New Numerical Algorithm for Modeling of Boson-Fermion Stars in Dilatonic Gravity*

T.L. Boyadjiev†, M.D. Todorov†, P.P. Fiziev§, S.S. Yazadjiev¶

Abstract

We investigate numerically a models of the static spherically symmetric boson-fermion stars in scalar-tensor theory of gravity with massive dilaton field. The proper mathematical model of such stars is interpreted as a nonlinear two-parametric eigenvalue problem with unknown internal boundary. We employ the Continuous Analogue of Newton Method (CANM) which leads on each iteration to two separate linear boundary value problems with different dimensions inside and outside the star, respectively. Along with them a nonlinear algebraic system for the spectral parameters - radius of the star $R_s$ and quantity $\Omega$ is solved also.

In this way we obtain the behaviour of the basic geometric quantities and functions describing dilaton field and matter fields which build the star.

Keywords. two-parametric nonlinear eigenvalue problem, Continuous Analogue of the Newton Method, mixed fermion-boson stars, scalar-tensor theory of gravity, massive dilaton field.

AMS subject classifications. 65C20, 65P20, 65P30, 8308, 83D05

1 Introduction

In this paper we present a numerical method for solving the equations of the general scalar-tensor theories of gravity including a dilaton potential term for the general case of mixed boson-fermion star. This method is an improvement of the method concerning the similar problem proposed in our recent work [3]. The original domain is splitted to two domains: inner inside the star and external outside the star. The solutions in these regions are obtained separately and after that they are matched. Before we go to the substantial part of this paper we will describe briefly the physical problem.

2 Main model

Boson stars are gravitationally bound macroscopic quantum states made up of scalar bosons [19, 23, 6, 14]. They differ from the usual fermionic stars in that they are only prevented from collapsing gravitationally by the Heisenberg uncertainty principle. For self-interacting boson field the mass of the boson star, even for small values of the coupling constant turns out to be of the order of Chandrasekhar’s mass when the boson mass is similar to a proton mass. Thus, the boson stars arise as possible candidates for non-baryonic dark matter in the universe and consequently as a possible solution of the one of outstanding problems in today’s astrophysics - the problem of nonluminous matter in the universe. Most of the stars are of primordial origin being formed from an original gas of fermions and

*This work was supported by Bulgarian National Scientific Fund, Contr. NoNo F610/99, MM602/96 and by Sofia University Research Fund, Contr. No. 245/99.
†Faculty of Mathematics and Computer Science, University of Sofia, 5 James Bourchier Blvd., 1164 Sofia, Bulgaria, E-mail: todorlb@fmi.uni-sofia.bg
‡Faculty of Applied Mathematics and Computer Science, Technical University of Sofia, 1756 Sofia, Bulgaria, E-mail: mmod@vmei.acad.bg
§Faculty of Physics, University of Sofia, 5, James Bourchier Blvd., 1164 Sofia, Bulgaria, E-mail: fiziev@phys.uni-sofia.bg
¶Faculty of Physics, University of Sofia, 5, James Bourchier Blvd., 1164 Sofia, Bulgaria, E-mail: yazad@phys.uni-sofia.bg
bosons in the early universe. That is why it should be expected that most stars are a mixture of both, fermions and bosons in different proportions.

Boson-fermion stars are also good model to learn more about the nature of strong gravitational fields not only in general relativity but also in the other theories of gravity.

The most natural and promising generalizations of general relativity are the scalar-tensor theories of gravity [1, 2, 3, 4, 5, 6, 7, 8]. In these theories the gravity is mediated not only by a tensor field (the metric of space-time) but also by a scalar field (the dilaton). These dilatonic theories of gravity contain arbitrary functions of the scalar field that determine the gravitational “constant” as a dynamical variable and the strength of the coupling between the scalar field and matter. It should be stressed that specific scalar-tensor theories of gravity arise naturally as low energy limit of string theory [15, 5, 12, 24, 21, 20] which is the most promising modern model of unification of all fundamental physical interactions.

Boson stars in scalar-tensor theories of gravity with massless dilaton have been widely investigated recently both numerically and analytically [25, 16, 26, 27, 7, 2, 30]. Mixed boson-fermion stars in scalar tensor theories of gravity however have not been investigated so far in contrast to general relativistic case where boson-fermion stars have been investigated [17].

In present paper we consider boson-fermion stars in the most general scalar-tensor theory of gravity with massive dilaton.

In the Einstein frame the field equations in presence of fermion and boson matter are:

\[ G^m_m = \kappa_s \left( \frac{B}{T_m} + \frac{F}{m} \right) + 2 \partial_m \varphi \partial^n \varphi - \partial^l \varphi \partial_l \varphi \delta^n_m + \frac{1}{2} U(\varphi) \delta^n_m, \]

\[ \nabla_m \nabla^m \varphi + \frac{1}{4} U'(\varphi) = -\frac{\kappa_s}{2} \alpha(\varphi) \left( \frac{B}{T} + \frac{F}{m} \right), \]

\[ \nabla_m \nabla^m \Psi + 2 \alpha(\varphi) \partial^l \varphi \partial_l \Psi = -2 A^2(\varphi) \frac{\partial \tilde{W}(\Psi^+ \Psi)}{\partial \Psi^+}, \]

\[ \nabla_m \nabla^m \Psi^+ + 2 \alpha(\varphi) \partial^l \varphi \partial_l \Psi^+ = -2 A^2(\varphi) \frac{\partial \tilde{W}(\Psi^+ \Psi)}{\partial \Psi^+}, \]

where \( \nabla_m \) is the Levi-Civita connection with respect to the metric \( g_{mn} \), \( m = 0, \ldots, 3; n = 0, \ldots, 3 \).

The constant \( \kappa_s \) is given by \( \kappa_s = 8 \pi G_s \) where \( G_s \) is the bare Newtonian gravitational constant. The physical gravitational “constant” is \( G_s A^2(\varphi) \) where \( A(\varphi) \) is a function of the dilaton field \( \varphi \) depending on the concrete scalar-tensor theory of gravity. \( \tilde{W}(\Psi^+ \Psi) \) is the potential of boson field. The dilaton potential \( U(\varphi) \) can be written in the form \( U(\varphi) = m_D^2 V(\varphi) \) where \( m_D \) is the dilaton mass and \( V(\varphi) \) is dimensionless function of \( \varphi \).

The function \( \alpha(\varphi) = \frac{d}{d\varphi} \ln A(\varphi) \) determines the strength of the coupling between the dilaton field \( \varphi \) and the matter. The functions \( \frac{B}{T} \) and \( \frac{F}{m} \) are correspondingly the trace of the energy-momentum tensor of the fermionic matter \( B T_m \) and bosonic matter \( F T_m \). Their explicit forms are

\[ \frac{B}{T_m} = \frac{1}{2} A^2(\varphi) \left( \partial_m \Psi^+ \partial^n \Psi + \partial_m \Psi \partial^n \Psi^+ \right) \]

\[ - \frac{1}{2} A^2(\varphi) \left[ \partial_l \Psi^+ \partial^l \Psi - 2 A^2(\varphi) \tilde{W}(\Psi^+ \Psi) \right] \delta^n_m, \]

\[ \frac{F}{T_m} = (\varepsilon + p) u_m u^n - p \delta^n_m. \]

Here \( \Psi \) is a complex scalar field describing the bosonic matter while \( \Psi^+ \) is its complex conjugated function. The energy density and the pressure of the fermionic fluid in the Einstein frame are \( \varepsilon = A^4(\varphi) \varepsilon \) and \( p = A^4(\varphi) p \) where \( \varepsilon \) and \( p \) are the physical energy density and pressure. Instead to give the equation of state of the fermionic matter in the form \( p = \tilde{p}(\varepsilon) \) it is more convenient to write it in a parametric form

\[ \tilde{\varepsilon} = \tilde{\varepsilon}_0 g(\mu) \quad \tilde{p} = \tilde{\varepsilon}_0 f(\mu) \]

where \( \tilde{\varepsilon}_0 \) is a properly chosen dimensional constant and \( \mu \) is dimensionless Fermi momentum.

\[ ^1 \text{In the present article we consider fermionic matter only in macroscopic approximation, i.e., after averaging quantum fluctuations of the corresponding fermion fields. Thus we actually consider standard classical relativistic matter.} \]
The physical four-velocity of the fluid is denoted by \( u_\mu \). The potential for the boson field has the form

\[
\dot{W}(\Psi^+\Psi) = -\frac{m_B^2}{2}\Psi^+\Psi - \frac{1}{4}\Lambda (\Psi^+\Psi)^2.
\]

Field equations together with the Bianchi identities lead to the local conservation law of the energy-momentum of matter

\[
\nabla_n T^F_m = \alpha(\varphi) \frac{F}{T} \partial_m \varphi.
\](5)

We will consider a static and spherically symmetric mixed boson-fermion star in asymptotic flat space-time. This means that the metric \( g_{mn} \) has the form

\[
ds^2 = \exp[\nu(R)] dt^2 - \exp[\lambda(R)] dR^2 - R^2 (d\theta^2 + \sin^2 \theta d\psi^2)
\](6)

where \( R, \theta, \psi \) are usual spherical coordinates. The field configuration is static when the boson field \( \Psi \) satisfies the relationship

\[
\Psi = \tilde{\sigma}(R) \exp(i\omega t).
\]

Here \( \omega \) is a real number and \( \tilde{\sigma}(R) \) is a real function. Taking into account the assumption that have been made the system of the field equation is reduced to a system of ordinary differential equations. Before we explicitly write the system we are going to introduce a rescaled (dimensionless) radial coordinate by \( r = m_B R \) where \( m_B \) is the mass of the bosons. From now on, a prime will denote a differentiation with respect to the dimensionless radial coordinate \( r \). After introducing the dimensionless quantities

\[
\Omega = \frac{\omega}{m_B}, \quad \sigma = \sqrt{\kappa_\gamma m}, \quad \Lambda = \frac{\tilde{\Lambda}}{\kappa_\gamma m_B^2}, \quad \gamma = \frac{m_D}{m_B}.
\]

and defining the dimensionless energy-momentum tensors as \( T^F_m = \frac{m_B}{m_B} T_m \) the components of the dimensionless energy-momentum tensor of the fermionic and bosonic matter become correspondingly

\[
\begin{align*}
T^B_0 &= b A^4(\varphi) g(\mu), & T^B_1 &= -b A^4(\varphi) f(\mu), \\
T^F_0 &= \frac{1}{2} A^2(\varphi) \left[ \Omega^2 \sigma^2(r) \exp(-\nu(r)) + \left( \frac{\partial \sigma}{\partial r} \right)^2 \exp(-\lambda(r)) \right] - A^4(\varphi) W(\sigma^2), \\
T^F_1 &= -\frac{1}{2} A^2(\varphi) \left[ \Omega^2 \sigma^2(r) \exp(-\nu(r)) + \left( \frac{\partial \sigma}{\partial r} \right)^2 \exp(-\lambda(r)) \right] - A^4(\varphi) W(\sigma^2).
\end{align*}
\]

Here the parameter \( b = \frac{\alpha_\gamma}{m_B} \) describes the relation between the Compton length of dilaton and the usual radius of neutron star in general relativity.

The functions \( T^B \) and \( T^F \) describing the trace of energy-momentum tensor have a form:

\[
\begin{align*}
\frac{B}{T} &= -A^2(\varphi) \left[ \Omega^2 \sigma^2(r) \exp(-\nu(r)) - \left( \frac{\partial \sigma}{\partial r} \right)^2 \exp(-\lambda(r)) \right] - 4A^4(\varphi) W(\sigma^2), \\
\frac{F}{T} &= b A^4(\varphi) [g(\mu) - 3 f(\mu)].
\end{align*}
\]

For the independent (dimensionless) radial coordinate \( r \) we have \( r \in [0, R_s] \cup [R_s, \infty) \) where \( 0 < R_s < \infty \) is the unknown radius of the fermionic part of the mixed boson-fermion star.

With all definitions we have given the main system of differential equations governing the structure of a static and spherically symmetric boson-fermion star can be written in the following form:

1. In the interior of the fermionic part of the star (the functions in this domain are subscribed by \( i \)
functions λ, µ formally that the function λ centre. The first three conditions in (8) guarantee the nonsingular
ity of the metrics and the functions σ, where
\[ \left\{ \begin{align*}
\lambda &= 1 - \exp(\lambda) + r \left\{ \exp(\lambda) \left[ \frac{F}{T_0} + \frac{B}{T_0} + \frac{1}{2} \gamma^2 V(\lambda) \right] + \left( \frac{d\varphi}{dr} \right)^2 \right\}, \\
\nu &= -1 - \exp(\lambda) - r \left\{ \exp(\lambda) \left[ \frac{F}{T_1} + \frac{B}{T_1} + \frac{1}{2} \gamma^2 V(\varphi) \right] - \left( \frac{d\varphi}{dr} \right)^2 \right\}, \\
\frac{d^2 \varphi}{dr^2} &= -2 \frac{d\varphi}{dr} + \frac{1}{2} (F_1 - F_2) \frac{d\varphi}{dr} \\
&\quad + \frac{1}{2} \exp(\lambda) \left[ \alpha(\varphi) \left( \frac{F}{T} + \frac{B}{T} + \frac{1}{2} \gamma^2 V'(\varphi) \right) \right], \\
\frac{d^2 \sigma}{dr^2} &= -2 \frac{d\sigma}{dr} + \left[ \frac{1}{2} (F_1 - F_2) - 2\alpha(\varphi) \frac{d\varphi}{dr} \right] \frac{d\sigma}{dr} \\
&\quad - \sigma \exp(\lambda) \left[ \Omega^2 \exp(-\nu) + 2A^2(\varphi)W'(\sigma^2) \right], \\
\frac{d\mu}{dr} &= -\frac{g(\mu) + f(\mu)}{f(\mu)} \left[ \frac{1}{2} F_2 + \alpha(\varphi) \frac{d\varphi}{dr} \right].
\end{align*} \right. \]

Here \( \lambda(r), \nu(r), \varphi(r), \sigma(r) \) and \( \mu(r) \) are unknown functions of \( r \) and \( \Omega \) is a unknown parameter. Having in mind the physical assumptions we have to solve the equations (7) under following boundary conditions:
\[ \begin{align*}
\lambda(0) &= \frac{d\varphi}{dr}(0) = \frac{d\sigma}{dr}(0) = 0, \\
\sigma(0) &= \sigma_c, \\
\mu(0) &= \mu_c,
\end{align*} \]

(8)

(9)

where \( \sigma_c \) and \( \mu_c \) are the values of density of the bosonic and fermionic matter, respectively at the star’s centre. The first three conditions in (8) guarantee the nonsingularity of the metrics and the functions \( \lambda(r), \varphi(r), \sigma(r) \) at the star’s centre.

2. In the external domain (subscribed by e) there is not fermionic matter, i.e., one can suppose formally that the function \( \mu(r) \equiv 0 \) if \( x \geq R_s \). The fermionic part of the energy-momentum tensor vanishes identically also and thus the differential equations with respect to the rest four unknown functions \( \lambda(r), \nu(r), \varphi(r) \) and \( \sigma(r) \) are:
\[ \begin{align*}
\frac{d\lambda}{dr} &= F_{1,e} \equiv 1 - \exp(\lambda) + r \left\{ \exp(\lambda) \left[ \frac{B}{T_0} + \frac{1}{2} \gamma^2 V(\lambda) \right] + \left( \frac{d\varphi}{dr} \right)^2 \right\}, \\
\frac{d\nu}{dr} &= F_{2,e} \equiv -1 - \exp(\lambda) - r \left\{ \exp(\lambda) \left[ \frac{B}{T_1} + \frac{1}{2} \gamma^2 V(\varphi) \right] - \left( \frac{d\varphi}{dr} \right)^2 \right\}, \\
\frac{d^2 \varphi}{dr^2} &= -2 \frac{d\varphi}{dr} + \frac{1}{2} (F_{1,e} - F_{2,e}) \frac{d\varphi}{dr} \\
&\quad + \frac{1}{2} \exp(\lambda) \left[ \alpha(\varphi) \left( \frac{B}{T} + \frac{1}{2} \gamma^2 V'(\varphi) \right) \right], \\
\frac{d^2 \sigma}{dr^2} &= -2 \frac{d\sigma}{dr} + \left[ \frac{1}{2} (F_{1,e} - F_{2,e}) - 2\alpha(\varphi) \frac{d\varphi}{dr} \right] \frac{d\sigma}{dr} \\
&\quad - \sigma \exp(\lambda) \left[ \Omega^2 \exp(-\nu) + 2A^2(\varphi)W'(\sigma^2) \right].
\end{align*} \]

(10)

As it is required by the asymptotic flatness of space-time the boundary conditions at the infinity are
\[ \begin{align*}
\nu(\infty) &= 0, \\
\varphi(\infty) &= 0, \\
\sigma(\infty) &= 0
\end{align*} \]

(11)

where we denote \( (\cdot)(\infty) = \lim_{r \to \infty} (\cdot)(r) \).
We seek for a solution \([\lambda(r), \nu(r), \varphi(r), \sigma(r), \mu(r), R_s, \Omega]\) subjected to the nonlinear ODEs (3) and (10), satisfying the boundary conditions (8), (11) and (13). At that we assume the function \(\mu(r)\) is continuous in the interval \([0, R_s]\), whilst the functions \(\lambda(r), \nu(r)\) are continuous and the functions \(\varphi(r), \sigma(r)\) are smooth in the whole interval \([0, \infty)\), including the unknown inner boundary \(r = R_s\).

The so posed BVP is a two-parametric eigenvalue problem with respect to the quantities \(R_s\) and \(\Omega\).

Let us emphasize that a number of methods for solving the free-boundary problems are considered in detail in [28, 22].

Here we aim at applying the new solving method to the above formulated problem. This method differs from that one proposed in [1] and for the governing field equations written in the forms (3) and (10) it possesses certain advantage.

3 Method of solution

At first we scale the variable \(r\) using the Landau transformation [28] and in this way we obtain a fixed computational domain. Namely

\[
x = \frac{r}{R_s}, x \in [0, 1] \cup [1, \infty).
\]

For our further considerations it is convenient to present the systems (3) and (10) in following equivalent form as systems of first order ODEs:

\[
- \mathbf{y}_i' + R_s \mathbf{F}_i(R_s x, \mathbf{y}_i, \Omega) = 0, \quad (12)
\]

\[
- \mathbf{y}_e' + R_s \mathbf{F}_e(R_s x, \mathbf{y}_e, \Omega) = 0 \quad (13)
\]

with respect to the unknown vector functions

\[
\mathbf{y}_i(x) \equiv (\lambda(x), \nu(x), \varphi(x), \xi(x), \sigma(x), \eta(x), \mu(x))^T,
\]

\[
\mathbf{y}_e(x) \equiv (\lambda(x), \nu(x), \varphi(x), \xi(x), \sigma(x), \eta(x))^T
\]

and right hand sides \(\mathbf{F}_i \equiv (F_1, F_2, \xi, F_3, \eta, F_4)^T, \mathbf{F}_e \equiv (F_1, F_2, \xi, F_3, \eta, F_4)^T\) where \((\cdot)'\) stands for differentiation towards the new variable \(x\).

For given values of the parameters \(R_s\) and \(\Omega\) the independent solving of the inner system (2) requires seven boundary conditions. At the same time we have at disposal only six conditions of the kind (3) and (8). In order to complete the problem we set additionally one more parametric condition (the value of someone from among the functions \(\lambda(x), \nu(x), \varphi(x), \xi(x), \sigma(x)\) or \(\eta(x)\)) at the point \(x = 1\).

Let us set for example

\[
\varphi_i(1) = \varphi_s \quad (14)
\]

where \(\varphi_s\) is a parameter. Then the boundary conditions (8), (11) and (14) of the inner BVP can be presented in the form:

\[
B_{0,i} \mathbf{y}_i(0) - D_{0,i} = 0, \quad B_{1,i} \mathbf{y}_i(1) - D_{1,i} (\varphi_s) = 0. \quad (15)
\]

Here the matrices \(B_{0,i} = diag(1, 0, 1, 1, 1, 1, 1), D_{0,i} = diag(0, 0, 0, 0, \sigma_s, 0, \mu_e), B_{1,i} = diag(0, 1, 0, 0, 0, 1, 1), D_{1,i} = diag(0, 0, \varphi_s, 0, 0, 0, 0)\).

Obviously the solution in the inner domain \(x \in [0, 1]\) depends not only on the variable \(x\), but it is a function of the three parameters \(R_s, \Omega, \varphi_s\) as well, i.e., \(\mathbf{y}_i = \mathbf{y}_i(x, \Omega, R_s, \varphi_s)\).

In the external domain \(x \geq 1\) the vector of solutions

\[
\mathbf{y}_e(x) \equiv (\lambda(x), \nu(x), \varphi(x), \xi(x), \sigma(x), \eta(x))^T
\]

is 6D. Thereupon six boundary conditions are indispensable to solving of the equation (13). At the same time the three boundary conditions (11) are only known. Let us consider that the solution \(\mathbf{y}_i(x)\)
in the inner domain \( x \in [0, 1] \) is knonwdledge. Then we postulate the rest three deficient conditions to be the continuity conditions at the point \( x = 1 \). The first of them is similar to the condition (14) and the else two we assign to arbitrary two functions from among \( \lambda(x), \nu(x), \xi(x), \sigma(x) \) and \( \eta(x) \), for example

\[
\lambda_e(1) = \lambda_i(1), \quad \varphi_e(1) = \varphi_s, \quad \sigma_e(1) = \sigma_i(1).
\]

It is convenient to present the boundary conditions in the external domain in matrix form again:

\[
B_{1,e} y_e(1) - D_{1,e}(\varphi_s) = 0, \quad B_{\infty,e} y_e(\infty) = 0
\]  

(16)

where the matrices \( B_{1,e} = \text{diag}(1, 0, 1, 0, 1, 0) \), \( D_{1,e} = \text{diag}(\lambda_i(1), 0, \varphi_s, 0, \sigma_i(1), 0) \), \( B_{\infty,e} = \text{diag}(0, 1, 1, 0, 1, 0) \).

Let the solutions \( y_i = y_i(x, \Omega, R_s, \varphi_s) \) and \( y_e = y_e(x, \Omega, R_s, \varphi_s) \) be supposed known. Generally speaking for given arbitrary values of the parameters \( R_s, \Omega \) and \( \varphi_s \) the continuity conditions with respect to the functions \( \nu(x), \xi(x) \) and \( \eta(x) \) at the point \( x = 1 \) are not satisfied. We choose the parameters \( R_s, \Omega \) and \( \varphi_s \) in such manner that the continuity conditions for the functions \( \nu(x), \xi(x) \) and \( \eta(x) \) to be held, i.e.,

\[
\nu_e(1, R_s, \Omega, \varphi_s) - \nu_i(1, R_s, \Omega, \varphi_s) = 0,
\]

\[
\xi_e(1, R_s, \Omega, \varphi_s) - \xi_i(1, R_s, \Omega, \varphi_s) = 0,
\]

\[
\eta_e(1, R_s, \Omega, \varphi_s) - \eta_i(1, R_s, \Omega, \varphi_s) = 0.
\]

(17)

These conditions should be interpreted as three nonlinear algebraic equations in regard to the unknown quantities \( R_s, \Omega \) and \( \varphi_s \). The usual way for solving of the above mentioned kind of equations (17) is by means of various iteration methods, for example Newton’s methods. The traditional technology similarly to the methods like shutting [33] requires separate treatment of the BVPs and the algebraic continuity equations and brings itself to additional linear ODEs for elements of the corresponding to (17) Jacobi matrix. These elements are functions of the variable \( x \) and they have to be known actually only at the point \( x = 1 \). The solving of the both nonlinear BVPs (12), (17) and (13), (16) along with the attached linear equations is another hard enough task.

At the present work using the CANM [12], [13] we propose a common treatment of both, differential and algebraic problems.

We suppose that the nonlinear spectral problem [12], [13], [16] and (17) has a “well separated” exact solution. Let the functions \( y_{i,0}(x), y_{e,0}(x) \) and the parameters \( R_{s,0}, \Omega_0, \varphi_{s,0} \) are initial approximations to this solution. The CANM leads to the following iteration process:

\[
y_{i,k+1}(x) = y_{i,k}(x) + \tau_k z_{i,k}(x), \quad (18)
\]

\[
y_{e,k+1}(x) = y_{e,k}(x) + \tau_k z_{e,k}(x), \quad (19)
\]

\[
R_{s,k+1} = R_{s,k} + \tau_k \rho_k, \quad (20)
\]

\[
\Omega_{k+1} = \Omega_k + \tau_k \omega_k, \quad (21)
\]

\[
\varphi_{s,k+1} = \varphi_{s,k} + \tau_k \phi_k. \quad (22)
\]

Here \( \tau_k \in (0, 1] \) is a parameter which can rule the convergence of iteration process. The increments \( z_{i,k}(x), z_{e,k}(x), \rho_k, \omega_k \) and \( \phi_k, k = 0, 1, 2, \ldots \) satisfy the linear ODEs (for sake of simplicity henceforth we will omit the number of iterations \( k \)):

\[
-z_i' + R_s \frac{\partial F_i}{\partial y_i} z_i + \left( R_s \frac{\partial F_i}{\partial R_s} + F_i \right) \rho + R_s \frac{\partial F_i}{\partial \Omega} \omega = y_i' - R_i F_i, \quad (23)
\]

\[
-z_e' + R_s \frac{\partial F_e}{\partial y_e} z_e + \left( R_s \frac{\partial F_e}{\partial R_s} + F_e \right) \rho + R_s \frac{\partial F_e}{\partial \Omega} \omega = y_e' - R_e F_e \quad (24)
\]

All the coefficients and right hand sides as well in the above two equations are known functions of the arguments \( x, R_s, \Omega \) by means of the solution from the previous iteration. We seek for the unknowns
\(z_i(x) = s_i(x) + \rho u_i(x) + \omega v_i(x) + \phi w_i(x),\) \(\tag{25}\)

\(z_e(x) = s_e(x) + \rho u_e(x) + \omega v_e(x) + \phi w_e(x).\) \(\tag{26}\)

Here \(s_i(x), u_i(x), v_i(x), w_i(x), s_e(x), u_e(x), v_e(x), w_e(x)\) are new unknown functions, which are defined in either, internal or external domains. Substituting for the decomposition \(25\) into equation \(23\) after reduction we obtain:

\[-s_i' + Q_i(x)s_i = y_i' - R_i F_i,\]

\[-u_i' + Q_i(x)u_i = -\left(F_i + R_s \frac{\partial F_i}{\partial R_s}\right),\]

\[-v_i' + Q_i(x)v_i = -R_s \frac{\partial F_i}{\partial \Omega}.,\]

\[-w_i' + Q_i(x)w_i = 0\]

where \(Q_i(x) \equiv R_s \frac{\partial F_i(R_s, x, y_i(x), \Omega)}{\partial y_i(x)}\) stands for a square matrix \((7 \times 7)\), which consists of the Frechet derivatives of operator \(F_i\) at the point \(y_i(x), R_s, \Omega)\).

Similarly applying the CANM to the boundary conditions \(15\) and taking into account the dependence of matrix \(D_{1,i}\) on the parameter \(\varphi_s\) yields:

\[B_{0,i} z_i(0) = D_{0,i} - B_{0,i} y_i(0), \quad B_{1,i} z_i(1) = D_{1,i} - B_{1,i} y_i(1) - D'_{1,i} \varphi_s.\]

By means of the decomposition \(26\) we obtain the following eight boundary conditions (four left + four right) for the equations \(27\):

\[B_{0,i} s_i(0) = D_{0,i} - B_{0,i} y_i(0), \quad B_{1,i} s_i(1) = D_{1,i} - B_{1,i} y_i(1),\]

\[B_{0,i} u_i(0) = 0, \quad B_{1,i} u_i(1) = 0,\]

\[B_{0,i} v_i(0) = 0, \quad B_{1,i} v_i(1) = 0,\]

\[B_{0,i} w_i(0) = 0, \quad B_{1,i} w_i(1) = -D'_{1,i} \varphi_s.\]

Let us now substitute for decomposition \(26\) into the linear equations for external domain \(24\). As result we obtain the following four vector equations with regard to the unknown functions \(s_e(x), u_e(x), v_e(x)\) and \(w_e(x)\) with eight boundary conditions (four left + four right):

\[-s_e' + Q_e(x)s_e = y_e' - R_e F_e,\]

\[-u_e' + Q_e(x)u_e = -\left(F_e + R_s \frac{\partial F_e}{\partial R_s}\right),\]

\[-v_e' + Q_e(x)v_e = -R_s \frac{\partial F_e}{\partial \Omega},\]

\[-w_e' + Q_e(x)w_e = 0.\]

Here \(Q_e(x) \equiv R_s \frac{\partial F_e(R_s, x, y_e(x), \Omega)}{\partial y_e(x)}\) is a square matrix \((6 \times 6)\) whose elements are Frechet’s derivatives of the operator \(F_e\) at the point \(y_e(x), R_s, \Omega)\).

The corresponding linear BC are obtained in the same way as \(28\) and they become:

\[B_{1,e} s_e(1) = D_{1,e} - B_{1,e} y_e(1), \quad B_{\infty,e} s_e(\infty) = -B_{\infty,e} y_e(\infty),\]

\[B_{1,e} u_e(1) = 0, \quad B_{\infty,e} u_e(\infty) = 0,\]

\[B_{1,e} v_e(1) = 0, \quad B_{\infty,e} v_e(\infty) = 0,\]

\[B_{1,e} w_e(1) = -D'_{1,e} \varphi_s, \quad B_{\infty,e} w_e(\infty) = 0.\]

In the end to compute the increments \(\{\rho, \omega, \phi\}\) of parameters \(R_s, \Omega\) and \(\varphi_s\) we use the three conditions \(17\).

Let the solutions of linear BVP \(27\), \(28\) and \(29\), \(30\) at the \(k\)th iteration stage are assumed to be known. For sake of the simplicity we introduce the vector \(\hat{y}(x) \equiv (\nu(x), \xi(x), \eta(x))^T\). For
two arbitrary functions \( h_i(x) \) and \( h_e(x) \), defined in left and right vicinity of the point \( x = 1 \), we set \( \Delta h \equiv h_e(1) - h_i(1) \). Then applying the CANM to the equations (17) and having in mind the decompositions (23), (24), we attain the vector equation
\[
\Delta \mathbf{u} \rho + \Delta \mathbf{v} \omega + \Delta \mathbf{w} \phi = - (\Delta \mathbf{y} + \Delta \mathbf{s}),
\]
which represents an algebraic system consisting of three linear scalar equations with respect to the three unknowns \( \rho, \omega \) and \( \phi \).

The general sequence of the algorithm can be recapitulated in the following way. Let us assume that the functions \( y_{i,k}(x) \), \( y_{e,k}(x) \), and parameters \( R_{s,k}, \Omega_k, \varphi_{s,k} \) are given for \( k \geq 0 \). We solve the linear BVPs (27), (28) and thus we compute the functions \( s_{e,k}(x), u_{i,k}(x), v_{i,k}(x), w_{i,k}(x) \) in the inner domain \( x \in [0,1] \). Then we solve the linear BVPs (29), (30) in the external domain \( x \in [1,\infty] \) and compute the functions \( s_{e,k}(x), u_{e,k}(x), v_{e,k}(x) \) and \( w_{e,k}(x) \). Next, to obtain the increments \( \rho_k, \omega_k \) and \( \phi_k \) we solve the linear algebraic system (31). So using the decompositions (23), (24) and then the formulae (13) - (22) we calculate the functions \( y_{i,k+1}(x), y_{e,k+1}(x) \), the radius of the star \( R_{s,k+1} \), the quantity \( \Omega_{k+1} \) and the parameter boundary condition \( \varphi_{s,k+1} \) as well at the new iteration stage \( k+1 \).

At every iteration \( k \) an optimal time step \( \tau_{opt} \) is determined in accordance with the Er- makov&Kalitkin formula (10)
\[
\tau_{opt} \approx \frac{\delta(0)}{\delta(0) + \delta(1)}
\]
where the residual \( \delta(\tau) \) is calculated as follows
\[
\delta(\tau_k) = \max \left[ \delta_f, (R_{s,k} + \tau_k \rho_k)^2, (\Omega_k + \tau_k \omega_k)^2, (\varphi_{s,k} + \tau_k \phi_k)^2 \right]
\]
and \( \delta_f \) is the Euclidean residual of right hand side of the first equations in the systems (27), (28) and (29), (30).

The criterion for termination of the iterations is \( \delta(\tau_{opt}) < \varepsilon \) where \( \varepsilon \sim 10^{-8} \pm 10^{-12} \) for some \( k \).

Taking into account the smoothness of sought solutions we solve the linear BVPs (27), (28) and (29), (30) employing Hermitean splines and spline collocation scheme of fourth order of approximation (11). At that we utilize essentially the important feature that everyone of the above mentioned two groups vector BVPs (inner and external) have one and the same left hand sides.

It is worth to note that the algebraic systems of linear equations and the system (11) as well become ill-posed in the vicinity of the “exact” solution, i.e., for sufficiently small residuals \( \delta \). That is why for small \( \delta \), for example if \( \delta < 10^{-3} \) (then \( \tau_{opt} \sim 1 \) usually), it is expedient to use the Newton-Kantorovich method when the respective matrices are fixed for some \( \delta \geq 10^{-3} \).

## 4 Some numerical results

For a purpose of illustrating we will consider and discuss some results obtained from numerical experiments. A detailed description and analysing of results from physical point of view will be object of another our paper.

In present article we consider concrete scalar-tensor model with functions (see Section 3)
\[
A(\varphi) = \exp\left(\frac{\varphi}{\sqrt{3}}\right), \quad V(\varphi) = \frac{3}{2}\left(1 - A^2(\varphi)\right)^2,
\]
\[
f(\mu) = \frac{1}{8}\left[(2\mu - 3)\sqrt{\mu + \mu^2} + 3 \ln \left(\sqrt{\mu + \sqrt{1 + \mu}}\right)\right],
\]
\[
g(\mu) = \frac{1}{8}\left[(6\mu + 3)\sqrt{\mu + \mu^2} - 3 \ln \left(\sqrt{\mu + \sqrt{1 + \mu}}\right)\right],
\]
\[
W(\sigma^2) = -\frac{1}{2} \left(\sigma^2 + \frac{1}{2} \Lambda \sigma^4\right).
\]
The quantities $b, \Lambda$ are given parameters. For completeness we note that in the concrete case the functions $f(\mu)$ and $g(\mu)$ represent in parametric form the equation of state of noninteracting neutron gas while the function $W(\sigma^2)$ describes the boson field with quartic self-interaction.

The calculated functions $\sigma(x)$, $\varphi(x)$, $\nu(x)$ and $\mu(x)$ are plotted correspondingly in Fig. 1, 2, 3 and 4 for the values of the parameters $\gamma = 0.1$, $\Lambda = 10$ and $b = 1$. The behaviour of the mentioned functions is typical for wider range of the parameters not only for those values presented in the figures. The function $\sigma(x)$ decreases rapidly from its central value $\sigma_c = 0.4$ (in the case under consideration) to zero, at that when dimensionless coordinate $x > 6$ the function does not exceed $10^{-4}$. Similarly the function $\nu(x)$ has most large derivative for $x \in (0, 9)$ after that it approaches slowly zero at infinity like $1/x$. For example when $x \approx 9$ the derivative $\nu'(x) \approx 10^{-2}$, while for $x > 27$ $\nu'(x) < 10^{-4}$, i.e., the asymptotical behaviour of calculated grid function and its derivative agrees very well with the theoretical prediction (see [3]). The function $\varphi(x)$ increases rapidly for $x < 4$, besides that it trends asymptotically to zero. Obviously the quantitative behaviour of $\varphi(x)$ for central value $\sigma_c = 0.4$ is determined by the dominance of the term $B$ over the term $F$ (see [3]). At last the function $\mu(x)$ is nontrivial in the inner domain $x \in [0, 1]$, i.e., inside the star. Here it varies monotonously and continuously from its central...
value (in the case under consideration) $\mu_c = 1.2$ till zero at $x = 1$ corresponding to the radius of the star.

From physical point of view it is important to know the mass of the boson-fermion star and the total number of particles (bosons and fermions) making up the star.

The dimensionless star mass can be calculated via the formula

$$M = \int_0^\infty r^2 \left( \frac{B}{T_0} + T_0^0 + \exp(-\lambda) \left( \frac{d\varphi}{dr} \right)^2 + \frac{\gamma^2}{2} V(\varphi) \right) dr.$$ 

The dimensionless rest mass of the bosons (total number of bosons times the boson mass) is given by

$$M_{RB} = \Omega \int_0^\infty r^2 A^2(\varphi) \exp \left( \frac{\lambda - \nu}{2} \right) \sigma^2 dr.$$
The dimensionless rest mass of the fermions is correspondingly

\[ M_{RF} = b \int_{0}^{\infty} r^2 A^3(\varphi) \exp \left( \frac{\lambda}{2} \right) n(\mu) dr \]

where \( n(\mu) \) is the density of the fermions. In the case we consider we have \( n(\mu) = \mu^3(x) \).

The dependencies of the star mass \( M \) (solid line) and the rest mass of fermions \( M_{RF} \) (dash line) on the central value \( \mu_c \) of the function \( \mu(x) \) are shown in configuration diagram on Fig. 5 for \( \lambda = 0, \gamma = 0.1, b = 1 \) and \( \sigma_c = 0.002 \). It should be pointed that for so small central value \( \sigma_c \) we have in practice pure fermionic star. On the figure it is seen that from small values of \( \mu_c \) to values near beyond the peak the rest mass is greater than the total mass of the star which means that the star is potentially stable.

![Figure 5: The star mass \( M \) and the rest fermion mass \( M_{RF} \) as functions of the central value \( \mu_c \) with parameters \( \lambda = 0, \gamma = 0.1, b = 1, \sigma_c = 0.002 \)](image)

On Fig. 6 the binding energy \( E_b = M - M_{RB} - M_{RF} \) is drawn against the rest mass of fermions \( M_{RF} \) for \( \lambda = 0, \gamma = 0.1, b = 1 \) and \( \sigma_c = 0.002 \). Fig. 6 is actually a bifurcation diagram. With increasing the central value of function \( \mu(x) \) one meets a cusp. The appearance of a cusp shows that

![Figure 6: The binding energy \( E \) versus the rest fermion mass \( M_{RF} \) with parameters \( \lambda = 0, \gamma = 0.1, b = 1, \sigma_c = 0.002 \)](image)
the stability of the star changes - one perturbation mode develops instability. Beyond the cusp the star is unstable and may collapse eventually forming black hole. The corresponding physical results for pure boson stars are considered in our recent paper [11].

Acknowledgments

We are grateful to Prof. I.V. Puzynin (JINR, Dubna, Russia) for helpful discussion.

References

[1] U. Ascher, J. Christiansen and R. D. Russell, *A collocation solver for mixed order systems of BVP*, Math. Comp., 33 (1979), pp. 659–679; *Collocation software for boundary-value ODE’s*, ACM Trans. Math. Soft., 7 (1981), pp. 209-222.

[2] J. Balakrishna and H. Shinkai, *Dynamical evolution of boson stars in Brans-Dicke theory*, Phys. Rev., D58 (1998), p.044016-1.

[3] T. L. Boyadjiev, M. D. Todorov, P. P. Fiziev and S. S. Yazadjiev, *Mathematical modeling of boson-fermion stars in the generalized scalar-tensor theory of gravity*, E-print: math.SC/9911118, submitted to J.Comp.Phys.

[4] C. Brans and R. Dicke, *Mach’s principle and a relativistic theory of gravitation*, Phys. Rev., 124 (1961), pp. 925–935.

[5] C. Callan, D. Friedan, E. Martinec and M. Perry, *Strings in background fields*, Nucl.Phys., B262 (1985), p.593.

[6] M. Colpi, S. Shapiro and I. Wasserman, *Boson stars: Gravitational equilibria of self-interacting scalar fields*, Phys. Rev. Lett., 57 (1986), p.2485.

[7] G. Comer and H. Shinkai, *Generation of scalar-tensor gravity effects in equilibrium state boson stars*, Class. Quant. Grav., 15 (1998), p.669.

[8] T. Damour and G. Esposito-Farese, *Tensor-multi-scalar theories of gravitation*, Class. Quantum Grav., 9 (1992), p.2093.

[9] R. Dicke, *Mach’s principle and invariance under transformation of units*, Phys. Rev., 125 (1962), p.2163.

[10] V. V. Ermakov and N. N. Kalitkin, *The optimal step and regularisation of the Newton’s method*, JVMiMF, 21(6) (1981), p.491 (in Russian).

[11] P. P. Fiziev, S. S. Yazadjiev, T. L. Boyadjiev and M. D. Todorov, *Boson stars in massive dilatonic gravity*, E-print: gr-qc/0001110; accepted in Phys. Rev. D.

[12] E. Fradkin and A. Tseytlin, *Effective field theory from quantized strings*, Phys.Lett., B158 (1985), p.316.

[13] M. K. Gavurin, *Nonlinear functional equations and continuous analogues of iterative methods*, Izvestia VUZ, Matematika, 14(6) (1958), p.18–31 (in Russian).

[14] M. Gleiser and R. Watkins, *Gravitational stability of stellar matter*, Nucl. Phys., B319 (1989), p.733.

[15] M. B. Green, J. H. Schwarz and E. Witten, *Superstring theory*, Cambridge University Press, Cambridge, 1987.

[16] M. Gunderson and L. Jensen, *Boson stars in the Brans-Dicke gravitational theories*, Phys. Rev., D48 (1993), p.5628.
[17] A. Henriques, A. Liddle and R. Moorhouse, Combined boson-fermion stars: configurations and stability, Nucl. Phys., B337 (1990), p.737.

[18] E. P. Jidkov, G. I. Makarenko and I. V. Puzynin, Continuous analog of Newton’s method in nonlinear physical problems, in Physics of Elementary Particles and Atomic Nuclei, Dubna, 1973, vol. 4, part I, pp. 127–166 (in Russian).

[19] D. Kaup Klein-Gordon Geon, Phys. Rev., 172 (1968), p.1331.

[20] J. Maharana and H. Schwarz, Noncompact symmetries in string theory, Nucl. Phys., B390 (1992), p.3.

[21] K. Meissner and G. Veneziano, Symmetries of cosmological superstring vacua, Phys. Lett., B267 (1991), p.33; Mod. Phys. Lett., A6 (1992), p.3398.

[22] W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery, Numerical recipes in Fortran, 2nd. ed., Cambridge University Press, Cambridge, 1992.

[23] R. Ruffini and S. Bonazzola, System of self-intersecting particles in general relativity and the concept of an equation of state, Phys. Rev., 187 (1969), p.1767.

[24] J. Sherk and J. Schwarz, Dual models and the geometry of space-time, Phys. Lett., B52 (1975), p.347.

[25] Z. Tao and X. Xue, Boson star in a gravitational theory with dilaton, Phys. Rev., D45 (1992), p.1878.

[26] D. Torres, Boson stars in general scalar-tensor gravitation: Equilibrium configurations, Phys. Rev., D56 (1997), p.3478.

[27] D. Torres, F. Shunk and A. Liddle, Brans-Dicke boson stars: Configurarions and stability through cosmic history, Class. Quant. Grav., 15 (1998), p.3701.

[28] P. N. Vabishchevich, (1987) Numerical methods for solving free-boundary problems, Publishing House of the Moscow State University, 1987 (in Russian).

[29] C. M. Will, Theory and Experiment in Gravitational Physics, Cambridge University Press, Cambridge, 1993.

[30] S. Yazadjiev, Tensor mass and particle number paek at the same location in the scalar-tensor gravity boson star models - an analytical proof, Class. Quant. Grav., 16 (1999), p.L63.

[31] Yu. S. Zavyalov, B. I. Kvasov and V. L. Mirosnichenko, Methods of spline-functions, Nauka, Moscow, 1998 (in Russian).

[32] Collected Algorithms of the ACM, http://www.acm.org/calgo/contents/.

[33] Numerical receipes Books on-line, http://www.ulib.org/webRoot/Books/Numerical_Recipes/.