Open problems for the superKdV equations

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Abstract

After a review of the basic results concerning the $N = 1, 2$ supersymmetric extensions of the Korteweg-de Vries equation, with a pedagogical presentation of the superspace techniques, we discuss some basic open problems mainly in relation with the $N = 2$ extensions.

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1. Introduction

Supersymmetry offers a powerful tool for widening the scope of integrability. The field of supersymmetric integrable systems turns out to be remarkably rich in addition to further displaying novel features such as conserved nonlocal ‘Poisson square roots’ of local conservation laws. Not surprisingly, it started with the extension of the Korteweg-de Vries (KdV) equation although by now many other equations have been supersymmetrized.

The $N = 1, 2$ (where $N$ refers to the number of supersymmetries) integrable supersymmetric versions of the KdV equation have been found about 10 years ago [1, 2, 3, 4]. The key points of this development were 1- the realization that supersymmetrization could be restricted to the space variable only and 2- that the crucial KdV structure whose core needs to be preserved is the KdV second hamiltonian structure. In Fourier components, the underlying hamiltonian operator is the Poisson bracket formulation of the Virasoro algebra [9], for which supersymmetric extensions were already known and could

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3 Some physical motivations for considering supersymmetric integrable systems are scattered in the footnotes 3, 4 and 7, while geometrical implications are alluded to in the conclusion.

4 The present restriction to $N \leq 2$ is partly motivated by the limitations of my own works. However there is also a physical motivation: these systems have found a remarkable application, albeit in their quantum formulation, in perturbed conformal field theory [5], in the context of which $N = 2$ is the maximal number of supersymmetries that is of real interest. Nevertheless, in some context, the $N = 3$ and 4 extensions may be physically relevant and the corresponding extensions of KdV have been considered; see for instance [6]. Our discussion will also be restricted to supersymmetric KdV extensions having an even Poisson brackets. A new $N = 2$ super KdV equation with an odd Poisson structure has been found recently in [7] but the resulting equations are somewhat less interesting in that the bosonic fields do not couple to the fermions in their time evolution. It would be of interest to see whether there are other ‘odd’ integrable extensions displaying fermionic interactions in the bosonic evolution equations.

5 Here are some comments on the literature concerning the $N = 1$ case. Already in 1984, Kupershmidt [8] has presented a simple fermionic (but not supersymmetric - see below) extension of the KdV equation by placing the emphasis on its hamiltonian formulation (this system has actually two local hamiltonian structures). This was an important step toward the formulation of the right supersymmetric extension given that both hamiltonian operators were supersymmetric invariant (the hamiltonian themselves were not) and could then serve in the formulation of a genuine supersymmetric system. The supersymmetric KdV equation was initially found independently of the work of Manin-Radul [2] on super KP hierarchy. It was realized afterwards that this general system has indeed a reduction to the supersymmetric KdV system.
then be used to construct the appropriate supersymmetric extension of KdV. A number of developments have occurred in the following years but recently the focus has moved toward the construction of extended super KdV hierarchies (in the way the Boussinesq equation generalizes KdV, that is, via higher order Lax operators) \cite{11}. However a certain number of problems associated to the $N = 1, 2$ KdV systems have remained unsolved and the goal of this presentation is to identify some of them. For this it is necessary to present a brief review of the supersymmetric formulation of the KdV equation and, for the benefit of those readers unfamiliar with super technologies, some manipulations will be worked out in some detail.

2. Supersymmetrization of the KdV equation: N=1

We will formulate the supersymmetric extension of the KdV equation in the superspace formalism. That amounts to extend the $x$ variable to a doublet $(x, \theta)$ where $\theta$ is a Grassmannian (or anticommuting) variable: $\theta^2 = 0$. Ordinary (i.e. commuting) fields $f(x)$ (functions of $x$ and $t$ in fact but the time dependence will generally be suppressed) will be replaced by superfields $F(x, \theta)$. Given that $\theta^2 = 0$, these superfields have a very simple Taylor expansion in terms of $\theta$:

$$F(x, \theta) = f(x) + \theta \gamma(x)$$

(2.1)

$f$ and $\gamma$ are called the component fields. $\gamma$ is said to be the super-partner of $f$ and vice-versa. In the present case, $F(x, \theta)$ is a bosonic superfield: it has the same ‘statistics’ (i.e. commuting or anticommuting character) as the field appearing in the $\theta$ independent term (here $f$); on the other hand, $\gamma$ is anticommuting, i.e. it is a fermionic field. In particular, $\gamma(x)\gamma(y) = -\gamma(y)\gamma(x)$ so that $\gamma(x)^2 = 0$; also for instance, $\theta \gamma = -\gamma \theta$. The final ingredient that we need is the superderivative

$$D = \theta \partial + \partial \theta$$

(2.2)

whose square is the usual space derivative: $D^2 = \partial$.

\footnote{6 The most important physical application of these constructions concerns conformal field theory: the corresponding Poisson structures yield classical versions of super $W$ algebras whose quantum form can be obtained via the quantization of the modified fields obtained through the Miura transformation (see e.g. \cite{10}).}
A supersymmetry transformation is nothing but a translation in superspace. Such a translation takes the form: $x \rightarrow x - \eta \theta$ and $\theta \rightarrow \theta + \eta$, where $\eta$ is a constant anticommuting parameter, supposed, in the following, to be very small. Consider then the effect of the translation in the superfield:

$$F(x, \theta) \rightarrow F(x - \eta \theta, \theta + \eta) = F(x, \theta) - \eta \theta \partial F(x, \theta) + \eta \partial \theta F(x, \theta)$$

$$\equiv F(x, \theta) + \delta \eta F(x, \theta)$$

$$= f(x) + \theta \gamma(x) + \delta \eta f(x) + \theta \delta \eta \gamma(x)$$

The second equality shows that $\delta \eta$ is bosonic so that it commutes with $\theta$. We read off the component-field transformations to be

$$\delta \eta f = \eta \gamma, \quad \delta \eta \gamma = \eta f_x$$

This is called a supersymmetry transformation; it relates a bosonic field to a fermionic field and vice-versa. It has the remarkable virtue of linking a field transformation to a (super)space translation. Note that two successive supersymmetry transformations lead to

$$\delta \eta \delta \eta' f = \eta' \eta f_x, \quad \delta \eta \delta \eta' \gamma = \eta' \eta \gamma_x$$

In other words, a translation in superspace, hence a supersymmetry transformation, is a sort of square root of an ordinary translation. Every local expression in the superfields and the superderivatives is manifestly supersymmetric invariant.

To supersymmetrize the KdV equation

$$u_t = -u_{xxx} + 6uu_x$$

one should then start by extending the $u$ field to a superfield. There are two ways of doing this: either as a fermionic superfield

$$u(x) \rightarrow \phi(x, \theta) = \theta u(x) + \xi(x)$$

or as a bosonic superfield

$$u(x) \rightarrow U(x, \theta) = u(x) + \theta \lambda(x)$$
It turns out that the first choice is the one that gives the interesting extension. The KdV equation is homogeneous with respect to the scaling gradation: in the normalization where \( \text{deg} \vartheta = 1 \), one finds that \( \text{deg} u = 2 \). The identity \( D^2 = \vartheta \) implies that \( \text{deg} D = 1/2 \), so that \( \text{deg} \theta = -1/2 \). For the superfield to be homogeneous, \( \xi \) must have degree \( 3/2 \).

Let us then proceed with a direct extension of the KdV equation, multiplying each term by \( \theta \) and rewriting the result in terms of superfields:

\[
\begin{align*}
    u_t &\rightarrow \phi_t \\
    u_{xxx} &\rightarrow \phi_{xxx} \\
    3uu_x &\rightarrow c\phi D\phi_x + (6 - c)\phi_x (D\phi)
\end{align*}
\]

where \( c \) is a free constant. We thus observe that the nonlinear term does not have a unique extension in terms of superfields. Therefore, this direct extension leaves us with a supersymmetric version of the KdV equation containing a free parameter:

\[
\phi_t = \phi_{xxx} + c(\phi D\phi)_x + (6 - 2c)\phi_x (D\phi)
\]  

(2.10)

It turns out that this equation is integrable only if \( c = 3 \). We call the resulting equation the super KdV equation, or sKdV for short. Its component version reads:

\[
\begin{align*}
    u_t &= -u_{xxx} + 6u_x - 3\xi_x x \\
    \xi_t &= -\xi_{xxx} + 3(u\xi)_x
\end{align*}
\]

(2.11)

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7 The other possibility would not have a second hamiltonian structure associated to the super Virasoro algebra as a simple dimensional (i.e. degree counting) analysis shows (it requires the introduction of an anticommuting field of degree 3/2 as \( \xi \) and not of degree 5/2 as \( \lambda \)). The degree counting is explained below. Notice that to a large extend, we try to reserve Greek letters for anticommuting variables or fields.

8 Actually the case \( c = 0 \) is also integrable but its leads to a somewhat trivial system in which the fermionic fields decouple from the bosonic equation which reduces then to the usual KdV equation. Nevertheless, this equation happens to be relevant in supersymmetric extensions of matrix models that describes superstrings in \( d < 3/2 \) dimensions, or equivalently, conformal field theories coupled to gravity [12].

9 Notice that the product \( \xi\xi_{xx} \) acts as a bosonic field: quite generally, a product of two fermions is a boson. It can be seen easily that passing an anticommuting variable in front of it does not induce an overall minus sign, e.g. \( \xi\xi_{xx} \theta = -\theta \xi\xi_{xx} = \theta \xi\xi_{xx} \). Notice moreover that this term is a total derivative: \( \xi\xi_{xx} = (\xi_x)_x \) since the extra resulting term is \( \xi_x \xi_x = 0 \).
It is not difficult to verify that the system (2.11) is invariant under the supersymmetry transformation \( \delta_\eta u = \eta \xi_x \) and \( \delta_\eta \xi = \eta u \). This is not so for the integrable fermionic extension proposed by Kupershmidt [8]:

\[
\begin{align*}
    u_t &= -u_{xxx} + 6u u_x - 3\xi \xi_{xx} \\
    \xi_t &= -4\xi_{xxx} + 6u \xi_x + 3u_x \xi
\end{align*}
\] (2.12)

With (2.11) being called the super KdV equation, it would be appropriate to call (2.12) the Kuper-KdV equation.

The integrability of (2.10) can be established in various ways. The most direct argument is that it has a Lax representation:

\[
    L_t = [-4L^{3/2}_+, L] \quad L = \partial^2 - \phi D
\] (2.13)

and the conservation laws are obtained as follows (the subscript gives the degree):

\[
    H_{2k+1} = \int dx d\theta \, s\text{Res}L^{(2k+1)/2}
\] (2.14)

For super pseudodifferential operators, the + projection and the super residue sRes are defined as follows

\[
    \Lambda = \sum_{k=-\infty}^{N} \alpha_i D^i, \quad \Lambda_+ = \sum_{k=0}^{N} \alpha_i D^i \quad s\text{Res}\Lambda = \alpha_{-1}
\] (2.15)

In the above expression for the conservation laws, we have also introduced the superintegration. The integration over the \( \theta \) variable is defined as follows:

\[
    \int d\theta 1 = 0 \quad \int d\theta \, \theta = 1
\] (2.16)

The integration over \( \theta \) is thus essentially equivalent to the differentiation with respect to \( \theta \). With these rules, the superintegration of a superderivative vanishes (with the usual rule that the ordinary integral of a total derivative vanishes):

\[
    \int dx d\theta [D\phi(x, \theta)] = \int dx d\theta (\theta \xi_x + u) = \int dx \xi_x = 0
\] (2.17)

\(^{10}\) Actually, the Lax operator is not unique: the choice \( L = \partial^2 + \phi D - (D\phi) \) (the formal adjoint of \( \partial^2 - \phi D \)) leads to completely equivalent results.
For instance, the second conservation law is\[11\]
\[H_3 = \int dxd\theta \, (\phi D\phi) = \int dx \, (u^2 - \xi_x)\] (2.18)

Another way of establishing the integrability is to supersymmetrize the Gardner transformation [13]. This extension is unique:
\[
\phi = \chi + \epsilon \chi_x + \epsilon^2 \chi D\chi
\] (2.19)
with \(\chi = \theta w + \sigma\). It maps a solution of the super Gardner equation\[12\]
\[
\chi_t = -\chi_{xxx} + 3(\chi D\chi)_x + \epsilon^2 3(D\chi)(\chi D\chi)_x
\] (2.20)
into a solution of the sKdV equation. Since \(\chi_t\) is a total superderivative, e.g.
\[
(D\chi)(\chi D\chi)_x = \frac{1}{6} D[(D\chi)^3] + \frac{1}{2}[\chi(D\chi)^2]_x
\] (2.21)
\(\int dxd\theta \, \chi\) is conserved and by inverting the super Gardner transformation (2.19), we recover an infinite number of conservation laws:
\[
\chi = \sum_{n=0}^{\infty} \epsilon^n h_n[\phi] \quad \Rightarrow \quad \frac{d}{dt} \int dxd\theta \sum_{n=0}^{\infty} \epsilon^n h_n[\phi] = 0
\] (2.22)
(where \(h_n[\phi]\) stands for a differential polynomial in \(\phi\)). Now the crucial point is that the sKdV equation is independent of \(\epsilon\) so that each separate power of \(\epsilon\) must be separately conserved. This produces an infinite number of conservation laws, half of which can be shown to be nontrivial, having a leading term \(\phi(D\phi)^k\); these are bound to be the \(H_{2k+1}\)

\[11\] Notice that for a fermionic variable, \(\xi_x\) is not a total derivative: a partial integration of \(\int dx \, \xi_x\) leads to \(-\int dx \, \xi_x \xi\) and the interchanges of the two terms generates another minus sign so that the original expression is recovered.

\[12\] The component form of this superfield equation reads:
\[
w_t = -w_{xxx} + 6w w_x - \sigma \sigma_{xx} + \epsilon^2 [6w^2 w_x - 3(\sigma \sigma_x w)_x]
\]
\[
\sigma_t = -\sigma_{xxx} + 3(\sigma w)_x + \epsilon^2 [3w(w \sigma)_x]
\]
Notice that \(\sigma_t\) is not a total derivative. The usual Gardner equation is recovered by setting the fermionic field \(\sigma = 0\) and the sKdV equation is the limiting case where \(\epsilon = 0\).
above [1]. Note that these are all bosonic (\(\chi\) is fermionic but the measure \(dxd\theta\) is also fermionic).

Finally, we point out that the sKdV equation is bihamiltonian, the two hamiltonian operators being \[1,14,15,16],

\[P_1 = \partial[D^3 - \phi]^{-1}\partial, \quad P_2 = -D^5 + 3\phi\partial + (D\phi)D + 2\phi_x\]  

(2.23)

Notice that \(P_1\) is a very complicated nonlocal hamiltonian operator, being essentially an infinite series: \([D^3 - \phi]^{-1} = D^{-3}[1 - D^{-3}\phi]^{-1}\). \(P_2\) is the direct supersymmetrization of the KdV second hamiltonian structure: \(-\partial^3 + 4u\partial + 2u_x\).

There is a remarkable feature of the super case that is not present for the usual KdV equation which is the presence of fermionic nonlocal conservation laws [17, 18]. The first few of them are

\[J_{1/2} = \int dxd\theta \ (D^{-1}\phi) = \int dx \ \xi \]
\[J_{3/2} = \int dxd\theta \ (D^{-1}\phi)^2 = \int dx \ u(\partial^{-1}\xi) \]
\[J_{5/2} = \int dxd\theta \ [(D^{-1}\phi)^3 - 6\partial^{-1}(\phi D\phi)] = \int dx \ [3\xi(\partial^{-1}u)^2 - 6\partial^{-1}(u^2 - \xi_x)] \]

(2.24)

They Poisson commute with the local bosonic conservation laws \(H_n\) but not among themselves:

\[\{J_{(4n+i)/2}, J_{(4m+i)/2}\} = H_{2(n+m)+i} \quad \text{with} \ i = 1, 3 \quad \{J_{(4n+1)/2}, J_{(4m+3)/2}\} = 0 \]  

(2.25)

The \(J_i\) are thus some sort of Poisson square roots of the usual conservation laws. The first fermionic nonlocal conservation law is actually local in terms of the component fields. It signals the presence of the supersymmetry invariance (and in particular \(\xi\) is not a conserved density for the Kuper-KdV equation). The infinite sequence can be generated from the first two conservation laws by the application of the recursion operator \(P_1^{-1}P_2\). But there is a more spectacular way of expressing them that makes manifest their supersymmetric origin: in superspace, \(\partial^2\) not only has a square root but it also has a fourth root: \((\partial^2)^{1/4} = D\). The

\[\text{The second hamiltonain structure has been found in the first two references and the first one in the last two. Here and below, the action of the derivatives is always delimited by parentheses, e.g., } D\phi = (D\phi) - \phi D.\]
fermionic nonlocal conservation laws are thus related to the super residues of the fourth root of odd powers of the Lax operators as

\[ J_{k/2} = \int dx d\theta \text{ sResL}^{k/4} \quad (k \text{ odd}) \]  

(2.26)

The sKdV equation represents the first example of an integrable system for which nonlocal conservation laws arise in such a clean form. In that respect, we introduce the first open problem (**OP**):

**OP-1:** Find an integrable deformation that reproduces the fermionic nonlocal conservation laws.

### 3. Supersymmetrization of the KdV equation: **N=2**

The next extension to be considered is the addition of an extra supersymmetry which amounts to add an extra anticommuting space dimension. We thus extend \( x \) to a triplet \( (x, \theta_1, \theta_2) \), with \( \theta_1^2 = \theta_2^2 = 0, \theta_1 \theta_2 = -\theta_2 \theta_1 \) and introduce two super derivatives:

\[
D_1 = \theta_1 \partial + \partial \theta_1, \quad D_2 = \theta_2 \partial + \partial \theta_2, \quad D_1^2 = D_2^2 = \partial, \quad D_1 D_2 = -D_2 D_1
\]  

(3.1)

The superfields are now functions of \( (x, \theta_1, \theta_2) \) (as well as \( t \)) and their Taylor expansion in terms of the anticommuting variables contain four terms. For instance the \( N = 2 \) KdV superfield will be written as:

\[
\Phi(x, \theta_1, \theta_2) = \theta_2 \theta_1 u(x) + \theta_1 \xi_1(x) + \theta_2 \xi_2(x) + v(x)
\]  

(3.2)

\( \xi_1 \) and \( \xi_2 \) are two fermionic fields and \( v \) is a new bosonic field of degree 1 (in a supersymmetric theory, the number of bosonic and fermionic fields must be the same). Notice that \( \Phi \) is a bosonic superfield.

One could then proceed to the direct supersymmetrization of the KdV equation and get a multiparameter \( N = 2 \) extension. However, a sounder approach, that reduces substantially the number of such free parameters, is to formulate the equation directly in terms of the \( N = 2 \) supersymmetric version of the second hamiltonian structure (which is expected to be the core structure underlying integrability of nontrivial KdV extensions):

\[
P_2 = D_1 D_2 \partial + 2 \Phi \partial - (D_1 \Phi) D_1 - (D_2 \Phi) D_2 + 2 \Phi_x
\]  

(3.3)
e.g. as
\[ \Phi_t = P_2 \frac{\delta}{\delta \Phi} \int dx d\theta_1 d\theta_2 \left[ \Phi(D_1 D_2 \Phi) + a \Phi^3 \right] \] (3.4)

This Hamiltonian is the direct generalisation of the KdV Hamiltonian \( \int dx u^2 \) and of the \( N = 1 \) version \( \int dx d\theta \phi(D\phi) \). However, the \( N = 2 \) generalisation is not unique and this introduces a free parameter \( a \). The resulting equation is
\[ \Phi_t = -\Phi_{xxx} + 3(\Phi D_1 D_2 \Phi)_x + \frac{(a - 1)}{2} (D_1 D_2 \Phi^2)_x + 3a \Phi^2 \Phi_x \] (3.5)

This is called the SKdV\(_a\) equation (the capital S is used for \( N = 2 \)). This system is integrable for exactly three values of \( a \) [3]: \( a = -2, 1, 4 \). For \( a = -2, 4 \), the Lax representation is standard: \( L_t = [-4L_+^{3/2}, L] \) with
\[ L_{a=4} = -(D_1 D_2 + \Phi)^2 \]
\[ L_{a=-2} = -\partial^2 + \sum_{i,j=1,2} \epsilon_{ij} D_i(D_1 D_2 + \Phi)D_j \] (3.6)

(with \( \epsilon_{12} = -\epsilon_{21} = 1 \)) while for \( a = 1 \) it is nonstandard [4]: \( L_t = [-4L_{\geq 1}^3, L] \) with
\[ L_{a=1} = \partial - \partial^{-1}[ (D_1 D_2 \Phi) - (D_2 \Phi)D_1 - (D_1 \phi)D_2 + \Phi D_1 D_2 ] \] (3.7)

In all cases, there is an infinite number of conservation laws given by
\[ H_{2k+1} = \int dd\theta_1 d\theta_2 \text{SRes}L^{(2k+1)/2} \] (3.8)

where the \( N = 2 \) version of the residue of a pseudodifferential operator is the coefficient of \( D_1 D_2 \partial^{-1} \).

Although there exists a Miura transformation, there are no known integrable (i.e., Gardner-type) deformations of it. This leads us to:

**OP-2:** Find an integrable deformation for SKdV\(_{-2,1,4}\) that reproduces their conservation laws.

For \( a = -2, 4 \), there are two independent towers of fermionic nonlocal conservation laws [18]: in each case, the first one is
\[ \int dx d\theta_1 d\theta_2 (D_i^{-1} \Phi) = \int dx \xi_i \quad (i = 1, 2) \] (3.9)
However, although the Lax operator has two distinct fourth roots, these have not yet been related to these fermionic conservation laws:

**OP-3:** For SKdV$_{-2,4}$, find the relation between the fermionic nonlocal conservation laws and the Lax operator.

The existence of these fermionic conservation laws is natural in that there are two supersymmetries. On the other hand, for the $a = 1$ case, the first fermionic laws (3.9) do not generate infinite towers: they are isolated.

**OP-4:** Why there are no infinite towers of fermionic conservation laws for SKdV$_1$?

The SKdV$_{-2,4}$ equations are both bihamiltonian: their first hamiltonian operator is [15]

$$P_1^{(a=4)} = \partial, \quad P_1^{(a=-2)} = (D_1 D_2 \partial^{-1} - D_1^{-1} \Phi D_1^{-1} - D_2^{-1} \Phi D_2^{-1}) D_1^{-1} \Phi D_1^{-1}$$ (3.10)

In that respect, the SKdV$_1$ stands as one of the rare example of classical integrable system which is not (known to be) bihamiltonian.

**OP-5:** Is SKdV$_1$ bihamiltonian?

Another very natural question is:

**OP-6:** Why is there exactly three integrable $N = 2$ super KdV extensions? Is there an underlying Lie algebraic interpretation for this threefold way (i.e, is this related to the existence of the three classical algebras)?

### 4. Concluding questions

There are further general questions that could be formulated in relation with super integrable systems. As stressed here, these equations are naturally formulated in superspace. Over the years, it became clear that a lot of structure is contained in the Painlevé test: for instance its truncation leads to Backlund transformations and the Lax operator [20]. Yet, no Painlevé analysis has been done directly in superspace.

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14 There are in fact also three distinct SKdV hierarchies but they generalize (in the Lax sens) the SKdV$_{2,4}$ equations; the SKdV$_1$ equation appears as a sort of isolated point. Interesting technical observations in relation with the bosonic truncation of the Lax operators are presented in [19].
OP-7: Is it possible to formulate the Painlevé analysis in superspace?

Moreover, little is known concerning the solutions of super integrable systems. The Darboux transformation in the $N = 1$ case has been worked out in [21] but nothing has been done at this point concerning the $N = 2$ cases.\footnote{Some solutions for the SKdV\textsubscript{a} equations have been reported in [22].}

Finally, unravelling the deep relations between geometry and soliton theory has been an important theme of this workshop; little is known on the super version of this connection. In particular, the super KdV equations have super Sine-Gordon relatives and these should lead to very interesting geometrical structures.

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