Abstract. In this paper, some known and novel properties of the Cauchy and signaling problems for the one-dimensional time-fractional diffusion-wave equation with the Caputo fractional derivative of order \(1 \leq \beta \leq 2\) are investigated. In particular, their response to a localized disturbance of the initial data is studied. It is known that whereas the diffusion equation describes a process where the disturbance spreads infinitely fast, the propagation velocity of the disturbance is a constant for the wave equation. We show that the time-fractional diffusion-wave equation interpolates between these two different responses in the sense that the propagation velocities of the maximum points, centers of gravity, and medians of the fundamental solutions to both the Cauchy and the signaling problems are all finite. On the other hand, the disturbance spreads infinitely fast and the time-fractional diffusion-wave equation is non-relativistic like the classical diffusion equation. In this paper, the maximum locations, the centers of gravity, and the medians of the fundamental solution to the Cauchy and signaling problems and their propagation velocities are described analytically and calculated numerically. The obtained results for the Cauchy and the signaling problems are interpreted and compared to each other.

1. Introduction

By the fractional diffusion-wave equation we mean a linear integro partial differential equation obtained from the classical diffusion or wave equation by replacing the first- or second-order time derivative by a fractional derivative (in the Caputo sense) of order \(\beta\) with \(0 < \beta \leq 2\). In our notations it reads

\[
\frac{\partial^\beta u}{\partial t^\beta} = \mathcal{D} \frac{\partial^2 u}{\partial x^2}, \quad u = u(x, t), \quad 0 < \beta \leq 2, \quad \mathcal{D} > 0,
\]

where \(\mathcal{D}\) denotes a positive constant with the dimension \(L^2 T^{-\beta}\), \(x\) and \(t\) are the space and time variables, and \(u = u(x, t)\) is the field variable, which is assumed to be a causal function of time, i.e. vanishing for \(t < 0\).

Recalling the definition of the Caputo fractional derivative, see e.g. Gorenflo and Mainardi [7], Podlubny [31], and setting for convenience, but without loss of generality, \(\mathcal{D} \equiv 1\), we get in explicit form the following integro-differential equations

\[
\frac{1}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} \left( \frac{\partial u}{\partial \tau} \right) d\tau = \frac{\partial^2 u}{\partial x^2}, \quad 0 < \beta \leq 1;
\]
(1.3) \[ \frac{1}{\Gamma(2-\beta)} \int_0^t (t-\tau)^{1-\beta} \left( \frac{\partial^2 u}{\partial \tau^2} \right) d\tau = \frac{\partial^2 u}{\partial x^2}, \quad 1 < \beta \leq 2. \]

The equations (1.2) and (1.3) can be properly referred to as the time-fractional diffusion and the time-fractional wave equation, respectively.

A fractional diffusion equation akin to (1.2) has been formerly introduced in 1986 by Nigmatullin [29] to describe diffusion in special types of porous media, which exhibit a fractal geometry. In 1995 Mainardi [19] has shown that the fractional wave equation (1.3) governs the propagation of mechanical diffusive waves in viscoelastic media which exhibit a simple power-law creep. This problem of dynamic viscoelasticity was formerly treated by Pipkin [30] and Kreiss and Pipkin [8] who however were unaware of the interpretation by fractional calculus. In our opinion, the above references provide some interesting and pioneering examples of the relevance of (1.1) in physics. Of course, any time some hereditary mechanisms of power-law type are present in diffusion or wave phenomena, an appearance of time-fractional derivatives in the evolution equations is expected.

In a series of papers [18, 19, 20, 21, 22, 23], Mainardi has pursued his analysis on the time-fractional diffusion-wave equation (1.1) based on Laplace transforms and special functions of Wright type. Other mathematical aspects of integro-differential equations akin to (1.1) based on the use of the integral transforms and special functions have been also treated in some relevant papers including those by Wyss [34], Schneider and Wyss [33], Fujita [2], Prüss [32], Mainardi, Luchko and Pagnini [25], Mainardi, Pagnini and Saxena [26], and more recently by Luchko [12, 14, 15]. Furthermore, mathematical aspects related to similarity properties and stable probability densities have been treated by Fujita [3], Engler [1], Mainardi and Tomirotti [28], Luchko and Gorenflo [9], Gorenflo, Luchko and Mainardi [4, 5], and more recently by Luchko, Mainardi and Povstenko [17], and by Luchko and Mainardi [16]. We also outline the papers [11, 13] by Luchko on the application of the maximum principle to the time-fractional diffusion equations. Of course, the above list of references is not exhaustive and mainly regards those that have attracted our attention.

In this paper, we consider Eq. (1.1) restricting out analysis to the case \( 1 \leq \beta \leq 2 \) that we refer to as the time-fractional diffusion-wave equation. Our main purpose is to point out some relevant properties of the related intermediate process that governs transition from pure diffusion (\( \beta = 1 \)) to pure wave propagation (\( \beta = 2 \)).

In the second section, we define the two basic boundary-value problems, referred to as the Cauchy problem and the Signaling problem, recalling for them the respective fundamental solutions (the Green functions). We outline a reciprocity relation between the Green functions themselves in the space-time domain. In view of this relation the Green functions can be expressed in terms of two interrelated auxiliary functions in the similarity variable \( r = |x|/t^{\beta/2} \). In some plots, the evolution of the fundamental solutions of both the Cauchy and Signaling problems for some values of the order of the time derivative is shown. These solutions exhibit a pulse-like pattern moving along the \( x \) axis that depends of the order of the time-fractional
derivative. This allows us to better recognize the processes intermediate between diffusion and wave propagation.

In the third section, we analyze the location and the evolution of the maximum of the pulse like patterns and its dependence of the order of the fractional derivative. In particular, we present both an analytical treatment of the maximum locations, maximum values, and the propagation velocities of the maximum points of the Green functions and their plots.

Then in the fourth section we consider the location of the center of gravity and of the median for these pulse-like patterns in order to compare their evolution with respect to that of the corresponding maximum.

Finally, in the last section some conclusive remarks are given.

2. Cauchy and signaling problems

As it is well known, the two basic boundary-value problems for the evolution equations of diffusion and wave type are the Cauchy and the Signaling problems. Extending the classical analysis to our fractional equation (1.1), and denoting by \( f(x) \) and \( h(t) \) two given, sufficiently well-behaved functions, the basic problems are thus formulated as following:

\[
(2.1) \text{Cauchy problem: } u(x, 0^+) = f(x), -\infty < x < +\infty, \\
u(\mp \infty, t) = 0, t > 0;
\]

\[
(2.2) \text{Signaling problem: } u(x, 0^+) = 0, 0 < x < +\infty, \\
u(0^+, t) = h(t), u(+\infty, t) = 0, t > 0.
\]

For \( 1 < \beta \leq 2 \), the initial value of the first-order time derivative of the field variable, \( \frac{\partial}{\partial t}u(x, 0^+) = g(x) \), is required in the above problems, since in this case the second time derivative appears in the integro-differential equation (1.3) and, consequently, two linearly independent solutions are to be determined. In what follows, we mainly limit ourselves to the case \( g(x) \equiv 0 \).

In view of our analysis, we find it convenient to put

\[
(2.3) \nu = \frac{\beta}{2}, \quad 0 < \nu < 1.
\]

For the Cauchy and Signaling problems, we introduce the so-called Green functions \( G_c(x, t; \nu) \) and \( G_s(x, t; \nu) \), which represent the fundamental solutions that are obtained when \( f(x) = \delta(x) \) and \( h(t) = \delta(t) \), where \( \delta \) denotes the Dirac \( \delta \)-function. It should be noted that the Green function for the Cauchy problem turns out to be an even function of \( x \), so \( G_c(x, t; \nu) = G_c(|x|, t; \nu) \).

For \( 0 < \nu \leq 1 \), the two Green functions are connected by the following reciprocity relation (see the already cited papers by Mainardi):

\[
(2.4) 2\nu x G_c(x, t; \nu) = t G_s(x, t; \nu) = F_\nu (r) = \nu r M_\nu (r),
\]

with the similarity variable

\[
(2.5) r = x/t^\nu > 0,
\]
with $x > 0$, $t > 0$, $r > 0$. Above the auxiliary functions $F_\nu(r)$ and $M_\nu(r)$ are Wright functions (of the second type) defined in the whole complex domain $z \in \mathbb{C}$ for $0 < \nu < 1$ as follows:

\begin{equation}
F_\nu(z) = \frac{1}{2\pi i} \int_{Ha} e^{\sigma} - z \sigma^\nu d\sigma = \sum_{n=1}^{\infty} \frac{(-z)^n}{n! \Gamma(-\nu n)},
\end{equation}

\begin{equation}
M_\nu(z) = \frac{1}{2\pi i} \int_{Ha} e^{\sigma} - z \sigma^\nu \frac{d\sigma}{\sigma^{1-\nu}} = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(-\nu n + (1-\nu))},
\end{equation}

where $Ha$ denotes the Hankel path properly defined for the representation of the reciprocal of the Gamma function.

In Fig. 1 some plots of the Green functions for both the Cauchy and signaling problems for different values of $\nu$ and for the fixed time instant $t = 1$ are presented. For numerical methods that were used to produce these plots we refer the interested reader to [6], [10], [16], and [17].

For more details regarding the auxiliary functions and their properties we refer the interested reader to the appendix F of the book by Mainardi [24].

3. Maximum locations of the Green functions

In this section, we deal with the maximum locations, maximum values, and propagation velocities of the maximum points of the Green functions $G_c$ and $G_s$. Our analysis follows the results presented in Luchko, Mainardi and Povstenko [17] for the Cauchy problem and in Luchko, Mainardi [16] for the signaling problem and is restricted to the most interesting and important case of $1/2 < \nu < 1$.

3.1. The Cauchy problem. In the paper by Fujita [2], a probabilistic proof of the fact that the Green function $G_c(x,t;\nu)$ of the Cauchy problem attains its maximum at the points

\begin{equation}
x_*(t) = \pm c_\nu t^\nu, \quad \nu = \beta/2
\end{equation}
for each \( t > 0 \), where \( c_\nu > 0 \) is a constant determined by \( \nu \), \( 1/2 < \nu < 1 \), has been presented for the first time. In Luchko, Mainardi and Povstenko [17], an analytical proof of this relation was given.

As mentioned in Fujita [2], the maximum point of the Green function \( G_c(x, t; \nu) \) propagates for \( t > 0 \) with a finite velocity \( V_c(t, \nu) \) that is determined by

\[
(3.2) \quad V_c(t, \nu) := x_\nu'(t) = \nu c_\nu t^{\nu - 1}.
\]

This formula shows that for every \( \nu \), \( 1/2 < \nu < 1 \) the propagation velocity of the maximum point of the Green function \( G_c \) is a decreasing function in \( t \) that varies from \( +\infty \) at time \( t = 0^+ \) to zero as \( t \to +\infty \).

For \( \nu = 1/2 \) (diffusion), the propagation velocity is equal to zero because of \( c_{1/2} = 0 \) whereas for \( \nu = 1 \) (wave propagation) it remains constant and is equal to \( c_1 = 1 \). In Fig. 2, some plots of the propagation velocity of the maximum point of the Green function \( G_c \) are given for different values of \( \nu \).

The maximum value \( G_c^*(t; \nu) \) of \( G_c(x, t; \nu) \) is given by the formula

\[
(3.3) \quad G_c^*(t; \nu) = m_\nu t^{-\nu}, \quad m_\nu = \frac{1}{2} M_\nu(c_\nu) = \frac{1}{\pi} \int_0^\infty E_{2\nu}(-\tau^2) \cos(c_\nu \tau) d\tau,
\]

\( E_\alpha \) being the Mittag-Leffler function (see [17]).

It follows from the relations (3.1) and (3.3) (and of course directly from the formula (2.4)) that the product

\[
(3.4) \quad G_c^*(t; \nu) \cdot x_\nu(t) = c_\nu m_\nu, \quad 0 < t < \infty,
\]

is a constant that depends only on \( \nu \), i.e., that the maximum locations and the corresponding maximum values specify a certain hyperbola for a fixed value of \( \nu \) and for \( 0 < t < \infty \).

In Fig. 2 we give some plots of the parametric curve \( (x_\nu(t), G_c^*(t; \nu)) \), \( 0 < t < \infty \) that is in fact a hyperbola for different values of \( \nu \). The vertex of the hyperbola tends to the point \((0, 0)\) as \( \nu \) tends to \( 1/2 \) (diffusion equation) and to infinity as \( \nu \to 1 \) (wave equation).

Finally, in Fig. 3 the maximum locations \( c_\nu \) and the maximum values \( m_\nu \) as well as their product are plotted for \( 1/2 < \nu < 1 \). As expected,
the product $c_\nu m_\nu$ is a monotonically increasing function that takes values between 0 (diffusion equation) and $+\infty$ (wave equation).

3.2. The Signaling problem. In this subsection, we present some results regarding the maximum location and the maximum value of the Green function $G_s$ for the signaling problem as a function in the spatial variable $x$ for the fixed values of $\nu$ and $t$ (see [16] for more details).

As in the previous subsection, the formula (2.4) is employed to get the following representation for the maximum location $x_s = x_s(t, \nu)$ of the Green function $G_s(x,t;\nu)$:

$$(3.5) \quad x_s(t, \nu) = Dt^{\nu} \quad \text{with} \quad D = x_s(1, \nu) = d_\nu.$$

Having determined the maximum location of $G_s$, we can now calculate the propagation velocity $V_s(t, \nu)$ of the maximum point. If follows from (3.5) that

$$(3.6) \quad V_s(t, \nu) = \frac{dx_s}{dt} = \nu d_\nu t^{\nu - 1}.$$

As we see, the propagation velocity $V_s(t, \nu)$ of the maximum point of the Green function $G_s$ is described by a formula of the same type as the one for the Green function $G_c$ (see the formula (3.2)). Its qualitative behavior is therefore very similar to that presented in Fig. 2 as one can see on the plots of Fig. 3. The only essential difference between the plots of Figs. 2 and 3 is in the curve for $\nu = 1/2$ (diffusion equation). Whereas the maximum location of the Green function for the Cauchy problem for the diffusion equation does not move with the time ($c_{1/2} = 0$), its velocity for the signaling problem is equal to $\frac{\sqrt{2}}{\sqrt{t}}$ (see the formula (3.6)).

The maximum value $G^*_s(t; \nu)$ of $G_s(x,t;\nu)$ in dependence of $t$ and $\nu$ is given by the following formula:

$$(3.7) \quad G^*_s(t; \nu) = \frac{n_\nu}{t} \quad \text{with} \quad n_\nu = F_\nu(d_\nu) = \frac{2}{\pi} \int_0^\infty \tau E_{2\nu,2\nu}(-\tau^2) \sin(d_\nu \tau) d\tau,$$

$E_{\alpha,\beta}$ being the generalized Mittag-Leffler function (see [16]). The maximum locations and the maximum values of the Green function $G_s(x,1;\nu)$ for the intermediate values of $\nu$, $1/2 < \nu < 1$ are presented in Fig. 4.
Let us finally note that it follows from the formulas (3.5) and (3.7) (and of course directly from the formula (2.4)) that the product $p_\nu$ of the maximum location and the maximum value of the Green function $G_s(x,t;\nu)$ is time-dependent

$$p_\nu = p_n u(t) = G_s^* (t;\nu) \cdot x_*(t) = d_\nu t^\nu n_\nu t^{-1} = d_\nu n_\nu t^{\nu-1}$$

and follows the formula of the same type as the one for the propagation velocities of the maximum locations for the Green functions $G_s$ and $G_c$.

4. Centers of gravity and medians of the Green functions $G_c$ and $G_s$

In this section, some new results regarding locations and velocities of the centers of gravity and medians of the Green functions both for the Cauchy and the signaling problems are presented. The key role in all calculations is played by the formula (2.4) that connects the Green functions $G_c$ and $G_s$ with the special functions of the Wright type. For the readers convenience we list here some formulas for the Mainardi function $M_\nu$ that are used in the further discussions.

The asymptotics of $M_\nu$ is described by the following formula (see e.g. [25]):

$$M_\nu(r) \sim A_0 Y^{-\nu-1/2} \exp (-Y), \quad r \to \infty,$$

$$A_0 = \frac{1}{\sqrt{2\pi} (1-\nu)^{\nu} \nu^{2\nu-1}}, \quad qY = (1 - \nu) (\nu^\nu r)^{1/(1-\nu)}.$$ 

As $r \to 0$, $M_\nu(r)$ evidently tends to $1/\Gamma(1 - \nu)$.

The known Mellin transform of the Wright function

$$\mathcal{M} \{ W_{\lambda,\mu}(-x); s \} = \frac{\Gamma(s)}{\Gamma(\mu - \lambda s)} \quad 0 < \Re(s), \quad \lambda < 1 \text{ or } 0 < \Re(s) < \Re(\mu)/2 - 1/4, \quad \lambda = 1$$
leads to the following formula for the Mellin transform of the Mainardi function $M_{\nu}$:

$$(4.3) \quad \mathcal{M}\{M_{\nu}(x); s\} = \frac{\Gamma(s)}{\Gamma(1 - \nu + \nu s)}, \quad 0 < \Re(s), \quad -1 < \nu.$$ 

Let us note that the Mellin integral transform of a sufficiently well-behaved function $f$ is defined as

$$(4.4) \quad \mathcal{M}\{f(x); s\} = f^*(s) = \int_{0}^{+\infty} f(x)x^{s-1}dx.$$ 

4.1. **Center of gravity of the Green function for the Cauchy problem.** Let us start with calculation of the center of gravity of the Green function $G_c$. Because $G_c$ is an even function, we restrict our attention to the function $G_c(r,t; \nu) = G_c(|x|, t; \nu)$, $r = |x| \geq 0$. The location $r_c^\nu(t)$ of the center of gravity of $G_c(r,t; \nu)$ is defined by the formula

$$(4.5) \quad r_c^\nu(t) = \frac{\int_{0}^{\infty} r G_c(r,t; \nu) dr}{\int_{0}^{\infty} G_c(r,t; \nu) dr}.$$ 

Using the formulas (2.4) and (4.3) and a linear variables substitution, we can calculate both integrals in (4.5) in explicit form. For the first integral we get

$$\int_{0}^{\infty} r G_c(r,t; \nu) dr = \int_{0}^{\infty} \frac{1}{2^\nu} \frac{M_{\nu}(r/t^\nu)}{2} dr = \frac{\Gamma(2)}{2 \Gamma(1 - \nu + 2\nu)} = \frac{t^\nu}{2 \Gamma(1 + \nu)}.$$ 

The second integral in (4.5) is evidently equal to $1/2$ because $G_c(x,t; \nu)$ is a probability density function (pdf) in $x$ evolving in time that is an even function (see e.g. [25]). Of course, this integral can be calculated explicitly following the same method we applied for the first integral.

The final formula for the location of the center of gravity of the Green function $G_c$ of the Cauchy problem is as follows:

$$(4.6) \quad r_c^\nu(t) = g_c(\nu) t^\nu, \quad g_c(\nu) = \frac{1}{\Gamma(1 + \nu)}.$$ 

For the diffusion equation with $\nu = 1/2$, the location $r_{c1/2}^\nu(t)$ of the center of gravity for a fixed $t$ is at the point $\sqrt{2t\pi}$ whereas for the wave equation we get as expected $r_{c1}^\nu(t) = t$.

As we can see from the formula (4.6), the location of the center of gravity is a power function in $t$ with the coefficient $g_c(\nu)$ that depends on $\nu$. On the other hand, $g_c(\nu)$ describes the location of the center of gravity of $G_c$ at the time instant $t = 1$. A plot of the function $g_c(\nu)$, $1/2 \leq \nu \leq 1$ is presented in Fig. 5. Because $\Gamma(1 + \nu)$ is a monotonically increasing function for $1/2 \leq \nu \leq 1$, the function $g_c(\nu)$ monotonically decreases from $\frac{2}{\sqrt{\pi}} \approx 1.1284$ (diffusion equation) to 1 (wave equation).

The velocity $V_c^\nu(t, \nu)$ of the center of gravity of $G_c$ is given by the formula

$$(4.7) \quad V_c^\nu(t, \nu) = \frac{d}{dt} r_c^\nu(t) = \frac{t^{\nu-1}}{\Gamma(\nu)},$$ 

$$\frac{1}{\sqrt{\pi}} \approx 1.1284.$$
so that again we obtain a formula of the same type as the one for velocities of the maximum locations of the Green functions (with a different coefficient). The plots of the velocity $V_c^2(t, \nu)$ for different values of $\nu$ look like the ones presented in Fig. 4 and are given in Fig. 5.

### 4.2. Center of gravity of the Green function for the Signaling problem

We consider now the location of the center of gravity of the Green function $G_s(x; 1; \nu)$ of the signaling problem. It is known (see e.g. [16]) that for a fixed $x > 0$ and for a fixed $\nu$, $1/2 \leq \nu < 1$, the Green function $G_s(x; t; \nu)$ is a one-sided stable probability density function (pdf) of the time variable $t > 0$. A prominent example is the function $G_s(x; t; 1/2)$ that for $x = 1$ is called the Lévy-Smirnov pdf. It follows from the formulas (2.4) and (4.1) that the location of the gravity center of $G_s$ with respect to the time variable $t$ is in infinity for all $x > 0$. That is why we consider the location of the gravity center of $G_s$ with respect to the spatial variable $x$ that is defined by the formula

$$r_s^a(t) = \frac{\int_0^\infty r G_s(r, t; \nu) \, dr}{\int_0^\infty G_s(r, t; \nu) \, dr}.$$

To evaluate the integrals in the formula (4.8), we again employ the relation (2.4), a linear variables substitution in the integrals, and the formula (4.3) for the Mellin transform of the Mainardi function $M_\nu$ and thus get the following results:

$$\int_0^\infty G_s(r, t; \nu) \, dr = \nu t^{\nu-1} \int_0^\infty u M_\nu(u) \, du = \frac{\nu}{\Gamma(1 + \nu)^\nu},$$

$$\int_0^\infty r G_s(r, t; \nu) \, dr = \nu t^{2\nu-1} \int_0^\infty u^2 M_\nu(u) \, du = \frac{2\nu}{\Gamma(1 + 2\nu)^2} t^{2\nu-1}.$$
the Green function $G_s$:

$$r^s_\nu(t) = \frac{\Gamma(\nu) t^{2\nu-1}}{\Gamma(1+2\nu)\Gamma(\nu)} = \frac{\Gamma(\nu)}{\Gamma(2\nu)} t^\nu = \frac{\sqrt{\pi} 2^{1-2\nu}}{\Gamma(\nu + \frac{1}{2})} t^{\nu}.$$  

The two known particular cases of this formula are $r^s_{1/2}(t) = \sqrt{\pi} t^{1/2}$ (diffusion equation) and $r^s_1(t) = t$ (wave equation).

At the time instant $t = 1$, we get the relation

$$g_s(\nu) := r^s_\nu(1) = \frac{\sqrt{\pi} 2^{1-2\nu}}{\Gamma(\nu + \frac{1}{2})}.$$  

A plot of the function $g_s(\nu)$ is presented in the Fig 6. As we can see, the function $g_s(\nu)$ monotonically decreases from the value $\sqrt{\pi} \approx 1.7725$ at the point $\nu = 1/2$ (diffusion equation) to the value 1 at the point $\nu = 1$ (wave equation). Surprisingly, the plot of $g_s(\nu)$ is very similar to a straight line, but of course $g_s(\nu)$ is not a linear function on the interval $1/2 \leq \nu \leq 1$.

The velocity $V^s_\nu(t, \nu)$ of the center of gravity of $G_s$ is calculated via the formula

$$V^s_\nu(t, \nu) = \frac{d}{dt} r^s_\nu(t) = \frac{\sqrt{\pi}^{2^{1-2\nu}}}{\Gamma(\nu + \frac{1}{2})} t^{\nu-1}.$$  

Some plots of the velocity $V^s_\nu(t, \nu)$ for different values of $\nu$ are presented in Fig. 6 (of course, they are very similar to those shown in Fig. 5).

### 4.3. Medians of the Green functions for the Cauchy and Signaling problems

Finally, we consider the locations of the medians of the Green functions $G_c$ and $G_s$.

The location $x = x^c_m$ of the median of the Green function $G_c(r, t; \nu)$, $r \geq 0$ can be determined from the equation

$$\int_0^{x^c_m} G_c(r, t; \nu) \, dr = \frac{1}{2} \int_0^{\infty} G_c(r, t; \nu) \, dr = \frac{1}{4}.$$  

\[\int_0^{x^c_m} G_c(r, t; \nu) \, dr = \frac{1}{2} \int_0^{\infty} G_c(r, t; \nu) \, dr = \frac{1}{4}.\]
Because
\[ \int_0^{x_m^c} G_c(r,t;\nu) \, dr = \int_0^{x_m^c} \frac{1}{2\nu} M_\nu(r/t^\nu) \, dr = \int_0^{x_m^c/t^\nu} \frac{1}{2} M_\nu(u) \, du, \]
the location of the median can be determined from the equation
\[ (4.12) \quad \int_0^{x_m^c/t^\nu} \frac{1}{2} M_\nu(u) \, du = \frac{1}{4}. \]
The integral \( \int_0^{x} M_\nu(u) \, du \) monotonically increases from 0 to 1 as \( x \) varies from 0 to \( +\infty \) and thus the equation (4.12) has a unique solution that of course depends on \( \nu \) and is denoted by \( m_c(\nu) \). Then we first get the relation
\[ x_m^c/t^\nu = m_c(\nu) \]
and then the formula
\[ (4.13) \quad x_m^c = m_c(\nu)t^\nu \]
for the location of the median. Once again, we see that the location of the median is a power function in \( t \) with the exponent \( \nu \) and the coefficient \( m_c(\nu) \) that corresponds to the location of the median at the time instant \( t = 1 \).

For the fixed \( x > 0 \) and \( \nu, 1/2 \leq \nu \leq 1 \), the location \( t = t_m^s \) of the median of the Green function \( G_s(x,t;\nu) \) of the signaling problem for the fixed \( x \) and \( \nu \) is determined from the equation
\[ \int_0^{t_m^s} G_s(x,t;\nu) \, dt = \frac{1}{2}. \]
Employing the same arguments as in the case of the Green function \( G_c \), we arrive at the equation
\[ (4.14) \quad \int_0^{+\infty} M_\nu(u) \, du = \frac{1}{2} \]
that evidently has the same solution \( m_c(\nu) \) as the equation (4.12) because of the fact that the integral \( \int_0^{x} M_\nu(u) \, du \) monotonically increases from 0 to 1 as \( x \) varies from 0 to \( +\infty \). It follows from (4.14) that
\[ x/(t_m^s)^\nu = m_c(\nu) \]
and
\[ (4.15) \quad t_m^s = m_s(\nu)x^{1/\nu}, \quad m_s(\nu) = \frac{1}{(m_c(\nu))^{1/\nu}}. \]

5. Conclusions and discussions
In this paper, we deal with some important properties of the Green functions of the Cauchy and signaling problems for the one-dimensional time-fractional diffusion-wave equation with the constant coefficients. It is known that these functions are relevant to characterize the evolution of the pulse-like initial data that appears as an intermediate process between diffusion and wave propagation. Except in the limiting case of the wave equation, the pulses propagate with infinite velocities that is typical for evolution equations of the parabolic type. These processes are common to refer to as the diffusive waves. In this paper, we show that the maximum locations and
the centers of gravity (in space) of the diffusive waves always propagate with a finite velocity that is determined by a power law in time. The exponent of the power law (related to the order of the fractional derivative in the time-fractional diffusion-wave equation) is the same for the Cauchy and signaling problems as expected from the similarity properties of the evolution equation. Whereas the location of the maxima and their velocities are determined numerically, the corresponding quantities for the centers of gravity are obtained analytically in a closed form. In the absence of a finite wave-front velocity that is typical for the standard waves, the velocities mentioned above can be interpreted as a sort of characteristic signal velocity of the diffusive waves and thus worthy to be investigated and calculated.

6. Acknowledgments

The authors appreciate constructive remarks and suggestions of the anonymous referees that helped to improve the manuscript.

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1 Professor of Mathematics, Department of Mathematics, Physics, and Chemistry, Beuth Technical University of Applied Sciences, 13353 Berlin, Germany.

E-mail address: luchko@beuth-hochschule.de

2 Professor of Mathematical Physics, Department of Physics and Astronomy, Bologna University, and INFN, 40126 Bologna, Italy.

E-mail address: francesco.mainardi@unibo.it; francesco.mainardi@bo.infn.it