Nonlinear resonances in $\delta$-kicked Bose-Einstein condensates

T.S. Monteiro,$^{1,}$ A. Rancon,$^{1}$ and J. Ruostekoski$^{2}$

$^1$Department of Physics and Astronomy, University College London, Gower Street, London WC1E 6BT, United Kingdom
$^2$School of Mathematics, University of Southampton, Southampton, SO17 1BJ, United Kingdom

(Dated: September 17, 2008)

We investigate the Quantum Resonance (QR) regime of a periodically kicked atomic Bose-Einstein condensate. We find that the clearest indicator of the nonlinear dynamics is a surprisingly abrupt cut-off which appears on the main “Talbot time” QR. We show that this is due to dephasing and excitation path combining both Beliaev and Landau processes, with some analogies to nonlinear self-trapping. Investigation of dynamical instability reveals further symptoms of nonlinearity such as a regime of exponential oscillations.

PACS numbers: 05.45.-a,05.45.Mt,03.75.Kk

The interaction between nonlinear dynamics and quantum dynamics has stimulated new domains of cold atom physics. For instance, the suppression of chaotic classical diffusion in a corresponding quantum system evolving under the linear Schrödinger equation has been investigated experimentally using cold atoms subjected to regularly spaced short pulses ($\delta$-kicks) from standing waves. But studies of the quantum $\delta$-kicked systems, which have as a classical analogue the famous nonlinear paradigm which is the Standard Map, had certain limitations: probing the dynamics close to the chaotic classical limit requires small values of Planck’s constant, $\hbar$. Even though $\hbar$ in the cold atom experiments is a scaled (and hence variable) effective value, the $\hbar \to 0$ regimes remained out of reach. However, an important advance was the realization in 2002 [1] that atoms kicked with periods which are a rational multiple of the Talbot time $T = 4\pi$ (so-named because of an analogy to the Talbot optical effect) represent a somewhat pathological case. They correspond to a classical ‘image’ system where the effective value of $\hbar$ is played by the dephasing from resonance and so can take very small values. This has stimulated a series of delicate experiments by several groups worldwide [2] probing the rich physics of both the Quantum Resonance (QR) regime as well as the closely related Quantum Accelerator (QA) regime (which is in effect, quantum resonant behavior in the presence of gravitational acceleration). Further theory [3] includes proposed applications such as the realization of a quantum random walk algorithm [4].

Since well-defined initial momenta are needed, most recent QR or QA experiments employed $\delta$-kicked Bose-Einstein condensates (BECs). But this in turn opens up a new and quite different possibility: namely the largely unexplored regime where nonlinear dynamics, arising from the many-body nature of the BEC, combine with the $\delta$-kicked quantum dynamics. To date, the deep understanding of collective excitations, acquired from other areas of BEC physics, has not been applied to the unique dynamical features of the $\delta$-kicked atoms.

A few theoretical studies have considered interactions in QR/QA: in [5] dephasing of resonant transport at the Talbot time as a function of nonlinearity parameter $g$ was found; in [6] exponential growth of non-condensate atoms was predicted for certain parameters at half the Talbot time; in [7] significant differences were reported for QA dynamics between attractive and repulsive interactions. Only recently, though, was it demonstrated [8] that an approach based on Bogoliubov phonon modes is essential; in particular, the onset of dynamical instability was attributed to parametric resonances. Parametric instabilities through periodic driving of collective modes have been investigated in several BEC studies [9]. Higher-order effects, resulting from phonon-phonon interactions, have been experimentally studied, e.g., in the context of excitation lifetimes [11] and in a nonlinear coupling between two phonon modes [12]. Beliaev and Landau (BL) couplings provide the dominant contribution to such nonlinear mode conversions; their importance was demonstrated in a series of recent experiments with BECs excited by optical Bragg pulses [13].

Here we have, for the first time, quantitatively investigated an interacting $\delta$-kicked BEC by mapping the position and properties of the resonances of all its low-lying modes. The leading Talbot time resonances, which appear to dephase and disappear at $T = 4\pi$, in fact are shifted and broadened (they are strong over $\Delta T \sim 1$). But our key finding is that they acquire an extraordinarily sharp cut-off at their maximum ($\Delta T \sim 10^{-3}$). We show this behavior is due to a nonlinear feedback process, originating from the resonant kicking and a combination of both Beliaev and Landau coupling. In other nonlinear resonances, we find novel features, not previously seen in other driven BECs, such as exponential oscillations and Fano-like profiles. We calculate the local Lyapunov exponents and model the Fano profiles quantitatively. The kicked-BEC experiments to date, have used effective values of the nonlinearity ($g \sim 0.5$) which are only slightly smaller than those ($g \gtrsim 1$) needed for the effects we find.

For non-interacting single-particle dynamics, an attractive features of the $\delta$-kicked system, for quantum
chaologists, is the extensively-investigated simple quantum map which stroboscopically evolves the system from kick $n$ to kick $n+1$. For example, expressing our quantum state in a momentum basis, $\psi(x,t) = \sum_{l=0}^{\infty} a_l(t) |l\rangle$, we write:

$$a((n+1)T) = U_{g=0}(T) a(t=nT),$$  \hspace{1cm} (1)

where $a(t=nT)$ is a vector with the amplitudes $a_l$. The atoms experience a kicking potential $V_{kick}(x,t) = K \cos x \sum \delta(t-nT)$. The corresponding unitary time evolution operator factors (exactly) into a free-evolution part $U_{free}$ and a kick part $U_{kick}$, i.e.,

$$U_{g=0}(T) = U_{free} U_{kick} = e^{-\frac{i\omega T}{2}} e^{-iK \cos x}.$$ \hspace{1cm} (2)

The first exponential term represents the free evolution under the kinetic energy operator in some units where the atom mass $M = 1$ and $h = 1$. Clearly, since the atomic momentum $l = 0, \pm 1, \pm 2...$ is quantized in units of the recoil momentum, $T = 4\pi$ implies $U_{free} = 1$ for all momentum states, so consecutive kicks add in phase. The result is a phase-matched absorption of energy from the field yielding ballistic transport. In contrast, the nonresonantly kicked atoms experience diffusive growth in energy.

We consider a uniform, tightly-confined, effectively 1D BEC with periodic boundary conditions. The corresponding field-free, many-body Hamiltonian, in a momentum representation, is:

$$H = \sum_{k,l} \epsilon_k a_k^\dagger a_l + \frac{g_{1D}}{2L} \sum_{l,j,m} a_j^\dagger a_k^\dagger a_m a_{j+l-m},$$ \hspace{1cm} (3)

where $\epsilon_k = \hbar^2 k^2/(2M)$ and $L$ is the BEC size. The 1D interaction constant $g_{1D} \simeq 2a_s \omega_\perp$ depends on the atomic scattering length $a_s$ and the transverse trap frequency $\omega_\perp$. In the presence of linearized perturbations around the macroscopically occupied $k = 0$ ground state, $H$ may be diagonalized up to quadratic order by the Bogoliubov transformation $a_k = u_k b_k - v_k b_k^\dagger$, for $k \neq 0$, with $u_k - v_k = 1/(u_k + v_k) = \sqrt{\hbar \omega_k}$, where $\hbar \omega_k = \epsilon_k/(\epsilon_k + 2g_{1D} n))^{1/2}$ and $n = N/L$ is the atom density. Then we can expand $H$ in the orders of $\sqrt{N}$: $H = const + H^{(2)} + H^{(3)} + H^{(4)}$ where $H^{(2)} = \sum_{k \neq 0} \hbar \omega_k b_k^\dagger b_k$ and the cubic part $H^{(3)}$ describes the leading order contribution to the interactions between phonons

$$H^{(3)} = \kappa \sum_{q,p} (\Gamma_{qp} b_q^\dagger b_p^\dagger b_{q+p} + \Delta_{qp} b_q^\dagger b_p b_{q+p} + H.c.),$$ \hspace{1cm} (4)

where $q,p,(q+p) \neq 0$ and $\kappa = \sqrt{N} g_{1D}/L$. The coefficient $\Gamma_{qp}$ represents a process in which three phonons are created or annihilated and $\Delta_{qp}$ a process in which one phonon with momentum $q + p$ decays into two phonons with momenta $q$ and $p$ (Belyaev term) or its inverse in which two merge to produce a third (Landau term). Since

the energy of the excitations around the ground state is positive, the processes described by $\Gamma_{qp}$ are suppressed by energy conservation. In terms of the Bogoliubov amplitudes we obtain

$$\Delta_{qp} = u_p u_q v_{q+p} + 2u_p v_q v_{q+p} - 2u_q v_p u_{q+p} - v_p v_q v_{q+p}$$

$$\Gamma_{qp} = u_q v_p v_{q+p} - u_p u_q v_{q+p}$$ \hspace{1cm} (5)

We assume macroscopic occupancy for the low-lying modes of interest and that phase fluctuations of the 1D condensate may be neglected. Hence we will treat the Bogoliubov mode amplitudes classically. The $\delta$-kicked map (2), in the presence of the quadratic Hamiltonian $H^{(2)}$ alone, requires only a straightforward modification to its free-evolution part:

$$U_{g}(T) = B^{-1} e^{-i\omega T} I B U_{kick}$$ \hspace{1cm} (6)

Where $B$ denotes the Bogoliubov transform, $\exp(-i\omega T)$ is a row vector with the mode frequencies and $I$ is the identity. The eigenvalues of the non-unitary matrix $U_{g}(T)$ indicate dynamical stability. It is easy to prove they in general come in quartets $\lambda, 1/\lambda, \lambda^*, 1/\lambda^*$. Then $|\lambda_{max}| > 1$ (where $\lambda_{max}$ is the largest eigenvalue and the local Lyapunov exponent) imply dynamical instability and exponential growth in the relevant modes (at least for short times).

We compare Eq. (6) to the numerical solutions of the 1D Gross-Pitaevski equation (GPE) (here rescaled to dimensionless units [1]) for an initially uniform BEC:

$$i\partial_t \psi(x,t) = \left[ -\frac{1}{2} \partial_x^2 + g|\psi(x,t)|^2 + K \cos x F_l \psi(x,t) \right]$$ \hspace{1cm} (7)

where $F_l = \sum \delta(t-nT)$. We study $g \simeq 0 - 10$, realistic with the current experiments [2, 16], for $K < 1$, which allows only the creation of discrete low-lying phonon excitations.

Fig. (a) shows GPE numerics; it maps the average BEC response, $< 1 - |\psi_0(t)|^2 > = \frac{1}{2} \sum_{n=1}^{N} 1 - |\psi_0(nT)|^2$ averaged over the first $t$ kicks, for $K = 0.5$. At $g \simeq 0$, the Talbot time resonance at $T = 4\pi$ is perfectly symmetrical, (as are the fractional resonances on either side). For $g \simeq 0 - 1$ an asymmetry develops, due here to the lifting of the degeneracy between the lowest modes: the main resonance splits into a mode 1 resonance $\omega_1 T \simeq 2\pi$ and mode 2 resonance $\omega_2 T \simeq 8\pi$. The mode 2 resonance rapidly decays away as the gap $\omega_2 - \omega_1$ increases: direct coupling $< 0 | U_{kick} | 2 > \sim J_2(K)$ with the condensate is small. A slight asymmetry was noted in the GPE numerics in Ref. [5] for the Talbot time resonance at $g \simeq 0.1$, which we now attribute to this regime.

But the most striking feature is the very sharp 'cut-off' appearing at $g \simeq 1$ (and still exists even at $g = 20$). It is also evident in the second harmonic of the resonance (upper half of the graph). Fig. (b) shows that even a grid of 100 points per unit of $T$ (each circle representing a GPE simulation for 30 kicks and $g = 5$) is too coarse to resolve
the cut-off (in comparison, such a grid could resolve the famously narrow $g = 0$ Talbot-time resonance).

The dotted line shows the Bogoliubov map \( \mathfrak{R} \) here incorrectly produces a symmetric resonance and fails to shift the resonance away from $\omega_1 T = 2\pi$. Hence we need to include the neglected phonon-phonon interaction terms between the kicks from $H^{(3)}$. The free-ringing exp $(-i\omega T)$ part of the map [Eq. 6] must be replaced by a set coupled equations following from the Heisenberg equations $d_t\hat{b}_k = -i\omega_k \hat{b}_k - i[\hat{b}_k, H^{(3)} / \hbar]$, for $k \neq 0$, where we replace $\hat{b}_k$ by the rescaled classical amplitudes $\tilde{b}_k = (\hat{b}_k / \sqrt{N})$. Figure 1(b) shows only four lowest excitations ($k = \pm 2, \pm 1$) provide an excellent quantitative agreement with the GPE, accurately reproducing the cut-off for small depletion ($\lesssim 10\%$) of the ground state.

We simplify further by transforming (by symmetry $\hat{b}_k = \hat{b}_{-k}$) to the basis $|n\rangle \rightarrow \frac{1}{\sqrt{2}}(|n\rangle + |-n\rangle)$, for $l \neq 0$, so that $\langle n|U_{kick}|n\rangle = U_{nl} = i^{-n} J_{n-l}(K) + i^{l+n} J_{l+n}(K)$ if $n, l > 0$, but $U_{00} = \sqrt{2}r^{-1}J_1(K)$ and $U_{00} = J_0(K)$. We then only need two coupled equations to accurately reproduce the cut-off. Moreover, neglecting the small $\Gamma_{GP}$ terms, we obtain (from the free-ringing plus BL terms) $d_t\hat{b}_1 = -i[\omega_1 + 2C_1 R(\hat{b}_2)]\hat{b}_1 + 2C_2\hat{b}_1^2\hat{b}_2$, $d_t\hat{b}_2 = -i\omega_2\hat{b}_2 - i[C_1|\hat{b}_1|^2 + C_2\hat{b}_1^2]$.

where $C_1 = \tilde{\kappa}(\Delta_{-1,2} + \Delta_{2,-1})$, $C_2 = \kappa\Delta_{1,1}$, and $\tilde{\kappa} = g/(2\sqrt{2}\pi)$ (note $\Delta_{-1,2} = \Delta_{1,-2}$, etc.). We can even obtain a reasonable cut-off if we set the smaller term $C_2 = 0$. Hence, the main effect of mode 2 is simply to provide a phase-shift on $\omega_1$. If we integrate $\hat{b}_2$ while keeping $|\hat{b}_1|$ constant, we can replace the full map by:

\begin{equation}
    d_t\hat{b}_1 \simeq -i\omega_1\hat{b}_1 + iA\sin^2(\omega_2 T / 2)|\hat{b}_1|^2\hat{b}_1 - iR(t),
\end{equation}

where $A = 4C_1^2 / \omega_2$, $R(t) = \sqrt{2}J_1(K)(u_1 - v_1)F(t)$. Fig 2(b) inset shows that we still get reasonable agreement with the full model for weak $K = 0.1$. In this case.
regime $|\tilde{b}_0(t)|^2 \simeq 1 - |\tilde{b}_1(t)|^2$. Writing $\tilde{b}_1 = \rho e^{\text{i}\theta}$, a phase space analysis in the $\rho, \theta$ plane reveals a separatrix curve which appears at the cut-off parameters and bounds the value of $\rho$. We may also describe the physical mechanism thus: if the $0 \rightarrow 1$ transition is initially only slightly off-resonant, the kicking starts populating mode 1 effectively; with the increasing mode 1 population, BL slightly off-resonant, the kicking starts populating mode 1 further into resonance. However, if the $0 \rightarrow 1$ transition is initially too far off-resonant (beyond the sharp cut-off value) the nonlinear feedback cycle cannot start and the population oscillations between the modes are suppressed. An analogous model of two-mode dynamics with continuous driving rather than $\delta$ kicks is reminiscent of a macroscopic self-trapping effect in a BEC in a double-well potential [17], but in the present work the cut-off is considerably sharper.

Fig. 2 maps the average probability of mode 2 (averaged over 100 kicks) for $K = 0.5$. The right hand side maps regions of dynamical instability $|\lambda_{\text{max}}| > 1$. We analyze dynamical stability by mapping the eigenvalues of $U_T(T)$ for all the resonances of the lowest 3 excited modes. We divide the resonances into (1) the “linear” family $L(n,l)$ (ie those which evolve from the linear case and converge at $g = 0$ to a rational fraction of the Talbot time. The resonance in Fig. 1(a) is the $L(1,1)$ (first resonance of mode $l = 1$). (2) The “nonlinear” resonances $N_n$ and $\nu_n$ which vanish in the absence of interactions, at $g = 0$; the $N_n$ correspond to $(\omega_1 + \omega_2)T \simeq 2\pi n$, while $\nu_n$ are somewhat analogous to “counter-propagating mode” resonances found in modulated traps [10] and imply $2\omega_n T \simeq 2\pi$. Contrary to the suggestion of [8] where no Liapunov exponents were calculated, we find that none of the $L(n,l)$ resonances have any $|\lambda| > 1$. They are all stable, including $L(1,1)$, by far the strongest of all. But counter-intuitively, they are associated with a much stronger BEC response, even after a very long-time (100 kicks) than the nonlinear resonances $N_n$ and $\nu_n$ which are unstable.

The reason for this is clear from Fig. 2(b). The mode 2 populations from the GPE (for both $N_1$ and $N_3$) grow exponentially for a finite time, then decay exponentially; the inset shows this behavior on a log-scale. The map with BL corrections here is quantitative for only the first 10-20 kicks, so we cannot model this behavior from $H^{(3)}$ alone. But it is tempting to attribute it to regimes where either the $\lambda, \lambda^*$ or the $1/\lambda, 1/\lambda^*$ eigenvalues are predominant. The cluster of interacting resonances $N_2, \nu_1$ and $\nu_2$ lies in a region of very low ground state depletion (it lies in the minimum of the dominant $L(1,1)$ tail). The map [4], corrected by BL terms $H^{(3)}$ in Eq. (4) (with the lowest 7 modes), reproduces very well the characteristic Fano-like profiles seen in all three peaks, while the uncorrected map produces only symmetric resonance profiles. To our knowledge neither the exponential oscillations, nor the Fano profiles are seen in comparable non-equilibrium BEC dynamics. While not fully understood, they indicate that the $\delta$-kicked systems offer new and experimentally accessible BEC dynamics.

While the Talbot time $g = 0$ resonances have been proposed for metrological applications (e.g., for measurement of gravity), the similarly sharp BEC cut-off suggests analogous possibilities as it provides a sharp excitation threshold. One could envisage a set of ring-BECs subject to kicks; at threshold kicking frequencies, a very small perturbation would determine whether a particular ring is left in an $l = 0$ state or acquires a large $l = 1$ component. Moreover, even under a very weak rotation, a BEC in a large ring would occupy a higher momentum (vortex) state $l$. Then the system could provide a sensitive probe of rotation for small changes in the resonance frequency between different angular momentum states.

* Electronic address: t.monteiro@ucl.ac.uk

[1] S. Fishman, I. Guarnieri, and L. Rebuzzi, Phys. Rev. Lett. 89, 008101 (2002); S.Wimberger, I.Guarnieri and S.Fishman Phys. Rev. Lett 92, 084102 (2004).
[2] M.K. Oberthaler, et al., Phys. Rev. Lett. 83, 4447 (1999); C. Ryu et al., ibid. 96, 160403 (2006); M. Sadgrove et al., ibid. 99, 043002 (2007); G. Behinain et al., ibid. 97, 244101 (2006); I. Dana et al., ibid. 100, 024103 (2008); J.F. Kanem et al., ibid. 98, 083004 (2007).
[3] A. Buchleitner et al., Phys. Rev. Lett. 96, 164101 (2006); M. Saunders et al., Phys. Rev. A 76, 043415 (2007); M. Lepers V. Zehul, and J-C Garreau, 77, 043628 (2008).
[4] Z-Y Ma, K Burnett, MB d’ArCY, SA Gardiner Phys Rev A 73, 013401 (2006).
[5] S. Wimberger, R. Mannella, O. Morsch, and E. Arimondo, Phys. Rev. Lett. 94, 130404 (2005).
[6] C. Zhang, J. Liu, M.G. Raizen, and Q. Niu, Phys. Rev. Lett. 92, 054101 (2004).
[7] L. Rebuzzi, R.Artuso, S.Fishman, I.Guarnieri, Phys. Rev. A 76, 031603(R) (2007).
[8] J. Reslen, C. E. Creffield and T. S. Monteiro, Phys. Rev. A 77, 043621 (2008).
[9] J.J. Garcia-Ripoll, V.M.Perez-Garcia and P Torres, Phys. Rev. Lett. 83, 1715 (1999); Yu Kagan and L.A. Manakova, ibid. (cond-mat/0609159) (2006).
[10] C.Tozzo, M.Kramer and P.Dalfovo, Phys.Rev.A 72 023613 (2005); P.Engels, C.Atherton and M.A.Hofer Phys. Rev. Lett. 98 095301 (2007).
[11] D.M. Stamper-Kurn et al., Phys. Rev. Lett. 81, 500 (1998).
[12] E. Hodby et al., Phys. Rev. Lett. 86, 2196 (2001).
[13] N.Katz, E.E.Rowen,R.Ozeri and N Davidson, Phys. Rev. Lett. 95, 220403 (2005) (2005). Phys. Rev. A 77, 033602 (2008); E.E.Rowen, N.Bar-Gill and N.Davidson Phys. Rev. Lett. 101, 010404 (2008).
[14] A. Sinatra, C. Lobo and Y. Castin J.Phys.B. At.Mol.Phys 35, 3599-3631 (2002).
We introduce dimensionless units $x = 2k_L x', t = 8\omega_r t'$, where the lattice recoil frequency $\omega_r \equiv \hbar k_L^2 / 2M$ and $g = \frac{N g_1 D}{(8\hbar \omega_r) L}$. Since $k_L L = \pi$, $L \to 2\pi$ and the BEC occupies $0 \leq x \leq 2\pi$. Also $2gn \to \frac{2N}{\pi}$ and the Bogoliubov spectrum $\omega^2 \to k^2 / 2(k^2 / 2 + g / \pi)$.

[15] K. Henderson et al., Europhys. Lett. 75, 392 (2006); Phys. Rev. Lett. 96, 150401 (2006).

[17] A. Smerzi et al., Phys. Rev. Lett. 79, 4950 (1997).