SU(3) phase states and finite Fourier transform

Brandon Zanette and Hubert de Guise

Department of Physics, Lakehead University, Thunder Bay, Canada
E-mail: hubert.deguise@lakeheadu.ca

Received 26 July 2011
Accepted for publication 23 August 2011
Published 17 February 2012
Online at stacks.iop.org/PhysScr/T147/014032

Abstract
We describe the construction of SU(3) phase operators using a Fourier-like transform on a hexagonal lattice. The advantages and disadvantages of this approach are contrasted with other results, in particular with the more traditional approach based on polar decomposition of operators.

PACS numbers: 02.10.Ud, 02.30.Nw, 03.65.Aa, 03.65.Ta

(Some figures may appear in colour only in the online journal)

1. Introduction: complementarity and the Fourier transform

The idea of complementarity in quantum mechanics goes back to Bohr and his attempt to explain wave–particle duality. The concept was sharpened by Pascual Jordan, who has stated [1] that:

For a given value of x, all values of p are equally possible.

This formulation automatically singles out the Fourier transform connecting operators such as $\hat{x}$ and $\hat{p}$ as their respective (generalized) eigenstates satisfy

$$\langle x | p \rangle \sim e^{i xp/\hbar} \quad \Rightarrow \quad |\langle x | p \rangle|^2 = \text{constant}. \quad (1)$$

The concept is not limited to continuous systems but also exists in finite dimensions. In this paper, we will discuss the construction of SU(3) phase operators, which are expected to be complementary to number operators. This paper emphasizes the importance of the finite Fourier transform and in particular investigates a new type of generalization of the Fourier transform that is constructed to preserve the symmetry of a hexagonal lattice, which is the natural (discrete) lattice to describe states appropriate for the description of a collection of three-level systems. Our approach should be contrasted with the approach of Dirac [2] which emphasizes polar decompositions and which has been applied to SU(2) and other systems in [3, 4].

2. Two examples

Consider first a spin-$\frac{1}{2}$ system, taking as operators the Pauli matrices $\sigma_x$, $\sigma_y$ and $\sigma_z$. The eigenstates $\{|+\rangle_z, |-\rangle_z\}$ of $\sigma_z$ and the eigenstates $\{|+\rangle_x, |-\rangle_x\}$ of $\sigma_x$ are complementary:

$$|\langle + | + \rangle_z|^2 = |\langle + | + \rangle_z|^2 = |\langle + | - \rangle_z|^2 = |\langle + | - \rangle_z|^2 = \frac{1}{2} = \text{constant}. \quad (2)$$

The eigenstates of $\sigma_z$ and $\sigma_x$ are related by a finite Fourier transform:

$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (3)$$

The operators $\sigma_z$ and $\sigma_x$ are said to be complementary. The same property holds for the pair $\sigma_y$ and $\sigma_z$ and for the pair $\sigma_y$ and $\sigma_x$. The transformation matrix connecting any two sets of eigenstates of the Pauli operators remains a finite Fourier transform.

A similar construction exists for a three-level system (or qutrit). Defining

$$\hat{Z} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{X} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \omega = e^{2\pi i/3}, \quad (4)$$

and writing their respective eigenstates as $\{|0_z\rangle, |1_z\rangle, |2_z\rangle\}$ and $\{|0_x\rangle, |1_x\rangle, |2_x\rangle\}$ we find for instance $|\langle 1_x | 0_x \rangle|^2 = |\langle 1_x | 0_x \rangle|^2 = \frac{1}{3}$ with all other such overlaps constant. Here again, the eigenstates of $\hat{X}$ and $\hat{Z}$ are related by a finite
Fourier transform:

$$F = \frac{1}{\sqrt{3}} \begin{pmatrix} \omega & 1 & \omega^2 \\ \omega & \omega^2 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (5)$$

This is the right time to mention some of the properties of the finite Fourier matrix $F$. It is unitary, which implies

$$\sum_j F^\dagger_{ij} F_{ij} = \delta_{ij}. \quad (6)$$

(This would be orthogonality under integration in the continuous case.) $F^4 = 1$, and its entries are characters of finite Abelian groups. In dimension $n$:

$$F_{jk} = e^{2\pi i jk/n} / \sqrt{n}. \quad (7)$$

Finally, $|F_{ij}|$ have constant magnitude, connecting with Jordan’s definition of complementarity.

### 3. SU(2) phase states

Following Dirac [2] and others [3], phase operators in su(2) (and other) systems are constructed by writing the matrix for $\hat{S}_x$ (or $\hat{S}_-$) in polar form, namely

$$\hat{S}_x \mapsto \sum_{m=0}^{j-1} c_m |j, m+1 \rangle \langle j, m| = E \cdot D, \quad (8)$$

where $D$ is diagonal and $E$ is a ‘phase’ part, containing entries that produce the shifting action of $\hat{S}_x$ on the basis states. The operator $E$ is expected on physical grounds to be complementary to the diagonal operator $\hat{S}_x$.

Geometrically, the set of eigenvalues $\{m; m = -j, -j+1, \ldots, j-1, j\}$ of $\hat{S}_x$ acting on number states $|jm\rangle$ are equidistant points on a line and the action of the ladder operators $\hat{S}_\pm$ takes a point $m$ to its neighbor $m \pm 1$. The action of $\hat{S}_x$ is pictorially represented in figure 1.

Because $|jj\rangle$ is killed by $\hat{S}_x$, the rank of $\hat{S}_x$ is one less than the dimension of the system, so $E$ is not completely defined: we can adjust the entries in one line. The usual choice makes $E$ cyclic ($E^{2j+1} = 1$), so it generates an Abelian group of order $2j + 1$:

$$E = e^{i\varphi} \mapsto \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} = \sum_{m=-j}^{j-1} |m+1\rangle \langle m| + |j\rangle \langle j|. \quad (9)$$

This $E$ is unitary and can be written in the form $E = e^{i\varphi}$, with $\varphi$ being the putative Hermitian phase operator. The eigenvectors of $E$ are eigenvectors of $\varphi$ and defined to be the SU(2) phase states. The components of the $m$th eigenvector $|\psi_m\rangle$ are just elements of a Fourier matrix $F$. Thus, the phase eigenstate $|\psi_m\rangle$ is given by

$$|\psi_m\rangle = \sum_{k} F_{mk} |jk\rangle = \frac{1}{\sqrt{2j+1}} \sum_{k} e^{2\pi i km/(2j+1)} |jk\rangle. \quad (10)$$

### 4. SU(3) and SU(3) phase states

#### 4.1. Geometry of SU(3) states

The algebra su(3) appears naturally in the construction of number-preserving transition operators for three-level systems. There are six transition operators, usually denoted by $C_{ij} = a_i a_j$ for $i \neq j \in \{1, 2, 3\}$, and two population differences $\hat{h}_1 = a_1 a_1 - a_2 a_2$ and $\hat{h}_2 = a_2 a_2 - a_3 a_3$. The states $|200\rangle$ and $|110\rangle$, for instance, respectively correspond to the pairs $(2,0)$ and $(0,1)$ of population differences. Pairs are located on a hexagonal lattice with basis vectors $\omega_1$ and $\omega_2$ as illustrated on the left of figure 2.

The action of $C_{ij}$ on lattice points is illustrated on the right of figure 2. Basis vectors $a_1$ and $a_2$ associated with the operators $\hat{C}_{12}$ and $\hat{C}_{23}$ are dual (reciprocal) to the lattice vectors $\omega_1$ and $\omega_2$, respectively, as illustrated. Using the hexagonal geometry, two points $(a, b)$ and $(c, d)$ corresponding to two pairs of population differences differ by an integer combination of the vectors $a_1$ and $a_2$. The action of $\hat{C}_{ij}$ on the state $|n_1 n_2 n_3\rangle$ is to translate the point $(n_1 - n_2, n_2 - n_3)$ by the vector associated with $\hat{C}_{ij}$ to the point $(n'_1 - n'_2, n'_2 - n'_3)$, so that, for instance,

$$\hat{C}_{12} |110\rangle \sim |200\rangle \iff \{0, 1\} \mapsto \{2, 0\} = a_1 + (0, 1). \quad (11)$$

The central ringed dot represents the two diagonal population difference operators $\hat{h}_1$ and $\hat{h}_2$. There should be one phase operator conjugate to each $\hat{h}_i$.

#### 4.2. Two solutions: boundaries

If we approach the construction of SU(3) phase states using polar decompositions, we are faced with an interesting problem. Because there are two basic shift directions, $a_1$ and $a_2$, each one of $\hat{C}_{12}$ and $\hat{C}_{23}$ will come with its own set of not necessarily mutually compatible boundary conditions.

In the simplest case of the states $|000\rangle$, $|010\rangle$ and $|001\rangle$, the shift matrices $E_{12}$ and $E_{23}$ that enter in the decompositions
As an alternative to the construction based on polar equations (14), one feature of this solution, already present in equation (13), is that the resulting phase operators do not commute:

\[ \hat{C}_{12} \neq \hat{C}_{23} \]

This in turn implies that the phases are not additive.

For the case of the states \(|100\rangle, |010\rangle\) and \(|001\rangle\), it is possible to find shift matrices \(E_{12}\) and \(E_{23}\) compatible with the polar decomposition of the respective operators so that \([E_{12}, E_{23}] = 0\) (figure 4). These matrices are

\[
E_{12} = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
o^2 & 0 & 0
\end{pmatrix}, \quad E_{23} = \begin{pmatrix}
0 & \omega^2 & 0 \\
0 & 0 & 1 \\
\omega & 0 & 0
\end{pmatrix}, \quad \omega = e^{\frac{2\pi i}{3}}.
\]

(15)

Note that \(E_{12}\) is just the operator \(\hat{X}\) of equation (4), while \(E_{23} = \hat{X}^2\). Clearly, \(E_{12}\) and \(E_{23}\) commute. However, we have not been able to find similar solutions for sets of states with \(n_1 + n_2 + n_3 > 1\).

4.3. Finite Fourier transform on a hexagonal lattice

As an alternative to the construction based on polar decomposition, we look for a finite Fourier transform (FFT) adapted to the discrete hexagonal symmetry natural to SU(3) states. Such an FFT was proposed in [5] and will be adapted to our needs.

We start with the physical states \(|n_1n_2n_3\rangle\). The procedure of [5] requires that the ‘data points’ be in the first hextant of the lattice, so we find a rigid displacement of the set of population differences \((n_1 - n_2, n_2 - n_3)\) corresponding to the physical states so that every pair \((n_1 - n_2, n_2 - n_3)\) is mapped to a single point in the first hextant (figure 4). One can show that the rigid displacement is a linear transformation comprising a translation, a rotation and a change of scale of the original pairs of points. The final result of the sequence is

\[ |n_1n_2n_3\rangle \mapsto (n_1 - n_2, n_2 - n_3) \mapsto (n_1, n_2). \]

(16)

An example of the result is given in figure 5. We obtain for each point \((a, b)\) in the first extant its orbit, i.e. the set of points obtained by considering reflections of \((a, b)\) through mirrors perpendicular to \(a_1\) and \(a_2\). Depending on the value of \(a\) and \(b\), an orbit may contain 1, 3 or 6 points. The orbits for the points (2, 0) and (1, 1) are illustrated in figure 6. Each orbit is labeled by its starting point \((a, b)\) in the first extant. There is the same number of orbits as the number of states. Each orbit is used to construct a so-called orbit function

\[
\chi_{(a,b)}(n_1, n_2) \sim \omega^{(2a+b)n_1 + (a+2b)n_2 + \omega^{(a-b)n_1 + (b+2a)n_2} + \omega^{(a+b)n_1 + (a-b)n_2} + \omega^{-(a+2b)n_1 + (a+b)n_2} + \omega^{-(2a+b)n_1 + (b+2a)n_2}},
\]

(17)

with \((n_1, n_2)\) points in the first hextant. The functions \(\chi\) are closely related to characters of elements of finite order of SU(3).

It is essential to rigidly translate the population differences of physical states. Two states \(|n_1n_2n_3\rangle, |n'_1n'_2n'_3\rangle\) that differ only by a permutation of \(n_1, n_2, n_3\) yield population differences \((n_1 - n_2, n_2 - n_3)\) and \((n'_1 - n'_2, n'_2 - n'_3)\) that are on the same orbit and so produce identical functions \(\chi\). It is only once the population differences have been translated to the first hextant that \((n_1, n_2)\) and \((n'_1, n'_2)\) will lie on different orbits.

The functions \(\chi\) need to be properly normalized and weighted as described in [5], but once this is done, they satisfy an orthogonality relation

\[
\sum_{n_1, n_2} \chi_{(a,b)}(n_1, n_2)^* \chi_{(a',b')}(n_1, n_2) \sim \delta_{aa'} \delta_{bb'}.
\]

(18)
The orbit functions can then be used to obtain a Fourier matrix

\[ F = \begin{pmatrix}
\chi_{\alpha_1, \beta_1} (s_1, s_2) & \cdots & \chi_{\alpha_1, \beta_1} (s_1, s_2) \\
\vdots & \ddots & \vdots \\
\chi_{\alpha_3, \beta_3} (s_1, s_2) & \cdots & \chi_{\alpha_3, \beta_3} (s_1, s_2)
\end{pmatrix} \]  \quad (19)

So defined, the matrix \( F \) immediately satisfies the majority of conditions given at the end of section 2. In particular, for the set of states \(|1\rangle, |0\rangle, |1\rangle\rangle, |0\rangle\rangle\), the matrix \( F \) is exactly the same as that of equation (5). However, for other states with \( n_1 + n_2 + n_3 > 1 \), the matrix \( F \) no longer contains entries of the same magnitude. For instance, using the states \(|n_1, n_2, n_3\rangle\rangle\) with \( n_1 + n_2 + n_3 = 2 \), we find

\[ F = \frac{1}{3} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
-\frac{1}{\sqrt{3}} & \omega & \omega^2 & -\frac{1}{\sqrt{3}} & \omega & \omega^2 \\
1 & 1 & 1 & 1 & 1 & 1 \\
-\frac{1}{\sqrt{3}} & \omega & \omega^2 & -\frac{1}{\sqrt{3}} & \omega & \omega^2 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\omega & -\frac{1}{\sqrt{3}} & \omega & -\frac{1}{\sqrt{3}} & \omega & -\frac{1}{\sqrt{3}}
\end{pmatrix}, \quad (20)

\[ \omega = e^{2i\pi/3}. \]

4.4. SU(3) phase states

Now define SU(3) phase states as transforms of the shifted population difference eigenstates:

\[ |\eta_1, \eta_2\rangle_{(n_1, n_2)} = \sum_{h_1, h_2} F_{(n_1, n_2), (h_1, h_2)} |h_1, h_2\rangle. \quad (21) \]

Phase ‘operators’ are conjugate to population difference operators:

\[ \hat{\eta}_1 = F \hat{h}_1 F^{-1}, \quad \hat{\eta}_2 = F \hat{h}_2 F^{-1}. \quad (22) \]

Since \( \hat{h}_1, \hat{h}_2 = 0 \), we recover \( [\hat{\eta}_1, \hat{\eta}_2] = 0 \): phases commute.
probability landscape is remarkably regular. The probability landscape is asymptotically flat, meaning that, in the large \( N \) limit, the phase states \( F|N00\rangle \), etc are asymptotically conjugate to the Fock states \( |n_1n_2n_3\rangle \).

### 4.6. \( su(3) \) phase operators

The phase operators \( \hat{\eta}_1, \hat{\eta}_2 \) of equation (22) generally have ‘complicated’ expressions. In spite of this, we have found the following observation to hold. If we evaluate the variances \( \Delta \eta_1 \) and \( \Delta \eta_2 \) using the physical states \( |n_1n_2n_3\rangle \), the smallest variances always occur for the states \( |N00\rangle, |0N0\rangle \) or \( |00N\rangle \). The landscape of variances of \( \hat{\eta}_1 \) evaluated in the physical states \( |n_1n_2n_3\rangle \) is illustrated in figure 9 for \( n_1 + n_2 + n_3 = 5 \) and 30.

An unanswered question (not discussed in this paper) is the difficulty in imposing correct cyclic boundary conditions on the phase operators themselves once they are exponentiated.

### 5. Conclusions

The polar decomposition of operators in SU(3) produces phase operators that are ambiguous and not unique: in general, non-commuting raising operators lead to a decomposition that produces non-commuting phase operators. Moreover, this approach produces an ‘exponential phase’ rather than a phase operator.

We can obtain Hermitian commuting ‘phase-like’ operators by using symmetry-adapted FFT. The procedure is mathematically systematic but not very intuitive, and we lose the connection with complementarity as defined by Jordon. With this approach the physical states \( |N00\rangle, |0N0\rangle \) and \( |00N\rangle \) stand out as having unexpected properties of asymptotic complementarity. The variances of the phase operators evaluated using those states are always the smallest.

### Acknowledgments

This work was supported in part by NSERC of Canada and Lakehead University.

### References

[1] Jammer M 1966 *The Conceptual Development of Quantum Mechanics* (New York: McGraw–Hill)

[2] Dirac P A M 1927 *Proc. R. Soc. Lond. A* 114 243

[3] Vourdas A 1990 *Phys. Rev. A* 41 1653

[4] Daoud M and Kibler M R 2011 *J. Math. Phys.* 52 082101

[5] Atoyan A and Patera J 2007 *J. Geom. Phys.* 57 745

[6] Kashuba I and Patera J 2007 *J. Phys. A: Math. Theor.* 40 4751