Augmented Synchronization of Power Systems

Peng Yang, Feng Liu, Senior Member, IEEE, Tao Liu, Member, IEEE, and David J. Hill, Fellow, IEEE

Abstract—Power system transient stability has been translated into a Lyapunov stability problem of the post-disturbance equilibrium for decades. Despite substantial results, conventional theories suffer from the stringent requirement of knowing the postdisturbance equilibrium a priori. In contrast, the wisdom from practice, which certifies stability by only the observation of converging frequencies and voltages, seems to provide an equilibrium-independent approach. Here, we formulate empirical wisdom by the concept of augmented synchronization and aim to bridge such a theory–practice gap. First, we derive conditions under which the convergence to augmented synchronization implies the convergence to the equilibrium set, laying the first theoretical foundation for empirical wisdom. Then, we reveal from what initial values the power system can achieve augmented synchronization. Our results open the possibility of an equilibrium-independent power system stability analytic that redefines the nominal motion as augmented synchronization rather than a certain equilibrium. Single-machine examples and the IEEE 9-bus system verify our results and illustrate promising implications.

Index Terms—AS-detectability, augmented synchronization, power system transient stability, region of attraction (RoA).

I. INTRODUCTION

TRANSIENT stability underlies functional operations of modern power grids, which usually span thousands of kilometers in open land and always suffer from various types of disturbances. It refers to the ability of a power system, for a given initial operating condition, to regain a (new) state of operating equilibrium after being subjected to a large disturbance such as short-circuit faults and sudden large load changes [1]. Although this descriptive definition covers the essence of transient stability, its interpretation diverges for theorists and engineering practitioners.

Theoretically, transient stability has been translated into the equilibrium stability problem in the sense of Lyapunov for decades [2]. Under this framework, a set of ordinary differential equations (ODEs) or differential algebraic equations (DAEs) are used to describe the postdisturbance dynamics of a power system [1]. Theorists are interested in whether a postdisturbance equilibrium is Lyapunov asymptotically stable, and if so, what is the region of attraction (RoA), i.e., from which initial point the system solution can converge to this equilibrium. This idea has led to the so-called direct methods that are based on Lyapunov functions or energy functions [3, 4, 5, 6] and can directly assess transient stability without time-consuming simulations. Despite substantial results, such theories assume that the postdisturbance equilibrium is known a priori. However, disturbances such as line tripping will alter the system dynamics, and hence, the postdisturbance equilibrium generally differs from the predisturbance one. Obtaining this new equilibrium in advance can be quite difficult in practice since it requires solving a huge set of nonlinear equations and multiple solutions may exist. The requirement of prior knowledge of the postdisturbance equilibrium can be even unrealistic when the postdisturbance equilibrium depends on initial points. In some cases, system trajectories may converge to an equilibrium set, but none of the equilibria is asymptotically stable, and hence, conventional direct methods fail. Indeed, taking a single stable equilibrium as the subject greatly restricts the capability of conventional transient stability theories. A rudimentary example will be shown in Section V, where these challenging issues can arise simply from interactions among subsystems.

Fortunately, power system engineering practitioners have already found an intuitive and effective way to circumvent the aforementioned dilemma. Instead of an equilibrium-dependent stability concept, they often interpret and assess transient stability differently. After a large disturbance, if all frequencies synchronize to around the nominal value (50 or 60 Hz), and all
voltages converge to steady-state values within a certain safety region, then the power system will be regarded to successfully regain an operating point and, hence, is transiently stable. This practical criterion demands no information on the postdisturbance equilibrium but only the observation of converging frequencies and voltages. This feature is appealing in practice as it is impossible to monitor all state variables while the measurement of frequencies and voltages are often easy to obtain [7]. Although this empirical wisdom works well, one intriguing and important question remains: Does the convergence of only frequencies and voltages guarantee the convergence of all states? After all, it is the latter, not the former, that the transient stability actually concerns.

A mismatch between “demand” and “supply” of theories exists as well. Although equilibrium-independent transient stability analytics has been advocated in the power system community for decades, which, to the best of our knowledge, dates back to the pioneering work by Willems in 1974 [8], the progress seems stagnant. Despite several mathematical concepts beyond Lyapunov stability have been proposed, e.g., partial stability [9], set stabilization [10], and contraction analysis [11], they rarely find proper applications in power systems (studies handling phase rotational symmetry are among few exceptions [8], [12], [13]). A more favorable concept is synchronization, which has been drawing increasing attention recently [13], [14], [15], [16], [17], [18], [19]. The state of synchronization essentially unifies different possible equilibria of a power system and, thus, may enable equilibrium-independent stability analytics. However, some existing works on this topic still considered equilibrium-dependent problems [13], [14], [15]. And most works concern only synchrony among generators and are built on network-reduced ODE models that assume constant voltages [15], [16], [17], [18]. Such models cannot capture the dynamical behavior of voltages, and fall short to capture heterogeneous devices in modern power grids. Many efforts have also been put into designing proper controllers to stabilize the system to some unknown equilibrium, e.g., the well-known droop control [20] and passivity-based control [21]. Despite the fruitful results on controller design and stability analysis [22], [23], [24], they restrict the specific dynamics of the controller and the underlying device, which limits their application scope. A more compatible and equilibrium-independent theory is still in need.

A clear gap between “demand” and “supply” of theories stands before us. On the one hand, practical experience indicates that we can assess transient stability in an equilibrium-independent fashion but without knowing why. On the other hand, the equilibrium-dependent Lyapunov stability theory often encounters limitations in practice. Here, we aim to bridge this theory–practice gap by introducing the concept of augmented synchronization, which means all frequencies synchronize and all voltages converge to steady states. Inspired by the practice wisdom, we redefine the nominal motion of power systems as an augmented synchronous state instead of an equilibrium. And we reinterpret power system transient stability as convergence to augmented synchronization after a disturbance rather than to any specified postdisturbance equilibrium. Our interpretation conforms to the physical definition of power system transient stability and, more importantly, will allow equilibrium-independent analytics.

To this end, we aim to answer two questions in this article. First, under what conditions does the convergence to augmented synchronization imply the convergence of all states? The answer will provide a theoretical justification for using augmented synchronization as a possible cornerstone in a transient stability analysis and also explain why the practice wisdom works. Second, given a system, from which initial points will the solution achieve augmented synchronization? Pursuing the answer is expected to stimulate a new equilibrium-independent approach to power system stability analysis without knowing the exact postdisturbance equilibrium a priori. Our main contributions are summarized as follows.

1) We formulate the long-observed practice wisdom by introducing the concept of augmented synchronization. Previous studies on power system synchronization [14], [15], [16], [17], [18] usually adopt network-reduced ODE models and omit voltage dynamics. In contrast, our formulation is built on the structure-preserving model described by DAEs, accounting for both phase and voltage dynamics. Such formulation provides a framework to define and study synchronization among heterogeneous dynamical devices. Then, we propose sufficient conditions under which the convergence to augmented synchronization implies the convergence of all states to the equilibrium set, which we refer to as augmented synchronization detectability. This concept is weaker than the most widely-used ones such as zero-state detectability [25] and output-to-state stability [26]. The major difference is that we do not expect to detect the states converging to any specific equilibrium. This result provides the first theoretical foundation for the long-observed practice wisdom that one may assess transient stability from only the observation of augmented synchronization.

2) We establish theorems that estimate the RoA to augmented synchronization. That is, the region starting from which the solution of a power system will asymptotically converge to augmented synchronization. This result extends the classical direct methods that estimate the RoA to a given postdisturbance equilibrium by the sublevel set of a Lyapunov function or an energy function [6], [27], [28]. Here, the challenging difference is that we do not have prior knowledge of the postdisturbance equilibrium, and hence, our result concerns the convergence to a set rather than to any specific working point. Our result is also different from the set stability theory [29] because the set corresponding to augmented synchronization is often not an invariant set. This result opens opportunities for power system stability analysis by redefining the “nominal motion” of a power system as augmented synchronization, rather than a single equilibrium as has been done for decades.

Combining the two results above is expected to cultivate an equilibrium-independent power stability analytic. It will provide a method to analyze power system transient stability without knowing the postdisturbance equilibrium. It will also show the
possibility to shift the focus of power system stability from the traditional equilibrium to the more general concept of synchronization. This could be a timely and more favorable approach, as power systems are evolving into more complex systems that operate in more volatile conditions, considering the burst of renewable generation integration.

The rest of the article is organized as follows. Section II formulates the problem and introduces the concept of augmented synchronization. Section III addresses the first problem, giving conditions under which convergence to augmented synchronization implies convergence to the equilibrium set. Section IV addresses the second problem, providing methods to estimate the RoA to augmented synchronization without prior knowledge of the postdisturbance equilibrium. Section V illustrates the potential implications of our results using rudimentary examples. Finally, Section VI concludes this article.

Notations: $\mathbb{R}$ is the set of (positive) real numbers. Let \( \text{col}(x_1, x_2) = (x_1^T, x_2^T)^T \) be a column vector in $\mathbb{R}^{n+m}$ with $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^m$. For a matrix $A \in \mathbb{R}^{n \times n}$, $\text{det}(A)$ denotes the determinant of $A$. For a symmetric matrix $A \in \mathbb{R}^{n \times n}$, $A \succ (\prec) 0$ means $A$ is positive definite (resp. negative definite). $I_n \in \mathbb{R}^{n \times n}$ denotes the all-one vector. For a vector $v \in \mathbb{R}^n$, $\|v\|$ denotes the 2-norm of $v$. And for a matrix $A \in \mathbb{R}^{n \times m}$, $\|A\|$ denotes the induced 2-norm of $A$. When the context is clear, we may use 0 to denote an all-zero vector of a proper dimension.

II. PROBLEM FORMULATION

A. Power System DAEs Model

We consider the following DAEs:

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, z) \\
\dot{x}_2 &= f_2(x_1, x_2, z) \\
0 &= g(x_2, z)
\end{align*}
\]

with compatible initial conditions \((x_{10}, x_{20}, z_0)\), i.e., \(0 = g(x_{20}, z_0)\). Here

\[
\begin{align*}
f_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m &\rightarrow \mathbb{R}^{n_1} \\
f_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m &\rightarrow \mathbb{R}^{n_2} \\
g : \mathbb{R}^{n_2} \times \mathbb{R}^m &\rightarrow \mathbb{R}^m
\end{align*}
\]

are all twice continuously differentiable w.r.t. \(x_1, x_2,\) and \(z\). Here, \(x_1 \in \mathbb{R}^{n_1}\) denotes the state variables that do not appear in the algebraic equations (1c), \(x_2 \in \mathbb{R}^{n_2}\) denotes the other state variables, and \(z \in \mathbb{R}^m\) denotes the algebraic variables. For the simplicity of notation, in the case when all state variables appear in \(g\), we write \(\dot{x}_1 = 0\) and all the following results in this article still apply. Collectively, let \(x := \text{col}(x_1, x_2) \in \mathbb{R}^n\) denote the vector of all state variables where \(n := n_1 + n_2\). Let \(f := \text{col}(f_1, f_2)\). For simplicity, sometimes we may shorten \((x_1, x_2, z)\) as \((x, z)\) and may write \(f(x, z), f_1(x, z),\) and \(f_2(x, z)\) representing \(f(x_1, x_2, z), f_1(x_1, x_2, z),\) and \(f_2(x_1, x_2, z)\), respectively.

We use \((x(t; x_0, z_0), z(t; x_0, z_0))\) to denote the solutions of (1) as a function of \(t\) and initial conditions \((x_0, z_0)\). When the initial conditions are irrelevant and clear, we may shorten the notation as \((x(t), z(t))\).

The algebraic equation (1c) restricts the states of the system to evolve only in the set defined as

\[
\mathcal{G} := \{(x, z) \in \mathbb{R}^n \times \mathbb{R}^m | g(x_2, z) = 0\}.
\]

Assumption 1: There exists an open connected set \(\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^m\) such that for any \((x, z) \in \mathcal{D}, \text{det}(\partial g/\partial z) \neq 0\), where \(\overline{\mathcal{D}}\) denotes the closure of \(\mathcal{D}\).

Let

\[
\mathcal{D}_G := \mathcal{D} \cap \mathcal{G}.
\]

Define the function

\[
h(x, z) := -\left(\frac{\partial g}{\partial z}(x_2, z)\right)^{-1} \frac{\partial g}{\partial x_2}(x_2, z)f_2(x, z)
\]

then \(h\) is a continuously differentiable function on \(\mathcal{D}\) by Assumption 1 and \(f_2, g \in C^2\). Consider the following ODEs system:

\[
\begin{align*}
\dot{x} &= f(x, z) \\
\dot{z} &= h(x, z).
\end{align*}
\]

For every \((x_0, z_0) \in \mathcal{D}\), the ODEs system has a unique solution defined on some interval \([0, t_+] \subset \mathbb{R}\). Clearly, \(g(x_2, z)\) is constant along such solution and, hence, can be considered as a first integral of (4). It was shown in [27] that any solution of (4) with \((x_0, z_0) \in \mathcal{D}_G\) is equivalent to the solution of (1) in \(\mathcal{D}_G\) with the same \((x_0, z_0)\). Note, however, the maximal \(t_+\) for a solution of (4) may be finite as the corresponding solution of (1) may leave \(\mathcal{D}_G\). While for solutions of (1) that always stay in \(\mathcal{D}_G\), it is equivalent to the corresponding solutions of (4) with \(t_+ = \infty\). These indicate that the DAEs system (1) can be imbedded in the ODEs system (4) with compatible initial conditions [27]. This allows us to apply results established for ODEs systems.

Furthermore, we assume the DAEs system (1) possesses at least one equilibrium in \(\mathcal{D}_G\), which obviously is also an equilibrium of the ODEs system (4). Define the equilibrium set

\[
\mathcal{E} := \{(x, z) \in \mathcal{D}_G | f(x, z) = 0\}.
\]

Assumption 2: \(\mathcal{E}\) is not empty.

In the context of power systems, the structure-preserving power system models, such as in [30], falls within the description of (1). Specifically, the algebraic variables of such a power system are the voltage phases \(\theta\) and the voltage magnitudes \(V\) of power network buses, i.e., \(z = \text{col}(\theta, V)\). The state variables model the dynamics of electric devices that connect to the network buses, including synchronous generators (SGs), inverters, loads, etc.. The algebraic equation (1c) consists of the power flow equations that dictate the interaction among devices. Hereinafter, we will focus on the structure-preserving power systems, but the problem and the presented results can also be relevant and applied to a wider class of systems that can be modeled as (1).

Remark 1: Assumption 1 is commonly used in studies on the power systems transient stability problem [31, 32, 33, 34, 35]. Together with the twice continuously differential property of \(f\) and \(g\), it guarantees the existence and uniqueness of the solution to (1) given any initial value \((x_0, z_0) \in \mathcal{D}_G\).
In fact, Assumption 1 brings the most simple situation of a DAEs system, i.e., index-1, where the algebraic equation can be solved, at least locally, for $z$ as a function of $x$ [36]. Here, the nonsingularity is required on the closure of $D$ to simplify technical details such that limit points in $D$ are also nonsingular. Note that we do not require $D_G$ being a positively invariant set, i.e., we do not require any solution starting in $D_G$ should stay in $D_G$ for all future time. But we do restrict the permissible equilibria in $D_G$. Solutions may leave $D_G$ at the so-called impasse surface defined by $\det(D'G) = 0$, at which the singularity-induced bifurcation takes place [37]. Solutions approaching the impasse surface are typically associated with the short-term voltage collapse in power systems [38], [39], which is beyond the scope of this article.

Remark 2: The structure-preserving model refers to an important class of power system models, which was first introduced in [30] for stability analysis. It represents the original network topology explicitly and allows for heterogeneous devices and high-order controllers or market dynamics. Our results apply to any structure-preserving power systems model that compactly forms a DAEs system as (1) with the algebraic variable $z$ being the voltages phase and magnitude and the algebraic equation (1c) being the network power flow. Please refer to [32], [33], [34], and [35] for different structure-preserving power systems that can all be unified as (1).

### B. Augmented Synchronization

Traditionally, power system transient stability is analyzed in the framework of Lyapunov stability. It concerns the asymptotic convergence of all states to an equilibrium. Given an equilibrium $(x^*, z^*) \in D_G$ of (1), one is interested in from which initial point $(x_0, z_0) \in D_G$, $(x(t), z(t)) \rightarrow (x^*, z^*)$ as $t \rightarrow \infty$, assuming the existence of the solution. However, practitioners often focus on the convergence to synchronous frequencies and steady voltages. It concerns neither the convergence of other states nor to which equilibrium they converge. Here, we formulate this interesting and effective experience by introducing the concept of augmented synchronization, defined as follows.

**Definition 1**: A solution of (1) is said to be an augmented synchronous solution if for all $t \geq 0$, $(x(t), z(t)) \in D_G$ and

$$
\dot{z}(t) = \text{col}(\dot{\theta}(t), \dot{V}(t)) = 0.
$$

**Definition 2**: A solution of (1) is said to be converging to augmented synchronization if for all $t \geq 0$, $(x(t), z(t)) \in D_G$ and

$$
\dot{z}(t) = \text{col}(\dot{\theta}(t), \dot{V}(t)) \rightarrow 0, \text{ as } t \rightarrow \infty.
$$

The prepositional adjective augmented distinguishes our definition from conventional frequency synchronization. It emphasizes that not only synchronous frequencies but also steady voltages are required. Generally, frequency synchronization holds if there is a common frequency $\omega_s \in \mathbb{R}$ such that $\dot{\theta}(t) = \omega_s I_n$. By working in a rotating framework, without loss of generality, we assume $\omega_s = 0$ and the frequencies in the system model represent the deviation from the nominal frequency.

**Remark 3**: Compared with previous studies on power system synchronization that were built on ODEs, our formulation takes a different perspective. We define synchronization by the algebraic variables instead of states. Physically, that means we regard synchronization as a property of electrical sinusoidal voltages at all buses across the grid rather than a property among generators. Such formulation accounts for the voltage dynamics and provides a unified framework to incorporate heterogeneous devices such as power and current dynamics, which cannot be handled in previous studies [14], [15], [16], [17], [18], since there is no phase or voltage state in such dynamics. Our formulation seems more compatible and desirable for future power systems, considering generators are gradually giving place to inverter-interfaced heterogeneous devices.

### C. Problem Statement

With the above formulations, we now re-state the two questions that we aim to answer in this article.

Consider a solution $(x(t), z(t))$ of (1) and the following two properties of the solution.

**Property 1**: The solution stays in $D_G$ for all $t \geq 0$ and it holds that $z(t)$ is bounded and $\dot{z}(t) \rightarrow 0$, as $t \rightarrow \infty$.

**Property 2**: The solution satisfies that

$$
(x(t), z(t)) \rightarrow \mathcal{E}, \text{ as } t \rightarrow \infty
$$

where the convergence to a set is defined in the sense of the distance to the set converging to zero.

First, we are interested in under what conditions the practice wisdom, which assesses transient stability by only observation of $z(t)$, i.e., Property 1, would imply the convergence to the equilibrium set, i.e., Property 2. Obviously, the former is a necessary condition for the latter, but generally, it is not sufficient. This problem is stated as follows.

**Problem 1**: Consider a solution $(x(t), z(t))$ of (1). Under what conditions does Property 1 of the solution imply Property 2?

Note that (5) implies the convergence of $\dot{x}(t)$ to zero, i.e.,

$$
\dot{x}(t) = f(x(t), z(t)) \rightarrow 0, \text{ as } t \rightarrow \infty
$$

which has a natural physical meaning from the engineering point of view, i.e., it requires the system to reach a steady state. But generally (6) does not imply (5) if the solution is not bounded. Note also since the solution stays in $D_G$, it is equivalent to study Problem 1 in the ODEs system (4).

Second, we are interested in under what conditions the power system (1) can achieve augmented synchronization. Specifically, we consider the following problem.

**Problem 2**: From which initial point $(x_0, z_0) \in D_G$ does the solution $(x(t), z(t))$ of (1) satisfy Property 1?

In fact, the classical equilibrium-based Lyapunov stability analysis provides an answer to this question with prior knowledge of the postdisturbance equilibrium. Here, we aim to tackle this question in an equilibrium-independent way.
The first question will be addressed in Section III and the second in Section IV.

**Remark 4:** Despite the specific background of power systems, the above two problems and our results apply to general DAEs system (1) satisfying Assumptions 1–2. Considering $z(t)$ and $\dot{z}(t)$ as the output of the system, Problem 1 essentially considers how we can infer certain properties of all states by certain observations of the output. This is relevant to the output-to-state stability (see Remark 5 for a more detailed discussion). Problem 2 concerns when the output achieves a certain property. It is closely related to the theory of output regulation and partial stability (see Remark 10 for a more detailed discussion). The central concept in our article, i.e., the augmented synchronization, is also relevant to the complex coupled-oscillator networks [40], especially the phase-amplitude oscillators that include both phase and amplitude dynamics [41], [42]. While we focus on power grids, our methodology that analyzes the augmented synchronization also apply to other generic complex phase-amplitude oscillator networks, which appear in multiple disciplines [43].

**III. WHY PRACTICE WISDOM WORKS**

This section provides our results of Problem 1. We find that what underlies the practice wisdom is a widely satisfied property of power systems, which we refer to as augmented synchronization detectability. In the following part, we present a rigorous definition of this concept and several checkable criteria with illustrative examples.

**A. Augmented Synchronization Detectability**

Throughout this section, we will assume that we have observed a solution of (1) that satisfies Property 1. It can be viewed as a certain detectability-type property of the solution if Property 1 implies Property 2. We introduce the concept of augmented synchronization detectability to capture this property, which is defined as follows.

**Definition 3:** We say a solution $(x(t), z(t))$ of (1) is augmented synchronization detectable (AS-detachable) if Property 1 of the solution implies Property 2.

**Remark 5:** Augmented synchronization detectability is closely related to the well-known concepts of output-to-state stability [26] and the zero-state detectability [25]. All three concepts are relevant to the property that one can infer the behavior of all states solely based on the observation of part of states or a certain function of states, e.g., the outputs. However, compared with zero-state detectability and output-to-state stability, AS-detachability is much weaker and requires neither $z(t)$ nor $x(t)$ converging to some specified points. Instead, it only focuses on the convergence of their time derivatives $\dot{z}(t)$ and $\dot{x}(t)$ to zero, which enables equilibrium-independent analytics.

The remainder of this section is devoted to finding checkable conditions under which AS-detachability holds, which provides the practice wisdom with a solid theoretical foundation.

**B. AS-Detectability for Nondegenerate Solutions**

We begin with a simple situation where the solution is nondegenerate, defined as follows.

**Definition 4:** Suppose $(x(t), z(t))$ is a solution of (1) that stays in $D_G$ for all $t \geq 0$. We say the solution is nondegenerate if $\frac{\partial g}{\partial x^2}$ has constant a full column rank on the solution, i.e.,

$$\text{rank} \left( \frac{\partial g}{\partial x^2}(x(t), z(t)) \right) = n_2 \quad \forall t \geq 0$$

and the matrix $(\frac{\partial g}{\partial x^2})^T \frac{\partial g}{\partial z}^2$ is bounded, i.e., $\exists M > 0$ s.t.

$$\left\| \frac{\partial g}{\partial x^2}(x(t), z(t)) \frac{\partial g}{\partial z}(x(t), z(t)) \right\| \leq M \quad \forall t \geq 0$$

where $\frac{\partial g}{\partial x^2} \dagger := (\frac{\partial g}{\partial x^2} \frac{\partial g}{\partial z}^T \frac{\partial g}{\partial z}^{-1} \frac{\partial g}{\partial z}^T$ denotes the left inverse. Otherwise, we say a solution is degenerate.

For nondegenerate solutions, a direct connection between $\dot{z}$ and $\dot{x}$ holds, as stated in the following Lemma.

**Lemma 1:** Consider a nondegenerate solution of (1) that satisfies Property 1. It holds that $\lim_{t \to \infty} \dot{x}_2(t) = 0$.

**Proof:** Since the solution is nondegenerate, it follows from (3) that

$$\dot{x}_2(t) = -\left( \frac{\partial g}{\partial x^2} \right) \frac{\partial g}{\partial z}(t) \dot{z}(t). \quad (7)$$

Since $\left\| \left( \frac{\partial g}{\partial x^2} \right) \frac{\partial g}{\partial z} \right\| < \infty$, $\dot{z}(t) \to 0$ implies $\dot{x}_2(t) \to 0$. \hfill \square

If all state variables appear in the algebraic equations, i.e., $x_1 = 0$, then nondegenerate solutions with bounded $x_2$ are AS-detachable.

**Theorem 2:** Suppose $x_1 = 0$. A nondegenerate solution of (1) is AS-detachable if $x_2(t)$ is bounded for all $t \geq 0$.

**Proof:** It follows from Property 1 and the boundedness of $x_2$ that the solution $(x_2(t), z(t))$ is bounded. Hence, it follows from [44, Lemma 4.1] that the $\omega$-limit set $\Lambda$ of the solution $(x_2(t), z(t))$ is a nonempty, compact, and invariant set. And $(x_2(t), z(t))$ approaches $\Lambda$ as $t \to \infty$. Let $(x_2^*, z^*)$ be an $\omega$-limit point of the solution. There is an increasing sequence of time $\{t_n\}$ such that $\lim_{n \to \infty} t_n = \infty$ and $\lim_{n \to \infty} (x_2(t_n), z(t_n)) = (x_2^*, z^*)$. Since the solution is nondegenerate, it follows from Lemma 1 and continuity of $f_2$ that

$$\lim_{n \to \infty} \dot{x}_2(t_n) = \lim_{n \to \infty} f_2(x_2(t_n), z(t_n)) = f_2(x_2^*, z^*) = 0.$$

This indicates the $\omega$-limit set only consists of equilibria, which implies Property 2. \hfill \square

When $x_1 \neq 0$, AS-detachability can be verified given two additional conditions of $x_1(t)$.

**Theorem 3:** Suppose $x_1 \neq 0$. A nondegenerate solution of (1) is AS-detachable if the following holds.

1) $x_1(t)$ and $x_2(t)$ are bounded for all $t \geq 0$.

1Let $(x(t), z(t))$ be a solution of (4). A point $(x^*, z^*) \in D_G$ is said to be an $\omega$-limit point, or a positive limit point, of the solution if there is a sequence of $\{t_n\}$, with $t_n \to \infty$ as $n \to \infty$, such that $(x(t_n), z(t_n)) \to (x^*, z^*)$, as $n \to \infty$. The set of all $\omega$-limit points of the solution, denoted as $\Lambda^\omega$, is called the $\omega$-limit set, or the positive limit set, of the solution.
2) No solution \((x(t), z(t))\) of (1) can stay identically in \(\mathcal{M}\) other than the solutions such that \(f_1(x(t), z(t)) \equiv 0\), where \(\mathcal{M}\) is defined as

\[
\mathcal{M} := \{(x, z) \in \mathcal{D} \cap f_2(x, z) = 0\}.
\]

**Proof:** Given Property 1, it follows from Lemma 1 and condition 1) that the solution \((x(t), z(t))\) is bounded. Hence, \((x(t), z(t))\) approaches its \(\omega\)-limit set \(L^+\) as \(t \to \infty\).

For any \((x^*, z^*) \in L^+\), it follows from the same argument in the previous proof that \(f_2(x^*, z^*) = 0\). Let \(\mathcal{T}\) be the largest invariant set of \(\mathcal{M}\). Thus,

\[
L^+ \subset \mathcal{T} \subset \mathcal{M}.
\]

Condition 2) further guarantees for any \((x^*, z^*) \in \mathcal{T}\), \(f(x^*, z^*) = 0\). Since the solution approaches \(L^+\) as \(t \to \infty\), it approaches \(\mathcal{T}\) as well. Hence, the solution approaches \(\mathcal{E}\) as \(t \to \infty\), which implies Property 2. □

We next show how to use the previous conditions to verify AS-detectability in two examples that are prevalent in modern power systems: 1) an inverter-interfaced power source and 2) an SG.

**Example 1:** Consider an inverter-interfaced power source connected to the infinite bus via a transmission line, as shown in Fig. 1. We consider the dynamics of angle droop and the voltage droop control, which is widely employed in inverter-interfaced devices [45], [46], [46], [47], [48]. It regulates the output power according to the deviation of the terminal phase and voltage via first-order dynamics.

The system dynamics read

\[
\begin{aligned}
\tau_1 \dot{P} &= -P + P_{\text{ref}} - d_1(\theta - \theta_{\text{ref}}) \\
\tau_2 \dot{Q} &= -Q + Q_{\text{ref}} - d_2(V - V_{\text{ref}})
\end{aligned}
\]  

where \(P\) and \(Q\) are the terminal output active and reactive power, respectively. \(V \angle \theta\) is the terminal complex voltage. \(P_{\text{ref}}, Q_{\text{ref}}, \theta_{\text{ref}},\) and \(V_{\text{ref}}\) are prespecified constant reference values. \(\tau_1 > 0\) and \(\tau_2 > 0\) are time constants. \(d_1 > 0\) and \(d_2 > 0\) are droop coefficients. The algebraic equations read

\[
\begin{aligned}
P - G_{11}V^2 - G_{12}V \cos \theta - B_{12}V \sin \theta &= 0 \\
Q + B_{11}V^2 - G_{12}V \sin \theta + B_{12}V \cos \theta &= 0.
\end{aligned}
\]

In terms of model (1), for this case, \(x_1 = 0, x_2 = (P, Q)^T,\) and \(z = (\theta, V)^T\). \(f\) and \(g\) are given by (9) and (10), respectively.

For this simple system, the practice wisdom suggests that when we observe the terminal voltage of the inverter approaching augmented synchronization, i.e., Property 1, then we may conclude that the system recovers its operating equilibrium after being subjected to a large disturbance, i.e., Property 2. The following proposition shows that every solution of this system is AS-detectable, and hence, the practice wisdom is always true.

**Proposition 1:** Every solution of the power system in Example 1 is AS-detectable.

**Proof:** It follows from (10) that

\[
\frac{\partial g}{\partial x_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

And one can verify that \(\frac{\partial g}{\partial z}\) is bounded for any bounded \(z\). Hence, any solution that satisfies Property 1 is nondegenerate. Given bounded \(z(t)\), it follows from (10) that \(P(t)\) and \(Q(t)\), i.e., \(x_2(t)\) must be bounded. Hence, by Theorem 2, any solution is AS-detectable. □

**Example 2:** Consider an SG connected to an infinite bus via a transmission line, as shown in Fig. 2.

The system dynamics read [49]

\[
\begin{aligned}
\delta &= \omega \\
M \dot{\omega} &= -D\omega - P^e + P^g \\
T_{d0} \dot{E}_{q}' &= -\frac{x_d}{x_d} E_{q}' + \frac{(x_d - x_q)V \cos(\delta - \theta)}{x_d} + E^f
\end{aligned}
\]

where \(E_{q}' \angle \delta\) is the \(q\)-axis transient internal complex voltage and \(V \angle \theta\) is the terminal complex voltage. \(\omega\) is the frequency derivation. \(M\) is the moment of inertia. \(D\) is the damping coefficient. \(T_{d0}\) is the \(d\)-axis open-circuit transient time constant. \(x_d, x_{d}', x_q, x_{q}'\) are the \(d\)-axis synchronous reactance, \(d\)-axis transient reactance, and \(q\)-axis synchronous reactance, respectively. For a realistic SG, \(x_d > x_{d}'\). \(P^g\) and \(E^f\) are constant parameters meaning the power generation and the excitation voltage, respectively. The terminal output active power \(P^e\) and reactive power \(Q^e\) are given by

\[
P^e = \frac{V^2 \sin(2(\delta - \theta))(x_{d}' - x_q)}{2x_qx_{d}'} + \frac{E_{q}'V \sin(\delta - \theta)}{x_{d}'}
\]

and

\[
Q^e = \frac{V^2 \cos(2(\delta - \theta))(x_{d}' - x_q)}{2x_qx_{d}'} + \frac{E_{q}'V \cos(\delta - \theta)}{x_{d}'} - \frac{x_{d}' + x_q}{2x_qx_{d}'} V^2.
\]

Power balance constraints at the terminal bus give the algebraic equations

\[
\begin{aligned}
P^e - G_{11}V^2 - G_{12}V \cos \theta - B_{12}V \sin \theta &= 0 \\
Q^e + B_{11}V^2 - G_{12}V \sin \theta + B_{12}V \cos \theta &= 0
\end{aligned}
\]

where \(G_{ij}\) and \(B_{ij}\) are elements of the admittance matrix corresponding to the transmission line.
In terms of model (1), for this case, we have $x_1 = \omega, x_2 = (\delta, E_q)^T$, and $z = (\theta, V)^T$. $f$ is given by (11); $g$ is given by substituting (12) and (13) into (14).

The practice wisdom suggests that observing the terminal voltage $z(t)$ of the generator may determine the transient stability of the system. By studying the AS-detectability, the following proposition provides a theoretical guarantee for such wisdom when the system’s solution is nondegenerate.

**Proposition 2**: Every nondegenerate solution with bounded $x_2(t)$ of the power system in Example 2 is AS-detectable.

**Proof**: For condition 1) in Theorem 3, it suffices to prove $\omega(t)$ is bounded when $x_2(t)$ and $z(t)$ are bounded. It follows from (14) and the boundedness of $z(t)$ that $P^2(t)$ is bounded. Therefore, there exists a real number $a > 0$ such that $|P^2(t)| < a$ for all $t \geq 0$. Substituting the bound into (11) yields that for all $t \geq 0$

$$-D\omega(t) + P^2 - a \leq M\omega(t) \leq -D\omega(t) + P^2 + a.$$  

(15)

Consider the following two first-order linear ODEs:

$$M\ddot{\omega}(t) = -D\dot{\omega}(t) + P^2 + a,$$

and

$$M\ddot{\omega}(t) = -D\dot{\omega}(t) + P^2 - a,$$ where $\ddot{\omega}(t) = \omega(t)$.

Their solutions are

$$\omega(t) = \left( \omega_0 - \frac{P^2 + a}{D} \right) e^{-\frac{a}{D}t} + \frac{P^2 + a}{D},$$

and

$$\omega(t) = \left( \omega_0 - \frac{P^2 - a}{D} \right) e^{-\frac{a}{D}t} + \frac{P^2 - a}{D},$$

respectively. It then follows from (15) and the comparison lemma [44, Lemma 3.4] that

$$\omega(t) \leq \omega(t) \leq \omega(t), \quad t \geq 0.$$  

Since $\omega(t)$ and $\omega(t)$ are bounded on $t \in [0, \infty)$, we conclude that $\omega(t)$ is bounded on $t \in [0, \infty)$, and hence condition 1) in Theorem 3 is satisfied.

For condition 2) in Theorem 3, it suffices to prove $f_1 = \dot{\omega}(t) = 0$ on the largest invariant set of $M$. Since $\delta(t) = \omega(t) = 0$ on $M$, it holds that $\omega(t) \equiv 0$ on the invariant set of $M$. Hence, $\omega(t) = 0$ on the invariant set of $M$. Hence, it follows from Theorem 3 that every nondegenerate solution of this system is AS-detectable.

**C. Compositional Conditions for Modularly Structured Power Systems**

Although we have only illustrated the usage of previous conditions on two single-machine systems, it should be noted that those conditions apply to general large-scale power systems. However, it might be a difficult task to determine the boundedness of $x_1(t)$ and the invariant set in $M$ for a general large system. In this section, we will propose a compositional approach to AS-detectability for modularly structured power systems, which requires only conditions of each low-dimensional subsystem.

Suppose a large-scale power system consisting of finite $n$ buses. Let $i \in \{1, \ldots, n\}$ be the bus index. We say the power system is modularly structured if there is a decomposition of variables and functions, denoted as $x_{1,i}, x_{2,i}, z_i, f_{1,i},$ and $f_{2,i}$ for $i = 1, \ldots, n$, such that: $x_i = co(x_{1,1}, \ldots, x_{1,n}), x_{2,i} = co(x_{2,1}, \ldots, x_{2,n}), z_i = co(z_1, \ldots, z_n)$, $f_{1,i} = co(f_{1,1}, \ldots, f_{1,n})$, and $f_{2,i} = co(f_{2,1}, \ldots, f_{2,n})$; and for each $i$, $f_{1,i}$ and $f_{2,i}$ are only functions of local variables, i.e., $f_{1,i}(x_{1,i}, x_{2,i}, z_i)$ and $f_{2,i}(x_{1,i}, x_{2,i}, z_i)$.

Hence, for a modular structure power system, (1a) and (1b) can be equivalently written as the composition of subsystems: for $i = 1, \ldots, n$

$$\dot{x}_{1,i} = f_{1,i}(x_{1,i}, x_{2,i}, z_i)$$

$$\dot{x}_{2,i} = f_{2,i}(x_{1,i}, x_{2,i}, z_i)$$  

(16)

Note, however, that all subsystems are still coupled via the algebraic equation $0 = g(x_{2,i}, z_i)$. With a little abuse of notation, we allow $x_{1,i} = 0$ and $f_{1,i} = 0$ for some $i$ to adapt to some subsystems, state variables of which are all involved in the algebraic equations.

The following theorem states that for a modular structure power system, conditions for AS-detectability of the entire system (1) can be decomposed into conditions of each subsystem (16).

**Theorem 4**: Suppose the power system (1) is modularly structured. Every nondegenerate solution with bounded $x_{2,i}(t)$ of (1) is AS-detectable if for each $i$ with $x_{1,i} \not\equiv 0$ the subsystem (16) regarding $z_i(t)$ as input satisfies the following.

1) The boundedness of $z_i(t)$ and $x_{2,i}(t)$ implies the boundedness of $x_{1,i}(t)$;

2) For any bounded input $z_i(t)$, no solution $x_{1,i}(t)$ of (16) exists, other than the solutions that $f_{1,i}(x_{1,i}(t), z_i(t)) \equiv 0$, such that $(x_{1,i}(t), z_i(t))$ stays identically in $\mathcal{M}_i$, where $\mathcal{M}_i$ is defined as

$$\mathcal{M}_i := \{ (x_i, z_i) \in \mathbb{R}^{n_i} \times \mathbb{R}^2 | f_{2,i}(x_i, z_i) = 0 \}.$$

**Proof**: Consider a nondegenerate solution $(x(t), z(t))$ of (1) that satisfies Property 1 and has bounded $x_{2}(t)$. It follows from Lemma 1 that $f_{2}(t) \rightarrow 0$ as $t \rightarrow \infty$. Hence, for each $i$, $z_i(t)$ and $x_{2,i}(t)$ are bounded and $f_{2,i}(t) \rightarrow 0$. By 1), $x_{1,i}(t)$ is bounded, and hence, $x_{1}(t)$ is bounded. It follows from the modular structure that $x_{1}(t)$ is also a solution to (16) under bounded input $z_i(t)$. Hence, by a similar argument, 2) implies that if the $\omega$-limit set of $(x_i(t), z_i(t))$ must be the equilibrium set, which implies Property 2. □

Theorem 4 provides a compositional approach to verify AS-detectability for large-scale modular structure power systems. Since conditions 1) and 2) are local and depend on subsystem $i$ only, it makes no difference to check them in an interconnected system or in the single-machine-infinite bus setting.

The modular structure is ubiquitous in power systems, which are widely studied in the study of power system stability (see, for example, [49], [50], [51]). Physically, this means each subsystem is controlled independently and interacts with others only by power flows. When interarea control is employed, one may still treat an independent area as one modular block. An example...
of using this modular property to verify AS-detectability of the IEEE 9-bus system will be presented in Section V-B.

Remark 6: By similar arguments as in Examples 1–2, one may verify that other power system dynamical devices satisfy conditions in Theorem 4. Combined with the ubiquitous modular structure of power systems, we conjecture that AS-detectability, at least for nondegenerate solutions, is a common property of power systems.

Remark 7: Technically, a solution of (1) that converges to an unstable equilibrium is, indeed, a solution that satisfies both Properties 1 and 2. Nevertheless, the probability of such a solution taking place in practice is zero, since the stable manifold of an unstable equilibrium has zero measure in the state space [52], [53]. Hence, an infinitesimal perturbation can cause the solution to diverge from the unstable equilibrium. This idea has led to the theory of a quasi-stability region, which practically regards the stable manifold of unstable equilibrium in the interior of the stability region, as part of the quasi-stability region [54], [55]. This fact, together with AS-detectability, provides a theoretical explanation for the long-observed practice wisdom that one may assess transient stability in practice solely based on the observation of augmented synchronization.

D. Physical Interpretation of Degeneration

Our previous results studied AS-detectability for nondegenerate solutions. Unfortunately, we cannot prove that nondegeneration is necessary for AS-detectability for a general power system, which will remain as our future work. This subsection will provide an interpretation of degeneration in the context of power systems, which helps to build insightful intuitions of AS-detectability.

It is reasonable to assume that $\frac{\partial g}{\partial z}$ is bounded when the solution is bounded, in most power system models. Therefore, degeneration as defined in Definition 4 often takes place when $\frac{\partial g}{\partial z}$ has a deficient column rank. This indicates that an infinitesimal change of $x_2$, given by the null space of $\frac{\partial g}{\partial z}$, will not change the value of the function $g$. Physically, since $g$ represents the power balance at buses, degeneration implies the terminal output power of some devices is locally irrelevant to their internal states $x_2$.

To further illustrate, consider for example the single-machine infinite bus power system as described in Example 2. It follows from (12) to (14) that

$$\frac{\partial g}{\partial x_2} = \begin{bmatrix} V^2 (x'_d - x_q) \cos(2(\delta - \theta)) + \frac{E'_q}{x_q} V \cos(\delta - \theta) - \frac{E'_q}{x_q} V \sin(\delta - \theta) \\ -\frac{V^2 (x'_d - x_q)}{x'_d} \sin(2(\delta - \theta)) \end{bmatrix}.$$  

It yields

$$\det \left( \frac{\partial g}{\partial x_2} \right) = \frac{V^2}{x'_d} \left( \frac{x'_d - x_q}{x_q} V \cos(\delta - \theta) + \frac{E'_q}{x_q} \right).$$

Hence if $V \neq 0$ and $\frac{x'_d - x_q}{x_q} V \cos(\delta - \theta) + \frac{E'_q}{x_q} \neq 0$ on the solution, the solution is nondegenerate if it is bounded.

Two possibilities of degeneration appear. First, $V = 0$ takes place at some point on the closure of the solution. Since $V = 0$ yields $\det(\frac{\partial g}{\partial x_2}) = 0$ as well, Assumption 1 does not hold for this solution either. Physically, this would only happen when the generator’s terminal undergoes a purely metallic short-circuit fault. It is natural that for solutions satisfying $V(t) = 0$ or $V(t) \to 0$, no conclusion of $x(t)$ can be made, even though we observe the convergence of $z(t)$.

Now we assume $V > 0$ and turn to the second possibility of degeneration, i.e.,

$$\frac{x'_d - x_q}{x_q} V \cos(\delta - \theta) + \frac{E'_q}{x_q} = 0. \quad (17)$$

Substituting (17) into (12) and (13), we obtain $P^e = 0$ and $Q^e = -\frac{1}{x_q} V^2$. This indicates that the terminal output power of the generator becomes independent of the system state $E'_q \angle \delta$, in which case the detectability from output to state is lost.

Similar results can be made for large-scale modular structured power systems that degeneration takes place when there exists one device whose output power becomes totally irrelevant to its internal states. Such scenarios should be rare in practical power systems.

E. Discussion on the Boundedness of $x_2(t)$

In some scenarios, the boundedness of $x_2$ could come from physical and engineered limits. For example, variables representing electrical quantities such as voltage, current, and electric power are naturally finite in real systems. Because $x_2$ and $z$ are closely related by the algebraic constraints (1c), sometimes the boundedness of $z$ and $z \to 0$, i.e., Property 1, can directly imply the boundedness of $x_2(t)$, as shown in Example 1. While in some cases, there may exist a solution that satisfies Property 1 but has unbounded $x_2(t)$. Consider for example the power system in Example 2. The algebraic constraints (12)–(14) cannot enforce $E'_q$ to be bounded if $V \to 0$. But $V \to 0$ violates Assumption 1 and can be excluded. It will be our future work to establish a more general approach to determining the boundedness of $x_2(t)$, which might require additional structure conditions imposed on the algebraic constraints (1c).

IV. WHEN AUGMENTED SYNCHRONIZATION

This section aims to answer Problem 2, i.e., under what conditions a power system can reach augmented synchronization? To this end, we consider a more general problem: given an output function $\eta(x, z)$, starting from what initial value $(x_0, z_0) \in D_G$, the solution of (1) would satisfy $\eta(x(t), z(t)) \to 0$ as $t \to \infty$. Clearly, solving the problem with $\eta = h$ would answer our second question. However, the choice of $\eta$ is not restricted to $h$, as long as $\eta \to 0$ implies $h \to 0$, which provides additional flexibility. For example, it is sometimes more convenient to set $\eta = f_2$ in power systems as will be shown in Section V.

Based on this idea, we introduce the RoA to $\eta(\eta$-RoA) defined as follows.

Definition 5: Consider the power system (1). Given a function $\eta : D \to \mathbb{R}^{n_\eta}$ for some dimension $n_\eta$, the $\eta$-RoA of (1) is
A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to be a $C$ function if it is strictly increasing and $\alpha(0) = 0$ [44].

We establish three theorems to provide estimations of $\eta$-RoA via three types of $V$-functions, which differ in specific requirements of $V$ and $\dot{V}$.

### A. Type I

**Theorem 6:** If there exists a scalar $C^1$ function $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, a vector function $\xi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ for some dimension $n_G$, and $K$ functions $\alpha$, $\beta$, and $\gamma$ such that for every $(x, z) \in D_G$, the following holds.

1. $\alpha(||\xi(x, z)||) \leq V(x, z) \leq \beta(||\xi(x, z)||)$.

2. $\dot{V}(x, z) \leq -\gamma(||\xi(x, z)||)$.

Then, for any $l \in \mathbb{R}$ such that $V_{l}^{-1} \subset D_G$, where $V_{l}^{-1}$ is defined as in (18), $V_{l}^{-1}$ is positively invariant and is an estimation of the $\eta$-RoA. Moreover, if on $D_G$, $||z|| \to \infty$ implies $V \to \infty$, then $z$ is bounded on $V_{l}^{-1}$.

**Proof:** Since $V$ is continuous and nonincreasing on $D_G$, $V_{l}^{-1}$ is positively invariant under (1). Hence, any solution starting in $V_{l}^{-1}$ will stay in $D_G$ for all $t \geq 0$. Since $V(x(t), z(t))$ is nonincreasing and bounded from below by zero, $V(x(t), z(t))$ must have a finite limit $v \geq 0$ as $t \to \infty$.

Now we prove $v = 0$. If $v > 0$, by 1), we have $0 < v \leq \beta(||\xi||)$ for all $t \geq 0$, which implies $||\xi|| \geq \beta^{-1}(v) > 0$. Hence, by 2), it holds that for all $t \geq 0$

$$\dot{V} \leq -\gamma(||\xi||) \leq -\gamma(\beta^{-1}(v)) < 0.$$  

Hence, for all $t \geq 0$

$$V(x(t), z(t)) = V(x_0, z_0) + \int_0^t \dot{V}(x(\tau), z(\tau))d\tau \leq V(x_0, z_0) - \gamma(\beta^{-1}(v))t.$$  

The right-hand side will eventually become negative, which contradicts $v > 0$. Hence, it holds that $0 \leq \alpha(||\eta||) \leq V \leq \infty$, as $t \to \infty$. It follows from the sandwich theorem that $\eta \to 0$, as $t \to \infty$, which proves the first claim.

For the second claim, it directly follows that for every finite $l$, $z$ must be bounded on $V_{l}^{-1}$, otherwise, $V \to \infty$ and $l$ cannot be bounded.

**Remark 9:** In usual Lyapunov-like functions [44], one would require the bounds of $V$ and $\dot{V}$ are $K$ functions of the same variable, e.g., $\alpha(||\eta||) \leq V \leq \beta(||\eta||)$ and $\dot{V} \leq -\gamma(||\eta||)$, which can be viewed as a special case of Theorem 6 when setting $\xi = \eta$. In that case, Theorem 6 implies $\{(x, z) \in V_{l}^{-1}||\eta(x, z)|| = 0\}$ is a positively invariant set. In fact, given $(x_0, z_0) \in V_{l}^{-1}$ and $\xi(x_0, z_0) = 0$, since $V$ is nonincreasing, 1) implies that for all $t \geq 0$

$$0 \leq V(x(t), z(t)) \leq V(x_0, z_0) \leq 0.$$  

Hence, $\{(x, z) \in V_{l}^{-1}||\eta(x, z)|| = 0\}$ is positively invariant. However, the set $\{(x, z) \in V_{l}^{-1}||\eta(x, z)|| = 0\}$ is often not positively invariant for power systems when $h = 0$ or $h = f_2$. For example, consider the simple power system in Example 2, in which case the set $\{f_2 = 0\} = \{\omega = 0, E_q' = 0\}$ is not invariant since on the set there exist points so that $\omega \neq 0$. Hence, to admit power systems applications, we introduce $\xi$ as a relaxation. Note that by
1) \( \xi = 0 \) implies \( \eta = 0 \). Hence, \( \{(x, z) \in V^{-1}_1|\xi(x, z) = 0\} \subset \{(x, z) \in V^{-1}_2|\eta(x, z) = 0\} \), which admits the former smaller set being positively invariant while the latter larger set not.

**Remark 10:** Theorem 6 is, to the best of our knowledge, a new result even though its formulation is based on similar concepts in the study of partial stability [56] and the pioneering work of Willems [8]. However, our result differs from partial stability, since we focus on the convergence of \( \eta \) rather than part of state variables. It also differs from Willems’s condition, as we do not require that \( V(x, z) \) vanish at the origin, which allows for equilibrium-independent analysis. Nevertheless, one important common feature among them is that they all differ from the classical Lyapunov function that requires \( V \) to be positive definite w.r.t. certain equilibrium. In fact, a function \( V(x, z) \) satisfying condition 1) in Theorem 6 is not necessarily a positive-definite function in \( D \), or even in \( D_G \); since we allow it to vanish when \( \xi(x, z) = 0 \). As will be illustrated in Section V, this property enables analysis without the prior knowledge of the targeted equilibrium.

A natural candidate for a type-I \( V \)-function is given in Krasovskii’s form as follows:

\[
V(x, z) = f(x, z)^T P f(x, z)
\]

where \( P = P^T > 0 \) is a constant positive-definite matrix.

It meets the condition 1) in Theorem 6 with \( \xi = f, \eta = f_z \), and 
\[
\lambda_{\min}(P)\|f_z\|^2 \leq V \leq \lambda_{\max}(P)\|f\|^2,
\]

where \( \lambda_{\min}(P) \) and \( \lambda_{\max}(P) \) denote the minimal and the maximal eigenvalue of \( P \), respectively. It yields

\[
\dot{V} = \frac{\partial V}{\partial x} \dot{x} + \frac{\partial V}{\partial z} \dot{z} = f^T(PJ(x, z) + J(x, z)^T P) f
\]

where

\[
J(x, z) = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial z} \left( \frac{\partial \eta}{\partial z} \right)^{-1} \frac{\partial \eta}{\partial x}.
\]

If one can verify that for all \((x, z) \in D_G\)

\[
PJ(x, z) + J(x, z)^T P < 0
\]

then \( V \) meets the condition 2) in Theorem 6.

In fact (20) can be relaxed in case \( \xi \neq f \). Section V will provide an example of using this type-I \( V \)-function to estimate the \( f_z \)-RoA of a power system.

**Remark 11:** Regions satisfying condition (20) is related to the concept of contracting region in the study of contraction analysis in DAEs systems [57]. One may further allow \( P \) to be state-dependent and the positive-definite matrix \( P(x, z) \) can serve as a metric of the space.

**B. Type II**

In some cases, it may be difficult to obtain a sign-definite bound condition on \( V \) as required in Theorem 6. It can be relaxed by restricting \( V \) to employ Barbalat’s lemma.

**Theorem 7:** If there exists a scalar \( C^1 \) function \( V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \), and a \( K \) function \( \gamma \) such that the following holds.

1) \( V(x, z) \) is bounded from below on \( D_G \).

2) \( \dot{V}(x, z) \leq -\gamma(|\eta(x, z)|) \) for every \((x, z) \in D_G\).

3) \( \dot{V}(x, z) \) is uniformly continuous in \( t \).

Then, for any \( l \in \mathbb{R} \) such that \( V_l^{-1} \subset D_G \), where \( V_l^{-1} \) is defined as in (18), \( V_l^{-1} \) is positively invariant and is an estimation of the \( \eta \)-RoA.

**Proof:** By the same argument as in the proof of Theorem 6, \( V_l^{-1} \) is positively invariant under (1) and \( V \) \((x(t), z(t)) \) must have a finite limit as \( t \to \infty \). Since \( V \) is uniformly continuous in \( t \), it follows from Barbalat’s Lemma [44, Lemma 8.2] that \( \dot{V} \to 0 \) as \( t \to \infty \). By 2), this implies \( \eta \to 0 \) as \( t \to \infty \), and hence \( V_l^{-1} \) is an estimation of the \( \eta \)-RoA. The second claim follows from the same argument as in the proof of Theorem 6. 

Often, instead of 3), it is more convenient to verify a stronger condition that is \( V(x, z) \) being bounded along the solution. An example of using type-II \( V \)-function will be present in Section V.

**C. Type III**

In some cases, it is only possible to construct a \( V(x, z) \) with a sign-constant (not sign-definite) derivative \( \dot{V} \leq 0 \). In this case, conditions of state convergence were obtained by LaSalle, Barbashin, and Krasovskii, by exploiting the properties of the \( \omega \)-limit set. Here, the same idea applies to \( \eta \)-RoA under an additional requirement of the set \( \{V(x, z) = 0\} \).

**Theorem 8:** Let \( R \subset D_G \) be a compact and positively invariant set under (1). Let \( V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) be a \( C^1 \) function, and let \( \Upsilon \) denote the largest invariant set of \( \{V(x, z) = 0\} \) if it holds that

1) \( V(x, z) \leq 0 \) for every \((x, z) \in R\);

2) \( \Upsilon \subset \{(x, z)|\eta(x, z) = 0\} \)

then, \( \Upsilon \) is an estimation of the \( \eta \)-RoA.

**Proof:** It follows from LaSalle’s Theorem [44, Th. 4.4] that every solution in \( \Upsilon \) approaches \( \Upsilon \) as \( t \to \infty \). Since \( \Upsilon \subset \{(x, z)|\eta(x, z) = 0\} \), every solution approaches \( \{(x, z)|\eta(x, z) = 0\} \) as \( t \to \infty \).

An example of the type-III \( V \)-function is the well-known energy functions for lossless multimachine power systems with \( \eta = f_z \) (see [58] and [6, Ch. 6] for example).

Unlike previous theorems, type-III does not require any bound condition of \( V(x, z) \). The closure of any bounded sublevel set of \( V \) is a valid candidate of the compact and positively invariant set \( R \), although the choice \( R \) does not have to be tied up with \( V \). For example, the closure of any bounded solution is a valid candidate of \( R \). Therefore, similar to the well-investigated energy function, the existence of a global type-III \( V \)-function excludes all bounded complex behaviors of power systems, e.g., sustained oscillation, chaos, etc.

**Remark 12:** The above three theorems to estimate the \( \eta \)-RoA have their counterparts that are well-known in the classical equilibrium-based stability theory, namely the Lyapunov function, Barbalat’s Lemma, and LaSalle’s invariance principle, respectively. The major differences among these three \( V \)-functions are exactly the same as the differences among their counterparts, which are the requirements for \( V(x, z) \) and its time derivative \( \dot{V}(x, z) \). Among these three, type I has the stringent sign-definite requirements for \( V \). Type II relaxes such a requirement by adding a uniformly continuous condition on \( \dot{V} \) as compensation. Type
III only needs \( \dot{V} \) to be sign-constant but an additional condition on the largest invariant set is required. These differences provide flexibility in application. One can always start with type I and seek alternatives in types II and III, according to the properties of the given system.

To end this section, we again emphasize that all three types of \( V \) functions do not require any knowledge of system equilibrium. And naturally, they concern the convergence of \( \eta \) to zero rather than the convergence of all states to any prespecified point. This distinguishes our theorems from the classical Lyapunov stability theory in terms of both what is requested and what is concluded.

V. ILLUSTRATIVE EXAMPLES

This section presents two examples to illustrate potential applications of our results in power systems. In each example, we would employ previous theorems to first justify AS-detectability and then construct a \( V \) function. We begin with a single-machine-single-load system with a proportional–integral (PI) regulator, in which case a type-II \( V \)-function is proposed that justifies the PI regulator can render the system to augmented synchronization from almost all initial points. In the second case, we consider the IEEE 9-bus system to demonstrate the modular property of AS-detectability and to show an estimation of \( f_2 \)-RoA by a type-I \( V \)-function. It is also an example of inherently nonisolated equilibria, in which case the classical point-based stability analysis cannot apply.

A. Single-Machine-Single-Load System

Consider the classical generator model connecting to a constant \( PQ \) load via a lossless transmission line (see Fig. 3).

The system is governed by [49]

\[
\begin{align*}
\dot{\delta} &= \omega \\
M \dot{\omega} &= -D \omega - P^e + P^g
\end{align*}
\] (21)

where \( P^g = P^g_0 + u \)

\[
\dot{P}^g = P^g_0 + u
\] (22)

where \( P^g_0 \) is the fixed mechanical power input and \( u \) is a simple PI regulator for some \( k_1 > 0 \) and \( k_2 > 0 \) defined as

\[
u(t) = -k_1 \omega(t) - k_2 \int_{0}^{t} \omega(\tau) d\tau
\] (23)

or equivalently in the differential form by introducing state \( \zeta \)

\[
\zeta = -k_2 \omega, \quad u = -k_1 \omega + \zeta.
\] (24)

The terminal output power reads

\[
P^e = \frac{E_V_1 \sin(\delta - \theta_1)}{x_d}, \quad Q^e = \frac{E_V_1 \cos(\delta - \theta_1)}{x_d} - \frac{1}{z_d} V^2 \] (25)

where \( E > 0 \) is the constant generator internal voltage. The power flow equations read

\[
\begin{align*}
0 &= P^e - B_1 V_1 V_2 \sin \theta_{12} \\
0 &= Q^e + B_1 V_1^2 + B_1 V_2 \cos \theta_{12} \\
0 &= P^d - B_2 V_2 \sin \theta_{21} \\
0 &= Q^d + B_2 V_2^2 + B_1 V_2 \cos \theta_{12}
\end{align*}
\] (26)

where \( V_1 \angle \theta_1 \) and \( V_2 \angle \theta_2 \) are complex voltages at buses 1 and 2, respectively. In this case, \( x_1 = \cos(\zeta, \omega) \), \( x_2 = \delta \), and \( z = \cos(\theta_1, V_1, V_2, V_2) \). is given by (21), (22), and (24); \( g \) is given by (25) and (26).

We first show this system satisfies conditions of Theorem 3 and hence any nondegenerate solution is AS-detectable. Since \( \zeta(t) = \zeta(0) - k_2 \int_{0}^{t} \omega(\tau) d\tau = \zeta(0) - k_2(\delta(t) - \delta(0)) \), bounded \( \delta(t) \) implies bounded \( \zeta(t) \). It follows from (21) to (23) that \( M \dot{\omega}(t) = -(D + k_1) \omega(t) + \zeta(t) - P^e(t) + P^g_0 \). Similar to the proof of Proposition 2, by invoking the comparison lemma we can prove the boundedness of \( \omega(t) \) given bounded \( z(t) \) and \( \delta(t) \). Hence, condition 1) of Theorem 3 is met. For any solution satisfying \( f_2 = \omega \equiv 0 \), it follows \( \dot{\omega} = 0 \) and \( \zeta = 0 \), which meets condition 2) of Theorem 3, and hence, any nondegenerate solution of this example is AS-detectable.

Next, we estimate the \( f_2 \)-RoA of the system by constructing a type-II \( V \)-function. Let

\[
Q(x, z) = \frac{1}{2} M \omega^2 + \frac{E_V_1(1 - \cos(\delta - \theta_1))}{x_d^2} + B_{12} V_1 V_2 (1 - \cos \theta_{12})
\]

Since \( B_{12} > 0 \) and \( E > 0 \), the function \( Q(x, z) \) is bounded from below on \( D_G \), provided \( V_1 > 0, V_2 > 0 \).

Direct calculation yields

\[
\left( \frac{\partial g}{\partial z} \right)^{-1} \frac{\partial g}{\partial \delta} = \begin{bmatrix} -1 & 0 & -1 & 0 \end{bmatrix}^T.
\]

Hence, by (4b), it holds \( \dot{z} = \cos(\omega, \omega, 0, 0) \). It holds

\[
\dot{Q}(x, z) = -D \omega^2 + \omega(P^d + П_0^q + u).
\]

Consider the following \( V \)-function

\[
V(x, z) = Q(x, z) + \frac{1}{2k_2} (\zeta + P^d + П_0^q)^2.
\]

Clearly, \( V(x, z) \) is lower bounded. It holds

\[
\frac{1}{dt} 2k_2 (\zeta + P^d + П_0^q)^2 = -\omega(u + P^d + П_0^q) - k_1 \omega^2.
\]

Hence

\[
\dot{V}(x, z) = -(D + k_1) \omega^2 = -(D + k_1) \| f_2 \|^2
\]

which verifies condition 2) in Theorem 7. To verify condition 3), we prove \( \dot{V} \) is bounded when \( \dot{V} \) is bounded. Direct calculation yields

\[
\dot{V}(x, z) = -2(D + k_1) \omega \omega
\]

\[
= -2(D + k_1) \omega(-D \omega + П_0^q + u - P^e).
\]
It is easy to show by contradiction arguments that $V$ being bounded implies $\omega$ and $u$ being bounded. Note also $|P| \leq EV_1/x_d'$. Hence, $\bar{V}$ is bounded in any sublevel set $V^{-1}_l$ with finite level value $l$. Hence, any $V_l^{-1} \subset D_G$ is a valid estimation of $f_{2,3}$-RoA of the system.

For illustration, this example was built on a single-machine system and an SG. However, we remark that the analysis applies to large systems and similar dynamical subsystems, e.g., the virtual SG with energy storage. Although the generator’s primary power input is conventionally assumed to be constant in transient stability analysis since the governor dynamics is usually slow, this example shows such an assumption can be relaxed, which might be necessary for microgrids with inverter-interfaced fast response power resources.

### B. IEEE 9-bus System

We now consider the IEEE 9-bus system as shown in Fig. 4. To demonstrate the capability of our results in heterogeneous settings, we modify the benchmark so that it consists of two different SGs, an inverter-interfaced power source, and three constant PQ loads. We assume the generator at bus 1 has a strong excitation control and hence is described by (21) together with the PI regulator. The generator at bus 2 is governed by the flux-decay model as in Example 2. The power source at bus 3 is modeled as in Example 1. The differential equations of each subsystem can be found in previous examples.

The algebraic equations consist of power balance constraints at each bus, which read for $i = 1, \ldots, 9$

$$\begin{align*}
P_i &= G_{ii} V_i^2 + \sum_{j \in N_i} V_i V_j (B_{ij} \sin \theta_{ij} + G_{ij} \cos \theta_{ij}) \\
Q_i &= -B_{ii} V_i^2 - \sum_{j \in N_i} V_i V_j (B_{ij} \cos \theta_{ij} - G_{ij} \sin \theta_{ij}).
\end{align*}$$

Network and load parameters were obtained from the MATPOWER package [59]. Table I reports the parameters of three dynamical subsystems.

| Bus | Parameters | Values |
|-----|------------|--------|
| 1   | $M, D, E, x_d^*$, $F_{q1}$, $x_q$, $x_d$, $F_{p1}$, $P_{p1}$ | 0.075, 0.032, 1.01, 0.061, 0.72, 0.02, 0.10 |
| 2   | $M, D, F_{q2}$, $x_q$, $x_d$, $x_d'$, $F_{p2}$, $P_{p2}$ | 0.02, 0.003, 6.00, 0.20, 0.896, 0.12, 1.63, 1.52 |
| 3   | $\tau_1, \tau_2, \delta_1, \delta_2$, $V_{ref}$, $Q_{ref,ref}$, $V_{ref}$ | 10.00, 10.00, 0.01, 0.01, 0.85, -0.0365, 0.0833, 1.00 |

Note that each bus in the IEEE 9-bus benchmark represents a control area in practice and the dynamics actually models aggregated devices in the area. The dynamics of both bus 1 and bus 3 have an integral of frequency deviation, representing the automatic generation control (AGC) units in the area [60, Ch. 11]. They are together responsible for restoring the system frequency to the nominal value.

1) **Nonisolated and Not Asymptotically Stable Equilibrium:** The collective state variables of the system read

$$x = \text{col}(\zeta, \omega_1, \delta_1, \omega_2, \delta_2, E_{q1}', P_3, Q_3).$$

The collective algebraic variable $z$ is of 18 dimensions and consists of $\theta_i$ and $V_i$ at each bus. Setting $\zeta = 0$, one equilibrium can be obtained, which reads

$$x_0 = \text{col}(0, 0, 0.0431, 0, 0.4756, 1.0288, 0.8500, -0.0365).$$

However, $x_0$ is not an isolated equilibrium. In fact, there is a continuous equilibria trajectory, along which it always holds that $\omega_1 = \omega_2 = 0$ but other states keep varying. Fig. 5 shows this equilibria trajectory as a function of $\zeta$.

Unlike the well-investigated phase rotational symmetry, the equilibria continuum here cannot be eliminated by simple coordinate transformation. Indeed, phase rotational symmetry does not exist in this example due to the power-angle droop at bus 3. The inherently nonisolated equilibria actually result from the interaction between the PI regulator at bus 1 and the power-angle droop at bus 3.
More importantly, any single equilibrium in this continuum is not Lyapunov asymptotically stable since any small perturbation along the direction tangent to the equilibria trajectory will make the system leave the initial equilibrium and settle in a new one. To illustrate, consider the projection of the system vector field onto the plane spanned by two orthogonal directions $s_1$ and $s_2$ around the equilibrium $x_0^*$. Here, we set $s_1 = \text{col}(1, 0, 1, 0, 1, 1, 1)/\sqrt{6}$ and $s_2 = \text{col}(0, 1, 0, 1, 0, 0, 0)/\sqrt{2}$. Hence, $s_2$ represents the variation of $\omega_1$ and $\omega_2$ while $s_1$ represents the combination of the others. Fig. 6 reports the projected vector field. It shows that all vectors point to the line $s_2 = 0$, but they do not point to any particular point. Note that the line $s_2 = 0$ is the projection of the equilibria trajectory on this plane. Now suppose the system is subject to a small disturbance that shifts the state from $x_0^*$ to $\tilde{x}_0$, which is represented by the red dot. From this initial point, the system solution can reach augmented synchronization but it converges to a different equilibrium $\tilde{x}^*$, which is represented by the green dot. This indicates that this equilibria trajectory as a whole has a certain stability property but no single equilibrium is Lyapunov asymptotically stable.

2) AS-Detectability: This 9-bus system provides an example of verifying AS-detectability in a modular manner. The system has exactly the modular structure as defined in Section III-C. Each subsystem satisfies conditions 1) and 2) in Theorem 4, which can be shown by the same argument as in Example 1-2 and in Section V-A. Therefore, any nondegenerate solution of this system is AS-detectable.

In this case, although $\omega(t)$ and $\dot{z}(t)$ may identically converge to 0, most other states such as $\zeta(t)$, $\theta(t)$, and $V(t)$ may converge to different values provided different initial points. Hence, one cannot single out any prespecified equilibrium to perform transient stability analysis. Our results provide an alternative way to handle this situation by checking the convergence of $\dot{z}(t)$ instead of all states.

3) Estimation of the $f_2$-RoA: To illustrate, let us estimate the $f_2$-RoA of the system by a type-I $V$-function in Krasovskii’s form. For some positive-definite matrix $P$, let $V(x, z) = f(x, z)^T P f(x, z)$. Note for subsystem 1, $\dot{\zeta}$ is linearly related to $\delta_1$. Hence, instead of letting

$$\xi = f = \text{col}(\zeta, \omega_1, \dot{\delta}_1, \dot{\omega}_2, \dot{\delta}_2, E_q, \dot{P}_3, \dot{Q}_3)$$

one can choose

$$\xi = \text{col}(\dot{\omega}_1, \dot{\delta}_1, \dot{\omega}_2, \dot{\delta}_2, E_q', \dot{P}_3, \dot{Q}_3).$$

This yields $f = A\xi$, where $A$ is a constant matrix and reads

$$A = \begin{bmatrix} 0 & -k_2 & 0_{1 \times 5} \\ I_{7 \times 7} \end{bmatrix}.$$ 

Hence, condition (20) can be relaxed as

$$A^T (P J(x, z) + J(x, z)^T P) A < 0. \tag{27}$$

\[ P = \begin{bmatrix} 1.136 & -0.049 & 0.026 & 0.225 & -0.049 & -0.050 & -0.507 & -0.066 \\ -0.049 & 0.719 & -0.151 & -0.237 & -0.041 & -0.077 & 0.005 & 0.052 \\ 0.026 & -0.151 & 0.497 & -0.060 & 0.131 & 0.350 & -0.053 & 0.174 \\ 0.225 & -0.237 & -0.060 & 1.070 & -0.146 & 0.096 & 0.006 & -0.047 \\ -0.049 & -0.041 & 0.131 & -0.146 & 0.461 & -0.302 & 0.073 & -0.143 \\ -0.050 & -0.077 & 0.350 & 0.096 & -0.302 & 0.651 & -0.012 & 0.304 \\ -0.507 & 0.005 & -0.053 & 0.006 & 0.073 & -0.012 & 0.530 & 0.038 \\ -0.066 & 0.052 & 0.174 & -0.047 & -0.143 & 0.304 & 0.038 & 0.463 \end{bmatrix} \tag{28} \]
As a practical approach is to solve (27), we first solve \( P \) at several random points around the predisturbance equilibrium \( (x_0^p, z_0^p) \), which is a standard LMI problem. Then, with this fixed \( P \) we guess a level value \( l \) and verify whether (27) holds over \( V_l^{-1} \). In our case, the numerical LMI solver yields a candidate of \( P \) as reported in (28) shown at the bottom of the previous page, such that (27) holds in \( V_l^{-1} \) with \( l = 4 \). Then, a conservative estimation of the \( f_2 \)-RoA is given by \( V_1^{-1} \), which takes 18.41% of the exact \( f_2 \)-RoA obtained by point-by-point simulations. Note that ideally, we should solve \( P > 0 \) and \( l > 0 \) simultaneously so that (27) holds for all \((x, z) \in V_1^{-1} \) and we should optimize \( P \) and \( l \) to enlarge the estimated RoA. Such a problem involves coupled nonlinear and nonconvex constraints. It is our ongoing work to develop effective algorithms for such a problem.

In this case, \( V_4^{-1} \) is an 8-dimensional manifold in the 26-dimensional Euclidean space. To illustrate, we project it onto the \( \zeta - \omega_1 \) plane as shown in Fig. 7. Two trajectories starting from different initial points were also projected onto the plane, which clearly shows the dependence of the converging point of \( \zeta \) on the initial point. It also shows the independence of the converging point of \( \omega_1 \), which should always converge to 0.

Figs. 8 and 9 give the solutions \( \theta(t), V(t) \) of the corresponding two trajectories, respectively. In both cases, the system approaches to the augmented synchronization, i.e., \( \dot{z}(t) \to 0 \) as \( t \to \infty \). However, the converging points of \( \theta(t) \) and \( V(t) \) are different. This again shows that augmented synchronization could be more desirable to power system transient stability analysis than equilibrium.

VI. CONCLUDING REMARKS

In this article, we redefined the "nominal motion" of power systems by introducing the concept of augmented synchronization. We derived conditions for augmented synchronization detectability and presented a compositional approach to verify these conditions in modularly structured power systems. That provides the long-standing practice wisdom with a solid theoretical foundation. Inspired by such wisdom, we further developed theorems to characterize augmented synchronization with different types of \( V \)-functions. These theorems extend the classical Lyapunov-based direct method and can provide estimations of RoA w.r.t. augmented synchronization rather than an equilibrium point.

Our results shed new light on power system stability analysis, which may open the possibility for an equilibrium-independent analytic that better fits the demand of future smart grids. As an initial step, the concept of augmented synchronization may also provide a better perspective to understand power system stability by rethinking what should be the cornerstone, i.e., the "nominal motion" in stability analysis.

One limitation of the proposed framework is the implicit assumption of access to full measurements of voltage angles and magnitudes at all network buses. It is among our future work to investigate whether the transient stability can be inferred based on partial measurements only, and how the topology of available measurement comes into play. Another limitation is that it might be hard to calculate \( \eta \)-RoA for large-scale power systems. It is among future work to develop efficient algorithms to implement our theory and to reduce the conservativeness of the estimation of \( \eta \)-RoA. Our ongoing works also include extending the theory to consider line dynamics.

ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers for their stimulating discussions and constructive suggestions.

REFERENCES

[1] P. Kundur et al., “Definition and classification of power system stability IEEE/CIGRE joint task force on stability terms and definitions,” IEEE Trans. Power Syst., vol. 19, no. 3, pp. 1387–1401, Aug. 2004.
[2] P. Kundur, N. J. Balu, and M. G. Lauby, Power System Stability and Control, vol. 7, New York, NY, USA: McGraw-Hill, 1994.
[3] M. Pai, Energy Function Analysis for Power System Stability. Berlin, Germany: Springer, 2012.
[4] A.-A. Fouad and V. Vittal, Power System Transient Stability Analysis Using the Transient Energy Function Method. London, U.K.: Pearson Education, 1995.
[5] H.-D. Chiang, F. Wu, and P. Varaiya, “Foundations of direct methods for power system transient stability analysis,” IEEE Trans. Circuits Syst., vol. TCAS-34, no. 2, pp. 160–173, Feb. 1987.
[6] H.-D. Chiang, Direct Methods for Stability Analysis of Electric Power Systems: Theoretical Foundation, BCU Methodologies, and Applications. Hoboken, NJ, USA: Wiley, 2011.
[7] A. Monticelli, “Electric power system state estimation,” Proc. IEEE, vol. 88, no. 2, pp. 262–282, Feb. 2000.
[8] J. Willems, “A partial stability approach to the problem of transient power system stability,” Int. J. Control, vol. 19, no. 1, pp. 1–14, 1974.
[9] V. I. Vorotnikov, “Partial stability and control: The state-of-the-art and development prospects,” Automat. Remote Control, vol. 66, no. 4, pp. 511–561, 2005.
[10] A. S. Shiriav and A. L. Fradkov, “Stabilization of invariant sets for nonlinear non-affine systems,” Automatica, vol. 36, no. 11, pp. 1709–1715, 2000.
[11] W. Lohmiller and J.-J. E. Slotine, “On contraction analysis for non-linear systems,” Automatica, vol. 34, no. 6, pp. 683–696, 1998.
[12] T. Jouini and Z. Sun, “Frequency synchronization of a high-order multiconverter system,” IEEE Trans. Control Netw. Syst., vol. 9, no. 2, pp. 1006–1016, Jun. 2022, doi: 10.1109/TCNS.2021.3128493.
[13] M. Colombino, D. Groß, J.-S. Brouillon, and F. Dörfler, “Global phase and magnitude synchronization of coupled oscillators with application to the control of grid-forming power inverters,” IEEE Trans. Autom. Control, vol. 64, no. 11, pp. 4496–4511, Nov. 2019.
[14] P. Yang, F. Liu, Z. Wang, S. Wu, and H. Mao, “Spectral analysis of network coupling on power system synchronization with varying phases and voltages,” in Proc. Chin. Control Decis. Conf., 2020, pp. 880–885.
[15] A. E. Motter, S. A. Myers, M. Anghel, and T. Nishikawa, “Spontaneous synchrony in power-grid networks,” Nature Phys., vol. 9, no. 3, pp. 191–197, 2013.
[16] L. Zhu and D. J. Hill, “Stability analysis of power systems: A network synchronization perspective,” SIAM J. Control Optim., vol. 56, no. 3, pp. 1640–1664, 2018.
[17] F. Dörfler, M. Chertkov, and F. Bullo, “Synchronization in complex oscillator networks and smart grids,” Proc. Nat. Acad. Sci. USA, vol. 110, no. 6, pp. 2005–2010, 2013.
[18] F. Dörfler and F. Bullo, “Synchronization and transient stability in power networks and nonuniform Kuramoto oscillators,” SIAM J. Control Optim., vol. 50, no. 3, pp. 1616–1642, 2012.
[19] F. Pagani and E. Mallada, “Global analysis of synchronization performance for power systems: Bridging the theory–practice gap,” IEEE Trans. Automat. Control, vol. 65, no. 7, pp. 3007–3022, Jul. 2020.
[20] J. W. Simpson-Porco, F. Dörfler, and F. Bullo, “Synchronization and power sharing for droop-controlled inverters in islanded microgrids,” Automatica, vol. 49, no. 9, pp. 2603–2611, 2013.
[21] D. Zonetti, G. Bergna-Diaz, and R. Ortega, “P+leaky I passivity-based control of power converters,” in Proc. IEEE 59th Conf. Decis. Control, 2020, pp. 854–859.
YANG et al.: AUGMENTED SYNCHRONIZATION OF POWER SYSTEMS

[22] D. Zonetti, G. Bergna-Díaz, R. Ortega, and N. Monshizadeh, “PID passivity-based droop control of power converters: Large-signal stability, robustness and performance,” *Int. J. Robust Nonlinear Control*, vol. 32, no. 3, pp. 1769–1795, 2022.

[23] G. Manoeur, R. G. Cafieri, J. Daafouz, and L. Grimaud, “Adaptive stabilization of switched affine systems with unknown equilibrium points: Application to power converters,” *Automatica*, vol. 99, pp. 82–91, 2019.

[24] J. W. Simpson-Porco, Q. Shafiee, F. Dörfler, J. C. Vasquez, J. M. Guerrero, and F. Bullo, “Secondary frequency and voltage control of islanded microgrids via distributed averaging,” *IEEE Trans. Ind. Electron.*, vol. 62, no. 11, pp. 7077–7085, Nov. 2015.

[25] C. I. Byrnes, A. Isidori, and J. C. Willems, “Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems,” *IEEE Trans. Autom. Control*, vol. 36, no. 11, pp. 1228–1240, Nov. 1991.

[26] E. D. Sontag and Y. Wang. “Output-to-state stability and detectability of nonlinear systems,” *Syst. Control Lett.*, vol. 29, no. 5, pp. 279–290, 1997.

[27] D. J. Hill and I. M. Mareels, “Stability theory for differential/algebraic systems with application to power systems,” *IEEE Trans. Circuits Syst.*, vol. 37, no. 11, pp. 1416–1423, Nov. 1990.

[28] H.-D. Chang, C.-C. Chu, and G. Cauley, “Direct stability analysis of electric power systems using energy functions: Theory, applications, and perspective,” *Proc. IEEE*, vol. 83, no. 11, pp. 1497–1529, Nov. 1995.

[29] A. L. Fradkov, I. V. Miroshnik, and V. O. Nikiforov, *Nonlinear and Adaptive Control of Complex Systems*, vol. 491. Berlin, Germany: Springer, 1999.

[30] A. R. Bergen and D. J. Hill, “A structure preserving model for power system stability analysis,” *IEEE Trans. Power Appl. Syst.*, vol. PAS-100, no. 1, pp. 25–35, Jan. 1981.

[31] M. Althoff, “Formal and compositional analysis of power systems using reachable sets,” *IEEE Trans. Power Syst.*, vol. 29, no. 5, pp. 2270–2280, Sep. 2014.

[32] R. J. Davy and I. A. Hiskens, “Lyapunov functions for multimechanic power systems with dynamic loads,” *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 44, no. 9, pp. 796–812, Sep. 1997.

[33] W. Dib, R. Ortega, A. Barabanov, and F. Lamnabhi-Lagarrigue, “A globally convergent controller for multi-machine power systems using structure-preserving models,” *IEEE Trans. Autom. Control*, vol. 54, no. 9, pp. 2179–2185, Sep. 2009.

[34] Y. Wan and F. Milano, “Nonlinear adaptive excitation control for structure preserving power systems,” *IEEE Trans. Power Syst.*, vol. 33, no. 3, pp. 3107–3117, May 2018.

[35] T. W. Stegink, C. De Persis, and A. J. Van der Schaft, “Stabilization of structure-preserving power networks with market dynamics,” *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 6737–6742, 2017.

[36] P. Kunkel and V. Mehrmann, *Papers OnLine*, vol. 2. Helsinki, Finland: Eur. Math. Soc., 2006.

[37] H. D. Nguyen, T. L. Vu, J.-J. Slotine, and K. Turitsyn, “Construction of analytical energy functions for network-preserving power system models,” *Automatica*, vol. 99, pp. 82–91, 2019.

[38] H.-D. Chang and L. Feki-H Ahmed, “Quasi-stability regions of nonlinear dynamical systems: Theory,” *IEEE Trans. Circuits Syst. I: Fundam. Theory Appl.*, vol. 43, no. 8, pp. 627–635, Aug. 1996.

[39] H.-D. Chang and L. Feki-H Ahmed, “Quasi-stability regions of nonlinear dynamical systems: Optimal estimations,” *IEEE Trans. Circuits Syst. I: Fundam. Theory Appl.*, vol. 43, no. 8, pp. 636–643, Aug. 1996.

[40] V. I. Vorotnikov, *Partial Stability and Control*. Berlin, Germany: Springer, 2012.

[41] H. D. Nguyen, T. L. Vu, J.-J. Slotine, and K. Turitsyn, “Optimal angle droop for power sharing enhancement with stability improvement in islanded microgrids,” *IEEE Trans. Smart Grid*, vol. 9, no. 5, pp. 5014–5026, Sep. 2018.

[42] N. Tsolas, A. Arapostathis, and P. Varaiya, “A structure preserving energy function for power system transient stability analysis,” *IEEE Trans. Circuits Syst.*, vol. 32, no. 10, pp. 1041–1049, Oct. 1985.

[43] J. Schiffer, R. Ortega, A. Astolfi, J. Raisch, and T. Sezi, “Conditions for stability of droop-controlled inverter-based microgrids,” *Automatica*, vol. 50, no. 10, pp. 2457–2469, 2014.

[44] R. YANG et al.: AUGMENTED SYNCHRONIZATION OF POWER SYSTEMS

[45] P. Yang, F. Liu, Z. Wang, and C. Shen, “Distributed stability conditions for power systems with heterogeneous nonlinear bus dynamics,” *IEEE Trans. Power Syst.*, vol. 35, no. 3, pp. 2313–2324, May 2020.

[46] H.-D. Chiang, M. Hirsch, and F. Wu, “Stability regions of nonlinear autonomous dynamical systems,” *IEEE Trans. Autom. Control*, vol. 33, no. 1, pp. 16–27, Jan. 1988.

[47] J. Zaborszky, G. Huang, B. Zheng, and T.-C. Leung, “On the phase portrait of a class of large nonlinear dynamical systems such as the power system,” *IEEE Trans. Autom. Control*, vol. 33, no. 1, pp. 4–15, Jan. 1988.

[48] C.-C. Chu and H.-D. Chiang, “Constructing analytical energy functions for network-preserving power system models,” *Circuits, Syst., Signal Process.*, vol. 24, no. 4, pp. 363–383, 2005.

[49] R. D. Zimmerman, C. E. Murillo-Sánchez, and R. J. Thomas, “MATPOWER: Steady-state operations, planning, and analysis tools for power systems research and education,” *IEEE Trans. Power Syst.*, vol. 26, no. 1, pp. 12–19, Feb. 2011.

[50] P. S. Kundur, *Power System Stability and Control*. New York, NY, USA: McGraw-Hill, 1994.

Peng Yang received the B.S.c. degree in electrical engineering, the B.S.c. degree in mathematics, and the Ph.D. degree in electrical engineering from Tsinghua University, Beijing, China, in 2017, 2018, and 2022, respectively. He is currently a System Operator with the State Grid Corporation of China, Beijing. His research interests include power system stability analysis and control. Dr. Yang was the winner of the 2020 Zhang Siyang (CCDC) Outstanding Young Paper Award. He is also the recipient of the 2018–2020 Best Paper Award of IEEE Transactions on Power Systems.

Feng Liu (Senior Member, IEEE) received the B.S.c. and Ph.D. degrees in electrical engineering from Tsinghua University, Beijing, China, in 1999 and 2004, respectively. He is currently an Associate Professor with the Electrical Engineering Department, Tsinghua University. From 2015 to 2016, he was a Visiting Associate with the California Institute of Technology, Pasadena, CA, USA. He is the author/coauthor of more than 30 peer-reviewed journal/conference papers and four books and holds more than 30 issued/pending patents. His research interests include stability analysis, optimal control, robust dispatch, and game-theory-based decision making in energy and power systems. Dr. Liu is an Associate Editor of several international journals, including *IEEE TRANSACTIONS ON SMART GRID* and *IEEE TRANSACTIONS ON POWER SYSTEMS*. He also served as a Guest Editor of the *IEEE TRANSACTIONS ON ENERGY CONVERSION*. 

Authorized licensed use limited to the terms of the applicable license agreement with IEEE. Restrictions apply.
Tao Liu (Member, IEEE) received the B.E. degree in automation from Northeastern University, Shenyang, China, in 2003, and the Ph.D. degree in engineering from the Australian National University (ANU), Canberra, ACT, Australia, in 2011.

From 2012 to 2015, he was a Postdoctoral Fellow with ANU, University of Groningen, and University of Hong Kong (HKU). He became a Research Assistant Professor with HKU in 2015 and is currently an Assistant Professor. His research interests include power system analysis and control, complex dynamical networks, distributed control, and event-triggered control.

David J. Hill (Fellow, IEEE) received the Ph.D. degree in electrical engineering from the University of Newcastle, Callaghan, NSW, Australia, in 1976.

Since 2022, he has been a Professor of Electrical Power and Energy Systems with Monash University, Melbourne, VIC, Australia. He is also a Professor Emeritus with The University of Sydney, Camperdown, NSW, Australia, and The University of Hong Kong, Hong Kong. From 2013 to 2020, he held the positions of Chair of Electrical Engineering and the Director of the Centre for Electrical Energy Systems, Department of Electrical and Electronic Engineering, The University of Hong Kong. He previously held positions at the University of Sydney including the Chair of Electrical Engineering from 1994 to 2002 and again in 2010–2013 along with an Australian Research Council Professorial Fellowship. He was the Foundation Director of the Centre for Future Energy Networks from 2010 to 2018 and a part-time Professor from 2013 to 2020. From 2005 to 2010, he was an ARC Federation Fellow with the Australian National University. He has also held academic and substantial visiting positions at the universities of Melbourne, California (Berkeley), Newcastle (Australia), Lund (Sweden), Munich, City University of Hong Kong, and New South Wales Sydney. He has held several honorary positions in Australia, Hong Kong, and Mainland China. He is also a consultant in the area of power and energy issues in Australia and internationally. His research interests include energy and power systems, control systems, complex networks and systems, learning systems, and stability analysis, mainly focusing on issues for future energy and power networks with the aim to bring science to accelerate the clean energy transition.

Dr. Hill is a Fellow of the Society for Industrial and Applied Mathematics, USA, the International Federation of Automatic Control, the Australian Academy of Science, the Australian Academy of Technological Sciences and Engineering, and the Hong Kong Academy of Engineering Sciences. He is also a Foreign Member of the Royal Swedish Academy of Engineering Sciences. He was the recipient of the 2021 IEEE Power and Energy Society Prabha S. Kundur Power System Dynamics and Control Award and was the 2022 IEEE Control System Society Hendrik W. Bode Lecture Prize winner.