Bipolar Lawson tau-surfaces and generalized Lawson tau-surfaces

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ABSTRACT. Recently Penskoi generalized the well known two-parametric family of Lawson tau-surfaces $\tau_{r,m}$ minimally immersed in spheres to a three-parametric family $T_{a,b,c}$ of tori and Klein bottles minimally immersed in spheres. It was remarked that this family includes surfaces carrying all extremal metrics for the first non-trivial eigenvalue of the Laplace-Beltrami operator on the torus and on the Klein bottle: the Clifford torus, the equilateral torus and surprisingly the bipolar Lawson Klein bottle $\tilde{\tau}_{3,1}$. In the present paper we show in Theorem 2 that this three-parametric family $T_{a,b,c}$ includes in fact all bipolar Lawson tau-surfaces $\tilde{\tau}_{r,m}$.

1 Introduction

Let $M$ be a closed surface and $g$ be a Riemannian metric on $M$. Let us consider the associated Laplace-Beltrami operator $\Delta : C^\infty(M) \to C^\infty(M)$,

$$\Delta f = -\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} (\sqrt{|g|} g^{ij} \frac{\partial f}{\partial x^j}).$$

The spectrum of $\Delta$ is non-negative and consists only of eigenvalues where each eigenvalue has a finite multiplicity and the associated eigenfunctions are smooth. Denote the eigenvalues of $\Delta$ by

$$0 = \lambda_0(M, g) < \lambda_1(M, g) \leq \lambda_2(M, g) \leq \lambda_3(M, g) \leq \ldots,$$

where eigenvalues are written with multiplicities.

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Let us fix the surface $M$ and consider $\Lambda_i(M,g)$ as a functional $g \mapsto \Lambda_i(M,g)$ on the space of all Riemannian metrics on $M$. The eigenvalues possess the following rescaling property,

$$\forall t > 0, \quad \lambda_i(M, tg) = \frac{\lambda_i(M,g)}{t}.$$ 

To get scale-invariant functionals on the space of Riemannian metrics one has to normalize the eigenvalue functionals. It is most natural to normalize the functionals by multiplying by the area,

$$\Lambda_i(M,g) = \lambda_i(M,g)\text{Area}(M,g).$$

The functionals $\Lambda_i(M,g)$ are invariant under the rescaling transformation $g \mapsto tg$.

If we consider the functional $\Lambda_i(M,g)$ over the space of Riemannian metrics $g$ with a fixed surface $M$, the question about the value of supremum $\sup \Lambda_i(M,g)$ is interesting. It is a very difficult question with a limited number of known results. It follows from Yang and Yau [21] and Korevaar [14] that this supremum is finite.

**Definition.** A metric $g_0$ on a fixed surface $M$ is called maximal for the functional $\Lambda_i(M,g)$ if

$$\sup \Lambda_i(M,g) = \Lambda_i(M,g_0),$$

where the supremum is taken over the space of Riemannian metrics $g$ on the fixed surface $M$.

Currently, we only know maximal metrics for $\Lambda_1(S^2, g)$, $\Lambda_1(\mathbb{R}P^2, g)$, $\Lambda_1(T^2, g)$, $\Lambda_2(S^2, g)$, and $\Lambda_1(\Sigma_2, g)$ where $\Sigma_2$ denotes a surface of genus 2. For more details please see the recent survey by Penskoi [17].

A problem in the study of $\Lambda_i(M,g)$-maximal metrics is that the functional $\Lambda_i(M,g)$ depends continuously on metric $g$ but is not differentiable. However, for any analytic deformation $g_t$, the left and right derivatives of the functional $\Lambda_i(M,g)$ with respect to $t$ exist (see Bando and Urakawa [1], Berger [2], El Soufi and Ilias [6]).

**Definition** ([5], [6], [16]). A Riemannian metric $g_0$ on a closed surface $M$ is called an extremal metric for the functional $\Lambda_i(M,g)$ if for any analytic deformation $g_t$ the following inequality holds,

$$\frac{d}{dt}\Lambda_i(M,g_t)\bigg|_{t=0^+} \cdot \frac{d}{dt}\Lambda_i(M,g_t)\bigg|_{t=0^-} \leq 0.$$ 

The mentioned metrics above are extremal since they are global maxima. However, extremal metrics are not necessarily maximal. El Soufi and Ilias...
proved in [5] that the only extremal metric for $\Lambda_1(\mathbb{T}^2, g)$ different from the maximal one is the metric on the Clifford torus.

Jakobson, Nadirashvili and Polterovich proved in [9] that the metric on the Klein bottle realized as the bipolar Lawson surface $\tilde{\tau}_{3,1}$ is extremal for $\Lambda_1(\mathbb{R}, g)$. Using this result El Soufi, Giacomini and Jazar proved in [4] that this metric is the unique extremal metric.

The following extremal metrics on families of tori and Klein bottles were investigated recently: Lapointe investigated metrics on bipolar Lawson surfaces $\tilde{\tau}_{r,m} \hookrightarrow \mathbb{S}^4$ in his 2008 paper [15], these surfaces are described below in Section 2; Penskoi investigated extremal metrics on Lawson surfaces $\tau_{m,n} \hookrightarrow \mathbb{S}^3$ and on Otsuki tori $O_{\mathbb{Q}} \hookrightarrow \mathbb{S}^3$ in his 2012 paper [18] and 2013 paper [19] respectively; Karpukhin investigated metrics on bipolar Otsuki tori $\tilde{O}_{\mathbb{Q}} \hookrightarrow \mathbb{S}^4$ and on a family of tori $M_{m,n} \hookrightarrow \mathbb{S}^5$ in his 2013 papers [12] and [11] respectively; and Karpukhin proved that the metrics on $\tau_{m,n}, \tilde{\tau}_{r,m}, O_{\mathbb{Q}}, \tilde{O}_{\mathbb{Q}},$ and $M_{m,n}$ are not maximal except metrics on $M_{1,1}$ (the equilateral torus) and $\tilde{\tau}_{3,1}$ in his 2013 paper [10]. Here $\hookrightarrow$ denotes an immersion.

In his paper [20], Penskoi introduced the following new three-parametric family of minimal surfaces in spheres, generalizing Lawson tau-surfaces, and investigated their extremal spectral properties.

**Theorem 1.** (Penskoi, [20]). Let $F_{a,b,c} : \mathbb{R}^2 \to \mathbb{S}^5 \subset \mathbb{R}^6$ be a three-parametric doubly-periodic immersion of the plane to the 5-dimensional sphere of radius 1 defined by the formula

$$F_{a,b,c}(x, y) = (\sin ax \tilde{\varphi}_1(y), \cos ax \tilde{\varphi}_1(y), \sin bx \tilde{\varphi}_2(y), \cos bx \tilde{\varphi}_2(y), \sin cx \tilde{\varphi}_3(y), \cos cx \tilde{\varphi}_3(y)),$$

where

$$\tilde{\varphi}_1(y) = \sqrt{\frac{b^2 + c^2 - a^2}{2(c^2 - a^2)}} \sin y, \quad \tilde{\varphi}_2(y) = \sqrt{\frac{a^2 + c^2 - b^2}{2(c^2 - b^2)}} \cos y,$$

$$\tilde{\varphi}_3(y) = \sqrt{\frac{a^2 + b^2 - c^2}{2(b^2 - c^2)}} \sqrt{1 - \frac{b^2 - a^2}{c^2 - a^2} \sin^2 y}$$

and

a) either $a, b, c$ are integers and $|c| > \sqrt{a^2 + b^2}$

b) or $a, b$ are nonzero integers and $|c| = \sqrt{a^2 + b^2}$.

Let $\mathcal{L} = \{(2\pi n, 2\pi m) | n, m \in \mathbb{Z}\}$ and $\tilde{F}_{a,b,c} : \mathbb{R}^2 / \mathcal{L} \to \mathbb{S}^5 \subset \mathbb{R}^6$ be the natural map induced by $F_{a,b,c}$. 
Let $S(a, b, c) = \frac{4\pi}{\sqrt{c^2-a^2}} \left(2(c^2-a^2)E\left(\sqrt{\frac{b^2-a^2}{c^2-a^2}}\right) - (c^2-a^2-b^2)K\left(\sqrt{\frac{b^2-a^2}{c^2-a^2}}\right)\right)$, where $K(\cdot)$ and $E(\cdot)$ are complete elliptic integrals of the first and second kind respectively as defined [7] by:

$$K(k) = \int_0^1 \frac{d\alpha}{\sqrt{1-k^2\alpha^2}}, \quad E(k) = \int_0^1 \frac{\sqrt{1-k^2\alpha^2}}{\sqrt{1-\alpha^2}}d\alpha.$$ 

Then the following statements hold:

1) The image $T_{a,b,c} = F_{a,b,c}(\mathbb{R}^2)$ is a minimal compact surface in the 5-dimensional sphere ($S^5$).

2) The case b) corresponds to Lawson tau-surfaces $\tau_{a,b} \cong T_{a,\pm\sqrt{a^2+b^2}}$. Distinct Lawson tau-surfaces correspond to unordered pairs $a, b \geq 1$ such that $(a, b) = 1$. The surface $T_{a,\sqrt{a^2+b^2}}$ is a Lawson torus $\tau_{a,b}$ if $a$ and $b$ are odd and $T_{a,-\sqrt{a^2+b^2}}$ is a Lawson Klein bottle $\tau_{a,b}$ if either $a$ or $b$ is even, where we assume $(a, b) = 1$.

3) In the case b) the metric induced on $\tau_{a,b}$ is extremal for the functionals $\Lambda_j(T^2, g)$ if $\tau_{a,b}$ is a Lawson torus or $\Lambda_j(K\mathbb{L}, g)$ if $\tau_{a,b}$ is a Lawson Klein bottle, where $j = \frac{2}{2} \left(\frac{a^2+b^2}{a}\right) + a + b - 1$ and $[\cdot]$ denotes the integer part.

The corresponding value of the functional is $\Lambda_j(\tau_{a,b}, g) = 8\pi aE\left(\frac{\sqrt{a^2+b^2}}{a}\right)$.

4) In the case a) for an integer $k \geq 1$ one has $T_{a,b,c} = T_{ka,kb,kc}$. Moreover, $T_{a,-b,c} = T_{a,b,-c} = T_{a,b,c}$ and $T_{b,a,c}$ is isometric to $T_{a,b,c}$. Hence, it is sufficient to consider non-negative integers $a, b, c$ satisfying conditions a) such that $(a, b, c) = 1$ and assume that $(a, b, c)$ and $(b, a, c)$ are equivalent.

5) In the case a) depending on the parity of $a, b$ and $c$ we have the following three subcases:

I) If $a$ and $b$ have different parity and $c$ is even then the surface $T_{a,b,c}$ is a Klein bottle and $\tilde{F}_{a,b,c} : \mathbb{R}^2/\mathcal{L} \to T_{a,b,c}$ is a double covering. The area of $T_{a,b,c}$ is equal to $\frac{1}{2}S(a, b, c)$.

II) If $a$ and $b$ are odd and $c$ is even then the surface $T_{a,b,c}$ is a torus and $\tilde{F}_{a,b,c} : \mathbb{R}^2/\mathcal{L} \to T_{a,b,c}$ is a double covering. The area of $T_{a,b,c}$ is equal to $\frac{1}{2}S(a, b, c)$.

III) Otherwise, the surface $T_{a,b,c}$ is a torus and $\tilde{F}_{a,b,c} : \mathbb{R}^2/\mathcal{L} \to T_{a,b,c}$ is a one-to-one map. The area of $T_{a,b,c}$ is equal to $S(a, b, c)$.

6) In the case a) the metric induced on the torus or the Klein bottle $T_{a,b,c}$ is extremal for the functional $\Lambda_j(T^2, g)$ or $\Lambda_j(K\mathbb{L}, g)$ respectively, where:

I) If $a$ and $b$ have different parity and $c$ is even then $j = a + b + c - 3$ except the case of $T_{a,0,c}$ where $j = a + c - 2$.

II) If $a$ and $b$ are odd and $c$ is even then $j = a + b + c - 3$.

III) Otherwise, $j = 2(a + b + c) - 3$ except the case of $T_{a,0,c}$ where $j = 2(a + c) - 2$ and the case of $T_{0,0,1}$ where $j = 1$.

The corresponding value of this functional $\Lambda_j(T_{a,b,c})$ is $S(a, b, c)$ in the subcases I) and II) and $2S(a, b, c)$ in the subcase III).

It was remarked in [20] that $\tau_{3,1}$ is isometric to $T_{1,0,2}$ but this proof was indirect and based on the uniqueness of the extremal metric for the first
eigenvalue of $\Delta$ on the Klein bottle, proved in [4].

The goal of the present paper is to show that in fact all bipolar Lawson surfaces $\tilde{\tau}_{r,m}$ are isometric to some $T_{a,b,c}$. The main result is the following theorem.

**Theorem 2.**

1. If $rm \equiv 0 \pmod{2}$ then the bipolar Lawson torus $\tilde{\tau}_{r,m}$ is isometric to the surface $T_{a,b,c}$ where $a = r - m$, $b = 0$, $c = r + m$.
2. If $rm \equiv 1 \pmod{4}$ then the bipolar Lawson torus $\tilde{\tau}_{r,m}$ is isometric to the surface $T_{a,b,c}$ where $a = \frac{r-m}{2}$, $b = 0$, $c = \frac{r+m}{2}$.
3. If $rm \equiv 3 \pmod{4}$ then the bipolar Lawson Klein bottle $\tilde{\tau}_{r,m}$ is isometric to the surface $T_{a,b,c}$ where $a = \frac{r-m}{2}$, $b = 0$, $c = \frac{r+m}{2}$.

2 Construction of bipolar Lawson surfaces

Let us now recall the construction of bipolar Lawson surface $\tilde{\tau}_{r,m}$ following Lapointe’s paper [15]. The Lawson surface $\tau_{r,m}$, with $r > m > 0$ and $(r, m) = 1$, is minimally immersed into $S^3$ by $I : \mathbb{R}^2 \to \mathbb{R}^4$, where

$$I(u, v) = (\cos ru \cos v, \sin ru \cos v, \cos mu \sin v, \sin mu \sin v).$$

The bipolar minimal surface $\tilde{\tau}_{r,m}$ of $\tau_{r,m}$ is the image of an exterior product of $I$ and $I^*$, where $I^*$ is a unit vector normal to $\tau_{r,m}$ and tangent to $S^3$,

$$I^*(u, v) = \frac{(m \sin ru \sin v, -m \cos ru \sin v, -r \sin mu \cos v, r \cos mu \cos v)}{\sqrt{r^2 \cos^2 v + m^2 \sin^2 v}}.$$

The explicit formula for $\tilde{I} = I \wedge I^* : \mathbb{R}^2 \to S^5 \subset \mathbb{R}^6$ is then

$$\tilde{I} = \frac{1}{\sqrt{r^2 \cos^2 v + m^2 \sin^2 v}} \begin{pmatrix}
-m \sin v \cos v \\
r \sin v \cos v \\
-r \cos^2 v \sin mu \cos ru + m \sin^2 v \sin ru \cos mu \\
r \cos^2 v \cos mu \sin ru + m \sin^2 v \cos ru \sin mu \\
-r \cos^2 v \sin mu \sin ru + m \sin^2 v \cos ru \cos mu \\
r \cos^2 v \cos mu \cos ru - m \sin^2 v \sin ru \sin mu
\end{pmatrix}.$$

It is known that $\tilde{\tau}_{r,m}$ actually lies in $S^4$, seen as an equator of $S^5$.

In [15] Lapointe proved that for the bipolar surface $\tilde{\tau}_{r,m}$ of a Lawson torus or Klein bottle $\tau_{r,m}$,

1. If $rm \equiv 0 \pmod{2}$, $\tilde{\tau}_{r,m}$ is a torus with an extremal metric for $\Lambda_{4r-2}$.
2. If $rm \equiv 1 \pmod{4}$, $\tilde{\tau}_{r,m}$ is a torus with an extremal metric for $\Lambda_{2r-2}$.
3. If $rm \equiv 3 \pmod{4}$, $\tilde{\tau}_{r,m}$ is a Klein bottle with an extremal metric for $\Lambda_{r-2}$.
The value of functional $\Lambda_{i}(\tau_{r,m})$ can be calculated as follows [15].

1. If $rm \equiv 0 \pmod{2}$, $\Lambda_{4r-2}(\tau_{r,m}) = 16\pi r E\left(\frac{\sqrt{r^2 - m^2}}{r}\right)$.
2. If $rm \equiv 1 \pmod{4}$, $\Lambda_{2r-2}(\tau_{r,m}) = 8\pi r E\left(\frac{\sqrt{r^2 - m^2}}{r}\right)$.
3. If $rm \equiv 3 \pmod{4}$, $\Lambda_{r-2}(\tau_{r,m}) = 4\pi r E\left(\frac{\sqrt{r^2 - m^2}}{r}\right)$.

3  Proof of Theorem 2.

3.1 Case $rm \equiv 0 \pmod{2}$

Let us prove that the bipolar Lawson surface $\tilde{\tau}_{r,m}$ when $rm \equiv 0 \pmod{2}$ is isometric to the surface $T_{a,b,c}$ where $a = r - m$, $b = 0$, and $c = r + m$.

The induced metric $g$ on $T_{a,b,c}$ is given by the formula [20]

$$
g = \frac{1}{2}(c^2 + (b^2 - a^2) \cos 2y)dx^2 + \frac{c^2 + (b^2 - a^2) \cos 2y}{2c^2 - a^2 - b^2 + (b^2 - a^2) \cos 2y}dy^2. $$

Set $b = 0$ and apply the change of variable $\sin y = \text{sn}(z, k)$, where $k = \frac{a}{\sqrt{a^2 - c^2}}$ and $\text{sn}(z, k)$ is a Jacobi elliptic function [7]. This implies

$$
g = \frac{1}{2}(c^2 - a^2 + 2a^2 \text{sn}^2(z, k)) \left(dx^2 + \frac{dz^2}{c^2 - a^2}\right). \quad (1)$$

Let us recall that the bipolar Lawson surface $\tilde{\tau}_{r,m}$ has the metric [15]

$$
\tilde{g} = \frac{(r^2 - (r^2 - m^2) \sin^2 v)^2 + r^2m^2}{r^2 - (r^2 - m^2) \sin^2 v} \left(du^2 + \frac{dv^2}{r^2 - (r^2 - m^2) \sin^2 v}\right).
$$

Begin by setting $r = \frac{a+c}{2}$, $m = \frac{c-a}{2}$ to rewrite the metric of $\tilde{\tau}_{r,m}$ as

$$
\tilde{g} = \frac{((a+c)^2 - 4ac \sin^2 v)^2 + (c^2 - a^2)^2}{(a+c)^2 - 4ac \sin^2 v} \left(\frac{du^2}{4} + \frac{dv^2}{(a+c)^2 - 4ac \sin^2 v}\right). \quad (1)
$$

Similarly, apply the change of variable $\sin v = \text{sn}(w, \tilde{k})$, where $\tilde{k} = \frac{2\sqrt{ac}}{a+c}$ arriving at the metric

$$
\tilde{g} = \frac{((a+c)^2 - 4ac \text{sn}^2(w, \tilde{k}))^2 + (c^2 - a^2)^2}{(a+c)^2 - 4ac \text{sn}^2(w, \tilde{k})} \left(\frac{du^2}{4} + \frac{dv^2}{(a+c)^2}\right). \quad (2)$$
The task is now to find the change of variable between metrics (1) and (2). Let us use the following transformation,

$$H_1(x, z) = \left(u, \frac{2\sqrt{c^2 - a^2}}{a + c}w + K(k)\right).$$

(3)

Then we obtain

$$g = \frac{1}{2}(c^2 - a^2 + 2a^2\text{sn}^2(z, k)) \left(dx^2 + \frac{dz^2}{c^2 - a^2}\right)$$

$$= \frac{1}{2} \left(c^2 - a^2 + 2a^2\text{sn}^2\left(\frac{2\sqrt{c^2 - a^2}}{a + c}w + K(k), k\right)\right) \left(du^2 + \frac{4dw^2}{(a + c)^2}\right)$$

$$= 2 \left(\frac{c^2 - a^2 + 2a^2\text{cn}^2\left(\frac{2\sqrt{c^2 - a^2}}{a + c}w, k\right)}{\text{dn}^2\left(\frac{2\sqrt{c^2 - a^2}}{a + c}w, k\right)}\right) \left(\frac{du^2}{4} + \frac{dw^2}{(a + c)^2}\right)$$

$$= 2 \left(\frac{c^2 - a^2 + 2a^2}{1 - k^2\text{sn}^2\left(\frac{2\sqrt{c^2 - a^2}}{a + c}w, k\right)}\right) \left(\frac{du^2}{4} + \frac{dw^2}{(a + c)^2}\right).$$

(4)

To continue, let $k' = \sqrt{1 - k^2}$. We use the following identities ([7], 13.22-23),

$$\text{sn}(k'u, \frac{ik}{k'}) = k'\frac{\text{sn}(u, k)}{\text{dn}(u, k)}, \quad \text{sn}(1 + k'u, \frac{1 - k'}{1 + k'}) = (1 + k')\frac{\text{sn}(u, k)\text{cn}(u, k)}{\text{dn}(u, k)}.$$ 

Let us now apply these identities to (4) and simplify,

$$g = 2 \left(\frac{c^2 - a^2 + 2a^2}{1 - k^2\frac{4(c^2 - a^2)\text{sn}^2(w, k) - 4(c^2 - a^2)\text{sn}^4(w, k)}{\text{dn}^2\left(\frac{2\sqrt{c^2 - a^2}}{a + c}w, k\right) - 4(c^2 - a^2)\text{sn}^2(w, k) - 4a^2\text{sn}^2(w, k) - 4a^2\text{sn}^4(w, k)}}{\text{dn}^2\left(\frac{2\sqrt{c^2 - a^2}}{a + c}w, k\right) - 4(c^2 - a^2)\text{sn}^2(w, k) - 4a^2\text{sn}^2(w, k) - 4a^2\text{sn}^4(w, k)}\right) \left(\frac{du^2}{4} + \frac{dw^2}{(a + c)^2}\right)$$

$$= \frac{(a + c)^2 - 4ac\text{sn}^2(w, k) + (c^2 - a^2)^2}{(a + c)^2 - 4ac\text{sn}^2(w, k) - 4a^2\text{sn}^2(w, k) - 4a^2\text{sn}^4(w, k)} \left(\frac{du^2}{4} + \frac{dv^2}{(a + c)^2}\right) = \tilde{g}.$$ 

When $rm \equiv 0 \mod 2$, $a = r - m$ and $c = r + m$ are both odd since $(r, m) = 1$. We have that $T_{a,b,c}$ is a torus and $\tilde{F}_{a,b,c} : \mathbb{R}^2 / \mathcal{L} \rightarrow T_{a,b,c}$ is a one-to-one map (Theorem 1). Apply change of variable $\sin y = \text{sn}(z, k)$, and we alternatively have $\tilde{F}_{a,b,c} : \mathbb{R}^2 / \tilde{\mathcal{L}} \rightarrow T_{a,b,c}$, where $\tilde{\mathcal{L}} = \{(2n\pi, 4mK(k)) | n, m \in \mathbb{Z}\}$. There is now a one-to-one correspondence between the rectangle $[0, 2\pi] \times [K(k), 5K(k)]$ and $T_{a,b,c}$. Our linear transformation (3) maps this rectangular domain as follows:
Let us prove that the bipolar Lawson surface \( \bar{r}_m \) has a one-to-one correspondence with \([0, 2\pi) \times [0, 2K(\tilde{k}))\).

Thus we obtained the desired isometry.

Using the values \( \Lambda_{2(a+c)-2}(T_{a,b,c}) \) and \( \Lambda_{4r-2}(\bar{\tau}_{r,m}) \) from [20] and [15], which are twice the areas of these surfaces, we can check that the areas of corresponding surfaces are the same. Note that \( S(a, b, c) = S(b, a, c) \) since \( T_{b,a,c} \approx T_{a,b,c} \).

\[
\Lambda_{2(a+c)-2}(T_{a,0,c}) = 2S(0, a, c) = \frac{8\pi}{c} \left[ 2c^2 E\left(\frac{a}{c}\right) - (c^2 - a^2)K\left(\frac{a}{c}\right) \right] \\
= 8\pi(a + c)E\left(\frac{2\sqrt{ac}}{a + c}\right) = \Lambda_{4r-2}(\bar{\tau}_{r,m}),
\]

which uses the identity \( E\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{2E(k)(1-k^2)K(k)}{1+k} \).

### 3.2 Case \( rm \equiv 1 \pmod{4} \)

Let us prove that the bipolar Lawson surface \( \bar{\tau}_{r,m} \) when \( rm \equiv 1 \pmod{4} \) is isometric to the three-parametric surface \( T_{a,b,c} \) where \( a = \frac{r-m}{2}, b = 0, \) and \( c = \frac{r+m}{2} \).

As before, after change of variable \( \sin y = \sin(z, k) \), the induced metric \( g \) on \( T_{a,b,c} \) is given by the formula

\[
g = \frac{1}{2}(c^2 - a^2 + 2a^2 \sin^2(z, k)) \left( dx^2 + \frac{dz^2}{c^2 - a^2} \right). \tag{5} \]

Let us set \( r = a + c, m = c - a \) to rewrite the metric of \( \bar{\tau}_{r,m} \) as

\[
\tilde{g} = \frac{((a + c)^2 - 4ac \sin^2 v)^2 + (c^2 - a^2)^2}{(a + c)^2 - 4ac \sin^2 v} \left( du^2 + \frac{dv^2}{(a + c)^2 - 4ac \sin^2 v} \right). 
\]

After the change of variable \( \sin v = \sin(w, \tilde{k}) \), the metric on \( \bar{\tau}_{r,m} \) becomes

\[
\bar{g} = \frac{((a + c)^2 - 4ac \sin^2(w, \tilde{k}))^2 + (c^2 - a^2)^2}{(a + c)^2 - 4ac \sin^2(w, \tilde{k})} \left( du^2 + \frac{dv^2}{(a + c)^2} \right). \tag{6} \]

The change of variable between metrics (5) and (6) is
\[ H_2(x, z) = \left( 2u, \frac{2\sqrt{c^2 - a^2}}{a + c}w + K(k) \right). \]  

(7)

Remark that the change of variable on \( z \) is the same as in the case \( rm \equiv 0 \) (mod 2).

When \( rm \equiv 1 \) (mod 4), \( a = \frac{r - m}{2} \) is even and \( c = \frac{r + m}{2} \) is odd. We have that \( T_{a,b,c} \) is again a torus and \( \hat{F}_{a,b,c} : \mathbb{R}^2 / \mathcal{L} \rightarrow T_{a,b,c} \) is a one-to-one map (Theorem 1). Let us again use the rectangle \([0, 2\pi) \times [K(k), 5K(k)]\) and \( \hat{F}_{a,b,c} : \mathbb{R}^2 / \hat{\mathcal{L}} \rightarrow T_{a,b,c} \), where \( \hat{\mathcal{L}} = \{(2n\pi, 4mK(k))|n, m \in \mathbb{Z}\} \). Now, our linear transformation (7) maps this rectangular domain as follows:

\[ H_2 ([0, 2\pi) \times [K(k), 5K(k)]) = [0, \pi) \times [0, \frac{2(a+c)K(k)}{\sqrt{a^2 - c^2}}) = [0, \pi) \times [0, 2K(\tilde{k})]. \]

Let us remark that when \( rm \equiv 1 \) (mod 4) and after change of variable \( \sin u = \sin(w, \tilde{k}) \), the bipolar Lawson torus \( \tilde{\tau}_{r,m} \) has a one-to-one correspondence with \([0, \pi) \times [0, 2K(\tilde{k})]\).

Thus we obtained the desired isometry.

Let us again use the values \( \Lambda_{2(a+c)-2}(T_{a,b,c}) \) and \( \Lambda_{2r-2}(\tilde{\tau}_{r,m}) \) to check that the areas of the corresponding surfaces are the same.

\[ \Lambda_{2(a+c)-2}(T_{a,0,c}) = 2S(0, a, c) = \frac{8\pi}{c} \left[ 2c^2 E \left( \frac{a}{c} \right) - (c^2 - a^2)K \left( \frac{a}{c} \right) \right] \]
\[ = 8\pi (a + c)E \left( \frac{2\sqrt{ac}}{a + c} \right) = \Lambda_{2r-2}(\tilde{\tau}_{r,m}). \]

3.3 Case \( rm \equiv 3 \) (mod 4)

Let us prove that the bipolar Lawson surface \( \tilde{\tau}_{r,m} \) where \( rm \equiv 3 \) (mod 4) is isometric to the three-parametric surface \( T_{a,b,c} \) where \( a = \frac{r - m}{2}, b = 0, \) and \( c = \frac{r + m}{2} \).

Let us remark that we need the same transformation (7) as the case \( rm \equiv 1 \) (mod 4).

When \( rm \equiv 3 \) (mod 4), \( a = \frac{r - m}{2} \) is odd and \( c = \frac{r + m}{2} \) is even. We have that \( T_{a,b,c} \) is a Klein bottle and \( \hat{F}_{a,b,c} : \mathbb{R}^2 / \mathcal{L} \rightarrow T_{a,b,c} \) is a double covering (Theorem 1). With change of variable \( \sin y = \sin(z, k) \) we now have as before a one-to-one correspondence between the rectangle \([0, \pi) \times [K(k), 5K(k)]\) and \( T_{a,b,c} \). Our linear transformation (7) maps this rectangular domain as follows:

\[ H_2 ([0, \pi) \times [K(k), 5K(k)]) = [0, \frac{\pi}{2}) \times [0, \frac{2(a+c)K(k)}{\sqrt{a^2 - c^2}}) = [0, \frac{\pi}{2}) \times [0, 2K(\tilde{k})]. \]
Let us remark that after change of variable \( \sin v = sn(w, \tilde{k}) \), the bipolar Lawson Klein bottle \( \tilde{\tau}_{r,m} \) has a one-to-one correspondence with \([0, \frac{\pi}{2}) \times [0, 2K(\tilde{k}))\).

Thus we obtained the desired isometry.

Finally, we can again use the values \( \Lambda_{a+c-2}(T_{a,b,c}) \) and \( \Lambda_{r-2}(\tilde{\tau}_{r,m}) \) to check that the areas of the corresponding surfaces are the same.

\[
\Lambda_{a+c-2}(T_{a,0,c}) = S(0, a, c) = \frac{4\pi}{c} \left[ 2c^2E\left(\frac{a}{c}\right) - (c^2 - a^2)K\left(\frac{a}{c}\right) \right]
= 4\pi(a + c)E\left(\frac{2\sqrt{ac}}{a + c}\right) = \Lambda_{r-2}(\tilde{\tau}_{r,m}).
\]

This completes the proof.

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