OPTIMAL GLOBAL ASYMPTOTIC BEHAVIOR OF THE SOLUTION TO A SINGULAR MONGE-AMPÈRE EQUATION

ZHIJUN ZHANG

School of Mathematics and Information Science
Yantai University, Yantai 264005, Shandong, China

Abstract. This paper is mainly concerned with the optimal global asymptotic behavior of the unique convex solution to a singular Dirichlet problem for the Monge-Ampère equation

$$\det D^2 u = b(x)g(-u), \quad u < 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = 0,$$

where $\Omega$ is a strict convex and bounded smooth domain in $\mathbb{R}^n$ with $n \geq 2$, $g \in C^1((0, \infty))$ is positive and decreasing in $(0, \infty)$ with $\lim_{s \to 0^+} g(s) = \infty$, $b \in C^\infty(\Omega)$ is positive in $\Omega$, but may vanish or blow up on the boundary properly. Our approach is based on the construction of suitable sub- and super-solutions.

1. Introduction. For a smooth, bounded and strict convex domain $\Omega$ of $\mathbb{R}^n$ ($n \geq 2$), we see that there is a function $\phi \in C^2(\overline{\Omega})$ with the properties [12]: $\phi(x) < 0$, $\forall x \in \Omega$, $\phi|_{\partial\Omega} = 0$, $\nabla \phi(x) \neq 0$, $\forall x \in \partial\Omega$, and $D^2 \phi(x)$ is positive definite on $\overline{\Omega}$, where $x = (x_1, x_2, \ldots, x_n) \in \overline{\Omega}$, $D^2 \phi = \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right)$ denotes the Hessian of $\phi(x)$. It follows that there exist $\delta_0 > 0$ and positive constants $c_1$ ($i = 1, 2$) such that

$$|\nabla \phi(x)| > 0, \quad \nabla \phi(x)(D^2 \phi(x))^{-1}(\nabla \phi(x))^T > 0, \quad \forall x \in \Omega_{\delta_0}, \quad (1)$$

and

$$c_1 d(x) \leq -\phi(x) \leq c_2 d(x), \quad \forall x \in \Omega, \quad (2)$$

where $\nabla \phi(x) = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \ldots, \frac{\partial \phi}{\partial x_n}\right)$ is the gradient of $\phi(x)$, $(\nabla \phi(x))^T$ denotes the transpose of $\nabla \phi(x)$, $(D^2 \phi(x))^{-1}$ denotes the inverse matrix of $D^2 \phi(x)$, $d(x) = \text{dist}(x, \partial\Omega)$, $x \in \Omega$ and $\Omega_{\delta_0} = \{x \in \Omega : d(x) < \delta_0\}$.

This paper is concerned with the global asymptotic behavior of the unique classical convex solution to the following singular Dirichlet problem for the Monge-Ampère equation

$$\det D^2 u = b(x)g(-u), \quad u < 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = 0,$$

where $\Omega$ is the above domain, $\det D^2 u$ is the so-called Monge-Ampère operator, $b$ satisfies

$(B_1): \quad b \in C^\infty(\Omega)$ is positive in $\Omega$;

2000 Mathematics Subject Classification. Primary: 35J75; Secondary: 35J96.

Key words and phrases. The Monge-Ampère equations, a singular boundary value problem, the unique convex solution, global asymptotic behavior.

The author is supported by NSF grant of P. R. China under grant 11571295.
(B₂): the problem
\[
det D^2u(x) = b(x), \quad u < 0, \quad x \in \Omega, \quad u|_{\partial \Omega} = 0, \tag{4}
\]
and \( g \) satisfies

\((g_1): g \in C^1((0, \infty)) \) is positive and decreasing in \((0, \infty)\) with \( \lim_{s \to 0^+} g(s) = \infty. \)

For convenience, we denote by \( \psi \) the solution to the following problem
\[
\int_0^{\psi(t)} (ng(\tau))^{-1/n} d\tau = t, \quad \forall \ t > 0. \tag{5}
\]

The problem (3) arises from a few geometric problems. If \( g(s) = s^{-(n+2)}, \ s > 0, \ b(x) \equiv 1 \) in \( \Omega, \) and \( u \) is a unique convex solution to problem (3), then the Legendre transform \( \hat{u} = x\nabla u(x) - u(x) \) of \( u \) is a complete affine hyperbolic sphere \((2, 3, 4, 8, 15), \) and \( \sum_{i,j=1}^n (-u) - u \frac{\partial^2 u(x)}{\partial x_i \partial x_j} \) gives the Hilbert metric (Poincare metric) in the convex domain \( \Omega \) \((16, 20)\). When \( g(s) = s^{-\gamma} (s > 0) \) with \( \gamma > 0, \) problem (3) may be obtained from \( L_p \)-Minkowski problem \((17)\) and the Minkowski problem in centro-affine geometry \((5), [8]-[11].\)

There are many papers which have been dedicated to resolving the global and boundary regularity and asymptotic behavior issues to the problem, see, for instance, \([3, 4, 7]-[12], [14, 15, 18, 19, 22, 26]\) and the references therein.

Some basic results are listed as follows.

When \( g(s) = s^{-(n+2)}, \ s > 0 \) and \( b(x) \equiv 1 \) in \( \Omega, \) Cheng and Yau \([3]\) established that if \( \Omega \) is a bounded convex then there exists a unique solution \( u \in C^\infty(\Omega) \cap C(\bar{\Omega}) \) to problem (3). Later, Lazer and McKenna \([12]\) considered problem (3) with \( g(s) = s^{-\gamma} (s > 0) \) for \( \gamma > 0, \ b \in C^\infty(\bar{\Omega}) \) which is positive on \( \bar{\Omega}. \) They showed that if \( \gamma > 1, \) then there exists a unique convex solution \( u \in C^2(\bar{\Omega}) \cap C(\bar{\Omega}) \) to problem (3). Moreover, \( u \) satisfies
\[
c_1(d(x))^{(n+1)/(n+\gamma)} \leq -u(x) \leq c_2(d(x))^{(n+1)/(n+\gamma)}, \quad x \in \Omega. \tag{6}
\]
In \([18], \) Mohammed extended the results in \([3] \) and \([12] \) for more general \( g \) and \( b, \)

and obtained the following results.

Let \( b \) satisfy \((B_1)\) and \( g \) be a positive nonincreasing smooth function in \((0, \infty).\)

(i₁) Problem (3) admits a convex solution \( u \in C^\infty(\bar{\Omega}) \cap C(\bar{\Omega}) \) if and only if \((B_2)\) holds;

(ii) if \( \lim_{s \to 0^+} g(s) = \infty, \) and \( \ b \in C(\bar{\Omega}) \) is positive on \( \bar{\Omega}, \) then problem (3) has a unique convex solution \( u \in C^\infty(\bar{\Omega}) \cap C(\bar{\Omega}) \) satisfying
\[
c_1\varphi(d(x)) \leq -u(x) \leq c_2\varphi(d(x)) \text{ and } |\nabla u(x)| \leq c_2 \frac{\varphi(d(x))}{d(x)} \text{ near } \partial \Omega,
\]
where \( \varphi \) is the solution to the following problem
\[
\int_0^{\psi(t)} (G(s))^{-1/(n+1)} ds = t, \quad t \in (0, t_0), \tag{7}
\]
\( G(s) = \int_s^{s_0} g(\tau)d\tau < \infty, \ s \in (0, s_0), \ s_0 \in (0, \infty]. \)

Recently, for \( g(s) = s^{-(n+2)}, \ s > 0, \) and \( b(x) \equiv 1 \) in \( \Omega, \) Jian and Li \([11]\) showed that if \( \Omega \) is a general bounded convex domain and \( u \in C^\infty(\bar{\Omega}) \cap C(\bar{\Omega}) \) is a convex solution to problem (3) then
\[
u \in C^{1/(n+1)}(\bar{\Omega}) \text{ and } ||u||_{C^{1/(n+1)}(\bar{\Omega})} \leq C(\text{diam}(\Omega), n),
\]
where \( \text{diam}(\Omega) \) is the diameter of \( \Omega \).

In this paper, we show optimal global asymptotic behavior of the unique convex solution to problem (3) under the following local structure conditions

\[(g_2)\]: there exists \( C_g \geq 0 \) such that

\[
\lim_{s \to 0^+} H'(s) \int_{0}^{s} \frac{d\tau}{H(\tau)} = -C_g, \quad H(s) := (ng(s))^{1/n}, \quad s > 0;
\]

\[(g_3)\]: there exists \( E_g \geq 0 \) such that

\[
\lim_{s \to \infty} H'(s) \int_{0}^{s} \frac{d\tau}{H(\tau)} = -E_g.
\]

We also give some results with regard to nonexistence and regularity of such solution.

Some basic examples of \( g \) in \((g_2)\) and \((g_3)\) are

\[(i_1)\]: when \( g(s) = s^{-\gamma} \) \((s > 0)\) with \( \gamma > 0 \), \( H'(s) \int_{0}^{s} \frac{d\tau}{H(\tau)} = \frac{\gamma}{\gamma+n}, \forall s > 0 \) and \( \psi(t) = (\frac{\gamma+n}{\gamma})^{\gamma/n} t^{\gamma/n} \)

\[(i_2)\]: when \( g(s) = s^{-\gamma_1} \) with \( \gamma_1 > 0 \) and \( s \in (0, 1) \), \( C_g = \frac{\gamma_1}{\gamma_1+n} \). Meanwhile, when \( s \) is large, \( g(s) = s^{-\gamma_2} \) with \( \gamma_2 > 0 \), \( E_g = \frac{\gamma_2}{\gamma_2+n} \)

\[(i_3)\]: when \( g(s) = (\ln s)^{\gamma_1} \) with \( \gamma_1 > 0 \) and \( s \in (0, 1/3) \), \( C_g = 0 \). Meanwhile, when \( s \) is large, \( g(s) = (\ln s)^{\gamma_2} \) with \( \gamma_2 > 0 \), \( E_g = 0 \)

\[(i_4)\]: when \( g(s) = \exp(s^{-\gamma_1}) \) with \( \gamma_1 > 0 \) and \( s \in (0, 1) \), \( C_g = 1 \). Meanwhile, when \( s \) is large, \( g(s) = \exp(-s^{\gamma_2}) \) with \( \gamma_2 > 0 \), \( E_g = 1 \)

A complete characterization of \( g \) in \((g_2)\) and \((g_3)\) are provided in Lemmas 2.1 and 2.2.

For convenience, let

\[
v = -\phi \text{ and assume } \max_{x \in \Omega} |\phi(x)| < 1, \quad (8)
\]

where \( \phi \) is given as in the beginning.

Our main results are summarized as follows.

**Theorem 1.1.** Let \( g(s) = s^{-\gamma}, s \in (0, \infty) \) with \( \gamma > 0 \), and \( b \) satisfy \((B_1)\).

\[(i_1)\]: If \( b \) satisfies the additional condition that there exist \( \sigma \leq -1-n \) and positive constant \( b_1 \) such that

\[
b(x) \geq b_1(v(x))^\sigma, \quad x \in \Omega,
\]

then problem (3) has no classical convex solution.

\[(i_2)\]: If \( b \) satisfies the additional condition that

\[(b_1)\]: there exist \( \sigma \in (-1-n, \gamma-1) \) and positive constants \( \beta_i \) \((i = 1, 2)\) such that

\[
b_1(v(x))^\sigma \leq b(x) \leq b_2(v(x))^\sigma, \quad x \in \Omega,
\]

then problem (3) has a unique classical convex solution \( u_\sigma \) satisfying

\[
m_1(v(x))^{\theta} \leq -u_\sigma(x) \leq M_1(v(x))^{\theta}, \quad x \in \Omega \text{, and } u_\sigma \in C^{1/\theta} (\bar{\Omega}), \quad (9)
\]

where \( \theta = \frac{n+1+\sigma}{n+\gamma} \), \( m_1 \) and \( M_1 \) satisfy

\[
m_1^{n+\gamma} \theta^n C_\theta = b_1, \quad M_1^{n+\gamma} \theta^n C_\theta = b_2, \quad (10)
\]

with

\[
C_\theta = \max_{x \in \Omega} \left( (-1)^n \det D^2 v(x) (v(x) + (1-\theta)\nabla v(x) (-D^2 v(x))^{-1} (\nabla v(x))^T) \right), \quad (11)
\]
and
\[
c_0 = \min_{x \in \Omega} \left( (-1)^n \det D^2 v(x) (v(x) + (1 - \theta) \nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x))^T) \right).
\]  
(12)

Moreover,
\[
\lim_{\sigma \to -1^{-n}} -u_\sigma(x) = +\infty,
\]
and
\[
\left( \frac{b_1(n + \gamma)^n}{c_0} \right)^{1/(n+\gamma)} \leq \liminf_{\sigma \to -1^{-n}} (n + 1 + \sigma)^{n/(n+\gamma)}(-u_\sigma(x)) \leq \limsup_{\sigma \to -1^{-n}} (n + 1 + \sigma)^{n/(n+\gamma)}(-u_\sigma(x)) \leq \left( \frac{b_2(n + \gamma)^n}{c_0} \right)^{1/(n+\gamma)}
\]
uniformly for \( x \in \Omega \) which is an arbitrary compact subset of \( \Omega \).

Where
\[
C_0 = \max_{x \in \Omega} \left( (-1)^n \det D^2 v(x) (v(x) + \nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x))^T) \right),
\]
and
\[
c_0 = \min_{x \in \Omega} \left( (-1)^n \det D^2 v(x) (v(x) + \nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x))^T) \right).
\]  
(13)

In particular, when \( \Omega = B_R \), which is a ball of radius \( R \) centered at the origin, \( r = |x| \), if the condition \((b_1)\) is replaced by
\[
(\text{b'}_1): \quad b(x) = b(r) = b_0(R^2 - r^2)^\sigma \quad (b_0 > 0) \quad \text{and} \quad n + 2 + 2\sigma = \gamma,
\]
then
\[
u_\sigma(x) = -(\frac{b_0}{R^2})^{1/(n+\gamma)}(R^2 - r^2)^{1/2}
\]
is the unique convex solution to problem (3).

(14)

\[-\int_\Omega u_\sigma(x) \det D^2 u_\sigma(x) dx < \infty\]
if and only if
\[n\gamma - (n + 1)\sigma < 2n + 1.\]

**Remark 1.1.** Let \( G(x, s) = \int_0^s g(x, v) dv \). For the more general regular Dirichlet problem
\[
det D^2 u = g(x, u), \quad u < 0, \quad x \in \Omega, \quad u|_{\partial \Omega} = 0,
\]  
(13)

Tso [23] first established the functional
\[
J_n(u) = -\frac{1}{n+1}\int_\Omega u(x) \det D^2 u(x) dx - \int_\Omega G(x, -u) dx,
\]
and showed that any critical point of \( J_n \) is a convex solution of problem (13) in a suitable convex space.

When \( \sigma = \gamma - 1 \) in Theorem 1.1, we have the following result.

**Theorem 1.2.** Let \( g(s) = s^{-\gamma}, \; s \in (0, \infty) \) with \( \gamma > 0 \), and \( b \) satisfy \((B_1)\). If \( b \) satisfies the condition that
\[
(\text{b}_2): \quad \text{there exist} \; \eta \geq -1 \; \text{and positive constants} \; b_i \; (i = 1, 2) \; \text{such that}
\]
\[
b_1(v(x))^{-1}(-\ln(v(x)))^\eta \leq b(x) \leq b_2(v(x))^{-1}(-\ln(v(x)))^\eta, \quad x \in \Omega,
\]
then problem (3) has a unique classical convex solution $u_\eta$ which satisfies
\[ \xi_1 v(x)(h(v(x)))^{1/(n+\gamma)} \leq -u_\eta(x) \leq -u_\psi(x)(h(v(x)))^{1/(n+\gamma)}, \]
where $\xi_1$ and $\xi_2$ are positive constants with $\xi_1 \leq \xi_2$, and
\[ h(t) = \begin{cases} (-\ln t)^{1+\eta}, & \eta > -1; \\ \ln(-\ln t), & \eta = -1. \end{cases} \]

For more general $g$, we have the following results.

**Theorem 1.3.** Let $g$ satisfy $(g_1)$-$\mathbf{(g}_3)$, $b$ satisfy $(B_1)$ and $(b_1)$. If
\[ \beta(1 - C_g) < 1 \text{ and } \beta = \frac{n+1+\sigma}{n} \leq 1, \]
then problem (3) has a unique classical convex solution $u_\sigma$ satisfying
\[ \psi(\xi_1 v^3(x)) \leq -u_\sigma(x) \leq -\psi(\xi_1 v^3(x)), \]
where $\psi$ is given as in (5), $\xi_3$ and $\xi_4$ are positive constants with $\xi_3 \leq \xi_4$.

When $b$ is in a borderline case near the boundary $\partial \Omega$, we have the following result.

**Theorem 1.4.** Let $g$ satisfy $(g_1)$-$\mathbf{(g}_3)$, $b$ satisfy $(B_1)$. If $b$ satisfies the additional condition that
\[ (b_3): \text{ there exist } \mu > 1 \text{ and positive constants } b_i (i = 1, 2) \text{ such that for } x \in \Omega \\ b_1(v(x))^{-1-n}(-\ln(v(x)))^{-n\mu} \leq b(x) \leq b_2(v(x))^{-1-n}(-\ln(v(x)))^{-n\mu}, \]
then problem (3) has a unique classical convex solution $u_\mu$ satisfying
\[ \psi(\xi_5(-\ln(v(x)))^{1-\mu}) \leq -u_\mu(x) \leq -\psi(\xi_5(-\ln(v(x)))^{1-\mu}), \]
where $\xi_5$ and $\xi_6$ are positive constants with $\xi_5 \leq \xi_6$.

In particular, when $g(s) = s^{-\gamma}$, $s \in (0, \infty)$ with $\gamma > 0$, $u_\mu$ satisfying
\[ m_2(-\ln(v(x)))^{n/(n+\gamma)} \leq -u_\mu(x) \leq M_2(-\ln(v(x)))^{n/(n+\gamma)}, \]
where
\[ C_2 = \max_{x \in \Omega} \left( (-1)^n \det D^2 v(x) \left( v(x) + \left( 1 - \frac{\mu + \gamma}{n + \gamma} \right)(-\ln(v(x)))^{-1}(\nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x))^T) \right) \right), \]
\[ c_2 = \min_{x \in \Omega} \left( (-1)^n \det D^2 v(x) \left( v(x) + \left( 1 - \frac{\mu + \gamma}{n + \gamma} \right)(-\ln(v(x)))^{-1}(\nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x))^T) \right) \right), \]
Moreover, $\lim_{\mu \to 1 -} -u_\mu(x) = +\infty$, and
\[ \langle \frac{C_0}{b_2} \rangle^{1/(n+\gamma)} \left( \frac{n + \gamma}{n} \right)^{n/(n+\gamma)} \leq \liminf_{\mu \to 1} \left( (-u_\mu(x))(\mu - 1)^{n/(n+\gamma)} \right) \]
\[ \leq \limsup_{\mu \to 1} \left( (-u_\mu(x))(\mu - 1)^{n/(n+\gamma)} \right) \leq \langle \frac{C_0}{b_1} \rangle^{1/(n+\gamma)} \left( \frac{n + \gamma}{n} \right)^{n/(n+\gamma)} \]
uniformly for $x \in \Omega_1$ which is an arbitrary compact subset of $\Omega$, where
\[
\bar{C}_0 = \max_{x \in \Omega} \left( (-1)^n \det D^2 v(x) \left( v(x) + (1 - (-\ln(v(x)))^{-1}) \nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x))^T \right) \right),
\]
\[
L_0 = \min_{x \in \Omega} \left( (-1)^n \det D^2 v(x) \left( v(x) + (1 - (-\ln(v(x)))^{-1}) \nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x))^T \right) \right).
\]

**Remark 1.2.** Let $b$ satisfy $(B_1)$. Cheng and Yau [3] showed that $(B_2)$ holds if for some $\sigma > -n - 1$ and $b_0 > 0$ such that
\[
b(x) \leq b_0(d(x))^{\sigma} \text{ in } \Omega.
\]
In [18], Mohammed proved that problem (4) has no convex solution if
\[
b(x) \geq b_0(d(x))^{-n-1} \text{ in } \Omega.
\]
These results have been improved by Yang and Chang [24] who showed that
\[
(i_1) \text{ problem (4) has no convex solution if } b(x) \geq b_0(d(x))^{-n-1} \text{ near } \partial \Omega;
\]
\[
(i_2) (B_2) \text{ holds if for some } \mu > 1 \text{ such that } b(x) \leq b_0(d(x))^{-n-1}(-\ln(d(x)))^{-\mu n} \text{ in } \Omega.
\]
Recently, Zhang and Du [25] provided a more general sufficient condition and a necessary condition for the existence of convex solution to problem (4).

The outline of this paper is as follows. In Section 2, we give some preliminaries. The proofs of Theorems 1.1-1.4 are provided in Section 3.

2. **Some preliminaries.** In this section, we present some basics of Karamata regular variation theory in order to show the complete characterization of $g$ in $(g_1)$-$(g_3)$ and the exact behavior near zero of $\psi$ in (5).

Incidentally, Cirstea and Trombetti [6] first introduced the theory to study boundary behavior of the blow-up boundary solutions of the Monge-Ampère equations.

**Definition 2.1** ([21], Definition 1.1). A positive continuous function $g$ defined on $(0, s_0]$, for some $s_0 > 0$, is called **regularly varying at zero** with index $\rho \in \mathbb{R}$, denoted by $g \in RVZ_{\rho}$, if for each $\xi > 0$,
\[
\lim_{s \to 0^+} \frac{g(\xi s)}{g(s)} = \xi^{\rho}.
\]
(19)

In particular, when $\rho = 0$, $g$ is called **slowly varying at zero**.

Clearly, if $g \in RVZ_{\rho}$, then $L(s) := g(s)/s^{\rho}$ is slowly varying at zero.

Some basic examples of slowly varying functions at zero are
\[
(i_1): \text{ every continuous function on } (0, s_0) \text{ which has a positive limit at zero;}
\]
\[
(i_2): (-\ln s)^{\gamma} \text{ and } (\ln(-\ln s))^{\gamma}, \quad \gamma \in \mathbb{R}, \ s \in (0, 1/3);
\]
\[
(i_3): \exp((\ln s)^{\gamma}), \quad 0 < \gamma < 1, \ s \in (0, 1).
\]

**Proposition 2.1** (Uniform convergence theorem, [21], Theorem 1.1). If $g \in RVZ_{\rho}$, then (19) holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2.$
Proposition 2.2 (The Karamata representation theorem, [21], Theorem 1.2). A function \( L \) is slowly varying at zero if and only if it may be written in the form

\[
L(s) = l(s) \exp \left( \int_s^{s_0} \frac{y(\tau)}{\tau} d\tau \right), \quad s \in (0, s_0],
\]

where the functions \( l \) and \( y \) are continuous and for \( s \to 0^+ \), \( y(s) \to 0 \) and \( l(s) \to c_0 \), with \( c_0 > 0 \).

Definition 2.2 ([21], p.7). We call that

\[
\hat{L}(s) = c_0 \exp \left( \int_s^{s_0} \frac{y(\tau)}{\tau} d\tau \right), \quad s \in (0, s_0],
\]

is normalized slowly varying at zero, and

\[
g(s) = s^\rho \hat{L}(s), \quad s \in (0, s_0],
\]

is normalized regularly varying at zero with index \( \rho \) (and denoted by \( g \in NRVZ_\rho \)).

Equivalently, a function \( g \in NRVZ_\rho \) if and only if

\[
g \in C^1(0, s_0) \quad \text{for some} \quad s_0 > 0 \quad \text{and} \quad \lim_{s \to 0^+} \frac{sg'(s)}{g(s)} = \rho.
\]

Proposition 2.3 ([1], Proposition 1.3.6). If functions \( L, L_1 \) are slowly varying at zero, then

(i): \( L^p \) for every \( \rho \in \mathbb{R} \), \( c_1 L + c_2 L_1 \) (\( c_1 \geq 0, c_2 \geq 0 \) with \( c_1 + c_2 > 0 \)), \( L \cdot L_1 \), \( L \circ L_1 \) (if \( L_1(s) \to 0 \) as \( s \to 0^+ \)), are also slowly varying at zero.

(ii): For every \( \varepsilon > 0 \) and \( s \to 0^+ \), \( s^{\varepsilon} L(s) \to 0 \) and \( s^{-\varepsilon} L(s) \to \infty \).

(iii): For \( \rho \in \mathbb{R} \) and \( s \to 0^+ \), \( \ln(L(s))/\ln s \to 0 \) and \( \ln(s^\rho L(s))/\ln s \to \rho \).

Proposition 2.4 (Asymptotic behavior, [1], Propositions 1.5.8 and 1.5.10). If a function \( L \) is slowly varying at zero, then for \( s_0 > 0 \) and \( s \to 0^+ \),

(i): \( \int_s^{s_0} \tau^p L(\tau) d\tau \cong (1 + p)^{-1} s^{1+p} L(s), \quad \text{for} \quad \rho > -1; \)

(ii): \( \int_s^{s_0} \tau^p L(\tau) d\tau \cong (-1 - p)^{-1} s^{1+p} L(s), \quad \text{for} \quad \rho < -1. \)

Similarly, for a positive continuous function \( f \) defined on \([S_0, \infty)\), for some \( S_0 > 0 \), we can give the definitions of regularly varying and normalized regularly varying at infinity and present some basic properties. Here we omit them.

Lemma 2.1 ([26], Lemma 2.2). Let \( g \) satisfy \((g_1)\).

(i): If \( g \) satisfies \((g_2)\), then \( C_g \leq 1 \);

(ii): \((g_2)\) holds with \( C_g \in (0, 1) \) if and only if \( g \in NRVZ_{-\gamma} \) with \( \gamma > 0 \). In this case \( \gamma = C_g/(1 - C_g) \);

(iii): \((g_2)\) holds with \( C_g = 0 \) if and only if \( g \) is normalized slowly varying at zero;

(iv): if \((g_2)\) holds with \( C_g = 1 \), then \( g \) grows faster than any \( s^{-p} \) (\( p > 1 \)) near zero;

(v): if \( g \in C^2(0, s_0) \) for some \( s_0 > 0 \) and

\[
g''(s) > 0, \quad \forall s \in (0, s_0); \quad \lim_{s \to 0^+} \frac{g(s)g''(s)}{(g'(s))^2} = 1,
\]

then \( g \) satisfies \((g_2)\) with \( C_g = 1 \).

Similarly, we have the following results.

Lemma 2.2. Let \( g \) satisfy \((g_1)\).

(i): If \( g \) satisfies \((g_3)\), then \( E_g \leq 1 \);
(i_2): (g_3) holds with \( E_g \in (0, 1) \) if and only if \( g \) is normalized regularly varying at infinity with index \(-\gamma n \) with \( \gamma > 0 \). In this case \( \gamma = E_g/(1-E_g) \);

(i_3): (g_3) holds with \( E_g = 0 \) if and only if \( g \) is normalized slowly varying at infinity;

(i_4): if (g_3) holds with \( E_g = 1 \), then \( g \) grows faster than any \( s^{-p} \) \((p > 1)\) at infinity;

(i_5): if \( g \in C^2(S_0, \infty) \) for some large \( S_0 > 0 \) and

\[
g''(s) > 0, \forall s \in (S_0, \infty); \quad \lim_{s \to \infty} \frac{g(s)g''(s)}{(g'(s))^2} = 1, \tag{25}\]

then \( g \) satisfies (g_3) with \( E_g = 1 \).

For completeness, we give its proof.

Proof. Recalling that \( H(s) = (ng(s))^{1/n} \), \( s > 0 \), and \( g \) satisfies (g_1), we see that

\[
0 < \int_0^s \frac{d\tau}{H(\tau)} \leq \frac{s}{H(s)}, \forall s > 0,
\]

i.e.,

\[
0 < H(s) \int_0^s \frac{d\tau}{H(\tau)} \leq s, \forall s > 0. \tag{26}\]

and thus

\[
\lim_{s \to 0^+} H(s) \int_0^s \frac{d\tau}{H(\tau)} = 0. \tag{27}\]

(i_1) Let

\[
I(s) = -H'(s) \int_0^s \frac{d\tau}{H(\tau)}, \forall s > 0.
\]

Integrating \( I(t) \) from 0 to \( s \) and using integration by parts, we obtain by (27) that

\[
\int_0^s I(t) dt = -H(s) \int_0^s \frac{d\tau}{H(\tau)} + s, \forall s > 0,
\]

i.e.,

\[
0 < \frac{H(s) \int_0^s \frac{d\tau}{H(\tau)}}{s} = 1 - \frac{\int_0^s I(t) dt}{s}, \forall s > 0.
\]

It follows from the l’Hospital’s rule that

\[
0 \leq \lim_{s \to \infty} \frac{H(s) \int_0^s \frac{d\tau}{H(\tau)}}{s} = 1 - \lim_{s \to \infty} I(s) = 1 - E_g. \tag{28}\]

So (i_1) holds.

(i_2) When (g_3) holds with \( E_g \in (0, 1) \), it follows from (28) that

\[
\lim_{s \to \infty} \frac{H(s)}{sH'(s)} = \lim_{s \to \infty} \frac{H(s) \int_0^s \frac{d\tau}{H(\tau)}}{sH'(s) \int_0^s \frac{d\tau}{H(\tau)}} = \frac{-1}{E_g} \lim_{s \to \infty} \frac{H(s) \int_0^s \frac{d\tau}{H(\tau)}}{s} = \frac{1 - E_g}{E_g},
\]

i.e., \( H \) is normalized regularly varying at infinity with index \(-E_g/(1-E_g)\). Then \( g \) is normalized regularly varying at infinity with index \(-nE_g/(1-E_g)\).

Conversely, when \( g \) is normalized regularly varying at infinity with index \(-\gamma \) with \( \gamma > 0 \), i.e., \( \lim_{s \to \infty} \frac{sg(s)}{g(s)} = -\gamma \) and there exist positive constant \( S_0 > 0 \) and \( \tilde{L} \) which is normalized slowly varying at infinity such that \( g(s) = s^{-\gamma} \tilde{L}(s), s \in (S_0, \infty), \)
and $H(s) = n^{1/n}s^{-\gamma/n}\hat{L}_1(s)$ with $\hat{L}_1(s) = \hat{L}^{1/n}(s)$. By using the result similar to Proposition 2.4 (i), we have

$$\lim_{s \to \infty} H'(s) \int_0^s \frac{d\tau}{H(\tau)} = -\lim_{s \to \infty} sH'(s) \lim_{s \to \infty} \frac{H(s) \int_0^s \frac{d\tau}{H(\tau)}}{s}$$

$$= \gamma \lim_{s \to \infty} \left( -\frac{n+\gamma}{n} \hat{L}_1(s) \int_0^s \tau^{\gamma/n}(\hat{L}_1(\tau))^{-1} d\tau \right) = \frac{\gamma}{n+\gamma} = E_g.$$

(i3) By $E_g = 0$ and (28), one can see that

$$\lim_{s \to \infty} sH'(s) = \lim_{s \to \infty} sH'(s) \int_0^s \frac{d\tau}{H(\tau)}$$

$$= \left( \lim_{s \to \infty} \frac{H(s)}{s} \int_0^s \frac{d\tau}{H(\tau)} \right)^{-1} \lim_{s \to \infty} sH'(s) \int_0^s \frac{d\tau}{H(\tau)} = 0,$$

i.e., $H$ is normalized slowly varying at infinity, so does $g$.

Conversely, when $H$ is normalized slowly varying at infinity, i.e., $\lim_{s \to \infty} \frac{sH'(s)}{H(s)} = 0$, it follows by (28) that

$$\lim_{s \to \infty} H'(s) \int_0^s \frac{d\tau}{H(\tau)} = \lim_{s \to \infty} sH'(s) \int_0^s \frac{d\tau}{H(\tau)} = 0.$$

(i4) By $E_g = 1$ and the proof of (i2), we see that $\lim_{s \to \infty} \frac{H(s)}{sH'(s)} = 0$, i.e., $\lim_{s \to \infty} \frac{sH'(s)}{H(s)} = -\infty$. Consequently, for an arbitrary $p > 1$, there exists $S_0 > 0$ such that

$$\frac{-H'(s)}{H(s)} > (p+1)s^{-1}, \quad \forall \ s \geq S_0.$$

Integrating the above inequality from $S_0$ to $s$, we obtain

$$\ln(H(S_0)) - \ln(H(s)) > (p+1)(\ln s - \ln S_0), \quad \forall \ s > S_0,$$

i.e.,

$$0 < H(s)s^p < \frac{H(S_0)S_0^{p+1}}{s}, \quad \forall \ s > S_0.$$

Letting $s \to \infty$, we see that $H$ grows faster than any $s^{-p}$ ($p > 1$) at infinity. So does $g$.

(i5) By a direct calculation and the l’Hospital’s rule, we see that

$$\lim_{s \to \infty} H'(s) \int_0^s \frac{d\tau}{H(\tau)} = \lim_{s \to \infty} \int_0^s \frac{d\tau}{H(\tau)} = -\lim_{s \to \infty} \frac{(H'(s))^2}{H(s)H''(s)}$$

$$= -\lim_{s \to \infty} \frac{ng(s)g''(s) - (n-1)(g'(s))^2}{ng(s)(g'(s))^2 - (n-1)} = -1.$$

\[\square\]

Lemma 2.3. ([26], Lemma 2.3) Let $g$ satisfy $(g_1)$ and $(g_2)$. Then we have

(i1): $\psi'(t) = H(\psi(t)) = (ng(\psi(t)))^{1/n}$, $\psi(t) > 0$ for $t > 0$, $\psi(0) = 0$, $\psi'(0) := \lim_{t \to 0^+} \psi'(t) = \lim_{t \to 0^+} (ng(\psi(t)))^{1/n} = \infty$, and $\psi''(t) = H(\psi(t))H'(\psi(t)) = \frac{g(\psi(t))}{(ng(\psi(t)))^{1/n}}, \ t > 0$;

(i2): $\psi'(t) > 0$ for $t > 0$.
Remark 2.1. When $g(s) = s^{-\gamma} (s \in (0, \infty))$ with $\gamma > 0$, $\Psi(t) = \frac{\gamma}{n+\gamma}$, $\forall t > 0$.

Proof. (i) It follows from (g1) that $\lim_{s \to \infty} \int_0^s \frac{ds}{H(t)} = \infty$. Thus (i1) holds from the definition of $\psi$.

(ii) From (g1)-(g3) and (i1), we see that the function $H'(s) \int_0^s \frac{ds}{H(t)}$ is bounded on $(0, \infty)$, and thus (i2) holds.

(iii) It follows from Lemma 2.3 and the result similar to Proposition 2.1 for a function which is slowly varying at infinity.

Lemma 2.4. Let $g$ satisfy (g1)-(g3). Then we have

(i1): $\lim_{t \to \infty} \psi(t) = \infty$;

(i2): $\sup_{t > 0} \Psi(t) < \infty$, where $\Psi(t) := -tH'(\psi(t)) = -\frac{tg'(\psi(t))}{g(\psi(t))} = -C_g$;

(i3): $\psi \in NRVZ_{-C_g}$ and $\psi' \in NRVZ_{-C_g}$.

Lemma 2.5 ([12], proof of Proposition 2.4). Let $\Omega$ be an open subset of $\mathbb{R}^n$ with $n \geq 2$, $f \in C^2(\mathbb{R})$, $v \in C^2(\Omega)$ and $D^2v(x)$ be reversible. We have

(i1): It holds for $x \in \Omega$

$$\det D^2f(v(x)) = (f'(v(x)))^{n-1} \det D^2v(x)(f'(v(x)))^2 + f''(v(x))\nabla v(x)(D^2v(x))^{-1}(\nabla v(x))^T.$$ (i2): If $D^2v(x)$ is positive definite on $\Omega$, $f'(s) > 0$ and $f''(s) \geq 0$ on $\mathbb{R}$, then $D^2f(v(x))$ is positive definite on $\Omega$.

Proof. We give a different proof here. Let $w(x) = f(v(x))$, we have

$$w_i(x) = f'(v(x))v_i(x), \quad w_{ij}(x) = f''(v(x))v_i(x)v_j(x) + f'(v(x))v_{ij}(x),$$

and

$$D^2w(x) = f'(v(x))D^2v(x) + f''(v(x))(v_i(x)v_j(x)) = f'(v(x))D^2v(x) + f''(v(x))(\nabla v(x))^T\nabla v(x).$$ (29)

(i1) It follows that

$$D^2w(x) = D^2v(x)(f'(v(x)))I + f''(v(x))(D^2v(x))^{-1}(\nabla v(x))^T\nabla v(x),$$

where $I$ is the identity matrix.

For the case of $f'(v(x_0)) = 0$, $x_0 \in \Omega$, since the rank of the matrix $(\nabla v(x_0))^T\nabla v(x_0)$ is 1 and $n \geq 2$, we see that $\det (\nabla v(x_0))^T\nabla v(x_0) = 0$, and the result holds.

For the case of $f'(v(x_0)) \neq 0$, $x_0 \in \Omega$, by using the basic fact: Let $A = A_{n \times m}$, $B = B_{m \times n}$, we have

$$\det (\lambda I_{n \times n} - AB) = \lambda^{n-m} \det (\lambda I_{m \times m} - BA), \quad \lambda \neq 0.$$ 

Now let $A = (D^2v(x_0))^{-1}(\nabla v(x_0))^T$ and $B = \nabla v(x_0)$, we have

$$\det D^2w(x_0) = \det D^2v(x_0)(f'(v(x_0)))^{n-1}(f'(v(x_0))) + f''(v(x_0))(\nabla v(x_0))(D^2v(x_0))^{-1}(\nabla v(x_0))^T.$$ (i2) Since the rank of the matrix $(\nabla v(x))^T\nabla v(x)$ ($x \in \Omega$) is 1 with $n \geq 2$ and $|\nabla v(x)|^2$ is its eigenvalue, we see that $(\nabla v(x))^T \cdot \nabla v(x)$ is positive semidefinite.
Thus, when $D^2v(x)$ is positive definite on $\Omega$, $f'(s) > 0$ and $f''(s) \geq 0$ on $\mathbb{R}$, we see from (29) that $D^2f(v(x))$ is positive definite on $\Omega$. □

Lemma 2.6 ([18], Theorem 2.1). Let $b \in C^\infty(\Omega)$ be positive on $\Omega$ and $g$ be a positive nonincreasing smooth function in $(0, \infty)$. Problem (3) admits a convex solution if and only if the problem (4) admits a convex solution.

Lemma 2.7 (The comparison principle, [19], Lemma 4.1). Let $\bar{\Omega}$ be a bounded convex domain in $\mathbb{R}^n$ with $n \geq 2$, $f \in C^1(\Omega \times (0, \infty), (0, \infty))$ be decreasing in $s$ for each $x \in \Omega$. If $u_1, u_2 \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfy the following conditions

(i) $u_1, u_2$ are convex functions on $\Omega$ with $u_1 < 0, u_2 < 0$ in $\Omega$;
(ii) $\det D^2u_1(x) \geq f(x, -u_1)$ and $\det D^2u_2(x) \leq f(x, -u_2), x \in \Omega$;
(iii) $u_2 \geq u_1$ on $\partial \Omega$;

then we have $u_2 \geq u_1$ in $\Omega$.

Lemma 2.8 ([10], Lemma 2.3). Let $\Omega$ be a bounded convex domain and $u \in C(\bar{\Omega})$ be a convex function in $\Omega$ with $u|_{\partial \Omega} = 0$. If there are $\nu \in (0, 1)$ and $C > 0$ such that

$$|u(x)| \leq C(d(x))^\nu, \forall x \in \Omega,$$

then $u \in C^\nu(\Omega)$ and $\|u\|_{C^\nu(\Omega)} \leq C(1 + (\text{diam}(\Omega))^{\nu})$.

Lemma 2.9 ([13], Lemma). Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^n$. We have

$$\int_{\Omega} (d(x))^\lambda \, dx < \infty$$

if and only if $\lambda > -1$.

3. Global asymptotic behavior. In this section, we prove Theorems 1.1-1.4.

Firstly, we see from Remark 1.2 that one of the assumptions (b$_1$), (b$_2$) and (b$_3$) implies that (B$_2$) holds. Moreover, (B$_2$) and (g$_1$) implies that problem (3) has a unique convex solution $u \in C^\infty(\Omega) \cap C(\Omega)$ (Lemmas 2.6 and 2.7). Thus, our main purpose in this section is to show global asymptotic behavior of such solution to problem (3) in the following.

Proof of Theorem 1.1. (i) Let $\varepsilon \in (0, 1)$ be an arbitrary constant. By using $\sigma \leq -1 - n$, (1), (2) and (8), we see that

$$\min_{x \in \bar{\Omega}} (-1)^n \det D^2v(x) > 0, \quad x \in \bar{\Omega},$$

$$\min_{x \in \bar{\Omega}} \left( v(x) + (1 - \varepsilon) \nabla v(x)(-D^2v(x))^{-1}(\nabla v(x))^T \right) > 0,$$

and

$$(v(x))^{\sigma - \varepsilon \gamma} \geq (v(x))^{-1 - n - \varepsilon \gamma} \geq (v(x))^{\nu - 1 - n}, \quad x \in \bar{\Omega}.$$

Let $\bar{u}_\sigma = -C_1\varepsilon^{-p}v^\varepsilon, \quad x \in \Omega$, where $p = \frac{n}{n+\gamma}$ and $C_1$ satisfies

$$C_1^{n+\gamma} \max_{x \in \bar{\Omega}} \left( (-1)^n \det D^2v(x)(v(x) + \nabla v(x)(-D^2v(x))^{-1}(\nabla v(x))^T) \right) = b_1.$$

We have from Lemma 2.5 and a direct computation that

$$\det D^2\bar{u}_\sigma(x) = C_1^n (v(x))^{n\varepsilon - n - 1} \varepsilon^{n(1-p)}(-1)^n \det D^2v(x)$$

$$\left( v(x) + (1 - \varepsilon) \nabla v(x)(-D^2v(x))^{-1}(\nabla v(x))^T \right) \leq b_1 C_1^{-\gamma}(v(x))^{\sigma - \varepsilon \gamma} \leq b(x)(-\bar{u}_\sigma(x))^{-\gamma}, \quad x \in \Omega,$$
i.e., $\bar{u}_\sigma = -C_1 \varepsilon^{-n} v^\sigma$ is a supersolution to problem (3) in $\Omega$. Thus if $u_\sigma$ were a classical convex solution of problem (3), it would then follow from Lemma 2.7 that

$$u_\sigma(x) \leq -C_1 \varepsilon^{-n} (v(x))^\sigma, \forall x \in \Omega.$$ 

Since $\varepsilon \in (0, 1)$ is an arbitrary constant, we see that

$$-\lim_{\varepsilon \to 0} u_\sigma(x) = +\infty, \forall x \in \Omega.$$ 

This is a contradiction. Thus problem (3) has no classical convex solution provided

$$\sigma \leq -1 - n.$$ 

(1) By using $\sigma \in (-1 - n, \gamma - 1)$ and $\theta = \frac{n+1+\sigma}{n+\gamma} \in (0, 1)$, we see that for $x \in \Omega$

$$\min_{x \in \Omega} \left( v(x) + (1 - \theta)\nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x))^T \right) > 0.$$ 

Let $\underline{u}_\sigma = -M_1 v^\theta(x), x \in \Omega$, where $M_1$ is given as in (10). We have from (b_1), Lemma 2.5 and a direct computation that

$$\det D^2 \underline{u}_\sigma(x) = M_1^n v(x)^{n\theta - n - 1} \theta^n (-1)^{n} \det D^2 v(x)(v(x) + (1 - \theta)\nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x))^T) \geq b_2 v^\sigma(x) M_1^{-\gamma} v^{-\gamma}(x) \geq b(x)(-\underline{u}_\sigma(x))^{-\gamma}, x \in \Omega,$$

i.e., $\underline{u}_\sigma = -M_1 v^\theta(x)$ is a subsolution to problem (3) in $\Omega$. Moreover, Lemma 2.5 (i_2) implies that $\underline{u}_\sigma$ is strictly convex on $\Omega$. In a similar way, we can show that $\bar{u}_\sigma = -m_1 v^\theta(x)$ is a supersolution to problem (3) in $\Omega$, where $m_1$ is given as in (10). Obviously, $\bar{u}_\sigma \leq \bar{u}_\sigma$ on $\Omega$. It follows from Lemmas 2.7 and 2.8 that the unique convex solution $u_\sigma$ of problem (3) is in the ordered interval $[\underline{u}_\sigma, \bar{u}_\sigma]$ and $u \in C^{1,\gamma}(\Omega)$. In particular, when $\Omega = B_R$, we can choose $v(x) = \frac{1}{2}(R^2 - |x|^2)$. In this case $-D^2 v(x) = -(D^2 v(x))^{-1} = I$, and

$$(-1)^n \det D^2 v(x) \cdot (v(x) + (1 - \theta)\nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x))^T) = \frac{1}{2} R^2,$$

if and only if $\theta = \frac{1}{2}$, i.e., $n + 2 + 2\sigma = \gamma$. Thus, when $b(x) = b(|x|) = b_0(R^2 - r^2)^\sigma$, $u_\sigma(x) = -\frac{b_0}{R^{1/2}} R^{1/(n+\gamma)} (R^2 - |x|^2)^{1/2}$ is the unique convex solution to problem (3).

(3) It follows from (i_2) and Lemma 2.9 that

$$\frac{b_1 m_1}{M_1} \int_{\Omega} (v(x))^{\sigma + (1-\gamma)} dx \leq \int_{\Omega} b(x)(-u_\sigma(x))^{1-\gamma} dx$$

$$\leq \frac{b_2 M_1}{m_1} \int_{\Omega} (v(x))^{\sigma + (1-\gamma)} dx < \infty$$

if and only if $n\gamma - (n + 1)\sigma < 2n + 1$.

Since

$$-\int_{\Omega} u_\sigma(x) \det D^2 u_\sigma(x) dx = \int_{\Omega} b(x)(-u_\sigma(x))^{1-\gamma} dx,$$

we see that (i_3) holds.

**Proof of Theorem 1.2.** (i_1) For $\eta > -1$, let $a = \frac{1+n}{n+\gamma}$. Without losing generality, we assume $\max_{x \in \Omega} |v(x)| < \exp(-a)$. It follows that

$$-\ln(v(x)) > a, \forall x \in \Omega.$$
Let \( \bar{u}_\eta = -\xi_1 v(x) (-\ln(v(x)))^\alpha \), where \( \xi_1 \) satisfies
\[
\xi_1^{n+\gamma} \min_{x \in \Omega} \left( (-1)^n \det D^2 v(x) \left( 1 - a(-\ln(v(x)))^{-1} \right)^{n-1} \left( v(x) (-\ln(v(x))) \right) \right.
\]
\[
(1 - a(-\ln(v(x)))^{-1}) + a(1 + (1 - a)(-\ln(v(x)))^{-1})
\]
\[
\nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x)^T) = b_1.
\]

We have from Lemma 2.5, (b_2) and a direct computation that
\[
\det D^2 \bar{u}_\eta(x) = \xi_1^n v^{-1}(x) (-1)^n \det D^2 v(x) (-\ln(v(x)))^{\alpha n-1}
\]
\[
(1 - a(-\ln(v(x)))^{-1})^{n-1} \left( v(x) (-\ln(v(x))) \right) \left( 1 - a(-\ln(v(x)))^{-1} \right)
\]
\[
+ a(1 + (1 - a)(-\ln(v(x)))^{-1}) \nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x)^T)
\]
\[
\leq b_1 \xi_1^{-\gamma} v^{\gamma-1}(x) (-\ln(v(x)))^{\alpha} v^{-\gamma}(x) (-\ln(v(x)))^{\alpha - \gamma}
\]
\[
\leq b(x)(-\bar{u}_\eta(x))^{-\gamma}, \ x \in \Omega,
\]
i.e., \( \bar{u}_\eta = -\xi_1 v(x) (-\ln(v(x)))^\alpha \) is a supersolution to problem (3) in \( \Omega \).

In a similar way, we can show that \( \underline{u}_\eta = -\xi_2 v(x) (\ln(-\ln(v(x))))^\alpha \) is a subsolution to problem (3) in \( \Omega \), where \( \xi_2 \) satisfies
\[
\xi_2^{n+\gamma} \max_{x \in \Omega} \left( (-1)^n \det D^2 v(x) \left( 1 - a(-\ln(v(x)))^{-1} \right)^{n-1} \left( v(x) (-\ln(v(x))) \right) \right.
\]
\[
(1 - a(-\ln(v(x)))^{-1}) + a(1 + (1 - a)(-\ln(v(x)))^{-1})
\]
\[
\nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x)^T) = b_2.
\]

Obviously, \( \underline{u}_\eta \leq \bar{u}_\eta \) on \( \bar{\Omega} \).

(\text{i}_2) For \( \eta = -1 \), let \( a = \frac{1}{1 + \gamma} \). Without losing generality, we assume
\[
\max_{x \in \Omega} |\phi(x)| < \exp(-c_0), \ \text{where} \ c_0 \ \text{is the unique solution to the equation}
\]
\[
t \ln t = a.
\]

It follows that
\[
\ln(-\ln(v(x))) > 0, \ a(-\ln(v(x)))^{-1}(\ln(-\ln(v(x))))^{-1} < 1, \ \forall x \in \Omega.
\]

Let \( \underline{u}_\eta = -\xi_2 v(x)(\ln(-\ln(v(x))))^\alpha \), where \( \xi_2 \) satisfies
\[
\xi_2^{n+\gamma} \min_{x \in \Omega} \left( (-1)^n \det D^2 v(x) \left( 1 - a(-\ln(v(x)))^{-1} \right)^{n-1} \left( v(x) (-\ln(v(x))) \right) \right.
\]
\[
(1 - a(-\ln(v(x)))^{-1}) + a(1 + (1 - a)(-\ln(v(x)))^{-1})
\]
\[
\nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x)^T) = b_2.
\]
We have from Lemma 2.5, (b_2) and a direct computation that
\[
\begin{align*}
\det D^3 u_0(x) &= \xi_3^2(v(x))^{-1} - (1)^n \det D^2 v(x)(-\ln(v(x)))^{-1} \\
&= (\ln(-\ln(v(x))))^{n-1} (1 - a(-\ln(v(x)))^{-1} (1 - a(-\ln(v(x)))^{-1})^{n-1} \\
&\geq b_2 v^{-1}(\ln(-\ln(v(x))))^{-1} v^{-1}(\ln(-\ln(v(x))))^{-1} v^{-1}(\ln(-\ln(v(x))))^{-1} \\
&\geq b(x)(-u_0(x))^{-\gamma}, x \in \Omega,
\end{align*}
\]
i.e., \( u_0 = -\xi_2 v(x)(\ln(-\ln(v(x))))^\alpha \) is a subsolution to problem (3) in \( \Omega \).

In a similar way, we can show that \( \bar{u}_\eta = -\xi v(x)(\ln(-\ln(v(x))))^\alpha \) is a supersolution to problem (3) in \( \Omega \), where \( \xi_1 \) satisfies
\[
\begin{align*}
\xi_1 n^{-1} \max_{x \in \Omega} (-1)^n \det D^2 v(x)(1 - a(-\ln(v(x)))^{-1} (1 - a(-\ln(v(x)))^{-1})^{n-1} \\
\geq b(x)(-\bar{u}_\eta(x))^{-\gamma}, x \in \bar{\Omega}.
\end{align*}
\]

Obviously, \( u_0 \leq \bar{u}_\eta \) on \( \bar{\Omega} \). Hence Lemma 2.7 implies the desired conclusion. \( \square \)

**Proof of Theorem 1.3.** It follows from Lemma 2.4 (i_3) that
\[
\lim_{t \to 0} \Psi(t) = \lim_{d(x) \to 0} \Psi(xv(x)^\beta) = C_g
\]
holds uniformly for \( \xi \in [c_1, c_2] \) with \( 0 < c_1 < c_2 \).

From (1) and (16), we see that there is a sufficiently small \( \delta_1 \in (0, \delta_0) \), which is independent of \( \xi \in [c_1, c_2] \), such that for \( x \in \Omega_{\delta_1} \),
\[
1 - \beta + \beta \Psi(xv^\beta(x)) \geq 0.
\]

Moreover, we have from Lemma 2.4 (i_2) that
\[
0 \leq \inf_{t \geq 0} \Psi(t) \leq \Psi(t) \leq \sup_{t \geq 0} \Psi(t) < \infty, \quad t > 0.
\]

Denote
\[
\bar{\Psi}(x) = \begin{cases}
\Psi(xv^\beta(x)), & d(x) \leq \frac{\delta_1}{2} \\
\sup_{t \geq 0} \Psi(t), & d(x) \geq \frac{\delta_1}{2};
\end{cases}
\]
and
\[
\Psi(x) = \begin{cases}
\Psi(xv^\beta(x)), & d(x) \leq \frac{\delta_1}{2} \\
\inf_{t \geq 0} \Psi(t), & d(x) \geq \frac{\delta_1}{2}.
\end{cases}
\]

Let \( \bar{u}_g = -\psi(x_4v^\beta(x)), \ x \in \Omega \), where \( \xi_4 \) satisfies
\[
\begin{align*}
n^\alpha \xi_4^n \max_{x \in \Omega} (-1)^n \det D^2 v(x)(v(x) + (1 - \beta + \beta \Psi(x)) \\
\geq b_2.
\end{align*}
\]
By using Lemma 2.5, (b1) and a direct computation, we show that for \( x \in \Omega \)
\[
\det D^2 u_\sigma (x) = n \beta^n \xi_3^n \sup_{x \in \Omega} \left( (-1)^n \det D^2 v(x) \left( v(x) + (1 - \beta + \beta \Psi(\xi_4 v^\beta(x))) \nabla v(x) (-D^2 v(x))^{-1} (\nabla v(x))^T \right) \right)
\geq b_2 v^\sigma(x) g(\psi(\xi_4 v^\beta(x))) \geq b(x) g(-\psi_\sigma(x)),
\]
i.e., \( u_\sigma = -\psi(\xi_4 v^\beta(x)) \) is a subsolution to problem (3) in \( \Omega \).

In a similar way, we can show that \( \bar{u}_\sigma = -\psi(\xi_3 v^\beta(x)) \) is a supersolution to problem (3) in \( \Omega \), where \( \xi_3 \) satisfies
\[
n \beta^n \xi_3^n \sup_{x \in \Omega} \left( (-1)^n \det D^2 v(x) \left( v(x) + (1 - \beta + \beta \Psi(\xi_4 v^\beta(x))) \nabla v(x) (-D^2 v(x))^{-1} (\nabla v(x))^T \right) \right).
\]

Obviously, \( \bar{u}_\sigma \geq u_\sigma \) on \( \Omega \). Hence the desired conclusion follows from Lemma 2.7.

The proof is finished. \( \square \)

Proof of Theorem 1.4. Without losing generality, we assume
\[
\max_{x \in \Omega} \phi(x) < \exp(-\mu), \text{ where } \mu > 1 \text{ is given as in (b3)}.
\]
It follows that
\[
-\ln(v(x)) > \mu, \text{ \forall } x \in \bar{\Omega}.
\]
Let \( u_\mu = -\psi(\xi_6 (-\ln(v(x)))^{1-\mu}) \), \( x \in \Omega \), where \( \xi_6 \) satisfies
\[
n (\mu - 1)^n \xi_6^n \min_{x \in \Omega} \left( (-1)^n \det D^2 v(x) \left( v(x) + (1 - \mu (-\ln(v(x)))^{-1} + (\mu - 1)(-\ln(v(x)))^{-1} \inf_{t > 0} \Psi(t) \nabla v(x) (-D^2 v(x))^{-1} (\nabla v(x))^T \right) \right) = b_2.
\]
Since
\[
\lim_{d(x) \to 0} \left( 1 - \mu (-\ln(v(x)))^{-1} + (\mu - 1)(-\ln(v(x)))^{-1} \Psi(\xi_6 (-\ln(v(x)))^{1-\mu}) \right) = 1
\]
holds uniformly for \( \xi_6 \in [c_1, c_2] \) with \( 0 < c_1 < c_2 \), we see that there is a sufficiently small \( \delta_1 \in (0, \delta_0) \), which is independent of \( \xi \in [c_1, c_2] \), such that for \( x \in \Omega_{\delta_1} \),
\[
\left( 1 - \mu (-\ln(v(x)))^{-1} + (\mu - 1)(-\ln(v(x)))^{-1} \Psi(\xi_6 (-\ln(v(x)))^{1-\mu}) \right) \nabla v(x) (-D^2 v(x))^{-1} (\nabla v(x))^T > 0.
\]

It follows that for \( x \in \Omega \)
\[
v(x) + (1 - \mu (-\ln(v(x)))^{-1} + (\mu - 1)(-\ln(v(x)))^{-1} \Psi(\xi_6 (-\ln(v(x)))^{1-\mu}) \nabla v(x) (-D^2 v(x))^{-1} (\nabla v(x))^T > 0.
\]

By using Lemma 2.5, (b3) and a direct computation, we have that for \( x \in \Omega \)
\[
\det D^2 u_\mu(x) = n (\mu - 1)^n \xi_6^n g(\psi(\xi_6 (-\ln(v(x)))^{1-\mu})(v(x))^{-\mu - 1} (-\ln(v(x)))^{-\mu n} \left( (-1)^n \det D^2 v(x) \left( v(x) + (1 - \mu (-\ln(v(x)))^{-1} + (\mu - 1)(-\ln(v(x)))^{-1} \Psi(\xi_6 (-\ln(v(x)))^{1-\mu}) \nabla v(x) (-D^2 v(x))^{-1} (\nabla v(x))^T \right) \right) \geq b_2 (v(x))^{-\mu - 1} (-\ln(v(x)))^{-\mu n} g(\psi(\xi_6 (-\ln(v(x)))^{1-\mu})) \geq b(x) g(-\psi_\mu(x)),
\]
i.e., \( u_\mu = -\psi(\xi_6 (-\ln(v(x)))^{1-\mu}) \) is a subsolution to problem (3) in \( \Omega \).
In a similar way, we can show that $\bar{u}_\mu = -\psi(\xi_5(-\ln(v(x)))^{1-\mu})$ is a supersolution to problem (3) in $\Omega$, where $\xi_5$ satisfies
\[
(n(\mu - 1)^n\xi_5^n \max_{x \in \Omega} \left( (-1)^n \det D^2 v(x) \left( v(x) + \left( 1 - \mu(-\ln(v(x)))^{-1} + \right. \right) \right)
\]
\[
(\mu - 1)(-\ln(v(x)))^{-1}\sup_{t > 0} \psi(t) \left( \nabla v(x)(-D^2 v(x))^{-1}(\nabla v(x))^T \right) \right) = b_1.
\]
Obviously, $\bar{u}_\mu \geq \underline{u}_\mu$ on $\Omega$. The rest of the proof is similar to that of Theorem 1.1 (i2) and the proof is omitted here. The proof is finished.

**Acknowledgments.** The author is greatly indebted to the anonymous referees for the very valuable suggestions and comments which improved the quality of the presentation.

**REFERENCES**

[1] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular Variation*, Encyclopedia of Mathematics and its Applications 27, Cambridge University Press, 1987.

[2] E. Calabi, Complete affine hypersurfaces I, *Symposia Math.*, 10 (1972), 19–38.

[3] S. Y. Cheng and S.-T. Yau, On the regularity of the Monge-Ampère equation det $((\partial^2 u/\partial x \partial v)) = F(x, u)$, *Comm. Pure Appl. Math.*, 30 (1977), 41–68.

[4] S. Y. Cheng and S.-T. Yau, Complete affine hypersurfaces I: The completeness of affine metrics, *Comm. Pure Appl. Math.*, 39 (1986), 839–866.

[5] Kai-Seng Chou and Xu-Jia Wang, The $L_p$-Minkowski problem and the Minkowski problem in centroaffine geometry, *Adv. Math.*, 205 (2006), 33–83.

[6] F.-C. Cirstea and C. Trombetti, On the Monge-Ampère equation with boundary blow-up: existence, uniqueness and asymptotics, *Cal. Var. Partial Diff. Equations*, 31 (2008), 167–186.

[7] F. Cui, H. Y. Jian and Y. Li, Boundary Hölder estimates for nonlinear singular elliptic equations, *J. Math. Anal. Appl.*, 470 (2019), 1185–1193.

[8] H. Y. Jian and Xu-Jia Wang, Bernstein theorem and regularity for a class of Monge-Ampère equation, *J. Diff. Geom.*, 93 (2013), 431–469.

[9] H. Y. Jian and Xu-Jia Wang, Optimal boundary regularity for nonlinear singular elliptic equations, *Adv. Math.*, 251 (2014), 111–126.

[10] H. Y. Jian, Xu-Jia Wang and Y. W. Zhao, Global smoothness for a singular Monge-Ampère equation, *J. Diff. Equations*, 263 (2017), 7250–7262.

[11] H. Y. Jian and Y. Li, Optimal boundary regularity for a singular Monge-Ampère equation, *J. Diff. Equations*, 264 (2018), 6873–6890.

[12] A. C. Lazer and P. J. McKenna, On singular boundary value problems for the Monge-Ampère Operator, *J. Math. Anal. Appl.*, 197 (1996), 341–362.

[13] A. C. Lazer and P. J. McKenna, On a singular elliptic boundary value problem, *Proc. Amer. Math. Soc.*, 111 (1991), 721–730.

[14] D. S. Li and S. S. Ma, Boundary behavior of solutions of Monge-Ampère equations with singular righthand sides, *J. Math. Anal. Appl.*, 454 (2017), 79–93.

[15] F. H. Lin and L. H. Wang, A class of fully nonlinear elliptic equations with singularity at the boundary, *J. Geom. Anal.*, 8 (1998), 583–598.

[16] C. Loewner and L. Nirenberg, Partial differential equations invariant under conformal or projective transformations, in *Contributions to Analysis (A Collection of Papers Dedicated to Lipman Bers)*, Academic Press, New York, 1974, 245–274.

[17] E. Lutwak, The Brunn-Minkowski-Firey theory I, Mixed volumes and the Minkowski problem, *J. Diff. Geom.*, 38 (1993), 131–150.

[18] A. Mohammed, Existence and estimates of solutions to a singular Dirichlet problem for the Monge-Ampère equation, *J. Math. Anal. Appl.*, 340 (2008), 1226–1234.

[19] A. Mohammed, Singular boundary value problems for the Monge-Ampère equation, *Nonlinear Anal.*, 70 (2009), 457–464.

[20] L. Nirenberg, Monge-Ampère equations and some associated problems in geometry, in *Proc. Internat. Congress of Mathematicians*, vol. 2, Vancouver, 1974, 275–279.
[21] E. Seneta, *Regular Varying Functions*, Lecture Notes in Math., vol. 508, Springer-Verlag, 1976.

[22] H. Sun and M. Q. Feng, Boundary behavior of $k$-convex solutions for singular $k$-Hessian equations, *Nonlinear Anal.*, **176** (2018), 141–156.

[23] Kaising Tso, On a real Monge-Ampère functional, *Invent. Math.*, **101** (1990), 425–448.

[24] H. T. Yang and Y. B. Chang, On the blow-up boundary solutions of the Monge-Ampère equation with singular weights, *Commun. Pure Appl. Anal.*, **11** (2012), 697–708.

[25] X. M. Zhang and Y. Du, Sharp conditions for the existence of boundary blow-up solutions to the Monge-Ampère equation, *Cal. Var. Partial Diff. Equations*, **57** (2018), 1–24.

[26] Z. J. Zhang, Refined boundary behavior of the unique convex solution to a singular Dirichlet problem for the Monge-Ampère equation, *Adv. Nonlinear studies*, **18** (2018), 289–302.

Received April 2019; revised June 2019.

E-mail address: zhangzj@ytu.edu.cn