A REPRODUCING KERNEL HILBERT SPACE FRAMEWORK FOR INVERSE SCATTERING PROBLEMS WITHIN THE BORN APPROXIMATION

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Abstract. In this work we develop a new reproducing kernel Hilbert space (RKHS) framework for inverse scattering problems using the Born approximation. We assume we have backscattered data of a field that is dependent on an unknown scattering potential. Our goal is to reconstruct or image this scattering potential. Assuming the scattering potential lies in a RKHS, we find that the imaging equation can be rewritten as the inner product of the desired unknown function with the adjoint of the forward operator applied to the kernel of the imaging operator. We therefore may choose the kernel of the imaging operator such that the adjoint applied to this kernel is precisely the reproducing kernel of the Hilbert space the reflectivity function lies in. In this way we are able to obtain an alternative definition of an ideal image. We will demonstrate this theory using synthetic aperture radar imaging as an example, though there are other applicable imaging modalities i.e. inverse diffraction and diffraction tomography [1, 6]. We choose SAR as it was the motivating application for this work. We will compare the RKHS ideal imaging technique to the standard microlocal analytic ideal image from backprojection theory. Note this method requires a variation of the standard SAR data model with the assumption of a full two dimensional data collection surface as opposed to a one dimensional flight path, however we are able to perform imaging with a single frequency and avoid the approximations made in the backprojection imaging operator derivation.

1. Introduction. In this paper we develop a new reproducing kernel Hilbert space framework for inverse scattering problems using the Born approximation. We assume we have backscattered data of some field that is dependent on an unknown scattering potential. Our goal is to reconstruct or image this scattering potential. We will use synthetic-aperture radar imaging as an example to demonstrate the theory developed in this work in terms of a real application.

There are two main mathematical tasks to consider for imaging, first developing a forward model and then creating a method to invert that model to obtain information about the unknown function of interest. We will assume our collected data is in the form of a scattered field which is related to an unknown function to be reconstructed. We will therefore assume our field of interest satisfies the inhomogeneous scalar wave equation in the time domain and the inhomogeneous Helmholtz equation in the frequency domain. Thus we may use a Green’s function

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solution to obtain an integral equation expression for the scattered field, known
as the Lippmann-Schwinger integral equation. It can be shown that the measured
data is equivalent to an integral operator acting on the desired unknown function.
We write this as

\[ D = \mathcal{F}(V) \]  

where \( D \) is the measured data, \( \mathcal{F} \) is known as the forward operator and incorporates
the Green’s function, and \( V \) is the unknown function to be reconstructed. This
model applies directly to some applications like diffraction tomography [6] and we
will also show that the standard form of the data/forward model for synthetic
aperture radar may be rewritten in this form. This new SAR data model turns out
to be a more general Fourier integral operator acting on the unknown reflectivity
function.

Now the second mathematical task, the inverse problem, is to reconstruct or
estimate the unknown function from the scattered field. In this work our goal is
to reformulate the inverse problem in a reproducing kernel Hilbert space (RKHS)
framework which allows us, under certain assumptions, to produce an ideal image in
a reproducing kernel Hilbert space sense. We will focus on comparing this method
with the standard image reconstruction technique for SAR imaging, filtered back-
projection. Filtered backprojection is one method of approximating an ‘ideal’ image
in the microlocal analytic sense by applying an approximate inverse operator to the
data above.

The author has previously explored the usefulness of RKHS in estimation and
detection of target reflectivity and/or location for SAR using the generalized likeli-
hood ratio test [18, 19]. It was found that collected SAR data is equal to an inner
product of the unknown reflectivity function and the covariance function of the
noise random process. We may therefore use an inverse of the covariance function
in order to filter out noise. This work differs from our work here where we use
the RKHS that represents the space of possible reflectivity functions instead of the
RKHS generated by the noise process and we write the image as an inner product
on this RKHS.

Reformulating the SAR inverse problem in a reproducing kernel Hilbert space
(RKHS) framework requires one to express the data and image equations in terms of
inner products. This follows from an assumption that the forward operator, as well
as the imaging operator, are continuous linear functionals acting on the unknown
function and data, respectively, where each lie in a RKHS. We may then make use
of the Reisz representation theorem to rewrite these linear functionals in terms of
inner products with unique elements of the Hilbert space utilized. This method of
writing the data and image expressions as inner products has been used before for
example in [2, 14, 21]. In Bertero et al. [2] they set up a general inverse problem
where the data is given in a similar way to (1) where \( \mathcal{F} \) is a general integral operator,
i.e.

\[ D(z) = \mathcal{F}(V(x)) = \int K(z, x)V(x)dx \]

which they re-express as an inner product of \( K \) and \( V \). In order to reconstruct \( V \)
they seek the normal solution or the least squares solution using this inner product
expression for the data. This is different than the inversion method in this work
where we will seek to design an imaging operator such that the kernel of the operator
is a reproducing kernel.
In [14] a similar data equation is used as in [2]. This work is more closely aligned with this current technique as they specifically utilize inner products on a RKHS. While they speak in the language of the Backus-Gilbert method [21], as opposed to backprojection, they also note the possibility of achieving an ideal imaging operator by attempting to make the kernel as close as possible to the reproducing kernel of the RKHS they are working in. They primarily address discrete inverse problems as opposed to our work here where we consider the continuum problem. They also define ideal images in terms of statistical quantities like minimizing bias and variance. They conclude they are unable to create the RK exactly and attempt an approximation using b-splines. This is different than our work presented here where we focus on attempting to recreate the RK precisely with the inverse operator. However their work provides some intriguing possibilities for future study when this current work is expanded to more realistic data collection scenarios which may require some approximation of the reproducing kernel.

We pause here to note that in this current work we will consider the case when the imaging scenario allows for a two-dimensional data collection surface. This is possible in certain applications, i.e. diffraction tomography [6], however future work is required to discretize the continuous analysis of this work in order to create a numerical reconstruction algorithm based on the RKHS framework presented here. We will discuss SAR imaging in more detail in this work. We note in SAR imaging typically only a one-dimensional flight path is available. At best we can typically assume a two dimensional array of antennas that move along a flight path. The analysis presented here requires the full $\mathbb{R}^2$ plane of antenna locations. Discretizing the image reconstruction for a finite set of antenna locations in the 2D plane is in progress. While the 2D antenna location surface assumption is limiting and may seem unrealistic in its current state, there is a perk of this formulation. Due to the assumed 2D data collection surface we no longer need to utilize multiple frequencies, i.e. a large bandwidth, to achieve the desired range resolution in the resulting image. A single transmitted frequency is all that is required to perform the image reconstruction. Single frequency SAR is desirable for many reasons, including a simplification of the needed radar hardware and a smaller chance of interference with other nearby transmitting devices. In fact bandwidth is becoming increasingly hard to come by, as the EM spectrum becomes more and more crowded. Finally many objects with frequency dependent reflectivities, i.e. dispersive objects, may cause issues for image reconstruction using wideband devices, and hence single frequency data may be considered preferable. While single frequency microwave antennas are still being studied, there has been much progress in developing antennas that produce two dimensional data collection surfaces, in particular dynamic metasurface antennas [15, 4]. These antennas are created to produce electrically large, linear transmit and receive apertures. They are able to modulate the radiation patterns of the antennas such that large portions of the data collection manifold are accessed even while using a single frequency. This new work in antenna design makes the image reconstruction method presented in this work applicable to SAR as well as desirable in certain imaging scenarios when little bandwidth is available for transmission or desirable for the image reconstruction task.

The paper is organized as follows. In section 2 we will summarize a derivation of the forward model mentioned briefly above. We then describe the RKHS imaging framework, assuming two different Hilbert space scenarios, in section 3. Next in section 4 we summarize the standard SAR forward model as well as the standard...
SAR inversion in section 5. In section 6 we reformulate the SAR image expression in terms of an inner product on a reproducing kernel Hilbert space which will allow us to develop an ideal image fidelity operator in the sense of reproducing kernels. We will show how we can reproduce the standard filtered backprojection image fidelity operator in this manner. In section 7 we develop a modified form for the SAR data as a Fourier integral operator acting on the reflectivity function which allows us to use the general reproducing kernel framework to develop an ideal image fidelity operator without the approximations necessary in the standard filtered backprojection derivation.

2. Forward model. We will consider inverse scattering problems associated with the inhomogeneous Helmholtz equation. In particular we will consider the case when the field of interest, \( \phi \), satisfies the following equation

\[
(\nabla^2 + k^2)\phi(x, \omega) = k^2 V(x)\phi(x, \omega)
\]

where \( V(x) \) may be thought of as the scattering potential associated with an object illuminated by the field \( \phi \). Note \( \omega \) is the angular frequency, \( k = \frac{2\pi\omega}{c} \) is the wavenumber, \( c \) is the speed of the field (e.g. the speed of light in a vacuum for SAR), and we assume \( x \in \mathbb{R}^3 \). The inverse problem is to recover \( V \) from the scattered field \( \phi^{sc} \) where \( \phi(x, \omega) = \phi^{in}(x, \omega) + \phi^{sc}(x, \omega) \).

We may rewrite this PDE as the Lippmann-Schwinger integral equation \[5\]

\[
\phi(x, \omega) = \phi^{in}(x, \omega) + \int G(x - y, \omega)k^2 V(y)\phi(y, \omega)dy
\]

where \( G \) is the three dimensional Green’s function for the Helmholtz equation.

If we assume the Born approximation \[5\] holds we may rewrite this as

\[
\phi^{sc}(x, \omega) = \int G(x - y, \omega)k^2 V(y)\phi^{in}(y, \omega)dy
\]

where we have subtracted the expression for the incident field from both sides.

We assume a simple model for the incident field, i.e. we assume it satisfies the following Helmholtz equation

\[
(\nabla^2 + k^2)\phi^{in}(x, \omega) = J(\omega, x)
\]

where \( J \) represents the waveform of the transmitting sensor. We again use the Green’s function solution of the wave equation to find an expression for the incident field

\[
\phi^{in}(x, \omega) = \int \frac{e^{-ik|x-y|}}{4\pi|x-y|}J(\omega, y)dy
\]

\[
\approx \frac{e^{-ik|x-x_0|}}{4\pi|x-x_0|}F_t(k, \widehat{x-x_0})
\]

where \( x_0 \) is the location of the transmitting device and \( \widehat{x-x_0} \) indicates we have taken the unit vector in the direction of \( x-x_0 \). We note we have used the far-field approximation \[5\]. We also note that \( F_t \) is proportional to the Fourier transform of \( J \).

We may insert this model for the incident field into the expression for the scattered field to obtain

\[
\phi^{sc}(x, \omega) = \int \frac{e^{-2ik|x-y|}}{|x-y|^2}A(k, x-y)V(y)dy
\]
where $A(k, x - y) = k^2 F_i(k, x - y)$. Note here we are assuming the transmitter and receiver are colocated and are limiting ourselves to problems where the available data is only backscattering data. The results to follow should hold if we have transmitter and receiver in different locations; we would simply need to change the value of the function $p$ found below.

We will now show this model may be rewritten as a pseudodifferential operator [16, 17] applied to the unknown function $V$. We seek a function $p$ that satisfies

$$
\frac{1}{(2\pi)^3} e^{-2ik|x|} A(k, x) = \int e^{i x \cdot \xi} p(k, \xi) d\xi.
$$

We find $p$ by taking the Fourier transform of the left-hand side above

$$
\frac{1}{(2\pi)^3} \int e^{-i x \cdot \xi} e^{-2ik|x|} A(k, x) dx
$$

We now express the collected data as

$$
d(k, x) = \mathcal{F}[T](k, x) = \int \int e^{i(x-y) \cdot \xi} p(k, \xi) d\xi V(y) dy
$$

where $p$ is defined as

$$
p(k, \xi) = \int \frac{1}{8\pi^2|\xi|} \left( \text{rect} \left( \frac{|\xi|}{2k} \right) + \frac{i}{2} \ln \left( \frac{(|\xi| - 2k)^2}{(|\xi| + 2k)^2} \right) \right) \hat{A}(k, \xi - \hat{\xi}) d\hat{\xi}.
$$

We note that our data model (82) is clearly in the form of an FIO [7] (in fact a pseudodifferential operator) if $A$ is such that $p \in C^\infty(\mathbb{R} \times \mathbb{R}^2)$. We will therefore assume $A$ is such that $p$ is a symbol of order 2 or less [16, 17].

We will now assume the object of interest lies on a flat plane, i.e. $V(y) = V(y) \delta(y_3)$ where $y \in \mathbb{R}^3$ now. In addition we assume that all possible measurement locations, i.e. $x$, also lie on a flat plane. Therefore we assume $x_3 = H$, some constant altitude, and now $x \in \mathbb{R}^2$. Therefore we rewrite our data expression:

$$
d(k, x) = \mathcal{F}[V](k, x) = \int \int e^{i(x-y) \cdot \hat{\xi}} e^{iH\xi_3} p(k, \xi) d\xi V(y) dy
$$

where $\hat{\xi} = (\hat{\xi}_1, \hat{\xi}_2)$. We may rewrite this further as

$$
d(k, x) = \mathcal{F}[V](k, x) = \int \int e^{i(x-y) \cdot \hat{\xi}} \hat{p}(k, \hat{\xi}) d\hat{\xi} V(y) dy.
$$

We note that this forward model applies to many applications in sensing. In particular it applies to diffraction tomography [6] and we will make the case that it applies to synthetic aperture radar imaging in certain scenarios as well.
3. Image reconstruction using an RKHS framework.

3.1. RKHS case 1. We now present the RKHS image reconstruction technique utilizing the forward model described above. We seek to demonstrate an imaging operator that forces the image fidelity operator, i.e. the composition of the forward and imaging operators, to be the reproducing kernel for the RKHS in which \( V \) lies.

We will assume that the potential \( V \) lies in the Sobolev space \( H^2(\mathbb{R}^2) \), which is a reproducing kernel Hilbert space, and that \( d \) lies in \( L^2(\mathbb{R}^2) \). Recall that \( H^2(\mathbb{R}^2) \) is defined as

\[
H^2(\mathbb{R}^2) = \left\{ f(x) \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(\xi)|^2 (1 + |\xi|^2)^2 d\xi < \infty \right. \right\}
\]

where

\[
F(\xi) = \frac{1}{(2\pi)^2} \int e^{-ix\cdot\xi} f(x) dx
\]

is the Fourier transform of \( f \). The inner product on this space is

\[
\langle f(x), g(x) \rangle_{H^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} (1 + |\xi|^2)^2 F(\xi) G(\xi) d\xi.
\]

This choice of space for \( V \) is somewhat arbitrary. However, \( H^2 \) is desirable as it is the Sobolev space of functions which are the least smooth while remaining an RKHS. We expect that the unknown potential functions will not be smooth, e.g. an object with defined edges. Another desirable property of \( H^2 \) is that its inner product may be written in terms of Fourier transforms which allows simpler analysis and makes clear the connection between the RKHS framework and standard imaging reconstruction algorithms.

We begin by providing some background on RKHS. The following definitions follow from [20].

**Definition.** An evaluation functional over the Hilbert space of functions on \( \mathcal{T}, \mathcal{H} \), is a linear functional \( L_t : \mathcal{H} \to \mathbb{R} \) that evaluates each function in the space at the point \( t \in \mathcal{T} \), or

\[
L_t f = f(t), \quad \forall f \in \mathcal{H}.
\]

**Definition.** A reproducing kernel Hilbert space (RKHS) \( \mathcal{H} \) is a Hilbert space of functions on \( \mathcal{T} \) with the property that, for each \( t \in \mathcal{T} \), the evaluation functional \( L_t \), which associates \( f \) with \( f(t) \), \( L_t f \to f(t) \) is a bounded linear functional. The boundedness means that there exists some \( M = M_t > 0 \) such that

\[
|L_t f| = |f(t)| \leq M ||f||_{\mathcal{H}} \quad \forall f \in \mathcal{H}.
\]

**Theorem.** If \( \mathcal{H} \) is a RKHS then for each \( t \in \mathcal{T} \) there exists a unique positive-definite function \( K_t \in H \) (called the representer of \( \mathcal{H} \) or the reproducing kernel) with the reproducing property

\[
L_t f = \langle K_t, f \rangle_{\mathcal{H}} = f(t) \quad \forall f \in \mathcal{H}.
\]

Note this theorem is a direct result of the Reisz Representation Theorem.

We will assume that the reconstructed image of \( V \in H^2(\mathbb{R}^2) \) is simply some linear operator applied to the data. That is, we assume \( I \) is of the form

\[
I(z) = K[d](z) = \int R(z, k, x)d(k, x) dx
\]
where $\mathcal{K}$ is known as the imaging operator. Note here, that in the calculation of $I$, we do not integrate over the wavenumber $k$ and therefore we only need a single transmitted frequency to perform imaging.

If we assume $R$ is such that $\mathcal{K}$ is a bounded linear functional from $L^2(\mathbb{R}^2)$ to $\mathbb{C}$ for each $z$, then we may rewrite the above expression as an inner product on $L^2(\mathbb{R}^2)$ using the Reisz Representation theorem. Therefore we have that there exists a unique function $Q$ in $L^2(\mathbb{R}^2)$, for each $z$, such that

\begin{equation}
I(z) = \langle d(k, x), Q(z, k, x) \rangle_{L^2(\mathbb{R}^2)} = \langle \mathcal{F}[V](k, x), Q(z, k, x) \rangle_{L^2(\mathbb{R}^2)}.
\end{equation}

We note if $\mathcal{F}$ is a bounded linear operator from $H^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ we may rewrite above again using the definition of the adjoint operator:

\begin{equation}
I(z) = \langle \mathcal{F}[V](k, x), Q(z, k, x) \rangle_{L^2(\mathbb{R}^2)} = \langle V(y), \mathcal{F}^*[Q](z, k, y) \rangle_{H^2(\mathbb{R}^2)}.
\end{equation}

It is simple to verify that the forward operator $\mathcal{F}$ is a bounded linear operator from $H^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ assuming $p$ is a symbol of order 2 or less, see Appendix 2 for a short proof.

We now seek the form of $\mathcal{F}^*$ from the definition in equation (20). We have that

\begin{equation}
\langle \mathcal{F}[V](k, x), Q(z, k, x) \rangle_{L^2(\mathbb{R}^2)} = \int \int e^{i\xi \cdot (x-y)} p(k, \xi) d\xi V(y) dy Q(z, k, x) dx
\end{equation}

\begin{equation}
= (2\pi)^2 \int e^{i\xi \cdot x} p(k, \xi) Q(z, k, x) V(\xi) d\xi dx
\end{equation}

\begin{equation}
= (2\pi)^2 \int \left\{ \int e^{i\xi \cdot x} Q(z, k, x) dx \right\} p(k, \xi) \frac{1}{(1 + |\xi|^2)^2} d\xi
\end{equation}

\begin{equation}
\times \hat{V}(\xi)(1 + |\xi|^2)^2 d\xi
\end{equation}

\begin{equation}
= \left\langle V, F^{-1} \left[ (2\pi)^2 \left( \int e^{-i\xi \cdot x} Q(z, k, x) dx \right) p(k, \xi) \frac{1}{(1 + |\xi|^2)^2} \right] \right\rangle_{H^2}
\end{equation}

where $F^{-1}$ is the inverse Fourier transform. It is clear now that

\begin{equation}
\mathcal{F}^*[Q(z, k, x)](z, k, y)
\end{equation}

\begin{equation}
= \int e^{iy \cdot \xi} \left[ \left(2\pi\right)^2 \int e^{-i\xi \cdot x} Q(z, k, x) dx \right] p(k, \xi) \frac{1}{(1 + |\xi|^2)^2} d\xi.
\end{equation}

We now argue if $Q$ may be chosen such that $\mathcal{F}^*[Q(z, k, x_0)](z, k, y) = K_z(y)$ where $K_z$ is the reproducing kernel for the Hilbert space $H^2(\mathbb{R}^2)$ then we obtain an unbiased or ideal image, that is $I(z) = T(z)$. First we must specify $K_z$. For $K_z$ to be the RKHS for $H^2(\mathbb{R}^2)$ $K_z$ should satisfy the following

\begin{equation}
\langle V(x), K_z(x) \rangle_{H^2(\mathbb{R}^2)} = V(z)
\end{equation}

which is equivalent to

\begin{equation}
\int \hat{V}(\xi) \hat{K}_z(\xi)(1 + |\xi|^2)^2 d\xi = \int e^{i\xi \cdot \hat{V}(\xi)} d\xi.
\end{equation}

We therefore see that

\begin{equation}
\hat{K}_z(\xi) = \frac{e^{-i\xi \cdot \xi}}{(1 + |\xi|^2)^2}
\end{equation}

or

\begin{equation}
K_z(x) = \int e^{i\xi \cdot x} \frac{e^{-i\xi \cdot \xi}}{(1 + |\xi|^2)^2} d\xi.
\end{equation}
Comparing this to the adjoint operator applied to $Q$ in equation (22) we desire a filter $Q$ such that

\begin{equation}
(27) \quad \frac{(2\pi)^4p(k, \xi)\hat{Q}(z, k, \xi)}{(1 + |\xi|^2)^2} = \frac{e^{-iz\cdot\xi}}{(1 + |\xi|^2)^2}
\end{equation}

and hence we have that $Q$ is defined by

\begin{equation}
(28) \quad Q(z, k, x) = \int e^{ix\cdot\xi}e^{-iz\cdot\xi} \frac{1}{(2\pi)^4p(k, \xi)} d\xi.
\end{equation}

If we insert this expression into the image equation (19) we obtain

\begin{align*}
I(z) &= \int d(k, x) \left( \int e^{i(z-x)\cdot\xi} \frac{1}{(2\pi)^4p(k, \xi)} d\xi \right) d\xi \\
&= \int \left( \int e^{i(z-x)\cdot\xi} p(k, \xi) d\xi V(y)dy \right) \left( \int e^{i(z-x)\cdot\xi} \frac{1}{(2\pi)^4p(k, \xi)} d\xi \right) d\xi \\
&= \int e^{iz\cdot\xi} e^{-iz\cdot\xi} p(k, \xi) \left( \int \frac{1}{(2\pi)^4p(k, \xi)} V(y)dy \right) dx d\xi d\xi d\xi \\
&= \int e^{iz\cdot\xi} e^{-iz\cdot\xi} p(k, \xi) \frac{1}{(2\pi)^2p(k, \xi)} V(y)dy d\xi d\xi \\
&= \int e^{iz\cdot\xi} \frac{1}{(2\pi)^2p(k, \xi)} V(y)dy d\xi \\
&= V(z)
\end{align*}

as desired.

We pause here to note that the result of the using $H^2(\mathbb{R}^2)$ as our RKHS leads to the standard inversion of a pseudodifferential operator found using symbol calculus [16, 17]. We therefore have the same issue of dealing with singularities in $1/p$. We may therefore augment the filter found above to contain a smooth cutoff function that is zero wherever singularities of $1/p$ exist. Doing this in practice is challenging and there are some methods to address this in the diffraction tomography literature [6] which will be the focus of future work as we attempt to create a numerical algorithm based on this approach. We also will aim to study different RKHS that may remove the issue of the singularities all together.

3.2. RKHS case 2. We next consider the case when the forward map $\mathcal{F} : H^2(\mathbb{R}^2) \to H^2(\mathbb{R}^2)$, i.e. we assume the data also lies in the Sobolev space. It was found in [18, 19] that radar data may be shown to also lie in a RKHS so we use this as an example of that case. It remains to be studied whether $d$ lies in the same RKHS as $V$.

In order to repeat the analysis above we require $\mathcal{F}$ to be a bounded linear operator from $H^2(\mathbb{R}^2)$ onto itself. See Appendix 2 for the proof of this. We now seek to define the adjoint of the forward operator. Again using the formal definition of the adjoint we have

\begin{equation}
\langle \mathcal{F}[V](k, x), Q(z, k, x) \rangle_{H^2} = \langle V(y), \mathcal{F}^*[Q](z, k, y) \rangle_{H^2}.
\end{equation}
Beginning with the left-hand side we have

$$\langle F[V](k, x), Q(z, k, x) \rangle_{H^2} = \frac{1}{(2\pi)^4} \int \left( \int e^{-ix}\xi F[V](k, x) dx \right)$$
\[
\times \left( \int e^{-ix}\xi Q(z, k, \tilde{x}) dx \right) (1 + |\xi|^2)^2 d\xi
\]
\[
= \frac{1}{(2\pi)^4} \int \left( \int e^{-ix}\xi \left[ \int e^{i\hat{\xi}(x-y)} P(k, \hat{\xi}) d\xi V(y) dy \right] dx \right)
\times \left( \int e^{-ix}\xi Q(z, k, \tilde{x}) dx \right) (1 + |\xi|^2)^2 d\xi.
\]
\[(31)\]

We first consider the $\xi$ integral, we have

$$\int e^{-ix}\xi (1 + |\xi|^2)^2 d\xi = \frac{1}{(2\pi)^2} \left\{ \delta(x - \tilde{x}) + \left( i\frac{2}{2\pi} \right)^4 \partial_x^4 \delta(x - \tilde{x}) + \left( i\frac{2}{2\pi} \right)^2 \partial_x^2 \delta(x - \tilde{x}) \right\}$$
\[
= P[\delta(x - \tilde{x})]
\]

where $P$ is the differential operator written out above. Plugging this result back into the inner product expression in (31) we have

$$\langle F[V](k, x), Q(z, k, x) \rangle_{H^2} = \frac{1}{(2\pi)^2} \int \left[ \int e^{i\hat{\xi}(x-y)} P(k, \hat{\xi}) d\xi V(y) dy \right]$$
\[
\times Q(z, k, \tilde{x}) P[\delta(x - \tilde{x})] d\tilde{x} d\tilde{\xi}
\]
\[
= \frac{1}{(2\pi)^2} \int \left[ \int e^{i\hat{\xi}(x-y)} P(k, \hat{\xi}) d\xi V(y) dy \right] P[Q(z, k, \tilde{x})]_{|\tilde{x}=x} d\tilde{x}
\]
\[
= \frac{1}{(2\pi)^2} \int \hat{V}(\hat{\xi}) \left[ \int e^{i\hat{\xi}x} \frac{P(k, \hat{\xi})}{(1 + |\xi|^2)^2} P[Q(z, k, \tilde{x})]_{|\tilde{x}=x} d\tilde{x} \right]
\times (1 + |\xi|^2)^2 d\xi
\]
\[(33)\]

where

$$\langle V(y), F^*[Q](z, k, y) \rangle_{H^2}$$

$$\langle F^*[Q](z, k, y) = F^{-1} \left\{ \frac{1}{(2\pi)^2} \int e^{-i\hat{\xi}x} \frac{P(k, \hat{\xi})}{(1 + |\xi|^2)^2} P[Q(z, k, \tilde{x})]_{|\tilde{x}=x} d\tilde{x} \right\}(y).$$

Note above we use integration by parts to go from line one to line two, assuming $Q$ has compact support in the third variable. This is highly likely in practice given that $A$ often has compact support and therefore we would define $Q$ to have the same support as $A$.

As in the first case considered we desire $F^*[Q](z, k, y) = K_x(y)$. We therefore have that their Fourier transforms must be equal as well, hence
which implies
\[ P\{Q(z, k, \hat{x})\}\big|_{\hat{x}=x} = \int e^{-i\xi \cdot (z-x)} \frac{1}{p(k, \xi)} d\xi. \]

If we use the integral definition of the differential operator \( P \) and compare to the expression above we have that
\[ e^{-i\xi \cdot z} p(k, \xi) = (2\pi)^2 (1 + |\xi|^2)^2 \tilde{Q}(z, k, \xi) \]
and therefore \( Q \) is given by
\[ Q(z, k, x) = \frac{1}{(2\pi)^2} \int e^{-i\xi \cdot (z-x)} \frac{1}{p(k, \xi)(1 + |\xi|^2)^2} d\xi. \]

We note in our true computation of the image we are calculating the integral
\[ I(z) = \int R(z, k, x) d(k, x) dx = \langle \mathcal{F}\{V\}(k, x), Q(z, k, x) \rangle_{H^2} \]
and it is therefore useful to derive \( R \) from the inner product of the data with \( Q \).

Writing out this expression in more detail we have
\[ \langle \mathcal{F}\{V\}(k, x), Q(z, k, x) \rangle_{H^2} = \int \tilde{d}(k, \xi) \tilde{Q}(z, k, \xi) (1 + |\xi|^2)^2 d\xi \]
\[ = \int \left( \frac{1}{(2\pi)^2} \int e^{-i\xi \cdot d(k, x)} dx \right) e^{i\xi \cdot z} \frac{1}{(2\pi)^2 p(k, \xi)(1 + |\xi|^2)^2} \]
\[ \times (1 + |\xi|^2)^2 d\xi \]
\[ = \frac{1}{(2\pi)^4} \int \left( \int e^{i\xi \cdot (z-x)} \frac{1}{p(k, \xi)} d\xi \right) d(k, x) dx \]
which implies
\[ R(z, k, x) = \frac{1}{(2\pi)^4} \int e^{i\xi \cdot (z-x)} \frac{1}{p(k, \xi)} d\xi \]
which is precisely the filter \( Q \) obtained in section 3.1.

4. Standard SAR data model. We now connect this method to the application of synthetic aperture radar. We first present a summary of the standard SAR forward model.

Radar waves are electromagnetic and thus their propagation is described by Maxwell’s equations. However, because most radar imaging scenarios involve propagation in dry air, we assume that the electromagnetic properties of free space hold. We therefore may utilize, in place of Maxwell’s equations, the free space wave equation for each component of the electric field vector \( E \) [5], i.e.
\[ \left( \nabla^2 - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) E(t, x) = j(t, x). \]

where \( j \) is a model for the current density on the transmitting antenna. Note we assume the wave speed is \( c_0 \), the constant speed of light in a vacuum, and therefore no scatterers are present.
When a scatterer is present the wave speed will change when the wave interacts with the object. We consider this a perturbation in the wave speed, which we express as

$$\frac{1}{c^2(x)} = \frac{1}{c^2_0} - T(x)$$

where $T$ is known as the scalar reflectivity function. We now assume that $E$ satisfies the following modified wave equation:

$$\left(\nabla^2 - \frac{1}{c^2(x)} \frac{\partial^2}{\partial t^2}\right)E(t, x) = j(t, x).$$

Note in reality the reflectivity function $T$ represents a measure of the reflectivity for the polarization measured by the antenna. For more information on the reflectivity function see [9].

Note the field above is the total electric field, i.e. $E^{\text{tot}} = E^{\text{in}} + E^{\text{sc}}$. Typically one assumes the incident field satisfies the following wave equation

$$\left(\nabla^2 - \frac{1}{c^2_0} \frac{\partial^2}{\partial t^2}\right)E^{\text{in}}(t, x) = j(t, x)$$

where $c_0$ is the speed of light in a vacuum. We subtract (45) from (44), while using the definition of the variable wave speed (43), to find an expression for the scattered field:

$$\left(\nabla^2 - \frac{1}{c^2_0} \frac{\partial^2}{\partial t^2}\right)E^{\text{sc}}(t, x) = -T(x)\frac{\partial^2}{\partial t^2}E^{\text{tot}}(t, x).$$

If we transform this equation into the frequency domain we obtain the inhomogeneous Helmholtz equation used in the general forward model. We then rewrite this in the form of the Lippmann-Schwinger integral equation and utilize the Born approximation to write

$$E^{\text{sc}}_{\text{B}}(\omega, x) = \int e^{-ik|x-y|} \frac{\omega^2 T(y)}{4\pi|x-y|} dy$$

where $k = \omega/c_0$.

Using the Green’s function solution to solve for the incident field and applying the far-field approximation we have the following SAR data equation

$$E^{\text{sc}}(\omega, x_0) \approx \int \frac{e^{-2ik|x_0-y|}}{(4\pi|y-x_0|)^2} \frac{A(\omega, y-x_0)}{1} T(y) dy$$

where $A$, the amplitude, is given by

$$A(\omega, y-x_0) = k^2 F_t(k, y-x_0) F_{\text{rec}}(k, y-x_0)$$

where $F_t$ is the radiation pattern of the antenna, and $F_{\text{rec}}$ is the reception pattern of the antenna. Note we have assumed that the receiving and transmitting antennas are colocated at $x_0$ and hence our system is monostatic.

Now when we consider the SAR modality, we must incorporate antenna motion in our model. If we are to use our method above we are assuming that the antenna is able to travel along the full $\mathbb{R}^2$ plane. More often it is assumed the antenna follows a path $\gamma \in \mathbb{R}^3$. This path is parametrized with a continuous parameter denoted $s$, called slow time, and therefore the position of the antenna at a given slow time $s$
is $\gamma(s)$. Therefore in the standard SAR model one replaces $x_0$ with $\gamma(s)$ in (48) to obtain the expression for the received mono-static SAR data

$$D(s, \omega) = \int e^{-2ik|\gamma(s)-y|}A(\omega, s, y)T(y)dy.$$  

We have assumed, as in the general model, for simplicity a flat topography where $y = (y_1, y_2)$, $y = (y, 0)$. Note that in this case we have shown there is a linear relationship between the data, $d$, and the target reflectivity $T$, in terms of the forward operator $\mathcal{F}$. Under mild conditions, the operator $\mathcal{F}$ is a Fourier Integral Operator (FIO) [18, 10, 16, 17], but not a pseudodifferential operator. The goal of SAR imaging is to reconstruct $T$ from knowledge of $d$. One common technique is based on the well known fact [10, 16, 17] that an FIO may be approximately inverted by another FIO. This inversion is called backprojection and is different from the method of inversion in the RKHS framework above. We briefly discuss backprojection in the next section to indicate the difference from the RKHS technique.

5. Filtered backprojection. Backprojection is one of the most traditional SAR imaging techniques. In the previous section we have made assumptions so that $\mathcal{F}$ is a Fourier integral operator and therefore one method of finding an approximate inverse is via another FIO. We will form an approximate inverse of $\mathcal{F}$ that is known as a filtered backprojection operator. We assume for simplicity that we are able to integrate over all of the ground plane in (50) and again over the full data collection manifold when forming the image (i.e. $\omega \in \mathbb{R}$, or $t \in \mathbb{R}$, and $s \in \mathbb{R}$). The case of backprojection given a limited ground plane and limited data collection manifold is discussed thoroughly in [5].

We will assume the imaging operator is of the form of the linear operator below

$$I(z) = \mathcal{K}[d](z) := \int e^{-i\omega(t-R_{s,z})}Q(z, s, \omega)d\omega ds dt$$

$$= \int e^{2ikR_{s,z}}Q(z, s, \omega)D(s, \omega)dsd\omega,$$

where $z = (z_1, z_2)$, $D(s, \omega)$ is the Fourier transform of the data in fast-time, $R_{s,z} = |\gamma(s) - z|$, and $Q$ is a filter which is to be determined. We note that defining the imaging operator to have the opposite sign in the phase of the forward operator is because we assume our inverse operator will look like the formal adjoint of $\mathcal{F}$, this portion of the operator performs what is known as backprojection. If we insert the equation for the data into above we have

$$I(z) = \mathcal{K}\mathcal{F}[T](z) = \int e^{2ik(R_{s,z}-R_{s,x})}Q(z, s, \omega)A(x, s, \omega)d\omega ds T(x)dx.$$  

Note we call the composition of the imaging operator $\mathcal{K}$ and the forward operator $\mathcal{F}$ the image fidelity operator.

We seek an ideal image, that is,

$$I(z) = \mathcal{K}\mathcal{F}[T](z) = T(z).$$

In SAR there is typically not much discussion of which function spaces are in use but it is implied typically that $T$ and $d$ lie in $L^2(\mathbb{R}^2)$. This implies that the image fidelity kernel should be the Dirac delta function

$$I(z) = \mathcal{K}\mathcal{F}[T](z) = \int \delta(z-x)T(x)dx = T(z).$$
We therefore aim to choose the filter $Q$ such that $\mathcal{K}\mathcal{F}$ is equal to, or as close as possible to, the delta function. One way to do this is to rewrite the delta function in terms of its Fourier transform, 1, i.e.

$$\delta(z - x) = \int e^{i(z-x)\cdot\xi}d\xi. \quad (55)$$

We therefore determine that the kernel of the image fidelity operator, and hence the filter $Q$, should satisfy the following equation

$$B(z, x) = \int e^{2ik(R_s, z - R_s, x)}Q(z, s, \omega)A(x, s, \omega)ds = \int e^{i(z-x)\cdot\xi}d\xi \quad (56)$$

Note the kernel of the image fidelity operator, i.e. $B$, is often called the point-spread function.

We now show that the image fidelity operator is a pseudodifferential operator and approximates the delta function. We aim to determine how close the phase of the image-fidelity operator is to that of a pseudodifferential operator. In order to do this we first apply the method of stationary phase to the $s$ and $\omega$ integrals. We introduce a large parameter $\beta$ via the change of variables $\omega = \beta\omega'$. The stationary phase theorem states that the main contribution to a highly oscillatory integral comes from the critical points of the phase, i.e. points satisfying certain critical conditions. In our case these conditions are

$$\begin{align*}
0 &= \nabla_\omega \Phi \propto \phi(s, z) - \phi(s, x) \\
0 &= \nabla_s \Phi \propto \nabla_s (\phi(s, z) - \phi(s, x))
\end{align*} \quad (57)$$

where $\Phi$ is the phase of the image fidelity operator. One of the solutions of the above equations is the critical point $x = z$. We will assume this is the only visible critical point or singularity. The other critical points lead to artifacts in the image (for example the standard right-left ambiguity issue which is well known in SAR imaging [5]), however their presence depends on the imaging set up. We may assume, for example, that our antenna is side looking and hence only the desired critical point is visible to the radar.

In the neighborhood of the critical point $x = z$, we next perform a Taylor expansion of the exponent. We utilize the following formula

$$f(z) - f(x) = \int_0^1 \frac{d}{d\mu}f(z + \mu(z - x))d\mu = (z - x) \cdot \int_0^1 \nabla f|_{z + \mu(z-x)}d\mu \quad (58)$$

where $f(z) = 2kR_{s, z}$. We use this formula to make what is commonly known as the Stolt change of variables

$$(s, \omega) \rightarrow \xi = \Xi(s, \omega, x, z) = \int_0^1 \nabla f|_{z + \mu(z-x)}d\mu. \quad (59)$$

After performing this change of variables we obtain this following expression for the kernel of the image-fidelity operator

$$B(z, x) = \int e^{i(z-x)\cdot\xi}Q(z, s(\xi), \omega(\xi))A(x, s(\xi), \omega(\xi))\eta(x, z, \xi)d\xi \quad (60)$$

where $\eta$ is the Jacobian resulting from the change of variables, sometimes referred to as the Beylkin determinant [3]. We now see clearly that the phase of $\mathcal{K}\mathcal{F}$ is of the form of a pseudodifferential operator, as well as the Fourier transform of the delta function. If we require $Q$ to satisfy symbol estimates [18], we can conclude that our image-fidelity operator is approximately a pseudodifferential operator.
We now seek to determine our filter $Q$. In order to do so we must first utilize a standard result in the theory of pseudodifferential operators \cite{16, 10}. This result states that a pseudodifferential operator can be written as
\begin{equation}
B[T](z) = \int e^{i(z-x)\cdot\xi} B(z, x, \xi) d\xi T(x) dx = \int e^{i(z-x)\cdot\xi} p(z, \xi) d\xi T(x) dx
\end{equation}
where $p(z, \xi) = e^{iz\cdot\xi} B(e^{iz}\cdot\xi)$, the symbol, has the following asymptotic expansion
\begin{equation}
p(z, \xi) \sim \sum_{\alpha \geq 0} \frac{i^{|\alpha|}}{\alpha!} D^{\alpha}_{\xi} D^{\alpha}_{x} B(z, x, \xi)|_{z=x}
\end{equation}
where $\alpha$ is a multi-index. In other words, the leading-order term of $p(z, \xi)$ is given by $B(z, z, \xi)$. For more information on this asymptotic expansion of the symbol see \cite{16}. Therefore we may write the leading-order term of (52) as
\begin{equation}
I(z) \approx \int e^{i(z-x)\cdot\xi} Q(\xi, z) A(\xi, z) \eta(z, z, \xi) d\xi T(x) dx.
\end{equation}
This now allows us to choose the filter $Q$ which makes the image fidelity operator as close to the delta function as possible. We see that our choice should be
\begin{equation}
Q(z, \xi) = \frac{1}{A(z, \xi) \eta(z, z, \xi)}.
\end{equation}

We note that this method forces $(s, k)$ to act as the spatial frequency when we perform the Stolt change of variables. This is not entirely intuitive as $s$ parametrizes a spatial variable itself, representing where the data was collected, and $k$ is $2\pi\omega/c$ where $\omega$ is the temporal frequency. The RKHS framework as described above avoids this, though it requires significantly more data, i.e. we need to replace the standard one-dimensional flight path with the full $\mathbb{R}^2$ plane of antenna locations which is not realistic currently. We also see we are only able to obtain an approximate inverse operator of our forward operator, which is demonstrated when we utilize the asymptotic expansion of the symbol of our operator above.

6. SAR image formation in the RKHS framework. We now show that using the standard SAR forward model with the RKHS framework for imaging still requires the Stolt change of variables and the approximation in finding the inverse of the forward operator. In the description of backprojection above we note that the ideal image would be of the form
\begin{equation}
I(z) = \int e^{i(z-x)\cdot\xi} d\xi T(x) dx = \int \delta(z - x) T(x) dx = T(z)
\end{equation}
where the second equality holds when we are able to integrate over the entire $\xi$ plane. This constraint arises in other contexts, for example when we seek to find the best linear unbiased estimator of $T$ \cite{11} we find that the unbiased constraint is given by
\begin{equation}
E[\hat{T}(z)] = I(z) = \int e^{i(z-x)\cdot\xi} d\xi T(x) dx = \int \delta(z - x) T(x) dx = T(z)
\end{equation}
where $\hat{T}$ is the estimator of $T$ and $E$ indicates we are taking the expected value. We seek to reinterpret this result/ideal image definition in the context of reproducing kernel Hilbert spaces. We define an ideal imaging operator to be that which ensures the image fidelity kernel is the reproducing kernel of the RKHS where $T$ lies.
Now we will work towards rewriting the image equation using an RKHS framework. We begin by assuming the reflectivity function $T(x)$ lies in some reproducing kernel Hilbert space $H$. We also assume the forward operator $F$ maps $H$ into $L^2(\mathbb{R}^2)$. Recall
\begin{equation}
D(s, \omega) = F[T](s, \omega) = \int e^{-2ikr_{x,s}y}A(\omega, s, y)T(y)dy.
\end{equation}
In particular we will assume $A$ is such that $F$ is a bounded linear operator from $H$ to $L^2(\mathbb{R}^2)$. Next we reconsider the image equation from before. We have that
\begin{equation}
I(z) = \int e^{2ikR_{x,s}y}Q(z, s, \omega)D(s, \omega)d\omega ds = K[D](z).
\end{equation}
If we assume $Q$ is such that $K$ is a continuous linear functional from $L^2(\mathbb{R}^2)$ into $\mathbb{C}$ for each $z$ then we may use the Reisz representation theorem to rewrite this as an inner product. In fact it is rather simple to see we may express
\begin{equation}
I(z) = \langle D(s, \omega), e^{-2ikR_{x,s}y}Q(z, s, \omega) \rangle_{L^2(\mathbb{R}^2)}.
\end{equation}
Note for simplicity of notation we will define $q = e^{-2ikr_{x,s}y}Q(z, s, \omega)$. Further we may rewrite $D$ in terms of the forward operator to obtain
\begin{equation}
I(z) = \langle F[T](s, \omega), q(z, s, \omega) \rangle_{L^2(\mathbb{R}^2)}.
\end{equation}
Now since we assumed $A$ is such that $F$ is a bounded linear operator from $H$ to $L^2(\mathbb{R}^2)$ then we can define the formal adjoint $F^*$ of the forward operator and rewrite the above equation:
\begin{equation}
I(z) = \langle \mathcal{F}[T](s, \omega), q(z, s, \omega) \rangle_{L^2(\mathbb{R}^2)} = \langle T(x), F^*[q(z, s, \omega)](x) \rangle_H
\end{equation}
Recall that we desire $I(z) = T(x)$ therefore we suppose that $F^*[q(z, s, \omega)](x)$ is the reproducing kernel of $H$. This gives us a new criterion for the ideal filter $Q$.

It is tempting to use $L^2(\mathbb{R}^2)$ as a starting guess for the Hilbert space that contains possible reflectivity functions. However, under this assumption we end up with the same criterion for $Q$ as in the microlocal case, i.e.
\begin{equation}
F^*[q(z, s, \omega)](x) = \delta(z - x).
\end{equation}
We note that $L^2(\mathbb{R}^2)$ is not a RKHS, this is due to the fact that the evaluator/reproducing kernel fails to belong to the space in question. We therefore consider alternative options. Alternatively we consider $T \in H^2(\mathbb{R}^2)$ where $H^2(\mathbb{R}^2)$ is a Sobolev space, as we used in the general case above. This choice leads to a result like that of microlocal analysis while utilizing a true RKHS. Investigating other RKHS options remains as future work.

We will first determine the formal adjoint of the forward operator and then determine the ideal filter $Q$. Returning to our expression for $I$ we have
\begin{equation}
I(z) = \langle \mathcal{F}[T](s, \omega), q(z, s, \omega) \rangle_{L^2(\mathbb{R}^2)} = \int \left( \int e^{-2ikR_{x,s}y}A(x, s, \omega)T(x)dx \right) q(z, s, \omega)dsd\omega
\end{equation}
\begin{equation}
= \int e^{2ikR_{x,s}y}A(x, s, \omega) \left( \int e^{ikx(x)}T(\xi)dx \right) q(z, s, \omega)dsd\omega
\end{equation}
where we have introduced $\hat{T}$, the Fourier transform of $T$ and rewritten $q = e^{-2ikR_{x,s}y}Q(z, s, \omega)$ on the last line. To rewrite this expression as an inner product on $H^2(\mathbb{R}^2)$
we perform the Stolt change of variables, from \((s, \omega)\) to \(\zeta\), and the symbol calculus as in the backprojection derivation above. We therefore obtain

\[
I(z) = \int e^{i(z-x)\xi} A(z, \xi) Q(z, \xi) \eta(z, \xi) \left( \int e^{i s \xi} \hat{T}(\xi) d\xi \right) d\zeta dx
\]

which in the backprojection derivation above. We therefore obtain

\[
I(z) = \int e^{iz\xi} A(z, \xi) Q(z, \xi) \eta(z, \xi) \left( \int e^{i(z-x)\xi} d\xi \right) d\zeta dx
\]

or

\[
= \int e^{iz\xi} A(z, \xi) Q(z, \xi) \eta(z, \xi) \left( \int e^{i(z-x)\xi} d\xi \right) d\zeta
\]

and therefore we may conclude that

\[
(74) \quad I(z) = \int e^{iz\xi} A(z, \xi) Q(z, \xi) \eta(z, \xi) \left( (1 + |\xi|^2)^2 \right) d\xi
\]

Now in order to achieve the ideal image we want \(F^*[q(z, \xi)](x) = K_\omega(x)\) where \(K_\omega\) is the reproducing kernel for \(H^2(\mathbb{R}^2)\). Recall that

\[
(75) \quad F^*[q(z, \xi)](x) = \int e^{iz\xi} \left( \frac{e^{-iz\xi} A(z, \xi) Q(z, \xi) \eta(z, \xi)}{(1 + |\xi|^2)^2} \right) d\xi.
\]

Comparing this equation with (75) we see that

\[
K_\omega(x) = \int e^{iz\xi} \frac{e^{-iz\xi}}{(1 + |\xi|^2)^2} d\xi.
\]

and therefore we may conclude that

\[
(74) \quad I(z) = \int e^{iz\xi} A(z, \xi) Q(z, \xi) \eta(z, \xi) \left( (1 + |\xi|^2)^2 \right) d\xi
\]

Note that because we use the inner product definition above we will be taking a complex conjugate of this \(Q\) before forming the image so it is the exact same filter as in backprojection (64). We observe that the resulting image relies on the Stolt change of variables and the approximation made with the symbol calculus. We conclude that we are unable to achieve an ideal image in the RKHS sense in this manner.

7. Modified FIO forward model. We now present a slightly modified model for the SAR data. This data model aims to eliminate the need for the Stolt change of variables and the approximation due to the symbol calculus in the image reconstruction algorithm. This model will ultimately match that of the general RKHS framework developed in section 2.

We begin with the expression for a Born approximated scattered electric field measured at the point \(x\) with wavenumber \(k\):

\[
(79) \quad E^sc(k, x) = \int e^{-2ik|x-y|} A(k, x - y) T(y) dy.
\]

We note that in the most general case \(x, y \in \mathbb{R}^3\). Also note we keep the geometrical spreading factors, i.e. the denominator above, outside of the amplitude term in this case. Therefore \(A\) in this model is slightly different from that of the standard SAR
model in section 4. Note that here we are assuming we are able to collect data in a subspace of $\mathbb{R}^3$, which we will limit to $\mathbb{R}^2$ shortly. Therefore we are, for now, assuming we have more data than a standard SAR data collection scenario which assumes we are only able to collect data along a one-dimensional flight path. This is obviously not realistic for many measurement scenarios but it is necessary for the analysis to follow. As mentioned in the introduction, new antenna design, e.g. dynamic metasurfaces, do allow for two-dimensional data collection surfaces. We will address the future work of considering a more standard measurement scenario in the conclusion.

Now we seek a function $p$ that satisfies

\begin{equation}
\frac{1}{(2\pi)^3} e^{-2ik|x|} A(k, x) = \int e^{ix \cdot \xi} p(k, \xi) d\xi.
\end{equation}

We find $p$ by taking the Fourier transform of the left-hand side above, just as we did in the general case, therefore

\begin{equation}
\frac{1}{(2\pi)^3} \int e^{-ix \cdot \xi} e^{-2ik|x|} \frac{1}{|x|^2} A(k, x) d\xi = \int \frac{1}{8\pi^2 |\xi|} \left( \text{rect} \left( \frac{|\xi|}{2k} \right) + \frac{i}{2} \ln \left( \frac{(|\xi| - 2k)^2}{(|\xi| + 2k)^2} \right) \right)
\end{equation}

\begin{equation}
\times \hat{A}(k, \xi - \hat{\xi}) d\hat{\xi}.
\end{equation}

We now express the collected data as

\begin{equation}
d(k, x) = F[T](k, x) = \int \int e^{i(x-y) \cdot \xi} p(k, \xi) d\xi T(y) dy
\end{equation}

where $p$ is defined as

\begin{equation}p(k, \xi) = \int \frac{1}{8\pi^2 |\xi|} \left( \text{rect} \left( \frac{|\xi|}{2k} \right) + \frac{i}{2} \ln \left( \frac{(|\xi| - 2k)^2}{(|\xi| + 2k)^2} \right) \right) \hat{A}(k, \xi - \hat{\xi}) d\hat{\xi}.
\end{equation}

We assume $A$ is such that $p$ is a symbol of order 2 or less, just as in the general case, which ensures $F$ is a pseudodifferential operator.

As before, we assume our scene of interest lies on a flat plane, i.e. $T(y) = T(y) \delta(y_3)$ where $y \in \mathbb{R}^2$ now. In addition we assume that all possible measurement locations, i.e. $x$, also lie on flat plane. We therefore assume $x_3 = H$, some constant altitude, and now $x \in \mathbb{R}^2$. Therefore we rewrite our data expression:

\begin{equation}d(k, x) = F[T](k, x) = \int \int e^{i(x-y) \cdot \xi} e^{iH \xi_3} \hat{p}(k, \xi) d\xi T(y) dy
\end{equation}

where $\hat{\xi} = (\xi_1, \xi_2)$. We may rewrite this further as

\begin{equation}d(k, x) = F[T](k, x) = \int \int e^{i(x-y) \cdot \xi} \hat{p}(k, \hat{\xi}) d\hat{\xi} T(y) dy.
\end{equation}

where $\hat{\xi}$ includes the $\xi_3$ integration. As before, for simplicity, we will drop the hat notation for $\xi$ and $p$ and simply assume from now on that $\xi \in \mathbb{R}^2$, i.e.

\begin{equation}d(k, x) = F[T](k, x) = \int \int e^{i(x-y) \cdot \xi} p(k, \xi) d\xi T(y) dy.
\end{equation}

We now have our data precisely in the general form from (14) and hence can apply the RKHS imaging technique outlined earlier, avoiding the Stolt change of variables and approximations of the backprojection filter derivation.
8. Conclusion. In this work we have reformulated a standard inverse problem stemming from the inhomogeneous Helmholtz equation with an unknown potential in a reproducing kernel Hilbert space framework. We note that if we assume the unknown function lies in the Sobolev space $H^2(\mathbb{R}^2)$ we are able to reconstruct an ideal image, assuming we have an infinite data collection manifold. While the current Sobolev space choice for the RKHS recreates the standard inversion for a pseudodifferential operator using symbol calculus, there is the possibility for future work determining an alternative RKHS that avoids the inversion of the symbol which typically has singularities to consider.

We demonstrate in this work how this technique is applicable to synthetic aperture radar imaging. We find that we can reproduce the ideal backprojection filter in the microlocal sense when we assume the standard SAR forward model and that the ground plane is $\mathbb{R}^2$ as well as an infinite data collection manifold. It was noted that this method, like backprojection, forces the slow time and wave number variables to act as the spatial frequencies and requires an approximation. We therefore seek to avoid the Stolt change of variables and the symbol calculus approximation. In order to accomplish this we have re-expressed the SAR forward model for the measured data in a new FIO format which is written in terms of the true spatial frequency variables without requiring the Stolt change of variables. We therefore may rewrite the SAR inverse problem in the RKHS framework utilizing the Sobolev space $H^2(\mathbb{R}^2)$ for the reflectivity function space and using $L^2(\mathbb{R}^2)$ as the data function space as well as $H^2(\mathbb{R}^2)$. In both scenarios we are able to avoid making the approximations needed in the microlocal backprojection derivation. We found that we were able to create an image fidelity kernel that is precisely the reproducing kernel of $H^2(\mathbb{R}^2)$.

Future work will focus on determining if this method will be practically useful for SAR imaging. It is important to note that this analysis here is entirely dependent on the assumptions that the ground plane is $\mathbb{R}^2$ and that we also are able to collect data on a surface, in fact all of $\mathbb{R}^2$, instead of simply on a one-dimensional flight path. Hence this makes the methods presented here limited in applicability based on commonly used radar systems. However, the development of dynamic metasurface antennas, capable of producing a data collection surface, make this method more realizable. The added benefit of these antennas and our technique presented in this work is that one may reconstruct images without a large bandwidth, in fact single frequency imaging is possible. Future work will address translating this work to a numerical image reconstruction algorithm which does not require the full $\mathbb{R}^2$ plane of antenna locations. In addition future work will address analysis with a more realistic data collection manifold, i.e. the typical assumption that we collect data along a one-dimensional flight path. This may require an assumption that the reflectivity functions lie in a one-dimensional subspace of $\mathbb{R}^2$. We note that if this extension is possible without requiring one to integrate over frequency or wave number, we may then utilize bandwidth in order to reconstruct frequency dependent target reflectivities. This would allow one to image dispersive objects while obtaining information about their frequency dependence.

Appendix 1. We consider

$$
\frac{1}{(2\pi)^3} \int e^{-ix \cdot \xi} e^{(-2ik|\xi|)} d\mathbf{x} = \frac{1}{(2\pi)^3} \int_0^{2\pi} \int_{-1}^1 \int_0^\infty e^{-i|\xi|\mu} e^{-2ikr} \frac{1}{r^2} r^2 dr d\mu d\phi
$$

(87)
where we changed to spherical coordinates \((r, \mu, \phi)\) where \(\mu = \cos(\theta)\) where \(\theta\) is the azimuthal angle and \(\phi\) is the elevation angle. Performing the \(\phi\) and \(\mu\) integrations yields

\[
\frac{1}{(2\pi)^2} \int_0^\infty e^{-2ikr} \text{sinc}(r|\xi|) dr = \frac{1}{(2\pi)^2} \int_0^\infty (\cos(2kr) - i \sin(2kr)) \text{sinc}(r|\xi|) dr.
\]

Considering the real term of the above integral first we have

\[
\int_0^\infty \cos(2kr) \frac{\sin(r|\xi|)}{r|\xi|} dr = \frac{1}{2|\xi|} \int_0^\infty \frac{\sin(r|\xi| - 2k) + \sin(r|\xi| + 2k)}{r} dr.
\]

Note that

\[
\frac{1}{r} = \int_0^\infty e^{-sr} ds.
\]

Inserting this into the previous line we obtain the following double integral for the real part of the Fourier transform of the square of the Green’s function

\[
\int_0^\infty \int_0^\infty \frac{1}{2|\xi|} \[\sin(r|\xi| - 2k) + \sin(r|\xi| + 2k)\] e^{-sr} drds.
\]

We note that using the known Laplace transforms of \(\sin\) we have that this is equivalent to

\[
\int_0^\infty \frac{1}{2|\xi|} \left[ \frac{|\xi| - 2k}{(|\xi| - 2k)^2 + s^2} + \frac{|\xi| + 2k}{(|\xi| + 2k)^2 + s^2} \right] ds
= \frac{1}{2|\xi|} \left[ -\text{sgn}(|\xi| - 2k) + \text{sgn}(|\xi| + 2k) \right]
= \frac{1}{2|\xi|} \text{rect}\left(\frac{|\xi|}{2k}\right).
\]

We now consider the imaginary term of (88)

\[
\int_0^\infty \frac{1}{|\xi|} \sin(2kr) \frac{\sin(r|\xi|)}{r} dr = \frac{1}{2|\xi|} \int_0^\infty \frac{[\cos(r|\xi| - 2k) - \cos(r|\xi| + 2k)]}{r} dr
= \int_0^\infty \int_0^\infty \frac{1}{2|\xi|} \left[ \cos(r|\xi| - 2k) - \cos(r|\xi| + 2k) \right] e^{-sr} drds
= \frac{1}{2|\xi|} \int_0^\infty \left[ \frac{s}{(|\xi| - 2k)^2 + s^2} - \frac{s}{(|\xi| + 2k)^2 + s^2} \right] ds
= \frac{1}{4|\xi|} \left. \ln \left( \frac{(|\xi| - 2k)^2 + s^2}{(|\xi| + 2k)^2 + s^2} \right) \right|_0^\infty
= \frac{1}{4|\xi|} \left[ \lim_{s \to \infty} \ln \left( \frac{(|\xi| - 2k)^2 + s^2}{(|\xi| + 2k)^2 + s^2} \right) - \frac{1}{4|\xi|} \ln \left( \frac{(|\xi| - 2k)^2}{(|\xi| + 2k)^2} \right) \right]
= -\frac{1}{4|\xi|} \ln \left( \frac{|\xi| - 2k}{|\xi| + 2k} \right).
\]

We therefore obtain the full Fourier Transform of the square of the Green’s function:

\[
\frac{1}{(2\pi)^3} \int e^{-ix \cdot \xi} e^{-2ik|\xi|} |\xi|^2 d\xi = \frac{1}{8\pi^2|\xi|} \left[ \text{rect}\left(\frac{|\xi|}{2k}\right) + i \frac{1}{2} \ln \left( \frac{|\xi| - 2k}{|\xi| + 2k} \right) \right].
\]
Appendix 2. We now argue that our forward operator, $\mathcal{F}$, is a bounded linear operator from $H^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$. We seek to show that
\begin{equation}
\|\mathcal{F}[T](k, x)\|_{L^2(\mathbb{R}^2)} \leq C\|T\|_{H^2(\mathbb{R}^2)}
\end{equation}
for some positive constant $C$. First note
\begin{align}
\|\mathcal{F}[T]\|_{L^2(\mathbb{R}^2)}^2 &= \int \int e^{i(x-y)\xi} p(k, \xi) d\xi T(y) dy d\xi \\
&= (2\pi)^4 \int e^{i(x-y)\xi} p(k, \xi) d\xi T(y) dy d\xi \\
&= (2\pi)^6 |p(k, \xi)|^2 |\hat{T}(\xi)|^2 d\xi \\
&\leq (2\pi)^6 |C(1 + |\xi|)^m|^2 \frac{1}{(1 + |\xi|^2)^2} |\hat{T}(\xi)|^2 (1 + |\xi|^2)^2 d\xi \\
&\leq \tilde{C} \int |\hat{T}(\xi)|^2 (1 + |\xi|^2)^2 d\xi \\
&= \tilde{C} \|T\|_{H^2(\mathbb{R}^2)}^2
\end{align}
where $\hat{T}$ is the Fourier transform of $T$ and where $m$ is the order of the symbol $p$ which we assume is $m \leq 2$. We note this requirement on $p$ is due to the fact
\[
\frac{(1 + |\xi|)^{2m}}{(1 + |\xi|^2)^2} \leq D
\]
for some constant $D$ whenever $m \leq 2$. We therefore conclude $F$ is a bounded linear operator from $H^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$.

Next we argue that our forward operator, $\mathcal{F}$, is a bounded linear operator from $H^2(\mathbb{R}^2)$ to $H^2(\mathbb{R}^2)$. We seek to show that
\begin{equation}
\|\mathcal{F}[T]\|_{H^2(\mathbb{R}^2)} \leq C\|T\|_{H^2(\mathbb{R}^2)}.
\end{equation}
We first consider the left hand side, i.e.
\begin{align}
\|\mathcal{F}[T]\|_{H^2(\mathbb{R}^2)} &= \int \int e^{-i(x-y)\xi} p(k, \xi) d\xi T(y) dy d\xi \\
&= \frac{1}{(2\pi)^4} \left\{ \int e^{-i(x-y)\xi} \left[ \int e^{i(x-y)\xi} p(k, \xi) d\xi T(y) dy \right] dx \right\} \\
&\times \left[ \int e^{i\xi} \left[ \int e^{-i(\xi\hat{\xi} - \hat{\xi})} p(k, \xi) d\xi \right] dx \right] (1 + |\xi|^2)^2 d\xi \\
&= \int \left[ \int e^{i\xi} (\xi\hat{\xi} - \hat{\xi}) p(k, \xi) d\xi \right] dx \\
&\times \left[ \int e^{-i\hat{\xi}} p(k, \xi) T(\xi) d\xi \right] dx
\end{align}
\[ (2\pi)^4 \int \delta(\xi - \hat{\xi})\delta(\hat{\xi} - \hat{\xi})p(k, \xi)\hat{T}(\xi)p(k, \xi)\hat{T}(\xi)d\xi d\hat{\xi} (1 + |\xi|^2)^2 d\xi \]

\[ = (2\pi)^4 \int |p(k, \hat{\xi})|^2 |\hat{T}(\hat{\xi})|^2 (1 + |\hat{\xi}|^2)^2 d\hat{\xi} \]

\[ \leq (2\pi)^4 \int C_k^2 |\hat{T}(\hat{\xi})|^2 (1 + |\hat{\xi}|^2)^2 d\hat{\xi} \]

\[ (98) \quad \leq \hat{C}_k \| T \|_{H^2} \]

where in the second to last line we must make the additional assumption on the symbol that for each \( k \)

\[ (99) \quad \sup_{\hat{\xi}} p(k, \hat{\xi}) \leq C_k \]

where \( C_k \) is a positive finite constant. When this assumption is satisfied we have that \( F \) is a bounded linear operator on \( H^2(\mathbb{R}^2) \) into itself.

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