Equivariant cohomology of real flag manifolds

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Abstract. Let \( P = G/K \) be a semisimple non-compact Riemannian symmetric space, where \( G = I_0(P) \) and \( K = G_p \) is the stabilizer of \( p \in P \). Let \( X \) be an orbit of the (isotropy) representation of \( K \) on \( T_p(P) \) (\( X \) is called a real flag manifold). Let \( K_0 \subset K \) be the stabilizer of a maximal flat, totally geodesic submanifold of \( P \) which contains \( p \). We show that if all the simple root multiplicities of \( G/K \) are at least 2 then \( K_0 \) is connected and the action of \( K_0 \) on \( X \) is equivariantly formal. In the case when the multiplicities are equal and at least 2, we will give a purely geometric proof of a formula of Hsiang, Palais and Terng concerning \( H^*(X) \). In particular, this gives a conceptually new proof of Borel’s formula for the cohomology ring of an adjoint orbit of a compact Lie group.

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1. Introduction

Let \( G/K \) be a non-compact symmetric space, where \( G \) is a non-compact connected semisimple Lie group and \( K \subset G \) a maximal compact subgroup. Then \( K \) is connected [He, Thm. 1.1, Ch. VI] and there exists a Lie group automorphism \( \tau \) of \( G \) which is involutive and whose fixed point set is \( G^\tau = K \). The involutive automorphism \( d(\tau)_e \) of \( g = \text{Lie}(G) \) induces the Cartan decomposition

\[ g = \mathfrak{k} \oplus \mathfrak{p}, \]

where \( \mathfrak{k} \) (the same as \( \text{Lie}(K) \)) and \( \mathfrak{p} \) are the \((+1)\)-, respectively \((-1)\)-eigenspaces of \((d\tau)_e \). Since \([\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \), the space \( \mathfrak{p} \) is \( \text{Ad}_G(K) := \text{Ad}(K) \)-invariant. The orbits of the action of \( \text{Ad}(K) \) on \( \mathfrak{p} \) are called real flag manifolds, or s-orbits. The restriction of the Killing form of \( g \) to \( \mathfrak{p} \) is an \( \text{Ad}(K) \)-invariant inner product on \( \mathfrak{p} \), which we denote by \( \langle \ , \ \rangle \).

Fix \( \mathfrak{a} \subset \mathfrak{p} \) a maximal abelian subspace. Recall that the roots of the symmetric space \( G/K \) are linear functions \( \alpha : \mathfrak{a} \to \mathbb{R} \) with the property that the space

\[ \mathfrak{g}_\alpha := \{ z \in \mathfrak{g} : [x,z] = \alpha(x)z \text{ for all } x \in \mathfrak{a} \} \]

is non-zero. The set \( \Pi \) of all roots is a root system in \((\mathfrak{a}^*, \langle \ , \ \rangle)\). Pick \( \Delta \subset \Pi \) a simple root system and let \( \Pi^+ \subset \Pi \) be the corresponding set of positive roots. For any \( \alpha \in \Pi^+ \) we have

\[ \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} = \mathfrak{t}_\alpha + \mathfrak{p}_\alpha, \]

where \( \mathfrak{t}_\alpha = (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) \cap \mathfrak{k} \) and \( \mathfrak{p}_\alpha = (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) \cap \mathfrak{p} \). We have the direct decompositions

\[ \mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \Pi^+} \mathfrak{p}_\alpha, \quad \mathfrak{t} = \mathfrak{t}_0 + \sum_{\alpha \in \Pi^+} \mathfrak{t}_\alpha, \]

where \( \mathfrak{t}_0 \) denotes the centralizer of \( \mathfrak{a} \) in \( \mathfrak{k} \). The multiplicity of a root \( \alpha \in \Pi^+ \) is

\[ m_\alpha = \dim \mathfrak{t}_\alpha + \dim \mathfrak{t}_{2\alpha}. \]
We note that this definition is slightly different from the standard one (see e.g. [Lo, Ch. VI, section 4]) which says that the multiplicity of $\alpha$ is just $\dim k_{\alpha}$.

Now $\mathfrak{t}_0$ is the Lie algebra of the Lie group $K_0 := C_K(\mathfrak{a})$ as well as of $K'_0 := N_K(\mathfrak{a})$. One can see that $K_0$ is a normal subgroup of $K'_0$; the Weyl group of the symmetric space $G/K$ is

$$W = K'_0/K_0.$$ 

It can be realized geometrically as the (finite) subgroup of $O(\mathfrak{a}, \langle , \rangle)$ generated by the reflections about the hyperplanes $\ker \alpha$, $\alpha \in \Pi^+$. 

Take $x_0 \in \mathfrak{a}$ and let $X = Ad(K)x_0$ be the corresponding flag manifold. The goal of our paper is to describe the cohomology, always with coefficients in $\mathbb{R}$, of $X$. The first main result concerns the action of $K_0$ on $X$.

**Theorem 1.1.** If the symmetric space $G/K$ has all root multiplicities $m_\alpha$, $\alpha \in \Delta$, strictly greater than 1 then:

(a) $K_0$ is connected;

(b) the action of $K_0$ on $X = Ad(K)x_0$ is equivariantly formal, in the sense that

$$H^*_K(X) \simeq H^*(X) \otimes H^*_K(\pt)$$

by an isomorphism of $H^*_K(\pt)$-modules;

(c) we have the isomorphisms of $\mathbb{R}$-vector spaces

$$H^*(X) \simeq \sum_{w \in W} H^{*-d_w}(w.x_0), \quad H^*_K(X) \simeq \sum_{w \in W} H^{*-d_w}_K(w.x_0).$$

Here

$$d_w = \sum m_\alpha$$

where the sum runs after all $\alpha \in \Pi^+$ such that $\alpha/2 \notin \Pi^+$ and the line segment $[x_0, w.x_0]$ crosses the hyperplane $\ker \alpha$.

**Remark.** Let $U$ be the (compact) Lie subgroup of $G^C$ whose Lie algebra is $\mathfrak{t} \oplus i\mathfrak{p}$. Then the manifold $X = Ad(K)x_0$ is the “real locus” [Go-Ho], [Bi-Gu-Ho] of an anti-symplectic involution on the adjoint orbit $Ad(U)x_0$ (see e.g. [Du, section 5]). The natural action of the torus $T := \exp(ia)$ on this orbit is Hamiltonian. In this way, $X$ fits into the more general framework of [Go-Ho] and [Bi-Gu-Ho]. But these papers investigate $X$ from the perspective of the action of $T_\mathbb{R} = T \cap K = T \cap K_0$, whereas we are interested here in the action on $X$ of a group which may be larger than $T_\mathbb{R}$, namely $K_0$.

In the second part of our paper we will deal with the ring structure of the usual cohomology of $X$, under the supplementary assumption that the symmetric space has all root multiplicities equal. By [He, Ch. X, Table VI], their common value can be only 2, 4 or 8. An important ingredient is the action of $W = K'_0/K_0$ on $X$ given by

$$(1) \quad hK_0. Ad(k)x_0 = Ad(k)Ad(h^{-1})x_0,$$

for any $h \in K'_0$ and $k \in K$. By functoriality, this induces an action of $W$ on $H^*(X)$. We also note that $W$ acts in a natural way on $\mathfrak{a}^*$. 

Theorem 1.2. Assume that $G/K$ is an irreducible non-compact symmetric space whose simple root multiplicities are equal to the same number, call it $m$, which is at least 2. Take $X = Ad(K)x_0$.

(i) If $x_0$ is a regular point of $\mathfrak{a}$, then there exists a canonical linear $W$-equivariant isomorphism $\Phi : \mathfrak{a}^* \rightarrow H^m(X)$. Its natural extension $\Phi : S(\mathfrak{a}^*) \rightarrow H^*(X)$ is a surjective ring homomorphism whose kernel is the ideal $\langle S(\mathfrak{a}^*)_W^+ \rangle$ generated by all nonconstant $W$-invariant elements of $S(\mathfrak{a}^*)$. Consequently we have the $\mathbb{R}$-algebra isomorphism

$$H^*(X) \simeq S(\mathfrak{a}^*)/\langle S(\mathfrak{a}^*)_W^+ \rangle.$$  

(ii) If $x_0$ is an arbitrary point in $\mathfrak{a}$, then we have the $\mathbb{R}$-algebra isomorphism

$$H^*(X) \simeq S(\mathfrak{a}^*)_{W_{x_0}}/\langle S(\mathfrak{a}^*)_W^+ \rangle,$$

where $W_{x_0}$ is the $W$-stabilizer of $x_0$.

Remark. Any real flag manifold $X = Ad(K)x$ with the canonical embedding in $(\mathfrak{p}, \langle , , \rangle)$ is an element of an isoparametric foliation [Pa-Te]. The topology of such manifolds, including their cohomology rings, has been investigated by Hsiang, Palais and Terng in [Hs-Pa-Te] (see also [Ma]). The formulas for $H^*(X)$ given by Theorem 1.2 have been proved by them in that paper. Even though we do use some of their ideas (originating in [Bo-Sa]), our proof is different: they rely on Borel’s formula [Bo] for the cohomology of a generic adjoint orbit of a compact Lie group, whereas we actually prove it.

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2. Symmetric spaces with multiplicities at least 2 and their $s$-orbits

Let $G/K$ be an arbitrary non-compact symmetric space, $x_0 \in \mathfrak{a}$ and $X = Ad(K)x_0$ the corresponding $s$-orbit. The latter is a submanifold of the Euclidean space $(\mathfrak{p}, \langle , , \rangle)$. The Morse theory of height functions on $X$ will be an essential instrument. The following proposition summarizes results from [Bo-Sa] or [Hs-Pa-Te] (see also [Ma]).

Proposition 2.1. (i) If $a \in \mathfrak{a}$ is a general vector (i.e. not contained in any of the hyperplanes $\ker \alpha$, $\alpha \in \Pi^+$), then the height function $h_a(x) = \langle a, x \rangle$, $x \in X$ is a Morse function. Its critical set is the orbit $W.x_0$.

(ii) Assume that $a$ and $x_0$ are contained in the same Weyl chamber in $\mathfrak{a}$. Then the index of $h_a$ at the critical point $w.x_0$ is

$$d_w = \sum m_\alpha$$

where the sum runs after all $\alpha \in \Pi^+$ such that $\alpha/2 \notin \Pi^+$ and the line segment $[a, wx_0)$ crosses the hyperplane $\ker \alpha$.

In the next lemma we consider the situation when all root multiplicities are at least 2.

Lemma 2.2. Assume that the root multiplicities $m_\alpha, \alpha \in \Delta$, of the symmetric space $G/K$ are all strictly greater than 1. Then:
(i) for any general vector \( a \in \mathfrak{a} \), the height function \( h_a : X \to \mathbb{R} \) is \( \mathbb{Z} \)-perfect,
(ii) the space \( K_0 \) is connected,
(iii) if \( X = \text{Ad}(K)x_0 \), then the orbit \( W.x_0 \) is contained in the fixed point set \( X^{K_0} \).

Proof. (i) According to [Ko, Theorem 1.1.4], there exists a metric on \( X \) such that if two critical points \( x \) and \( y \) can be joined by a gradient line, then \( x = s_\gamma y \), where \( \gamma \in \Pi^+ \). By [2], the difference of the indices of \( x \) and \( y \) is different from \( \pm 1 \). Because the stable and unstable manifolds intersect transversally [Ko, Corollary 2.2.7], the Morse complex of \( h_a \) has all boundary operators identically zero, hence \( h_a \) is \( \mathbb{Z} \)-perfect.

(ii) Take \( a \in \mathfrak{a} \) a general vector. The height function \( h_a \) on \( \text{Ad}(K)a \) is \( \mathbb{Z} \)-perfect. From [2] we deduce that \( H_1(\text{Ad}(K)a, \mathbb{Z}) = 0 \), thus \( \text{Ad}(K)a \) is simply connected. On the other hand, the stabilizer \( C_K(a) \) is just \( K_0 \) (see e.g. [Bo-Sa]). Because \( K/K_0 \) is simply connected and \( K \) is connected, we deduce that \( K_0 \) is connected.

(iii) The height function \( h_a \) is \( \text{Ad}(K_0) \)-invariant, thus \( \text{Crit}(h_a) = W.x_0 \) is also \( \text{Ad}(K_0) \)-invariant. The result follows from the fact that \( K_0 \) is connected. \( \square \)

We are now ready to prove Theorem 1.1

Proof of Theorem 1.1 Point (a) was proved in Lemma 2.2 (ii).

(b) According to [Gu-Gi-Ka, Proposition C.25] it is sufficient to show that \( H^*_K(X) \) is free as a \( H^*_K(\mathfrak{a}) \)-module. In order to do that we consider the height function \( h_a : X \to \mathbb{R} \) corresponding to a general \( a \in \mathfrak{a} \). We use the same arguments as in the proof of Lemma 2.2 (i). The function \( h_a \) is a \( K_0 \)-invariant. By the same reasons as above, the \( K_0 \)-equivariant Morse complex [Au-Br, Sections 5 and 6] has all boundary operators identically zero. Thus \( H^*_K(X) \) is a free \( H^*_K(\mathfrak{a}) \)-module (with a basis indexed by \( \text{Crit}(h_a) = W.x_0 \)).

(c) The space \( H_*(X) \) has a basis \( \{[X_{w.x_0}] : w \in W \} \), where \( X_{w,x_0} \) is some \( d_w \)-dimensional cycle in \( X, w \in W \). The evaluation pairing \( H^*(X) \times H_*(X) \to \mathbb{R} \) is non-degenerate; consider the basis of \( H^*(X) \) dual to \( \{[X_{w,x_0}] : w \in W \} \), which gives one element of degree \( d_w \) for each \( w.x_0 \). The result follows. \( \square \)

3. Cohomology of s-orbits of symmetric spaces with uniform multiplicities at least 2

Throughout this section \( G/K \) is a non-compact irreducible symmetric space whose simple root multiplicities are all equal to \( m \), where \( m \geq 2; x_0 \in \mathfrak{a} \) is a regular element and

\[ X = \text{Ad}(K)x_0 \cong K/K_0 \]

is the corresponding real flag manifold. There are three such symmetric spaces; their compact duals are (see e.g. [Hs-Pa-Te, Section 3]):

1. any connected simple compact Lie group \( K \); we have \( m = 2 \); the flag manifold is \( X = K/T \), where \( T \) is a maximal torus in \( K \);
2. \( SU(2n)/Sp(n) \) where \( m = 4 \); the flag manifold is \( X = Sp(n)/Sp(1)^n \);
3. \( E_6/F_4 \) where \( m = 8 \); the flag manifold is \( X = F_4/\text{Spin}(8) \).
Let $\Delta = \{\gamma_1, \ldots, \gamma_l\}$ be a simple root system of $\Pi$. To each $\gamma_j$ corresponds the distribution $E_j$ on $X$, defined as follows: its value at $x_0$ is
\[ E_j(x_0) = [\xi_j, x_0] \]
and $E_j$ is $K$-invariant, i.e.
\[ E_j(Ad(k)x_0) = Ad(k)E_j(x_0), \]
for all $k \in K$.

A basis of $H_m(X)$ can be obtained as follows: Assume that $x_0$ is in the (interior of the) Weyl chamber $C \subset a$ which is bounded by the hyperplanes $\ker \gamma_j$, $1 \leq j \leq l$. The Weyl group $W$ is generated by $s_j$, which is the reflection of $a$ about the wall $\ker \gamma_j$, $1 \leq j \leq l$. For each $1 \leq j \leq l$ we consider the Lie subalgebra $k_0 + k_{\gamma_j}$ of $k$; denote by $K_j$ the corresponding connected subgroup of $K$. It turns out that the orbit $Ad(K_j)x_0$ is a round $m$-dimensional metric sphere in $(p, \langle \ , \rangle)$. To any $x = Ad(k)x_0 \in X$ we attach the round sphere
\[ S_j(x) = Ad(k)Ad(K_j)x_0. \]

The spheres $S_j$ are integral manifolds of the distribution $E_j$. We denote by $[S_j]$ the homology class carried by any of the spheres $S_j(x)$, $x \in X$. It turns out that $S_1(x_0), \ldots, S_l(x_0)$ are cycles of Bott-Samelson type (see [Bo-Sa], [Hs-Pa-Te]) for the index $m$ critical points of the height function $h_a$, thus $[S_1], \ldots, [S_l]$ is a basis of $H_m(X)$.

The following result concerning the action of $W$ on $H_m(X)$ was proved in [Hs-Pa-Te, Corollary 6.10] (see also [Ma, Theorem 2.1.1]):

**Proposition 3.1.** We can choose an orientation of the spheres $S_j$, $1 \leq j \leq l$, such that the linear isomorphism $a \rightarrow H_m(X)$ determined by
\[ \gamma_j := \frac{2\gamma_j}{\langle \gamma_j, \gamma_j \rangle} \mapsto [S_j]; \]
$1 \leq j \leq l$, is $W$-equivariant.

We need one more result concerning the action of $W$ on $H^*(X)$:

**Lemma 3.2.** Let $x \in a$ be an arbitrary element, $C = C_K(x)$ its centralizer in $K$, and let
\[ p : X = K/K_0 \rightarrow Ad(K)x = K/C \]
be the natural map induced by the inclusion $K_0 \subset C$. Then the map $p^* : H^*(Ad(K)x) \rightarrow H^*(X)$ is injective. Its image is
\[ p^*H^*(Ad(K)x) = H^*(X)^{W_x} \]
where the right hand side denotes the set of all $W_x$-invariant elements of $H^*(X)$. Here $W_x$ denotes the $W$-stabilizer of $x$. In particular, the only elements in $H^*(X)$ which are $W$-invariant are those of degree 0, i.e.
\[ H^*(X)^W = H^0(X). \]
Proof. The map \( p : K/K_0 \to K/C \) is a fibre bundle. The fiber \( C/K_0 \) is an \( s \)-orbit of the symmetric space \( C_G(x)/C_K(x) \). The latter has all root multiplicities equal to \( m \), as they are all root multiplicities of some roots of \( G/K \). By Theorem 1.1 (ii), \( C/K_0 \) can have non-vanishing cohomology groups only in dimensions which are multiples of \( m \). The same can be said about the cohomology of the space \( K/C \). Because \( m \in \{ 2, 4, 8 \} \), the spectral sequence of the bundle \( p : K/K_0 \to K/C \) collapses, which implies that \( p^* \) is injective.

The map \( p \) is \( W \)-equivariant with respect to the actions of \( W \) on \( Ad(K)x_0 \), respectively \( Ad(K)x \) defined by (1). Thus if \( w \in W_x \), then \( w|_{Ad(K)x} \) is the identity map, hence we have \( p \circ w = p \). This implies the inclusion
\[
p^*H^*(Ad(K)x) \subset H^*(X)^{W_x}.
\]

On the other hand, the action of \( W \) on \( X \) defined by (1) is free, as the \( Ad(K) \) stabilizer of the general point \( x_0 \) reduces to \( K_0 \). Consequently we have
\[
H^*(X)^{W_x} = H^*(X/W_x)
\]
and
\[
\chi(X/W_x) = \frac{\chi(X)}{|W_x|} = \frac{|W|}{|W_x|},
\]
where \( \chi \) denotes the Euler-Poincaré characteristic. It follows from Theorem 1.1 (c) that
\[
\dim H^*(X)^{W_x} = \frac{|W|}{|W_x|} = \dim H^*(Ad(K)x).
\]
Now we use that \( p^* \) is injective.

In order to prove the last statement of the lemma, we take \( x = 0 \in a \). Let us consider the Euler class \( \tau_i = e(E_i) \in H^m(X), 1 \leq i \leq l \). We will prove that:

**Lemma 3.3.** (i) The cohomology classes \( \tau_i, 1 \leq i \leq l \) are a basis of \( H^m(X) \).

(ii) The linear isomorphism \( \Phi : a^* \to H^m(X) \) determined by
\[
\gamma_i \mapsto e(E_i),
\]
\( 1 \leq i \leq l \), is \( W \)-equivariant.

**Proof.** By Proposition 3.1 we know that
\[
s_{i*}[S_j] = [S_j] - d_{ji}[S_i],
\]
where
\[
d_{ji} = 2 \frac{\langle \gamma_j, \gamma_i \rangle}{\langle \gamma_i, \gamma_i \rangle}.
\]
Denote by \( \langle \cdot, \cdot \rangle \) the evaluation pairing \( H^2(M) \times H_2(M) \to \mathbb{R} \). Consider \( \alpha_j \in H^2(M) \) such that \( \langle \alpha_j, [S_i] \rangle = \delta_{ij}, 1 \leq i, j \leq l \). Take the expansion
\[
\tau_i = \sum_{j=1}^l t_{ij} \alpha_j.
\]
The automorphism $s_i$ of $X$ maps the distribution $E_i$ onto itself and changes its orientation (since so does the antipodal map on an $m$-dimensional sphere). Thus

$$s_i^*(\tau_i) = -\tau_i.$$ Consequently we have

$$t_{ij} = \langle \tau_i, [S_j] \rangle = -s_i^*(\tau_i)E_j = -\langle \tau_i, s_iw[S_j] \rangle = -\langle \tau_i, [S_j] - d_{ji}[S_i] \rangle = -t_{ij} + 2d_{ji}$$

which implies $t_{ij} = d_{ji}$. By Proposition 3.1, the matrix $(d_{ij})$ is the Cartan matrix of the root system dual to $\Pi$, hence it is non-singular. Consequently $\tau_i$, $1 \leq i \leq l$ is a basis of $H^m(X)$. Again by Proposition 3.1 we have

$$\langle s_j^*(\tau_i), [S_k] \rangle = \langle \tau_i, [S_k] - d_{kj}[S_j] \rangle = t_{ik} - d_{kj}t_{ij} = t_{ik} - t_{jk}d_{ji},$$

thus

$$s_j^*(\tau_i) = \tau_i - d_{ji}\tau_j.$$ It remains to notice that $d_{ji}$ can also be expressed as

$$d_{ji} = 2\frac{\langle \gamma_i; \gamma_j \rangle}{\langle \gamma_j; \gamma_j \rangle}.$$}

We are now ready to prove Theorem 1.2

Proof of Theorem 1.2 (i) Consider the ring homomorphism $\Phi : S(\mathfrak{a}^*) \to H^*(X)$ induced by $\gamma_i \mapsto e(E_i)$, $1 \leq i \leq l$. By Lemma 3.3, $\Phi$ is $W$-equivariant and from Lemma 3.2 we deduce that $\langle S(\mathfrak{a}^*)_W \rangle \subset \ker \Phi$. By Lemma 3.4 (see below), it is sufficient to prove that

$$\Phi(\prod_{\alpha \in \Pi^+} \alpha) \neq 0.$$ To this end, we will describe explicitly $\Phi(\alpha)$, for $\alpha \in \Pi^+$. Write $\alpha = w.\gamma_j$, where $w \in W$. The latter is of the form $w = hK_0$, with $h \in K_0'$. The image of $S_j(x_0)$ by the automorphism $w$ of $X$ is

$$w(S_j(x_0)) = \text{Ad}(K_j).\text{Ad}(h^{-1}).x_0 = \text{Ad}(h^{-1}).\text{Ad}(hK_j.h^{-1}).x_0$$

$$= \text{Ad}(h^{-1}).\text{Ad}(K_\alpha).x_0 = \text{Ad}(h^{-1}).S_\alpha(x_0) = S_\alpha(\text{Ad}(h^{-1}).x_0)$$

$$= S_\alpha(w.x_0).$$

Here $K_\alpha$ is the connected subgroup of $K$ of Lie algebra $\mathfrak{k}_0 + \mathfrak{t}_\alpha$ and $S_\alpha(x_0) := \text{Ad}(K_\alpha)x_0$ is a round metric sphere through $x_0$; for any $x = \text{Ad}(k)x_0 \in X$ we have $S_\alpha(x) := \text{Ad}(k)S_\alpha(x_0)$, which is an integral manifold of

$$E_\alpha(x) = \text{Ad}(k)[\mathfrak{k}_\alpha, x_0].$$ It is worth mentioning in passing that the spheres $S_\alpha$ and the distributions $E_\alpha$ are the curvature spheres, respectively curvature distributions of the isoparametric submanifold $X \subset \mathfrak{p}$ (see the remark following Theorem 1.2 in the introduction). Thus the differential of $w$ satisfies $(dw)(E_j) = E_\alpha$, which implies

$$e(E_j) = w^*e(E_\alpha).$$
Consequently
\[ \Phi(\alpha) = \Phi(w \cdot \gamma_j) = w^{-1} \cdot \Phi(\gamma_j) = (w^{-1})^* (e(E_j)) = e(E_\alpha). \]

We deduce that
\[ \Phi\left( \prod_{\alpha \in \Pi^+} \alpha \right) = \prod_{\alpha \in \Pi^+} e(E_\alpha) = e\left( \sum_{\alpha \in \Pi^+} E_\alpha \right). \]

On the other hand,
\[ \sum_{\alpha \in \Pi^+} E_\alpha(x_0) = \sum_{\alpha \in \Pi^+} [\mathfrak{f}_\alpha, x_0] = [\mathfrak{f}, x_0] = T_{x_0} X \]
thus
\[ \sum_{\alpha \in \Pi^+} E_\alpha = TX. \]
It follows that
\[ \Phi\left( \prod_{\alpha \in \Pi^+} \alpha \right) = e(TX), \]
which is different from zero, as
\[ e(TX)([X]) = \chi(X) = |W|, \]
where \( \chi(X) \) is the Euler-Poincaré characteristic of \( X \).

(ii) We apply Lemma 3.2.

The following lemma has been used in the proof:

**Lemma 3.4.** ([Hi, Lemma 2.8]) Let \( I \) be a graded ideal of \( S(\mathfrak{a}^*) \) which is also a vector subspace and such that \( \langle S(\mathfrak{a}^*)^+ \rangle \subset I \). We have \( I = \langle S(\mathfrak{a}^*)^+ \rangle \) if and only if
\[ \prod_{\alpha \in \Pi^+} \alpha \notin I. \]

A proof of this lemma can also be found in the appendix.

4. Appendix: Proof of Lemma 3.4

The goal of this appendix is to provide a proof of Lemma 3.4, which is stated without a proof in [Hi]. As mentioned in the introduction, the Weyl group \( W \) can be realized as the group of orthogonal transformations of \( \mathfrak{a} \) generated by the reflections \( s_\alpha, \alpha \in \Pi^+ \). In fact, if \( \{\gamma_1, \ldots, \gamma_l\} \) is a simple root system, then \( W \) is generated by \( s_i := s_{\gamma_i}, 1 \leq i \leq l \). Denote by \( w_0 \) the longest element of \( W \), where the length is measured with respect to the generating set \( \{s_1, \ldots, s_l\} \). We will use the notations
\[ S := S(\mathfrak{a}^*), \quad I_W := \langle S(\mathfrak{a}^*)^+ \rangle. \]
First of all we note that the action of \( W \) on the polynomial ring \( S \) is given by
\[ (w.f)(x) = f(w^{-1}.x), \]
where \( w \in W, f \in S, x \in \mathfrak{a} \). This action preserves the grading of \( S \), hence the ideal \( I_W \) generated by the nonconstant \( W \)-invariant polynomials is also graded. The most prominent example of a polynomial which is not \( W \)-invariant is
\[ d = \prod_{\alpha \in \Pi^+} \alpha. \]
In fact $d$ is skew-invariant, in the sense that $w.d = (-1)^{l(w)}d$, for any $w \in W$.

If $\alpha \in \Pi^+$, we consider the operator $\Delta_\alpha : \mathcal{S} \to \mathcal{S}$ defined as follows:

$$\Delta_\alpha(f) = \frac{f - s_\alpha.f}{\alpha},$$

$f \in \mathcal{S}$. Note that $f - s_\alpha.f$ vanishes on the space $\ker \alpha$, hence $\Delta_\alpha(f)$ is really a polynomial. The following result is straightforward:

**Lemma 4.1.** If $w \in W$, $\alpha \in \Pi^+$, $f, g \in \mathcal{S}$, then we have:

(a) $\Delta_\alpha(fg) = \Delta_\alpha(f)g + s_\alpha(f)\Delta_\alpha(g)$;

(b) $\Delta_\alpha(I_W) \subset I_W$.

To any $w \in W$ we can associate the operator $\Delta_w : \mathcal{S} \to \mathcal{S}$, which has degree $-l(w)$, and is defined as follows: take $w = s_{i_1} \cdots s_{i_k}$ a reduced expression and put $\Delta_w = \Delta_{\gamma_1} \cdots \Delta_{\gamma_k}$. We note that $\Delta_w$ does not depend on the choice of the reduced expression (see e.g. [Hi, Proposition 2.6]). The operators obtained in this way have the following property (see [Hi, Lemma 3.1]):

$$\Delta_{w'} \circ \Delta_w = \begin{cases} 
\Delta_{ww'}, & \text{if } l(ww') = l(w) + l(w') \\
0, & \text{otherwise}
\end{cases}$$

(3)

A classical result which goes back to Chevalley, says that the ideal $I_W$ is generated by $l$ homogeneous polynomials, which are algebraically independent. Let $d_1, \cdots, d_l$ denote their degrees. It follows that the Poincaré polynomial of $\mathcal{S}/I_W$ is:

$$P(\mathcal{S}/I_W) = \sum_{k=0}^{\infty} (\dim \mathcal{S}^k - \dim I_W^k)t^k = \prod_{j=1}^{l} (1 + t + \cdots + t^{d_j-1}).$$

Combined with the fact that $d_1 + \cdots + d_l = N + l$ (see for instance [Hu, Theorem 3.9]), this tells us that $I^k = \mathcal{S}^k$, for $k \geq N + 1$. The same polynomial can be expressed as (see [Hu, Theorem 3.15]):

$$P(\mathcal{S}/I_W) = \sum_{w \in W} t^{l(w)}.$$  

We deduce that $\dim \mathcal{S}^k - \dim I_W^k$ equals the number of $w \in W$ with $l(w) = k$, $0 \leq k \leq N$. The following result describes a direct complement of $I_W^k$ in $\mathcal{S}^k$:

**Proposition 4.2.** For any $0 \leq k \leq N$, the elements $\Delta_w(d)$, $w \in W$, $l(w) = N - k$ are linearly independent and span a direct complement of $I_W^k$ in $\mathcal{S}^k$.

**Proof.** The number of elements of $W$ of length $k$ equals the number of elements of length $N - k$, hence we only have to prove that the polynomials $\Delta_w(d)$, where $l(w) = N - k$ are linearly independent and their span intersected with $I_W$ is $\{0\}$. To this end, it is sufficient to show that if

$$\sum_{l(w) = N - k} \lambda_w \Delta_w(d) \in I_W^k$$

then all $\lambda_w$ must vanish. Indeed, if we fix $v \in W$ with $l(v) = N - k$, then by (3), we have

$$\Delta_{w_{0}v^{-1}} \left( \sum_{l(w) = N - k} \lambda_w \Delta_w(d) \right) = \lambda_v.$$

The left hand side of this equation is in $I_W^0$, hence it must be 0.
We are ready to prove Lemma 3.4.

Proof of Lemma 3.4 We prove by induction on $k$ that $I^k_W = I^k$, $0 \leq k \leq N$. Things are clear for $k = N$: $I^N_W$ equals $I^N$ because $I^N_W \subset I^N \neq S^N$ and the codimension of $I^N_W$ in $S^N$ is 1 (see Proposition 4.2). Now, from $I^{k+1}_W = I^{k+1}$ we deduce that $I^k = I^k_W$. Suppose that we have

$$f := \sum_{l(w) = N - k} \lambda_w \Delta_w(d) \in I^k,$$

where $\lambda_w \in \mathbb{R}$, not all of them equal to 0. We will prove by induction on $m \in \{0, \ldots, k\}$ the following claim

Claim. For any $h_m \in S^m$ and any $\alpha_1, \ldots, \alpha_m \in \Pi^+$, we have

$$h_m \Delta_{\alpha_1} \circ \cdots \circ \Delta_{\alpha_m} (f) \in I^k.$$

For $m = 0$, this is trivial. Suppose it is true for a certain $m$ and prove it for $m + 1$. If $h_m \in S^m$, $\alpha_1, \ldots, \alpha_m \in \Pi^+$, $h$ an arbitrary homogeneous polynomial of degree 1, and $\alpha$ a positive root, then we have

$$hh_m \Delta_{\alpha_1} \circ \cdots \circ \Delta_{\alpha_m} (f) \in I^{k+1} = I^{k+1}_W,$$

hence its image by $\Delta_{\alpha}$ is in $I^k_W \subseteq I^k$. We deduce that

$$\Delta_{\alpha}(h)m \Delta_{\alpha_1} \circ \cdots \circ \Delta_{\alpha_m} (f) + s_{\alpha}(h)\Delta_{\alpha}(h_m)\Delta_{\alpha_1} \circ \cdots \circ \Delta_{\alpha_m} (f) + s_{\alpha}(h)s_{\alpha}(h_m)\Delta_{\alpha} \circ \Delta_{\alpha_1} \circ \cdots \circ \Delta_{\alpha_m} (f)$$

is in $I^k$, consequently $s_{\alpha}(hh_m)\Delta_{\alpha} \circ \Delta_{\alpha_1} \circ \cdots \circ \Delta_{\alpha_m} (f) \in I^k$. Since any $h_{m+1} \in S^{m+1}$ is a linear combination of polynomials of the form $s_{\alpha}(hh_m)$, the claim is proved.

We deduce that for any $v \in W$ with $l(v) = k$, and any $h_k \in S^k$ we have that

$$h_k \Delta_v(f) \in I^k.$$

Fix now $w \in W$ with $l(w) = N - k$ and take $v := w_0w^{-1}$. Then $\Delta_v(f) = \lambda_w$ by (1), hence $\lambda_w h_k \in I^k$, for any $h_k \in S^k$. But then $\lambda_w$ must vanish, since $I^k \neq S^k$ (if they were equal, from $k \leq N$ we would deduce $I^N = S^N$, which is false). We conclude that $f = 0$, which is a contradiction. This finishes the proof. \qed
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