Two Dimensional Quantum Well of Gluons in Color Ferromagnetic Quark Matter

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We have recently pointed out that color magnetic field is generated in dense quark matter, i.e. color ferromagnetic phase of quark matter. Using light cone quantization, we show that gluons occupying the lowest Landau level under the color magnetic field effectively form a two dimensional quantum well (layer), in which infinitely many zero modes of the gluons are present. We discuss that the zero modes of the gluons form a quantum Hall state by interacting repulsively with each other, just as electrons do in semiconductors. Such a ferromagnetic quark matter with the layer structure of the gluons is a possible origin of extremely strong magnetic field observed in magnetars.

I. INTRODUCTION

Dense quark matters have been extensively analyzed and have been shown to possess various interesting phases. Most of these phases arise from the dynamical effects of quarks, e.g. the condensation of diquark pairs. The phases with the diquark condensation are called as color superconducting phases\[1\]. The phases have been argued to arise in sufficiently dense quark matters, where perturbative approximations are reliable owing to the asymptotic freedom of QCD. (Non interacting free quark gas forms a Fermi surface, but it is unstable against to an attractive force produced by one gluon exchange between a diquark channel, even if it is fairly weak. This instability leads to the condensation of the diquark pairs. Since the pairs carry color charges, color superconducting states arise.) On the other hand, quark matters are composed of not only quarks but also gluons. Our previous analyses\[2\] of the gluons in the dense quark matters have revealed a color ferromagnetic phase in which color magnetic field, \(B\), is generated spontaneously. The generation happens by not alignment of quark spins\[3\], but the vacuum fluctuation of gluons at one loop order; an effective potential of color magnetic field obtained with one loop approximation shows a non-trivial minimum\[4\] at \(B \neq 0\). The loop expansions are also reliable in the dense quark matters. Thus, we must determine which phase arises, the color superconducting phase or the color ferromagnetic phase in the sufficiently dense quark matter. We have shown\[2, 5\] that the color superconducting phase is favored in the limit of infinitely dense quark matter. We have also shown that the color ferromagnetic phase appears as an energetically more favorable state than the color superconducting state as we decrease the baryon densities of the quark matters. Recent analyses\[6\] of the instability of the gapless color superconductors suggest the existence of the phase transition from the color superconducting state to the color ferromagnetic state with decreasing the baryon density.

The quark matter may be present in the core of neutron stars. For instance, neutron stars with mass \(\simeq 1.6M_\odot\), and radius \(\simeq 10\text{km}\), have the average density \(\simeq 2.7\rho_n\), where \(\rho_n\) represents the normal nuclear density approximately given by \(2.8 \times 10^{14}\text{g/cm}^3\). If the density in the core of the neutron stars reaches the density \(\simeq 6\rho_n\), nucleons overlap and they would melt into quark matter. Models\[7\] of neutron stars involving only nuclear components predict that the density at the center reaches \((6 \sim 10)\rho_n\), or more. Hence, it is very probable that the cores of neutron stars involve quark matter. Such a quark matter would not be sufficiently dense for the color superconducting states to appear. But it would be sufficiently dense for the color ferromagnetic states to appear. We have shown in the previous paper\[8\] that extremely strong electromagnetically magnetic fields (\(\sim 10^{15}\text{Gauss}\)) observed in magnetars can be produced by the color ferromagnetic quark matter. Thus, the detail examination of such a quark matter is an important issue. In the present paper we show that the gluons form effectively a two dimensional quantum well in the color ferromagnetic quark matter. The presence of the quantum well is an important ingredient for the generation of the strong magnetic field in the magnetars.

As was shown about 30 years ago, the naive ferromagnetic state with color magnetic field is unstable\[9, 10\] in QCD. Gluons occupying the lowest Landau level under the color magnetic field have imaginary energies, that is, the gluons are unstable. The situation is very similar to the case of Higgs models in which Higgs fields have imaginary energies in a naive false vacuum without any condensation of the fields. The real vacuum is a condensed state of the Higgs field. Thus, we expect that such unstable gluons condense to form a stable ground state under the color magnetic field. But the problem of finding the stable ground state\[2, 10, 11\] is not so easy because there are infinitely many unstable gluons characterized by their angular momenta, \(m (\infty > m \geq 0)\) in the lowest Landau level. All of them condense to form an appropriate ground state. In the Higgs model, only the spatially uniform component of the
Higgs field condenses to form a uniform ground state. Thus, it is easy to find the classical solution of the Higgs field representing the ground state. In this paper we show that the unstable gluons condense to form effectively a two dimensional quantum well. Namely, they make a layer perpendicular to the color magnetic field. Furthermore, we show that there are infinitely many excited states of gluons with zero energy (zero modes) in the quantum well. In other words, there exist infinitely many degenerate ground states of the gluons. Each of the zero modes induces a color magnetic field to screen partially the original one. Thus, it apparently seems to indicate expelling the magnetic field or squeezing it by exciting the zero modes just like superconductors. This is because the condensed state of the unstable gluons is expected to show a Meissner effect just as in Higgs models. But, expelling or squeezing the magnetic field is energetically unfavorable in the gauge theory. We show that contrary to superconducting states, a quantum Hall state of the gluons comes out as a gapped stable state in the quantum well.

In condensed matter physics two dimensional quantum wells are fabricated by connecting two semiconductors, for example, GaAs and AlGaAs[12]. The junction is a surface with small width (quantum Hall state of the gluons comes out as a gapped stable state in the quantum well. In the longitudinal direction, any excitation energies in the direction are finite and much larger than ones we are concerned with. On the other hand, excitations energies in the two dimensional transverse directions start to grow at zero energy, in other words, the excitations are gapless. This is a typical feature of electrons confined in the two dimensional quantum well fabricated in semiconductors.

As we will show, the similar excitation spectra of the gluons to those of the electrons are obtained as a result of the condensation of the unstable gluons under the color magnetic field. This implies that the gluons form effectively a two dimensional quantum well in the ferromagnetic dense quark matter.

In the next section, we introduce a light cone formulation[13, 14] of SU(2) gauge theory whose gauge fields are referred as gluons. We extract "zero modes"[15, 16] by using finite volume in the longitudinal direction and neglecting the zero modes[17]. This leads to a simple form of Hamiltonian with the use of the light cone gauge, but we loose a merit of the light cone formulation; the real vacuum is a Fock vacuum. We do not address the spontaneous generation of the color magnetic field in the light cone formulation. We simply assume the presence of the field. In the section (3) we pick up only states of gluons in the lowest Landau level. Only the states are relevant to the formation of the ground states of the gluons. In the section (4) we examine the classical and quantum structures of the ground states. We find that the ground states are not coherent states, but approximate eigenstates of number operators of the gluons. In the section (5) we analyze excitation energies in the ground states. We find that two dimensional quantum well is formed effectively. We also find that there are infinitely many states with zero energy in the well. In the section (6) we examine the effects of such states on the structure of the ground states. Although each of the states partially screen the color magnetic field, their excitations do not completely screen the field but form a fractional quantum Hall state in the well. In the last section (7) we summarize our results.

II. LIGHT CONE FORMULATION OF SU(2) GAUGE THEORY

First of all, we explain our notations in the light cone quantization[13, 14] of gauge fields. We use the light cone time coordinate, \( x^+ = (x^0 + x^3)/\sqrt{2} \) and longitudinal coordinate, \( x^- = (x^0 - x^3)/\sqrt{2} \). Transverse coordinates are denoted by \( x^i \), or \( \vec{x} \). We assume a finite length, \( -L \leq x^- \leq L \), in the longitudinal space and impose a periodic boundary condition such that \( A^a_i(x^- = L) = A^a_i(x^- = -L) \). Then, corresponding momentum becomes discrete denoted by \( p^+_n = n\pi/L \) with integer \( n \). Light cone components of gauge fields, \( A^+, A^-, A_i \), are defined similarly.

Then, the Hamiltonian, \( H \), with the light cone gauge, \( A^+ = 0 \), is given by

\[
H = \frac{1}{4} F_{ij}^a F^{a}_{ij} + \frac{g^2}{2} \rho^a \rho^a - \frac{1}{\partial^2} \rho^a,
\]

with field strength, \( F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a + g \epsilon_{abc} A_b^c A_j^c \), where color indices \( a \) run from 1 to 3 and space indices, \( i, j \) run from 1 to 2. Color charge, \( \rho^a \), is defined by

\[
\rho^a \equiv (D_i A_i)^a = \rho^a_{\text{quark}} = (\partial_i \delta^{ab} + g \epsilon^{abc} A_i^b) A_i^b + \rho^a_{\text{quark}}
\]

where \( \rho_{\text{quark}} \) denotes the contribution of quarks. Here we treat it classically and assume it being spatially uniform and pointing to \( \sigma_3 \) in color space. \( A_i^a \) denotes the gauge potential of color magnetic field generated spontaneously,
which is assumed to direct into $\sigma_3$ in color SU(2); $B = \partial_1 A^B_1 - \partial_2 A^B_2$. In this paper we do not address a question of the spontaneous generation of the magnetic field, $B$, in the light cone formulation. We simply assume the presence of $B$.

In the above equations we have neglected a dynamical gauge potential, $A_i^{\alpha=3}$ aside from the classical one, $A_i^B$ since it does not couple directly with $A_i^B$. We have only taken dynamical gauge fields, $A_i^{a=1,2}$ perpendicular in color space to the color magnetic field. They form Landau levels under $B$.

We make a comment that our treatment of "zero mode" in the light cone quantization is similar to the one used by Thorn [17]. We quantize gauge fields in the finite volume, $-L \leq x^- \leq L$, and neglect "zero modes" of the fields. Consequently, Hamiltonian becomes a simple form involving at most quartic terms of creation or annihilation operators in addition to quadratic ones. As has been shown in a two dimensional model of scalar field [17] and a Higgs model of complex scalar field [18], the true ground states can be gripped even if we neglect the "zero modes" of the fields, at least in the limit of $L \to \infty$. We assume that it also holds in the gauge theory. We may justify neglecting the "zero modes" in the analysis of dense quark matter as follows. That is, our concern is not the real vacuum, but a ground state of gluons in dense quark matter. The "zero modes" may play an important role in the real vacuum of strongly interacting gluons. But, they may not play such a role in a ground state of gluons weakly interacting with each other in the dense quark matter. Typical energy scale of quarks and gluons in the quark matter is given by the chemical potential of the quarks and is much larger than $\Lambda_{QCD}$. In such dense quark matter the "zero modes" do not play an important role for realizing the ground state. It is similar to the case of QCD at high energy scattering where the typical energy scale is much higher than $\Lambda_{QCD}$. Hence, the "zero modes" do not play an important role for realizing so-called color glass condensates [19]. Therefore, it is reasonable to neglect the "zero modes" of the gluons in the quark matter.

In the light cone gauge only dynamical variables are transverse components, $A_i^b$ of gauge fields. This can be expressed in terms of creation and annihilation operators,

$$A_i^b = \sqrt{\frac{\pi}{L}} \sum_{p^+ > 0} \frac{1}{2\pi p^+} \left( a_{i,p^+}^b (x^+, \vec{x}) e^{-ip^+ x^-} + a_{i,p^+}^{b\dagger} (x^+, \vec{x}) e^{ip^+ x^-} \right),$$

(3)

with $p^+ = \pi n/L$ ( integer, $n \geq 1$ ), where operators, $a_{i,p^+}^{\dagger}$, satisfy the commutation relations,

$$[a_{i,p^+}^b (x^+, \vec{x}), a_{j,k^+}^{b\dagger} (x^+, \vec{y})] = \delta_{ij} \delta^{bc} \delta_{p^+, k^+} \delta(\vec{x} - \vec{y})$$

with other commutation relations being trivial. As we have mentioned, we have neglected the "zero modes", $p^+ = 0$, of the gauge fields.

Then, the gauge fields satisfy the equal time, $x^+$, commutation relation,

$$[\partial_-, A_i^b (x^+, x^-, \vec{x}), A_j^{\dagger b} (x^+, y^-, \vec{y})] = -i \delta_{ij} \delta^{ab} \delta(\vec{x} - \vec{y}) \left( \delta(x^- - y^-) - \frac{1}{2L} \right),$$

(4)

where the last factor, $1/2L$, in the right hand side of the equation comes from neglecting the "zero modes" of the gauge fields.

We should mention that the second term in $H$ represent a Coulomb interaction. It is derived by solving a constraint equation, $\partial_+ A^- = \rho$, that is, Gauss law associated with the light cone gauge condition, $A^- = 0$. In order to assure that the gauge field, $A^-$ is periodic in $x^-$, the zero mode of $\rho$ ($\rho \propto \sum_{n=\text{integer}} \rho_n e^{i \pi n x^- / L}$) must vanish; $\rho_{n=0} = 0$.

Then, the operation of $1/(-\partial_+^2)$ is well defined. The condition of $\rho_{n=0} = 0$ implies that the color charge, $\int_{-L}^{L} dx^- \rho$, vanishes and is consistent with our postulate, $A_{-n}^- = 0$.

We now rewrite the Hamiltonian in terms of "charged vector fields", $\Phi_i = (A_i^1 + i A_i^2)/\sqrt{2}$, which are decomposed into spin parallel ($\Phi_p = (\Phi_1 + \Phi_2)/\sqrt{2}$) and anti-parallel components ($\Phi_{ap} = (\Phi_1 - \Phi_2)/\sqrt{2}$). These fields transform as Abelian charged fields under the $U(1)$ gauge transformation, $A_i \to U^† A_i U + U^† \partial U$ with $U = \exp(i \theta \sigma_3)$.

Then, using the fields, $\Phi_p$ ($\Phi_{ap}$), we obtain the following Hamiltonian,

$$H = \frac{1}{2} \partial^2 + \Phi_p^† (-\vec{D}^2 - 2gB) \Phi_p + \Phi_{ap}^† (-\vec{D}^2 + 2gB) \Phi_{ap}$$

$$+ \frac{g^2}{2} \left( |\Phi_p|^2 - |\Phi_{ap}|^2 \right)^2 + \frac{g^2}{2} \rho \left( \frac{1}{-\partial_+^2} \right) \rho,$$

(5)

with $\vec{D} = \vec{\partial} + ig A^B$, where $\rho$ is given by

$$\rho = i (\Phi_p^† \partial_- \Phi_p - \partial_- \Phi_p^† \Phi_p + \Phi_{ap}^† \partial_- \Phi_{ap} - \partial_- \Phi_{ap}^† \Phi_{ap}) + \rho_{\text{quark}},$$

(6)
The first term in eq(5) represents the classical energy of the color magnetic field, and second (third) term does the kinetic energy of the charged gluons with spin parallel (anti-parallel) under the color magnetic field, \( B \). The terms, \( \pm 2g |\Phi_{p,n}|^2 \), represent anomalous magnetic moments of the charged gluons. The forth term represents the energy of the repulsive self-interactions. The last term represents the Coulomb energy coming from the second term with \( \rho^\alpha=3 \) component in eq(1).

**III. GLUONS IN THE LOWEST LANDAU LEVEL**

As we have mentioned, the ground state of the gluons in the system is determined by the gluons occupying the lowest Landau level. Since eigenstates of the operator \( \vec{D}^2 \) are classified by Landau levels, the second and the third terms in the Hamiltonian of eq(5) can be rewritten as,

\[
\sum_{n=0,1,2,}\left(\Phi^\dagger_{p,n}(2n-1)gB\Phi_{p,n} + \Phi^\dagger_{a,p,n}(2n+3)gB\Phi_{a,p,n}\right),
\]

where the fields, \( \Phi_{p,n} \) (\( \Phi_{a,p,n} \)) denote operators in the Landau level specified by integer \( n \). (We have implicitly assumed integration over the transverse directions in the above equation.)

Now, we take only the field, \( \Phi_{p,n=0} \) in the lowest Landau level, \( n=0 \), that is, the component having negative kinetic energy. Obviously, the other components of the gluons have positive energies so that the ground state is the state with no such gluons. The gluons with negative kinetic energies play important roles for the formation of the ground state, at least, in the limit of strong magnetic field, \( B \). The gluons correspond to the unstable gluon in our previous discussions\[2, 9\] with the use of the time-like quantization. (The unstable gluons in the time-like formulation have imaginary energies, while the gluons corresponding to the unstable gluons can have real, but negative energies in the light cone formulation.) Therefore, we obtain the following reduced Hamiltonian for analyzing the ground state of the system,

\[
H_r(\Phi) = -gB|\Phi|^2 + \frac{g^2}{2} |\Phi|^4 + \frac{g^2}{2} \rho_r \frac{1}{(-\partial^2)} \rho_r,
\]

with \( \rho_r \equiv i(\Phi^\dagger \partial_\Phi - \partial_\Phi \Phi^\dagger) + \rho_{quark} \), where we have put \( \Phi \equiv \Phi_{p,n=0} \) for simplicity. The field, \( \Phi \), can be expressed by using creation and annihilation operators,

\[
\Phi = \sqrt{\frac{\pi}{L}} \sum_{p>0, m=0,1,2,} \frac{1}{\sqrt{2\pi p}}(a_{p,m}\phi_m(\vec{x})e^{-ipx} + b^\dagger_{p,m}\phi^*_m(\vec{x})e^{ipx}),
\]

where simplified notation such as \( x = x^- \) and \( p = p^+ \) is exploited and will be used below. \( \phi_m(\vec{x}) = g_m z^m \exp(-|z|^2/4l^2) \) represents the normalized eigenfunction of \( \vec{D}^2 \) with angular momentum, \( m \), around the color magnetic field in the lowest Landau level; \( f d^2\vec{x} \phi_m^\dagger \phi_n = \delta_{m,n} \) with \( z = x_1 + ix_2 \), and \( g_m^2 = \frac{1}{4m(|2m|^2+1)} \) with \( l^2 = 1/gB \). \( a_{p,m} \) and \( b_{p,m} \) satisfy the commutation relations; \( [a_{p,m}, a^\dagger_{k,n}] = \delta_{p,k} \delta_{m,n}, \quad [b_{p,m}, b^\dagger_{k,n}] = \delta_{p,k} \delta_{m,n}, \) and others = 0.

**IV. GROUND STATE**

When we express the first term in eq(5) in terms of the operators, \( a_{p,m} \) and \( b_{p,m} \),

\[
\int_{-L}^{L} dx \int d^2\vec{x} : -gB|\Phi|^2 := -gB \sum_{p>0, m} \frac{1}{p}(a^\dagger_{p,m}a_{p,m} + b^\dagger_{p,m}b_{p,m}),
\]

we find that there exist states with lower energies than a trivial Fock vacuum, \( |\text{vac}\rangle; a_{p,m}|\text{vac}\rangle = b_{p,m}|\text{vac}\rangle = 0 \). Namely, the gluons in the lowest Landau level are produced spontaneously to form a state with lower energy than that of the vacuum. (This fact that the Fock vacuum is not a real ground state results from neglecting "zero modes" in the light cone quantization\[14, 18\].) The production of the gluons is limited by the second term in eq(5) representing repulsion among the gluons. This is similar to the case of Higgs model. Actually, the Hamiltonian in eq(5) looks
like the one of Higgs model. But, there are several differences we should note between these two models. The first one is that the field, \( \Phi \) has no spatially uniform components; it is written in terms of the wave functions, \( \phi_m \), in the lowest Landau level. On the other hand, Higgs field has a spatially uniform component with zero momentum. The second one is that there are infinitely many degenerate unperturbative states in eq(11). For instance, the state, \( |p,m\rangle = a_{p,m}^\dagger |\text{vac}\rangle \), is degenerate with the states, \( |p,n\rangle \ (n \neq m) \). On the other hand, there is no such degeneracy in Higgs model. We will see below that the presence of the infinitely many degenerate unperturbative states give rise to the degeneracy in the ground state.

According to the standard procedure, we minimize the classical energy of the field, in order to find the ground state of the Hamiltonian in eq(8). First, we minimize the Coulomb energy in eq(8). Since the energy is positive semi definite, the field configuration giving zero Coulomb energy minimizes the energy. Such a field is given by \( \Phi_c = \exp(-ip_0x)\phi(\vec{x}) \) with arbitrary \( p_0 = n_0\pi/L \neq 0 \). That is, the dependence of the longitudinal momentum of the field is determined by minimizing the Coulomb energy. (We will discuss below more appropriate treatment of the Coulomb energy in the ground state.)

Inserting the field, \( \Phi_c = \exp(-ip_0x)\sum_{n=0,1,2,\ldots} a_n\phi_n(\vec{x}) \), into the remaining two terms in Hamiltonian, eq(8), we find that

\[
\frac{H_r(\Phi_c)}{2L} = \int \frac{dx^-d\vec{x}}{2L} \left( -gB|\Phi_c|^2 + g^2|\Phi_c|^4 \right) = -gB \sum_n |a_n|^2 + \frac{g^2}{2} \sum_n a_n^*a_n a_n^*a_n H_{n_1,n_2,n_3,n_4} \tag{11}
\]

where we have put

\[
\Phi_c = \exp(-ip_0x) \sum_{n=0,1,2,\ldots} a_n\phi_n(\vec{x}), \quad \text{and} \quad H_{n_1,n_2,n_3,n_4} = \int d\vec{x} \phi_n^*\phi_n^*\phi_n^*\phi_n.
\tag{12}
\]

Hence, the configurations minimizing the energy of eq(11) must satisfy

\[
0 = -gB a_n + g^2 a_n^*a_n H_{n_1,n_2,n_3,n_4} \equiv \sum_m T_{n,m} a_m \tag{13}
\]

with \( T_{n,m} = -gB\delta_{n,m} + g^2 \int d\vec{x} |\Phi_c|^2 \phi_n^*\phi_m \). For non-trivial solutions of \( a_n \) to exist, the determinant of \( T \) must vanish; \( \det T = 0 \).

One of such solutions is given by

\[
\Phi_c = a \exp(-ip_0x)\phi_0(\vec{x}), \quad \text{with} \quad a = \frac{\sqrt{4\pi}}{g}. \tag{14}
\]

There are other solutions degenerate with this solution. Namely, the classical ground state is not determined uniquely. This is due to the presence of infinitely degenerate unperturbative states as we have pointed above.

The color charge density, \( \rho_r \), and the longitudinal momentum, \( P_L \), of these solutions are given by

\[
\rho_r = 2p_0|\phi(\vec{x})|^2 + \rho_{\text{quark}} \quad \text{and} \quad P_L = 2Lp_0^2 \sum_m |a_m|^2, \tag{15}
\]

The color charge, \( Q = \int_{-L}^{L} dx^- \rho_r \) must vanish for the operator, \( 1/\partial^2 \) to be well defined. Furthermore, In order for \( H_r(\Phi_c)P_L \) to be finite as \( L \to \infty, p_0 \) goes to zero in the limit, just as \( p_0 \propto 1/L \to 0 \). Note that the quantity of \( H_r(\Phi_c)P_L \) is invariant under longitudinal boosts, just like masses of particles.

Here we wish to comment that as far as the classical solutions of the ground state are concerned, the color charge, \( Q \) does not vanish. Since \( |\phi(\vec{x})|^2 = \sum_m a_m^*a_m \phi_m^*(\vec{x})\phi_m(\vec{x}) \) is never uniform in \( \vec{x} \), the color charge of gluons does not cancel the color charge of quarks, \( \rho_{\text{quark}} \), which is assumed to be uniform. The classical solutions correspond to coherent states. Thus, the fact that the classical color charge, \( Q \) does not vanish, implies that the real ground state is not a coherent state of the field, \( \Phi \).

More appropriate treatment of the ground state is in the following. Gluons are produced due to the term, \( -gB|\Phi|^2 \), in the Hamiltonian, but they never condense to form the classical solutions as discussed just above. We speculate that they form a state with approximately definite number of particles, or anti-particles created by the operators, \( a_{p_0,m}^\dagger \) or \( b_{p_0,m}^\dagger \), that is, the eigenstates of the number operators; \( a_{p_0,m}^\dagger a_{p_0,m}(G) = p_0a(p_0,m)(G) \) and \( b_{p_0,m}^\dagger b_{p_0,m}(G) = p_0b(p_0,m)(G) \). Here, we denote the number of the particles, \( a \), or anti-particles, \( b \) as \( p_0a(p_0,m) \) or \( p_0b(p_0,m) \),
respectively. Assuming that the ground state is composed of such eigenstates of the number operators, we evaluate the expectation value of the Hamiltonian,

\[
\langle G | H_c | G \rangle = -gB \sum_{p>0,m} \left( a(p,m) + b(p,m) \right) + \\
+ \frac{g^2}{2L} \sum_{p,q>0,m,n} \left( a(p,m)a(q,n) + b(p,m)b(q,n) + 2a(p,m)b(q,n) \right) N_{m,n} \\
+ \frac{g^2}{2L} \sum_{p\neq q>0} \left( \frac{p+q}{p-q} \right)^2 \sum_{m,n} \left( a(p,m)a(q,n) + b(p,m)b(q,n) \right) N_{m,n} \\
+ \frac{g^2}{2L} \sum_{p,q>0} \left( \frac{p-q}{p+q} \right)^2 \sum_{m,n} 2a(p,m)b(q,n) N_{m,n},
\]

(16)

with \( H_{m,n} \equiv H_{m,n,n} = (m+n)!(\pi m! n!)^{2m+n+2} 2^{m+n+1} \), where the Hamiltonian is normal-ordered. The condition of \( p \neq q \) in the third term comes from the regularization of the operator, \( 1/\partial^2 \) in the Coulomb energy.

In eq(16) the first term represents the kinetic energy in the Landau level and the second term does the energy of the repulsion between gluons. These two terms are denoted by \( E \). The third and forth terms represent the Coulomb energy between gluons and are denoted by \( E_{\text{Coulomb}} \). Obviously, these terms are non negative except for the first one. Therefore, we can find a ground state with the lowest energy, \( \langle G | H_c | G \rangle \equiv E + E_{\text{Coulomb}} \), by minimizing the first two terms, \( E \) and the last two terms, \( E_{\text{Coulomb}} \), respectively.

It is easy to minimize the first two terms, i.e. \( E \). The minimum of \( E \) is given by a set of values, \( c^+(m) \equiv \sum_{p>0} (a(p,m) + b(p,m)) \), since \( E = -gB \sum_{m} c^+(m) + g^2/(2L) \sum_{m,n} c^+(m) H_{m,n} c^+(n) \). Thus, minimizing the energy, \( E \), does not determine the distribution of the longitudinal momentum, \( p = p^+ \), in \( a(p,m) \) or \( b(p,m) \). It simply gives the summation over the momentum, \( p \), namely, \( c^+(m) \propto gBL/g^2 \). The dependence on \( p \) is determined only by minimizing the Coulomb energy, \( E_{\text{Coulomb}} \). The energy, \( E_{\text{Coulomb}} \geq 0 \), can be minimized easily by assuming that the ground state depends only on a single momentum, \( p = p_0 \), that is, \( a(p,m) \propto \delta_{p,p_0} \) and \( b(q,n) \propto \delta_{q,p_0} \). Namely, the ground state involves only gluons with the single longitudinal momentum, \( p_0 \). This distribution of the momentum leads to the minimum, \( E_{\text{Coulomb}} = 0 \). ( Any other distributions with the dependence on various momenta give rise to higher energies ( \( > 0 \) ). ) This fact is very result used when we have discussed classical solutions, \( \Phi_c \propto \exp(-ip_0x) \), of the ground state.

We should note that there are infinitely many solutions, \( c^+(m) = a(p_0,m) + b(p_0,m) \), minimizing the energy, \( E(c^+) + E_{\text{Coulomb}} \), in eq(16). This is because \( c^+(m) = a(p_0,m) + b(p_0,m) = a(p_0,m) - 1/p_0 + b(p_0,m) + 1/p_0 \). Namely, the creation of an anti-particle with quantum number, \( m \), along with the annihilation of a particle with quantum number, \( m \), does not change the energy, \( E(c^+) \), with \( E_{\text{Coulomb}} = 0 \). This implies the presence of infinitely many excitations with zero energy in a ground state \( |G\rangle \).

Using the state, \( |G\rangle \), the color charge is given by \( Q = \int_L L dx - \langle G | \rho_c | G \rangle = 2p_0 \sum_m \left( a(p_0,m) - b(p_0,m) \right) |\phi_m(x)|^2 + 2L \rho_{\text{quark}} \). In order for this charge to be uniform, it is sufficient to take \( a(p_0,m) - b(p_0,m) \) as independent of \( m \), since \( \sum_m |\phi_m(x)|^2 = gB/2\pi \). Putting \( c^-(p_0) \equiv \left( a(p_0,m) - b(p_0,m) \right) \), we can rewrite the color charge such as \( Q = p_0 e^{-i(p_0)gB/\pi} + 2L \rho_{\text{quark}} \). Therefore, the color charge can vanish when we take \( p_0 e^{-i(p_0)} = -2\pi L \rho_{\text{quark}}/gB \). The assumption that the ground state is an approximate eigenstate of the number operators is consistent with the condition of the color charge, \( Q \), to vanish.

The longitudinal momentum also vanishes in the limit of \( L \to \infty \). The longitudinal momentum is given such as \( P_L = p_0 \sum_m \left( p_0 a(p_0,m) + p_0 b(p_0,m) \right) = p_0^2 \sum_m c^+(m) \). That is, it is the sum of momentum, \( p_0 \), each particles carries. We remind you that the number of the particles with \( p = p_0 \) and angular momentum, \( m \), is given by \( p_0 a(p_0,m) + p_0 b(p_0,m) \). Since \( c^+(m) (p_0) \) goes to infinity (zero) such as \( c^+ \propto L \) (\( p_0 \propto 1/L \)) in the limit of \( L \to \infty \), \( P_L \) goes to zero.

In the above discussion about classical solutions we have assumed that the ground state is a coherent state of the field operator, \( \Phi \). But, this leads to the nonvanishing color charge although it must vanish for the regularity of \( 1/\partial^2 \). In the more appropriate argument, we have assumed that the ground state is an eigenstate of the number operators. This leads to the color charge to vanish. When the number, \( p_0 a(p_0,m) + p_0 b(p_0,m) \), of the particles in each state specified by \( m \) is sufficiently large, we may use approximately the coherent state for such a state. Then, the ground state can be represented by the classical solutions, although \( \langle G | \Phi_c | G \rangle = 0 \). In the case classical Coulomb energy should be taken to vanish for consistency. In the next section, we examine excitation energies by using the classical ground state.
V. EXCITATION ENERGIES ON THE GROUND STATE, $(\Phi) = \Phi_c$

Now, we wish to examine the excitation energies and show that the gluons form effectively a two dimensional quantum well. Supposing that the classical solutions, $\Phi_c$, approximately describe the ground state, we put the field operator such as $\Phi = \Phi_c + \delta \Phi$ in the Hamiltonian, $H_r := H_0 + H_{\text{Coulomb}}$, and diagonalizing it by taking only quadratic terms of $\delta \Phi$, 

$$H_0 = \int_{-L}^L dx(-gB|\Phi_c + \delta \Phi|^2 + \frac{\hbar^2}{2m}|\vec{\Phi}|^2) 
\simeq \int_{-L}^L dx(-gB|\Phi_c|^2 + \frac{\hbar^2}{2m}|\vec{\Phi}|^2) 
- \sum_{p>0} \frac{gB}{p} (\alpha_p^\dagger \alpha_p + \beta_p^\dagger \beta_p) + \sum_{p>0} \frac{2\hbar^2}{p} (\alpha_p^\dagger \alpha_p + \beta_p^\dagger \beta_p) + \sum_{p>0} \frac{2\hbar^2}{p} (\alpha_p^\dagger \beta_p e^{i2p_0 x} \Phi_c^2 + h.c.)$$

and 

$$H_{\text{Coulomb}} = \frac{g^2}{2} \sum_{0<p<\infty} \frac{1}{\sqrt{2\hbar p}} \sum_{n,m} \frac{1}{\sqrt{2\hbar p}} \Phi_c |(p+q+q_0)(p-q)| \delta_{2p_0-p+q} \Phi_c$$

with 

$$\delta \Phi = \sqrt{\frac{\hbar}{2\pi}} \sum_{0<p<\infty} \frac{1}{\sqrt{2\hbar p}} (\alpha_p^\dagger \bar{x}) e^{-ipx} + \beta_p^\dagger \bar{x} e^{ipx}$$

where 

$$\alpha_p(\bar{x}) \equiv \sum_{m=0,1,2} \alpha_{p,m} \phi_m(\bar{x})$$

It is interesting to see that the modes of $\alpha_p$ and $\beta_p$ with different longitudinal momenta do not mix with each other in $H_0$ and in the first term of $H_{\text{Coulomb}}$. The mixing between the modes with different momenta only arises owing to the interactions represented by the remaining two terms in $H_{\text{Coulomb}}$. Furthermore, the modes with $p = p_0$ does not mix with the other modes with $p \neq p_0$ even if we take into account all of the interactions in eq(17) and eq(18). This implies that the "transverse modes" with $p = p_0$ decouples from the other "longitudinal modes" with $p \neq p_0$.

It is easy to diagonalize the transverse modes, 

$$\int d\bar{x} \frac{1}{p_0} (-gB + 2g^2|\Phi_c|^2)(\alpha_{p_0,0}^\dagger \alpha_{p_0} + \beta_{p_0,0}^\dagger \beta_{p_0}) + g^2(e^{i2p_0 x} \Phi_c^2 \alpha_{p_0,0} \beta_{p_0} + h.c.) \equiv \int d\bar{x} \frac{1}{p_0} \left( -gB + 3g^2|\Phi_c|^2 \alpha_{p_0,0}^\dagger \alpha_{p_0} + (-gB + g^2|\Phi_c|^2) \beta_{p_0}^\dagger \beta_{p_0} \right)$$

with $\alpha'_{p_0} \equiv (\alpha_{p_0} + \beta_{p_0})/\sqrt{2}$ and $\beta'_{p_0} \equiv (\alpha_{p_0} - \beta_{p_0})/\sqrt{2}$. Here, the second term can be rewritten as 

$$\int d\bar{x} \frac{1}{p_0} (-gB + g^2|\Phi_c|^2) \beta_{p_0}^\dagger \beta_{p_0} = \frac{1}{p_0} \sum_{n,m} T_{n,m} b_n^* b_m^*$$

where $T_{n,m}$ has been defined in eq(13) and $\beta_{p_0}$ has been expanded in term of the eigenfunctions of the lowest Landau level; $\beta_{p_0} \equiv \sum_m b_m^* \phi_n(\bar{x})$. Since det$T = 0$ as explained in eq(13), the transverse modes are gapless. On the other hand, as the first term in eq(20) can be rewritten as $(-gB + 3g^2|\Phi_c|^2) \equiv 3(-gB + g^2|\Phi_c|^2) + 2gB$, the eigenvalues of the operator, $\int d\bar{x} (-gB + 3g^2|\Phi_c|^2)$, are positive definite. Thus, the corresponding modes are gapped.
Here we should make a comment. Rigorously speaking, we have simply shown the existence of zero eigen value of $T$, but have not yet shown that the eigenvalues are positive semi definite. The condition of the semipositivity must hold since the Hamiltonian is bounded below. At least, we can choose appropriate solutions, $\Phi_c$, for the eigenvalues of $T(\Phi_c)$ to be positive semi definite. In the discussion we have assumed that we take such classical solutions.

We now proceed to diagonalize the Hamiltonian in the longitudinal components, $\alpha_{p \neq p_0}$ and $\beta_{p \neq p_0}$. Before analyzing the problem, we should note that the reference momentum, $p_0$, characterizing the ground state must go to zero as $L \to \infty$. Thus, we consider only such modes with components of $p \gg p_0$. Then, it is easy to extract a Hamiltonian involving the longitudinal modes from $H_0 + H_{\text{Coulomb}}$.

\[
\int d\vec{x} \left(-gB + 3g^2|\Phi_c|^2\right) \sum_{p > p_0} \frac{1}{b} (\alpha_p^\dagger \alpha_p + \beta_p^\dagger \beta_p).
\] (22)

As we have made a comment just above, the eigenvalues of the operator, $\int d\vec{x} \left(-gB + 3g^2|\Phi_c|^2\right)$, are positive definite. Thus, we find that the longitudinal modes have gap energies on the ground state, $\langle \Phi \rangle = \Phi_c$.

It is important to note that the gaps of the longitudinal modes arise due to the Coulomb interaction. Actually, if we neglect the interaction, a Hamiltonian of the longitudinal modes is the same as the one of the transverse modes, which is given in eq(20). Thus, the gaps do not arise.

We have discussed the cases of the modes with $p = p_0$ and $p \gg p_0$. Here, we should make a comment the case of modes with $p \sim p_0$, but $p \neq p_0$. In this case the modes couple with other modes with different longitudinal momentum so that the diagonalization of the relevant Hamiltonian is very difficult. But we can diagonalize the Hamiltonian only by taking the modes with $p_\pm = p_0 \pm \pi/L$ and neglecting the other modes. These modes receive large Coulomb energies due to the terms, $\sim 1/(p_\pm - p_0)^2$. Then, we find that there is no gapless mode. This is not exact treatment, but a simple exercise to grip real excitation energies. It makes us speculate that the energy gaps exist even if we diagonalize the Hamiltonian including all modes with $p \sim p_0$.

Up to now, we have examined excitation energies by using the classical ground state, $\Phi_c$. On the other hand, by using the quantum ground state $|G\rangle$, we have given a plausible argument in the previous paper[13] that there arise the gap energies in the longitudinal direction. In the argument the Coulomb energy also plays an important role for the generation of the gap.

The fact that there exist the gapless modes in the transverse directions, while no gapless modes in the longitudinal direction implies that gluons are localized in a two dimensional layer. They can move easily in the transverse plane, but can not move in the longitudinal direction as far as we are concerned with sufficiently low energies. This is very similar to the case of electrons confined in a two dimensional quantum well fabricated in semiconductors. Electrons can move only within the two dimensional quantum well as far as we are concerned with low energies or low temperatures.

VI. ZERO MODES AND QUANTUM HALL STATES OF GLUONS

We have shown that there are zero modes in the transverse directions, in other words, the ground state, $\langle \Phi \rangle \neq 0$, is degenerate with those states involving the excitations of the zero modes. As we can see soon below, there are infinitely many zero modes. It suggests that the ground state discussed above is unstable. In general, the excitations of the zero modes lead to a unique stable ground state owing to the residual interactions, which have been neglected in the above approximation. Thus, we need to answer the question what is the real stable ground state. We will find that the real ground state is a quantum Hall state of the zero mode gluons[2,12].

In order to do so, by taking a simple classical ground state, $\Phi_c$, we first show explicitly that there are infinitely many zero modes. After that, we examine the effects of the zero mode excitations. ( We have already shown that there are infinitely many zero modes by using the quantum ground state $|G\rangle$. In this section we analyze classical zero mode solutions around the classical ground state. )

The zero modes are given by the solutions of the following equations,

\[
\sum_{m=0,1,2,\ldots} T_{n,m} a_m = \sum_{m=0,1,2,\ldots} \left(-gB \delta_{n,m} + g^2 \int d\vec{x} |\Phi_c|^2 \phi_n^* \phi_m \right) a_m = 0,
\] (23)

with $\delta \Phi_0 = \sum_{m=0,1,2,\ldots} a_m \phi_m$, where $\Phi_c$ represents a classical solution of a ground state, which also satisfies the same equation as the equation (23). As an example we consider a field configuration such as $\Phi = a_0 \phi_0$. Obviously, it is a solution in eq(23) with $a_0 = \sqrt{4\pi/g^2}$. It is not, however, a solution representing a ground state, because we expect that the ground state should be spatially uniform. Although we have mentioned the absence of such a spatially
uniform field in the lowest Landau level, we may consider an approximate solution which is spatially uniform, that is, $\Phi_{const} = \text{const.} \times e^{-ip\omega x}$. Using the approximate classical ground state, we solve the equation for the zero modes,

$$
\sum_m T_{n,m} a_m = \sum_m (-gB + g^2|\Phi_{const}|^2) \delta_{n,m} a_m = (-gB + g^2|\Phi_{const}|^2)a_n = 0.
$$

(24)

Hence, we find that any configurations of $\delta\Phi_0$ satisfy the equation if we take $|\Phi_{const}| = gB/g^2$. The configuration, $|\Phi_{const}| = gB/g^2$, is just the field configuration minimizing the potential energy, $-gB|\Phi|^2 + g^2|\Phi|^4/2$ when we neglect the limitation of the field such as it occupies the lowest Landau level. Therefore, any modes in the lowest Landau level can be zero modes. Obviously, there are infinitely many independent zero modes.

We proceed to show that the excitations of these zero modes partially screen the color magnetic field, $B$. The result is expected naively from the fact that the ground state is a condensed state of the color charged field, $\Phi$, just like a color superconductor. Thus, it apparently seems that the color magnetic field is ejected or squeezed in the condensed state of the field, $\Phi$. But this is not true as we will show below.

In order to see the partial screening, we calculate color magnetic field generated by the zero modes. As an explicit example, we take a zero mode described by the wave function, $\phi_m = g_m z^m \exp(-|z|^2gB/4)$. The mode generates a color current in the transverse directions given by

$$
\delta \bar{J} = ig\langle \left( \delta \Phi_0^\dagger \bar{D} \delta \Phi_0 - (\bar{D} \delta \Phi_0)^\dagger \delta \Phi_0 \right) \rangle \quad \text{with} \quad \bar{D} \equiv \bar{\partial} + igA^B
$$

(25)

where the expectation value is calculated by using an eigenstate of the operator of the zero mode. The zero mode, $\delta \Phi_0$, is given by

$$
\delta \Phi_0 = \left( \frac{a_{p_0,m} \phi_m e^{-ip_0 x} + b_{p_0,m}^* \phi_m^* e^{ip_0 x}}{\sqrt{2\pi Lp_0}} \right)_{zero} = \left( \frac{b_{p_0,m}' \phi_m e^{-ip_0 x} - b_{p_0,m}^* \phi_m^* e^{ip_0 x}}{2\sqrt{2\pi Lp_0}} \right)
$$

(26)

with an operator of non zero mode, $a_{p_0,m}' = (a_{p_0,m} + b_{p_0,m})/\sqrt{2}$, and the operator of the zero mode, $b_{p_0,m}' = (a_{p_0,m} - b_{p_0,m})/\sqrt{2}$. This color current induces a color magnetic field, $\delta B$, assumed to point to the longitudinal direction. The field satisfies the Maxwell equation,

$$
J_z \equiv J_1 + iJ_2 = ig\langle \left( \delta \Phi_0^\dagger D_z \delta \Phi_0 - (D_z \delta \Phi_0)^\dagger \delta \Phi_0 \right) \rangle = -i\partial_z \delta B,
$$

(27)

where we have assumed the rotational symmetry of the induced magnetic field around the longitudinal direction and have used the following notations,

$$
D_z \equiv \partial_1 + i\partial_2 + ig(A_1^B + iA_2^B) = \partial_z - gBz/2
$$

$$
D_z \equiv \partial_1 - i\partial_2 + ig(A_1^B - iA_2^B) = \partial_z + gBz/2,
$$

(28)

with $z \equiv x_1 + ix_2$ and a gauge $A_1^B = -Bx_2/2$ and $A_2^B = Bx_1/2$. It is easy to evaluate the current of $J_z$,

$$
J_z = -i\frac{gB g_m z^m |z|^{2m}}{2\pi Lp_0} \exp(-\frac{gB|z|^2}{2})N_m,
$$

(29)

with the number of the zero mode, $N_m \equiv \langle b_{m}^* b_{m} \rangle$ in the state. Thus, the color magnetic field induced by the current is

$$
B(|\vec{x}|) = \delta B + B = -\frac{g^2Bg_m^2}{4\pi Lp_0} \left( \frac{2}{gB} \right)^m \Gamma(m + 1, gB|\vec{x}|^2 / 2) N_m + B
$$

(30)

where $\Gamma(a,z)$ is the incomplete gamma function; $\Gamma(a,z) = \int_z^\infty t^{a-1}e^{-t}dt$. The value of $Lp_0$ is finite as $L \to \infty$. The first term represents the induced magnetic field, $\delta B$, and the second one does the original magnetic field, $B$. Therefore, we find that the excitations of the zero modes partially screen the color magnetic field.
The formation of the two-dimensional quantum well in the ferromagnetic quark matter is a necessary condition for the realization of a quantum Hall state of gluons. The state can be realized in such a quantum well as in the case of semiconductors. We have discussed that since there are infinitely many excited states with zero energy (zero for the realization of a quantum Hall state of gluons. The state can be realized in such a quantum well as in the excitation spectra implies that the gluons form the two-dimensional quantum well.

The formation of the two-dimensional quantum well in the ferromagnetic quark matter effectively form a two-dimensional quantum well. Namely, the gluons with a longitudinal momentum, \(p_0^+\), characterizing the ground state, decouple with the other gluons with longitudinal momenta, \(p^+ \neq p_0^+\). The gluons with \(p^+ \gg p_0^+\) which also do not couple with other gluons with different momentum, \(q^+ \neq p^+\), are gapped; their excitation energies start to grow at a finite non-zero energy. The gluons with \(q^+ \neq p_0^+\) couple with the other gluons with different momentum, \(q^+ \neq p^+\). Although we have not discussed in detail the case of the gluons with the momentum, \(p^+ \sim p_0^+\), a simple exercise shows that such gluons also possess gap energies. This feature of the excitation spectra implies that the gluons form the two-dimensional quantum well.

The formation of the two-dimensional quantum well in the ferromagnetic quark matter is a necessary condition for the realization of a quantum Hall state of gluons. The state can be realized in such a quantum well as in the case of semiconductors. We have discussed that since there are infinitely many excited states with zero energy (zero modes), they are produced without any energy costs to form a quantum Hall state owing to the repulsive interaction between them. The quantum Hall state has an energy gap, i.e., no zero modes. Thus, it is the stable ground state of the gluons in the quantum quark matter.

Such a quark matter may be present in neutron stars. We may suppose that the quantum layers parallel to each other are formed in the quark matter of the neutron stars. That is, the quark matter has a domain structure in which many parallel layers are involved and the color magnetic field points to a direction in the domain. Then, there is a quantum Hall state of gluons in each layer. The state possesses the color charge density of the gluons, \(\langle \rho_c \rangle_d\), must satisfy the conditions on the filling factors, \(\frac{2\pi \langle \rho_c \rangle_d}{qB}\),

\[
\frac{2\pi \langle \rho_c \rangle_d}{qB} = \frac{2\pi (\delta \Phi_0^0 \mathcal{P} \Phi_0 - \mathcal{P} \delta \Phi_0^0 \Phi_0) |_{\Phi_0}}{qB} = \frac{1}{2n} = 1/2, 1/4, \ldots
\]

with integer, \(n\), where \(l_d\) denotes a width of the quantum well. The width is given approximately by the inverse of the gap energy in the longitudinal direction. The appearance of the even number in the denominator is due to the fact that gluons are bosons. It is odd number if gluons were fermions. Such a color charge density may arise automatically by generating the zero modes for the quantum Hall state to be formed. Therefore, the quantum Hall state of the gluons naturally arises owing to the excitations of the zero mode gluons in the two-dimensional quantum well. Please refer to our previous papers for the more detailed treatment of the quantum Hall states by the use of Chern-Simons gauge theory.

VII. SUMMARY AND DISCUSSION

Using the light cone quantization with neglecting "zero modes", we have shown that gluons in the ferromagnetic quark matters effectively form a two-dimensional quantum well. Namely, the gluons with a longitudinal momentum, \(p_0^+\), characterizing the ground state, decouple with the other gluons with longitudinal momenta, \(p^+ \neq p_0^+\). The gluons with \(p^+ \gg p_0^+\) which also do not couple with other gluons with different momentum, \(q^+ \neq p^+\), are gapped; their excitation energies start to grow at a finite non-zero energy. The gluons with \(p^+ \sim p_0^+\) couple with the other gluons with different momentum, \(q^+ \neq p^+\). Although we have not discussed in detail the case of the gluons with the momentum, \(p^+ \sim p_0^+\), a simple exercise shows that such gluons also possess gap energies. This feature of the excitation spectra implies that the gluons form the two-dimensional quantum well.

The formation of the two-dimensional quantum well in the ferromagnetic quark matter is a necessary condition for the realization of a quantum Hall state of gluons. The state can be realized in such a quantum well as in the case of semiconductors. We have discussed that since there are infinitely many excited states with zero energy (zero modes), they are produced without any energy costs to form a quantum Hall state owing to the repulsive interaction between them. The quantum Hall state has an energy gap, i.e., no zero modes. Thus, it is the stable ground state of the gluons in the quark matter.

Such a quark matter may be present in neutron stars. We may suppose that the quantum layers parallel to each other are formed in the quark matter of the neutron stars. That is, the quark matter has a domain structure in which many parallel layers are involved and the color magnetic field points to a direction in the domain. Then, there is a quantum Hall state of gluons in each layer. The state possesses the color charge of the gluons. The sign of the charge is common in all layers. The amount of the color charge in a layer can be determined by the condition eq(31) on the filling factor. It must be compensated by the color charge of quarks due to the condition of the color neutrality. Therefore, the gas of the quarks carries the color charge and rotates around the color magnetic field. Since the gas of quarks also carries electric charges, its rotation induces an electromagnetic magnetic field. (On the other hand,
electrons whose electric charges compensate the charge of quarks, do not rotate around the color magnetic field so that they do not produce an electromagnetical magnetic field. Thus, the electromagnetical magnetic field is generated spontaneously in the domain. The domain may be extended all over the quark matter in neutron stars. This is a possible origin of the strong magnetic field observed in magnetars.

Recent analyses of gapless color superconductors indicate the existence of the phase transition from the color superconducting state to the color ferromagnetic state with decreasing the baryon density. Actually, it has been shown that a color magnetic instability in the color superconductivity arises, that is, the instability such that external color magnetic field can penetrate the color superconductor. In other words, the coefficient of the kinetic term, $|\vec{D}\Phi_d|^2$ of Higgs fields, $\Phi_d$, representing diquark pairs becomes negative. Furthermore, it has been shown that the spontaneous generation of color magnetic field (spatially inhomogeneous gauge fields) arises. These results have been obtained only by the analyses of quark dynamics. But, it strongly suggests that we need to include appropriately the quantum effects of the gluons along with those of the quarks, just as we have done in the present paper.

We also wish to point out an intriguing relationship. That is, a relation between color glass condensate of gluons in nucleons and quantum Hall state of gluons in quark matter. It is quite plausible that the color ferromagnetic quark matter arises through the phase transition from hadronic phase when we increase the baryon density of nuclear matter. Then, it is natural to ask how a gluonic specific state of color glass condensate in nucleons transforms into the gluonic state mentioned above, namely, the quantum Hall state of gluons. Since the saturation momentum of the color glass condensate increases with baryon density, the gluons with even large $x = p^+/P_L \sim 1$, may constitute the color glass condensate in the dense quark matter. On this point, we have argued a similarity between the color glass condensate and the quantum Hall state of gluons. We wish to clarify the relation much more in near future.

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