Abstract

The existence of run-away solutions in classical and non-relativistic quantum electrodynamics is reviewed. It is shown that the less singular high energy behavior of relativistic spin $\frac{1}{2}$ quantum electrodynamics precludes an analogous behavior in that theory. However, a Landau-like anomalous pole in the photon propagation function or in the electron-massive photon forward scattering amplitude would generate a new run-away, characterized by an energy scale $\omega \sim m_e \exp \frac{1}{\alpha}$. This contrasts with the energy scale $\omega \sim \frac{m_e}{\alpha}$ associated with the classical and non-relativistic quantum run-aways.
I. INTRODUCTION

Almost a hundred years ago, Lorentz calculated the self force on a charged particle, that is, the force exerted on the particle by its own radiation field. In an expansion in powers of the radius $b$ of the particle, he found for the force $\vec{f}$

$$\vec{f} = -\delta m \ddot{\vec{y}} + \frac{2}{3} \frac{e^2}{4\pi} \vec{y} + O(b) \ ,$$

(1.1)

where $\vec{y}$ is the position vector of the particle, $\delta m$ is a function of $b$ that goes like $1/b$ for small $b$, the $\vec{y}$ coefficient is independent of $b$ ($e$ is the charge of the particle) and the higher terms all go to zero as $b \to 0$. In order to deal with the problem of the divergence of $\delta m$ as $b \to 0$, Lorentz invented mass renormalization. Applying Newton’s second law

$$m_0 \ddot{\vec{y}} = -\delta m \ddot{\vec{y}} + \frac{2}{3} \frac{e^2}{4\pi} \vec{y} + O(b) \quad (1.2)$$

or, with $m = m_0 + \delta m$ fixed and $b \to 0$,

$$m \ddot{\vec{y}} = \frac{2}{3} \frac{e^2}{4\pi} \vec{y} \quad (1.3)$$

a finite equation of motion. However, the divergence problem returns through the back door: (1.3) has a run-away solution

$$\vec{y} = \vec{y}_0 + \left( \vec{v}_0 - \vec{a}_0 \frac{e^2}{6\pi m} \right) t + \vec{a}_0 \left( \frac{e^2}{6\pi m} \right)^{2} \left( \exp \frac{6\pi m t}{e^2} - 1 \right) \quad (1.4)$$

with a run-away time of $\tau = \frac{2}{3} \cdot \frac{e^2}{4\pi me^3}, \sim 10^{-23}$ seconds for electrons.

A Lorentz covariant equation which reduces to (1.3) in the instantaneous rest system of the particle was proposed by Dirac; however, the run-away problem was not removed. Of course, in the relativistic model it is the momentum, not the velocity, that runs away.

The clearest algebraic insight into the problem, both classical and quantum, is obtained by finding an upper half-plane pole in the Laplace transform $\vec{y}(\omega)$ of $\vec{y}(t)$:

$$\vec{y}(\omega) = \int_0^\infty dt \ e^{i\omega t} \vec{y}(t) \ .$$

(1.5)

Here $\omega = \omega_1 + i\omega_2$, with $\omega_2$ positive and large enough to make the integral converge. In Appendix A we take that path and, following Moniz and Sharp, show that in a suitably cut-off classical theory there are no run-aways, no matter how small the cut-off radius, provided $m_0$, the bare mass, is positive. We then show that the mass renormalization, carried out for $b \to 0$ as described above, leads to run-aways.

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2H. A. Lorentz, *Theory of Electrons*, 2nd edition (Dover, New York, 1952, p. 49); 1st edition, 1909

3P. A. M. Dirac, *Proc. Roy. Soc.* A167, 148 (1938)

4E. J. Moniz and D. H. Sharp, *Phys. Rev.* D10, 1133 (1974); *Phys. Rev.* D15, 2850 (1977)
In Appendix B, we obtain the same result in a non-relativistic quantum theory. Thus quantum effects per se do not cure the disease, nor does the addition of a confining oscillator potential.

In Appendices A and B we considered, and in the following we consider, the simplest problem that shows the run-away effect: a charged (in the limit) point particle, which we shall call an electron, is in a wave-packet state with approximate momentum $\vec{p}$ and energy $E$ for times $t \leq 0$. At $t = 0$, a weak spatially uniform external electric field $\vec{E}(t)$ is switched on, and we follow the electron’s motion, calculating the mean value of the coordinate $\vec{y}$, or of the velocity $\vec{v}(t) = \dot{\vec{y}}(t)$.

The solution for $\vec{y}(\omega)$ is given for classical motion in (A.20) of Appendix A and for the non-relativistic quantum case by (B.16) of Appendix B:

$$\vec{y}(\omega) = -\frac{e\vec{E}(\omega)/\omega^2}{m_0 + \frac{2\epsilon^2}{(2\pi)^2} \int \frac{dk}{k^2 - \omega^2} |f(\vec{k})|^2} \ . \quad (1.6)$$

Here $f(\vec{k})$ is the cut-off function, which must go to zero as $k \to \infty$ fast enough to make the integral converge. $\vec{E}(\omega)$ is the Laplace transform of $\vec{E}(t)$,

$$\vec{E}(\omega) = \int_0^\infty dt \ e^{i\omega t} \vec{E}(t) \ ,$$

and $\vec{y}(\omega)$ in the quantum case represents the mean value of the variable. The absence (in (1.4) of an upper half $\omega$ plane pole in the cut-off theory, and its emergence following Lorentz’s renormalization procedure, are shown in Appendix A. The presence of such a pole in the Laplace transform produces an exponential run-away in time, via the inversion formula

$$\vec{y}(t) = \frac{1}{2\pi} \int_{-\infty + i\omega_2}^{+\infty + i\omega_2} d\omega \ e^{-i\omega t} \vec{y}(\omega ) \ . \quad (1.7)$$

The upper half plane pole appears at a frequency $\hbar\omega = \frac{3\epsilon}{2} \frac{m_0 c^2}{\alpha}$, where $i = \sqrt{-1}$ and $\alpha$ is the fine structure constant. We point out here that for weak coupling the energy $\frac{m_\alpha}{\alpha}$ is deep in the relativistic region where the equations used to describe the system are certainly not valid. Since relativistic spin $\frac{1}{2}$ quantum electrodynamics is known to hold accurately in that energy region, we presume that there will be no analogous pole in that theory. In the following we formulate a way of studying that issue, and indeed find no analogous pole in the relativistic spin $\frac{1}{2}$ theory. However, if a pole of the type suggested by Landau, Abrikosov and Khalatnikov exists, either in the photon Green’s function or in the electron photon forward scattering amplitude, there would be a new type of run-away with the characteristic frequency $\hbar\omega \sim imc^2 \exp 1/\alpha$, high enough so that there appears to be no way in which present day experiments can rule it out. Of course, such a pole is already a symptom of a diseased theory.

In the following sections we will be calculating the mean value of the electron velocity, to lowest order in the external field. Of course, the limitation of the electron’s velocity to $c$ does not automatically occur in this approximation. However, if the electron momentum does run away, the limitation of the velocity to $c$ will appear as a non-linear effect of the external field. Correspondingly, the linear approximation to a run-away momentum would be a run-away velocity, for which we search.
II. SPINOR QUANTUM ELECTRODYNAMICS

We work in the Schroedinger representation with a state vector \( \Psi(t) \) which satisfies the equation

\[
-\frac{1}{i} \frac{\partial \Psi}{\partial t} = (H + H')\Psi .
\]  

(2.1)

Here \( H \) is the electron-photon Hamiltonian, and \( H' \) the interaction Hamiltonian with the spatially uniform external field:

\[
H' = -\int \vec{\mathbf{j}}(\vec{x}) \cdot \vec{A}^e(t) d\vec{x}
\]

(2.2)

where

\[
\vec{E}^e = -\frac{\partial \vec{A}^e}{\partial t}
\]

(2.3)

and \( \vec{\mathbf{j}}(\vec{x}) \) is the electron current density

\[
\vec{\mathbf{j}}(\vec{x}) = e\psi^\dagger(\vec{x})\vec{\alpha}\psi(\vec{x})
\]

(2.4)

where \( e, \psi, \vec{\alpha} \) and \( \vec{x} \) have their usual meaning.

We take for the velocity operator

\[
\vec{v} = \frac{\int d\vec{x}' \vec{\mathbf{j}}(\vec{x}')}{\langle \int d\vec{x} \rho(\vec{x}) \rangle}
\]

(2.5)

where \( \rho \) is the charge density

\[
\rho = e\psi^\dagger(\vec{x})\psi(\vec{x}) .
\]

(2.6)

For \( t < 0 \), the electron is in a state \( \Psi_0 \) which satisfies the equation

\[
-\frac{1}{i} \frac{\partial \Psi_0}{\partial t} = H\Psi_0 .
\]

(2.7)

\( \Psi_0 \) may be expressed as a superposition of stationary states of momentum \( \vec{p} \) and spin \( s \):

\[
\Psi_0 = \sum_s \int \phi(\vec{p}, s) \Psi_{\vec{p}, s} d\vec{p}
\]

(2.8)

where

\[
H\Psi_{\vec{p}, s} = E(\vec{p})\Psi_{\vec{p}, s} .
\]

(2.9)

The state vector normalization is

\[5\]

Charge renormalization must be taken into account here. This is discussed in Section III, Eq. (3.16).

\[6\]

We ignore problems associated with infra-red singularities.
\[(\Psi_{\vec{p}',s'}, \Psi_{\vec{p},s}) = \delta_{s's}(2\pi)^3 \delta(\vec{p}' - \vec{p}) \, , \quad (2.10)\]

so that the wave-packet state \(\Psi_0\) is normalized to one with
\[
\sum_s \int d\vec{p}\phi(\vec{p}, s)|^2 \cdot (2\pi)^3 = 1 \, . \quad (2.11)\]

The mean value of the velocity vector in the state \((2.8)\) is
\[
\sum_{s', s} \int \phi^*(\vec{p}', s')d\vec{p}' \left( \Psi_{\vec{p}',s'} \frac{\tilde{j}(\vec{x})}{\int d\vec{x}' \langle \rho \rangle} \Psi_{\vec{p},s} \right) \times \phi(\vec{p}, s)d\vec{p}d\vec{x} \, . \quad (2.12)\]

The matrix element of \(\int \tilde{j}(\vec{x})d\vec{x}\) is
\[
\int (\Psi_{\vec{p}',s'} \cdot \tilde{j}(\vec{x}) \Psi_{\vec{p},s}) d\vec{x} = (2\pi)^3 \delta(\vec{p}' - \vec{p})u^*(\vec{p}', s') e\vec{\alpha} u(\vec{p}, s) \, ; \quad (2.13)\]

since \(\vec{p}' = \vec{p}\), there is no form factor or anomalous magnetic term. The spinors \(u(\vec{p}, s)\) satisfy the Dirac equation
\[
(\vec{\alpha} \cdot \vec{p} + \beta m) \, u(\vec{p}, s) = E(\vec{p}) \, u(\vec{p}, s) \, . \quad (2.14)\]

The matrix-element in \((2.13)\) has the value
\[
u^*(\vec{p}, s')\vec{\alpha} u(\vec{p}, s) = \delta_{s's} \vec{p}/E \, , \quad (2.15)\]

giving for the mean value \((2.12)\)
\[
(\Psi_0, \vec{v}\Psi_0) = \sum_s \int |\phi(\vec{p}, s)|^2 \vec{p}/E \, d\vec{p} \, (2\pi)^3 \, . \quad (2.16)\]

We can, in this calculation and others that involve momentum conserving operators, dispense with the wave packet \(\phi\), replace the state \(\Psi_0\) by \(\Psi_{\vec{p}}\), and leave the \(d\vec{x}\) integral undone, thereby eliminating the delta function in \((2.13)\), and replacing the wave packet average in \((2.10)\) by the single value \(\vec{p}/E\). From here on we will assume that this has been done. For the real electron, the wave-packet average must be taken.

We solve \((2.1)\) for small \(\vec{A}\) by expanding in powers of \(\vec{A}\). With
\[
\Psi = \Psi_0 + \Psi_1 + \ldots \, . \quad (2.17)\]

we have
\[
- \frac{1}{i} \frac{\partial \Psi_0}{\partial t} = H\Psi_0 \quad (2.18)\]

and
\[
- \frac{1}{i} \frac{\partial \Psi_1}{\partial t} = H\Psi_1 + H'\Psi_0 \quad (2.19)\]

so that
\[
\Psi_1 = -i \int_0^t dt' e^{iH(t'-t)}H'(t')\Psi_0(t') \, . \quad (2.20)\]
As just described, we replace $\Psi_0$ by $\Psi_{\vec{p}, s}$ and leave the $d\vec{x}$ integral in (2.5) undone. The mean value of the velocity is then given by

$$\langle \vec{v} \rangle = \vec{p}/E + \delta \vec{v}$$  \hspace{1cm} (2.21)

with

$$\delta \vec{v} = -i \left( \Psi_p, \vec{v} \int_0^t dt' e^{iH(t'-t)}H'(t')\Psi_p \right) + \text{c.c.}$$ \hspace{1cm} (2.22)

for $t > 0$ and $\delta \vec{v}$ understood to be zero for $t < 0$.

We make (2.22) more explicit:

$$\delta v^i = i \left( \Phi_p, v^i \int_0^t dt' e^{i(H-E)(t'-t)}j^k(\vec{x}')\Phi_p \right) A_k^c(t')d\vec{x}' + \text{c.c.}$$ \hspace{1cm} (2.23)

where $\Phi_p = \Psi_p(t = 0) = e^{iE_p t}\Psi_p$. To compare with the results from Appendices A and B we take the Laplace transform of (2.23). We note first that the Laplace transform of $\int_0^t dt' e^{i(H-E)(t'-t)}A_k^c(t')$ is

$$\int_0^\infty dt e^{i\omega t} \int_0^t dt' e^{i(H-E)(t'-t)}A_k^c(t') = \int_0^\infty dt e^{i\omega t} A_k^c(t') \int_0^\infty dt e^{i\omega (t-t')} e^{i(H-E)(t'-t)}$$

$$= -\frac{1}{i(\omega - (H - E))} A_k^c(\omega)$$ \hspace{1cm} (2.24)

(where $A_k^c(\omega)$ is the Laplace transform of $A_k^c(t)$) so that the Laplace transform in (2.23) is

$$\delta v^i(\omega) = -\left( \Phi_p, v^i \frac{1}{\omega - (H - E)} j^k(\vec{x}')\Phi_p \right) A_k^c(\omega) + \text{contribution of complex conjugate}$$ \hspace{1cm} (2.25)

or in all

$$\delta v^i(\omega) = -\left( \Phi_p, \left[ v^i \frac{1}{\omega - (H - E)} j^k(\vec{x}') - j^k(\vec{x}') \frac{1}{\omega + H - E} v^i \right] \Phi_p \right) A_k^c(\omega).$$ \hspace{1cm} (2.26)

If we took the expression (2.26) at face value, we would immediately conclude that there could be no upper half plane singularities in $\delta v^i(\omega)$, since $H$ has only real eigenvalues. However, since the theory is not finite, we are not justified in drawing that conclusion. We must rather deal with the mass and charge renormalized theory, which in practice means order by order perturbation theory in $\alpha$. In the next section we will show how the expression $\delta v^i(\omega)$ in (2.26) can be calculated order by order from Feynman diagrams.

Before we turn to that, we exploit (2.26) to calculate the zero'th order (in the fine structure constant) motion of the electron in the weak external field. There are two classes of intermediate states: one particle (which makes no contribution), and three particles (two electrons and a positron). One finds for $\delta v^i(\omega)$

\[7\] We will see in Appendix C and Section III how the calculation can be done directly from Feynman diagrams.
\[ \delta v^\ell (\omega) = -e \left[ \frac{u^* \alpha^\ell \Lambda^+(\vec{p}) \alpha^k u}{\omega} - \frac{u^* \alpha^k \Lambda^- \alpha^\ell u}{\omega - 2E} + \omega \leftrightarrow -\omega, \ell \leftrightarrow k \right] A^c_\ell (\omega) . \]  

(2.27)

Here \( \Lambda^+(\vec{p}) \) is the positive energy projection operator with momentum \( \vec{p} \), \( \Lambda^- (\vec{p}) \) the negative energy projection operator of momentum \( \vec{p} \) (corresponding to positron momentum \( -\vec{p} \)):

\[ \Lambda^\pm (\vec{p}) = \frac{E \pm (\vec{\alpha} \cdot \vec{p} + \beta m)}{2E} . \]  

(2.28)

The first term in (2.27) is symmetric in \( \ell \) and \( k \), and hence gives zero when \( \omega \leftrightarrow -\omega \). To calculate the second term, we need

\[ u^* \alpha^k \Lambda^- \alpha^\ell u = u^* \alpha^k \left( \frac{1}{2} - \frac{H_D}{2E} \right) \alpha^\ell u \]

\[ = u^* \left( \alpha^k \alpha^\ell - \frac{p^k p^\ell}{E^2} \right) u \]

\[ = \left( \delta^{kl} - \frac{p^k p^l}{E^2} \right) + i \sigma_{kl} \]  

(2.29)

so that

\[ \delta v^\ell (\omega) = e \left\{ \frac{1}{\omega - 2E} - \frac{1}{\omega + 2E} \right\} A^c_\ell (\omega) . \]

(2.31)

We see in (2.31) the structure that will guide us in the search for run-away poles. The amplitudes that are odd in interchange of \( k \) and \( \ell \) are odd in \( \omega \), to all orders in \( \alpha \); those that are even in \((k, \ell)\) interchange are even in \( \omega \) to all orders in \( \alpha \). Higher order odd amplitudes in spinor electrodynamics go like \( \frac{1}{\omega} \) (at high \( \omega \)) except for logarithms; higher order even amplitudes go like \( \frac{1}{\omega^2} \) except for logarithms. There is therefore no way for a second order term to be of the same order as a zero’th order term (the sign of a possible pole) at \( \omega \sim \frac{m}{\alpha} \), as in the non-relativistic case; rather, it must be of order \( \omega \sim \exp(1/\alpha) \). Indeed, the pole conjectured by Landau, Abrikosov and Khalatnikov would arise in just that way.

We translate (2.31) into time dependence:

\[ \delta v^\ell (t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \ e^{-i\omega t} \delta v^\ell (\omega) , \quad t \geq 0 . \]  

(2.32)

We close the \( \omega \) integration in the upper or lower half plane as called for, and find

\[ \delta v^\ell (t) = \left( \delta^{kl} - \frac{p^k p^l}{E^2} \right) \frac{e}{E} \int_0^t \left\{ 1 - \cos[2E(t' - t)] \right\} E^c_k (t') dt' \]

\[ + e \sigma_{kl} \frac{1}{E} \int_0^t \sin[2E(t' - t)] E^c_k (t') dt' . \]  

(2.33)

We recognize in (2.33) the classical relativistic first order equation for \( \delta v^\ell \). This follows by neglecting the zitterbewegung like sine and cosine averages of the external electric field.
— a valid neglect for fields whose characteristic time is much longer than \( \hbar/E \) (which is \( \sim 10^{-21} \) seconds for electrons).

Finally, we wish here to find the \( \omega \) dependence of \( \delta v^\ell \) at high \( \omega \). From (2.26), it is

\[
\delta v^\ell(\omega) \to -\frac{1}{\omega} \left( \phi_p, [v^\ell, j^k]\phi_p \right) A^\ell_k(\omega)
\]

(2.34)

\[
= -\frac{2ieA^\ell_k(\omega)}{\omega} \left( \phi_p, \psi^\dagger(x) \sigma^{\ell k} \psi(x) \phi_p \right).
\]

(2.35)

The matrix element in (2.35) is of course finite in zero'th order, as shown in (2.31). In second order, it would be finite except for vacuum polarization effects. We shall see in Section III that the correctly renormalized Feynman amplitude for \( \delta v^\ell \) implies that the expression (2.26) must carry an explicit factor \( \frac{1}{Z_3^2} \), where \( Z_3 \) is the photon propagator renormalization constant

\[
Z_3 = 1 - \frac{\alpha}{3\pi} \log \frac{\Lambda^2}{m^2} + \text{finite} + \text{higher order in } \alpha
\]

(2.36)

with \( \Lambda \) a momentum cut-off. The finiteness of (2.26) together with the divergence in \( Z_3^{-2} \) implies a high \( \omega \) dependence,

\[
\delta v^\ell \to -\frac{2}{\omega} ieA^\ell_k(\omega) \psi^\dagger \sigma^{\ell k} \psi \left( 1 + 2 \frac{\alpha}{3\pi} \log \frac{\omega^2}{m^2} \right).
\]

(2.37)

Using the technique discussed in Section III, we can calculate the high \( \omega \) behavior of \( \delta v^\ell \) explicitly, and find agreement with (2.37).

### III. LOWEST ORDER RADIATIVE CORRECTION

We consider first the mean value of the velocity operator, without the accelerating electric field. The stationary state \( \phi_p \) is given as

\[
\phi_p = U(0, -\infty) \chi_p
\]

(3.1)

where \( \chi_p \) is the bare (or interaction representation) one particle state, and \( U(t_1, t_2) \) is the unitary operator which transforms that state in time:

\[
-\frac{1}{i} \frac{\partial}{\partial t_2} U(t_2, t_1) = H_I(t_2) U(t_2, t_1)
\]

(3.2)

and

\[
U(t_1, t_1) = 1.
\]

(3.3)

\( H_I \) is the interaction Hamiltonian expressed in terms of interaction representation operators:

\[
H_I = \int d^4 \bar{x} \left[ -eA_\mu(x) \bar{\psi}(x) i\gamma^\mu \psi(x) - \delta m \bar{\psi}(x) \psi(x) \right].
\]

(3.4)

The free Hamiltonian \( H_0 \) is expressed in terms of the correct mass of the electron; hence the necessity for the subtraction of \( \delta m \bar{\psi} \psi \) in (3.4).

We wish to calculate the velocity operator in the Schroedinger state \( \psi_p \). It is
\[ \langle v^k \rangle = (\Psi_p, v^k \Psi_p) = (\Phi_p, e^{iHt}v^k e^{-iHt} \Phi_p) = (U(0, -\infty) \chi_p, e^{iHt}e^{-iHt}v^k(t)e^{iHt}e^{-iHt}U(0, -\infty) \chi_p) . \] (3.5)

Here we recognize the operator

\[ e^{iHt}e^{-iHt} = U(t, 0) \] (3.6)

so that \( v^k(t) \) is the velocity expressed in the interaction representation, and

\[ \langle v^k \rangle = \left( S \chi_p, U(\infty, t)v^k(t)U(t, -\infty) \chi_p \right) \] (3.7)

where the \( S \) matrix \( S = U(\infty, -\infty) \), and where we have used the unitarity of \( U \) and its group property:

\[ U(t_3, t_2)U(t_2, t_1) = U(t_3, t_1) \] (3.8)

The \( S \) matrix is diagonal on one particle states. Since there is no self-energy correction, it must be a phase, which is canceled by disconnected diagrams.

In (3.7) we see \( \langle v^k \rangle \) expressed in a way which allows us to make use of the Dyson technique and Feynman diagrams. The 2nd order result is expressed in the diagrams shown in Fig. 1.

\begin{align*}
\text{(a)} & \quad + \quad \text{(b)} & \quad + \quad \text{(c)} & \quad + \quad \text{(d)}
\end{align*}

Fig. 1

Here the initial and final lines represent the wave functions \( \overline{u}(p) \ldots u(p) \); the dot \( \bullet \) represents the operator \( i\gamma^\mu \); and so forth. The diagrams (1b) and (1c) have the \( \delta m \) correction subtracted from them, and are multiplied by \( \frac{1}{2} \) to reproduce the correct normalization of the incoming state. The sum of all such diagrams is

\[ \langle v^\mu \rangle = \sqrt{Z_2 \overline{u}} \frac{1}{Z_1} i\gamma^\mu u \sqrt{Z_2} = \overline{u} i\gamma^\mu u \] (3.9)

since \( Z_2 = Z_1 \). Thus, \( \langle v^\mu \rangle \) is both ultra-violet and infra-red finite, and equal to its expected free particle value,

\[ \langle v^\mu \rangle = \frac{p^\mu}{E} = \left( \frac{\overline{p}}{E}, 1 \right) \] (3.10)

The closed fermion loop diagrams cancel because \( \overline{f} \) and \( \rho \) carry the same charge renormalization.

We turn next to the effect of the external field as given by (2.23).

We consider the Feynman amplitude

\[ G^{\ell k}(q^0) = i \int_{-\infty}^{\infty} dx^0 e^{iq^0x^0} \int d\vec{x} \left( \Phi_p, j^\ell(\vec{x}, x^0) j^k(\vec{y}, 0) \right) \Phi_p \] (3.11)

which we can readily calculate, order by order, following Dyson.
\[ G^{\ell k}(q^0) = i \int_{-\infty}^{\infty} dx^0 e^{iq^0 x^0} \int d\vec{x} \left( \chi_{p, \left( U(\infty, -\infty) j^\ell(\vec{x}, x_0) j^k(\vec{y}, 0) \right) \right) \chi_p \right). \] (3.12)

\[ G^{\ell k}(q^0) \] in (3.11) and (3.12) is the forward scattering amplitude of a photon of four momentum \( q^\mu = (q^0, \vec{0}) \) on an electron of momentum \( \vec{p} \). By considering the form (3.11) we see that

\[ G^{\ell k}(q^0) = - \left( \Phi_p, \int d\vec{x} j^\ell(\vec{x}, 0) \frac{1}{q^0 + i\epsilon - (H - E)} j^k(\vec{y}, 0) - j^k(\vec{y}, 0) \frac{1}{q^0 - i\epsilon + (H - E)} \int d\vec{x} j^\ell(\vec{x}, 0) \Phi_p \right) \] (3.13)

which has the same structure as (2.26) except that \( q^0 - i\epsilon \) in the second term of (3.13) is in the lower half plane, whereas \( \omega \) in (2.26) is in the upper half plane. We therefore obtain the correct formula (2.26) from (3.12) by continuing \( q^0 \) into the upper half plane and setting it equal to \( \omega \).

The use of Feynman diagrams makes clear the finiteness and high \( q^0 \) behavior of (2.26). We illustrate with the 2nd order (in \( \alpha \)) terms. The diagrams (with the exception of vacuum polarization) are:

![Feynman diagrams](image)

Fig. 2

In Figs. (2a) and (2c), the self-mass is subtracted, and the remainder diagrams multiplied by \( \frac{1}{2} \). In Fig. (2b), the self-mass is subtracted. In what remains, the ultra-violet divergences cancel (using \( Z_2 = Z_1 \) again) by adding

\[ \frac{1}{2}(2a) + \frac{1}{2}(2c) + (2b) + (2d) + (2e). \]

The infra-red divergence cancels by adding \( \frac{1}{2}(2a) + \frac{1}{2}(2b) + (2f) \). The high \( q^0 \) dependence of each of the diagrams (1b), (1d), (1c) and (1f) is \( \frac{1}{q^0} \log q^0 \). However, when the four diagrams are added the logarithms cancel, leading to a high \( \omega \) dependence in the odd amplitudes

\[ v^\ell(\omega) \sim \frac{1}{\omega} \]

which therefore cannot in weak coupling produce the upper half plane singularity which classically leads to the run-away solutions. Note however that the above simple asymptotic calculation only shows a cancellation in the \((k, \ell)\) odd amplitude. The logarithms in the \((k, \ell)\) even amplitudes fail to cancel, leaving the possibility of a singularity of order \( \omega \sim me^{\frac{1}{\gamma}} \) in the even amplitudes, and a run-away \( x \sim \exp(me^{\frac{1}{\gamma}}) \) in the even amplitudes. We should call this a Landau run-away, as opposed to a Lorentz run-away. The second order results of these calculations are given in Appendix C.

The relativistic theory brings in a new effect, the electron-positron vacuum polarization. To take this into account we must add two more diagrams to those of Fig. 2:
We study the effect of the diagrams of Fig. 3 on the renormalization of (2.26). We replace $v^\ell$ by $e_0 \int d\vec{x} L^\ell(\vec{x},0) \langle \rho \rangle$ where $e_0$ is the bare charge and $L^\ell(\vec{x}, t)$ the lepton current:

$$L^\ell(\vec{x}, t) = \psi^+(\vec{x}) \alpha^\ell \psi(x).$$  \hspace{1cm} (3.14)

Eq. (2.26) becomes

$$\delta v^\ell(\omega) = - e_0 \int_{<\rho> / e_o} \frac{1}{\omega - (H - E)} e_0 L^k(\vec{y}, 0) + \text{crossed term} \left[ \Phi_p \right] A_{k}^{e,u}$$  \hspace{1cm} (3.15)

where $A_{k}^{e,u}$ is the unrenormalized external vector potential.

In the combination $\frac{L^\ell}{<\rho> / e_o}$ the numerator operator will produce a factor $Z_3 q^2 D_{FC}(q^2)$, where $D_{FC}$ is the finite photon propagation function, normalized to $\frac{1}{q^2}$ as $q^2 \to 0$. The denominator is $Z_3$. The calculation of the numerator matrix element will automatically generate the factor $Z_3$; the denominator must be put in by hand.

On the right side, the unrenormalized external field is produced by an external current with a charge $e_0$. The renormalized external field would be produced by a charge $e$. Therefore $A_{k}^{e,u} = e_0 e A_{k}^{e}$, where $A_{k}^{e}$ is the renormalized external potential. The matrix element containing $e_0 L^k A_{k}^{e,u} = e_0^2 L^k A_{k}^{e}$ will therefore produce a finite factor $q^2 D_{FC}(q^2) A_{k}^{e}$; correspondingly the matrix element of $L^k$ will produce a renormalization factor $q^2 D_{FC}(q^2) Z_3$. We may therefore rewrite (3.13) as the finite expression

$$\delta v^\ell(\omega) = - e_0 \frac{1}{Z_3^2} \left( \Phi_p \left[ L^\ell \left( \frac{1}{\omega - (H - E)} L^k + \text{crossed term} \right) \Phi_p \right] A_{k}^{e} \right),$$  \hspace{1cm} (3.16)

justifying the discussion following Eq. (2.25).

It is straightforward to show directly from the diagrams in Fig. 3 that the logarithmic dependence of $D_{FC}(q^2)$ at high $q^2$ translates into the expected logarithmic behavior of $\delta v^\ell(\omega)$, as given in (2.36) and (2.37). (See Appendix C.)

It is possible that the series of radiative corrections to $D_{F}$ sum to a pole, as conjectured by Landau, Abrikosov and Khalatnikov. If so, the pole is at space like $q^2$, i.e. $(q^0)^2 < 0$ and $\omega^2 < 0$, producing a new class of run-away solutions.

We emphasize that the above discussion is specifically for spin $\frac{1}{2}$ particles. What happens to the classical run-away solutions in the relativistic quantum electrodynamics of a spin zero particle is an interesting question, but has not been dealt with here.

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APPENDIX A: CLASSICAL RUN-AWAY

We consider the simplest problem: an electron is placed at \( \vec{y} = 0 \) with \( \dot{\vec{y}} = 0 \). At \( t = 0 \), an external spatially constant electric field \( \vec{E}_e(t) \) is turned on. We solve the equations of motion exactly in the weak \( \vec{E}_e \) limit. We work with a cut-off theory, with a cut-off function \( f(\vec{x}) \). The equations of motion are

\[
m_0 \ddot{\vec{y}} = \int e \left[ \vec{E}(\vec{x}, t) + \dot{\vec{y}} \times \vec{B}(\vec{x}, t) + e\vec{E}_e(t) \right] f(\vec{x} - \vec{y}) \, d\vec{x} , \tag{A.1}
\]

\[
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{A.2}
\]

\[
\nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + e\dot{\vec{y}}(t) f(\vec{x} - \vec{y}) \tag{A.3}
\]

\[
\nabla \cdot \vec{E} = e f(\vec{x} - \vec{y}) \tag{A.4}
\]

and

\[
\nabla \cdot \vec{B} = 0 . \tag{A.5}
\]

In the Coulomb gauge, the scalar potential is

\[
\phi(\vec{x}) = \frac{e}{4\pi} \int \frac{f(\vec{x}' - \vec{y})}{|\vec{x}' - \vec{x}|} \, d\vec{x}' \tag{A.6}
\]

and the force coming from it is

\[
\vec{f}_\phi = -\frac{e^2}{4\pi} \int f(\vec{x} - \vec{y}) \nabla_x \int \frac{d\vec{x}' f(\vec{x}' - \vec{y})}{|\vec{x}' - \vec{x}|} \, d\vec{x} = 0 \tag{A.7}
\]

identically.

The vector potential is transverse, and given by

\[
\nabla^2 \vec{A} - \frac{\partial^2 \vec{A}}{\partial t^2} = -e\dot{\vec{y}}_\perp(t) f(\vec{x} - \vec{y}(t)) \tag{A.8}
\]

where \( \dot{\vec{y}}_\perp \) is defined by Fourier transforming (A.8):

\[
-\vec{k}^2 \vec{a}(\vec{k}) - \frac{\partial^2 \vec{a}(\vec{k})}{\partial t^2} = -e\dot{\vec{y}}_\perp f(\vec{k}) e^{-i\vec{k} \cdot \vec{y}} \tag{A.9}
\]

and

\[
\dot{\vec{y}}_\perp = \dot{\vec{y}} - \hat{k} \hat{k} \cdot \dot{\vec{y}} . \tag{A.10}
\]

Here,

\[
\vec{a}(\vec{k}) = \int e^{-i\vec{k} \cdot \vec{x}} \vec{A}(\vec{x}) d\vec{x}
\]

and

\[
f(\vec{k}) = \int e^{-i\vec{k} \cdot \vec{x}} f(\vec{x}) \, d\vec{x} . \tag{A.11}
\]

We solve (A.9) for \( \vec{a} \) with the boundary condition (for convenience only)

\[
\vec{a}(\vec{k}, t) = 0 , \quad t = 0 \quad \text{and} \quad \dot{\vec{a}}(\vec{k}, t) = 0 , \quad t = 0 . \tag{A.12}
\]

The solution is
\[ \vec{a}(\vec{k}, t) = e \int_0^t dt' \frac{\sin k(t - t')}{k} \frac{\dot{\vec{y}}(t')}{k} e^{-i\vec{k} \cdot \vec{y}(t')} f(\vec{k}) \]  
(A.13)

so with
\[ \vec{A}(\vec{x}, t) = \frac{1}{(2\pi)^3} \int d\vec{k} e^{i\vec{k} \cdot \vec{x}} \vec{a}(\vec{k}, t), \]  
(A.14)

\[ m_0 \ddot{\vec{y}} = e\vec{E}^e(t) + e^2 \int_0^t dt' \int \frac{d\vec{k}}{(2\pi)^3} e^{i\vec{k} \cdot (\vec{y}(t) - \vec{y}(t'))} |f(\vec{k})|^2 \]  
\[ \left\{ -\cos k(t - t') \vec{y}_\perp(t') + \vec{y}(t) \times (i\vec{k} \times \vec{y}(t')) \frac{\sin k(t - t')}{k} \right\}. \]  
(A.15)

This is an exact integral equation for the electron motion. If we replace \( m_0 \ddot{\vec{y}} \) by \( m_0 \ddot{\vec{y}} \sqrt{1 - \vec{y}^2} \) it would be a relativistic equation were it not for the cut-off function \(|f(\vec{k})|^2\). If we set \(|f(\vec{k})|^2 = 1\), the equation becomes divergent (near \( t' = t \)) and has no solution. In view of these difficulties it makes no sense to study this equation further, or to try to make it relativistic. However, in the weak field limit we can and will solve the equation exactly, in order to study the way the run-away solutions appear. In this limit \( y(t) - \vec{y}(t') \) and \( \vec{y} \) are of first order in \( \vec{E}^e \) and \( \vec{y}^k \times \vec{y}^e \) is of second order. Therefore the equation becomes
\[ m_0 \ddot{\vec{y}} = e\vec{E}^e - e^2 \int_0^t dt' \int \frac{d\vec{k}}{(2\pi)^3} \cos k(t - t') |f(\vec{k})|^2 \vec{y}_\perp(t') \]  
(A.16)

and can be solved by Laplace transform. Remembering our boundary conditions \( \vec{y}(0) = \vec{y}(\omega) = \vec{y}(0) = 0 \), we have, for \( \vec{y}(\omega) \),
\[ -m_0 \omega^2 \vec{y}(\omega) = e\vec{E}^e(\omega) + \frac{2}{3} e^2 \int \frac{d\vec{k}}{(2\pi)^3} |f(\vec{k})|^2 (-i\omega \vec{y}^e(\omega)) \int_0^\infty dt e^{i\omega t} (-\cos kt) \]  
(A.17)

\[ = e\vec{E}^e(\omega) + \frac{2}{3} \omega^2 e^2 \int \frac{d\vec{k}}{(2\pi)^3} |f(\vec{k})|^2 \vec{y}(\omega), \]  
(A.18)

where
\[ \vec{E}^e(\omega) = \int_0^\infty dt e^{i\omega t} \vec{E}^e(t). \]  
(A.19)

So we find
\[ \vec{y}(\omega) = \frac{-e\vec{E}^e(\omega)/\omega^2}{m_0 + \frac{2}{3} e^2 \int \frac{d\vec{k}|f(\vec{k})|^2}{(2\pi)^3 (k^2 - \omega^2)}}. \]  
(A.20)

\( \vec{y}(\omega) \) in (A.20) has no upper half-plane singularities for \( m_0 > 0 \). This follows from the observation that the imaginary part of \( \frac{1}{k^2 - \omega^2} \) is
\[ \text{Im} \frac{1}{k^2 - \omega^2} = \frac{2\omega_1 \omega_2}{(k^2 - (\omega_1^2 - \omega_2^2))^2 + 4\omega_1^2 \omega_2^2}, \]  
(A.21)

which is different from zero in the upper half \( \omega \) plane unless \( \omega_1 = 0 \). Here \( \omega_1 \) and \( \omega_2 \) are the real and imaginary parts of \( \omega \). When \( \omega_1 = 0 \), the denominator in (A.20) is always positive. Thus there are no run-aways.
Renormalization changes that. Replace the integral
\[
\frac{2}{3} e^2 \int \frac{d\vec{k}}{(2\pi)^3} \frac{|f(\vec{k})|^2}{k^2 - \omega^2}
\]
by
\[
\frac{2}{3} e^2 \int \frac{d\vec{k}}{(2\pi)^3} |f(k)|^2 \left( \frac{1}{k^2 - \omega^2} - \frac{1}{k^2} \right) + \delta m
\] (A.22)
where
\[
\delta m = \frac{2}{3} e^2 \int \frac{d\vec{k}}{(2\pi)^3} |f(k)|^2 ,
\] (A.23)
and note that the integral in (A.22) is now convergent in the limit $|f|^2 \to 1$. In that limit, the denominator $D$ in (A.20) becomes
\[
D = m_0 + \delta m + \frac{2}{3} e^2 \int \frac{d\vec{k}}{(2\pi)^3} \frac{\omega^2}{(k^2 - \omega^2)k^2}
\] (A.24)
\[
= m + \frac{2}{3} e^2 \frac{i\omega}{4\pi}
\] (A.25)
and the run-away pole appears at
\[
\omega = im \cdot \frac{6\pi}{e^2} .
\] (A.26)

APPENDIX B: RUN-AWAY IN NON RELATIVISTIC QUANTUM THEORY

In this appendix we show that the classical formula (A.20) holds exactly, including arbitrary external field strength, in a non-relativistic quantum theory in which we neglect recoil. The neglect of recoil has an internal consistency, since recoil can only be properly taken into account in a fully relativistic theory. The point of this exercise is to show that quantum theory in itself does not cure the run-away problem, since it almost identically reproduces the classical effect.

We can solve this model exactly since the neglect of recoil makes the Heisenberg equations of motion linear, and hence soluble, with the same solution as the classical case. Note that the neglect of recoil, i.e. of the $\vec{y}$ dependence of $\vec{A}(\vec{y})$, is exactly the approximation we made following (A.10) to arrive at a simply soluble classical problem there.

The Hamiltonian is
\[
H = \frac{(\vec{p} - e\vec{A})^2}{2m_0} + \sum_{k,\lambda} k \left( \frac{q_{k,\lambda}^2 + \pi_{k,\lambda}^2}{2} \right) - e\vec{y} \cdot \vec{E}^e(t)
\] (B.1)
where $q_{k,\lambda}$ and $\pi_{k,\lambda}$ are radiation oscillators:
\[
[p_{k,\lambda}, q_{k',\lambda'}] = \frac{1}{i} \delta_{kk'} \delta_{\lambda\lambda'}
\] (B.2)
and
\[ \vec{A} = \sum_{k,\lambda} q_k \vec{e}_k \sqrt{k} \]  \hspace{1cm} (B.3)

A cut-off function can be introduced here, as in the classical calculation of Appendix A. We save space and time by simply inserting it into the final answer, (B.17).

The Coulomb Hamiltonian
\[ H_C = \frac{e^2}{4\pi} \int \frac{f(\vec{x} - \vec{y})f(\vec{x}' - \vec{y})d\vec{x}d\vec{x}'}{|\vec{x} - \vec{x}'|} \] \hspace{1cm} (B.4)

is independent of \( \vec{y} \) and hence does not enter into the equations of motion. These are
\[ \dot{\vec{p}} = -\nabla_y H = e\vec{E}(t) \] \hspace{1cm} (B.5)
\[ \dot{\vec{y}} = \nabla_y H = \frac{\vec{p} - e\vec{A}}{m} \] \hspace{1cm} (B.6)
\[ \dot{q}_{k,\lambda} = \frac{\partial H}{\partial \pi_{k,\lambda}} = k\pi_{k,\lambda} \] \hspace{1cm} (B.7)

and
\[ \dot{\pi}_{k,\lambda} = -\frac{\partial H}{\partial q_{k,\lambda}} = -kq_{k,\lambda} + \dot{\vec{y}} \cdot \left( e \frac{\partial \vec{A}}{\partial q_{k,\lambda}} \right) \] \hspace{1cm} (B.8)

Although (B.5)–(B.8) are operator equations, we will in what follows only need mean values. We therefore can set all operators to zero at \( t = 0 \), reproducing the classical boundary condition of Appendix A. With that understanding, we carry out a Laplace transform. As before, we call
\[ A(\omega) = \int_0^\infty dt e^{i\omega t} A(t) \] \hspace{1cm} (B.9)

for any variable \( A(t) \). The resulting equations are
\[ -i\omega \vec{p}(\omega) = e\vec{E}(\omega) \] \hspace{1cm} (B.10)
\[ -i\omega \vec{y}(\omega) = \frac{\vec{p}(\omega) - e\vec{A}(\omega)}{m} \] \hspace{1cm} (B.11)
\[ -i\omega q_{k,\lambda} = k\pi_{k,\lambda} \] \hspace{1cm} (B.12)
\[ -i\omega \pi_{k,\lambda} = -kq_{k,\lambda} - i\omega \vec{y} \cdot \frac{e\vec{e}_k}{\sqrt{k}} \] \hspace{1cm} (B.13)

with solution
\[ \pi_{k,\lambda} = -\frac{i\omega}{k} q_{k,\lambda} \] \hspace{1cm} (B.14)
\[ q_{k,\lambda} = -\frac{e\sqrt{k}\vec{e}_{k,\lambda} \cdot \vec{y}}{i\omega(1 - k^2/\omega^2)} \] \hspace{1cm} (B.15)

and
\[ \vec{y} = -\frac{e\vec{E}}{\omega^2} \frac{1}{m_0 + \frac{2\alpha^2}{3} \sum_{k^2 - \omega^2} \frac{1}{k^2}} \] \hspace{1cm} (B.16)

or, with the introduction of the cut-off function,
\[ \vec{y}(\omega) = \frac{-e \vec{E}^\gamma(\omega)}{\omega^2 \left[ m_0 + \frac{2}{3} e^2 \int \frac{d\vec{k}}{(2\pi)^3} |f(\vec{k})|^2 \right]} , \]  
(B.17)

the same formula we found for the classical theory. Of course \( \vec{y}(\omega) \) here represents the mean value of the operator. Note finally that adding a harmonic binding potential does not remove the run-away pole.

**APPENDIX C: LEADING RADIATIVE CORRECTION**

The amplitude we must calculate is

\[ M^{\mu\nu}(p, q) = i \int d^4 y <p'|(j^\mu(0)j^{\nu}(y))_+ e^{i\vec{q} \cdot \vec{y}}|p> \]  
(C.1)

taken in the forward scattering limit, \( p' = p \), and evaluated by continuing \( q^0 = w \) to the upper half plane, as shown in Eq. (3.13).

In lowest order

\[ M^{\mu\nu} = M_0^{\mu\nu} = -e^2 \bar{u} \gamma^\mu \frac{1}{i \gamma \cdot (p + q) + m} \gamma^\nu u + \text{crossed term} \]  
(C.2)

This formula is equivalent to Eq. (2.31), which results from taking the \( q_i = 0 \), \( (\mu, \nu) = (i, j) \) limit of (C.2).

The possibility of a Landau like pole will appear in lowest order by the coherent positive addition to (C.2) of a term coming from the next order in \( \alpha \), and going like \( \log \omega^2 \) or \( \log \omega^2 \omega^2 \) at large \( \omega \).

The possible asymptotic covariant amplitudes are limited by the conservation law \( q_\nu M^{\nu\nu} = 0 \). Those that contribute to \( M^{\mu\nu} \) in the next order are

\[ T_1^{\mu\nu} = \bar{u} \frac{(\gamma^\mu i \gamma \cdot q \gamma^\nu - \gamma^\nu i \gamma \cdot q \gamma^\mu)}{q^2} u \]  
(C.3)

which goes like \( 1/\omega \) at large \( \omega \);

\[ T_2^{\mu\nu} = \frac{1}{E q^2} \left( p^\mu p^\nu - (p^\mu q^\nu + p^\nu q^\mu) \frac{q \cdot p}{q^2} + g^{\mu\nu} \frac{(p \cdot q)^2}{q^2} \right) \]  
(C.4)

and

\[ T_3^{\mu\nu} = \frac{m^2}{E q^2} \left( \frac{q^\mu q^\nu}{q^2} - g^{\mu\nu} \right) \]  
(C.5)

both of which go like \( 1/\omega^2 \) at large \( \omega \).

One finds easily, to order \( 1/\omega^2 \),

\[ M_0^{\mu\nu} = e^2 (T_1^{\mu\nu} + 2T_2^{\mu\nu}) \]  
(C.6)

The next order correction comes from the diagrams in Fig. 2 and Fig. 3. It is, again to order \( 1/\omega^2 \), and keeping only terms with a factor \( \log \omega^2 \) at large \( \omega \):
\[ \delta M^{\mu\nu} = \frac{2\alpha}{3\pi} \log q^2 M_0^{\mu\nu} - \frac{4\alpha}{3\pi} e^2 \log q^2 T_2^{\mu\nu} + \frac{2\alpha}{3\pi} e^2 \log q^2 T_3^{\mu\nu} . \] (C.7)

The first term in (4.7) comes from the Feynman diagrams in Fig. 3, which in a different context signal the Landau ghost.

Finally, we point out two curiosities that emerge on inspecting Eq. (C.7). First, since \( T_1 \sim 1/\omega \), we see that the leading term in \( 1/\omega \) has no first order logarithmic radiative correction, as implied earlier by Eq. (2.37). Second, we see that the radiative corrections to the \( 1/\omega^2 \) terms in the amplitude \( T_2^{\nu} \) coming from the diagram of Fig. 2 precisely cancel those coming from the Landau diagram of Fig. 3. Whether these are more than a numerical accident is not known to the author.