Topology and purity for torsors

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ABSTRACT. We study the homotopy theory of the classifying space of the complex projective linear groups to prove that purity fails for $\text{PGL}_{p}$-torsors on regular noetherian schemes when $p$ is a prime. Extending our previous work when $p = 2$, we obtain a negative answer to a question of Colliot-Thélène and Sansuc, for all $\text{PGL}_{p}$. We also give a new example of the failure of purity for the cohomological filtration on the Witt group, which is the first example of this kind of a variety over an algebraically closed field.

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1. INTRODUCTION

Let $X$ be a regular noetherian integral scheme, let $G$ be a smooth reductive group scheme over $X$, and let $K$ be the function field of $X$. Consider the injective map

\[ \text{im} \left( H^1_{\text{ét}}(X, G) \to H^1_{\text{ét}}(\text{Spec } K, G) \right) \to \bigcap_{x \in X^{(1)}} \text{im} \left( H^1_{\text{ét}}(\text{Spec } \mathcal{O}_{X, x}, G) \to H^1_{\text{ét}}(\text{Spec } K, G) \right), \]

where the intersection is over all codimension-1 points of $X$. Colliot-Thélène and Sansuc ask in [13, Question 6.4] whether this map is surjective. When it is, we say that purity holds for $H^1_{\text{ét}}(X, G)$.

Purity trivially holds for $H^1_{\text{ét}}(X, G)$ when $G$ is special in the sense of Serre, for example if $G = \text{SL}_n$, since $H^1_{\text{ét}}(\text{Spec } K, G)$ is a single point in this case. It holds for $H^1_{\text{ét}}(X, G)$ where $G$ is a finite type $X$-group scheme of multiplicative type by [13, Corollaire 6.9]. It is also known to hold in many cases when $X$ is the spectrum of a regular local ring containing a field of characteristic 0. With this assumption, purity was proven for $H^1_{\text{ét}}(X, G)$ when $G$ is a split group of type $A_n$, a split orthogonal or special orthogonal group, or a split spin group by Panin [31] and also when $G = G_2$ by Chernousov and Panin [12]. The local purity conjecture asserts that purity holds for $H^1_{\text{ét}}(X, G)$ whenever $X$ is the spectrum of a regular noetherian integral semi-local ring and $G$ is a smooth reductive algebraic $X$-group scheme. Finally, purity holds for $H^1_{\text{ét}}(X, G)$ if the Krull dimension of $X$ is at most 2 by [13, Theorem 6.13].
Purity is often considered along with another property, the so-called injectivity property, which is said to hold when $H^1_{ét}(X, G) \to H^1(\mathcal{O}_X, G)$ has trivial kernel for all $U \subseteq X$ containing $X^{(1)}$. In fact, Grothendieck and Serre conjectured that this map is always injective when $X$ is the spectrum of a regular local ring $R$ and $G$ is a reductive $X$-group scheme. This has been proved recently using affine Grassmannians by Fedorov and Panin [19] when $R$ contains an infinite field following partial progress by many other mathematicians. They prove more strongly that $H^1_{ét}(X, G) \to H^1(\mathcal{O}_X, G)$ is injective. The injectivity property for torsors is usually only sensible when $X$ is in fact the spectrum of a local ring: otherwise it typically fails, even for $G = G_m$.

When $X$ is neither local nor low-dimensional and $G$ is a non-special semisimple algebraic group, no results were known about purity for torsors until our paper [3], which showed that purity fails for $\text{PGL}_2$-torsors on smooth affine complex 6-folds in general. It is the purpose of this paper to use $p$-local homotopy theory to extend our previous result to $\text{PGL}_p$ for all $p$.

**Theorem 1.1.** Let $p$ be a prime. Then, there exists a smooth affine complex variety $X$ of dimension $2p + 2$ such that purity fails for $H^1_{ét}(X, \text{PGL}_p)$.

We outline the proof. Recall first that $\text{PGL}_p$-torsors correspond to degree-$p$ Azumaya algebras, and write $Br_{top}(X(C))$ for the topological Brauer group, which classifies topological Azumaya algebras up to Brauer equivalence [25]. Let $X$ be a smooth complex variety such that $H^2(X(C), \mathbb{Z}) = 0$. In this case, by [2, Lemma 6.3], there is an isomorphism $Br(X) \cong Br_{top}(X(C)) = H^0(X(C), \mathbb{Z})_{\text{tors}}$. Because $H^2(X(C), \mathbb{Z}) = 0$, topological Azumaya algebras of degree $n$ and exponent $m$ on $X(C)$ are classified by homotopy classes of maps $X(C) \to \text{BP}(m, n)$, where $\text{P}(m, n) = \text{SL}_n(C)/\mu_m$.

In order to prove the theorem, we must construct a complex affine variety, $X$. First, following Totaro [34], we take high dimensional algebraic approximations, $X$, to the classifying spaces $\text{BP}(p, pq)$, where $q > 1$ is prime to $p$. These resemble universal examples of varieties equipped with Azumaya algebra $A$ of degree $pq$ and having obstruction class of order $p$ in $H^2_{ét}(X, \mu_{pq})$. Let $\alpha$ be the Brauer class of $A$ on $X$. The exponent of $\alpha$ is $p$. Comparing the $p$-local homotopy type of $\text{BP}(p, pq)$ to that of $\text{BP}_{G_{pq}}(C)$, we find that there is a non-vanishing obstruction in $H^{2p+2}(X(C), \mathbb{Z}/p)$ to the existence of a degree-$p$ Azumaya algebra on $X$ with the same Brauer class as $A$. Second, we replace $X$ by a homotopy-equivalent smooth affine variety using Jouanolou’s device [28]. Third, we use the affine Lefschetz theorem [24, Introduction, Section 2.2] to cut down to a smooth affine $2p + 2$-dimensional variety where the obstruction in $H^{2p+2}(X(C), \mathbb{Z}/p)$ persists. By using an unpublished preprint of Ekedahl [18], it is possible to construct smooth projective complex examples of this nature as well, although we will not emphasize this last point in our paper.

Let $K$ be the function field of the $2p + 2$-dimensional affine variety $X$ alluded to in the previous paragraph. The theorem is deduced from the properties of $X$ as follows. The Brauer class $\kappa_K \in Br(K)$ has exponent $p$ and index dividing $pq$. Its index is therefore $p$ by a result of Brauer [23, Proposition 4.5.13], and it is represented by a division algebra $D$ of degree $p$ over $K$. If $P \in X^{(1)}$, then $\alpha$ restricts to a class $\sigma_P \in Br(\mathcal{O}_{X, P})$. Since $D$ is unramified along $\mathcal{O}_{X, P}$ and since $\mathcal{O}_{X, P}$ is a discrete valuation ring, it follows that any maximal order in $D$ over $\mathcal{O}_{X, P}$ is in fact an Azumaya algebra (see the proof of [7, Proposition 7.4]). Thus, the class of $D$ is in the target of the map of (9), but by our choice of $X$, the class of $D$ is not in the source of that map.

We make the following conjecture.

**Conjecture 1.2.** Let $G$ be a non-special semisimple $k$-group scheme. Then there exists a smooth affine $k$-variety $X$ such that purity fails for $H^1_{ét}(X, G)$. 
Our theorem proves the conjecture for \( G = \text{PGL}_p \) over \( \mathbb{C} \), and since the schemes in question may all be defined over \( \mathbb{Q} \), the conjecture is actually settled for \( \text{PGL}_p \) over any field of characteristic 0.

We explore three other points in the paper. First, in Section 3.3, we show that, in contrast to the global case, purity holds for \( \text{PGL}_n \)-torsors over regular noetherian integral semilocal rings \( R \), at least if we restrict our attention to those torsors whose Brauer class has order invertible in \( R \). This is a generalization of equivalent results of Ojanguren [29] and Panin [31] in characteristic 0.

Second, in Section 3.4, we give a topological perspective that explains why we expect purity to fail for \( H^1_{\text{et}}(X, \text{PGL}_n) \) for all \( n \).

Third, in Section 3.5, we give examples where purity fails for \( I^2(X)/I^3(X) \) where \( I^* \) is the filtration on the Witt group induced by the cohomological filtration on \( W(\mathbb{C}(X)) \) and \( X \) is a certain smooth affine complex 5-fold. Previous examples of a different, arithmetic nature were produced by Parimala and Sridharan [32], but these were explained by Auel [3] as failing to take into account quadratic modules with coefficients in line bundles. Our examples have \( \text{Pic}(X) = 0 \).

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2. Topology

In [3], we used knowledge of both the low-degree singular cohomology of \( \text{BPGL}_2(\mathbb{C}) \) and of the low-degree Postnikov tower of \( \text{BPGL}_2(\mathbb{C}) \) to produce counterexamples to the existence of Azumaya maximal orders in unramified division algebras. This is equivalent to showing that purity fails for \( \text{PGL}_2 \) over \( \mathbb{C} \). At the time we wrote [3], we did not know how to extend our results to other primes, because our argument relied on the accessibility of the low-degree Postnikov tower of \( \text{BPGL}_2(\mathbb{C}) \). While remarkable calculations have been made by Vezzosi [37] and Vistoli [38] on the cohomology of \( \text{BPGL}_p(\mathbb{C}) \) for odd primes \( p \), the problem of determining the Postnikov tower up the necessary level, \( 2p + 1 \), was beyond us. By using a \( p \)-local version of our arguments in [3] we bypass our ignorance to prove similar results.

We prove a result in this section about self-maps of \( \tau_{\leq 2p+1}\text{BPGL}_p(\mathbb{C}) \), the \( 2p + 1 \) stage in the Postnikov tower of \( \text{BPGL}_p(\mathbb{C}) \). Our theorem is in some sense related to the important results of Jackowski, McClure, and Oliver [27] about maps \( \text{BG} \to \text{BH} \) when \( G \) and \( H \) are compact Lie groups, and especially about self-maps of \( \text{BG} \). For the applications to algebraic geometry we have in mind, one must use finite approximations to \( \text{BPGL}_p(\mathbb{C}) \) \( \simeq \text{BPU}_p \), where the results of [27] do not immediately apply. For more on the relationship of our work to [27], see Section 3.4.

The group \( \text{PGL}_p(\mathbb{C}) \) and other classical groups are always equipped with the classical topology.

2.1. The \( p \)-local cohomology of some Eilenberg-MacLane spaces. We fix a prime number \( p \). The \( p \)-local cohomology of a space \( X \) is the singular cohomology of \( X \) with coefficients in \( \mathbb{Z}_p(\mathbb{C}) \). In the next few lemmas, we compute the low-degree \( p \)-local cohomology of \( K(\mathbb{Z}, n) \), up to the first \( p \)-torsion. These results are both straightforward and classical, being corollaries of the all-encompassing calculations of Cartan [11] for instance. We include proofs here for the sake of completeness.

Recall that \( K(\mathbb{Z}, 2) \simeq \mathbb{CP}^\infty \) and that \( H^*(K(\mathbb{Z}, 2), \mathbb{Z}) \cong \mathbb{Z}[[t_2]] \), where \( \deg(t_2) = 2 \). In general, there is a canonical class \( t_n \in H^n(K(\mathbb{Z}, n), \mathbb{Z}) \) representing the identity map. We will use the Serre spectral sequences for the fiber sequences \( K(\mathbb{Z}, n) \to \ast \to K(\mathbb{Z}, n + 1) \) as well as the multiplicative structure in the spectral sequences.
Lemma 2.1. For $1 \leq k \leq 2p + 4$, the $p$-local cohomology of $K(Z, 3)$ is

$$H^k(K(Z, 3), Z(p)) \cong \begin{cases} \mathbb{Z}_p & \text{if } k = 3, \\ \mathbb{Z}/p & \text{if } k = 2p + 2, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. We can choose $t_3$ so that $d_3(t_3) = t_3$ in the Serre spectral sequence for $K(Z, 2) \to \to \to K(Z, 3)$:

$$E_2^{pq} = H^p(K(Z, 3), H^q(K(Z, 2), Z(p))) \Rightarrow H^{p+q}(\ast, Z(p)).$$

Then, $d_3(t_3^n) = n\alpha^{n-1}t_3$ and it follows that $d_3(t_3^n)$ is a generator of $E_2^{2n-2,3}$ for $1 \leq n < p$. For $4 \leq k \leq 2p + 1$ the cohomology group $H^k(K(Z, 3), Z(p))$ vanishes since there are no possible non-zero differentials hitting it. The first point on the $q$-axis where $d_3$ is not surjective is $d_3 : E_3^{2p,0} \to E_3^{2p-2,3}$ where the cokernel is $\mathbb{Z}/p$, see Figure 1. In order for the sequence to converge to zero, this non-zero cokernel must support a non-zero de-parting differential; since $H^k(K(Z, 2), Z(p)) = 0$ for $4 \leq k \leq 2p + 1$, the differential $d_{2p-1}$ induces an isomorphism $\mathbb{Z}/p \to H^{2p+2}(K(Z, 3), Z(p))$. Let $j_p$ be a generator of $H^{2p+2}(K(Z, 3), Z(p))$. In terms of total degree, the next non-zero term in the spectral sequence is $E_3^{2p+2,2} = \mathbb{Z}/p \cdot t_2|p$. Thus, the next potentially non-zero $p$-local cohomology group of $K(Z, 3)$ is $H^{2p+5}(K(Z, 3), Z(p))$. \qed

![Figure 1. The E3-page of the Serre spectral sequence associated to K(Z, 2) → * → K(Z, 3).](image-url)

The next two lemmas have proofs conceptually similar to the preceding proof.

Lemma 2.2. For $0 \leq k \leq 2p + 5$, the $p$-local cohomology of $K(Z, 4)$ is

$$H^k(K(Z, 4), Z(p)) \cong \begin{cases} \mathbb{Z}_p & \text{if } k = 0 \mod 4, \\ \mathbb{Z}/p & \text{if } k = 2p + 3, \\ 0 & \text{otherwise}. \end{cases}$$
The Postnikov tower is the sequence of natural maps with the fiber of \(\tau \pi \mathbb{P} \mathcal{X} \) of a path-connected space \(\tau \mathbb{P} \mathcal{X} \). For this reason the group \(H^{2p+2}(K(\mathbb{Z}, 3), \mathbb{Z}(p))\) survives to the \(E_{2p+3}\)-page of the spectral sequence, and the differential

\[
d_{2p+3} : \mathbb{Z}/p \cong H^{2p+2}(K(\mathbb{Z}, 3), \mathbb{Z}(p)) \to H^{2p+3}(K(\mathbb{Z}, 4), \mathbb{Z}(p))
\]

is an isomorphism. The next potential non-zero torsion class in the spectral sequence is in \(H^{2p+5}(K(\mathbb{Z}, 3), \mathbb{Z}(p))\), which shows that the other cohomology groups vanish in the range indicated. \(\square\)

Lemma 2.3. The cohomology groups \(H^k(K(\mathbb{Z}, n), \mathbb{Z}(p))\) are torsion-free for \(0 \leq k \leq 2p + 3\) and \(n \geq 5\). In this range they are isomorphic to a polynomial algebra over \(\mathbb{Z}(p)\) with a single generator \(\iota_n\) in degree \(n\) if \(n\) is even or an exterior algebra over \(\mathbb{Z}(p)\) with a single generator \(\iota_n\) in degree \(n\) if \(n\) is odd.

Proof. This follows inductively as in the previous two lemmas. The important point is that the first \(p\)-torsion in \(H^i(K(\mathbb{Z}, n-1), \mathbb{Z}(p))\) is in degree \(2p + (n - 1) - 1\). No differential exiting it can be non-zero until the differential \(d_{2p+(n-1)}\), which produces \(p\)-torsion in \(H^{2p+n-1}(K(\mathbb{Z}, n), \mathbb{Z}(p))\). If \(n \geq 5\), then \(2p + n - 1 \geq 2p + 4\). \(\square\)

2.2. The \(p\)-local homotopy type of \(\mathbb{B}L_p(C)\). Now we harness the computations of the previous section to study the \(p\)-local homotopy type of truncations of \(\mathbb{B}G_{\mathbb{C}}(C)\). If \(X\) is a connected topological space, we will write \(\tau_{\leq n} X\) for the \(n\)th stage in the Postnikov tower of a path-connected space \(X\). Thus, \(\tau_{\leq n} X\) is a topological space such that

\[
\pi_i(\tau_{\leq n} X) \cong \begin{cases} 
\pi_i(X) & \text{if } i \leq n, \\
0 & \text{otherwise.}
\end{cases}
\]

The Postnikov tower is the sequence of natural maps

with the fiber of \(\tau_{\leq n} X \to \tau_{\leq n-1} X\) identified with \(K(\pi_n X, n)\). In good cases, such as when the action of \(\pi_1 X\) on \(\pi_n X\) for \(n \geq 1\) is trivial, the extension

\[
K(\pi_n X, n) \to \tau_{\leq n} X
\]

is classified by the \(k\)-invariant

\[
k_{n-1} : \tau_{\leq n-1} X \to K(\pi_n X, n + 1)
\]
in the sense that $\tau_{\leq n}X$ is the homotopy fiber of $k_{n-1}$. This $k$-invariant is a cohomology class in $H^{n+1}(\tau_{\leq n-1}X, \pi_nX)$. When it vanishes, the fibration is trivial.

There is a $p$-localization functor $L_p$ that takes a topological space $X$ and produces a space $L_pX$ whose homotopy groups are $\mathbb{Z}(p)^{\infty}$-modules. For the theory of localization of CW complexes, we refer to the monograph of Bousfield and Kan [10]. This functor takes fiber sequences to fiber sequences when the base is simply connected by the principal fibration lemma [10, Chapter III]. Since the $\mathbb{Z}(p)$-localization of an Eilenberg-MacLane space $K(\pi, n)$ is $K(\pi \otimes_{\mathbb{Z}} \mathbb{Z}(p), n)$, for which see [10, page 65], it follows that application of $L_p$ commutes with the formation of Postnikov towers of simply-connected spaces.

Now, we consider the $p$-local homotopy type of certain stages in the Postnikov tower of $\text{BSL}_n(C)$. By Bott periodicity [9] the $p$-local homotopy groups of $\text{BSL}_n(C)$ for $1 \leq i \leq 2n + 1$ are

$$\pi_i\left(L_p(\text{BSL}_n(C))\right) \cong \pi_i(\text{BSL}_n(C)) \otimes_{\mathbb{Z}} \mathbb{Z}(p) \cong \begin{cases} \mathbb{Z}(p) & \text{if } i \text{ is even and } i \geq 4, \\ \mathbb{Z}/(n!) \otimes_{\mathbb{Z}} \mathbb{Z}(p) & \text{if } i = 2n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

**Proposition 2.4.** The localization $L_p\tau_{\leq 2p} \text{BSL}_n(C)$, where $n \geq p$, is a generalized Eilenberg–MacLane space:

\begin{equation}
L_p\tau_{\leq 2p} \text{BSL}_n(C) \cong K(\mathbb{Z}(p), 4) \times K(\mathbb{Z}(p), 6) \times \cdots \times K(\mathbb{Z}(p), 2p).
\end{equation}

**Proof.** We prove this by induction on $j$. The base case when $j = 1$ is trivial. For the induction step, suppose that $L_p\tau_{\leq 2j} \text{BSL}_n(C)$ is

$$K(\mathbb{Z}(p), 4) \times \cdots \times K(\mathbb{Z}(p), 2j)$$

for some $1 \leq j < p$. The extension

$$K(\mathbb{Z}(p), 2j + 2) \to L_p\tau_{\leq 2j + 2} \text{BSL}_n(C) \to L_p\tau_{\leq 2j} \text{BSL}_n(C)$$

is classified by the $k$-invariant

$$k_{2j} \in H^{2j + 3}(K(\mathbb{Z}(p), 4) \times \cdots \times K(\mathbb{Z}(p), 2j), \mathbb{Z}(p)).$$

By Lemmas 2.2 and 2.3, this cohomology group must vanish, since $j < p$. Hence $k_{2j} = 0$ and the extension is trivial. \hfill \Box

Before we prove the next proposition, we need a well-known lemma. Recall that an $n$-equivalence is a map such that $\pi_k(f) : \pi_k(X) \to \pi_k(Y)$ is an isomorphism for $0 \leq k < n$ and a surjection for $k = n$.

**Lemma 2.5.** Let $f : X \to Y$ be an $n$-equivalence. Then, for any coefficient abelian group $A$, the induced map

$$f^* : H^k(Y, A) \to H^k(X, A)$$

is an isomorphism for $0 \leq k \leq n - 1$ and an injection for $k = n$.

**Proof.** This follows most easily from the Serre spectral sequence for the fibration sequence $F \to X \to Y$. Since the fiber is $n$-connected, the groups $H^k(F, A)$ vanish for $k < n$. The first nontrivial extension problem in the spectral sequence takes the form

$$0 \to H^0(Y, A) \to H^0(X, A) \to H^n(F, A)$$

which proves the result. \hfill \Box

The previous proposition asserts that $L_p\tau_{\leq 2p} \text{BSL}_p(C)$ is a generalized Eilenberg–MacLane space, the following asserts that the $\tau_{\leq 2p}$ appearing there is sharp, and the nontriviality of the extension can be detected after pulling the extension back along an inclusion $K(\mathbb{Z}(p), 4) \to L_p\tau_{\leq 2p} \text{BSL}_p(C)$.
Proposition 2.6. Fix a map
\[ i: K(\mathbb{Z}/p, 4) \to L_{(p)}\tau_{\leq 2p} BSL_p(\mathbb{C}) \cong \prod_{j=2}^{p} K(\mathbb{Z}/(p), 2j) \]
splitting the projection map. Write \( k_{2p} \in H^{2p+2}(L_{(p)}\tau_{\leq 2p} BSL_p(\mathbb{C}), \mathbb{Z}/p) \) for the \( k \)-invariant of the extension
\[
K(\mathbb{Z}/p, 2p+1) \xrightarrow{i} L_{(p)}\tau_{\leq 2p+1} BSL_p(\mathbb{C})
\]
\[
\xrightarrow{L_{(p)}\tau_{\leq 2p} BSL_p(\mathbb{C}).}
\]

Then \( k_{2p} \) is of order \( p \), and moreover \( i^*(k_{2p}) \) is a generator for \( H^{2p+2}(K(\mathbb{Z}/(p), 4), \mathbb{Z}/p) \cong \mathbb{Z}/p \).

Proof. There is a quotient relationship arising from the Postnikov extensions
\[
H^{2p+2}(L_{(p)}\tau_{\leq 2p} BSL_p, \mathbb{Z}/p) / (k_{2p}) \cong H^{2p+2}(L_{(p)}\tau_{\leq 2p+1} BSL_p, \mathbb{Z}/p).
\]

In order to determine \( k_{2p} \), we compare \( H^{2p+2}(L_{(p)}\tau_{\leq 2p} BSL_p, \mathbb{Z}/p) \) with \( H^{2p+2}(L_{(p)}\tau_{\leq 2p+1} BSL_p, \mathbb{Z}/p) \).

Since the functorial map
\[
H^{2p+2}(L_{(p)}\tau_{\leq 2p} BSL_p, \mathbb{Z}/p) \hookrightarrow H^{2p+2}(L_{(p)} BSL_p, \mathbb{Z}/p)
\]
is injective by Lemma 2.5, there is an injection
\[
H^{2p+2}(L_{(p)}\tau_{\leq 2p} BSL_p, \mathbb{Z}/p) / (k_{2p}) \hookrightarrow H^{2p+2}(L_{(p)}\tau_{\leq 2p+1} BSL_p, \mathbb{Z}/p).
\]

To determine \( k_{2p} \), therefore, it will be enough to compare \( H^{2p+2}(L_{(p)}\tau_{\leq 2p} BSL_p, \mathbb{Z}/p) \) with \( H^{2p+2}(L_{(p)} BSL_p, \mathbb{Z}/p) \).

We switch temporarily to \( \mathbb{Z}_{(p)} \)-coefficients. The map \( L_{(p)}\tau_{\leq 2p} BSL_p \to L_{(p)} BSL_p \) is a \( 2p + 1 \)-equivalence. By Lemma 2.5, the map of rings
\[
H^i(L_{(p)}\tau_{\leq 2p} BSL_p, \mathbb{Z}_{(p)}) \to H^i(L_{(p)} BSL_p, \mathbb{Z}_{(p)}) = \mathbb{Z}_p[c_2, c_3, \dotsc, c_p]
\]
is an isomorphism when \( i \leq 2p \), and an injection, and hence an isomorphism, when \( i = 2p + 1 \). By Lemmas 2.2 and 2.3, the ring \( H^i(L_{(p)}\tau_{\leq 2p} BSL_p, \mathbb{Z}_{(p)}) \) is isomorphic to a polynomial ring on generators in degrees 4, 6, 8, \ldots, \( 2p \) in the range where \( i \leq 2p + 2 \), so that it follows that
\[
H^{2p+2}(L_{(p)}\tau_{\leq 2p} BSL_p, \mathbb{Z}_{(p)}) \cong H^{2p+2}(L_{(p)} BSL_p, \mathbb{Z}_{(p)})
\]
is an isomorphism as well. From Lemmas 2.2 and 2.3 we also deduce that
\[
H^{2p+3}(L_{(p)}\tau_{\leq 2p} BSL_p(\mathbb{C}), \mathbb{Z}_{(p)}) \cong \mathbb{Z}/p \cdot \rho
\]
where \( i^*(\rho) \) is a generator of \( H^{2p+3}(K(\mathbb{Z}, 4), \mathbb{Z}_{(p)}) \cong \mathbb{Z}/p \).

Considering the long exact sequence in cohomology associated to the sequence
\[
0 \to \mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)} \to \mathbb{Z}/p \to 0
\]
of coefficients, we deduce the existence of a decomposition
\[
H^{2p+2}(L_{(p)}\tau_{\leq 2p} BSL_p, \mathbb{Z}/p) = \mathbb{Z}/p \otimes_{\mathbb{Z}_{(p)}} H^{2p+2}(L_{(p)}\tau_{\leq 2p} BSL_p, \mathbb{Z}_{(p)}) \oplus \mathbb{Z}/p \cdot \sigma
\]
where the unreduced Bockstein homomorphism of \( \sigma \), denoted \( \beta_p(\sigma) \), is the \( p \)-torsion class \( \rho \).

The integral cohomology of \( BSL_p \) is torsion-free, and it follows that
\[
H^{2p+2}(L_{(p)} BSL_p, \mathbb{Z}/p) = \mathbb{Z}/p \otimes_{\mathbb{Z}_{(p)}} H^{2p+2}(L_{(p)} BSL_p, \mathbb{Z}_{(p)}).
\]
The isomorphism (5) and the presentations (6) and (7) applied to the comparison (4) show that \( k_{2p} = u \sigma \) where \( u \) is a unit. By naturality, \( \beta_{2p}(i^*(k_{2p})) = u i^*(\rho) \), and in particular, \( i^*(k_{2p}) \neq 0 \).
Corollary 2.7. A map \( h : L_{(p)} \pi_{\leq 2p+1} \text{BSL}_p(C) \to L_{(p)} \pi_{\leq 2p+1} \text{BSL}_p(C) \) that induces an isomorphism on \( \pi_4 \left( L_{(p)} \pi_{\leq 2p+1} \text{BSL}_p(C) \right) \cong \mathbb{Z}_{(p)} \), also induces an isomorphism on \( \pi_{2p+1} \left( L_{(p)} \pi_{\leq 2p+1} \text{BSL}_p(C) \right) \cong \mathbb{Z}/p. \)

Proof. Let \( i : K(Z_{(p)}, 4) \to L_{(p)} \pi_{\leq 2p} \text{BSL}_p(C) \) again denote a map splitting the projection onto \( K(Z_{(p)}, 4) \) in Proposition 2.4. Let \( \tau_{\leq 2p} : \tau_{\leq 2p} \text{BSL}_p(C) \to \tau_{\leq 2p} \text{BSL}_p(C) \) be the truncation of \( h \). This map fits into a commutative diagram

\[
\begin{array}{ccc}
K(Z_{(p)}, 4) & \xrightarrow{i} & L_{(p)} \pi_{\leq 2p} \text{BSL}_p(C) \\
\downarrow \cong & & \downarrow \tau_{\leq 2p} \\
K(Z_{(p)}, 4) & \xrightarrow{i} & L_{(p)} \pi_{\leq 2p} \text{BSL}_p(C)
\end{array}
\]

where the map \( B_{h*} \) is the result of applying a functorial classifying-space construction to the endomorphism of \( K(\pi_{2p+1} \left( L_{(p)} \text{BSL}_p(C) \right), 2p+1) \cong \mathbb{Z}/p, 2p+1 \) arising from the map \( h \) on \( \pi_{2p+1} \text{BSL}_p(C) \). The map \( K(Z_{(p)}, 4) \to K(Z_{(p)}, 4) \) is the composition of \( i \) with \( h \) and the projection, and is a weak equivalence since \( i \) and \( h \) and the projection all induce isomorphisms on \( \pi_4 \), by hypothesis. Since \( i^*(\tau_{2p}) \neq 0 \) is a generator of \( \mathbb{H}^{2p+2}(K(Z_{(p)}, 4), \mathbb{Z}/p) \), commutativity of the diagram proves that \( h \) is an equivalence, as claimed. \( \square \)

2.3. The \( p \)-local homotopy type of \( \text{BPGL}_p(C) \). There is a fiber sequence, obtained by truncating a sequence associated to the defining quotient \( \text{SL}_p(C)/\mu_p = \text{PGL}_p(C) \), of the form

\[
\begin{array}{ccc}
\tau_{\leq 2p+1} \text{BSL}_p(C) & \to & \tau_{\leq 2p+1} \text{BPGL}_p(C) \\
\downarrow & & \downarrow \\
K(Z_{(p)}, 2) & \to & K(Z_{(p)}, 2)
\end{array}
\]

The main theorem concerns itself with maps \( f : \tau_{\leq 2p+1} \text{BPGL}_p(C) \to \tau_{\leq 2p+1} \text{BPGL}_p(C) \) that induce isomorphisms on \( \pi_2(\tau_{2p+1} \text{BPGL}_p(C)) \), these maps fit into diagrams

\[
\begin{array}{ccc}
\tau_{\leq 2p+1} \text{BSL}_p(C) & \xrightarrow{f} & \tau_{\leq 2p+1} \text{BPGL}_p(C) \\
\downarrow & & \downarrow \\
\tau_{\leq 2p+1} \text{BSL}_p(C) & \to & \tau_{\leq 2p+1} \text{BPGL}_p(C)
\end{array}
\]

in which the right-hand square commutes up to homotopy. The map, \( f \), making the left-hand square commute up to homotopy exists, but is not unique. We refer to such a map as a lift of the map \( f \).

Since \( \pi_{2p+1} \left( \text{BSL}_p(C) \right) \cong \mathbb{Z}/(p!) \), it follows that \( \pi_{2p+1} \left( \text{BPGL}_p(C) \right) \cong \mathbb{Z}/(p!) \), and hence that

\[
\pi_{2p+1} \left( L_{(p)} \text{BPGL}_p(C) \right) \cong \mathbb{Z}/(p!) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \cong \mathbb{Z}/p.
\]

The following lemma is a technical ingredient in Theorem 2.9.

Lemma 2.8. Let \( f : \tau_{\leq 2p+1} \text{BPGL}_p(C) \to \tau_{\leq 2p+1} \text{BPGL}_p(C) \) be a map that induces an isomorphism

\[
\pi_2(f) : \pi_2(\tau_{\leq 2p+1} \text{BPGL}_p(C)) \to \pi_2(\tau_{\leq 2p+1} \text{BPGL}_p(C)) = \mathbb{Z}/p.
\]

Any lift \( \tilde{f} : \tau_{\leq 2p+1} \text{BSL}_p(C) \to \tau_{\leq 2p+1} \text{BSL}_p(C) \) of \( f \) has the property that the \( p \)-localization

\[
\pi_4(L_{(p)} \tilde{f}_*) : \pi_4(\text{BSL}_p(C)) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \to \pi_4(\text{BSL}_p(C)) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}
\]

is an isomorphism.
Proof. The first nontrivial fiber sequence appearing in the Postnikov tower of $\text{BPGL}_p(C)$ is

$$K(Z, 4) \longrightarrow \tau_{\leq 4} \text{BPGL}_p(C) \longrightarrow K(Z/p, 2).$$

In [2] we proved that the $K(Z, 4)$–bundle above is classified by a map $K(Z/p, 2) \rightarrow K(Z, 5)$ that represents a generator of the group $H^5(K(Z/p, 2), Z) \cong Z/p^\epsilon$, where $\epsilon = 1$ unless $p = 2$ in which case $\epsilon = 2$. If $f$ induces an isomorphism on $\pi_2$, it must also induce a map $f_*$ on $\pi_4 \left( \tau_{\leq 2p+1} \text{BPGL}_p(C) \right) \cong Z$ such that the funtorially-derived diagram

$$K(Z/p, 2) \longrightarrow K(Z, 5)$$

commutes. Here the horizontal arrows induce isomorphisms on all homotopy groups, $\pi_i$, $i \geq 3$, and the result follows. 

We are now in a position to prove the main topological theorem of the paper.

**Theorem 2.9.** Let $f : \tau_{\leq 2p+1} \text{BPGL}_p(C) \rightarrow \tau_{\leq 2p+1} \text{BPGL}_p(C)$ be a map that induces an isomorphism

$$\pi_2(f) : \pi_2 \left( \tau_{\leq 2p+1} \text{BPGL}_p(C) \right) \rightarrow \pi_2 \left( \tau_{\leq 2p+1} \text{BPGL}_p(C) \right) \cong Z/p.$$

Then

$$\pi_{2p+1} \left( L(p)f \right) : \pi_{2p+1} \left( L(p) \tau_{\leq 2p+1} \text{BPGL}_p(C) \right) \rightarrow \pi_{2p+1} \left( L(p) \tau_{\leq 2p+1} \text{BPGL}_p(C) \right)$$

is an isomorphism.

**Proof.** Suppose $f$ is a map meeting the hypothesis of the theorem. Choose a lift, $\tilde{f} : \tau_{\leq 2p+1} \text{BSL}_p(C) \rightarrow \tau_{\leq 2p+1} \text{BSL}_p(C)$. By Lemma 2.8, the map $\tilde{f}$ is an isomorphism on $\pi_4 \left( L(p) \tau_{\leq 2p+1} \text{BSL}_p(C) \right)$, and therefore by Corollary 2.7, $\pi_{2p+1} \left( L(p)\tilde{f} \right)$ is an isomorphism.

Since the projection $L(p) \tau_{\leq 2p+1} \text{BSL}_p(C) \rightarrow L(p) \tau_{\leq 2p+1} \text{BPGL}_p(C)$ induces an isomorphism on all higher homotopy groups $\pi_i$ where $i \geq 3$, it follows that $f_*$ is an isomorphism on $\pi_{2p+1} \left( L(p) \tau_{\leq 2p+1} \text{BPGL}_p(C) \right)$, as claimed. 

3. Purity

We consider purity in this section, giving two applications of algebraic topology to algebraic purity questions. The first uses the machinery of Section 2 to show that purity fails in general for $\text{PGL}_p$ torsors, while the second uses [2, Theorem D] to show that purity fails for the cohomological filtration on the Witt group.
3.1. Definitions. Let \( \mathcal{F} : \mathcal{C}^{\text{op}} \rightarrow \text{Sets} \) be a presheaf on some category of schemes \( \mathcal{C} \). We will suppress any mention of the category \( \mathcal{C} \) throughout, and we will assume that all necessary localizations of an object \( X \) in \( \mathcal{C} \) are also in \( \mathcal{C} \). Suppose that \( X \) is a regular noetherian integral scheme in \( \mathcal{C} \), and let \( K \) be the function field of \( X \). If the natural map

\[
\text{im}(\mathcal{F}(X) \rightarrow \mathcal{F}(\text{Spec } K)) \rightarrow \bigcap_{p \in X^{(1)}} \text{im}(\mathcal{F}(\text{Spec } \mathcal{O}_{X,p}) \rightarrow \mathcal{F}(\text{Spec } K))
\]

is a bijection, where \( X^{(1)} \) denotes the set of codimension 1 points of \( X \), then we say that \textit{purity} holds for \( \mathcal{F}(X) \).

Example 3.1. If \( X \) is a regular noetherian integral scheme with an ample line bundle such that \( \mathcal{O} \subseteq \Gamma(X, \mathcal{O}_X) \), then purity holds for \( \text{Br}(X) \). In particular, purity holds for \( \text{Br}(X) \) for smooth quasi-projective schemes over fields of characteristic 0. This follows from two facts. First, it is a theorem of Gabber and de Jong [16] that if \( X \) has an ample line bundle, then \( \text{Br}(X) = H^2_{\text{ét}}(X, G_m)_{\text{tors}} \). Second, Gabber has shown (see Fujiwara [20]) that \( H^2_{\text{ét}}(X, G_m)_{\text{tors}} \) satisfies purity when \( X \) is a regular scheme and when each positive integer is invertible in \( X \). The case of smooth affine schemes over fields had been handled previously by Hoobler [26], following Auslander and Goldman’s work on the 2-dimensional affine situation [7, Proposition 6.1], while Gabber [22] had proved the result in characteristic 0 with an added excellence condition. Gabber [21] proved purity for \( H^2_{\text{ét}}(X, G_m)_{\text{tors}} \) without the excellence hypothesis when \( \dim X \leq 3 \); hence, in combination with the \( \text{Br} = \text{Br}' \) result above, purity holds for the Brauer group when \( \dim X \leq 3 \) and \( X \) has an ample line bundle. If \( X \) is an arbitrary regular noetherian integral scheme, then purity holds for \( \text{Br}(X)' \), the part of the Brauer group containing the \( m \)-torsion for all \( m > 0 \) invertible in \( X \). This follows from purity for \( H^2_{\text{ét}}(X, \mu_n) \) when \( n \) is prime to \( p \). See Fujiwara [20] together with [4, Exposée XIV, Section 3] or [14, Theorem 3.8.2].

Currently unknown is whether purity holds for \( \text{Br}(X) \) for every regular noetherian integral scheme \( X \). The results above should be contrasted to what happens for degree 3 cohomology classes: for any integer \( n > 1 \), there are smooth projective complex varieties \( X \) such that purity fails for \( H^3_{\text{ét}}(X, \mathbb{Z}/n) \). See [15, Section 5] for an overview, or Totaro [35] and Schoen [33] for examples. It is not hard to see that unramified cohomology is homotopy invariant [36, Theorem 1.3], so it follows by using Jouanolou’s device [28] that there are smooth affine complex varieties where purity fails for \( H^2_{\text{ét}}(X, \mathbb{Z}/n) \) as well.

3.2. Purity for torsors. Let \( X \) be a regular noetherian integral scheme, and let \( G \) be a smooth reductive group scheme over \( X \). In [13, Question 6.4], Colliot-Thélène and Sansuc ask whether purity holds for \( H^1_{\text{ét}}(X, G) \). As stated in the introduction, many examples are known where purity fails in the special case where \( X = \text{Spec } R \) is the spectrum of a regular local ring \( R \). In the non-local case, the list of known results is shorter. As far as the authors are aware, prior to the present paper, it contained only the negative result of [3] for \( G = \text{PGL}_2 \), the case of groups for which \( H^1(\text{Spec } K, G) \) is trivial, and the following two theorems:

Theorem 3.2 ([13, Corollaire 6.9]). Purity holds for \( H^1_{\text{ét}}(X, G) \) for all regular noetherian integral schemes \( X \) and all finite type \( X \)-group schemes of multiplicative type \( G \).

Theorem 3.3 ([13, Theorem 6.13]). Purity holds for \( H^1_{\text{ét}}(X, G) \) for all regular noetherian integral 2-dimensional schemes \( X \) and all smooth reductive \( X \)-group schemes \( G \).

To prove our main theorem, we need a standard lemma.

Lemma 3.4. Let \( R \) be a discrete valuation ring, and let \( \alpha \in \text{Br}(R) \subseteq \text{Br}(K) \) be a Brauer class. If \( D \) is a central simple algebra over \( K \), the fraction field of \( R \), with Brauer class \( \alpha \), then every maximal order \( A \) in \( D \) is Azumaya over \( R \).
Proof. A maximal order $A$ is in particular reflexive. Since a reflexive module on a regular noetherian domain of dimension at most 2 is projective, $A$ is projective. The lemma now follows from the argument in the second paragraph of the proof of [7, Proposition 7.4]. \qed

The goal of this paper is to show that Theorem 3.3 does not extend to higher-dimensional schemes. The method is based on [3], augmented by the results of Section 2.

**Theorem 3.5.** Let $p$ be a prime. Then, there exists a smooth affine complex variety $X$ of dimension $2p + 2$ such that purity fails for $H^2_{\text{et}}(X, \text{PGL}_p)$.

**Proof.** Let $q > 1$ be an integer prime to $p$. Let $V$ be an algebraic representation of the complex algebraic group $\text{P}(p, pq) = \text{SL}_{pq} / \mu_p$ such that the non-free locus $S \subseteq V$ has codimension at least $p + 2$. Then, since $V - S$ is $2p + 2$-connected, it follows that the natural map $(V - S) / \text{P}(p, pq) \to \text{BP}(p, pq)$ is a $2p + 3$-equivalence. We can replace $(V - S) / \text{P}(p, pq)$ by an affine scheme using Jouanolou’s device [28], and then we can use the affine Lefschetz theorem [24, Introduction, Section 2.2] to cut down to a $2p + 2$-dimensional scheme $X$. There is a natural map $X \to \text{BP}(p, pq)$ determining an algebraic Azumaya algebra on $X$ of degree $pq$ and order $p$. Let $\alpha \in \text{Br}(X)$ be the associated Brauer class. As $X$ is a smooth variety, prime divisors of the order of $\alpha$ and the index of $\alpha$ must be the same [2, Proposition 6.1], so it follows that $\text{ind}(\alpha) = p$. In particular, over the function field $K$ of $X$, there is a $\text{PGL}_p$-torsor with class $\alpha$. This is also true over every codimension 1 local ring $\mathcal{O}_{X,p}$ of $X$ by Lemma 3.4. Because of the $2p + 2$-equivalence $X \to \text{BP}(p, pq)$, it follows that $H^2(X, \mathbb{Z}/p) \cong H^2(X, \mathbb{Z}/p)$ is isomorphic to $\mathbb{Z}/p$. Let $\gamma$ be the obstruction class of $X \to \text{BP}(p, pq)$ in $H^2(X, \mathbb{Z}/p)$. If $P : X \to \text{BPGL}_p(C)$ is a torsor representing $\alpha$, then the associated obstruction class in $H^2(X, \mathbb{Z}/p)$ must also be equal to $\gamma$.

We consider the composition

$$
\tau_{\leq 2p+2}\text{BPGL}_p(C) \to \tau_{\leq 2p+2}\text{BP}(p, pq) \to \tau_{\leq 2p+2}X \to \tau_{\leq 2p+2}\text{BPGL}_p(C),
$$

where the first arrow is the $2p + 2$-truncation of the $q$-fold block sum map $\text{BPGL}_p(C) \to \text{BP}(p, pq)$, the second arrow is a homotopy inverse to the homotopy equivalence $\tau_{\leq 2p+2}X \to \tau_{\leq 2p+2}\text{BP}(p, pq)$, and the third arrow is the truncation of $P$. By our choices in the previous paragraph, this composition induces an isomorphism on $\pi_{2q}$, and hence on $\pi_{2p+1}$, by Theorem 2.9 (applied to the further truncation $\tau_{\leq 2p+1}$ of the composition). But $\pi_{2p+1}\text{BP}(p, pq) = 0$, which is a contradiction. \qed

The theorem implies, in particular, that on $X$ there is an unramified degree-$p$ division algebra over $K$ that does not extend to an Azumaya algebra on $X$. The case $p = 2$ was proved first in [3].

**Corollary 3.6.** Let $p$ be a prime, then there exists a smooth affine complex variety $X$ of dimension $2p + 2$ and an unramified division algebra $D$ over $C(X)$ of degree $p$ that contains no Azumaya maximal order on $X$.

**Scholium 3.7.** Let $p$ be a prime and let $n_1, \ldots, n_k$ be integers greater than $p$ such that $\gcd\{n_i, n_j\} = p$. There is a smooth affine complex variety $X$ of dimension $2p + 2$ and a Brauer class $\alpha \in \text{Br}(X)$ of exponent $p$ such that there are Azumaya algebras of degrees $n_1, \ldots, n_k$ in the class $\alpha$, but no Azumaya algebra of degree $p$.

**Proof.** The proof is largely the same as that of the theorem, but using the algebraic group

$$
\text{SL}_{n_1} \times \cdots \times \text{SL}_{n_k} / \mu_p,
$$

where $\mu_p$ is embedded diagonally in each of the groups $\text{SL}_{n_i}$. \qed
3.3. Local purity. In contrast to the global failure of purity for $\text{PGL}_p$-torsors exhibited above, in this section, we give a proof that purity holds for $H^1_{\text{et}}(X, \text{PGL}_n)$ when $X$ is the spectrum of a regular local ring $R$ and the Brauer class has order invertible in $X$. Our result is a slight generalization of a recent theorem of Ojanguren [29] and of the local purity result for $\text{PGL}_n$ in characteristic 0 due to Panin [31].

To prove the theorem, we recall first a result of DeMeyer, which is also used by both Ojanguren and Panin.

**Theorem 3.8** (DeMeyer [17, Corollary 1]). Suppose that $R$ is an integral semi-local ring and that $\alpha \in \text{Br}(R)$. Then, there exists a unique Azumaya algebra $A$ with class $\alpha$ having no idempotents besides 0 and 1. Moreover, any other Azumaya algebra with class $\alpha$ is of the form $M_r(A)$ for some $r$.

We now prove our local purity result. Define $H^1(X, \text{PGL}_n)^n$ to be the set of $\text{PGL}_n$-torsors for which the order of the associated Brauer class in $\text{Br}(X)$ is invertible in $X$.

**Theorem 3.9.** If $R$ is a regular noetherian integral semi-local ring, then purity holds for $H^1(\text{Spec } R, \text{PGL}_n)^n$.

**Proof.** Let $K$ be the function field of $R$, and let $D$ be a degree $n$ central simple algebra in

$$\bigcap_{\text{ht } P = 1} \text{im} \left( H^1_{\text{et}}(\text{Spec } R_P, \text{PGL}_n)^n \to H^1_{\text{et}}(\text{Spec } K, \text{PGL}_n)^n \right).$$

Let $m$ be the order of $[D] \in \text{Br}(K)$. Because $D$ lifts to every codimension 1 local ring, so does the Brauer class. Since $m$ is invertible in $R$ and hence in these local rings, this Brauer class lifts to a Brauer class $\alpha \in \text{Br}(R)$ by purity for $\text{Br}(R)^n$ (see Example 3.1).

By DeMeyer’s theorem, there exists an Azumaya algebra $A$ with Brauer class $\alpha$ such that every other Azumaya algebra in the class $\alpha$ is isomorphic to $M_r(A)$ for some $r$. In particular, $\text{ind}(\alpha) = \text{deg}(A)$, where, if $X$ is a scheme and $\alpha \in \text{Br}(X)$, we define $\text{ind}(\alpha)$ to be the gcd of the degrees of all Azumaya algebras with class $\alpha$. On the other hand, by [2, Proposition 6.1], the index of $\alpha$ can be computed either over $R$ or over $K$. Thus, $\text{ind}(\alpha)$ divides $\text{deg}(D)$. Therefore $D \cong M_r(A_K)$ for some integer $r > 0$. It follows that $M_r(A)$ is a class in $H^1_{\text{et}}(\text{Spec } R, \text{PGL}_n)^n$ that restricts to $D$, which shows that purity holds for $H^1_{\text{et}}(\text{Spec } R, \text{PGL}_n)^n$. \hfill $\square$

3.4. Canonical factorization. In this section a theorem we prove a theorem that we view as evidence for Conjecture 1.2 for all $\text{PGL}_n$.

Let $m > 1$ divide $n$. Both $\text{BP}(m, n)$ and $\text{BPGL}_m(C)$ are equipped with canonical maps to $K(Z/m, 2)$. Moreover, a topological $\text{PGL}_n(C)$ bundle, $P \to X$, may be lifted to a $P(m, n)$ bundle if and only if the associated obstruction class $\delta_0(P)$ in $H^2(X, Z/n)$ is $m$-torsion. A canonical factorization of Azumaya algebras with structure group $P(m, n)$ is a factorization $\text{BP}(m, n) \to \text{BPGL}_m \to K(Z/m, 2)$. The existence of such a factorization would give, for every Azumaya algebra $A$ of degree $n$ and $m$-torsion obstruction class, a canonical Azumaya algebra $B$ of degree $m$ with the same obstruction class in $H^2(X, Z/m)$. Unsurprisingly, no such canonical Azumaya algebra can exist.

**Theorem 3.10.** If $n > m$, then there is no canonical factorization $\text{BP}(m, n) \to \text{BPGL}_m(C) \to K(Z/m, 2)$.

**Proof.** Suppose that $\text{BP}(m, n) \to K(Z/m, 2)$ factors through $\text{BPGL}_m(C) \to K(Z/m, 2)$. Let $\text{BPGL}_m(C) \to \text{BP}(m, n)$ be the map induced block-summation. Write $f : \text{BP}(m, n) \to \text{BP}(m, n)$ for the composition. This map induces an isomorphism $H^2(\text{BP}(m, n), Z/m) \cong Z/m$, and is in particular not nullhomotopic.

As $\text{BP}(m, n)$ is homotopy equivalent to $\text{BSU}_m/\mu_m$, there is a complete description of the homotopy-classes of self-maps $\text{BP}(m, n) \to \text{BP}(m, n)$ due to Jackowski, McClure, and Oliver [27, Theorem 2]. That theorem says we can factor $f$ as $\text{BS} \circ \psi^k$, where $\alpha$ is an outer automorphism of $P(m, n)$, and $\psi^k$ is an unstable Adams operation on $\text{BP}(m, n)$, for some $k \geq 0$ prime to the order of the Weyl group of $P(m, n)$, which is $n!$. The map $\psi^k$ induces
multiplication by \(k^i\) on \(H^{2i}(\text{BP}(m, n), \mathbb{Q})\). In particular a map \(\text{BP}(m, n) \to \text{BP}(m, n)\) is either nullhomotopic or induces an isomorphism on rational cohomology.

The rational cohomology of \(\text{BP}(m, n)\) is
\[
H^*(\text{BP}(m, n), \mathbb{Q}) \cong \mathbb{Q}[c_2, \ldots, c_n], \quad c_i \in H^{2i}(\text{BP}(m, n), \mathbb{Q}),
\]
while that of \(\text{BPGL}_m(\mathbb{C})\) is
\[
H^*(\text{BPGL}_m(\mathbb{C}), \mathbb{Q}) \cong \mathbb{Q}[c_2, \ldots, c_n], \quad c_i \in H^{2i}(\text{BP}(m, n), \mathbb{Q}).
\]
In particular,
\[
\dim H^{2m+2}(\text{BP}(m, n), \mathbb{Q}) = \dim H^{2m+2}(\text{BPGL}_m(\mathbb{C}), \mathbb{Q}) + 1,
\]
so that \(f\) cannot induce an isomorphism on rational cohomology, and must be nullhomotopic, a contradiction. \(\square\)

The argument above motivated the authors’ work both in this paper and in [3]. In order to construct algebraic counterexamples, however, it is necessary to employ complex algebraic varieties \(X\) that approximate \(\text{BPGL}_m(\mathbb{C})\) in the sense that there exists a map \(X(\mathbb{C}) \to \text{BPGL}_m(\mathbb{C})\) induced by an algebraic \(\text{PGL}_m\)-torsor on \(X\) and inducing an isomorphism on homotopy groups in a range dimensions, and for these we cannot bring the strength of [27] to bear. We have made do with \textit{ad hoc} arguments that furnish obstructions in known, bounded dimension to maps \(\text{BPGL}_m(\mathbb{C}) \to \text{BPGL}_m(\mathbb{C})\). For instance, the topological plank in the argument proving that purity fails for \(H^1_{\text{et}}(X, \text{PGL}_p)\) is an obstruction to a map
\[
\tau_{\leq 2p+1} \text{BP}(p, pq) \to \tau_{\leq 2p+1} \text{BPGL}_p(\mathbb{C})
\]
that induces an isomorphism on Brauer group. This obstruction depends on Theorem 2.9, which describes a restriction on maps
\[
\tau_{\leq 2p+1} \text{BPGL}_p \to \tau_{\leq 2p+1} \text{BPGL}_p,
\]
in that it says a map inducing an isomorphism on \(\tau_2\) must necessarily also induce an isomorphism on the \(p\)-primary part of \(\tau_{2p+1}\). To prove Conjecture 1.2 for all \(\text{PGL}_m\), one might only have to find an obstruction to the existence of maps \(X \to \text{BPGL}_m\) where \(X\) approximates \(\text{BP}(m, mq)\), with \(q > 1\) prime to \(m\).

3.5. The Witt group. Our second application of topology to purity is to give a new example where purity fails for the cohomological filtration on the Witt group.

Example 3.11. Local purity is known for the Witt group \(W(\text{Spec } R)\) whenever \(R\) is a regular local ring containing a field of characteristic not 2 by work of Ojanguren and Panin [30].

Given the positive results for the Brauer group, it is natural to ask the following question.

Question 3.12. Does purity hold for \(W(X)\) when \(X\) is an regular excellent noetherian integral scheme having no points of characteristic 2?

It is known that purity holds for \(W(X)\) when \(X\) is a regular noetherian separated integral scheme of Krull dimension at most 4 and 2 is invertible in \(\Gamma(X, \mathcal{O}_X)\) by Balmer-Walter [8, Corollary 10.3]. However, Totaro [36] showed that the injectivity property fails for the Witt group: there is a smooth affine complex 5-fold such that \(W(X) \to W(K)\) is not injective. Thus, it might be natural to guess that the purity property fails as well. For an extensive overview of results on purity for the Witt group, see Auel [6].

Let \(\Gamma(X)\) be the ideal of \(W(X)\) generated by even-dimensional quadratic spaces. There is a discriminant map \(\Gamma^3(X) \to H^1_{\text{et}}(X, \mu_2)\). Let \(\Gamma^2(X)\) be the kernel. There is a map \(\Gamma^2(X) \to 2 \text{Br}(X)\), called the Clifford invariant map. Denote by \(\Gamma(X)\) the kernel. It is known that purity fails for \(\Gamma^2(X) / \Gamma^3(X)\). The first examples were due to Parimala and Sridharan [32], who showed that it fails for some affine bundle torsors over smooth projective \(p\)-adic curves. We include another example below, which uses a smooth affine variety we constructed in [2], giving the first examples over \(\mathbb{C}\).
Example 3.13. Let $X$ be the smooth affine 5-dimensional variety over $\mathbb{C}$ constructed in [2, Theorem D], having a Brauer class $\alpha \in \text{Br}(X)$ of exponent 2 that is not in the image of the Clifford invariant map $I^2(X) \to 2 \text{Br}(X)$. Consider the commutative diagram

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
I^2(X) & \to & I^3(K) \\
\downarrow & & \downarrow \\
I^2(X) & \to & \bigoplus_{p \in X(1)} I^2(k(p)) \\
\downarrow & & \downarrow \\
0 & \to & 2 \text{Br}(X) \\
\downarrow & & \downarrow \\
0 & \to & 2 \text{Br}(K) \\
\end{array}
\]

where the columns and the bottom row are exact, and where $I^2(X)$ (resp. $I^3(X)$) maps into the kernel of the map $I^2(K) \to \bigoplus I^2(k(p))$ (resp $I^3(K) \to \bigoplus I^2(k(p))$). The image of $\alpha$ in $2 \text{Br}(K)$ is in the image of the map $I^2(K) \to 2 \text{Br}(K)$ by Merkurjev’s theorem; say it is the Clifford invariant of $\sigma \in I^2(K)$. Then, $\sigma$ is unique up to an element of $I^2(K)$. On the other hand, the ramification classes $\overline{\partial_p(\sigma)}$ are all in $I^2(k(p))$. Hence, $\overline{\sigma} \in I^2(K)/I^3(K)$ is unramified. But, by construction, it is not in the image of $I^2(X)/I^3(K)$.

This is the first such example known for a variety over an algebraically closed field. It has the added advantage that it is not explained by the presence of line-bundle valued quadratic forms, as explained in [2, Section 7].

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