Dedicated to J. Bernstein on the occasion of his 70th birthday

BERNSTEIN COMPONENTS VIA BERNSTEIN CENTER

ALEXANDER BRAVERMAN AND DAVID KAZHDAN (WITH AN APPENDIX BY R. BEZRUKAVNIKOV)

Abstract. Let $G$ be a reductive $p$-adic group. Let $\Phi$ be an invariant distribution on $G$ lying in the Bernstein center $\mathcal{Z}(G)$. We prove that $\Phi$ is supported on compact elements in $G$ if and only if it defines a constant function on every component of the set $\mathrm{Irr}(G)$; in particular, we show that the space of all elements of $\mathcal{Z}(G)$ supported on compact elements is a subalgebra of $\mathcal{Z}(G)$. Our proof is a slight modification of the argument from Section 2 of [6], where our result is proven in one direction.

1. Introduction

1.1. Components of $\mathrm{Irr}(G)$. In this paper $G$ denotes the set of point of a connected reductive algebraic group over a local non-archimedian field $K$. We shall denote by $\mathcal{M}(G)$ the category of smooth complex representations of $G$. This category is equivalent to the category of unital modules over the Hecke algebra $\mathcal{H}(G)$. We let $\mathrm{Irr}(G)$ denote the set of isomorphism classes irreducible objects of $\mathcal{M}(G)$. Bernstein and Zelevinsky defined a decomposition of the set $\mathrm{Irr}(G)$ of irreducible representations of $G$ into a union of certain components $\Omega$; this decomposition in fact defines a decomposition of $\mathcal{M}(G)$ into a product of the corresponding categories. The set $\mathcal{C}(G)$ of components of $\mathrm{Irr}(G)$ is in one-to-one correspondence with pairs $(M,\sigma)$ where $M$ is a Levi subgroup of $G$ and $\sigma$ is a cuspidal representation of $M$ (the data of $(M,\sigma)$ is uniquely determined by $\Omega$ up to natural equivalence relation generated by conjugation and multiplying $\sigma$ by an unramified character of $M$). An element $\pi \in \mathrm{Irr}(G)$ lies in $\Omega(M,\sigma)$ if and only if there exists a parabolic subgroup $P$ containing $M$ as a Levi subgroup and an unramified character $\chi$ of $M$ such that $\sigma$ is a subquotient of the induced representation $\text{Ind}_P^G(\sigma \otimes \chi)$ (here we use the natural map $P \to M$ in order to view $\sigma \otimes \chi$ as a representation of $P$). Every $\Omega(M,\sigma)$ is equipped with a map to an irreducible affine algebraic variety $\overline{\Omega}(M,\sigma)$. The variety $\overline{\Omega}(M,\sigma)$ is in fact a quotient of the torus of unramified characters of $M$ by a finite group. The above map has finite fibers and is generically one-to-one. We shall say that a function $f : \Omega(M,\sigma) \to \mathbb{C}$ is regular if it comes from a regular function on $\overline{\Omega}(M,\sigma)$. We shall say that a function $f : \mathrm{Irr}(G) \to \mathbb{C}$ is regular if it is regular when restricted to every component.

1.2. Bernstein center. Let $\mathcal{Z}(G)$ be the center of the category $\mathcal{M}(G)$. It is easy to see that it consists of all invariant distributions $\Phi$ on $G$ such that for any $h \in \mathcal{H}(G)$ we have $\Phi \ast h \in \mathcal{H}(G)$. It is enough to test the above condition for all $h = e_K$ where $K$ is an open compact subgroup of $K$ and $e_K$ is the Haar measure on it.

By Schur-Quillen lemma any $\Phi \in \mathcal{Z}(G)$ defines a function on the set $\mathrm{Irr}(G)$. Bernstein proved (cf. [1]) that in this way we get an isomorphism between $\mathcal{Z}(G)$ and the algebra of regular functions on $\mathrm{Irr}(G)$. Thus $\mathcal{Z}(G)$ has both "geometric" (in terms of distributions on $G$) and "spectral" (in terms of functions on $\mathrm{Irr}(G)$) description. The relationship between
these two descriptions tends to be quite non-trivial. This note is devoted to one particular aspect of this relationship. Namely, we are going to prove the following

**Theorem 1.3.** Let $\mathcal{Z}_{comp}(G)$ denote the subspace of $\mathcal{Z}(G)$ consisting of distributions supported on compact elements. Similarly, let $\mathcal{Z}_{lc}(G)$ denote the subalgebra of $\mathcal{Z}(G)$ consisting of those elements $\Phi$ for which $f(\Phi)$ is a locally constant function (i.e. a constant function when restricted to every Bernstein component of $\text{Irr}(G)$). Then $\mathcal{Z}_{comp}(G) = \mathcal{Z}_{lc}(G)$.

Theorem 1.3 has the following surprising corollary (in fact, technically we are first going to prove the corollary and then deduce Theorem 1.3 from it, but historically our starting conjectural point was the assertion of Theorem 1.3):

**Corollary 1.4.** $\mathcal{Z}_{comp}(G)$ is a subalgebra of $\mathcal{Z}(G)$.

Corollary 1.4 is surprising since the set of compact elements of $G$ is not closed under multiplication. We believe that Corollary 1.4 is actually a part of a more general statement. While we are not sure what this statement really is, at least we believe in the following:

**Conjecture 1.5.** Let $\mathcal{Z}_{tunip}(G)$ denote the subspace of $\mathcal{Z}(G)$ consisting of distributions supported on topologically unipotent elements. Then $\mathcal{Z}_{tunip}(G)$ is a subalgebra of $\mathcal{Z}(G)$.

1.6. **Relation to the work of J.-F. Dat.** Theorem 1.3 is in fact not completely new – the inclusion $\mathcal{Z}_{lc}(G) \subset \mathcal{Z}_{comp}(G)$ was essentially proved by J.-F. Dat (cf. Section 2 of [6]). Namely, it is shown in loc. cit. that every idempotent in $\mathcal{Z}(G)$ is supported on compact elements. Hence if for every $\Omega \in \mathcal{C}(G)$ we denote by $E_\Omega$ the element of $\mathcal{Z}(G)$ for which the function $f(E_\Omega)$ is equal to 1 on $\Omega$ and is equal to 0 on any other component, then $\mathcal{E}_\Omega \in \mathcal{Z}_{comp}(G)$. On the other hand, any $\Phi \in \mathcal{Z}(G)$ such that $f(\Phi)$ is constant on every $\Omega$ is locally on $G$ a linear combination of the distributions $E_\Omega$, hence $\Phi$ is supported on compact elements. The main observation of this note is that a mild adaptation of Dat’s argument also proves the converse statement.

1.7. **A variant.** In fact the inclusion $\mathcal{Z}_{lc}(G) \subset \mathcal{Z}_{comp}(G)$ has the following stronger version. Given an element $g \in G$ we can define in a standard way a parabolic subgroup $P_g$ of $G$ and a strictly dominant element $\lambda_g \in Z(M_g)/Z(M_g)^0$ (the latter group is always a lattice and we shall denote the multiplication there by $+$; also, ”strictly dominant” means that the adjoint action of $\lambda_g$ contract the unipotent radical of $P_g$ to the unit element). Here we denote by $M_g$ the Levi group of $P_g$; also $Z(M)$ stands for the center of $M$ and $Z(M)^0$ is its maximal compact subgroup. Namely, $P_g$ consists of all $x \in G$ such that $\lim_{n \to \infty} g^n x g^{-n}$ exists.

Also the image of $g$ under the natural map $P_g \to M_g$ must be compact modulo center and hence $g$ defines an element in $Z(M)/Z(M) \cap M^0 = Z(M)/Z(M)^0$ which we call $\lambda_g$. Note that $P_g = G$ if and only if $g$ is compact modulo center. Moreover, we have $P_g = G, \lambda_g = 0$ if and only if $g$ is compact.

Let now $\mathcal{P}(G)$ denote the set of conjugacy classes of pairs $(P, \lambda)$ as above. Then the above construction produces a decomposition

$$ G = \bigsqcup_{(P, \lambda) \in \mathcal{P}(G)} G_{P, \lambda} \quad (1.1) $$

and each $G_{P, \lambda}$ is an open subset of $G$ invariant under conjugation.
Let now \( D(G) \) denote the space of distributions on \( G \); let also \( D_{\text{inv}}(G) \subset D(G) \) be the space of invariant distributions. Then (1.1) produces a decomposition

\[
D_{\text{inv}}(G) = \prod_{(P,\lambda) \in P(G)} D_{P,\lambda}^{\text{inv}}.
\]

(1.2)

Here \( D_{P,\lambda}^{\text{inv}} \) consists of all invariant distributions supported on \( G_{P,\lambda} \). We can now formulate

**Theorem 1.8.** Let \( \Phi \in \mathcal{Z}_{\text{lc}}(G) \). Then convolution with \( \Phi \) preserves the decomposition (1.2) (i.e. preserves each \( D_{P,\lambda}^{\text{inv}} \)).

This result is due to R. Bezrukavnikov and we reproduce its proof in the Appendix. Theorem [1.8] implies the inclusion \( \mathcal{Z}_{\text{lc}}(G) \subset \mathcal{Z}_{\text{comp}}(G) \). Namely let \( \delta \) denote the delta-distribution at the unit element of \( G \). Obviously \( \delta \in D_{G,0}^{\text{inv}} \), hence \( \Phi = \Phi \ast \delta \in D_{G,0}^{\text{inv}} \), i.e. \( \Phi \) is supported on compact elements. Our proof of Theorem [1.8] is somewhat simpler than the proof of the inclusion \( \mathcal{Z}_{\text{lc}}(G) \subset \mathcal{Z}_{\text{comp}}(G) \) from [6]. However, we still need the arguments of [6] in order to prove the opposite inclusion.

1.9. **An example.** For a rational number \( r \in \mathbb{Q}_{\geq 0} \) Moy and Prasad (cf. [7]) define a subset \( \text{Irr}_{\leq r}(G) \) of \( \text{Irr}(G) \) called "representations of depth \( \leq r \)". The set \( \text{Irr}_{\leq r}(G) \) is a union of components of \( \text{Irr}(G) \). Let \( \Phi_r \in \mathcal{Z}_G \) be the projector to \( \text{Irr}_{\leq r}(G) \); in other words \( \Phi_r \) is the element of \( \mathcal{Z}(G) \) such that \( f(\Phi_r)(\pi) = 1 \) if \( \pi \in \text{Irr}_{\leq r}(G) \) and \( f(\Phi_r)(\pi) = 0 \) otherwise. According to Theorem [1.3] \( \Phi_r \) should be concentrated on compact elements. In [3] the authors give an explicit formula for \( \Phi_r \) which indeed shows this explicitly. In fact, the main result of [3] implies a much stronger restriction to on the support of \( \Phi_r \). It would be interesting to include this restriction into a general theorem in the style of Theorem [1.3].

1.10. **Geometric and spectral support.** We conclude the introduction with yet another conjecture which contains Theorem [1.3] as a special case. To simplify the discussion we shall assume that \( G \) is a split.

Let us assume that \( G = G(K) \) where \( G \) is the corresponding split algebraic group defined over \( \mathbb{Z} \). Let \( \Lambda \) denote the coweight lattice of \( G \); we shall denote by \( \Lambda^+ \) the set of dominant coweights. Also, for \( \lambda, \mu \in \Lambda \) we shall write \( \lambda \geq \mu \) if \( \lambda - \mu \) is a sum of positive coroots of \( G \). Let \( K = G(O) \). Then by Cartan decomposition the double quotient \( K\backslash G/K \) is in natural bijection with the set \( \Lambda^+ \) of dominant coweights of \( G \). For each \( \lambda \in \Lambda^+ \) we shall denote by \( G^\lambda \) the corresponding double coset. We set

\[
G^{\leq \lambda} = \bigcup_{\mu \in \Lambda^+, \mu \leq \lambda} G^\mu.
\]

We now define

\[
\mathcal{Z}_{\text{geom}}^\lambda(G) = \{ \Phi \in \mathcal{Z}(G) \mid \text{supp}(\Phi) \subset \text{Ad}G \cdot (G^{\leq \lambda}) \}.
\]

(1.3)

Note that \( \mathcal{Z}_{\text{geom}}^{=0}(G) = \mathcal{Z}_{\text{comp}}(G) \).

On the hand, let \( P, M \) be a parabolic subgroup of \( G \) and its Levi subgroup (both defined over \( K \)). Let \( \Lambda_M \) be the cocharacter lattice of \( M/\left[M,M\right] \). Then \( \Lambda_M \) is a sublattice of \( \Lambda \) and \( \mathbb{C}[\Lambda_M] \) is the algebra of regular functions on the set of unramified characters of \( M \) (for an element \( \lambda \in \Lambda_M \) we shall denote by \( e^\lambda \) the corresponding element of \( \mathbb{C}[\Lambda_M] \)). Fix a cuspidal representation \( \sigma \) of \( M \) with unitary central character. Then every \( \Phi \in \mathcal{Z}(G) \) defines an
element $f_\sigma(\Phi) \in \mathbb{C}[\Lambda_M]$. Namely, for an unramified $\psi : M \to \mathbb{C}^*$ we define $f_\sigma(\Phi)(\psi)$ to be the scalar by which $\Phi$ acts in $i_{GM}(\sigma \otimes \psi)$ where $i_{GM}$ stands for unitary induction from $P$ to $G$. We now define

$$Z^{\leq \lambda}_{\text{spectral}}(G) = \{ \Phi \in \mathcal{Z}(G) | f_\sigma(\Phi) = \sum_{\mu \leq \lambda} a_\mu e^\mu. \} \quad (1.4)$$

It is easy to see that $\Phi \in Z^{\leq 0}_{\text{spectral}}(G)$ if and only if $f(\Phi)$ is constant on every component of $\text{Irr}(G)$. We can now formulate

**Proposition 1.11.** For any $\lambda \in \Lambda^+$ we have $Z^{\leq \lambda}_{\text{geom}}(G) = Z^{\leq \lambda}_{\text{spectral}}(G)$.

Proposition 1.11 reduces to Theorem 1.3 when $\lambda = 0$. The proof of Proposition 1.11 easily follows from Theorem 4.4.

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## 2. Proof of Corollary 1.4

2.1. $Z_{\text{comp}}$ and $\overline{\mathcal{H}}$. Let $\mathcal{H}(G)$ denote the Hecke algebra of $G$. In what follows we shall choose a Haar measure on $G$ and we are going to identify $\mathcal{H}(G)$ with the space of locally constant compactly supported functions on $G$. Let $\overline{\mathcal{H}}(G) = \mathcal{H}(G)/[\mathcal{H}(G), \mathcal{H}(G)]$. Obviously $\mathcal{Z}(G)$ acts on $\overline{\mathcal{H}}$. For any $\Phi \in \mathcal{Z}(G)$ we shall denote by $\overline{\Phi}$ the corresponding endomorphism of $\overline{\mathcal{H}}$.

Following [6] let us denote by $G^0$ the subgroup of $G$ generated by all the open compact subgroups of $G$. We also denote by $G_c$ the set of compact elements modulo center. Then $G^0_c = G^0 \cap G_c$ is the set of compact elements of $G$.

For an open subset $X$ of $G$ we denote by $1_X$ the characteristic function of $X$. Then multiplication (not convolution!) by $1_X$ is an endomorphism of $\mathcal{H}$ which (abusing the notation) we shall also denote by the same symbol. Moreover, if $X$ is invariant under conjugation, then multiplication by $1_X$ descends to endomorphism of $\overline{\mathcal{H}}$, which we shall denote by $\overline{1}_X$. Then $\overline{1}_{G^0}, \overline{1}_{G_c}$ and $\overline{1}_{G^0_c}$ are well-defined and we have $\overline{1}_{G^0} = 1_{G_c} \circ 1_{G^0}$. Also, all of these endomorphisms commute with each other.

Below is the main result of this Section.

**Theorem 2.2.** We have $\Phi \in Z_{\text{comp}}(G)$ if and only if $\overline{\Phi}$ commutes with $\overline{1}_{G^0}$.

It is clear that Theorem 2.2 implies Corollary 1.4.

2.3. **Proof of Theorem 2.2: the "if" direction.** This is the easy part of Theorem 2.2. Note that two elements $h_1, h_2 \in \mathcal{H}$ have the same image in $\overline{\mathcal{H}}$ iff for any invariant distribution $\mathcal{E}$ on $G$ we have $\mathcal{E}(h_1) = \mathcal{E}(h_2)$. In particular, if $\overline{h_1} = \overline{h_2}$, then $1_{h_1}(e) = 1_{h_2}(e)$ where $e$ is the unit element of $G$. 


Let us now assume that we are given \( \Phi \in \mathcal{Z}(G) \) such that \( \Phi \) commutes with \( \tilde{1}_{G^0} \). Then for any \( h \in H \) we have \( \Phi \star (h|_{G^0}) = (\Phi \star h)|_{G^0} \). Let \( \tilde{h}(g) = h(g^{-1}) \). Then we have
\[
\Phi(\tilde{h}) = (\Phi \star h)(e) = (\Phi \star h)|_{G^0}(e) = (\Phi \star (h|_{G^0}))(e).
\]
Hence \( \Phi(\tilde{h}) = 0 \) if \( \text{supp}(\tilde{h}) = \text{supp}(h) \subset G \setminus G_c^0 \), which means that \( \text{supp} \Phi \subset G^0 \).

2.4. Let us now start proving the opposite direction. Namely, let \( \Phi \in \mathcal{Z}(G) \). We want to show that \( \Phi \) commutes with \( \tilde{1}_{G_c^0} \). For this it is enough to prove that \( \Phi \) commutes with \( \tilde{1}_{G^0} \) and \( \tilde{1}_{G_c} \). Let us first prove that \( \Phi \) commutes with \( \tilde{1}_{G^0} \). For this it is enough to prove that \( \Phi \) commutes with \( \tilde{1}_{G^0} \). In other words, we need to prove that for any \( h \in H \) we have \( \Phi \star (h|_{G^0}) = (\Phi \star h)|_{G^0} \). But this is obvious since \( G^0 \) is a subgroup of \( G \) and \( \tilde{1}_{G^0} \subset \text{supp} \Phi \).

2.5. **Induction and restriction.** Let us choose a split Cartan subgroup \( T \) of \( G \) and a Borel subgroup \( B \) of \( G \) with unipotent radical \( U \). We denote by \( \overline{B} \) the opposite Borel subgroup of \( G \). Then we have a notion of standard Levi subgroup \( M \) of \( G \). We shall use the notation \( M < G \) to indicate that \( M \) is a standard Levi subgroup of \( G \). To any such \( M \) there corresponds a pair of parabolic subgroups \( P, \overline{P} \), where \( P \cap \overline{P} = M \) and \( B \subset P, \overline{B} \subset \overline{P} \). Also for any such \( M \) we have maps (cf. [6] and references therein): \( i_{GM}^Z, i_{GM}^{\overline{Z}} : \mathcal{Z}(M) \rightarrow \mathcal{Z}(G), \ r_{GM}^Z, r_{GM}^{\overline{Z}} : \mathcal{Z}(G) \rightarrow \mathcal{Z}(M), \ r_{GM}^\overline{Z}, r_{GM}^{\overline{Z}} : \overline{\mathcal{H}}(M) \rightarrow \overline{\mathcal{H}}(G), \ r_{GM}^\overline{Z}, r_{GM}^{\overline{Z}} : \overline{\mathcal{H}}(G) \rightarrow \overline{\mathcal{H}}(M) \). These maps satisfy the following properties:

- For any \( \Phi \in \mathcal{Z}(G) \), \( f \in \overline{\mathcal{H}}(M) \) we have \( \Phi \star i_{GM}^\overline{Z}(f) = i_{GM}^\overline{Z}(\Phi \star f) \).
- For any \( \Phi \in \mathcal{Z}(G) \), \( h \in \overline{\mathcal{H}}(G) \) we have \( r_{MG}^\overline{Z}(\Phi \star h) = r_{MG}^\overline{Z}(\Phi \star r_{MG}^\overline{Z}(h)) \).

2.6. **Clozel’s formula.** The main ingredient of the argument of Section 2 of [4] (and also of our proof of Theorem 2.2) is the following formula due to Clozel.

**Proposition 2.7.** For any standard Levi \( M \) there exists a function \( \chi_M : M/M^0 \rightarrow \mathbb{C} \) such that for any \( h \in \overline{\mathcal{H}}(G) \) we have
\[
h = \sum_{M < G} i_{GM}^\overline{Z}(\chi_M \cdot (\tilde{1}_{M_c}(r_{GM}^\overline{Z}(h)))). \tag{2.1}
\]
Moreover, \( \chi_G = 1 \).
2.8. **End of the proof.** We can now finish the proof of Theorem 2.2. By induction we can assume that for any Levi subgroup $M$ of $G$ which is different from $G$ and any $\Phi \in Z_{\text{comp}}(M)$ we have $\overline{1}_{M_c}(\Psi \star f) = \Psi \star (\overline{1}_{M_c}(f))$ for any $f \in \mathcal{H}(M)$. Let now $\Phi \in Z_{\text{comp}}(G), h \in \mathcal{H}(G)$. Then we have

$$\overline{1}_{G_c}(\Phi \star h) = \Phi \star h - \sum_{M < G, M \neq G} \overline{r}_{GM}^Z(\chi_M \cdot (\overline{1}_{M_c}(\overline{1}_{G_c}(\Phi \star h))))) =$$

$$= \Phi \star h - \sum_{M < G, M \neq G} \overline{r}_{GM}^Z(\chi_M \cdot (\overline{1}_{M_c}(\overline{1}_{G_c}(\Phi \star h))))$$

$$= \Phi \star h - \sum_{M < G, M \neq G} \overline{r}_{GM}^Z(\chi_M \cdot (\overline{1}_{M_c}(\overline{1}_{G_c}(\Phi \star h))))$$

Here $\dagger$ follows from the fact that $\overline{r}_{GM}^Z(\Phi) \in Z_{\text{comp}}(M)$ and from Section 2.4. The equality $\ddagger$ follows from the induction hypothesis.

3. **Proof of Theorem 1.3**

3.1. To finish the proof of Theorem 1.3 we need to show that for any $\Phi \in Z_{\text{comp}}(G)$ the function $f(\Phi)$ is constant on every $\Omega \in \mathcal{C}(G)$. Let us recall that we denote by $E_{\Omega}$ the element of $Z(G)$ for which the function $f(E_{\Omega})$ is equal to 1 on $\Omega$ and is equal to 0 on any other component. Then by [6] we have $E_{\Omega} \in Z_{\text{comp}}(G)$ and by Corollary 1.4 we also have $\Phi \star E_{\Omega} \in Z_{\text{comp}}(G)$ and it is enough to prove that $f(\Phi \star E_{\Omega})$ is constant on $\Omega$. Let $Z_{\Omega}(G)$ denote the subalgebra of $Z(G)$ consisting of elements $\Phi$ such that $f(\Phi)$ is equal to 0 on any $\Omega' \neq \Omega$ and let $Z_{\Omega,\text{comp}}(G) = Z_{\text{comp}}(G) \cap Z_{\Omega}(G)$. We already know that $E_{\Omega} \in Z_{\Omega,\text{comp}}(G)$. Clearly, our assertion follows from the following

**Proposition 3.2.** $Z_{\Omega,\text{comp}}(G) = C \cdot E_{\Omega}$.

We claim that Proposition 3.2 follows from the following:

**Lemma 3.3.** $\dim Z_{\Omega,\text{comp}}(G) < \infty$.

Indeed, $Z_{\Omega,\text{comp}}(G)$ is a subalgebra of $Z_{\Omega}(G)$ which is isomorphic to the algebra of functions on an irreducible algebraic variety $\overline{\Omega}$. Hence the only finite-dimensional subalgebra of it consists of constants. So, it remains to prove Lemma 3.3.

**Proof.** Let $\mathcal{D}(G^0)$ denote the space of distributions on $G^0$. Consider the subspace $D_{\Omega}$ of $\mathcal{D}(G^0)$ generated by distributions of the form $\text{ch}_\pi |_{G^0}$ where $\pi$ is an irreducible representation in $\Omega$ and $\text{ch}_\pi$ is its character. Then it follows immediately from the Corollary in Section 3.1 in [2] that $\mathcal{D}_{\Omega}$ is finite-dimensional.
On the other hand, by Plancherel formula for every $\Omega \in C(G)$ there exists a measure $d\pi_\Omega$ on $\Omega$ such that

$$\delta_G = \sum_{\Omega \in C(G)} \int_{\pi \in \Omega} \text{ch}_\pi \, d\pi_\Omega.$$  \hspace{1cm} (3.1)

Here $\delta_G$ denotes the $\delta$-distribution at the unit element of $G$.

Convolving this with a central element $\Phi \in Z(G)$ we get

$$\Phi = \sum_{\Omega \in C(G)} \int_{\pi \in \Omega} f(\Phi)(\pi) \text{ch}_\pi \, d\pi_\Omega.$$  \hspace{1cm} (3.2)

If $\Phi \in Z_{\Omega,\text{comp}}$ we get

$$\Phi = \int_{\pi \in \Omega} f(\Phi)(\pi) \text{ch}_\pi \, |G|^0 \, d\pi_\Omega.$$  \hspace{1cm} (3.3)

Since the LHS of (3.3) is concentrated on $G^0$, the same is true for RHS. Thus we get

$$\Phi = \int_{\pi \in \Omega} f(\Phi)(\pi) \text{ch}_\pi \, |G|_0 \, d\pi_\Omega.$$  \hspace{1cm} (3.4)

Hence $\Phi \in D_{\Omega,\text{comp}}$, i.e. $Z_{\Omega,\text{comp}} \subset D_{\Omega,\text{comp}}$, which implies that dim $Z_{\Omega,\text{comp}}$ is finite-dimensional.

4. Appendix: Proof of Theorem 1.8 (by R. Bezrukavnikov)

4.1. Decomposition of $\overline{\mathcal{H}}(G)$. We have an obvious perfect pairing between $D^{inv}(G)$ and $\overline{\mathcal{H}}(G)$. We claim that there is a decomposition

$$\overline{\mathcal{H}}(G) = \bigoplus_{(P,\lambda) \in \mathcal{P}(G)} \overline{\mathcal{H}}(G)_{P,\lambda},$$  \hspace{1cm} (4.1)

which is compatible with (1.2) by means of the above pairing. Namely, we let $\overline{\mathcal{H}}(G)_{P,\lambda}$ to be the image of $\overline{\mathcal{H}}(G)_{P,\lambda}$ where the latter consists of functions supported on $G_{P,\lambda}$. The fact that (4.1) holds is clear.

4.2. Spectral description of $\overline{\mathcal{H}}(G)$. The space $\overline{\mathcal{H}}(G)$ admits the following well-known description. Let $\pi \in \mathcal{M}(G)$ be a finitely generated representation and let $E$ be an endomorphism of $\pi$. It is well-known (cf. e.g. [8]) that we can associate to the pair $(\pi, E)$ and element $[\pi, E]$ of $\mathcal{H}(G)$. Moreover, $\mathcal{H}(G)$ is isomorphic to the $\mathbb{C}$-span of symbols $[\pi, E]$ subject to the relations:

a) Let $\pi_1, \pi_2 \in \mathcal{M}(G)$ and let $u \in \text{Hom}(\pi_1, \pi_2), v \in \text{Hom}(\pi_2, \pi_1)$. Then $[\pi_1, v u] = [\pi_2, u v]$.

b) $[\pi_1, E_1] + [\pi_3, E_3] = [\pi_2, E_2]$ for a short exact sequence $0 \rightarrow \pi_1 \rightarrow \pi_2 \rightarrow \pi_3 \rightarrow 0$ which is compatible with the endomorphisms $E_i \in \text{End}(\pi_i)$.

c) $[\pi, c_1 E_1 + c_2 E_2] = c_1 [\pi, E_1] + c_2 [\pi, E_2]$, where $c_i \in \mathbb{C}$ and $E_i \in \text{End}(\pi)$.

The action of $Z(G)$ on $\overline{\mathcal{H}}(G)$ can also be described in these terms. Namely, let $\Phi \in Z(G)$. Then $\Phi \cdot [\pi, E] = [\pi, E \circ \pi(\Phi)]$.

In addition, let $\rho$ be an admissible representation of $G$. Then we have

$$\langle [\pi, E], \text{ch}_\rho \rangle = \sum_i (-1)^i \text{Tr}(E, \text{Ext}^i(\pi, \rho)).$$  \hspace{1cm} (4.2)
In view of the Trace Paley-Wiener theorem (cf. [2]), (4.2) defines $[\pi, E]$ uniquely.

4.3. **Spectral description of** $\mathcal{H}_{P,\lambda}$. Let now $P$, $\overline{P}$ be a pair of opposite parabolic subgroups with $M = P \cap \overline{P}$. Let $\lambda \in Z(M)/Z(M^0)$ such that $(\overline{P}, \lambda) \in \mathcal{P}(G)$. Let also $\sigma$ be a finitely generated representation of $M$. Set

$$\pi = i_{GP}(\sigma). \quad (4.3)$$

Let us now choose a uniformizer $t$ of our local field. Then any $\lambda \in Z(M)/Z(M^0)$ lifts naturally to an element $t^\lambda \in Z(M)$. Hence it defines an endomorphism of $\sigma$ and thus also of $\pi$. We shall denote this endomorphism by $E_\lambda$.

**Theorem 4.4.** The subspace $\mathcal{H}_{P,\lambda}$ is spanned by elements $[\pi, E_\lambda]$ as above (here again $\overline{P}$ denotes a parabolic subgroup which is opposite to $P$).

**Remark.** The element $[\pi, E_\lambda]$ actually depends on the choice of $t$; however, it is easy to see that the span of all the $[\pi, E_\lambda]$ does not.

**Proof.** For $(P, \lambda) = (G, 0)$ this is the "abstract Selberg principle" (cf. [11]). The case $P = G$ and arbitrary $\lambda$ is completely analogous.

Let us now take arbitrary $P$ and $\lambda$. Let $\sigma$ be a finitely generated representation of the Levi group $M$ as above and $\lambda$ - a strictly dominant cocharacter of $\pi$. Then we have a natural identification $t^\lambda M_c^0/Ad(M) = G_{P,\lambda}/Ad(G)$. Hence we get a natural isomorphism between $\mathcal{H}(M)_{M,0}$ and $\mathcal{H}(G)_{P,\lambda}$. Indeed, if an element of $\mathcal{H}(M)$ is represented by some $h \in H(M)$ supported on $M_c^0 = M_{M,0}$, then let us denote by $h_\lambda$ the corresponding element of $\mathcal{H}(M)$ supported on $M_{M,\lambda} = t^\lambda \cdot M_c^0$. For an open compact subgroup $K$ of $G$ let us denote by $h_{\lambda,K}$ the result of averaging of $h_\lambda$ with respect to the adjoint action of $K$. Its image in $\mathcal{H}(G)$ is independent of $K$ and the assignment $h \bmod[\mathcal{H}(M), \mathcal{H}(M)] \mapsto h_{\lambda,K} \bmod[\mathcal{H}(G), \mathcal{H}(G)]$ is the desired isomorphism. Let us denote it by $\eta_{P,\lambda}$.

Now in order to finish the proof it is enough to show that for $\pi$ as in (4.3) we have

$$[\pi, E_\lambda] = \eta_{P,\lambda}([\sigma, \text{Id}]). \quad (4.4)$$

Let $h = [\sigma, \text{Id}]$. Then it is easy to see that

$$[\sigma, t^\lambda] = h_\lambda. \quad (4.5)$$

Now to prove (4.4) it is enough (by Trace Paley-Wiener theorem) to check that both the LHS and the RHS of (4.4) have the same inner product with $ch_\rho$ where $\rho$ stands for a generic irreducible representation of $G$. But we have

$$\text{Ext}^i_G(i_{GP}(\sigma), \rho) = \text{Ext}^i_M(\sigma, r_{GP}(\rho)).$$

Hence $\langle [\pi, E_\lambda], ch_\rho \rangle = \langle [\sigma, \lambda], r_{GP}(\rho) \rangle$ and (4.4) follows from (4.5) and from the Casselman formula for the character of $r_{GP}(\rho)$ (cf. [5]) which says for any $g \in G$ such that $P_g = \overline{P}$ we have $ch_\rho(g) = ch_{r_{GP}(\rho)}(g)$.

$\square$

**Corollary 4.5.** Theorem 1.8 holds.
Proof. It is enough to show that the action of any \( \Phi \in Z_{lc}(G) \) preserves each \( \mathcal{H}_{\mathbb{P},\lambda} \). Let us consider an element \( [\pi,E] \) as above; without loss of generality we may assume that all irreducible subquotients of \( \sigma \) lie in one component of \( \mathcal{C}(M) \). But then all irreducible subquotients of \( \pi \) as in lie in one component \( \Omega \in \mathcal{C}(G) \) and it follows that \( \Phi \star [\pi,E_{\lambda}] = [\pi, f(\Phi)|_{\Omega} \cdot E_{\lambda}] = f(\Phi)|_{\Omega} \cdot [\pi,E_{\lambda}] \) (note that \( f(\Phi)|_{\Omega} \in \mathbb{C} \) as \( \Phi \in Z_{lc}(G) \)). Hence the span of all the \( [\pi,E_{\lambda}] \) is preserved by the convolution with \( \Phi \). \( \square \)

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R.B.: Department of Mathematics, Massachusetts Institute of Technology
A.B.: Department of Mathematics, University of Toronto, Perimeter Institute for Theoretical Physics and Department of Mathematics, Brown University
D.K.: Department of Mathematics, Hebrew University of Jerusalem