Radiative Corrections to the Mass of the Kink using an Alternative Renormalization Procedure

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We compute the lowest order quantum radiative correction to the mass of the kink in \( \phi^4 \) theory in 1+1 dimensions using an alternative renormalization procedure. We have introduced this procedure in our earlier works, and we utilize it here to compute the mass of the kink up to and including the one-loop radiative correction. This renormalization program is systematic and self-contained in which the breaking of the translational invariance and the topological nature of the problem, due to the presence of the kink, is automatically taken into account. We use the standard mode number cutoff in conjunction with the above program. Our procedure adds a small correction to the previously established results.

I. INTRODUCTION

The quantum radiative corrections to the mass of the solitons have been of great interest since the 1970’s, and has had a long, complicated, and at times controversial history. In 1974, Dashen et al.\[1\] computed the one-loop correction to the mass of the bosonic kink in \( \phi^4 \) field theory for the first time. In that article they used the mode number cutoff method with a continuous form for the phase shifts of the scattering states. After Dashen, several other authors have used similar methods to compute similar corrections for analogous problems \[2,3\]. Ever since Dashen’s work, several other methods have been invented or used for analogous problems, the most important five of which are the following. First, the energy momentum cutoff using discontinuous form of the scattering states phase shifts \[3,4,5\]. The above two approaches have sparked remarkable controversies \[6,7,8\]. Second, the derivative expansion of the effective action using summation of the series for the exactly solvable cases, which embeds an analytic continuation, or Padé approximation or Borel summation formula for the approximately solvable cases \[9\]. Third, the scattering phase shift method in which the change in the density of states due to the presence of the disturbance is represented by the scattering phase shifts \[10\]. Fourth, the dimensional regularization technique in which the zero point energy of the free vacuum is subtracted and dimensional regularization is used \[11,12\]. Fifth, the zeta function regularization technique which completely bypasses the explicit subtraction of free vacuum energy \[13\]. Obviously all these methods have eventually confirmed the DHN result. As a side note we should mention that analogous corrections to the mass of the bosonic kink have been computed in supersymmetric models (see for example \[11,14,15,16,17\]).

The main issue that we discussed in previous papers \[18,19,20\] is that the presence of nontrivial boundary conditions or strong nontrivial (position dependent) backgrounds such as solitons, which could also affect the boundary conditions, are in principle nonperturbative effects. Therefore they define the overall structure and the properties of the theory and obviously cannot be ignored or even taken into account perturbatively. We believe that the renormalization program should be self-contained, and automatically take into account the aforementioned conditions in a self-consistent manner. An additional supporting argument along these lines is the fact that the presence of either nontrivial boundary conditions or nontrivial backgrounds or both break the translational symmetry of the system. Obviously the breaking of the translational symmetry has many manifestations. Most importantly all the \( n \)-point functions of the theory will have in general nontrivial position dependence in the coordinate representation. The procedure to deduce the counterterms from the \( n \)-point functions in a renormalized perturbation theory is standard and has been available for over half a century. This, as we have shown, will inevitably lead to uniquely defined position dependent counterterms. Therefore, the radiative corrections to all the input parameters of the theory, will be in general position dependent.

We have used this program for the one-loop radiative correction to the Casimir effect within \( \phi^4 \) theory \[19,20\]. In this letter we calculate the one-loop quantum correction to the mass of the \( \phi^4 \) kink in 1+1 dimensions, which is analogous to the Casimir problem with the kink as its static background using our proposed renormalization program \[18\]. We choose the kink as our example because it has simultaneously two nontrivial properties which are its spatial dependence and boundary conditions. The starting part of our computation parallels closely Dashen’s work. That is we use the mode number cutoff with continuous phase shifts. However, the counterterms that we use are different than all the ones that we have encountered so far. For this category of problems, in all the papers that we are aware of, the authors use free counterterms, by which we mean the ones derived and appropriate for the free case, i.e. cases with no nontrivial boundary conditions or spatial backgrounds. The difference between our position dependent counterterm and the position independent free one, relevant to the trivial sector, turns out to be small terms proportional to the lowest
lying localized state distributions. This difference, depicted in [18], will lead to a small correction to the well established DHN result [1]. Our method can be compared with the mean field approach applied to this problem [21].

We have organized the paper in four sections as follows. In Section II we set up the usual problem of $\phi^4$ theory for a real scalar field in 1+1 dimensions, in the spontaneously broken phase. We find the static background solutions which include the trivial and the kink sectors. We also exhibit the quantum fluctuations in both sectors, the latter of which includes two normalized states, one zero mode state reflecting the invariance of the system under translation of the kink and the other a true bound state. Then in Section III we calculate the one-loop radiative correction to the kink mass by subtracting the vacuum energies of the two sectors. The part of this energy which does not depend on the counterterms is calculated using the mode number cutoff. We then calculate the contribution from the mass counterterms. When we add up all the contributions, we find an extra term which is due to our nontrivial counterterm in the kink sector. Finally in Section IV we compare our methods and results to some earlier work, and conjecture on some possible implications of our method.

II. KINK SOLUTIONS AND THEIR QUANTUM FLUCTUATIONS

In this section we shall very briefly state the standard results for the static background solutions and their quantum fluctuations for the bosonic $\phi^4$ theory. For a comprehensive review of the standard materials, see for example [3]. We start with the Lagrangian density for a neutral massive scalar field, within $\phi^4$ theory, appropriate for the spontaneously broken symmetry phase in 1+1 dimensions:

$$\mathcal{L} = \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - U[\phi(x)],$$

where $U[\phi] = \frac{\lambda}{4} \left( \phi^2 - \frac{\mu^2}{\lambda^2} \right)^2$. The Euler-Lagrange equation can be easily obtained and is a second-order nonlinear PDE with the following solutions: Two non-topological static solutions $\phi_{\text{vac}}(x) = \pm \mu_0 / \sqrt{\lambda_0}$, and two topological static ones $\phi_{\text{kink}}(x) = \pm \mu_0 / \sqrt{\lambda_0} \tanh(\mu_0(x - x_0) / \sqrt{2})$ which are called kink and antikink, respectively. The presence of $x_0$ indicates the translational invariance, and this will lead to a zero mode. The total kink energy, sometimes called the classical kink mass can be easily calculated and is given by $M_{\text{cl}} = \frac{2\sqrt{\pi} \mu_0^4}{\lambda_0}$. In order to find the quantum corrections to this mass, we have to make a functional Taylor expansion of the potential about the static solutions which yields the stability equation

$$-\nabla^2 + \frac{d^2U}{d\phi^2}\bigg|_{\phi_{\text{static}}(x)} \eta(x) = \omega^2 \eta(x),$$

where we have defined $\phi = \phi_{\text{static}} + \eta$. The results in the trivial sector are the following continuum states $\eta(x) = \exp(ikx)$ with $\omega^2 = k^2 + 2\mu_0^2$. In the kink sector we have the following two bound states and continuum states:

$$\eta_0(z) = \sqrt{3\mu_0} \frac{1}{8 \cosh^2 z},$$
$$\eta_B(z) = \sqrt{3\mu_0} \frac{1}{4 \cosh z},$$
$$\eta_q(z) = \frac{e^{iqz}}{N_q} \left[ -3 \tanh^2 z + 1 + q^2 + 3iq \tanh z \right],$$

where $m_0 = \sqrt{2}\mu_0$, $\omega_0^2 = 0$, $\omega_B^2 = \frac{3}{4}m_0^2$ and $\omega_q^2 = m_0^2 \left( \frac{q^2}{4} + 1 \right)$. Here $N_q = 16\frac{\mu_0^2}{m_0^2} (\omega_q^2 - \omega_B^2)$, and $z = m_0x/2$. The continuum states $\eta_q(z)$ have the following asymptotic behavior for $x \to \pm \infty$,

$$\eta_q(z) \to \exp[iqz \pm \frac{i}{2}\delta(q)],$$

where $\delta(q) = -2 \arctan\left[ \frac{3q}{2 - q^2} \right]$ is the phase shift for the scattering states. We believe the phase shifts should be in principle defined to be continuous functions of their arguments. This is particularly apparent in their use in the Levinson theorem (see for example [22]). For this particular case the phase shift is illustrated in figure I.

III. FIRST ORDER RADIATIVE CORRECTION TO THE KINK MASS

In this section we calculate the first order quantum correction to the kink mass. As is well known, this is analogous to the Casimir problem for this case. That is, the exact kink mass is the difference between the vacuum energies in
FIG. 1: A direct calculation of the phase shift yields the graph on the top. However in all physical applications that we are aware of the proper form to use is a continuous as the one illustrated on the bottom.

the presence and absence of the kink. To calculate this effect we set up renormalized perturbation theory. We should mention that in these 1+1 dimensional problems, one usually chooses a minimal renormalization scheme defined at all loops by \[ Z_\lambda = 1, \quad Z_\eta = 1 \quad \text{and} \quad m_0^2 = m^2 - \delta m^2. \]

The sufficiency of these conditions is supported by the fact that for any theory of a scalar field in two dimensions with nonderivative interactions, all divergences that occur in any order of perturbation theory can be removed by normal-ordering the Hamiltonian \[ m_0^2 \]. However, relaxing the first condition will lead to some extra finite contributions \[ \delta m^2 \]. Here, for mere comparison reasons, we want to stay focused on the renormalization program with the above stated conditions but with nontrivial mass counterterm \[ \delta m^2 \].

Now we can split the expression for mass of the kink into the following two parts,

\[
M = (E_{\text{kink}} - E_{\text{vac}}) + (\Delta E_{\text{kink}} - \Delta E_{\text{vac}}),
\]

where the first part is in the vacuum sector of each, and the second part is due to the counterterms. We put our solutions in a box of length \( L \) and impose periodic boundary conditions. The continuum limit is reached by taking \( L \) to infinity and the sum turns into an integral. Now we use the usual mode number cutoff as advocated by R.F. Dashen \[ \lambda \] to calculate the first part of Eq. (6). In this method one subtracts the energies of the bound states in the presence of solitons from the same number of lowest lying quasi-continuum states in the vacuum of the trivial sector. Then one subtracts the remaining quasi-continuum states from each other in ascending order. In this case our two lowest lying states are the aforementioned two normalizable states which are to be subtracted from \( \omega_{1,1}' \). Then we subtract the quasi-continuum \( q_n \) from the remaining \( k_{n+1} \) one by one. The periodic boundary condition implies,

\[
k_{n+1}L - 2\pi = 2n\pi = q_n - \frac{mL}{2} + \delta(q_n).
\]

The first part of Eq. (6) can be easily calculated as follows

\[
E_{\text{kink}} - E_{\text{vac}} = M_{\text{cl}} + \frac{1}{2}(\sum \omega - \sum \omega')
\]

\[
= \frac{m^3}{3\lambda} + \frac{1}{2} \left[ \omega_0 + \omega_B - (\omega_1' + \omega_{-1}') + 2 \sum_{n=1}^{N} (\omega_n - \omega_{n+1}') \right]
\]

\[
= \frac{m^3}{3\lambda} + \frac{\sqrt{2}m}{4} - m + \sum_{n=1}^{N} \left[ m\left(\frac{q_0^2}{4} + 1\right)^{1/2} - (k_{n+1}^2 + m^2)^{1/2} \right]
\]

\[
= \frac{m^3}{3\lambda} + \frac{\sqrt{2}m}{4} - \frac{3m}{2\pi} - \frac{3m}{4\pi} \int_{-\infty}^{\infty} \frac{k^2 + m^2/2}{\sqrt{k^2 + m^2(k^2 + m^2)}} dk,
\]

where in the last step we have taken the continuum limit, performed an integration by parts taking the appropriate boundary values of the phase shift into account, as explained earlier. Note that the last term is logarithmically divergent.
Now we calculate the second part of Eq. (6):

$$\Delta E_{\text{kink}} - \Delta E_{\text{vac}} =$$

$$-\frac{1}{2} \int_{-\infty}^{\infty} dx \left[ \delta m_{kink}^2 \phi_{kink}^2(x) - \delta m_{\text{vac}}^2 \phi_{\text{vac}}^2(x) \right],$$

(9)

where $\delta m_{kink}^2$ and $\delta m_{\text{vac}}^2$ are the mass counterterms in the kink and vacuum backgrounds, respectively, and are calculated below. We first start with the mass counterterms in vacuum background. The procedure for obtaining this quantity is well known, e.g. by setting the tadpole equal to zero. The result is

$$\delta m_{\text{vac}}^2 = \frac{3\lambda}{4\pi} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{k^2 + m^2}},$$

(10)

which is logarithmically divergent.

The appropriate mass counterterm in the kink background is

$$\delta m_{kink}^2 = \frac{\lambda}{\sqrt{3}\pi m^2} \eta_{B}^2(x) - \frac{3\lambda}{\pi m} \eta_{0}^2(x) + \frac{3\lambda}{4\pi} \int_{-\infty}^{\infty} dk \frac{1}{\sqrt{k^2 + m^2}},$$

(11)

which as expected earlier is different from mass counterterm in the trivial sector, i.e. the last term in Eq. (11). In fact it has extra finite $x$-dependent terms due to the presence of the bound states, and obviously this difference tends to zero as $x \to \pm \infty$. However, note that introducing an $x$-dependent counterterm into the Lagrangian does not introduce any unaccounted-for interactions to the system, since Eq. (5) is always valid. An alternative reasoning is that the kink solution also tends to either of the trivial vacuum states as $x \to \pm \infty$. To complete the calculation we need to calculate Eq. (9) by inserting the expressions for $\delta m_{kink}^2$ and $\delta m_{\text{vac}}^2$ into it. The result is

$$\Delta E_{kink} - \Delta E_{\text{vac}} =$$

$$-\frac{1}{2} \int_{-\infty}^{\infty} dx \left[ \delta m_{\text{vac}}^2 \left( \phi_{kink}^2(x) - \phi_{\text{vac}}^2(x) \right) + \left( \frac{\lambda}{\sqrt{3}\pi m^2} \eta_{B}^2(x) - \frac{3\lambda}{\pi m} \eta_{0}^2(x) \right) \phi_{kink}^2(x) \right],$$

(12)

Inserting the expressions obtained in Eqs. (8,12) into Eq. (6) we obtain the following expression for $M$:

$$M = \frac{m^3}{3\lambda} + \frac{\sqrt{3m}}{4} - \frac{3m}{2\pi} \sqrt{3\pi - 3} \frac{m}{20\pi},$$

(13)

$$-\frac{3m}{4\pi} \int_{0}^{\infty} \frac{2k^2 + m^2}{\sqrt{k^2 + m^2}(k^2 + m^2 + \frac{m^2}{4})} dk + \frac{3m}{2\pi} \int_{0}^{\infty} \frac{dk}{\sqrt{k^2 + m^2}}.$$

The logarithmic divergences cancel and the final result is:

$$M = \frac{m^3}{3\lambda} + \frac{m}{4\sqrt{3}} - \frac{3m}{2\pi} \frac{\sqrt{3\pi - 3}}{20\pi}.$$  

(14)

Our results adds a small correction to the well established DHN result. This correction is given by the last term in Eq. (14).

IV. CONCLUSION

In this paper we have calculated the one-loop radiative correction to the mass of the kink using a systematic renormalization approach. We have argued that the presence of a kink with a fixed position breaks the translational symmetry of the system and this has profound consequences. In particular all of the $n$-point functions of the theory, the counterterms, and the renormalized parameters of the theory will in general become position dependent. This is where our procedure differs from the more established ones. In particular in Eq. (11) we have shown explicitly the difference between $\delta m_{kink}^2$ and $\delta m_{\text{vac}}^2$. This has led to a small correction to the well established result. The main issue that we want to emphasis here is that the presence of nontrivial boundary conditions or strong nontrivial backgrounds, such as solitons, which could also affect the boundary conditions are in principle nonperturbative effects. Therefore they define the overall structure and the properties of the theory and obviously cannot be ignored or even taken into account perturbatively. We believe that the solution to the problem should be self-contained and the renormalization procedure be done self-consistently with the nature of the problem. Our approach can be considered an alternative which takes into account the aforementioned issues, as compared to the more established approaches. Two lines of research seem...
interesting to us at this point. First analogous reasonings could also affect the results in the SUSY cases, where as far as we know free counterterms are used invariably (see for example [14, 17] and references therein). Second the perturbative corrections presented here may change if one uses the RG running of coupling constants [25].

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