A REMARK ON THE FOURIER TRANSFORM OF \( l^p \)-BALLS

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Abstract. We re-examine through an example the connection between the curvature of the boundary of a set, and the decay at infinity of the Fourier transform of its characteristic function. Let \( B_p \subset \mathbb{R}^2 \) denote the unit ball of \( \mathbb{R}^2 \) in the \( l^p \)-norm. It is a consequence of a classical result of Hlawka that for each \( p \in (1, 2] \), there exists \( C(p) > 0 \) such that

\[
|\hat{\chi}_{B_p}(\omega)| \leq \frac{C(p)}{|\omega|^{3/2}} \quad (\omega \in \mathbb{R}^2, |\omega| \text{ large}).
\]

The above estimate does not hold for \( p = 1 \). Thus, one expects that \( C(p) \to \infty \) as \( p \to 1^+ \); we determine the sharp asymptotic behaviour of \( C(p) \) as \( p \to 1^+ \).

1. Introduction

For \( K \subset \mathbb{R}^2 \) the characteristic function \( \chi_K \) of \( K \) is defined by \( \chi_K(x) = 1 \) if \( x \in K \) and \( \chi_K(x) = 0 \) if \( x \notin K \). There is an interesting connection between the geometry of \( K \) (more specifically the curvature of the boundary \( \partial K \)) and the rate of decay of the Fourier transform \( \hat{\chi}_K(\omega) \) as \( |\omega| \to \infty \). (See (2.1) below for the definition of the Fourier transform that we shall use.)

Recall that the curvature of \( \partial K \) is \( \kappa = \kappa(s) = |\gamma''(s)| \) where \( \gamma(s) \) is the arc length parametrization of \( \partial K \). If \( K \) is compact and convex and the boundary \( \partial K \) is piece-wise smooth and has everywhere non-zero curvature, then there exists a constant \( C > 0 \) such that for all \( \omega \in \mathbb{R}^2 \) with \( |\omega| \) large, the following holds

\[
|\hat{\chi}_K(\omega)| \leq \frac{C}{|\omega|^{3/2}},
\]

where \( |\cdot| \) denotes the usual euclidean norm. The condition that \( \kappa \neq 0 \) everywhere is also necessary. The estimate (1.1) is due to Hlawka [4] (see also [3, 6, 7, 8]). Results of the type (1.1) are closely related to the quite well-developed theory of Fourier transforms of surface-carried measures, see [10, Chapter VIII, §5.7–§5.13]. We also mention that the estimate (1.1) has some intriguing applications in number theory, e.g., in connection to error estimates for the Gauss circle problem. The paper [5] offers an engaging survey of results of this type.

In this note, we shall consider (1.1) for a specific family of subsets of \( \mathbb{R}^2 \): the unit ball in different \( l^p \)-norms. Our aim is to determine how the multiplicative constant \( C \) of (1.1) depends on \( p \). Often when one has a family of estimates depending on a parameter, the asymptotic behaviour (with respect to said parameter) of the involved multiplicative constants holds some interesting information. A well-known example of this phenomenon is the limiting behaviour of the fractional Sobolev embedding first studied in [2].

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Recall that the $l^p$-norm on $\mathbb{R}^2$ is given by
\[
\|(x, y)\|_p \equiv (|x|^p + |y|^p)^{1/p} \quad (1 \leq p < \infty).
\]
Note that $\| \cdot \|_2 = | \cdot |$, the euclidean norm. For $1 \leq p < \infty$ we denote by $B_p$ the unit ball in $l^p$-norm, i.e.
\[
B_p = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\|_p \leq 1\}.
\]
Let $E = \{ (\pm 1, 0), (0, \pm 1) \}$. Clearly $\partial B_p$ is piece-wise smooth and one can show that the curvature of $\partial B_p$ is non-zero everywhere on $B_p \setminus E$, and undefined on $E$. Hence, ignoring the exceptional set $E$, we expect that (1.1) holds for $K = B_p$ ($1 < p \leq 2$). (It does, see Theorem 1.2 below.) On the other hand, for $p = 1$ the curvature of $\partial B_1$ satisfies $\kappa = 0$ everywhere on $\partial B_1 \setminus E$. Thus, we expect (1.1) to be false for $B_1$. This is true, see Remark 2.3 below.

Since (1.1) holds for $1 < p \leq 2$ and fails for $p = 1$, it is natural to suppose that if
\[
(1.2) \quad C(p) = \sup_{\omega \in \mathbb{R}^2} |\omega|^{3/2}|\hat{\chi}_{B_p}(\omega)|,
\]
then $C(p) \to \infty$ as $p \to 1+$. In this way, the following problem presents itself.

**Problem 1.1.** Determine the asymptotic behaviour of $C(p)$ as $p \to 1+$.

The main result of this note is the solution of Problem 1.

**Theorem 1.2.** Let $1 < p \leq 2$. There is an absolute constant $C_1$ such that
\[
(1.3) \quad \sup_{\omega \in \mathbb{R}^2} |\omega|^{3/2}|\hat{\chi}_{B_p}(\omega)| \leq \frac{C_1}{\sqrt{p - 1}}.
\]
The estimate (1.3) is sharp in the following sense: there is an absolute constant $C_2$ such that for any $p \in (1, 2)$, there exists a sequence $\{\omega_n\} \subset \mathbb{R}^2$ with $|\omega_n| \to \infty$ such that
\[
(1.4) \quad |\omega_n|^{3/2}|\hat{\chi}_{B_p}(\omega_n)| \geq \frac{C_2}{\sqrt{p - 1}} + o(1),
\]
where $o(1)$ means a term tending to 0 as $n \to \infty$.

**Remark 1.3.** The constants $C_1, C_2$ can be taken to be
\[
C_1 = 12(2)^{1/4} \approx 14.270 \ldots, \quad C_2 = 27/4\sqrt{\pi} \approx 5.961 \ldots.
\]

**Remark 1.4.** The estimate (1.3) has a "blow-up" of the order $(p - 1)^{-1/2}$ as $p \to 1+$, and (1.4) shows that this order is essentially sharp.

The proof of Theorem 1.2 is elementary. It uses in particular some well-known tools from oscillatory integrals, such as van der Corput’s lemma and the method of stationary phase.

Despite the rather simplistic nature of Theorem 1.2 we believe it might be an interesting addition (albeit certainly not of central importance) to the study of the influence of curvature on the behaviour of the Fourier transform. In particular, one can speculate that Theorem 1.2 might be a special case of a more general phenomenon. For any sufficiently smooth curve $C$, denote by $\kappa(P)$ the curvature at the point $P \in C$. Then we have
\[
\min_{P \in \partial B_p} \kappa(P) = (p - 1)2^{1/p - 1/2}.
\]
Hence, the factor \((p - 1)^{-1/2}\) of (1.3) and (1.4) can be interpreted in terms of the minimum value of the curvature of \(\partial B_p\). Theorem 1.2 then suggests that the following conjecture might be true.

**Conjecture 1.5.** Let \(K \subset \mathbb{R}^2\) be a convex compact set with piece-wise smooth boundary. Set

\[
\nu = \min_{P \in \partial K} \kappa(P),
\]

then there exists an absolute constant \(C_1 > 0\) such that

\[
\sup_{\omega \in \mathbb{R}^2} |\omega|^{3/2} |\hat{\chi}_K(\omega)| \leq \frac{C_1}{\sqrt{\nu}}.
\]

Moreover, (1.4) is sharp in the sense that there exists an absolute constant \(C_2 > 0\) and a sequence \(\{\omega_n\} \subset \mathbb{R}^2\) with \(|\omega_n| \to \infty\) such that

\[
|\omega_n|^{3/2} |\hat{\chi}_K(\omega_n)| \geq \frac{C_2}{\sqrt{\nu}} + o(1).
\]

We have made no attempt to either prove or disprove the above conjecture for any \(K \neq B_p\). We still found it worthwhile to state the conjecture here. If Conjecture 1.5 is true, then it would provide another look at the condition that the curvature is non-vanishing in order for (1.1) to hold.

2. **Auxiliary results**

Let \(\omega = (\alpha, \beta) \in \mathbb{R}^2\). The Fourier transform of a function \(f \in L^1(\mathbb{R}^2)\) is defined by

\[
\hat{f}(\omega) = \hat{f}(\alpha, \beta) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} e^{-i(x\alpha + y\beta)} f(x, y) dxdy.
\]

We first derive some simple formulae for the Fourier transform (2.1) of \(\chi_{B_p}\).

**Lemma 2.1.** Define \(\varphi_p : [0, 1] \to [0, 1]\) by

\[
\varphi_p(x) = (1 - x^p)^{1/p}.
\]

If \(\beta \neq 0\), then

\[
\hat{\chi}_{B_p}(\alpha, \beta) = \frac{2}{\pi \beta} \int_0^1 \cos(\alpha x) \sin(\beta \varphi_p(x)) dx.
\]

If \(\alpha \neq 0\), then

\[
\hat{\chi}_{B_p}(\alpha, \beta) = \frac{2}{\pi \alpha} \int_0^1 \cos(\beta y) \sin(\alpha \varphi_p(y)) dy.
\]

**Proof.** We prove (2.3) first. By Fubini’s theorem,

\[
2\pi \hat{\chi}_{B_p}(\alpha, \beta) = \iint_{B_p} e^{-i(x\alpha + y\beta)} dxdy = \int_{-1}^1 e^{-ix\alpha} \left( \int_{-(1-|x|^p)^{1/p}}^{(1-|x|^p)^{1/p}} e^{-iy\beta} dy \right) dx
\]

Using Euler’s formula for complex exponentials together with the fact that \(\cos\) is even and \(\sin\) is odd, we get

\[
\int_{-1}^1 e^{-ix\alpha} \left( \int_{-(1-|x|^p)^{1/p}}^{(1-|x|^p)^{1/p}} e^{-iy\beta} dy \right) dx = 2 \int_{-1}^1 e^{-ix\alpha} \left( \int_0^{\varphi_p(|x|)} \cos(y\beta) dy \right) dx,
\]
where $\varphi_p$ is defined by (2.2). Computing the inner integral yields

$$
2\pi \hat{\chi}_{B_p}(\alpha, \beta) = \frac{1}{\beta} \int_{-1}^{1} e^{-ix\alpha} \sin(\beta \varphi_p(|x|)) dx = \frac{4}{\beta} \int_{0}^{1} \cos(\alpha x) \sin(\beta \varphi_p(x)) dx
$$

where the last equality follows by again using Euler’s formula together with the fact that $x \mapsto \sin(\beta \varphi_p(|x|))$ is an even function for any $\beta$. To prove (2.4), one simply reverses the order of integration.

**Remark 2.2.** Below, we shall use (2.3) and (2.4) to estimate $\hat{\chi}_{B_p}(\alpha, \beta)$. It is clear from either formula that $\hat{\chi}_{B_p}$ is even in both arguments, in the sense that $\hat{\chi}_{B_p}(\pm \alpha, \pm \beta) = \hat{\chi}_{B_p}(\alpha, \beta)$ for any permutations of signs of the arguments at the right-hand side. Hence, we may assume that $\alpha, \beta \geq 0$. Furthermore, thanks to the symmetry of (2.3) and (2.4), we may without loss of generality assume that $\beta \geq \alpha$.

**Remark 2.3.** Using Lemma 2.1, it is easy to compute $\hat{\chi}_{B_1}(\alpha, \beta)$. Assume that $\beta \geq \alpha > 0$, the case $\alpha = 0$ is even simpler. The calculation only uses (2.3) and the trigonometric identity

$$
2 \cos(x) \sin(y) = \sin(y + x) + \sin(y - x).
$$

Indeed,

$$
\hat{\chi}_{B_1}(\alpha, \beta) = \frac{2}{\pi \beta} \int_{0}^{1} \cos(\alpha x) \sin(\beta(1 - x)) dx = \frac{1}{\pi \beta} \int_{0}^{1} (\sin(x(\alpha - \beta) + \beta) + \sin(\beta - x(\alpha + \beta))) dx = \cos(\beta) - \cos(\alpha) \left( \frac{1}{\alpha - \beta} - \frac{1}{\alpha + \beta} \right) = \frac{2(\cos(\beta) - \cos(\alpha))}{\pi(\beta^2 - \alpha^2)}
$$

Since $|\cos(\beta) - \cos(\alpha)| \leq |\beta - \alpha|$, we clearly have

$$
|\hat{\chi}_{B_1}(\alpha, \beta)| \leq \frac{2}{\pi} \frac{1}{|\alpha \beta|}.
$$

The estimate (2.6) is of course only useful for large $|\alpha \beta|$. In general, (2.6) cannot be improved. Indeed, take

$$
\alpha_n = 2\pi n + \frac{\pi}{2}, \quad \beta_n = 2\pi n + \frac{\pi}{2} + \varepsilon.
$$

for $n \in \mathbb{N}$ and $\varepsilon > 0$. Then $\beta_n - \alpha_n = \varepsilon$ and $\cos(\beta_n) - \cos(\alpha_n) = \varepsilon + O(\varepsilon^3)$, by Taylor’s theorem. By taking $\varepsilon > 0$ small enough, we have

$$
|\hat{\chi}_{B_1}(\alpha_n, \beta_n)| = \left| \frac{-2 \varepsilon + O(\varepsilon^3)}{\pi \varepsilon (\alpha_n + \beta_n)} \right| \geq \frac{1}{\pi |(\alpha_n, \beta_n)|}.
$$

The next lemma provides us with some useful facts about the function (2.2).

**Lemma 2.4.** Let $\varphi_p$ be defined by (2.2). The function $|\varphi''_p|$ has a unique minimum $x^* = x^*(p)$ on $[0, 1]$. The minimum value satisfies

$$
|\varphi''_p(x^*)| = \min_{0 \leq x \leq 1} |\varphi''_p(x)| = (p - 1)m(p),
$$

where
where $m(p)$ is a decreasing function on $[1, 2]$ with $m(1) = 4$, $m(2) = 1$. Further,

\[(2.8)\]
\[\frac{-1}{\varphi_p'(x^*)} \geq 1\]

and

\[(2.9)\]
\[\lim_{p \to 1^+} \frac{-1}{\varphi_p'(x^*)} = 1.\]

Proof. We have

\[(2.10)\]
\[\varphi_p'(x) = -x^{p-1}(1 - x^p)^{1/p-1},\]

\[(2.11)\]
\[\varphi_p''(x) = -(p - 1)x^{p-2}(1 - x^p)^{1/p-2},\]

and

\[(2.12)\]
\[\varphi_p^{(3)}(x) = -(p - 1)x^{p-3}(1 - x^p)^{1/p-3}(x^p(p + 1) + p - 2).\]

It is clear that the third derivative (2.12) has only the zero

\[(2.13)\]
\[x^* \equiv x^*(p) = \left(\frac{2 - p}{p + 1}\right)^{1/p}\]
on $(0, 1)$. Further, $\varphi_p^{(3)}(x) > 0$ for $x < x^*$ and $\varphi_p^{(3)}(x) < 0$ for $x > x^*$. Hence, $\varphi_p''$ has a unique interior maximum at $x^*$. Since $\varphi_p''$ is strictly negative, it follows that $|\varphi_p''|$ has a unique interior minimum at $x^*$. Inserting the expression (2.13) into (2.11) and taking absolute value yields

\[|\varphi_p''(x^*)| = (p - 1)(2 - p)^{1-2/p}(2p - 1)^{1/p-2}(p + 1)^{1+1/p}.\]

Define

\[m(p) = (2 - p)^{1-2/p}(2p - 1)^{1/p-2}(p + 1)^{1+1/p} \quad \text{for } p \in [1, 2),\]

and

\[m(2) = \lim_{p \to 2^-} m(p).\]

Clearly $m(1) = 4$. By logarithmic differentiation,

\[\frac{dm}{dp} = \frac{m(p)}{p^2} (-\ln(p + 1) - \ln(2p - 1) + 2 \ln(2 - p)) < 0\]

for $p \in (1, 2)$. To demonstrate $m(2) = 1$, we calculate the limit defining $m(2)$:

\[\lim_{p \to 2^-} m(p) = \lim_{p \to 2^-} (2 - p)^{1-2/p} \times 3^{-3/2} \times 3^{3/2}\]

\[= \lim_{p \to 2^-} \exp \left( \left(1 - \frac{2}{p}\right) \ln(2 - p) \right)\]

\[= \lim_{t \to 0^+} \exp \left( \frac{-t \ln(t)}{2 - t} \right) = \exp(0) = 1.\]

To prove (2.8), we insert the expression (2.13) into (2.10) and get

\[\varphi_p'(x^*) = -\left(\frac{2 - p}{2p - 1}\right)^{1-1/p} \quad \text{implies} \quad \frac{-1}{\varphi_p'(x^*)} = \left(\frac{2p - 1}{2 - p}\right)^{1-1/p}.\]

Since $p \geq 1$, we have $(2p - 1)/(2 - p) \geq 1$, thus proving (2.8). Finally, (2.9) follows by direct evaluation.
We shall need van der Corput’s lemma, which is a fundamental tool in the theory of oscillatory integrals, see e.g. [10, Chapter VIII, §1.2]. See also [9] and the references given therein for a discussion concerning sharp constants. We mention that Lemma 2.5 below is a special case of van der Corput’s lemma, sufficient for our purpose of deriving (1.3).

**Lemma 2.5** (van der Corput’s lemma). Let \( \psi : [a, b] \to \mathbb{R} \) be smooth on \((a, b)\) and let \( r > 0 \).

If \( \psi \) is monotone and \(|\psi'(x)| \geq \lambda\) for all \( x \in (a, b) \), then

\[
\left| \int_a^b \sin(r\psi(x)) \, dx \right| \leq \frac{2}{r\lambda}.
\]

If \(|\psi''(x)| \geq \lambda\) for all \( x \in (a, b) \), then

\[
\left| \int_a^b \sin(r\psi(x)) \, dx \right| \leq \frac{6}{\sqrt{r\lambda}}.
\]

**Remark 2.6.** Below we shall use Lemma 2.5 for both the interval \([a, b] = [0, 1]\) as well as certain sub-intervals of \([0, 1]\); it is important that the constants at the right-hand sides of (2.14) and (2.15) are independent of the interval \([a, b]\).

The next lemma is closely related to the ’method of stationary phase’ [10, Chapter VIII, §1.3]. We shall use it as a kind of ’reverse van der Corput inequality’ to obtain (1.4). Rather than deriving Lemma 2.7 from general methods presented in e.g. [10], we give a self-contained proof.

**Lemma 2.7.** Let \( \psi : [0, 1] \to \mathbb{R} \) be a smooth function on \((0, 1)\). Assume that there exists \( x_0 \in (0, 1) \) such that \( \psi(x_0) = \psi'(x_0) = 0 \),

\[
|\psi''(x_0)| = \min_{0 \leq x \leq 1} |\psi''(x)| = \lambda > 0,
\]

and \( \psi^{(3)} \) is bounded in a neighbourhood of \( x_0 \). Then

\[
\left| \int_0^1 \sin(r\psi(x)) \, dx \right| = \frac{\sqrt{\pi}}{\sqrt{r\lambda}} + o(r^{-1/2})
\]

as \( r \to \infty \).

**Proof.** Without loss of generality, we may assume in the proof that \( \psi''(x) > 0 \) for \( x \in [0, 1] \), so that the absolute values can be dropped from (2.10). If \( \psi''(x) < 0 \) for \( x \in [0, 1] \), then we simply replace \( \psi \) with \(-\psi\) and observe that the left-hand side of (2.17) remains unchanged, since the sine function is odd.

Take small \( \varepsilon > 0 \) (to be specified later). Write \([0, 1] = \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3\) where \( \mathcal{J}_1 = [0, x_0 - \varepsilon] \), \( \mathcal{J}_2 = [x_0 - \varepsilon, x_0 + \varepsilon] \) and \( \mathcal{J}_3 = [x_0 + \varepsilon, 1] \). Denote further

\[
I_i = \int_{\mathcal{J}_i} \sin(r\psi(x)) \, dx, \quad i = 1, 2, 3.
\]

The main term is \( I_2 \); we estimate it first. By Taylor’s formula,

\[
\psi(x) = \frac{\lambda(x - x_0)^2}{2} + O((x - x_0)^3).
\]
Using (2.18), a change of variable, and the mean value theorem, we get
\[
I_2 = \int_{-\varepsilon}^{\varepsilon} \sin \left( \frac{r \lambda x^2}{2} + O(rx^3) \right) dx
\]
(2.19)
\[
= \int_{-\varepsilon}^{\varepsilon} \sin \left( \frac{r \lambda x^2}{2} \right) dx + O(\varepsilon^4)
\]
Performing the change of variable \( t = x \sqrt{r \lambda} \) in the integral at the right-hand side of (2.19) gives
(2.20)
\[
I_2 = \sqrt{\frac{2}{r \lambda}} \int_{-C \varepsilon \sqrt{r}}^{C \varepsilon \sqrt{r}} \sin(t^2) dt + O(r\varepsilon^4),
\]
where \( C = \sqrt{\lambda/2} \). We proceed to estimate \( I_1 \) and \( I_3 \). Since \( \psi''(x) > 0 \) for all \( x \in [0, 1] \), the derivative \( \psi' \) is increasing. Hence, \( \psi'(x) \leq \psi'(x_0 - \varepsilon) < 0 \) for \( x \in J_1 \).
Furthermore,
\[
-\psi'(x_0 - \varepsilon) = \int_{x_0 - \varepsilon}^{x_0} \psi''(x) dx \geq \lambda \varepsilon
\]
whence
\[
\psi'(x) \leq -\lambda \varepsilon, \quad x \in J_1.
\]
Consequently, \( \psi' \) satisfies the lower bound \( |\psi'(x)| \geq \lambda \varepsilon \) for \( x \in J_1 \). Since \( \psi' \) also is increasing on \( J_1 \), (2.14) and Remark 2.6 yield
(2.21)
\[
|I_1| \leq \frac{2}{r \lambda \varepsilon}.
\]
A similar argument shows that (2.21) also holds for \( I_3 \). Hence, by (2.20) and (2.21)
\[
\int_0^1 \sin(r \psi(x)) dx = \sqrt{\frac{2}{r \lambda}} \int_{-C \varepsilon \sqrt{r}}^{C \varepsilon \sqrt{r}} \sin(t^2) dt + O(r \varepsilon^4) + I_1 + I_3
\]
where \( I_1, I_3 \) satisfy the bound (2.21). To deal with the first term of the above equality, we use asymptotics for the Fresnel integral (see [1, Chapter 7]):
(2.22)
\[
\int_{-m}^{m} \sin(x^2) dx = \sqrt{\frac{\pi}{2}} + O \left( \frac{1}{m} \right)
\]
as \( m \to \infty \). Hence, under the assumption that \( \varepsilon \sqrt{r} \) is large, (2.22) yields
(2.23)
\[
\sqrt{\frac{2}{r \lambda}} \int_{-C \varepsilon \sqrt{r}}^{C \varepsilon \sqrt{r}} \sin(t^2) dt = \sqrt{\frac{\pi}{r \lambda}} + O \left( \frac{1}{r \lambda \varepsilon} \right).
\]
Using (2.20), (2.21) and (2.23), we get
\[
\int_0^1 \sin(r \psi(x)) dx = \sqrt{\frac{\pi}{r \lambda}} + O(r \varepsilon^4) + O \left( \frac{1}{r \lambda \varepsilon} \right),
\]
again under the assumption that \( \varepsilon \sqrt{r} \) is large. Take \( \varepsilon = r^{-7/16} \), in this case \( \varepsilon \sqrt{r} = r^{1/16} \to \infty \). Further, as \( r \to \infty \),
\[
re^4 = r^{1-7/4} = r^{-3/4} = o(r^{-1/2})
\]
and
\[
\frac{1}{r \varepsilon} = \frac{1}{r^{1-7/16}} = \frac{1}{r^{9/16}} = o(r^{-1/2}).
\]
This concludes the proof of (2.17). \( \square \)
3. Proof of Theorem 1.2

In this section, we prove our main result, starting with (1.3). We shall use (2.3), by Remark 2.2, we may take \( \beta \geq \alpha \). Furthermore, proving (1.3) in the case \( \alpha = 0 \) is easier, hence we assume

\[
\beta \geq \alpha > 0.
\]

It is useful to express the frequency variable \( \omega = (\alpha, \beta) \) in terms of polar coordinates, i.e.

\[
\alpha = r \cos(\theta), \quad \beta = r \sin(\theta).
\]

Then (3.1) translates to

\[
r > 0, \quad \theta \in [\pi/4, \pi/2).
\]

Abusing notation slightly, we write

\[
\hat{\chi}_{Bp}(r, \theta) = \hat{\chi}_{Bp}(r \cos(\theta), r \sin(\theta)),
\]

and by (2.3)

\[
\hat{\chi}_{Bp}(r, \theta) = \frac{2}{\pi r \sin(\theta)} \int_0^1 \cos(r \cos(\theta) x) \sin(r \sin(\theta) \phi_p(x)) \, dx.
\]

Using (2.5), we get

\[
(3.2) \quad \hat{\chi}_{Bp}(r, \theta) = \frac{1}{\pi r \sin(\theta)} \int_0^1 \left[ \sin(r \psi_p(x, \theta)) + \sin(r \tilde{\psi}_p(x, \theta)) \right] \, dx
\]

for \((r, \theta) \in (0, \infty) \times [\pi/4, \pi/2)\) where

\[
\psi_p(x; \theta) = \cos(\theta) x + \sin(\theta) \varphi_p(x),
\]

\[
\tilde{\psi}_p(x; \theta) = -\cos(\theta) x + \sin(\theta) \varphi_p(x).
\]

**Remark 3.1.** The functions \( \psi_p, \tilde{\psi}_p \) are considered as functions of \( x \in [0, 1] \); \( \theta \) is viewed as a parameter.

**Proof of (3.2).** By (3.2),

\[
|\omega|^{3/2} |\hat{\chi}_{Bp}(\omega)| = \sqrt{\pi} \left| \int_0^1 \left[ \sin(r \psi_p(x, \theta)) + \sin(r \tilde{\psi}_p(x, \theta)) \right] \, dx \right|
\]

Using the above identity, the triangle inequality and the fact that \( 2^{-1/2} \leq \sin(\theta) \leq 1 \) (by (3.1)), we see that it suffice to show that the existence of a constant \( C_1 > 0 \) such that

\[
(3.3) \quad \sqrt{\pi} \left| \int_0^1 \sin(r \psi_p(x; \theta)) \, dx \right| + \sqrt{\pi} \left| \int_0^1 \sin(r \tilde{\psi}_p(x; \theta)) \, dx \right| \leq \frac{C_1}{\sqrt{p - 1}}
\]

for any \( p \in (1, 2] \) and \((r, \theta) \in (0, \infty) \times [\pi/4, \pi/2)\). We only estimate the first term at the left-hand side (3.3); the argument is the same for the second term. Note that

\[
\min_{0 \leq x \leq 1} \left| \psi_p''(x; \theta) \right| = \left| \sin(\theta) \right| \min_{0 \leq x \leq 1} \left| \varphi_p''(x) \right| \geq \frac{m(p)}{\sqrt{2}} \sqrt{p - 1} \geq \frac{(p - 1)}{\sqrt{2}},
\]

by (2.7). Hence, by (2.15)

\[
\sqrt{\pi} \left| \int_0^1 \sin(r \psi_p(x; \theta)) \, dx \right| \leq \sqrt{\pi} \frac{6}{\sqrt{r(p - 1)/\sqrt{2}}} = \frac{6^{1/4}}{\sqrt{p - 1}}
\]
Thus, we have shown that (3.3) holds for any $p \in (1, 2]$ and any $(r, \theta) \in (0, \infty) \times [\pi/4, \pi/2]$, with $C_1 = 12(2)^{1/4}$; this concludes the proof of (1.3). □

**Proof of 1.4.** Fix $p \in (1, 2)$, we shall describe how to construct the sequence $\{\omega_n\}$ of (1.4). Let $x^* \in (0, 1)$ be the point provided by Lemma 2.4. Define

$$\theta^* = \arctan \left( \frac{-1}{|\varphi_p'(x^*)|} \right)$$

by (2.8) we have $\theta^* \in [\pi/4, \pi/2]$. Note also that by taking $\theta = \theta^*$ in $\psi_p(x; \theta)$, we have $\psi_p'(x^*; \theta^*) = 0$. Indeed,

$$\psi_p'(x^*; \theta^*) = \cos(\theta^*) + \sin(\theta^*)\varphi_p'(x^*) = \cos(\theta^*) + \sin(\theta^*) \frac{-1}{\tan(\theta^*)} = 0.$$

Define now $\psi(x) = \psi_p(x; \theta^*) - \psi_p(x^*; \theta^*)$, by construction $\psi$ satisfies the conditions of Lemma 2.17 with $x_0 = x^*$, and

$$\min_{0 \leq x \leq 1} |\psi''(x)| = |\psi''(x^*)| = \sin(\theta^*)(p-1)m(p) \leq 4\sin(\theta^*)(p-1).$$

Hence,

$$\int_0^1 \sin(r\psi(x))dx = \frac{\sqrt{\pi}}{\sqrt{r \sin(\theta^*)(p-1)m(p)}} + o(r^{-1/2})$$

(3.4)

$$\geq \frac{2\sqrt{\pi}}{\sqrt{r \sin(\theta^*)(p-1)}} + o(r^{-1/2}).$$

Further,

$$\tilde{\psi}_p'(x; \theta^*) = -\cos(\theta^*) + \sin(\theta^*)\varphi_p'(x),$$

and since $\varphi_p'(x) < 0$, $\tilde{\psi}_p(x; \theta^*)$ is an decreasing function of $x \in [0, 1]$. Furthermore,

$$\tilde{\psi}_p'(x; \theta^*) \leq -\cos(\theta^*).$$

By (2.9), $\theta^* \to \pi/4$ as $p \to 1+$. Hence, we may assume that $p$ is sufficiently close to 1 in order to have $\theta^* \leq \pi/3$. Therefore, $\tilde{\psi}_p'(x; \theta^*) \leq -\cos(\theta^*) \leq -1/2$ so $|\tilde{\psi}_p'(x; \theta^*)| \geq 1/2$. By (2.15)

(3.5)

$$\left| \int_0^1 \sin(r\tilde{\psi}_p(x; \theta^*))dx \right| \leq \frac{4}{r}.$$ 

Using (3.2), (3.5) together with the definition of $\psi$, we have

$$r \sin(\theta^*) \bar{\chi}_{B_p}(r; \theta^*) = \int_0^1 \sin(r\psi(x) + r\psi_p(x^*; \theta^*))dx + \int_0^1 \sin(r\tilde{\psi}_p(x; \theta^*))dx = \int_0^1 \sin(r\psi(x) + r\psi_p(x^*, \theta^*))dx + O \left( \frac{1}{r} \right)$$

Take now $r_n = 2\pi n/\psi_p(x^*, \theta^*)$, then

$$\sin(r_n\psi(x) + r_n\psi_p(x^*, \theta^*)) = \sin(r_n\psi(x))$$
and by (3.3)

\[ r_n \sin(\theta^*) \hat{\chi}_{B_p}(r_n, \theta^*) = \int_0^1 \sin(r_n \psi(x))dx + O\left(\frac{1}{r_n}\right) \]

\[ \geq \frac{2\sqrt{\pi}}{\sqrt{r_n \sin(\theta^*)(p - 1)}} + o(r_n^{-1/2}) + O(r_n^{-1}) \]

Consequently, since \( \sin(\theta^*) \geq 1/\sqrt{2} \), we have

\[ r_n^{3/2} \hat{\chi}_{B_p}(r_n, \theta^*) \geq \frac{2(2^{1/2})^{3/2}\sqrt{\pi}}{\sqrt{p - 1}} + o(1) \]

as \( n \to \infty \). Finally, setting \( \omega_n = r_n(\cos(\theta^*), \sin(\theta^*)) \), the above relation states exactly that

\[ |\omega_n|^{3/2}\hat{\chi}_{B_p}(\omega_n)| \geq \frac{C_2}{\sqrt{p - 1}} + o(1) \]

which proves (1.4) with \( C_2 = 2^{1+3/4}\sqrt{\pi} \).

\[ \square \]

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