Make Workers Work Harder: Decoupled Asynchronous Proximal Stochastic Gradient Descent

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Abstract

Asynchronous parallel optimization algorithms for solving large-scale machine learning problems have drawn significant attention from academia to industry recently. This paper proposes a novel algorithm, decoupled asynchronous proximal stochastic gradient descent (DAP-SGD), to minimize an objective function that is the composite of the average of multiple empirical losses and a regularization term. Unlike the traditional asynchronous proximal stochastic gradient descent (TAP-SGD) in which the master carries much of the computation load, the proposed algorithm off-loads the majority of computation tasks from the master to workers, and leaves the master to conduct simple addition operations. This strategy yields an easy-to-parallelize algorithm, whose performance is justified by theoretical convergence analyses. To be specific, DAP-SGD achieves an $O(\log T/T)$ rate when the step-size is diminishing and an ergodic $O(1/\sqrt{T})$ rate when the step-size is constant, where $T$ is the number of total iterations.

1 Introduction

A majority of classical machine learning tasks can be formulated as solving a general regularized optimization problem:

$$\min_{x \in \mathbb{R}^m} P(x) = f(x) + h(x),$$

where $f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x)$.

Given $n$ samples, $f_i(x)$ represents the empirical loss of the $i^{th}$ sample with regard to the decision variable $x$, and $h(x)$ corresponds to a (usually non-smooth) regularization term. Our goal is to find the optimal solution, defined as $x^*$, which minimizes the summation of the averaged empirical loss and the regularization term over the whole dataset.

With the enormous growth of data size $n$ and model complexity, asynchronous parallel algorithms \cite{1, 2, 3, 4, 5, 6} have become an important tool and received significant successes for solving large scale machine learning problems in the form of (1). Asynchronous parallel algorithms distribute computation on multi-core systems (shared memory architecture) or multi-machine system (parameter server architecture), whose computation power generally scales up with the increasing number of cores or machines. As a consequence, effective design and implementation of asynchronous parallel algorithms is critical for large scale machine learning.

Numerous efforts have been devoted to this topic. Among them, asynchronous stochastic gradient descent is proposed in \cite{1, 2}, and its performance is guaranteed by theoretical convergence analyses. An asynchronous proximal gradient descent algorithm is designed on the parameter server architecture in \cite{3} with a distributed optimization software provided. Convergence rate of asynchronous stochastic gradient descent with a non-convex objective is analyzed in \cite{4}. Apart from work on asynchronous gradient descent and its proximal...
variant, much attention has also been attracted to asynchronous alternating direction method of multipliers (ADMM) [5], asynchronous stochastic coordinate ascent [7, 8, 9, 10, 11, 12] and asynchronous dual stochastic coordinate ascent [13].

The traditional asynchronous proximal stochastic gradient method (TAP-SGD) that solves (1) works as follows. The workers (multiple cores or machines) access samples, compute the gradients of their corresponding empirical losses, and send to the master. The master fuses the gradients and runs a proximal step on the regularization term (more details are given in Section 2). However, the performance of this paradigm is restricted when the proximal operator is not an element-wise operation. For this case, running proximal steps can be time-consuming, and the computation in the master becomes the bottleneck of the whole system. We note that this is common for many popular regularization terms, as shown in Section 2. To avoid this difficulty, one has to design a customized parallel computation for every single regularization term, which makes the framework inflexible. For the sake of speeding up computation and simplifying algorithm design, we expect to design an alternative algorithm that is easier to parallelize.

In light of this issue, this paper develops a decoupled asynchronous proximal stochastic gradient descent (DAP-SGD), which off-loads the majority of computation tasks (especially the proximal steps) from the master to workers, and leaves the master to conduct simple addition operations. This algorithmic framework is suitable for many master/worker architectures including the single machine multi-core system (shared memory architecture) where the master is the parameter updating thread and the workers correspond to other threads processing samples, and the multi-machine system (parameter server architecture) where the master is the central machine for storing and updating parameters and the workers represent those machines for storing and processing samples.

The main contributions of this paper are highlighted as follows:

- The proposed DAP-SGD algorithm off-loads the computation bottleneck from the master to workers. To be more specific, DAP-SGD allows workers to evaluate the proximal operators (work harder) and the master only needs to do element-wise addition operations, which is easy to parallelize.

- Convergence analysis is provided for DAP-SGD. DAP-SGD achieves an $O(\log T/T)$ rate when the step-size is diminishing and an ergodic $O(1/\sqrt{T})$ rate when the step-size is constant, where $T$ is the number of total iterations.

2 Traditional Asynchronous Proximal Stochastic Gradient Descent (TAP-SGD)

We start from the synchronous proximal stochastic gradient descent (P-SGD) algorithm that solves (1). P-SGD only requires the gradient of one sample in a single iteration. Hence in large scale optimization problems, it is a preferred surrogate for proximal gradient descent [14, 15], which requires computing gradients of all samples in a single iteration. The recursion of P-SGD is

$$x_{t+1} = \text{Prox}_{\eta_t, h}(x_t - \eta_t \nabla f_{t_i}(x_t)),$$

where $\text{Prox}_{\eta, h}(x) = \arg\min_y \|y - x\|_2^2/(2\eta) + h(y)$ denotes a proximal operator, while $\eta_t$ is the step-size and $i_t$ is the index of the selected sample in the $t$th iteration.

The traditional asynchronous proximal stochastic gradient descent (TAP-SGD) algorithm is an asynchronous variant of P-SGD, as summarized in Algorithm 1. The master is the main updating processor, while the workers provide the gradients of the samples. Every worker receives the parameter (namely, decision variables) $x$ from the master, computes the gradient of one random sample $\nabla f_{t_i}(x)$ and sends it to the master. Obviously, when one worker is computing and sending its gradient, the master may update the parameter using the gradients sent by the other workers in the previous time period. As a consequence, the gradients received at the master are often delayed, causing the main difference between P-SGD and TAP-SGD. In the master, the delayed gradient received at the $i^{th}$ iteration is denoted by $\nabla f_{t_i}(x_{d(t)})$ where $i_t$ indexes the selected sample, $x_{d(t)}$ refers to that the parameter is the one from the $d(t)^{th}$ iteration, and $d(t) \in [t - \tau, t]$ where $\tau$ stands for the maximum delay of the system. Therefore, we can write the recursion
of TAP-SGD as

\[ x_{t+1} = \text{Prox}_{\eta, h}(x_t - \eta_t \nabla f_{i_t}(x_{d(t)})) . \]  

\[ \text{Algorithm 1: Asynchronous Proximal Stochastic Gradient Descent (AP-SGD)} \]

**Input:** Initialization \( x_0 \), \( t = 0 \), dataset with \( n \) samples in which the loss function of the \( i^{th} \) sample is denoted by \( f_i(x) \), regularization term \( h(x) \), maximum number of iterations \( T \), number of workers \( S \), step-size in the \( t^{th} \) iteration \( \eta_t \), maximum delay \( \tau \)

**Output:** \( x_T \)

**Procedure of each worker** \( s \in [1,...,S] \)

1. repeat
   2. Uniformly sample \( i \) from \([1,...,n]\);
   3. Obtain the parameter \( x \) from the master (shared memory or parameter server);
   4. Evaluate the gradient of the \( i^{th} \) sample over parameter \( x \), denoted by \( \nabla f_i(x) \);
   5. Send \( \nabla f_i(x) \) to the master;
5. until procedure of master ends

**Procedure of master**

1. for \( t = 0 \) to \( T - 1 \) do
   2. Get a gradient \( \nabla f_{i_t}(x_{d(t)}) \) (the delay \( t - d(t) \) is bounded by \( \tau \));
   3. Update the parameter with the proximal operator \( x_{t+1} = \text{Prox}_{\eta_t, h}(x_t - \eta_t \nabla f_{i_t}(x_{d(t)})) \);
   4. \( t = t + 1 \);

Observe that the updating procedure of the master is the computational bottleneck of the TAP-SGD algorithm. When the proximal step is time-consuming to calculate, the workers must wait for a long time to receive updated parameters, which significantly degrades the performance of the system. To avoid this difficulty, one has to design a customized parallel computation for every single regularization term, which makes the framework inflexible. In a multi-machine system with multiple masters, such parallelized proximal operators will also cause complicated network communications between masters.

**Coupled Proximal Operators**

In practice, many widely used (usually non-smooth) regularization terms are associated with coupled proximal operators, which lead to high computational complexity, including group lasso regularization [16], fused lasso regularization [17], nuclear norm regularization [18, 19], etc.

The proximal operator of group lasso regularization \( h(x) = \lambda \sum_{i=1}^{g} \| x_{k_i:(k_{i+1}-1)} \|_2 \):

\[
\text{Prox}_{\eta, h}(x) = \text{argmin}_y \frac{1}{2\eta} \| y - x \|_2^2 + \lambda \sum_{i=1}^{g} \| y_{k_i:(k_{i+1}-1)} \|_2 .
\]  

(4)

Here \( g \) is the number of groups and \( k_1 = 1 < ... k_i < k_{i+1}... < k_g+1 = m + 1 \). The closed-form solution of the proximal operator above is

\[
[\text{Prox}_{\eta, h}(x)]_{k_i:(k_{i+1}-1)} = x_{k_i:(k_{i+1}-1)} \left( 1 - \frac{\lambda}{\| x_{k_i:(k_{i+1}-1)} \|_2} \right)_+. 
\]  

(5)

For the group lasso regularization, the proximal operator is separated into \( g \) groups. When partitions of groups are unbalanced, it will be hard to speed up the computation with parallelization.

The proximal operator of simplified fused lasso regularization \( h(x) = \lambda \sum_{i=1}^{m-1} \| x_i - x_{i+1} \|_1 \):

\[
\text{Prox}_{\eta, h}(x) = \text{argmin}_y \frac{1}{2\eta} \| y - x \|_2^2 + \lambda \sum_{i=1}^{m-1} \| y_i - y_{i+1} \|_1
\]

\[ = y - R^T z^* , \]

(6)
where $R = \begin{bmatrix} 1 & -1 & 0 & \ldots & 0 \\ 0 & 1 & -1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & -1 \end{bmatrix} \in \mathbb{R}^{(m-1) \times m},$

$z^* = \arg \min \frac{1}{2} \| R^T z \|^2_2 - \langle R^T z, y \rangle.$

For the simplified fused lasso regularization, the proximal operator (6) has a closed form solution. However, solving $z^*$ involves a subproblem that is time-consuming.

The proximal operator of nuclear norm regularization $h(X) = \lambda \| X \|_*$:

$$\text{Prox}_{\eta,h}(X) = \arg \min_Y \frac{1}{2\eta} \| Y - X \|_F^2 + \lambda \| Y \|_*,$$

where $X = U\Sigma V^T$ calculated from singular value decomposition, $\sigma_i$ is the $i$th singular value of $X$, $\hat{\sigma}_i = \max(\sigma_i - \eta \lambda, 0)$ is the $i$th element of $\hat{\sigma}$, and $\hat{\Sigma} = \text{Diag}(\hat{\sigma})$. For the nuclear norm regularization, the proximal operator (7) involves singular value decomposition, which is challenging especially for large scale problems.

As discussed above, evaluating the proximal operator can be a computational bottleneck and limits the performance of TAP-SGD. This motivates us to design a novel asynchronous parallel algorithm, which decouples and distributes the calculation of the proximal operator to the workers.

3 Decoupled Asynchronous Proximal Stochastic Gradient Descent (DAP-SGD)

The key idea of the decoupled asynchronous proximal stochastic gradient descent (DAP-SGD) algorithm is to off-load the computational bottleneck from the master to the workers. The master no longer takes care of the proximal operators; instead, it only needs to conduct element-wise addition operations. On the other hand, the workers must work harder: they evaluate the proximal operators independently, without caring about the parallel mechanism.

The procedure of DAP-SGD is summarized in Algorithm 2. Each worker evaluates the proximal operator and sends update information (namely, innovation) $\Delta = x' - x$ to the master. In the master, the delayed update information $\Delta_d(t) = x'_d(t) - x_d(t)$ is used to modify the parameter $x$. Obviously, parameter updating in the master is no longer the computational bottleneck of the system, since it only involves element-wise addition operations.

The recursion of DAP-SGD is

$$x'_d(t) = \text{Prox}_{\eta,h}(x_d(t) - \eta d(t) \nabla f_d(t)(x_d(t))),$$

$$x_{t+1} = x_t + x'_d(t) - x_d(t).$$

Comparing the recursions of TAP-SGD (3) and DAP-SGD (8), we can observe that the DAP-SGD recursion (8) splits the proximal operator and parameter updating step [1]. This is the reason we call the proposed algorithm “decoupled”. The benefit of decoupling is that the computational bottleneck (for example, the unbalanced partitioned groups in [4], the subproblem in [6], and the singular value decomposition in [7]) no longer lies in the master. The workers conduct these operations, which improves the performance of the system. Below, we further analyze the convergence properties of DAP-SGD theoretically.

4 Convergence Analysis

This section gives theorems that establish the convergence properties of DAP-SGD. The detailed proofs are presented in the appendix. We start from some basic assumptions.

The first two assumptions are about the properties of the averaged empirical cost $f(x)$.

\[^1\text{Note that both TAP-SGD and DAP-SGD can support mini-batch updating.}\]
Algorithm 2: Decoupled Asynchronous Proximal Stochastic Gradient Descent (DAP-SGD)

Input: Initialization $\mathbf{x}_0$, $t = 0$, dataset with $n$ samples in which loss function of the $i$th sample is denoted by $f_i(\mathbf{x})$, regularization term $h(\mathbf{x})$, maximum number of iterations $T$, number of workers $S$, step-size in the $t$th iteration $\eta_t$, maximum delay $\tau$

Output: $\mathbf{x}_T$ 

Procedure of each worker $s \in [1, ..., S]$

1 repeat
2 \hspace{1em} Uniformly sample $i$ from $[1, ..., n]$;
3 \hspace{1em} Obtain parameter $\mathbf{x}$ and step-size $\eta$ from master (shared memory or parameter server);
4 \hspace{1em} Evaluate the gradient of the $i$th sample over parameter $\mathbf{x}$, denoted by $\nabla f_i(\mathbf{x})$
5 \hspace{1em} Evaluate the proximal operator $\mathbf{x}' = \text{Prox}_{\eta h}(\mathbf{x} - \eta \nabla f_i(\mathbf{x}))$
6 \hspace{1em} Send update information $\Delta = \mathbf{x}' - \mathbf{x}$ to the master;
7 until procedure of master end

Procedure of master
1 for $t = 0$ to $T - 1$ do
2 \hspace{1em} Get $\Delta_{d(t)} = \mathbf{x}'_{d(t)} - \mathbf{x}_{d(t)}$ from one worker (the delay $t - d(t)$ is bounded by $\tau$);
3 \hspace{1em} Update parameter with $\mathbf{x}_{t+1} = \mathbf{x}_t + \Delta_{d(t)}$
4 \hspace{1em} $t = t + 1$

Assumption 1 Lipschitz continuous gradient of $\nabla f(\mathbf{x})$: The function $f(\mathbf{x})$ is differentiable and its gradient $\nabla f(\mathbf{x})$ is Lipschitz continuous with constant $L$. Namely, the following two equivalent inequalities hold:

$$f(\mathbf{x}) \leq f(\mathbf{y}) + \langle \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle + \frac{L}{2} \| \mathbf{x} - \mathbf{y} \|_2^2, \quad \forall \mathbf{x}, \mathbf{y},$$

and

$$\frac{1}{L} \| \nabla f(\mathbf{x}) - \nabla f(\mathbf{y}) \|_2^2 \leq \langle \nabla f(\mathbf{x}) - \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle \leq L \| \mathbf{x} - \mathbf{y} \|_2^2, \quad \forall \mathbf{x}, \mathbf{y}. \quad (10)$$

Assumption 2 Strong convexity of $f(\mathbf{x})$: The function $f(\mathbf{x})$ is strongly convex with constant $\mu$. Namely, the following inequality holds:

$$f(\mathbf{x}) \geq f(\mathbf{y}) + \langle \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle + \frac{\mu}{2} \| \mathbf{x} - \mathbf{y} \|_2^2, \quad \forall \mathbf{x}, \mathbf{y}. \quad (11)$$

The next assumption bounds the variance of sampling a random gradient $\nabla f_i(\mathbf{x})$ to replace the true gradient $\nabla f(\mathbf{x})$.

Assumption 3 Bounded variance of gradient evaluation: The variance of a selected gradient is bounded by a constant $C_f$: 

$$\mathbb{E} \| \nabla f_i(\mathbf{x}) - \nabla f(\mathbf{x}) \|_2^2 \leq C_f, \quad \forall \mathbf{x}. \quad (12)$$

The last two assumptions are about the properties of the regularization term $h(\mathbf{x})$.

Assumption 4 Convexity of $h(\mathbf{x})$: The function $h(\mathbf{x})$ is convex. Namely, the following inequality holds:

$$h(\mathbf{x}) \geq h(\mathbf{y}) + \langle \partial h(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle, \quad \forall \mathbf{x}, \mathbf{y}, \quad (13)$$

where $\partial h(\mathbf{x})$ stands for any subgradient of $h(\mathbf{x})$.

Assumption 5 Bounded subgradient of $h(\mathbf{x})$: The squared subgradient of $h(\mathbf{x})$ is bounded by a constant $C_h$

$$\| \partial h(\mathbf{x}) \|_2^2 \leq C_h. \quad (14)$$
An immediate result from Assumption 5 is that, $\nabla f(x^*)$ is also bounded where $x^*$ is the optimal solution to (1), as given in the following corollary.

**Corollary 1** Bounded gradient of $f(x)$ at the optimum: Let $x^* = \text{argmin}_x f(x) + h(x)$ be the optimal solution to (1), then we have

$$\|\nabla f(x^*)\|_2^2 = \|\partial h(x^*)\|_2^2 \leq C_h.$$  

Assumptions 1, 2, 3 and 4 are common in the convergence analysis of stochastic gradient descent algorithms [1, 2, 3, 20, 21]. Assumption 5 is due to the (usually non-smooth) regularization term $h(x)$, and is reasonable for many non-smooth regularization terms such as $L_1$ regularization, group lasso, fused lasso and nuclear norm, etc. Next we provide the constant upper bounds of subgradients for these non-smooth regularization terms. In the following part, $\partial$ denotes the set of subderivatives, and with a slight abuse of notation, also denotes any element (namely, subgradient) in the set.

**Upper bound of subgradient for $L_1$ regularization $\|x\|_1$:**

$$\|\partial\|x\|_1\|_2 \leq m.$$  

**Upper bound of subgradient for group lasso regularization $\sum_{i=1}^g \|x_{k_i:(k_{i+1}-1)}\|_2$:**

$$\left\|\partial \sum_{i=1}^g \|x_{k_i:(k_{i+1}-1)}\|_2 \right\| \leq g,$$

where

$$\partial \|x_{k_i:(k_{i+1}-1)}\|_2 = \begin{cases} \frac{1}{\|x_{k_i:(k_{i+1}-1)}\|} x_{k_i:(k_{i+1}-1)} & \text{if } x_{k_i:(k_{i+1}-1)} \neq 0, \\ \{g||g||_2 \leq 1\} & \text{if } x_{k_i:(k_{i+1}-1)} = 0. \\ 

\right.$$

**Upper bound of subgradient for simplified fused lasso regularization $\sum_{i=1}^{m-1} \|x_i - x_{i+1}\|_1 = \|Rx\|_1$:**

$$\|\partial\|Rx\|_2\|_2 = \|R^T SGN(Rx)\|_2 \leq \sum_i \|R_{i,:}\|_2 \|SGN(Rx)\|_2 \leq (m - 1) \sum_i \|R_{i,:}\|_2 \leq \sqrt{2m(m-1)},$$

where $SGN \{x\}$ is a function whose output is within $[-1, 1]$.

**Upper bound of subgradient of nuclear norm regularization $\|X\|_*$, $X \in \mathbb{R}^{m \times q}$, $d = \min(m, q)$:**

$$\|\partial\|X\|_*\|_F \leq \|UU^T\|_F + \|W\|_F \leq \|U\|_F \|V^T\|_F + \|W\|_F \leq \text{rank}(X)^2 + d \leq d^2 + d,$$

where $\partial\|X\|_* = \{UV^T + W|W| \in \mathbb{R}^{m \times q}, U^TW = 0, WV = 0, \|W\|_2 \leq 1, X = U\Sigma V^T\}.$

Under the assumptions given above, we prove that DAP-SGD achieves an $O(\log T/T)$ rate when the step-size is diminishing (Theorem 1) and an ergodic $O(1/\sqrt{T})$ rate when the step-size is constant (Theorem 2), where $T$ is the number of total iterations. The proofs of the theorems are given in the appendix.

**Theorem 1** Suppose that the cost function of (1) satisfies the following conditions: $f(x)$ is strongly convex with constant $\mu$ and $h(x)$ is convex; $f(x)$ is differentiable and $\nabla f(x)$ is Lipschitz continuous with constant $L$; $\mathbb{E}\|\nabla f_i(x) - \nabla f(x)\|_2^2 \leq C_f$; $\|\partial h(x)\|_2^2 \leq C_h$. Define the optimal solution of (1) as $x^*$. At time $t$, set the step-size of the DAP-SGD recursion (8) as $\eta_t = O(1/t)$. Then the iterate generated by (8) at time $T$, denoted by $x_T$, satisfies

$$\mathbb{E}\|x_T - x^*\|_2^2 \leq O\left(\frac{\log T}{T}\right).$$

**Theorem 2** Suppose that the cost function of (1) satisfies the following conditions: $f(x)$ is strongly convex with constant $\mu$ and $h(x)$ is convex; $f(x)$ is differentiable and $\nabla f(x)$ is Lipschitz continuous with constant $L$; $\mathbb{E}\|\nabla f_i(x) - \nabla f(x)\|_2^2 \leq C_f$; $\|\partial h(x)\|_2^2 \leq C_h$. Define the optimal solution of (1) as $x^*$. At time $t$, fix the
Figure 1: Comparison of TAP-SGD and DAP-SGD in terms of time and number of iterations. The Y-axis shows the log distance between the solution generated by an algorithm and the optimal solution, denoted by $\log \|x - x^*\|^2_2$. Results of $L_1$, group lasso, simplified fused lasso and nuclear norm regularized objectives are shown in columns from left to right, respectively. Top and bottom rows correspond to the results regarding time and number of iterations, respectively.

5 Experiments

We compare the proposed DAP-SGD algorithm with TAP-SGD in a consistent way without assuming the data is sparse. The implementation is based on the single machine multi-core system (shared memory architecture). Both algorithms are implemented in C++ and run on a multi-core server. Singular value decomposition (SVD) is calculated by eigen\textsuperscript{2}. The parameters are locked while they are being updated. The lock operation will slow down the computation; however it guarantees that the implementation conforms to the algorithm and its corresponding convergence analysis.

Without loss of generality, we choose the least square loss with a non-smooth regularization term as the optimization objective:

$$\min_{x \in \mathbb{R}^m} P(x) = f(x) + h(x) = \frac{1}{n} \sum_{i=1}^{n} \left[ \|x^T s_i - y_i\|^2_2 + \lambda \|x\|^2_F \right] + h(x).$$

In the case of nuclear norm regularization, the loss function $f(x)$ becomes the multi-target least square loss $f(X) = \frac{1}{n} \sum_{i=1}^{n} \left[ \|X^T s_i - y_i\|^2_2 + \lambda \|X\|^2_F \right]$ correspondingly.

In the implementation TAP-SGD, the proximal operator of the $L_1$ regularized objective can be parallelized easily, while the proximal operators of group lasso, simplified fused lasso and nuclear norm are not parallelized due to their coupled and non-element-wise operations. On the other hand, the procedure of the master in the proposed DAP-SGD only involves simple element-wise operations.

\[^2\text{eigen.tuxfamily.org}\]
This paper proposes a novel decoupled asynchronous proximal stochastic gradient descent (DAP-SGD) algorithm for optimizing a composite objective function. By off-loading computation from the master to workers,
the proposed DAP-SGD algorithm becomes easy to parallelize. DAP-SGD is suitable for many master-worker architectures, including single machine multi-core systems and multi-machine systems. We further provide theoretical convergence analyses for DAP-SGD, with both diminishing and fixed step-sizes.
**References**

[1] F. Niu, B. Recht, C. Re, S. J. Wright, Hogwild: A lock-free approach to parallelizing stochastic gradient descent, in: Proceedings of Advances in Neural Information Processing Systems 24, December 12-14, 2011, Granada, Spain, 2011, pp. 693–701.

[2] A. Agarwal, J. C. Duchi, Distributed delayed stochastic optimization, in: Proceedings of Advances in Neural Information Processing Systems 24, December 12-14, 2011, Granada, Spain, 2011, pp. 873–881.

[3] M. Li, D. G. Andersen, A. J. Smola, K. Yu, Communication efficient distributed machine learning with the parameter server, in: Proceedings of Advances in Neural Information Processing Systems 27, December 8-13 2014, Montreal, Quebec, Canada, 2014, pp. 19–27.

[4] X. Lian, Y. Huang, Y. Li, J. Liu, Asynchronous parallel stochastic gradient for nonconvex optimization, in: Proceedings of Advances in Neural Information Processing Systems 28, December 7-12, 2015, Montreal, Quebec, Canada, 2015, pp. 2737–2745.

[5] R. Zhang, J. T. Kwok, Asynchronous distributed ADMM for consensus optimization, in: Proceedings of the 31th International Conference on Machine Learning, ICML 2014, Beijing, China, 21-26 June 2014, 2014, pp. 1701–1709.

[6] H. R. Feyzmahdavian, A. Aytekin, M. Johansson, A delayed proximal gradient method with linear convergence rate, in: IEEE International Workshop on Machine Learning for Signal Processing, MLSP 2014, Reims, France, September 21-24, 2014, pp. 1–6.

[7] J. Liu, S. J. Wright, C. Ré, V. Bittorf, S. Sridhar, An asynchronous parallel stochastic coordinate descent algorithm, Journal of Machine Learning Research 16 (2015) 285–322.

[8] J. Liu, S. J. Wright, Asynchronous stochastic coordinate descent: Parallelism and convergence properties, SIAM Journal on Optimization 25 (1) (2015) 351–376.

[9] O. Fercoq, P. Richtárik, Accelerated, parallel, and proximal coordinate descent, SIAM Journal on Optimization 25 (4) (2015) 1997–2023.

[10] J. Mareček, P. Richtárik, M. Takáč, Distributed block coordinate descent for minimizing partially separable functions, in: Numerical Analysis and Optimization, Springer, 2015, pp. 261–288.

[11] M. Hong, A distributed, asynchronous and incremental algorithm for nonconvex optimization: An admm based approach, arXiv preprint arXiv:1412.6058.

[12] Y. Zhou, Y. Yu, W. Dai, Y. Liang, E. Xing, On convergence of model parallel proximal gradient algorithm for stale synchronous parallel system, in: International Conference on Artificial Intelligence and Statistics (AISTATS), 2016.

[13] C. Hsieh, H. Yu, I. S. Dhillon, Passcode: Parallel asynchronous stochastic dual co-ordinate descent, in: Proceedings of the 32nd International Conference on Machine Learning, ICML 2015, Lille, France, 6-11 July 2015, 2015, pp. 2370–2379.

[14] A. Beck, M. Teboulle, A fast iterative shrinkage-thresholding algorithm for linear inverse problems, SIAM journal on imaging sciences 2 (1) (2009) 183–202.

[15] N. Parikh, S. P. Boyd, Proximal algorithms., Foundations and Trends in optimization 1 (3) (2014) 127–239.

[16] J. Friedman, T. Hastie, R. Tibshirani, A note on the group lasso and a sparse group lasso, arXiv preprint arXiv:1001.0736.

[17] J. Liu, L. Yuan, J. Ye, An efficient algorithm for a class of fused lasso problems, in: Proceedings of the 16th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, Washington, DC, USA, July 25-28, 2010, 2010, pp. 323–332.
[18] S. Ji, J. Ye, An accelerated gradient method for trace norm minimization, in: Proceedings of the 26th Annual International Conference on Machine Learning, ICML 2009, Montreal, Quebec, Canada, June 14-18, 2009, 2009, pp. 457–464.

[19] J. Cai, E. J. Candès, Z. Shen, A singular value thresholding algorithm for matrix completion, SIAM Journal on Optimization 20 (4) (2010) 1956–1982.

[20] Y. Nesterov, Introductory lectures on convex optimization: A basic course, Vol. 87, Springer Science & Business Media, 2013.

[21] A. Nemirovski, A. Juditsky, G. Lan, A. Shapiro, Robust stochastic approximation approach to stochastic programming, SIAM Journal on Optimization 19 (4) (2009) 1574–1609.
Appendix for Make Workers Work Harder: Decoupled Asynchronous Proximal Stochastic Gradient Descent

Theorem 1 Suppose that the cost function of (1) satisfies the following conditions: \( f(x) \) is strongly convex with constant \( \mu \) and \( h(x) \) is convex; \( f(x) \) is differentiable and \( \nabla f(x) \) is Lipschitz continuous with constant \( L \); \( \mathbb{E}\|\nabla f_i(x) - \nabla f(x)\|_2^2 \leq C_f \); \( \|\partial h(x)\|_2^2 \leq C_h \). Define the optimal solution of (1) as \( x^* \). At time \( t \), set the step-size of the DAP-SGD update (8) as \( \eta_t = O(1/t) \). Then the iterate generated by (8) at time \( T \), denoted by \( x_T \), satisfies

\[
\mathbb{E}\|x_T - x^*\|_2^2 \leq O\left( \frac{\log T}{T} \right). \tag{23}
\]

Proof of Theorem 1: From the DAP-SGD update \( x_{t+1} = x_t + x'_{d(t)} - x_{d(t)} \), we have

\[
\mathbb{E}\|x_{t+1} - x^*\|_2^2 = \mathbb{E}\|x_t - x^* + x'_{d(t)} - x_{d(t)}\|^2
= \mathbb{E}\|x_t - x^*\|_2^2 + \mathbb{E}\|x'_{d(t)} - x_{d(t)}\|^2 + 2\mathbb{E}\left(\langle x'_{d(t)} - x_{d(t)}, x_t - x^* \rangle \right)
\]

\[
\geq \mathbb{E}\|x_t - x^*\|_2^2 + \mathbb{E}\|x'_{d(t)} - x_{d(t)}\|^2 + 2\mathbb{E}\left(\langle x'_{d(t)} - x_{d(t)}, x_t - x_d(t) \rangle \right).
\tag{24}
\]

Below we bound the value of \( Q_1 \) from above. Recalling the update of \( x'_{d(t)} \) in (8) of the paper, which is

\[
x'_{d(t)} = \text{Prox}_{\eta, h}(x_{d(t)} - \eta_{d(t)} \nabla f_{a(t)}(x_{d(t)}))
= \argmin_y \frac{1}{2\eta_{d(t)}} \|y - (x_{d(t)} - \eta_{d(t)} \nabla f_{a(t)}(x_{d(t)}))\|_2^2 + h(y),
\tag{25}
\]

we have

\[
\frac{1}{\eta_{d(t)}}(x_{d(t)} - x'_{d(t)}) - \nabla f_{a(t)}(x_{d(t)}) \in \partial h(x'_{d(t)}).
\tag{26}
\]

Because \( f(x) \) is convex (right now we do not need to use its strong convexity) and \( h(x) \) is also convex, we have the following lower bound for the optimal value

\[
P(x^*) \triangleq f(x^*) + h(x^*) 
\geq f(x_{d(t)}) + \langle \nabla f(x_{d(t)}), x^* - x_{d(t)} \rangle + h(x'_{d(t)}) + \langle \partial h(x'_{d(t)}), x^* - x'_{d(t)} \rangle.
\tag{27}
\]

With a slight abuse of notation, here and thereafter \( \partial h(x'_{d(t)}) \) stands for any subgradient. Hence we substitute the one given in \( \text{(25)} \) into \( \text{(27)} \) and obtain

\[
P(x^*) \geq f(x_{d(t)}) + \langle \nabla f(x_{d(t)}), x^* - x_{d(t)} \rangle 
+ h(x'_{d(t)}) + \left( \frac{1}{\eta_{d(t)}}(x_{d(t)} - x'_{d(t)}) - \nabla f_{a(t)}(x_{d(t)}), x^* - x'_{d(t)} \right).
\tag{28}
\]

On the other hand, \( \nabla f(x) \) being Lipschitz continuous with constant \( L \) implies

\[
f(x'_{d(t)}) \leq f(x_{d(t)}) + \langle \nabla f(x_{d(t)}), x'_{d(t)} - x_{d(t)} \rangle + \frac{L}{2} \|x'_{d(t)} - x_{d(t)}\|_2^2.
\tag{29}
\]

Substituting \( \text{(24)} \) into \( \text{(28)} \)

\[
P(x^*) \geq f(x'_{d(t)}) - \langle \nabla f(x_{d(t)}), x'_{d(t)} - x_{d(t)} \rangle - \frac{L}{2} \|x'_{d(t)} - x_{d(t)}\|_2^2 + \langle \nabla f(x_{d(t)}), x^* - x_{d(t)} \rangle 
+ h(x'_{d(t)}) + \left( \frac{1}{\eta_{d(t)}}(x_{d(t)} - x'_{d(t)}) - \nabla f_{a(t)}(x_{d(t)}), x^* - x'_{d(t)} \right).
\tag{30}
\]
Noticing that by definition $P(x'_{d(t)}) \triangleq f(x'_{d(t)}) + h(x'_{d(t)})$ and reorganizing the terms of (30), we obtain

$$-[P(x'_{d(t)}) - P(x^*)] \geq \left\langle \nabla f(x_{d(t)}) - \nabla f(x'_{d(t)}), x^* - x_{d(t)} \right\rangle + \frac{1}{\eta_{d(t)}} \left\langle x_{d(t)} - x'_{d(t)}, x^* - x_{d(t)} \right\rangle$$

$$+ \frac{1}{\eta_{d(t)}} \|x_{d(t)} - x'_{d(t)}\|^2 - \frac{L}{2} \|x_{d(t)} - x'_{d(t)}\|^2.$$

Assuming that $\eta_t \leq 1/L$ for any $t$ (this assumption holds according to the step-size rule given later), (31) yields

$$-[P(x'_{d(t)}) - P(x^*)] \geq \left\langle \nabla f(x_{d(t)}) - \nabla f(x'_{d(t)}), x^* - x_{d(t)} \right\rangle + \frac{1}{\eta_{d(t)}} \left\langle x_{d(t)} - x'_{d(t)}, x^* - x_{d(t)} \right\rangle$$

$$+ \frac{1}{2\eta_{d(t)}} \|x_{d(t)} - x'_{d(t)}\|^2.$$

Taking expectation on both sides of (32) and reorganizing terms, we have

$$- \mathbb{E}[P(x'_{d(t)}) - P(x^*)] + \mathbb{E} \left\langle \nabla f_{i_{d(t)}}(x_{d(t)}) - \nabla f(x_{d(t)}), x^* - x_{d(t)} \right\rangle$$

$$\geq \frac{1}{\eta_{d(t)}} \mathbb{E} \left\langle x_{d(t)} - x'_{d(t)}, x^* - x_{d(t)} \right\rangle + \frac{1}{2\eta_{d(t)}} \mathbb{E} \|x_{d(t)} - x'_{d(t)}\|^2.$$

Define $\hat{x}'_{d(t)} \triangleq \text{Prox}_{\eta,h}(x_{d(t)} - \eta_{d(t)} \nabla f(x_{d(t)}))$ as an approximation of $x'_{d(t)} \triangleq \text{Prox}_{\eta,h}(x_{d(t)} - \eta_{d(t)} \nabla f_{i_{d(t)}}(x_{d(t)}))$. Because the random variable $i_{d(t)}$ is independent with $x^*$ and $\hat{x}'_{d(t)}$, while $\mathbb{E} [\nabla f_{i_{d(t)}}(x_{d(t)})] = \nabla f(x_{d(t)})$, it holds $\mathbb{E} \langle \nabla f_{i_{d(t)}}(x_{d(t)}) - \nabla f(x_{d(t)}), x^* - \hat{x}'_{d(t)} \rangle = 0$. Hence, $Q_2$ can be upper bounded by

$$Q_2 = \mathbb{E} \left\langle \nabla f_{i_{d(t)}}(x_{d(t)}) - \nabla f(x_{d(t)}), x^* - x'_{d(t)} \right\rangle$$

$$\leq \mathbb{E} \left\| \nabla f_{i_{d(t)}}(x_{d(t)}) - \nabla f(x_{d(t)}) \right\|_2 \|\hat{x}'_{d(t)} - x'_{d(t)}\|_2,$$

where the last inequality comes from the Cauchy-Schwarz inequality. Further, the non-expansive property of proximal operators [8] implies

$$\|\hat{x}'_{d(t)} - x'_{d(t)}\|_2 = \|\text{Prox}_{\eta,h}(x_{d(t)} - \eta_{d(t)} \nabla f(x_{d(t)})) - \text{Prox}_{\eta,h}(x_{d(t)} - \eta_{d(t)} \nabla f_{i_{d(t)}}(x_{d(t)}))\|_2$$

$$\leq \eta_{d(t)} \|\nabla f_{i_{d(t)}}(x_{d(t)}) - \nabla f(x_{d(t)})\|_2.$$

Combining (34) and (35) yields an upper bound of $Q_2$ as

$$Q_2 \leq \eta_{d(t)} \mathbb{E} \|\nabla f_{i_{d(t)}}(x_{d(t)}) - \nabla f(x_{d(t)})\|_2 \leq \eta_{d(t)} C_f,$$

where the last inequality is due to the assumption of bounded variance $\mathbb{E} \|\nabla f(x) - \nabla f(x)\|_2 \leq C_f$.

Substituting (36) into (33), we have

$$- \mathbb{E}[P(x'_{d(t)}) - P(x^*)] + \eta_{d(t)} C_f$$

$$\geq \frac{1}{\eta_{d(t)}} \mathbb{E} \left\langle x_{d(t)} - x'_{d(t)}, x^* - x_{d(t)} \right\rangle + \frac{1}{2\eta_{d(t)}} \mathbb{E} \|x_{d(t)} - x'_{d(t)}\|^2.$$

Now we end up with an upper bound of $Q_1$ as

$$Q_1 \triangleq \mathbb{E}[\|x'_{d(t)} - x_{d(t)}\|^2 + 2 \mathbb{E} \langle x'_{d(t)} - x_{d(t)}, x_{d(t)} - x^* \rangle$$

$$\leq -2 \eta_{d(t)} \mathbb{E}[P(x'_{d(t)}) - P(x^*)] + 2 \eta_{d(t)}^2 C_f.$$
Therefore

$$Q_1 \leq -2\eta_{d(t)}\mathbb{E}[P(x_t) - P(x^*)] - 2\eta_{d(t)}\mathbb{E}[P(x'_{d(t)}) - P(x_t)] + 2\eta_{d(t)}^2C_f.$$ \hspace{1cm} (39)

The second line comes from the inequality

$$P(x_t) - P(x^*) \geq \frac{\mu}{2}\|x_t - x^*\|_2^2,$$ \hspace{1cm} (40)

which is due to the facts that $x^*$ is the optimal solution of $P(x) = f(x) + h(x)$, $f(x)$ is strongly convex with constant $\mu$, and $h(x)$ is convex.

Substituting (39) into (24), we have

$$\mathbb{E}\|x_{t+1} - x^*\|_2^2 \leq (1 - \mu\eta_{d(t)})\mathbb{E}\|x_t - x^*\|_2^2 + 2\eta_{d(t)}\mathbb{E}[P(x'_{d(t)}) - P(x'_{d(t)})]$$

$$+ 2\eta_{d(t)}\sum_{p=1}^{t-d(t)} \mathbb{E}[P(x_{t-p+1}) - P(x_{t-p})] + 2\eta_{d(t)}^2C_f + 2\mathbb{E}\left\langle x'_{d(t)} - x_{d(t)}, x_t - x_{d(t)} \right\rangle.$$ \hspace{1cm} (41)

We proceed to bound the terms $Q_3$, $Q_4$, and $Q_5$.

Because $f(x)$ and $h(x)$ are convex as well as the norm of $\partial h(x)$ is bounded, we have the following basic inequality

$$P(x) - P(y) = f(x) - f(y) + h(x) - h(y)$$

$$\leq \langle \nabla f(x), x - y \rangle + \langle \partial h(x), x - y \rangle$$

$$\leq \|\nabla f(x)\|_2\|x - y\|_2 + \|\partial h(x)\|_2\|x - y\|_2$$

$$\leq \|\nabla f(x)\|_2\|x - y\|_2 + \sqrt{C_h}\|x - y\|_2$$

$$= (\|\nabla f(x)\|_2 + \sqrt{C_h})\|x - y\|_2.$$ \hspace{1cm} (42)

In (42), the second line comes from the convexity of $f(x)$ and $h(x)$, while the third line comes from the Cauchy-Schwarz inequality. Replacing $x$ by $x_{d(t)}$ and $y$ by $x'_{d(t)}$ in (42), we have

$$Q_3 = \mathbb{E}\left[ P(x_{d(t)}) - P(x'_{d(t)}) \right] \leq \mathbb{E}\left[ (\|\nabla f(x_{d(t)})\|_2 + \sqrt{C_h})\|x_{d(t)} - x'_{d(t)}\|_2 \right].$$ \hspace{1cm} (43)

Applying the expression of $x_{d(t)} - x'_{d(t)}$ in (20) into (43) yields

$$Q_3 \leq \eta_{d(t)}\mathbb{E}\left[ (\|\nabla f(x_{d(t)})\|_2 + \sqrt{C_h})\|f_{i_{d(t)}}(x_{d(t)}) + \partial h(x'_{d(t)})\|_2 \right]$$

$$\leq \frac{1}{2}\eta_{d(t)}\mathbb{E}\|\nabla f(x_{d(t)})\|_2^2 + \frac{1}{2}\eta_{d(t)}C_h + \eta_{d(t)}\mathbb{E}\|f_{i_{d(t)}}(x_{d(t)}) + \partial h(x'_{d(t)})\|_2^2.$$ \hspace{1cm} (44)

Due to the inequalities

$$\frac{1}{2}\|\nabla f(x_{d(t)})\|_2^2 \leq \|\nabla f(x_{d(t)}) - \nabla f(x^*)\|_2^2 + \|\nabla f(x^*)\|_2^2.$$ \hspace{1cm} (45)

and

$$\|f_{i_{d(t)}}(x_{d(t)}) + \partial h(x'_{d(t)})\|_2^2 \leq 2\|f_{i_{d(t)}}(x_{d(t)})\|_2^2 + 2\|\partial h(x'_{d(t)})\|_2^2$$

$$\leq 2\|f_{i_{d(t)}}(x_{d(t)})\|_2^2 + 2\|\partial h(x'_{d(t)})\|_2^2.$$ \hspace{1cm} (46)

Therefore
The inequality (44) turns to

$$Q_3 \leq 9\eta_{d(t)}E\|\nabla f(x_{d(t)}) - \nabla f(x^*)\|_2^2 + 9\eta_{d(t)}E\|\nabla f(x^*)\|_2^2 + 4\eta_{d(t)}E\|\nabla f_{i_d(t)}(x_{d(t)}) - \nabla f(x_{d(t)})\|_2^2 + 2\eta_{d(t)}E\|\partial h(x'_{d(t)})\|_2^2 + \frac{1}{2}\eta_{d(t)}C_h. \quad (47)$$

Considering Lipschitz continuity of $\nabla f(x)$, $\|\nabla f(x^*)\|_2 \leq C_h$ from Corollary 1, $E\|\nabla f_i(x) - \nabla f(x)\|_2 \leq C_f$, as well as $\|\partial h(x)\|_2 \leq C_h$, (47) further turns to

$$Q_3 \leq 9\eta_{d(t)}L^2E\|x_{d(t)} - x^*\|_2^2 + 4\eta_{d(t)}C_f + \frac{23}{2}\eta_{d(t)}C_h. \quad (48)$$

Similar to the derivation of (47), we have

$$Q_4 = E\left[ P(x_{t-p+1}) - P(x_{t-p}) \right] \leq E \left[ \|\nabla f(x_{t-p+1})\|_2 + \sqrt{C_h}\|x_{t-p+1} - x_{t-p}\|_2 \right] \leq \eta_{d(t-p)}E \left[ \|\nabla f(x_{t-p+1})\|_2 + \sqrt{C_h}\|\nabla f_{i_{t-p}}(x_{d(t-p)}) + \partial h(x'_{d(t-p)})\|_2 \right] \leq \frac{1}{2}\eta_{d(t-p)}E\|\nabla f(x_{t-p+1})\|_2^2 + \frac{1}{2}\eta_{d(t-p)}C_h + \eta_{d(t-p)}E\|\nabla f_{i_{t-p}}(x_{d(t-p)}) + \partial h(x'_{d(t-p)})\|_2^2. \quad (49)$$

Using the inequalities (see (45) and (46))

$$\frac{1}{2}\|\nabla f(x_{t-p+1})\|_2^2 \leq \|\nabla f(x_{t-p+1}) - \nabla f(x^*)\|_2^2 + \|\nabla f(x^*)\|_2^2, \quad (50)$$

and

$$\|\nabla f_{i_{t-p}}(x_{d(t-p)}) + \partial h(x'_{d(t-p)})\|_2^2 \leq 4\|\nabla f_{i_{t-p}}(x_{d(t-p)})\|_2^2 + 8\|\nabla f(x_{d(t-p)}) - \nabla f(x^*)\|_2^2 + \frac{2\|\nabla f(x_{d(t-p)})\|_2^2}{2}, \quad (51)$$

(49) yields

$$Q_4 \leq \eta_{d(t-p)}E\|\nabla f(x_{t-p+1}) - \nabla f(x^*)\|_2^2 + 9\eta_{d(t-p)}E\|\nabla f(x^*)\|_2^2 + 8\eta_{d(t-p)}E\|\nabla f(x_{t-p}) - \nabla f(x^*)\|_2^2 + 4\eta_{d(t-p)}E\|\nabla f_{i_{t-p}}(x_{d(t-p)})\|_2^2 + 2\eta_{d(t-p)}C_h \leq \eta_{d(t-p)}L^2E\|x_{t-p+1} - x^*\|_2^2 + 8\eta_{d(t-p)}L^2E\|x_{d(t-p)} - x^*\|_2^2 + 4\eta_{d(t-p)}C_f + \frac{23}{2}\eta_{d(t-p)}C_h. \quad (52)$$

Again, the last line of (52) utilizes Lipschitz continuity of $\nabla f(x)$, $\|\nabla f(x^*)\|_2 \leq C_h$ from Corollary 1, $E\|\nabla f_i(x) - \nabla f(x)\|_2 \leq C_f$, as well as $\|\partial h(x)\|_2 \leq C_h$.

For the term $Q_5$, we use the Cauchy-Schwarz inequality followed by the substitution of (50) and get

$$Q_5 = E\left( x'_{d(t)} - x_{d(t)} \right) \leq E \left( \|x'_{d(t)} - x_{d(t)}\|_2 \|x_t - x_{d(t)}\|_2 \right) \leq \eta_{d(t)}E \left( \|\nabla f_{i_{d(t)}}(x_{d(t)}) + \partial h(x'_{d(t)})\|_2 \|x_t - x_{d(t)}\|_2 \right). \quad (53)$$

Further relaxing (50) by the triangle inequality yields

$$Q_5 \leq \eta_{d(t)} \sum_{p=1}^{t-d(t)} \mathbb{E} \left( \|\nabla f_{i_{d(t)}}(x_{d(t)}) + \partial h(x'_{d(t)})\|_2 \|x_{t-p+1} - x_{t-p}\|_2 \right). \quad (54)$$

Since the maximum delay is $\tau$, we have

$$Q_5 \leq \eta_{d(t)} \sum_{p=1}^{t-d(t)} \mathbb{E} \left( \|\nabla f_{i_{d(t)}}(x_{d(t)}) + \partial h(x'_{d(t)})\|_2 \|x_{t-p+1} - x_{t-p}\|_2 \right). \quad (55)$$
Though not straightforward, we can show that under the step-size rule given by (59), (58) yields
\[ Q_5 \leq \eta_d(t) \sum_{p=1}^{\tau} \eta_d(t-p) \mathbb{E} \left( \| \nabla f_{d(t)}(x_d(t)) + \partial h(x'_{d(t)}) \|_2 \| \nabla f(x_d(t)) + \partial h(x'_{d(t)}) \|_2 \right). \] 
(56)

Following the similar routines as those in [26] and [32], eventually we reach
\[ Q_5 \leq 4\eta_d(t)L^2 \sum_{p=1}^{\tau} \eta_d(t-p) \mathbb{E}\|x_d(t) - x^*\|_2^2 + 4\eta_d(t)L^2 \sum_{p=1}^{\tau} \eta_d(t-p) \mathbb{E}\|x_d(t-p) - x^*\|_2^2 \]
\[ + 4\eta_d(t) \sum_{p=1}^{\tau} \eta_d(t-p) C_f + 10\eta_d(t) \sum_{p=1}^{\tau} \eta_d(t-p) C_h \] 
(57)

Substituting [26], [32] and [34] into (48), we have
\[ \mathbb{E}\|x_{t+1} - x^*\|_2^2 \leq (1 - \mu \eta_d(t)) \mathbb{E}\|x_t - x^*\|_2^2 + \left( 8\eta_d(t)L^2 \sum_{p=1}^{\tau} \eta_d(t-p) + 18\eta_d(t)^2 L^2 \right) \mathbb{E}\|x_d(t) - x^*\|_2^2 \]
\[ + 2\eta_d(t)L^2 \sum_{p=1}^{\tau} \eta_d(t-p) \mathbb{E}\|x_{t-p+1} - x^*\|_2^2 + 24\eta_d(t)L^2 \sum_{p=1}^{\tau} \eta_d(t-p) \mathbb{E}\|x_d(t-p) - x^*\|_2^2 \]
\[ + \left( 16\eta_d(t) \sum_{p=1}^{\tau} \eta_d(t-p) + 8\eta_d(t)^2 \right) C_f + \left( 43\eta_d(t) \sum_{p=1}^{\tau} \eta_d(t-p) + 23\eta_d(t)^2 \right) C_h. \]
(58)

Define the step-size rule
\[ \eta_t = \frac{1}{\mu(t + 1) + u} = O \left( \frac{1}{t} \right), \] 
(59)

where \( u \) is a positive constant satisfying:
- \( u > (2\tau - 1)\mu \) such that \( \eta_t \leq \eta_d(t) \);
- \( u \) is large enough such that \( \min(\mu/(4C_1\tau), 1/L) \geq \eta_t \), where \( C_1 \) is a constant we give below.

Define two constants
\[ C_1 = \left( 2L^2 - \mu + u \right) \frac{\mu + u}{\mu + u - 2\mu \tau} + 48\tau L^2 + 8\tau L^2 - \mu + u \frac{\mu + u}{\mu + u - 2\mu \tau} \] 
\[ + 18L^2, \]
and
\[ C_2 = \frac{(\mu + u)^2}{(\mu + u - 2\mu \tau)^2}. \]

Though not straightforward, we can show that under the step-size rule given by (59), (58) yields
\[ \mathbb{E}\|x_{t+1} - x^*\|_2^2 \leq (1 - \mu \eta_t) \mathbb{E}\|x_t - x^*\|_2^2 + C_1 \sum_{p=0}^{\tau} \eta_t^2 \mathbb{E}\|x_{t-p} - x^*\|_2^2 + C_2 \eta_t^2. \] 
(60)

For the ease of presentation, we define \( a_t = \mathbb{E}\|x_t - x^*\|_2^2 \) and will analyze its rate. Rewrite (48) to
\[ a_{t+1} \leq (1 - \mu \eta_t)a_t + C_1 \sum_{p=0}^{\tau} \eta_t^2 a_{t-p} + C_2 \eta_t^2. \] 
(61)
Applying telescopic cancellation to \( t = 0 \) to \( t = T - 1 \) yields

\[
a_T \leq a_0 - \sum_{t=0}^{T-1} \mu \eta_t a_t + C_1 \sum_{t=0}^{T-1} 2 \eta_t^2 \eta_t a_{t+1} + C_2 \sum_{t=0}^{T-1} \eta_t^2
\]

\[
\leq a_0 - \sum_{t=0}^{T-1} (2C_1 \eta_t^2) a_t + C_2 O(1).
\]

As we can verify, \( \mu/(4C_1) \geq \eta_t \), meaning that

\[
\sum_{t=0}^{T-1} (2C_1 \eta_t^2) a_t \geq \frac{1}{2} \sum_{t=0}^{T-1} \mu \eta_t a_t.
\]

Combining (62) and (63), we have

\[
\frac{1}{2} \sum_{t=0}^{T-1} \mu \eta_t a_t \leq a_0 - a_T + C_2 O(1),
\]

which, along with the step-size rule (59), implies that

\[
\sum_{t=0}^{T-1} \mu(t+1) + u a_t \leq 2 \left( a_0 + C_2 O(1) \right)
\]

Further define \( C_3 = u/(u - \mu \tau) \) such that

\[
\frac{\mu(t+1) + u}{(\mu(t - p + 1) + u)^2} \leq \frac{C_3}{\mu(t - p + 1) + u}.
\]

Substituting the step-size rule (59) into (61), we have

\[
a_{t+1} \leq \left(1 - \frac{\mu}{\mu(t+1) + u}\right) a_t + C_1 \sum_{p=0}^{2r} \frac{1}{(\mu(t - p + 1) + u)^2} a_{t-p} + \frac{1}{(\mu(t+1) + u)^2} C_2,
\]

and consequently

\[
(\mu(t+1) + u) a_{t+1} \leq (\mu t + u) a_t + C_1 \sum_{p=0}^{2r} \frac{\mu(t+1) + u}{(\mu(t - p + 1) + u)^2} a_{t-p} + \frac{1}{\mu(t+1) + u} C_2
\]

\[
\leq (\mu t + u) a_t + C_1 C_3 \sum_{p=0}^{2r} \frac{1}{(\mu(t - p + 1) + u)^2} a_{t-p} + \frac{1}{\mu(t+1) + u} C_2.
\]

Applying telescopic cancellation again to (67) from \( t = 0 \) to \( t = T - 1 \), we have

\[
(\mu t + u) a_T \leq u a_0 + C_1 C_3 \sum_{t=0}^{T-1} \frac{1}{(\mu(t - p + 1) + u)} a_{t-p} + \frac{1}{(\mu(t+1) + u)} C_2
\]

\[
\leq u a_0 + 2C_1 C_3 \frac{1}{(\mu(t+1) + u)} a_T + \frac{1}{(\mu(t+1) + u)} C_2.
\]

Substituting (65) into (68) yields

\[
(\mu t + u) a_T \leq u a_0 + \frac{4}{\mu} C_1 C_3 \tau (a_0 + C_2 O(1)) + C_2 O(\log T),
\]

(69)
and consequently
\[ a_T \leq \frac{u a_0 + \frac{4}{\mu} C_1 C_3 \tau (a_0 + C_2 O(1)) + C_2 O(\log T)}{\mu T + u} = O \left( \frac{\log T}{T} \right), \] (70)
which completes the proof.

**Theorem 2** Suppose that the cost function of (1) satisfies the following conditions: \( f(x) \) is strongly convex with constant \( \mu \) and \( h(x) \) is convex; \( f(x) \) is differentiable and \( \nabla f(x) \) is Lipschitz continuous with constant \( L \); \( E \| \nabla f(x) - \nabla f(x') \|^2 \leq C_f \); \( \| \partial h(x) \|^2 \leq C_h \). Define the optimal solution of (1) as \( x^* \). At time \( t \), fix the step-size of the DAP-SGD recursion (8) \( \eta_t = O \left( \frac{1}{\sqrt{T}} \right) \), where \( T \) is the maximum number of iterations. Define the iterate generated by (8) at time \( t \) as \( x_t \). Then the running average iterate generated by (8) at time \( T \), denoted by \( \bar{x}_T = \frac{1}{T+1} \sum_{t=0}^{T} x_t \), satisfies

\[ E \| \bar{x}_T - x^* \|^2 \leq O \left( \frac{1}{\sqrt{T}} \right). \] (71)

**Proof of Theorem 2:** We start from (58) in the proof of Theorem 1. Define the step-size rule
\[ \eta_t = \eta = \frac{1}{v \sqrt{T}}, \] (72)
where \( v \) is a positive constant such that \( \min(\mu/(4C_4 \tau), 1/L) \geq \eta \). Defining constants
\[ C_4 = (2 + 56\tau) L^2, \]
and
\[ C_5 = (16\tau + 8) C_f + (43\tau + 23) C_h, \]
followed by manipulating (58), we have (similar to the inequality (61)) the following result
\[ a_{t+1} \leq (1 - \mu \eta) a_t + C_4 \eta^2 \sum_{p=0}^{2\tau} a_{t-p} + C_5 \eta^2 \] (73)

Applying telescopic cancellation to (73) from \( t = 0 \) to \( t = T \) yields
\[ a_{T+1} \leq a_0 - \sum_{t=0}^{T} \mu \eta a_t + C_4 \eta^2 \sum_{t=0}^{T} \sum_{p=0}^{2\tau} a_{t-p} + C_5 (T+1) \eta^2 \]
\[ \leq a_0 - \sum_{t=0}^{T} (\mu \eta - 2C_4 \eta^2 \tau) a_t + C_5 (T+1) \eta^2. \] (74)

Since \( \mu/(4C_4 \tau) \geq \eta \) such that
\[ \sum_{t=0}^{T} (\mu \eta - 2C_4 \eta^2 \tau) a_t \geq \frac{\mu \eta}{2} \sum_{t=0}^{T} a_t, \] (75)
(74) implies
\[ \frac{\mu \eta}{2} \sum_{t=0}^{T} a_t \leq a_0 - a_{T+1} + C_5 (T+1) \eta^2, \] (76)
and consequently

\[ \frac{\mu \eta}{T + 1} \sum_{t=0}^{T} a_t \leq \frac{2a_0 + 2C_5(T + 1)\eta^2}{T + 1}. \quad (77) \]

According to Jensen’s inequality, we have

\[ \frac{\mu \eta}{T + 1} \sum_{t=0}^{T} a_t = \frac{\mu \eta}{T + 1} \sum_{t=0}^{T} \mathbb{E} \| x_t - x^* \|_2^2 \]
\[ \geq \mu \eta \mathbb{E} \left\| \frac{1}{T + 1} \sum_{t=0}^{T} x_t - x^* \right\|_2^2 \]
\[ = \mu \eta \mathbb{E} \| \bar{x}_T - x^* \|_2^2. \quad (78) \]

Substituting (78) and the step-size rule (72) into (77), we have

\[ \mathbb{E} \| \bar{x}_T - x^* \|_2^2 \leq \frac{2a_0 v \sqrt{T} + 2C_5(T + 1)\frac{1}{\sqrt{T}}}{\mu(T + 1)} = O\left( \frac{1}{\sqrt{T}} \right), \quad (79) \]

which completes the proof.\[\]

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