INFINITE VERSIONS OF SOME PROBLEMS
FROM FINITE COMPLEXITY THEORY

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Abstract. Recently, several authors have explored the connections between NP-complete problems for finite objects and the complexity of their analogs for infinite objects. In this paper, we will categorize infinite versions of several problems arising from finite complexity theory in terms of their recursion theoretic complexity and proof theoretic strength. These infinite analogs can behave in a variety of unexpected ways.

Startling parallels exist between the computational complexity of certain graph theoretic problems and the recursion theoretic complexity and proof theoretic strength of their infinite analogs. For example, the problem of deciding which finite graphs have an Euler path is known to be P-time computable [9], and Beigel and Gasarch [4] have shown that the problem of deciding which infinite recursive graphs have an Euler path is arithmetical. By contrast, the problem of deciding which finite graphs have Hamilton paths is NP-complete [8], and Harel [6] has shown that the problem of deciding which infinite recursive graphs have a Hamilton graph is $\Sigma_1^1$ complete. Thus, the possibly greater computational complexity is paralleled by a demonstrable increase in recursion theoretic complexity. This pattern can also be seen through an application of the techniques of reverse mathematics. The

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existence of a function that decides which graphs have Euler paths is provably equivalent to $\text{ACA}_0$, while the existence of a similar function for Hamilton paths is equivalent to the much stronger axiom system $\Pi_1^1-\text{CA}_0$.

Unfortunately, other graph theoretic problems do not demonstrate this parallelism. We have selected some examples to illustrate two general themes. First, different infinite formulations of a fixed finite problem can have different recursion theoretic complexities. This would seem to indicate that the use of a preferred infinite formulation might lead to natural parallels between finite complexity and recursion theoretic complexity. However, the behavior of infinite analogs is not so easily tamed. Indeed, similar formulations of infinite versions of problems with different finite complexities may have the same recursion theoretic complexity.

**Variability due to translations.**

This section contains examples illustrating our first theme. The problem of determining which finite graphs are 3-chromatic is NP-complete [8]. Extrapolating from the problem of finding Hamilton paths, we would expect infinite analogs of the 3-coloring problem to be $\Sigma^1_1$ complete. However, the actual recursion theoretic complexity depends on the formulation of the infinite analog, as demonstrated by the following three theorems. Our notation is patterned after that of Soare [12].

**Theorem 1.** (Beigel and Gasarch [1]) *The set of indices of 3-chromatic recursive graphs is $\Pi^0_1$ complete.*

*Proof.* Let $\mathcal{G}_1$ denote the set of indices of 3-chromatic recursive graphs. Note that $x \in \mathcal{G}_1$ if and only if every finite subgraph of the graph with index $x$ is 3-chromatic. Thus, $\mathcal{G}_1$ is a $\Pi^0_1$ definable subset of the set of indices of recursive graphs.

To show that $\mathcal{G}_1$ is $\Pi^0_1$ complete, let $\mathcal{G}_0$ denote the set of indices of recursive graphs which are not 3-chromatic. It suffices to show that $(K, \bar{K}) \leq_1 (\mathcal{G}_0, \mathcal{G}_1)$. Here $K = \{ e : e \in W_e \}$ is the self-halting set.
For each $e \in \omega$, define the graph $G_e$ as follows. The vertex set of $G_e$ is $\omega$, and for $m < n$, the edge $(m, n)$ is in $G_e$ if and only if ${e}(e)$ halts by stage $m$. For every $e$, $G_e$ is recursive. By the s-m-n Theorem, there is a 1-1 recursive function $f$ such that for every $e$, $f(e)$ is an index for $G_e$.

Note that if $e \in K$, $G_e$ contains an infinite clique. In this case, $G_e$ is not 3-chromatic, so $f(e) \in G_0$. On the other hand, if $e \in \overline{K}$, $G_e$ has no edges. Such a graph is certainly 3-chromatic, so $f(e) \in G_1$. Thus, $f$ witnesses that $(K, \overline{K}) \leq_1 (G_0, G_1)$, as desired. □

**Theorem 2.** The set of indices of recursive graphs with finitely colorable connected components is $\Pi^0_3$ complete.

**Proof.** Let $G_1$ denote the set of indices of recursive graphs with finitely colorable connected components. Suppose that $x$ is the index of a graph $G$. Then $x \in G_1$ if and only if for every vertex $v$ of $G$, there is an integer $k$ such that every finite connected subgraph of $G$ containing $v$ is $k$-chromatic. Thus, $G_1$ is a $\Pi^0_3$ definable subset of the set of indices of recursive graphs.

To show that $G_1$ is $\Pi^0_3$ complete, let $G_0$ denote the set of indices of those recursive graphs which have connected components that are not finitely colorable. It suffices to show that $(\text{Cof}, \overline{\text{Cof}}) \leq_1 (G_0, G_1)$. Here, $\text{Cof} = \{e : W_e \text{ is cofinite}\}$.

For each $e \in \omega$, define the graph $G_e$ as follows. $G_e$ will contain vertices labeled $v_{m,n}$ for each $m$ and $n$ in $\omega$, and some additional unlabeled vertices. For each $m$, the vertex $v_{m,0}$ will be included in a complete graph on $m + 1$ vertices. For every $m$ and $j$ all edges of the form $(v_{m,j}, v_{m,j+1})$ will be included in $G_e$. Finally, the edge $(v_{m,j}, v_{m+1,j})$ will be included in $G_e$ if and only if ${e}(m)$ halts by stage $j$. For every $e$, $G_e$ is recursive. By the s-m-n Theorem, there is a 1-1 recursive function $f$ such that for every $e$, $f(e)$ is an index for $G_e$.

Note that if $e \in \text{Cof}$, then there is a $j$ such that the vertices $\{v_{m,n} : m > j\}$ are all in the same connected component. Consequently, arbitrarily large complete finite subgraphs are contained in this component, and it is not finitely colorable. Thus, if $e \in \text{Cof}$, $f(e) \in G_0$. □
Now suppose that $e \in C_{\text{of}}$ and $C$ is a connected component of $G_e$. $C$ must contain a vertex of the form $v_{m,0}$. Since $e \in C_{\text{of}}$, there is a least $j$ greater than $m$ such that $\{e\}(j)$ never halts. Consequently, $C$ cannot contain any vertex $v_{n,k}$ such that $n > j$. This ensures that $C$ is $j + 1$-chromatic, so $f(e) \in G_1$. Thus, $f$ witnesses that $(C_{\text{of}}, C_{\text{of}}) \leq_1 (G_0, G_1)$, as desired. □

For the next proof, we will need the following notation for finite sequences of natural numbers. Assuming a recursive bijection between $\omega$ and $\omega^\omega$, we will use a Greek letter (usually $\sigma$ or $\tau$) to denote both a sequence and its integer code. The formula $\sigma \subseteq \tau$ means that $\sigma$ is a (not necessarily proper) initial segment of $\tau$. Thus, $T$ is a tree if whenever $\tau \in T$ and $\sigma \subseteq \tau$, then $\sigma \in T$.

Given an arbitrary index $e$, $\{e\}$ may or may not be the characteristic function for a recursive tree. To streamline our discussion, consider the following auxiliary function.

**Definition 3.** For $e \in \omega$, the partial recursive function $\eta_e$ is defined by:

$$
\eta_e(\tau) = \begin{cases} 
1 & \text{if } \forall \sigma \subseteq \tau (\{e\}(\sigma) = 1), \\
0 & \text{if } \{e\}(\tau) = 0 \land [\forall \sigma \subseteq \tau (\{e\}(\sigma) = 0 \lor \{e\}(\sigma) = 1] \\
& \land [\forall \sigma \subseteq \tau \forall \alpha \subseteq \sigma (\{e\}(\alpha) = 0 \rightarrow \{e\}(\sigma) = 0)], \\
\uparrow & \text{otherwise.}
\end{cases}
$$

Naïvely, $\eta_e$ approximates the characteristic function of a tree. In particular, $\eta_e$ is total if and only if $e$ is the index of a recursive tree. Note that by the s-m-n Theorem, there is a 1-1 recursive function which maps each $e$ to an index for $\eta_e$.

**Theorem 4.** (TIRZA HIRST and HARREL [7]) The set of indices of recursive graphs with colorings which use one color infinitely often is $\Sigma_1^1$ complete.

*Proof.* Let $\mathcal{G}$ denote the set of indices of recursive graphs with colorings that use one color infinitely often. Note that $x \in \mathcal{G}$ if and only if there is a function $\chi$ mapping the vertices of the graph with index $x$ into $\omega$, such that $\chi$ maps neighboring vertices to different values,
and 0 appears infinitely often in the range of \( \chi \). This statement can be formalized using a single existential set quantifier followed by an arithmetical formula, so \( \mathcal{G} \) is \( \Sigma^1_1 \) definable.

To show that \( \mathcal{G} \) is \( \Sigma^1_1 \) complete, we will show that \( \mathcal{T} \leq_1 \mathcal{G} \), where \( \mathcal{T} \) denotes the set of indices of recursive trees which are not well-founded. With each \( e \in \omega \), we associate a partial recursive graph, \( G_e \). The vertex set for \( G_e \) consists of (codes for) elements of \( \omega^{<\omega} \). For every \( \sigma, \tau \in \omega^{<\omega} \), the characteristic function for the edge set of \( G_e \) is defined by

\[
E_e(\sigma, \tau) = \begin{cases} 
0 & \text{if } \eta_e(\sigma) = 1 \land \eta_e(\tau) = 1 \land (\sigma \subseteq \tau \lor \tau \subseteq \sigma), \\
1 & \text{if } \eta_e(\sigma) = 1 \land \eta_e(\tau) = 1 \land \neg(\sigma \subseteq \tau \lor \tau \subseteq \sigma), \\
1 & \text{if } \eta_e(\sigma) \downarrow \land \eta_e(\tau) \downarrow \land (\eta_e(\sigma) = 0 \lor \eta_e(\tau) = 0), \\
\uparrow & \text{otherwise.}
\end{cases}
\]

Roughly, we connect \( \sigma \) and \( \tau \) by an edge if they are incomparable nodes on the tree or if one of them is not in the tree, ignoring those nodes whose status is suspect. By the \textit{s-m-n} Theorem, there is a 1-1 recursive function \( f \) such that for every \( e \), \( f(e) \) is an index for \( G_e \).

If \( e \in \mathcal{T} \), then \( e \) is the index of a recursive tree containing an infinite path \( P \). Consequently, \( f(e) \) is the index of a recursive graph. We can color this graph by mapping every node of \( P \) to 0, and mapping all other nodes to their integer codes. Since 0 is used infinitely often in this coloring, \( f(e) \in \mathcal{G} \).

Now suppose \( e \notin \mathcal{T} \). If \( e \) is not the index of a recursive tree, then \( f(e) \) is not the index of a recursive graph, so \( f(e) \notin \mathcal{G} \). If we suppose that \( e \) is the index of a recursive tree \( T \), then \( T \) is well-founded. Suppose, by way of contradiction, that there is a coloring of the associated recursive graph \( G_e \) that uses 0 infinitely often. All the nodes of \( G_e \) that are colored 0 correspond to comparable nodes of \( T \), contradicting the claim that \( T \) is well-founded. Again, we have \( f(e) \notin \mathcal{G} \), completing the proof that \( \mathcal{T} \leq_1 \mathcal{G} \).

The techniques of reverse mathematics can be used to draw a distinction between the first two of our infinite analogs and the third. The following two results make use of the axiom systems \textbf{RCA}_0 (Recursive Comprehension Axiom), \textbf{ACA}_0 (Arithmetical
Comprehension Axiom), and $\Pi^1_1-\text{CA}_0$ ($\Pi^1_1$ Comprehension Axiom). For a brief overview of reverse mathematics, see Simpson [11]

**Theorem 5 (RCA$_0$).** The following are equivalent:

1. $\text{ACA}_0$.
2. For any sequence of graphs $\langle G_i : i \in \omega \rangle$, there is a function $s : \omega \to 2$ such that $s(i) = 1$ if and only if $G_i$ is 3-chromatic.
3. For any sequence of graphs $\langle G_i : i \in \omega \rangle$, there is a function $s : \omega \to 2$ such that $s(i) = 1$ if and only if every connected component of $G_i$ is finitely colorable.

*Proof.* To prove $(1) \to (2)$ and $(1) \to (3)$, it suffices to show that the function $s$ is arithmetically definable in $\langle G_i : i \in \omega \rangle$. For $(2)$, imitating the first paragraph of the proof of Theorem 1 yields a $\Pi^0_1$ defining formula. Similarly, for $(3)$, imitating the proof of Theorem 2 yields a $\Pi^0_3$ defining formula.

By Lemma 2.7 of [10], to prove that $(2) \to (1)$ and $(3) \to (1)$, it suffices to show that given any injection $g : \omega \to \omega$, RCA$_0$ can prove the existence of a sequence of graphs $\langle G_i : i \in \omega \rangle$ such that the range of $g$ is $\Delta^0_1$ definable in the associated function $s$. Fix $g$ and assume RCA$_0$. We will define a sequence of graphs that works for both $(2)$ and $(3)$. Let $G_n$ have $\omega$ as its vertex set. For every $j \in \omega$, include the edge $(j, j+1)$ in $G_n$. For $j < k$, add the edge $(j, k)$ to $G_n$ if and only if $\exists t \leq j (g(t) = n)$. The sequence $\langle G_i : i \in \omega \rangle$ is $\Delta^0_1$ definable in $g$, so RCA$_0$ proves it exists. Let $s$ be as in $(2)$ or $(3)$. Then $s(n) = 1$ if and only if $n$ is not in the range of $g$. Thus, the range of $g$ is $\Delta^0_1$ definable in $s$, as desired. □

**Theorem 6 (RCA$_0$).** The following are equivalent:

1. $\Pi^1_1-\text{CA}_0$.
2. For any sequence of graphs $\langle G_i : i \in \omega \rangle$, there is a function $s : \omega \to 2$ such that $s(i) = 1$ if and only if $G_i$ has a coloring in which one color is used infinitely often.

*Proof.* To prove that $(1) \to (2)$, it suffices to note that the function $s$ is $\Sigma^1_1$ definable in $\langle G_i : i \in \omega \rangle$, and so exists by $\Pi^1_1-\text{CA}_0$. To prove the converse, we will use the fact that
$\Pi_1^1-\text{CA}_0$ is equivalent to the existence of a function that decides which members of a sequence of trees are well founded. (This is an easy consequence of Lemma 6.1 in [3].) Assume $\text{RCA}_0$, and suppose that $\langle T_i : i \in \omega \rangle$ is a sequence of trees. With each tree $T_n$, we associate a graph $G_n$ as follows. The vertices of $G_n$ are the nodes of $T_n$, and two vertices of $G_n$ are connected if and only if the associated nodes are incomparable in the tree ordering. The sequence $\langle G_i : i \in \omega \rangle$ is $\Delta^0_1$ definable in $\langle T_i : i \in \omega \rangle$, and so exists by $\text{RCA}_0$. Let $s$ be as in (2). Then $s(i) = 1$ if and only if $G_i$ contains an infinite collection of pairwise disconnected vertices, which occurs if and only if $T_i$ is not well founded. Thus (2) implies $\Pi_1^1-\text{CA}_0$, completing the proof. \[\square\]

**Other Variability.**

From the results in the preceding section, it is clear that the recursion theoretic strength of infinite analogs depends in part on their formulation. As shown by Harel and Tirza Hirst [7], adoption of a standardized translation yields interesting parallels between finite complexity and recursion theoretic complexity for restricted classes of problems. However, for broader classes of problems, the parallels break down. In this section, we will consider three problems of diverse finite complexity that all have $\Sigma^1_1$ complete infinite analogs, thus illustrating our second theme.

Consider the following three variants of the subgraph isomorphism problem:

**P1.** Given a pair of finite graphs, $H$ and $G$, determine if $H$ is isomorphic to a subgraph of $G$.

**P2.** For a fixed finite graph $H$, given a finite graph $G$, determine if $H$ is isomorphic to a subgraph of $G$.

**P3.** For a fixed finite graph $G$, given a finite graph $H$, determine if $H$ is isomorphic to a subgraph of $G$.

**P1** is the familiar form of the subgraph isomorphism problem, and is known to be NP complete [2]. One algorithm for solving **P2** and **P3** consists of enumerating all functions...
from \( H \) into \( G \), and checking each one to see if it is the desired isomorphism. The number of functions to check is bounded by \(|G|^{|H|}\), where \(|G|\) denotes the number of vertices of \( G \). Since \( H \) is fixed in \( \textbf{P2} \), the number of functions to check is a constant power of \(|G|\). Furthermore, the number of steps required to check each function is bounded by a constant based on the fixed value \(|H|\). Thus, \( \textbf{P2} \) can be solved in a number of steps which is bounded by a polynomial in \(|G|\). In \( \textbf{P3} \), \( G \) is fixed, and we can discard any graphs \( H \) such that \(|H| > |G|\), so the number of steps required to solve an instance of \( \textbf{P3} \) is bounded by a constant based on the fixed value \(|G|\). Summarizing, the complexity of three problems ranges from NP complete to constant time computable.

Compared to the coloring problem in §1, these subgraph isomorphism problems have very straightforward infinite analogs. Despite the variation in the computational complexity of the finite problems, their infinite analogs are all \( \Sigma^1_1 \) complete, as is shown in the following three theorems.

**Theorem 7.** (Tirza Hirst and Harel [7]) *The set of indices of ordered pairs of recursive graphs, \((H,G)\), such that \( H \) is isomorphic to a subgraph of \( G \) is \( \Sigma^1_1 \) complete.*

**Proof.** Let \( \mathcal{G} \) be the set of indices of ordered pairs of recursive graphs such that the first graph is isomorphic to a subgraph of the second. Since \( x \in \mathcal{G} \) if and only if an appropriate isomorphism exists, it is easy to see that \( \mathcal{G} \) is \( \Sigma^1_1 \) definable.

To prove that \( \mathcal{G} \) is \( \Sigma^1_1 \) complete, we will show that \( \mathcal{T} \leq_1 \mathcal{G} \), where \( \mathcal{T} \) denotes the set of indices of recursive trees which are not well founded. With each \( e \in \omega \), we associate a pair of partial recursive graphs, \( H_e \) and \( G_e \). \( H_e \) is a countably infinite linear graph with a triangle attached at one end. To be precise, the vertex set of \( H_e \) is \( \{v_n : n \in \omega\} \) and the edge set is \( \{(v_0, v_2)\} \cup \{(v_n, v_{n+1}) : n \in \omega\} \). If \( e \) is the index of a recursive tree \( T \), then \( G_e \) consists of a copy of \( T \) with a triangle attached to the root, and a collection of disconnected vertices. In general, the vertex set for \( G_e \) consists of \( \{v_0, v_1, v_2\} \) and (codes for) the elements of \( \omega^{<\omega} \). Let \( \sigma_0 \) denote the code for the empty sequence. The edge \((v_0, \sigma_0)\) and the three edges of the form \((v_i, v_j)\) where \( i \neq j \) are included in \( G_e \). For every \( \sigma \) and \( \tau \)
in $\omega^{<\omega}$, the edge $(\sigma, \tau)$ is included in $G_e$ if and only if

$$\eta_e(\sigma) = \eta_e(\tau) = 1 \land \sigma \subseteq \tau \land \neg \exists \alpha (\sigma \subsetneq \alpha \subsetneq \tau),$$

where $\eta_e$ is the function defined in §1. By the $s-m-n$ theorem, there is a recursive 1-1 function $f$ such that for every $e$, $f(e)$ is an index for the pair $(H_e, G_e)$.

If $e \in T$, then $e$ is the index of a recursive tree containing an infinite path $P$. In this case, $H_e$ is isomorphic to the subgraph of $G_e$ consisting of the base triangle and a copy of $P$. Thus $f(e) \in \mathcal{G}$.

Now suppose that $e \notin T$. If $e$ is not the index of a recursive tree, then $G_e$ is not a recursive graph, so $f_e \notin \mathcal{G}$. If $e$ is the index of a recursive tree $T$, then $T$ is well founded. The graph $G_e$ is a copy of $T$ with a triangle attached to its base. Any isomorphism mapping $H_e$ into $G_e$ must map the triangle in $H_e$ into the triangle in $G_e$, and the linear portion of $H_e$ to an infinite path in the copy of $T$. Since $T$ is well founded, no such isomorphism exists. Thus $f(e) \notin \mathcal{G}$, completing the proof that $T \leq_1 \mathcal{G}$. □

**Theorem 8.** There is a recursive graph $H$, such that the set of indices of recursive graphs containing a subgraph isomorphic to $H$ is $\Sigma^1_1$ complete.

**Proof.** In the proof of Theorem 7, $H_e$ is a fixed recursive graph defined without reference to $e$. Any recursive 1-1 function mapping $e$ to an index for the graph $G_e$ (defined as in the proof of Theorem 7) witnesses the desired 1-reduction. □

**Theorem 9.** There is a recursive graph $G$, such that the set of indices of recursive graphs that are isomorphic to a subgraph of $G$ is $\Sigma^1_1$ complete.

**Proof.** We begin the proof by constructing the recursive graph $G$. This graph will consist of a countable collection of subgraphs $\langle G_e : e \in \omega \rangle$, where each $G_e$ consists of a tree-like substructure together with some spurious disconnected subgraphs.

For each $e \in \omega$, $G_e$ will be constructed from cycles labeled $C(e, \sigma, k)$ for each non-empty $\sigma \in \omega^{<\omega}$ and each $k \in \omega$. The cycle $C(e, \sigma, k)$ consists of $2(e+1) + 2$ vertices joined to
make a circular graph. We designate two vertices of \( C(e, \sigma, k) \) as \( v_{e, \sigma, k}^0 \) and \( v_{e, \sigma, k}^1 \), and require that the paths joining them contain \( e + 2 \) edges. To give a concrete example, \( C(1, \sigma, k) \) looks like a hexagon, with the bottom vertex labeled \( v_{1, \sigma, k}^0 \) and the top vertex labeled \( v_{1, \sigma, k}^1 \).

The tree-like substructure of \( G_e \) consists of a triangular base with a vertex labeled \( t_0 \), and branches consisting of linked cycles. We say that a cycle \( C(e, \sigma, k) \) is exact if \( k \) is the least integer such that 1) \( \eta_e(\tau) \downarrow \) by stage \( k \) for every \( \tau \) which is an initial subsequence of \( \sigma \) or has a code less than \( \sigma \), and 2) \( \eta_e(\sigma) = 1 \). (Here \( \eta_e \) is the function defined in §1.) Edges are added to \( G_e \) by the following two rules. Connect \( v_{e, \sigma, k}^0 \) to \( t_0 \) if and only if \( C(e, \sigma, k) \) is an exact cycle and \( \sigma \) is a sequence of length 1. Connect \( v_{e, \sigma, k}^1 \) to \( v_{e, \tau, j}^0 \) if and only if \( C(e, \sigma, k) \) and \( C(e, \tau, j) \) are exact cycles and \( \tau = \sigma * \langle m \rangle \) for some \( m \in \omega \). Cycles which are not exact are spurious; they are included in \( G_e \), but are never connected to the tree-like substructure.

Let \( G \) be the union of all the \( G_e \)'s. \( G \) is recursive, since the rules for adding edges involve only bounded computations. Furthermore, if \( e \) is the code of a recursive tree \( T \), then the tree-like substructure of \( G_e \) can be mapped into \( T \) by identifying exact cycles with corresponding nodes. Viewing the cycles as nodes, the substructure is well founded if and only if \( T \) is a well founded tree. If \( e \) is not the code of a recursive tree, \( \eta_e \) is not total, and the tree-like substructure of \( G_e \) is finite.

Let \( \mathcal{G} \) be the set of indices of recursive graphs that are isomorphic to a subgraph of \( G \). Since \( x \in \mathcal{G} \) if and only if an isomorphism exists, it is easy to see that \( \mathcal{G} \) is \( \Sigma_1^1 \) definable. To prove that \( \mathcal{G} \) is \( \Sigma_1^1 \) complete, we will show that \( \mathcal{T} \leq_1 \mathcal{G} \), where \( \mathcal{T} \) denotes the set of indices of recursive trees which are not well founded. With each \( e \in \omega \), we associate a recursive graph \( H_e \) consisting of a countable linear graph with each node replaced by a \( 2(e + 1) + 2 \) cycle and with a triangle attached at one end. More precisely, \( H_e \) contains a triangle with one vertex labeled \( t_0 \), and (copies of) the cycles \( C(e, \langle 0 \rangle, k) \) for each \( k \in \omega \). To the edges already specified, we add the edge \( (t_0, v_{e, \langle 0 \rangle, k}^0) \) and the edges \( (v_{e, \langle 0 \rangle, k}^1, v_{e, \langle 0 \rangle, k+1}^0) \) for each
By the s-m-n Theorem, there is a recursive 1-1 function $f$ such that for every $e$, $f(e)$ is an index for $H_e$.

If $e \in \mathcal{T}$, then $e$ is the index of a recursive tree containing an infinite path $P$. In this case, $H_e$ is isomorphic to the subgraph of $G_e$ consisting of the base triangle and a copy of $P$ with nodes replaced by cycles. Thus $f(e) \in \mathcal{G}$.

Now suppose that $e \notin \mathcal{T}$. Note that because the size of the cycles varies with $e$, if $H_e$ is isomorphic to a subgraph of $G$, then $H_e$ is isomorphic to a subgraph of $G_e$. Since $e \notin \mathcal{T}$, $G_e$ consists of disconnected cycles and a well founded tree-like substructure. If $H_e$ is isomorphic to a subgraph of $G_e$, then the tree-like substructure of $G_e$ contains an infinite path, yielding a contradiction. Thus $f(e) \notin \mathcal{G}$ completing the proof that $\mathcal{T} \leq_1 \mathcal{G}$. □

Using the reverse mathematics framework, the preceding three theorems can be lumped together into a single equivalence result.

**Theorem 10 (RCA$_0$).** The following are equivalent:

1. $\Pi^1_1$–CA$_0$.

2. For any sequence of ordered pairs of graphs, $\langle (H_i, G_i) : i \in \omega \rangle$, there is a function $s : \omega \rightarrow 2$ such that $s(i) = 1$ if and only if $H_i$ is isomorphic to a subgraph of $G_i$.

3. For any graph $H$, and any sequence of graphs $\langle G_i : i \in \omega \rangle$, there is a function $s : \omega \rightarrow 2$ such that $s(i) = 1$ if and only if $H$ is isomorphic to a subgraph of $G_i$.

4. For any graph $G$, and any sequence of graphs $\langle H_i : i \in \omega \rangle$, there is a function $s : \omega \rightarrow 2$ such that $s(i) = 1$ if and only if $H_i$ is isomorphic to a subgraph of $G$.

**Proof.** To prove that (1) implies (2), (3), or (4), it suffices to note that the function $s$ is $\Sigma^1_1$ definable in the appropriate sequence of graphs. Since (3) is a special case of (2), we need only show that (3)$\rightarrow$(1) and (4)$\rightarrow$(1) to complete the proof. As in the proof of Theorem 6, we will determine which members of a sequence of trees are well founded. For the remainder of the proof, assume RCA$_0$ and let $\langle T_i : i \in \omega \rangle$ be a sequence of trees.

To prove that (3)$\rightarrow$(1), we use a simplified version of the construction in the proof of
Theorem 8. As in that proof, let $H$ be a countable linear graph with a triangle attached to one end. For each $n \in \omega$, let $G_n$ be a copy of $T_n$, with a triangle attached to the root. The graph $H$ and the sequence $\langle G_i : i \in \omega \rangle$ are $\Delta^0_1$ definable in $\langle T_i : i \in \omega \rangle$, so $\text{RCA}_0$ proves that they exist. Let $s$ be as in (3). Then $s(i) = 1$ if and only if $H$ is isomorphic to a subgraph of $G_i$, which occurs if and only if $T_i$ has an infinite path. Thus (3) implies $\Pi^1_1 - \text{CA}_0$.

To prove that (4)$\rightarrow$(1), we use a simplified version of the proof of Theorem 9. As in that proof, let $H_n$ consist of a linear graph with each node replaced by a $2(n+1)+2$ cycle, and with a triangle attached to one end. The graph $G$ consists of subgraphs $G_n$ for each $n \in \omega$, where each $G_n$ is a copy of $T_n$ with non-base nodes replaced by $2(n+1)+2$ cycles, and a triangle attached to the base node. The graph $G$ and the sequence $\langle H_i : i \in \omega \rangle$ are $\Delta^0_1$ definable in $\langle T_i : i \in \omega \rangle$, so $\text{RCA}_0$ proves that they exist. If $s$ is as in (4), then $s(i) = 1$ if and only if $H_i$ is isomorphic to a subgraph of $G$, which occurs if and only if $H_i$ is isomorphic to a subgraph of $G_i$. Finally, $H_i$ is isomorphic to a subgraph of $G_i$ if and only if $T_i$ is not well founded, so (4) implies $\Pi^1_1 - \text{CA}_0$, completing the proof. □

Although infinite analogs are useful for studying restricted classes of problems, the preceding examples indicate that, in a general setting, their behavior does not necessarily parallel that of the associated finite problems. However, examination of results in finite complexity can provide motivation for appealing results in recursion theory and reverse mathematics.

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