1 Introduction

The purpose of this note is to show that there exist infinitely many commensurability classes of finite volume hyperbolic 3-manifolds whose fundamental group contains a subgroup which is locally free but not free. This gives a strong answer to a question of Krogholler given in the Problem List in Niblo and Roller [26].

Theorem 5.1: There exist infinitely many commensurability classes of finite volume hyperbolic 3-manifolds whose fundamental groups contain a subgroup which is locally free but not free.

There are three main observations which make up the proof. The first observation, discussed in Section 2, is that convex co-compact subgroups of a fundamental group of a hyperbolic 3-manifold persist in the approximates given by hyperbolic Dehn surgery. This result is stated formally as Proposition 2.1.

The second observation, discussed in Section 4, is that a collection of distinct hyperbolic 2- or 3-manifolds of uniformly bounded volume contains infinitely many commensurability classes. This result is stated formally as Proposition 4.1. Such collections of hyperbolic 3-manifolds are generated by hyperbolic Dehn surgery.

The third observation, discussed in the proof of Theorem 5.1, is a general construction of finite volume hyperbolic 3-manifolds which is a direct application of Thurston’s hyperbolization theorem for Haken atoroidal 3-manifolds. The material in this Section is well-known, though to my knowledge the construction given has never been put to paper and so is included for the sake of completeness.

At this point, given the prevalence of hyperbolic 3-manifolds whose fundamental groups contain subgroups which are locally free but not free, I would like to put forward two complementary conjectural pictures. One is that the result of Theorem 5.1 holds not just for infinitely many commensurability classes, but in fact holds for all finite volume hyperbolic 3-manifolds.
Conjecture 1.1 The fundamental group of every finite volume hyperbolic 3-manifold contains a subgroup which is locally free but not free.

Though there is no direct evidence for Conjecture 1.1, it is not unplausible, given Theorem 5.1, together with Proposition 3.1, which states that the property of containing a subgroup which is locally free but not free is an invariant of a commensurability class.

The second conjecture is based on the as yet unrealized hope that the property of containing a subgroup which is locally free but not free can be used to distinguish between certain classes of finite volume hyperbolic 3-manifolds. We note here that Reid [29] has shown that there are infinitely many commensurability classes of hyperbolic 3-manifolds which fiber over the circle.

Conjecture 1.2 The fundamental group of a finite volume hyperbolic 3-manifold \( N \) contains a subgroup which is locally free but not free if and only if \( N \) does not have a finite cover which fibers over the circle.

Though there is no direct evidence to support Conjecture 1.2, one piece of heuristic evidence is that there are other algebraic properties of the fundamental groups of finite volume hyperbolic 3-manifolds which conjecturally fail if and only if the 3-manifold in question is commensurable to one that fibers over the circle. One such is the finitely generated intersection property, for a discussion, see Canary [7] and the references contained therein.

Note that if Conjecture 1.2 were to hold true, then all the hyperbolic 3-manifolds constructed in this note would be counterexamples to the conjecture of Thurston [35] that a finite volume hyperbolic 3-manifold has a finite cover which fibers over the circle.

We close this introduction by giving some definitions. A Kleinian group is a discrete subgroup of \( \text{PSL}_2(\mathbb{C}) \), which we view as acting both on the Riemann sphere \( \overline{\mathbb{C}} \) by Möbius transformations and on real hyperbolic 3-space \( \mathbb{H}^3 \) by isometries, where the two actions are linked by the Poincaré extension. For the duration of this note, we assume that all Kleinian groups are torsion-free. Note though that by Selberg’s lemma [30], which implies that a finitely generated Kleinian group contains a torsion-free subgroup of finite index, that the results of this note hold for Kleinian groups with torsion as well.

The action of an infinite Kleinian group \( \Gamma \) partitions \( \overline{\mathbb{C}} \) into two sets, the domain of discontinuity \( \Omega(\Gamma) \), which is the largest open subset of \( \overline{\mathbb{C}} \) on which \( \Gamma \) acts discontinuously, and the limit set \( \Lambda(\Gamma) \). If \( \Lambda(\Gamma) \) contains two or fewer points, \( \Gamma \) is elementary, otherwise \( \Gamma \) is non-elementary. For a non-elementary Kleinian group \( \Gamma \), the limit set \( \Lambda(\Gamma) \) can also be described as the smallest non-empty closed subset of \( \overline{\mathbb{C}} \) invariant under \( \Gamma \). We refer the reader to Maskit [20] or Matsuzaki and Taniguchi [22] for the basic elements of the theory of Kleinian groups.

Two Kleinian groups \( \Gamma \) and \( \Gamma' \) are commensurable if their intersection \( \Theta = \Gamma \cap \Gamma' \) has finite index in both \( \Gamma \) and \( \Gamma' \). Equivalently, two hyperbolic 3-manifolds \( N \) and \( N' \) are commensurable if there exists a hyperbolic 3-manifold \( P \) which is a finite cover of both \( N \) and \( N' \). In the latter case, there are then realizations of both \( \pi_1(N) \) and \( \pi_1(N') \) as commensurable Kleinian groups.
A finitely generated Kleinian group is *convex co-compact* if the convex core of its associated hyperbolic 3-manifold $\mathbb{H}^3/\Gamma$ is compact. Recall that the *convex core* of a hyperbolic 3-manifold $\mathcal{N}$ is the minimal convex submanifold of $\mathcal{N}$ whose inclusion into $\mathcal{N}$ is a homotopy equivalence. Equivalently, a Kleinian group $\Gamma$ is convex co-compact if and only if the associated 3-manifold $(\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ is compact.

More generally, a finitely generated Kleinian group is *geometrically finite* if the convex core of its associated hyperbolic 3-manifold has finite volume. This is one of several equivalent definitions of geometric finiteness; the interested reader is referred to Bowditch [6] for a complete discussion.

A Kleinian group $\Gamma$ is *minimally parabolic* if every parabolic element of $\Gamma$ is contained in a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup of $\Gamma$.

An *incompressible surface* in a 3-manifold $M$ is an embedded orientable surface $S$ in $M$ for which $\pi_1(S)$ is infinite and for which $\pi_1(S)$ injects into $\pi_1(M)$ under the inclusion map. If $M$ is compact, we say that $M$ has *incompressible boundary* if each component of \( \partial M \) is an incompressible surface. A surface $S$ in $M$ is *essential* if it is incompressible and is not homotopic into $\partial M$.

Let $M$ be a compact 3-manifold and let $S$ be a (possibly disconnected) compact subsurface of $\partial M$. Say that $S$ is *an-annular* if there does not exist an essential annulus $A$ in $M$ both of whose boundary components are contained in $S$. A compact 3-manifold $M$ is *acylindrical* if $\partial M$ is an-annular.

A compact 3-manifold $M$ is *irreducible* if every embedded 2-sphere in $M$ bounds a 3-ball in $M$, and $M$ is *atoroidal* if $M$ does not contain an essential torus.

A compact 3-manifold $M$ is *hyperbolizable* if there exists a hyperbolic 3-manifold $N = \mathbb{H}^3/\Gamma$ homeomorphic to the interior of $M$. Note that a hyperbolizable 3-manifold $M$ is necessarily orientable, irreducible, and atoroidal. The converse holds for compact 3-manifolds which contain an incompressible surface, by the hyperbolization theorem of Thurston, see Morgan [23]. Also, since the universal cover $\mathbb{H}^3$ of $N$ is contractible, the fundamental group of $M$ is isomorphic to $\Gamma$. For a discussion of the basic theory of 3-manifolds, we refer the reader to Hempel [12].

Given a compact, hyperbolizable 3-manifold $M$, a *uniformization* of $M$ is a Kleinian group $\Gamma$ so that $N = \mathbb{H}^3/\Gamma$ is homeomorphic to the interior of $M$. For the purposes of this note, we restrict our attention to *minimally parabolic, geometrically finite uniformizations*. Note that if $\partial M$ contains no tori and if $\Gamma$ is a minimally parabolic, geometrically finite uniformization of $M$, then $\Gamma$ contains no $\mathbb{Z} \oplus \mathbb{Z}$ subgroups.

This paper was written while I was visiting Rice University, and I like to thank the department there for its hospitality. I would also like to thank Graham Niblo, Alan Reid, Joe Masters, and Dick Canary for helpful conversations about the ideas in this note.

## 2 Persistence of convex co-compact subgroups

We begin this Section by describing the operation of hyperbolic Dehn surgery. Let $M$ be a compact hyperbolizable 3-manifold whose boundary $\partial M$ contains a non-empty collection of tori.
Let $\mathcal{T} = \{T_1, \ldots, T_n\}$.

For each $1 \leq k \leq n$, choose a meridian-longitude system $(m_k, l_k)$ on $T_k$. Given a pair $(p_k, q_k)$ of relatively prime integers, form a new manifold by attaching a solid torus $V$ to $M$ by an orientation-reversing homeomorphism $g : \partial V \to T_k$ so that, if $c$ is the meridian of $V$, then $g(c)$ is a $(p_k, q_k)$ curve on $T_k$ (with respect to the chosen meridian-longitude system). If we write the $n$-tuple of pairs as $(p, q) = ((p_1, q_1), \ldots, (p_n, q_n))$, we say that $M(p, q)$ is obtained from $M$ by $(p, q)$-Dehn filling along $\mathcal{T}$.

Let $\Gamma$ be a minimally parabolic, geometrically finite Kleinian group uniformizing $M$, so that the hyperbolic 3-manifold $N = \mathbb{H}^3/\Gamma$ is homeomorphic to the interior of $M$. Let $\{(p_j, q_j) = ((p_j^1, q_j^1), \ldots, (p_j^n, q_j^n))\}$ be a sequence of $n$-tuples of pairs of relatively prime integers such that, for each $1 \leq k \leq n$, $\{(p_j^k, q_j^k)\}$ converges to $\infty$ as $j \to \infty$.

The generalized hyperbolic Dehn surgery theorem states that for all sufficiently large $j$, there exists a representation $\beta_j : \Gamma \to \text{PSL}_2(\mathbb{C})$ so that $\beta_j(\Gamma)$ is a minimally parabolic, geometrically finite uniformization of $M(p_j, q_j)$. We refer to $\beta_j$ as the hyperbolic Dehn surgery representation of $\Gamma$ associated to $(p_j, q_j)$. Moreover, the sequence $\{\beta_j\}$ of representations of $\Gamma$ converges to the identity representation of $\Gamma$.

This form of the generalized hyperbolic Dehn surgery theorem is due to Comar 5. There is also a version due to Bonahon and Otal 6. The original hyperbolic Dehn surgery theorem, which applies in the case that the boundary of $M$ is equal to a union of tori, is due to Thurston; the interested reader is directed to Thurston’s lecture notes 13 or to Benedetti and Petronio 3. There is also the survey article of Gromov 11 discussing convergence and volumes of hyperbolic 3-manifolds.

The main result of this Section is the observation that convex co-compact subgroups of $\Gamma$ persist in the approximates $\beta_j(\Gamma)$ given by the hyperbolic Dehn surgery theorem.

**Proposition 2.1** Let $M$ be a compact hyperbolizable 3-manifold and let $\{T_1, \ldots, T_n\}$ be a non-empty collection of toroidal components of $\partial M$. Let $\Gamma$ be a minimally parabolic, geometrically finite uniformization of $M$.

Let $(m_k, l_k)$ be a meridian-longitude system for $T_k$. Let $\{(p_j, q_j) = ((p_j^1, q_j^1), \ldots, (p_j^n, q_j^n))\}$ be a sequence of $n$-tuples of pairs of relatively prime integers such that, for each $k$, $\{(p_j^k, q_j^k)\}$ converges to $\infty$ as $j \to \infty$. Let $\beta_j$ be the hyperbolic Dehn surgery representation of $\Gamma$ associated to $(p_j, q_j)$.

Let $\Phi$ be a convex co-compact subgroup of $\Gamma$. Then, for all $j$ sufficiently large, $\beta_j(\Phi)$ is a convex co-compact subgroup of $\beta_j(\Gamma)$. Moreover, $\beta_j : \Phi \to \beta_j(\Phi)$ is an isomorphism.

**Proof** We begin the proof of Proposition 2.1 with a definition. Let $\varphi_1, \ldots, \varphi_j$ be a generating set for $\Phi$. An $\varepsilon$-deformation of $\Phi$ with respect to this generating set is a representation $\psi$ of $\Phi$ into $\text{PSL}_2(\mathbb{C})$ such that $|\varphi - \psi(\varphi)| < \varepsilon$ for $1 \leq k \leq m$. Here, for an element $m$ of $\text{PSL}_2(\mathbb{C})$, we take $|m|$ to be the matrix norm of a matrix representative of $m$.

By the quasiconformal stability theorem of Marden 17, there exists a constant $\varepsilon_0 > 0$ so that an $\varepsilon$-deformation $\psi$ of $\Phi$ for $\varepsilon < \varepsilon_0$ is induced by a quasiconformal homeomorphism of $\mathbb{C}$. In particular, $\psi$ is an isomorphism from $\Phi$ to $\psi(\Phi)$, and $\psi(\Phi)$ is convex co-compact.
Since \(\{\beta_j\}\) converges to the identity representation of \(\Gamma\) into \(\text{PSL}_2(\mathbb{C})\), we have that \(\{\beta_j(\varphi_k)\}\) converges to \(\varphi_k\) for each \(1 \leq k \leq m\). In particular, for \(\varepsilon = \frac{1}{2}\varepsilon_0\), there exists \(N > 0\) so that the restriction \(\beta_j|_{\Phi}\) of \(\beta_j\) to \(\Phi\) is an \(\varepsilon\)-deformation of \(\Phi\) for \(j \geq N\), and we are done. \(\square\)

We can recast this discussion slightly. Suppose as above that \(M\) is a compact, hyperbolizable 3-manifold, that \(\{T_1, \ldots, T_n\}\) is a collection of toroidal components of \(\partial M\), and that \(\Gamma\) is a minimally parabolic, geometrically finite uniformization of \(M\).

Let \(\mathcal{P}\) denote the set of all pairs \((p, q)\) of relatively prime integers, together with \(\infty\). Identify the pair \((p, q)\) with \(p + qi \in \mathbb{C}\), so that we can view \(\mathcal{P}\) as a subset of the Riemann sphere \(\overline{\mathbb{C}}\) with its usual topology.

For each \(n\)-tuple of pairs \((p, q)\) = \(((p_1, q_1), \ldots, (p_n, q_n))\) in \(\mathcal{P}^n\), let \(M(p, q)\) denote the manifold obtained by performing \((p_k, q_k)\) Dehn surgery along \(T_k\), relative to a chosen meridian-longitude system on \(T_k\). (If the Dehn surgery coefficient in \(\mathcal{P}\) for \(T_k\) is \(\infty\), then we delete \(T_k\) from \(M\).)

Then, the generalized hyperbolic Dehn surgery theorem can be rephrased as saying that there exists a neighborhood \(U\) of \((\infty, \ldots, \infty)\) in \(\overline{\mathbb{C}}^n\) so that for every point \((p, q)\) in \(U \cap \mathcal{P}^n\), there exists a representation \(\beta : \Gamma \to \text{PSL}_2(\mathbb{C})\) so that \(\beta(\Gamma)\) is a minimally parabolic, geometrically finite uniformization of \(M(p, q)\). Also, the Dehn surgery representations converge to the identity representation of \(\Gamma\) into \(\text{PSL}_2(\mathbb{C})\) as \((p, q)\) converges to \((\infty, \ldots, \infty)\).

In the case that we are performing Dehn surgery along a single torus component \(T\) of \(\partial M\), we can use this discussion to obtain a variant of a Theorem of Wu [37]. Building on work of Culler, Gordon, Luecke, and Shalen [8], Wu proves the following.

**Theorem 2.2 (Wu [37])** Let \(M\) be a compact orientable 3-manifold, let \(T\) be a toroidal component of \(\partial M\), and let \(S\) be a component of \(\partial M - T\). Suppose that \(S\) is incompressible in \(M\) and that \(S \cup T\) is an-annular. Then, there are at most three Dehn surgeries along \(T\) so that \(S\) is not incompressible in the resulting 3-manifold.

Using Proposition [2.1], we can give a non-quantitative proof of Theorem 2.2 for hyperbolizable 3-manifolds. As above, let \(M\) be a compact hyperbolizable 3-manifold with a single toroidal boundary component \(T\), and let \(\Gamma\) is a minimally parabolic, geometrically finite uniformization of \(M\). Let \(S\) be a incompressible component of \(\partial M - T\). By the generalized hyperbolic Dehn surgery theorem, all but finitely many of the 3-manifolds obtained from \(M\) by performing Dehn surgery along \(T\) are hyperbolizable.

Suppose that \(S\) has genus 1, so that \(S\) is itself a toroidal component of \(\partial M\). Then, \(S\) is a component of the boundary of the hyperbolizable 3-manifold obtained by performing hyperbolic Dehn surgery along \(T\) for all but finitely many Dehn surgery coefficients. The observation that boundary tori of hyperbolizable 3-manifolds are necessarily incompressible completes this case.

Suppose that \(S\) has genus at least two, and let \(\Phi\) be a choice of conjugacy class of \(\pi_1(S)\) in \(\Gamma = \pi_1(M)\). The hypotheses that \(\Gamma\) is a minimally parabolic, geometrically finite uniformization of \(M\), that \(S\) is incompressible, and that \(S \cup T\) is an-annular together imply that \(\Phi\) is a convex...
co-compact quasifuchsian subgroup of \( \Gamma \). By Proposition 2.1, the Dehn surgery representation restricts to an isomorphism of \( \Phi \) with convex co-compact image for all but finitely many Dehn surgeries along \( T \).

Though we cannot obtain the quantitative information in this case that Wu obtains in the general case, we are also not restricted to subgroups of \( \Gamma \) corresponding to incompressible surfaces in \( M \), but can instead show that every convex co-compact subgroup of \( \Gamma \) persists in all but finitely many of the Dehn fillings along \( T \). In particular, we have that a non-quantative version of Theorem 2.2 for immersed surfaces holds.

Also, we are not restricted to working in hyperbolizable 3-manifolds with a single cusp, but have a non-quantative version of Theorem 2.2 for an embedded or immersed quasifuchsian surface in an \( n \)-cusped finite volume hyperbolic 3-manifold, and in fact in any hyperbolizable 3-manifold uniformized by a geometrically finite, minimally parabolic Kleinian group, a class which includes many infinite volume hyperbolic 3-manifolds.

### 3 Locally free groups

A group \( G \) is **locally free** if every finitely generated subgroup of \( G \) is free. Every free group is locally free, as all subgroups of free groups are free by Grushko’s theorem. Another standard example of a locally free group is the additive group \( \mathbb{Q} \) of rational integers, in which every finitely generated subgroup is infinite cyclic.

There is an example of Maskit [21] of a Kleinian group which is locally free but not free. However, this example is infinitely generated, and does not seem to reside inside a finitely generated Kleinian group.

A fourth example, which we make heavy use of in this paper, is contained in the group

\[
G_0 = \langle a, b, t \mid t \cdot a \cdot t^{-1} = [b, a] \rangle,
\]

where \([b, a] = b \cdot a \cdot b^{-1} \cdot a^{-1}\) is the commutator of \( b \) and \( a \). Namely, the subgroup

\[
H_0 = \langle t^m \cdot a \cdot t^{-m}, t^m \cdot b \cdot t^{-m}, m \in \mathbb{Z} \rangle
\]

of \( G_0 \) is locally free but not free. We refer the Reader to Maskit [20], Chapter VIII.E.9, and to Freedman and Freedman [18] for a more detailed discussion of this group, including the proof that \( H_0 \) is locally free but not free.

We note here that Maskit realizes \( G_0 \) as a convex co-compact Kleinian group \( \Gamma_0 \), and that the boundary of the associated compact 3-manifold \( M_0 = (H^3 \cup \Omega(\Gamma_0))/\Gamma_0 \) is an incompressible surface of genus two.

We close this Section with the observation that the property of a Kleinian group containing a subgroup which is locally free but not free is preserved under commensurability.

**Proposition 3.1** Suppose that \( \Phi_1 \) and \( \Phi_2 \) are commensurable Kleinian groups. Then, \( \Phi_1 \) contains a subgroup which is locally free but not free if and only if \( \Phi_2 \) contains a subgroup which is locally free but not free.
Proof Let Ξ be a subgroup of Φ₁ which is locally free but not free, and let Θ = Φ₁ ∩ Φ₂. We first show that Θ contains a subgroup which is locally free but not free, namely the intersection Θ ∩ Ξ.

Every finitely generated subgroup of Θ ∩ Ξ is a finitely generated subgroup of Ξ, and so is free. In particular, this shows that Θ ∩ Ξ is locally free.

To see that Θ ∩ Ξ is not free, we first note that since Θ has finite index in Φ₁, Θ ∩ Ξ has finite index in Ξ. If Θ ∩ Ξ is free, then Ξ is a torsion-free group containing a free subgroup of finite index, and so by a Theorem of Stallings [31] and Swan [33], we have that Ξ is free, a contradiction.

Hence, Θ contains the subgroup Θ ∩ Ξ which is locally free but not free. Since Θ is a subgroup of Φ₂, we see that Φ₂ contains a subgroup which is locally free but not free. □

4 A criterion for non-commensurability

The purpose of this Section is to discuss criteria for a collection of hyperbolic 3-manifolds to contain infinitely many commensurability classes. For a collection of finite volume hyperbolic 3-manifolds, one such criterion is given by an upper bound on the volume.

Proposition 4.1 Let M be a collection of infinitely many distinct finite volume hyperbolic 3-manifolds. Suppose there exists some constant K > 0 so that vol(N) ≤ K for all N in M. Then, no commensurability class in M contains infinitely many manifolds. In particular, M contains infinitely many commensurability classes.

Proof We begin the proof of Proposition 4.1 with a definition. For a Kleinian group Γ, define the commensurability subgroup comm(Γ) of Γ in PSL₂(C) to be

\[ \text{comm}(Γ) = \{ g ∈ PSL₂(C) | Γ \text{ and } gΓg^{-1} \text{ are commensurable.} \}. \]

Note that Γ ⊂ comm(Γ). Moreover, if Γ and Γ' are commensurable, then comm(Γ) = comm(Γ').

Suppose there exist infinitely many manifolds \( N_n = H^3/Γ_n, n ≥ 1, \) in M which are commensurable. Since \( Γ_n \) and \( Γ_m \) are commensurable, we have that \( \text{comm}(Γ_n) = \text{comm}(Γ_m) \) for all \( n, m ≥ 1. \) Set \( Θ = \text{comm}(Γ_n). \)

Though we do not give a precise definition here, we note that there exists a special class of Kleinian groups, the arithmetic Kleinian groups, which roughly are Kleinian groups defined by number theory. For a detailed discussion of arithmeticity, we refer the interested Reader to the forthcoming book of Maclachlan and Reid [15].

For the purposes of this note, it suffices to make use of a major result of Margulis [18], see also the discussion in Zimmer [38], which states that a finite co-volume Kleinian group Γ is arithmetic if and only if Γ has infinite index in comm(Γ). Note that arithmeticity of Γ implies that comm(Γ) is not discrete, and hence is dense in PSL₂(C).

We now apply a result of Borel [9], which states that given a constant \( K > 0, \) there exist only finitely many arithmetic Kleinian groups Φ with co-volume \( \text{vol}(H^3/Φ) \) at most \( K. \) In particular,
only finitely many of the $\Gamma_n$ can be arithmetic, and so there must exist some $j \geq 1$ for which $\Gamma_j$ is not arithmetic. In particular, $\Gamma_j$ is a subgroup of finite index in $\Theta = \text{comm}(\Gamma_j)$, and so $\Theta$ is discrete.

We now make use of a fact about the set of volumes of hyperbolic 3-manifolds, namely that there is a minimum volume hyperbolic 3-manifold, and so there exists some $C > 0$ so that $\text{vol}(N_n) \geq C$ for all $n$. In particular, the index of $\Gamma_n$ in $\Theta$ is bounded above by

$$[\Theta : \Gamma_n] = \frac{\text{vol}(H^3/\Theta)}{\text{vol}(N_n)} \leq \frac{\text{vol}(H^3/\Theta)}{C}$$

for all $n \geq 1$.

In particular, infinitely many of the $\Gamma_n$ have the same index in $\Theta$. Since $\Theta$ is finitely generated, it has only finitely many subgroups of a given finite index, and so infinitely many of the $\Gamma_n$ are equal, a contradiction. This contradiction completes the proof.

At this point, there are several remarks to make.

**Remark 4.2** Note that Proposition 4.1 also holds, with the same hypotheses, for hyperbolic 2-manifolds, as the results of Margulis and Borel both apply to $\text{PSL}_2(\mathbb{R})$, as does the lower bound of the volume (in this case, area) of the quotient manifold (in this case, surface).

**Remark 4.3** In the case that $N = H^3/\Gamma$ is a finite volume hyperbolic 3-manifold with a single cusp, there is an alternative proof of Proposition 4.1 using work of Long and Reid [14], strengthening an earlier observation of Hodgson. Namely, for each $D \in \mathbb{N}$, there are (up to conjugacy) only finitely many discrete (not necessarily faithful) representations $\rho$ of $\Gamma$ into $\text{PSL}_2(\mathbb{C})$ for which the degree of the invariant trace field of $\rho(\Gamma)$ has degree at most $D$. Here, the invariant trace field of $\rho(\Gamma)$ is the extension of $\mathbb{Q}$ generated by the traces of the squares of the elements of $\rho(\Gamma)$.

It is a result of Reid [29], that the invariant trace field of a finite co-volume Kleinian group is an invariant of the commensurability class. Also, the Thurston hyperbolic Dehn surgery theorem implies that the degree of the invariant trace field goes to infinity under Dehn surgery, which then implies that Dehn surgery on a singly cusped, finite volume hyperbolic 3-manifold yields infinitely many commensurability classes.

I would like to thank Joe Masters and Alan Reid for bringing this to my attention.

**Remark 4.4** For hyperbolic $n$-manifolds with $n \geq 4$, Proposition 4.1 is a trivial consequence of a result of Wang [36], which states that for each $K > 0$, there are only finitely many isometry classes of hyperbolic $n$-manifolds of volume at most $K$.

One application of Proposition 4.1 is to the collections of hyperbolic 3-manifolds obtained from a given finite volume, cusped hyperbolic 3-manifold $N$. 

\[ \boxed{} \]
Corollary 4.5 Let $M$ be a compact hyperbolizable 3-manifold and suppose that $\partial M$ is a union of tori $\partial M = T_1 \cup \cdots \cup T_j$. Let $\mathcal{M}(M)$ denote the set of all finite volume hyperbolic 3-manifolds obtained from $M$ by performing Dehn surgery along some or all of the tori in $\partial M$. Then, at most finitely many of the manifolds in $\mathcal{M}(M)$ are commensurable.

Proof This Corollary follows immediately from Proposition 4.1 and the fact, due to Thurston, that the volumes of the manifolds in $\mathcal{M}(M)$ are bounded by $\operatorname{vol}(N)$; for a discussion of this fact, see for instance Gromov [11] or Benedetti and Petronio [3]. $\square$

Corollary 4.5 itself has a simple Corollary, follows immediately from the observation that for each $n \geq 0$, there exists a finite volume hyperbolic 3-manifold with $n+1$ cusps. For $n = 0$, Corollary 4.6 is due to Maclachlan and Reid [16] using arithmetic techniques.

Corollary 4.6 For $n \geq 0$, the collection $\mathcal{M}_n$ of all finite volume hyperbolic 3-manifolds with $n$ cusps contains infinitely many commensurability classes.

There is also a variant of Proposition 4.1 for a collection of geometrically finite, infinite volume hyperbolic 3-manifolds. First, recall that if $\Gamma$ is a geometrically finite, infinite co-volume Kleinian group, then $\Omega(\Gamma)$ is non-empty.

Proposition 4.7 Let $\mathcal{M}$ be a collection of infinitely many distinct geometrically finite, infinite volume hyperbolic 3-manifolds. Suppose there exists a constant $K > 0$ so that $\operatorname{area}(\Omega(\Gamma)/\Gamma) \leq K$ for all $N = \mathbb{H}^3/\Gamma$ in $\mathcal{M}$. Then, no commensurability class in $\mathcal{M}$ contains infinitely many manifolds. In particular, $\mathcal{M}$ contains infinitely many commensurability classes.

Proof Suppose there exist infinitely many elements $N_1, N_2, \ldots, N_m, \ldots$ of $\mathcal{M}$ which are pairwise commensurable. Write $N_m = \mathbb{H}^3/\Gamma_m$, so that $\Gamma_m \cap \Gamma_{m+1}$ has finite index in both $\Gamma_m$ and $\Gamma_{m+1}$. Let $\operatorname{comm}(\Gamma_m)$ be the commensurability subgroup of $\Gamma_m$. Since $\Gamma_m$ and $\Gamma_{m+1}$ are commensurable for each $m$, we have that $\operatorname{comm}(\Gamma_m) = \operatorname{comm}(\Gamma_p)$ for $m \neq p$. Set $\Theta = \operatorname{comm}(\Gamma_m)$.

We first show that $\Lambda(\Gamma_m) = \Lambda(\Theta)$. Since $\Gamma_m$ is a subgroup of $\Theta$, we immediately have that $\Lambda(\Gamma_m) \subset \Lambda(\Theta)$.

For the opposite inclusion, let $g \in \Theta = \operatorname{comm}(\Gamma_m)$ be any element. Since $\Gamma_m$ and $g \Gamma_m g^{-1}$ are commensurable, we have that $\Lambda(\Gamma_m) = \Lambda(g \Gamma_m g^{-1})$. Since $\Lambda(g \Gamma_m g^{-1}) = \Lambda(\Gamma_m)$, we have that $g(\Lambda(\Gamma_m)) = \Lambda(\Gamma_m)$, and so $\Lambda(\Gamma_m)$ is a non-empty closed subset of $\mathbb{C}$ invariant under $\Theta$. Since $\Theta$ contains $\Gamma_m$, no smaller non-empty subset of $\mathbb{C}$ can be invariant under $\Theta$, and so $\Lambda(\Gamma_m) = \Lambda(\Theta)$.

There are now two cases. In the case that $\Lambda(\Gamma_m)$ is not a circle in $\mathbb{C}$, it is a standard fact, see for instance Sullivan [32], that

$$\Xi = \operatorname{stab}_{\operatorname{PSL}_2(\mathbb{C})}(\Lambda(\Gamma_m))$$

is a discrete subgroup of $\operatorname{PSL}_2(\mathbb{C})$ with $\Lambda(\Xi) = \Lambda(\Gamma_m)$.

Since each $\Gamma_m$ is non-elementary, there is a canonical metric of constant curvature $-1$ on $\Omega(\Gamma_m) = \Omega(\Theta) = \Omega(\Xi)$ which is invariant under $\Xi$, and so descends to the hyperbolic surfaces
\(\Omega(\Xi)/\Gamma_m, \Omega(\Xi)/\Theta,\) and \(\Omega(\Xi)/\Xi.\) By the Ahlfors finiteness theorem \[1\], \(\Omega(\Xi)/\Gamma_m\) has finite area. Since \(\Omega(\Xi)/\Gamma_m\) covers both \(\Omega(\Xi)/\Theta\) and \(\Omega(\Xi)/\Xi,\) both of these hyperbolic surfaces have finite area. Since there is a lower bound on the area of a hyperbolic surface, this implies that each \(\Gamma_m\) has finite index in \(\Xi,\) and so \(\Xi\) is finitely generated and discrete. The index \([\Xi : \Gamma_m]\) is bounded by the quotient of the areas \(\text{area}(\Omega(\Xi)/\Xi)/\text{area}(\Omega(\Gamma_m)/\Xi),\) and so as in the proof of Proposition \[1.1\] infinitely many of the \(\Gamma_m\) must be equal.

In the case that \(\Lambda(\Xi)\) is a circle in \(\mathbb{C},\) the stabilizer of \(\Lambda(\Xi)\) in \(\text{PSL}_2(\mathbb{C})\) is conjugate to \(\text{PSL}_2(\mathbb{R}),\) and so we are done by Remark \[4.2.\]

We close this Section by noting that the conclusion and proof of Proposition \[4.7\] also hold for collections of finitely generated but geometrically infinite Kleinian groups, as long as it is assumed that the domains of discontinuity of the Kleinian groups in the collection are non-empty.

5 The construction

We are now ready to give the proof of Theorem \[5.1.\]

**Theorem 5.1** There exist infinitely many commensurability classes of finite volume hyperbolic 3-manifolds whose fundamental groups contain a subgroup which is locally free but not free.

**Proof** We begin by recalling the example of Maskit from Section \[3\] of a convex co-compact Kleinian group \(\Gamma_0\) which contains a subgroup which is locally free but not free. The boundary of the compact 3-manifold \(M_0 = (\mathbb{H}^3 \cup \Omega(\Gamma_0))/\Gamma_0\) corresponding to \(\Gamma_0\) is a connected surface of genus two which is incompressible in \(M_0.\)

Let \(M_1\) be a compact hyperbolizable acylindrical 3-manifold whose boundary \(\partial M_1\) is the union of a torus \(T\) and a surface \(S\) of genus two. Such manifolds are common. To construct one, begin with a compact hyperbolizable acylindrical 3-manifold \(M\) whose boundary is a closed surface of genus two. Such manifolds are known to exist by work of Kojima and Miyamoto \[13\], see also Paoluzzi and Zimmermann \[27\]. Here, we are using the fact that the condition that \(M\) be acylindrical is equivalent to the existence of a hyperbolic structure with totally geodesic boundary on \(M.\)

It is a result of Myers \[25\] that for each element \(\gamma\) in \(\pi_1(M),\) there exists a simple closed curve \(c\) in \(M\) representing \(\gamma\) so that \(M - N(c)\) is hyperbolizable, where \(N(c)\) is a regular neighborhood of \(c\) in the interior \(\text{int}(M)\) of \(M.\)

We now consider how to choose \(\gamma\) so that \(M - N(c)\) is acylindrical. Let \((m, l)\) be a meridian-longitude system on the torus \(\partial N(c),\) as described in Section \[2.\] Choose \(\gamma\) so that \(\gamma\) is not homotopic into \(\partial M.\) Suppose there exists an essential annulus \(A\) in \(M - N(c)\) so that one component of \(\partial A\) lies in \(\partial M\) and the other component of \(\partial A\) lies in \(T = \partial N(c).\) Write \(\partial A = B \cup C,\) where \(B\) lies in \(\partial M\) and \(C\) lies in \(T.\)

Express \(C\) in the chosen meridian-longitude system, so that it is a \((p, q)\) curve on \(T,\) where \(p, q\) are relatively prime.
If $C$ is a $(1, 0)$ curve on $T = \partial N(c)$, then $C$ bounds a disc $D$ in $N(c)$. Then, the union $A \cup D$ is a properly embedded disc in $M$ whose boundary $B$ is a homotopically non-trivial curve on $\partial M$, which contradicts the incompressibility of $\partial M$.

If $C$ is a $(p, q)$ curve on $T$ for $q \neq 0$, then $C$ is homotopic to $c^q$ in $M$. However, the acylindricality of $M$ implies that if $c^q$ is homotopic into $\partial M$, then $c$ is homotopic into $\partial M$, a contradiction. To see this, we use a result of Maskit [19], which states that if $\Gamma$ is a finitely generated Kleinian group and if $\Delta$ and $\Delta'$ are two components of $\Omega(\Gamma)$, then

$$\partial \Delta \cap \partial \Delta' = \Lambda(\text{stab}_\Gamma(\Delta) \cap \text{stab}_\Gamma(\Delta')).$$

So, if $c^q$ is homotopic into $\partial M$, then $\gamma^q$ stabilizes a component $\Delta$ of $\Omega(\Gamma)$, where $\Gamma$ is a realization of $\pi_1(M)$ as a Kleinian group. In particular, the fixed points of $\gamma$ lie in $\partial \Delta$. Since $M$ is acylindrical, the components of $\Omega(\Delta)$ have disjoint closures, see for instance Anderson [2]. So, it cannot be that $\gamma(\Delta) \neq \Delta$, as then the two components $\Delta$ and $\gamma(\Delta)$ of $\Omega(\Gamma)$ would have disjoint components with intersecting closures. So, $\gamma(\Delta) = \Delta$, and so $\gamma$ and hence $c$ are homotopic into $\partial M$.

The acylindricality of $M$ implies that there is no essential annulus in $M - N(c)$ with both boundary components in $\partial M$, as then would then give an essential annulus in $M$. Also, there can be no essential annulus in $M - N(c)$ with both boundary components in $\partial N(c)$. Hence, there are no essential annuli in $M$.

Let $h : \partial M_0 \to S$ be an orientation-reversing homeomorphism, and consider the 3-manifold $M = M_0 \cup_h M_1$.

We now make use of Thurston’s hyperbolization theorem for 3-manifolds, see Morgan [22], which states that a compact, orientable, irreducible, atoroidal 3-manifold with non-empty boundary is hyperbolizable. As mentioned in the Introduction, this part of the argument is standard and is included for the sake of completeness.

Note that $M$ is compact, as both $M_0$ and $M_1$ are compact, and is orientable, as $h$ is orientation-reversing. Let $X$ be the image of $\partial S = h(\partial M_0)$ in $M$.

$M$ is irreducible: Let $S$ be an embedded $S^2$ in $M$. First, isotope $S$ so that $S \cap X$ is the finite union of disjoint simple closed curves. Let $c$ be an innermost curve on $S$, meaning that one of the components of $S - c$ contains no curve in $S \cap X$.

Note that $c$ bounds a closed disc $D$ in $S$, namely the closure in $S$ of the component of $S - c$ which contains no curve in $S \cap X$. Since the interior of $D$ is disjoint from $X$, we have that $D$ is contained in $M_k$, where either $k = 0$ or $k = 1$. Since $X$ is incompressible in $M_k$ by assumption, $\partial D$ is a homotopically trivial curve in $M_k$. So, we can isotope $D$ into $S$ and thereby get rid of $c$.

Repeating this argument for each curve in $S \cap X$ in turn, working from the innermost out, we can isotope $S$ into either $M_0$ or $M_1$. Since both $M_k$ for $k = 0$ and $k = 1$ are irreducible, $S$ then bounds a 3-ball in $M_k$ and hence in $M$, and so $M$ is irreducible as well.

$M$ is atoroidal: This argument is very similar to the argument that $M$ is irreducible. Let $T$ be an incompressible torus in $M$, and isotope $T$ so that $X \cap T$ is the finite union of disjoint simple
closed curves. Again performing an innermost disc argument, we can isotope away all the curves in \( X \cap T \) which bound a disc in \( T \).

Hence, we can assume that all the curves in \( X \cap T \) are homotopically non-trivial curves on the torus \( T \). Since both \( X \) and \( T \) are embedded surfaces in \( M \), the curves in \( X \cap T \) are parallel on \( T \). Moreover, since \( X \) separates \( M \), there must be an even number of curves in \( X \cap T \). If \( X \cap T \) is empty, then \( T \) is contained in \( M_k \) for either \( k = 0 \) or \( k = 1 \). Since both \( M_0 \) and \( M_1 \) are hyperbolizable, \( T \) is then homotopic in \( M_k \) into \( \partial M_k \).

If \( X \cap T \) is non-empty, then consider the closure \( A \) of a component of \( T - (X \cap T) \) contained in \( M_1 \). Since the curves in \( X \cap T \) are homotopically non-trivial on \( T \), \( A \) is an annulus. Since \( T \) is incompressible in \( M \), the boundary curves are homotopically non-trivial curves in \( \partial M_1 \), and so \( A \) is an incompressible annulus in \( M_1 \). Since \( M_1 \) is acylindrical, we can homotope \( A \) into \( \partial M_1 \), and hence into \( M_0 \).

Doing this for each component of \( T - (X \cap T) \) which lies in \( M_1 \), we can homotope all of \( T \) into \( M_0 \). Since \( M_0 \) is atoroidal, we can homotope \( T \) into a toroidal component of \( \partial M_0 \), which is also a toroidal component of \( \partial M \). Hence, \( M \) is atoroidal.

Hence, we have a hyperbolizable 3-manifold, namely \( M \), whose boundary is a torus. The Kleinian group \( \Gamma \) uniformizing \( M \) is necessarily minimally parabolic and geometrically finite. Also, since \( \Gamma \) contains a quasiconformal conjugate of \( \Gamma_0 \), we see that \( \Gamma \) contains a convex co-compact subgroup which in turn contains a subgroup which is locally free but not free. If we now perform Dehn surgery along \( \partial M \), we may combine Proposition 2.1 and Proposition 4.1 to see that there exist infinitely many commensurability classes of co-closed Kleinian groups which contain a subgroup which is locally free but not free.

\[ \square \]

Note that there is an extraordinary amount of flexibility in the construction given in the proof of Theorem 5.1, specifically in the choice of the 3-manifold \( M_1 \) and in the choice of the gluing map \( h : \partial M_0 \to S \).

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