On the Topological Charges of Affine Toda Solitons

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Abstract

We provide a proof of a formula conjectured in [1] for some coefficients relevant in the principal vertex operator construction of a simply-laced affine algebra \( \hat{\mathfrak{g}} \). These coefficients are important for the study of the topological charges of the solitons of affine Toda theories, and the construction of representations of non-simply-laced \( \hat{\mathfrak{g}} \) and their associated Toda solitons.
1 Introduction

In this letter we provide a simple proof of a conjectured formula of [1], whose notation we will be following, for some phases $\epsilon(\tau_j, i)$ which appear in the description of the principal vertex operator construction [2] of the level one representations of a simply-laced affine Lie algebra $\hat{g}$. This can be thought of as an affinisation of the finite algebra $g$. We will explain the significance of the $\tau_j$ and $i$ below.

These phases have several uses [1]:

• They appear in the formula for the vertex operator representation of the Kac-Moody algebra valued $\hat{F}_i(z)$, thus

$$
\rho_j(\hat{F}_i(z)) = \epsilon(\tau^{-1}_j, i) \exp \left( \sum_{N>0} \frac{\gamma_i \cdot q([N]) z^N \hat{E}_{-N}}{N} \right)
\times \exp \left( \sum_{N>0} \frac{\gamma_i \cdot q([N]) z^{-N} \hat{E}_{N}}{N} \right)
$$

where the $N$ run over the set of positive exponents (see e.g. chapter 14 of [3]) of $\hat{g}$. These representations are important for the calculation of the soliton solutions of an affine Toda theory. In [1] it was shown that these are generated by choosing a constant element of the Kac-Moody group which appears in a specialisation of the Leznov-Saveliev [4] general solution of these theories to be a product of exponentials of the form $\exp(Q\hat{F}_i(z))$. The index $i$ is a positive integer $\leq r$, where $r$ is the rank of $g$. It labels the species of the soliton generated by the exponential.

• In [1] it was shown that the principal vertex operator construction can in fact be used to evaluate the solitons solutions of affine Toda theories with non-simply-laced $\hat{g}$, by identifying such $\hat{g}$ as the subalgebras of some simply-laced algebras upon which some outer automorphism acts trivially. We form the $\hat{F}^{<i>}(z)$ for this subalgebra as linear combinations of the original $\hat{F}_i(z)$. To determine the precise combination of these we need to know the expansion of $\hat{F}_i(z)$ over the generators of the Cartan subalgebra of $\hat{g}$, and to do this we need to know the $\epsilon$.

• They are the characters of the irreducible one-dimensional representations of the abelian group $W_0$, which is the subgroup of the diagram automorphisms of the Dynkin diagram of $\hat{g}$ which become inner automorphisms when projected to automorphisms of the finite algebra $g$. We shall describe $W_0$ further below.

• Solitons of an affine Toda theory can be thought of as the solutions of least energy which interpolate the degenerate vacua of the theory. The possible such vacua belong to the co-weight lattice $\Lambda^*_{W}$ of $g$, and the difference between the solution at $x = -\infty$ and $x = \infty$, known as the topological charge, lies in this lattice. It has been a problem since the original discovery of solitons solutions by Hollowood [5] to determine these
topological charges. It turns out \[1\] that the \(\epsilon(\tau_j, i)\) associated to an \(\hat{F}^i(z)\) tell us which coset of \(\Lambda^*_W\) by the co-root lattice \(\Lambda^*_R\) the topological charge of the one soliton solution generated by \(\exp(Q\hat{F}^i(z))\) belongs to.

### 1.1 Definitions and Notation

We shall now proceed to the derivation of a formula for the \(\epsilon\), for which we require some notation. Let us fix a Cartan subalgebra of \(g\), which we shall call \(H'\). There is an element \(T_3\) of \(H'\) whose adjoint action grades the step-operators of \(g\) corresponding to particular roots by the height of those roots. This is the principal gradation. We may convert it to a multiplicative gradation of the algebra using the Adjoint action of the element \(S\) defined by

\[ S = \exp(2\pi iT_3/h). \]  

(1.2)

Here \(h\) is the Coxeter number of \(g\) which can be defined to be one more than the height of highest root of \(g\). It can be shown (see e.g. \[3\]) that \(\text{AdS}\) is actually the inner automorphism of \(g\) corresponding to a Coxeter element \(w_C\) of the Weyl group, defined with respect to a second Cartan subalgebra \(H_0\). Such a subalgebra is said \[4\] to be in apposition, and \(H_0 \cap H' = 0\). Following \[4\] we bicolour the points of the Dynkin diagram of \(g\) alternately black and white so that no points of like colour are adjacent. Let us denote a product of Weyl reflections in ‘black’ simple roots by \(w_B\) and \(w_W\) as a product of reflections in the ‘white’ roots. The value of this, \[5\] is that

\[ w_C = w_Bw_W. \]  

(1.3)

For later use let us define \(c(i) = \pm 1\), as \(i\) is white or black, and \(\delta_{iB} = 1\) if \(i\) is black and zero otherwise, with the complementary definition for \(\delta_{iW}\).

When the Dynkin diagram of \(\hat{g}\) has a symmetry \(\tau\) there is a natural lift of this to an outer automorphism of \(\hat{g}\), which can be projected onto \(g\) by setting the loop parameter to one for example. Of these automorphisms some will become inner when projected. This subgroup we denote by \(W_0\). We now give some results proved in \[1\].

- \(|W_0|\) is equal to the number of points \(j\) on the Dynkin diagram of \(\hat{g}\) which can be related to the point 0 by a symmetry \(\tau\).

- For each such point \(j\) there is exactly one \(\tau_j \in W_0\) such that \(\tau(0) = j\).

- This serves to label the elements of \(W_0\).

- Let us denote the realisation of \(\tau_j\) as an inner automorphism of \(g\) by \(\text{Ad}T_j\). Then

\[ T_jST_j^{-1} = S \exp \left( -2\pi i \frac{2\lambda \tau(0) \cdot H'}{\alpha(0)^2} \right). \]  

(1.4)

- \(\text{Ad}T_j\) acts trivially on \(H_0\) and so \(T_j \in \exp H_0\).
2 Calculation of the phases $\epsilon$

We want to calculate directly the action of the $\tau \in W_0$ on the $\hat{F}_i(z)$. Because the $\tau_j$ can be realised as inner automorphisms $\text{Ad} T_j$ in $\mathfrak{g}$ we can determine this action if we explicitly solve for $T_j$. We already know that $T_j \in \exp H_0$ and so writing $T_j = \exp(2\pi i Y_j \cdot H_0)$ and using the definition of the $\epsilon$

$$T_j E_{\gamma_k} T_j^{-1} = \epsilon(\tau_j, k) E_{\gamma_k}, \quad (2.1)$$

we find that

$$\epsilon(\tau_j, k) = e^{2\pi i Y \cdot \gamma_k}. \quad (2.2)$$

$E_{\gamma_k}$ are the step-operators with respect to $H_0$, which yield $\hat{F}_k(z)$ when lifted to $\hat{g}$ in an appropriate fashion. It is sufficient to use only $\gamma_k = c(k) \alpha_k$ to obtain $\hat{F}_k(z)$ whose modes span $\hat{g} (\emptyset, 3)$.

Rearranging equation 1.4 yields

$$ST_j S^{-1} = T e^{2\pi i (2\lambda_j \cdot H'/\alpha_j^2)} = T e^{2\pi i (2\lambda_j \cdot H_0/\alpha_j^2)}. \quad (2.3)$$

It is important to note the small but significant difference between the right-hand sides of this expression. In one case the central element is expanded over $H'$, and the second over $H_0$. The equality of these expressions follows precisely because the element is central, so that any conjugation of $\mathbf{G}$ sending $\exp H'$ to $\exp H_0$ will leave the central elements unchanged. We can then use 2.3 to find

$$\exp (2\pi i (wC - 1)(Y_j)) = \exp \left(2\pi i \frac{2\lambda_j \cdot H_0}{\alpha_j^2}\right), \quad (2.4)$$

and so solving this expression we find that

$$(wC - 1) Y_j \in \frac{2\lambda_j}{\alpha_j^2} + \Lambda_R^*, \quad (2.5)$$

yielding

$$Y_j = (wC - 1)^{-1} \frac{2\lambda_j}{\alpha_j^2} \text{ mod } \Lambda_R^*. \quad (2.6)$$

Note that $wC - 1$ is invertible since none of the eigenvalues of $w$ is unity.

Let us examine explicitly the action of $(wC - 1)$ on the basis of $H_0^*$ provided by the fundamental weights of $\mathfrak{g}$. Let us drop the subscript. We find, using 1.3

$$(w - 1) \lambda_i = -\alpha_i, \quad i \black;$$

$$= -w (w^{-1} - 1) \lambda_i = w(\alpha_i), \quad i \white. \quad (2.7)$$

Inverting these expressions produces

$$(w - 1)^{-1} \alpha_i = \begin{cases} -\lambda_i, & i \black; \\ \lambda_i - \alpha_i, & i \white. \end{cases} \quad (2.8)$$
To find the action of \((w - 1)^{-1}\) on \(\lambda_j\), we need to expand it over the simple roots, as \(2.8\) gives us the images of these. The relevant expansion can easily be checked to be

\[
\lambda_j = K_{ji}^{-1} \alpha_i,
\]

and so substituting in we get

\[
(w - 1)^{-1} \lambda_j = - \sum_i K_{ji}^{-1} \lambda_i + \sum_i K_{ji}^{-1} (\lambda_i - \alpha_i).
\]

(2.10)

Defining the traceless Cartan matrix \(\hat{K}_{ji} = K_{ji} - 2\delta_{ji}\) we can then rearrange the white sum to make the above look more symmetrical

\[
\sum_i K_{ji}^{-1} (\lambda_i - \alpha_i) = \sum_i K_{ji}^{-1} \left( -\lambda_i - \hat{K}_{il} \lambda_l \right)
\]

\[
= - \sum_i K_{ji}^{-1} \lambda_i + \sum_{i,l} K_{ji}^{-1} \hat{K}_{il} \lambda_l \delta_{il} \delta_{iB}
\]

\[
= - \sum_i K_{ji}^{-1} \lambda_i - \sum_{i,l} K_{ji}^{-1} (K_{ij} - 2\delta_{il}) \lambda_l \delta_{iB}
\]

\[
= - \delta_{jB} \lambda_j + \sum_i K_{ji}^{-1} (2\delta_{iB} - \delta_{iW}) \lambda_i.
\]

(2.11)

In this calculation the crucial point is that \(\hat{K}_{ij}\) contains only cross terms between white and black indices, and so we are able to replace a sum over purely white indices by a sum over all of them. Now we can rewrite \(2.10\) in the form

\[
(w - 1)^{-1} \lambda_j = - \delta_{jB} \lambda_j + \sum_i K_{ji}^{-1} c(i) \lambda_i.
\]

(2.12)

Using \(2.12\) and \(2.6\) in \(2.2\) we find that

\[
\epsilon(\tau_j, k) = \exp \left( 2\pi i \frac{K_{\tau k}^{-1} \alpha_k^2}{\alpha_k^2} \right).
\]

(2.13)

Substituting \(K_{\tau k}^{-1} = 2\lambda_\tau \cdot \lambda_k / \alpha_k^2\) finally yields

\[
\epsilon(\tau_j, k) = \exp \left( 2\pi i \frac{2\lambda_\tau \cdot \lambda_k}{\alpha_k^2} \right),
\]

(2.14)

which proves the conjecture of [1], in a corrected form.1

3 Discussion

It is important to have a proof of the formula \(2.14\), as it has since been used by Kneipp and Olive [9] to derive an interesting identity in a given representation of the Kac-Moody group which encapsulates the crossing of soliton into antisoliton. Of course \(2.14\) was also important for the work of [1], where it appeared to rationalise a number of things known at the time.

1The reason the sign of the exponential differs is due to an incorrect assignment of eigenvectors in [1].
It would be interesting to know if this sort of argument could be applied to the more general vertex operator constructions described in [10] and used to construct non-abelian Toda theories and their soliton-like solutions in [11].

Finally we note that a variety of interesting identities for the Coxeter element and its eigenvectors are already known (see for example [12], [13], [14]).

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References

[1] D.I. Olive, N. Turok, and J.W.R. Underwood. Affine Toda solitons and vertex operators. preprint Imperial/TP/92-93/29, SWAT/92-93/5, PUP-TH-93/1392, 1993.

[2] V.G. Kac, D.A. Kazhdan, J. Lepowsky, and R.L. Wilson. Realisation of the basic representation of the Euclidean Lie algebras. Advances in Mathematics, 42:83, 1981.

[3] V.G. Kac. Infinite-dimensional Lie algebras. Cambridge University Press, 1990.

[4] A.N. Leznov and M.V. Saveliev. Group-Theoretical Methods for Integration of Nonlinear Dynamical Systems, volume 15 of Progress in Physics. Birkhauser-Verlag, Basel, 1992.

[5] T.J. Hollowood. Solitons in affine Toda field theories. Nucl. Phys., B384:523, 1992.

[6] A. Fring, H.C. Liao, and D.I. Olive. The mass spectrum and coupling in affine Toda field theory. Phys. Lett., B266:82, 1991.

[7] B. Kostant. The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group. American Journal of Mathematics, 81:973, 1959.

[8] D.I. Olive, N. Turok, and J.W.R. Underwood. Solitons and the energy-momentum tensor for affine Toda theory. Nucl. Phys. B, to appear, 1993.

[9] M.A.C. Kneipp and D.I. Olive. Crossing and antisolitons in affine Toda theories. preprint SWAT/92-93/6, 1993.

[10] V.G. Kac and D.H. Peterson. 112 constructions of the basic representation of the loop group of $E_8$. In Proceedings of the Conference ‘Anomalies, Geometry, Topology’, Argonne. World Scientific, March 1985.
[11] J.W.R. Underwood. Aspects of non-abelian Toda theories. *preprint Imperial/TP/92-93/30*, 1993.

[12] P.E. Dorey. Root systems and purely elastic S-matrices. *Nucl. Phys.*, B358:654, 1991.

[13] P.E. Dorey. Root systems and purely elastic S-matrices 2. *Nucl. Phys.*, B374:741, 1992.

[14] A. Fring and D.I. Olive. The fusing rule and scattering matrix of affine Toda theory. *Nucl. Phys.*, 379B:429, 1992.