Control of Tripod-Scheme Cold-Atom Wavepackets by Manipulating a non-Abelian Vector Potential

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Tripod-scheme cold atoms interacting with laser beams have attracted considerable interest for their role in synthesizing effective non-Abelian vector potentials. Such effective vector potentials can be exploited to realize an all-optical imprinting of geometric phases onto matter waves. By working on carefully designed extensions of our previous work, we show that coherent lattice structure of cold-atom sub-wavepackets can be formed and that the non-Abelian Aharonov-Bohm effect can be easily manifested via the translational motion of cold atoms. We also show that by changing the frame of reference, effects due to a non-Abelian vector potential may be connected with a simple dynamical phase effect, and that under certain conditions it can be understood as an Abelian geometric phase in a different frame of reference. Results should help design better schemes for the control of cold-atom matter waves.

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I. INTRODUCTION

As one of many significant developments in using cold-atoms to achieve quantum simulations, tripod-scheme cold atoms interacting with three laser beams have attracted considerable interest for their role in synthesizing effective non-Abelian vector potentials [1-6]. Specifically, for cold-atoms with a tripod internal level structure and in the presence of plane-wave laser fields, a simple space-independent (as long as the atoms are well inside all the laser beams) non-Abelian vector potential can be generated. This result has led to a number of interesting predictions, such as negative anomalous reflection [7], cold-atom analog of Datta-Das transistor [9], negative refraction [10] and cold-atom analog of the so-called Zitterbewegung oscillation [11-13].

Recently, we showed that if the laser beams interacting with a tripod-scheme cold-atom are displaced slowly, then it is possible to actively imprint a non-Abelian vector potential induced phase [14] onto the matter wave [15]. Because the phases thus obtained are not induced by the translational motion of the cold atom itself, but by active manipulation of laser-matter interaction, the imprinting of quantum phases onto matter waves results in a new type of coupling between internal and translational motions. Indeed, as shown in Ref. [15], due to the phase imprinting a wavepacket may develop many interesting interference patterns and may even be splitted into many equal-weight copies along a straight line. Because by construction the phase imprinting due to the laser-beam displacement is insensitive to the details of the laser beams (such as the laser intensity, actual speed of the moving laser beam, etc), the active displacement of laser beams can potentially provide a robust means of controlling matter wave propagation. Further, a direct experimental observation of the phases due to a non-Abelian vector potential can now be reduced to the observation of the translational motion of cold atoms.

The purpose of this work is twofold. First, we consider a few more scenarios of laser-beam displacement, thus extending and strengthening our early work [15]. In one scenario where the laser-beams are moved along a square, we show that it is possible to split a wavepacket into equal-weight copies on a square lattice. We further show that it is also possible to form a triangular lattice of cold-atom sub-wavepackets. In another scenario where the laser beams are moved along a circle, we show that one may force a cold-atom wavepacket to “dance” along a circle of a different radius or redirect the propagation direction of that wavepacket. As seen below, these new scenarios are carefully designed for our second and more important purpose, namely, to address two issues of more fundamental interest. The first issue is on a simple but intriguing demonstration of non-Abelian Aharonov-Bohm effect via the translational motion of cold atoms. The second issue is on how our perspectives of quantum phases may change as we change the frame of reference. We shall explicate that, by changing the frame of reference, i.e., by changing from the laboratory frame to a reference frame moving with the laser beams, the effect due to an effective non-Abelian vector potential may be connected with a dynamical phase effect, and under certain conditions part of this effect may be understood as an Abelian geometric phase obtained from a different frame of reference. These results, based on concrete and explicit examples, may enhance our understanding of the dynamics of tripod-scheme cold atoms and motivate experiments on the associated control of cold-atom wavepackets.
This paper is organized as follows. For self-completeness, in Sec. II we provide some necessary details about the effective non-Abelian vector potential realized by tripod-scheme atoms. In Sec. III we show how moving laser beams along a circle can lead to formation of a square lattice of cold-atom sub-wavepackets. An extension of this approach can lead to the formation of a triangular lattice of cold-atom sub-wavepackets. In analyzing the results we show that the physical effects induced by the non-Abelian vector potential can be connected to pure dynamical phases in another reference frame. In the same section we also discuss how such laser manipulation can serve as a direct approach to the observation of the non-Abelian Aharonov-Bohm effect. In Sec. IV, we provide an interesting relation between an evolution matrix as an integral of the non-Abelian vector potential in the laboratory frame and an Abelian geometric phase in the frame that moves with the laser beams. In so doing we consider the displacement of laser beams along a circle and the resultant motion of cold-atom wavepackets. Section V concludes this work.

II. NON-ABELIAN VECTOR POTENTIALS REALIZED BY TRIPOD-SCHEME ATOMS

Tripod-scheme atoms refer to four-level atoms interacting with three laser fields [16]. We denote the four internal levels as $|n\rangle$, $n = 0 - 3$. Each of the three transitions $|0\rangle \leftrightarrow |1\rangle$, $|0\rangle \leftrightarrow |2\rangle$, and $|0\rangle \leftrightarrow |3\rangle$ is coupled by one laser field. This coupling scheme can be realized if states $|1\rangle$, $|2\rangle$, and $|3\rangle$ are degenerate magnetic sub-levels and the three coupling fields have different polarizations. For convenience we adopt the same configuration as in Ref. [7], where two laser beams are counter-propagating along the $x$-axis and the third laser beam is along the $z$-axis. The associated internal Hamiltonian under RWA is given by

$$H_{RWA,4} = \sum_{n=1}^{3} \Omega_n |n\rangle\langle n| + h.c.,$$

with

$$\Omega_1 = \frac{\Omega_0 \sin(\xi)}{\sqrt{2}} e^{-ik_x x},$$

$$\Omega_2 = \frac{\Omega_0 \sin(\xi)}{\sqrt{2}} e^{ik_x x},$$

$$\Omega_3 = \Omega_0 \cos(\xi) e^{ik_z z},$$

where the parameter $\xi$ is set to satisfy $\cos(\xi) = \sqrt{2} - 1$, as in Ref. [7], and $k_x$ is the wavevector of the laser fields.

The Hamiltonian $H_{RWA,4}$ has two degenerate states with a null eigenvalue. We denote these two degenerate states as $|D_{1(2)}\rangle$ and it is straightforward to find their spatial dependence as follows [3],

$$|D_1\rangle = (|\tilde{1}\rangle - |\tilde{2}\rangle)e^{-i\kappa' z}/\sqrt{2},$$

$$|D_2\rangle = \left[\cos(\xi) (|\tilde{1}\rangle + |\tilde{2}\rangle)/\sqrt{2} - \sin(\xi)|3\rangle\right] e^{-i\kappa' z},$$

where

$$\kappa' \equiv k_x [1 - \cos(\xi)],$$

$$|\tilde{1}\rangle \equiv |1\rangle e^{ik_x (x+z)},$$

$$|\tilde{2}\rangle \equiv |2\rangle e^{-ik_x (x-z)}.$$  

When the laser field strengths are sufficiently large, $\Omega$ can be large compared to any possible two-photon detuning induced by fluctuations in laser frequencies and/or Doppler shift. Then if the initial state is in the dark-state subspace and if the system parameters are changing slowly, the internal atomic state can evolve within the two-dimensional dark-state subspace. As such we focus on the time evolution in the dark-state subspace only. Clearly then, the dynamical phase contributed by the internal energy is always zero because $H_{RWA,4}|D_{1(2)}\rangle = 0$. Any state therein can be spanned as $c_1|D_1\rangle + c_2|D_2\rangle$. This expansion henceforth defines a dark state representation. In this representation, the mechanical momentum operator becomes [7, 8]

$$P^D = -i\hbar \vec{\nabla} + \hbar \kappa (\sigma_x \hat{e}_x + \sigma_z \hat{e}_z),$$

where $\sigma_{x,z}$ are Pauli matrices, $\kappa = \cos(\xi)k_x$, $\vec{\nabla}$ represents the gradient in the dark-state representation, and $\hat{e}_x$ and $\hat{e}_z$ are the unit vectors along $x$ and $z$. As in Ref. [7], we also introduce an additional constant shift to state $|3\rangle$, which
accommodates a detuning of the third laser from resonance by \( V_s = \hbar (k_l^2 \sin^2(\xi)/2m \) (\( m \) is the mass of atom). Then the total effective Hamiltonian becomes

\[
H_{\text{eff}}^D = \frac{(P_D)^2}{2m}.
\] (10)

This effective Hamiltonian and the explicit form of \( P_D \) in Eq. (9) make it clear that a tripod-scheme atom interacting with three laser beams effectively synthesize a non-Abelian vector potential, whose \( x \)-component is given by \( \hbar \kappa \sigma_x \), \( z \)-component is given by \( \hbar \kappa \sigma_z \), and \( y \)-component is zero.

The eigenstates of \( H_{\text{eff}}^D \) are

\[
|\Psi_{D,\pm}\rangle = |g^\pm_k\rangle e^{i\mathbf{k} \cdot \mathbf{R}},
\] where \( \mathbf{R} \) is the spatial coordinate (for simplicity the plane-wave normalization factor \((2\pi\hbar)^{3/2} \) is not included), and \( |g^\pm_k\rangle \) denotes two-component internal-state vectors in the dark subspace. Note that \( |g^\pm_k\rangle \) depends on the direction of the wavevector \( \mathbf{k} \) via its dependence on \( \varphi_k \), the angle between the \( x \)-axis and the atom wavevector \( \mathbf{k} \). From Eq. (9), it is straightforward to show that the eigenstates \( |\Psi_{D,\pm}\rangle \) have momenta \((k \pm \kappa \hat{\mathbf{k}})\hbar \) respectively, with their energy eigenvalues \(( (k \pm \kappa)\hbar )^2/2m \). Note also that throughout this paper \( \hat{\mathbf{k}} \) represents a unit vector along the vector \( \mathbf{k} \) and \( k \) represents the modulus of \( \mathbf{k} \).

III. WAVEPACKET CONTROL BY MANIPULATING A NON-ABELIAN VECTOR POTENTIAL

In Ref. [15], we proposed to control the matter wave of tripod-scheme atoms by displacing the laser beams along a straight line. By considering two different frames of reference, in this section we will first re-analyze our results in Ref. [15]. We then generalize the scheme in [15] by displacing the lasers along a square, as illustrated in Fig. 1. As another extension we also consider laser displacement along an equilateral triangle.

A. Displacing laser-beams along a straight line: two perspectives

1. Perspective from the laboratory frame

Consider a laser-beam movement along a straight line, e.g., along the path \( A - B \) in Fig. 1. As what we did in Ref. [15], it is natural to first analyze this problem in the laboratory frame. Without loss of generality we assume that the initial state is given by

\[
|\Psi_{\text{lab}}\rangle = |g_k(\varphi_k = \pi/2)\rangle e^{i\mathbf{k}_0 \cdot \mathbf{R}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\mathbf{k}_0 \cdot \mathbf{R}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\] (12)

where we have assumed that the wavevector \( \mathbf{k}_0 \) associated with the spatial part of the total wavefunction is zero [17]. Denoting \( x = v_d t \), where \( v_d \) is the speed of the laser beam displacement, then at each spatial point \( \mathbf{R} \) in the laboratory frame the evolution of \((c_1, c_2)\) is determined by

\[
\frac{id}{dx} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} i \langle D_1 | \frac{\partial}{\partial x} | D_1 \rangle & i \langle D_1 | \frac{\partial}{\partial x} | D_2 \rangle \\ i \langle D_2 | \frac{\partial}{\partial x} | D_1 \rangle & i \langle D_2 | \frac{\partial}{\partial x} | D_2 \rangle \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = - \hat{G}_x \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},
\] (13)

where \( \hat{G}_x \) is defined as a \( 2 \times 2 \) matrix associated with the displacement along the \( x \) direction. This expression is expected because it reflects the \( x \)-component of the non-Abelian vector potential in our dark-state representation.
FIG. 1: One scenario of laser beam displacement along a square for the control of wavepackets of tripod-scheme cold-atoms. As explained in the text, a cloud of tripod scheme cold-atoms is simultaneously illuminated by three laser beams. All the laser beams are moved either clockwise or counter-clockwise from the starting point $A$, along the square with the length of each side given by $a$.

(See Eq. (5)). One tends to call the evolution matrix determined by the above equation as a non-Abelian geometric phase. However, because here the involved path in the parameter space is not necessarily closed or cyclic, such an evolution matrix is clearly representation-dependent. Nevertheless, if we move the laser beams such that $e^{ik_{x,t}r} = 1$, then because the parameters for both the internal Hamiltonian and the dark states are subject to cyclic changes, the evolution matrix associated with Eq. (13) becomes gauge-independent and therefore can be regarded as a non-Abelian geometric phase. In either case, the physical effects discussed below are based on the actual solution to Schrödinger’s equation and are hence representation-independent.

In addition to this effect due to the non-Abelian vector potential, a simple dynamical phase also accumulates during the laser manipulation. In the laboratory frame, the energy of any state $(c_1, c_2)^T e^{i\mathbf{k}_0 \mathbf{R}} = (c_1 | D_1 \rangle + c_2 | D_2 \rangle) e^{i\mathbf{k}_0 \mathbf{R}}$ is contributed by two terms. The first term is the internal energy of the dark subspace, which is zero at all times. The second term is the eigenvalues $[(k_0 \pm \kappa)\hbar]^2/2m$ of the eigenstates given in Eq. (11). In the case of $k_0 = 0$, this term is given by

$$E_{\text{kin}}^{D} = \frac{(-i\hbar \langle D_{1(2)} | \frac{\partial}{\partial \mathbf{R}} | D_{1(2)} \rangle)^2}{2m} = \frac{\hbar^2 \kappa^2}{2m},$$

which is independent of $c_1$ or $c_2$. Hence, so long as the laser displacement is sufficiently slow, the internal state will remain in the dark subspace and the kinetic energy will be necessarily given by $E_{\text{kin}}^{D}$ for $k_0 = 0$, which is independent of $c_1$ or $c_2$.

Based on Eq. (13) and the dynamical phase determined by $E_{\text{kin}}^{D}$, one then finds the state evolution during the passage $A-B$, i.e.,

$$|\Psi(t)\rangle = e^{i\hat{G}_{x,t} v d t} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) e^{i\phi_d}$$

$$= \frac{1}{2} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) e^{iv_d \kappa t} e^{i\phi_d} + \frac{1}{2} \left( \begin{array}{c} 1 \\ -1 \end{array} \right) e^{-iv_d \kappa t} e^{i\phi_d}$$

$$= \frac{1}{2} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) e^{-i\omega_1 t} e^{i\phi_d} + \frac{1}{2} \left( \begin{array}{c} 1 \\ -1 \end{array} \right) e^{-i\omega_2 t} e^{i\phi_d},$$

where

$$\phi_d = -\frac{\hbar \kappa^2}{2m}$$

is the overall dynamical phase induced by the kinetic energy term $E_{\text{kin}}^{D}$, and

$$\omega_1 = -v_d \kappa;$$

$$\omega_2 = v_d \kappa$$

(17)
are two frequencies due to the non-Abelian vector potential. Because, at a time $t$, the total distance of laser displacement in the $x$ direction is $x = v_d t$, the phase factors $\pm v_d x t$ can also be written as $\pm \kappa x$, depending only on the distance of laser displacement. This implies the robustness of this phase acquired by the process, thus manifesting its geometric nature. Equation (15) also indicates that the acquired phases $\pm \kappa a$ point B are space-independent, and hence it will maintain $k_0 = 0$ as the wavevector for the spatial part of the total wavefunction. This further justifies our treatment here.

Equation (15) for $k_0 = 0$ actually indicates the existence of two group velocities. This can be well understood by finding the derivatives of the frequencies $\omega_{1(2)}$ with respect to the atomic wavevector $k_0$ in the laboratory frame. To that end we consider a nonzero but small $\delta k_0$ in the $x$ direction in the laboratory frame. We then choose another frame of reference that moves with atom with a velocity of $h \delta k_0 \hat{e}_x / m$. In the new frame of reference the wavevector becomes zero again, the laser beams are displaced with a velocity $\tilde{v}_d = v_d - h \delta k_0 / m$, and Eq. (15) can be directly applied with the modified moving speed $\tilde{v}_d$. Since

$$\frac{d \tilde{v}_d}{d k_0} = \frac{h}{m},$$

(18)

the two group velocities can then be obtained using Eq. (18)

$$v_{1G}^{\text{lab}} = \frac{\partial \omega_1}{\partial k_0} = \frac{\partial (-\tilde{v}_d \kappa)}{\partial \tilde{v}_d} \cdot \frac{d \tilde{v}_d}{d k_0} = h \kappa / m,$$

(19)

$$v_{2G}^{\text{lab}} = \frac{\partial \omega_2}{\partial k_0} = \frac{\partial (\tilde{v}_d \kappa)}{\partial \tilde{v}_d} \cdot \frac{d \tilde{v}_d}{d k_0} = -h \kappa / m.$$

Because these two group velocities associated with the two components of the evolving state in Eq. (15) are different, an initial wavepacket of the tripod-scheme cold atoms is expected to split into two, each of the sub-wavepackets possesses an internal state $\frac{1}{\sqrt{2}} (1,1)^T$ or $\frac{1}{\sqrt{2}} (1,-1)^T$. This result is nicely confirmed by numerical simulation results shown in Fig. 2(a). In particular, all our numerical experiments are based on the full Hamiltonian $-\hbar^2 \nabla^2 / 2m + H_{\text{RWA}4} + V_s$, and the initial state is chosen as a Gaussian wave packet instead of a plane wave. From Fig. 2(a), it is seen that when the point $B$ is reached, two sub-wavepackets located at $x = \pm \frac{t_B}{m} v_d z = 0$ emerge, where $t_X$ stands for the time when point $X$ is reached. Because the length of square shown in Fig. 1 is given by $a$, we have $t_B = a / v_d$.

2. Perspective from the frame of reference that moves with the laser beams

In the frame of reference that moves with the laser beams, called laser frame below, all the three laser beams are static by construction, but the atom momentum will be changed. Because the Hamiltonian in Eq. (10) and its eigenstates in Eq. (11) implicitly assume fixed laser-beams, this laser frame can be treated in a straightforward manner. For example, if the laser beams are moving with a speed $v_d$ along the $x$-direction, then an atom momentum $\mathbf{p}$ in the laboratory frame will assume a modified momentum

$$\mathbf{p}_{\text{laser}} = \mathbf{p} - m v_d \hat{e}_x,$$

(20)

in the laser frame. As a consequence, a wavevector $\mathbf{k}_0$ in the laboratory frame will change to

$$\mathbf{k}_{0,\text{laser}} = \mathbf{p}_L / \hbar = \mathbf{k}_0 - m v_d \hat{e}_x / \hbar,$$

(21)

in the laser frame. For the sake of comparison with our early results, we set $k_0 = 0$, and an initial internal state same as that in Eq. (12). Thus the total wavefunction in the laser frame is given by

$$|\Psi_{\text{laser}}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i \frac{m v_d}{\hbar} \hat{e}_x \cdot \mathbf{r}}.$$

(22)

The state in Eq. (22) in the laser frame is not an eigenstate of the Hamiltonian. Nevertheless, it can be written as a superposition of two eigenstates $|\Psi^{D,+}\rangle$ and $|\Psi^{D,-}\rangle$ defined previously by Eq. (11). Its time evolution can then be obtained by use of the two energy eigenvalues associated with $|\Psi^{D,+}\rangle$ and $|\Psi^{D,-}\rangle$ (with $\mathbf{k} = \mathbf{k}_{0,\text{laser}}$). Specifically,

$$|\Psi_{\text{laser}}(t)\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i \frac{m v_d}{\hbar} \hat{e}_x \cdot \mathbf{r}} e^{-i E_+ t / \hbar}$$

$$+ \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i \frac{m v_d}{\hbar} \hat{e}_x \cdot \mathbf{r}} e^{-i E_- t / \hbar}.$$

(23)
where $E_{+}$ and $E_{-}$ are the two eigenvalues $\left(\{k_{0,\text{laser}} \pm \kappa\}\hbar\right)^{2}/2m$, with $k_{0,\text{laser}} = -\frac{mv_{d}}{\hbar}$. One can obtain the same expression for the wavefunction in the laser frame more formally, by use of the well-known Galilean transformation for wavefunctions, namely, $|\Psi(t)\rangle = |\Psi_{\text{laser}}(t)\rangle e^{i\frac{mv_{d}x + m\kappa^{2}}{2\hbar}t}$.

Interestingly, the two phase factors in Eq. (23), i.e., $e^{-iE_{+}\tau/\hbar}$ and $e^{-iE_{-}\tau/\hbar}$, are purely dynamical phase factors. This is in contrast to our early perspective in the laboratory frame (where a geometric phase may arise). The two group velocities corresponding to the two components of the state in Eq. (23), both along the $x$ direction, can then be obtained as follows,

$$
v_{+}^{\text{laser}} = \frac{\partial E_{+}/\hbar}{\partial k_{0,\text{laser}}} = -v_{d} + \hbar\kappa/m;
$$

$$
v_{-}^{\text{laser}} = \frac{\partial E_{-}/\hbar}{\partial k_{0,\text{laser}}} = -v_{d} - \hbar\kappa/m. \tag{24}
$$

We can now return to the laboratory frame. Because the laboratory frame is moving with the laser frame at a velocity $-v_{d}\hat{e}_{x}$, the two group velocities in the laboratory frame are also along the $x$ direction, and they are given by

$$
v_{+}^{\text{lab}} = v_{+}^{\text{laser}} + v_{d} = \hbar\kappa/m;
$$

$$
v_{-}^{\text{lab}} = v_{-}^{\text{laser}} + v_{d} = -\hbar\kappa/m. \tag{25}
$$

This result, which is based entirely on dynamical phase consideration, is exactly the same as those in Eq. (15) obtained by considering the integral of the non-Abelian vector potential in the laboratory frame. Consistent with a previous finding that a geometric phase may not be Galilean invariant \cite{18}, it is still intriguing to see from our concrete example that the effect of a non-Abelian vector potential (which leads to a gauge-independent geometric phase for cyclic processes) may be interpreted totally as that of a dynamical phase in a different frame of reference. That is, the quantum phases and their consequences can be traced back to either an integral of the vector potential in laboratory frame induced by laser beam displacement or to the dynamical phases in the laser frame (i.e., the time integral of the energy eigenvalues). In retrospect, this is not entirely surprising. The phase due to the vector potential is generated by active parameter manipulation associated with laser-field displacement. In a frame of reference in which laser fields are not moving, there is no such phase induced: observable effects (such as wavepacket splitting) must be understandable in terms of other evolution effect, which is purely the dynamical phase here.

**B. Formation of a square or triangular lattice of cold-atom sub-wavepackets**

When the laser beams are displaced along the path $A - B$ in Fig. 1, an initial wavepacket can split into two parts with group velocities $\pm \hbar\kappa/m$ in the $x$ direction. Here we first generalize this scheme by displacing the laser beams along the square shown in Fig. 1.

The two wavepackets shown in Fig. 2(a) are associated with the two components shown in the last line of Eq. (13), with the only difference being that for wavepacket simulations a Gaussian profile is imposed on the initial state and hence each component does not extend to infinity. Because our plane-wave consideration is seen to describe the actual wavepacket dynamics very well, we continue to use plane waves to understand the process along the square. In addition, because the two wavepackets in Fig. 2(a) are already separated when the point $B$ is reached, we only need to treat each term in Eq. (15) separately, rather than treating the two sub-wavepackets as a whole.

Consider then the second stage of a clockwise laser beam displacement along the square, which is along the path $B - C$ shown in Fig. 1, with a velocity $v_{d}$ in the $-z$ direction. The effect of the non-Abelian vector potential can be obtained from

$$
\frac{i}{\hbar} \frac{d}{dz} \begin{pmatrix} c_{1} \\ c_{2} \end{pmatrix} = \begin{pmatrix} i\langle D_{1}|\frac{\partial}{\partial z}D_{1}\rangle & i\langle D_{1}|\frac{\partial}{\partial z}D_{2}\rangle \\ i\langle D_{2}|\frac{\partial}{\partial z}D_{1}\rangle & i\langle D_{2}|\frac{\partial}{\partial z}D_{2}\rangle \end{pmatrix} \begin{pmatrix} c_{1} \\ c_{2} \end{pmatrix}
$$

$$
= -\begin{pmatrix} \kappa & 0 \\ 0 & -\kappa \end{pmatrix} \begin{pmatrix} c_{1} \\ c_{2} \end{pmatrix}
$$

$$
= -\hat{G}_{z} \begin{pmatrix} c_{1} \\ c_{2} \end{pmatrix}, \tag{26}
$$

where a $2 \times 2$ matrix $\hat{G}_{z}$ is defined. This effect now reflects the $z$-component of the non-Abelian vector potential. Because of the special form of the dark states that have two wavevectors $\vec{k}_{+}$ and $\vec{k'}_{+}$, here the parameter manipulation is always noncyclic, in the sense that the dark states do not return to their initial form after the laser displacement.
As such, the evolution matrix as a solution to Eq. (26) is necessarily representation-dependent. For this reason, in our fixed representation for the two dark states $|D_1\rangle$ and $|D_2\rangle$, we do not call the evolution matrix determined by Eq. (26) as a non-Abelian geometric phase, but only as an effect due to the non-Abelian vector potential defined above. (To remove such a representation-dependence one can adopt a unique gauge proposed in Ref. [19] based on the concept of parallel frames via re-defining the dark states. However, it is found that the resulting non-Abelian gauge potential is too complicated. Alternatively, we can simply remove the factor $e^{-i\kappa z}$ from the dark states defined previously. With the newly defined dark states, our parameter manipulation will be cyclic and the obtained solution to Schrödinger’s equation remains unchanged). We then proceed by assuming that the initial internal state is either the first or the second component of Eq. (15). Apart from an overall phase when the point $B$ is reached, the first component of Eq. (15) will evolve to

$$|\Psi_{1,C}(t_C)\rangle = \frac{1}{2} e^{-i\tilde{\varphi}_d \varphi_d(t_C-t_B)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\delta_d}$$

when point $C$ is reached, where $\delta_d$ is again a common dynamical phase for the internal states $(1,0)^T$ and $(0,1)^T$. To seek the group velocities for the two new components in Eq. (27), we also need to find the derivatives of the two frequencies $\pm \tilde{\nu}_d \kappa$ with respect to the wavevector $k_0$ in the laboratory frame. In the same manner as we derive Eqs. (15) and (19), we may consider a $\delta k_0$ in the $z$ direction and take advantage of a frame moving at the velocity $\hbar \delta k_0 \tilde{v}_d/m$ relative to the laboratory frame. Denote again $\tilde{v}_d$ as the velocity of the laser beams in the frame with a
FIG. 3: (Color Online) (a) A schematic plot showing that the laser-beam displacement is along an equilateral triangle with the side length \(a = 100/\kappa\). The laser beams are moved either clockwise or counter-clockwise from the starting point \(A\). The initial wavepacket is placed at the origin with the internal state taken the same as that shown in Eq. (12). (b) The final distribution of sub-wavepackets, obtained from numerical simulation of the tripod-scheme cold-atom dynamics when all the laser beams have been slowly displaced clockwise along the triangle for one cycle. The calculations are based on the full Hamiltonian

\[
-\hbar^2 \frac{1}{2m} \nabla^2 + H_{\text{RWA}} + V_s
\]

(In real units, we take \(m = 10^{-25}\) kg, \(\kappa \sim 10^6\) m\(^{-1}\)). The time interval to move across each side of the triangle is \(20\frac{m}{\hbar \kappa^2} \sim 0.03\) s. \(x\) and \(z\) are in units of \(1/\kappa\). (c) Same as in panel (b), but the displacement of the laser beams is counter-clockwise.

zero wavevector for the spatial part of the total wavefunction, we obtain

\[
\frac{d\tilde{v}_d}{dk_0} = \frac{\hbar}{m},
\]

and the two new group velocities along the \(z\) direction,

\[
v_{1G}^{\text{lab}} = \frac{\partial(\tilde{v}_d \kappa)}{\partial \tilde{v}_d} \cdot \frac{d\tilde{v}_d}{dk_0} = \frac{\hbar \kappa}{m};
\]

\[
v_{2G}^{\text{lab}} = \frac{\partial(-\tilde{v}_d \kappa)}{\partial \tilde{v}_d} \cdot \frac{d\tilde{v}_d}{dk_0} = -\frac{\hbar \kappa}{m}.
\]

These two group velocities correspond to the internal states \((1, 0)^T\) and \((0, 1)^T\), respectively. It can then be predicted that the wavepacket associated with the first term in the last line of Eq. (15) will further split into two parts in the \(z\) direction, due to the different group velocities \(\pm \hbar \kappa/m\). In the same manner we can predict that the wavepacket associated with the second term in the last line of Eq. (15) will also split into two parts. As a consequence, when the point \(C\) is reached, there should be altogether four sub-wavepackets located at \(x = \pm \frac{\hbar \kappa}{m} \frac{a}{v_d}\), \(z = \pm \frac{\hbar \kappa}{m} \frac{a}{v_d}\). The numerical results shown in Fig. 2(b) nicely confirm our predictions.

As the laser beams are displaced further along the square shown in Fig. 1, in principle one can split the wavepacket into \(2^n\) copies on the \(x-z\) plane, forming a beautiful square lattice of matter waves. Figure 2(c) and 2(d) show the numerical results of the distribution of sub-wavepackets when the points \(D\) and \(A\) are reached, respectively, in a clockwise order. Remarkably, when the laser beams are moved back to the initial position \(A\), there are altogether 16 sub-wavepackets formed. It is interesting to examine the relative quantum phases between each sub-wavepacket. If their relative phases can be easily calculated and are independent of the details of the manipulation process, then
a coherent lattice of matter wavepackets is created and they should be useful for atom optics applications such as atomic interferometry. This is indeed the case here. In particular, because of a common dynamical phase $\phi_d$, the relative phases between the sub-wavepackets can be easily determined by examining the imprinted geometric phases, say $\pm v_d \kappa t$ shown in Eqs. \(15\) and \(27\). Take the four sub-wavepackets in Fig. 2(b) as an example. Other than a common overall phase, their quantum phases are found to be $-2\alpha$, $0$, $0$, and $2\alpha$, for the upper-left, upper-right, lower-left, and lower-right sub-wavepackets, respectively.

It should also be pointed out that, after displacing the laser beams along a square, a closed path in the parameter space is formed. As such, the overall effect due to the non-Abelian vector potential becomes that of a gauge-independent non-Abelian geometric phase. Further, because this manipulation is achieved by displacing the laser beams along different straight lines, our discussions above also indicate that this geometric phase effect can be equally understood as arising from the associated dynamical phases in several laser frames.

The formation of a square lattice of cold-atom sub-wavepackets makes it clear that it should be equally possible to form other types of lattices. A simple consideration shows that if we move the laser beams along an arbitrary direction, the wavepacket splitting, if it occurs, must be along the same direction. As a result, if we move the laser beams along an equilateral triangle, it should be possible to form a triangular lattice of cold-atom sub-wavepackets. This is confirmed by our results shown in Fig. 3. In particular, Fig. 3(b) [Fig. 3(c)] depicts the distribution of sub-wavepackets, with unequal weights, after the laser beams are moved one clockwise (counter-clockwise) cycle along an equilateral triangle shown in Fig. 3(a). By repeating this process more sub-wavepackets can be generated.

C. Non-Abelian Aharonov-Bohm effect

In Refs. \[20, 21\], the non-Abelian Aharonov-Bohm effect is demonstrated by showing different internal states corresponding to two different paths from a common starting point to a common ending point, or by showing the non-commutability between different navigation paths. To make connection between our results and the context of non-Abelian Aharonov-Bohm effect, here we consider in detail what happens to a wavepacket if the laser beams are non-commutability between different navigation paths. To make connection between our results and the context of non-Abelian Aharonov-Bohm effect can manifest clearly on the translational motion of cold atoms, rather than on their internal states considered in Refs. \[20, 21\].
FIG. 4: The scheme of laser beam displacement along a circle with a radius $r_L$, with a starting point $E$. The atom wavepacket will be forced to dance along a circle with a radius $r_A$ during the process of circular laser beam displacement. The initial group velocity $v_i$ of a cold-atom wavepacket in the $z$ direction will be changed to $v_f$ when the point $F$ is reached.

IV. RELATION BETWEEN A NON-ABELIAN GEOMETRIC PHASE AND AN ABELIAN GEOMETRIC PHASE

Results in the previous section show that by considering different frames of reference, new insights into the dynamics induced by laser beam displacement may be obtained. In this section we extend this idea and attempt to make a connection between the non-Abelian vector potential induced phase and an Abelian geometric phase. As a concrete example, we consider a situation where all the laser beams are slowly moved along a circle of a radius $r_L$. After one cycle, a closed path in the parameter space is formed, and as such the integral of the non-Abelian vector potential gives rise to a non-Abelian geometric phase that is gauge invariant.

A. Perspective from the laboratory frame

As shown in Fig. 4, we slowly displace the laser beams from the starting point $E$ along the circle $C_L$ clockwise in the laboratory frame, with the initial $(t = 0)$ direction of the laser displacement in the $z$ direction. The moving speed of the laser beams is still denoted by $v_d$. The initial state in the laboratory frame is again chosen as that in Eq. (12).

In a circular laser displacement process, the direction of the laser displacement always changes, with an angular velocity $v_d/r_L$. At time $t$, the laser beams are moving in the direction $\sin\left(\frac{v_d}{r_L}t\right)\hat{e}_x + \cos\left(\frac{v_d}{r_L}t\right)\hat{e}_z$. Let $s = v_d t$, at time $t$ we have

$$\frac{dx}{ds} = \sin\left(\frac{v_d}{r_L}t\right),$$

$$\frac{dz}{ds} = \cos\left(\frac{v_d}{r_L}t\right).$$

(32)

Combining Eqs. (13), (20) and (32), one can easily find the evolution of $(c_1, c_2)$ at time $t$ satisfies

$$i\frac{d}{dt}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -\hbar \kappa v_d \begin{pmatrix} \cos\left(\frac{v_d}{r_L}t\right) & \sin\left(\frac{v_d}{r_L}t\right) \\ -\sin\left(\frac{v_d}{r_L}t\right) & \cos\left(\frac{v_d}{r_L}t\right) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$  

(33)

As explained in the previous section for $k_0 = 0$, the dynamical phase $\phi_d$ given by Eq. (14) is just an overall phase for any internal state in the dark subspace. $\phi_d$ can hence be neglected.

Interestingly, though Eq. (33) clearly describes the effect of a changing non-Abelian vector potential, it however assumes the very same form as a time-dependent Schrödinger equation of a two-level system. As a result its solution can be obtained by treating Eq. (33) as a Schrödinger equation for a time-dependent pseudo Hamiltonian

$$H_p(t) = -\hbar \kappa v_d \begin{pmatrix} \cos\left(\frac{v_d}{r_L}t\right) & \sin\left(\frac{v_d}{r_L}t\right) \\ -\sin\left(\frac{v_d}{r_L}t\right) & \cos\left(\frac{v_d}{r_L}t\right) \end{pmatrix}.$$  

(34)
The slow movement of the laser beams induces a slow change of this pseudo Hamiltonian. The eigenstates of $H_p(t)$, denoted $|1(2)\rangle$, are

$$
|1\rangle = |g_{k(t)}^+\rangle; \\
|2\rangle = |g_{k(t)}^-\rangle;
$$

(35)

where $|g_{k(t)}^{+(-)}\rangle$ are the internal state defined in Eq. (11), with the angle between the $x$ axis and $k(t)$ given by

$$
\varphi_{k(t)} = \frac{\pi}{2} - \frac{v_d}{r_L}t. \tag{36}
$$

The energy eigenvalues of $|1(2)\rangle$ as eigenstates of $H_p(t)$ are

$$
E_1 = -\hbar \kappa v_d; \\
E_2 = \hbar \kappa v_d. \tag{37}
$$

Since $E_1 - E_2 \neq 0$, the two eigenstates $|1(2)\rangle$ are necessarily non-degenerate. Further, because at $t = 0$, the initial state we adopt here (given by Eq. (12)) is exactly the eigenstate $|1\rangle$ with $\varphi_{k(t)} = \pi/2$, the time evolution of this initial state with a slowly changing pseudo Hamiltonian $H_p(t)$ becomes a non-degenerate adiabatic problem. That is, so long as the change of this pseudo Hamiltonian is sufficiently slow, then the adiabatic theorem for a non-degenerate spectrum guarantees that the time-evolving state will remain on the instantaneous eigenstate $|g_{k(t)}^+\rangle$ afterwards.

Roughly speaking, a sufficiently slowly changing “$H_p(t)$” requires

$$
\frac{|\langle 1(t)|2(t)\rangle|}{|E_1(t) - E_2(t)|} \ll 1. \tag{38}
$$

Substituting Eqs. (35), (11) and (37) into Eq. (38), we find that the above adiabatic condition, when applied to the pseudo Hamiltonian $H_p(t)$, reduces to

$$
r_L \gg \frac{1}{\kappa}. \tag{39}
$$

If this additional condition is satisfied in the laser field manipulation, then according to the standard adiabatic theorem for a non-degenerate spectrum, at time $t_F$ when an arbitrary point $F$ along the circle is reached, the initial state $|\hat{g}_{k(t=0)}^+\rangle(\varphi_{k(t=0)} = \frac{\pi}{2}) = (1, 0)^T$ will evolve to the instantaneous eigenstate $|g_{k(t_F)}^+\rangle$ multiplied by a geometric phase factor, namely,

$$
|\Psi_F\rangle = |g_{k(t_F)}^+\rangle(\varphi_{k(t_F)} = \alpha)e^{i\beta} = \frac{1}{2} \left(1 - ie^{i\alpha}(1 - i + e^{i\alpha}) e^{i\beta}, \tag{40}
$$

where $\alpha$ is the angle between $x$ axis and the tangent line of the circle at the ending point $F$ shown in Fig. 4. The phase $\beta$ is given by the following expression that is characteristic of an Abelian geometric phase,

$$
\beta = \int_{\varphi = \alpha}^{\varphi = \frac{\pi}{2}} \langle g_{k}^+ | \frac{\partial}{\partial k} | g_{k}^+ \rangle \cdot dk. \tag{41}
$$

Because this geometric phase arises from considering a pseudo Hamiltonian, it could be called as a pseudo Berry-like Abelian geometric phase. Recalling that the two-component final state given by Eq. (40) after moving the laser beams along one cycle is the result of non-Abelian geometric phases (33) imprinted onto the initial state $(1, 0)^T$, all the components $1 - ie^{i\varphi_k}$, $-i + e^{i\varphi_k}$ and the phase $\beta$ comprise the total effect of the non-Abelian vector potential. In this sense $\beta$ is just one part of a total non-Abelian geometric phase.

Figure 4 illustrates what can be predicted from this picture afforded by the pseudo Hamiltonian defined above. When $t = 0$ the laser beams are displaced in the $z$ direction and the initial state is $|\hat{g}_{k(t=0)}^+\rangle(\varphi_{k(t=0)} = \frac{\pi}{2})$; when the ending point $F$ is reached at $t = t_F$, the laser beams are displaced in the indicated $\alpha$ direction and the final state is $|\hat{g}_{k(t=t_F)}^+\rangle(\varphi_{k(t=t_F)} = \alpha)$. The group velocity of the time-evolving state can be obtained using the same method as that used for deriving Eqs. (19) and (29). It is found that the magnitude of the group velocity is fixed at $\hbar \kappa/m$, but its direction changes from $\varphi_{k(t=0)} = \pi/2$ to $\varphi_{k(t=t_F)} = \alpha$. Thus, the instantaneous laser displacement direction
always coincides with the group velocity of the wavepacket. Now if the speed of laser displacement \( v_d \) is a constant, then one deduces that the atom wavepacket must dance on a circle \( C_A \) with a different radius \( r_A \), which satisfies
\[
\frac{r_A}{r_L} = \frac{\hbar \kappa}{mv_d}.
\] (42)

This also suggests that one can easily change the final group velocity of a wavepacket. For example, if we need to change the initial group velocity \( v_i \) in the \( z \) direction to the \( \alpha \) direction as shown in Fig. 4, we can displace the laser beams along the circle \( C_L \) to point \( F \) and then along the tangent line at point \( F \).

All these predictions have been verified by our numerical simulations. In particular, in our simulations the initial state is chosen as a Gaussian wavepacket instead of a plane wave considered in Eq. (12). As an example we initially locate the wavepacket at \( x = -50/\kappa, z = 0 \). The laser beams are displaced with a velocity \( 1.5\hbar \kappa/m \) along the circle \( C_L \) of radius \( r_L \approx 75/\kappa \). As shown in Figs. 5(a) and 5(c), the wavepacket indeed dances on a circle of radius \( r_A \approx 50/\kappa \), satisfying the relation (42). By contrast, Fig. 5(b) depicts the wavepacket at \( t = 50m/(\hbar \kappa^2) \), if the laser beams are displaced not along the circle \( C_L \) but in the \( z \) direction. The vertical displacement of the wavepacket indeed shows that the initial group velocity of the initial state is in the \( z \) direction. Figure 5(d) depicts the wavepacket after the lasers are displaced along the circle \( C_L \) to point \( F \) and then along the tangent line for a duration of \( 50m/(\hbar \kappa^2) \). The result in Fig. 5(d) demonstrates that the final group velocity is indeed in the direction of the tangent line at point \( F \).

### B. Perspective from a different frame of reference

In the previous subsection we have argued that one part of a non-Abelian geometric phase, namely, the \( \beta \) phase factor in Eq. (41), can be regarded as a pseudo Abelian geometric phase. In this section, we revisit this connection between a non-Abelian geometric and an Abelian geometric phase in the laser frame of reference. As shown below, in this new frame of reference the pseudo Abelian geometric phase \( \beta \) becomes a truly Abelian geometric phase, i.e., an Abelian geometric phase without introducing a pseudo Hamiltonian.

As seen before, in the laser frame the wavevector for the spatial part of the total wavefunction becomes \( \mathbf{k}_{0,\mathrm{laser}} = -v_d m/\hbar \). Note that now the direction of the laser beam displacement is continuously changing with time. That is,
\[
\mathbf{v}_d = v_d \left[ \cos \left( \frac{v_d}{r_L} t \right) \hat{e}_z + \sin \left( \frac{v_d}{r_L} t \right) \hat{e}_x \right],
\] (43)
with its magnitude fixed at \( v_d \). Note also that in the laser frame the wavevector for the translational part of the total wavefunction should be given by \( \mathbf{k}_{0,\text{laser}} \). One then finds that in the laser frame the total Hamiltonian in Eq. (10) can be explicitly written as

\[
H_{D}^{\text{eff}} = \frac{\hbar^2 (k^2_{0,\text{laser}} + \kappa^2)}{2m} - \hbar \kappa v_d \left( \frac{\cos(\frac{\mathbf{k}_{0,\text{laser}}}{m} t)}{\sin(\frac{\mathbf{k}_{0,\text{laser}}}{m} t)} - \cos(\frac{\mathbf{k}_{0,\text{laser}}}{m} t) \right).
\]

The above Hamiltonian assumes the very same form, except the constant term \( \frac{\hbar^2 (k^2_{0,\text{laser}} + \kappa^2)}{2m} \), as the pseudo Hamiltonian in Eq. (34). Clearly then, the adiabatic evolution of the eigenstate of \( H_{D}^{\text{eff}} \) will require exactly the same adiabatic condition as Eq. (39). Under this adiabatic condition, the final result of the adiabatic process viewed in the laser frame should be given by the state in Eq. (40) multiplied by a component due to \( \mathbf{k}_{0,\text{laser}} \), i.e.,

\[
|\Psi_{f,\text{laser}}\rangle = \frac{1}{2} \left( 1 - ie^{i\alpha} \right) e^{i\beta} e^{i\mathbf{k}_{0,\text{laser}} \cdot \mathbf{R}}.
\]

Returning to the laboratory frame, the wavevector for the spatial part of the total wavefunction changes from \( \mathbf{k}_{0,\text{laser}} \) to \( \mathbf{k}_0 = 0 \), so the above state \( |\Psi_{f,\text{laser}}\rangle \) reduces exactly to that obtained in Eq. (40). The two perspectives from different frames of reference are thus in agreement. Because here \( H_{D}^{\text{eff}} \) stands for a real Hamiltonian for the translational motion in the laser frame, the pseudo geometric phase \( \beta \), which is part of the total non-Abelian geometric phase, can now be understood as a truly Abelian geometric phase in the laser frame. Thus an interesting relation between non-Abelian and Abelian geometric phases is established, i.e., in some cases, part of non-Abelian geometric phase can be interpreted as an Abelian geometric phase from a different frame of reference.

V. CONCLUDING REMARKS

To conclude, we have considered two interesting extensions of our previous study \cite{15} of quantum control of matter waves in tripod-scheme cold-atom systems. The results have enhanced our understandings on a number of issues and should motivate further experimental and theoretical interests in the dynamics of tripod-scheme cold atoms.

In particular, by displacing the laser beams along a square, it is shown that the effect of a non-Abelian vector potential can split an initial wavepacket into \( 2^n \) copies that form a coherent square lattice. It is also shown that even a triangular lattice of cold-atom sub-wavepackets can be formed by moving the laser beams along a triangle. Coherence between the sub-wavepackets thus generated makes them potentially useful for atom optics applications. We further discussed how to clearly manifest a non-Abelian Aharonov-Bohm effect by comparing the wavepacket profiles obtained by clockwise and counter-clockwise paths along a square. Examining the same process from a different frame of reference that moves with the laser beams, we find a remarkable connection between an integral of a non-Abelian vector potential (which becomes a gauge-independent non-Abelian geometric phase for cyclic processes) and simple dynamical phase expressions in the laser frame.

By displacing the laser beams along a circle, we show that a non-Abelian vector potential effect can force the atom wavepacket to dance on a circle of a different radius and can also redirect the propagation direction of that wavepacket. Comparing the perspective in the laboratory frame and in the frame that moves with the laser beams, it is shown that one part of the associated non-Abelian geometric phase in one frame of reference can be understood as an Abelian geometric phase in another frame of reference.

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This is clear, because the actual mechanical momentum is shifted from $-i\hbar \nabla$ by an effective vector potential derived in Ref. [7].