Order Restricted Bayesian Analysis of a Simple Step Stress Model

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**Abstract**

In this article we consider a simple step stress set up under the cumulative exposure model assumption. At each stress level the lifetime distribution of the experimental units are assumed to follow the generalized exponential distribution. We provide the order restricted Bayesian inference of the model parameters by considering the fact that the expected lifetime of the experimental units are larger in lower stress level. Analysis and the related results are extended to different censoring schemes also. The Bayes estimates and the associated credible intervals of the unknown parameters are constructed using importance sampling technique. We perform extensive simulation experiments both for the complete and censored samples to see the performances of the proposed estimators. We analyze two simulated and one real data sets for illustrative purposes. An optimal value of the stress changing time is obtained by minimizing the total posterior coefficient of variations of the unknown parameters.

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**1 Introduction**

Nowadays, since the products are highly reliable, it is very difficult to get sufficient failure time data in a normal condition during a reasonable experimental time. The accelerated life testing (ALT) procedures are proposed to overcome this problem. The ALT method has been introduced in a reliability experiment mainly to obtain more failures in a shorter interval of
time. In an ALT experiment, units are put into a higher stress level than the usual that ensures early failure of the experimental units. Interested readers are referred to Nelson (1980) and Bagdonavicius and Nikulin (2002) for an exposure to different ALT models. The step stress life test (SSLT) model is a special type of the ALT model in which stress level can be changed during the experiment. In a conventional SSLT, the stress levels are changed at pre-fixed time points. Hence, in a conventional SSLT experiment, \( n \) experimental units are placed into life testing experiment at an initial stress level \( S_1 \) and then the stress level changes to \( S_2, S_3, \ldots, S_m \) at prefixed time points \( \tau_1, \tau_2, \ldots, \tau_{m-1} \), respectively. If \( m = 2 \), i.e., in case of only two stress levels, the experiment is known as the simple SSLT experiment.

The data collected from such an SSLT experiment, may then be extrapolated to estimate the underlying distribution of failure times under normal stress level. To connect the distributions of lifetime under different stress levels various models have been proposed in the literature. One such model was introduced by Sediakin (1966), and it is known as the cumulative exposure model (CEM). The CEM relates the distributions of lifetime under different stress levels by assuming that the residual life of the experimental units depends only on the cumulative exposure that the units have experienced, with no memory of how this exposure was accumulated. Latter this model was extensively studied by Nelson (1980). Interested readers are referred to a review article by Balakrishnan (2009) or the recent monograph by Kundu and Ganguly (2017), and the references cited therein.

In this paper we consider a simple step stress model when the lifetime distribution of experimental units follow generalized exponential (GE) distribution with the common shape parameter \( \alpha \) but different scale parameters \( \theta_1 \) and \( \theta_2 \) at the two different stress levels. From now on it is assumed that a GE distribution with the shape parameter \( \alpha > 0 \) and scale parameter \( \lambda > 0 \), has the following probability density function (PDF)

\[
f(t; \alpha, \lambda) = \alpha \lambda (1 - e^{-\lambda t})^{\alpha-1} e^{-\lambda t}; \quad t \geq 0,
\]

zero otherwise, and it will be denoted by GE(\( \alpha, \lambda \)). The GE distribution was first considered by Gupta and Kundu (1999) as an alternative to the well known gamma or Weibull distributions. It is also an extension of the exponential distribution, and it also can have increasing or decreasing hazard functions similar to the gamma and Weibull distributions. The GE distribution has a decreasing density function if the shape parameter is less than one and the density function becomes unimodal if the shape parameter is greater than one. This distribution has a very good interpretation in case of integer shape parameter. If the shape parameter is an integer, this distribution
represents the lifetime of a parallel system where each component follows independent exponential distribution. It is observed, see for example Gupta and Kundu (2001), that there are many cases where GE provides a better fit than the gamma or Weibull distribution. Interested readers are referred to the article by Nadarajah (2011) for a survey on the GE distribution and the recent monograph by Al-Hussaini and Ahsanullah (2015) for the development of the different exponentiated distributions. It may be mentioned that Abdel-Hamid and AL-Hussaini (2009) considered the inference of the parameters of a GE distribution for simple SSLT model for Type-I censored data.

In a step stress model the basic assumption is that the expected lifetime of units under higher stress level is shorter than under the lower stress level. Therefore, this information can be incorporated by considering the order restriction on the scale parameters as \( \theta_1 < \theta_2 \). It seems although for a step-stress model, the order restricted inference is a natural choice, not much work has been done along this line mainly due to analytical difficulty. The order restricted inference for an exponential step stress model was first considered by Balakrishnan et al. (2009) in case of Type-I and Type-II censored data. It is observed that for exponential model, although the maximum likelihood estimators (MLEs) of the unknown parameters can be obtained in explicit forms, the associated exact confidence intervals cannot be obtained in explicit form. Bayesian inference seems to be a reasonable choice in this case. Samanta et al. (2017) developed the order restricted Bayesian inference for exponential simple step stress model. They obtained the Bayes estimates and the associated credible intervals of the unknown parameters under the squared error loss function based on importance sampling technique. The results have been developed for different censoring schemes also.

The main aim of this paper is to provide the Bayesian inference on order restricted parameters of a GE distribution for a simple SSLT model. It is assumed that at the two different stress levels the lifetime distributions of the items follow \( \text{GE}(\alpha, \theta_1) \) and \( \text{GE}(\alpha, \theta_2) \), respectively with \( \theta_1 < \theta_2 \). Moreover, it is assumed that it satisfies the CEM assumptions. We consider the Bayesian inference on the unknown parameters under a fairly flexible prior assumptions (the details of the priors will be provided in the next section). First we consider the complete sample, and provide the Bayes estimates and the associated credible intervals (CRIs) based on importance sampling technique. The necessary theoretical results for the convergence of the corresponding importance sampling procedure are also provided. The results are extended for different other censoring schemes, namely for Type I censoring, Type II censoring, Type I hybrid censoring scheme (HCS), introduced
by Epstein (1954), and for Type II hybrid censoring scheme, introduce by Childs et al. (2003), also. Extensive Monte Carlo simulations are performed for complete and censored samples to see the performance of the proposed method, and they are quite satisfactory. Two simulated and one real data sets have been analyzed for illustrative purposes.

Finally we consider the ‘optimal’ simple SSLT model under the same assumptions. Similar to the idea proposed by Zhang and Meeker (2005), we propose to choose the ‘optimal’ value of \( \tau_1 \), so that the sum of the posterior coefficient of variations of \( \alpha \), \( \theta_1 \) and \( \theta_2 \) is minimum. Since the posterior coefficient of variations of the unknown parameters cannot be obtained in explicit forms, we use Lindley’s approximation for the posterior coefficient of variations, and provide a methodology to choose the ‘optimum’ \( \tau_1 \). A small table with the ‘optimal’ values of \( \tau_1 \) is provided for different sample sizes and for different parameter values.

The rest of the paper is organized as follows. In Section 2, we provide the model and the necessary prior assumptions. The Bayesian inference of the unknown parameters for complete sample is provided in Section 3, and for different censoring schemes the results are provided in Section 4. Simulation and data analysis results are reported in Section 5. In Section 6 we consider the optimality of the simple SSLT model, and finally we conclude the paper in Section 7.

## 2 Model Assumption and Prior Information

Consider the simple step-stress model with two stress levels \( S_1 \) and \( S_2 \). Suppose \( n \) items are put into an experiment under the stress level \( S_1 \) and the stress level is changed to \( S_2 \) at a pre-fixed time \( \tau_1 \). The failure times, denoted by \( t_{1:n} < t_{2:n} < t_{3:n} < \ldots < t_{n:n} \), of the unit placed on the test are recorded chronologically. It is assumed that the lifetimes have a generalized exponential distribution under both the stress levels, with the common shape parameter \( \alpha \) and different scale parameters, say \( \theta_1 \) and \( \theta_2 \) under stress level \( S_1 \) and \( S_2 \), respectively. It is further assumed that the lifetime satisfies CEM assumptions. Hence, the cumulative distribution function (CDF) of the lifetime is given by

\[
F(t) = \begin{cases} 
(1 - e^{-\theta_1 t})^\alpha & \text{if } 0 < t \leq \tau_1 \\
(1 - e^{-\theta_2 (t + \frac{\theta_1}{\theta_2} \tau_1 - \tau_1)})^\alpha & \text{if } \tau_1 < t < \infty,
\end{cases}
\tag{2.1}
\]

and the corresponding PDF is given by

\[
f(t) = \begin{cases} 
\alpha \theta_1 (1 - e^{-\theta_1 t})^{\alpha - 1} e^{-\theta_1 t} & \text{if } 0 < t \leq \tau_1 \\
\alpha \theta_2 (1 - e^{-\theta_2 (t + \frac{\theta_1}{\theta_2} \tau_1 - \tau_1)})^{\alpha - 1} e^{-\theta_2 (t + \frac{\theta_1}{\theta_2} \tau_1 - \tau_1)} & \text{if } \tau_1 < t < \infty.
\end{cases}
\tag{2.2}
\]
The purpose of an ALT procedure is to increase the stress level which ensures the early failure of the experimental units. Hence, it is reasonable to assume that the mean lifetime at the stress level \( S_1 \) is larger than that at the stress level \( S_2 \), i.e.,

\[
\frac{1}{\theta_2} \left[ \psi(\alpha + 1) - \psi(1) \right] < \frac{1}{\theta_1} \left[ \psi(\alpha + 1) - \psi(1) \right],
\]

(2.3)

where \( \psi(\cdot) \) is the digamma function. From Eq. 2.3, it follows that \( \theta_1 < \theta_2 \). We use this information in our prior assumption as follows. Let us assume \( \theta_1 = \beta \theta_2 \), where \( 0 < \beta < 1 \). Suppose the prior belief of the experimenter is measured by the function \( \pi(\alpha, \theta_2, \beta) \), which is given by

\[
\pi(\alpha, \theta_2, \beta) = \pi_1(\alpha) \pi_2(\theta_2) \pi_3(\beta).
\]

It is assumed that \( \alpha \sim \text{Gamma}(a_0, b_0) \), \( \theta_2 \sim \text{Gamma}(a_1, b_1) \), \( \beta \sim \text{Beta}(a_2, b_2) \) and they are independently distributed. The joint prior distribution of \((\alpha, \theta_2, \beta)\), is given by

\[
\pi(\alpha, \theta_2, \beta) \propto \beta^{a_2-1} (1 - \beta)^{b_2-1} e^{-a_0 \alpha} \alpha^{b_0-1} e^{-a_1 \theta_2} \theta_2^{b_1-1}.
\]

(2.4)

### 3 Posterior Analysis and Bayesian Inference

Based on the joint prior distribution (2.4), and under the CEM assumptions, the joint posterior distribution of \( \alpha, \theta_2 \) and \( \beta \) is given by

\[
l(\beta, \theta_2, \alpha|\text{Data}) \propto \beta^{n_1+a_2-1} (1 - \beta)^{b_2-1} \theta_2^{n+b_1-1} e^{-A_1(\beta) \theta_2} \alpha^{n+b_0-1} e^{-A_2(\beta, \theta_2) \alpha}
\]

\[
\times \prod_{i=1}^{n_1} (1 - e^{-\beta \theta_2 t_{i:n}})^{-1} \prod_{i=n_1+1}^{n} (1 - e^{-\theta_2 (t_i - \tau_1 + \tau_1 \beta)})^{-1},
\]

(3.1)

where \( n_1 \) denotes the number of failures till \( \tau_1 \), and

\[
A_1(\beta) = a_1 + \beta \sum_{i=1}^{n_1} t_{i:n} + \sum_{i=n_1+1}^{n} (t_{i:n} - \tau_1 + \tau_1 \beta),
\]

\[
A_2(\beta, \theta_2) = a_0 - \sum_{i=1}^{n_1} \log(1 - e^{-\beta \theta_2 t_{i:n}}) - \sum_{i=n_1+1}^{n} \log(1 - e^{-\theta_2 (t_{i:n} - \tau_1 + \tau_1 \beta)}).
\]

Therefore, the Bayes estimate of some parametric function of \((\beta, \theta_2, \alpha)\), say \( g(\beta, \theta_2, \alpha) \), under the squared error loss function is

\[
\hat{g}_B(\beta, \theta_2, \alpha) = E_{\beta, \theta_2, \alpha|\text{Data}}(g(\beta, \theta_2, \alpha))
\]
\[
\frac{\int_0^1 \int_0^\infty \int_0^\infty g(\beta, \theta_2, \alpha)l(\beta, \theta_2, \alpha|\text{Data})d\alpha d\theta_2 d\beta}{\int_0^1 \int_0^\infty \int_0^\infty l(\beta, \theta_2, \alpha|\text{Data})d\alpha d\theta_2 d\beta}, \tag{3.2}
\]
provided the expectation exists. In general (3.2) cannot be obtained in explicit form. One can use approximation procedure like Lindley’s approximation or Tierney and Kadane’s Method. However, the associated CRI

**Algorithm 1**

**Step 1.** Generate $\beta_1$ from Uniform(0, 1), $\theta_{21}$ from Gamma($n + b_1$, $A_1(\beta_1)$), and $\alpha_1$ from Gamma($n + b_0$, $A_2(\beta_1, \theta_{21})$) distribution.

**Step 2.** Repeat Step 1, $N$ times to obtain ($\beta_1$, $\theta_{21}$, $\alpha_1$), ..., ($\beta_N$, $\theta_{2N}$, $\alpha_N$), where $\beta_i$, $\theta_{2i}$ and $\alpha_i$ is the generation of $\beta$, $\theta_2$ and $\alpha$ at $i$-th ($i = 1, \ldots, N$) replication respectively.

**Step 3.** Calculate $g_i = g(\beta_i, \theta_{2i}, \alpha_i)$ and $w_i = \frac{h(\beta_i, \theta_{2i}, \alpha_i)}{\sum_{j=1}^N h(\beta_j, \theta_{2j}, \alpha_j)}$ for $i = 1, \ldots, N$.

**Step 4.** The approximate value of Eq. 3.2 can be obtained as $\sum_{i=1}^N w_i g_i$.

**Step 5.** Rearrange $(g_1, w_1)$, $(g_2, w_2)$, ..., $(g_N, w_N)$ as $(g_1, w_{(1)})$, $(g_2, w_{(2)})$, ..., $(g_N, w_{(N)})$ where $g_1 \leq g_2 \leq \ldots \leq g_N$. Note that $w_{(i)}$’s are not ordered, they are just associated with $g_{(i)}$’s.

**Step 6.** A 100(1$-$\gamma)% CRI for $g(\beta, \theta_2, \alpha)$ can be obtain as ($g_{j_1}$, $g_{j_2}$), where $j_1$ and $j_2$ satisfy

$$j_1, j_2 \in \{1, 2, \ldots, N\}, \quad j_1 < j_2, \quad \sum_{i=j_1}^{j_2} w_{(i)} \leq 1 - \gamma < \sum_{i=j_1}^{j_2+1} w_{(i)}.$$

The 100(1$-$\gamma)% height posterior density (HPD) CRI of $g(\beta, \theta_2, \alpha)$ becomes $(g_{(j_1^*)}, g_{(j_2^*)})$, where $1 \leq j_1^* < j_2^* \leq N$ satisfy

$$\sum_{i=j_1^*}^{j_2^*} w_{(i)} \leq 1 - \gamma < \sum_{i=j_1^*}^{j_2^*+1} w_{(i)}, \quad \text{and} \quad g_{(j_2^*)} - g_{(j_1^*)} \leq g_{(j_2)} - g_{(j_1)},$$

for all $j_1$ and $j_2$ satisfying (3.3).
cannot be constructed using these techniques. Hence, we propose to use importance sampling technique to compute the Bayes estimates and the associated CRIs intervals of the unknown parameters. Note that posterior density of \((\beta, \theta_2, \alpha)\) can be written as

\[
l(\beta, \theta_2, \alpha \mid \text{Data}) \propto h(\beta, \theta_2, \alpha)l_1(\beta)l_2(\theta_2 \mid \beta)l_3(\alpha \mid \theta_2, \beta), \tag{3.4}
\]

where

\[
h(\beta, \theta_2, \alpha) = \beta^{n_1+a_2-1}(1 - \beta)^{b_2-1} [A_1(\beta)]^{-(n+b_1)} [A_2(\beta, \theta_2)]^{-(n+b_0)} 
\prod_{i=1}^{n_1} (1 - e^{-\theta_2 t_i})^{-1} \prod_{i=n_1+1}^{n} (1 - e^{-\theta_2 (t_i - \tau_1 + \tau \beta)})^{-1},
\]

\[
l_1(\beta) = 1 \quad \text{for } 0 < \beta < 1,
\]

\[
l_2(\theta_2 | \beta) = \frac{[A_1(\beta)]^{n+b_1}}{\Gamma(n+b_1)} \theta_2^{n+b_1-1} e^{-A_1(\beta) \theta_2} \quad \text{for } \theta_2 > 0,
\]

\[
l_3(\alpha | \theta_2, \beta) = \frac{[A_2(\beta, \theta_2)]^{n+b_0}}{\Gamma(n+b_0)} \alpha^{n+b_0-1} e^{-A_2(\beta, \theta_2) \alpha} \quad \text{for } \alpha > 0.
\]

Using Eq. 3.4, Algorithm 1 can following algorithm can be executed to compute the Bayes estimate and the associated credible interval of some parametric function \(g(\beta, \theta_2, \alpha)\) of \(\beta, \theta_2\) and \(\alpha\), as given in Eq. 3.2.

### 4 Different Censoring Schemes and Posterior Analysis

Due to the experimental time and budget restrictions, the experimenter often terminates the experiment before the last unit fails. This is known as censoring in the statistical terminology. In this section we discuss different censoring schemes and associated posterior analysis based on the same prior and model assumptions. Consider the following general notations for different censoring schemes. \(n_1^*\) = number of failure before \(\tau_1\); \(n_2^*\) = number of failure between \(\tau_1\) and \(\tau^*\); \(\tau^*\) = termination time of the experiment; \(n^*\) = total number of failure before \(\tau^*\).

#### 4.1. Type-I Censoring

In Type-I censoring scheme we stop the experiment at a prefix time, say \(\tau_2\) and the number of observations failed under stress level \(S_1\) and \(S_2\) are \(n_1\) and \(n_2\) respectively. In this case observed data are one of the forms

\((a) \quad \{\tau_1 < t_{1:n} < \ldots < t_{n_2:n} < \tau_2\},
\)

\((b) \quad \{t_{1:n} < t_{2:n} < \ldots < t_{n_1:n} < \tau_1 < t_{n_1+1:n} < \ldots < t_{n_1+n_2:n} < \tau_2\},
\)

\((c) \quad \{t_{1:n} < t_{2:n} < \ldots < t_{n_1:n} < \tau_1 < \tau_2\}.
\)

Under Type-I censoring scheme posterior distribution can be written as

\[
l(\beta, \theta_2, \alpha | \text{Data}) \propto h_1(\beta, \theta_2, \alpha)l_1(\beta)l_2(\theta_2 | \beta)l_3(\alpha | \theta_2, \beta), \tag{4.1}
\]
where
\[
\begin{align*}
h_1(\beta, \theta_2, \alpha) &= \beta n_1^* + a_2 - 1 (1 - \beta)^{b_2 - 1} [A_1(\beta)]^{-(n^* + b_1)} [A_2(\beta, \theta_2)]^{-(n^* + b_0)} \\
&\quad \times [A_3(\beta, \theta_2, \alpha)]^{(n^*)} \\
&\quad \prod_{i=1}^{n_1^*} (1 - e^{-\beta \theta_2 t_i})^{-1} \prod_{i=n_1^*+1}^{n^*} (1 - e^{-\theta_2 (t_i - \tau_1 + \tau_1 \beta)})^{-1}, \\
l_1(\beta) &= 1 \quad \text{for } 0 < \beta < 1, \\
l_2(\theta_2|\beta) &= \frac{[A_1(\beta)]^{n^* + b_1} \theta_2^{n^* + b_1 - 1} e^{-A_1(\beta) \theta_2}}{\Gamma(n^* + b_1)} \quad \text{for } \theta_2 > 0, \\
l_3(\alpha|\theta_2, \beta) &= \frac{[A_2(\beta, \theta_2)]^{n^* + b_0} \alpha^{n^* + b_0 - 1} e^{-A_2(\beta, \theta_2) \alpha}}{\Gamma(n^* + b_0)} \quad \text{for } \alpha > 0, \\
A_1(\beta) &= a_1 + \beta \sum_{i=1}^{n_1^*} t_{i:n} + \sum_{i=n_1^*+1}^{n^*} (t_{i:n} - \tau_1 + \tau_1 \beta), \\
A_2(\beta, \theta_2) &= a_0 - \sum_{i=1}^{n_1^*} \log(1 - e^{-\beta \theta_2 t_i}) - \sum_{i=n_1^*+1}^{n^*} \log(1 - e^{-\theta_2 (t_i - \tau_1 + \tau_1 \beta)}), \\
A_3(\beta, \theta_2, \alpha) &= 1 - \{1 - e^{-\theta_2 (\tau^* - \tau_1 + \tau_1 \beta)}\}^\alpha.
\end{align*}
\]

Here \(\tau^* = \tau_2\) and in case (a) \(n_1^* = 0, n_2^* = n_2\), in case (b) \(n_1^* = n_1, n_2^* = n_2\), and in case (c) \(n_1^* = n, n_2^* = 0\).

The Bayes estimate and the associated HPD credible interval of any parametric function of \((\beta, \theta_2, \alpha)\) can be obtain using the same algorithm as discussed in case of complete data.

4.2. Type-II Censoring In this censoring scheme the life testing experiment is terminated when the \(r\)th (prefixed number) failure occurs, i.e., the total number of failure is fixed but the termination time of the experiment is random. Available data under this censoring scheme is one of the forms.

(a) \(\{\tau_1 < t_{1:n} < \ldots < t_{r:n}\}\),
(b) \(\{t_{1:n} < t_{2:n} < \ldots < t_{n_1:n} < \tau_1 < t_{n_1+1:n} < \ldots < t_{r:n}\}; n_1 < r\),
(c) \(\{t_{1:n} < t_{2:n} < \ldots < t_{r:n} < \tau_1 < \tau_2\}\).

Based on Type-II censored data, the posterior analysis is same as that of Type-I censoring scheme with \(\tau^* = t_{r:n}\), \(n^* = r\) and in case (a) \(n_1^* = 0, n_2^* = r\); in case (b) \(n_1^* = n_1, n_2^* = r - n_1\); in case (c) \(n_1^* = r, n_2^* = 0\). All other expressions and the following analysis are same as the Type-I censoring scheme.

4.3. Type-I Hybrid Censoring The termination time in Type-I HCS is \(\tau^* = \min\{t_{r:n}, \tau_2\}\). Let \(n_1\) and \(n_2\) be the number of failures under stress level \(S_1\) and \(S_2\), respectively. Available data under this censoring scheme is one of the forms

(a) \(\{\tau_1 < t_{1:n} < \ldots < t_{r:n}\} \text{ if } t_{r:n} \leq \tau_2\),
(b) \(\{t_{1:n} < t_{2:n} < \ldots < t_{n_1:n} < \tau_1 < t_{n_1+1:n} < \ldots < t_{r:n}\} \text{ if } t_{r:n} < \tau_2, n_1 < r\),
(c) \(\{t_{1:n} < t_{2:n} < \ldots < t_{r:n} < \tau_1 < \tau_2\} \text{ if } t_{r:n} < \tau_1\),
(d) \(\{\tau_1 < t_{1:n} < \ldots < t_{n_2:n} < \tau_2\} \text{ if } t_{r:n} > \tau_2\).
\( t_1:n < \ldots < t_{n_1}:n < \tau_1 < t_{n_1+1}:n < \ldots < t_{n_1+n_2}:n < \tau_2 \) if \( t_r:n > \tau_2, n_1 < r \),

\( t_1:n < \ldots < t_{n_1}:n < \tau_1 < t_{n_1+1}:n < \ldots < t_{n_1+n_2}:n < \tau_2 \) if \( t_r:n > \tau_2 \).

Based on Type-I Hybrid censored data, the posterior analysis is same as that of Type-I censoring scheme with, for case (a) \( n_1^* = 0, n_2^* = r \), for case (b) \( n_1^* = n_1, n_2^* = r - n_1 \), for case (c) \( n_1^* = r, n_2^* = 0 \), for case (d) \( n_1^* = 0, n_2^* = n_2 \), for case (e) \( n_1^* = n_1, n_2^* = n_2 \), and for case (f) \( n_1^* = n_1, n_2^* = 0 \). All other expressions and the following analysis are same as the Type-I censoring scheme.

### 4.4. Type-II Hybrid Censoring

In Type-II HCS the experiment is terminated at \( \tau^* = \max\{t_r:n, \tau_2\} \). In this case the experimental time and the number of failures both are random but it ensures at least \( r \) failures from the experiment. Let \( n_1 \) and \( n_2 \) be the number of failures under stress level \( S_1 \) and \( S_2 \), respectively. Available data under this censoring scheme is one of the forms

- (a) \( \{\tau_1 < t_1:n < \ldots < t_r:n\} \) if \( t_r:n \geq \tau_2 \),
- (b) \( \{t_1:n < t_2:n < \ldots < t_{n_1}:n < \tau_1 < t_{n_1+1}:n < \ldots < t_{r:n}\} \) if \( t_r:n \geq \tau_2, n_1 < r \),
- (c) \( \{\tau_1 < t_1:n < \ldots < t_{n_2}:n < \tau_2\} \) if \( t_r:n < \tau_2 \),
- (d) \( \{t_1:n < \ldots < t_{n_1}:n < \tau_1 < t_{n_1+1}:n < \ldots < t_{n_1+n_2}:n < \tau_2\} \) if \( t_r:n < \tau_2, n_1 < r \),
- (e) \( \{t_1:n < \ldots < t_{n_1}:n < \tau_1 < \tau_2\} \) if \( t_r:n < \tau_2 \).

Based on the Type-II Hybrid censored data, the posterior analysis is same as that of the Type-I censoring scheme with, for case (a) \( n_1^* = 0, n_2^* = r \),

| \( n \) | \( \tau \) | \( \alpha \) | \( \text{AE} \) | \( \text{MSE} \) | \( \theta_1 \) | \( \text{AE} \) | \( \text{MSE} \) | \( \theta_2 \) | \( \text{AE} \) | \( \text{MSE} \) |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 5 | 0.7598 | 0.2177 | 0.1285 | 0.0061 | 0.2811 | 0.0758 |
| | 7 | 0.7669 | 0.2552 | 0.1246 | 0.0051 | 0.3040 | 0.1416 |
| | 9 | 0.7639 | 0.2163 | 0.1201 | 0.0041 | 0.3437 | 0.5255 |
| 20 | 5 | 0.6772 | 0.0633 | 0.1180 | 0.0027 | 0.2315 | 0.0102 |
| | 7 | 0.6745 | 0.0554 | 0.1157 | 0.0024 | 0.2394 | 0.0155 |
| | 9 | 0.6711 | 0.0545 | 0.1144 | 0.0021 | 0.2531 | 0.0272 |
| 30 | 5 | 0.6544 | 0.0331 | 0.1155 | 0.0018 | 0.2218 | 0.0056 |
| | 7 | 0.6483 | 0.0316 | 0.1125 | 0.0014 | 0.2207 | 0.0050 |
| | 9 | 0.6522 | 0.0294 | 0.1115 | 0.0013 | 0.2306 | 0.0229 |
| 40 | 5 | 0.6491 | 0.0235 | 0.1151 | 0.0015 | 0.2172 | 0.0038 |
| | 7 | 0.6427 | 0.0201 | 0.1119 | 0.0012 | 0.2161 | 0.0045 |
| | 9 | 0.6421 | 0.0201 | 0.1113 | 0.0010 | 0.2217 | 0.0062 |
| 50 | 5 | 0.6424 | 0.0173 | 0.1137 | 0.0012 | 0.2123 | 0.0026 |
| | 7 | 0.6406 | 0.0162 | 0.1127 | 0.0010 | 0.2160 | 0.0034 |
| | 9 | 0.6380 | 0.0152 | 0.1114 | 0.0009 | 0.2184 | 0.0045 |
for case (b) \( n_1^* = n_1, n_2^* = r - n_1 \), for case (c) \( n_1^* = 0, n_2^* = n_2 \), for case (d) \( n_1^* = n_1, n_2^* = n_2 \), for case (e) \( n_1^* = n_1, n_2^* = 0 \). All other expressions and the following analysis are same as the Type-II censoring scheme.

5 Simulation and Data Analysis

5.1. Simulation In this section first we perform some simulation experiments on complete data to evaluate the performances of proposed method. In this simulation study we consider almost non-informative priors on \( \alpha, \beta \) and \( \theta_2 \), i.e., \( a_0 = 0.0001, b_0 = 0.0001, a_1 = 0.0001, b_1 = 0.0001, a_2 = 1 \) and \( b_2 = 1 \) as suggested by Congdon (2006). Results are obtained on 5000 replications with \( N = 15000 \). The Bayes estimates and the associated mean square errors (MSEs) for different parameter values are obtained and they are presented in Tables 1, 2 and 3. As expected, the MSEs of Bayes estimates decrease as \( n \) increases. Also we provide the 95% symmetric and HPD CRI of the different parameters in Tables 4, 5 and 6. It has been observed that most of the cases average estimates (AE) are overestimated for all the parameters. Hence, we also consider the left sided CRIs in simulation study.

We have further performed some simulation experiments based on Type-I and Type-II censored data. We have taken the same parameter values and the priors. The order restricted Bayes estimates and the associated MSEs of Type-I and Type-II censored data are presented in Tables 14 and 15, respectively. 95% CRIs of censored data are provided in Tables 16 and 17.

Table 2: AEs and MSEs of \( \alpha, \theta_1 \), and \( \theta_2 \) based on 5000 simulations with \( \alpha = 1.0, \theta_1 = 0.1 \), and \( \theta_2 = 0.2 \) for different values of \( n \) and \( \tau \)

| \( n \) | \( \tau \) | \( \alpha \) | \( \theta_1 \) | \( \theta_2 \) |
|----|----|----|----|----|
|    |    | AE  | MSE | AE  | MSE | AE  | MSE |
| 10 | 5  | 1.3876 | 0.9952 | 0.1245 | 0.0048 | 0.2438 | 0.0214 |
|    | 7  | 1.3850 | 1.0298 | 0.1222 | 0.0042 | 0.2574 | 0.0417 |
|    | 9  | 1.3498 | 0.8898 | 0.1183 | 0.0036 | 0.2710 | 0.1230 |
| 20 | 5  | 1.1687 | 0.2340 | 0.1148 | 0.0022 | 0.2152 | 0.0049 |
|    | 7  | 1.1596 | 0.2240 | 0.1130 | 0.0019 | 0.2204 | 0.0065 |
|    | 9  | 1.1377 | 0.2006 | 0.1098 | 0.0016 | 0.2250 | 0.0105 |
| 30 | 5  | 1.1179 | 0.1374 | 0.1125 | 0.0016 | 0.2084 | 0.0029 |
|    | 7  | 1.1159 | 0.1330 | 0.1099 | 0.0013 | 0.2093 | 0.0031 |
|    | 9  | 1.1149 | 0.1277 | 0.1090 | 0.0012 | 0.2126 | 0.0042 |
| 40 | 5  | 1.1024 | 0.0981 | 0.1117 | 0.0014 | 0.2059 | 0.0021 |
|    | 7  | 1.0934 | 0.0890 | 0.1091 | 0.0010 | 0.2060 | 0.0022 |
|    | 9  | 1.0778 | 0.0781 | 0.1068 | 0.0008 | 0.2070 | 0.0027 |
| 50 | 5  | 1.0864 | 0.0746 | 0.1108 | 0.0012 | 0.2043 | 0.0016 |
|    | 7  | 1.0739 | 0.0653 | 0.1080 | 0.0009 | 0.2050 | 0.0018 |
|    | 9  | 1.0676 | 0.0633 | 0.1067 | 0.0007 | 0.2052 | 0.0022 |
Table 3: AEs and MSEs of $\alpha$, $\theta_1$, and $\theta_2$ based on 5000 simulations with $\alpha = 1.5$, $\theta_1 = 0.1$, and $\theta_2 = 0.2$ for different values of $n$ and $\tau$

| $n$ | $\tau$ | $\alpha$ | AE | MSE  | $\theta_1$ | AE | MSE  | $\theta_2$ | AE | MSE  |
|-----|--------|----------|-----|------|------------|-----|------|------------|-----|------|
| 10  | 5      | 2.0073   | 1.6746 | 0.1180 | 0.0030 | 0.2228 | 0.0089 |
|     | 7      | 2.0745   | 1.9250 | 0.1167 | 0.0030 | 0.2309 | 0.0113 |
|     | 9      | 2.0925   | 2.1423 | 0.1142 | 0.0026 | 0.2395 | 0.0380 |
| 20  | 5      | 1.7279   | 0.4353 | 0.1100 | 0.0016 | 0.2081 | 0.0032 |
|     | 7      | 1.7431   | 0.4939 | 0.1077 | 0.0014 | 0.2090 | 0.0036 |
|     | 9      | 1.7316   | 0.5466 | 0.1080 | 0.0013 | 0.2168 | 0.0047 |
| 30  | 5      | 1.6468   | 0.2595 | 0.1052 | 0.0011 | 0.2020 | 0.0018 |
|     | 7      | 1.6424   | 0.2727 | 0.1050 | 0.0010 | 0.2023 | 0.0019 |
|     | 9      | 1.6461   | 0.2619 | 0.1057 | 0.0009 | 0.2065 | 0.0026 |
| 40  | 5      | 1.6035   | 0.1714 | 0.1048 | 0.0010 | 0.2003 | 0.0014 |
|     | 7      | 1.5871   | 0.1629 | 0.1033 | 0.0008 | 0.1986 | 0.0014 |
|     | 9      | 1.5937   | 0.1617 | 0.1029 | 0.0007 | 0.2014 | 0.0018 |
| 50  | 5      | 1.5662   | 0.1269 | 0.1027 | 0.0009 | 0.1980 | 0.0011 |
|     | 7      | 1.5718   | 0.1320 | 0.1023 | 0.0007 | 0.1982 | 0.0011 |
|     | 9      | 1.5667   | 0.1205 | 0.1022 | 0.0006 | 0.2000 | 0.0014 |

Tables 14 to 17 are provided in the Appendix A.2. Censored data simulation results are very similar to that of complete data. In all the cases the parameter estimates are very consistent and the coverage percentages (CP) are very close to the nominal values. Also average lengths (AL) of CRIs are gradually decreases as sample size increases.

5.2. Data Analysis

5.2.1. Simulated Data Analysis. Here we consider the analysis of two simulated data sets; one the shape parameter is less than one and other it is greater than one. Data presented in Table 7 is generated from Eq. 2.1 with $\alpha = 0.6$, $\theta_1 = 0.1 \theta_2 = 0.2$, $n = 20$ and $\tau_1 = 5$. Artificially we have created Type-I and Type-II censored data by taking $\tau_2 = 8$ and $r = 16$, respectively. Prior assumptions are same as considered in simulation study. For Type-I censored data the Bayes estimates of $\alpha$, $\theta_1$, and $\theta_2$ under the squared error loss function are 0.6995, 0.1032, and 0.2747, respectively. In case of Type- II censored data Bayes estimates of $\alpha$, $\theta_1$, and $\theta_2$ are 0.6244, 0.0840 and 0.2659 respectively. Different CRIs for both Type-I and Type-II of censoring schemes are given in Table 8.

We analyze another data presented in Table 9 which is generated from the Eq. 2.1 with $\alpha = 1.5$. All other parameter values are same as the first data set. Here also we have considered Type-I and Type-II censored data. The Bayes estimates of $\alpha$, $\theta_1$, and $\theta_2$ in Type-I censoring are 1.2787, 0.1109, and 0.2269, respectively. In Type-II censored data Bayes estimates of $\alpha$, $\theta_1$,
Table 4: CPs and ALs of 95% CRI for $\alpha$, $\theta_1$ and $\theta_2$ based on 5000 simulations with $\alpha = 0.6$, $\theta_1 = 0.1$, and $\theta_2 = 0.2$ for different values of $n$ and $\tau$

| $n \times \tau$ | $\alpha$ |  |  |  |  |  |  |  |  |  |  |  |  |
|-----------------|---------|---|---|---|---|---|---|---|---|---|---|---|---|
|                 | CP      | AL | CP | AL | CP | AL | CP | AL | CP | AL | CP | AL | CP | AL |
| 10 5            | 95.36   | 1.3426 | 96.04 | 1.3458 | 95.06 | 1.2343 | 97.66 | 0.2657 | 97.26 | 0.2728 | 96.40 | 0.2504 | 95.70 | 0.5171 | 94.94 | 0.5722 | 95.18 | 0.5090 |
| 7               | 94.88   | 1.3446 | 95.54 | 1.3312 | 94.52 | 1.2265 | 97.48 | 0.2489 | 97.12 | 0.2497 | 95.86 | 0.2298 | 95.58 | 0.6246 | 95.36 | 0.7248 | 95.18 | 0.6178 |
| 9               | 95.14   | 1.3254 | 95.42 | 1.3088 | 94.82 | 1.2037 | 96.96 | 0.2354 | 97.54 | 0.2331 | 95.82 | 0.2144 | 96.10 | 0.8630 | 96.20 | 1.0492 | 95.90 | 0.8578 |
| 20 5            | 95.82   | 0.9474 | 96.42 | 0.8134 | 95.84 | 0.7687 | 97.60 | 0.2114 | 96.66 | 0.1927 | 96.06 | 0.1813 | 95.48 | 0.2932 | 94.58 | 0.3044 | 94.10 | 0.2824 |
| 7               | 96.18   | 0.9395 | 95.94 | 0.7905 | 96.16 | 0.7485 | 97.40 | 0.2003 | 95.58 | 0.1749 | 95.22 | 0.1646 | 95.30 | 0.3314 | 94.78 | 0.3510 | 94.84 | 0.3220 |
| 9               | 96.00   | 0.9388 | 95.50 | 0.7772 | 95.78 | 0.7377 | 97.54 | 0.1954 | 95.36 | 0.1650 | 95.28 | 0.1550 | 95.86 | 0.4657 | 95.68 | 0.5419 | 95.90 | 0.4578 |
| 30 5            | 95.72   | 0.8218 | 95.64 | 0.6303 | 95.44 | 0.5992 | 97.68 | 0.1918 | 95.34 | 0.1590 | 95.00 | 0.1506 | 94.86 | 0.2311 | 93.64 | 0.2345 | 92.96 | 0.2192 |
| 7               | 95.80   | 0.8124 | 95.28 | 0.6040 | 95.24 | 0.5758 | 97.22 | 0.1799 | 95.10 | 0.1406 | 94.36 | 0.1330 | 95.36 | 0.2476 | 95.14 | 0.2550 | 94.42 | 0.2371 |
| 9               | 95.86   | 0.8162 | 95.34 | 0.5979 | 95.28 | 0.5701 | 97.08 | 0.1743 | 94.10 | 0.1310 | 93.80 | 0.1234 | 95.40 | 0.2860 | 95.80 | 0.3010 | 94.82 | 0.2769 |
| 40 5            | 96.26   | 0.7632 | 95.62 | 0.5341 | 95.68 | 0.5085 | 97.26 | 0.1809 | 94.20 | 0.1381 | 93.50 | 0.1310 | 94.88 | 0.1961 | 92.98 | 0.1952 | 92.10 | 0.1833 |
| 7               | 96.22   | 0.7558 | 95.74 | 0.5111 | 95.78 | 0.4877 | 97.42 | 0.1699 | 95.26 | 0.1211 | 93.16 | 0.1147 | 95.06 | 0.2123 | 94.50 | 0.2145 | 93.66 | 0.2010 |
| 9               | 96.16   | 0.7543 | 95.30 | 0.4993 | 95.08 | 0.4765 | 97.10 | 0.1647 | 92.52 | 0.1115 | 92.32 | 0.1050 | 94.78 | 0.2383 | 95.20 | 0.2454 | 93.66 | 0.2284 |
| 50 5            | 96.56   | 0.7210 | 95.56 | 0.4658 | 95.48 | 0.4432 | 97.62 | 0.1721 | 93.14 | 0.1225 | 92.70 | 0.1162 | 94.98 | 0.1707 | 92.82 | 0.1677 | 91.72 | 0.1578 |
| 7               | 96.34   | 0.7211 | 95.10 | 0.4487 | 95.16 | 0.4276 | 97.88 | 0.1646 | 91.18 | 0.1084 | 91.16 | 0.1023 | 95.38 | 0.1922 | 94.26 | 0.1909 | 93.32 | 0.1793 |
| 9               | 96.26   | 0.7207 | 95.28 | 0.4392 | 95.14 | 0.4184 | 97.36 | 0.1583 | 90.82 | 0.0983 | 90.62 | 0.0924 | 95.00 | 0.2098 | 94.90 | 0.2123 | 93.64 | 0.1981 |
Table 5: CPs and ALs of 95% CRI for $\alpha$, $\theta_1$ and $\theta_2$ based on 5000 simulations with $\alpha = 1$, $\theta_1 = 0.1$, and $\theta_2 = 0.2$ for different values of $n$ and $\tau$

| $n \times \tau$ | CP | AL | CP | AL | CP | AL | CP | AL | CP | AL | CP | AL | CP | AL |
|-----------------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 10 5            | 96.08 | 2.7174 | 97.08 | 2.8248 | 95.30 | 2.5520 | 98.76 | 0.2508 | 98.54 | 0.2558 | 96.82 | 0.2368 | 95.68 | 0.3918 | 95.14 | 0.3921 | 94.96 | 0.3921 |
| 20 5            | 96.38 | 1.8108 | 97.34 | 1.6592 | 95.50 | 1.5566 | 98.40 | 0.2051 | 97.92 | 0.1907 | 96.08 | 0.1807 | 94.96 | 0.2664 | 95.18 | 0.2441 | 93.78 | 0.2339 |
| 30 5            | 95.96 | 1.5564 | 96.84 | 1.3035 | 94.76 | 1.2371 | 97.86 | 0.1883 | 97.68 | 0.1632 | 95.12 | 0.1557 | 94.18 | 0.2225 | 95.10 | 0.1930 | 93.32 | 0.1862 |
| 40 5            | 95.82 | 1.4405 | 96.26 | 1.1248 | 94.98 | 1.0746 | 97.56 | 0.1792 | 97.08 | 0.1467 | 93.98 | 0.1403 | 94.32 | 0.1985 | 94.66 | 0.1651 | 93.64 | 0.1597 |
| 50 5            | 96.10 | 1.3578 | 96.04 | 1.0029 | 93.98 | 1.0746 | 97.56 | 0.1792 | 97.08 | 0.1467 | 93.98 | 0.1403 | 94.32 | 0.1985 | 94.66 | 0.1651 | 93.64 | 0.1597 |
| 50 7            | 96.10 | 1.3578 | 96.04 | 1.0029 | 93.98 | 1.0746 | 97.56 | 0.1792 | 97.08 | 0.1467 | 93.98 | 0.1403 | 94.32 | 0.1985 | 94.66 | 0.1651 | 93.64 | 0.1597 |
| 50 9            | 96.38 | 1.3421 | 96.26 | 0.9439 | 94.98 | 1.0746 | 97.56 | 0.1792 | 97.08 | 0.1467 | 93.98 | 0.1403 | 94.32 | 0.1985 | 94.66 | 0.1651 | 93.64 | 0.1597 |
| 50 9            | 94.10 | 1.3281 | 95.44 | 0.8582 | 93.98 | 1.0746 | 97.56 | 0.1792 | 97.08 | 0.1467 | 93.98 | 0.1403 | 94.32 | 0.1985 | 94.66 | 0.1651 | 93.64 | 0.1597 |
Table 6: CPs and ALs of 95% CRI for $\alpha_1$, $\theta_1$ and $\theta_2$ based on 5000 simulations with $\alpha = 1.5$, $\theta_1 = 0.1$, and $\theta_2 = 0.2$ for different values of $n$ and $\tau$

| $n$ | $\tau$ | $\alpha_{\text{CP}}$ | $\alpha_{\text{AL}}$ | $\alpha_{\text{Left CRI}}$ | $\alpha_{\text{Symmetric CRI}}$ | $\alpha_{\text{HPD CRI}}$ | $\theta_1_{\text{CP}}$ | $\theta_1_{\text{AL}}$ | $\theta_1_{\text{Left CRI}}$ | $\theta_1_{\text{Symmetric CRI}}$ | $\theta_1_{\text{HPD CRI}}$ | $\theta_2_{\text{CP}}$ | $\theta_2_{\text{AL}}$ | $\theta_2_{\text{Left CRI}}$ | $\theta_2_{\text{Symmetric CRI}}$ | $\theta_2_{\text{HPD CRI}}$ |
|-----|--------|----------------|----------------|----------------------------|-------------------------------|----------------------------|----------------|----------------|----------------------------|-------------------------------|----------------------------|----------------|----------------|----------------------------|-------------------------------|----------------------------|
| 10  | 5      | 96.14 4.0881 | 97.82 4.2513 | 96.02 3.8536           | 99.10 0.2352                  | 99.28 0.2377                  | 98.14 0.2217 | 94.80 0.3361 | 95.02 0.3148                  | 95.36 0.2992                  |
| 7   | 95.36  4.1680 | 97.86 4.3001 | 95.54 3.8968           | 98.62 0.2207                  | 98.78 0.2161                  | 96.52 0.2019 | 95.60 0.3597 | 95.72 0.3398                  | 94.28 0.3215                  |
| 9   | 96.10  4.1478 | 97.00 4.2346 | 95.00 3.8512           | 98.08 0.2083                  | 97.96 0.1984                  | 95.40 0.1860 | 95.38 0.3998 | 95.62 0.3933                  | 94.22 0.3629                  |
| 20  | 5      | 96.04 2.7793 | 97.86 2.5945 | 95.84 2.4207           | 98.24 0.1944                  | 98.90 0.1802                  | 96.82 0.1701 | 94.04 0.2493 | 95.42 0.2049                  | 93.90 0.1969                  |
| 7   | 96.62  2.7795 | 97.74 2.5297 | 95.82 2.3624           | 97.90 0.1799                  | 98.30 0.1579                  | 95.64 0.1491 | 94.18 0.2603 | 94.76 0.2145                  | 93.40 0.2063                  |
| 9   | 96.10  2.7503 | 97.20 2.4378 | 94.50 2.2821           | 97.20 0.1730                  | 97.70 0.1442                  | 94.70 0.1363 | 94.40 0.2834 | 95.80 0.2392                  | 94.70 0.2294                  |
| 30  | 5      | 97.20 2.3413 | 98.10 2.0186 | 96.40 1.8986           | 98.60 0.1744                  | 99.10 0.1528                  | 97.60 0.1441 | 93.60 0.2133 | 94.40 0.1608                  | 92.50 0.1545                  |
| 7   | 95.60  2.3496 | 97.20 1.9468 | 94.70 1.8403           | 97.60 0.1634                  | 98.10 0.1325                  | 95.10 0.1253 | 92.20 0.2220 | 94.50 0.1691                  | 93.30 0.1628                  |
| 9   | 95.50  2.3318 | 96.50 1.8576 | 93.30 1.7544           | 98.00 0.1571                  | 95.80 0.1184                  | 93.50 0.1119 | 93.70 0.2369 | 94.50 0.1842                  | 93.30 0.1774                  |
| 40  | 5      | 96.80 2.1207 | 97.80 1.6984 | 95.90 1.6066           | 97.60 0.1647                  | 98.10 0.1356                  | 94.90 0.1280 | 90.50 0.1923 | 92.20 0.1355                  | 90.50 0.1301                  |
| 7   | 96.50  2.1081 | 97.40 1.6193 | 94.60 1.5290           | 97.00 0.1536                  | 97.40 0.1162                  | 94.40 0.1098 | 92.40 0.2004 | 94.10 0.1427                  | 93.30 0.1377                  |
| 9   | 95.46  2.1156 | 96.72 1.5526 | 94.06 1.4738           | 96.56 0.1467                  | 96.80 0.1025                  | 93.58 0.0968 | 92.40 0.2136 | 94.54 0.1563                  | 92.90 0.1508                  |
| 50  | 5      | 96.84 1.9516 | 97.96 1.4749 | 96.18 1.3982           | 97.14 0.1560                  | 97.90 0.1228                  | 95.28 0.1159 | 91.32 0.1780 | 92.76 0.1186                  | 90.92 0.1138                  |
| 7   | 96.52  1.9784 | 97.20 1.4123 | 94.86 1.3377           | 97.06 0.1462                  | 96.90 0.1039                  | 93.78 0.0980 | 90.96 0.1873 | 93.28 0.1259                  | 91.46 0.1212                  |
| 9   | 95.94  1.9781 | 96.38 1.3479 | 94.36 1.2799           | 96.46 0.1404                  | 96.60 0.0913                  | 93.24 0.0862 | 91.66 0.1991 | 93.78 0.1384                  | 92.48 0.1335                  |
Table 7: Type-I and Type-II censored data for analysis with $\alpha = 0.6$

| Censoring Scheme     | Stress Level | Data          |
|----------------------|--------------|---------------|
| Type-I and Type-II   | $S_1$        | 0.0185 0.0763 | 1.0137 1.2043 | 1.3411 1.3968 | 2.6797 3.4931 |
| Type-I               | $S_2$        | 5.1680 5.2476 | 5.4308 5.9575 | 7.2580 7.5416 | 7.7453         |
| Type-II              | $S_2$        | 5.1680 5.2476 | 5.4308 5.9575 | 7.2580 7.5416 | 7.7453 8.0116 |

and $\theta_2$ are 1.2147, 0.1041, and 0.2220, respectively. 90%, 95% and 99% CRIs for both Type-I and Type-II censoring schemes are reported in Table 10.

5.2.2. **Solar Lighting Device Data Set.** A simple step stress test was conducted in order to assess the reliability characteristics of a solar lighting device. Thirty five (35) devices are put on a life test at the normal operating temperature 293K, and then the stress factor temperature is changed to 353K at the time point $\tau_1 = 5$ (in hundred hours). The experiment was terminated at the time point $\tau_2 = 6$ (in hundred hours). Thirty one (31) failures occur before $\tau_2$ and among them fifteen (15) devices are failed at first stress and remaining sixteen (16) devices are failed at second stress level. The data are presented in Table 11.

We analyze the solar lighting data set based on the assumptions that at any stress level lifetime of devices follow GE distribution. We have obtained the order restricted Bayes estimates and different CRIs of model parameters. The order restricted Bayes estimates of $\alpha$, $\theta_1$ and $\theta_2$ are respectively 1.4434, 0.1810 and 1.7921. CRIs of parameters are presented in Table 12.

Now one natural question whether the GE distribution fits the data set or not. We have used the Kolmogorov-Smirnov (K-S) statistic, which quantifies the distance between the empirical distribution function of the data set and the cumulative distribution function of the fitted distribution function, for that purpose. The K-S distance and associated $p$-value are 0.2070 and 0.1212, respectively. It indicates that we cannot reject the null hypothesis at the 10% level of significance that the data are coming from a GE distribution.

6. **Optimality of Test Plan**

In the previous section we have obtained the Bayes estimates of the unknown parameters when the stress changing time $\tau_1$ is pre-fixed. In this section we consider the problem of choosing the optimal value of $\tau_1$, for a simple step-stress experiment. We obtain an optimal value of $\tau_1$ by minimizing the sum of posterior coefficient of variations of $\alpha$, $\theta_1$ and $\theta_2$. Since explicit form of the Eq. 3.2 cannot be obtained, we have used Lindley’s approximation to calculate the posterior coefficient of variations of the unknown parameters. See Appendix A.1 for the detailed derivations of the Lindley’s approximation. By minimizing sum of the posterior coefficient of variations, an optimal value of $\tau_1$ can be obtained by using Algorithm 2.
| Parameters | Left Symmetric HPD | Type-I Censored data |
|-----------|-------------------|----------------------|
| α | 90% | 0.1186 | 0.9681 | 0.4207 | 1.1003 | 0.4129 | 1.0244 | 0.0852 | 0.8891 | 0.3713 | 1.0242 |
| 95% | 0.1186 | 1.1003 | 0.4050 | 1.1396 | 0.4152 | 1.1396 | 0.0852 | 1.0242 | 0.2738 |
| 99% | 0.1186 | 1.2732 | 0.3015 | 1.3605 | 0.3015 | 1.2732 | 0.0852 | 1.2267 | 0.3015 |
| θ₁ | 90% | 0.0001 | 0.1472 | 0.0539 | 0.1676 | 0.0561 | 0.1564 | 0.0001 | 0.1323 | 0.0375 |
| 95% | 0.0001 | 0.1676 | 0.0561 | 0.1704 | 0.0484 | 0.1704 | 0.0001 | 0.1550 | 0.0375 |
| 99% | 0.0001 | 0.2098 | 0.0484 | 0.2219 | 0.0466 | 0.2219 | 0.0001 | 0.1983 | 0.0270 |
| θ₂ | 90% | 0.1041 | 0.3981 | 0.1469 | 0.4496 | 0.1500 | 0.4118 | 0.1041 | 0.3981 | 0.1469 |
| 95% | 0.1041 | 0.4496 | 0.1252 | 0.4973 | 0.1163 | 0.4535 | 0.1041 | 0.3981 | 0.1469 |
| 99% | 0.1041 | 0.5548 | 0.1113 | 0.5845 | 0.1113 | 0.5548 | 0.1041 | 0.5548 | 0.1113 |
Table 9: Type-I and Type-II censored data for analysis with $\alpha = 1.5$

| Censoring Scheme | Stress Level | Data     |
|------------------|--------------|----------|
| Type-I and Type-II | $S_1$ | 0.6277 0.7266 2.2977 2.8450 3.0599 3.3134 |
| Type-I           | $S_2$ | 5.1058 5.4453 5.5445 6.3469 7.1927 7.2401 7.5872 |
| Type-II          | $S_2$ | 5.1058 5.4453 5.5445 6.3469 7.1927 7.2401 7.5872 8.0156 8.0383 10.7256 |

Algorithm 2

Step 1. For given $\alpha, \theta_1, \theta_2, \tau_1$ and $n$ generate data from CEM.

Step 2. Obtain the posterior variance of all the parameters using Lindley’s approximation as explained in Appendix A.1.

Step 3. Repeat Step 1 and Step 2, $N$ times and take the average of variances.

Step 4. Calculate the coefficients of variation for Bayes estimates of $\alpha, \theta_1, \theta_2$.

Coefficient of Variation = $\frac{\text{posterior standard deviation}}{\text{posterior mean}}$

Step 5. Take the sum of coefficients of variation for Bayes estimates of $\alpha, \theta_1$ and $\theta_2$.

Step 6. Repeat Step 1 - Step 5 for different values of $\tau_1$ within its range.

Step 7. Choose $\tau_1$ for which the sum of coefficients of variation is minimum.

We have obtained numerically the optimal values of the stress changing times for different sample sizes and for different parameter values. It has been observed that the posterior variance of $\alpha$ is decreasing with the increase of $\tau_1$. As expected the posterior variance of $\theta_1$ has a decreasing trend and the posterior variance of $\theta_2$ increases with $\tau_1$. However, if we consider total dispersion of three parameters in terms of coefficient of variation, it is initially decreasing and then increasing as $\tau_1$ increases. Hence, we have obtained a point where the total dispersion is minimum and which is the optimal value of the stress changing time $\tau_1$. The experimental results and the plots of the sum of the coefficient of variations are given below (Table 13 and Figs. 1, 2 and 3).

7. Conclusion

This paper we have considered the ordered restricted Bayesian inference of the unknown parameters of the GE distributions when the data are coming from a step-stress model. It is assumed that the lifetime distribution satisfies the CEM assumptions. We have assumed a fairly flexible priors on the ordered parameters, and based on that we propose to use importance sampling technique to compute the Bayes estimates and the associated credible intervals. Extensive simulation experiments are performed for different sample
Table 10: CRIs for the unknown parameters for data in Table 9

| Parameters | Level | Type-I Censored data | Type-II Censored data |
|------------|-------|----------------------|-----------------------|
|            |       | Left | Symmetric | HPD | Left | Symmetric | HPD |
| α          | 90%   | 0.1246 | 1.9342 | 0.6761 | 2.1827 | 0.6668 | 2.0429 | 0.1286 | 1.8628 | 0.6028 | 2.1363 | 0.4506 | 1.8835 |
|           | 95%   | 0.1246 | 2.1827 | 0.6378 | 2.4342 | 0.5816 | 2.2610 | 0.1286 | 2.1363 | 0.4769 | 2.4069 | 0.4360 | 2.1514 |
|           | 99%   | 0.1246 | 2.7845 | 0.5143 | 3.0832 | 0.4637 | 2.8129 | 0.1286 | 2.8159 | 0.4506 | 3.1361 | 0.4248 | 2.8789 |
| θ₁         | 90%   | 0.0004 | 0.1805 | 0.0518 | 0.2056 | 0.0424 | 0.1830 | 0.0001 | 0.1776 | 0.0311 | 0.2037 | 0.0135 | 0.1777 |
|           | 95%   | 0.0004 | 0.2056 | 0.0494 | 0.2297 | 0.0424 | 0.2115 | 0.0001 | 0.2037 | 0.0210 | 0.2264 | 0.0135 | 0.2038 |
|           | 99%   | 0.0004 | 0.2572 | 0.0347 | 0.2748 | 0.0330 | 0.2579 | 0.0001 | 0.2565 | 0.0135 | 0.2789 | 0.0135 | 0.2574 |
| θ₂         | 90%   | 0.0585 | 0.3322 | 0.1061 | 0.3784 | 0.1135 | 0.3816 | 0.0690 | 0.3192 | 0.1183 | 0.3523 | 0.1013 | 0.3287 |
|           | 95%   | 0.0585 | 0.3784 | 0.0764 | 0.4248 | 0.0585 | 0.3784 | 0.0690 | 0.3523 | 0.1015 | 0.3819 | 0.1013 | 0.3731 |
|           | 99%   | 0.0585 | 0.4746 | 0.0585 | 0.5132 | 0.0585 | 0.4746 | 0.0690 | 0.4209 | 0.0774 | 0.4490 | 0.0690 | 0.4209 |
Table 11: Solar lighting device dataset

| Stress Level | Data          |
|--------------|---------------|
| $S_1$        | 0.140 0.783 1.324 1.582 1.716 1.794 1.883 2.293 2.660 2.674 2.725 |
|              | 3.085 3.924 4.396 4.612 4.892 |
| $S_2$        | 5.002 5.022 5.082 5.147 5.238 5.244 5.247 5.305 5.337 5.407 |
|              | 5.408 5.445 5.483 5.717 |

Table 12: CRIs for the unknown parameters for data in Table 11

| Parameters | Level | Left Symmetric HPD |
|------------|-------|--------------------|
|            | LL    | UL     | LL    | UL     |
| $\alpha$  | 90%   | 0.1948 | 2.0435 | 0.8623 | 2.2657 | 0.8249 | 2.1474 |
|           | 95%   | 0.1948 | 2.2657 | 0.8149 | 2.4153 | 0.7491 | 2.3247 |
|           | 99%   | 0.1948 | 2.6514 | 0.6694 | 2.8292 | 0.6200 | 2.6677 |
| $\theta_1$| 90%   | 0.0003 | 0.2609 | 0.1028 | 0.2856 | 0.0982 | 0.2654 |
|           | 95%   | 0.0003 | 0.2856 | 0.1009 | 0.3051 | 0.0989 | 0.2942 |
|           | 99%   | 0.0003 | 0.3284 | 0.0797 | 0.3480 | 0.0797 | 0.3413 |
| $\theta_2$| 90%   | 0.1357 | 2.4480 | 1.1284 | 2.6483 | 1.0702 | 2.5322 |
|           | 95%   | 0.1357 | 2.6483 | 1.0273 | 2.8873 | 0.9308 | 2.7295 |
|           | 99%   | 0.1357 | 3.1060 | 0.8437 | 3.2785 | 0.7681 | 3.1655 |

Table 13: Optimal value of $\tau_1$ for different $n$ and $\alpha$ with $\theta_1 = 0.1, \theta_2 = 0.2$

| $\alpha$ | n   | Optimal value of $\tau_1$ |
|----------|-----|--------------------------|
| 0.6      | 20  | 3.6                      |
| 0.6      | 30  | 6.4                      |
| 0.6      | 40  | 7.4                      |
| 0.6      | 50  | 7.2                      |
| 1.0      | 20  | 8.4                      |
| 1.0      | 30  | 8.2                      |
| 1.0      | 40  | 9.4                      |
| 1.0      | 50  | 10.0                     |
| 1.5      | 20  | 10.8                     |
| 1.5      | 30  | 13.0                     |
| 1.5      | 40  | 13.4                     |
| 1.5      | 50  | 13.4                     |

Figure 1: Plot of total coefficient of variation for different values of $\tau_1$ with parameter values $\alpha = 0.6, \theta_1 = 0.1$ and $\theta_2 = 0.2$
sizes and for different parametric values. It is observed that the proposed method works quite well in practice. Finally we consider choosing the optimal value for the stress changing time. We choose the optimal value $\tau_1$, so that the sum of the posterior coefficient of variations is minimum. Since the posterior coefficient of variations cannot be obtained in explicit forms, we suggest to use Lindley’s approximation to compute the posterior coefficient of variations. A small table is provided for optimal values of $\tau_1$, for different sample sizes and for different parametric values mainly for practical uses.

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Appendix A

A.1. Three Parameters Lindley’s Approximation For the three parameter case, using the notation $(\lambda_1, \lambda_2, \lambda_3) = (\alpha, \theta_2, \beta)$ Lindley’s approximation of Bayes estimator of any function $g(\lambda_1, \lambda_2, \lambda_3)$ can be given by

$$E_{\lambda_1, \lambda_2, \lambda_3 | \text{Data}} (g(\lambda_1, \lambda_2, \lambda_3)) = g(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3) + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} u_{ij} \sigma_{ij}$$

$$+ \sum_{i=1}^{3} \sum_{j=1}^{3} u_{ij} \rho_{ij} \sigma_{ij}$$

$$+ \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} L_{ijk} U_{k} \sigma_{ij}, \quad (A.1)$$
where

\[ L_{ijk} = \frac{\delta^3 L}{\delta \lambda_i \delta \lambda_j \delta \lambda_k}; \quad i, j, k = 1(1)3 \] and L is log likelihood of the data;

\[ u_i = \frac{\delta g}{\delta \lambda_i}; \quad i = 1(1)3; \]

\[ u_{ij} = \frac{\delta^2 g}{\delta \lambda_i \delta \lambda_j}; \quad i, j = 1(1)3; \]

\[ \sigma_{ij} = (i, j)^{th} \text{ element of the inverse of the matrix having elements}((-L_{ij})); \]

\[ \rho_i = \frac{\delta \log \pi}{\delta \lambda_i}; \]

\[ U_k = \sum_{i=1}^{3} u_i \sigma_{ki}; \quad k = 1(1)3, \]

\[ \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3 \text{ are MLEs of } \lambda_1, \lambda_2, \lambda_3 \text{ respectively and all of the quantities are evaluated at } (\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3). \]

The log likelihood function of the data under order restriction of parameter is given by

\[ L(t_1, t_2, \ldots t_n | \alpha, \theta_2, \beta) = \log(n!) + n \log(\alpha) + n_1 \log(\beta) + n \log(\theta_2) - \beta \theta_2 \sum_{k=1}^{n_1} t_k 
\]

\[ + (\alpha - 1) \sum_{k=n_1+1}^{n} \log(1 - e^{-\beta \theta_2 t_k}) - \theta_2 \sum_{k=n_1+1}^{n} (t_k + \beta \tau_1 - \tau_1) \]

\[ + (\alpha - 1) \sum_{k=n_1+1}^{n} \log(1 - e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)}). \quad (A.2) \]

The MLEs of \( \alpha, \theta_2, \) and \( \beta \) can be obtain by maximizing \((A.2)\).

\[ \frac{\delta L}{\delta \alpha} = 0 \Rightarrow \hat{\alpha} = \frac{-n}{\sum_{k=1}^{n_1} \ln(1 - e^{-\beta \theta_2 t_k}) + \sum_{k=n_1+1}^{n} \ln(1 - e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)})}. \quad (A.3) \]

\[ \frac{\delta L}{\delta \theta_2} = 0 \Rightarrow \frac{n}{\theta_2} - \sum_{k=1}^{N_1} \beta t_k + (\alpha - 1) \sum_{k=1}^{N_1} \frac{\beta t_k e^{-\theta_2 t_k}}{1 - e^{-\theta_2 t_k}} - \sum_{k=N_1+1}^{n} (t_k + \beta \tau_1 - \tau_1) \]

\[ + (\alpha - 1) \sum_{k=N_1+1}^{n} \frac{(t_k + \beta \tau_1 - \tau_1) e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)}}{1 - e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)}} = 0. \quad (A.4) \]

For known \( \beta(0 < \beta < 1) \), the estimate of \( \theta_2 \) can be obtain by solving \((A.4)\) numerically and hence an estimate of \( \alpha \) from \((A.3)\). The value of \( \beta \) between 0 and 1 and the corresponding estimates of \( \alpha \) and \( \theta_2 \), for which likelihood is maximum will be the MLEs of \( \beta, \alpha \) and \( \theta_2 \) respectively.

\[ L_{11} = \frac{\delta^2 L}{\delta \alpha^2} = -\frac{n}{\alpha^2}, \]
\[ L_{12} = \frac{\delta^2 l}{\delta \alpha \delta \beta} = \frac{n_1}{\delta \alpha \delta \beta} + \sum_{k=1}^{n_1} \frac{\beta t_k e^{-\beta \theta_2 t_k}}{1 - e^{-\beta \theta_2 t_k}} \left( t_k + \beta \tau_1 - \tau_1 \right) \frac{1}{1 - e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)}}, \]

\[ L_{13} = \frac{\delta^2 l}{\delta \alpha \delta \beta} = \sum_{k=1}^{n_1} \frac{\beta t_k e^{-\beta \theta_2 t_k}}{1 - e^{-\beta \theta_2 t_k}} + \sum_{k=1}^{n_1} \frac{\theta_2 \tau_1 e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)}}{1 - e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)}}, \]

\[ L_{22} = \frac{\delta^2 l}{\delta \theta_2 \delta \beta} = -\frac{n_1}{\delta \theta_2} + (\alpha - 1) \sum_{k=1}^{n_1} \frac{\theta_2^2 t_k e^{-\beta \theta_2 t_k}}{1 - e^{-\beta \theta_2 t_k}} \left( t_k (1 - e^{-\beta \theta_2 t_k}) - \beta \theta_2 t_k e^{-\beta \theta_2 t_k} \right) - \alpha (n - n_1) \tau_1 \]

\[ L_{33} = \frac{\delta^2 l}{\delta \beta^2} = -\frac{n_1}{\delta \beta^2} - (\alpha - 1) \sum_{k=1}^{n_1} \frac{\theta_2^2 t_k e^{-\beta \theta_2 t_k}}{1 - e^{-\beta \theta_2 t_k}} \left( t_k (1 - e^{-\beta \theta_2 t_k}) - \beta \theta_2 t_k e^{-\beta \theta_2 t_k} \right) \left( t_k + \beta \tau_1 - \tau_1 \right) \frac{1}{1 - e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)}}, \]

\[ L_{111} = \frac{\delta^3 l}{\delta \alpha^3} = \frac{2n}{\alpha^3}, \quad L_{112} = \frac{\delta^3 l}{\delta \alpha^2 \delta \beta} = 0, \quad L_{113} = \frac{\delta^3 l}{\delta \alpha \delta \beta^2} = 0, \]

\[ L_{122} = \frac{\delta^3 l}{\delta \alpha \delta \theta_2 \delta \beta} = -n_1 \sum_{k=1}^{n_1} \frac{\beta t_k e^{-\beta \theta_2 t_k}}{1 - e^{-\beta \theta_2 t_k}} \left( t_k (1 - e^{-\beta \theta_2 t_k}) - \beta \theta_2 t_k e^{-\beta \theta_2 t_k} \right) - \sum_{k=1}^{n_1} \frac{(t_k + \beta \tau_1 - \tau_1)^2 e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)}}{1 - e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)}}, \]

\[ L_{123} = \frac{\delta^3 l}{\delta \alpha \delta \theta_2 \delta \beta} = \sum_{k=1}^{n_1} \left[ \frac{t_k (1 - e^{-\beta \theta_2 t_k}) - \theta_2 t_k e^{-\beta \theta_2 t_k}}{1 - e^{-\beta \theta_2 t_k}} - t_k \right] \]

\[ L_{133} = \frac{\delta^3 l}{\delta \alpha \delta \beta^2} = \sum_{k=1}^{n_1} \frac{\theta_2^2 t_k e^{-\beta \theta_2 t_k}}{1 - e^{-\beta \theta_2 t_k}} \left( t_k + \beta \tau_1 - \tau_1 \right) e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)} \left( 1 - e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)} \right) - \sum_{k=1}^{n_1} \frac{\theta_2^2 t_k e^{-\beta \theta_2 t_k}}{1 - e^{-\beta \theta_2 t_k}} \left( t_k + \beta \tau_1 - \tau_1 \right) e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)} \left( 1 - e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)} \right), \]

\[ L_{222} = \frac{\delta^3 l}{\delta \theta_2^3} = 2n \theta_2^3 + (\alpha - 1) \sum_{k=1}^{n_1} \frac{\beta^3 t_k e^{-\beta \theta_2 t_k} (1 + e^{\beta \theta_2 t_k})}{1 - e^{-\beta \theta_2 t_k}} \left[ t_k + \beta \tau_1 - \tau_1 \right] \frac{1}{1 - e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)}}, \]

\[ + (\alpha - 1) \sum_{k=1}^{n_1} \frac{(t_k + \beta \tau_1 - \tau_1)^3 e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)} (1 + e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)})}{1 - e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)}}, \]

\[ \delta \theta_2 \delta \beta \delta \tau_1 = \frac{\theta_2}{\beta \tau_1} \left[ \frac{\theta_2^2}{\beta \tau_1} - \frac{1}{\beta \tau_1} \right] \frac{1}{1 - e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)}}, \]

\[ \delta \theta_2 \delta \beta \delta \tau_2 = \frac{\theta_2}{\beta \tau_2} \left[ \frac{\theta_2^2}{\beta \tau_2} - \frac{1}{\beta \tau_2} \right] \frac{1}{1 - e^{-\theta_2 (t_k + \beta \tau_2 - \tau_2)}}, \]

\[ \delta \theta_2 \delta \beta \delta \tau_3 = \frac{\theta_2}{\beta \tau_3} \left[ \frac{\theta_2^2}{\beta \tau_3} - \frac{1}{\beta \tau_3} \right] \frac{1}{1 - e^{-\theta_2 (t_k + \beta \tau_3 - \tau_3)}}, \]

\[ \delta \theta_2 \delta \beta \delta \tau_4 = \frac{\theta_2}{\beta \tau_4} \left[ \frac{\theta_2^2}{\beta \tau_4} - \frac{1}{\beta \tau_4} \right] \frac{1}{1 - e^{-\theta_2 (t_k + \beta \tau_4 - \tau_4)}}, \]

\[ \delta \theta_2 \delta \beta \delta \tau_5 = \frac{\theta_2}{\beta \tau_5} \left[ \frac{\theta_2^2}{\beta \tau_5} - \frac{1}{\beta \tau_5} \right] \frac{1}{1 - e^{-\theta_2 (t_k + \beta \tau_5 - \tau_5)}}, \]

\[ \delta \theta_2 \delta \beta \delta \tau_6 = \frac{\theta_2}{\beta \tau_6} \left[ \frac{\theta_2^2}{\beta \tau_6} - \frac{1}{\beta \tau_6} \right] \frac{1}{1 - e^{-\theta_2 (t_k + \beta \tau_6 - \tau_6)}}, \]

\[ \delta \theta_2 \delta \beta \delta \tau_7 = \frac{\theta_2}{\beta \tau_7} \left[ \frac{\theta_2^2}{\beta \tau_7} - \frac{1}{\beta \tau_7} \right] \frac{1}{1 - e^{-\theta_2 (t_k + \beta \tau_7 - \tau_7)}}, \]

\[ \delta \theta_2 \delta \beta \delta \tau_8 = \frac{\theta_2}{\beta \tau_8} \left[ \frac{\theta_2^2}{\beta \tau_8} - \frac{1}{\beta \tau_8} \right] \frac{1}{1 - e^{-\theta_2 (t_k + \beta \tau_8 - \tau_8)}}, \]

\[ \delta \theta_2 \delta \beta \delta \tau_9 = \frac{\theta_2}{\beta \tau_9} \left[ \frac{\theta_2^2}{\beta \tau_9} - \frac{1}{\beta \tau_9} \right] \frac{1}{1 - e^{-\theta_2 (t_k + \beta \tau_9 - \tau_9)}}, \]

\[ \delta \theta_2 \delta \beta \delta \tau_{10} = \frac{\theta_2}{\beta \tau_{10}} \left[ \frac{\theta_2^2}{\beta \tau_{10}} - \frac{1}{\beta \tau_{10}} \right] \frac{1}{1 - e^{-\theta_2 (t_k + \beta \tau_{10} - \tau_{10})}}, \]
order restricted bayesian analysis of a simple sslt

\[ L_{233} = \frac{\delta^3 L}{\delta \theta_2^3 \delta \beta} = -(\alpha - 1) \sum_{k=1}^{n_1} \frac{2 \beta^2 t_k^2 e^{-\beta \theta_2 t_k}}{(1 - e^{-\beta \theta_2 t_k})^2} + (\alpha - 1) \sum_{k=1}^{n_1} \frac{\beta^2 \theta_2 t_k^2 e^{-\beta \theta_2 t_k} (1 + e^{-\beta \theta_2 t_k})}{(1 - e^{-\beta \theta_2 t_k})^3} - (\alpha - 1) \sum_{k=n_1+1}^{n} \frac{2 \tau_1 (t_k + \beta \tau_1 - \tau_1) e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)}}{(1 - e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)}(1 - e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)})^2} + (\alpha - 1) \sum_{k=n_1+1}^{n} \frac{\theta_2 \tau_1 (t_k + \beta \tau_1 - \tau_1)^2 e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)(1 + e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)})}{(1 - e^{-\theta_2 (t_k + \beta \tau_1 - \tau_1)})^3.}

\]

Note that \( L_{ij} \) does not depends on the order of appearance of \( i, j \) and \( k \).

\[ \rho_1 = \frac{\delta \log(\pi)}{\delta \alpha} = \frac{b_0 - 1}{\alpha} - a_0; \quad \rho_2 = \frac{\delta \log(\pi)}{\delta \theta_2} = \frac{b_1 - 1}{\theta_2} - a_1 \]

and \( \rho_3 = \frac{\delta \log(\pi)}{\delta \beta} = \frac{a_2 - 1}{\beta} - \frac{b_2 - 1}{1 - \beta} \)

To obtain the posterior variance of the parameters we need to take below assumptions on the function \( g(\alpha, \theta_2; \beta) \).

(a) To calculate posterior variance of \( \alpha \): \( g = \alpha \) and \( g = \alpha^2 \).

(b) To calculate posterior variance of \( \theta_1 \): \( g = \beta \theta_2 \) and \( g = \beta^2 \theta_2^2 \).

(c) To calculate posterior variance of \( \theta_2 \): \( g = \theta_2 \) and \( g = \theta_2^2 \).

In case (a) if \( g = \alpha \), \( u_1 = 1 \), \( u_2 = u_3 = 0 \), \( u_{ij} = 0 \), \( i, j = 1(1)3 \).

If \( g = \alpha^2 \), \( u_{11} = 2 \alpha \), \( u_2 = u_3 = 0 \), \( u_{ij} = 0 \) for \( i, j = 1(1)3 \) and \( (i, j) \neq (1, 1) \).

In case (b) if \( g = \beta \theta_2 \), \( u_1 = 0 \), \( u_2 = \beta \), \( u_3 = \theta_2 \), \( u_{23} = 1 \), \( u_{ij} = 0 \) for \( i, j = 1(1)3 \) and \( (i, j) \neq (2, 3) \).
Table 14: AEs and MSEs of $\alpha$, $\theta_1$, and $\theta_2$ based on 5000 simulations with $\alpha = 1.5$, $\theta_1 = 0.1$, and $\theta_2 = 0.2$ for different values of $n$, $\tau_1$ and $\tau_2$ of Type-I censored data

| $n$ | $\tau_1$ | $\tau_2$ | $\alpha$ AE | $\alpha$ MSE | $\theta_1$ AE | $\theta_1$ MSE | $\theta_2$ AE | $\theta_2$ MSE |
|-----|----------|----------|-------------|-------------|-------------|-------------|-------------|-------------|
| 20  | 7        | 13       | 1.8329      | 0.8966      | 0.1114      | 0.0018      | 0.2079      | 0.0043      |
|     | 9        | 13       | 1.8214      | 0.8455      | 0.1096      | 0.0016      | 0.2140      | 0.0062      |
|     | 9        | 15       | 1.8132      | 0.7975      | 0.1080      | 0.0015      | 0.2103      | 0.0051      |
| 30  | 7        | 13       | 1.7538      | 0.4748      | 0.1111      | 0.0014      | 0.2052      | 0.0029      |
|     | 9        | 13       | 1.7314      | 0.4446      | 0.1088      | 0.0011      | 0.2067      | 0.0038      |
|     | 9        | 15       | 1.6914      | 0.3702      | 0.1071      | 0.0010      | 0.2059      | 0.0030      |
| 40  | 7        | 13       | 1.7024      | 0.3247      | 0.1106      | 0.0011      | 0.2040      | 0.0020      |
|     | 9        | 13       | 1.6671      | 0.2831      | 0.1078      | 0.0008      | 0.2043      | 0.0028      |
|     | 9        | 15       | 1.6541      | 0.2562      | 0.1064      | 0.0008      | 0.2004      | 0.0021      |
| 50  | 7        | 13       | 1.6716      | 0.2261      | 0.1105      | 0.0009      | 0.2017      | 0.0016      |
|     | 9        | 13       | 1.6693      | 0.2228      | 0.1099      | 0.0007      | 0.2029      | 0.0022      |
|     | 9        | 15       | 1.6395      | 0.1909      | 0.1075      | 0.0007      | 0.2011      | 0.0019      |

If $g = \beta^2 \theta_2^2$, $u_1 = 0$, $u_2 = 2\beta^2 \theta_2$, $u_3 = 2\beta \theta_2^2$, $u_{11} = u_{12} = u_{13} = 0$, $u_{22} = 2\beta^2$, $u_{23} = 4\beta \theta_2$, $u_{33} = 2\theta_2^2$.

In case (c) if $g = \theta_2$, $u_2 = 1$, $u_1 = u_3 = 0$ $u_{ij} = 0$ for $i, j = 1(1)3$.

If $g = \theta_2^2$, $u_2 = 2\theta_2$, $u_1 = u_3 = 0$, $u_{22} = 2$, $u_{ij} = 0$ for $i, j = 1(1)3$ and $(i, j) \neq (2, 2)$.

Note that $u_{ij} = u_{ji}$ for all $i, j = 1(1)3$.

Now posterior variance of the parameters can be obtain by using the Eq. A.1.

A.2. Simulation Results

Table 15: AEs and MSEs of $\alpha$, $\theta_1$, and $\theta_2$ based on 5000 simulations with $\alpha = 1.5$, $\theta_1 = 0.1$, and $\theta_2 = 0.2$ for different values of $n$, $\tau_1$ and $r$ of Type-II censored data

| $n$ | $\tau_1$ | $r$   | $\alpha$ AE | $\alpha$ MSE | $\theta_1$ AE | $\theta_1$ MSE | $\theta_2$ AE | $\theta_2$ MSE |
|-----|----------|-------|-------------|-------------|-------------|-------------|-------------|-------------|
| 20  | 7        | 15    | 1.9245      | 1.2123      | 0.1148      | 0.0022      | 0.2273      | 0.0107      |
|     | 9        | 15    | 1.8758      | 0.9971      | 0.1115      | 0.0017      | 0.2406      | 0.0400      |
|     | 9        | 17    | 1.8348      | 0.8609      | 0.1104      | 0.0017      | 0.2227      | 0.0081      |
| 30  | 7        | 23    | 1.7902      | 0.5586      | 0.1133      | 0.0016      | 0.2142      | 0.0040      |
|     | 9        | 23    | 1.7353      | 0.4554      | 0.1096      | 0.0012      | 0.2220      | 0.0088      |
|     | 9        | 27    | 1.7147      | 0.3786      | 0.1079      | 0.0011      | 0.2090      | 0.0033      |
| 40  | 7        | 32    | 1.7028      | 0.3272      | 0.1105      | 0.0012      | 0.2081      | 0.0026      |
|     | 9        | 32    | 1.6905      | 0.3092      | 0.1079      | 0.0009      | 0.2083      | 0.0031      |
|     | 9        | 36    | 1.6659      | 0.2506      | 0.1073      | 0.0009      | 0.2061      | 0.0023      |
| 50  | 7        | 42    | 1.6759      | 0.2408      | 0.1094      | 0.0010      | 0.2049      | 0.0018      |
|     | 9        | 42    | 1.6440      | 0.2089      | 0.1073      | 0.0007      | 0.2049      | 0.0021      |
|     | 9        | 45    | 1.6284      | 0.1846      | 0.1058      | 0.0007      | 0.2036      | 0.0018      |
Table 16: CPs and ALs of 95\% CRI for $\alpha$, $\theta_1$ and $\theta_2$ based on 5000 simulations with $\alpha = 1.5$, $\theta_1 = 0.1$, and $\theta_2 = 0.2$ for different values of $n$, $\tau_1$ and $\tau_2$ of Type-I censored data

| $n$ | $\tau_1$ | $\tau_2$ | Left CRI | Symmetric CRI | HPD CRI | Left CRI | Symmetric CRI | HPD CRI | Left CRI | Symmetric CRI | HPD CRI | Left CRI | Symmetric CRI | HPD CRI |
|-----|----------|----------|----------|--------------|---------|----------|--------------|---------|----------|--------------|---------|----------|--------------|---------|
| 20  | 7        | 13       | 95.64    | 3.1864      | 97.12   | 3.0328   | 94.28      | 2.7679   | 97.78    | 0.1947      | 98.14   | 0.1743   | 95.78      | 0.1629   | 94.24     | 0.2726      | 95.38   | 0.2530   |
| 9   | 13       | 95.44    | 3.1317   | 96.28      | 2.9208  | 93.46    | 2.6716     | 96.80    | 0.1840   | 97.00      | 0.1561  | 93.88    | 0.1455     | 95.08   | 0.3055   | 96.02      | 0.2982  | 94.20     | 0.2800     |
| 9   | 15       | 95.16    | 3.0708   | 96.30      | 2.8365  | 93.02    | 2.6166     | 96.64    | 0.1803   | 97.22      | 0.1541  | 93.68    | 0.1445     | 94.70   | 0.2871   | 95.92      | 0.2680  | 93.96     | 0.2547     |
| 30  | 7        | 13       | 96.40    | 2.7229     | 97.18   | 2.3466   | 93.80      | 2.1774   | 98.04    | 0.1802     | 97.44   | 0.1466   | 94.94      | 0.1375  | 94.02     | 0.2326      | 95.40   | 0.2043   |
| 9   | 13       | 95.63    | 2.6491   | 96.20      | 2.2070  | 92.80    | 2.0526     | 97.23    | 0.1688   | 96.57      | 0.1271  | 94.13    | 0.1187     | 94.40   | 0.2556   | 95.70      | 0.2392  | 93.63     | 0.2268     |
| 9   | 15       | 96.53    | 2.5628   | 96.00      | 2.1399  | 93.47    | 2.0030     | 96.83    | 0.1670   | 97.37      | 0.1293  | 94.47    | 0.1219     | 94.47   | 0.2461   | 95.77      | 0.2192  | 94.13     | 0.2103     |
| 40  | 7        | 13       | 96.47    | 2.4529     | 96.37   | 1.9535   | 93.23      | 1.8261   | 97.73    | 0.1709     | 97.20   | 0.1283  | 94.63      | 0.1207  | 94.67     | 0.2092      | 95.53   | 0.1772   |
| 9   | 13       | 95.30    | 2.3766   | 95.70      | 1.8153  | 92.60    | 1.6960     | 97.17    | 0.1595   | 96.67      | 0.1088  | 94.50    | 0.1015     | 93.90   | 0.2285   | 95.50      | 0.2077  | 92.93     | 0.1978     |
| 9   | 15       | 96.10    | 2.3392   | 96.20      | 1.7972  | 93.67    | 1.6899     | 96.87    | 0.1584   | 96.93      | 0.1123  | 93.93    | 0.1058     | 93.57   | 0.2171   | 95.97      | 0.1867  | 93.53     | 0.1800     |
| 50  | 7        | 13       | 96.80    | 2.2919     | 96.63   | 1.7071   | 93.57      | 1.5995   | 97.83    | 0.1650     | 97.13   | 0.1159  | 94.80      | 0.1089  | 94.17     | 0.1911      | 95.13   | 0.1573   |
| 9   | 13       | 96.37    | 2.2664   | 95.47      | 1.6042  | 92.33    | 1.5057     | 98.13    | 0.1562   | 97.07      | 0.0979  | 95.27    | 0.0914     | 94.30   | 0.2102   | 95.37      | 0.1871  | 93.73     | 0.1784     |
| 9   | 15       | 96.23    | 2.2150   | 96.23      | 1.5880  | 93.33    | 1.4986     | 97.23    | 0.1544   | 96.77      | 0.1019  | 93.50    | 0.0962     | 94.20   | 0.2030   | 95.33      | 0.1693  | 93.67     | 0.1636     |
Table 17: CPs and ALs of 95% CRI for $\alpha$, $\theta_1$ and $\theta_2$ based on 5000 simulations with $\alpha = 1.5$, $\theta_1 = 0.1$, and $\theta_2 = 0.2$ for different values of $n$, $\tau_1$ and $r$ of Type-II censored data

| $n$ | $\tau_1$ | $\tau_2$ | Left CRI | Symmetric CRI | HPD CRI | Left CRI | Symmetric CRI | HPD CRI | Left CRI | Symmetric CRI | HPD CRI |
|-----|----------|----------|----------|---------------|---------|----------|---------------|---------|----------|---------------|---------|
| 20  | 7        | 96.30    | 3.3992   | 96.98        | 3.2367  | 94.28    | 2.9648        | 97.80   | 0.2018   | 97.64        | 0.1819  |
| 9   | 15       | 95.76    | 3.2467   | 96.10        | 3.0317  | 93.20    | 2.7877        | 97.40   | 0.1878   | 97.26        | 0.1617  |
| 9   | 17       | 95.50    | 3.0823   | 96.16        | 2.8396  | 93.02    | 2.6304        | 97.48   | 0.1838   | 97.06        | 0.1584  |
| 30  | 7        | 96.28    | 2.7953   | 96.82        | 2.4164  | 93.64    | 2.2468        | 98.02   | 0.1845   | 97.14        | 0.1509  |
| 9   | 23       | 96.38    | 2.6635   | 96.06        | 2.2360  | 93.52    | 2.0851        | 97.44   | 0.1715   | 96.58        | 0.1321  |
| 9   | 27       | 96.08    | 2.5628   | 96.16        | 2.1224  | 93.24    | 2.0007        | 96.88   | 0.1671   | 96.58        | 0.1307  |
| 40  | 7        | 96.54    | 2.4596   | 96.36        | 1.9713  | 93.48    | 1.8483        | 97.62   | 0.1721   | 97.06        | 0.1319  |
| 9   | 32       | 96.46    | 2.4068   | 95.78        | 1.8552  | 92.76    | 1.7464        | 97.06   | 0.1610   | 96.60        | 0.1148  |
| 9   | 36       | 96.26    | 2.3261   | 96.50        | 1.7818  | 93.58    | 1.6901        | 97.28   | 0.1591   | 96.44        | 0.1157  |
| 50  | 7        | 96.46    | 2.2933   | 96.38        | 1.7292  | 93.18    | 1.6310        | 97.42   | 0.1650   | 96.80        | 0.1208  |
| 9   | 42       | 96.22    | 2.2202   | 95.74        | 1.6018  | 92.68    | 1.5161        | 97.14   | 0.1549   | 96.30        | 0.1045  |
| 9   | 45       | 96.48    | 2.1732   | 96.44        | 1.5640  | 93.70    | 1.4855        | 97.32   | 0.1525   | 96.48        | 0.1046  |
References

ABDELMHAMID, A. H. and AL-HUSSAINI, E. K. (2009). Estimation in step-stress accelerated life tests for the exponentiated exponential distribution with type-I censoring. *Computational Statistics and Data Analysis*, 53: 1328–1338.

AL-HUSSAINI, E. K. and AHSANULLAH, M. (2015). *Exponentiated distributions*. AP, Paris, France.

BAGDONAVICIIUS, V. B. and NIKULIN, M. (2002). *Accelerated life models: modeling and statistical analysis*. Chapman and Hall CRC Press, Boca Raton, Florida.

BALAKRISHNAN, N. (2009). A synthesis of exact inferential results for exponential step-stress models and associated optimal accelerated life-tests. *Metrika*, 69: 351–396.

BALAKRISHNAN, N., BEUTNER, E. and KATERI, M. (2009). Order restricted inference for exponential step-stress models. *IEEE Transactions on Reliability*, 58: 132–142.

CHILDS, A., CHANDRASEKAR, B., BALAKRISHNAN, N. and KUNDU, D. (2003). Exact likelihood inference based on Type-I and Type-II hybrid censored samples from the exponential distribution. *Annals of the Institute of Statistical Mathematics*, 55: 319–330.

CONGDON, P. (2006). *Bayesian Statistical Modeling*. Wiley, New York.

EPSTEIN, B. (1954). Truncated life-test in exponential case. *Annals of Mathematical Statistics*, 25: 555–564.

GUPTA, R. D. and KUNDU, D. (1999). Generalized exponential distribution. *Australian and New Zealand Journal of Statistics*, 41: 173–188.

GUPTA, R. D. and KUNDU, D. (2001). Exponentiated exponential distribution: An alternative to gamma and weibull distributions. *Biometrical Journal*, 43: 117–130.

KUNDU, D. and GANGULY, A. (2017). *Analysis of step-stress models: existing methods and recent developments*. Elsevier/ Academic Press, London, UK.

NADARAJAH, S. (2011). The exponentiated exponential distribution. *Advance in Statistical Analysis*, 95: 219–251.

NELSON, W. B. (1980). Accelerated life testing: step-stress models and data analysis. *IEEE Transactions on Reliability*, 29: 103–108.

SAMANTA, D., GANGULY, A., KUNDU, D. and MITRA, S. (2017). Order restricted Bayesian inference for exponential simple step-stress model. *Communication in Statistics - Simulation and Computation*, 46: 1113–1135.

SEDIKAIN, N. M. (1966). On one physical principle in reliability theory. *Technical Cybernetics*, 3: 80–87.

ZHANG, Y. and MEeker, W. Q. (2005). Bayesian life test planning for weibull distribution. *Metrika*, 61: 237–249.

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