Fast Triangle Counting through Wedge Sampling

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ABSTRACT

Graphs and networks are used to model interactions in a variety of contexts, and there is a growing need to be able to quickly assess the qualities of a graph in order to understand its underlying structure. Some of the most useful metrics are triangle based and give a measure of the connectedness of “friends of friends.” Counting the number of triangles in a graph has, therefore, received considerable attention in recent years. We propose new sampling-based methods for counting the number of triangles or the number of triangles with vertices of specified degree in an undirected graph and for counting the number of each type of directed triangle in a directed graph. The number of samples depends only on the desired relative accuracy and not on the size of the graph. We present extensive numerical results showing that our methods are often much better than the error bounds would suggest. In the undirected case, our method is generally superior to other approximation approaches; in the undirected case, ours is the first approximation method proposed.

Keywords

triangle counting, directed triangle counting, clustering coefficient, Hoeffding’s inequality

1. INTRODUCTION

Over the last decade, graphs and networks have emerged as the standard for modeling interactions between entities in a wide variety of applications. Graphs are used to model infrastructure networks, the world wide web, computer traffic, molecular interactions, ecological systems, epidemics, co-authors, citations, and social interactions, among others. Despite the differences in the motivating applications, some topological structures have emerged to be important across all these domains. The most prevalent, and arguably the most important, of these topological structures is the triangle (3-clique). Many networks, especially social networks, are known to have an abundance of triangles, which can be explained by homophily (people become friends with those similar to themselves) and transitivity (friends of friends become friends). This abundance of triangles, along with the information they reveal, motivates metrics such as clustering coefficient and the transitivity ratio [28].

We show that the total number of triangles $t$, can be estimated by sampling a fixed number $p$ of wedges and checking if they are closed. A wedge is simply a length-2 path, and a triangle is a length-3 cycle. We let $p$ be the total number of wedges. We can create an estimate $\hat{t}$ such that

$$\Pr \left\{ |t - \hat{t}| \leq \epsilon p/3 \right\} \leq \delta.$$

In Tab. 1, we show how many samples are needed to estimate the total number of triangles, $t$, within an accuracy of $\epsilon p/3$, i.e., the proportion of 1/3 of the total wedges, at 99.9% confidence ($\delta = 0.001$). The size of the sample is independent of the size of the graph, although each sample requires the expense of checking existence of an edge. Not only is our proposed method extremely efficient, but it also has easy-to-compute error bounds.

| Accuracy ($\epsilon$) | Samples |
|-----------------------|---------|
| 0.10                  | 380     |
| 0.05                  | 1,520   |
| 0.01                  | 38,005  |
| 0.05                  | 152,018 |
| 0.001                 | 3,800,451 |

Table 1: Number of sampled wedges required for various accuracies at 99.9% confidence.

Our contributions enable fast computation of triangles and related metrics in both undirected and directed graphs. Specifically, we present

- a new sampling-based approach for undirected graphs for estimating the number of triangles and the clustering coefficient;
- a new sampling-based approach for undirected graphs for quickly estimating the number of triangles having at least one node of degree $d$ (or, more generally, at least one nodes in a set $D$), as well as the degree-wise clustering coefficients;
- a new sampling-based approach for directed graphs for estimating counts of directed triangles;
- precise error bounds based on known quantities for all of the above estimates; and
extensive numerical results confirming the accuracy of our method and the bounds as well as comparisons to other approaches.

We show that our sampling-based approach for counting triangles is more accurate and at least as fast as competing approximation approaches. To the best of our knowledge, ours is the first approximation approach in the regime of directed graphs.

Given an estimate of the number of triangles for directed or undirected graphs, we can compute metrics that are of use in a variety of contexts. The clustering coefficient measures how tightly the neighbors of a vertex are connected amongst themselves. At the global level, this property is an indicator of how tightly the communities of the graph are connected and may help to predict the behavior of individuals in the network. For instance, Coleman [11] and Portes [20] use the clustering coefficient to predict to likelihood of going against social norms. Burt, on the other hand, underlined the importance of nodes that can serve as a bridge between various communities [7] and tied this observation to the number of open triangles in a vertex [8]. Welles et al. studied the variance of clustering coefficients for different demographics groups and found that adolescents are more likely to have connected friends than adults and are even more likely to terminate connections with friends that are not connected to their other friends [15].

Triangles have also been used in graph mining applications such as spam detection [3]. Eckman and Moses [14] interpreted the clustering coefficients as a curvature and showed that connected regions of high curvature on the WWW characterized common topics.

Directed triangles are important motifs for comparing and characterizing graphs [18, 19, 12, 21, 5]. For graph databases, exploiting frequent patterns have also been proposed for efficient query processing [24, 29].

In our earlier work, we have used distribution of degree-wise clustering coefficients as the driving force for a new generative model, BlockTwo-Level Erdős-Rényi [23]. In this work, we have not only looked at the clustering coefficient, but also how the clustering coefficients related to the degree distribution, which motivates our algorithms in §4.

1.1 Sketch of Results

We present an extremely efficient sampling technique for estimating the number of triangles and clustering coefficient. Recall that the clustering coefficient of an undirected graph \( G = (V, E) \) is given by

\[
c = \frac{3t}{p} \equiv \frac{3 \times \text{total number of triangles}}{\text{total number of wedges}}.
\]

(1)

In Fig. 1, for example, \( \{3, 4, 6\} \) and \( \{3, 4, 5\} \) are two wedges centered at 4. We say a wedge is \textit{closed} if it is part of a triangle; otherwise, we say the wedge is \textit{open}. Thus, \( \{3, 4, 6\} \) is an open wedge, while \( \{3, 4, 5\} \) is a closed wedge. We can interpret \( c \) as the probability that a random wedge is closed.

Suppose we pick a sequence of \( j = 1, \ldots, k \) random wedges; let \( X_j \) be a random variable associated with the \( j \)th random wedge such that \( X_j = 1 \) if the wedge is closed and \( X_j = 0 \) if it is open. Define \( X = \frac{1}{k} \sum_{j=1}^{k} X_j \). It is easy to see that

\[
c = E[X].
\]

We show that \( c \) can be estimated to very high accuracy by taking a constant number of wedges and checking if they are closed. Specifically, we prove

\[
\Pr \{|\hat{X} - c| \geq \epsilon \} \leq \delta
\]

using \( k = [0.5 \epsilon^{-2} \ln(2/\delta)] \) samples. Note that the number of samples does not depend on the size of the graph. For instance, it requires fewer than 2,000 samples to have an absolute error of 0.05 with 99.9% confidence; fewer than 40,000 samples is needed for an absolute error of 0.01 at 99.9% confidence.

This translates directly to an estimate of the number of triangles, i.e., if we define \( \hat{t} = \hat{X}p/3 \), then

\[
\Pr \{|\hat{t} - t| \geq \epsilon p/3 \} \leq \delta.
\]

Hence, we can bound the error in our estimate of the number of triangles as a fraction \( \epsilon \) of the total number of wedges with confidence given by \( \delta \).

Through extensive numerical studies, we show that our proposed algorithm is much faster than direct enumeration and has less variance (and tighter bounds) than previously proposed approximation approaches.

We also extend this basic premise to computing the degree-wise clustering coefficients and triangles as well as counting directed triangles in a directed graph.

1.2 Related Work

The enumeration algorithms for finding triangles are either the node- or edge-centric. The node-centric algorithm iterates over all nodes and, for each node \( v \), checks all pairs among the neighbors of \( v \) for being connected. The edge-centric algorithm, on the other hand, goes over all edges \((u, v)\) and seeks common neighbors of \( u \) and \( v \). Chiba and Nishizeki [9] proposed a node-centric algorithm that orders the vertices by degree and processes each edge only once, by its lower degree vertex. They showed that this algorithm runs in \( O(m\alpha(G)) \)-time, where \( m \) is the number of edges, and \( \alpha(G) \) is the arboricity of the graph \( G \) (arboricity is defined as the minimum number of forests into which its edges can be partitioned and can be considered as a measure of how dense the graph is). Schank and Wagner [22] used the same idea for their forward algorithm. Cohen [10] and Suri and Vassilvitskii [25] independently proposed the same idea. Latapy proved that the forward algorithm runs in \( O(m^{3/2}) \)-time and proposed improvements that reduce the search space [17]. Latapy also showed that the runtime of this algorithm becomes \( O(mn^{1/\alpha}) \) for graphs with power-law degree distributions, where \( \alpha \) is the power-law coefficient and \( n \) is the number of vertices [17]. More recently, Berry et al. improved this bound to \( O(m) \) when the power-law coefficient is at least 7/3 [4].

To cope with the ever increasing data sizes, streaming algorithms have been proposed to count the number of tri-
angles [2, 3, 13, 6]. The work by Buriol et al. [6] is particularly important for this paper since their sampling strategy is similar to what we use for estimating undirected triangles. Despite the similar sampling approach, the error and confidence bounds in two studies are different and their work focuses only on the number of triangles, where we extend this sampling approach to directed triangles and distribution of triangles.

Another sampling-based approach was proposed by Tsourakakis et al. [27]. Their algorithm, Doulion, reduces the size of the graph by randomly sparsifying the graph. Specifically, a smaller graph is constructed by keeping each edge in the original graph with a given probability \( \rho \). Then the number of triangles in the original graph is estimated by multiplying the number of triangles of the small graph by \( \rho^3 \). The error bounds of this algorithm rely on two parameters that we cannot know in advance. The first parameter is the number of triangles, which is what we are trying to compute, hence the algorithm offers little guidance about the quality of an estimation or what would be a good number of samples if fairly low for error rates of 0.1 or 0.01.

Our results derive from the following well-known result by Hoeffding on the accuracy of estimating the mean from a few random samples. We make no assumptions on the distribution of the random variables.

**Theorem 1 (Hoeffding [16]).** Let \( X_1, X_2, \ldots, X_k \) be independent random variables with \( 0 \leq X_i \leq 1 \) for all \( i = 1, \ldots, k \). Define \( \bar{X} = \frac{1}{k} \sum_{i=1}^{k} X_j \). Let \( \mu = \mathbb{E}[\bar{X}] \). Then for \( \varepsilon \in (0, 1 - \mu) \), we have
\[
\Pr \{ |\bar{X} - \mu| \geq \varepsilon \} \leq 2 \exp(-2k\varepsilon^2).
\]

Note that the requirement that \( \varepsilon < 1 - \mu \) is for convenience. If \( \varepsilon > 1 - \mu \), then the implication is that \( |\bar{X}| \geq 1 \), which violates the assumption of the theorem. In other words, if \( \varepsilon \) is too large, then the probability is zero. We use this more convenient corollary in the proofs of our theorems.

**Corollary 2.** For positive \( \varepsilon, \delta \), set \( k = \lceil 0.5 \varepsilon^{-2} \ln(2/\delta) \rceil \). Let \( X_1, X_2, \ldots, X_k \) be independent random variables with \( 0 \leq X_i \leq 1 \) for all \( i = 1, \ldots, k \). Define \( \bar{X} = \frac{1}{k} \sum_{i=1}^{k} X_j \). Let \( \mu = \mathbb{E}[\bar{X}] \). Then,
\[
\Pr \{ |\bar{X} - \mu| \geq \varepsilon \} \leq \delta.
\]

**Proof.** Let \( 2 \exp(-2\varepsilon^2k) = \delta \), solve for \( k \), and apply Thm. 1. \( \square \)

3. **COUNTING TRIANGLES**

We first consider the problem of counting all triangles in an undirected graph. This is closely related to estimating the clustering coefficient.

Our goal is to estimate \( t \), the total number of triangles, in an undirected graph \( G = (V, E) \). Let \( n = |V| \) and \( m = |E| \). Without loss of generality, assume the vertices are indexed by \( i = 1, \ldots, n \). Let \( d_i \) denote the degree of vertex \( i \). The number of wedges centered at node \( i \) is given by
\[
p_i = \binom{d_i}{2} = \frac{d_i(d_i - 1)}{2}.
\]

Note that \( \binom{0}{2} = \binom{2}{2} = 0 \). Let \( W \) denote the set of all wedges in \( G \). The total number of wedges is \( p = |W| = \sum p_i \).

We derive a result on the accuracy of estimating the clustering coefficient.

**Theorem 3.** (Clustering Coefficient). For \( \varepsilon, \delta > 0 \), set \( k = \lceil 0.5 \varepsilon^{-2} \ln(2/\delta) \rceil \). For \( j = 1, \ldots, k \), choose wedge \( w_j \) uniformly at random (with replacement) from \( W \) and let \( X_j \) be defined as
\[
X_j = \begin{cases} 
1, & \text{if } w_j \text{ is closed}, \\
0, & \text{otherwise}.
\end{cases}
\]

Define \( \bar{X} = \frac{1}{k} \sum_{i=1}^{k} X_j \). Then
\[
\Pr \{ |\bar{X} - c| \geq \varepsilon \} \leq \delta,
\]
where \( c \) is the clustering coefficient defined in (1).

**Proof.** Recall that \( c \) is the proportion of wedges that are closed. Thus, it is straightforward to observe that \( c = \mathbb{E}[X] \).

The proof follows directly from Cor. 2. \( \square \)

**Corollary 4.** (Counting Triangles). Let the conditions of Thm. 3 hold. Define \( t = \bar{X}p/3 \) and \( \varepsilon = \varepsilon p/3 \). Then
\[
\Pr \{ |\bar{X} - t| \geq \varepsilon \} \leq \delta.
\]

**Proof.** Since \( t = cp/3 \) per (1), this corollary follows immediately from Thm. 3. \( \square \)

Observe that the number of samples, \( k \), does not depend on the size of the graph. We say that \( \varepsilon \) is the error and \( 1 - \delta \) is the confidence. Fig. 3 shows the number of samples needed for different error rates. We show three different curves for different confidence levels. Increasing the confidence has minimal impact on the number of samples. The number of samples if fairly low for error rates of 0.1 or 0.01.
but it increases with the inverse square of the desired error. Nonetheless, the three million samples required for an error rate of $\varepsilon = 0.01$ at 99.9% confidence requires only a few seconds of calculations on most serial machines.

We design an algorithm for estimating the clustering coefficient and number of triangles and analyze its complexity. The basic premise is to select a number of wedges uniformly at random and check whether or not each is closed. There are numerous ways that this can be implemented. For instance, we can select vertex $i$ with probability equal to $p_i / p$ and then select two of its neighbors uniformly at random without replacement. In this case, the overall probability of selecting a particular wedge is $p_i / p \times 1 / \binom{d_i}{2} = 1 / p$. The implementation we describe in Alg. 1 directly chooses a wedge at random. However, we do not explicitly enumerate all wedges. Instead, we have an implicit mapping of each random number to each particular wedge. To make this work cleanly, each wedge is actually listed twice as $(i, j, k)$ and $(i, k, j)$, and we will pick a random number in $\{1, \ldots, 2p\}$. We consider the algorithm in some detail.

- **Step 1** is actually the most expensive step, costing $O(m)$ to calculate the degrees of all vertices.
- **Step 2** just computes the number of wedges per degree, and $p$ is the total number of wedges.
- **Step 3** calculates the edges of the “wedge bins” corresponding to each vertex. For vertex $i$, its bin size is $2p_i$, since each wedge appears twice. If vertex $i$ has $p_i = 0$, then it will never be selected by the requirement that $z_{i+1} > r$ (strict inequality).
- **Steps 8–15** are converting the random number $r$ selected in Step 7 into an actual wedge. Note that Step 8 can be performed using a binary search at a cost of $O(\log n)$. The cost of Steps 14 and 15 are $O(1)$, when the standard adjacency list format is used to store the graph.
- **Step 16** requires checking the existence of an edge in the graph. The cost of this is $O(\log m)$.

We may conclude that the total cost of the method is $O(m)$ for preprocessing the degree distributions and $O(k \log m)$ for checking the closure of $k$ wedges.

We compare this method to exact enumeration and the Doulion method in §6.

**Algorithm 1 Hoeffding Triangle Estimate**

Given error $\varepsilon$ and failure probability $\delta$.

1. Calculate degree $d_i$ for $i = 1, \ldots, n$.
2. Set $p_i = d_i(d_i - 1)/2$ for $i = 1, \ldots, n$, and $p = \sum_i p_i$.
3. Set $z_1 = 1$ and $z_{i+1} = 2 \sum_i p_i' + 1$ for $i = 1, \ldots, n$.
4. $k \leftarrow \lceil 0.5 \varepsilon^2 \ln(2/\delta) \rceil$.
5. $cnt \leftarrow 0$.
6. for $j = 1, \ldots, k$ do
7. $r \leftarrow \text{Uniform}[1, \ldots, 2p]$.
8. Find $i$ such that $z_i \leq r < z_{i+1}$.
9. $\ell' \leftarrow \lceil (r - z_i)/(d_i - 1) \rceil + 1$.
10. $\ell'' \leftarrow (r - z_i) - (d_i - 1)(\ell' - 1) + 1$.
11. if $\ell'' \geq \ell'$ then
12. $\ell'' \leftarrow \ell'' + 1$.
13. end if
14. $i' \leftarrow$ index of $\ell'$th neighbor of $j$.
15. $i'' \leftarrow$ index of $\ell''$th neighbor of $j$.
16. if $(i', i'') \in E$ then
17. $cnt \leftarrow cnt + 1$.
18. end if
19. end for
20. $\hat{c} = cnt / k$. \textbf{Estimate for $c$.}
21. $\hat{t} = p\left(\frac{1}{2} cnt\right) / k$. \textbf{Estimate for $t$.}

**4. Counting Triangles Per Degree**

Here we consider the problem of counting a subset of triangles in a graph, i.e., all those that contain a node of some specified degree. Likewise, we consider the problem of estimating the clustering coefficient for all wedges centered at nodes of specified degree. In this case, the estimate clustering coefficient does not lead directly to an estimate for the number of triangles. However, both use the same basic data.

The node-level clustering coefficient (first used in [28]) is

$$c_i = t_i / p_i = \text{number of triangles incident to node } i / \text{number of wedges centered at node } i.$$

The degree-wise clustering coefficient, $c_d$, is the average of $c_i$ for nodes of degree $d$. Define $V_d = \{ i \in V \mid d_i = d \}$. Let $n_d = |V_d|$. Then we can write $c_d$ as

$$c_d = \frac{1}{n_d} \sum_{i \in V_d} c_i.$$

Let $W_d$ be the set of wedges centered at a node of degree $d$. We partition $W_d$ into four disjoint subsets as follows:

- $W_{d,0} = \{ w \in W_d \mid w \text{ open} \}$,
- $W_{d,1} = \{ w \in W_d \mid w \text{ closed and has 1 degree-}d \text{ node} \}$,
- $W_{d,2} = \{ w \in W_d \mid w \text{ closed and has 2 degree-}d \text{ nodes} \}$,
- $W_{d,3} = \{ w \in W_d \mid w \text{ closed and has 3 degree-}d \text{ nodes} \}$.

The total number of wedges centered at a degree $d$ node is

$$p_d = |W_d| = n_d \binom{d}{2}.$$

Define $p_{d,q} = |W_{d,q}|$ for $q = 0, 1, 2, 3$. Clearly, $p_d = \sum_q p_{d,q}$.

It is easy to show that (2) can be rewritten as

$$c_d = \frac{p_{d,1} + p_{d,2} + p_{d,3}}{p_d}.$$

We are also interested in the number of triangles having one
or more degree-$d$ nodes, denoted by $t_d$. Observe that

$$t_d = p_{d,1} + \frac{1}{2}p_{d,2} + \frac{1}{3}p_{d,3},$$  \hspace{1cm} (3)$$

since for each triangle there is either one wedge in $W_{d,1}$, two wedges in $W_{d,2}$ or three wedges in $W_{d,3}$.

**Theorem 5** (Degree-wise Clustering Coefficient). For $\varepsilon, \delta > 0$, set $k = \lceil 0.5\varepsilon^{-2}\ln(2/\delta) \rceil$. For $j = 1, \ldots, k$, choose wedge $w_j$ uniformly at random (with replacement) from $W_d$ and let $X_j$ be defined as

$$X_j = \begin{cases} 1, & \text{if } w_j \text{ is closed}, \\ 0, & \text{otherwise}. \end{cases}$$

Define $\bar{X} = \frac{1}{k} \sum_{j=1}^{k} X_j$. Then

$$\Pr\{|\bar{X} - c_d| \geq \varepsilon\} \leq \delta,$$

where $c_d$ is the degree-wise clustering coefficient from (2).

**Proof.** Observe that $c_d = E[X]$ since it is the probability that a random wedge in $W_d$ is closed. The proof follows immediately from Cor. 2. \hfill \square

**Algorithm 2** Hoeffding Degree-$d$ Triangle Estimate

Given set of degrees $D$, error $\varepsilon$, and failure probability $\delta$.

1: Calculate degree $d_i$ for $i = 1, \ldots, n$
2: Set $p_i = \begin{cases} d_i(d_i - 1)/2, & \text{if } d_i \in D, \\ 0, & \text{otherwise}, \end{cases}$ for $i = 1, \ldots, n$.
3: Set $p_D = \sum_{i=1}^{n} p_i$.
4: Set $z_1 = 1$ and $z_{i+1} = 2 \sum_{i'=1}^{i} p_{i'} + 1$ for $i = 1, \ldots, n$.
5: $k \leftarrow \lceil 0.5\varepsilon^{-2}\ln(2/\delta) \rceil$.
6: $\text{cnt1} \leftarrow 0$, $\text{cnt2} \leftarrow 0$, $\text{cnt3} \leftarrow 0$.
7: for $j = 1, \ldots, k$ do
8: $r \leftarrow \text{Uniform}[1, 2p_D]$
9: Find $i$ such that $z_i \leq r < z_{i+1}$.
10: $\ell' \leftarrow [(r - z_i)/(d_i - 1)] + 1$.
11: $\ell'' \leftarrow (r - z_i) - (d_i - 1)(\ell' - 1) + 1$.
12: if $\ell'' \geq \ell'$ then
13: $\ell'' \leftarrow \ell' + 1$.
14: end if
15: $i' \leftarrow \text{index of } \ell'\text{th neighbor of } j$
16: $i'' \leftarrow \text{index of } \ell''\text{th neighbor of } j$
17: if $(i', i'') \in E$ then
18: if $d_{i'} \in D$ and $d_{i''} \in D$ then
19: $\text{cnt3} \leftarrow \text{cnt3} + 1$
20: else if $d_{i'} \in D$ or $d_{i''} \in D$ then
21: $\text{cnt2} \leftarrow \text{cnt2} + 1$
22: else
23: $\text{cnt1} \leftarrow \text{cnt1} + 1$
24: end if
25: end if
26: end for
27: $\hat{c}_D = (\text{cnt1} + \text{cnt2} + \text{cnt3})/k$ \hspace{1cm} \triangleright \text{Estimate for } c_D
28: $\hat{t}_D = p_D(\text{cnt1} + \frac{1}{2}\text{cnt2} + \frac{1}{3}\text{cnt3})/k$ \hspace{1cm} \triangleright \text{Estimate for } t_D

**Theorem 6** (Degree-wise Triangle Count). For $\varepsilon, \delta > 0$, set $k = \lceil 0.5\varepsilon^{-2}\ln(2/\delta) \rceil$. For $j = 1, \ldots, k$, choose wedge $w_j$ uniformly at random (with replacement) from $W_d$ and let $Y_j$ be defined as

$$Y_j = \begin{cases} 1, & \text{if } w \in W_{d,1}, \\ \frac{2}{3}, & \text{if } w \in W_{d,2}, \\ \frac{1}{3}, & \text{if } w \in W_{d,3}, \\ 0, & \text{if } w \in W_{d,0} \text{ (open)} . \end{cases}$$

Let $\bar{Y} = \frac{1}{k} \sum_{j=1}^{k} Y_j$. Define $\hat{t} = \bar{Y} \cdot p_d$ and $\hat{\varepsilon} = \varepsilon p_d$. Then

$$\Pr\{|\hat{t} - t_d| \geq \hat{\varepsilon}\} \leq \delta,$$

where $t_d$ is the number of triangles having one or more vertices of degree $d$.

**Proof.** We claim $E[\bar{Y}] = E[Y] = t_d/p_d$. Suppose that $w$ is selected from $W_d$ uniformly at random. Observe that

$$E[Y] = \Pr\{w \in W_{d,1}\} + \Pr\{w \in W_{d,2}\} + \Pr\{w \in W_{d,3}\}$$

$$= 1 \cdot \frac{p_{d,1}}{p_d} + \frac{1}{2} \cdot \frac{p_{d,2}}{p_d} + \frac{1}{3} \cdot \frac{p_{d,3}}{p_d}$$

$$= t_d/p_d,$$

per (3). Hence, from Cor. 2 we have

$$\Pr\{|\hat{Y} - t_d/p_d| \geq \hat{\varepsilon}\} \leq \delta,$$

and the theorem follows by multiplying the inequality by $p_d$. \hfill \square

The algorithm to compute the degree-wise clustering coefficient and triangle count is shown in Alg. 2 in essence. We have generalized the idea here for any set of specified degrees $D \subseteq \{1, \ldots, d_{\text{max}}\}$. If $D = \{1, \ldots, d_{\text{max}}\}$, it is easy to see that this is equivalent to Alg. 1. There are three counts corresponding to the number of closed wedges with 1, 2, and 3 vertices with degrees in $D$, respectively. If only interested in the clustering coefficient, then there is no need to split the counts. In Step 3, we define $p_D$ to be the number of wedges with a node of degree $d \in D$ at their center. Similarly, we define $c_D$ to be the average of all $c_i$ such that $d_i \in D$ and $t_D$ to be the number of triangles with at least one vertex $i$ such that $d_i \in D$.

5. COUNTING DIRECTED TRIANGLES

Counting triangles in directed graphs is considerably more difficult because there are seven types of directed triangles (up to isomorphism); see Fig. 4. Nonetheless, the same principles apply.

![Figure 4: All different directed triangles](image-url)
the out-degree, in-degree, and bi-degree, respectively. These are denoted by \(d^+_i, d^-_i,\) and \(d^\pm_i\).

Given these three edges types, there are six different types of wedges, labeled by lower case Roman numerals in Fig. 5. For any wedge type \(\gamma \in \Gamma \equiv \{i, ii, iii, iv, v, vi\},\) define

\[
P_i(\gamma) = \text{no. of wedges of type } \gamma \text{ centered at node } i, \quad \text{and} \quad p(\gamma) = \sum_{i=1}^{n} P_i(\gamma) = \text{number of wedges of type } \gamma.
\]

The formulas for calculating \(P_i(\gamma)\) are given in Tab. 2.

![Figure 5: All different directed wedges](image)

Table 2: Number of wedges per node for each wedge type

| \(\gamma\) | i | ii | iii | iv | v | vi |
|---|---|---|---|---|---|---|
| \(P_i(\gamma)\) | \((\ell^+_{i})\) | \(d^+_i d^-_i\) | \((\ell^-_{i})\) | \(d^+_i d^+_i\) | \(d^+_i d^-_i\) | \((\ell^\pm_{i})\) |

Finally, we come back to the seven different types of triangles, labeled by lowercase letters in Fig. 4. For any triangle type \(\sigma \in \Sigma \equiv \{a, b, c, d, e, f, g\},\) define

\[
t(\sigma) = \text{number of triangles of type } \sigma.
\]

We let \(\omega(\gamma, \sigma)\) be the number of triangles of type \(\gamma\) is a triangle of type \(\sigma\). These values are listed in Tab. 3. We also define \(\Gamma_\sigma = \{ \gamma \in \Gamma \mid \omega(\gamma, \sigma) > 0 \}\), i.e., the subset of wedges participating in triangle type \(\sigma\).

Table 3: Number of each wedge type per triangle type

| Wedge types (\(\gamma\)) | \(\omega\) |
|---|---|
| a | 1 |
| b | 1 |
| c | 1 |
| d | 1 |
| e | 1 |
| f | 1 |
| g | 3 |

| Triangle types (\(\sigma\)) | \(i\) | ii | iii | iv | v | vi |
|---|---|---|---|---|---|---|
| a | 1 |
| b | 1 |
| c | 2 |
| d | 1 |
| e | 1 |
| f | 1 |
| g | 3 |

We define \(W(\gamma)\) to be the set of wedges of type \(\gamma\). We partition it into eight subsets as follows. Let

\[
W(\gamma, 0) = \{ w \in W(\gamma) \mid w \text{ is open} \},
\]

\[
W(\gamma, \sigma) = \{ w \in W(\gamma) \mid w \text{ closes to be of type } \sigma \}.
\]

Then we can write the number of triangles of type \(\sigma\) as

\[
t(\sigma) = \frac{|W(\gamma, \sigma)|}{\omega(\gamma, \sigma)} \quad \text{for any } \gamma \in \Gamma_\sigma.
\]

**Theorem 7** (Directed Triangle Count). Assume we wish to count triangles of type \(\gamma\). Choose \(\gamma \in \Gamma_\sigma\). For \(\varepsilon, \delta > 0,\) set \(k = \lceil 0.5 \varepsilon^{-2} \ln(2/\delta) \rceil\). For \(j = 1, \ldots, k\), choose wedge \(w_j\) uniformly at random (with replacement) from \(W(\gamma)\) and let \(X_j\) be defined as

\[
X_j = \begin{cases} 1, & \text{if } w_j \text{ closes to form a triangle of type } \sigma \\ 0, & \text{otherwise.} \end{cases}
\]

Define \(\hat{X} = \frac{1}{k} \sum_{j=1}^{k} X_j\). Define \(\hat{t} = \hat{X} p(\gamma) / \omega(\gamma, \sigma)\) and \(\hat{\varepsilon} = \varepsilon p(\gamma) / \omega(\gamma, \sigma)\). Then

\[
\Pr \{|\hat{t} - t(\sigma)| \geq \hat{\varepsilon}\} \leq \delta.
\]

The proof follows the same principals as the previous theorems and so is omitted.

We will not write down the full algorithm for all scenarios because it is too complex to be easily represented in pseudocode. Instead, we focus on triangle type \(d\) and wedge type \(ii\) as a representative. This algorithm is presented in Alg. 3. In Step 1, we calculate just the in- and out-degrees, but we omit the bi-degrees since they are not used explicitly for finding this triangle type. Recall, however, that these in- and out-degree counts exclude any bidirectional edges. In general, we recommend choosing wedge type \(\gamma \in \Gamma_\sigma\) with the lowest total wedge count so that the sampling will visit a larger fraction of the set, but this specific choice is not necessary from a theoretical point of view.

Algorithm 3 Hoefding Type (d) Triangle Estimate

Given error \(\varepsilon\) and failure probability \(\delta\).

1: \(\text{Calculate degree } d^+_i \text{ and } d^-_i \text{ for } i = 1, \ldots, n\)
2: \(\text{Set } p_i = d^+_i d^-_i \text{ for } i = 1, \ldots, n, \text{ and } p = \sum_i p_i\)
3: \(\text{Set } z_1 = 1 \text{ and } z_{i+1} = \sum_{i=1}^{p} p_i + 1 \text{ for } i = 1, \ldots, n\)
4: \(k \leftarrow \lfloor 0.5 \varepsilon^{-2} \ln(2/\delta) \rfloor\)
5: \(\text{cnt} \leftarrow 0\)
6: for \(j = 1, \ldots, k\) do
7: \(r \leftarrow \text{Uniform}[\{1, \ldots, p\}]\)
8: \(i^\prime \leftarrow \text{index of } r\text{'th out-neighbor of } j\)
9: \(i'' \leftarrow \text{index of } r\text{'th in-neighbor of } j\)
10: if \((i', i'') \in E \text{ and } (i'', i') \in E\) then
11: \(\text{cnt} \leftarrow \text{cnt} + 1\)
12: end if
13: end for
14: \(\hat{t} = p \cdot \text{cnt} / k\)

> Estimate for \(t\)

**6. EXPERIMENTAL RESULTS**

6.1 Experimental Setup

We have implemented all our algorithms in C, and have run our experiments on a computer equipped with a 2.3GHz Intel core i7 processor with 4 cores and 256KB L2 cache (per core), 8MB L3 cache, and 8GB memory. We have performed our experiments on 21 graphs chosen out of the SNAP data set [30]. In all cases, edge weights and self-edges are omitted. For the undirected tests, we ignore direction on the edges. From this collection, we have chosen matrices with higher number of triangles. The properties of these matrices are presented in Tab. 4.
Below, we compare algorithms with the forward (enumeration) algorithm [9, 22, 10, 25] and Doulion approach [27]. For the forward algorithm, we have ordered vertices according to their degrees and used the vertex numbering as a tie-breaker. For the Doulion approach, we have used the forward algorithm after down-selecting the edges.

### 6.2 Counting Triangles

In Tab. 4, we summarize experiments on 21 graphs from the SNAP collection [30]. Recall that $n$ is the number of vertices, $m$ is the number of edges, and $p$ is the number of wedges, and $t$ is the number of triangles. We compare the following methods:

- **Enumeration (E)** - Enumerates all triangles, being clever to look at only one wedge per triangle rather than three [9, 22, 10, 25].
- **Doulion (D)** - Estimates the number of triangles by working with a reduced graph; Edges are selected from the original graph with probability $p$ [27]. We use $p = 1/25$ (labeled D25) and $p = 1/10$ (labeled D10).
- **Hoeffding (H)** - This is our proposed approach. Here we have used $k = 26,500$ samples, corresponding to an error of $\varepsilon = 0.01$ at 99% confidence. This means we expect the difference between our estimate and the real answer to be no more than 1% of $p/3$.

Note that the enumeration approach gives the true number of triangles ($t$). We show the estimate $\hat{t}$, computed by each approximation method, as well as the error, which is shown as a percentage of 1/3 of the total number of wedges (the maximum number of possible triangles if every wedge were closed), i.e.,

$$\text{error} = 100|\hat{t} - t|/(p/3).$$

For Hoeffding, we expect

$$\text{error} = 100|\hat{t} - t|/(p/3) \leq 100\varepsilon = 1,$$

with 99% confidence. Indeed, the maximum error is 0.49, well under the bound. As expected, D10 is generally better than D25 (due to high variance, it is occasionally worse) since it uses a larger sample of the graph. Hoeffding is generally as good or better than Doulion. On average, the error of Doulion is much larger than that of Hoeffding. We could use a higher value of $p$ in Doulion and save more edges, but then it would take more time.

The timing comparisons are also shown. It is worth noting that 90–99% of the time for the Doulion and Hoeffding methods is just reading the graph. Nevertheless, our objective is to show that the Hoeffding method is at least as fast as the Doulion methods while achieving better accuracy. For graphs with a large number of wedges (e.g., as-skitter), the estimation methods are an order of magnitude faster than direct enumeration.

The clustering coefficient is directly proportional to the number of triangles, so we do not include it in Tab. 4.

![Fig. 7](image_url) shows the convergence of the clustering coefficient estimate as the number of samples increases. The dashed line shows the error bars at 99.9% confidence. Indeed, it is always possible to increase the number of samples, adding to those already completed, in order to further reduce the error bound. Given the number of samples computed and the desired confidence, it is possible to determine the error bars, as we show here. The level of confidence does not change them much.

### 6.3 Counting Triangles per Degree

One of the unique benefits of our approach is the derivation of a method to count only triangles with a specified degree as well as the clustering coefficient for a specified degree. For instance, the BTER model of [23] can accurately capture the degree-wise clustering coefficients, but these are prohibitive to compute for large graphs because it requires enumerating all triangles.

In Fig. 7, we compare true and predicated clustering coefficients by degree. We use just $k = 6,622$ samples per
degree, since that gives an absolute accuracy of \( \varepsilon = 0.02 \) with 99% confidence.

In Tab. 5, we compare are predictions of triangles by degree with the actual counts computed by enumeration. We show the predicted error range for the Hoeffding estimate, and the actual difference (shown in the last column) is always well within the bound. A nice feature of our algorithm is that it can be adapted to any set of triangle degrees, so we show the set of triangles that have at least one degree being in the set \( \{3, 4, 5\} \).

### 6.4 Counting Directed Triangles

In Tab. 6, we show the results of our method for counting directed triangles. We specify the number of directed edges, which may be different than the undirected versions considered in Tab. 4. For the directed triangles, we consider only the Type (d) triangle (see Fig. 4). We use \( k = 3,800,451 \) samples, corresponding to \( \varepsilon = 0.001 \) and \( \delta = 0.001 \) (99.9% confidence). We have only tested this for relatively small triangles for which we can also do direct enumeration to compare the results. Observe that the estimates are typically an order of magnitude more accurate than the estimated bounds (which are not unreasonable in the first place).

### 7. CONCLUSIONS

We have developed a novel approach to very fast estimation of the number of triangles in a graph. The approach is premised on sampling wedges and using Hoeffding’s inequality (Thm. 1) to bound the estimation error. The bulk of the work for our Hoeffding method is the preprocessing to determine the degree (or in-, out-, and bi-degree for directed graphs) of each vertex. From these values, we can directly calculate the total number of wedges (or each directed wedge) and from that compute exact error bounds for estimating the number of triangles.

In our experimental results, we showed that our Hoeffding estimation approach is more accurate than Doulion’s method and at least as fast in terms of computation time.

We have also showed that it is extremely accurate in terms
of counting the number of triangles of specified degree or for calculating degree-wise clustering coefficients. To the best of our knowledge, ours is the first estimation method for calculating counts of directed triangles, and it is extremely accurate in our experiments.

A major advantage of our Hoeffding method is that it can be easily implemented in a distributed framework. In a Hadoop MapReduce framework, for example, we may assume that every node knows its neighbors (this can be done but is a little more complicated when the neighbor list is too big to fit in a single mapper) and so can randomly select some wedges to check for closure. If the list of wedges to check is small (which will generally be the case), the distributed cache can be employed and mappers can check for wedge closure. One can also consider checking for closure in the reducers, but it causes considerably more message traffic.

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