the null hypothesis the shifted Borel distribution is considered, the performance of both tests is rather satisfactory even with $n = 30$, but the proposed test seems to be superior. If the geometric distribution is specified under $H_0$, the power of both tests generally decreases and heavily deteriorates for zero-inflated alternative distributions, with the only remarkable exception for the chi-squared test when the zero-inflated binomial distribution is considered. Finally, a further decrease in the power of both tests occurs for the one-parameter Bell distribution, with $Z_n$ reaching unbiasedness against the generalized Hermite alternatives only for $n = 50$, even thought in this case both tests show a very poor performance.

As to the contiguous alternatives, for each distribution specified under the null hypothesis, shrinking mixtures are obtained according to (8). In particular, the component $X_i$ is a shifted Borel r.v., a Geometric r.v., a one-parameter Bell r.v., all having $\mu = 1$, respectively, while $Y_i \sim B(4, 0.25)$ and $\lambda$ varies from 0 to $\sqrt{n}$ by 0.5. Figure 6 shows that the empirical power is rather satisfactory for both tests, with the best performance achieved with shifted Borel component. However, the empirical power of $Z_n$ is higher for any value of $\lambda$ and any component.

Finally, Figure 7 shows the empirical power under alternatives obtained by means of the binomial thinning. Since the thinning operator preserves the law in most of the cases, we report the Borel case in which the law is not preserved and the Geometric case in which the law is well-known to be preserved. In both cases $\mu$ is fixed at 15 and $\lambda$ varies from 0 to 6.5 by 0.5. Coherently with the theoretical results, in the Geometric case the empirical power of both tests remains constantly close to the nominal level. On the other hand, in the Borel case the empirical power of both tests start to increase when $\lambda$ increases but $Z_n$ performs better than $Q_n$. 

\[ \text{Figure 6. Empirical power under contiguous distributions for } n = 50. \text{ In the abscissa } \lambda \text{ values.} \]
### Table 1. Empirical power (percent) with 5% nominal significance level and \( n = 30 \).

| Alternative | \( \mu \times 0.5 \) | \( \mu \times 1 \) | \( \mu \times 2 \) | \( \mu \times 3 \) | \( \mu \times 4 \) | \( \mu \times 5 \) |
|-------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( P(0.5) \) | 56.0 | 47.0 | 25.2 | 12.4 | 18.0 | 10.9 |
| \( P(1) \) | 94.7 | 91.4 | 57.7 | 38.9 | 39.3 | 20.2 |
| \( P(2) \) | 100.0 | 100.0 | 88.0 | 77.5 | 54.4 | 26.1 |
| \( MP(1, 2) \) | 99.0 | 97.8 | 62.4 | 43.4 | 31.7 | 14.3 |
| \( MP(1, 3) \) | 99.0 | 97.9 | 45.7 | 34.7 | 13.5 | 6.4 |
| \( MP(1, 4) \) | 98.3 | 96.8 | 26.6 | 26.6 | 5.3 | 6.7 |
| \( B(4, 0.25) \) | 99.4 | 98.9 | 85.9 | 72.4 | 73.6 | 50.4 |
| \( B(30, 0.1) \) | 100.0 | 100.0 | 97.8 | 97.2 | 60.2 | 34.4 |
| \( NB(4, 0.75) \) | 94.2 | 89.5 | 39.8 | 24.8 | 17.4 | 8.4 |
| \( NB(10, 0.9) \) | 94.7 | 90.7 | 51.2 | 32.1 | 30.2 | 13.5 |
| \( GH(1, 1.25, 2) \) | 100.0 | 100.0 | 43.0 | 62.3 | 4.0 | 6.1 |
| \( GH(1, 1.5, 2) \) | 100.0 | 100.0 | 44.7 | 69.8 | 3.3 | 7.2 |
| \( ZB(5, 0.9, 0.2) \) | 93.3 | 100.0 | 2.8 | 100.0 | 33.6 | 100.0 |
| \( ZNB(5, 0.9, 0.1) \) | 36.7 | 30.4 | 11.7 | 6.6 | 7.9 | 5.5 |
| \( ZP(1, 0.2) \) | 65.2 | 60.3 | 18.3 | 12.7 | 9.8 | 5.4 |

### Table 2. Empirical power (percent) with 5% nominal significance level and \( n = 50 \).

| Alternative | \( \mu \times 0.5 \) | \( \mu \times 1 \) | \( \mu \times 2 \) | \( \mu \times 3 \) | \( \mu \times 4 \) | \( \mu \times 5 \) |
|-------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( P(0.5) \) | 77.8 | 70.4 | 37.0 | 24.7 | 28.9 | 16.8 |
| \( P(1) \) | 99.6 | 99.0 | 79.7 | 64.2 | 60.4 | 34.9 |
| \( P(2) \) | 100.0 | 100.0 | 98.7 | 96.4 | 78.7 | 50.6 |
| \( MP(1, 2) \) | 100.0 | 100.0 | 83.7 | 70.0 | 49.5 | 25.0 |
| \( MP(1, 3) \) | 100.0 | 100.0 | 68.3 | 54.1 | 19.6 | 9.0 |
| \( MP(1, 4) \) | 100.0 | 99.9 | 42.4 | 40.4 | 5.4 | 7.8 |
| \( B(4, 0.25) \) | 100.0 | 100.0 | 75.5 | 68.5 | 10.1 | 6.0 |
| \( B(30, 0.1) \) | 100.0 | 100.0 | 100.0 | 100.0 | 87.1 | 69.2 |
| \( NB(4, 0.75) \) | 99.7 | 98.4 | 58.9 | 41.2 | 26.5 | 12.7 |
| \( NB(10, 0.9) \) | 99.7 | 98.9 | 72.3 | 54.3 | 47.1 | 24.6 |
| \( GH(1, 1.25, 2) \) | 100.0 | 100.0 | 67.9 | 85.6 | 5.4 | 8.7 |
| \( GH(1, 1.5, 2) \) | 100.0 | 100.0 | 72.4 | 91.6 | 5.0 | 9.2 |
| \( ZB(5, 0.9, 0.2) \) | 99.2 | 100.0 | 3.4 | 100.0 | 53.0 | 100.0 |
| \( ZNB(5, 0.9, 0.1) \) | 53.3 | 48.3 | 15.7 | 10.1 | 10.9 | 6.7 |
| \( ZP(1, 0.2) \) | 86.0 | 81.2 | 26.7 | 20.3 | 14.5 | 7.8 |
Figure 7. Empirical power under $\alpha$-fraction distributions for $n = 50$. In the abscissa $\lambda$ values.

7. Concluding remarks

Many discrete probability distributions have been introduced with the aim of describing and modelling count data, which are encountered in several fields of applications, and therefore goodness-of-fit tests for count distributions are essential. While a huge literature deals with testing continuous distributions, a few proposals have been developed for testing the fit of discrete distributions. Among these, many are tailored to deal with particular distributions and so they are of limited applicability, while there is an emerging need of tests allowing to specify an extremely broad class of distributions under the null hypothesis. Undoubtedly, the chi-squared test is the most widely adopted, notwithstanding the arbitrariness of its implementation due to the requirement on the frequency minimum values.

We propose a goodness-of-fit test, based on the p.g.f., for all the families of distributions indexed by a $k$-variate parameter having finite moment of order $2k$. The test statistic has a simple expression, depending on the empirical p.g.f. only through the probability of zero occurrence, and an asymptotic normal distribution. Moreover, the test is omnibus, being consistent with respect to all the alternative distributions. As to the test performance, the empirical significance level is always very close to the nominal one and the empirical power is rather satisfactory even for very moderate sample sizes, also compared to that of the chi-squared test. The test shows some criticalities when an inflation of zeroes occurs, which could be overcome introducing a family of test statistics, indexed by a parameter $h$, depending on the cumulative distribution function at $h$ instead of the probability in zero. Further research will be devoted to this last issue and to the use of alternative estimators involved in the implementation of the statistic, in order to improve the performance of the test.

References

[1] Baccini, A., Barbesi, L. and Stracqualursi, L. (2016). Random variate generation and connected computational issues for the Poisson–Tweedie distribution. Computational Statistics 31 729–748.
[2] BARABESI, L., BECATTI, C. and MARCHESELLI, M. (2018). The tempered discrete Linnik distribution. Statistical Methods & Applications 27 45–68.

[3] BARABESI, L. and PRATELLI, L. (2014). A note on a universal random variate generator for integer-valued random variables. Statistics and Computing 24 589–596.

[4] BATSIDIS, A., JIMÉNEZ-GAMERO, M. D. and LEMONTE, A. J. (2020). On goodness-of-fit tests for the Bell distribution. Metrika 83 297–319.

[5] BELL, E. T. (1934). Exponential numbers. The American Mathematical Monthly 41 411–419.

[6] BORÉL, E. (1942). Sur l’emploi du théorème de Bernoulli pour faciliter le calcul d’une infinité de coefficients. Application au problème de l’attentea un guichet. CR Acad. Sci. Paris 214 452–456.

[7] CASTELLARES, F., FERRARI, S. L. and LEMONTE, A. J. (2018). On the Bell distribution and its associated regression model for count data. Applied Mathematical Modelling 56 172–185.

[8] CASTELLARES, F., LEMONTE, A. J. and MORENO-ARENAS, G. (2020). On the two-parameter Bell–Touchard discrete distribution. Communications in Statistics—Theory and Methods 49 4834–4852.

[9] CHASTELLARES, F., LEMONTE, A. J. and MORENO-ARENAS, G. (2020). On the two-parameter Bell–Touchard discrete distribution. Communications in Statistics—Theory and Methods 49 4834–4852.

[10] D’HAR, S. S., DASSIOS, A. and BERGSMA, W. (2016). A study of the power and robustness of a new test for independence against contiguous alternatives. Electronic Journal of Statistics 10 330–351.

[11] EL-SHAARAWI, A. H., ZHU, R. and JOE, H. (2011). Modelling species abundance using the Poisson–Tweedie family. Environmetrics 22 152–164.

[12] GIBBONS, J. D. and CHAKRABORTI, S. (2020). Nonparametric statistical inference. CRC press.

[13] GUPTA, N. and HENZE, N. (2000). Recent and classical goodness-of-fit tests for the Poisson distribution. Journal of Statistical Planning and Inference 90 207–225.

[14] JANSÓN, S. and LUCZAK, M. J. (2008). Susceptibility in subcritical random graphs. Journal of mathematical physics 49 125207.

[15] JIMÉNEZ-GAMERO, M. and BATSIDIS, A. (2017). Minimum distance estimators for count data based on the probability generating function with applications. Metrika 80 503–545.

[16] JIMÉNEZ-GAMERO, M.-D. and ALBA-FERNÁNDEZ, M. (2019). Testing for the Poisson–Tweedie distribution. Mathematics and Computers in Simulation 164 146–162.

[17] JOHNSON, N. L., KEMP, A. W. and KOTZ, S. (2005). Univariate discrete distributions 444. John Wiley & Sons.

[18] KALEMKERIAN, J. and FERNÁNDEZ, D. (2020). An independence test based on recurrence rates. Journal of Multivariate Analysis 178 104624.

[19] KIM, J. (2020). Implementation of a goodness-of-fit test through Khmaladze martingale transformation. Computational Statistics 35 1993–2017.

[20] KOCHELRAKOTA, S. and KOCHELRAKOTA, K. (1986). Goodness of fit tests for discrete distributions. Communications in statistics-theory and methods 15 815–829.

[21] LECAM, L. (1960). Locally asymptotically normal families of distributions. Univ. California Publ. Statist. 3 37–98.

[22] LEE, S. and SEO, B. (2021). Omnibus goodness of fit test based on quadratic distance. Journal of Statistical Computation and Simulation 1–21.

[23] MARCHESELLI, M., BACCINI, A. and BARABESI, L. (2008). Parameter estimation for the discrete stable family. Communications in Statistics—Theory and Methods 37 815–830.
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[26] MEINTANIS, S. and NIKITIN, Y. Y. (2008). A class of count models and a new consistent test for the Poisson distribution. Journal of statistical planning and inference 138 3722–3732.

[27] MESELIDIS, C. and KARAGRIGORIOU, A. (2020). Statistical inference for multinomial populations based on a double index family of test statistics. Journal of Statistical Computation and Simulation 90 1773–1792.

[28] NAKAMURA, M. and PÉREZ-ABREU, V. (1993). Use of an empirical probability generating function for testing a Poisson model. Canadian Journal of Statistics 21 149–156.

[29] PUIG, P. and WEISS, C. H. (2020). Some goodness-of-fit tests for the Poisson distribution with applications in Biodosimetry. Computational statistics & data analysis 144 106878.

[30] RAYNER, J. C., THAS, O. and BEST, D. J. (2009). Smooth tests of goodness of fit: using R. John Wiley & Sons.

[31] RUEDA, R., O’REILLY, F. and PÉREZ-ABREU, V. (1991). Goodness of fit for the Poisson distribution based on the probability generating function. Communications in Statistics—Theory and Methods 20 3093–3110.

[32] RUEDA, R. and O’REILLY, F. (1999). Tests of fit for discrete distributions based on the probability generating function. Communications in Statistics—Simulation and Computation 28 259–274.

[33] SIM, S. and ONG, S. (2010). Parameter estimation for discrete distributions by generalized Hellinger-type divergence based on probability generating function. Communications in Statistics—Simulation and Computation® 39 305–314.

[34] STEUETEL, F. W. and VAN HARN, K. (2003). Infinite divisibility of probability distributions on the real line. CRC Press.

[35] R CORE TEAM (2021). R: A Language and Environment for Statistical Computing R Foundation for Statistical Computing, Vienna, Austria.

[36] TOUCHARD, J. (1933). Propriétés arithmétiques de certains nombres récurrents. Secrétariat de la société scientifique.

[37] TSYLOVA, E. and EKGAUZ, E. Y. (2017). Using probabilistic models to study the asymptotic behavior of Bell numbers. Journal of Mathematical Sciences 221 609–615.

[38] VAN DER VAART, A. W. (2000). Asymptotic statistics 3. Cambridge university press.