A Lattice Calculation of the Isgur-Wise Function

C. Bernard\textsuperscript{(1)}, Y. Shen\textsuperscript{(2)} and A. Soni\textsuperscript{(3)}

\textsuperscript{(1)}Physics Department, Washington University, St. Louis, MO 63130, USA
\textsuperscript{(2)}Physics Department, Boston University, Boston, MA 02215, USA
\textsuperscript{(3)}Physics Department, Brookhaven National Laboratory, Upton, NY 11973, USA

Abstract

We calculate the Isgur-Wise function ($\xi_0$) by measuring the heavy-heavy meson transition matrix element on the lattice in the quenched approximation. The standard Wilson action is used for both the heavy and the light quarks. Our numerical results are compared with various model calculations and experimental data. In particular, using a linear fit, we find $\rho^2 = 1.24 \pm .26 \pm .26$. (The slope of $\xi_0$ at the zero-recoil point is $-\rho^2$.) Using instead the parametrization of $\xi_0$ suggested by Neubert and Rieckert and fitting the lattice data up to $v \cdot v' \sim 1.2$, we find $\rho^2_{NR} = 1.41 \pm .19 \pm .19$
1 Introduction

The physics of heavy quark systems has been one of the active research topics in the last few years (for reviews, see [1]). Consider a meson system consisting of a heavy quark \( Q \) and a light quark \( \bar{q} \). When \( m_Q \to \infty \) the heavy quark essentially becomes a static color source. It can be described by a simple heavy quark effective field theory (HQEFT) with flavor and spin symmetries. Because of these new flavor and spin symmetries in HQEFT, the meson decay matrix element can be simplified and different decay processes can be related to each other. For example, in the case of \( B \to D \) decay there are two independent form factors in the full theory. However, if we assume both \( B \) and \( D \) mesons are heavy enough, using HQEFT the number of unknown form factor can be reduced to one. We have [1]

\[
<D(v')|\bar{c}\gamma_\nu b|B(v)> = \sqrt{m_B m_D} C_{cb}(\mu) \xi_0 (v' \cdot v; \mu) (v + v')_\nu ,
\]

where \( v, v' \) are the four-velocity and \( m_B, m_D \) are the \( B \) and \( D \) meson mass, respectively. The constant \( C_{cb}(\mu) \) comes from integrating the full QCD contribution from the heavy quark mass scale down to a renormalization scale \( \mu \ll m_D \)

\[
C_{cb}(\mu) = \left[ \frac{\alpha_s(m_D)}{\alpha_s(m_B)} \right]^{6/(33 - 2N)} \left[ \frac{\alpha_s(m_B)}{\alpha_s(\mu)} \right]^{a(v' \cdot v')},
\]

where \( a(v \cdot v') \) is a slowly varying function of \( v' \cdot v \) which vanishes at \( v = v' \) [1]. The Isgur-Wise function \( \xi_0 (v' \cdot v; \mu) \) represents the interactions between the light degrees of freedom in the heavy meson system. It thus has to be calculated using nonperturbative methods.

On the lattice the heavy meson system can be studied in two different approaches. One is to keep the heavy quark dynamical by using the standard Wilson action. This may require extrapolation to the physical heavy meson mass of interest. The alternative is to integrate out the heavy quark first and derive an effective action including only the light degrees of freedom and then perform numerical simulation using this effective action [2]. Here we stay with the first approach (see [3] for reviews of the second approach). We reported preliminary results from our approach at the Lattice ’92 conference [4].

It is important to note that using the flavor symmetry of HQEFT the Isgur-Wise function relevant to the \( B \to D \) decay of Eq. (1) can be obtained also from the \( D \to D \) elastic scattering matrix element [1]

\[
<D(v')|\bar{c}\gamma_\nu c|D(v)> = m_D C_{cc}(\mu) \xi_0 (v' \cdot v; \mu) (v + v')_\nu ,
\]

where

\[
C_{cc}(\mu) = \left[ \frac{\alpha_s(m_c)}{\alpha_s(\mu)} \right]^{a(v' \cdot v')}.
\]

This of course requires that the \( D \) meson be sufficiently heavy for the onset of the heavy quark limit (HQL).

Conventionally \( B \to D \) (here we use \( B \) and \( D \) as generic names for heavy pseudoscalar mesons, they do not necessarily represent the physical \( B \) and \( D \) mesons) transition matrix
elements can be parametrized as

\[ <D(p')|V_\nu|B(p)> = f_+(q^2)(p'+p)_\nu + f_-(q^2)(p-p')_\nu , \]

where \( q^2 = (p'-p)^2 \) is the momentum transfer between the initial and final states and \( V_\nu \) is a vector current. One can also use an equivalent form

\[ <D(p')|V_\nu|B(p)> = f_+(q^2) \left[ (p'+p)_\nu - \frac{m_B^2 - m_D^2}{q^2}(p-p')_\nu \right] + f_0(q^2) \frac{m_B^2 - m_D^2}{q^2}(p-p')_\nu , \]

with relation between the form factors \( f_+, f_- \) and \( f_0 \)

\[ (m_B^2 - m_D^2)f_0(q^2) = (m_B^2 - m_D^2)f_+(q^2) + q^2f_-(q^2) . \]

For the elastic scattering process \((m_B = m_D)\), we have \( f_-(q^2) = 0 \), which follows from current conservation. Using the relation \( v_\nu = p_\nu/m \), Eq. (5) becomes

\[ <D(p')|V_\nu|B(p)> = m_Df_+(q^2)(v' + v)_\nu . \]

Comparing this with Eq. (5) one finds the simple relation between \( f_+ \) and \( \xi_0 \)

\[ f_+ = C_{cc}\xi_0 \text{ .} \]

The lattice calculation method for \( f_+ \) has been well established \[6\]. We will use that method through equation (9) and deduce \( \xi_0 \).

## 2 Considerations in the Lattice Calculation

Choosing the lattice parameters. HQEFT becomes valid when the heavy quark mass \( m_Q \gg \Lambda_{QCD} \). Therefore we need to choose the lattice parameters to satisfy this constraint. For example, at \( \beta = 6.0 \) the inverse lattice spacing \( a^{-1} \approx 2.0 \text{ GeV} \). If we take the physical value \( \Lambda_{QCD} \approx 0.2 \text{ GeV} \), then in lattice units \( \Lambda_{QCD} = 0.1 \). We need to choose the heavy quark mass \( am_Q \) to be much larger than 0.1. However, one would expect large lattice spacing artifacts for \( am_Q \gtrsim 1 \). At \( \beta = 6.0 \) this means that the physical mass of the heavy quark is limited to be in the range of the \( D \) meson or smaller. At larger \( \beta \) values, the lattice spacing is smaller, and we can therefore accommodate mesons with larger physical masses on the lattice. In this work we choose the heavy quark mass in the range \( 1.5 - 3 \text{ GeV} \) by tuning the hopping parameter \( \kappa_Q \) for the heavy quark \( Q \). On the other hand, the light quark \( q \) in the heavy meson should have a mass \( m_q \leq \Lambda_{QCD} \). This can be achieved by changing the hopping parameter \( \kappa_q \) for the light quark and then extrapolating to the chiral limit \( \kappa_q \rightarrow 0 \), where the extrapolated pion mass is approximately matched to the physical pion mass \( m_\pi = 140 \text{ MeV} \).

How far can \( v \cdot v' \) change on the lattice? In the lattice calculations for the matrix element, we always have either the initial or the final particle at rest. Thus

\[ v \cdot v' = \begin{cases} \frac{E_D}{m_D} & \text{if } v' = (0, 0, 0, 1) \\ \frac{E'_D}{m_D} & \text{if } v = (0, 0, 0, 1) \end{cases} \]
with \( E_D = \sqrt{m_D^2 + \vec{p}^2} \). For a spatial lattice \( L = 24 \), we have injected momenta

\[
\vec{p} = \frac{2\pi}{L} (1, 0, 0), \frac{2\pi}{L} (1, 1, 0). \tag{11}
\]

Since \( m_D \approx 1.0 \) in lattice units, one gets \( v \cdot v' \approx 1.034 \) and 1.066 respectively. To get larger values for \( v \cdot v' \) one needs to inject larger lattice momenta which would in turn introduce large statistical noise in the matrix element calculations. Thus \( v \cdot v' \) can not be much larger than \( \sim 1.1 \) in current simulations.

**Removing the lattice artifacts; normalization.** Since the heavy meson mass is near 1 in lattice units the lattice artifacts could be significant. Ultimately the lattice artifacts can be brought under control either by comparing data at different \( \beta \) values or by using improved lattice actions. However, for simulations at a given \( \beta \) value there are several ways to check the size of the lattice artifacts. Also, for the purpose of this work, we are able to get rid of most of the undesirable effects by imposing the conservation of heavy quark flavor current, as we will explain below.

For the elastic scattering process the matrix element has the simple form given in Eq. (8). This is derived from Eq. (5) which is based on the assumption that the theory is Lorentz invariant. The lattice theory, however, is not exactly Euclidean rotational invariant due to the finite lattice spacing \( a \). Thus \( f_- \) is not exactly zero. The amplitude of \( f_- \) (or \( f_-/f_+ \)) gives a measure for the violation of the Euclidean invariance on the lattice.

We can also estimate the size of the lattice artifacts by checking the simulation results against known continuum matrix element values at some special points. For example, when both the initial and the final D mesons are at rest, \( v = v' = (0, 0, 0, 1) \), the continuum matrix elements of \( \bar{c}\gamma_4c \) is known because of the quark flavor current conservation

\[
<D|\bar{c}\gamma_4c|D> = 2m_D. \tag{12}
\]

At this so-called “zero-recoil point” we have

\[
\xi_0(1) = 1. \tag{13}
\]

Both Eqs. (3) and (12) will have \( O(a) \) corrections on the lattice which can come from different origins.

Part of the \( O(a) \) effect can be disposed of, to a good approximation, by including a normalization factor

\[
<\psi(x)\bar{\psi}(0)>_{\text{cont}} = 2\kappa u_0 e^m <\psi(x)\bar{\psi}(0)>_{\text{latt}}, \tag{14}
\]

where

\[
e^m = 1 + \frac{1}{u_0} \left( \frac{1}{2\kappa} - \frac{1}{2\kappa_{cr}} \right), \tag{15}
\]

with \( u_0 \) the “tadpole improvement” factor.
Another correction comes from the use of (nonconserved) local vector current \( V_\nu = \bar{c} \gamma_\nu c \) on the lattice \[8\]. This effect can be corrected by introducing a rescaling factor \( Z_{V}^{\text{loc}} \) in the vector current. In perturbation theory \( Z_{V}^{\text{loc}} \) is calculated to be \[9\]

\[
Z_{V}^{\text{loc}} = 1 - 27.5 \frac{g^2}{16\pi^2}.
\]

In general the size of lattice artifacts will be momentum dependent so the \( O(a) \) correction will be different for Eq. \[12\] and Eq. \[8\]. However, since we are using only small momentum injections, we assume that the leading \( O(a) \) correction is approximately independent of \( p, p' \) and the Lorentz index \( \nu \). Thus by enforcing Eq. \[12\] as the normalization condition for the matrix element \( <D(v')|V_\nu|D(v)> \), corrections due to both the \( \exp(m) \) factors and \( Z_{V}^{\text{loc}} \) factors are taken care of automatically (together with any other multiplicative \( O(a) \) factors).

**Corrections due to finite heavy quark mass.** At what mass scale the heavy quark limit sets in is an important question. The charm quark mass is not that much larger than \( \Lambda_{QCD} \). If we choose the \( \rho \)-meson mass \( m_\rho \) as the typical scale for QCD, it becomes even less clear if \( m_c \) is heavy enough. In lattice calculations of the matrix elements the size of the corrections to the symmetry limit (i.e., \( m_Q \to \infty \)) have been observed to be process dependent. For example, the heavy meson decay constant \( f_P \) is found to have a mass correction that is as large as 50% at \( D \)-meson mass scale \[10, 11\]; while the form factor \( A_1(q^2) \) at the zero-recoil point \( q^2 = q_{\text{max}}^2 \) for \( D \to K^* \) decay shows very little mass dependence even for fairly small values of the heavy quark mass \[12\]. Both observations are consistent with theoretical expectations. In HQEFT the leading correction due to the heavy quark mass is in general \( O(1/m_Q) \), as is the case for \( f_P \). However, in the case of \( D \to K^* \), the \( O(1/m_Q) \) correction vanishes exactly at the zero-recoil point and the finite mass correction there is at most \( O(1/m_Q^2) \)[13]. Thus, in lattice calculations, the observations of the linear \( O(1/m_Q) \) correction for \( f_P \) calculation and little mass dependence of \( A_1(q_{\text{max}}^2) \) for \( D \to K^* \) decay seem to indicate that the finite mass corrections are under control within HQEFT at a scale \( \sim m_c \) and that numerical simulation results in this range can be used and extrapolated reliably to obtain the heavy quark limit results.

In principle, for data obtained at fixed masses one may estimate the size of \( O(1/m_Q) \) corrections in the following way: the matrix element can be in general written as

\[
<A'(v')|V_\nu|B(v)> = \sqrt{m_A m_B C_{AB}} \left[ \tilde{f}_+(v \cdot v')(v' + v)_\nu + \tilde{f}_-(v \cdot v')(v' - v)_\nu \right].
\]

This parametrization is equivalent to the conventional parametrization Eq. \[3\]; we have relations

\[
\tilde{f}_+ = \frac{1}{2 C_{AB}} \left[ \sqrt{m_A/m_B} (f_+ + f_-) + \sqrt{m_B/m_A} (f_+ - f_-) \right],
\]

\[
\tilde{f}_- = \frac{1}{2 C_{AB}} \left[ \sqrt{m_A/m_B} (f_+ + f_-) - \sqrt{m_B/m_A} (f_+ - f_-) \right],
\]

In the \( m_Q \to \infty \) limit because of the constraints on the effective heavy quark fields

\[
\gamma_\nu v_\nu h_v = h_v, \quad \bar{h}_v \gamma_\nu v_\nu = \bar{h}_v,
\]
we have \[ f_- = 0. \] (21)

At finite \( m_Q \), \( f_- \) is nonvanishing. Thus the value of \( f_- \) or the ratio

\[
\frac{f_-}{f_+} = \frac{1 + f_-/f_+ - \frac{m_B}{m_A}(1 - f_-/f_+)}{1 + f_-/f_+ + \frac{m_B}{m_A}(1 - f_-/f_+)},
\]

(22)
gives a measure of the \( O(1/m_Q) \) correction to the leading order HQEFT. Unfortunately, for the elastic scattering process considered in this paper this relation is not useful because in this case we have \( m_A = m_B \) so that \( f_-/f_+ = f_-/f_+ = 0 \) in the continuum. However, in principle, equation (22) may be used to check the finite mass correction in computations of \( B \to D \) decay.

3 Numerical results

We use the standard Wilson action for the propagation of both heavy and light quarks and work in the quenched approximation. The gauge configurations used are listed in Table 1. The techniques for measuring the two-point function and the three-point matrix elements are standard [5, 6]. Here we only list the results for the measured \( m_D, f_+(0) \) and \( f_+(q^2) \) in Tables 2-7. To increase the statistics, we have used symmetry properties of the Green functions and averaged over \( \pm t \) and \( \pm \vec{p} \) whenever possible.

Estimating the size of the lattice artifacts. Data for \( f_+(0) \) and \( f_+(q^2) \) listed in Tables 2-7 are direct numerical results. They include all the lattice artifacts. For example, comparing to the known continuum value \( f_+(0) = 1 \), we observe that the lattice artifacts at \( \beta = 6.0 \) are typically 20%–40% at \( \kappa_Q = 0.118 \) and less than 10% at \( \kappa_Q = 0.135 \). From the computation of the ratio \( f_-/f_+ \) we find the violation of Euclidean invariance is typically 5%–15% for \( \kappa_Q = 0.118 \) and 3%–10% for \( \kappa_Q = 0.135 \). Note that the reduction of the lattice artifacts when \( \kappa_Q \) is changed from 0.118 to 0.135 agrees with our intuitive expectations that those lattice artifacts that are due to the heavy quark mass approaching the lattice ultra-violet cutoff should decrease with a decrease in mass of the heavy quark. One may try to use the factors \( Z_{V}^{\text{loc}} \) and \( e^m \) to remove some of the artifacts. At \( \beta = 6.0 \) we get from the perturbative calculation, Eq. (14), that \( Z_{V}^{\text{loc}} \approx 0.7 \) (using the “boosted” effective gauge coupling \( g^2 \approx 1.7 \) as suggested in ref [14]). The factor \( e^m \) is about 2 for \( \kappa_Q = .118 \) and 1.5 for \( \kappa_Q = .135 \). Including both these factors the corrected \( f_+(0) \) become around one within errors. This is in agreement with other observations [10] that \( Z_{V}^{\text{loc}} \) and \( e^m \) factors seem to account for the largest part of the lattice artifacts.

Normalization and the Isgur-Wise function. As explained in Section 2, we define \( \xi_0(v \cdot v') = f_+(q^2)/f_+(0) \) in an effort to correct for possible lattice artifacts. We list the results in the last two columns of Tables 2–7. This method for deducing \( \xi_0 \) has the significant

\footnote{We have also computations at \( \beta = 5.7 \) with results basically consistent with data listed here within large errors. However, the heavy quark mass used (corresponding to \( \kappa = 0.094 \)) is extremely high for that \( \beta \) with potentially very large lattice artifacts. They are therefore not used in this work.}
advantage that it is free of uncertainties in factors such as $Z_{V,e}^{loc}$ and $e^m$. Note that in Eq. (9) there is a factor $C_{cc}$ in the connection between $f_+$ and $\xi_0$. This factor comes from integrating out the QCD effects from the heavy quark scale down to a light scale $\mu$. For the current calculation, however, the heavy quark and hard gluon effects in QCD are included dynamically, and we are therefore determining the combination $C_{cc}\xi_0$ at arbitrary scale $\mu$. For simplicity we choose $\mu = m_D$ and set $C_{cc} = 1$ according to Eq. (4).

Finite volume effects and the dependence on the light quark mass. In an effort to keep finite volume effects under control, we first study our data with light quark mass ($m_q$) in the range of the s-quark mass ($m_s$). On the lattice we choose $\kappa = 0.154$ at $\beta = 6.0$ and $\kappa = 0.149$ at $\beta = 6.3$. The physical meson masses at these $\beta$ and $\kappa$ values are in the range of $0.5$–$1$ GeV, corresponding to bound states of $\bar{s}s$. The results for the Isgur-Wise function are plotted in Fig. 1. All data seem to fall on a smooth curve. In particular, data for the $16^3$ and $24^3$ lattices at $\beta = 6.0$ agree to a very good approximation (i.e., well within statistical errors) assuring us that in this limit finite size effects on these lattices are small. Since the physical volume of the $24^3$ lattice at $\beta = 6.3$ is the same as the physical volume of the $16^3$ lattice at $\beta = 6.0$, we expect the finite size effects on the $6.3$ lattice also to be small.

As discussed in Section 2, for the elastic scattering of physical $D$ mesons, one should extrapolate the light quark hopping parameter to the chiral limit $\kappa \to \kappa_{cr}$. By taking this limit on a finite lattice, we expect not only a shift of the Isgur-Wise function due to the change of the light quark mass, but also a finite volume effect. The former effect is physical but the latter one is due to the limitations of a finite lattice size and should be brought under control by carefully comparing results on different lattice sizes. In practice, however, it is hard with a limited set of data to separate the light quark mass dependence from the finite size effect. Since, as Table 8 shows, the physical linear size of all our lattices are roughly in the range of $(100$ MeV)$^{-1}$, as the light quark mass gets smaller than $m_s$ we expect finite size effects to become important, especially on the two smaller lattices. Indeed, comparison of Fig. 2 with Fig. 1 shows that all the data is shifted up in the chiral limit. The size of shift is the smallest on the $\beta = 6.0, 24^3$ lattice, which has the largest physical volume. Therefore, the $\beta = 6.0, 24^3$ lattice is the most suitable one for extracting the dependence of the Isgur-Wise function on the light quark mass. Comparison of the numbers in Tables 4 and 5 corresponding to $\kappa = 0.154$ with those from the extrapolated case (i.e., for $\kappa = 0.157$) indicates that the form factors and the ratio $f_+/f_+(0)$ change very little (i.e., $\leq 2\%$) at a fixed momentum $\vec{p}$. Most of the change in the Isgur-Wise function comes from the change of the D meson mass (with the light quark mass) which in turn shifts the value of $v \cdot v'$. We note that the light quark mass dependence of the Isgur-Wise function has also been studied using chiral perturbation theory \cite{15} and model calculations \cite{16}. It was found that the shift due to the light quark mass is small: at $y = 1.2$ the shift in $\xi_0$ ranges from $\approx -2\%$ to about $7\%$.

Comparison with model calculations and experimental data. As discussed in the previous two paragraphs, we quote the $m_q \sim m_s$ data as our final results. They are listed in Tables 2–7 (At $\beta = 6.0, \kappa_q = 0.154$ on the $16^3$ and $24^3$ lattices and at $\beta = 6.3, \kappa_q = 0.149$ on the $24^3$ lattice) and plotted in Fig. 1. For comparison we also plotted the theoretical upper bound (dashed line) on the Isgur-Wise function derived from algebraic sum rules by Bjorken \cite{17}. 

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The lower curve (dotted line) is the lower bound derived in Ref. [18]. All of our data appear to be, within errors, below the upper bound of Bjorken. The solid curve going through the data points is a fit to the form [20]

$$\xi_0(y) = \frac{2}{y+1} \exp\left[-\left(2\rho^2_{NR} - 1\right)\frac{y-1}{y+1}\right], \quad y = v \cdot v' .$$

(23)

Taking into account the correlations in the data, we find, after fitting, $\rho^2_{NR} = 1.41(19)$ with $\chi^2/dof \approx 13/12$. A similar fit to the data (which, as explained in the preceding paragraph, is in this case only from the $\beta = 6.0$, $24^3$ lattice) in the chiral limit gives $\rho^2_{NR} = 1.09(28)$ with $\chi^2/dof \approx 1.4/4$. However, this shift between the central values (1.41 vs. 1.09) cannot be taken as a reliable reflection of the light quark mass dependence of the Isgur-Wise function. Indeed, holding $\rho^2_{NR} = 1.41$ fixed also provides a good fit (with $\chi^2 \approx 2.6/4$) to the $\beta = 6.0$, $24^3$ data in the chiral limit. Thus it is reasonable to assume that the shift of the Isgur-Wise function from $m_q = m_s$ to the chiral limit is within the quoted statistical errors. So far as other systematic effects such as the residual $O(a)$ effects and $O(1/m_Q)$ effects are concerned, we expect them to be rather small, since we are able to get a good universal fit to all the data. We therefore estimate the total systematic error to be at most the same size as the statistical error, and quote $\rho^2_{NR} = 1.41 \pm 0.19 \pm 0.19$ as our result for $m_q \sim m_s$ as well as in the chiral limit.

Close to $v \cdot v' = 1$ the Isgur-Wise function can be parametrized as

$$\xi_0(v \cdot v') = 1 - \rho^2(v \cdot v' - 1) + O((v \cdot v' - 1)^2) .$$

(24)

If we fit our data points to this linear form for $v \cdot v' < 1.06$ we get $\rho^2 = 1.24(26)$ with $\chi^2/dof = 2.6/5$. Once again, that value of $\rho^2$ also provides an acceptable fit ($\chi^2 \approx 1.8/3$) for the three data points with $v \cdot v' < 1.06$ (of the $\beta = 6.0$, $24^3$ lattice) in the chiral limit. Therefore for the linear fit we quote the result as $\rho^2 = 1.24 \pm 0.26 \pm 0.26$. We list our linear fit result in Table 9 along with $\rho^2$ values estimated by other authors. We note that lattice and various model calculations seem to be in rough agreement within errors. However, the value of Blok and Shifman tends to be rather low compared to the lattice result.

In Fig. 3 we plot our lattice results taken from Fig. 1 against experimental data from ARGUS [26]. We have normalized the experimental graph with respect to $|V_{cb}|/\sqrt{\tau_B/(1.32 \text{ psec.})} = 0.047$. For comparison we also plotted the theoretical upper bound [17] (dashed line), our fit to Eq. (23) (solid line) and the curve obtained in Ref. [18] (dotted line). It is gratifying that the lattice data has much smaller errors than the experimental results in the region $v \cdot v' < 1.15$.

4 Conclusions

We have computed the Isgur-Wise function on the lattice from the elastic scattering matrix element. The calculation is done using the standard Wilson fermion formulation for both

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2Recent work brings into question the validity of this lower bound for the Isgur-Wise function [19]. However, one may take the result of Ref. [18] as a model calculation.
heavy and light quarks in the quenched approximation. We find that all of our numerical results are within the theoretical upper bound\cite{17}. The Isgur-Wise function ($\xi_0$) is deduced by taking ratios of form factors. In this approach the lattice artifacts appear to be under control. Although the heavy meson mass in our calculation is typically order one in lattice units, the residual $O(a)$ effects are apparently small, as indicated by direct comparison of the data at $\beta = 6.0$ and $\beta = 6.3$. Further checks can also be done with an improved action scheme \cite{27}. The lattice meson mass we used is in the range of physical D meson. Thus there are potential $O(1/m_Q)$ corrections. Within the limited range accessible in our simulations we have not observed strong $m_Q$ dependence.

Our best lattice results are obtained when the light quark mass is set at the scale of s-quark mass. Data from our largest lattice seems to indicate that the Isgur-Wise function changes by less than our statistical error between $m_q \sim m_s$ and the chiral limit. A model independent linear fit using data close to $v \cdot v' = 1$ gives the slope $\rho^2 = 1.24 \pm .26 \pm .26$. Comparison of the slope $\rho^2$ with continuum model calculations are given in Table 9. A direct comparison to the ARGUS experimental results \cite{26} is given in Fig. 3. Our results are also in good agreement with the preliminary results of UKQCD Collaboration \cite{27}.

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| $\beta$ | $L^3T$ | # conf. | fitting range |
|-------|--------|---------|---------------|
| 6.0   | $16^339$ | 19      | $10 < t < 15$ |
| 6.0   | $24^339$ | 8       | $10 < t < 15$ |
| 6.3   | $24^361$ | 20      | $15 < t < 25$ |

**Table 1:** List of gauge configurations used in this work.

| $\kappa_q$ | $m_D$   | $f_+(0)$ | $\bar{p}$ | $f_+$  | $v \cdot v'$ | $\xi_0$   |
|-----------|--------|----------|---------|--------|-------------|----------|
| 0.152     | 1.237(6) | 0.69(12)  | (1, 0, 0) | 0.65(12) | 1.0408(4)   | 0.953(22) |
|           |         |          | (1, 1, 0) | 0.62(14) | 1.0790(8)   | 0.911(57) |
| 0.154     | 1.202(8) | 0.74(15)  | (1, 0, 0) | 0.72(16) | 1.0434(5)   | 0.974(30) |
|           |         |          | (1, 1, 0) | 0.70(18) | 1.0842(10)  | 0.945(79) |
| 0.155     | 1.186(8) | 0.77(19)  | (1, 0, 0) | 0.76(21) | 1.0445(6)   | 0.983(42) |
|           |         |          | (1, 1, 0) | 0.74(24) | 1.0869(11)  | 0.95(11)  |
| $\kappa_{cr}$ | 1.151(8) | 0.82(19)  | (1, 0, 0) | 0.82(21) | 1.0567(8)   | 1.00(4)   |
|           |         |          | (1, 1, 0) | 0.81(24) | 1.1103(15)  | 0.99(11)  |

**Table 2:** Data and extrapolation to $\kappa_{cr}$ on the $\beta = 6.0, 16^3 \times 39$ lattice. The heavy quark hopping parameter is set to $\kappa_Q = 0.118$. $\kappa_{cr} \approx 0.157$.

| $\kappa_q$ | $m_D$   | $f_+(0)$ | $\bar{p}$ | $f_+$  | $v \cdot v'$ | $\xi_0$   |
|-----------|--------|----------|---------|--------|-------------|----------|
| 0.152     | 0.885(6) | 1.00(12)  | (1, 0, 0) | 0.89(12) | 1.085(1)    | 0.883(29) |
|           |         |          | (1, 1, 0) | 0.77(14) | 1.160(2)    | 0.771(72) |
| 0.154     | 0.847(7) | 1.03(15)  | (1, 0, 0) | 0.92(16) | 1.0937(14)  | 0.894(40) |
|           |         |          | (1, 1, 0) | 0.79(18) | 1.1749(25)  | 0.77(9)   |
| 0.155     | 0.829(7) | 1.06(19)  | (1, 0, 0) | 0.94(20) | 1.0980(15)  | 0.894(40) |
|           |         |          | (1, 1, 0) | 0.78(22) | 1.1841(27)  | 0.74(12)  |
| $\kappa_{cr}$ | 0.791(7) | 1.08(19)  | (1, 0, 0) | 0.98(20) | 1.116(2)    | 0.90(5)   |
|           |         |          | (1, 1, 0) | 0.79(22) | 1.222(4)    | 0.73(12)  |

**Table 3:** Data and extrapolation to $\kappa_{cr}$ on the $\beta = 6.0, 16^3 \times 39$ lattice. The heavy quark hopping parameter is set to $\kappa_Q = 0.135$. $\kappa_{cr} \approx 0.157$. 
Table 4: Data and extrapolation to $\kappa_{cr}$ on the $\beta = 6.0, 24^3 \times 39$ lattice. The heavy quark hopping parameter is set to $\kappa_Q = 0.118$. $\kappa_{cr} \approx 0.157$.

| $\kappa$  | $m_D$     | $f_+(0)$  | $\vec{p}$ | $f_+$  | $v \cdot v'$ | $\xi_0$ |
|-----------|-----------|-----------|------------|--------|--------------|---------|
| 0.152     | 1.241(8)  | 0.61(13)  | (1,0,0)    | 0.60(14)| 1.0185(2)    | 0.973(20)|
|           |           |           | (1,1,0)    | 0.57(14)| 1.0366(4)    | 0.934(35)|
| 0.154     | 1.202(10) | 0.64(13)  | (1,0,0)    | 0.63(14)| 1.0201(3)    | 0.975(20)|
|           |           |           | (1,1,0)    | 0.60(14)| 1.0396(7)    | 0.938(42)|
| 0.155     | 1.182(11) | 0.66(12)  | (1,0,0)    | 0.64(12)| 1.0209(4)    | 0.973(20)|
|           |           |           | (1,1,0)    | 0.62(12)| 1.0412(8)    | 0.936(49)|
| $\kappa_{cr}$ | 1.144(10) | 0.69(13)  | (1,0,0)    | 0.67(14)| 1.0258(4)    | 0.98(2) |
|           |           |           | (1,1,0)    | 0.65(14)| 1.0511(9)    | 0.94(5) |

Table 5: Data and extrapolation to $\kappa_{cr}$ on the $\beta = 6.0, 24^3 \times 39$ lattice. The heavy quark hopping parameter is set to $\kappa_Q = 0.135$. $\kappa_{cr} \approx 0.157$.

| $\kappa$  | $m_D$     | $f_+(0)$  | $\vec{p}$ | $f_+$  | $v \cdot v'$ | $\xi_0$ |
|-----------|-----------|-----------|------------|--------|--------------|---------|
| 0.152     | 0.889(7)  | 0.96(10)  | (1,0,0)    | 0.90(10)| 1.0413(6)    | 0.944(17)|
|           |           |           | (1,1,0)    | 0.84(10)| 1.0802(12)   | 0.873(33)|
| 0.154     | 0.847(8)  | 0.99(10)  | (1,0,0)    | 0.94(10)| 1.0458(8)    | 0.949(19)|
|           |           |           | (1,1,0)    | 0.87(11)| 1.089(3)     | 0.883(40)|
| 0.155     | 0.826(9)  | 1.01(10)  | (1,0,0)    | 0.96(10)| 1.0476(10)   | 0.948(18)|
|           |           |           | (1,1,0)    | 0.90(11)| 1.093(5)     | 0.884(43)|
| $\kappa_{cr}$ | 0.784(9)  | 1.05(10)  | (1,0,0)    | 1.00(10)| 1.0543(12)   | 0.95(2) |
|           |           |           | (1,1,0)    | 0.90(11)| 1.106(2)     | 0.89(4) |
| $\kappa_q$ | $m_D$   | $f_+(0)$ | $\vec{p}$ | $f_+$   | $v \cdot v'$ | $\zeta_0$ |
|---------|---------|----------|-----------|---------|---------------|------------|
| 0.148   | 0.928(4)| 0.74(8)  | (1, 0, 0) | 0.69(8) | 1.0350(3)     | 0.929(14)  |
|         |         |          | (1, 1, 0) | 0.66(9) | 1.0708(5)     | 0.89(4)    |
| 0.149   | 0.906(4)| 0.78(10) | (1, 0, 0) | 0.73(10)| 1.0373(3)     | 0.937(16)  |
|         |         |          | (1, 1, 0) | 0.71(11)| 1.0752(5)     | 0.91(5)    |
| 0.150   | 0.884(4)| 0.86(13) | (1, 0, 0) | 0.83(13)| 1.0401(3)     | 0.960(23)  |
|         |         |          | (1, 1, 0) | 0.84(17)| 1.0805(7)     | 0.98(9)    |
| 0.1507  | 0.868(9)| 0.99(18) | (1, 0, 0) | 0.99(19)| 1.0423(6)     | 0.993(32)  |
|         |         |          | (1, 1, 0) | 1.05(27)| 1.0836(17)    | 1.06(12)   |
| $\kappa_{cr}$ | 0.848(9)| 1.02(18) | (1, 0, 0) | 1.04(27)| 1.0466(6)     | 1.02(3)    |
|         |         |          | (1, 1, 0) | 1.12(27)| 1.0912(12)    | 1.12(12)   |

Table 6: Data and extrapolation to $\kappa_{cr}$ on the $\beta = 6.3, 24^3 \times 61$ lattice. The heavy quark hopping parameter is set to $\kappa_Q = 0.125$. $\kappa_{cr} \approx 0.1516$.

| $\kappa_q$ | $m_D$   | $f_+(0)$ | $\vec{p}$ | $f_+$   | $v \cdot v'$ | $\zeta_0$ |
|---------|---------|----------|-----------|---------|---------------|------------|
| 0.148   | 0.587(2)| 1.16(7)  | (1, 0, 0) | 0.99(6) | 1.0894(6)     | 0.85(3)    |
|         |         |          | (1, 1, 0) | 0.86(12)| 1.1743(10)    | 0.74(11)   |
| 0.149   | 0.562(3)| 1.18(8)  | (1, 0, 0) | 1.05(8) | 1.0994(11)    | 0.89(3)    |
|         |         |          | (1, 1, 0) | 0.98(16)| 1.193(2)      | 0.78(12)   |
| 0.150   | 0.537(3)| 1.22(12) | (1, 0, 0) | 1.09(11)| 1.1128(14)    | 0.89(5)    |
|         |         |          | (1, 1, 0) | 1.06(21)| 1.216(3)      | 0.87(19)   |
| 0.1507  | 0.519(4)| 1.27(18) | (1, 0, 0) | 1.20(17)| 1.125(2)      | 0.95(6)    |
|         |         |          | (1, 1, 0) | 1.33(34)| 1.235(3)      | 1.05(28)   |
| $\kappa_{cr}$ | 0.496(4)| 1.31(18) | (1, 0, 0) | 1.27(17)| 1.131(2)      | 0.98(6)    |
|         |         |          | (1, 1, 0) | 1.48(34)| 1.249(4)      | 1.08(28)   |

Table 7: Data and extrapolation to $\kappa_{cr}$ on the $\beta = 6.3, 24^3 \times 61$ lattice. The heavy quark hopping parameter is set to $\kappa_Q = 0.140$. $\kappa_{cr} \approx 0.1516$.  

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| $\beta$ | $a^{-1}$ | $L$ | $(aL)^{-1}_{\text{phys}}$ | $\kappa_Q$ | $m_D$ |
|-------|---------|-----|-------------------------|-----------|-------|
| 6.0   | 2.0 GeV | 16  | 125 MeV                 | 0.118     | 2.3 GeV |
| 6.0   | 2.0 GeV | 24  | 83 MeV                  | 0.118     | 2.3 GeV |
| 6.0   | 2.0 GeV | 24  | 83 MeV                  | 0.135     | 1.6 GeV |
| 6.3   | 3.2 GeV | 24  | 125 MeV                 | 0.125     | 2.7 GeV |
| 6.3   | 3.2 GeV | 24  | 125 MeV                 | 0.140     | 1.3 GeV |

Table 8: Physical values of the lattice spacing $a$, lattice volume $aL$ and “D” meson mass $m_D$.

| Lattice                          | $\xi_0$ |
|----------------------------------|---------|
| Rosner [21]                      | 1.59(43)|
| Neubert [22]                     | 1.42(60)|
| Burdman [23]                     | 1.08(10)|
| Jin, Huang and Dai [24]          | 1.05(20)|
| Blok and Shifman [25]            | 0.65(15)|
| Bjorken [17]                     | $0.25 < \rho^2$ |
| de Rafael and Taron [18]         | $\rho^2 < 1.42$ |

Table 9: Comparison of the lattice calculation, using a linear fit to $\xi_0$, with various model calculations for $\rho^2$ at $v \cdot v' = 1$. 
Figure Captions

Fig. 1: The Isgur-Wise function is plotted against $v \cdot v'$. The lattice data are obtained for $m_q \sim m_s$. The parameters are: $\beta = 6.0, \kappa_Q = 0.118, \kappa_q = 0.154$ on $16^3 \times 39$ lattice (open triangle), $\beta = 6.0, \kappa_Q = 0.135, \kappa_q = 0.154$ on $16^3 \times 39$ lattice (solid triangle), $\beta = 6.0, \kappa_Q = 0.118, \kappa_q = 0.154$ on $24^3 \times 39$ lattice (open circle), $\beta = 6.0, \kappa_Q = 0.135, \kappa_q = 0.154$ on $24^3 \times 39$ lattice (solid circle), $\beta = 6.3, \kappa_Q = 0.125, \kappa_q = 0.149$ on $24^3 \times 61$ lattice (open square), and $\beta = 6.3, \kappa_Q = 0.140, \kappa_q = 0.149$ on $24^3 \times 61$ lattice (solid square). Only statistical errors are shown. The horizontal errors on the data points are smaller than symbol sizes. The dashed curve is the upper bound on the Isgur-Wise function [17]. The solid curve is a fit of the lattice data to Eq. (23). The dotted curve is derived in Ref. [18].

Fig. 2: The data for the Isgur-Wise function obtained after extrapolations to the chiral limit. The parameters are: $\beta = 6.0, \kappa_Q = 0.118$, on $16^3 \times 39$ lattice (open triangle), $\beta = 6.0, \kappa_Q = 0.135$, on $16^3 \times 39$ lattice (solid triangle), $\beta = 6.0, \kappa_Q = 0.118$, on $24^3 \times 39$ lattice (open circle), $\beta = 6.0, \kappa_Q = 0.135$, on $24^3 \times 39$ lattice (solid circle), $\beta = 6.3, \kappa_Q = 0.125$, on $24^3 \times 61$ lattice (open square), and $\beta = 6.3, \kappa_Q = 0.140$, on $24^3 \times 61$ lattice (solid square). The data points can be compared to the corresponding ones in Fig. 1. For reference we have shown the upper bound (dashed curve) on the Isgur-Wise function [17] and the curve (dotted) from Ref. [18].

Fig. 3: Comparison with experimental data. The open circles are ARGUS experiment data from Ref. [20] (We have normalized the graph with respect to $|V_{cb}|\sqrt{\tau_B/1.32ps} = 0.047$). The crosses are the lattice results from Fig. 1. The solid, dashed and dotted curves have the same meaning as in Fig. 1. The horizontal errors on the lattice data points are smaller than symbol sizes. (Note that the experimental point around $v \cdot v' = 1.1$ is shifted slightly for clarity.)