ON WEYL SUMS OVER PRIMES IN SHORT INTERVALS

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1. Introduction

In this note we pursue bounds for exponential sums of the form

\[ f_k(\alpha; x, y) = \sum_{x < n \leq x+y} \Lambda(n) e(\alpha n^k), \]

where \( k \geq 2 \) is an integer, \( 2 \leq y \leq x \), \( \Lambda(n) \) is von Mangoldt’s function, and \( e(z) = e^{2\pi iz} \).

When \( y = x^\theta \) with \( \theta < 1 \), such exponential sums play a central role in applications of the Hardy–Littlewood circle method to additive problems with almost equal prime unknowns (see [6, 7, 9]). When \( \alpha \) is closely approximated by a rational number with a small denominator (i.e., when \( \alpha \) is on a “major arc”), Liu, Lü and Zhan [5] bounded \( f_k(\alpha; x, x^\theta) \) using methods from multiplicative number theory. Their result, which generalizes earlier work by Ren [8], can be stated as follows.

**Theorem 1.** Let \( k \geq 1 \), \( 7/10 < \theta \leq 1 \) and \( 0 < \rho \leq \min\{(8\theta - 5)/(6k + 6), (10\theta - 7)/15\} \).

Suppose that \( \alpha \) is real and that there exist integers \( a \) and \( q \) satisfying

\[ 1 \leq q \leq P, \quad (a, q) = 1, \quad |q\alpha - a| \leq x^{-k+2(1-\theta)}P, \]

with \( P = x^{2k\rho} \). Then, for any fixed \( \varepsilon > 0 \),

\[ f_k(\alpha; x, x^\theta) \ll x^{\theta-\rho+\varepsilon} + x^{\theta+\varepsilon} \Xi(\alpha)^{-1/2}, \]

where \( \Xi(\alpha) = q + x^{k-2(1-\theta)}|q\alpha - a| \).

For a given \( P \), let \( \mathcal{M}(P) \) denote the set of real \( \alpha \) that have rational approximations of the form (1.2), and let \( \mathcal{m}(P) \) denote the complement of \( \mathcal{M}(P) \). In the terminology of the circle method, \( \mathcal{M}(P) \) is a set of major arcs and \( \mathcal{m}(P) \) is the respective set of minor arcs. The main goal of this note is to bound \( f_k(\alpha; x, x^\theta) \), \( k \geq 3 \), on sets of minor arcs by extending a theorem of the author [4, Theorem 1], which gives the best known bound for \( f_k(\alpha; x, x) \). We first state our result for cubic sums.

**Theorem 2.** Let \( \theta \) be a real number with \( 4/5 < \theta \leq 1 \) and suppose that \( 0 < \rho \leq \rho_3(\theta) \), where

\[ \rho_3(\theta) = \min \left( \frac{1}{11}(2\theta - 1), \frac{1}{30}(14\theta - 11), \frac{1}{6}(5\theta - 4) \right). \]

Then, for any fixed \( \varepsilon > 0 \),

\[ \sup_{\alpha \in \mathcal{m}(P)} |f_3(\alpha; x, x^\theta)| \ll x^{\theta-\rho+\varepsilon} + x^{\theta+\varepsilon} P^{-1/2}. \]
We remark that when $\theta = 1$, Theorem 2 recovers the bound
\[
\sup_{\alpha \in \mathbb{M}(P)} |f_3(\alpha; x, x)| \ll x^{13/14+\varepsilon} + x^{1+\varepsilon}P^{-1/2},
\]
which is the essence of the cubic case of [4, Theorem 3]. In the case $k \geq 4$, our estimates take the following form.

**Theorem 3.** Let $k \geq 4$ be an integer and $\theta$ be a real number with $1 - (k+2)^{-1} < \theta \leq 1$. Suppose that $0 < \rho \leq \rho_k(\theta)$, where
\[
\rho_k(\theta) = \min \left( \frac{1}{6} \sigma_k(3\theta - 1), \frac{1}{6}((k+2)\theta - (k+1)) \right),
\]
with $\sigma_k$ defined by $\sigma_k^{-1} = \min(2^{k-1}, 2k(k-2))$. Then, for any fixed $\varepsilon > 0$,
\[
(1.4) \quad \sup_{\alpha \in \mathbb{M}(P)} |f_k(\alpha; x, x^\theta)| \ll x^{\theta-\rho+\varepsilon} + x^{\theta+\varepsilon}P^{-1/2}.
\]

When $\theta = 1$ and $k \leq 7$, this theorem also recovers the respective cases of [4, Theorem 3]. On the other hand, when $k \geq 8$, (1.4) is technically new even in the case $\theta = 1$, as we use the occasion to put on the record an almost automatic improvement of the theorems in [4] that results from a recent breakthrough by Wooley [11, 12].

**Notation.** Throughout the paper, the letter $\varepsilon$ denotes a sufficiently small positive real number. Any statement in which $\varepsilon$ occurs holds for each positive $\varepsilon$, and any implied constant in such a statement is allowed to depend on $\varepsilon$. The letter $p$, with or without subscripts, is reserved for prime numbers. As usual in number theory, $\mu(n)$, $\tau(n)$ and $\|x\|$ denote, respectively, the Möbius function, the number of divisors function and the distance from $x$ to the nearest integer. We write $(a, b) = \gcd(a, b)$, and we use $m \sim M$ as an abbreviation for the condition $M < m \leq 2M$.

## 2. Auxiliary results

When $k \geq 3$, we define the multiplicative function $w_k(q)$ by
\[
w_k(p^{ku+v}) = \begin{cases} kp^{-u-1/2}, & \text{if } u \geq 0, v = 1, \\ p^{-u-1}, & \text{if } u \geq 0, v = 2, \ldots, k. \end{cases}
\]

By the argument of [10, Theorem 4.2], we have
\[
(2.1) \quad \sum_{1 \leq x \leq q} e(\frac{ax^k}{q}) \ll qw_k(q) \ll q^{1-1/k}
\]
whenever $k \geq 2$ and $(a, q) = 1$. We also need several estimates for sums involving the function $w_k(q)$. We list those in the following lemma.
Lemma 2.1. Let \( w_k(q) \) be the multiplicative function defined above. Then the following inequalities hold for any fixed \( \epsilon > 0 \):

(2.2) \[
\sum_{q \sim Q} w_k(q)^j \ll \begin{cases} 
Q^{-1+\epsilon} & \text{if } k = 3, j = 4, \\
Q^{-1+k} & \text{if } k \geq 4, j = k;
\end{cases}
\]

(2.3) \[
\sum_{n \sim N} w_k \left( \frac{q}{(q, n^j)} \right) \ll q^\epsilon w_k(q) N \quad (1 \leq j \leq k);
\]

(2.4) \[
\sum_{n \sim N} w_k \left( \frac{q}{(q, R(n, h))} \right) \ll q^\epsilon w_k(q) N + q^\epsilon;
\]

where \( R(n, h) = ((n + h)^k - n^k) / h \).

Proof. See Lemmas 2.3 and 2.4 and inequality (3.11) in Kawada and Wooley [3]. □

Lemma 2.2. Let \( k \geq 3 \) be an integer and let \( 0 < \rho \leq \sigma_k \), where \( \sigma_k = \min(2^{k-1}, 2k(k-2)) \). Suppose that \( y \leq x \) and \( x^k \leq y^{k+1-2\rho} \). Then either

(2.5) \[
\sum_{x < n \leq x+y} e(\alpha n^k) \ll y^{1-\rho+\epsilon},
\]

or there exist integers \( a \) and \( q \) such that

(2.6) \[
1 \leq q \leq y^{k\rho}, \quad (a, q) = 1, \quad |q\alpha - a| \leq x^{1-k}y^{k\rho-1},
\]

and

(2.7) \[
\sum_{x < n \leq x+y} e(\alpha n^k) \ll \frac{w_k(q)y}{1 + xy^{k-1}|\alpha - a/q|} + x^{k/2+\epsilon}y^{(1-k)/2}.
\]

Proof. By Dirichlet’s theorem on Diophantine approximation, there exist integers \( a \) and \( q \) with

(2.8) \[
1 \leq q \leq y^{k-1}, \quad (a, q) = 1, \quad |q\alpha - a| \leq y^{1-k}.
\]

When \( q > y \), we rewrite the sum on the left of (2.6) as

\[
\sum_{1 \leq n \leq y} e(\alpha n^k + \alpha_{k-1}n^{k-1} + \cdots + \alpha_0),
\]

where \( \alpha_j = \binom{k}{j} \alpha [x]^{k-j-1} \). Hence, (2.5) follows from Weyl’s bound

\[
\sum_{1 \leq n \leq y} e(\alpha n^k + \alpha_{k-1}n^{k-1} + \cdots + \alpha_0) \ll y^{1-\sigma_k+\epsilon}.
\]

Under (2.8), this follows from [10, Lemma 2.4] when \( \sigma_k = 2^{1-k} \) and from Wooley’s recent improvement [12] of Vinogradov’s mean-value theorem otherwise. When \( q \leq X \), we deduce (2.7) from [10, Lemmas 6.1 and 6.2] and (2.1). Thus, at least one of (2.5) and (2.7) holds. The lemma follows on noting that when conditions (2.6) fail, inequality (2.5) follows from (2.7) and the hypothesis \( x^k \leq y^{k+1-2\rho} \). □

The following lemma is a slight variation of [11, Lemma 6]. The proof is the same.
Lemma 2.3. Let $q$ and $N$ be positive integers exceeding 1 and let $0 < \delta < \frac{1}{2}$. Suppose that $q \nmid a$ and denote by $S$ the number of integers $n$ such that

$$N < n \leq 2N, \quad (n, q) = 1, \quad \|an^k/q\| < \delta.$$  

Then

$$S \ll \delta q^\varepsilon (q + N).$$

3. Multilinear Weyl sums

We write

$$\delta = x^{\theta - 1}, \quad L = \log x, \quad T = (x, x + x^\theta).$$

We also set

$$Q = \left(\delta x^{k-2\rho}\right)^{k/(2k-1)}.$$

Recall that, by Dirichlet’s theorem on Diophantine approximations, every real number $\alpha$ has a rational approximation $a/q$, where $a$ and $q$ are integers subject to

$$1 \leq q \leq Q, \quad (a, q) = 1, \quad |\alpha - a/q| < (qQ)^{-1}.$$  

Lemma 3.1. Let $k \geq 3$ and $0 < \rho < \sigma_k/(2 + 2\sigma_k)$. Suppose that $\alpha$ is real and that there exist integers $a$ and $q$ such that (3.2) holds with $Q$ given by (3.1). Let $|\xi_n| \leq 1$, $|\eta_n| \leq 1$, and define

$$S(\alpha) = \sum_{m \sim M} \sum_{mn \in I} \xi_m \eta_ne \left(\alpha mn^k\right).$$

Then

$$S(\alpha) \ll x^{\theta - \rho + \varepsilon} + \frac{w_k(q)^{1/2}x^{\theta + \varepsilon}}{(1 + \delta^2x^k|\alpha - a/q|)^{1/2}},$$

provided that

$$\delta^{-1} \max \left(\delta^{2\rho/k}, \delta^{-k}x^{4\rho}, \left(\delta^{2k-2}x, k-1+4k\rho\right)^{1/(2k-1)}\right) \ll M \ll x^{2\rho}.$$

Proof. Set $H = \delta M$ and $N = xM^{-1}$ and define $\nu$ by $H^\nu = x^{2\rho}L^{-1}$. By (3.3), we have $\nu < \sigma_k$. For $n_1, n_2 \leq 2N$, let

$$\mathcal{M}(n_1, n_2) = \{m \in (M, 2M] : mn_1, mn_2 \in I\}.$$  

By Cauchy’s inequality and an interchange of the order of summation,

$$|S(\alpha)|^2 \ll x^\theta M + MT_1(\alpha),$$

where

$$T_1(\alpha) = \sum_{n_1 < n_2} \left| \sum_{m \in \mathcal{M}(n_1, n_2)} e \left(\alpha \left(n_2^k - n_1^k\right) m^k\right) \right|.$$  

Let $\mathcal{N}$ denote the set of pairs $(n_1, n_2)$ with $n_1 < n_2$ and $\mathcal{M}(n_1, n_2) \neq \emptyset$ for which there exist integers $b$ and $r$ such that

$$1 \leq r \leq H^\nu, \quad (b, r) = 1, \quad |r \left(n_2^k - n_1^k\right) \alpha - b| \leq H^\nu(\delta M)^{-1}.$$  

We remark that $\mathcal{N}$ contains $O(\delta N^2)$ pairs $(n_1, n_2)$. Since $\nu < \sigma_k$ and $M^k \leq H^{k+1-2\nu}$, we can apply Lemma 2.2 with $\rho = \nu, x = M$ and $y = H$ to the inner summation in $T_1(\alpha)$. We get

$$T_1(\alpha) \ll x^{2\theta - 2\rho + \varepsilon} M^{-1} + T_2(\alpha),$$

where
where

\[ T_2(\alpha) = \sum_{(n_1, n_2) \in \mathcal{N}} \frac{w_k(r)H}{1 + \delta M^k \left| (n_2^k - n_1^k)\alpha - b/r \right|}. \]

We now change the summation variables in \( T_2(\alpha) \) to

\[ d = (n_1, n_2), \quad n = n_1/d, \quad h = (n_2 - n_1)/d. \]

We obtain

\[ T_2(\alpha) \ll \sum_{dh \leq \delta N} \sum_n \frac{w_k(r)H}{1 + \delta M^k |hd^kR(n, h)\alpha - b/r|}, \tag{3.7} \]

where \( R(n, h) = ((n + h)^k - n^k)/h \) and the inner summation is over \( n \) with \((n, h) = 1\) and \((nd, (n + h)d) \in \mathcal{N}\). For each pair \((d, h)\) appearing in the summation on the right side of \( \text{(3.7)} \), Dirichlet’s theorem on Diophantine approximation yields integers \( b_1 \) and \( r_1 \) with

\[ 1 \leq r_1 \leq x^{-2k}\rho(\delta M^k), \quad (b_1, r_1) = 1, \quad |r_1hd^k\alpha - b_1| \leq x^{2k}\rho(\delta M^k)^{-1}. \tag{3.8} \]

As \( R(n, h) \leq 3^kN^{k-1} \), combining \( \text{(3.3)}, \text{(3.5)} \) and \( \text{(3.8)} \), we get

\[
|b_1rR(n, h) - br_1| \\leq r_1 H^{k\nu}(\delta M^k)^{-1} + rR(n, h)x^{2k}\rho(\delta M^k)^{-1} \\leq L^{-k} + 3^k\delta^{-1}x^{k-1+4k}\rho M^{1-2k}L^{-k} < 1.
\]

Hence,

\[ \frac{b}{r} = \frac{b_1R(n, h)}{r_1}, \quad r = \frac{r_1}{(r_1, R(n, h))}. \tag{3.9} \]

Combining \( \text{(3.7)} \) and \( \text{(3.9)} \), we obtain

\[ T_2(\alpha) \ll \sum_{dh \leq \delta N} \frac{H}{1 + \delta M^k N_d^{k-1} |hd^k\alpha - b_1/r_1|} \sum_{n \sim N_d} \sum_{(n, h) = 1} w_k \left( \frac{r_1}{(r_1, R(n, h))} \right), \]

where \( N_d = Nd^{-1} \). Using \( \text{(2.24)} \), we deduce that

\[ T_2(\alpha) \ll \delta x^{\theta + \varepsilon} + T_3(\alpha), \tag{3.10} \]

where

\[ T_3(\alpha) = \sum_{dh \leq \delta N} \frac{r_1^\varepsilon w_k(r_1)HN_d}{1 + \delta M^k N_d^{k-1} |hd^k\alpha - b_1/r_1|}.
\]

We now write \( \mathcal{H} \) for the set of pairs \((d, h)\) with \( dh \leq \delta N \) for which there exist integers \( b_1 \) and \( r_1 \) subject to

\[ 1 \leq r_1 \leq x^{2k}, \quad (b_1, r_1) = 1, \quad |r_1hd^k\alpha - b_1| \leq x^{-k+1+2k}\rho H^{-1}. \tag{3.11} \]

We have

\[ T_3(\alpha) \ll x^{2\theta - 2\rho + \varepsilon} M^{-1} + T_4(\alpha), \tag{3.12} \]

where

\[ T_4(\alpha) = \sum_{(d, h) \in \mathcal{H}} \frac{r_1^\varepsilon w_k(r_1)HN_d}{1 + \delta M^k N_d^{k-1} |hd^k\alpha - b_1/r_1|}. \]
For each $d \leq \delta N$, Dirichlet’s theorem on Diophantine approximation yields integers $b_2$ and $r_2$ with

$$1 \leq r_2 \leq \frac{1}{2} x^{k-1-2k\rho} H, \quad (b_2, r_2) = 1, \quad \left| r_2 d^k \alpha - b_2 \right| \leq 2 x^{-k+1+2k\rho} H^{-1}.$$  

Combining (3.11) and (3.13), we obtain

$$|b_2 r_1 h - b_1 r_2| \leq (r_2 + 2r_1 h) x^{-k+1+2k\rho} H^{-1} \leq \frac{1}{2} + 2 x^{-k+2+4k\rho} M^{-2} < 1,$$

whence

$$\frac{b_1}{r_1} = \frac{hb_2}{r_2}, \quad r_1 = \frac{r_2}{(r_2, h)}.$$  

We write $Z_d = \delta M^{k} N_{d}^{k-1} |d^k \alpha - b_2 / r_2|$ and we use (2.3) to get

$$T_4(\alpha) \ll x^{2\theta - 2\rho + \varepsilon} M^{-1} + T_5(\alpha),$$

where

$$T_5(\alpha) = \sum_{d \in \mathcal{D}} \frac{w_k(r_2) x^{2\theta + \varepsilon} M^{-1}}{d^2 (1 + \delta^2 (x/d)^k |d^k \alpha - b_2 / r_2|)}$$

and $\mathcal{D}$ is the set of integers $d \leq x^{2\rho}$ for which there exist integers $b_2$ and $r_2$ with

$$1 \leq r_2 \leq x^{2\rho}, \quad (b_2, r_2) = 1, \quad \left| r_2 d^k \alpha - b_2 \right| \leq \delta^{-2} x^{-k+2k\rho}.$$  

Combining (3.11), (3.2) and (3.15), we deduce that

$$\left| r_2 d^k a - b_2 q \right| \leq r_2 d^k Q^{-1} + q \delta^{-2} x^{-k+2k\rho} \leq x^{4k\rho} Q^{-1} + \delta^{-2} x^{-k+2k\rho} Q < 1,$$

whence

$$\frac{b_2}{r_2} = \frac{d^k a}{q}, \quad r_2 = \frac{q}{(q, d^k)}.$$  

Thus, recalling (2.3), we get

$$T_5(\alpha) \ll \frac{x^{2\theta + \varepsilon} M^{-1}}{1 + \delta^2 x^k |\alpha - a/q|} \sum_{d \leq x^{2\rho}} w_k \left( \frac{q}{(q, d^k)} \right) d^{-2} \ll \frac{w_k(q) x^{2\theta + \varepsilon} M^{-1}}{1 + \delta^2 x^k |\alpha - a/q|}.$$  

The lemma follows from (3.3), (3.4), (3.6), (3.10), (3.12), (3.14) and (3.16).

\[ \square \]

**Lemma 3.2.** Let $k \geq 3$ and $0 < \rho < \sigma_k$. Suppose that $\alpha$ is real and that there exist integers $a$ and $q$ such that (3.2) holds with $Q$ given by (3.1). Let $|\xi_{m_1m_2}| \leq 1$, and define

$$S(\alpha) = \sum_{m_1 \sim M_1} \sum_{m_2 \sim M_2} \sum_{m_1 m_2 n \in \mathcal{I}} \xi_{m_1m_2} e \left( \alpha (m_1 m_2 n)^k \right).$$

Then

$$S(\alpha) \ll x^{\theta - \rho + \varepsilon} + \frac{w_k(q) x^{\theta + \varepsilon}}{1 + \delta x^k |\alpha - a/q|},$$

where $\mathcal{I}$ is the set of integers $m_1 m_2 n \in I$ satisfying

$$\sum_{m_1 m_2 n \in \mathcal{I}} \xi_{m_1m_2} e \left( \alpha (m_1 m_2 n)^k \right) = 0.$$
provided that

\[(3.17) \quad M_1^{2k-1} \ll \delta x^{-(2k+1)p}, \quad M_1 M_2 \ll \min(\delta x^{1-p/\sigma_k}, \delta^{k+1} x^{1-2p}), \quad M_1 M_2^2 \ll \delta^{1/k} x^{1-2p}.\]

**Proof.** Set \(N = x(M_1 M_2)^{-1}\) and \(H = \delta N\) and define \(\nu\) by \(H^\nu = x^p L^{-1}\). Note that, by \((3.17)\), we have \(\nu < \sigma_k\). We denote by \(\mathcal{M}\) the set of pairs \((m_1, m_2)\), with \(m_1 \sim M_1\) and \(m_2 \sim M_2\), for which there exist integers \(b_1\) and \(r_1\) with

\[(3.18) \quad 1 \leq r_1 \leq H^\nu, \quad (b_1, r_1) = 1, \quad |r_1 (m_1 m_2)^k \alpha - b_1| \leq H^\nu (\delta N^k)^{-1}.\]

We apply Lemma 2.2 to the summation over \(n\) and get

\[(3.19) \quad S(\alpha) \ll x^\theta - \rho + \varepsilon + T_1(\alpha),\]

where

\[T_1(\alpha) = \sum_{(m_1, m_2) \in \mathcal{M}} \frac{w_k(r_1) H}{1 + \delta N^k |(m_1 m_2)^k \alpha - b_1/r_1|}.\]

For each \(m_1 \sim M_1\), we apply Dirichlet’s theorem on Diophantine approximation to find integers \(b\) and \(r\) with

\[(3.20) \quad 1 \leq r \leq x^{-k^p (\delta N^k)}, \quad (b, r) = 1, \quad |rm_1^k \alpha - b| \leq x^{k^p (\delta N^k)^{-1}}.\]

By \((3.17), (3.18)\) and \((3.20)\),

\[
|b_1 r - bm_2^k r_1| \leq r H^\nu (\delta N^k)^{-1} + r_1 m_2^k x^{k^p (\delta N^k)^{-1}} \\
\leq L^{-k} + 2^{k} \delta^{-1} x^{-k + 2k^p (M_1 M_2^2)^k} L^{-k} < 1,
\]

whence

\[
\frac{b_1}{r_1} = \frac{m_1^k b}{r}, \quad r_1 = \frac{r}{(r, m_2^k)}.
\]

Thus, by \((2.3)\),

\[(3.21) \quad T_1(\alpha) \ll \sum_{m_1 \sim M_1} \frac{H}{1 + \delta (M_2 N)^k |m_1^k \alpha - b/r|} \sum_{m_2 \sim M_2} w_k \left( \frac{r}{(r, m_2^k)} \right) \sum_{m_2 \sim M_2} w_k \left( \frac{r}{(r, m_2^k)} \right)
\ll \sum_{m_1 \sim M_1} \frac{r^c w_k(r) H M_2}{1 + \delta (M_2 N)^k |m_1^k \alpha - b/r|}.
\]

Let \(\mathcal{M}_1\) be the set of integers \(m \sim M_1\) for which there exist integers \(b\) and \(r\) with

\[(3.22) \quad 1 \leq r \leq x^{k^p L^{-1}}, \quad (b, r) = 1, \quad |rm_1^k \alpha - b| \leq \delta^{-1} x^{-k + k^p} M_1^k L^{-1}.\]

From \((3.21)\),

\[(3.23) \quad T_1(\alpha) \ll x^{\theta - \rho + \varepsilon} + T_2(\alpha),\]

where

\[T_2(\alpha) = \sum_{m \in \mathcal{M}_1} \frac{r^c w_k(r) H M_2}{1 + \delta (M_2 N)^k |m^k \alpha - b/r|}.
\]

We now consider two cases depending on the size of \(q\) in \((3.2)\).
Case 1: \( q \leq \delta x^{k - k\rho}M_1^{-k} \). In this case, we estimate \( T_2(\alpha) \) as in the proof of Lemma 3.1. Combining (3.1), (3.2), (3.17) and (3.22), we obtain
\[
|r m^k a - b q| \leq q \delta^{-1} x^{-k + k\rho} M_1^k L^{-1} + r m^k Q^{-1} \\
\leq L^{-1} + 2^k x^{k\rho} M_1^k Q^{-1} L^{-1} < 1.
\]

Therefore,
\[
\frac{b}{r} = \frac{m^k a}{q}, \quad r = \frac{q}{(q, m^k)},
\]
and by (2.3),
(3.24) \[ T_2(\alpha) \ll \frac{q^\varepsilon H M_2}{1 + \delta x^k |\alpha - a/q|} \sum_{m \sim M_1} w_k \left( \frac{q}{(q, m^k)} \right) \ll \frac{w_k(q)x^\theta + \varepsilon}{1 + \delta x^k |\alpha - a/q|}.
\]

Case 2: \( q > \delta x^{k - k\rho}M_1^{-k} \). We remark that in this case, the choice (3.1) implies that \( M_1 \geq x^{\rho} \).

By a standard splitting argument,
(3.25) \[ T_2(\alpha) \ll \sum_{d | q} \sum_{m \sim M_d(R, Z)} \frac{w_k(r)H M_2 x^\varepsilon}{1 + \delta(M_2 N)^k(RZ)^{-1}},
\]
where
(3.26) \[ 1 \leq R \leq x^{k\rho} L^{-1}, \quad \delta x^{k - k\rho} M_1^{-1} L \leq Z \leq \delta(x/M_1)^k,
\]
and \( M_d(R, Z) \) is the subset of \( M_1 \) containing integers \( m \) subject to
\[
(m, q) = d, \quad r \sim R, \quad |r m^k \alpha - b| < Z^{-1}.
\]

We now estimate the inner sum on the right side of (3.25). We have
(3.27) \[ \sum_{m \sim M_d(R, Z)} w_k(r) \ll \sum_{r \sim R} w_k(r) S_0(r),
\]
where \( S_0(r) \) is the number of integers \( m \sim M_1 \) with \( (m, q) = d \) for which there exists an integer \( b \) such that
(3.28) \[ (b, r) = 1 \quad \text{and} \quad |r m^k \alpha - b| < Z^{-1}.
\]

Since for each \( m \sim M_1 \) there is at most one pair \( (b, r) \) satisfying (3.28) and \( r \sim R \), we have
(3.29) \[ \sum_{r \sim R} S_0(r) \leq \sum_{m \sim M_1 \atop (m, q) = d} 1 \ll M_1 d^{-1} + 1.
\]

Hence,
(3.30) \[ \sum_{r \sim R \atop (q, rd^k) = q} w_k(r) S_0(r) \ll R^{-1/k} (M_1 d^{-1} + 1) \ll M_1 q^{-1/k} + 1,
\]
on noting that the sum on the left side is empty unless \( Rd^k \gg q \).

When \( (q, rd^k) < q \), we make use of Lemma 2.3. By (3.2), (3.26) and (3.28),
(3.31) \[ S_0(r) \leq S(r),
\]
where we \( S(r) \) is the number of integers \( m \) subject to
\[
m \sim M_1 d^{-1}, \quad (m, q_1) = 1, \quad \left\| ard^{k-1} m^k / q_1 \right\| < \Delta,
\]
with \( q_1 = q d^{-1} \) and \( \Delta = Z^{-1} + 2^{k+1} R M_1^k (q Q)^{-1} \). Since (3.17) implies \( M_1 \leq \delta x^{k-k_0} M_1^{-k} < q \), we obtain
\[
(3.32) \quad S(r) \ll \Delta q^k d^{-1} (M_1 + q) \ll \Delta q^{1+\varepsilon}.
\]
Combining (3.31) and (3.32), we get
\[
(3.33) \quad S_0(r) \ll \Delta q^{1+\varepsilon}.
\]
We now apply Hölder’s inequality, (2.2), (3.29), and (3.33) and obtain
\[
(3.34) \quad \sum_{r \sim R, (q, rd^k) < q} w_3(r) S_0(r) \ll \left( \Delta q^{1+\varepsilon} \right)^{1/4} \left( \sum_{r \sim R} w_3(r)^4 \right)^{1/4} \left( \sum_{r \sim R} S_0(r) \right)^{3/4}
\ll \Delta^{1/4} q^{1+\varepsilon} R^{-1/4} M_1^{3/4}.
\]
Similarly, when \( k \geq 4 \), we have
\[
(3.35) \quad \sum_{r \sim R, (q, rd^k) < q} w_k(r) S_0(r) \ll \left( \Delta q^{1+\varepsilon} \right)^{1/k} \left( \sum_{r \sim R} w_k(r)^k \right)^{1/k} \left( \sum_{r \sim R} S_0(r) \right)^{1-1/k}
\ll \Delta^{1/k} q^{1/k+\varepsilon} R^{(1-k)/k^2} M_1^{(k-1)/k}.
\]
Combining (3.27), (3.30), (3.34) and (3.35), we deduce
\[
(3.36) \quad \sum_{m \in M_2(R, Z)} w_3(r) \ll \Delta^{1/4} q^{1+\varepsilon} R^{-1/4} M_1^{3/4} + M_1 q^{-1/3} + 1
\]
and
\[
(3.37) \quad \sum_{m \in M_2(R, Z)} w_k(r) \ll \Delta^{1/k} q^{1/k+\varepsilon} R^{(1-k)/k^2} M_1^{(k-1)/k} + M_1 q^{-1/k} + 1
\]
for \( k \geq 4 \).

Substituting (3.36) into (3.25), we get
\[
T_2(\alpha) \ll \frac{x^{\theta+\varepsilon} M_1^{-1/4}}{1 + \delta(M_2 N)^3 (R Z)^{-1}} \left( \frac{Q}{R Z} + \frac{M_1^2}{Q} \right)^{1/4} + x^{\theta+\varepsilon} q^{-1/3} + x^{\theta+\varepsilon} M_1^{-1}
\ll (\delta^3 x M_1^2 Q)^{1/4+\varepsilon} + x^{\theta+\varepsilon} (M_1^2 Q^{-1})^{1/4} + x^{\theta+\varepsilon} M_1 + x^{\theta-\rho+\varepsilon}.
\]
The hypotheses of the lemma ensure that
\[
M_1 \leq \min \left( \delta^{1/2} x^{3/2-2\rho} Q^{-1/2}, Q^{1/2} x^{-2\rho}, x^{\theta-2\rho} \right),
\]
and so when \( k = 3 \),
\[
(3.38) \quad T_2(\alpha) \ll x^{\theta-\rho+\varepsilon}.
\]
When \( k \geq 4 \), by (3.25) and (3.37),
\[
T_2(\alpha) \ll \frac{x^{\theta+\varepsilon}M^{-1/k}R^{1/k^2}}{1 + \delta(M_2N)^k(RZ)^{-1}} \left( \frac{Q}{RZ} + M_1^{1/k} \right)^{1/k} + x^{\theta+\varepsilon}q^{-1/k} + x^{\theta+\varepsilon}M_1^{-1}
\]
\[
\ll \left( x^{\rho}Q(\delta M_1)^{k-1} \right)^{1/k+\varepsilon} + x^{\theta+\varepsilon} \left( x^{\rho}M_1^{k-1}Q^{-1} \right)^{1/k} + x^{\theta+\varepsilon}M_1 + x^{\theta-\rho+\varepsilon},
\]
and using (3.1) and (3.17), we find that (3.38) holds in this case as well.

The desired estimate follows from (3.19), (3.23), (3.24) and (3.38).

\[
\square
\]

4. Proof of Theorems 2 and 3

In this section we deduce the main theorems from Lemmas 3.1 and 3.2 and Heath-Brown’s identity for \( \Lambda(n) \). We apply Heath-Brown’s identity in the following form [2, Lemma 1]: if \( n \leq X \) and \( J \) is a positive integer, then

\[
(4.1) \quad \Lambda(n) = \sum_{j=1}^{J} \binom{J}{j} (-1)^j \sum_{n_1 \cdots n_j \leq X^{1/J}} \mu(n_1) \cdots \mu(n_j)(\log n_{2j}).
\]

Let \( \alpha \in \mathfrak{m}(P) \). By Dirichlet’s theorem on Diophantine approximation, there exist integers \( a \) and \( q \) such that (3.2) holds with \( Q \) given by (3.1). Let \( \beta \) be defined by

\[
x^\beta = \min \left( \delta^2x^{1-2\rho(a+1)}, \delta^{k+2}x^{1-6\rho}, (\delta^{2k}x^{-k-(8k-2)\rho})^{1/(2k-1)} \right),
\]
and suppose that \( \rho \) and \( \delta \) are chosen so that

\[
(4.2) \quad \delta^{-1}x^{\beta+2\rho} \geq 2x^{1/3}.
\]

We apply (4.1) with \( X = x + x^\beta \) and \( J \geq 3 \) chosen so that \( x^{1/J} \leq x^\beta \). After a standard splitting argument, we have

\[
(4.3) \quad \sum_{n \in \mathbb{Z}} \Lambda(n) e\left( \alpha n^k \right) \ll \sum_{N} \left| \sum_{n \in \mathbb{Z}} c(n; N) e\left( \alpha n^k \right) \right|,
\]
where \( N \) runs over \( O(L^{2J-1}) \) vectors \( N = (N_1, \ldots, N_{2J}), \ j \leq J, \) subject to

\[
N_1, \ldots, N_j \ll x^{1/J}, \quad x \ll N_1 \cdots N_{2j} \ll x,
\]
and

\[
c(n; N) = \sum_{n_1 \cdots n_{2j}} \mu(n_1) \cdots \mu(n_j)(\log n_{2j}).
\]

In fact, since the coefficient \( \log n_{2j} \) can be removed by partial summation, we may assume that

\[
c(n; N) = L \sum_{n_1 \cdots n_{2j}} \mu(n_1) \cdots \mu(n_j),
\]
where \( N_i < N'_i \leq 2N_i \) (in reality, \( N'_i = 2N_i \) except for \( i = 2j \)). We also assume (as we may) that the summation variables \( n_{j+1}, \ldots, n_{2j} \) are labeled so that \( N_{j+1} \leq \cdots \leq N_{2j} \). Next, we
show that each of the sums occurring on the right side of (4.3) satisfies the bound
\[ (4.4) \quad \sum_{n \in I} c(n; N)e(\alpha n^k) \ll x^{\theta - \rho + \varepsilon} + \frac{w_k(q)^{1/2}x^{\theta + \varepsilon}}{(1 + \delta^2 x^k|\alpha - a/q|)^{1/2}}. \]

The analysis involves several cases depending on the sizes of $N_1, \ldots, N_{2j}$.

**Case 1:** $N_1 \cdots N_j \gg \delta^{-1} x^{2\rho}$. Since none of the $N_i$'s exceeds $x^\theta$, there must be a set of indices $S \subset \{1, \ldots, j\}$ such that
\[ (4.5) \quad \delta^{-1} x^{2\rho} \leq \prod_{i \in S} N_i \leq \delta^{-1} x^{\theta + 2\rho}. \]

Hence, we can rewrite $c(n; N)$ in the form
\[ (4.6) \quad c(n; N) = \sum_{m r = n, M \geq M} \xi_m \eta_r, \]
where $|\xi_m| \leq \tau(m)^c$, $|\eta_r| \leq \tau(r)^c$, and $M = \prod_{i \notin S} N_i$. By (4.5), $M$ satisfies (4.3), so (4.4) follows from Lemma 3.1.

**Case 2:** $N_1 \cdots N_j < \delta^{-1} x^{2\rho}$, $j \leq 2$. When $j = 1$, (4.4) follows from Lemma 3.2 with $M_1 = N_1$, $M_2 = 1$ and $N = N_2$. When $j = 2$, we have
\[ N_3 \leq (x/N_1 N_2)^{1/2} \leq x^{1/2}, \quad N_1 N_2 N_3 \leq (x N_1 N_2)^{1/2} \leq \delta^{-1} x^{1/2 + \rho}, \]
\[ (N_1 N_2)^2 N_3 \leq x^{1/2} (N_1 N_2)^{3/2} \leq \delta^{-2} x^{1/2 + 3\rho}. \]

Hence, we can deduce (4.4) from Lemma 3.2 with $M_1 = N_3$, $M_2 = N_1 N_2$ and $N = N_4$, provided that
\[ (4.7) \quad x^{k - 1/2} \leq \delta x^{k - (2k + 1)\rho}, \quad \delta^{-2} x^{1/2 + 3\rho} \leq \delta^{1/k} x^{1 - 2\rho}, \]
\[ (4.8) \quad \delta^{-1} x^{1/2 + \rho} \leq \delta \min \left( x^{1 - \rho/\sigma_k}, \delta^k x^{1 - 2\rho} \right). \]

**Case 3:** $N_1 \cdots N_j < \delta^{-1} x^{2\rho}$, $j \geq 3$. In this case, we have
\[ N_{j+1}, \ldots, N_{2j-2} \leq 2x^{1/3} \leq \delta^{-1} x^{\beta + 2\rho}. \]

**Case 3.1:** $N_1 \cdots N_{2j-2} \geq \delta^{-1} x^{2\rho}$. Let $r$ be the least index with $N_1 \cdots N_r \geq \delta^{-1} x^{2\rho}$. We can use the product $N_1 \cdots N_r$ in a similar fashion to the product $N_1 \cdots N_j$ in Case 1 to represent $c(n; N)$ in the form (4.6). Thus, we can appeal to Lemma 3.1 to show that (4.4) holds again.

**Case 3.2:** $N_1 \cdots N_{2j-2} < \delta^{-1} x^{2\rho}$. Then we are in a similar situation to Case 2 with $j = 2$, with the product $N_1 \cdots N_{2j-2}$ playing the role of $N_1 N_2$ in Case 2. Thus, we can again use Lemma 3.2 to obtain (4.4).

By the above analysis,
\[ (4.9) \quad \sum_{n \in I} A(n)e(\alpha n^k) \ll x^{\theta - \rho + \varepsilon} + \frac{w_k(q)^{1/2}x^{\theta + \varepsilon}}{(1 + \delta^2 x^k|\alpha - a/q|)^{1/2}}, \]
provided that conditions \((4.2), (4.7)\) and \((4.8)\) hold. Altogether, those conditions are equivalent to the inequality

\[
x^\rho \ll \min \left( \left( \delta^3x^2 \right)^{\sigma_k/6}, \left( \delta^2x \right)^{1/(4k+2)}, \left( \delta^2x \right)^{(1+\sigma_k)}, \delta(k+2)/6x^{1/6}, \delta(k+1)/4x^{1/6}, \delta(2k+1)/5kx^{1/10}, \delta^4/(4k)x(k+1)/(12k) \right).
\]

We have

\[
\delta(k+2)/6x^{1/6} \leq \delta(k+1)/4x^{1/6}, \quad \left( \delta^2x \right)^{1/(4k+2)} \leq \delta^4/(4k)x(k+1)/(12k),
\]

\[
(\delta^3x^2)^{\sigma_k/6} \leq \left( \delta^2x \right)^{(1+\sigma_k)} \quad \text{when } \delta \geq x^{-1/3},
\]

so the third, fifth and seventh terms in the above minimum are superfluous. Recalling the definition of \(\delta\), we conclude that \((4.9)\) holds whenever

\[
(4.10) \quad \rho \leq \min \left( \frac{\sigma_k(3\theta - 1)}{6}, \frac{2\theta - 1}{4k+2}, \frac{(k+2)\theta - k - 1}{6}, \frac{(4k+2)\theta - 3k - 2}{10k} \right).
\]

The latter minimum is exactly the function \(\rho_k(\theta)\) defined in the statements of Theorems \([2]\) and \([3]\). Indeed, when \(k = 3\), the first term in the minimum is always larger than the second, so it can be discarded and we are left with \(\rho_3(\theta)\). On the other hand, when \(k \geq 4\), the second and fourth terms in the minimum are superfluous. Therefore, \((4.10)\) is a direct consequence of the hypotheses of the theorems and the proof of \((4.9)\) is complete.

If either \(q \geq x^{2k\rho}\) or \(|\alpha - a| \geq \delta^{-2}x^{-k-2k\rho}\), we can use \([2.1]\) to show that the second term on the right side of \((4.9)\) is smaller than the first. Thus,

\[
(4.11) \quad \sup_{\alpha \in \mathbb{M}(x^{2k\rho})} \left| f_k(\alpha; x, x^\theta) \right| \ll x^{\theta - \rho + \varepsilon}.
\]

This establishes the theorems when \(P \geq x^{2k\rho}\). When \(P < x^{2k\rho}\), Theorem \([1]\) gives

\[
\sup_{\alpha \in \mathbb{M}(P) \cap \mathbb{M}(x^{2k\rho})} \left| f_k(\alpha; x, x^\theta) \right| \ll x^{\theta - \rho + \varepsilon} + x^{\theta + \varepsilon} P^{-1/2},
\]

which in combination with \((4.11)\) establishes the theorems in the case \(P < x^{2k\rho}\). \(\square\)

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