Hölder continuity of solutions for a class of drift-diffusion equations

Quoc-Hung Nguyen, Yannick Sire and Le Xuan Truong

Abstract

We provide several regularity results for non-homogeneous drift-diffusion equations with applications to general dissipative SQG. Our results unify in a rather simple way several previously known results. We build the estimates on an algebraic identity (for the refined energy argument) which relates any integral operator with pure powers of the laplacian.

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1 Introduction

We consider drift-diffusion equations involving rather general integral operators (with bounded elliptic measurable kernels). We aim at proving Hölder regularity of solutions whenever the right hand side has enough integrability. We apply also our results to the dissipative SQG equation, i.e. whenever the divergence-free vector-field in the transport term is related in a non-trivial way to the solution of the equation by a Darcy law.

The standard dissipative SQG equation has been introduced by Constantin, Majda and Tabak (see [CMT94] for instance) as a toy model for regularity in three-dimensional flows. Besides qualitative properties of the equations, such as long-time behaviour, a major issue is the regularity of weak solutions. This is the issue we address here for a general version of drift-diffusion equations.

The standard critical dissipative SQG equation is

\[
\partial_t u + B \cdot \nabla u + (-\Delta)^{1/2} u = 0 \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{R},
\]

where $B$ is divergence free and related to $u$ by $B = \nabla \perp (-\Delta)^{-1/2} u$. The regularity of (weak) solutions has been established in [CV10] for data satisfying an energy type inequality (see also [KNV07, KN09]). Several generalizations of this equation can be considered, where the Fourier
multiplier \((-\Delta)^{1/2}\) is replaced by \((-\Delta)^s\), yielding sub critical whenever \(s > 1/2\) (see [CF08]) and super critical whenever \(s < 1/2\) (see [CW09, CW99]) equations. One could also consider the case of bounded domains such as in [CI16]. Several tools have been developed to handle drift-diffusion PDEs, whenever the diffusion is given by a Fourier multiplier. The mathematical difficulties substantially increase when one considers other types of integral operators like the ones described below, which are not Fourier multipliers.

In the present work, we consider the following general drift-diffusion equation for \(d \geq 2\)

\[
(1.2) \quad \partial_t u + \mathbf{B} \cdot \nabla u + \mathcal{L}_t u = g \text{ in } \mathbb{R}^d \times \mathbb{R} = \mathbb{R}^{d+1},
\]

where \(g \in L^1(\mathbb{R}^{d+1})\) and \(\mathbf{B} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d\) is a vector field satisfying \(\text{div}(\mathbf{B}) = 0\). Furthermore, the integral operator \(\mathcal{L}_t\) (\(t \in \mathbb{R}\)) is defined by

\[
\mathcal{L}_t u(x) = \text{P.V.} \int_{\mathbb{R}^d} (u(x) - u(y))K_t(x,y)dy,
\]

where the kernel has some structure we will discuss below but is of homogeneity \(-2s\) similarly to the fractional laplacian (hence ensuring some ellipticity). Notice that parabolic equations in such generality (for the diffusion) have been considered for instance in [CCV11]. For the general operator we are interested in, there is of course no extension as a local operator in one more dimension. This fact, already present in [CCV11], introduces several non trivial difficulties and one has to develop a somehow intrinsically nonlocal/integral method to obtain Hölder regularity.

Contrary to [CCV11], we actually use the extension in one more dimension by relating energy estimates (the crucial part of De Giorgi’s argument) to the ones obtained for pure powers of the laplacian. This allows in particular to simplify the proof in [CCV11]. We focus on the most difficult case of the critical/supercritical regime \(s \leq 1/2\).

As mentioned previously, in order to tackle the case of general integral diffusion as described above, we take here a different route from for [CCV11], bridging in a natural way the \(s\)-harmonic extension of a function \(u\) to its Gagliardo semi-norms, which control from above and below the kernel \(K_t\). This approach is new and we believe could be used successfully in other contexts. We now explain this link since this is the core of the argument. In proving Hölder regularity by De Giorgi’s method, the crucial step is to go from an \(L^2\) type estimate to local boundedness, via a special energy inequality obtained by testing the equation with suitable truncations of the unknown. Using a trick we describe below, one can relate the purely nonlocal energy inequality coming from \(\mathcal{L}_t\) to an energy inequality in terms of \((-\Delta)^s\), which can be ”localized" using the argument of Caffarelli and Silvestre [CS07]. For a function \(f : \mathbb{R}^d \to \mathbb{R}\), consider the extension function \(f^* : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}\) that satisfies

\[
(1.3) \quad \begin{cases}
\text{div}(z^{1-2s} \nabla f^*(x,z)) = 0, \\
f^*(x,0) = f(x).
\end{cases}
\]

Then it is well-known by now [CS07] that in the weak sense

\[-\Delta^s f = \lim_{z \to 0^-} \left( - z^{1-2s} \partial_z f^* \right).\]

Furthermore, the extension \(f^*\) can be determined by using Poisson formula

\[f^*(x,z) = \int_{\mathbb{R}^d} P(x-y,z)f(y)dy,\]

where \(P\) is the Poisson kernel for equation (1.3) and is defined by

\[P(x,z) = c_{d,s} \frac{z^{2s}}{(|x|^2 + |z|^2)^{1+s}} = \frac{1}{2^d} P\left(\frac{x}{z},1\right),\]
where \( c_{d,s} \) is a normalizing constant such that \( \| P(\cdot, 1) \|_{L^1(\mathbb{R}^d)} = 1 \). In this paper, we will denote the extension \((u(\cdot, t))^*\) of \( u(\cdot, t) \) by

\[
u^*(x, z, t) = (u(\cdot, t))^*(x, z).
\]

As previously mentioned, the starting point of the proof of Hölder regularity by De Giorgi’s method is an energy-type inequality for truncations of the weak solution \( u \). The following lemma, of algebraic nature, relates the energy associated to \( L_t \) to the one associated to the fractional laplacian.

**Lemma 1.1.** Let \( h \) be a positive function in \( \mathbb{R}^d \) and \( f \in H^s(\mathbb{R}^d) \) for \( s \in (0, 1) \). Then the following estimate holds

\[
\int_{\mathbb{R}^d} h^2 f^+ L_t f \, dx + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f^+(x) f^+(y) \frac{(h(x) - h(y))^2}{|x - y|^{d+2s}} \, dx \, dy \\
\sim \int_{\mathbb{R}^d} h^2 f^+ (\Delta)^s f \, dx + C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f^+(x) f^+(y) \frac{(h(x) - h(y))^2}{|x - y|^{d+2s}},
\]

where \( f^+ \) is the positive part of \( f \).

Lemma 1.1 ensures that one can always compare the quadratic forms \( \int_{\mathbb{R}^d} h^2 f^+ L_t f \, dx \) and

\[
\int_{\mathbb{R}^d} h^2 f^+ (\Delta)^s f \, dx
\]

by the Gagliardo semi-norms, which are under control by Sobolev inequalities. The \( s \)-harmonic extension described above then allows to control the term \( \int_{\mathbb{R}^d} h^2 f^+ L_t f \, dx \) in a local fashion as in [CV10].

**Main results**

We now describe the precise assumptions on the kernel and our main results. We introduce the space \( X^s \) which is defined by

\[
X^s = \begin{cases} 
BMO(\mathbb{R}^d) & \text{if } s = 1/2, \\
C^{1-2s}(\mathbb{R}^d) & \text{if } 0 < s < 1/2.
\end{cases}
\]

Moreover, we always assume that \( B \) is divergence free almost-everywhere in time \( t \in \mathbb{R} \). The kernel \( K_t \) satisfies

\[
K_t : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to [0, \infty), \quad (x, y, t) \mapsto K_t(x, y)
\]

is assumed to satisfy the following conditions for a given \( s \in (0, 1/2) \):

(i) \( K_t \) is a measurable function;

(ii) for every \( t \in \mathbb{R} \), \( K_t \) is symmetric which means that

\[
K_t(x, y) = K_t(y, x), \quad \text{for a. e. } (x, y, t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R};
\]

(iii) there exists a constant \( \Lambda > 1 \) such that

\[
\Lambda^{-1} \leq |x - y|^{d+2s} K_t(x, y) \leq \Lambda, \quad \text{for a. e. } (x, y) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R};
\]
(iv) for a. e. \( x \in \mathbb{R}^d \)

\[
\left| \int_{S^{d-1}} \theta K_t(x, x + \rho \theta) d\mathcal{H}^{d-1}(\theta) \right| \leq \Lambda \rho^{-d}.
\]

In the following we drop the dependence in the time variable \( t \) in \( \mathcal{L}_t \). Such operators have been considered in many works and clearly generalize the case of the fractional laplacian. See in particular [CCV11] for an equation related to ours.

Our first theorem is an Hölder estimate for solutions of the equation (1.2).

**Theorem 1.2.** Assume that the divergence free vector field \( \mathbf{B} \) satisfies the conditions

\[
\sup_{(x,t) \in \mathbb{R}^{d+1}} \left| \int_{x+B_1} B(y, t) dy \right| \leq M_1 \quad \text{and} \quad \| \mathbf{B} \|_{L^\infty(\mathbb{R}; \mathbb{R}^d)} \leq M_2,
\]

for some \( M_1, M_2 \geq 1 \) and \( g \) belongs to the class

\[
L^1(\mathbb{R}^{d+1}) \cap L^q(\mathbb{R}^{d+1}),
\]

for some \( q > (d+1)/2s \). Let \( u \) be a weak solution to (1.2) with

\[
u \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^d)) \cap L^2(\mathbb{R}; H^s(\mathbb{R}^d)).
\]

Then there exists \( \delta_0 \) depending only on \( d, s, \Lambda, q \) and \( M_2 \) such that the following holds true

- **Global in time:**

\[
\| u \|_{C^s(Q_{\delta_0}(x,t))} + \| u \|_{L^\infty(\mathbb{R}^{d+1})} \leq C(M_2) \left( \| g \|_{L^1(\mathbb{R}^{d+1})} + \| g \|_{L^2(\mathbb{R}^{d+1})} \right),
\]

- **Local in time:**

\[
\sup_{x \in \mathbb{R}^d, t > 1} \| u \|_{C^s(Q_{\delta_0}(x,t))} + \| u \|_{L^\infty(\mathbb{R}^d \times (1, \infty))} \leq C(M_2) \left( \| u(0) \|_{L^2(\mathbb{R}^d)} + \| g \|_{L^1(\mathbb{R}^{d+1})} + \| g \|_{L^q(\mathbb{R}^{d+1})} \right),
\]

for some \( \alpha \in (0, 1 - (d+1)/2sq) \).

In the following we use the notations

\[
X_1 = (t_1, x_1), \quad \|X_1 - X_2\|_s = |x_1 - x_2| + |t_1 - t_2|^{\frac{1}{s}},
\]

\[
\mathcal{R}_1^\lambda := \left\{ X_1 = (x_1, t_1), X_2 = (x_2, t_2) \in \mathbb{R}^{d+1}_+: \begin{cases} t_1, t_2 > \lambda, \\
\|X_1 - X_2\|_s \leq \frac{C(M_2, M_1) \lambda}{|\log(\lambda)|^{1+1}},
\end{cases} \right\},
\]

\[
\mathcal{R}_2^\lambda := \left\{ X_1 = (x_1, t_1), X_2 = (x_2, t_2) \in \mathbb{R}^{d+1}_+: \begin{cases} t_1, t_2 > \lambda^{2s}, \\
\|X_1 - X_2\|_s \leq C(M_3) \min\{\lambda, \lambda^{2s}\},
\end{cases} \right\}.
\]

From Theorem 1.2 we derive the following corollary.

**Corollary 1.3.** Assume \( g = 0 \) in \( \mathbb{R}^d \times (0, \infty) \). Then,

\[
\| u(t) \|_{L^\infty(\mathbb{R}^d)} \leq C t^{-\frac{d}{2s}} \| u(0) \|_{L^2(\mathbb{R}^d)}.
\]

Moreover,
• If $s = \frac{1}{2}$ and $B$ satisfies
\[
\sup_{(x,t) \in \mathbb{R}^{d+1}} \left| \int_{x+B_1} B(y,t) dy \right| \leq M_1, \quad \|B\|_{L^\infty(\mathbb{R}^d;BMO(\mathbb{R}^d))} \leq M_2, \quad (M_1, M_2 \geq 1)
\]
then, for any $\lambda \in (0, \frac{1}{2})$, we have
\[
(1.8) \quad \sup_{x_1, x_2 \in \mathbb{R}^d} \frac{|u(x_1) - u(x_2)|}{\|x_1 - x_2\|^\alpha \lambda^d} \leq C(M_2)\lambda^{-d/2-\alpha}\|u(0)\|_{L^2(\mathbb{R}^d)}.
\]

• If $0 < s < \frac{1}{2}$ and $B$ satisfies
\[
\|B\|_{L^\infty(\mathbb{R}^d;L^\infty(\mathcal{C}^{1-2s}(\mathbb{R}^d)))} \leq M_3, \quad (M_3 \geq 1)
\]
then, for any $\lambda \in (0, \frac{1}{2})$, we have
\[
(1.9) \quad \sup_{x_1, x_2 \in \mathbb{R}^d} \frac{|u(x_1) - u(x_2)|}{\|x_1 - x_2\|^\alpha \lambda^d} \leq C(M_3)\lambda^{-d/2-\alpha}\|u(0)\|_{L^2(\mathbb{R}^d)}.
\]

We now draw some conclusion in the case of dissipative SQG, namely
\[
(1.10) \quad \partial_t u + B \cdot \nabla u + \mathcal{L}_t u = g \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^d,
\]
where $s = \frac{1}{2}$ and $B = \nabla \perp \mathcal{L}_t^{-1} u$. Notice that the latter is of order zero and is a Calderon-Zygmund operator.

Corollary 1.4. Let $u$ be a weak solution of \((1.10)\). If $g \in (L^1 \cap L^q)(\mathbb{R}^{d+1})$ for $q > d+1$, then $u$ is Hölder continuous in space and time.

Proof. Proposition 2.1 gives that $u(t) \in L^\infty_t(\mathbb{R}^d;L^\infty_x \cap L^2_x(\mathbb{R}^d))$; hence by Calderon-Zygmund estimate one has $B \in L^\infty_t(BMO_x \cap L^2_x)$. Applying Theorem 1.2 gives actually that $u$ is $C^\alpha$. \(\square\)

Throughout this work we denote
\[
B_r = [-r, r]^d, \quad Q_r = B_r \times [-r, 0], \quad Q_r = Q_r \cap [0, \infty), \quad B_{r,0} = B_r \times (0, \infty), \quad B_{r,r} = B_{r,r} \cap [0, \infty), \quad Q_r = Q_r \cap [0, \infty).
\]

2 From $L^2$ to $L^\infty$

This section is devoted to the first step of De Giorgi’s method in proving the Hölder continuity of solutions. We prove that weak solution is actually bounded. Our result is the following:

Proposition 2.1. Let $q > \frac{d+1}{2s}$ and assume that $g$ belongs to the class
\[
L^{\frac{2(d+1)}{d+1+2s}}(\mathbb{R}^{d+1}) \cap L^q(\mathbb{R}^{d+1}).
\]
Then there exists a unique solution of the equation \((1.2)\) satisfying following estimates
\[
(2.1) \quad \sup_{t \in \mathbb{R}} \int_{\mathbb{R}^d} u(t)^2 dx + \int_{\mathbb{R}^d} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 dx dt \leq C\|g\|_{L^{\frac{2(d+1)}{d+1+2s}}(\mathbb{R}^{d+1})}^2,
\]
\[
(2.2) \quad \|u\|_{L^\infty(\mathbb{R}^{d+1})} \leq C\|g\|_{L^q(\mathbb{R}^{d+1})}^{\frac{2(g-d-2s)}{2(d+1)} + \frac{qd}{2(d+1)+2s}} \|g\|_{L^\infty(\mathbb{R}^{d+1})}^{\frac{2g}{2(d+1)+2s}}.
\]
Proof. By a standard argument, we get the existence and uniqueness of a solution \( u \) for the equation (1.2) satisfying the estimate (2.1).

In order to prove (2.2) we only need to show that

\[
\| u^+ \|_{L^\infty([1,\infty) \times \mathbb{R}^d)} \leq C,
\]

provided that

\[
\| g \|_{L^{2(d+1)}(\mathbb{R}^{d+1})} + \| g \|_{L^q(\mathbb{R}^{d+1})} \leq 1.
\]

We shall use the De Giorgi’s iteration. For the constant \( \lambda > 1 \) that will be chosen later, we consider the levels

\[
C_k = \lambda(1 - 2^{-k})
\]

and the truncation functions \( u_k = (u - C_k)_+ \). By using \( u_k \) as test function of (1.2), we obtain:

\[
U_k \leq C \int_{\mathbb{R}^d} u_k(\tau)^2 \, dx + C \int_{T_{k-1}}^\infty \int_{\mathbb{R}^d} u_k |g| \, dx \, dt,
\]

for any \( \tau \in [T_{k-1}, T_k] \), where \( T_k = 1 - 2^{-k} \) and \( U_k \) is defined by

\[
U_k = \sup_{t \geq T_k} \int_{\mathbb{R}^d} u_k(t)^2 \, dx + \int_{T_k}^\infty \int_{\mathbb{R}^d} \left| (-\Delta)^{d/2} u_k \right|^2 \, dx \, dt.
\]

Taking the mean value in \( \tau \) on \([T_{k-1}, T_k]\) we find,

\[
U_k \leq 2^k C \| u_k \|^2_{L^2([T_{k-1}, \infty) \times \mathbb{R}^d)} + C \| u_k \|_{L^q([T_{k-1}, \infty) \times \mathbb{R}^d)},
\]

where

\[
q' = \frac{q}{q-1} < \frac{d + 2s}{d + 1 - 2s}.
\]

As in [CV10] we can control the right-hand side of this inequality by \( U_{k-1} \). Indeed, by using Sobolev’s inequality we get

\[
\| u_{k-1} \|^2_{L^{2(d+1)}(\mathbb{R}^{d+1})} \leq C U_{k-1}.
\]

On the other hand, it is noted that

\[
1_{\{u_k > 0\}} \leq \left( \frac{2^k}{\lambda} u_{k-1} \right)^{\frac{4s}{d+1-2s}} \quad \text{and} \quad 1_{\{u_k > 0\}} \leq \left( \frac{2^k}{\lambda} u_{k-1} \right)^{-q' + \frac{2(d+1)}{d+1-2s}}.
\]

These imply that

\[
\| u_k \|^2_{L^2([T_{k-1}, \infty) \times \mathbb{R}^d)} \leq \left( \frac{2^k}{\lambda} \right)^{\frac{4s}{d+1-2s}} \| u_{k-1} \|^2_{L^\infty([T_{k-1}, \infty) \times \mathbb{R}^d)},
\]

and

\[
\| u_k \|_{L^q([T_{k-1}, \infty) \times \mathbb{R}^d)} \leq \left( \frac{2^k}{\lambda} \right)^{\frac{2(d+1)}{(d+1-2s)q'}} \| u_{k-1} \|_{L^{\frac{2(d+1)}{(d+1-2s)q'}}([T_{k-1}, \infty) \times \mathbb{R}^d)}.
\]

We combine above estimates and obtain

\[
U_k \leq C \left( \frac{2^k}{\lambda} \right)^{\frac{d+1}{d+1-2s}} U_{k-1}^{\frac{d+1}{d+1-2s}} + C \left( \frac{2^k}{\lambda} \right)^{\frac{2(d+1)}{(d+1-2s)q'}} U_{k-1}^{\frac{d+1}{d+1-2s} q'}, \quad \forall k \geq 1.
\]
Thanks to (2.1), one has $U_0 \leq C$. Since
\[
\frac{d + 1}{d + 1 - 2s} > 1 \quad \text{and} \quad \frac{d + 1}{(d + 1 - 2s)q'} > 1,
\]
so for $\lambda > 1$ large enough, $U_k$ converges to 0. This means $u \leq \lambda$ for almost everywhere in $[1, \infty) \times \mathbb{R}^d$ and we find (2.3). The proof is complete. \qed

**Remark 2.2.** From the proof of the inequality (2.3), we find that if
\[
\|u(0)\|_{L^2(\mathbb{R}^d)} + \|g\|_{L^{\frac{2(d+1)}{d+1+2s}}(\mathbb{R}^{d+1})} + \|g\|_{L^q(\mathbb{R}^{d+1})} \leq 1,
\]
then there holds
\[
\|u\|_{L^\infty([1, \infty) \times \mathbb{R}^d)} \leq C.
\]

### 3 Oscillation lemma

This section is concerned with reducing the oscillation of the solution, one of the most difficult tasks in proving the Hölder continuity of bounded solutions. In the critical case, Caffarelli and Vasseur used the harmonic extension $u^\ast$ of solution $u$ in $\mathbb{R}^d \times \mathbb{R}^+$. Namely, they expressed the fractional Laplacian $(-\Delta)^{\frac{s}{2}}u$ as the normal derivative of the harmonic extension $u^\ast$ on the boundary $\mathbb{R}^d$. Then thanks to the good properties of harmonic functions, they obtained the following diminishing oscillation result: if $u^\ast \leq 2$ in a box centered at the origin then $u^\ast$ satisfies
\[
u^\ast < 2 - \lambda^\ast
\]
in a smaller box, for some $\lambda^\ast > 0$ which depends on the BMO-norm of the vector field $B$. From this diminishing oscillation result, the Hölder continuity of $\nu^\ast$ is proved by construction a suitable sequence of functions and by using the natural scaling invariance.

In the supercritical case, there is a change in the scaling invariance. Following the idea of Caffarelli and Vasseur with some modifications, Constantin and Wu [CW09] also derived a similar result.

It seems that the approach of Caffarelli and Vasseur can not be used directly for the general kernel case because we have no information relating our operator in terms of some extension. In our case we then use an algebraic result to compare our integral operator with the standard fractional Laplacian. This allows us to continue using $s$-harmonic extension to obtain the following diminishing oscillation result:

**Proposition 3.1.** Assume that the vector field $B$ and the function $g$ satisfy following conditions
\[
\|B\|_{L^\infty((-4,0);L^{d/s}(B_2))} \leq M_0, \quad \|g\|_{L^4(Q_4)}^2 \leq \tilde{\varepsilon}_0,
\]
where $\tilde{\varepsilon}_0 < \varepsilon_0$ ($\varepsilon_0$ is the small constant defined in 3.4). Then there are positive constants $\lambda^\ast, \varepsilon_2$ depending only on $s, d, M_0$ such that, for every solution $u$ of (1.2), the following holds true:

If $u$ satisfies
\[
u^\ast \leq 2 \quad \text{in} \quad Q_1^1, \quad \int_{B_1^1} \frac{(u(x,t) - 2)_+}{|x|^{d+2s}} \, dx \leq \varepsilon_2 \, \forall t \in \mathbb{R},
\]
\[
L^\frac{d+2}{2} (\{(x,z,t) \in Q_1^1 : u^\ast(x,z,t) \leq 0\}) \geq \frac{|B_1|}{4(1 - s)},
\]
then we have
\[
u^\ast \leq 2 - \lambda^\ast \quad \text{in} \quad Q^\ast_{1/16}.
\]
The proof of this proposition is related to three propositions: a local energy inequality, a result on the diminishing oscillation for \( u^\ast \) under conditions the local \( L^2 \)-norm of \( u \) and \( u^\ast \) are small. Third proposition shows the sufficient conditions to obtain the smallness of the local \( L^2 \)-norm of \( u \) and \( u^\ast \).

**Proposition 3.2.** Let \( t_1, t_2 \) be real numbers with \( t_1 < t_2 \). Assume that the vector field \( B \) with divergence free satisfies the condition

\[
\|B\|_{L^\infty(t_1,t_2;L^d/(B_2))} \leq M_0,
\]

and let \( \phi, \psi \) be cut-off functions in \( \mathbb{R}^d \) and \( \mathbb{R} \) respectively such that

\[
1_{B_1} \leq \phi \leq 1_{B_2} \quad \text{and} \quad 1_{(-1,1)} \leq \psi \leq 1_{(-2,2)}.
\]

If \( u \) be a solution to (1.2) with

\[
u \in L^\infty(t_1,t_2;L^2(\mathbb{R}^d)), \quad (-\Delta)^{s/2}u \in L^2((t_1,t_2) \times \mathbb{R}^d),
\]

then there exists a positive constant \( C = C(d, \Lambda) \) such that

\[
\frac{d}{dt} \int_{B_2} (\phi u_+)^2 dx + \int_{B_2} z^{-1-2s} |\nabla (\eta u_+^*)|^2 dx dz 
\]

\[
\leq C \int_{\mathbb{R}^d \times \mathbb{R}^d} u_+(x) u_+(y) \left( \frac{\phi(x) - \phi(y)}{|x - y|^{d+2s}} \right)^2 \, dx dy
\]

\[
+ C \int_{B_2} z^{-1-2s}(|\nabla \eta u_+^*|^2) dx dz + C M_0 \int_{B_2} |\nabla \phi|^2 u_+^2 dx + C \int_{B_2} \phi^2 u_+ |g| dx,
\]

for every \( t_1 \leq t \leq t_2 \). Here we denote \( \eta = \phi \psi \).

**Proof.** Using \( \phi^2 u_+ \) as test function of equation (1.2), one has

\[
\frac{d}{dt} \int_{B_2} \phi^2 u_+^2 dx + \int_{B_2} \phi^2 u_+ \mathcal{L}_u u dx = \int_{B_2} B \cdot \nabla \phi \frac{u_+^2}{2} dx + \int_{B_2} \phi^2 u_+ g dx.
\]

By using Holder’s inequality it follows that

\[
\left| \int_{B_2} B \cdot \nabla \phi \frac{u_+^2}{2} dx \right| \leq \varepsilon \|\phi u_+\|_{L^{\frac{2d}{d+2s}}(B_2)}^2 + \frac{1}{\varepsilon} \| (B \cdot \nabla \phi) u_+ \|_{L^{\frac{2d}{d+2s}}(B_2)}^2
\]

\[
\leq \varepsilon \|\phi u_+\|_{L^{\frac{2d}{d+2s}}(B_2)}^2 + \frac{M_0}{\varepsilon} \int_{B_2} |\nabla \phi|^2 u_+^2 dx.
\]

The first term in the RHS can be estimated by using the trace theorem and the Sobolev embedding as follows

\[
\|\phi u_+\|_{L^{\frac{2d}{d+2s}}(B_2)}^2 \leq C \|1_{B_2} \phi u_+\|_{H^s(\mathbb{R}^d)}^2
\]

\[
= C \int_0^\infty \int_{\mathbb{R}^d} z^{1-2s} |\nabla (1_{B_2} \phi u_+^*)|^2 dx dz
\]

\[
\leq C \int_0^\infty \int_{\mathbb{R}^d} z^{1-2s} |\nabla [1_{B_2}^* \eta u_+^*]|^2 dx dz
\]

\[
= C \int_{B_2^*} z^{1-2s} |\nabla [\eta u_+^*]|^2 dx dz.
\]
Hence it implies

\[
\left(3.9\right) \quad \left| \int_{B_2} B \cdot \nabla \phi \frac{u^2_+}{2} dx \right| \leq \varepsilon \int_{B_2^2} z^{1-2s} |\nabla[\eta u^+_+]|^2 dx dz + C \varepsilon M_0 \int_{B_2} |\nabla \phi|^2 (u_+^2)^2 dx.
\]

Next, thanks to Lemma 1.1, it follows that

\[
\int_{B_2} \phi^2 u_+ \mathcal{L}_t u dx \geq C_0 \int_{B_2} \phi^2 u_+ (-\Delta)^{s/2} dx - C_1 \int_{\mathbb{R}^d \times \mathbb{R}^d} u_+(x) u_+(y) (\phi(x) - \phi(y))^2 \frac{|x - y|^{d+2s}}{|x - y|^{d+2s}} dx dy.
\]

On the other hand, we have

\[
\begin{align*}
0 &= \int_{B_2^2} \eta^2 u^+_+ \text{div}(z^{1-2s} \nabla u^*) dx dz \\
&= - \int_{B_2^2} z^{1-2s} \nabla(\eta^2 u^+_+) \nabla u^* dx dz + \int_{B_2} \phi^2 u_+ (-\Delta)^{s/2} dx \\
&= - \int_{B_2^2} z^{1-2s} |\nabla(\eta u^+_+)|^2 dx dz + \int_{B_2} z^{1-2s} |\nabla \eta|^2 (u^+_+)^2 dx dz + \int_{B_2} \phi^2 u_+ (-\Delta)^{s/2} dx.
\end{align*}
\]

Hence we imply

\[
\begin{align*}
\left(3.10\right) \quad \int_{B_2} \eta^2 |u^+| \mathcal{L}_t u &\geq C_0 \int_{B_2^2} z^{1-2s} |\nabla(\eta u^+_+)|^2 dx dz - C_0 \int_{B_2^2} z^{1-2s} |\nabla \eta|^2 u^+_+ dx dz \\
&- C_1 \int_{\mathbb{R}^d \times \mathbb{R}^d} u_+(x) u_+(y) (\phi(x) - \phi(y))^2 \frac{|x - y|^{d+2s}}{|x - y|^{d+2s}} dx dy.
\end{align*}
\]

By choosing \(\varepsilon\) small enough and by combining (3.8), (3.9), (3.10) yield (3.7).

\[\square\]

**Remark 3.3.**

\(\circ\) This proposition allows us to control the \(L^2 L^2_{x,z}\)-norm of the gradient of \(u^*\). However, it gives a control in \(L^\infty L^2_{x,z}\)-norm of \(u^*\) only on the trace \(z = 0\). In order to obtain the regularity results, it remains to control this norm of \(u^*\) in \(z > 0\).

\(\circ\) In order to control the term

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} u_+(x) u_+(y) (\phi(x) - \phi(y))^2 \frac{|x - y|^{d+2s}}{|x - y|^{d+2s}} dx dy
\]

we need to add the condition \(\int_{B_4} (u(x,t) - 2)_+ dx \leq 2, \forall t \in \mathbb{R}\) in the sequel estimates.

**Proposition 3.4.** Let \(q > (d + 1)/2\). Assume that the vector field \(B\) with divergence free satisfies

\[
\left(3.11\right) \quad \|B\|_{L^\infty(-4,0; L^{4/(2s)}(B_2))} \leq M_0.
\]

There exist constants \(\varepsilon_0 = \varepsilon_0(s,d,\Lambda,M_0) > 0\) and \(\lambda = \lambda(s,d) \in (0,\frac{1}{2})\) such that for any solution \(u\) to equation (1.2), the following property holds true:

If

\[
\left(3.12\right) \quad \int_{B_4} \frac{(u(x,t) - 2)_+}{|x|^{d+2s}} dx \leq 2, \forall t \in \mathbb{R},
\]

then

\[
\left(3.13\right) \quad u^* \leq 2 \text{ in } Q_4,
\]

and

\[
\left(3.14\right) \quad \int_{Q_1} z^{1-2s}(u^+_+)^2 dx dz ds + \int_{Q_4} \eta^2 dx ds + \|g\|_{L^s(Q_4)} \leq \varepsilon_0,
\]

then

\[
\left(3.15\right) \quad u_+ \leq 2 - \lambda \text{ on } Q_1.
\]
Proof. We follow the proof of [CV10, Lemma 6]. For \( k = 0, 1, 2, \ldots \), we set
\[
C_k = 2 - \lambda (1 + 2^{-k}),
\]
\[
u_k = (u - C_k)^+, \quad u_k^* = (u^* - C_k)^+.
\]
By using the energy inequality \((3.7)\) with the test function \( \phi_k \psi u_k \), where
\[
\diamond \phi_k \text{ is a cut off function in } x \text{ such that }
1_{R_{1+2^{-k}-1/2}} \leq \phi_k \leq 1_{R_{1+2^{-k}}}, \quad |\nabla \phi_k| \leq C 2^k,
\]
\[
\diamond \psi \text{ is a fixed cut-off function in } z \text{ only with } 1_{(-1,1)} \leq \psi \leq 1_{(-2,2)},
\]
it follows that for any \( t_1 \in [-4, -2] \) we have
\[
A_k \leq \sup_{t_2 \in [-1-2^{-k}, 0] \setminus [\delta, 1]} \left( \int_{B_2} \phi_k(t_2) dx + \int_{t_1}^{t_2} \int_{B_2^*} z^{1-2s} |\nabla (\eta_k u_k^*)|^2 dxdzdt \right)
\]
\[
\leq C \int_{-4}^{0} \int \phi_k(x) u_k(y) \frac{(\phi_k(x) - \phi_k(y))^2}{|x - y|^{d+2s}} dxdydt
\]
\[
+ \int_{B_2} \phi_k^2 u_k^2(t_1) dx + C \int_{-4}^{0} \int_{B_2^*} z^{1-2s} |\nabla \eta_k|^2 u_k^* dxdzdt
\]
\[
+ C M_0 \int_{-4}^{0} \int_{B_2} |\nabla \phi_k|^2 u_k^2 dxds + \int_{-4}^{0} \int_{B_2} \phi_k^2 u_k^* dx.
\]
Here \( A_k \) is defined by
\[
A_k = \int_{-1-2^{-k}}^{0} \int_0^{5^k} \int \phi_k^2 u_k^* dx + \sup_{t \in [-1-2^{-k}, 0]} \int |\nabla \phi_k u_k|^2 dx,
\]
for some constant \( \delta > 0 \) that will be chosen later.

Thanks to the conditions \((3.14)\) and the facts that \(|\nabla \phi_k| \leq C 2^k\) and
\[
\int_{-4}^{0} \int_{B_2} \phi_k^2 u_k^* dx \leq 2 \left( \int_{Q_4} u_k^2 dx + \|g\|_{L_2(Q_4)}^2 \right),
\]
we obtain
\[
A_k \leq C (1 + M_0) 2^{2k} \epsilon_0 + C \int_{-4}^{0} \int_{B_4} \int_{B_4} u_k(x) u_k(y) \frac{(\phi_k(x) - \phi_k(y))^2}{|x - y|^{d+2s}} dxdydt.
\]
On the other hand we note that
\[
\int_{-4}^{0} \int_{B_4} \int_{B_4} u_k(x) u_k(y) \frac{(\phi_k(x) - \phi_k(y))^2}{|x - y|^{d+2s}} dxdydt \leq C \int_{-4}^{0} \int_{B_4} \int_{B_4} \frac{u_k(x) u_k(y)}{|x - y|^{d-2s+2s}} dxdydt
\]
\[
\leq C \|u_k\|_{L^2(Q_4)}^2 \leq C \epsilon_0,
\]
and
\[
\int_{-4}^{0} \int_{B_4} \int_{B_4} u_k(x) u_k(y) \frac{(\phi_k(x) - \phi_k(y))^2}{|x - y|^{d+2s}} dxdydt \leq \int_{-4}^{0} \int_{B_4} \int_{B_4} u_k(x) u_k(y) \frac{u_k(y)}{|x - y|^{d+2s}} dxdydt
\]
\[
\leq C \int_{-4}^{0} \int_{B_4} |y|^{d+2s} \int_{B_2} u_k(x) dx dt \leq C \sqrt{\epsilon_0}.
\]
Here we used (3.12) and (3.14) in two last inequalities. Hence it follows that

\[(3.16) \quad A_k \leq C \left[ (1 + M_0)2^{2k}\varepsilon_0 + \sqrt{\varepsilon_0} \right]. \]

In the following we shall prove that there exist \(0 < \delta < 1\) and \(M > 1\) such that the following estimates hold true:

\[(3.17) \quad A_k \leq M^{-k}, \]

\[(3.18) \quad \eta_k u^*_k = 0 \quad \text{for} \quad \delta^k \leq z \leq \min\{2, \delta^{k-1}\}, \]

for every \(k \geq 0\).

**Step 1.** (Existence of \(\delta\) and \(M\))

Let \(\xi_1\) and \(\xi_2\) be the solutions of respectively following problems

\[
\begin{cases}
\text{div}(z^{1-2s}\xi_1) = 0 & \text{in } B_4^*, \\
\xi_1 = 2 & \text{in } \partial B_4^* \setminus \{z = 0\}, \\
\xi_1 = 0 & \text{in } \partial B_4^* \cap \{z = 0\},
\end{cases}
\]

and

\[
\begin{cases}
\text{div}(z^{1-2s}\xi_2) = 0 & \text{in } [0, \infty) \times [0, 1], \\
\xi_2(0, z) = 2 & 0 \leq z \leq 1, \\
\xi_2(x, 0) = \xi_2(x, 1) = 0 & 0 < x < \infty.
\end{cases}
\]

From the maximum principle we can choose a constant \(\lambda \in (0, 1/2)\) such that

\[(3.19) \quad \xi_1(x, z) \leq 2 - 4\lambda \quad \text{in } B_2^*. \]

Moreover, there are also constants \(c_1 > 0\) and \(\sigma_0\) satisfying

\[(3.20) \quad |\xi_2(x, z)| \leq c_1 e^{-\sigma_0 x} \quad \text{in } [0, \infty) \times [0, 1]. \]

Let \(c_2\) and \(\theta_1, \theta_2 > 1\) be constants that will be chosen in later. It is not difficult to check that (see more details in [CV10, Lemma 7]) there are constants \(\delta \in (0, 1)\) and \(M > 1\) satisfying

\[(3.21) \quad 2dc_1 \exp \left( -\frac{2^{-k}}{4(\sqrt{2} + 1)\delta^k} \right) \leq \lambda 2^{-k-2}, \]

\[(3.22) \quad M^{-k/2}\delta^{\frac{d(k+1)}{2}}||P(\cdot, 1)||_{L^2(\mathbb{R}^d)} \leq \lambda 2^{-k-2}, \]

and

\[(3.23) \quad M^{-k} \geq c_2^k \left( M^{-\theta_1(k-3)} + M^{-\theta_2(k-3)} \right), \]

for all \(k \geq 12d.\)

**Step 2.** (Initial step)

From (3.16) we can choose \(\varepsilon_0\) sufficiently small so that (3.17) is verified for all \(0 \leq k \leq 12d.\) Next, we shall prove the equality (3.18) with \(k = 0.\) Similarly as in proof of [CV10, Lemma 6] we can use the maximum principle to obtain

\[u^* \leq (1_{B_4^*}u_+) \ast P(\cdot, z) + \xi_1 \quad \text{in } B_4^* \times \mathbb{R}^+. \]
On the other hand, it follows from Young’s inequality and (3.14) that, for $t \in [-4, 0]$ and $z \geq 1$ that
\[
\|(1_{B_4} u_+ ) \ast P(\cdot, z)\|_{L^\infty(B_4)} \leq C \|P(\cdot, z)\|_{L^2(\mathbb{R}^d)} \|(1_{B_4} u_+ )\|_{L^2(\mathbb{R}^d)} \\
\leq C \|P(\cdot, 1)\|_{L^2(\mathbb{R}^d)} \|1_{B_4} u_+ \|_{L^2(B_4)} \leq C \sqrt{\varepsilon_0}.
\]
So thanks to (3.19) we can choose $\varepsilon_0$ small enough so that
\[
u^* \leq 2 - 2\lambda \quad \text{in} \quad B_4 \times [1, \infty) \times [-4, 0],
\]
which implies $\eta_0 u_0^*$ vanishes for $1 \leq z \leq 2$.

**Step 3.** (Propagation of Properties (3.17) and (3.18))

Let us assume that (3.17) and (3.18) hold true for $k = \alpha$. We shall prove that (3.18) is also true at $k = \alpha + 1$. To do this we consider the function $\zeta_\alpha$ as follow
\[
\zeta_\alpha(x, z) = \sum_{i=1}^d \hat{\xi}_i(x, z) + \left( (\eta_m u_m) \ast P(\cdot, z) \right)(x).
\]
Here, for $x = (x_1, x_2, \ldots, x_d)$, the function $\hat{\xi}_i$ is defined by
\[
\hat{\xi}_i(x, z) = \xi_2 \left( \frac{-x_i + x^+}{\delta^m}, \frac{z}{\delta^m} \right) + \xi_2 \left( \frac{x_i - x^-}{\delta^m}, \frac{z}{\delta^m} \right),
\]
with $x^+ = 1 + 2^{-1/2}$, $x^- = -x^+$. Then we can see that $\zeta_\alpha$ satisfies the equation
\[
\text{div}(z^{1-2s} \nabla \zeta_\alpha) = 0 \quad \text{in} \quad Q^\delta_m \:= B_{1+2^{-m-1/2}} \times (0, \delta^m).
\]
It is clear that $\zeta_\alpha$ vanishes on the side $z = \delta^m$ thanks to the induction assumptions and the definition of $\xi_2$. It is $\eta_m u_m \ast P(\cdot, \cdot)$ on the side $z = 0$. Moreover, on the side $x_i = x^+$ or $x_i = x^-$ then it is bigger than 2. Hence by maximum principle we have
\[
u_m^* \leq \zeta_\alpha \quad \text{in} \quad Q^\delta_m.
\]
On the other hand, it follows from Step 1 that for every $x \in B_{1+2^{-m-1}}$ and $z \in [0, \delta^m]$ we have
\[
\zeta_m(x, z) \leq 2dc_1 \exp \left( -\frac{2^{-m}}{4(\sqrt{2} + 1)\delta^m} \right) + \left( (\eta_m u_m) \ast P(\cdot, z) \right)(x) \\
\leq \lambda 2^{-m-2} + \left( (\eta_m u_m) \ast P(\cdot, z) \right)(x).
\]
Hence, for $x \in B_{1+2^{-m-1}}$ and $z \in [0, \delta^m],
\[
u_m^* \leq \left( \nu_m^* - \lambda 2^{-m-2} \right)_+ \leq \left( (\eta_m u_m) \ast P(\cdot, z) - \lambda 2^{-m-2} \right)_+.
\]
This implies that
\[
\eta_{m+1} u_{m+1}^* \leq \eta_{m+1} \left( (\eta_m u_m) \ast P(\cdot, z) - \lambda 2^{-m-2} \right)_+.
\]
From step 1 and the fact $A_m \leq M^{-m}$, we can use Hölder’s inequality and (3.22) to find that
\[
\left( \eta_m u_m \right) \ast P(\cdot, z) \leq A_1^{1/2} \left( P(\cdot, z) \right)_{L^2(\mathbb{R}^d)} \leq \frac{M^{-m/2}}{\delta^{(m+1)d/2}} \left( P(\cdot, 1) \right)_{L^2(\mathbb{R}^d)} \leq \lambda 2^{-m-2},
\]
for $\delta^m \leq z \leq \delta^m$. Combining this with (3.24) yields
\[
\eta_{m+1} u_{m+1}^* = 0 \quad \text{for} \quad \delta^{m+1} \leq z \leq \delta^m.
\]
Let $m \geq 12d + 1$. Assume that (3.17) is true for $k = m - 3, m - 2, m - 1$ and (3.18) is true for $k = m - 3$. In order to prove the estimate (3.17) holds true at $k = m$ it is enough to show that

$$
(3.25) \quad A_m \leq c_2^m \left( A_{m-3}^{\theta_1} + A_{m-3}^{\theta_2} \right), \quad \theta_1, \theta_2 > 1.
$$

for some $\theta_1, \theta_2 > 1$. Since the functions

$$
\eta_{m-3} u_{m-3}^* 1_{\{0 < z < \delta^{m-3}\}} \quad \text{and} \quad (\eta_{m-3} u_{m-3})^*
$$

have the same trace at $z = 0$ we imply

$$
\int_0^{\delta^{m-3}} \int_{\mathbb{R}^d} z^{1 - 2s} \left| \nabla (\eta_{m-3} u_{m-3}^*) \right|^2 \, dx dz = \int_0^{\infty} \int_{\mathbb{R}^d} z^{1 - 2s} \left| \nabla (\eta_{m-3} u_{m-3}^* 1_{\{0 < z < \delta^{m-3}\}}) \right|^2 \, dx dz \geq \int_0^{\infty} \int_{\mathbb{R}^d} z^{1 - 2s} \left| \nabla (\eta_{m-3} u_{m-3})^* \right|^2 \, dx dz = \int_{\mathbb{R}^d} \left| (-\Delta)^{s/2} (\phi_{m-3} u_{m-3}) \right|^2 \, dx.
$$

By integrating with respect to $t$ over $[-1 - 2^{-m-3}, 0]$ we obtain

$$
\int_{-1 - 2^{-m-3}}^{0} \int_0^{\delta^{m-3}} \int_{\mathbb{R}^d} z^{1 - 2s} \left| \nabla (\eta_{m-3} u_{m-3}^*) \right|^2 \, dx dz dt \geq \int_{-1 - 2^{-m-3}}^{0} \int_0^{\delta^{m-3}} \left| \nabla (\eta_{m-3} u_{m-3}^*) \right|^2 \, dx dz dt.
$$

Hence, by using the Sobolev’s inequality it follows from the definition of $A_k$ that

$$
A_{m-3} \geq C \| \phi_{m-3} u_{m-3} \|_{L^{2(d+1)}(\mathcal{O}_{m-3})}, \quad \text{with} \quad \mathcal{O}_k = \mathbb{R}^d \times [-1 - 2^{-k}, 0].
$$

On the other hand, from (3.24) we have

$$
\eta_{m-2} u_{m-2}^* \leq \left[ (\eta_{m-3} u_{m-3}^*)^* P(z) \right] \eta_{m-2} \quad \text{on} \quad \mathcal{O}_{m-3}^*,
$$

with $\mathcal{O}_k^* = \mathbb{R}^d \times [0, \delta^k] \times [-1 - 2^{-k}, 0]$. Hence it follows from Young’s inequality that

$$
\| \eta_{m-2} u_{m-2}^* \|_{L^{2(d+1)}(\mathcal{O}_{m-2}^*)}^2 \leq \| P(1) \|_{L^1(\mathbb{R}^d)}^2 \| \eta_{m-3} u_{m-3} \|_{L^{2(d+1)}(\mathcal{O}_{m-3})}^2.
$$

Thus we obtain

$$
A_{m-3} \geq C \left( \| \eta_{m-2} u_{m-2}^* \|_{L^{2(d+1)}(\mathcal{O}_{m-2}^*)}^2 + \| \eta_{m-3} u_{m-3} \|_{L^{2(d+1)}(\mathcal{O}_{m-3})}^2 \right)
$$

$$
(3.26) \quad \geq C \left( \| \eta_{m-2} u_{m-1}^* \|_{L^{2(d+1)}(\mathcal{O}_{m-2})}^2 + \| \eta_{m-2} u_{m-1} \|_{L^{2(d+1)}(\mathcal{O}_{m-3})}^2 \right).
$$

Next, by the definition of $A_k$ we have

$$
A_m \leq \sup_{t_2 \in [1 - 2^{-m}, 0]} \left( \int_{\mathbb{R}^d} \phi_{m-2}^2(t_2)^2 \, dx + \int_{t_2}^{t_1} \int_0^{\delta^m} z^{1 - 2s} \left| \nabla (\eta_m u_m^*) \right|^2 \, dx dz dt \right) 13.
$$
for any $t_1 \in [-1 - 2^{-m+1}, -1 - 2^{-m}]$. Then we can apply Proposition 3.2 to obtain

$$A_m \lesssim \int_{-1-2^{-m+1}}^{0} \int_{B_2} z^{1-2s} |\nabla \eta_m|^2 (u_m^*)^2 dx dz dt$$

$$+ M_0 \int_{-1-2^{-m+1}}^{0} \int_{B_2} |\nabla \varphi_m|^2 u_m^2 dx ds + \int_{-1-2^{-m+1}}^{0} \int_{B_2} \varphi_m^2 u_m^2 dx dt$$

$$+ \int_{-1-2^{-m+1}}^{0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u_m(x) u_m(y) \frac{(\varphi_m(x) - \varphi_m(y))^2}{|x-y|^{d+2s}} dxdydt.$$  

Using Hölder's inequality we have

$$\int_{-1-2^{-m+1}}^{0} \eta_m^2 u_m |g| dx dt \leq ||g||_{L^q(\mathbb{R}^d)} \left( \int_{-1-2^{-m+1}}^{0} \int_{\mathbb{R}^d} (\eta_{m-1} u_m)^{q'} dx ds \right)^\frac{1}{q}.$$

So we obtain

$$A_m \lesssim \int_{-1-2^{-m+1}}^{0} \int_{B_2} z^{1-2s} |\nabla \eta_m|^2 (u_m^*)^2 dx dz dt$$

$$+ M_0 \int_{-1-2^{-m+1}}^{0} \int_{B_2} |\nabla \varphi_m|^2 u_m^2 dx ds + \int_{-1-2^{-m+1}}^{0} \int_{B_2} \varphi_m^2 u_m^2 dx dt$$

$$+ \left( \int_{-1-2^{-m+1}}^{0} \int_{\mathbb{R}^d} (\eta_{m-1} u_m)^{q'} dx ds \right)^\frac{1}{q}$$

$$+ \int_{-1-2^{-m+1}}^{0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u_m(x) u_m(y) \frac{(\varphi_m(x) - \varphi_m(y))^2}{|x-y|^{d+2s}} dxdydt.$$  

It is easy to see that

$$1_{\{\eta_{m-1} u_m > 0\}} \leq 1_{\{\eta_{m-2} u_{m-1} > \frac{1}{2} m\}}.$$  

Hence, on the set $\{\eta_{m-1} u_m > 0\}$, we have

$$\left( \frac{2m}{\lambda} \eta_{m-1} u_m \right)^{q_1} \leq \left( \frac{2m}{\lambda} \eta_{m-2} u_{m-1} \right)^{q_1} \leq \left( \frac{2m}{\lambda} \eta_{m-2} u_{m-1} \right)^{q_2},$$

for any $0 < q_1 < q_2$. So, as the proof of (27), by (3.26), one has

$$(3.27) \quad A_m \leq C m \left( A_{m-3}^{\theta_1} + A_{m-3}^{\theta_2} \right)$$

$$+ C \int_{-1-2^{-m+1}}^{0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u_m(x) u_m(y) (\varphi_m(x) - \varphi_m(y))^2 |x-y|^{d+1} dxdydt,$$

for some $\theta_1, \theta_2 > 1$. Now we estimate the last term in (3.27). To do this, we note that

$$J_1 := \int_{-1-2^{-m+1}}^{0} \int_{B_{1+2^{-m+1/2}}}^{0} \int_{B_{1+2^{-m+1/2}}}^{0} u_m(x) u_m(y) (\varphi_m(x) - \varphi_m(y))^2 |x-y|^{d+2s} dxdydt$$

$$\leq C \int_{-1-2^{-m+1}}^{0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi_m(x) u_m(x) \varphi_m(y) u_m(y) |x-y|^{d+2s-2} dxdydt$$

$$\leq C 2^m \int_{-1-2^{-m+1}}^{0} \int_{B_{1+2^{-m+1/2}}}^{0} \int_{B_{1+2^{-m+1/2}}}^{0} \varphi_{m-1}(x) u_m(x) \varphi_{m-1}(y) u_m(y) |x-y|^{d+2s-2} dxdydt$$

$$\leq C 2^m \int_{-1-2^{-m+1}}^{0} \||\varphi_{m-1} u_m||^2_{L^2} dt.$$
So by the same way as above we obtain

\[(3.28)\]

\[J_1 \leq C \frac{2^{m(d+1)/d}}{\chi^{2s/(d+1-2s)}} A^{d+1}_{m-3} \cdot \]

Moreover, since \(\phi_m(y) = 0\) for all \(y \in B^c_{1+2-m+1/2}\) it follows that

\[(3.29)\]

\[
\begin{align*}
J_2 &= \int_{-1-2-m+1}^{0} \int_{B^c_{1+2-m+1/2}} \int_{B_{1+2-m+1/2}} u_m(x) u_m(y) \frac{(\phi_m(x) - \phi_m(y))^2}{|x-y|^{d+2s}} dxdydt \\
&\leq \int_{-1-2-m+1}^{0} \int_{B^c_{1+2-m+1/2}} \int_{B_{1+2-m}} \phi_{m-1}(x) u_m(x) u_m(y) \frac{dy}{|x-y|^{d+2s}} dxdydt \\
&\leq C \int_{-1-2-m+1}^{0} \int_{B^c_{1+2-m+1/2}} \int_{B_{1+2-m}} u_m(y) \frac{dy}{y^{d+1}} \int_{B_{1+2-m}} \phi_{m-1}(x) u_m(x) dxdt \\
&\leq C \|\phi_{m-1} u_m\| L^1([-1-2-m+1,0] \times \mathbb{R}^d) \leq C \left( \frac{2^{m}}{\chi} \right) \frac{2^{(d+1)/(d+1-2s)-1}}{A^{d+1}_{m-3}} .
\end{align*}
\]

Here we used (3.12) in the third inequality. Thus, combining (3.27), (3.28) and (3.29) one get

\[
A_m \leq C^m \left( A^{\theta_1}_{m-3} + A^{\theta_2}_{m-3} \right) + C(J_1 + J_2) \\
\leq C^m \left( A^{\theta_1}_{m-3} + A^{\theta_2}_{m-3} + A^{d+1}_{m-3} \right) .
\]

This implies (3.25) for some for some \(c_2, \theta_1, \theta_2 > 1\). The proof is complete. \qed

**Proposition 3.5.** Let \(q > (d + 1)/2s\). Assume that \(g \in L^q(Q_4)\) and the vector field \(B\) with divergence free satisfy following conditions

\[(3.30)\]

\[
\|B\|_{L^\infty([-4,0],L^{d/q}(B_2))} \leq M_0 \quad \text{and} \quad \|g\|_{L^q(Q_4)} \leq 1 .
\]

Let \(u\) be a solution of the problem (1.2) such that

\[(3.31)\]

\[
u^* \leq 2 \quad \text{in} \quad Q^*_4, \quad \int_{B^*_4} \frac{u(x,t) - 2}{|x|^{d+2s}} dx \leq 2 \quad \text{for} \quad t \in [-4,0] ,
\]

\[
\mathcal{L}^{d+2}_{\omega} (\{(x,z,t) \in Q^*_4 : u^*(x,z,t) \leq 0\}) \geq \mathcal{L}^{d+2}_{\omega} (Q^*_4) / 2 .
\]

Then we can find a positive constant \(C = C(s,d,M_0,\Lambda)\) such that the following statement holds:

For each \(\varepsilon_1 > 0\) there exists a positive constant \(\delta_1 = \delta_1(s,d,M_0,\Lambda,\varepsilon_1)\) so that if

\[
\mathcal{L}^{d+2}_{\omega} (\{(x,z,t) \in Q^*_4 : 0 < u^*(x,s,t) < 1\}) \leq \delta_1
\]

then we have

\[(3.32)\]

\[
\int_{Q_4} (u - 1)^2 dxdt + \int_{Q^*_4} z^{1-2s}(u^* - 1)^2 dxdt \leq C_\varepsilon^*. 
\]

Here \(\mathcal{L}^{d+2}_{\omega}\) is measure in space \(\mathbb{R}^{d+2}\) with respect to the weighted measure \(\omega dx dz := z^{1-2s} dx dz\).

In order to prove this lemma we need the following weighted version of De Giorgi’s isoperimetric inequality which was proven in [CW09]. Its unweighted version was given in [CV10]).
Proposition 3.6. ([CW09, Lemma 3.5]) Let \( p > \frac{1-s}{s} \) and let \( f \) be a function defined in \( B^*_r \) such that
\[
\mathcal{K} := \int_{B^*_r} z^{1-2s} |\nabla f|^2 \, dx \, dz < \infty.
\]
We denote
\[
\mathcal{A} = \{(x, z) \in B^*_r : f(x, z) \leq 0\},
\]
\[
\mathcal{B} = \{(x, z) \in B^*_r : f(x, z) \geq 1\},
\]
\[
\mathcal{C} = \{(x, z) \in B^*_r : 0 < f(x, z) < 1\},
\]
and \( \mathcal{L}_{\omega}^{d+1} \) is measure in space \( \mathbb{R}^{d+1} \) with respect to the weighted measure \( \omega \, dx \, dz := z^{1-2s} \, dx \, dz \). Then we have
\[
\mathcal{L}_{\omega}^{d+1}(\mathcal{A}) \mathcal{L}_{\omega}^{d+1}(\mathcal{B}) \leq C r^d \mathcal{K}^{\frac{1}{d}} \left( \mathcal{L}_{\omega}^{d+1}(\mathcal{C}) \right)^{\frac{1}{p}},
\]
where
\[
\varrho = 1 + \frac{1}{2} \left( d + 1 - \frac{p+1}{p-1} (1-2s) \right) \left( 1 - \frac{1}{p} \right).
\]

Proof of Lemma 3.5. We use the similar arguments as in [CV10].

It follows from the energy inequality (3.7) of Proposition 3.2 and assumptions (3.30), (3.31) and Holder’s inequality that there exists a positive constant \( C \) such that
\[
(3.33) \quad \int_{Q^*_4} z^{1-2s} |\nabla u^*_+|^2 \, dx \, dz \, dt \leq C.
\]
Let \( \varepsilon_1 \ll 1 \). We put
\[
\Upsilon = 4\varepsilon_1^{-1} \int_{Q^*_4} z^{1-2s} |\nabla u^*_+|^2 \, dx \, dz \, dt,
\]
\[
\mathbb{I}_1 = \left\{ t \in [-4, 0] : \int_{B^*_4} z^{1-2s} |\nabla u^*_+|^2 \, dx \, dz \leq \Upsilon \right\},
\]
and for every \( t \in \mathbb{I}_1 \),
\[
\mathcal{A}(t) = \{(x, z) \in B^*_4 : u^*(x, z, t) \leq 0\},
\]
\[
\mathcal{B}(t) = \{(x, z) \in B^*_4 : u^*(x, z, t) \geq 1\},
\]
\[
\mathcal{C}(t) = \{(x, z) \in B^*_4 : 0 < u^*(x, z, t) < 1\}.
\]
Given a fixed number \( p > (1-s)s^{-1} \).

Claim 1. If \( \mathbb{I} \) is the set defined by
\[
\mathbb{I} = \mathbb{I}_1 \cap \mathbb{I}_2 \quad \text{with} \quad \mathbb{I}_2 := \left\{ t \in [-4, 0] : \left( \mathcal{L}_{\omega}^{d+1}(\mathcal{C}(t)) \right)^{\frac{1}{p}} \leq \varepsilon_1^{1+\frac{p}{2}} \right\}.
\]
Then we have
\[
(3.34) \quad \mathcal{L}^1([-4, 0] \setminus \mathbb{I}) \leq \frac{\varepsilon_1}{2},
\]
provided that we choose \( \delta_1 \) such that \( \delta_1 < \varepsilon_1^{2p(1+p/s)+2} \). This is because the following inequalities hold true
\[
(3.35) \quad \mathcal{L}^1([-4, 0] \setminus \mathbb{I}_1) \leq \Upsilon^{-1} \int_{Q^*_4} z^{1-2s} |\nabla u^*_+|^2 \, dx \, dz \, dt \leq \frac{\varepsilon_1}{4}.
\]
\begin{align}
    \mathcal{L}^1([-4, 0] \setminus I_2) & \leq \varepsilon_1^{-2p(1+p/s)} \mathcal{L}_\omega^{d+2}(\{(x, z, t) : 0 < u^* < 1\}) \leq \varepsilon_1^{-2p(1+p/s)} \delta_1 \leq \varepsilon_1^4,
\end{align}

thanks to Chebyshev’s inequality.

**Claim 2.** If \( t \in I \) such that \( \mathcal{L}_\omega^{d+1}(A(t)) \geq 1/4 \) then

\begin{align}
    \int_{B^*_4} z^{1-2s}(u^*_+)^2 \, dx \, dz & \leq \varepsilon_1^2 \quad \text{and} \quad \int_{B_4} u^*_+ \, dx \leq \varepsilon_1^{1-s}. \tag{3.37}
\end{align}

Indeed, we use first the Proposition 3.6 to obtain

\begin{align}
    \mathcal{L}_\omega^{d+1}(B(t)) & \leq C \mathcal{Y}^{1/2} \left( \mathcal{L}_\omega^{d+1}(C(t)) \right)^{1/p} \left( \mathcal{L}_\omega^{d+1}(A(t)) \right)^{-1} \leq C \varepsilon_1^{1/2+p/s}. \tag{3.36}
\end{align}

Hence, since \( u^* \leq 2 \) in \( Q_4^* \) we have

\begin{align}
    \int_{B^*_4} z^{1-2s}(u^*_+)^2 \, dx \, dz & \leq 4 \left( \mathcal{L}_\omega^{d+1}(B(t)) + \mathcal{L}_\omega^{d+1}(C(t)) \right) \leq C \varepsilon_1^{1-s} \leq \varepsilon_1^4. \tag{3.38}
\end{align}

On the other hand, for any \( z \), it is clear that

\begin{align}
    \int_{B_4} u^*_+ \, dx = \int_{B_4} (u^*_+(x, z))^2 \, dx - 2 \int_0^z \int_{B_4} u^*_+ \partial_z u^* \, dx \, dz.
\end{align}

Then by integrating in \( z \) on \([0, 1]\) this implies

\begin{align}
    \int_{B_4} u^*_+ \, dx & \leq \int_{B^*_4} (u^*_+)^2 \, dx dz + 2 \int_{B^*_4} |u^*_+| |\partial_z u^*| \, dx dz. \tag{3.39}
\end{align}

From Hölder’s inequality and the fact that \( u^* \leq 2 \) we have

\begin{align}
    \int_{B^*_4} (u^*_+)^2 \, dx dz & \leq 2^{2-s} \int_{B^*_4} z^{(2s-1)\sigma} (u^*_+)^\sigma \, dx dz \\
    & \leq 2^{2-s} \left( \int_{B^*_4} z^{(2s-1)\sigma} \, dx dz \right)^{\frac{\sigma}{\sigma}} \left( \int_{B^*_4} z^{1-2s}(u^*_+)^2 \, dx dz \right)^{\frac{s}{\sigma}},
\end{align}

where \( \sigma < (1-s)^{-1} \). Since \( ps > 1 \) it follows that

\begin{align}
    \int_{B^*_4} (u^*_+)^2 \, dx dz & \leq C \varepsilon_1^{\frac{1}{ps}} \leq \varepsilon_1^{1/p}. \tag{3.40}
\end{align}

Similarly we have from Hölder’s inequality

\begin{align}
    2 \int_{B_4^*} |u^*_+| |\partial_z u^*| \, dx dz & \leq 2^{2-1/p} \int_{B^*_4} z^{1-2s} |u^*_+|^\frac{1}{p} \left( \int_{B^*_4} z^{1-2s} |\partial_z u^*|^\frac{1}{p} \, dx dz \right)^{\frac{p-1}{p}} \, dx dz \\
    & \leq C \left( \int_{B^*_4} z^{1-2s} |\partial_z u^*|^2 \, dx dz \right)^{\frac{1}{2}} \left( \int_{B^*_4} z^{1-2s} |u^*|^2 \, dx dz \right)^{\frac{1}{2p}} \left( \int_{B^*_4} z^{(1-2s)(p+1)} \, dx dz \right)^{\frac{p}{2p}} \tag{3.38}
\end{align}

provided that \( \frac{(1-2s)(p+1)}{p-1} < 1 \). This implies

\begin{align}
    \int_{B_4^*} |u^*_+| |\partial_z u^*| \, dx dz & \leq C \varepsilon_1^{\frac{1}{p} + \frac{1}{2p} + \frac{1}{2p}}. \tag{3.41}
\end{align}
And therefore we obtain from (3.39), (3.40) and (3.41) that
\[ \int_{B_4} u_+^2 dx \leq \varepsilon_1^{\frac{1}{t^+}} + C\varepsilon_1^{-\frac{1}{t^+} + \frac{1}{t^+} + \frac{1}{t^+}} \leq \varepsilon_1^{\frac{1}{t^+}}. \]

So, we get (3.37) for \( \varepsilon_1 > 0 \) small enough.

**Claim 3.** If \( t \in I \cap [-1, 0] \) then \( \mathcal{L}_{\omega}^{d+1}(A(t)) \geq \frac{1}{4} \).

Indeed, since
\[ \mathcal{L}_{\omega}^{d+2} \left( \{ (x, z, t) \in Q^*_1 : u^*(x, z, t) \leq 0 \} \right) \geq \mathcal{L}_{\omega}^{d+2}(Q^*_1) / 2, \]
there exists \(-4 \leq t_0 \leq -1\) such that \( \mathcal{L}_{\omega}^{d+1}(A(t_0)) \geq 1/4 \). Thus, by using Claim 2 it follows that
\[ \int_{B_4} u_+^2(x, t_0) dx \leq \varepsilon_1^{\frac{1}{t^+}}. \]

Let \( r > 0 \) be a small number. Combining this estimate and the energy inequality (3.7) of Proposition 3.2 and assumptions (3.30), (3.31) and Holder’s inequality we deduce that, for any \( t \geq t_0 \),
\[ \int_{B_1} u_+^2(x, t) dx \leq \int_{B_4} u_+^2(x, t_0) dx + C(t - t_0 + (t - t_0)^{\frac{s-1}{s}}), \]

So for \( t - t_0 \leq \delta^* = \left( \min \{ 1, (10^{10}C)^{-1} \} \right) \frac{10^8}{t^+} \), we have
\[ (3.42) \quad \int_{B_1} u_+^2(x, t) dx \leq 10^{-9}, \text{ for all } 0 \leq t \leq t_0 + \delta^*. \]

for \( \varepsilon_1 > 0 \) small enough. It is also noted that \( \delta^* \) do not depend on \( \varepsilon_1 \). Hence we can suppose that \( \varepsilon_1 \ll \delta^* \).

Next, we have
\[ u_+^* = u_+ + \int_0^z \partial_\zeta u_+^* d\zeta. \]

This implies that
\[ z^{1-2s}(u_+^*)^2 \leq 2z^{1-2s}u_+^2 + 2z^{1-2s}\left( \int_0^z \partial_\zeta u_+^* d\zeta \right)^2 \leq 2z^{1-2s}u_+^2 + \frac{z}{s} \int_0^z z^{1-2s}|\nabla u_+^*|^2 d\zeta. \]

Thus, for given \( t \in I \) such that \( t_0 \leq t \leq t_0 + \delta^* \), by integrating in \((x, z)\) on \( B_1 \times [0, \varepsilon_1^{1/s}] \) we get
\[ \int_0^{\varepsilon_1^{1/s}} \int_{B_1} z^{1-2s}(u_+^*)^2 dx dz \leq \frac{\varepsilon_1^{2(1-s)/s}}{1 - s} \int_{B_1} u_+^2 dx + \frac{\varepsilon_1^{2/s}}{2s} \int_0^{\varepsilon_1^{1/s}} \int_{B_1} z^{1-2s}|\nabla u_+^*|^2 dz dz \leq 10^{-8} \varepsilon_1^{2(1-s)/s} + C(s)\varepsilon_1^{2/s-1} \leq 10^{-7} \varepsilon_1^{2(1-s)/s}. \]

Using Chebyshev’s inequality again, it follows that
\[ \mathcal{L}_{\omega}^{d+1} \left( \{ x \in B_1, z \in [0, \varepsilon_1^{1/s}] : u_+^* \geq 1 \} \right) \leq 10^{-7} \varepsilon_1^{2(1-s)/s}. \]

Since \( \mathcal{L}_{\omega}^{d+1}(C(t)) \leq \varepsilon_1^{2p(1+\tilde{p})} \) we have
\[ \mathcal{L}_{\omega}^{d+1}(A(t)) \geq \frac{\varepsilon_1^{2(1-s)/s}}{2(1-s)} \mathcal{L}_{d+1}(B_1) - 10^{-7} \varepsilon_1^{2(1-s)/s} - \varepsilon_1^{2p(1+\tilde{p})} \geq \frac{\varepsilon_1^{2(1-s)/s}}{4}. \]
for \( \varepsilon_1 > 0 \) small enough.

Using again Proposition 3.6 we get
\[
\mathcal{L}^{d+1}_\omega(B(t)) \leq C \frac{\frac{1}{d} \mathcal{L}^{d+1}_\omega(C(t))^{\frac{1}{d}}}{\mathcal{L}^{d+1}_\omega(A(t))} \leq C \varepsilon_1^{\frac{1}{d}} + 2^{(1-s)/s} \leq C \sqrt{\varepsilon_1}.
\]

So
\[
\mathcal{L}^{d+1}_\omega(A(t)) \geq \mathcal{L}^{d+1}_\omega(B_1^*) - \mathcal{L}^{d+1}_\omega(B(t)) - \mathcal{L}^{d+1}_\omega(C(t)) \geq \frac{1}{4}.
\]

In summary, we have proved that \( \mathcal{L}^{d+1}_\omega(A(t)) \geq 1/4 \) for every \( t \in I \cap [t_0, t_0 + \delta^*] \). It is easy to find from (3.34) that on \([t_0 + \delta^*/2, t_0 + \delta^*]\) there exists \( t_1 \in I \). Hence we can find an increasing sequence \( \{t_n\}_{n=1}^\infty \) such that

(3.43) \[ t_0 + \frac{n\delta^*}{2} \leq t_n \leq 0, \]

(3.44) \[ \mathcal{L}^{d+1}_\omega(A(t)) \geq \frac{1}{4} \text{ if } t \in [t_n, t_n + \delta^*] \cap I \cap [t_n, t_{n+1}] \cap I. \]

Therefore we conclude that \( \mathcal{L}^{d+1}_\omega(A(t)) \geq 1/4 \) on \( I \cap [-1, 0] \).

**The estimate (3.32) holds true.**

From Claim 3 and Proposition 3.6 we imply that, for each \( t \in I \cap [-1, 0] \),
\[
\mathcal{L}^{d+1}_\omega(B(t)) \leq 4C \varepsilon_1^{\frac{1}{d} + \frac{2}{s}} \leq \frac{\varepsilon_1}{16}
\]
which leads to
\[
\mathcal{L}^{d+2}_\omega((\{(x, z, t) \in Q_1^* : u^* \geq 1\}) \leq \frac{\varepsilon_1}{16} + \mathcal{L}^1([-4, 0] \setminus I) \leq \varepsilon_1.
\]

Hence, by \((u^* - 1)_+ \leq 1, \)
\[
(3.45) \int_{Q_1^*} z^{1-2s}(u^* - 1)_+^2 dxdzdt \leq \varepsilon_1.
\]

For every fixed \( x \) and \( t \), we have
\[
u = u^* - \int_0^z \partial_z u^* d\bar{z}.
\]

Hence, for any \( z \),
\[
z^{1-2s}(u - 1)_+^2 \leq 2z^{1-2s}(u^* - 1)_+^2 + 2z^{1-2s}\left(\int_0^z |\nabla(u^*)^2| d\bar{z}\right)^2
\]
\[
\leq 2z^{1-2s}(u^* - 1)_+^2 + z \int_0^z \bar{z}^{1-2s}|\nabla(u^*)|^2 d\bar{z}.
\]

By taking the average in \( z \) on \([0, \sqrt{\varepsilon_1}]\) we get
\[
\frac{\varepsilon_1^{1-2s}}{2(1-s)}(u - 1)_+^2 \leq \frac{2\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1}} \int_0^{\sqrt{\varepsilon_1}} z^{1-2s}(u^* - 1)_+^2 dz + \sqrt{\varepsilon_1} \int_0^{\sqrt{\varepsilon_1}} z^{1-2s}|\nabla(u^*)|^2 dz.
\]

Therefore, by Integrating with respect to \((x, t)\) on \( B_1 \times [-1, 0] \) it follows from (3.33) and (3.45) that
\[
\frac{\varepsilon_1^{1-2s}}{2(1-s)} \int_{Q_1} (u - 1)_+^2 dxdt \leq 2\sqrt{\varepsilon_1}.
\]

So,
\[
\int_{Q_1} (u - 1)_+^2 dxdt \leq C \sqrt{\varepsilon_1}.
\]

The results follows from this and (3.45). The proof is complete. \( \square \)
Proof of Lemma 3.1. Put $\varepsilon_1 = \varepsilon_0^2/16C^2$ with $C$ is the constant defined in Lemma 3.5. Let us take the constants $\lambda$ and $\delta_1$ be associated to $\varepsilon_0$ as in Lemma 3.4 and to $\varepsilon_1$ as in Lemma (3.5), respectively.

Define $\bar{u}_0 = u$ and

$$\bar{u}_k = 2(\bar{u}_{k-1} - 1), \quad \text{for every } k \in \mathbb{N}, \ k \leq K_0 = [1 + |B_1|(4(1-s)\delta_1)^{-1}].$$

Then, for every $k$,

- $\bar{u}_k = 2^k(u - 2) + 2$,
- $\bar{u}_k$ is solution (1.2) with data $2^k g$ and satisfies $(\bar{u}_k)^* \leq 2$. Moreover, we also have

$$\int_{B_1^+} \frac{(\bar{u}_k(x) - 2)_+}{|x|^{d+2s}}dxdt \leq 1,$$

and

$$\mathcal{L}^{d+2}_\omega \{ (x, z, t) \in Q_1^* : (\bar{u}_k)^*(x, z, t) \leq 0 \} \geq \frac{|B_1|}{4(1-s)},$$

where we choose $\varepsilon_2 > 0$ such that $2^{K_0}\varepsilon_2 = 1$.

Now we consider two following cases:

**Case 1:** There is $k_0 \leq K_0$ such that

$$\mathcal{L}^{d+2}_\omega \{ (x, z, t) \in Q_1^* : 0 < (\bar{u}_{k_0})^*(x, z, t) < 1 \} \leq \delta_1.$$

By applying Lemma (3.5) for solution $\bar{u}_{k_0}$ with data $2^{k_0} g$, and then Lemma 3.4 for solution $\bar{u}_{k_0+1}$ with data $2^{k_0+1} g$, we obtain

$$\bar{u}_{k_0+1} \leq 2 - \lambda \text{ on } Q_1^{1/8}.$$ 

Thus,

$$u \leq 2 - 2^{-k_0-1}\lambda \leq 2 - 2^{-K_0}\lambda \text{ in } Q_{1/8}.$$

**Case 2:** For all $k \leq K_0$, we have

$$\mathcal{L}^{d+2}_\omega \{ (x, z, t) \in Q_1^* : 0 < (\bar{u}_k)^*(x, z, t) < 1 \} > \delta_1.$$

Then for every $k \leq K_0$,

$$\mathcal{L}^{d+2}_\omega \{ (x, z, t) \in Q_1^* : (\bar{u}_k)^* < 0 \} = \mathcal{L}^{d+2}_\omega \{ (x, z, t) \in Q_1^* : (\bar{u}_{k-1})^* < 1 \} \geq \delta_1 + \mathcal{L}^{d+2}_\omega \{ (x, z, t) \in Q_1^* : (\bar{u}_{k-1})^* \leq 0 \}.$$

Thus, we get

$$\mathcal{L}^{d+2}_\omega \{ (x, z, t) \in Q_1^* : (\bar{u}_{K_0})^* < 0 \} \geq \frac{|B_1|}{4(1-s)},$$

and it leads to $(\bar{u}_{K_0})^* \leq 0$ almost everywhere, which means

$$u^* \leq 2 - 2^{-K_0}.$$

Then, as the proof of [CV10, Proposition 9], we get (3.6).  \qed

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4 Proof of the main results

This section is devoted proving the main results, Theorem 1.2 and Corollary 1.3. Firstly, we shall prove the following Theorem which leads to Theorem 1.2 as a direct consequence.

**Theorem 4.1.** Assume that the vector field $\mathbf{B}$ satisfies the conditions

\[
\sup_{x \in \mathbb{R}^d, t > -2} \left| \int_{x + B_1} \mathbf{B}(y, t) dy \right| \leq M_1 \quad \text{and} \quad \|\mathbf{B}\|_{L^\infty([-2, \infty); \mathbb{X})} \leq M_2,
\]

for some $M_1, M_2 \geq 1$. There exists $\varepsilon_0 > 0$ such that if we have

\[
\|u(-2)\|_{L^2(\mathbb{R}^d)} + \|g\|_{L^1(\mathbb{R}^d \times (-2, \infty))} + \|g\|_{L^2(\mathbb{R}^d \times (-2, \infty))} \leq \varepsilon_0,
\]

then the following estimate holds true

\[
\sup_{|x_1 - x_2| + |t_1 - t_2|^{1/2s} \leq \rho_0, t_1 > -1/2} \frac{|u(x_1, t_1) - u(x_2, t_2)|}{(|x_1 - x_2| + |t_1 - t_2|^{1/2s})^\alpha} \leq C,
\]

where $\rho_0 = C(M_2)/M_1 \leq 1/10$.

**Proof.** By Remark (2.2), one has

\[
\sup_{t > -2} \int_{\mathbb{R}^d} u(t)^2 dx + \int_{-2}^\infty \int_{\mathbb{R}^d} (-\Delta)^{s/2} u(t)^2 dx dt + \|u\|_{L^\infty(\mathbb{R}^d \times (-1, \infty))}^2 \leq C\varepsilon_0^2.
\]

In particular,

\[
|u(x, t)| \leq 1 \quad \text{in} \quad \mathbb{R}^d \times (-1, 0),
\]

for $\varepsilon_0 > 0$ small enough.

Now we put $b_{-1} \equiv \mathbf{B}$ and, for $k \geq 0$,

\[
b_k(y, t) = \sigma^{2s-1} b_{k-1}(\sigma y + \sigma^2 x_k(t)), \sigma^2 t) - \sigma^{2s-1} \dot{x}_k(t),
\]

where $\sigma < 1$ is a constant that will be chosen later, $x_k$ is the solution to problem

\[
\dot{x}_k(t) = \frac{1}{E^d(B_1)} \int_{B_1} b_{k-1}(\sigma y + \sigma^2 x_k(t)), \sigma^2 t) dy, \quad x_k(0) = 0.
\]

We also define the sequences $\{g_k\}_{k=-1}^\infty$ and $\{F_k\}_{k=-1}^\infty$ defined by recursively as follows:

- $g_{-1} \equiv g, \quad F_{-1} \equiv u$ and
- for every $k \geq 0$,

\[
g_k(y, t) = \tilde{\lambda} \sigma^{2s} g_{k-1} \left( \sigma y + \sigma^2 x_k(t), \sigma^2 t \right),
\]

\[
F_k(y, t) = \frac{8}{8 - \lambda^*} \left[ F_{k-1} \left( \sigma y + \sigma^2 x_k(t), \sigma^2 t \right) + \lambda^*/4 \right], \quad \text{or}
\]

\[
F_k(y, t) = \frac{8}{8 - \lambda^*} \left[ F_{k-1} \left( \sigma y + \sigma^2 x_k(t), \sigma^2 t \right) - \lambda^*/4 \right],
\]

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where $\lambda^* \in (0, 10^{-10})$ is the constant defined in Lemma 3.1.

Clearly, for any $k = 0, 1, 2, \ldots$, the function $F_k$ satisfies

$$\partial_t F_k + b_k \nabla F_k + \mathcal{L}_{k,t} F_k = g_k,$$

where

$$\mathcal{L}_{j,t} f(y) = \int_{\mathbb{R}^d} (f(y) - f(y')) K_{j,t}(y, y') dy',$$

with

$$K_{j,t}(y, y') = \sigma^{d+2s} K_{j-1,t}(\sigma y - \sigma^2 x_j(t), \sigma y' - \sigma^2 x_j(t)).$$

It is necessary to note that, for all $y, y' \in \mathbb{R}^d$, we have $K_{j,t}(y, y') = K_{j,t}(y', y)$ and

$$\frac{\Lambda^{-1}}{|y - y'|^{d+2s}} \leq K_{j,t}(y, y') \leq \frac{\Lambda}{|y - y'|^{d+2s}}.$$

On the other hand, for $k \geq 0$, we also have

$$\text{div} b_k = 0, \quad \int_{B_4} b_k(y, t) dy = 0, \quad \|b_k(\cdot, t)\|_{X_s} = \|B(\cdot, \sigma^{2s(k+1)} t)\|_{X_s}.$$

In particular, it leads to

$$\|b_k\|_{L^\infty((-1,0]; L^3(B_4))} \leq C M_2, \text{ for all } k \geq 0, q \geq 1.$$

Moreover, one can check that

$$g_k(y, t) = \left(\frac{8}{8 - \lambda^*}\right)^{k+1} \sigma^{2s(k+1)} g \left(\sum_{m=0}^{k} \sigma^{k+2s-m} x_m(t), \sigma^{2s(k+1)} t\right),$$

and

$$F_k(y_1, t_1) - F_k(y_2, t_2) = \left(\frac{8}{8 - \lambda^*}\right)^{k+1} u \left(\sum_{m=0}^{k} \sigma^{k+2s-m} x_m(t_1), \sigma^{2s(k+1)} t_1\right) - \left(\frac{8}{8 - \lambda^*}\right)^{k+1} u \left(\sum_{m=0}^{k} \sigma^{k+2s-m} x_m(t_2), \sigma^{2s(k+1)} t_2\right).$$

**Step 1: (Boundedness of functions $x_k(t)$)** We shall prove that, for any $k \geq 0$, the following estimate holds true

$$\sup_{t \in [-1,0]} |x_k(t)| + \sup_{t \in [-1,0]} |\dot{x}_k(t)| \lesssim \Theta(k, s, \sigma, M_1, M_2),$$

where the constant $\Theta = \Theta(k, s, \sigma, M_1, M_2)$ given by

$$\Theta = \begin{cases} 
M_1 + M_2 \log \sigma, & \text{if } s = \frac{1}{2}, \ k = 0, \\
M_2 \log \sigma, & \text{if } s = \frac{1}{2}, \ k > 0, \\
M_1 + M_2, & \text{if } s < \frac{1}{2}, \ k = 0, \\
M_2^{\frac{1}{2}}, & \text{if } s < \frac{1}{2}, \ k > 0.
\end{cases}$$

First we consider the case $0 < s < 1/2$. Since $B \in L^\infty(\mathbb{R}; C^{1-2s}(\mathbb{R}^d))$ one has

$$|\dot{x}_0(t)| = \left| \int_{B_4} B(\sigma y + \sigma^{2s} x_0(t), \sigma^{2s} t) dy \right|
= \left| \int_{\sigma^{2s} x_0(t) + B_{4s}} B(y, \sigma^{2s} t) dy \right| \leq \left| \int_{\sigma^{2s} x_0(t) + B_1} B(y, \sigma^{2s} t) dy \right|$$

$$+ \left| \int_{\sigma^{2s} x_0(t) + B_{4s}} B(y, \sigma^{2s} t) dy - \int_{\sigma^{2s} x_0(t) + B_1} B(y, \sigma^{2s} t) dy \right|.$$
This implies
\[ |\dot{x}_0(t)| \lesssim M_1 + (1 + \sigma^{-1-2s})M_2. \]
Combining this and \( x_k(0) = 0 \) we obtain (4.7) for case \( k = 0 \) and \( s < 1/2 \).

For any \( k \geq 1 \), by using the fact \( \int_{B_4} b_{k-1}(y, \sigma^2 t)dy = 0 \), it follows that
\[
|\dot{x}_k(t)| = \left| \int_{\sigma^2 x_k(t) + B_{4\sigma}} b_{k-1}(y, \sigma^2 t)dy - \int_{B_4} b_{k-1}(y, \sigma^2 t)dy \right| \lesssim (1 + |x_k(t)|^{-2s})M_2.
\]

Since \( x_k(0) = 0 \) we get
\[
\sup_{t \in [-1,0]} |x_k(t)| \lesssim M_2 + M_2 \left( \sup_{t \in [-1,0]} |x_k(t)| \right)^{1-2s}
\]
which implies \( \sup_{t \in [-1,0]} |x_k(t)| \lesssim M_2^{1/2s} \). This will give (4.7) for case \( k \geq 1 \) and \( s < 1/2 \).

Next we consider the case \( s = 1/2 \). For \( \delta < 1 \) and \( \xi \in \mathbb{R}^d \), we have
\[
(4.9) \quad \left| \int_{\xi + B_\delta} B(y, t)dy \right| \leq \left| \int_{\xi + B_1} B(y, t)dy \right| + \int_{\xi + B_\delta} B(y, t)dy - \int_{\xi + B_{2\log_2 \delta}} B(y, t)dy
\]
\[
+ \sum_{j=1}^{[\log_2 \delta]} \left| \int_{\xi + B_{2j}} B(y, t)dy - \int_{\xi + B_{2j-1}} B(y, t)dy \right|
\]
\[
\leq M_1 + 2^d (1 - \log \delta) M_2 \lesssim M_1 + |\log_2 \delta| M_2.
\]

It follows from (4.9) that
\[
|\dot{x}_0(t)| = \left| \int_{B_4} B(\sigma y + \sigma x_0(t), \sigma t)dy \right| = \left| \int_{\sigma x_0(t) + B_{4\sigma}} B(y, \sigma t)dy \right| \lesssim M_1 + |\log \sigma| M_2.
\]
Combining this and \( x_0(0) = 0 \) we get (4.7). Now for \( k \geq 1 \), since \( \int_{B_4} b_{k-1}(y, t)dy = 0 \) it follows
\[
\left| \int_{\sigma x_k(t) + B_{4\sigma}} b_{k-1}(y, \sigma t)dy \right| \leq \left| \int_{\sigma x_k(t) + B_{4\sigma}} b_{k-1}(y, \sigma t)dy - \int_{B_{4\sigma}} b_{k-1}(y, \sigma t)dy \right|
\]
\[
+ \left| \int_{B_{4\sigma}} b_{k-1}(y, \sigma t)dy - \int_{B_4} b_{k-1}(y, \sigma t)dy \right|.
\]
On the other hand, by using [Aus, Corollary 5.1.10] we have
\[
\left| \int_{\sigma x_k(t) + B_{4\sigma}} b_{k-1}(y, \sigma t)dy - \int_{B_{4\sigma}} b_{k-1}(y, \sigma t)dy \right| \lesssim \log_2 (2 + |x_k(t)|),
\]
and, similarly as (4.9),
\[
\left| \int_{B_{4\sigma}} b_{k-1}(y, \sigma t)dy - \int_{B_4} b_{k-1}(y, \sigma t)dy \right| \leq (1 - \log_2 \sigma) M_2.
\]
Hence we can obtain
\[
|\dot{x}_k(t)| = \left| \int_{\sigma x_k(t) + B_{4\sigma}} b_{k-1}(y, \sigma t)dy \right| \lesssim \left( |\log \sigma| + \log (2 + |x_k(t)|) \right) M_2.
\]
Thanks to \(x_k(0) = 0\) we deduce that
\[
\sup_{t \in [-1,0]} \left| x_k(t) \right| \lesssim \left\{ \log \sigma + \log(2 + \sup_{t \in [-1,0]} \left| x_k(t) \right|) \right\} M_2
\]
which implies
\[
\sup_{t \in [-1,0]} \left| x_k(t) \right| \lesssim \log \sigma M_2 + 2 M_2 \log M_2.
\]
So one gets
\[
\sup_{t \in [-1,0]} \left| x_k(t) \right| + \sup_{t \in [-1,0]} \left| \dot{x}_k(t) \right| \lesssim \log \sigma M_2 + 2 M_2 \log M_2.
\]

**Step 2: (Control the functions \(F_k\))** For any \(k \geq 0\) we shall prove that
\begin{equation}
|F_k(x, t)| \leq 2 + c_0 \lambda^s \left( |\sigma^{1-2s}x + x_k(t)|^{2s} - 1 \right)_+ \quad \text{in } \mathbb{R}^d \times (-1,0),
\end{equation}
and
\begin{equation}
|F_k(x, t)| \leq 2 \quad \text{in } Q_{1/16},
\end{equation}
where \(c_0\) will be chosen later. To do this we use the induction argument.

Let \((x, t) \in \mathbb{R}^d \times (-1,0)\). It follows from (4.4) that
\[
|F_0(x, t)| \leq \frac{8}{8 - \lambda^s} \left( 1 + \frac{\lambda^s}{4} \right) \leq 2 \leq 2 + c_0 \lambda^s (|\sigma^{1-2s}x + x_0(t)|^{2s} - 1)_+.
\]

Similarly, by the definition of \(F_1\) we imply
\[
|F_1(t, x)| \leq \frac{8}{8 - \lambda^s} \left\{ \frac{8}{8 - \lambda^s} \left( 1 + \frac{\lambda^s}{4} \right) + \frac{\lambda^s}{4} \right\} \leq 2
\leq 2 + c_0 \lambda^s (|\sigma^{1-2s}x + x_1(t)|^{2s} - 1)_+.
\]

Assume that (4.10) and (4.11) hold for any \(k \leq k_1\) for \(k_1 \geq 1\). We will need to prove
\begin{equation}
|F_{k_1+1}(x, t)| \leq 2 + c_0 \lambda^s (|\sigma^{1-2s}x + x_{k_1+1}(t)|^{2s} - 1)_+.
\end{equation}
and
\begin{equation}
|F_{k_1+1}(x, t)| \leq 2 \quad \text{in } Q_{1/16}.
\end{equation}

Indeed, it follows from Step 2 and the induction assumption that
\[
|P(F_{k_1})(x, z)| \leq \int_{\mathbb{R}^d} P(x - y, z) \left[ 2 + c_0 \lambda^s (|\sigma y + \sigma^{2s}x_{k_1}(t)|^{2s} - 1)_+ \right] dy
\]
\[
\leq 2 + c_0 \lambda^s \int_{\mathbb{R}^d} P(x - y, z) |\sigma^{1-2s}y + x_{k_1}(t)|^{2s} dy
\]
\[
\leq 2 + c_0 \lambda^s \int_{\mathbb{R}^d} P(x - y, z) \left( |y|^{2s} + |x_{k_1}(t)|^{2s} \right) dy.
\]

By (4.7)-(4.8) and the definition of Poisson kernel it follows that
\[
|P(F_{k_1})(x, z)| \leq 2 + c_0 \lambda^s \left( C + \Theta^{2s} \right) \leq 2 + 2c_0 \lambda^s C(M_2) \log \sigma|.
\]
Moreover, we also have,

\[ (4.14) \quad \int_{B_{\varepsilon}^+} \frac{(F_{k_1}(x,t) - 2)_+}{|x|^{d+2s}} \, dx \leq c_0 \lambda^* C(M_2) |\log \sigma|. \]

Since \( \|g_k\|_{L^q(Q_4)} \leq \|g\|_{L^q(Q_4)} \) we can apply Lemma (3.1) with \( c_0 \lambda^* C(M_2) |\log \sigma| \leq \varepsilon_2 \) (here \( \varepsilon_2 \) is the constant in Lemma (3.1)) to the function

\[ (x,t) \mapsto \frac{F_{k_1}(x,t)}{1 + c_0 \lambda^* C(M_2) |\log \sigma|} \]

and get that

\[ F_{k_1} \leq (2 - \lambda^*)(1 + c_0 \lambda^* C(M_2) |\log \sigma|) \text{ in } Q_{1/16}, \]

or

\[ -F_{k_1} \leq (2 - \lambda^*)(1 + c_0 \lambda^* C(M_2) |\log \sigma|) \text{ in } Q_{1/16}. \]

Taking \( c_0 = \varepsilon_2 \left( 4C(M_2) |\log \sigma| \right)^{-1} \) yields

\[ F_{k_1} \leq 2 - \frac{\lambda^*}{2} \text{ in } Q_{1/16}, \]

or

\[ -F_{k_1} \leq 2 - \frac{\lambda^*}{2} \text{ in } Q_{1/16}. \]

In the case \( F_{k_1} \leq 2 - \lambda^*/2 \) in \( Q_{1/16} \) we shall set

\[ F_{k_1+1}(y,t) = \frac{8}{8 - \lambda^*} \left( F_{k_1}(\sigma y + \sigma^{2s} x_{k_1+1}(t)), \sigma^{2s} t \right) + \lambda^*/4, \]

and if \( -F_{k_1} \leq 2 - \lambda^*/2 \) in \( Q_{1/16} \), then we set

\[ F_{k_1+1}(y,t) = \frac{8}{8 - \lambda^*} \left( F_{k_1}(\sigma y + \sigma^{2s} x_{k_1+1}(t)), \sigma^{2s} t - \lambda^*/4 \right). \]

Clearly,

\[ |F_{k_1+1}(y,t)| \leq 2 \text{ if } |\sigma^{1-2s} y + x_{k_1+1}(t)| \leq \frac{1}{16\sigma^{2s}}. \]

Since (4.10) holds for \( k = k_1 \),

\[ |F_{k_1+1}(y,t)| \leq \frac{8}{8 - \lambda^*} \left( 2 + c_0 \lambda^* \left( |\sigma^{1-2s}(\sigma y + \sigma^{2s} x_{k_1+1}(t))| + x_{k_1}(\sigma^{2s} t)|^{2s/3} - 1 \right)_+ + \lambda^*/4 \right). \]

Thus, it remains to show that, for any \( t \in [-1,0] \) and \( y \in \mathbb{R}^d \) such that

\[ |z| > \frac{1}{16\sigma^{2s}}, \]

we have

\[ (4.15) \quad G(z, t) \leq 2 + c_0 \lambda^* \left( |z|^{2s/3} - 1 \right)_+. \]

Here

\[ G(z,t) := \frac{8}{8 - \lambda^*} \left( 2 + c_0 \lambda^* \left( |\sigma z + x_{k_1}(\sigma^{2s} t)|^{2s/3} - 1 \right)_+ + \lambda^*/4 \right), \]

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with $z = \sigma^{1-2s}y + x_{k_1+1}(t)$. Indeed, if $|z| > \frac{1}{16\sigma^s}$ then

$$G(y, t) = 2 + \frac{4\lambda^*}{8 - \lambda^*} \left\{ 1 + 2c_0 \left( |\sigma z + x_{k_1}(\sigma^{2s}t)\right)^{2s/3} - 1 \right\}^+$$

This implies from (4.8) by choosing $\sigma = \sigma(M_2) \ll 1$ that

$$G(y, t) \leq 2 + \frac{1}{2}c_0\lambda^*|z|^{2s/3} \leq 2 + c_0\lambda^*(|z|^{2s/3} - 1)^+$$

This implies (4.15).

**Step 3:** By (4.13) in Step 2, we find that

$$\text{osc}_{Q_{\frac{1}{8}}} F_k \leq 4, \quad \text{for } k \geq 0.$$  

So, thanks to (4.6) one has for any $(y_1, t_1), (y_2, t_2) \in Q_{1/16}$

$$u\left( \sigma^{k+1}y_1 + \sum_{m=0}^{k} \sigma^{k+2s-m}x_m(t_1), \sigma^{2s(k+1)}t_1 \right) - u\left( \sigma^{k+1}y_2 + \sum_{m=0}^{k} \sigma^{k+2s-m}x_m(t_2), \sigma^{2s(k+1)}t_2 \right)$$

$$\leq 4 \left( 1 - \frac{\lambda^*}{8} \right)^{k+1}.$$  

If $s = 1/2$ the from (4.7) we imply that, if $-\sigma^{k+1} < t < 0,$

$$\left| \sum_{m=0}^{k} \sigma^{k+2s-m}x_m(t) \right| \leq \sigma^{k+1} \left( \sigma^{k+2s}(M_1 + |\log(\sigma)|) + \sum_{m=1}^{k} \sigma^{k+2s-m}\log(\sigma) \right) C(M_2)$$

$$\leq \frac{\sigma^{k+1}}{32},$$

provided that $\sigma \ll M_2 1$. Here we note that this is also true when $0 < s \leq 1/2.$ Hence it follows from (4.17) that

$$u(y_1, t_1) - u(y_2, t_2) \leq 4 \left( 1 - \frac{\lambda^*}{8} \right)^{k+1},$$

for any $y_1, y_2 \in B_{\sigma^{k+1}/32}$ and any $t_1, t_2 \in \left( -\sigma^{2s(k+1)}/16, 0 \right).$ By choosing

$$\sigma = \sigma(M_2) \ll 1, \quad k \geq k_0 := \left\lceil \frac{\log M_1}{\log \sigma} \right\rceil + 1,$$

we get for any $\rho < \rho_0 := \sigma^{k_0+1}/32 \sim C(M_2)/M_1$

$$\text{osc}_{B_{\rho}} u \leq C\rho^\alpha,$$

where

$$\alpha = \left| \log \left( 1 - \frac{\lambda^*}{8} \right) \right| / \left| \log \sigma \right|.$$  

This gives

$$\sup_{|x_1 - x_2| + |t_1 - t_2|^{1/(2s)} < \rho_0, \rho_0 > -1/2} \left| u(x_1, t_1) - u(x_2, t_2) \right| \leq C.$$  

The proof of our theorem is completed.
Proof of Corollary 1.3. By Remark 2.2, one has
\[ \|u(1)\|_{L^\infty(\mathbb{R}^d)} \leq C\|u(0)\|_{L^2(\mathbb{R}^d)}. \]
Since \( u_\lambda(t, x) = \lambda^{-2s} u(\lambda^{2s} t, \lambda x) \) \((\lambda > 0)\) is also solution to (1.2) with drift term
\[ B_\lambda(t, x) = \lambda^{1-2s} B(\lambda^{2s} t, \lambda x), \]
it follows that
\[ \|u_\lambda(1)\|_{L^\infty(\mathbb{R}^d)} \leq C\|u_\lambda(0)\|_{L^2(\mathbb{R}^d)}. \]
This implies, for \( t = \lambda^{2s} \),
\[ \|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq C t^{-\frac{d}{2s}}\|u(0)\|_{L^2(\mathbb{R}^d)}, \]
and we obtain (1.7).

By Theorem 1.2, we have
\[ \sup_{|x_1-x_2|+|t_1-t_2|^{1/(2s)} \leq \rho_0} \left\| \frac{|u(x_1, t_1) - u(x_2, t_2)|}{|x_1 - x_2| + |t_1 - t_2|^{1/(2s)}} \right\| \leq C(M_2)\|u(0)\|_{L^2(\mathbb{R}^d)}, \]
where \( \rho_0 = C_2(M_1)/M_1 \). We apply this estimate for \( u_\lambda(t, x) = \lambda^{-2s} u(\lambda^{2s} t, \lambda x) \) and
\[ B_\lambda(t, x) = \lambda^{1-2s} B(\lambda^{2s} t, \lambda x) \]
to derive
\[ \sup_{|x_1-x_2|+|t_1-t_2|^{1/(2s)} \leq c_0} \left\| \frac{|u(\lambda^{2s} t_1, \lambda x_1) - u(\lambda^{2s} t_2, \lambda x_2)|}{|x_1 - x_2| + |t_1 - t_2|^{1/(2s)}} \right\| \leq C(M_2)\lambda^{-d/2}\|u(0)\|_{L^2(\mathbb{R}^d)}, \]
with
\[ c_0 = \frac{C(M_2, M_3)}{\lambda^{1-2s} \sup_{(x,t) \in \mathbb{R}^{d+1}_+} \left| \int_{x+B_\lambda} B(y,t)dy \right| + 1}. \]

Case \( s = 1/2 \): Since
\[ \sup_{(x,t) \in \mathbb{R}^{d+1}_+} \left| \int_{x+B_\lambda} B(y,t)dy \right| \leq C(\log(|\lambda|) + 1)M_2 + M_1, \]
it implies that
\[ c_0 \geq \frac{C(M_2, M_1)}{\log(|\lambda|) + 1}, \]
and hence
\[ \sup_{|x_1-x_2|+|t_1-t_2| \leq C(M_2, M_1)\lambda} \left\| \frac{|u(t_1, x_1) - u(t_2, x_2)|}{|x_1 - x_2| + |t_1 - t_2|^{1/(2s)}} \right\| \leq C(M_2)\lambda^{-d/2-\alpha}\|u(0)\|_{L^2(\mathbb{R}^d)}. \]
This implies (1.8).

Case \( 0 < s < 1/2 \): Since
\[ c_0 \geq \frac{C(M_3)}{\lambda^{1-2s} + 1}, \]
we can deduce that
\[ \sup_{|x_1-x_2|+|t_1-t_2|^{1/(2s)} \leq C(M_3)\min\{\lambda, \lambda^{2s}\}, \; t_1, t_2 > \lambda^{2s}} \left\| \frac{|u(t_1, x_1) - u(t_2, x_2)|}{|x_1 - x_2| + |t_1 - t_2|^{1/(2s)}} \right\| \leq C(M_2)\lambda^{-d/2-\alpha}\|u(0)\|_{L^2(\mathbb{R}^d)}. \]
This implies (1.9). The proof is complete. \( \square \)
Appendix A

We state here a technical lemma which implies the crucial lemma (Lemma 1.1.)

**Lemma A.** For any $a_1, a_2 \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}_+$, there holds

(4.1) \[(b_1 a_1 - b_2 a_2)^2 + |a_1||a_2|(b_1 - b_2)^2 \sim (a_1 - a_2)(b_1^2 a_1 - b_2^2 a_2) + C|a_1||a_2|(b_1 - b_2)^2\]

and

(4.2) \[(b_1 a_1 - b_2 a_2)(b_1 a_1^+ - b_2 a_2^+) + (|a_1| |a_2^+ + |a_2|)(b_1 - b_2)^2 \sim (a_1 - a_2)(b_1^2 a_1^+ - b_2^2 a_2^+) + C(|a_1| |a_2^+ + |a_2|)(b_1 - b_2)^2\]

for some constant $C > 0$.

**Proof.** In order to prove (4.1), without loss of generality, we can assume that $|a_1| \geq |a_2|$. Then we have

\[
(a_1 - a_2)(b_1^2 a_1 - b_2^2 a_2) = (a_1 - a_2)(b_1^2 a_1 - b_1 b_2 a_2 + b_1 b_2 a_2 - b_2^2 a_2) \\
= (b_1 a_1 - b_2 a_2)(b_1 a_1 - b_2 a_2) + (b_2 a_1 - b_2 a_2) a_2(b_1 - b_2) \\
= (b_1 a_1 - b_2 a_2)^2 + (b_2 - b_1) a_2 a_2(b_1 - b_2) \\
+ (b_1 a_1 - b_2 a_2) a_2(b_1 - b_2) - a_1 a_2(b_1 - b_2)^2.
\]

Thanks to Holder’s inequality, we get that

\[
|(a_1 - a_2)(b_1^2 a_1 - b_2^2 a_2) - (b_1 a_1 - b_2 a_2)^2| \leq \frac{1}{10}(b_1 a_1 - b_2 a_2)^2 + C(|a_2|^2 + |a_1||a_2|)(b_1 - b_2)^2.
\]

This implies (4.1).

Since (4.1) holds true, we only need to check (4.2) in case $a_1 \geq 0 > a_2$. Indeed, one has

\[
(a_1 - a_2)(b_1^2 a_1^+ - b_2^2 a_2^+) = (b_1 a_1 - b_2 a_2)b_1 a_1^+ \\
= (b_1 a_1 - b_2 a_2)b_1 a_1^+ + (b_2 - b_1) a_2 b_1 a_1^+ \\
= (b_1 a_1 - b_2 a_2) b_1 a_1^+ - (b_2 - b_1)^2 a_2 a_1^+ + (b_2 - b_1) a_2 a_1^+ b_2.
\]

Hence

\[
|(a_1 - a_2)(b_1^2 a_1^+ - b_2^2 a_2^+) - (b_1 a_1 - b_2 a_2) b_1 a_1^+| \\
\leq (b_2 - b_1)^2|a_2| a_1^+ + \min\{|(b_2 - b_1) a_2 a_1^+ b_2|, |(b_2 - b_1) a_2 b_1 a_1^+|\} \\
\leq C|a_2| a_1^+(b_1 - b_2)^2 + \frac{1}{10}|a_2| a_1^+ b_1 b_2 \\
\leq C|a_2| a_1^+(b_1 - b_2)^2 + \frac{1}{10}(b_1 a_1 - b_2 a_2) b_1 a_1^+.
\]

It follows (4.2) in case $a_1 \geq 0 > a_2$. \qed

**Proof of Lemma 1.1.** We have

\[
\int_{\mathbb{R}^d} h^2 f^+ \mathcal{L}_t f \, dx = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (h(x)^2 f^+(x) - h(y)^2 f^+(y) (f(x) - f(y))) K_t(x, y) \, dx \, dy,
\]

\[
\int_{\mathbb{R}^d} h^2 f^+ (-\Delta)^s f \, dx = c_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (h(x)^2 f^+(x) - h(y)^2 f^+(y) (f(x) - f(y))) \frac{dx \, dy}{|x - y|^{d+2s}}.
\]

Thus, the result follows from Lemma 4.1. The proof is complete. \qed
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