Comment on ”Infrared freezing of Euclidean QCD observables”

Irinel Caprini

National Institute of Physics and Nuclear Engineering,
Bucharest POB MG-6, R-077125 Romania

Jan Fischer†

Institute of Physics, Academy of Sciences of the Czech Republic,
CZ-182 21 Prague 8, Czech Republic

Abstract

Recently, P. M. Brooks and C.J. Maxwell [Phys. Rev. D74 065012 (2006)] claimed that the Landau pole of the one-loop coupling at $Q^2 = \Lambda^2$ is absent from the leading one-chain term in a skeleton expansion of the Euclidean Adler $D$ function. Moreover, in this approximation one has continuity along the Euclidean axis and a smooth infrared freezing, properties known to be satisfied by the ”true” Adler function. We show that crucial in the derivation of these results is the use of a modified Borel summation, which leads simultaneously to the loss of another fundamental property of the true Adler function: the analyticity implied by the Källen-Lehmann representation.

PACS numbers: 12.38.Bx, 12.38.Cy, 12.38.Aw

Keywords: QCD, renormalons, analytic properties

* caprini@theory.nipne.ro
† fischer@fzu.cz
I. INTRODUCTION

In confined gauge theories like QCD, causality and unitarity imply that the Green functions and the physical amplitudes are analytic functions of the complex energy variables, with singularities at the hadronic unitarity thresholds \([1]\). In particular, the Adler function \(D(Q^2)\) (related to the polarisation amplitude by \(D(Q^2) = -Q^2d\Pi/dQ^2 - 1\)) is a real analytic function in the complex \(Q^2\) plane cut along the negative real axis from the threshold \(-4M^2_\pi\) for hadron production to \(-\infty\). This property is implemented by the well-known Källen-Lehmann representation

\[
D(Q^2) = \frac{Q^2}{\pi} \int_{4M^2_\pi}^{\infty} \frac{R(s)ds}{(s+Q^2)^2},
\]

where \(R(s)\) is related to the observable cross section \(\sigma_{e^+e^- \rightarrow \text{hadrons}}\). From (1) it follows in particular that \(D(Q^2)\) is continuous in the Euclidean region \(Q^2 > 0\) and vanishes at \(Q^2 = 0\).

The renormalization-group improved expansion of the Adler function in massless QCD does not satisfy all the properties contained in the above representation. The finite-order expansion

\[
D_{PT}^{(N)}(Q^2) = \sum_{n=0}^{N} d_n a^{n+1}(Q^2)
\]

is plagued by the unphysical (Landau) pole at \(Q^2 = \Lambda^2\), present in the one loop running coupling

\[
a(Q^2) = \frac{\alpha_s(Q^2)}{\pi} = \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)}.
\]

A modified perturbative QCD series ("analytic perturbation theory"), which implements the Källen-Lehmann representation (1) at each finite order, has been proposed in [2, 3].

Beyond finite orders, the observables can be defined by a summation of the Borel type. The Borel transform \(B(u)\) of the Adler function has singularities on the real axis of the \(u\)-plane [4]: the ultraviolet (UV) renormalons along the range \(u \leq -1\), and the infrared (IR) renormalons along \(u \geq 2\) (we adopt the definition of the Borel transform used in [4]). While the Borel transform is, for a wide class of functions, uniquely determined once all the perturbation expansion coefficients are explicitly given, the determination of the function having a given perturbative (asymptotic) expansion is, actually, infinitely ambiguous; not only due to the singularities, but because the contour of the Borel-type integral can be also varied, without affecting the expansion coefficients of the perturbation series.
In Ref. [6] the authors use two different Borel summations of the perturbation series in the Euclidean region: for positive coupling, $a(Q^2) > 0$, they choose the integration contour along the positive (IR renormalon) axis,

$$\mathcal{D}_{PT}(Q^2) = \frac{1}{\beta_0} \int_0^\infty e^{-u/(\beta_0a(Q^2))} B(u) \, du, \quad a(Q^2) > 0,$$

while for negative coupling the integral is taken instead along the negative (UV renormalon) axis:

$$\mathcal{D}_{PT}(Q^2) = \frac{1}{\beta_0} \int_{-\infty}^0 e^{-u/(\beta_0a(Q^2))} B(u) \, du, \quad a(Q^2) < 0.$$  

As shown in [6], the summation based on the above definitions can be expressed as:

$$\mathcal{D}_{PT}(Q^2) = \int_0^\infty d\tau \omega_D(\tau)a(\tau Q^2),$$

in terms of the characteristic function $\omega_D(\tau)$ defined by Neubert [5]. Regulating with the Principal Value the singularity of $a(\tau Q^2)$ at $\tau = \Lambda^2/Q^2$, and taking into account the continuity of the characteristic function $\omega_D(\tau)$ at $\tau = 1$, the authors of [6] conclude from (6) that the contribution of the leading chain of the skeleton expansion of the Adler function is finite and continuous along the whole spacelike axis $Q^2 > 0$ and approaches a zero limit at $Q^2 = 0$.

Therefore, in [6] it is shown that by a suitable summation of a class of diagrams in perturbative QCD, one recovers a property of the true Adler function, which follows from the representation (1). Unfortunately, it turns out that another fundamental property implied by same representation (1), namely analyticity in the complex plane, is simultaneously lost. In the present Comment, we prove this by calculating the Adler function in the complex energy plane with the Borel prescription adopted in [6]. The calculation uses the technique described in [7], based on the inverse Mellin transform of the Borel function.

II. CHARACTERISTIC FUNCTION AND INVERSE MELLIN TRANSFORM

As shown in [5], the function $\omega_D$ appearing in [6] is the inverse Mellin transform of the Borel function $B(u)$:

$$\omega_D(\tau) = \frac{1}{2\pi i} \int_{u_0-i\infty}^{u_0+i\infty} du \, B(u) \, \tau^{-1}.$$  

(7)
The inverse relation
\[ B(u) = \int_0^\infty d\tau \omega_D(\tau) \tau^{-u}, \]  
(8)
defines the function \( B(u) \) in a strip parallel to the imaginary axis with \(-1 < \text{Re} \, u < 2\), where it is assumed to be analytic.

The function \( \omega_D(\tau) \) was calculated in \([5]\) in the large-\( \beta_0 \) approximation. The result was rederived in \([6]\). Using (7), the calculation is based on residues theorem: for \( \tau < 1 \) the integration contour is closed on the right half-\( u \)-plane, and the result is the sum over the residues of the infrared renormalons; for \( \tau > 1 \) the integration contour is closed on the left half-\( u \)-plane, and the result contains the residues of the ultraviolet renormalons. The residues of the IR and UV renormalons satisfy some symmetry properties \([6]\), but their contributions are not equal. Therefore \( \omega_D(\tau) \) has different analytic expressions, depending on whether \( \tau \) is less or greater than 1. Following Ref. \([6]\), we denote the two branches of \( \omega_D \) by \( \omega^{IR}_D \) and \( \omega^{UV}_D \), respectively. According to the above discussion, it follows from (7) that
\[ \omega^{IR}_D(\tau) = \frac{1}{2\pi i} \left[ \int_{C_+} du B(u) \tau^{-u-1} - \int_{C_-} du B(u) \tau^{-u-1} \right], \]
(9)
where \( C_\pm \) are two parallel lines going from 0 to \(+\infty\) slightly above and below the real positive axis, and
\[ \omega^{UV}_D(\tau) = \frac{1}{2\pi i} \left[ \int_{C'_+} du B(u) \tau^{-u-1} - \int_{C'_-} du B(u) \tau^{-u-1} \right], \]
(10)
where \( C'_\pm \) are two lines going from 0 to \(-\infty\) slightly above and below the real negative axis.

The explicit expressions of \( \omega^{IR}_D \) and \( \omega^{UV}_D \) in the large-\( \beta_0 \) approximation are given in Eq. (80) of \([5]\) (see also Eq. (2.19) of \([7]\), where \( \omega^{IR}_D \) is denoted by \( \tilde{w}^{(\leq)}_D \), and \( \omega^{UV}_D \) by \( \tilde{w}^{(>)}_D \)). As shown in \([5]\), the function \( \omega_D(\tau) \) and its first three derivatives are continuous at \( \tau = 1 \). Moreover, the explicit expressions given in \([5, 7]\) imply that \( \omega^{IR}_D(\tau) \) and \( \omega^{UV}_D(\tau) \) are both analytic functions in the \( \tau \)-complex plane cut along the real negative axis \( \tau < 0 \).

### III. ADLER FUNCTION IN THE COMPLEX PLANE

A closed representation of the Adler function \( \mathcal{D}_{PT}(Q^2) \) for complex values of \( Q^2 \) in terms of the characteristic function was derived in \([7]\). The function \( \mathcal{D}_{PT}(Q^2) \) was defined for large \( |Q^2| \) by a Borel-Laplace integral along the IR axis, while the expression for low \( Q^2 \)
was obtained by analytical continuation. In the present Comment we use the technique presented in [7], adapted for the choice of the Borel-Laplace integral made in [6]. For clarity, we shall present the calculation in some detail.

As in Ref. [6] we work in the $V$-scheme, where all the exponential dependence in the Borel-Laplace integrals (4) and (5) is absorbed in the running coupling, and denote by $\Lambda^2_V$ the corresponding QCD scale parameter. Let us consider $Q^2$ complex, first such that $|Q^2| > \Lambda^2_V$. Since in this case $\text{Re} a(Q^2) > 0$ we use the choice (4) of the Borel-Laplace integral with the principal value ($PV$) prescription, taking

$$D_{PT}(Q^2) = \frac{1}{2}[D^{(+)}(Q^2) + D^{(-)}(Q^2)],$$

where $D^{(\pm)}(Q^2)$ are defined as

$$D^{(\pm)}(Q^2) = \frac{1}{\beta_0} \int_{C_{\pm}} \exp[-u/(\beta_0 a(Q^2))] B(u) \, du.$$  

(12)

Here $C_{\pm}$ are two parallel lines slightly above and below the real positive axis, introduced already in Eq. (9).

Following [7], we pass from the integrals along the contours $C_{\pm}$ to integrals along a line parallel to the imaginary axis, where the representation (8) is valid. This can be achieved by rotating the integration contour from the real to the imaginary axis, provided the contribution of the circles at infinity is negligible. We consider first a point in the upper half of the energy plane, for which $Q^2 = |Q^2|e^{i\phi}$ with a phase $0 < \phi < \pi$. Taking $u = \mathcal{R}e^{i\theta}$ on a large semi-circle of radius $\mathcal{R}$, the relevant exponential appearing in the integrals (12) is

$$\exp\left\{-\mathcal{R} \left[ \ln \left( \frac{|Q^2|}{\Lambda^2_V} \right) \cos \theta - \phi \sin \theta \right] \right\}.$$  

(13)

For $|Q^2| > \Lambda^2_V$, the exponential is negligible at large $\mathcal{R}$ for $\cos \theta > 0$ and $\sin \theta < 0$, i.e. for the fourth quadrant of the complex $u$-plane. The integration contour defining $D^{(-)}(Q^2)$ can be rotated to the negative imaginary $u$-axis, where the representation (8) is valid. This leads to the double integral

$$D^{(-)}(Q^2) = \frac{1}{\beta_0} \int_0^{-i\infty} du \int_0^{\infty} d\tau \omega_D(\tau) \exp \left[ -u \left( \ln \left( \frac{|Q^2|}{\Lambda^2_V} \right) + i\phi \right) \right].$$  

(14)

The order of integrations over $\tau$ and $u$ can be interchanged, since for positive $\phi$ the integral over $u$ is convergent and can be easily performed. Expressed in terms of the complex variable
\[ D^{(-)}(Q^2) = \frac{1}{\beta_0} \int_0^\infty d\tau \frac{\omega_D(\tau)}{\ln(\tau Q^2/\Lambda_V^2)} = \int_0^\infty d\tau \omega_D(\tau)a(\tau Q^2). \tag{15} \]

We evaluate now the function \( D^{(+)}(Q^2) \) given by the integral along the contour \( C_+ \) above the real axis. The rotation of the integration contour to the positive imaginary axis is not allowed, because along the corresponding quarter of a circle \( \sin \theta > 0 \), and the exponent \( \int_0^\infty d\tau \omega_D(\tau) \) does not vanish at infinity for \( 0 < \phi \). As explained in [7], we must perform again a rotation to the negative imaginary \( u \) axis, for which the contribution of the circle at infinity vanishes. But in this rotation the contour crosses the positive real axis, and hence picks up the contributions of the IR renormalon singularities located along this line. This can be evaluated by comparing the expression (9) of the function \( \omega^{IR}_D(\tau) \) with the definition (12) of the functions \( \sigma^{(\pm)}_D \): they are connected by the change of variable \( \tau = \exp\left[-1/(\beta_0 a(Q^2))\right] \). It follows that \( D^{(+)} \) can be expressed in terms of \( D^{(-)} \) as

\[ D^{(+)} = D^{(-)} + \frac{2\pi i}{\beta_0} \frac{\Lambda_V^2}{Q^2} \omega^{IR}_D(\Lambda_V^2/Q^2). \tag{16} \]

The relations (11), (15) and (16) completely specify the function \( D_{PT}(Q^2) \) for \(|Q^2| > \Lambda_V^2\), in the upper half plane \( \text{Im } Q^2 > 0 \):

\[ D_{PT}(Q^2) = \int_0^\infty d\tau \omega_D(\tau)a(\tau Q^2) + \frac{i\pi}{\beta_0} \frac{\Lambda_V^2}{Q^2} \omega^{IR}_D(\Lambda_V^2/Q^2). \tag{17} \]

Using the same method, the function \( D_{PT}(Q^2) \) can be calculated in the lower half of the energy plane, where \( Q^2 = |Q^2|e^{i\phi} \) with \(-\pi < \phi < 0\). In this case, the integral along \( C_+ \) can be calculated by rotating the contour up to the positive imaginary \( u \) axis, while for the integration along \( C_- \) one must first pass across the real axis and then rotate towards the positive imaginary axis. Combining the results, we obtain the following expression for the Adler function for complex \( Q^2 \) with \(|Q^2| > \Lambda_V^2\):

\[ D_{PT}(Q^2) = \int_0^\infty d\tau \omega_D(\tau)a(\tau Q^2) \pm \frac{i\pi}{\beta_0} \frac{\Lambda_V^2}{Q^2} \omega^{IR}_D(\Lambda_V^2/Q^2), \tag{18} \]

where the \( \pm \) signs correspond to \( \text{Im } Q^2 > 0 \) and \( \text{Im } Q^2 < 0 \), respectively. We recall that the first term in (18) is given by the integration with respect to \( u \), while the last term is produced by the residues of the infrared renormalons picked up by crossing the positive axis of the Borel plane.
We consider now \( |Q^2| < \Lambda^2_V \), when \( \text{Re} a(Q^2) < 0 \). Following [6] we use the definition (5) of the Borel-Laplace integral along the negative axis. In this case the integral is not defined due to the UV renormalons. The Principal Value prescription will be given by (11), where the \( D^{(\pm)} \) are now

\[
D^{(\pm)}(Q^2) = \frac{1}{\beta_0} \int_{C'_\pm} e^{-u/(\beta_0 a(Q^2))} B(u) \, du ,
\]

(19)

\( C'_\pm \) being the two parallel lines above and below the negative \( u \)-axis defined in (10).

We apply then the same techniques as above, by rotating the contours \( C'_\pm \) towards the imaginary axis in the \( u \) plane, where the representation (8) is valid. If the exponential (13) decreases we can make the rotation. If not, we must first cross the real axis and perform the rotation. The calculations proceed exactly as before, with the difference that now one picks up the contribution of the UV renormalons, according to the relation (10). This leads to the expression of the Adler function for \( |Q^2| < \Lambda^2_V \)

\[
D_{PT}(Q^2) = \int_0^\infty d\tau \, \omega_D(\tau) a(\tau Q^2) \pm \frac{i\pi}{\beta_0} \frac{\Lambda^2_V}{Q^2} \omega_D^{UV}(\Lambda^2_V/Q^2) ,
\]

(20)

where the signs correspond to \( \text{Im} Q^2 > 0 \) and \( \text{Im} Q^2 < 0 \), respectively.

We show now that the limit of the expressions (18) and (20) when \( Q^2 \) is approaching the Euclidean axis coincides with (6). Consider first that \( Q^2 \) tends to the real positive axis from above, in the region \( |Q^2| > \Lambda^2_V \), when \( D_{PT}(Q^2) \) has the expression (18). The integrand has a pole at \( \tau = \Lambda^2_V/Q^2 \). Writing explicitly the real and the imaginary part of the integral we obtain, for real \( Q^2 > \Lambda^2 \):

\[
D_{PT}(Q^2 + i\epsilon) = \text{Re} \left[ \int_0^\infty d\tau \, \omega_D(\tau) a(\tau Q^2) \right] - \frac{i\pi}{\beta_0} \frac{\Lambda^2_V}{Q^2} \left[ (\omega_D(\Lambda^2_V/Q^2) - \omega_D^{IR}(\Lambda^2_V/Q^2)) \right] .
\]

(21)

But for \( \Lambda^2_V/Q^2 < 1 \), the function \( \omega_D \) coincides with \( \omega_D^{IR} \), so the last term in (21) vanishes: the imaginary part of the integral in (18) is exactly compensated by the additional term.

For \( Q^2 < \Lambda^2_V \), we obtain from (20)

\[
D_{PT}(Q^2 + i\epsilon) = \text{Re} \left[ \int_0^\infty d\tau \, \omega_D(\tau) a(\tau Q^2) \right] - \frac{i\pi}{\beta_0} \frac{\Lambda^2_V}{Q^2} \left[ (\omega_D(\Lambda^2_V/Q^2) - \omega_D^{UV}(\Lambda^2_V/Q^2)) \right] ,
\]

(22)

in the same way. Again the last term in this relation vanishes, since for \( \Lambda^2_V/Q^2 > 1 \) the function \( \omega_D \) is equal to \( \omega_D^{UV} \). Moreover, one can easily see that the expressions of \( D_{PT}(Q^2 - i\epsilon) \), obtained for \( Q^2 \) approaching the Euclidean axis from the lower half plane, differ from
and (22) only by the sign in front of the last term, which again vanishes. Thus, for all \( Q^2 > 0 \), the functions (18) and (20) approach the same expression

\[
D_{PT}(Q^2 \pm i\epsilon) = \text{Re} \left[ \int_0^\infty d\tau \omega_D(\tau) a(\tau Q^2) \right].
\]

(23)

This coincides with the PV regulated integral of the Cauchy type (6) which, as shown in [6], is finite and satisfies the infrared freezing. Moreover, since \( \omega_D(\tau) \) is holomorphic (infinitely differentiable) for all \( \tau > 0 \) except \( \tau = 1 \), the right-hand side of (23) has all derivatives defined at \( Q^2 > 0 \), except at \( Q^2 = \Lambda^2 \), where only the first three derivatives exist [5]. This means that (18) and (20) define in fact analytic functions in the regions \( |Q^2| > \Lambda_V^2 \) and \( |Q^2| < \Lambda_V^2 \), respectively. In this way we have obtained, following the approach of Ref. [6], two expressions, (18) and (20), which represent \( D_{PT}(Q^2) \) in terms of analytic functions for \( |Q^2| > \Lambda_V^2 \) and \( |Q^2| < \Lambda_V^2 \) respectively.

But the success is illusory, because \( \omega^{IR}_D(\tau) \) and \( \omega^{UV}_D(\tau) \) are two different analytic functions. The expressions (18) and (20) show that \( D_{PT}(Q^2) \) coincides with a certain analytic function in the region \( |Q^2| > \Lambda_V^2 \), but with another analytic function in the region \( |Q^2| < \Lambda_V^2 \). So, the Adler function obtained with the two different Borel representations adopted in [6] is not analytic, but only piecewise analytic. This is in evident conflict with the principle of analyticity implemented by the Källen-Lehmann representation (1).

IV. DISCUSSION

We have shown by explicit calculation that the Borel prescription adopted in [6] is in conflict with analyticity, which is a general property considered fundamental in field theory.
This result implies that the infrared freezing of the Euclidean observables achieved in [6] has had a price, being possible only at the expense of analyticity. The loss is not only of an academic interest: the analytical continuation is the only technique to obtain the Minkowskian observables form the Euclidean ones, and all theoretical predictions in field theory are based on it. Moreover, the simple model for the complete Adler function proposed in [6] cannot represent the physical observable: although it is finite in the Euclidean region and exhibits infrared freezing, it is not consistent with the analyticity properties implied by the Källen-Lehmann representation.

Note also that analyticity is repeatedly invoked by the authors themselves (for instance, the term ”analytical continuation” or its verbal analog are mentioned at least eight times in [6], in particular in Sections VI and VII, where the Minkowskian ratio $\mathcal{R}$ is discussed). Analytical continuation is unavoidable even if a smearing procedure is used in the Minkowskian region.

It is worth emphasizing that the result of Ref. [6] is not an intrinsic or natural property of the leading one-chain term in the skeleton expansion of QCD, but the consequence of a specific, but questionable hypothesis. A step of crucial importance in [6] is the ad-hoc redefinition of the Borel integral in the region where the running coupling $a(Q^2)$ becomes negative. In Ref. [6], this redefinition originates in a particular utilization of the function $\text{Ei}(z)$. The authors expressed the Borel integrals, cf. Eqs. (28) and (29) of [6], in terms of $\text{Ei}(z)$ depending only on the ratio $z = a/z_n$, where $a$ is the coupling and the $z_n$ are the positions of renormalons. With the conventional definition of the Principal Value of $\text{Ei}(z)$, a branch cut is located at $a > 0$ and $z_n < 0$, or at $a < 0$ and $z_n > 0$. This implicitly selects a specific form of the Borel integral: for $a > 0$, it is taken along the positive, and for $a < 0$, along the negative real semiaxis, respectively. But this definition is not the only possibility. Note that, as pointed out in Ref. [6] (Section VII), for $Q^2 < \Lambda^2_{\text{V}}$ the expression (3) is not the solution of the renormalization-group equation. We have shown that the use of these two different Borel-type integrals defining one single function in two different regions is responsible for the loss of analyticity.

Incidentally, the authors of Ref. [6] admit that the function $\text{Ei}(z)$ regulated by the Principal Value does not give a reasonable result the for Minkowskian observable $\mathcal{R}$. In Section VI they adjust the result by hand, by introducing additional ad-hoc terms (see Eqs. (89)-(92) of [6] and [8]). These ambiguous procedures are avoided if analyticity is preserved.
and analytic continuation is performed in a consistent way. 

Acknowledgments

We acknowledge interesting discussions with Chris Maxwell and thank Stan Glazek for useful comments. This work was supported by the CEEX Program of Romanian ANCS under Contract Nr.2-CEx06-11-92, and by the Ministry of Education of the Czech Republic, Project Nr. 1P04LA211.

[1] R. Oehme, π-N Newsletters 7 (1992) 1; Int. J. Mod. Phys. A10, 1995 (1995).
[2] D. V. Shirkov, Eur. Phys. J. C22, 331 (2001).
[3] For a recent review, see: D. V. Shirkov and I.L. Solovtsov, Ten years of Analytic Perturbation Theory in QCD, hep-ph/0611229.
[4] See for instance the review: M. Beneke, Phys. Rep. 317, 1 (1999).
[5] M. Neubert, Phys. Rev. D51 5924 (1995).
[6] P. M. Brooks, and C.J. Maxwell, Phys. Rev. D74 065012 (2006).
[7] I. Caprini and M. Neubert, JHEP 03, 007 (1999).
[8] D. M. Howe and C.J. Maxwell, Phys. Rev. D 70, 014002 (2004).
[9] I. Caprini and J. Fischer, Phys. Rev. D 71, 094017 (2005).