Extremal properties of contraction semigroups
on $C_0$

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Abstract
For any complex Banach space $X$, let $J$ denote the duality mapping of $X$. For any unit vector $x$ in $X$ and any $(C_0)$ contraction semigroup $(T_t)_{t>0}$ on $X$, Baillon and Guerre-Delabriere proved that if $X$ is a smooth reflexive Banach space and if there is $x^* \in J(x)$ such that $|(T(t)x, J(x))| \to 1$ as $t \to \infty$, then there is a unit vector $y \in X$ which is an eigenvector of the generator $A$ of $(T_t)_{t>0}$ associated with a purely imaginary eigenvalue. They asked whether this result is still true if $X$ is replaced by $c_0$. In this article, we show the answer is negative.

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Let $X$ be a complex Banach space. The duality mapping $J$ of $X$ is a (multivalued) function from $X \setminus \{0\}$ into $X^*$ which is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = 1 = \|x^*\|\}.$$

Goldstein [5] proved the following theorem.

**Theorem 1** Let $A$ be a generator of a $C_0$ contraction semigroup $T = \{e^{tA} : t \geq 0\}$ on a complex Banach space $X$. If $X$ is a Hilbert space, then for any unit vector $x \in X$,

$$\lim_{t \to \infty} |\langle T_t x, J(x) \rangle| = 1$$

implies that $Ax = i\lambda x$ for some real number $\lambda$. (Note: since any Hilbert space is a smooth Banach space, $J(x)$ is singleton for $x \neq 0$.)

He also showed that Theorem 1 is not true if $X$ is replaced by an $L_\infty$-space. In [6], the author proved that Theorem 1 holds if and only if $X$ is strictly convex (also see [3]). We shall note: the counter-example in [3], the generator of the $C_0$-semigroup has no eigenvector. On the other hand, the counter-example in [6], the generator has an eigenvector vector associated with a pure imaginary eigenvalue. In [11], Baillon and Guerre-Delabriere considered the following question.

**Question 1** Let $(T_t)_{t>0}$ be a $C_0$ contraction semigroup on a complex Banach space $X$. Suppose that $x$ is a unit vector in $X$ such that

$$\lim_{t \to \infty} |\langle T_t x, x^* \rangle| = 1 = \|x^*\|.$$ 

Find a necessary and sufficient condition so that there is a unit vector $y \in X$ which is an eigenvector of $A$ associated with a purely imaginary eigenvalue.

They proved the following theorems.

**Theorem 2** Let $(T_t)_{t>0}$ be a $C_0$ contraction semigroup on a complex Banach space $X$. Let $x$ be a unit vector in $X$. If there is $x^* \in J(x)$ such that

$$\lim_{t \to \infty} |\langle T_t x, x^* \rangle| = 1,$$

then there is $y^* \in J(x)$ such that $|\langle T_t x, y^* \rangle| = 1$ for all $t > 0$. 

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Theorem 3 Let $X$ be a reflexive smooth complex Banach space and let $(T_t)_{t>0}$ be a semigroup of contractions on $X$. If there are a unit vector $x \in X$ and $x^* \in J(x)$ such that

$$\lim_{t \to \infty} |\langle T_t x, x^* \rangle| = 1,$$

then there are a unit vector $y \in X$ and $\omega \in \mathbb{R}$ such that for all $t > 0$, $T_t y = e^{i\omega t} y$. Hence, $y$ is an eigenvector of the generator of $(T_t)_{t>0}$ associated with an eigenvalue $i\omega$.

They also showed Theorem 3 is not true if one replaces $X$ by $\ell_1$ or $L_1$. They asked whether Theorem 3 is still true if $X$ is replaced by $c_0$ or any (non-smooth) reflexive Banach space. Recently, Ruess showed the answer is affirmative if $X$ is reflexive. Indeed, he proved the following strong theorem.

Theorem 4 Assume that $(T_t)_{t>0}$ of a uniformly bounded $C_0$-semigroup on $X$. If there exist $x \in X$ and $x^* \in X^*$ such that $\{T_t x : t \in \mathbb{R}\}$ is weakly compact and $\lim_{t \to \infty} |\langle T_t x, x^* \rangle| > 0$, then there is a $y \in X \setminus \{0\}$ which is an eigenvector of the generator of $(T_t)_{t>0}$ corresponding to a pure imaginary number.

In this article, we consider the contraction $C_0$-semigroups on $c_0$ and show the answer is negative.

Example 5 Let $\{e_1, e_2, \cdots\}$ be the natural basis of $c_0$. $A : c_0 \to c_0$ is defined by

$$A(e_i) = \begin{cases} \sum_{k=2}^{\infty} \frac{1}{k} e_k & \text{if } i = 1, \\ -\frac{1}{i} e_i & \text{otherwise.} \end{cases}$$

Then

$$T_t(e_i) = e^{tA}(e_i) = \begin{cases} e_1 + \sum_{k=2}^{\infty} (1 - e^{-\frac{t}{k}}) e_k & \text{if } i = 1, \\ e^{-\frac{t}{i}} e_i & \text{otherwise.} \end{cases}$$

So $(T_t)_{t>0}$ is a $C_0$ semigroup of contractions on $c_0$. It is easy to see that

$$\langle e^{tA}(e_1), e_1^* \rangle = 1,$$

and $A$ does not have any eigenvector associated with any purely imaginary eigenvalue (or 0). (Here $e_1^* \in J(e_1)$ and $e_1^*$ is $e_1$, veiwed as a member of $\ell_1 = c_0^*$.) On the other hand, for any $k \geq 2$, $e_k$ is an eigenvector of $A$ associated with eigenvalue $-\frac{1}{k}$. 


It is known that if $T$ is an isometry on $\ell_p$, $1 \leq p < \infty$, $p \neq 2$, then the images $Tx$ and $Ty$ of any two disjoint elements $x, y$ (this is $|x| \land |y| = 0$) are disjoint (\cite{P} p. 416 Lemma 23). Let $(T_t)_{t > 0}$ be an isometric semigroup on $\ell_1$. Since $\lim_{t \downarrow 0} Te_i = e_i$, Fleming, Goldstein, and Jamison \cite{FGJ1, FGJ2} proved there exists a sequence $\{\omega_n\}$ of $\mathbb{R}$ such that $T_t(e_n) = e^{i\omega_n t}e_n$.

Hence, there is no counter-examples of isometric semigroups on $\ell_1$. On the other hand, there is an isometry $T(\sum_{i=1}^{\infty} a_i e_i) = \frac{a_1 + a_2}{2} e_1 + \sum_{i=1}^{\infty} a_i e_{i+1}$ on $c_0$ which does not preserve disjoint support. It is natural to ask whether we can improve the Example \cite{FGJ2} to be $C_0$-isometric semigroup. The following theorem shows the answer is negative.

**Theorem 6** Let $(T_t)_{t > 0}$ be a $C_0$ isometric semigroup on $c_0$. Let $\{e_k\}$ be the natural basis of $c_0$. Then for any $k \in \mathbb{N}$, there is $\omega_k \in \mathbb{R}$ such that $T_t e_k = e^{i\omega_k t} e_k$.

**Proof.** Let $\{e_k^*\}$ be the natural basis of $\ell_1 = (c_0)^*$ and let $k$ be any fixed natural number. Since $\lim_{t \downarrow 0} T_t e_k = e_k$, there is $\delta_k > 0$ such that if $0 < t \leq \delta_k$, then $|\langle T_t e_k, e_j^* \rangle| < \frac{1}{2}$ for all $j \neq k$.

But $T_t$ is an isometry. This implies $|\langle T_t e_k, e_k^* \rangle| = 1$.

We claim that if $0 < t \leq \delta_k$, then $\langle T_t e_j, e_k^* \rangle = 0$ for all $j \neq k$. Suppose it is not true. There are $j \neq k$, and $0 < t \leq \delta_k$, such that $\langle T_t e_j, e_k^* \rangle \neq 0$.

So
$$\langle T_t (\langle T_t e_k, e_k^* \rangle e_k + \frac{\langle T_t e_j, e_k^* \rangle}{|\langle T_t e_j, e_k^* \rangle|} e_j), e_k^* \rangle = 1 + |\langle T_t e_j, e_k^* \rangle| > 1.$$

But
$$\left\| \langle T_t e_j, e_k^* \rangle e_k + \frac{\langle T_t e_j, e_k^* \rangle}{|\langle T_t e_j, e_k^* \rangle|} e_j \right\| = 1.$$
This contradicts that $T_t$ is an isometry. We proved our claim.

We note: the claim proved that if $0 < t < \delta_k$ and $x \in c_o$ such that $\langle x, e_k^* \rangle = 0$, then $\langle T_t x, e_k^* \rangle = 0$. Now, let $t$ be any positive real number. There exist integer $n \geq 0$ and $0 \leq s < \delta_k$ such that

$$ t = n \delta_k + s. $$

But $T_t = (T_{\delta_k})^n \circ T_s$. This implies that if $\langle x, e_k^* \rangle = 0$, then

$$ \langle T_t x, e_k^* \rangle = 0 $$

for all $t > 0$. So for any $k \in \mathbb{N}$, there is a function $\gamma_k(t)$ such that

$$ T_t e_k = \gamma_k(t) e_k, $$

But $T_t$ is an isometry. We must have $|\gamma_k(t)| = 1$ for all $t > 0$. By the $C_0$ semigroup property of $(T_t)_{t>0}$, it is easy to see that $\gamma_k(t) = e^{i\omega_k t}$ for some $\omega_k \in \mathbb{R}$. The proof is complete. \( \Box \)

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