A modified Lin equation for the energy balance in isotropic turbulence

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Abstract

At sufficiently large Reynolds numbers, turbulence is expected to exhibit scale-invariance in an intermediate (‘inertial’) range of wavenumbers, as shown by power-law behaviour of the energy spectrum and also by a constant rate of energy transfer through wavenumber. However, there is an apparent contradiction between the definition of the energy flux (i.e. the integral of the transfer spectrum) and the observed behaviour of the transfer spectrum itself. This is because the transfer spectrum $T(k)$ is invariably found to have a zero-crossing at a single point (at $k = k_*$, say), implying that the corresponding energy flux cannot have an extended plateau but must instead have a maximum value at $k = k_*$. This behaviour was formulated as a paradox and resolved by the introduction of filtered/partitioned transfer spectra, which exploited the symmetries of the triadic interactions (J. Phys. A: Math. Theor., 41:75501, 2008). In this paper we consider the more general implications of that procedure for the spectral energy balance equation, also known as the Lin equation. It is argued that the resulting modified Lin equations (and their corresponding Navier-Stokes equations) offer a new starting point for both numerical and theoretical methods, which may lead to a better understanding of the underlying energy transfer processes in turbulence. In particular the filtered-partitioned transfer spectra could provide a basis for a hybrid approach to the statistical closure problem, with the different spectra being tackled using different methods.
1 Introduction

We have previously written about the scale-invariance paradox and shown how it may be resolved by the introduction of filtered-partitioned forms of the transfer spectra [1], [2]. In the present paper we carry on this work to show how the underlying symmetries of the triadic interactions in wavenumber space also have implications for any more general study of the Lin equation. We have remarked elsewhere that to treat the Lin equation as purely a local energy balance equation is to be in danger of failing to realize that it is actually a highly non-local equation which couples all modes together. It is in fact the basis of the cascade picture of turbulent energy transfer, and it is important to always bear in mind that the transfer spectrum can be written as an integral over all wavenumbers of a term containing the triple-moment. In the present work we will argue that it is desirable to extend this scrutiny to the filtered-partitioned forms of the transfer spectrum in order to achieve a fuller understanding of the basic energy transfer processes.

This paper is organized as follows. We begin by stating the Lin equations and making some observations about the conventional interpretation of its role as an energy balance in wavenumber. Next we remind ourselves about the scale-invariance paradox and how it may be resolved. Then we move on to discussing the ways in which the Lin equation can be modified in order to clarify its role.

2 The Lin equation

We begin with the (by now) well-known spectral energy balance equation in its most familiar form, thus:

\[
\left( \frac{\partial}{\partial t} + 2\nu k^2 \right) E(k, t) = T(k, t),
\]

(1)

where \( E(k, t) \) is the energy spectrum, \( T(k, t) \) is the energy transfer spectrum and \( \nu \) is the kinematic viscosity. A full derivation and discussion will be found in the book [2]. We will also follow the growing practice of referring to it as the Lin equation.

Now let us integrate each term of (1) with respect to wavenumber, from zero up to some arbitrarily chosen wavenumber \( \kappa \):

\[
\frac{\partial}{\partial t} \int_0^\kappa dk E(k, t) = \int_0^\kappa dk T(k, t) - 2\nu \int_0^\kappa dk k^2 E(k, t).
\]

(2)

The energy transfer spectrum may be written as

\[
T(k, t) = \int_0^\infty dj S(k, j; t),
\]

(3)

where, as is well known, \( S(k, j; t) \) can be expressed in terms of the triple moment. Its antisymmetry under interchange of \( k \) and \( j \) guarantees energy conservation in the form:

\[
\int_0^\infty dk T(k, t) = 0.
\]

(4)

With some use of the antisymmetry of \( S \), along with equation (4), equation (2) may be written as

\[
\frac{\partial}{\partial t} \int_0^\kappa dk E(k, t) = - \int_0^\kappa dk \int_0^\kappa dj S(k, j; t) - 2\nu \int_0^\kappa dk k^2 E(k, t).
\]

(5)
Figure 1: A schematic view of the energy transfer in isotropic turbulence. The input spectrum $I(k)$ can represent either the work spectrum $W(k)$ or $-\partial E(k,t)/\partial t$; or the combined effects of both terms. All the other symbols have their usual meaning as defined in the text.

In this familiar form, the integral of the transfer term is readily interpreted as the net flux of energy from wavenumbers less than $\kappa$ to those greater than $\kappa$, at any time $t$. This the well known basis for the energy cascade.

It is usual to introduce a specific symbol $\Pi$ for this energy flux, thus:

$$\Pi(\kappa,t) = \int_{\kappa}^{\infty} dk T(k,t) = -\int_{0}^{\kappa} dk T(k,t),$$

where the second equality follows from (4).

In order to consider the stationary case, we may introduce an input spectrum $W(k)$. It is also convenient to introduce the dissipation spectrum $D(k,t)$ such that:

$$D(k,t) = 2\nu k^2 E(k,t).$$

With these introductions, and some rearrangement, we may write the energy balance equation as:

$$\frac{\partial E(k,t)}{\partial t} = W(k) + T(k,t) - D(k,t).$$

Figure (1) illustrates the general form of the energy transfers involved.

It should be noted that this general schematic form applies both to the stationary case and the case of free decay, with the input term $I(k)$ being interpreted as appropriate to each case.

3 The paradox and its resolution

The inertial range of wavenumbers is defined as being where the time derivative (or input term) and the viscous term are negligible. Hence, from equation (8), it follows that the criterion for an inertial range of wavenumbers can be taken as the vanishing of the
Figure 2: The expected behaviour of $T(k)$, on the basis of elementary calculus, to correspond to the scale invariance of $\Pi(k)$. The fact that $T(k)$ does not behave like that is the scale-invariance paradox.

Transfer spectrum; and, from equation (6), the constancy of the flux. In other words, for wavenumbers $\kappa$ in the inertial range we might expect to have have:

$$T(\kappa, t) = 0 \quad \text{and} \quad \Pi(\kappa, t) = \varepsilon. \quad (9)$$

Scale invariance, can be summed up as the observation that the energy spectrum takes the form of a power law (which is in itself scale-free) and that there is a constant rate of energy transfer over a range of wavenumbers, which must necessarily be equal to the rate of energy dissipation. In practice, the second criterion of equation (9) is widely used to identify the inertial range. This criterion was first put forward in 1941 by Obukhov [3] and first used to derive the famous $-5/3$ spectrum using dimensional analysis by Onsager in 1945 [4]. More recently, the books by Leslie [5] and McComb [6],[2] all follow Kraichnan [7], and cite the criterion $\Pi = \varepsilon$; as does work by, for instance, Bowman [8], Thacker [9], and Falkovich [10]. However, the first criterion given in equation (9) only holds for a single wavenumber and this fact is the scale-invariance paradox.

There are two inertial-range criteria in (9); and, by elementary calculus, they seem to be equivalent. This point is illustrated in Fig. (2). It shows an extended region where the flux is constant and also the transfer spectrum is zero. This makes an appealingly simple picture of spectral energy transfers but unfortunately it is wrong. The transfer spectrum always passes through zero at a single point as illustrated in Fig. (1).

So, let us consider again equation (5) for the transfer of energy from low wavenumbers to high. Now we wish to draw attention to the fact that, although the first term on the right hand side correctly represents the integral over wavenumber $k$ of the transfer
spectrum from zero up to $\kappa$, nevertheless the integrand is not actually $T(k)$ (from now on, we shall suppress time arguments in the interests of conciseness). In fact the integrand represents some part of $T(k)$, because the internal integration with respect to the dummy variable $j$ has been truncated at $j = \kappa$.

In order to clarify this situation, it will be found helpful to introduce low- and high-pass filtering operations, based on a cut-off wavenumber $k = \kappa$, on the Fourier components of the velocity field. These operations are used for the study of spectral mode elimination in the context of large-eddy simulation and its associated subgrid modelling; see, for example, [16] and references therein. We are thus led to introduce transfer spectra which have been filtered with respect to $k$ and which have had their integration over $j$ partitioned at the filter cut-off, i.e. $j = \kappa$.

Beginning with the Heaviside unit step function, defined by:

$$H(x) = \begin{cases} 1 & \text{for } x > 0; \\ 0 & \text{for } x < 0. \end{cases}$$

we may define low-pass and high-pass filter functions, thus:

$$\theta^-(x) = 1 - H(x),$$

and

$$\theta^+(x) = H(x).$$

We may then decompose the transfer spectrum, as given by (3), into four constituent parts,

$$T^{--}(k|\kappa) = \theta^-(k - \kappa) \int_0^\kappa dj S(k, j);$$

$$T^{--}(k|\kappa) = \theta^+(k - \kappa) \int_0^\kappa dj S(k, j);$$

$$T^{--}(k|\kappa) = \theta^+(k - \kappa) \int_0^\kappa dj S(k, j);$$

and

$$T^{++}(k|\kappa) = \theta^+(k - \kappa) \int_0^\kappa dj S(k, j),$$

such that the overall requirement of energy conservation is satisfied:

$$\int_0^\infty dk \left[ T^{--}(k|\kappa) + T^{--}(k|\kappa) + T^{--}(k|\kappa) + T^{++}(k|\kappa) \right] = 0.\tag{18}$$

It is readily verified that the individual filtered/partitioned transfer spectra have the following properties:

$$\int_0^\kappa dk T^{--}(k|\kappa) = 0;$$

$$\int_0^\kappa dk T^{--}(k|\kappa) = -\Pi(\kappa);$$

$$\int_\kappa^\infty dk T^{--}(k|\kappa) = \Pi(\kappa);$$

and
Figure 3: The behaviour of the filtered-partitioned transfer spectra: the paradox resolved!

and

$$\int_{\kappa}^{\infty} dk T^{++}(k|\kappa) = 0. \quad (22)$$

Equation (2) may be rewritten in terms of the filtered/partitioned transfer spectrum as:

$$\frac{d}{dt} \int_{0}^{\kappa} dk E(k, t) = -\int_{0}^{\kappa} \int_{\kappa}^{\infty} dk T^{--}(k|\kappa) - 2\nu_{0} \int_{0}^{\kappa} dk k^{2} E(k, t). \quad (23)$$

We note from equation (19) that $T^{--}(k|\kappa)$ is conservative on the interval $0 \leq k \leq \kappa$, and hence does not appear in (23), while $T^{--}(k|\kappa)$ has been replaced by $-T^{++}(k|\kappa)$, using (20) and (21).

Filtered and partitioned transfer spectra have been measured, using DNS, in the context of spectral large-eddy simulation. In particular, Zhou and Vahala [17] found that the resolvable-scales energy transfer spectrum $T^{<\kappa}(k)$ (i.e. $T^{--}(k|\kappa)$ in our notation) is conservative on the interval $0 \leq k \leq \kappa$, in agreement with our equation (19); while the resolvable-subgrid transfer spectrum (i.e. our $T^{++}(k|\kappa)$) is zero over a range of wavenumbers. Similar behaviour has also been found in the more detailed investigation by McComb and Young [18].

As we have previously pointed out in [1], experimentalists, who do not have access to partitioned versions of the transfer spectrum, will still find pragmatic procedures, such as the Lumley criterion for the inertial range [14], useful. However, those working with DNS or analytical theory, can avoid the paradox by changing their definition of energy.
fluxes, from those given by \([6]\), to the forms\(^1\)

\[
\Pi(\kappa, t) = \int_0^\infty dk T^-(k, \kappa, t) = -\int_0^\infty dk T^+(k, \kappa, t),
\]

where \(T^-(k, \kappa, t)\) is defined by \([16]\) and \(T^+(k, \kappa, t)\) by \([13]\). This is equivalent to \([6]\); but, unlike it, avoids the paradox. This resolution of the paradox is shown schematically in Fig. \((3)\).

4 Modifications to the Lin equation

In view of the above discussion, the obvious step now is to filter the energy spectrum in the same way as we have done for the transfer spectrum, and consider low-\(k\) and high-\(k\) forms of the Lin equation. In order to do this we make the decomposition:

\[
E(k, t) = E^-(k, \kappa, t) + E^+(k, \kappa, t),
\]

where \(E^-\) is defined for \(k \leq \kappa\) and \(E^+\) is defined for \(k \geq \kappa\). Trivially, we can also do this for the input spectrum \(W(k)\) and dissipation spectrum \(D(k, t)\), and equation \((8)\) can be written in low-\(k\) and high-\(k\) forms respectively, as:

\[
\frac{\partial E^-(k, \kappa, t)}{\partial t} = W^-(k, \kappa) + T^-\big(\kappa, k, t\big) + T^+(k, \kappa, t) - D^-(k, \kappa, t), \quad \text{for } k \leq \kappa; \tag{26}
\]

and

\[
\frac{\partial E^+(k, \kappa, t)}{\partial t} = W^+(k, \kappa) + T^-\big(\kappa, k, t\big) + T^+(k, \kappa, t) - D^+(k, \kappa, t), \quad \text{for } k \geq \kappa. \tag{27}
\]

For this decomposition to be meaningful, the Reynolds number must be large enough for the inertial flux to be equal to the dissipation, in accordance with the second criterion of equation \((9)\). As we increase the Reynolds number beyond this critical value, we have an increasing range of wavenumbers \(k\) which satisfy that criterion, and this is the inertial range. We shall denote this range by

\[
k_{\text{bot}} \leq k \leq k_{\text{top}} \equiv \text{the inertial range of wavenumbers},
\]

where we now have to define \(k_{\text{bot}}\) and \(k_{\text{top}}\). For sake of simplicity, we will consider stationary turbulence and omit the time variables.

First, we need to consider the nature of the forcing spectrum \(W(k)\). In formulating the turbulence problem according to the tenets of statistical physics, this is normally taken to arise from the introduction of random stirring forces, which are assumed to be of white noise form. In particular, the forcing spectrum is taken to be peaked near the origin in wavenumber space, so that the turbulence that results from it is due to the Navier-Stokes equation, and not specifically related to the forcing. We should note that a different view was taken from the late 1970s onwards, in connection with the application of renormalization group methods to the Navier-Stokes equation. See either of the books \([6]\) or \([2]\) for a general discussion of this point.

\(^1\)We should mention that these forms are exactly equivalent to Kraichan’s original definition of what he called the transport power \([7]\). In later work \([19]\), his definition of the transport power was equivalent to equation \([6]\) in the present paper.
Accordingly, for theoretical approaches to the statistical closure problem, and also for
direct numerical simulation, we should choose a form of forcing spectrum $W(k)$ which
satisfies the conditions:
\[
\int_0^\infty dk W(k) = \varepsilon W \approx \int_0^{k_{\text{bot}}} dk W(k),
\]
where the equality defines $\varepsilon_W$, while the approximate equality defines $k_{\text{bot}}$, which we take
to be the lower limit of the inertial range.

In general, we would require $k_{\text{bot}}$ to be very much smaller than the Kolmogorov
dissipation wavenumber $k_d$ which is generally taken as being an indicator of the dissipation
range of wavenumbers. Experimenters have usually taken the the upper limit of the
inertial range to be about $0.1k_d - 0.2k_d$. In fact we will define $k_{\text{top}}$ by another approximate
equality, thus:
\[
\int_0^\infty dk D(k) = \varepsilon \approx \int_{k_{\text{top}}}^\infty dk D(k),
\]
where the equality is the conventional definition of the dissipation rate, and the approx-
imate equality defines the upper limit of the inertial range $k_{\text{top}}$.

With these points in mind, we may simplify the low-wavenumber and high-wavenumber
forms of the Lin equation, respectively (26) and (27), to:
\[
\frac{\partial E^-(k|\kappa,t)}{\partial t} = W(k) + T^-(k|\kappa,t) + T^+(k|\kappa,t), \quad \text{for } k \leq \kappa; \quad (30)
\]
and
\[
\frac{\partial E^+(k|\kappa,t)}{\partial t} = T^-(k|\kappa,t) + T^{++}(k|\kappa,t) - D(k,t), \quad \text{for } k \geq \kappa. \quad (31)
\]
That is, for sufficiently high Reynolds numbers, and an appropriate choice of stirring
forces, we may simplify matters by treating the input spectrum as being confined to
the low-wavenumber region and the dissipation spectrum as being confined to the high-
wavenumber region. Deriving the flux balance equations from (30) and (31), and invoking
equations (28) and (29), we obtain the final flux balances as:
\[
\varepsilon_W - \Pi(\kappa) = 0 \quad \text{for } k \leq \kappa; \quad (32)
\]
and
\[
\Pi(\kappa) - \varepsilon = 0 \quad \text{for } k \geq \kappa. \quad (33)
\]

Reminding ourselves that the transfer spectrum has its single zero crossing at $k = k_*$,
we may define the maximum value of the inertial flux as
\[
\Pi_{\text{max}} = \Pi(k_*) = \varepsilon_T, \quad (34)
\]
and at the same time introduce the useful symbol $\varepsilon_T$ for the maximum flux. Since $k_*$
must lie within the inertial range, we can write the general criterion for the existence of
the inertial range as:
\[
\Pi(\kappa) = \varepsilon_T = \varepsilon_W = \varepsilon. \quad (35)
\]
For completeness it should be noted that this analysis is readily extended to the case of
free decay, if we replace $\varepsilon_W$ by the energy decay rate $\varepsilon_D$. Further details may be found
in [2].
5 Conclusion

Provided we are faced with the ideal situation, where the input and the output (i.e. dissipation) are well separated in wavenumber space, equations (30) and (31) may provide a new, and one might hope, productive basis for the study of the energy transfers in isotropic turbulence. The corresponding partitioned-filtered Navier-Stokes equations are readily deduced and may be studied by direct numerical simulation as a four-component composite dynamical system, where the four components correspond to the four filtered-partitioned transfer spectra.

Also, there is a growing use of hybrid approaches in fluid dynamics problems, and the closure problem could be approached in such a way by using different methods to tackle the different filtered-partitioned transfer spectra. For instance, in the low-\(k\) system, we might use the local energy transfer theory [20] for \(T^{-\rightarrow}(k)\), and renormalization group methods [21] for \(T^{-\rightarrow+}(k)\); or, conceivably, the other way round! It would require investigation.

For the ideal situation just discussed, where we have the input and output (or, production and dissipation) ranges of wavenumber well separated, we need to choose the input spectrum \(W(k)\) to be peaked near the origin; and also we need the Reynolds number to be reasonably high. If for some reason, we cannot satisfy these conditions, then we must resort to equations (26) and (27). However, even so, we must still have the Reynolds number large enough for the condition for the existence of an inertial range to be satisfied.

Lastly, I should emphasise that Fig. (3) is very much a schematic indication of how this graph should look, based on the small amount of information available to us. The behaviour of these filtered-partitioned transfer spectra was studied in the 1990s in the context of subgrid modelling and renormalization group methods: see [1] for references. Computers have advanced a lot since then, so we end with a plea to the effect that this field of study should be revived in the context of later work. An informal introduction to this topic may be found in the post of 23 July on the following weblog: blogs.ed.ac.uk/physics-of-turbulence/.

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