Renormalization Reductions for Systems with Delay

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The renormalization method which is a type of perturbation method is extended to a tool to study weakly nonlinear time-delay systems. For systems with order-one delay, we show that the renormalization method leads to reduced systems without delay. For systems with order-one and large-delay, we propose an extended renormalization method which leads to reduced systems with delay. In some examples, the validities of our perturbative results are confirmed analytically and numerically. We also compare our reduced equations with reduced ones obtained by another perturbation method.

§1. Introduction

Many nonlinear dynamical systems in various scientific disciplines are influenced by the finite propagation time of signals in feedback loops. A typical physical system is provided by a laser system where the output light is reflected and fed back to the cavity.1–3) Time delays also occur in other situations. For example, in a traffic flow model including a driver's reaction time,4) in biology due to physiological control mechanisms,5) or in economy where the finite velocity of information processing has to be taken into account.6) Furthermore, realistic models in population dynamics or in ecology include the duration for the replacement of the resources.7) In some situations, such as lasers and electro-mechanical systems,8) systems with large-delay appear. For this reason, we need to develop a mathematical tool to study them, especially for weakly nonlinear systems as a first step. The main difficulty peculiar to systems with delay is its dimensionality. Due to a delayed arrangement in a given system, $x(t - r) = \exp(-r \partial/\partial t)x(t)$, the dimension of the phase space is high.

Suppose we add a perturbation term to a given system, the system is not guaranteed to be structurally stable. So the perturbation result is, if computed naively, plagued with singularities such as secular terms. It has been recognized that these singularities in the result of the naive perturbation method can be renormalized away by the modification (renormalization) of parameters associated with the unperturbed system.9) The modified parameters are governed by the renormalization equations that turn out to be slow-motion equations (reduced equations). It is important that the prescription of the method does not depend on the details of the system under study. To obtain a more useful and sophisticated tool to study weakly nonlinear systems, reformulated versions10)–16) of the original method9) have been proposed. It is noted that there are a variety of applications of renormalization methods to physical systems, such as plasma physics,17) general relativity,18) and quantum optics,19) in addition to studies in standard nonlinear dynamical systems. Although the reformulated version of the renormalization method that we employ here is easily
applied to non-chaotic systems\textsuperscript{16,20} and chaotic maps,\textsuperscript{21} we do not know whether or not the renormalization method can be applied to systems with delay.

The purpose of this paper is to show that the reformulated renormalization method can be applied to weakly nonlinear systems with delay. For systems with order-one delay, the method leads to reduced systems without delay. For systems with large-delay, the method should be extended and the application of the extended method leads to reduced systems with delay. Our extended method can also be applied to systems with order-one delay, the resultant reduced equations are different from those obtained by the use of the conventional renormalization method. As mentioned, a time-delay term of a given system makes the dimension of the phase space high. Even in such a case, our method can lead to reduced equations.

The organization of this paper is as follows. In the next section (\S\textsuperscript{2}), we show that our conventional renormalization method can lead to reduced equations for systems with order-one delay. The validities of our analyzes are shown. In \S\textsuperscript{3}, we propose an extended version of our reformulated renormalization method so that we deal with a system with large-delay. The definition of large-delay is to be given in the beginning of the section. We show that the extended method can also be applied to systems with order-one delay. The validity of the extended method is also discussed. Finally, in \S\textsuperscript{4}, we discuss the features of our methods and conclude our study.

\section*{\S\textsuperscript{2}. Conventional Renormalization Method}

In this section, using the conventional renormalization method, we first analyze a linear system that has an oscillatory solution, and show that our perturbative analysis is in agreement with the exact solution analytically. By the conventional method we mean the method proposed in Ref.\textsuperscript{16}). Next, we study some classes of weakly nonlinear systems using our renormalization method. Our classes include a nonlinear oscillator, a laser model, systems with many degrees of freedom, and spatially extended systems. In some examples, we show that our analyzes are valid by comparing with the numerical simulation or the previous studies.

\subsection*{2.1. Linear system}

To show that our renormalization method\textsuperscript{16}) can be applied to systems with delay, we consider the following system as an example,

\begin{equation}
\frac{d^2 x(t)}{dt^2} + \omega^2 x(t) + \varepsilon x(t - r) = 0,
\end{equation}

where $\omega (\in \mathbb{R})$ is a parameter, $r (\in \mathbb{R})$ represents the time-delay, and $\varepsilon (\in \mathbb{R})$ is a small parameter ($|\varepsilon| \ll 1$). In this system, there is an oscillatory solution that is analytically expressible without any approximation. The exact solution is written as

\begin{equation}
x(t) = A \exp(it\sqrt{\omega^2 - \varepsilon}) + c.c.,
\end{equation}

under the condition

\begin{equation}
r = \frac{\pi}{\sqrt{\omega^2 - \varepsilon}}.
\end{equation}
Here c.c. represents for the complex conjugate terms of the preceding expression, $A(\in \mathbb{C})$ is the integration constant.

Let us derive a perturbation solution of (2.1) using our renormalization method. In this perturbative analysis, we do not use the exact solution (2.2). As well as in the case of a differential equation without delay, we first find the naive perturbation solution, $x(t) = x^{(0)}(t) + \varepsilon x^{(1)}(t) + \varepsilon^2 x^{(2)}(t) + O(\varepsilon^3)$. This naive perturbation solution is obtained by solving the following equations,

$$L x^{(0)}(t) = 0, \quad L x^{(j)}(t) = -x^{(j-1)}(t - r), (j = 1, 2, ...)
$$

$$L x(t) := \left(\frac{d^2}{dt^2} + \omega^2\right) x(t).
$$

The solutions are obtained as following

$$x^{(0)}(t) = A e^{i\omega t} + \text{c.c.},$$

$$x^{(1)}(t) = \frac{iA}{2\omega} e^{i\omega(t-r)} + \text{c.c.},$$

$$x^{(2)}(t) = -\frac{A}{8\omega^2} \left( t^2 - 2rt + \frac{i}{\omega} t \right) e^{i\omega(t-2r)} + \text{c.c.},$$

where $A(\in \mathbb{C})$ is the integration constant of the solution of the unperturbed system, $x^{(0)}(t)$. Note that the solutions $x^{(j)}(t), (j \geq 1)$ contain the terms const.exp($i\omega t$) and const.exp($-i\omega t$). We assume that these terms are included in $A e^{i\omega t}$ and its complex conjugate term in $x^{(0)}(t)$. Apparently, the validity of the naive perturbation solution is invalid in the regime $t > O(1/\varepsilon)$, due to the secular terms ($\propto \varepsilon t, \propto \varepsilon^2 t^2$ etc.).

The renormalization method removes the secular behavior in a systematic way. We define the renormalized variable $\bar{A}(t)$ up to $O(\varepsilon^2)$,

$$\bar{A}(t) := A + \varepsilon \frac{iA}{2\omega} t e^{-i\omega r} + \varepsilon^2 \frac{-A}{8\omega^2} \left( t^2 - 2rt + \frac{i}{\omega} t \right) e^{-2i\omega r} + O(\varepsilon^3). \quad (2.4)
$$

Note that this definition is a form of a near-identity transformation at the constant $A^{13}$ and that the naive perturbation solution is expressed in terms of the renormalized variable,

$$x(t) = \bar{A}(t) \exp(i\omega t) + \text{c.c.} + O(\varepsilon^3). \quad (2.5)
$$

We construct the equation which $\bar{A}(t)$ should satisfy perturbatively. Such an equation is our renormalization equation. From Eq. (2.4), we obtain the following two relations,

$$\bar{A}(t + \sigma) - \bar{A}(t) = \varepsilon \frac{iA}{2\omega} e^{-i\omega r}$$

$$+ \varepsilon^2 \frac{-A}{8\omega^2} \left( 2t\sigma + \sigma^2 - 2r\sigma + \frac{i}{\omega} \sigma \right) e^{-2i\omega r} + O(\varepsilon^3). \quad (2.6)
$$

and

$$A = \bar{A}(t) - \varepsilon \frac{i\bar{A}(t)}{2\omega} t e^{-i\omega r} + O(\varepsilon^2), \quad (2.7)$$
where \( \sigma (\in \mathbb{R}) \) is a parameter. Substituting Eq. (2.7) into Eq. (2.6), we have the approximate closed relation

\[
\frac{\tilde{A}(t + \sigma) - \tilde{A}(t)}{\sigma} = \varepsilon \frac{i \tilde{A}(t)}{2\omega} e^{-i \omega r} + \varepsilon^2 \tilde{A}(t) \left( \frac{r}{4\omega^2} - \frac{i}{8\omega^3} \right) e^{-2i \omega r} + \mathcal{O}(\sigma, \varepsilon^3).
\]

The renormalization equation is obtained in the limit \( \sigma \to 0 \) as

\[
\frac{d\tilde{A}(t)}{dt} = \varepsilon \frac{i \tilde{A}(t)}{2\omega} e^{-i \omega r} + \varepsilon^2 \tilde{A}(t) \left( \frac{r}{4\omega^2} - \frac{i}{8\omega^3} \right) e^{-2i \omega r}.
\]

(2.8)

The solution of Eq. (2.8) is given by

\[
\tilde{A}(t) = \tilde{A}(0) e^{\phi(t)},
\]

(2.9)

\[
\phi(t) := t \left\{ \varepsilon \frac{i}{2\omega} e^{-i \omega r} + \varepsilon^2 \left( \frac{r}{4\omega^2} - \frac{i}{8\omega^3} \right) e^{-2i \omega r} \right\}.
\]

(2.10)

To compare the solution of our renormalization method with the exact solution (2.2), we restrict ourselves to the approximate solution imposed on the condition (2.3). Using Eq. (2.3), we can rewrite Eq. (2.9) as

\[
\tilde{A}(t) = \tilde{A}(0) \exp \left( \varepsilon \frac{-it}{2\omega} + \varepsilon^2 \frac{-it}{8\omega^3} + \mathcal{O}(\varepsilon^3) \right).
\]

(2.11)

In terms of \( x(t) \), we obtain the approximate solution using Eqs. (2.3) and (2.10),

\[
x(t) = \tilde{A}(0) \exp \left\{ it \left( \omega - \varepsilon \frac{1}{2\omega} - \varepsilon^2 \frac{1}{8\omega^3} + \mathcal{O}(\varepsilon^3) \right) \right\} + \text{c.c.} + \mathcal{O}(\varepsilon^3).
\]

(2.11)

In fact, due to the relation

\[
\sqrt{\omega^2 - \varepsilon} = \omega - \varepsilon \frac{1}{2\omega} - \varepsilon^2 \frac{1}{8\omega^3} + \mathcal{O}(\varepsilon^3),
\]

Eq. (2.11) is the same as Eq. (2.2) up to \( \mathcal{O}(\varepsilon^2) \).

2.2. **Nonlinear single oscillator**

Let us consider a nonlinear equation including a time-delay term, which is to show that our renormalization method can be applied to such a system. The system which we study here is

\[
\frac{dx(t)}{dt} + \alpha x(t) + \beta x(t - r) = \varepsilon (\gamma_1 x(t) - \gamma_3 x^3(t)),
\]

(2.12)

where \( \alpha, \beta, \gamma_1 (\in \mathbb{R}) \) and \( \gamma_3 (\in \mathbb{R}) \) are parameters. The value of \( r (\in \mathbb{R}) \) represents the time-delay, and \( \varepsilon (\in \mathbb{R}) \) is the small parameter. It is noted here that the unperturbed system has an analytically expressible oscillatory solution when the following condition is satisfied,

\[
r = \frac{\arccos(-\alpha/\beta)}{\sqrt{\beta^2 - \alpha^2}}, \quad (\beta^2 > \alpha^2).
\]

(2.13)
We restrict ourselves to the case that this condition is satisfied. The oscillatory solution of the unperturbed system is given by
\[ x^{(0)}(t) = Ae^{i\omega t} + \text{c.c.}, \]
where \( \omega := \sqrt{\beta^2 - \alpha^2} \), \( A \in \mathbb{C} \) is the integration constant, and the relation \( i\omega + \alpha = -\beta \exp(-i\omega r) \) is satisfied. Although we can analytically express the exact solution of the unperturbed system, it is difficult to express the exact solution analytically in the case \( \varepsilon \neq 0 \). To investigate the effect of the perturbation term, we construct the renormalization equation for Eq. (2.12). The procedure to obtain the renormalization equation is the same as that described in §2.1.

The naive perturbation solution, \( x(t) = x^{(0)}(t) + \varepsilon x^{(1)}(t) + \mathcal{O}(\varepsilon^2) \), is obtained by solving the following equations,
\[ L_r x^{(0)}(t) = 0, \quad L_r x^{(1)}(t) = \gamma_1 x^{(0)}(t) - \gamma_3 x^{(0)}(t)^3, \]
\[ L_r x(t) := \frac{dx(t)}{dt} + \alpha x(t) + \beta x(t - r). \]

The solutions are given by
\[ x^{(0)}(t) = Ae^{i\omega t} + \text{c.c.}, \]
\[ x^{(1)}(t) = t \frac{\gamma_1 A - 3\gamma_3 |A|^2 A}{1 + r(\alpha + i\omega)} e^{i\omega t} + \text{c.c.}. \]

The naive perturbation solution includes the secular term. To remove the secular term, we define the renormalized variable \( \tilde{A}(t) \).
\[ \tilde{A}(t) := A + \varepsilon t \frac{\gamma_1 A - 3\gamma_3 |A|^2 A}{1 + r(\alpha + i\omega)}. \]

The renormalization equation is obtained as
\[ \frac{d\tilde{A}(t)}{dt} = \varepsilon \frac{\gamma_1 \tilde{A}(t) - 3\gamma_3 |\tilde{A}(t)|^2 \tilde{A}(t)}{1 + r(\alpha + i\omega)}. \]

This system can be split into two parts: dynamics described by its amplitude and phase,
\[ \frac{dR^2}{dt} = -3\gamma_3 QR^2 \left( R^2 - \frac{\gamma_1}{3\gamma_3} \right), \quad (2.14) \]
\[ \frac{d\phi}{dt} = \frac{3\varepsilon \gamma_3 \omega}{(1 + r\alpha)^2 + (r\omega)^2} \left( R^2 - \frac{\gamma_1}{3\gamma_3} \right), \quad (2.15) \]

Here, \( \tilde{A}(t) := R(t)e^{i\phi(t)} \) and \( Q := 2\varepsilon(1 + r\alpha)/(1 + r\alpha)^2 + (r\omega)^2 \). It turns out that, from Eqs. (2.14) and (2.15), \( R = R_\ast, (R_\ast := \sqrt{\gamma_1/(3\gamma_3)}) \) is a stable fixed point when \( \gamma_1/\gamma_3 > 0 \) and \( \gamma_3 Q > 0 \). It is thus expected that the limit-cycle oscillation appears when these conditions are satisfied. The amplitude of this limit-cycle oscillation is given by \( 2R_\ast \) due to the relation, \( x(t) \approx 2R(t) \cos(\omega t + \phi(t)) \). Fig. 1 shows that our analysis is valid.
2.3. Lang-Kobayashi phase equation

In this subsection, we show that our analysis can lead to a set of reduced equations which describe dynamics of semiconductor lasers with feedback. In Ref.22), they have analyzed the following delay equation

$$\Phi'' + \omega \xi \Phi'' + \Phi' - \Delta + \omega \Lambda_1 \cos[\Phi(S - \Theta) - \Phi(S)] = 0,$$  

(2.16)

where $\xi, \Delta, \Lambda_1 (\in \mathbb{R})$ are parameters, $\omega (\in \mathbb{R})$ is the small parameter, and the prime means differentiation with respect to $S$. This equation is obtained from the Lang-Kobayashi equations in the following conditions;\(^{22)}\) the small ratio of the phonon and carrier lifetimes and the relatively large value of the linewidth enhancement factor. We derive a reduced equation from Eq. (2.16) using our method and compare our result with that obtained by using the multiple-scale method in Ref.22).

First, the naive perturbation solution, $\Phi(S) = \Phi^{(0)}(S) + \omega \Phi^{(1)}(S) + \mathcal{O}(\omega^2)$, is obtained by solving the following equations

$$L \Phi^{(0)}(S) = \Delta, \quad (2.17)$$

$$L \Phi^{(1)}(S) = -\xi \Phi^{(0)}(S) - \Lambda_1 \cos[\Phi^{(0)}(S - \Theta) - \Phi^{(0)}(S)], \quad (2.18)$$

$$L \Phi(S) := \left(\frac{d^3}{dS^3} + \frac{d}{dS}\right) \Phi(S).$$

The solution to the unperturbed system (2.17), $\Phi^{(0)}(S)$, is obtained as

$$\Phi^{(0)}(S) = \frac{A}{2} e^{i(S+v)} + \text{c.c.} + S\Delta + B,$$  

(2.19)

where $A, v (\in \mathbb{R})$, and $B (\in \mathbb{R})$ are the integration constants. Substituting Eq. (2.19) into Eq. (2.18), we obtain the following equation

$$L \Phi^{(1)}(S) = -\xi A \cos(S + v) - \Lambda_1 J_0(D) \cos(\Theta \Delta)$$

where $J_0(D)$ is the Bessel function of the first kind of order zero.
This naive perturbation solution includes the secular term $\tilde{s}$ (\ref{eq:secular}). We define the renormalized variables and we have used the following relation in deriving Eq. \ref{eq:naive_solution},

$$D_C = \text{reduced ones obtained by the multiple-scale method. Using the decomposition}$$

The solution of Eq.\ref{eq:naive_solution} is given by

$$e^{iz\sin\theta} = J_0(z) + 2i \sum_{n=1,3,\ldots} J_n(z) \sin(n\theta) + 2 \sum_{n=2,4,\ldots} J_n(z) \cos(n\theta).$$

The solution of Eq.\ref{eq:naive_solution} is given by

$$\Phi^{(1)}(S) = -\frac{\xi A S}{2} \cos(S + v) - A_1 J_0(D) S \cos(\Theta \Delta)$$

$$+ S A_1 J_1(D) \sin(\Theta \Delta) \sin(-\Theta/2 + S + v)$$

$$- A_1 \left\{ \cos(\Theta \Delta) \sum_{n=2,4,\ldots} J_n(D) \frac{e^{in(-\Theta/2+S+v)}}{in(1 - n^2)} \right.$$  

$$- \sin(\Theta \Delta) \sum_{n=3,5,\ldots} J_n(D) \frac{e^{in(-\Theta/2+S+v)}}{n(1 - n^2)} + \text{c.c.} \right\}. \quad \text{\ref{eq:phi1}}$$

This naive perturbation solution includes the secular terms ($\propto \omega S$).

Next, we remove these secular terms using our renormalization method. We define the renormalized variables $\tilde{C}(S) (\in \mathbb{C})$ and $\tilde{B}(S) (\in \mathbb{R})$ as follows

$$\tilde{C}(S) := A - \frac{\omega S}{2} \left\{ \xi A + 2i A_1 J_1(D(A)) \sin(\Theta \Delta) e^{-i\Theta/2} \right\},$$

$$\tilde{B}(S) := B - \omega S A_1 J_0(D(A)) \cos(\Theta \Delta).$$

The set of the renormalization equations up to $O(\omega)$ is obtained as

$$\frac{d\tilde{C}(S)}{dS} = -\frac{\omega}{2} \xi \tilde{C}(S) - i\omega A_1 J_1(D(|\tilde{C}(S)|)) \sin(\Theta \Delta) e^{-i\Theta/2}, \quad \text{\ref{eq:reduced_c}}$$

$$\frac{d\tilde{B}(S)}{dS} = -\omega A_1 J_0(D(|\tilde{C}(S)|)) \cos(\Theta \Delta). \quad \text{\ref{eq:reduced_b}}$$

Here, we compare the renormalization equations (\ref{eq:reduced_c}) and (\ref{eq:reduced_b}) with the reduced ones obtained by the multiple-scale method. Using the decomposition $\tilde{C}(S) = \tilde{A}(S)e^{i\tilde{\varphi}(S)}, (A(S) \in \mathbb{R}, \tilde{\varphi}(S) \in \mathbb{R})$, we obtain

$$\frac{d\tilde{A}(S)}{dS} = -\frac{\omega \xi}{2} \tilde{A}(S) - \omega A_1 \sin(\Theta \Delta) J_1(D(\tilde{A}(S))) \sin(\Theta/2), \quad \text{\ref{eq:reduced_a}}$$

$$\frac{d\tilde{\varphi}(S)}{dS} = -\frac{\omega A_1}{\tilde{A}(S)} \sin(\Theta \Delta) J_1(D(\tilde{A}(S))) \cos(\Theta/2). \quad \text{\ref{eq:reduced_phi}}$$
When we introduce the slow variable \( \zeta := \omega S \), the renormalization equations (2.23), (2.24) and (2.25) become the reduced equations derived in Ref.22). This comparison shows that our analysis is consistent with one by a traditional perturbation method, and that our method can lead to the reduced equations from a physical system.

2.4. Weakly nonlinear lattice

In this subsection, we show that our method leads to a discrete complex Ginzburg-Landau equation from a weakly nonlinear lattice with delay. In this lattice system, the finite propagation time of motion from the nearest oscillators is taken into account. Studying the derived reduced system, we predict the stability of a trivial solution, and this prediction is confirmed numerically.

The weakly nonlinear oscillator which we study here is given by

\[
\frac{dx_j(t)}{dt} = p_j(t),
\]

\[
\frac{dp_j(t)}{dt} = -\Omega^2 x_j(t) + \varepsilon \left\{ \nu \left( x_{j+1}(t-r) + x_{j-1}(t-r) - 2x_j(t) \right) - \alpha x_j^3(t) \right\},
\]

(2.26)

where \( \alpha, \nu \in \mathbb{R} \) are parameters, \( \varepsilon \in \mathbb{R} \) is the small parameter, and \( r \in \mathbb{R} \) represents the time-delay. The variables \( x_j(t) \in \mathbb{R} \) and \( p_j(t) \in \mathbb{R} \) denote the displacement and momentum of the single oscillator located at lattice site \( j \in \mathbb{Z} \) respectively. It is noted that this given system becomes a Hamiltonian system when \( r = 0 \). Using the conventional renormalization method, we derive the reduced system here.

First, the naive perturbation solutions \( x_j(t) = x_j^{(0)}(t) + \varepsilon x_j^{(1)}(t) + \mathcal{O}(\varepsilon^2) \) are obtained as

\[
x_j(t) \approx A_j e^{i\Omega t} + \varepsilon \frac{t e^{i\Omega t}}{2i\Omega} \left\{ \nu \left( e^{-i\Omega r} (A_{j+1} + A_{j-1}) - 2A_j \right) - 3\alpha |A_j|^2 A_j \right\} + \text{c.c.}
\]

(2.28)

Here \( A_j \in \mathbb{C} \) are the integration constants, and the higher harmonic terms in \( x_j^{(1)}(t) \) are omitted.

Second, from Eq.(2.28), the renormalized variables are defined as

\[
\tilde{A}_j(t) := A_j + \varepsilon \frac{t}{2i\Omega} \left\{ \nu \left( e^{-i\Omega r} (A_{j+1} + A_{j-1}) - 2A_j \right) - 3\alpha |A_j|^2 A_j \right\}.
\]

From the definitions, we have the relation \( x_j(t) \approx \tilde{A}_j(t) e^{i\Omega t} + \text{c.c.} \), and the renormalization equations

\[
\frac{d\tilde{A}_j(t)}{dt} = \frac{\varepsilon}{2i\Omega} \left\{ \nu \left( e^{-i\Omega r} (\tilde{A}_{j+1}(t) + \tilde{A}_{j-1}(t)) - 2\tilde{A}_j(t) \right) - 3\alpha |\tilde{A}_j(t)|^2 \tilde{A}_j(t) \right\}.
\]

(2.29)
The system (2.29) becomes the discrete nonlinear Schrödinger equation, a Hamiltonian system, in the case $r = 0$. When $r \neq 0$, Eq. (2.29) is the discrete complex Ginzburg-Landau equation.

In the rest of this subsection, we clarify a part of the phase space for the derived system (2.29). Using the renormalization equations, we predict the behavior of motion in the original system and confirm it numerically. Here we restrict ourselves to the conditions $x_{j+N}(t) = x_j(t)$ with $N$ being the number of the oscillators, this conditions lead to $\tilde{A}_{j+N}(t) = \tilde{A}_j(t)$. There is the trivial solution $\tilde{A}_j(t) = 0$ in Eq. (2.29). We show that the uniform solution, expressed as $\tilde{A}_j(t) \equiv \tilde{A}(t)$, (for any $j$), can be viewed as one of the local stable manifolds of the fixed point $\tilde{A}_j = 0$ when a certain condition is satisfied. To do this, we study the linear stability for $\tilde{A}_j = 0$. Substituting $\tilde{A}_j(t) = \tilde{a}_j(t), (|\tilde{a}_j(t)| \ll 1)$ into Eq. (2.29) we have the linearized equation of motion in Fourier space

$$
\frac{d\tilde{b}_k(t)}{dt} = -\frac{i\varepsilon\nu}{\Omega} \left\{ -1 + e^{-i\Omega r} \cos \left( \frac{2\pi k}{N} \right) \right\} \tilde{b}_k(t),
$$

where

$$
\tilde{b}_k(t) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{-2\pi kj/N} \tilde{a}_j(t), \quad \tilde{a}_j(t) := \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi kj/N} \tilde{b}_k(t),
$$

with ($k = 0, ..., N - 1$). From Eq. (2.30), we can predict which modes increase or decrease in time. The absolute value of $b_k(t)$ decreases to zero as time evolves when

$$
\frac{\varepsilon\nu}{\Omega} \sin(\Omega r) \cos \left( \frac{2\pi k}{N} \right) > 0,
$$

and this condition with $k = 0$ gives that the uniform solution can be viewed as the local stable manifold of $\tilde{A}_j = 0$. Fig. 2 shows that our analysis for the given system, via the renormalization equation, is valid.

2.5. Spatially extended system (1)

To show that our renormalization method is also useful in the case that a spatially extended system modeled by including unperturbed terms with delayed arguments, we consider the following system,

$$
\frac{\partial u(t, x)}{\partial t} + \frac{\pi}{2r} u(t - r, x) = \varepsilon \left( \alpha u(t, x)^3 + \nu \frac{\partial^2 u(t, x)}{\partial x^2} \right),
$$

where $\alpha, \nu (\in \mathbb{R})$ are parameters, $\varepsilon (\in \mathbb{R})$ is the small parameter, and $r (\in \mathbb{R})$ represents the time-delay. Although the system is described by a delay partial differential equation, our prescription is not changed.

First, the naive perturbation solution, $u(t, x) = u^{(0)}(t, x) + \varepsilon u^{(1)}(t, x) + \mathcal{O}(\varepsilon^2)$, is obtained by solving the following equations,

$$
L_r u^{(0)}(t, x) = 0, \quad L_r u^{(1)}(t, x) = \alpha u^{(0)^3}(t, x) + \nu \frac{\partial^2 u^{(0)}(t, x)}{\partial x^2},
$$

$$
L_r u(t, x) := \frac{\partial u(t, x)}{\partial t} + \frac{\pi u(t - r, x)}{2r}.
$$
Fig. 2. The time sequence of the system described by Eqs. (2.26) and (2.27). The values of the parameters are $\Omega = 0.5$, $\alpha = 1$, $\nu = 1.01$, $r = 1$ and $\varepsilon = 0.01$. The number of the oscillators $N$ is three, and $x_{j,N}(t) = x_j(t)$. The initial conditions are $x_j(t) = 0.1, p_j(t) = 0, (j = 0, 1, 2)$ for $-r \leq t \leq 0$, which correspond to the uniform solution. The amplitudes of $x_j(t), (j = 0, 1, 2)$ decrease to zero as time evolves, which we can predict using the condition (2.31) with $k = 0$ [See text]. The numerical simulation method is given in the caption to Fig. 1.

The solutions are given by

$$u^{(0)}(t, x) = A(x)e^{i\pi t/(2r)} + c.c.,$$

$$u^{(1)}(t, x) = \frac{3\alpha|A(x)|^2A(x) + \nu \frac{\partial^2 A(x)}{\partial x^2}}{1 + i\frac{\pi}{2}}te^{i\pi t/(2r)},$$

where $A(x)(\in \mathbb{C})$ is an arbitrary differentiable function of $x$. The naive perturbation solution includes the secular term.

Next, to remove the secular behavior, the renormalized variable $\tilde{A}(t, x)(\in \mathbb{C})$ is defined as

$$\tilde{A}(t, x) := A(x) + \varepsilon \frac{t}{1 + i\frac{\pi}{2}}\left(3\alpha|A(x)|^2A(x) + \nu \frac{\partial^2 A(x)}{\partial x^2}\right).$$

Finally, the renormalization equation, which $\tilde{A}(t, x)$ should satisfy, is derived as

$$\frac{\partial \tilde{A}(t, x)}{\partial t} = \frac{\varepsilon}{1 + i\frac{\pi}{2}}\left(3\alpha|\tilde{A}(t, x)|^2\tilde{A}(t, x) + \nu \frac{\partial^2 \tilde{A}(t, x)}{\partial x^2}\right).$$

This is the well-known complex Ginzburg-Landau equation.

2.6. Spatially extended system (2)

To show that the renormalization method is useful in the case that a spatially extended system modeled by including perturbation terms with delayed arguments, we consider the following system,

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + 1\right)u(t, x) = \varepsilon au^3(t - r, x),$$

(2.33)

where $a(\in \mathbb{R})$ is a parameter, $r(\in \mathbb{R})$ is the time-delay, and $\varepsilon(\in \mathbb{R}, |\varepsilon| \ll 1)$ is the small parameter. Along with the procedure for partial differential equations proposed in Ref.16), we can derive the reduced equation.
First, the naive perturbation solution \( u(t, x) = u^{(0)}(t, x) + \varepsilon u^{(1)}(t, x) + \mathcal{O}(\varepsilon^2) \) is obtained by solving the following equations,

\[
Lu^{(0)}(t, x) = 0, \quad Lu^{(1)}(t, x) = au^{(0)}(t-r, x),
\]

\[
Lu(t, x) := (\partial^2_t - \partial^2_{xx} + 1)u(t, x).
\]

We set \( u^{(0)}(t, x) \) as

\[
u^{(0)}(t, x) = Ae^{i(kx - \omega t)} + \text{c.c.},
\]

where \( A(\in \mathbb{C}) \) is the integration constant, and \( \omega^2 = k^2 + 1 \), with \( k(\in \mathbb{R}) \) being a parameter. A secular solution of \( u^{(1)}(t, x) \) is found to be

\[
u^{(1)}(t, x) = 3a|A|Ae^{i\omega r}(p_{10}t + p_{01}x)e^{i(kx - \omega t)} + \text{c.c.},
\]

where \( p_{10}(\in \mathbb{C}) \) and \( p_{01}(\in \mathbb{C}) \) are parameters. The values of these parameters are restricted by the condition

\[
2(\omega p_{10} + kp_{01}) = i. \quad (2.34)
\]

Next, the renormalized variables \( \tilde{A}(t, x)(\in \mathbb{C}) \) is defined as

\[
\tilde{A}(t, x) := A + \varepsilon 3a|A|^2Ae^{i\omega r}(p_{10}t + p_{01}x).
\]

From this definition \( 2.35 \), we have

\[
\partial_t \tilde{A}(t, x) = 3a\varepsilon|\tilde{A}(t, x)|^2\tilde{A}(t, x)e^{i\omega r}p_{10}, \quad (2.36)
\]

\[
\partial_x \tilde{A}(t, x) = 3a\varepsilon|\tilde{A}(t, x)|^2\tilde{A}(t, x)e^{i\omega r}p_{01}. \quad (2.37)
\]

Finally, eliminating \( p_{10} \) and \( p_{01} \) from Eqs.(2.34), (2.36) and (2.37), we have the following renormalization equation,

\[
\left( \frac{\partial}{\partial t} + \frac{d\omega}{dk} \frac{\partial}{\partial x} \right) \tilde{A}(t, x) = i\frac{3a}{2}\varepsilon|\tilde{A}(t, x)|^2\tilde{A}(t, x)e^{i\omega r}.
\]

\section{Extended Renormalization Method}

In this section, we propose an extended renormalization method which can lead to a reduced equation with delay from a given system with large- or order-one delay. In this paper, by large-delay we mean that the delayed arguments are of order \( 1/\varepsilon^\alpha \), \( (\alpha > 0) \) with \( \varepsilon \) being the small parameter associated with the given weakly nonlinear system. We show that our reduced equations are consistent with those obtained by the multiple-scale method in Refs.23), 24).

3.1. Linear system

The model which we study here is

\[
\frac{d^2x(t)}{dt^2} + \omega^2x(t) + \varepsilon x \left( t - \frac{r}{\varepsilon^\alpha} \right) = 0, \quad (3.1)
\]

where \( r/\varepsilon^\alpha \) represents large-delay with \( \varepsilon(\in \mathbb{R}) \) being a small parameter, and \( \alpha(\geq 0) \) is a parameter. When \( \alpha = 0 \), this equation is the same as Eq.(2.1). When \( \alpha = 1 \) and
\( \omega = 1 \), the system (3.1) was analyzed using the multiple-scale method in Ref23. We analyze this system (3.1) using the extended renormalization method and compare the result with that in the previous work.

First, we obtain the perturbation solution, \( x(t) = x^{(0)}(t) + \varepsilon x^{(1)}(t) + \varepsilon^2 x^{(2)}(t) + \mathcal{O}(\varepsilon^3) \), by solving the following equations

\[
L x^{(0)}(t) = 0, \quad (3.2)
\]
\[
L x^{(1)}(t) = -x^{(0)} \left( t - \frac{r}{\varepsilon^\alpha} \right), \quad (3.3)
\]
\[
L x^{(2)}(t) = -x^{(1)} \left( t - \frac{r}{\varepsilon^\alpha} \right), \quad L x(t) := \left( \frac{d^2}{dt^2} + \omega^2 \right) x(t).
\]

In deriving these equations, the magnitude of \( r/\varepsilon^\alpha \) in the delayed argument is treated as a large value, this treatment corresponds to the non-standard expansion in the previous study.23)

The solution of Eq.(3.2) is

\[
x^{(0)}(t) = A^{(0)} \exp(i\omega t) + \text{c.c.}, \quad (3.4)
\]

where \( A^{(0)}(\in \mathbb{C}) \) represents the contribution to the solution \( x^{(0)}(t) \) except for the fast motion \( \exp(i\omega t) \). We assume that the solution \( x^{(0)} \) at \( t - r/\varepsilon^\alpha \) is expressed as

\[
x^{(0)} \left( t - \frac{r}{\varepsilon^\alpha} \right) = A \left( -\frac{r}{\varepsilon^\alpha} \right) e^{-i\omega r/\varepsilon^\alpha} \exp(i\omega t) + \text{c.c.}.
\]

Here \( A(-r/\varepsilon^\alpha) \) represents the contribution to \( x^{(0)}(t-r/\varepsilon^\alpha) \), except for \( e^{-i\omega r/\varepsilon^\alpha} \exp(i\omega t) \). This implies that the argument of \( A(t) \) is only affected by a large time shift. At this stage, we do not know the functional form of \( A(t) \). The existence of \( A(t) \) is the most fundamental assumption in this extended method. It is noted that the equation which \( A(t) \) should perturbatively satisfy is our extended renormalization equation. This extended reduced equation is constructed by removing the secular behavior coming from the resonance between the frequency in the operator \( L \) and \( \propto \exp(i\omega t) \) in the forcing terms. When the delay \( r \) becomes zero, the extended method corresponds to the conventional method. Substituting this solution \( x^{(0)}(t) \) into Eq.(3.3), we obtain

\[
L x^{(1)}(t) = -A \left( -\frac{r}{\varepsilon^\alpha} \right) e^{-i\omega r/\varepsilon^\alpha} \exp(i\omega t) + \text{c.c.}.
\]

The solution is given by

\[
x^{(1)}(t) = \frac{it}{2\omega} A \left( -\frac{r}{\varepsilon^\alpha} \right) e^{-i\omega r/\varepsilon^\alpha} \exp(i\omega t) + \text{c.c.}.
\]

At the next order in \( \varepsilon \), we obtain

\[
x^{(2)}(t) = \left\{ -\frac{t^2}{8\omega^3} + \left( \frac{-i}{8\omega^3} + \frac{r}{4\omega\varepsilon^\alpha} \right) t \right\} A \left( -\frac{2r}{\varepsilon} \right) e^{-2\omega r/\varepsilon^\alpha} \exp(i\omega t) + \text{c.c.}.
\]

We observe the secular behavior as we have already seen in the case that the delay is not large.
Renormalization Reductions for Systems with Delay

Next, we remove the secular behavior. To do this, we define the extended renormalized variable,

\[
\tilde{A}(t) := A(0) + \varepsilon \frac{it}{2\omega} A \left( -\frac{r}{\varepsilon^\alpha} \right) e^{-\omega r / \varepsilon^\alpha} \\
+ \varepsilon^2 \left\{ \frac{-t^2}{8\omega^3} + \left( -\frac{i}{8\omega^3} + \frac{r}{4\omega^2 \varepsilon^\alpha} \right) t \right\} A \left( -\frac{2r}{\varepsilon^\alpha} \right) e^{-2\omega r / \varepsilon^\alpha}. \quad (3.5)
\]

This definition is a form of a near-identity transformation at the function \(A(0)\), instead of that at the constant \(A\) in the conventional renormalization method. From this definition of \(\tilde{A}(t)\), we obtain

\[
\frac{\tilde{A}(t + \sigma) - \tilde{A}(\sigma)}{\sigma} = \frac{\varepsilon i}{2\omega} A \left( -\frac{r}{\varepsilon^\alpha} \right) e^{-\omega r / \varepsilon^\alpha} \\
+ \varepsilon^2 \left\{ \frac{-2t}{8\omega^3} + \left( -\frac{i}{8\omega^3} + \frac{r}{4\omega^2 \varepsilon^\alpha} \right) t \right\} A \left( -\frac{2r}{\varepsilon^\alpha} \right) e^{-2\omega r / \varepsilon^\alpha} + O(\sigma, \varepsilon^3), \quad (3.6)
\]

where \(\sigma (\in \mathbb{R})\) is a parameter whose value is of smaller than \(r/\varepsilon^\alpha\). The inverse of Eq.(3.5) is derived as

\[
A(0) = \tilde{A}(t) - \varepsilon \frac{it}{2\omega} A \left( -\frac{r}{\varepsilon^\alpha} \right) e^{-\omega r / \varepsilon^\alpha} + O(\varepsilon^2).
\]

Using the above expression, we obtain the following relation

\[
A \left( -m \frac{r}{\varepsilon^\alpha} \right) = \tilde{A} \left( t - m \frac{r}{\varepsilon^\alpha} \right) \\
- \varepsilon \frac{i(t - mr / \varepsilon^\alpha)}{2} \tilde{A} \left( t - m \frac{r}{\varepsilon^\alpha} \right) e^{-\omega r / \varepsilon^\alpha} + O(\varepsilon^2). \quad (3.7)
\]

with \(m (\in \mathbb{N})\). We substitute Eq.(3.7) into Eq.(3.6) and take the limit \(\sigma \to 0\), we obtain our extended renormalization method which \(\tilde{A}(t)\) should perturbatively satisfy,

\[
\frac{d\tilde{A}(t)}{dt} = \varepsilon \frac{i}{2\omega} e^{-i\omega r / \varepsilon^\alpha} \tilde{A} \left( t - \frac{r}{\varepsilon^\alpha} \right) + \varepsilon^2 \frac{-i}{8\omega^3} e^{-2\omega r / \varepsilon^\alpha} \tilde{A} \left( t - 2 \frac{r}{\varepsilon^\alpha} \right). \quad (3.8)
\]

In Eq.(3.8) there are delay terms, and this renormalization equation in the case of \(\alpha = 1\) and \(\omega = 1\) is equivalent to reduced equations derived in Ref.23, where numerical simulation and some analysis have shown that the reduced system reproduces the behavior of slow motion in the original system. In the case that \(\alpha = 0\) and \(r\) is given by Eq.(2.3), we can show that one of the solutions to Eq.(3.8) up to \(O(\varepsilon^2)\) is given by Eq. (2.10).

Here we compare this extended renormalization method with the conventional method discussed in [2] for this system (3.1). When we use the conventional method we cannot obtain a reduced equation. To see this, we use the conventional method.
The naive perturbation solutions are
\[ x^{(0)}(t) = A e^{i\omega t} + \text{c.c.} \]
\[ x^{(1)}(t) = \frac{it}{2\omega} A e^{-i\omega r/\varepsilon^\alpha} e^{i\omega t} + \text{c.c.} \]
\[ x^{(2)}(t) = \left\{ -\frac{t^2}{8\omega^3} + \left( \frac{-i}{8\omega^3} + \frac{r}{4\omega \varepsilon^\alpha} \right) t \right\} A e^{-2i\omega r/\varepsilon^\alpha} e^{i\omega t} + \text{c.c.} \]

The renormalized variable is defined as
\[ \tilde{A}(t) := A + \varepsilon \frac{it}{2\omega} A e^{-i\omega r/\varepsilon^\alpha} + \varepsilon^2 \left\{ -\frac{t^2}{8\omega^3} + \left( \frac{-i}{8\omega^3} + \frac{r}{4\omega \varepsilon^\alpha} \right) t \right\} A e^{-2i\omega r/\varepsilon^\alpha}. \]

The renormalization equation up to \( \mathcal{O}(\varepsilon) \) becomes
\[ \frac{d\tilde{A}(t)}{dt} = \varepsilon \frac{i}{2\omega} e^{-i\omega r/\varepsilon^\alpha} \tilde{A}(t), \quad (3.9) \]
and that up to \( \mathcal{O}(\varepsilon^2) \) becomes
\[ \frac{d\tilde{A}(t)}{dt} = \varepsilon \frac{i}{2\omega} e^{-i\omega r/\varepsilon^\alpha} \tilde{A}(t) + \varepsilon^2 \left( \frac{r}{4\varepsilon^\alpha} - \frac{i}{8\omega^3} \right) e^{-2i\omega r/\varepsilon^\alpha} \tilde{A}(t). \quad (3.10) \]

Since the magnitudes of the terms calculated as higher-order correction in Eq. (3.10) are \( \mathcal{O}(\varepsilon^2) \) and \( \mathcal{O}(\varepsilon^{2-\alpha}) \), this approximation is in contradiction with Eq. (3.9), except for the case of \( \alpha = 0 \). When \( \alpha = 0 \), Eq. (3.10) becomes Eq. (2.8), and there is no contradiction only in the case \( \alpha = 0 \). We conclude that, for systems with large-delay, the extended renormalization method should be used.

### 3.2. Nonlinear system

We consider a weakly nonlinear system with large-delay which appears in optics. In Ref. 24), they have analyzed the system with optoelectronic feedback, and the system is described as
\[ \frac{dx(s)}{ds} = -y(s) - \varepsilon^2 x(s) \left( 1 + \frac{2P}{1 + 2P} y(s) \right), + \varepsilon^2 C \left\{ 1 + y \left( s - \frac{\Theta}{\varepsilon^2} \right) \right\} \]
\[ \frac{dy(s)}{ds} = (1 + y(s)) x(s), \]
where \( s \) is the scaled time \( C, P, \Theta (\in \mathbb{R}) \) are parameters, and \( \varepsilon (\in \mathbb{R}) \) is the small parameter. The solution which we focus on is the small amplitude regime, described by the following assumption
\[ x(s) = \varepsilon x^{(1)}(s) + \varepsilon^2 x^{(2)}(s) + \varepsilon^3 x^{(3)}(s) + \mathcal{O}(\varepsilon^4), \]
\[ y(s) = \varepsilon y^{(1)}(s) + \varepsilon^2 y^{(2)}(s) + \varepsilon^3 y^{(3)}(s) + \mathcal{O}(\varepsilon^4). \]

We construct the reduced equation using our extended method, and compare the result with that reported in Ref. 24).
First the naive perturbation problems are

\[
\frac{dx^{(1)}(s)}{ds} = -y^{(1)}(s), \quad \frac{dy^{(1)}(s)}{ds} = x^{(1)}(s),
\]

\[
\frac{dx^{(2)}(s)}{ds} = -y^{(2)}(s) + C, \quad \frac{dy^{(2)}(s)}{ds} = x^{(2)}(s) + x^{(1)}(s)y^{(1)}(s),
\]

\[
\frac{dx^{(3)}(s)}{ds} = -y^{(3)}(s) - x^{(1)}(s) + Cy^{(1)}(s - \theta), \quad \frac{dy^{(3)}(s)}{ds} = x^{(3)}(s) + y^{(2)}(s)x^{(1)}(s) + y^{(1)}(s)x^{(2)}(s),
\]

where \( \theta = \Theta/\varepsilon^2 \). The solutions \( x^{(1)}(s), x^{(2)}(s) \) and \( x^{(3)}(s) \) are given by

\[
x^{(1)}(s) = A(0)e^{is} + \text{c.c.}, \quad x^{(2)}(s) = -\frac{i}{3}A(0)^2e^{2is} + \text{c.c.},
\]

\[
x^{(3)}(s) = \frac{s}{2}\left(iCA(0) - \frac{i}{3}|A(0)|^2 A(0) - A(0) - iCA(-\theta)\right)e^{is} + \text{c.c.} + \text{higher harmonics}.
\]

The definition of the renormalized variable is

\[
\tilde{A}(t) := A(0) + \varepsilon^2 s\left(iCA(0) - \frac{i}{3}|A(0)|^2 A(0) - A(0) - iCA(-\theta)\right).
\] (3.11)

The renormalization equation is derived from Eq. (3.11) as

\[
\frac{d\tilde{A}(s)}{ds} = \frac{\varepsilon^2}{2}\left(iC\tilde{A}(s) - \frac{i}{3}|\tilde{A}(s)|^2\tilde{A}(s) - \tilde{A}(s) - iC\tilde{A}(s - \theta)\right).
\] (3.12)

The renormalization equation (3.12) is the reduced equation derived in Ref.24). Some analytical analyses in Ref.24) have shown where bifurcation points are. Again, we confirm that our extended method gives the same results given by the use of the multiple-scale method.

§4. Discussion and Conclusions

In this section, we discuss the features of both our conventional and extended renormalization methods, and then we conclude our study.

The prescription of the conventional renormalization method for systems with order-one delay is not different from ones without delay. The standard prescription leads to the reduced equation from a given weakly nonlinear system. The conventional method removes the secular terms from naive perturbation series by accounting for their effect with renormalized variables. Derived reduced systems are always ones without delay. Being without delay in a reduced equation means that the dimension of phase space for the original system can be reduced perturbatively. This reduction provides us the approximate structure of phase space. Systems to
which we can apply this method are weakly nonlinear ones with order-one delay. In this sense, the conventional method is restricted.

The prescription of our proposed extended renormalization method also removes the secularity. The basic assumption for this method is that we can introduce an unknown function contributing the naive perturbation solution, instead of the integration constant in the use of the conventional method. Although a rigorous mathematical meaning of the extended method has not yet given in this paper, we have checked the validity of our method through various examples. Derived reduced systems using this extended method are always ones with delay. This means that the dimensions of phase space for both the original and reduce systems are high. The advantage of our reduction method is that a steady state in the reduced system corresponds to a periodic one in the given system, which provides us some bifurcation analyzes. Using extended method, we can deal with systems whose delay time is of order \(1/\varepsilon^{\alpha}\), \((\alpha \geq 0)\) where \(\varepsilon\) is the small parameter appearing in the original system under study.

For both the renormalization methods, terms in the reduced equation arise from secular terms appearing in the naive perturbation analysis. This implies that higher harmonics in the naive perturbation analysis does not contribute to the reduced equation in the first order approximation. In this sense, the reduced equation can be obtained from a wide class including the original system. Compared to the multiple-scale method, our methods do not need scaled variables. While in the course of the derivation of a reduced system using our methods, we need the analytical expressions of the naive perturbation solutions so that we define the renormalized variables. Although the procedures of our methods are systematic, the application of our methods are restricted by this disadvantage.

In this paper, we have shown that the renormalization method can be extended to a tool to study systems with delay, and that the method gives reduced systems successfully. Combining the previous studies of the renormalization method with the present study, we expect that our renormalization method includes all the asymptotic analyzes. Furthermore, we believe that the application of the renormalization method can help elucidate the behavior of time-delayed systems in a non-chaotic regime.

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