Kerr-Schild Symmetries

Bartolomé Coll†, Sergi R. Hildebrandt‡, and José M. M. Senovilla∗§

†Systèmes de Référence Spatio-temporels, Observatoire de Paris-CNRS UMR 8630, 61, avenue de l’Observatoire, Paris F-75014 France
‡Institut d’Estudis Espacials de Catalunya, IEEC/CSIC, Edifici Nexus 201, Gran Capità 2-4, 08034 Barcelona, Spain
§Departamento de Física Teórica, Facultad de Ciencias, Universidad del País Vasco, Apartado 644, 48080, Bilbao, Spain.

Abstract

We study continuous groups of generalized Kerr-Schild transformations and the vector fields that generate them in any \(n\)-dimensional manifold with a Lorentzian metric. We prove that all these vector fields can be intrinsically characterized and that they constitute a Lie algebra if the null deformation direction is fixed. The properties of these Lie algebras are briefly analyzed and we show that they are generically finite-dimensional but that they may have infinite dimension in some relevant situations. The most general vector fields of the above type are explicitly constructed for the following cases: any two-dimensional metric, the general spherically symmetric metric and deformation direction, and the flat metric with parallel or cylindrical deformation directions.

1 Introduction

The classical Kerr-Schild Ansatz \([1]\), in which one considers metrics of the form \(\tilde{g} = \eta + 2H\ell \otimes \ell\), where \(\eta\) is the Minkowski metric and \(\ell\) is a null 1-form, was very successful in finding exact solutions of the vacuum Einstein field equations, and Kerr-Schild type of metrics had been studied before with other aims \([2]\). As is well known, the celebrated Kerr metric was in fact originally presented in its Kerr-Schild form \([3]\), and the general Kerr-Schild vacuum solution was explicitly found \([1, 4]\). The Ansatz was also successfully applied to the Einstein-Maxwell equations \([4, 5]\) and to the case of null radiation \([6]\). The Kerr-Schild metrics were also analyzed on theoretical grounds, see for instance \([7]\), and a review with the main results can be found in \([8]\).

The Kerr-Schild Ansatz was soon generalized to the case in which the base metric is not flat \([9, 10, 11, 12]\). Thus, two metrics \(\tilde{g}\) and \(g\) are linked by a generalized Kerr-Schild relation if there exist a function \(H\) and a null 1-form \(\ell\) such that

\[
\tilde{g} = g + 2H\ell \otimes \ell.
\]

A possible physical interpretation of this relation has been recently put forward in \([13]\). Again, many exact solutions to Einstein’s field equations have been found by using the generalized Kerr-Schild Ansatz. Several examples are given in \([14]\) for vacuum and Einstein-Maxwell and in \([15]\) for perfect fluids. The general vacuum to vacuum generalized Kerr-Schild metric was also solved in \([16]\).

In this paper, we take a point of view which seems to have not been adopted hitherto, namely, that the above formula is the deformation that a transformation of the spacetime produces on the metric,
and we will simply use the term *Kerr-Schild transformation*. In this sense, Kerr-Schild transformations are on the same footing as isometries (which leave the metric invariant, \( \tilde{g} = g \)), or conformal transformations (\( \tilde{g} = \Psi g \)). As in the latter cases, in many situations the interesting point is not the existence of a discrete transformation, but the existence of a continuous group of such transformations admitted by the given metric. This is our aim, so that we shall consider continuous groups of Kerr-Schild transformations \([17]\), or *Kerr-Schild groups*. As in the case of Killing or conformal vector fields which generate the afore-mentioned classical transformations, we shall show that such groups are generated by what we call *Kerr-Schild vector fields* \( \xi \), solutions to the equations

\[
\mathcal{L}_\xi g = 2h \ell \otimes \ell, \quad \mathcal{L}_\xi \ell = m \ell
\]

and that they form a Lie algebra. However, the Kerr-Schild groups are associated to the metric structure of the spacetime \( g \) as well as to a field of null directions \( \ell \). Among other implications of this fact, we shall prove that the Lie algebra of Kerr-Schild vector fields can be of infinite dimension \([18]\).

The paper is organized as follows. In Section 2 Kerr-Schild vector fields are introduced and the basic definitions are given. Section 3 is devoted to the general properties of such fields. We provide the general equations that they must satisfy independently of the null direction \( \ell \), and make some considerations about the structure of the set of all such vector fields in a given spacetime. In particular, we also show that they form a Lie algebra for each direction \( \ell \), and that these Lie algebras can be infinite dimensional. In Section 4 we present several explicit examples: a) we find the general solution for the case of an arbitrary 2-dimensional metric, which depends on 4 arbitrary functions, two for each possible null direction; b) the solution for the case of a parallel null direction in flat \( n \)-dimensional spacetime is given, and shown to depend on \( n \) arbitrary functions of one variable and on \((n - 2)(n - 3)/2\) arbitrary constants; c) the general solution for the case in which the metric as well as the deformation direction are spherically symmetric is explicitly found. Several cases appear and some well-known metrics arise naturally; and d) the case of a cylindrical deformation direction in flat spacetime is analyzed, with some surprising results about the local character of the solutions. Finally, the last Section contains some conclusions and the possible lines for additional work.

## 2 Kerr-Schild vector fields

Let \((V_n, g)\) be an \( n \)-dimensional manifold with a metric \( g \) of Lorentzian signature \((-,+,. . . ,+\)). Indices in \( V_n \) run from 0 to \( n - 1 \) and are denoted by Greek small letters. The tensor and exterior products are denoted by \( \otimes \) and \( \wedge \), respectively, boldface letters are used for 1-forms and arrowed symbols for vectors, and the exterior differential is denoted by \( d \). The pullback of any application \( \phi \) is \( \phi^* \) and the Lie derivative with respect to the vector field \( \xi \) is written as \( \mathcal{L}_\xi \). Equalities by definition are denoted by \( \equiv \), and the end of a proof is signalled by \( \blacksquare \).

**Definition 1 (Kerr-Schild group)** A one-parameter group of transformations \( \{\phi_s\} \) of \( V_n \), \( s \in \mathbb{R} \), is called a *Kerr-Schild group* if the transformed metric is of the form

\[
\phi_s^*(g) = g + 2H_s \ell \otimes \ell,
\]

where \( \ell \) is a null 1-form field and \( H_s \) are functions over \( V_n \).

It is easily seen that the group structure \( (\phi_s \phi_r = \phi_{s+r}) \) requires a transformation law of the form

\[
\phi_s^*(\ell) = M_s \ell
\]

where \( M_s \) is a function over \( V_n \) for each \( s \).

Let us denote by \( \xi \) the infinitesimal generator of such a group. By writing \( h \equiv dH_s/ds|_{s=0} \), \( m \equiv dM_s/ds|_{s=0} \), a standard calculation leads to
Proposition 1 (Kerr-Schild equations) The generator \( \vec{\xi} \) of a Kerr-Schild group satisfies the equations

\[
\mathcal{L}_{\vec{\xi}} g = 2h \ell \otimes \ell, \\
\mathcal{L}_{\vec{\xi}} \ell = m \ell
\]

where \( h \) and \( m \) are two functions over \( V_n \).

Note that the set of equations (2) is nothing but the guarantee that the form of (1) is stable under Lie derivatives of arbitrary order \( p \), that is, \( \mathcal{L}^p_{\vec{\xi}} g = 2h \otimes \ell \otimes \ell \), where \( h[p] = \mathcal{L}^{p-1}_{\vec{\xi}} h + 2mh[p-1] \).

The usual results on differential equations ensure that, conversely, any \( \vec{\xi} \) satisfying (1-2) generates a Kerr-Schild group which is generically local, so that it will define a local Kerr-Schild group of local transformations.

It is convenient to know the contravariant version of equations (1-2),

\[
\mathcal{L}_{\vec{\xi}} g^{-1} = -2h \mathring{\ell} \otimes \mathring{\ell}, \\
\mathcal{L}_{\vec{\xi}} \mathring{\ell} = [\mathring{\xi}, \mathring{\ell}] = m \mathring{\ell},
\]

and also their expressions with index notation

\[
\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 2h \ell_\alpha \ell_\beta, \\
\xi_\rho \nabla_\rho \ell_\alpha + \ell_\rho \nabla_\alpha \xi_\rho = m \ell_\alpha.
\]

As \( \ell \) is null, the function \( h \) is not an invariant of the tensor \( 2h \ell \otimes \ell \), which can be equally characterized by any other pair \( h' = A^{-2}h, \ell' = A\ell \), where \( A \) is a non-vanishing \( C^\infty \) function, so that equations (2) become \( L_{\mathring{\xi}} \mathring{\ell}' = m' \mathring{\ell}' \), with \( m' = m + L_{\mathring{\xi}} \log |A| \).

It is worth to remark that, in contrast with the classical isometries or conformal transformations, the Kerr-Schild groups take into account the metric deformation \( \mathcal{L}_{\vec{\xi}} g \) with regard to a given null direction \( \ell \). In this sense, and in order to be precise, we give the following

Definition 2 (Kerr-Schild vector fields) Any solution \( \vec{\xi} \) of the Kerr-Schild equations (1-2) will be called a Kerr-Schild vector field (KSVF) with respect to the deformation direction \( \ell \). The functions \( h \) and \( m \) are the gauges of the metric \( g \) and of the deformation \( \ell \), respectively.

Obviously, any Killing vector field which leaves invariant the deformation direction \( \ell \) is also a Kerr-Schild vector field with \( h = 0 \). Thus, as is usual in similar contexts, we define

Definition 3 (Proper Kerr-Schild vector fields) A non-zero Kerr-Schild vector field \( \vec{\xi} \) will be called proper if its metric deformation \( \mathcal{L}_{\vec{\xi}} g \) is non-vanishing.

In other words, \( \vec{\xi} \neq 0 \) is a proper KSVF if the corresponding metric gauge is non-zero, \( h \neq 0 \). The zero vector \( \vec{\xi} = 0 \) is also considered to be a proper KSVF for any deformation direction.

3 General properties of Kerr-Schild vector fields

A first, straightforward, property of Kerr-Schild vector fields is

Proposition 2 Two metrics related by a Kerr-Schild transformation, \( \tilde{g} = g + 2H\ell \otimes \ell \), admit the same KSVFs with respect to \( \ell \).

Corollary 1 Every KSVF \( \vec{\xi} \) of a metric \( g \) is a Killing vector field of a Kerr-Schild transformed metric \( \tilde{g} \) of \( g \).
Now, if the Kerr-Schild equations (1-2) hold, then it is very simple to check (3-4). Conversely, assume such that a proper KSVF, so that equations (3-4) hold and we can set $\ell$ and $t$.

**Proof:** For any related Kerr-Schild metric $\tilde{g}$, one has $\mathcal{L}_{\xi} \tilde{g} = 2\tilde{h} \ell \otimes \ell$ with $\tilde{h} = h + \mathcal{L}_\xi H + 2mH$, and thus the equation $\tilde{h} = 0$ admits local solutions in the unknown $H$. \hfill \blacksquare

Notice that this does not mean in general that the set of KSVFs is the isometry algebra of some Kerr-Schild related metric, because the solutions $H$ are in general different for each $\xi$. The conditions under which there exists a common solution $H$ for all $\xi$, that is to say, a new Kerr-Schild metric for which all KSVFs are Killing fields, will be given elsewhere.

The natural question arises whether or not the set of all KSVFs for a metric, regardless of their deformation directions, can be characterized in some sense. This is answered in the following

**Theorem 1** A vector field $\xi$ in $(V_n, g)$ is a KSVF for some deformation direction if and only if

\begin{align}
\mathcal{L}_\xi g \times \mathcal{L}_\xi g &= 0, \\
(\mathcal{L}_\xi \mathcal{L}_\xi g) \wedge \mathcal{L}_\xi g &= 0.
\end{align}

**Proof:** We are using the notation $(t \times T)_{\mu\nu} = t_{\mu\rho}T_{\rho\nu}$ for the inner product of any two rank-2 tensors $t$ and $T$. On the other hand, the second equation (4) simply means that there exists some function $\Psi$ such that

$$\mathcal{L}_\xi \mathcal{L}_\xi g = \Psi \mathcal{L}_\xi g.$$  

Now, if the Kerr-Schild equations (1-2) hold, then it is very simple to check (3-4). Conversely, assume that (3) are satisfied. As is known (see e.g. (5)), any 2-index symmetric tensor $t$ with the property $t \times t = 0$ must have the form $t = 2h \ell \otimes \ell$ for some null 1-form $\ell$ and some function $h$, so that from (3) it follows (1) at once. Using this, the equation (5) readily gives

$$h \left( \mathcal{L}_\xi \ell \otimes \ell + \ell \otimes \mathcal{L}_\xi \ell \right) = \left( h\mathcal{L}_\xi - \mathcal{L}_\xi h \right) \ell \otimes \ell$$

which leads to (2) if $h \neq 0$ or is empty if $h = 0$. \hfill \blacksquare

**Corollary 2** If $\xi$ is a proper KSVF of the metric $g$, then its deformation direction can be explicitly constructed as

$$\ell' \equiv \left( (\mathcal{L}_\xi g)(\bar{u}, \bar{u}) \right)^{-1/2} i(\bar{u})\mathcal{L}_\xi g$$

where $\bar{u}$ is an arbitrary timelike vector.

**Proof:** By $i(\bar{u})T$ we mean the usual contraction $(i\bar{u})T)_{\mu} = u^{\rho}T_{\rho\mu}$. To prove (3), assume that $\xi$ is a proper KSVF, so that equations (3) hold and we can set $\mathcal{L}_\xi g = 2\epsilon \ell' \otimes \ell'$ with $\epsilon = \pm 1$. Then, by contracting with $\bar{u}$ once and twice, the expression (6) follows.

Theorem 1 allows to define a well-posed initial-value problem for KSVFs by considering equations (4) (or equivalently (3)) as the evolution system and the remaining set (3) as the constraint equations for the initial data set. In fact, we have

**Corollary 3** The system of equations for the general KSVFs of $g$ is involutive: if the equations (3) are satisfied in an open set $\Omega \subseteq V_n$ and the equations (3) hold on a hypersurface $\Sigma \subseteq \Omega$ non-tangent to $\xi$, then (3) are satisfied all over $\Omega$.

**Proof:** Let us see the evolution of the constraint equations (3) under the action of $\mathcal{L}_\xi$. Note that

$$\mathcal{L}_\xi (\mathcal{L}_\xi g \times \mathcal{L}_\xi g) = (\mathcal{L}_\xi \mathcal{L}_\xi g) \times \mathcal{L}_\xi g + \mathcal{L}_\xi g \times (\mathcal{L}_\xi \mathcal{L}_\xi g) - \mathcal{L}_\xi g \times \mathcal{L}_\xi g \times \mathcal{L}_\xi g$$

so that using (3) it follows

$$\mathcal{L}_\xi (\mathcal{L}_\xi g \times \mathcal{L}_\xi g) = 2\Psi (\mathcal{L}_\xi g \times \mathcal{L}_\xi g) - \mathcal{L}_\xi g \times (\mathcal{L}_\xi g \times \mathcal{L}_\xi g)$$
which proves the assertion, because this is a first order ODE for the constraint and its unique solution with zero initial condition vanishes.

It is interesting to remark that many other well-known sets of equations are also constraints for (3), such as the cases of Killing or conformal vector fields. In this sense, the KSVFs satisfy a set of evolution equations which is common to Killing or conformal vectors, and they differ from each other in the constraints for the initial data. Furthermore, systems of the type

\[(\mathcal{L}_\xi)\psi = \Psi(\mathcal{L}_\xi)\psi\]

were considered some years ago by Papadopoulos [20], so that one could say that the proper KSVFs of a metric are of Papadopoulos type with \(p = 2\) constrained to satisfy the conditions (3).

As is well-known, the Killing or the conformal vector fields provide constants of motion along geodesic curves (null geodesics for the conformal fields). As we are going to prove now, this also holds in an appropriate sense for the KSVFs. Let us remind first that a differentiable curve \(\gamma\) is called a subgeodesic with respect to the vector field \(\vec{p}\) (see e.g. [21]) if its tangent vector \(\vec{v}\) satisfies \(\nabla_{\vec{v}} \vec{v} = av + \lambda \vec{p}\) for some \(a\) and \(\lambda\). As with the case of geodesics curves, one can always choose an affine parametrization along \(\gamma\) such that one can set \(a = 0\). For our purposes, only a subset of the subgeodesics are of interest.

**Definition 4 (\(\ell\)-parametrized subgeodesics)** Any subgeodesic with \(\lambda = (\ell_{\mu} v^\mu)^2\) and an affine parametrization will be called an affinely \(\ell\)-parametrized subgeodesic.

Let us remark that this definition is given for arbitrary \(\vec{p}\), and only the scalar \(\lambda\) is restricted. The affinely \(\ell\)-parametrized subgeodesics with tangent vector \(\vec{v}\) are the solutions to the ordinary differential equations

\[\frac{dv^\mu}{d\tau} + \left(\Gamma_{\nu\rho}^\mu + \rho^\mu \ell_{\nu} \ell_{\rho}\right) v^\nu v^\rho = 0\]

which only need as initial conditions the value of \(\vec{v}\) at any given point. Then, a typical calculation of \(\nabla_{\vec{v}}(\vec{\xi} \cdot \vec{v})\) leads to

**Proposition 3** Let \(\vec{v}\) be the tangent vector of an affinely \(\ell\)-parametrized subgeodesic \(\gamma\) and \(\vec{\xi}\) a KSVF with respect to \(\ell\) and metric gauge \(h\). Then, \(\vec{\xi} \cdot \vec{v}\) is constant along \(\gamma\) whenever \(\vec{\xi} \cdot \vec{p} + h = 0\). ■

Let us remark that the condition \(\vec{\xi} \cdot \vec{p} + h = 0\) is very weak in the sense that it is not very restrictive. For instance, for any proper KSVF and any field of directions \(\vec{P}\) non-orthogonal to \(\vec{\xi}\) the above condition simply fixes the appropriate factor which must multiply \(\vec{P}\) to define the subgeodesics with respect to that direction. In other words, by simply choosing \(\vec{p} = -h \vec{P}/(\vec{\xi} \cdot \vec{P})\) the condition holds.

Nevertheless, there are important differences between the classical Killing or conformal vector fields and the KSVFs. To start with, the set \(\mathcal{K}\) of all KSVFs for a given metric does not have the structure of a vector space, as is obvious from the non-linear character of the relation (3) or directly from the Kerr-Schild equations if several deformations directions are taken into account. However, one can define \(\mathcal{K}_\ell\) as the set of all KSVFs with regard to \(\ell\). Obviously, \(\mathcal{K}_\ell = \mathcal{K}_{\ell'}\) for any other \(\ell' = A\ell\), so that the sensible thing to do is to consider \(\mathcal{K}_\ell\) only for the direction defined by \(\ell\) or \(\ell'\). To that end, let us denote by \(\mathcal{C}_\ell\) the null congruence of integral curves of \(\ell\), (so that \(\mathcal{C}_\ell \equiv \mathcal{C}_{\ell'}\).) Then, the set \(\mathcal{K}\) can be written as the union

\[\mathcal{K} = \bigcup_{\mathcal{C}_\ell} \mathcal{K}_\ell.\]

The interesting point here is that each of the \(\mathcal{K}_\ell\) is a vector space and, in fact, one has the following result.

**Proposition 4** The set \(\mathcal{K}_\ell\) of solutions \(\vec{\xi}\) to the equations (4) form a Lie algebra, hereafter called the Kerr-Schild algebra with respect to \(\mathcal{C}_\ell\).
Proof: By construction, the set $\mathcal{K}_\ell$ has an evident vector space structure because if $\vec{\zeta}$ and $\vec{\zeta}'$ are any two KSVFs with regard to the same $\ell$, then any linear combination with constant coefficients $c_1\vec{\zeta} + c_2\vec{\zeta}'$ is also a KSVF with regard to the same deformation direction. Let us denote by $h_{\vec{\zeta}}, m_{\vec{\zeta}}$ and $h_{\vec{\zeta}'}, m_{\vec{\zeta}'}$ the gauge functions associated to $\vec{\zeta}$ and $\vec{\zeta}'$, respectively. The identity

$$L_{[\vec{\zeta},\vec{\zeta}]} = L_{\vec{\zeta}} L_{\vec{\zeta}} - L_{\vec{\zeta}'} L_{\vec{\zeta}'}$$

applicable to any tensor field, immediately leads to

$$L_{[\vec{\zeta},\vec{\zeta}]} g = 2h_{[\vec{\zeta},\vec{\zeta}]} \ell \otimes \ell,$$

with

$$h_{[\vec{\zeta},\vec{\zeta}]} = L_{\vec{\zeta}} h_{\vec{\zeta}} - L_{\vec{\zeta}'} h_{\vec{\zeta}'} + 2(h_{\vec{\zeta}'} m_{\vec{\zeta}} - h_{\vec{\zeta}} m_{\vec{\zeta}'})$$

$$m_{[\vec{\zeta},\vec{\zeta}]} = L_{\vec{\zeta}} m_{\vec{\zeta}} - L_{\vec{\zeta}'} m_{\vec{\zeta}'}$$

where the Kerr-Schild equations (1-2) for $\vec{\zeta}$ and $\vec{\zeta}'$ have been used.

It is interesting to observe that the equations (2), which were necessary to ensure the local group property in the one-parameter case, are also sufficient to produce the Lie algebra structure in the multidimensional case.

Despite the above, the set $\mathcal{K}$ is not the direct sum of the $\mathcal{K}_\ell$’s for all $\mathcal{C}_\ell$ because one has $\mathcal{K}_\ell \cap \mathcal{K}_k \neq \{\vec{0}\}$ in general. Still, $\mathcal{K}$ can be expressed as a simple direct sum sometimes as follows from the following results.

Lemma 3.1 If a KSVF belongs to two different Kerr-Schild algebras, then it is a Killing vector field.

Proof: If $\vec{\zeta} \in \mathcal{K}_\ell \cap \mathcal{K}_k$ with $\mathcal{C}_\ell \neq \mathcal{C}_k$, then $L_{\vec{\zeta}} g = 2h_{\ell} \ell \otimes \ell = 2f_k \otimes k$ with $k \wedge \ell \neq 0$, from where $h = f = 0$.

Proposition 5 The set $\mathcal{K}$ is the following disjoint union

$$\mathcal{K} = \bigsqcup_{\mathcal{C}_\ell} \hat{\mathcal{K}}_\ell \sqcup \mathcal{K}il$$

where $\mathcal{K}il$ is the Lie algebra of Killing vector fields in $(V_n, g)$ and each $\hat{\mathcal{K}}_\ell$ is the subset of $\mathcal{K}_\ell$ formed by the proper KSVFs with regard to $\mathcal{C}_\ell$. Thus, if there are no Killing vectors in the spacetime, the set $\mathcal{K}$ is the direct sum of Lie algebras

$$\mathcal{K}il = \{\vec{0}\} \implies \mathcal{K} = \bigoplus_{\mathcal{C}_\ell} \hat{\mathcal{K}}_\ell = \bigoplus_{\mathcal{C}_\ell} \mathcal{K}_\ell.$$  

Proof: The proof is immediate from the above Lemma, because $\mathcal{K}il \cap \hat{\mathcal{K}}_\ell = \{\vec{0}\}$ for all $\mathcal{C}_\ell$.

The above result does not say that we have a direct sum of finite-dimensional Lie algebras. Of course, we know that $\mathcal{K}il$ is always of finite dimension, but we do not know yet about the other algebras $\mathcal{K}_\ell$. As we are going to prove now, they are generically finite-dimensional, but there are some special degenerate cases in which some of them are infinite-dimensional. First, we need a Lemma identifying the cases when a KSVF leaves invariant every single integral curve of $\mathcal{C}_\ell$.

Lemma 3.2 There are proper KSVFs $\vec{\xi}$ tangent to its deformation direction $\ell$, $\xi \wedge \ell = 0$, if and only if $\ell$ is geodesic, shear-free, expansion-free, and the 1-form $\alpha$ appearing in

$$L_{\vec{\xi}} g = a \otimes \ell + \ell \otimes a = \mathcal{L}_{\ell} g = a$$

has the form

$$\alpha = -d \log |\mu| + \frac{\ell}{\mu}$$

for some functions $\mu, h$. 

Proof: If $\vec{\xi} = \mu \vec{\ell}$, the invariance of $\mathcal{C}_\ell$ is trivial, so that to prove the Lemma only equations (1) must be checked. They readily lead to (2), with $a$ given in (3). Equations (2) can be rewritten as
\[
\nabla_\nu \ell_\mu + \nabla_\mu \ell_\nu = \ell_\mu a_\nu + \ell_\nu a_\mu
\]
which characterize the geodesic, shear-free and expansion-free null 1-forms $\vec{\ell}$, see e.g. [8].

From the well-known Goldberg-Sachs theorem and its generalizations (see e.g. [8]), the existence of geodesic and shear-free null congruences is severely restricted in arbitrary spacetimes, so that the possibility above is rather exceptional. Nevertheless, these exceptions are of great interest, as they include many of the simpler and/or physically relevant spacetimes, see the next section.

Now we can prove the infinite-dimensional character of some of the Lie algebras $K_\ell$.

**Theorem 2** Two vector fields $\vec{\xi}$ and $\rho \vec{\xi}$ with $d\rho \neq 0$ are KSVFs with respect to the same deformation direction $\vec{\ell}$ if and only if $\vec{\xi} \wedge \vec{\ell} = 0$ and $\ell$ is integrable $\ell \wedge d\ell = 0$ satisfying (7-8). Then the functions $\rho$ are those of the ring generating $\ell$, that is to say, such that $\ell \wedge d\rho = 0$.

**Proof:** The Kerr-Schild equations (1) for both $\vec{\xi}$ and $\rho \vec{\xi}$ imply that $\vec{\xi} = \mu \vec{\ell}$ and $d\rho = \sigma \vec{\ell} \neq 0$. The second of these conditions implies that the null $\vec{\ell}$ is irrotational (and therefore geodesic) with $\vec{\ell} \wedge d\rho = 0$, while the first one says that we are in the situation of Lemma 3.2, so that (7-8) hold. Conversely, if $\ell$ satisfies (7-8) then the vector field $\vec{\xi} = \mu \vec{\ell}$ is a KSVF with regard to $\ell$ and metric gauge $h$. As $\ell$ is also hypersurface-orthogonal we have $du \wedge \ell = 0$ for some non-vanishing function $u$. Then, for any function $\rho(u)$ we have
\[
\hat{\mathcal{L}}_{(\rho \vec{\xi})} g = \hat{\mathcal{L}}_{(\rho \mu \ell)} g \propto \ell \otimes \ell
\]
which proves the result. 

Notice that this result means that all the vector fields $\rho \mu \ell$ are KSVFs for arbitrary $\rho$, as long as $\rho$ is in the ring generating $\ell$. That is to say, these KSVFs depend on an arbitrary function $\rho(u)$, with $\ell \propto du$, and thus the corresponding Lie algebra $K_\ell$ has infinite dimension. Other infinite-dimensional algebras associated to a metric structure exist such as, for example, curvature collineations [22], but they are directly related to the partly antisymmetric Riemann tensor in degenerate cases, and not to the regular symmetric metric $g$, as in the present case. Explicit examples of infinite-dimensional Kerr-Schild algebras will be presented in the next section.

### 4 Explicit examples of Kerr-Schild vector fields

In this Section, explicit expressions for the KSVFs in several situations of relevance and interest are given. Some implications on the corresponding (generalized) Kerr-Schild related metrics are derived and briefly commented.

#### 4.1 General two-dimensional spacetime $(V_2, g)$

The most general line-element for $n = 2$ can be locally written as
\[
\begin{align*}
\ &ds^2 = 2e^f \, du \, dv, \quad f = f(u,v) \\
\end{align*}
\]
and there are only two inequivalent null directions given by $du$ and $dv$. From a theoretical point of view, it is enough to find the KSVFs associated with the deformation direction $\vec{\ell} = du$ (say), and then the solutions for the deformation direction $dv$ are analogous interchanging $u$ with $v$. We have

**Proposition 6** For any 2-dimensional metric, the most general KSVF with respect to $\vec{\ell} = du$ is given by
\[
\vec{\xi} = a(u) \frac{\partial}{\partial u} + B(u,v; f; a(u), b(u)) \frac{\partial}{\partial v}
\]
where \(a(u), b(u)\) are two arbitrary functions and \(B\) is the general solution of

\[
\frac{\partial}{\partial v}(e^f B) = -\frac{\partial}{\partial u}(e^f a).
\]

The deformation and metric gauges are then given by

\[
m = \dot{a}, \quad h = e^f \frac{\partial B}{\partial u}
\]

where a dot means derivative with respect to the argument.

Proof: From (2) one easily gets

\[
\mathcal{L}_\xi du = d(L_\xi u) = mdu
\]

which fixes the component of \(\xi\) along \(\partial/\partial u\) as an arbitrary function \(a(u)\) and gives the first equation of (11). Now, the remaining Kerr-Schild equations (1) are equivalent to

\[
\mathcal{L}_\xi (e^f dv) + \dot{a} e^f dv = hdu
\]

which can be rewritten as

\[
dB = e^{-f} \left( hdu - (\dot{a} + L_\xi f) dv \right).
\]

This is the desired result. Let us note that one only has to solve the part of the above equation giving the derivative \(\partial B/\partial v\), which depends on the arbitrary integrating function \(b(u)\), and then \(h\) is simply isolated as written in (11).

Thus, the solution in this case depends on two arbitrary functions of one variable \(u\). This is an explicit example in which the Kerr-Schild algebra has infinite dimension.

Similarly, one can derive the general solution for the other possible deformation direction \(dv\), getting

\[
\xi = A(u, v; f; c(v), d(v)) \frac{\partial}{\partial u} + d(v) \frac{\partial}{\partial v}
\]

where now \(c\) and \(d\) are arbitrary functions of \(v\), and the corresponding metric gauges are \(m = \dot{d}\) and \(h = e^f \partial A/\partial v\). These KSVFs are proper if and only if \(\partial A/\partial v \neq 0\), and analogously for (10), so that we have also obtained the following result.

**Corollary 4** The set \(K\) of all KSVFs of any two-dimensional spacetime \((V_2, g)\) can be written as the disjoint union

\[K = \hat{K}_{du} \sqcup \hat{K}_{dv} \sqcup Kil\]

where \(\hat{K}_{du}\) is the set of all vector fields of the form (11) with \(\partial B/\partial u \neq 0\), \(\hat{K}_{dv}\) is the set of all vector fields of the form (12) with \(\partial A/\partial v \neq 0\), and \(Kil\) is the Lie algebra of Killing vector fields.

Therefore, the set \(K\) can be completely and explicitly constructed for \(n = 2\) in general, and it depends on four arbitrary functions. This is the maximum freedom one can attain in two dimensions, so that the above Corollary suggests the validity of the following theorem, which can be certainly proven.

**Theorem 3** Any two-dimensional Lorentzian \(g\) is a Kerr-Schild transformed metric of the flat two-dimensional Minkowski metric.

Proof: Starting with the general metric \(g\) given by (9), one can set \(d\tilde{v} \equiv -Hdu + e^f dv\) for some \(H\) as long as the integrability conditions \(\partial H/\partial v = -e^f \partial f/\partial u\) hold. This has always solution for \(H\), so that we have

\[
ds^2 = 2dud\tilde{v} + 2Hdu^2
\]

which is the desired result as \(2dud\tilde{v}\) is obviously flat.

Obviously, the combination of this Theorem with Prop.2 allows to obtain the expressions (10,12) in a simple way.
Corollary 5 Any pair of 2-dimensional Lorentzian metrics are related by a Kerr-Schild transformation with respect to any of the two possible null deformation directions.

These nice simple results are analogous to the similar well-known ones for conformal transformations and conformal vector fields for $n=2$.

4.2 Flat n-dimensional spacetime with parallel deformation direction

Let us take flat $n$-dimensional Minkowski spacetime with Cartesian coordinates $\{x^\mu\}$, and let us pick up any covariantly constant null direction $\ell$. By adapting the coordinate system, we can always choose $\ell = du$ with $u \equiv (x^0 + x^1)/\sqrt{2}$. Let us define another null coordinate $v \equiv (x^1 - x^0)/\sqrt{2}$ so that the line-element becomes $\text{d}s^2 = 2 \text{d}u \text{d}v + \sum_i (\text{d}x^i)^2$, where Latin small indices will take values $i, j, \ldots = 2, \ldots, n-1$.

In order to solve the Kerr-Schild equations (1-2) we can use a method similar to that of the previous two-dimensional case. Thus, (2) immediately leads to $\pounds_{\vec{\xi}} u = a(u)$ with $m = \dot{a}$. The contravariant form of (2) partly restricts further the form of $\vec{\xi}$ and can be used before attacking the first group of Kerr-Schild equations (1). Notice that the part of equations coming from the 2-planes $\{u, v\}$ are similar to the 2-dimensional case of the previous subsection with $f = 0$, so that a part of the equations is already solved. Then, the remaining part can be easily integrated and we have the following result

Proposition 7 The KSVFs corresponding to a covariantly constant deformation direction $\ell = du$ in flat spacetime are of the form

$$\vec{\xi} = a(u) \frac{\partial}{\partial u} + \left[b(u) - \dot{a}(u)v - \dot{c}_i(u)x^i\right] \frac{\partial}{\partial v} + \left[c_i(u) + \epsilon_{ij}x^j\right] \frac{\partial}{\partial x^i},$$

where $a, b$ and $c_i$ are arbitrary functions of $u$, $\epsilon_{ij} = -\epsilon_{ji}$ are arbitrary constants, and the sum over repeated indices is to be understood. Their associated deformation and metric gauge functions are given by

$$m = \dot{a}, \quad h = \dot{b} - \dot{a}v - \dot{c}_i x^i.$$

Thus, this Kerr-Schild algebra is uniquely characterized by the generating set $\{a, b, c_i, \epsilon_{ij}\}$ formed by $n$ arbitrary functions of $u$ and $(n-2)(n-3)/2$ arbitrary constants. Given that we are in the case of maximum degeneracy, in the sense that the metric has zero curvature and the deformation direction has vanishing covariant derivative, it seems plausible that the above is the maximum freedom one can have for a single Kerr-Schild algebra in general dimension $n$.

A direct evaluation leads to its derived algebra structure

Proposition 8 The Lie bracket $[\vec{\xi}, \vec{\zeta}]$ of two KSVFs of type (13) characterized by the generating sets $\{a, b, c_i, \epsilon_{ij}\}$ and $\{\tilde{a}, \tilde{b}, \tilde{c}_i, \epsilon_{ij}\}$ respectively, is another KSVF with regard to $\ell = du$ whose corresponding generating set reads

$$\begin{align*}
\vec{\eta} &= \dot{\tilde{a}} - \tilde{a} \dot{a} \\
\vec{\gamma} &= (\tilde{a}b - \tilde{b} \dot{a}) + \dot{\tilde{c}}_i \dot{c}_i - c_i \dot{c}_i \\
\vec{c}_i &= \dot{\tilde{a}} \dot{c}_i - \tilde{a} \ddot{c}_i + \ddot{c}_i c_k - \epsilon_{ik} \ddot{c}_k \\
\vec{c}_{ij} &= \epsilon_{kj} \ddot{c}_k - \dot{c}_{ij} \epsilon_{ik}. \quad (14)
\end{align*}$$
Let us notice that the KSVFs (13) are Killing fields if \( h = 0 \), which gives a Lie algebra of dimension
\[
3 + 2(n - 2) + \frac{(n - 2)(n - 3)}{2} = 2 + \frac{n(n - 1)}{2}.
\]
A basis of this algebra is constituted by the \( n \) translations together with the \((n - 1) + (n - 2)(n - 3)/2\) rotations leaving invariant the \((n - 1)(n - 2)/2\) two-planes containing \( \ell \) and their orthogonal vectors.

Expressions (14) are useful to find special subalgebras. For instance, one directly sees that the subalgebra of KSVFs defined by \( \epsilon_{ij} = 0 \) is an ideal, so that our Kerr-Schild algebra is not simple. This particular ideal is formed by Killing vectors, but this is not a general property. In the present case it is simply due to the particular form of flat spacetime. This will be clear from the following results.

**Proposition 9** Any \( g \) which is a Kerr-Schild transformed metric of flat spacetime with respect to a Minkowskian covariantly constant null deformation direction \( \ell \) has the general solution (13) for the KSVFs with regard to \( \ell \).

**Proof:** This follows from Proposition 2, where the new metric gauge \( \tilde{h} \) is given by \( \tilde{h} = h + \xi \cdot H + 2mH \) (the other gauge function being invariant, \( \tilde{m} = m \)).

The spacetimes of the last Proposition have line-elements of type
\[
ds^2 = 2dudv + \sum (dx^i)^2 + 2H(u, v, x^k) du^2
\]
and therefore they do not admit Killing vector fields in general. Thus, all the vector fields included in expression (13) are proper KSVFs for the generic metric above. In the particular case with \( \partial H/\partial v = 0 \) we have the classical pp-waves metrics (in dimension \( n \)), see e.g. [8, 23]. These have a null Killing vector field along \( \ell \).

### 4.3 Spherically symmetric spacetime and deformation (\( n = 4 \))

Let us consider the most general spherically symmetric spacetime in the standard case of \( n = 4 \). As is well known, there are only two independent spherically symmetric null directions, which are usually called the radial null directions. The congruences they define are always hypersurface-orthogonal, and thus we can select two null coordinates \( u, v \) such that \( du \) and \( dv \) point along these two radial null directions. Completing the coordinate system with the angular variables \( \theta, \phi \), the most general line-element for such a spacetime can be written in the following simple form
\[
ds^2 = 2e^f dudv + r^2d\Omega^2, \quad f(u, v), \quad r(u, v)
\]
where the functions \( f \) and \( r \) are independent of the angular coordinates and \( d\Omega^2 \) is the line-element of the standard unit 2-sphere. The function \( r \) has a clear geometrical meaning: \( 4\pi r^2 \) is the area of the 2-dimensional spheres defined by constant values of \( u \) and \( v \), which are the orbits of the \( \text{SO}(3) \) group of motions. Concerning the function \( f \), it can be related to the invariantly defined mass function \( M(u, v) \) by means of
\[
M(u, v) = \frac{r}{2} \left( 1 - 2e^{-f} \frac{\partial r}{\partial u} \frac{\partial r}{\partial v} \right).
\]
The selected coordinate system is preferable to the standard Schwarzschild coordinates for two reasons: it is clearly adapted to the null deformation directions we are going to study; and it allows to study spacetimes with event or Cauchy horizons. Thus, for instance, the maximal Kruskal extension of Schwarzschild spacetime can be described with a single coordinate system of the above type.

We have restricted ourselves to the case \( n = 4 \) for the sake of simplicity and clarity, but it is evident that the analysis can be performed in general \( n \) by simply substituting the metric of the standard \((n - 2)\)-sphere for \( d\Omega^2 \) and \( \text{SO}(n - 1) \) for the isometry group.

In order to get the general solution for KSVFs with respect to a radial null deformation direction let us take, without loss of generality, \( \ell = du \). Then, as a first simple step we obtain the next result.
Lemma 4.1 The Killing vector fields of the group SO(3) for the general spherically symmetric space-time

\[ \vec{J} = c_1 \sin(\phi + \phi_0) \frac{\partial}{\partial \theta} + (c_1 \cot \theta \cos(\phi + \phi_0) + c_2) \frac{\partial}{\partial \phi} \]  

are KSVFs with respect to \( \ell = du \).

Proof: It is enough to see that the vectors \( \vec{J} \) leave \( C_\ell \) invariant, which is true because

\[ \mathcal{L}_J \ell = \mathcal{L}_J du = d(\mathcal{L}_u) = 0, \]

so that in fact we also have vanishing deformation gauge \( m = 0 \) for the KSVFs (16).

Therefore, the Lie algebra \( K_\ell \) has at least three free parameters \( c_1, c_2, \phi_0 \). In order to complete the algebra \( K_\ell \) we can proceed as follows. We notice that the \( \{u, v\} \)-part of the line-element (15) is identical to the general 2-dimensional metric of subsection 4.1. Thus, our general solution will be of the form (16) but restricted to satisfy the conditions coming from the angular part of the metric. This can be easily shown to lead to the intuitive condition

\[ \mathcal{L}_\xi r = 0 \]

which combined with (16) gives the remaining solutions for \( K_\ell \). There appear several cases depending on the specific form of \( r \). They are summarized as follows.

Proposition 10 The KSVFs of the most general spherically symmetric metric with respect to a radial deformation direction \( \ell = du \) are given by the Killing fields (16) together with the following:

1. If \( r = \text{const.} \), the vector fields of the form (16), with the deformation and metric gauges appearing in (17).
2. If \( \partial r/\partial v = 0 \) but \( r \) is not constant, the vector fields of the form

\[ \vec{\xi} = e^{-f} b(u) \frac{\partial}{\partial v} \]

where \( b(u) \) is an arbitrary function and the gauges are

\[ m = 0, \quad h = \dot{b} - b \frac{\partial f}{\partial u}. \]

3. If \( \partial r/\partial v \neq 0 \), and if

\[ e^f = \frac{F(r)}{q(u)} \frac{\partial r}{\partial v} \]

for some functions \( F(r) \) and \( q(u) \), the vector fields of type

\[ \vec{\xi} = q(u) \left( \frac{\partial}{\partial u} - \frac{\partial r/\partial u}{\partial r/\partial v} \frac{\partial}{\partial v} \right) \]

with the following gauges

\[ m = \dot{q}, \quad h = -\frac{F(r)}{q(u)} \frac{\partial r}{\partial v} \frac{\partial}{\partial u} \left( q(u) \frac{\partial r/\partial u}{\partial r/\partial v} \right). \]  

\[ (17) \]

\[^1\] Of course, a similar reasoning can be applied to any metric in which there is a well-defined 2-dimensional subpart, such as the cases of decomposable spacetimes, warped products, or the general metric with an isometry group acting on spacelike \((n-2)\)-orbits.
In case 1 the Lie algebra $K_\ell$ is generated by two arbitrary functions and three constants, and in case 2 by one arbitrary function and the three constants. Notice that in case 3, and despite what it may seem, the solution depends on just four constants, as the function $q(u)$ appears explicitly in the metric (actually, this function could be set to a constant locally). In this last case, the fourth KSVF is proper in general, but there are some important cases in which it is in fact a Killing vector field. To find them, from (17), it is necessary that

$$q(u)\frac{\partial r/\partial u}{\partial r/\partial v} = p(v)$$

where $p(v)$ is an arbitrary function of $v$. There are two cases now. If $p(v) = 0$, then $r = r(v)$ and the line element reads simply

$$ds^2 = 2\frac{\tilde{F}(v)}{q(u)}du dv + r^2(v)d\Omega^2$$

where $\tilde{F}(v) \equiv F(r(v))\dot{r}$. Notice that the $\{u,v\}$-part of the metric is flat, and the fourth KSVF is in fact a null Killing vector field given by $\xi = q(u)\partial_u$. The second possibility is defined by $p(v) \neq 0$. In this case, the function $r$ must have the form $r = r(P(v) + Q(u))$ where $\dot{P} = 1/p$ and $\dot{Q} = 1/q$ and the fourth KSVF which is a Killing reads

$$\xi = q(u)\partial_u - p(v)\partial_v.$$ 

These spacetimes can be characterized by the property that the mass function depends only on $r$: $M = M(r)$. Then, it is easily seen that the above KSVF is timelike or spacelike depending on the sign of $1 - 2M/r$, as was to be expected. This set of spacetimes includes all spherically symmetric metrics with a static region, such as Minkowski spacetime, Schwarzschild vacuum solution and its maximal Kruskal extension, Schwarzschild interior solution, all static spherically symmetric perfect fluids, Reissner-Nordström charged solution and its maximal extension, Einstein static universe, de Sitter spacetime, and many others.

As a simple but illustrative example of a physical case in which the fourth KSVF of case 3 is proper, we can take the Vaidya radiative solution [24], which is a Kerr-Schild transformed metric of flat spacetime, as is known [8, 10]. In our coordinates, the Vaidya solution is given by

$$F(r) = C = \text{const.}, \quad M = M(u), \quad \frac{\partial r}{\partial u} = \frac{1}{2q(u)} \left(1 - \frac{2M(u)}{r}\right)$$

where the mass $M(u)$ is an arbitrary function of $u$ (the Schwarzschild metric is contained as the case $\dot{M} = 0$.) The proper KSVF reads

$$\xi = q(u)\partial_u + e^{-f} \left(1 - \frac{2M(u)}{r}\right)\partial_v$$

and its metric gauge is

$$h = \frac{C}{q(u)} \frac{\dot{M}}{r}.$$ 

4.4 Flat spacetime with cylindrical deformation direction ($n = 4$)

In all previous cases, the congruence $C_\ell$ defined by the deformation direction $\ell$ was irrotational (and therefore geodesic) and shear-free. Now, we are going to present a simple case of a shearing deformation direction, given by a cylindrical null direction in Minkowski. Again, for the sake of simplicity we assume $n = 4$, but the results can be straightforwardly generalized to any $n$.

By using a classical cylindrical coordinate system $\{t, \rho, \varphi, z\}$, a cylindrical null direction in Minkowski spacetime is given by $\ell = d(t + \rho)$. We can select advanced and retarded null coordinates, $u = \ldots$
\((t + \rho)/\sqrt{2}\) and \(v = (t - \rho)/\sqrt{2}\), so that \(\ell\) writes now as \(\ell = du\). The Kerr-Schild equations for this null direction can also be explicitly integrated, their general solution being

\[
\vec{\xi} = (c_0 u + c_1)\partial_u + (2c_0 u - c_0 v + c_1)\partial_v + (c_0 \varphi + c_2)\partial_\varphi + c_3\partial_z,
\]

where \(c_\alpha\) are four arbitrary constants. For any value of them one has \(h = -2c_0\) and \(m = c_0\) for the gauges.

This case presents an interesting feature, unusual in General Relativity: the above general solution corresponds to proper KSVFs but, due to the presence of the term \(\varphi\partial_\varphi\), they are local vector fields (they would be bivalued after a complete revolution). Only when \(c_0 = 0\) they become global, but then they reduce to Killing vector fields with \(m = 0\). Denoting by \(\vec{\xi}_\alpha\) the KSVF obtained from the above general solution by setting the constants equal to zero except for the \(\alpha\)-th, we have

**Proposition 11** For a cylindrical Kerr-Schild deformation in flat spacetime, proper KSVFs are necessarily local, and form a four-dimensional Lie algebra, their non vanishing brackets being

\[
[\vec{\xi}_0, \vec{\xi}_1] = -c_0 \vec{\xi}_1, \quad [\vec{\xi}_0, \vec{\xi}_2] = -c_0 \vec{\xi}_2.
\]

Global KSVFs are necessarily Killing vector fields, and reduce to the static cylindrical symmetry which is Abelian (three-dimensional translation abstract group).

**5 Conclusion**

As we have seen, the notion of Kerr-Schild vector fields seems to be meaningful and, in fact, it leads to a structure richer than that of the classical Killing or conformal fields. As we have seen, one can define the set of all KSVFs in the spacetime and give the general equations for them, independently of the deformation direction \(\ell\). However, this set has not even the structure of a vector space in general. Nevertheless, the KSVFs with respect to \(\ell\) constitute a Lie algebra. These are generically finite-dimensional, even though they can be of infinite dimension in some particular cases which are of relevance. So far, one knows very little about the structure of these Kerr-Schild infinite algebras. We have shown that they are not simple, but a formal proof that they do not admit Abelian ideals is lacking, as well as the characterization of the possible Abelian subalgebras.

Many questions remain open. For instance, the necessary and sufficient integrability conditions of the Kerr-Schild equations, or the construction of a complete set of geometrical objects which are invariant under Kerr-Schild transformations, and under KSVFs. In this sense, we have proved the result that any two 2-dimensional metrics are Kerr-Schild transformed of each other, and of the flat metric, with respect to any of the two possible null deformation directions. This is analogous to the result that establishes the conformally flat character of any 2-dimensional metric. But the corresponding result for general dimension \(n\) is still open. On the other hand, and analogously to the case of Killing fields, which become conformal fields by a conformal transformation, we have seen that Killing fields leaving invariant a null deformation direction \(\ell\) become KSVFs by a Kerr-Schild transformation. However, we do not know the analogue for KSVFs of the Defrise-Carter theorem for conformal transformations, namely, how to control the number of KSVFs that may become Killing fields by a suitable Kerr-Schild transformation. Some of these open problems will be considered elsewhere.

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References

[1] Kerr, R. P., and Schild, A. (1965), in Proceedings of the Galileo Galilei Centenary Meeting on General Relativity, Problems of Energy and Gravitational Waves, G. Barbera, ed., Comitato Nazionale per le Manifestazione Celebrative, Florence, pp. 222-233.

[2] Trautman A. (1962), in Recent Developments on General Relativity, 459-463 (Pergamon Press, New York).

[3] Kerr, R.P. (1963), Phys. Rev. Lett. 11 237-238.

[4] Kerr, R.P., and Schild A. (1969), Proc. Symp. Appl. Math. 17 199; Debney, G.C., Kerr, R.P., and Schild A. (1969), J. Math. Phys. 10 1842-1854; Urbantke, H. (1972), Acta. Phys. Austr. 35 396; Debney, G.C. (1973), N. Cim. Lett. 8 337.

[5] Vaidya, P.C. (1972), Tensor N.S. 24 315-321; Debney, G.C. (1974), J. Math. Phys. 15 992-997; Bhatt, P.V., and Vaidya S.K. (1991), Class. Quantum Grav. 8 1717-1722.

[6] Vaidya, P.C. (1973) Tensor N.S. 27 276-280; Urbantke, H. (1975), Acta Phys. Austr. 41 1; Herlt, E. (1980) Gen. Rel. Grav. 12 1-7.

[7] Mas, L. (1969) C.R. Acad. Sci. Paris A 268 441-444; (1970), Ph. D. Thesis, Universitat Autònoma de Barcelona; Debney G.C. (1972), Tensor N.S. 24 227-230; Gürses, M. and Gürsey, F. (1975), J. Math. Phys. 16 2385-2390; Kerr, R.P., and Wilson, W.B. (1979), Gen. Rel. Grav. 10 273-281;

[8] Kramer, D., Stephani, H., Herlt, E., and MacCallum, M. A. H. (1980). Exact solutions of Einstein's Field Equations (Cambridge University Press, Cambridge).

[9] Thompson, A.H. (1966), Tensor N.S. 17 92-95.

[10] Taub, A. H. (1981), Ann. Phys. 134 326-372.

[11] Bilge, A.K., and Gürses, M. (1982), in “XI International Colloquium on Group Theoretical Methods in Physics”, M. Cerdaroglu and E. İnönü, (Springer Verlag, Istanbul), pp. 252-255; (1986), J. Math. Phys. 27 1819-1833.

[12] Nahmad-Achar, E. (1988), J. Math. Phys. 29 1879-1884.

[13] Coll, B., (1999), in Relativity and Gravitation in General. Proceedings of the Spanish Relativity Meeting in Honour of the 65th Birthday of L Bel, J. Martín, E. Ruiz, F. Atrio, and A. Molina, eds., pp. 91-98 (World Scientific, Singapore).

[14] Talbot, C.J. (1969), Commun. Math. Phys. 13 45-61; Mastronikola, K.E. (1987), Class. Quantum Grav. 4 L179-L184; Kupeli, A.H. (1988), Class. Quantum Grav. 5 401-408; Fels, M., and Held, A. (1989), Gen. Rel. Grav. 21 61-68.

[15] Martin, J., and Senovilla, J.M.M., (1986), J. Math. Phys. 27 265-270 (erratum p.2209); Senovilla, J.M.M., (1987), Class. Quantum Grav. 4 1449-1455; Martín-Pascual, F, and Senovilla J.M.M. (1988), J. Math. Phys. 29 937-944; Senovilla, J.M.M., and Sopuerta, C.F. (1994), Class. Quantum Grav. 11 2073-2083.

[16] Gergely, L.Á., and Perjés, Z. (1993), Phys. Lett. A 181 345-348; (1994), 35 2438-2447, 2448-2462.

[17] All our geometrical considerations being local, it has to be understood that transformations and groups of transformations are both local. In fact, without ad hoc global assumptions on the spacetime, one cannot ensure the existence of global transformations, and consequently, one can only infer the existence of possible pseudogroups on the whole manifold.
Riemannian and Symplectic manifolds show that strict or conformal invariance of symmetric and antisymmetric regular tensors is respectively finite and infinite dimensional. It is interesting to observe that the non-homogeneous character of the equations in the Kerr-Schild case is responsible for the possibility of having infinite dimensions, despite the symmetry and the regular character of the tensor $g$ involved.

Penrose, R., and Rindler, W. (1984) *Spinors and spacetime*, vol.1, pp.328-331 (Cambridge Univ. Press, Cambridge).

Papadopoulos, A.D. (1983), Tensor N.S. 40, 135-143; (1985) Tensor N.S. 42, 90-92.

Schouten, J.A. (1954) *Ricci Calculus* (Springer-Verlag, Berlin).

Katzin, G. H., Levine, J., and Davis, W. R. (1969). *J. Math. Phys.* 10, 617.

Ehlers, J., and Kundt, W. (1962), in *Gravitation: An Introduction to Current Research*, ed. L. Witten (Wiley, New York.)

Vaidya, P.C. (1951), *Proc. Indian Acad. Sci. A* 33 264; (1953), *Nature* 171 260.

Defrise-Carter L. (1975), *Commun. Math. Phys.* 40 273-282.

Hildebrandt, S.R., submitted to GRG.