Revolutionaries and Spies

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Abstract

Let $G = (V, E)$ be a graph and let $r, s, k$ be natural numbers. “Revolutionaries and Spies”, $G(G, r, s, k)$, is the following two-player game. The sets of positions for player 1 and player 2 are $V^r$ and $V^s$ respectively. Each coordinate in $p \in V^r$ gives the location of a “revolutionary” in $G$. Similarly player 2 controls $s$ “spies”. We say
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$u, u' \in V(G)^n$ are adjacent, $u \sim u'$, if for all $1 \leq i \leq n$, $u_i = u'_i$ or \{ $u_i, u'_i$ \} $\in E(G)$. In round 0 player 1 picks $p_0 \in V^r$ and then player 2 picks $q_0 \in V^s$. In each round $i \geq 1$ player 1 moves to $p_i \sim p_{i-1}$ and then player 2 moves to $q_i \sim q_{i-1}$. Player 1 wins the game if he can place $k$ revolutionaries on a vertex $v$ in such a way that player 1 cannot place a spy on $v$ in his following move. Player 2 wins the game if he can prevent this outcome.

Let $k(G, r, s)$ be the maximum value of $k$ such that player 1 can win $\mathcal{G}(G, r, s, k)$. We show that if $G$ is an acyclic graph with at least $s + 1$ vertices, then $k(G, r, s) = \left\lfloor \frac{r}{s+1} \right\rfloor$.

Let $s(G, r, k)$ be the minimum $s$ such that player 2 can win $\mathcal{G}(G, r, s, k)$. We show that $\lim \inf_{r \to \infty} s(\mathbb{Z}^2, r, 2)/r \geq 3/4$. Here moves in $\mathbb{Z}^2$ are “king” moves.

1 Introduction

Let $G = (V, E)$ be a graph, possibly infinite, and let $r, s, k$ be natural numbers. "Revolutionaries and Spies"; $\mathcal{G}(G, r, s, k)$, is the following two-player game, invented by Beck [1]. In round 0, player 1 places $r$ markers called revolutionaries on the vertices of $G$. Then player 2 places $s$ markers called spies on the vertices. There is no restriction on the number of spies and revolutionaries that may be placed on a vertex. For $i \geq 1$, round $i$ begins with player 1 moving each revolutionary either to a vertex adjacent to its current vertex or by leaving it at its current vertex. Round $i$ ends with player 2
moving his spies in the same fashion. Player 1 has a meeting of size $k$ at a vertex $v$ if $k$ or more revolutionaries are present at that vertex. A set of vertices is guarded if a spy is present at some vertex in the set. Player 1 wins $G(G, r, s, k)$ if he has a strategy to achieve an unguarded meeting of size $k$ by the end of some round $i$. Otherwise player 2 has a strategy to block this and we say player 2 wins $G(G, r, s, k)$.

Let $k(G, r, s)$ be the maximum value of $k$ such that player 1 wins $G(G, r, s, k)$. We define $G(G, r, s)$ to be $G(G, r, s, k_0)$ where $k_0 = k(G, r, s)$. An optimum strategy for player 1 in $G(G, r, s)$ is one eventually achieving an unguarded meeting of size $k_0$. Similarly an optimum strategy for player 2 is one preventing a meeting of size $k_0 + 1$. We sometimes describe these just as player 1’s (player 2’s) strategies in $G(r, s)$. Let $s(G, r, k)$ be the minimum value $s$ for which it is possible for player 2 to win $G(G, r, s, k)$.

We record the following trivial observation.

**Lemma 1.1** If $G$ has at least $s + 1$ vertices, $k(G, r, s) \geq \left\lfloor \frac{r}{s+1} \right\rfloor$. Otherwise, $k(G, r, s) = 0$.

**Proof:** In the first case, player 1 can win at the end of round 0 by placing at least $\left\lfloor \frac{r}{s+1} \right\rfloor$ revolutionaries on each of $s + 1$ vertices. In the second case, player 2 can win by maintaining a spy at each vertex.

This trivial lower bound is attained on the following classes of graphs.
Theorem 1.2 If $G$ is a tree and has at least $s+1$ vertices then $k(G,r,s) = \left\lfloor \frac{r}{s+1} \right\rfloor$.

If $v, w \in V(G)$ let $d_G(v, w)$ be the distance between $v$ and $w$ in $G$, i.e. the minimum length of an $v$-$w$ path in $G$. If no such path exists we define $d_G(v, w) = +\infty$. Note $d_G(v, v) = 0$.

Let $G$ and $H$ be graphs. The strong product of $G$ and $H$, denoted $G \boxtimes H$, is the graph with vertex set $V(G \boxtimes H) = V(G) \times V(H)$. Vertices $(g, h)$ and $(g', h')$ are adjacent in $G \boxtimes H$ if and only if $(g, h) \neq (g', h'), \ d_G(g, g') \leq 1$, and $d_H(h, h') \leq 1$. We denote by $Z$ the graph $G = (V, E)$ with $V = \mathbb{Z}$ and $E = \{\{i, i+1\} : i \in \mathbb{Z}\}$. For $d \geq 1$, let $\mathbb{Z}^{\boxtimes d}$ be the $d$-fold strong product of $\mathbb{Z}$ with itself.

We also study revolutionaries and spies on $\mathbb{Z}^{\boxtimes d}$, primarily for $d = 2$. Perhaps one of the most basic (yet non-trivial) quantities to study is the threshold $s_d(r) := s(\mathbb{Z}^{\boxtimes d}, r, 2)$.

Theorem 1.3 We have $\lim \inf_{r \to \infty} \frac{s_d(r)}{r} \geq \frac{3}{4}$.

Note that the best such result obtainable using Lemma 1.1 is $\lim \inf_{r \to \infty} \frac{s_d(r)}{r} \geq \frac{1}{2}$.

The organization of the paper is as follows. We present a number of basic definitions and results in Section 2. Then we prove Theorem 1.2 in Section 3 and Theorem 1.3 in Section 4. We outline a few directions for further research in the concluding section.
2 Basic Results

Lemma 2.1 Let $G$ be a graph and let $r, r', s, s' \in \mathbb{N}$ with $r \leq r'$ and $s \leq s'$.

Then

1. $k(G, r, 0) = r$ and $k(G, r, r) = 0$.

2. $k(G, r', s) \geq k(G, r, s)$, and

3. $k(G, r, s') \leq k(G, r, s)$.

Proof: To prove $k(G, r, r) = 0$, player 2 can match each spy with a unique revolutionary. Each spy is then moved to its matched revolutionary’s position at the end of each round. To prove statement 2, first note that player 2 has a strategy to prevent an unguarded meeting of size $K = k(r', s) + 1$ in $G(r', s)$. Player 2 can then prevent a meeting of this size in $G(r, s)$ by “pretending” that player 1 has an additional $r' - r$ revolutionaries left fixed at some arbitrary fixed vertex. Player 2 then moves according to the strategy in $G(r', s)$.

Similarly, to prove statement 3, we see that player 2 can keep $s' - s$ spies fixed at some arbitrary vertex and then play his optimum strategy in $G(r, s)$ with his remaining $s$ spies to prevent a meeting of size $k(G, r, s) + 1$ in $G(r, s')$.

If $U, W \subseteq V(G)$, let $d_G(U, W) = \min_{u \in U, w \in W} d_G(u, w)$. If $W = \{v\}$ we write $d_G(S, v)$. Let $B_G(S, r) := \{v \in V(G) : d_G(S, v) \leq r\}$. 
Lemma 2.2 Let $r_1, \ldots, r_l, s_1 \ldots s_l \in \mathbb{N}$. Let $r = \sum_{i=1}^{\ell} r_i$ and $s = \sum_{i=1}^{\ell} s_i$. Then the following statements hold.

1. For all graphs $G$, $k(G, r, s) \leq \sum_{i=1}^{\ell} k(G, r_i, s_i)$.

2. Let $n \geq 0$ and suppose $G_1, \ldots, G_{\ell}$ are subgraphs of a graph $G$ with $B_G(V(G_i), n)$ for $1 \leq i \leq \ell$ pairwise disjoint. Suppose that for all $1 \leq i \leq \ell$, player 1 has a strategy to win $G(G, r_i, s_i, k_i)$ in at most $n$ rounds with all revolutionaries starting and remaining in $V(G_i)$. Then $k(G, r, s + \ell - 1) \geq \min_{1 \leq i \leq \ell} k_i$.

Proof: To prove statement 1, player 2 partitions player 1’s revolutionaries into groups of sizes $r_i$ for $1 \leq i \leq \ell$ and also partitions the spies into groups of size $s_i$. Thereafter he simultaneously uses his $i$th group of spies to prevent a meeting of size $k(G, r_i, s_i) + 1$ amongst the $i$th group of revolutionaries, for each $1 \leq i \leq \ell$. Clearly, player 1 cannot achieve a meeting of size $1 + \sum_{i=1}^{\ell} k(G_i, r_i, s_i)$.

To prove statement 2, we first note that the proof of Lemma 2.1 statement 3 gives for each $1 \leq i \leq \ell$ a “unified” strategy for player 1 to achieve an unguarded meeting of size $k_i$ by the end of round $n$, in $G(G, r_i, t)$ for all $t \leq s_i$, using the same starting position in $G_i$. Player 1 pretends that there are $s_i - t$ spies fixed at some arbitrary vertex and then plays his strategy for $G(G, r_i, s_i, k_i)$ while player 2 moves his $t$ spies. The initial position for player 1 is the same for all $t \leq s_i$. Player 1 makes these uniform initial placements of $r_i$ revolutionaries in each $G_i$. For some $i$, there must be $t \leq s_i$ spies placed
in $B_G(V(G_i), n)$. Thus player 1 can achieve an unguarded meeting of size $k_i$ in $G_i$ in $n$ rounds.

The following statements are all easy corollaries of Lemma 2.2 statement 1.

**Corollary 2.3** Let $r, s \in \mathbb{N}$ then

1. $k(G, r, s) \leq k(G, r - a, s - a)$ for all $a \leq r, s$.

2. $k(G, r + r', s) \leq k(G, r, s) + r'$.

3. $k(G, ar, as) \leq ak(G, r, s)$ for all $a \geq 0$.

**Lemma 2.4** For all graphs $G$ and for all $R, r, s \in \mathbb{N}$ we have $k(G, R, s) \geq \left\lfloor \frac{R}{r} \right\rfloor k(G, r, s)$.

**Proof:** Let $R = R_1 + \cdots + R_r$ where $R_i \leq R_{i+1}$ for all $1 \leq i \leq r - 1$ and $R_r \leq R_1 + 1$. Let player 1 partition his revolutionaries into $r$ groups of sizes $R_i$ and then place each member of the $i$th group on the position of revolutionary $i$ in some optimum strategy for $G(r, s)$. This strategy allows player 1 to achieve an unguarded meeting of size at least $R_1k(G, r, s)$ in $G(G, R, s)$. Note that $R_1 = \left\lfloor \frac{R}{r} \right\rfloor$.

The next result follows directly from the definition of $k(G, r, s)$. 
Lemma 2.5 Suppose $G$ is a graph whose components are $\{G_i : i \in I\}$. Then

$$k(G, r, s) = \max_{f} \min_{g} \max_{i \in I} k(G_i, f(i), g(i))$$

where functions $f, g : I \to \mathbb{N}$ satisfy $r = \sum_{i} f(i)$, $s = \sum_{i \in I} g(i)$ respectively.

Let $\mathcal{X} : V(G) \to \mathbb{R}$. We define $\mathcal{X}(S) := \sum_{v \in S} \mathcal{X}(v)$ for subsets $S \subseteq V(G)$ and $\mathcal{X}(H) := \mathcal{X}(V(H))$ for subgraphs $H \subseteq G$. The weight of $\mathcal{X}$ is $\mathcal{X}(G)$. We say $\mathcal{X}$ is finite if $\mathcal{X}$ has finite weight. If $\mathcal{X} : V(G) \to \mathbb{N}$ we call $\mathcal{X}$ a position. Let $\mathcal{P}(G, m)$ denote the set of all the positions of weight $m$ on $G$. The set of all possible placements of $r$ revolutionaries in $G(G, r, s)$ is in one-to-one correspondence with the functions $\mathcal{R}$ in $\mathcal{P}(G, r)$. Namely, for each vertex $v$, let $\mathcal{R}(v)$ be the number of revolutionaries present at $v$. Similarly, we let $\mathcal{P}(G, s)$ represent the possible placements of the spies in $G(G, r, s)$. If $\mathcal{X}, \mathcal{X}' \in \mathcal{P}(G, m)$ and $\mathcal{X}'$ is one move from $\mathcal{X}$ then we denote this by $\mathcal{X}' \sim \mathcal{X}$.

Let $G$ be a graph and let $v$ be a vertex in $G$. Let $N(v) = N_G(v) = \{w \in V(G) : \{v, w\} \in E(G)\}$. Let $D(G) = \bigcup_{v \in V(G)} \{(v, w) : w \in N_G(v)\}$.

Lemma 2.6 Let $G = (V, E)$ be a graph and let $m \geq 0$. Let $\mathcal{X}, \mathcal{X}' \in \mathcal{P}(G, m)$. $\mathcal{X} \sim \mathcal{X}'$ if and only if there is a function $\mathcal{Q} : V \times V \to \mathbb{N}$ satisfying the following properties.

1. $\mathcal{Q}(v, w) = 0$ for all $v, w$ with $\{v, w\} \notin E$.

2. For all $v \in V$, $\mathcal{X}(v) \geq \sum_{w \in V} \mathcal{Q}(v, w)$.

3. For all $v \in V$, $\mathcal{X}'(v) = \mathcal{X}(v) + \sum_{w \in V}(\mathcal{Q}(w, v) - \mathcal{Q}(v, w))$. 

**Proof:** If \( Q \) satisfies the three properties above it is clear that \( X' \) is one move from \( X \), as one can move from \( X \) to \( X' \) by moving \( Q(v, w) \) revolutionaries from \( v \) to \( w \) for all \( v, w \in V \).

Suppose now that \( X' \) is one move from \( X \). If \( m = 0 \) the function \( Q = 0 \) satisfies the properties. If \( m = 1 \), then \( Q \) defined by \( Q(v, w) = 1 \) if \( X(v) = 1 \) and \( X'(w) = 1 \) and \( Q = 0 \) otherwise, satisfies the properties. For \( m > 1 \) let \( X = \sum X_i \) and \( X' = \sum X'_i \) where for each \( i \), \( X_i, X'_i \in \mathcal{P}(G, 1) \) and \( X_i \) is one move from \( X'_i \). For each \( i \), let \( Q_i \) satisfy the properties above with respect to \( X_i, X'_i \). Let \( Q' = \sum Q_i \). It is clear that \( Q' \) satisfies the properties with respect to \( X \) and \( X' \).

\[ \square \]

If \( Q \) satisfies the properties in Lemma 2.6 then we call \( Q \) a **move** from \( X \) to \( X' \).

Hall’s theorem gives another characterization of when two positions are connected by a single move. Given \( X \subseteq V(G) \) we define \( X^{(1)} := \{ x' \in V(G) : \exists x \in X \ d_G(x', x) \leq 1 \} \).

**Theorem 2.7** Let \( G = (V, E) \) be a graph and let \( m \geq 0 \). Let \( X, X' \in \mathcal{P}(G, m) \). Then the following are equivalent.

1. \( X \sim X' \)

2. \( \forall X \subseteq V(G) \ X(X) \leq X'(X^{(1)}) \)

3. \( \forall X \subseteq V(G) \ X'(X) \leq X(X^{(1)}) \)
Proof: Note that it suffices to show statements 1 and 2 are equivalent. If we move from position $X'$ to position $X$, then $\mathcal{X}(X)$, the number of revolutionaries in $X$ at the end of the move, must be less than or equal to $\mathcal{X}'(X^{(1)})$, the number of revolutionaries that may reach $X$ in one move. It remains to show that statement 2 implies statement 1.

Let $U = \{(1, x_1), \ldots, (r, x_r)\}$ be a listing of the revolutionaries in $X$ in some fixed order, i.e. for all $i \in [r]$ revolutionary $i$ is at vertex $x_i$. Similarly let $W = \{(1, x'_1), \ldots, (r, x'_r)\}$ be some listing of the revolutionaries in $X'$. Let $B(X, X') = (U \cup W, E)$ be the bipartite graph with bipartition $(U, W)$ such that $\{(i, x_i), (j, x'_j)\} \in E$ if and only if $d_G(x_i, x'_j) \leq 1$. Clearly, $X \sim X'$ if and only if $B$ has a perfect matching, i.e. each revolutionary in $X$ has a unique target revolutionary in $X'$ to which it can move. It is well known that this perfect matching exists if and only if Hall’s condition holds: $|N(S)| \geq |S|$ for all $S \subseteq U$, where $N(S) = \{w \in W : \exists s \in S \{w, s\} \in E(B)\}$. We show that statement 2 implies Hall’s condition.

Let $S = \{(i_1, x_{i_1}), \ldots, (i_t, x_{i_t})\}$ be an arbitrary subset of $U$. Let $X = \{x_{i_1}, \ldots, x_{i_t}\}$. Note that we may have $|X| < t$ as there may be repetitions amongst the $x_{i_j}$. We have $\mathcal{X}(X) = \sum_{x \in X} |\{(i, x_i) \in U : x_i = x\}| \geq \sum_{x \in X} |\{(i, x_i) \in S : x_i = x\}| = |S|$. These expressions for $\mathcal{X}(X)$ and $|S|$ are derived by partitioning the revolutionaries counted according to the vertex on which they lie. Similarly we get

$$N(S) = \bigcup_{x' \in X^{(1)}} \{(j, x'_j) \in W : x'_j = x'\}.$$
Using this expression we get \( \mathcal{X}'(X^{(1)}) = \sum_{x' \in X^{(1)}} |\{(j, x'_j) \in W : x'_j = x'\}| = |N(S)| \). Thus \( \mathcal{X}(X) \leq \mathcal{X}'(X^{(1)}) \) implies \( |S| \leq |N(S)| \) as desired.

Given \( X \in \mathcal{P}(G, m) \) let \( \text{supp}(X) := \{ v \in V(G) : X(v) > 0 \} \). The proof of Lemma 2.7 shows that \( \mathcal{X} \sim \mathcal{X}' \) if and only if \( \mathcal{X}(X) \leq \mathcal{X}'(X^{(1)}) \) for all \( X \subset \text{supp}(X) \).

If \( e = \{v, w\} \in E(T) \) we let \( T(v, w) \) be the component of \( T - e \) containing \( w \). If \( U \subseteq N(v) \) let \( T(v, U) = \bigcup_{u \in U} T(v, u) \). Note that if \( V_0 = V(T(v, U)) \) then \( V_0^{(1)} = V_0 \cup \{v\} \). We write \( T(v, U) + v \) for \( T[V_0 \cup \{v\}] \).

Suppose \( T \) is a tree. To prove \( \mathcal{X} \sim \mathcal{X}' \) on \( T \) it suffices to only check \( \mathcal{X}(X) \leq \mathcal{X}'(X^{(1)}) \) on a subclass of vertex sets \( X \).

**Theorem 2.8** Let \( T = (V, E) \) be a tree and let \( m \geq 0 \). Let \( \mathcal{X}, \mathcal{X}' \in \mathcal{P}(G, m) \).

Then the following are equivalent.

1. \( \mathcal{X} \sim \mathcal{X}' \)

2. \( \forall v \in V(G) \forall U \subseteq N(v) \mathcal{X}(T(v, U)) \leq \mathcal{X}'(T(v, U) + v) \).

We require the following lemma. For \( A, B \subseteq V(G) \) let \( A^c = V(G) \setminus A \), and \( (A, B) = \{(a, b) \in A \times B : \{a, b\} \in E(G)\} \).

**Lemma 2.9** Let \( T \) be a tree and let \( X \subseteq V(G) \).

1. If \( T[X] \) is a tree,

\[
T - T[X] = \bigcup_{(v, w) \in (X^c \times X)} T(v, w) = \bigcup_{v \in X} T(v, N(v) \setminus X).
\]


2. If $T[X^{(1)}]$ is a tree,
   $$T - T[X^{(1)}] = \bigcup_{v \in X^{(1)} \setminus X} T(v, N(v) \setminus X^{(1)}).$$

3. If $T[X^{(1)}]$ is a tree,
   $$T - T[X] = \bigcup_{v \in X^{(1)} \setminus X} (T(v, N(v) \setminus X^{(1)}) + v).$$

**Proof:** Statement 1 follows from basic graph theory arguments showing that the edges of $(X, X^c)$ are in one-to-one correspondence with the components of $T - T[X]$. Statement 2 then follows from 1 with the observation that $T(v, N(v) \setminus X^{(1)}) = \emptyset$ if $x \in X$. Statement 3 follows easily from 2.

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**Proof of Lemma 2.8** By Lemma 2.7 it suffices to show that statement 2 implies statement 1. Suppose statement 2 holds. By Lemma 2.7 it suffices to prove $X'(X) \leq X'(X^{(1)})$ for all $X \subseteq V(G)$. Suppose $T[X^{(1)}]$ is a tree. Then by Lemma 2.9

$$X'(T) - X'(X^{(1)}) = \sum_{v \in X^{(1)} \setminus X} X'(T(v, N(v) \setminus X^{(1)})) \leq \sum_{v \in X^{(1)} \setminus X} X''(T(v, N(v) \setminus X^{(1)} + v) = X''(T) - X'(X) \text{ or } X'(X) \leq X''(X^{(1)}).

For the general case, write $T[X^{(1)}] = \bigcup_{i \in I} T_i$ as a disjoint union of trees. Let $X_i = V(T_i) \cap X$. Then $T[X_i^{(1)}] = T_i$. By the preceding argument, we have $X'(X_i) = \sum_i X'(X_i) \leq \sum_i X''(X_i^{(1)}) = X''(X^{(1)}).

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If $G$ is a graph the $n$th power of $G$ is the graph $G^n$ with $V(G^n) = V(G)$ and $E(G^n) := \{ \{v, w\} : 0 < d_G(v, w) \leq n\}$. Let $X, X' \in \mathcal{P}(G, m)$. Clearly
\( \mathcal{X}' \sim \mathcal{X} \) in \( G^n \) if and only if there is a sequence of positions \( \mathcal{X}_i \in \mathcal{P}(G, m) \) for \( 0 \leq i \leq n \) with \( \mathcal{X}_0 = \mathcal{X} \) and \( \mathcal{X}_n = \mathcal{X}' \) such that \( \mathcal{X}_i \sim \mathcal{X}_{i-1} \) in \( G \) for all \( 1 \leq i \leq n \).

**Lemma 2.10** For all graphs \( G \) and all \( n, r, s \in \mathbb{N} \), \( k(G^n, r, s) \leq k(G, r, s) \).

**Proof:** If player 2 has a strategy to prevent an unguarded meeting of size \( k+1 \) in \( G \), then it can prevent such a meeting in \( G^n \) as follows. Initially, player 2 places the spies according to an optimum strategy in \( \mathcal{G}(G, r, s) \). Suppose \( \mathcal{R} \) and \( \mathcal{S} \) are the positions of the revolutionaries and the spies, respectively, at the beginning of some round. Suppose player 1 moves from \( \mathcal{R} \) to \( \mathcal{R}' \) in \( G^n \). Player 2 constructs a sequence of positions \( \mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_n = \mathcal{R}' \) with \( \mathcal{R}_{i+1} \sim \mathcal{R}_i \) in \( G \) for \( 0 \leq i < n \). Let \( \mathcal{S}_0 = \mathcal{S} \) and for \( 1 \leq i \leq n \) let \( \mathcal{S}_i \) be the position played by the spies in response to \( \mathcal{R}_i \) according to the strategy in \( \mathcal{G}(G, r, s) \). Clearly \( \mathcal{S}' = \mathcal{S}_n \) is one move from \( \mathcal{S} \) in \( G^n \) and continues to prevent an unguarded meeting of size \( k + 1 \) by \( \mathcal{R}' \).

**Lemma 2.11** For graphs \( G, H \), we have

\[
k(G \boxast H, r, s) \geq \max(k(G, r, s), k(H, r, s)).
\]

**Proof:** Fix \( h_0 \in V(H) \). Given \( \mathcal{X}_0 \in \mathcal{P}(G, m) \) we define \( \mathcal{X}_1 = \mathcal{L}(\mathcal{X}_0) \in \mathcal{P}(G \boxast H, m) \) as follows. For all \( (g, h) \in V(G \boxast H) \), \( \mathcal{X}_1((g, h)) = \mathcal{X}(g)1_{h=h_0} \) where
$1_{h=h_0} = 1$ if $h = h_0$, 0 otherwise. Given $X_1 \in \mathcal{P}(G \boxtimes H, m)$ we define $X_0 = P(X_1) \in \mathcal{P}(G, m)$ by $X_0(g) = X_1(\{g\} \times H)$.

Let $X_0, X'_0 \in \mathcal{P}(G, m)$ and let $X_1 = L(X_0), X'_1 = L(X'_0)$. Since $G \times \{h_0\}$ is an isomorphic copy of $G$ in $G \boxtimes H$, it is clear that $X_0 \sim X'_0$ implies $X_1 \sim X'_1$ in $G \boxtimes H$.

Let $X_1, X'_1 \in \mathcal{P}(G \boxtimes H, m)$ and let $X_0 = P(X_1), X'_0 = P(X'_1)$. Suppose $X_1 \sim X'_1$. Then for all $X \subseteq V(G)$ we have $X_0(X) = X_1(X \times H) \leq X'_1((X \times H)^{(1)}) = X'_1(X^{(1)} \times H) = X'_0(X^{(1)})$ and so by Lemma 2.7 $X_0 \sim X'_0$.

Let $k = k(G \boxtimes H, r, s)$. It suffices to prove $k(G, r, s) \leq k$.

Given a position $R_1 \in \mathcal{P}(G \boxtimes H, r)$ for player 1, let $S_1 = S_1(R_1) \in \mathcal{P}(G \boxtimes H, s)$ be an optimum response for player 2 in $G(G \boxtimes H, r, s)$. That is, by playing $S_1$ player 2 will prevent a meeting of size $k + 1$. We claim that if $R_0 \in \mathcal{P}(G, r)$ is a position for player 1 then it is always possible for player 2 to play $S_0 = S_0(R_0) := P(S_1(L(R_0))) \in \mathcal{P}(G, s)$ and that this will prevent a meeting of size $k + 1$ in $G(G, r, s)$. Let $R_0, R'_0 \in \mathcal{P}(G, r)$ and let $S_0 = S_0(R_0), S'_0 = S_0(R'_0)$. It is clear from the definitions that $S_0, S'_0 \in P(G, s)$. It is also clear from the statements proven in the previous paragraphs that if $R_0 \sim R'_0$, then $S_0 \sim S'_0$. Furthermore if $R_0(g) > k$ then $L(R_0)((g, h_0)) = R_0(g) > k$ and $S_1(L(R_0))((g, h_0)) \geq 1$, as $S_1$ is a response that prevents meetings of size $k + 1$. Thus $S_0(R_0) = P(S_1(L(R_0)))(g) \geq 1$. 

\[ \square \]
3 Proof of Theorem 1.2

As a warmup, we prove

**Theorem 3.1** For all $r, s \in \mathbb{N}$, $k(\mathbb{Z}, r, s) = \left\lfloor \frac{r}{s+1} \right\rfloor$.

We need a lemma first. Given $\mathcal{X} \in \mathcal{P}(\mathbb{Z}, m)$ let $f_i(\mathcal{X})$ be the $i$th order statistic of $\mathcal{X}$, i.e. $f_i(\mathcal{X}) = j$ if and only if $\mathcal{X}((−\infty, j]) \geq i$ and $\mathcal{X}((−\infty, j)) < i$.

**Lemma 3.2** If $\mathcal{X}, \mathcal{X}' \in \mathcal{P}(\mathbb{Z}, m)$ and $\mathcal{X}' \sim \mathcal{X}$ then $|f_i(\mathcal{X}') - f_i(\mathcal{X})| \leq 1$ for all $1 \leq i \leq m$.

**Proof:** Suppose $f_i(\mathcal{X}) = j$. Since $\mathcal{X}((−\infty, j]) \geq i$, $i \leq \mathcal{X}((−\infty, j]) \leq \mathcal{X}'((−\infty, j+1])$ and $f_i(\mathcal{X}') \leq j+1$. Also since $\mathcal{X}((−\infty, j)) < i$, $\mathcal{X}'((−\infty, j-1)) \leq \mathcal{X}((−\infty, j)) < i$ and $f_i(\mathcal{X}') \geq j - 1$.

Proof of Theorem 3.1. Clearly we have $k(\mathbb{Z}, r, s) \geq k := \left\lfloor \frac{r}{s+1} \right\rfloor$, by Lemma 3.1. Since $r < r_0 := k(s + 1) + s$, $k(\mathbb{Z}, r, s) \leq k(\mathbb{Z}, r_0, s)$, by Lemma 2.1. Thus it suffices to show $k(\mathbb{Z}, r_0, s) \leq k$. If player 1’s position is $\mathcal{R} \in \mathcal{P}(\mathbb{Z}, r_0)$, player 2’s strategy is to play spy $i$ at position $f_i(k+1)(\mathcal{R})$ for all $1 \leq i \leq s$. Since $r_0 = s(k + 1) + k$, any vertex at which there is a meeting of $k + 1$ revolutionaries must contain a spy. Let $\mathcal{R}' \sim \mathcal{R}$ be player 1’s new position. Clearly, by Lemma 3.2, each spy’s new position is one move from its old position, because $|f_i(k+1)(\mathcal{R}') - f_i(k+1)(\mathcal{R})| \leq 1$.
To prove Theorem 1.2 we first use Lemma 3.3 below to reduce to the case in which $G$ is a tree and $r = ks + k + s$.

**Lemma 3.3** Let $r, s, k \in \mathbb{N}$. Let $G$ be a graph. Consider the following statements.

1. If $G$ is a tree and $r = ks + k + s$, then $k(G, r, s) \leq \left\lfloor \frac{r}{s+1} \right\rfloor$.

2. If $G$ is an acyclic graph then $k(G, r, s) \leq \left\lfloor \frac{r}{s+1} \right\rfloor$.

3. If $G$ is the power of an acyclic graph then $k(G, r, s) \leq \left\lfloor \frac{r}{s+1} \right\rfloor$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

**Proof:** Assume (1). Write $G = \bigcup_{i \in I} T_i$ as a disjoint union of trees $T_i$. Suppose player 1 decides to use $r_i$ of his revolutionaries in $T_i$, where $r = \sum_{i \in I} r_i$. Let $k = \left\lfloor \frac{r}{s+1} \right\rfloor$. Note that this means that $r \leq ks + k + s$ and $\left\lfloor \frac{r}{s+1} \right\rfloor \leq s$. For each $i \in I$ let $s_i = \left\lfloor \frac{r_i}{k+1} \right\rfloor$. Clearly $\sum_{i \in I} s_i \leq \left\lfloor \frac{r}{k+1} \right\rfloor \leq s$. Thus player 2 may use $s_i$ spies in each $T_i$. Since $r_i \leq (k + 1)s_i + k$, $k(T_i, r_i, s_i) \leq k(G_i, ks_i + k + s_i, s_i)$ $\leq$ $k$ by (1). Player 2 can thus simultaneously play a strategy to prevent a meeting of size $k + 1$ in each $T_i$. (Compare with Lemma 2.5)

Assume (2). By Lemma 2.10 $k(G^n, r, s) \leq k(G, r, s) \leq k$ for any power $G^n$ of a graph $G$. Thus (2) implies (3).

[End of proof]
If $\mathcal{R} \in \mathcal{P}(G, r)$ is a position for player 1 and $k \in \mathbb{N}$ we define $\overline{\mathcal{R}} : 2^V \rightarrow \mathbb{N}$ by $\overline{\mathcal{R}}(S) := \left\lceil \frac{\mathcal{R}(S)}{k+1} \right\rceil$ for subsets $S \subseteq V(G)$. Note that $\overline{\mathcal{R}}(S)$ gives the number of meetings of size $k+1$ that player 2 can theoretically form. We will shortly construct a spies’ response function $S = S(\mathcal{R})$ with the strong property that $S(H) \geq \overline{\mathcal{R}}(H)$ for all subtrees $H$, thereby guarding all meetings. We’ll later show that this property can be maintained as the revolutionaries move. Note that $\overline{\mathcal{R}}(S)$ depends on $\mathcal{R}$ and $k$ but we will suppress these variables as they will be thought of as fixed in our proofs.

Let $T$ be a tree (finite or countably infinite). If $e = \{v, w\} \in E(T)$ recall that $T(v, w)$ is the component of $T - e$ containing $w$. For positions $\mathcal{R}$, $\mathcal{S}$, or for any auxiliary function $\overline{\mathcal{R}} = \overline{\mathcal{R}}(\mathcal{R}, k)$, we define $\mathcal{R}(v, w) := \mathcal{R}(T(v, w))$, $\mathcal{S}(v, w) := \mathcal{S}(T(v, w))$, and $\overline{\mathcal{R}}(v, w) := \overline{\mathcal{R}}(T(v, w))$. Given a position $\mathcal{R} \in \mathcal{P}(T, ks + k + s)$ we define the spies’ response function $S = S(\mathcal{R}, k)$ by $S(v) := s - \sum_{w \sim v} \overline{\mathcal{R}}(v, w)$. Note that since $\mathcal{R}$ is finite, the sum in the definition of $S(v)$ has only finitely many non-zero terms.

**Lemma 3.4** Let $T$ be a finite or countably infinite tree. Let $r, s, k \in \mathbb{N}$ satisfy $r = ks + k + s$. Let $\mathcal{R}, \mathcal{R}' \in \mathcal{P}(T, r)$ and $S = S(\mathcal{R}, k), S' = S(\mathcal{R}', k)$. Then the following statements hold for all $v \in V(T)$, $\{v, w\} \in E(T)$, and all finite subtrees $H$ of $T$.

1. $s = \overline{\mathcal{R}}(v, w) + \overline{\mathcal{R}}(w, v)$

2. $S(H) = s - \sum_{(v, w) \in (V(H), V(H)^c)} \overline{\mathcal{R}}(v, w) \geq \overline{\mathcal{R}}(H)$

3. $S(v, w) = \overline{\mathcal{R}}(v, w)$
4. \( S \in \mathcal{P}(T, s) \)

5. If \( R \sim R' \) then \( S \sim S' \).

**Proof:** To prove 1, we note that \( r = R(v, w) + \mathcal{R}(w, v) \) implies \( \mathcal{R}(v, w) = \mathcal{R}(v, w)(k + 1) + i \) and \( \mathcal{R}(w, v) = \mathcal{R}(w, v)(k + 1) + j \) where \( 0 \leq i, j \leq k \). Then \( s(k + 1) + k = r = \mathcal{R}(v, w) + \mathcal{R}(w, v) \) is the same as \( (k - i - j) = (k + 1)(\mathcal{R}(v, w) + \mathcal{R}(w, v) - s) \). Thus \( (k + 1)(k - i - j) \). Since \( |k - i - j| \leq k \), \( k - i - j = 0 \) and \( s = \mathcal{R}(v, w) + \mathcal{R}(w, v) \).

We now prove statement 2.

\[
S(H) = \sum_{v \in V(H)} (s - \sum_{w \sim v} \mathcal{R}(v, w))
\]

\[
= |V(H)|s - \sum_{\{v, w\} \in E(H)} (\mathcal{R}(v, w) + \mathcal{R}(w, v)) - \sum_{(v, w) \in (V(H), V(H)^c)} \mathcal{R}(v, w)
\]

\[
= s - \sum_{(v, w) \in (V(H), V(H)^c)} \mathcal{R}(v, w)
\]

by statement 1 and the fact that \( |E(H)| = |V(H)| - 1 \). Since \( r = \mathcal{R}(H) + \sum_{(v, w) \in (V(H), V(H)^c)} \mathcal{R}(v, w) \) we may divide by \( k + 1 \) and take floors to get \( s \geq \mathcal{R}(H) + \sum_{(v, w) \in (V(H), V(H)^c)} \mathcal{R}(v, w) \) or \( S(H) \geq \mathcal{R}(H) \) as claimed.

We get statement 3 by applying statement 2 to \( H = T(v, w) \). As \( (w, v) \) is the only edge in \( (V(H), V(H)^c) \) we get \( S(v, w) = s - \mathcal{R}(w, v) = \mathcal{R}(v, w) \).

Clearly statement 4 is true if \( T \) consists of a single vertex. Otherwise \( T \) contains an edge \( \{v, w\} \) and \( S(T) = S(v, w) + S(w, v) = \mathcal{R}(v, w) + \mathcal{R}(w, v) = s \) by statements 1 and 3.
We now prove statement 5. Since $\mathcal{R} \sim \mathcal{R}'$, Lemma 2.8 implies that for all $v \in V(T)$ and all $U \subseteq N(v)$, $\mathcal{R}(T(v, U)) \leq \mathcal{R}'(T(v, U) + v)$. Thus $\sum_{u \in U} \mathcal{R}(T(v, u)) = \mathcal{R}(T(v, U)) \leq \mathcal{R}'(T(v, U) + v)$. Dividing by $k + 1$ and taking floors we get $S(T(v, U)) = \sum_{u \in U} S(T(v, u)) \leq \mathcal{R}'(T(v, U) + v)$. The last inequality in the previous statement follows from statement 2. Thus by Lemma 2.8 $S \sim S'$.

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**Theorem 3.5** Let $T$ be a tree and let $r, s, k \in \mathbb{N}$ satisfy $r = ks + k + s$. Then $k(T, r, s) \leq k$.

**Proof of Theorem 3.5** Player 2 can prevent an unguarded meeting of size $k + 1$ in $G(T, r, s)$ by always playing $S(\mathcal{R}, k) \in \mathcal{P}(T, s)$ in response to player 1's move $\mathcal{R} \in \mathcal{P}(T, r)$. By Lemma 3.4, statements 2, 4 and 5, we have $S(\mathcal{R}, k) \in \mathcal{P}(T, s)$, $S(\mathcal{R}, k)(v) \geq \mathcal{R}(v)$ which implies all meetings of size $k + 1$ or more are guarded, and $S(\mathcal{R}, k) \sim S'(\mathcal{R}', k)$ if $\mathcal{R} \sim \mathcal{R}'$.

\begin{flushright}
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**Corollary 3.6** Theorem 1.2 holds.

**Proof:** Use Lemma 3.3 to reduce Theorem 1.2 to Theorem 3.5.

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4 Proof of Theorem 1.3.

Recall that $a = (a_1, \ldots, a_d), b = (b_1, \ldots, b_d) \in \mathbb{Z}^d$ are adjacent if and only if $a \neq b$ and $|a_i - b_i| \leq 1$ for all $1 \leq i \leq d$.

Lemma 4.1 If $m > 2k$, $k(\mathbb{Z}^d, m, m - 2k) \leq k$.

Proof: Clearly $k(\mathbb{Z}^d, 2k + 1, 1) \geq \left\lceil \frac{2k+1}{2} \right\rceil = k$ by Lemma 1.1. We give a strategy for the spies showing that $k(\mathbb{Z}^d, 2k + 1, 1) \leq k$. Suppose player 1 is in position $\mathcal{R}$. Fix $1 \leq i \leq d$. Let $\mathcal{R}_i(j) := \mathcal{R}(\{x \in \mathbb{Z}^d : x_i = j\}) \in \mathcal{P}(\mathbb{Z}, r)$ be the projection of $\mathcal{R}$ onto the $i$th coordinate axis. Let $c_i = f_{k+1}(\mathcal{R}_i)$ (see Lemma 3.2). Player 2’s response to $\mathcal{R}$ is to move his spy to the vertex $c = (c_i : 1 \leq i \leq d)$. By Lemma 3.2 this is a playable strategy for player 2. Furthermore it guards all meetings of size $k + 1$ or more. Clearly, if at least $k + 1$ revolutionaries are at a single vertex $c' \in \mathbb{Z}_d$ then $c_i = c_i'$ for all $i$ and $c = c'$, the spy is there also. This suffices to prove the theorem as Lemma 2.2 implies $k(\mathbb{Z}^d, m, m - 2k) \leq k(\mathbb{Z}^d, 2k+1, 1) + k(\mathbb{Z}^d, m - 2k - 1, m - 2k - 1) = k + 0 = k$.

Theorem 1.3 will follow from this next theorem, see Corollary 1.3 below.

Theorem 4.2 We have $k(\mathbb{Z}^2, 8, 5) = 2$.

Proof: By Lemma 2.2 and Lemma 4.1 $k(\mathbb{Z}^2, 8, 5) \leq k(\mathbb{Z}^2, 7, 5) + k(\mathbb{Z}^2, 1, 0) \leq 1 + 1 = 2$. We give a strategy for player 1, showing $k(\mathbb{Z}^2, 8, 5) \geq$
2. In the first round, player 1 will place his revolutionaries as in Figure 1. In all of our figures X’s represent revolutionaries while O’s represent spies. If 2 revolutionaries may reach a vertex in \( n \) rounds, there must be a spy within distance \( n \) of \( v \) to guard that potential meeting. We often describe this by saying that a spy must prevent a \( \text{win} \) at \( v \) in \( n \) rounds. The point \( v \) itself may be called a (potential) win. In all of our figures the center point is position \((0,0)\). The inner box is defined to be \( B_1 := [-1,1] \times [-1,1] \) and the outer box is \( B_2 := [-2,2] \times [-2,2] \). We call spies placed in \( B_1 \) \textit{inner} spies and those placed outside \( B_2 \) \textit{outer} spies.

Case 1: Suppose player 2 initially places at most 1 spy outside \( B_2 \).

Then either all spies must have their \( x \)-coordinates strictly less than 3 or all spies must have their \( x \)-coordinates strictly greater than \(-3\). If the former
holds, the revolutionaries at \((3, 3)\) and \((3, -3)\) can win at \((6, 0)\) in 3 rounds. The latter case is symmetric.

We now assume player 2 initially places at least 2 spies outside of \(B_2\).

Case 2: Suppose player 2 initially places at most 1 spy within \(B_1\).

At most 3 spies may be placed within \(B_2 \setminus B_1\). No matter how these spies are placed, at least one of the points \((0, 1)\), \((1, 0)\), \((-1, 0)\), or \((0, -1)\) (say \((0, 1)\), without loss of generality) must be at least distance 2 from every spy in \(B_2 \setminus B_1\). If the revolutionaries at \((-1, 1)\) and \((1, 1)\) move to \((0, 1)\) while, simultaneously, the revolutionaries at \((-1, -1)\) and \((1, -1)\) move to \((0, 0)\), player 1 will achieve two meetings of size 2 in one move. Only a spy in \(B_1\) can guard the meeting at \((0, 0)\) in time, so the meeting at \((0, 1)\) will be unguarded.

We now assume player 2 places at least 2 spies within \(B_1\).

Case 3: Suppose player 2 initially places exactly 3 spies outside \(B_2\). (Note that this means there are no spies in \(B_2 \setminus B_1\).)

For each point \(p = (6, 0), (-6, 0), (0, 6),\) or \((0, -6)\) there is a pair of revolutionaries (among those at \((-3, -3)\), \((-3, 3)\), \((3, -3)\), and \((3, 3)\)) that can form a meeting of size 2 at \(p\) in 3 rounds. No spy within \(B_2\) can guard any such win. Thus one of the spies outside \(B_2\) must guard two of them. By symmetry, we assume that one of these outer spies guards \((6, 0)\) and \((0, 6)\). The only spot this spy can occupy is \((3, 3)\). (Note that a single spy cannot
guard both \((0, \pm 6)\) or both \((\pm 6, 0)\) in time.)

For all further sub-cases of case 3 we will assume that one of the spies outside \(B_2\) is placed at \((3, 3)\).

Case 3a: Suppose player 2 does not initially place the 2 spies within \(B_1\) on opposite corners of \(B_1\).

We define the sets \(\{1\} \times [-1, 1], \{-1\} \times [-1, 1], [-1, 1] \times \{1\}, \) and \([-1, 1] \times \{-1\}\) to be the east, west, north, and south internal walls of \(B_1\), respectively. Similarly, we define the sets \(\{3\} \times [-3, 3], \{-3\} \times [-3, 3], [-3, 3] \times \{3\}, \) and \([-3, 3] \times \{-3\}\) to be the east, west, north, and south external walls of \(B_2\), respectively. At least one of the 4 internal walls must not contain a spy.

The two spies in \(B_1\) can not both be in the same internal wall, otherwise the 2 revolutionaries in the opposite internal wall can win in one round. For example, if both spies in \(B_1\) are in the north internal wall, the two revolutionaries in the south internal wall can win at \((0, -1)\) in 1 round. The other cases are symmetric.

By symmetry about the line \(y = x\), we may now assume that either the east internal wall is unguarded or the south internal wall is unguarded.

Case 3a1: Suppose the east internal wall is unguarded.

A spy must be placed on the east outer wall to be able to block wins \((2, 0)\) and \((2, -2)\) in one round. (See Figure 2). Thus this spy must be placed at \((3, -1)\). (If instead two spies were used to guard these wins then
$w$ after round 0. ($-6,0$ would be unguarded.) This forces the remaining spy outside $B_2$ to
be placed at $(-3,-3)$ to guard the wins at $(0,-6)$ and $(-6,0)$. Now an
internal spy must be placed at $(-1,1)$ to guard against the win at $(-2,2)$. 
The other internal spy must be placed at $(0,-1)$; it cannot be on the east
or west internal walls, and it must guard the win at $(0,-2)$. Player 1 can
now win in 2 rounds by moving the revolutionaries at $(-3,3)$ and $(1,1)$ to
$w = (-1,3)$ while simultaneously moving the revolutionaries $(-1,1)$ and
$(-1,-1)$ to $(-2,0)$. Only the spy at $(-1,1)$ can guard either of these meet-
ings in time.

Case 3a$_2$. Suppose the south internal wall is unguarded.

By reasoning similar to the beginning of case 3a$_1$ an outer spy must be
placed at $(1,-3)$ to guard wins at $(0,-2)$ and $(2,-2)$. (If two spies were used

Figure 2: Case 3a$_1$ after round 0.
to guard these wins then \((-6, 0)\) would remain unguarded.) (See Figure 3).

Thus the remaining outer spy must be placed in \(L_1 = \{-3\} \times [-3, -1]\) to guard the wins at \((-2, -2)\) and \((-6, 0)\). An internal spy must be placed at \((-1, 1)\) to guard a win at \((-2, 2)\). The other internal spy must be placed at \((1, 0)\) to guard a win at \((2, 0)\). Player 1 may now win by simultaneously moving the revolutionaries at \((-1, 1)\) and \((-1, -1)\) to \((-1, 0)\) and moving those at \((1, 1)\) and \((-3, 3)\) to \(w = (-1, 3)\). The spy at \((-1, 1)\) is the only spy that can guard either of these meetings in time.

We now assume that the 2 spies in \(B_1\) are located at opposite corners; either at \((-1, 1)\) and \((1, -1)\), Case 3b1, or at \((-1, -1)\) and \((1, 1)\), Case 3b2.

Case 3b1: Suppose the 2 revolutionaries in \(B_1\) are at \((-1, 1)\) and \((1, -1)\).
Figure 4: Case 3$b_1$ after round 0.

One outer spy must be placed in $L_2 = (\{-3\} \times [-3,-1]) \cup ([3, -1] \times \{-3\})$ to guard a win at $(-2, -2)$. (See Figure 4) The remaining outer spy can not guard both $w = (-1, 3)$ and $w' = (3, -1)$ within 2 moves. By symmetry, we may assume the outer spy does not guard $w$. The winning strategy for player 1 is then identical to case 3$a_2$.

Case 3$b_2$: Suppose the 2 revolutionaries in $B_1$ are at $(1, 1)$ and $(-1, -1)$.

In this case the sets $L_3 = (\{-3\} \times [1, 3]) \cup ([3, -1] \times \{3\})$, and $L_4 = (\{3\} \times [-3, -1]) \cup ([1, 3] \times \{-3\})$ must each contain an outer spy to guard wins at $(2, -2)$ and $(-2, 2)$, respectively. (See Figure 5) Furthermore these spies must be placed at points $(-3, 1)$ and $(1, -3)$ to guard wins at $v = (-3, -1)$ and $v' = (-1, -3)$ within 2 moves, otherwise an argument similar to that of case 3$a_2$ gives player 1 a win. For example if $v$ is not guarded, revolutionaries
at \((-1, 1)\) and \((-3, -3)\) can win at \(v\) in 2 moves.

Player 1’s strategy is to move the revolutionary at \((-1, 1)\) to \((0, 0)\) and
the one at \((1, 1)\) to \((2, 1)\) while leaving all other revolutionaries unchanged.
To compensate, player 2 must leave the spy at \((3, 3)\) in place to guard \((6, 0)\)
and \((0, 6)\). (See Figure 5.) At the end of its move, the spy at \((-3, 1)\) must
be somewhere in \(L_5 = [-4, -3] \times [0, 2]\) as it must guard \((-6, 0)\). Similarly
the spy at \((1, -3)\) must remain in \(L_6 = [0, 2] \times [-4, -3]\).

The spy at \((-1, -1)\) must stay in place to guard wins at \((-2, -2)\) and
\((0, 0)\) in one move. (Note: the spy at \((-1, -1)\) must guard \((0, 0)\) as the spy
at \((1, 1)\) must guard against the potential meeting of the revolutionaries from
\((2, 1)\) and \((1, -1)\) at the point \((2, 0)\).) The spy at \((1, 1)\) must move to the
point \((1, 0)\) to guard wins at \((0, 0)\), \((1, -1)\), and \((2, 0)\). The spies' positions
at the end of round 1 are illustrated in Figure 6.
Player 1’s strategy is to simultaneously move \((0, 0)\) and \((-1, -1)\) to \((-1, 0)\), \((1, -1)\) and \((-3, -3)\) to \((-1, -3)\) and \((2, 1)\) and \((3, -3)\) to \((4, -1)\). Only the spy at \((-1, -1)\) can guard the first meeting and consequently only the spy in \(L_6\) can guard either of the other two meetings. Thus player 1 wins.

We now assume there are exactly two spies outside \(B_2\). As these must guard \((6, 0), (0, 6), (-6, 0),\) and \((0, -6)\) we can exploit symmetry and assume they are at \((3, 3)\) and \((-3, -3)\).

Case 4:

Each of \(C_1 = [-2, -1] \times [1, 2]\) and \(C_2 = [1, 2] \times [-2, -1]\) must contain a spy in order to guard wins at \((-2, 2)\) and \((2, -2)\). (See Figure 7) By symmetry about \(y = x\) and \(y = -x\), we may assume the coordinates \((x, y)\) of the remaining spy satisfy \(-x \leq y \leq x, \ 0 \leq x \leq 2\).
Figure 7: Case 4: partial location of spies after round 0.

Case 4a: Suppose \((x, y) \neq (1, 1)\).

Player 1 may win by simultaneously moving revolutionaries from \((-1, 1)\) and \((-1, -1)\) to \((-2, 0)\) and those from \((-3, 3)\) and \((1, 1)\) to \(w = (-1, 3)\). Only the spy in \(C_1\) can guard either of these wins in time.

Case 4b: Suppose \((x, y) = (1, 1)\).

Recall that at least 2 spies are in \(B_1\). By symmetry we may assume one of those spies is in \(C_1 \cap B_1\), hence at \((-1, 1)\). The other spy must be located in \(R_0 = \{1\} \times [-2, -1]\) as it must guard wins at \((2, -2)\) and \((0, -1)\). (See Figure 8)

Player 1’s strategy is to move the revolutionary at \((-1, -1)\) to \((-2, -2)\) and the one at \((-1, 1)\) to \((0, 0)\), keeping the other revolutionaries in place.
Player 2 must keep the spies at (3, 3) and (−3, 3) fixed to continue guarding wins at (±6, 0) and (0, ±6). Besides these two spies, only the spy in $R_0$ (if it were located at (1, −2)) could be moved to help guard (0, −6), but that spy can not assist as it must also guard against the potential meeting of the revolutionaries at (−2, −2) and (1, −1) at the point (0, −3).

Let $(a, b)$ be the position of the spy in $R_0$ in round 0 and $(a', b')$, its position in round 1. We have $a' = 1$, since this spy must guard (2, −2) and also the meeting of (−2, −2) and (1, −1) at (−1, −3). We must have $b' \leq −2$ as this spy must guard the meeting of the revolutionaries at (−2, −2) and (3, −3) at (1, −5). Thus, player 2 must have a spy in $R_1 = \{1\} \times [−3, −2]$ at the end of round 1. (See Figure 9)

The spy located originally at (1, 1) cannot decrease its $x$-coordinate because it must guard (2, 0). This forces the spy originally located at (−1, 1)
to decrease its $y$-coordinate to guard $(-1, -1)$. This same spy must also decrease its $x$-coordinate to guard against the meeting of the revolutionaries from $(-3, 3)$ and $(-2, -2)$ at $(-5, 1)$. This forces the spy at $(1, 1)$ to move to $(1, 0)$ to guard wins at $(2, 0)$, $(0, 1)$, and $(0, -1)$. Note that the spy in $R_1$ cannot help guard these since it independently must guard the meeting at $(1, -5)$ by revolutionaries from $(-2, -2)$ and $(3, -3)$.

Now player 1 can win at $w = (-1, 3)$ in 2 moves.

\[\]  

**Corollary 4.3** Theorem 4.2 implies Theorem 1.3.

**Proof:** The winning strategy for player 1 in Theorem 4.2 does not involve movement of the revolutionaries outside of the box $[-6, 6]^2$ and takes no more than 5 moves to achieve a meeting of size 2. Letting $G_i$, $1 \leq i \leq \ell$ be
sufficiently separated copies of $[-6,6]^2$ in $\mathbb{Z}^{2d}$ and applying Lemma 2.2 gives $k(\mathbb{Z}^{2d}, 8\ell, 6\ell - 1) \geq k(\mathbb{Z}^{2d}, 8, 5) = 2$. Thus for all $\ell \geq 1$ and all $0 \leq i < 8$ we have $s_2(8\ell + i) \geq 6\ell$. Hence $\liminf_{n \to \infty} s_2(n)/n \geq \frac{3}{4}$. By Lemma 2.11, $k(\mathbb{Z}^{2d}, r, s) \geq k(\mathbb{Z}^{2d}, r, s)$ for $d \geq 2$ and so $\liminf_{n \to \infty} s_d(n)/n \geq \frac{3}{4}$ for $d \geq 2$.

5 Conclusion

It would be of interest to get tight bounds on $k(\mathbb{Z}^{2d}, r, s)$. It would be also interesting to extend the result on trees to some larger class of graphs, perhaps characterizing those graphs for which $k(G, r, s) = \lfloor \frac{r}{s+1} \rfloor$. One natural class that can be proposed is chordal graphs, but simple examples show that one can have $k(G, r, s) > \lfloor \frac{r}{s+1} \rfloor$ when $G$ is chordal.

Continuous versions of this problem can be considered. For example one could play the game in the plane, where each agent has the power to move to points within a Euclidean distance of 1 from their current position. This particular variant was suggested by Beck [1].

References

[1] J. Beck, personal communication, 1994.