A Generalised CRI Iteration Method for Complex Symmetric Linear Systems

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Abstract. A generalisation of the combination method of real and imaginary parts for complex symmetric linear systems based on the introduction of an additional parameter is proposed. Sufficient conditions for the convergence of the method are derived. Numerical examples show the efficiency of this algorithm.

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1. Introduction

Let \( i = \sqrt{-1} \) be the imaginary unit, \( p, q \in \mathbb{R}^n \) and \( b = p + iq \in \mathbb{C}^n \). We consider the system of linear equations

\[
Ax = b, \quad x = y + iz, \quad y, z \in \mathbb{R}^n
\]  

(1.1)

with a nonsingular complex symmetric matrix \( A \in \mathbb{C}^{n \times n} \) of the form

\[
A = W + iT.
\]

Assume that \( W, T \in \mathbb{R}^{n \times n} \) are symmetric positive semidefinite matrices. Complex symmetric systems arise in various applications, including wave propagation [34], FFT-based solutions of time-dependent PDEs [11], diffuse optical tomography [1], numerical methods for time-dependent Schrödinger equation [15], Maxwell’s equations [23], molecular scattering [31], structural dynamics [25], modelling of electrical power systems [24], lattice quantum chromodynamics [27], quantum chemistry, eddy current problems [3], discretisation of self-adjoint integro-differential equations of environmental modelling [28], and so on [2, 4, 6, 9, 14].

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In recent years, the solution of matrix equations attracts more and more interest \cite{12, 16-20, 26, 32, 35}. Iterative algorithms are a powerful and very successful technique in linear systems and system identification. Various iteration methods have been used for solving the complex linear system (1.1) — e.g. the conjugate orthogonal conjugate gradient method, the complex symmetric method and quasi-minimal residual method \cite{12, 26, 35}. These methods are directly applicable to the complex linear system (1.1). However, one can avoid using complex arithmetic by rewriting complex systems of linear equations as $2 \times 2$ block real linear systems \cite{2, 10, 13, 39}. Instead of solving the original complex linear system (1.1), Salkuyeh et al. \cite{33} applied the generalised successive overrelaxation (GSOR) iterative method to an equivalent real system. In addition, Hezari et al. \cite{21} presented a preconditioned GSOR iterative method and studied conditions when the spectral radius of the iteration matrix of the preconditioned GSOR method is smaller than that of the GSOR method, also finding optimal iteration parameters.

Since $W$ and $T$ in (1.1) are real symmetric matrices, the Hermitian $H$ and skew-Hermitian $S$ parts of the complex symmetric matrix $A$ are

$$H = \frac{1}{2} (A + A^H) = W \quad \text{and} \quad S = \frac{1}{2} (A - A^H) = iT.$$  

Using the special structure of the complex matrix $A \in \mathbb{C}^{n \times n}$ and the Hermitian and skew-Hermitian splitting (HSS) method \cite{7}, Bai et al. \cite{4} developed a modified HSS (MHSS) iteration method, which converges unconditionally to the unique solution of the Eq. (1.1). Besides, in order to accelerate the convergence rate of the MHSS method, Bai et al. \cite{5} proposed the following preconditioned MHSS (PMHSS) method.

**PMHSS Iteration Method.** Let $x^{(0)} \in \mathbb{C}^n$ be an arbitrary initial guess. For $k = 0, 1, 2, \ldots$, until the sequence of iterates $\{x^{(k)}\}_{k=0}^\infty \subset \mathbb{C}^n$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$(\alpha V + W)x^{(k+1/2)} = (\alpha V - iT)x^{(k)} + b,$$

$$(\alpha V + T)x^{(k+1)} = (\alpha V + iW)x^{(k+1/2)} - ib,$$  \hspace{1cm} (1.2)

where $\alpha$ is a given positive constant and $V \in \mathbb{R}^{n \times n}$ is a prescribed symmetric positive definite matrix.

If $V$ is the identity matrix, then the PMHSS method becomes the MHSS method. Bai et al. \cite{5} proved that for any initial guess, the PMHSS iteration method converges to the unique solution of (1.1). Hezari et al. \cite{22} presented a new stationary matrix splitting iteration method, called Scale-Splitting, to solve the complex system (1.1). Convergence theory and spectral properties of the corresponding preconditioned matrix have been also established. Zheng et al. \cite{40} exploited the symmetry of the PMHSS method and employed scaling technique to reconstruct complex linear system (1.1). Moreover, it was shown that the double-step scale splitting iteration method proposed in \cite{40} is unconditionally convergent and converges faster than the PMHSS method. Liao and Zhang \cite{30} introduced a block multiplicative preconditioner and the corresponding block multiplicative iteration method for complex symmetric linear algebraic systems.
Recently, Wang et al. [37] considered a novel iteration method for the system (1.1) and established its convergence for symmetric positive semidefinite matrices $W$ and $T$ without assuming that one of them is positive definite. This iteration method, which uses a combination of real and imaginary parts is abbreviated as CRI and can be described as follows.

**CRI Iteration Method.** Given an initial vector $x^{(0)} \in \mathbb{C}^n$, for $k = 0, 1, 2, \ldots$, until the sequence of iterates $\{x^{(k)}\}$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$(aT + W)x^{(k+1/2)} = (a - i)Tx^{(k)} + b,$$

$$(aW + T)x^{(k+1)} = (a + i)Wx^{(k+1/2)} - ib,$$

(1.3)

where $a > 0$ is a given constant.

In this work we consider a generalised CRI (GCRI) iteration method for complex symmetric linear system (1.1) and establish its convergence for symmetric positive semidefinite matrices $W$ and $T$ without assuming that one of them is positive definite. We also estimate the spectral radius of the GCRI method and find parameters, which minimise the corresponding upper bound. Numerical experiments show the effectiveness of the GCRI iteration method.

Throughout the paper, we use the following notation. Let $\mathbb{C}^{n \times n}$ ($\mathbb{R}^{n \times n}$) denote the set of all $n \times n$ complex (real) matrices and $\mathbb{C}^n := \mathbb{C}^{n \times 1}$ ($\mathbb{R}^n := \mathbb{R}^{n \times 1}$). Moreover, $B^H$ and $B^{-1}$ refer to the conjugate transpose and the inverse of the matrix $B$, respectively. The spectral radius of a square matrix $B$ is denoted by $\rho(B)$ and null($B$) represents the null space of $B$. Let $|a|$ denote the modulus of a complex number $a$.

The remainder of the paper is organised as follows. In Section 2, we introduce the GCRI iteration method. The convergence of the GCRI method for complex linear system (1.1) is discussed in Section 3. The results of numerical experiments presented in Section 4 demonstrate feasibility and effectiveness of this method. Finally, conclusions are drawn in Section 5.

## 2. A Generalised CRI Method

In this section, we introduce a generalised CRI iteration method, which is actually a two-parameter alternating-direction iterative method.

A number of efficient iteration methods have been recently developed for linear system (1.1). New parameters are introduced in existing methods. Thus using HSS [7] and preconditioned HSS (PHSS) [8] methods, Yang et al. [38] presented a two-parameter generalised PHSS (GPHSS) method for large sparse non-Hermitian positive definite linear systems and show that under certain conditions the GPHSS method converges to the unique solution of the corresponding linear system. For singular linear systems, Li et al. [29] proposed a generalised HSS method with two parameters and discussed its semi-convergence and quasi-optimal parameters. Dehghan et al. [14] generalised the MHSS method by replacing parameter $\alpha$ in the second half-step of the MHSS scheme by another parameter $\beta$ and also constructed a generalised preconditioned MHSS (GPMHSS) method [14], using two differ-
ent parameters $\alpha$ and $\beta$ in the PMHSS scheme. For large complex symmetric linear systems, Wang et al. [36] proposed an accelerated GPMHSS method. These ideas motivated us to consider an iteration method for (1.1) combining real and imaginary parts such that in the second half-step of the scheme CRI (1.3) the parameter $\alpha$ is replaced by another parameter $\beta$. Numerical experiments show that for properly chosen parameters, the GCRI iteration method performs better than CRI and PMHSS iteration methods.

The GCRI iteration method can be described as follows:

$$
(\alpha T + W)x^{(k+1/2)} = (\alpha - i)Tx^{(k)} + b, \\
(\beta W + T)x^{(k+1)} = (\beta + i)Wx^{(k+1/2)} - ib.
$$

(2.1)

It is easily seen that (2.1) is equivalent to the method

$$
x^{(k+1)} = \mathcal{T}(\alpha, \beta)x^{(k)} + \mathcal{G}(\alpha, \beta)b,
$$

(2.2)

where

$$
\mathcal{T}(\alpha, \beta) = (\alpha - i)(\beta + i)(\beta W + T)^{-1}W(\alpha T + W)^{-1}T,
$$

(2.3)

$$
\mathcal{G}(\alpha, \beta) = (\beta W + T)^{-1}(\beta W - i\alpha T)(\alpha T + W)^{-1}. 
$$

The iteration scheme (2.2) is also coming from the splitting

$$
A = \mathcal{M}(\alpha, \beta) - \mathcal{N}(\alpha, \beta),
$$

where

$$
\mathcal{M}(\alpha, \beta) = (\alpha T + W)(\beta W - i\alpha T)^{-1}(\beta W + T),
\mathcal{N}(\alpha, \beta) = (\alpha - i)(\beta + i)(\alpha T + W)(\beta W - i\alpha T)^{-1}W(\alpha T + W)^{-1}T.
$$

Since $\mathcal{T}(\alpha, \beta) = \mathcal{M}(\alpha, \beta)^{-1}\mathcal{N}(\alpha, \beta)$, the GCRI method converges if and only if the spectral radius of the iteration matrix $\mathcal{T}(\alpha, \beta)$ is smaller than one. We also note that for the complex symmetric linear system (1.1), the splitting matrix $\mathcal{M}(\alpha, \beta)$ can be used as a preconditioner. Later on we analyse the convergence of this method for complex linear system (1.1).

### 3. Convergence of GCRI Method

In this section, we consider the convergence of the method (2.1) for the system (1.1) and derive an estimate for $\rho(\mathcal{T}(\alpha, \beta))$ along with optimal parameters minimizing the estimate obtained.

**Lemma 3.1** (cf. Wang et al. [37]). Let $W \in \mathbb{R}^{n \times n}$ and $T \in \mathbb{R}^{n \times n}$ be symmetric positive semidefinite matrices such that $\text{null}(W) \cap \text{null}(T) = \{0\}$. Then, there exists a nonsingular matrix $P \in \mathbb{R}^{n \times n}$ such that

$$
W = P^T \Lambda P \quad \text{and} \quad T = P^T \tilde{\Lambda} P.
$$

Here, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\tilde{\Lambda} = \text{diag}(\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n)$ and $\lambda_i, \tilde{\lambda}_i$ satisfy the conditions

$$
\lambda_i + \tilde{\lambda}_i = 1, \quad \lambda_i \geq 0, \quad \tilde{\lambda}_i \geq 0, \quad i = 1, 2, \ldots, n.
$$

(3.1)
Lemma 3.1 is used in the proof of the following theorem.

**Theorem 3.1.** Let \( \alpha \) and \( \beta \) be positive constants, \( \lambda_{\min} \) and \( \lambda_{\max} \) be, respectively, the smallest nonzero and the largest nonunit terms \( \lambda_i \) in (3.1) and \( A = W + iT \in \mathbb{C}^{n \times n} \) be a nonsingular matrix with symmetric positive semidefinite matrices \( W, T \in \mathbb{R}^{n \times n} \). If the parameters \( \alpha \) and \( \beta \) satisfy the inequalities

\[
\alpha > \frac{1 - 2\lambda_{\min}}{2\lambda_{\min}(1 - \lambda_{\min})}, \quad \beta > \frac{2\lambda_{\max} - 1}{2\lambda_{\max}(1 - \lambda_{\max})},
\]

then the GCRI method converges.

**Proof.** Let us show that the spectral radius of the iteration matrix \( \mathcal{F}(\alpha, \beta) \) satisfies the inequality

\[
\rho(\mathcal{F}(\alpha, \beta)) < 1. \tag{3.2}
\]

This will imply the convergence of the GCRI method.

Indeed, it follows from (2.3) and Lemma 3.1 that

\[
\mathcal{F}(\alpha, \beta) = (\alpha - i)(\beta + i)(\beta W + T)^{-1}W(\alpha T + W)^{-1}T \\
= (\alpha - i)(\beta + i)P^{-1}(\beta \Lambda + \tilde{\Lambda})^{-1}(\alpha \tilde{\Lambda} + \Lambda)^{-1}P.
\]

Considering the matrix

\[
\tilde{\mathcal{F}}(\alpha, \beta) = (\alpha - i)(\beta + i)(\beta \Lambda + \tilde{\Lambda})^{-1}(\alpha \tilde{\Lambda} + \Lambda)^{-1}\tilde{\Lambda},
\]

we observe that \( \mathcal{F}(\alpha, \beta) \) and \( \tilde{\mathcal{F}}(\alpha, \beta) \) are similar, hence they have the same eigenvalues. Therefore,

\[
\rho(\mathcal{F}(\alpha, \beta)) = \rho(\tilde{\mathcal{F}}(\alpha, \beta))
\]

\[
= \rho((\alpha - i)(\beta + i)(\beta \Lambda + \tilde{\Lambda})^{-1}(\alpha \tilde{\Lambda} + \Lambda)^{-1} \tilde{\Lambda})
\]

\[
= \max_{1 \leq i \leq n} \left\{ \frac{|(\alpha - i)\tilde{\lambda}_i|}{|\beta \lambda_i + \tilde{\lambda}_i|}, \frac{|(\beta + i)\lambda_i|}{|\alpha \lambda_i + \lambda_i|} \right\}
\]

\[
\leq \max_{1 \leq i \leq n} \frac{|(\alpha - i)\tilde{\lambda}_i|}{\alpha \lambda_i + \lambda_i}, \max_{1 \leq i \leq n} \frac{|(\beta + i)\lambda_i|}{\beta \lambda_i + \tilde{\lambda}_i}.
\]

In order to obtain inequality (3.2), it suffices to show that

\[
\frac{|(\alpha - i)\tilde{\lambda}_i|}{\alpha \lambda_i + \lambda_i} < 1, \quad \frac{|(\beta + i)\lambda_i|}{\beta \lambda_i + \tilde{\lambda}_i} < 1. \tag{3.3}
\]

Relations (3.1) and simple calculations ensure that the left inequality in (3.3) is equivalent to the estimate

\[
2\alpha \lambda_i(1 - \lambda_i) > 1 - 2\lambda_i. \tag{3.4}
\]

We now consider two cases.
Case 1. If $0 < \lambda_i \leq 1/2$, then the inequality (3.4) is valid if and only if

$$\alpha > \frac{1 - 2\lambda_i}{2\lambda_i(1 - \lambda_i)}. \quad (3.5)$$

Introducing the function

$$f(\lambda) = \frac{1 - 2\lambda}{2\lambda(1 - \lambda)},$$

and computing its derivative, we obtain

$$\frac{d}{d\lambda} f(\lambda) = \frac{-4(\lambda - 1/2)^2 - 1}{4\lambda^2(1 - \lambda)^2} < 0.$$

Thus $f(\lambda)$ is strictly monotone decreasing function. Therefore, according to the inequality (3.5), one can chose $\alpha$ so that

$$\alpha > \max_{1 \leq i \leq n, 0 < \lambda_i \leq 1/2} \frac{1 - 2\lambda_i}{2\lambda_i(1 - \lambda_i)} = \frac{1 - 2\lambda_{\min}}{2\lambda_{\min}(1 - \lambda_{\min}).}$$

Case 2. If $1/2 < \lambda_i < 1$, then for any $\alpha > 0$ the inequality

$$\alpha > \frac{1 - 2\lambda_i}{2\lambda_i(1 - \lambda_i)}$$

holds.

The above considerations yield

$$\alpha > \frac{1 - 2\lambda_{\min}}{2\lambda_{\min}(1 - \lambda_{\min}).}$$

Analogously, one can show that if

$$\beta > \frac{2\lambda_{\max} - 1}{2\lambda_{\max}(1 - \lambda_{\max}).}$$

then the second inequality in (3.3) holds. This completes the proof. \qed

Remark 3.1. If there are $k$ and $t \in \{1, 2, \ldots, n\}$ such that $\lambda_k = 0$ or $\lambda_t = 1 - \lambda_t = 0$ — i.e. if $\lambda_t = 1$, then

$$\rho(\mathcal{F}(\alpha, \beta)) = 0$$

for any $\alpha > 0$, $\beta > 0$. Therefore, the generalised CRI iteration method is unconditionally convergent.
Finding the parameters $\alpha$ and $\beta$, which minimise the spectral radius of the iteration matrix is a difficult problem. Therefore, we minimise the corresponding upper bound for $\rho(\mathcal{F}(\alpha, \beta))$. It follows from the proof of Theorem 3.1 that

$$
\rho(\mathcal{F}(\alpha, \beta)) = \rho(\mathcal{F}(\frac{\alpha}{\lambda_i} - 1, \frac{\beta}{\lambda_i} - 1)) = \max_{1 \leq i \leq n} \left| \frac{(\alpha - i)(\beta + i)\lambda_i}{(\alpha + i)\lambda_i + \lambda_i} \right|
$$

Thus under assumptions of Theorem 3.1, the parameters $\alpha^*$ and $\beta^*$ satisfy the equations

$$
\alpha^* = \frac{1}{\lambda_{\min}} - 1, \quad \beta^* = \frac{1}{1 - \lambda_{\max}} - 1.
$$

**Proof.** Consider the functions

$$
f_{\lambda_i}(\alpha) = \frac{|(\alpha - i)\lambda_i|}{\alpha\lambda_i + \lambda_i}, \quad g_{\lambda_i}(\beta) = \frac{|(\beta + i)\lambda_i|}{\beta\lambda_i + \lambda_i}.
$$

It follows from the equations $\lambda_i = 1 - \lambda_i$, $i = 1, 2, \ldots, n$ in (3.1) that

$$
f_{\lambda_i}(\alpha) = \frac{\sqrt{1 + \alpha^2(1 - \lambda_i)}}{\alpha(1 - \lambda_i) + \lambda_i}, \quad g_{\lambda_i}(\beta) = \frac{\sqrt{1 + \beta^2\lambda_i}}{\beta\lambda_i - \lambda_i + 1}.
$$

Therefore,

$$
(\alpha^*, \beta^*) = \arg\min_{\alpha > 0} \arg\min_{\beta > 0} \left\{ \max_{0 \leq i \leq n} f_{\lambda_i}(\alpha) \cdot \max_{0 \leq i \leq n} g_{\lambda_i}(\beta) \right\}
$$

Thus $\alpha^*$ and $\beta^*$ can be independently computed as

$$
\alpha^* = \arg\min_{0 \leq i \leq n} f_{\lambda_i}(\alpha), \quad \beta^* = \arg\min_{0 \leq i \leq n} g_{\lambda_i}(\beta).
$$

To find $\alpha^*$, we differentiate the function $f_{\lambda_i}(\alpha)$ and note that

$$
\frac{d}{d\lambda_i}f_{\lambda_i}(\alpha) = -\frac{\sqrt{1 + \alpha^2}}{(\alpha + (1 - \alpha)\lambda_i)^2} < 0.
$$

Hence, $f_{\lambda_i}(\alpha)$ is a strictly monotone decreasing function of $\lambda_i$, so that

$$
\max_{0 \leq i \leq n} f_{\lambda_i}(\alpha) = f_{\lambda_{\min}}(\alpha) = \frac{\sqrt{1 + \alpha^2(1 - \lambda_{\min})}}{\alpha + (1 - \alpha)\lambda_{\min}}.
$$
Hence
\[ \alpha^* = \arg\min_{\alpha > 0} \max_{0 \leq i \leq n} f_{\lambda_i}(\alpha) = \arg\min_{\alpha > 0} \frac{\sqrt{1 + \alpha^2 (1 - \lambda_{\min})}}{\alpha + (1 - \alpha)\lambda_{\min}}. \]

Simple calculations show that
\[ \frac{d}{d\alpha} f_{\lambda_{\min}}(\alpha) = \frac{(\alpha \lambda_{\min} - (1 - \lambda_{\min})) (1 - \lambda_{\min})}{\sqrt{1 + \alpha^2} \cdot ((1 - \lambda_{\min}) \alpha + \lambda_{\min})^2}. \]

We now consider two situations.

Case 1. If \( \alpha \lambda_{\min} - (1 - \lambda_{\min}) \geq 0 \) — i.e. if \( \alpha \geq 1/\lambda_{\min} - 1 \), then \( f_{\lambda_{\min}}(\alpha) \) is a strictly monotone increasing function.

Case 2. If \( \alpha \lambda_{\min} - (1 - \lambda_{\min}) < 0 \) — i.e. if \( \alpha < 1/\lambda_{\min} - 1 \), then \( f_{\lambda_{\min}}(\alpha) \) is strictly monotone decreasing function.

Therefore, the function \( f_{\lambda_{\min}}(\alpha) \) attains its minimum at the point \( \alpha^* = 1/\lambda_{\min} - 1 \).

Using the same approach, we obtain \( \beta^* = 1/(1 - \lambda_{\max}) - 1 \).

**Remark 3.2.** If there are \( k \) and \( t \in \{1, 2, \ldots, n\} \) such that \( \lambda_k = 0 \) or \( \tilde{\lambda}_t = 1 - \lambda_t = 0 \) — i.e. if \( \lambda_t = 1 \), then \( \sigma(\alpha, \beta) = 0 \) for any \( \alpha > 0, \beta > 0 \).

### 4. Numerical Experiments

In this section, we present some experiments to illustrate the effectiveness of the GCRI iteration method for complex symmetric linear systems and compare it with CRI and PMHSS iteration methods. All computations are performed in MATLAB environment on PC with Intel (R) Core (TM) i5 CPU 2.50GHz and 4.00 GB memory. In all methods, the sparse Cholesky factorisation is used in two half-steps comprising each iteration. Moreover, the zero vector is always chosen as the initial guess and the stopping criterion is

\[ \text{RES} := \frac{||r^{(k)}||_2}{||r^{(0)}||_2} \leq 10^{-6}, \]

where \( ||r^{(k)}||_2 \) and \( ||r^{(0)}||_2 \) refer to \( k \)-th and initial residual vectors, respectively. In tables, the items IT and CPU, respectively, denote the number of iteration steps and elapsed CPU time in seconds.

Tables 1-3 present numerical results concerning iteration steps, CPU time and relative residuals in PMHSS, CRI and GCRI iteration methods for nonsingular complex symmetric linear systems. In actual computations, \( \alpha \) and \( \beta \) are experimentally computed optimal parameters minimising the total number of iteration steps for the corresponding methods, with \( \alpha \) ranging from 0.01 to 10 and \( \beta \) from 0.01 to 5. For GCRI iteration method, we also choose the parameters \( \alpha^* \) and \( \beta^* \) minimising the upper bound \( \sigma(\alpha, \beta) \) of \( \rho(T(\alpha, \beta)) \). Here, \( \alpha_{\text{exp}} \) and \( \beta_{\text{exp}} \) denote the experimentally optimal parameters. To provide experimentally optimal parameters, we plot the iterative steps of three methods for Example 4.1 in Fig. 1 and for Example 4.2 in Fig. 2.
Example 4.1 (cf. Refs. [4, 5, 9]). Consider the linear system
\[
\left[ \left( -\omega^2 M + K \right) + i(\omega C_V + C_H) \right] x = b,
\]
where \( \omega \) is the driving circular frequency, \( M \) and \( K \) are inertia and stiffness matrices, and \( C_V \) and \( C_H \) are the viscous and the hysteretic damping matrices, respectively. We take \( C_H = \mu K \), where \( \mu \) is a damping coefficient, \( M = I \), \( C_V = 10 I \), and \( K \) the five-point centered difference matrix approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions on the uniform mesh in the unit square \([0, 1] \times [0, 1]\) with the mesh-size \( h = 1/(m+1) \). The matrix \( K \in \mathbb{R}^{n \times n} \) has the tensor-product form
\[
K = I \otimes V_m + V_m \otimes I,
\]
where \( V_m = \frac{1}{h^2} \text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m \times m} \). Thus \( K \) is an \( n \times n \), \( n = m^2 \) block-tridiagonal matrix. For \( \mu = 2 \) and \( \mu = 5 \), we set \( \omega = \pi \) and the right-hand side vector \( b = (1 + i) A_1 \), \( A_1 = (1, 1, \ldots, 1)^T \) and the system is normalised by multiplying it by \( h^2 \).

Numerical results for PMHSS, CRI and GCRI iteration methods are presented in Tables 1 and 2. Note that for \( \alpha = \alpha^* \) and \( \beta = \beta^* \) and for the experimental optimal parameters \( \alpha = \alpha_{\text{exp}}, \beta = \beta_{\text{exp}} \), the GCRI method requires less iteration steps than the other two methods. We also observe that the GCRI method with the optimal parameters \( \alpha^*, \beta^* \) and with the experimentally optimal parameters \( \alpha_{\text{exp}}, \beta_{\text{exp}} \) requires the same numbers of iterations. On the other hand, for any method, the size of the problem does not strongly influence the number of iterations but iteration steps may change along with the parameter \( \mu \).

Table 1: Example 4.1. Numerical results for various methods, \( \mu = 2 \).

| methods | \( n \) | \( 16^2 \) | \( 32^2 \) | \( 64^2 \) | \( 128^2 \) | \( 256^2 \) |
|---------|------|--------|--------|--------|--------|--------|
| GCRI    | \( \alpha^* \) | 2.100775 | 2.031222 | 2.009329 | 2.003003 | 2.0012 |
|         | \( \beta^* \) | 0.488982 | 0.497006 | 0.49925 | 0.4997 | 0.50015 |
|         | IT    | 15 | 15 | 16 | 16 | 16 |
|         | CPU   | 0.012314 | 0.031902 | 0.279727 | 1.852795 | 16.875878 |
|         | RES   | 6.7548e-07 | 9.2424e-07 | 4.0967e-07 | 4.2338e-07 | 4.2800e-07 |
|         | \( \alpha_{\text{exp}} \) | 2.1 | 2.029 | 2.0093 | 2.003 | 2 |
|         | \( \beta_{\text{exp}} \) | 0.48 | 0.494 | 0.4993 | 0.4997 | 0.5 |
|         | IT    | 15 | 15 | 16 | 16 | 16 |
|         | CPU   | 0.006989 | 0.028804 | 0.236957 | 1.657327 | 15.887745 |
|         | RES   | 6.7524e-07 | 9.2417e-07 | 4.0967e-07 | 4.2338e-07 | 4.2800e-07 |
| CRI     | \( \alpha_{\text{exp}} \) | 1 | 1 | 1 | 1 | 1 |
|         | IT    | 17 | 17 | 18 | 18 | 18 |
|         | CPU   | 0.007486 | 0.030161 | 0.274499 | 1.904455 | 17.071447 |
|         | RES   | 7.4499e-07 | 9.3059e-07 | 4.4387e-07 | 4.5398e-07 | 4.5682e-07 |
| PMHSS   | \( \alpha_{\text{exp}} \) | 2.2 | 1.81 | 1.54 | 1.431 | 1.42 |
|         | IT    | 25 | 23 | 22 | 21 | 21 |
|         | CPU   | 0.009465 | 0.036353 | 0.284932 | 2.062621 | 19.50357 |
|         | RES   | 6.2217e-07 | 7.3717e-07 | 6.4720e-07 | 9.2303e-07 | 8.8329e-07 |
Table 2: Example 4.1. Numerical results for various methods, $\mu = 5$.

| methods | n  | 16$^2$ | 32$^2$ | 64$^2$ | 128$^2$ | 256$^2$ |
|---------|----|--------|--------|--------|---------|---------|
| GCRI    | $\alpha$ | 5.169031 | 5.053269 | 5.016847 | 5.006006 | 5.002401 |
|         | $\beta$ | 0.197175 | 0.199328 | 0.19976 | 0.199904 | 0.200048 |
|         | IT    | 9 9 9 9 9 |
|         | CPU   | 0.009323 | 0.026343 | 0.168485 | 1.16657 | 10.742965 |
|         | RES   | 2.6937e-07 | 3.2801e-07 | 3.5015e-07 | 3.5708e-07 | 3.5903e-07 |
|         | $\alpha_{exp}$ | 5.169 | 5.0533 | 5.0168 | 5.006 | 4.99 |
|         | $\beta_{exp}$ | 0.1972 | 0.1993 | 0.1998 | 0.1999 | 0.2 |
|         | IT    | 9 9 9 9 9 |
|         | CPU   | 0.005386 | 0.020355 | 0.154896 | 1.047814 | 9.63585 |
|         | RES   | 2.6937e-07 | 3.2801e-07 | 3.5015e-07 | 3.5708e-07 | 3.5903e-07 |
| CRI     | $\alpha_{exp}$ | 1 1 1 1 1 |
|         | IT    | 11 11 11 11 11 |
|         | CPU   | 0.006204 | 0.021189 | 0.17258 | 1.234347 | 11.293065 |
|         | RES   | 5.8446e-07 | 6.9927e-07 | 7.4169e-07 | 7.5479e-07 | 7.5845e-07 |
| PMHSS   | $\alpha_{exp}$ | 1.521 | 1.713 | 2.08 | 2.22 | 2.24 |
|         | IT    | 28 27 26 25 25 |
|         | CPU   | 0.010793 | 0.040213 | 0.321155 | 2.3883 | 22.390161 |
|         | RES   | 9.1772e-07 | 6.3628e-07 | 6.2124e-07 | 9.1999e-07 | 9.0055e-07 |

Figure 1: Example 4.1. Number of iterations versus $\alpha$ and $\beta$, $n = 64 \times 64$. 
Example 4.2 (cf. Dehghan et al. [14]). Consider the system

$$Ax = (W + iT)x = b,$$

where $$b(j) = 55i + 90$$, $$j = 1, 2, \ldots, n$$, and $$W_{n \times n}, T_{n \times n}$$ are symmetric positive definite Toeplitz matrices of the form

$$W_{n \times n} = \begin{bmatrix}
100 & 5 & -2 & 1.5 & 10 & 0 \\
5 & \ddots & \ddots & \ddots & \ddots & \ddots \\
-2 & \ddots & \ddots & \ddots & \ddots & \ddots \\
1.5 & \ddots & \ddots & \ddots & \ddots & \ddots \\
10 & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 10 & 1.5 & -2 & 5 & 0
\end{bmatrix},$$

$$T_{n \times n} = \begin{bmatrix}
20 & 2 & -4 & -2 & 0 \\
2 & \ddots & \ddots & \ddots & \ddots \\
-2 & \ddots & \ddots & \ddots & \ddots \\
-4 & \ddots & \ddots & \ddots & \ddots \\
-2 & \ddots & \ddots & \ddots & \ddots \\
0 & -4 & -2 & 2 & 20
\end{bmatrix},$$

Numerical results for PMHSS, CRI and GCRI iteration methods are presented in Table 3. Note that for $$\alpha = \alpha^*, \beta = \beta^*$$ and for the experimental optimal parameters $$\alpha = \alpha_{exp}, \beta = \beta_{exp}$$, the GCRI method requires less iteration steps than the other two methods. We also observe that the GCRI method with the optimal parameters $$\alpha^*, \beta^*$$ and with the experimentally optimal parameters $$\alpha_{exp}, \beta_{exp}$$ requires the same numbers of iterations. On the other hand, in all methods, the size of the problem does not strongly influence the number of iterations.
5. Conclusion

In this work we developed a generalised combination method of real and imaginary parts based on the introduction of a new parameter in the combination method of real and imaginary parts for the complex symmetric linear system (1.1). Sufficient conditions for the convergence of the method are derived and numerical examples show the efficiency of this algorithm.

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