THE ZASSENHAUS FILTRATION, MASSEY PRODUCTS,
AND REPRESENTATIONS OF PROFINITE GROUPS

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Abstract. We consider the \( p \)-Zassenhaus filtration \((G_n)\) of a profinite group \( G \). Suppose that \( G = S/N \) for a free profinite group \( S \) and a normal subgroup \( N \) of \( S \) contained in \( S_\cap \). Under a cohomological assumption on the \( n \)-fold Massey products (which holds, e.g., if \( G \) has \( p \)-cohomological dimension \( \leq 1 \)), we prove that \( G_{n+1} \) is the intersection of all kernels of upper-triangular unipotent \((n+1)\)-dimensional representations of \( G \) over \( \mathbb{F}_p \). This extends earlier results by Mináč, Spira, and the author on the structure of absolute Galois groups of fields.

1. Introduction

Let \( p \) be a fixed prime number. The \( p \)-Zassenhaus filtration of a profinite group \( G \) is the fastest descending sequence of closed subgroups \( G_n, n = 1, 2, \ldots, \) of \( G \) such that \( G_1 = G, G_p \leq G_{ip}, \) and \([G_i, G_j] \leq G_{i+j}\) for \( i, j \geq 1 \). Thus \( G_n = G_{[n/p]} \prod_{i+j=n}[G_i, G_j] \) for \( n \geq 2 \). Here given closed subgroups \( H, K \) of \( G \), we write \([H, K] \) (resp., \( H^p \)) for the closed subgroup generated by all commutators \([\sigma, \tau] \) (resp., \( p \)-th powers \( \sigma^p \)), with \( \sigma \in H, \tau \in K \). This filtration has been studied since the middle of the 20th century both from the group-algebra and Lie algebra viewpoints ([Jen41], [Laz65], [Zas39], [DDMS, Ch. 11–12]), and had important Galois-theoretic applications, e.g., to the Golod–Šafarevič problem [Koc02, §7.7], mild groups, and Galois groups of restricted ramification ([Lab06], [LM11], [Mor04], [Vog05]).

In this paper we interpret the \( p \)-Zassenhaus filtration from the viewpoint of linear representations over \( \mathbb{F}_p \), and use this to explain and generalize several known facts on the structure of absolute Galois groups of fields. Our first main result characterizes the filtration for free pro-\( p \) groups:

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Theorem A. Let $\bar{S}$ be a free pro-$p$ group and $n \geq 1$. Then $\bar{S}_n$ is the intersection of all kernels of linear representations $\rho: \bar{S} \to \text{GL}_n(\mathbb{F}_p)$.

We may clearly replace here $\text{GL}_n(\mathbb{F}_p)$ by its $p$-Sylow subgroup $U_n(\mathbb{F}_p)$, consisting of all upper-triangular $n \times n$ unipotent matrices over $\mathbb{F}_p$. In this reformulation, Theorem A extends to general profinite groups $G$, whose defining relations lie sufficiently high in the $p$-Zassenhaus filtration, and which satisfy a certain cohomological condition, to be explained below.

Specifically, let $n \geq 2$, and suppose that $G$ can be presented as $S/N$ for a free profinite group $S$ and a closed normal subgroup $N$ of $S$ contained in $S_n$ (this can be slightly relaxed - see Remark 11.3). The cohomological condition we assume is on the $n$-fold Massey products in the mod-$p$ profinite cohomology ring $H^\bullet(G/G_n) = H^\bullet(G/G_n, \mathbb{Z}/p)$. These are maps $H^1(G/G_n)^n \to H^2(G/G_n)$, that generalize the usual cup product (which is essentially the case $n = 2$). According to their general construction (recalled in [3], the Massey products are multi-valued maps. Yet, it was shown by Vogel [Vog05] that in our situation they are well-defined maps (see [8]). Moreover, they are multi-linear. Let $H^2(G/G_n)_{n-Massey}$ be the subgroup of $H^2(G/G_n)$ generated by the image of this map. We say that $G$ satisfies the $n$-th Massey kernel condition if the kernel of the inflation map $\text{inf}: H^2(G/G_n)_{n-Massey} \to H^2(G)$ is generated by $n$-fold Massey products from $H^1(G/G_n)^n$. When $n = 1$ we declare the condition to be true by definition. We prove:

Theorem A'. If $G$ satisfies the $n$-th Massey kernel condition, then $G_{n+1}$ is the intersection of all kernels of representations $G \to U_{n+1}(\mathbb{F}_p)$.

The assumptions of Theorem A’ are satisfied when $G$ has $p$-cohomological dimension $\leq 1$ (Corollary [11,4]). Hence Theorem A is a special case of Theorem A’.

Theorem A’ is elementary for $n = 1$. For $n = 2, p = 2$ it was proved in [EM11a], generalizing a result of Minác and Spira [MSp96] (see also [NQD12]). For $n = 2, p > 2$ it was proved in [EM11b] Example 9.5(1)]. Moreover, it has the following remarkable Galois-theoretic application ([12]): Assume that $G = G_K$ is the absolute Galois group of a field $K$ containing a root of unity of order $p$. Then $G_3$ is the intersection of all normal open subgroups $M$ of $G$ such that $G/M$ embeds in $U_3(\mathbb{F}_p)$. The proof of this fact uses the Merkurjev–Suslin theorem [MS82]. Note that $U_3(\mathbb{F}_p)$ is either the dihedral group $D_4$ (when $p = 2$) or the Heisenberg group $H_{p^3}$ (when $p > 2$).
The proof of Theorem A’ is based on an alternative description of the \( p \)-Zassenhaus filtration in terms of Magnus algebra \( \mathbb{F}_p\langle\langle X_A \rangle\rangle \) of formal power series over \( \mathbb{F}_p \) in non-commuting variables \( X_a \), where \( a \) ranges over a basis \( A \) of \( S \). The Magnus homomorphism \( \Lambda: S \rightarrow \mathbb{F}_p\langle\langle X_A \rangle\rangle^\times \) is defined by \( a \mapsto 1 + X_a \) (see [5]). Then \( S_n \) is the preimage under \( \Lambda \) of the multiplicative group of all formal power series \( 1 + \sum_{|w| \geq n} c_w X_w \) (Proposition 6.2).

Behind the proof of Theorem A’ is also a key observation from [EM11b], relating such intersection results with duality principles between profinite groups and subgroups of second cohomology groups (see Proposition 6.1 for the precise statement). In our case we consider the subgroup \( H^2(S/S_n)_{n-\text{Massey}} \) of \( H^2(S/S_n) \) and prove:

**Theorem B.**  
(a) There is a natural perfect pairing  
\[ S_n/S_{n+1} \times H^2(S/S_n)_{n-\text{Massey}} \rightarrow \mathbb{Z}/p. \]

(b) There is a natural exact sequence  
\[ 0 \rightarrow H^2(S/S_n)_{n-\text{Massey}} \hookrightarrow H^2(S/S_n) \rightarrow H^2(S/S_{n+1}). \]

See Corollary 10.4 for a generalization of these facts to profinite groups \( G \) as above.

Finally, a method of Dwyer [Dwy75] expresses the central extensions corresponding to elements of \( H^2(S/S_n)_{n-\text{Massey}} \) by means of linear representations into \( U_n(\mathbb{Z}/p) \) (see [5], leading to Theorem A’.

Fundamental connections between the \( p \)-Zassenhaus filtration and Massey products were earlier observed and studied in the works of Morishita [Mor04, Mor12] and Vogel [Vog05]; see also [Sta65, §5]. In fact, in the case \( p = 2, n = 3 \), related connections were earlier studied in [GLMS03] under a different terminology (this was pointed out to me by Ján Mináč). Among the other recent important works on Massey products in Galois theory and arithmetic geometry are those by Sharifi [Sha07], Wickelgren, Hopkins ([Wic09], [Wic12], [HW12]), and Gärtnert [Gär12].

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2. Preliminaries on bilinear pairings

First we recall some terminology and facts on bilinear maps. We fix a commutative ring \( R \). A bilinear map \( (\cdot, \cdot): A \times B \rightarrow R \) of \( R \)-modules \( A \) and \( B \) is non-degenerate (resp., a perfect pairing) if the induced \( R \)-module homomorphisms \( A \rightarrow \text{Hom}_R(B, R) \) and \( B \rightarrow \text{Hom}_R(A, R) \) are
injective (resp., bijective). We say that a diagram of bilinear pairings and $R$-homomorphisms

\begin{equation}
\begin{array}{ccc}
A & \times & B \\
\downarrow & & \downarrow \\
A' & \times & B' \\
\end{array}
\end{equation}

commutes if $(\alpha(a'), b) = (a', \beta(b))'$ for every $a' \in A'$ and $b \in B$.

The proofs of the next two lemmas are straightforward:

**Lemma 2.1.** Let $(I, \leq)$ be a filtered set and for every $i \in I$ let $A_i \times B_i \to R$ be a perfect pairing. Let $\alpha_{ij} : A_j \to A_i$, $\beta_{ij} : B_i \to B_j$, for $i, j \in I$ with $i \leq j$, be homomorphisms which commute with the pairings, and are compatible in the natural sense. Then there is an induced perfect pairing $\lim \leftarrow \bigoplus A_i \times \lim \rightarrow \bigoplus B_i \to R$.

**Lemma 2.2.** Suppose that (2.1) commutes and the induced maps $A \to \text{Hom}_R(B, R)$, $B' \to \text{Hom}_R(A', R)$ are injective. Then (2.1) induces a non-degenerate bilinear map $\text{Im}(\alpha) \times \text{Im}(\beta) \to R$.

**Lemma 2.3.** Assume that every $R$-module is semi-simple. Suppose that the diagram (2.1) commutes, $(\cdot, \cdot)$ is non-degenerate, and the induced map $A' \to \text{Hom}_R(B', R)$ is surjective. Then (2.1) induces a non-degenerate bilinear map $\text{Coker}(\alpha) \times \text{Ker}(\beta) \to R$.

**Proof.** Set $A'' = \text{Coker}(\alpha)$ and $B'' = \text{Ker}(\beta)$. The existence of an induced well-defined bilinear map $A'' \times B'' \to R$ is straightforward. It is also immediate that the induced map $B'' \to \text{Hom}_R(A'', R)$ is injective.

Finally, there is a commutative diagram with exact rows

\begin{equation}
\begin{array}{ccc}
A' & \to & A \\
\downarrow & & \downarrow \\
\text{Hom}_R(B', R) & \to & \text{Hom}_R(B, R) \\
\downarrow & & \downarrow \\
\text{Hom}_R(B'', R) & \to & \text{Hom}_R(B', R)
\end{array}
\end{equation}

where the exactness at $\text{Hom}_R(B, R)$ is by the semi-simplicity. By the snake lemma, the right vertical map is injective. \hfill \Box

3. **Massey products**

Recall that a **differential $\mathbb{Z}$-graded algebra** (DGA) over a ring $R$ is a graded $R$-algebra $C^* = \bigoplus_{r \in \mathbb{Z}} C^r$ equipped with homomorphisms $\partial^r : C^r \to C^{r+1}$ such that $(C^*, \bigoplus_{r \in \mathbb{Z}} \partial^r)$ is a complex satisfying the Leibnitz rule $\partial^{r+s}(ab) = \partial^r(a)b + (-1)^r a \partial^s(b)$ for $a \in C^r$, $b \in C^s$. Let $H^i = \text{Ker}(\partial^i)/\text{Im}(\partial^{i-1})$
be the $i$-th cohomology group of $C^\bullet$. For an $i$-cocycle $c$ let $[c]$ be its cohomology class in $H^i$. Thus $[c] = [c']$ will mean that $c, c'$ are homogenous cocycles of equal degree and are cohomologous.

We fix an integer $n \geq 2$. We consider systems $c_{ij} \in C^1$, where $1 \leq i \leq j \leq n$ and $(i,j) \neq (1,n)$. For any $i,j$ satisfying $1 \leq i \leq j \leq n$ (including $(i,j) = (1,n)$) we may define

$$\tilde{c}_{ij} = - \sum_{r=i}^{j-1} c_{ir}c_{r+1,j} \in C^2.$$  

One says that $(c_{ij})$ is a **defining system of size** $n$ in $C^\bullet$ if $\tilde{c}_{ij} = \partial c_{ij}$ for every $1 \leq i \leq j \leq n$ with $(i,j) \neq (1,n)$. We also say that the defining system $(c_{ij})$ is **on** $c_{11}, \ldots, c_{nn}$. Note that then $c_{ii}$ is a 1-cocycle, $i = 1, 2, \ldots, n$. Further, $\tilde{c}_{1n}$ is a 2-cocycle (Fenn, Kraines).

**Lemma 3.1.** Suppose that for every cocycles $c_1, \ldots, c_n \in C^1$ there is a defining system of size $n$ on $c_1, \ldots, c_n$ in $C^\bullet$. Let $1 \leq k < n$. Then for every cocycles $c_1, \ldots, c_k \in C^1$ there is a defining system of size $k$ on $c_1, \ldots, c_k$ such that in addition $[\tilde{c}_{1k}] = 0$.

**Proof.** Choose arbitrary cocycles $c_{k+1}, \ldots, c_n \in C^1$ and a defining system $(c_{ij})$ on $c_1, \ldots, c_k, c_{k+1}, \ldots, c_n$. Then $(c_{ij}), 1 \leq i \leq j \leq k, (i,j) \neq (1,k)$, is a defining system on $c_1, \ldots, c_k$. Moreover, $\tilde{c}_{1k} = \partial c_{1k}$, so $[\tilde{c}_{1k}] = 0$. \qed

The main objective of this section is Proposition 3.3 below. Its proof will be based on the following fact from Fenn, Lemma 6.2.7, which is a variant of Kra66, Lemma 20] (with different sign conventions). We note that while the latter results are stated in a topological setting, their proofs are at the level of general DGAs. A self-contained exposition of these results, also for cocycles of higher degrees, as well as of the multi-linearity of the Massey product (see below), is given in Efr12.

**Proposition 3.2** (Fenn, Kraines). Assume that for every $1 \leq k < n$ and every defining system $(d_{ij})$ of size $k$ in $C^\bullet$ one has $[\tilde{d}_{1k}] = 0$. Let $(c_{ij}), (c'_{ij})$ be defining systems of size $n$ in $C^\bullet$ with $[c_{ii}] = [c'_{ii}], i = 1, 2, \ldots, n$. Then $[\tilde{c}_{1n}] = [\tilde{c}'_{1n}]$.

**Proposition 3.3.** Suppose that for every cocycles $c_1, \ldots, c_n \in C^1$ there is a defining system on $c_1, \ldots, c_n$ in $C^\bullet$. Then for any two defining systems $(c_{ij}), (c'_{ij})$ of size $n$ with $[c_{ii}] = [c'_{ii}], i = 1, 2, \ldots, n$, one has $[\tilde{c}_{1n}] = [\tilde{c}'_{1n}]$.

**Proof.** We argue by induction on $n$. For $n = 2$ this is immediate from the definition of $\tilde{c}_{12}, \tilde{c}'_{12}$.
For arbitrary \( n \), the assumption holds also for smaller values of \( n \), by Lemma 3.1. We obtain the last sentence of the proposition for smaller values of \( n \). We need to verify the assumption of Proposition 3.2. So let \( 1 < k < n \), and consider a defining system \((d_{ij})\) of size \( k \) in \( C^*\). Lemma 3.1 again yields a defining system \((d'_{ij})\) of size \( k \) on \( d_{11}, \ldots, d_{kk} \) such that in addition \([d'_{1k}] = 0\). Hence \([d_{1k}] = [d'_{1k}] = 0\), as required. \( \square \)

Under the assumptions of Proposition 3.3, define the Massey product

\[
\langle \cdot, \ldots, \cdot \rangle \colon H^1 \times \cdots \times H^1 \to H^2,
\]

as follows: given cocycles \( c_1, \ldots, c_n \in C^1 \) we take a defining system \((c_{ij})\) in \( C^*\) with \( c_{11} = c_1, \ldots, c_{nn} = c_n \). As remarked above, \( \tilde{c}_{1n} \) is a 2-cocycle. We set \( \langle [c_1], \ldots, [c_n] \rangle = [\tilde{c}_{1n}] \). By Proposition 3.3 this map is well-defined. Moreover, it is multi-linear ([Fen83 Lemma 6.2.4], [Efr12]).

4. Formal power series

Let \((A, \leq)\) be a set, considered as an alphabet, and \( A^*\) be the set of all finite words on \( A \). We write \( \emptyset \) for the empty word, and \(|w|\) for the length of the word \( w \).

We fix a commutative unital ring \( R \) and noncommuting variables \( X_a, a \in A \). The collection \( X_A \) of all formal expressions \( X_w = X_{a_1} \cdots X_{a_n} \), with \( w = (a_1, \ldots, a_n) \in A^* \), forms a monoid under concatenation. Let \( R\langle \langle X_A \rangle \rangle \) be the ring of all formal power series \( \sum_{w \in A^*} c_w X_w \), with \( c_w \in R \). We denote by \( R\langle \langle X_A \rangle \rangle^\times \) its multiplicative group.

For a positive integer \( n \) let \( V_{n,R} \) be the subset of \( R\langle \langle X_A \rangle \rangle^\times \) consisting of all power series \( \sum_w c_w X_w \) such that \( c_\emptyset = 1 \) and \( c_w = 0 \) for every \( w \) with \( 1 \leq |w| < n \). Note that any \( 1 + \alpha \in V_{n,R} \) has an inverse \( \sum_{k=0}^{\infty} (-1)^k \alpha^k \) in \( V_{n,R} \), so \( V_{n,R} \) is a subgroup of \( R\langle \langle X_A \rangle \rangle^\times \). We write \( V_{n,R}^m \) for the subgroup of \( V_{n,R} \) generated by all \( m \)-th powers, and \([V_{n,R}, V_{k,R}]\) for the subgroup of \( V_{k,R} \) generated by all commutators \([\gamma, \delta] = \gamma^{-1} \delta^{-1} \gamma \delta \), with \( \gamma \in V_{n,R}, \delta \in V_{k,R} \).

**Lemma 4.1.**

(a) \([V_{n,R}, V_{k,R}] \leq V_{n+k,R};\)

(b) If \( mR = 0 \), then \( V_{n,R}^m \leq V_{n+1,R} \).

**Proof.** For (a) let \( 1 + \alpha \in V_{n,R} \) and \( 1 + \beta \in V_{k,R} \). Then

\[
[1 + \alpha, 1 + \beta] = (1 + \alpha + \alpha^2 - + \cdots) (1 + \beta + \beta^2 - + \cdots) (1 + \alpha)(1 + \beta) \in 1 + R\langle \langle X_A \rangle \rangle \alpha \beta R\langle \langle X_A \rangle \rangle + R\langle \langle X_A \rangle \rangle \beta \alpha R\langle \langle X_A \rangle \rangle \leq V_{n+k,R}.
\]

For (b), use the (non-commutative) binomial formula. \( \square \)
5. Free profinite groups

We recall from [FJ08, §17.4] the following terminology and facts on free profinite groups.

Let $G$ be a profinite group and $A$ a set. A map $\varphi: A \to G$ converges to 1 if for every open normal subgroup $M$ of $G$, the set $A \setminus \varphi^{-1}(M)$ is finite.

Let $S$ be a profinite group. We say that $S$ is a free profinite group on basis $A$ with respect to a map $\iota: A \to S$ if

(i) $\iota: A \to S$ converges to 1 and $\iota(A)$ generates $S$;
(ii) For every profinite group $G$ and a continuous map $\varphi: A \to G$ converging to 1, there is a unique continuous homomorphism $\hat{\varphi}: S \to G$ with $\varphi = \hat{\varphi} \circ \iota$ on $A$.

A free profinite group on $A$ exists, and is unique up to a continuous isomorphism. We denote it by $S_A$. Necessarily, $\iota$ is injective, and we identify $A$ with its image in $S_A$. The group $S_A$ is projective, whence has cohomological dimension $\leq 1$ [NSW08, Cor. 3.5.16].

Now let $R$ be a profinite unital ring. The map $\sum_w c_w X_w \mapsto (c_w)_w$ identifies $R\langle\langle X_A \rangle\rangle$ with $R^A$. This induces on the additive group of $R\langle\langle X_A \rangle\rangle$ a profinite topology. Moreover, the multiplication map in $R\langle\langle X_A \rangle\rangle$ is continuous, making it a profinite topological ring.

Now take $A$ finite. The ring $R\langle\langle X_A \rangle\rangle$ is profinite ring, so $R\langle\langle X_A \rangle\rangle^\times$ is a profinite group. Hence the map $a \mapsto 1 + X_a$, $a \in A$, extends to the (continuous) Magnus homomorphism $\Lambda_{S_A,R}: S_A \to R\langle\langle X_A \rangle\rangle^\times$.

This generalizes to arbitrary $A$ as follows: Let $B$ be a finite subset of $A$. The map $\varphi: A \to S_B$, given by $a \mapsto a$ for $a \in B$, and $a \mapsto 1$ for $a \in A \setminus B$, converges to 1. It extends to a unique continuous epimorphism $S_A \to S_B$. Also, there is a continuous $R$-algebra epimorphism $R\langle\langle X_A \rangle\rangle \to R\langle\langle X_B \rangle\rangle$, given by $X_b \mapsto X_b$ for $b \in B$ and $X_a \mapsto 0$ for $a \in A \setminus B$. Then

$$S_A = \lim_{\leftarrow} S_B, \quad R\langle\langle X_A \rangle\rangle = \lim_{\leftarrow} R\langle\langle X_B \rangle\rangle, \quad R\langle\langle X_A \rangle\rangle^\times = \lim_{\leftarrow} R\langle\langle X_B \rangle\rangle^\times,$$

where $B$ ranges over all finite subsets of $A$. We now define $\Lambda_{S_A,R}: S_A \to R\langle\langle X_A \rangle\rangle^\times$ to be the inverse limit of the maps $\Lambda_{S_B,R}$. Thus $\Lambda_{S_A,R}(a) = 1 + X_a$ for $a \in A$.

From now on we abbreviate $S = S_A$. For $\sigma \in S$ we set

$$\Lambda_{S,R}(\sigma) = \sum_{w \in A^*} \epsilon_{w,R}(\sigma) X_w$$

with $\epsilon_{w,R}(\sigma) \in R$. Thus $\epsilon_{\emptyset,R}(\sigma) = 1$. 

For a positive integer $n$ let $S_{n,R} = \Lambda_{S,R}^{-1}(V_{n,R})$. It is closed subgroup of $S$, and $S_{1,R} = S$. Also, $S_{n,R} = \varprojlim (S_B)_{n,R}$, with $B$ ranging over all finite subsets of $A$. The next lemma records a few well-known facts.

**Lemma 5.1.** Let $w \in A^*$.

(a) For $\sigma, \tau \in S$ one has $\epsilon_{w,R}(\sigma\tau) = \sum \epsilon_{w_1,R}(\sigma)\epsilon_{w_2,R}(\tau)$, where the sum is over all $w_1, w_2 \in A^*$ with $w = w_1w_2$.

(b) For $n = |w|$, the restriction $\epsilon_{w,R}: S_{n,R} \to R$ is a homomorphism.

(c) For $a \in A$, the map $\epsilon_{a,R}: S \to R$ is a homomorphism.

(d) $\epsilon_{a,R}(a) = 1$ for $a \in A$ and $\epsilon_{a,R}(a') = 0$ for $a, a' \in A$ distinct.

**Proof.** (a) is a restatement of $\Lambda_{S,R}(\sigma\tau) = \Lambda_{S,R}(\sigma)\Lambda_{S,R}(\tau)$. (b) follows from (a), and (c) is a special case of (b). (d) is immediate from the definition. □

6. The filtration $S_{n,Z/m}$

Let $S = S_A$ be as before, and assume that $R = \mathbb{Z}/m$ for an integer $m \geq 2$. Lemma 4.1 implies that

$$S_{n,Z/m}[S, S_{n,Z/m}] \leq S_{n+1,Z/m}. \tag{6.1}$$

For $n = 1$, this is an equality:

**Lemma 6.1.** $S^m[S, S] = S_{2,Z/m}$.

**Proof.** By (6.1), $S^m[S, S] \leq S_{2,Z/m}$.

For the converse, we may use an inverse limit argument to assume that $A$ is finite. Let $\bar{S} = S/S^m[S, S]$ and let $\bar{\Lambda}_{S,Z/m}: \bar{S} \to V_{1,Z/m}/V_{2,Z/m}$ be the homomorphism induced by $\Lambda_{S,Z/m}$. Use Lemma 4.1 to obtain that $V_{1,Z/m}/V_{2,Z/m}$ is a free $\mathbb{Z}/m$-module on generators $(1 + X_a)V_{2,Z/m}$, $a \in A$. Since $\bar{S}$ is abelian of exponent $m$, we may define a homomorphism $\lambda: V_{1,Z/m}/V_{2,Z/m} \to \bar{S}$ by mapping this generator to the image $\bar{a}$ of $a$ in $\bar{S}$. Then $(\lambda \circ \bar{\Lambda}_{S,Z/m})(\bar{a}) = \bar{a}$ for $a \in A$, implying that $\lambda \circ \bar{\Lambda}_{S,Z/m} = \text{id}_S$ and $\bar{\Lambda}_{S,Z/m}$ is injective. But $S_{2,Z/m}/S^m[S, S]$ is mapped trivially by $\bar{\Lambda}_{S,Z/m}$, and therefore is trivial. □

Now let $m = p$ prime. For a profinite group $G$ let $I_p(G)$ be the augmentation ideal in the complete group ring $\mathbb{F}_p[[G]]$, i.e., the closed ideal generated by all elements $g - 1$, with $g \in G$. When $G$ is finite (whence discrete), a theorem of Jennings and Brauer ([Jen41 Th. 5.5], [DDMS Th. 12.9]) identifies $G \cap (1 + I_p(G)^n)$ with the $n$th term $G_n$ in the Zassenhaus filtration of $G$, $n \geq 1$. An inverse limit argument extends this to arbitrary profinite groups $G$. 


The map \( 1 + \sum_{|w| \geq n} c_w X_w \mapsto (c_w)_{|w|=n} \) induces a group isomorphism \( V_{n,F_p}/V_{n+1,F_p} \xrightarrow{\sim} \prod_{w \in A, |w|=n} F_p \). When \( A \) is finite, \( V_{1,F_p}/V_{n,F_p} \) is therefore a finite \( p \)-group for every \( n \), so \( V_{1,F_p} = \lim_{\leftarrow} V_{1,F_p}/V_{n,F_p} \) is a pro-\( p \) group.

**Proposition 6.2.** \( S_n = S_{n,F_p} \).

*Proof.* By an inverse limit argument, we may assume that the basis \( A \) is finite. Let \( \bar{S} \) be the maximal pro-\( p \)-quotient of \( S \). It is a free pro-\( p \)-group. By the definition of the Zassenhaus filtration, \( S/S_n \) has a \( p \)-power exponent. Hence the preimage of \( \bar{S}_n \) under the epimorphism \( S \to \bar{S} \) is \( S_n \).

Also, \( \Lambda_{S,F_p} \) breaks via a homomorphism \( \Lambda_{S,F_p} : \bar{S} \to V_{1,F_p} \). It extends to a continuous \( F_p \)-algebra homomorphism \( \Lambda_{S,F_p} : F_p[[\bar{S}]] \to F_p[[\langle X_A \rangle]] \), which by [Koc02, Th. 7.16], is an isomorphism. Let \( J = V_{1,F_p} - 1 \) be the ideal of all power series in \( F_p[[\langle X_A \rangle]] \) with constant term 0. Using the identity \( \sigma \tau - 1 = (\sigma - 1)(\tau - 1) + (\sigma - 1) + (\tau - 1) \) for \( \sigma, \tau \in \bar{S} \) we see that \( \Lambda_{S,F_p} \) maps \( I_{F_p} (\bar{S}) \) onto \( J \). Therefore it maps \( 1 + I_{F_p} (\bar{S})^n \) bijectively onto \( 1 + J^n = V_{n,F_p} \).

By the Jennings–Brauer theorem, \( S_n \) is therefore the preimage of \( V_{n,F_p} \) in \( \bar{S} \) under \( \Lambda_{S,F_p} \). We conclude that \( S_n \) is the preimage of \( V_{n,F_p} \) in \( S \) under \( \Lambda_{S,F_p} \), i.e., \( S_n = S_{n,F_p} \). \( \square \)

**Lemma 6.3.** Let \( c_1, \ldots, c_n : S = S_A \to F_p \) be continuous homomorphisms and \( B = \{ b_1, \ldots, b_n \} \) a set of size \( n \). There is a continuous homomorphism \( \phi : S_A \to S_B \) such that \( c_i F_{(b_i),F_p} \circ \phi = c_i, \) \( i = 1, 2, \ldots, n \), where \( c_i \) denotes the Magnus coefficient with respect to \( S_B \).

*Proof.* An inverse limit argument reduces this to the case where \( A \) is finite. For every \( a \in A \) and \( 1 \leq i \leq n \) choose \( \hat{c}_i(a) \in \mathbb{Z} \) with \( c_i(a) = \hat{c}_i(a) \pmod{p\mathbb{Z}} \). The map \( A \to S_B, a \mapsto \prod_{i=1}^n b_i^{\hat{c}_i(a)} \), extends to a continuous homomorphism \( \phi : S_A \to S_B \). Then \( \epsilon_{(b_i),F_p} (\phi(a)) = c_i(a) \) for every \( a \in A \) and \( 1 \leq i \leq n \), and the assertion follows. \( \square \)

7. Unipotent Matrices

Consider integers \( n \geq 2 \) and \( d \geq 0 \), and let \( R \) be a (discrete) finite ring. Let \( T_{n,d}(R) \) be the set of all \( n \times n \) matrices \( (a_{ij}) \) over \( R \) with the \((1,n)\) entry omitted and such that \( a_{ij} = 0 \) for \( j - i \leq d - 1 \) (in particular, \((a_{ij})\) is upper-triangular). It is an \( R \)-algebra with respect to the standard operations. Note that \( T_{n,d}T_{n,d'} \subseteq T_{n,d+d'} \). Furthermore, for every entry \((i,j) \neq (1,n)\) we have \( j - i \leq n - 2 \). Hence \( T_{n,d} = \{0\} \) for \( n - 1 \leq d \). We denote the \( n \times n \) identity matrix with the \((1,n)\) entry omitted by \( I_n \).

Let \( U_n(R) \) be the group of all upper-triangular unipotent \( n \times n \) matrices over \( R \). Let \( \bar{U}_n(R) = I_n + T_{n,1}(R) \) be the group of all unipotent (punctured)
matrices in $T_{n,0}$. Let $\pi : U_n(R) \to \bar{U}_n(R)$ be the obvious forgetful epimorphism. Its kernel consists of all matrices in $U_n(R)$ which are zero except for the main diagonal and at the entry $(1, n)$, and is therefore isomorphic to the additive group of $R$. We obtain a central extension of groups

$$0 \to R \to U_n(R) \xrightarrow{\pi} \bar{U}_n(R) \to 1.$$ 

We endow $U_n(R), \bar{U}_n(R)$ with the discrete topologies.

For the rest of this section we set $S = S_A$ and $R = \mathbb{Z}/m$, with $m \geq 2$.

**Proposition 7.1.** Every continuous homomorphism $\gamma : S \to \bar{U}_n(R)$ is trivial on $S_{n-1,R}$.

**Proof.** An inverse limit argument reduces this to the case where $A$ is finite. For $w = (a_1, \ldots, a_d) \in A^*$ we set $M_w = \prod_{j=1}^d (\gamma(a_j) - I_n) \in T_{n,d}$ (where $M_{\emptyset} = I_n$). Since $T_{n,d} = \{0\}$ for $n - 1 \leq d$, we may therefore define a unital $R$-algebra homomorphism $h : R\langle\langle X_A \rangle\rangle \to T_{n,0}(R)$ by

$$h\left(\sum_{w \in A^*} c_w X_w\right) = \sum_{\substack{w \in A^* \mid |w| \leq n-2}} c_w M_w.$$ 

Note that $h$ is continuous with respect to the profinite topology on $R\langle\langle X_A \rangle\rangle$ and the discrete topology on $T_{n,0}(R)$. The restriction of $h$ to $V_{1,R}$ is a continuous group homomorphism $h : V_{1,R} \to U_n(R) \subseteq T_{n,0}(R)$. One has $\gamma = h \circ \Lambda_{S,R}$ on $A$, whence on $S$. For $\sigma \in S_{n-1,R}$ this gives $\gamma(\sigma) = (h \circ \Lambda_{S,R})(\sigma) = I_n$. \qed

**Corollary 7.2.** Every continuous homomorphism $\gamma : S \to U_n(R)$ is trivial on $S_{n,R}$.

**Proof.** Embed $U_n(R)$ in $\bar{U}_{n+1}(R)$ and use Proposition 7.1. \qed

Given a continuous homomorphism $\bar{\gamma} : S/S_{n-1,R} \to \bar{U}_n(R)$ we write $U_n(R) \times_{\bar{U}_n(R)} (S/S_{n-1,R})$ for the fiber product with respect to $\pi$ and $\bar{\gamma}$.

**Lemma 7.3.** Let $N$ and $M_0$ be closed normal subgroup of $S$ such that $N \leq S_{n-1,R} \cap M_0$. The following conditions are equivalent:

1. there exist a continuous homomorphism $\bar{\gamma} : S/S_{n-1,R} \to \bar{U}_n(R)$ and a continuous homomorphism

$$\hat{\Phi} : S/N \to U_n(R) \times_{\bar{U}_n(R)} (S/S_{n-1,R})$$ 

which commutes with the projections to $S/S_{n-1,R}$, and such that $M_0/N = \text{Ker}(\hat{\Phi})$.

2. there exists a continuous homomorphism $\Phi : S/N \to U_n(R)$ such that $M_0/N = \text{Ker}(\Phi) \cap (S_{n-1,R}/N)$;
There is a closed normal subgroup $M$ of $S$ containing $N$ such that $S/M$ embeds in $U_n(R)$ and $M_0 = M \cap S_{n-1,R}$.

Proof. (1)$\Rightarrow$(2): For $i = 1, 2$ let $pr_i$ be the projection on the $i$-th coordinate of the fiber product, and $pr : S \to S/N$ the natural map. For $\Phi$ as in (1), we set $\Phi = pr_1 \circ \hat{\Phi}$. We get a commutative diagram:

\[
\begin{array}{ccc}
S & \xrightarrow{pr} & S/N \\
& \swarrow_{\Phi} & \\
U_n(R) \times_{U_n(R)} (S/S_{n-1,R}) & \xrightarrow{pr_2} & S/S_{n-1,R} \\
& \downarrow^{pr_1} & \\
U_n(R) & \xrightarrow{\pi} & \bar{U}_n(R).
\end{array}
\]

Further, $M_0/N = \text{Ker}(\hat{\Phi}) = \text{Ker}(\Phi) \cap (S_{n-1,R}/N)$.

(2)$\Rightarrow$(1): Given $\Phi$ as in (2), the homomorphism $\pi \circ \Phi \circ pr : S \to \bar{U}_n(R)$ factors via a continuous homomorphism $\bar{\gamma} : S/S_{n-1,R} \to \bar{U}_n(R)$, by Proposition 7.1. Thus the outer part of the diagram above commutes. The universal property of the fiber product yields a continuous homomorphism $\hat{\Phi}$ making the two triangles commutative. We have

\[M_0/N = \text{Ker}(\Phi) \cap (S_{n-1,R}/N) = \text{Ker}(\hat{\Phi}).\]

(2)$\Leftrightarrow$(3): Take $\text{Ker}(\Phi) = M/N$. \(\square\)

One has the following important connection between homomorphisms as discussed above and words:

Lemma 7.4. Let $w = (a_1, \ldots, a_n) \in A^*$. Define maps $\gamma_1, \gamma_2 : S \to U_{n+1}(R)$ by

\[\gamma_1(\sigma)_{ij} = \epsilon(a_i, \ldots, a_{j-1}, R)(\sigma), \quad \gamma_2(\sigma)_{ij} = (-1)^{j-i} \epsilon(a_i, \ldots, a_{j-1}, R)(\sigma)\]

for $\sigma \in S$ and $i < j$ (the other entries being obvious). Then $\gamma_1, \gamma_2$ are continuous group homomorphisms.

Proof. For $\gamma_1$ use Lemma 5.1(a). For $\gamma_2$ observe that the map $(a_{ij}) \mapsto ((-1)^{j-i}a_{ij})$ is an automorphism of $U_{n+1}(R)$, and compose it with $\gamma_1$. \(\square\)

8. Massey products for inhomogenous cochains

Let $G$ be a profinite group which acts trivially and continuously on the unital finite (discrete) ring $R$. The complex $(C^\bullet(G, R), \partial)$ of continuous inhomogenous $G$-cochains into the additive group of $R$, and $C^\infty(G, R) = 0$.
for $r < 0$, is a DGA with the cup product \cite[Ch. I, §2]{NSW08}. We recall that for $c \in C^1(G, R)$ and $\sigma, \tau \in G$ one has $(\partial c)(\sigma, \tau) = c(\sigma) + c(\tau) - c(\sigma \tau)$. Thus the 1-cocycles are the continuous homomorphisms $c: G \to R$. We now focus on defining systems in $C^\bullet(G, R)$.

As observed by Dwyer \cite[§2]{Dwy75} in the discrete context, one may view defining systems of size $n$ in $C^\bullet(G, R)$ as continuous homomorphisms $G \to \check{U}_{n+1}(R)$, as follows. Define a bijection between the systems of 1-cochains $c_{ij} \in C^1(G, R)$, $1 \leq i \leq j \leq n$, $(i, j) \neq (1, n)$, and the continuous maps $\bar{\gamma}: G \to \check{U}_{n+1}(R)$ by

$$\bar{\gamma}(\sigma)_{ij} = (-1)^{j-i} c_{i,j-1}(\sigma)$$

for $\sigma \in G$ and $1 \leq i < j \leq n + 1$, $(i, j) \neq (1, n + 1)$ (where the other entries are obvious). Under this bijection one has for $\sigma, \tau \in G$,

$$\tilde{c}_u(\sigma, \tau) = - \sum_{r=i}^{l-1} c_r(\sigma)c_{r+1,l}(\tau) = (-1)^{l-i} \sum_{k=i+1}^l \bar{\gamma}(\sigma)_{ik}\bar{\gamma}(\tau)_{k,l+1}. \quad (8.1)$$

**Lemma 8.1.** $\bar{\gamma}$ is a homomorphism if and only if $(c_{ij})$ is a defining system of size $n$ for $C^\bullet(G, R)$.

**Proof.** The map $\bar{\gamma}$ is a homomorphism if and only if for every $\sigma, \tau \in G$ and $1 \leq i \leq l \leq n$ with $(i, l) \neq (1, n)$,

$$\bar{\gamma}(\sigma\tau)_{i,l+1} = \bar{\gamma}(\sigma)_{i,l+1} + \sum_{k=i+1}^l \bar{\gamma}(\sigma)_{ik}\bar{\gamma}(\tau)_{k,l+1} + \bar{\gamma}(\sigma)_{i,l+1}.$$

By (8.1), this means that $c_u(\sigma\tau) = c_u(\tau) - \tilde{c}_u(\sigma, \tau) + c_u(\sigma)$. Equivalently, $\tilde{c}_u(\sigma, \tau) = (\partial c_u)(\sigma, \tau)$, i.e., $(c_{ij})$ is a defining system. \qed

**Remark 8.2.** The same formula gives a bijection between the systems $c_{ij} \in C^1(G, R)$, $1 \leq i \leq j \leq n$, and the continuous maps $\gamma: G \to U_{n+1}(R)$. Moreover, $\gamma$ is a homomorphism if and only if $(c_{ij})$ is a defining system such that in addition $\tilde{c}_1 = \partial c_1$.

The following fact (with different sign conventions) is stated without a proof in \cite[p. 182, Remark]{Dwy75}; see also \cite[§2.4]{Wic12}.

**Proposition 8.3.** Let $\bar{\gamma}: G \to \check{U}_{n+1}(R)$ correspond to a defining system $(c_{ij})$ as above. The central extension associated with $(-1)^{n-1}\tilde{c}_1$ is

$$0 \to R \to U_{n+1}(R) \times \check{U}_{n+1}(R) G \to G \to 1,$$

where the fiber product is with respect to $\pi$ and $\bar{\gamma}$.
Proof. Since \((-1)^{n-1}c_{1n}\) is a 2-cocycle, \(B = R \times G\) is a group with respect to the product \((r, \sigma) \ast (s, \tau) = (r + s + (-1)^{n-1}c_{1n}(\sigma, \tau), \sigma \tau)\). Then the central extension corresponding to \((-1)^{n-1}c_{1n}\) is [NSW08, Th. 1.2.4]
\[
0 \to R \to B \to G \to 1.
\]

The map \(h: U_{n+1}(R) \times \hat{U}_{n+1}(R) \to B, ((a_{ij}), \sigma) \mapsto (a_{1,n+1}, \sigma)\), is a bijection commuting with the projections to \(G\). To show that \(h\) is a homomorphism, take \(((a_{ij}), \sigma), (b_{ij}), \tau) \in U_{n+1}(R) \times \hat{U}_{n+1}(R) \to B\). Thus
\[
a_{ij} = \bar{\gamma}(\sigma)_{ij} = (-1)^{j-i}c_{i,j-1}(\sigma), \quad b_{ij} = \bar{\gamma}(\tau)_{ij} = (-1)^{j-i}c_{i,j-1}(\tau)
\]
for \(1 \leq i < j \leq n + 1\), \((i, j) \neq (1, n + 1)\). By (8.1), \(\sum_{k=2}^{n} a_{1k}b_{k,n+1} = (-1)^{n-1}c_{1n}(\sigma, \tau)\). Hence
\[
h(((a_{ij}), \sigma)(b_{ij}), \tau)) = \left(\sum_{k=1}^{n+1} a_{1k}b_{k,n+1}, \sigma \tau\right)
= \left(a_{1,n+1} + b_{1,n+1} + (-1)^{n-1}c_{1n}(\sigma, \tau), \sigma \tau\right) = h\left(((a_{ij}), \sigma)\right) \ast h\left(((b_{ij}), \tau)\right).
\]

For the rest of the paper we set again \(R = \mathbb{Z}/m\). Let \(S = S_A\) and let \(N\) be a normal closed subgroup of \(S\) contained in \(S_{n,\mathbb{Z}/m}\), with \(n \geq 2\).

**Proposition 8.4.** Given continuous homomorphisms \(c_1, \ldots, c_n: S \to \mathbb{Z}/m\), there is a defining system \((\bar{c}_{ij})\) of size \(n\) in \(C^\bullet(S/N, \mathbb{Z}/m)\) with \(c_i = \inf_S(\bar{c}_i), i = 1, 2, \ldots, n\).

Proof. Let \(S_B\) be a free profinite group on a set \(B = \{b_1, b_2, \ldots, b_n\}\) of \(n\) elements. We write \(\epsilon^B_{u,\mathbb{Z}/m}\) for the corresponding Magnus coefficients. Lemma 6.3 yields a continuous homomorphism \(\phi: S \to S_B\) such that \(\epsilon^B_{b_i,\mathbb{Z}/m} \circ \phi = c_i, i = 1, 2, \ldots, n\). We define a map \(\gamma': S_B \to U_{n+1}(\mathbb{Z}/m)\) by \(\gamma'(\sigma')_{ij} = (-1)^{j-i}\epsilon'_{(b_i, \ldots, b_{j-1}),\mathbb{Z}/m}(\sigma')\) for \(i < j\) and \(\sigma' \in S_B\). By Lemma 7.4 \(\gamma'\) is a continuous homomorphism. The composition \(\gamma = \pi \circ \gamma' \circ \phi: S \to \hat{U}_{n+1}(\mathbb{Z}/m)\) is also a continuous homomorphism. By Proposition 7.7 it factors via a continuous homomorphism \(\bar{\gamma}: S/N \to \hat{U}_{n+1}(\mathbb{Z}/m)\):
Let \((\tilde{c}_{ij})\) be the defining system of size \(n\) on \(C^\bullet(S/N, \mathbb{Z}/m)\) associated with \(\tilde{\gamma}\), in the sense of Lemma 8.1. Then for \(\sigma \in S\) and \(1 \leq i \leq n\),
\[
(\inf_S(\tilde{c}_{ii})(\sigma)) = -\gamma_{i,i+1}(\sigma) = -\gamma_{i,i+1}'(\phi(\sigma)) = \epsilon_{(b_i), \mathbb{Z}/m}^\Phi(\phi(\sigma)) = c_i(\sigma). \quad \square
\]

For a profinite group \(G\) acting trivially on \(\mathbb{Z}/m\), let \(H^i(G) = H^i(G, \mathbb{Z}/m)\) be the \(i\)-th cohomology group corresponding to the DGA \(C^\bullet(G, \mathbb{Z}/m)\) over \(\mathbb{Z}/m\). In view of Proposition 8.4, the assumption of Proposition 3.3 is satisfied for \(C^\bullet(S/N, \mathbb{Z}/m)\). Consequently, as explained in [3] there is a well-defined Massey product
\[
\langle \cdot, \ldots, \cdot \rangle: H^1(S/N)^n \to H^2(S/N).
\]

This was earlier shown using a different method by Vogel [Vog05 Th. A3] for \(m = p\) prime. We write \(H^2(S/N)_{n-\text{Massey}}\) for the image of this map.

**Example 8.5.** For \(n = 2\) and \(\chi_1, \chi_2 \in H^1(S/N)\) we have by construction \(\langle \chi_1, \chi_2 \rangle = -\chi_1 \cup \chi_2 \in H^2(S/N)\), where \(\cup\) denotes the cup product. Thus in the terminology of [CEM12], \(H^2(S/N)_{2-\text{Massey}} = H^2(S/N)_{\text{dec}}\).

For \(a \in A\) we may consider \(\epsilon(a), z/m \in H^1(S)\) also as an element of \(H^1(S/N)\) (see Lemma 6.1). With this convention we have

**Lemma 8.6.** The Massey products \(\psi_w = \langle \epsilon(a_1), z/m, \ldots, \epsilon(a_n), z/m \rangle\), where \(w = (a_1, \ldots, a_n) \in A^*\), generate \(H^2(S/N)_{n-\text{Massey}}\).

**Proof.** In view of Lemma 5.1(d), \(\epsilon(a), z/m\), where \(a \in A\), generate \(H^1(S/N) = \text{Hom}(S/N, \mathbb{Z}/m)\). Now use the multi-linearity of the Massey product. \(\square\)

### 9. Cohomological duality

Let \(G\) be a profinite group acting trivially on \(\mathbb{Z}/m\) and let \(N\) be a closed normal subgroup of \(G\). One has the 5-term exact sequence for the lower cohomology groups with coefficients in \(\mathbb{Z}/m\) [NSW08 Prop. 1.6.7]:

\[
0 \to H^1(G/N) \xrightarrow{\inf} H^1(G) \xrightarrow{\text{res}} H^1(N)^G \xrightarrow{\text{trg}} H^2(G/N) \xrightarrow{\inf} H^2(G).
\]

When \(N \leq G^m[G, G]\), the inflation map \(\inf: H^1(G/N) \to H^1(G)\) is surjective, so the transgression map \(\text{trg}\) identifies \(H^1(N)^G\) with \(\text{Ker}(H^2(G/N) \xrightarrow{\inf} H^2(G))\). Therefore [EM11a Cor. 2.2] gives a non-degenerate bilinear map

\[
(\cdot, \cdot)': N/N^m[G, N] \times \text{Ker}(H^2(G/N) \xrightarrow{\inf} H^2(G)) \to \mathbb{Z}/m,
\]

\[
(\sigma N^m[G, N], \alpha)' = (\text{trg}^{-1}(\alpha))(\sigma).
\]

We will need the following result from [EM11b Prop. 3.2]. Note that while it is stated for \(m\) a prime power, this is not needed in its proof.
Proposition 9.1. Let $T, T_0$ be closed normal subgroups of $G$ such that $T^m[G, T] \leq T_0 \leq T \leq G^m[G, G]$. Let $H$ be a subgroup of $H^2(G/T)$ and $H_0$ a set of generators of $H \cap \text{trg}(H^1(T)^G) = \text{Ker}(\inf : H \to H^2(G))$. The following conditions are equivalent:

(a) $(\cdot, \cdot)'$ (for $N = T$) induces a non-degenerate bilinear map
$$T/T_0 \times \text{Ker}(H \xrightarrow{\inf} H^2(G)) \to \mathbb{Z}/m;$$
(b) there is an exact sequence
$$0 \to \text{Ker}(H^2(G/T) \xrightarrow{\inf} H^2(G/T_0)) \to H \xrightarrow{\inf} H^2(G);$$
(c) $T_0 = \bigcap \text{Ker}(\Psi)$, with $\Psi$ ranging over all homomorphisms with a commutative diagram

$$\begin{array}{c}
\Psi \\
G \\
\omega \\
\end{array} \xymatrix{ 0 \ar[r] & \mathbb{Z}/m \ar[r] & C \ar[r] & G/T \ar[r] & 1, }
$$

where $\omega$ is a central extension associated with some element of $H_0$.

We now restrict ourselves to the case where $G$ is a free profinite group $S = S_A$. Let $N$ be a normal subgroup of $S$ contained in $S_{n,\mathbb{Z}/m}$, where $n \geq 2$. Then $N \leq S_{2,\mathbb{Z}/m} = S^m[S, S]$, by Lemma 6.1. As $H^2(S) = 0$, we obtain a non-degenerate bilinear map

$$(9.1) \quad (\cdot, \cdot)': N/N^m[S, N] \times H^2(S/N) \to \mathbb{Z}/m.$$

Variants of the following fundamental fact were proved by Dwyer [Dwy75 Prop. 4.1], Fenn and Sjerve [FS84 Th. 6.6], Morishita [Mor04 Cor. 2.2.3], Vogel [Vog05 Th. A3] and Wickelgren [Wic09 Prop. 2.3.7]. We prove it here in our terminology and setup. Let $\psi_w$ be as in Lemma 8.6.

Theorem 9.2. For $\sigma \in N$ and $w \in A^*$ one has $(\bar{\sigma}, \psi_w)' = \epsilon_{w,\mathbb{Z}/m}(\sigma)$.

Proof. Let $w = (a_1, \ldots, a_n)$. Lemma 7.1 gives a continuous homomorphism $\gamma: S \to U_{n+1}(\mathbb{Z}/m)$, where $\gamma(\sigma)_{ij} = (-1)^{j-i} \epsilon(a_{i-1}, a_{j-1}, a_j, a_i) \mathbb{Z}/m(\sigma)$ for $\sigma \in S$, $i < j$. By Proposition 7.1, it induces a continuous homomorphism $\bar{\gamma}: S/N \to \hat{U}_{n+1}(\mathbb{Z}/m)$ such that $\bar{\gamma} \circ \lambda = \pi \circ \gamma$, where $\lambda: S \to S/N$ and $\pi: U_{n+1}(\mathbb{Z}/m) \to \hat{U}_{n+1}(\mathbb{Z}/m)$ are the natural epimorphisms.

Let $(c_{ij})$ and $(\bar{c}_{ij})$ correspond to $\gamma$ and $\bar{\gamma}$, respectively, under the bijections of Lemma 8.1 and Remark 8.2. Thus
$$c_{ij} \in C^1(S, \mathbb{Z}/m), \quad \bar{c}_{ij} \in C^1(S/N, \mathbb{Z}/m), \quad c_{ij} = \inf_S(\bar{c}_{ij}),$$
and \((\bar{c}_{ij})\) is a defining system of size \(n\) in \(C^\bullet(S/N, \mathbb{Z}/m)\). Furthermore, \(\bar{c}_{1n} = \partial c_{1n}\). By construction, \(\epsilon_{(a_1),z/m} = \bar{c}_{ii}\) as elements of \(H^1(S/N), \ i = 1, 2, \ldots, n\), and \(\epsilon_{w,z/m} = (-1)^n \gamma_{1,n+1} = c_{1n}\) in \(H^1(S)\).

Now by the definition of the transgression [NSW08, Prop. 1.6.5], \([\bar{c}_{1n}] = \text{trg}[c_{1n}|_N]\). Altogether,

\[
\psi_w = \langle \epsilon_{(a_1),z/m}, \ldots, \epsilon_{(a_n),z/m} \rangle = \langle \bar{c}_{11}, \ldots, \bar{c}_{mn} \rangle = [\bar{c}_{1n}] = \text{trg}[c_{1n}|_N].
\]

Therefore \((\bar{\sigma}, \psi_w) = c_{1n}(\sigma) = \epsilon_{w,z/m}(\sigma)\). 

\section{10. Proof of Theorem B}

As before, let \(S = S_A\) and \(N\) a closed normal subgroup of \(S\) with \(N \leq S_{n,z/m}, n \geq 2\). By Lemma 6.1, \(N \leq S_{2,z/m} = S^m[S,S]\).

\textbf{Theorem 10.1.}  \(\text{(a) \ (\cdot, \cdot)'} \text{ induces a perfect pairing}

\[
NS_{n+1,z/m}/S_{n+1,z/m} \times H^2(S/N)_{n-Massey} \rightarrow \mathbb{Z}/m.
\]

(b) There is a natural exact sequence

\[
0 \rightarrow H^2(S/N)_{n-Massey} \hookrightarrow H^2(S/N) \overset{\text{inf}}{\rightarrow} H^2(S/(N \cap S_{n+1,z/m})).
\]

\textbf{Proof.} (a) In view of Lemma 5.1(b), there is a \(\mathbb{Z}/m\)-bilinear map

\[
S_{n,z/m} \times \bigoplus_{w \in \mathbb{A}^*} \mathbb{Z}/m \rightarrow \mathbb{Z}/m, \quad (\sigma, (\bar{r}_w)) = \sum_{|w|=n} \bar{r}_w \epsilon_{w,z/m}(\sigma)
\]

with left kernel \(S_{n+1,z/m}\). Let \(\psi_w\) be as in Lemma 8.6. We consider the diagram of bilinear maps

\[
\begin{array}{ccc}
S_{n,z/m}/S_{n+1,z/m} & \times & \bigoplus_{w \in \mathbb{A}^*, |w|=n} \mathbb{Z}/m \\
\downarrow \quad i \quad & & \downarrow \psi \\
N/N^m[S,N] & \times & H^2(S/N) \\
\end{array}
\]

\overset{(\cdot, \cdot)'}{\rightarrow} \mathbb{Z}/m,

where \(i\) is induced by inclusion (noting that \(N^m[S,N] \leq S_{n+1,z/m}\), by (6.1)), and \(\psi((\bar{r}_w)_w) = \sum_w \bar{r}_w \psi_w\). By Theorem 9.2, the diagram commutes, and by (9.1), the lower map is non-degenerate. Lemma 2.2 gives a non-degenerate bilinear map as in (a), and it remains to show that it is perfect.

When the basis \(A\) of \(S\) is finite, the \(\mathbb{Z}/m\)-modules in the upper row of (10.1) are finite. Therefore so are the \(\mathbb{Z}/m\)-modules in the bilinear map (a), so its non-degeneracy implies its perfectness.

When \(A\) is infinite we write \(S = \varprojlim S_B\), where \(B\) ranges over all finite subsets of \(A\), and let \(\pi_B: S \rightarrow S_B\) be the associated projection. Then

\[
NS_{n+1,z/m}/S_{n+1,z/m} = \varprojlim \pi_B(N)(S_B)_{n+1,z/m}/(S_B)_{n+1,z/m}
\]
and by the functoriality of the Massey product,

\[ H^2(S/N)_{n-Massey} = \lim_{\to} H^2(S_B/\pi_B(N))_{n-Massey}. \]

We now use the perfectness in the finite basis case and Lemma 2.1.

(b) As remarked above, \( N^m[S, N] \leq N \cap S_{n+1, Z/m} \) and \( N \leq S^m[S, S] \).

We may now apply Proposition 9.1 for \( G = S, T = N, T_0 = N \cap S_{n+1, Z/m}, H = H^2(S/N)_{n-Massey} \) and use (a). Note that \( H^2(S) = 0. \) \( \square \)

Taking here \( N = S_{n, Z/m} \) and \( m = p \) prime, we get Theorem B. See also [Koc02, §7.8] and [NSW08, Prop. 3.9.13] for related facts in the case \( n = 2. \)

From Theorem 10.1(a) we deduce:

**Corollary 10.2.** Let \( N, M \) be normal closed subgroups of \( S \) with \( N \leq M \leq S_{n, Z/m} \). The following conditions are equivalent:

1. \( \inf: H^2(S/M)_{n-Massey} \to H^2(S/N)_{n-Massey} \) is an isomorphism;
2. \( M \leq NS_{n+1, Z/m} \).

From now on we assume that \( m = p \) is prime. Recall that \( S_n = S_{n, Z/p} \) (Proposition 6.2).

**Theorem 10.3.**

(a) \( (\cdot, \cdot)' \) induces a non-degenerate bilinear map

\[ S_n/NS_{n+1} \times \text{Ker}(H^2(S/S_n)_{n-Massey} \xrightarrow{\inf} H^2(S/N)) \to \mathbb{Z}/p. \]

(b) The kernels of the following inflation maps coincide:

\[ H^2(S/S_n) \to H^2(S/NS_{n+1}), H^2(S/S_n)_{n-Massey} \to H^2(S/N). \]

**Proof.** (a) Theorem 10.1(a) gives a commutative diagram of perfect pairings

\[ \begin{array}{ccc}
S_n/S_{n+1} & \times & H^2(S/S_n)_{n-Massey} \\
\downarrow & & \downarrow \text{inf} \\
NS_{n+1}/S_{n+1} & \times & H^2(S/N)_{n-Massey} \\
\end{array} \to \mathbb{Z}/p. \]

Since every \( \mathbb{F}_p \)-linear space is semi-simple, we may now apply Lemma 2.3.

(b) This follows from (a) and Proposition 9.1 with

\( G = S/N, T = S_n/N, T_0 = NS_{n+1}/N, H = H^2(S/S_n)_{n-Massey}. \) \( \square \)

Now let \( G \) be a profinite group and \( \bar{G} = G(p) \) its maximal pro-\( p \) quotient. Suppose that \( \bar{G} = S/N \) for a free profinite group \( S \) and a closed normal subgroup \( N \) of \( S \) with \( N \leq S_n, n \geq 2. \)
Corollary 10.4. (a) $(\cdot, \cdot)'$ induces a non-degenerate bilinear map
\[ G_n/G_{n+1} \times \text{Ker}(H^2(G/G_n)_{n-\text{Massey}} \overset{\text{inf}}{\rightarrow} H^2(G)) \rightarrow \mathbb{Z}/p. \]

(b) The kernels of the following inflation maps coincide:
\[ H^2(G/G_n) \rightarrow H^2(G/G_{n+1}), \ H^2(G/G_n)_{n-\text{Massey}} \rightarrow H^2(G). \]

Proof. The 5-term sequence of lower cohomologies implies that $\text{inf}: H^2(\bar{G}) \rightarrow H^2(G)$ is injective. Since $G/G_k$ has a finite $p$-power exponent, $G/G_k \cong \bar{G}/\bar{G}_k$ and $G_k/G_{k+1} \cong \bar{G}_k/\bar{G}_{k+1}$ canonically for every $k$. Therefore we may replace $G$ by $\bar{G}$ to assume that $G = S/N$. Let $\lambda: S \rightarrow G$ be the projection map. Then $\lambda(S_k) = G_k$ for every $k$. Moreover, $S/S_n \cong G/G_n$ and $S_n/NS_{n+1} \cong G_n/G_{n+1}$. We now apply Theorem 10.3.

See [Bog92, Lemma 3.3] in the case $n = 2$. Corollary 10.2 gives, by a similar reduction to the maximal pro-$p$ quotient:

Corollary 10.5. Let $\bar{M}$ be a normal subgroup of $G$ contained in $G_n$. Then $\text{inf}: H^2(G/\bar{M})_{n-\text{Massey}} \rightarrow H^2(G)_{n-\text{Massey}}$ is an isomorphism if and only if $\bar{M} \leq G_{n+1}$. In particular, $H^2(G/G_{n+1})_{n-\text{Massey}} \cong H^2(G)_{n-\text{Massey}}$.

For $n = 2$ this was proved in [EM11b, Cor. 5.2, Th. A], extending earlier results from [CEM12].

11. Proof of Theorem A’

Let again $S = S_A$, $m = p$ prime, and $N$ a normal subgroup of $S$ contained in $S_n$. The following theorem is an equivalent form of Theorem A’ when we take $G = S/N$. We note that while $H^2(S/S_n)_{n-\text{Massey}}$ is generated by (elementary) $n$-fold Massey products, this property need not be inherited by its subgroups.

Theorem 11.1. Let $n \geq 1$. When $n \geq 2$ we assume that
\[ \text{Ker}(H^2(S/S_n)_{n-\text{Massey}} \overset{\text{inf}}{\rightarrow} H^2(S/N)) \]
is generated by $n$-fold Massey products. Then $NS_{n+1} = \bigcap M$, where $M$ ranges over all open normal subgroups of $S$ containing $N$ such that $S/M$ embeds as a subgroup of $U_{n+1}(\mathbb{Z}/p)$.

Proof. We argue by induction on $n$. For $n = 1$ we have by Lemma 6.1, $S_2 = S^p[S, S]$, so $S/NS_2$ is an elementary abelian $p$-group. Consequently, $NS_2 = \bigcap M$, where $M$ ranges over all open normal subgroups of $S$ containing $N$ such that $S/M \cong \mathbb{Z}/p \cong U_2(\mathbb{Z}/p)$.

Let $n \geq 2$. We assume the assertion for $n - 1$ and prove it for $n$. 

When \( n \geq 3 \) Theorem 10.1(b) implies that

\[
\inf: H^2(S/S_{n-1})_{(n-1)\text{-Massey}} \to H^2(S/S_n)
\]

is the zero map, so trivially, its kernel is generated by \((n-1)\text{-fold Massey products}\. We may therefore apply the induction hypothesis for \( n-1 \) and \( N = S_n \) (also when \( n = 2 \), to get \( S_n = \bigcap M \), where \( M \) ranges over all open normal subgroups of \( S \) containing \( S_n \) such that \( S/M \) embeds as subgroup of \( U_n(\mathbb{Z}/p) \).

Next, we have already noted in the proof of Theorem 10.3(b) that (a) and (b) of Proposition 9.1 hold with \( G = S/N \), \( T = S_n/N \), \( T_0 = NS_{n+1}/N \), \( H = H^2(S/S_n)_{n\text{-Massey}} \). By assumption, the set \( H_0 \) of all \( n \)-fold Massey products in \( \text{Ker}(\inf: H \to H^2(G)) \) generates this kernel. Therefore (c) of Proposition 9.1 also holds in this setup.

By Proposition 8.3, the central extensions corresponding to \( n \)-fold Massey products in \( H \) are (up to signs) as in the lower part of the diagram

\[
0 \to \mathbb{Z}/p \to U_{n+1}(\mathbb{Z}/p) \times \hat{c}_{n+1}(\mathbb{Z}/p) (S/S_n) \xrightarrow{pr_2} S/S_n \to 1,
\]

where the fiber product is with respect to the projection \( \pi: U_{n+1}(\mathbb{Z}/p) \to U_{n+1}(\mathbb{Z}/p) \) and some continuous homomorphism \( \hat{\gamma}: S/S_n \to U_{n+1}(\mathbb{Z}/p) \). The corresponding Massey product is in the kernel of \( \inf: H^2(S/S_n) \to H^2(G) \) (i.e., belongs to \( H_0 \)) if and only if there is a continuous homomorphism \( \hat{\Phi} \) making the diagram commutative \([Hoc68, 1.1]\). We therefore conclude from (c) of Proposition 9.1 that

\[
NS_{n+1}/N = \bigcap \text{Ker}(\hat{\Phi}),
\]

where \( \hat{\Phi} \) ranges over all continuous homomorphisms making the diagram commutative for some \( \hat{\gamma} \). By Lemma 7.3, the subgroups Ker(\( \hat{\Phi} \)) are exactly the quotients \((M \cap S_n)/N \), where \( M \) is a normal open subgroup of \( S \) containing \( N \) such that \( S/M \) embeds in \( U_{n+1}(\mathbb{Z}/p) \). Hence, \( NS_{n+1} = (\bigcap M) \cap S_n \).

Since \( U_n(\mathbb{Z}/p) \) embeds as a subgroup of \( U_{n+1}(\mathbb{Z}/p) \), and by what we have seen earlier in the proof, \( S_n \) contains \( \bigcap M \), so in fact \( NS_{n+1} = \bigcap M \). \( \square \)

**Remark 11.2.** In our general setup, the projection \( S \to G \) induces an isomorphism \( S/S_n \cong G/G_n \). When it further induces an isomorphism \( S/S_{n+1} \cong G/G_{n+1} \), i.e., \( N \leq S_{n+1} \), Theorem 10.1(b) shows that the assumption about the kernel in Theorem 11.1 is satisfied. This example is not exhaustive (see Corollary 12.3).
Remark 11.3. As in Corollary 10.4, we may replace in Theorem A’ the group $G$ by its maximal pro-$p$ quotient $G(p)$ to assume that $G(p)$ (but not necessarily $G$) has a presentation $S/N$ with $N \leq S_n$.

Corollary 11.4. Let $G$ be a profinite group with $cd_p(G) \leq 1$. Then $G_n = \bigcap \rho \Ker(\rho)$, where $\rho$ ranges over all representations of $G$ in $U_n(F_p)$.

Proof. The maximal pro-$p$ quotient $G(p)$ of $G$ is a free pro-$p$ group [NSW08, Prop. 3.5.3 and Prop. 3.5.9]. We may therefore replace $G$ by $G(p)$, to assume that $G$ is a free pro-$p$ group.

As $H^2(G) = 0$, all Massey kernel conditions are satisfied. Take a free profinite group $S$ on the same basis as $G$ and let $N = \Ker(S \to S' = G)$. Since $S/S_n$ has a $p$-power exponent, $N \leq S_n$ for every $n$. Now apply Theorem A’. □

12. Examples

We conclude by examining Theorem A’ in low degrees.

Example 12.1. When $n = 1$ Theorem A’ is just the elementary fact that $G^p[G,G] = \bigcap M$, where $M$ ranges over all open normal subgroups of $G$ with $G/M \cong \{1\}, \mathbb{Z}/p$.

Example 12.2. Let $n = 2$. Assume that the kernel of $\inf: H^2(S/S_2)_{\text{Massey}} = H^2(G/S_2)_{\text{dec}} \to H^2(G)$ is generated by cup products (see Example 8.5).

When $p = 2$, $U_3(\mathbb{Z}/2)$ is the dihedral group $D_4$ of order 8. Hence in this case Theorem A’ asserts that

$$G_3 = \bigcap \{ \bar{M} \mid \bar{M} \subseteq G, G/\bar{M} \cong \{1\}, \mathbb{Z}/2, \mathbb{Z}/4, D_4 \}.$$  

This recovers [EM11a, Cor. 11.3], which in turn generalizes [MSp96, Cor. 2.18] (see below).

When $p > 2$, $U_3(\mathbb{Z}/p)$ is the unique nonabelian group $H_{p^3}$ of order $p^3$ and exponent $p$ (also called the Heisenberg group). Its subgroups are $\{1\}$, $\mathbb{Z}/p$, $(\mathbb{Z}/p)^2$ and $U_3(\mathbb{Z}/p)$ itself. Moreover, when $G/\bar{M} \cong (\mathbb{Z}/p)^2$, we may write $\bar{M} = \bar{M}_1 \cap \bar{M}_2$ with $G/\bar{M}_i \cong \mathbb{Z}/p$, $i = 1, 2$. Therefore

$$G_3 = \bigcap \{ \bar{M} \mid \bar{M} \subseteq G, G/\bar{M} \cong \{1\}, \mathbb{Z}/p, H_{p^3} \}.$$  

This recovers [EM11b, Example 9.5(1)].

Finally we recover [MSp96 Cor. 2.18] and [EM11b Th. D] (see also [NQD12]):
**Corollary 12.3.** Let $K$ be a field containing a root of unity of order $p$ and $G = G_K$ its absolute Galois group. Then $G_3 = \bigcap \bar{M}$, where $\bar{M}$ ranges over all open normal subgroups of $G$ such that $G/\bar{M}$ embeds in $U_3(\mathbb{Z}/p)$.

**Proof.** The injectivity of the Galois symbol in degree 2, which is a part of the Merkurev–Suslin theorem ([MS82], [GS06]), implies that the kernel of $\text{inf} : H^2(S/S^p[S,S])_{\text{dec}} = H^2(G/G^p[G,G])_{\text{dec}} \to H^2(G)$ is generated by cup products (see [Bog91], [EM11a, Prop. 3.2]). Now apply Theorem A’. □

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