THE PRICING OF LOOKBACK OPTIONS AND BINOMIAL APPROXIMATION

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Abstract. Refining a discrete model of Cheuk and Vorst we obtain a 
closed formula for the price of a European lookback option at any time 
between emission and maturity. We derive an asymptotic expansion of 
the price as the number of periods tends to infinity, thereby solving a 
problem posed by Lin and Palmer. We prove, in particular, that the 
price in the discrete model tends to the price in the continuous Black-
Scholes model. Our results are based on an asymptotic expansion of the 
binomial cumulative distribution function that improves several recent 
results in the literature.

1. Introduction

Cheuk and Vorst [4] have proposed a discrete model for the pricing of 
European lookback options with floating strike; they suppose implicitly that 
they evaluate the price at emission. In this paper we will refine their model 
in order to price the option at any given time after emission, and we derive 
an asymptotic expansion for the price as the number of time intervals tends 
to infinity. This solves completely a problem posed by Lin and Palmer [14]; 
in the special case of the price at emission the problem has already been 
treated by the second author [11].

Lookback options give the holder the right to buy (for the call), respectively to sell (for the put), the underlying asset at maturity for its lowest, respectively highest, price during its lifetime. The payoff functions at maturity are therefore given by

\[ S_T - \min_{t \leq T} S_t \quad \text{and} \quad \max_{t \leq T} S_t - S_T, \]

respectively.

In the traditional continuous model (i.e., when the underlying asset price follows a Wiener process with drift, as proposed by Black and Scholes [2] and Merton [15]), Goldman, Sosin and Gatto [10] derived a formula for the price under the assumption that the spot rate \( r \) is non-zero. Babbs [1] obtained the price for \( r = 0 \) by passing to the limit.

A discrete model for the price of a lookback option was proposed by Hull and White [13], see also Hull [12], who based it on the familiar binomial model of Cox, Ross and Rubinstein (CRR) [6], see Figure 1.1. However,

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since lookback options are path-dependent, Hull and White had to subdivide each node into different states. This problem was overcome by Cheuk and Vorst [4] who proposed an equivalent tree (CV) in which every node corresponds to a single state, see Figure [1.2] (for a call).

Neither Hull and White, nor Cheuk and Vorst obtained a closed formula for the prices in their model. Such a formula was first obtained by Föllmer and Schied [9] and, by a different method, by the second author [11] (see also the discussion in [11, Appendix A]). However, all these papers only evaluate the price of the option at emission.

The aim of this paper is two-fold. In Section 2 we will derive a closed formula, in the discrete model, for the price at any time after emission of a European lookback option with floating strike. In Section 4 we show that the price in the discrete model converges to the price in the continuous model, and we derive an asymptotic expansion. This answers a problem posed by Lin and Palmer [14] and generalizes the earlier results in [11]. In order to derive these asymptotics we need a refinement of the known asymptotic expansions of the binomial cumulative distribution function, which will be achieved in Section 3. The final section is devoted to numerical examples.

2. Lookback options with floating strike

In the discrete model of Cheuk and Vorst, the traditional CRR tree, see Figure [1.1] is still used for modelling the evaluation of the underlying.

We first introduce the usual notations for the various parameters: \(T\) is the time from emission to maturity, \(t\) with \(0 \leq t < T\) is the present time, \(\tau = T - t\) is the remaining time until maturity, and \(S_t\) is the value of the underlying at time \(t\). Moreover, \(r \geq 0\) is the spot rate and \(\sigma > 0\) is the volatility of the underlying asset.

Now, the binomial tree for the underlying is only built for the time interval \([t, T]\) (the prices before \(t\) being known). This interval is divided equally into \(n\) subintervals. At each node, the price may either increase by a factor \(u\) or it may decrease by a factor \(d\), where
\[
u = e^{\sigma \sqrt{\tau/n}} \quad \text{and} \quad d = u^{-1} = e^{-\sigma \sqrt{\tau/n}}.
\]
The probability of an increase is given by
\[
p = \frac{e^{r \tau/n} - d}{u - d};
\]
we take \(n\) sufficiently large so that \(0 < p < 1\).

Let us first consider \(t = 0\), and let us concentrate on call options. For pricing a lookback option, Cheuk and Vorst have introduced a second tree.
In the CV tree, the level $j$ at time $t_m$ ($0 \leq m \leq n$) denotes the difference in powers of $u$ between the value $S_{t_m}$ at time $t_m$ and the lowest value of the underlying since emission. In other words, $j$ is the non-negative integer such that

$$S_{t_m} = \left( \min_{t^* \leq t_m} S_{t^*} \right) u^j.$$  

As shown in [4] and made explicit in [11, Theorem 2.1], the price of the option at emission is then given by

$$C_{n}^{fl}(0) = S_0 \sum_{j=0}^{n} (1 - u^{-j}) \sum_{k=0}^{n} \Lambda_{j,k,n} q^k (1 - q)^{n-k},$$

where

$$(2.2) \quad q = pue^{-r\tau/n} = \frac{u - e^{-r\tau/n}}{u - d}$$

(which lies in (0,1) if $p$ does) and $\Lambda_{j,k,n}$ is the number of paths in the CV tree from the initial node $(0,0)$ to level $j$ at maturity that have exactly $k$ upward jumps. It was shown in [11, Theorem 2.1] that

$$\Lambda_{j,k,n} = \begin{cases} \binom{n}{k-j} - \binom{n}{k-j-1} & \text{if } j \leq k \leq \lfloor \frac{n+j}{2} \rfloor, \\ 0 & \text{else.} \end{cases}$$

In this paper we are interested in finding the price of a lookback (call) option at any given time $t$ ($0 \leq t < T$). In that case, the market prices $S_{t^*}$, $0 \leq t^* \leq t$, of the underlying between emission and time $t$ are known. Of course, the price at time $t$ is not necessarily the minimal value; there is in fact an initial level $j_0$ that is defined by

$$S_t = \left( \min_{t^* \leq t} S_{t^*} \right) u^{j_0}.$$ 

In the sequel we will write for brevity

$$M_t = \min_{t^* \leq t} S_{t^*}.$$ 

Then we have

$$(2.3) \quad j_0 = \frac{\log(S_t/M_t)}{\sigma \sqrt{\tau/n}}.$$ 

Clearly, $j_0 \geq 0$ is not necessarily an integer, and the initial node is located at position $(0,j_0)$. This leads to a modified CV tree (see Figure 1.4, where $m$ corresponds to time $t_m$). Note that after sufficiently many downward jumps one reaches a new minimum and thus level 0. From there on, all levels are integers.

It can be shown as in [4] that

$$(2.4) \quad C_{n}^{fl}(t) = S_t \sum_{j \in J} (1 - u^{-j}) \sum_{k=0}^{n} \Lambda_{j_0,k,n} q^k (1 - q)^{n-k},$$

where $J$ is the set of possible levels at maturity, $q$ is given by (2.2) and, as before, $\Lambda_{j_0,k,n}$ is the number of paths from the initial node $(0,j_0)$ to level $j$ at maturity that have exactly $k$ upward jumps. It remains to evaluate these numbers.
Put abstractly, we have the following problem. Let \( j_0 \geq 0 \) be a positive real number and \( n \in \mathbb{N} \). We create a graph with an initial node at \((0, j_0)\), and from each node \((m, j_m)\) at period \( m, 0 \leq m < n \), we create two nodes at period \( m + 1 \) given by

\[(m + 1, j_m + 1) \text{ up} \]

and

\[(m + 1, \max(j_m - 1, 0)) \text{ down,} \]

with connecting edges. Let

\[ \Lambda_{j_0}^{j,k,n} \]

denote the number of paths from the initial node \((0, j_0)\) to the final node \((n, j)\) that has exactly \( k \) upward jumps. The following result may then also be of independent interest.

**Lemma 2.1.** Let \( j_0 \geq 0 \) and \( n \in \mathbb{N} \). Then

\[
\Lambda_{j_0}^{j,k,n} = \begin{cases} 
\binom{n}{k}, & \text{if } j = j_0 + 2k - n \\
\binom{n}{k} - \binom{n}{k + \lfloor j_0 \rfloor + 1}, & \text{if } j = j_0 + 2k - n \\
\binom{n}{k - j} - \binom{n}{k - j - 1}, & \text{if } 0 \leq j \leq n - \lfloor j_0 \rfloor - 1 \\
& \text{and } j \leq k \leq \lfloor \frac{n - \lfloor j_0 \rfloor - 1 + j}{2} \rfloor.
\end{cases}
\]

All the other values of \( \Lambda_{j_0}^{j,k,n} \) are zero.

**Proof.** (I) We will first consider the situation where \( j_0 \) is not an integer, see Figure 2.1. In that case there are three disjoint classes of terminal nodes. The proof will be done in three steps devoted to the nodes in each of these classes.

(1) The first class is given by the end-points of simple solid lines. These nodes are exactly those for which the set of predecessors (going back to
lookback options and binomial approximation

$m = 0$) is the same as in a traditional binomial tree. It is then easy to count the number of paths. To arrive at such a node there are at least $k = n - \lfloor j_0 \rfloor$ and at most $k = n$ upward jumps, and there are \( \binom{n}{k} \) paths with exactly $k$ upward jumps. The corresponding terminal levels are $\bar{j} = j_0 + k - (n - k) = j_0 + 2k - n$. Thus,

$$\Lambda_{j_0}^{\bar{j},k,n} = \binom{n}{k}$$

for $j = j_0 + 2k - n$ and $n - \lfloor j_0 \rfloor \leq k \leq n$. We note for later use that the levels $\bar{j}$ in this class range from $j_0 + (n - 2\lfloor j_0 \rfloor)$ to $n$. Thus, $\Lambda_{j_0}^{\bar{j},k,n}$

(2.5) $k = \frac{n + j - j_0}{2}$.

(2) The second class is constituted of all the remaining terminal nodes that belong to the traditional binomial tree starting at $(0,0)$; in Figure 2.1 they are indicated by dashed lines. We have here a partial binomial tree: some of the paths of the traditional binomial tree leading to such nodes have been lost. Indeed, some paths would have had at least one intermediate node at level zero. The possible terminal levels range from $\{j_0\}$ (if $\lfloor j_0 \rfloor + n$ is even) or from $\{j_0\} + 1$ (if $\lfloor j_0 \rfloor + n$ is odd) to $j_0 + n - 2\lfloor j_0 \rfloor - 2$ in steps of two; moreover there are such nodes only when $n \geq \lfloor j_0 \rfloor + 2$. We can rewrite the lower bound in both cases as $j_0 + n - 2\lfloor \frac{n + j_0}{2} \rfloor$. By (2.5), each level corresponds to a unique number $k$ of upward jumps, and the possible values of $k$ for nodes of the second class range from $n - \lfloor \frac{n + j_0}{2} \rfloor$ to $n - \lfloor j_0 \rfloor - 1$.

![Figure 2.2. The reflection principle](image)

Now, the number of paths leading to a node of the second class at level $j$ is again $\binom{n}{k}$, but we have to withdraw all the lost paths. The number of lost paths is obtained by using the reflection principle, see Figure 2.2. It states that there is a one-to-one correspondence between the paths in a binomial tree connecting the nodes $(0,a)$ and $(n,b)$ (with $a, b \in \mathbb{N}_0$) in touching or crossing the $x$-axis and the paths connecting the nodes $(0,-a)$ and $(n,b)$. Our case is equivalent to the situation where $a = \lfloor j_0 \rfloor + 1$ and $b = j - \{j_0\} + 1 = j - j_0 + \lfloor j_0 \rfloor + 1$. It remains to count the number of paths from $(0,-a)$ to $(n,b)$. The sum of the number of upward jumps ($U$) with the number of downward jumps ($D$) is the number of periods $n$, while the
difference between $U$ and $D$ is the overall increase, that is, $a + b$. We thus have that $U = \frac{n + a + b}{2}$. Therefore, the number of lost paths is
\[
\left(\frac{n}{n + a + b}\right) = \left(\frac{n + 2|j_0| + j - j_0 + 2}{n + 2|j_0| + j - j_0 + 2}\right) = \left(\frac{n}{k + |j_0| + 1}\right),
\]
where we have used (2.5). Thus,
\[
\Lambda_{j, k, n}^{j_0} = \left(\frac{n}{k}\right) - \left(\frac{n}{k + |j_0| + 1}\right)
\]
for $j = j_0 + 2k - n$ and $n - \left\lfloor \frac{n + j_0}{2} \right\rfloor \leq k \leq n - |j_0| - 1$.

(3) The third class of nodes is constituted by the remaining nodes; they are indicated in Figure 2.1 by double lines (either solid or dashdotted ones). They are the terminal nodes of a smaller Cheuk-Vorst tree with the initial node at $(|j_0| + 1, 0)$. Moreover, there are such nodes only when $n \geq |j_0| + 1$. The possible terminal levels range from $j = 0$ to $j = n - |j_0| - 1$.

Unlike in the two previous cases, the paths joining the initial node $(0, j_0)$ with a terminal level $j$ will not have a fixed number of upward jumps, just like in the traditional Cheuk-Vorst tree. We will prove that the number of paths with exactly $k$ upward jumps arriving at a level $j$ in the third class is given by
\[
\Lambda_{j, k, n}^{j_0} = \left(\frac{n}{k - j}\right) - \left(\frac{n}{k - j - 1}\right)
\]
for $0 \leq j \leq n - |j_0| - 1$ and $j \leq k \leq \left\lfloor \frac{n - |j_0| - 1}{2} \right\rfloor$, and that no other values of $k$ are possible.

We prove the claim by induction on $n$, $n \geq |j_0| + 1$. The result is trivial for $n = |j_0| + 1$. For $n = |j_0| + 2$, we can arrive at level $j = 0$ (with $k = 0$) or at $j = 1$ (with $k = 1$). In both cases, there is exactly one possible path.

Now let $n \geq |j_0| + 3$. Again, for $j = n - |j_0| - 1$ (with $k = n - |j_0| - 1$) and $j = n - |j_0| - 2$ (with $k = n - |j_0| - 2$) the result is trivial. In both cases there is only one possible path, which are respectively $|j_0| + 1$ downs followed by $n - |j_0| - 1$ ups and $|j_0| + 2$ downs followed by $n - |j_0| - 2$ ups.

It remains to discuss the case when $0 \leq j \leq n - |j_0| - 3$. If $j = 0$, there are two downward paths leading to it from period $n - 1$ if $n + |j_0|$ is even, and three downward paths otherwise.

(i) If $n + |j_0|$ is even, they come from the nodes located at levels 0 or 1 at period $n - 1$.

If $k = 0$, the only possible path comes from $j = 0$ and so
\[
\Lambda_{0,0,n}^{j_0} = \Lambda_{0,0,n-1}^{j_0} = 1.
\]

If $1 \leq k \leq \frac{n - |j_0| - 2}{2}$, then we have
\[
\Lambda_{0,k,n}^{j_0} = \Lambda_{0,k,n-1}^{j_0} + \Lambda_{1,k,n-1}^{j_0} = \left(\frac{n - 1}{k}\right) - \left(\frac{n - 1}{k - 1}\right) + \left(\frac{n - 1}{k - 1}\right) - \left(\frac{n - 1}{k - 2}\right) = \left(\frac{n}{k}\right) - \left(\frac{n}{k - 1}\right).
\]

There are no paths for $k > \frac{n - |j_0| - 2}{2}$. 

(ii) If $n + \lfloor j_0 \rfloor$ is odd, there are additional downward paths coming from the partial binomial tree. By \eqref{2.3}, these paths did exactly $\frac{n - \lfloor j_0 \rfloor - 1}{2}$ ups before entering the Cheuk-Vorst tree. They make therefore no contribution if $0 \leq k \leq \frac{n - \lfloor j_0 \rfloor - 3}{2}$, and so we argue as in (i).

If $k = \frac{n - \lfloor j_0 \rfloor - 1}{2}$, it is impossible to come from level $j = 0$ at period $n - 1$ and so we have with (2) that

$$
\Lambda_{0,k,n}^{j_0} = \Lambda_{1,k,n-1}^{j_0} + \Lambda_{\{j_0\},k,n-1}^{j_0}
$$

$$=
\begin{pmatrix}
\frac{n - 1}{k - 1} - \frac{n - 1}{k - 2} + \frac{n - 1}{k} - \left( \frac{n - 1}{k + \lfloor j_0 \rfloor + 1} \right)
\end{pmatrix}
$$

$$=
\begin{pmatrix}
\frac{n - 1}{k - 1} - \frac{n - 1}{k - 2} + \frac{n - 1}{k} - \left( \frac{n - 1}{k - 1} \right)
\end{pmatrix}
$$

There are no paths for $k > \frac{n - \lfloor j_0 \rfloor - 1}{2}$.

If $1 \leq j \leq n - \lfloor j_0 \rfloor - 3$, there are two paths leading to it from period $n - 1$, one upward path coming from the node located at level $j - 1$ and one downward path coming from the node located at level $j + 1$. If $0 \leq k < j$ there are no paths with $k$ ups. If $k = j$, the only possible path comes from $j - 1$ and so

$$\Lambda_{j,j,n}^{j_0} = \Lambda_{1,j-1,n-1}^{j_0} = 1.$$

If $j + 1 \leq k \leq \left\lfloor \frac{n - \lfloor j_0 \rfloor - 1 + j}{2} \right\rfloor$, then we have

$$\Lambda_{j,k,n}^{j_0} = \Lambda_{j-1,k-1,n-1}^{j_0} + \Lambda_{j+1,k,n-1}^{j_0}
$$

$$=
\begin{pmatrix}
\frac{n - 1}{k - j} - \frac{n - 1}{k - j - 1} + \frac{n - 1}{k - j - 1} - \left( \frac{n - 1}{k - j - 2} \right)
\end{pmatrix}
$$

$$=
\begin{pmatrix}
\frac{n}{k - j} - \frac{n}{k - j - 1}
\end{pmatrix}
$$

There are no paths for $k > \left\lfloor \frac{n - \lfloor j_0 \rfloor - 1 + j}{2} \right\rfloor$.

This proves the lemma in the case when $j_0$ is not an integer.

(II) The situation changes slightly when $j_0$ is an integer, see Figure 2.3.

The three classes of nodes are defined as before: Those for which the set of predecessors is the same as in the traditional binomial tree starting from $(0, j_0)$ (indicated by solid lines); the remaining terminal nodes of the traditional binomial tree (indicated by dashed lines or by thick solid lines); and the terminal nodes of the smaller Cheuk-Vorst tree starting at $(j_0 + 1, 0)$ (indicated by double lines, thick solid lines, or the dash-dotted line). Note, however, that some terminal nodes belong to both the second and the third class.

(1) For the first class, we apply the same argument as before.

(2) For the nodes in the second class we only count the paths that are inside the binomial tree. As before, there are such nodes only when $n \geq j_0 + 2$, the possible number $k$ of upward paths ranges from $n - \lfloor \frac{n + j_0}{2} \rfloor$ to $n - j_0 - 1$, and the terminal level is given by $j = j_0 + 2k - n$. The number of corresponding paths (inside the binomial tree) is $\binom{n}{k}$ minus the number of
Figure 2.3. CV tree for the call with $n = 5$ and $j_0 = 1$

lost paths. But now the lost paths are those paths in the binomial tree that hit level $-1$ at some point. This is equivalent to counting the paths from $(0, j_0 + 1)$ to $(n, j + 1)$ that hit level 0, and the number of such paths is, as before, $\binom{n}{k+j_0+1}$. This confirms the lemma for the stated values of $j$ and $k$.

(3) Finally, for the nodes of the third class we count those paths that at least once follow a path outside the binomial tree (in this way we do not count paths twice when the node also belongs to the second class). The argument as before confirms the third alternative in the lemma.

We note that if the terminal level $j$ belongs to both the second and the third class then the paths inside the binomial tree have exactly $k = \frac{n + j - j_0}{2}$ upward jumps (whence $n + j - j_0$ is even), while the paths outside this tree have at most

$$\left\lfloor \frac{n - j_0 - 1 + j}{2} \right\rfloor < \frac{n + j - j_0}{2}$$

upward jumps. This shows that for such levels $j$ there is no conflict in the statement of the lemma.

Combining the formula (2.4) for the price of a lookback call with Lemma 2.1 we obtain the following.

**Theorem 2.2.** Let $0 \leq t < T$ and $n \in \mathbb{N}$. The price of a European lookback call option with floating strike at time $t$ is given by

$$C_{n}^{fl}(t) = S_t (V_1 - V_2 + V_3),$$

where

$$V_1 = \sum_{k=k_{\min}}^{n} (1 - u^{n-j_0-2k}) \binom{n}{k} q^k (1 - q)^{n-k},$$

$$V_2 = \sum_{k=k_{\min}}^{n-\lfloor j_0 \rfloor - 1} (1 - u^{n-j_0-2k}) \binom{n}{k + \lfloor j_0 \rfloor + 1} q^k (1 - q)^{n-k},$$

$$V_3 = \sum_{j=0}^{\lfloor j_0 \rfloor - 1} (1 - u^{-j}) \sum_{k=j}^{k_{\max}} \left[ \binom{n}{k-j} - \binom{n}{k-j-1} \right] q^k (1 - q)^{n-k}.$$
with \( j_0 \) given by (2.3), \( q \) given by (2.2), \( k_{\text{min}} = n - \lfloor \frac{n+j_0}{2} \rfloor \) and \( k_{\text{max}} = \lfloor \frac{n-j_0-1+1}{2} \rfloor \).

\[ S_{t_m} = \left( \max_{t^* \leq t_m} S_{t^*} \right) u^{-j}. \]

The initial level \( j_0 \) (at initial time \( t \)) of the tree satisfies

\[ S_t = \left( \max_{t^* \leq t} S_{t^*} \right) u^{-j_0}, \]

so that

\[ (2.7) \quad j_0 = \frac{\log(\max_{t^* \leq t} S_{t^*}/S_t)}{\sigma \sqrt{\tau/n}}. \]

It can be shown as in Cheuk and Vorst [4] that the price at time \( t \) of the European lookback put option with floating strike is given by

\[ P^\text{fl}_n(t) = S_t \sum_{j \in J} (u^j - 1) \sum_{k=0}^{n} \Lambda_{j,k,n}^{j_0} (1-q)^k q^{n-k}, \]

where \( J \) is the set of possible levels (taken positively) at maturity, \( q \) is given by (2.2) and \( \Lambda_{j,k,n}^{j_0} \) is the number of paths from the initial node \((0,j_0)\) to level \( j \) at maturity that have exactly \( k \) upward jumps. Since we have evaluated these numbers of paths in Lemma [2.1] we obtain the following.

**Theorem 2.3.** Let \( 0 \leq t < T \) and \( n \in \mathbb{N} \). The price of a European lookback put option with floating strike at time \( t \) is given by

\[ P^\text{fl}_n(t) = S_t(V_1 - V_2 + V_3), \]
where

\[ V_1 = \sum_{k=k_{\min}}^{n} (u^{n+2k-n} - 1) \binom{n}{k} (1 - q)^k q^{n-k}, \]

\[ V_2 = \sum_{k=k_{\min}}^{n-[j_0]-1} (u^{n+2k-n} - 1) \binom{n}{k + [j_0] + 1} (1 - q)^k q^{n-k}, \]

\[ V_3 = \sum_{j=0}^{n-[j_0]-1} (u^j - 1) \sum_{k=j}^{k_{\max}} \left[ \binom{n}{k-j} - \binom{n}{k-j-1} \right] (1 - q)^k q^{n-k} \]

with \( j_0 \) given by (2.7), \( q \) given by (2.2), \( k_{\min} = n - \left\lfloor \frac{n+j_0}{2} \right\rfloor \) and \( k_{\max} = \left\lfloor \frac{n-[j_0]-1}{2} \right\rfloor \).

3. Asymptotic expansions of the binomial cumulative distribution function

In order to obtain asymptotic expansions for the price of lookback options as the number of steps \( n \) tends to infinity we first need to derive an asymptotic expansion of the binomial cumulative distribution function

\[ B_{n,p_n}(j_n) = \sum_{k=0}^{j_n} \binom{n}{k} p_n^k (1 - p_n)^{n-k} \]

or, equivalently, for the complementary binomial cumulative distribution function

\[ B_{n,p_n}^*(j_n) = \sum_{k=j_n+1}^{n} \binom{n}{k} p_n^k (1 - p_n)^{n-k}, \]

and that with a lower error term than in the expansions found in the literature. Moreover, our method allows us to treat any sequence \((p_n)\) satisfying the rather weak assumption that

\[ 0 < \liminf_{n \to \infty} p_n \leq \limsup_{n \to \infty} p_n < 1. \]

Our approach relies on the work of Uspensky [16] who represented binomial probabilities in a convenient analytical form. Chang and Palmer [3, Lemma 1] refined this result in order to obtain an approximation of order \( o(n^{-1}) \) in the case when \( p_n \to 1/2 \). Recently, Lin and Palmer [14, Lemma C.1] provided an estimate of order \( O(n^{-2}) \) if \( p_n = 1/2 + O(n^{-1/2}) \).

Chang and Palmer [3] have used their approximation in order to obtain an asymptotic expansion for digital and European options with a remainder term \( O(n^{-3/2}) \) (see [14, Theorem 1.1]), while Lin and Palmer [14] treated barrier options with the same precision. The second author [11] recently evaluated lookback options with floating strike, limiting himself to the price at emission (that is, at \( t = 0 \)). Moreover, when the spot rate \( r \) equals 0 then he only obtained a remainder term \( O(n^{-1}) \) (see [11, Remark 4.1]).

In order to obtain an asymptotic expansion for lookback options with floating strike with a remainder term \( O(n^{-3/2}) \), and that for any spot rate \( r \geq 0 \) and for the price at any time \( t \geq 0 \), we will need an approximation
of the binomial cumulative distribution with remainder term $O(n^{-5/2})$, as provided by the following theorem.

In the sequel, $C > 0$ denotes a generic constant, which may have a different value at each occurrence. Note also that, for the sake of readability, we will often drop the index $n$.

Let $\Phi$ denote the standard normal cumulative distribution function.

**Theorem 3.1.** Suppose that $p = p_n$ satisfies

$$0 < \lim inf_{n \to \infty} p_n \leq \lim sup_{n \to \infty} p_n < 1.$$ 

If $q = q_n = 1 - p_n$ and $0 \leq j = j_n \leq n$, then

$$\sum_{k=0}^{j} \binom{n}{k} p^k q^{n-k} = \Phi(y) + e^{-\frac{y^2}{2}} \left( \frac{P_1}{\sqrt{V}} + \frac{P_2}{V^{3/2}} + \frac{P_3}{V^{1/2}} + \frac{P_4}{V^{3/2}} \right) + O\left(\frac{1}{n^{5/2}}\right)$$

as $n \to \infty$, where $V = npq$, $y = \frac{j - np + \frac{1}{2}}{\sqrt{V}}$ and

- $P_1 = \frac{1}{6} (q - p)(1 - y^2)$,
- $P_2 = y\left(12(-3 + 7y^2 - y^4) - \frac{pq}{36}(-3 + 11y^2 - 2y^4)\right)$,
- $P_3 = (q - p)\left(\frac{pq}{360}(123 + 129y^2 - 384y^4 + 95y^6 - 5y^8) - \frac{pq}{1248}(3 + 69y^2 - 399y^4 + 145y^6 - 10y^8)\right)$,
- $P_4 = y\left(\frac{1}{8400}(4293 - 1359y^2 + 6165y^4 - 1971y^6 + 185y^8 - 5y^{10}) + \frac{pq}{8400}(3105 + 1395y^2 - 7794y^4 + 2979y^6 - 325y^8 + 10y^{10}) + \frac{pq^2}{8880}(135 - 1035y^2 + 7947y^4 - 4167y^6 + 560y^8 - 20y^{10})\right)$.

As mentioned above, the proof of this result will be based on an analytical representation of binomial probabilities due to Uspensky [16, p. 121].

**Theorem (Uspensky).** Let $0 < p < 1$, $q = 1 - p$ and $0 \leq j \leq n$ be fixed numbers. Then

$$\sum_{k=0}^{j} \binom{n}{k} p^k q^{n-k} = J(y) - J(y'),$$

where

$$y = \frac{j - np + \frac{1}{2}}{\sqrt{V}} \quad \text{and} \quad y' = \frac{-np + \frac{1}{2}}{\sqrt{V}}$$

with $V = npq$; here the function $J$ is defined by

$$J(y) = \frac{1}{2\pi} \int_0^{\pi} \rho^2 \sin(y\sqrt{V}\varphi - \chi) \, d\varphi, \quad y \in \mathbb{R},$$

where

$$\rho = |pe^{i\varphi} + q|, \quad \omega = \arg(pe^{i\varphi} + q) \quad \text{and} \quad \chi = n\omega - np\varphi.$$

We will derive Theorem 3.1 from Uspensky’s representation by employing and refining his ideas (see [16, pp. 121–129]). For this we will need two preliminary lemmas.
Lemma 3.2. Suppose that $p = p_n$ satisfies
\[ 0 < \liminf_{n \to \infty} p_n \leq \limsup_{n \to \infty} p_n < 1. \]
Let $q = q_n = 1 - p_n$ and $V = npq$. For a fixed constant $M > 0$, let $\varphi$ be a positive number such that $\varphi \leq M/V^{1/4}$. Then we have for $R = |pe^{i\varphi} + q|^n$ that
\[
R = e^{R_2 \varphi^2} \left( 1 + R_4 \varphi^4 + R_6 \varphi^6 + \frac{1}{2} R_4^2 \varphi^8 + \sum_{k=1}^{3} O(n^k \varphi^{2k+6}) \right)
\]
as $n \to \infty$, where
\[
R_2 = -\frac{1}{4} V, \\
R_4 = \frac{1}{4} V \left( \frac{1}{6} - pq \right), \\
R_6 = -\frac{1}{6} V (\frac{1}{120} - \frac{1}{4} pq + p^2 q^2).
\]
Proof. Let $\rho = |pe^{i\varphi} + q|$, so that $R = \rho^n$. We then have that
\[
\rho = (p^2 + 2pq \cos \varphi + q^2)^{1/2} = \left( 1 - 4pq \sin^2 \frac{\varphi}{2} \right)^{1/2}.
\]
This gives us that
\[
\log \rho = \frac{1}{2} \log \left( 1 - 4pq \sin^2 \frac{\varphi}{2} \right)
\]
\[
= -2pq \sin^2 \frac{\varphi}{2} - 4(pq)^2 \sin^4 \frac{\varphi}{2} - \frac{32}{3} (pq)^3 \sin^6 \frac{\varphi}{2} - \delta,
\]
where $\delta = 32(1 - \eta)^{-4}(pq)^4 \sin^8 \frac{\varphi}{2}$ for some real number $\eta$ between 0 and $4pq$. Since $\lim inf_{n \to \infty} p_n, \lim inf_{n \to \infty} q_n > 0$ we have that $V = npq \to \infty$ and therefore $0 \leq \varphi \leq M/V^{1/4} \to 0$ as $n \to \infty$. Thus
\[
0 \leq \delta \leq C p^4 q^4 \varphi^8
\]
for $n$ sufficiently large.

Now, in order to obtain bounds for $\log \rho$, we use suitable Taylor expansions about 0. First, we have
\[
\sin^2 \frac{\varphi}{2} = \frac{1}{4} \varphi^2 - \frac{1}{48} \varphi^4 + \frac{1}{1440} \varphi^6 - \frac{1}{80640} \varphi^8 + \frac{1}{7257600} \varphi^{10} + o(\varphi^{10}).
\]
Thus we have
\[
\frac{1}{4} \varphi^2 - \frac{1}{48} \varphi^4 + \frac{1}{1440} \varphi^6 - \frac{1}{80640} \varphi^8 \leq \sin^2 \frac{\varphi}{2} \leq \frac{1}{4} \varphi^2 - \frac{1}{48} \varphi^4 + \frac{1}{1440} \varphi^6
\]
for $n$ sufficiently large. Next we deduce from
\[
\sin^4 \frac{\varphi}{2} = \frac{1}{16} \varphi^4 - \frac{1}{96} \varphi^6 - \frac{1}{1280} \varphi^8 - \frac{17}{483840} \varphi^{10} + o(\varphi^{10})
\]
that
\[
\frac{1}{16} \varphi^4 - \frac{1}{96} \varphi^6 \leq \sin^4 \frac{\varphi}{2} \leq \frac{1}{16} \varphi^4 - \frac{1}{96} \varphi^6 + \frac{1}{1280} \varphi^8
\]
for $n$ sufficiently large. Finally, we obtain from
\[
\sin^6 \frac{\varphi}{2} = \frac{1}{64} \varphi^6 - \frac{1}{256} \varphi^8 + \frac{7}{15360} \varphi^{10} + o(\varphi^{10})
\]
Now, the definitions of $R$

By combining (3.9) and (3.10), we see that

$$R$$

for $n$ sufficiently large. From (3.3), (3.4), (3.5), (3.6), we find upper and lower bounds for the expression (3.2):

$$\log \rho \leq -2pq \left( \frac{1}{4} \varphi^2 - \frac{1}{48} \varphi^4 + \frac{1}{1440} \varphi^6 - \frac{1}{80640} \varphi^8 \right) - 4p^2 q^2 \left( \frac{1}{16} \varphi^4 - \frac{1}{96} \varphi^6 \right) - \frac{32}{3} p^3 q^3 \left( \frac{1}{64} \varphi^6 - \frac{1}{256} \varphi^8 \right)

= \frac{R_2}{n} \varphi^2 + \frac{R_4}{n} \psi^4 + \frac{R_6}{n} \varphi^6 + \frac{1}{24} pq \left( \frac{1}{1680} + p^2 q^2 \right) \varphi^8$$

and

$$\log \rho \geq -2pq \left( \frac{1}{4} \varphi^2 - \frac{1}{48} \varphi^4 + \frac{1}{1440} \varphi^6 - \frac{1}{80640} \varphi^8 \right) - 4p^2 q^2 \left( \frac{1}{16} \varphi^4 - \frac{1}{96} \varphi^6 \right) - \frac{32}{3} p^3 q^3 \left( \frac{1}{64} \varphi^6 - \frac{1}{256} \varphi^8 \right)

= \frac{R_2}{n} \varphi^2 + \frac{R_4}{n} \varphi^4 + \frac{R_6}{n} \varphi^6 - p^2 q^2 \left( \frac{1}{320} + p^2 q^2 \right) \varphi^8.$$

By combining (3.7) and (3.8), we establish that

$$e^{R_2 \varphi^2 + R_4 \varphi^4 + R_6 \varphi^6} e^{\Delta_1} \leq R = \rho^n \leq e^{R_2 \varphi^2 + R_4 \varphi^4 + R_6 \varphi^6} e^{\Delta_2},$$

where

$$\Delta_1 = -Vpq \left( \frac{1}{320} + C p^2 q^2 \right) \varphi^8 \leq 0, \quad \Delta_2 = \frac{1}{24} V \left( \frac{1}{1680} + p^2 q^2 \right) \varphi^8 \geq 0.$$

This clearly implies that

$$\left| R - e^{R_2 \varphi^2 + R_4 \varphi^4 + R_6 \varphi^6} \right| \leq e^{R_2 \varphi^2 + R_4 \varphi^4 + R_6 \varphi^6} \left( e^{\Delta_2} - e^{\Delta_1} \right).$$

Using the facts that $e^x \leq 1 + 2x$ for $0 \leq x \leq 1$ and $e^x \geq 1 + x$, we have for large $n$ that

$$e^{\Delta_2} - e^{\Delta_1} \leq 2 \Delta_2 - \Delta_1 \leq CV \varphi^8;$$

note that $\Delta_2 \to 0$ as $n \to \infty$. It follows that

$$\left| R - e^{R_2 \varphi^2 + R_4 \varphi^4 + R_6 \varphi^6} \right| \leq C e^{R_2 \varphi^2 + R_4 \varphi^4 + R_6 \varphi^6} V \varphi^8,$$

in other words,

$$R = e^{R_2 \varphi^2 + R_4 \varphi^4 + R_6 \varphi^6} + O(e^{R_2 \varphi^2 + R_4 \varphi^4 + R_6 \varphi^6} V \varphi^8).$$

Now, the definitions of $R_4$ and $R_6$ and the fact that $0 \leq \varphi \leq M/V^{1/4}$ imply that

$$R_4 \varphi^4 + R_6 \varphi^6 \leq C.$$

By combining (3.9) and (3.10), we see that

$$R = e^{R_2 \varphi^2} \left( e^{R_4 \varphi^4 + R_6 \varphi^6} + O(n \varphi^8) \right).$$

We rewrite the second exponential term with a Taylor expansion about 0 to obtain for some $\eta$ between 0 and $R_4 \varphi^4 + R_6 \varphi^6$,

$$e^{R_4 \varphi^4 + R_6 \varphi^6} = 1 + (R_4 \varphi^4 + R_6 \varphi^6) + \frac{1}{2} (R_4 \varphi^4 + R_6 \varphi^6)^2 + \frac{1}{6} e^{\eta (R_4 \varphi^4 + R_6 \varphi^6)^3},$$

where

$$\eta = \frac{R_4 \varphi^4 + R_6 \varphi^6}{R_4 \varphi^4 + R_6 \varphi^6}.$$
hence with (3.10)

\[(3.12) \quad e^{R_4 \phi^4 + R_6 \phi^6} = 1 + R_4 \phi^4 + R_6 \phi^6 + \frac{1}{2} R_4^2 \phi^8 + O(n^2 \phi^{10}) + O(n^3 \phi^{12}).\]

Combining (3.11) and (3.12) we obtain the result. □

**Lemma 3.3.** Suppose that \( p = p_n \) satisfies
\[0 < \lim \inf_{n \to \infty} p_n \leq \lim \sup_{n \to \infty} p_n < 1.\]

Let \( q = q_n = 1 - p_n \) and \( V = npq \). Then, for any non-negative integer \( m \), there exists a constant \( C > 0 \) such that
\[I_m := \int_0^\infty e^{-\frac{1}{2}V \phi^2} \phi^m \, d\phi \leq C n^{-(m+1)/2}.\]

Moreover, suppose that \( \tau = \tau_n > 0 \) satisfies \( \tau^{-1} = O(n^\alpha) \) with \( \alpha < 1/2 \). Then for any integers \( m \) and \( k \), there exists a constant \( C > 0 \) such that
\[I^*_m := \int_{\tau}^\infty e^{-\frac{1}{2}V \phi^2} \phi^m \, d\phi \leq C n^{-k}.\]

**Proof.** We first note that by our hypotheses there is some \( \delta > 0 \) such that, for large \( n \),
\[(3.13) \quad V \geq \delta n \quad \text{and} \quad \tau \sqrt{V} \geq \delta n^{1/2-\alpha}.\]

We set \( x = \sqrt{V} \phi \). Then the integrals become
\[I_m = V^{-(m+1)/2} \int_0^\infty e^{-\frac{1}{2}x^2} x^m \, dx\]
and
\[I^*_m = V^{-(m+1)/2} \int_{\tau \sqrt{V}}^\infty e^{-\frac{1}{2}x^2} x^m \, dx.\]

The integrands can be bounded by a function \( Ce^{-x} \) if \( m \geq 0 \), or if \( m < 0 \) and \( x \geq 1 \). Since, by (3.13), \( \tau \sqrt{V} \to \infty \) as \( n \to \infty \), we obtain that
\[I_m \leq CV^{-(m+1)/2}\]
and
\[I^*_m \leq CV^{-(m+1)/2}e^{-\tau \sqrt{V}}.\]

Thus we deduce with (3.13) that
\[I_m \leq Cn^{-(m+1)/2}\]
and, for any \( k \),
\[I^*_m \leq Cn^{-k}.\] □

We can now prove our main result.

**Proof of Theorem 3.1.** The proof will be divided into several steps.

(1) In view of Uspensky’s representation stated above the value \( J(y) \) plays a crucial role, see (3.1). We will split the integral into two parts. Let
\[\tau = V^{-1/4},\]
so that \( \tau^{-1} = O(n^{1/4}) \). Throughout the proof we suppose that \( n \) is large enough to have that \( \tau \leq \frac{\pi}{2} \). Then

\[
J^*(y) := \left| \frac{1}{2\pi} \int_{\tau}^{\pi} \rho^n \frac{\sin(y\sqrt{V\varphi - \chi})}{\sin \frac{\varphi}{2}} \, d\varphi \right| \leq \frac{1}{2} \int_{\tau}^{\pi} \rho^n \, d\varphi.
\]

With (3.2) we have that

\[
n \log \rho \leq -2V \sin^2 \frac{\varphi}{2}.
\]

Applying the fact that \( \sin^2 \frac{\varphi}{2} \geq \frac{\varphi}{\pi} \) for \( 0 \leq \varphi \leq \pi \), we obtain that

\[
J^*(y) \leq \frac{1}{2} \int_{\tau}^{\pi} e^{-2V \sin^2 \frac{\varphi}{2}} \, d\varphi \leq \frac{1}{2} \int_{\tau}^{\pi} e^{-2V \frac{\varphi}{2}^2} \, d\varphi \leq \frac{1}{2} \int_{\tau}^{\infty} e^{-2V \frac{\varphi}{2}^2} \, d\varphi.
\]

Substituting \( x = \frac{\varphi}{\pi} \) and applying Lemma 3.3, we have that

(3.14)

\[
J^*(y) \leq \frac{1}{2} \int_{\frac{\varphi}{\pi}}^{\infty} e^{-\frac{1}{2}V x^2} \, dx = O \left( \frac{1}{n^{5/2}} \right)
\]

since \( (\frac{2}{\pi} \tau)^{-1} = O(n^{1/4}) \). From (3.1) and (3.14) we then obtain that

(3.15)

\[
J(y) = \frac{1}{2\pi} \int_{0}^{\tau} \rho^n \frac{\sin(y\sqrt{V\varphi - \chi})}{\sin \frac{\varphi}{2}} \, d\varphi + O \left( \frac{1}{n^{5/2}} \right).
\]

(2) Looking at the integrand of \( J(y) \) we now want to estimate \( \sin(a - \chi) \) in powers of \( \varphi \), where \( a \in \mathbb{R} \); recall that

\[
\omega = \arg(pe^{i\varphi} + q) \quad \text{and} \quad \chi = n\omega - np\varphi.
\]

By Taylor expansion we have that

(3.16)

\[
\sin(a - \chi) = \sin(a) - \cos(a)\chi - \frac{1}{2}\sin(a)\chi^2 + \frac{1}{6}\cos(a)\chi^3 + \frac{1}{24}\sin(a)\chi^4 - \frac{1}{120}\cos(a - \eta)\chi^5
\]

for some \( \eta \) between 0 and \( \chi \). Since by (3.15) it suffices to assume that \( 0 \leq \varphi \leq \tau \leq \frac{\pi}{2} \) we see that

\[
\omega = \arctan \frac{p \sin \varphi}{p \cos \varphi + q},
\]

so that

\[
\chi = n \arctan \frac{p \sin \varphi}{p \cos \varphi + q} - np\varphi.
\]

Now,

\[
\frac{d}{d\varphi} \left( \frac{\chi(\varphi)}{n} \right) = \frac{1}{2} - p + \frac{p - q}{2} \frac{1}{1 + 2pq(\cos \varphi - 1)},
\]

where we have used that \( p + q = 1 \). Note that, as \( x \to 0 \),

\[
\frac{1}{1 + 2pqx} = 1 - 2pqx + (2pq)^2 x^2 + O(x^3),
\]

where the constant in the big-O condition does not depend on \( n \) since \( pq \) is bounded in \( n \). Thus the Taylor expansion

\[
\cos \varphi - 1 = -\frac{\varphi^2}{2} + \frac{\varphi^4}{24} + O(\varphi^6)
\]
gives us that
\[
\frac{d}{d\varphi} \left( \frac{\chi(\varphi)}{n} \right) = \frac{1}{2} - p + \frac{p - q}{2} \left( 1 - 2pq \left( -\frac{\varphi^2}{2} + \frac{\varphi^4}{24} \right) + (2pq)^2 \frac{\varphi^4}{4} + O(\varphi^6) \right)
\]
\[
= \frac{1}{2}pq(p - q)\varphi^2 - \frac{1}{24}pq(p - q)(1 - 12pq)\varphi^4 + O(\varphi^6)
\]
and hence
(3.17) \quad \chi = \chi_3\varphi^3 + \chi_5\varphi^5 + nO(\varphi^7)

with
\[
\chi_3 = \frac{1}{6}V(p - q),
\]
\[
\chi_5 = -\frac{1}{120}V(p - q)(1 - 12pq).
\]

Applying (3.16) and (3.17) we obtain
\[
\sin(a - \chi) = \sin(a) - \cos(a)(\chi_3\varphi^3 + \chi_5\varphi^5 + nO(\varphi^7))
\]
\[
- \frac{1}{2} \sin(a)(\chi_3\varphi^3 + \chi_5\varphi^5 + nO(\varphi^7))^2 + \frac{1}{6} \cos(a)(\chi_3\varphi^3 + nO(\varphi^5))^3
\]
\[
+ \frac{1}{24} \sin(a)(\chi_3\varphi^3 + nO(\varphi^5))^4 - \frac{1}{120} \cos(a - \eta)(nO(\varphi^3))^5,
\]

and hence
(3.18) \quad \sin(a - \chi) = \sin(a) - \cos(a)\chi_3\varphi^3 - \cos(a)\chi_5\varphi^5 - \frac{1}{2} \sin(a)\chi_3^2\varphi^6
\]
\[- \sin(a)\chi_3\chi_5\varphi^8 + \frac{1}{6} \cos(a)\chi_3^3\varphi^9 + \frac{1}{24} \sin(a)\chi_3^4\varphi^{12}
\]
\[+ \sum_{k=1}^{5} O(n^k\varphi^{2k+5}).
\]

(3) Next, using Laurent expansion, we have that
\[
\frac{1}{\sin \frac{\varphi}{2}} = 2 \varphi + \frac{1}{12} \varphi^3 + \frac{7}{2880} \varphi^5 + O(\varphi^7),
\]
which together with Lemma 3.2 and the fact that \( R = \rho^n \) gives that
\[
\rho^n \sin \frac{y\sqrt{\varphi}}{2} = e^{R_2\varphi^2} \left( \frac{2}{\varphi} + \frac{1}{12} \varphi + \left[ \frac{7}{2880} + 2R_4 \right] \varphi^3 + \left[ \frac{1}{12} R_4 + 2R_6 \right] \varphi^5 + R_4^2 \varphi^7 + \sum_{k=0}^{3} O(n^k\varphi^{2k+5}) \right).
\]

We can now rewrite the integrand of (3.15). Setting
\[
y\sqrt{\varphi} = \alpha \varphi = a
\]
we obtain, by combining (3.18) and (3.19),
\[
\rho^n \sin(y\sqrt{\varphi} - \chi) =
\]
\[
e^{R_2\varphi^2} \left( \frac{2}{\varphi} \sin(\alpha \varphi) + \sum_{k=1}^{11} J_k \varphi^k + \sum_{k=1}^{8} O(n^k\varphi^{2k+1}) \right),
\]
where \( J_{10} = 0 \) and
obtained in (3.20).

Using the definitions of the coefficients \(J\), we obtain with Lemma 3.3 that

\[
J = \left[ \frac{7}{2880} + 2R_4 \right] \sin(\alpha \varphi),
\]

\[
J = -2R_4 \chi_3 \cos(\alpha \varphi),
\]

\[
J = \left[ \frac{7}{2880} + 2R_4 \right] \sin(\alpha \varphi),
\]

\[
J = \left[ \frac{7}{2880} + 2R_4 \right] \sin(\alpha \varphi),
\]

\[
J = -2R_4 \chi_3 \cos(\alpha \varphi),
\]

\[
J = \frac{1}{12} R_4 + 2R_6 - \chi_3^2 \sin(\alpha \varphi),
\]

\[
J = \frac{1}{12} R_4 + 2R_6 - \chi_3^2 \sin(\alpha \varphi),
\]

\[
J = \frac{1}{12} R_4 + 2R_6 - \chi_3^2 \sin(\alpha \varphi).
\]

(4) We will estimate (3.15) using the form of the integrand what we have obtained in (3.20).

We begin with the error terms. Applying Lemma 3.3, we have that

\[
\left| \int_0^T e^{R_2 \varphi^2} f_k \, d\varphi \right| \leq Cn^k \int_0^\infty e^{-\frac{1}{10} V \varphi^2} \varphi^{2k+4} \, d\varphi = O(1/\sqrt{n^{5/2}}),
\]

with \( f_k = O(n^k \varphi^{2k+4}) \), \( k \) an integer between 1 and 8.

For the main part in (3.20) we can replace the integral on \( [0, \tau] \) by one on \( [0, \infty) \) with an error of at most \( O(n^{-5/2}) \). Indeed,

\[
J^{**}(y) := \left| \int_0^\infty e^{R_2 \varphi^2} \left( \frac{2}{\varphi} \sin(\alpha \varphi) + \sum_{k=1}^{11} J_k \varphi^k \right) d\varphi \right|
\]

\[
\leq \int_0^\infty e^{-\frac{1}{10} V \varphi^2} \left( \frac{2}{\varphi} + \sum_{k=1}^{11} |J_k| \varphi^k \right) d\varphi.
\]

Using the definitions of the coefficients \( J_k \), and noting that \( \tau^{-1} = O(n^{1/4}) \), we obtain with Lemma 3.3 that

\[
J^{**}(y) = O \left( \frac{1}{\sqrt{n^{5/2}}} \right).
\]

Applying (3.20), (3.21) and (3.22) to (3.15), we derive that

\[
J(y) = \frac{1}{2\pi} \int_0^\infty e^{R_2 \varphi^2} \frac{2}{\varphi} \sin(\alpha \varphi) d\varphi + \frac{1}{2\pi} \sum_{k=1}^{11} \int_0^\infty e^{R_2 \varphi^2} J_k \varphi^k d\varphi + O \left( \frac{1}{\sqrt{n^{5/2}}} \right)
\]

with \( \alpha = y \sqrt{V} \) and \( R_2 = -\frac{4}{V} \).

It remains to evaluate these integrals, which we have relegated to the Appendix A. After simplification, using in particular that \( (p-q)^2 = 1 - 4pq \), we obtain that

\[
J(y) = \Phi(y) \left( -\frac{1}{2} + \frac{e^{-\frac{1}{2} y^2}}{\sqrt{2\pi}} \left( P_1(y) + P_2(y) + P_3(y) + P_4(y) \right) + O \left( \frac{1}{\sqrt{n^{5/2}}} \right) \right)
\]

with

\[
P_1(y) = \frac{1}{4} (q - p)(1 - y^2),
\]

\[
P_2(y) = \frac{y}{2\sqrt{2\pi}} \left( \frac{1}{15} (-3 + 7y^2 - y^4) - \frac{y}{32\sqrt{2\pi}} (-3 + 11y^2 - 2y^4) \right),
\]

\[
P_3(y) = (q - p) \left( 123 + 129y^2 - 384y^4 + 95y^6 - 5y^8 \right) - \frac{p}{32\sqrt{2\pi}} \left( 3 + 69y^2 - 399y^4 + 145y^6 - 10y^8 \right),
\]

\[
P_4(y) = \frac{y}{15520 \sqrt{2\pi}} \left( -4293 + 1359y^2 + 6165y^4 - 1971y^6 + 185y^8 - 5y^{10} \right) + \frac{p}{32\sqrt{2\pi}} \left( 3105 + 1395y^2 - 7794y^4 + 2979y^6 - 325y^8 + 10y^{10} \right)
\]

\[
+ \frac{y^2}{30560 \sqrt{2\pi}} \left( 135 - 1035y^2 + 7947y^4 - 4167y^6 + 560y^8 - 20y^{10} \right).
\]
Finally, by Uspensky, we have that
\[(3.24) \quad \sum_{k=0}^{j} \binom{n}{k} p^k q^{n-k} = J(y) - J(y'),\]
where
\[y = \frac{j - np + \frac{1}{4}}{\sqrt{V}} \quad \text{and} \quad y' = -\frac{np + \frac{1}{2}}{\sqrt{V}}.\]

Since \(0 < \lim \inf_{n \to \infty} p_n \leq \lim \sup_{n \to \infty} p_n < 1\) there are \(\delta > 0\) and \(C > 0\) such that, for large \(n\),
\[\delta n \leq V \leq n \quad \text{and} \quad \delta \sqrt{n} \leq |y'| \leq C \sqrt{n}.\]

It follows that for each \(k = 1, \ldots, 4\)
\[(3.25) \quad e^{-\frac{1}{2} |y'|^2} P_k(y') = O\left(\frac{1}{n^{5/2}}\right).\]
Moreover, for \(x \leq -2\), \(-\frac{1}{2} x^2 \leq x\). Therefore, if \(n\) is sufficiently large, then
\[(3.26) \quad \Phi(y') \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y'} e^x dx = \frac{1}{\sqrt{2\pi}} e^{y'} = O\left(\frac{1}{n^{5/2}}\right),\]
so that
\[(3.27) \quad J(y') = -\frac{1}{2} + O\left(\frac{1}{n^{5/2}}\right).\]

Now the theorem follows from (3.24), (3.23) and (3.27).

We can deduce a first easy corollary.

**Corollary 3.4.** Suppose that \(p = p_n \to p_0\) with \(0 < p_0 < 1\) and that \(j = j_n\) satisfies \(\frac{j}{n} \to j_0\). Then
\[\sum_{k=0}^{j} \binom{n}{k} p^k (1-p)^{n-k} = O\left(\frac{1}{n^{5/2}}\right) \quad \text{if} \ j_0 < p_0,\]
\[\sum_{k=0}^{j} \binom{n}{k} p^k (1-p)^{n-k} = 1 + O\left(\frac{1}{n^{5/2}}\right) \quad \text{if} \ j_0 > p_0.\]

*Proof.* From our assumptions it follows that
\[\frac{y}{\sqrt{n}} = \frac{j - np + \frac{1}{4}}{n\sqrt{p(1-p)}} \to \frac{j_0 - p_0}{\sqrt{p_0(1-p_0)}}.\]
Then as in (3.25) we have that, for \(k = 1, \ldots, 4\),
\[e^{-\frac{1}{2} y'^2} P_k(y') = O\left(\frac{1}{n^{5/2}}\right).\]
Moreover, if \(j_0 < p_0\) then as in (3.26) we have that
\[\Phi(y) = O\left(\frac{1}{n^{5/2}}\right),\]
while if \(j_0 > p_0\), we have that
\[\Phi(y) = 1 - \Phi(-y) = 1 + O\left(\frac{1}{n^{5/2}}\right).\]
The claim now follows from Theorem 3.1. □

The second corollary will be crucial in our applications to lookback options in the next section.

**Corollary 3.5.** Suppose that

\[ p = \frac{1}{2} + \frac{\alpha}{\sqrt{n}} + \frac{\beta}{n} + \frac{\gamma}{n^{3/2}} + \frac{\delta}{n^2} + \frac{\epsilon}{n^{5/2}} + O\left(\frac{1}{n^5}\right) \]

and

\[ j = \frac{n}{2} + a\sqrt{n} + \frac{1}{2} + b_n + \frac{c}{\sqrt{n}} + \frac{d}{n} + \frac{e}{n^{3/2}} + O\left(\frac{1}{n^3}\right), \]

where \((b_n)_n\) is a bounded sequence. Then

\[
\sum_{k=j}^{n} \binom{n}{k} p^k (1-p)^{n-k} = \Phi(A) + \frac{e^{-\frac{B_n^2}{2\pi}}}{\sqrt{n}} \frac{C_0 - C_2 B_n^2}{n} + \frac{D_0 - D_1 B_n - D_3 B_n^2}{n^{3/2}} + \frac{E_0 - E_1 B_n - E_2 B_n^2 + E_4 B_n^4}{n^2} + O\left(\frac{1}{n^{5/2}}\right),
\]

where

- \(C_0 = 2\alpha^2 A - (1 - A^2)(A - 8\alpha)/12 + C\),
- \(C_2 = A/2\),
- \(D_0 = 4\alpha^3 A + 2(1 - A^2)\beta + D\),
- \(D_1 = (8\alpha A - 1)/6 - (1 - A^2)(A^2 - 8\alpha A + 24\alpha^2 - 3)/12 + AC\),
- \(D_3 = (1 - A^2)/6\),
- \(E_0 = 2(\beta^2 + 2\alpha\gamma) A + (1 - A^2)(6\alpha^2 C + 2\gamma)/3 + (3 - A^2)(6\alpha^3 - 2C)\alpha A/3 + (A^4 - 4A^2 + 1)(16\alpha^3 - C)/12 - (5A^6 - 53A^4 + 33A^2 + 171)A/1440 + (5A^6 - 41A^4 + 21A^2 + 27)\alpha/90 - (7A^4 - 40A^2 + 15)\alpha^2 A/18 - AC^2/2 + E\),
- \(E_1 = 4\alpha^3 A/3 + (1 - A^2)(2A - 12\alpha)\beta + AD\),
- \(E_2 = 2\alpha^2 A + (1 - A^2)(C + 2\alpha A)/2 - (A^4 - 8A^2 + 9)A/24 + (A^4 - 6A^2 + 3)\alpha + E\),
- \(E_3 = (3 - A^2) A/24\),

with \(A = 2(\alpha - a), B_n = 2(\beta - b_n), C = 2(\gamma - c), D = 2(\delta - d), E = 2(\epsilon - e)\).

**Proof.** The proof is based on the same ideas as Lemma 3.1 of [11]. As it is very computational, we will just recall the main steps for obtaining the result.

We first note that by Theorem 3.1,

\[
\sum_{k=j}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1 - \sum_{k=0}^{j-1} \binom{n}{k} p^k (1-p)^{n-k} = 1 - \Phi(y) - \frac{e^{-\frac{1}{2} y^2}}{\sqrt{2\pi}} \left( \frac{P_1}{V} + \frac{P_2}{V^{3/2}} + \frac{P_3}{V^2} + \frac{P_4}{V^{3/2}} \right) + O\left(\frac{1}{n^{5/2}}\right),
\]

with \(y = \Phi(\sqrt{\frac{n}{V}})^{1/2}\), where \(P_1, P_2, P_3, P_4\) and \(V\) are as in that theorem. The goal is then to give an asymptotic expansion of each term in this sum.

We begin with \(1 - \Phi(y) = \Phi(-y)\). This term is an integral that we will decompose into two parts: one from \(-\infty\) to \(A\) (giving \(\Phi(A)\)) and second one from \(A\) to \(-y\). This splitting is motivated by the convergence of \(y\) to \(-A\).
as $n$ tends to infinity. Then we find an asymptotic expansion for the second integral by using a Taylor expansion about $A$ in which we substitute $y$ by its asymptotic expansion.

We finish with all the other terms. For each of them, we substitute $V$ and $y$ by their respective asymptotic expansions. □

Remark 3.6. We have stated the result for

$$B_{n,p}^*(j-1) = \sum_{k=j}^{n} \binom{n}{k} p^k(1-p)^{n-k},$$

in line with the results in [3], [14] and [11]. The corresponding result for the binomial cumulative distribution function

$$B_{n,p}(j) = \sum_{k=0}^{j} \binom{n}{k} p^k(1-p)^{n-k},$$

is obtained by writing $j$ in the form

$$j = \frac{n}{2} + a\sqrt{n} - \frac{1}{2} + b_n + \frac{e}{n^{3/2}} + O\left(\frac{1}{n^2}\right)$$

and taking 1 minus the result given in the corollary.

4. Asymptotics of the price for lookback options

In this section we combine the results of the previous sections in order to derive asymptotic expansions for the price of lookback options. We will use the notation of Section 2. In addition we adopt the following notations that are in line with common usage in the literature:

- $d_1 = \frac{1}{\sigma \sqrt{\tau}} \left( \log \left( \frac{S_t}{M_t} \right) + (r + \frac{\sigma^2}{2})\tau \right)$,
- $d_2 = \frac{1}{\sigma \sqrt{\tau}} \left( \log \left( \frac{S_t}{M_t} \right) + (r - \frac{\sigma^2}{2})\tau \right) = d_1 - \sigma \sqrt{\tau}$,
- $d_3 = \frac{1}{\sigma \sqrt{\tau}} \left( -\log \left( \frac{S_t}{M_t} \right) + (r - \frac{\sigma^2}{2})\tau \right) = -d_1 + \frac{2r}{\sigma} \sqrt{\tau}$,
- $d_4 = \frac{1}{\sigma \sqrt{\tau}} \left( -\log \left( \frac{S_t}{M_t} \right) + (r + \frac{\sigma^2}{2})\tau \right) = d_3 + \sigma \sqrt{\tau}$,

and we will write

$$\kappa_n = \{j_0\}(1 - \{j_0\}),$$

where, as before, $j_0 = \frac{\log(S_t/M_t)}{\sigma \sqrt{\tau}/n}$.

We first note that the price $C_{n}^{fl}$ of a lookback call as a function of $n$ shows some (mild) oscillations, see Figure 4.1, which are caused by the fact that the non-integer part $\{j_0\}$ of the initial level $j_0$ varies with $n$.

This reminds one of the (much more violent) oscillations for European vanilla options. To deal with them, Diener and Diener [7] have used asymptotic expansions of the form

$$C_n = c_0 + c_1 \frac{\sqrt{n}}{n} + c_2 \frac{1}{n} + O\left(\frac{1}{n^{3/2}}\right),$$

where the coefficients $c_1$ et $c_2$ are allowed to be bounded functions of $n$. The variability of these coefficients capture the observed oscillations.

In order to obtain such asymptotic expansions in our setting we will need the following.
Lemma 4.1. Let \( a_0, a_1, a_2 \in \mathbb{R} \), and let \( \eta = \eta(n) \) be a bounded function of \( n \). Then

\[
\left( 1 + \frac{a_1}{\sqrt{n}} + \frac{a_2}{n} + O\left( \frac{1}{n^{3/2}} \right) \right)^\eta = 1 + \frac{a_1 \eta}{\sqrt{n}} + \frac{a_2 \eta - \frac{1}{2} \eta(1 - \eta)a_1^2}{n} + O\left( \frac{1}{n^{3/2}} \right).
\]

Proof. We have that, for large \( n \),

\[
\left( 1 + \frac{a_1}{\sqrt{n}} + \frac{a_2}{n} + O\left( \frac{1}{n^{3/2}} \right) \right)^\eta = \exp \left( \eta \log \left( 1 + \frac{a_1}{\sqrt{n}} + \frac{a_2}{n} + O\left( \frac{1}{n^{3/2}} \right) \right) \right)
\]

\[
= \exp \left( \eta \left( \frac{a_1}{\sqrt{n}} + \frac{a_2 - \frac{1}{2} a_1^2}{n} + O\left( \frac{1}{n^{3/2}} \right) \right) \right)
\]

\[
= \exp \left( \frac{a_1 \eta}{\sqrt{n}} + \frac{(a_2 - \frac{1}{2} a_1^2)\eta}{n} + O\left( \frac{1}{n^{3/2}} \right) \right)
\]

\[
= 1 + \frac{a_1 \eta}{\sqrt{n}} + \frac{(a_2 - \frac{1}{2} a_1^2)\eta}{n} + \frac{a_1^2 \eta^2}{2} + O\left( \frac{1}{n^{3/2}} \right),
\]

which confirms the assertion. It is important to note here that each big-O condition contains a constant that is absolute. \( \square \)

In essence, the lemma says that we may expand the function \( (1 + \frac{a_1}{\sqrt{n}} + \frac{a_2}{n} + O\left( \frac{1}{n^{3/2}} \right))^\eta \) as if \( \eta \) was a constant. Diener and Diener \cite{Diener1974} therefore speak of a frozen parameter.

We first consider the asymptotic expansion of the lookback call. Its price in the Black-Scholes model is well known. If \( r > 0 \) then Goldman, Sosin and Gatto \cite{Goldman1978} found that

\[
C_{BS}^t(t) = S_t - S_t \theta_1 B_1 - M_t B_2 + S_t (1 - \theta_2) B_3,
\]

where

\[
\theta_1 = 1 + \frac{\sigma^2}{2r},\]

\[
\theta_2 = 1 - \frac{\sigma^2}{2r},\]

\[
B_1 = \Phi(-d_1),
\]
By passing to the limit in (4.1), Babbs [1] obtained the price in the case
\( r = 0 \) as
\[
C_{BS}^I(t) = S_t - S_t B_1 - M_t B_2 - S_t (B_3 - B_4),
\]
where
\[
B_3 = \log \left( \frac{S_t}{M_t} + \frac{\sigma^2 t}{2} \right) \Phi(-d_1),
\]
\[
B_4^* = \sigma \sqrt{t} \frac{d_2 - d_1}{\sqrt{2\pi}}.
\]

**Theorem 4.2.** Let \( 0 \leq t < T \) and \( n \in \mathbb{N} \). The price at time \( t \) of the European lookback call option with floating strike in the \( n \)-period CRR binomial model satisfies the following:

(i) if \( r > 0 \) then
\[
C_n^I(t) = C_{BS}^I(t) - S_t \frac{\sigma \sqrt{t}}{2} \left( \theta_1 B_1 + \theta_2 B_3 \right) \frac{1}{\sqrt{n}}
\]
\[
- \left[ S_t \frac{\sigma^2 t}{12} \left( (\theta_1 + 2) B_1 + (\theta_2 + 2 - T_1) B_3 \right) - M_t T_2 B_4 \right] \frac{1}{n}
\]
\[
+ O \left( \frac{1}{n^{3/2}} \right),
\]
where \( B_4 = \sigma \sqrt{T} \left( \frac{S_t}{M_t} \right) \left( 1 - 2r/\sigma^2 \right) ^{1/2} \frac{e^{-\frac{1}{8}(d_1^2+d_2^2)}}{\sqrt{2\pi}}, \quad T_1 = \frac{12r}{\sigma^2} \theta_2 \kappa_n - (1 + \frac{4r^2}{\sigma^2}) \log \frac{S_t}{M_t}, \)
and \( T_2 = \frac{1}{2} + \kappa_n + \frac{d_4}{6\sigma \sqrt{t}} \log \frac{S_t}{M_t} \);

(ii) if \( r = 0 \), then
\[
C_n^I(t) = C_{BS}^I(t) - S_t \frac{\sigma \sqrt{t}}{2} \left( 2 B_1 + B_3^* - B_4^* \right) \frac{1}{\sqrt{n}}
\]
\[
- \left[ S_t \frac{\sigma^2 t}{6} \left( 3 + 3 \kappa_n - \frac{\sigma^2 t}{4} \right) B_1 + (B_3^* - S_t T_2^* B_4^*) \right] \frac{1}{n}
\]
\[
+ O \left( \frac{1}{n^{3/2}} \right),
\]
where \( T_2^* = \frac{1}{2} + \kappa_n + \frac{\sigma^2 t}{12} - \frac{d_2}{6\sigma \sqrt{t}} \log \frac{S_t}{M_t} \).

**Proof.** The first step of the proof consists in writing (2.6) as a combination of (complementary) binomial cumulative distribution functions.

As for \( V_1 \) we note that, by (2.1) and (2.2),
\[
u^{n-j_0-2k} q^k (1 - q)^{n-k} = u^{-j_0} e^{-rt} p^k (1 - p)^{n-k},
\]
so that
\[
V_1 = B_{n,q}^* (j_1 - 1) - \frac{M_t}{S_t} e^{-rt} B_{n,p}^* (j_1 - 1),
\]
where \( j_1 = n - \lfloor \frac{n-j_0}{2} \rfloor \).

For \( V_2 \) we first make a change of index \( k \to k + \lfloor j_0 \rfloor + 1 \) and then proceed as for \( V_1 \) to obtain that
\[
V_2 = Q^{-\lfloor j_0 \rfloor} B_{n,q}^* (j_2 - 1) - \frac{M_t}{S_t} e^{-rt} P^{-\lfloor j_0 \rfloor} B_{n,p}^* (j_2 - 1),
\]
where

\[ Q = \frac{q}{1-q}, \quad P = \frac{p}{1-p} \]

and \( j_2 = j_1 + |j_0| + 1 \).

For \( V_3 \) we proceed as in \[11\]. We first split the inner sum and then change indices, \( k \to k - j \) and \( k \to k - j - 1 \). Next we interchange the two double sums that have appeared, noting that \( 0 \leq j \leq n - |j_0| - 1 \) and \( 0 \leq k \leq \left\lfloor \frac{n-|j_0| - 1}{2} \right\rfloor \) is equivalent to \( 0 \leq k \leq \left\lfloor \frac{n-|j_0| - 1}{2} \right\rfloor \) and \( 0 \leq j \leq n - |j_0| - 1 - 2k \); and that \( 0 \leq j \leq n - |j_0| - 1 \) and \( 0 \leq k \leq \left\lfloor \frac{n-|j_0| - 1}{2} \right\rfloor - 1 \) is equivalent to \( 0 \leq k \leq \left\lfloor \frac{n-|j_0| - 1}{2} \right\rfloor - 1 \) and \( 0 \leq j \leq n - |j_0| - 3 - 2k \). Note that \( \left\lfloor \frac{n-|j_0| - 1}{2} \right\rfloor = j_1 - 1 \).

Now, if \( r > 0 \), then the geometric series that arise as inner sums have ratio different from 1, so that by continuing as in \[11\] we obtain

\[
V_3 = \frac{Q(1-d)}{(Q-1)(Qd-1)} (B_{n,q}(j_3) - QB_{n,q}(j_3 - 1)) + \frac{Q^{-|j_0|-1}}{Q-1} (QB_{n,1-q}(j_3) - B_{n,1-q}(j_3 - 1)) + e^{-r\tau} \frac{(Qd)^{-|j_0|-1}}{d(1-Qd)} (PB_{n,1-p}(j_3) - B_{n,1-p}(j_3 - 1))
\]

with \( j_3 = j_1 - 1 \), where we have used that \( u^{-1} = d \).

However, if \( r = 0 \), then \( Q = u \) so that geometric series with ratio 1 appear. Using \( k(n) = n(k^{-1}) \), the calculations then lead us to

\[
V_3 = \left( \left\lfloor j_0 \right\rfloor - n - \frac{1}{u-1} \right) (B_{n,q}(j_3) - uB_{n,q}(j_3 - 1)) - 2uB_{n,q}(j_3 - 1)
\]

\[+ \frac{u^{-|j_0|-1}}{u-1} (uB_{n,p}(j_3) - B_{n,p}(j_3 - 1)) + 2uq(B_{n,1,q}(j_3 - 1) - uB_{n-1,q}(j_3 - 2)),
\]

where we have used that \( 1 - q = p \) in this situation. We note that the formulas for \( V_1 \) and \( V_2 \) are not affected, but one may replace \( Q \) by \( u \) and \( P \) by \( d \) in \( V_2 \).

The second step of the proof is to expand each term using Corollary 3.5 and Remark 3.6. We note that we use Lemma 4.1 with \( \eta = \{j_0\} \) to expand a power containing \( \left\lfloor j_0 \right\rfloor \). For example, we write

\[Q^{-|j_0|-1} = Q^{\{j_0\}} Q^{-j_0-1}\]

and expand separately. We then obtain the result after a long series of calculations and simplifications. \( \Box \)

We turn to the asymptotics of the put price. We have to replace the running minimum by the running maximum

\[M_t := \max_{t' \leq t} S_{t'}, \]

and the initial level becomes

\[j_0 = \frac{\log(M_t/S_0)}{\sigma \sqrt{\tau/n}} (> 0).\]
With this re-interpretation, the numbers $d_1$, $d_2$, $d_3$, $d_4$ and $\kappa_n$ keep their meaning.

For $r > 0$, Goldman, Sosin and Gatto [10] found that

$$P_{BS}^{fl}(t) = -S_t + S_t \theta_1 B_1 + M_t B_2 - S_t (1 - \theta_2) B_3$$

with

$$B_1 = \Phi(d_1),$$
$$B_2 = e^{-r\tau} \Phi(-d_2),$$
$$B_3 = e^{-r\tau} \left( \frac{S_t}{M_t} \right)^{-2r/\sigma^2} \Phi(-d_3),$$

and, for $r = 0$, Babbs [1] obtained that

$$P_{BS}^{fl}(t) = -S_t + S_t B_1 + M_t B_2 + S_t (B_3^* + B_4^*)$$

with

$$B_3^* = (\log \frac{S_t}{M_t} + \frac{\sigma^2 \tau}{2}) \Phi(d_1),$$
$$B_4^* = \sigma \sqrt{\tau} \left( \frac{S_t}{M_t} \right)^{-d_3/2 \sqrt{2\pi}}.$$

**Theorem 4.3.** Let $0 \leq t < T$ and $n \in \mathbb{N}$. The price at time $t$ of the European lookback put option with floating strike in the $n$-period CRR binomial model satisfies the following:

(i) if $r > 0$ then

$$P_n^{fl}(t) = P_{BS}^{fl}(t) - S_t \sigma \sqrt{\tau} \left( \frac{\theta_1 B_1 + \theta_2 B_3}{2} \right) \frac{1}{\sqrt{n}}$$
$$+ \left[ S_t \frac{\sigma^2 \tau}{12} \left( \left( \theta_1 + 2 \right) B_1 + \left( \theta_2 + 2 - T_1 \right) B_3 \right) + M_t T_2 B_4 \right] \frac{1}{n}$$
$$+ O\left( \frac{1}{n^{3/2}} \right),$$

where $B_4 = \sigma \sqrt{\tau} \left( \frac{S_t}{M_t} \right)^{-2r/\sigma^2} \left( \frac{\Theta_3 + d_4}{\sqrt{2\pi}} \right)$, $T_1 = \frac{12r}{\sigma^2} \theta_2 \kappa_n - (1 + \frac{4r^2}{\sigma^2}) \log \frac{S_t}{M_t}$

and $T_2 = \frac{1}{2} + \kappa_n + \frac{d_4}{6 \sigma \sqrt{\tau}} \log \frac{S_t}{M_t}$;

(ii) if $r = 0$ then

$$P_n^{fl}(t) = P_{BS}^{fl}(t) - S_t \sigma \sqrt{\tau} \left( 2B_1 + B_3^* + B_4^* \right) \frac{1}{\sqrt{n}}$$
$$+ \left[ S_t \frac{\sigma^2 \tau}{6} \left( 3 + 3 \kappa_n - \frac{\sigma^2 \tau}{4} \right) B_1 + B_3^* \right] \frac{1}{n}$$
$$+ O\left( \frac{1}{n^{3/2}} \right),$$

where $T_2^* = \frac{1}{2} + \kappa_n + \frac{\sigma^2 \tau}{12} - \frac{d_2}{6 \sigma \sqrt{\tau}} \log \frac{S_t}{M_t}$.

The proof is quite similar to the proof of Theorem 4.2 and is therefore omitted.

**Remark 4.4.** In case the option is valued at emission we have that $S_t = M_t$ and thus $\kappa_n = 0$. The formulas obtained here then reduce to the formulas previously found by the second author [11].
Remark 4.5. In Theorems 4.2 and 4.3 we have that the coefficients of $\frac{1}{n}$ are bounded functions of $n$. In fact, these coefficients are affine functions of $\kappa_n = \{j_0\}(1 - \{j_0\})$, which is bounded in $n$. The function $x \mapsto x(1 - x)$ is therefore responsible for the parabola-like oscillations in Figure 4.1.

Remark 4.6. The coefficients for $r = 0$ are the limits of those for $r > 0$.

5. Numerical examples

In this part we give a numerical illustration of Theorems 4.2 and Theorem 4.3. These results tell us that
\[
C_{n}^{fl} = C_{BS}^{fl} + \frac{C_1}{\sqrt{n}} + \frac{C_2}{n} + O\left(\frac{1}{n^{3/2}}\right)
\]
and
\[
P_{n}^{fl} = P_{BS}^{fl} + \frac{P_1}{\sqrt{n}} + \frac{P_2}{n} + O\left(\frac{1}{n^{3/2}}\right)
\]
with certain constants $C_1$, $P_1$ and functions $C_2$, $P_2$ that are bounded in $n$. For example, for the call we should find that $(C_{n}^{fl} - C_{BS}^{fl})\sqrt{n}$ and $(C_{n}^{fl} - C_{BS}^{fl} - C_1/\sqrt{n})n$ almost coincide respectively with $C_1$ and $C_2$ for large $n$.

This will be considered in the four tables below. We choose as values $S_0 = 80$, $\sigma = 0.2$ and $\tau = 1.27$. For each type of option we produce an example with a positive value for the spot rate ($r = 0.08$) and an example with $r = 0$.

For the call, we take $M_t = 60$ as the minimal price of the underlying (see Tables 1 and 2). The results obtained are consistent with Theorem 4.2.

| Number of periods $n$ | 1,000 | 5,000 | 10,000 | 50,000 | 100,000 |
|-----------------------|-------|-------|--------|--------|---------|
| $C_{n}^{fl}$          | 26.3647 | 26.3765 | 26.3794 | 26.3832 | 26.3842 |
| $C_{BS}^{fl}$         | 26.3864 | 26.3864 | 26.3864 | 26.3864 | 26.3864 |
| $(C_{n}^{fl} - C_{BS}^{fl})\sqrt{n}$ | -0.6866 | -0.6987 | -0.7004 | -0.7040 | -0.7050 |
| $C_1$                 | -0.7071 | -0.7071 | -0.7071 | -0.7071 | -0.7071 |
| $(C_{n}^{fl} - C_{BS}^{fl} - C_1/\sqrt{n})n$ | 0.6491 | 0.5931 | 0.6658 | 0.6868 | 0.6746 |
| $C_2$                 | 0.6640 | 0.5961 | 0.6635 | 0.6808 | 0.6681 |

Table 1. Example for the call ($r > 0$)

We take the same parameters for the put but here we consider the maximal price of the underlying at time $t$ as $M_t = 100$ (see Tables 3 and 4). The results are consistent with Theorem 4.3.

6. Conclusion

The main goal of our paper is to derive an asymptotic expansion in powers of $n^{-1/2}$ for the price of European lookback options with floating strike. In order to achieve this we had to refine a discrete model of Cheuk and Vorst. Their tree only worked for options evaluated at emission. In our work we consider the price of the option at any time between emission and maturity.
This more general situation turned out to require a new type of tree that mixes a partial binomial tree with a Cheuk-Vorst tree. Counting the number of paths in this tree, we derive a closed formula for the option price.

In order to describe the asymptotic behaviour of this formula we need an asymptotic expansion of the binomial cumulative distribution function with a smaller error term than known so far in the literature. Following the work of Chang, Lin and Palmer [3], [14], we base our work on an integral representation of the binomial cumulative distribution function that is due to Uspensky [16].
We procure explicit formulas for the coefficients of $n^{-1/2}$ and $n^{-1}$ in the asymptotic expansion, both for the call and the put, and for any values of the parameters; in particular, we allow the spot rate to be zero. These formulas confirm the convergence to the Black Scholes prices. Our results are tested on random examples.

Several issues can be proposed as a follow-up to our work. One can continue the study on lookback options; so far no asymptotic expansions are known for the fixed-strike case. Cheuk and Vorst [4] built a one-state tree which can be the basis for this work. However, the price deduced from their tree cannot, a priori, be written using a binomial cumulative distribution function. It will be necessary to provide an asymptotic expansion for some new type of functions.

Another possibility is to look at Asian options for which the payoff is determined by the average value of the underlying. The average can be taken to be geometric or arithmetic. Again there are fixed-strike and floating strike options. So far, no closed form for the price is known in the case of arithmetic averages [5]. Thus determining the asymptotic expansion of the price obtained by an equivalent CRR tree could allow to find a closed-form solution also in the arithmetic case.

Appendix A. Some integrals

We evaluate here some integrals that are needed in the proof of Theorem 3.1. Recall that $R_2 = -\frac{1}{2} V$ and $\alpha = y\sqrt{V}$. The first integral is a Fourier sine transform:

$$\frac{1}{2\pi} \int_0^\infty e^{R_2 \varphi^2} \frac{2}{\varphi} \sin(\alpha \varphi) \, d\varphi = \frac{1}{\pi} \int_0^\infty e^{-\frac{1}{2} x^2} \sin(yx) \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^y e^{-\frac{1}{2} x^2} \, dx = \Phi(y) - \frac{1}{2}$$

with $\Phi$ the standard normal cumulative distribution function; see [16, pp. 128–129], [8, p. 73].

The remaining integrals are Fourier sine or Fourier cosine transforms of the functions $x \mapsto x^m e^{-\frac{1}{2} x^2}$, $m \geq 1$. It is well known that their values involve the Hermite polynomials $H_m$:

$$\frac{1}{\pi} \int_0^\infty x^m e^{-\frac{1}{2} x^2} \begin{cases} \sin(yx) & \text{if } m \text{ is odd} \\ \cos(yx) & \text{if } m \text{ is even} \end{cases} \, dx = (-1)^{\lfloor m/2 \rfloor} e^{-\frac{1}{2} y^2} \sqrt{2\pi} H_m(y),$$

see [8, p. 15, p. 74]. For $m = 1$ we thus have

$$\frac{1}{2\pi} \int_0^\infty e^{R_2 \varphi^2} J_1 \varphi \, d\varphi = \frac{1}{24V} \frac{1}{\pi} \int_0^\infty x e^{-\frac{1}{2} x^2} \sin(yx) \, dx$$

$$= \frac{1}{24V} e^{-\frac{1}{2} y^2} \sqrt{2\pi} y.$$
For \( m = 2 \),
\[
\frac{1}{2\pi} \int_0^\infty e^{R_2 \varphi^2} J_2 \varphi^2 \, d\varphi = -\frac{p-q}{6\sqrt{V}} \frac{1}{\pi} \int_0^\infty x^2 e^{-\frac{1}{2}x^2} \cos(yx) \, dx
\]
\[
= \frac{p-q}{6\sqrt{V}} \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}} (y^2 - 1).
\]
For the following values of \( m \) we obtain
\[
\frac{1}{2\pi} \int_0^\infty e^{R_2 \varphi^2} J_m \varphi^m \, d\varphi = \frac{J_m^* (p-q)}{2V^{3/2}} \frac{1}{\pi} \int_0^\infty x^m e^{-\frac{1}{2}x^2} \cos(yx) \, dx
\]
\[
= \frac{J_m^* (p-q)}{2V^{3/2}} e^{-\frac{1}{2}y^2} \frac{1}{\sqrt{2\pi}} (3 - 6y^2 + y^4)
\]
with \( J_3^* = \frac{7}{2880} + \frac{1}{5} V (\frac{1}{9} - pq) \);
\[
\frac{1}{2\pi} \int_0^\infty e^{R_2 \varphi^2} J_4 \varphi^4 \, d\varphi = \frac{J_6^* (p-q)}{2V^{3/2}} \frac{1}{\pi} \int_0^\infty x^4 e^{-\frac{1}{2}x^2} \cos(yx) \, dx
\]
\[
= \frac{J_6^* (p-q)}{2V^{3/2}} e^{-\frac{1}{2}y^2} \frac{1}{\sqrt{2\pi}} y(15 - 10y^2 + y^4)
\]
with \( J_6^* = \frac{1}{60} (\frac{1}{6} - pq) - \frac{1}{4} (\frac{1}{120} - \frac{1}{4} pq + p^2 q^2) - \frac{1}{36} V (p-q)^2 \);
\[
\frac{1}{2\pi} \int_0^\infty e^{R_2 \varphi^2} J_6 \varphi^6 \, d\varphi = \frac{J_6^* (p-q)}{2V^{3/2}} \frac{1}{\pi} \int_0^\infty x^6 e^{-\frac{1}{2}x^2} \cos(yx) \, dx
\]
\[
= \frac{J_6^* (p-q)}{2V^{3/2}} e^{-\frac{1}{2}y^2} \frac{1}{\sqrt{2\pi}} (15 - 45y^2 + 15y^4 - y^6)
\]
with \( J_6^* = \frac{1}{135} (\frac{1}{6} - pq) \);
\[
\frac{1}{2\pi} \int_0^\infty e^{R_2 \varphi^2} J_7 \varphi^7 \, d\varphi = \frac{J_7^*}{2V^{2}} \frac{1}{\pi} \int_0^\infty x^7 e^{-\frac{1}{2}x^2} \cos(yx) \, dx
\]
\[
= \frac{J_7^*}{2V^{2}} e^{-\frac{1}{2}y^2} \frac{1}{\sqrt{2\pi}} y(105 - 105y^2 + 21y^4 - y^6)
\]
with \( J_7^* = \frac{1}{10} (\frac{1}{6} - pq)^2 + \frac{1}{500} (p-q)^2 (1 - 12pq) - \frac{1}{864} (p-q)^2 \);
\[
\frac{1}{2\pi} \int_0^\infty e^{R_2 \varphi^2} J_8 \varphi^8 \, d\varphi = \frac{(p-q)^3}{1296V^{3/2}} \frac{1}{\pi} \int_0^\infty x^8 e^{-\frac{1}{2}x^2} \cos(yx) \, dx
\]
\[
= \frac{(p-q)^3}{1296V^{3/2}} e^{-\frac{1}{2}y^2} \sqrt{2\pi} H_8(y)
\]
with $H_8(y) = 105 - 420y^2 + 210y^4 - 28y^6 + y^8$;

$$\frac{1}{2\pi} \int_0^\infty e^{2r\varphi^2} J_9 \varphi^9 \, d\varphi = \frac{J_9^* (p-q)^2}{2V^2} \frac{1}{\pi} \int_0^\infty x^9 e^{-\frac{1}{2}x^2} \sin(yx) \, dx$$

$$= \frac{J_9^* (p-q)^2}{2V^2} \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}} H_9(y)$$

with $J_9^* = -\frac{1}{114}(\frac{1}{6} - pq)$ and $H_9(y) = y(945 - 1260y^2 + 378y^4 - 36y^6 + y^8)$;

$$\frac{1}{2\pi} \int_0^\infty e^{2r\varphi^2} J_{11} \varphi^{11} \, d\varphi = \frac{(p-q)^4}{31104V^2} \frac{1}{\pi} \int_0^\infty x^{11} e^{-\frac{1}{2}x^2} \sin(yx) \, dx$$

$$= -\frac{(p-q)^4}{31104V^2} \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}} H_{11}(y)$$

with $H_{11}(y) = y(-10395 + 17325y^2 - 6930y^4 + 990y^6 - 55y^8 + y^{10})$.

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