ON LINE ARRANGEMENTS OVER FIELDS WITH $1-ad$ STRUCTURE

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Abstract. In this article we explore local and global gonality principle present in the line arrangements of the plane. We prove two main results, Theorems 1.1, 1.2. Firstly we associate invariants such as permutation cycles and local cycles at infinity with $2-standard$ consecutive structures (refer to Definitions 5.9, 5.11, 5.26) to a line arrangement (refer to Definition 2.3) which has global cyclicity (refer to Definition 5.7) over fields with $1-ad$ structure (refer to Definition 2.1) to describe the gonality structures (refer to Definition 3.1) in Theorem 1.1 when there exists a local permutation chart where the intersections points corresponding to simple transpositions satisfy One Sided Property 1.1. We construct a graph of isomorphism of classes of line arrangements over fields with $1-ad$ structure using the associated invariants and Elementary Collineation Transformations (ECT) in Theorem 1.2 and in Note 5.32. Secondly here we prove another main theorem of the article, the representation Theorem 1.2 where we represent each isomorphism class with lines having a given set of distinct slopes. We also prove two isomorphism Theorems 5.35, 5.36 for the line arrangement collineation maps using quadrilateral substructures (refer to Definition 5.33) based on the theme of central points of triples of intersection points. At the end of the article we ask some open questions on line-folds (refer to Definition 6.1).

1. Introduction

The line arrangements (refer to Definition 2.3) has been studied by various authors like A. Dimca [1], S.Papadima, A.D.R Choudary, A.Suciu, P. Orlik, H. Terao [2] in various contexts over fields $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ and finite fields $\mathbb{F}_q, q$ a prime power. This field has applications in fields like Combinatorics, Braids and Configurations Spaces, Computer Science and Physics. Here we study the features of a finite set of linear inequalities in two variables over fields with $1-ad$ structure (refer to Definition 2.1) exploring the local and global gonality principle with a purpose to associate invariants which describe the gonality structures (refer to Definition 3.1) in a line arrangement (refer to Definition 2.3).

1.1. Main Results. We prove two main results Theorem 1.1 and Theorem 1.2 in this article. We state the theorems here but refer the reader to the required definitions in the article.

Exploring the local and global gonality principle the first main Theorem 1.1 gives a combinatorial description of the polygonal regions formed in a line arrangement by associating combinatorial invariants which respect geometrical notions. The following Theorem 1.1 is proved after the statement of Definition 5.26. Now we state the theorem.

Theorem 1.1 (Criterion for Existence of a Local $k-$Gonality and Sufficiency). Let $\mathbb{F}$ be a field with $1-ad$ structure (refer to Definition 2.1). Let $\mathcal{L}_n = \{L_1, L_2, \ldots, L_n\}$ be a line arrangement (refer to Definition 2.3) in the plane $\mathbb{F}^2$.

- (Existence: Local $2-standard$ consecutive structure) Suppose
  \[ L_{i_1} \rightarrow L_{i_2} \rightarrow L_{i_3} \rightarrow \ldots \rightarrow L_{i_{k-1}} \rightarrow L_{i_k} \rightarrow L_{i_1} \]
is a $k$-gonality (refer to Definition [3.1]) in the arrangement then there exists a local permutation coordinate chart such that the local cycle at infinity (refer to Definitions [5.9, 5.20]) obtained by deleting the subscripts in

\[ \{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_k\} \]

lies in $T_k$ (refer to Definition 5.20).

- (Sufficiency: One Sided Property) If there exists a local coordinate chart such that the intersection points corresponding to simple transpositions lie on only one half side for each of the lines $L_{ij} : j = 1, 2, \ldots, k$ then the lines

\[ L_i \rightarrow L_{i_2} \rightarrow L_{i_3} \rightarrow \ldots \rightarrow L_{i_{k-1}} \rightarrow L_{i_k} \rightarrow L_i \]

form a $k$-gonality and if the local cycle chart respects the slope property (refer to Definition 5.25) (we also say that the chart preserves the orientation here) then this $k$-gonality is given in this anti-clockwise cyclic manner.

The second main Theorem 1.2 is regarding a representation of a line arrangement isomorphically, (refer to Definition 4.4), by some set of lines forming a line arrangement with a given set of distinct slopes, of same cardinality, which is useful to pick an element in the same isomorphism class by fixing a finite set of slopes. The representation theorem is proved after the proof of Lemma 5.31. Now we state the theorem.

**Theorem 1.2 (Representation Theorem).**

Let $\mathbb{F}$ be a field which has the 1–ad structure. Let

\[ \{m_1, m_2, \ldots, m_n\} \subset \mathbb{F} \cup \{\infty\} \]

be a set of $n$ distinct slopes. In any isomorphism class (refer to Definition 4.4) of line arrangement $\mathcal{L}_n^m$, there exists a set of lines which represents exactly this slope set.

1.2. Structure of the paper. Here we mention about the structure of this paper by mentioning about the results proved in various sections.

In Section 2 we define the features of a line arrangement over a field with 1–ad structure and prove Lemma 2.7 which counts the number of sets of strict inequalities which have solutions over the same field arising out of all possibilities in a Venn-Diagram.

In Section 3 we define the gonality structures (refer to Definition 3.1) and two possible orders (refer to Definition 3.2) the slope order and the order of intersection points on a line induced by other lines. The slope order is used for defining 2–standard consecutive structures which respect slope property (refer to Definitions 5.11, 5.25) and the order of intersection points is used in proving isomorphism Theorem 4.6 for line arrangements.

In Section 4 we define line arrangement collineation isomorphism (refer to Definition 4.2) and isomorphism between two line arrangements (refer to Definition 4.4) to prove the isomorphism Theorem 4.6 using the order of intersection points on a line with the remaining lines.

Section 5 is the important section of this article. In this section we define the important structures associated to a line arrangement and then describe the gonality structures by describing the combinatorial structure of the cycle at infinity when a certain type of gonality exists and also give a sufficient criterion as to when a certain subset of lines form a gonality. In this section we prove our first main Theorem 1.1, the local analogue after proving the global version in Theorems 5.14, 5.17, 5.19. We also describe in Theorems 5.22, 5.24 the possible gonalities present when there is global cyclicity (refer to Definition 5.7) and also give a count of them. This is done by identifying certain topological, geometric and combinatorial features of line arrangement when there is global cyclicity.

In the same section we describe the graph of isomorphism classes in Theorems 5.30, 1.2 and in Note 5.32. Later in this section we describe quadrilateral structures as unique nook
point structures and prove that any bijection of two line arrangements which is identity on subscripts where the lines are indexed with increasing angles or slopes in the order $0 \rightarrow -\infty \rightarrow \infty \rightarrow 0$ preserves the nook points of quadrilateral substructures if and only if it is actually an isomorphism of line arrangements. At the end of this section we also give a list invariants for a line arrangement found.

In the final Section 6 we define Line-Folds (refer to Definition 6.1) and prove a theorem on counting the number of regions formed by the complement of the zero set of a polynomial in two variables which factorizes completely into linear factors. Later in this action we ask some open questions regarding line-folds.

2. Line Arrangements over Fields with $1 - ad$ Structure

We begin the section with the theme of Venn diagrams on how lines divide the plane into various regions over a field with $1 - ad$ structure.

2.1. Theme of Venn Diagrams, Regions and $1 - ad$ Structure on the Field. The theme of Venn diagrams is a very well known combinatorial way of partitioning an union of $n$-sets and its complement into $2^n$ sets. We do have a partition of the plane associated to $n$-lines in a plane. We define below in the next few sections more precisely this partition by seeking solutions to a set of $n$-inequalities.

2.1.1. $1 - ad$ Structure on the Field and $1 - ad$ Structured Subsets of $F$ and the Plane $F^2$. First we define a $1 - ad$ structure on a field $F$.

**Definition 2.1.** Let $(F, \leq)$ be a totally ordered field. We say $F$ has a $1 - ad$ structure if in addition the total order satisfies the following properties.

- If $x, y, z \in F$ then $x \leq y \Rightarrow x + z \leq y + z$.
- If $x, y \in F$ then $x \geq 0, y \geq 0 \Rightarrow xy \geq 0$.

We say a set $S \subset F$ is a $1 - ad$ structured subset if there exists an element $a \in S$ such that $x \in S \iff x \geq a$ or $x \in S \iff x \leq a$. We say that in both cases $S$ has a $1 - ad$ structure at the point $a$ with respect to the set $S$. Let $F^+ = \{x \mid x \geq 0\}, F^- = \{x \mid x \leq 0\}$.

We say a set in $S \subset F^2$ has a $1 - ad$ structure of dimension one if there exists points $v, w \in F^2$ such that $S = \{v + tw \mid t \geq 0, t \in F\}$. We say a set $S \subset F^2$ is a set with $1 - ad$ structure of dimension 2 if there exists an affine functional $f : F^2 \rightarrow F$ such that $S = \{(x, y) \in F^2 \mid f(x, y) \in F^+\}$. We say that the set $S$ has $1 - ad$ structure at each point of $(x, y) \in S$ such that $f(x, y) = 0$. Also see the below note.

**Note 2.2.**

- In linear programming problems, sets with $1 - ad$ structures naturally arise. Also the definition of the region $R$ given below happens to be the intersection of inverse images of $1 - ad$ structured subsets of the field $F$ with $1 - ad$ structure at the origin.
- In the case of manifolds (Euclidean Spaces) over reals, the $1 - ad$ structured subsets of dimension $n$ are the half spaces of dimension $n$.
- “$ad$” stands for adjacency. $1 - ad$ means there is only one side. For example locally the point $a \in R$ in the interval $[a, \infty] \subset R$ has only one side.
- It is possible to develop a notion (not in this article) called $(p - 1) - ad$ structure for a finite field $\mathbb{F}_p$ with $p$ points. Note here we say the origin $0 \in \mathbb{F}_p$ as $p - 1$ adjacent sides. Each side of 0 contains just one other point.
2.1.2. **Definition of Region Using the 1−ad Structure.** We start this section with a definition.

**Definition 2.3** (Lines in Generic Position in the Plane $\mathbb{F}^2$ or Line Arrangement).
Let $\mathbb{F}$ be a field with 1−ad structure. We say a finite set $\mathcal{L}_n^\mathbb{F} = \{L_1, L_2, \ldots, L_n\}$ of lines in $\mathbb{F}^2$ are in generic position or forms a line arrangement if the following two conditions hold.

1. No two lines are parallel.
2. No three lines are concurrent.

In this case we say that $\mathcal{L}_n^\mathbb{F}$ is a line arrangement. We denote the line arrangement by $\mathcal{L}_n^\mathbb{F}$ if the field $\mathbb{F} = \mathbb{R}$.

The most general theme is that of Venn Diagrams. A Venn Diagram on $n$−sets gives rise to $2^n$−disjoint sets. In the case of a line arrangement $\mathcal{L}_n^\mathbb{F}$ each line $L : ax + by = c, a, b, c \in \mathbb{F}$ gives rise to two “regions” given by

$$R_1 = \{(x, y) \mid ax + by \leq c\}, R_2 = \{(x, y) \mid ax + by \geq c\}$$

both of which just include the line $L$ in common.

**Definition 2.4** (Definition of a Region (a Polygonal Region)).
Let $\mathcal{L}_n^\mathbb{F}$ be a line arrangement. Suppose an equation for $L_i$ is given by $a_i x + b_i y = c_i$ with $a_i, b_i, c_i \in \mathbb{F}$.

A region $R$ is defined to be a set of solutions to these inequalities

$$\{(x, y) \in \mathbb{F}^2 \mid a_i x + b_i y \leq, \geq c_i\}$$

A region $R$ has non-empty interior if there exists $(x, y) \in R$ which satisfies strict inequalities.

**Definition 2.5** (Definition of Bounded/Unbounded Regions).
We say the region $R$ is unbounded if there exists $v, w \in R$ such that either $\{v + t(w - v) \mid t \geq 0\} \subset R$ or $\{v + t(w - v) \mid t \leq 0\} \subset R$. Otherwise we say the region $R$ is bounded.

**Note 2.6.** Out of the $2^n$−possibilities (exponential in $n$) of sets of inequalities, in the case of line arrangements we will see in the next lemma the sets that have solutions in $x, y$ over the field $\mathbb{F}$ are a sparse sub-collection of possibilities of inequalities having polynomial $(O(n^2))$−cardinality.

We mention the following two lemmas [2.7,2.9] without proof as they are straight forward.

**Lemma 2.7** (Count of Number of Regions). Let $\mathbb{F}$ be a field with 1−ad structure. Let $\mathcal{L}_n^\mathbb{F}$ be a line arrangement. Then we have $2n$ unbounded regions, $\binom{n-1}{2}$ bounded regions and $\binom{n+1}{2} + 1$ total number of regions. In particular a set of lines generated by $n$−points in the plane has at most

$$\frac{n^4 - 2n^3 + 3n^2 - 2n + 8}{8}$$

regions.

Figure[1] represents intersection regions at an intersection point over a field which has 1−ad structure.

**Definition 2.8** (Definition of Crossing Number).
Let $\mathbb{F}$ be a field with 1−ad structure. Let $\mathcal{L}_n^\mathbb{F}$ be a line arrangement. Let $R_1, R_2$ be two different polygonal regions formed by the line arrangement. Let $L$ be a new line which is generically placed meeting the regions $R_1$ and $R_2$. Then we define the crossing number $C(R_1, R_2)$ between the regions $R_1$ and $R_2$ as the number of intersection vertices of the line.
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Figure 1. The Regions at an Intersection

$L$ between an interior point of $R_1$ and an interior point of $R_2$ with the original given set of lines. This is well defined because of the following lemma.

**Lemma 2.9** (Crossing Number between Two Regions).
Let $\mathbb{F}$ be a field with $1–ad$ structure. Let $L_n^\mathbb{F}$ be a line arrangement. Let $R_1, R_2$ be two different polygonal regions. Then the crossing number $C(R_1, R_2)$ is well defined and it is independent of the choice of the generic line used for the definition.

3. Definition of Gonality Structures and Orders

In this section we introduce gonality structures and orders. First we begin with a definition.

**Definition 3.1** (Definition of a $k$–Gonality).
Let $n \geq 3, k \geq 2, k \leq n$ be positive integers.
Let $L_n^\mathbb{F} = \{L_1, L_2, \ldots, L_n\}$ be a line arrangement. We say a subset $\{L_{i_1}, L_{i_2}, \ldots, L_{i_k}\}$ form a bounded $k$–gonality if there exists a bounded region $R$ and end vertices $v_{i_j}, w_{i_j} \in L_{i_j} \subset \mathbb{F}^2, 1 \leq i \leq n$ such that
- $L_i \cap R \neq \emptyset \Rightarrow i = i_j$ for some $1 \leq j \leq k$.
- $L_{i_j} \cap R = [v_{i_j}, w_{i_j}] = \{(1 - t)v_{i_j} + tw_{i_j} \mid 0 \leq t \leq 1\}$.
- The vertices $v_{i_j}, w_{i_j}$ are consecutive i.e. there is no intersection vertex in the line segment $(v_{i_j}, w_{i_j})$.

We say the $k$–gonality is unbounded if there exists an unbounded region $R$ which satisfies the same conditions except for only two of the lines $L_{i_1}, L_{i_2}, 1 \leq j_1 \neq j_2 \leq k$ there is exactly one intersection vertex in each of the sets $L_{i_{j_1}} \cap R, L_{i_{j_2}} \cap R$ as the end vertex which need not be common.

Now below we define two orders the slope order on all the lines and the intersection order with respect to a line on the remaining lines.

**Definition 3.2** (Two Definitions of Orders on the Lines: Slope Order, Order Induced by Lines).
Let $\mathbb{F}$ be a field with $1–ad$ structure. Assigning a linear coordinate system to a line as
$$\{v + tw \mid t \in \mathbb{F}\}$$
we obtain an order of the intersection points and hence on the set of other lines. We obtain two possibilities which are mutually inverses of each other. The slope order is defined as
usual using the total order on \( \mathbb{F} \) and using positive elements and negative elements in \( \mathbb{F} \) corresponding to positive and negative slopes. Also the vertical line corresponds to infinite slope lies in between positive slopes and negative slopes as first positives and then negatives in accordance with angles.

\[ +0 \rightarrow \infty \rightarrow -\infty \rightarrow -0. \]

4. Isomorphism Theorem of Line Arrangements

In this section we state and prove an isomorphism theorem for line arrangements. We begin with definitions.

**Definition 4.1.** Let \( \mathbb{F} \) be a field. Let \( \mathcal{P}_1, \mathcal{P}_2 \) be two sets of points in the plane. We say a map \( \phi : \mathcal{P}_1 \rightarrow \mathcal{P}_2 \) is a collineation if for any three points \( P_1, P_2, P_3 \) which are collinear the points \( \phi(P_1), \phi(P_2), \phi(P_3) \) are collinear. If in addition the map \( \phi \) is a bijection then we say \( \phi \) is a collineation isomorphism.

**Definition 4.2** (Line Arrangement Collineation Isomorphism).
Let \( \mathbb{F} \) be a field. Let \( (\mathcal{L}^F_n)_1, (\mathcal{L}^F_n)_2 \) be two line arrangements. For \( k = 1, 2 \) let \( \mathcal{P}_k = \{ L_i \cap L_j \mid L_i, L_j \in (\mathcal{L}^F_n)_k, 1 \leq i < j \leq n \} \). We say

\[ T : (\mathcal{L}^F_n)_1 \rightarrow (\mathcal{L}^F_n)_2 \]

is a line arrangement collineation isomorphism if \( T : \mathcal{P}_1 \rightarrow \mathcal{P}_2 \) is bijective and if \( P_1, P_2, P_3 \) are collinear by some line \( L_k \in (\mathcal{L}^F_n)_1 \) then \( T(P_1), T(P_2), T(P_3) \) is collinear by some line \( M_s \in (\mathcal{L}^F_n)_2 \) with \( 1 \leq s, t \leq n \).

**Example 4.3.** A permutation of lines gives rise to a line arrangement collineation automorphism of a line arrangement. For \( n = 3 \) any collineation automorphism is a line arrangement collineation isomorphism as this holds true vacuously.

**Definition 4.4.** Let \( \mathbb{F} \) be a field equipped with \( 1 - \text{ad} \) structure. Let \( (\mathcal{L}^F_n)_1, (\mathcal{L}^F_n)_2 \) be two line arrangements. We say that they are isomorphic if there exists a piece-wise linear automorphism of \( \mathbb{F}^2 \) which takes one line arrangement to another.

**Note 4.5.** A piece-wise linear automorphism which takes one line arrangement \( (\mathcal{L}^F_n)_1 \) to another \( (\mathcal{L}^F_n)_2 \) by taking intersection points to intersection points and lines to lines is not only a line arrangement collineation isomorphism but also takes regions which are \( k \)-gons to corresponding regions which are \( k \)-gons for \( k = 3, 4, \ldots, n \). It also preserves the adjacency of the intersection vertices.

Now we state the isomorphism theorem below and mention its proof.

**Theorem 4.6** (Line Arrangement Isomorphism Theorem).
Let \( \mathbb{F} \) be a field equipped with \( 1 - \text{ad} \) structure. Let \( (\mathcal{L}^F_n)_1, (\mathcal{L}^F_n)_2 \) be two line arrangements. Let

\[ \phi : (\mathcal{L}^F_n)_1 \rightarrow (\mathcal{L}^F_n)_2 \]

be a line arrangement collineation isomorphism such that \( \phi \) preserves the order of intersection points on each line. Then there exists a piece-wise linear automorphism of the plane which takes one line arrangement to the other.

**Proof.** Let \( \phi \) be a line arrangement collineation isomorphism preserving the orders of intersection vertices on each line. Then the following holds.

1. **(Bijection on the set of Lines):** The line arrangement collineation isomorphism induces a bijection on the lines.
2. **(Preserves Adjacency):** If there are \( k \)-intermediate intersection vertices between two intersection vertices on a line then there are also \( k \)-intermediate intersection vertices between the image of the two intersection vertices on the image line.
(3) (Preservation of Central point): This can be derived from the previous step. If there are three points \(P_1, P_2, P_3\) on a line \(L\) such that \(P_2\) is in between \(P_1\) and \(P_3\) then \(\phi(P_2)\) is in between \(\phi(P_1)\) and \(\phi(P_3)\) on \(\phi(L)\).

(4) (Preservation of Sidedness): We have only two sides of any line in the plane. Suppose we have a partition of intersection points

\[ \mathcal{P} = \mathcal{P}_L \coprod \mathcal{P}_{\text{Side } L_1} \coprod \mathcal{P}_{\text{Side } L_2} \]

and

\[ \mathcal{P} = \mathcal{P}_{\phi(L)} \coprod \mathcal{P}_{\text{Side } \phi(L_1)} \coprod \mathcal{P}_{\text{Side } \phi(L_2)} \]

where \(L_1, L_2\) represent either sides of the line \(L\) and \(\phi(L_1), \phi(L_2)\) represent either sides of the line \(\phi(L)\) then we have

- \(\phi(\mathcal{P}_L) = \mathcal{P}_{\phi(L)}\) i.e. the points on the line \(L\) goes to points on the line \(\phi(L)\).
- Either \(\phi(\mathcal{P}_{\text{Side } L_1}) = \mathcal{P}_{\text{Side } \phi(L_1)}\) and \(\phi(\mathcal{P}_{\text{Side } L_2}) = \mathcal{P}_{\text{Side } \phi(L_2)}\).
- Or the other way \(\phi(\mathcal{P}_{\text{Side } L_1}) = \mathcal{P}_{\text{Side } \phi(L_2)}\) and \(\phi(\mathcal{P}_{\text{Side } L_2}) = \mathcal{P}_{\text{Side } \phi(L_1)}\).

Here \(\mathcal{P}_{\text{Side } L_1}, \mathcal{P}_{\text{Side } L_2}\) are the discrete half sides of the discrete line \(\mathcal{P}_L\) Similarly for the line \(\mathcal{P}_{\phi(L)} = \phi(\mathcal{P}_L)\).

(5) (Preservation of the Regions): Regions are mapped to regions as the regions are formed by intersection points of lines as finite intersections of discrete half sides of the lines.

- If a half side of one line intersects with a half side of another line then the corresponding intersection of the half sides in the image is also non-empty.
- More importantly for a bounded region the number of half sides (without repetition) whose intersection is considered is the same as the number of intersection vertices which forms the region which is the intersection set provided we include in the intersection the adjacent intersection vertices of the lines whose half sides are considered.
- For an unbounded region the number of half sides (without repetition) whose intersection is considered is one more than the number of intersection vertices which forms the region which is the intersection set provided we include in the intersection the adjacent intersection vertices of the lines whose half sides are considered.
- The map \(\phi\) preserves all these properties of the regions.

Finally since the regions are convex and these regions occupy as a jigsaw puzzle for the plane, we have, after further similar triangulation, by convex extension, there is a piece-wise linear bijection of the two dimensional plane \(\mathbb{F}^2\) to \(\mathbb{F}^2\). This proves the theorem that the two line arrangements are isomorphic. We note however that this piece-wise linear bijection need not be an orientation preserving map of the plane \(\mathbb{F}^2\).

\[\square\]

5. On some Structures in a Line Arrangement

We begin with a definition of a simplicial map/path, adjacent simplicial map/path, jordan curve in this section.

**Definition 5.1** (Simplicial Map/Path, Adjacent Simplicial Map/Path). Let \((\mathbb{F}, \leq)\) denote a field equipped with a 1–ad structure. Let

\[ \mathcal{L}_n^\mathbb{F} = \{L_1, L_2, \ldots, L_n\} \]

be a line arrangement. Let \(\mathcal{P} = \{L_i \cap L_j \mid 1 \leq i < j \leq n\}\). Let \(X = L_1 \cup L_2 \cup \ldots \cup L_n\). For \(x, y \in \mathbb{F}\) let \([x, y] = \{z \in \mathbb{F} \mid x \leq z \leq y\}\). We say a map \(\sigma : [0, 1] \rightarrow \mathbb{F}\) is simplicial if \(\sigma(t) = (1 - t)\sigma(0) + t\sigma(1)\). We say a map \(\sigma : [0, 1] \rightarrow \mathbb{F}^2\) is simplicial if there exists
v = σ(0), w = σ(1) ∈ ℝ^2 such that σ(t) = (1 − t)v + tw. We say two vertices v, w ∈ P are adjacent if the following holds.

- There exists a subscript 1 ≤ i ≤ n such that v, w ∈ L_i.
- The simplicial map σ : [0, 1] → X given by σ(t) = (1 − t)v + tw has the property that

\[ σ([0, 1]) \cap P = \{v, w\}. \]

We say σ is an adjacent simplicial map if σ(0), σ(1) ∈ P and are adjacent.

Definition 5.2 (Polygonal Jordan Curve).
Let F be a field with 1 − ad structure. Let [0, 1] = \{x ∈ F | 0 ≤ x ≤ 1\}. We say a curve σ : [0, 1] → ℝ^2 is piece-wise simplicial if there exists a finite sequence of vertices \(v_0, v_1, v_2, \ldots, v_n \in ℝ^2\) and field elements \(0 = t_0 < t_1 < t_2 < \ldots < t_n = 1\) such that

\[ σ|_{[t_i, t_{i+1}]}(t) = (1 − t)v_i + tv_{i+1}, 1 ≤ i ≤ n − 1. \]

We say the piece-wise simplicial curve is closed if \(v_n = v_0\). We say the curve is jordan if it is piece-wise simplicial, closed and injective on [0, 1].

We introduce the next two definitions regarding any polygon made up of lines. The usefulness of the first definition lies in identifying convex polygons in a line arrangement.

Definition 5.3 (2-Standard Consecutive Structure). Let F be a field with 1 − ad structure. We say a certain set of slopes

\( (m_1, m_2, \ldots, m_n) ∈ (F \cup \{∞\})^n = (FF_F)^n \)

has a 2-standard consecutive structure if the following occurs.

- \(0 ≤ m_1 < m_2 < \ldots < m_i ≤ ∞\)
- \(m_{i+1} < m_{i+2} < \ldots < m_j ≤ 0\)
- \(0 < m_{j+1} < m_{j+2} < \ldots < m_k ≤ ∞\)
- \(m_{k+1} < m_{k+2} < \ldots < m_n < 0\)

for some \(1 ≤ i < j < k ≤ n\). The last sequence of slopes may be empty. i.e. \(k = n\). If the slopes \(m_i : 1 ≤ i ≤ n\) arise from a line arrangement then the slopes are distinct as any two distinct lines meet. The structure is considered as 2-standard by referring to usual angles instead of slopes.

Theorem 5.4. Let F be a field with 1 − ad structure. Let \(F_F^n\) be a line arrangement. Let T be a closed traversal of distinct vertices except the first and last one by adjacent simplicial paths such that no two consecutive paths cyclically considered lie on the same line and there exists a consecutive set up to cyclic permutation of slopes which satisfy a 2-standard consecutive structure. Then the interior domain of the jordan curve T is a region.

Proof. Using the 2-standard consecutive structure and the fact that no two consecutive paths lie on the same line we have that the interior domain of the jordan curve satisfies that it lies on only one side of each line and hence it is a region. This proves the theorem. This 2-standard consecutive structure is needed for existence of solutions to a set of linear inequalities in a line arrangement by identifying the regions uniquely with respect to the set of inequalities.

In Figure 2 we depict a domain whose boundary simplicial lines of the colored region satisfy all properties of Theorem 5.4 except the 2-standard consecutive structure for slopes. We note that the movement at an intersection point is from one simplicial line to another simplicial line and the domain in the figure is not a region. The cyan colored triangular region is enclosed on all the three sides by green, yellow and orange regions.
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Now we introduce the second definition. This second definition is useful in classifying the cycles at infinity of a line arrangement

$\mathcal{L}_n^F = \{L_1, L_2, \ldots, L_n\}$

which has an $n-$gonality in this anti-clockwise cyclic order

$L_1 \rightarrow L_2 \rightarrow \ldots \rightarrow L_n \rightarrow L_1$

**Definition 5.5 (Opposite Vertex of a Side in a Convex Polygon).**

Let $\mathcal{L}_n^F = \{L_1, L_2, \ldots, L_n\}$ be a line arrangement. Let

$\{L_{i_1} \rightarrow L_{i_2} \rightarrow \ldots \rightarrow L_{i_r} \rightarrow L_{i_1}\}$

be a convex polygon with $r-$sides in this anticlockwise cyclic order. We assume that by rotation (such rotations exist) that $L_{i_1}$ has the least non-negative slope. Suppose the $2-$standard consecutive structure for this polygon is given by

- $0 \leq m_1 < m_2 < \ldots < m_i \leq \infty$
- $m_{i+1} < m_{i+2} < \ldots < m_j < 0$
- $0 < m_{j+1} < m_{j+2} < \ldots < m_k \leq \infty$
- $m_{k+1} < m_{k+2} < \ldots < m_r < 0$

for some $1 \leq i < j < k \leq n$ where $m_j$ is the slope of the line $L_{i_j}$. Then the opposite vertex of $L_{i_1}$ is defined to be $L_{i_j} \cap L_{i_{j+1}}$. We just note that

$m_j < 0 \leq m_1 < m_{j+1}$.

The definition of the opposite vertex for the remaining sides of the polygon is similar using rotations and making the given side as having least non-negative slope with $X-$axis.

**Note 5.6.** Let $\mathbb{F}$ be a field with $1-\text{ad structure}$. Hence $\mathbb{F}$ is of characteristic zero with $\mathbb{Q} \subset \mathbb{F}$. Let

$\mathcal{TAN} = \{m \in \mathbb{F}^+ | 1 + m^2 = \Box\}$.

Given a slope $m > 0$ there exists an $m_1 \in \mathcal{TAN}$ such that $0 \neq m_1 < \frac{m}{2}$.

5.1. **On Global Cyclicity.** We introduce this section with the main definition.

**Definition 5.7 (Existence of Global Cyclicity).**

Let $\mathcal{L}_n^F = \{L_1, L_2, \ldots, L_n\}$ be a line arrangement. We say that there exists global cyclicity if all the lines give rise to an $n-$gonality (refer to Definition 3.1).
Now we give a criterion as to when a global cyclicity exists in a line arrangement. Before we state the theorem we need another definition. In general for an arbitrary field $F$ with $1 - ad$ structure the equation

$$x^2 = a > 0$$

need not have solutions. For example we consider the field of rationals. So taking the intersection of lines with a circle of large radius is not preferred to define the cycle at infinity. Instead we have solutions for any two linear equations whose lines are not parallel.

**Definition 5.8 (Orientation of the plane).**

We further introduce the notion of quadrants and hence the standard forms of lines.

- **First Quadrant:** \( \{(x, y) \in F^2 \mid x > 0, y > 0\} \).
- **Second Quadrant:** \( \{(x, y) \in F^2 \mid x < 0, y > 0\} \).
- **Third Quadrant:** \( \{(x, y) \in F^2 \mid x < 0, y < 0\} \).
- **Fourth Quadrant:** \( \{(x, y) \in F^2 \mid x > 0, y < 0\} \).

An orientation of the plane (more precisely an orientation at the origin) with respect to the quadrants is given by

\[ I \rightarrow II \rightarrow III \rightarrow IV \rightarrow I \]

We make an important observation that any line not parallel to any one of the axes meets only three quadrants and misses exactly one of them. Their equations are given as follows in standard forms with orientations.

- \( IV, I, II : \frac{x}{a} + \frac{y}{b} = 1, a > 0, b > 0 \)
- \( II, III, IV : \frac{x}{a} + \frac{y}{b} = 1, a < 0, b < 0 \)
- \( I, II, III : \frac{x}{a} + \frac{y}{b} = 1, a < 0, b > 0 \)
- \( III, IV, I : \frac{x}{a} + \frac{y}{b} = 1, a > 0, b < 0 \)

Now we fix orientations for these lines by saying that the line

\[ ax + by = c > 0, abc \neq 0 \]

is oriented such that the origin is on the left side of the line. If \( ab = 0 \) then the orientation can be coherently induced as well on these lines with this definition provided \( c \neq 0 \) given as \( I, II \) or \( II, III \) or \( III, IV \) or \( IV, I \).

**Definition 5.9 (Cycle (Local Cycle) at infinity: An Element of the Symmetric Group).**

Let $L_n^F$ be a line arrangement. Let $L$ be any line with slope different from that of the lines of the arrangement and not passing through origin. We assume that the line $L$ is generic to obtain the line arrangement $L_n^F \cup \{L\}$ and right side (non origin side) of the discrete half sides of $L$ is empty. Then the cycle at infinity is defined as the sequence of subscripts of the lines, a permutation

\[ (i_1i_2\ldots i_n) \in S_n \]

corresponding to the intersections of the lines in $L_n^F$ with $L$ in the direction of the orientation of $L$.

A local cycle with respect to a subset $A \subset L_n^F$ is the cycle at infinity obtained by dropping subscripts of the lines not in the set $A$. Also refer to Definition 5.26.

Now we introduce a structure on a permutation as follows.
Definition 5.10. We say an \( n \)-cycle \( (a_1 = 1, a_2, \ldots, a_n) \) is an \( i \)-standard cycle if there exists a way to write the integers \( a_i : i = 1, \ldots, n \) as \( i \) sequences of inequalities as follows:

\[
\begin{align*}
a_{11} &< a_{12} < \ldots < a_{1j_1} \\
a_{21} &< a_{22} < \ldots < a_{2j_2} \\
a_{31} &< a_{32} < \ldots < a_{3j_3} \\
&\vdots \\
a_{i1} &< a_{i2} < \ldots < a_{ij_i}
\end{align*}
\]

where \( \{a_{st} \mid 1 \leq s \leq i, 1 \leq t \leq j_s\} = \{a_1, a_2, \ldots, a_n\} = \{1, 2, \ldots, n\}, j_1 + j_2 + \ldots + j_i = n \) and \( i \) is minimal .i.e. there exists no smaller integer with such property and \( a_{s(t+1)} \) occurs to the right of \( a_{st} \) for every \( 1 \leq s \leq i \) and \( 1 \leq t \leq j_s - 1 \) in this cycle arrangement \( (a_1 = 1, a_2, \ldots, a_n) \).

Definition 5.11. We say an \( n \)-cycle \( (a_1 = 1, a_2, \ldots, a_n) \) is a consecutive \( i \)-standard cycle or a \( i \)-standard consecutive cycle if we have

\[
a_{s1} < a_{s2} < \ldots < a_{sj_s}
\]

and in addition \( a_{st} = a_{s1} + (t - 1), 1 \leq t \leq j_s, 1 \leq s \leq i \) where \( \{a_{st} \mid 1 \leq s \leq i, 1 \leq t \leq j_s\} = \{a_1, a_2, \ldots, a_n\} = \{1, 2, \ldots, n\}, j_1 + j_2 + \ldots + j_s = n \) and \( a_{s(t+1)} \) occurs to the right of \( a_{st} \) for every \( s = 1, \ldots, i \) and \( 1 \leq t \leq j_s - 1 \) in this cycle arrangement \( (a_1 = 1, a_2, \ldots, a_n) \) and \( i \) is minimal .i.e. there exists no smaller integer with such property.

Example 5.12. For example if we consider the \( 5 \)-cycle \( (1, 4, 5, 2, 3) \) it is a \( 2 \)-standard consecutive cycle. However it has the following two \( 2 \)-standard structures.

- \( 1 < 4 < 5, 2 < 3 \) (not consecutive).
- \( 1 < 2 < 3, 4 < 5 \) (consecutive).

Now we prove a lemma on the existence and uniqueness of the \( i \)-standard consecutive structure on an \( n \)-cycle.

Lemma 5.13 (Existence and Uniqueness of the Consecutive \( i \)-Standard Structure on an \( n \)-cycle).
The consecutive \( i \)-standard structure exists on an \( n \)-cycle and is unique.

Proof. We prove this by induction on \( i, n \) as follows. If \( i = n = 1 \) then there is nothing to prove. The position of the element \( n \) is uniquely determined as it should appear in one of them at the end and \( (n - 1) \) appears before \( n \) if \( (n - 1) \) appears before \( n \) in the \( n \)-cycle and appears as a single element of standardness if \( (n - 1) \) appears after \( n \). Now we remove \( n \) from the cycle. The remaining cycle is either \( i \)-standard on \( (n - 1) \)-elements or \( (i - 1) \) standard on \( (n - 1) \)-elements. This proves the lemma.

We can actually build this structure in an unique way for the given \( n \)-cycle as follows. Write 1 first. Then write 1 < 2 if it appears later or write 2 as single element of standardness if it appears before. Then write 3 next to 2 if it appears after 2 or write as a single element of standardness if it appears before 2 and so on. \[ \blacksquare \]

Now we state the theorem.

Theorem 5.14. Let \( L_n^F = \{L_1, L_2, \ldots, L_n\} \) be a line arrangement which gives rise to an \( n \)-gonality in this anticlockwise manner

\[
L_1 \rightarrow L_2 \rightarrow \ldots \rightarrow L_n \rightarrow L_1
\]

then the cycle at infinity has a \( 2 \)-standard consecutive structure.
Proof. Assume by rotation that $L_1$ has the least non-negative slope. Suppose

$L_i \cap L_{i+1}$

is the opposite vertex for $L_1$. Then on the $n-$ cycle at infinity we have the following unique 2–standard consecutive structure.

- $1 < 2 < \ldots < i$
- $i + 1 < i + 2 < \ldots < n$

This proves the theorem. ■

We mention a note below.

Note 5.15. The converse need not be true. Consider a line arrangement $\mathcal{L}_n^\mathbb{P}$ with $n = 5$ where there exist a 2–standard consecutive structure on the cycle at infinity however the arrangement does not have a pentagonality. After a parallel translation of the lines we have a pentagonality structure.

However we have the following theorem for the converse. We need a definition.

Definition 5.16 (Simple Transposition).

Let $S_n$ be a symmetric group on $n-$letters $\{1, 2, \ldots, n\}$. We say a transposition is simple if it is of the form $(i(i+1))$ where $1 \leq i \leq n - 1$ or the transposition is $(n1)$ which is also denoted by $n(n+1) \equiv 1 \mod n$.

Now we state the theorem about existence of global cyclicity.

Theorem 5.17 (Existence of Global Cyclicity).

Let $\mathcal{L}_n^\mathbb{P}$ be a line arrangement. For any of the lines $L_i : i = 1, 2, \ldots, n$ the intersection vertices corresponding to simple transpositions apart from $((i-1)i), (i(i+1))$ lie on only one discrete half side of $L_i$. Then there exists an $n-$gonality giving rise to global cyclicity.

Proof. We need to prove that the vertices $((i-1)i), (i(i+1))$ on the line $L_i$ are adjacent. Suppose we have $L_i \cap L_j$ as an intermediate vertex then the vertex $(i(i+1))$ and the vertex $((i-1)i)$ are on either side of the line $L_j$ which is a contradiction. Now the proof is immediate as the vertices $\{L_i \cap L_{i+1} \mid 1 \leq i \leq n\}$ form a region. However the existence of 2–standard consecutive structure does not guarantee the clockwise or anticlockwise cyclicity of the lines $L_i : i = 1, 2, \ldots, n$. ■

Example 5.18. For $n = 4$, if the cycle at infinity is $(1324)$ then we have both possibilities of $4-$gonalities of lines

$L_1 \rightarrow L_2 \rightarrow L_3 \rightarrow L_4 \rightarrow L_1, L_1 \rightarrow L_4 \rightarrow L_3 \rightarrow L_2 \rightarrow L_1.$

Now we prove a theorem regarding the global cyclicity which says that the opposite vertices of the line segments of the global gonality and the cycle at infinity which respects the slope property (refer to Definition 5.25) can be obtained from one other and each one of them determines the line arrangement up to isomorphism.

Theorem 5.19 (Cycle at Infinity and the Opposite Vertices of sides of the Global Gonality).

Let

$\mathcal{L}_n^\mathbb{P} = \{L_1 \rightarrow L_2 \rightarrow \ldots \rightarrow L_n \rightarrow L_1\}$

be a line arrangement in a plane giving rise to an $n-$gonality in this anticlockwise cyclic order. Then the cycle at infinity having the 2–standard structure which respects the slope property determines uniquely the opposite vertex for any side in the $n-$gon. Conversely if we
know the opposite vertex for any side in this $n$–gon then the cycle at infinity is determined uniquely and its $2$–standard structure respects the slope property (refer to Definition [5.25]).

Proof. Respecting the slope property is a given, once the cycle at infinity is determined. The cycle at infinity determines uniquely the opposite vertex for any side in the $n$–gon as we can use the $2$–standard structures which respects the slope property on each of the cyclically permuted conjugate cycles of the cycle at infinity.

Conversely if we know the opposite vertex for any side we need to determine the slope ordering of the lines $L_1, L_2, \ldots, L_n$. First we can assume without loss of generality that $L_1$ has the least non-negative slope with respect to $X$–axis. This slope order gets determined because of the following reason. If we know the opposite vertex for a side on a line $L$ of the arrangement then the sequence of intersections gets determined including the endpoints using the opposite vertex on the line $L$. Hence the complete gonality structure gets determined using isomorphism Theorem 4.6. Hence this determines the cycle at infinity as well.

Definition 5.20 (2-standard consecutive $n$–cycles).
Let $T_n \subset S_n$ be the set of $2$–standard consecutive $n$–cycles in $S_n$.

Lemma 5.21.
• We have $$\#(T_n) = 2^{n-1} - n.$$ • The number of non-isomorphic global gonality structures arising from a line arrangement $L_n^F$ is given by $$\#(\mathbb{Z}/n\mathbb{Z} \circ T_n).$$

Proof. This follows by counting the cardinality of $T_n$ and the isomorphism class does not change under cyclic renumbering of the subscripts of the lines.

5.1.1. On the Gonality Function when there is Global Cyclicity. Now we prove below a theorems which give the gonality functions of the bounded regions and the unbounded regions when there is global cyclicity.

Theorem 5.22 (A Theorem on Bounded Gonalities).
Let $n \geq 3$ be a positive integer. Let $L_n^F$ be a line arrangement which give rise to a global $n$–gonality say in the anticlockwise order $$L_1 \rightarrow L_2 \rightarrow \ldots \rightarrow L_n \rightarrow L_1.$$ Let $k$ denote the number of lines $L_i : i = 1, 2, \ldots, n$ such that the following occurs. For each line $L_i$ the vertices corresponding to simple transpositions lie on one discrete half side of $L_i$ and the vertex $L_{i-1} \cap L_{i+1}$ lies on the other side of $L_i$. Then we have
• $k$–triangular regions.
• $1$–bounded $n$–gonality.
• $\binom{n-1}{2} - k - 1$ quadrilaterals.

Proof. This theorem follows because of the observation that the triangles if they exist are adjacent to the global gonality and the rest of the bounded regions are all quadrilaterals apart from the $n$–gonality.

Definition 5.23 (The inner coordinates of a point on a line).
Let $L_n^F$ be a line arrangement. Let $P = L_i \cap L_j$. Then the point $P$ acquires a pair of inner coordinates of the form $\pm a \mod n, \pm b \mod n$ depending on the position of the point on the lines $L_i, L_j$. These are called inner coordinates of the point. It is well defined upto a sign modulo $n$ and if the lines are oriented then it is a well defined pair of integers in the set $\{1, 2, \ldots, n-1\} \times \{1, 2, \ldots, n-1\}$. We say a point $P$ is an outer point if one of the inner
coordinates of the pair is $\pm 1 \mod n$. Otherwise it is called a non-outer point. We say a point $P$ is an extreme point if the inner coordinates is given by $(\pm 1 \mod n, \pm 1 \mod n)$.

**Theorem 5.24** (A Theorem on Unbounded Gonalities).
Let $n \geq 3$ be a positive integer. Let $L_2^F$ be a line arrangement which give rise to a global $n-$gonality. Let $R_i : 1 \leq i \leq \binom{n-1}{2}$ be the set of bounded regions. Since the plane is coherently orientable at every point let $T$ denote a cyclic traversal of the boundary line segments of the domain $\bigcup_{i=1}^{\binom{n-1}{2}} R_i$ where all the regions are coherently oriented. Let $r$ denote the number of extreme points. Let $k$ denote the non-outer points of cyclic traversal $T$ of the boundary line segments. Then we have

- $r-$unbounded 2–gonalities (One extreme point).
- $k-$unbounded 4–gonalities (One non-outer point in between two outer points on the traversal $T$).
- $(2n - r - k)-$unbounded 3–gonalities (Two adjacent outer points on the traversal $T$).
- There are no unbounded gonality higher than unbounded 4–gonalities.

**Proof.** This theorem follows from the observation that unbounded 2–gonalities are formed by extreme points. The unbounded 4–gonalities are formed due to non-outer points on the cyclic traversal $T$. If $L_{a_i} \cap L_{b_i}$ and $L_{a_{i+1}} \cap L_{b_{i+1}}$ are two consecutive extreme points in the clock-wise order then a non-outer point if it exists i.e. $a_i \neq b_{i+1}$ in between them is given by $L_{a_i} \cap L_{b_{i+1}}$. Remaining points will be outer points on these two lines $L_{a_i}, L_{b_{i+1}}$. This observation also proves that the remaining unbounded gonality are all unbounded 3–gonality and there are no higher $> 4$ unbounded gonality.

5.2. **On Local Gonality Structures and Local Gonality Cycles at Infinity.** As a continuation from the previous Example 5.18 we prove a theorem of existence of local gonality structures with respect to the 2–standard structures on local cycles at infinity in a local permutation chart in this section. First we need a definition.

**Definition 5.25** (Slope Property).
We say that the 2–standard consecutive structure on a permutation $n-$cycle associated to a line arrangement $L_2^F$ respects the slope property if the following occurs. If the 2-standard consecutive structure is given by

- $1 < 2 < 3 < \ldots < j$.
- $j + 1 < j + 2 < \ldots < n$.

then we have modulo a rotation of the plane $\mathbb{R}^2$

- $0 \leq m_1 < m_2 < \ldots < m_i \leq \infty$
- $m_{i+1} < m_{i+2} < \ldots < m_j \leq 0$
- $0 < m_{j+1} < m_{j+2} < \ldots < m_k \leq \infty$
- $m_{k+1} < m_{k+2} < \ldots < m_n < 0$

with $m_j < 0 \leq m_1 < m_{j+1}$. 


In Example 5.18 if in addition the 2–standard consecutive structure respects the slope property then the clockwise 4–gonality can be eliminated.

In the definition below we introduce the concept of a local cycle at infinity and local permutation (coordinate) charts for local cycles at infinity.

**Definition 5.26** (Definition of a Local Cycle, Permutation (Coordinate) Charts for the Local Cycles at Infinity).

Let \( \sigma \in S_n \) be an \( n \)-cycle. Let \( A^c \subset \{1, 2, \ldots, n\} \) be a subset of cardinality \( n - k \). By dropping the letters in the subset \( A^c \) in the cycle \( \sigma \) we obtain a cycle \( \sigma_k \), an element in the symmetric group on \( k \)-letters. This is called a local cycle at infinity on \( k \)-letters. If \( A = \{i_1, i_2, \ldots, i_k\} \) in some order. Let \( \tau = (i_1 \ i_2 \ldots \ i_k \ 1 \ 2 \ldots \ k) \). Then a cycle

\[ \tau \sigma_k \tau^{-1} \]

is the local cycle of \( \sigma_k \) at infinity in a local permutation (coordinate) chart. Usually one considers the local cycles where the order of the numbers \( i_1, i_2, \ldots, i_k \) to construct the cycle \( \tau \) is such that there exists a local \( k \)-gonality

\[ L_{i_1} \rightarrow L_{i_2} \rightarrow \ldots \rightarrow L_{i_k} \rightarrow L_{i_1} \]

in this anticlockwise order among just these \( k \)-lines.

Now we prove the first main Theorem 1.1 a more general theorem of local gonality structures.

**Proof.** The proof is short after having proved Theorems 5.14, 5.17. The first main theorem follows from Theorems 5.14, 5.17 by applying locally. ■

**Note 5.27** (Global and Local Gonality Principle).

Once global cyclicity structure is known then in a general situation these “global structures” are present as “local structures” among the lines of their respective structures and these local structures embed in a certain way in the general situation.

### 5.3. Finite Graph of Isomorphism Classes.

We prove in this section a representation Theorem 1.2 for the isomorphism classes. First we observe that for any given two line arrangements a bijection of the lines give rise to a line arrangement collineation isomorphism. So these types of collineation maps are not useful to distinguish the isomorphism classes of line arrangements.

We begin with the definition of an Elementary Collineation Transformation (ECT) of two line arrangements.

**Definition 5.28** (Definition of ECT).

Let \((L_n^F)_1, (L_n^F)_2\) be two line arrangements in the plane such that

\[ (L_n^F)_1 \cap (L_n^F)_2 = \{L_1, L_2, \ldots, L_{k-1}, L_{k+1}, \ldots, L_n\} \]

In addition the \( k^{th} \)-line is given by

\[ (L_k)_1 : ax + by = c_1 \in (L_n^F)_1, (L_k)_2 : ax + by = c_2 \in (L_n^F)_2 \] with \( c_1 < c_2 \).

Consider the three lines

\[ L_i, L_j, (L_k)_1 \in (L_n^F)_1 \]

with intersection vertices \( L_j \cap L_i, (L_k)_1 \cap L_j, L_i \cap (L_k)_1 \)

in the anticlockwise order
and the three lines
\[ L_i, L_j, (L_k)_2 \in (L^F_n)_2 \]
with intersection vertices \( L_i \cap L_j, (L_k)_2 \cap L_i, L_j \cap (L_k)_2 \)
in the clockwise order

We assume that in the space \( \mathbb{F}^2 \) the parallel strip
\[ \{(x, y) \mid c_1 < ax + by < c_2\} \]
contains only one intersection point \( L_i \cap L_j \) of both the line arrangements. Then the ECT
\[ E_{ijk} : (L^F_n)_1 \rightarrow (L^F_n)_2 \]
is defined to be the transformation induced by mapping the lines \( L_i \rightarrow L_i \) for \( 1 \leq i \leq n, i \neq k \) and by mapping the line \( (L_k)_1 \) to \( (L_k)_2 \) on the intersection points.

**Lemma 5.29** (Swap of the Inner Coordinates, Changes in Gonalities).
With the notations as in Definition 5.28 if the points and inner coordinates in the anticlockwise order are given by
\[
L_j \cap L_i = (B_j = Q_j - 1, A_i = X_i + 1),
(L_k)_1 \cap L_j = ((P_k)_1 = (Y_k)_1 - 1, Q_j = B_j + 1),
L_i \cap (L_k)_1 = (X_i = A_i - 1, (Y_k)_1 = (P_k)_1 + 1)
\]
then after applying \( E_{ijk} \) we obtain the following points with inner coordinates in the clockwise order (so that the orientation of the lines are unchanged) as
\[
L_i \cap L_j = (X_i = A_i - 1, Q_j = B_j + 1),
(L_k)_2 \cap L_i = ((P_k)_1 = (Y_k)_1 - 1, A_i = X_i + 1),
L_j \cap (L_k)_2 = (B_j = Q_j - 1, (Y_k)_1 = (P_k)_1 + 1)
\]
The gonalities change in this manner from
\[
(n_1, n_2, n_3, n_4, n_5, n_6) \rightarrow (n_1 \pm 1, n_2 \mp 1, n_3 \pm 1, n_4 \mp 1, n_5 \pm 1, n_6 \mp 1)
\]
with bounded gonalities remain bounded and unbounded gonalities remain unbounded. The triangle \( L_i L_j (L_k)_1 \) becomes \( (L_k)_2 L_j L_i \).

**Proof.** The proof is immediate. \( \blacksquare \)

Now we prove the following theorem on line arrangements with global cyclicity which differ by parallel translations.

**Theorem 5.30** (Transitivity on the \( n \)-cycles which have 2-standard consecutive structures by parallel translations).
Let \( L^F_n \) be a line arrangement which gives rise to an \( n \)-gonality with a cycle \( \tau \) at infinity having a 2-standard consecutive structure which respects slope property with the anticlockwise \( n \)-gonality
\[ L_1 \rightarrow L_2 \rightarrow \ldots \rightarrow L_n \rightarrow L_1. \]
Let \( \sigma \) be another \( n \)-cycle having a 2-standard consecutive structure. Assume that \( L_1 \) has the least non-negative slope. Then we can move the lines
\[ L_2, L_3, \ldots, L_n \]
by parallel translations into another line arrangement which also gives rise to an \( n \)-gonality which after a permutation of subscripts
\[ 2, 3, \ldots, n \]
has the cycle at infinity $\sigma$ having a 2-standard consecutive structure which respects slope property with the anticlockwise $n$-gonality

$$L_1 \rightarrow L_2 \rightarrow \ldots \rightarrow L_n \rightarrow L_1.$$  

Proof. Consider the $n$-cycles $\tau$ and $\sigma$ which have the 2-standard consecutive structures. We claim that there exists a parallel translation of any set of lines which gives rise to an $n$-gonality which gives any $n$-cycle with a 2-standard consecutive structure where the $n$-gon is given by

$$L_1 \rightarrow L_2 \rightarrow \ldots \rightarrow L_n \rightarrow L_1$$

in this anticlockwise order. To observe this fact first we consider over the field of reals in which we consider arbitrary $n$ distinct angles in $[0, \pi)$ in the increasing order corresponding to $n$-lines in the real plane.

$$0 = \theta_1 < \theta_2 < \ldots < \theta_n < \pi$$

It does not matter what the exact angles are, however what matters is the order of the angles with respect to subscripts. Now the possibilities of the $n$-gons are precisely all the possibilities which satisfy the following.

- $0 = \alpha_1 < \alpha_2 < \ldots < \alpha_i < \pi$
- $0 < \alpha_{i+1} < \alpha_{i+2} < \ldots < \alpha_n < \pi$.

where $\{\alpha_1 = 0, \alpha_2, \ldots, \alpha_n\} = \{\theta_1 = 0, \theta_2, \ldots, \theta_n\}$. The lines with slopes $\alpha_i : i = 1, 2, \ldots, n$ gives an anticlockwise $n$-gon

$$L_1 \rightarrow L_2 \rightarrow \ldots \rightarrow L_n \rightarrow L_1$$

where $L_i$ makes an angle $\alpha_i$ with respect to $X-$axis. The permutation $\lambda$ of the subscripts corresponding to $\theta_i = \alpha_{\lambda(i)}$ are precisely those $\lambda$ which have the 2-standard consecutive structures. This observation can be extended to any field $F$ which has a 1-ad structure. This proves the theorem.

Lemma 5.31 (ECT Application Lemma).

Let $L_n^F$ be a line arrangement. Let $E_{ijk}$ be an elementary collineation transformation (ECT). Suppose $L_iL_jL_k$ is a triangular region of the line arrangement with vertices $L_i \cap L_j, L_k \cap L_i, L_j \cap L_k$ oriented anticlockwise. Then by moving all the other lines $L_t$ for $t \neq i, j, k$ parallely away from the fixed triangle $L_iL_jL_k$ we can make the ECT $E_{ijk}$ applicable with the required condition on the parallel strip that arises from $L_k$.

Proof. This is straight forward.

Now we prove the Representation Theorem 1.2.

Proof. First we observe that if we have two line arrangements with global cyclicity and has the same cycle at infinity then they are isomorphic. Now using the previous Theorem 5.30 we can move from one cycle at infinity to another using parallel translations and also by applying elementary collineation transformations and their inverses using Lemma 5.31. So from an arbitrary line arrangement which may not have a global gonality we apply ECT’s to obtain an arrangement which has a global gonality. This proves the theorem.

Note 5.32. Now we have a finite graph with possibly multiple edges and loop edges on isomorphism classes of line arrangements $L_n^F$. The vertices of this graph are the isomorphism classes and the edges are denoted by elementary collineation transformations which can be made applicable for an isomorphism class to go to another isomorphism class.
5.4. On Quadrilateral Structures as Unique Nook Point Structures, Automorphism group of the Quadrilateral Structure. We begin this section with definition of the quadrilateral structure and the nook point.

**Definition 5.33** (Quadrilateral Structure, Nook Point, Extreme Nook Point, End Points, Central Pair).

A quadrilateral structure \( Q \) is defined to be any four line arrangement \( L_n \) in the plane. The nook point is defined to be the unique point none of whose inner coordinates is \( \pm 1 \mod 4 \). The opposite vertex of the nook point is defined to be the extreme nook point. The remaining two extreme points are called end points. The remaining two points are defined as central pair of points.

**Note 5.34.** Let \( A_{\text{col}}(Q) \) denote the line arrangement collineation automorphism group of \( Q \) and let \( A_{\text{Nook}}(Q) \) denote the line arrangement collineation group of automorphisms which preserve the nook points. Then we have

\[
A_{\text{Nook}}(Q) \cong \mathbb{Z}_2 \times \mathbb{Z}_2.
\]

We prove the following isomorphism theorem which preserve the pair of central points and the nook point and hence the extreme nook point. So the group here is \( \mathbb{Z}_2 \) and hence non-trivial. This non-triviality gives rise to the following two non-trivial theorems 5.35 5.36.

**Theorem 5.35.** Let \( (L_n^1), (L_n^2) \) be two line arrangements. Let

\[
\phi : (L_n^1) \rightarrow (L_n^2)
\]

be a line arrangement collineation isomorphism. Suppose for every quadrilateral substructure of the line arrangement the map \( \phi \) preserves nook points and the pair of central points with respect to the image quadrilateral substructure. Then \( \phi \) is an isomorphism of line arrangements. Also conversely any isomorphism of the line arrangement preserves the nook points and the central points of any quadrilateral substructure and its image substructure.

**Proof.** If the nook point and central points are preserved for every quadrilateral substructure then the isomorphism has the property that for any three intersection points \( P_1, P_2, P_3 \) on a line of the line arrangement if \( P_2 \) is in between \( P_1, P_3 \) then \( \phi(P_2) \) is in between \( \phi(P_1) \) and \( \phi(P_3) \). So it preserves the order of intersection points on each line. This proves the theorem. ■

**Theorem 5.36** (Preservation of Nook Points Under Isomorphisms which preserve Slope Order).

Let \( \mathbb{F} \) be a field with 1–ad structure. Let \( (L_n)^1 = \{L_1^1, L_2^1, \ldots, L_n^1\} \), \( (L_n)^2 = \{L_1^2, L_2^2, \ldots, L_n^2\} \) are two line arrangements in the plane where the lines are indexed in the order of increasing slopes \( 0 \rightarrow \infty \rightarrow -\infty \rightarrow 0 \). Then a bijection \( \phi : (L_n)^1 \rightarrow (L_n)^2 \) which is identity on the subscripts is an isomorphism of line arrangements if and only if for any pair of four subsets \( \{L_t^1, L_i^1, L_t^2, L_i^2\} \), \( t = 1, 2 \) the map \( \phi \) preserves the nook points. Moreover in this case there is a cyclic renumbering of any one of the arrangements such that the nook point intersection of any pair of corresponding four subsets is identical.

**Proof.** The proof is similar to the previous theorem based on the theme of central point of three points on any line. ■

5.5. Invariants of Line Arrangements. The following is a list of invariants of any type for line arrangements.

- The planarity crossing numbers.
- The slopes of the equations of the lines.
• The cycle at infinity, local cycles at infinity with 2-standard consecutive structures which respects the slope property.
• The nook points of quadrilateral substructures.
• The gonality structures and the opposite vertices of the sides of any gonality.
• The modified simplicial homology groups.

6. Line-Folds

In this section we define line-folds and later ask some open questions.

Definition 6.1 (Line-Folds).
Let $\mathbb{F}$ be a field with $1-ad$ structure. We say a set of lines $L_n = \{L_1, L_2, \ldots, L_n\}$ in the plane $\mathbb{F}^2$ is called a line-fold.

We say two line-folds $(L_n^1, (L_n^2)$ are isomorphic if there is a piecewise linear bijection of the plane which map one line-fold to another line-fold. We define the following for a line-fold.

Definition 6.2. Let $L_n^1$ be a line-fold. We say a point is $k-fold$ concurrency point if it is the intersection concurrency of $k$-lines. The intersection points of the line-fold is called the zero skeleton of the line-fold. The cardinality of the line-fold $L_n^1$ is defined to be $n$. A gonality of a line-fold is defined to be a convex region given by a set of $n-$inequalities similarly as before. It is bounded if there does not exist a subset which has a $1-ad$ structure of dimension one. It is unbounded otherwise.

6.1. On the Complement of Zero Sets of Certain Polynomials. Here we prove a theorem on the exact number of regions present in the complement of the zero set of a polynomial which corresponds to a line-fold. The author Prof. Milnor J. has obtained bounds for the betti numbers associated to the complement of the zero set of polynomials in the article [3]. Now we state theorem which has its generalizations to higher dimensions as follows.

Theorem 6.3. Let $\mathbb{F}$ be a field with $1-ad$ structure. Let $f(x, y) \in \mathbb{F}[x, y]$ be a polynomial which factorizes completely into linear factors over the field $\mathbb{F}$ corresponding to lines in the plane $\mathbb{F}^2$. Let $l_1, l_2, \ldots, l_r$ be the number of lines in each equivalence class under the parallel equivalence relation. Let $p = (x_0, y_0) \in \mathbb{F}^2$ and let $\mathcal{M}_p = (x - x_0, y - y_0)$ denote its corresponding maximal ideal. Let $k_p$ denote the order of vanishing of $f_{red}$ at $p$ i.e. $f_{red} \in \mathcal{M}_p^{k_p} \setminus \mathcal{M}_p^{k_p+1}$ where $f_{red}$ is the reduced polynomial associated to $f$. Let $d$ be the degree of the reduced polynomial $f_{red}$. Then we have

- The total number of regions is given by
  \[ 1 + d + \left(\frac{d}{2}\right) - \sum_{p \in \mathbb{F}^2} \binom{k_p - 1}{2} - \sum_{i=1}^{r} \binom{l_i}{2}. \]
- If $l_i = 1, 1 \leq i \leq r$ then the total number of bounded regions is given by
  \[ \binom{d - 1}{2} - \sum_{p \in \mathbb{F}^2} \binom{k_p - 1}{2}. \]
- Also in this case the total number of unbounded regions is given by $2d$.

Proof. This follows by subtracting the number of bounded local gonality forms by non-degenerate perturbation of the concurencies of order $k_p$ when $k_p > 2$. This proves the theorem.
6.2. **Open Questions on Line-Folds.** We have the following combinatorial, topological, geometric and number theoretic open questions.

**Question 6.4.**

1. Classify two line-folds up to isomorphism by associating invariants?
2. How many isomorphism classes of line-folds whose cardinality is $n$ are there?
3. How many isomorphism classes of line arrangements whose cardinality is $n$ are there?
4. What are the possible cardinalities of the zero skeleton of a line-fold whose cardinality is $n$?
5. What are the possible cardinalities of the zero skeleton of a line-fold whose cardinality is $n$ which has no parallel lines?
6. How many non-isomorphic line-folds are there passing through a given set of $n$--generic points?

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