Projective moduli space of semistable principal sheaves for a reductive group

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_in the sixtieth anniversary of Silvio Greco, to Nadia and Silvio._

1 Introduction

This contribution to the homage to Silvio Greco is mainly an announcement of results to appear somewhere in full extent, explaining their development from our previous article [G-S1] on conic bundles.

In [N-S] and [S1] Narasimhan and Seshadri defined stable bundles on a curve and provided by the techniques of Geometric Invariant Theory (GIT) developed by Mumford [Mu] a projective moduli space of the stable equivalence classes of semistable bundles. Then Gieseker [Gi] and Maruyama [MaI] [MaII] generalized this construction to the case of a higher-dimensional projective variety, obtaining again a projective moduli space by also allowing torsion-free sheaves. Ramanathan [R1] [R2] has provided the moduli space of semistable principal bundles on a connected reductive group $G$, thus generalizing the Narasimhan and Seshadri notion and construction, which then becomes the particular case $G = GL(n, \mathbb{C})$.

Faltings [Fa] has considered the moduli stack of principal bundles on semistable curves. For $G$ orthogonal or symplectic he considers a torsion-free sheaf with a quadratic form, and he also defines a notion of stability. For general reductive group $G$ he uses the approach of loop groups. Sorger

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had considered a similar problem. He works on a curve \( C \) (not necessarily smooth) on a smooth surface \( S \), and constructs the moduli space of torsion free sheaves on \( C \) together with a symmetric form taking values on the dualizing sheaf \( \omega_C \).

In the talk “open problems on principal bundles” closing the conference on “vector bundles on algebraic curves and Brill-Noether theory” at Bad Honnef 2000, prof. Narasimhan proposed the problem of generalizing the work of the late Ramanathan to the case of higher-dimensional varieties and to the case of positive characteristics. We solve the first problem by providing a suitable definition of principal sheaf on a higher-dimensional projective variety \( X \) over the complex field, and a definition of their (semi)stability, which in case \( \dim X = 1 \) is that of Ramanathan, and for which a projective moduli space can be obtained.

We start by recalling our notion of (semi)stable conic bundles, i.e. symmetric \((2,0)\)-tensors \( \varphi : E \otimes E \to \mathcal{O}_C \) of rank \( E = 3 \) on an algebraic curve \( C \), and their projective moduli space, notion and moduli space which have been generalized to the case of \((s,0)\) tensors on a curve by Schmitt with the purpose of dealing with (semi)stable objects \( \varphi : E^\rho \to M \), where \( E^\rho \) is the vector bundle associated to a vector bundle \( E \) and an arbitrary representation \( \rho \) of \( G = Gl(n, \mathbb{C}) \), and \( M \) is a line bundle. In case the symmetric \((2,0)\) tensor is of maximal rank at all points and \( \det(E) \cong \mathcal{O}_C \), i.e. the case when \((E, \varphi)\) is just a principal \( SO(3, \mathbb{C})\)-bundle, our notion of (semi)stability is drastically simplified and becomes equivalent to Ramanathan’s notion of (semi)stability. We then generalize to higher dimension, with techniques of Simpson and Huybrechts-Lehn, the notion and coarse projective moduli space of (semi)stable \((s,0)\)-tensors, by allowing \( E \) to be a torsion free sheaf and those symmetric or antisymmetric and nowhere degenerate provide thus the moduli space of principal sheaves on \( G = O(n, \mathbb{C}) \), \( Sp(n, \mathbb{C}) \), \( SO(2n + 1, \mathbb{C}) \), the remaining classical group \( SO(2n, \mathbb{C}) \) requiring a special treatment which fortunately does not alter the notion of (semi)stability.

Then we cope with the problem of an arbitrary connected reductive group \( G \), by defining principal sheaves as \((2,1)\) tensors, i.e. torsion free sheaves \( E \) and \( \varphi : E \otimes E \to E^{**} \), which on the points of the open set \( U_E \) where \( E \) is locally free are isomorphic to the structure tensor \( \varphi_{g'} : g' \otimes g' \to g' \) of the Lie algebra \( g' \) tangent to the commutator \( G' = [G,G] \), together with a \( G \to Aut(g') \) reduction of the associated principal bundle on \( U_E \). The (semi)stability is defined as just the one of the \((2,1)\) tensor \((E, \varphi)\) and leads to a coarse projective moduli space, reducing to Ramanathan’s (semi)stability and moduli space in case \( \dim X = 1 \).

This announcement note consists mainly of the precise definitions and statements of such objects and results.
2 Conic bundles.

Let $X$ be a complete, smooth, connected curve, and fix a positive rational $\tau > 0$. A conic bundle on $X$ of degree $d$ is a rank 3 symmetric $(2, 0)$-tensor on $X$, i.e. a vector bundle $E$ of rank 3 and degree $d$, together with a nonzero homomorphism

$$\varphi : E^2 = \text{Sym}^2 E \to L$$

Where $L$ is a line bundle. We say it is (semi)stable if

1) For any subbundle $F \subseteq E$, it is

$$\frac{\deg F - c_\varphi(F)\tau}{\text{rank } F} \leq \frac{\deg E - 2\tau}{\text{rank } E}$$

where

$$c_\varphi(F) = \begin{cases} 
2, & \text{if } \varphi(F^2) \neq 0 \text{ (i.e. } F \text{ not isotropic)} \\
1, & \text{if } \varphi(F^2) = 0 \text{ and } \varphi(FE) \neq 0 \\
0, & \text{if } \varphi(FE) = 0 \text{ (i.e. } F \text{ singular)} 
\end{cases}$$

2) For all critical flags $F_1 \subseteq F_2 \subseteq E$, i.e. $F_1$ of rank 1, $F_2$ of rank 2, $\varphi(F_1F) \neq 0$, $\varphi(F_2F_2) \neq 0$ and $\varphi(F_1F_2) = 0$ (at a general point of $X$ these are a point of the conic and its tangent line), the following inequality holds

$$\deg F_1 + \deg F_2 \leq \deg E$$

(By the expression (semi)stable we always mean both semistable and stable, and then by the symbol ($\leq$) we mean $\leq$ and $<$, respectively). As usual, there is a notion of stable equivalence classes of semistable objects (see [G-S1] for the definition), and then it is proved in [G-S1], by the use of GIT, the following

**Theorem 1.** There is a projective coarse moduli space of stable equivalence classes of semistable conic bundles of degree $d$ and parameter $\tau$, on a smooth, complete, connected curve.

If $\det(E) \cong \mathcal{O}_X$, $L \cong \mathcal{O}_X$ and $\varphi$ is nowhere degenerate, i.e. such that $\text{rank } \varphi(x) = 3$ for all $x \in X$, which amounts to a principal $SO(3)$-bundle on $X$, then the condition 2), independent of the parameter $\tau$ is enough for the definition of (semi)stability, thus leading to a projective coarse moduli space as in Theorem 1.

Recall that in [R1], [R2], a definition of (semi)stable principal bundle $P$ on a curve, for a connected, reductive group $G$ was already given: if for all
reduction $P(H)$ of $P$ to a maximal parabolic subgroup $H \subseteq G$, the vector bundle $P(H, \mathfrak{h})$ associated to $P$ by the adjoint representation of $H$ in its tangent Lie algebra $\mathfrak{h}$, has

$$\deg P(H, \mathfrak{h}) (\leq) 0$$

In fact Ramanathan obtains in [R2] a projective coarse moduli space of stable equivalence classes of semistable principal $G$-bundles of fixed topological type and our result for $SO(3)$-bundles on $X$ becomes a particular case of Ramanathan’s result, because it is proved in [G-S1] that condition 2 is equivalent to the notion of Ramanathan.

Rank 2 bundles correspond, after projectivization, to geometrically ruled surfaces, and properties of the (semi)stable objects have been largely studied since their definition in [N-S] and [S1]. Our definition of (semi)stable conic bundles opens analogous problems. For instance we would like to express here the following conjecture. If has been proved in [C-S], for a semistable scroll of $\mathbb{P}^r$ of degree $d$ and irregularity $q$ which is special (i.e. $r$ distinct of Riemann-Roch number $d + 1 - 2q$), the existence of a hyperplane containing $r - 1$ lines of the ruling, which amounts to the upper bound $d - (r - 1)$ for the degree of a unisecant curve of the ruled surface, a problem posed by Severi in [Se] (the analogous bound being trivial in the nonsemistable case). Most probably, for a special semistable conic bundle of $\mathbb{P}^r$ there is a hyperplane containing $\left\lfloor \frac{r-2}{2} \right\rfloor$ of its conics, thus leading to an analogous upper bound of the minimal degree of a bisecant curve of the surface (and so on).

3 Principal sheaves for a classical group

Let $X$ be a smooth, projective complex variety of dimension $n$.

Definition 2. A tensor field, or just a tensor, on $X$, is a pair $(E, \varphi)$ consisting of a torsion free sheaf $E$ and an homomorphism

$$\varphi : \otimes^s E \to \mathcal{O}_X,$$

the rank and Chern classes of the tensor being called those of $E$. Let $\sigma$ be a positive rational polynomial of degree at most $n-1$ (i.e. rational coefficients, and positive leading coefficient). The tensor is said to be $\delta$-(semi)stable if for all weighted filtration $(E., m.)$ of $E$, i.e. subsheaves $E_1 \subset \ldots \subset E_t \subset E_{t+1} = E$ and positive integers $m_1, \ldots, m_t$, it is

$$\sum m_t (r \chi_{E_t} - r \chi_{E}) + \delta \mu(E., m., \varphi)(\leq) 0$$
where \( r, r_i, \chi_E, \chi_{E_i} \) are the ranks and Hilbert polynomials of \( E, E_i \), and \( \mu \) is defined as

\[
\mu = \min \{ \lambda_{i_1} + \ldots + \lambda_{i_s} \mid \varphi(E_{i_1} \otimes \ldots \otimes E_{i_s}) \neq 0 \}
\]

where \( \lambda_1 < \ldots < \lambda_s \) are integers with \( \lambda_i - \lambda_{i-1} = m_i \) and

\[
\sum \lambda_i \text{rank}(E_i / E_{i-1}) = 0.
\]

In [G-S2] the definition is slightly more general, and we prove the following

**Theorem 3.** There is a coarse projective moduli space of \( \delta \)-stable equivalence classes of \( \delta \)-semistable tensors on a projective variety \( X \), of fixed Chern classes and rank.

The proof has two parts: first, show that the family consisting of such objects is bounded (remark that for \( \delta \)-semistable \( (E, \varphi) \), the torsion free sheaf \( E \) needs not be semistable). Second, proceed with the techniques of Simpson [S] and Huybrechts-Lehn [H-L], starting by considering an integer \( m \gg 0 \) such that all torsion free sheaves in the family are generated by global sections and have \( H^0(E(m)) = \chi_E(m) \). For each member of the family choose an isomorphism \( \beta \) of \( H^0(E(m)) \) with a fixed complex vector space \( V \) of dimension \( \chi(E(m)) \), thus obtaining a quotient

\[
V \otimes \mathcal{O}_X(-m) \simeq H^0(E(m)) \otimes \mathcal{O}_X(-m) \longrightarrow E
\]

inducing, for \( l \) high enough, a quotient

\[
q : V \otimes H^0(\mathcal{O}_X(l-m)) \longrightarrow H^0(E(l)).
\]

Consider also the induced homomorphism

\[
\psi : V^{\otimes s} \longrightarrow H^0(E(m)^{\otimes s}) \longrightarrow H^0(\mathcal{O}_X(sm))
\]

We then obtain an element of

\[
\mathbb{P}\left( \bigwedge^{\chi_E(l)}(V^* \otimes H^0(\mathcal{O}_X(l-m))^*) \right) \times \mathbb{P}\left( V^{* \otimes s} \otimes H^0(\mathcal{O}_X(sm)) \right)
\]

which we consider included in projective space by the linear system \( |\mathcal{O}(n_1, n_2)| \) with

\[
\frac{n_2}{n_1} = \frac{\chi_E(l)\delta(m) - \delta(l)\chi_E(m)}{\chi_E(m) - s\delta(m)}
\]

This assignation embeds in a projective space \( \mathbb{P} \) the scheme \( R \) of triples \( (E, \varphi, \beta) \), with \( (E, \varphi) \) being a \( \delta \)-semistable tensor of the given rank and Chern classes and \( \beta \) a choice of basis as above. Quotienting by GIT with the natural action of \( Sl(V) \) on \( R \), induced from its natural action on \( \mathbb{P} \), we obtain the wanted projective coarse moduli space.
**Definition 4.** Let $G = O(r, \mathbb{C})$ or $Sp(r, \mathbb{C})$. A principal $G$-sheaf on $X$ is a tensor $\varphi : E \otimes E \to \mathcal{O}_X$ symmetric or antisymmetric which induces an isomorphism $E|_U \to E^*_|_U$ on the open set $U$ where $E$ is locally free. We call it (semi)stable if for all isotropic subsheaves $F \subseteq E$ it is

$$\chi_F + \chi_{F^\perp} \leq \chi_E$$

**Theorem 5.** For any positive polynomial $\delta$ of degree at most $n - 1$, a principal $G$-sheaf on $X$ ($G = O(r, \mathbb{C})$ or $Sp(r, \mathbb{C})$) is $\delta$-(semi)stable if and only if it is (semi) stable, so there is a coarse projective moduli space of stable-equivalence classes of semistable principal $G$-sheaves.

The remaining classical group $G = SO(r, \mathbb{C})$. Define a principal $SO(r, \mathbb{C})$-sheaf to be a triple $(E, \varphi, \psi)$, where $(E, \varphi)$ is a principal $O(r, \mathbb{C})$-sheaf and $\varphi$ is an isomorphism between $\text{det}(E)$ and $\mathcal{O}_X$ such that $\text{det}(\varphi) = \psi^2$. Note that for each $O(r, \mathbb{C})$-sheaf $(E, \varphi)$, there is at most two distinct $SO(r, \mathbb{C})$-sheaves, namely $(E, \varphi, \psi)$ and $(E, \varphi, -\psi)$. If $\text{rank}(E)$ is odd, these two objects are isomorphic. This is why for $SO(2m + 1, \mathbb{C})$ we can forget the third datum $\psi$. But if $\text{rank}(E)$ is even, these two objects might not be isomorphic. With the same definition of (semi)stability as in Definition 4, Theorem 5 still holds in this case (i.e. the added datum does not alter the GIT notion of stability) so we obtain a coarse projective moduli space in the case $G$ is any classical group.

## 4 Principal sheaves on a reductive group

Tensors considered in Section 3 were all $(s, 0)$ tensors, but with the same machinery we could have worked with (semi)stability and coarse projective moduli space of $(s, 1)$-tensors. In particular we need in this section $(2, 1)$-tensors $\varphi : E \otimes E \to E^{**}$, for which $\delta$-(semi)stability is defined by the fact that for all weighted filtration $(E_1 \subset \ldots \subset E_t, m_1, \ldots, m_t > 0)$ of $E$, it is

$$\sum m_i(r\chi_{E_i} - r_i\chi_E) + \delta \mu(E., m., \varphi) \leq 0$$

where

$$\mu = \min \{\lambda_i + \lambda_j - \lambda_k | 0 \neq \overline{\varphi} : E_i \otimes E_j \to E^{**}/E_{k-1}^{**}\}$$

For fixed value of rank and Chern classes, there is a projective coarse moduli space of stable equivalence classes of $\delta$-semistable $(2, 1)$ tensors on $X$. 


Definition 6. Let $X$ be a projective variety, and $G$ an algebraic group. A principal $G$-sheaf $\mathcal{P}$ is a triple $(E, \varphi, \xi)$ where $(E, \varphi)$ is a $(2,1)$-tensor on $X$

$$\varphi : E \otimes E \rightarrow E^{**}$$

such that for the points $x$ of the open set $U_E$ where $E$ is locally free, $\varphi(x)$ is isomorphic to the structure tensor $\varphi_{g'} : g' \otimes g' \rightarrow g'$ of the Lie algebra $g'$ tangent to the commutator subgroup $G' = [G, G]$ (in particular, there is an associated $\text{Aut}(g')$-bundle $P_{U_E}$ on $U_E$), and $\xi$ is a reduction of $P_{U_E}$ to $G$, via $\text{Ad} : G \rightarrow \text{Aut}(g')$.

Obviously, if $E$ is locally free, we recover the usual notion of principal $G$-bundle.

Definition 7. Let $G$ be a connected reductive group. We say that a principal $G$-sheaf $\mathcal{P} = (E, \varphi, \xi)$ is semistable if $E$ is semistable. We say it is strictly semistable if there is not a Lie algebra filtration, i.e.

$$E_1 \subseteq \ldots \subseteq E_t \subseteq E_{t+1} = E$$

such that $[E_i, E_j] \subseteq E_{i+j}^{**}$ with all

$$\frac{\chi_{E_i}}{r_i} = \frac{\chi_E}{r}$$

Theorem 8. There is a projective coarse moduli stable of equivalence classes of semistable principal $G$-sheaves on $X$ of fixed topological type.

Comment on the proof: It is a very long proof, parallel to the proof of Ramanathan [R2], which will appear published elsewhere. Because of the nondegeneracy of the Killing form of the semisimple Lie algebra $g'$, the factor $\mu(E, m, \varphi)$ is always nonpositive, so for a polynomial $\delta$ of degree zero, and small with respect to the invariants of $X$, $E$, our notion of (semi)stability of $\mathcal{P} = (E, \varphi, \xi)$ is equivalent to the $\delta$-(semi)stability of the $(2,1)$-tensor $(E, \varphi)$. It does not assure the existence of a moduli space, because it must also be checked that the extra datum of reduction $\xi$ does not alter the (semi)stability in the sense of GIT of the corresponding point of the $\text{Sl}(V)$-acted projective space, which is the main bulk of the proof.

Finally, we need some considerations on root spaces in order to re-state (semi)stability in a way which is equivalent, but more convenient to check that it coincides with Ramanathan’s (semi) stability when $\dim X = 1$. Recall from [Bo] that a $t$-root decomposition

$$g' = \bigoplus_{\alpha \in R_t \cup \{0\}} g^{(\alpha)}$$
of the Lie algebra $\mathfrak{g}'$ arises whenever an abelian algebra $\mathfrak{t} \subseteq \mathfrak{g}'$ is given, not necessarily a Cartan algebra, in particular for the center $\mathfrak{t} = \mathfrak{z}(\mathfrak{h}')$ of the Levi component $\mathfrak{l}(\mathfrak{h}')$ of any parabolic subalgebra $\mathfrak{h}' \subset \mathfrak{g}'$. In this case a system of simple $\mathfrak{t}$-roots (or decomposition $R_\mathfrak{t} = R_\mathfrak{t}^+ \cup R_\mathfrak{t}^-$) is naturally given, so the set $R_\mathfrak{t} \cup \{0\}$ has a natural partial ordering ($\alpha \leq \beta$ if $\beta$ is the sum of $\alpha$ with a sum of simple $\mathfrak{t}$-roots). Denote $\mathfrak{g}'_{(\leq \alpha)} = \oplus_{\beta \leq \alpha} \mathfrak{g}_\beta'$ and analogously $\mathfrak{g}'_{(< \alpha)}$. We also write $R_{\mathfrak{h}'}$ for $R_{\mathfrak{t}}$. Both are invariant by the adjoint action of $\mathfrak{h}'$, thus by the inner automorphism action of the corresponding parabolic subgroup $H'$ of the group $G'$, so the analogous subalgebras $\mathfrak{g}'_{(\leq \alpha)}$ and $\mathfrak{g}'_{(< \alpha)}$ of the Lie algebra $\mathfrak{g}'$ are also $H'$-invariant.

Let $\mathcal{P} = (E, \varphi, \xi)$ be a principal sheaf on $X$, having on $U_E$ a further $H \hookrightarrow G$ reduction to a parabolic subgroup $H$, let $H' = H \cap G'$, and let $\alpha \in R_{\mathfrak{h}'} \cup \{0\}$ where $\mathfrak{h}' = \text{Lie}(H')$ as before. We define $E_{(\leq \alpha)}$ and $E_{(< \alpha)}$ as the saturated extensions to $X$ of the vector bundles on $U_E$ associated to this reduction by the above representation of $H'$ on $\mathfrak{g}'_{(\leq \alpha)}$ and $\mathfrak{g}'_{(< \alpha)}$, and define $E^\alpha$ as $E_{(\leq \alpha)}/E_{(< \alpha)}$.

**Proposition 9.** A semistable principal $G$-sheaf $\mathcal{P} = (E, \varphi, \xi)$ on $X$ is stable if and only if for all reductions of $\mathcal{P}|_{U_E}$ to a maximal parabolic subgroup $H$ and $\alpha \in R_{\mathfrak{h}'} \cup \{0\}$, it is

$$\frac{\chi_{E^\alpha}}{\text{rank}(E^\alpha)} < \frac{\chi_E}{\text{rank}(E)}$$

**The case dim $X=1$.** In this case we have $U_E = X$, then a principal sheaf is equivalent to a principal bundle. The polynomials $\chi_{E^\alpha}$ and $\chi_E$ in Proposition 9 can be replaced by $\deg E^\alpha$, $\deg E$, and, being $\deg E = 0$, the strict semistability of a semistable principal bundle amounts to $\deg E^\alpha < 0$ for all $\alpha$ as in Proposition 9. A short argument shows the following

**Proposition 10.** If $\dim X = 1$, a principal bundle $\mathcal{P} = (E, \varphi, \xi)$ is (semi)stable (Definition 7) if and only if for all reductions $P(H)$ to a maximal parabolic subgroup $H$ of $G$, we have $\deg(P(H, \mathfrak{h}))(\leq 0)$, where $P(H, \mathfrak{h})$ is the vector bundle associated to $P(H)$ by the adjoint representation of $H$ in its Lie algebra $\mathfrak{h}$.

Therefore, in case $\dim X = 1$, we obtain exactly that our notion of (semi)stability coincides with Ramanathan’s definition [R1, Remark 2.2].

**Remark 11.** In the case of $G$ classical group, it would be interesting to compare the definitions of sections 3 and 4. If $\dim X = 1$, then they coincide.
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