SECTIONAL CURVATURE FOR RIEMANNIAN MANIFOLDS WITH DENSITY

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Abstract. In this paper we introduce two new notions of sectional curvature for Riemannian manifolds with density. The trace of one of our weighted sectional curvatures is the Bakry-Emery Ricci tensor, which appears in the gradient Ricci soliton equation, while the trace of the other is the modified Ricci tensor that arises in the conformal Einstein equation. Under both notions of curvature we classify the constant curvature manifolds. We also prove generalizations of the theorems of Cartan-Hadamard, Synge, and Bonnet-Myers as well as a generalization of the (non-smooth) 1/4-pinched sphere theorem. The main idea is to modify the radial curvature equation and second variation formula and then apply the techniques of classical Riemannian geometry to these new equations.

1. Introduction

In this paper we are interested in the geometry of a Riemannian manifold \((M, g)\) with a smooth positive density function, \(e^{-f}\). A theory of Ricci curvature for these spaces goes back to Lichnerowicz \[Lic70, Lic71\] and was later developed by Bakry-Emery \[BE85\] and many others. It has turned out to be integral to developments in both Ricci flow and optimal transport and has thus experienced an explosion of results in the last few years, which we will not try to reference here. A notion of weighted scalar curvature also comes up in Perelman’s work \[Per\] and is related to his functionals for the Ricci flow, also see \[Lot07, CGY06, CGY11\]. The weighted Gauss curvature and the weighted Gauss Bonnet theorem in dimension two has also been studied in \[CHH+06, CM11\].

We introduce two new concepts of sectional curvature for manifolds with density. Given a fixed point \(p\) and direction vector \(V \in T_p M\), we let \(R^V : T_p M \to T_p M\) be the symmetric operator given by the formula

\[
R^V(U) = R(U, V)V - \nabla_U \nabla_V V - \nabla_V \nabla_U V - \nabla_{[U, V]} V.
\]

Following \[Pet06\] we will call \(R^V\) the \textit{directional curvature operator} in the direction of \(V\). \(R^V\) is also called the \textit{tidal force operator} in general relativity; see Chapters 8 and 12 of \[QN83\] for a discussion of the physical meaning. Note that \(R^V(V) = 0\), so we can think of \(R^V\) as acting on the orthogonal complement of \(V\) in \(T_p M\) which we denote by \(V^\perp\). If we normalize \(U\) and \(V\) to be unit length with \(U \in V^\perp\) then \(\sec(U, V) = g(R^V(U), U)\) where \(\sec(U, V)\), the sectional curvature of the plane spanned by \(U\) and \(V\).

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We define the weighted directional curvature operators, $R^V_f$ and $\overline{R}^V_f$, on $V^\perp$ as

$$R^V_f(U) = R^V(U) + \text{Hess}_f(V,V)U$$

$$\overline{R}^V_f(U) = R^V(U) + (\text{Hess}_f(V,V) + df(V)^2)U$$

and the weighted sectional curvatures, $\sec^V_f(U)$ and $\overline{\sec}^V_f(U)$ for unit perpendicular vectors $V$ and $U$ as

$$\sec^V_f(U) = g(R^V_f(U),U) = \sec(V,U) + \text{Hess}_f(V,V)$$

$$\overline{\sec}^V_f(U) = g(\overline{R}^V_f(U),U) = \sec(V,U) + \text{Hess}_f(V,V) + (df(V))^2$$

$$= \sec(V,U) + \frac{\text{Hess}_u}{u}(V,V)$$

where $u = e^f$.

We will see that these notions of sectional curvature come naturally from at least three places: the radial curvature equation, the second variation of energy formula, and formulae for Killing fields. Moreover, if we trace the weighted directional curvature operators over $V^\perp$ we obtain natural weighted Ricci curvatures

$$\sum_{i=1}^{n-1} g(R^V_f(E_i),E_i) = \text{Ric}(V,V) + (n-1)\text{Hess}_f(V,V)$$

$$\sum_{i=1}^{n-1} g(\overline{R}^V_f(E_i),E_i) = \text{Ric}(V,V) + (n-1)\text{Hess}_f(V,V) + (n-1)df \otimes df(V,V).$$

Up to rescaling $f$, the first tensor is sometimes called the Bakry-Emery Ricci tensor and the equation $\text{Ric} + (n-1)\text{Hess}_f = (n-1)cg$ for some constant $c$ is the gradient Ricci soliton equation. The equation $\text{Ric} + (n-1)\text{Hess}_f + (n-1)df \otimes df = \psi g$ where $\psi$ is a function on $M$ is also well studied as the conformal Einstein equation.

Note the obvious inequality $\overline{\sec}^V_f \geq \sec^V_f$. While the weighted directional curvature operators are symmetric operators on $V^\perp$, the asymmetrical placement of $U$ and $V$ emphasizes that $\sec^V_f(U) \neq \sec^V_f(V)$ (and similarly for $\overline{\sec}^V_f$). That is, the weighted sectional curvatures depend on the ordered pair $(V,U)$ and not only on the plane spanned by the vectors. This asymmetry arises naturally from the radial curvature equation and the second variation of energy formula because the equations depend on a chosen preferred geodesic or distance function. Also see Proposition 2.

We will discuss the motivation for these definitions coming from the radial curvature equation in section two. For the rest of the introduction we will try to motivate the reader by discussing some of the results.

The most fundamental fact about sectional curvature is that constant curvature characterizes the classical Euclidean, spherical, and hyperbolic geometries. Constant weighted sectional curvature also characterizes the spaces of constant curvature with natural densities.

We will write $\sec_f = \psi$ where $\psi$ is a real valued function on $M$ to mean that $\sec^V_f(U) = \psi(p)$ for all $p \in M$ and $U,V \in T_pM$ s.t. $|U| = |V| = 1$ and $g(U,V) = 0$. Equivalently we could write $R^V_f = \psi I$, $\forall V$ s.t. $|V| = 1$, where $I$ denotes the identity map on $V^\perp$. We define the condition $\overline{\sec}^V_f = \psi$ as well as $\sec_f \geq (\leq)\psi$, etc. similarly.
Recall that, in dimension larger than two, by Schur’s lemma the sectional curvature is equal to a function if and only if it is constant. When the weighted curvature is equal to a function we also have the following complete picture.

**Theorem 1.1 (Constant Curvature).** Let \((M^n, g, f)\) be a simply connected, complete Riemannian manifold with smooth density of dimension \(n > 2\). If \(\sec f = \psi\) or \(\overline{\sec}_f = \psi\) for some function \(\psi\), then \((M, g)\) has constant sectional curvature. Moreover,

1. If \(\sec f = c\) and \(f\) is non-constant, then \(g\) is the flat metric on \(\mathbb{R}^n\) and \(f\) is either a linear function and \(c = 0\) or \(f(x) = \frac{c}{2}||x - v||^2 + A\) where \(v \in \mathbb{R}^n\) and \(A \in \mathbb{R}\).
2. If \(\overline{\sec}_f = c\) and \(f\) is non-constant, then \(c = 0\) and \(u = e^f\) is the restriction of a coordinate function from the appropriate canonical embedding \(S^n \rightarrow \mathbb{R}^{n+1}, \mathbb{R}^n \rightarrow \mathbb{R}^{n+1},\) or \(H^n \rightarrow \mathbb{R}^{n+1}\).
3. \(\sec f = \psi\) if and only if \(\nabla f\) is a conformal field on \((M, g)\).
4. \(\overline{\sec}_f = \psi\) if and only if \(e^{-2f}g\) is also a metric of constant curvature.

**Remark 1.2.** The conclusion (1) of Theorem 1.1 is equivalent to \((M, g, f)\) being a constant curvature gradient Ricci soliton. Conclusion (4) of Theorem 1.1 says that the condition \(\overline{\sec}_f = \phi\) characterizes the conformal changes between two constant curvature metrics.

**Remark 1.3.** The conformal fields on spaces of constant curvature as well as the conformal changes between spaces of constant curvature are completely classified, in fact they are known in the Einstein case, see [Bri25, KR09].

**Remark 1.4.** We have stated the result for simply connected complete metrics to avoid a few pathological examples. In fact, many of the examples in (2) do not give complete examples of manifolds with density because the functions \(u = e^f\) vanish. The results are true locally as is discussed in Section 3. Also note that an equivalent formulation of (2) is that \(u\) is a first eigenfunction of the Riemannian Laplacian of \((M, g)\).

We also generalize some of the foundational results of the study of metrics of both non-negative and non-positive curvature to the weighted setting. First we generalize the Cartan-Hadamard theorem to the case where \(\overline{\sec}_f \leq 0\).

**Theorem 1.5 (Weighted Cartan-Hadamard Theorem).** If a complete Riemannian manifold admits a smooth function \(f\) such that \(\overline{\sec}_f \leq 0\), then \(M\) does not have any conjugate points. In particular, if \(M\) is simply connected then it is diffeomorphic to \(\mathbb{R}^n\).

Applying the result to the universal cover gives the following standard corollary.

**Corollary 1.6.** If a compact Riemannian manifold admits a function \(f\) such that \(\overline{\sec}_f \leq 0\), then \(\pi_1(M)\) is infinite and \(\pi_n(M) = 0\) for all \(n > 1\).

**Remark 1.7.** The lack of conjugate points also implies much more about the fundamental group, see [CS86].

**Remark 1.8.** There are simple examples of metrics with conjugate points that satisfy the weaker condition \(\sec f \leq 0\). We also prove a conjugate point estimate for the condition \(\overline{\sec}_f \leq K, K > 0\), but the result depends on bounds on the density \(f\), see Theorem 4.3.
In the case of positive curvature we also have the following basic results.

Theorem 1.9. Suppose a compact Riemannian manifold admits a smooth function $f$ such that $\sec f > 0$ then

1. If $M$ is even dimensional then every Killing field has a zero.
2. If $M$ is even dimensional and orientable, then $M$ is simply connected.
3. If $M$ is odd-dimensional, then $M$ is orientable.

Remark 1.10. Since $\sec f \geq \sec f$, these results are also true for $\sec f > 0$. In the classical case (1) is a result of Berger, while (2) and (3) are commonly referred to as Synge’s theorem.

We also prove the following generalization of the homeomorphic $1/4$-pinched sphere theorem. Our generalization will depend on the maximum and minimum of $u = e^f$, which we denote by $u_{\max}$ and $u_{\min}$.

Theorem 1.11. If $(M, g, f)$ is complete, simply connected and there is a smooth function $f$ and a constant $L$ such that

\[ \frac{1}{4} \left( \frac{u_{\max}}{u_{\min}} \right)^2 < L \leq \sec f \leq \left( \frac{u_{\min}}{u_{\max}} \right)^2, \]

then $M$ is homeomorphic to the sphere.

Remark 1.12. In this paper, we prove that the manifold is homotopic to the sphere. From the resolution of the Poincare conjecture, this gives the conclusion that the manifold is homeomorphic, but a more direct proof should be possible. The missing ingredient is a Alexandrov-Toponogov type triangle inequality for the weighted curvature, see Remark 3.6.

Remark 1.13. We do not know to what extent this theorem is optimal. Note that the hypothesis implies that $\frac{u_{\max}}{u_{\min}} \leq (4)^{1/4} \approx 1.414$, so the result only applies to small densities. Some other pinching phenomena for manifolds with density will also be examined in [WW09]. It would also be interesting to investigate a weighted version of the smooth $1/4$-pinched sphere theorem.

In addition to generalizing the classical results about sectional curvature to the weighted setting, another reason to study sectional curvature for manifolds with density is that understanding sectional curvature will enhance our understanding of weighted Ricci curvature. In this vein we prove the following theorem which generalizes a result from [WW09].

Theorem 1.14. If a complete Riemannian manifold supports a bounded function $f$ such that

\[ \text{Ric} + (n - 1)\text{Hess} f + (n - 1)df \otimes df \geq (n - 1)kg \]

for some $k > 0$, then $M$ is compact with finite fundamental group and

\[ \text{diam} M \leq \frac{u_{\max}}{u_{\min}} \sqrt[k]{k}, \]

Remark 1.15. In [WW09] a diameter bound was proven under the stronger hypothesis $\text{Ric} + (n - 1)\text{Hess} f \geq (n - 1)k$. There are simple examples showing that $f$ being bounded is a necessary assumption for $M$ to be compact. Also see [Mor06] for a similar result for the weighted diameter.

We also obtain the following generalization of a classical theorem of Bochner.
Theorem 1.16. If a compact Riemannian manifold admits a smooth function $f$ such that
\[ \text{Ric} < - (\Delta f + |\nabla f|^2) \ g, \]
then the isometry group is finite.

Remark 1.17. Lott has also proven that if a compact manifold has a function satisfying $\text{Ric} < -\text{Hess} f$ for a function $f$ then the isometry group is finite [Lot03]. His result follows from a Bochner formula for the Laplacian on 1-forms, while Theorem 1.16 follows from a formula for the Laplacian of the norm of a Killing field.

Remark 1.18. We prove many of the results mentioned above more generally in the context of a Riemannian manifold $(M, g)$ equipped with a vector field $X$, but we have only considered the gradient case where $X = \nabla f$ in the introduction for simplicity.

The paper is organized as follows. In the next section we discuss the motivation for the definitions which come from the Bakry-Emery Ricci curvatures and the radial curvature equation. We also discuss the relationship between our curvature and the curvature of the conformal change and a construction for Riemannian submersions called vertical warping. In section 3 we discuss the case of constant weighted curvature and prove Theorem 1.1 in section 4 we discuss conjugate radius estimates and prove the weighted Cartan Hadamard theorem; in section 5 we consider the second variation of energy formula and prove parts (2) and (3) of Theorem 1.9; in section 6 we prove Theorem 1.14; and in section 7 we discuss Theorem 1.11. In the final section we consider Killing Fields and prove part (1) of Theorem 1.9 and Theorem 1.16.

The purpose of this paper is to introduce the new notions of weighted sectional curvature and show that some of the basic results of Riemannian geometry can be generalized to this setting. Many of our arguments can be generalized to prove more technical results. These will appear in a future paper.

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2. Motivation and the fundamental equations

In this section we first discuss a motivation for the definitions of the weighted Ricci and scalar curvatures in terms of Bochner formulas and then show how a similar approach yields the definitions of $\text{sec}_f$ and $\text{sec}_f$.

2.1. Ricci and scalar curvature for manifolds with density. Recall the Bochner formula for the Riemannian Laplacian
\[ \frac{1}{2} \Delta |\nabla u|^2 = |\text{Hess} u|^2 + \text{Ric}(\nabla u, \nabla u) + g(\nabla \Delta u, \nabla u) \quad u \in C^3(M). \]
If $\text{Ric} \geq k$ and the dimension of $M$ is less than or equal to $n$, by Cauchy-Schwarz the Bochner formula becomes
\[ \frac{1}{2} \Delta |\nabla u|^2 \geq \frac{(\Delta u)^2}{n} + k|\nabla u|^2 + g(\nabla \Delta u, \nabla u) \quad u \in C^3(M). \]
For a smooth density, $f$ we consider the weighted (or drift or $f$-)Laplacian $\Delta_f = \Delta - D\nabla f$, then a simple calculation gives the Bochner formula

$$\frac{1}{2} \Delta_f |\nabla u|^2 = |\text{Hess} u|^2 + \text{Ric} (\nabla u, \nabla u) + \text{Hess} f (\nabla u, \nabla u) + g (\nabla \Delta_f u, \nabla u) \quad u \in C^2(M).$$

The $N$-Bakry Emery Ricci tensor is defined to be $\text{Ric}_f^N = \text{Ric} + \text{Hess}_f - \frac{df \otimes df}{N}$. If $\text{Ric}_f^N \geq k$ one can show that

$$\frac{1}{2} \Delta_f |\nabla u|^2 \geq \frac{(\Delta u)^2}{n + N} + k |\nabla u|^2 + g (\nabla \Delta_f u, \nabla u).$$

This looks exactly like the Bochner formula for a $n + N$ dimensional manifold. We can also consider the case where $N = \infty$, then we have the Bakry-Emery Ricci tensor $\text{Ric}_f = \text{Ric} + \text{Hess}_f$, and $\text{Ric}_f \geq k$ gives

$$\frac{1}{2} \Delta_f |\nabla u|^2 \geq k |\nabla u|^2 + g (\nabla \Delta_f u, \nabla u).$$

From these Bochner formulas one can prove versions of many comparison results for lower bounds on $\text{Ric}_f^N$ or $\text{Ric}_f$. All of the classical results generalize to the $\text{Ric}_f^N$ case but with all of the dimension dependent constants now depending on the synthetic dimension $n + N$ (see [Qia97] and Remark 2.2). We can think of $\text{Ric}_f$ as being an infinite dimensional (or dimension-less) condition and thus the results for lower bounds on $\text{Ric}_f$ are weaker, see for example [Mor05, WW09, MW12].

Remark 2.1. Bakry-Emery, in fact, extended the concept of a Bochner formula to much more general diffusion operators, not necessarily on manifolds. Operators that satisfy inequalities analogous to (2.1) are said to satisfy curvature dimension inequalities of curvature $k$ and dimension $n$. Lower bounds on Ricci curvature and curvature dimension inequalities can also be characterized in terms of optimal transport, even in the the non-smooth setting, see [vRS05, LV09, Stu06, Stu06b].

The tensor $\text{Ric}_f$ also comes up in the Ricci flow, where $\text{Ric}_f = \lambda g$ is the gradient Ricci soliton equation. In section 1.3 of [Per] Perelman introduces a weighted scalar curvature which can be motivated from the Bochner formula for the Dirac operator acting on spinors in an analogous way to the above. This weighted scalar curvature is also the integrand of the $F$-functional which he uses to give a gradient interpretation of the Ricci flow. Other definitions of weighted scalar curvature have been studied in [Lot07] and [CGY06]. Generalizations of Perelman’s functionals to the $N$-Bakry-Emery Ricci tensor case are also studied in [Cas12].

2.2. The radial curvature equation. Now to consider sectional curvature we examine the special case of the Bochner formula applied to a distance function. Fix $p \in M$ and let $r(x) = d(p, x)$. The function $r$ is smooth on $M \setminus C_p$ where $C_p$ denotes the cut locus of $p$. On $M \setminus C_p$ introduce geodesic polar coordinates $(r, \theta)$ where $\theta \in S^{n-1}$. The Bochner formula applied to the function $r$ then gives

$$\frac{\partial}{\partial r} \Delta_f r = -|\text{Hess} r|^2 - \text{Ric}(\partial_r, \partial_r) - \text{Hess} f (\partial_r, \partial_r).$$

The weighted Laplacian is also related to the weighted volume by the equation

$$L_{\partial_r} (e^{-f} \text{dvol}) = (\Delta_f r) e^{-f} \text{dvol}.$$

Putting these two equations together we can then see that control on $\text{Ric} + \text{Hess}_f$ gives some control on the measure $e^{-f} \text{dvol}$. 
The corresponding equations for a distance function that involve sectional curvature are the fundamental equations.

\begin{align}
L_{\partial_r} g &= 2\text{Hess}r \\
(2.5) \quad (\nabla_{\partial_r} \text{Hess}r)(X,Y) + \text{Hess}^2 r(X,Y) &= -R(X,\partial_r,\partial_r,Y)
\end{align}

Where \(\text{Hess}^2 r\) is the operator square of \(\text{Hess} r\), namely if \(S\) is a dual \((1,1)\)-tensor to \(\text{Hess} r\), \(\text{Hess} r(X,Y) = g(S(X),Y)\), then \(\text{Hess}^2 r = g(S(S(X),Y))\). Note that if we trace these equations we get equations (2.2) and (2.3). (2.5) is called the radial curvature equation.

For a manifold with density, the weighted sectional curvatures will control the growth of \(e^{-2f}g\) along a geodesic \(\gamma\). Consider the equation

\[
L_{\partial_r} (e^{-2f}g) = 2e^{-2f}(\text{Hess}r - g(\nabla f,\partial_r)g).
\]

We then obtain a weighted version of the radial curvature equation by considering the derivative of the right hand side of this equation. Set \(Hfr = \text{Hess}r - g(\nabla f,\partial_r)g\), then

\[
(\nabla_{\partial_r} (Hfr))(X,Y) = (\nabla_{\partial_r} \text{Hess}r)(X,Y) - \text{Hess} f(\partial_r,\partial_r)g(X,Y)
\]

\[
= -\text{Hess}^2 r(X,Y) - R(X,\partial_r,\partial_r,Y) - \text{Hess} f(\partial_r,\partial_r)g(X,Y)
\]

\[
= -\text{Hess}^2 r(X,Y) - \mathcal{R}^{J}_{J}(X,Y)
\]

Where \(\mathcal{R}^{J}_{J}(X,Y) = g(R^{\partial_r}(X),Y)\) and \(R^{\partial_r}(X)\) is the weighted directional curvature operator defined in the introduction. Note the similarity of this equation to (2.5).

We can make these equations more concrete by considering Jacobi fields. For a Jacobi field \(J\) along a unit speed radial geodesic, \(\gamma(r)\), with \(J \perp \dot{\gamma}\) the fundamental equations are

\[
\partial_r |J|^2 = 2\text{Hess}r(J,J)
\]

\[
\partial_r (\text{Hess}r(J,J)) = \text{Hess}^2 r(J,J) - R(J,\partial_r,\partial_r,J).
\]

The interesting thing here in the weighted case is that the curvatures \(\text{sec}_f\) appear. Let \(\mathcal{R}^{J}_{J}(X,Y) = g(\mathcal{R}^{\partial_r}(X),Y)\), then we have

\[
\partial_r (e^{-2f}|J|^2) = 2e^{-2f}(Hfr(J,J))
\]

\[
\partial_r (Hfr(J,J)) = D_{\partial_r} (\text{Hess}r(J,J)) - \text{Hess} f(\nabla r,\nabla r)|J|^2 - 2g(\partial_r,\nabla f)g(\dot{J},J)
\]

\[
= \text{Hess}^2 r(J,J) - 2g(\nabla r, X)g(J,J) - \mathcal{R}^{J}_{J}(J,J)
\]

\[
= (Hfr)^2(J,J) - \mathcal{R}^{J}_{J}(J,J).
\]

Which now looks even closer to the radial curvature equation (2.5).

Jacobi fields are the variation fields produced by variations of geodesics. So we can think of Jacobi fields as measuring the rate of the spreading of geodesics and of the fundamental equations as showing that sectional curvature controls this spreading. Thus, the weighted sectional curvatures control the rate of spreading of geodesics in a weighted sense by controlling the derivative of \(e^{-2f}|J|^2\) along geodesics.

Remark 2.2. In the case of weighted Ricci curvature, the N-Bakry Emery Ricci tensor has the interpretation as being the Ricci tensor applied to vectors tangent to \(M\) in a warped product metric of the form

\[
g_M + e^{-2f}g_F
\]
Where $F$ is a manifold of dimension $N - \dim(M)$. Considering warped products, however, does not lead to a definition of weighted sectional curvature because the submanifolds $M \times \{p\}$ are totally geodesic so the sectional curvatures measured with respect to the submanifold and ambient space are the same.

**Remark 2.3.** The approach above is similar to a construction in the theory of Riemannian submersions called vertical warping. Given a Riemannian submersion $\pi : N \to B$, the tangent space splits orthogonally into the distributions of vertical and horizontal fields. For a vector field $U$ write $U = U^h + U^v$ where $U^v$ is a vertical field and $U^h$ is a horizontal field, then we can define a new metric $g_f$ on $N$ via the formula
\[
g_f(U, V) = g(U^h, V^h) + e^{-2f}g(U^v, V^v).
\]
This constructs a new metric on $M$ such that $\pi$ is still a Riemannian submersion. This is referred to as vertical warping in [GW09] and such deformations are also found, for example, in [PW] where the warping is called an orthogonal conformal change.

To see how this relates to our weighted curvatures, view the distance function $r$ as a Riemannian submersion $r : M \setminus \{p\} \to \mathbb{R}$. The horizontal distribution is spanned by $\partial_r$, and the vertical distribution consists of the tangent spaces of the distance spheres. We can write $g = dr^2 + g_r$ where $g_r$ is the induced metric on the distance spheres. Vertical warping then produces the new metric $g_{f,r} = dr^2 + e^{-2f}g_r$. The second fundamental form of the distance spheres for this metric is exactly $e^{-2f}H_f$. However, the curvature of the metric $g_{f,r}$ is different from the weighted curvatures. In fact, the curvatures of the metrics $g_{f,r}$ have terms that will also depend on $\text{Hess} r$ so they depend on the function $r$ and not just on the given direction $\nabla r(p)$.

**Remark 2.4.** Sectional curvature is often introduced through the second variation of the energy formula. We have first motivated the definition of the weighted sectional curvature through the radial curvature equation because it is closer to the approach of Bakry-Emery and Lichnerowicz in the Ricci curvature case. We consider the second variation of energy formula in section 5. The weighted curvature also comes up in considering equations for Killing fields, as we will show in section 8.

### 2.3. Relationship to the conformal change.

The weighted curvatures are also different from the sectional curvatures of the conformal metric $h = e^{-2f}g$. The weighted sectional curvatures come from considering the Lie derivative of the weighted metric $e^{-2f}g$ along a geodesic for the metric $g$ while the conformal curvatures would come from differentiating along the geodesics of $h$. The formula for the $(4,0)$-curvature tensor of $h$ in terms of the curvature of $g$ is
\[
R^h = e^{-2f} \left( R^g + \left( \text{Hess}^g f + df \otimes df - \frac{1}{2} |df|^2 g \right) \circ g \right)
\]
Where $\circ$ denotes the Nomizu-Kulkarni product. In fact, this formula can be re-interpreted in the following way.

**Proposition 2.5.** Let $(M, g, f)$ be a Riemannian manifold with density and let $h = e^{-2f}g$ then
\[
\left( \mathcal{R}^f \right)^Z_f (U, U) = e^{2f} \left( \mathcal{R}^g \right)_U (Z, Z)
\]
In particular,
\[
\text{sec}_{f,r}(U, V) = e^{-2f} \left( \text{sec}_{r}(V, U) \right)
\]
Remark 2.6. This proposition shows that the map \((g, f) \mapsto (e^{-2f}g, -f)\) is an involution on the space of Riemannian metrics with density that preserves the conditions \(\text{sec} f = \phi, \text{sec} f \geq 0\) or \(\text{sec} f \leq 0\).

Proof. Let \(U, V\) be orthogonal vectors in \(g\). Then we have

\[
e^{2f}R^h(U, V, V, U) = R(U, V, V, U) + \text{Hess}^h f(U, U)g(V, V) + \text{Hess}^f g(U, U)\]
\[
+ df(U)^2 g(V, V) + df(V)^2 g(U, U) - |df|^2 g(U, U)g(V, V)
\]

Which gives us

\[
\mathbf{R}^V_f(U, U) = e^{2f} \left( R^h(U, V, V, U) - \text{Hess}^h f(U, U)g(U, U) + df(V)^2 h(U, U) \right)
\]
\[
= e^{2f} \left( R^h(U, V, V, U) + df(V)^2 h(U, U) \right)
\]
\[
= e^{2f} \left( \mathbf{R}^h_f(V, V) \right).
\]

Where we have used the formula for the Hessian under the change of metrics

\[
\text{Hess}^h f(U, V) = \text{Hess}^g f(U, V) + 2df(U)df(V) - |df|^2 g
\]

\[
\square
\]

2.4. The non-gradient case. To simplify the exposition we have only considered the gradient case above. However, many results will hold more generally in the setting of a smooth manifold \((M, g)\) equipped with a vector field \(X\). In this case we define

\[
R^V_X(U) = R^V(U) + \frac{1}{2} L_X g(V, V)U
\]
\[
\mathbf{R}^V_X(U) = R^V(U) + \left( \frac{1}{2} L_X g(V, V) + g(X, V)^2 \right) U
\]
\[
\text{sec}^V_X(U) = \text{sec}(V, U) + \frac{1}{2} L_X g(V, V)
\]
\[
\text{sec}^V_X(U) = \text{sec}(V, U) + \frac{1}{2} L_X g(V, V) + (g(X, V))^2
\]

While there is no globally defined function \(f\) with \(\nabla f = X\), many of our results will involve only arguments along a fixed geodesic \(\gamma\) in which case we can always find an anti-derivative for \(X\) along the geodesic by simply defining

\[
f_\gamma(t) = \int_0^t g(X, \dot{\gamma}) dt.
\]

We can then still make sense of the equations above along \(\gamma\), replacing \(e^{-2f}\) with \(e^{-2f_\gamma}\). On the other hand, there will be times where we require a global anti-derivative. For example this will come up when \(\gamma : [0, r_0] \to M\) is a closed geodesic. Then there is no guarantee that \(f_\gamma\) will have the same value at 0 and \(r_0\) unless \(X\) is a gradient field.
3. Constant Curvature

In this section we establish that our definitions of constant sectional curvature characterize natural canonical Riemannian manifolds with density in dimension larger than two.

First we consider the case \( \sec_X = \psi \) for some function \( \psi \). In dimension two we always have \( \sec = \phi \) and so \( \sec_X = \psi \) if and only if \( X \) is a conformal field. An obvious example in higher dimensions is a constant curvature metric with \( X \) a conformal field.

It is, in fact easy to see that these are the only examples. The idea is to exploit the lack of symmetry in the weighted sectional curvatures \( \sec_X(U,V) \neq \sec_X(V,U) \).

**Theorem 3.1.** Suppose that \((M^n, g)\) has \( n > 2 \). There is a vector field \( X \) on \((M,g)\) such that \( \sec_X = \psi \) for some function \( \psi : M \to \mathbb{R} \) if and only if \((M,g)\) is a space of constant curvature and \( X \) is a conformal field on \((M,g)\). Moreover, if \( \psi = K \) is constant then either \( X \) is a Killing field or \((M,g)\) is isometric to a domain of Euclidean space and \( X \) is a homothetic field satisfying \( L_X g = K g \).

**Proof.** Let \( U, V \) be perpendicular unit vectors in \( T_p M \), then

\[
\psi = \sec_X^U(V) = \sec(U,V) + L_X g(U,U)
\]

\[
\psi = \sec_X^V(U) = \sec(V,U) + L_X g(V,V)
\]

Since \( \sec(U,V) = \sec(V,U) \), we have \( L_X g(U,U) = L_X g(V,V) \), showing that \( X \) is a conformal field, \( L_X g = \phi g \), \( \phi : M \to \mathbb{R} \). Then, letting \( \{E_i\}_{i=1}^{n-1} \) be an orthonormal basis for the orthogonal complement of \( U \) we have

\[
(n-1)\psi = \sum_{i=1}^{n-1} \sec_X^U(E_i) = \text{Ric}(U,U) + (n-1)\phi
\]

So that \( \text{Ric} = (n-1)(\psi - \phi)g \). By Schur’s lemma, \( \psi - \phi \) must be constant, showing the metric has constant curvature.

This also shows that if \( \psi = K \) is constant if and only if \( \phi \) is. If \( \phi \) is zero, then \( X \) is Killing and \((M,g)\) has constant curvature \( K \). If \( \phi \neq 0 \), then \( X \) is a non-Killing homothetic field. The existence of such a field implies that \((M,g)\) is isometric to a domain in Euclidean space. \( \square \)

In the case \( \sec_X = \psi \) the same proof gives the following result.

**Lemma 3.2.** Suppose that \((M^n, g, f)\) has \( n > 2 \) and there is constant \( K \) such that \( \sec_x = \psi \) then \((M,g)\) has constant curvature \( \rho \) and \( X \) satisfies

\[
L_X g + X^\sharp \otimes X^\sharp = (\psi - \rho)g.
\]

When \( X = \nabla f \) this gives us the following theorem.

**Theorem 3.3.** \((M^n, g, f)\) has \( n > 2 \) and \( \sec_f = \psi \) if and only if \( g \) and \( h = e^{-2f} g \) have constant curvature. Moreover, \( \phi = Ke^f \) for some constant \( K \).

**Proof.** \( g \) has constant curvature by Lemma 3.2 and from Proposition 2.5 \( \sec^h_f = \psi e^{-2f} \). Therefore applying Lemma 3.2 to \( h \) tells us that \( h \) also has constant curvature. Conversely, if \( g \) and \( h \) are both constant curvature, the equation for the curvature tensor under conformal change shows that \( \text{Hess} u \) is a function times the metric, which implies that \( \sec_f = \phi \).
As we mention above, the conformal changes between Riemannian metrics with constant curvature, are completely classified. We delve briefly into the proof of this fact in order to obtain the final statement of the theorem. We have that \( \text{Hess} u = ( \psi - \rho ) u g \) where \( \rho \) is the curvature of the metric. The non-constant solutions to this equation are easily classified. To see this use a lemma of Brinkmann-Tashiro which states that if one has a non-constant solution to \( \text{Hess} u = \phi g \) for some function \( \phi \), then the metric must be of the form \( g = dr^2 + (u'(t))^2 g_N \) where \( u \) is a function of \( t \) and \( g_N \) is some fixed metric. Brinkmann [Bri25] showed that this is true locally and Tashiro [Tas65] showed it is true globally when the metric is complete, also see [OS92, JW].

Once we have these coordinates we can compute that \( \text{Hess} u = u'' g \) where prime denotes derivatives in the \( t \) direction. So we have that \( u \) is a solution to \( u'' = (\psi - \rho) u \). Differentiating this equation gives us \( u''' = (\psi u)' - \rho u' \). On the other hand the sectional curvature in these coordinates is given by \( \text{sec}(\partial_r, X) = -\frac{u'''}{u'} \).

Since \( \rho \) is also the sectional curvature these two equations combine to give us \( (\psi u)' = 0 \), i.e. \( \psi = Ku^{-1} \) for some constant \( K \).

In particular, we can see that if \( \overline{\text{sec}} f = K \) a constant and \( f \) is non constant, then \( K \) must be zero and we get the following classification in terms of the curvature \( \rho \).

**Example 3.4.** Suppose that \( (M^n, g, f) \) has \( n > 2 \) and \( \overline{\text{sec}} f = 0 \). If \( f \) is non-constant, then after normalizing \( \rho \) to be 1, 0, or -1 and possibly re-parametrizing \( r \) and rescaling the metric \( g_N \) below, the only possibilities are

1. \( \rho = 1, g = dr^2 + \sin^2(r) g_N \), where \( g_N \) is a metric of constant curvature 1, and \( u = \cos(r) \).
2. \( \rho = 0, g = dr^2 + g_N \) where \( g_N \) is a flat metric, and \( u = Ar \).
3. \( \rho = -1 \) and either
   a. \( g = dr^2 + \sinh^2(r) g_N \) where \( g_N \) is a metric of constant curvature 1, and \( u = \cosh(r) \).
   b. \( g = dr^2 + e^{2r} g_N \) where \( g_N \) is a flat metric, and \( u = e^r \).
   c. \( g = dr^2 + \cosh^2(r) g_N \) where \( g_N \) is a metric of constant curvature -1, and \( u = \sinh(r) \).

**Remark 3.5.** Many of the examples listed in the previous theorem do not give complete examples since we also must require that \( u = e^f \) is positive. In fact, the only complete examples we obtain are the (3a) and (3b) cases. In the case of (3a) if the metric is complete it must be hyperbolic space while in (3b) the metric could be a nontrivial noncompact quotient of hyperbolic space.

**Remark 3.6.** These examples already show that there is no obvious Toponogov triangle comparison type theorem for the conditions \( \overline{\text{sec}} f \geq 0 \) or \( \leq 0 \) as the hemisphere and hyperbolic space both admit densities with constant zero curvature. It also shows that \( \overline{\text{sec}} f \geq 0 \) or \( \leq 0 \) does not imply a triangle comparison theorem for
the metric \( h = e^{-2f} g \) since if \( g \) is the hemisphere then \( h \) is the hyperbolic space with the opposite curvature and vice-versa.

4. Conjugate Radius estimates

In this section we discuss Jacobi field estimates. First we discuss the simplest Jacobi field comparison, the Cartan-Hadamard Theorem and then we prove a theorem for the case of a positive upper bound on weighted curvature.

4.1. Weighted Cartan Hadamard theorem. The Cartan-Hadamard theorem states that manifolds with non-positive sectional curvature do not have conjugate points. First we show through an example that this theorem is not true for \( \sec X \leq 0 \).

Example 4.1. Suppose \((M, g)\) is a complete, noncompact Riemannian manifold with positive sectional curvature. By a theorem of Gromoll-Meyer \cite{GM69}, the manifold has a pole, i.e. there is at least one point \( O \) such that the exponential map at \( O \) is a diffeomorphism. Let \( f(x) = \frac{1}{2} d^2(x, O) \), then \( f \) is a smooth function. Moreover, since the manifold has positive curvature, standard comparison geometry shows that \( \text{Hess} f \leq g \). If the sectional curvature is also bounded above by a positive constant \( C \), then we have \( \sec_{-C} f \leq 0 \). However, any non-flat noncompact Riemannian manifold with positive sectional curvature has conjugate points. Concrete examples are the paraboloid or Hamilton’s cigar metric.

On the other hand, we show that the stronger condition \( \overline{\sec} X \leq 0 \) does imply the non-existence of conjugate points.

Theorem 4.2. Suppose that a manifold \((M, g, X)\) satisfies \( \overline{\sec} X \leq 0 \), then \((M, g)\) has no conjugate points.

Proof. This essentially follows from the fundamental equations mentioned in section 2. Let \( \gamma : [0, t_0] \to M \) be a unit speed geodesic and \( J \) a Jacobi field along \( \gamma \) which is perpendicular to \( \dot{\gamma} \). Let \( f = f_\gamma \) be the function

\[
f_\gamma(t) = \int_0^t g_{\gamma(r)}(X, \dot{\gamma}(r)) dr
\]

then we have

\[
\frac{d}{dt} \left( \frac{1}{2} e^{-2f} |J|^2 \right) = e^{-2f} \left( g \left( \dot{J} - g(X, \dot{\gamma})J, J \right) \right)
\]

and

\[
\frac{d}{dt} \left( g \left( \dot{J} - g(X, \dot{\gamma})J, J \right) \right) = g(\ddot{J}, J) + g(\dot{J}, \ddot{J}) - \frac{1}{2} L_X g(\dot{\gamma}, \dot{\gamma}) g(J, J) - 2g(X, \dot{\gamma}) g(\dot{J}, J)
\]

\[
= -R(J, \dot{\gamma}, \dot{\gamma}, J) - \frac{1}{2} L_X g(\dot{\gamma}, \dot{\gamma}) g(J, J) + g(\dot{J}, \ddot{J}) - 2g(X, \dot{\gamma}) g(\dot{J}, J)
\]

\[
= -R(J, \dot{\gamma}, \dot{\gamma}, J) - \frac{1}{2} L_X g(\dot{\gamma}, \dot{\gamma}) g(J, J) + |\dot{J} - g(X, \dot{\gamma})J|^2 - g(X, \dot{\gamma})^2 g(J, J)
\]

\[
\geq |\dot{J} - g(X, \dot{\gamma})J|^2 - \overline{\sec} X |J|^2.
\]

Then the assumption \( \overline{\sec} X \leq 0 \) gives us that \( \frac{d}{dt} \left( g \left( \dot{J} - g(X, \dot{\gamma})J, J \right) \right) \geq 0 \). If \( J(0) = 0 \), this implies that \( g \left( \dot{J} - g(X, \dot{\gamma})J, J \right) \geq 0 \), which gives us that \( \frac{d}{dt} \left( \frac{1}{2} e^{-2f} |J|^2 \right) \geq 0 \). Thus, the only way \( J(0) = J(t_0) = 0 \) is if \( J(t) = 0 \) for all \( 0 \leq t \leq t_0 \) and so there are no conjugate points.

\[\square\]
4.2. Positive Upper bound. Now we consider the case $\text{sec}_{X} \leq K$, for a positive constant $K$. Recall that if Riemannian manifold satisfies $\text{sec} \leq K$ for some $K > 0$ then any two conjugate points are distance greater than or equal to $\frac{\pi}{\sqrt{K}}$ apart. We generalize this result to the condition $\text{sec}_{f} \leq K$.

To do so we fix some notation. Given a fixed parametrized geodesic $\gamma$ we let $u = e^{f_{\gamma}}$ and let $u_{\max}$ and $u_{\min}$ be the maximum and minimum of $u$ on the geodesic. While the function $f_{\gamma}$ depends on the parametrization of $\gamma$ we note that the ratio $\frac{u_{\min}}{u_{\max}}$ does not. We then have a conjugate radius estimate in terms of $\frac{u_{\min}}{u_{\max}}$.

**Theorem 4.3.** If $\gamma$ is a geodesic such that $\text{sec}_{X}(\dot{\gamma}, E) \leq K$ for all $|E| = 1$, $E \perp \dot{\gamma}$ then the distance between any two conjugate points of $\gamma$ is greater than or equal to $\frac{u_{\min}}{u_{\max}} \cdot \frac{\pi}{\sqrt{K}}$.

**Remark 4.4.** Note that this estimate gives another proof of the Weighted Cartan-Hadamard theorem by applying it for $K \to 0$ for a fixed geodesic $\gamma$ with $\text{sec}_{f}(\dot{\gamma}, E) \leq 0$. In particular, since there are metrics with $\text{sec}_{f} \leq K$ that have conjugate points, no such estimate is true for $\text{sec}_{f} \leq K$.

Let $J(t)$ a Jacobi field along $\gamma$ with $J(0) = 0$ and let $\phi = \ln(\frac{1}{2}e^{-2f_{\gamma}}|J|^{2})$. If $J(a) = 0$ then $\phi \to -\infty$ at $a$. The derivative of $\phi$ is

$$\frac{d\phi}{dt} = \frac{2}{|J|^{2}} \left( g(\dot{J}, J) - g(X, \dot{\gamma})|J|^{2} \right)$$

Define $\lambda(t) = \frac{1}{2}e^{2f_{\gamma}} \frac{d\phi}{dt}$. The reason for this choice of $\lambda$ is the following equation.

**Lemma 4.5.** If $\text{sec}_{X} \leq K$ then $\lambda(t)$ satisfies the Riccati inequality

$$\frac{d\lambda}{dt} \geq -\frac{\lambda^{2}}{u^{2}} - Ku^{2}$$

where $u = e^{f}$.

**Proof.**

$$\frac{d\lambda}{dt} = u^{2} \left( \frac{d}{dt} \left( g(\dot{J}, J) - g(X, \dot{\gamma})|J|^{2} \right) \right) \geq \frac{2}{|J|^{2}} \left( g(\dot{J}, J) - g(X, \dot{\gamma}) \right)^{2} e^{-2f_{\gamma}}|J|^{4}$$

$$= u^{2} \left( \frac{|\dot{J} - g(X, \dot{\gamma})J|^{2} - 2 \left( g(\dot{J} - g(X, \dot{\gamma})J, J) \right)^{2}}{|J|^{4}} - \frac{\text{sec}_{X}^{2} |J|^{4}}{|J|^{4}} \right)$$

$$\geq -\frac{\lambda^{2}}{u^{2}} - Ku^{2}$$

Where we have used the formula

$$\frac{d}{dt} \left( g \left( J - g(X, \dot{\gamma})J, J \right) \right) \geq |J - g(X, \dot{\gamma})J|^{2} - \frac{\text{sec}_{X}^{2}}{|J|^{4}}$$

and Cauchy-Schwarz.

**Proof of Theorem 4.3.** We have

$$\frac{d\lambda}{dt} \geq -\frac{\lambda^{2}}{u^{2}} - Ku^{2} \geq -\frac{\lambda^{2}}{u_{\min}^{2}} - Ku^{2}_{\max}.$$
In general, we can then get a lower bound for $\lambda$ in terms of the solution to the corresponding Ricatti equation. To see this explicitly, let $\varepsilon > 0$, then we have

$$\int_t^t \frac{\dot{\lambda}}{\lambda^2 u_{\min}^2 + Ku_{\max}^2} dt \geq -t + \varepsilon$$

Integrating gives us

$$\arctan\left(\frac{\lambda(t)}{u_{\min}u_{\max}\sqrt{k}}\right) \geq \arctan\left(\frac{\lambda(\varepsilon)}{u_{\min}u_{\max}\sqrt{k}}\right) - \frac{u_{\max}\sqrt{K}}{u_{\min}}(t - \varepsilon)$$

As $\lambda \to -\infty$ as $\varepsilon \to 0$ and tangent is an increasing function we obtain

$$\arctan\left(\frac{\lambda(t)}{u_{\min}u_{\max}\sqrt{k}}\right) \geq -\frac{\pi}{2} - \frac{u_{\max}\sqrt{K}}{u_{\min}}t$$

$$\lambda(t) \geq (u_{\min}u_{\max}\sqrt{k}) \cot\left(\frac{u_{\max}\sqrt{K}}{u_{\min}}t\right)$$

This shows that $\lambda$ can not diverge to $-\infty$ for $t < \frac{u_{\min}u_{\max}}{u_{\max}\sqrt{K}}$. □

5. Second variation of energy formula and Synge’s theorem

We now discuss how the weighted curvatures appear in the formula for the second variation of energy of a path. Recall that the energy of a path $c : [a, b] \to \mathbb{R}$ in a Riemannian manifold is

$$E(c) = \frac{1}{2} \int_a^b |\dot{c}|^2 dt$$

where $\dot{c}$ here and below will denote derivative in the $t$ direction. The critical curves of the energy are the geodesics and the formula for the second variation of energy of geodesic is

$$\frac{d^2 E}{ds^2}|_{s=0} = \int_a^b |\dot{\gamma}|^2 - R(V, \dot{\gamma}, \dot{\gamma}, V)dt + g\left(\frac{\partial^2 \dot{\gamma}}{\partial s^2}, \dot{\gamma}\right)$$

$$\left|_{a}^{b}\right| = I_{[a,b]}(V, V) + g\left(\frac{\partial^2 \dot{\gamma}}{\partial s^2}, \dot{\gamma}\right)$$

where $\dot{\gamma} : [a, b] \times (-\varepsilon, \varepsilon) \to M$ is a variation of the geodesic $\gamma(t) = \bar{\gamma}(t, 0)$, $V(t) = \frac{\partial \gamma}{\partial s}|_{s=0}$ is the variation field, and

$$I_{[a,b]}(V, V) = \int_a^b |\dot{\gamma}|^2 - R(V, \dot{\gamma}, \dot{\gamma}, V)dt$$

is the index form. Recall from section two that weighted directional curvature operators along $\gamma$ are

$$R_X^\gamma(U, V) = R(U, \dot{\gamma}, \dot{\gamma}, V) + \frac{1}{2}L_X g(\dot{\gamma}, \dot{\gamma}) g(U, V)$$

$$\overline{R}^\gamma_X(U, V) = R(U, \dot{\gamma}, \dot{\gamma}, V) + \frac{1}{2}L_X g(\dot{\gamma}, \dot{\gamma}) g(U, V) + g(X, \dot{\gamma})^2 g(U, V).$$
and that the weighted sectional curvatures are given by $\sec^w_X(U) = R^w_X(U, U)$ and $\sec^w_X(U) = R^w_X(U, U)$ where $U$ is a unit vector perpendicular to $\bar{\gamma}$. We can modify the formula for the index form to involve the weighted directional curvature operators.

**Proposition 5.1.** For the triple $(M, g, X)$ we have the following formulas for the Index form along a geodesic $\gamma$.

\[
(5.1) I_{[a,b]}(V, V) = \int_a^b |\dot{V}|^2 - R^w_X(V, V) - 2g(\dot{\gamma}, X)g(V, \dot{V})dt + g(\dot{\gamma}, X)|V|^2 \bigg|_a^b
\]

\[
(5.2) = \int_a^b |\dot{V} - g(\dot{\gamma}, X)V|^2 - R^w_X(V, V)dt + g(\dot{\gamma}, X)|V|^2 \bigg|_a^b
\]

**Proof.** To obtain the first formula we write

\[
I_{[a,b]}(V, V) = \int_a^b |\dot{V}|^2 - R^w_X(V, V) + \frac{1}{2}L_X(\dot{\gamma}, \dot{\gamma})|V|^2 dt
\]

\[
= \int_a^b |\dot{V}|^2 - R^w_X(V, V) + \left( \frac{d}{dt}g(\dot{\gamma}, X) \right)|V|^2 dt
\]

\[
= \int_a^b |\dot{V}|^2 - R^w_X(V, V) - g(\dot{\gamma}, X)\frac{d}{dt}|V|^2 + \frac{d}{dt}(g(\dot{\gamma}, X)|V|^2) dt
\]

\[
= \int_a^b |\dot{V}|^2 - R^w_X(V, V) - 2g(\dot{\gamma}, X)g(V, \dot{V})dt + g(\dot{\gamma}, X)|V|^2 \bigg|_a^b
\]

To incorporate the strongly weighted curvature into the equation we complete the square

\[
|\dot{V} - g(\dot{\gamma}, X)V|^2 = |\dot{V}|^2 - 2g(\dot{\gamma}, X)g(V, \dot{V}) + g(\dot{\gamma}, X)^2|V|^2
\]

to obtain (5.2).

\[\square\]

**Remark 5.2.** It is interesting that in the case where $X = \nabla f$, the weighted sectional curvatures $\sec^w_X(\dot{\gamma})$ and $\sec^w_f(\dot{\gamma})$ also appear in the second variation formula for the weighted energy at a weighted geodesic, see [Mor06,Mor09]. In this paper we focus on the formulas involving the classical energy because then we can apply all of the machinery of classical Riemannian geometry involving injectivity radius, exponential maps, etc. However, it would be interesting and natural to also investigate the relationship between the weighted sectional curvatures and the weighted energy and geodesics.

Our first application of these formulas will be to generalize Synge’s theorem to the weighted setting. We have the following lemma for parallel fields around closed geodesics.

**Lemma 5.3.** Let $(M, g, X)$ be a Riemannian manifold equipped with a smooth vector field $X$ which contains a closed geodesic $\gamma$ which supports a unit parallel field perpendicular to $\bar{\gamma}$. If either $\sec_X > 0$, or $X = \nabla f$ and $\sec_f > 0$, then there is smooth closed curve which is homotopic to $\gamma$ and has shorter length.

**Proof.** First consider the case $\sec_X > 0$. For a parallel field $V$ along a geodesic $\bar{\gamma}$ implies

\[
\frac{d^2E}{ds^2} \bigg|_{s=0} = - \int_a^b R^w_X(V, V)dt + g(\dot{\gamma}, X) \bigg|_a^b + g \left( \frac{\partial^2 \gamma}{\partial s^2}, \dot{\gamma} \right) \bigg|_a^b.
\]
If the geodesic is closed then the boundary terms cancel and from \( \sec_X > 0 \) we obtain

\[
\frac{d^2E}{ds^2}\bigg|_{s=0} = - \int_a^b R_X^\gamma(V,V)dt < 0
\]

Which shows that the closed curve obtained from the variation has smaller length than the original closed geodesic.

When \( X = \nabla f \) and \( \sec f > 0 \), let \( Y = e^fV \), then

\[
\dot{Y} = g(X,\dot{\gamma})e^fV = g(X,\dot{\gamma})Y
\]

Applying (5.2) to the variation field \( Y \) we also get that the boundary terms cancel and we obtain

\[
\frac{d^2E}{ds^2}\bigg|_{s=0} = - \int_a^b \overline{R}_X^\gamma(Y,Y)dt < 0
\]

Again showing that there is a closed curve with smaller length. \( \square \)

The proof of Synge’s theorem now goes exactly as in the classical case.

**Theorem 5.4** (Synge’s Theorem). Suppose that \( M \) is a compact manifold supporting a vector field \( X \) such that either \( \sec_X > 0 \), or \( X = \nabla f \) and \( \sec f > 0 \), then

1. If \( M \) is even dimensional and orientable, then \( M \) is simply connected.
2. If \( M \) is odd-dimensional, then \( M \) is orientable

**Proof.** The argument of Synge proceeds by contradiction and shows that if the topological conclusions do not hold then there is a closed geodesic with a parallel field which minimizes length in its homotopy class, see e.g. Theorem 26 of [Pet06]. Applying Lemma 5.3 then gives the desired contradiction in the weighted setting. \( \square \)

## 6. Diameter Estimate

Now we use the second variation of energy formula to prove a diameter estimate. We consider the trace of \( \overline{R}_X^\gamma(U,V) \) which we will denote by

\[
\overline{\text{Ric}}_X(\dot{\gamma},\dot{\gamma}) = \text{Ric}(\dot{\gamma},\dot{\gamma}) + \frac{n-1}{2}L_Xg(\dot{\gamma},\dot{\gamma}) + (n-1)g(X,\dot{\gamma})^2.
\]

Along a geodesic \( \gamma: [0,l] \to M \) with a proper variation, \( V \), (5.2) becomes

\[
\frac{d^2E}{ds^2}\bigg|_{s=0} = \int_0^l |\dot{V} - g(\dot{\gamma},X)V|^2 - \overline{\text{Ric}}_X^\gamma(V,V)dt
\]

Choose \( E \) to be a unit length parallel field along \( \gamma \) such that \( E \perp \dot{\gamma} \), let \( \phi(t) \) be a function such that \( \phi(0) = 0 \) and \( \phi(l) = 0 \), and let \( V = \phi e^fE \). Then we have

\[
\dot{V} - g(\dot{\gamma},X)V = \dot{\phi}e^fE
\]

Plugging \( V \) into the second variation formula then gives

\[
\frac{d^2E}{ds^2}\bigg|_{s=0} = \int_0^l e^{2f} \left( (\phi)^2 - \phi^2 \overline{\text{Ric}}_X^\gamma(V,V) \right)dt
\]

\[
= - \int_0^l e^{2f} \left( \phi\dot{\phi} + 2\dot{f}\phi + \phi^2\overline{\text{Ric}}_X^\gamma(V,V) \right)dt
\]
In order to obtain the Ricci curvature we let $E_i$ be an orthonormal basis of parallel fields of the orthogonal complement of $\dot{\gamma}$. Then, summing the second variations and assuming $\text{Ric}_X(\dot{\gamma}, \dot{\gamma}) \geq (n-1)k$ we obtain

$$\sum_{i=1}^{n-1} \frac{d^2 E_i}{ds^2}|_{s=0} = -(n-1) \int_0^t e^{2f} \left( \ddot{\phi} \phi + 2 \dot{f} \phi \dot{\phi} + \frac{\phi^2}{n-1} \text{Ric}_X(\dot{\gamma}, \dot{\gamma}) \right) dt$$

$$\leq -(n-1) \int_0^t e^{2f} \left( \ddot{\phi} \phi + 2 \dot{f} \phi \dot{\phi} + k \phi^2 \right) dt$$

As comparison functions we consider functions $\phi$ such that the right hand side vanishes.

**Definition 6.1.** Given a geodesic $\gamma$ and a constant $k$, Define $\phi_{f,\gamma,k}$ to be the solution to

$$(6.1) \quad \ddot{\phi} + 2 \dot{f} \phi + k \phi = 0 \quad \phi(0) = 0 \quad \phi'(0) = 1.$$ 

Let $L_{f,\gamma,k}$ to be the smallest positive number such that $\phi(L_{\gamma,k}) = 0$ and define $L_{f,\gamma,k} = \infty$ if $\phi_{f,\gamma,k}$ is positive for all time.

We have the following comparison principle in terms of $L_{f,\gamma,k}$.

**Lemma 6.2.** Suppose that $\gamma$ is a geodesic such that $\overline{\text{Ric}}_X(\dot{\gamma}, \dot{\gamma}) \geq (n-1)k$ then $\gamma$ does not minimize past the length $L_{f,\gamma,k}$.

**Proof.** For the proof we set $L_{f,\gamma,k} = L$ and $\phi_{f,\gamma,k} = \phi$. Let $l > L$ and define $\psi(t) = \phi \left( \frac{L}{l} t \right)$ along the geodesic $\gamma(t)$, then we have $\psi(0) = \psi(l) = 0$ and applying the second variation of arclength formula to the field $V = \psi e^{2f} E_i$ we obtain

$$\sum_{i=1}^{n-1} \frac{d^2 E_i}{ds^2}|_{s=0} = \int_0^l e^{2f} \left( \ddot{\psi}(t)^2 - \frac{\psi^2(t)}{n-1} \overline{\text{Ric}}_X(\dot{\gamma}, \dot{\gamma}) \right) dt$$

$$\leq \int_0^l e^{2f} \left( \ddot{\phi} \phi + k \phi(t)^2 \right) dt$$

$$= \int_0^l e^{2f} \left( \left( \frac{L}{l} \right)^2 \left( \ddot{\phi} \left( \frac{L}{l} t \right) \right)^2 - k \phi \left( \frac{L}{l} t \right)^2 \right) dt$$

$$< \int_0^l e^{2f} \left( \left( \ddot{\phi} \left( \frac{L}{l} t \right) \right)^2 - k \phi \left( \frac{L}{l} t \right)^2 \right) dt$$

$$= \frac{l}{L} \int_0^L e^{2f} \left( \ddot{\phi} (r)^2 - k \phi (r)^2 \right) dr$$

$$= \frac{l}{L} \int_0^L e^{2f} \left( \ddot{\phi} \phi + 2 \dot{f} \phi \dot{\phi} + k \phi^2 \right) dr$$

$$= 0$$

Where we have used the linear change of variables $r = \frac{L}{l} t$. Since the sum of the variations is negative, at least one of the variations has negative second variation, showing the $\gamma$ is not minimizing. \qed

The comparison function $\phi$ is also related to the Ricatti equation we encountered in Section 4.
Proposition 6.3. If a positive function $\phi$ is a solution to $\ddot{\phi} + 2f \dot{\phi} + k \phi = 0$ then the function $\lambda(t) = \frac{e^{2f} \dot{\phi}}{\phi}$ satisfies

$$\dot{\lambda} = -\frac{\lambda^2}{u^2} - ku^2$$

Proof. This just follows from direct calculation

$$\dot{\lambda} = e^{2f} \left( \frac{\ddot{\phi} + 2f \dot{\phi}}{\phi} - \frac{(\dot{\phi})^2}{\phi^2} \right)$$

$$= e^{2f} \left( -k - \frac{(\dot{\phi})^2}{\phi^2} \right)$$

$$= -\frac{\lambda^2}{u^2} - ku^2.$$ 

We note that, even when $k > 0$ it is possible to have $L_{f,k} = \infty$, a simple example is when $f = 1$, then $\phi(t) = te^{-t}$. It is thus a relevant question to find conditions on $f$ which imply that $L_{f,k}$ is finite. Using the Ricarti equation we can see it is finite when $f$ is bounded. Combined with Lemma 6.2 the following result implies the diameter estimate mentioned in the introduction.

Lemma 6.4. Suppose $f$ is bounded along a geodesic $\gamma$ and $k > 0$ then $L_{f,k} \leq \frac{u_{\text{max}}}{u_{\text{min}}} \cdot \frac{\pi}{\sqrt{k}}$

Proof. We argue as in the conjugate radius estimate section, we have

$$\dot{\lambda} = -\frac{\lambda^2}{u^2} - ku^2$$

$$\leq -\frac{\lambda^2}{u_{\text{max}}^2} - ku_{\text{min}}^2$$

We then obtain

$$\lambda(t) \leq u_{\text{min}} u_{\text{max}} \sqrt{k} \cot \left( \frac{u_{\text{min}}}{u_{\text{max}}} \sqrt{k} t \right)$$

since the right hand side goes to $-\infty$ at $t_0 = \frac{u_{\text{max}}}{u_{\text{min}}} \frac{\pi}{\sqrt{k}}$. $\lambda$ must go to $-\infty$ at some earlier time, but the only way this is possible is if $\phi$ goes to zero at some $t < t_0$. □

7. Pinching

In this section we present the proof of Theorem 1.11. What we will show is that the conjugate radius and second variation estimates we already have combined with classical methods give a proof that any such manifold is a homotopy sphere. We will go into less detail in many of the arguments in this section and instead reference the textbooks [dC92, Pet06, Kli82].

For submanifolds $A$ and $B$ in $M$ define the path space as

$$\Omega_{A,B}(M) = \{ \gamma : [0,1] \rightarrow M, \gamma(0) = A, \gamma(1) = B \}$$

We consider the Energy $E : \Omega_{A,B}(M) \rightarrow \mathbb{R}$ and variation fields tangent to $A$ and $B$ at the end points. The critical points are then the geodesics perpendicular to $A$ and $B$ and we say that the index of such a geodesic is $\geq k$ if there is a $k$-dimensional space of variation fields along the geodesic which have negative second variation.
The proof of the diameter estimate in the previous section also gives the following index estimate for sectional curvature.

**Lemma 7.1.** Suppose that $\sec_X \geq k$, then if $\gamma$ is a geodesic of length longer than $L_{\gamma,k}$ than the index of $\gamma$ is greater than or equal to $(n-1)$.

**Proof.** The argument in the previous section allows us to construct a variation with negative second derivative out of each parallel field along $\gamma$. \hfill \Box

Combining this with Lemma 6.4 gives the following generalization of a sphere theorem of Berger which is Theorem 33 in [Pet06].

**Theorem 7.2.** If a compact Riemannian manifold has $\sec_f \geq K$ and

$$\text{inj}(M,g) \geq \frac{u_{\max} \pi}{u_{\min} 2\sqrt{K}}$$

Then $M$ is a homotopy sphere.

**Proof.** Under the hypothesis, every geodesic loop $\gamma$ such that $\gamma(0) = \gamma(l) = p$ must have length greater than or equal to $u_{\min} \frac{\pi}{2\sqrt{K}}$. Then Lemma 6.4 combined with Lemma 7.1 implies that every geodesic in $\Omega_{p,p}$ has index greater than or equal to $(n-1)$. This then implies that $M$ is $(n-1)$ connected and thus a homotopy sphere see Theorems 32 and 33 of [Pet06] along with Theorem 2.5.16 of [Kli82]. \hfill \Box

This shows that the key to proving a sphere theorem is to prove injectivity radius estimates. In the even dimensional case an injectivity radius estimate follows from Theorem 4.3 and Lemma 5.3.

**Theorem 7.3.** Suppose that $M$ is a compact even dimensional simply connected manifold such that $0 \leq \sec_f \leq L$ then $\text{inj}(M,g) \geq \frac{u_{\min} \pi}{u_{\max} \sqrt{L}}$.

**Proof.** Suppose that $\text{inj}(M,g) < \frac{u_{\min} \pi}{u_{\max} \sqrt{L}}$. Then from Theorem 4.3 the conjugate radius is larger than the injectivity radius. This tells us that there is a closed geodesic of length $\frac{1}{2}\text{inj}_M$. From the proof of Synge’s theorem, when the manifold is orientable and even-dimensional it is possible to construct a parallel field along the geodesic and from Lemma 5.3 there is a variation which decreases the length of this closed curve. However, it is possible to show that this leads to conjugate points of smaller distance apart, a contradiction, see the proof of Theorem 30 of [Pet06]. \hfill \Box

The odd dimensional case is more difficult where the injectivity radius estimate is due to Klingenberg in the classical case. However, from what we have already proved, Klingenberg’s arguments carry over to the weighted setting. First we have the homotopy lemma.

**Lemma 7.4** (Klingenberg’s homotopy lemma). Suppose that a Riemannian manifold $(M,g)$ has the property that no geodesic segment of length less than $\pi$ contains a conjugate point. Suppose that $p,q \in M$ such that $p$ and $q$ are joined by two distinct geodesics $\gamma_0$ and $\gamma_1$ which are homotopic. Then there exists a curve in the homotopy $\alpha_{t_0}$ such that

$$\text{length}(\alpha_{t_0}) \geq 2\pi - \min\{\text{length}(\gamma_i)\}$$

**Proof.** This is usually stated with the conjugate point estimate replaced with the condition $\sec \leq 1$. However, as is pointed out in 2.6.5 of [Kli82], the lemma holds with the same proof in this generality. \hfill \Box
Now we can prove the injectivity radius estimate in all dimensions.

**Theorem 7.5.** Suppose that \((M, g, f)\) is complete simply connected and satisfies
\[
\frac{1}{4} \left( \frac{u_{\text{max}}}{u_{\text{min}}} \right)^2 \leq L \leq \text{sec}_f \leq \left( \frac{u_{\text{min}}}{u_{\text{max}}} \right)^2
\]
then \(\text{inj}(M, g) \geq \pi\)

**Proof.** Since \(\text{sec}_f \leq \left( \frac{u_{\text{min}}}{u_{\text{max}}} \right)^2\) Theorem 4.3 shows that the conjugate radius is less than or equal to \(\pi\) so that we can apply the homotopy lemma. On the other hand, Lemmas 6.4 and 7.1 show that \(\text{sec}_f > \frac{1}{4} \left( \frac{u_{\text{min}}}{u_{\text{max}}} \right)^2\) implies that any geodesic of length longer than \(\pi/2\) has index greater than or equal to 2. These are the only two elements about curvature used in the proof of the injectivity radius estimate, see for example the proof of Proposition 3.1 on page 276 of [dC92]. □

The proof of Theorem 1.11 now follows as Theorem 7.5 and Theorem 7.2 showing the manifold is a homotopy sphere.

### 8. Killing Fields

In this section we augment the previous considerations involving Jacobi fields and the second variation of energy formula by showing that the weighted sectional curvatures also come up naturally in formulas for Killing fields. Recall that for a Killing field \(V\) on a Riemannian manifold \((M, g)\) we have the following.

\[
\frac{1}{2} \nabla \left( |V|^2 \right) = -\nabla_V V
\]

\[
\frac{1}{2} \text{Hess} \left( |V|^2 \right) (Y, Y) = |\nabla_Y V|^2 - R(Y, V, V, Y)
\]

Now suppose we have a smooth manifold with smooth density \((M, g, f)\) and consider the function
\[
h = \frac{1}{2} e^{-2f} |V|^2.
\]

then we have the following formulas.

**Lemma 8.1.** Let \(Y\) be a tangent vector, then
\[
\nabla h = -e^{-2f} (\nabla_V V + |V|^2 \nabla f)
\]
\[
\text{Hess}_h(Y, Y) = -2df \otimes dh(Y, Y) + |\nabla_Y (e^{-f} V)|^2 - e^{-2f} (R(V, Y, Y, V) + |V|^2 \text{Hess}_f(Y, Y) + |V|^2 df(Y)^2)
\]

**Proof.** For the first equation, from the product rule we have
\[
dh = e^{-2f} \left( -|V|^2 df + d \left( \frac{1}{2} |V|^2 \right) \right)
\]
So that
\[
dh(Y) = -e^{-2f} (g(\nabla_V V, Y) + g(\nabla f, Y)|V|^2)
\]
Differentiating this equation then gives us
\[
\text{Hess}_h = e^{-2f} \left( 2|V|^2 df \otimes df - 2d \left( \frac{1}{2} |V|^2 \right) \otimes df - 2df \otimes d \left( \frac{1}{2} |V|^2 \right) - |V|^2 \text{Hess}_f + \text{Hess} \left( \frac{1}{2} |V|^2 \right) \right).
\]
Theorem 8.2. Suppose \((M, g)\) is a compact even dimensional manifold, if there is a function \(f\) such that \(\sec_f > 0\) then every Killing field has a zero.

Remark 8.3. From section 2 we know that the conformal metric \(e^{-2f}g\) also has positive weighted curvature with respect to the density \(-f\). This shows that Killing fields for the conformal metric also must have zeros. In terms of the original metric \(g\) these are fields satisfying \(LVg = 2df(V)g\).

Proof. The argument is by contradiction. If there is a vector field \(V\) which does not have a zero then the function \(h\) has a non-zero minimum at a point \(p\). At \(p\), we then have \(dh = 0\) which implies from the previous lemma that
\[g(\nabla V Y) = -g(\nabla f Y)|V|^2 \quad \forall Y \in T_p M\]

In particular, setting \(Y = V\) and using the skew-symmetry of \(\nabla V\) we obtain \(g(\nabla f V) = 0\) at \(p\).

Consider the skew symmetric endomorphism on \(A : T_p M \to T_p M\) given by
\[A(w) = \nabla w V + g(w, V)\nabla f - g(w, \nabla f) V\]

Then, using that \(V \perp \nabla f\) at \(p\) we can see that \(V|_p\) is in the null space of \(A\) as
\[A(V|_p) = (\nabla V + |V|^2 \nabla f)|_p = \nabla h|_p = 0\]

If the dimension of the manifold is even, then we know that \(A\) has another zero eigenvector for \(A\) which is perpendicular to \(V\), call it \(w\). Then we have
\[0 = A(w) = \nabla w V - g(w, \nabla f) V\]

Which implies that
\[\nabla w (e^{-f} V) = e^{-f} A(w) = 0\]

Plugging this into the formula for the Hessian of \(h\) in the previous lemma gives us
\[\text{Hess}(w, w) = -e^{-2f} \left( R(w, V, V, w) + |V|^2 \text{Hess}(w, w) + |V|^2 g(\nabla f, w)^2 \right)\]

The assumption \(\sec_f > 0\) then shows that \(\text{Hess}(w, w) < 0\), which is a contradiction to \(p\) being a minimum. \(\square\)
It is also interesting that, if we trace the equation for the Hessian of $h$ we obtain

$$
\Delta h + 2g(\nabla f, \nabla h) = |\nabla (e^{-f}V)|^2 - e^{-2f} (\text{Ric}(V, V) + (\Delta f + |\nabla f|^2) g)
$$

and not the tensor $\text{Ric}_f(V, V)$ on the right hand side. This leads to Theorem 1.16 mentioned in the introduction.

**Theorem 8.4.** Let $(M, g)$ is a compact oriented smooth manifold that admits a smooth function $f$ such that

$$\text{Ric} \leq -(\Delta f + |\nabla f|^2)g.$$  

If $V$ is a Killing field then $e^{-f}V$ is a parallel field. Furthermore, if $\text{Ric} < -(\Delta f + |\nabla f|^2)g$ then the isometry group is finite.

**Proof.** When $\text{Ric} \leq -(\Delta f + |\nabla f|^2)g$ we have

$$\Delta h + 2g(\nabla f, \nabla h) \geq |\nabla (e^{-f}V)|^2.$$  

Then

$$0 = \int_M (\Delta h + 2g(\nabla f, \nabla h)) e^{2f} d\text{vol}_g \geq \int_M |\nabla (e^{-f}V)|^2 e^{2f} d\text{vol}_g$$

so $e^{-f}V$ is parallel. When $\text{Ric} < -(\Delta f + |\nabla f|^2)g$ we obtain that $\int_M |\nabla (e^{-f}V)|^2 e^{2f} d\text{vol}_g < 0$ which shows there can be no Killing fields, implying the isometry group is finite. \qed

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