Interference Relay Channels – Part II: Power Allocation Games

Elena Veronica Belmega, Student Member, IEEE, Brice Djeumou, Student Member, IEEE, and Samson Lasaulce, Member, IEEE

Abstract

In the first part of this paper we have derived achievable transmission rates for the (single-band) interference relay channel (IRC) when the relay implements either the amplify-and-forward, decode-and-forward or estimate-and-forward protocol. Here, we consider wireless networks that can be modeled by a multi-band IRC. We tackle the existence issue of Nash equilibria (NE) in these networks where each information source is assumed to selfishly allocate its power between the available bands in order to maximize its individual transmission rate. Interestingly, it is possible to show that the three power allocation (PA) games (corresponding to the three protocols assumed) under investigation are concave, which guarantees the existence of a pure NE after Rosen [3]. Then, as the relay can also optimize several parameters e.g., its position and transmit power, it is further considered as the leader of a Stackelberg game where the information sources are the followers. Our theoretical analysis is illustrated by simulations giving more insights on the addressed issues.

Index Terms

Cognitive radio, game theory, interference relay channel, Nash equilibrium, power allocation games, relay channel, Stackelberg game.

I. INTRODUCTION

A possible way to improve the performance of a network comprising pairs of source-destination links, interfering with each other, is to add some relaying nodes in the network. In the first part of this two-
part paper, we have studied the special case of a network composed of two source nodes, each of them sending private messages to its respective destination with the help of a common relay node. The emphasis was on the relaying protocol and determination of achievable transmission rates. In this second part paper, we assume that there are several relaying nodes operating in non-overlapping frequency bands and that the source nodes have to allocate their powers between the available frequency bands in order to maximize their transmission rates. Therefore, the focus in this part is on the decentralized power allocation (PA) problem. One of our motivations for analyzing such a scenario is to acquire a better understanding of a system where two cognitive transmitters, each of them communicating with its respective receiver, are offered the opportunity to re-exploit frequency bands (unused locally in space or time) in which a relaying node is available in order to further increase the performance of the system. As the transmitters are assumed to be decision makers that can freely choose their own resource allocation policies while selfishly maximizing their transmission rate and as the communications interfere in each band, the resource allocation problem can be modeled by a non-cooperative game. Then, several key questions arise: What is the influence of the relaying protocol on the game formulation? Is there a predictable state (equilibrium) at which the system will effectively operate?

Before describing the structure of this paper, presenting the system model and tackling these issues, we will mention the closest contributions to those presented in this paper. As mentioned in the Abstract, the network under consideration can be modeled as a set of parallel (or orthogonal) interference relay channels (IRCs). In Part I, we have mentioned that this channel has been introduced by [4][5]. In these works, the problem is not formulated as a game and, therefore, no equilibrium analysis is conducted. Additionally, the authors study in detail only the case of the decode-and-forward (DF) protocol while, here, we exploit the results derived in Part I for the three dominant classes of protocols i.e., amplify-and-forward (AF), DF and also estimate-and-forward (EF). The closest work concerning the game-theoretic approach we adopt is the work by Yu et al. [6]. In [6], the authors studied the distributed PA game in frequency selective interference channels. The problem we present here is a generalization of the problem addressed by Yu et al. for which a relay (implementing a certain protocol) is available on each band or sub-carrier.

This paper is structured as follows. Sec. II describes the system model. In Sec. III, IV and V we analyze the existence of a Nash equilibrium (NE) in the non-cooperative PA game when zero-delay scalar AF (ZDSAF), DF and EF protocols are respectively considered. Sec. III is itself divided into two sub-sections, one dedicated to the general ZDSAF protocol and the other to a special case where the amplification gains...
at the relay nodes are independent of the power of the signals they receive. Sec. [V] also comprises two special cases, one case where the game is due to the power allocation between the bands and another one where it is due to the power allocation between the fine and coarse messages in the DF protocol (following the comments made in Part I). Then, in Sec. [VI] we formulate the PA problem as a Stackelberg game by introducing the relay as an additional player of a hierarchical game in order to know how the relay parameters should be tuned in practice. Sec. [VII] provides, in particular, simulations showing the importance of choosing the amplification factor properly for the ZDSAF protocol and optimally locating the relay. Summarizing remarks and possible extensions are given in Sec. [VIII]

II. System Model

The system under investigation comprises two source nodes $S_1, S_2$, transmitting their private messages to their respective destination nodes $D_1, D_2$. To this end, each source can exploit $Q$ non-overlapping frequency bands (the notation $(q)$ will be used to refer to band $q \in \{1, \ldots, Q\}$) which are assumed to be unitary. The signals transmitted by $S_1$ and $S_2$ in band $(q)$, denoted by $X_{1}^{(q)}$ and $X_{2}^{(q)}$, respectively, are assumed to be independent and subject to power constraints:

$$\forall i \in \{1, 2\}, \sum_{q=1}^{Q} \mathbb{E}|X_{i}^{(q)}|^2 \leq P_{i}. \quad (1)$$

For $i \in \{1, 2\}$, we denote by $\theta_{i}^{(q)}$ the fraction of power that is used by $S_i$ for transmitting in band $(q)$ that is $\mathbb{E}|X_{i}^{(q)}|^2 = \theta_{i}^{(q)} P_{i}$. On each band $(q)$, a relaying node $R^{(q)}$ is available. Each relaying node is assumed to operate in the full-duplex mode. The transmit power at each relay is also subject to a constraint $\mathbb{E}|X_{r}^{(q)}|^2 \leq P_{r}^{(q)}$. Note that the relay transmit power in band $(q)$ is subject to an individual constraint, which implicitly means that the relays are not co-located (operating in the same frequency band) Otherwise, for a single relay operating in $Q$ bands, the power constraint would be $\sum_{q=1}^{Q} \mathbb{E}|X_{r}^{(q)}|^2 \leq P_{r}$, which is not what we assume in this paper. With these notations, the baseband signal received by $D_1, D_2$ and $R^{(q)}$ in band $(q)$ express as:

$$\begin{align*}
Y_{1}^{(q)} &= h_{11}^{(q)} X_{1}^{(q)} + h_{21}^{(q)} X_{2}^{(q)} + h_{r1}^{(q)} X_{r}^{(q)} + Z_{1}^{(q)} \\
Y_{2}^{(q)} &= h_{12}^{(q)} X_{1}^{(q)} + h_{22}^{(q)} X_{2}^{(q)} + h_{r2}^{(q)} X_{r}^{(q)} + Z_{2}^{(q)} \\
Y_{r}^{(q)} &= h_{1r}^{(q)} X_{1}^{(q)} + h_{2r}^{(q)} X_{2}^{(q)} + Z_{r}^{(q)} \quad (2)
\end{align*}$$

where $Z_{i}^{(q)} \sim \mathcal{N}(0, N_{i}^{(q)})$, $i \in \{1, 2, r\}$, represents the Gaussian complex noise on band $(q)$ and, for all $i, j \in \{1, 2, r\}$, $h_{ij}^{(q)}$ is the channel gain between $S_i$ and $D_j$ in band $(q)$, with the convention $h_{rr} = 0$. 

April 16, 2009 DRAFT
As justified in Part I, we consider a realistic situation where only large scale propagation effects can be taken into account by the users to optimize their rates. Thus the channel gains are considered to be static. Concerning channel state information (CSI), we will always assume coherent communications between each transmitter-receiver pair \((S_i, D_i)\) whereas, at the transmitters, the information assumptions will be contextdepending and deduced from the latter. At \(D_1\) and \(D_2\), single-user decoding (SUD) will always be assumed. It is a realistic assumption in a framework where devices operate in unlicensed bands in an a priori noncoordinated manner. At the relays \(R^{(q)}, q \in \{1, ..., Q\}\), the reception scheme implemented will depend on the protocol assumed. The expression of the signals \(X_r^{(q)}\) depend on the respective protocol and will therefore also be explicated in the corresponding sections. At last, we will use the same specific notations as in Part I: The capacity function for complex signals is denoted by \(C(x) \triangleq \log_2 (1 + x)\); for any real \(a \in [0, 1]\), the quantity \(\overline{a}\) will stand for \(\overline{a} = 1 - a\); the notation \(-i\) will mean that \(-i = 1\) if \(i = 2\) and \(-i = 2\) if \(i = 1\).

III. EQUILIBRIUM ANALYSIS FOR THE ZDSAF PROTOCOL

In this section, we assume that the relay nodes implement the ZDSAF protocol, which has already been described in Part I. One of the nice features of the (analog) ZDSAF protocol is that relays are very easy to be deployed since they can be used without any change on the existing (non-cooperative) communication system. The amplification factor/gain for the relay \(R^{(q)}\) will be denoted by \(a_r^{(q)}\). We have discussed the optimal choice of this parameter in Part I. In this part, we consider two choices for \(a_r^{(q)}\): (A) the case where it is chosen to saturate the transmit power constraint at \(R^{(q)}\) that is, \(a_r^{(q)} = \sqrt{\frac{P_r^{(q)}}{\mathbb{E}|Y_r^{(q)}|^2}} = \sqrt{\frac{P_r^{(q)}}{|h_1^{(q)}|^2 P_1 \theta_1^{(q)} + |h_2^{(q)}|^2 P_2 \theta_2^{(q)} + N_r^{(q)}}} \triangleq \tilde{a}_r^{(q)}\); (B) the case where it is a constant w.r.t. \(\theta_1^{(q)}\) and \(\theta_2^{(q)}\), in which case it will be denoted by \(A_r^{(q)}\). In this section, we prove the existence of an NE in the most general case, i.e, case (A). The reason for analyzing case (B) is at least twofold. First, in practice, it corresponds to the situation where the relay is an analog repeater, which is very easy to be implemented. The second motivation is technical. While the determination of the selfish PA policies for the different sources at the equilibrium is always possible numerically, case (B) allows one to analytically determine the number of Nash equilibria and corresponding policies. Before treating cases (A) and (B) in detail, we now describe the non-cooperative PA game with ZDSAF. Indeed, one of our goals is to know how each selfish transmitter is going to allocate its available power between the different bands, given the fact that the transmitters are able to observe each other and react accordingly. This situation of interaction can be modeled by a non-cooperative game where: (i) the players of the game are the two
information sources or transmitters; (ii) the strategy of transmitter $i$ consists in choosing $\theta_i = (\theta_i^{(1)}, \ldots , \theta_i^{(Q)})$ in its strategy set $\mathcal{A}_i = \left\{ \theta_i \in [0,1]^Q \mid \sum_{q=1}^Q \theta_i^{(q)} \leq 1 \right\}$; (iii) the utility (or payoff) function of user $i \in \{1,2\}$ is its achievable transmission rate given by $u_i^{AF}(\theta_i, \theta_{-i}) = \sum_{q=1}^Q R_i^{(q),AF}(\theta_i^{(q)}, \theta_{-i}^{(q)})$ where $R_i^{(q),AF}$ is the rate user $i$ obtains by using band $(q)$ when the ZDSAF protocol is used by the relay $\mathcal{R}^{(q)}$. The latter quantity has been shown to be (in Part I):

$$R_i^{(q),AF} = C(\eta_i^{(q),AF})$$

where

$$\eta_i^{(q),AF} = \frac{\left| a_r^{(q)} h_{ir}^{(q)} r_{ir}^{(q)} + h_{ii}^{(q)} \right|^2 \rho_i^{(q)} \theta_i^{(q)}}{\left| a_r^{(q)} h_{jr}^{(q)} r_{jr}^{(q)} + h_{jj}^{(q)} \right|^2 \rho_j^{(q)} N_i^{(q)} + \rho_i^{(q)} N_j^{(q)} + 1} ,$$

with $\forall i \in \{1,2\}$, $j = -i$ and $\rho_i^{(q)} = \frac{P_i}{N_i^{(q)}}$; in case (A), $a_r^{(q)} = a_r^{(q)}(\theta_1^{(q)}, \theta_2^{(q)})$ and, in case (B), $a_r^{(q)} = A_r^{(q)}$.

Without loss of generality and for sake of clarity we will assume in Sec. II that $\forall (i, q) \in \{1,2,3\} \times \{1,\ldots , Q\}$, $N_i^{(q)} = N$, $P_i^{(q)} = P_r$ and we introduce the quantities $\rho_i = \frac{P_i}{N}$. Additionally, we suppose that the game is played once (one-shot game), the users are rational (each selfish player does what is best for itself) and the game is with complete information that is, every player knows the triple $\mathcal{G}^{AF} = (\mathcal{K}, (\mathcal{A}_i)_{i \in \mathcal{K}}, (u_i^{AF})_{i \in \mathcal{K}})$, where $\mathcal{K} = \{1,2\}$ is used to refer to the set of players. From now on, we will call state of the network the vector of power fractions that the users allocate to the IRCs i.e., $\theta = (\theta_1, \theta_2)$.

In distributed networks where users are selfish and free decision makers who interact with each other, a desirable feature for the network is the existence of an equilibrium or a stable operating state of the system. In this respect, the Nash equilibrium [7] corresponds to a state of the network from which the users do not have any incentive to deviate unilaterally, because otherwise they would lose in terms of utility; this translates mathematically by the following definition.

**Definition 3.1:** [Nash equilibrium] The state $(\theta^*_1, \theta^*_2)$ is a pure NE of the game $\mathcal{G}^{AF}$ if $\forall i \in \mathcal{K}$, $\forall \theta'_i \in \mathcal{A}_i$, $u_i(\theta^*_1, \theta^*_2) \geq u_i(\theta'_1, \theta^*_2)$.

It turns out that the existence of such a stable state is guaranteed in both scenarios (cases (A) and (B)) under investigation. Proving this is the purpose of the following two sub-sections.
A. Case of full power regime

In this section, \( a_r^{(q)} = a_r^{(q)}(\theta_1^{(q)}, \theta_2^{(q)}) \) \( \forall q \in \{1, \ldots, Q\} \). Under this assumption we can state the following existence theorem.

**Theorem 3.2:** [Existence of an NE for ZDSAF when \( a_r^{(q)} = a_r^{(q)}(\theta_1^{(q)}, \theta_2^{(q)}) \)] There exists at least one pure NE in the PA game \( G^{AF} \).

The proof of this theorem is provided in Appendix A. Here, we will just give a few comments on it. It is based on Theorem 1 in [3]. The latter theorem states that in a game with a finite number of players, if for every player (i) the strategy set is convex and compact, 2) its utility is continuous in the vector of strategies and 3) concave in its own strategy, then the existence of at least one pure NE is guaranteed. In our setup it is easy to check that conditions 1) and 2) are met. Verifying condition 3) for the utility \( u_i \) is however more involving. Indeed, it can be checked that the second-order derivative of \( R_i^{(q),AF} \) w.r.t. \( \theta_i^{(q)} \) is intractable. It turns out that proving that this second-order derivative is non-positive is possible if a proper change of variables is made before the sign analysis, as shown in the Appendix A.

The proved theorem indicates, in particular, that whatever the values of the channel gains \( h_{ij}, (i, j) \in \{1, 2, r\}^2 \), there exists an equilibrium. As a consequence, if some relays are added in the network, the transmitters will adapt their PA policies accordingly and, whatever the locations of the relays, the selfish behavior of the transmitters will drive the network to an equilibrium. This is a nice property for the system under investigation. The question to know which equilibrium is also an interesting issue to be treated. We will treat it partially in this paper and leave it as an extension of this work. In the next sub-section we analyze a useful and simple case where the equilibria can be determined.

B. Case of fixed amplification gains

In this section, we assume \( Q = 2 \) and that each transmitter saturates its power constraint that is, \( \forall i \in \{1, 2\}, \theta_i^{(1)} + \theta_i^{(2)} = 1 \). Thus, for the sake of clarity, we rename the power fractions in bands (1) and (2) as: \( \theta_1^{(1)} = \theta_1 \) and \( \theta_2^{(1)} = \theta_2 \) in the remaining of this section. More importantly, we suppose that \( \forall q \in \{1, 2\}, a_r^{(q)} = a_r \in [0, \tilde{a}_r(1, 1)] \), a fixed constant. This choice is interesting in terms of both, the simplicity of the relays (simple repeaters) and the information assumptions. It is true that choosing the amplification gain \( a_r = \tilde{a}_r^{(q)}(\theta_1, \theta_2) \), as above, allows the relay nodes to exploit all their available powers. However, it also assumes the presence of a mechanism to estimate the power of the received signals at these relay nodes. While this can be easy for a digital relay transceiver that knows the possible training sequences used by
the sources, it might be impossible if the relay is a simple analog power amplifier without automatic gain control. Furthermore, the case $a_r^{(q)} = A_r^{(q)}$ has a very interesting convergence property which allows to relax some information assumptions at the transmitters. To understand this and determine the Nash equilibria, consider the best responses (BRs) of the different players. The BR of player $i$ to player $j$ is defined by $BR_i(\theta_j) = \arg \max u_i(\theta_i, \theta_j)$. In general, it is a correspondence but in our case it is just a function. The equilibrium points precisely correspond to the intersection of the BRs of the two users. In this case, using the Lagrangian functions to impose the power constraint, it can be checked that:

$$BR_i(\theta_j) = \begin{cases} F_i(\theta_j) & \text{if } 0 < F_i(\theta_j) < 1 \\ 1 & \text{if } F_i(\theta_j) \geq 1 \\ 0 & \text{otherwise} \end{cases}$$  \tag{5}$$

where $j = -i$, $F_i(\theta_j) \triangleq -\frac{c_i}{c_{ii}} \theta_j + \frac{d_i}{c_{ii}}$ is an affine function of $\theta_j$; for $(i, j) \in \{(1, 2), (2, 1)\}$, $c_{ii} = 2|A_r^{(1)}| h_{ri} h_{ir} + h_{ii}|^2 A_r^{(2)} g_{ri} g_{ir} + g_{ii} |^2 \rho_i$; $c_{ij} = |A_r^{(1)}| h_{ri} h_{ir} + h_{ii}|^2 |A_r^{(2)} g_{ri} g_{jr} + g_{ji}|^2 \rho_j + |A_i^{(1)}| h_{ri} h_{jr} + h_{ji}|^2 |A_i^{(2)} g_{ri} g_{ir} + g_{ii}|^2 \rho_j$; $d_i = |A_r^{(1)}| h_{ri} h_{ir} + h_{ii}|^2 |A_r^{(2)} g_{ri} g_{ir} + g_{ii}|^2 \rho_i + |A_i^{(2)}| g_{ri} g_{jr} + g_{ji}|^2 \rho_j + A_i^{(2)} |g_{ri}|^2 + |A_i^{(2)} g_{ri} g_{ir} + g_{ii}|^2 (A_r^{(1)}| h_{ri}|^2 + 1)$. By studying the intersection points between BR$_1$ and BR$_2$, one can prove the following theorem (the proof is provided in Appendix B).

**Theorem 3.3 (Number of Nash equilibria):** For the game $G^{AF}$ with fixed amplification gains at the relays, (i.e., $\frac{\partial a_r^{(q)}}{\partial y^{(q)}} = 0$), there can be a unique NE, two NE, three NE or an infinite number of NE, depending on the channel gains $h_{ij}$, $(i, j) \in \{1, 2, r\}^2$.

If the channel gains are thought of as the realizations of continuous random variables, it is easy to prove that the probability of observing the necessary conditions on the channel gains for having two NEs or an infinite number of NEs is zero. Said otherwise, if one puts the relays in arbitrary places, there will be, with probability one, either one or three NE, depending on the channel gains. When the channel gains are such that the NE is unique, the unique NE can be shown to be:

$$\theta^{NE} = \theta^* = \left(\frac{c_{22} d_1 - c_{12} d_2}{c_{11} c_{22} - c_{12} c_{21}}, \frac{c_{11} d_2 - c_{21} d_1}{c_{11} c_{22} - c_{12} c_{21}}\right)$$ \tag{6}$$

When there are three NE, it is a priori impossible to predict the NE that will be effectively observed in the one-shot game. In fact, in practice, in a context of cognitive transmitters, it is possible to predict the equilibrium of the network. In practice, a multi-band IRC with cognitive transmitters will work as follows. First, there is no reason why the sources should start transmitting at the same time. Thus, one transmitter, say $i$, will be alone and using a certain PA policy. The transmitter coming after, namely $S_{-i}$, will sense...
its environment and play its BR to what it observes. As a consequence, user $i$ will move to a new policy, maximizing its BR to what transmitter $-i$ has played. User $i$ will then re-adapt its policy and so on. The key question is: does this procedure converge? This is where the nice property of the BRs come into play. It can be checked that this procedure will converge to one of the three NE predicted by the one-shot game.

The observed NE can be predicted by knowing the initial network state that is, the PA policy played by the first player starting to transmit; this will be illustrated in Sec. VII. In fact, even though the BRs are just piecewise affine and not affine (as in a Cournot duopoly [8]), the game has the same convergence property as a Cournot duopoly. To implement such an iterative procedure (called the Cournot tâtonnement process in [9]), it can be checked that the transmitters need to know less network parameters than in the original game where the amplification factor saturates the constraint. In fact, the needed parameters can be acquired by realistic sensing techniques or feedback mechanisms based on standard estimation procedures, which is particularly easy in fast fading environment since the path loss can be considered to be constant during a period of time long enough to store a large number of channel realizations.

IV. EQUILIBRIUM ANALYSIS FOR THE DF PROTOCOL

In this section, the relays are assumed to be able to reliably decode the messages sent by the two information sources. In Gaussian relay channels, the DF protocol can perform better than the AF and EF protocols in terms of transmission rate; this typically occurs when the relay is close to the source relatively to the destination [10]. The principle of the DF protocol is reviewed in Part I. Here we will just mention that, in the Gaussian relay channel, the source superposes two codewords corresponding to two messages i.e., a coarse message reliably decodable by the destination alone and a fine message that the destination reliably decodes with the help of the relay node. This explains the following structure for the signals transmitted by $S_1$ and $S_2$ in the type of networks we consider in this paper: $X_i^{(q)} = X_{i,0}^{(q)} + \sqrt{\tau_i^{(q)} \theta_i^{(q)} P_i} X_{r,i}^{(q)}$ where the signals $X_{i,0}^{(q)}$ and $X_{r,i}^{(q)}$ are Gaussian and independent. The first signal $X_{i,0}^{(q)}$ precisely corresponds to the coarse message while the second one corresponds to the fine message. Indeed, at the relay $R^{(q)}$, the transmitted signal writes as: $X_r^{(q)} = X_{r,1}^{(q)} + X_{r,2}^{(q)}$. We see that the problem of resources allocation is more complex than in the case of AF. There are actually three power allocation problems instead of one: (1) like AF, each information source needs to allocate its power between the $Q$ different bands; (2) in each band, each source needs to tune its cooperation degree $\tau_i^{(q)}$ with the relay that is, it has to allocate its power in band (q) between the coarse and fine signals; (3) each relay has to allocate its power between the two cooperation signals.
intended for $\mathcal{D}_1$ and $\mathcal{D}_2$. The parameters $\theta_i^{(q)}$, $\tau_i^{(q)}$ and $\nu^{(q)}$ translate these three PA problems respectively: $X_i^{(q)} \sim \mathcal{N}(0, \theta_i^{(q)} P_i)$, $X_{i,0}^{(q)} \sim \mathcal{N}(0, (1 - \tau_i^{(q)} \theta_i^{(q)} P_i)$ and $X_{r,1}^{(q)} \sim \mathcal{N}(0, \nu^{(q)} P_r^{(q)})$. In this paper, the authors restrict their attention to two special cases of PA games and let the general case where the strategy of a user is $(\theta_i, \tau_i) = (\theta_i^{(1)}, ..., \theta_i^{(Q)}, \tau_i^{(1)}, ..., \tau_i^{(Q)})$ as an interesting and significant extension of this paper. First, we consider the PA game (1) by supposing the cooperation degrees and PA policies at the relays to be fixed (Sec. IV-A). Second, we analyze the PA game (2) where only the cooperation degrees can be tuned, the other parameters being assumed to be fixed (Sec. IV-B). The first game allows one to be coherent with the PA game with AF while the second game allows one to study the game introduced in Part I. In both sub-sections we assume that each relay implements SUD. This choice is fair for the transmitters, does not involve any additional signalling and allows us to cope with the constraint on the sum-rate on each multiple access channel formed by $(\mathcal{D}_1, \mathcal{D}_2) - \mathcal{R}^{(q)}$, which make the game more complex to be played and is therefore left as an extension of this paper. Also, in both sub-sections the utility of a transmitter will have the same expression that is, $\forall i \in \{1, 2\}$, $u_i^{DF} = \sum_{q=1}^{Q} R_i^{(q),DF}$ where

$$
\begin{align*}
R_1^{(q),DF} &= \min \left\{ R_{1,1}^{(q),DF}, R_{1,2}^{(q),DF} \right\}, \\
R_2^{(q),DF} &= \min \left\{ R_{2,1}^{(q),DF}, R_{2,2}^{(q),DF} \right\},
\end{align*}
$$

(7)

$$
\begin{align*}
R_{1,1}^{(q),DF} &= C \left( \frac{|h_{11}^{(q)}|^2 (1 - \tau_1^{(q)}) \theta_1^{(q)} P_1}{|h_{2r}^{(q)}|^2 (1 - \tau_2^{(q)}) \theta_2^{(q)} P_2 + N_r^{(q)}} \right), \\
R_{2,1}^{(q),DF} &= C \left( \frac{|h_{12}^{(q)}|^2 (1 - \tau_2^{(q)}) \theta_2^{(q)} P_2}{|h_{1r}^{(q)}|^2 (1 - \tau_1^{(q)}) \theta_1^{(q)} P_1 + N_r^{(q)}} \right), \\
R_{1,2}^{(q),DF} &= C \left( \frac{|h_{11}^{(q)}|^2 \theta_1^{(q)} P_1 + |h_{r1}^{(q)}|^2 \nu^{(q)} P_i^{(q)} + 2Re(h_{i1}^{(q)} h_{r1}^{(q),*}) \sqrt{\tau_1^{(q)} \theta_1^{(q)} P_1 \nu^{(q)} P_i^{(q)}}}{|h_{21}^{(q)}|^2 \theta_2^{(q)} P_2 + |h_{r1}^{(q)}|^2 \nu^{(q)} P_i^{(q)} + 2Re(h_{i2}^{(q)} h_{r1}^{(q),*}) \sqrt{\tau_2^{(q)} \theta_2^{(q)} P_2 \nu^{(q)} P_i^{(q)}} + N_r^{(q)}} \right), \\
R_{2,2}^{(q),DF} &= C \left( \frac{|h_{22}^{(q)}|^2 \theta_2^{(q)} P_2 + |h_{r2}^{(q)}|^2 \nu^{(q)} P_i^{(q)} + 2Re(h_{i2}^{(q)} h_{r2}^{(q),*}) \sqrt{\tau_2^{(q)} \theta_2^{(q)} P_2 \nu^{(q)} P_i^{(q)}} + N_r^{(q)}}}{|h_{12}^{(q)}|^2 \theta_1^{(q)} P_1 + |h_{r2}^{(q)}|^2 \nu^{(q)} P_i^{(q)} + 2Re(h_{i1}^{(q)} h_{r2}^{(q),*}) \sqrt{\tau_1^{(q)} \theta_1^{(q)} P_1 \nu^{(q)} P_i^{(q)}} + N_r^{(q)}} \right),
\end{align*}
$$

(8)

and $(\nu^{(q)}, \tau_1^{(q)}, \tau_2^{(q)}) \in [0, 1]^3$, and $\mathcal{D}^{(q)} = 1 - \nu^{(q)}$. Note that we have not indicated the arguments of $u_i^{DF}$ i.e., the strategies of the users, since these are context-dependent. In Sec. IV-A, the strategy of user $i$ is $\underline{\theta}_i = (\theta_i^{(1)}, ..., \theta_i^{(Q)})$ whereas in Sec. IV-B it is $\underline{\tau}_i = (\tau_i^{(1)}, ..., \tau_i^{(Q)})$.

A. Allocating the power between the bands

Here we assume that the cooperation degrees $(\underline{\tau}_1, \underline{\tau}_2)$ are fixed and the strategy for user $i$ consists in choosing $\underline{\theta}_i$. The PA game is similar to that encountered with the AF protocol. Therefore, here again, the
problem for proving the existence of an NE in this game is to show the concavity of \( u_{i}^{DF} \) w.r.t. \( \theta_{i} \). In contrast with AF, this task can be checked to be very easy for the rates achieved by using DF. Having a concave game in the sense of Rosen [3], the following existence theorem follows.

**Theorem 4.1: [Existence of an NE for the DF protocol]**

The game defined by \( G^{DF} = (\mathcal{K}, (A_{i})_{i \in \mathcal{K}}, (u^{DF}_{i}(\bar{\theta}_{i}, \bar{\theta}_{-i}))_{i \in \mathcal{K}}) \) with \( \mathcal{K} = \{1, 2\} \) and \( A_{i} = \left\{ \theta_{i} \in [0, 1]^{Q} \mid \sum_{q=1}^{Q} \theta_{i}^{(q)} \leq 1 \right\} \), has always at least one pure NE.

While the existence is easy to prove, the uniqueness remains a though problem. For example, the diagonally strict concavity condition of [3], which is a sufficient condition for uniqueness for concave games, is not trivial to be checked.

**B. Allocating the power between the coarse and fine signals**

We now suppose that the power fractions \( \theta_{1} \) and \( \theta_{2} \) allocated to the \( Q \) bands are fixed (for example \( \theta_{1}^{(q)} = \frac{1}{Q} \) and \( \theta_{2}^{(q)} = \frac{1}{Q} \), which corresponds to a uniform PA). We therefore obtain exactly the game we have highlighted in Part I (for the case \( Q = 1 \)). Among the three dominant classes of protocols (AF, DF, EF) we see that the DF protocol is the only protocol that introduces a game in a natural manner that is, even if there is no multiple antenna transmitters, multi-carrier systems, multi-slot transmissions, etc. Fortunately, the game where each transmitter tunes their cooperation degree with the relays \( \mathcal{R}^{(q)} \) has at least one equilibrium, as stated in the following theorem.

**Theorem 4.2: [Existence of an NE for the DF protocol]**

The game defined by \( G^{DF} = (\mathcal{K}, (A_{i})_{i \in \mathcal{K}}, (u^{DF}_{i}(\tau_{i}, \tau_{-i}))_{i \in \mathcal{K}}) \) with \( \mathcal{K} = \{1, 2\} \) and \( A_{i} = [0, 1]^{Q} \), has always at least one pure NE.

The proof also follows from [3].

**V. Equilibrium analysis for the EF protocol**

In this section, we make the same assumptions as in Sec. [IV] concerning the reception schemes and PA policies at the relays: we assume that each relay \( \mathcal{R}^{(q)}, D_{1} \) and \( D_{2} \) implement single-user decoding and the PA policy at each relay i.e., \( \nu^{(q)} \) is fixed. Each relay now implements the EF protocol. The principle of the EF protocol in the relay channel is to send an approximated version of the relay’s observation signal to the receiver. In Part I, we have seen that, in IRCs, the EF protocol can have at least two natural variants depending on whether the relay constructs a unique estimate of the signal it receives that is decodable by
both receivers or two estimates, each of them being decodable by one or two receivers. We have called the two corresponding schemes the single-level compression scheme and bi-level compression scheme. Both because of the lack of space and for simplicity reasons, we treat here the case of EF with two compression levels. The single-level compression scheme implies a more complicated expression of the compression noise level that makes the concavity issue strenuous to deal with. Under this assumption, the utility for user \( i \in \{1, 2\} \) can be expressed as follows: 

\[
\begin{align*}
R_{1}^{(q), \text{EF}} & = C \left( \frac{h_{12}^{(q)} \theta_{1}^{(q)} P_{2} + h_{11}^{(q)} + N_{u_{w,z,1}}^{(q)}}{h_{12}^{(q)} + N_{u_{w,z,1}}^{(q)}} \right) \left( \frac{h_{12}^{(q)} \theta_{2}^{(q)} P_{2} + h_{11}^{(q)} + N_{u_{w,z,2}}^{(q)}}{h_{12}^{(q)} + N_{u_{w,z,2}}^{(q)}} \right) \left( \frac{h_{12}^{(q)} \theta_{1}^{(q)} P_{1} + h_{11}^{(q)} + N_{u_{w,z,2}}^{(q)}}{h_{12}^{(q)} + N_{u_{w,z,2}}^{(q)}} \right), \\
R_{2}^{(q), \text{EF}} & = C \left( \frac{h_{12}^{(q)} \theta_{1}^{(q)} P_{1} + h_{11}^{(q)} + N_{u_{w,z,1}}^{(q)}}{h_{12}^{(q)} + N_{u_{w,z,1}}^{(q)}} \right) \left( \frac{h_{12}^{(q)} \theta_{2}^{(q)} P_{2} + h_{11}^{(q)} + N_{u_{w,z,2}}^{(q)}}{h_{12}^{(q)} + N_{u_{w,z,2}}^{(q)}} \right) \left( \frac{h_{12}^{(q)} \theta_{2}^{(q)} P_{2} + h_{11}^{(q)} + N_{u_{w,z,2}}^{(q)}}{h_{12}^{(q)} + N_{u_{w,z,2}}^{(q)}} \right) \left( \frac{h_{12}^{(q)} \theta_{1}^{(q)} P_{1} + h_{11}^{(q)} + N_{u_{w,z,2}}^{(q)}}{h_{12}^{(q)} + N_{u_{w,z,2}}^{(q)}} \right) \right).
\end{align*}
\]

\[\nu(q) \in [0, 1], \quad A(q) = \left| h_{11}^{(q)} \theta_{1}^{(q)} P_{1} + h_{21}^{(q)} \theta_{2}^{(q)} P_{2} + N_{u_{w,z,1}}^{(q)} \right|, \quad A_{1}^{(q)} = h_{11}^{(q)} \theta_{1}^{(q)} P_{1} + h_{21}^{(q)} \theta_{2}^{(q)} P_{2} \quad \text{and} \quad A_{2}^{(q)} = h_{12}^{(q)} \theta_{1}^{(q)} P_{1} + h_{22}^{(q)} \theta_{2}^{(q)} P_{2}. \]

What is interesting with EF is that, here again, one can prove that the utility is concave for every user. This is the purpose of the following theorem.

**Theorem 5.1:** [Existence of an NE for the bi-level compression EF protocol] The game defined by \( G_{\text{EF}} = (K, (A_{i})_{i \in K}, (u_{i}^{\text{EF}}(\theta_{i}, \theta_{-i}))_{i \in K}) \) with \( K = \{1, 2\} \) and \( A_{i} = \left\{ \theta_{i} \in [0, 1]^{Q} : \sum_{q=1}^{Q} \theta_{i}^{(q)} \leq 1 \right\} \), has always at least one pure NE.

To be able to apply Theorem of Rosen [3], we have to prove that the utility \( u_{i}^{\text{EF}} \) is concave w.r.t. \( \theta_{i} \). The problem is less simple than for DF because the compression noise \( N_{u_{w,z,i}}^{(q)} \), which appears in the denominator of the capacity function in Eq. (9), also depends on the strategy \( \theta_{-i} \) of transmitter \( i \). It turns out that it is still possible to prove the desired result as shown in Appendix C.

**VI. STACKELBERG FORMULATION**

We have mentioned that a strong motivation for studying IRCs is to be able to introduce relays in a network with non-coordinated and interfering pairs of terminals. For example, relays could be introduced by an operator aiming at improving the performance of the communications of his subscribers. In such a
scenario, the operator acts as a player and more precisely as a game leader in the sense of [11]. In [11], the author introduced what is called nowadays a Stackelberg game. This type of hierarchical games comprises one leader who plays in the first step of the game and several players (the followers) who observe the leader’s strategy and choose their actions accordingly. In our context, the game leader is the operator/engineer who chooses the parameters of the relays and the followers are the cognitive transmitters that adapt their PA policy to what they observe. What are these parameters? In the preceding sections we have mentioned some of them:

1) the location of each relay;
2) in the case of AF, the amplification gain of each relay;
3) in the case of DF and EF, the power allocation policy between the two cooperative signals at each relay i.e., the parameter $\nu^{(q)}$.

Therefore, the group of relays can be thought of as a player who maximizes its own utility. This utility can be the individual utility of a given transmitter (picture one WiFi subscriber who wants to increase his downlink throughput by locating his cellular phone somewhere in his apartment while his neighbor can also exploit the same spectral resources) or the network sum-rate (in the case of an operator). In the latter case, the operator possesses some degrees of freedom to make the Nash equilibrium more efficient. In the next section, we will use implicitly the Stackelberg formulation to determine the best location for a relay in a given scenario.

VII. Simulation results

First, we show that in the PA game with ZDSAF, one can have three possible Nash equilibria. For a given typical scenario, which does not need to be given here but can be found in [2], Fig. 1 represents the BRs of the two transmitters. We see that there are three intersection points and therefore three Nash equilibria. As explained in Sec. [11], the effectively observed NE in a one-shot game is not predictable without any additional assumptions but the Cournot tatâtonnement procedure converges towards a given NE which can be predicted from the sole knowledge of the starting point of the game, namely $\theta_1^0$ or $\theta_2^0$.

In the rest of this section we focus on the Stackelberg formulation where the strategy of the leader is respectively the relay amplification factor, position and power allocation between the cooperative signals. The system considered is composed of an IRC in parallel with an interference channel (IC) [12], thus $Q = 2$. All the simulations provided are obtained by applying the Cournot tatâtonnement procedure. The simulation
setup is as follows. The source and destination nodes are located in fixed locations in the region $[-L, L]^2$ of a plane, with $L = 10$ m, such that the relative distances between the nodes are: $d_{11} = 6.52$ m, $d_{12} = 8.32$ m, $d_{21} = 6.64$ m, $d_{22} = 6.73$ m. We assume a path loss model for the channel gains $|g_{ij}|$, $|h_{ij}|$. For the path loss model we take $|h_{ij}| = \left( \frac{d_{ij}}{d_0} \right)^{-\gamma_{ij}}$ and $|g_{ij}| = \left( \frac{d_{ij}}{d_0} \right)^{-\gamma_{ij}}$ for $(i, j) \in \{1, 2, r\}^2$ where $d_0 = 1$ m is a reference distance. As in Part I, to avoid any divergence for the path loss in a plane, with relay is not in the plane but at $z$ as follows. For this purpose, consider Fig. 3(b) which is a temperature image representing the values of $P_i$ on one of the the segment between $S_1$ and $D_i$. We observe that there are two local maximum that actually correspond to the points that maximize the individual rate of a given user.

**Optimal relay amplification gain for ZDSAF.** First we consider the ZDSAF relaying scheme assuming a fixed amplification gain $a_r = A_r$ (Sec. III-B). We want to analyze the influence of the value of the amplification factor, $A_r \in [0, \tilde{a}_r(1, 1)]$, on the achievable network sum-rate at the NE. This is what Fig. 2 shows for the following scenario: $\epsilon = 0.5$ m, $P_1 = 20$ dBm, $P_2 = 23$ dBm, $P_r = 22$ dBm, $N_1 = 10$ dBm, $N_2 = 9$ dBm, $N_r = 7$ dBm, $\gamma^{(1)} = \gamma^{(2)} = 2$. We observe that the optimal value is $A_r^* = 0.05$ and is not equal to the one saturating the relay power constraint $\tilde{a}_r(1, 1) = 0.17$. This result illustrates for the sum-rate what we have proved analytically in Part I for the individual rate of a given user.

**Optimal relay location for ZDSAF.** Now, we consider the ZDSAF when the relay full power regime is assumed, $a_r = \tilde{a}_r(\theta_1, \theta_2)$ (Sec. III-A) and study the relay location problem. Fig. 3(a) represents the achievable network sum-rate as a function of the relay position $(x_R, y_R) \in [-L, L]^2$ for the scenario: $P_1 = 20$ dBm, $P_2 = 17$ dBm, $P_r = 22$ dBm, $N_1 = 10$ dBm, $N_2 = 9$ dBm, $N_r = 7$ dBm, $\gamma^{(1)} = 2.5$ and $\gamma^{(2)} = 2$. We observe that there are two local maximum that actually correspond to the points that maximize the individual achievable rates. Many simulation results have confirmed that, when the source nodes are sufficiently far away from each other, maximizing the individual rate of either user at the NE amounts to locating the relay on one of the the segment between $S_1$ and $D_i$. This interesting and quite generic observation can be explained as follows. For this purpose, consider Fig. 3(b) which is a temperature image representing the values of $\theta_1$ and $\theta_2$ for different relay positions in $[-L, L]^2$. The region where $(\theta_1, \theta_2) = (1, 0)$ (resp. $(\theta_1, \theta_2) = (0, 1)$) is the region around $S_1$ (resp. $S_2$). We see that the intersection between these regions corresponds to small area. This quite general observation shows that the selfish behavior of the users leads to self-regulating the interference in the network. Said otherwise the selfish behavior of each transmitter leads it to leave the relay to the other transmitter when it is too far from it. Thus, when one transmitter uses the relay, it is often alone and sees no interference. In these conditions, by considering the path loss effects it can be proved that the
optimal relay position is on the segment between the considered source and destination nodes. This also explains why the position that maximizes the network sum-rate is also on one of the segments from \( S_i \) to \( D_i \).

**Optimal relay power allocation at the relay for DF and EF.** For the DF protocol we fix the cooperation degrees \( \tau_1 = 0 \) and \( \tau_2 = 0 \). In Fig. 4 we plot the achievable sum-rate at the equilibrium as a function of the relay power allocation policy is \( \nu \in [0, 1] \) (with the convention \( \nu = \nu^{(1)} \)) for the scenario: \( x_R = 0 \) m, \( y_R = 0 \) m, \( P_1 = 22 \) dBm, \( P_2 = 17 \) dBm, \( P_r = 23 \) dBm, \( N_1 = 7 \) dBm, \( N_2 = 9 \) dBm, \( N_r = 0 \) dBm, \( \gamma^{(1)} = 2.5 \) and \( \gamma^{(2)} = 2 \). We observe that, for both protocols, the optimal power allocation \( \nu^* = 1 \), meaning that the relay allocates all its available power to the better receiver, \( D_1 \). In this case the relay is in very good conditions and can therefore reliably decode the source messages. This explains why DF outperforms EF which is in agreement with the observations we have made in Part I. We have observed that, in general, the network sum-rate is not concave w.r.t. \( \nu \in [0, 1] \) and that the optimal power allocation lies on the borders \( \nu^* \in \{0, 1\} \) for both relaying protocols. From Fig. 4 we also see that the fair PA policy that is, \( \nu = \frac{1}{2} \) can lead to a relatively significant performance loss.

**VIII. Conclusion**

In this paper we have introduced and studied a channel model which is very useful to analyze scenarios where each wireless device operating in unlicensed bands is offered the opportunity to exploit additional resources to increase its own transmission rate. Here the additional resources consist of spectrum plus cooperation power. We have analyzed in detail the problem of decentralized PA for the three dominant classes of relaying protocols (AF, DF, EF) and shown the existence of an equilibrium for the network for all these protocols. There can be multiple equilibria as proved for the game with AF. Fortunately, we have proved that in the case of AF with fixed amplification gain, the Cournot tâtonnement converges to an NE that be predicted from the sole knowledge of the starting point of the game. This useful analysis needs to be extended to the case of the DF and EF protocols. We have seen that our problem can also be formulated as a Stackelberg game where the relay is the game leader and optimizes its amplification gain when AF is used, its location, and its power allocation between the two cooperative signals when DF or EF is implemented. This analysis is useful when adding some relaying nodes in a pre-existing network in order to improve its performance. In particular it has shown how to optimally locate the relay in a simple scenario and how the selfish behavior of the users leads to self-regulating the interference in the network. Important issues like
the uniqueness issue of the NE for DF and EF (under appropriate sufficient conditions), the convergence issue towards an NE in an iterative observation-reaction procedure need to be addressed; the latter issue is fundamental for cognitive transmitters where realistic spectrum sensing algorithms have to be assumed. From a broader perspective much works remains to be done to analyze the general problem of competition in cooperative channels.

APPENDIX A

PROOF OF CONCAVITY OF \( u_i^{AF} \) W.R.T. \( \theta_i^{(q)} \)

We have to prove that \( R_i^{(q)} \) is also concave w.r.t. \( \theta_i^{(q)} \). We will further consider only user 1 and prove the concavity of \( R_1^{(q)} \) w.r.t. \( \theta_1^{(q)} \) (for user 2 the proof is identical). Let us denote \( \Phi_1(\theta_1^{(q)}) = 1 + \gamma_1^{(q)} \). The second derivative of \( R_1^{(q)} \) w.r.t. \( \theta_1^{(q)} \) can be written as

\[
\frac{\partial^2 R_1^{(q)}}{\partial (\theta_1^{(q)})^2} = \Phi_1'' - (\Phi_1')^2.
\]

In what follows we will show that \( \Phi_1'' - (\Phi_1')^2 \leq 0 \). For sake of clarity we denote by \( \lambda = \rho_1 |h_1^{(q)}|^2 \), \( \alpha = \frac{h_1^{(q)} h_i^{(q)}}{h_1^{(q)}} \), \( \beta = |h_{r_1}^{(q)}|^2 \), \( \gamma = h_{2v}^{(q)} \sqrt{\theta_2^{(q)}} \rho_2 \), \( \delta = h_{2v}^{(q)} h_{r_1}^{(q)} \sqrt{\theta_2^{(q)}} \rho_2 \), \( \varepsilon_1 = \sqrt{\frac{\rho_2}{h_1^{(q)} \theta_2^{(q)} \rho_2 + 1}} \), \( \varepsilon_2 = \frac{|h_{r_1}^{(q)}|^2 \rho_1}{h_{2v}^{(q)} \theta_2^{(q)} \rho_2 + 1} \), and we define the function

\[
x(\theta_1^{(q)}) = \varepsilon_1 (1 + \varepsilon_2 \theta_1^{(q)})^{-1/2}.\]

We can now write \( \Phi_1(\theta_1^{(q)}) = f(x(\theta_1^{(q)})) \) with

\[
f(x) = 1 + \frac{\lambda (1 + \alpha x)^2 (\varepsilon_1^2 - x^2)}{\varepsilon_2 x^2 [\gamma + \delta x]^2 + \beta x^2 + 1}.
\]  

The first derivative of \( \Phi_1 \) is given by \( \Phi'(\theta_1^{(q)}) = \frac{df}{dx} x' \) and the second derivative of \( \Phi_1 \) is given by \( \Phi''(\theta_1^{(q)}) = \frac{d^2 f}{dx^2} (x')^2 + \frac{df}{dx} x'' \), where \( x' \) and \( x'' \) are the first and second order derivatives of \( x(\theta_1^{(q)}) \).

Let us define the functions \( \mathcal{N} (\cdot) \) and \( \mathcal{M} (\cdot) \) that give the numerator and the denominator of a fraction, respectively. One can show that:

\[
\mathcal{N}(f(x)) = -2\lambda x \varepsilon_2 (1 + \alpha x) \left[ (\alpha x^3 + \varepsilon_1^2) \left( (\gamma + \delta x)^2 + \beta x^2 + 1 \right) + \frac{1}{\varepsilon_2 x^2} (\varepsilon_1^2 - x^2) (\delta (\gamma + \delta x) + \beta x) \right],
\]

and that \( \mathcal{M}_1(x) \triangleq \mathcal{M}(f'(x)) = \varepsilon_2^2 x^4 \left[ (\gamma + \delta x)^2 + \beta x^2 + 1 \right]^2. \) We note \( \mathcal{N}_1(x) \triangleq \mathcal{N}(f'(x)), \mathcal{N}_0(x) \triangleq \mathcal{N}(f(x)), \mathcal{M}_0(x) \triangleq \mathcal{M}(f(x)) \) and then define \( \mathcal{N}_2(x) \triangleq \frac{d\mathcal{N}_1}{dx} \) and \( \mathcal{M}_2(x) \triangleq \frac{d\mathcal{M}_1}{dx}. \)

We obtain that

\[
\mathcal{N}_2 = -2\lambda \varepsilon_2 [(1 + 2\alpha x)(\varepsilon_1^2 + \alpha x^3) + 3\alpha x^3 (1 + \alpha x)] \left[ (\gamma + \delta x)^2 + \beta x^2 + 1 \right] - 2\lambda \varepsilon_2 x (3 + 2\alpha x)(1 + \alpha x)(\varepsilon_1^2 - x^2) [\delta (\gamma + \delta x) + \beta x] - 2\lambda x^2 \varepsilon_2 (1 + \alpha x)^2 (\varepsilon_1^2 - x^2) (\delta^2 + \beta).
\]  

\( \gamma \) and \( \delta \) are given by (11).
and

\[ \mathcal{M}_2 = 4 \varepsilon_2^2 x^3 (\gamma + \delta x)^2 + \beta x^2 + 1^2 + 4 \varepsilon_2^2 x^4 [\delta (\gamma + \delta x) + \beta x]. \]  

(13)

Considering all these definitions we further obtain \( \frac{\partial^2 f}{\partial x^2} = \frac{N_2 M_1 - N_1 M_2}{M_1^2} \) and also that:

\[
\Phi''_1 \Phi - (\Phi'_1)^2 = \frac{d^2}{dx^2} \left( f x'' - \frac{df}{dx} (x')^2 \right) + f \frac{d^2 f}{dx^2} (x')^2 \\
= \frac{N_1}{M_1} \left[ \frac{N_0}{M_0} x'' - \frac{N_1}{M_1} (x')^2 \right] + \frac{N_0}{M_0} \frac{N_0 M_1 - N_1 M_0}{M_1^2} (x')^2 \\
= \frac{N_0 N_1 (M_1 x'' - M_2 (x')^2) + (N_0 N_2 M_1 - N_2 M_0) (x')^2}{M_0 M_1^2}.
\]

To show that \( \Phi''_1 \Phi - (\Phi'_1)^2 \leq 0 \) it is sufficient to show that both \( \Delta_1 = M_1 x'' - M_2 (x')^2 \) and \( \Delta_2 = N_0 N_2 M_1 - N_1^2 M_0 \) are negative.

Knowing that \( x' = \frac{\varepsilon_2}{2} x^3 \) and \( x'' = \frac{3 \varepsilon_2}{4} x^5 \), one can easily verify that:

\[
\Delta_1 = -\frac{\varepsilon_2}{4} x^9 \left[ 5 (\delta^2 + \beta) x^2 + 6 \gamma \delta x + \gamma^2 + 1 \right] \left[ (\gamma + \delta x)^2 + \beta x^2 + 1 \right] \\
\leq 0.
\]

(14)

Furthermore, observing that \( N_0 \geq 0, M_0 \geq 0, M_1 \geq 0 \) and that \( N_2 \leq 0 \) we have \( \Delta_2 \leq 0 \). In conclusion \( \frac{\partial^2 R_{1(q)}}{\partial (\theta_1^{(q)})^2} \leq 0 \) and thus \( R_{1(q)} \) is a concave function of \( \theta_1^{(q)} \).

APPENDIX B

NUMBER OF NASH EQUILIBRIA IN THE GAME \( G^{AF} \)

Before discussing these situations in detail, let us first observe that the two functions \( F_i(\theta_j) \) are decreasing w.r.t. \( \theta_j \) and also \( F_i(0) = \frac{d}{c_{ii}}, \quad F_i(\theta_j^*) = 0 \) where \( \theta_j^* = \frac{d}{c_{jj}} \).

1) If \( d_1 \leq 0 \) and \( d_2 \leq 0 \), then the BR are constants \( BR_i(\theta_j) = 0 \) and thus the NE is unique \( \Theta_{1NE}, \Theta_{2NE} = (0, 0) \), for all \( c_{ii} \geq 0, c_{ji} \geq 0 \).

2) If \( d_1 \leq 0 \) and \( d_2 > 0 \), then it can be checked that the NE is unique, for all \( c_{ii} \geq 0, c_{ji} \geq 0 \): \( \Theta_{1NE} = 0 \) and

\[
\Theta_{2NE} = \frac{d_2}{c_{22}} \quad \text{if} \quad d_2 < c_{22}, \\
1 \quad \text{otherwise}.
\]

3) If \( d_1 > 0 \) and \( d_2 \leq 0 \), then, similarly to the previous item, we have a unique NE, for all \( c_{ii} \geq 0, c_{ji} \geq 0 \): \( \Theta_{2NE} = 0 \) and

\[
\Theta_{1NE} = \frac{d_1}{c_{11}} \quad \text{if} \quad d_1 < c_{11}, \\
1 \quad \text{otherwise}.
\]
4) If $d_1 > 0$ and $d_2 > 0$, we have to take into consideration the parameters $c_{ii} \geq 0$, $c_{ji} \geq 0$.
   a) If $F_1(1) \geq 1$ and $F_2(1) \geq 1$, then we have $d_1 \geq c_{12} + c_{11}$ and $d_2 \geq c_{21} + c_{22}$. In this case the BR are constants i.e., $BR_i(\theta_j) = 1$ and thus the NE is unique $(\theta_1^{\text{NE}}, \theta_2^{\text{NE}}) = (1, 1)$.
   b) If $F_1(1) \geq 1$ and $F_2(1) < 1$, then we have $d_1 \geq c_{12} + c_{11}$ and $d_2 < c_{21} + c_{22}$. Here also the NE is unique and $\theta_2^{\text{NE}} = 1$ and
   \[ \theta_2^{\text{NE}} = \begin{cases} \frac{d_2 - c_{21}}{c_{22}}, & \text{if } d_2 > c_{22}, \\ 0, & \text{otherwise}. \end{cases} \]
   c) If $F_1(1) < 1$ and $F_2(1) \geq 1$, then we have $d_1 < c_{12} + c_{11}$ and $d_2 \geq c_{21} + c_{22}$. Here also the NE is unique and $\theta_2^{\text{NE}} = 1$ and
   \[ \theta_1^{\text{NE}} = \begin{cases} \frac{d_1 - c_{12}}{c_{11}}, & \text{if } d_1 > c_{11}, \\ 0, & \text{otherwise}. \end{cases} \]
   d) If $F_1(1) < 1$ and $F_2(1) < 1$, then we'll have $d_1 < c_{12} + c_{11}$ and $d_2 < c_{21} + c_{22}$. This case is the most demanding one and will be treated in detail separately.

At this point an important observation is in order. The discussed scenarios, for which we have determined the unique NE, have a simple geometric interpretation. If the intersection point $(\theta_1^*, \theta_2^*)$ is such that either $\theta_1^* \in \mathbb{R} \setminus [0, 1]$ or $\theta_2^* \in \mathbb{R} \setminus [0, 1]$ then the NE is unique and differs from this point $((\theta_1^{\text{NE}}, \theta_2^{\text{NE}}) \neq (\theta_1^*, \theta_2^*))$. The case 4.(d) corresponds to the case where the intersection point $(\theta_1^*, \theta_2^*) \in [0, 1]^2$ is an NE point. Now we are interested in finding whether this intersection point is the unique NE or there are more than one NE. If $0 < d_1 < c_{11} + c_{12}$ and $0 < d_2 < c_{22} + c_{21}$ we have the following situations:

1) If $c_{11}c_{22} = c_{21}c_{12}$, then the curves described by $\theta_i = F_i(\theta_j)$ are parallel.
   a) If $d_1 = d_2$, then the curves are superposed. In this special case we have an infinity of NE that can be characterized by $(\theta_1^{\text{NE}}, \theta_2^{\text{NE}}) \in T$ where:
   \[ T = \left\{ (\theta_1, \theta_2) \in [0, 1]^2 \mid \theta_1 = F_1(\theta_2^{\text{NE}}) \right\}. \]
   b) If $d_1 \neq d_2$, then the two lines are only parallel. In this case it can be checked that the NE is unique. In order to explicit the exact relation of the NE, one has to consider all scenarios in function of the sign of the following four relations $F_i(0) - 1$ and $\theta_j^* - 1$, $i \in \{1, 2\}$. We will explicit only one of them. Let us assume that $F_i(0) - 1 < 0$ and $\theta_j^* < 0$ which means that $d_1 < \min\{c_{12}, \frac{c_{11}c_{21}}{c_{22}}\}$ and $d_2 < \min\{c_{21}, c_{22}\}$. Here we have two sub-cases:
If \( \frac{d_1}{c_{12}} < \frac{d_2}{c_{22}} \), then the NE is characterized by \( \theta_1^{\text{NE}} = 0 \) and \( \theta_2^{\text{NE}} = \frac{d_2}{c_{22}} \).

If \( \frac{d_1}{c_{12}} > \frac{d_2}{c_{22}} \), then the NE is characterized by \( \theta_1^{\text{NE}} = \frac{d_1 c_{22}}{c_{12} c_{21}} \) and \( \theta_2^{\text{NE}} = 0 \).

2) Consider \( c_{11} c_{22} \neq c_{21} c_{12} \). Here we have to consider all cases in function of the sign of the four relations \( F_i(0) - 1 \) and \( \theta_j^0 - 1 \), \( i \in \{1, 2\} \). We will focus on only one of them. Let us assume that \( F_i(0) - 1 < 0 \) and \( \theta_j^0 < 0 \) and thus \( d_1 < \min\{c_{12}, c_{11}\} \) and \( d_2 < \min\{c_{21}, c_{22}\} \). Here we have four sub-cases:

- If \( \frac{d_1}{c_{12}} < \frac{d_1}{c_{11}} \) and \( \frac{d_2}{c_{22}} > \frac{d_2}{c_{21}} \), then the NE is unique: \( \theta_1^{\text{NE}} = \theta_1^* \) and \( \theta_2^{\text{NE}} = \theta_2^* \).

- If \( \frac{d_1}{c_{12}} > \frac{d_1}{c_{11}} \) and \( \frac{d_2}{c_{22}} < \frac{d_2}{c_{21}} \), then there are three different NE:

\[
(\theta_1^{\text{NE}}, \theta_2^{\text{NE}}) \in \{(\theta_1^*, \theta_2^*), (0, \frac{d_2}{c_{22}}), (\frac{d_1}{c_{11}}, 0)\}.
\]

- If \( \frac{d_1}{c_{12}} = \frac{d_1}{c_{11}} \) and \( \frac{d_2}{c_{22}} < \frac{d_2}{c_{21}} \), then there are only two different NE: \( (\theta_1^{\text{NE}}, \theta_2^{\text{NE}}) \in \{(0, \frac{d_2}{c_{22}}), (\frac{d_1}{c_{11}}, 0)\} \).

- If \( \frac{d_1}{c_{12}} > \frac{d_1}{c_{11}} \) and \( \frac{d_2}{c_{22}} = \frac{d_2}{c_{21}} \), then there are two NE: \( (\theta_1^{\text{NE}}, \theta_2^{\text{NE}}) \in \{(\frac{d_1}{c_{11}}, 0), (0, \frac{d_2}{c_{22}})\} \).

In conclusion, the cases where there are multiple NE are: 1) either when the lines \( \theta_i = F_i(\theta_i) \) are superposed and the game has an infinity of NE or 2) when the lines have a unique intersection point \( (\theta_1^*, \theta_2^*) \) that lies inside \([0, 1] \times [0, 1]\). In the latter cases there can be one, two or three different NE.

**APPENDIX C**

**PROOF OF CONCAVITY OF \( u_i^{\text{EF}} \) W.R.T. \( \theta_i \)**

We want to prove that for each user \( R_i^{(q)} \) is concave w.r.t. \( \theta_i^{(q)} \). Consider w.l.o.g. the case of user 1. We follow a similar approach to the AF case, based on the second derivative of \( R_1^{(q)} \) given in Eq. (9). After some manipulations we obtain the following relation:

\[
\frac{d^2 R_i^{(q)}}{d(\theta_i^{(q)})^2} = M_1 - M_2 \text{ with } M_k = \frac{N M_k}{D M_k}, \ k \in \{1, 2\}
\]

where (for sake of clarity we have denoted \( h_i^{(q)} \) by \( h_{ij} \)):

\[
\begin{align*}
NM_1 &= 2 \frac{\Lambda_4 h_{11}^2 P_1}{\nu(q) h_i^{(q)} P_i(q) \Lambda_4} - 2 \frac{\Lambda_4 \Lambda_7 (h_{21}^2 \theta_i^{(q)} P_2 + \nu(q) h_{11}^2 P_i^{(q)} + N_1^{(q)})}{\Lambda_4 \nu(q) h_i^{(q)} h_{11}^2 P_i^{(q)} + N_1^{(q)}} + 2 \frac{\Lambda_6 \Lambda_7^2 (h_{21}^2 \theta_i^{(q)} P_2 + \nu(q) h_{11}^2 P_i^{(q)} + N_1^{(q)})^2}{\Lambda_4^3 (\nu(q) h_i^{(q)} P_i^{(q)})^2}, \\
NM_2 &= \left[ \frac{\Lambda_7 h_{11}^2 \theta_i^{(q)} P_1}{\nu(q) h_i^{(q)} P_i^{(q)}} + \Lambda_5 h_{11}^2 P_1 + (h_{21}^2 \theta_i^{(q)} P_2 + \nu(q) h_{11}^2 P_i^{(q)} + N_1^{(q)}) h_{11}^2 P_1 \right] \frac{1}{\Lambda_4} \\
&\quad - \frac{\Lambda_6 \Lambda_7 (h_{21}^2 \theta_i^{(q)} P_2 + \nu(q) h_{11}^2 P_i^{(q)} + N_1^{(q)})^2}{\Lambda_4^2 (\nu(q) h_i^{(q)} P_i^{(q)})}, \\
DM_1 &= 1 + \frac{\Lambda_6}{\Lambda_4}, \\
DM_2 &= DM_1^2.
\end{align*}
\]
Therefore we obtain the desired result

\[
\begin{align*}
\Lambda_1 &= h_{11} h_{1r} \theta_1^{(q)} P_1 + h_{21} h_{2r} \theta_2^{(q)} P_2, \\
\Lambda_2 &= h_{11}^2 \theta_1^{(q)} P_1 + h_{21}^2 \theta_2^{(q)} P_2 + \nu^{(q)} h_{r1}^2 P_r^{(q)} + N_1^{(q)}, \\
\Lambda_3 &= h_{11}^2 \theta_1^{(q)} P_1 + h_{21}^2 \theta_2^{(q)} P_2 + N_r^{(q)}, \\
\Lambda_4 &= \left( N_1^{(q)} + \frac{\Lambda_3 \Lambda_2 - \Lambda_1^2}{\nu^{(q)} h_{r1}^2 P_r^{(q)}} \right) \left( h_{21}^2 \theta_2^{(q)} P_2 + \nu^{(q)} h_{r1}^2 P_r^{(q)} + N_1^{(q)} \right) + h_{21}^2 \theta_2^{(q)} P_2 \nu^{(q)} h_{r1}^2 P_r^{(q)} + N_1^{(q)}, \\
\Lambda_5 &= h_{2r}^2 \theta_2^{(q)} P_2 + N_r^{(q)} + \frac{\Lambda_3 \Lambda_2 - \Lambda_1^2}{\nu^{(q)} h_{r1}^2 P_r^{(q)}}, \\
\Lambda_6 &= \Lambda_5 h_{11}^2 \theta_1^{(q)} P_1 + \left( h_{21}^2 \theta_2^{(q)} P_2 + \nu^{(q)} h_{r1}^2 P_r^{(q)} + N_1^{(q)} \right) h_{11}^2 \theta_1^{(q)} P_1, \\
\Lambda_7 &= h_{11}^2 P_1 \Lambda_2 + \Lambda_3 h_{11}^2 P_1 - 2 \Lambda_1 h_{11} h_{1r} P_1, \\
\Lambda_8 &= \frac{\Lambda_3 h_{11}^2 \theta_1^{(q)} P_1}{\nu^{(q)} h_{r1}^2 P_r^{(q)}} + \Lambda_5 h_{11}^2 P_1 + \left( h_{21}^2 \theta_2^{(q)} P_2 + \nu^{(q)} h_{r1}^2 P_r^{(q)} + N_1^{(q)} \right) h_{11}^2 P_1.
\end{align*}
\]

(16)

First note that \( \Lambda_3 \Lambda_2 - \Lambda_1^2 = \left( \frac{\nu^{(q)} h_{r1}^2 P_r^{(q)} + N_1^{(q)}}{h_{11}^2} \right) \Lambda_3 + \Lambda_2 N_1^{(q)} + (h_{11} h_{1r} - h_{21} h_{2r}) \theta_1^{(q)} \theta_2^{(q)} P_2 \geq 0 \) and thus we have that all the terms \( \Lambda_k \geq 0, k \in \{1, \ldots, 8\} \). Also we can easily see from Eq. (15) that \( M_2 > 0, DM_1 > 0 \). Thus if we prove that \( N M_1 \leq 0 \) the concavity of \( R_1^{(q)} \) will be guaranteed. In this purpose we plug the expressions of \( \Lambda_4, \Lambda_5, \Lambda_7, \Lambda_8 \) into Eq. (15) and obtain that \( N M_1 = \frac{N M_1}{D N M_1} \) with

\[
\begin{align*}
N N M_1 &= -2 P_1^2 \nu^{(q)} h_{r1}^2 P_r^{(q)} \left[ (h_{11} h_{21} - h_{11} h_{2r})^2 \theta_2^{(q)} P_2 + h_{11}^2 \left( \nu^{(q)} h_{r1}^2 P_r^{(q)} + N_1^{(q)} \right) + h_{11}^2 N_r^{(q)} \right] \\
&+ h_{11}^2 h_{21}^2 \theta_2^{(q)} P_2^2 + h_{11}^2 \left( \nu^{(q)} h_{r1}^2 P_r^{(q)} + N_1^{(q)} \right)^2 + 2 h_{11}^2 h_{21}^2 \theta_2^{(q)} P_2 \left( \nu^{(q)} h_{r1}^2 P_r^{(q)} + N_1^{(q)} \right) \\
&+ h_{2r}^2 \theta_2^{(q)} P_2 \left( \nu^{(q)} h_{r1}^2 P_r^{(q)} + N_1^{(q)} \right) \left( h_{21}^2 \theta_2^{(q)} P_2 + \nu^{(q)} h_{r1}^2 P_r^{(q)} + N_1^{(q)} \right)
\end{align*}
\]

and

\[
\begin{align*}
D N M_1 &= \left[ (h_{11} h_{21} - h_{11} h_{2r})^2 \theta_2^{(q)} P_1 \theta_2^{(q)} P_2 \left( h_{21}^2 \theta_2^{(q)} P_2 + \nu^{(q)} h_{r1}^2 P_r^{(q)} + N_1^{(q)} \right) \right] \\
&+ N_1^{(q)} \nu^{(q)} h_{r1}^2 P_r^{(q)} h_{21}^2 \theta_2^{(q)} P_2 + N_1^{(q)} \nu^{(q)} h_{r1}^2 P_r^{(q)} \left( \nu^{(q)} h_{r1}^2 P_r^{(q)} + N_1^{(q)} \right) \\
&+ h_{11}^2 \theta_2^{(q)} P_1 h_{21}^2 \theta_2^{(q)} P_2 \left( \nu^{(q)} h_{r1}^2 P_r^{(q)} + N_1^{(q)} \right) + h_{11}^2 \theta_1^{(q)} P_1 \left( \nu^{(q)} h_{r1}^2 P_r^{(q)} + N_1^{(q)} \right)^2 \\
&+ h_{2r}^2 \theta_2^{(q)} P_2 \left( \nu^{(q)} h_{r1}^2 P_r^{(q)} + N_1^{(q)} \right) h_{11}^2 + h_{2r}^2 \theta_2^{(q)} P_2 \left( \nu^{(q)} h_{r1}^2 P_r^{(q)} + N_1^{(q)} \right)^2 \\
&+ N_1^{(q)} h_{11}^2 \theta_1^{(q)} P_1 h_{21}^2 \theta_2^{(q)} P_2 + N_r^{(q)} h_{11}^2 \theta_2^{(q)} P_2 \left( \nu^{(q)} h_{r1}^2 P_r^{(q)} + N_1^{(q)} \right) + N_r^{(q)} h_{21}^2 \theta_2^{(q)} P_2^2 \\
&+ 2 N_r^{(q)} h_{21}^2 \theta_2^{(q)} P_2 \left( \nu^{(q)} h_{r1}^2 P_r^{(q)} + N_1^{(q)} \right) + N_1^{(q)} \nu^{(q)} h_{r1}^2 P_r^{(q)} + N_1^{(q)} \right)^2 \\
&+ h_{2r}^2 \theta_2^{(q)} P_2 \left( \nu^{(q)} h_{r1}^2 P_r^{(q)} + N_1^{(q)} \right) \nu^{(q)} h_{r1}^2 P_r^{(q)} \right]^3
\end{align*}
\]

Therefore we obtain the desired result \( N M_1 \leq 0 \) and thus \( M_1 \geq 0 \), which implies that \( \frac{d^2 R_1^{(q)}}{d q^{(1)}} \leq 0 \), whatever the channel parameters.
REFERENCES

[1] E. V. Belmega, B. Djeumou, and S. Lasaulce, “What happens when cognitive terminals compete for a relay node?” in IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), Taipei, Taiwan, Apr. 2009.
[2] ——, “Resource allocation games in interference relay channels,” in IEEE International Conference on Game Theory for Networks (Gamenets), Istanbul, Turkey, May 2009.
[3] J. Rosen, “Existence and uniqueness of equilibrium points for concave n-person games,” Econometrica, vol. 33, pp. 520–534, 1965.
[4] O. Sahin and E. Erkip, “On achievable rates for interference relay channel with interference cancellation,” in Proc. IEEE Annual Asilomar Conference on Signals, Systems and Computers (invited paper), Pacific Grove, California, Nov. 2007.
[5] ——, “Achievable rates for the gaussian interference relay channel,” in Proc. IEEE Global Communications Conference (GLOBECOM’07), Washington D.C., USA, pp. 786–787, Nov. 2007.
[6] W. Yu, G. Ginis, and J. M. Cioffi, “Distributed multiuser power control for digital subscriber lines,” IEEE J. Sel. Areas Commun., vol. 20, no. 5, pp. 1105–1115, Jun. 2002.
[7] J. F. Nash, “Equilibrium points in n-points games,” Proc. of the Nat. Academy of Science, vol. 36, no. 1, pp. 48–49, Jan. 1950.
[8] A. Cournot, Recherches sur les principes mathématiques de la la théorie des richesses, (Re-edited by Mac Millan in 1987) 1838.
[9] H. Moulin, “Dominance solvability and cournot stability,” Mathematical Social Sciences, vol. 7, pp. 83–102, 1984.
[10] G. Kramer, M. Gastpar, and P. Gupta, “Cooperative strategies and capacity theorems for relay networks,” IEEE Trans. Inf. Theory, vol. 51, no. 9, pp. 3037–3067, Sep. 2005.
[11] H. von Stackelberg, The theory of the market economy. Oxford, England: Oxford University Press, 1952.
[12] A. B. Carleial, “Interference channels,” IEEE Trans. Inf. Theory, vol. 24, no. 1, pp. 60–70, Jan. 1978.
Fig. 1. Best replies for a system composed of an IC in band (1) and IRC in band (2) when the ZDSAF protocol is assumed (fixed amplification factor). The number of equilibria is generally three as indicated the figure.

Fig. 2. ZDSAF relaying protocol with fixed amplification gain. Achievable network sum-rate at the NE as a function of $A_r \in [0, \tilde{a}_r]$ for $L = 10m$, $\varepsilon = 0.5m$, $P_1 = 20dBm$, $P_2 = 23dBm$, $P_r = 22dBm$, $N_1 = 10dBm$, $N_2 = 9dBm$, $N_r = 7dBm$, $\gamma^{(1)} = \gamma^{(2)} = 2$. The optimal amplification gain $A_r^* = 0.05 \leq \tilde{a}_r(1,1) = 0.17$ meaning that saturating the relay power constraint is suboptimal.
Fig. 3. ZDSAF relaying protocol, full power regime. $L = 10m$, $\epsilon = 1m$, $P_1 = 20$dBm, $P_2 = 17$dBm, $P_r = 22$dBm, $N_1 = 10$dBm, $N_2 = 9$dBm, $N_r = 7$dBm, $\gamma^{(1)} = 2.5$ and $\gamma^{(2)} = 2$. (a) Achievable network sum-rate at the NE as a function of $(x_R, y_R) \in [-L, L]^2$ (the optimal relay position $(x^*_R, y^*_R) = (-1.2, 1.7)$ lies on the segment between $S_1$ and $D_1$). (b) Power allocation policies at the NE $(\theta_1^{NE}, \theta_2^{NE})$ as a function of $(x_R, y_R) \in [-L, L]^2$ (the regions where the uses allocate their power to IRC are almost non overlapping.).
Fig. 4. EF vs. DF relaying protocol. Achievable network sum-rate at the NE as a function of $\nu \in [0, 1]$ for $L = 10\text{m}$, $\epsilon = 1\text{m}$, $P_1 = 22\text{dBm}$, $P_2 = 17\text{dBm}$, $P_r = 23\text{dBm}$, $N_1 = 7\text{dBm}$, $N_2 = 9\text{dBm}$, $N_r = 0\text{dBm}$, $\gamma^{(1)} = 2.5$ and $\gamma^{(2)} = 2$. The optimal relay PA $\nu^* = 1$ is in favor of the better user and outperforms the uniform relay PA $\nu = 0.5$ for both EF and DF.