Butterfly Effect and Spatial Structure of Information Spreading in a Chaotic Cellular Automaton

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In this work, we provide a basic description of such information-spreading in a minimal chaotic setting: the Kauffmann Cellular Automaton (KCA). Our system choice is motivated by the fact that KCA are minimal chaotic many-body models [27] which display a rich phenomenology, including a phase transition as a function of a tuning parameter, a probability $p$. They exhibit universal scaling, e.g. of the directed percolation universality class [28], and lend themselves to analytical insight [29]. Additionally, KCA have a wide range of applicability [30–36]: initially introduced to study fitness landscapes of biological systems and gene expression [37], they now appear also in optimisation problems [38], random mapping models [39], and most pertinently in the emergence of chaos [40].

Surprisingly, despite decades of research on chaotic CA, see Ref. [41] for a review, the dynamics of chaos has only

\begin{equation}
D(x, t) \propto e^{-\mu(v - v_b) t} = e^{-\lambda(v)t}.
\end{equation}

Intriguingly, a unifying framework of VDLEs with $\lambda(v) = \mu(v - v_b)^\beta$ captures the spatio-temporal structure of information spreading in many-body quantum, semiclassical and classical chaotic systems [1, 3, 5, 20, 26].

In this work, we provide a basic description of such information-spreading in a minimal chaotic setting: the Kauffmann Cellular Automaton (KCA). Our system choice is motivated by the fact that KCA are minimal chaotic many-body models [27] which display a rich phenomenology, including a phase transition as a function of a tuning parameter, a probability $p$. They exhibit universal scaling, e.g. of the directed percolation universality class [28], and lend themselves to analytical insight [29]. Additionally, KCA have a wide range of applicability [30–36]: initially introduced to study fitness landscapes of biological systems and gene expression [37], they now appear also in optimisation problems [38], random mapping models [39], and most pertinently in the emergence of chaos [40].
been investigated in terms of the global Hamming distance but a local diagnostic has thus far been missing. Here, we construct an OTOC analogue for KCA and explore the VDLE phenomenology. This analogue enables us to uncover the ballistic spatio-temporal structure of perturbation spreading in the chaotic phase, Fig. 1(b), in contrast to the decay of such “damage” spreading in the frozen phase, Fig. 1(a). We develop a full microscopic theory of the VDLE, recovering the functional form Eq. (1) including an analytical calculation of the exponent β. Thus, we provide the tools for describing the sensitivity of chaotic many-body systems to perturbations through the framework of VDLEs.

**KCA Model and Classical OTOC Analogue.** We focus on a generic dissipative dynamical system called an NK model. Concretely, a local KCA is a system of N Boolean elements \( \sigma(x, t) = \pm 1 \) which evolve in discrete time steps through rules which depend upon 2K nearest neighbours of each site in 1D [41]. Our KCA system evolves under a set of (annealed) local rules \( \{ f_{x,t} \} \):

\[
s(x, t + 1) = f_{x,t} [\sigma(x - K, t), ..., \sigma(x, t), ..., \sigma(x + K, t)],
\]

which are random with probability \( p \) in space and time:

\[
f_{x,t} = \begin{cases} 
+1 & \text{with probability } p \\
-1 & \text{with probability } 1 - p.
\end{cases}
\]

In a pioneering work, Derrida and Stauffer [42] showed that KCA display a chaotic-to-frozen phase transition controlled by the parameter \( p \), see Fig. 1 inset. The two phases are distinguished by the decay or spread of localised perturbations diagnosed with the global Hamming distance,

\[
H(t) = \frac{1}{2N} \left\langle \sum_x |\sigma^A(x, t) - \sigma^B(x, t)| \right\rangle_p,
\]

between two copies of the system \( \sigma^{A/B}(x, t) \) which differ by a single inverted site in the initial state at \( t = 0 \). This measure is then ensemble-averaged over realisations with the same probability \( p \). The distance grows linearly in the chaotic phase and decays to zero in the frozen phase, see Fig. 1.

Analogous to the classical Heisenberg chain [20], we take the classical OTOC as the local distance between two copies of the same system which only differ by a local perturbation of the initial conditions, thus leading to the *decorrelator*

\[
D(x, t) = \frac{1}{2} \left[ 1 - \langle \sigma^A(x, t) \cdot \sigma^B(x, t) \rangle_p \right].
\]

It is nothing but a *local Hamming distance*, which is related to the global distance by \( H(t) = \frac{1}{N} \sum_x D(x, t) \).

**Numerical Results.** In Fig. 1 we show the spatio-temporal evolution of the decorrelator \( D(x, t) \) for representative values of \( p \) below and above \( p_c \). First, in the frozen phase \( D(x, t) \) initially spreads but then decays in time and space to zero. This attenuation of the local decorrelator reflects the vanishing of the long time value of the Hamming distance in the frozen phase. Second, in the chaotic phase \( D(x, t) \) spreads with an apparent light-cone structure. Again, from the sum rule the long-time value \( D_0 = D(x = 0, t \to \infty) \) within the light-cone can be identified as the order parameter associated with the phase transition, see Fig. 1 inset.

Because of the locality of KCA rules, the speed of the damage spreading measured with \( D(x, t) \) is necessarily bounded by the maximum velocity \( v_{\text{max}} = K \) but the actual spread is slower, with the butterfly velocity \( v_b < K \). To demonstrate the presence of \( v_b \), we plot the behavior of the decorrelator along rays of constant velocity. In Fig. 2 we see that, after
This establishes the presence of a butterfly velocity \( v \). A transient effect, there is a ray (blue lines, corresponding to lines) of the boundary random walk model.

A small window around \( v \) nature of the variables the scaling behavior only appears in a pickup in time of the OTOC analogue. Because of the discrete after disorder-averaging one may still observe an exponential decay of the OTOC in time, with \( v \) “rays” of constant velocity \( v > v_b \) for \( v < v_b \). As shown in Fig. 2, the black dotted line is the Gaussian profile predicted by the random walk model. The earliest two times \( t = 50, 100 \) show slight deviations from the equilibrium/long-time limit. Upper inset: unscaled boundary probability density at \( t = 50, 100, ..., 300 \). Lower inset: Raw data of boundary positions plotted against time. The red line is the average boundary, with the velocity \( v = x/t = 2.9 \).

Next, we investigate whether such behaviour follows the general VDLE phenomenology of Eq. (1). We note that in contrast to previous work on Heisenberg spin chains, the presence of Lyapunov exponents in KCA is far from obvious. In contrast to previous work on Heisenberg spin chains, the presence of Lyapunov exponents in KCA is far from obvious. In particular, the numerical results establish the presence of a butterfly velocity \( v_b \) for all \( v > v_\text{max} \) and is an efficient way of extracting \( v_b \) as a function of \( p \), see blue dots in the inset of Fig. 4.

Boundary Random Walk Model. We can now develop a microscopic statistical model of the butterfly dynamics starting from the microscopic KCA rules. We only sketch the main idea and relegate details to the Supplementary Material. The basic ingredient for our random-walk model is the expectation of the outwards move of a damage site in each time-step: The furthest the boundary could move outwards in one time-step is \( K \), with probability \( p(K) = p_d = 2p(1-p) \); and the probability of moving \( x < K \) steps is \( p(x) = p^K_x p_d \), where \( p_x = p^2 + (1-p)^2 \). From this we obtain the mean and variance of the boundary steps and the Central Limit Theorem (CLT) allows us to obtain the full distribution (valid in the long time and \( p \gg p_c \) limit) as a Gaussian.

This model correctly predicts various aspects of the decorrelator. First, we obtain the long-time value of \( D(x,t) \) inside the light-cone as given by \( D_0 = 2p(1-p) \) which corresponds to the spins pointing up and down with random probability \( p \) and \( 1-p \), see the dashed line in the inset of Fig. 1(a).

Second, the model predicts the butterfly velocity, given by the following closed-form expression in \( p \) which can be read-
away from It agrees with the full model’s butterfly velocity at large 
Appendix for is obtained by numerically evaluating the full 
FIG. 4. Main figure: scaling collapse performed for a range of 
ily expanded into a power series

\[ v_b(p) = K - \frac{p_s}{p_d} = K - 1 - 2 \sum_{j=1}^{\infty} 4^j \left( \frac{1}{2} - p \right)^{2j}. \] (6)

It agrees with the full model’s butterfly velocity at large \( p \) away from \( p_c \), as shown in the inset of Fig. 4.

Third, the standard deviation of the boundary distribution is

\[ \sigma^2(t) = t \left( \frac{p_s}{p_d} + \frac{p_s^2}{p_d^2} \right) \equiv \frac{t}{2\mu^2(p)}. \] (7)

which confirms the Gaussian form of the boundary random walk with a variance that scales linearly with time. As detailed in the Supplementary Material, the cumulative distribution of the boundary then allows us to obtain the complete functional form of the OTOC as a function of \( v \), governed by its inverse width \( \mu(p) \). Figure 2 shows the predicted analytic form of the decorrelator (dashed lines) compared to the numerical data (solid lines). This quantitative agreement in the long-time limit confirms that the boundary controls the dynamics of chaos spreading of the full KCA model.

Most notably, in the long-time limit the analytical model recovers the linear decay of the decorrelator in time, described by a VDLE for \( v \) close to \( v_b \):

\[ \ln D(x = vt, t) = -\lambda(v)t = -\mu(p)(v - v_b)^2 t, \] (8)

with an exponent \( \beta = 2 \). Therefore, in this regime we expect a data collapse around \( v_b \) by plotting \( \ln(D)/t \) against \( \sqrt{\mu(p)(v - v_b)} \). Indeed, as shown in Fig. 4, the data of these two variables for a range of probabilities fall onto the single curve of the analytical prediction (black dashed line).

Discussion. Given the discrete nature of the dynamics, the quantitative agreement between our model and the full numerical simulation is somewhat surprising but highlights the universal features of chaos spreading. Crucially, our analysis is only valid after sample averaging and in the long time limit where the effects of the discrete perturbation and dynamics have been smoothed out. In particular, when approaching the critical point \( p_c \) from above we see systematic deviations due to fluctuations.

Our work on a minimal classical model is inspired by recent developments in the study of chaos in quantum many-body systems where OTOCs have become a powerful quantitative tool. One important prediction in that context is that the butterfly velocity is bounded at high temperatures \( v_b(T \to \infty) \sim v_{LR} \) where \( v_{LR} \) is the Lieb-Robinson velocity which is the upper limit of information propagation in short-range interacting non-relativistic quantum systems [26, 44]. For KCA one may connect the model parameter \( p \) to a temperature \( T \) via \( p = e^{-1/T}/(e^{1/T} + e^{-1/T}) \) [45]. Then from Eq.6 we can obtain the full temperature dependence of the butterfly velocity, and the high temperature limit is

\[ v_b(T \to \infty) \approx K - 1 - \frac{2}{T^2}. \] (9)

It confirms that also in our classical many-body system the maximum velocity of information spreading \( v_b(T \to \infty) = K - 1 \), is always less than the maximum \( v_{max} = K \) allowed by the local dynamics.

Conclusion and Outlook. We have constructed a local diagnostic of information spreading for a one-dimensional random CA in analogy with recent semi-classical versions of OTOCs. We demonstrated that it displays ballistic propagation characterised by a butterfly velocity and exponential growth in time captured by a VDLE. We developed a random walk model of the boundary of information spreading which permits the calculation of the full functional form of the classical OTOCs, including the exponent \( \beta \) of the VDLE.

An obvious extension of our work is to consider the 2-dimensional KCA where an even richer phenomenology is expected. Even though our boundary random walk model proposed here is simple, it has many physical and mathematical aspects that remain unexplored. For example, if the random walk is approximated to have continuous walk distance which still follows the same probability distribution, can it have a closed form solution for early times without having to invoke the CLT? Alternatively, it would be worthwhile to explore a Langevin like description of the boundary which is particularly useful while considering KCA in higher dimensions. For example, in 2D KCA, we expect a diffusion-like process for the perturbation which could potentially percolate across the lattice. In particular, it is interesting to investigate if this falls into any percolation universality class where established critical exponents could be linked to the exponent of the classical
OTOCS.

Generally we expect our decorrelator and theoretical tools to be applicable to other stochastic models with discrete variables, which are widely used to describe the dynamics of both quantum and classical many-body systems [46]. We expect that different variants of the boundary theory developed here should be able to predict the scaling forms in these systems as well. In particular, it would be interesting to see them applied to models with charge or dipole conservation rules [47, 48] where conjectured critical exponents could potentially be derived analytically.

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[49] The use of $-\infty$ as the lower limit of sum is a useful approximation for large $K$ as it provides a simple closed-form expression for $v_b$ and $\sigma$.

Supplementary Material

In this numerical analysis we perform a study of the average velocity of the boundary paths, which are observed to obey a Gaussian distribution (in the long-time limit). The central value of this distribution is in agreement with the probability-dependent butterfly velocity $v_b(p)$ for all $p > p_c$, indicating that there is only one characteristic velocity which governs chaos spreading. Moreover, the width of this Gaussian distribution spreads with $\sqrt{t}$ dependence, indicating its random-walk-like nature. A data collapse of the distributions in boundary position is provided in Fig. 3, showing good agreement with a Gaussian profile. There is residual skewness due to the initial perturbation which vanishes in the long-time limit.

We are motivated by this result to provide a probabilistic model of the boundary that predicts the butterfly velocity, and later we will use this model to recover the functional form of the VDLE $\lambda(v)$. This model involves calculating the expectation value of the furthest site outwards in each time-
The furthest the boundary could move outwards in one time-step is $K$, with probability $p(K) = p_d = 2p(1-p)$. The probability of moving $x$ steps is $p(x) = p_s^{K-x} p_d$, where $p_s = p^2 + (1-p)^2$.

A quick verification of this model is that the long-time value of $D(x,t)$ inside the light-cone, is given by $D_0 = 2p(1-p)$ which corresponds the spins pointing up and down with random probability $p$ and $1-p$.

Now we will determine the scaling form of the OTOCs from this Gaussian boundary model, which is expected to be valid in the long-time limit and at high $p$. The long-time requirement is to ensure the distribution profile converges to Gaussian under the CLT which conventionally requires $t > 30$. The requirement on high $p$ fixes the system in the chaotic phase and away from the point of phase transition, thus reduces finite-size effects and statistical fluctuations in numerical simulation.

The comparison between calculated $v_b$ and the numerical simulation is shown in the inset of Fig. 4 which demonstrates close agreement at high $p$. This model only works away from criticality, which is expected since close to $p_c$, there will arise correlations which span the system and reduce the velocity of edge proliferation.

One may evaluate moments of this probability density to calculate the time evolution of the boundary. The expectation of the outwards movement of the boundary in a single time-step is approximated by [49].

$$\langle \Delta x \rangle = \sum_{x=-\infty}^{K} xp(x) = \sum_{x=-\infty}^{K} xp_s^{K-x} p_d \quad (10)$$

so that the spacial profile as a function of time is $\langle x(t) \rangle = t(\Delta x)$.

By the CLT, one expects this profile to approach a Gaussian at large times, with a variance given by $\sigma^2(t) = t \left[ \langle (\Delta x)^2 \rangle - \langle \Delta x \rangle^2 \right]$. One may evaluate the moments of the probability density using the following trick

$$\langle \Delta x^n \rangle = \sum_{x=-\infty}^{K} x^n p_s^{K-x} p_d$$

$$= \left( \frac{d}{d \log p_s} + K \right) \sum_{x=0}^{\infty} e^{2 \log p_s x} \cdot (11)$$

This expression may be evaluated using the geometric series, and gives the surprisingly simple expression for the butterfly velocity

$$v_b = \frac{(x(t))}{t} = K - \frac{p_s}{p_d}, \quad (12)$$

and the time-dependent width of the Gaussian $G(x,t)$

$$\sigma^2(t) = t \left( \frac{p_s}{p_d} + \frac{p_s^2}{p_d^2} \right) = \frac{t}{2\mu^2(p)}. \quad (13)$$

For any one realisation, the boundary will trace a biased random walk in time, and inside of its chaos spreading will be its own scrambling region with expected value $D_0$ for the OTOC. The OTOC therefore acts like the cumulative probability density function of the boundary’s probability density, weighted such that it has a central value of $D_0 = 2p(1-p)$:

$$D(x,t) = D_0 \int_x^{\infty} G(x',t) \, dx'. \quad (14)$$

In our approximate model at high $p$ and late times we may take the Gaussian limit, therefore deriving the following form for the OTOC in terms of the error function $\text{erf}(x)$

$$D(x,t) = \frac{D_0(p)}{2} \left[ 1 - \text{erf} \left( \frac{x - v_b(p)t}{\sqrt{2}\sigma(t)} \right) \right]$$

$$= p(1-p) \left[ 1 - \text{erf} \left( \frac{(v - v_b(p))\sqrt{\mu(p)t}}{\sqrt{2} \sigma(t)} \right) \right] \quad (15)$$

where we have used $x = vt$ and $\sigma^2(t) = t/2\mu^2(p)$. By taking the series expansion of the error function in $v - v_b$ for large $x$ we recover an exponential decay of $D(x,t)$; in logarithmic form this is (when $v > v_b$)

$$\ln D(v,t) = \ln p(1-p) - \frac{1}{2} \ln \mu(v - v_b)^2 \pi t - \mu(v - v_b)^2 t. \quad (16)$$

Comparing this result to the general form of the scaling forms predicted by previous works on spin chains,

$$\ln D(v,t) \sim -\mu(v - v_b)^3 t, \quad (17)$$

one may identify the same behaviour in the long-time limit.