QUANTIZED DUAL GRADED GRAPHS

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Abstract. We study quantized dual graded graphs, which are graphs equipped with linear operators satisfying the relation $DU - qUD = rI$. We construct examples based upon: the Fibonacci poset, permutations, standard Young tableau, and plane binary trees.

1. Introduction

Fomin’s dual graded graphs [Fom] and Stanley’s differential posets [Sta] are constructions developed to understand and generalize the enumerative consequences of the Robinson-Schensted algorithm. The key relation in these constructions is $DU - UD = rI$, where $U, D$ are up-down operators acting on the graphs or posets. In this article we develop some of the basic theory of quantized dual graded graph, which are equipped with up-down operators $U, D$ satisfying the $q$-Weyl relation $DU - qUD = rI$. One of the motivations for the current work were the signed differential posets developed in [Lam], which correspond to the relation $DU + UD = rI$. Thus quantized dual graded graphs specialize to usual dual graded graphs at $q = 1$, and to signed differential posets (or their dual graded graph equivalent) at $q = -1$.

The central enumerative identity in the subject developed by Fomin and Stanley is

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$

where the sum is over partitions of $n$, and $f^\lambda$ is the number of standard Young tableau of shape $\lambda$. The corresponding analogue (Theorem 4) for a quantized dual graded graph $(\Gamma, \Gamma')$ reads

$$\sum_v f_v^\Gamma(q) f_v^{\Gamma'}(q) = r^n [n]_q!$$

where the sum is over vertices of height $n$, the polynomials $f_v^\Gamma(q)$ and $f_v^{\Gamma'}(q)$ are weighted enumerations of paths in $\Gamma$ and $\Gamma'$, and $[n]_q!$ is the $q$-analogue of $n!$.

We explicitly construct examples of quantized dual graded graphs and interpret (1). These examples are based on various combinatorial objects:
the Fibonacci poset, permutations, standard Young tableau, and plane binary trees. Unfortunately, we have been unable to quantize Young’s lattice. More examples will be given in joint work [BLL] with Bergeron and Li, where in some cases a representation theoretic explanation for the identities $DU - qUD = I$ and (1) will be given.

2. Quantized dual graded graphs

Let $\Gamma = (V, E, m)$ be a graded graph with edge weights $m(v, w) \in \mathbb{N}[q]$. That is, $\Gamma$ is a directed graph with a height function $h : V \to \mathbb{N}$ such that if $(v, w) \in E$ then $h(w) = h(v) + 1$. Furthermore, each edge has a weight $m(v, w) \in \mathbb{N}[q]$ which is a non-zero polynomial in $q$ with nonnegative coefficients. We shall assume that $\Gamma$ is locally finite, so that for each $v$, there are finitely many edges entering and leaving. Because each edge has a weight, we shall assume that there are no multiple edges.

Let $C(q)[V]$ be the $C(q)$-vector space of formal linear combinations of the vertex set $V$. A linear operator on $C(q)[V]$ is continuous if it is compatible with arbitrary linear combinations. Define continuous linear operators $U, D : C(q)[V] \to C(q)[V]$ by

$$U(v) = \sum_{w : (v, w) \in E} m(v, w) w$$

$$D(w) = \sum_{v : (v, w) \in E} m(v, w) v.$$

and extending by linearity and continuity. We define a pairing $(., .) : C(q)[V] \times C(q)[V] \to \mathbb{C}(q)$ by $(v, w) = \delta_{v, w}$ for $v, w \in E$. Then $U$ and $D$ are adjoint with respect to this pairing.

Let $(\Gamma = (V, E, m), \Gamma' = (V', E', m'))$ be a pair of graded graphs with the same vertex set. Then $(\Gamma, \Gamma')$ is a pair of quantized dual graded graphs (qDGG for short) if we have the identity

$$D_{\Gamma'} U_{\Gamma} - q U_{\Gamma} D_{\Gamma'} = rI$$

for some positive integer $r \in \{1, 2, 3, \ldots\}$, called the differential coefficient. In the sequel, we will often write $U$ and $D$ for $U_{\Gamma}$ and $D_{\Gamma'}$. When $q = 1$, we obtain the dual graded graphs of [Fom], which are equipped with the relation $DU - UD = rI$. We should note that Fomin also considered the more general relation $DU = f(UD)$ for arbitrary functions $f$; however, he did not focus on (2) where $q$ is a parameter.

If $(\Gamma(q), \Gamma'(q))$ are a pair of quantized dual graded graphs then we say that $(\Gamma(q), \Gamma'(q))$ is a quantization of $(\Gamma(1), \Gamma'(1))$. The basic example of a dual graded graph is Young’s lattice of partitions, ordered by containment; see [Fom, Sta]. The following is the basic problem for quantized dual graded graphs.

**Problem 1.** Find a quantization of Young’s lattice.
In [LS], we constructed dual graded graphs from the strong (Bruhat) and weak orders of the Weyl group of a Kac-Moody algebra. The dual graded graphs constructed this way include Young’s lattice, and closely related graphs such as the shifted Young’s lattice.

**Problem 2.** Find a quantization of Kac-Moody dual graded graphs.

**Remark 1.** Equation (2) specializes to $DU + UD = I$ when $q = -1$ and $r = 1$. Graphs satisfying this relation were studied in [Lam]. More specifically, in [Lam] we studied only such graphs, called signed differential posets, which arose from labeled posets. The examples constructed in the present paper can also be specialized at $q = -1$, giving what would be called “signed dual graded graphs”. The main example in [Lam] was the construction of a signed differential poset structure on Young’s lattice. Since we have been unable to quantize Young’s lattice, we have stopped short of explicitly writing the examples in the current article using the notation in [Lam].

### 3. $q$-Derivatives and Enumeration on Quantized Dual Graded Graphs

Let $f(t) = \sum_{n \geq 0} a_n t^n \in \mathbb{C}[[t]]$ be a formal power series in one variable. Define the $q$-derivative as follows:

$$f_q(t) = \sum_{n \geq 0} [n]_q a_n t^{n-1}.$$

Here $[n]_q = 1 + q + \ldots + q^{n-1}$ denotes the $q$-analogue of $n$. We also set $[n]_q! := [n]_q [n-1]_q \cdots [2]_q [1]_q$. Let $U, D$ be formal, non-commuting variables satisfying the relation $DU - qUD = r$. We assume that $U$ and $D$ commute with the variable $q$. The following Lemma explains the relationship between the relation $DU - qUD = r$ and $q$-derivatives.

**Lemma 3.** Suppose $f(U) \in \mathbb{C}[[U]]$ is a formal power series in the variable $U$. Then $Df(U) = r f(qU) + f(U)D$.

**Proof.** By linearity and continuity it suffices to prove the statement for $f(U) = U^n$. For $n = 0$, the formula is trivially true. The inductive step follows from the calculation

$$DU^n = (r[n-1]_q U^{n-2} + q^{n-1} U^{n-1} D)U$$

$$= r[n-1]_q U^{n-1} + q^{n-1} U^{n-1} (r + qUD)$$

$$= r[n]_q U^{n-1} + q^n U^n D,$$

using $[n]_q = [n-1]_q + q^{n-1}$.

We now suppose $(\Gamma, \Gamma')$ is a qDGG with a unique minimum (source) $\emptyset$, which we assume has height $h(\emptyset) = 0$. Let us denote the weight generating
function of paths in $\Gamma$ from $\emptyset$ to a vertex $v \in V$ by $f^n_v = (U^n \emptyset, v)$, where $n = h(v)$. It is not difficult to see that we have

$$(D^n U^n \emptyset, \emptyset) = \sum_{v : h(v) = n} f^n_v f^n_{v'}.$$ 

The following is an analogue of [Fom, Corollary 1.5.4]; see also [Sta].

**Theorem 4.** Let $(\Gamma, \Gamma')$ be a qDGG with a unique minimum $\emptyset$. Then

$$\sum_{v : h(v) = n} f^n_v f^n_{v'} = r^n[n]_q!.$$ 

**Proof.** By Lemma 3 we have

$$D^n U^n \emptyset = D^{n-1}(r[n]_q U^{n-1} + q^n U^n D) \emptyset = r[n]_q D^{n-1} U^{n-1} \emptyset$$

from which the result follows by induction. $\square$

More generally, let $f(\emptyset \to v \to w)$ denote the weight generating function of paths beginning at $\emptyset$, going up to $v$ in $\Gamma$, then going down to $w$ in $\Gamma'$. For fixed $w$ with $h(w) = m \leq n$, we then have

$$\sum_{v : h(v) = n} f(\emptyset \to v \to w) = (D^{n-m} U^n \emptyset, w) = r^{n-m}([n]_q[n-1]_q \cdots [m+1]_q) f^w_v.$$ 

Other path generating function problems can be solved by studying the “normal ordering problem” for the relation $DU - qUD = r$, that is, the problem of rewriting a word in the letters $U$ and $D$ as a linear combination of terms $U^i D^j$. We shall not pursue this direction here, but see for example [Vaz].

4. q-REFLECTION

Let $(\Gamma_n = (V, E, m), \Gamma'_n = (V, E', m'))$ be a pair of graded graphs with height function taking values in $[0, n]$, and such that (2) holds for some fixed $r$, when applied to all vertices $v$ such that $h(v) < n$. We call such a pair a partial qDGG of height $n$. We will construct a partial qDGG $(\Gamma_{n+1}, \Gamma'_{n+1})$ of height $n+1$, and such that they agree with $(\Gamma_n, \Gamma'_n)$ up to height $n$.

Let us write $V_i = \{v \mid h(v) = i\}$. The height $n+1$ vertices of (both) $\Gamma_{n+1}$ and $\Gamma'_{n+1}$ will be given by the set $V_{n+1} = \{v^1, v^2, \ldots, v^r \mid v \in V_n\} \cup \{w' \mid w \in V_{n-1}\}$. There will be two kinds of edges. For $\Gamma_{n+1}$, we construct

1. $r$ edges $(v, v^1), (v, v^2), \ldots, (v, v^r)$ for each $v \in V_n$ which have weight $m(v, v') := 1$.

2. An edge $(v, w')$ for each edge $(w, v)$ of $\Gamma'$, where $v \in V_n$ and $w \in V_{n-1}$. This edge has weight $m(v, w') := q m'(w, v)$.

And for $\Gamma'_{n+1}$, we construct

1. $r$ edges $(v, v^1), (v, v^2), \ldots, (v, v^r)$ for each $v \in V_n$ which have weight $m'(v, v') := 1$.

2. An edge $(v, w')$ for each edge $(w, v)$ of $\Gamma'$, where $v \in V_n$ and $w \in V_{n-1}$. This edge has weight $m'(v, w') := m(w, v)$.
We omit the proof of the following, which is the same as the corresponding result for differential posets [Sta] or signed differential posets [Lam].

**Proposition 5.** Suppose $(\Gamma_n, \Gamma'_n)$ is a partial qDGG of height $n$ and differential coefficient $r$. Then $(\Gamma_{n+1}, \Gamma'_{n+1})$ is a partial qDGG of height $n+1$ and differential coefficient $r$. Furthermore, $(\Gamma, \Gamma') = \lim_{n \to \infty} (\Gamma_n, \Gamma'_n)$ is a well-defined qDGG with differential coefficient $r$.

5. THE QUANTIZED FIBONACCI POSET

Let $(\Gamma, \Gamma')$ be a qDGG. If the edge sets of $\Gamma$ and of $\Gamma'$ are identical and in addition every edge weight $m(v, w)$ (and $m'(v, w)$) of $\Gamma$ (and $\Gamma'$) is a single power $q^i$ then we call $(\Gamma, \Gamma')$ a quantized differential poset. For then, $\Gamma(1)$ would be a differential poset in the sense of Stanley [Sta].

**Remark 2.** We could insist that $\Gamma = \Gamma'$ as graded graphs, but then in the construction of a quantization of the Fibonacci differential posets we would need to use half powers of $q$.

Define $\Gamma_0 = \Gamma'_0$ to be the graded graph with a single vertex $\emptyset$ with height 0. For each $r \in \{1, 2, \ldots\}$, we define the quantized $r$-Fibonacci poset to be the corresponding qDGG $(\text{Fib}_{(r)}, \text{Fib}'_{(r)})$ obtained from $(\Gamma_0, \Gamma'_0)$ via Proposition 5. The qDGG $(\text{Fib}_{(r)}, \text{Fib}'_{(r)})$ is a quantization of the Fibonacci differential poset of Stanley [Sta], or the Young-Fibonacci graph of Fomin [Fom]. We now describe $(\text{Fib}_{(r)}, \text{Fib}'_{(r)})$ explicitly, suppressing the parameter $r$ in most of the notation.

The vertex set $V$ of $(\text{Fib}_{(r)}, \text{Fib}'_{(r)})$ consists of words $w$ in the letters $1_1, 1_2, \ldots, 1_r, 2$ with height function given by summing the letters in the word (all the $1$'s have the same value). In the notation of the $q$-reflection algorithm, the vertices $v^1, \ldots, v^r$ are obtained from $v$ by prepending $1_1, 1_2, \ldots, 1_r$ respectively; the vertices $w^s$ are obtained from $w$ by prepending the letter 2. The edges $(v, w)$ are of one of the two forms:

1. $v$ is obtained from $w$ by removing the first 1 (one of the letters $1_1, 1_2, \ldots, 1_r$);
2. $v$ is obtained from $w$ by changing a 2 to one of the 1's, such that all letters to the left of this 2 is also a 2.

In either case, let $s(v, w)$ denote the number of letters preceding the letter which is changed or removed to go from $w$ to $v$. The edges $m(v, w)$ of form (1) have edge weight $m(v, w) = m'(v, w)q^{s(v, w)}$ in both $\text{Fib}_{(r)}$ and $\text{Fib}'_{(r)}$.

The edges of form (2) have edge weight $m(v, w) = q^{s(v, w)+1}$ in $\text{Fib}_{(r)}$, and edge weight $m'(v, w) = q^{s(v, w)}$ in $\text{Fib}'_{(r)}$.

For the rest of this section, we will restrict ourselves to $r = 1$, and write 1 instead of $1_1$. We now describe the weight of a path from $\emptyset$ to a word $w$ in $\text{Fib} = \text{Fib}_{(1)}$ or $\text{Fib}' = \text{Fib}'_{(1)}$. Given a word $w \in \text{Fib}$ one has a snakeshape ([Fom]) consisting of a series of columns of height one or two. For example,
for \( w = 21121 \) we have the shape

\[
\begin{array}{ccc}
& & 1 \\
& 2 & \\
3 & & \\
\end{array}
\]

Given such a snakeshape \( \lambda \), following Fomin we say that a Young-Fibonacci-tableau of shape \( \lambda \) is a bijective filling of \( \lambda \) with the numbers \{1, 2 \ldots, n\} so that:

1. In any height two column the lower number is smaller.
2. To the right of a height two column containing the numbers \( a \) and \( b \) none of the numbers in \([a, b]\) occur.
3. To the right of a height one column containing the number \( a \), no numbers greater than \( a \) occur.

For each number \( i \in \{1, 2 \ldots, n\} \), let \( p_i(T) \) denote the position of the column containing \( i \) in \( T \), counting from the left with the leftmost column being 0. Then set

\[
wt(T) = \prod_{i \in \text{lower row}} q^{p_i(T)} \prod_{i \in \text{upper row}} q^{p_i(T)+1} \quad \text{and} \quad wt'(T) = \prod_i q^{p_i(T)}.
\]

Fomin described a bijection between Young-Fibonacci-tableau \( T \) of shape \( \lambda = \lambda(w) \) and paths from \( \emptyset \) to \( w \) in Fib (or Fib'). For example, the tableau

\[
\begin{array}{ccc}
3 & & \\
2 & 7 & 6 \\
& 5 & 4 & 1 \\
\end{array}
\]

corresponds to the path \( \emptyset \rightarrow 1 \rightarrow 11 \rightarrow 21 \rightarrow 211 \rightarrow 221 \rightarrow 2121 \rightarrow 21121 \).

**Lemma 6.** Under this bijection the weight of path is equal to \( wt(T) \) in Fib, and equal to \( wt'(T) \) in Fib'.

**Proof.** This is straightforward, using the description of the bijection on p.394.

Thus we have \( f^w_{\text{Fib}} = \sum_T wt(T) \) and \( f^w_{\text{Fib}'} = \sum_T wt'(T) \) where the sum is over Young-Fibonacci tableau with shape \( \lambda(w) \). It is not clear whether there is a simple way to write the identity that results from Theorem 4.

6. The qDGG on permutations

Let \( V = \sqcup_{n \geq 0} S_n \) be the disjoint union of all permutations equipped with the height function \( h(w) = n \) if \( w \in S_n \). Define a graded graph \( \text{Perm} \) with vertex set \( V \) and edge set \( E \) consisting of edges \((v, w)\) whenever \( v \in S_{n-1} \) is obtained from \( w \in S_n \) by deleting the letter \( n \); define \( m(v, w) := q^{n-s} \), where \( 1 \leq s \leq n \) is the position of the letter \( n \) in \( w \). Define \( \text{Perm}' \) with the same vertex set and edges \((v, w)\) whenever \( v \in S_{n-1} \) is obtained from \( w \in S_n \) by deleting the first letter, followed by reducing all letters greater than the deleted letter by one; define \( m(v, w) := 1 \) always.

For example, in \( \text{Perm} \) there is an edge from 4123 to 41523 with weight \( q^3 \). In \( \text{Perm}' \) there is an edge from 1423 to 41523 with weight 1. The following result is a straightforward verification of the definitions.
Proposition 7. The pair \((\text{Perm}, \text{Perm}')\) is a qDGG with differential coefficient \(r = 1\).

Let \(\text{inv}(w)\) denote the number of inversions of a permutation \(w\). For the pair \((\text{Perm}, \text{Perm}')\), we have

\[ f^w_{\text{Perm}} = q^{\text{inv}(w)} \quad \text{and} \quad f^w_{\text{Perm}'} = 1. \]

Thus Theorem 4 expresses the identity (see [EC1])

\[ \sum_{w \in S_n} q^{\text{inv}(w)} = [n]_q! . \]

7. The qDGG on tableaux

Let \(Y_n\) denote the set of standard Young tableau \(P\) of size \(n\) with any shape (see [EC2]). We assume the reader is familiar with tableaux, and with Schensted insertion.

Let \(V = \cup_{i \geq 0} Y_i\) with the obvious height function. Define \(\text{Tab}\) to be the graded graph with vertex set \(V\), and edges \((P, P') \in Y_n \times Y_{n+1}\) whenever there is some \(k \in \{1, 2, \ldots, n+1\}\) so that \(P'\) is obtained from \(P\) by first increasing the numbers greater than or equal to \(k\) inside \(P\) by 1, and then Schensted inserting \(k\); declare \(m(P, P') = q^{n+1-k}\). Define \(\text{Tab}'\) to be the graded graph with vertex set \(V\) and edges \((P, P') \in Y_n \times Y_{n+1}\) whenever \(P'\) is obtained from \(P\) by removing \(n\); declare \(m(P, P') = 1\). The following result is straightforward.

Proposition 8. The pair \((\text{Tab}, \text{Tab}')\) is a qDGG with differential coefficient \(r = 1\).

Fix a standard Young tableau \(P \in Y_n\). There is a bijection from the set of paths \(p\) from the empty tableau \(\emptyset\) to \(P\) in \(\text{Tab}\), to the set of standard Young tableau of shape equal to the shape of \(T\). The bijection is obtained by taking the sequence of shapes encountered along \(p\), or equivalently, by taking the recording tableau of the sequence of Schensted insertions given by \(p\). The following Lemma is immediate.

Lemma 9. Suppose \(p\) is a path from \(\emptyset\) to \(P\), corresponding to a standard Young tableau \(Q\). Then the weight of \(p\) in \(\text{Tab}\) is equal to \(q^{\text{inv}(w(P,Q))}\), where \(w(P,Q) \iff (P,Q)\) under the Robinson-Schensted bijection.

It follows that Theorem 4 applied to Proposition 8 gives (3), with the terms labeled by permutations \(w\) on the left hand side grouped according to the insertion tableau of \(w\).

8. The qDGG on plane binary trees

A plane binary tree is a tree \(T\) embedded into the plane which has three kinds of vertices: (a) a unique root node \(r\) which has exactly 1 child, (b) a number of internal nodes with two children, and (c) a number of leaves with no children. The leaves are numbered \(\{0, 1, \ldots, n\}\) from left to right, where
$n$ is the number of internal nodes. Let $T_n$ denote the set of plane binary trees with $n$ internal nodes. By definition, $T_0$ consists of the tree $\emptyset$, which has a root $r$, no internal nodes, and a single leaf $0$.

We now describe a number of combinatorial operations on plane binary trees; see [AS] for further details. Given two plane binary trees $T_1 \in T_p$ and $T_2 \in T_q$ we can **graft** a new plane binary tree $T_1 \vee T_2 \in T_{p+q+1}$ by placing $T_1$ to the left of $T_2$ in the plane, identifying the two root nodes $r_1$ and $r_2$ to form a new internal node, and attaching a new root to this internal node:

Given a tree $T \in T_p$ and a position $i \in \{0, 1, \ldots, p\}$ indexing a leaf $v \in T$ we can **splice** $T$ at $v$ to obtain two trees $T_1 \in T_i$ and $T_2 \in T_p - i$ as follows: draw the unique path $P$ from $v$ to the root $r$. Then the edges of $T$ weakly to the left of $P$ form the tree $T_1$, while the edges of $T$ weakly to the right of $P$ form the tree $T_2$. Note that every internal node of $T$ is “given” to either $T_1$ or $T_2$. The following tree has been spliced at the *-ed leaf:

We write $SG(T, i) = T_1 \vee T_2$ to denote the composition of splicing and grafting.

Given a non-empty tree $T \in T_p$, we can obtain another tree $T^* \in T_{p-1}$ from $T$ by removing the leftmost (or 0) leaf $v$ and erasing the node $w$ which is joined to $v$:

Let $V = \bigcup_{i \geq 0} T_i$, with the obvious height function $h : V \to \mathbb{N}$. Define a graded graph $\text{Tree}$ with vertex set $V$, and edges $(T, T')$ whenever $T' = SG(T, i)$ for some $i$; declare that $m(T', T) := q^i$. Define a graded graph $\text{Tree}'$ with vertex set $V$, and edges $(T^*, T)$ for every $T \neq \emptyset$; declare that $m(T^*, T) := 1$.

**Proposition 10.** The pair $(\text{Tree}, \text{Tree}')$ is a qDGG with differential coefficient $r = 1$.

**Proof.** Let $T \in T_p$. Let $T' = SG(T, i)$, where $i \in \{1, 2, \ldots, p\}$. Then it is not difficult to see that $(T'^*) = SG(T^*, i - 1)$. This cancels out all the
terms in $(D_{\text{Tree}}U_{\text{Tree}} - qU_{\text{Tree}}D_{\text{Tree}})T$ except for the one corresponding to $SG(T, 0)^* = T$ which has coefficient $q^0 = 1$. □

To describe the identity of Theorem 4 explicitly, let us define a linear extension of $T \in \mathcal{T}_p$ to be a bijective labeling $e : T \to \{1, 2, \ldots, p\}$ of the internal nodes of $T$ with $\{1, 2, \ldots, p\}$, so that children are labeled with numbers bigger than those of their ancestors. Let $E(T)$ denote the set of linear extensions of $T$. Also, let us say that an internal node $v$ is to the left (resp. to the right) of an internal node $w$ if $v$ belongs to the left (resp. right) branch and $w$ belongs to the right (resp. left) branch of their closest (youngest) common ancestor.

If $e$ is a linear extension of $T \in \mathcal{T}_p$, then we may define a permutation $w_e \in S_p$ by reading the labels of the internal nodes from left to right. It is well known (see for example [LR]) that as $T$ varies over $\mathcal{T}_p$ and $e$ varies over $E(T)$ we obtain every $w \in S_p$ exactly once in this way. For example, the following are the three linear extensions of the same tree:

\[
\begin{align*}
2 & \rightarrow 3 & 4 & \rightarrow 1 \\
1 & \rightarrow 2 & 3 & \rightarrow 4 \\
1 & \rightarrow 3 & 4 & \rightarrow 2
\end{align*}
\]

corresponding to the permutations 2134, 3124, and 4123.

**Lemma 11.** Let $T \in \mathcal{T}_p$. Then

\[ f^{T}_{\text{Tree}} = \sum_{e \in E(T)} q^{\text{inv}(w_e)} \text{ and } f^{T}_{\text{Tree}'} = 1. \]

**Proof.** The claim for $T'$ is clear. For Tree, we will describe a bijection between $E(T)$ and paths from $\emptyset$ to $T$.

Let $e'$ be a linear extension of $T'$ and suppose that $T = T_1 \lor T_2$ is obtained from grafting a splice of $T'$. We may treat $T_1$ and $T_2$ as subtrees of $T'$, and in particular restrict $e'$ to $T_1$ and $T_2$. Thus we may define a labeling $e$ (depending on $e'$, $T'$, $T_1$, and $T_2$) of $T$ by declaring it to be equal to $e' + 1$ on $T_1 \cup T_2$, and equal to 1 on the new internal node present in $T$ but absent in $T'$. It is straightforward to see that $e \in E(T)$. Conversely, given $e \in E(T)$, it is easy to recover $e'$ and $T'$ by comparing the labels along the leftmost branch of $T_2$ with the labels along the rightmost branch of $T_1$.

Recursively applying this procedure we obtain the desired bijection between $E(T)$ and paths from $\emptyset$ to $T$. Finally, the number of new inversions created in each step of this procedure is equal to the number of internal nodes of $T_1$, which in turn is the exponent of $q$ in $m(T', T)$. This completes the proof. □
Thus Theorem 2 for \((\text{Tree}, \text{Tree}')\) amounts to grouping together the terms of the left hand side of \((3)\) into Catalan number many terms.

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